AUTOMORPHISMS OF HYPERBOLIC GROUPS AND GRAPHS OF GROUPS

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Abstract. Using the canonical JSJ splitting, we describe the outer automorphism group \( \text{Out}(G) \) of a one-ended word hyperbolic group \( G \). In particular, we discuss to what extent \( \text{Out}(G) \) is virtually a direct product of mapping class groups and a free abelian group, and we determine for which groups \( \text{Out}(G) \) is infinite. We also show that there are only finitely many conjugacy classes of torsion elements in \( \text{Out}(G) \), for \( G \) any torsion-free hyperbolic group.

More generally, let \( \Gamma \) be a finite graph of groups decomposition of an arbitrary group \( G \) such that edge groups \( G_e \) are rigid (i.e. \( \text{Out}(G_e) \) is finite). We describe the group of automorphisms of \( G \) preserving \( \Gamma \), by comparing it to direct products of suitably defined mapping class groups of vertex groups.

1. Introduction and statement of results

The basic idea for understanding the outer automorphism group \( \text{Out}(G) \) of a torsion-free one-ended word hyperbolic group \( G \) is due to Rips and Sela [13, 14]. Since \( G \) has a canonical JSJ splitting, automorphisms of \( G \) may be seen on this splitting: \( \text{Out}(G) \) is virtually generated by Dehn twists along edges of the splitting, together with mapping class groups of punctured surfaces (automorphism groups of “quadratically hanging” subgroups).

Giving a precise description of \( \text{Out}(G) \), however, requires a careful study of these generators. This is what we do here, both for hyperbolic groups and for splittings of arbitrary groups.

Let us first illustrate this on a simple example. Let \( \Sigma \) be a 2-torus with one open disc removed. Let \( G \) be the fundamental group of the 2-complex \( K \) obtained by gluing three copies \( \Sigma_1, \Sigma_2, \Sigma_3 \) of \( \Sigma \) along their boundary circle \( C \). All automorphisms of \( G \) are realized by homeomorphisms of \( K \). Since homeomorphisms may permute the \( \Sigma_i \)'s, or reverse orientation, we restrict to homeomorphisms mapping each \( \Sigma_i \) to itself in an orientation-preserving way; this amounts to passing to a finite index subgroup \( \text{Out}_0(G) \subset \text{Out}(G) \).

Restricting homeomorphisms to each \( \Sigma_i \) gives an epimorphism \( \rho : \text{Out}_0(G) \to \prod_{i=1}^{3} \text{MCG}(\Sigma_i) \), where \( \text{MCG}(\Sigma_i) \cong \text{Out}(F_2) \cong SL(2, \mathbb{Z}) \) is the mapping class group of the punctured torus, i.e. the group of isotopy classes of orientation-preserving
homeomorphisms of $\Sigma_i$. The kernel of $\rho$ is generated by $D_1, D_2, D_3$, with $D_i$ the (automorphism induced by a) Dehn twist in an annulus parallel to the boundary in $\Sigma_i$. The product of the $D_i$'s being isotopic to the identity, $\ker \rho$ is isomorphic to $\mathbb{Z}^2$.

The epimorphism $\rho$ does not have a section (basically because the map $\text{Aut}(F_2) \to \text{Out}(F_2)$ has none). In geometric terms, there is no canonical way of gluing elements of $\text{MCG}(\Sigma_i)$ along $C$, because we consider homeomorphisms of $\Sigma_i$ up to an isotopy which is free on the boundary. We therefore introduce $\text{MCG}^\partial(\Sigma_i)$, the group of homeomorphisms of $\Sigma_i$ fixing the boundary pointwise, up to isotopy relative to the boundary. This group is a nontrivial central extension of $\text{MCG}(\Sigma_i)$ by $\mathbb{Z}$ (generated by the twist $D_i$).

Now we can glue along $C$, thus getting an epimorphism $\lambda : \prod_{i=1}^3 \text{MCG}^\partial(\Sigma_i) \to \text{Out}_0(G)$, whose kernel is isomorphic to $\mathbb{Z}$ (corresponding to the relation $D_1D_2D_3 = 1$).

The discussion of this example is summed up by the following commutative triangle of nontrivial central extensions:

\[
\begin{array}{ccc}
\mathbb{Z}^3 & \xrightarrow{\pi} & \prod \text{MCG}(\Sigma_i) \\
\longrightarrow & & \longrightarrow \\
\Pi \text{MCG}^\partial(\Sigma_i) & \xrightarrow{\lambda} & \text{Out}_0(G) \\
\lambda \downarrow & & \downarrow \rho \\
\mathbb{Z}^2 & \xleftarrow{\mathbb{Z}} & \\
\end{array}
\]

Now consider an arbitrary one-ended hyperbolic group $G$ (assumed to be torsion-free for the moment). We use the JSJ splitting of $G$ constructed by Bowditch [4], because it is completely canonical and thus invariant under all automorphisms of $G$.

It is a finite graph of groups $\Gamma$, with $\pi_1(\Gamma) = G$. There are three types of vertices. Let $V_1$ denote the set of vertices of $\Gamma$ with cyclic vertex group (we use the same 1, 2, 3 numbering as in [4]). Let $V_2$ be the set of vertices $v$ corresponding to “MHF subgroups”: the vertex group $G_v$ is the fundamental group of a compact surface $\Sigma_v$, with edge groups being fundamental groups of boundary components (in the example studied above $V_1$ has one element, corresponding to $C$, and $V_2$ has three elements, corresponding to the $\Sigma_i$'s). Let $V_3$ consist of the remaining vertices of $\Gamma$, which we call rigid. Let $\mathcal{E}$ be the set of (unoriented) edges of $\Gamma$. Every edge joins a vertex of $V_1$ to a vertex of $V_2 \cup V_3$, leading to $\mathcal{E} = \mathcal{E}_2 \cup \mathcal{E}_3$.

Given a compact surface $\Sigma$, we denote by $\text{MCG}(\Sigma)$ the (pure) mapping class group of the surface with punctures, that is the group of homeomorphisms of $\Sigma$
fixing the boundary pointwise, modulo those that are freely isotopic to the identity (the boundary may “turn” during the isotopy). We also define $\text{MCG}^\partial(\Sigma)$, the mapping class group of the surface with boundary, dividing only by homeomorphisms isotopic to the identity relatively to the boundary. It is a central extension of $\text{MCG}(\Sigma)$ by $\mathbb{Z}^b$, where $b$ is the number of boundary components of $\Sigma$.

As suggested in [4], the $\text{Aut}(G)$-invariant JSJ splitting allows a simple proof of the following result due to Sela.

**Theorem 1.1 ([14, Theorem 1.9]).** Let $G$ be a torsion-free one-ended hyperbolic group. There is an exact sequence

$$1 \to \mathbb{Z}^n \to \text{Out}'(G) \to \prod_{v \in V_2} \text{MCG}(\Sigma_v) \to 1,$$

where $\text{Out}'(G)$ has finite index in $\text{Out}(G)$, and $n = |\mathcal{E}| - |V_1|$ is “the number of edges in the essential JSJ decomposition of $G$”.

This central extension, however, is not trivial in general, so one cannot claim, as in [14], that $\text{Out}(G)$ is virtually a direct product of mapping class groups and a free abelian group. The next result leads to such a product structure, but only under an additional hypothesis.

**Theorem 1.2.** Let $G$ be a torsion-free one-ended hyperbolic group. Let $r$ be the number of vertices of $V_1$ connected only to vertices of $V_2$, and $q = |\mathcal{E}_3| - |V_1| + r$. The group $\text{Out}(G)$ is virtually a direct product $\mathbb{Z}^q \times M$, where $M$ is the quotient of $\prod_{v \in V_2} \text{MCG}^\partial(\Sigma_v)$ by a central subgroup isomorphic to $\mathbb{Z}^r$.

**Corollary 1.3.** If every cyclic vertex of $\Gamma$ is connected to at least one rigid vertex, then $\text{Out}(G)$ is virtually a direct product of a free abelian group and mapping class groups $\text{MCG}^\partial(\Sigma_v)$ of surfaces with boundary.

There are statements similar to Theorems 1.1 and 1.2 if $G$ is allowed to have torsion, they will be given in section 5. The main differences are that $\text{MCG}(\Sigma_v)$ and $\text{MCG}^\partial(\Sigma_v)$ are now mapping class groups of 2-orbifolds, and only edges of $\Gamma$ whose group has infinite center must be considered. Our study allows us to prove the following result (see section 6). It was conjectured in [10] (we thank G. Swarup for suggesting this application).

**Theorem 1.4.** Let $G$ be a one-ended hyperbolic group. Then $\text{Out}(G)$ is infinite if and only if $G$ splits over a virtually cyclic subgroup with infinite center, as an arbitrary HNN extension or as an amalgam of groups with finite center.

Paulin’s theorem [11], together with Rips theory [2], provides a splitting over a virtually cyclic subgroup, but with no control on the center. Torsion-free hyperbolic groups with infinitely many ends always have $\text{Out}(G)$ infinite. Allowing torsion makes things much more complicated, see [12] for the case of virtually free groups.

We also prove (see section 7):
Theorem 1.5. If \( G \) is a torsion-free hyperbolic group, there are only finitely many conjugacy classes of torsion elements in \( \text{Out}(G) \).

This will be used in [9] to give a proof of Shor’s theorem on fixed subgroups [15].

