QUASINORMS IN SEMILINEAR ELLIPTIC PROBLEMS

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Abstract. In this note we examine the a priori and a posteriori analysis of discontinuous Galerkin finite element discretisations of semilinear elliptic PDEs with polynomial nonlinearity. We show that optimal a priori error bounds in the energy norm are only possible for low order elements using classical a priori error analysis techniques. We make use of appropriate quasinorms that results in optimal energy norm error control.

We show that, contrary to the a priori case, a standard a posteriori analysis yields optimal upper bounds and does not require the introduction of quasinorms. We also summarise extensive numerical experiments verifying the analysis presented and examining the appearance of layers in the solution.

1. Introduction

Let \( \Omega \subset \mathbb{R}^d \) with \( d \geq 1 \) be an open Lipschitz domain and consider the problem: find \( u \in H^1_0(\Omega) \), such that

\[
-\Delta u + |u|^{p-2} u = f \quad \text{in } \Omega \\
u = 0 \quad \text{on } \partial \Omega.
\]

(1.1)

This class of equation is sometimes referred to as the Lane-Emden-Fowler equation and are related to problems with critical exponents [CdFM96]. Furthermore, they arise in the theory of boundary layers of viscous fluids [Won75].

We are particularly interested in the class of PDE (1.1) because of its application to the analysis of numerical schemes posed for the KdV-like equation

(1.2)

\[
u_t - \left( |u|^{p-2} u \right)_x + u_{xxx} = 0.
\]

Indeed, solutions of (1.2) posed over a 1-dimensional domain satisfy

(1.3)

\[
0 = \frac{d}{dt} \mathcal{J}[u], \quad \text{with } \mathcal{J}[u] = \int_{\Omega} \frac{1}{2} |u_x|^2 + \frac{1}{p} |u|^p \, dx
\]

and energy minimisers of (1.3) satisfy (1.1) with \( f = 0 \) and appropriate boundary conditions. In [JPP18, JP18] a conservative Galerkin scheme was proposed for (1.2) and the a priori and a posteriori analysis of this scheme requires quasi-optimal approximation of the finite element solution of (1.1) and optimal a posteriori estimates. Hence our goal in this work is the derivation and a priori and a posteriori bounds of Galerkin discretisations of (1.1).

We proceed as follows: In \$2\$ we introduce notation and the model problem. We give some insight as to its properties that we use in subsequent sections and propose a discontinuous Galerkin finite element approximation. In \$3\$ we give a classical a priori analysis based on arguments in [Cia78]. We show that in the energy norm, the analysis is suboptimal for high polynomial degrees and large values of \( p \). In \$4\$ we modify the notion of a quasinorm from the works of [LB96] to enable an optimal a priori error estimate to be shown. In \$5\$ we derive an a posteriori estimate and finally, in \$6\$, we showcase some numerical experiments.
2. Problem setup

In this section we formulate the model problem, fix notation and give some basic assumptions. Weakly, we may consider the PDE (1.1) as: find \( u \in H^1_0(\Omega) \), such that

\[
\mathcal{A}(u, v) + \mathcal{B}(u; u, v) = \langle f, v \rangle \quad \forall \ v \in H^1_0(\Omega),
\]

where \( \langle \cdot, \cdot \rangle \) denotes the \( L^2 \) inner product and the bilinear form \( \mathcal{A} : H^1_0(\Omega) \times H^1_0(\Omega) \rightarrow \mathbb{R} \) is given by

\[
\mathcal{A}(u, v) := \int_{\Omega} \nabla u \cdot \nabla v \, dx.
\]

The semilinear form \( \mathcal{B} \) is given by

\[
\mathcal{B}(w; u, v) := \int_{\Omega} |w|^{p-2} uv \, dx.
\]

It is straightforward to verify this problem admits a unique solution.

2.1. Proposition (A priori bound 1). Let \( f \in H^{-1}(\Omega) \) and \( u \in H^1_0(\Omega) \) solve (2.1). Then we have

\[
\frac{1}{2} \| \nabla u \|_{L^2(\Omega)}^2 + \| u \|_{L^p(\Omega)}^p \leq \| f \|_{H^{-1}(\Omega)}^2.
\]

Proof. Using a standard energy argument, take \( v = u \) in (2.1), then

\[
\frac{1}{2} \| \nabla u \|_{L^2(\Omega)}^2 + \| u \|_{L^p(\Omega)}^p = \langle f, u \rangle \leq \| f \|_{H^{-1}(\Omega)} \| \nabla u \|_{L^2(\Omega)} \leq \frac{1}{2} \left( \| f \|_{H^{-1}(\Omega)}^2 + \| \nabla u \|_{L^2(\Omega)}^2 \right),
\]

as required. \( \square \)

