GRAVITATIONAL COLLAPSE WITH DARK ENERGY AND DARK MATTER IN HOŘAVA-LIFSHITZ GRAVITY

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In this work, the collapsing process of a spherically symmetric star, made of dust cloud, is studied in Hořava Lifshitz gravity in the background of Chaplygin gas dark energy. Two different classes of Chaplygin gas, namely, New variable modified Chaplygin gas and generalized cosmic Chaplygin gas are considered for the collapse study. Graphs are drawn to characterize the nature and to determine the possible outcome of gravitational collapse. A comparative study is done between the collapsing process in the two different dark energy models. It is found that for open and closed universe, collapse proceeds with an increase in black hole mass, the only constraint being that, relatively smaller values of $\Lambda$ has to be considered in comparison to $\lambda$. But in case of flat universe, possibility of the star undergoing a collapse in highly unlikely. Moreover it is seen that the most favourable environment for collapse is achieved when a combination of dark energy and dark matter is considered, both in the presence and absence of interaction. Finally, it is to be seen that, contrary to our expectations, the presence of dark energy does not really hinder the collapsing process in case of Hořava-Lifshitz gravity.

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I. INTRODUCTION

Cosmology has been a subject that has always attracted human mind and as a result it has been an ever-developing field. Extensive research has been carried out both theoretical and experimental right from its early days. The most remarkable and significant discovery in recent past in the field of cosmology is the discovery of the fact, that our universe is undergoing an accelerated expansion \[1, 2\]. This event has shaken the traditional theories of cosmology right from their roots. So Einstein’s equation needed some serious revisions in order to account for the observed cosmic acceleration.

As a result modified gravity theories came into existence. Loop quantum gravity, brane-gravity, Hořava-Lifshitz gravity \[3–7\], etc. are some of the modified gravity theories that was developed in the recent past. In 2009, a new four dimensional gravity theory was proposed by Hořava. The theory is devoid of full diffeomorphism invariance but it has UV completeness. As a matter of fact, the theory has three dimensional general covariance and time re-parameterization invariance. In fact it is a non-relativistic renormalizable quantum gravity theory possessing higher spatial derivatives.

Some cosmologists thought that the nature of the content of the universe is more responsible than its geometry for the recent cosmic acceleration. As a result the concept of dark energy (DE) \[8\] was developed. It is basically a mysterious negative pressure component which violates the strong energy condition i.e. \(\rho + 3p < 0\), thus causing accelerated expansion. Till date numerous DE models have been proposed. Chaplygin gas model is one such form of DE that has gained enormous popularity in the cosmological society. Many Chaplygin gas models have appeared in the scene, with extensive research. The earliest form was pure Chaplygin gas (CG) \[9, 10\], which got modified into generalized Chaplygin gas (GCG) \[11–15\] and then into modified Chaplygin gas (MCG) \[16, 17\]. As time passed variable Chaplygin gas (VMCG) \[18\] and new variable modified Chaplygin gas (NVMCG) \[19\] came into existence. In 2003, Gonzalez-Diaz \[20\] introduced the generalized cosmic Chaplygin gas (GCCG) model \[21\]. The speciality of the model being that it can be made stable and free from unphysical behaviours even when the vacuum fluid satisfies the phantom energy condition. Another interesting feature of this model is that it does not drive the universe towards the Big-Rip singularity, unlike the other models of Chaplygin gas.
The Equation of state (EoS) of NVMCG is given by \[19\],

\[ p = A(a)\rho - B(a)\frac{\rho^\alpha}{\rho^\alpha}, \quad 0 \leq \alpha \leq 1 \] (1)

Here we consider \( A(a) = A_0 a^{-n} \) and \( B(a) = B_0 a^{-m} \), where \( A_0, B_0, \alpha, m \) and \( n \) are positive constants. The motivation of the above choices has been discussed in ref \[19\].

The Equation of state (EoS) of the GCCG model is \[20, 21\]

\[ p = -\rho - \alpha\left[ C + \left\{ \rho^{(1+\alpha)} - C \right\}^{-\omega} \right] \] (2)

where \( C = \frac{A}{1+\omega} - 1 \), with \( A \) being a constant that can take on both positive and negative values, and \(-l < \omega < 0\), \( l \) being a positive definite constant, which can take on values larger than unity. GCCG can explain the evolution of the universe starting from the dust era to \( \Lambda CDM \), radiation era, matter dominated quintessence and lastly phantom era.

Recently cosmic coincidence problem and fine tuning problem have crippled many dark energy models. In this regard interacting DE models have been proposed \[6, 22–29\]. The interacting models presents a scenario of co-existence of DE and DM thus explaining the present day universe in a far effective way.

Gravitational collapse is one of the most important problem in classical general relativity for decades. The study of gravitational collapse began with the pioneering work of Oppenheimer and Snyder in 1939 \[30\], where they studied the gravitational collapse of dust. From Penrose’s Cosmic censorship hypothesis (CCH) \[31\], we are made to believe that any forms of gravitational collapse will resulting in a singularity is destined to form Black holes. But the inquisitive mind would like to know whether, and under what initial conditions, gravitational collapse results in black hole (BH) formation. Moreover, one would like to know if there are physical collapse solutions that lead to naked singularities (NS), thus violating the CCH. In last few years, there have been extensive studies on gravitational collapse in order to investigate the nature of the singularities \[32–35\].

The collapsing process of a spherically symmetric star, made of dust cloud, in the background of dark energy was studied for Einstein’s gravity \[36–38\], RSII Brane world model \[39\], DGP Brane gravity and Loop Quantum gravity \[40\]. It was found that the presence of DE hinders the collapsing procedure upto certain extent. Motivated by these works, we intend to study the nature and outcome of gravitational collapse of a star made up of DM in the background of DE (of different forms), in Hořava-Lifshitz gravity \[41–45\].

