On entropic quantities related to the classical capacity of infinite dimensional quantum channels

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1 Introduction

In the study of the classical capacity of finite dimensional quantum channels the three quantities play a basic role, namely, the output entropy, its convex hull and their difference, called the $\chi$-function. The Holevo capacity of a channel with constraints defined by some subset of states is equal to the maximal value of the $\chi$-function on this subset [11]. The proof of the existence of optimal ensembles is based on the analysis of the convex hull of the output entropy [24], [19]. It also makes possible to apply convex analysis approach to the additivity problem [2] and provides an equivalent formulation of the strong additivity of the Holevo capacity for two channels [11] in terms of the superadditivity of the convex hull of the output entropy for these channels.

In this paper we consider generalizations of the $\chi$-function and of the convex hull of the output entropy to the infinite dimensional case developing results in [20], [12].

It is shown that the $\chi$-function of an arbitrary channel is a concave lower semicontinuous function on the whole state space with natural chain properties (propositions 1-2), having continuous restriction to any set of continuity of the output entropy (proposition 7). This implies continuity of the $\chi$-function for the Gaussian channels with the power constraint (corollary 3 and notes below). For the $\chi$-function the analog of Simon’s dominated convergence theorem for quantum entropy [22] (corollary 1) is also obtained.

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These results provide the proof of new version of theorem 2 in [20], where it is stated that the subadditivity of the $\chi$-function for all finite dimensional channels implies the subadditivity of the $\chi$-function for all infinite dimensional channels.

Since in the finite dimensional case the convex hull of the output entropy is a continuous function it coincides with its convex closure [17] (lower envelope in terms of [1]). In the infinite dimensional case this coincidence does not hold and it seems reasonable to consider the convex closure of the output entropy instead of its convex hull. The explicit integral representations of the convex closure of the output entropy of an arbitrary infinite dimensional channel is obtained and its properties are explored (propositions 3-6, corollary 2). The main technical problem here is non-compactness of the state space which makes impossible to apply the general theory of integral representation on convex compact sets [1,5]. The main ingredient of this consideration is the criterion of compactness of a subset of measures as well as other results obtained in [12]. It is shown that the convex closure of the output entropy coincides with the convex hull of the output entropy on the convex set of states with finite output entropy. Thus the representation of the $\chi$-function as a difference between the output entropy and its convex closure remains valid on this set. Similarly to the case of the $\chi$-function, it is shown that the convex closure of the output entropy has continuous restriction to any set of continuity of the output entropy (proposition 7).

The obtained properties of the convex closure of the output entropy make it possible to generalize to the infinite dimensional case the convex duality approach to the additivity problem proposed in [2] (the theorem in section 6).

A very important particular case of the convex closure (= convex hull) of the output entropy of a finite dimensional channel is the notion of the entanglement of formation (EoF) [7] of a state in bipartite system. Indeed, the EoF coincides with the convex closure of the output entropy of a partial trace channel from the state space of bipartite system onto the state space of single subsystem. It seems natural to define entanglement of formation of a state in tensor product of two infinite dimensional systems in the same way as the convex closure of the output entropy of a partial trace channel. This definition guarantees such properties of the EoF as convexity, lower semicontinuity on the whole state space and continuity on the subsets with constrained mean energy. It is shown that this definition coincides with the conventional definition of the EoF considered in [7,8] for all states in...
bipartite system having marginal states with finite entropy.

2 Preliminaries

Let \( \mathcal{H} \) be a separable Hilbert space, \( \mathcal{B}(\mathcal{H}) \) the algebra of all bounded operators in \( \mathcal{H} \), \( \mathcal{K}(\mathcal{H}) \) the Banach space of all trace-class operators with the trace norm \( \| \cdot \|_1 \) and \( \mathcal{S}(\mathcal{H}) \) the closed convex subset of \( \mathcal{K}(\mathcal{H}) \) consisting of all density operators (states) in \( \mathcal{H} \), which is complete separable metric space with the metric defined by the norm. We shall use the fact that convergence of a sequence of states to a state in the weak operator topology is equivalent to convergence of this sequence to this state in the trace norm [4].

A finite collection \( \{ \pi_i, \rho_i \} \) of states \( \rho_i \) with the corresponding probabilities \( \pi_i \) is conventionally called ensemble. The state \( \bar{\rho} = \sum_i \pi_i \rho_i \) is called the average state of the ensemble.

We refer to [6], [18] for definitions and facts concerning probability measures on separable metric spaces. In particular we denote \( \text{supp}(\pi) \) support of measure \( \pi \) as defined in [18]. Following [12] we consider an arbitrary Borel probability measure \( \pi \) on \( \mathcal{S}(\mathcal{H}) \) as generalized ensemble and the barycenter

\[
\bar{\rho}(\pi) = \int_{\mathcal{S}(\mathcal{H})} \rho \pi(d\rho).
\]

of the measure \( \pi \) as the average state of this ensemble. In this notations the conventional ensembles correspond to measures with finite support.

Denote by \( \mathcal{P} \) the convex set of all probability measures on \( \mathcal{S}(\mathcal{H}) \) equipped with the topology of weak convergence [6] and by \( \mathcal{P}_A \) the convex set of all probability measures with barycenters contained in \( A \subseteq \mathcal{S}(\mathcal{H}) \). It is easy to see (due to the result of [4]) that \( \pi \mapsto \bar{\rho}(\pi) \) is a continuous mapping from \( \mathcal{P} \) onto \( \mathcal{S}(\mathcal{H}) \).

We refer to [1], [17] for definitions and facts from convex analysis. For reader’s convenience all the necessary information is presented in the Appendix.

In what follows \( \log \) denotes the function on \([0, +\infty)\), which coincides with the usual logarithm on \((0, +\infty)\) and vanishes at zero. If \( A \) is a positive finite rank operator in \( \mathcal{H} \), then the entropy is defined as

\[
H(A) = \text{Tr}A \left( I \log \text{Tr}A - \log A \right),
\]
where $I$ is the unit operator in $\mathcal{H}$. If $A, B$ two such operators then the relative entropy is defined as

$$H(A \parallel B) = \text{Tr}(A \log A - A \log B + B - A)$$

(2)

provided $\text{ran} A \subseteq \text{ran} B$, and $H(A \parallel B) = +\infty$ otherwise (throughout this paper ran denotes the closure of the range of an operator in $\mathcal{H}$).

This definitions can be extended to arbitrary positive $A, B \in \mathfrak{T}(\mathcal{H})$ in the following way:

$$H(A) = \lim_{n \to +\infty} H(P_n AP_n); \quad H(A \parallel B) = \lim_{n \to +\infty} H(P_n AP_n \parallel P_n BP_n),$$

where $\{P_n\}$ is an arbitrary sequence of finite dimensional projectors monotonously increasing to the unit operator $I$. In [13] it is shown that the both sequences in the above limit expressions are nondecreasing and that these limits coincide with the values of the entropy and of the relative providing by the conventional definitions.

We denote by $\sum_i \pi_i \rho_i$ a finite convex decomposition as distinct from countable decomposition $\sum_i \pi_i \rho_i$.

3 The $\chi$-function

Let $\Phi : \mathfrak{S}(\mathcal{H}) \mapsto \mathfrak{S}(\mathcal{H}')$ be an arbitrary quantum channel. The output entropy $H_\Phi(\rho) \equiv H(\Phi(\rho))$ of the channel $\Phi$ is nonnegative lower semicontinuous concave function on the set $\mathfrak{S}(\mathcal{H})$. For given $\rho \in \mathfrak{S}(\mathcal{H})$ the quantity $\chi\Phi(\rho)$ (the Holevo capacity of the $\{\rho\}$-constrained channel $\Phi$ [11],[20]) is defined as

$$\chi\Phi(\rho) = \sup_{\sum_i \pi_i \rho_i = \rho} \sum_i \pi_i H(\Phi(\rho_i) \parallel \Phi(\rho)).$$

(3)

It is shown in [12] that

$$\chi\Phi(\rho) = \sup_{\pi \in \mathcal{P}(\rho)} \int_{\mathfrak{S}(\mathcal{H})} H(\Phi(\sigma) \parallel \Phi(\rho)) \pi(\sigma) d\sigma,$$

(4)

where $\mathcal{P}(\rho)$ is the set of all probability measures on $\mathfrak{S}(\mathcal{H})$ with the barycenter $\rho$, and that under the condition $H_\Phi(\rho) < +\infty$ the supremum in (4) is achieved on some measure supported by pure states.
Definition 1. A measure $\pi_0$ with the barycenter $\rho_0$ supported by the set of pure states and such that

$$\chi_{\Phi}(\rho_0) = \int_{\mathcal{S}(\mathcal{H})} H(\Phi(\sigma)\|\Phi(\rho_0))\pi_0(d\sigma)$$

is called a $\chi_{\Phi}$-optimal measure for a state $\rho_0$.

