Gauge Field Theory Coherent States (GCS) : I.
General Properties

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Abstract

In this article we outline a rather general construction of diffeomorphism covariant coherent states for quantum gauge theories.

By this we mean states $\psi_{(A,E)}$, labelled by a point $(A,E)$ in the classical phase space, consisting of canonically conjugate pairs of connections $A$ and electric fields $E$ respectively, such that
(a) they are eigenstates of a corresponding annihilation operator which is a generalization of $A - iE$ smeared in a suitable way,
(b) normal ordered polynomials of generalized annihilation and creation operators have the correct expectation value,
(c) they saturate the Heisenberg uncertainty bound for the fluctuations of $\hat{A}, \hat{E}$ and
(d) they do not use any background structure for their definition, that is, they are diffeomorphism covariant.

This is the first paper in a series of articles entitled “Gauge Field Theory Coherent States (GCS)” which aim at connecting non-perturbative quantum general relativity with the low energy physics of the standard model. In particular, coherent states enable us for the first time to take into account quantum metrics which are excited everywhere in an asymptotically flat spacetime manifold as is needed for semi-classical considerations.

The formalism introduced in this paper is immediately applicable also to lattice gauge theory in the presence of a (Minkowski) background structure on a possibly infinite lattice.

1 Introduction

Quantum General Relativity (QGR) has matured over the past decade to a mathematically well-defined theory of quantum gravity. In contrast to string theory, by definition QGR is a manifestly background independent, diffeomorphism invariant and non-perturbative theory. The obvious advantage is that one will never have to postulate the existence of a non-perturbative extension of the theory, which in string theory has been called the still unknown M(ystery)-Theory.

The disadvantage of a non-perturbative and background independent formulation is, of course, that one is faced with new and interesting mathematical problems so that one cannot just go ahead and “start calculating scattering amplitudes”: As there is no

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background around which one could perturb, rather the full metric is fluctuating, one is not doing quantum field theory on a spacetime but only on a differential manifold. Once there is no (Minkowski) metric at our disposal, one loses familiar notions such as causality, locality, Poincaré group and so forth, in other words, the theory is not a theory to which the Wightman axioms apply. Therefore, one must build an entirely new mathematical apparatus to treat the resulting quantum field theory which is drastically different from the Fock space picture to which particle physicists are used to.

As a consequence, the mathematical formulation of the theory was the main focus of research in the field over the past decade. The main achievements to date are the following (more or less in chronological order):

i) **Kinematical Framework**

The starting point was the introduction of new field variables for the gravitational field which are better suited to a background independent formulation of the quantum theory than the ones employed until that time. In its original version these variables were complex valued, however, currently their real valued version, considered first in for classical Euclidean gravity and later in for classical Lorentzian gravity, is preferred because to date it seems that it is only with these variables that one can rigorously define the kinematics and dynamics of Euclidean or Lorentzian quantum gravity. These variables are coordinates for the infinite dimensional phase space of an SU(2) gauge theory subject to further constraints besides the Gauss law, that is, a connection and a canonically conjugate electric field. As such, it is very natural to introduce smeared functions of these variables, specifically Wilson loop and electric flux functions. (Notice that one does not need a metric to define these functions, that is, they are background independent). This had been done for ordinary gauge fields already before in and was then reconsidered for gravity (see e.g. ).

The next step was the choice of a representation of the canonical commutation relations between the electric and magnetic degrees of freedom. This involves the choice of a suitable space of distributional connections and a faithful measure thereon which, as one can show, is -additive. The proof that the resulting Hilbert space indeed solves the adjointness relations induced by the reality structure of the classical theory as well as the canonical commutation relations induced by the symplectic structure of the classical theory can be found in . Independently, a second representation of the canonical commutation relations, called the loop representation, had been advocated (see e.g. and references therein) but both representations were shown to be unitarily equivalent in (see also for a different method of proof).

This is then the first major achievement: The theory is based on a rigorously defined kinematical framework.

ii) **Geometrical Operators**

The second major achievement concerns the spectra of positive semi-definite, self-adjoint geometrical operators measuring lengths, areas and volumes of curves, surfaces and regions in spacetime. These spectra are pure point (discrete) and imply a discrete Planck scale structure. It should be pointed out that the discreteness is, in contrast to other approaches to quantum gravity, not put in by hand but it is a prediction!

iii) **Regularization- and Renormalization Techniques**

The third major achievement is that there is a new regularization and renormalization technique for diffeomorphism covariant, density-one-valued operators at
our disposal which was successfully tested in model theories \cite{23}. This technique can be applied, in particular, to the standard model coupled to gravity \cite{24, 25} and to the Poincaré generators at spatial infinity \cite{26}. In particular, it works for Lorentzian gravity while all earlier proposals could at best work in the Euclidean context only (see, e.g. \cite{12} and references therein). The algebra of important operators of the resulting quantum field theories was shown to be consistent \cite{27}. Most surprisingly, these operators are \textit{UV and IR finite}! Notice that this result, at least as far as these operators are concerned, is stronger than the believed but unproved finiteness of scattering amplitudes order by order in perturbation theory of the five critical string theories, in a sense we claim that the perturbation series converges. The absence of the divergences that usually plague interacting quantum fields propagating on a Minkowski background can be understood intuitively from the diffeomorphism invariance of the theory: “short and long distances are gauge equivalent”. We will elaborate more on this point in future publications.

iv) \textit{Spin Foam Models}

After the construction of the densely defined Hamiltonian constraint operator of \cite{21, 22}, a formal, Euclidean functional integral was constructed in \cite{28} and gave rise to the so-called spin foam models (a spin foam is a history of a graph with faces as the history of edges) \cite{29}. Spin foam models are in close connection with causal spin-network evolutions \cite{30}, state sum models \cite{31} and topological quantum field theory, in particular BF theory \cite{32}. To date most results are at a formal level and for the Euclidean version of the theory only but the programme is exciting since it may restore manifest four-dimensional diffeomorphism invariance which in the Hamiltonian formulation is somewhat hidden.

v) Finally, the fifth major achievement is the existence of a rigorous and satisfactory framework \cite{33, 34, 35, 36, 37, 38, 39} for the quantum statistical description of black holes which reproduces the Bekenstein-Hawking Entropy-Area relation and applies, in particular, to physical Schwarzschild black holes while stringy black holes so far are under control only for extremal charged black holes.

Summarizing, the work of the past decade has now culminated in a promising starting point for a quantum theory of the gravitational field plus matter and the stage is set to pose and answer physical questions.

The most basic and most important question that one should ask is: \textit{Does the theory have classical general relativity as its classical limit?} Notice that even if the answer is negative, the existence of a consistent, interacting, diffeomorphism invariant quantum field theory in four dimensions is already a quite non-trivial result. However, we can claim to have a satisfactory quantum theory of Einstein’s theory only if the answer is positive.

To settle this issue we have launched an attack based on coherent states which has culminated in a series of papers called “Gauge Field Theory Coherent States” \cite{40, 41, 42, 43, 44, 45} and this paper is the first one in this collection which is going to be extended further.

The organization of this series is the following:

I) \textit{General Properties}

In this paper we describe a fairly general method to generate families of diffeomorphism covariant coherent states with the usual desired properties such as annihilation operator eigenstate nature, expectation value reproduction for annihilation and creation operators and saturation of the Heisenberg uncertainty bound. If certain
analytical conditions are met, overcompleteness can be established as well. The construction is based on the so-called configuration space complexifier method described in detail in [46]. The latter work arose as an abstraction of the results of Hall [47] who chose a very special, but very convenient configuration space complexification for the case that the configuration space is a compact, connected Lie group. Hall’s results were later generalized to diffeomorphism invariant gauge theories in [48]. In this paper we focus on general properties of such states for a general complexification such as gauge invariance and diffeomorphism covariance. Besides such physical features also analytical properties are addressed and it is a mixture of the two that will determine one’s choice of the complexification. In fact, in the remainder of this series we will mostly deal with a generalization of the complexification chosen by Hall. Our main reason for this choice is simply mathematical convenience: The spectrum of the operator that generates the configuration space complexification is explicitly known and sufficiently simple. This allows us to get started, but it should be kept in mind that other choices are available that may prove physically more interesting later on in our programme.

II) **Peakedness Properties**

Associated with the configuration space complexification is a so-called coherent state transform and both of [47, 48] focussed on the unitarity of that transform while the properties of the coherent states themselves remained untouched. Moreover, it remained unclear how the complexified connection \( A^C \) looks like in terms of the coordinates \((A, E)\) of the real phase space and without this an interpretation of the label \( A^C \) of the coherent state and thus expectation values, fluctuations and so forth remain veiled. Here, \( A \) is a connection for a compact gauge group and \( E \) is a canonically conjugate electric field. To fill both of these gaps is the purpose of the second paper [40] in this series. First of all, we find the expected result, namely that roughly speaking \( A^C = A - iE \) in a suitably smeared sense. Secondly, we analyze in detail the peakedness properties of the coherent states for diffeomorphism invariant gauge theories in the configuration –, momentum – and the Segal-Bargmann representation. We find that these states are very sharply peaked at the point \( A, E \) or \((A, E)\) respectively of the configuration –, momentum – and phase space respectively. That paper also contains extensive graphics to demonstrate these peakedness properties pictorially and while there are important differences, the states display the essential Gaussian decay of the harmonic oscillator coherent states.

III) **Ehrenfest Theorems**

In the third paper [41] of this series we prove Ehrenfest theorems for our coherent states. That is, we show that the expectation value not only of normal ordered polynomials of creation and annihilation operators but of all polynomials of the elementary operators associated with \( \hat{A}, \hat{E} \) equals, to leading order in \( \hbar \), precisely the labels \( A, E \) of the coherent state. This result can be extended to certain operators that are non-polynomial in the basic ones and that appear in the Hamiltonian constraint of quantum general relativity coupled to matter [21, 24, 25]. Moreover, we show that commutators between these operators divided by \( i\hbar \) have an expectation value which equals to leading order in \( \hbar \) the corresponding Poisson bracket evaluated at the label \((A, E)\) of the coherent state. Together, these results imply that the classical limit of the Hamiltonian constraint operator and its infinitesimal quantum dynamics correspond to its classical counterparts.

Both of [10, 11] mainly deal with \( G = U(1), SU(2) \) but we sketch how all the results can be extended to groups of higher rank, an issue which we will examine in detail
IV) Infinite Tensor Product and Thermodynamical Limit

The states that one considered in Quantum General Relativity until now are labelled by piecewise analytic, finite graphs (an extension to finite collections of smooth curves with controlled intersection properties is possible, see later on). However, finite graphs are suitable to describe semiclassical physics on physically interesting spacetimes only if the underlying manifold is spatially compact. The most interesting applications, flat space or an entire black hole spacetime (and not only the horizon region) cannot be treated with finite graphs. To extend the framework it turns out that piecewise analytical, countably infinite graphs together with the framework of the Infinite Tensor Product (ITP) construction introduced by von Neumann \[50\] more than sixty years ago are appropriate. To the best of the knowledge of the author, the first time that truly infinite graphs and infinite tensor product states were considered in QGR in the context of a Hilbert space structure, was in section 3.2 of \[26\] which dealt with the asymptotic Poincaré group of asymptotically flat spacetimes, however, the overall mathematical framework of such constructions was not described there. In \[42\] we deliver this structure and embed it into our coherent states framework. In particular, we are able to connect mathematical notions with physical ones, an example being the following:

A state \(f\) in the infinite tensor product Hilbert space over an infinite graph which is a direct product of normalized states, one for each edge of the graph, generates so-called strong and weak equivalence classes of so-called \(C_0\)-sequences. It turns out that the corresponding \(C_0\)-vector plays the role of a cyclic vector (vacuum state) for a Fock-like tiny closed subspace of the complete ITP Hilbert space, called an \(f\)-adic incomplete ITP. Fock-like spaces corresponding to different strong and weak equivalence classes are mutually orthogonal. Those Fock-like spaces that correspond to the same weak class but different strong classes are unitarily equivalent while those that correspond to different strong and weak classes are unitarily inequivalent. This way the ITP gives rise to an uncountably infinite number of mutually unitarily inequivalent representations of the canonical commutation relations. The representation theory of operator algebras becomes especially interesting, the enveloping framework being that of factors of von Neumann algebras.

Generically, incomplete ITP’s generated by different weak equivalence classes correspond to physical situations which differ drastically with respect to certain physical quantities such as energy, volume or topology. For instance, the Ashtekar-Isham-Lewandowski Hilbert space based on finite graphs describes finite volume and/or compact topology while a \(C_0\) vector of infinite volume can be constructed by using our coherent states, appropriate to approximate a flat Minkowski space geometry. The two Hilbert spaces are mutually orthogonal closed subspaces within our complete ITP Hilbert space corresponding to different weak classes. The vacuum underlying the Ashtekar-Isham-Lewandowski Hilbert space via the GNS construction is based on a \(C_0\) vector which equals unity for every edge of any possible graph. It can be shown that such a state, in the context of non-compact topologies, is a pure quantum vacuum in the sense that it describes metrics of almost everywhere zero spatial volume.

It should be clear from these considerations that the ITP is possibly able to describe all physically different situations at once and might enable us to describe topology change within canonical quantum general relativity and therefore to get rid off the embedding spacetime manifold that one started with classically!
The infinite tensor product opens the gate to a plethora of other physical and mathematical disciplines, such as thermodynamics and statistical field theory, Tomita-Takesaki (or modular) theory necessary to classify the appearing types of type III factors of von Neumann algebras etc.

V) Higgs Fields and Fermions
The framework described so far is sufficient for pure quantum gauge theories coupled to quantum general relativity only. By combining the framework of [25] with the infinite tensor product construction and existing results for coherent states for fermions (e.g. [51] and references therein) we can extend the framework to all matter of the standard model including possible supersymmetric extensions. The details are described in [43].

VI) Photons and Gravitons
Most of the criticism directed towards quantum general relativity coming from the particle physics community is that the programme, being manifestly non-perturbative by construction, seems to be infinitely far away from any established perturbative results such as (free) quantum field theory on curved backgrounds (widely believed to be the first approximation to full quantum gravity), perturbative quantum (super)gravity (non-renormalizable) and perturbative quantum superstring theory. In [44] we make a first contact with these programmes. Namely, we try to construct a map between the perturbative Photon or Graviton Hilbert spaces and a fully non-perturbative incomplete $f$-adic ITP subspace where the $C_0$-vector corresponding to $f$ is a best approximation state to the Minkowski space solution of the Einstein-Maxwell equations. This work is aimed at demonstrating how perturbative notions such as particles can be absorbed into our fully non-perturbative programme.

VII) The Non-Perturbative $\gamma$-Ray Burst Effect
Many serious theorists and experimentalists nowadays discuss the possibility to actually measure quantum gravity effects, a prominent example being the so-called $\gamma$-ray burst experiment (see, e.g. [52, 53]). In all these types of experiments one exploits the fact that the incredibly tiny quantum gravity effects may accumulate over vast periods of time of the order of the age of the universe to a measurable size. In particular, the theoretical mechanism of the $\gamma$-ray burst effect can be roughly described as follows: the quantum metric depends on canonically conjugate magnetic and electric degrees of freedom and thus the Heisenberg uncertainty obstruction tells us that there is no state that can describe the Minkowski vacuum exactly. In other words, there is no Poincaré invariant state in the theory, the best one can do is to construct a coherent state peaked on Minkowski space. The expectation value of the Einstein-Maxwell-Hamiltonian with respect to the gravitational field will therefore include corrections to the classical Minkowski metric which give rise to Poincaré invariance violating dispersion relations. Thus, if one could measure the arrival times of $\gamma$-ray photons of different energies they should differ by an amount proportional to the travelling time from the source.