The paper is organized as follows: We study the general formulation of the collapsing process in section 2. Section 3 is dedicated to the study of gravitational collapse in Hořava-Lifshitz gravity.
Collapse of dark matter and dark energy with and without interactions are discussed. Two types of dark energy i.e., new variable modified Chaplygin gas and generalized cosmic Chaplygin gas are considered for study of collapse. Section 4 deals with the detailed graphical analysis of the plots generated. Finally the paper ends with a conclusion in section 5.

II. GENERAL FORMULATION OF THE COLLAPSING PROCESS

The flat, homogeneous and isotropic FRW model of the universe is described by the line element

\[ ds^2 = dt^2 + a^2(t) \left[ dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right] \]  

(3)

The energy conservation equation is given by

\[ \dot{\rho}_T + 3 \frac{\dot{a}}{a} (\rho_T + p_T) = 0 \]  

(4)

with total density and pressure, \( \rho_T = \rho_M + \rho_E \) and \( p_T = p_M + p_E \).

The interaction \( Q(t) \) between DM and DE can be expressed as

\[ \dot{\rho}_M + 3 \frac{\dot{a}}{a} \rho_M = Q \]  

(5)

\[ \dot{\rho}_E + 3 \frac{\dot{a}}{a} (\rho_E + p_E) = -Q \]  

(6)

Now, if we consider gravitational collapse of a spherical cloud consists of above DM and DE distribution and is bounded by the surface \( \Sigma : r = r_\Sigma \) then the metric on it can be written as

\[ ds^2 = dT^2 - R^2(T) \{d\theta^2 + \sin^2 \theta d\phi^2\} \]  

(7)

Thus on \( \Sigma : T = t \) and \( R(T) = r_\Sigma a(T) \) where \( R(r, t) \equiv ra(t) \) is the geometrical radius of the two spheres \( t, r = \text{constant} \). Also the total mass of the collapsing cloud is given by

\[ M(T) = m(r, t) \bigg|_{r=r_\Sigma} = \frac{1}{2} r^3 a^2 \bigg|_{\Sigma} = \frac{1}{2} R(T) \dot{R}^2(T) \]  

(8)

The apparent horizon is defined as

\[ R_{\alpha\beta} R^{\alpha\beta} = 0, \ i.e., \ r^2 \dot{a}^2 = 1 \]  

(9)

So if \( T = T_{AH} \) be the time when the whole cloud starts to be trapped then

\[ \dot{R}^2(T_{AH}) \bigg|_{\Sigma} = r_\Sigma^2 \dot{a}^2(T_{AH}) = 1 \]  

(10)
As it is usually assumed that the collapsing process starts from regular initial data so initially at \( t = t_i \) \((< T_{AH})\), the cloud is not trapped i.e.,

\[
\mathcal{r}_\Sigma^2 \dot{a}^2(t_i) < 1, \quad (\mathcal{r}_\Sigma \dot{a}(t_i) > -1)
\]  

(11)

Thus if equation (10) has any real solution for \( T_{AH} \) satisfying (11) then black hole (BH) will form, otherwise the collapsing process leads to a naked singularity (NS). So the gravitational collapse and consequently the formation of a BH solely depends upon the nature of root obtained from equation (10). If any real solution for \( T_{AH} \) exists for equation (10) then apparent horizon will be formed and thus a BH. If there is no real solution the collapse is destined to result in a NS.

III. GRAVITATIONAL COLLAPSE IN HÓRAVA-LIFSHITZ GRAVITY

Here we briefly review the scenario where the cosmological evolution is governed by Hořava-Lifshitz (HL) gravity. The dynamical variables are the lapse and shift functions, \( N \) and \( N_i \) respectively, and the spatial metric \( g_{ij} \). In terms of these fields the full metric is written as \[46, 47\]

\[
ds^2 = -N^2 dt^2 + g_{ij} (dx^i + N^i dt) (dx^j + N^j dt)
\]

(12)

where the indices are raised and lowered using \( g_{ij} \). The scaling transformation of the coordinates reads: \( t \rightarrow \ell^3 t \) and \( x^i \rightarrow \ell x^i \).

The action of HL gravity is given by \[48\]

\[
I = dt \int d^3x \left( L_0 + L_1 + L_m \right)
\]

\[
L_0 = \sqrt{g} N \left[ \frac{2}{\kappa^2} (K_{ij} K^{ij} - \lambda K^2) + \frac{\kappa^2 \mu^2}{8 (1 - 3\lambda)} \left( \Lambda R - \frac{3}{2} \Lambda^2 \right) \right]
\]

\[
L_1 = \sqrt{g} N \left[ \kappa^2 \mu^2 \frac{(1 - 4\lambda)}{32 (1 - 3\lambda)} R^2 - \frac{\kappa^2}{2 \omega^4} \left( C_{ij} - \frac{\mu \omega^2}{2} R_{ij} \right) \left( C^{ij} - \frac{\mu \omega^2}{2} R^{ij} \right) \right]
\]

(13)

where \( \kappa^2, \lambda, \mu, \omega \) and \( \Lambda \) are constant parameters, and \( C_{ij} \) is Cotton tensor (conserved and traceless, vanishing for conformally flat metrics). The first two terms in \( L_0 \) are the kinetic terms, others in \((L_0 + L_1)\) give the potential of the theory in the so-called “detailed-balance” form, and \( L_m \) stands for the Lagrangian of other matter field. Comparing the action to that of the general relativity, one can see that the speed of light and the cosmological Newton’s constant are
\[ c = \frac{\kappa^2 \mu}{4} \sqrt{\frac{\Lambda}{1 - 3\lambda}}, \quad G_c = \frac{\kappa^2 c}{16\pi (3\lambda - 1)} \] (14)

It should be noted that when \( \lambda = 1 \), \( L_0 \) reduces to the usual Lagrangian of Einstein’s general relativity. Thus when \( \lambda = 1 \), the general relativity is approximately recovered at large distances.