Note that $H_{\Phi}(\rho) = +\infty$ does not imply $\chi_{\Phi}(\rho) = +\infty$. Indeed, it is easy to construct FI-channel $\Phi^1$ such that $H_{\Phi}(\rho) = +\infty$ for any $\rho \in \mathcal{S}(\mathcal{H})$. On the other hand, by the monotonicity property of the relative entropy [14]

$$\sum_i \pi_i H(\Phi(\rho_i)\|\Phi(\rho)) \leq \sum_i \pi_i H(\rho_i\|\rho) \leq \log \dim \mathcal{H} < +\infty$$

for arbitrary ensemble $\{\pi_i, \rho_i\}$, and hence $\chi_{\Phi}(\rho) \leq \log \dim \mathcal{H} < +\infty$ for any $\rho \in \mathcal{S}(\mathcal{H})$.

For arbitrary state $\rho$ such that $H_{\Phi}(\rho) < +\infty$ the $\chi$-function has the following representation

$$\chi_{\Phi}(\rho) = H_{\Phi}(\rho) - \text{conv} H_{\Phi}(\rho), \quad (5)$$

where $\text{conv} H_{\Phi}(\rho)$ is a convex hull of the output entropy (see the Appendix)

$$\text{conv} H_{\Phi}(\rho) = \inf_{\sum_i \pi_i \rho_i = \rho} \sum_i \pi_i H_{\Phi}(\rho_i). \quad (6)$$

In the finite dimensional case the output entropy $H_{\Phi}(\rho)$ and its convex hull $\text{conv} H_{\Phi}(\rho)$ are continuous concave and convex functions on $\mathcal{S}(\mathcal{H})$ correspondingly and the representation [5] is valid for all states. It follows that in this case the function $\chi_{\Phi}(\rho)$ is continuous and concave on $\mathcal{S}(\mathcal{H})$.

In the infinite dimensional case the output entropy $H_{\Phi}(\rho)$ is only lower semicontinuous and, hence, the function $\chi_{\Phi}(\rho)$ is not continuous even in the case of the noiseless channel $\Phi$, for which $\chi_{\Phi}(\rho) = H_{\Phi}(\rho)$. But it turns out that the function $\chi_{\Phi}(\rho)$ for arbitrary channel $\Phi$ has properties similar to the properties of the output entropy $H_{\Phi}(\rho)$.

Proposition 1. The function $\chi_{\Phi}(\rho)$ is nonnegative concave and lower semicontinuous function on $\mathcal{S}(\mathcal{H})$.

\footnote{FI-channel is defined in [20] as a channel from finite dimensional system into infinite dimensional one.}
The proof of this proposition is based on the following lemma.

**Lemma 1.** Let \( \{ \pi_i, \rho_i \} \) be an arbitrary ensemble of \( m \) states with the average state \( \rho \) and let \( \{ \rho_n \} \) be an arbitrary sequence of states converging to the state \( \rho \). There exists the sequence \( \{ \pi_i^n, \rho_i^n \} \) of ensemble of \( m \) states such that

\[
\lim_{n \to +\infty} \pi_i^n = \pi_i, \quad \lim_{n \to +\infty} \rho_i^n = \rho_i, \quad \text{and} \quad \rho_n = \sum_{i=1}^{m} \pi_i^n \rho_i^n.
\]

**Proof.** Without loss of generality we may assume that \( \pi_i > 0 \) for all \( i \).

Let \( \mathcal{D} \subseteq \mathcal{H} \) be the support of \( \rho = \sum_{i=1}^{m} \pi_i \rho_i \) and \( P \) be the projector onto \( \mathcal{D} \). Since \( \rho_i \leq \pi_i^{-1} \rho \) we have

\[
0 \leq A_i \equiv \rho^{1/2} \rho_i \rho^{-1/2} \leq \pi_i^{-1} I,
\]

where we denote by \( \rho^{-1/2} \) the generalized (sometimes called Moore-Penrose) inverse of the operator \( \rho^{1/2} \) (equal 0 on the orthogonal complement to \( \mathcal{D} \)).

Consider the sequence \( B_i^n = \rho^{1/2} A_i \rho_i^{1/2} + \rho_i^{1/2} (I_{\mathcal{H}} - P) \rho_i^{1/2} \) of operators in \( \mathcal{B} \). Since \( \lim_{n \to +\infty} \rho_n = \rho = P \rho \) in the trace norm, we have

\[
\lim_{n \to +\infty} B_i^n = \rho^{1/2} A_i \rho_i^{1/2} = \rho_i
\]

in the weak operator topology. The last equality implies \( A_i \neq 0 \). Note that

\[
\text{Tr} B_i^n = \text{Tr} A_i \rho_n + \text{Tr} (I_{\mathcal{H}} - P) \rho_n < +\infty \quad \text{and hence}
\]

\[
\lim_{n \to +\infty} \text{Tr} B_i^n = \text{Tr} A_i \rho = \text{Tr} \rho_i = 1.
\]

Denote by \( \rho_i^n = (\text{Tr} B_i^n)^{-1} B_i^n \) a state and by \( \pi_i^n = \pi_i \text{Tr} B_i^n \) a positive number for each \( i \), then \( \lim_{n \to +\infty} \pi_i^n = \pi_i \) and \( \lim_{n \to +\infty} \rho_i^n = \rho_i \) in the weak operator topology and hence, by the result in \([4]\), in the trace norm. Moreover,

\[
\sum_{i=1}^{m} \pi_i^n \rho_i^n = \sum_{i=1}^{m} \pi_i B_i^n = \rho_i^{1/2} \rho^{-1/2} \sum_{i=1}^{m} \pi_i \rho_i \rho_i^{-1/2} \rho_i^{1/2} + \rho_i^{1/2} (I_{\mathcal{H}} - P) \rho_i^{1/2} = \rho_n. \quad \Box
\]

**Proof of the proposition.** Nonnegativity of the \( \chi \)-function is obvious. Show first the concavity property of the \( \chi \)-function. Note that for convex set of states with finite output entropy this concavity easily follows from \([5]\). But to prove concavity on the whole state space we will use a different approach.
Let \( \rho \) and \( \sigma \) be arbitrary states. By definition for arbitrary \( \varepsilon > 0 \) there exist ensembles \( \{\pi_i, \rho_i\}_{i=1}^n \) and \( \{\mu_j, \sigma_j\}_{j=1}^m \) with the average states \( \rho \) and \( \sigma \) correspondingly such that \( \sum_i \pi_i H(\Phi(\rho_i)\|\Phi(\rho)) > \chi_\Phi(\rho) - \varepsilon \) and \( \sum_j \mu_j H(\Phi(\sigma_j)\|\Phi(\sigma)) > \chi_\Phi(\sigma) - \varepsilon \). Taking the mixture

\[
\{(1 - \eta)\pi_1 \rho_1, \ldots, (1 - \eta)\pi_n \rho_n, \eta \mu_1 \sigma_1, \ldots, \eta \mu_m \sigma_m\}, \quad \eta \in [0,1]
\]

of the above two ensembles we obtain the ensemble with the average state \((1 - \eta)\rho + \eta \sigma\). By using Donald’s identity \(3\) we have

\[
\chi_\Phi((1 - \eta)\rho + \eta \sigma) \geq (1 - \eta) \sum_i \pi_i H(\Phi(\rho_i)\|\Phi((1 - \eta)\rho + \eta \sigma))
\]

\[
+ \eta \sum_j \mu_j H(\Phi(\sigma_j)\|\Phi((1 - \eta)\rho + \eta \sigma)) = (1 - \eta) \sum_i \pi_i H(\Phi(\rho_i)\|\Phi(\rho))
\]

\[
+ (1 - \eta) H(\Phi(\rho)\|\Phi((1 - \eta)\rho + \eta \sigma)) + \eta \sum_j \mu_j H(\Phi(\sigma_j)\|\Phi(\sigma))
\]

\[
+ \eta H(\Phi(\sigma)\|\Phi((1 - \eta)\rho + \eta \sigma)) \geq (1 - \eta) \sum_i \pi_i H(\Phi(\rho_i)\|\Phi(\rho))
\]

\[
+ \eta \sum_j \mu_j H(\Phi(\sigma_j)\|\Phi(\sigma)) \geq (1 - \eta) \chi_\Phi(\rho) + \eta \chi_\Phi(\sigma) - 2\varepsilon,
\]

where nonnegativity of the relative entropy was used. Since \( \varepsilon \) can be arbitrary small the concavity property of the \( \chi \)-function is established.

To prove lower semicontinuity of the \( \chi \)-function we have to show

\[
\liminf_{n \to +\infty} \chi_\Phi(\rho_n) \geq \chi_\Phi(\rho_0),
\]

for arbitrary state \( \rho_0 \) and arbitrary sequence \( \rho_n \) converging to this state \( \rho_0 \).

