The Hausdorff dimension of fractal sets and fractional quantum Hall effect

Wellington da Cruz

Departamento de Física,

Universidade Estadual de Londrina, Caixa Postal 6001,
Cep 86051-970 Londrina, PR, Brazil
(February 11, 2022)

Abstract

We consider Farey series of rational numbers in terms of fractal sets labeled by the Hausdorff dimension with values defined in the interval $1 < h < 2$ and associated with fractal curves. Our results come from the observation that the fractional quantum Hall effect-FQHE occurs in pairs of dual topological quantum numbers, the filling factors. These quantum numbers obey some properties of the Farey series and so we obtain that the universality classes of the quantum Hall transitions are classified in terms of $h$. The connection between Number Theory and Physics appears naturally in this context.

PACS numbers: 47.53.+n, 02.10.De; 05.30.-d; 05.70.Ce; 11.25.Hf

Keywords: Fractal sets; Farey series; Hausdorff dimension; Fractional quantum Hall effect; Fractal distribution function; Fractons

E-mail address: wdacruz@exatas.uel.br
I. INTRODUCTION

We have obtained from physical considerations about fractional quantum Hall effect, a mathematical result related to the Farey series of rational numbers [1]. According to our approach the FQHE occurs in pairs of dual topological quantum numbers, the filling factors. These parameters characterize the quantization of the Hall resistance in some systems of the condensed matter under lower temperatures and intense external magnetic fields. The filling factor is defined by $f = N\frac{\phi}{\phi_0}$, where $N$ is the electron number, $\phi_0$ is the quantum unit of flux and $\phi$ is the flux of the external magnetic field throughout the sample. In our formulation the filling factor gets its topological character from the parameter $h$ to be defined.

We can check that all experimental data for the occurrence of FQHE satisfies a symmetry principle discovered by us, that is, the duality symmetry between universal classes $h$ of particles or quasiparticles with any value of spin [1–5]. For example, we have the dual filling factors

$$(f, \tilde{f}) = \left(\frac{1}{3}, \frac{2}{3}\right), \left(\frac{5}{3}, \frac{4}{3}\right), \left(\frac{1}{5}, \frac{4}{5}\right), \left(\frac{2}{5}, \frac{3}{5}\right), \left(\frac{2}{7}, \frac{5}{7}\right), \left(\frac{3}{7}, \frac{4}{7}\right), \left(\frac{4}{9}, \frac{5}{9}\right) \text{ etc.}$$

On the other hand, in Ref. [1] we have considered the filling factors into equivalence classes labeled by $h$ and so, we can estimate the occurrence of FQHE, just taking into account a fractal spectrum and a duality symmetry between the classes. A relation between equivalence classes $h$ and the modular group [6, 1] for the quantum phase transitions of the FQHE was also noted. As a consequence, the parameter $h$ classifies the universality class of these transitions. Another approach in the literature [7] has considered Cantor sets in connection with the FQHE and some relation with the Farey sequences.
II. HAUSDORFF DIMENSION, FRACTAL SETS AND FAREY SERIES

Now, considering fractal curves in connection with quantum paths, we have defined a fractal spectrum, which relates the parameter $h$ and the spin $s$ of the particle through the spin-statistics relation defined by $\nu = 2s = 2\frac{\phi}{\phi_0}$, where $\phi$ is the magnetic flux associated to the charge-flux system. Follows that,

\begin{align*}
    h - 1 &= 1 - \nu, \quad 0 < \nu < 1; \quad h - 1 = \nu - 1, \quad 1 < \nu < 2; \\
    h - 1 &= 3 - \nu, \quad 2 < \nu < 3; \quad h - 1 = \nu - 3, \quad 3 < \nu < 4; \\
    h - 1 &= 5 - \nu, \quad 4 < \nu < 5; \quad h - 1 = \nu - 5, \quad 5 < \nu < 6; \\
    h - 1 &= 7 - \nu, \quad 6 < \nu < 7; \quad h - 1 = \nu - 7, \quad 7 < \nu < 8;
\end{align*}

etc.

where $h$ is a fractal parameter or Hausdorff dimension defined into the interval $1 < h < 2$. According to our approach there is a correspondence between $f$ and $\nu$, numerically $f = \nu$.