The field equations are

\[ H^2 + \frac{k}{a^2} = \frac{8\pi G_c}{3} (\rho_T + \rho_D) \] (15)

and

\[ \dot{H} + \frac{3}{2} H^2 + \frac{k}{2a^2} = -4\pi G_c (\rho_T + p_D) \] (16)

where

\[ \rho_D = \frac{3\kappa^2 \mu^2 k^2}{8(3\lambda - 1)a^4} + \frac{3\kappa^2 \mu^2 \Lambda^2}{8(3\lambda - 1)} = \frac{1}{16\pi G_c} \left( \frac{3k^2}{\Lambda a^4} + 3\Lambda \right) \] (17)

and

\[ p_D = \frac{\kappa^2 \mu^2 k^2}{8(3\lambda - 1)a^4} - \frac{3\kappa^2 \mu^2 \Lambda^2}{8(3\lambda - 1)} = \frac{1}{16\pi G_c} \left( \frac{k^2}{\Lambda a^4} - 3\Lambda \right) \] (18)

Using eqns. (12), (15) and (16) in (13) and (14), we get,

\[ H^2 + \frac{k}{a^2} = \frac{k^2 c}{6(3\lambda - 1)} \rho_T + \frac{1}{2} \left( \frac{k^2}{\Lambda a^4} + \Lambda \right) \] (19)

and

\[ \dot{H} + \frac{3}{2} H^2 + \frac{k}{2a^2} = -\frac{k^2 c}{4(3\lambda - 1)} \rho_T - \frac{1}{4} \left( \frac{k^2}{\Lambda a^4} - 3\Lambda \right) \] (20)

In the following subsections, we shall discussed the collapse of dark matter and dark energy in the form of new variable modified Chaplygin gas and generalized cosmic Chaplygin gas separately and also combination of dark matter and dark energy with and without interactions.
A. Collapse with Dark Matter

Here $\rho_M \neq 0$, $p_E = \rho_E = 0$. From the conservation equation (3), we get

$$\rho_M = \frac{C_0}{a^3}$$

(21)

Now using this relation in eqn.(17), we have,

$$\dot{a} = \sqrt{\frac{k^2 c C_0}{6a (3\lambda - 1)} + \frac{1}{2} \left( \frac{k^2}{\Lambda a^4} + \Lambda \right) a^2 - k}$$

(22)

Therefore the expressions for the time gradient of the geometrical radius of the collapsing cloud becomes,

$$\dot{R}(T) = r \Sigma \sqrt{\frac{k^2 c C_0}{6a (3\lambda - 1)} + \frac{1}{2} \left( \frac{k^2}{\Lambda a^4} + \Lambda \right) a^2 - k}$$

(23)

The mass of the collapsing cloud is given by

$$M(T) = \frac{1}{2} r \Sigma \left[ \frac{k^2 c C_0}{6(3\lambda - 1)} + \frac{1}{2} \left( \frac{k^2}{\Lambda a^4} + \Lambda \right) a^3 - ka \right]$$

(24)

From the above solutions it is evident that as $T \rightarrow \infty$, $a \rightarrow \infty$, $\rho_M \rightarrow 0$, $\dot{R}(T) \rightarrow 0$, $M(T) \rightarrow 0$. So we see that there is a tendency of matter density being diminished as time passes, and finally it tends towards zero. The time for formation of apparent horizon is given by the real root of the equation, $\dot{R}^2(T_{AH}) |_{\Sigma} = r_\Sigma^2 \dot{a}^2(T_{AH}) = 1$ as given by equation (10). Thus the corresponding expression for HL gravity is,

$$r_\Sigma^2 \left[ \frac{k^2 c C_0}{6(3\lambda - 1)} + \frac{1}{2} \left( \frac{k^2}{\Lambda a^4} + \Lambda \right) a^2 - k \right] = 0$$

(25)

B. Collapse with Dark Energy in the form of New Variable Modified Chaplygin Gas

Here DE in the form of NVMCG is considered. So, $\rho_M = 0$, $p = A(a)\rho - \frac{B(a)}{\rho^n}$, $0 \leq \alpha \leq 1$ as given in equation (1). The solution for density of NVMCG is obtained as [19],

$$\rho_{nvmcg} = a^{-3} \exp \left( \frac{3A_0 a^{-n}}{n} \right) \left[ D_0 + \frac{B_0}{A_0} \left( \frac{3A_0 (1 + \alpha)}{n} \right)^{\frac{3(1 + \alpha) + n - m}{n}} \times \Gamma \left( \frac{m - 3(1 + \alpha)}{n} \right) \right]^{\frac{1}{1 + \alpha}}$$

(26)
Fig. 1: The time derivative of the radius is plotted against time for open universe (Red), flat universe (Blue) and closed universe (Black). The other parameters are considered as
\[ \Lambda = 10, \ r = 10, \ \lambda = 100000, \ c = 10, \ C_0 = 10. \]

Fig 2: The mass of the collapsing cloud is plotted against time for open universe (Red), flat universe (Blue) and closed universe (Black). The other parameters are considered as
\[ \Lambda = 10, \ r = 10, \ \lambda = 100000, \ c = 10, \ C_0 = 10. \]

where \( D_0 \) is the integration constant and \( \Gamma(s, t) \) is the upper incomplete gamma function.