The challenge is now to precisely compute these corrections from our fully non-perturbative framework, in particular, what is the precise power of the Planck mass that the effect is proportional to. This is the subject of [45] which will improve the pioneering work [44] in two respects: First, the latter was based on so-called weave states [55] which, however, approximate only half of the number of degrees of freedom and, secondly, in contrast to our coherent states the existence of weave itself with the assumed semi-classical properties was not proved to exist.
To compute the effect exactly turns out to be a hard piece of analysis due to the non-linear, even non-analytic (interacting) nature of the theory, a property which carries over to our coherent states. In particular, the complicated spectrum of the volume operator makes the enterprise not an easy one. On the other hand, it is absolutely crucial to know the precise spectrum and not only of, say, its main series: If one would do the same with the area operator then, as has been beautifully demonstrated in [39], one would reach the conclusion that the black hole Hawking radiation spectrum is discrete rather than the quasi-continuous one of a black body, in other words, the spectrum has direct bearing on observation!

It is at this point that super-computers may enter the stage as analytic computations start becoming too hard and lengthy. Notice, however, that in contrast to usual perturbation series in perturbative quantum field theory the computational error is always under good control. The series that we are dealing with are manifestly absolutely converging and there are precise estimates on the error that one creates when keeping only the dominant terms. We will display such error controlled estimates in the next two issues of this series.

VIII) **The Classical Limit**

As an immediate application of coherent states and the ITP framework one can now precisely prove in detail [50] how it happens that the Hamiltonian constraint constructed in [21] obeys the correct quantum algebra.

More work is in progress. The following list of projects associated with our coherent states represents just the tip of the iceberg, in principle it would would be interesting to repeat all perturbative calculations that have been performed so far with our non-perturbative tools and to provide the error bars.

A) To relate standard perturbative quantum field theory on curved backgrounds with non-perturbative quantum general relativity one would like to understand why the UV singularities of the former have disappeared in the latter. The naive answer is that the renormalization group has been absorbed into the diffeomorphism group (large and small momenta are gauge related) but one would like to understand this and related notions like bare and renormalized charges, effective actions, renormalization transformations, Epstein-Glaser formalism and the importance of Hadamard states for quantum field theory on curved backgrounds etc. in more detail from the non-perturbative point of view. In particular, it would be nice to map the usual Feynman rules into our framework. This research project will be started in [57, 58].

B) An ever fascinating research object has been the black hole. The coherent states provide a natural new setting in which to study quantum black holes and Hawking radiation, in principle one “just” has to take the coherent state that approximates a Kruskal spacetime together with its excitations in order to provide the Kruskal–spacetime–adic incomplete closed ITP Hilbert space structure (that is, a vacuum and excitations). Notice that while the Bekenstein–Hawking entropy has been successfully computed in both canonical quantum gravity and string theory as mentioned above, what would be new here is that one can treat the full spacetime in a Hilbert space context and not only its near horizon structure (charges). Also, there are a priori no constraints such as (near-) stationarity or extremality of the black hole. Finally, one would like to understand what happens to the classical singularity theorems, the information paradoxon, cosmic censorship etc. in the quantum theory. These and related issues will be the topic of [19].

C)
As already mentioned, von Neumann algebras and their representation theory appear quite naturally in the Infinite Tensor Product construction. For the latter, the decomposition of a von Neumann algebra into factors is of particular importance and the basic tool to characterize factors of type III, which typically appear in quantum field theory, is provided by modular theory. This brings us into close contact with algebraic quantum field theory, although presumably in a generalized setting, since the notion of locality plays, almost by definition, a less dominant role in a diffeomorphism invariant quantum field theory. These and related issues will be examined in [60].

D) The most effective way to derive a path integral formulation for kinematically linear field theories from the Hamiltonian formulation of the theory is via coherent states, see e.g. [51] and references therein. Thus, it is natural to expect this to be the case also for our coherent states. This may bring us into contact with the formerly mentioned spin foam models [28, 29, 30, 31, 32] which have recently attracted quite some attention after the appearance of [21, 22] and will be studied in [61].

E) Finally, our coherent states are pure states. The semi-classical behaviour of such states may yet be improved by superimposing them to a so-called mixed state which makes use of random lattices. For weaves, such a framework already exists and has been studied in [62]. We intend to combine both frameworks in [63].

This article is assembled as follows:

In section two we recall the classical and quantum kinematics of diffeomorphism invariant gauge field theories.

In section three we recall the complexifier method to generate Bargmann-Segal representations for general theories and gauge theories in particular. We comment on the physical and mathematical requirements to be imposed on the complexifier, that is, the canonical generator of the transform that complexifies the configuration space and identifies it with the phase space. In three related subsections we propose three candidate families of coherent states for gauge theories. The first one leads to an actual complex connection, the second only to a complexified holonomy without underlying complex connection and the third one maps the problem at hand in principle to coherent states for an (in)finite collection of uncoupled harmonic oscillators. We describe the advantages and disadvantages of these states as compared to each other. All of this will be done mostly for gauge – and diffeomorphism variant coherent states.

In sections four and five respectively we will deal with the issue of how to construct gauge – and diffeomorphism invariant coherent states respectively. Some of these can even be chosen to be annihilated by the Hamiltonian constraint.

Finally, in section six we display a simple example for gauge invariant coherent states with an actual complex connection in 2+1 gravity and study some of their peakedness properties.

## 2 Kinematical Structure of Diffeomorphism Invariant Quantum Gauge Theories

In this section we will recall the main ingredients of the mathematical formulation of (Lorentzian) diffeomorphism invariant classical and quantum field theories of connections
with local degrees of freedom in any dimension and for any compact gauge group. See \(\text{[10, 64]}\) and references therein for more details.

### 2.1 Classical Theory

Let \(G\) be a compact gauge group, \(\Sigma\) a \(D\)-dimensional manifold admitting a principal \(G\)-bundle with connection over \(\Sigma\). Let us denote the pull-back to \(\Sigma\) of the connection by local sections by \(A_a^i\) where \(a, b, c, \ldots = 1, \ldots, D\) denote tensorial indices and \(i, j, k, \ldots = 1, \ldots, \dim(G)\) denote indices for the Lie algebra of \(G\). Likewise, consider a density-one vector bundle of electric fields, whose pull-back to \(\Sigma\) by local sections (their Hodge dual is a \(D - 1\) form) is a Lie algebra valued vector density of weight one. We will denote the set of generators of the rank \(N - 1\) Lie algebra of \(G\) by \(\tau_i\) which are normalized according to \(\text{tr}(\tau_i \tau_j) = -N \delta_{ij}\) and \([\tau_i, \tau_j]\) = \(2 f_{ij}^k \tau_k\) defines the structure constants of \(\text{Lie}(G)\).

Let \(F_i^a\) be a Lie algebra valued vector density test field of weight one and let \(f_a^i\) be a Lie algebra valued covector test field. We consider the smeared quantities

\[
F(A) := \int_\Sigma d^D x F_i^a A_a^i \quad \text{and} \quad E(f) := \int_\Sigma d^D x E_i^a f_a^i
\]

(2.1)

While both of them are diffeomorphism covariant it is only the latter which is gauge covariant and this is one motivation to consider the singular smearings discussed below. The choice of the space of pairs of test fields \((F, f) \in \mathcal{S}\) depends on the boundary conditions on the space of connections and electric fields which in turn depends on the topology of \(\Sigma\) and will not be specified in what follows.

Let the set of all pairs of smooth functions \((A, E)\) on \(\Sigma\) such that (2.1) is well defined for any \((F, f) \in \mathcal{S}\) be denoted by \(M\). We define a topology on \(M\) through the following globally defined metric:

\[
d_{\rho, \sigma}[(A, E), (A', E')] := \sqrt{-\frac{1}{N} \int_\Sigma d^D x \sqrt{\det(\rho)} \rho^{ab} \text{tr}([A_a^i - A'_a^i][A_b^j - A'_b^j]) + \frac{\sigma_{ab} \text{tr}([E^a - E'^a][E^b - E'^b])}{\sqrt{\det(\sigma)}}}
\]

(2.2)

where \(\rho_{ab}, \sigma_{ab}\) are fiducial metrics on \(\Sigma\) of everywhere Euclidean signature. Their fall-off behaviour has to be suited to the boundary conditions of the fields \(A, E\) at spatial infinity. Notice that the metric (2.2) on \(M\) is gauge invariant. It can be used in the usual way to equip \(M\) with the structure of a smooth, infinite dimensional differential manifold modelled on a Banach (in fact Hilbert) space \(\bar{E}\) where \(\mathcal{S} \times \mathcal{S} \subset \bar{E}\). (It is the weighted Sobolev space \(H^2_0 \times H^2_0\) in the notation of [53]).

Finally, we equip \(M\) with the structure of an infinite dimensional symplectic manifold through the following strong (in the sense of [54]) symplectic structure

\[
\Omega((f, F), (f', F'))_m := \int_\Sigma d^D x [F_i^a f_a'^{i'} - F_i^a' f_a^{i'}](x)
\]

(2.3)

for any \((f, F), (f', F') \in \bar{E}\). We have abused the notation by identifying the tangent space to \(M\) at \(m\) with \(\bar{E}\). To prove that \(\Omega\) is a strong symplectic structure one uses standard Banach space techniques. Computing the Hamiltonian vector fields (with respect to \(\Omega\)) of the functions \(E(f), F(A)\) we obtain the following elementary Poisson brackets

\[
\{E(f), E(f')\} = \{F(A), F'(A)\} = 0, \quad \{E(f), A(F)\} = F(f)
\]

(2.4)

As a first step towards quantization of the symplectic manifold \((M, \Omega)\) one must choose a polarization. As usual in gauge theories, we will use a particular real polarization,
specifically connections as the configuration variables and electric fields as canonically conjugate momenta. As a second step one must decide on a complete set of coordinates of $M$ which are to become the elementary quantum operators. The analysis just outlined suggests to use the coordinates $E(f), F(A)$. However, the well-known immediate problem is that these coordinates are not gauge covariant. Thus, we proceed as follows:

Let $\Gamma_0^\omega$ be the set of all piecewise analytic, finite, oriented graphs $\gamma$ embedded into $\Sigma$ and denote by $E(\gamma)$ and $V(\gamma)$ respectively its sets of oriented edges $e$ and vertices $v$ respectively. Here finite means that $E(\gamma)$ is a finite set. (One can extend the framework to $\Gamma_0^\omega$, the restriction to webs of the set of piecewise smooth graphs $\Omega \subset \Gamma^\omega$ but the description becomes more complicated and we refrain from doing this here). It is possible to consider the set $\Gamma^\omega_\sigma$ of piecewise analytic, infinite graphs with an additional regularity property ([42]) but for the purpose of this paper it will be sufficient to stick to $\Gamma_0^\omega$. The subscript $0$ as usual denotes “of compact support” while $\sigma$ denotes “$\sigma$-finite”.

We denote by $h_e(A)$ the holonomy of $A$ along $e$ and say that a function $f$ on $A$ is cylindrical with respect to $\gamma$ if there exists a function $f_\gamma$ on $G[E(\gamma)]$ such that $f = p^*_\gamma f_\gamma = f_\gamma \circ p_\gamma$ where $p_\gamma(A) = \{h_e(A)\}_{e \in E(\gamma)}$. Holonomies are invariant under reparameterizations of the edge and in this article we assume that the edges are always analytic preserving diffeomorphic images from $[0,1]$ to a one-dimensional submanifold of $\Sigma$. Gauge transformations are functions $g : \Sigma \mapsto G; x \mapsto g(x)$ and they act on holonomies as $h_e \mapsto g(e(0))h_eg(e(1))^{-1}$.

Next, given a graph $\gamma$ we choose a polyhedronal decomposition $P_\gamma$ of $\Sigma$ dual to $\gamma$. The precise definition of a dual polyhedronal decomposition can be found in [64] but for the purposes of the present paper it is sufficient to know that $P_\gamma$ assigns to each edge $e$ of $\gamma$ an open “face” $S_e$ (a polyhedron of codimension one embedded into $\Sigma$) with the following properties:

(1) the surfaces $S_e$ are mutually non-intersecting,
(2) only the edge $e$ intersects $S_e$, the intersection is transversal and consists only of one point which is an interior point of both $e$ and $S_e$,
(3) $S_e$ carries the orientation which agrees with the orientation of $e$.

Furthermore, we choose a system $\Pi_\gamma$ of paths $p_e(x) \subset S_e, x \in S_e, e \in E(\gamma)$ connecting the intersection point $p_e = e \cap S_e$ with $x$. The paths vary smoothly with $x$ and the triples $\gamma, P_\gamma, \Pi_\gamma$ are such that if $\gamma, \gamma'$ are diffeomorphic, so are $P_\gamma, P_{\gamma'}$ and $\Pi_\gamma, \Pi_{\gamma'}$, see [64] for details.

With these structures we define the following function on $(M, \Omega)$

$$P^e_i(A, E) := -\frac{1}{N} \text{tr}(\tau_i h_e(0, 1/2) \int_{S_e} h_{p_e(x)} E(x) h_{p_e(x)}^{-1} h_e(0, 1/2)^{-1})$$

(2.5)

where $h_e(s, t)$ denotes the holonomy of $A$ along $e$ between the parameter values $s < t$, $\ast$ denotes the Hodge dual, that is, $\ast E$ is a $(D - 1)$-form on $\Sigma$, $E^a := E^a_i \tau_i$ and we have chosen a parameterization of $e$ such that $p_e(e(1/2))$.

Notice that in contrast to similar variables used earlier in the literature the function $P^e_i$ is gauge covariant. Namely, under gauge transformations it transforms as $P^e_i \mapsto g(e(0))P^e_i g(e(0))^{-1}$, the price to pay being that $P^e_i$ depends on both $A$ and $E$ and not only on $E$. The idea is therefore to use the variables $h_e, P^e_i$ for all possible graphs $\gamma$ as the coordinates of $M$.

The problem with the functions $h_e(A)$ and $P^e_i(A, E)$ on $M$ is that they are not differentiable on $M$, that is, $Dh_e, DP^e_i$ are nowhere bounded operators on $E$ as one can easily see. The reason for this is, of course, that these are functions on $M$ which are not properly smeared with functions from $S$, rather they are smeared with distributional test functions with support on $e$ or $S_e$ respectively. Nevertheless one would like to base the quantization of the theory on these functions as basic variables because of their gauge and
diffeomorphism covariance. Indeed, under diffeomorphisms \( h_e \mapsto h_{\varphi^{-1}(e)} \), \( P^e_i \mapsto P^e_{\varphi^{-1}(e)} \) where the latter notation means that \( P^e_{\varphi^{-1}(e)} \) is labelled by \( \varphi^{-1}(S_e) \), \( \varphi^{-1}(\Pi_\gamma) \). We proceed as follows.

**Definition 2.1** By \( \bar{M}_\gamma \) we denote the direct product \( [G \times \text{Lie}(G)]^{[E(\gamma)]} \). The subset of \( \bar{M}_\gamma \) of pairs \( (h_e(A), P^e_i(A, E))_{e \in E(\gamma)} \) as \( (A, E) \) varies over \( M \) will be denoted by \( (\bar{M}_\gamma)_{|M} \). We have a corresponding map \( p_\gamma : M \mapsto \bar{M}_\gamma \) which maps \( M \) onto \( (\bar{M}_\gamma)_{|M} \).

Notice that the set \( (\bar{M}_\gamma)_{|M} \) is in general a proper subset of \( \bar{M}_\gamma \), depending on the boundary conditions on \( (A, E) \), the topology of \( \Sigma \) and the “size” of \( e, S_e \). For instance, in the limit of \( e, S_e \to e \cap S_e \) but holding the number of edges fixed, \( (\bar{M}_\gamma)_{|M} \) will consist of only one point in \( M \). This follows from the smoothness of the \( (A, E) \).

We equip a subset \( M_\gamma \) of \( \bar{M}_\gamma \) with the structure of a differentiable manifold modelled on the Banach space \( \mathcal{E}_\gamma = \mathbb{R}^{2 \dim(G)[E(\gamma)]} \) by using the natural direct product manifold structure of \( [G \times \text{Lie}(G)]^{[E(\gamma)]} \). While \( \bar{M}_\gamma \) is a kind of distributional phase space, \( M_\gamma \) satisfies appropriate regularity properties induced by (2.2).

