Regret Pruning for Learning Equilibria in Simulation-Based Games

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Abstract

In recent years, empirical game-theoretic analysis (EGTA) has emerged as a powerful tool for analyzing games in which an exact specification of the utilities is unavailable. Instead, EGTA assumes access to an oracle, i.e., a simulator, which can generate unbiased noisy samples of players’ unknown utilities, given a strategy profile. Utilities can thus be empirically estimated by repeatedly querying the simulator. Recently, various progressive sampling (PS) algorithms have been proposed, which aim to produce PAC-style learning guarantees (e.g., approximate Nash equilibria with high probability) using as few simulator queries as possible. A recent work by Areyan Viqueira, Cousins, and Greenwald introduces a pruning technique called regret-pruning which further minimizes the number of simulator queries placed in PS algorithms which aim to learn pure Nash equilibria. In this paper, we address a serious limitation of this original regret pruning approach – it is only able to guarantee that true pure Nash equilibria of the empirical game are approximate equilibria of the true game, and is unable to provide any strong guarantees regarding the efficacy of approximate pure Nash equilibria. This is a significant limitation since in many games, pure Nash equilibria are computationallly intractable to find, or even non-existent. We introduce three novel regret pruning variations. The first two variations generalize the original regret pruning approach to yield guarantees for approximate pure Nash equilibria of the empirical game. The third variation goes further to even yield strong guarantees for all approximate mixed Nash equilibria of the empirical game. We use these regret pruning variations to design two novel progressive sampling algorithms, PS-REG+ and PS-REG-M, which experimentally outperform the previous state-of-the-art algorithms for learning pure and mixed equilibria, respectively, of simulation-based games.

Introduction

Game theory is the standard conceptual framework used to analyze strategic interactions among rational agents in multi-agent systems. A game comprises a collection of players, each with a set of strategies and a utility function, mapping strategy profiles (i.e., combinations of strategies) to values. Traditionally, game-theoretic analysis presumes complete access to a game’s structure, including the utility functions. In recent years, empirical game-theoretic analysis (EGTA) has emerged as a powerful tool for analyzing games in which such an exact specification of the utilities is unavailable. Instead, EGTA assumes access to an oracle, i.e., a (stochastic) simulator, which produces unbiased noisy samples of players’ unknown utilities given a strategy profile (Wellman, 2006; Tuyls et al., 2020; Areyan Viqueira, Cousins, and Greenwald, 2020). Such games are called simulation-based games (Vorobeychik and Wellman, 2008), or black-box games (Picheny et al., 2016), and their empirical counterparts, which are derived from simulation data, are called empirical games. Simulation-based games have been studied in many practical settings including trading agent analyses in supply chains (Vorobeychik, Kiekintveld, and Wellman, 2006; Jordan, Kiekintveld, and Wellman, 2007); ad auctions (Jordan and Wellman, 2010; Areyan Viqueira et al., 2019), and energy markets (Ketter, Peters, and Collins, 2013); designing network routing protocols (Wellman, Kim, and Duong, 2013); strategy selection in real-time games (Tavares et al., 2016); and the dynamics of RL algorithms, like AlphaGo (Tuyls et al., 2018).

A typical EGTA goal is to produce PAC-style learning guarantees (e.g., approximate Nash equilibria with high probability) with minimal query complexity, i.e., the number of simulation queries placed (Tuyls et al., 2020; Areyan Viqueira, Cousins, and Greenwald, 2020; Cousins et al., 2022). This goal has led to the development of progressive sampling algorithms (Areyan Viqueira, Cousins, and Greenwald, 2020; Cousins et al., 2022), which place simulation queries in progressive batches, until the desired guarantee is reached. On top of progressive sampling, two papers introduce various pruning techniques, to further minimize query complexity. One of these techniques, well-estimated pruning, prunes strategy profiles whose utilities are very likely to already be sufficiently close to the true utilities (Cousins et al., 2022). This technique is useful for learning a variety of game properties, including regret, pure or mixed Nash equilibria, welfare-maximizing outcomes, and more. A second technique, called regret pruning, is intended for use only when learning pure Nash equilibria, as it prunes strategy profiles that are highly unlikely to be best responses, and hence unlikely to be necessary for finding pure Nash equilibria (Areyan Viqueira, Cousins, and Greenwald, 2020).

In this paper, we focus predominantly on regret pruning. Areyan Viqueira, Cousins, and Greenwald (2020) claim that their regret pruning criterion can be used to learn an empirical game which, with high probability, satisfies a certain dual pure Nash containment guarantee – all pure Nash equilibria
of the simulation-based game are approximate equilibria of
the empirical game, and all approximate pure Nash equilibria
of the empirical game are approximate equilibria of the
simulation-based game. We show via a direct counterexample,
however, that their progressive sampling algorithm using
regret pruning fails to satisfy this second inclusion. Rather,
their algorithm only yields the guarantee that all true pure
Nash equilibria of the empirical game are approximate equi-
libria of the simulation-based game, and is unable to directly
yield any non-trivial guarantee regarding approximate pure
Nash equilibria of the empirical game. This difference is
similar, since in many games, computing a pure Nash equilib-
rium is computationally intractable (e.g., it is NP-complete
in graphical games (Gottlob, Greco, and Scarcello 2005)), and
sometimes such an equilibrium does not even exist. In such
cases, an approximate pure Nash equilibrium is the best that
can be hoped for, but Areyan Viqueira, Cousins, and Green-
wald’s progressive sampling algorithm using regret pruning
yields no guarantees for such equilibria.

In response to this limitation of the original regret pruning
technique, we design three new variations of regret pruning
(and new corresponding progressive sampling algorithms).
At the cost of a slightly tighter regret pruning criterion, our
first regret pruning variation yields the guarantee regard-
ing approximate equilibria of the empirical game which
Areyan Viqueira, Cousins, and Greenwald’s progressive sampling algorithm using regret pruning yields no guarantees for such equilibria.

The third regret pruning variation is a particularly signifi-
cant contribution, as it is one of the first pruning techniques
beyond simple well-estimated pruning for learning mixed
equilibria. The only available alternative is rationalizabil-
ity pruning introduced by Areyan Viqueira, Cousins, and
Greenwald (2020), which requires the use of a computa-
tionally expensive iterative dominance algorithm, has a very
tight pruning criterion which often prunes few to no strategy
profiles in practice, and most importantly, like the original
regret pruning technique, only yields guarantees regarding true mixed Nash equilibria of the empirical game. In contrast,
our novel third regret pruning variation utilizes a pruning
criterion which is very cheap to compute, can prune a very
significant number of simulation queries (which we confirm experimentally), and yields Nash containment guarantees for
approximate mixed equilibria of the empirical game.

In addition to presenting the guarantees progressive sam-
pling algorithms using these novel variations of regret pruning
can satisfy, we also derive sample complexity bounds
for these new algorithms. In particular, we present PAC-style
upper bounds on the number of samples our progressive sam-
ping algorithms will take to prune each respective strategy
profile. Finally, we conclude by demonstrating experi-
mentally that our novel progressive sampling algorithms which
incorporate both well-estimated pruning and novel regret
pruning variations significantly outperform Cousins et al.’s progressive sampling algorithm which used well-estimated
pruning alone, in some cases requiring up to 50% fewer sim-
ulation queries to learn equilibria of similar quality.

Related Works

The EGTA literature, while relatively young, is growing
rapidly, with researchers actively contributing methods for
myriad game models. Some of these methods are designed for
normal-form games (Cousins et al. 2022; Areyan Viqueira,
Cousins, and Greenwald 2020; Areyan Viqueira et al. 2019; Tavares et al. 2016; Fearnley et al. 2015; Vorobeychik
and Wellman 2008), and others, for extensive-form games
(Marchesi, Trovò, and Gatti 2020; Gatti and Restelli 2011; Zhang and Sandholm 2021). Most methods apply to games
with finite strategy spaces, but some apply to games with
infinite strategy spaces (Marchesi, Trovò, and Gatti 2020; Vorobeychik, Wellman, and Singh 2007; Wiedenbeck,
Yang, and Wellman 2018). A related line of work aims to
empirically design mechanisms via EGTA methodologies
(Vorobeychik, Kiekintveld, and Wellman 2006; Areyan
Viqueira et al. 2019).

The progressive sampling algorithms and pruning tech-
niques which we design in this paper extend work done by
Cousins et al. 2022; Areyan Viqueira, Cousins, and Green-
wald 2020; Areyan Viqueira et al. 2019 in designing al-
gorithms for learning equilibria in normal-form simulation-
based games with finite strategy spaces.

Learning Framework

We begin with the standard definition of standard normal-
form games, and some related properties. We then introduce
our formal model of simulation-based games and empirical
games. Finally, we state the concentration inequalities we use
to guide pruning in our progressive sampling algorithms.

Basic Game Theory

Definition 1 (Normal-Form Game). A normal-form game
\( \Gamma = (P, \{ S_p \}_{p \in P}, u) \) consists of a set of players \( P \), each
with a corresponding pure strategy set \( S_p \). We define \( S = S_1 \times \cdots \times S_{|P|} \) to be the pure strategy profile space, and
then \( u : S \rightarrow \mathbb{R}^{|P|} \) is a vector-valued utility function (equiv-
antly, a vector of \(|P|\) scalar utility functions \( u_p \)).

Given an NFG \( \Gamma \), we denote by \( S^p \) the set of distributions
over \( S_p \); this set is called player \( p \)'s mixed strategy set. We define \( S^o = S_1^o \times \cdots \times S_{|P|}^o \) to be the mixed strategy profile
space, and, overloading notation, we write \( u(s) \) to denote
the expected utility of a mixed strategy profile \( s \in S^o \). Each
pure strategy profile \( s \in S \) is contained in the mixed strategy
profile space \( S^o \), represented by the profile with each mixed
strategy concentrated entirely at the respective pure strategy.

Given player \( p \) and strategy profile \( s \in S^o \), the set
\( A[p, s] = \{ \{ s_1, \ldots, s_{p-1}, t, s_{p+1}, \ldots, s_{|P|} \} \mid t \in S_p \} \) con-
tains all adjacent strategy profiles, meaning those in which
the strategies of all players \( q \neq p \) are fixed at \( s_q \), while player
\( p \)'s strategy may vary across their pure strategy set.
Definition 2 (Regret). A player $p$'s regret at strategy profile $s \in S^o$ is defined as $	ext{Reg}_p(s; u) \doteq \sup_{s' \in \text{Adj}_p, s} u_p(s') - u_p(s)$. We further define $	ext{Reg}(s; u) \doteq \max_{p \in P} \text{Reg}_p(s; u)$.

A strategy profile $s \in S$ is player $p$'s best response if $\text{Reg}_p(s; u) = 0$ (i.e., the player does not regret choosing this strategy profile as opposed to an adjacent one). We say it is an $\varepsilon$-best response if $\text{Reg}_p(s; u) \leq \varepsilon$.

A strategy profile $s \in S^o$ is an $\varepsilon$-Nash equilibrium if it is an $\varepsilon$-best response for each player $p \in P$ (i.e., if $\text{Reg}(s; u) \leq \varepsilon$). If $s$ corresponds to a pure strategy profile, then we call it an $\varepsilon$-pure Nash equilibrium ($\varepsilon$-PNE); otherwise, we call it an $\varepsilon$-mixed Nash equilibrium ($\varepsilon$-MNE). A 0-PNE is simply called a PNE, and a 0-MNE is called an MNE. The set of $\varepsilon$-pure (resp. mixed) Nash equilibria is denoted $E_\varepsilon(u)$ (resp. $E^\varepsilon(u)$), and the set of pure (resp. mixed) Nash equilibria is denoted $E(u)$ (resp. $E^\varepsilon(u)$).

Formal Model of Simulation Based Games

In simulation-based games, we assume access to a simulator $\mathcal{S}(\cdot)$, which can be queried to produce unbiased noisy samples of the players’ utilities when $s \in S$ is played. We denote such a sample by $\hat{u}(s) \sim \mathcal{S}(s)$, where $\hat{u}(s)$ is a $|P|$-vector comprising utilities for each player.

