Noncommutative Spacetime in Very Special Relativity

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Abstract

Very Special Relativity (VSR) framework, proposed by Cohen and Glashow \cite{1}, demonstrated that a proper subgroup of the Poincaré group, (in particular ISIM(2)), is sufficient to describe the spacetime symmetries of the so far observed physical phenomena. Subsequently a deformation of the latter, DISIM\textsubscript{b}(2), was suggested by Gibbons, Gomis and Pope \cite{2}. In the present work, we introduce a novel Non-Commutative (NC) spacetime structure, underlying the DISIM\textsubscript{b}(2). This allows us to construct explicitly the DISIM\textsubscript{b}(2) generators, consisting of a sector of Lorentz rotation generators and the translation generators. Exploiting the Darboux map technique, we construct a point particle Lagrangian that lives in the NC phase space proposed by us and satisfies the modified dispersion relation proposed by Gibbons et. al. \cite{2}. It is interesting to note that in our formulation the momentum algebra becomes non-commutative.

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1 Introduction

Einstein’s Special Relativity (SR) theory invokes that all the physical theories as well as all the observables remain invariant or covariant under the Poincaré symmetry. Poincaré symmetry is implemented by the Poincaré group, consisting of the Lorentz transformations (boosts plus rotations) along with the spacetime translations. In mathematical terminology, the Poincaré group is the isometry group of the (3 + 1)-dimensional Minkowski spacetime. In the particle physics sector, at low energy scales (QED + QCD), parity (P), charge conjugation (C) and time reversal (T) are individually good symmetries of nature. However, for higher energies, there is evidence of CP violation. From a purely theoretical point of view, one may even consider the breaking of Poincaré symmetry at such high energy scales. So it might be possible to describe the spacetime symmetry of all the observed physical phenomena considering some proper subgroups of the Lorentz group along with the spacetime translations. The underlying criterion is that these subgroups, together with either of P, T or CP, can be enlarged to the full Lorentz group. The generic models based on these smaller subgroups are restricted by the principle of Very Special Relativity (VSR), proposed by Cohen and Glashow [1]. The authors of [1] identified these VSR subgroups up to isomorphism as $T(2)$ (2-dimensional translations), $E(2)$ (3-parameter Euclidean motion), $HOM(2)$ (3-parameter orientation preserving transformations) and $SIM(2)$ (4-parameter similitude group). The semi-direct product of the $SIM(2)$ group with the spacetime translation group gives a 8-dimensional subgroup of the Poincaré group called $ISIM(2)$.

Stringent observational bounds on the CP violation put a constraint that the deviation of VSR from SR should be very small. Thus it is very difficult to observe the effects of VSR theory in the physical scenarios. Fortunately, in case of $SIM(2)$, there are no invariant vector or tensor fields (the so called “spurion fields”), although an invariant null direction is present. This particular VSR theory therefore appears to be compatible with all the current experimental limits on violations of Poincaré invariance [1, 3].

VSR phenomenology has been investigated in [4]. In attempting to construct a quantum
field theory based on the above VSR subgroups, Sheikh-Jabbari and Tureanu [5] noticed a problem: all the above proper subgroups allow only one-dimensional representations and hence can not represent the nature faithfully. However, the authors of [5] provided an ingenious resolution of the representation problem: they generalize the normal products of operators as deformed or twisted coproducts [6]. The resulting novel forms of NC spacetimes were further studied as arena of generalized particle dynamics [7].

In their work, Gibbons et al [2] proposed that gravity may be incorporated in the \( ISIM(2) \) invariant VSR theory by taking deformations of the \( ISIM(2) \) group such that the spacetime translations become non-commutative. In fact, in [2], the authors showed that there exists a one-parameter family of continuous deformations of the group \( ISIM(2) \), which they denoted by \( DISIM_b(2) \). For any values of the deformation parameter \( b \), the group \( DISIM_b(2) \) is an 8-dimensional subgroup of the 11-dimensional Weyl group (semi-direct product of the Poincaré group with the dilation). Interestingly, Gibbons et. al. [2] arrived at a \( DISIM_b(2) \)-invariant point particle Lagrangian \( L \) for a particle with mass \( m \):

