Abstract

The extended persistence diagram introduced by Cohen-Steiner, Edelsbrunner, and Harer is an invariant of real-valued continuous functions, which are \(F\)-tame in the sense that all open interlevel sets have degree-wise finite-dimensional cohomology with coefficients in a fixed field \(F\). We show that relative interlevel set cohomology (RISC), which is based on the Mayer–Vietoris pyramid by Carlsson, de Silva, and Morozov, categorifies this invariant. More specifically, we define an abelian Frobenius category \(\text{pres}(\mathcal{J})\) of presheaves, which are presentable in a certain sense, such that the RISC \(h(f)\) of an \(F\)-tame function \(f: X \to \mathbb{R}\) is an object of \(\text{pres}(\mathcal{J})\), and moreover the extended persistence diagram of \(f\) uniquely determines – and is determined by – the corresponding element \([h(f)] \in K_0(\text{pres}(\mathcal{J}))\) in the Grothendieck group \(K_0(\text{pres}(\mathcal{J}))\) of the abelian category \(\text{pres}(\mathcal{J})\). As an intermediate step we show that \(\text{pres}(\mathcal{J})\) is the abelianization of the (localized) category of complexes of \(F\)-linear sheaves on \(\mathbb{R}\), which are tame in the sense that sheaf cohomology of any open interval is finite-dimensional in each degree. This yields a close link between derived level set persistence by Curry, Kashiwara, and Schapira and the categorification of extended persistence diagrams.

1 Introduction

Based on prior findings by [CdM09, BEMP13] we introduced the notion of relative interlevel set cohomology (RISC) over a fixed field \(F\) in [BBF21] as an invariant of real-valued continuous functions, which are \(F\)-tame in the sense that all open interlevel sets have degree-wise finite-dimensional cohomology with coefficients in \(F\). More specifically, for an \(F\)-tame function \(f: X \to \mathbb{R}\) its RISC is a functor \(h(f): \mathcal{M}^\circ \to \text{Vect}_F\) defined on the opposite poset \(\mathcal{M}^\circ\) of a sublattice \(\mathcal{M} \subset \mathbb{R}^\circ \times \mathbb{R}\) in the shape of an infinite strip depicted in Fig. 1.1. Here \(\mathbb{R}^\circ \times \mathbb{R}\) is the lattice which is given as the product of the reals \(\mathbb{R}\) with the opposite order and the reals \(\mathbb{R}\) with the ordinary \(\leq\)-order. The values of \(h(f): \mathcal{M}^\circ \to \text{Vect}_F\) are all relative cohomology groups of the form \(H^n(f^{-1}(I, C); F)\) for some \(n \in \mathbb{Z}\), some open interval \(I \subseteq \mathbb{R}\) and some subset \(C \subseteq I\), that is the complement of a closed interval in \(\mathbb{R}\). Now taking the difference

\[(I, C) \mapsto I \setminus C\]
yields a bijection between the set of all non-empty intervals in \( \mathbb{R} \) and the set of all such pairs \((I, C)\) with \(I \neq C\). Moreover, for any pair of open subspaces \((U, V)\) of \(\mathbb{R}\) with \(U \setminus V = I \setminus C\) we have
\[
H^n(f^{-1}(U, V); \mathbb{F}) \cong H^n(f^{-1}(I, C); \mathbb{F})
\]
by excision. Furthermore, given any pair of open subspaces \((U, V)\) of \(\mathbb{R}\) such that any connected component of \(U\) contains finitely many connected components of \(V\) we have
\[
H^n(f^{-1}(U, V); \mathbb{F}) \cong \prod_{(I, C)} H^n(f^{-1}(I, C); \mathbb{F}),
\]
where the product ranges over all pairs \((I, C)\) as above with \(I \setminus C\) a connected component of \(U \setminus V\). Thus, the RISC \(h(f): \mathcal{M}^o \rightarrow \text{Vect}_\mathbb{F}\) contains enough information to obtain the cohomology \(H^n(f^{-1}(U, V); \mathbb{F})\) for any such pair \((U, V)\). As it turns out, the support of \(h(f): \mathcal{M}^o \rightarrow \text{Vect}_\mathbb{F}\) is bounded above by the diagonal \(\{(t, t) \mid t \in \mathbb{R}\} \subset \mathbb{R}^2\). Moreover, we show in \cite[Section 2]{BBF21} that \(h(f): \mathcal{M}^o \rightarrow \text{Vect}_\mathbb{F}\) is point-wise finite-dimensional (pfd), cohomological (Definition \ref{def:cohomological}), and sequentially continuous (Definition \ref{def:sequentially_continuous}). Throughout this document – excluding the appendix – we denote the full subcategory of pfd functors \(\mathcal{M}^o \rightarrow \text{Vect}_\mathbb{F}\) that are cohomological, sequentially continuous, and have bounded above support by \(\mathcal{J}\). In \cite[Theorem 3.5]{BBF21} we show in particular that any functor in \(\mathcal{J}\) decomposes into a (potentially infinite) direct sum of indecomposables, whose supports each have the shape of a maximal axis-aligned rectangle in \(\mathcal{M}\) as shown in Fig. \ref{fig:1.1}. We denote such an indecomposable by \(B_v: \mathcal{M}^o \rightarrow \text{Vect}_\mathbb{F}\), where \(v \in \text{int} \mathcal{M}\) inside the interior of \(\mathcal{M}\) is the upper left corner of the rectangular support of \(B_v\), see also Definition \ref{def:indecomposable}. In \cite[Section 3.2.2]{BBF21} we also show that for an \(\mathbb{F}\)-tame function \(f: X \rightarrow \mathbb{R}\) the multiplicities of indecomposables \(B_v: \mathcal{M}^o \rightarrow \text{Vect}_\mathbb{F}\) in \(h(f): \mathcal{M}^o \rightarrow \text{Vect}_\mathbb{F}\) determine the extended persistence diagram of \(f\) due to \cite{CSEH09} and vice versa. Thus, the extended persistence diagram provides a complete isomorphism invariant for the class of functors obtained as RISC. Now the full subcategory \(\mathcal{J}\) is not an abelian subcategory of the category of all functors \(\mathcal{M}^o \rightarrow \text{Vect}_\mathbb{F}\). This raises the question about the smallest full abelian subcategory of \(\text{Vect}_\mathbb{F}^{\mathcal{M}^o}\) containing \(\mathcal{J}\). Now such a subcategory has to contain in particular all cokernels of natural transformations

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**Figure 1.1:** The sublattice \(\mathcal{M} \subset \mathbb{R}^n \times \mathbb{R}\) is shaded in gray and the support of the indecomposable cohomological functor \(B_v: \mathcal{M}^o \rightarrow \text{Vect}_\mathbb{F}\) is shaded in blue.
in $\mathcal{J}$. This leads us to considering the closure of $\mathcal{J}$ by cokernels, consisting of all functors $F : M^\circ \to \text{Vect}_F$ that fit into an exact sequence
\[ H \to G \to F \to 0 \] (1.1)
with $H, G$ functors in $\mathcal{J}$. As in Definition D.1 we refer to the exact sequence (1.1) as a presentation of $F$ and we say that $F$ is $\mathcal{J}$-presentable. The full subcategory of all $\mathcal{J}$-presentable functors is denoted by $\text{pres}(\mathcal{J})$. By the following theorem, which we prove in Section 6, the subcategory of all $\mathcal{J}$-presentable functors is an abelian subcategory. To state the theorem, we recall that an abelian category is Frobenius if it has enough projectives and enough injectives and if both coincide.

**Theorem 1.1.** The category of $\mathcal{J}$-presentable functors $\text{pres}(\mathcal{J})$ is an abelian subcategory of $\text{ Vect}_M^\circ$, which is also Frobenius with $\mathcal{J}$ being the subcategory of projective-injectives.

Now for any point $v \in M$, there is an associated simple functor
\[ S_v : M^\circ \to \text{Vect}_F, \quad u \mapsto \begin{cases} F & u = v \\ \{0\} & \text{otherwise} \end{cases} \] (1.2)
with all internal maps necessarily trivial. Suppose $F : M^\circ \to \text{Vect}_F$ is some $\mathcal{J}$-presentable functor and let $v \in \text{int} M$ be a point inside the interior of $M$. Even though $S_v : M^\circ \to \text{Vect}_F$ is not $\mathcal{J}$-presentable, both $F$ and $S_v$ are functors vanishing on $\partial M$. Now let $C$ be the category of functors $M^\circ \to \text{Vect}_F$ vanishing on $\partial M$. Then $C$ is an abelian subcategory of $\text{ Vect}_F^\circ$. As any functor in $\mathcal{J}$ is projective in $C$ by [BBF21, Corollary 3.6], any projective resolution of $F : M^\circ \to \text{Vect}_F$ by $\mathcal{J}$-presentable functors also is a projective resolution of $F$ as a functor in $C$ by Theorem 1.1. Thus, we may use a projective resolution of $F : M^\circ \to \text{Vect}_F$ in $\text{pres}(\mathcal{J})$ to compute the Ext vector spaces $\text{Ext}_C^n(F, S_u)$ with respect to $C$ for all $n \in \mathbb{N}_0$. (Here it is crucial, that we work in $C$ and not all of $\text{ Vect}_F^\circ$, as this category has other projectives.) Moreover, as all $\mathcal{J}$-presentable functors are pfd, the Ext vector space $\text{Ext}_C^n(F, S_u)$ is finite-dimensional for each $n \in \mathbb{N}_0$.

**Definition 1.2.** For any $\mathcal{J}$-presentable functor $F : M^\circ \to \text{Vect}_F$ we define its $n$-th Betti function to be
\[ \beta^n(F) : \text{int} M \to \mathbb{N}_0, \quad u \mapsto \dim F \text{ Ext}_C^n(F, S_u) \]
for any $n \in \mathbb{N}_0$.

In Section 6 we show the following proposition, which illustrates how above defined Betti functions are closely related to the bigraded Betti numbers of [LW19, Definition 2.1]. (They are not quite the same, because of our condition on all functors to vanish on $\partial M$.)

**Proposition 1.3.** Any $\mathcal{J}$-presentable functor $F : M^\circ \to \text{Vect}_F$ admits a (potentially infinite) minimal projective resolution
\[ \cdots \to P_n \to \cdots \to P_2 \to P_1 \to P_0 \to F \to 0 \]
by functors $P_n : M^\circ \to \text{Vect}_F$ in $\mathcal{J}$. Moreover, the multiplicity of $B_u : M^\circ \to \text{Vect}_F$ in $P_n$ is equal to $\beta^n(F)(u) = \dim \text{ Ext}_C^n(F, S_u)$ for each $u \in \text{int} M$ and $n \in \mathbb{N}_0$. 

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Furthermore, we show with Corollary 6.6 that all Betti functions satisfy a certain finiteness constraint. We refer to all functions $\int M \to N_0$ satisfying this constraint as admissible Betti functions and we denote the commutative monoid of all admissible Betti functions by $B$; this is Definition 4.6. Then we show in Section 7 that the alternating sum of functions - we call it the Euler function in Definition 7.5 -

$$\chi(F) := \sum_{n \in N_0} (-1)^n \beta^n(F) : \text{int } M \to \mathbb{Z}, u \mapsto \sum_{n \in N_0} (-1)^n \dim \operatorname{Ext}^0_n(F, S_u)$$

is well-defined and moreover, that the point-wise absolute value

$$|\chi(F)|: \text{int } M \to N_0, u \mapsto |\chi(F)(u)|$$

is an admissible Betti function as well. In Definition 7.6 we refer to such functions as admissible Euler functions and we denote the abelian group of all admissible Euler functions by $G$ as it is the Grothendieck group of the commutative monoid $B$ up to unique isomorphism. Then we show with Proposition 7.10 that $\chi : \text{Ob}(\text{pres}(J)) \to G(B)$ is an additive invariant, hence there is a homomorphism

$$[\chi] : K_0(\text{pres}(J)) \to G(B), [F] \mapsto \chi(F)$$

of abelian groups. Finally we obtain the following result.

**Theorem 1.4.** The group homomorphism $[\chi] : K_0(\text{pres}(J)) \to G(B)$ is an isomorphism.

Now suppose that $f : X \to \mathbb{R}$ is an $\mathbb{F}$-tame function. Then $h(f) : M^0 \to \text{Vect}_{\mathbb{F}}$ is a functor in $J$, hence $h(f)$ is its own minimal projective resolution. So by Proposition 1.3 the function $\chi(h(f)) = \beta^0(h(f)) : \text{int } M \to \mathbb{Z}$ counts for each point $u \in \text{int } M$ inside the interior of $M$ the multiplicity of $B_u : M^0 \to \text{Vect}_{\mathbb{F}}$. As noted above, the multiplicities of functors $B_u : M^0 \to \text{Vect}_{\mathbb{F}}$ for $u \in \text{int } M$ determine the extended persistence diagram of $f : X \to \mathbb{R}$ by CSEH09 and vice versa. Thus, we may think of the function

$$\chi(h(f)) : \text{int } M \to \mathbb{Z}$$

as the extended persistence diagram of $f : X \to \mathbb{R}$ and of $G(B) \cong K_0(\text{pres}(J))$ as “the abelian group of extended persistence diagrams” in quotes since not every function in $G(B)$ can be obtained as the extended persistence diagram of some real-valued continuous function. Now Maz12 Definition 1.8] provides three notions of a categorification for an abelian group $G$. In some sense the strongest of these three notions is that of a pair of an abelian category $\mathcal{A}$ and a group isomorphism $G \xrightarrow{\cong} K_0(\mathcal{A})$ from $G$ to the Grothendieck group $K_0(\mathcal{A})$. As we provide the functor $h$ from the category of $\mathbb{F}$-tame functions to the abelian category $\text{pres}(\mathcal{J})$ such that $[h(f)] \in K_0(\text{pres}(\mathcal{J}))$ is a faithful representation of the extended persistence diagram of $f : X \to \mathbb{R}$, we think of $\text{pres}(\mathcal{J})$ as the categorification of extended persistence diagrams. This is also in close analogy to the following categorification of the Euler characteristic: Given a topological space $X$ the Euler characteristic $\chi(X) \in \mathbb{Z}$ uniquely determines $[\Delta_\bullet(X)] \in K_0(\text{Ab})$, which is the element in the Grothendieck group $K_0(\text{Ab})$ of the category of abelian groups $\text{Ab}$ corresponding to the singular chain complex $\Delta_\bullet(X)$.

Throughout this work we make extensive use of homological algebra and sheaf theory. For most results needed from these two areas we will draw on references to chapters 1 and 2 of KS90 respectively.
1.1 Equivalence of RISC and Derived Level Set Persistence

Before we show Theorem 1.1 we need to provide an intermediate result, which is interesting in its own right. In Section 3.3 we recall derived level set persistence by [Cur14, KS18] as a functor

$$R(-)_* F(-) : (\text{Top/}\mathbb{R})^\circ \to D^+ (\mathbb{R}), \ (f : X \to \mathbb{R}) \mapsto Rf_* F_X$$

from the opposite category of the category of topological spaces over the reals Top/\mathbb{R} to the bounded below derived category $D^+ (\mathbb{R})$ of $\mathbb{F}$-linear sheaves on the real numbers. Then we describe a cohomological functor $h_{\mathbb{R}} : D^+ (\mathbb{R}) \to \text{Vect}_F^{M^\circ}$ from the derived category $D^+ (\mathbb{R})$ to the category of functors $M^\circ \to \text{Vect}_F$. With Proposition 3.34 we show that RISC factors through $h_{\mathbb{R}}$ and derived level set persistence up to natural isomorphism $\zeta$:

$$(1.3) \quad (\text{lcContr/}\mathbb{R})^\circ \xrightarrow{R(-)_* F(-)} D^+ (\mathbb{R}) \xrightarrow{h_{\mathbb{R}}} \text{Vect}_F^{M^\circ} \xleftarrow{\zeta} \text{Vect}_F^{M^\circ}$$

Here (lcContr/\mathbb{R})^\circ denotes the full subcategory of all functions on locally contractible spaces. Then we consider the full triangulated subcategory $D^+_t (\mathbb{R})$ of the derived category $D^+ (\mathbb{R})$ consisting of all complexes of sheaves $F$, which are tame in the sense that all open intervals $I \subseteq \mathbb{R}$ have degree-wise finite-dimensional sheaf cohomology $H^n (I; F)$ with coefficients in $F$. With Corollary 5.8 from Section 5 we provide a statement, which immediately implies the equivalence of categories $D^+_t (\mathbb{R})$ and $\mathcal{J}$:

**Theorem 1.5.** The functor $h_{\mathbb{R}}$ (defined in Section 3.3) restricts to an equivalence of categories $h_{\mathbb{R},t} : D^+_t (\mathbb{R}) \to \mathcal{J}$.

In particular the diagram (1.3) restricts to the diagram

$$(1.3) \quad (\text{lcContr/}\mathbb{R})^\circ_t \xrightarrow{R(-)_* F(-)} D^+_t (\mathbb{R}) \xrightarrow{h_{\mathbb{R},t}} \mathcal{J}$$

with the lower horizontal functor an equivalence of categories, where (lcContr/\mathbb{R})^\circ_t denotes the full subcategory of all $\mathbb{F}$-tame functions on locally contractible spaces, see also Definition 3.35. Now $D^+_t (\mathbb{R})$ is a full triangulated subcategory of the derived category $D^+ (\mathbb{R})$, while $\mathcal{J}$ is a full subcategory of the abelian category of $\mathcal{J}$-presentable functors. We may put this into perspective with Proposition E.1 which is a slight generalization of a result
by [Kra07, Section 4.2]. Using Theorem 1.5 (or rather a generalization thereof) and Proposition E.1 we show in Section 6 that the composition of functors

\[ D^+_{\mathbb{R}} \xrightarrow{h_{\mathbb{R}}^*} \mathcal{J} \hookrightarrow \text{pres}(\mathcal{J}) \]  

(1.4)

is the abelianization of \( D^+_{\mathbb{R}} \), i.e. (1.4) is the universal or initial cohomological functor on the triangulated category \( D^+_{\mathbb{R}} \); this is Corollary 6.3. Thus, we obtain a close link between derived level set persistence and the categorification of extended persistence diagrams.

### 1.2 Related Work

Originally extended persistence diagrams have been defined by [CSEH09] in terms of ranks of internal maps of corresponding (extended) persistence modules [ZC05]. Moreover, these ranks of internal maps form an invariant in their own right, the rank invariant of multi-dimensional persistence modules by [CZ09]. The correspondence between the rank invariant and persistence diagrams has been generalized by [Pat18]. Now persistence modules form an abelian category themselves and under suitable finiteness assumptions any 1-dimensional extended persistence module decomposes into a direct sum of indecomposables. As it turns out, these indecomposables are in a one-to-one correspondence with the vertices of the corresponding extended persistence diagram. In particular, the extended persistence diagram determines the isomorphism class of the corresponding extended persistence modules and vice versa. Now the Grothendieck group of the abelian category of extended persistence modules (with suitable finiteness assumptions) is generated by the isomorphism classes. Thus, the corresponding element in the Grothendieck group is determined by the extended persistence diagram as well. However, the converse is not true, as the category of extended persistence modules has too many short exact sequences. For example, if we consider a persistence module \( M \) provided as a representation of an \( A_n \)-quiver, then the corresponding element \([M]\) in the Grothendieck group determines the dimension vector of \( M \) and vice versa.

One way to eliminate this mismatch between the Grothendieck group and extended persistence diagrams is to “omit” some of the exact sequences when forming the Grothendieck group. This leads to the notion of a Quillen exact category [Qui73], which can be defined as an abelian category with a certain class of distinguished short exact sequences, and the corresponding generalized notion of an associated Grothendieck group [Swa71]. With this notion this mismatch between the Grothendieck group and extended persistence diagrams can be eliminated by considering the Quillen exact category with respect to the class of split short exact sequences of extended persistence modules. A similar approach has been used by [BOO21] in the multi-dimensional setting. There the authors introduce their notion of rank-exact short exact sequences, which coincide with split short exact sequences in the 1-dimensional setting (under suitable finiteness assumptions). Further invariants of multi-dimensional persistence modules defined in terms of Grothendieck groups of Quillen exact categories were introduced by [BBH21]. In the present work we propose another approach to eliminate the mismatch between the Grothendieck group and (extended) persistence diagrams. Instead of constraining ourselves to short exact sequences of extended persistence modules (or of sheaves on the reals) that split, we define the cohomological functor (1.4)
from the full subcategory $D^+_t(\mathbb{R})$ of the derived category $D^+(\mathbb{R})$ to $\mathcal{J}$-presentable functors and then we consider the bona fide Grothendieck group of $\text{pres}(\mathcal{J})$. (We note that we have to constrain ourselves to some subcategory of tame objects since otherwise, the Grothendieck group would be trivial by the Eilenberg swindle.) In future work we intend to explore similar techniques to obtain invariants of multi-dimensional persistence modules in terms Grothendieck groups of abelian categories as well, whereas in the present work we focus on $\mathbb{F}$-tame functions, which we consider to be one of several sweet spots within the realm of persistence theory.

Originally persistence diagrams were introduced as multisets by [CSEH07], which we may think of as functions taking values in the natural numbers $\mathbb{N}_0$ or non-negative functions to the integers $\mathbb{Z}$. This notion of a persistence diagram has been generalized by [Pat18] to functions taking values in an abelian group. One particular generalization of a persistence diagram by [Pat18] is defined for persistence modules taking values in an abelian category $\mathcal{A}$. For such a persistence module its associated persistence diagram is a function taking values in the Grothendieck group $K_0(\mathcal{A})$. In the present work we consider extended persistence diagrams themselves as elements of the Grothendieck group $K_0(\text{pres}(\mathcal{J}))$.

Now in order to show that $h_{R,t}: D^+_t(\mathbb{R}) \to \mathcal{J}$ is an equivalence of categories, we employ similar techniques as [Hap88] used to describe derived categories of Dynkin quivers. More specifically, we may think of the category of $\mathbb{F}$-linear sheaves on the reals as a continuous counterpart to the category of representations of an $A_n$-quiver with alternating orientations. Now [Hap88] associates to any quiver an $\mathbb{F}$-linear mesh category in such a way that any two quivers of type $A_n$ have isomorphic mesh categories. As a result of this theory, the derived category of a Dynkin quiver is equivalent to the category of projective $\mathbb{F}$-linear pfd presheaves on the mesh category with finite support. In particular the derived categories of any two quivers of type $A_n$ are equivalent. In the present work we use the poset $\mathbb{M} \subset \mathbb{R}^0 \times \mathbb{R}$ in place of the mesh category. To be more precise, the quotient category $\mathbb{F}\mathbb{M}/\partial \mathbb{M}$ of the linearization $\mathbb{F}\mathbb{M}$ modulo $\partial \mathbb{M}$ can be seen as a continuous counterpart of the mesh category of an $A_n$-quiver. Moreover, the category of $\mathbb{F}$-linear presheaves on $\mathbb{F}\mathbb{M}/\partial\mathbb{M}$ is equivalent to the category $\mathcal{C}$ of all functors $\mathbb{M}^0 \to \text{Vect}_\mathbb{F}$ vanishing on $\partial \mathbb{M}$, which we work with throughout. Furthermore, in the same way that [Hap88] abstracts over the orientation of arrows of say an $A_n$-quiver, we abstract over the specific topology on the real numbers. Of course, we cannot use any topology, but there is a whole family of topologies we can use in place of the Euclidean topology. This is also closely related to the way in which [CdM09, BEMP13] abstract over the orientations of zigzags. Moreover, for finitely indexed persistence modules as in [CdM09], the results by [Hap88] imply that the derived category of ordinary persistence modules and the derived category of zigzag persistence modules (with corresponding vertex sets) are equivalent. This functorial extension of the correspondence of barcodes (i.e. isomorphism classes) by [CdM09] through the equivalence by [Hap88] has been investigated by [HY20]. In the present work we make use of sheaf theory to extend this equivalence (on the level of 1-categories) to the continuously indexed setting.

We also note that [BGO19] obtained a closely related invariant of continuous functions and of sheaves on the reals, namely Mayer–Vietoris systems. If we consider the tessellation of $\mathbb{M}$ shown in Fig. 1.2 then we may think of the restriction of $h(f): \mathbb{M}^0 \to \text{Vect}_\mathbb{F}$ for some function $f: X \to \mathbb{R}$ to each tile, as a layer of the Mayer–Vietoris pyramid introduced by
Figure 1.2: The tessellation of $\mathcal{M}$ induced by $T$ and $D$.

Following [CDM09] we may further subdivide each pyramid into their north, south, west, and east faces. Roughly speaking, the Mayer–Vietoris system associated to $f: X \to \mathbb{R}$ can be obtained as the pointwise dual of the restriction of $h(f): \mathcal{M}^\circ \to \text{Vect}_F$ to the subposet $P \subset \mathcal{M}$ that is the union of all south faces – one from each pyramid. Furthermore, the authors of [BGO19] provide a functor $\Psi: D^+(\mathbb{R}) \to \mathcal{M}–\mathcal{V}(\mathbb{R})$, where $\mathcal{M}–\mathcal{V}(\mathbb{R})$ is the category of Mayer–Vietoris systems. In a similar way, provided an object $F$ of $D^+(\mathbb{R})$ we may think of $\Psi(F): P \to \text{Vect}_F$ as the pointwise dual of $h_{\mathbb{R}}(F)|_P: P^\circ \to \text{Vect}_F$. Now thinking of Mayer–Vietoris systems as a subcategory of $\text{Vect}_P^\circ$, we may consider the cohomological functor

$$D^+(\mathbb{R}) \xrightarrow{\Psi} \mathcal{M}–\mathcal{V}(\mathbb{R}) \hookrightarrow \text{Vect}_P^\circ.$$

(1.5)

Considering that

$$D^+_t(\mathbb{R}) \xrightarrow{h_{\mathbb{R},t}} \mathcal{J} \hookrightarrow \text{pres}(\mathcal{J})$$

is the universal or initial cohomological functor on the triangulated category $D^+_t(\mathbb{R})$ by Corollary [6.3], the restriction of (1.5) to $D^+_t(\mathbb{R})$ has to factor through the latter universal cohomological functor. This factorization in turn is provided by restriction to $P \subset \mathcal{M}$ and pointwise dualization as described above. We also note that while $h_{\mathbb{R},t}: D^+_t(\mathbb{R}) \to \mathcal{J}$ is an equivalence of categories by Theorem [1.5], the functor $\Psi: D^+(\mathbb{R}) \to \mathcal{M}–\mathcal{V}(\mathbb{R})$ is not. An explicit example of ours showing that $\Psi$ is not faithful is provided in [BGO19, Remark 4.9]. In Example [3.33] we show that this particular example may appear in nature as well.

### 1.3 Outline

In Section 2 we describe a family of topological spaces, which can be seen as a continuous counterpart to the family of $\mathbb{A}_n$-quivers. Then we consider the associated categories of $\mathbb{F}$-linear sheaves on these topological spaces as continuously indexed counterparts to representations of $\mathbb{A}_n$-quivers. Moreover, we provide a sheaf-theoretical counterpart $\iota$ to Happel’s embedding [Hap88, Section I.5.6] of the mesh category of a quiver into its derived category, see also [KS16, Theorem 2.2]. Then with Proposition [2.13] we provide the property of the functor $\iota$, that is the most fundamental to our work.
In Section 3 we describe how \( \iota \) can be used to obtain cohomological functors \( M^\circ \to \text{Vect}_F \).

In Section 3.1 we introduce a \textit{tameness} assumption for objects in each of the categories we considered up to this point and we show that tameness is an invariant under most of the functors we consider between any two of these categories. In Sections 3.2 and 3.3 we connect our developments up to this point with derived level set persistence by [Cur14, KS18].

In Section 4 we show that any \( J \)-presentable functor admits a projective cover by a functor in \( J \) drawing upon theory and techniques of [Kra15, Section 3]. Later this result is strengthened to Proposition 1.3.

In Section 5 we provide an equivalence of categories, which eventually implies the equivalence of categories from Theorem 1.5. To this end, we show that certain infinite coproducts of sheaves satisfy the universal property of the product in Sections 5.1 and 5.2.

In Section 6 we show that the composition of functors 
\[
D^+_t(R) \xrightarrow{h_t} J \hookrightarrow \text{pres}(J)
\]
is the abelianization of \( D^+_t(R) \). This result and its proof are inspired by [Kra07, Section 4.2]. Finally, in Section 7 we first show that the Euler function \( \chi(F) : \text{int} M \to \mathbb{Z} \) is an additive invariant and then we prove Theorem 1.4. This concludes our paper and supports the idea that RISC is a categorification of extended persistence diagrams.

2 A Sheaf-Theoretical Happel Functor

Let \( R \) and \( R^\circ \) denote the posets given by the orders \( \leq \) and \( \geq \) on \( R \), respectively. Then we may form the product poset \( R^\circ \times R \) whose underlying set is the Euclidean plane. Let \( l_0 \) and \( l_1 \) be two lines of slope \(-1\) in \( R^\circ \times R \) with \( l_1 \) sitting above \( l_0 \) as shown in Fig. 2.1. Moreover, let \( \mathbb{M} \) be the convex hull of \( l_0 \) and \( l_1 \), then \( \mathbb{M} \) is a sublattice of \( R^\circ \times R \). The \textit{central} automorphism \( T : \mathbb{M} \to \mathbb{M} \) with the following defining property will be essential to many of our constructions (also see Fig. 2.1):

Let \( u \in \mathbb{M} \), \( h \) be the horizontal line through \( u \), let \( g_0 \) be the vertical line through \( u \), let \( h_1 \) be the horizontal line through \( T(u) \), and let \( g_1 \) be the vertical line through \( T(u) \). Then the lines \( l_0 \), \( h \), and \( g_1 \) intersect in a common point, and the same is true for the lines \( l_1 \), \( g_0 \), and \( h_1 \).