The expressions for relevant physical quantities are

\[
\dot{R}(T) = -r_\Sigma \sqrt{\frac{k^2 c a^2 \rho_{nvmcg}}{6(3\lambda - 1)}} + \frac{1}{2} a^2 \left( \frac{k^2}{a^4 + \Lambda} \right) - k \tag{27}
\]

and

\[
M(T) = \frac{1}{2} a r_\Sigma^3 \left[ \frac{k^2 c a^2 \rho_{nvmcg}}{6(3\lambda - 1)} + \frac{1}{2} a^2 \left( \frac{k^2}{a^4 + \Lambda} \right) - k \right] \tag{28}
\]

The limiting values of the physical parameters are as follows:

**Case1**

When \( a \to 0 \) : \( \rho_{nvmcg} \to \infty \), \( \dot{R} \to -\infty \), \( M(T) \to \infty \). Obviously, \( \frac{k^2 c a^2 \rho_{nvmcg}}{6(3\lambda - 1)} + \frac{1}{2} a^2 \left( \frac{k^2}{a^4 + \Lambda} \right) \geq k_{max} = 1. \)

**Case2**

When \( a \to \infty \) : \( \rho_{nvmcg} \to 0 \), \( \dot{R} \to -\infty \), \( M(T) \to \infty \).
Fig 3: The time derivative of the radius is plotted against time for open universe (Red), flat universe (Blue) and closed universe (Black). The other parameters are considered as 
\[ \Lambda = 1000, \ r = 10, \ \lambda = 1000000, \ c = 10, \ A_0 = 1/3, B_0 = 3, D_0 = 10, \alpha = 1/2, n = 1, m = 1. \]

Fig 4: The mass of the collapsing cloud is plotted against time for open universe (Red), flat universe (Blue) and closed universe (Black). The other parameters are considered as 
\[ \Lambda = 1000, \ r = 10, \ \lambda = 1000000, \ c = 10, \ A_0 = 1/3, B_0 = 3, D_0 = 10, \alpha = 1/2, n = 1, m = 1. \]

The cloud will start untrapped at the instant given by the real roots of the following equation and gradually start to be trapped,

\[ r^2_\Sigma \left[ \frac{k^2c a^2 \rho_{nvmc}}{6(3\lambda - 1)} + \frac{1}{2} a^2 \left( \frac{k^2}{\Lambda a^4} + \Lambda \right) - k \right] = 1 \]  
\[ (29) \]

C. Effect of a combination of dark matter and New variable modified Chaplygin gas

1. Case I : \( Q = 0 \) i.e., No Interaction Between Dark Matter And Dark Energy:

In this case we consider the DE and DM to co-exist in a non-interacting scenario. The energy density of DM is given by eqn.(20) and the energy density of DE is given by eqn.(25). Therefore in this case the total energy can be considered as the sum total of the energy densities of DE and DM, i.e. \( \rho_T = \rho_{nvmc} + \rho_M \). Using this in the field equation for HL gravity, i.e. eqn.(17),we get the expressions of the relevant parameters as given below.

The time gradient of the geometrical radius of the collapsing star is given by,

\[ \dot{R}(T) = -r_\Sigma \sqrt{\frac{k^2c a^2}{6(3\lambda - 1)}} \left( \rho_{nvmc} + \frac{C_0}{a^4} \right) + \frac{1}{2} a^2 \left( \frac{k^2}{\Lambda a^4} + \Lambda \right) - k \]  
\[ (30) \]
Fig 5: The time derivative of the radius is plotted against time for open universe (Red), flat universe (Blue) and closed universe (Black). The other parameters are considered as
\[ \Lambda = 100, \quad r = 10, \quad \lambda = 100000, \quad c = 10, \quad A_0 = 1/3, \quad B_0 = 3, \quad C_0 = 5, \quad D_0 = 10, \quad a = 1/2, \quad n = 1, \quad m = 1. \]

Fig 6: The mass of the collapsing cloud is plotted against time for open universe (Red), flat universe (Blue) and closed universe (Black). The other parameters are considered as
\[ \Lambda = 100, \quad r = 10, \quad \lambda = 100000, \quad c = 10, \quad A_0 = 1/3, \quad B_0 = 3, \quad C_0 = 5, \quad D_0 = 10, \quad a = 1/2, \quad n = 1, \quad m = 1. \]

and

\[ M(T) = \frac{1}{2} a r^{3/2} \left[ \frac{k^2 c a^2}{6 (3\lambda - 1)} \left( \rho_{nvmcg} + \frac{C_0}{a^3} \right) + \frac{1}{2} a^2 \left( \frac{k^2}{\Lambda a^4} + \Lambda \right) - k \right] \] (31)

In this case the limiting values of the physical parameters are given as follows,

**Case 1**

When \( a \to 0 \): \( \rho_{nvmcg} \to \infty \), \( \dot{R} \to -\infty \), \( M(T) \to \infty \). Obviously, \( \frac{k^2 c a^2}{6 (3\lambda - 1)} \left( \rho_{nvmcg} + \frac{C_0}{a^3} \right) + \frac{1}{2} a^2 \left( \frac{k^2}{\Lambda a^4} + \Lambda \right) \geq k_{max} = 1 \)

**Case 2**

When \( a \to \infty \): \( \rho_{nvmcg} \to 0 \), \( \dot{R} \to -\infty \), \( M(T) \to \infty \).