\[
L = -m(\dot{x}^2)^{\frac{1}{2}+b}(-\alpha \dot{x})^b,
\]

which is of the Finsler form, and was first proposed by Bogoslovsky in a different context [8]. This observation prompted the authors of [2] to argue that deforming VSR theory one arrives at Finsler geometry, thus maintaining the commutativity of spacetime translations. The constant vector \( \alpha_\mu \) in (1) denotes the invariant direction mentioned before that breaks the full Poincaré invariance. The induced dispersion relation obtained from (1) is

\[
p^2 + m^2(1-b^2)\left(\frac{-\alpha p}{m(1-b)}\right)^{\frac{2}{1+b}} = 0.
\]

In this perspective our work bears a special significance, since we have explicitly shown that starting from a first-order Lagrangian with an underlying noncommutative spacetime structure, one can still arrive at the same Finsler Lagrangian (1). In particular, the momentum algebra in our model are indeed noncommutative.
Furthermore, Finsler geometry as a natural generalization of Riemann geometry may provide new sight on modern physics. The models based on Finsler geometry are claimed to explain recent astronomical observations which are not explained in the framework of Einstein’s gravity. For example, the flat rotation curves of spiral galaxies can be deduced naturally without invoking dark matter [9] and the anomalous acceleration in solar system observed by Pioneer 10 and 11 spacecrafts can be accounted for [10].

Our paper is organized as follows: In Section 2, we describe the $DISIM_b(2)$ algebra given by Gibbons [2]. In Section 3, the underlying non-commutative spacetime structure representing the $DISIM_b(2)$ algebra is derived explicitly. This is one of the major result of our paper. In Section 4, we introduce a reparametrization invariant first-order (phase space) point-particle Lagrangian and it is shown in detail that starting with this Lagrangian, the non-commutative spacetime structure specified above can be exactly reproduced using the well-known method of Dirac constraint analysis [11]. In Section 5, we proceed to formulate a second-order (coordinate space) particle Lagrangian. Notably we recover the second-order Lagrangian [11], as proposed in [8, 2]. Our paper ends with a conclusion in Section 6.

2 Deformed $ISIM(2)$ or $DISIM_b(2)$ Algebra

In [2], the authors have considered the VSR theory based on $ISIM(2)$, one of the subgroups suggested by Cohen and Glashow [1]. In their attempt to incorporate gravity into this VSR framework, the authors of [2] further considered a 1-parameter family of deformations ($b$ being the deformation parameter), called $DISIM_b(2)$, under which the VSR theory is invariant. The group $DISIM_b(2)$ has the following eight generators:

$$J_{+-}, J_{+i}, J_{ij}, p_+, p_-, p_i ; \quad (i, j) = 1, 2$$

(3)
where $J$-s are the rotation and boost generators and $p$-s are the generators for the translations. Here we used the usual light-cone coordinates

$$x_\pm = \frac{1}{\sqrt{2}}(x_0 \pm x_3), \quad x_i, \quad i = 1, 2$$

and the metric $g_{\mu\nu}$ is of the form

$$g_{\mu\nu} = \text{diag}(1, -1, -1, -1).$$

The $DISIM_b(2)$ algebra is explicitly given by [2]:

\[
\begin{align*}
\left[J_{+-}, p_\pm\right] &= -(b \pm 1)p_\pm, & \left[J_{+-}, p_i\right] &= -bp_i, & \left[J_{+-}, J_{+i}\right] &= -J_{+i}, \\
\left[J_{12}, p_i\right] &= \epsilon_{ij}p_j, & \left[J_{12}, J_{+i}\right] &= \epsilon_{ij}J_{+j}, & \left[J_{+i}, p_-\right] &= p_i, & \left[J_{+i}, p_+\right] &= g_{ij}p_+, \quad (4)
\end{align*}
\]

where we have chosen $\epsilon_{12} = -1$.

In [2], if the deformation parameter $b$ vanishes, we recover the usual canonical algebra. In [2], on the basis of this non-canonical $DISIM_b(2)$ algebra (2), the Finsler Lagrangian (1) was obtained. However, the symplectic structure between the phase space variables $(x, p)$ was not explicitly specified in (2).