Now let \( \Gamma \) be a finite graph of groups decomposition of an arbitrary group \( G \). We consider the group \( \text{Out}^\Gamma(G) \subset \text{Out}(G) \) consisting of automorphisms preserving \( \Gamma \). This group was thoroughly studied by Bass-Jiang [1]. Here we focus on a subgroup \( \text{Out}^\Gamma_1(G) \subset \text{Out}^\Gamma(G) \), which we squeeze between products of automorphism groups associated to vertex groups.

Let \( V \) be the vertex set of \( \Gamma \). For \( v \in V \) we consider the vertex group \( G_v \), with incident edge groups \( G_e \). We let \( \text{MCG}(G_v) \subset \text{Out}(G_v) \) be the group of automorphisms which act on each \( G_e \) as conjugation by some \( g_e \in G_v \). We also define an extension \( \text{MCG}^\partial(G_v) \), by keeping track of the elements \( g_e \) (see section 4). These groups are an algebraic generalization of the groups \( \text{MCG}(\Sigma_v) \) and \( \text{MCG}^\partial(\Sigma_v) \) considered above.

Theorem 1.6. There are natural epimorphisms \( \lambda : \prod_{v \in V} \text{MCG}^\partial(G_v) \to \text{Out}^\Gamma_1(G) \) and \( \rho_1 : \text{Out}^\Gamma_1(G) \to \prod_{v \in V} \text{MCG}(G_v) \), with \( \text{Out}^\Gamma_1(G) \subset \text{Out}^\Gamma(G) \) and \( \rho_1 \circ \lambda \) equal to the natural projection.

If \( \text{Out}(G_e) \) is finite for every edge group \( G_e \), then \( \text{Out}^\Gamma_1(G) \) has finite index in \( \text{Out}^\Gamma(G) \).

The kernels of \( \lambda \) and \( \rho_1 \) may be described explicitly. In particular, our main technical result (Proposition 3.1) is a presentation for \( T = \ker \rho_1 \), the group of twists around edges of \( \Gamma \).

2. Automorphisms of graphs of groups

Most results in this section follow from [1], but our exposition will be self-contained.

Generalities.

The following notation will be used throughout the paper. If \( H \) is a subgroup of a group \( G \), then \( Z_G(H) \) and \( N_G(H) \) denote the centralizer and normalizer of \( H \) in \( G \), and \( Z(H) \) is the center of \( H \). For \( m \in G \), denote by \( i_m \) the inner automorphism \( g \mapsto mgm^{-1} \). If \( \alpha \in \text{Aut}(G) \), let \( \hat{\alpha} \) be its class in \( \text{Out}(G) \). We say that \( \alpha \) represents \( \hat{\alpha} \).

Let \( G \) be the fundamental group of a finite graph of groups \( \Gamma \), with vertex set \( V \), vertex groups \( G_v \), and edge groups \( G_e \). Let \( E \) be the set of oriented edges \( e \) of \( \Gamma \), and \( E_v \) the set of edges \( e \) with origin \( o(e) = v \). Let \( \mathcal{E} \) be the set of unoriented edges. The incident edge groups \( G_e \), for \( e \in E_v \), and their conjugates in \( G_v \), will be called peripheral subgroups of \( G_v \).

Let \( T \) be the Bass-Serre \( G \)-tree associated to \( \Gamma \). We simply write \( g \) for the automorphism of \( T \) associated to \( g \in G \), and \( G_x \) for the stabilizer of \( x \in T \). Choosing a fundamental domain, we identify each vertex \( v \) of \( \Gamma \) with a vertex of \( T \).
whose stabilizer is $G_v$. Similarly, we often write $e$ for an edge of $T$ with stabilizer $G_e$.

We assume that $\Gamma$ is minimal: $G_e$ is a proper subgroup of $G_v$ for every terminal vertex $v$ (equivalently, $\pi_1(\Gamma')$ is a proper subgroup of $\pi_1(\Gamma)$ for every proper connected subgraph $\Gamma'$). We also assume that $\Gamma$ is not a mapping torus: we say that $\Gamma$ is a mapping torus if $\Gamma$ is topologically a circle and all inclusions from edge group to vertex group are onto (equivalently, $T = \mathbb{R}$ and $G$ acts by translations); if $\Gamma$ is a mapping torus, then $G$ is a semi-direct product $G_v \rtimes \varphi \mathbb{Z}$, with $\varphi \in \text{Aut}(G_v)$.

These assumptions imply that the centralizer of the image of $G$ in $\text{Aut}(T)$ is trivial. In particular, $Z(G)$ acts trivially on $T$.

Let $\text{Aut}^\Gamma(G) \subset \text{Aut}(G)$ and $\text{Out}^\Gamma(G) \subset \text{Out}(G)$ be the groups of automorphisms preserving $\Gamma$. Topologically, $G$ is the fundamental group of a graph of spaces, and automorphisms in $\text{Out}^\Gamma(G)$ are induced by homeomorphisms preserving such a structure. Algebraically, the condition is best expressed in terms of $T$: an automorphism $\alpha \in \text{Aut}(G)$ belongs to $\text{Aut}^\Gamma(G)$ if and only if there exists an automorphism $H_\alpha$ of $T$ conjugating the action of $G$ with the action twisted by $\alpha$, in the sense that $\alpha(g)H_\alpha = H_\alpha g$ for every $g \in G$ (if $G$ fixes no end of $T$, this is the same as saying that the length function satisfies $\ell \circ \alpha = \ell$).

With our assumptions, $H_\alpha$ is unique and $\alpha \mapsto H_\alpha$ defines an action of $\text{Aut}^\Gamma(G)$ on $T$, with $i_m$ acting as $m$ for $m \in G$. In the situation of [4], this action is induced by the action of $\text{Aut}(G)$ on $\partial G$.

The homomorphism $\rho : \text{Out}_0\Gamma(G) \to \prod_{v \in V} \text{Out}(G_v)$.

The map $H_\alpha$ induces an automorphism of the finite graph $\Gamma = T/G$, and we let $\text{Out}_0\Gamma(G) \subset \text{Out}^\Gamma(G)$ be the finite index subgroup consisting of automorphisms acting as the identity on $\Gamma$.

Note that for $x \in T$ the stabilizer of $H_\alpha x$ is $\alpha(G_x)$, and points in the same $G$-orbit have conjugate stabilizers. If $\hat{\alpha} \in \text{Out}_0\Gamma(G)$ and $v \in V$, then $\alpha(G_v)$ is a conjugate of $G_v$ and we can define $\rho_v(\hat{\alpha}) \in \text{Out}(G_v)$. More precisely, choose $m \in G$ such that $H_\alpha v = mv$ (recall that we identify $v$ with a vertex of $T$). Then $\alpha(G_v) = G_{mv} = mG_vm^{-1}$, so that $\beta = i_m^{-1}\alpha$ maps $G_v$ to itself. It is easily checked that the class of $\beta$ in $\text{Out}(G_v)$ depends only on $\hat{\alpha}$, and we obtain a homomorphism $\rho_v : \text{Out}_0\Gamma(G) \to \text{Out}(G_v)$.

We also define $\rho = \prod \rho_v : \text{Out}_0\Gamma(G) \to \prod_{v \in V} \text{Out}(G_v)$. In the rest of this section we study the image and kernel of $\rho$.

Extending vertex automorphisms.

Elements in the image of $\rho_v$ preserve the peripheral structure of $G_v$ (if $e$ is an edge of $T$ containing $v$, then $H_\beta e = g_e e$ for some $g_e \in G_v$; the stabilizer of $H_\beta e$ is $\beta(G_e)$, and also $g_e G_v g_e^{-1}$). The converse is not necessarily true: if $G = \mathbb{Z} \ast_2 \mathbb{Z}$, the nontrivial automorphism of $\mathbb{Z}$ does not always extend to $G$.

Define the mapping class group $\text{MCG}(G_v) \subset \text{Out}(G_v)$ by restricting to automorphisms which act on each edge group $G_e$, $e \in E_v$, as conjugation by some $g_e \in G_v$.

**Proposition 2.1.** The product $\prod_{v \in V} \text{MCG}(G_v)$ is contained in the image of $\rho$. 

Proof. Given $\beta \in \text{Aut}(G_v)$ with $\hat{\beta} \in MCG(G_v)$, we extend it to $\alpha \in \text{Aut}(G)$ with $\rho_v(\hat{\alpha}) = \hat{\beta}$ and $\rho_w(\hat{\alpha}) = 1$ for $w \neq v$. For each $e \in E_v$, fix $g_e \in G_v$ such that $\beta(g) = g_e g g_e^{-1}$ for $g \in G_e$.

For concreteness we start with the elementary cases. If $G = G_v \ast_{G_e} H$, we define $\alpha$ on $H$ as conjugation by $g_e$. If $G$ is an HNN-extension $< G_v, t \mid t \varphi(g) t^{-1} = g$ if $g \in G_e >$, we have $\beta(g) = g_e g g_e^{-1}$ and $\beta(\varphi(g)) = g_f \varphi(g) g_f^{-1}$ for $g \in G_e$, and we define $\alpha(t) = g_e t g_f^{-1}$.

In the general case, recall that $G$ is a subgroup of a path group $\pi(\Gamma)$ generated by the $G_v$'s and the edges of $\Gamma$ (see e.g. [1]). We define $\alpha$ on the generators of $\pi(\Gamma)$ in the following way: it equals $\beta$ on $G_v$, it is the identity on the other vertex groups and on edges not incident to $v$, and an edge $e$ with origin $v$ is mapped to $g_e e$ (to $g_e e g_e^{-1}$ if the opposite edge $\bar{e}$ also has origin $v$). One checks that this is compatible with the relations defining $\pi(\Gamma)$, and that the subgroup $G$ is invariant.

