Additivity in Isotropic Quantum Spin Channels

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Abstract

We prove additivity of the minimum output entropy and the Holevo capacity for rotationally invariant quantum channels acting on spin–1/2 and spin–1 systems. The physical significance of these channels and their relations to other known channels is also discussed.

1 Introduction

Information is transmitted through channels which are inherently noisy. A measure of the efficiency of a channel is hence given by the maximal rate at which information can be reliably (that is, without distortion) transferred through it. For a classical communications channel, this rate defines its capacity. In contrast, there are various different capacities of a quantum channel. This is because the structure of a quantum channel, $\Phi$, is much more complex than its classical counterpart, and in addition, there is a lot of flexibility in its use. A quantum channel can be used to transmit either classical or quantum information; there are various ways of encoding (decoding) the input (output) of such a channel; and it can be used in conjunction with additional resources e.g. shared entanglement between the sender and the receiver. There are various quantities characterizing a quantum channel, which can be used to describe how efficiently information can be transmitted through it. Two such quantities are the minimum output entropy $h(\Phi)$ and the Holevo capacity $\chi(\Phi)$ of the channel.

Mathematically, a quantum channel $\Phi$ is defined as a completely positive trace-preserving map on the set of density matrices $\rho$ acting on a Hilbert space $\mathcal{H}$. This definition can be extended to the $\ast$-algebra $\mathcal{B}(\mathcal{H})$ of all complex $d \times d$ matrices, $d < \infty$ being the dimension of $\mathcal{H}$. A
quantum channel $\Phi$ is said to be memoryless if its repeated use is given by the tensor product 
$\Phi \otimes \Phi \otimes \cdots \otimes \Phi$, since in this case the action of each use of the channel is identical, and it is 
independent for different uses.

The minimum output entropy of a channel $\Phi$ is defined as

$$h(\Phi) := \min_{\rho} S(\Phi(\rho)),$$  \hspace{1cm} (1)

where the minimization is over all possible input states of the channel. Here $S(\sigma) = -\text{tr}(\sigma \log \sigma)$, 
the von Neumann entropy of the output $\sigma = \Phi(\rho)$ of the channel. The von Neumann entropy 
$S(\sigma)$ is zero if and only if $\sigma$ is a pure state density matrix. Hence, the minimum output entropy 
gives a measure of the extent to which the output of a channel deviates from a pure state.

The Holevo capacity of a memoryless channel $\Phi$ is defined as

$$\chi(\Phi) = \max_{p_j, \rho_j} \left[ S\left( \sum_j p_j \rho_j \right) - \sum_j p_j S\left( \Phi(\rho_j) \right) \right],$$  \hspace{1cm} (2)

where the maximum is over all finite ensembles of states $\rho_j$ taken with probabilities $p_j$. It is known to represent the asymptotic capacity of the quantum channel for transmission of classical information through it, under the restriction that each input state used can be expressed as a tensor product (see e.g. \cite{9}).

It is conjectured (see e.g. \cite{10}) that $h(\Phi)$ and $\chi(\Phi)$ are additive for general product channels, i.e., for any two channels $\Phi_1$ and $\Phi_2$

$$h(\Phi_1 \otimes \Phi_2) = h(\Phi_1) + h(\Phi_2),$$  \hspace{1cm} (3)

and

$$\chi(\Phi_1 \otimes \Phi_2) = \chi(\Phi_1) + \chi(\Phi_2).$$  \hspace{1cm} (4)

The product channel $\Phi_1 \otimes \Phi_2$, for $\Phi_1 = \Phi_2 = \Phi$, describes two consecutive uses of a memoryless channel $\Phi$. In this case the additivity relations (3) and (4) reduce to

$$h(\Phi \otimes \Phi) = 2h(\Phi) \quad (a); \quad \chi(\Phi \otimes \Phi) = 2\chi(\Phi). \quad (b)$$  \hspace{1cm} (5)

These have a natural generalisation to additivity under $n$ uses of the channel:

$$h(\Phi^{\otimes n}) = nh(\Phi) \quad ; \quad \chi(\Phi^{\otimes n}) = n\chi(\Phi).$$  \hspace{1cm} (6)

In this paper we prove the additivity relations (5) for isotropic spin channels. These were first 
introduced in the context of Quantum Information Theory in \cite{3}, and are given by the following 
bistochastic completely positive maps acting on spin–$s$ systems:

$$\Phi_s(\rho) = \frac{1}{s(s+1)} \sum_{k=1}^{3} S_k \rho S_k.$$  \hspace{1cm} (7)
Here $S_1, S_2, S_3$ are spin operators which satisfy the commutation relations $[S_i, S_j] = i\epsilon_{ijk} S_k$, and provide an irreducible representation $\{U_s(n), n \in \mathbb{R}^3, |n| \leq 1\}$ of the SU(2) group on a Hilbert space of dimension $d = 2s + 1$. Moreover, they belong to a family of channels referred to as Casimir channels [8], which can be defined starting from any compact Lie group. Here we focus attention on the cases $s = 1/2$ and $s = 1$ of the channel $\Phi_s$.

