Canonical Bases for Subspaces of a Vector Space, and 5-Dimensional Subalgebras of Lie Algebra of Lorentz Group

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Abstract

Canonical bases for subspaces of a vector space are introduced as a new effective method to analyze subalgebras of Lie algebras. This method generalizes well-known Gauss-Jordan elimination method.

Keywords: Vector space; Subspaces; Lie algebras; Subalgebras

Introduction

This article has two parts. In Part I, the canonical bases for 5-dimensional subspaces of a 6-dimensional vector spaces are introduced, and all of them are found. Then the nonequivalent canonical bases are classified in Theorem 1. The corresponding procedure involves nonequivalent canonical bases for the (n–1)-dimensional subspaces of vector spaces of dimension n>6 can be constructed in the way similar to this in Part I. This new method of canonical bases helps to study all objects associated with subspaces of vector spaces.

In Part II, this method is applied to study subalgebras of Lie algebra of Lorentz group. It’s a fact that a classification problem of subalgebras of low dimensional real Lie algebras was discussed during 1970-1980 years. That classification of subalgebras of all real Lie algebras of dimension n ≤ 4 only was obtained in the form of representative classes of subalgebras considering under their groups of inner automorphisms [2,3].

The subalgebras of real Lie algebras of dimension n ≥ 5 were not classified before. As a step of the further classification, the 5-dimensional hypothetical subalgebras of 6-dimensional Lie algebra of Lorentz group are investigated in Part II [4]. The corresponding procedure involves nonequivalent canonical bases from Part I. It is proved that Lie algebra of Lorentz group has no connected 5-dimensional subgroups. This means also that Lorentz group has no 5-dimensional subalgebras.

Part I

Canonical bases for 5-dimensional subspaces of a 6-dimensional vector space

Let
c = a_1 c_1 + a_2 c_2 + a_3 c_3 + a_4 c_4 + a_5 c_5 + a_6 c_6, \quad \vec{b} = b_1 c_1 + b_2 c_2 + b_3 c_3 + b_4 c_4 + b_5 c_5 + b_6 c_6, \quad \vec{d} = d_1 c_1 + d_2 c_2 + d_3 c_3 + d_4 c_4 + d_5 c_5 + d_6 c_6, \quad \vec{f} = f_1 c_1 + f_2 c_2 + f_3 c_3 + f_4 c_4 + f_5 c_5 + f_6 c_6,

\begin{align}
\vec{a} &= a_1 c_1 + a_2 c_2 + a_3 c_3 + a_4 c_4 + a_5 c_5 + a_6 c_6, \\
\vec{b} &= b_1 c_1 + b_2 c_2 + b_3 c_3 + b_4 c_4 + b_5 c_5 + b_6 c_6, \\
\vec{d} &= d_1 c_1 + d_2 c_2 + d_3 c_3 + d_4 c_4 + d_5 c_5 + d_6 c_6, \\
\vec{f} &= f_1 c_1 + f_2 c_2 + f_3 c_3 + f_4 c_4 + f_5 c_5 + f_6 c_6,
\end{align}

\begin{align}
\vec{J} &= f_1 c_1 + f_2 c_2 + f_3 c_3 + f_4 c_4 + f_5 c_5 + f_6 c_6 \quad \text{is a general basis for arbitrary 5-dimensional subspace S of a 6-dimensional vector space V with its standard basis } \{e_1, e_2, e_3, e_4, e_5, e_6\}. \quad \text{We associate the next matrix } M \text{ with the basis (I)}
\end{align}

\begin{align}
M = \begin{bmatrix}
a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\
b_1 & b_2 & b_3 & b_4 & b_5 & b_6 \\
c_1 & c_2 & c_3 & c_4 & c_5 & c_6 \\
d_1 & d_2 & d_3 & d_4 & d_5 & d_6 \\
f_1 & f_2 & f_3 & f_4 & f_5 & f_6
\end{bmatrix}
\end{align}

Definition 1

The basis (I) is called canonical if its associated matrix M is in reduced row echelon form.

Definition 2

Two bases are called equivalent if they generate the same subspace of a given vector space, and two bases are nonequivalent if they generate different subspaces.

We start our transformation procedure for the basis (I) to find all canonical nonequivalent bases for the subspace S.