This will be referred to as the extension construction. It is simply an algebraic translation of the following topological observation: if $K_0$ is a subcomplex of codimension 0 of a complex $K$, every self-homeomorphism of $K_0$ which equals the identity on $\partial K_0$ has a canonical extension to $K$.

Twists and bitwists.

The extension construction depends on the choice of the elements $g_e$. Each $g_e$ may be right-multiplied by an element $z$ of the centralizer $Z_{G_e}(G_v)$. This changes $\alpha$ by $D_z$, the extension of $\beta = 1$ relative to the choice $g_e = z$ (and $g_f = 1$ for $f \neq e$).

We call $D_z$ the twist by $z$ around $e$ (near $v$). It may also be defined, more directly, as follows. Let $e \in E_v$ and $z \in Z_{G_v}(G_e)$. If $e$ separates $\Gamma$, we have an amalgam and $D_z$ acts as the identity on the factor containing $G_v$ and as conjugation by $z$ on the other. In the case of an HNN-extension, $D_z$ maps the stable letter $t$ to $z t$. If $z$ is a generator of $G_e \simeq \mathbb{Z}$, then $D_z$ is induced by a Dehn twist on an annulus.

We denote by $\mathcal{T}$ the subgroup of $\text{Out}^\Gamma(G)$ generated by all twists $\hat{D}_z$. Extensions of a given $\hat{\beta}$ associated to different choices of the $g_e$'s differ by an element of $\mathcal{T}$.

We will see that $\mathcal{T}$ has finite index in ker $\rho$ when all groups $\text{Out}(G_e)$ are finite. In general, though, describing ker $\rho$ requires bitwists, which we now define.

Let $e$ be an edge of $\Gamma$ with endpoints $v, w$ (possibly equal). Suppose that $z \in G_v$ normalizes the image of $G_e$, that $z' \in G_w$ normalizes the image of $G_e$, and that $z, z'$ have the same action on $G_e$ (if $v \neq w$, this means that $z^{-1} z'$ centralizes $G_e$). We define the bitwist $D_{z, z'}$ around $e$ as conjugation by $z$ on one factor and by $z'$ on the other in the case of an amalgam, as mapping $t$ to $z t z'^{-1}$ in the HNN case. For instance, amalgamate two Klein bottle groups $< t a t_i^{-1} = a^{-1} >$ along the cyclic group $< a >$. Then $D_{t_1, t_2}$ fixes $t_1, t_2$ and maps $a$ to $a^{-1}$.

Of course twists are bitwists, and (bi)twists around distinct edges commute in $\text{Out}(G)$. Also note that bitwists belong to $\mathcal{T}$ when edge groups are trivial, and
when $G$ is a torsion-free hyperbolic group and edge groups are cyclic (because no nontrivial element of $G$ is conjugate to its inverse).

The kernel of $\rho$.

**Proposition 2.2.** The kernel of $\rho$ is generated by bitwists around the edges of $\Gamma$.

*Proof.* Fix a maximal subtree $\Gamma_0 \subset \Gamma$, which we identify with a lift to $T$. Let $e = vw$ be an edge of $\Gamma_0$. Any $\hat{\alpha} \in \ker \rho$ has a representative $\alpha$, equal to the identity on $G_v$, with $H_\alpha v = v$. The image of the edge $e$ by $H_\alpha$ is an edge in the same $G$-orbit, which we can write $z^{-1}e$ for some $z \in G_v$. Note that $z$ normalizes $G_e$ since the stabilizer of $H_\alpha e = z^{-1}e$ is both $\alpha(G_e) = G_e$ and $z^{-1}G_e z$. Now $\alpha(G_w)$ is the stabilizer of $H_\alpha w = z^{-1}w$, equal to $z^{-1}G_w z$. Since $\rho_w(\hat{\alpha})$ is trivial, there exists $z' \in G_w$ such that $\alpha(g) = z'^{-1}g z'^{-1}z$ for $g \in G_w$. We conclude that $\hat{\alpha} \circ \hat{D}_{z,z'}^{-1}$ has a representative $\alpha'$ inducing the identity on $\langle G_v, G_w \rangle$, with $H_{\alpha'}$ equal to the identity on $e$.

Repeating this argument, we can compose any $\hat{\alpha} \in \ker \rho$ by bitwists and obtain $\hat{\alpha}'$ with a representative $\alpha'$ equal to the identity on the fundamental group of $\Gamma_0$, with $H_{\alpha'}$ equal to the identity on $\Gamma_0$. If $\Gamma_0 = \Gamma$ we are done since $\hat{\alpha}$ is a product of bitwists. Otherwise we apply a similar argument to the HNN generators. \qed

Rigid edge groups.

**Proposition 2.3.** Suppose that $\text{Out}(G_e)$ is finite for every edge group $G_e$ of $\Gamma$. Then $\prod_{v \in V} \text{MCG}(G_v)$ has finite index in the image of $\rho$, and the group of twists $T$ has finite index in $\ker \rho$.

*Proof.* For the first assertion, it suffices to show that $\text{MCG}(G_v)$ has finite index in the image of $\rho_v$ for every $v \in V$. Since any $\varphi$ in the image of $\rho_v$ preserves the peripheral structure of $G_v$, one can associate to $\varphi$ and $e \in E_v$ a right coset of $\text{Out}(G_e)$ modulo the image of $\text{N}_{G_v}(G_e)$. If $\varphi$ and $\varphi'$ define the same coset for every $e \in E_v$, then $\varphi$ and $\varphi'$ belong to the same coset modulo $\text{MCG}(G_v)$.

The second assertion follows from the fact that $\hat{D}_{z,z'}$ belongs to $T$ if $z$ and $z'$ act on $G_e$ by an inner automorphism. \qed

3. The Group of Twists $T$

We view $T$ as a quotient of $\prod_{v \in V} \left( \prod_{e \in E_v} Z_{G_v}(G_e) \right) = \prod_{e \in E} Z_{G_{\phi(e)}}(G_e)$. We denote by $\theta$ the quotient map. Generators for $\ker \theta$ will be given by vertex and edge relations, coming from centers of vertex groups and edge groups: twisting by $z \in Z(G_v)$ around all edges with origin $v$, or by $z \in Z(G_e)$ at both ends of $e$, produces an inner automorphism.

More formally, we embed $Z(G_v)$ into $\prod_{e \in E} Z_{G_{\phi(e)}}(G_e)$ by embedding it diagonally into $\prod_{e \in E_v} Z_{G_e}(G_e)$. Given an edge $e$ with endpoints $v, w$, we have embeddings of $G_e$ into $G_v$ and $G_w$, therefore a diagonal embedding of $Z(G_e)$ into
\[ Z_{G_v}(G_e) \times Z_{G_u}(G_e) \]  
Putting all these embeddings together gives a map \[ j : \prod_{v \in V} Z(G_v) \times \prod_{e \in E} Z(G_e) \to \prod_{e \in E} Z_{G_{\alpha(e)}}(G_e) \]  
(recall that \( E \) is the set of unoriented edges and \( E \) is the set of oriented edges). Note that only two terms contribute to a given factor \( Z_{G_{\alpha}(e)}(G_e) \), namely \( Z(G_v) \) and \( Z(G_e) \). The image of \( j \) is central and contained in \( \ker \theta \).

**Proposition 3.1.** Assume that \( \Gamma \) is minimal and is not a mapping torus. The group of twists \( T \) is the quotient of \( \prod_{e \in E} Z_{G_{\alpha(e)}}(G_e) \) by the vertex and edge relations: there is an exact sequence

\[ \{1\} \to Z(G) \overset{i}{\to} \prod_{v \in V} Z(G_v) \times \prod_{e \in E} Z(G_e) \overset{j}{\to} \prod_{e \in E} Z_{G_{\alpha(e)}}(G_e) \overset{\theta}{\to} T \to \{1\}. \]

**Proof.** Elements of \( Z(G) \) act trivially on \( T \), hence belong to every \( Z(G_v) \) and \( Z(G_e) \). We define \( i \) by mapping \( n \in Z(G) \) to \( n \) in every \( Z(G_v) \) and \( n^{-1} \) in every \( Z(G_e) \). It is easy to check that the image of \( i \) equals \( \ker j \).

Given \( z = (z_e)_{e \in E} \) with \( \theta(z) \) trivial, we must now show that \( z \) belongs to the image \( J \) of \( j \). The proof is by induction on the number of edges of \( \Gamma \), we divide it into four steps.

- **First step:** graphs with one edge.
  
  First suppose that \( \Gamma \) is an amalgam \( G = G_1 *_{G_e} G_2 \). Consider two elements \( z_i \in Z_{G_i}(G_e) \) such that the automorphism equal to conjugation by \( z_i \) on \( G_i \) is inner. There exists \( m \in G \) such that \( u_i = m^{-1}z_i \) commutes with \( G_i \). Since \( \Gamma \) is minimal we have \( N_G(G_i) = G_i \), and therefore \( u_i \in Z(G_i) \). Furthermore \( m = z_1u_1^{-1} = z_2u_2^{-1} \in Z(G_e) \), showing that \( (z_1, z_2) \) belongs to \( J \) since \( z_1 = mu_i \) with \( m \in Z(G_e) \) and \( u_i \in Z(G_i) \).

  Now consider an HNN extension \( G = G_1 *_{G_e} G_2 \). Suppose that an automorphism equal to the identity on \( G_1 \) and sending the stable letter \( t \) to \( z_1tz_2^{-1} \) is a conjugation \( i_m \).