The channel $\Phi_s$ satisfies the following covariance (or rotational invariance) relation

$$\Phi_s(U_s(n)\rho U_s^*(n)) = U_s(n)\Phi_s(\rho)U_s^*(n).$$

This property leads to certain simplifications in proving additivity relations for these channels. This is because for channels satisfying (3), the minimum output entropy $h(\Phi_s)$ and the Holevo capacity $\chi(\Phi_s)$ are linearly related [11]:

$$\chi(\Phi_s) = \log d - h(\Phi_s).$$

Therefore, the additivity (5a) of $h(\Phi_s)$ implies the additivity (5b) for such channels.

For $s = 1/2$, the above channel reduces to a unital qubit channel:

$$\Phi(\rho) := \Phi_{1/2}(\rho) = \frac{1}{3} \sum_{k=1}^{3} \sigma_k \rho \sigma_k,$$

where $\sigma_1, \sigma_2, \sigma_3$ are the three Pauli matrices $\sigma_x, \sigma_y$ and $\sigma_z$ respectively. The additivity of the minimum output entropy and the Holevo capacity for unital qubit channels was proved in [12]. Our paper gives an alternative proof of the additivity relations (5) for the product channel $\Phi \otimes \Phi$, based on the method developed in [4]. It provides an explicit illustration of the sufficient condition for this additivity, which was given in [5] (see Section 2 for more details).

The channel (9) can be viewed as the familiar depolarizing channel, in the limit in which a qubit is subject to errors represented by the Pauli matrices $\sigma_x, \sigma_y$ and $\sigma_z$, each with probability $1/3$. The channel has yet another significance. It is known that an unknown quantum state cannot be perfectly copied (no–cloning theorem, [6]) or perfectly complemented [7]. In [7], Buzek et al. defined the Universal–NOT (U–NOT) gate, which generates $M$ output qubits in a state which is as close as possible to the perfect complement of the states of $N$ identically prepared input qubits. The channel (9) can be shown to provide the optimal realization of the U–NOT gate for $N = 1$. This can be seen as follows. The state of a qubit is given in the Bloch representation as $\rho = \frac{1}{2}(I + \vec{s} . \vec{\sigma})$, where $\vec{s} = (s_1, s_2, s_3)$ is its Bloch vector. The angle between the Bloch vectors of the state $\rho$ and its complement $\rho^\perp$ is $\pi$. In fact, if $\rho$ is a pure state, the points corresponding to $\rho$ and $\rho^\perp$ are antipodes on the Bloch sphere. It follows from [7] that for $N = 1$, the optimal U–NOT gate converts the input state $\rho$ of a qubit to the state

$$\rho^\perp + \frac{1}{3} I \equiv \frac{1}{2} \left(I + \vec{s}' . \vec{\sigma}\right),$$

where $\vec{s}' = -(1/3)\vec{s}$. In other words the gate causes a shrinking of the Bloch vector (by a factor $1/3$) in addition to a rotation by an angle $\pi$. Using the Bloch representation of $\rho$ we find that the output $\Phi(\rho)$ of the channel (9) is indeed given by (10).
The channel (7) for \( s = 1 \) is given by

\[
\Phi_1(\rho) = \frac{1}{2} \sum_{k=1}^{3} S_k \rho S_k,
\]

where \( S_k, k = 1, 2, 3 \) are the corresponding spin operators. The underlying Hilbert space is three–dimensional. We show in Section 4 that \( \Phi_1 \) is equivalent to the channel \( \Phi_d \) defined by

\[
\Phi_d(\mu) := \frac{1}{d-1} (I \text{Tr} \mu - \mu^T), \tag{12}
\]

for the choice \( d = 3 \). Here \( \mu \in B(\mathcal{H}) \), where \( \mathcal{H} \simeq \mathbb{C}^d \), \( \mu^T \) denotes the transpose of the matrix \( \mu \), and \( I \) denotes the \( d \times d \) unit matrix. The channels \( \Phi_1 \) and \( \Phi_3 \) are equivalent in the sense that, for any \( \mu \in \mathbb{C}^3 \), \( \Phi_1(\mu) = \Phi_3(\mu) \). The additivity relations (6) for the minimum output entropy and the Holevo capacity of the channel \( \Phi_d \), for any arbitrary dimension \( 2 \leq d < \infty \), was proved in [14, 15]. Additivity (5) for the product channel was also proved in [4]. The equivalence between the channels \( \Phi_1 \) and \( \Phi_3 \) therefore imply the validity of these additivity relations for \( \Phi_1 \) as well.