The cloud will start untrapped at the instant given by the real roots of the following equation and gradually start to be trapped,

\[ r^2 \left[ \frac{k^2 c a^2}{6 (3\lambda - 1)} \left( \rho_{nvmcg} + \frac{C_0}{a^3} \right) + \frac{1}{2} a^2 \left( \frac{k^2}{\Lambda a^4} + \Lambda \right) - k \right] = 1 \] (32)
2. Case II: $Q \neq 0$ i.e., Interaction Between Dark Matter And Dark Energy:

Here we will use the assumption given by Cai and Wang in [36, 37]. We consider

$$\frac{\rho_{\text{numcg}}}{\rho_M} = C a^{3n'}$$  \hspace{1cm} (33)

where $C > 0$ and $n'$ are arbitrary constants. We solve the conservation equations (3) and (4) and get the following expression for $\rho_T$ where $\rho_T = \rho_{\text{numcg}} + \rho_M$,

$$\rho_T = \exp \left( \frac{-a^{-n}}{n'(3n' - n)} \left( a^n (n - 3n') \log \left( \frac{C + a^{-3n'}}{C (1 + Ca^{3n'})} \right) + 3A n'Ca^{3n'} F_1 \left( 1 - \frac{n}{3n'}, 1, 2 - \frac{n}{3n'}, -Ca^{3n'} \right) \right) \right)$$

$$\left[ C_1 + \int_1^t - \frac{1}{1 + Ca(u)^{3n'}} 3C^{-\alpha} \exp \left( \frac{a(u)^{-n}}{n'(3n' - n)} \left( a(u)^n (n - 3n') \log \left( \frac{C + a(u)^{-3n'}}{C (1 + Ca(u)^{3n'})} \right) + 3ACn'a(u)^{3n'} F_1 \left( 1 - \frac{n}{3n'}, 1, 2 - \frac{n}{3n'}, -Ca^{3n'} \right) \right) \right] \right]$$

where $2F_1$ is the hypergeometric function and $C_1$ is the integration constant. Using the relation $\rho_T = \rho_{\text{numcg}} + \rho_M$. The expression for interaction is given by,

$$Q = \frac{-3\rho_T}{(1 + C a^{3n'})^2} \left[ 1 + Ca^{3n'} + A_0Ca^{n+3n'} - B_0C^{-\alpha}a^{m-3n'\alpha} \left( \frac{1 + Ca^{3n'}}{\rho_T} \right)^{\alpha+1} + Cn'a^{3n'} \right] \times$$

$$\sqrt{\frac{k^2c\rho_T}{6(3\lambda - 1)} + \frac{1}{2} \left( \frac{k^2}{\Lambda a^4 + \Lambda} - \frac{k}{a^2} \right)} + 3 \sqrt{\frac{k^2c\rho_T}{6(3\lambda - 1)} + \frac{1}{2} \left( \frac{k^2}{\Lambda a^4 + \Lambda} - \frac{k}{a^2} \left( \frac{\rho_T}{1 + Ca^{3n'}} \right) \right)}$$  \hspace{1cm} (35)

Using the relation $\rho_T = \rho_{\text{MCG}} + \rho_M$ we get

$$\rho_{\text{numcg}} = \frac{Ca^{3n'}\rho_T}{1 + Ca^{3n'}}$$  \hspace{1cm} (36)

The gradient of scale factor is given by,

$$\dot{a} = -\sqrt{\frac{k^2ca^2}{6(3\lambda - 1)} + \frac{a^2}{2} \left( \frac{k^2}{\Lambda a^4 + \Lambda} - k \right)}$$  \hspace{1cm} (37)

where $\rho_T$ is given by eqn.(33).
The corresponding expressions for the time derivative of geometrical radius is given by,

\[ \dot{R} = -r \sqrt{\frac{k^2 c a^2}{6 (3\lambda - 1)} \rho_T + \frac{a^2}{2} \left( \frac{k^2}{\Lambda a^4} + \Lambda \right) - k} \]  

(38)

The expression for the mass is given by,

\[ M(T) = \frac{1}{2} a^2 r^2 \left[ \frac{k^2 c a^2}{6 (3\lambda - 1)} \rho_T + \frac{a^2}{2} \left( \frac{k^2}{\Lambda a^4} + \Lambda \right) - k \right] \]

(39)

where \( \rho_T \) is given by eqn.(33).

In this case the limiting values of the physical parameters are given as follows,

**Case1**

When \( a \to 0 \): \( \rho_T \to 0, \quad \dot{R} \to -\infty, \quad M(T) \to \frac{1}{4} \frac{k^2}{\Sigma} \). Obviously, \( \frac{k^2 c a^2}{6 (3\lambda - 1)} (\rho_T) + \frac{1}{2} a^2 \left( \frac{k^2}{\Lambda a^4} + \Lambda \right) \geq k_{max} = 1 \).

**Case2**

When \( a \to \infty \): \( \rho_{nec} \to 0, \quad \dot{R} \to -\infty, \quad M(T) \to \infty \).

The cloud will start untrapped at the instant given by the real roots of the following equation and gradually start to be trapped,

\[ r^2 \left[ \frac{k^2 c a^2}{6 (3\lambda - 1)} (\rho_T) + \frac{1}{2} a^2 \left( \frac{k^2}{\Lambda a^4} + \Lambda \right) - k \right] = 1 \]  

(40)

**D. Collapse with Dark Energy in the form of Generalized cosmic Chaplygin Gas**

Here DE in the form of GCCG is considered. So, \( \rho_M = 0, \quad p = -\rho_{GCCG}^{\alpha} \left[ C' + \left( \rho_{GCCG}^{1+\alpha} - C' \right)^{-\omega} \right] \) as given in equation (2) . The solution for density is given by \[ 21 \]

\[ \rho_{GCCG} = \left[ C' + \left\{ 1 + \frac{B'}{a^3(1+\alpha)(1+\omega)} \right\} \right]^{\frac{1}{1+\omega}}, \quad B' is the integration constant. \]  

(41)

The expressions for the other physical quantities are given below,

\[ \dot{R}(T) = -r \sqrt{\frac{k^2 c a^2 \left[ C' + \left\{ 1 + \frac{B'}{a^3(1+\alpha)(1+\omega)} \right\} \right]^{\frac{1}{1+\omega}}}{6 (3\lambda - 1)}} + \frac{1}{2} a^2 \left( \frac{k^2}{\Lambda a^4} + \Lambda \right) - k \]  