  Assume for a moment that \( m \in G_1 \). We then write \( u = z_2^{-1}m = t^{-1}z_1^{-1}mt \in G_1 \cap t^{-1}G_1t = G_e \), so that \( u_1 = mту^{-1}t^{-1} \) and \( u_2 = mту^{-1}t^{-1} \) with \( m \in Z(G_1) \) and \( u^{-1} \in Z(G_e) \) (note that \( u \) commutes with \( G_e \) because \( m \) and \( z_2 \) do). This expresses that \( (z_1, z_2) \) belongs to \( J \).

  Now we show \( m \in G_1 \). This is clear if both inclusions \( G_e \hookrightarrow G_1 \) are proper, since \( N_G(G_1) = G_1 \). If an inclusion \( G_e \hookrightarrow G_1 \) is onto, write \( G = \langle G_1, t \mid tgt^{-1} = \varphi(g) \rangle \) if \( g \in G_1 \rangle \), and \( m = t^{-k}gt^k \) with \( g \in G_1 \), \( k \geq 0 \), and \( k \) minimal. From \( z_1tz_2^{-1} = mту^{-1}t^{-1} \) we get \( g = t^kz_1(tz_2^{-1}t^{-1})t^{-k}(tgt^{-1}) \), so that \( g \in tG_1t^{-1} \) if \( k > 0 \). Minimality of \( k \) then implies \( k = 0 \). Writing that \( m = gt^k \) centralizes \( G_1 \) shows that \( \varphi|G_1 \) is onto. Since \( \Gamma \) is not a mapping torus, we have \( t = 0 \), and \( m \in G_1 \).

- **Second step:** \( \Gamma \) has more than one edge and there is an edge \( e \) such that each component of \( \Gamma \setminus \{e\} \) is a minimal graph of groups but not a mapping torus.
First assume that \( e = v_1v_2 \) separates. We write \( G = G_1 \ast_{G_e} G_2 \), where \( G_i \) is the fundamental group of a subgraph \( \Gamma_i \). By minimality \( G_e \) is a proper subgroup of \( G_1 \) and \( G_2 \), and therefore \( N_G(G_i) = G_i \).

A twist around an edge of \( \Gamma_i \) may be represented by an automorphism sending \( G_i \) to \( G_i \) and equal to the identity on \( G_{3-i} \). Similarly, a twist around \( e \) near \( v_i \) may be represented by an automorphism sending \( G_i \) to \( G_i \) by an inner automorphism and equal to the identity on \( G_{3-i} \). With these representatives, consider the product \( D_i \) of all twists by \( z_f \) around edges \( f \) contained in \( \Gamma_i \). It sends \( G_i \) to itself, and because \( \theta(z) = 1 \) it coincides on \( G_i \) with an inner automorphism of \( G \). Since \( N_G(G_i) = G_i \), we deduce that \( D_i \) restricts to an inner automorphism of \( G_i \). Applying the induction hypothesis to \( \Gamma_1 \) and \( \Gamma_2 \), we can then change \( z_\tilde{e} \) by an element of \( J \) so as to make every \( z_f \) trivial for \( f \subset \Gamma_1 \cup \Gamma_2 \) (this may change \( z_e, z_\tilde{w} \) by an element of the center of \( G_{v_1} \) or \( G_{v_2} \)).

We have now reduced to the automorphism conjugating each \( G_i \) by some \( z_i \in Z_{G_e}(G_e) \). We have seen in the first step of the proof that \( z_i = mu_i \) with \( m \in Z(G_e) \) and \( u_i \in Z(G_i) \). The element \( u_i \) fixes a point in the Bass-Serre tree of \( \Gamma_i \) (it belongs to \( G_{v_i} \)). Since \( \Gamma_i \) is minimal, \( u_i \) acts trivially on the tree and therefore belongs to every edge and vertex group of \( \Gamma_i \). We conclude that \( z_\tilde{e} \) is the image by \( j \) of the element of \( \prod_{v \in V} Z(G_v) \times \prod_{f \in E} Z(G_f) \) whose component in \( Z(G_v) \) is \( u_i \) for \( v \in \Gamma_i \), and whose component in \( Z(G_f) \) is \( u_i^{-1} \) if \( f \subset \Gamma_i \) and \( m \) if \( f = e \).

The argument when \( e \) does not separate is similar, using representatives sending \( \pi_1(\Gamma \setminus \{e\}) \) to itself (by the identity for twists around \( e \)). Minimality of \( \Gamma_1 \) implies that both inclusions \( G_e \hookrightarrow G_1 \) are proper, so \( N_G(G_1) = G_1 \).

The proof is now complete for graphs such that all edge groups are proper subgroups of vertex groups. The general case requires one more argument.

- **Third step: graphs with a collapsible edge.**

We prove the result for a graph \( \Gamma \) containing a vertex \( v \) of valence 2, with adjacent edges \( e, f \), such that \( G_e = G_v \). We shall make \( \tilde{z} \) trivial by multiplying it by elements of \( J \) (i.e. applying the edge and vertex relations).

Call \( u \) (resp. \( w \)) the other vertex of \( e \) (resp. \( f \)). Let \( \Gamma' \) be the graph of groups obtained by collapsing \( e \), i.e. replacing \( e \cup f \) by a single edge \( g \) with group \( G_f \).

The automorphism \( \theta(\tilde{z}) \) twists by some \( z_1 \) around \( e \) near \( u \), by \( z_2 \) near \( v \), by \( z_3 \) around \( f \) near \( v \), by \( z_4 \) near \( w \). This is the same automorphism as twisting around \( g \) in \( \Gamma' \), by \( z_1z_2^{-1}z_3 \) near \( u \) and by \( z_4 \) near \( w \) (and twisting as in \( \Gamma \) around edges other than \( g \)). Since \( \Gamma' \) is minimal, not a mapping torus, we may use the induction hypothesis and kill \( z_1z_2^{-1}z_3, z_4 \), and all twists around edges other than \( g \), by applying all edge and vertex relations of \( \Gamma' \). If we use only those relations that exist in \( \Gamma \), we cannot use the relation corresponding to the edge \( g \) and we can only replace \( z_1z_2^{-1}z_3 \) and \( z_4 \) by some \( m \in Z(G_f) \).

Going back to \( \Gamma \), the relations let us replace \( (z_1, z_2, z_3, z_4) \) by \( (mz_3^{-1}z_2, z_2, z_3, m) \) and make all other \( z \)'s trivial. The element \( mz_3^{-1}z_2 \) commutes with \( G_e \), and belongs to \( G_e \) because all three factors do. It is then easy to make \( mz_3^{-1}z_2, z_2, z_3, m \) trivial by using the edge relations across \( e \) and \( f \) and the vertex relation at \( v \).

- **Fourth step: conclusion.**
Let $\Gamma$ be any graph with more than one edge, satisfying the hypotheses of the proposition. We may assume that there is no vertex $v$ as in step 3. There is then an edge $e$ with $\Gamma \setminus \{e\}$ minimal. We argue as in step 2.

If no component of $\Gamma \setminus \{e\}$ is a mapping torus, we are done. If $e$ separates and $G_i$ is a mapping torus, we observe that the restriction of $D_i$ to $G_i$ is conjugation by an element of $G_v$, because all representatives used in step 2 send $G_1$ (resp. $G_2$) to itself. Thus step 1 applies (since mapping tori were ruled out only to prove $m \in G_1$).

There are now two cases left. First, $\Gamma$ may consist of two loops at a vertex $v$, with all four edge groups mapping onto $G_v$. Then $G = \langle G_v, t_1, t_2 \mid t_i g t_i^{-1} = \varphi_i(g) \rangle$ if $g \in G_v$. If an automorphism sending $t_i$ to $z_i t_i z_i'$ is $i_m$, then $m \in G_v$ because it normalizes $< G_v, t_i >$ and we apply step 1. The last case is when $\Gamma$ has 3 edges, each going from a vertex $v$ to a vertex $w$ (and all edge groups map onto vertex groups). We then use vertex relations to kill twists at both ends of one of the edges, and we are reduced to the previous case. \hfill $\Box$

Remark. Trivial examples show that the minimality hypothesis is necessary. In the case of a mapping torus, the result fails precisely in the following situation: $\varphi \in \text{Aut}(G_1)$ satisfies $\varphi^\ell = i_g$ for some $\ell \geq 2$ and $g \in G_1$ such that $g^{-1} \varphi(g) \in Z(G_1)$ but there is no $n \in Z(G_1)$ with $g^{-1} \varphi(g) = n^{-1} \varphi(n)$. We do not know whether this situation is possible.

4. Relative automorphism groups

Many arguments in the beginning of this section are simple computations which are left to the reader.

Let $H$ be a group, with a finite collection of (peripheral) subgroups $\mathcal{H} = \{H_1, \ldots, H_k\}$ (some groups may be repeated). For instance, $H$ is a vertex group in a graph of groups, and the $H_i$’s are the incident edge groups.

Let $\text{Aut}_\mathcal{H}(H)$ be the group of automorphisms which act on each $H_i$ as conjugation by some $a_i \in H$, and let $MCG(H)$ be its image in $\text{Out}(H)$.