Definition 3 (Simulation-Based Game). A simulation-based game $\Gamma_{\mathcal{S}} \doteq \langle P, S, \mathcal{S} \rangle$ consists of a set of players $P$, a pure strategy profile space $S \doteq S_1 \times \cdots \times S_P$, and a simulator $\mathcal{S}(\cdot)$ that produces noisy samples $\hat{u}(s) \sim \mathcal{S}(s)$ upon simulation of a strategy profile $s \in S$.

Corresponding to each simulation-based game $\Gamma_{\mathcal{S}}$ is an expected normal-form game.

Definition 4 (Expected Normal-Form Game). Given a simulation-based game $\Gamma_{\mathcal{S}} \doteq \langle P, S, \mathcal{S} \rangle$, we define the “underlying” utility function $u : S \to \mathbb{R}^{|P|}$ by $u(s) \doteq E_{\hat{u}(s) \sim \mathcal{S}(s)}[u(s)]$. The expected game corresponding to $\Gamma_{\mathcal{S}}$ is then the normal-form game $(P, S, u)$. Overloading notation, we also let $\hat{\Gamma}_{\mathcal{S}}$ denote this (unknown) expected normal-form game.

Since we do not have direct access to the expected normal-form game, its utilities must be learned by repeatedly querying the simulator. The resulting empirical estimate of the expected game is called an empirical game.

Definition 5 (Empirical Normal-Form Game). Given a simulation-based game $\Gamma_{\mathcal{S}} \doteq \langle P, S, \mathcal{S} \rangle$, let $\hat{u}^{(1)}(s), \ldots, \hat{u}^{(n)}(s) \sim \mathcal{S}(s)$ denote the sample utilities produced by $n_s > 0$ queries to the simulator at strategy profile $s \in S$. We define the empirical utility function $\hat{u} : S \to \mathbb{R}^{|P|}$ by $\hat{u}(s) \doteq \frac{1}{n_s} \sum_{i=1}^{n_s} \hat{u}^{(i)}(s)$ for all $s \in S$, and the ensuing empirical normal-form game by $\hat{\Gamma}_{\mathcal{S}} \doteq \langle P, S, \hat{u} \rangle$.

From here onwards, let $\hat{\Gamma}_{\mathcal{S}} \doteq \langle P, S, \hat{u} \rangle$ be an arbitrary simulation-based game with underlying utility function $\hat{u}$, and let $\hat{\Gamma}_{\mathcal{S}} \doteq \langle P, S, \hat{u} \rangle$ be a corresponding empirical game. Using our formalization, we can now present one of the foundational results of EGTA (Tuyls et al. 2020).

Lemma 1. If $|u_p(s) - \hat{u}_p(s)| \leq \varepsilon$ for all $(p, s) \in \mathcal{I}$, then $E(u) \subseteq E_2(\hat{u}) \subseteq E_4(u)$ and $E^\varepsilon(u) \subseteq E^\varepsilon_2(\hat{u}) \subseteq E^\varepsilon_4(u)$, or more generally, $E_\varepsilon(u) \subseteq E_{2+\gamma}(\hat{u})$ and $E^\varepsilon(u) \subseteq E^{2+\gamma}(\hat{u})$ for all $\gamma \geq 0$ (resp. for mixed equilibria).

This result can be understood as stating that given a sufficiently strong approximation $\Gamma_{\mathcal{S}}$ of a simulation-based game $\Gamma_{\mathcal{S}}$, we can approximate pure (resp. mixed) Nash equilibria in $\Gamma_{\mathcal{S}}$ with perfect recall – all pure (resp. mixed) Nash equilibria in $\Gamma_{\mathcal{S}}$ are approximate Nash equilibria in $\Gamma_{\mathcal{S}}$ – and with approximately perfect precision – all approximate pure (resp. mixed) Nash equilibria in $\Gamma_{\mathcal{S}}$ are approximate pure (resp. mixed) Nash equilibria in $\Gamma_{\mathcal{S}}$. Lemma 1 is one of the primary motivations for EGTA’s pursuit of designing efficient algorithms for learning strong approximations of simulation-based games, as it guarantees that the better an approximation an empirical game is of a simulation-based game, the more strategically representative the empirical game will be of the underlying simulation-based game.

Tail Bounds

Next, we state the tail bounds upon which our novel regret pruning techniques and progressive sampling algorithms depend. These are the same bounds derived and used by Cousins et al. for a more thorough discussion of them see Cousins et al. (2022). For all subsequent results, we make the following “bounded utilities” assumption.

Assumption 1 (Bounded Utilities). For each strategy profile $s \in S$, the sample utilities produced via $\mathcal{S}(s)$ lie on the bounded interval $[a_s, b_s]$ for some fixed $a_s, b_s \in \mathbb{R}$. We define $c := \sup_{s \in S}(b_s - a_s)$.

The most straightforward tail bound for mean-estimation is Hoeffding’s Inequality, which was used by Tuyls et al. (2020). We use Hoeffding’s inequality to bound each individual utility, combined with a union bound to yield a guarantee for all utilities.

Theorem 1 (Hoeffding’s Inequality). Let $\hat{\Gamma}_{\mathcal{S}} \doteq \langle P, S, \hat{u} \rangle$ be an empirical game. Then, with probability at least $1 - \delta$, for all $(p, s) \in \mathcal{I}$, it holds that

$$|u_p(s) - \hat{u}_p(s)| \leq c \frac{\ln \left(\frac{2|\mathcal{I}|}{\delta}\right)}{2n_s} \leq \varepsilon_p^u(s) \; .$$

Let $\mathcal{V}_p$ denote $\bigvee_{(p, s) \in \mathcal{I}} |u_p(s)|$ for all $(p, s) \in P \times S$. Since Hoeffding’s Inequality assumes a worst-case variance on the utilities (i.e., $\mathcal{V}_p(s) = c^2/4$), when variances are small, it yields a very loose bound. When the variances of utilities are known, Bennett’s inequality provides a non-uniform, variance-sensitive guarantee.

Theorem 2 (Bennett’s Inequality). Let $\hat{\Gamma}_{\mathcal{S}} \doteq \langle P, S, \hat{u} \rangle$ be an empirical game. Then, with probability at least $1 - \delta$, for all $(p, s) \in \mathcal{I}$, it holds that

$$|u_p(s) - \hat{u}_p(s)| \leq c \frac{\ln \left(\frac{2|\mathcal{I}|}{\delta}\right)}{3n_s} + \sqrt{2\mathcal{V}_p(s) \ln \left(\frac{2|\mathcal{I}|}{\delta}\right)} \; .$$
Of course, since our only access to the utilities of the simulation-based game are via the simulator, we do not know their variances. Cousins et al. (2022) circumvent this limitation by deriving an “empirical Bennett’s inequality” depending on empirical estimates of the true utility variances.

**Theorem 3 (Empirical Bennett’s Inequality).** Let \( \bar{\Gamma}_{\gamma} \doteq \langle P, S, \bar{u} \rangle \) be an empirical game. Let \( \kappa_5 \doteq \left( \frac{1}{\gamma^2} + \frac{1}{2\ln(\frac{1}{\gamma^2})} \right) \).

For all \((p, s) \in \mathcal{I}\), define
\[
\hat{v}_p(s) = \frac{1}{m-1} \sum_{j=1}^{m} (u_p(s; y_j) - \bar{u}_p(s; Y))^2;
\]
\[
\varepsilon^p(s) = \frac{2\gamma^2 \ln(\frac{1}{\gamma^2})}{3m} + \sqrt{\kappa_5 \left( \frac{c^2 \ln(\frac{3}{\gamma^2})}{m-1} \right)^2 + \frac{4\gamma^2 \varepsilon_p(s) \ln(\frac{3}{\gamma^2})}{m}};
\]
\[
\hat{\varepsilon}_p(s) = \frac{c \ln(\frac{3}{\gamma^2})}{3m} + \sqrt{2(\hat{v}_p(s) + \varepsilon^p(s)) \ln(\frac{3}{\gamma^2})}.
\]

Then, with probability at least \(1 - \delta\), for all \((p, s) \in \mathcal{I}\), it holds that \(|u_p(s) - \hat{u}_p(s)| \leq \hat{\varepsilon}_p(s)\).

This empirical Bennett guarantee forms the basis for our progressive sampling algorithms.

**Progressive Sampling Algorithms**

Finally, we present the general class of progressive sampling (PS) algorithms (see Algorithm 1) for learning simulation-based games. As the name suggests, progressive sampling algorithms work by progressively sampling utilities, pruning those which are sufficiently estimated for the relevant learning goal at hand. One core component of progressive sampling algorithms is the sampling schedule. On each iteration, a progressive sampling algorithm will collect the number of samples dictated by the sampling schedule for each active (i.e., unpruned) utility index. It will then use the samples to update the empirical game and to compute new utility deviation bounds (which in our case will be dependent on the tail bounds introduced earlier). Notice that the utility deviation bounds must each have individual failure probability \(\frac{\delta}{m^2}\), as opposed to just \(\frac{\delta}{m}\) as in Theorem 3. This is to ensure that, via an additional union bound, all pruned indices will, with high probability (w.h.p.), have been pruned justifiably with respect to the true game. Finally, the empirical game and utility deviation bounds will be used to inform which utility indices can be pruned on the current iteration. Progressive sampling algorithms terminate once either all utility indices are pruned, or the sampling schedule is exhausted.

In this paper, we focus on PS algorithms for learning equilibria. On the basis of Lemma 4, one sufficient condition for pruning an index \((p, s) \in \mathcal{I}\) is that it is estimated (w.h.p.) to within some target error \(\varepsilon\). If this is the only pruning criteria used, then upon termination of the algorithm, if all indices have been pruned, the resulting empirical game will (w.h.p.) satisfy \(|u_p(s) - \hat{u}_p(s)| \leq \varepsilon\) for all \((p, s) \in \mathcal{I}\), and will thus (w.h.p.) satisfy the pure and mixed dual Nash containment results in Lemma 4. Cousins et al. (2022) design precisely this algorithm, using Hoeffding (Theorem 1) and empirical Bennett (Theorem 3) bounds to inform their pruning (pruning once the guarantee corresponding to an index is tighter than the target error \(\varepsilon\), and carefully crafting a sampling schedule which guarantees that all indices will be pruned prior to its exhaustion. They refer to this pruning approach as well-estimated pruning (Cousins et al. 2022). We show an example of how the pruning criteria may be implemented in Algorithm 1 Line 15.

**Regret Pruning**

Having covered the requisite background, we can now begin our discussion of regret pruning and present our novel regret-pruning variations. As discussed earlier, Lemma 4 immediately suggests well-estimated pruning as a pruning approach, and this approach was used to design the PS algorithm presented in Cousins et al. (2022) (which we henceforth refer to as PS-WE for Progressive Sampling with Well-Estimated Pruning).

**Theorem 4.** If PS-WE(\(\Gamma_{\chi}, \mathcal{S}, \mathcal{I}, \delta, \varepsilon\)) returns an empirical utility function \(\bar{u}\), then with probability at least \(1 - \delta\), for all \(\gamma \geq 0\), it holds that
1. \(E_{\gamma}(u) \subseteq E_{2\delta + \gamma}(u)\) and \(E_{\gamma}(\hat{u}) \subseteq E_{2\delta + \gamma}(u)\)
2. \(E_{\gamma}(u) \subseteq E_{2\delta + \gamma}(\bar{u})\) and \(E_{\gamma}(\hat{u}) \subseteq E_{2\delta + \gamma}(\bar{u})\).