We also note that \( T \) is a glide reflection along the bisecting line between \( l_0 \) and \( l_1 \), and the amount of translation is the distance of \( l_0 \) and \( l_1 \). Moreover, as a space, \( M/\langle T \rangle \) is a Möbius strip; see also [CdM09].

Now let \( Q \subset \mathbb{M} \) be a closed proper downset and let \( q := \partial Q \) be the boundary of \( Q \) in \( \mathbb{M} \). Then \( D := Q \setminus T^{-1}(Q) \) is a fundamental domain of \( \mathbb{M} \) with respect to the action of \( \langle T \rangle \), see Fig. 2.2 for an example. Moreover, \( T \) and \( D \) induce the tessellation of \( \mathbb{M} \) shown in Fig. 2.3.

Now let \( \gamma := [0, \infty) \times (-\infty, 0] \subset R^\circ \times R \) be the downset of the origin. As in [KS90, Section 3.5] let \( R^\circ_\gamma \) be the plane endowed with the \( \gamma \)-topology [KS90, Definition 3.5.1]. Moreover, let \( q_\gamma \) be the subspace of \( R^\circ_\gamma \), whose underlying set is \( q \), and let

\[
\phi_\gamma : q \to q_\gamma, \quad t \mapsto t
\]
be the natural continuous map. Fixing a field $\mathbb{F}$ throughout this document, we consider the category $\text{Sh}(q_\gamma)$ of $\mathbb{F}$-linear sheaves on $q_\gamma$. Let $\partial q := q \cap \partial M$ and let $j : \partial q \hookrightarrow q$ be the inclusion. Moreover, let $\text{Sh}(q_\gamma, \partial q)$ be the full subcategory of $\mathbb{F}$-linear sheaves in $\text{Sh}(q_\gamma)$ vanishing on $\partial q$. Now considering the unit $\eta^j : \text{id} \to j_* \circ j^{-1}$ of the adjunction $j^{-1} \dashv j_*$ (viewed as a functor from $\text{Sh}(q_\gamma)$ to homomorphisms in $\text{Sh}(q_\gamma)$) and taking the kernel yields a coreflection $\flat = \ker \circ \eta^j : \text{Sh}(q_\gamma) \to \text{Sh}(q_\gamma, \partial q)$:

$$\text{Sh}(q_\gamma, \partial q) \xrightarrow{\flat} \text{Sh}(q_\gamma).$$

**) Lemma 2.1.** If $F$ is a flabby sheaf on $q_\gamma$, then the unit $\eta^j_F : F \to j_* j^{-1} F$ is an epimorphism.

**Proof.** Let $U \subseteq q_\gamma$ be an open subset. If $U \cap \partial q$ contains just a single point $t$, then we have
the commutative diagram

\[
\begin{array}{ccc}
F(U) & \xrightarrow{\eta_{F,U}} & (j_*j^{-1}F)(U) = (j^{-1}F)(\{t\}) \\
\downarrow & & \downarrow \\
F(V_0 \cup V_1) & \xrightarrow{\eta_{F,V_0 \cup V_1}} & (j_*j^{-1}F)(V_0 \cup V_1) \\
\cong & & \cong \\
F(V_0) \oplus F(V_1) & \xrightarrow{\eta_{F,V_0} \oplus \eta_{F,V_1}} & (j_*j^{-1}F)(V_0) \oplus (j_*j^{-1}F)(V_1)
\end{array}
\]

hence \( \eta_{F,U} \) is an epimorphism at \( U \). Now suppose \( \partial q \subset U \). If \( \partial q \) is discrete, then we can find disjoint open subsets \( V_0, V_1 \subset U \) with \( \partial q \subset V_0 \cup V_1 \) and each containing a single point of \( \partial q \). Moreover, we have the commutative diagram

with the lower horizontal map an epimorphism by our reasoning above. If \( \partial q \) is not discrete, then \( U = q_\gamma \) and moreover, \( \eta_{F,q_\gamma} : F(q_\gamma) \to (j_*j^{-1}F)(q_\gamma) = (j^{-1}F)(\partial q) \) is even an isomorphism.

\[\square\]

**Lemma 2.2.** The induced adjunction on the level of derived categories

\[
\begin{array}{ccc}
D^+(\text{Sh}(q_\gamma, \partial q)) & \xrightarrow{R_*} & D^+(\text{Sh}(q_\gamma)) \\
\xleftarrow{R^*} & & \xrightarrow{R_*}
\end{array}
\]

is coreflective as well.
Proof. By Lemma\textsuperscript{A.1} it suffices to show that all sheaves vanishing on $\partial q$ are $♭$-acyclic. To this end, we again view $\eta^j$ as an exact functor from $\text{Sh}(q_\gamma)$ to homomorphisms in $\text{Sh}(q_\gamma)$. In particular the subcategory of flabby sheaves is $\eta^j$-injective \cite[Definition 1.8.2]{KS90}. Suppose that $F$ is a flabby sheaf on $q_\gamma$. Then $\eta^j(F) = \eta^j_F$ is an epimorphism by Lemma\textsuperscript{2.1}. Moreover, as $\text{Sh}(q_\gamma)$ is additive, the full subcategory of all epimorphisms in the arrow category is injective with respect to taking kernels \cite[Definition 1.8.2]{KS90}. From this we obtain

$$R♭ ∼ = R\ker(D^+(\eta^j))$$

by \cite[Proposition 1.8.7]{KS90} and by exactness of $\eta^j$.

Now suppose that $G$ is an arbitrary sheaf vanishing on $\partial q$. Then $j^{-1}G = 0$, and so $\eta^j_G: G \to j_*j^{-1}G \cong 0$ is an epimorphism. We thus get

$$R♭(G) ∼ = R\ker(D^+(\eta^j)(G)) = R\ker(\eta^j_G) \cong \ker(\eta^j_G) = G.$$

In particular $G$ is $♭$-acyclic. (Here the second isomorphism of the previous equation follows from the fact that $\eta^j_G: G \to j_*j^{-1}G$ is a ker-injective object of the arrow category.)

From this point onwards we write $D^+(q_\gamma)$ for the derived category $D^+(\text{Sh}(q_\gamma))$ and we write $D^+(q_\gamma, \partial q)$ for the full subcategory of complexes of sheaves in $D^+(q_\gamma)$, whose cohomology sheaves vanish on $\partial q$. Then we obtain the following in conjunction with Lemma\textsuperscript{A.3}.

**Corollary 2.3.** The category $D^+(q_\gamma, \partial q)$ is a triangulated coreflective subcategory of $D^+(q_\gamma)$ with coreflector $R♭: D^+(q_\gamma) \to D^+(q_\gamma, \partial q)$.

Now let $\breve{q} := q_\gamma \setminus \partial q$ and let $i: \breve{q} \hookrightarrow q_\gamma$ be the corresponding subspace inclusion. Then we have

$$i! = b \circ i_*: \text{Sh}(\breve{q}) \to \text{Sh}(q_\gamma, \partial q).$$

Moreover, let $\breve{q} \subseteq q_\gamma$ be the smallest open subset of $q_\gamma$ containing $\breve{q}$ and let $i_1: \breve{q} \hookrightarrow \breve{q}$ and $i_2: \breve{q} \to q_\gamma$ be the corresponding inclusions.

**Lemma 2.4.** The adjunction

$$\begin{array}{ccc}
\text{Sh}(\breve{q}) & \xleftarrow{i_!} & \text{Sh}(\breve{q}) \\
\downarrow{i^\sim} & & \uparrow{i^\sim} \\
\text{Sh}(\breve{q}) & \cong & \text{Sh}(\breve{q})
\end{array}$$

is an exact adjoint equivalence.

**Proof.** The inclusion $i_1: \breve{q} \hookrightarrow \breve{q}$ induces a lattice isomorphism between the topologies of $\breve{q}$ and $\breve{q}$. Moreover, this isomorphism preserves all covers.

**Lemma 2.5.** For any sheaf $F$ on $q_\gamma$ the naturally induced map

$$(b \circ \eta^j)_F: bF \to b(i_*i^{-1}F) = i_!i^{-1}F$$

is an isomorphism.
Proof. By Lemma 2.4 it suffices to check that

$$(\flat \circ \eta^i)_{F}: \flat F \to \flat i^{-1} F$$

is an isomorphism. This in turn follows from Lemma C.1.

Composing the adjunctions

$$\text{Sh}(q, \partial q) \dashv \text{Sh}(\hat{q})$$

we obtain the adjunction

$$\text{Sh}(q, \partial q) \dashv \text{Sh}(\hat{q}). \tag{2.3}$$

Lemma 2.6. If $\partial q$ is closed in $q$, then (2.3) is an exact adjoint equivalence. 

Proof. By definition [KS90, Page 93] of the functor $(-)_{\hat{q}}: \text{Sh}(q) \to \text{Sh}(q, \partial q)$, $F \mapsto F_{\hat{q}}$ we have $b = (-)_{\hat{q}}$. With this the result follows from Lemma C.2.

Now whenever $\partial q$ is closed in $q$, we define the composition

$$\varepsilon^i_F: i_{i^{-1}} F \xrightarrow{\varepsilon_F} F$$

for any sheaf $F$ on $q$, harnessing Lemma 2.5 where $\varepsilon_F: \flat F \to F$ is the counit of the adjunction (2.1).

Lemma 2.7. If $\partial q$ is closed in $q$, then we have the $F$-linear adjunction

$$\text{Sh}(q) \dashv \text{Sh}(\hat{q})$$

with counit $\varepsilon^i: i \circ i^{-1} \to \text{id}$. 

Proof. Let $F$ be a sheaf on $q$, let $G$ be a sheaf on $q$, and let $\varphi: i_{i^{-1}} G \to G$ be a sheaf homomorphism. We have to show there is a unique sheaf homomorphism $\varphi^i: F \to i^{-1} G$ such that the diagram

$$\begin{array}{ccc}
i_{i^{-1}} F & \xrightarrow{\varphi} & G \\
i \downarrow \varphi^i & & \downarrow \varepsilon_G \\
i_{i^{-1}} G & \xleftarrow{(b \circ \eta^i)_G} & \flat G \\
\end{array} \tag{2.4}$$

commutes. Now the adjunction (2.1) yields a unique sheaf homomorphism $\varphi^i: i_{i^{-1}} F \to \flat G$ as indicated in (2.4). Moreover, the homomorphism $(b \circ \eta^i)_G: \flat G \to i^{-1} G$ is an isomorphism.
by Lemma 2.5 hence the vertical arrow on the left hand side of (2.4) exists uniquely as indicated. Furthermore, \( \iota_1 : \text{Sh}(q) \to \text{Sh}(q_\gamma) \) is fully faithful by Lemma 2.6 and this implies the claim.

We now construct a functor

\[ \iota_0 : D \to \text{Sh}(q_\gamma, \partial q) \]

from \( D \) to the category of \( \mathbb{F} \)-linear sheaves on \( q_\gamma \) vanishing on \( \partial q \). This will be our first step towards the construction of a functor

\[ \iota : \mathbb{M} \to D^+(q_\gamma, \partial q), \]

which assigns an indecomposable object \( \iota(u) \) of \( D^+(q_\gamma, \partial q) \) to each \( u \in \text{int} \mathbb{M} \). This functor \( \iota \) is closely related to a construction due to [Hap88, Section I.5.6], see also [KST16, Theorem 2.2]. For \( u \in D \) we define the locally closed subset

\[ Z(u) := q \cap (\uparrow u) \cap \text{int}(\downarrow T(u)). \]

Here \( \uparrow u \) denotes the upset of \( u \) and \( \text{int}(\downarrow T(u)) \) denotes the interior of the downset of \( T(u) \).

With this we may use the construction from [KS90, Page 93] to define

\[ \iota_0(u) := F_{Z(u)}. \]

For \( u \preceq v \in D \) the homomorphism

\[ \iota_0(u \preceq v) : F_{Z(u)} \to F_{Z(v)} \]

is uniquely determined by the property that it induces the identity on stalks whenever the corresponding stalks are isomorphic to \( \mathbb{F} \). We note the following two properties of \( \iota_0 \), which we will need later.

**Lemma 2.8.** For \( u, v \in D \) we have

\[ \text{Hom}_{\text{Sh}(q_\gamma)}(\iota_0(u), \iota_0(v)) = \langle \iota_0(u \preceq v) \rangle \cong \begin{cases} \mathbb{F} & v \in (\uparrow u) \cap \text{int}(\downarrow T(u)) \\ \{0\} & \text{otherwise.} \end{cases} \]
Lemma 2.9. Let $F$ be an $\mathcal{F}$-linear sheaf on $q_\gamma$ and let $u \in D$. Then we have a natural $\mathcal{F}$-linear isomorphism

$$\text{Hom}_{\text{Sh}(q_\gamma)}(\iota_0(u), F) \cong \Gamma_{q \cap (\uparrow T(u))}(q \cap \text{int}(\downarrow T(u)); F).$$

**Proof.** We have

$$\text{Hom}_{\text{Sh}(q_\gamma)}(\iota_0(u), F) = \text{Hom}_{\text{Sh}(q_\gamma)}(F_Z(u), F)$$

$$\cong \Gamma(q_\gamma; \text{Hom}(F_Z(u), F))$$

$$= \Gamma(q_\gamma; \Gamma_Z(F_Z(u)))$$

$$= \Gamma(q_\gamma; \Gamma_{q \cap \text{int}(\downarrow T(u))}(\Gamma_{q \cap (\uparrow T(u))}(F)))$$

$$= \Gamma_{q \cap (\uparrow T(u))}(q \cap \text{int}(\downarrow T(u)); F).$$

Here $\text{Hom}$ denotes the internal homomorphism functor of sheaves as defined in [KS90, Definition 2.2.7]. The second equality follows from [KS90, (2.2.6)]. The isomorphism follows from [KS90, Proposition 2.3.9.(ii)] and the last equality from [KS90, Proposition 2.3.9.(iii) and (2.3.12)].

Now let $u \in D$. As $\phi_\gamma : q \to q_\gamma$ is continuous, the subset $Z(u) = \phi_\gamma^{-1}(Z(u)) \subseteq q$ is locally closed in $q$ as well. Moreover, we have the pair of adjoint functors

$$\text{Sh}(q_\gamma) \rightleftarrows \text{Sh}(q) : \phi_\gamma^{-1} \circ \iota_0 \cong \phi_\gamma^{-1} \text{F}_Z(u) \cong \text{F}_Z(u).$$

between categories of $\mathcal{F}$-linear sheaves on $q$ and $q_\gamma$. Thus, we have

$$(\phi_\gamma^{-1} \circ \iota_0)(u) = \phi_\gamma^{-1} \text{F}_Z(u) \cong \text{F}_Z(u).$$

Furthermore, the adjunction unit

$$(\eta \circ \iota_0)_u : \text{F}_Z(u) \to \phi_\gamma^{-1} \text{F}_Z(u) \cong \phi_\gamma^{-1} \text{F}_Z(u)$$

is an isomorphism, hence $\phi_\gamma^{-1}$ is fully faithful when restricted to the essential image of $\iota_0$.

It will be convenient to have a description of $\phi_\gamma^{-1} \circ \iota_0$ in terms of complexes of sheaves that are $\Gamma(I, -)$-acyclic [KS90, Exercise I.19] for any connected open subset $I \subseteq q$. More specifically, we resolve each sheaf $\text{F}_Z(u)$ in $\text{Sh}(q)$ for $u \in D$ by sheaves of the form $\text{F}_A$, where $A \subseteq q$ is closed with at most two connected components as opposed to being merely locally closed. Such a sheaf $\text{F}_A$ is $\Gamma(I, -)$-acyclic as $\text{F}_A|_I$ can be written as the pushforward of the locally constant sheaf on $A \cap I$ along the inclusion $A \cap I \to I$ and by the homotopy invariance of sheaf cohomology [KS90, Proposition 2.7.5]. To this end, we have the following generalization of [BBF20, Proposition 2.1.1].

**Proposition 2.10.** Let $\mathcal{P}$ denote the set of pairs of closed subspaces of $q$. Then there is a unique order-reversing map

$$\rho : D^\circ \to \mathcal{P}$$

with the following three properties:
(1) For any $t \in q$ we have $\rho(t) = (q \cap \uparrow t, \partial q \cap \uparrow t)$, where $\partial q := q \cap \partial M$ and $\partial M := l_0 \cup l_1$.

(2) For any $u \in D \cap \partial M$ the two components of $\rho(u)$ are identical.

(3) For any axis-aligned rectangle contained in $D^o$ the corresponding joins and meets are preserved by $\rho$. (A join in $D^o$ is a meet in $D$ and vice versa.)

More concretely, the map $\rho$ can be described by the formula

$$\rho(u) = (\rho_0(u), \rho_1(u)), \quad \text{where} \quad \begin{cases} \rho_0(u) = q \cap \uparrow u & \text{and} \\ \rho_1(u) = q \setminus \text{int}(\downarrow T(u)) \end{cases}, \quad (2.6)$$

for any $u \in D^o$. Moreover, we have

$$Z(u) = \rho_0(u) \setminus \rho_1(u). \tag{2.7}$$

Now for $u \in D$ let $\kappa_0(u)$ be the complex of sheaves with

$$(\kappa_0(u))^n = \begin{cases} F_{\rho_0(u)} & n = 0 \\ F_{\rho_1(u)} & n = 1 \\ 0 & \text{otherwise} \end{cases}$$

and the differential

$$\delta^0_u : F_{\rho_0(u)} \to F_{\rho_1(u)}$$

being the homomorphism which induces the identity on all stalks isomorphic to $F$. Similarly, we may extend $\kappa_0$ to a functor

$$\kappa_0 : D \to C^+(q) := C^+(\text{Sh}(q))$$

from $D$ to the category of complexes of $F$-linear sheaves on $q$. In conjunction with $\text{(2.7)}$ we obtain

$$H^0 \circ \kappa_0 \cong \phi_\gamma^{-1} \circ \iota_0 \quad \text{and} \quad H^1 \circ \kappa_0 \cong 0. \tag{2.8}$$

This equation yields the following two lemmas.

**Lemma 2.11.** For any axis-aligned rectangle $u \preceq v_1, v_2 \preceq w \in D$ as shown in Fig. 2.5 the sequence

$$0 \to \iota_0(u) \xrightarrow{1 \ 1} \iota_0(v_1) \oplus \iota_0(v_2) \xrightarrow{1 \ -1} \iota_0(w) \to 0 \tag{2.9}$$

is exact. Here we use $1$ as a shorthand for the corresponding induced map, e.g. $\iota_0(u \preceq v_1)$.

**Proof.** By Proposition 2.10(3) the sequence of complexes

$$0 \to \kappa_0(u) \xrightarrow{1 \ 1} \kappa_0(v_1) \oplus \kappa_0(v_2) \xrightarrow{1 \ -1} \kappa_0(w) \to 0$$
is exact. In conjunction with (2.8) and the Snake Lemma we obtain the exactness of
\[ 0 \to \phi^{-1}_{\gamma_0}(u) \xrightarrow{(1)} \phi^{-1}_{\gamma_0}(v_1) \oplus \phi^{-1}_{\gamma_0}(v_2) \xrightarrow{(1 -1)} \phi^{-1}_{\gamma_0}(w) \to 0. \]
As we can check the exactness of (2.9) on the level of stalks, this is sufficient.

Lemma 2.12. For any \( u \in D \) the derived unit
\[ \left( \eta^{D^+(q_\gamma)} \circ \iota_0 \right)_u : \mathbb{F}_{Z(u)} \to R\phi\gamma_\ast \phi^{-1}_{\gamma_0}\mathbb{F}_{Z(u)} \cong R\phi\gamma_\ast \mathbb{F}_{Z(u)} \]
is an isomorphism of the derived category \( D^+(q_\gamma) \), where we view \( \mathbb{F}_{Z(u)} \) as a complex concentrated in degree 0.

Proof. As \( (\eta \circ \iota_0)_u \) is an isomorphism it suffices to show that \( \phi^{-1}_{\gamma_0}\mathbb{F}_{Z(u)} \cong \mathbb{F}_{Z(u)} \) is \( \phi_{\gamma_0^\ast} \)-acyclic by Lemma A.1 from Appendix A. To this end, it suffices to check that \( \mathbb{F}_{Z(u)} \in \text{Sh}(q) \) is \( \Gamma(U; -) \)-acyclic for each open subset \( U \) of some basis \( \mathcal{B} \) of \( q_\gamma \) by Lemma A.2. Now let \( \mathcal{B} := \{ q \cap \text{int}(\downarrow v) \mid T(u) \not\subseteq v \} \) be our choice of a basis and let \( U = q \cap \text{int}(\downarrow v) \in \mathcal{B} \). As \( U \) is connected, \( \kappa_0(u) \) is a \( \Gamma(U; -) \)-acyclic resolution of \( \mathbb{F}_{Z(u)} \). Moreover, as \( T(u) \not\subseteq v \), the differential \( \Gamma(U; \delta^0_{\mathcal{U}}) \) is surjective, hence \( \mathbb{F}_{Z(u)} \) is \( \Gamma(U; -) \)-acyclic.

The short exact sequence (2.9) yields a particular distinguished triangle
\[ \iota_0(u) \xrightarrow{(1)} \iota_0(v_1) \oplus \iota_0(v_2) \xrightarrow{(1 -1)} \iota_0(w) \xrightarrow{\partial'} \iota_0(u)[1] \quad (2.10) \]
in the derived category \( D^+(q_\gamma) \), where we view each sheaf as a complex concentrated in degree 0. Now let \( \Sigma := (-)[1] \) and let
\[ R_D := \{(w, \hat{u}) \in D \times T(D) \mid w \preceq \hat{u} \preceq T(w)\} \]
as in [BBF21, Definition A.8]. As shown in Fig. 2.5 any pair \((w, \hat{u}) \in R_D\) determines an axis-aligned rectangle \(u \preceq v_1, v_2 \preceq w \in D\) with \(T(u) = \hat{u}\). Moreover, the collection of homomorphisms \(\partial'\) for the corresponding triangles (2.10) forms a natural transformation as in the diagram

\[
\begin{array}{ccc}
R_D & \xrightarrow{pr_1} & D \\
pr_2 \downarrow & \downarrow \delta' & \downarrow \iota_0 \\
T(D) & \xrightarrow{\Sigma_{\psi \iota_0} \Gamma^{-1}} & D^+(q)_{\gamma},
\end{array}
\]

where we consider \(R_D\) as a subposet of \(D \times T(D)\) with the product order and with \(pr_1: R_D \to D\) and \(pr_2: R_D \to T(D)\) being the canonical projections from \(R_D\) to the first and second component, respectively. Thus, [BBF21, Proposition A.14] yields a functor

\[\iota: \mathcal{M} \to D^+(q_{\gamma}, \partial q)\]

that is a strictly stable extension of \(\iota_0: D \to \operatorname{Sh}(q_{\gamma}, \partial q) \rightarrow D^+(q_{\gamma}, \partial q)\) in the sense that

\[\iota \circ T = \Sigma \circ \iota = \iota(-)[1]\]

and \(\iota|_D = \iota_0\).

We show the following counterpart to Lemma 2.8.

**Proposition 2.13.** For \(u, v \in \mathcal{M}\) we have

\[
\operatorname{Hom}_{D^+(q)}(\iota(u), \iota(v)) = \langle \iota(u \preceq v) \rangle \cong \begin{cases} 
\mathbb{F} & v \in (\uparrow u) \cap \operatorname{int}(\downarrow T(u)) \\
\{0\} & \text{otherwise.}
\end{cases}
\]

The long exact sequence from the following lemma is one ingredient to our proof of this proposition.

**Lemma 2.14.** For all \(u, w \in D\) there is a long exact sequence

\[
\begin{array}{ccc}
\longrightarrow & \operatorname{Hom}_{D^+(q)}(\mathbb{F} Z(u), \mathbb{F} Z(w)[3]) & \longrightarrow 0 & \longrightarrow \cdots, \\
\longrightarrow & \operatorname{Hom}_{D^+(q)}(\mathbb{F} Z(u), \mathbb{F} Z(w)[2]) & \longrightarrow 0 & \longrightarrow 0 \\
\longrightarrow & \operatorname{Hom}_{D^+(q)}(\mathbb{F} Z(u), \mathbb{F} Z(w)[1]) & \longrightarrow \coker \Gamma(U; \delta^0_w) & \longrightarrow \coker \Gamma(V; \delta^0_w) \\
0 & \longrightarrow & \operatorname{Hom}_{D^+(q)}(\mathbb{F} Z(u), \mathbb{F} Z(w)) & \longrightarrow \Gamma(U; \mathbb{F} Z(w)) & \longrightarrow \Gamma(V; \mathbb{F} Z(w))
\end{array}
\]

where \(U := q \cap \operatorname{int}(\downarrow T(u))\) and \(V := q \setminus (\uparrow u)\).
Proof. We have $V = U \setminus Z(u)$ and thus there is a distinguished triangle

$$R\Gamma_{Z(u)} (F_{Z(w)}) \to R\Gamma_U (F_{Z(w)}) \to R\Gamma_V (F_{Z(w)}) \to R\Gamma_{Z(u)} (F_{Z(w)}) \oplus [1]$$

by [KS90] Equation (2.6.32). Applying the cohomological functor [KS90] Definition 1.5.2 $H^0(q; -)$ to this triangle we obtain the long exact sequence

$$\longrightarrow H^2_{Z(u)}(q; F_{Z(w)}) \to H^2_U(q; F_{Z(w)}) \to \cdots \to$$

$$\longrightarrow H^1_{Z(u)}(q; F_{Z(w)}) \to H^1_U(q; F_{Z(w)}) \to H^1_V(q; F_{Z(w)}) \to$$

$$0 \to H^0_{Z(u)}(q; F_{Z(w)}) \to H^0_U(q; F_{Z(w)}) \to H^0_V(q; F_{Z(w)}) \to$$

Now by [KS90] Equation (2.6.9) we have

$$H^n_{Z(u)}(q; F_{Z(w)}) \cong \text{Hom}_{D^+(q)}(F_{Z(u)}, F_{Z(w)}[n]).$$

By substituting the corresponding terms in above exact sequence we obtain an exact sequence whose left column coincides with the sequence $\text{(2.11)}$. Moreover, by [KS90] Proposition 2.3.9.(iii) and Remark 2.6.9] we have

$$H^0_U(q; F_{Z(w)}) \cong H^0(U; F_{Z(w)}|U) \cong H^0(U; F_{Z(w)})$$

and similarly

$$H^n_U(q; F_{Z(w)}) \cong H^n(V; F_{Z(w)})$$

for all $n \in \mathbb{N}_0$. Furthermore, as $U$ is open and connected and $V$ is a disjoint union of at most two connected open subsets of $q$, the complex $\kappa_0(w)$ is a resolution of $F_{Z(w)}$ by $\Gamma(U; -)$-acyclic and $\Gamma(V; -)$-acyclic sheaves by (2.8). Thus, we may replace each term in the center and right column of above exact sequence by $H^n(\Gamma(U; \kappa_0(w)))$ and $H^n(\Gamma(V; \kappa_0(w)))$, respectively. Simplifying these terms using (2.8) for each $n \in \mathbb{N}_0$ we obtain the long exact sequence $\text{(2.11)}$.

To make use of the long exact sequence from the previous lemma, we need to understand the map $\text{coker} \Gamma(U; \delta^0_w) \to \text{coker} \Gamma(V; \delta^0_w)$ from this sequence.

Lemma 2.15. Suppose we have $u, w \in D$ with $w \preceq u \nsubseteq T(w)$ and $U$ and $V$ as in the previous lemma, then the linear map $\text{coker} \Gamma(U; \delta^0_w) \to \text{coker} \Gamma(V; \delta^0_w)$ is an isomorphism.

Proof. We distinguish between two cases. If $q \subseteq (w)$, then two connected components of $\rho_1(w)$ are contained in the same connected component of $\rho_0(w)$. As $u \nsubseteq [w, T(w)]$ this property still holds after we intersect both sets with $U$ or $V$. Thus, when we apply the cokernel to the map of maps $\Gamma(U; \delta^0_w) \to \Gamma(V; \delta^0_w)$, we obtain the identity $K \xrightarrow{id} K$ (up to isomorphism of maps). If $q \nsubseteq (w)$ both cokernels vanish. \qed
Using the previous two lemmas we show the following “vanishing theorem”.