We define $\text{Aut}_{\mathcal{H}}^\partial(H)$ as the set of $(\alpha; a_1, \ldots, a_k)$, where $\alpha \in \text{Aut}_\mathcal{H}(H)$, $a_i \in H$, and $\alpha(g) = a_i g a_i^{-1}$ for $g \in H_i$. If $\alpha$ acts on $H_i$ as conjugation by $a_i$, and $\beta$ acts as conjugation by $b_i$, then $\alpha \beta$ acts as conjugation by $\alpha(b_i) a_i$. We therefore define

$$(\alpha; a_1, \ldots, a_k)(\beta; b_1, \ldots, b_k) = (\alpha \beta; \alpha(b_1) a_1, \ldots, \alpha(b_k) a_k).$$

This makes $\text{Aut}_{\mathcal{H}}^\partial(H)$ into a group, with an exact sequence

$$\{1\} \to \prod_{i=1}^k Z_H(H_i) \to \text{Aut}_{\mathcal{H}}^\partial(H) \to \text{Aut}_{\mathcal{H}}(H) \to \{1\}.$$

The map $g \mapsto (i_g; g, \ldots, g)$ embeds $H$ as a normal subgroup of $\text{Aut}_{\mathcal{H}}^\partial(H)$ and we define $MCG_{\mathcal{H}}^\partial(H)$ as the quotient (note that $MCG(H)$ and $MCG_{\mathcal{H}}^\partial(H)$ depend on $\mathcal{H}$). Up to isomorphism, $\text{Aut}_{\mathcal{H}}^\partial(H)$ and $MCG_{\mathcal{H}}^\partial(H)$ do not change if each $H_i$ is replaced by a conjugate. One easily checks:
Lemma 4.1. There is an exact sequence

\[ \{1\} \to \left( \prod_{i=1}^{k} \mathbb{Z} \left( H_i \right) \right) / \mathbb{Z}(H) \to MCG^\partial(H) \xrightarrow{\pi_H} MCG(H) \to \{1\}, \]

with \( \mathbb{Z}(H) \) embedded diagonally into \( \prod \mathbb{Z} H(H_i) \). If \( Z_H(H_i) \subset H_i \) for every \( i \), this extension is central.

Examples.

- If \( k = 0 \), then \( MCG^\partial(H) = \text{Out}(H) \). If \( k = 1 \), the map \( (\alpha; a) \mapsto i_a^{-1}\alpha \) identifies \( MCG^\partial(H) \) with the group of automorphisms of \( H \) equal to the identity on \( H_1 \).

- Suppose the \( H_i \)'s are all trivial. Then \( MCG(H) = \text{Out}(H) \). If \( k = 1 \), we have \( MCG^\partial(H) = \text{Aut}(H) \) and the exact sequence of Lemma 4.1 is simply \( \{1\} \to H/\mathbb{Z}(H) \to \text{Aut}(H) \to \text{Out}(H) \to \{1\} \). If \( k \geq 1 \), we get \( \{1\} \to H^k/\mathbb{Z}(H) \to MCG^\partial(H) \to \text{Out}(H) \to \{1\} \), with \( \mathbb{Z}(H) \) embedded diagonally into \( H^k \).

- If \( H = \mathbb{Z} \) and at least one \( H_i \) is nontrivial, then \( MCG^\partial(H) \cong \mathbb{Z}^{k-1} \), the quotient of \( \mathbb{Z}^k \) by \( \mathbb{Z} \) embedded diagonally.

- If \( H \) is the fundamental group of a compact surface \( \Sigma \) with \( \chi(\Sigma) < 0 \), and \( \mathcal{H} \) comes from the boundary curves, then \( MCG^\partial(H) = MCG^\partial(\Sigma) \), the group of homeomorphisms fixing \( \partial\Sigma \) pointwise, up to isotopy relative to \( \partial\Sigma \). It is a central extension of \( MCG(\Sigma) = MCG(H) \) by \( \mathbb{Z}^k \), where \( k \) is the number of boundary components.

In general this extension is not trivial. For instance, the “lantern relation” (see e.g. [8, section 4]) implies that the kernel \( \mathbb{Z}^k \) is contained in the derived subgroup of \( MCG^\partial(\Sigma) \) whenever \( \Sigma \) has genus \( \geq 2 \). When \( \Sigma \) is a once-punctured torus the extension, though not trivial, is trivial above a finite index subgroup of \( MCG(\Sigma) \). This is probably not true for surfaces of high genus.

Returning to a graph of groups \( \Gamma \) as in section 2, we consider \( \text{Aut}^\partial_H(G_v) \) and \( MCG^\partial(G_v) \) relative to the collection of incident edge groups \( (k \) is the valence of \( v \) in \( \Gamma \)). We denote by \( \pi_v : MCG^\partial(G_v) \to MCG(G_v) \) the natural projection, and we define \( \pi = \prod_{v \in V} \pi_v : \prod MCG^\partial(G_v) \to \prod MCG(G_v) \).

The extension construction of section 2 may be viewed as a homomorphism \( \text{Aut}^\partial_H(G_v) \to \text{Aut}(G) \). It induces \( \lambda_v : MCG^\partial(G_v) \to \text{Out}_0^\Gamma(\Sigma) \), with \( \rho_v \circ \lambda_v = \pi_v \). The images of \( \lambda_v \) and \( \lambda_w \) commute for \( v \neq w \), so that we get \( \lambda : \prod_{v \in V} MCG^\partial(G_v) \to \text{Out}_0^\Gamma(G) \). We call \( \text{Out}_1^\Gamma(G) \) the image of \( \lambda \).
Proposition 4.2. There is a commutative triangle of exact sequences

\[
\begin{array}{ccc}
\prod_{e \in E} \frac{Z_{G_v}(G_e)}{Z(G_v)} & \xrightarrow{\kappa} & \prod MCG^\partial(G_v) \\
N & \xrightarrow{\lambda} & \prod MCG(G_v) \\
\mathcal{T} & \xrightarrow{\rho_1} & \text{Out}^\Gamma_1(G)
\end{array}
\]

where \(\text{Out}^\Gamma_1(G)\) is a subgroup of \(\text{Out}^\Gamma(G)\), all products are taken over \(V\), and \(\mathcal{T}\) is the group of twists.

If all groups \(\text{Out}(G_e)\) are finite, then \(\text{Out}^\Gamma_1(G)\) has finite index in \(\text{Out}^\Gamma(G)\). If all edge groups are trivial, then \(\text{Out}^\Gamma_1(G) = \text{Out}^\Gamma_0(G)\) and \(\lambda\) is an isomorphism.

Proof. We have seen that \(\pi\) factors as \(\rho_1 \circ \lambda\), with \(\rho_1\) the restriction of \(\rho\) to \(\text{Out}^\Gamma_1(G)\). The kernel of \(\pi\) is given by Lemma 4.1, and the extension construction shows \(\ker \rho_1 = \mathcal{T}\). Note that \(N\) is the kernel of the natural epimorphism \(\kappa : \prod_{e \in E_v} \frac{Z_{G_v}(G_e)}{Z(G_v)} \to \mathcal{T}\).

The group \(\text{Out}^\Gamma_1(G)\) is mapped by \(\rho\) onto \(\prod MCG(G_v)\), and it contains \(\mathcal{T}\). If all groups \(\text{Out}(G_e)\) are finite, Proposition 2.3 implies that \(\text{Out}^\Gamma_1(G)\) has finite index in \(\text{Out}^\Gamma_0(G)\), hence in \(\text{Out}^\Gamma(G)\). If every \(G_e\) is trivial, then \(MCG(G_v) = \text{Out}(G_v)\), and \(\mathcal{T} = \ker \rho\) by Proposition 2.2. Thus \(\text{Out}^\Gamma_1(G) = \text{Out}^\Gamma_0(G)\). Furthermore \(\lambda\) is injective, because \(N = \ker \lambda = \ker \kappa\) is generated by centers of edge groups by Lemma 3.1. \(\square\)

5. Hyperbolic groups

In this section \(G\) is a one-ended hyperbolic group. The elementary subgroups of \(G\) are the finite or virtually cyclic (= 2-ended) subgroups. Recall (see [10]) that there are two types of infinite elementary groups, those with infinite center (which map onto \(\mathbb{Z}\)) and those with finite center (which map onto the infinite dihedral group). The normalizer of a virtually cyclic subgroup is a maximal elementary subgroup. If \(G\) is torsion-free, nontrivial elementary subgroups are cyclic.

Let \(\Gamma\) be the JSJ splitting of \(G\), as constructed by Bowditch [4]. Edge groups of \(\Gamma\) are virtually cyclic. There are three types of vertices:

1. Elementary vertices, whose group is a maximal elementary subgroup.
2. Orbifold vertices, whose group is a “hanging Fuchsian group”, the fundamental group of a 2-orbifold.
3. Rigid vertices, whose group $G_v$ cannot be split any further over an elementary subgroup. By Paulin’s theorem [11] and Rips theory [2], the group $\text{MCG}(G_v)$ is finite (we need a relative version of these results, which follows from Theorem 9.6 of [2] since applying [11] to automorphisms of $\text{MCG}(G_v)$ gives a tree in which every peripheral subgroup of $G_v$ fixes a point).

We denote by $V_1$, $V_2$, $V_3$ the corresponding sets of vertices of $\Gamma$. Every edge joins an elementary vertex to an orbifold or rigid vertex, we denote $E = E_2 \cup E_3$ the set of geometric edges. We shall add the symbol $\infty$ to indicate that we restrict to edges or elementary vertices whose group has infinite center (orbifold and rigid vertex groups always have finite center). The notation $|A|$ denotes the number of elements of a finite set $A$.

Edge groups may fail to be maximal as elementary subgroups of $G$, but it is easy to see that they are maximal elementary in the corresponding rigid or orbifold vertex group (otherwise their normalizer would not be elementary). If $v \in V_2 \cup V_3$, we have $Z_{G_v}(G_e) = Z(G_e)$ for $e \in E_v$, and $\text{MCG}^\partial(G_v)$ is a central extension of $\text{MCG}(G_v)$ by Lemma 4.1.