(42)
Fig 7: The time derivative of the radius is plotted against time for open universe (Red), flat universe (Blue) and closed universe (Black). The other parameters are considered as
\[ \Lambda = 1000, \; r = 10, \; \lambda = 100000000, \; c = 10, \; C = 10, \; A_0 = 1/3, \; B_0 = 3, \; \alpha = 1/2, \; n = 1, \; m = 1, \; n' = 2. \]

and

\[ M(T) = \frac{1}{2} a r_0^3 \left[ \frac{k^2 c a^2 \left[ C' + \left\{ 1 + \frac{B'}{a^{3(1+\alpha)(1+\omega)}} \right\} \frac{1}{1+\omega} \right]^{\frac{1}{1+\alpha}}}{6(3\lambda - 1)} + \frac{1}{2} a^2 \left( \frac{k^2}{\Lambda a^4} + \Lambda \right) - k \right] \] (43)

The limiting values of the physical parameters are as follows:

**Case 1**

When \( a \to 0 \): \( a \to 0 \): \( \rho_{GCCG} \to \infty \), for \( 1 + \omega > 0 \); \( \rho_{GCCG} \to (C' + 1)^{\frac{1}{1+\alpha}} \), for \( 1 + \omega < 0 \),

\[ \dot{R} \to -\infty, \quad M(T) \to \infty. \]

Obviously,

\[ \frac{k^2 c a^2 \left[ C' + \left\{ 1 + \frac{B'}{a^{3(1+\alpha)(1+\omega)}} \right\} \frac{1}{1+\omega} \right]^{\frac{1}{1+\alpha}}}{6(3\lambda - 1)} + \frac{1}{2} a^2 \left( \frac{k^2}{\Lambda a^4} + \Lambda \right) \geq k_{\text{max}} = 1. \]

**Case 2**

When \( a \to \infty \): \( \rho_{GCCG} \to \infty \), for \( 1 + \omega < 0 \); \( \rho_{GCCG} \to (C' + 1)^{\frac{1}{1+\alpha}} \), for \( 1 + \omega > 0 \),

\[ \dot{R} \to -\infty \quad \text{for} \quad 1 + \omega < 0; \quad \dot{R} \to -r_0 a \sqrt{\frac{k^2 c (C' + 1)^{\frac{1}{1+\alpha}}}{6(3\lambda - 1)} + \frac{1}{2} - k} \quad \text{for} \quad 1 + \omega > 0, \]

\[ M(T) \to \infty \quad \text{for} \quad 1 + \omega < 0; \quad M(T) \to \frac{1}{2} a^3 r_0^3 \left[ \frac{k^2 (C' + 1)^{\frac{1}{1+\alpha}}}{6(3\lambda - 1)} + \frac{1}{2} - k \right] \quad \text{for} \quad 1 + \omega > 0. \]
Fig. 9: The time derivative of the radius is plotted against time for open universe (Red), flat universe (Blue) and closed universe (Black). The other parameters are considered as $\Lambda = 100$, $r = 10$, $\lambda = 1000000$, $\omega = -2$, $c = 10$, $C' = 5$, $B' = 3$, $\alpha = 1$.

Fig. 10: The mass of the collapsing cloud is plotted against time for open universe (Red), flat universe (Blue) and closed universe (Black). The other parameters are considered as $\Lambda = 100$, $r = 10$, $\lambda = 1000000$, $\omega = -2$, $c = 10$, $C' = 5$, $B' = 3$, $\alpha = 1$.

The cloud will start untrapped at the instant given by the real roots of the following equation and gradually start to be trapped,

$$r^2_{\Sigma} \left[ \frac{k^2 c a^2}{6(3\lambda - 1)} \left( C' + \frac{1 + \frac{B'}{a^2(1+\alpha)(1+\omega)}}{1+\omega} \right)^{\frac{1}{1+\alpha}} + \frac{1}{2} a^2 \left( \frac{k^2}{\Lambda a^4} + \Lambda \right) - k \right] = 1 \quad (44)$$

E. Effect of a combination of dark matter and Generalized cosmic Chaplygin gas

1. Case I : $Q = 0$ i.e., No Interaction Between Dark Matter And Dark Energy:

In this case we consider the DE and DM to co-exist in a non-interacting scenario. The energy density of DM is given by eqn.(20) and the energy density of DE is given by eqn.(40). Therefore in this case the total energy can be considered as the sum total of the energy densities of DE and DM, i.e. $\rho_T = \rho_{GCCG} + \rho_M$. Using this in the field equation for HL gravity, i.e. eqn.(17), we get the expressions of the relevant parameters as given below.
The time gradient of the geometrical radius of the collapsing star is given by,

\[ \dot{R}(T) = -r \Sigma \frac{k^2ca^2}{6(3\lambda - 1)} \left[ C' + \left\{ 1 + \frac{B'}{a^3(1+\alpha)(1+\omega)} \right\} \right]^{\frac{1}{1+\omega}} + \frac{C_0}{a^3} \right]^{\frac{1}{1+\omega}} + \frac{1}{2}a^2 \left( \frac{k^2}{\Lambda a^4} + \Lambda \right) - k \]  

and

\[ M(T) = \frac{1}{2}a^3r^3 \Sigma \frac{k^2ca^2}{6(3\lambda - 1)} \left[ C' + \left\{ 1 + \frac{B'}{a^3(1+\alpha)(1+\omega)} \right\} \right]^{\frac{1}{1+\omega}} + \frac{C_0}{a^3} \right]^{\frac{1}{1+\omega}} + \frac{1}{2}a^2 \left( \frac{k^2}{\Lambda a^4} + \Lambda \right) - k \]  

In this case the limiting values of the physical parameters are given as follows,

**Case1**

When \( a \to 0 \): \( \rho_{GCCG} \to \infty \), for \( 1+\omega > 0 \); \( \rho_{GCCG} \to (C' + 1)^{\frac{1}{1+\alpha}} \), for \( 1+\omega < 0 \),

\[ \dot{R} \to -\infty \], \( M(T) \to \infty \).