Our main motivation stems from the fact that the non-canonical algebra of the generators postulated in [2] appears in a purely algebraic way. Hence our aim is to understand the appearance of this algebra at a more physical and fundamental level. In fact, from our previous experience, we know that a generalized version of Nambu-Goto type action (as in (1)) and modified dispersion relation (as in (2)) are usually connected to an NC phase space (example in other contexts can be found in [12]).

In the following sections we show that the algebra (2) can be induced by a new and interesting form of Non-Commutative (NC) spacetime (or more accurately NC phase space). This NC phase space algebra allows us to explicitly construct the $DISIM_b(2)$ generators in terms of phase space degrees of freedom. Finally, armed with this novel phase space, we go on to construct a point particle Lagrangian in a systematic and physically transparent way.
In the process we recover the Lagrangian proposed by Gibbons et. al. in [2]. An added bonus in our scheme, which makes the study of DISIM$_b$(2) [2] all the more attractive is that the momenta $p_\mu$ become noncommutative.

3 The Underlying NC Phase Space Algebra

In this section we will disclose the novel NC phase space algebra which will turn out to be fairly involved, but we will see that it can be recast in a more convenient and symmetric form. Since there is no unique way to recover the algebra, we fall back to the canonical prescription for the angular momentum structure and postulate the generators to be

$$J_{\mu\nu} = x_\mu p_\nu - x_\nu p_\mu. \quad (5)$$

Obviously, to satisfy the deformed algebra [2] a non-canonical or NC $(x,p)$ phase space algebra is required. We posit it to be of the following form:

$$[x_+, p_-] = 1 - \frac{b}{2}, \quad [x_-, p_+] = 1 + \frac{b}{2}, \quad [x_-, p_-] = b \left( -\frac{x_- + p_-}{x_+ + p_+} \right),$$

$$[x_+, p_+] = 0, \quad [x_-, x_+] = -\frac{b}{2} \frac{x_+}{p_+}, \quad [p_-, p_+] = -\frac{b}{2} \frac{p_+}{x_+}, \quad [x_-, p_i] = \frac{b}{2} \frac{p_i}{x_+},$$

$$[p_- x_i] = \frac{b}{2} \frac{x_i}{x_+}, \quad [x_-, x_i] = -\frac{b}{2} \frac{x_i}{p_+}, \quad [p_-, p_i] = -\frac{b}{2} \frac{p_i}{x_+},$$

$$[x_+, p_i] = [p_+, x_i] = [x_+, x_i] = [p_+, p_i] = 0, \quad [x_i, p_j] = g_{ij}. \quad (6)$$

The above algebra satisfies the Jacobi identity. It is straightforward to convince oneself that the algebra (3) together with the definition of angular momentum (5) reproduces the DISIM$_b$(2) algebra (2). This is the first important result of our paper.

Our next objective is to search for a canonical representation of the NC algebra (3) which will be vital in our systematic derivation of the particle Lagrangian. That this map (between NC and canonical variables) is in principle derivable follows from Darboux’s theorem, which states that it is possible at least locally to construct an invertible map between NC and canonical (commutative) phase space variables. It should also be stressed that in general
the Darboux map might be very involved and difficult to construct explicitly, and we are not aware of a unique or systematic method for deriving this map.

In the present case, the explicit form of the Darboux map between the noncommutative phase space variables \((x_\mu, p_\mu)\) and the canonical phase space variables \((X_\mu, P_\mu)\) with

\[
[X_\mu, P_\nu] = g_{\mu\nu}, \quad [X_\mu, X_\nu] = [P_\mu, P_\nu] = 0
\]

is given by

\[
x_+ = X_+ , \quad x_i = X_i , \quad p_+ = P_+ , \quad p_i = P_i ,
\]

\[
x_- = X_- + \frac{b (X P)}{2 P_+} , \quad p_- = P_- - \frac{b (X P)}{2 X_-}
\]

(7)

where \((X P) = X_\mu P^\mu\). The inverse Darboux map can be readily obtained,

\[
X_+ = x_+ , \quad X_i = x_i , \quad P_+ = p_+ , \quad P_i = p_i ,
\]

\[
X_- = x_- - \frac{b (x p)}{2 p_+} , \quad P_- = p_- + \frac{b (x p)}{2 x_-}
\]

