Numerical analyses of $\mathcal{N} = 2$ supersymmetric quantum mechanics with cyclic Leibniz rule on lattice

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Abstract

We study a cyclic Leibniz rule, which provides a systematic approach to lattice supersymmetry, using a numerical method with a transfer matrix. The computation is carried out in $\mathcal{N} = 2$ supersymmetric quantum mechanics with the $\phi^6$-interaction for weak and strong couplings. The computed energy spectra and supersymmetric Ward-Takahashi identities are compared with those obtained from another lattice action. We find that a model with the cyclic Leibniz rule behaves similarly to the continuum theory compared with the other lattice action.
I. INTRODUCTION

The difficulty in lattice supersymmetry (SUSY) is originated from the lack of Leibniz rule \[1\]. Since any local lattice difference operator does not obey Leibniz rule \[2, 3\], it is difficult to realize the full SUSY within a local lattice theory \[1, 4–6\]. Several approaches in which part of SUSY is kept on the lattice and the full symmetry is restored at the continuum limit have been proposed so far \[7–20\]. Those are, however, the same in a sense that, without getting into details about the algebraic structure of a lattice Leibniz rule, nilpotent SUSY are realized on the lattice in various ways. The deep understanding of the lattice Leibniz rule could help us to define a lattice model naturally keeping as many symmetries as possible and to study higher dimensional SUSY theories without fine tunings, or with less fine tunings.

In Ref.\[21\], another type of the lattice Leibniz rule was proposed in $\mathcal{N} = 2$ SUSY quantum mechanics (QM) \[22, 23\], which keeps a part of symmetries exactly. The indices of the new rule appear cyclically \[4\] and we refer to it as a cyclic Leibniz rule (CLR) in this paper as well as the authors of Ref.\[21\] did. The CLR has many solutions and the general solution for a symmetric difference operator has been studied in Ref.\[24\]. $\mathcal{N} = 4$ SUSY QM and $\mathcal{N} = 2$ SYK model are also defined on the lattice such that the half SUSY is exactly kept \[25, 26\]. For those models, the exact invariance of half symmetry naturally leads to the CLR although there is another lattice formulation with an exact symmetry in $\mathcal{N} = 2$ SUSY QM \[8\]. Furthermore a kind of non-renormalization theorem holds for the CLR action of the $\mathcal{N} = 4$ case such that any finite correction to the F-term is prohibited \[25\]. We can say that the CLR keeps various natural properties of SUSY at a perturbative level, however its non-perturbative property which will be important to extend the CLR formulations to higher dimensions is still unknown.

In this paper, we propose a lattice action with the CLR for a backward difference operator and study its non-perturbative property using numerical computations. We present a solution of the CLR for any interaction term. Numerical computations are carried out for the $\phi^6$-interaction for which SUSY is unbroken. We do not employ the standard Monte-Carlo method used in previous studies of SUSY QM \[8, 17, 27, 29\] but a direct computational method on the basis of a transfer matrix \[30, 31\], see also \[32–36\] for related numerical methods. The obtained energy spectra show that the cut-off dependence of the CLR action

\[4\] The difference between the standard Leibniz rule and the cyclic Leibniz rule is shown in section III A.

See \[33\] and \[34\] for the expressions as a product rule.
is smaller than another lattice action defined by Catterall and Gregory (CG) in Ref. [8]. Numerical results of the SUSY Ward-Takahashi identities (WTIs) also tell us that full symmetry is restored more rapidly than the CG action for the weak and strong couplings.

This paper is organized as follows. In section II we introduce the continuum and the lattice theories of $\mathcal{N} = 2$ SUSY QM. The continuum theory is given in the Euclidean path integral formulation in section II A and the lattice theory is introduced in section II B. The CG lattice action is then presented in section II C. We formulate the CLR for the backward difference operator showing a solution for any superpotential and mention a relation between the CLR and the standard Leibniz rule in section III. Section IV presents the numerical results. In section IV A we briefly explain the computational method based on the transfer matrix [30]. Then, using computational parameters given in section IV B we show the numerical results of energy spectra in section IV C and those of SUSY WTIs in section IV D. We summarize in section V. Appendix A is devoted to study more about the CLR and appendix B shows the results of weak coupling expansion of several lattice actions.

II. SUSY QM AND THE LATTICE THEORY

$\mathcal{N} = 2$ supersymmetric quantum mechanics is defined in the Euclidean path integral formulation according to [22, 23, 37]. We then present a naive lattice approach to SUSY QM and introduce a known improved lattice action [8].

A. $N=2$ SUSY QM

With an euclidean time $t$, the action of $\mathcal{N} = 2$ SUSY QM is given by

$$S = \int_0^\beta dt \left\{ \frac{1}{2} (\partial_t \phi)^2 + \frac{1}{2} W^2(\phi) + \bar{\psi} \partial_t \psi + \bar{\psi} W'(\phi) \psi \right\},$$

(1)

where $\phi(t)$ is a real bosonic variable and $\bar{\psi}(t), \psi(t)$ are one-component fermionic variables. Those variables satisfy the periodic boundary condition such as $\phi(\beta) = \phi(0)$. The superpotential $W(\phi)$ is any function of $\phi$, which determines the physical behavior of this model. The partition function is defined as

$$Z_P = \int D\phi D\bar{\psi} D\psi e^{-S}$$

(2)

which is the path integral form of the Witten index.
The classical action is invariant under two SUSY transformations,
\[
\begin{align*}
\delta \phi &= \epsilon \psi - \bar{\epsilon} \bar{\psi} \\
\delta \psi &= \bar{\epsilon} (\partial_t \phi - W) \\
\delta \bar{\psi} &= -\epsilon (\partial_t \phi + W),
\end{align*}
\] (3)
where \( \epsilon \) and \( \bar{\epsilon} \) are global Grassmann parameters. The Leibniz rule is needed to show that the action (1) is invariant under these transformations.

The Witten index \( \Delta \) is defined by
\[
\Delta \equiv \text{Tr}(e^{-\beta \hat{H}} (-1)^{\hat{F}}),
\] (4)
with the quantum Hamiltonian,
\[
\hat{H} = \frac{1}{2} \hat{p}^2 + \frac{1}{2} W^2(\hat{q}) + \frac{1}{2} W'(\hat{q}) \left[ \hat{\psi}^\dagger, \hat{\psi} \right],
\] (5)
where \( \hat{q} \) and \( \hat{p} \) are the position and momentum operator and \( \hat{\psi}^\dagger \) and \( \hat{\psi} \) are the creation and annihilation operators, which satisfy \([\hat{p}, \hat{q}] = -i\) and \(\{\hat{\psi}, \hat{\psi}^\dagger\} = 1\). Here \( \hat{F} \equiv \hat{\psi}^\dagger \hat{\psi} \) is the fermion number operator. The trace is a summation over all possible normalized states of the system.

We can also write
\[
\Delta = \text{Tr}(e^{-\beta \hat{H}_-}) - \text{Tr}(e^{-\beta \hat{H}_+}),
\] (6)
where \( \hat{H}_\pm = \frac{1}{2} \hat{p}^2 + \frac{1}{2} W^2(\hat{q}) \pm \frac{1}{2} W'(\hat{q}) \) are the Hamiltonians of bosonic (-) and fermionic (+) sectors, respectively. The Witten index does not depend on \( \beta \) because all non-zero eigenmodes in \( \hat{H}_\pm \) form pairs and only \( \beta \)-independent zero modes contribute to \( \Delta \). It is well-known that \( \Delta \) is zero (non-zero) when SUSY is broken (unbroken) in this model. We study a SUSY unbroken case with \( \Delta = 1 \), given by \( W(\phi) \simeq \lambda \phi^3 \) for \( |\phi| \to \infty \) in this paper.

**B. Lattice theory**

The lattice theory is defined on a lattice whose coordinate is given by \( t = na \ (n \in \mathbb{Z}) \). Lattice bosonic and fermionic variables, which live on the sites, are expressed as \( \phi_n \) and \( \psi_n \), respectively. It is assumed that all variables satisfy the periodic boundary condition,
\[
\phi_{n+N} = \phi_n, \quad \psi_{n+N} = \psi_n, \quad \bar{\psi}_{n+N} = \bar{\psi}_n,
\] (7)
where \( N \) is the lattice size with \( \beta = Na \).

The difference operator \( \nabla \) acts on a lattice variable \( \varphi_n \) as \( \nabla \varphi_n \equiv \sum_m \nabla_{nm} \varphi_m \) and its transpose is \((\nabla^T)_{nm} \equiv \nabla_{mn}\). Throughout this paper, \( \nabla_+ \) and \( \nabla_- \) denote a simple forward and a backward difference operator, respectively:

\[
\begin{align*}
\nabla_+ \varphi_n &\equiv \frac{\varphi_{n+1} - \varphi_n}{a}, \\
\nabla_- \varphi_n &\equiv \frac{\varphi_n - \varphi_{n-1}}{a}.
\end{align*}
\]

Note that \((\nabla_+)^T = -\nabla_-\).