Suppose that at least one coefficient from a_1, b_1, c_1, d_1, f_1 is not zero. Without any loss in the generality, let a_1 ≠ 0. Perform the linear operation \( \bar{\vec{a}} = a_1 \vec{a} \), and the operations \( \bar{\vec{b}} = b_2 \vec{b} + b_3 \vec{c} + b_4 \vec{d} + b_5 \vec{f}, \quad \bar{\vec{d}} = d_1 \vec{d} + d_2 \vec{e} + d_3 \vec{f}, \quad \bar{\vec{f}} = f_1 \vec{f} + f_2 \vec{g} + f_3 \vec{h} + f_4 \vec{i} + f_5 \vec{j} + f_6 \vec{k} \). Then the following basis is obtained

\begin{align}
\bar{\vec{a}} &= a_1 c_1 + a_2 c_2 + a_3 c_3 + a_4 c_4 + a_5 c_5 + a_6 c_6, \\
\bar{\vec{b}} &= b_2 c_2 + b_3 c_3 + b_4 c_4 + b_5 c_5 + b_6 c_6, \\
\bar{\vec{d}} &= d_2 c_2 + d_3 c_3 + d_4 c_4 + d_5 c_5 + d_6 c_6, \\
\bar{\vec{f}} &= f_2 c_2 + f_3 c_3 + f_4 c_4 + f_5 c_5 + f_6 c_6,
\end{align}

Remark

The first components of vectors \( \bar{\vec{a}}, \bar{\vec{b}}, \bar{\vec{d}}, \bar{\vec{f}} \) are changed as the result of the operations performed but all other components of them are saved just for convenience. This idea will be used throughout of Part I.

Suppose now that at least one coefficient from b_1, c_1, d_1, f_1 in the basis (a) is not zero. Without any loss in generality, let b_1 ≠ 0. Perform the first linear operation \( \bar{\vec{b}} = b_1 \vec{b} \), and the operations \( \bar{\vec{a}} = a_1 \bar{\vec{a}} – a_1 \bar{\vec{b}}, \quad \bar{\vec{c}} = c_1 \bar{\vec{c}}, \quad \bar{\vec{f}} = f_1 \bar{\vec{f}} \). Then the following new basis is obtained

\begin{align}
\bar{\vec{a}} &= a_1 c_1 + a_2 c_2 + a_3 c_3 + a_4 c_4 + a_5 c_5 + a_6 c_6, \\
\bar{\vec{b}} &= b_1 c_1 + b_2 c_2 + b_3 c_3 + b_4 c_4 + b_5 c_5 + b_6 c_6, \\
\bar{\vec{c}} &= c_1 c_1 + c_2 c_2 + c_3 c_3 + c_4 c_4 + c_5 c_5 + c_6 c_6, \\
\bar{\vec{d}} &= d_1 \bar{\vec{d}} + d_2 \bar{\vec{e}} + d_3 \bar{\vec{f}},
\end{align}

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Suppose that at least one coefficient among $c_i, d_i, f_i$ in the basis (1) is not zero. Again, without any loss in the generality, let $c_i \neq 0$. Perform the first operation $c_i/c_i$, and the operations $\bar{a} - a_i \bar{c}, \bar{b} - b_i \bar{c}, \bar{d} - d_i \bar{c}$, $\bar{f} - f_i \bar{c}$ then. We obtain the following basis
\begin{align*}
\bar{b} &= e_1 + b_i e_i + b_i e_i + b_i e_i, \\
\bar{c} &= e_1 + c_i e_i + c_i e_i, \\
\bar{f} &= f_i e_i + f_i e_i.
\end{align*}
(2)

Suppose now that at least one coefficient from $d_i, f_i$ in the basis (2) is not zero. Let $d_i \neq 0$. Perform the operation $\bar{d} / d_i$ first, and then the operations $\bar{a} - a_i \bar{d}, \bar{b} - b_i \bar{d}, \bar{c} - c_i \bar{d}, \bar{f} - f_i \bar{d}$. The new transformed basis is
\begin{align*}
\bar{a} &= e_1 + a_i e_i, \\
\bar{b} &= e_1 + b_i e_i, \\
\bar{c} &= e_1 + c_i e_i, \\
\bar{f} &= f_i e_i + f_i e_i.
\end{align*}
(3)