For arbitrary \( \varepsilon > 0 \) let \( \{\pi_i, \rho_i\} \) be an ensemble with the average \( \rho_0 \) such that

\[
\sum_i \pi_i H(\Phi(\rho_i)\|\Phi(\rho_0)) \geq \chi_\Phi(\rho_0) - \varepsilon.
\]

By lemma 1 there exists the sequence of ensembles \( \{\pi_i^n, \rho_i^n\} \) of fixed size such that

\[
\lim_{n \to +\infty} \pi_i^n = \pi_i, \quad \lim_{n \to +\infty} \rho_i^n = \rho_i, \quad \text{and} \quad \rho_n = \sum_{i=1}^m \pi_i^n \rho_i^n.
\]

By definition we have

\[
\liminf_{n \to +\infty} \chi_\Phi(\rho_n) \geq \liminf_{n \to +\infty} \sum_i \pi_i^n H(\Phi(\rho_i^n)\|\Phi(\rho_n))
\]

\[
\geq \sum_i \pi_i H(\Phi(\rho_i)\|\Phi(\rho_0)) \geq \chi_\Phi(\rho_0) - \varepsilon,
\]

for arbitrary state \( \rho_0 \) and arbitrary sequence \( \rho_n \) converging to this state \( \rho_0 \).
where lower semicontinuity of the relative entropy [23] was used. This implies [7] (due to the freedom of the choice of \( \varepsilon \)). □

The similarity of the properties of the functions \( \chi_{\Phi}(\rho) \) and \( H_{\Phi}(\rho) \) is stressed by the following analog of Simon’s dominated convergence theorem for quantum entropy [22].

**Corollary 1.** Let \( \rho_n \) be a sequence of states in \( \mathcal{S}(\mathcal{H}) \), converging to the state \( \rho \) and such that \( \lambda_n \rho_n \leq \rho \) for some sequence \( \lambda_n \) of positive numbers, converging to 1. Then

\[
\lim_{n \to +\infty} \chi_{\Phi}(\rho_n) = \chi_{\Phi}(\rho).
\]

**Proof.** The condition \( \lambda_n \rho_n \leq \rho \) implies decomposition \( \rho = \lambda_n \rho_n + (1 - \lambda_n) \rho' \), where \( \rho' = (1 - \lambda_n)^{-1}(\rho - \lambda_n \rho_n) \). By concavity of the \( \chi \)-function we have

\[
\chi_{\Phi}(\rho) \geq \lambda_n \chi_{\Phi}(\rho_n) + (1 - \lambda_n) \chi_{\Phi}(\rho') \geq \lambda_n \chi_{\Phi}(\rho_n),
\]

which implies \( \limsup_{n \to +\infty} \chi_{\Phi}(\rho_n) \leq \chi_{\Phi}(\rho) \). This and lower semicontinuity of the \( \chi \)-function completes the proof. □

**Remark.** Corollary 1 provides the possibility to approximate the value \( \chi_{\Phi}(\rho) \) for arbitrary state \( \rho \) by the sequence \( \chi_{\Phi}(\rho_n) \), where \( \rho_n \) is the sequence of finite rank approximations of the state \( \rho \) defined by \( \rho_n = \text{Tr}(P_n \rho)^{-1} P_n \rho \), where \( P_n \) is the spectral projector of the state \( \rho \), corresponding to \( n \) maximal eigenvalues. This possibility is used in the proof of new version of theorem 2 in [20]. □

By exploring the properties of the convex closure of the output entropy in section 4 we will establish in section 5 the continuity of the restriction of the \( \chi \)-function to any set of continuity of the output entropy.

We shall also use the following chain properties of the \( \chi \)-function.

**Proposition 2.** Let \( \Phi : \mathcal{S}(\mathcal{H}) \mapsto \mathcal{S}(\mathcal{H}') \) and \( \Psi : \mathcal{S}(\mathcal{H}') \mapsto \mathcal{S}(\mathcal{H}'') \) be two channels. Then

\[
\chi_{\Psi \circ \Phi}(\rho) \leq \chi_{\Phi}(\rho) \quad \text{and} \quad \chi_{\Psi \circ \Phi}(\rho) \leq \chi_{\Psi}(\Phi(\rho)), \quad \forall \rho \in \mathcal{S}(\mathcal{H})
\]

**Proof.** The first inequality follows from the monotonicity property of the relative entropy [14] and [3], while the second one is a direct corollary of the definition (3) of the \( \chi \)-function. □

### 4 The convex closure of the output entropy

In the finite dimensional case the output entropy is finite and the \( \chi \)-function can be represented by (5) as a difference between the output entropy \( H_{\Phi}(\rho) \)
and its convex hull \( \text{conv} H_\Phi(\rho) \). In this case the function \( \text{conv} H_\Phi(\rho) \) is continuous and hence it is closed in terms of convex analysis (see the Appendix). This implies that the function \( \text{conv} H_\Phi(\rho) \) coincides with the convex closure \( \text{conv} H_\Phi(\rho) \) of the output entropy \( H_\Phi(\rho) \).

In the infinite dimensional case the function \( \text{conv} H_\Phi(\rho) \) is not closed even in the case of noiseless channel \( \Phi \). Indeed, \( \text{conv} H_\Phi(\rho) = +\infty \) for any state \( \rho \) with \( H_\Phi(\rho) = +\infty \) (see the proof of lemma 2 below), but such a state \( \rho \) can be represented as a limit of a sequence \( \rho_n \) of finite rank states, for which \( \text{conv} H_\Phi(\rho_n) = 0. \) It follows that \( \text{conv} H_\Phi(\rho) \) is not lower semicontinuous.

It seems natural to suppose that in the finite dimensional case the role of the function \( \text{conv} H_\Phi(\rho) \) is played by the function \( \hat{H}_\Phi(\rho) \). The aim of this section is to confirm this conjecture by exploring properties of the function \( \text{conv} H_\Phi(\rho) \) and its relation to the \( \chi \)-function. First of all we will obtain an explicit representation for \( \hat{H}_\Phi(\rho) \).

Consider the function

\[
\hat{H}_\Phi(\rho) = \inf_{\pi \in P_{\{\rho\}}} \int_{\mathcal{G}(\mathcal{H})} H_\Phi(\rho) \pi(d\rho) \leq +\infty
\]

where \( P_{\{\rho\}} \) is the set of all probability measures with the barycenter \( \rho \). It is easy to see that \( \hat{H}_\Phi(\rho) \leq H_\Phi(\rho) \) for all states \( \rho \) in \( \mathcal{G}(\mathcal{H}) \). By considering properties of the function \( \hat{H}_\Phi \) we will establish that \( \hat{H}_\Phi = \text{conv} H_\Phi \) (proposition 5 below).

It was mentioned in the previous section that in the definition of the \( \chi \)-function the supremum over all measures coincides with the supremum over all measures with finite support (conventional ensembles). In contrast to this we have the following

**Lemma 2.** The equality \( \hat{H}_\Phi(\rho) = \inf \sum_i \pi_i \rho_i = \rho \sum_i \pi_i H_\Phi(\rho_i) = \text{conv} H_\Phi(\rho) \) takes place only if either \( H_\Phi(\rho) < +\infty \) or \( H_\Phi(\rho) = +\infty \).

**Proof.** If \( H_\Phi(\rho) < +\infty \) then \( \chi_\Phi(\rho) = H_\Phi(\rho) - \text{conv} H_\Phi(\rho) \). By proposition 1 and corollary 1 in \[12\] we have \( \chi_\Phi(\rho) = H_\Phi(\rho) - \hat{H}_\Phi(\rho) \) and hence \( \hat{H}_\Phi(\rho) = \text{conv} H_\Phi(\rho) \).

If \( H_\Phi(\rho) = +\infty \) then \( \text{conv} H_\Phi(\rho) = +\infty \) since by general properties of entropy the set of states with finite entropy is convex. \[23\] \[ \square \]

Lemma 2 implies that \( \hat{H}_\Phi(\rho) < \text{conv} H_\Phi(\rho) \) for any state \( \rho \) such that \( H_\Phi(\rho) = +\infty \) while \( \hat{H}_\Phi(\rho) < +\infty \). Note that the set of such states is nonempty. For example, in the case of noiseless channel \( \Phi \) it is easy to see
that $\hat{H}_{\Phi}(\rho) = 0$ for any $\rho$, but the set of states $\rho$ with $H_{\Phi}(\rho) < +\infty$ is the subset of the first category of $\mathcal{S}(\mathcal{H})$ \[23\].

Our first goal is to show that the infimum in the definition of the $\hat{H}_{\Phi}(\rho)$ can be taken over all measures supported by the set of pure states. For this purpose it is useful to consider the following partial order on the set $\mathcal{P}$. Denote by $\mathcal{S}$ the set of all convex continuous bounded function on $\mathcal{S}(\mathcal{H})$. We say that $\mu \succ \nu$ if and only if

$$\int_{\mathcal{S}(\mathcal{H})} f(\rho) \mu(d\rho) \geq \int_{\mathcal{S}(\mathcal{H})} f(\rho) \nu(d\rho) \quad \text{for all } f \in \mathcal{S}.$$ 

The partial order of this type is widely used in \[1,2\], where measures on compact convex set are considered. The compactness makes it possible to establish antisymmetrical property of this partial order, which is not needed in our consideration.