In order to proceed and to give \( M_\gamma \) a symplectic structure derived from \( (M, \Omega) \) one must regularize the elementary functions \( h_e, P^e_i \) by writing them as limits (in which the regulator vanishes) of functions which can be expressed in terms of the \( F(A), E(f) \). Then one can compute their Poisson brackets with respect to the symplectic structure \( \Omega \) at finite regulator and then take the limit pointwise on \( M \). The result is the following well-defined strong symplectic structure \( \Omega_\gamma \) on \( M_\gamma \).

\[
\begin{align*}
\{h_e, h_e'\}_\gamma &= 0 \\
\{P^e_i, h_e'\}_\gamma &= \delta^e_i \frac{\tau_2 \partial}{2} h_e \\
\{P^e_i, P^e_j\}_\gamma &= -\delta^{ee'} f_{ij}^k P^e_k 
\end{align*}
\]

(2.6)

Since \( \Omega_\gamma \) is obviously block diagonal, each block standing for one copy of \( G \times \text{Lie}(G) \), to check that \( \Omega_\gamma \) is non-degenerate and closed reduces to doing it for each factor together with an appeal to well-known Hilbert space techniques to establish that \( \Omega_\gamma \) is a surjection of \( \mathcal{E}_\gamma \). This is done in [64] where it is shown that each copy is isomorphic with the cotangent bundle \( T^* G \) equipped with the symplectic structure (2.6) (choose \( e = e' \) and delete the label \( e \)).

Now that we have managed to assign to each graph \( \gamma \) a symplectic manifold \( (M_\gamma, \Omega_\gamma) \) we can quantize it by using geometric quantization. This can be done in a well-defined way because the relations (2.8) show that the corresponding operators are non-distributional. This is therefore a clean starting point for the regularization of any operator of quantum gauge field theory which can always be written in terms of the \( \hat{h}_e, P^e_i, e \in E(\gamma) \) if we apply this operator to a function which depends only on the \( h_e, e \in E(\gamma) \).

The question is what \( (M_\gamma, \Omega_\gamma) \) has to do with \( (M, \Omega) \). In [63] it is shown that there exists a partial order \( \prec \) on the set \( \mathcal{L} \) of triples \( l = (\gamma, P_\gamma, \Pi_\gamma) \). In particular, \( \gamma \prec \gamma' \) means \( \gamma \subset \gamma' \) and \( \mathcal{L} \) is a directed set so that one can form a generalized projective limit \( M_\infty \) of the \( M_\gamma \) (we abuse notation in displaying the dependence of \( M_\gamma \) on \( \gamma \) only rather than on \( l \)). For this one verifies that the family of symplectic structures \( \Omega_\gamma \) is self-consistent in the sense that if \( (\gamma, P_\gamma, \Pi_\gamma) \prec (\gamma', P_\gamma, \Pi_\gamma) \) then \( \{p^\gamma_{\gamma'} \{f, g\} \}_\gamma = \{p^\gamma_{\gamma'} f, p^\gamma_{\gamma'} g\} \gamma \) for any \( f, g \in C^\infty(M_\gamma) \) and \( p_{\gamma_\gamma} : M_\gamma \mapsto M_\gamma \) is a system of natural projections, more precisely, of (non-invertible) symplectomorphisms.

Now, via the maps \( p_\gamma \) of definition 2.1 we can identify \( M \) with a subset of \( M_\infty \). Moreover, in [63] it is shown that there is a generalized projective sequence \( (\gamma_{m}, P_{\gamma_{m}}, \Pi_{\gamma_{m}}) \) such that \( \lim_{m \to \infty} p^\gamma_{\gamma_{m}} \Omega_{\gamma_{m}} = \Omega \) pointwise in \( M \). This displays \( (M, \Omega) \) as embedded into
a generalized projective limit of the \((M, \Omega)\), intuitively speaking, as \(\gamma\) fills all of \(\Sigma\), we recover \((M, \Omega)\) from the \((M, \Omega_\gamma)\). Of course, this works with \(\Gamma^0_\gamma\) only if \(\Sigma\) is compact, otherwise we need the extension to \(\Gamma^0_\gamma\).

It follows that quantization of \((M, \Omega)\), and conversely taking the classical limit, can be studied purely in terms of \((M, \Omega_\gamma)\) for all \(\gamma\). The quantum kinematical framework is given in the next subsection.

### 2.2 Quantum Theory

Let us denote the set of all smooth connections by \(\mathcal{A}\). This is our classical configuration space and we will choose for its coordinates the holonomies \(h_e(A), \ e \in \gamma, \ \gamma \in \Gamma^0_\gamma\). \(\mathcal{A}\) is naturally equipped with a metric topology induced by (2.2).

Recall the notion of a function cylindrical over a graph from the previous subsection. A particularly useful set of cylindrical functions are the so-called spin-network functions \([69, 70, 13]\). A spin-network function is labelled by a graph \(\gamma\). Clearly \(\tilde{\Phi}\) finite linear combinations of spin-network functions over an arbitrary finite collection of graphs. One can show that these functions are linearly independent. From now on we denote by \(\Phi_\gamma\) finite linear combinations of spin-network functions over \(\gamma\), by \(\Phi\) the finite linear combinations of elements from any possible \(\Phi_\gamma\), \(\gamma' \subset \gamma\) a subgraph of \(\gamma\) and by \(\Phi\) the finite linear combinations of spin-network functions over an arbitrary finite collection of graphs.

The set \(\Phi\) of finite linear combinations of spin-network functions forms an Abelian * algebra of functions on \(\mathcal{A}\). By completing it with respect to the sup-norm topology it becomes an Abelian C* algebra \(\mathcal{B}\) (here the compactness of \(G\) is crucial). The spectrum \(\mathcal{A}\) of this algebra, that is, the set of all algebraic homomorphisms \(\mathcal{B} \to \mathbb{C}\) is called the quantum configuration space. This space is equipped with the Gel’fand topology, that is, the space of continuous functions \(C^0(\mathcal{A})\) on \(\mathcal{A}\) is given by the Gel’fand transforms of elements of \(\mathcal{B}\). Recall that the Gel’fand transform is given by \(\tilde{f}(A) := \tilde{A}(f) \forall A \in \mathcal{A}\). It is a general result that \(\mathcal{A}\) with this topology is a compact Hausdorff space. Obviously, the elements of \(\mathcal{A}\) are contained in \(\mathcal{A}\) and one can show that \(\mathcal{A}\) is even dense \([7]\). Generic elements of \(\mathcal{A}\) are, however, distributional.

The idea is now to construct a Hilbert space consisting of square integrable functions on \(\mathcal{A}\) with respect to some measure \(\mu\). Recall that one can define a measure on a locally compact Hausdorff space by prescribing a positive linear functional \(\chi_\mu\) on the space of continuous functions thereon. The particular measure we choose is given by \(\chi_{\mu_0}(\tilde{T}_I) = 1\) if \(I = \{p, 0, \tilde{I}\}\) and \(\chi_{\mu_0}(\tilde{T}_I) = 0\) otherwise. Here \(p\) is any point in \(\Sigma\), \(0\) denotes the trivial representation and \(\tilde{1}\) the trivial contraction matrix. In other words, (Gel’fand transforms of) spin-network functions play the same role for \(\mu_0\) as Wick-polynomials do for Gaussian measures and like those they form an orthonormal basis in the Hilbert space \(\mathcal{H} := L^2(\mathcal{A}, d\mu_0)\) obtained by completing their finite linear span \(\Phi\).

An equivalent definition of \(\mathcal{A}\), \(\mu_0\) is as follows:

\(\mathcal{A}\) is in one to one correspondence, via the surjective map \(H\) defined below, with the set \(\mathcal{A} := \text{Hom}(\mathcal{X}, G)\) of homomorphisms from the groupoid \(\mathcal{X}\) of composable, holonomically independent, analytical paths into the gauge group. The correspondence is explicitly given by \(\mathcal{A} \ni \tilde{A} \mapsto H_\tilde{A}(e) \in \text{Hom}(\mathcal{X}, G)\) where \(\mathcal{X} \ni e \mapsto H_\tilde{A}(e) := \tilde{A}(h_e) = \tilde{h}_e(A) \in G\) and \(\tilde{h}_e\) is...
the Gel’fand transform of the function $\mathcal{A} \ni A \mapsto h_e(A) \in G$. Consider now the restriction of $\mathcal{A}$ to $\mathcal{X}_\gamma$, the groupoid of composable edges of the graph $\gamma$. One can then show that the projective limit of the corresponding cylindrical sets $\overline{\mathcal{A}}_\gamma := \text{Hom}(\mathcal{X}_\gamma, G)$ coincides with $\overline{\mathcal{A}}$. Moreover, we have $\{\{H(e)\}_{e \in E(\gamma)}; H \in \overline{\mathcal{A}}_\gamma\} = \{\{H_A(e)\}_{e \in E(\gamma)}; A \in \overline{\mathcal{A}}\} = G^{\ell(E(\gamma))}$.

Let now $f \in B$ be a function cylindrical over $\gamma$ then

$$\chi_{\mu_0}(\tilde{f}) = \int_{\mathcal{A}} d\mu_0(\tilde{A}) \tilde{f}(\tilde{A}) = \int_{G^{\ell(E(\gamma))}} \otimes_{e \in E(\gamma)} d\mu_H(h_e) f_\gamma(\{h_e\}_{e \in E(\gamma)})$$

where $\mu_H$ is the Haar measure on $G$. As usual, $\mathcal{A}$ turns out to be contained in a measurable subset of $\overline{\mathcal{A}}$ which has measure zero with respect to $\mu_0$.

Let $\Phi_\gamma$, as before, be the finite linear span of spin-network functions over $\gamma$ and $\mathcal{H}_\gamma$ its completion with respect to $\mu_0$. Clearly, $\mathcal{H}$ itself is the completion of the finite linear span $\Phi$ of vectors from the mutually orthogonal $\Phi_\gamma$. Our basic coordinates of $M_\gamma$ are promoted to operators on $\mathcal{H}$ with dense domain $\Phi$. As $h_e$ is group-valued and $P^e$ is real-valued we must check that the adjointness relations coming from these reality conditions as well as the Poisson brackets (2.6) are implemented on our $\mathcal{H}$. This turns out to be precisely the case if we choose $\hat{h}_e$ to be a multiplication operator and $\hat{P}^e = i\hbar \kappa X^e_j/2$ where $X^e_j = X_j(h_e)$ and $X^j(h)$, $h \in G$ is the vector field on $G$ generating left translations into the $j$–th coordinate direction of $\text{Lie}(G) \equiv T_h(G)$ (the tangent space of $G$ at $h$ can be identified with the Lie algebra of $G$) and $\kappa$ is the coupling constant of the theory. For details see [11], [47].

3 Coherent States from a Coherent State Transform

In the first subsection of this section we will recall the state of the art of families of coherent state transforms which have been defined in the literature already. We point out advantages and disadvantages of one transform as compared to another as well as general properties of every transform and draw attention to some gaps that were left over. In the subsequent subsection we show how some of these gaps can be filled.

3.1 Review of Known Results

The first construction of coherent states that are relevant for the quantization of cotangent bundles over connected compact Lie groups $G$ is due to Hall [17] who showed how to construct a unitary map between the Hilbert space $L_2(G, d\mu_H)$ and a Hilbert space consisting of square integrable holomorphic functions of the complexification $G^\mathbb{C}$ of $G$ with respect to some measure $\nu$ that he explicitly constructed. In [18] these results were applied to our graph theoretic framework, namely one needs to repeat Hall’s construction, roughly speaking, for every holonomy associated with the various edges of a graph and to glue them together in a cylindrically consistent way. In [10] finally, Hall’s construction was generalized suitably and made applicable to very general phase spaces taking into account also some dynamical aspects. We will now outline the main idea, following [10]:

Central to the subject is the choice of a complex polarization of the classical phase space. In other words, we must choose the analogue of $z = x - ip$ of the harmonic oscillator. This is equivalent to choosing a certain generator $C$ (called complexifier in [10]) of the associated complex symplectomorphism which in the case of the harmonic oscillator consists of the the map $(x, p) \mapsto (z, p)$ and is easily seen to be $C = p^2/2$ if, as
usual, the symplectic structure is defined by $\{p, x\} = 1$. Namely we have

$$z = x + i\{x, C\} = \sum_{n=0}^{\infty} \frac{i^n}{n!}\{x, C\}_n$$

where the multiple Poisson bracket is inductively defined by $\{f, g\}_0 = f$, $\{f, g\}_{n+1} = \{\{f, g\}_n, g\}$. It is important for the existence of the coherent state transform that the polarization is a positive Kähler polarization, in other words, that the generator $C$ is a positive function on the phase space. We will see this in a moment.

The next step consists in the quantization. Following the rule that Poisson brackets be replaced by commutators times $1/(i\hbar)$ and phase space functions by operators in a suitable ordering we obtain for the harmonic oscillator

$$\hat{z} = \sum_{n=0}^{\infty} \frac{\hbar^{-n}}{n!}\{\hat{x}, \hat{C}\}_n = \hat{W}_t \hat{x} \hat{W}_t^{-1}$$

where we have defined

$$\hat{W}_t := e^{-\frac{C}{\hbar}}$$

where in this case $t = \hbar$ so that $\hat{W}_t = \exp(\hbar \Delta/2)$ where $\Delta = (\partial/\partial x)^2$ is the Laplacian on $\mathbb{R}$. Notice that with our conventions $\hat{p} = i\hbar \partial/\partial x$. One can check that in the case of the harmonic oscillator this gives correctly the annihilation operator $\hat{z} = \hat{x} - i\hat{p}$. We see that the generator $C$ naturally gives rise to the map $\hat{W}_t$ which due to the positiveness of the operator $\hat{C}$ defines a self-adjoint contraction semi-group of bounded operators.

It is for this reason that the following map, called the kernel of the coherent state transform, is well-defined

$$\rho^t(y, x) := (\hat{W}_t \delta_y)(x)$$

which for the harmonic oscillator is easily seen to be the standard heat kernel on $\mathbb{R}$, $\delta_y$ being the $\delta$ distribution with respect to $dx$ supported at $x$.

The coherent states themselves arise as the analytic continuation of the kernel, that is

$$\psi^t_z(x) := \rho^t(y, x)_{y \rightarrow z}$$

which exists, again, because the operator $\hat{C}$ is positive. It can be shown for the harmonic oscillator that the naturally arising map

$$\hat{U}_t := \hat{K}\hat{W}_t,$$ 

where $\hat{K}$ denotes analytic continuation, is a unitary map between $\mathcal{H} = L^2(\mathbb{R}, dx)$ and $\mathcal{H}^\mathbb{C} = L^2(\mathbb{C}, d\nu_t) \cap \text{Hol}(\mathbb{C})$ where the latter denotes the space of square integrable holomorphic functions on $\mathbb{C}$ with respect to a measure $\nu_t$ which is constructed from $\rho_t$. For the case of the harmonic oscillator this latter Hilbert space is the familiar Bargmann-Segal-Fock space.

In [47] Hall observed that the case of the harmonic oscillator can be naturally extended to the case of a cotangent bundle over a connected compact Lie group $G$, once the following substitutions are made:

$\mathbb{R} \rightarrow G$, $dx \rightarrow d\mu_H(h)$, $\mathbb{C} \rightarrow G^\mathbb{C}$, $\Delta \rightarrow \Delta_G$ where $G^\mathbb{C}$ is the complexification of $G$ and $\Delta_G$ denotes the Laplace-Beltrami operator on $G$. In particular, he constructed the map $\hat{U}_t$ and the measure $\nu_t$. What he did not analyze, except for phase space bounds, are the anlytical properties of the states $\psi_g(h)$ of $L^2$, that is, peakedness and Ehrenfest properties. Here and in what follows we will always take $h \in G$, $g \in G^\mathbb{C}$.