At least one coefficient from $f_i, f_i$ is not zero in the basis (3). If $f_i \neq 0$, then perform the operation $\bar{f} / f_i$ first, and the operations $\bar{a} - a_i \bar{f}, \bar{b} - b_i \bar{f}, \bar{c} - c_i \bar{f}, \bar{d} - d_i \bar{f}$ after the first one. The following canonical basis is obtained
\begin{align*}
\bar{a} &= e_1 + a_i e_i, \\
\bar{b} &= e_1 + b_i e_i, \\
\bar{c} &= e_1 + c_i e_i, \\
\bar{f} &= f_i e_i + f_i e_i.
\end{align*}
(a)

If $f_i = 0$, then perform operation $\bar{f} / f_i$, and the operations $\bar{b} - b_i \bar{f}, \bar{c} - c_i \bar{f}, \bar{d} - d_i \bar{f}$ after the first one. The new basis is obtained
\begin{align*}
\bar{a} &= e_1 + a_i e_i, \\
\bar{b} &= e_1 + b_i e_i, \\
\bar{c} &= e_1 + c_i e_i, \\
\bar{f} &= f_i e_i + f_i e_i.
\end{align*}
\begin{align*}
\bar{b} &= e_1 + b_i e_i, \quad \bar{c} = e_1 + c_i e_i, \quad \bar{d} = e_1 + d_i e_i, \\
\bar{f} &= f_i e_i + f_i e_i.
\end{align*}
(b)

The last basis is equivalent to the basis (a) if $f_i \neq 0$. So, $f_i = 0$, and the new canonical basis is obtained
\begin{align*}
\bar{a} &= e_1 + a_i e_i, \\
\bar{b} &= e_1 + b_i e_i, \\
\bar{c} &= e_1 + c_i e_i, \\
\bar{f} &= f_i e_i + f_i e_i.
\end{align*}
(a)

1. Suppose that both coefficients $d_i, f_i$ at the basis (2) are zero. We have
\begin{align*}
\bar{a} &= e_1 + a_i e_i, \\
\bar{b} &= e_1 + b_i e_i, \\
\bar{c} &= e_1 + c_i e_i, \\
\bar{f} &= f_i e_i + f_i e_i.
\end{align*}
(4)

Suppose that at least one coefficient from $d_i, f_i$ at (4) is not zero. It’s easy to see that the alternative case with $d_i = 0$ and $f_i = 0$ is impossible because the corresponding vectors $\bar{d} = d_i e_i, \bar{f} = f_i e_i$ are linearly dependent. Let $d_i \neq 0$. Perform operation $\bar{d} / d_i$ first, and the operations $\bar{a} - a_i \bar{d}, \bar{b} - b_i \bar{d}, \bar{f} - f_i \bar{d}, \bar{f} - f_i \bar{d}$ next. The following basis is obtained
\begin{align*}
\bar{a} &= e_1 + a_i e_i, \\
\bar{b} &= e_1 + b_i e_i, \\
\bar{c} &= e_1 + c_i e_i, \\
\bar{f} &= f_i e_i + f_i e_i.
\end{align*}
(5)

It’s obvious that $f_i \neq 0$ for the vector $\bar{f}$ at the last basis. Perform the operation $\bar{f} / f_i$ first, and the operations $\bar{a} - a_i \bar{f}, \bar{b} - b_i \bar{f}, \bar{c} - c_i \bar{f}, \bar{d} - d_i \bar{f}$ after the first one. We obtain the new canonical basis
\begin{align*}
\bar{a} &= e_1 + a_i e_i, \\
\bar{b} &= e_1 + b_i e_i, \\
\bar{c} &= e_1 + c_i e_i, \\
\bar{d} &= e_1 + d_i e_i,
\end{align*}
(6)
\[ a = a_1 e_1 + a_2 e_2 + a_3 e_3 + a_4 e_4 + a_5 e_5, \quad b = b_1 e_1 + b_2 e_2 + b_3 e_3 + b_4 e_4 + b_5 e_5, \]
\[ c = c_1 e_1 + c_2 e_2 + c_3 e_3 + c_4 e_4 + c_5 e_5. \]  
(6)

It's obvious that 3 vectors \( \vec{a}, \vec{b}, \vec{c}, \vec{d}, \vec{f} \) at (6) are located at the same plane determined by vectors \( e_1, e_2, \). So, they are linearly dependent that contradicts the fact that all vectors \( \vec{a}, \vec{b}, \vec{c}, \vec{d}, \vec{f} \) are linearly independent. So, the case \( \epsilon_4 = 0, d_5 = 0 \) doesn't generate any canonical basis.