The paper is organized as follows. In Section 2 we briefly summarize the key idea behind the method developed in [4], with emphasis on the sufficient condition for additivity of the minimum output entropy of product channels. In Section 3 we give an explicit proof of the additivity relation (6) for the channel (9), i.e., for the case \( s = 1/2 \). In Section 4 the equivalence between the channels \( \Phi_1 \) and \( \Phi_3 \) is proved by two different methods. The additivity relations (6) follow as a consequence.

## 2 A sketch of the method

Our aim is to prove the additivity relations (6) for product channels \( \Phi \otimes \Phi \), for the cases in which \( \Phi \) is defined through (7) and (9). The concavity of the von Neumann entropy implies that the minimum value of the output entropy is necessarily achieved for pure input states, which correspond to the extreme points of the convex set of input states. This allows the minimisations in the definitions (11) of the minimum output entropy to be restricted to pure input states alone. Hence, the minimum output entropies of a single channel \( \Phi \) on \( B(\mathcal{H}) \), and the product channel \( \Phi \otimes \Phi \) on \( B(\mathcal{H} \otimes \mathcal{H}) \), is equivalently given by

\[
h(\Phi) = \min_{|\psi\rangle \in \mathcal{H}, ||\psi||=1} S(\Phi(|\psi\rangle\langle\psi|)),
\]

\[
h(\Phi \otimes \Phi) = \min_{|\psi_{12}\rangle \in \mathcal{H}_1 \otimes \mathcal{H}_2, ||\psi_{12}||=1} S((\Phi \otimes \Phi)(|\psi_{12}\rangle\langle\psi_{12}|)).
\]

Here \( |\psi_{12}\rangle\langle\psi_{12}| \) is a pure state of a bipartite system with the Hilbert space \( \mathcal{H}_1 \otimes \mathcal{H}_2 \), where \( \mathcal{H}_1 \simeq \mathcal{H}_2 \simeq \mathcal{H} \). In order to prove (6), it is sufficient to show that the minimum in (14) is achieved on unentangled vectors \( |\psi_{12}\rangle = |\psi_1\rangle \otimes |\psi_2\rangle \in \mathcal{H}_1 \otimes \mathcal{H}_2 \). This follows from the fact that the von Neumann entropy of a tensor product state is additive.
The starting point of our analysis is the Schmidt decomposition

$$|ψ_{12}⟩ = \sum_{α=1}^{d} \sqrt{λ_α}|α; 1⟩|α; 2⟩,$$

where \(\{|α; i⟩\}\) denotes an orthonormal basis in the Hilbert space \(H_i, i = 1, 2\). The Schmidt coefficients form a probability distribution:

$$λ_α ≥ 0 ; \sum_{α=1}^{d} λ_α = 1,$$

and hence the Schmidt vector \(λ := (λ_1, ..., λ_d)\) varies in the \((d - 1)\)-dimensional simplex \(Σ_d\), defined by these constraints.

In terms of the Schmidt decomposition, the input to the product channel is given by

$$|ψ_{12}⟩⟨ψ_{12}| = \sum_{α, β=1}^{d} \sqrt{λ_α λ_β}|α; 1⟩⟨β; 1| ⊗ |α; 2⟩⟨β; 2|, \quad (16)$$

and its output is given by

$$σ_{12}(λ) := (Φ ⊗ Φ)(|ψ_{12}⟩⟨ψ_{12}|) = \sum_{α, β=1}^{d} \sqrt{λ_α λ_β}Φ(|α; 1⟩⟨β; 1|) ⊗ Φ(|α; 2⟩⟨β; 2|). \quad (17)$$

Note that the extreme points (vertices) of \(Σ_d\) correspond to unentangled vectors \(|ψ_{12}⟩ = |ψ_1⟩ ⊗ |ψ_2⟩ \in H ⊗ H\). Hence, a sufficient condition for the additivity of the minimum output entropy can be stated as follows:

For every choice of the bases \(\{|α; 1⟩\}\) and \(\{|α; 2⟩\}\) in (17), the von Neumann entropy, \(S(σ_{12}(λ))\), of the channel output \(σ_{12}(λ)\), attains its minimum at the vertices of \(Σ_d\).

\( (18)\)

We show in Section 3 that this sufficient condition for additivity indeed holds for the channel \(Φ_{1/2}(ρ)\) (defined by (9)). It is a consequence of the fact that in this case \(S(σ_{12}(λ))\) is a concave function of the Schmidt vector \(λ\). For the channel \(Φ_1(ρ)\) (defined by (11)) as well the sufficient condition (18) is satisfied and the additivity relations (5) hold. This is proved in Section 4 by showing that \(Φ_1\) is equivalent to a channel for which the additivity relations have been previously established.
3 Additivity for the channel $\Phi_{1/2}$

In this section we explicitly prove the validity of the condition (18), and hence of the additivity relations (5), for the channel $\Phi_{1/2}$ defined by (9).