In [48], Hall’s results were applied to the case of a quantum gauge field theory. That is, one applies the coherent state transform as generated by the Laplace Beltrami operator
to each copy of the group $G$ associated with the edges $e$ of a graph of a cylindrical function and obtains a function cylindrical over the same graph but with the holonomies taking values in the complexified gauge group. Thus, coherent states become functions of $g_e \in G^\mathfrak{q}, e \in E(\gamma)$. While this gives a satisfactory mathematical framework for the construction of measures on $G^\mathfrak{q}$, the physics of this map was not understood: namely, not only do we need square integrable functions on $G^\mathfrak{q}$ but we also need to know what the complex connection is which gives rise to the complexified holonomies, that is, we need to know the map $(A, E) \mapsto A^\mathfrak{q}$ that expresses the complex connection as a function of the real phase space. Otherwise, for instance expectation values which will be functions of the $g_e$ cannot be interpreted in terms of the $(A, E)$ and thus semi-classical analysis cannot be developed because, say solutions to the Einstein equations, are formulated in terms of the latter.

In order to determine $A^\mathfrak{q}$ one must determine the classical limit of the operator which on cylindrical functions reduces to $\Delta_\gamma := \sum_{e \in E(\gamma)} \Delta(h_e)$ where $h_e$ is the holonomy of the real connection of the $G$–bundle along the edge $e$ of $\gamma$. The problem is, that such a classical limit does not exist!

To see this, notice that $[17]$ roughly $-\Delta(h_e) \propto (\dot{E}(S_e)_i/h)^2$ where $S_e$ are mutually disjoint analytic surfaces each of which intersects the graph only in one point which is an interior point of both $e$ and $S_e$ (for definiteness, that intersection can be chosen transversal). However, it is not possible to write down a single operator which reproduces $\Delta_\gamma$ for every $\gamma$ and has a classical limit as a well-defined function on the classical phase space $M$. Namely, suppose first that $\Sigma$ is compact. Since the graph $\gamma$ is arbitrary we may consider a net of finer and finer graphs $\gamma_\epsilon$ which in the limit $\epsilon \to 0$ fill all of $\Sigma$. Let us choose the $\gamma_\epsilon$ to be such that $\gamma_\epsilon \subset \gamma_{\epsilon'}$ for $\epsilon < \epsilon'$ and to be (subsets of) cubic lattices of spacing $\epsilon$ with respect to some spatial background metric. If $V$ is the volume of $\Sigma$ as measured by that metric, then in $D$ spatial dimension one will have an order of $V/\epsilon^D$ vertices in $\gamma_\epsilon$ each of which accounts for $D$ surfaces of area of order $\epsilon^{D-1}$. We see that in the classical limit for sufficiently small $\epsilon$, using the smoothness of the classical fields

$$\Delta_{\gamma_\epsilon} \to [-\epsilon^{2(D-1)} \sum_{v \in V(\gamma_\epsilon)} \sum_{l=1}^{D} [E_i^a(v)n^I_a(v)]^2] [1 + O(\epsilon)] \quad (3.7)$$

where the sum runs over the vertices of $\gamma_\epsilon$, $n^I_a(v)$ is the normal of the surface $S_{e^I(v)}$ and $e^I(v)$ is an edge of $\gamma_\epsilon$ that starts at $v$ and runs into the $I$th coordinate direction. This object has a chance to converge in the limit $\epsilon \to 0$ to a well-defined classical quantity only if $2(D-1) = D$, i.e. $D = 2$, so that in fact an integral results. For $D < 2$ this object diverges and for $D > 2$ it approaches zero for generic field configurations. One could replace $-\Delta_\gamma$ by $\sum_{e \in E(\gamma)} (-\Delta_e)^{D/(2(D-1))}$ in order to fix this (the eigenvalues would still behave as $j^{D/(D-1)} > j$), however, while this operator now does have a suitable classical limit at least for the net $\gamma_\epsilon$, it is no longer diffeomorphism covariant because it carries the sign of the background metric in the definition of the normals $n^I_a(x)$. If $\Sigma$ is not compact then (3.7) diverges even in $D = 2$ (or its just described replacement in any $D$) because in gravity the field $E$ does not decay at spatial infinity.

In conclusion, there seems to be no classical limit of the cylindrically defined operator $\Delta_\gamma$ as a well-defined, diffeomorphism covariant function on $M$ and therefore the interpretation of the $g_e$ remains obscure. This state of affairs is clearly unsatisfactory and there are basically two ways out:

Option 1) : One has to choose a different generator of the transform which actually comes from a well-defined function on $M$.

Option 2) : One gives up the requirement to have a complex continuum connection $A^\mathfrak{q}$ altogether and is satisfied with an interpretation of $g_e$ in terms of $h_e$ and certain other
functions of $A, E$ smeared over some surfaces $S_e$. Since the latter functions can be interpreted in terms of $(A, E)$ one also arrives at an interpretation of $g_e$ and this is sufficient in order to do semi-classical physics.

In the next two sections we will describe both options in detail.

Remark:
Before closing this section we would like to point out that a great deal of properties of the coherent states can be obtained already at this point, even if the interpretational issue raised above is not yet answered. Namely, let $\hat{C}_\gamma$ be the cylindrical projections of any complexifier and

$$\psi_{t, \gamma, \vec{g}} := (e^{-t\hat{C}_\gamma} \delta_{\gamma, \vec{h}})|_{\vec{h} \to \vec{g}}$$

(3.8)

where $\vec{g} = \{g^e\}_{e \in E(\gamma)}$ and similar for $\hat{h}$. Moreover, define the annihilation and creation operators respectively $(A, B, C, \ldots$ are group indices)

$$\hat{g}_{AB}^e := e^{-t\hat{C}_\gamma} \hat{h}_{\gamma, \vec{h}}^{eAB} e^{t\hat{C}_\gamma}$$

and

$$\hat{g}_{AB}^{e\dagger}$$

(3.9)

respectively. Then, without specifying $\hat{C}_\gamma$ at all, the following properties are automatically satisfied (obviously all of this is also theory independent, in the relations below, with the obvious changes, $\vec{h}$ could be any configuration coordinates for its cotangent bundle and $\vec{g}$ their analytical continuations):

\begin{itemize}
  \item[a)] *Eigenvalue Property*

  The coherent states (3.8) are eigenstates of any of the annihilation operators (3.9)

  $$[\hat{g}_{AB}^e \psi_{t, \gamma, \vec{g}}](\vec{h}) = [e^{-t\hat{C}_\gamma} \hat{h}_{\gamma, \vec{h}}^{eAB} \delta_{\gamma, \vec{h}}](\vec{h})|_{\vec{h} \to \vec{g}}$$

  $$= [e^{-t\hat{C}_\gamma} \hat{h}_{\gamma, \vec{h}}^{eAB} \delta_{\gamma, \vec{h}}](\vec{h})|_{\vec{h} \to \vec{g}} = \hat{g}_{AB}^e \psi_{t, \gamma, \vec{g}}(\vec{h})$$

  (3.10)

  simply because the $\delta$-distribution is a generalized eigenfunction of the multiplication operator in the configuration representation.

  \item[b)] *Expectation Values for Normal Ordered Operators*

  From a) it is trivial to see that

  $$\langle \psi_{t, \gamma, \vec{g}}, P(\{\vec{g}^\dagger, \vec{g}\}) \psi_{t, \gamma, \vec{g}} \rangle||\psi_{t, \gamma, \vec{g}}||^2 = P(\{\vec{g}, \vec{g}\})$$

  (3.11)

  where $P$ is any normal ordered polynomial of the creation and annihilation operators (annihilation operators to the right).

  \item[c)] *Saturation of the Unquenched Heisenberg Uncertainty Relation*

  Define the symmetric operators

  $$\hat{x}_{AB}^e := \frac{1}{2}(\hat{g}_{AB}^e + (\hat{g}_{AB}^e)^\dagger), \quad \hat{y}_{AB}^e := \frac{1}{2i}(\hat{g}_{AB}^e - (\hat{g}_{AB}^e)^\dagger)$$

  (3.12)

  then again with a) it is trivial to see that for the fluctuations we find

  $$\frac{\langle \psi_{t, \gamma, \vec{g}}(\hat{x}_{AB}^e - x_{AB}^e)^2 \psi_{t, \gamma, \vec{g}} \rangle}{||\psi_{t, \gamma, \vec{g}}||^2} = \frac{\langle \psi_{t, \gamma, \vec{g}}((\hat{y}_{AB}^e - y_{AB}^e))^2 \psi_{t, \gamma, \vec{g}} \rangle}{||\psi_{t, \gamma, \vec{g}}||^2}$$

  $$= \frac{1}{2} \frac{\langle \psi_{t, \gamma, \vec{g}}[\hat{x}_{AB}^e, \hat{y}_{AB}^e] \psi_{t, \gamma, \vec{g}} \rangle}{||\psi_{t, \gamma, \vec{g}}||^2}$$

  (3.13)
Reproducing Property

The connection between the coherent state transform \( \hat{U}_t : \mathcal{H}_\gamma \mapsto \mathcal{H}_\gamma^\mathbf{C} \), defined analogously to (3.6), and the coherent states is summarized by the following reproducing property, valid for any \( \psi \in \mathcal{H}_\gamma \):

\[
(\hat{U}_t \psi)(\vec{g}) = \langle \psi_{\gamma, \vec{g}^*}, \psi \rangle \tag{3.14}
\]

where \( g \mapsto g^* \) is the unique involution on \( G^\mathbf{C} \) that preserves \( G \) (this formula can be proved by using the expansion of the group \( \delta \) distribution in terms of characters according to the Peter&Weyl theorem, see e.g. [40]).

The additional properties that one would like the coherent states to possess and which do not directly follow from the general form (3.8) are the following, for which we need now the expression for \( g_e \) in terms of \( A,E \):

d) **Peakedness Properties**

We want the coherent states (3.8) to be peaked in the configuration representation at \( A \), in the momentum representation at \( E \) and in the Bargmann-Segal representation \( H_\mathbf{C} \) (the image of \( \mathcal{H} \) under \( \hat{U}_t \) to be defined for general \( \hat{C} \) along the lines outlined in [46]) at \( (A,E) \). For instance, if with respect to \( \gamma \) we take as elementary configuration coordinates the holonomies \( h_e \) and as elementary momentum coordinates the \( E_i(S_e) \) considered above and if we know the explicit formula \( g_e(\{h_e', E_i(S_e')\}) \) which is supposed to be invertible, then we want the probability amplitudes for the coherent state with label \( \vec{g} \) in the configuration –, momentum – and Segal-Bargmann Hilbert spaces respectively to be peaked at \( h_e(\vec{g}) \), \( [E(S_e)](\vec{g}) \), \( \vec{g} \) respectively. Notice that if we take \( \vec{g} \in G_{[E(\gamma)]} \) then as \( t \to 0 \) \( \psi_{\gamma, \vec{g}}(\vec{h}) \) on \( G_{[E(\gamma)]} \) is supported at \( \vec{g} = \vec{h} \) for any choice of complexifier \( \hat{C}_\gamma \) since by its very definition \( \psi_{\gamma, \vec{g}} \) approaches \( \delta(\vec{g}, \vec{h}) \) as \( t \to 0 \).

e) **Ehrenfest Property**

While expectation values of normal ordered polynomials of alternation operators already have the correct expectation values without quantum corrections, we want that to leading order in \( t \) or \( \bar{h} \) also the elementary operators associated with \( h_e, E(S_e) \) as well as their various commutators divided by \( i\hbar \) have the correct expectation value guaranteeing the correct infinitesimal quantum dynamics. The fact that the alternation operators do have the correct expectation values makes it plausible that also this property can be verified for any \( \hat{C}_\gamma \).

f) **Overcompleteness**

The coherent states should be overcomplete in order to be able to approximate any possible physical situation. Overcompleteness follows automatically if the coherent state transform \( \hat{U}_t : \mathcal{H} \mapsto \mathcal{H}_\gamma^\mathbf{C} \) is unitary since then that map is onto. More precisely, due to the reproducing property (see e.g. [40]) :

\[
1_{\mathcal{H}_\gamma} = \int_{(G^\mathbf{C})_{[E(\gamma)]}} d\nu_t(\vec{g})|\psi_{\gamma, \vec{g}^*}^t| <\psi_{\gamma, \vec{g}^*}^t> \tag{3.15}
\]

A method for a constructive proof for general \( \hat{C} \), up to analytical details, is given in [46]. Namely, the measure \( \nu_t \) can be uniquely determined if the operator \( \hat{W}_t \) is well-defined and if the cylindrical family of measures constructed in [46] can be extended to a \( \sigma \)-additive measure on the projective limit of the cylindrical projections of spaces of complex quantum connections that one can define in analogy to [48]. Overcompleteness is actually also rather plausible for general \( \hat{C} \) by inspection because these states arise as the “evolution” under \( \hat{W}_t \) of the \( \delta \) distributions. Now the
latter provide a complete basis of generalized functions and $\tilde{W}_t$ is invertible on a dense domain of $\tilde{W}_t^{-1}$ (the inverse is certainly not bounded).

h) **Diffeomorphism Covariance**

The coherent states should, as all the other states of the Hilbert space, transform covariantly under the diffeomorphism group. This will be the case provided that the operator $\hat{C}_\gamma$ is itself diffeomorphism covariant (does not make use of any background structure), specifically, $\hat{U}(\varphi) \hat{C}_\gamma \hat{U}(\varphi)^{-1} = \hat{C}_{\varphi^{-1}(\gamma)}$ where $\text{Diff}(\Sigma) \ni \varphi \mapsto \hat{U}(\varphi)$ is the unitary representation of the diffeomorphism group described in [10].

### 3.2 Option 1) : The Volume Operator as the Complexifier

In this section we modify the coherent state transform by choosing a different complexifier. We will argue now that (a suitable power of) the “volume” of a region $R \subset \Sigma$

$$V(R) := \int_R d^Dx \sqrt{\text{det}(q)(x)}$$

(3.16)

is the most natural candidate. In case that $\Sigma$ is compact or that classically the fields vanish sufficiently fast at spatial infinity as in Yang-Mills theory, we will take $R = \Sigma$ in the sequel. Otherwise, we will take $R$ to be a bounded region to begin with and send $R \to \Sigma$ only after all calculations have been performed. Here,

$$\text{det}(q) := \sqrt{\text{det}(E^a_i E^b_j)}$$

(3.17)

and (3.16) is called the volume functional because in the case of general relativity $E^a_i = \sqrt{\text{det}(q)} e^a_i$ where $e^a_i$ is the D-bein field and $q_{ab}$ is the D-metric intrinsic to $\Sigma$.

The reasons are as follows:

(i) As it is clear from the discussion in the previous section, it is important that $C$ is a positive semi-definite function on the phase space as this translates into a positive definite operator upon quantization. The volume has this property.

(ii) Notice that even in the case of gauge theories on a background metric the electric field is a Lie algebra valued vector density of weight one. Therefore, $E^a_i E^b_j = \text{det}(q) q^{a\bar{b}}$ is in general a gauge invariant tensor density of weight two. Hence, (3.17) is a scalar density of weight two which can be constructed without any background structure and therefore the volume functional is **diffeomorphism invariant** if $R = \Sigma$ and **diffeomorphism covariant** if $R \subset \Sigma$ ! This is important in order to obtain diffeomorphism covariant coherent states in the case of diffeomorphism invariant quantum field theories of connections.

(iii) As we want to start with a Hilbert space which consists of square integrable functions of connections for which the connection operator is a multiplication operator, it is natural to consider an operator which is entirely constructed from the electric field operator so that the analogue of $z$ is given by $Z^j_a = A^j_a + if^j_a(E)$. The volume density is the simplest scalar density of weight one entirely constructed from electric fields.

(iv) Using the symplectic structure $\{ A^j_a(x), E^b_j(y) \} = -\kappa \delta^b_a \delta^*_j \delta(x,y)$ where $\kappa$ is the coupling constant and

$$C(R) := \frac{1}{\lambda \kappa} V(R)^n$$

(3.18)
where $n \geq 1$ is a positive real number and $\lambda$ is a positive, possibly dimensionful, parameter so chosen that for $x \in \mathbb{R}$

$$f^j_a(x) := \{C(R), A^j_a(x)\} = \frac{nV(R)^{n-1}}{\lambda} \frac{\partial z^{(D-1)}}{\partial E^a_j} = nV(R)^{n-1} \frac{e^i_a}{\lambda(D-1)}$$

has dimension of inverse length we easily see that $E^a_j$ can be reconstructed from $f^j_a$ and therefore the complex connection $Z^j_a$ together with its complex conjugate contains full phase space information. The field $e^i_a$ is the co-$D$-bein in general relativity.