4. Suppose, in opposition to the Step 1, that the second coefficients \( b_2, c_2, d_2, f_2 \) at the basis (7) are zero. We obtain the following basis

\[ a = a_1 e_1 + a_2 e_2 + a_3 e_3 + a_4 e_4 + a_5 e_5, \]
\[ b = b_1 e_1 + b_2 e_2 + b_3 e_3 + b_4 e_4 + b_5 e_5, \]
\[ c = c_1 e_1 + c_2 e_2 + c_3 e_3 + c_4 e_4 + c_5 e_5, \]
\[ \bar{d} = d_1 e_1 + d_2 e_2 + d_3 e_3 + d_4 e_4 + d_5 e_5, \]
\[ \bar{f} = f_1 e_1 + f_2 e_2 + f_3 e_3 + f_4 e_4 + f_5 e_5. \]  
(7)

Consider coefficients \( b_1, c_1, d_1, f_1 \) at the basis (7). Suppose that at least one of them is not zero. Without any loss in the generality, let \( b_1 \neq 0 \). Perform the operation \( \bar{b} = b_1 \), first, and then the operations \( a - a b_1, \bar{c} = -c b_1, \bar{d} = -d b_1, \bar{f} = -f b_1 \). The following basis is obtained

\[ a = a_1 e_1 + a_2 e_2 + a_3 e_3 + a_4 e_4 + a_5 e_5, \quad \bar{b} = b_1 e_1 + b_2 e_2 + b_3 e_3 + b_4 e_4 + b_5 e_5, \]
\[ c = c_1 e_1 + c_2 e_2 + c_3 e_3 + c_4 e_4 + c_5 e_5, \]
\[ \bar{d} = d_1 e_1 + d_2 e_2 + d_3 e_3 + d_4 e_4 + d_5 e_5, \]
\[ \bar{f} = f_1 e_1 + f_2 e_2 + f_3 e_3 + f_4 e_4 + f_5 e_5. \]  
(8)

At least one coefficient among \( d_1, f_1 \) in the basis (8) is not zero. If both coefficients \( d_1, f_1 \) are zero, then vectors \( \bar{d} = d_1 e_1, \bar{f} = f_1 e_1 \) are linearly dependent but it's impossible for (8) to be a basis. Let \( d_1 \neq 0 \). Perform the operation \( \bar{d} = d_1 \), first, and the operations \( a - a \bar{d}, \bar{b} = b_1 \bar{d}, \bar{c} = -c \bar{d}, \bar{d} = -d \bar{d}, \bar{f} = -f \bar{d} \) after the first one. The following basis is obtained

\[ a = a_1 e_1 + a_2 e_2 + a_3 e_3 + a_4 e_4 + a_5 e_5, \]
\[ \bar{b} = b_1 e_1 + b_2 e_2 + b_3 e_3 + b_4 e_4 + b_5 e_5, \]
\[ \bar{c} = c_1 e_1 + c_2 e_2 + c_3 e_3 + c_4 e_4 + c_5 e_5, \]
\[ \bar{d} = d_1 e_1 + d_2 e_2 + d_3 e_3 + d_4 e_4 + d_5 e_5, \]
\[ \bar{f} = f_1 e_1 + f_2 e_2 + f_3 e_3 + f_4 e_4 + f_5 e_5. \]  
(9)

It's obvious that \( f_2 = 0 \) at the last basis, and it generates the new canonical basis

\[ a = a_1 e_1 + a_2 e_2 + a_3 e_3 + a_4 e_4 + a_5 e_5, \quad b = b_1 e_1 + b_2 e_2 + b_3 e_3 + b_4 e_4 + b_5 e_5, \]
\[ c = c_1 e_1 + c_2 e_2 + c_3 e_3 + c_4 e_4 + c_5 e_5. \]  
(10)