The condition presented in Lemma 4 (i.e., \(|u_p(s) - \hat{u}_p(s)| \leq \varepsilon\) for all \((p, s) \in \mathcal{I}\)), however, is not the only condition to yield these kinds of Nash containment results. Consider the PS algorithm designed in Areyan Viqueira, Cousins, and Greenwald (2020) (which we refer to as PS-REG-0 for Progressive Sampling with Regret Pruning; the 0 will distinguish this algorithm from our new variations). Unlike PS-WE, PS-REG-0 uses uniform utility deviation bounds – they simply bound all utility deviations by \(\varepsilon\) defined in Theorem 3. PS-REG-0 uses well-estimated pruning (though the authors do not explicitly call it that), but it additionally uses what the authors call “regret pruning” (Areyan Viqueira, Cousins, and Greenwald 2020). The authors claim that though using this pruning approach does not guarantee that the condition in Lemma 4 is met upon termination of the algorithm, it nonetheless guarantees that (w.h.p.) the pure Nash containment result \(E(u) \subseteq E_{2\varepsilon}(u) \subseteq E_{2\varepsilon}(\bar{u})\) is satisfied. In this section, we present a counterexample which shows that their pruning approach does not in fact guarantee this pure Nash containment result. We show that, instead, their pruning approach is only able to guarantee a weaker Nash containment result. Furthermore, we also present 3 novel variations of the regret pruning criterion presented in Areyan Viqueira, Cousins, and Greenwald (2020) which each have varying benefits and costs in comparison to the original. The first variation is a generalization of the original regret pruning criterion with respect to a hyper-parameter \(\gamma^* \geq 0\). We show that when \(\gamma^* = 0\), this variation is identical to the regret pruning criterion from PS-REG-0. When \(\gamma^* = 2\varepsilon\), however, we show that, at the cost of taking slightly longer to prune indices, this new variation yields the stronger...
Algorithm 1: General Progressive Sampling Algorithm

1: procedure PS($\Gamma_Y$, $\mathcal{D}$, $\mathcal{I}$, $\varepsilon$, $\delta$, $\varepsilon$)
2:     input: Conditional game $\Gamma_Y$, condition distribution $\mathcal{D}$, index set $\mathcal{I}$, failure probability $\delta \in (0, 1)$, target error $\varepsilon > 0$
3:     Initialize empirical utilities $\hat{u}_p^{(0)}(s) = 0$ for $(p, s) \in \mathcal{I}$
4:     Initialize utility deviation bounds $\hat{\varepsilon}_p^{(0)}(s) = \infty$ for $(p, s) \in \mathcal{I}$
5:     Initialize active utility index set $\mathcal{I}^{(0)} \leftarrow \mathcal{I}$
6:     Initialize a sampling schedule $m_1, \ldots, m_T$ and cumulative sample size $M_0 \leftarrow 0$
7:     for $t = 1, \ldots, T$ do
8:         for $s \in S$ do
9:             Determine unpruned player indices $P(s; \mathcal{I}^{(t)}) \leftarrow \{p \in P \mid (p, s) \in \mathcal{I}^{(t)}\}$ at strategy profile $s$
10:            Query simulator for utilities of unpruned players: $\{\hat{u}_p(s) \mid p \in P(s; \mathcal{I}^{(t)})\} \leftarrow \mathcal{D}(s; P(s; \mathcal{I}^{(t)}))$
11:            Update empirical utilities $\hat{u}_p^{(t)}(s) \leftarrow \frac{M_{t-1}}{M_{t-1} + m_t} \cdot \hat{u}_p^{(t-1)}(s) + \frac{m_t}{M_{t-1} + m_t} \cdot \hat{u}_p(s)$ for $p \in P(s; \mathcal{I}^{(t)})$
12:            Compute new utility deviation bounds $\hat{\varepsilon}_p^{(t)}(s)$ for $p \in P(s; \mathcal{I}^{(t)})$, each with failure probability $\delta/|\mathcal{I}|$
13:     end for
14:     Update cumulative sample size $M_t \leftarrow M_{t-1} + m_t$
15:     Prune any indices in $\mathcal{I}^{(t)}$ which do not require further estimation (e.g., well-estimated pruning): $\mathcal{I}^{(t)} \leftarrow \{(p, s) \in \mathcal{I}^{(t-1)} \mid \hat{\varepsilon}_p^{(t)}(s) > \varepsilon\}$, i.e., prune indices that have met the target $\varepsilon$ error guarantee
16:     if all indices in $\mathcal{I}^{(t)}$ are pruned (i.e., $\mathcal{I}^{(t)} = \emptyset$) then
17:         return empirical utilities $\hat{u}^{(t)}$
18:     end if
19:     end for
20: end procedure

pure Nash containment guarantee which are intended their pruning criterion to meet (i.e., $E(u) \subset E_{2\varepsilon}(\hat{u}) \subset E_{4\varepsilon}(u)$). The second variation takes advantage of non-uniform utility deviation bounds to yield the same guarantee as the first variation, while pruning indices significantly sooner in practice than otherwise. Finally, the third variation modifies the second variation, yielding a mixed Nash containment guarantee in addition to the same pure Nash guarantee, at the cost of a slightly tighter pruning criterion than variation 2.

Old Regret Pruning

We begin by presenting the regret pruning criterion used in PS-REG-0 from Areyan Viqueira, Cousins, and Greenwald (2020). Since PS-REG-0 uses uniform utility deviation bounds we simply use $\hat{\varepsilon}^{(t)}$ to denote the utility deviation bound on $\hat{u}^{(t)}$ at iteration $t$. PS-REG-0 prunes an index $(p, s) \in \mathcal{I}$ on an iteration $t$ if any of the following holds:

1. $\hat{\varepsilon}^{(t)} \leq \varepsilon$ (well-estimated pruning)
2. $\text{Reg}_p^{(s)}(\hat{u}^{(t)}) \geq 2\hat{\varepsilon}^{(t)}$ (regret pruning).

Areyan Viqueira, Cousins, and Greenwald (2020) claim that PS-REG-0 satisfies the following guarantee.

Claim 1. If PS-REG-0($\Gamma_Y$, $\mathcal{D}$, $\mathcal{I}$, $\delta$, $\varepsilon$) returns an empirical utility function $\hat{u}$, then with probability at least $1 - \delta$, it holds that $E(u) \subseteq E_{2\varepsilon}(\hat{u}) \subseteq E_{4\varepsilon}(u)$.

We present a counter-example that shows that the second inclusion $E_{2\varepsilon}(\hat{u}) \subseteq E_{4\varepsilon}(u)$ does not necessarily hold.

Counterexample 1. Consider a two-player game $\Gamma$ in which player $A$ has two pure strategies, $a_1$ and $a_2$, while player $B$ has just one pure strategy $b$. Define player $A$’s utility function by $u_A(a_1, b) = 2$ and $u_A(a_2, b) = 1$. Suppose we run PS-REG-0 with target error $\varepsilon = 0.2$ and get the following:

| Iteration | $u_A^{(1)}(a_1, b) = 2.5$ | $u_A^{(1)}(a_2, b) = 1.45$ | $\hat{\varepsilon}^{(1)} = 0.5$ |
|-----------|-------------------|-------------------|-------------------|

$\Rightarrow$ Index $(A, (a_2, b))$ regret pruned

(since $\hat{\varepsilon}^{(1)} = 0.5 < \frac{\text{Reg}_A((a_2, b); \hat{u}^{(1)})}{2} = 0.525$)

| Iteration | $u_A^{(2)}(a_1, b) = 1.8$ | $u_A^{(2)}(a_2, b) = 1.45$ | $\hat{\varepsilon}^{(2)} = 0.2$ |
|-----------|-------------------|-------------------|-------------------|

$\Rightarrow$ Index $(A, (a_1, b))$ well-pruned;

PS-REG-0 terminates with $\hat{u}_A = \hat{u}_A^{(2)}$

We have that $(a_2, b) \in E_{2\varepsilon}(\hat{u})$, since $\text{Reg}_A((a_2, b); \hat{u}) = 1.8 - 1.45 = 0.35 < 2\varepsilon = 0.4$ and $\text{Reg}_B((a_2, b); \hat{u}) = 0$. But we have $(a_2, b) \notin E_{4\varepsilon}(u)$, since $\text{Reg}_A((a_2, b); u) = 2 - 1 = 1 > 4\varepsilon = 0.8$. Hence, we have $E_{2\varepsilon}(\hat{u}) \not\subseteq E_{4\varepsilon}(u)$. Since all utility deviation guarantees have been held, this is not a failure case. Thus, Claim 1 cannot be true.

Though the second inclusion in Claim 1 cannot be guaranteed, we show that an alternative inclusion does hold (w.h.p.).
Theorem 5. If PS-REG-0(Γ, Ξ, δ, ε) returns an empirical utility function  ě, then with probability at least 1 − δ, we have that ıt(ě) ≤ E_{ε}(ě) and E(ě) ⊆ E_{ε}((u).

From this guarantee, we see that if a game analyst is using PS-REG-0 to learn an approximate pure Nash equilibrium of the simulation-based game, they will need to compute a (true) pure Nash equilibrium of the resulting empirical game. Of course, in many games, computing a pure Nash equilibrium is computationally intractable (e.g., it is NP-complete in graphical games (Gottlob, Greco, and Scarcello 2003)); the best that can be hoped for is an approximate pure Nash equilibrium. Furthermore, even if the simulation-based game has a pure Nash equilibrium, it is not then guaranteed that the empirical game will also have a pure Nash equilibrium, but rather only that it will have a 2ε-pure Nash equilibrium. A limitation of PS-REG-0 is that it is not able to provide any guarantees regarding the efficacy of approximate pure Nash equilibria from the empirical game in the true game.

New Regret Pruning Variations

We now introduce our novel regret pruning criteria. We begin by presenting a variation which resolves the limitation observed in PS-REG-0 of lacking guarantees regarding empirical approximate Nash equilibria. This variation is derived on the basis of a stronger version of Lemma 1.

Lemma 2. Let γ * ≥ 0. If  |u_p(s) −  u_p(s)| ≤ ε for all (p, s) ∈ I satisfying Reg_p(s; u) = 0 or Reg_p(s; u) ≤ γ *, then E(s) ⊆ E_{ε}(s) and E_{γ}(s) ⊆ E_{ε+γ}(s) for all 0 ≤ γ ≤ γ *.

Whereas Lemma 1 required all indices to be well-estimated, Lemma 2 requires only indices with sufficiently low regret in both the true game and empirical game to be well-estimated. This gives room for indices with provably high regret to be regret-pruned. Of course, Areyan Viqueira, Cousins, and Greenwald (2020) also observed, this potential for regret pruning seems to come at the cost of any guarantees regarding mixed Nash equilibria. On the basis of Lemma 1, we design our first regret pruning variation.

Theorem 6. Consider a PS algorithm, PS-REG, using uniform utility deviation bounds, which conducts well-estimated pruning and on each iteration t, also regret-prunes any index (p, s) ∈ I which satisfies

\[ \text{Reg}_p(s;  u^{(t)}) > \max\{2\hat{ε}^{(t)}, γ * + ε + \hat{ε}^{(t)}\} \]

If PS-REG(Γ, Ξ, δ, ε, γ *) returns an empirical utility function  ě, then with probability at least 1 − δ, it holds that ıt(ě) ⊆ E_{ε}(ě) and E_{γ}(ě) ⊆ E_{ε+γ}(ě) for all 0 ≤ γ ≤ γ *.

Notice that when γ * = 0, PS-REG is identical to PS-REG-0, even yielding the same exact guarantees. PS-REG is thus a generalization of PS-REG-0 to cases where γ * > 0. When γ * > 0, PS-REG yields, at the cost of potentially reduced pruning, a stronger guarantee upon termination than PS-REG-0, ensuring that even an approximate empirical pure Nash equilibrium (so long as it is at worst a γ *-pure Nash equilibrium) will be an approximate pure Nash equilibrium in the true game. In practice, the parameter γ * can be set to the smallest value for which the game analyst is still certain they will be able to compute a γ *-pure Nash equilibrium of the empirical game. If we set γ * = 2ε, we get the dual pure Nash containment guarantee. E(ě) ⊆ E_{ε}(ě) ⊆ E_{2ε}(ě), which PS-REG-0 was originally designed to meet.