**Lemma 2.16.** If we have \( u, v \in \mathcal{M} \) with \( v \notin (\uparrow u) \cap \text{int}(\downarrow T(u)) \), then

\[
\text{Hom}_{\mathcal{D}^+} (\iota(u), \iota(v)) \cong \{0\}.
\]

**Proof.** By Lemma 2.12 it suffices to show that

\[
\text{Hom}_{\mathcal{D}^+} (\phi^{-1}_\gamma \iota(u), \phi^{-1}_\gamma \iota(v)) \cong \{0\}.
\]

Without loss of generality we assume \( u \in D \). For

\[
v \notin D \cup T(D) \cup T^2(D)
\]

we have

\[
\text{Hom}_{\mathcal{D}^+} (\phi^{-1}_\gamma \iota(u), \phi^{-1}_\gamma \iota(v)) \cong \text{Hom}_{\mathcal{D}^+} (F_{Z(u)}, F_{Z(w)}[n])
\]
By Lemma 2.14 it suffices to show that the map

\[ \text{Hom}_D(\gamma(\phi^{-1}(u)), \phi^{-1}(v)) \cong \text{Hom}_D(\mathbb{F}_{Z(w)}, \mathbb{F}_{Z(v)}) \cong \{0\}. \]

By Lemma 2.14 it suffices to show that the map

\[ \psi: \text{coker } \Gamma(U; \delta_w^0) \to \text{coker } \Gamma(V; \delta_w^0) \]

is an epimorphism. Now in order for coker \( \Gamma(V; \delta_w^0) \) to be non-zero, there need to be at least two connected components in \( V \cap \rho_1(w) \) which lie in the same connected component of the superset \( V \cap \rho_0(w) \). Moreover, for \( V \cap \rho_1(w) \) to have at least two connected components we need to have \( w \leq u \). If \( u \nleq T(w) \), then \( \psi \) is an isomorphism by Lemma 2.15. If \( u \leq T(w) \), then the two components of \( V \cap \rho_1(w) \) lie in different components of \( V \cap \rho_0(w) \) as illustrated by Fig. 2.6 and thus coker \( \Gamma(V; \delta_w^0) \cong \{0\} \), hence \( \psi \) is surjective.

Now suppose we have \( v \in T(D) \), then let \( w := T^{-1}(v) \) and let \( U \) and \( V \) be as in the previous two lemmas. We have to show that

\[ \text{Hom}_D(q) \left( \phi^{-1}(u), \phi^{-1}(v) \right) \cong \text{Hom}_D(q) \left( \mathbb{F}_{Z(u)}, \mathbb{F}_{Z(v)} [2] \right) \cong \{0\}. \]

By Lemma 2.14 it suffices to show that the map

\[ \varphi: \Gamma(U; \mathbb{F}_{Z(u)}) \to \Gamma(V; \mathbb{F}_{Z(v)}) \]

is an epimorphism and that

\[ \psi: \text{coker } \Gamma(U; \delta_w^0) \to \text{coker } \Gamma(V; \delta_w^0) \]

is injective. If \( u \nleq v \), then \( Z(w) \) is not a closed subset of \( V \) and thus \( \Gamma(V; \mathbb{F}_{Z(w)}) \cong \{0\} \), hence \( \varphi \) is surjective. Moreover, \( \psi \) is an isomorphism by Lemma 2.15. Finally, we consider the case \( u \leq v \). By symmetry we may assume without loss of generality that the \( y \)-coordinate of \( v \) is at least as large as the \( y \)-coordinate of \( T(u) \), see also Fig. 2.7. In this case \( U \cap \rho_1(w) \) has at most one connected component and thus coker \( \Gamma(U; \delta_w^0) \cong \{0\} \), hence \( \psi \) is injective. Moreover, \( q \) has to intersect at least one of the three colored triangles in Fig. 2.7. (It may happen that the blue and the red triangle overlap, in which case the green triangle is empty.) Here each triangle is closed at the right edge and open at the top edge in the sense that for each triangle, the vertical edge on the right is part of the triangle, but not the horizontal edge at the top. If \( q \) intersects the blue triangle, then \( Z(w) \) and \( V \) are disjoint, hence \( \Gamma(V; \mathbb{F}_{Z(w)}) \cong \{0\} \) and thus \( \varphi \) is surjective. If \( q \) intersects the green triangle, then \( V \cap Z(w) \) is not a non-empty closed subset of \( V \), hence we have \( \Gamma(V; \mathbb{F}_{Z(w)}) \cong \{0\} \) in this case as well and thus \( \varphi \) is surjective. If \( q \) intersects the red triangle, then \( U \cap Z(w) \) is a closed subset of \( U \) and \( V \cap Z(w) \) has at most one connected component, hence \( \varphi \) is surjective. \( \square \)

Now we can deduce Proposition 2.13 from this “vanishing theorem”.

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Figure 2.8: The axis-aligned rectangle in $D$ determined by $u$ and $v$.

**Proof of Proposition 2.13.** Without loss of generality we assume $u \in D$. If

$$v \notin (\uparrow u) \cap \text{int}(\downarrow T(u)),$$

then the statement follows from the previous Lemma 2.16. Now suppose we have

$$v \in (\uparrow u) \cap \text{int}(\downarrow T(u)).$$

If $v \in D$, the statement follows from Lemma 2.8. For $v \in T(D)$ let $w_1, w_2 \in D$ be as in Fig. 2.8. Now by construction of $\iota$ we have $\iota(u \preceq v) = \partial'$, where $\partial'$ is the boundary map of the triangle

$$\iota\left(T^{-1}(v)\right) \xrightarrow{\begin{pmatrix} 1 \\ 1 \end{pmatrix}} \iota(w_1) \oplus \iota(w_2) \xrightarrow{\begin{pmatrix} 1 \\ -1 \end{pmatrix}} \iota(u) \xrightarrow{\partial'} \iota(v)[1]$$

associated to the short exact sequence

$$0 \to \iota_0\left(T^{-1}(v)\right) \xrightarrow{\begin{pmatrix} 1 \\ 1 \end{pmatrix}} \iota_0(w_1) \oplus \iota_0(w_2) \xrightarrow{\begin{pmatrix} 1 \\ -1 \end{pmatrix}} \iota_0(u) \to 0.$$

Applying the cohomological functor $\text{Hom}_{D^{\tau\gamma}(q)}(\iota(u), -)$ to this triangle we obtain the exact
sequence

\[ \text{Hom}_{D^+}(q, \iota(u), \iota(w_1)) \oplus \text{Hom}_{D^+}(q, \iota(u), \iota(w_2)) \]

\[ \downarrow \]

\[ \text{Hom}_{D^+}(q, \iota(u), \iota(u)) \]

\[ \downarrow \]

\[ \text{Hom}_{D^+}(q, \iota(u), \iota(u \leq v)) \]

\[ \downarrow \]

\[ \text{Hom}_{D^+}(q, \iota(u), \iota(v)) \]

\[ \downarrow \]

\[ \text{Hom}_{D^+}(q, \iota(u), (\iota \circ T)(w_1)) \oplus \text{Hom}_{D^+}(q, \iota(u), (\iota \circ T)(w_2)). \]

By the previous Lemma 2.16 this exact sequence starts with \( \{0\} \) and ends with \( \{0\} \), hence \( \text{Hom}_{D^+}(q, \iota(u), \iota(u \leq v)) \) is an isomorphism. Moreover,

\[ \text{Hom}_{D^+}(q, \iota(u), \iota(u)) = \langle \iota(u \leq u) \rangle = \langle \text{id} \circ \iota(u) \rangle \cong F \]

by Lemma 2.8 and thus

\[ \text{Hom}_{D^+}(q, \iota(u), \iota(v)) = \langle \iota(u \leq v) \rangle \cong F. \]

\[ \square \]

3 Induced Cohomological Functors on \( \mathbb{M} \)

We consider a bounded below chain complex \( F \) of sheaves on \( q \) as an object of the derived category \( D^+(q) \). Then we have the functor

\[ h_\gamma(F) := \text{Hom}_{D^+}(q, \iota(-), F): \mathbb{M}^o \to \text{Vect}_F \]

from \( \mathbb{M} \) to the category of vector spaces over \( F \). As \( \iota(u) \cong 0 \) for all \( u \in \partial \mathbb{M} \) the functor \( h_\gamma(F) \) vanishes on \( \partial \mathbb{M} \).

**Lemma 3.1.** The functor \( h_\gamma(F): \mathbb{M}^o \to \text{Vect}_F \) has bounded above support.

**Proof.** Since \( F \) is a bounded below chain complex of sheaves, there is an integer \( n \in \mathbb{Z} \), such that \( F^k \cong 0 \) for all \( k < -n \). Thus, we have \( h_\gamma(F)(u) \cong \{0\} \) for all \( u \in \mathbb{M} \setminus T^n(Q) \). \[ \square \]

We will shortly see that the functor \( h_\gamma(F): \mathbb{M}^o \to \text{Vect}_F \) satisfies the exactness properties of the following definition.

**Definition 3.2.** We say that a functor \( G: \mathbb{M}^o \to \text{Vect}_F \) vanishing on \( \partial \mathbb{M} \) is cohomological, if for any axis-aligned rectangle with one corner lying on \( l_1 \) and the other corners
Figure 3.1: The linear subposet given by the orbits of $u$, $v$, and $w$. The region shaded in dark grey is our fundamental domain $D$.

$u \preceq v \preceq w \in \mathcal{M}$, the long sequence

$$\cdots \rightarrow G(T(u)) \rightarrow$$

$$\rightarrow G(w) \rightarrow G(v) \rightarrow G(u)$$

$$\rightarrow G(T^{-1}(w)) \rightarrow \cdots .$$

is exact; see also Fig. 3.1 or [BBF21, Definition C.1].

In [BBF21, Proposition C.2] we also provide the following useful characterization of cohomological functors.

**Proposition 3.3.** A functor $G: \mathcal{M}^\circ \rightarrow \text{Vect}_F$ vanishing on $\partial \mathcal{M}$ is cohomological iff for any axis-aligned rectangle $u \preceq v_1, v_2 \preceq w \in D$ as shown in Fig. 2.5 the long sequence

$$\cdots \rightarrow G(T(u)) \rightarrow$$

$$\rightarrow G(w) \rightarrow G(v_1) \oplus G(v_2) \xrightarrow{(1 \ -1)} G(u)$$

$$\rightarrow G(T^{-1}(w)) \rightarrow \cdots .$$

is exact.
Lemma 3.4. The functor $h_\gamma(F) : M^0 \to \text{Vect}_F$ is cohomological.

Proof. If we apply the cohomological functor $\text{Hom}_{D^+(q_\gamma)}(-, F) : D^+(q_\gamma) \to \text{Vect}_F$ to the distinguished triangle (2.10) for $u \leq v_1, v_2 \leq w \in D$ as in Fig. 2.5, then we obtain the long exact sequence

$$\cdots \to h_\gamma(F)(T(u)) \to h_\gamma(F)(w) \to h_\gamma(F)(v_1) \oplus h_\gamma(F)(v_2) \overset{(1-1)}{\to} h_\gamma(F)(u) \to \cdots.$$  

Thus, the long sequence (3.1) is exact for $G := h_\gamma(F) = \text{Hom}_{D^+(q_\gamma)}(\iota(-), F)$ and hence $h_\gamma(F) : M^0 \to \text{Vect}_F$ is cohomological. \qed

We also have the following primitive type of a cohomological functor, see also Fig. 1.1.

Definition 3.5 (Contravariant Block). For $v \in M$ we define

$$B_v : M^0 \to \text{Vect}_F, u \mapsto \begin{cases} F & u \in (\downarrow v) \cap \text{int}(\uparrow T^{-1}(v)) \\ \{0\} & \text{otherwise} \end{cases},$$

where $\text{int}(\uparrow T^{-1}(v))$ is the interior of the upset of $T^{-1}(v)$ in $M$. The internal maps are identities whenever both domain and codomain are $F$, otherwise they are zero.

Lemma 3.6 (Yoneda). Let $G : M^0 \to \text{Vect}_F$ be a functor vanishing on $\partial M$ and let $v \in \text{int} M := M \setminus \partial M$. Then the evaluation at $1 \in F = B_v(v)$ yields a linear isomorphism

$$\text{Nat}(B_v, G) \cong G(v),$$

where $\text{Nat}(B_v, G)$ denotes the vector space of natural transformations from $B_v$ to $G$.

Using the Yoneda Lemma 3.6 we may rephrase Proposition 2.13 as follows.

Corollary 3.7. The unique natural transformation

$$B_v \to (h_\gamma \circ \iota)(v) = \text{Hom}_{D^+(q_\gamma)}(\iota(-), \iota(v))$$

sending $1 \in F = B_v(v)$ to $\text{id}_{\iota(v)} \in \text{Hom}_{D^+(q_\gamma)}(\iota(v), \iota(v))$ is a natural isomorphism.

The following proposition provides a description of $h_\gamma(F)$ in terms of local sheaf cohomology.

Proposition 3.8. Let $u \in D$ and let $n \in \mathbb{Z}$, then we have a natural isomorphism

$$h_\gamma(F)(T^{-n}(u)) = \text{Hom}_{D^+(q_\gamma)}((\iota \circ T^{-n})(u), F) \cong H^n_{q \cap \text{int}(\downarrow T(u))}(q \cap \text{int}(\downarrow T(u)); F).$$
Proof. By Lemma 2.9 we have
\[
\text{Hom}_{D^+}(q, (t \circ T^{-n})(u), F) = \text{Hom}_{D^+}(q, t_0(u)[-n], F) \\
= \text{Hom}_{D^+}(q, t_0(u), F[n]) \\
\cong H^n_{q}(T(u) \cap \text{int}(T(u)); F).
\]

Finally we note that the assignment \( F \mapsto h_\gamma(F) = \text{Hom}_{D^+}(q, \iota_0(u), F) \) is functorial in \( F \). Therefore we obtain the functor \( h_\gamma : D^+(q) \to \text{Vect}_F^M \). Moreover, we take note of the following.

**Lemma 3.9.** For each complex of sheaves \( F \) of the derived category \( D^+(q) \) the counit \( \varepsilon_F : R^\flat(F) \to F \) of the adjunction

\[
D^+(q, 0) \xrightarrow{\varepsilon} D^+(q, 0)
\]

is mapped to a natural isomorphism \( h_\gamma(\varepsilon_F) : h_\gamma(R^\flat(F)) \to h_\gamma(F) \).

**Proof.** Let \( u \in M \), then \( \iota(u) \) is an object of \( D^+(q, 0) \). Thus, the linear map

\[
\text{Hom}_{D^+(q, 0)}(\iota(u), F) : \text{Hom}_{D^+(q, 0)}(\iota(u), R^\flat(F)) \to \text{Hom}_{D^+(q, 0)}(\iota(u), F)
\]

is an isomorphism.

Let \( h_{\gamma, 0} : D^+(q, 0) \to \text{Vect}_F^M \) be the restriction of \( h_\gamma \) to \( D^+(q, 0) \). With this we may rephrase Lemma 3.9 more concisely:

**Corollary 3.10.** The commutative square

\[
D^+(q, 0) \xrightarrow{R^\flat} D^+(q, 0)
\]

of categories and functors satisfies the dual Beck–Chevalley condition as in Definition B.2.

**Corollary 3.11.** We have the diagram
with all 2-cells natural isomorphisms, where \( D^+(\hat{q}) \) is the bounded-below derived category of the category of sheaves on \( \hat{q} \) with the subspace topology inherited from \( q_\gamma \).

**Proof.** The triangular 2-cell at the top is an isomorphism by (2.2) and Lemma A.2 and the square 2-cell at the bottom is an isomorphism by Corollary 3.10 or Lemma 3.9. \( \square \)

In addition to the commutative square (3.2) we may also consider the commutative square

\[
\begin{array}{ccc}
D^+(\hat{q}) & \xrightarrow{Ri_*} & D^+(q_\gamma) \\
\downarrow h_\gamma \circ Ri_* & & \downarrow h_\gamma \\
\text{Vect}_{M^o} & \xrightarrow{\cong} & \text{Vect}_{\hat{q}}^{M^o}.
\end{array}
\]

**Lemma 3.12.** The commutative square (3.3) satisfies the Beck–Chevalley condition as in Definition B.1.

**Proof.** By Lemma 2.4 it suffices to show that the square

\[
\begin{array}{ccc}
D^+(\hat{q}) & \xrightarrow{Ri_{12}^*} & D^+(q_\gamma) \\
\downarrow h_\gamma \circ Ri_{12}^* & & \downarrow h_\gamma \\
\text{Vect}_{M^o} & \xrightarrow{\cong} & \text{Vect}_{\hat{q}}^{M^o}
\end{array}
\]

satisfies the Beck–Chevalley condition. Now (3.4) satisfies the Beck–Chevalley condition iff the square

\[
\begin{array}{ccc}
D^+(\hat{q}) & \xrightarrow{Ri_{12}^*} & D^+(q_\gamma) \\
\downarrow H^2_{Z(u;\cdot)} & \cong & \downarrow H^2_{Z(u;\cdot)} \\
\text{Vect}_{\hat{q}} & \xrightarrow{\cong} & \text{Vect}_{\hat{q}}
\end{array}
\]

satisfies the Beck–Chevalley condition for any \( u \in D \) and any \( n \in \mathbb{Z} \). This in turn follows from Corollary C.4. \( \square \)

Now if \( \partial q \) is closed in \( q_\gamma \), then \( D^+(q_\gamma, \partial q) \) and \( D^+(\hat{q}) \) are equivalent by Lemma 2.6. So dealing with both of these categories separately is somewhat redundant in this case.

**Lemma 3.13.** If \( \partial q \) is closed in \( q_\gamma \), then the square diagram

\[
\begin{array}{ccc}
D^+(q_\gamma) & \xrightarrow{i^{-1}} & D^+(\hat{q}) \\
\downarrow h_\gamma & & \downarrow h_\gamma \circ Ri_* \\
\text{Vect}_{M^o} & \xrightarrow{\cong} & \text{Vect}_{\hat{q}}^{M^o}
\end{array}
\]

satisfies both (the ordinary and the dual) Beck–Chevalley conditions, where \( \eta^i : \text{id} \to Ri_* \circ i^{-1} \) is the unit of the adjunction \( i^{-1} \dashv Ri_* \).
Proof. By Lemma 3.12, the natural transformation \( h_\gamma \circ \eta^i : h_\gamma \to h_\gamma \circ Ri_s \circ i^{-1} \), which is the mate of the identity natural transformation in the square (3.3), is a natural isomorphism, hence (3.9) satisfies the dual Beck–Chevalley condition. Moreover, by Lemma 2.7 the functor \( i^{-1} : D^+(q_\gamma) \to D^+(q) \) also has the left adjoint \( Ri_t : D^+(q) \to D^+(q_\gamma) \). Now \( h_\gamma \circ Ri_t = h_\gamma \circ \circ Ri_s \) and moreover, the natural transformation

\[
\eta^i \circ h_\gamma = \eta^i \circ Ri_s \circ i^{-1},
\]

is a natural isomorphism by Lemma 3.9. Thus, the square diagram (3.9) satisfies the Beck–Chevalley condition iff the composition of square diagrams

\[
\begin{array}{c}
D^+(q_\gamma) \xrightarrow{i^{-1}} D^+(q) \\
\downarrow h_\gamma \\
\text{Vect}_F^M \end{array} \quad \begin{array}{c}
D^+(q) \xrightarrow{i^{-1}} D^+(q) \\
\downarrow h_\gamma \circ Ri_t \\
\text{Vect}_F^M \end{array}
\]

satisfies the Beck–Chevalley condition. Now for a sheaf \( F \) on \( q_\gamma \) we have the commutative diagram

\[
\begin{array}{ccc}
\♭ F & \xrightarrow{(b \circ \eta^i)^F} & i^i^{-1} F \\
\downarrow \eta^i_F & & \downarrow \eta^i \\
F & \xrightarrow{\epsilon^i_F} & i^i^{-1} F,
\end{array}
\]

hence the composition of square diagrams (3.6) is the square diagram

\[
\begin{array}{c}
D^+(q_\gamma) \xrightarrow{i^{-1}} D^+(q) \\
\downarrow h_\gamma \\
\text{Vect}_F^M \end{array} \quad \begin{array}{c}
D^+(q) \xrightarrow{i^{-1}} D^+(q) \\
\downarrow h_\gamma \circ Ri_t \\
\text{Vect}_F^M \end{array}
\]

Moreover, the natural transformation \( h_\gamma \circ \epsilon^i : h_\gamma \circ Ri_t \circ i^{-1} \to h_\gamma \) is just the mate of the identity natural transformation of the commutative square

\[
\begin{array}{c}
D^+(q_\gamma) \xleftarrow{Ri_t} D^+(q) \\
\downarrow h_\gamma \\
\text{Vect}_F^M \end{array} \quad \begin{array}{c}
D^+(q) \xrightarrow{i^{-1}} D^+(q) \\
\downarrow h_\gamma \circ Ri_t \\
\text{Vect}_F^M \end{array}
\]

and thus (3.7) satisfies the Beck–Chevalley condition.

**Proposition 3.14.** The functor \( h_{\gamma,0} : D^+(q_\gamma, \partial q) \to \text{Vect}_F^M \) is conservative. In other words, if \( \psi : G \to F \) is a homomorphism of \( D^+(q_\gamma, \partial q) \) with \( h_{\gamma}(\psi) : h_{\gamma}(G) \to h_{\gamma}(F) \) a natural isomorphism, then \( \psi : G \to F \) is an isomorphism as well.
Figure 3.2: The fundamental domain $D$ partitioned into four regions.

Proof. Suppose $\psi : G \to F$ is a homomorphism of $D^+(q_\gamma, \partial q)$ with $h_\gamma(\psi) : h_\gamma(G) \to h_\gamma(F)$ a natural isomorphism and let $n \in \mathbb{Z}$ be arbitrary. It suffices to show that the induced linear map

$$H^n(U ; \psi) : H^n(U ; G) \to H^n(U ; F)$$

is an isomorphism for each open subset $U$ of some basis $\mathcal{B}$ of $q_\gamma$. To this end, let $\mathcal{B} := \{ q \cap \text{int}(\downarrow (T(u))) \mid u \in D \}$.

If $\partial q$ is discrete as a subposet of $\mathbb{M}$, then $\mathcal{B}$ is a basis of $q_\gamma$. Otherwise, $\mathcal{B} \cup \{ q \}$ is a basis of $q_\gamma$ and both $H^n(F ; q)$ and $H^n(G ; q)$ vanish. So in either case, it suffices to check that $H^n(U ; \psi)$ is an isomorphism for each $U \in \mathcal{B}$.

Now let $u \in D$ be contained in the region of $D$ shaded in red in Fig. 3.2 and let $U := q \cap \text{int}(\downarrow (T(u)))$. Then we have

$$H^n(U ; \psi) \cong H^n_q(U ; \psi) \cong \text{Hom}_{D^+(q_\gamma)}((T^{-n} \circ \iota)(u), \psi) \cong (h_\gamma(\psi) \circ T^{-n})(u)$$

as linear maps by Proposition 3.8 and thus $H^n(U ; \psi)$ is an isomorphism.

Now suppose that $u = (x, y) \in D$ is a point on the vertical line segment separating the red from the cyan region in Fig. 3.2, presuming the cyan region is non-empty. (Otherwise there is no need to treat this case.) Then for all $s < x$ the open subsets $q \cap \text{int}(\downarrow (T(s, y)))$ are identical. So let $U := q \cap \text{int}(\downarrow (T(s, y)))$ for some (and thus any) $y < x$. Now for $s < x$ we have the exact sequence

$$H^{n-1}(q \cap ((s, \infty) \times \mathbb{R}) ; \psi) \to H^n_{q \cap \{T(s, y)\}}(U ; \psi) \to H^n(U ; \psi) \to H^n(q \cap ((s, \infty) \times \mathbb{R}) ; \psi)$$

in the category of linear maps by [KS90, Equation (2.6.32)]. As both $F$ and $G$ vanish on $\partial q$, taking the direct limit of this sequence over all $s < x$ we obtain the exact sequence

$$0 \to \lim_{s < x} H^n_{q \cap \{T(s, y)\}}(U ; \psi) \to H^n(U ; \psi) \to 0.$$ 

Moreover, Proposition 3.8 implies

$$H^n_{q \cap \{T(s, y)\}}(U ; \psi) \cong \text{Hom}_{D^+(q_\gamma)}((T^{-n} \circ \iota)(s, y), \psi) \cong (h_\gamma(\psi) \circ T^{-n})(s, y),$$

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of D

We say that a bounded below complex is an isomorphism as well.

As before let \( \dot{\gamma} \) be the corresponding subspace inclusion.

**Lemma 3.16.** Let \( I \subseteq q_{\gamma} \) and any integer \( n \in \mathbb{Z} \), then \( F \) is tame.

**Definition 3.15.** We say that a bounded below complex \( F \) of sheaves on \( q_{\gamma} \) is **tame** if \( h_{\gamma}(F) : M^0 \to \text{Vect}_F \) is pointwise finite-dimensional (pfd). Similarly, we say that an object \( F \) of \( D^+(\dot{q}) \) is **tame** if \( R_{i*}F \) is tame. We denote the full subcategories of tame complexes in the derived categories \( D^+(q_{\gamma}) \) and \( D^+(\dot{q}) \) by \( D^+_t(q_{\gamma}) \) and \( D^+_t(\dot{q}) \) respectively. Furthermore, we denote the intersection of the two full subcategories \( D^+_t(q_{\gamma}) \) and \( D^+(q_{\gamma}, \partial \dot{q}) \) by \( D^+_t(q_{\gamma}, \partial \dot{q}) \).

**Lemma 3.16.** Let \( F \) be an object of \( D^+(q_{\gamma}) \). If \( \dim_F H^n(I; F) < \infty \) for any connected open subset \( I \subseteq q_{\gamma} \) and any integer \( n \in \mathbb{Z} \), then \( F \) is tame.

**Proof.** Let \( u \in D \) and let \( n \in \mathbb{Z} \). We show \( \dim_F (h_{\gamma}(F) \circ T^{-n})(u) < \infty \). To this end, let \( I := q \cap \text{int}(\downarrow T(u)) \) and \( J := q \cap (\uparrow u) \). By Proposition 3.8 it suffices to show that \( \dim_F H^n_I(J; F) < \infty \). We consider the fragment

\[
H^{n-1}(I; F) \rightarrow H^{n-1}(I \setminus J; F) \rightarrow H^n_I(J; F) \rightarrow H^n(I; F) \rightarrow H^n(I \setminus J; F)
\]
of the corresponding long exact sequence for local sheaf cohomology, see for example \cite[Equation (2.6.32) and Remark 2.6.10]{KS90}. Now \( I \) is connected and \( I \setminus J \) is the disjoint union of at most two connected open sets. By our assumptions on \( F \) all four cohomology spaces surrounding \( H^n_q(I; F) \) in above exact sequence are finite-dimensional. As a result, \( H^r_q(I; F) \) is finite-dimensional as well.

Now let \( F \) be an object of \( D^+(\hat{q}) \).

**Lemma 3.17.** Let \( I \subseteq \xi_q \) be a connected open subset. Then we have \( H^n(I; R_i F) \cong H^n(I \setminus \partial q; F) \) for all integers \( n \in \mathbb{Z} \).

**Proof.** This follows from Lemma \ref{lemma} with \( \Gamma(I; -) : \text{Sh}(q_\gamma) \to \text{Vect}_\mathbb{F} \) in place of \( H : B \to A \) and \( i_* : \text{Sh}(q) \to \text{Sh}(q_\gamma) \) in place of \( G : C \to B \).

**Corollary 3.18.** For any \( n \in \mathbb{Z} \) we have \( \dim_q H^n(I; F) < \infty \) for all connected open subsets \( I \subseteq \hat{q} \) iff we have \( \dim_q H^n(I; R_i F) < \infty \) for all connected open subsets \( I \subseteq q_\gamma \).

**Lemma 3.19.** If \( \dim_q H^n(I; F) < \infty \) for all connected open subsets \( I \subseteq \hat{q} \) and all integers \( n \in \mathbb{Z} \), then \( F \) is tame. If \( \partial q \) is closed in \( q_\gamma \), then the converse is true as well.

**Proof.** The first implication follows from Corollary \ref{corollary} and Lemma \ref{lemma}. Now suppose \( \partial q \) is closed in \( q_\gamma \), let \( I \subseteq \hat{q} \) be a connected open subset, let \( n \in \mathbb{Z} \), and suppose that \( h_\gamma(R_i F) : \mathbb{M} \to \text{Vect}_\mathbb{F} \) is pfd. As \( \partial q \) is closed in \( q_\gamma \) there is some point \( u \in D \) with \( \rho_1(u) = q \setminus I \) and \( \rho_0(u) = q \). By Proposition \ref{proposition} (2.6), and Lemma \ref{lemma} this implies
\[
(h_\gamma(R_i F) \circ T^{-n})(u) \cong H^n(I; R_i F) \cong H^n(I; F),
\]

hence \( H^n(I; F) \) is finite-dimensional.

**Definition 3.20.** We say that a functor \( G : \mathbb{M} \to \text{Vect}_\mathbb{F} \) is **sequentially continuous**, if for any increasing sequence \((u_k)_{k=1}^\infty \) in \( \mathbb{M} \) converging to \( u \) the natural map
\[
G(u) \to \varprojlim_k G(u_k)
\]
is an isomorphism, see also \cite[Definition 2.4]{BBF21}.

**Proposition 3.21.** Let \( F \) be an object of \( D^+(\hat{q}_\gamma) \) and suppose that \( h_\gamma(F) : \mathbb{M} \to \text{Vect}_\mathbb{F} \) is pfd. Then \( h_\gamma(F) \) is sequentially continuous.