Obviously,

\[ \frac{k^2ca^2}{6(3\lambda - 1)} \left[ C' + \left\{ 1 + \frac{B'}{a^3(1+\alpha)(1+\omega)} \right\} \right]^{\frac{1}{1+\omega}} + \frac{C_0}{a^3} \right]^{\frac{1}{1+\omega}} + \frac{1}{2}a^2 \left( \frac{k^2}{\Lambda a^4} + \Lambda \right) \geq k_{max} = 1. \]

**Case2**

When \( a \to \infty \): \( \rho_{GCCG} \to \infty \), for \( 1+\omega < 0 \); \( \rho_{GCCG} \to (C' + 1)^{\frac{1}{1+\alpha}} \), for \( 1+\omega > 0 \),

\[ \dot{R} \to -\infty \] for \( 1+\omega < 0 \); \( \dot{R} \to -r \Sigma \sqrt{\frac{k^2c(C' + 1)^{\frac{1}{1+\alpha}}}{6(3\lambda - 1)} + \frac{\Lambda}{2} - k} \) for \( 1+\omega > 0 \)

\[ M(T) \to \infty \] for \( 1+\omega < 0 \); \( M(T) \to \frac{1}{2}a^3r^3 \Sigma \left[ \frac{k^2c(C' + 1)^{\frac{1}{1+\alpha}}}{6(3\lambda - 1)} + \frac{\Lambda}{2} - k \right] \) for \( 1+\omega > 0 \).

The cloud will start untrapped at the instant given by the real roots of the following equation and gradually start to be trapped,

\[ r^2 \Sigma \left[ \frac{k^2ca^2}{6(3\lambda - 1)} \left[ C' + \left\{ 1 + \frac{B'}{a^3(1+\alpha)(1+\omega)} \right\} \right]^{\frac{1}{1+\omega}} + \frac{C_0}{a^3} \right]^{\frac{1}{1+\omega}} + \frac{1}{2}a^2 \left( \frac{k^2}{\Lambda a^4} + \Lambda \right) - k \right] = 1 \]  

2. **Case II**: \( Q \neq 0 \) i.e., Interaction Between Dark Matter And Dark Energy:

Just like the case of NVMCG, here also we consider

\[ \frac{\rho_{GCCG}}{\rho_M} = C_2a^{3\omega''} \]  

(48)
where $C_2 > 0$ and $n''$ are arbitrary constants. Here $\rho_T = \rho_{GCCG} + \rho_M$. Using this relation we get,

$$\rho_{GCCG} = \frac{C_2 a^{3n''} \rho_T}{1 + C_2 a^{3n''}}, \quad \rho_M = \frac{\rho_T}{1 + C_2 a^{3n''}} \quad (49)$$

Solving the conservation equations we get the expression for $\rho_T$ as,

$$\rho_T = \frac{C_3 \left( 1 + C_2 a^{3n''} \right)^{-\frac{1}{n''}}}{a^{3}} \quad (50)$$

where $C_3$ is the integration constant. The expression for interaction is given by,

$$Q = \frac{3\rho_T}{1 + C_2 a^{3n''}} \sqrt{\frac{k^2 c \rho_T}{6(3\lambda - 1)}} + \frac{1}{2} \left( \frac{k^2}{\Lambda a^4 + \Lambda} \right) - \frac{k}{a^2} \left[ 1 - \frac{1}{1 + C_2 a^{3n''}} \left( 1 + C_2 a^{3n''} - C_2 a^{-3n''} \alpha \left( \frac{\rho_T}{1 + C_2 a^{3n''}} \right)^{-\alpha-1} \left( C' + \left( \frac{C_2 a^{3n''} \rho_T}{1 + C_2 a^{3n''}} \right)^{1+\alpha} - C' \right)^{-\frac{1}{n''}} + n'' C_2 a^{3n''} \right] \right] \quad (51)$$

where $\rho_T$ is given by eqn. (49). The expressions for the other physical quantities are,

$$\dot{R} = -r \sqrt{\frac{k^2 c a^2}{6(3\lambda - 1)}} \rho_T + \frac{a^2}{2} \left( \frac{k^2}{\Lambda a^4 + \Lambda} \right) - k \quad (52)$$
The expression for the mass is given by,

\[ M(T) = \frac{1}{2} a^2 r_\Sigma^3 \left[ \frac{k^2 c a^2}{6 (3 \lambda - 1)} \rho_T + \frac{a^2}{2} \left( \frac{k^2}{\Lambda a^4} + \Lambda \right) - k \right] \]  

(53)

where \( \rho_T \) is given by eqn.(49). In this case the limiting values of the physical parameters are given as follows,

**Case 1**

When \( a \to 0 \) : \( \rho_T \to \infty \), \( \dot{R} \to -\infty \), \( M(T) \to \infty \). Obviously, \( \frac{k^2 c a^2}{6 (3 \lambda - 1)} (\rho_T) + \frac{1}{2} a^2 \left( \frac{k^2}{\Lambda a^4} + \Lambda \right) \geq k_{max} = 1 \).

**Case 2**

When \( a \to \infty \) : \( \rho_T \to C_3 C_2^{\frac{1}{2}} \), \( \dot{R} \to -r_\Sigma a \sqrt{\frac{k^2 c}{6 (3 \lambda - 1)} C_3 C_2^{\frac{1}{2}}} + \Lambda \), \( M(T) \to \frac{1}{2} a^4 r_\Sigma^3 \left[ \frac{k^2 c}{6 (3 \lambda - 1)} C_3 C_2^{\frac{1}{2}} + \Lambda \right] \).