**Theorem 5.1.** There is an exact sequence

$$1 \to T \to \text{Out}_2(G) \xrightarrow{\rho_2} \prod_{v \in V_2} \text{MCG}(G_v) \to 1,$$

where $\text{Out}_2(G)$ has finite index in $\text{Out}(G)$. The group of twists $T$ is virtually $\mathbb{Z}^n$, with $n = |E^\infty| - |V_1^\infty|$. If $G$ is torsion-free, then $T$ is free abelian of rank $|E| - |V_1|$ and the extension is central.

**Proof.** Since $\text{MCG}(G_v)$ is finite if $v \in V_1 \cup V_3$, we define $\text{Out}_2(G)$ as the kernel of the map $\text{Out}_1(G) \to \prod_{v \in V_1 \cup V_3} \text{MCG}(G_v)$. The exact sequence then comes from Proposition 4.2.

We use Proposition 3.1 to compute the rank of $T$. Since $Z_{G_v}(G_e) = Z(G_e)$ if $e$ is an edge and $v$ is its rigid or orbifold vertex, we see that $T$ is the quotient of $T' = \prod_{v \in V_1} \frac{\prod_{e \in E_v} Z_{G_v}(G_e)}{Z(G_v)}$ by a finite group (corresponding to orbifold and rigid vertex relations).

First suppose that $G$ is torsion-free. Then $T = T'$ because $Z(G_v)$ is trivial if $v \notin V_1$. Of course $Z_{G_v}(G_e) = Z(G_e) = G_v = \mathbb{Z}$ for $v \in V_1$, and the result follows, recalling that $Z(G_v)$ is embedded diagonally in $\prod Z_{G_v}(G_e)$.

If $G$ has torsion, we obtain that $T'$, hence also $T$, is virtually $\mathbb{Z}^n$ with $n = |E^\infty| - |V_1^\infty|$ because $Z_{G_v}(G_e)$ is infinite if and only if $G_e$ has infinite center. The extension is central as soon as $\text{MCG}^\partial(G_v)$ is a central extension of $\text{MCG}(G_v)$ for every $v$, in particular if $G$ is torsion-free. \qed

Proposition 4.2 also yields the exact sequence

$$\{1\} \to N \to A \times \prod_{v \in V_2} \text{MCG}^\partial(G_v) \to \text{Out}_2(G) \to \{1\},$$

13
where \( A = \prod_{v \not\in V_2} \ker \pi_v = \prod_{v \not\in V_2} \frac{\prod_{e \in E_v} Z_{G_v}(G_e)}{Z(G_o)} \) is virtually abelian. Note that \( N \) is also virtually abelian (because it is a subgroup of \( \ker \pi \)). We shall now use the edge and vertex relations in order to replace \( A \) and \( N \) by simpler groups.

We define a set \( Q \) of edges as follows. We consider \( E_\infty \) (edges \( e \) connected to a rigid vertex, with \( Z(G_e) \) infinite), and for each \( v \in V_1 \) whose group has infinite center we discard one edge of \( E_\infty \) adjacent to \( v \) (if there is at least one). We view \( Q \) as a set of oriented edges, going from \( V_1 \) to \( V_3 \). Note that \( q = |Q| \) equals \( |E_\infty| - |V_1| + r \), where \( r \) is the number of vertices of \( |V_1| \) connected only to orbifold vertices.

Now consider the map \( \psi : \prod_{e \in Q} Z_{G_o(e)}(G_e) \times \prod_{v \in V_2} \prod_{e \in E_v} Z_{G_v}(G_e) \to \text{Out}_2(G) \) obtained by mapping the first factor to \( A \) and applying \( \lambda \).

**Lemma 5.2.** The image of \( \psi \) has finite index. The images of the two factors have finite intersection. The intersection of \( \ker \psi \) with the first (resp. second) factor is finite (resp. virtually \( \mathbb{Z}^r \)).

**Proof.** The first factor maps into \( \mathcal{T} \), the kernel of \( \rho_2 : \text{Out}_2(G) \to \prod_{v \in V_2} \text{MCG}(G_v) \). The image by \( \psi \) of the second factor is mapped onto the whole of \( \prod_{v \in V_2} \text{MCG}(G_v) \) by \( \rho_2 \), and the intersection of \( \psi^{-1}(\mathcal{T}) \) with the second factor is the subgroup \( \prod_{v \in V_2} \frac{\prod_{e \in E_v} Z_{G_v}(G_e)}{Z(G_v)} \). It therefore suffices to prove the statements of the lemma for the map

\[
\prod_{e \in Q} Z_{G_o(e)}(G_e) \times \prod_{v \in V_2} \frac{\prod_{e \in E_v} Z_{G_v}(G_e)}{Z(G_v)} \to \mathcal{T},
\]

or for

\[
\prod_{e \in Q} Z_{G_o(e)}(G_e) \times \prod_{v \in V_2} \prod_{e \in E_v} Z_{G_v}(G_e) \to \mathcal{T}
\]

since \( Z(G_v) \) is finite for \( v \in V_2 \). But this follows directly from Proposition 3.1, as we now explain.

In this argument, we neglect finite groups (and we don’t distinguish between a group and a finite index subgroup). We know that \( \mathcal{T} \) is generated by all groups \( Z_{G_o(e)}(G_e) \). The edge relations reduce the generating set to \( \prod_{e \in Q} Z_{G_o(e)}(G_e) \times \prod_{v \in V_2} \prod_{e \in E_v} Z_{G_v}(G_e) \), where \( Q \) is the set of oriented edges from \( V_1 \) to \( V_3 \) whose group has infinite center. Now we have to consider vertex relations at vertices \( v \in V_1^\infty \). Vertices connected to at least one rigid vertex let us reduce from \( Q \) to \( Q \), while vertices connected only to orbifold vertices generate the desired \( \mathbb{Z}^r \) in the second factor. \( \square \)
Theorem 5.3. Let \( r \) be the number of vertices of \( |V_1^\infty| \) connected only to orbifold vertices. Let \( q = |E_3^\infty| - |V_1^\infty| + r \) and \( s = |E_2^\infty| - r = n - q \). The group \( \text{Out}(G) \) is virtually a direct product \( \mathbb{Z}^q \times M \), where \( M \) fits in exact sequences

\[
\{1\} \to Z_r \to \prod_{v \in V_2} \text{MCG}^0(G_v) \to M \to \{1\}
\]

\[
\{1\} \to Z_s \to M \to \prod_{v \in V_2} \text{MCG}(G_v) \to \{1\}
\]

with \( Z_r \) (resp. \( Z_s \)) virtually \( \mathbb{Z}^r \) (resp. \( \mathbb{Z}^s \)). If \( G \) is torsion free, then \( Z_r = \mathbb{Z}^r \), \( Z_s = \mathbb{Z}^s \), and the extensions are central.

Proof. This follows from Lemma 5.2, with \( M \) the image by \( \psi \) of the second factor, and \( \mathbb{Z}^q \) the image of a \( \mathbb{Z}^q \) of finite index in \( \prod_{e \in Q} \mathbb{Z}_{G_{o(e)}}(G_e) \). The group \( Z_s \) is virtually \( \mathbb{Z}^s \) because \( \mathbb{Z}^q \times \mathbb{Z}^s \) has finite index in \( T \), which is virtually \( \mathbb{Z}^{q+s} \).

If \( G \) is torsion-free, then \( Z_r \) is contained in the free abelian group \( \prod_{v \in V_2} \frac{\prod_{e \in E_v} \mathbb{Z}_{G_v}(G_e)}{Z(G_v)} \) and \( Z_s \) is contained in \( T = \mathbb{Z}^n \). The extensions are central by Lemma 4.1 and Theorem 5.1. \( \square \)

Corollary 5.4. Suppose \( G \) is torsion free. If every cyclic vertex is connected to at least one rigid vertex, then \( \text{Out}(G) \) is virtually a direct product of a free abelian group with mapping class groups \( \text{MCG}^0(\Sigma_v) \) of surfaces with boundary. \( \square \)

6. HYPERBOLIC GROUPS WITH \( \text{Out}(G) \) INFINITE

In this section we prove Theorem 1.4. The following fact is quite general (compare [10, Proposition 2.1]).

Proposition 6.1. Let \( G \) be any group. Suppose \( G = G_1 \ast_H G_2 \) or \( G = G_1 \ast_H \), where \( H \) has infinite center but \( G_1, G_2 \) have finite center. Then \( \text{Out}(G) \) is infinite.

This may be viewed as a special case of Proposition 3.1 (see also [10] for the case of an amalgam), and proves one direction of Theorem 1.4 (note that \( G_1 \) has finite center if \( G = G_1 \ast_H \) is hyperbolic and \( H \) is virtually cyclic).

Conversely, suppose \( G \) is one-ended, hyperbolic, with \( \text{Out}(G) \) infinite. Consider the exact sequence of Theorem 5.1. Since \( G \) splits as in Theorem 1.4 if \( n > 0 \), we may assume that there exists \( v \in V_2 \) with \( \text{MCG}(G_v) \) infinite (note that this argument requires Proposition 2.3, but not Proposition 3.1).

The group \( G_v \) maps into the group of isometries of \( H^2 \) with finite kernel (see [4]). We start by assuming this kernel to be trivial.