One limitation of both PS-REG and PS-REG-0 is that they both depend only on uniform utility deviation bounds. The next algorithm and regret pruning variation takes advantage of non-uniform utility deviation bounds (Theorem 3) to prune potentially more (and in practice significantly more; see Figure 1) indices via both well-estimated pruning and regret-pruning, while yielding the same guarantees as PS-REG. In the following result, we use  Reg_p(s; u) to denote

\[ \text{Reg}_p(s; u) = \sup_{s' \in A_{σ_p, s}} (u_p(s') - v_p(s')) - (u_p(s) + v_p(s)). \]

Theorem 7. Consider a PS algorithm, PS-REG+, using non-uniform utility deviation bounds, which conducts well-estimated pruning and on each iteration t, also regret-prunes any index (p, s) ∈ I which satisfies

\[ \text{Reg}_p(s; u^{(t)}) > \max\{0, γ * + ε - \hat{ε}^{(t)}(s)\} \]

If PS-REG+(Γ, Ξ, δ, ε, γ *) returns an empirical utility function  ě, then with probability at least 1 − δ, it holds that ıt(ě) ⊆ E_{ε}(ě) and E_{γ}(ě) ⊆ E_{ε+γ}(ě) for all 0 ≤ γ ≤ γ *.

Notice again that when γ * = 0, PS-REG+ can also reasonably be called PS-REG+.

Since all the aforementioned regret pruning variations derive from the result presented in Lemma 3, they are only able to provide guarantees regarding pure equilibria. This is the most glaring limitation of the new PS algorithms presented so far, since many games do not even have strong approximate pure Nash equilibria which the algorithms could potentially be used to learn. In contrast, PS-WE derives from Lemma 1 and thus yields mixed Nash containment guarantees, but of course does not allow for regret-pruning. We now present a lemma which serves as a middle ground between Lemma 2 and Lemma 1.

Lemma 3. If  |u_p(s) −  u_p(s)| ≤ max\{ε, \text{Reg}_p(s; u)\} for all (p, s) ∈ I, then for all 0 ≤ γ ≤ 2ε, it holds that

\[ E(ě) ⊆ E_{ε}(ě) \]

and for all γ ≥ 0, it holds that \[ E(ě) ⊆ E_{γ}(ě) \]

The condition in Lemma 3 is looser than that in Lemma 1 opening up the potential for regret-pruning, but is (strictly) tighter than the condition in Lemma 2 when γ * = 2ε (i.e., empirical games which satisfy the condition in Lemma 3 necessarily satisfy the condition in Lemma 2 when γ * = 2ε, but the converse does not hold; see Appendix for proof), which allows for the additional mixed Nash containment.
guarantee. Notice further that unlike Lemma 2, Lemma 3 does not depend on any additional parameter $\gamma^*$ and yields guarantees for all $\gamma \geq 0$, rather than just $\gamma \in [0, \gamma^*)$. We use Lemma 3 to derive yet another regret pruning variation.

**Theorem 8.** Consider a PS algorithm, PS-REG-M, using non-uniform utility deviation bounds, which conducts well-estimated pruning and on each iteration $t$, also regret-prunes any index $(p, s) \in \mathcal{I}$ which satisfies

$$\text{Reg}_p^f(s; \hat{u}^{(t)}) > \varepsilon + \beta_p^f(s).$$

If PS-REG-M$(\Gamma_y, \mathcal{D}, \mathcal{I}, \delta, \varepsilon)$ returns an empirical utility function $\hat{u}$, then with probability at least $1 - \delta$, for all $0 \leq \gamma \leq 2\varepsilon$, it holds that

$$E(u) \subseteq E_{2\varepsilon}(\hat{u}) \text{ and } E_{\gamma}(\hat{u}) \subseteq E_{2\varepsilon + \gamma}(u),$$

and for all $\gamma \geq 0$, it holds that

$$E_0(u) \subseteq E_{2\varepsilon}(\hat{u}) \text{ and } E_{\gamma}(\hat{u}) \subseteq E_{2\varepsilon + \gamma}(u).$$

The mixed Nash containment guarantee achieved by PS-REG-M is a $\frac{1}{2}$ factor looser than that achieved by PS-WE (Lemma 1). As a result, slightly stronger approximate empirical mixed Nash equilibria will need to be computed when using PS-REG-M in order to guarantee an equally strong approximate true mixed Nash equilibrium as in PS-WE.

**Efficiency Bounds and Correctness**

Using similar proof techniques to those used in Cousins et al. (2022) to derive efficiency bounds for PS-WE, we derive upper bounds on the number of samples each utility index requires prior to being pruned by PS-REG+ and PS-REG-M, respectively. In the following results, suppose that the utility deviation bounds being used by the mentioned PS algorithms are the minimum of Hoeffding bounds (Theorem 1) and empirical Bennett bounds (Theorem 3).

**Theorem 9.** When running PS-REG+$(\Gamma_y, \mathcal{D}, \mathcal{I}, \delta, \varepsilon, \gamma^*)$, with probability at least $1 - \frac{1}{4}$, the index $(p, s) \in \mathcal{I}$ will be pruned prior to the first iteration $t$ with cumulative sample size $M_t \geq$

$$2 + 2 \ln 3 |\mathcal{I}| T \delta \min \left\{ \frac{10c}{2\varepsilon} + \frac{25\|v_p(\text{Adj}_p, s)\|_\infty}{(\text{Reg}_p(s; u) - \gamma)^2}, \frac{25\|v_p(\text{Adj}_p, s)\|_\infty}{(\text{Reg}_p(s; u) - \gamma)^2} \right\}$$

(defaulting to the second option when $\text{Reg}_p(s; u) \leq \gamma^*$).

When running PS-REG-M$(\Gamma_y, \mathcal{D}, \mathcal{I}, \delta, \varepsilon)$, with probability at least $1 - \frac{1}{4}$, the index $(p, s) \in \mathcal{I}$ will be pruned prior to the first iteration $t$ with cumulative sample size $M_t \geq$

$$2 + 2 \ln 3 |\mathcal{I}| T \delta \min \left\{ \frac{12.5c}{2\varepsilon} + \frac{25\|v_p(\text{Adj}_p, s)\|_\infty}{(\text{Reg}_p(s; u) - \gamma)^2}, \frac{25\|v_p(\text{Adj}_p, s)\|_\infty}{(\text{Reg}_p(s; u) - \gamma)^2} \right\}$$

queries at profile $s$ (defaulting to the second option when $\text{Reg}_p(s; u) \leq \varepsilon$).

The above result reinforces the idea that care is required when choosing a sampling schedule for these algorithms. If the marginal sample size $m_t$ is small at each iteration $t$, then very few queries will be wasted from when an index is ready to be pruned when it is actually pruned by the algorithm. On the other hand, if very small marginal sample sizes are used, then a very large schedule length $T$ will be required to reach a sufficiently large cumulative sample size to prune all utility indices. But a larger schedule length $T$ yields looser utility deviation bounds, thus resulting in all indices requiring more queries to be pruned than otherwise. Hence, there is a trade-off in designing a sampling schedule between keeping marginal sample sizes small and keeping the schedule length small.

We discuss our particular choice of sampling schedule in the next section, and in greater detail in the Appendix. In the following result, we use Hoeffding’s Inequality to derive an upper bound on the requisite total cumulative sample size a sampling schedule needs to ensure that all utility indices will be pruned prior to its exhaustion.

**Theorem 10.** Suppose that the total samples $M_T$ allocated in the sampling schedule is greater than or equal to the maximum number of samples needed to prune an arbitrary index, i.e.,

$$M_T = \sum_{t=1}^{T} m_t \geq \frac{c^2 \ln 2 |\mathcal{I}| T}{2 \varepsilon^2}.$$ 

Then for each algorithm $PS \in \{\text{PS-WE}, \text{PS-REG-0}, \text{PS-REG}, \text{PS-REG+}, \text{PS-REG-M}\}$, it is guaranteed that $PS(\Gamma_y, \mathcal{D}, \mathcal{I}, \delta, \varepsilon)$ will terminate and return an empirical game with the guarantees corresponding to the respective algorithm.

**Experiments**

In this section, we experimentally explore the behavior of all the aforementioned PS algorithms. When choosing a sampling schedule for each algorithm, we follow Cousins et al. (2022) in designing sampling schedules which begin at sample size $\alpha$, a lower bound on the minimum number of samples needed to prune an arbitrary index, and end with a cumulative sample size $\omega$, an upper bound on the maximum number of samples needed to prune an arbitrary index. For all algorithms, we use the upper bound from Theorem 10 for $\omega$, i.e., $\omega = \frac{c^2 \ln 2 |\mathcal{I}| T}{2 \varepsilon^2}$, thus guaranteeing that each algorithm will return an empirical game satisfying the respective guarantees of the algorithm upon termination. For algorithms using regret pruning, we set $\alpha$ to be a lower bound on the number of samples to estimate a zero-variance utility to (w.h.p.) within $\varepsilon$ error (since no regret pruning can occur prior to at least one index achieving such an error guarantee; see Appendix for proof). For PS-WE, we follow Cousins et al. (2022) in setting $\alpha$ to be a lower bound on the number of samples needed to estimate a zero-variance utility to (w.h.p.) within a target error $\varepsilon$, though we improve their lower bound by a small constant factor. Finally, while our PS-WE sampling schedule has a geometric sampling schedule (i.e., geometrically increasing cumulative sample size) as in Cousins et al. (2022), using such a geometric schedule for our regret pruning algorithms results in too many iterations spent on very
For our first experiment (Figure 1), we compare the query complexity of each algorithm for each target error $\varepsilon \in \{\frac{1}{5}, \frac{2}{5}, \ldots, \frac{1}{100}\}$. All algorithm runs are conducted on a single randomly-generated two-player zero-sum game with non-uniform additive noise.

Figure 1: Average query complexity vs $1/\varepsilon$ for 10 runs of each algorithm for each target error $\varepsilon \in \{\frac{1}{5}, \frac{2}{5}, \ldots, \frac{1}{100}\}$. All algorithm runs are conducted on a single randomly-generated two-player zero-sum game with non-uniform additive noise.

Query Complexity vs. Target Error

In the following experiments, we test our algorithms on two-player random zero-sum games (generated via the game-generator GAMUT (Nudelman et al. 2004)) with 40 actions for each player and utility values in the range $[-2, 2]$. In order to emulate a noisy simulator, we add noise to each sample utility value. For each utility index $(p, s) \in \mathcal{I}$, we sample a variance modifier $\nu_{p,s} \sim \text{Beta}(1.5, 3)$, and then each time the simulator is queried for $u(p, s)$, we set $u(p, s) = u(p, s) + N(\nu_{p,s})$ where $N(\nu_{p,s})$ is a scaled and shifted Bernoulli random variable, generating either $10\nu_{p,s}$ or $-10\nu_{p,s}$ with equal probability. Notice then that our final utility value for these random zero-sum simulation-based games is $c = 24$. Sampling variance modifiers from Beta$(1.5, 3)$ ensures that our utility indices have a wide range of noise variables with mostly moderate variance, but with some noise variables having particularly high variance and some particularly low variance.

For our first experiment (Figure 1), we compare the query complexities (i.e., the number of simulation queries placed prior to termination) of our algorithms for varying target errors $\varepsilon$. We begin by generating a two-player random zero-sum game $\Gamma$ (and variance modifiers $\nu_{p,s}$ for each utility index $(p, s) \in \mathcal{I}$). For each target error $\varepsilon \in \{\frac{1}{5}, \frac{2}{5}, \ldots, \frac{1}{100}\}$, we then run each of our aforementioned PS algorithms (with failure probability $\delta = 0.05$) on $\Gamma$ a total of 10 times and plot the average query complexity across those runs. For each algorithm, we then connect these average query complexities for each target error by a line plot in Figure 1.