The partition function with a lattice action \( S \) is defined by

\[
Z_P \equiv \int D\bar{\psi} D\psi D\phi \ e^{-S},
\]

where

\[
\begin{align*}
\int D\phi &\equiv \prod_n \int_{-\infty}^{\infty} \frac{d\phi_n}{\sqrt{2\pi a}}, \\
\int D\bar{\psi} D\psi &\equiv \int \prod_n d\bar{\psi}_n d\psi_n.
\end{align*}
\]

Here each Grassmann measure is an anti-commuting derivative as \( d\psi_n \equiv \partial/\partial \psi_n \) and \( d\bar{\psi}_n \equiv \partial/\partial \bar{\psi}_n \).

We now consider a naive lattice action,

\[
S_{naive} = a \sum_n \left\{ \frac{1}{2} (\nabla_- \phi_n)^2 + \frac{1}{2} W^2(\phi_n) + \bar{\psi}_n \nabla_- \psi_n + \bar{\psi}_n W'(\phi_n) \psi_n \right\}
\]

which is obtained by replacing \( \phi(t), \psi(t), \bar{\psi}(t) \) and \( \partial_t \) of (11) by the corresponding lattice variables \( \phi_n, \psi_n, \bar{\psi}_n \) and \( \nabla_- \) and replacing the integral by the summation over lattice site. This action is not invariant under a naive lattice SUSY transformation defined by the same replacement of the variables for (11).

SUSY which is broken at \( O(a) \) in (13) is classically restored in the continuum limit \( a \to 0 \), however such a restoration does not occur at the quantum level. As seen in later sections, modifying \( O(a) \) interactions of the lattice action, we can keep only either one of two SUSY transformations parametrized by \( \epsilon \) and \( \bar{\epsilon} \) at a finite lattice spacing, and SUSY is restored in the quantum continuum limit for such a lattice model.
C. Catterall-Gregory lattice model

Before discussing the CLR, we review a lattice action proposed by Catterall and Gregory [8]:

\[ S_{CG} = S_{naive} + a \sum_n \nabla_- \phi_n W(\phi_n), \] (14)

where \( \nabla_- \) is the backward difference operator defined in (9). Note that the added term is a kind of surface term which vanishes in the naive continuum limit.

We can show that, in the free limit given by \( W(\phi) = m a \phi \), \( S_{CG} \) is invariant under the lattice SUSY transformations,

\[
\begin{align*}
\delta \phi_n &= \epsilon \psi_n - \bar{\epsilon} \bar{\psi}_n \\
\delta \psi_n &= \bar{\epsilon} (\nabla_+ \phi_n - W(\phi_n)) \\
\delta \bar{\psi}_n &= -\epsilon (\nabla_- \phi_n + W(\phi_n)).
\end{align*}
\] (15)

For interacting cases, it is not invariant under the whole transformations (15) but invariant under part of SUSY, \( \delta_\epsilon = \delta|_{\epsilon=0} \):

\[ \delta_\epsilon S_{CG} = 0. \] (16)

This is because the extra term of R.H.S. in (14) provides \( -\delta_{\epsilon} S_{naive} \) for any finite lattice spacing. The remaining \( \bar{\epsilon} \) symmetry in (15) is restored in the quantum continuum limit as shown in Refs. [8, 17, 27, 30] and also in section IV D of this paper.

III. CYCLIC LEIBNIZ RULE FOR BACKWARD DIFFERENCE OPERATOR

We propose an alternative lattice action with the cyclic Leibniz rule (CLR) for the backward difference operator and show a solution of the CLR for any superpotential. It is straightforward to extend the results to the case of the forward difference operator.

A. Lattice action with the CLR

The CLR for the symmetric difference operator is proposed in Ref. [21]. As an straightforward extension of Ref. [21], we introduce a lattice action with the CLR for the backward operator:

\[ S_{CLR} = a \sum_n \left\{ \frac{1}{2} (\nabla_- \phi_n)^2 + \frac{1}{2} (W_n)^2 + \bar{\psi}_n \nabla_- \psi_n + \sum_m \bar{\psi}_n W'_{nm} \psi_m \right\}, \] (17)
where \( W_n \) is a local function of the boson variables \( \phi \) and \( W'_{nm} = \frac{\partial W_n}{\partial \phi_m} \). We now assume that \( W_n \) satisfies the CLR,

\[
\sum_n \{ W_n (\nabla_-)_{nm} + \nabla_- \phi_n W'_{nm} \} = 0.
\]

As explained in the next section, a desirable local solution is

\[
W_n = \frac{U(\phi_n) - U(\phi_{n-1})}{\phi_n - \phi_{n-1}},
\]

where \( U(\phi) = \int^\phi d\phi' W(\phi') \). The lattice action (17) classically reproduces the continuum one as \( a \to 0 \) since \( W_n = W(\phi_n) + O(a) \).

The importance of CLR is understood by considering a half lattice SUSY transformation,

\[
\delta_\epsilon \phi_n = \epsilon \psi_n \\
\delta_\epsilon \psi_n = 0 \\
\delta_\epsilon \bar{\psi}_n = -\epsilon (\nabla_- \phi_n + W_n).
\]

The lattice action (17) with any solution of (18) is invariant under (20) because

\[
\delta_\epsilon S_{CLR} = \epsilon a \sum_n X_n \psi_n = 0,
\]

where

\[
X_n \equiv - \sum_m \{ W_m (\nabla_-)_{mn} + W'_m \nabla_- \phi_m \}
\]

which vanishes as long as \( W_n \) satisfies the CLR (18).

The other half transformation of \( N = 2 \) is broken on the lattice in general, which is restored at the continuum limit as seen in section IV D. However, in the free theory, it still remains as an exact symmetry because the free lattice action with the solution (19) is invariant under

\[
\delta_\epsilon \bar{\psi}_n = 0
\]

Note that \( W_n \neq W(\phi_n) \) in general because \( W_n \) may contain \( \phi_m \) with \( m \neq n \) as long as the correlation rapidly vanishes for \( |m - n| \to \infty \). See (A5) of appendix A 2 for the strict definition of the locality condition.
Note that $W_{n+1}$ is used in $\delta \psi_n$ instead of $W_n$. We can actually show that

$$
\delta \tilde{e} S_{CLR} = \tilde{e} \left\{ a \sum_n X_n \tilde{\psi}_n + a \sum_{n,m} Y_{nm} (W_n \tilde{\psi}_m - \tilde{\psi}_m \nabla - \phi_n) + a \sum_{nmk} Z_{nmk} \tilde{\psi}_n \tilde{\psi}_k \psi_m \right\},
$$

(24)

where

$$
Y_{nm} \equiv W'_{m,n-1} - W'_{n,m},
$$

$$
Z_{nmk} \equiv \frac{\partial^2 W_n}{\partial \phi_k \partial \phi_m}.
$$

(25)

Although we have $X_n = 0$ from the CLR, $Y_{nm}$ and $Z_{nmk}$ do not vanish for a generic superpotential. However, for the free theory with the solution (19),

$$
W_n = \frac{m}{2} (\phi_n + \phi_{n-1}),
$$

(26)

it is easy to show that $Y_{mn}$, $Z_{nmk}$ and (24) vanish.

**B. A solution of CLR for the backward difference operators**

We show that (19) is a local and well-defined solution of (18) for a generic superpotential. Once the solution is given, the lattice CLR action retains an exact SUSY as seen in the previous section.

Let us first consider the free theory. For the backward operator $(a \nabla -)_{nm} = \delta_{nm} - \delta_{n,1,m}$, we take an ansatz solution within the nearest neighbor interactions, $W_n = d_0 \phi_n + d_1 \phi_{n-1} + d_2 \phi_{n+1}$. It is then found that $d_0 = d_1 = 1/2, d_2 = 0$ is a solution of (18), for which (26) is obtained.

It is not easy to apply such a straightforward way to a generic superpotential. We derive another representation of (18) to find a solution. Rescaling $\phi_n$ of (18) as $u \phi_n$ with a parameter $u \in [0, 1]$ and using the chain rule for $\partial_n$, we obtain

$$
\frac{\partial}{\partial u} \sum_n \{ u \nabla - \phi_n W_n |_{\phi \to u \phi} \} = 0.
$$

(27)

Integrating (27) from $u = 0$ to $u = 1$, we find a condition that means a vanishing surface term,

$$
\sum_n \nabla - \phi_n W_n = 0.
$$

(28)

This condition is equivalent to (18) because (18) can also be derived from (28) differentiating (28) with respect to $\phi_m$. 

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The relation (28) is easily solved by a local function (19). All we have to do is check whether or not $W_n$ given by (19) is a well-defined function that coincides with $W(\phi_n)$ as $a \to 0$. By integrating $\partial_u U(\phi_n - u\nabla - \phi_n)$ from $u = 0$ to $u = 1$ and using the chain rule for $\partial_u$, we have

$$U(\phi_n) - U(\phi_{n+1}) = (\phi_n - \phi_{n+1}) \int_0^1 du \, W(\phi_n - au\nabla - \phi_n).$$

(29)

The division in (19) is well-defined because the integral of R.H.S. is well-defined for any configuration of $\phi_m$. Since the integral is $W(\phi_n)$ up to $O(a)$, we can immediately show that $W_n = W(\phi_n) + O(a)$.