Let us first compute the von Neumann entropy $S[\Phi] \equiv S(\Phi(\rho))$ of the output of a single channel $\Phi := \Phi_{1/2}$:

$$
\Phi(\rho) = \frac{1}{3} \sum_{k=1}^{3} \sigma_k \rho \sigma_k
= \frac{1}{3} \sum_{k=1}^{3} \sigma_k \left( \frac{1}{2} \sum_{i=0}^{3} s_i \sigma_i \right) \sigma_k
= \frac{1}{2} - \frac{1}{6} \sum_{i=1}^{3} s_i \sigma_i,
$$

(19)

The last line follows from the fact that for $i = 1, 2, 3$,

$$
\sum_{k=1}^{3} \sigma_k \sigma_i \sigma_k = -\sigma_i.
$$

(20)

The above identity (20) can be easily obtained by explicit computation. Hence, $\Phi(\rho)$ is given by the matrix

$$
\begin{pmatrix}
\frac{1}{2} - \frac{s_3}{6} & -\frac{s_1}{6} + \frac{i s_2}{6} \\
-\frac{s_1}{6} - \frac{i s_2}{6} & \frac{1}{2} + \frac{s_3}{6}
\end{pmatrix}.
$$

(21)

To compute $S[\Phi]$ we need to compute the eigenvalues of the above matrix. The characteristic equation is given by

$$
\lambda^2 - \frac{1}{4} - \frac{1}{36} (s_1^2 + s_2^2 + s_3^2) = 0
$$

(22)

which reduces to

$$
\lambda^2 - \lambda + \frac{2}{9} = 0
$$

(23)

since $s_1^2 + s_2^2 + s_3^2 = 1$. The eigenvalues are found to be equal to 2/3 and 1/3. This yields

$$
S[\Phi] = \log 3 - \frac{2}{3} \log 2.
$$

(24)

Let us now calculate the entropy of the output $\sigma_{12}$ ($\lambda$) of the product channel $\Phi_{1/2} \otimes \Phi_{1/2}$ . Note that for the channel $\Phi_{1/2}$ (i.e., for $s = 1/2$), the covariance relation (8) reduces to invariance with respect to any arbitrary unitary transformation $U$:

$$
\Phi(U \rho U^*) = U \Phi(\rho) U^*.
$$

(25)

This allows us to choose for $\{ |\alpha; j \rangle ; j = 1, 2 \}$ in (15), the canonical basis in $\mathbb{C}^2$. Hence, we write

$$
|\psi_{12} \rangle = \sum_{\alpha=1}^{2} \sqrt{\lambda_\alpha} |\alpha \rangle |\alpha \rangle, \quad |\psi_{12} \rangle \langle \psi_{12} | = \sum_{\alpha, \beta=1}^{2} \sqrt{\lambda_\alpha \lambda_\beta} |\alpha \rangle \langle \beta | \otimes |\alpha \rangle \langle \beta |.
$$
and

$$\sigma_{12}(\lambda) = \sum_{\alpha,\beta=1}^{2} \sqrt{\lambda_{\alpha} \lambda_{\beta}} \Phi(\{\alpha\} \langle \beta\}) \otimes \Phi(\{\alpha\} \langle \beta\}).$$  \hspace{1cm} (26)$$

Let us first compute \( \Phi(\{\alpha\} \langle \beta\}). \) For this we express the basis vectors \(|\alpha\rangle\) in terms of the vectors \(|0\rangle\) and \(|1\rangle\) of the computational basis:

$$|\alpha\rangle = \sum_{i=0}^{1} U_{\alpha i} |i\rangle \; ; \; \langle \beta\| = \sum_{j=0}^{1} \langle j| U_{\beta j},$$

where \( U_{\gamma i} \) denote the elements of a unitary matrix and \( \overline{U_{\gamma i}} \) denotes their complex conjugates. Hence

$$\Phi(\{\alpha\} \langle \beta\}) = \frac{1}{3} \left[ \sum_{k=1}^{3} \sigma_{k} |\alpha\rangle \langle \beta\| \sigma_{k} = \frac{1}{3} \sum_{k=1}^{3} \sum_{i=0}^{1} \sum_{j=0}^{1} \sigma_{k} \{ U_{\alpha i} |i\rangle \langle j| U_{\beta j} \} \sigma_{k} = \frac{1}{3} \sum_{k=1}^{3} \sum_{i,j=0}^{1} \sigma_{k} |i\rangle \langle j| \sigma_{k} \right]$$

$$= \frac{1}{3} \left[ U_{\alpha 0} \overline{U_{\beta 0}} \left( \sum_{k=1}^{3} \sigma_{k} |0\rangle \langle 0| \sigma_{k} \right) + U_{\alpha 1} \overline{U_{\beta 1}} \left( \sum_{k=1}^{3} \sigma_{k} |1\rangle \langle 1| \sigma_{k} \right) \right.$$

$$\left. + U_{\alpha 0} \overline{U_{\beta 1}} \left( \sum_{k=1}^{3} \sigma_{k} |0\rangle \langle 1| \sigma_{k} \right) + U_{\alpha 1} \overline{U_{\beta 0}} \left( \sum_{k=1}^{3} \sigma_{k} |1\rangle \langle 0| \sigma_{k} \right) \right].$$