**Proof.** Let \((u_k)_{k=1}^\infty \) be an increasing sequence in \( \mathbb{M} \) converging to \( u \in \mathbb{M} \). Without loss of generality we assume that \((u_k)_{k=1}^\infty \) is contained in a single tile \( T^{-n}(D) \) for some \( n \in \mathbb{Z} \). As \( h_\gamma(F) \) is pfd the projective system
\[
\left\{ H_{\gamma \setminus (\{u_k\})}^{n-1}(q \cap \text{int}(\downarrow T(u_k)); F) \right\}_{k=1}^\infty
\]
satisfies the Mittag-Leffler condition by Proposition \ref{proposition}. Thus, the natural map
\[
H_{\gamma \setminus (\{u\})}^{n}(q \cap \text{int}(\downarrow T(u)); F) \to \varprojlim_k H_{\gamma \setminus (\{u_k\})}^{n}(q \cap \text{int}(\downarrow T(u_k)); F)
\]
is an isomorphism by \cite[Proposition 2.7.1.(ii)]{KS90}. In conjunction with Proposition \ref{proposition} this implies the sequential continuity of \( h_\gamma(F) \).
As already noted in Section 1, we denote the category of finite-dimensional vector spaces over \( F \) by \( \text{vect}_F \) and by \( \mathcal{J} \) the full subcategory of pfd functors \( M^\circ \to \text{vect}_F \) that are cohomological, sequentially continuous, and have bounded above support. We summarize the results of this subsection so far:

**Proposition 3.22.** Let \( F \) be an object of \( D^+(q_{\gamma}) \) or of \( D^+(q) \). Then \( F \) is tame iff \( h_{\gamma}(F): M^\circ \to \text{Vect}_F \) respectively \( h_{\gamma}(Ri_*F): M^\circ \to \text{Vect}_F \) is in \( \mathcal{J} \).

**Proof.** This follows directly from Lemmas 3.1, 3.4, and Proposition 3.21.

Now by Corollary 3.10 the coreflector \( R\flat: D^+(q_{\gamma}) \to D^+(q_{\gamma}, \partial q) \) restricts to a coreflector for the full subcategory inclusion \( D^+_t(q_{\gamma}, \partial q) \to D^+_t(q_{\gamma}) \). With this we obtain the following.

**Lemma 3.23.** The commutative square

\[
\begin{array}{ccc}
D^+_t(q_{\gamma}, \partial q) & \longrightarrow & D^+_t(q_{\gamma}) \\
\downarrow & & \downarrow \\
D^+(q_{\gamma}, \partial q) & \longrightarrow & D^+(q_{\gamma})
\end{array}
\]

of full replete subcategory inclusions satisfies the dual Beck–Chevalley condition.

Now let

\( h_{\gamma,t}: D^+_t(q_{\gamma}) \to \mathcal{J} \quad \text{and} \quad h_{\gamma,0,t}: D^+_t(q_{\gamma}, \partial q) \to \mathcal{J} \)

be the corresponding restrictions of \( h_{\gamma}: D^+(q_{\gamma}) \to \text{Vect}_M^M \).

**Lemma 3.24.** The commutative square

\[
\begin{array}{ccc}
D^+_t(q_{\gamma}, \partial q) & \longrightarrow & D^+_t(q_{\gamma}) \\
\downarrow & & \downarrow \\
\mathcal{J} & \longrightarrow & \mathcal{J}
\end{array}
\]

satisfies the dual Beck–Chevalley condition.

**Proof.** This follows directly from Corollary 3.10.

**Lemma 3.25.** The derived adjunction

\[
D^+(q_{\gamma}) \cup D^+(q) \longrightarrow D^+(q_{\gamma}, \partial q)
\]

restricts to an adjunction

\[
D^+_t(q_{\gamma}) \cup D^+_t(q) \longrightarrow D^+_t(q_{\gamma}, \partial q).
\]

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Proof. By Definition 3.15 the derived functor \( R_i^* : D^+(\dot{q}) \to D^+(q_\gamma) \) restricts to a functor \( D^+(i^{-1}) \to D^+(q_\gamma) \). Now suppose \( F \) is an object of \( D^+(q_\gamma) \). We have to show that \( D^+(i^{-1})(F) \) is an object of \( D^+(\dot{q}) \). With some abuse of notation we write \( i^{-1}F \) for \( D^+(i^{-1})(F) \). By Definition 3.15 and Corollary 3.10 it suffices to show that \( R\circ Ri_*i^{-1}F \) is an object of \( D^+(q_\gamma) \). Now by Lemma A.2 (2.2), and Lemma 2.5 we have

\[
R\circ Ri_*i^{-1}F \cong R(i \circ i_* i^{-1})(F) = R(i \circ i^{-1})(F) \cong RF,
\]

which is in \( D^+(q_\gamma) \) by Corollary 3.10 or Lemma 3.23.

Lemma 3.26. The commutative square

\[
\begin{tikzcd}
D^+_t(\dot{q}) \ar{r}{R_i^*} \ar{d}{h_{\gamma,t \circ Ri_*}} & D^+_t(q_\gamma) \ar{d}{h_{\gamma,t}} \\
J \ar{r}{J} & J
\end{tikzcd}
\]

satisfies the Beck–Chevalley condition.

Proof. This follows directly from Lemmas 3.25 and 3.12.

By Lemmas 3.23, 3.25, and A.2 we may compose the adjunctions

\[
\begin{tikzcd}
D^+_t(q_\gamma, \partial q) \ar{rr}{R_i^*} \ar{rr} & & D^+_t(q_\gamma) \ar{rr}{R_i^*} \ar{rr} & & D^+_t(\dot{q}) \ar{rr}{D^+(i^{-1})} & & D^+_t(q_\gamma, \partial q)
\end{tikzcd}
\]

to obtain the adjunction

\[
\begin{tikzcd}
D^+_t(q_\gamma, \partial q) \ar{rr}{R_i^*} \ar{rr} & & D^+_t(q_\gamma) \ar{rr}{R_i^*} \ar{rr} & & D^+_t(\dot{q}) \ar{rr}{D^+(i^{-1})} & & D^+_t(q_\gamma, \partial q)
\end{tikzcd}
\]

(3.8)

Lemma 3.27. If \( \partial q \) is closed in \( q_\gamma \), then the adjunction (3.8) is an adjoint equivalence.

Proof. This follows directly from Lemma 2.6.

Lemma 3.28. If \( \partial q \) is closed in \( q_\gamma \), then the square diagram

\[
\begin{tikzcd}
D^+_t(q_\gamma) \ar{r}{i^{-1}} \ar{d}{h_{\gamma,t \circ Ri_*}} & D^+_t(\dot{q}) \ar{d}{h_{\gamma,t \circ Ri_*}} \\
J \ar{r}{J} & J
\end{tikzcd}
\]

(3.9)

satisfies both (the ordinary and the dual) Beck–Chevalley conditions.

Proof. This follows directly from Lemma 3.13.
We also note that \( J \) is an additive \( \mathbb{F} \)-linear subcategory of \( \text{Vect}^\mathbb{F}_p \). We end this section with a proof that \( D^+_\mathbb{F}_t(q, \partial q) \) is a triangulated subcategory of \( D^+(q, \gamma) \). To this end, we show the following auxiliary lemma.

**Lemma 3.29.** The category of pfd sequentially continuous functors \( \mathbb{M}^o \rightarrow \text{Vect}_p \) is a weak Serre subcategory of \( \text{Vect}^\mathbb{F}_p \), i.e. for any exact sequence

\[
F_1 \rightarrow F_2 \rightarrow F_3 \rightarrow F_4 \rightarrow F_5
\]

with \( F_i: \mathbb{M}^o \rightarrow \text{Vect}_p \) pfd sequentially continuous for \( i = 1, 2, 4, 5 \) the functor \( F_3: \mathbb{M}^o \rightarrow \text{Vect}_p \) is pfd sequentially continuous as well.

**Proof.** The statement that \( F_3: \mathbb{M}^o \rightarrow \text{Vect}_p \) is pfd follows point-wise from the analogous property of the full subcategory inclusion \( \text{vect}_p \rightarrow \text{Vect}_p \) of finite-dimensional vector spaces in \( \text{Vect}_p \). Now let \( (u_k)_{k=1}^\infty \) be an increasing sequence in \( \mathbb{M} \) converging to \( u \). As inverse limits of finite-dimensional vector spaces are exact, both rows of the commutative diagram

\[
\begin{array}{cccccc}
F_1(u) & \rightarrow & F_2(u) & \rightarrow & F_3(u) & \rightarrow & F_4(u) & \rightarrow & F_5(u) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\lim_k F_1(u_k) & \rightarrow & \lim_k F_2(u_k) & \rightarrow & \lim_k F_3(u_k) & \rightarrow & \lim_k F_4(u_k) & \rightarrow & \lim_k F_5(u_k)
\end{array}
\tag{3.10}
\]

are exact. Moreover, as the functors \( F_i: \mathbb{M}^o \rightarrow \text{Vect}_p \) for \( i = 1, 2, 4, 5 \) are sequentially continuous, all four non-center vertical maps of (3.10) are isomorphisms. With this it follows from the five lemma that \( F_3(u) \rightarrow \lim_k F_3(u_k) \) is an isomorphism as well. \( \square \)

**Corollary 3.30.** The category \( D^+_\mathbb{F}_t(q, \gamma) \) is a triangulated subcategory of \( D^+(q, \gamma) \).

**Proof.** Suppose

\[
F \rightarrow G \rightarrow H \rightarrow F[1]
\]

is a distinguished triangle with \( F \) and \( G \) in \( D^+_\mathbb{F}_t(q, \gamma) \). We have to show that \( h_\gamma(H): \mathbb{M}^o \rightarrow \text{Vect}_p \) is a functor in \( J \). By Lemma 3.29 the functor \( h_\gamma(H) \) is cohomological. Thus, it remains to show \( h_\gamma(H) \) is pfd and sequentially continuous. As \( h_\gamma: D^+(q, \gamma) \rightarrow \text{Vect}^\mathbb{F}_p \) is cohomological, we obtain the exact sequence

\[
h_\gamma(F) \rightarrow h_\gamma(G) \rightarrow h_\gamma(H) \rightarrow h_\gamma(F[1]) \rightarrow h_\gamma(G[1])
\]

with \( h_\gamma(F), h_\gamma(G), h_\gamma(F[1]) \), and \( h_\gamma(G[1]) \) functors in \( J \) by assumption. In conjunction with Lemma 3.29 we obtain that \( h_\gamma(H): \mathbb{M}^o \rightarrow \text{Vect}_p \) is pfd and sequentially continuous as well. \( \square \)

**Corollary 3.31.** The category \( D^+_\mathbb{F}_t(q, \partial q) \) is a triangulated subcategory of \( D^+(q, \gamma) \).

**Proof.** This follows from Corollaries 2.3 and 3.30 \( \square \)

**Corollary 3.32.** The category \( D^+_\mathbb{F}_t(q) \) is a triangulated subcategory of \( D^+(q) \).

**Proof.** The derived functor \( R\iota_*: D^+(q) \rightarrow D^+(q, \gamma) \) is triangulated, hence the result follows from Definition 3.15 and Corollary 3.30 \( \square \)
3.2 Alternative Construction of Induced Cohomological Functors

Let $G$ be a bounded below complex of flabby sheaves on $q\gamma$. In the following we provide an alternative construction of $h\gamma(G) : \mathcal{M}^\circ \to \mathcal{V}ect\mathcal{F}$, which will be useful later. To this end, we define the functor

$$\tilde{F}' : D \to C^+(\mathcal{V}ect\mathcal{F})^\circ, \ u \mapsto \Gamma_{q\cap(\uparrow u)}(q \cap \text{int}(\downarrow u); G),$$

where the internal maps are induced by inclusions. Post-composing $\tilde{F}' : D \to C^+(\mathcal{V}ect\mathcal{F})^\circ$ with the opposite graded cohomology functor

$$H^\bullet : C^+(\mathcal{V}ect\mathcal{F})^\circ \to (\mathcal{V}ect\mathcal{Z}\mathcal{F})^\circ, \ C \mapsto H^\bullet(C)$$

we obtain the functor

$$F' : D \to (\mathcal{V}ect\mathcal{Z}\mathcal{F})^\circ, \ u \mapsto H^\bullet_{q\cap(\uparrow u)}(q \cap \text{int}(\downarrow u); G)$$

as $G$ is a complex of flabby sheaves. Now for $n \in \mathbb{Z}$ and $u \in D$ we have an isomorphism

$$\varphi^\circ_n : h\gamma(G)(T^{-n}(u)) \xrightarrow{\cong} H^\bullet_{q\cap(\uparrow u)}(q \cap \text{int}(\downarrow u); G) = (F'(u))^n \quad (3.11)$$

by Proposition 3.8, which is natural in $u$. Now in order to use this to connect our original construction of $h\gamma(G) : \mathcal{M}^\circ \to \mathcal{V}ect\mathcal{F}$ with this alternative approach, let $h^\#(G) : \mathcal{M} \to (\mathcal{V}ect\mathcal{Z}\mathcal{F})^\circ$ be the transform of $(h\gamma(G))^\circ : \mathcal{M} \to \mathcal{V}ect\mathcal{F}$ under the 2-adjunction from [BBF21, Lemma A.6]. Then the family of isomorphisms $\{\varphi^\circ_n\}_{u \in D, n \in \mathbb{Z}}$ assembles to a natural isomorphism

$$\varphi' : F' \xrightarrow{\cong} h^\#(G)|_D.$$

Now the missing ingredient, in order to obtain a strictly stable functor $F : \mathcal{M} \to (\mathcal{V}ect\mathcal{Z}\mathcal{F})^\circ$ from $F' : D \to (\mathcal{V}ect\mathcal{Z}\mathcal{F})^\circ$ is a natural transformation as in the diagram

$$\begin{array}{ccc}
R_D & \xrightarrow{\text{pr}_1} & D \\
\downarrow \text{pr}_2 & & \downarrow F' \\
T(D) & \xrightarrow{\Sigma \circ F' \circ T^{-1}} & (\mathcal{V}ect\mathcal{Z}\mathcal{F})^\circ
\end{array} \quad (3.12)$$

according to [BBF21, Proposition A.14], where

$$R_D := \{(w, \hat{u}) \in D \times T(D) \mid w \preceq \hat{u} \preceq T(w)\}$$

as in [BBF21, Definition A.8]. To this end, let $(w, \hat{u}) \in R_D$, let $u := T^{-1}(\hat{u})$, and let $v_1, v_2 \in [u, w]$ be the lower left respectively the upper right vertex of $[u, w]$ as shown in Fig. 2.5. If we now instantiate Corollary C.7 with

$$X_1 := q \cap \text{int}(\downarrow v_1), \quad A_1 := q \setminus (\uparrow v_1),$$

$$X_2 := q \cap \text{int}(\downarrow v_2), \quad \text{and} \quad A_2 := q \setminus (\uparrow v_2),$$

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then we obtain the short exact sequence

\[
0 \longrightarrow \tilde{F}'(w) \longrightarrow \tilde{F}'(v_1) \oplus \tilde{F}'(v_2) \underset{(1-1)}{\longrightarrow} \tilde{F}'(u) = (\tilde{F}' \circ T^{-1})(\hat{u}) \longrightarrow 0
\]

of cochain complexes in \( \text{Vect}_F \). By the zig-zag lemma, this short exact sequence yields a differential

\[
\delta'_{(w,\hat{u})} : (\Sigma \circ F' \circ T^{-1})(\hat{u}) \to F'(w).
\]

Now let

\[
\partial'_{(w,\hat{u})} := (\delta'_{(w,\hat{u})})^\circ : F'(w) \to (\Sigma \circ F' \circ T^{-1})(\hat{u})
\]

be the corresponding homomorphism in the opposite category \( (\text{Vect}_F)^\circ \) for all \( (w, \hat{u}) \in R_D \). Then \( \partial' \) is a natural transformation as in (3.12). As it turns out, the diagram

\[
\begin{array}{ccc}
F' \circ \text{pr}_1 & \xrightarrow{\varphi^0 \circ \text{pr}_1} & h_{\gamma}^!(G) \circ \text{pr}_1 \\
\downarrow \partial' & & \downarrow \partial(h_{\gamma}^!(G),D) \\
\Sigma \circ F' \circ T^{-1} \circ \text{pr}_2 & \xrightarrow{\Sigma \circ \varphi^0 \circ T^{-1} \circ \text{pr}_2} & h_{\gamma}^!(G) \circ \text{pr}_2
\end{array}
\]

of functors and natural transformations commutes, where \( \partial(h_{\gamma}^!(G),D) \) is defined as in [BBF21, Definition A.9]. By [BBF21, Proposition A.14] this determines a unique strictly stable functor \( F : M \to (\text{Vect}_F)^\circ \) as well as a unique strictly stable natural transformation \( \varphi : F \to h_{\gamma}^!(G) \). Moreover, as \( \varphi \) is strictly stable and as its restriction \( \varphi|_D = \varphi' \) to the fundamental domain \( D \) is a natural isomorphism, \( \varphi : F \to h_{\gamma}^!(G) \) is a natural isomorphism as well. With this we obtain the natural isomorphism

\[
ev^0 \circ \varphi^0 : h_{\gamma}(G) \to \ev^0 \circ F^0,
\]

where \( \ev^0 : \text{Vect}_F \to \text{Vect}_F, M^\bullet \mapsto M^0 \) is the evaluation at 0 as in [BBF21, Lemma A.6].

### 3.3 Connection to Derived Level Set Persistence and RISC

The following form of derived level set persistence has been introduced by [Cur14, KS18]. Let \( f : X \to \mathbb{R} \) continuous function. Then we may consider the derived pushforward \( Rf_*F_X \) as an object of the derived category \( D^+(\mathbb{R}) \) of \( \mathbb{R} \)-linear sheaves on \( \mathbb{R} \). This construction can be made into a contravariant functor in the following way. For a commutative triangle

\[
\begin{array}{ccc}
X & \xrightarrow{\varphi} & Y \\
\downarrow f & \searrow & \downarrow g \\
\mathbb{R} & \to & \mathbb{R}
\end{array}
\]

of topological spaces, we consider the unit

\[
\eta_{F_Y}^\varphi : F_Y \to R\varphi_*\varphi^{-1}F_Y \simeq R\varphi_*F_X
\]
of the derived adjunction $\varphi^{-1} \dashv R\varphi_*$ at $F_Y$. Applying the functor $Rg_*$ to $\eta_{\varphi^X}$ we obtain the homomorphism

$$R\varphi_* F_\varphi : Rg_* F_Y \xrightarrow{(Rg_* \eta_{\varphi^X})_Y} Rg_* R\varphi_* F_X \simeq R(g \circ \varphi)_* F_X = Rf_* F_X,$$

see also [KS90 (2.7.4)]. This way we obtain the functor

$$R(-)_* F(-) : (\text{Top}/R)^\circ \to D^+(R), (f : X \to R) \mapsto Rf_* F_X$$

from the opposite category of the category of topological spaces over the reals $\text{Top}/R$ to the derived category $D^+(R)$. As a note of caution, we point out that this notation is not being used consistently across the literature as [BGO19] use the same notation for the functor $\text{Top}/R \to D^+(R)$.

Below we provide the Example 3.33 which also shows that the functor (3.15) and the functor we denote as $R(-)_* F(-)$ are not naturally isomorphic.

We now assume we are in the setting of [BBF21 Section 2]. More specifically, we assume $l_0$ and $l_1$ intersect the $x$-axis in $-\pi$ and $\pi$ respectively and that $Q = \downarrow \text{Im } \Delta$, where

$$\Delta = \Delta \circ \arctan : \mathbb{R} = \left[-\infty, \infty\right] \to \mathbb{M}, t \mapsto (\arctan t, \arctan t).$$

Then the restriction $\Delta|_R : \mathbb{R} \to q_\gamma$ yields an embedding of $\mathbb{R}$ onto $\mathbb{q} \subset q_\gamma$ as shown in Fig. 3.3. The induced tessellation of for this particular choice of $Q \subset \mathbb{M}$ is shown in Fig. 1.2. Moreover, we have $q_\gamma = \text{Im } \Delta \cong \mathbb{R}$. Thus, by post-composing the functor $R(\Delta|_R)_* : D^+(\mathbb{R}) \to D^+(q_\gamma)$ with $h_\gamma$ we obtain the functor

$$h_\mathbb{R} : D^+(\mathbb{R}) \xrightarrow{R(\Delta|_R)_*} D^+(q_\gamma) \xrightarrow{h_\gamma} \text{Vect}_{\mathbb{F}}^{\mathbb{M}^\circ}.$$

Furthermore, by composing $R(-)_* F(-) : (\text{Top}/R)^\circ \to D^+(R)$ and $h_\mathbb{R} : D^+(R) \to \text{Vect}_{\mathbb{F}}^{\mathbb{M}^\circ}$ we obtain a functor

$$(\text{Top}/R)^\circ \xrightarrow{R(-)_* F(-)} D^+(R) \xrightarrow{h_\mathbb{R}} \text{Vect}_{\mathbb{F}}^{\mathbb{M}^\circ}.$$
Figure 3.4: The geometric realization $Y := |5 * C_4|$ on the left hand side and $X := |A|$ on the right hand side, making $g: Y \to \mathbb{R}$ and $f: X \to \mathbb{R}$ the corresponding height functions of the depicted embeddings into $\mathbb{R}^3$.

Now in [BBF21, Section 2] we have already provided the functor $h: (\text{Top}/\mathbb{R})^\circ \to \text{Vect}_F^{M_\circ}$. In the following we show that these two functors are naturally isomorphic when restricted to functions on locally contractible spaces. More specifically, let $\text{lcContr}$ denote the full subcategory of locally contractible topological spaces. Before we construct a natural isomorphism $\zeta$ as in the diagram

\[
\begin{array}{c}
\text{lcContr}/\mathbb{R}^\circ \\
\downarrow R(-), F(-) & \downarrow h \\
D^+_{\mathbb{R}} & \downarrow h_k \\
\text{Vect}_F^{M_\circ}, \end{array}
\]

we illustrate the behavior of all three of these functors with an example.

Example 3.33. Let $C_4$ be the cyclic graph on four vertices $\{1, 2, 3, 4\}$. Moreover, let $5 * C_4$ be the abstract simplicial cone over $C_4$ with 5 as the tip of the cone, let $A$ be the full subcomplex of $5 * C_4$ spanned by the four vertices $\{1, 2, 3, 5\}$, let $X := |A|$ and $Y := |5 * C_4|$ be the corresponding geometric realizations, and let $\varphi: X \to Y$ be the corresponding inclusion. Furthermore, let $a < b < c$, let $g: Y \to \mathbb{R}$ be the unique simplexwise linear function with

\[
g(1) = g(3) = g(4) = a, \quad g(2) = b, \quad \text{and} \quad g(5) = c,
\]

and let $f := g|_X: X \to \mathbb{R}$ be the restriction of $g: Y \to \mathbb{R}$ to $X$. Then the diagram (3.14) commutes; see also Fig. 3.4. Now let

\[
\tilde{a} := \arctan a, \quad \tilde{b} := \arctan b, \quad \text{and} \quad \tilde{c} := \arctan c.
\]
Then we have

\[ h(g) \cong B(\bar{c}, \bar{a}) \oplus B(\pi - \bar{b}, \bar{c} - 2\pi) \]

and

\[ h(f) \cong B(\bar{c}, \bar{a}) \oplus B(\pi - \bar{b}, \bar{a}) \]

or, in other words

\[(\beta^0 \circ h)(g) = 1\{(\bar{c}, \bar{a}), (\pi - \bar{b}, \bar{c} - 2\pi)\} \quad \text{and} \quad (\beta^0 \circ h)(f) = 1\{(\bar{c}, \bar{a}), (\pi - \bar{b}, \bar{a})\},\]

where \(1\{(\bar{c}, \bar{a}), (\pi - \bar{b}, \bar{c} - 2\pi)\} : \text{int } \mathbb{M} \to \mathbb{N}_0\) is the indicator function of the subset \(\{(\bar{c}, \bar{a}), (\pi - \bar{b}, \bar{c})\} \subset \text{int } \mathbb{M}\); see also Fig. 3.5. We consider the natural transformation

\[ \eta := B(\pi - \bar{b}, \bar{c} - 2\pi) \leq (\pi - \bar{b}, \bar{a}) : B(\pi - \bar{b}, \bar{c} - 2\pi) \to B(\pi - \bar{b}, \bar{a}).\]

As is shown in Fig. 3.5 the supports of \(B(\pi - \bar{b}, \bar{a})\) and \(B(\pi - \bar{b}, \bar{c} - 2\pi)\) have a non-empty intersection. For any \(u\) in this intersection, \(\eta_u\) maps \(B(\pi - \bar{b}, \bar{c} - 2\pi)(u) = F\) identically onto \(B(\pi - \bar{b}, \bar{a})(u) = F\). As it turns out, we have the commutative diagram

\[
\begin{array}{ccc}
B(\bar{c}, \bar{a}) \oplus B(\pi - \bar{b}, \bar{c} - 2\pi) & \cong & h(g) \\
B(\bar{c}, \bar{a}) \oplus \eta & \downarrow & h(\varphi) \\
B(\bar{c}, \bar{a}) \oplus B(\pi - \bar{b}, \bar{a}) & \cong & h(f),
\end{array}
\]

when the isomorphisms (3.16) are chosen appropriately. Now

\[
Rg_* \mathbb{F}_Y \cong \mathbb{F}_{[a,c]} \oplus \mathbb{F}_{[b,c]}[-1] \cong \bigwedge^{-1} \iota(\bar{c}, \bar{a}) \oplus \bigwedge^{-1} \iota(\pi - \bar{b}, \bar{c} - 2\pi)
\]

and

\[
Rf_* \mathbb{F}_X \cong \mathbb{F}_{[a,c]} \oplus \mathbb{F}_{[a,b]} \cong \bigwedge^{-1} \iota(\bar{c}, \bar{a}) \oplus \bigwedge^{-1} \iota(\pi - \bar{b}, \bar{a}).
\]
Moreover, considering the short exact sequence
\[ 0 \to \mathcal{F}_{[a,b]} \to \mathcal{F}_{[a,c]} \to \mathcal{F}_{[b,c]} \to 0 \]
of sheaves we obtain the distinguished triangle
\[ \mathcal{F}_{[a,b]} \to \mathcal{F}_{[a,c]} \to \mathcal{F}_{[b,c]} \to \mathcal{F}_{[a,b]}[1]. \]
Furthermore, we have commutative diagrams
\[
\begin{array}{ccc}
\triangle^{-1}_{\ell} \left( \pi - \bar{b}, \bar{c} - 2\pi \right) & \xrightarrow{=} & \mathcal{F}_{[b,c]}[-1] \\
\downarrow & & \downarrow \\
\triangle^{-1}_{\ell} \left( \pi - \bar{b}, \bar{a} \right) & \xrightarrow{=} & \mathcal{F}_{[a,b]}
\end{array}
\]
and
\[
\begin{array}{ccc}
\mathcal{F}_{[a,c]} \oplus \mathcal{F}_{[b,c]}[-1] & \xrightarrow{=} & Rg_* \mathcal{F}_Y \\
\downarrow \& & \downarrow \& \\
\mathcal{F}_{[a,c]} \oplus \mathcal{F}_{[a,b]} \xrightarrow{=} & Rf_* \mathcal{F}_X,
\end{array}
\]
when the isomorphisms \((3.18)\) are chosen appropriately. Now the functor \(h_\mathbb{R} : D^+(\mathbb{R}) \to \text{Vect}_\mathbb{F}^{\mathcal{M}^0}\) maps the commutative square \((3.19)\) to a square isomorphic to \((3.17)\). Moreover, Fig. 3.5 shows that the intersection of the supports of \(B(\pi - \bar{b}, \bar{a})\) and \(B(\pi - \bar{b}, \bar{c} - 2\pi)\) is disjoint from the regions of \(\mathcal{M}\) covered by the south faces of the Mayer–Vietoris pyramids \([CdM09]\) in the tessellation Fig. 1.2. This way we obtain a geometric explanation of the phenomenon described in \([BGO19, \text{Remark 4.9}]\).