C. CLR v.s. Leibniz rule

The difference between the CLR and the standard Leibniz rule (LR) is discussed here. In the continuum theory, LR for $\partial_t$ is $\partial_t W(\phi) = W'(\phi)\partial_t \phi$. So a naive lattice LR is introduced as

$$\text{LR} : \sum_m \{\nabla_{nm}W_m - W'_{nm}(\nabla \phi)_m\} = 0,$$

(30)

for $W_n$ that is a local function of bosonic variables. Here we again use $W'_{nm} = \partial W_n / \partial \phi_m$. We find that the CLR is different from LR in general since

$$\text{CLR} : \sum_m \{-\nabla^T_{nm}W_m - W'_{nm}(\nabla \phi)_m\} = 0.$$

(31)

Note that $W'$ in the second term is transposed.

The CLR coincides with LR if $W'_{nm} = W'_{mn}$ for $\nabla^T = -\nabla$ (symmetric difference operators), which corresponds to the case that the lattice action is invariant under both of two SUSY transformations [21]. However, the no-go theorem [2] tells us that LR does not hold for any difference operator and any interacting cases with keeping the locality principle. It is therefore difficult to realize the full SUSY transformation exactly on the lattice. The CLR cannot be realized with a non-trivial solution in this case.

The similar argument holds for the backward difference operator $\nabla_-$. Suppose that $W_n$ is a solution of the CLR and $\delta_\epsilon S = 0$. Using $W'_{nm} = W'_{n,m-1}$ from $Y_{nm} = 0$, we can show that the CLR coincides with LR for $\nabla_+$ since $\nabla^T = -\nabla_+$ and $\sum_m W'_{nm}(\nabla_+ \phi)_m = \sum W_{nm}(\nabla_+ \phi)_m$. The no-go theorem again tells us that one cannot find a solution of the CLR so that the lattice action (17) is invariant under both of $\delta_\epsilon$ and $\delta_\epsilon$.  

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The lattice rules (30) and (31) can also be expressed as a product rule of lattice variables. As an example, let us consider a lattice superpotential,

\[ W_{n}^{e,g.} \equiv \sum_{m,k} M_{nmk} \phi_{m} \phi_{k}, \quad (32) \]

as a discretization of \( W^{e,g.}(\phi(x)) = \phi^2(x) \). Then the (two-body) LR can be expressed as

\[ \sum_{n} \left\{ \nabla_{na} M_{bnc} - \nabla_{bn} M_{nca} + \nabla_{nc} M_{ban} \right\} = 0, \quad (33) \]

while the (two-body) CLR is

\[ \sum_{n} \left\{ \nabla_{na} M_{nbc} + \nabla_{nb} M_{nca} + \nabla_{nc} M_{nab} \right\} = 0. \quad (34) \]

The name of cyclic Leibniz rule comes from a cyclicity of the indices \( a, b, c \). In appendix A, an explicit solution for the \( m \)-body CLR is also given.

IV. NUMERICAL RESULTS

Numerical computation is carried out for the CLR action (17) with the periodic boundary conditions for the superpotential,

\[ W = m\phi + \lambda m^2 \phi^3, \quad (35) \]

where \( \lambda \) is the dimensionless coupling constant and \( m \) is the mass. Supersymmetry is kept unbroken since the Witten index is nonzero for this potential. The energy spectra and the SUSY Ward Takahashi identities are evaluated at two coupling constants \( \lambda = 0.001 \) (weak) and \( \lambda = 1 \) (strong). We compare the results with those obtained from the CG action (14) to understand the dependence of the results on the lattice spacing.

A. Numerical methods

We begin with giving the CLR lattice action used in the actual computations:

\[ S_{CLR} = a \sum_{n} \left\{ \frac{1}{2}(\nabla_{-} \phi_{n})^2 + \frac{1}{2}(W_{n})^2 + \bar{\psi}_{n} \nabla_{-} \psi_{n} + \sum_{m} \bar{\psi}_{nm} W'_{nm} \psi_{m} \right\}, \quad (36) \]
where

\[ W_n = \frac{ma}{2}(\phi_n + \phi_{n-1}) + \frac{(ma)^2 \lambda}{4}(\phi_n^3 + \phi_{n-1}^2 + \phi_n^2 + \phi_{n-1}^3), \quad (37) \]

\[ W'_{nm} = \frac{ma}{2}(\delta_{nm} + \delta_{n-1,m}) + \frac{(ma)^2 \lambda}{4}\left\{ (3\phi_n^2 + 2\phi_n\phi_{n-1} + \phi_{n-1}^2)\delta_{nm} + (\phi_n^2 + 2\phi_n\phi_{n-1} + 3\phi_{n-1}^2)\delta_{n-1,m}\right\}. \quad (38) \]

As shown in section III A, the action (36) is invariant under a single SUSY transformation (20) thanks to the CLR (18).

The partition function and the correlation functions are expressed in terms of transfer matrices. It is straightforward to show that, integrating out the fermionic variables, the partition function (10) with (36) is given as

\[ Z_P = \int D\phi \left\{ \prod_{n=1}^{N} (1 + A_{\phi_n\phi_{n-1}})e^{-L_{\phi_n\phi_{n-1}}} - \prod_{n=1}^{N} (1 - A_{\phi_{n-1}\phi_n})e^{-L_{\phi_{n-1}\phi_n}} \right\}, \quad (39) \]

where

\[ A_{\alpha\beta} \equiv \frac{ma}{2} + \frac{(ma)^2 \lambda}{4}(3\alpha^2 + 2\alpha\beta + \beta^2), \quad (40) \]

\[ L_{\alpha\beta} \equiv \frac{1}{2}(\alpha - \beta)^2 + \frac{1}{8}\left( ma(\alpha + \beta) + \frac{(ma)^2 \lambda}{2}(\alpha^3 + \alpha^2\beta + \alpha\beta^2 + \beta^3)\right)^2, \quad (41) \]

because \( S_B = \sum_{n=1}^{N} L_{\phi_n\phi_{n-1}} \) and \( W'_nm = A_{\phi_n\phi_{n-1}}\delta_{n,m} + A_{\phi_{n-1}\phi_n}\delta_{n-1,m} \). Note that \( A_{\alpha\beta} \) and \( L_{\alpha\beta} \) are infinite dimensional matrices since \( \alpha, \beta \in \mathbb{R} \).

In order to define finite dimensional matrices, each path integral measure of (39) is discretized by the Gauss-Hermite quadrature. For a function \( f(x) \), the Gauss-Hermite quadrature formula is given by an approximation of the integral:

\[ \int_{-\infty}^{\infty} dx f(x) \approx \sum_{x \in S_K} g_K(x)f(x), \quad (42) \]

where \( S_K \) is a set of roots of \( K \)-th Hermite polynomial \( H_K \) and the weight \( g_K(x) \) is

\[ g_K(x) = \frac{2^{K-1}K!\sqrt{\pi}}{K^2H_{K-1}^2(x)}e^{x^2}. \quad (43) \]

Since \( K \) is the order of the approximation, the sum of (42) is expected to reproduce the integral of L.H.S. as \( K \to \infty \).

We can express \( Z_P \) using finite dimensional matrices \( T_\pm \) as

\[ Z_P \approx \text{tr}(T_{-}^N) - \text{tr}(T_{+}^N), \quad (44) \]
discretizing all path integral measures (11) by the quadrature:

\[ \int D\phi \approx \frac{1}{(2\pi)^{N/2}} \sum_{\phi_1 \in S_K} \cdots \sum_{\phi_N \in S_K} g_K(\phi_1) \cdots g_K(\phi_N). \]  

(45)

Here, for \( \alpha, \beta \in S_K \),

\[ (T_-)_{\alpha\beta} \equiv (1 + A_{\alpha\beta}) R_{\alpha\beta}, \]

(46)

\[ (T_+)_{\alpha\beta} \equiv (1 - A_{\beta\alpha}) R_{\alpha\beta}, \]

(47)

\[ R_{\alpha\beta} \equiv \sqrt{g_K(\alpha)g_K(\beta)} \frac{2\pi}{e^{-L_{\alpha\beta}}}. \]

(48)

A comparison with (6) tells us that \( T_- \) and \( T_+ \) are a bosonic and fermionic transfer matrix, respectively. The trace of \( X \) means

\[ \text{tr}(X) \equiv \sum_{\alpha \in S_K} X_{\alpha\alpha}, \]

(49)

where \( X_{\alpha\beta} \) is a matrix with \( \alpha, \beta \in S_K \).

Similarly, any correlation function is given in terms of the transfer matrices. We basically follow Ref.\[30\] to derive the expressions. The two point correlation function of the bosonic variable is

\[ \langle \phi_j \phi_k \rangle \approx \frac{1}{Z} \text{Tr} \left\{ T_-^{N-k+j} D T_-^{k-j} D - T_+^{N-j+k} D T_+^{k-j} D \right\}, \]

(50)

for \( 0 \leq j \leq k \leq N \). Here \( D \) represents an operator insertion, which is defined as

\[ D_{\alpha\beta} \equiv \alpha \delta_{\alpha\beta}. \]

(51)

The boson two-point function is exactly the same formula as that of the CG action \[30\]. On the other hand, the fermion two-point function is slightly different:

\[ \langle \bar{\psi}_j \psi_k \rangle \approx \frac{1}{Z} \text{tr} \left\{ R T_-^{k-j-1} T_+^{N+j-k} \right\}, \]

(52)

for \( 0 \leq j \leq k \leq N \).