If \( f_2 = 0 \) at the basis (8), perform the operation \( \bar{f} = f_1 \), first, and then the operations \( a - a \bar{f}, \bar{b} = b_1 \bar{f}, \bar{c} = -c \bar{f}, \bar{d} = -d \bar{f} \) at the basis (8). The following basis obtained

\[ a = a_1 e_1 + a_2 e_2 + a_3 e_3 + a_4 e_4 + a_5 e_5, \quad b = b_1 e_1 + b_2 e_2 + b_3 e_3 + b_4 e_4 + b_5 e_5, \]
\[ c = c_1 e_1 + c_2 e_2 + c_3 e_3 + c_4 e_4 + c_5 e_5, \]
\[ \bar{d} = d_1 e_1 + d_2 e_2 + d_3 e_3 + d_4 e_4 + d_5 e_5, \]
\[ \bar{f} = f_1 e_1 + f_2 e_2 + f_3 e_3 + f_4 e_4 + f_5 e_5. \]  
(11)

It's obvious that \( d_2 \neq 0 \) in the last basis. So, the following canonical basis is generated

\[ a = a_1 e_1 + a_2 e_2 + a_3 e_3 + a_4 e_4 + a_5 e_5, \quad b = b_1 e_1 + b_2 e_2 + b_3 e_3 + b_4 e_4 + b_5 e_5, \]
\[ c = c_1 e_1 + c_2 e_2 + c_3 e_3 + c_4 e_4 + c_5 e_5, \]
\[ \bar{d} = d_1 e_1 + d_2 e_2 + d_3 e_3 + d_4 e_4 + d_5 e_5, \]
\[ \bar{f} = f_1 e_1 + f_2 e_2 + f_3 e_3 + f_4 e_4 + f_5 e_5. \]  
(12)
Consider coefficients $d_i, f_i$ in the basis (3). At least one of them is not zero. Otherwise, the vectors $\vec{a} = a_i e_i$ and $\vec{f} = f_i e_i$ are linearly dependent but it’s impossible. Suppose that $d_i \neq 0$. Perform the operation $\vec{a} = d_i e_i$ first, and the operations $\vec{b} = b_i \vec{a}, \vec{c} = c_i \vec{a}, \vec{f} = f_i \vec{a}$ after the first one. The following basis is obtained

$$
\vec{a} = e_i + a_i c_i e_i, \quad \vec{b} = e_i + b_i c_i e_i, \quad \vec{c} = e_i + c_i e_i, \quad \vec{d} = e_i + d_i e_i, \quad \vec{f} = \vec{e}_i + f_i \vec{e}_i.
$$

The coefficient $f_i$ is not zero in the basis (4). Perform the operation $\vec{f} / f_i$ in the basis (4) first, and the linear operations $a_i = a_i f_i, b_i = b_i f_i, c_i = c_i f_i, d_i = d_i f_i$ after the first one. We obtain the following new canonical basis

$$
\vec{a} = e_i, \quad \vec{b} = e_i, \quad \vec{c} = e_i, \quad \vec{d} = e_i, \quad \vec{f} = \vec{e}_i.
$$

If $f_i \neq 0$ in the basis (4), we obtain the same canonical basis (b.).

2. Suppose that all coefficients $a_i, b_i, c_i, d_i, f_i$ are zero in the basis (b). The following possible basis is obtained

$$
\vec{a} = a_i e_i + a_i c_i e_i + a_i e_i + a_i e_i, \quad \vec{b} = b_i e_i + b_i e_i + b_i e_i + b_i e_i, \quad \vec{c} = c_i e_i + c_i e_i + c_i e_i + c_i e_i, \quad \vec{d} = d_i e_i + d_i e_i + d_i e_i + d_i e_i, \quad \vec{f} = \vec{f}_i + f_i \vec{e}_i.
$$

These 5 vectors $\vec{a}, \vec{b}, \vec{c}, \vec{d}, \vec{f}$ in (5) are linearly independent because they are located at the same 4-dimensional subspace generated by vectors $e_i, e_i, e_i, e_i$. So, the system (5) of the vectors $\vec{a}, \vec{b}, \vec{c}, \vec{d}, \vec{f}$ doesn’t produce any canonical basis.

The research performed for the set of coefficients $a_i, b_i, c_i, d_i, f_i$ in the cases A and B produces 6 canonical bases $a_i$, $b_i$, $c_i$, $d_i$, $f_i$.