Now let \(f : X \to \mathbb{R}\) be a continuous function with \(X\) a locally contractible topological space. Moreover, let \(C^\bullet\) be the presheaf of singular cochains with coefficients in \(\mathbb{F}\) on \(X\) and let \(\epsilon : \mathcal{F}_X \to C^\bullet\) be the embedding of \(\mathcal{F}_X\) as the subpresheaf of 0-cocycles of \(C^\bullet\). By \([Sel16]\) there is a complex \(\mathcal{F}\) of flabby sheaves on \(X\) together with a quasi-isomorphism of complexes of presheaves \(\tilde{\psi} : C^\bullet \to \mathcal{F}\) such that \(\tilde{\psi} \circ \epsilon : \mathcal{F}_X \to \mathcal{F}\) is a quasi-isomorphism of complexes of
sheaves. For \( u \in D \) we consider the commutative diagram

\[
\begin{array}{ccc}
0 & \rightarrow & (C^* \circ f^{-1} \circ \Delta^{-1})(\text{int}(\downarrow T(u)), M \setminus \uparrow u)) \\
\downarrow & & \downarrow \\
0 & \rightarrow & \Gamma_{(\Delta \circ f)^{-1}(\uparrow u)}((\Delta \circ f)^{-1}(\text{int}(\downarrow T(u))); \mathcal{F}) \\
\downarrow & & \downarrow \\
(C^* \circ f^{-1} \circ \Delta^{-1})(\text{int}(\downarrow T(u))) & \rightarrow & \tilde{\psi}(\Delta \circ f)^{-1}(\text{int}(\downarrow T(u))) \rightarrow (\mathcal{F} \circ f^{-1} \circ \Delta^{-1})(\text{int}(\downarrow T(u))) \\
\downarrow & & \downarrow \\
(C^* \circ f^{-1} \circ \Delta^{-1})(M \setminus \uparrow u)) & \rightarrow & (\mathcal{F} \circ f^{-1} \circ \Delta^{-1})(M \setminus \uparrow u)) \\
\downarrow & & \downarrow \\
0 & \rightarrow & 0.
\end{array}
\]

(3.20)

By definition of the relative singular cochain complex \((C^* \circ f^{-1} \circ \Delta^{-1})(\text{int}(\downarrow T(u)), M \setminus \uparrow u))\) and the set of local sections \(\Gamma_{(\Delta \circ f)^{-1}(\uparrow u)}((\Delta \circ f)^{-1}(\text{int}(\downarrow T(u))); \mathcal{F})\) both columns are exact. In particular, the horizontal dashed arrow exists as indicated. Moreover, the lower two horizontal arrows are quasi-isomorphisms of cochain complexes as \(\tilde{\psi}: C^* \rightarrow \mathcal{F}\) is a quasi-isomorphism of complexes of presheaves. Thus, the dashed arrow is a quasi-isomorphism of cochain complexes as well. Taking the \(Z\)-graded cohomology of the cochain complex \((C^* \circ f^{-1} \circ \Delta^{-1})(\text{int}(\downarrow T(u)), M \setminus \uparrow u))\) we obtain

\[
(H^* \circ f^{-1} \circ \Delta^{-1})(\text{int}(\downarrow T(u)), M \setminus \uparrow u)) = h^\#(f)(u),
\]

(3.21)

where \(h(f): M^\circ \rightarrow \text{Vect}_F\) is the relative interlevel set cohomology (RISC) of \(f: X \rightarrow \mathbb{R}\) as defined in [BBF21] Section 2 and \(h^\#(f): M^\circ \rightarrow \text{Vect}^F\) is the transform of \(h(f)\) under the 2-adjunction from [BBF21] Lemma A.6. (Strictly speaking, we should only use the term RISC, when \(f: X \rightarrow \mathbb{R}\) is \(\mathbb{F}\)-tame, so we may still adapt this notion to more general settings.) Moreover, we have

\[
\Gamma_{(\Delta \circ f)^{-1}(\uparrow u)}((\Delta \circ f)^{-1}(\text{int}(\downarrow T(u))); \mathcal{F}) = \Gamma_{g \cap (\uparrow u)}(g \cap \text{int}(\downarrow T(u)); (\Delta \circ f)_* \mathcal{F}) = \Gamma_{g \cap (\uparrow u)}(g \cap \text{int}(\downarrow T(u)); G),
\]

where \(G := (\Delta \circ f)_* \mathcal{F}\). As both \(\mathcal{F}\) and \(G\) are complexes of flabby sheaves, this implies that

\[
H^*_{(\Delta \circ f)^{-1}(\uparrow u)}((\Delta \circ f)^{-1}(\text{int}(\downarrow T(u))); \mathcal{F}) \cong H^*_{g \cap (\uparrow u)}(g \cap \text{int}(\downarrow T(u)); G) = F'(u),
\]

(3.22)

where

\[
F': D \rightarrow \left(\text{Vect}^Z_{\mathcal{F}}\right)^\circ, u \mapsto H^*_{g \cap (\uparrow u)}(g \cap \text{int}(\downarrow u); G)
\]

as in the previous Section 3.2. Altogether, (3.21), the dashed arrow in (3.20), and (3.22) yield a natural isomorphism

\[
\psi': F' \xrightarrow{\cong} h^\#(f)|_D.
\]

Now in order to apply [BBF21] Proposition A.14] to extend \(\psi'\) to a strictly stable natural isomorphism

\[
\psi: F \rightarrow h^\#(f),
\]
where \( F : M \to \left( \text{Vect}_F \right)^\circ \) is defined as in the previous Section 3.2, we have to show that the diagram

\[
\begin{array}{ccc}
F \circ \text{pr}_1 & \xrightarrow{\psi \circ \text{pr}_1} & h_0^\#(f) \circ \text{pr}_1 \\
\downarrow \varphi & & \downarrow \partial(h_0^\#(f), D) \\
F \circ \text{pr}_2 & \xrightarrow{\psi \circ \text{pr}_2} & h_0^\#(f) \circ \text{pr}_2
\end{array}
\tag{3.23}
\]

of functors and natural transformations commutes, where \( \partial(h_0^\#(f), D) \) is defined as in [BBF21, Definition A.9]. To this end, it suffices to check the commutativity of (3.23) for all pairs \((w, \hat{u}) \in R_D\) with the same \(x\)- or the same \(y\)-coordinate and both within the square \((-\frac{\pi}{2}, \frac{\pi}{2})^2\). We treat the case, where \(w\) and \(\hat{u}\) have the same \(y\)-coordinate, the other case is similar. In this case there are real numbers \(a < b < c\) with

\[
\hat{u} = (\arctan a, \arctan b) \quad \text{and} \quad w = (\arctan c, \arctan b).
\]

Then we have

\[
\left( h_0^\#(f) \circ \text{pr}_1 \right)(w) = (H^\bullet \circ f^{-1})(\mathbb{R}, \mathbb{R} \setminus [a, b])
\]

and

\[
\left( h_0^\#(f) \circ \text{pr}_2 \right)(\hat{u}) = (H^\bullet^{-1} \circ f^{-1})(a, b).
\]

Moreover,

\[
\partial(h_0^\#(f), D)_{(w, \hat{u})} : (H^\bullet^{-1} \circ f^{-1})(a, b) \to (H^\bullet \circ f^{-1})(\mathbb{R}, \mathbb{R} \setminus [a, b])
\tag{3.24}
\]

is the differential of the Mayer–Vietoris sequence associated to the square-shaped sublattice

\[
\begin{array}{c}
\begin{array}{c}
\xymatrix{f^{-1}(\mathbb{R}, \mathbb{R} \setminus [b, c]) & f^{-1}((\infty, b), (\infty, b))} \\
\xymatrix{f^{-1}((a, \infty), (c, \infty)) & f^{-1}((a, b), \emptyset)}
\end{array}
\end{array}
\]

of pairs of open subspaces of \(X\). Now considering the right hand side of (3.23) we have

\[
(F \circ \text{pr}_1)(w) \cong H^\bullet_{f^{-1}(\mathbb{R}, \mathbb{R} \setminus [b, c])}(X; \mathcal{F})
\]

and

\[
(F \circ \text{pr}_2)(\hat{u}) \cong H^\bullet_{f^{-1}(\mathbb{R}, \mathbb{R} \setminus [a, b])}(X; \mathcal{F}).
\]

Under these isomorphisms the map \( \delta'_{(w, \hat{u})} : (F' \circ \text{pr}_2)(\hat{u}) \to (F' \circ \text{pr}_1)(w) \) corresponds to the differential associated to the short exact sequence

\[
0 \to \Gamma_{f^{-1}(\mathbb{R}, \mathbb{R} \setminus [b, c])}(X; \mathcal{F}) \to \Gamma_{f^{-1}(\mathbb{R}, \mathbb{R} \setminus [a, \infty])}(X; \mathcal{F}) \to \Gamma(f^{-1}(\mathbb{R}, \mathbb{R} \setminus [a, b])\setminus \emptyset)) \to 0 \tag{3.25}
\]

of cochain complexes in \( \text{Vect}_F \). Now in order to show the commutativity of (3.23) at \((w, \hat{u}) \in R_D\) we show that the Mayer–Vietoris differential (3.24) can be realized as the
differential associated to a short exact sequence of cochain complexes as well. To this end, we consider the sublattice

\[
\begin{align*}
&((\mathbb{R}, \mathbb{R} \setminus [b, c]) \leftarrow (\mathbb{R} \setminus [b, c], \mathbb{R} \setminus [b, c]) \leftarrow ((-\infty, b), (-\infty, b)) \quad (3.26) \\
&\uparrow \quad \uparrow \quad \uparrow \\
&((a, \infty), (a, b) \cup (c, \infty)) \leftarrow ((a, b) \cup (c, \infty), (a, b) \cup (c, \infty)) \leftarrow ((a, b), (a, b)) \\
&\uparrow \quad \uparrow \quad \uparrow \\
&((a, \infty), (c, \infty)) \leftarrow ((a, b) \cup (c, \infty), (c, \infty)) \leftarrow ((a, b), \emptyset).
\end{align*}
\]

Now all inclusions of (3.26) other than those of the lower left square induce isomorphisms in singular cohomology by excision. Thus, the Mayer–Vietoris sequence associated to the outer square of (3.26) is isomorphic to the Mayer–Vietoris sequence associated to the lower left square of (3.26). Moreover, the Mayer–Vietoris sequence associated to the lower left square of (3.26) is the same as the long exact sequence associated to the triple \( f^{-1}((a, \infty), (a, b) \cup (c, \infty), (c, \infty)) \), which is the long exact sequence associated to the short exact sequence

\[
\begin{align*}
0 \\
\downarrow \\
(C^* \circ f^{-1})((a, \infty), (a, b) \cup (c, \infty)) \\
\downarrow \\
(C^* \circ f^{-1})((a, \infty), (c, \infty)) \\
\downarrow \\
(C^* \circ f^{-1})((a, b) \cup (c, \infty), (c, \infty)) \\
\downarrow \\
0
\end{align*}
\]

of cochain complexes in \( \text{Vect}_F \). Furthermore, the short exact sequence (3.28) is isomorphic to the short exact sequence

\[
\begin{align*}
0 \\
\downarrow \\
\Gamma_{f^{-1}([b,c])}(f^{-1}((a, \infty)); \mathcal{F}) \\
\downarrow \\
\Gamma_{f^{-1}([a,c])}(f^{-1}((a, \infty)); \mathcal{F}) \\
\downarrow \\
\Gamma_{f^{-1}([a,c])}(f^{-1}((a, b) \cup (c, \infty)); \mathcal{F}) \\
\downarrow \\
0
\end{align*}
\]

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by the sheaf condition. Now the presheaf homomorphism \( \tilde{\psi} : C^\bullet \to F \) induces a homomorphism 

\[
\begin{array}{ccc}
0 & \rightarrow & 0 \\
\downarrow & & \downarrow \\
(C^\bullet \circ f^{-1})((a, \infty), (a, b) \cup (c, \infty)) & \rightarrow & \Gamma_{f^{-1}((b, c); F)}(f^{-1}((a, \infty)); F) \\
\downarrow & & \downarrow \\
(C^\bullet \circ f^{-1})((a, \infty), (c, \infty)) & \rightarrow & \Gamma_{f^{-1}((a, c); F)}(f^{-1}((a, \infty)); F) \\
\downarrow & & \downarrow \\
(C^\bullet \circ f^{-1})((a, b) \cup (c, \infty), (c, \infty)) & \rightarrow & \Gamma_{f^{-1}((a, c) \cup (c, \infty); F)}(f^{-1}((a, \infty)); F) \\
0 & & 0 \\
\end{array}
\]  

(3.29)

of short exact sequences. Moreover, the horizontal arrow at the top of (3.29) induces the homomorphism 

\[ \psi'_{w} : F(w) \to h_0^{#\circ}(f)(w) \] 

under the aforementioned isomorphisms. Similarly the horizontal arrow at the bottom of (3.29) induces the homomorphism 

\[ (\Sigma \circ \psi' \circ T^{-1})_{\hat{u}} : F(\hat{u}) \to h_0^{#\circ}(f)(\hat{u}). \] 

Thus, the diagram 

\[
\begin{array}{ccc}
F(w) & \xrightarrow{\psi'} & h_0^{#\circ}(f)(w) \\
\downarrow_{\partial'(w, \hat{u})} & & \downarrow_{\partial(h_0^{#\circ}(f), D)(w, \hat{u})} \\
F(\hat{u}) & \xrightarrow{(\Sigma \circ \psi' \circ T^{-1})_{\hat{u}}} & h_0^{#\circ}(f)(\hat{u}) \\
\end{array}
\]  

(3.30)

commutes. By a similar argument (3.30) commutes for any pair \((w, \hat{u}) \in R_D\) with identical \(x\)-coordinates and both points within the square \((-\pi/2, \pi/2)^2\). We just need to keep track of an additional sign, as in this case the long exact sequence of the associated triple “passes through the second direct summand of the Mayer-Vietoris sequence”. Now for an arbitrary pair \((w, \hat{u}) \in R_D\) we can always find a pair \((w', \hat{u}') \in R_D\) with the same \(x\)- or the same \(y\)-coordinate and both within the square \((-\pi/2, \pi/2)^2\) and \(w \leq w' \leq \hat{u}' \leq \hat{u}\), hence the diagram (3.23) commutes. As a result, \(\psi' : F' \xrightarrow{\sim} h_0^{#\circ}(f)|_D\) extends to a unique strictly stable natural transformation \(\psi : F \to h_0^{#\circ}(f)\) by [BBF21, Proposition A.14]. Moreover, as \(\psi' : F' \xrightarrow{\sim} h_0^{#\circ}(f)|_D\) is a natural isomorphism, \(\psi : F \to h_0^{#\circ}(f)\) is a natural isomorphism as well. In conjunction with the previous Section 3.2 we obtain the span 

\[
h_0^{#\circ}(f) \xrightarrow{\psi} F \xrightarrow{\psi'} h_0^{#\circ}(G)
\]

of strictly stable functors and and strictly stable natural isomorphisms. Applying the op-positization 2-functor to this span and using the 2-adjunction from [BBF21] Lemma A.6] we obtain a natural isomorphism 

\[
h(f) \cong h_\gamma(G).
\]  

(3.31)
Moreover, as $\tilde{\psi} \circ \epsilon: \mathcal{F}_X \to \mathcal{F}$ is a quasi-isomorphism of complexes of sheaves, we have a quasi-isomorphism

$$G = (\bigtriangleup \circ f)_* \mathcal{F} \simeq R(\bigtriangleup \circ f)_* \mathcal{F}_X \simeq R\bigtriangleup_* Rf_* \mathcal{F}_X$$

and hence

$$h_\gamma(G) \cong h_\gamma(R\bigtriangleup_* Rf_* \mathcal{F}_X) \cong h_\mathbb{R}(Rf_* \mathcal{F}_X).$$

(3.32)

Combining (3.31) and (3.32) we obtain the natural isomorphism

$$\zeta_f: h(f) \cong h_\mathbb{R}(Rf_* \mathcal{F}_X),$$

which is natural in $f: X \to \mathbb{R}$. In summary we obtain the following.

**Proposition 3.34.** There is a natural isomorphism $\zeta$ as in the diagram

$$\begin{array}{ccc}
(lcContr/\mathbb{R})^o & \overset{R(-), \mathcal{F}(-)}{\longrightarrow} & D^+(\mathbb{R}) \\
\overset{h}{\downarrow} & & \overset{h_\mathbb{R}}{\downarrow} \\
\overset{h}{\bigtriangleup} & & \overset{\mathbb{R}}{\bigtriangleup} \\
D^+(\mathbb{R}) & \longrightarrow & \text{Vect}^M_{\mathcal{F}}.
\end{array}$$

Now in [BBF21, Definition 2.2] we have also defined the following notion.

**Definition 3.35.** We say that a function $f: X \to \mathbb{R}$ is $\mathcal{F}$-tame if the singular cohomology $H^n(f^{-1}(I); \mathcal{F})$ is finite-dimensional for all open intervals $I \subseteq \mathbb{R}$ and any integer $n \in \mathbb{Z}$. Moreover, we denote the full subcategory of all $\mathcal{F}$-tame functions on locally contractible spaces by $(lcContr/\mathbb{R})_t$.

Analogously we may define the following notion for objects of $D^+(\mathbb{R})$.

**Definition 3.36.** We say that an object $F$ of $D^+(\mathbb{R})$ is tame if $H^n(I; F)$ is finite-dimensional for any open interval $I \subseteq \mathbb{R}$ and any integer $n \in \mathbb{Z}$. Moreover, we denote the full subcategory of tame objects in $D^+(\mathbb{R})$ by $D^+_t(\mathbb{R})$.

By Lemma 3.19 and Proposition 3.22 the functor $h_\mathbb{R}: D^+(\mathbb{R}) \to \text{Vect}^M_{\mathcal{F}}$ restricts to a functor $h_{\mathbb{R},t}: D^+_t(\mathbb{R}) \to \mathcal{F}$. In conjunction with Proposition 3.34 we obtain the diagram

$$\begin{array}{ccc}
(lcContr/\mathbb{R})_t^o & \overset{R(-), \mathcal{F}(-)}{\longrightarrow} & D^+_t(\mathbb{R}) \\
\overset{h_{\mathbb{R},t}}{\downarrow} & & \overset{\mathbb{R}}{\bigtriangleup} \\
\overset{h}{\bigtriangleup} & & \overset{\mathbb{R}}{\bigtriangleup} \\
D^+_t(\mathbb{R}) & \longrightarrow & \mathcal{F}.
\end{array}$$
4 Projective Covers of $J$-Presentable Functors

Let $F : \mathcal{M}^o \to \text{vect}_\mathcal{F}$ be a pfd sequentially continuous functor.

**Lemma 4.1.** Suppose $G \hookrightarrow F$ is a proper subfunctor of $F : \mathcal{M}^o \to \text{vect}_\mathcal{F}$. Then, there is a point $w \in \mathcal{M}$ and a commutative triangle of the form

$$
\begin{array}{ccc}
G & \longrightarrow & F \\
\downarrow & & \downarrow \\
0 & \longrightarrow & S_w.
\end{array}
$$

**Corollary 4.2.** Any proper subfunctor of $F : \mathcal{M}^o \to \text{vect}_\mathcal{F}$ is contained point-wise in a maximal subfunctor of $F$.

**Proof of Lemma 4.1.** Since the point-wise inclusion $G \hookrightarrow F$ is proper, there is a point $u := (x, y) \in \mathcal{M}$ such that $G(u)$ is a proper subspace of $F(u)$. Thus, there is a linear form $\alpha \in (F(u))^*$ with $\alpha|_{G(u)} = 0$. As $F : \mathcal{M}^o \to \text{vect}_\mathcal{F}$ is pfd and sequentially continuous, the dual covariant functor $F^* : \mathcal{M} \to \text{vect}_\mathcal{F}$ is sequentially cocontinuous, i.e. for any increasing sequence $(v_k)_{k=1}^{\infty} \in \mathcal{M}$ converging to $v \in \mathcal{M}$ the natural map

$$
\lim_{k \to} F^*(v_k) \to F^*(v)
$$

(4.1)

is an isomorphism. We consider the restriction of $F^* : \mathcal{M}^o \to \text{vect}_\mathcal{F}$ to the maximal horizontal line segment $h \subset \mathcal{M}$ through $u$. Since $F^*$ vanishes on $l_0$, the linear form $\alpha$ has to die at some point $v \in h$. Let $\beta := F^*(u \preceq v)(\alpha)$ be the pullback of $\alpha$ to $F(v)$. As $F^* : \mathcal{M}^o \to \text{vect}_\mathcal{F}$ is sequentially cocontinuous we have $\beta \neq 0$ (the linear form $\alpha$ dies hard so to speak). Now let $g \subset \mathcal{M}$ be the maximal vertical line segment through $v$. As $F^*$ vanishes on $l_1$, the linear form $\beta$ has to die at some point $w \in g$. Let $\gamma := F^*(v \preceq w)(\beta)$ be the pullback of $\beta$ to $G(w)$. Again, $\gamma \neq 0$ as $F^*$ is sequentially cocontinuous. Now $\gamma \in F^*(v)$ determines a point-wise surjective family of maps $\rho : F \to S_w$. As all internal pullbacks of $\gamma$ are trivial, the family of maps $\rho : F \to S_w$ is a natural transformation. $\square$

Now let $p : I \to \text{int} \mathcal{M}$ be a map of sets with

$$
\#p^{-1}(u) = \dim_\mathcal{F} \text{Nat}(F, S_u)
$$

for any $u \in \text{int} \mathcal{M}$. For $i \in I$ we now also write $S_i := S_{p(i)} : \mathcal{M}^o \to \text{Vect}_\mathcal{F}$. Then $(\text{Nat}(F, S_i))_{i \in I}$ is an indexed family of non-empty sets. Let $(\alpha_i)_{i \in I}$ be a choice function of this indexed family of sets such that $(\alpha_i)_{i \in p^{-1}(u)}$ is a basis of $\text{Nat}(F, S_u)$ for any $u \in \text{int} \mathcal{M}$. As $F : \mathcal{M}^o \to \text{vect}_\mathcal{F}$ is pfd we have $\bigoplus_{i \in I} S_i = \prod_{i \in I} S_i$. Let

$$
\theta : F \to \bigoplus_{i \in I} S_i
$$

be the natural transformation induced by the family $(\alpha_i)_{i \in I}$, then $\theta : F \to \bigoplus_{i \in I} S_i$ is an epimorphism since $(\alpha_i)_{i \in p^{-1}(u)}$ is a basis of $\text{Nat}(F, S_u)$ for any $u \in \text{int} \mathcal{M}$. Moreover, let

$$
\text{rad} F : \mathcal{M}^o \to \text{vect}_\mathcal{F}
$$

be the point-wise intersection of all maximal subfunctors of $F : \mathcal{M}^o \to \text{vect}_\mathcal{F}$.
Lemma 4.3. We have the point-wise inclusion \( \ker \theta \hookrightarrow \text{rad } F \) of subfunctors.

Proof. By Lemma 4.1 we know that any maximal subfunctor \( G \hookrightarrow F \) of \( F: \mathcal{M}^o \to \text{vect}_F \) yields a short exact sequence

\[
0 \longrightarrow G \longrightarrow F \longrightarrow S_u \longrightarrow 0 \tag{4.2}
\]

for some \( u \in \text{int } \mathbb{M} \). Since \((\alpha_i)_{i \in p^{-1}(u)}\) is a basis of \( \text{Nat}(F, S_u) \) there is a linear combination

\[ \alpha = \sum_{i \in p^{-1}(u)} \lambda_i \alpha_i. \]

Now let \( \eta: S_u \to S_u^{#p^{-1}(u)} \) be the natural transformation induced by the family \((\lambda_i \text{id}_{S_u})_{i \in p^{-1}(u)}\), then we have the commutative diagram

\[
\begin{array}{ccc}
G & \xrightarrow{\alpha} & S_u \\
\theta \downarrow & & \downarrow \eta \\
\bigoplus_{i \in I} S_i & \xleftarrow{\eta} & S_u^{#p^{-1}(u)}. \\
\end{array}
\]

Since \( \eta: S_u \to S_u^{#p^{-1}(u)} \) and the horizontal arrow at the bottom are both natural monomorphisms the kernel of \( \theta: F \to \bigoplus_{i \in I} S_i \) is contained in the kernel of \( \alpha: F \to S_u \), which is \( G: \mathcal{M}^o \to \text{vect}_F \) by the exactness of (4.2).

Corollary 4.4. The epimorphism \( \theta: F \to \bigoplus_{i \in I} S_i \) is essential.

The proof of this corollary is almost identical to the proof of a similar statement in [Kra15, Lemma 3.3.(2)].

Proof. Let \( G \hookrightarrow F \) be a subfunctor of \( F: \mathcal{M}^o \to \text{vect}_F \) with \( F = G + \ker \theta \). If \( G \neq F \), then there is a maximal subfunctor \( G' \hookrightarrow F \) of \( F \) containing \( G: \mathcal{M}^o \to \text{vect}_F \) point-wise by Corollary 4.2. Thus we have

\[ F = G + \ker \theta \hookrightarrow G + \text{rad } F \hookrightarrow G' \]

by Lemma 4.3. This is a contradiction and therefore \( F = G \). It follows that \( \theta: F \to \bigoplus_{i \in I} S_i \) is essential. \( \square \)

Recall that \( \mathcal{C} \) is the category of functors \( \mathcal{M}^o \to \text{Vect}_F \) vanishing on \( \partial \mathcal{M} \). By [BBF21, Corollary 3.6] any functor in \( \mathcal{J} \) is projective in \( \mathcal{C} \). Thus, we may view \( \mathcal{J} \) as a full replete additive subcategory of projectives in \( \mathcal{C} \). In Appendix D we provide some auxiliary results in this particular context that we will need here. Now suppose \( F: \mathcal{M}^o \to \text{Vect}_F \) is some \( \mathcal{J} \)-presentable functor in the sense of Definition D.1. This implies in particular, that \( F: \mathcal{M}^o \to \text{Vect}_F \) is pfd, sequentially continuous by the Mittag-Leffler condition, has bounded above support, and vanishes on \( \partial \mathcal{M} \). So above constructions apply to \( F: \mathcal{M}^o \to \text{Vect}_F \). Now at this point, we haven’t shown Theorem 1.1 yet. So we may not use the Betti functions.
of Definition 1.2 yet. However, as the Ext$^0_C$-functor is naturally isomorphic to the Nat-functor and since we know that $F: M^\circ \to \text{Vect}_F$ is pfd, we can use the 0-th Betti function $\beta^0(F): \text{int}\, M \to \mathbb{N}_0$. Moreover, by our choice of $p: I \to \text{int}\, M$ above we have
\[ \beta^0(F)(u) = \dim_F \text{Nat}(F, S_u) = \# p^{-1}(u) \] (4.3)
for any $u \in \text{int}\, M$.

**Lemma 4.5.** For any $u \in \text{int}\, M$ we have
\[ \sum_{v \in (\uparrow u) \cap (\downarrow T(u))} \beta^0(F)(v) < \infty. \]

*Proof.* Since $F: M^\circ \to \text{Vect}_F$ is $J$-presentable, there is some epimorphism $\psi: P \to F$ with $P$ a functor in $J$. Moreover, as Nat($-, S_u$) is a left-exact functor for any $u \in \text{int}\, M$ we have the point-wise inequality $\beta^0(F) \leq \beta^0(P)$. Thus, it suffices to show
\[ \sum_{v \in (\uparrow u) \cap (\downarrow T(u))} \beta^0(P)(v) < \infty \]
for any $u \in \text{int}\, M$. By [BBF21, Theorem 3.5] the cohomological functor $P: M^\circ \to \text{Vect}_F$ is naturally isomorphic to a direct sum of contravariant blocks $B_v, v \in \text{int}\, M$. Moreover, we have
\[ \dim_F \text{Nat}(B_v, S_u) = \dim_F \text{Nat}(S_v, u) = \begin{cases} 1 & u = v \\ 0 & u \neq v \end{cases} \]
by the Yoneda Lemma 3.6, hence $\beta^0(P)(v)$ is the multiplicity of $B_v$ in $P$. Now let $u \in \text{int}\, M$, then we have
\[ \dim_F B_v(u) = \begin{cases} 1 & v \in (\uparrow u) \cap (\downarrow T(u)) \\ 0 & \text{otherwise} \end{cases} \]
and thus,
\[ \sum_{v \in (\uparrow u) \cap (\downarrow T(u))} \beta^0(P)(v) = \dim_F P(u) < \infty \]
since $P: M^\circ \to \text{Vect}_F$ is pfd. \qed

**Definition 4.6.** We say that a function $b: \text{int}\, M \to \mathbb{N}_0$ is an admissible Betti function if its support is bounded above and
\[ \sum_{v \in (\uparrow u) \cap (\downarrow T(u))} b(v) < \infty \]
for all $u \in \text{int}\, M$. We denote the commutative monoid of admissible Betti functions by $\mathbb{B}$.

In particular $\beta^0(F): \text{int}\, M \to \mathbb{N}_0$ is an admissible Betti function by Lemma 4.5. We will see in Corollary 6.6 below that all higher Betti functions are admissible as well.

**Lemma 4.7.** For an admissible Betti function $b: \text{int}\, M \to \mathbb{N}_0$ the direct sum
\[ \bigoplus_{v \in \text{int}\, M} B_v^{b(v)}: M^\circ \to \text{Vect}_F \]
is a functor in $J$. 48
Proof. As each contravariant block is cohomological the direct sum $\bigoplus_{v \in \text{int } \mathbb{M}} B_v^{(v)}$ is a cohomological functor as well. Moreover, any upper bound for the support of $\bigoplus_{v \in \text{int } \mathbb{M}} B_v^{(v)}$ is an upper bound for the support of $\bigoplus_{v \in \text{int } \mathbb{M}} B_v^{(v)}$. Next, we show the direct sum $\bigoplus_{v \in \text{int } \mathbb{M}} B_v^{(v)}$ is pdf. To this end, let $u \in \text{int } \mathbb{M}$. Since

$$\dim \bigoplus_{v \in \text{int } \mathbb{M}} B_v^{(v)}(u) = \begin{cases} 1 & v \in (\uparrow u) \cap \text{int}(\downarrow T(u)) \\ 0 & \text{otherwise} \end{cases}$$

for any $v \in \text{int } \mathbb{M}$, we have

$$\dim \bigoplus_{v \in \text{int } \mathbb{M}} B_v^{(v)}(u) = \sum_{v \in (\uparrow u) \cap \text{int}(\downarrow T(u))} b(v) < \infty.$$  

Now that $\bigoplus_{v \in \text{int } \mathbb{M}} B_v^{(v)}$ is pdf we have $\bigoplus_{v \in \text{int } \mathbb{M}} B_v^{(v)} = \prod_{v \in \text{int } \mathbb{M}} B_v^{(v)}$. With this sequential continuity follows from the commutativity of limits.