**Proposition 3.** For arbitrary state $\rho_0$ there exist a measure $\pi_0$ in $\mathcal{P}_{\{\rho_0\}}$ supported by pure states such that

$$\hat{H}_{\Phi}(\rho_0) = \int_{\mathcal{S}(\mathcal{H})} H_{\Phi}(\rho) \pi_0(d\rho).$$

The measure $\pi_0$ can be chosen to be a measure with support consisting of $n^2$ atoms (ensemble of $n^2$ pure states) if and only if the state $\rho_0$ has finite rank $n$.

**Proof.** In the proof of the theorem in \[12\] it was shown that the functional

$$\pi \mapsto \int_{\mathcal{S}(\mathcal{H})} H_{\Phi}(\rho) \pi(d\rho) \quad (8)$$

is well defined and lower semicontinuous on the set $\mathcal{P}$ equipped with the topology of weak convergence. By proposition 2 in \[12\] the set $\mathcal{P}_{\{\rho_0\}}$ is compact. Hence the above functional achieves its minimum on this set at some point $\pi_*$, i.e.

$$\hat{H}_{\Phi}(\rho_0) = \int_{\mathcal{S}(\mathcal{H})} H_{\Phi}(\rho) \pi_*(d\rho). \quad (9)$$

To show that among all such measures $\pi_*$ there exists a measure $\pi_0$ supported by pure states we will use the following two simple properties of the introduced partial order:
1. Let \( \{\mu_n\} \) and \( \{\nu_n\} \) be two sequences in \( \mathcal{P} \) weakly converging to measures \( \mu \) and \( \nu \) correspondingly and such that \( \mu_n \succ \nu_n \) for all \( n \). Then \( \mu \succ \nu \);

2. If \( \mu \succ \nu \) then
\[
\int_{\mathcal{S}(\mathcal{H})} g(\rho) \mu(d\rho) \geq \int_{\mathcal{S}(\mathcal{H})} g(\rho) \nu(d\rho)
\]
for every function \( g \) which can be represented as a pointwise limit of monotous sequence of functions in \( \mathcal{S} \).

By lemma 1 in [12] there exists the sequence \( \pi_n \) of measures in \( \mathcal{P}(\rho_0) \) with finite supports, weakly converging to \( \pi^* \). Decomposing each atom of the measure \( \pi_n \) into pure states we obtain (as in the proof of the theorem in [12]) the measure \( \hat{\pi}_n \) with the same barycenter supported by the set of pure states. It is easy to see by definition that \( \hat{\pi}_n \succ \pi_n \). By compactness of the set \( \mathcal{P}(\rho_0) \) there exists subsequence \( \hat{\pi}_{n_k} \) converging to some measure \( \pi_0 \) supported by the set of pure states due to theorem 6.1 in [13]. Since \( \hat{\pi}_{n_k} \succ \pi_{n_k} \), the above property 1 of the partial order \( \succ \) implies \( \pi_0 \succ \pi^* \).

By lemma 4 in [13] the convex function \( -H(\Phi) \) is a pointwise limit of the monotonous sequence of bounded continuous functions \( -H(P_n \Phi(\rho)P_n) \), where \( \{P_n\} \) is an arbitrary sequence of finite dimensional projectors strongly increasing to the unit operator \( I \). By noting that \( H(A) = -\text{Tr}A \log A + \text{Tr}A \log \text{Tr}A = \text{Tr}AH(A/\text{Tr}A) \) we see that the functions \( -H(P_n \Phi(\rho)P_n) \) are convex and hence lie in \( \mathcal{S} \) for all \( n \). By the above property 2 of the partial order \( \succ \) (with \( g(\rho) = -H(\Phi(\rho)) \)) and [9] we have
\[
\check{H}_\Phi(\rho_0) = \int_{\mathcal{S}(\mathcal{H})} H(\Phi(\rho)) \pi^*(d\rho) \geq \int_{\mathcal{S}(\mathcal{H})} H(\Phi(\rho)) \pi_0(d\rho).
\]

The definition of the function \( \check{H}_\Phi \) implies equality in the above inequality.

Let us prove the last statement of the proposition. If the state \( \rho_0 \) has infinite rank then any measure in \( \mathcal{P}(\rho_0) \) supported by the set of pure states has infinite support.

Let the state \( \rho_0 \) have finite rank \( n \), \( \mathcal{H}_0 = \text{supp}\rho_0 \) be an \( n \)-dimensional Hilbert space and \( \Phi_0 \) be a FI-subchannel of the channel \( \Phi \), corresponding to the subspace \( \mathcal{H}_0 \) (see [20]).

If \( H_{\Phi_0}(\rho_0) = H(\Phi(\rho_0)) \leq +\infty \) then by lemma 6 in [20] the function \( H_{\Phi_0}(\rho) \) is continuous on the compact set \( \mathcal{S}(\mathcal{H}_0) \). This makes it possible to apply
lemma A-2 in [23] to show existence of ensemble of \((\dim \mathcal{H}_0)^2\) states with the average \(\rho_0\), optimal in the sense of the definition of the function \(\text{conv} H_{\Phi_0}\), which obviously coincides with the restriction of the function \(\text{conv} H_{\Phi}\) to the subset \(\mathcal{G}(\mathcal{H}_0)\) of the set \(\mathcal{G}(\mathcal{H})\). By lemma 2 the restriction of the function \(\text{conv} H_{\Phi}\) to the subset \(\mathcal{G}(\mathcal{H}_0)\) coincides with the restriction of the function \(\hat{H}_{\Phi}\) to this subset.

If \(H_{\Phi_0}(\rho_0) = H_\Phi(\rho_0) = +\infty\) then \(\hat{H}_{\Phi}(\rho_0) = +\infty\) and hence any decomposition of the state \(\rho_0\) is optimal. To show this note first that \(H_{\Phi}(\rho_0) = +\infty\) implies \(H_\Phi(\sigma) = +\infty\) for arbitrary state \(\sigma\) such that \(\text{supp} \sigma = \text{supp} \rho_0 = \mathcal{H}_0\). Indeed, for such a state \(\sigma\) there is a positive number \(\lambda_\sigma\) such that \(\lambda_\sigma \sigma \geq \rho_0\). Nonnegativity of the relative entropy implies

\[
\lambda_\sigma \text{Tr} \Phi(\sigma)(-\log \Phi(\sigma)) \geq \text{Tr} \Phi(\rho_0)(-\log \Phi(\sigma)) \geq \text{Tr} \Phi(\rho_0)(-\log \Phi(\rho_0)) = +\infty.
\]

Suppose \(\hat{H}_{\Phi}(\rho_0) < +\infty\). Then there exist a measure \(\pi\) with the barycenter \(\rho_0\) such that the function \(H_\Phi(\rho)\) is finite \(\pi\)-almost everywhere. Let \(\mathcal{F}\) be a subset of \(\mathcal{G}(\mathcal{H}_0)\) such that \(H_{\Phi}(\rho)\) is finite on the set \(\mathcal{F}\) and \(\pi(\mathcal{F}) = 1\). The equality \(\rho_0 = \int_\mathcal{F} \rho \pi(d\rho)\) implies that the linear hull of the set of subspaces \(\{\text{supp} \rho\} \rho \in \mathcal{F}\) coincides with \(\mathcal{H}_0\) and hence there exists a finite collection \(\{\rho_i\}_{i=1}^n\) of states in \(\mathcal{F}\) such that \(\text{supp}(\rho_i) = \mathcal{H}_0\). Since the state \(\rho_n = \frac{1}{n} \sum_{i=1}^n \rho_i\) is a finite convex combination of the states \(\rho_i, i = 1, n\) with \(H_\Phi(\rho_i) < +\infty\) for all \(i = 1, n\) we conclude that \(H_\Phi(\rho_n) < +\infty\) [23]. But this contradicts to the previous observation. □

**Definition 2.** A measure \(\pi_0\) with the properties stated in proposition 3 is called an \(\hat{H}_\Phi\)-optimal measure for a state \(\rho_0\).

It is easy to see that the set of \(\hat{H}_\Phi\)-optimal measures coincides with the set of \(\chi_\Phi\)-optimal measures for arbitrary state \(\rho\) such that \(H_\Phi(\rho) < +\infty\).

The other important property of the function \(\hat{H}_\Phi(\rho)\) is stated in the following proposition.

**Proposition 4.** The function \(\hat{H}_\Phi(\rho)\) is convex and lower semicontinuous on \(\mathcal{G}(\mathcal{H})\). The function \(\hat{H}_\Phi(\rho)\) is closed in the sense of convex analysis (see the Appendix).

**Proof.** To prove convexity of the function \(\hat{H}_\Phi(\rho)\) it is sufficient to note that

\[
\lambda P_{\{\rho_1\}} + (1 - \lambda) P_{\{\rho_2\}} \subseteq P_{\{\lambda \rho_1 + (1 - \lambda) \rho_2\}}
\]

for arbitrary states \(\rho_1, \rho_2\) and \(\lambda \in [0, 1]\).