(8)

For later convenience, we stick to a manifestly covariant framework by introducing two constant null vectors \(\alpha_\mu = (\frac{1}{\sqrt{2}}, 0, 0, -\frac{1}{\sqrt{2}})\) and \(\beta_\mu = (\frac{1}{\sqrt{2}}, 0, 0, \frac{1}{\sqrt{2}})\), satisfying the relations \(\alpha^2 = \beta^2 = 0\), \((\alpha\beta) = 1\). Thus in our notation, \((\alpha x) = \frac{x_0 + x_3}{\sqrt{2}} \equiv x_+\), \((\beta x) = \frac{x_0 - x_3}{\sqrt{2}} \equiv x_-\).

With the help of the above, the Darboux and inverse Darboux maps (3, 3) can be written in the covariant form:

\[
x_\mu = X_\mu + \frac{b (X P)}{2 (\alpha P)} \alpha_\mu , \quad p_\mu = P_\mu - \frac{b (X P)}{2 (\alpha X)} \alpha_\mu
\]

(9)

\[
X_\mu = x_\mu - \frac{b (x p)}{2 (\alpha p)} \alpha_\mu , \quad P_\mu = p_\mu + \frac{b (x p)}{2 (\alpha x)} \alpha_\mu.
\]

(10)

Furthermore, we rewrite our new NC phase space algebra (3) in a covariant form as well,

\[
[x_\mu, x_\nu] = \frac{b}{2(\alpha p)} (\alpha_\nu x_\mu - \alpha_\mu x_\nu) , \quad [p_\mu, p_\nu] = \frac{b}{2(\alpha x)} (\alpha_\nu p_\mu - \alpha_\mu p_\nu)
\]

\[
[x_\mu, p_\nu] = g_{\mu\nu} - \frac{b}{2(\alpha x)} \alpha_\nu x_\mu + \frac{b}{2(\alpha p)} \alpha_\mu p_\nu.
\]

(11)
The expressions (3) give the underlying non-commutative phase space structure of the DISIM$_b(2)$-invariant Very Special Relativity (VSR) theory. We observe a very pleasing symmetrical structure in (9, 10, 3) under the exchange between $x_\mu$ and $p_\mu$.

The infinitesimal transformations of $x_\mu$ and $p_\mu$ and the NC Lorentz algebra follow easily,

$$[J_{\mu\nu}, x_\rho] = g_{\mu\rho} x_\nu - g_{\nu\rho} x_\mu - \frac{b}{2(\alpha p)} x_\rho (\alpha_\mu p_\nu - \alpha_\nu p_\mu) - \frac{b}{2(\alpha x)} (\alpha_\mu x_\nu - \alpha_\nu x_\mu) x_\rho$$

$$[J_{\mu\nu}, p_\rho] = g_{\mu\rho} p_\nu - g_{\nu\rho} p_\mu + \frac{b}{2(\alpha p)} p_\rho (\alpha_\mu p_\nu - \alpha_\nu p_\mu) + \frac{b}{2(\alpha x)} (\alpha_\mu x_\nu - \alpha_\nu x_\mu) p_\rho$$  \hspace{1cm} (12)

$$[J_{\mu\nu}, J_{\rho\sigma}] = g_{\mu\rho} J_{\nu\sigma} - g_{\nu\rho} J_{\mu\sigma} + g_{\mu\sigma} J_{\rho\nu} - g_{\nu\sigma} J_{\rho\mu}$$

$$- \frac{b^2}{4(\alpha x)(\alpha p)} (\alpha_\mu \alpha_\rho J_{\nu\sigma} - \alpha_\nu \alpha_\rho J_{\mu\sigma} + \alpha_\mu \alpha_\sigma J_{\rho\nu} - \alpha_\nu \alpha_\sigma J_{\rho\mu}) .$$ \hspace{1cm} (13)

It is interesting to note that the Lorentz algebra (3) is deformed at $O(b^2)$ only and all the above relations will reduce to (4) in the relevant sector of the algebra, to match [2].