2.2. Proposition (A priori bound 2). Let \( f \in L^q(\Omega) \), where \( q = \frac{p}{p-1} \) and \( u \in H^1_0(\Omega) \) solve (2.1) then we have

\[
\| \nabla u \|_{L^2(\Omega)}^2 + \frac{1}{q} \| u \|_{L^p(\Omega)}^p \leq \frac{1}{q} \| f \|_{L^q(\Omega)}^q.
\]

Proof Again, take \( v = u \) in (2.1), then

\[
\| \nabla u \|_{L^2(\Omega)}^2 + \frac{1}{q} \| u \|_{L^p(\Omega)}^p = \langle f, u \rangle \leq \| f \|_{L^q(\Omega)} \| u \|_{L^p(\Omega)} \leq \frac{1}{p} \| u \|_{L^p(\Omega)}^p + \frac{1}{q} \| f \|_{L^q(\Omega)}^q,
\]

as required. \( \square \)

2.3. Remark (Behaviour of the a priori bounds in \( p \)). Note that the bounds given in Propositions 2.1 and 2.2 behave the same as \( p \) increases, since

\[
\lim_{p \to \infty} \frac{1}{q} = 1,
\]

however the bound given in Proposition 2.2 blows up as \( p \) decreases, indeed

\[
\lim_{p \to 1} \frac{1}{q} = \infty.
\]

We will only consider the case \( p \geq 2 \) in this work.

2.4. Discretisation. Let \( \mathcal{T} \) be a regular subdivision of \( \Omega \) into disjoint simplicial elements. We assume that the subdivision \( \mathcal{T} \) is shape-regular \([\text{Cia78}, \text{p.124}]\), is \( \overline{\Omega} = \bigcup_K K \) and that the elemental faces are points (for \( d = 1 \)), straight lines (for \( d = 2 \)) or planar (for \( d = 3 \)) segments; these will be, henceforth, referred to as facets. By \( \Gamma \) we shall denote the union of all \((d-1)\)-dimensional facets associated with the subdivision \( \mathcal{T} \) including the boundary. Further, we set \( \Gamma_i := \Gamma \setminus \partial \Omega \).

For a nonnegative integer \( k \), we denote the set of all polynomials of total degree at most \( k \) by \( \mathbb{P}^k(K) \). For \( k \geq 1 \), we consider the finite element space

\[
V^k_h := \{ v \in L^2(\Omega) : v|_K \in \mathbb{P}^k(K), \forall K \}.
\]
Further, let \( K^+, K^- \) be two (generic) elements sharing a facet \( e := \partial K^+ \cap \partial K^- \subset \Gamma_i \) with respective outward normal unit vectors \( \mathbf{n}^+ \) and \( \mathbf{n}^- \) on \( e \). For a function \( v : \Omega \to \mathbb{R} \) that may be discontinuous across \( \Gamma_i \), we set \( v^+ := \lim_{e \searrow e^+} v \mathbf{e} \subset \partial K^+ \), \( v^- := \lim_{e \searrow e^-} v \mathbf{e} \subset \partial K^- \), and we define the jump by
\[
[v] := v^+ \mathbf{n}^+ + v^- \mathbf{n}^- ;
\]
where \( e \in \partial K \cap \partial \Omega \), we set \( [v] := v^+ \mathbf{n}^+ \). Also, we define \( h_K := \text{diam}(K) \) and we collect them into the element-wise constant function \( h : \Omega \to \mathbb{R} \), with \( h|_K = h_K \), \( K \), \( h|_e = (h_{K^+} + h_{K^-})/2 \) for \( e \subset \Gamma_i \) and \( h|_e = h_K \) for \( e \subset \partial K \cap \partial \Omega \). We assume that the families of meshes considered in this work are locally quasi-uniform. Note that this restriction can be relaxed by following arguments as in \([\text{GMP}18]\).

For \( s > 0 \), we define the broken Sobolev space \( H^s(\mathcal{T}) \), by
\[
H^s(\mathcal{T}) := \{ w \in L^2(\Omega) : w|_K \in H^s(K), K \in \mathcal{T} \},
\]
along with the broken (element-wise) gradient and Laplacian \( \nabla_h \equiv \nabla_h(\mathcal{T}) \) and \( \Delta_h \equiv \Delta_h(\mathcal{T}) \).

We consider the interior penalty (IP) discontinuous Galerkin discretisation of (2.2), reading: find \( u_h \in V^k_h \) such that
\[
\mathcal{A}_h(u_h, v_h) + \mathcal{B}(u_h; u_h, v_h) = \langle f, v_h \rangle \quad \forall v_h \in V^k_h ,
\]
where
\[
\mathcal{A}_h(u_h, v_h) = \int_\Omega \nabla_h u_h \cdot \nabla_h v_h \, dx - \int_{\Gamma} (\|v\|^2_h \cdot \{P_{k-1}(\nabla u_h)\} + \|u_h\|^2 \cdot \{P_{k-1}(\nabla v_h)\} - \sigma \|u_h\| \cdot \|v_h\| ) \, ds,
\]
where \( \sigma > 0 \) is the, so-called, discontinuity penalisation parameter given by
\[
\sigma := C_\sigma \frac{k^2}{h} .
\]
Notice the nonstandard projection operator appearing in the definition. This is to ensure that \( \mathcal{A}_h \) is well defined over \( H^1(\Omega) \times H^1(\Omega) \).