The cloud will start untrapped at the instant given by the real roots of the following equation and gradually start to be trapped,

\[ r_\Sigma^2 \left[ \frac{k^2 c a^2}{6 (3 \lambda - 1)} (\rho_T) + \frac{1}{2} a^2 \left( \frac{k^2}{\Lambda a^4} + \Lambda \right) - k \right] = 1 \]  

(54)

where \( \rho_T \) is given by eqn. (49).

**IV. GRAPHICAL ANALYSIS**

In this section we analyze the plots in detail. **First of all it should be mentioned that in all cases, the collapsing scenario is realized for smaller values of \( \Lambda \) and relatively much higher values of \( \lambda \).** In fig.1, a plot between the time derivative of black hole radius, \( \dot{R} \) against time is provided for the collapsing scenario of dark matter. Trajectories for open (Red), flat (Blue) and closed (Black) universe is shown in a comparative scenario. It is seen that for flat universe although the trajectory remains in the negative region yet its increasing magnitude puts a lot of question in the feasibility of the collapse. For open and closed universe, a perfectly collapsing scenario is realized, with closed universe presenting the most ideal conditions for gravitational collapse. In fig.2, black hole mass, \( M \) is plotted against time for an universe filled with dark matter. Here also three different trajectories for open, flat and closed universe are shown. We see
Fig 13 : The time derivative of the radius is plotted against time for open universe (Red), flat universe (Blue) and closed universe (Black). The other parameters are considered as 
\[ \Lambda = 10, \ r = 10, \ \lambda = 100000000, \ C_3 = 2, \ C_2 = 5, \ c = 10, \ n'' = 2, \alpha = 1. \]

Fig 14 : The mass of the collapsing cloud is plotted against time for open universe (Red), flat universe (Blue) and closed universe (Black). The other parameters are considered as 
\[ \Lambda = 10, \ r = 10, \ \lambda = 100000000, \ C_3 = 2, \ C_2 = 5, \ c = 10, \ n'' = 2, \alpha = 1. \]

that in case of open and closed universe the mass increases, with the progression of collapse. The mass is maximum in case of open universe. But in case of flat universe the result is completely different. The collapse progresses with a decrease in black hole mass. This is really strange.

In fig. 3 and 4, respectively \( \dot{R} \) and \( M \) are plotted against time for an universe filled with NVMCG type dark energy. The results are quite similar to the previous case of dark matter collapse, the only difference being that, the maximum mass is obtained in case of closed universe, unlike the previous section. **If we compare the plots for NVMCG and DM it is evident that introduction of DE does not affect the collapsing procedure in case of closed universe, whereas in case of flat and open universe some effect is visible.** In figs. 5 and 6, we plot \( \dot{R} \) and \( M \) for the collapsing system when the universe is filled with a combination of NVMCG and DM in a non-interacting scenario. Here we see that collapse is favoured the most in case of open universe. It is also seen that due to the co-existence of DE and DM, the trajectories for open and closed universe almost coincide with each other thus showing identical outcomes of collapse. Interaction between DE and DM is considered and plots are generated as shown in figs. 7 and 8. From fig.7, it is quite clear that this is the most favourable environment for collapse to occur as is evident from the steeply decreasing slopes of \( \dot{R} \). It is quite obvious that interaction between DE and DM is largely responsible for this. The mass of the black holes increase quite
steeply in case of open and closed universe, but in case of flat universe there is a decrease in mass just like the previous sections.

In figs. 9 and 10, the plots are obtained for an universe filled with GCCG as the DE. In this case as well tendency for collapse is much less in case of flat universe compared to open and closed universe. The plot for BH mass presents an identical scenario as NVMCG. In figs. 11 and 12 the plots are generated for a combination of GCCG and DM in a non-interacting scenario. Again the striking feature is that the best collapsing scenario is obtained for open universe unlike the previous case. Finally considering interaction between GCCG and DM, figs. 13 and 14 are obtained. If we closely look at the figures 5,6,7,8 and 11,12,13,14, we see that in case of GCCG the best collapsing scenario is obtained in case of the combination of DE and DM in the absence of interaction, much unlike NVMCG where the ideal conditions were obtained in the presence of interaction. This is a really interesting result. Since the collapsing process for both NVMCG and GCCG are identical, this difference can be phenomenally attributed to the difference in their own internal mechanism as DE.

V. CONCLUSIONS

Here we have studied the gravitational collapse of a spherically symmetric dust cloud of finite radius, filled with homogeneous and isotropic fluid. Two different types of DE fluids were considered for our study, namely new variable modified Chaplygin gas and generalized cosmic Chaplygin gas. The gravitational collapse of a star filled with DM was studied in the background of DE, in Hořava-Lifshitz gravity. The two different models of DE was considered first independently and then in combination with DM, with and without interaction. Both the studies are compared and an attempt to obtain a meaningful result is made. Relevant parameters (time derivative of radius and mass of the collapsing cloud) were calculated and their variations with time was plotted. A detailed graphical analysis was done to get a clear understanding of the results obtained in the two different models.

It was seen that for open and closed universe gravitational collapse was accompanied with an increase of BH mass for both NVMCG and GCCG. The above fact is quite understandable and expected. But for flat universe occurrence of collapse is highly unlikely. In case of open and closed universe, the only constraint on collapse is that the values of \( \lambda \)
should be relatively much higher than the values of $\Lambda$. For flat universe, there is a bleak possibility of collapse irrespective of any conditions. The possible reasons are not very clear for the time being. For a speculation we can attribute it to the quantum effects of the Hořava-Lifshitz gravity. But actually it remains an open question for the time being. The hindrance in gravitational collapse due to the presence of DE is not really prominent in our study, which was a major concern in [40]. Again it is a surprising feature of the Hořava-Lifshitz gravity, that cosmic acceleration is not really weakening the gravitational collapse, as it really did in case of LQC and DGP brane in Rudra et al [40]. This is really unexpected and the true reason remains an open question for the time being.

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