The model of the kernel of the Lindley integral equation based on selective functions

I Kartashevskiy¹

¹Povolzhskiy State University of Telecommunications and Informatics (PSUTI), L. Tolstoy str. 23, Samara, Russia, 443010
e-mail: g.sharikoff@ya.ru

Abstract. Here considered a model of the kernel of the Lindley integral equation for a queuing system G/G/1. An exact solution of Lindley equation can be obtained using the Fourier transform under analytic continuation to the half-plane of the complex function originally defined on the real axis. However, it is quite difficult to obtain such solution, therefore, one resorts to finding approximate solutions based on the procedure of “degeneration” of the kernel when the kernel is factorized according to its variables. The method based on the use of selective functions is used. As an example, a special case is considered when the arrival time intervals have a gamma distribution, and the service time is constant. The result is shown for the kernel with the exponential form.

1. Introduction

Multimedia traffic at the input of telecommunication devices has a non-Poisson distribution [1]. Therefore, the calculation of the parameters of the telecommunication network’s nodes is possible when choosing the queuing system G/G/1 as their mathematical model [2,3]. Now there are no simple methods for calculating the characteristics of nodes described by the queuing systems G/G/1 or G/G/n.

It is known [4] that the research of a single-channel queuing system with general input GI/G/1 is closely related with the solution of the Lindley integral equation (LIE). LIE can be written in the form

\[ F(y) = \int_0^y K(y-x)dF(x), \quad y > 0. \]  

where \( F(y) \) is cumulative distribution function (CDF) of waiting time, and \( K(y) \) is the kernel of the integral equation written as

\[ K(u) = \int_0^u B(u+t)dA(t), \]  

where \( B(t) \) is the CDF of service time, and \( A(t) \) is the CDF of arrival time.

There are several different ways to solve this equation. Since the Lindley equation is a Wiener-Hopf equation, appropriate techniques can be used [5]. The solution by the spectral method using the approximation of the arrival and service distributions by fading exponentials [6] or use of Gauss-Kristoffel method proposed in the work [7]. There are also methods based on spectral decomposition using generalized hyper-exponential distributions [8,9], methods for discrete time [10] with the use of generalizedSchur decomposition, and others [11].

This paper is devoted to a method for solving the Lindley integral equation using the selection functions for a queuing system of the form G/G/1.
2. Preliminary definitions

Equation (1) belongs to the type of Wiener-Hopf equations whose solution in the general case is rather difficult to obtain [5]. Therefore, we can transform it to an integral Fredholm equation of the second kind with differentiation (1) with respect to the variable $y$:

$$\frac{dF(y)}{dy} = \frac{dK(y)}{dy} F(0) + \int_{0}^{\infty} K'(y-x) f(x) dx,$$

(3)

Considering that $\frac{dF(y)}{dy} = f(y)$ and dividing both sides of equation (3) by the probability that the queuing system is empty ($F(0)$), we obtain

$$\phi(y) = K'(y) + \int_{0}^{\infty} K'(y-x) \phi(x) dx,$$

(4)

where $\frac{f(y)}{F(0)} = \phi(y)$. $F(0)$ for single-channel system determines through the utilization of the server as $F(0) = 1 - \rho$, where $\rho < 1$ is an utilization coefficient. Wherein:

$$K'(u) = \int_{0}^{\infty} b(u+t) a(t) dt.$$

(5)

Functions $f(y), b(t)$ and $a(t)$ in the expressions (3) - (5) are the corresponding probability densities functions (PDF).

To specify the method of solving equation (4) we consider a $\Gamma/D/1$ system, where $\Gamma$ is the Gamma distribution and $D$ is the constant duration of service denoted as $t_0$. Then $b(t) = \delta(\tau-t_0)$.

The PDF of the gamma distribution has the form

$$a(t) = t^{k-1} \exp\left(-\frac{t}{\theta}\right) \left(\frac{\theta}{\Gamma(k)}\right)^{\frac{k}{t}}, \quad t \geq 0,$$

For definiteness, we choose $\theta = 2, k = 0.5$ (Gamma distribution is “heavy-tailed” with $k = 0.5$) and then $a(t) = \exp\left(-\frac{t}{2}\right) \left(2\pi t\right)^{-1/2}, \quad t \geq 0$.