2.5. Definition (Mesh dependent norms). We introduce the mesh dependent \( H^1 \) norm to be
\[
|w|_{ad}^2 := \|\nabla_h w\|^2_{L^2(\Omega)} + \|\sqrt{\sigma} \|w\|^2_{L^2(\Gamma)} .
\]
Note that the the bilinear form (2.2) satisfies boundedness and coercivity properties for \( C_\sigma \) chosen large enough \([\text{EG}04, \text{c.f.}]\), that is
\[
\mathcal{A}_h(u_h, v_h) \leq \tilde{C}_B |u_h|_{ad} |v_h|_{ad} ,
\]
\[
\mathcal{A}_h(u_h, v_h) \geq \tilde{C}_C |u_h|^2_{ad} \quad \forall u_h, v_h \in V^k_h .
\]

3. Classical a priori analysis

In this section we examine analysis based on classical arguments such as those used in \([\text{Cia}78]\) for the \( p \)-Laplacian.

3.1. Lemma (Properties of \( \mathcal{B}(\cdot; \cdot; \cdot) \), cf. \([\text{Cia}78, \text{§5.3}]\)). There exist constants
\[
(1) \quad C_L > 0 \text{ such that}
\]
\[
\mathcal{B}(u - u_h; u - u_h, u - u_h) \leq C_L (\mathcal{B}(u; u, u - u_h) - \mathcal{B}(u_h; u_h, u - u_h))
\]
(2) \quad \( C_U > 0 \text{ such that}
\]
\[
\mathcal{B}(u; u, u - v_h) - \mathcal{B}(u_h; u_h, u - v_h) \leq C_U \|u - u_h\|_{L^p(\Omega)} \|u - v_h\|_{L^q(\Omega)} .
\]

3.2. Theorem. Let \( u \in H^2(\Omega) \cap H^1_0(\Omega) \) solve (1.1) and \( u_h \in V^k_h \) be the finite element approximation of (2.11) then for \( k \geq 1 \) we have
\[
|u - u_h|^2_{ad} + \|u - u_h\|^p_{L^p(\Omega)} \leq C \inf_{v_h \in V^k_h} \left( |u - v_h|^2_{ad} + \|u - v_h\|^p_{L^p(\Omega)} \right) ,
\]
where \( q = \frac{p}{p-1} \) is the Sobolev conjugate of \( p \).
We will not prove this here for brevity but, as illustrated in Table 2, the bound is optimal only when \( p = 3.4 \).

**Remark**

Further, (3.5)

Note that (3.4) and (3.5) or all (3.6) into (3.4) yields the desired result. □

### 3.3. Corollary

Choosing \( v_h = I_h u \), the Clément interpolant of \( u \), in Theorem 3.2 and under further smoothness requirements, that \( u \in W^{k+1,p}(\Omega) \), we see that

\[
|u - u_h|_{dG}^2 + \|u - u_h\|_{L^p(\Omega)}^2 \leq C (h^{2k} |u|_{H^{k+1}(\Omega)} + h^{(k+1)q} |u|_{W^{k+1,p}(\Omega)}) .
\]

### 3.4. Remark (Optimality of Corollary 3.3)

Notice that the bound given in Corollary 3.3 depends upon \( q = \frac{p}{p-1} \). Notice, as shown in Table 1, the energy error bounds are optimal only if \( p = 2 \) for all \( k \) or \( k = 1 \) or all \( p \).

**Table 1.** In the following table we examine the optimality of the finite element approximation in the energy norm. The numerical values in the table correspond to \( \min \left( k, \frac{(k+1)q}{2} \right) \) and are coloured green or red depending upon whether the bound is optimal, in the function approximation sense, or suboptimal respectively.

| \( k \) | \( p = 2 \) | \( p = 3 \) | \( p = 4 \) | \( p = 5 \) | \( p \to \infty \) |
|---|---|---|---|---|---|
| 1 | 2 | 3/2 | 4/3 | 6/4 | 1 |
| 2 | 3 | 9/4 | 2 | 15/8 | 3/2 |
| 3 | 4 | 3 | 8/3 | 5/2 | 2 |
| 4 | 5 | 15/4 | 10/3 | 25/8 | 5/2 |

### 3.5. Remark (Dual bounds)

This lack of optimality propagates further when consider bounds based on duality approaches. Indeed, using the dual problem

\[
-\Delta z + (p-1) u^{p-2} z = u - u_h \text{ in } \Omega
\]

\[
z = 0 \text{ on } \partial \Omega,
\]

we are able to show

\[
\|u - u_h\|_{L^2(\Omega)} \leq C \left( h |u - u_h|_{dG} + \|u - u_h\|_{L^p(\Omega)}^2 \right) .
\]