Notice that $Z^j_a = A^j_a - i f^j_a$ really transforms as a $G$-connection under gauge transformations since $\delta Z = -d\Lambda + [\Lambda, A] + i[\Lambda, e] = -d\Lambda + i[\Lambda, Z]$ so that the coherent state transform is both diffeomorphism covariant and gauge covariant.

Finally, to be useful, it is necessary that one can quantize the generator. But this is the case for the volume functional in any dimension along the lines of [16, 18, 19, 20]. Moreover, on the Hilbert space that we have chosen in section 2 the spectrum of that operator is entirely discrete and, although very complicated, explicitly known at least in terms of matrix elements [20, 11].

Upon quantization $\hat{E}^a_i = i\hbar\kappa\delta/\delta A^i_a$ and the generator takes the following form on cylindrical functions

$$\hat{C}(R) = \frac{(\hbar\kappa)^n R_n}{\lambda\kappa} \hat{\nu}$$

where $\hat{\nu}$ is a dimensionless operator constructed from invariant vector fields corresponding to the copies of the group associated with the edges of graphs. The coherent state transform is then generated by

$$W_t = e^{-te^i}$$

where $t = (\hbar\kappa)^n R_n^{-1}/\lambda$ is a dimensionless parameter which vanishes as $\hbar \rightarrow 0$. For instance, for general relativity in $3 + 1$ dimensions, $\hbar\kappa$ is the Planck area.

Next, we define coherent states in analogy to (3.5). The idea is to define coherent states graphwise, which means that the state approximates a certain point in the classical phase space on that graph only. We can do this for every graph which is contained in the region $R$. In case that $R \neq \Sigma$ this does not exclude the possibility to have graphs which run to spatial infinity: We can use the asymptotic structure available and allow only such graphs which run to spatial infinity inside fixed “thin tubes” of $R$ which have finite Lebesgue measure. Notice that these complications would not be necessary if we would choose $n = 1$. In general we cannot choose $n = 1$ for reasons explained below, see also the model described in section 6.

The fundamental definition is

$$\psi^i_{\gamma,Z}(A) := (\hat{W}_t^i)\delta^i_{\gamma,A'}(A) |_{A' \rightarrow Z}$$

where the $\delta$ distribution in (3.22) is defined by

$$\delta^i_{\gamma,A'}(A) := \sum_{j,j'} T^i_{\gamma,j,j'}(A)T^{j,j'}_{\gamma,j,j'}(A')$$

and where the sum is over all possible not necessarily gauge invariant spin-network functions on that graph $\gamma$ if we work at the non-gauge invariant level while it is over all possible
gauge invariant functions only if we work at the gauge invariant level. It is important to stress that in (3.23) we include only spin-network states whose vector of representations \( \vec{j} \) does not contain a zero entry.

We thus obtain coherent states \( \psi_{\gamma,Z}^t \) which have the property to be orthogonal, 
\[
< \psi_{\gamma,Z}^t, \psi_{\gamma',Z'}^t >= 0, \text{ if their underlying graphs are different, } \gamma \neq \gamma'.
\]
We also define coherent states of a different type,
\[
\Psi_{\gamma,Z}^t := \sum_{\gamma' \subset \gamma} \psi_{\gamma',Z}^t
\]  
(3.24)
where the sum extends over all subgraphs of \( \gamma \) which can be obtained from \( \gamma \) by deleting edges of \( \gamma \) in all possible ways (if the state is to be gauge invariant then the sum extends over closed subgraphs only). The idea is not to take inner products of states (3.24) with different \( \gamma \) but only between those with the same \( \gamma \) but different \( Z, Z' \). In other words, one first restricts the Hilbert space \( \mathcal{H} \) to the completion \( \mathcal{H}_\gamma \) of the span of spin-network states over closed subgraphs of \( \gamma \) and then one lets \( \gamma \) grow. Recall that given two piecewise analytic graphs, their union is still a piecewise analytic graph. (We can not immediately say, an (in)finite number of \( N = (e_1, \ldots, e_n) \), \( n < \infty \), given as a finite collection of edges, is played by the mode label \( \vec{k} = (k_1, \ldots, k_n) \), \( n < N \), given by a finite collection of non-negative integers. The analogues of the states \( \Psi_{\gamma,Z}^t = \Psi_{\gamma,\vec{g}}^t \) with \( \vec{g} = (h_{e_1}(Z), \ldots, h_{e_n}(Z)) \) is given by the coherent state for \( n \) uncoupled harmonic oscillators \( \Psi_{\vec{k},\vec{z}}^t \), with an array of complex numbers \( \vec{z} = (z_{k_1}, \ldots, z_{k_n}) \) corresponding to the mode vector \( \vec{k} \). The projective limit of taking the “biggest possible graph” corresponds to taking the (in)finite direct product limit \( \vec{k} \to (1, 2, \ldots, N) \) and one obtains the full coherent state \( \Psi_Z^t, Z = (z_1, z_2, \ldots, z_N) \). One does not compute inner products between states with different \( \vec{k} \) but only with different \( \vec{z} \) for the same \( \vec{k} \) which models the properties of \( \Psi_Z^t \) on its cylindrical projections \( \Psi_{\vec{k},\vec{z}}^t \). The analogues of the states \( \psi_{\gamma,Z}^t \) are the states \( \psi_{\vec{k},\vec{z}}^t = \Psi_{\vec{k},\vec{z}}^t - \Omega < \Omega, \Psi_{\vec{k},\vec{z}}^t \) where we have taken out the vacuum mode so that \( < \psi_{\vec{k},\vec{z}}^t, \psi_{\vec{k}',\vec{z}}^t > = 0 \) for \( \vec{k} \neq \vec{k}' \).

Notice that the restriction of \( \Psi_{\gamma,Z}^t(A) \) to real valued \( Z = A' \) is the “heat kernel” \( \rho_{\gamma,t}(A, A') \) for the “heat equation”
\[
[\partial/\partial t + \hat{v}]\rho_{\gamma,t}(A, A') = 0 \text{ such that } \rho_{\gamma,0}(A, A') = \delta_\gamma(A, A'). \quad (3.25)
\]
As the volume operator is an essentially self-adjoint, positive semi-definite operator with discrete spectrum which leaves the subspace of \( \mathcal{H} \) spanned by spin-network states of given \( \gamma, \vec{j} \) invariant, we can diagonalize it and define another orthonormal basis of eigenstates \( T_{\gamma,\lambda,n} \) of \( \hat{v} \) where \( \lambda \) labels the eigenvalue and the integer \( n \) its degeneracy. We can then write (3.23) alternatively as
\[
\delta_{\gamma,A}(A') := \sum_{\lambda,n} T_{\gamma,\lambda,n}(A') T_{\gamma,\lambda,n}(A) \quad (3.26)
\]
which allows us to explicitly compute the coherent states as
\[
\psi_{Z,\gamma,t}(A) = \sum_{\lambda,n} e^{-t\lambda} T_{\gamma,\lambda,n}(Z) T_{\gamma,\lambda,n}(A). \quad (3.27)
\]
The function \((3.27)\) is to be understood in the following sense: Given a point in the classical phase space \(A, E\), compute the \(G^\mathbf{J}\) connection \(Z = A - i f(E)\) and from this its holonomies \(h^\mathbf{J}_e := h_e(Z)\) for each edge \(e\) of \(\gamma\). Then insert these elements of \(G^\mathbf{J}\) into the eigenfunctions appearing in the series in \((3.27)\).

Several points of worry arise when looking at \((3.27)\):

(i) Does the series in \((3.27)\) converge, in the sup-norm topology with respect to \(G^n\), where \(n\) denotes the number of edges of \(\gamma\)? This will, in particular, not be the case if one of the \(\lambda\) has infinite multiplicity. The volume operator as defined in \([16, 18, 19, 20]\), however, has presumably precisely this property for the zero eigenvalue, at least in the case of general relativity in 3+1 dimensions which requires \(G = SU(2)!\) Thus, in this case, in order to make sense of \((3.27)\) we must discard the zero volume eigenstates even from the kinematical Hilbert space. (In particular this has to be done at the gauge non-invariant level). This is quite satisfactory because the classical phase space can be viewed as a cotangent bundle over smooth, signature \((+, \ldots, +)\) \(D\)-metrics for which vanishing volume, that is, vanishing determinant of the three-metric, is not allowed. That the signature is \((+, \ldots, +)\) is guaranteed if we restrict to states with non-vanishing expectation value for the area operator \([16, 17]\) for every surface that intersects the graph.

But even if all eigenvalues have finite multiplicity, the series does not necessarily converge: while \(T_{\gamma,j,n}\) is a bounded function of \(G^n\), it is not any longer so of \((G^\mathbf{J})^n\) because that group is not compact. What is needed, roughly speaking, is the following: we can decompose the \(T_{\gamma,j,n}\) in terms of spin-network functions which turns the above series into a series over \(\vec{j}, \vec{J}\). The coefficient of \(T_{\gamma,j,\vec{J}}(Z)\) is of the form \(e^{-t\lambda(j,\vec{J})}\) times something that grows at most linearly with \(T_{\gamma,j,\vec{J}}(Z)\) grows exponentially with \(\vec{j}, \vec{J}\) for \(Z\) in the non-compact directions of \(G^\mathbf{J}\). Thus, for the series to converge it would be sufficient if

\[
\lambda(\vec{j}, \vec{J}) \geq c\left(\sum_{e \in E(\gamma)} j_e + \sum_{\nu \in V(\gamma)} J_\nu\right)^{1+\epsilon} \tag{3.28}
\]

where \(c\) is a positive number independent of \(\vec{j}, \vec{J}\) and \(\epsilon\) can be any positive number. The criterion \((3.28)\) is a condition on the spectrum of \(\hat{v}\) which needs to be checked to hold. This is the reason why we have allowed for a power \(n\) different from \(n = 1\) in \((3.18)\); by taking \(n\) sufficiently large we can guarantee that criterion \((3.28)\) is satisfied.

More precisely we have the following:
Looking at \((3.16), (3.18)\) and the explicit expression for the volume operator as derived in \([16, 18, 19, 20]\) we infer that the electric fields get, roughly speaking, replaced by right invariant vector fields \(X^i_e := X^i(h_e)\) on the various copies of \(G\) corresponding to the edges of \(\gamma\). As those act on spin-network functions roughly by multiplication by \(j_e\), we find the eigenvalues of the volume operator to be of the form

\[
\lambda(\vec{j}, \vec{J}) = (P_{2D}(\vec{j}, \vec{J}))^{n/(2(D-1))} \tag{3.29}
\]

where \(P_{2D}\) is a homogenous, positive polynomial of degree \(2D\) which depends non-trivially on all the variables \(\vec{j}, \vec{J}\). Obviously, taking \(n > 2(D-1)\) we have good chances to satisfy \((3.28)\). Presumably, \(n > D - 1\) will be sufficient since because of gauge invariance the \(\vec{j}, \vec{J}\) do not have independent ranges. For instance, for \(SU(2)\), due to gauge invariance the sum of all but one, say \(j_{e_0}\), of those \(j_e\) that correspond to \(e\)'s which meet at a common vertex must always exceed the value of \(j_{e_0}\). Moreover, the value of \(J_\nu\) is bounded by the sum of all those \(j_e\). These relations hold for all of the vertices and thus there is a good chance that we can estimate \((3.29)\) as

\[
\lambda(\vec{j}, \vec{J}) \geq c(\max(\vec{j}, \vec{J}))^{n/(D-1)} \tag{3.30}
\]
which would be sufficient for any $D$. However, this must be checked in the case at hand. In particular, it could happen that the function $\lambda(\vec{j}, \vec{J})$ has "degenerate directions" in which case even large $n$ would not help to make the series converge.

(ii) Even if the series converges, are these coherent states square integrable? We easily see that the convergence of the series is sufficient for this to be the case. Namely, if the series converges, we compute the norm as

$$||\psi_Z||^2 = \sum_{\lambda, n} e^{-2t\lambda}|T_{\gamma, \lambda, n}(Z)|^2$$

which then certainly converges as well.

(iii) Finally, is the generalized projective limit $\Psi^t_Z$ of the states $\Psi^t_{\gamma, Z}$ square integrable? There is no, not even partial answer to this question available at the moment, however, notice that even the uncountably infinite direct product limit of an uncountably infinite number of harmonic oscillators is square integrable. This follows immediately from the Kolmogorov theorem \cite{72} for an uncountably infinite tensor product of probability measure (here: Gaussian measures) Hilbert spaces. Thus, the normalizability of $\Psi^t_Z$ is indeed conceivable.

(iv) If we can then verify the properties (a)-(h) mentioned above, what we will have achieved is that we have states that are peaked on a classical configuration $Z$ in the sense that the operator $\hat{g}^e$ corresponding to $g^e = h^e(Z)$ has expectation value $\bar{g}^e$, saturates the Heisenberg uncertainty bound etc. However, since all these properties (a)-(h) are verified for $g^e$ only, we must ask whether we can reconstruct $Z$ from all the $h^e(Z)$, that is, whether the holonomies separate the points on the space of smooth complexified connections.

This is a non-trivial question due to the presence of so-called null-rotations for non-compact gauge groups and amounts to proving a Giles’ theorem \cite{73} for non-compact gauge groups. At least for $SU(2)^\Phi = SL(2, \mathbb{C})$, this has been answered affirmatively in an appropriate sense in \cite{74} and we believe the proof to be valid generally for complexifications of compact connected gauge groups. If we work at the gauge non-invariant level, the proof is obvious since we just have to consider the limit of infinitesimal open paths.

We now argue that the coherent states (3.24) so constructed have very good chances to satisfy all the properties (d)-(h) mentioned above, assuming that there are no convergence problems even at the gauge non-invariant level. We will indicate the necessary modifications of the analysis when we restrict to the gauge invariant sector. The analysis is in fact quite general and can be generalized to the quantization of any field theory with a generalized projective structure, once a choice of the complexifier $C$ and a choice of polarization of the classical phase space has been made.

(d) The way in which these states are localized is obscure at the moment. In this paper we will just outline how one might prove this property. First of all, notice that the coherent states become, for real connections $Z$, just $\delta$ distributions on $\gamma$ in the semi-classical limit as $\hbar \to 0$ (that is, $t \to 0$, see (3.21), (3.25)). Thus, in the connection representation, the state $|Z, \gamma, t >$ is certainly peaked at $A = \Re(Z)$ as $t \to 0$ for $\Im(Z) = 0$ for any $\gamma$. What happens if $\Im(Z) \neq 0$ is unclear at the moment. Next, we want to study the state $|Z, \gamma, t >$ in the momentum or electric field representation which is nothing else than the spin-network representation (see \cite{20}). Now, the representation (3.27), with the $T_{\gamma, \lambda, n}$ written in the spin-network basis, is not immediately useful in order to study the behaviour of the state in the limit $t \to 0$ because the exponential terms become unity in
the limit \( t \to 0 \), that is, the convergence of the series worsens in the limit \( t \to 0 \). The idea is to use a Poisson summation formula which exists for all compact gauge groups \([3, 4]\) and which should transform the series into a series with coefficients of the form \( \exp(-\lambda(t, \vec{J})/t^\alpha) \) where \( \alpha \) is a positive number. In the limit \( t \to 0 \) then the leading term would be the one with \( \lambda \) closest to zero and this would be the peakedness property in the electric field representation. We will actually use this method in the next paper of this series \([50]\) for the original heat kernel complexifier.

(e) To prove the Ehrenfest property is very much like proving the peakedness property in the Bargmann Segal representation and also should be based on the Poisson summation formula. The reason is that expectation values of polynomials in the basic operators can be expanded, using overcompleteness of the coherent states, as a polynomial in the matrix elements between normalized coherent states \( \xi^t_{\gamma, \vec{g}} \) where the extra variables \( \vec{g}' \) as in (3.13) are integrated over with respect to \( d\nu_t(\vec{g}')||\psi^t_{\gamma, \vec{g}}||^2 \). But then the Ehrenfest property follows once we find for any elementary operator \( \hat{O} \) that

\[
< \xi^t_{\gamma, \vec{g}}, \hat{O} \xi^t_{\gamma, \vec{g}} > = O(\vec{g}) < \xi^t_{\gamma, \vec{g}}, \xi^t_{\gamma, \vec{g}} > (1 + O(t))
\]

which in turn should be easy to establish if the overlap function on the right hand side of this equality is peaked at \( \vec{g} = \vec{g}' \). But the latter property is just the same as the peakedness property in the Segal-Bargmann representation which can be seen generally from the reproducing property.