Suppose that the function \(\hat{H}_\Phi(\rho)\) is not lower semicontinuous. This implies the existence of a sequence \(\rho_n\) converging to some state \(\rho_0\) and such
that
\[
\lim_{n \to +\infty} \hat{H}_\Phi(\rho_n) < \hat{H}_\Phi(\rho_0).
\] (10)

By proposition 3 for each \( n = 1, 2, \ldots \) there exists a measure \( \pi_n \) in \( \mathcal{P}_{\{\rho_n\}} \) such that
\[
\hat{H}_\Phi(\rho_n) = \int_{\mathcal{S}(\mathcal{H})} H_\Phi(\rho) \pi_n(d\rho).
\]
Let \( \mathcal{A} = \{\rho_n\}_{n=0}^{+\infty} \) be compact subset of \( \mathcal{S}(\mathcal{H}) \). By proposition 2 in \[12\] the set \( \mathcal{P}_\mathcal{A} \) is compact. Since \( \{\pi_n\} \subset \mathcal{P}_\mathcal{A} \) there exists subsequence \( \pi_{n_k} \) converging to some measure \( \pi_0 \). Continuity of the mapping \( \pi \mapsto \bar{\rho}(\pi) \) implies \( \pi_0 \in \mathcal{P}_{\{\rho_0\}} \).

By lower semicontinuity of the functional (8) we obtain
\[
\hat{H}_\Phi(\rho) \leq \int_{\mathcal{S}(\mathcal{H})} H_\Phi(\rho_0) \pi_0(d\rho) \leq \liminf_{k \to +\infty} \int_{\mathcal{S}(\mathcal{H})} H_\Phi(\rho) \pi_{n_k}(d\rho) = \lim_{k \to +\infty} \hat{H}_\Phi(\rho_{n_k}),
\]
which contradicts to (10).

**Proposition 5.** The function \( \hat{H}_\Phi \) coincides with the convex closure \( \text{conv} H_\Phi \) of the output entropy \( H_\Phi \) and hence lemma 2 implies
\[
\{\text{conv} H_\Phi(\rho) = \text{conv} H_\Phi(\rho) < +\infty\} \leftrightarrow \{H_\Phi(\rho) < +\infty\}.
\]

**Proof.** By proposition 4
\[
\hat{H}_\Phi(\rho) \leq \text{conv} H_\Phi(\rho) \leq \text{conv} H_\Phi(\rho) \leq H_\Phi(\rho), \quad \forall \rho \in \mathcal{S}(\mathcal{H}).
\] (11)

By lemma 2 \( \hat{H}_\Phi(\rho) \) coincides with \( \text{conv} H_\Phi(\rho) \) for arbitrary state \( \rho \) with finite \( H_\Phi(\rho) \). This and \[11\] imply \( \hat{H}_\Phi(\rho) = \text{conv} H_\Phi(\rho) \) for all such states.

Let \( \rho \) be an arbitrary state with finite \( \hat{H}_\Phi(\rho) \). By lemma 3 below there exists a sequence \( \rho_n \) of states with finite \( H_\Phi(\rho_n) \) converging to the state \( \rho \) and such that \( \lim_{n \to +\infty} \hat{H}_\Phi(\rho_n) = \hat{H}_\Phi(\rho) \). By the above observation \( \hat{H}_\Phi(\rho_n) = \text{conv} H_\Phi(\rho_n) \) for all \( n \). Since the function \( \text{conv} H_\Phi(\rho) \) is closed and convex it is lower semicontinuous \[17\]. It follows
\[
\text{conv} H_\Phi(\rho) \leq \liminf_{n \to +\infty} \text{conv} H_\Phi(\rho_n) = \lim_{n \to +\infty} \hat{H}_\Phi(\rho_n) = \hat{H}_\Phi(\rho)
\]
This and \[11\] imply \( \hat{H}_\Phi(\rho) = \text{conv} H_\Phi(\rho) \) for arbitrary state \( \rho \). □
Lemma 3. For arbitrary state $\rho_0$ with $\hat{H}_\Phi(\rho_0) < \infty$ there exists a sequence $\rho_n$ of finite rank states converging to the state $\rho_0$ and such that

$$H_\Phi(\rho_n) < +\infty \quad \text{and} \quad \lim_{n \to +\infty} \hat{H}_\Phi(\rho_n) = \hat{H}_\Phi(\rho_0).$$

Proof. Let $\pi_0$ be an $\hat{H}$-optimal measure for the state $\rho_0$ supported by the set of pure states (proposition 3). Since any probability measure on the complete separable metric space $\mathcal{S}(\mathcal{H})$ is tight \cite{6,18} for arbitrary $n \in \mathbb{N}$ there exists compact subset $\mathcal{K}_n$ of $\text{Extr}(\mathcal{S}(\mathcal{H}))$ such that $\pi_0(\mathcal{K}_n) > 1 - 1/n$. Compactness of the set $\mathcal{K}_n$ implies decomposition $\mathcal{K}_n = \bigcup_{i=1}^{m(n)} \mathcal{A}_i^n$, where $\{\mathcal{A}_i^n\}_{i=1}^{m(n)}$ is a finite collection of disjoint measurable subsets with diameter less than $1/n$. Without loss of generality we may assume that $\pi_0(\mathcal{A}_i^n) > 0$ for all $i$ and $n$. By construction compact set $\mathcal{A}_i^n$ lies within some closed ball $\mathcal{B}_i^n$ of diameter $1/n$ for all $i$ and $n$.

By assumption

$$\hat{H}_\Phi(\rho_0) = \int_{\mathcal{S}(\mathcal{H})} H_\Phi(\rho) \pi_0(d\rho) < +\infty$$

and hence the function $H_\Phi(\rho)$ is finite $\pi_0$-almost everywhere. Since the function $H_\Phi(\rho)$ is lower semicontinuous it achieves its finite minimum on the compact set $\mathcal{A}_i^n$ of positive measure at some pure state $\rho_i^n \in \mathcal{A}_i^n$. Consider the state $\rho_n = (\pi_0(\mathcal{K}_n))^{-1} \sum_{i=1}^{m(n)} \pi_0(\mathcal{A}_i^n) \rho_i^n$. We want to show that

$$\|\rho_n - \rho_0\|_1 \leq 3/n \quad \text{(12)}$$

The state $\hat{\rho}_i^n = (\pi_0(\mathcal{A}_i^n))^{-1} \int \rho \pi_0(d\rho)$ lies in $\mathcal{B}_i^n$ (see the proof of lemma 1 in \cite{12}). It follows that $\|\rho_i^n - \hat{\rho}_i^n\|_1 \leq 1/n$. By noting that $\pi_0(\mathcal{K}_n) = \sum_{i=1}^{m(n)} \pi_0(\mathcal{A}_i^n)$ we have

$$\|\rho_n - \rho_0\|_1 = \|((\pi_0(\mathcal{K}_n))^{-1} \sum_{i=1}^{m(n)} \pi_0(\mathcal{A}_i^n) \rho_i^n - \sum_{i=1}^{m(n)} \int \rho \pi_0(d\rho) - \int_{\mathcal{S}(\mathcal{H}) \setminus \mathcal{K}_n} \rho \pi_0(d\rho))\|_1$$

$$\leq \sum_{i=1}^{m(n)} \pi_0(\mathcal{A}_i^n) \|((\pi_0(\mathcal{K}_n))^{-1} \rho_i^n - \hat{\rho}_i^n\|_1 + \| \int_{\mathcal{S}(\mathcal{H}) \setminus \mathcal{K}_n} \rho \pi_0(d\rho)\|_1$$

$$\leq (1 - \pi_0(\mathcal{K}_n)) + \sum_{i=1}^{m(n)} \pi_0(\mathcal{A}_i^n) \|\rho_i^n - \hat{\rho}_i^n\|_1 + \pi_0(\mathcal{S}(\mathcal{H}) \setminus \mathcal{K}_n) < 3/n,$
which implies (12).

By the choice of the states \( \rho^n_i \) we have \( H_\Phi(\rho^n_i) \leq H_\Phi(\rho) \) for all \( \rho \) in \( A^n_i \). It follows that

\[
\hat{H}_\Phi(\rho_n) \leq (\pi_0(K_n))^{-1} \sum_{i=1}^{m(n)} \pi_0(A^n_i) H_\Phi(\rho^n_i)
\]

\[
\leq (\pi_0(K_n))^{-1} \sum_{i=1}^{m(n)} \int_{A^n_i} H_\Phi(\rho) \pi_0(d\rho)
\]

\[
\leq (\pi_0(K_n))^{-1} \int_{\mathcal{S}(\mathcal{H})} H_\Phi(\rho) \pi_0(d\rho) = (\pi_0(K_n))^{-1} \hat{H}_\Phi(\rho_0).
\]

This implies \( \limsup_{n \to +\infty} \hat{H}_\Phi(\rho_n) \leq \hat{H}_\Phi(\rho_0) \). But \( \lim_{n \to +\infty} \rho_n = \rho_0 \) due to (12) and by proposition 4 we have \( \liminf_{n \to +\infty} \hat{H}_\Phi(\rho_n) \geq \hat{H}_\Phi(\rho_0) \). Hence there exists \( \lim_{n \to +\infty} \hat{H}_\Phi(\rho_n) = \hat{H}_\Phi(\rho_0) \).