### 4 Particle Lagrangian: First-Order Form

Enough of kinematics! Let us now consider dynamics in this NC phase space scenario. The first and foremost problem is to construct a point particle model that will live in this new phase space described by (3). Since the dynamics is in phase space, a first-order form is most suitable for our purpose. Once the Darboux map relating the NC phase space variables to the corresponding canonical variables is derived, one can conveniently construct a dynamical model in the NC phase space applying the following prescription: start with a known (canonical) action, exploit the Darboux map to express the action in terms of the NC phase space variables and then study the dynamics. Though in principle the Darboux map exists, in practice it is not always possible to derive such a Darboux map. In our work, we are able to find out the explicit expression for the Darboux map which helps us to derive the Lagrangian for this VSR model. This scheme has been exploited before in [12].
We want to define a Lagrangian \( L \) such that we can reproduce the above non-commutative structure (3) from its kinetic part. So we start with

\[
L = P_\mu \dot{X}^\mu - \frac{\lambda}{2} \left( P^2 + A(-\alpha P) \frac{2b}{1+\bar{b}} \right),
\]

(14)

where \( X_\mu \) and \( P_\mu \) represent canonical phase space variables, \( \dot{X}_\mu = \frac{\partial X_\mu}{\partial \tau} \) and \( A \) is an arbitrary constant. The Lagrange multiplier \( \lambda \) enforces the mass-shell constraint. The arbitrary constant \( A \) is determined from the mass-shell constraint as

\[
A = -\frac{P^2}{(-\alpha P) \frac{2b}{1+\bar{b}}},
\]

since the ratio \( \frac{P^2}{(-\alpha P) \frac{2b}{1+\bar{b}}} \) is invariant under the DISIM\(_b\) algebra given by (2) and (3),

\[
\left[ J_{\mu\nu}, \frac{P^2}{(-\alpha P) \frac{2b}{1+\bar{b}}} \right] = 0 ; \quad \mu, \nu = +, -, i.
\]

(15)

It is important to keep in mind that \( J_{\mu\nu} = x_\mu p_\nu - x_\nu p_\mu \), as defined before, and one way of computing (15) is to convert \( J_{\mu\nu} \) to the canonical coordinates by using the Darboux map (9). Obviously \( J_{\mu\nu} \) has a non-canonical structure in \( X - P \) (canonical) space:

\[
J_{\mu\nu} = X_\mu P_\nu - X_\nu P_\mu + \frac{b(XP)}{2(\alpha P)} (\alpha_\mu P_\nu - \alpha_\nu P_\mu) + \frac{b(XP)}{2(\alpha X)} (\alpha_\mu X_\nu - \alpha_\nu X_\mu).
\]

The Lagrangian (14) is exactly of the same form as given in [2].

Applying the inverse Darboux map (10) on (14) we get the required first-order Lagrangian:

\[
L = p_\mu \hat{x}^\mu + \frac{b(xp)(\alpha \hat{x})}{2(\alpha x)} + \frac{b(xp)(\alpha \hat{p})}{2(\alpha p)} - \frac{\lambda}{2} \left( p^2 + \frac{b(xp)(\alpha p)}{(\alpha x)} + A(-\alpha x) \frac{2b}{1+\bar{b}} \right).
\]

(16)

Before proceeding further, let us quickly recall the main features of Dirac’s constraint analysis in Hamiltonian formulation. One starts by computing the conjugate momentum \( p = \frac{\partial L}{\partial q} \) of a generic variable \( q \) and identifies the relations that do not contain time derivatives as (Hamiltonian) constraints. A constraint is classified as First Class when it commutes with all other constraints and a set of constraints are Second Class when they do not commute. First Class constraints generate gauge invariance which is not of our concern here. However, for systems containing second class constraints, one has to replace the Poisson brackets by Dirac
brackets, to properly incorporate the Second Class constraints. If \([\psi^i_\rho, \psi^j_\sigma]^{-1}\) is the \((ij)\)-th element of the inverse constraint matrix where \(\psi^i(q, p)\) is a set of Second Class constraints, then the Dirac bracket between two generic variables \([A(q, p), B(q, p)]_D\) is given by

\[
[A, B]_D = [A, B] - [A, \psi^i_\rho]([\psi^i_\rho, \psi^j_\sigma]^{-1})[\psi^j_\sigma, B],
\]

where \([\ , \ ]\) denotes Poisson brackets. Let us now study the constraint structure of our model \((16)\). Our aim is to show that the NC algebra \((3)\) proposed by us is realized by the Dirac brackets in this particle model.