The fractal curve is continuous and nowhere differentiable, it is self-similar, it does not depend on the scale and has fractal dimension just in that interval. Given a closed path with length $L$ and resolution $R$, the fractal properties of this curve can be determined by

\[ h - 1 = \lim_{R \to 0} \frac{\ln L/l}{\ln R}, \]

where $l$ is the usual length for the resolution $R$ and the curve is covering with $l/R$ spheres of diameter $R$.

Farey series $F_n$ of order $n$ is the increasing sequence of irreducible fractions in the range $0$ to $1$ whose denominators do not exceed $n$. They satisfy the properties

P1. If $\nu_1 = \frac{p_1}{q_1}$ and $\nu_2 = \frac{p_2}{q_2}$ are two consecutive fractions $\frac{p_1}{q_1} > \frac{p_2}{q_2}$, then $|p_2q_1 - q_2p_1| = 1$.

P2. If $\frac{p_1}{q_1}, \frac{p_2}{q_2}, \frac{p_3}{q_3}$ are three consecutive fractions $\frac{p_1}{q_1} > \frac{p_2}{q_2} > \frac{p_3}{q_3}$, then $\frac{p_2}{q_2} = \frac{p_1 + p_3}{q_1 + q_3}$.

P3. If $\frac{p_1}{q_1}$ and $\frac{p_2}{q_2}$ are consecutive fractions in the same sequence, then among all fractions between the two, $\frac{p_1 + p_2}{q_1 + q_2}$ is the unique reduced fraction with the smallest denominator.

Let us consider the fractal spectrum and the duality symmetry between the sets $h$, defined by $\tilde{h} = 3 - h$, so we have the following
Theorem:

The elements of the Farey series $F_n$ of the order $n$, belong to the fractal sets, whose Hausdorff dimensions are the second fractions of the fractal sets. The Hausdorff dimension has values within the interval $1 < h < 2$, which are associated with fractal curves.