The transfer matrices \( T_\pm \) can be improved by rescaling the bosonic variables before the discretization of the measures. According to Ref.\[30\], applying the quadrature after rescaling \( \phi \) as \( \phi \to \phi/s \) (\( s \in \mathbb{R} \)), we have

\[ (T_-^{(s)})_{\alpha\beta} \equiv (1 + A_{\alpha\beta}^{(s)}) R_{\alpha\beta}^{(s)}, \]

(53)

\[ (T_+^{(s)})_{\alpha\beta} \equiv (1 - A_{\beta\alpha}^{(s)}) R_{\alpha\beta}^{(s)}, \]

(54)

\[ R_{\alpha\beta}^{(s)} \equiv \sqrt{g_K^{(s)}(\alpha)g_K^{(s)}(\beta)} \frac{2\pi s^2}{e^{-L^{(s)}_{\alpha\beta}}}. \]

(55)

\[ 6 \text{ The formula for the CG action given in } [30] \text{ is reproduced because } T_+ = R \text{ in the case.} \]
where \( A^{(s)}_{\alpha\beta} \equiv A_{\alpha(s)\beta(s)} \) and \( \mathcal{L}^{(s)}_{\alpha\beta} \equiv \mathcal{L}_{\alpha(s)\beta(s)} \) with \( \alpha(s) \equiv \alpha/s \) and \( \beta(s) \equiv \beta/s \). The partition function and the correlation functions are then given by the same formulas as (44), (49), and (50) with \( T^{(s)}_{\pm} \) and \( R^{(s)} \) instead of \( T_\pm \) and \( R \). The operator insertion \( D \) is also replaced by \( D^{(s)} = D/s \). The trace is still given by (19). We can obtain computational results with a high precision by tuning the rescaling parameter \( s \) such that the Witten index \( Z_P = 1 \) is realized as accurate as possible.

### B. Computational parameters

Table I shows the parameters used in our computations of the CLR action. We employ two representative coupling constants, \( \lambda = 0.001 \) as a weak coupling and \( \lambda = 1 \) as a strong coupling. The rescaling parameter \( s \) should be tuned for each parameter set such that the Witten index \( Z_P = 1 \) is reproduced as accurate as possible, as done in Ref. [30]. The matrix sizes \( K \) used for the SUSY WTI are smaller than those for the mass spectra to reduce the computational cost. This is because the SUSY WTIs are evaluated by performing the direct matrix product several times while the mass spectra are evaluated by diagonalizing \( T_\pm \) once. Similarly, we use the same lattice sizes with a slightly different \( s \) for the CG action.

We take \( m\beta = 30 \) that is large enough to obtain the numerical results with a negligible finite \( \beta \) effect because \( e^{-\beta E_1} < O(10^{-13}) \) for the first excited energy \( E_1/m \geq 1 \). The lattice spacing is shown as rounded numbers, which is uniquely determined from the lattice size \( N \) for fixed \( m\beta \) as \( ma = m\beta/N(= 30/N) \). For instance, \( ma = 0.017964 \ldots \) for \( N = 1670 \) is denoted as \( ma = 0.018 \) in the table but we use \( ma = 30/N \) in the actual computations without loss of digit.

Figure 1 shows the results of \( Z_P \) against \( \beta \) for several \( s \). Although \( Z_P \) is analytically shown to be unity even on the lattice [21], the numerical results depend on \( \beta \). The deviations from \( Z_P = 1 \) are systematic errors which come from the finite \( K \)-effect. We can decrease the errors tuning \( s \) for fixed \( K \). We find that \( s = 0.68 \) leads to \( |Z_P - 1| < O(10^{-9}) \) for \( K = 150 \) in the case of \( ma = 0.01 \) and \( \lambda = 1 \). Each parameter has a different value of \( s \) so that \( Z_P = 1 \) is realized within \( O(10^{-9}) \) as shown in Table I.
| \( \lambda = 0.001 \) | \( \lambda = 1 \) |
|---|---|
| **Energy spectra** | **SUSY WTI** | **Energy spectra** | **SUSY WTI** |
| \( \text{am} \) | \( s \) | \( N \) | \( K \) | \( \text{am} \) | \( s \) | \( N \) | \( K \) | \( \text{am} \) | \( s \) | \( N \) | \( K \) |
| 0.020 | 0.47 | 1500 | 150 | 0.600 | 1.39 | 50 | 40 | 0.020 | 0.97 | 1500 | 150 | 0.600 | 2.93 | 50 | 40 |
| 0.019 | 0.46 | 1580 | 150 | 0.500 | 1.26 | 60 | 40 | 0.019 | 0.95 | 1580 | 150 | 0.500 | 2.68 | 60 | 40 |
| 0.018 | 0.45 | 1670 | 150 | 0.400 | 1.13 | 75 | 40 | 0.018 | 0.92 | 1670 | 150 | 0.400 | 2.46 | 75 | 40 |
| 0.017 | 0.44 | 1770 | 150 | 0.300 | 0.97 | 100 | 40 | 0.017 | 0.90 | 1770 | 150 | 0.300 | 2.08 | 100 | 40 |
| 0.016 | 0.42 | 1880 | 150 | 0.250 | 0.89 | 120 | 40 | 0.016 | 0.87 | 1880 | 150 | 0.250 | 1.89 | 120 | 40 |
| 0.015 | 0.41 | 2000 | 150 | 0.200 | 0.79 | 150 | 40 | 0.015 | 0.84 | 2000 | 150 | 0.200 | 1.69 | 150 | 40 |
| 0.014 | 0.40 | 2140 | 150 | 0.150 | 0.68 | 200 | 40 | 0.014 | 0.81 | 2140 | 150 | 0.150 | 1.47 | 200 | 40 |
| 0.013 | 0.38 | 2310 | 150 | 0.100 | 0.56 | 300 | 40 | 0.013 | 0.78 | 2310 | 150 | 0.100 | 1.18 | 300 | 40 |
| 0.012 | 0.37 | 2500 | 150 | 0.080 | 0.51 | 375 | 40 | 0.012 | 0.75 | 2500 | 150 | 0.080 | 1.06 | 375 | 40 |
| 0.011 | 0.36 | 2730 | 150 | 0.060 | 0.49 | 500 | 50 | 0.011 | 0.72 | 2730 | 150 | 0.060 | 0.91 | 500 | 50 |
| 0.010 | 0.34 | 3000 | 150 | 0.050 | 0.44 | 600 | 50 | 0.010 | 0.68 | 3000 | 150 | 0.050 | 0.83 | 600 | 50 |
| 0.009 | 0.33 | 3330 | 150 | 0.040 | 0.44 | 750 | 60 | 0.009 | 0.65 | 3330 | 150 | 0.040 | 0.74 | 750 | 60 |
| 0.008 | 0.33 | 3750 | 170 | 0.030 | 0.41 | 1000 | 70 | 0.008 | 0.61 | 3750 | 150 | 0.030 | 0.64 | 1000 | 40 |
| 0.007 | 0.30 | 4290 | 170 | 0.025 | 0.34 | 1200 | 70 | 0.007 | 0.57 | 4290 | 150 | 0.025 | 0.65 | 1200 | 50 |
| 0.006 | 0.27 | 5000 | 170 | 0.020 | 0.32 | 1500 | 80 | 0.006 | 0.53 | 5000 | 150 | 0.020 | 0.57 | 1500 | 50 |
| 0.005 | 0.27 | 6000 | 200 | 0.015 | 0.33 | 2000 | 100 | 0.005 | 0.48 | 6000 | 150 | 0.015 | 0.58 | 2000 | 70 |
| 0.004 | 0.24 | 7500 | 200 | 0.010 | 0.29 | 3000 | 120 | 0.004 | 0.46 | 7500 | 170 | 0.010 | 0.47 | 3000 | 70 |
| 0.003 | 0.40 | 10000 | 170 | 0.002 | 0.32 | 15000 | 170 | 0.003 | 0.40 | 10000 | 170 | 0.002 | 0.32 | 15000 | 170 |
| 0.001 | 0.23 | 30000 | 200 |  |  |  |  |  |  |  |  |  |  |  |  |  |

TABLE I. Parameters used in the numerical computations of the CLR system. Left and right tables are ones for a weak coupling \( \lambda = 0.001 \) and for a strong coupling \( \lambda = 1 \), respectively.