Proof. By Lemma 4.4 the function $\beta^0(F) : \text{int } \mathbb{M} \to \mathbb{N}_0$ is an admissible Betti function and $\# p^{-1}(v) = \beta^0(F)(v)$ for any $v \in \text{int } \mathbb{M}$ by (4.3). Thus, $\bigoplus_{i \in I} B_i : \mathbb{M}^\circ \to \text{Vect}_\mathcal{F}$ is a functor in $\mathcal{F}$.

We now construct a projective cover $\varphi : \bigoplus_{i \in I} B_i \to F$. To this end, let $\nu_i : B_i \to S_i$ be the unique natural transformation sending $1 \in F = B_i(p(i))$ to $1 \in F = S_i(p(i))$ provided by the Yoneda Lemma 3.6. As $\bigoplus_{i \in I} B_i$ is projective in $\mathcal{C}$ by the Yoneda Lemma 3.6, there is a “section” $\varphi : \bigoplus_{i \in I} B_i \to F$ as in the commutative triangle

$$\begin{array}{ccc}
\bigoplus_{i \in I} B_i & \\
\theta \downarrow & \\
\bigoplus_{i \in I} \nu_i & \xrightarrow{\varphi} F
\end{array} \quad (4.4)$$

Lemma 4.9. The natural transformation $\varphi : \bigoplus_{i \in I} B_i \to F$ is a natural epimorphism.

Proof. Considering the commutative triangle (4.4) and the fact that $\theta : F \to \bigoplus_{i \in I} S_i$ is an essential epimorphism by Corollary 4.4 we see that $\varphi : \bigoplus_{i \in I} B_i \to F$ is an epimorphism as well.

Corollary 4.10. The natural epimorphism $\bigoplus_{i \in I} \nu_i : \bigoplus_{i \in I} B_i \to \bigoplus_{i \in I} S_i$ is essential.

Proof. Above we defined $\theta : F \to \bigoplus_{i \in I} S_i$ to be the natural transformation induced by the choice function $(\alpha_i)_{i \in I}$ of the indexed family $(\text{Nat}(F, S_i))_{i \in I}$. As $\bigoplus_{i \in I} B_i$ is pdf by Corollary 4.8 we may consider above constructions in the special case where $F = \bigoplus_{i \in I} B_i$. Moreover, since $\bigoplus_{i \in I} B_i$ is pdf we also have $\bigoplus_{i \in I} B_i = \prod_{i \in I} B_i$ so we may define

$$\alpha_i := \text{pr}_i \circ \nu_i$$
for \( i \in I \), where \( \text{pr}_i : \bigoplus_{i \in I} B_i \to B_i \) is the projection to the \( i \)-th summand. For this particular choice for the family \((\alpha_i)_{i \in I}\) we obtain \( \theta = \bigoplus_{i \in I} \upsilon_i \), hence \( \bigoplus_{i \in I} \upsilon_i \) is an essential epimorphism by Corollary 4.13.

**Proposition 4.11.** The natural transformation \( \varphi : \bigoplus_{i \in I} B_i \to F \) is a projective cover of \( F : M^0 \to \text{vect}_F \) in the category of \( J \)-presentable functors \( \text{pres}(J) \).

**Proof.** We consider the commutative triangle (4.4). By Lemma 4.9 the natural transformation \( \varphi : \bigoplus_{i \in I} B_i \to F \) is an epimorphism. Moreover, \( \bigoplus_{i \in I} \upsilon_i : \bigoplus_{i \in I} B_i \to \bigoplus_{i \in I} S_i \) is essential by Corollary 4.10, hence \( \varphi : \bigoplus_{i \in I} B_i \to F \) is essential by [Kra15, Lemma 3.1]. Furthermore, the direct sum \( \bigoplus_{i \in I} B_i : M^0 \to \text{Vect}_F \) is projective in \( C \) by the Yoneda Lemma 3.6 and thus it is also projective in \( \text{pres}(J) \) by Corollaries 4.8 and D.4. In particular the 0-th Betti function \( \beta^0(F) : \text{int} M \to \mathbb{N}_0 \) determines the isomorphism class of the domains of projective covers of the \( J \)-presentable functor \( F : \text{int} M \to \text{vect}_F \).

**Corollary 4.12.** If \( F : M^0 \to \text{Vect}_F \) is projective in \( \text{pres}(J) \), then \( \varphi : \bigoplus_{i \in I} B_i \to F \) is a natural isomorphism.

**Proof.** As \( F : M^0 \to \text{Vect}_F \) is projective, the identity natural transformation \( \text{id}_F : F \to F \) is a projective cover as well. By the uniqueness of projective covers [Kra15, Corollary 3.5] the natural transformation \( \varphi : \bigoplus_{i \in I} B_i \to F \) is a natural isomorphism.

**Corollary 4.13.** If \( F : M^0 \to \text{Vect}_F \) is a functor in \( J \), then \( \varphi : \bigoplus_{i \in I} B_i \to F \) is a natural isomorphism.

**Proof.** By [BBF21] Corollary 3.6 and Corollary D.4 the functor \( F : M^0 \to \text{Vect}_F \) is projective itself, and thus the result follows with Corollary 4.12.

**Corollary 4.14.** The additive category \( J \) is the subcategory of projectives in \( \text{pres}(J) \).

**Proof.** This follows from Proposition 4.11 Corollary 4.8 and Corollary D.5. Alternatively, we may also conclude this from Corollaries 4.12 and 4.13.

## 5 Equivalence of \( D^+_t(q_\gamma, \partial q) \) and \( J \)

Let \( p : I \to \text{int} \mathbb{M} \) be some map of sets such that the assignment

\[
\text{int} \mathbb{M} \to \mathbb{N}_0, \ u \mapsto \# p^{-1}(u)
\]

is an admissible Betti function. Again we write \( B_i := B_{p(i)} : M^0 \to \text{vect}_F \) for any \( i \in I \).

**Proposition 5.1.** The direct sum \( \bigoplus_{i \in I} (\iota \circ p)(i) \) together with the projections

\[
\text{pr}_j : \bigoplus_{i \in I} (\iota \circ p)(i) \to (\iota \circ p)(j)
\]

for \( j \in I \) satisfies the universal property of the product in \( D^+_t(q_\gamma, \partial q) \).
Our proof of this proposition is involved enough so that we defer it to its own Section 5.2.

**Remark 5.2.** If \( \partial q \) is not closed in \( q_\gamma \), then it may well happen that the derived product \( R \prod_{i \in I} (\iota \circ p)(i) \), which is the product in \( D^+(q_\gamma) \), is not in \( D^+(q_\gamma, \partial q) \). Only after applying the coreflection \( R \theta \) to the derived product \( R \prod_{i \in I} (\iota \circ p)(i) \) we obtain the product in \( D^+(q_\gamma, \partial q) \), which is conveniently isomorphic to the direct sum \( \bigoplus_{i \in I} (\iota \circ p)(i) \) by Proposition 5.1.

**Corollary 5.3.** The direct sum \( \bigoplus_{i \in I} (\iota \circ p)(i) \) is preserved by \( h_\gamma : D^+(q_\gamma) \to \text{Vect}^{M^o}_F \).

**Proof.** As \( \iota(u) \) is in \( D^+(q_\gamma, \partial q) \) for any \( u \in \text{int} \, M \), the functor
\[
h_\gamma : D^+(q_\gamma) \to \text{Vect}^{M^o}_F, \quad F \mapsto \text{Hom}_{D^+(q_\gamma)}(\iota(-), F)
\]
maps the direct sum \( \bigoplus_{i \in I} (\iota \circ p)(i) \) to the direct product \( \prod_{i \in I} (h_\gamma \circ \iota \circ p)(i) : M^o \to \text{Vect}_F \) by Proposition 5.1. Moreover, we have \( (h_\gamma \circ \iota \circ p)(i) \cong B_i \) by Corollary 3.7 hence
\[
\prod_{i \in I} (h_\gamma \circ \iota \circ p)(i) \cong \prod_{i \in I} B_i.
\]

Furthermore, the direct sum \( \bigoplus_{i \in I} B_i \) is pfd by Corollary 4.8 and thus \( \prod_{i \in I} B_i = \bigoplus_{i \in I} B_i \).

Invoking Corollary 3.7 once more we obtain
\[
\bigoplus_{i \in I} B_i \cong \bigoplus_{i \in I} (h_\gamma \circ \iota \circ p)(i). \tag*{\Box}
\]

**Corollary 5.4.** The direct sum \( \bigoplus_{i \in I} (\iota \circ p)(i) \) is tame.

**Proof.** By Corollary 5.3 the direct sum \( \bigoplus_{i \in I} (\iota \circ p)(i) \) is mapped to \( \bigoplus_{i \in I} (h_\gamma \circ \iota \circ p)(i) \).

Moreover, we have \( (h_\gamma \circ \iota \circ p)(i) \cong B_i \) by Corollary 3.7 hence
\[
\bigoplus_{i \in I} (h_\gamma \circ \iota \circ p)(i) \cong \bigoplus_{i \in I} B_i,
\]
which is in \( \mathcal{J} \) by Lemma 4.7. \( \tag*{\Box} \)

Now suppose \( F \) is an object of \( D^+_t(q_\gamma, \partial q) \) and let \( p : I \to \text{int} \, M \) be some map of sets such that
\[
\# p^{-1}(u) \approx (\beta^0 \circ h_\gamma)(F)(u) = \dim_F \text{Nat}(h_\gamma(F), S_u)
\]
for any \( u \in \text{int} \, M \). By Corollary 4.1 we have a natural isomorphism \( \varphi : \bigoplus_{i \in I} B_i \to h_\gamma(F) \).

For each \( j \in I \) let \( \psi_j : (\iota \circ p)(j) \to F \) be the image of \( 1 \in \mathbb{F} = B_j(p(j)) \) under the composition of linear maps
\[
\mathbb{F} = B_j(p(j)) \longrightarrow \bigoplus_{i \in I} B_i(p(j)) \xrightarrow{\varphi_{p(j)}} h_\gamma(F)(p(j)) = \text{Hom}_{D^+(q_\gamma)}((\iota \circ p)(j), F).
\]

Now the family \( (\psi_i)_{i \in I} \) is a choice function for the indexed family
\[
(\text{Hom}_{D^+(q_\gamma)}((\iota \circ p)(i), F))_{i \in I}.
\]
As such, the family \( (\psi_i)_{i \in I} \) induces a homomorphism
\[
\psi : \bigoplus_{i \in I} (\iota \circ p)(i) \to F
\]
in the derived category \( D^+(q_\gamma) \). Now let \( G := \bigoplus_{i \in I} (\iota \circ p)(i) \), we aim to show that \( \psi : G \to F \) is an isomorphism. We use the following lemma as a stepping stone towards this goal.
Lemma 5.5. The natural transformation \( h_\gamma(\psi) : h_\gamma(G) \to h_\gamma(F) \) is a natural isomorphism.

Proof. We consider the commutative diagram

\[
\begin{array}{ccc}
\bigoplus_{i \in I} (h_\gamma \circ \iota \circ p)(i) & \longrightarrow & h_\gamma(G) \\
\bigoplus_{i \in I} B_i & \longrightarrow & h_\gamma(F)
\end{array}
\]

of functors \( M^o \to \text{Vect}_F \) and natural transformations. By Corollary 5.3, the horizontal arrow at the top is a natural isomorphism. Moreover, the vertical arrow on the left is a natural isomorphism by Corollary 3.7. Furthermore, the natural transformation \( \varphi : \bigoplus_{i \in I} B_i \to h_\gamma(F) \) is a natural isomorphism by Corollary 4.13. As a result \( h_\gamma(\psi) : h_\gamma(G) \to h_\gamma(F) \) is an isomorphism as well.

In conjunction with Proposition 3.14 we obtain the following.

Corollary 5.6. The homomorphism \( \psi : G \to F \) is an isomorphism.

Theorem 5.7. The functor \( h_{\gamma,0,t} : D^+_1(q_\gamma, \partial q) \to \mathcal{J} \) is an equivalence of \( \mathcal{F} \)-linear categories.

Proof. First we show that \( h_{\gamma,0,t} \) is essentially surjective. To this end, let \( F : M^o \to \text{vect}_F \) be a functor in \( \mathcal{J} \). By Corollary 4.13 the functor \( F \) is naturally isomorphic to a direct sum \( \bigoplus_{i \in I} B_{p(i)} : M^o \to \text{vect}_F \) for some map of sets \( p : I \to \text{int} M \) with \( # p^{-1}(u) = \beta_0(F)(u) \) for any \( u \in \text{int} M \). Now \( \bigoplus_{i \in I} B_{p(i)} \cong \bigoplus_{i \in I} (h_\gamma \circ \iota \circ p)(i) \) by Corollary 3.7. Furthermore, \( h_\gamma \left( \bigoplus_{i \in I} (\iota \circ p)(i) \right) \cong \bigoplus_{i \in I} (h_\gamma \circ \iota \circ p)(i) \) by Corollary 5.3 hence \( h_{\gamma,0,t} : D^+_1(q_\gamma, \partial q) \to \mathcal{J} \) is essentially surjective.

Next we show \( h_{\gamma,0,t} \) is fully faithful. To this end, let \( F \) and \( G \) be objects of \( D^+_1(q_\gamma, \partial q) \), then we have

\[
F \cong \bigoplus_{i \in I} (\iota \circ p_1)(i) \quad \text{and} \quad G \cong \bigoplus_{j \in J} (\iota \circ p_2)(j)
\]

for some maps of sets \( p_1 : I \to \text{int} M \) and \( p_2 : J \to \text{int} M \) with

\[
# p_1^{-1}(u) = (\beta_0 \circ h_\gamma)(F)(u) \quad \text{and} \quad # p_2^{-1}(u) = (\beta_0 \circ h_\gamma)(G)(u)
\]

for any \( u \in \text{int} M \) by Corollary 5.6. Moreover, by Proposition 5.1 the direct sum \( \bigoplus_{j \in J} (\iota \circ p_2)(j) \) satisfies the universal property of the corresponding product in \( D^+_1(q_\gamma, \partial q) \). Furthermore, the functor \( h_\gamma : D^+(q_\gamma, \partial q) \to \text{Vect}_F \) preserves the direct sums in (5.1) by Corollary 5.3. As any pfd direct sum of functors is a direct product as well we obtain the commutative diagram

\[
\begin{array}{ccc}
\text{Hom}_{D^+(q_\gamma)}(F,G) & \cong & \prod_{i \in I, j \in J} \text{Hom}_{D^+(q_\gamma)}((\iota \circ p)(i),(\iota \circ p)(j)) \\
\downarrow h_\gamma & & \downarrow \prod_{i \in I, j \in J} h_\gamma \\
\text{Nat}(h_\gamma(F), h_\gamma(G)) & \cong & \prod_{i \in I, j \in J} \text{Nat}((h_\gamma \circ \iota \circ p)(i),(h_\gamma \circ \iota \circ p)(j))
\end{array}
\]
correspondence with corresponding natural transformations with codomain \( h \) the functor \( h \) Now in order to prove Proposition 5.1 we will first show with Proposition 5.14 below that To complete our proof of Theorem 5.7 we still need to provide a proof of Proposition 5.1.

5.1 Partial Faithfulness Corollary 5.8. \( \psi \) this end, we note that for a homomorphism \( \psi \): \( \iota(u) \rightarrow \iota(v) \), where \( u, v \in \text{int } M \), the natural transformation

\[
h_\gamma (\psi) : (h_\gamma \circ \iota)(u) \rightarrow (h_\gamma \circ \iota)(v)
\]

sends the identity

\[
id_{\iota(u)} \in (h_\gamma \circ \iota)(u)(u) = \text{Hom}_{D^+(q_\gamma)}(\iota(u), \iota(u))
\]

to

\[
\psi \in (h_\gamma \circ \iota)(v)(u) = \text{Hom}_{D^+(q_\gamma)}(\iota(u), \iota(v)).
\]

By Corollary 3.7 and the Yoneda Lemma 3.6 this describes a one-to-one correspondence between \( \text{Hom}_{D^+(q_\gamma)}(\iota(u), \iota(v)) \) and \( \text{Nat}((h_\gamma \circ \iota)(u), (h_\gamma \circ \iota)(v)) \).

**Corollary 5.8.** If \( \partial q \) is closed in \( q_\gamma \), then the composition of functors

\[
D_i^+ (q) \xrightarrow{R_{i^*}} D_i^+(q_\gamma) \xrightarrow{h_\gamma \circ \iota} \mathcal{F}
\]

yields an equivalence of categories.

**Proof.** This follows in conjunction with Corollary 3.11 and Lemma 3.27 \( \square \)

5.1 Partial Faithfulness

To complete our proof of Theorem 5.7 we still need to provide a proof of Proposition 5.1. Now in order to prove Proposition 5.1.4 below that the functor \( h_\gamma, o : D^+(q_\gamma, \partial q) \rightarrow \text{Vect}_F^{M^o} \) is at least partially full and partially faithful in the sense that the family of homomorphisms with codomain \( (\iota \circ T^{-n}) (t) \) is in one-to-one correspondence with corresponding natural transformations with codomain \( (h_\gamma \circ \iota \circ T^{-n})(t) \) for any \( n \in \mathbb{Z} \) and \( t \in q \setminus \partial q \). To this end, let \( t \in q \setminus \partial q \) and let \( n \in \mathbb{Z} \). For any \( u \in T^{-n}(D) \) we define the following subsets of \( q \):

\[
I(u) := q \cap \text{int} (\downarrow T^{n+1}(u)) = q \setminus (\rho_1 \circ T^n)(u),
\]

\[
C(u) := q \setminus (\uparrow T^n(u)) = q \setminus (\rho_0 \circ T^n)(u),
\]

and

\[
Z(u) := I(u) \setminus C(u) = (\rho_0 \circ T^n)(u) \setminus (\rho_1 \circ T^n)(u).
\]

Moreover, we define the functors

\[
\mathbb{F}_I[-n] : T^{-n}(D) \rightarrow C^+(q_\gamma), \ u \mapsto \mathbb{F}_I(u)[-n],
\]

\[
\mathbb{F}_C[-n] : T^{-n}(D) \rightarrow C^+(q_\gamma), \ u \mapsto \mathbb{F}_C(u)[-n],
\]

and

\[
\mathbb{F}_Z[-n] : T^{-n}(D) \rightarrow C^+(q_\gamma), \ u \mapsto \mathbb{F}_Z(u)[-n],
\]

where \( C^+(q_\gamma) := C^+(\text{Sh}(q_\gamma)) \) is the category of bounded below cochain complexes of sheaves on \( q_\gamma \). We note that

\[
\mathbb{F}_Z[-n] = \iota|_{T^{-n}(D)}, \tag{5.2}
\]
By [KS90, Proposition 2.3.6.(v)] we have the short exact sequence
\[ 0 \rightarrow \mathbb{F}_C[-n] \rightarrow \mathbb{F}_I[-n] \rightarrow \mathbb{F}_Z[-n] \rightarrow 0 \]
of functors \( T^{-n}(D) \rightarrow C^+(q_\gamma) \). Thus, we obtain the sequence
\[ \mathbb{F}_C[-n] \rightarrow \mathbb{F}_I[-n] \rightarrow \mathbb{F}_Z[-n] \rightarrow \mathbb{F}_C[1-n] \]  
(5.3)
of functors \( T^{-n}(D) \rightarrow D^+(q_\gamma) \), which is point-wise a distinguished triangle in \( D^+(q_\gamma) \), see also [KS90, Equation (2.6.33)]. Now let \( q_0 \) be the unique point of intersection of \( q \) and \( l_0 \). Similarly let \( q_1 \) be the unique point of intersection of \( q \) and \( l_1 \). We note that \( t \in q \) lies on the vertical line through \( q_0 \) iff \( q_0 \preceq t \). Similarly \( t \) lies on the horizontal line through \( q_1 \) iff \( q_1 \preceq t \). For \( u \in T^{-n}(D) \) we write \( C_0(u) \subseteq C(u) \) for the connected component of \( C(u) \) containing \( q_0 \) if it exists and otherwise we set \( C_0(u) = \emptyset \). We define \( C_1(u) \subseteq C(u) \) analogously as well as the functors

\[ \mathbb{F}_{C_0}[-n]: T^{-n}(D) \rightarrow C^+(q_\gamma), \ u \mapsto \mathbb{F}_{C_0(u)}[-n] \]
and \( \mathbb{F}_{C_1}[-n]: T^{-n}(D) \rightarrow C^+(q_\gamma), \ u \mapsto \mathbb{F}_{C_1(u)}[-n] \).

Now let \( m = n, n - 1 \) and let
\[ P: (T^{-n}(D))^\circ \rightarrow \text{vect}_\mathbb{F}, \ u \mapsto \text{Hom}_{D^+(q_\gamma)}(\mathbb{F}_{I(u)}[-n], (u \circ T^{-n})(t)) \].

Then \( P \) is isomorphic to the functor
\[ P': (T^{-n}(D))^\circ \rightarrow \text{vect}_\mathbb{F}, \ u \mapsto \begin{cases} \mathbb{F} & u \in \text{int}(\uparrow T^{-n-1}(t)) \\ \{0\} & \text{otherwise,} \end{cases} \]
whose internal maps are identities whenever both domain and codomain are \( \mathbb{F} \) and zero otherwise. For any functor \( F: (T^{-n}(D))^\circ \rightarrow \text{Vect}_\mathbb{F} \) we have
\[ \text{Nat}(F, P') \cong \left( \lim_{u \in \text{int}(\uparrow T^{-n-1}(t))} F(u) \right)^* . \]

As direct limits are exact as well as the dual space functor, the functor \( \text{Nat}(-, P') \cong \text{Nat}(-, P) \) is exact as well. Now suppose \( F \) is an object of \( D^+(q_\gamma) \). In the following we use \( \text{Hom}(\mathbb{F}_{C_0}[-m], F) \) as a short hand for the functor
\[ (T^{-n}(D))^\circ \rightarrow \text{Vect}_\mathbb{F}, \ u \mapsto \text{Hom}_{D^+(q_\gamma)}(\mathbb{F}_{C_0(u)}[-m], F) \]
and similarly for \( \mathbb{F}_{C_1}[-m], \mathbb{F}_I[-n], \mathbb{F}_C[-n], \) or \( \mathbb{F}_Z[-n] \) in place of \( \mathbb{F}_{C_0}[-m] \).

**Lemma 5.9.** We have
\[ \text{Nat}(\text{Hom}(\mathbb{F}_{C_0}[-m], F), P) \cong \begin{cases} \lim_{q_0 \in U} H^m(U; F) & q_0 \preceq t \\ \{0\} & \text{otherwise.} \end{cases} \]

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Similarly we have
\[
\text{Nat}(\text{Hom}(\mathbb{F}_C[-m], F), P) \cong \begin{cases} \lim_{q_1 \in U} H^n(U; F) & q_1 \leq t \\ \{0\} & \text{otherwise.} \end{cases}
\]

**Corollary 5.10.** For any object \( F \) of \( D^+(q_1, \partial q) \) we have \( \text{Nat}(\text{Hom}(\mathbb{F}_C[-m], F), P) \cong \{0\} \).

**Proof.** We have
\[
\text{Nat}(\text{Hom}(\mathbb{F}_C[-m], F), P) = \text{Nat}(\text{Hom}(\mathbb{F}_C[-m] \oplus \mathbb{F}_C[-m], F), P) = \text{Nat}(\text{Hom}(\mathbb{F}_C[-m], F) \oplus \text{Hom}(\mathbb{F}_C[-m], F), P) = \text{Nat}(\text{Hom}(\mathbb{F}_C[-m], F), P) \oplus \text{Nat}(\text{Hom}(\mathbb{F}_C[-m], F), P) = \{0\} \oplus \{0\}
\]
by Lemma 5.9.

**Lemma 5.11.** The functor \( \text{Hom}(\mathbb{F}_I[-n], -) \) induces an isomorphism
\[
\text{Hom}_{D^+(q_r)}(F, (t \circ T^{-n})(t)) \xrightarrow{\cong} \text{Nat}(\text{Hom}(\mathbb{F}_I[-n], F), P).
\]

**Proof.** The set of open neighbourhoods of \( t \in q \) of the form \( I(u) \) for some \( u \in T^{-n}(D) \) forms a neighbourhood basis for \( t \). Thus, the induced map
\[
\lim_{u \in T^{-n}(D)} H^n(I(u); F) \xrightarrow{\cong} \lim_{t \in U} H^n(U; F)
\]
is an isomorphism. With this we consider the commutative diagram
\[
\begin{array}{ccc}
\text{Hom}_{D^+(q_r)}(F, (t \circ T^{-n})(t)) & \xrightarrow{\text{Hom}(\mathbb{F}_I[-n], -)} & \text{Nat}(\text{Hom}(\mathbb{F}_I[-n], F), P) \\
\downarrow & & \downarrow \\
\left( \lim_{t \in U} H^n(U; F) \right)^* & \xrightarrow{\cong} & \left( \lim_{u \in T^{-n}(D)} H^n(I(u); F) \right)^*
\end{array}
\]
Here the vertical map on the left hand side takes any homomorphism \( \psi: F \to (t \circ T^{-n})(t) \) to the family of maps \( \{H^n(U; \psi): H^n(U; F) \to \mathbb{F} \mid t \in U\} \) and then to the naturally induced map of type \( \lim_{t \in U} H^n(U; F) \to \mathbb{F} \). By Lemma C.5 this map is an isomorphism. The vertical map on the right hand side takes any natural transformation \( \eta: \text{Hom}(\mathbb{F}_I[-n], F) \to P \) to the family of maps \( \{\eta_u: H^n(I(u); F) \to \mathbb{F} \mid t \in I(u)\} \) and then to the naturally induced map of type
\[
\lim_{u \in T^{-n}(D)} H^n(I(u); F) \to \mathbb{F}.
\]
As \( P(u) \cong \{0\} \) for all \( u \in T^{-n}(D) \) with \( t \notin I(u) \), this vertical map on the right hand side is an isomorphism. As a result, we obtain that the horizontal map at the top is an isomorphism as well.

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Now let
\[ B := \text{Hom}(\mathbb{F}^n[-n], (\iota \circ T^{-n})(t)) : (T^{-n}(D))^o \to \text{vect}_\mathbb{F}, \]
then we have
\[ B = (h_\gamma \circ \iota \circ T^{-n})(t)|_{T^{-n}(D)} \cong B_{T^{-n}(D)} \]
by (5.2) and Corollary 3.7. We consider
\[ \text{Nat}(\mathbb{F}^n[-n], (\iota \circ T^{-n})(t)) \]
as a functor from the category of functors \((T^{-n}(D))^o \to \text{Vect}_\mathbb{F}\) vanishing on \(\partial M \cap T^{-n}(D)\)
to the category of vector spaces over \(\mathbb{F}\).

**Lemma 5.12.** The natural transformation
\[ B = \text{Hom}(\mathbb{F}^n[-n], (\iota \circ T^{-n})(t)) \to \text{Hom}(\mathbb{F}^n[-n], (\iota \circ T^{-n})(t)) = P \]
is a universal element of the functor \(\text{Nat}(\mathbb{F}^n[-n], (\iota \circ T^{-n})(t))\). In other words, for any natural transformation \(\eta : F \to P\) with \(F\) vanishing on \(\partial M \cap T^{-n}(D)\) there is a unique natural transformation \(\eta' : F \to B\) such that the diagram
\[
\begin{array}{ccc}
B & \to & P \\
\downarrow & & \downarrow \\
F & \nearrow & P
\end{array}
\]
commutes.

**Proof.** As \(P \cong P'\) we have the short exact sequence
\[ 0 \to B \to P \to \text{Ran}_{\partial M} P|_{\partial M} \to 0, \]
where \(\text{Ran}_{\partial M} P|_{\partial M}\) is the right Kan extension of the restriction of \(P\) to \(\partial M \cap T^{-n}(D)\)
along the inclusion \(\partial M \cap T^{-n}(D) \hookrightarrow T^{-n}(D)\). As \(\text{Ran}_{\partial M} F|_{\partial M} \cong 0\) for any functor \(F : (T^{-n}(D))^o \to \text{Vect}_\mathbb{F}\) vanishing on \(\partial M \cap T^{-n}(D)\) the result follows.

**Lemma 5.13.** For any functor \(F : M^o \to \text{Vect}_\mathbb{F}\) the restriction map
\[ \text{Nat}(F, B_{T^{-n}(t)}) \cong \text{Nat}(F, (h_\gamma \circ \iota \circ T^{-n})(t)) \to \text{Nat}(F|_{T^{-n}(D)}, B) \]
is an isomorphism.