The (convex core of the) quotient orbifold \( \Sigma_v = H^2/G_v \) is homeomorphic to a compact surface, with a singular set which may be one-dimensional: this phenomenon is responsible for the existence of virtually cyclic subgroups of \( G_v \) with finite center. The required splitting of \( G \) will come from a splitting of \( G_v \), given by an essential 2-sided simple closed curve \( C \subset \Sigma_v \) disjoint from the singular set.
If \( G_v \) is torsion-free, then \( \Sigma_v \) is a regular compact hyperbolic surface with infinite mapping class group. Thus \( \Sigma_v \) cannot be a pair of pants or a twice-punctured projective plane. On all other surfaces we can find the required essential 2-sided curve \( C \) (essential means that, if \( C \) separates, then both complementary regions have negative Euler characteristic).

If \( G_v \) has torsion, then \( \Sigma_v \) is an orbifold. By [7], generalized to the nonorientable case in [6], the group \( \text{MCG}(G_v) \) is commensurable with the mapping class group of a regular compact surface \( \Sigma_0 \) obtained from \( \Sigma_v \) by removing an open neighborhood of the singular set. An essential curve \( C \subset \Sigma_0 \) yields the desired splitting of \( G_v \).

Up to now we have assumed that \( G_v \) acts effectively on \( H^2 \). If not, there is a finite normal subgroup \( F \subset G_v \) with \( G_v' = G_v/F \subset \text{Isom}(H^2) \). As pointed out in [4], \( F \) is the unique maximal finite normal subgroup of \( G_v \). Automorphisms of \( G_v \) therefore induce automorphisms of \( G_v' \). The map \( \text{Aut}(G_v) \to \text{Aut}(G_v') \), hence also the map \( \text{MCG}(G_v) \to \text{MCG}(G_v') \), is at most \( |F|^k \)-to-one if \( G_v \) may be generated by \( k \) elements, and therefore \( \text{MCG}(G_v') \) is infinite. The splitting of \( G_v' \) constructed above lifts to a splitting of \( G_v \) and extends to a splitting of \( G \) (the splitting of \( G_v \) is over a group that maps onto \( \mathbb{Z} \), hence has infinite center).

This completes the proof of Theorem 1.4.

**Remark.** We sketch a proof which does not use the orbifold theory of [7] and [6]. View \( \Sigma_v \) as the quotient of a surface \( \Sigma \) by a finite group \( \Omega \). An algebraic argument shows that the mapping class group of \( \Sigma \) contains an element of infinite order commuting with \( \Omega \). Represent it by a homeomorphism consisting of pseudo-Anosov homeomorphisms of subsurfaces and Dehn twists along disjoint curves. Because of \( \Omega \)-equivariance, the support of these maps is disjoint from the one-dimensional components of the branching locus of \( \Omega \), and we can construct a curve \( C \).

### 7. Automorphisms of finite order

This section is devoted to the proof of Theorem 1.5.

**Algebraic preliminaries.**

We denote by \( \mathcal{C} \) the class of groups with only finitely many conjugacy classes of torsion elements.

**Lemma 7.1.** Let \( B \) be a group. If one of the following holds, then \( B \in \mathcal{C} \).

1. \( B = A \times A' \), or \( B = A * A' \), with \( A, A' \in \mathcal{C} \).
2. \( B \) is a subgroup of finite index of \( A \), and \( A \in \mathcal{C} \).
3. There is an epimorphism \( p : B \to A \), with \( A \in \mathcal{C} \), such that \( p^{-1}(F) \in \mathcal{C} \) for every finite subgroup \( F \subset A \).
4. \( B \) contains a normal subgroup \( H \) with \( H \) hyperbolic and \( B/H \in \mathcal{C} \).

**Proof.** (1), (2), (3) are simple observations. By (3), it suffices to prove (4) when \( B/H \) is finite. But then \( B \) is hyperbolic, hence in \( \mathcal{C} \). \( \square \)
Remark. Unfortunately, there is no converse to (2). Let \( A \) be the wreath product \( \mathbb{Z} \wr \mathbb{Z} \), with commuting generators \( a_i \ (i \in \mathbb{Z}) \) and an extra generator \( t \) conjugating \( a_i \) to \( a_{i+1} \). Let \( B \) be the semi-direct product \( A \rtimes \mathbb{Z}_2 \), where \( \varphi \) is the involution sending \( a_i \) to \( a_i^{-1} \) and \( t \) to \( t \). Then \( B \) is a finitely generated group with infinitely many conjugacy classes of elements of order 2, but \( A \) is a torsion-free subgroup of index 2.

**Proposition 7.2.** Let \( B \) be a group. If one of the following holds (with \( n \) a positive integer), then \( B \in \mathcal{C} \).

(i) There is an exact sequence \( \{1\} \to K \to B \to A \to \{1\} \), with \( K \) virtually \( \mathbb{Z}^n \) and \( A \in \mathcal{C} \).

(ii) \( B \) is the semi-direct product \( A^n \rtimes S_n \), with the symmetric group \( S_n \) acting on \( A^n \) by permuting the factors, and \( A \in \mathcal{C} \).

**Proof.** We first prove (i) when \( K = \mathbb{Z}^n \) and \( A \) is finite. For every \( a \in A \), choose a lift \( b_a \in B \) of finite order, if there is one. Also fix an integer \( q \) such that \( b_a^q = 1 \) for every torsion element \( b_a \in B \).

Given \( b \) of finite order, we write \( b = tb_a \) with \( a \in A \) and \( t \in \mathbb{Z}^n \), and we consider the relation \( (tb_a)^q = 1 \). The action of \( b_a \) on \( \mathbb{Z}^n \) by conjugation is given by a matrix \( P \in GL(n, \mathbb{Z}) \), with \( P^q = 1 \), and \( (tb_a)^q = tP(t)P^2(t) \ldots P^{q-1}(t)b_a^q \). We get \( tP(t)P^2(t) \ldots P^{q-1}(t) = 1 \).

Over the rationals we would conclude that \( t \) may be written \( s^{-1}P(s) \) for some \( s \), because the restriction of \( P \) to the subspace generated by \( t, P(t), P^2(t), \ldots, P^{q-1}(t) \) does not have 1 as an eigenvalue. Here we obtain \( t = s^{-1}P(s)t' \) with \( t' \) belonging to some finite set (depending on \( a \)). We then write \( b = tb_a = s^{-1}P(s)t'b_a = s^{-1}t'b_a.s \). Since there are finitely many couples \( (b_a, t') \), we get \( B \in \mathcal{C} \).

We have now proved that every group which is virtually \( \mathbb{Z}^n \) belongs to \( \mathcal{C} \). The general case of (i) follows from assertion (3) of Lemma 7.1.

To prove (ii), write \( b \in B \) as \( (a_1, \ldots, a_n)\sigma \) with \( a_i \in A \) and \( \sigma \in S_n \). First suppose that \( \sigma \) is an \( n \)-cycle, say the standard \( n \)-cycle. For \( c_i \in A \) we have \( \sigma(c_i^{-1}, \ldots, c_n^{-1}) = (c_2^{-1}, \ldots, c_n^{-1}, c_1^{-1})\sigma \) and

\[
(c_1, \ldots, c_n)(a_1, \ldots, a_n)\sigma(c_1, \ldots, c_n)^{-1} = (c_1a_1c_2^{-1}, \ldots, c_{n-1}a_{n-1}c_n^{-1}, c_na_n^{-1})\).
\]

It follows that \( b \) is conjugate to \( (a, 1, \ldots, 1)\sigma \), with \( a = a_1 \ldots a_n \), and that \( (a, 1, \ldots, 1)\sigma \) and \( (a', 1, \ldots, 1)\sigma \) are conjugate whenever \( a \) and \( a' \) are conjugate in \( A \). Also note that \( [(a, 1, \ldots, 1)\sigma]^n = (a, \ldots, a) \), and therefore \( (a, 1, \ldots, 1)\sigma \) has finite order if and only if \( a \) has finite order. We deduce that \( B \) contains only finitely many classes of torsion elements whose associated permutation is an \( n \)-cycle.

The general case follows easily. Write \( \sigma = \sigma_1 \ldots \sigma_k \) as a product of cycles with disjoint support, including cycles of length 1. This induces a decomposition of \( b \) as a product of commuting factors \( b = (\theta_1\sigma_1)(\theta_2\sigma_2) \ldots (\theta_k\sigma_k) \), where the \( j \)-th component of \( \theta_i \in A^n \) is \( a_j \) if \( j \) belongs to the support of \( \sigma_i \) and 1 otherwise. We conclude by arguing as above in each factor separately. \( \square \)
Groups with one end.

Let $G$ be a torsion-free one-ended hyperbolic group, and $\Gamma$ its JSJ splitting as before. We associate to $\hat{\alpha} \in \text{Out}(G)$ the induced permutation $\sigma$ on the vertex set $V$ of $\Gamma$. This defines $\mu : \text{Out}(G) \to S(V)$, and we call $S(\Gamma)$ its image.

In section 2 we restricted to automorphisms $\alpha$ acting trivially on $V$, and we defined $\rho_v(\hat{\alpha}) \in \text{Out}(G_v)$ for $v \in V$. In the general case, we obtain a set of isomorphisms $\eta_v(\hat{\alpha}) : G_v \to G_{\sigma(v)}$, each defined up to composition with inner automorphisms. These isomorphisms preserve the peripheral structure (the set of conjugacy classes of edge groups), but in general not every isomorphism preserving the peripheral structure arises in this way.

We view this construction as a homomorphism $\eta$ from $\text{Out}(G)$ to a group $\Lambda$ defined as follows. An element of $\Lambda$ consists of $\sigma \in S(\Gamma)$ and isomorphisms $\eta_v(\hat{\alpha}) : G_v \to G_{\sigma(v)}$ preserving the peripheral structure, defined up to inner automorphisms. The group law on $\Lambda$ is defined in the obvious way.