C. Energy spectra

The energy spectra of lattice SUSY quantum mechanics are read from two transfer matrices \( T_\pm \) associated with two Hamiltonians \( \hat{H}_\pm \) as \( T_\pm \approx e^{-a\hat{H}_\pm} \). The energy eigenvalues of
FIG. 1. Partition function with the periodic boundary condition against $\beta$ for the CLR action. We use several $s$ with fixed $K = 150$ for $ma = 0.01, \lambda = 1$.

the bosonic and fermionic states $E_n^B$ and $E_n^F$ are thus obtained from the $n$-th eigenvalue of $T_\pm$: $(T_-)_n = e^{-aE_n^B}$ and $(T_+)_n = e^{-aE_n^F}$. We use numerical diagonalizations of $T_\pm$ to evaluate $(T_\pm)_n$. The non-zero eigenvalues are degenerate between $\hat{H}_+$ and $\hat{H}_-$ and only $\hat{H}_-$ has a zero mode for the superpotential \((35)\). We expect that $T_\pm$ have the same spectra even on the lattice thanks to the exact SUSY.

1. Weak coupling results

Table I shows the ten smallest energy eigenvalues obtained from the CLR action for $\lambda = 0.001$ at a lattice spacing $ma = 0.01$. The central values are ones obtained for $K = 150$ and the errors are estimated from the largest difference among the results with $K = 140, 150, \ldots, 200$. The spectra look like ones of the harmonic oscillator, $E_n = nm (n = 1, 2, \cdots)$, since $\lambda = 0.001$ is sufficiently small. As we expected, $E_n^B$ and $E_n^F$ coincide with each other within the errors. The same degeneracies are observed for the other lattice spacings.

Figure 2 shows the lowest five eigenvalues against the lattice spacing $ma$. Since the
| n | $E_n^B/m$ | $E_n^F/m$ |
|---|---|---|
| 0 | 0.000000000001(3) | |
| 1 | 1.001498936(1) | 1.00149893546(2) |
| 2 | 2.00598024(3) | 2.005980230(1) |
| 3 | 3.0134265(5) | 3.01342635(3) |
| 4 | 4.023822(5) | 4.0238202(4) |
| 5 | 5.03716(4) | 5.037146(5) |
| 6 | 6.0535(3) | 6.05340(4) |
| 7 | 7.073(1) | 7.0726(2) |
| 8 | 8.097(5) | 8.095(1) |
| 9 | 9.13(2) | 9.122(5) |
| 10 | 10.18(4) | 10.16(1) |

TABLE II. Energy eigenvalues obtained from the CLR action for $\lambda = 0.001$ at $ma = 0.01$.

difference between $E_n^B$ and $E_n^F$ are sufficiently smaller than the systematic errors from finite $K$ effect, we plotted only $E_n^F$ as $E_n$ in the figure. As we can see, the cut-off dependence of the CLR action is milder than that of CG action.

Tables III and IV show the fit results of the lowest five energy eigenvalues for the CLR and CG actions, respectively. For the continuum extrapolation, we employ a quadratic polynomial,

$$E/m = a_0 + a_1(ma) + a_2((ma)^2).$$  \hspace{1cm} (56)

Two actions reproduce the same $a_0$, which is $E_n/m$ at the continuum limit, within the errors. The CLR action behaves rather similar to the continuum theory in comparison with the CG action as suggested from small values of $a_1$.

The weak coupling expansion of the first excited energy is demonstrated in appendix B, in which the quantum corrections to the masses are evaluated from the correlation functions. We find that, for $E_1 = E_1^F = E_1^B$, the one-loop result of the CLR action is

$$\frac{E_1^{CLR}}{m} = 1 + \frac{3}{2} \lambda - \frac{1}{2}(ma\lambda + O((ma)^2, \lambda^2)),$$

(57)

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FIG. 2. Five lowest energy eigenvalues against the lattice spacing $ma$ for $\lambda = 0.001$. The results of CLR (circles) show a better convergence than the CG results (triangles). The solid lines represent the fit results shown in Tables III and IV.

| $E_n/m$ | $E_1/m$ | $E_2/m$ | $E_3/m$ | $E_4/m$ | $E_5/m$ |
|-------|-------|-------|-------|-------|-------|
| $a_0$ | 1.001495535(1) | 2.00597334(3) | 3.0134165(4) | 4.023814(4) | 5.0372(4) |
| $a_1$ | -0.0004999(4) | -0.00101(1) | -0.0017(2) | -0.004(2) | -0.02(2) |
| $a_2$ | 0.08400(4) | 0.1702(9) | 0.27(1) | 0.5(2) | 2(1) |

TABLE III. Fit results of $E_n$ for the CLR action with $\lambda = 0.001$.

| $E_n/m$ | $E_1/m$ | $E_2/m$ | $E_3/m$ | $E_4/m$ | $E_5/m$ |
|-------|-------|-------|-------|-------|-------|
| $a_0$ | 1.0014954(2) | 2.0059733(4) | 3.0134161(9) | 4.023806(1) | 5.037131(3) |
| $a_1$ | -0.50221(6) | -1.0089(1) | -1.5202(3) | -2.0357(3) | -2.557(1) |
| $a_2$ | 0.330(3) | 0.667(7) | 1.02(2) | 1.36(2) | 1.8(1) |

TABLE IV. Fit results of $E_n$ for the CG action with $\lambda = 0.001$. 

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while one of the CG action is

\[
\frac{E_{1}^{CG}}{m} = 1 + \frac{3}{2} \lambda - \frac{ma}{2} - \frac{1}{2} ma \lambda + O((ma)^{2}, \lambda^{3}).
\] (58)

Both one-loop results coincide with one of the continuum theory, \(E^{\text{cont}}/m = 1 + \frac{3}{2} \lambda\), as \(a \to 0\). The CG action has a large discretization error due to the third term of \(O(ma)\) in (58), while the \(O(a)\)-term starts from \(O(\lambda ma)\) in the CLR action, which is much smaller than \(O(ma)\) for \(\lambda = 0.001\).

Figure 3 shows the numerical results of \(E_{1}\) with the perturbative ones (57) and (58) for \(ma \leq 0.02\). The numerical results nicely reproduce the perturbation theory shown by the dotted lines and the relative errors are of the order of \(10^{-6}\) that is the same size of \(\lambda^{2}\). Although a linear \(ma\) dependence is seen in the CG-results, the CLR-results perfectly reproduce the continuum theory for this range of \(ma\) since the third term of (57) is negligibly small for \(\lambda = 0.001\).

FIG. 3. Continuum limit of \(E_{1}\) for \(\lambda = 0.001\). The solid lines represent the fit results and the dotted lines are the perturbative results.
2. Strong coupling results

Table V shows the ten smallest energy eigenvalues obtained from the CLR action for $\lambda = 1$ at a fixed $ma = 0.01$. The central values are again ones evaluated for $K = 150$ and the errors are estimated from the largest difference among the results for $K = 140, 150, \cdots, 200$. The energy spectra have large quantum corrections compared to Figure II for the weak coupling $\lambda = 0.001$. $E_n^B$ and $E_n^F$ coincide with each other within the errors as well as the case of the weak coupling.

| $n$ | $E_n^B/m$       | $E_n^F/m$       |
|-----|----------------|----------------|
| 0   | 0.0000000000(2)| 1.682687275(2) |
| 1   | 1.682687275(2) | 1.682687274859(4)|
| 2   | 4.365387624(8) | 4.36538762319(6)|
| 3   | 7.622118414(4) | 7.6221184119(5)|
| 4   | 11.3640034(2)  | 11.364003389(3)|
| 5   | 15.5273615(7)  | 15.52736144(2)|
| 6   | 20.068372(3)   | 20.06837202(9)|
| 7   | 24.954588(9)   | 24.9545871(4)|
| 8   | 30.16073(3)    | 30.160725(2)|
| 9   | 35.66638(9)    | 35.666371(6)|
| 10  | 41.4546(3)     | 41.45459(2)|

TABLE V. Energy eigenvalues obtained from the CLR action for $\lambda = 1$ at $ma = 0.01$.

Figure 4 shows the lowest five energy eigenvalues against $ma$ for $\lambda = 1$. We also show Figure 5 which focuses on $E_1$ for $\lambda = 1$ for a comparison with Figure 3. The obtained $E_n^F$ is again plotted as $E_n$ since $E_n^F = E_n^B$ within the sufficiently small errors of $O(10^{-8})$. The cut-off dependence of the CLR action is milder than that of CG action as well as the weak coupling shown in Figure 2.

Tables VI and VII show the fit results of $E_n$ with a quadratic function. The same $a_0$ which is $E/m$ in the continuum limit are obtained between the CLR and CG actions. As a visible difference between Figure 2 and Figure 4 can be seen, the coefficients $a_1$ and $a_2$ are systematically larger than those for the weak coupling, which are shown in Tables III and
In the strong coupling region, we can confirm that the $O(a)$ dependence of $E_1$ obtained for the CLR action is still smaller than that of the CG action.