For example, consider the Farey series of order 6 for any interval

| $h$ | 2 | $\frac{11}{6}$ | $\frac{9}{5}$ | $\frac{7}{4}$ | $\frac{5}{3}$ | $\frac{3}{2}$ | $\frac{7}{5}$ | $\frac{5}{4}$ | $\frac{4}{3}$ | $\frac{6}{5}$ | $\frac{7}{6}$ | $\frac{1}{1}$ |
|-----|---|---|---|---|---|---|---|---|---|---|---|---|
| $0 < \nu < 1$ | $\frac{2}{1}$ | $\frac{1}{6}$ | $\frac{1}{5}$ | $\frac{1}{4}$ | $\frac{1}{3}$ | $\frac{2}{5}$ | $\frac{1}{2}$ | $\frac{3}{5}$ | $\frac{2}{3}$ | $\frac{3}{4}$ | $\frac{4}{5}$ | $\frac{5}{6}$ | $\frac{1}{1}$ |
| $1 < \nu < 2$ | $\frac{2}{1}$ | $\frac{11}{6}$ | $\frac{9}{5}$ | $\frac{7}{4}$ | $\frac{5}{3}$ | $\frac{3}{2}$ | $\frac{7}{5}$ | $\frac{4}{3}$ | $\frac{5}{4}$ | $\frac{6}{5}$ | $\frac{7}{6}$ | $\frac{1}{1}$ |
| $2 < \nu < 3$ | $\frac{2}{1}$ | $\frac{13}{6}$ | $\frac{11}{5}$ | $\frac{9}{4}$ | $\frac{7}{3}$ | $\frac{5}{2}$ | $\frac{13}{6}$ | $\frac{8}{3}$ | $\frac{11}{4}$ | $\frac{14}{5}$ | $\frac{17}{6}$ | $\frac{3}{1}$ |
| $3 < \nu < 4$ | $\frac{4}{1}$ | $\frac{23}{12}$ | $\frac{19}{9}$ | $\frac{15}{6}$ | $\frac{11}{4}$ | $\frac{18}{6}$ | $\frac{7}{2}$ | $\frac{17}{5}$ | $\frac{19}{6}$ | $\frac{15}{4}$ | $\frac{16}{5}$ | $\frac{19}{6}$ | $\frac{3}{1}$ |
| $4 < \nu < 5$ | $\frac{4}{1}$ | $\frac{25}{12}$ | $\frac{21}{9}$ | $\frac{17}{6}$ | $\frac{13}{4}$ | $\frac{22}{6}$ | $\frac{9}{2}$ | $\frac{23}{5}$ | $\frac{14}{4}$ | $\frac{19}{6}$ | $\frac{24}{5}$ | $\frac{29}{6}$ | $\frac{5}{1}$ |
| $5 < \nu < 6$ | $\frac{6}{1}$ | $\frac{35}{12}$ | $\frac{29}{9}$ | $\frac{23}{6}$ | $\frac{17}{4}$ | $\frac{28}{6}$ | $\frac{11}{2}$ | $\frac{27}{5}$ | $\frac{16}{3}$ | $\frac{21}{4}$ | $\frac{26}{5}$ | $\frac{31}{6}$ | $\frac{5}{1}$ |
| $6 < \nu < 7$ | $\frac{6}{1}$ | $\frac{37}{12}$ | $\frac{31}{9}$ | $\frac{25}{6}$ | $\frac{19}{4}$ | $\frac{32}{6}$ | $\frac{13}{2}$ | $\frac{33}{5}$ | $\frac{20}{3}$ | $\frac{27}{4}$ | $\frac{34}{5}$ | $\frac{41}{6}$ | $\frac{7}{1}$ |
| $7 < \nu < 8$ | $\frac{8}{1}$ | $\frac{47}{12}$ | $\frac{39}{9}$ | $\frac{31}{6}$ | $\frac{21}{4}$ | $\frac{23}{3}$ | $\frac{38}{6}$ | $\frac{15}{2}$ | $\frac{37}{5}$ | $\frac{22}{3}$ | $\frac{29}{4}$ | $\frac{36}{5}$ | $\frac{43}{6}$ | $\frac{7}{1}$ |
| $8 < \nu < 9$ | $\frac{8}{1}$ | $\frac{49}{12}$ | $\frac{41}{9}$ | $\frac{33}{6}$ | $\frac{25}{4}$ | $\frac{19}{3}$ | $\frac{42}{6}$ | $\frac{17}{2}$ | $\frac{43}{5}$ | $\frac{26}{3}$ | $\frac{35}{4}$ | $\frac{44}{5}$ | $\frac{53}{6}$ | $\frac{9}{1}$ |
| $9 < \nu < 10$ | $\frac{10}{1}$ | $\frac{59}{12}$ | $\frac{49}{9}$ | $\frac{39}{6}$ | $\frac{29}{4}$ | $\frac{48}{6}$ | $\frac{19}{2}$ | $\frac{47}{5}$ | $\frac{28}{3}$ | $\frac{37}{4}$ | $\frac{46}{5}$ | $\frac{55}{6}$ | $\frac{9}{1}$ |
| $10 < \nu < 11$ | $\frac{10}{1}$ | $\frac{61}{12}$ | $\frac{51}{9}$ | $\frac{41}{6}$ | $\frac{31}{4}$ | $\frac{52}{6}$ | $\frac{21}{2}$ | $\frac{53}{5}$ | $\frac{32}{3}$ | $\frac{43}{4}$ | $\frac{54}{5}$ | $\frac{65}{6}$ | $\frac{11}{1}$ |
| $11 < \nu < 12$ | $\frac{12}{1}$ | $\frac{71}{12}$ | $\frac{59}{9}$ | $\frac{47}{6}$ | $\frac{36}{3}$ | $\frac{58}{6}$ | $\frac{23}{2}$ | $\frac{57}{5}$ | $\frac{34}{3}$ | $\frac{45}{4}$ | $\frac{56}{5}$ | $\frac{67}{6}$ | $\frac{11}{1}$ |
| $12 < \nu < 13$ | $\frac{12}{1}$ | $\frac{73}{12}$ | $\frac{61}{9}$ | $\frac{49}{6}$ | $\frac{37}{3}$ | $\frac{62}{6}$ | $\frac{25}{2}$ | $\frac{63}{5}$ | $\frac{38}{3}$ | $\frac{51}{4}$ | $\frac{64}{5}$ | $\frac{77}{6}$ | $\frac{13}{1}$ |
| $13 < \nu < 14$ | $\frac{14}{1}$ | $\frac{83}{12}$ | $\frac{69}{9}$ | $\frac{55}{6}$ | $\frac{41}{3}$ | $\frac{68}{6}$ | $\frac{27}{2}$ | $\frac{67}{5}$ | $\frac{40}{3}$ | $\frac{53}{4}$ | $\frac{66}{5}$ | $\frac{79}{6}$ | $\frac{13}{1}$ |
| $14 < \nu < 15$ | $\frac{14}{1}$ | $\frac{85}{12}$ | $\frac{71}{9}$ | $\frac{57}{6}$ | $\frac{43}{3}$ | $\frac{72}{6}$ | $\frac{29}{2}$ | $\frac{73}{5}$ | $\frac{44}{3}$ | $\frac{59}{4}$ | $\frac{74}{5}$ | $\frac{89}{6}$ | $\frac{15}{1}$ |
| $15 < \nu < 16$ | $\frac{16}{1}$ | $\frac{95}{12}$ | $\frac{79}{9}$ | $\frac{63}{6}$ | $\frac{47}{3}$ | $\frac{78}{6}$ | $\frac{31}{2}$ | $\frac{77}{5}$ | $\frac{46}{3}$ | $\frac{61}{4}$ | $\frac{76}{5}$ | $\frac{91}{6}$ | $\frac{15}{1}$ |
| $16 < \nu < 17$ | $\frac{16}{1}$ | $\frac{97}{12}$ | $\frac{81}{9}$ | $\frac{65}{6}$ | $\frac{49}{3}$ | $\frac{82}{6}$ | $\frac{33}{2}$ | $\frac{83}{5}$ | $\frac{50}{3}$ | $\frac{67}{4}$ | $\frac{84}{5}$ | $\frac{101}{6}$ | $\frac{17}{1}$ |
| $17 < \nu < 18$ | $\frac{18}{1}$ | $\frac{107}{12}$ | $\frac{89}{9}$ | $\frac{71}{6}$ | $\frac{53}{3}$ | $\frac{88}{6}$ | $\frac{35}{2}$ | $\frac{87}{5}$ | $\frac{52}{3}$ | $\frac{69}{4}$ | $\frac{86}{5}$ | $\frac{103}{6}$ | $\frac{17}{1}$ |
| ... | ... | ... | ... | ... | ... | ... | ... | ... | ... | ... | ... | ... | ... |
Then, we obtain *fractal sets* labeled by the Hausdorff dimension