In Figure 1, we see that, as expected, PS-REG obtains its guarantees for approximate pure equilibria of the empirical game at the cost of a slightly greater query complexity than PS-REG-0. In a similar vein, PS-REG+ with $\gamma^* = 2\varepsilon$ also requires a greater number of queries than PS-REG+ with $\gamma^* = 0$ (i.e., PS-REG-0+) in order to yield its guarantees for $\gamma^*$-pure equilibria of the empirical game. We also observe that PS-REG-0 and PS-REG with $\gamma^* = 2\varepsilon$, which both use uniform utility deviation bounds, consume significantly more queries than PS-WE, which takes advantage of non-uniform bounds but conducts no regret pruning. This suggests that utilizing non-uniform utility deviation bounds is crucial for designing query efficient progressive sampling algorithms. This idea is further reinforced when looking at PS-REG+ and PS-REG-M, both of which use regret pruning criteria that take advantage of non-uniform utility deviation bounds, and outperform PS-WE by a very significant margin, especially when the target error $\varepsilon$ is small. Another particularly surprising result is that PS-REG-M only consumes marginally more queries than PS-REG+ with $\gamma^* = 2\varepsilon$, despite yielding strong mixed Nash containment guarantees in return.

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Since our first experiment only tests on a single randomly generated simulation-based game, it is possible that the generated game was just particularly amenable to regret pruning. For our second experiment (Figure 2), we observe the proportion of additional queries our new PS algorithms are able to save on average (across 10 random zero-sum simulation-based games) in comparison to PS-WE. This time, we run each PS algorithm (again with failure probability $\delta = 0.05$) only once for each generated game.

In Figure 2 we observe that the comparative performance of the algorithms in Figure 1 remain consistent across many different random two-player zero-sum games. We further observe that past a certain turning point, progressive sampling algorithms which use regret pruning techniques save a greater proportion of queries with respect to PS-WE as smaller target errors $\varepsilon$ are used. When the target error $\varepsilon$ is very small, PS-REG+ (both $\gamma^* = 0$ and $\gamma^* = 2\varepsilon$) and PS-REG-M are able to obtain their respective guarantees while saving more than 50% and 40%, respectively, of the queries used by PS-WE.

**Conclusion**

In this paper, we address a serious limitation of Areyan Viqueira, Cousins, and Greenwald’s progressive sampling algorithm with regret pruning – it is only able to yield guarantees regarding true pure Nash equilibria of the empirical game. We design two novel primary progressive sampling algorithms for practitioners to use to learn equilibria in simulation-based games: PS-REG+ and PS-REG-M. PS-REG+ combines well-estimated pruning with a novel regret pruning variation which is modified to ensure the algorithm yields pure Nash containment guarantees for approximate $\gamma^*$-pure Nash equilibria of the empirical game and to take advantage of non-uniform utility deviation bounds to prune utility indices as soon as possible. When using PS-REG+, a game analyst will set $\gamma^*$ according to the weakest approximate pure Nash equilibria for which they desire guarantees. PS-REG-M also incorporates well-estimated pruning and a novel regret pruning variation, except unlike PS-REG+, it yields strong Nash containment guarantees for all approximate pure or mixed Nash equilibria of the empirical game, at the cost of a slightly greater query complexity. Both PS-REG+ and PS-REG-M significantly outperform PS-WE, the prior state-of-the-art algorithm for learning equilibria in simulation-based games. In light of this, game analysts seeking such an algorithm should use PS-REG+ if they only aim to learn pure Nash equilibria and should otherwise use PS-REG-M.

In this work, we have only applied our progressive sampling algorithms and pruning techniques to normal-form games. In future work, we aim to extend this methodology to other game models such as extensive-form games. Additionally, EGTA algorithms for learning equilibria of simulation-based game have thus far been completely detached from algorithms for computing equilibria. A game analyst must first use EGTA algorithms to learn a sufficiently strong approximation of the simulation-based game, and then must compute equilibria of this empirical game. Future work can incorporate variance-sensitive and regret-sensitive progressive sampling techniques into an existing game-solving algorithm to make an EGTA-aware game-solving algorithm.

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Appendix

Notation

To improve readability of the following proofs, we introduce a few notational short-hands.

1. Given a utility index \((p, s) \in \mathcal{I}\) and a utility function \(u\), let \(s_u^*\) denote the best response of player \(p\) to the opponents’ strategies in \(s\), i.e., \(s_u^* \equiv \sup_{s' \in \text{Adj}_{p,s}^*} u_p(s')\).

2. Given a mixed strategy profile \(s \in \mathcal{S}^p\), player \(p \in P\), and strategy \(t \in S_p\), we let \(s_t\) denote the strategy profile \(s' \in \text{Adj}_{p,s}\) satisfying \(s'_p = t\), and let \(P(t|s)\) denote the probability that mixed strategy profile \(s\) assigns to strategy \(t\).

Nash Containment Lemmas

We begin by proving the three different Nash containment lemmas.

Lemma 1. If \(|u_p(s) − \hat{u}_p(s)| ≤ \varepsilon\) for all \((p, s) \in \mathcal{I}\), then

\[
E(u) \subseteq E_{2\varepsilon}(\hat{u}) \subseteq E_{4\varepsilon}(u) \quad \text{and} \quad E^\gamma(u) \subseteq E_{2\varepsilon}(\hat{u}) \subseteq E_{4\varepsilon}(u) \, ,
\]

or more generally, \(E_\gamma(u) \subseteq E_{2\varepsilon+\gamma}(\hat{u})\) and \(E_\gamma(\hat{u}) \subseteq E_{2\varepsilon+\gamma}(u)\) for all \(\gamma \geq 0\) (resp. for mixed equilibria).

Proof. Suppose that \(|u_p(s) − \hat{u}_p(s)| ≤ \varepsilon\) for all \((p, s) \in \mathcal{I}\). Let \(\gamma \geq 0\) be arbitrary, and suppose that \(s \in E_\gamma(u)\). Then we have that

\[
\begin{align*}
\text{Reg}_p(s; \hat{u}) &= \hat{u}_p(s^*_u) − \hat{u}_p(s) \\
&\leq (u_p(s^*_u) + \varepsilon) − (u_p(s) − \varepsilon) \\
&\leq (u_p(s) + \gamma + \varepsilon) − (u_p(s) − \varepsilon) \\
&= 2\varepsilon + \gamma,
\end{align*}
\]

and hence \(s \in E_{2\varepsilon+\gamma}(\hat{u})\). Since \(s\) was arbitrary, we have that \(E_\gamma(u) \subseteq E_{2\varepsilon+\gamma}(\hat{u})\). By completely analogous reasoning, we see that \(E_\gamma(\hat{u}) \subseteq E_{2\varepsilon+\gamma}(u)\). \(\square\)

Lemma 2. Let \(\gamma^* \geq 0\). If \(|u_p(s) − \hat{u}_p(s)| ≤ \varepsilon\) for all \((p, s) \in \mathcal{I}\) satisfying \(\text{Reg}_p(s; u) = 0\) or \(\text{Reg}_p(s; \hat{u}) ≤ \gamma^*\), then \(E(u) \subseteq E_{\varepsilon}(\hat{u})\) and \(E_\gamma(\hat{u}) \subseteq E_{\varepsilon+\gamma}(u)\) for all \(0 ≤ \gamma ≤ \gamma^*\).

Proof. Suppose that \(|u_p(s) − \hat{u}_p(s)| ≤ \varepsilon\) for all \((p, s) \in \mathcal{I}\) satisfying \(\text{Reg}_p(s; u) = 0\) or \(\text{Reg}_p(s; \hat{u}) ≤ \gamma^*\). Suppose that \(s \in E(u)\). Then we have that

\[
\begin{align*}
\text{Reg}_p(s; \hat{u}) &= \hat{u}_p(s^*_u) − \hat{u}_p(s) \\
&\leq (u_p(s^*_u) + \varepsilon) − (u_p(s) − \varepsilon) \\
&\leq (u_p(s) + \varepsilon) − (u_p(s) − \varepsilon) \\
&= 2\varepsilon,
\end{align*}
\]

and hence \(s \in E_{2\varepsilon}(\hat{u})\). Since \(s\) was arbitrary, we have that \(E(u) \subseteq E_{2\varepsilon}(\hat{u})\). Now let \(\gamma \in [0, \gamma^*]\) and suppose instead that \(s \in E_\gamma(\hat{u})\). Then we have that

\[
\begin{align*}
\text{Reg}_p(s; u) &= u_p(s^*_u) − u_p(s) \\
&\leq (u_p(s^*_u) + \varepsilon) − (u_p(s) − \varepsilon) \\
&\leq (u_p(s) + \gamma + \varepsilon) − (u_p(s) − \varepsilon) \\
&= 2\varepsilon + \gamma,
\end{align*}
\]

and hence \(s \in E_{2\varepsilon+\gamma}(u)\). Since \(s\) was arbitrary, we have that \(E_\gamma(\hat{u}) \subseteq E_{2\varepsilon+\gamma}(u)\) and we are done. \(\square\)

Before proving the third Nash containment lemma, we prove an accessory lemma.

Lemma 4. Let \(\varepsilon > 0\) be arbitrary. Suppose that for all \((p, s) \in \mathcal{I}\), it holds that

\[
|u_p(s) − \hat{u}_p(s)| ≤ \max \left\{ \varepsilon, \frac{\text{Reg}_p(s; \hat{u})}{2} \right\}.
\]

Then for all \((p, s) \in \mathcal{I}\) satisfying \(\text{Reg}_p(s; u) = 0\), it must hold that \(|u_p(s) − \hat{u}_p(s)| ≤ \varepsilon\).
Proof. Suppose that \((p, s) \in \mathcal{I}\) satisfies \(\text{Reg}_p(s; \hat{u}) = 0\). Since \(\text{Reg}_p(s^*_u; \hat{u}) = 0\), we have that \(|u_p(s^*_u) - \hat{u}_p(s^*_u)| \leq \varepsilon\), which implies that \(u_p(s^*_u) \geq \hat{u}_p(s^*_u) - \varepsilon\). But by definition, we have that \(u_p(s) \geq u_p(s^*_u)\), and hence, it must hold that \(u_p(s) \geq \hat{u}_p(s^*_u) - \varepsilon\). By hypothesis, we also have that \(u_p(s) \leq \hat{u}_p(s) + \max \left\{ \varepsilon, \frac{\text{Reg}_p(s; \hat{u})}{2} \right\}\). Chaining these two inequalities, we have that

\[
\hat{u}_p(s^*_u) - \varepsilon \leq \hat{u}_p(s) + \max \left\{ \varepsilon, \frac{\text{Reg}_p(s; \hat{u})}{2} \right\}
\]

\[\iff \hat{u}_p(s^*_u) - \hat{u}_p(s) \leq \varepsilon + \max \left\{ \varepsilon, \frac{\text{Reg}_p(s; \hat{u})}{2} \right\} \]

\[\iff \text{Reg}_p(s; \hat{u}) \leq \varepsilon + \max \left\{ \varepsilon, \frac{\text{Reg}_p(s; \hat{u})}{2} \right\} .\]

If \(\frac{\text{Reg}_p(s; \hat{u})}{2} > \varepsilon\), then the second term in the max wins out, but solving for \(\text{Reg}_p(s; \hat{u})\) immediately yields that \(\text{Reg}_p(s; \hat{u}) \leq 2\varepsilon\), a contradiction. Hence, it must hold that \(\frac{\text{Reg}_p(s; \hat{u})}{2} \leq \varepsilon\), and hence that \(|u_p(s) - \hat{u}_p(s)| \leq \varepsilon\).

Lemma 3. If \(|u_p(s) - \hat{u}_p(s)| \leq \varepsilon\) for all \((p, s) \in \mathcal{I}\), then for all \(0 \leq \gamma \leq 2\varepsilon\), it holds that

\[E(u) \subseteq E_{2\varepsilon}(\hat{u})\] and \(E_\gamma(u) \subseteq E_{2\varepsilon+\gamma}(u)\).