(f) As already said, the (over)completeness of the coherent states in the kinematical Hilbert space \( \mathcal{H} = L_2(\mathcal{A}, d\mu_0) \) would follow trivially if one could establish that the map (3.6), generalized to our context, is a unitary map between \( \mathcal{H} \) and a suitable \( L_2 \) space of holomorphic functions of complex connections with respect to a measure \( \nu_t \) because then the map \( \hat{W}_t \) would be onto, in particular. In addition, the general comments from the previous section apply.

(h) The coherent states of this section are diffeomorphism covariant by their very construction.

This concludes the general outline of how one might construct coherent states for quantum gauge theories from a coherent state transform which can also be interpreted in terms of a complex connection \( A^\mathbb{C} \). In \([50]\) we will, however, not use the volume operator as the complexifier for the following reasons:

i) The spectrum of the volume operator is not explicitly known. This lack of knowledge makes analytical proofs very hard although a numerical method is of course possible.

ii) More serious is the following observation: Unless \( V(R) \) itself is a polynomial function of the \( E_i(S) \), then even classically the \( g^t_{AB}, \bar{g}^t_{AB} \) do not form a * Poisson algebra for \( D > 2 \). This becomes obvious from the fact that while \( \{A^\mathbb{C}_a(x), A^\mathbb{C}_b(y)\} = \{E^a_j(x), E^b_k(y)\} = 0, \{A^\mathbb{C}_a(x), E^b_k(y)\} = -\delta^b_a \delta^j_k \delta(x, y) \) (the complexifier induces a canonical transformation) we have

\[
\{A^\mathbb{C}_a(x), A^\mathbb{C}_b(y)\} = \delta(x, y) \frac{\partial^2 V(R)}{\partial E^a_j(x) \partial E^b_k(x)} + \text{more}
\]

where “more” is non-distributional. Thus, since the connections are only smeared in one spatial direction inside a holonomy functional, it follows that for \( D > 2 \) the Poisson bracket \( \{g_{AB}(A^\mathbb{C}), \bar{g}_{AB}(A^\mathbb{C})\} \) is necessarily distributional or even ill-defined and does not lie in the original Poisson algebra any longer. This means that the fluctuations of the
\( \hat{x}_{AB}, \hat{g}_{AB} \) are ill-defined if the Ehrenfest property holds because the right hand side of (3.13) will then be proportional to \( \{ g_{AB}(A^0), \hat{g}_{AB}(A^0) \} \) to first order in \( t \). Whether or not this is bad is unclear, after all it is unnecessary to work with \( \hat{g}_{AB} \) itself. On the other hand, due to the eigenvalue property and the similarity with the creation and annihilation operator algebra it would be very convenient to have the \( \hat{g}_{AB} \) at one’s disposal.

Due to these difficulties we will turn to option ii) in the remainder of this paper and the subsequent issues of this series. It should be kept in mind, however, that option i) exists. Its obvious advantage is that one has an actual complex connection which implies that one can work entirely with graphs and never needs the additional dual polyhedronal decompositions which are a source of ambiguity.

### 3.3 Option 2) : The Heat Kernel Complexifier

In this section we will be satisfied with obtaining \( g_e \) as a definite function of the functions \( h_e, P_e \) described in section 2.4. We do not require that \( g_e \) is itself the holonomy along \( e \) for some complex connection \( A^0 \).

The results of this section hold for arbitrary compact, semisimple connected gauge groups and direct products of such with Abelian ones.

As we want to bring in Planck’s constant \( \hbar \) as a measure of closeness to classical physics, we need to spend a few moments on dimensionalities as in the previous section for the volume functional. The dimension of the time coordinate \( x^0 \) is taken to be the same as that of the spatial coordinates \( x^a \), namely \( [x^0] = [x^a] = \text{cm}^1 \) which can always be achieved by absorbing an appropriate power of the speed of light into the coupling constant \( \kappa \) of the theory.

We will take our connection one-form to be of dimension \( [A] = \text{cm}^{-1} \) so that its holonomy is dimensionless. In \( D + 1 \) spacetime dimensions the kinetic term of the classical action is given by

\[
A_{\text{kin}} = \frac{1}{\kappa} \int_R dt \int_{\Sigma} d^Dx \ E_i^a(x) \dot{A}_i^a(x)
\]

and its dimension is that of an action, that is, \( [A_{\text{kin}}] = [\hbar] \). In Yang-Mills theories the electric field is a first derivative of \( A_i^a \) and thus has dimension \( [E_i^a] = \text{cm}^{-2} \). In general relativity the metric components, the D-beins and also \( [E_i^a] = \text{cm}^0 \) are dimensionfree. It follows that in Yang-Mills (YM) theory the Feinstructur constant

\[
\alpha := \hbar \kappa \quad (3.32)
\]

has dimension \( [\alpha] = \text{cm}^{D-3} \) and in general relativity (GR) \( [\alpha] = \text{cm}^{D-1} \).

Let now \( \gamma \) be a graph and consider the symplectic manifold \((M_\gamma, \Omega_\gamma)\) introduced in section 2.4 with its canonical coordinates \( h_e, P_e : e \in E(\gamma) \). The electric flux variable (2.5) then has dimension \( [P_e] = \text{cm}^{D-3} \) in YM and \( \text{cm}^{D-1} \) in GR respectively and in general let \( [P_e] = \text{cm}^{n_D} \). Let now \( a \) be an arbitrary but fixed constant with the dimension of a length, \( [a] = \text{cm}^1 \), say \( a = 1 \text{cm} \) if \( n_D \neq 0 \) and let \( a \) be dimensionfree otherwise. Then we introduce the dimensionfree quantity

\[
p_e^i := \frac{P_e^i}{a^{n_D}} \quad (3.33)
\]

where \( n_D = n'_D \) if \( n'_D \neq 0 \) and \( n_D = 1 \) otherwise. Notice that a natural choice for a dimensionful constant in general relativity in any \( D > 1 \) would be \( a = 1 / \sqrt{|\Lambda|} \) where \( \Lambda \) is the (supposed to be non-vanishing) cosmological constant.
On the other hand, it is $E_i^a / \kappa$ which is canonically conjugate to $A_i^a$ rather than $E_i^a$ itself, therefore the brackets (2.6) get modified into

$$\{ h_e, h_e' \}_\gamma = 0$$
$$\{ \frac{P^e_i}{\kappa}, h_e' \}_\gamma = \delta_{ii}^e \frac{\tau_i}{2} h_e$$
$$\{ \frac{P^e_i}{\kappa}, \frac{P^e_j}{\kappa} \}_\gamma = -\delta_{ij}^e f_{ij}^k \frac{P^e_k}{\kappa}$$

We are now ready to define the complexifier for the symplectic manifold $M_\gamma$, it is given by

$$C_\gamma := \frac{1}{2\kappa a^{n_D}} \sum_{e \in E(\gamma)} \delta_{ij}^e P^e_i P^e_j$$

and since $C_\gamma$ is gauge invariant it will pass to the reduced phase space. Using the partial order $\prec$ of [64] or section 2.1 it is immediately clear that $C_\gamma$ defines a self-consistently defined function on the $M_\gamma$, that is, for $\gamma \prec \gamma'$ we have $\{ p_{\gamma, \gamma'}^e, C_\gamma, p_{\gamma', \gamma}^e f_{ij} \} = p_{\gamma, \gamma'}^e \{ C_\gamma, f_{ij} \}$ for any $f_{ij} \in C^\infty(M_\gamma)$.

We can explicitly compute the complexified holonomy and complexified momenta for any compact, semi-simple gauge group $G$. Since $\{ P^e_i, C_\gamma \} = 0$ (gauge invariance of $C_\gamma$) we have

$$\{ h_e, C_\gamma \}_\gamma = -P^e_i \frac{\tau_i}{2 a^{n_D}} h_e = -p^e_i \frac{\tau_i}{2} h_e$$
$$\{ h_e, C_\gamma \}_\gamma(2) = \frac{1}{a^{2n_D}} P^e_i P^e_j \frac{\tau_i \tau_j}{4} h_e = (-p^e_j \frac{\tau_j}{2})^2 h_e$$

where we define generally $p^e := \sqrt{|p^e_j|^2}$. In the second line of (3.36) we have made use of the fact that $G$ is semi-simple so that the structure constants are completely skew and so $\{ p^e_j, C_\gamma \} = 0$.

We therefore conclude that the complexification of $h_e$ is given by

$$h^q_e := g_e = \sum_{n=0}^{\infty} \frac{i^n}{n!} \{ h_e, C \}(n)$$
$$= \left[ \sum_{n=0}^{\infty} \frac{i^n}{n!} (-p^e_j \frac{\tau_j}{2})^n \right] h_e$$
$$= e^{-i \tau_j p^e_j / 2} h_e$$

and similarly $P^e_i = P^e_i$. Thus we have established the following.

**Lemma 3.1**

The complexification of the holonomy for compact and semisimple $G$ is given directly as a left polar decomposition, where the right unitary factor is the holonomy of the compact gauge group while the left positive definite hermitian factor is just the exponential of $-ip^e_j \tau_j / 2$.

For $G = U(1)$ the generator $\tau_j / 2$ has to be replaced by the imaginary unit $i$.

Notice that (3.37) makes sense since $p^e_j$ is dimensionless. Moreover, we have naturally stumbled on the diffeomorphism [47]

$$\Phi : T^*(G) \mapsto G^\Phi; (p^i, h) \mapsto g := H h = e^{-ip^i \tau_i / 2} h$$

The diffeomorphism (3.38) has a further consequence : $(T^*(G), \omega)$ is a symplectic manifold while $G^\Phi$ is a complex manifold. Thus, $T^*(G)$ is a symplectic manifold with a complex
structure which, as one can show ([17, 54] and references therein), is $\omega$-compatible. In fact, $\omega$ is just given by (3.34) with $P^c_i$ replaced by $p_i$ and the label $e = e'$ dropped. Therefore, $T^*(G)$ is in fact a Kähler manifold and a Segal-Bargmann representation (wave functions depending on $g$) corresponds to a positive Kähler polarization [51].

Finally, let us compute the Segal-Bargmann transform corresponding to $C_\gamma$ as in [17]. As follows from the previous section, we have in the connection representation (wave functions depending on the $h_e$)

$$\hat{P}_j^e = \frac{i\hbar \kappa}{2} X_j^e \text{ where } X_j^e = X_j(h_e),$$

and $X_j(h)$ denotes the right invariant vector fields on $G$ at $h$, that is $X_j(h) := \text{tr}(\tau_j h^T \partial / \partial h)$. Thus, the coherent state transform is (following the notation of [46])

$$\hat{W}_{\gamma t} := e^{-\frac{C_\gamma}{\kappa}} = e^{\frac{t}{2} \Delta_{\gamma}}$$

where we have defined the Laplacian on $\gamma$ by

$$\Delta_{\gamma} = \sum_{e \in E(\gamma)} \Delta_e, \quad \Delta_e = \frac{1}{4} \delta^{ij} X_i^e X_j^e$$

and the heat kernel time parameter has the following interpretation in terms of the fundamental constants of the theory

$$t := \frac{\hbar \kappa}{a^{D+D}}.$$  

(3.42)

Notice that $a$ is just a parameter that we have put in by hand to make things dimensionless, for instance, it could be 1cm in quantum general relativity in $D+1 = 4$ spacetime dimensions or $a = 10^5$ for Yang-Mills in $D+1 = 4$ and thus is “large”. The semiclassical limit $\hbar \to 0$ thus corresponds to $t \to 0$. That $t$ is a tiny positive real number will be crucial in all the estimates that we are going to perform in this and the next paper of this series.

The factor of $1/4$ in the definition of $\Delta_e$ relative to $(X_i^e)^2$ is due to the factor of $1/2$ in the second Poisson bracket of (3.34) and it is the same factor which gives $-\Delta_e$ the standard spectrum $j(j+1); \; j = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$ for the case of $G = SU(2)$.

We can also explicitly compute the quantum operator corresponding to $g^e_e$ in (3.34) for arbitrary $G$. We have

$$\hat{g}_e = e^{t \Delta_{\gamma}/2} \hat{h}_e^{-t \Delta_{\gamma}/2} = \sum_{n=0}^{\infty} \frac{(-t)^n}{2^n n!} [\hat{h}_e, \Delta_e]_{(n)}$$

$$-\left[\hat{h}_e, \Delta_e\right] = \frac{1}{4} (X_i^e \tau_i \hat{h}_e + \tau_i \hat{h}_e X_i^e) = X_i^e \tau_i \hat{h}_e - \frac{(\tau_i)^2}{4} \hat{h}_e$$

(3.43)

Since $\Delta_{\gamma}$ commutes with $X_i^e$ we immediately find

$$\hat{g}_e = e^{t X_i^e \tau_i \frac{\tau_i^2}{4} \hat{h}_e} = e^{-i \phi^e_i \frac{\tau_i \tau_i^2}{4} \hat{h}_e} = e^{-i \phi^e_i \frac{\tau_i}{\hbar}} e^{i \frac{\tau_i^2}{4} \hat{h}_e}$$

(3.44)

since $i t X_i^e / 2 = \hat{p}_j^e$ and in the third step we used that the matrix $\tau_i^2$ commutes with $\tau_i$. Since the $\hat{p}_j^e$ are not mutually commuting the exponential in (3.44) cannot be defined by the spectral theorem, however, we can define it through Nelson’s analytic vector theorem.

Thus, we find precisely the quantization of the polar decomposition (3.37) up to a factor of $e^{-\tau_i^2/8}$ which tends to unity linear in $t \to 0$ as to be expected. Notice that one obtains the first line of (3.43) from (3.37) if one replaces everywhere $\{\ldots\}$ by $[\ldots]/(i\hbar)$ and phase space
functions by operators which holds, of course, by the very construction of the map $\tilde{W}_t$.

This accomplishes our goal to write $g_e$ as a function of the $h_e, P_e$ and thus an interpretation of $g_e$ is indeed possible. As we will discuss all the properties of the corresponding coherent states in great detail in [40, 41] we will refrain from commenting on them here. As we will see, these states in fact enjoy all the properties (a)–(h) that we wanted them to satisfy. In particular, they are diffeomorphism covariant since, in contrast to [48], we have simply managed to interpret $\hat{C}_\gamma$ as a function of the diffeomorphism covariant functions $h_e, P_e$.

We restrict ourselves here to pointing out that the states constructed there will be mainly discussed at the gauge non-invariant and diffeomorphism non-invariant level only. There are two good reasons for this restriction. First of all, both the gauge group and the diffeomorphism group are represented unitarily on the Hilbert space [10] and thus expectation values of gauge – and diffeomorphism invariant operators are in fact gauge – and diffeomorphism invariant. It follows that no redundant information is produced as far as expectation values are concerned which is enough for semi-classical considerations. Secondly, while the gauge transformations generated by the Hamiltonian constraint are not unitarily represented, what we can do is to investigate whether the infinitesimal dynamics of quantum general relativity as advertised in [21, 22] reduces to that of classical general relativity as $t \rightarrow 0$. This would give faith into the proposal [21, 22] and as we will see, the answer is indeed affirmative [56].

More ambitiously, however, one may ask whether it is not possible to work directly at the gauge – and diffeomorphism invariant level. The next two sections outline what can be said about this issue.