The rest of the proof consists in showing that $\Lambda \in C$, the image of $\eta$ has finite index in $\Lambda$, and $\ker \eta$ is virtually $\mathbb{Z}^n$. These facts imply Theorem 1.5 (for $G$ one-ended) by 7.1.2 and 7.2.i.

The kernel of the natural projection $\pi : \Lambda \to S(\Gamma)$ is $\prod_{v \in V} A_v$, with $A_v \subset \text{Out}(G_v)$ consisting of automorphisms preserving the peripheral structure. The group $A_v$ is finite if $v$ is a rigid or cyclic vertex. If $v$ is a surface vertex, then $A_v$ is the full mapping class group of the surface (boundary components may be permuted, orientation may be reversed). In all cases $A_v \in C$.

The reason for introducing the abstract group $\Lambda$ is that the map $\pi$ always has a section: for every orbit $V_i$ of the action of $S_v$ on $V$, choose isomorphisms compatible with the peripheral structures) of the groups $G_v, v \in V_i$, with a fixed model, and use these identifications to construct the section.

We now know that $\Lambda$ is a semidirect product $\prod_{v \in V} A_v \rtimes S(\Gamma)$. It embeds as a finite index subgroup into $\prod_i \Lambda_i$, with $\Lambda_i = (\prod_{v \in V_i} A_v) \rtimes S(V_i)$. Proposition 7.2.ii implies $\Lambda_i \in C$, and $\Lambda \in C$ by 7.1.1 and 7.1.2.

Now consider $\eta : \text{Out}(G) \to \Lambda$. By Proposition 2.1 it contains $\prod_{v \in V} \text{MCG}(G_v)$, which has finite index in $\prod_{v \in V} A_v$. Thus the image of $\eta$ has finite index. Its kernel contains $\ker \rho$ with finite index (an element of $\ker \eta$ belongs to $\ker \rho$ if and only if it acts trivially on the set of edges of $\Gamma$). But $\ker \rho$ is virtually $\mathbb{Z}^n$ by Proposition 2.3 and Theorem 5.1.

This completes the proof of Theorem 1.5 for one-ended groups.

Groups with infinitely many ends.

We now write $G = G_1 * \cdots * G_p * F_q$ with $G_i$ one-ended and $F_q$ free of rank $q$. We argue by induction on the Kurosh rank $r(G) = p + q$.

The relative train track maps of Bestvina-Handel [3] have been generalized to free products in [5], so that we can represent any $\hat{\alpha} \in \text{Out}(G)$ by a relative train track map $\varphi : X \to X$. We assume that $\hat{\alpha}$ has finite order, and we use $\varphi$ to construct an $\alpha$-invariant splitting of $G$.

The map $\varphi$ is a self-homotopy equivalence of a complex $X$ obtained by attaching connected complexes $X_i$ with $\pi_1(X_i) \simeq G_i$ onto vertices of a finite connected graph.
\( \Delta \) with \( \pi_1(\Delta) \simeq F_q \). Let \( \Delta_t \) (resp. \( \Delta'_t \)) be the union of edges (resp. open edges) in the top stratum. The complement of \( \Delta'_t \) in \( X \) is mapped to itself by \( \varphi \).

Since \( \hat{\alpha} \) has finite order, \( \Delta_t \) cannot be exponentially growing. It follows that the (unoriented) edges of \( \Delta_t \) are cyclically permuted (modulo the lower strata); they may be numbered \( e_1, \ldots, e_k \) in such a way that \( \varphi(e_i) \) consists of \( e_{i+1} \), possibly preceded and followed by a path disjoint from \( \Delta'_t \).

To obtain an \( \alpha \)-invariant graph of groups \( \Omega_{\alpha} \), we first make \( \Delta \) into a graph of groups in the obvious way (vertex groups are free products of \( G_i \)'s), and then we collapse all edges not in \( \Delta_t \).

Edge groups of \( \Omega_{\alpha} \) are trivial. Vertex groups \( G_v \) are free products of \( G_i \)'s and \( Z \)'s with Kurosh rank \( r(G_v) < r(G) \), so \( \text{Out}(G_v) \in \mathcal{C} \) by induction. Furthermore, there are only finitely many possible \( \Omega_{\alpha} \)'s, up to isomorphism. This means that there exist finitely many graph of groups decompositions \( \Omega_i \) of \( G \), such that every torsion element of \( \text{Out}(G) \) is conjugate to an element of an \( \text{Out}^{\Omega_i}(G) \). It therefore suffices to show:

**Proposition 7.3.** Suppose that \( G \) is the fundamental group of a finite graph of groups \( \Omega \) with trivial edge groups. Assume that vertex groups are torsion-free hyperbolic groups \( G_v \) with \( \text{Out}(G_v) \in \mathcal{C} \). Then \( \text{Out}^{\Omega}(G) \in \mathcal{C} \).

**Proof.** As usual, we denote by \( \text{Out}^\Omega(G) \) the kernel of the natural map \( \psi : \text{Out}^\Omega(G) \to \text{Aut}(\Omega) \). Let \( L \) denote the image. Let \( V \) be the vertex set of \( \Omega \).

By Proposition 4.2, we have \( \text{Out}^\Omega(G) = \prod_{v \in V} \text{MCG}^\partial(G_v) \). Here \( \text{MCG}^\partial(G_v) \) is the automorphism group of the vertex group \( G_v \) relative to the family \( \mathcal{H} \) consisting of the trivial group repeated \( n_v \) times (\( n_v = |E_v| \) being the valence of \( v \) in \( \Omega \)). It fits in an exact sequence

\[
\{1\} \to (G_v)^{n_v}/Z(G_v) \to \text{MCG}^\partial(G_v) \to \text{Out}(G_v) \to \{1\}
\]

(see the second example in section 4). If \( G_v = Z \), then \( \text{MCG}^\partial(G_v) \) is virtually \( Z^{n_v-1} \). If \( G_v \neq Z \), the center of \( G_v \) is trivial and the kernel is simply \( (G_v)^{n_v} \); the natural action of \( \mathcal{S}(E_v) \) on \( \text{MCG}^\partial(G_v) \) permutes the factors of \( (G_v)^{n_v} \).

Let \( V_i \) be the orbits of \( V \) under the action of \( L \). For each \( v \in V_i \), we choose an identification of \( G_v \) with a fixed group \( G_i \), and also a numbering of edges adjacent to \( v \) (i.e. a bijection \( E_v \to \{1, \ldots, n_v\} \)). This gives canonical isomorphisms \( \text{MCG}^\partial(G_v) \to \text{MCG}^\partial(G_w) \) for \( v, w \) in the same \( V_i \), and provides a section to the map \( \psi : \text{Out}^\Omega(G) \to L \).

We now have an action of \( L \) on \( \prod_{v \in V} \text{MCG}^\partial(G_v) = \ker \psi \), which is a restriction of the natural action of the semi-direct product \( (\prod_{v \in V} \mathcal{S}(E_v)) \rtimes \prod_i \mathcal{S}(V_i) \). In other words, \( \text{Out}^\Omega(G) \) is a finite index subgroup of \( \left( \prod_{v \in V} \text{MCG}^\partial(G_v) \right) \rtimes \left( \prod_{v \in V} \mathcal{S}(E_v) \right) \rtimes \prod_i \mathcal{S}(V_i) \). It suffices to show that this larger group is in \( \mathcal{C} \).

We may rewrite it as \( \prod_i \left( \prod_{v \in V_i} \left( \text{MCG}^\partial(G_v) \rtimes \mathcal{S}(E_v) \right) \rtimes \mathcal{S}(V_i) \right) \), and by 7.1.1
and 7.2.ii we need only show \( MCG^\partial(G_v) \rtimes S(E_v) \in \mathcal{C} \). We may assume \( G_v \neq \mathbb{Z} \), since otherwise the group is virtually \( \mathbb{Z}^{n_v-1} \), hence in \( \mathcal{C} \) by 7.2.i.

With the notations of section 4, the map \((\alpha; a_1, \ldots, a_k) \mapsto (\alpha; a_1, \ldots, (\alpha; a_k))\) induces \( \zeta: MCG^\partial(G_v) \to \prod_{i=1}^{n_v} \text{Aut}(G_v) \) (recall that \( MCG^\partial = \text{Aut} \) when \( \mathcal{H} \) consists of one copy of the trivial group). Since \( G_v \) has trivial center, \( \zeta \) is an embedding. In general, its image does not have finite index.

Now let \( F \) be a finite subgroup of \( \text{Out}(G_v) \), and \( MCG^\partial_F(G_v), \text{Aut}_F(G_v) \) its preimages. The images of the embeddings \( MCG^\partial_F(G_v) \to \prod_{i=1}^{n_v} \text{Aut}_F(G_v) \) and \( MCG^\partial_F(G_v) \rtimes S(E_v) \to (\prod_{i=1}^{n_v} \text{Aut}_F(G_v)) \rtimes S(E_v) \) have finite index (all groups are virtually \((G_v)^{n_v}\)). The group \( \text{Aut}_F(G_v) \) is hyperbolic, hence in \( \mathcal{C} \). By 7.1.2 and 7.2.ii, we deduce that \( MCG^\partial_F(G_v) \rtimes S(E_v) \in \mathcal{C} \) for every finite \( F \subset \text{Out}(G_v) \).

Applying 7.1.3 to the map from \( MCG^\partial_F(G_v) \rtimes S(E_v) \) to \( \text{Out}(G_v) \), we get the required result \( MCG^\partial(G_v) \rtimes S(E_v) \in \mathcal{C} \).

\[ \square \]

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