By the construction the state \( \rho_n \) (for all \( n \)) is a finite convex combination of pure states \( \rho^n_i \) with finite output entropy \( H_\Phi(\rho^n_i) \). By general properties of entropy [23] it follows that \( H_\Phi(\rho_n) < +\infty \) for all \( n \). □

Since the set \( \mathfrak{B}(\mathcal{H}) \) can be identified with the dual space for \( \mathfrak{F}(\mathcal{H}) \), considered as a complex Banach space the set \( \mathfrak{B}_h(\mathcal{H}) \) of all hermitian operators can be identified with the dual space for real Banach space \( \mathfrak{R}_h(\mathcal{H}) \) of all hermitian trace class operators. The nonnegative lower semicontinuous function \( H_\Phi(\rho) \) on \( \mathfrak{S}(\mathcal{H}) \) can be extended to the lower semicontinuous function \( \hat{H}_\Phi(\rho) \) on \( \mathfrak{R}_h(\mathcal{H}) \) by ascribing the value +\( \infty \) to arbitrary operator in \( \mathfrak{R}_h(\mathcal{H}) \setminus \mathfrak{S}(\mathcal{H}) \). Hence the Fenchel transform of the function \( H_\Phi(\rho) \) (see the Appendix) is defined on the set \( \mathfrak{B}_h(\mathcal{H}) \) of all hermitian operators by

\[
H_\Phi^*(A) = \sup_{\rho \in \mathfrak{R}_h(\mathcal{H})} (\text{Tr} A \rho - \hat{H}_\Phi(\rho)) = \sup_{\rho \in \mathfrak{S}(\mathcal{H})} (\text{Tr} A \rho - H_\Phi(\rho)).
\]

(13)

Double Fenchel transform \( H_\Phi^{**}(\rho) \) is defined on the set \( \mathfrak{R}_h(\mathcal{H}) \) by

\[
H_\Phi^{**}(\rho) = \sup_{A \in \mathfrak{B}_h(\mathcal{H})} (\text{Tr} A \rho - H_\Phi^*(A)).
\]

(14)

Since the function \( \hat{H}_\Phi(\rho) \) is nonnegative its convex closure \( \overline{\text{conv}} \hat{H}_\Phi(\rho) \) coincides with its double Fenchel transform \( H_\Phi^{**}(\rho) \) (see the Appendix). By noting that the restriction of \( \overline{\text{conv}} \hat{H}_\Phi(\rho) \) to the set \( \mathfrak{S}(\mathcal{H}) \) coincides with \( \overline{\text{conv}} H_\Phi(\rho) \) proposition 5 implies the following result.

**Corollary 2.** \( H_\Phi(\rho) = H_\Phi^{**}(\rho) = \sup_{A \in \mathfrak{B}_h(\mathcal{H})} \inf_{\sigma \in \mathfrak{S}(\mathcal{H})} (H_\Phi(\sigma) + \text{Tr} A(\rho - \sigma)) \)

for arbitrary state \( \rho \in \mathfrak{S}(\mathcal{H}) \).
Let us consider the set $\hat{H}^{-1}_\Phi(0) = \{ \rho \in \mathcal{S}(\mathcal{H}) | \hat{H}_\Phi(\rho) = 0 \}$. Note that the set $H^{-1}_\Phi(0) = \{ \rho \in \mathcal{S}(\mathcal{H}) | H_\Phi(\rho) = 0 \}$ is closed subset of $\mathcal{S}(\mathcal{H})$ due to lower semicontinuity of the quantum entropy \[23\].

**Proposition 6.** The set $\hat{H}^{-1}_\Phi(0)$ coincides with the convex closure of the set $H^{-1}_\Phi(0) \cap \text{Extr}\mathcal{S}(\mathcal{H})$.

**Proof.** Let $\rho \in \text{conv}(H^{-1}_\Phi(0) \cap \text{Extr}\mathcal{S}(\mathcal{H}))$. Then there exist a sequence of states $\rho_n \in \text{conv}(H^{-1}_\Phi(0) \cap \text{Extr}\mathcal{S}(\mathcal{H}))$ converging to $\rho$. By definition $\hat{H}_\Phi(\rho_n) = 0$. Lower semicontinuity and nonnegativity of the function $\hat{H}_\Phi$ (proposition 4) implies $\hat{H}_\Phi(\rho) = 0$.

Let $\rho \in \hat{H}^{-1}_\Phi(0)$. By proposition 3 the state $\rho$ is a barycenter of a particular measure $\pi_0$ supported by the set of pure states and such that $H_\Phi(\rho) = 0$ for $\pi_0$-almost all $\rho$. By using arguments from the proof of theorem 6.3 in [18] it is easy to see that this measure $\pi_0$ can be approximated by the sequence of measures $\pi_n$ with finite support within the set of pure states and such that $H_\Phi(\rho) = 0$ for $\pi_n$-almost all $\rho$. This implies that for each $n$ all atoms of the measure $\pi_n$ are pure states in $H^{-1}_\Phi(0)$. By continuity of the mapping $\pi \mapsto \bar{\rho}(\pi)$ the state $\rho = \bar{\rho}(\pi_0)$ is a limit of the sequence $\bar{\rho}(\pi_n)$ of states in $\text{conv}(H^{-1}_\Phi(0) \cap \text{Extr}\mathcal{S}(\mathcal{H}))$. □

### 5 On continuity of the functions $\chi_\Phi$ and $\hat{H}_\Phi$

It follows from lemma 2 that

$$\chi_\Phi(\rho) = H_\Phi(\rho) - \hat{H}_\Phi(\rho), \quad (15)$$

for all states with finite output entropy. This expression remains valid in the case $H_\Phi(\rho) = +\infty$ if $\hat{H}_\Phi(\rho) < +\infty$. Indeed, by substituting $\hat{H}_\Phi$-optimal measure $\pi$ for the state $\rho$ in expression (4) in [12] it is easy to see that $\chi_\Phi(\rho) = +\infty$.

The properties of the functions $\chi_\Phi$ and $\hat{H}_\Phi$ obtained in the previous sections allow to relate the continuity of these functions to the continuity of the output entropy $H_\Phi$.

**Proposition 7.** If the restriction of the output entropy $H_\Phi(\rho)$ to a particular subset $\mathcal{A} \subseteq \mathcal{S}(\mathcal{H})$ is continuous then the restrictions of the functions $\chi_\Phi(\rho)$ and $\hat{H}_\Phi(\rho)$ to the subset $\mathcal{A}$ are continuous as well.

Let $\{ \rho_n \}$ be a sequence of states converging to a state $\rho_0$ such that $\lim_{n \to +\infty} \hat{H}_\Phi(\rho_n) = \hat{H}_\Phi(\rho_0)$. Let $\pi^*_n$ be an $\hat{H}_\Phi$-optimal measure for the state $\rho_n$. □
for all \( n = 1, 2, \ldots \). Then the set of partial limits of the sequence \( \{ \pi_n^* \}_{n=1}^{+\infty} \) is nonempty and consists of \( \hat{H}_\Phi \)-optimal measures for the state \( \rho_0 \).

**Proof.** The first assertion of the proposition follows from lower semi-continuity of the function \( \chi_\Phi(\rho) \) (proposition 1), lower semi-continuity of the function \( \hat{H}_\Phi(\rho) \) (proposition 4) and expression (15).

Let \( \{ \rho_n \} \) and \( \pi_n^* \) be sequences mentioned in the second statement of the proposition. Since the set \( \{ \rho_n \}_{n=0}^{\infty} \) is compact the set \( \mathcal{P}_n \{ \{ \rho_n \}_{n=0}^{\infty} \} \) is compact by proposition 2 in [12]. Hence the sequence \( \pi_n^* \subseteq \mathcal{P}_n \{ \{ \rho_n \}_{n=0}^{\infty} \} \) has partial limits.

Let \( \pi_0 \) be a limit of some subsequence \( \{ \pi_{n_k}^* \}_{k=1}^{+\infty} \) of the sequence \( \{ \pi_n^* \}_{n=1}^{+\infty} \). By lower semi-continuity of the functional (8) we have

\[
\hat{H}_\Phi(\rho_0) = \lim_{k \to +\infty} \hat{H}_\Phi(\rho_{n_k}) = \lim_{k \to +\infty} \int_{\mathcal{S}(\mathcal{H})} H_\Phi(\rho)\pi_{n_k}^*(d\rho) \geq \int_{\mathcal{S}(\mathcal{H})} H_\Phi(\rho)\pi_0(d\rho),
\]

which implies \( \hat{H}_\Phi \)-optimality of the measure \( \pi_0 \).