\[
\left\{ \frac{1}{6}, \frac{11}{6}, \frac{13}{6}, \frac{23}{6}, \frac{25}{6}, \frac{35}{6}, \frac{37}{6}, \frac{47}{6}, \frac{49}{6}, \frac{59}{6}, \frac{61}{6}, \ldots \right\}_{h = \frac{3}{2}},
\]

\[
\left\{ \frac{1}{5}, \frac{9}{5}, \frac{11}{5}, \frac{19}{5}, \frac{21}{5}, \frac{29}{5}, \frac{31}{5}, \frac{39}{5}, \frac{41}{5}, \frac{51}{5}, \ldots \right\}_{h = \frac{4}{5}},
\]

\[
\left\{ \frac{1}{4}, \frac{3}{4}, \frac{7}{4}, \frac{9}{4}, \frac{15}{4}, \frac{17}{4}, \frac{23}{4}, \frac{25}{4}, \frac{31}{4}, \frac{33}{4}, \ldots \right\}_{h = \frac{5}{4}},
\]

\[
\left\{ \frac{1}{3}, \frac{5}{3}, \frac{11}{3}, \frac{13}{3}, \frac{23}{3}, \frac{29}{3}, \frac{35}{3}, \frac{37}{3}, \frac{43}{3}, \frac{49}{3}, \ldots \right\}_{h = \frac{6}{5}},
\]

\[
\left\{ \frac{1}{2}, \frac{5}{2}, \frac{9}{2}, \frac{11}{2}, \frac{17}{2}, \frac{21}{2}, \frac{23}{2}, \frac{29}{2}, \frac{31}{2}, \frac{35}{2}, \ldots \right\}_{h = \frac{7}{5}}.
\]

Observe that the sets are dual sets and, in particular, we have a fractal selfdual set, with

Hausdorff dimension \( h = \frac{3}{2} \). Thus, in this way we can extract for any Farey series of rational numbers, *taking into account the fractal spectrum and the duality symmetry between sets,* fractal sets whose Hausdorff dimension is the second fraction of the set.