Following the standard procedure of First-Order formalism, we consider \(x_\mu\) and \(p_\mu\) as two independent variables and obtain the two sets of constraints \(\psi^1_\mu\) and \(\psi^2_\mu\):

\[
\psi^1_\mu \equiv \pi^p_\mu + \frac{b}{2} x_\mu - \frac{b}{2} \frac{(xp)}{\alpha p} \alpha_\mu,
\]

\[
\psi^2_\mu \equiv \pi^x_\mu - \frac{b}{2} \frac{(xp)}{\alpha x} \alpha_\mu - (1 - \frac{b}{2}) p_\mu,
\]

where \(\pi^x_\mu\) and \(\pi^p_\mu\) are the momenta conjugate to \(x^\mu\) and \(p^\mu\) respectively, satisfying the commutation relation

\[
[x_\mu, \pi^x_\nu] = [p_\mu, \pi^p_\nu] = g_{\mu\nu}.
\]

The constraint matrix and its inverse are given by

\[
[\psi^i_\mu, \psi^j_\nu] = \begin{pmatrix}
\frac{b}{2\alpha p} (\alpha_\nu x_\mu - \alpha_\mu x_\nu) & g_{\mu\nu} + \frac{b}{2\alpha x} \alpha_\nu x_\mu - \frac{b}{2\alpha p} \alpha_\mu p_\nu \\
-g_{\mu\nu} - \frac{b}{2\alpha x} \alpha_\mu x_\nu + \frac{b}{2\alpha p} \alpha_\nu p_\mu & \frac{b}{2\alpha x} (\alpha_\nu p_\mu - \alpha_\mu p_\nu)
\end{pmatrix}
\]

(19)

\[
[\psi^i_\mu, \psi^j_\sigma]^{-1} = \begin{pmatrix}
\frac{b}{2\alpha x} (\alpha_\sigma p_\nu - \alpha_\nu p_\sigma) & -g_{\nu\sigma} + \frac{b}{2\alpha p} \alpha_\sigma x_\nu - \frac{b}{2\alpha p} \alpha_\nu p_\sigma \\
g_{\nu\sigma} - \frac{b}{2\alpha x} \alpha_\sigma x_\nu + \frac{b}{2\alpha p} \alpha_\nu P_\sigma & \frac{b}{2\alpha x} (\alpha_\sigma x_\nu - \alpha_\nu x_\sigma)
\end{pmatrix}.
\]

(20)

Thus the constraints \((11)\) are Second Class, that is, they do not commute between themselves under the Poisson bracket. For this DISIM6(2) invariant VSR scenario, using \((20)\) we obtain the following Dirac brackets

\[
[x_\mu, p_\nu]_D = \frac{b}{2(\alpha p)} (\alpha_\nu x_\mu - \alpha_\mu x_\nu), \quad [p_\mu, p_\nu]_D = \frac{b}{2(\alpha x)} (\alpha_\nu p_\mu - \alpha_\mu p_\nu),
\]

10
\[ [x_\mu, p_\nu]_D = g_{\mu\nu} - \frac{b}{2(\alpha x)} \alpha_\nu x_\mu + \frac{b}{2(\alpha p)} \alpha_\mu p_\nu \]  

(21)

which is exactly of the same form as (3). Thus we are able to provide a first-order Lagrangian whose kinetic part gives rise to a NC structure between the phase space variables (3). Obviously this NC structure correctly reproduces the relevant sector of the \( DISIM_b(2) \) algebra given by (2), as discussed in the previous section.

This concludes the first part of our objective as we have been able to provide a particle model that is a dynamical realization of the \( DISIM_b(2) \) symmetry. The NC phase space structure emerges inherently and need not be imposed in an \( ad \ hoc \) way. Note that the dispersion relation in (16), obtained from the first-order Lagrangian (16), is different from the one (2) derived in [2]. However, this new dispersion relation and symplectic structure (3) conspire to reproduce the action (1) given in [2]. This is explicitly established in the following section, where we provide the equivalent coordinate space second-order (or Nambu-Goto) Lagrangian.