We will not prove this here for brevity but, as illustrated in Table 2, the bound is optimal only when \( p = 2 \).
In addition, it is bounded \([EL05]\) in that for any \(\theta > 0\)
\[
\|v\|_{\theta, p} := \int_\Omega |v|^p (|w| + |v|)^{p-2} \, dx.
\]
This satisfies the usual properties of a norm, in that
\[
\|v\|_{\theta, p} \geq 0 \quad \text{and} \quad \|v\|_{\theta, p} = 0 \iff v = 0.
\]
However, the usual triangle inequality is replaced by
\[
\|v_1 + v_2\|_{\theta, p} \leq C\left(\|v_1\|_{\theta, p} + \|v_2\|_{\theta, p}\right),
\]
where \(C = C(v_1, v_2, w, p)\).

#### 4. A priori analysis based on quasi norms

In this section we will examine the use of quasinorms to rectify the gap in the a priori analysis.

**4.1. Definition** (Quasinorm). Let \(v \in L^p(\Omega)\), \(p \geq 2\), then for any \(w \in L^p(\Omega)\) we define the quasinorm
\[
\|v\|_{L^p(\Omega)} := \int_\Omega |v|^p \, dx.
\]
This satisfies the usual properties of a norm, in that
\[
\|v\|_{L^p(\Omega)} \geq 0 \quad \text{and} \quad \|v\|_{L^p(\Omega)} = 0 \iff v = 0.
\]
However, the usual triangle inequality is replaced by
\[
\|v_1 + v_2\|_{L^p(\Omega)} \leq C\left(\|v_1\|_{L^p(\Omega)} + \|v_2\|_{L^p(\Omega)}\right),
\]
for \(v, w \in L^p(\Omega)\), \(p \geq 2\) and any \(w \in L^p(\Omega)\). The key property that the quasinorm satisfies that allows for optimal a priori treatment is that the semilinear form is coercive with respect to it, that is
\[
\mathcal{B}(u; u, u - v) - \mathcal{B}(v; v, u - v) \geq C_C \|u - v\|^2_{L^p(\Omega)}.
\]
In addition, it is bounded \([EL05]\) in that for any \(\theta > 0\) there exists a \(\gamma > 0\) such that
\[
|\mathcal{B}(u; u, w) - \mathcal{B}(v; v, w)| \leq C_B \left(\theta^\gamma \|u - v\|^2_{L^p(\Omega)} + \theta \|w\|^2_{L^p(\Omega)}\right),
\]
where
\[
\gamma = \begin{cases} 1 & \text{if } \theta < 1 \\ \frac{1}{\theta} & \text{if } \theta \geq 1. \end{cases}
\]

**4.2. Remark** (Properties of the quasinorm). As can be seen from the definition, the quasinorm is related to the \(L^p\) norm through
\[
\|v\|_{L^p(\Omega)} \leq \|v\|_{\theta, p} \leq C\|v\|_{L^p(\Omega)},
\]
for \(v \in L^p(\Omega), p \geq 2\) and any \(w \in L^p(\Omega)\). The key property that the quasinorm satisfies that allows for optimal a priori treatment is that the semilinear form is coercive with respect to it, that is
\[
\mathcal{B}(u; u, u - v) - \mathcal{B}(v; v, u - v) \geq C_C \|u - v\|^2_{L^p(\Omega)}.
\]
In addition, it is bounded \([EL05]\) in that for any \(\theta > 0\) there exists a \(\gamma > 0\) such that
\[
|\mathcal{B}(u; u, w) - \mathcal{B}(v; v, w)| \leq C_B \left(\theta^\gamma \|u - v\|^2_{L^p(\Omega)} + \theta \|w\|^2_{L^p(\Omega)}\right),
\]
where
\[
\gamma = \begin{cases} 1 & \text{if } \theta < 1 \\ \frac{1}{\theta} & \text{if } \theta \geq 1. \end{cases}
\]

It was the lack of a sufficiently sharp boundedness property that led to suboptimality in the analysis presented in section 3. The key observation to rectify the suboptimality is to measure the error in
\[
|u - u_h|^2_{H^1} + \|u - u_h\|^2_{H^1},
\]
rather than the energy norm.