With the selected $b(t)$ and $a(t)$ we can write

$$K'(u) = \left(2\pi\right)^{-1/2} \exp\left(-\frac{t_0-u}{2}\right) \left(t_0-u\right)^{-1/2}.$$

(6)

The form of the kernel (6) with the value $t_0 = 4$ is shown in Figure 1.
3. Lindley integral equation solution using selective functions

As shown in [12], an exact solution of equation (4), which belongs to the class of convolution equations, can be obtained using the Fourier transform under analytic continuation to the half-plane of the complex function originally defined on the real axis. However, it is quite difficult to obtain such solution, therefore, one resorts to finding approximate solutions [4] based on the procedure of "degeneration" of the kernel when the kernel is factorized according to its variables. One such method is based on the use of selective functions [14].

Due to the fact that the function \( \phi(y) \) is the PDF of the random intervals of request’s waiting time in the queue, the search for the solution of equation (4) should be performed for the \( y \geq 0 \) values when the condition \( x \geq y \) is satisfied. On the other hand, it follows from expression (6) and Fig. 1 that, the kernel exists also for negative values of the argument. For example, with \( u = -4 \) a kernel has a very small value of 0.0026. To preserve almost all the kernel values that determine the form of the sought PDF we shift the kernel along the abscissa axis by a \( t_0 \) value to the right, and carry out the integration in equation (4) on the interval \([0, 2t_0]\) where the kernel represented with an error permissible for our purposes. Now equation (4) can be rewritten in the following form:

\[
\phi(y) = K'(y) + \int_0^{2t_0} K'(y-x)\phi(x)dx. \tag{7}
\]

Representing the kernel on an interval \([0, 2t_0]\) in a degenerate form with use of selective functions can be performed as follows [13].

Denote \( K'(x,t) = k(x,t) \), then

\[
k(x,t) = \sum_{i=1}^n [g_i(x)t + h_i(x)]s(t_i,t_{i+1}), \tag{8}
\]

Where the selection function \( s(t_i,t_{i+1}) \) is defined as

\[
s(t_i,t_{i+1}) = \begin{cases} 0, & t < t_i, \\ 1, & t_i < t < t_{i+1}, \\ 0, & t > t_{i+1}. \end{cases} \tag{9}
\]

And it’s values at the points \( t_i \) and \( t_{i+1} \) are written as

\[
s(t_i,t_{i+1}) = s(t_i,t_{i+1}) = 0.5(\lim_{x \to x_{i+1}^-} + \lim_{x \to x_{i+1}^+}) = 0.5(\lim_{x \to x_i^-} + \lim_{x \to x_i^+}) = 0.5. \tag{10}
\]

Substituting (8) into (7) we can obtain

\[
\phi(y) = K'(y) + \sum_{i=1}^n [g_i(x)\beta_i + h_i(x)v_i], \tag{11}
\]

where

\[
\beta_i = \int_0^{2t_0} t\phi(t)s(t_i,t_{i+1})dt, \quad v_i = \int_0^{2t_0} \phi(t)s(t_i,t_{i+1})dt. \tag{12}
\]

According to [13], if we multiply both parts of equation (10) first by \( t \cdot s(t_i,t_{i+1}) \) and integrate on the interval \([0, 2t_0]\), then multiply by \( s(t_i,t_{i+1}) \) and re-integrate, we can obtain a system of algebraic equations

\[
\begin{cases}
\beta_j - \sum_{i=1}^n (\beta_i e_{ji} + v_i d_{ji}) = k_{1j}, \\
v_j - \sum_{i=1}^n (\beta_i z_{ji} + v_i r_{ji}) = k_{2j},
\end{cases} \tag{13}
\]

where
After the values \( \beta_j \) and \( \nu_j \) are found from the solution of the system of linear algebraic equations, equation (9) gives the solution of the original task.