5 Particle Lagrangian: Second-Order Form

In the previous section, we have proposed a first-order particle Lagrangian compatible with the \( DISIM_b(2) \) invariant VSR theory. We proceed further to get a second-order Lagrangian. The Euler-Lagrange equations of motion for \( x_\mu \) and \( p_\mu \) obtained from the Lagrangian \( L \) in (16) are respectively given by:

\[
\dot{p}_\mu + \frac{b((\dot{x}p) + (x\dot{p}))}{2(\alpha x)} \alpha_\mu - \frac{b(\alpha \dot{x})}{2(\alpha x)} p_\mu - \frac{b(\alpha \dot{p})}{2(\alpha p)} p_\mu + \frac{\lambda b(\alpha p)}{2(\alpha x)} x_\mu - \frac{\lambda b(xp)}{2(\alpha x)} p_\mu - \frac{\lambda b(xp)}{2(\alpha p)} x_\mu = 0
\]

(22)

\[
\dot{x}_\mu + \frac{b(\alpha \dot{x})}{2(\alpha x)} x_\mu + \frac{b(\alpha \dot{p})}{2(\alpha p)} x_\mu - \lambda p_\mu - \frac{\lambda b(\alpha p)}{2(\alpha x)} x_\mu - \frac{\lambda b(xp)}{2(\alpha x)} x_\mu - \frac{b((\dot{x}p) + (x\dot{p}))}{2(\alpha p)} x_\mu + \frac{\lambda A b(-\alpha p)}{1 + b} \alpha_\mu = 0.
\]

(23)

Taking the dot product with \( \alpha_\mu \) of both the equations (22) and (5) we obtain the following relations

\[
(\alpha \dot{x}) = \lambda(\alpha p)
\]

(24)
\[(\alpha \dot{p}) = 0. \quad (25)\]

We also obtain the following two relations from (22) and (5):

\[(\dot{x}p) + (xp) = -\frac{\lambda A}{1 + b} (-\alpha p) \frac{2b}{1 + b} \quad (26)\]

\[(xp) = \frac{(x\dot{x})}{\lambda(1 + \frac{b}{2})} + \frac{Ab(-\alpha \dot{x}) \frac{b+1}{b+1} \lambda \frac{1}{1+b} (\alpha x)}{2(1+b)(1 + \frac{b}{2})}. \quad (27)\]

Substituting the relations (24), (25), (26) and (27) in (5), we finally obtain the expression for the momenta \(p_\mu\) as

\[p_\mu = \frac{\dot{x}_\mu}{\lambda} - \frac{b(x\dot{x})\alpha_\mu}{2\lambda(1 + \frac{b}{2})(\alpha x)} + \frac{Ab(-\alpha \dot{x}) \frac{b+1}{b+1} \lambda \frac{1}{1+b} \alpha_\mu}{2(1+b)(1 + \frac{b}{2})}. \quad (28)\]

Putting back this expression for \(p_\mu\) in the Lagrangian \(L\) (16) we obtain the second-order Lagrangian in terms of \(x_\mu, \dot{x}_\mu\) and \(\lambda\):

\[L = \frac{\dot{x}^2}{2\lambda} - \frac{A(-\alpha \dot{x}) \frac{b+1}{b+1} \lambda \frac{1}{1+b}}{2}. \quad (29)\]

Using the equation of motion for \(\lambda\) obtained from (29), we eliminate \(\lambda\) from the Lagrangian. The final expression for the second-order Lagrangian becomes

\[L = -A \frac{1+b}{1+b} (1-b) \frac{b+1}{b+1} (1+b)^{-\frac{1+b}{1+b}} (-\dot{x}^2) \frac{1+b}{1+b} (-\alpha \dot{x})^b. \quad (30)\]

Using the definition of conjugate momenta

\[p_\mu = \frac{\partial L}{\partial \dot{x}_\mu},\]

obtained from the Lagrangian (30), the mass-shell condition turns out to be:

\[p^2 + A(-\alpha p) \frac{2b}{1 + b} = 0. \quad (31)\]

This mass-shell condition (31) is invariant under DISIM\(_b\)(2) algebra.