![Graph](image_url)

**FIG. 4.** Five lowest energy eigenstates against $ma$ for $\lambda = 1$. The results of CLR and CG are shown as the circles and the triangles, respectively. The solid lines represent the fit results shown in Tables VI and VII.

|    | $E_1/m$   | $E_2/m$   | $E_3/m$   | $E_4/m$   | $E_5/m$   |
|----|-----------|-----------|-----------|-----------|-----------|
| $a_0$ | 1.6865004(6) | 4.371816(2) | 7.630953(5) | 11.374845(7) | 15.53978(1) |
| $a_1$ | -0.3907(3) | -0.684(1)  | -0.985(2)  | -1.282(3)  | -1.575(4)  |
| $a_2$ | 0.94(2)   | 4.08(8)    | 10.2(2)    | 19.8(2)    | 33.3(3)    |

**TABLE VI.** Fit results of $E_n$ for the CLR action with $\lambda = 1$.

**D. SUSY WT identities**

The CLR action has an exact SUSY parametrized by $\epsilon$ in (20) while the other $\bar{\epsilon}$ SUSY is broken at finite lattice spacing for any interacting case. The correct mass spectra shown in
|     | $E_1/m$ | $E_2/m$ | $E_3/m$ | $E_4/m$ | $E_5/m$ |
|-----|--------|--------|--------|--------|--------|
| $a_0$ | 1.686500(3) | 4.37181(1) | 7.63095(4) | 11.37483(8) | 15.5398(1) |
| $a_1$ | -1.898(1) | -6.422(6) | -13.30(2) | -22.43(3) | -33.75(6) |
| $a_2$ | 3.05(9) | 12.6(4) | 31(1) | 58(5) | 95(5) |

TABLE VII. Fit results of $E_n$ for the CG action with $\lambda = 1$.

![Comparison of $E_1$ for different actions](image)

FIG. 5. Continuum limit of $E_1$ for $\lambda = 1$.

The previous section imply that the broken $\bar{\epsilon}$ symmetry is restored in the continuum limit. Testing the SUSY WTIs, we study the restoration of the full SUSY.

To this end, we first define the SUSY WTIs on the lattice. The broken $\bar{\epsilon}$ transformation cannot be uniquely defined on the lattice because one can add several terms that vanish in the continuum limit to the transformation. Here, for the CLR action, we employ (23) as a lattice $\bar{\epsilon}$ transformation, which is an exact symmetry in the free theory. Correspondingly, we use (15) for the CG action, whose $\bar{\epsilon}$-transformation is exactly kept in the free case of (14).

We can show that

$$\langle \delta (\phi_n \bar{\psi}_N + \psi_n \phi_N) \rangle = \epsilon R_n + \bar{\epsilon} \bar{R}_n,$$

(59)
where

\[ R_n \equiv \langle \psi_n \bar{\psi}^N \rangle - \langle \phi_n (\nabla - \phi)^N \rangle - \langle \phi_n W^N_n \rangle, \quad (60) \]

\[ \bar{R}_n \equiv \langle \psi_n \bar{\psi}^N \rangle - \langle \phi_n (\nabla - \phi)^N \rangle - \langle W_{n+1} \phi^N_n \rangle, \quad (61) \]

for the CLR action. For the CG action, \( W^N_n \) and \( W_{n+1} \) of (60) and (61) are replaced by \( W(\phi^N_n) \) and \( W(\phi_n) \), respectively. The second term of \( \bar{R}_n \) is actually found as \( \langle (\nabla + \phi)^n \phi^N_n \rangle \) which can be written as the same form as the second term of \( R_n \) using the translational invariance. Note that the third term is the only difference between \( R_n \) and \( \bar{R}_n \).

For any interacting case, we have \( R_n = 0 \) since the \( \epsilon \)-transformation is an exact symmetry of the lattice actions. However, \( \bar{R}_n \) does not vanish at any finite lattice spacing for the interacting cases even if it vanishes for the free theory. If the \( \epsilon \)-symmetry is restored at a quantum continuum limit, \( \bar{R}_n \) should approach zero as \( a \to 0 \). We evaluate \( \bar{R}_n \) numerically to confirm whether the second SUSY WTI is restored in the continuum limit or not, as already done for the CG action in Ref. [17].

Figure 6 shows \( \langle \phi_n \phi^N_n \rangle \) and \( \langle \psi_n \bar{\psi}^N \rangle \) for \( \lambda = 1 \) and \( ma = 0.2 \). When \( N \) is sufficiently large, as confirmed in the figure, \( \langle \phi_n \phi^N_n \rangle \) and \( \langle \psi_n \bar{\psi}^N \rangle \) behave as

\[ \langle \phi_n \phi^N_n \rangle \approx C(e^{-anE_1} + e^{-a(N-n)E_1}), \quad (62) \]

\[ \langle \psi_n \bar{\psi}^N \rangle \approx De^{-anE_1}, \quad (63) \]

for \( 1 \ll n \ll N \). Here \( C \) and \( D \) are some constants that depend on the lattice spacing. Similarly, using the translational invariance, the other correlation functions in \( R_n \) and \( \bar{R}_n \) are expected to be

\[ \langle \phi_n \nabla - \phi^N_n \rangle \approx C_1(e^{-anE_1} - e^{-a(N-n-1)E_1}), \quad (64) \]

\[ \langle \phi_n W^N_n \rangle \approx C_2 e^{-anE_1} + C_3 e^{-a(N-n-1)E_1}, \quad (65) \]

\[ \langle W_{n+1} \phi^N_n \rangle \approx C_3 e^{-anE_1} + C_2 e^{-a(N-n-1)E_1}, \quad (66) \]

for \( 1 \ll n \ll N \). Here \( C_1 = C(1 - e^{-aE_1})/a \) and \( C_2, C_3 \) are some constants that depend on the lattice spacing. Note that it is possible to ignore the contribution from the second excited state for \( 1 \ll n \ll N \). We can immediately show that

\[ C_1 = C_3 = D - C_2 \quad (67) \]

from \( R_n = 0 \) and the second WTI holds if and only if \( C_2 \to C_3 \) as \( a \to 0 \).
FIG. 6. \(\langle \phi_n \phi_N \rangle\) and \(\langle \psi_n \bar{\psi}_N \rangle\) obtained from the CLR action for \(\lambda = 1\) and \(ma = 0.2\). The x-axis denotes the lattice site \(n\) and the y-axis shows the numerical values of the correlators in the logarithmic scale.

Figure 7 shows the cancellation among three correlation functions in \(R_n\) (Left) and \(\bar{R}_n\) (Right) for \(\lambda = 1\) and \(ma = 0.2\). In Figure 8, the similar plots obtained for the CG action are shown. As we expected, the other correlators show the behavior of (64), (65) and (66). The cancellation for \(n < N/2\) is realized in a different way from that of \(n > N/2\). As suggested from (68)- (69), the sum of two bosonic correlators (denoted as crosses) cancels the fermion correlator (denoted as squares) for \(1 \ll n \ll N/2\) while two bosonic correlators cancel each other out for \(N/2 \ll n \ll N\) since the fermion correlator is approximately zero compared with the others.

Since each term of \(R_n\) and \(\bar{R}_n\) is very small for \(n \simeq N/2\), we normalize them to observe the effect of the breaking term clearly:

\[
S_n \equiv \frac{R_n}{|\langle \psi_n \bar{\psi}_N \rangle| + |\langle \phi_n (\nabla - \phi) \rangle_N| + |\langle \phi_n W_N \rangle|} \tag{68}
\]

\[
\bar{S}_n \equiv \frac{\bar{R}_n}{|\langle \psi_n \bar{\psi}_N \rangle| + |\langle \phi_n (\nabla - \phi) \rangle_N| + |\langle W_{n+1} \phi_N \rangle|} \tag{69}
\]

Note again that \(W_N\) and \(W_{n+1}\) of (68) and (69) are replaced by \(W(\phi_N)\) and \(W(\phi_n)\), respectively, for the CG action. It is immediately found that \(S_n = 0\) for any \(n\) since \(R_n = 0\).
FIG. 7. Three correlation functions in $R_n$ (left) and $\bar{R}_n$ (right) for the CLR action. The cancellations among them are clearly observed.

FIG. 8. Three correlation functions in $R_n$ (left) and $\bar{R}_n$ (right) for the CG action. The cancellations are observed as well as the CLR case shown in Figure 7.

The asymptotic behavior of $\bar{S}_n$ can be understood from (63), (64) and (66). For sufficiently large $N$, it can be shown that $\bar{S}_n$ behaves as constants:

\[
S_n \approx h_1 \equiv \frac{C_2 - C_3}{2|C_3| + |D|}, \quad (1 \ll n \ll N/2) \tag{70}
\]

and

\[
S_n \approx h_2 \equiv \frac{C_3 - C_2}{|C_2| + |C_3|}, \quad (N/2 \ll n \ll N). \tag{71}
\]

We have

\[
h_2 = -2h_1 + O(h_1^2), \tag{72}
\]
when $C_2$ and $C_3$ have the same sign. The similar identities as (70), (71) and (72) hold for the CG action.

In Figure 9 and Figure 10, $S_n$ and $\tilde{S}_n$ are plotted against $n$. As we expected, $S_n$ vanishes as numerical results while $\tilde{S}_n$ has two plateaux corresponding to $h_1$ and $h_2$. We should note that the scale of the $y$-axis for the CLR action is rather smaller than that of the CG action. The value of $\tilde{S}_n$ rapidly changes from $h_1$ to $h_2$ around $n = N/2$ as a result of the cancellation of three correlation functions.

![Graph showing $S_n$ and $\tilde{S}_n$ for the CLR action with $\lambda = 1$ and $ma = 0.2$.]