Henceforth, we will use the notation
\[
C_C := \min(C_C, \tilde{C}_C) \quad C_B := \max(C_B, \tilde{C}_B).
\]

**4.3. Proposition** (A priori bound 3). Let \(f \in H^{-1}(\Omega)\) and \(u \in H^1_0(\Omega)\) solve (2.1) then
\[
\|\nabla u\|^2_{L^2(\Omega)} + 2^{3-p} \|u\|^2_{L^p(\Omega)} \leq \|f\|^2_{H^{-1}(\Omega)}.
\]
Proof Notice that
\begin{equation}
\| \nabla u \|^2_{L^2(\Omega)} + 2^{2-p} \| u \|^2_{(u,p)} = A(u,u) + B(u;u,u) = \langle f, u \rangle \leq \frac{1}{2} \left( \| f \|^2_{H^{-1}(\Omega)} + \| \nabla u \|_{L^2(\Omega)} \right),
\end{equation}
as required. \hfill \Box

4.4. Theorem. Let \( u \in H^2(\Omega) \cap H^1_0(\Omega) \) solve (1.1) and \( u_h \) be the finite element approximation of (2.11) then for \( k \geq 1 \) we have
\begin{equation}
\| u - u_h \|_{dG}^2 + \| u - u_h \|_{(u,p)}^2 \leq C \inf_{v_h \in V_h} \left( \| u - v_h \|_{dG}^2 + \| u - v_h \|_{(u,p)}^2 \right).
\end{equation}

Proof Making use of the coercivity of \( B \) we have
\begin{equation}
C_C \left( \| u - u_h \|_{dG}^2 + \| u - u_h \|_{(u,p)}^2 \right) \leq A(u - u_h, u - u_h) + B(u;u,u - u_h) - B(u_h;u_h,u - u_h)
= A(u - u_h, u - u_h) + B(u;u,u - v_h) - B(u_h;u_h,u - v_h),
\end{equation}
for any \( v_h \in V_h \), using Galerkin orthogonality. Now, through (4.6) and (2.15) we have
\begin{equation}
C_C \left( \| u - u_h \|_{dG}^2 + \| u - u_h \|_{(u,p)}^2 \right) \leq \frac{C_C}{2} \| u - u_h \|_{dG}^2 + \frac{C_B}{2C_C} \| u - v_h \|_{dG}^2
+ C_B \left( \theta \gamma \| u - u_h \|_{(u,p)}^2 + \theta \| u - v_h \|_{(u,p)}^2 \right).
\end{equation}
Choosing \( \theta = \min \left( \frac{C_C}{2C_B}, \frac{1}{2} \right) \) then \( \gamma = 1 \). Rearranging the inequality yields the desired result. \hfill \Box

4.5. Lemma. Let \( p \geq 2 \) and \( v \in W^{k+1,p}(\Omega) \) then
\begin{equation}
\inf_{v_h \in V_h} \| v - v_h \|_{(v,p)} \leq C h^{k+1} \| v \|_{W^{k+1,p}(\Omega)}.
\end{equation}

Proof Using the property of the quasinorm given in Remark 4.2 we have
\begin{equation}
\| v - v_h \|_{(v,p)}^2 \leq C \| v - v_h \|_{L^p(\Omega)}^2,
\end{equation}
and the result follows from best approximation in \( L^p(\Omega) \). \hfill \Box

4.6. Corollary. Under the conditions of Theorem 4.4 suppose that \( u \in H^{k+1}(\Omega) \cap H^1_0(\Omega) \cap W^{k,p}(\Omega) \), then
\begin{equation}
\left( \| u - u_h \|_{dG}^2 + \| u - u_h \|_{(u,p)}^2 \right)^{1/2} \leq C h^k \left( \| u \|_{H^{k+1}(\Omega)} + \| u \|_{W^{k,p}(\Omega)} \right).
\end{equation}

4.7. Remark (Optimality of Corollary 4.6). Notice that the bound given in Corollary 4.6 is optimal regardless of the choice of \( p \) for smooth enough \( u \).

4.8. Remark (Dual bounds). By modifying the dual problem to
\begin{equation}
-\Delta z + (p-1) u^{p-2} z = (u - u_h) (|u| + |u - u_h|)^{p-2} \text{ in } \Omega
\end{equation}
\( z = 0 \) on \( \partial \Omega \),
one can also show optimal a priori bounds for the quasinorm error.

5. A posteriori error analysis
In this section we derive a reliable a posteriori estimator.

5.1. Proposition (A priori bound 4). Let \( f \in H^{-1}(\Omega) \) and \( u \in H^1_0(\Omega) \) solve (2.1) and \( w \in H^1_0(\Omega) \) solve
\begin{equation}
A(w,v) + B(w;w,v) = \langle f, v \rangle \quad \forall v \in H^1_0(\Omega),
\end{equation}
for some \( A \in H^{-1}(\Omega) \) then
\begin{equation}
\| \nabla u - \nabla w \|^2_{L^2(\Omega)} + 2^{2-p} \| u - w \|^2_{W^{k,p}(\Omega)} \leq \| A \|^2_{H^{-1}(\Omega)}.
\end{equation}
Proof Through the definitions of $u$ and $w$, we have the relation that

$$\mathcal{A}(u - w, v) + \mathcal{B}(u; u, v) - \mathcal{B}(w; w, v) = \langle \mathcal{R}, v \rangle \quad \forall v \in H_0^1(\Omega).$$

Hence, choosing $v = u - w$

$$\|\nabla u - \nabla w\|_{L^2(\Omega)}^2 + C_L \|u - w\|_{L^p(\Omega)}^p = \langle \mathcal{R}, u - w \rangle \leq \frac{1}{2} \left( \|\mathcal{R}\|_{H^{-1}(\Omega)}^2 + \|\nabla u - \nabla w\|_{L^2(\Omega)}^2 \right),$$

as required.