Proof. Suppose that \(|u_p(s) - \hat{u}_p(s)| \leq \varepsilon\) for all \((p, s) \in \mathcal{I}\). We will first show the pure Nash containment result, and then show the mixed Nash containment result. Suppose that \(s \in E(u)\). Then we have that

\[
\text{Reg}_p(s; \hat{u}) = \hat{u}_p(s^*_u) - u_p(s)
\]

\[
\leq (u_p(s^*_u) + \varepsilon) - (u_p(s) - \varepsilon)
\]

\[
\leq (u_p(s) + \varepsilon) - (u_p(s) - \varepsilon)
\]

\[= 2\varepsilon,\]

and hence \(s \in E_{2\varepsilon}(\hat{u})\). Since \(s\) was arbitrary, we have that \(E(u) \subseteq E_{2\varepsilon}(\hat{u})\).

Now let \(\gamma \in [0, 2\varepsilon]\) suppose instead that \(s \in E_\gamma(\hat{u})\). Then we have that

\[
\text{Reg}_p(s; \hat{u}) = \hat{u}_p(s^*_u) - u_p(s)
\]

\[
\leq (\hat{u}_p(s^*_u) + \varepsilon) - (\hat{u}_p(s) - \varepsilon)
\]

\[
\leq (\hat{u}_p(s) + \gamma + \varepsilon) - (\hat{u}_p(s) - \varepsilon)
\]

\[= 2\varepsilon + \gamma,\]

and hence \(s \in E_{2\varepsilon+\gamma}(\hat{u})\). Since \(s\) was arbitrary, we have that \(E_\gamma(\hat{u}) \subseteq E_{2\varepsilon+\gamma}(\hat{u})\), and we have shown the pure Nash containment result.

Now we will prove the mixed Nash containment result. Suppose that \(s \in E^o(u)\). Then we have that

\[
\text{Reg}_p(s; \hat{u}) = \hat{u}_p(s^*_u) - \hat{u}_p(s)
\]

\[
= \hat{u}_p(s^*_u) - \sum_{t \in S_p} \hat{u}_p(s|t) \mathbb{P}[t|s]
\]

\[
\leq (u_p(s^*_u) + \varepsilon) - \sum_{t \in S_p} \left( u_p(s|t) - \max \left\{ \varepsilon, \frac{\text{Reg}_p(s; \hat{u})}{2} \right\} \right) \mathbb{P}[t|s]
\]

\[\leq (u_p(s) + \varepsilon) - \sum_{t \in S_p} \left( u_p(s|t) - \max \left\{ \varepsilon, \frac{\text{Reg}_p(s|t; \hat{u})}{2} \right\} \right) \mathbb{P}[t|s]
\]

Since \(s \in E^o(u)\), we have \(u_p(s) \geq u_p(s^*_u)\).
we have that

\[ \text{Theorem 4.} \]

\[ \text{Algorithm Correctness Proofs} \]

approximations (e.g., the empirical utilities can be arbitrarily low), even if the condition in Lemma 2 holds.

\[ (\text{implication to be false.} \]

show this by proving the forward implication from Lemma 3’s condition to Lemma 2’s condition, and then showing the reverse implication to be false.

Solving the above for \( \text{Reg}_{p}(s; \hat{u}) \), we get that \( \text{Reg}_{p}(s; \hat{u}) \leq 4\varepsilon \), and hence \( s \in E_{4\varepsilon}^{o}(\hat{u}) \). Since \( s \) was arbitrary, we have that \( E_{\varepsilon}(u) \subseteq E_{4\varepsilon}^{o}(\hat{u}) \).

Now instead let \( \gamma \geq 0 \), and suppose that \( s \in E_{\gamma}^{o}(\hat{u}) \). Then we have that

\[ \text{Reg}_{p}(s; u) = u_{p}(s_{u}^*) - u_{p}(s) \]

\[ = u_{p}(s_{u}^*) - \sum_{t \in S_{p}} u_{p}(s_{t}) P[t|s] \]

\[ \leq (\hat{u}_{p}(s_{u}^*) + \varepsilon) - \sum_{t \in S_{p}} \left( \hat{u}_{p}(s_{t}) - \max \left\{ \varepsilon, \frac{\text{Reg}_{p}(s_{t}; \hat{u})}{2} \right\} \right) P[t|s] \]

\[ \leq (\hat{u}_{p}(s) + \gamma + \varepsilon) - \sum_{t \in S_{p}} \left( \hat{u}_{p}(s_{t}) - \max \left\{ \varepsilon, \frac{\text{Reg}_{p}(s_{t}; \hat{u})}{2} \right\} \right) P[t|s] \]

\[ = \gamma + \varepsilon + \sum_{t \in S_{p}} P[t|s] \max \left\{ \varepsilon, \frac{\text{Reg}_{p}(s_{t}; \hat{u})}{2} \right\} \]

\[ \leq \gamma + \varepsilon + \sum_{t \in S_{p}} P[t|s] \left( \varepsilon + \frac{\text{Reg}_{p}(s_{t}; \hat{u})}{2} \right) \]

\[ = \gamma + 2\varepsilon + \frac{\text{Reg}_{p}(s; \hat{u})}{2} \]

\[ \leq \frac{3\gamma}{2} + 2\varepsilon , \]

and hence \( s \in E_{2\varepsilon + \frac{3\gamma}{2}}^{o}(u) \). Since \( s \) was arbitrary, we have that \( E_{\gamma}^{o}(\hat{u}) \subseteq E_{2\varepsilon + \frac{3\gamma}{2}}^{o}(u) \).

In the text, we claim that the condition in Lemma\( [5] \) is strictly tighter than the condition in Lemma\( [2] \) when \( \gamma = 2\varepsilon \). We show this by proving the forward implication from Lemma\( [5] \)’s condition to Lemma\( [2] \)’s condition, and then showing the reverse implication to be false.

Suppose the condition from Lemma\( [2] \), i.e., that \( |u_{p}(s) - \hat{u}_{p}(s)| \leq \max \left\{ \varepsilon, \frac{\text{Reg}_{p}(s; \hat{u})}{2} \right\} \) for all \( (p, s) \in D \). Then by Lemma\( [4] \), we have that \( |u_{p}(s) - \hat{u}_{p}(s)| \leq \varepsilon \) for all \( (p, s) \in D \) with \( \text{Reg}_{p}(s; u) = 0 \). But by the hypothesis, we directly have that for all \( (p, s) \in D \) with \( \text{Reg}_{p}(s; \hat{u}) \leq 2\varepsilon \), it holds that \( |u_{p}(s) - \hat{u}_{p}(s)| \leq \max \left\{ \varepsilon, \frac{\text{Reg}_{p}(s; \hat{u})}{2} \right\} \leq \max\{\varepsilon, \varepsilon\} = \varepsilon \). Hence, the condition from Lemma\( [2] \) is satisfied.

But obviously the converse does not hold, since indices with \( \text{Reg}_{p}(s; u) > 0 \) and \( \text{Reg}_{p}(s; \hat{u}) > 2\varepsilon \) can have arbitrarily bad approximations (e.g., the empirical utilities can be arbitrarily low), even if the condition in Lemma\( [2] \) holds.

**Algorithm Correctness Proofs**

A crucial component to all of our progressive sampling algorithm correctness proofs is the use of a union bound to ensure that (w.h.p.) all pruning that occurs is justified with respect to the true game. We see this line of reasoning in the following correctness proof for PS-WE.

**Theorem 4.** If PS-WE\( (\Gamma_{Y}, D, \mathcal{I}, \varepsilon) \) returns an empirical utility function \( \hat{u} \), then with probability at least \( 1 - \delta \), for all \( \gamma \geq 0 \), it holds that

1. \( E_{\gamma}(u) \subseteq E_{2\varepsilon + \gamma}(\hat{u}) \) and \( E_{\gamma}(\hat{u}) \subseteq E_{2\varepsilon + \gamma}(u) \)
2. \( E_{\gamma}^{o}(u) \subseteq E_{2\varepsilon + \gamma}(\hat{u}) \) and \( E_{\gamma}^{o}(\hat{u}) \subseteq E_{2\varepsilon + \gamma}(u) \).
Proof. Recall that for each index \((p, s) \in \mathcal{I}\) and iteration \(t \in \{1, \ldots, T\}\), the scalar \(\hat{\varepsilon}_p^{(t)}(s)\) is an upper bound on \(|u_p(s) - \hat{u}_p^{(t)}(s)|\) with probability at least \(1 - \frac{\delta}{|\mathcal{I}|}\). Thus, via a union bound, with probability \(1 - \delta\), it holds for all indices \((p, s) \in \mathcal{I}\) and iterations \(t \in \{1, \ldots, T\}\) if \(u_p^{(t)}(s)\) has been computed (i.e., the index hasn’t been pruned on a prior iteration), it satisfies \(|u_p(s) - \hat{u}_p^{(t)}(s)| \leq \hat{\varepsilon}_p^{(t)}(s)\).

Now suppose that PS-WE returns an empirical utility function \(\hat{u}\). This implies that PS-WE managed to prune all utility indices prior to the exhaustion of its sampling schedule. Combining this with the result from the previous paragraph, we get that with probability at least \(1 - \delta\), for all \((p, s) \in \mathcal{I}\), if we let \(t\) denote the iteration on which \((p, s)\) was well-estimated pruned, it holds that

\[
|u_p(s) - \hat{u}_p(s)| = |u_p(s) - \hat{u}_p^{(t)}(s)| \leq \hat{\varepsilon}_p^{(t)}(s) \leq \varepsilon.
\]

The correctness guarantee then follows from Lemma 1.

As the same union bound argument is applied in all of the following correctness proofs, we do not repeat it again. We begin by proving the correctness of our PS-REG+ algorithm, and show PS-REG and PS-REG-0 to simply be special cases of this algorithm. Before proving the correctness of PS-REG+, we prove another accessory lemma.

**Lemma 5.** Suppose that for all \((p, s) \in \mathcal{I}\), it holds that \(|u_p(s) - \hat{u}_p(s)| \leq \varepsilon_p(s)\). Then for all \((p, s) \in \mathcal{I}\), we have that

\[
\sup_{s' \in \text{Adj}_{p,s}} \left(\hat{u}_p(s') - \varepsilon_p(s')\right) \leq u_p(s_u) \leq \sup_{s' \in \text{Adj}_{p,s}} \left(\hat{u}_p(s') + \varepsilon_p(s')\right)
\]

**Proof.** Let \((p, s) \in \mathcal{I}\). We have that

\[
u_p(s_u) = \sup_{s' \in \text{Adj}_{p,s}} u_p(s') \geq \sup_{s' \in \text{Adj}_{p,s}} (\hat{u}_p(s') - \varepsilon_p(s')).
\]

The second inequality holds by analogous reasoning.

**Theorem 7.** Consider a PS algorithm, PS-REG+, using non-uniform utility deviation bounds, which conducts well-estimated pruning and on each iteration \(t\), also regret-prunes any index \((p, s) \in \mathcal{I}\) which satisfies

\[
\text{Reg}^{\uparrow}_p(s, \hat{u}^{(t)}) > \max\{0, \gamma^* + \varepsilon - \hat{\varepsilon}_p^{(t)}(s)\}.
\]

If PS-REG\((\Gamma, \mathcal{D}, \mathcal{I}, \delta, \varepsilon, \gamma^*)\) returns an empirical utility function \(\hat{u}\), then with probability at least \(1 - \delta\), it holds that \(E(u) \subseteq E_{2\varepsilon}(\hat{u})\) and \(E_{\gamma}(\hat{u}) \subseteq E_{2\varepsilon + \gamma}(u)\) for all \(0 \leq \gamma \leq \gamma^*\).