Let's demonstrate the procedure for obtaining a "degenerate" kernel according to expressions (7) and (8) using the example of a kernel in the form (6). The expanded expression (8) with the kernel shift of \( t_0 \) and \( n = 2 \) will have the form

\[
\left[ g_1(x) \cdot t_0 + \frac{2}{\pi} \exp \left( -\frac{2t_0 - x + t}{2} \right) (2t_0 - x) \right] = \left[ g_1(x) t + h_1(x) \right] \text{si}(t, t_1, t_2) + \left[ g_2(x) t + h_2(x) \right] \text{si}(t, t_2, t_3). \quad (14)
\]

Focusing on the shifted view of the kernel for the value \( t_0 = 4 \) we choose the time moments \( t_1 = 0 \) and \( t_2 = t_0 \) for \( \text{si}(t, t_1, t_2) \). Now, from the first term on the right side of (12), taking into account the definition and properties of the function \( \text{si}(t, t_1, t_2) \) with the \( t_1 = 0 \) we immediately obtain

\[
h_1(x) = \left( \frac{2}{\pi} \right)^{1/2} \exp \left( -\frac{2t_0 - x}{2} \right) (2t_0 - x)^{-1/2}
\]

It follows from condition \( t_2 = t_0 \) that

\[
\left[ g_1(x) \cdot t_0 + \frac{2}{\pi} \exp \left( -\frac{2t_0 - x}{2} \right) (2t_0 - x) \right] = \left( \frac{2}{\pi} \right)^{-1/2} \exp \left( -\frac{3t_0 - x}{2} \right) (3t_0 - x)^{1/2}.
\]

And then

\[
g_1(x) = \frac{2}{t_0} (2\pi)^{1/2} \left[ \exp \left( -\frac{3t_0 - x}{2} \right) (3t_0 - x)^{-1/2} - \exp \left( -\frac{2t_0 - x}{2} \right) (2t_0 - x)^{-1/2} \right]
\]

For the second term on the right side of (12), taking into account that \( t_2 = t_0 \) and \( t_3 = 2t_0 \), we can write the system of equations

\[
\begin{align*}
\left[ g_2(x) t_0 + h_2(x) \right] \text{si}(t_0, t_2, 2t_0) & = (2\pi)^{-1/2} \exp \left( -\frac{3t_0 - x}{2} \right) (3t_0 - x)^{1/2} \\
\left[ g_2(x) 2t_0 + h_2(x) \right] \text{si}(2t_0, t_2, 2t_0) & = (2\pi)^{-1/2} \exp \left( -\frac{4t_0 - x}{4} \right) (4t_0 - x)^{1/2}
\end{align*}
\]

Solving this it is possible to receive functions \( g_2(x) \) and \( h_2(x) \).

4. Determining the coefficients

It is necessary to pay attention to the feature of the computation of the integrals that determine the coefficients of equations (12). For example, the coefficient \( c_{11} \) is
\[ c_{11} = \int_0^{2t_0} t \cdot g_1(t) \sin(t, t_1, t_2) dt \]

With considering the properties of the function \( \sin(t, t_1, t_2) \) the integral can be represented in the form
\[
c_{11} = \int_0^{2t_0} t \cdot g_1(t) \sin(t, t_1, t_2) dt = \left[ \frac{1}{2} \text{sgn}(0 - t_1) \int_0^{t_1} t \cdot g_1(t) dt + \text{sgn}(2t_0 - t_1) \int_0^{2t_0} t \cdot g_1(t) dt \right] - \text{sgn}(0 - t_2) \int_0^{t_2} t \cdot g_1(t) dt - \text{sgn}(2t_0 - t_2) \int_0^{2t_0} t \cdot g_1(t) dt ,
\]

where
\[
\text{sgn}(x) = \begin{cases} 
1, & x > 0, \\
0, & x = 0, \\
-1, & x < 0. 
\end{cases}
\]
is another selective function [14].

Analysis of the expression for the \( c_{11} \) considering properties of the function \( \text{sgn}(x) \) shows that the value of \( c_{11} \) can be calculated in the form \( c_{11} = \int_0^{t_0} t \cdot g_1(t) dt \). Similar arguments using expressions (13) and (14) can be carried out with reference to the calculation of other coefficients.