To invoke the results of Proposition 5.1 we require an object $w \in H^1(\Omega)$. The dG solution $u_h \notin H^1(\Omega)$, so we make use of an appropriate postprocessor as an intermediate quantity.

5.2. Lemma ([KP03]). Let $\mathcal{N}$ denote the set of all Lagrange nodes of $V_h^k$, and $\mathcal{E} : V_h^k \to V_h^k \cap H_0^1(\Omega)$ be defined on the conforming Lagrange nodes $\nu \in \mathcal{N}$ by

$$\mathcal{E}(\nu)(\nu) := \left\{ \begin{array}{ll} \|\nu\|^{-1} \sum_{K \in \mathcal{K}_\nu} v|_K(\nu), & \nu \in \Omega; \\ 0, & \nu \in \partial \Omega, \end{array} \right.$$ 

with $\omega_\nu := \bigcup_{K \in \mathcal{T}, \nu \in K} K$, the set of elements sharing the node $\nu \in \mathcal{N}$ and $|\omega_\nu|$ their cardinality. Then, the following bound holds

$$\sum_{K \in \mathcal{T}} \|v - \mathcal{E}(\nu)|_K \nabla^\mathcal{N}(K) \leq C_\alpha \sum_{\nu \in \mathcal{T}} \|h^{1/2 - \alpha} \|v\|_{L^2(e)}^2,$$

with $\alpha = 0, 1, C_\alpha \equiv C_\alpha(k) > 0$ a constant independent of $h, v$ and $\mathcal{T}$, but depending on the shape-regularity of $\mathcal{T}$ and on the polynomial degree $k$.

5.3. Proposition. The reconstruction $\mathcal{E}(u_h)$ satisfies the perturbed PDE

$$\mathcal{A}(\mathcal{E}(u_h), v) + \mathcal{B}(\mathcal{E}(u_h); \mathcal{E}(u_h), v) = \langle f - \mathcal{R}, v \rangle \quad \forall v \in H_0^1(\Omega),$$

with

$$\mathcal{R}(\nu) = \mathcal{A}(\mathcal{E}(u_h) - \mathcal{E}(u_h), v) + \mathcal{B}(u_h; u_h, v) - \mathcal{B}(\mathcal{E}(u_h); \mathcal{E}(u_h), v)$$

$$+ \langle f, v - v_h \rangle - \mathcal{A}(u_h, v - v_h) - \mathcal{B}(u_h; u_h, v - v_h) \quad \forall v_h \in V_h^k.$$

5.4. Theorem. Let $f \in H^{-1}(\Omega)$ and $u \in H_0^1(\Omega)$ solve (2.1) and $u_h \in V_h^k$ solve (2.11). Further let $\mathcal{E}(u_h) \in H_0^1(\Omega)$ denote the reconstruction operator given in Lemma 5.2. Then,

$$\|\nabla u - \nabla \mathcal{E}(u_h)\|_{L^2(\Omega)}^2 + 2C_L \|u - \mathcal{E}(u_h)\|_{L^p(\Omega)} \leq C \sum_{K \in \mathcal{T}} \sum_{\nu \in \partial \Omega} \eta_R^2 + \sum_{\nu \in \partial \Omega} \eta_j^2,$$

where

$$\eta_R^2 := \left\| h \left( f + \Delta u_h - |u_h|^{p-2} u_h \right) \right\|_{L^2(K)}^2$$

$$\eta_j^2 := \left\| h^{1/2} \left\| \nabla u_h \right\|_{L^2(e)}^2 + \left\| h^{-1/2} \left\| u_h \right\|_{L^2(e)}^2 \right\|_{L^2(e)}^2$$

Proof. It suffices to determine an upper bound for $\|\mathcal{R}\|_{H^{-1}(\Omega)}$. To that end, by Proposition 5.3

$$\langle \mathcal{R}, v \rangle = \mathcal{A}(u_h - \mathcal{E}(u_h), v) + \mathcal{B}(u_h; u_h, v) - \mathcal{B}(\mathcal{E}(u_h); \mathcal{E}(u_h), v)$$

$$+ \langle f, v - v_h \rangle - \mathcal{A}(u_h, v - v_h) - \mathcal{B}(u_h; u_h, v - v_h).$$
and we proceed to bound the terms individually. Firstly,