**Proof.** Suppose that PS-REG+ returns an empirical utility function \(\hat{u}\), thus guaranteeing that all indices have been pruned. Suppose that an index \((p, s) \in \mathcal{I}\) is regret-pruned on iteration \(t\). Then, we have (w.h.p.) that

\[
\text{Reg}_p(s, u) = u_p(s_u) - u_p(s) \\
\geq \sup_{s' \in \text{Adj}_{p,s}} \left(\hat{u}_p(s') - \varepsilon_p(s')\right) - \left(\hat{u}_p(s) + \varepsilon_p(s)\right) \quad \text{Lemma 5 + Definition of } \varepsilon_p^{(t)}(s) \\
= \text{Reg}_p^{\uparrow}(s, \hat{u}^{(t)}) > 0 \quad \text{Definition of Reg}^{\uparrow} + \text{Pruning Criterion}.
\]

We also have (w.h.p.) that

\[
\text{Reg}_p(s, \hat{\Gamma}) = u_p(s_u) - u_p(s) \\
= \hat{u}_p(s_u) - \hat{u}_p^{(t)}(s) \\
\geq u_p(s_u) - \hat{u}_p^{(t)}(s) \\
\geq (u_p(s_u) - \varepsilon) - \hat{u}_p^{(t)}(s) \\
\geq \left(\sup_{s' \in \text{Adj}_{p,s}} \left(\hat{u}_p^{(t)}(s') - \varepsilon_p^{(t)}(s')\right) - \varepsilon\right) - \hat{u}_p^{(t)}(s) \quad \text{Lemma 5} \\
= \text{Reg}_p^{\uparrow}(s, \hat{\Gamma}^{(t)}) + \varepsilon_p^{(t)}(s) - \varepsilon \\
> (\gamma^* + \varepsilon - \hat{\varepsilon}_p^{(t)}(s)) + \varepsilon_p^{(t)}(s) - \varepsilon \\
= \gamma^*.
\]

Since \((p, s)\) was pruned on iteration \(i\)

\[
\hat{u}_p(s_u) \geq \hat{u}_p(s_u^*) \text{ by definition}
\]

\((p, s_u^*)\) cannot have been regret pruned since \(\text{Reg}_p(s_u^*, \hat{\Gamma}^{(t)}) = 0\). Hence, it must have been WE pruned.
Since \((p, s)\) was arbitrary, we have (w.h.p.) that all regret-pruned indices have positive corresponding regret in the true game and greater than \(\gamma^*\) corresponding regret in the empirical game. But since all indices have been pruned, this implies that, with probability at least \(1 - \delta\), all indices \((p, s) \in \mathcal{I}\) with \(\text{Reg}_p(s, \Gamma_{\gamma'}) = 0\) or \(\text{Reg}_p(s, \hat{\Gamma}_{\gamma'}) \leq \gamma^*\) will be well-estimated pruned by PS-REG+ prior to termination, and will hence satisfy \(|u_p(s) - \underline{u}_p(s)| \leq \varepsilon\). The correctness result then follows from Lemma 2.

**Theorem 6.** Consider a PS algorithm, PS-REG, using uniform utility deviation bounds, which conducts well-estimated pruning and on each iteration \(t\), also regret-prunes any index \((p, s) \in \mathcal{I}\) which satisfies

\[
\text{Reg}_p(s; \hat{u}^{(t)}) > \max\{2\hat{\varepsilon}^{(t)}, \gamma^* + \varepsilon + \hat{\varepsilon}^{(t)}\}.
\]

If PS-REG-0(\(\Gamma_{\gamma'}, \mathcal{D}, \mathcal{I}, \delta, \varepsilon\)) returns an empirical utility function \(\hat{u}\), then with probability at least \(1 - \delta\), it holds that \(E(\hat{u}) \subseteq E_{2\varepsilon}(\hat{u})\) and \(E_{\gamma}(\hat{u}) \subseteq E_{2\varepsilon + \gamma}(u)\) for all \(0 \leq \gamma \leq \gamma^*\).

**Proof.** Since PS-REG uses uniform utility deviation bounds, we have that for each index \((p, s) \in \mathcal{I}\) and iteration \(t\), it holds that \(\hat{\varepsilon}^{(t)}(s) = \hat{\varepsilon}^{(t)}\). If we applied the regret pruning technique from PS-REG+ to such an algorithm, the pruning criterion would simplify as follows:

\[
\text{Reg}_p(s; \hat{u}^{(t)}) > \max\{0, \gamma^* + \varepsilon - \hat{\varepsilon}^{(t)}(s)\}
\]

\[
\Leftrightarrow \sup_{s' \in \text{Adj}_p, s} \left(\hat{u}_p^{(t)}(s') - \hat{\varepsilon}_p^{(t)}(s')\right) - \left(\hat{u}_p^{(t)}(s) + \hat{\varepsilon}_p^{(t)}(s)\right) > \max\{0, \gamma^* + \varepsilon - \hat{\varepsilon}_p^{(t)}(s)\}
\]

\[
\Leftrightarrow \left[\sup_{s' \in \text{Adj}_p, s} \left(\hat{u}_p^{(t)}(s') - \hat{u}_p^{(t)}(s)\right) - 2\hat{\varepsilon}_p^{(t)} \right] > \max\{0, \gamma^* + \varepsilon - \hat{\varepsilon}_p^{(t)}\}
\]

\[
\Leftrightarrow \sup_{s' \in \text{Adj}_p, s} \left(\hat{u}_p^{(t)}(s') - \hat{u}_p^{(t)}(s)\right) > 2\hat{\varepsilon}_p^{(t)} + \max\{0, \gamma^* + \varepsilon - \hat{\varepsilon}_p^{(t)}\}
\]

\[
\Leftrightarrow \text{Reg}_p(s; \hat{u}^{(t)}) > \max\{2\hat{\varepsilon}_p^{(t)}, \gamma^* + \varepsilon + \hat{\varepsilon}_p^{(t)}\}.
\]

But this is precisely the pruning criterion of PS-REG. Thus, PS-REG+ is simply a generalization of PS-REG to cases with non-uniform utility deviation bounds, and PS-REG must then yield the same guarantees as PS-REG+.

**Theorem 5.** If PS-REG-0(\(\Gamma_{\gamma'}, \mathcal{D}, \mathcal{I}, \delta, \varepsilon\)) returns an empirical utility function \(\hat{u}\), then with probability at least \(1 - \delta\), we have that \(E(\hat{u}) \subseteq E_{2\varepsilon}(\hat{u})\) and \(E(\hat{u}) \subseteq E_{2\varepsilon}(u)\).

**Proof.** Consider the regret pruning criterion in PS-REG when \(\gamma^* = 0\). We know that the first iteration \(t\) on which \(\hat{\varepsilon}^{(t)} \leq \varepsilon\), all indices will be well-estimated pruned and the algorithm will terminate. Thus, regret pruning will only occur on iterations on which \(\hat{\varepsilon}^{(t)} > \varepsilon\). But then the regret pruning criterion simplifies to \(\text{Reg}_p(s; \hat{u}^{(t)}) > \max\{2\hat{\varepsilon}_p^{(t)}, \gamma^* + \varepsilon\} = 2\hat{\varepsilon}_p^{(t)}\), which is precisely the pruning condition of PS-REG-0. Thus, PS-REG is simply a generalization of PS-REG-0 to \(\gamma^* > 0\), and hence PS-REG-0 must yield the same guarantees as PS-REG when \(\gamma^* = 0\).

**Theorem 8.** Consider a PS algorithm, PS-REG-M, using non-uniform utility deviation bounds, which conducts well-estimated pruning and on each iteration \(t\), also regret-prunes any index \((p, s) \in \mathcal{I}\) which satisfies

\[
\text{Reg}_p^\gamma(s; \hat{u}^{(t)}) > \varepsilon + \hat{\varepsilon}_p^{(t)}(s).
\]

If PS-REG-M(\(\Gamma_{\gamma'}, \mathcal{D}, \mathcal{I}, \delta, \varepsilon\)) returns an empirical utility function \(\hat{u}\), then with probability at least \(1 - \delta\), for all \(0 \leq \gamma \leq 2\varepsilon\), it holds that

\[
E(\hat{u}) \subseteq E_{2\varepsilon}(\hat{u})\]

and for all \(\gamma \geq 0\), it holds that

\[
E(\hat{u}) \subseteq E_{4\varepsilon}(\hat{u})\]

**Proof.** Suppose an index \((p, s) \in \mathcal{I}\) is regret-ruled on iteration \(i\). In the proof of Theorem 7 we see that \(\text{Reg}_p(\hat{u}^{(t)}, \hat{\varepsilon}_p^{(t)}) > 0\) implies that \(\text{Reg}_p(s, \Gamma_{\gamma'}) > 0\). Hence, this pruning criteria is also guaranteed not to regret prune any index \((p, s', s) \in \mathcal{I}\) satisfying \(\text{Reg}_p(s, \Gamma_{\gamma'}) = 0\). Further following the proof of Theorem 7, we have that \(\text{Reg}_p(s, \Gamma_{\gamma'}) \geq \text{Reg}_p(\hat{u}^{(t)}, \hat{\varepsilon}_p^{(t)}) - \varepsilon\), which when combined with our pruning criteria, yields (w.h.p.) that

\[
\text{Reg}_p(s, \hat{\varepsilon}_p^{(t)}) > 2\hat{\varepsilon}_p^{(t)}(s) \geq \varepsilon,\]

\[
\text{Reg}_p(s, \hat{\varepsilon}_p^{(t)}) \geq 2\hat{\varepsilon}_p^{(t)}(s) \geq 2|u_p(s) - \underline{u}_p(s)|.
\]
Since \((p, s)\) was arbitrary, we have that \(|u_p(s) - \hat{u}_p(s)|\lesssim \frac{\text{Reg}_p(s, \hat{\Gamma}_r)}{2}\) for each index \((p, s)\in\mathcal{I}\) that is regret-pruned. Since the remaining indices must all be well-estimated, we have (w.h.p.) that for all \((p, s)\in\mathcal{I}\), it holds that
\[
|u_p(s) - \hat{u}_p(s)| \leq \max \left\{ \epsilon, \frac{\text{Reg}_p(s, \hat{\Gamma}_r)}{2} \right\}.
\]
The conclusion then follows from Lemma 3.

### Efficiency Bounds

(Cousins et al. 2022) derive high-probability sample complexity bounds for their empirical Bennett tail bounds. We state these sample complexity results below, and use them to derive our efficiency bounds for PS-REG+ and PS-REG-M.

**Lemma 6.** Consider an index \((p, s)\in\mathcal{I}\). If the sample size \(m_s \geq 2 + 2 \ln \frac{3 |\mathcal{I}| T}{\delta} \left( \frac{25}{2} \epsilon^2 + \frac{\text{Reg}_p(s, u)}{\epsilon^4} \right)\), then with probability at least \(1 - \frac{\delta}{3}\), it will hold that \(\tilde{\epsilon}_p^*(s) \leq \epsilon\).

**Theorem 9.** When running PS-REG+\((\Gamma, \mathcal{D}, \mathcal{I}, \delta, \epsilon, \gamma^*)\), with probability at least \(1 - \frac{\delta}{3}\), the index \((p, s)\in\mathcal{I}\) will be pruned prior to the first iteration \(t\) with cumulative sample size \(M_t \geq 2 + 2 \ln \frac{3 |\mathcal{I}| T}{\delta} \left( \frac{10c}{5c} \frac{\text{Reg}_p(s, u)}{\epsilon^4} + \frac{25\|v_p(\text{Adj}_p, s)\|_\infty}{\text{Reg}_p(s, u) - \epsilon^4} \right)\).

When running PS-REG-M\((\Gamma, \mathcal{D}, \mathcal{I}, \delta, \epsilon)\), with probability at least \(1 - \frac{\delta}{3}\), the index \((p, s)\in\mathcal{I}\) will be pruned prior to the first iteration \(t\) with cumulative sample size \(M_t \geq 2 + 2 \ln \frac{3 |\mathcal{I}| T}{\delta} \left( \frac{12.5c}{5c} \frac{\text{Reg}_p(s, u)}{\epsilon^4} + \frac{25\|v_p(\text{Adj}_p, s)\|_\infty}{\text{Reg}_p(s, u) - \epsilon^4} \right)\).