(27)

Now,

$$\sum_{k=1}^{3} \sigma_{k} |0\rangle \langle 0| \sigma_{k} = 2I - |0\rangle \langle 0| \; ; \; \sum_{k=1}^{3} \sigma_{k} |1\rangle \langle 1| \sigma_{k} = 2I - |1\rangle \langle 1|$$

$$\sum_{k=1}^{3} \sigma_{k} |0\rangle \langle 1| \sigma_{k} = -|0\rangle \langle 1| \; ; \; \sum_{k=1}^{3} \sigma_{k} |1\rangle \langle 0| \sigma_{k} = -|1\rangle \langle 0|,$$

(28)

Substituting the above relations in the RHS of (27) yields

$$\Phi(\{\alpha\} \langle \beta\}) = \frac{1}{3} \left[ 2I \left( U_{\alpha 0} \overline{U_{\beta 0}} + U_{\alpha 1} \overline{U_{\beta 1}} \right) - U_{\alpha 0} \overline{U_{\beta 0}} |0\rangle \langle 0| - U_{\alpha 1} \overline{U_{\beta 1}} |1\rangle \langle 1| \right.$$  

$$\left. - U_{\alpha 0} \overline{U_{\beta 1}} |0\rangle \langle 1| - U_{\alpha 1} \overline{U_{\beta 0}} |1\rangle \langle 0| \right]$$

$$= \frac{1}{3} \left[ 2I \delta_{\alpha \beta} - |\alpha\rangle \langle \beta| \right].$$

(29)

The last line follows from the relation

$$|\alpha\rangle \langle \beta\| = U_{\alpha 0} \overline{U_{\beta 0}} |0\rangle \langle 0| + U_{\alpha 1} \overline{U_{\beta 1}} |1\rangle \langle 1| + U_{\alpha 0} \overline{U_{\beta 1}} |0\rangle \langle 1| + U_{\alpha 1} \overline{U_{\beta 0}} |1\rangle \langle 0|. \hspace{1cm} (30)$$
Substituting (29) on the RHS of (26) we get

$$
\sigma_{12}(\lambda) = \frac{1}{9} \sum_{\alpha,\beta=1}^{2} \sqrt{\lambda_{\alpha} \lambda_{\beta}} (2I \delta_{\alpha\beta} - |\alpha\rangle \langle \beta|) \otimes (2I \delta_{\alpha\beta} - |\alpha\rangle \langle \beta|)
$$

$$
= \frac{1}{9} \left( \sum_{\alpha,\beta=1}^{2} |\alpha\beta\rangle \langle \alpha\beta| (4 - 2\lambda_{\alpha} - 2\lambda_{\beta}) + \sum_{\alpha,\beta=1}^{2} \sqrt{\lambda_{\alpha} \lambda_{\beta}} |\alpha\alpha\rangle \langle \beta\beta| \right).
$$

(31)

In the above we have used the completeness relations

$$
I = \sum_{\alpha=1}^{2} |\alpha\rangle \langle \alpha|, \quad I \otimes I = \sum_{\alpha,\beta=1}^{2} |\alpha\beta\rangle \langle \alpha\beta|.
$$

The matrix $9\sigma_{12}(\lambda)$ can be written in the form of a more general matrix

$$
A = \sum_{j=1}^{n} \mu_j |j\rangle \langle j| + \sum_{j,k=1}^{n} \sqrt{\eta_j \eta_k} |j\rangle \langle k|,
$$

by identifying $j$ with a pair $(\alpha, \beta)$ and setting

$$
\mu_j \equiv \mu_{\alpha\beta} = 4 - 2\lambda_{\alpha} - 2\lambda_{\beta}; \quad \eta_j \equiv \eta_{\alpha\beta} = \lambda_{\alpha} \delta_{\alpha\beta}, \quad \alpha, \beta = 1, 2.
$$

(32)