FIG. 9. $S_n$ and $\tilde{S}_n$ for the CLR action with $\lambda = 1$ and $ma = 0.2$.

Figures 11 shows the lattice spacing dependence of $h_1$ and $h_2$ for $\lambda = 1$ and the numerical values are shown in Table VIII for the convenience of further studies. Figure 12 shows the same plot for $\lambda = 0.001$. We evaluate $h_1$ and $h_2$ at $n = N/5$ and $n = 4N/5$, respectively. It can be seen that $h_1$ and $h_2$ approach zero as $a \to 0$. Consequently, the second SUSY WTI holds in the continuum limit, that is, full SUSY is restored in the quantum continuum limit at low energy region $1 \ll n \ll N$. The breaking effect $h_1$ and $h_2$ of the CLR action are significantly smaller than the CG action even for the strong coupling. Thus we can conclude that the CLR shows a good behavior that is similar to the continuum theory at a non-perturbative level.
FIG. 10. $S_n$ and $\tilde{S}_n$ for the CG action with $\lambda = 1$ and $ma = 0.2$.

FIG. 11. Lattice spacing dependence of $h_1$ and $h_2$ for $\lambda = 1$. We plot $h_1$ and $\tilde{h}_2 = -h_2/2$, which are evaluated at $n = N/5$ and $n = 4N/5$, as circles and diamonds for the CLR action and triangles and squares for the CG action.
FIG. 12. Lattice spacing dependence of $h_1$ and $h_2$ for $\lambda = 0.001$. We plot $h_1$ and $\tilde{h}_2 = -h_2/2$, which are evaluated at $n = N/5$ and $n = 4N/5$, as circles and diamonds for the CLR action and triangles and squares for the CG action.

V. SUMMARY AND DISCUSSION

The property of the cyclic Leibniz rule has been studied in $\mathcal{N} = 2$ SUSY QM beyond the perturbation theory. We have defined the lattice action on the basis of the CLR with the backward difference operator giving a solution for any superpotential. The numerical computations have been carried out using the transfer matrix representation of the partition function and the correlation functions. Tuning the rescaling parameter, the energy spectra and SUSY Ward-Takahashi identities are obtained in a high accuracy. We have compared them with those of the Catterall-Gregory action.

Although the number of exact symmetry is the same between the CLR and the CG actions, the CLR action provides a milder cut-off dependence of energy spectra for both weak and strong couplings. In the weak coupling limit, the $\mathcal{O}(a)$ term does not appear in the energy spectra for the CLR action but does for the CG action. Even for the strong coupling, we have observed that the coefficient of $\mathcal{O}(a)$ term for the CLR action is smaller than the CG action. The lattice SUSY WTIs have shown the same tendency in the cut-off
In the $N = 4$ case with the CLR, the number of exact SUSY is greater than the other lattice formulation. We can expect that a lattice theory with the CLR is highly improved and behaves much similar to the continuum theory. The results shown in this paper could be useful to construct the SUSY action with a modified Leibniz rule in higher dimensions.

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Appendix A: More about the CLR

1. Solutions for other difference operators

The solution of the CLR for the forward difference operator and a symmetric difference operator $\nabla_S = \frac{1}{2}(\nabla_+ - \nabla_-)$ are presented. For any difference operator $\nabla$, the CLR is defined in the same manner as (31). By repeating the same procedures as in section III B we find that (31) can be written as

$$\sum_n \nabla \phi_n W_n = 0. \quad (A1)$$

It is then easy to find a local solution of (A1):

$$W_n = \begin{cases} 
U(\phi_{n+1}) - U(\phi_n) \\ \frac{\phi_{n+1} - \phi_n}{\phi_{n+1} - \phi_{n-1}} 
\end{cases} \quad \text{for} \ \nabla = \nabla_+$$

$$W_n = \begin{cases} 
U(\phi_{n+1}) - U(\phi_n) \\ \frac{\phi_{n+1} - \phi_{n-1}}{\phi_{n+1} - \phi_{n-1}} 
\end{cases} \quad \text{for} \ \nabla = \nabla_S \quad (A2)$$

The same discussions as mentioned in section III B tell us that $W_n$ is well-defined local function that reproduces $W(\phi_n)$ up to $O(a)$.

2. The $m$-body CLR

We now consider $W(\phi) = \sum_{m=0}^{\infty} c_m \phi^m$ with coupling constants $c_m$. Then the lattice superpotential $W_n$ is also expressed as an expansion,

$$W_n \equiv \sum_{\ell=0}^{\infty} c_{\ell} [\phi]_n^\ell \quad (A3)$$

with

$$[\phi]_n^\ell \equiv \sum_{m_1, m_2, \ldots, m_\ell} M_{n, m_1, m_2, \ldots, m_\ell} \phi_{m_1} \phi_{m_2} \cdots \phi_{m_\ell}. \quad (A4)$$

Here we assume that $M_{n, m_1, m_2, \ldots, m_\ell}$ is totally symmetric for $m_1, m_2, \ldots, m_\ell$ except for the first index $n$ and $[1]_n^\ell = 1$ as an overall normalization. The locality condition is strictly

\text{7} The simplest example of $M$ (but it is not a solution of CLR) is $M_{n, m_1, m_2, \ldots, m_\ell} = \delta_{nm_1} \delta_{nm_2} \cdots \delta_{nm_\ell}$.

Then the lattice action (17) coincides with the naive one owing to $W_n = W(\phi_n)$ and $W'_{nm} = W'(\phi_n)\delta_{nm}$.

We can express a scattering of lattice variables around the site $n$ by $M_{n, m_1, m_2, \ldots, m_\ell}$. 

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defined as

\[ |M_{n,m_1,m_2,\ldots,m_\ell}| < C \exp\{-\rho|n - m_k|\}, \tag{A5} \]

where \(C\) and \(\rho > 0\) are some positive constants for \(k = 1, \ldots, \ell\). The summation in (A4) is well-defined because it is absolutely convergent for (A5).

The CLR in (18) is shown to be

\[ \sum_n \left\{ \nabla_{nk} M_{n,n_1,n_2,\ldots,n_{m-1},n_m} + \nabla_{nn_1} M_{n,n_2,n_3,\ldots,n_m,k} + \cdots + \nabla_{nn_m} M_{n,k,n_1,\ldots,n_{m-2},n_{m-1}} \right\} = 0, \tag{A6} \]

which is referred to as \(m\)-body CLR. It is easy to show that (A6) is equivalent to (18). We should note that the indices \(k, n_1, n_2, \ldots, n_m\) cyclically appear in (A6). This is the reason why we called (18) the cyclic Leibniz rule.

The solutions of the \(m\)-body CLR for the backward difference operator can be read from (19) using

\[ M_{n,m_1,m_2,\ldots,m_\ell} = \frac{1}{\ell!} \left. \frac{\partial^\ell W_n}{\partial \phi_{m_1} \partial \phi_{m_2} \cdots \partial \phi_{m_\ell}} \right|_{c_m=1,\phi=0}. \tag{A7} \]

We have

\[ M_{n,m} = \frac{1}{2} (\delta_{nm} + \delta_{n-1,m}), \tag{A8} \]
\[ M_{n,m,k} = \frac{1}{6} \left( 2 \delta_{nm} \delta_{nk} + \delta_{n-1,m} \delta_{nk} + \delta_{nm} \delta_{n-1,k} + 2 \delta_{n-1,m} \delta_{n-1,k} \right), \tag{A9} \]
\[ M_{n,m,k,l} = \frac{1}{12} \left( 3 \delta_{nm} \delta_{nk} \delta_{nl} + \delta_{n-1,m} \delta_{nk} \delta_{nl} + \delta_{nm} \delta_{n,k+1} \delta_{nl} + \delta_{nm} \delta_{nk} \delta_{n-1,l} + \delta_{n-1,m} \delta_{n,k} \delta_{n-1,l} + 3 \delta_{n-1,m} \delta_{n-1,k} \delta_{n-1,l} \right), \tag{A10} \]

and so on.

The explicit forms of \(M_{n,m_1,m_2,\ldots,m_\ell}\) for the forward difference operator are ones obtained by replacing the lattice site \(n - 1\) by \(n + 1\) in (A8), (A9) and (A10). Those for the symmetric difference operator \(\nabla_S = \frac{1}{2}(\nabla_+ + \nabla_-)\) are also obtained by the similar replacement of the lattice site.

**Appendix B: Weak coupling expansion**

The weak coupling expansion of the first excited energy are presented at one-loop order for the naive, the CG and the CLR actions. We perform the lattice perturbation theory
on the infinite volume lattice. The first excited energy are evaluated as effective masses obtained from the two-point correlation functions. In this section, we assume $m > 0$ and basically take $a = 1$ except for final results of the effective masses.