### Proof
Each of our efficiency bounds is presented as a minimum over two bounds, the first corresponding to regret pruning and the second to well-estimated pruning. It is clear that the second is a direct consequence of Lemma 6. We show the regret pruning bounds, beginning with PS-REG+.

Recall that PS-REG+ prunes an index \((p, s)\in\mathcal{I}\) on an iteration \(t\) if \(\text{Reg}_p(s, \hat{u}^{(t)}) > \max\{0, \gamma^* + \epsilon - \hat{\epsilon}_p(s)\}\). We have (w.h.p.) that
\[
\text{Reg}_p(s, \hat{u}^{(t)}) = \sup_{s' \in \text{Adj}_p, s} \left( \hat{u}_p^{(t)}(s') - \hat{\epsilon}_p^{(t)}(s') \right) - \left( \hat{u}_p^{(t)}(s) + \hat{\epsilon}_p^{(t)}(s) \right)
\]
\[
> \left[ \sup_{s' \in \text{Adj}_p, s} \hat{u}_p^{(t)}(s') \right] - \left[ \sup_{s' \in \text{Adj}_p, s} \hat{\epsilon}_p^{(t)}(s') \right] - \left( \hat{u}_p^{(t)}(s) + \hat{\epsilon}_p^{(t)}(s) \right)
\]
\[
> \left[ \sup_{s' \in \text{Adj}_p, s} u_p(s) - \hat{\epsilon}_p^{(t)}(s) \right] - \left[ \sup_{s' \in \text{Adj}_p, s} \hat{\epsilon}_p^{(t)}(s') \right] - \left( u_p(s) + 2\hat{\epsilon}_p^{(t)}(s) \right)
\]
\[
= \text{Reg}_p(s, u) - 2\hat{\epsilon}_p(s) - 2\sup_{s' \in \text{Adj}_p, s} \hat{\epsilon}_p(s') + \hat{\epsilon}_p(s).
\]

Hence, a strictly tighter pruning criterion for PS-REG+ would be pruning an index \((p, s)\in\mathcal{I}\) when
\[
\text{Reg}_p(s, u) - \gamma^* > 2\hat{\epsilon}_p(s) + 2\sup_{s' \in \text{Adj}_p, s} \hat{\epsilon}_p(s') - \gamma^* + \max\{0, \gamma^* + \epsilon - \hat{\epsilon}_p(s)\}
\]
\[
= 2\sup_{s' \in \text{Adj}_p, s} \hat{\epsilon}_p(s') + \max\left\{2\hat{\epsilon}_p(s) - \gamma^*, \epsilon + \hat{\epsilon}_p(s)\right\}.
\]
We can make the pruning criterion even tighter by increasing the right-hand side:

$$2 \sup_{s' \in \text{Adj}_p,s} \hat{\varepsilon}_p^{(t)}(s') + \max \left\{ 2\hat{\varepsilon}_p^{(t)}(s) - \gamma^*, \varepsilon + \hat{\varepsilon}_p^{(t)}(s) \right\} \leq 4 \sup_{s' \in \text{Adj}_p,s} \hat{\varepsilon}_p^{(t)}(s').$$

Hence, the latest (w.h.p.) an index \((p, s) \in \mathcal{I}\) will be regret-pruned by PS-REG+ is when

$$\sup_{s' \in \text{Adj}_p,s} \hat{\varepsilon}_p^{(t)}(s') < \frac{\text{Reg}_u(s; u) - \gamma^*}{4}.$$ 

Our result then follows via Lemma \[\text{Lemma 6}\] by analogous reasoning, we have that a strictly tighter regret pruning criterion than the one in PS-REG-M would be pruning an index \((p, s) \in \mathcal{I}\) when

$$\text{Reg}_u(s; u) - \varepsilon > 3\hat{\varepsilon}_p^{(t)}(s) + 2 \sup_{s' \in \text{Adj}_p,s} \hat{\varepsilon}_p^{(t)}(s').$$

Similar to before, we can make the pruning criterion even tighter by increasing the right hand side:

$$3\hat{\varepsilon}_p^{(t)}(s) + 2 \sup_{s' \in \text{Adj}_p,s} \hat{\varepsilon}_p^{(t)}(s') \leq 5 \sup_{s' \in \text{Adj}_p,s} \hat{\varepsilon}_p^{(t)}(s')$$

Hence, the latest (w.h.p.) an index \((p, s) \in \mathcal{I}\) will be regret pruned by PS-REG-M is when

$$\sup_{s' \in \text{Adj}_p,s} \hat{\varepsilon}_p^{(t)}(s') < \frac{\text{Reg}_u(s; u) - \varepsilon}{5}.$$ 

Once again, our result follows from Lemma \[\text{Lemma 6}\].

**Theorem 10.** Suppose that the total samples \(M_T\) allocated in the sampling schedule is greater than or equal to the maximum number of samples needed to prune an arbitrary index, i.e.,

$$M_T \geq T \sum_{t=1}^{T} m_t \geq \frac{c^2 \ln \left( \frac{2|\mathcal{I}| T}{\delta} \right)}{2\varepsilon^2}.$$ 

Then for each algorithm \(\text{PS} \in \{\text{PS-WE}, \text{PS-REG-0}, \text{PS-REG}, \text{PS-REG+}, \text{PS-REG-M}\}\),

it is guaranteed that \(\text{PS}(\Gamma, Y, \mathcal{I}, \delta, \varepsilon)\) will terminate and return an empirical game with the guarantees corresponding to the respective algorithm.

**Proof.** Recall that all the aforementioned progressive sampling algorithms use well-estimated pruning. By Hoeffding’s Inequality (Theorem 1), we have that on iteration \(T\) of algorithm \(\text{PS}\), for all \((p, s) \in \mathcal{I}\), it holds that

$$\hat{\varepsilon}_p^{(T)}(s) \leq c \sqrt{\frac{\ln \left( \frac{2|\mathcal{I}| T}{\delta} \right)}{M_T}} \leq c \sqrt{\ln \left( \frac{2|\mathcal{I}| T}{\delta} \right)} \cdot \frac{2\varepsilon^2}{c^2 \ln \left( \frac{2|\mathcal{I}| T}{\delta} \right)} = \varepsilon,$$

and thus all indices which remain active until iteration \(T\) will be pruned on iteration \(T\). Hence, the aforementioned algorithms are guaranteed to prune all indices prior to the exhaustion of the sampling schedule, and thus will return an empirical game satisfying the respective guarantees of the algorithm.

**Sampling Schedule**

Our sampling schedule is derived via a sample complexity lower bound for the empirical Bennett tail bounds presented in Theorem 3 (Cousins et al. 2022) lower bound the empirical Bennett bounds via the zero-variance case of Bennett’s inequality (Theorem 2). We, however, use a tighter lower bound which we derive below.

**Lemma 7.** Consider an index \((p, s) \in \mathcal{I}\). If \(\varepsilon_p^B(s) \leq \varepsilon\), then it must hold that the sample size \(m_s > \left( \frac{1}{3} + \sqrt{\frac{4+2\sqrt{3}}{3}} \right) \cdot \frac{\varepsilon \ln \left( \frac{2|\mathcal{I}|}{\delta} \right)}{\varepsilon} \).
Proof. We have that
\[
\varepsilon_p^8(s) = \frac{2c^2 \ln \left( \frac{3|I|}{8} \right)}{3m} + \frac{2c^2 \ln \left( \frac{3|I|}{8} \right)}{3m} + \frac{1}{3} \left( \frac{1}{3} + \frac{3}{2 \ln \left( \frac{3|I|}{8} \right)} \right) \left( \frac{c^2 \ln \left( \frac{3|I|}{8} \right)}{m - 1} \right)^2 + \frac{2c^2 \hat{\nu}_p(s) \ln \left( \frac{3|I|}{8} \right)}{m} .
\]

We further have that
\[
\varepsilon_p^8(s) > \frac{2c^2 \ln \left( \frac{3|I|}{8} \right)}{3m} + \frac{1}{3} \left( \frac{1}{3} \right)^2 c^2 \ln \left( \frac{3|I|}{8} \right) .
\]

We thus have that
\[
\varepsilon_p^8(s) > \frac{2 + \sqrt{3}}{3} . c^2 \ln \left( \frac{3|I|}{8} \right) .
\]

The conclusion follows directly. \qed

Thus, our sampling schedule for PS-WE begins at \( \alpha = \left( \frac{1}{3} + \sqrt{\frac{1 + 4 + \sqrt{3}}{3}} \right) \cdot c \ln \left( \frac{3|I|}{8} \right) \) and ends at a cumulative sample size that is at least \( \omega = \frac{c^2 \ln \left( \frac{3|I|}{8} \right)}{2|I|} \) to satisfy Theorem \( 10 \). Following (Cousins et al., 2022), we then use a schedule with a geometrically increasing cumulative sample size with a geometric factor \( \beta \) (for our experiments, we use \( \beta = 1.1 \)). Our schedule length is then \( T \approx \left\lfloor \log_\beta \left( \frac{\omega}{n} \right) \right\rfloor \). The first sample size is defined by \( m_1 = \alpha \beta \), and each following sample size is defined by \( m_t = \alpha \beta^t - m_{t-1} \).

Of course, for all of our regret pruning algorithms, it may be possible for regret pruning to occur prior to at least one index \( (p, s) \in \mathcal{I} \) achieving \( \varepsilon_p^8(s) \leq \varepsilon \). Regret pruning cannot, however, happen prior to at least one index \( (p, s) \in \mathcal{I} \) achieving \( \varepsilon_p^8(s) \leq \frac{\varepsilon}{2} \). This can be seen by looking at the loosest regret pruning criterion we discuss, that used in PS-REG+ with \( \gamma^* = 0 \).

An index \( (p, s) \in \mathcal{I} \) is regret-pruned on iteration \( t \) if \( \text{Reg}_p^+(s; \hat{u}^{(t)}) > 0 \). Notice that if \( \varepsilon_p^8(s) > \frac{\varepsilon}{2} \) for all \( (p, s) \in \mathcal{I} \), then for any given index \( (p, s) \in \mathcal{I} \), we have that
\[
\text{Reg}_p^+(s; \hat{u}^{(t)}) = \sup_{s' \in \text{Adj}_p, s} \left( \hat{u}_p^{(t)}(s') - \varepsilon_p^8(s') \right) - \left( \hat{u}_p^{(t)}(s) + \varepsilon_p^8(s) \right) < \text{Reg}_p^+(s; \hat{u}^{(t)}) - c,
\]
and hence \( (p, s) \) will be regret-pruned only if it holds that \( \text{Reg}_p^+(s; \hat{u}^{(t)}) - c > \text{Reg}_p^+(s; \hat{u}^{(t)}) > 0 \). But the latter is impossible since \( \text{Reg}_p^+(s; \hat{u}^{(t)}) \leq c \) by definition. Hence, no index can be regret-pruned (by any of our regret pruning criteria) prior to at least one index being an estimation guarantee of at least \( \frac{\varepsilon}{2} \).

Based on the above, we start our sampling schedule for all our regret pruning algorithms on \( \alpha' = \left( \frac{1}{3} + \sqrt{\frac{1 + 4 + \sqrt{3}}{3}} \right) \cdot 2 \ln \left( \frac{3|I|}{8} \right) \). But, as argued in the text, using a schedule with a strictly geometrically increasing schedule, we waste too many iterations on small sample sizes and yield a schedule length that is too large. Hence, we instead fix the schedule length to be 1.5 times the schedule length used for PS-WE. We then occupy the final two-thirds of our schedule with the same sample sizes used in the sampling schedule for PS-WE, and occupy the first third of our schedule with one that has linearly increasing cumulative sample size beginning at \( \alpha' \) and ending at \( \alpha \) (from above).