The eigenvalues of the matrix $9\sigma_{12}(\lambda)$ are the roots of the equation

$$
0 = \prod_{\alpha,\beta=1}^{\alpha \neq \beta} (4 - 2\lambda_{\alpha} - 2\lambda_{\beta} - \gamma) \left[ 1 + \sum_{\alpha',\beta'=1}^{2} \frac{\lambda_{\alpha'} \delta_{\alpha'\beta'}}{(4 - 2\lambda_{\alpha'} - 2\lambda_{\beta'} - \gamma)} \right]
$$

$$
= \prod_{\alpha,\beta=1}^{\alpha \neq \beta} (4 - 2\lambda_{\alpha} - 2\lambda_{\beta} - \gamma) \left[ \prod_{\alpha'=1}^{2} (4 - 4\lambda_{\alpha'} - \gamma) \right] \left[ 1 + \sum_{\alpha''=1}^{2} \frac{\lambda_{\alpha''}}{(4 - 4\lambda_{\alpha''} - \gamma)} \right].
$$

(33)

Eq. (33) yields the following equations:

$$
(4 - 2\lambda_{\alpha} - 2\lambda_{\beta} - \gamma) \equiv 2 - \gamma = 0, \quad \alpha \neq \beta, \quad \alpha, \beta = 1, 2.
$$

(34)

since the Schmidt coefficients satisfy $\lambda_{\alpha} + \lambda_{\beta} = 1$. Hence the matrix $9\sigma_{12}(\lambda)$ has 2 eigenvalues equal to 2 and the matrix $\sigma_{12}(\lambda)$ has two eigenvalues equal to $2/3$.

The roots of the equation

$$
\prod_{\alpha=1}^{2} (4 - 4\lambda_{\alpha} - \gamma) \left[ 1 + \sum_{\alpha'=1}^{d} \frac{\lambda_{\alpha'}}{(4 - 4\lambda_{\alpha'} - \gamma)} \right] = 0,
$$

(35)

give the remaining 2 eigenvalues of the matrix $9\sigma_{12}(\lambda)$. These are obtained as follows. Eq. (35) can be written as

$$
(4 - 4\lambda_{1} - \gamma)(4 - 4\lambda_{2} - \gamma) + \lambda_{1}(4 - 4\lambda_{2} - \gamma) + \lambda_{2}(4 - 4\lambda_{1} - \gamma) = 0,
$$

(36)
which reduces to the quadratic equation \( \gamma^2 - 5\gamma + (8\lambda_1\lambda_2 + 4) = 0 \). This has roots \( \left[ \frac{5}{2} \pm \left( \sqrt{\frac{9}{2} - 32\lambda_1\lambda_2} \right)/2 \right] \). Hence the corresponding eigenvalues of the matrix \( \sigma_{12}(\lambda) \) are \( \left[ \frac{5}{18} \pm \frac{\sqrt{9 - 32\lambda_1\lambda_2}}{18} \right] \).

The 4 eigenvalues of \( \sigma_{12}(\lambda) \) are therefore given by

\[
\begin{align*}
r_1 &= \frac{2}{9}, \quad r_2 = \frac{2}{9}, \quad r_3 = \frac{5}{18} + \frac{\sqrt{9 - 32\lambda_1\lambda_2}}{18}, \quad r_4 = \frac{5}{18} - \frac{\sqrt{9 - 32\lambda_1\lambda_2}}{18}. 
\end{align*}
\]

(37)

The output entropy \( S(\sigma_{12}(\lambda)) := S(\Phi_1 \otimes \Phi_2)(|\psi\rangle\langle\psi|) \) of the product channel is given by

\[
S(\sigma_{12}(\lambda)) = -\sum_{i=1}^{4} r_i \log r_i.
\]

Since \( \lambda_2 = 1 - \lambda_1 \) we write

\[
f_1(\lambda_1) \equiv r_3 := \frac{5}{18} + \frac{\sqrt{9 - 32\lambda_1(1 - \lambda_1)}}{18},
\]

and

\[
f_2(\lambda_1) \equiv r_4 := \frac{5}{18} - \frac{\sqrt{9 - 32\lambda_1(1 - \lambda_1)}}{18}.
\]

Hence,

\[
S(\sigma_{12}(\lambda)) \equiv S(\sigma_{12}(\lambda_1)) := T_1 + T_2(\lambda_1),
\]

(38)

where \( T_1 \) is a constant:

\[
T_1 = -(r_1 \log r_1 + r_2 \log r_2) = -\frac{4}{9} \log \left( \frac{2}{9} \right)
\]

(39)

and

\[
T_2(\lambda_1) = -(r_3 \log r_3 + r_4 \log r_4) = -(f_1(\lambda_1) \log f_1(\lambda_1) + f_2(\lambda_1) \log f_2(\lambda_1)).
\]

(40)

Note that \( f_1(\lambda_1) + f_2(\lambda_1) = 10/18 \) and \( f_2'(\lambda_1) = -f_1'(\lambda_1) \), where the prime denotes differentiation with respect to \( \lambda_1 \). The sufficient condition (18) is satisfied, if the output entropy \( S(\sigma_{12}(\lambda)) \) is a concave function of the Schmidt vector \( \lambda \). To prove concavity, it suffices to show that

\[
S''(\sigma_{12}(\lambda_1)) := \frac{d^2 S(\sigma_{12}(\lambda_1))}{d\lambda_1^2} < 0.
\]