\[
\mathcal{S}_1 = \sum_{K \in \mathcal{F}} \int_K (\nabla u_h - \nabla \delta'(u_h)) \cdot \nabla v \, dx - \sum_{e \in \Gamma} [u_h] \{ P_{k-1}(\nabla v) \} \, ds \\
\leq \sum_{K \in \mathcal{F}} \| \nabla u_h - \nabla \delta'(u_h) \|_{L^2(K)} \| \nabla v \|_{L^2(K)} + \sum_{e \in \Gamma} \left\| h^{-1/2} [u_h] \right\|_{L^2(e)} \left\| h^{1/2} \{ P_{k-1}(\nabla v) \} \right\|_{L^2(e)} \leq (C_1^{1/2} + C_{\text{dim,F}}^{1/2} C_{\text{trace}}^{1/2}) \left( \sum_{K \in \mathcal{F}} \left\| h^{-1/2} [u_h] \right\|_{L^2(K)}^2 \right)^{1/2} \left( \sum_{K \in \mathcal{F}} \| \nabla v \|_{L^2(K)}^2 \right)^{1/2} \leq C \left( \sum_{K \in \mathcal{F}} \sum_{e \in \partial K} \eta^2_e \right)^{1/2} \| \nabla v \|_{L^2(\Omega)},
\]

where \( C_{\text{dim,F}}^{1/2} \) is a constant depending on the dimension and the triangulation and \( C_{\text{trace}} \) is the constant from a trace estimate. The second term can be controlled by

\[
\mathcal{S}_2 = \int_\Omega \left( |u_h|^{p-2} u_h - |\delta'(u_h)|^{p-2} \delta'(u_h) \right) v \\
\leq C(u_h, \delta'(u_h), p) \sum_{K \in \mathcal{F}} \| u_h - \delta'(u_h) \|_{L^2(K)} \| v \|_{L^2(K)}.
\]

Similarly, for \( K \in \mathcal{F} \),

\[
\mathcal{S}_3 = \int_{\Omega} f(v - v_h) - \nabla u_h \cdot (\nabla v - \nabla v_h) - |u_h|^{p-2} u_h (v - v_h) \, dx + \int_{\Gamma} [u_h] \cdot \{ P_{k-1}(\nabla v - \nabla v_h) \} \, ds \\
+ \int_{\Gamma} [v - v_h] \{ \nabla u_h \} \cdot \sigma [u_h] \cdot [v - v_h] \, ds \\
= \int_{\Omega} (f + \Delta u_h - |u_h|^{p-2} u_h) (v - v_h) \, dx - \int_{\Gamma} [\nabla u_h] \{ v - v_h \} \, ds \\
+ \int_{\Gamma} [u_h] \cdot \{ P_{k-1}(\nabla v - \nabla v_h) \} \cdot \sigma [u_h] \cdot [v - v_h] \, ds.
\]

Splitting the integrals elementwise and making use of the Cauchy-Schwarz inequality we see

\[
\mathcal{S}_3 \leq \sum_{K \in \mathcal{F}} \left\| h \left( f + \Delta u_h - |u_h|^{p-2} u_h \right) \right\|_{L^2(K)} \left\| h^{-1} (v - v_h) \right\|_{L^2(K)}
\]

Similarly, for the second,

\[
- \sum_{e \in \Gamma} \int_e [\nabla u_h] \{ v - v_h \} \, ds \leq \sum_{K \in \mathcal{F}} \sum_{e \in \partial K} h^{1/2} \| \nabla u_h \|_{L^2(e)} \| h^{-1/2} \{ v - v_h \} \|_{L^2(e)} \| h^{1/2} \{ P_{k-1}(\nabla v - \nabla v_h) \} \|_{L^2(e)} \]