**Proposition 5.14.** For any object \(F\) of \(D^+(q_\gamma, \partial q)\) the natural map
\[ \text{Hom}_{D^+(q_\gamma)}(F, (\iota \circ T^{-n})(t)) \xrightarrow{h_\gamma} \text{Nat}(h_\gamma(F), (h_\gamma \circ \iota \circ T^{-n})(t)) \]
is an isomorphism.
Proof. We consider the commutative diagram

\[
\begin{array}{ccc}
\text{Nat}(h_\gamma(F), (h_\gamma \circ t \circ T^{-n})(t)) & \xrightarrow{h_\alpha} & \text{Nat}(\text{Hom}(F_I[-n], F), P) \\
\text{Nat}(\text{Hom}(F_C[1-n], F), P) & \xrightarrow{\text{Nat}(\text{Hom}(F_Z[-n], F), B)} & \text{Nat}(\text{Hom}(F_Z[-n], F), P) \\
\end{array}
\]

By Lemma 5.13 the vertical map on the lower left hand side is an isomorphism. Moreover, by Lemma 5.12 the horizontal map at the bottom is an isomorphism. Furthermore, the diagonal map on the upper right hand side induced by the functor Hom(F_I[-n], -) is an isomorphism by Lemma 5.11. Thus, it suffices to show that the vertical map on the right hand side

\[\text{Nat}(\text{Hom}(F_I[-n], F), P) \to \text{Nat}(\text{Hom}(F_I[-n], F), P)\]

is an isomorphism. To this end, we post-compose the point-wise distinguished triangle (5.3) of functors with the cohomological functor Hom_{D^+(q_\gamma)}(-, F) to obtain the exact sequence

\[\text{Hom}(F_C[1-n], F) \to \text{Hom}(F_Z[-n], F) \to \text{Hom}(F_I[-n], F) \to \text{Hom}(F_C[-n], F)\]

of functors \((T^{-n}(D))^o \to \text{Vect}_F\). Now we apply the exact functor Nat(-, P) to this exact sequence to obtain the exact sequence

\[\begin{array}{ccc}
\text{Nat}(\text{Hom}(F_I[-n], F), P) & \xrightarrow{\text{Nat}(\text{Hom}(F_Z[-n], F), P)} & \text{Nat}(\text{Hom}(F_I[-n], F), P) \\
\text{Nat}(\text{Hom}(F_C[-n], F), P) & \xrightarrow{\text{Nat}(\text{Hom}(F_I[-n], F), P)} & \text{Nat}(\text{Hom}(F_C[1-n], F), P) \\
\end{array}\]

in turn. By Corollary 5.10 we have

\[\text{Nat}(\text{Hom}(F_C[-n], F), P) \cong \{0\} \cong \text{Nat}(\text{Hom}(F_C[1-n], F), P)\]

and thus the vertical map in the center of (5.4) has to be an isomorphism. \square

5.2 Products Vanishing on \(\partial q\)

In the following we provide a proof of Proposition 5.1 by considering the induced maps on cohomology stalks of the direct sum and the coreflection of the derived product into \(D^+(q_\gamma, \partial q)\). To this end, let \(t \in q \setminus \partial q\) and let \(n \in \mathbb{Z}\) as in Section 5.1.
Lemma 5.15. For a family of objects \( \{ F_i \mid i \in I \} \) in \( D^+(q, \partial q) \) the naturally induced map

\[
\text{Nat} \left( h_{\gamma} \left( \bigoplus_{i \in I} F_i \right), (h_{\gamma} \circ \iota \circ T^{-n})(t) \right) \rightarrow \text{Nat} \left( \bigoplus_{i \in I} h_{\gamma} (F_i), (h_{\gamma} \circ \iota \circ T^{-n})(t) \right)
\]

is an isomorphism.

Proof. We consider the commutative diagram

\[
\text{Hom}_{D^+(q, \partial q)}(\bigoplus_{i \in I} F_i, (\iota \circ T^{-n})(t)) \xrightarrow{h_{\gamma}} \text{Nat} \left( h_{\gamma} \left( \bigoplus_{i \in I} F_i \right), (h_{\gamma} \circ \iota \circ T^{-n})(t) \right)
\]

\[
\downarrow
\]

\[
\text{Nat} \left( \bigoplus_{i \in I} h_{\gamma} (F_i), (h_{\gamma} \circ \iota \circ T^{-n})(t) \right)
\]

\[
\prod_{i \in I} \text{Hom}_{D^+(q, \partial q)}(F_i, (\iota \circ T^{-n})(t)) \xrightarrow{\prod_{i \in I} h_{\gamma}} \prod_{i \in I} \text{Nat} \left( h_{\gamma} (F_i), (h_{\gamma} \circ \iota \circ T^{-n})(t) \right).
\]

By Proposition 5.14 the two horizontal maps are isomorphisms. Moreover, the vertical map on the left hand side and the vertical map on the lower right hand side are both isomorphisms by the universal property of the direct sum. Thus, the vertical map on the upper right hand side is an isomorphism as well.

Now let \( p : I \rightarrow \text{int} \mathbb{M} \) be some map of sets such that the assignment

\[
\text{int} \mathbb{M} \rightarrow \mathbb{N}_0, \ u \mapsto \# p^{-1}(u)
\]

is an admissible Betti function. Again we write \( B_i := B_{p(i)} : \mathbb{M}^\circ \rightarrow \text{vect}_F \) for any \( i \in I \).

Proof of Proposition 5.1. Let

\[
\kappa : \bigoplus_{i \in I} (\iota \circ p)(i) \rightarrow R \prod_{i \in I} (\iota \circ p)(i)
\]

be the naturally induced homomorphism from the direct sum to the derived product. Then there is a unique homomorphism

\[
\kappa^\# : \bigoplus_{i \in I} (\iota \circ p)(i) \rightarrow R \mathfrak{B} R \prod_{i \in I} (\iota \circ p)(i)
\]

such that the diagram

\[
\begin{array}{ccc}
\bigoplus_{i \in I} (\iota \circ p)(i) & \xrightarrow{\kappa^\#} & R \mathfrak{B} R \prod_{i \in I} (\iota \circ p)(i) \\
\kappa & \downarrow & \\
R \mathfrak{B} R \prod_{i \in I} (\iota \circ p)(i) & \\
\end{array}
\]

(5.5)

commutes. We show that the induced map on cohomology stalks

\[
\lim_{i \in U} H^n(U; \kappa^\#) : \lim_{i \in U} H^n \left( U; \bigoplus_{i \in I} (\iota \circ p)(i) \right) \rightarrow \lim_{i \in U} H^n \left( U; R \mathfrak{B} R \prod_{i \in I} (\iota \circ p)(i) \right)
\]

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is an isomorphism. By Lemma 3.9 we may as well show that the induced map

\[\text{Hom}_{D^+(X)} \left(R\mathcal{R} \prod_{i \in I}(t \circ p)(i), (t \circ T^{-n})(t)\right)\]

\[\xrightarrow{\text{Hom}_{D^+(X)}(\kappa^\# \circ t \circ T^{-n})(t)}\]

\[\text{Hom}_{D^+(X)} \left(\bigoplus_{i \in I}(t \circ p)(i), (t \circ T^{-n})(t)\right)\]

is an isomorphism. Proposition 5.14 in turn implies that it suffices to show that the induced map

\[\text{Nat} \left(h_\gamma \left(R\mathcal{R} \prod_{i \in I}(t \circ p)(i)\right), (h_\gamma \circ t \circ T^{-n})(t)\right)\]

\[\xrightarrow{\text{Nat}(h_\gamma(\kappa^\#), (h_\gamma \circ t \circ T^{-n})(t))}\]

\[\text{Nat} \left(h_\gamma \left(R\mathcal{R} \prod_{i \in I}(t \circ p)(i)^\#\right), (h_\gamma \circ t \circ T^{-n})(t)\right)\]

is an isomorphism. To this end, we apply the functor

\[\text{Nat}(h_\gamma(-), (h_\gamma \circ t \circ T^{-n})(t)) : D^+(q_\gamma) \to \text{Vect}_F\]

to the commutative triangle \((5.5)\) to obtain the commutative diagram

\[\text{Nat} \left(h_\gamma \left(R\mathcal{R} \prod_{i \in I}(t \circ p)(i)\right), (h_\gamma \circ t \circ T^{-n})(t)\right)\]

\[\xrightarrow{\text{Nat}(h_\gamma(\kappa^\#), (h_\gamma \circ t \circ T^{-n})(t))}\]

\[\text{Nat} \left(h_\gamma \left(R\mathcal{R} \prod_{i \in I}(t \circ p)(i)^\#\right), (h_\gamma \circ t \circ T^{-n})(t)\right)\]

By Lemma 3.9 the upper left diagonal map \(\text{Nat} \left(h_\gamma \left(\prod_{i \in I}(t \circ p)(i)\right), (h_\gamma \circ t \circ T^{-n})(t)\right)\) is an isomorphism. Thus, it suffices to show that the curved map on the left hand side \(\text{Nat} \left(h_\gamma(\kappa), (h_\gamma \circ t \circ T^{-n})(t)\right)\) is an isomorphism. To this end, we consider the commutative diagram

\[\begin{array}{ccc}
h_\gamma \left(\bigoplus_{i \in I}(t \circ p)(i)\right) & \xrightarrow{h_\gamma(\kappa)} & h_\gamma \left(R\mathcal{R} \prod_{i \in I}(t \circ p)(i)\right) \\
\bigoplus_{i \in I}(h_\gamma \circ t \circ p)(i) & \xrightarrow{h_\gamma(\kappa)} & \prod_{i \in I}(h_\gamma \circ t \circ p)(i) \\
\bigoplus_{i \in I} B_{p(i)} & \xrightarrow{h_\gamma(\kappa)} & \prod_{i \in I} B_{p(i)} \\
\end{array}\]
Here the equality at the bottom follows from Lemma 4.7. Moreover, the two vertical arrows in the second row are natural isomorphisms by Corollary 3.7. By the universal property of the derived product the vertical arrow on the upper right hand side is a natural isomorphism as well. Furthermore, if we apply the functor

$$\text{Nat}(-, (h_\gamma \circ \iota \circ T^{-n})(t)) : \text{Vect}^M \rightarrow \text{Vect}_F$$

(5.6) to the vertical map on the upper left hand side, then we obtain an isomorphism by Lemma 5.15. Thus applying the functor (5.6) to the horizontal arrow at the top yields an isomorphism as well and this implies the result.

6 Abelianization of $D_t^+(q_\gamma, \partial q)$

Recall that $C$ is the category of functors $M^0 \rightarrow \text{Vect}_F$ vanishing on $\partial M$.

Proposition 6.1. The category of $J$-presentable functors $\text{pres}(J)$ is an abelian subcategory of $C$. Moreover, the cohomological functor

$$D_t^+(q_\gamma, \partial q) \xrightarrow{h_\gamma} J \hookrightarrow \text{pres}(J)$$

is the abelianization of $D_t^+(q_\gamma, \partial q)$. In other words, the precomposition functor $(-) \circ h_{\gamma,0,t}$ from the category of exact functors $\text{pres}(J) \rightarrow A$ to the category of cohomological functors $D_t^+(q_\gamma, \partial q) \rightarrow A$ is an equivalence of categories for any abelian category $A$.

Proof. By Corollary 3.31 the category $D_t^+(q_\gamma, \partial q)$ is indeed triangulated. Now let $F : D_t^+(q_\gamma, \partial q) \rightarrow C$ be the composition of the two functors $h_{\gamma,0,t} : D_t^+(q_\gamma, \partial q) \rightarrow J$ and the inclusion functor $J \hookrightarrow C$. Then $J$ is the essential image of $F : D_t^+(q_\gamma, \partial q) \rightarrow C$ by Theorem 5.7. With this the result follows directly from Proposition E.1.

Corollary 6.2. If $\partial q$ is closed in $q_\gamma$, then the composition of functors

$$D_t^+(\dot{q}) \xrightarrow{Ri_a} D_t^+(q_\gamma) \xrightarrow{h_\gamma} J \hookrightarrow \text{pres}(J)$$

is the abelianization of $D_t^+(\dot{q})$.

Proof. This follows in conjunction with Corollary 3.11 Lemma 3.27 and Corollary 3.32.

Corollary 6.3. The composition of functors

$$D_t^+(\mathbb{R}) \xrightarrow{h_\gamma} J \hookrightarrow \text{pres}(J)$$

is the abelianization of $D_t^+(\mathbb{R})$.

Proof of Theorem 1.1. By Proposition 6.1 the category of $J$-presentable functors $\text{pres}(J)$ is an abelian subcategory of $C$, which is an abelian subcategory of $\text{Vect}_F^M$. Moreover, Proposition 6.1 further implies that $\text{pres}(J)$ is the abelianization of the triangulated category $D_t^+(q_\gamma, \partial q)$. As such, $\text{pres}(J)$ is Frobenius by [Kra07, Section 4.2]. Furthermore, a $J$-presentable functor is projective in $\text{pres}(J)$ iff it is in $J$ by Corollary 4.14. Thus, $J$ is the subcategory of projectives in $\text{pres}(J)$. 

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Moreover, Theorem 1.1 has the following corollary.

**Corollary 6.4.** The triangulated category $D_t^+(q_-, \partial q)$ has split idempotents.

*Proof.* By [Kra07, Section 4.3] and Proposition 6.1, the subcategory of projectives in $\text{pres}(J)$ has split idempotents, which is $J$ by Theorem 1.1. Furthermore, $J$ and $D_t^+(q_-, \partial q)$ are equivalent as $\mathbb{F}$-linear categories by Theorem 6.1.

We end this section with a proof of Proposition 1.3.

**Lemma 6.5.** Let $\varphi: P \to F$ be a projective cover in $\text{pres}(J)$ and let $\kappa: K \to P$ be its kernel so we have the short exact sequence

$$0 \to K \xrightarrow{\kappa} P \xrightarrow{\varphi} F \to 0.$$

Then $\beta^{n+1}(F) = \beta^n(K)$ for all $n \in \mathbb{N}_0$.

*Proof.* Let $u \in \text{int} M$ be a point inside the interior of $M$. As $\text{Nat}(-, S_u)$ is left-exact as a functor on $\mathcal{C}$ and since $P: M^\circ \to \text{Vect}_F$ is projective in $\mathcal{C}$ by Theorem 1.1 and [BBF21, Corollary 3.6] we obtain the long exact sequence

\[
\begin{align*}
\cdots \to \text{Ext}^2_C(F, S_u) &\to \{0\} \to \text{Ext}^2_C(K, S_u) \\
&\to \text{Nat}(F, S_u) \to \text{Nat}(P, S_u) \to \text{Nat}(K, S_u) \\
&\to \text{Ext}^1_C(F, S_u) \to \{0\} \to \cdots.
\end{align*}
\]

Moreover, by Proposition 4.11 we have

$$\dim \text{Nat}(F, S_u) = \beta^0(F)(u) = \beta^0(P)(u) = \dim \text{Nat}(P, S_u),$$

hence the naturally induced map $\text{Nat}(F, S_u) \to \text{Nat}(P, S_u)$ on the lower left hand side of (6.1) is an isomorphism. In conjunction with (6.1) we obtain natural isomorphisms

$$\text{Ext}^n_C(K, S_u) \to \text{Ext}^{n+1}_C(F, S_u)$$

for all $n \in \mathbb{N}_0$.

*Proof of Proposition 1.3.* By Proposition 4.11 and Theorem 1.1 any $J$-presentable functor has a minimal projective resolution

$$\cdots \to P_n \to \cdots \to P_2 \to P_1 \to P_0 \to F \to 0$$

by functors $P_n: M^\circ \to \text{Vect}_F$ in $J$. The statement about the multiplicities of indecomposables and the Betti functions follows by induction from Lemma 6.5 and Proposition 4.11.

Moreover, Proposition 1.3 has the following corollary.
Corollary 6.6. Let $F: \mathcal{M}^o \to \text{Vect}_F$ be a $\mathcal{J}$-presentable functor. Then $\beta^n(F): \int \mathcal{M} \to \mathbb{N}_0$ is an admissible Betti function for any $n \in \mathbb{N}_0$.

Proof. Let $n \in \mathbb{N}_0$ and let $u \in \int \mathcal{M}$. Then $\beta^n(F)(u)$ is the multiplicity of the indecomposable $B_u: \int \mathcal{M} \to \text{Vect}_F$ in some projective $P: \mathcal{M}^o \to \text{Vect}_F$ by Proposition 1.3, which is the same as $\beta^0(P)(u)$. Thus, $\beta^n(F) = \beta^0(P)$, which is an admissible Betti function by Lemma 4.5. \hfill \Box

7 Categorification of Persistence Diagrams

We show that the sequence of Betti functions is actually determined by the first four of these. To this end, the following notion will be useful.

Definition 7.1. Let
\[ \cdots \xrightarrow{\delta_{n+1}} P_n \xrightarrow{\delta_n} \cdots \xrightarrow{\delta_2} P_2 \xrightarrow{\delta_1} P_1 \xrightarrow{\delta_0} P_0 \xrightarrow{\epsilon} F \to 0 \] (7.1)
be a projective resolution of $F: \mathcal{M}^o \to \text{Vect}_F$ in $	ext{pres}(\mathcal{J})$. We say that the projective resolution (7.1) is equivariant, if $P_{n+3} = P_n \circ T$ and $\delta_{n+3} = -\delta_n \circ T$ for all $n \in \mathbb{N}_0$.

Lemma 7.2. Any $\mathcal{J}$-presentable functor $F: \mathcal{M}^o \to \text{Vect}_F$ has an equivariant projective resolution.

Proof. By Definition D.1 and Theorem 5.7 we may choose a presentation
\[ h_\gamma(Y) \xrightarrow{h_\gamma(\delta)} h_\gamma(X) \to F \to 0 \]
with $X$ and $Y$ objects of $D^+(q_\gamma, \partial q)$. Then we may form homotopy pullback squares
\[ \begin{array}{ccc} X[-1] & \xrightarrow{\theta} & Z \xrightarrow{\kappa} 0 \\ \downarrow & & \downarrow \\ 0 & \xrightarrow{0} & Y \xrightarrow{\delta} X. \end{array} \]
As $h_\gamma(X[-1]) \cong h_\gamma(X) \circ T$ we may set
\[ P_{3n} := h_\gamma(X) \circ T^n, \quad P_{3n+1} := h_\gamma(Y) \circ T^n, \quad P_{3n+2} := h_\gamma(Z) \circ T^n, \]
\[ \delta_{3n+1} := (-1)^n h_\gamma(\delta) \circ T^n, \quad \delta_{3n+2} := (-1)^n h_\gamma(\kappa) \circ T^n, \quad \delta_{3n+3} := (-1)^n h_\gamma(\theta) \circ T^n \]
for all $n \in \mathbb{N}_0$. Moreover, as $h_\gamma: D^+(q_\gamma) \to \text{Vect}_F^{\mathcal{M}^o}$ is cohomological and as $	ext{pres}(\mathcal{J})$ is an abelian subcategory of $	ext{Vect}_F^{\mathcal{M}^o}$ by Theorem 1.1 (or by Proposition 6.1) the sequence (7.1) is indeed exact. Theorem 1.1 further implies that $P_n: \mathcal{M}^o \to \text{Vect}_F$ is projective for each $n \in \mathbb{N}_0$. \hfill \Box

Now let $F: \mathcal{M}^o \to \text{Vect}_F$ be a $\mathcal{J}$-presentable functor. We obtain the following corollary.
Corollary 7.3. We have $\beta^{n+3}(F) = \beta^n(F) \circ T$ for all $n \geq 1$.

As $F : \mathcal{M}^\circ \to \mathbb{Vect}$ has bounded above support as a $\mathcal{J}$-presentable functor, we obtain yet another corollary.

Corollary 7.4. For any $u \in \text{int} \mathcal{M}$ we have $\beta^n(F)(u) = 0$ for almost all $n \in \mathbb{N}_0$.

With this the following is a sound definition.

Definition 7.5. We define the Euler function of $F : \mathcal{M}^\circ \to \mathbb{Vect}$ to be

$$\chi(F) := \sum_{n \in \mathbb{N}_0} (-1)^n \beta^0(P_n) : \text{int} \mathcal{M} \to \mathbb{Z}, u \mapsto \sum_{n \in \mathbb{N}_0} (-1)^n \dim \text{Ext}^n_{\mathcal{C}}(F, S_u).$$

Now Corollary 7.3 further implies that for any bounded region in $\mathcal{M}$ the restriction of $\beta^n(F) : \mathcal{M} \to \mathbb{N}_0$ to this region vanishes for almost all $n \in \mathbb{N}_0$. Thus, the point-wise absolute value

$$|\chi(F)| : \text{int} \mathcal{M} \to \mathbb{N}_0, u \mapsto |\chi(F)(u)|$$

is an admissible Betti function.

Definition 7.6. We say that a function $\mu : \text{int} \mathcal{M} \to \mathbb{Z}$ is an admissible Euler function if the point-wise absolute value

$$|\mu| : \text{int} \mathcal{M} \to \mathbb{N}_0, u \mapsto |\mu(u)|$$

is an admissible Betti function. Moreover, we denote the abelian group of admissible Euler functions by $G(\mathbb{B})$. Then the inclusion $\mathbb{B} \hookrightarrow G(\mathbb{B})$ satisfies the universal property of the Grothendieck group of the commutative monoid $\mathbb{B}$.

In particular $\chi(F) : \text{int} \mathcal{M} \to \mathbb{Z}$ is an admissible Euler function. Next we show that $\chi : \text{Ob}(\text{pres}(\mathcal{J})) \to G(\mathbb{B})$ is an additive invariant of $\text{pres}(\mathcal{J})$. To this end, we choose an equivariant projective resolution

$$\ldots \xrightarrow{\delta_{n+1}} P_n \xrightarrow{\delta_n} \ldots \xrightarrow{\delta_3} P_2 \xrightarrow{\delta_2} P_1 \xrightarrow{\delta_1} P_0 \xrightarrow{\epsilon} F \to 0.$$

Then we obtain the following counterpart to Euler’s polyhedron formula.

Lemma 7.7. We have $\chi(F) = \sum_{n \in \mathbb{N}_0} (-1)^n \beta^0(P_n)$.

Proof. Let $u \in \text{int} \mathcal{M}$. Then the cochain complex $\text{Nat}(P_\bullet, S_u)$ is finite as $P_n(u) \cong \{0\}$ for almost all $n \in \mathbb{N}_0$, and thus the Euler characteristic of $\text{Nat}(P_\bullet, S_u)$ can be computed in terms of its cohomology $\text{Ext}^n_{\mathcal{C}}(F, S_u)$. As a result, we obtain

$$\sum_{n \in \mathbb{N}_0} (-1)^n \beta^0(P_n)(u) = \sum_{n \in \mathbb{N}_0} (-1)^n \dim \text{Nat}(P_n, S_u)$$

$$= \sum_{n \in \mathbb{N}_0} (-1)^n \dim \text{Ext}^n_{\mathcal{C}}(F, S_u)$$

$$= \chi(F)(u).$$
Now in order to harness this lemma to show the additivity of \( \chi : \text{Ob}(\text{pres}(J)) \to G(\mathcal{B}) \), we make the following observation about \( \beta^0 : \text{Ob}(\text{pres}(J)) \to \mathcal{B} \).

**Lemma 7.8.** The restriction of \( \beta^0 : \text{Ob}(\text{pres}(J)) \to \mathcal{B} \) to functors in \( J \) is an additive invariant.

**Proof.** As all functors in \( J \) are projective by Corollary 4.14 or Theorem 1.1, all short exact sequences in \( J \) split. Thus, the result follows from the additivity of the functor \( \text{Nat}(-, S_u) : \text{pres}(J) \to \text{Vect}_F \) for any \( u \in \text{int} M \).

Now suppose

\[
0 \to F \to G \to H \to 0
\]

is a short exact sequence of \( J \)-presentable functors. Moreover, suppose we have equivariant projective resolutions

\[
\cdots \to P_n \to \cdots \to P_2 \to P_1 \to P_0 \to F \to 0
\]

and

\[
\cdots \to R_n \to \cdots \to R_2 \to R_1 \to R_0 \to H \to 0.
\]

**Lemma 7.9 (Horseshoe Lemma).** There exist an equivariant projective resolution

\[
\cdots \to Q_n \to \cdots \to Q_2 \to Q_1 \to Q_0 \to G \to 0
\]

and a short exact sequence of chain complexes

\[
0 \to P_\bullet \to Q_\bullet \to R_\bullet \to 0
\]

such that the diagram

\[
\begin{array}{ccc}
P_0 & \longrightarrow & Q_0 \\
\downarrow & & \downarrow \\
F & \longrightarrow & G
\end{array} \quad \begin{array}{ccc}
& & \longrightarrow \\
& & \downarrow \\
& & R_0 \\
& & \downarrow \\
& & G \\
& & \longrightarrow \\
& & H
\end{array}
\]

commutes.

**Proof.** If we apply the construction from the proof of the ordinary horseshoe lemma, then we obtain an equivariant chain complex.

**Proposition 7.10.** The function \( \chi : \text{Ob}(\text{pres}(J)) \to G(\mathcal{B}) \) is an additive invariant of \( \text{pres}(J) \).

**Proof.** This result follows directly from Lemmas 7.7, 7.9, and 7.8.

In particular we obtain the group homomorphism

\[
[\chi] : K_0(\text{pres}(J)) \to G(\mathcal{B}), [F] \mapsto \chi(F).
\]

Next we show that \( [\chi] \) is an isomorphism. Now \( J \) being the subcategory of projectives in \( \text{pres}(J) \) by Theorem 1.1 it is a Quillen exact category as well. Thus, we may also consider the Grothendieck group \( K_0(J) \) of \( J \). Moreover, the restriction of \( \beta^0 : \text{Ob}(\text{pres}(J)) \to \mathcal{B} \) to
functors in \( J \) is an additive invariant by Lemma 7.8, hence we obtain the group homomorphism
\[
[\beta^0]: K_0(J) \to G(\mathcal{B}), \ [P] \mapsto \beta^0(P).
\]
Furthermore, since \( \beta^n(P) = 0 \) for any \( n \geq 1 \) and any projective \( P \), we obtain the commutative triangle
\[
\begin{array}{ccc}
K_0(J) & \longrightarrow & K_0(\text{pres}(J)) \\
[\beta^0] & \downarrow & \phi \\
G(\mathcal{B}) & \\
\end{array}
\] (7.2)
of abelian groups. Thus, in order to show that \( \chi: K_0(\text{pres}(J)) \to G(\mathcal{B}) \) is an isomorphism it suffices to show that \( [\beta^0]: K_0(J) \to G(\mathcal{B}) \) is an isomorphism and that the upper vertical map in (7.2) induced by the subcategory inclusion \( J \hookrightarrow \text{pres}(J) \) is an epimorphism.

**Lemma 7.11.** The group homomorphism \( [\beta^0]: K_0(J) \to G(\mathcal{B}) \) is an isomorphism.

**Proof.** By Lemma 4.7, the group homomorphism \( [\beta^0]: K_0(J) \to G(\mathcal{B}) \) is surjective. Now suppose \( P: M^o \to \text{Vect}_F \) is a functor in \( J \) such that \( \beta^0(P) = [\beta^0](P) = 0 \). Then the zero natural transformation \( 0 \to P \) is a projective cover of \( P \) by Proposition 1.3 and thus \( P \cong 0 \).

**Lemma 7.12.** The group homomorphism \( K_0(J) \to K_0(\text{pres}(J)) \) induced by the full subcategory inclusion \( J \hookrightarrow \text{pres}(J) \) is an isomorphism.

**Proof.** By Lemma 7.11, the homomorphism \( [\beta^0]: K_0(J) \to G(\mathcal{B}) \) is injective. In conjunction with the commutativity of (7.2), this implies the injectivity of \( K_0(J) \to K_0(\text{pres}(J)) \). Now let \( F: M^o \to \text{Vect}_F \) be a \( J \)-presentable functor and let
\[
\cdots \to P_n \xrightarrow{\delta_n} P_{n+1} \xrightarrow{\delta_n} P_{n+2} \xrightarrow{\delta_n} P_{n+3} \to \cdots
\]
be an equivariant projective resolution of \( F \). Then both
\[
\bigoplus_{n=0}^\infty P_{2n} \quad \text{and} \quad \bigoplus_{n=0}^\infty P_{2n+1}
\]
are functors in \( J \) and thus it suffices to show that
\[
\left[ \bigoplus_{n=0}^\infty P_{2n} \right] = [F] + \left[ \bigoplus_{n=0}^\infty P_{2n+1} \right]
\]
as elements of \( K_0(\text{pres}(J)) \). To this end, we consider the exact sequence
\[
\begin{array}{c}
\bigoplus_{n=0}^\infty P_{2n+3} \xrightarrow{\varphi_3} \bigoplus_{n=0}^\infty P_{2n+2} \xrightarrow{\varphi_2} \bigoplus_{n=0}^\infty P_{2n+1} \xrightarrow{\varphi_1} \bigoplus_{n=0}^\infty P_{2n},
\end{array}
\] (7.3)
where
\[
\varphi_3 := \bigoplus_{n=0}^\infty \delta_{2n+3}, \quad \varphi_2 := \bigoplus_{n=0}^\infty \delta_{2n+2}, \quad \text{and} \quad \varphi_1 := \bigoplus_{n=0}^\infty \delta_{2n+1}.
\]
First we note that
\[ \text{coker } \varphi_1 \cong \text{coker } \delta_1 \oplus \text{coker } \varphi_3 \cong F \oplus \text{coker } \varphi_3. \] (7.4)
Moreover, by the exactness of (7.3) we have \( \text{coker } \varphi_3 \cong \text{Im } \varphi_2 = \text{ker } \varphi_1. \) In conjunction with (7.4) we obtain \( \text{coker } \varphi_1 \cong F \oplus \text{ker } \varphi_1 \) and hence
\[ [\text{coker } \varphi_1] = [F] + [\text{ker } \varphi_1]. \]
From this equation in turn we obtain
\[
\left[ \bigoplus_{n=0}^{\infty} P_{2n} \right] = [\text{Im } \varphi_1] + [\text{coker } \varphi_1] \\
= [\text{Im } \varphi_1] + [F] + [\text{ker } \varphi_1] \\
= [F] + [\text{ker } \varphi_1] + [\text{Im } \varphi_1] \\
= [F] + \left[ \bigoplus_{n=0}^{\infty} P_{2n+1} \right]. \]

Proof of Theorem 1.4. This follows directly from Lemmas 7.11 and 7.12 and the commutativity of the triangle (7.2).