We find that

\[
S''(\sigma_{12}(\lambda)) = T_2''(\lambda_1) = f_1''(\lambda_1) \log \left[ \frac{f_2(\lambda_1)}{f_1(\lambda_1)} \right] - \frac{5}{9} \left( f_1'(\lambda_1) \right)^2 \frac{1}{f_1(\lambda_1)f_2(\lambda_1)}.
\]

(41)

Now,

\[
f_1''(\lambda_1) = \frac{16}{9} (9 - 32\lambda_1(1 - \lambda_1))^{-3/2} > 0,
\]

(42)
whereas $\log\left[\frac{f_2(\lambda_1)}{f_1(\lambda_1)}\right] < 0$, since $f_2(\lambda_1) < f_1(\lambda_1)$. Hence, the first term on the RHS of (41) is negative. Moreover,

$$ f_1(\lambda_1)f_2(\lambda_1) = \left(\frac{5}{18}\right)^2 - \frac{1}{(18)^2} (9 - 32\lambda_1(1 - \lambda_1)) = \frac{1}{(18)^2} (16 + 32\lambda_1(1 - \lambda_1)) > 0. $$

Therefore $S''(\sigma_{12}(\lambda)) < 0$ and the output entropy is a concave function of the Schmidt coefficients. The concavity of $S(\sigma_{12}(\lambda))$ implies that it attains its minimal value at the vertices of the simplex defined by

$$ \lambda_1, \lambda_2 > 0 \quad ; \quad \lambda_1 + \lambda_2 = 1. $$

Hence the sufficient condition for additivity of the minimum output entropy is satisfied and the additivity relation (5a) therefore holds.

This can also be explicitly verified as follows. At any vertex of this simplex, we have $\lambda_1\lambda_2 = 0$, and hence the eigenvalues in (37) reduce to

$$ \frac{2}{9}, \frac{2}{9}, \frac{4}{9}, \frac{1}{9} $$

and the output entropy is

$$ S(\sigma_{12}(\lambda)) = -\left[\frac{2}{9}\log\frac{2}{9} + \frac{4}{9}\log\frac{4}{9} + \frac{1}{9}\log\frac{1}{9}\right] $$

$$ = 2\log 3 - \frac{4}{3}\log 2 \equiv 2 \times S[\Phi], \quad (43) $$

where $S[\Phi]$ is given by (24). This proves the additivity relation (5a).

As discussed in the Section 2, the covariance relation (25) implies that the Holevo capacity $\chi(\Phi)$ is also additive for this channel, i.e. (5b) holds.

4 Additivity for the channel $\Phi_1(\rho)$

In this section we study the channel

$$ \Phi_1(\rho) = \frac{1}{2} \sum_{k=1}^3 S_k \rho S_k. \quad (44) $$

In the basis $\{|s,m\}; s = 1, m = -1, 0, 1\}$, the spin operators $S_k$ are represented by the matrices

$$ S_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}; \quad S_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}; \quad S_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \quad (45) $$

However, if instead of the basis $\{|s,m\}$, we use the basis $\{|a\}, |b\}, |c\}$, where

$$ |a\rangle = -\frac{1}{\sqrt{2}} (|1\rangle - | - 1\rangle); \quad |b\rangle = \frac{i}{\sqrt{2}} (|1\rangle + | - 1\rangle); \quad |c\rangle = |0\rangle, $$

$$ S_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \quad (45) $$

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$$ S_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \quad (45) $$

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$$ |a\rangle = -\frac{1}{\sqrt{2}} (|1\rangle - | - 1\rangle); \quad |b\rangle = \frac{i}{\sqrt{2}} (|1\rangle + | - 1\rangle); \quad |c\rangle = |0\rangle, $$
the spin operators are represented by the following matrices:

\[ S'_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}; \quad S'_2 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}; \quad S'_3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{47} \]

which are just the matrices representing the infinitesimal generators of rotations on the components of a vector. It will prove to be convenient for our analysis to use the representation (47) of the spin operators.