**Corollary 3.** Let \( H' \) be a positive unbounded operator on the space \( \mathcal{H}' \) with discrete spectrum of finite multiplicity such that

\[
\text{Tr} \exp(-\beta H') < +\infty \quad \text{for all} \quad \beta > 0.
\]

Then the restrictions of the functions \( \chi_\Phi(\rho) \) and \( \hat{H}_\Phi(\rho) \) to the subset \( \mathcal{A}_{h'} = \{ \rho \in \mathcal{S}(\mathcal{H}) \mid \text{Tr} \Phi(\rho)H' \leq h' \} \) are continuous for all \( h' \geq 0 \).

**Proof.** In the proof of proposition 3 in [12] it is established that the condition of the corollary implies continuity of the restriction of the function \( H_\Phi(\rho) \) to the subset \( \mathcal{A}_{h'} \).

As it is mentioned in [12] the condition of corollary 3 is fulfilled for Gaussian channels with the power constraint of the form \( \text{Tr} \rho H \leq h \), where \( H = R^T \epsilon R \) is the many-mode oscillator Hamiltonian with nondegenerate energy matrix \( \epsilon \) and \( R \) are the canonical variables of the system.

The second part of proposition 7 implies that \( \hat{H}_\Phi \)-optimal (=\( \chi_\Phi \)-optimal) measure for arbitrary state with finite output entropy can be obtained as a limit point of any sequence of \( \hat{H}_\Phi \)-optimal (=\( \chi_\Phi \)-optimal) measures for finite rank approximations of this state.

**Corollary 4.** Let \( \rho_0 \) be a state such that \( H_\Phi(\rho_0) < +\infty \), \( P_n \) be a spectral projector of \( \rho_0 \), corresponding to the maximal \( n \) eigenvalues, and \( \pi_n^* \) be an \( \hat{H}_\Phi \)-optimal (=\( \chi_\Phi \)-optimal) measure with finite support (conventional ensemble of \( n^2 \) states) for the finite rank state \( \rho_n = (\text{Tr} P_n \rho_0)^{-1} P_n \rho_0 \) for all \( n \in \mathbb{N} \). Then any partial limit of the sequence \( \{ \pi_n^* \} \) is \( \hat{H}_\Phi \)-optimal (=\( \chi_\Phi \)-optimal) measure for the state \( \rho_0 \).
Proof. By using the dominated convergence theorem for quantum entropy \([22]\) it is easy to see that

\[
\lim_{n \to +\infty} H_\Phi(\rho_n) = H_\Phi(\rho_0).
\]

Hence we can apply proposition 7 with \(A = \{\rho_n\}_{n=0}^{+\infty}\). □

6 The case of tensor product

In \([2]\) the convex duality approach to the additivity problem in the finite dimensional case was proposed. The results of previous sections provide a generalization of this approach to the infinite dimensional case.

For a channel \(\Phi : \mathfrak{S}(\mathcal{H}) \mapsto \mathfrak{S}(\mathcal{H}')\) and an operator \(A \in \mathfrak{B}_+(\mathcal{H})\) we introduce the following output purity of the channel \([21]\)

\[
\nu_H(\Phi, A) = \inf_{\rho \in \mathfrak{S}(\mathcal{H})} (H_\Phi(\rho) + \text{Tr} A \rho).
\]

Note that this characteristic is a generalization of the minimal output entropy of the channel \(\Phi\) defined by

\[
H_{\text{min}}(\Phi) = \inf_{\rho \in \mathfrak{S}(\mathcal{H})} H(\Phi(\rho)) = \nu_H(\Phi, 0).
\]

The concavity of the quantum entropy implies that infinitum in \((16)\) and \((17)\) can be taken over all pure states \(\rho\) in \(\mathfrak{S}(\mathcal{H})\).

Let \(\Psi : \mathfrak{S}(\mathcal{K}) \mapsto \mathfrak{S}(\mathcal{K}')\) be another channel. By considering product states it is easy to obtain the subadditivity property of the above output purity for tensor product channel \(\Phi \otimes \Psi\) with respect to the Kronecker sum:

\[
\nu_H(\Phi \otimes \Psi, A \otimes I + I \otimes B) \leq \nu_H(\Phi, A) + \nu_H(\Psi, B).
\]

The additivity of the minimal output entropy for the channels \(\Phi\) and \(\Psi\) means \([9]\)

\[
H_{\text{min}}(\Phi \otimes \Psi) = H_{\text{min}}(\Phi) + H_{\text{min}}(\Psi),
\]

which is equivalent to equality in \((18)\) with \(A = \lambda I_\mathcal{H}\) and \(B = \mu I_\mathcal{K}, \lambda, \mu \in \mathbb{R}\).

The Holevo capacity of the \(A\)-constrained channel \(\Phi\) is defined by \([20]\), \([12]\)

\[
\bar{C}(\Phi; A) = \sup_{\sum_i \pi_i \rho_i \in A} \sum_i \pi_i H(\Phi(\rho_i) \| \Phi(\bar{\rho})).
\]

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The strong additivity of the $\chi$-capacity for channels $\Phi$ and $\Psi$ means
\[
\bar{C}(\Phi \otimes \Psi; A \otimes B) = \bar{C}(\Phi; A) + \bar{C}(\Psi; B).
\]
for arbitrary sets $A \subseteq \mathcal{S}(\mathcal{H})$ and $B \subseteq \mathcal{S}(\mathcal{K})$ such that $H_\Phi(\rho) < +\infty$ for all $\rho$ in $A$ and $H_\Psi(\sigma) < +\infty$ for all $\sigma$ in $B$.

In the finite dimensional case the strong additivity the $\chi$-capacity for channels $\Phi$ and $\Psi$ implies additivity of the minimal output entropy for these channels \[21\]. But in the infinite dimensional case this implication is an open problem due to the existence of pure ”superentangled” states, whose partial traces have infinite entropy (see remark 3 in \[20\]). Denote $\omega^\mathcal{H} := \text{Tr}_{\mathcal{K}} \omega$ and $\omega^\mathcal{K} := \text{Tr}_{\mathcal{H}} \omega$ for arbitrary state $\omega$ in $\mathcal{S}(\mathcal{H} \otimes \mathcal{K})$.

**Theorem.** Let $\Phi : \mathcal{S}(\mathcal{H}) \mapsto \mathcal{S}(\mathcal{H}')$ and $\Psi : \mathcal{S}(\mathcal{K}) \mapsto \mathcal{S}(\mathcal{K}')$ be arbitrary channels. Statements (i)-(ii) are equivalent and imply (iii)-(v):

(i) For all $\omega \in \mathcal{S}(\mathcal{H} \otimes \mathcal{K})$
\[
\hat{H}_{\Phi \otimes \Psi}(\omega) \geq \hat{H}_{\Phi}(\omega^\mathcal{H}) + \hat{H}_{\Psi}(\omega^\mathcal{K});
\]

(ii) For all $A \in \mathfrak{B}_+(\mathcal{H})$ and $B \in \mathfrak{B}_+(\mathcal{K})$
\[
\nu_H(\Phi \otimes \Psi, A \otimes I + I \otimes B) = \nu_H(\Phi, A) + \nu_H(\Psi, B);
\]

(iii) For all $\rho \in \mathcal{S}(\mathcal{H})$ and $\sigma \in \mathcal{S}(\mathcal{K})$
\[
\hat{H}_{\Phi \otimes \Psi}(\rho \otimes \sigma) = \hat{H}_{\Phi}(\rho) + \hat{H}_{\Psi}(\sigma);
\]

(iv) The strong additivity of the $\chi$-capacity \[21\] holds for the channels $\Phi$ and $\Psi$;

(v) Additivity of the minimal output entropy \[19\] holds for arbitrary subchannels\(^2\) $\Phi_0$ and $\Psi_0$ of the channels $\Phi$ and $\Psi$ correspondingly.

**Proof.** (i) $\Leftrightarrow$ (ii) By proposition 5 the function $\hat{H}_\Phi$ is the convex closure of the function $H_\Phi$. The Fenchel transform $H_\Phi^*$ of $H_\Phi$ is defined on the set

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\(^2\)The notion of subchannel is defined in \[20\].
\[ B_{h}(H) \] of all hermitian operators by \(^{13}\). By lemma 1 in \(^{2}\) the strong superadditivity of the function \( \hat{H}_{\phi} \) is equivalent to the subadditivity of the Fenchel transform \( H_{\phi}^{*} \) with respect to the Kronecker sum:

\[ H_{\phi}^{*}(A \otimes I_{K} + I_{H} \otimes B) \leq H_{\phi}^{*}(A) + H_{\phi}^{*}(B), \quad \forall A \in B_{h}(H), \forall B \in B_{h}(K). \]

By the definition of \( H_{\phi}^{*} \) the last inequality is equivalent to

\[
\sup_{\sigma \in \mathcal{S}(H \otimes K)} (\text{Tr} A \omega^{H} + \text{Tr} B \omega^{K} - H(\Phi \otimes \Psi(\omega))) \leq \sup_{\rho \in \mathcal{S}(H)} (\text{Tr} A \rho - H(\Phi(\rho))) + \sup_{\sigma \in \mathcal{S}(K)} (\text{Tr} B \sigma - H(\Psi(\sigma)))
\]

for all \( A \in B_{h}(H) \) and \( B \in B_{h}(K) \).