To find other canonical bases, we should repeat the similar evaluations considering the following five sets of coefficients $(a_i, b_i, c_i, d_i, f_i)$, $(a_i, b_i, c_i, d_i, f_i)$, $(a_i, b_i, c_i, d_i, f_i)$, $(a_i, b_i, c_i, d_i, f_i)$, $(a_i, b_i, c_i, d_i, f_i)$, $(a_i, b_i, c_i, d_i, f_i)$.

The basis (1) is equivalent to the basis $(a_i, b_i, c_i, d_i, f_i)$ for all 6 bases, respectively. The bases $(a_i, b_i, c_i, d_i, f_i)$ are obviously equivalent to the basis (a). The basis (c) is equivalent to the basis (a) if $a_i = 0$, and (c) is equivalent to the basis (b) if $g_i = 0$. The bases (e) – (e) are different. The basis (e) is equivalent to the basis (a) if $a_i = 0$, and (e) is a particular case of (g) if $a_i = 0$.

The bases (g) – (g) are equivalent to the bases (a), (a), (a), (a), (a), (a). The basis (c) is equivalent to the basis (a) if $a_i = 0$, and (c) is a particular case of (g) if $a_i = 0$. The bases (g) – (g) are equivalent to the bases (a), (a), (a), (a), (a), (a).
and \((i)\) is equivalent to \((a)\) if \(f_0 \neq 0\). The basis \((i)\) is a particular case of the basis \((a)\) if \(a \neq 0\), and the basis \((i)\) is equivalent to the basis \((a)\) if \(a = 0\). The basis \((i)\) is a particular case of the basis \((a)\) if \(a \neq 0\), and \((i)\) is equivalent to the basis \((a)\) if \(a = 0\). The basis \((i)\) is a particular case of the basis \((a)\) if \(a \neq 0\), and \((i)\) is equivalent to the basis \((a)\) if \(a = 0\). The basis \((i)\) is a particular case of the basis \((a)\) if \(a \neq 0\), and \((i)\) is equivalent to the basis \((a)\) if \(a = 0\). The basis \((i)\) is a particular case of the basis \((a)\) if \(a \neq 0\), and \((i)\) is equivalent to the basis \((a)\) if \(a = 0\). The basis \((i)\) is a particular case of the basis \((a)\) if \(a \neq 0\), and \((i)\) is equivalent to the basis \((a)\) if \(a = 0\).

The analysis performed above implies the following statement.

**Theorem 1**

Each basis of any 5-dimensional subspace in a 6-dimensional vector space is equivalent to one and only one of the following 6 canonical bases

\[
\begin{align*}
\mathbf{a} &= \mathbf{e}_1 + \mathbf{e}_2, \\
\mathbf{b} &= \mathbf{e}_3 + \mathbf{e}_4, \\
\mathbf{c} &= \mathbf{e}_5 + \mathbf{e}_6, \\
\mathbf{f} &= \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3 + \mathbf{e}_4 + \mathbf{e}_5 + \mathbf{e}_6.
\end{align*}
\]

This group is not compact, not abelian, and

\[
\begin{align*}
\text{vol} \mathbf{L} &= \pi \cdot \text{vol} \mathbf{L}, \\
\text{vol} \mathbf{L} &= \pi \cdot \text{vol} \mathbf{L}.
\end{align*}
\]

**Part II**

5-dimensional subalgebras of Lie algebra of Lorentz group

Introduction: Lorentz group is the group of transformations of Minkowski space-time \(\mathbb{R}^4\). This group is not compact, not abelian, and not connected 6-dimensional real Lie group. The identity component of Lorentz group is the group \(SO(3,1)\). This component contains the generators for boosts along \(x\)-, \(y\)-, and \(z\)-axis, and it contains the generators for rotations in Minkowski space-time [4]. Lie algebra of the group \(SO(3,1)\) is 6-dimensional real Lie algebra denoted below by \(L\) that has the following standard basis:

| \(i\) | \(e_1\) | \(e_2\) | \(e_3\) | \(e_4\) | \(e_5\) | \(e_6\) |
|------|--------|--------|--------|--------|--------|--------|
| 1    | 1 0 0 0 | 0 0 1 0 | 0 0 0 0 | 0 0 0 0 | 0 0 0 0 | 0 0 0 0 |
| 2    | 0 1 0 0 | 0 1 0 0 | 0 0 0 0 | 0 0 0 0 | 0 0 0 0 | 0 0 0 0 |
| 3    | 0 0 1 0 | 1 0 0 0 | 0 0 0 0 | 0 0 0 0 | 0 0 0 0 | 0 0 0 0 |
| 4    | 0 0 0 1 | 0 0 0 0 | 0 0 0 0 | 0 0 0 0 | 0 0 0 0 | 0 0 0 0 |
| 5    | 0 0 0 0 | 0 0 0 0 | 0 0 0 0 | 0 0 0 0 | 0 0 0 0 | 0 0 0 0 |
| 6    | 0 0 0 0 | 0 0 0 0 | 0 0 0 0 | 0 0 0 0 | 0 0 0 0 | 0 0 0 0 |

The Lie product of any two square matrices \(A, B\) is defined by \([A, B] = AB - BA\). For the standard basis of Lie algebra of Lorentz group, the non-zero products are

\[
\begin{align*}
[\mathbf{e}_1, \mathbf{e}_2] &= \mathbf{e}_3, \\
[\mathbf{e}_1, \mathbf{e}_3] &= \mathbf{e}_2, \\
[\mathbf{e}_2, \mathbf{e}_4] &= \mathbf{e}_5, \\
[\mathbf{e}_3, \mathbf{e}_5] &= \mathbf{e}_4, \\
[\mathbf{e}_1, \mathbf{e}_4] &= -\mathbf{e}_6, \\
\end{align*}
\]

To determine which 5-dimensional subspace \(h\) of the given Lie algebra \(L\) is a subalgebra of \(L\), we will check the condition \([h, h] \subset h\) applying to the nonequivalent canonical bases that are described in the Theorem 1.

Let the subspace \(h\) be generated by the canonical basis \((a)\). Compute all products between vectors \(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{f}\) in this basis. Utilizing the table of products \((\ast)\), we have

\[
\begin{align*}
[\mathbf{a}, \mathbf{b}] &= \mathbf{e}_1 + \mathbf{e}_2, \\
[\mathbf{a}, \mathbf{c}] &= \mathbf{e}_3 + \mathbf{e}_4, \\
[\mathbf{a}, \mathbf{d}] &= \mathbf{e}_5 + \mathbf{e}_6, \\
[\mathbf{b}, \mathbf{c}] &= \mathbf{e}_7 + \mathbf{e}_8, \\
[\mathbf{b}, \mathbf{d}] &= \mathbf{e}_9 + \mathbf{e}_10, \\
[\mathbf{c}, \mathbf{d}] &= \mathbf{e}_11 + \mathbf{e}_12, \\
[\mathbf{a}, \mathbf{f}] &= -\mathbf{a}, \\
[\mathbf{b}, \mathbf{f}] &= -\mathbf{b}, \\
[\mathbf{c}, \mathbf{f}] &= -\mathbf{c}, \\
[\mathbf{d}, \mathbf{f}] &= -\mathbf{d}.
\end{align*}
\]

So, \(a, b, c, d, f\) are linearly independent. The equation \(\mathbf{a} = \mathbf{b} = \mathbf{c} = \mathbf{d} = \mathbf{f} = 0\) has no solution in the set of all real numbers. This means that no 5-dimensional subalgebra of Lie algebra \(L\) with the basis \((a)\) exists.

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[\mathbf{a}, \mathbf{c}] &= \mathbf{e}_3 + \mathbf{e}_4, \\
[\mathbf{a}, \mathbf{d}] &= \mathbf{e}_5 + \mathbf{e}_6, \\
[\mathbf{b}, \mathbf{c}] &= \mathbf{e}_7 + \mathbf{e}_8, \\
[\mathbf{b}, \mathbf{d}] &= \mathbf{e}_9 + \mathbf{e}_10, \\
[\mathbf{c}, \mathbf{d}] &= \mathbf{e}_11 + \mathbf{e}_12, \\
[\mathbf{a}, \mathbf{f}] &= \mathbf{a}, \\
[\mathbf{b}, \mathbf{f}] &= \mathbf{b}, \\
[\mathbf{c}, \mathbf{f}] &= \mathbf{c}, \\
[\mathbf{d}, \mathbf{f}] &= \mathbf{d}.
\end{align*}
\]