Remark:

The reader may wonder what happens with the quantization ambiguity labelled by the Immirzi parameter $\beta$ (e.g. [76]) if one combines the quantum theory with the semi-classical considerations started in this paper. It is easy to see that the ambiguity, expectedly, does not affect the classical limit. To see this, recall that the canonical pair is given by $A_\beta = \Gamma + \beta K, E/(\kappa \beta)$ where $\Gamma$ is the spin connection associated with $E$ and $K$ is related to the extrinsic curvature. Now, for instance, the area of a surface $S$ with normal co-vector $n_a$ is given by

$$A(S) = \int_S d^2 x \sqrt{E^a_j E^b_k n_a n_b} = \kappa \beta \int_S d^2 x \sqrt{E^a_j E^b_k \frac{\kappa \beta}{\kappa \beta} n_a n_b}$$

and the area operator in the theory with label $\beta$ will be of the form $\hat{A}_\beta(S) = \beta \hat{A}_1(S)$ where $\hat{A}_1(S)$ has the standard spectrum of, say [17]. Now the Immirzi parameter also modifies the classicality parameter $t = \beta \kappa \hbar / a^2$ and the definition of the momenta $P_{\beta}^e(E) = P_{1}^e(E)/\beta$. Consider now a coherent state peaked at $E$. In the $\beta$-theory the coherent state will then be labelled by $P_{\beta}^e(E)$ and the expectation value of the area operator, which in terms of $P_{\beta}^e$ is of the form $\hat{A}_\beta(S) = \beta \sum e \sqrt{P_{\beta \beta}^e P_{\beta \beta}^e}$, will be by construction

$$< \hat{A}_\beta(S) > = \beta \sum e \sqrt{P_{\beta \beta}^e P_{\beta \beta}^e} = \sum e \sqrt{P_{1j}^e P_{1j}^e},$$

that is, independent of $\beta$.

4 Coherent States Directly for Gauge Invariant Quantities

There are two possibilities for constructing gauge invariant coherent states. The first possibility consists in group avaraging the gauge-variant coherent states of [40] by means
of the group averaging method [10] applied to the gauge group which means quantizing before reducing. Precisely, such states will be constructed as

$$\Psi_{\gamma, \vec{g}}^t(\vec{h}) = \int_{G(V(\gamma))} \prod_{v \in V(\gamma)} d\mu_H(u_v) \psi_{\gamma, \vec{g}}^t(\{u_{e(0)} h_i u_{e(1)}^{-1} \}_{e \in E(\gamma)})$$

(4.1)

where we have assumed that the parameterizations of edges are such that parameter values 0, 1 respectively correspond to start and end respectively. An interesting feature of the state (4.1) is that it is separately invariant under gauge transformations of both $\vec{h}$, $\vec{g}$, a property that is not shared by the diffeomorphism group averaged coherent states of the next section. In order to qualify as a state on the reduced phase space with respect to the Gauss constraint one would have to restrict $\vec{g}$, in addition, to the constraint surface which for the variables $h_e$, $P_e$ is explicitly described in [12]. The properties of the states (4.1) will be studied in some detail in [10] so that we can pass on to the second possibility.

This second approach to gauge invariant coherent states is the following one, consisting in reducing before quantizing:

One directly constructs gauge invariant configuration and momentum operators on the constraint surface of the Gauss constraint which leave the space of cylindrical gauge invariant functions over a given graph invariant. Next, one constructs from those new operators with canonical commutation relations and thus has mapped the problem to that of the construction of coherent states for the quantization of a particle moving in a finite number of dimensions for which a natural answer is given by the usual harmonic oscillator coherent states.

We will now outline this idea in some detail:

First we must determine suitable, independent, gauge invariant configuration and momentum operators on a given graph.

Consider a graph $\gamma$ with $E = |E(\gamma)|$ edges and $V = |V(\gamma)|$ vertices. If the gauge group is $N-$dimensional then for each vertex we have $N$ gauge degrees of freedom which allows us to fix $NV$ of the $NE$ independent components of the $E$ holonomies $h_e$, $e \in E(\gamma)$. This reveals that the number of physical configuration degrees of freedom associated with a graph $\gamma$ is given by $D(\gamma) = N(E - V)$. (We are considering here generic graphs with only at least four-valent vertices in order to have non-vanishing volume; the formula is not correct for the remaining degenerate graphs, for instance the graph consisting of only a single loop still has $r$ degrees of freedom while $E = V = 2$ with $r$ the rank of the group. We also consider only semi-simple Lie groups for definiteness).

Before we construct a suitable set of such $D$ configuration observables, let us check that the number of gauge invariant momentum observables also equals $D = N(E - V)$. A suitable set of gauge invariant quantum operators that can be obtained from electrical field operators alone consists of a maximal set of mutually commuting, gauge invariant operators constructed from the left or right invariant vector fields $L^i X^i_e, R^i X^i_e$ on the various copies of the group associated with the edges $e$ of the graph. Such a choice of invariants corresponds to the choice of a “recoupling scheme for the associated angular momenta”.

Let us outline this for $G = SU(2)$:

We can construct the $E$ Laplacians $\Delta_e = (R^i X^i_e)^2$ and for each $n(v)$-valent vertex $v$ we can construct further mutually commuting $n(v) - 3$ invariants given by the squares of the operators $(R^i X^i_{e_1}) + (R^i X^i_{e_2})$, $(R^i X^i_{e_3}) + (R^i X^i_{e_4})$, $\ldots$, $(R^i X^i_{e_{n(v) - 1}})$. By gauge invariance $(R^i X^i_{e_1}) + (R^i X^i_{e_{n(v)}}) = 0$ so that $(R^i X^i_{e_1}) + (R^i X^i_{e_{n(v) - 1}}) = - (R^i X^i_{e_{n(v)}})$ is not another independent quantity. The choice of these recoupling momenta corresponds to the choice of a recoupling scheme. Now notice that each edge is connected to two vertices.
Thus the number of recoupling degrees of freedom is given by $\sum_{v \in V(\gamma)}(n(v) - 3) = 2E - 3V$ which amounts together with the $E$ Laplacians to precisely $D = N(E - V) = 3(E - V)$ momentum degrees of freedom as well.

In the case of a general group, similar arguments apply.

We now come back to the problem of the construction of quantum observables with canonical commutation relations from the basic holonomy and membrane variables $h_e(A)$ and $P^e(A, E)$ respectively.

Let us first consider the configuration space operators. Notice that by the Euler relation \([7t]\) there are $L(\gamma) = E - V + 1$ generators (based at an arbitrary but fixed vertex $p$ of $\gamma$) of the homotopy group $\pi_p(\gamma)$ of $\gamma$. Thus, choosing a set of such generators one can construct $D$ independent configuration degrees of freedom by forming $D$ traces of holonomies along those loops and their compositions (and products of those if $r = \text{rank}(G) > 1$). However, one must be careful that the ranges of these traces (of products of holonomies along the various generators) in the set of real numbers do not depend on each other. Let us outline this for $G = SU(2)$:

Choose generators $\alpha_1, .., \alpha_L$ of $\pi(\gamma)$ and define

$$t_I = \frac{1}{2} tr(h_{\alpha_I}), \; I = 1, .., L$$

and since the $\alpha_I$ are independent we have that the $t_I$ take independently values in $[-1, 1]$.

Notice that so far we did not capture any information about the unit vectors $n_I$ in the representation $h_{\alpha_I} = t_I 1 + \tau J n_J^I \sqrt{1 - t_I^2}$. The scalar products $n_I^I n_J^J$ are certainly gauge invariant but they cannot be all independent. Pick one of the generators, say $\alpha_1$, and decompose the $n_J$, $J = 2, .., L$ into unit vectors parallel and orthogonal $b_J$, $J = 2, .., L$ to $n_1$

$$n_J = t_{L+J-1} n_1 + \sqrt{1 - t_{L+J-1}^2} b_J, \; J = 2, .., L$$

where the parameters $t_J$ again take independent values in $[-1, 1]$. We can obtain them in terms of traces as

$$t_{L+J-1} = \frac{t_1 t_J - t_{1} tr(h_{\alpha_1 \alpha_J})}{\sqrt{1 - t_1^2 \sqrt{1 - t_J^2}}}$$

Finally, we can also decompose $b_K$, $K = 3, .., L$ into unit vectors parallel and orthogonal $c_K$ to, say, $b_2$

$$b_K = t_{2L+K-3} b_2 + \sqrt{1 - t_{2L+K-3}^2} c_K$$

where, of course, $c_K = \epsilon_K c_3, \epsilon_K = \pm 1, K = 4, .., L$. Clearly, $n_1, b_2, c_3$ form an orthonormal basis in $\mathbb{R}^3$. In terms of traces again:

$$t_{2L+K-3} = \frac{t_{2L+K-3} - t_{1} tr(h_{\alpha_1 \alpha_2 \alpha_K})}{\sqrt{1 - t_1^2 \sqrt{1 - t_{2L+K-3}^2}}} - t_{L+1} t_{L+K-1}$$

Similarly, we could also express the $L - 3$ discrete variables $\epsilon_M, L = 4, .., M$ in terms of traces along the lines given above but we will not display the explicit formulae here. Rather, by means of the following trick we can get rid of them: define $t'_{2L+K-3} := t_{2L+K-3}$ and new parameters $t'_{2L+K-3}, K = 4, .., L$ by

$$t'_ {2L+K-3} = 2t_{2L+K-3}^2 - 1$$

$$t'_{2L+K-3} = 2t_{2L+K-3}^2 - 1 + \epsilon_K \sqrt{1 - (t'_{2L+K-3})^2} = 2 \sqrt{1 - t_{2L+K-3}^2 t_{2L+K-3}}$$

with, again, $t'_{2L+K-3} \in [-1, 1]$.

Obviously, the above equations \((1.2)-(4.7)\) define precisely $3(L - 1) = 3(E - V)$ continuous gauge invariant parameters $t_I, I = 1, .., D$ with independent range in $[-1, 1]$. The
map between the selected traces and these variables is singular but the subset of the space $[-1,1]^D$ where this map is singular is of Lebesgue measure zero and thus is irrelevant for $L_2$ functions. In any case, all traces of loops on $\gamma$ can be written as definite functions of the $D$ variables $t_I$ since any such function is a polynomial in the quantities $n^J_i n^J_j$ and we just need to substitute (4.3), (1.3).

From now on we will assume that we have constructed precisely $D(\gamma)$ independent, gauge invariant configuration variables $t_I$, $I = 1,..,D$ for every graph $\gamma$ with range in $[-1,1]$ along lines similar as above. This suggests the following strategy:

We would like to map the problem at hand to the problem of $D$ uncoupled harmonic oscillators. We achieve this by defining new variables

$$x_I := \text{arctanh}(t_I) = \frac{1}{2} \ln \left( \frac{1 + t_I}{1 - t_I} \right) \Leftrightarrow t_I = \tanh(x_I) \quad (4.8)$$

which take values in the whole real line. We can now consider the Hilbert space $\mathcal{H}_\gamma = L_2(\mathbb{R}^D, d^Dx)$ and construct the usual coherent states associated with the annihilation operators $\hat{z}_I = \hat{x}_I + i\frac{\hbar}{2} \hat{p}_I$ where $\hat{p}_I = -i\hbar \partial / \partial x_I$ and $t$ is a dimensionless parameter.

This is, however, not the end of the story. Namely, in order to interprete these coherent states in terms of the original quantities, we must make the connection with the classical theory. For the configuration variables the interpretation is obvious through the formulae (4.2)-(4.6). For the momentum variables this is less obvious. The way to proceed is to first express the operators $\hat{p}_I$ in terms of right invariant vector fields on functions cylindrical with respect to $\gamma$ and then to express the latter in terms of the phase space variables. In order to do that we write

$$\partial x_I = \frac{\partial t_J}{\partial x_I} \frac{\partial \theta_{e,i}}{\partial t_J} \quad (4.9)$$

where $\theta_{e,i} = \theta_{e,i}(t_I, t_\mu)$, $e \in E(\gamma)$, $i = 1,..,N$, $\mu = 1,..,NE - D$ are the $NE$ angle parameters which coordinatize the $E$ copies of $G$ and which we can think of as functions of the $t_I$ and remaining gauge degrees of freedom $t_\mu$. Now, there exists a map

$$\partial \theta_{e,i} = F_{ij}(\theta_{e,k})(R X_e^j) \quad (4.10)$$

which generically (that is, almost everywhere) is also non-singular and which allows us to write (4.3) in the form

$$\partial x_I = F_{I,e}(\{h_e\}_{e \in E(\gamma)})(R X_e^j) . \quad (4.11)$$

The final step consists in expressing the right invariant vector fields in terms of electric fields in the form of membrane operators which has been done in section 2.1 where they have been called $P^e_j$.

We can then finally think of $\partial x_I$ as a definite function of the $\{\hat{h}_e, \hat{P}^e_i\}_{e \in E(\gamma)}$ with an obvious classical limit. Of course, the formula (4.11) is far from simple.

Notice that in the course of the construction we have defined a new Hilbert space $\mathcal{H}_\gamma = L_2(\mathbb{R}^{D(\gamma)}, d^{D(\gamma)}x)$ which, however, is unitarily equivalent to the projection of the kinematical Hilbert space $\mathcal{H}$ of section 3 to the space of functions cylindrical over $\gamma$ (after integrating out gauge degrees of freedom) which also shows that these Hilbert spaces are cylindrically consistent so that they line up to a big Hilbert space in the projective limit, unitarily equivalent to $\mathcal{H}$.

We close this section with a number of comments:

(i) The advantage of this approach as compared to the one outlined in the previous section is that we are guaranteed to fulfill all the requirements (a)-(h) without going through considerable amount of functional analytic work since we can just copy all the results.
known from the harmonic oscillator coherent states.

(ii) A disadvantage is that the coherent states so constructed in terms of the $x_I$ are not easily expressed in terms of the gauge invariant spin-network functions in terms of which the spectra of important operators, such as the geometrical ones \([15, 16, 17, 18, 19, 20]\), are well known.

(iii) Finally, the reader may ask why we did not work at the gauge non-invariant level to begin with, obtain harmonic oscillator kind of coherent states for the gauge-variant quantities and only then solve the Gauss constraint. While this would simplify the analysis considerably since all the gauge angles $\theta^I_e$ could be taken as independent configuration variables and we could relate the conjugate derivative operators much more easily to the right invariant vector fields, unfortunately the gauge invariant subspace of the coherent states constructed from the gauge non-invariant quantities $\theta^I_e$ is not explicitly known. The only known procedure is to write them in terms of non-gauge invariant spin-network functions and then to keep only the gauge invariant combinations (this can be done alternatively by integrating those states over the gauge degrees of freedom as in \((4.1)\)). However, the coherent states are an infinite superposition of harmonic oscillator eigenstates each of which is an infinite superposition of spin-network states (in the $L_2$ sense) because the relation between the $\theta_e, I$ and the $h_e$ are not at all polynomial. Thus, the amount of work to be done to solve the Gauss constraint is considerably larger, if possible at all, than to define gauge invariant coherent states directly.

(iv) Finally, the complications mentioned in ii) of course also apply if one works entirely with gauge variant variables $\theta^I_e$ mentioned in iii) without caring about the Gauss constraint, the only simplification as compared to iii) is that the construction of the $t_I$ is not necessary.

To summarize, the coherent states defined in sections 3.2, 3.3 may reveal the required properties (a)-(h) less obviously, on the other hand, the operators that appear in applications have a much simpler action on these than on the ones that were constructed in the present section. Thus altogether, at least for analytical purposes the set of states of section 3.3 seems to be preferred.

5 Diffeomorphism Invariant Coherent States

Given a coherent state $\psi_{t, \gamma, Z}$ we can group average it with respect to the diffeomorphism constraint \([10]\) and obtain (we discard certain technicalities that come from graph symmetry factors, see \([27]\), whose notation we follow, for details)

$$\eta_{Diff} \cdot \psi_{\gamma, \vec{g}} = \sum_{\lambda, n} e^{-t \lambda} T_{\gamma, \lambda, n}(\vec{g}) [T_{\gamma, \lambda, n}] \text{ and } \eta_{Diff} \cdot \Psi_{\gamma, \vec{g}} = \sum_{\gamma' \in \gamma} \eta_{Diff} \cdot \psi_{\gamma', \vec{g}}$$

(5.1)

where $[\psi]$ denotes the orbit of the state $\psi$ under $Diff(\Sigma)$, typically

$$[T_{\gamma, \lambda, n}] = \sum_{\gamma' \in [\gamma]} T_{\gamma', \lambda, n}.$$  

(5.2)

where $[\gamma]$ denotes the orbit of $\gamma$. Here, as in section 4.1 we have written coherent states in terms of eigenfunctions $T_{\gamma, \lambda, n}$ of a general complexifier with eigenvalue $\lambda$ and degeneracy level $n$ each of which can be decomposed in terms of spin-network functions with non-trivial dependence on every edge of that graph. This requirement is very important.
in order for group averaging to be well-defined and thus excludes, in particular, the possibility to average infinite graphs as we will see in [12], at least not without some kind of renormalization as discussed there, see [52] for a general discussion.