If we identify the arbitrary parameter \(A\) as

\[A = m \frac{b}{1+b} (1+b)(1-b) \frac{1+b}{1+b},\]
the Lagrangian (30) takes the form

\[ L = -m(\dot{x}^2)^{\frac{1}{4}} (-\alpha \dot{x})^b, \]

as given in [2]. From the Lagrangian (32), we recover the \(DISIM_b(2)\) invariant mass-shell condition,

\[ p^2 + m^2 (1 - b^2) \left( \frac{(-\alpha p)}{m(1 - b)} \right) \frac{\vartheta}{\vartheta + b} = 0. \]

This is the modified dispersion relation proposed by Gibbons et al in [2].

6 Conclusion

In this paper, we have considered \(DISIM_b(2)\) introduced in [2], which is a deformation of a particular VSR group, \(ISIM(2)\), the largest possible subgroup of the Poincaré group compatible with VSR theory. We invoke a non-commutative spacetime structure in this \(DISIM_b(2)\) invariant VSR scenario, where the momentum algebra becomes non-commutative. We also propose a first-order Lagrangian that, after performing the Dirac constraint analysis, reproduces the non-commutative Lie bracket structure between the phase space variables. Further, we obtained a second-order Lagrangian which is of the Finsler form suggested in [8, 2].

The present work suggests that it is possible to attribute a non-commutative momentum algebra with a Finslerian spacetime. But it should be noted that these momenta are not the translation generators since \([x_\mu, p_\nu]\) contains an extension.

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References

[1] A. G. Cohen and S. L. Glashow, Phys. Rev. Lett. 97, 021601 (2006) [arXiv: hep-ph/0601236].

[2] G. W. Gibbons, J. Gomis and C. N. Pope, Phys. Rev. D 76, 081701 (2007) [arXiv: hep-th/0707.2174]; G. W. Gibbons, J. Gomis and C. N. Pope [arXiv: hep-th/0910.3220].

[3] A. Dunn and T. Mehen [arXiv: hep-ph/0610202].

[4] A. E. Bernardini, Phys. Rev. D 75, 097901 (2007) [arXiv: 0706.3932]; A. E. Bernardini and R. Rocha, Europhys. Lett. 81, 40010 (2008) [arXiv: hep-th/0701092]; A. P. Kouretsis, M. Stathakopoulos and P. C. Stavrinos [arXiv: 0810.3267]; S. Das, S. Mohanty [arXiv: 0902.4549]; X. Li, Z. Chang and X. Mo [arXiv: 1001.2667].

[5] M. M. Sheikh-Jabbari and A. Tureanu, Phys. Rev. Lett. 101, 261601 (2008) [arXiv: 0806.3699], [arXiv: 0811.3670].

[6] M. Chaichian, P. P. Kulish, K. Nishijima and A. Tureanu, Phys. Lett. B 604 (2004) 98 [arXiv: hep-th/0408069]; M. Chaichian, P. Presnajder and A. Tureanu, Phys. Rev. Lett. 94 (2005) 151602 [arXiv: hep-th/0409096].

[7] S. Ghosh and P. Pal, Phys. Rev. D 80, 125021 (2009) [arXiv: hep-th/0906.2072].

[8] G. Y. Bogoslovsky [arXiv: gr-qc/0706.2621].

[9] Z. Chang and X. Li, Phys. Lett. B 668, 453 (2008).

[10] J. D. Anderson et al., Phys. Rev. Lett. 81 2858, (1998); Phys. Rev. D 65 082004, (2002); Mod. Phys. Lett. A17 875, (2002).

[11] P. A. M. Dirac, Lectures on Quantum Mechanics, Belfer Graduate School of Science, Yeshiva University, New York, 1964.
[12] S. Mignemi, Phys. Rev. D 68, 065029 (2003) [arXiv: gr-qc/0304029]; A. A. De-
riglazov, JHEP 0303 (2003) 021; S. Ghosh and P. Pal, Phys. Rev. D 75, 105021 (2007)
[arXiv:hep-th/0702159]; S. Ghosh and P. Pal, Phys. Rev. D 80, 125021 (2009) [arXiv:
hep-th/0906.2072].