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A General Facts on Derived Adjunctions

Let

\[ \begin{array}{ccc}
\mathcal{B} & \xrightarrow{G} & \mathcal{C} \\
\downarrow{F} & & \downarrow{H} \\
\mathcal{C} & \xrightarrow{\eta} & \mathcal{A}
\end{array} \]

be an adjunction of abelian categories. Moreover, we assume that the left adjoint \( F \) is exact (or equivalently left-exact) and that \( \mathcal{C} \) has enough injectives.

Lemma A.1. For any object \( X \in \mathcal{B} \) the derived unit

\[ \eta^\mathbb{D^+}(\mathcal{B})_X : X \xrightarrow{\eta_X} (G \circ F)(X) \to (R G \circ F)(X) \]

is an isomorphism of the derived category \( \mathbb{D^+}(\mathcal{B}) \) iff the ordinary unit \( \eta_X \) is an isomorphism and \( F(X) \) is \( G \)-acyclic, i.e., \( R^k \eta(G) = 0 \) for all \( k \neq 0 \).

Lemma A.2. Let \( H : \mathcal{B} \to \mathcal{A} \) be a left exact functor that has a right derived functor \( RH : D^+(\mathcal{B}) \to D^+(\mathcal{A}) \). Then the derived functors \( RH \) and \( RG \) compose as \( R(H \circ G) \cong RH \circ RG \).

Lemma A.3. If the adjunction \( D^+(F) = LF \dashv RG \) is \([\text{coreflective}] \) then the essential image of \( D^+(F) \) is the full subcategory \( D^+_{F(\mathcal{B})}(\mathcal{C}) \) of complexes whose cohomology objects are in the essential image of \( F \). In particular, \( D^+_{F(\mathcal{B})}(\mathcal{C}) \) is a triangulated subcategory of \( D^+(\mathcal{C}) \).
B The Beck–Chevalley Condition

Suppose we have a square

\[
\begin{array}{ccc}
C_1 & \xrightarrow{G_1} & C_2 \\
\downarrow H_1 & & \downarrow H_2 \\
C_3 & \xleftarrow{G_2} & C_4
\end{array}
\]  

of categories and functors that commutes up to a natural isomorphism \( \zeta : H_2 \circ G_1 \Rightarrow G_2 \circ H_1 \). Moreover, suppose \( G_1 : C_1 \to C_2 \) and \( G_2 : C_3 \to C_4 \) have left adjoints \( F_1 : C_2 \to C_1 \) and \( F_2 : C_4 \to C_3 \), respectively. Then we also have the square diagram

\[
\begin{array}{ccc}
C_1 & \xleftarrow{F_1} & C_2 \\
\downarrow H_1 & & \downarrow H_2 \\
C_3 & \xleftarrow{F_2} & C_4
\end{array}
\]

where \( \xi : F_2 \circ H_2 \Rightarrow H_1 \circ F_1 \), defined as the composition

\[
\begin{align*}
F_2 \circ H_2 \\
\downarrow & \\
F_2 \circ H_2 \circ \eta^\dagger \\
\downarrow & \\
F_2 \circ G_2 \circ \eta \\
\downarrow & \\
F_2 \circ G_2 \circ H_1 \circ F_1 \\
\downarrow & \\
H_1 \circ F_1
\end{align*}
\]

is the so called mate of \( \zeta : H_2 \circ G_1 \Rightarrow G_2 \circ H_1 \).

**Definition B.1 (Beck–Chevalley Condition).** We say that the square diagram (B.1) satisfies the Beck–Chevalley condition if \( \xi : F_2 \circ H_2 \Rightarrow H_1 \circ F_1 \) is a natural isomorphism.

Morally, a commutative square of categories and functors satisfies the Beck–Chevalley condition, if the horizontal arrows have left adjoints and the corresponding square with left adjoints commutes as well, which is not precisely the same as Definition [B.1] but it is close enough for intuition.

There is a dual version of the Beck–Chevalley condition as well, which involves right adjoints in place of left adjoints. To this end, we consider the square diagram

\[
\begin{array}{ccc}
C_1 & \xrightarrow{F_1} & C_2 \\
\downarrow H_1 & & \downarrow H_2 \\
C_3 & \xrightarrow{F_2} & C_4
\end{array}
\]  

(B.2)
with $\zeta: F_2 \circ H_1 \Rightarrow H_2 \circ F_1$ a natural isomorphism. Moreover, suppose $F_1: C_1 \to C_2$ and $F_2: C_3 \to C_4$ have right adjoints $G_1: C_2 \to C_1$ and $G_2: C_4 \to C_3$, respectively. Then we have the square diagram

\[
\begin{array}{ccc}
C_1 & \xrightarrow{G_1} & C_2 \\
H_1 \downarrow & \nearrow & \downarrow H_2 \\
C_3 & \xleftarrow{G_2} & C_4
\end{array}
\]

with $\xi: H_1 \circ G_1 \Rightarrow G_2 \circ H_2$ provided by the composition

\[
H_1 \circ G_1 \\
\eta_2 \circ H_1 \circ G_1 \\
G_2 \circ F_2 \circ H_1 \circ G_1 \\
G_2 \circ H_2 \circ F_1 \circ G_1 \\
G_2 \circ H_2 \circ G_1
\]

\[
0 \xrightarrow{0} (\ker \circ \eta^j)_F \xrightarrow{\eta_F} F \xrightarrow{\eta_F} j_* j^{-1} F \\
0 \xrightarrow{0} (\ker \circ \eta^j \circ i_* \circ i^{-1})_F \xrightarrow{(\eta' \circ \eta^{-1})_F} F \xrightarrow{(\eta' \circ \eta^{-1})_F} j_* j^{-1} i_* i^{-1} F
\]

**Definition B.2 (Dual Beck–Chevalley Condition).** We say that the square diagram (B.2) satisfies the *dual Beck–Chevalley condition* if $\xi: H_1 \circ G_1 \Rightarrow G_2 \circ H_2$ is a natural isomorphism.

## C General Facts on Sheaves

Let $X$ be a topological space and let $\text{Sh}(X)$ be the category of sheaves on $X$ with values in the category of abelian groups. We write $D^+(X) := D^+(\text{Sh}(X))$ for the bounded below derived category of $\text{Sh}(X)$.

**Lemma C.1.** Let $i: U \hookrightarrow X$ and $j: A \hookrightarrow X$ be inclusions with $U$ open and $A \cup U = X$. If $F$ is a sheaf on $X$, then the naturally induced map on kernels

\[
0 \xrightarrow{0} (\ker \circ \eta^j)_F \xrightarrow{\eta_F} F \xrightarrow{\eta_F} j_* j^{-1} F \\
0 \xrightarrow{0} (\ker \circ \eta^j \circ i_* \circ i^{-1})_F \xrightarrow{(\eta' \circ \eta^{-1})_F} F \xrightarrow{(\eta' \circ \eta^{-1})_F} j_* j^{-1} i_* i^{-1} F
\]

is an isomorphism.

**Lemma C.2.** Let $U$ be an open subset of $X$, let $C := X \setminus U$, let $i: U \hookrightarrow X$ be the inclusion, and let $\text{Sh}(X,C)$ be the full subcategory of sheaves vanishing on $C$. We write $(-)_U: \text{Sh}(X) \to \text{Sh}(X,C)$, $F \mapsto F_U$ for the corresponding functor defined in [KS90, Page 93]. Then we have $i_* = (-)_U \circ i_*$ and moreover, the composition of adjunctions

\[
\begin{array}{ccc}
\text{Sh}(U,C) & \xrightarrow{(\cdot)_U} & \text{Sh}(X) \\
\downarrow i^* & & \downarrow \iota \\
\text{Sh}(U) & \xrightarrow{i_*} & \text{Sh}(X)
\end{array}
\]
yields an exact adjoint equivalence:

\[
\begin{array}{ccc}
\text{Sh}(U, C) & \xrightarrow{\iota^*} & \text{Sh}(U) \\
\text{Sh}(U) & \xleftarrow{\iota_*} & \text{Sh}(X)
\end{array}
\]

Now let \(U, V \subseteq X\) be open subsets, let \(i: U \hookrightarrow X\) be the inclusion, let \(A \subseteq X\) be a closed subset, let \(Z := V \cap A\), and suppose that \(Z \subseteq U\).

**Lemma C.3.** The commutative square

\[
\begin{array}{ccc}
\text{Sh}(U) & \xrightarrow{i^*} & \text{Sh}(X) \\
\Gamma_Z(U; \cdot) & \downarrow & \Gamma_Z(X; \cdot) \\
\text{Ab} & \xrightarrow{\text{Ab}} & \text{Ab}
\end{array}
\]

satisfies the Beck–Chevalley condition as defined in Definition [B.1] where Ab is the category of abelian groups.

**Proof.** Let \(F\) be a sheaf on \(X\). Then we have the composition of isomorphisms

\[
\Gamma_Z(X; F) \cong \Gamma_Z(V; F) \\
\cong \Gamma_Z(U \cap V; F) \\
= \Gamma_Z(U \cap V; F|_U) \\
\cong \Gamma_Z(U \cap V; i^{-1}F) \\
\cong \Gamma_Z(U; i^{-1}F),
\]

which is the mate of the identity natural transformation. \(\square\)

This lemma has the following corollary.

**Corollary C.4.** The square diagram

\[
\begin{array}{ccc}
D^+(U) & \xrightarrow{Ri^*} & D^+(X) \\
H_Z(U; \cdot) & \downarrow & H_Z(X; \cdot) \\
\text{Ab} & \xrightarrow{\text{Ab}} & \text{Ab}
\end{array}
\]

satisfies the Beck–Chevalley condition.

Now let \(\text{Sh}(X)\) be the category of \(F\)-linear sheaves on \(X\) for some fixed field \(F\), let \(F\) be an object of \(D^+(X) := D^+(\text{Sh}(X))\), let \(n \in \mathbb{Z}\), let \(x \in X\) be a point of \(X\), and let \(S_x\) be the skyscraper sheaf at \(x\). We consider the map

\[
\text{Hom}_{D^-(X)}(F, S_x[-n]) \to \left( \lim_{x \in U} H^n(U; F) \right)^* \quad (C.1)
\]

which takes any homomorphism \(\psi: F \to S_x[-n]\) to the family of maps \(\{H^n(U; \psi): H^n(U; F) \to F \mid x \in U\}\) and then to the naturally induced map of type \(\lim_{x \in U} H^n(U; F) \to F\).
Lemma C.5. The map (C.1) is an isomorphism.

Proof. As $S_x$ is injective, as $H^n(U; S_x[-n]) \cong \{0\}$ for all opens $U$ excluding $x$, and by the exactness of the dual space functor (or the UCT for cohomology) the map (C.1) is an isomorphism.

\[\text{Lemma C.6. If } F \text{ is a flabby sheaf on } X, \text{ then the commutative diagram} \]

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & \Gamma_{\overline{A}_3}(X_3; F) & \longrightarrow & \Gamma_{\overline{A}_1}(X_1; F) \oplus \Gamma_{\overline{A}_2}(X_2; F) & \overset{(1-1)}{\longrightarrow} & \Gamma_{\overline{A}_0}(X_0; F) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \Gamma(X_3; F) & \longrightarrow & \Gamma(X_1; F) \oplus \Gamma(X_2; F) & \overset{(1-1)}{\longrightarrow} & \Gamma(X_0; F) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \Gamma(A_3; F) & \longrightarrow & \Gamma(A_1; F) \oplus \Gamma(A_2; F) & \overset{(1-1)}{\longrightarrow} & \Gamma(A_0; F) & \longrightarrow & 0 \\
0 & & 0 & & 0 & & 0 & & 0
\end{array}
\]

has exact rows and columns, where $\overline{A}_i$ is the complement $X \setminus A_i$ of $A_i$ in $X$ for $i = 0, 1, 2, 3$.

Proof. The columns are exact as $F$ is a flabby sheaf and by the definition \cite{KS90} (2.3.14) of $\Gamma_{\overline{A}_i}(X_i; F)$ for $i = 0, 1, 2, 3$. The two bottom rows are exact by the sheaf condition and as $F$ is a flabby sheaf. By the nine lemma the top row is exact as well.

\[\text{Corollary C.7. If } G \text{ is a complex of flabby sheaves on } X, \text{ then we have the short exact sequence} \]

\[
0 \longrightarrow \Gamma_{\overline{A}_3}(X_3; G) \longrightarrow \Gamma_{\overline{A}_1}(X_1; G) \oplus \Gamma_{\overline{A}_2}(X_2; G) \overset{(1-1)}{\longrightarrow} \Gamma_{\overline{A}_0}(X_0; G) \longrightarrow 0
\]

of cochain complexes.
Proposition C.8 (Mayer–Vietoris for Local Sheaf Cohomology). If $F$ is an object of $D^+(X)$, then there is a long exact sequence

$$0 \rightarrow H^n_{\mathcal{A}_3}(X_3; F) \rightarrow H^n_{\mathcal{A}_1}(X_1; F) \oplus H^n_{\mathcal{A}_2}(X_2; F) \rightarrow H^n_{\mathcal{A}_0}(X_0; F) \rightarrow \cdots$$

for some $n \in \mathbb{Z}$.

Proof. This follows directly from Corollary C.7 and the zig-zag lemma. 

D Presentable Objects

Let $\mathcal{C}$ be an abelian category and let $\mathcal{J}$ be a full replete additive subcategory of projectives in $\mathcal{C}$.

Definition D.1. We say that an object $X$ of $\mathcal{C}$ is $\mathcal{J}$-presentable if there is an exact sequence (called presentation)

$$Q \rightarrow P \rightarrow X \rightarrow 0$$

with $P$ and $Q$ in $\mathcal{J}$. We write $\text{pres}(\mathcal{J})$ for the full subcategory of $\mathcal{J}$-presentable objects in $\mathcal{C}$.

Next we show that the category of $\mathcal{J}$-presentable objects of $\mathcal{C}$ is closed under cokernels. This Lemma D.2 and the post-ceding Lemma D.6 provide a slight generalization of the very first result from [Kra07, Section 4.1] with essentially the same proof. Another closely related result is [KS06, Exercise 8.23].

Lemma D.2. The cokernel of a homomorphism between $\mathcal{J}$-presentable objects is again $\mathcal{J}$-presentable.

Proof. Let

$$X \xrightarrow{\varphi} Y \xrightarrow{\psi} Z \rightarrow 0$$

be an exact sequence with $X$ and $Y$ being $\mathcal{J}$-presentable. We choose presentations

$$P_1 \rightarrow P_0 \rightarrow X \rightarrow 0$$

and

$$Q_1 \xrightarrow{\delta} Q_0 \xrightarrow{\epsilon} Y \rightarrow 0$$



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as in Definition D.1. As $P_0$ is projective in $C$ there is a homomorphism $\tilde{\varphi}: P_0 \to Q_0$ such that the square

$$
\begin{array}{ccc}
P_0 & \longrightarrow & Q_0 \\
\downarrow & & \downarrow \\
X & \longrightarrow & Y
\end{array}
$$

commutes. We aim to show that the sequence

$$
P_1 \oplus Q_1 \xrightarrow{(\tilde{\varphi},\delta)^\flat} Q_0 \xrightarrow{\psi \circ \epsilon} Z \to 0
$$

is exact. To this end, let $\xi: Q_0 \to W$ be a homomorphism such that

$$
\xi \circ (\tilde{\varphi},\delta)^\flat = 0. \quad (D.2)
$$

We consider the commutative diagram

$$
\begin{array}{ccc}
Q_1 & \xrightarrow{\delta} & P_1 \oplus Q_1 \\
\downarrow & & \downarrow \quad (\tilde{\varphi},\delta)^\flat \\
P_0 & \xrightarrow{\tilde{\varphi}} & Q_0 \\
\downarrow & \downarrow \epsilon & \downarrow \psi \\
X & \xrightarrow{\varphi} & Y & \xrightarrow{\psi} & Z \\
& & & \xi & \\
& & & W
\end{array}
$$

(D.3)

Now (D.2) implies in particular that $\xi \circ \delta = 0$, hence there is a unique natural transformation $\xi': Y \to W$ such that $\xi' \circ \epsilon = \xi$ as indicated in (D.3). Now

$$
\xi' \circ \epsilon \circ \tilde{\varphi} = \xi \circ \tilde{\varphi} = 0
$$

by (D.2). Moreover, as the vertical arrow on the left hand side of (D.3) is an epimorphism $\xi' \circ \varphi = 0$ as well. By the exactness of (D.1) there is a unique natural transformation $\xi'': Q_0 \to W$ such that $\xi'' \circ \psi \circ \epsilon = \xi'$ as indicated in (D.3).

**Corollary D.3.** A homomorphism of $\mathcal{J}$-presentable objects is an epimorphism in $\text{pres}(\mathcal{J})$ iff it is an epimorphism in $C$.

This corollary has yet another corollary.

**Corollary D.4.** A $\mathcal{J}$-presentable object $P$, which is projective in $C$, is also projective in $\text{pres}(\mathcal{J})$.

**Corollary D.5.** If any $\mathcal{J}$-presentable object $X$ of $C$ admits a projective cover $P \to X$ by an object $P$ in $\mathcal{J}$, then $\mathcal{J}$ is the subcategory of projectives in $\text{pres}(\mathcal{J})$.  

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Proof. Suppose $Q$ is an object of $\mathcal{J}$. Then $Q$ is projective in $C$, hence it is projective in $\text{pres}(\mathcal{J})$ by Corollary D.4. Now suppose $Q$ is projective in $\text{pres}(\mathcal{J})$ and let $\varphi: P \to Q$ be a projective cover with $P$ in $\mathcal{J}$. Then $P$ is projective in $\text{pres}(\mathcal{J})$ by Corollary D.4, hence $\varphi: P \to Q$ also is a projective cover in $\text{pres}(\mathcal{J})$. Moreover, the identity $\text{id}_Q: Q \to Q$ is a projective cover as well and thus $P \cong Q$ by the uniqueness of projective covers [Kra15, Corollary 3.5].

Lemma D.6. Suppose that for any homomorphism $\psi: Q \to P$ with $Q$ and $P$ in $\mathcal{J}$ there is an exact sequence

$$R \to Q \xrightarrow{\psi} P$$

with $R$ in $\mathcal{J}$. Then the full subcategory $\text{pres}(\mathcal{J})$ of $C$ is an abelian subcategory.

Proof. By Lemma D.2 the cokernel of a natural transformation of $\mathcal{J}$-presentable objects is again a $\mathcal{J}$-presentable object. Thus, it suffices to show that the kernel of a homomorphism of $\mathcal{J}$-presentable objects is again $\mathcal{J}$-presentable. To this end, let

$$0 \to X \xrightarrow{\varphi} Y \xrightarrow{\psi} Z$$

be an exact sequence with $Y$ and $Z$ being $\mathcal{J}$-presentable. By Definition D.1 we may choose presentations

$$Q_1 \xrightarrow{\alpha} Q_0 \to Y \to 0$$

and

$$R_1 \xrightarrow{\delta} R_0 \to Z \to 0$$

with $Q_0$, $Q_1$, $R_0$, and $R_1$ objects of $\mathcal{J}$. Since $Q_0$ is projective there is a homomorphism $\hat{\psi}: Q_0 \to R_0$ such that the square

$$
\begin{array}{ccc}
Q_0 & \xrightarrow{\hat{\psi}} & R_0 \\
\downarrow & & \downarrow \\
Y & \xrightarrow{\psi} & Z
\end{array}
$$

commutes. By our assumptions on $\mathcal{J}$ there is an exact sequence

$$P_0 \xrightarrow{\kappa_0} Q_0 \oplus R_1 \xrightarrow{\varphi, \delta} R_0.$$  \hfill (D.5)

Using this assumption on $\mathcal{J}$ once more we also obtain an exact sequence

$$P_1 \xrightarrow{\kappa_1} P_0 \oplus Q_1 \xrightarrow{\text{pr}_1, \varphi, \delta} Q_0,$$  \hfill (D.6)

where $\text{pr}_1: Q_0 \oplus R_1 \to Q_0$ is the projection to the first summand $Q_0$. We consider the
where \(pr_i\) denotes the corresponding projection for \(i = 1, 2\). As the center column of (D.7) is a complex and as (D.4) is exact, the vertical dashed arrow on the left hand side of (D.7) exists as indicated. Moreover, the center and the right column of (D.7) are exact. Using the exact sequences (D.5) and (D.6) it follows from a diagram chase in (D.7) that the left column is exact as well, hence \(X\) is \(\mathcal{J}\)-presentable.

\[ \begin{array}{c}
P_1 \\ \downarrow \text{pr}_1 \circ \kappa_1 \\
\cdots \\
X \\
\downarrow \varphi \\
0 \\
\end{array} \quad \begin{array}{c}
\begin{array}{cccc}
\text{pr}_2 \\
\alpha \\
\psi \\
\end{array} \\
\begin{array}{cccc}
\downarrow \\
\downarrow \\
\downarrow \\
\end{array} \\
\begin{array}{cccc}
Q_1 \\
Q_0 \\
Y \\
Z \\
\end{array} \\
\end{array} \quad \begin{array}{c}
\begin{array}{c}
\text{pr}_1 \circ \kappa_1 \\
\varphi \\
\psi \\
\end{array} \\
R_0 \\
\end{array} \quad (D.7) \]

\[ \begin{array}{c}
0 \\
0 \\
0 \\
\end{array} \]

\section{Abelianization of Triangulated Categories}

The following Proposition E.1 is a slight generalization of the first lemma from [Kra07, Section 4.2] with essentially the same proof.

\textbf{Proposition E.1.} Let \(\mathcal{C}\) be an abelian category, let \(\mathcal{T}\) be a triangulated category, and let \(F: \mathcal{T} \to \mathcal{C}\) be a fully faithful cohomological functor, such that \(F(X)\) is projective for all objects \(X\) of \(\mathcal{T}\). Moreover, let \(\mathcal{J}\) be the essential image of \(F: \mathcal{T} \to \mathcal{C}\). Then the category \(\text{pres}(\mathcal{J})\) of \(\mathcal{J}\)-presentable objects is an abelian subcategory of \(\mathcal{C}\) and the functor

\[ \mathcal{T} \to \text{pres}(\mathcal{J}), \left\{ \begin{array}{c}
X \mapsto F(X) \\
\varphi \mapsto F(\varphi)
\end{array} \right. \]

is the abelianization of \(\mathcal{T}\). In other words, the precomposition functor \((-) \circ F\) from the category of exact functors \(\text{pres}(\mathcal{J}) \to \mathcal{A}\) to the category of cohomological functors \(\mathcal{T} \to \mathcal{A}\) is an equivalence of categories for any abelian category \(\mathcal{A}\).

\textbf{Proof.} Now suppose we have a homomorphism \(\psi: Q \to P\) in \(\mathcal{C}\) with \(Q\) and \(P\) in \(\mathcal{J}\). Since \(F: \mathcal{T} \to \mathcal{C}\) is fully faithful there is a homomorphism \(\tilde{\psi}: \tilde{Q} \to P\) with \(F(\tilde{\psi}) \cong \psi\). Moreover, there exists a distinguished triangle

\[ \tilde{R} \xrightarrow{\varphi} \tilde{Q} \xrightarrow{\psi} \tilde{P} \to \tilde{R}[1] \]

in \(\mathcal{T}\) by TR2 and TR3. Writing \(R := F(\tilde{R})\) and \(\varphi: R \xrightarrow{F(\varphi)} F(\tilde{Q}) \xrightarrow{\sim} Q\) we obtain the exact sequence

\[ R \xrightarrow{\varphi} Q \xrightarrow{\psi} P, \]

\[ \]
since \( F: \mathcal{T} \to \mathcal{C} \) is cohomological. Thus, \( \text{pres}(\mathcal{J}) \) is an abelian subcategory of \( \mathcal{C} \) by Lemma D.6. Now let \( \mathcal{A} \) be another abelian category and let \( G: \mathcal{T} \to \mathcal{A} \) be a cohomological functor. We construct a preimage \( \overline{G}: \text{pres}(\mathcal{J}) \to \mathcal{A} \) of \( G \) under \((-) \circ F \) as follows. Suppose \( X \) is \( \mathcal{J} \)-presentable. As \( F: \mathcal{T} \to \mathcal{C} \) is fully faithful we may choose a presentation

\[
F(P_1) \xrightarrow{F(\delta)} F(P_0) \to X \to 0
\]

with \( P_0 \) and \( P_1 \) objects of \( \mathcal{T} \). With this we define \( \overline{G}(X) \) to be the cokernel of \( G(\delta): G(P_1) \to G(P_0) \). It remains to show that \( \overline{G}: \text{pres}(\mathcal{J}) \to \mathcal{A} \) is exact. To this end, let

\[
0 \to X \to Y \to Z \to 0
\]

be a short exact sequence of \( \mathcal{J} \)-presentable objects in \( \mathcal{C} \). Again, we may choose presentations

\[
F(P_1) \to F(P_0) \to X \to 0
\]

and

\[
F(Q_1) \xrightarrow{\delta} F(Q_0) \xrightarrow{\epsilon} Z \to 0
\] (E.1)

since \( F: \mathcal{T} \to \mathcal{C} \) is fully faithful. As \( F(Q_0) \) is projective in \( \mathcal{C} \) there is a homomorphism \( \sigma: F(Q_0) \to Y \) such that the diagram

\[
\begin{array}{ccc}
F(Q_0) & \xrightarrow{\sigma} & Y \\
\downarrow & & \downarrow \\
Z & \xrightarrow{\epsilon} & 0
\end{array}
\]

commutes. In conjunction with the nine lemma this yields a commutative diagram

\[
\begin{array}{ccccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\downarrow & & & & \downarrow & & & & \downarrow \\
0 & \to & U & \to & V & \to & W & \to & 0 \\
\downarrow & & & & \downarrow & & & & \downarrow \\
0 & \to & F(P_0) & \to & F(P_0 \oplus Q_0) & \to & F(Q_0) & \to & 0
\end{array}
\] (E.2)

with exact rows and columns. By the exactness of [E.1] at \( F(Q_0) \) there is a homomorphism \( \xi: F(Q_1) \to W \) such that the diagram

\[
\begin{array}{ccc}
0 & \to & W \\
\downarrow & & \downarrow \\
F(Q_1) & \xrightarrow{\delta} & F(Q_0)
\end{array}
\]
commutes. Moreover, as $F(Q_1)$ is projective there is a homomorphism $\sigma': F(Q_0) \to V$ such that the diagram

$$
\begin{array}{ccc}
F(Q_1) & \xrightarrow{\sigma'} & V \\
\downarrow & & \downarrow \\
W & \xrightarrow{\varepsilon} & 0
\end{array}
$$

commutes. In conjunction with (E.2) we obtain the commutative diagram

$$
\begin{array}{ccc}
0 & \to & F(P_1) & \to & F(P_1 + Q_1) & \to & F(Q_1) & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & F(P_0) & \to & F(P_0 + Q_0) & \to & F(Q_0) & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & X & \to & Y & \to & Z & \to & 0
\end{array}
$$

(E.3)

with exact rows and columns and the upper two rows split-exact. As $G: T \to A$ is cohomological it is an additive functor in particular. As a result the functor $\overline{G}: \text{pres}(J) \to A$ maps the upper two rows of (E.3) to short exact sequences in $A$. Now the cokernel seen as a functor from the category of homomorphisms in $A$ to $A$ is right-exact, hence the sequence

$$
\overline{G}(X) \to \overline{G}(Y) \to \overline{G}(Z) \to 0
$$

is exact. Finally we show the exactness of

$$
0 \to \overline{G}(X) \to \overline{G}(Y) \to \overline{G}(Z) \to 0
$$

(E.4)

as well. As $F: T \to C$ is cohomological we may use the diagram (E.3) to obtain the commutative diagram

$$
\begin{array}{ccc}
0 & \to & 0 & \to & 0 & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & F(P_1[1]) & \to & F(P_1[1] + Q_1[1]) & \to & F(Q_1[1]) & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & F(P_0[1]) & \to & F(P_0[1] + Q_0[1]) & \to & F(Q_0[1]) & \to & 0
\end{array}
$$

(E.5)

with exact rows and columns and the lower two rows split-exact. As $G: T \to A$ additive the functor $\overline{G}: \text{pres}(J) \to A$ maps the lower two rows of (E.5) to short exact sequences in $A$. Moreover, as the kernel is left-exact as a functor from the category of homomorphisms in $A$ to $A$ and as $G: T \to A$ is cohomological the sequence (E.4) is exact.