and third term

\[
\sum_{e \in \Gamma} \int_e [u_h] \cdot \{ P_{k-1}(\nabla v - \nabla v_h) \} \, ds \leq \sum_{K \in \mathcal{F}} \sum_{e \in \partial K} h^{1/2} \| u_h \|_{L^2(e)} \| h^{1/2} \{ P_{k-1}(\nabla v - \nabla v_h) \} \|_{L^2(e)} \]
For the final term
\begin{equation}
(5.17) \sum_{e \in \mathcal{T}} \int_{e} \sigma \left[ u_h \cdot (v - v_h) \right] \, ds \leq C_{\sigma} \sum_{K \in \mathcal{T}} \sum_{e \subset \partial K} \left\| h^{-1/2} u_h \right\|_{L^2(e)} \left\| h^{-1/2} (v - v_h) \right\|_{L^2(e)}
\end{equation}
Collecting (5.13)–(5.17) we have
\begin{equation}
(5.18) \mathcal{A}_3 \leq C \left( \sum_{K \in \mathcal{T}} \left[ \eta_R + \sum_{e \subset \partial K} \eta_j \right] \Phi(v - v_h) \right),
\end{equation}
where
\begin{equation}
(5.19) \Phi(w) = \max \left( \left\| h^{-1} w \right\|_{L^2(K)}, \max_{e \subset \Gamma} \left\| h^{-1/2} w \right\|_{L^2(e)}, \max_{e \subset \Gamma} \left\| h^{1/2} \left\{ P_{k-1}(\nabla w) \right\} \right\|_{L^2(e)}, \max_{e \subset \Gamma} \left\| h^{-1/2} \left\| w \right\|_{L^2(e)} \right) \right).
\end{equation}
Choosing \( v_h = P_0 v \) and in view of the approximation properties and stability of the \( L^2 \) projector we have that
\begin{equation}
(5.20) \Phi(v - v_h) \leq C \left\| \nabla v \right\|_{L^2(\hat{K})},
\end{equation}
where \( \hat{K} \) denotes the patch of \( K \). Using a discrete Cauchy-Schwarz inequality
\begin{equation}
(5.21) \mathcal{A}_3 \leq C \left( \sum_{K \in \mathcal{T}} \left[ \eta_R^2 + \sum_{e \subset \partial K} \eta_j^2 \right] \right)^{1/2} \left( \sum_{K \in \mathcal{T}} \left\| \nabla v \right\|_{L^2(\hat{K})}^2 \right)^{1/2}
\end{equation}
hence, making use of (5.11) and (5.12), we have
\begin{equation}
(5.22) \langle \mathcal{R}, v \rangle = \mathcal{A}_1 + \mathcal{A}_2 + \mathcal{A}_3 \leq C \left( \sum_{K \in \mathcal{T}} \left[ \eta_R + \sum_{e \subset \partial K} \eta_j^2 \right] \right)^{1/2} \left\| \nabla v \right\|_{L^2(\Omega)},
\end{equation}
where the constant \( C \) depends only upon the above regularity of the mesh, \( p \) and \( u_h \). The result follows by dividing through by \( \left\| \nabla v \right\|_{L^2(\Omega)} \) and taking the supremum over all possible \( 0 \neq v \in H_0^1(\Omega) \). \( \square \)

5.5. **Corollary.** Making use of the triangle inequality, one may show under the conditions of Theorem 5.4 the following result holds:
\begin{equation}
(5.23) \left\| u - u_h \right\|_{dG}^2 + 2C_L \left\| u - u_h \right\|_{L^p(\Omega)}^p \leq C \sum_{K \in \mathcal{T}} \left[ \eta_R^2 + \sum_{e \subset \partial K} \eta_j^2 \right].
\end{equation}

6. Numerical experiments

We now illustrate the performance of the scheme through a series of numerical experiments.

6.1. **Test 1 – Asymptotic behaviour approximating a smooth solution.** As a first test, we consider the domain \( \Omega = [0, 1]^2 \). We fix \( f \) such that the exact solution is given by
\begin{equation}
(6.1) u(x, y) = \sin (\pi x) \sin (\pi y),
\end{equation}
and approximate \( \Omega \) through a uniformly generated, criss-cross triangular type mesh to test the asymptotic behaviour of the numerical approximation. The results are summarised in Figure 1 (A) – (D), and confirm the theoretical findings in Sections 4 and 5.

More specifically, we consider the case \( k = 1, 2, p = 4, 8 \) and show that convergence measured the \( L^p \) norm, the \( (u, p) \) quasinorm and the dG norm are all optimal. Notice that the fact the \( L^p \) norm is optimal is contrary to the analysis. This is a well known fact [Pry18, KP18]. In addition, the a posteriori estimator is of optimal rate with an effectivity index of just under 10.
6.2. Test 2 – Behaviour of an adaptive scheme for various values of $p$. We consider the domain $\Omega = [0, 1]^2$ and fix $f = 1000$ in this case there is no known solution. However, examining the energy functional (1.3) one can see that a minimiser has to 'balance' the $L^2$ norm of its derivative with the $L^p$ norm of the function. For large $p$ this almost translates into control of the $\text{ess sup}$ which causes boundary layers to appear.

We approximate $\Omega$ through a uniformly generated, criss-cross triangular type initial mesh consisting of 4 elements. We run an adaptive algorithm of SOLVE, ESTIMATE, MARK, REFINE type, where SOLVE consists of solving the formulation (2.11), ESTIMATE is done through the evaluation of the estimator given in Corollary 5.5, MARK is a maximum strategy with 50% of the elements marked for refinement at each iteration and REFINE is a newest vertex bisection.

The results are summarised in Figure 2 (A) – (D) where we consider the case $k = 1, p = 4, 8, 12, 16, 20$ and examine the solution and underlying adaptive mesh.

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Figure 2. Solutions of the dG scheme (2.11) for Test 2 and the underlying meshes generated through the adaptive algorithm.

(a) $p = 2$. The mesh consists of 924,085 elements after 13 iterations.

(b) $p = 4$. The mesh consists of 361,680 elements after 13 iterations.

(c) $p = 8$. The mesh consists of 339,392 elements after 13 iterations.

(d) $p = 12$. The mesh consists of 340,176 elements after 13 iterations.

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