To obtain the numerical values of the integrals that determine the coefficients of the system of equations (11), tabular integrals are used [15]:
\[
\int_A^\mu x^{-1/2} \exp(-x) dx = \sqrt{\pi} \left[ \Phi(\mu) - \Phi(\lambda) \right]
\]
\[
\int_A^\infty \sqrt{x} \exp(-x) dx = \left[ \sqrt{\lambda} \exp(-\lambda) - \sqrt{\mu} \exp(-\mu) \right] + \frac{1}{2} \sqrt{\pi} \left[ \Phi(\mu) - \Phi(\lambda) \right],
\]
where \( \Phi(x) = \left( \frac{2}{\pi} \right)^{1/2} \int_0^x \exp(-t^2) dt \).

The numerical solution of system (11) allows for the chosen \( t_0 = 4 \) from equation (10) to obtain an expression for the function \( \varphi(x) \):
\[
\varphi(x) = -4,26 \exp \left( -\frac{x-12}{2} \right) (12-x)^{-1/2} + 5,32 \exp \left( -\frac{x-16}{2} \right) (16-x)^{-1/2} + 0,42 \exp \left( -\frac{x-8}{8} \right) (8-x)^{-1/2}
\]

Figure 2. Function \( \varphi(x) \).
The verification realized by substituting (15) into equation (4) and the numerical solution of the equation (4) confirmed the validity of the result obtained. The graph of the function $\varphi(x)$ is shown in Figure 2.

As follows from equation (4), the PDF of the waiting time in the queue should be determined (with the known system utilization $\rho$) in the form $f(x) = (1-\rho)\varphi(x)$

5. Conclusion
Here was considered a method for solving the Lindley integral equation using the selection functions for a queuing system of the form G/G/1. When the arrival time intervals have a gamma distribution, and the service times constant the result is obtained and verified with numerical solution of equation (4).

However, this result is shown for the kernel with the exponential form. For the skew-symmetric form of the kernel solution can not be found.

6. References
[1] Paxson V and Floyd S 1995 Wide-area traffic: the failure of Poisson modeling IEEE/ACM Trans. Networking 3 226-244
[2] Boxma O and Cohen J 1999 Heavy-traffic analysis for the GI/G/1 queue with heavy-tailed distributions Queueing Systems 33 177 DOI:10.1023/A:1019124112386
[3] Chu T, Phan H and Zepernick H 2014 Delay analysis for cognitive ad hoc networks using multi-channel medium access control IET Communications 8 1083-1093
[4] Kleinrock L 1975 Queueing Systems, volume I: Theory (New York: Wiley)
[5] Zabreyko P, Koshelev A 1975 Integral Equations – areference text (Leyden: Noordhoff Int Publ)
[6] Kartashevskiy V, Kirieva N, Buranova M and Chupakhina L 2015 Study of queuing system G/G/1 with an arbitrary distribution of time parameter system Proc. Int. Conf. Problems of Infocommunications Science and Technology (Kharkiv) 145-148 DOI: 10.1109/INFOCOMMST.2015.7357297
[7] Kartashevsky V, Kozyreva N and Makarov I 2016 The numerical method for analysis of arbitrary type queuing systems application Proc. Int. Conf. Problems of Infocommunications Science and Technology (Kharkiv) pp 109-111 DOI: 10.1109/INFOCOMMST.2016.7905350I
[8] Tarasov V 2016 Analysis of queues with hyperexponential arrival distributions Probl Inf Transm 52 14 DOI:10.1134/S0032946016010038
[9] Li J 1997 An Approximation Method for the Analysis of GI/G/1 Queues Operations Research 45(1) 140-144
[10] Akar N 2006 A matrix analytical method for the discrete time Lindley equation using the generalized Schur decomposition Proc. Workshop on Tools for solving structured Markov chains (Pisa) p 12 DOI:10.1145/1190366.1190377
[11] Vlasiou M and Adan I 2007 Exact solution to a Lindley-type equation on a bounded support Operations Research Letters 35(1) 105-113
[12] Gakhov F and Cherskii Yu 1978 Equations of Convolution Type (Moscow: Nauka) (in Russian)
[13] Kantorovich L and Krylov V 1958 Approximate methods of higher analysis (New York: Interscience Publishers)
[14] Mishchenko V 2012 The theory of selective functions in practical applications (Moscow: Medisont) (in Russian)
[15] Gradstein I and Ryjik I 1963 Tables of integrals (Moscow: GIFML)