Noting invariance of the previous inequality after changing \( A \) and \( B \) on \( A \pm \| A \| I_{H} \) and \( B \pm \| B \| I_{K} \) correspondingly and using \(^{18}\) we obtain that

(i) \( \Rightarrow \) (ii).

(i) \( \Rightarrow \) (iii) The inequality " \( \geq \) " follows from (i) while the inequality " \( \leq \) " can be deduced from the definition of the function \( \hat{H}_{\Phi \otimes \Psi} \) by considering measures on \( \mathcal{S}(H \otimes K) \) supported by product states;

(i) \( \Rightarrow \) (iv) It follows from theorem 1 in \(^{20}\);

(i) \( \Rightarrow \) (v) Let \( \Phi_{0} \) and \( \Psi_{0} \) be subchannels of the channels \( \Phi \) and \( \Psi \) corresponding to the subspaces \( H_{0} \subseteq H \) and \( K_{0} \subseteq K \). It is easy to see that property (i) for the channels \( \Phi \) and \( \Psi \) implies the same property for its subchannels \( \Phi_{0} \) and \( \Psi_{0} \). Due to (i) for arbitrary state \( \omega \) in \( \mathcal{S}(H_{0} \otimes K_{0}) \) we have

\[
H(\Phi_{0} \otimes \Psi_{0}(\omega)) \geq \hat{H}_{\Phi_{0}}(\omega^{H_{0}}) + \hat{H}_{\Psi_{0}}(\omega^{K_{0}}) \geq H_{\text{min}}(\Phi_{0}) + H_{\text{min}}(\Psi_{0})
\]

This implies inequality " \( \geq \) " in \(^{19}\). Since the converse inequality is obvious, the equality in \(^{19}\) is proved. \( \square \)

### 7 On definition of the EoF

Entanglement is a specific feature of composed quantum systems. One of the measures of entanglement of a state of a bipartite system is the entanglement

\(^{3}\)Formally the considered functions do not satisfy the condition of this lemma, but it is easy to see that all arguments in the proof remain valid in our case.
of formation (EoF) [7]. In the finite dimensional case it is defined as
\[
E_F(\rho) = \min_{\sum_i \pi_i \rho_i = \rho} \sum_i \pi_i H_\Phi(\rho_i),
\]
where \( \Phi \) is the partial trace channel from the state space of bipartite system onto the state space of a marginal subsystem. In term of convex analysis this definition means that the EoF coincides with the convex hull of the output entropy of the partial trace channel. Continuity of the EoF established in [16] implies that it coincides with the convex closure of the output entropy of the partial trace channel in this case.

A natural generalization of EoF to the infinite dimensional case was considered in [8] and it was defined by
\[
E^1_F(\rho) = \inf_{\sum_i \pi_i \rho_i = \rho} \sum_i \pi_i H_\Phi(\rho_i),
\]
where infimum is over all countable decomposition of a state \( \rho \) into pure states and \( \Phi \) is the partial trace channel.

An alternative approach to the definition of the EoF was considered in [15] in the case of tensor product of two systems with one of them finite dimensional. In this spirit we can define the EoF in the general case by
\[
E^2_F(\rho) = \hat{H}_\Phi(\rho) = \inf_{\pi \in \mathcal{P}(\rho)} \int_{\mathcal{S}(\mathcal{H})} H_\Phi(\rho) \pi(d\rho),
\]
where \( \Phi \) is the partial trace channel.

Propositions 4 and 5 imply that \( E^2_F \) is a convex lower semicontinuous function which coincides with the convex closure of the output entropy of the partial trace channel. Proposition 3 shows that the infimum in the above expression is achieved at some measure supported by a set of pure states while proposition 6 implies the following natural property of \( E^2_F \):
\[
\{ E^2_F(\rho) = 0 \} \iff \text{state} \ \rho \ \text{is nonentangled}
\]
where the set of nonentangled states is defined as the convex closure of all product pure states. Indeed, for the partial trace channel \( \Phi \) the set \( H^{-1}_\Phi(0) \cap \text{Extr}_\mathcal{S}(\mathcal{H}) \) coincides with the set of all product pure states. Proposition 7 and proposition 3 in [8] implies that \( E^2_F \) is trace norm continuous on the subsets of states with constrained mean energy.
An interesting question is the relations between $E_F^1$ and $E_F^2$. By proposition 3 we have

$$E_F^1(\rho) \geq E_F^2(\rho)$$

for all states $\rho$. Since an arbitrary state can be represented as a countable convex combination of pure states lemma 2 and concavity of the output entropy imply

$$E_F^1(\rho) = E_F^2(\rho)$$

for all states $\rho$ having partial traces with finite entropy. It is easy to see that (22) obviously holds for all nonentangled and all pure states (for which $\hat{H}_\Phi$ coincides with $H_\Phi$). Note that lemma 3 implies

$$E_F^2(\rho) = \lim_{\varepsilon \to +0} \inf_{\sum_i \pi_i \rho_i \in U_\varepsilon(\rho)} \sum_i \pi_i H_\Phi(\rho_i),$$

where $U_\varepsilon(\rho)$ is a $\varepsilon$-vicinity of the state $\rho$ and the infimum is over all (finite) ensembles of pure states. But the validity of equality (22) for mixed states having partial traces with infinite entropy remains an open problem.

8 Appendix

Here the notions from the convex analysis used in the main text are presented, following [17]. Let $f$ be an arbitrary real valued function defined on closed convex subset $X$ of some locally convex Hausdorff topological space. Consider the subset $\text{epi}(f) = \{(x, \lambda) \in X \times \mathbb{R} | \lambda \geq f(x)\}$ of the set $X \times \mathbb{R}$. Note that a function $f$ is uniquely determined by the corresponding set $\text{epi}(f)$. A function $f$ is called convex if the set $\text{epi}(f)$ is a convex subset of $X \times \mathbb{R}$ and it is called closed if the set $\text{epi}(f)$ is a closed subset of $X \times \mathbb{R}$. If a function $f$ does not take the value $-\infty$ then its convexity means that

$$f(\lambda x_1 + (1 - \lambda)x_2) \leq \lambda f(x_1) + (1 - \lambda)f(x_2), \quad \forall x_1, x_2 \in X, \forall \lambda \in [0, 1].$$

Each closed function $f$ is lower semicontinuous in the sense that the set defined by the inequality $f(x) \leq \lambda$ is a closed subset of $X$ for arbitrary $\lambda \in \mathbb{R}$ and, conversely, each lower semicontinuous function $f$ is closed. It is possible to show that the lower semicontinuity of a function $f$ means that

$$\liminf_{n \to +\infty} f(x_n) \geq f(x_0)$$
for arbitrary sequence \( \{x_n\} \) converging to \( x_0 \).

Let \( f \) be an arbitrary function on \( X \). The convex hull \( \text{conv} f \) of the function \( f \) is defined by

\[
\text{conv} f(x) = \inf_{(x,\lambda) \in \text{conv}(\text{epi}(f))} \lambda,
\]

where the symbol \( \text{conv} \) in the right side means the convex hull of a set. This is equivalent to the following representation

\[
\text{conv} f(x) = \inf_{\sum_i \pi_i x_i = x} \sum_i \pi_i f(x_i), \quad \pi_i > 0, \quad \sum_i \pi_i = 1.
\]

It follows that \( \text{conv} f \) is the greatest convex function majorized by \( f \). The convex closure \( \overline{\text{conv}} f \) of the function \( f \) is defined by

\[
\text{epi}(\overline{\text{conv}} f) = \overline{\text{conv}}(\text{epi}(f)),
\]

where the symbol \( \overline{\text{conv}} \) in the right side means the closure of the convex hull of a set. It follows that \( \overline{\text{conv}} f \) is the greatest convex and closed function majorized by \( f \). This implies

\[
\overline{\text{conv}} f(x) \leq \text{conv} f(x) \leq f(x), \quad \forall x \in X.
\]

If \( f \) is a continuous function on compact convex set \( X \) then \( \overline{\text{conv}} f = \text{conv} f \) \cite{1}.

For arbitrary real valued function \( f \) on locally convex real linear topological space \( X \) the Fenchel transform \( f^* \) is a function on the dual space \( X^* \) defined by

\[
f^*(y) = \sup_{x \in X} (\langle y, x \rangle - f(x)), \quad \forall y \in X^*.
\]

The double Fenchel transform \( f^{**} \) is a function on the space \( X \) defined by

\[
f^{**}(x) = \sup_{y \in X^*} (\langle y, x \rangle - f^*(y)), \quad \forall x \in X.
\]

By Fenchel’s theorem \( f^{**}(x) = \overline{\text{conv}} f \) for arbitrary function \( f \) which does not take the value \(-\infty\). This implies that in this case \( \overline{\text{conv}} f \) coincides with the upper bound of the set of all affine continuous functions majorized by \( f \).

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