By another method, we have obtained for *fractons* or charge-flux systems, that is, particles with any value of spin defined in two-dimensional multiply connected space, a *fractal distribution function* [1,2]
\[ n = \frac{1}{\mathcal{Y}[\xi] - h} \]  \hspace{1cm} (7)

where \( \mathcal{Y}[\xi] \) satisfies the equation

\[ \xi = \left\{ \mathcal{Y}[\xi] - 1 \right\}^{h - 1} \left\{ \mathcal{Y}[\xi] - 2 \right\}^{2 - h} \]  \hspace{1cm} (8)

and \( \xi = \exp \left\{ (\epsilon - \mu)/KT \right\} \) has the usual definition.

This quantum-geometrical description of the statistical laws of nature is associated with a fractal von Neumann entropy per state in terms of the average occupation number

\[ S_G[h, n] = K \left[ [1 + (h - 1)n] \ln \left( \frac{1 + (h - 1)n}{n} \right) - [1 + (h - 2)n] \ln \left( \frac{1 + (h - 2)n}{n} \right) \right]. \]  \hspace{1cm} (9)

An interesting point is that the solutions for the algebraic equations given by the Eq.(8) are of the form

\[ \mathcal{Y}_h[\xi] = f[\xi] + \tilde{h} \]

or

\[ \mathcal{Y}_\tilde{h}[\xi] = g[\xi] + h. \]

The functions \( f[\xi] \) and \( g[\xi] \) at least for third, fourth degrees algebraic equation differ by signals \( \pm \) in some terms of their expressions. Observe also that the solution for a given \( h \) receives its dual \( \tilde{h} \) as a constant. We can conjecture if this result gives us some information about these classes of algebraic equations. In Ref. [3] we have also defined a topological concept termed fractal index related to the Rogers dilogarithm function and the concept of central charge associated with the conformal field theories. The fractal index can assume rational values and are related to the algebraic numbers which come from of the algebraic equation for each value of \( h \) by the Eq.(8). The connection between Physics and Number Theory is manifest because, on the one hand, we have the FQHE characterized in terms of a geometrical parameter related to the fractal curves and, on the other hand, the dilogarithm function appears in various branches of mathematics besides number theory, such as [8]:

\[ 6 \]
exactly solvable models, algebraic K-theory, hyperbolic manifolds, low dimensional topology etc. Now, as was introduced in [3], we have also established a connection between fractal geometry and Rogers dilogarithm function.

III. CONCLUSIONS

In this Letter we have determined an algorithm for computation of the Hausdorff dimension of any fractal set related to the Farey series. This mathematical result is related to the observation that the FQHE occurs in pairs of dual topological filling factors [1]. In this way, these quantum numbers are classified in classes labeled by the parameter $h$ and satisfy the properties of the Farey sequences. Thus, the universality classes of the quantum Hall transitions are classified in terms of $h$ and this result is obtained from our analysis considering the modular group for these transitions [1,6]. We have also, in our context, made a connection between Physics and Number Theory relating the fractal geometry and dilogarithm function through of the concept of fractal index introduced by us [3]. We have also the possibility of fractal sets with irrational values for the Hausdorff dimension.

ACKNOWLEDGMENTS

The author thanks the referee by the comments and suggestions.
REFERENCES

[1] W. da Cruz, Int. J. Mod. Phys. 2000;A15:3805-3828.

[2] W. da Cruz, Physica 2002;A313:446-452.

[3] W. da Cruz and R. de Oliveira, Mod. Phys. Lett. 2000;A15:1931-1939.

[4] W. da Cruz, J. Phys: Cond. Matter. 2000;12:L673-L675.

[5] W. da Cruz, Mod. Phys. Lett. 1999;A14:1933-1936.

[6] B. P. Dolan, J. Phys. 1999;A32:L243-L248; Nucl. Phys. 1999;B554:487-513.

[7] M. S. El Naschie, Chaos, Solitons and Fractals, 1999;10:567-580; Chaos, Solitons and Fractals, 1996;4:499-518.

[8] A. Kirillov, Prog. Theor. Phys. Suppl. 1995;118:61-142.