So, \(a, b, c, d, f\) are linearly independent. The equation \(\mathbf{a} = \mathbf{b} = \mathbf{c} = \mathbf{d} = \mathbf{f} = 0\) has no solution in the set of all real numbers. This means that no 5-dimensional subalgebra of Lie algebra \(L\) with the basis \((a)\) exists.
Consider all products between vectors \((a, b, c, d, f)\) in this basis. Utilizing the table of products \((*)\) we have
\[
\begin{align*}
(a, b) &= \left( e_1 + a e_2 e_3 - b e_4, e_5 \right) = e_1 - b e_4, a_1 = x_1 b + x_2 c + x_3 d + x_4 f, \\
(b, c) &= \left( e_2 + b e_3, e_4 - c e_5 \right) = e_2 - c e_5, a_2 = y_1 a + y_2 b + y_3 c + y_4 d + y_5 f, \\
(c, d) &= \left( e_3 + c e_4 - d e_5, e_5 \right) = e_3 - d e_5, a_3 = x_1 a + x_2 b + x_3 c + x_4 d + x_5 f, \\
(d, f) &= \left( e_4 + d e_5, e_5 \right) = e_4 - d e_5, a_4 = x_1 a + x_2 b + x_3 c + x_4 d + x_5 f, \\
\end{align*}
\]
Let the subspace \(h_i\) be generated by the canonical basis \((a_i)\). Consider all products between vectors \(\bar{a}_i, \bar{b}_i, \bar{c}_i, \bar{d}_i, \bar{f}_i\) in this basis. Utilizing the table of products \((*)\), we have
\[
\begin{align*}
\bar{a}_i, \bar{b}_i, \bar{c}_i, \bar{d}_i, \bar{f}_i &= \left( e_1 + a e_2 e_3 - b e_4, e_5 \right) = e_1 - b e_4, a_1 = x_1 b + x_2 c + x_3 d + x_4 f, \\
\bar{b}_i, \bar{c}_i, \bar{d}_i, \bar{f}_i &= \left( e_2 + b e_3, e_4 - c e_5 \right) = e_2 - c e_5, a_2 = y_1 a + y_2 b + y_3 c + y_4 d + y_5 f, \\
\bar{c}_i, \bar{d}_i, \bar{f}_i &= \left( e_3 + c e_4 - d e_5, e_5 \right) = e_3 - d e_5, a_3 = x_1 a + x_2 b + x_3 c + x_4 d + x_5 f, \\
\bar{d}_i, \bar{f}_i &= \left( e_4 + d e_5, e_5 \right) = e_4 - d e_5, a_4 = x_1 a + x_2 b + x_3 c + x_4 d + x_5 f, \\
\end{align*}
\]
Let the subspace \(h_i\) be generated by the canonical basis \((a_i)\). Consider all products between vectors \(\bar{a}_i, \bar{b}_i, \bar{c}_i, \bar{d}_i, \bar{f}_i\) in this basis. Utilizing the table of products \((*)\), we have
\[
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\bar{a}_i, \bar{b}_i, \bar{c}_i, \bar{d}_i, \bar{f}_i &= \left( e_1 + a e_2 e_3 - b e_4, e_5 \right) = e_1 - b e_4, a_1 = x_1 b + x_2 c + x_3 d + x_4 f, \\
\bar{b}_i, \bar{c}_i, \bar{d}_i, \bar{f}_i &= \left( e_2 + b e_3, e_4 - c e_5 \right) = e_2 - c e_5, a_2 = y_1 a + y_2 b + y_3 c + y_4 d + y_5 f, \\
\bar{c}_i, \bar{d}_i, \bar{f}_i &= \left( e_3 + c e_4 - d e_5, e_5 \right) = e_3 - d e_5, a_3 = x_1 a + x_2 b + x_3 c + x_4 d + x_5 f, \\
\bar{d}_i, \bar{f}_i &= \left( e_4 + d e_5, e_5 \right) = e_4 - d e_5, a_4 = x_1 a + x_2 b + x_3 c + x_4 d + x_5 f, \\
\end{align*}
\]