Note that the matrices \( S'_k \) are related to the matrices \( S_1, S_2 \) and \( S_3 \) via a unitary transformation

\[ \vec{S}' = V \vec{S} V^*, \quad \text{where} \quad \vec{S}' = (S'_1, S'_2, S'_3); \quad \vec{S} = (S_1, S_2, S_3), \]

where

\[ V = \begin{pmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ -\frac{i}{\sqrt{2}} & 0 & -\frac{i}{\sqrt{2}} \\ 0 & 1 & 0 \end{pmatrix}. \tag{48} \]

The output \( \sigma_{12}(\lambda) \) of the product channel, in this case, is given by (17), with \( \Phi \) replaced by \( \Phi_1 \). To compute it, let us first evaluate \( \Phi(|\alpha; i\rangle \langle \beta|, i = 1, 2 \). For notational simplicity we omit the index \( i \) and express the basis vectors in terms of the basis \(|-1\rangle, |0\rangle, |1\rangle\) as follows:

\[ |\alpha\rangle = \sum_{i=-1}^{1} U_{\alpha i} |i\rangle; \quad \langle \beta| = \sum_{j=-1}^{1} \langle j| U_{\beta j}, \]

where once again \( U_{\gamma i} \) denote the elements of a unitary matrix. We obtain

\[ \Phi(|\alpha\rangle \langle \beta|) = \frac{1}{2} \sum_{i,j=-1}^{1} U_{\alpha i} U_{\beta j} \left( \sum_{k=1}^{3} S'_k |i\rangle \langle j| S'_k \right), \]

which is analogous to (27) of the previous section. Using the relations

\[ S'_1 |1\rangle = 0, \quad S'_1 |0\rangle = i |1\rangle - 1, \quad S'_1 |-1\rangle = -i |0\rangle, \]
\[ S'_2 |1\rangle = -i |1\rangle + 1, \quad S'_2 |0\rangle = 0, \quad S'_2 |-1\rangle = i |1\rangle, \]
\[ S'_3 |1\rangle = i |0\rangle, \quad S'_3 |0\rangle = -i |1\rangle, \quad S'_3 |-1\rangle = 0, \]

we obtain

\[ \Phi(|\alpha\rangle \langle \beta|) = \frac{1}{2} \left( \lambda \delta_{\alpha \beta} - |\alpha\rangle \langle \alpha| \right), \tag{49} \]

where the entries of the vector \( |\alpha\rangle \) are complex conjugates of the corresponding entries of vector \( |\alpha\rangle \). Note that the RHS of (49) is identical to the corresponding expression for the channel \( \tilde{\Phi}_d \) for the choice \( d = 3 \). This channel is defined by its action on any \( \mu \in \mathbb{C}^d \) as follows:

\[ \tilde{\Phi}_d(\mu) := \frac{1}{d-1} \left( \text{Tr} \mu - \mu^T \right), \tag{50} \]
and was studied in detail [4]. In the latter, $|\alpha\rangle\langle\alpha|$ was replaced by $|\alpha\rangle\langle\alpha|$ because the basis $\{|\alpha\rangle\}$ was chosen to be real. Eq. (12) therefore implies that the channel $\Phi_1$ is equivalent to the channel $\tilde{\Phi}_3$ of (50).

There is an alternative way of demonstrating this equivalence between the two channels. It is easy to see that $\tilde{\Phi}_3(\rho)$ for the channel defined by (50), can be expressed as (see e.g. [17]):

$$\tilde{\Phi}_3(\rho) = \frac{1}{4} \sum_{i,j=1}^{3} B_{(ij)} \rho B_{(ij)}^*,$$

(51)

where $B_{(ij)} = |j\rangle\langle i| - |i\rangle\langle j|$ and $\{|i\rangle, i = 1, 2, 3\}$ denotes a complete set of orthonormal basis vectors of the Hilbert space $\mathcal{H} \simeq \mathbb{C}^d$. Since $B_{(ii)} = 0$ for all $i$, and $B_{(ij)} = -B_{(ji)}$ for $j \neq i$, $\tilde{\Phi}_3(\rho)$ can be expressed as follows:

$$\tilde{\Phi}_3(\rho) = \frac{1}{2} \sum_{i} A_i \rho A_i^*,$$

(52)

where $A_1 := B_{(12)}$, $A_2 := B_{(23)}$ and $A_3 := iB_{(31)}$. Note that the operators $A_i$, $i = 1, 2, 3$ satisfy the following commutation relations: $[A_i, A_j] = i\epsilon_{ijk} A_k$, which is identical to that satisfied by the spin operators $S_k$ (or $S'_k$) of the channel $\Phi_1$ of (11). This again proves the equivalence of the channels $\Phi_1$ and $\tilde{\Phi}_3$.

As mentioned in the Introduction, the Holevo capacity and the minimum output entropy of the channel $\tilde{\Phi}_d$ have been proved to be additive for any arbitrary dimension $2 \leq d < \infty$ [14, 15, 4, 5]. The equivalence of the channels $\Phi_1$ and $\tilde{\Phi}_3$ implies the validity of the additivity relations for the channel $\Phi_1$ as well. Moreover, in [5] it was shown that the sufficient condition (18) was satisfied for the channel $\tilde{\Phi}_d$. Hence it is also satisfied for the channel $\Phi_1$.

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