1. Perturbative calculation on the infinite volume lattice

The free part of a lattice action $S$ can be expressed in the momentum space as

$$S_{\text{free}} = \int_{-\pi}^{\pi} \frac{dp}{2\pi} \left\{ \frac{1}{2} D_0^{-1}(p) \phi(p) \phi(-p) + S_0^{-1}(p) \bar{\psi}(p) \psi(-p) \right\},$$

(B1)

where $D_0(p)$ and $S_0(p)$ are bare propagators of the boson and the fermion, respectively. The concrete form of $D_0(p)$ and $S_0(p)$, which depends on $S_{\text{free}}$, are obtained by the Fourier transformation for a lattice variable $\varphi_n$:

$$\varphi(p) = \sum_{n \in \mathbb{Z}} e^{ipn} \varphi_n,$$

(B2)

$$\varphi_n = \int_{-\pi}^{\pi} \frac{dp}{2\pi} e^{-ipn} \varphi(p),$$

(B3)

with a useful identity $\delta_{n0} = \int_{-\pi}^{\pi} \frac{dp}{2\pi} e^{ipn} (n \in \mathbb{Z})$. Note that $\varphi(p + 2\pi m) = \varphi(p)$ for $m \in \mathbb{Z}$.

The two-point correlation functions are defined as

$$D_{kl} \equiv \langle \phi_k \phi_l \rangle = \int_{-\pi}^{\pi} \frac{dp}{2\pi} D(p) e^{ip(k-l)},$$

(B4)

$$S_{kl} \equiv \langle \psi_k \bar{\psi}_l \rangle = \int_{-\pi}^{\pi} \frac{dp}{2\pi} S(p) e^{ip(k-l)},$$

(B5)

where $D(p)$ and $S(p)$ are the full propagators. We have $D_{mn} = D_{m-n,0}$ and $S_{mn} = S_{m-n,0}$ as a result of the translational invariance. The free two-point correlation functions $(D_0)_{kl}$ and $(S_0)_{kl}$ are calculated from (B4) and (B5) with $D_0(p)$ and $S_0(p)$ using the complex integral with $z = e^{ip}$.

The full propagators can be evaluated in the weak coupling expansion from $D_0, S_0$ and the boson and the fermion self energies $\Pi_{kl}$ and $\Sigma_{kl}$. As well-known, $D_{kl}$ is given by an infinite series,

$$D_{kl} = D_{0,kl} - (D_0 \Pi D_0)_{kl} + (D_0 \Pi D_0 \Pi D_0)_{kl} - \ldots.$$  

(B6)

Thus we have

$$D_{kl} = \left( \frac{1}{D_0^{-1} + \Pi} \right)_{kl}.$$  

(B7)
Similarly,
\[
S_{kl} = \left( \frac{1}{S_{0}^{-1} + \Sigma} \right)_{kl}.
\]  
(B8)

Once \( \Pi_{kl} \) and \( \Sigma_{kl} \) are evaluated at the n-loop level, \( D_{kl} \) and \( S_{kl} \) are obtained at the same order.

The effective masses \( m_{B}^{\text{eff}} \) and \( m_{F}^{\text{eff}} \) are read from the large distance behavior of \( D_{kl} \) and \( S_{kl} \). For \( |k - l| \gg 1 \),
\[
D_{kl} \approx C e^{-m_{B}^{\text{eff}}|k-l|},
\]  
(B9)
\[
S_{kl} \approx C' \theta_{k,l} e^{-m_{F}^{\text{eff}}|k-l|}.
\]  
(B10)

with
\[
\theta_{k,l} \equiv \begin{cases} 
1 & \text{for } k \geq l \\
0 & \text{for } k < l.
\end{cases}
\]  
(B11)

At one-loop level, the self-energies provide the shifts of masses \( \Delta m \) in \( D_{0}^{-1}(p) \) and \( S_{0}^{-1}(p) \) via (B7) and (B8). The one-loop effective masses \( m_{0,\text{eff}}^{B,F} \) are actually obtained from the formulas of tree level effective masses \( m_{0,\text{eff}}^{B,F} \) with \( m \to m + \Delta m \).

2. The naive action

We begin with the case of the naive action \([13]\) whose \( D_{0}(p) \) and \( S_{0}(p) \) are given by
\[
D_{0}(p) \equiv \frac{1}{2(1 - \cos p) + m^2},
\]  
(B12)
\[
S_{0}(p) \equiv \frac{1}{1 - e^{-ip} + m}.
\]  
(B13)

The free boson propagator in the position space is evaluated from \([54]\):
\[
D_{0,kl} = \int dz \frac{z^{k-l}}{z^2 - (m^2 + 2)z + 1},
\]  
(B14)
for \( z = e^{ip} \). It is easily shown that
\[
D_{0,kl} = e^{-m_{0,\text{eff}}^{B,F}|k-l|}
\]  
(B15)
where
\[
m_{0,\text{eff}}^{B} = -\log \left( \frac{1 + m^2}{2} - m \sqrt{1 + \frac{m^2}{4}} \right).
\]  
(B16)
Similarly,

\[ S_{0,kl} = \theta_{k,l} \frac{e^{-m_{0,\text{eff}} |k-l|}}{1 + m} \]  

(B17)

where

\[ m_{0,\text{eff}} = \log (1 + m), \]  

(B18)

and \( \theta_{k,l} \) is given by (B11).

At one-loop level, the boson and fermion self energies are obtained as

\[ \Pi(p) = 6 \lambda m^2 \left( \frac{1}{\sqrt{1 + m^2 4}} - \frac{1}{1 + m} \right), \]  

(B19)

\[ \Sigma(p) = \frac{3 \lambda m}{2 \sqrt{1 + m^2 4}}. \]  

(B20)

The one-loop self energies provide different corrections to the mass \( m \rightarrow m + \Delta m_{B,F} \) where \( \Delta m_B \) and \( \Delta m_F \) are identified from (B19) and (B20), respectively.

The one-loop effective masses are obtained by inserting \( m + \Delta m_{B,F} \) into (B16) and (B18):

\[ E_B^1 \frac{m}{m} = 1 + 3 \lambda ma + \frac{(-1 - 81 \lambda)m^2 a^2}{24} + O(\lambda^2, m^3 a^3) \]  

(B21)

\[ E_F^1 \frac{m}{m} = 1 + \frac{3 \lambda}{2} - \frac{(2 + 6 \lambda)ma}{4} + O(\lambda^2, m^2 a^2). \]  

(B22)

We should note that \( E_B^1 \) is different from \( E_F^1 \) even in the continuum limit \( ma \rightarrow 0 \) as a result of the one-loop effect although they coincide with each other at the tree level with \( \lambda = 0 \).

3. The CG action

The free propagators of the CG action are

\[ D_0^{CG}(p) = \frac{1}{2(1 - \cos p) + m^2 + 2m(1 - \cos p)}. \]  

(B23)

\[ S_0^{CG}(p) = \frac{1}{1 - e^{-ip} + m}. \]  

(B24)

The similar calculation as done around (B14) tells us that the effective masses are degenerated as

\[ m_{0,\text{eff}}^B = m_{0,\text{eff}}^F = \log (1 + m), \]  

(B25)

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at the tree level.

The self energies are calculated at the one-loop level as

\[ \Pi(p) = \Delta m[2m + 2(1 - \cos p)], \quad (B26) \]
\[ \Sigma(p) = \Delta m, \quad (B27) \]

where

\[ \Delta m \equiv \frac{3\lambda m}{2 + m}. \quad (B28) \]

These give the same correction to the boson mass and the fermion mass up to \( O(\lambda) \). The one-loop effective masses are evaluated from (B25) with \( m + \Delta m \). We thus obtain that

\[ \frac{E_1}{m} = 1 + \frac{3\lambda}{2} - \frac{ma}{2} - \frac{\lambda(ma)^2}{2} - \frac{9\lambda(ma)^2}{4} + O(\lambda^2, (ma)^3), \quad (B29) \]

for \( E_1 \equiv m_{\text{eff}}^B = m_{\text{eff}}^F \) owing to an exact SUSY.

4. The CLR action

The free propagators of the CLR action are given by

\[ D_0^{\text{CLR}}(p) \equiv \frac{1}{2(1 - \cos p) + m^2(1 + \cos p)/2}, \quad (B30) \]
\[ S_0^{\text{CLR}}(p) \equiv \frac{1}{1 - e^{-ip} + m(1 + e^{-ip})/2}. \quad (B31) \]

The tree level effective masses are

\[ m_{\text{eff}}^B = m_{\text{eff}}^F = \log \left( \frac{1 + \frac{m}{2}}{1 - \frac{m}{2}} \right), \quad (B32) \]

which are degenerated between the boson and the fermion.

The one-loop self energies are given by

\[ \Pi(p) = 2m\Delta m(1 + \cos p), \quad (B33) \]
\[ \Sigma(p) = \Delta m \left( \frac{1 + e^{-ip}}{2} \right), \quad (B34) \]

where

\[ \Delta m = \frac{\lambda m(m + 6)}{2(m + 2)}. \quad (B35) \]
The one-loop effective masses are read from (B32) with \( m + \Delta m \). The first excited energies for the bosonic and fermionic states are thus obtained as \( E_1 \equiv m_{\text{eff}}^B = m_{\text{eff}}^F \):

\[
\frac{E_1}{m} = 1 + \frac{3\lambda}{2} - \frac{\lambda ma}{2} + \frac{(ma)^2}{12} + \frac{5\lambda(ma)^2}{8} + O(\lambda^2, (ma)^3),
\] (B36)

owing to an exact SUSY.

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