If $\hat{C}$ is diffeomorphism invariant, as it is the case for $\hat{v}$ above then $\lambda$ is a diffeomorphism invariant quantity.

Although the state (5.1) is certainly diffeomorphism invariant, being a linear combination of diffeomorphism invariant states, it depends not only on $[\gamma]$ and the equivalence class of complex holonomies under diffeomorphisms $[\vec{g}]$, but explicitly on the representatives. In other words, while under diffeomorphisms also $g_e \to \phi \cdot g_e = g_{e^{-1}(\phi)}$, (5.2) is not invariant under mapping $\vec{g}$ by a diffeomorphism in contrast to what happened in (4.1) with respect to the gauge group. This is unsatisfactory because on the diffeomorphism invariant Hilbert space $\mathcal{H}_{Diff}$, which is the Cauchy completion of states of the form $\eta_{Diff} \cdot f$, $f \in Cyl$ under the inner product

$$< \eta_{Diff} \cdot f, \eta_{Diff} \cdot g >_{Diff} := [\eta_{Diff} \cdot f](g)$$

where the latter denotes the application of the distribution $\eta_{Diff} \cdot f$ to the test function $g$, the inner product between diffeomorphism invariant coherent states should depend only on $[\vec{g}], [\vec{g}']$ and not on the representatives. In particular, this leads to the following problem: Suppose that $(A, E)$ and $(A', E')$ are diffeomorphic points of the classical phase space and compute from these $g_e = g_e(A, E)$ as in section 3.1 or 3.2 and similar for the primed quantities. Then if these quantities differ in the range of $\gamma$ then the inner product

$$< \eta_{Diff} \cdot \psi^t_{\gamma, \vec{g}}, \eta_{Diff} \cdot \psi^t_{\gamma, \vec{g}'} >_{Diff} = \sum_{\lambda, n} e^{-2t\lambda} T_{\gamma, \lambda, n}(\vec{g}) T_{\gamma, \lambda, n}(\vec{g}')$$

will be small by the very definition of a coherent state, that is, these states are almost orthogonal with respect to $< ..., >_{Diff}$. This is certainly not what we want.

The reason for this is, of course, that there are too many of the states $\eta_{Diff} \cdot \Psi_{\gamma, Z}$. We should identify all those that are labelled by those $\vec{g}'$ which lie in the same equivalence class under diffeomorphisms as $\vec{g}$. This can be done by choosing a representant $Z_0([Z])$ in every equivalence class $[Z]$ where $Z$, as before, stands for phase space points $(A, E)$ or an actual complex connection depending on whether we choose coherent states based on option 2) or 1). Notice that this is not, in general, equivalent to fixing a gauge because choosing a representant is possible also if there does not exist a global gauge fixing condition as it is typically the case in field theories.

One might think that one could alternatively define diffeomorphism invariant coherent states by heat kernel evolution, followed by analytical continuation, of the $\delta$ distribution with respect to $< ..., >_{Diff}$ given by (notice that $T_{[\gamma], \lambda, n}(A) = T_{[\gamma], \lambda, n}([A])$)

$$\delta_{[\gamma], [A]}([A']) := \sum_{\lambda, n} T_{[\gamma], \lambda, n}(A) \overline{T_{[\gamma], \lambda, n}(A')},$$

however, the resulting state

$$\psi^t_{[\gamma], [Z]}([A]) = \sum_{\lambda, n} e^{-t\lambda} T_{[\gamma], \lambda, n}(Z) \overline{T_{[\gamma], \lambda, n}(A)},$$

is no longer normalizable with respect to $< ..., >_{Diff}$ so that we are forced to adopt the above strategy.

To summarize, we pick arbitrary but fixed representant functions

$$\gamma_0 : [\Gamma_0^0] \mapsto \Gamma_0^0; [\gamma] \mapsto \gamma_0([\gamma]) \quad \text{and}$$
$$Z_0 : M_{Diff} \mapsto M; [Z] \mapsto Z_0([Z])$$

(5.7)
from the sets of equivalence classes under diffeomorphisms of piecewise analytical graphs and from the phase space $M_{\text{Diff}}$ reduced with respect to the diffeomorphism constraint to the full phase space $M$ respectively and we define diffeomorphism invariant coherent states by

$$\Psi_{\gamma, Z}^t := \eta_{\text{Diff}} \cdot \Psi_{\gamma_0([\gamma]), Z_0([Z])}^t.$$  

(5.8)

The function $\gamma_0$ is necessary on top of $Z_0$ since a coherent state on $\gamma$ depends on $Z$ only at $\gamma$ and not everywhere. The inner product between these states is given through

$$<\psi_{\gamma, Z}^t, \psi_{\gamma, Z}^t>_{\text{Diff}} = <\psi_{\gamma_0([\gamma]), Z_0([Z])}^t, \psi_{\gamma_0([\gamma]), Z_0([Z])}^t>$$

(5.9)

using the orthogonality of the $\psi_{\gamma, Z}^t$ for different $\gamma$. Notice that in the last line we just have the kinematical inner product on $\mathcal{H}$. It follows from (5.9) immediately that the diffeomorphism invariant coherent states so defined are localized in the same way as the kinematical ones are. The Ehrenfest properties cannot be verified because we would need a complete set of observables on the Hilbert space $\mathcal{H}_{\text{Diff}}$ but it is sufficient to know that these states are peaked on $[Z]$ for every $[\gamma]$ in order to make semi-classical approximations. Moreover, it also follows from (5.18) that the group average of the projective limit $\Psi_Z^t$ coherent state is normalizable with respect to $<.,.>_{\text{Diff}}$ if and only if $\Psi_Z^t$ is normalizable with respect to $<.,.>$.

As we have explicitly indicated in (5.8), the coherent states depend on the representant functions (5.7). But

$$\eta_{\text{Diff}} \cdot \Psi_{\gamma, Z}^t = \eta_{\text{Diff}} \cdot \Psi_{\gamma_0([\gamma]), \varphi_0^Z}^t.$$  

(5.10)

where $\varphi_0(\gamma_0([\gamma])) = \gamma$ and $\varphi_0^Z$ is the action of diffeomorphisms on phase space points $Z$. Thus, it is only the relation between $\gamma_0$ and $Z_0$ which makes a difference (has a diffeomorphism invariant meaning) because in the pair $\gamma_0'$, $Z_0'$ we can always replace $\gamma_0'$ by $\gamma_0$ at the price of changing $Z_0'$. In other words, if we fix $\gamma_0$ once and for all as we can without loss of generality, then our choice of diffeomorphism invariant coherent states is entirely labelled by $Z_0$. This choice is to be interpreted as a choice of basis of diffeomorphism invariant coherent states. The inner products between members of different bases have no definite locality properties as we have shown in (5.4). But this is in general true for different sets of coherent states even in systems with only a finite number of degrees of freedom. After all, the requirement of localization does not determine a coherent state uniquely, not even up to unitary equivalence because all that is required is that the inner product between such states is unity if their labels coincide and is “small” otherwise where the notion of smallness depends on the basis.

Thus the dependence of the states on $Z_0$ is not a bad but in fact an expected property.

Notice further that some of these diffeomorphism invariant coherent states also lie in the kernel of the Hamiltonian constraint operator defined in [21, 22, 27] : we just have to choose $[\gamma]$ in such a way that the range of the Hamiltonian constraint in the set of linear combinations of spin-network functions cannot contain a spin-network state whose underlying graph lies in the class $[\gamma]$. As shown in [27], there are an infinite number of such states. This observation may be a starting point for the construction of semiclassical states which lie in the kernel of all three types of constraints : the Gauss-, Diffeomorphism- and Hamiltonian constraint.

6 Model for Gauge Invariant Coherent States : Euclidean 2+1 gravity

As we have mentioned in section 3.1, the volume operator qualifies best as a complexifier in $D = 2$. For Euclidean 2+1 gravity we have $D = 2$ and $G = SU(2)$. The volume
operator in two dimensions was derived in [23]. The spectrum of that operator for at most three-valent vertices was also computed analytically there. In this section we focus on a Hilbert space for this theory which has finite linear combinations of spin-network states on at most three-valent graphs as a dense subset because otherwise the spectrum is only known numerically.

There are two cases to consider: either (A) no two of the three edges $e_1, e_2, e_3$ meeting at a vertex are co-linear or (B) there is a co-linear pair, say $e_1, e_2$ (the third case, that all three edges are co-linear is excluded because the volume would vanish). Let $\vec{j} = (j_1, j_2, j_3)$ be the spins with which the three edges are coloured (for at most three-valent graphs the space of vertex-contractors is one dimensional and thus $J_v = 1$ is suppressed in what follows; this is also the reason why these spin-network states are eigenstates of the volume operator).

The square of the eigenvalues of the volume operator for a given vertex in the two cases are [23]

\[
\lambda_v(\vec{j}) = \frac{9}{4}[2(\Delta_1 \Delta_2 + \Delta_2 \Delta_3 + \Delta_3 \Delta_1) - (\Delta_1^2 + \Delta_2^2 + \Delta_3^2)] - \frac{1}{2}[\Delta_1 + \Delta_2 + \Delta_3] \quad \text{(A)}
\]

\[
\lambda_v(\vec{j}) = 2(\Delta_1 \Delta_2 + \Delta_2 \Delta_3 + \Delta_3 \Delta_1) - (\Delta_1^2 + \Delta_2^2 + \Delta_3^2) - \Delta_3 \quad \text{(B)}
\]

where $\Delta_i = -j_i(j_i+1)$. At first sight it seems that in this case we can evey take $n = 1$ since the $\Delta$’s appear squared in leading order which would be sufficient to make the series of the coherent state converge. However, this is not the case: For instance we can consider the case that $j_1 = \text{const.}$ and $j_2 \to \infty$. Then, due to gauge invariance $j_3$ is of the same order as $j_2$ and therefore the leading order of the square bracket in (6.1) is only $j_2^2$. We choose $n = 2$ in what follows. It is then easy to see that in this case the complex connection is explicitly given by $Z_a^j = A_a^j - if_a^j$ where

\[
 f_a^j \propto \frac{V}{\sqrt{\det(q)}} \epsilon_{jkl} \epsilon_{abc}(\epsilon_{kmn} \epsilon_{cd} E_m^c E_n^d) E_l^b
\]

We will for our example analyze only the simplest non-trivial graph, a kink (or double kink) $\alpha$ with two edges and one (or two) two-valent vertices. This corresponds to, say $j_2 = 0$, in (6.1) and $j := j_1 = j_3$. Then we obtain the simple eigenvalue $\lambda_j = \lambda_v^2 = -\Delta = j(j+1)$, in other words, on this graph the volume operator reduces to (two times) the square root of the Laplacian on the copy of the group corresponding to $h := h_\alpha, \alpha = e_1 \circ e_3^{-1}$. A complete orthonormal basis of gauge invariant spin-network functions is given by the characters $\chi_n(h) = tr(\pi_n/2(h)), n = 0, 1, 2, \ldots$

On the kink, the coherent state is simply given by

\[
\Psi_{g,\alpha,t}(h) = \sum_{n=0}^{\infty} e^{-\frac{i}{2} \lambda_n} \chi_n(g) \chi_n(h)
\]

where $g = h_\alpha(Z), \lambda_n = n(n+2) = 4(\lambda_j)_{n=2j}$. The series (6.2) converges for any $g \in SL(2, \mathbb{C})$ as shown in [47].

We wish to show that the state (6.2) diagonalizes the gauge invariant operator

\[
\hat{T}^q := \hat{W}_t \hat{W}_T^{-1}, \hat{W}_t = e^{t\Delta}, \hat{T} = tr(\hat{h})
\]

Denoting $T_n = tr(h^n), n = 0, 1, \ldots, T = T_1$ we notice the identity

\[
\chi_n = \left\{ \begin{array}{ll}
1 + T_2 + T_4 + \ldots + T_N & : n \text{ even} \\
T_1 + T_3 + T_5 + \ldots + T_N & : n \text{ odd}
\end{array} \right.
\]
from which follows that $T \chi_n = \chi_{n+1} + \chi_{n-1}$ by using the $SU(2)$ Mandelstam identity $TT_n = T_{n+1} + T_{n-1}$. It is understood that $T_{-1} = 0$. Let $T^\Phi = tr(g)$, then

$$
(\hat{T}^\Phi \Psi_{g,t})(h) = e^{t\Delta} \sum_n \chi_n(g) T \chi_n(h) = e^{t\Delta} \sum_n \chi_n(g) [\chi_{n+1}(h) + \chi_{n-1}(h)]
$$

$$
= \sum_n \chi_n(g) [e^{-t\lambda_{n+1}} \chi_{n+1}(h) + e^{-t\lambda_{n-1}} \chi_{n-1}(h)]
$$

$$
= \sum_n [\chi_{n+1}(g) + \chi_{n-1}(g) e^{-t\lambda_n} \chi_n(h)] = T^\Phi \psi_g(h).
$$

(6.5)

From this and the general discussion in section 3 it easily follows that $\hat{T}^\Phi$ and $(\hat{T}^\Phi)^\dagger$ respectively have expectation values $T^\Phi$ and $\overline{T^\Phi}$ respectively, moreover, we have the uncertainty relation

$$
< (\Delta \hat{x})^2 > < (\Delta \hat{y})^2 > \geq \left| < [\hat{x}, \hat{y}] > \right|^2 / 4,
$$

(6.6)

with $\hat{x} = \frac{1}{2}(\hat{T}^\Phi + (\hat{T}^\Phi)^\dagger)$, $\hat{y} = \frac{1}{2i}(\hat{T}^\Phi - (\hat{T}^\Phi)^\dagger)$.

The inner product between two coherent states is given by (we suppress the label $\alpha$ in what follows)

$$
< \Psi^t_t, \Psi^t_{g'} > = \sum_n e^{-2t\lambda_n} \chi_n(g) \chi_n(g')
$$

(6.7)

We will now show that the overlap integral

$$
I^t(g, g') := \frac{|< \Psi^t_t, \Psi^t_{g'} >|^2}{< \Psi^t_t, \Psi^t_t > < \Psi^t_{g'}, \Psi^t_{g'} >}
$$

(6.8)

decays exponentially fast with $|tr(g) - tr(g^\Phi)|$ as $t \to 0$, i.e. in the classical limit $\hbar \to 0$.

The proof uses the Euler–MacLaurin estimate for the difference between a series and its replacement by an integral $\sum_{[0,\infty]}$ which turns out to vanish in our limit $t \to 0$.

To begin with, recall that the characters are explicitly given by

$$
\chi_n(h) = \sin((n + 1)\phi) \sin(\phi)
$$

where $2 \cos(\phi) := tr(h)$, $\phi \in [0, \pi]$

(6.9)

for any $h \in SU(2)$. Formula (6.9) is entire analytic in $\phi$ and is readily extended to $g \in SL(2, \mathbb{C}) = SU(2)^\Phi$

$$
\chi_n(g) = \sin((n + 1)\theta) \sin(\theta)
$$

where $2 \cos(\theta) := tr(g)$, $\theta = \phi - is$, $\phi \in [0, \pi]$, $s \in \mathbb{R}$.

(6.10)

Since the characters are class functions, we can always rotate $g$ into a maximal torus of $SU(2)$ and so we can think of $g$ as given by $g = \exp(\theta \tau_3)$. Notice that the Weyl subgroup acts on the torus by $\theta \to -\theta$ and indeed (6.9), (6.10) are still invariant under it. We can therefore restrict, without loss of generality to $s \in [0, \infty]$. In general, $\theta = \pm \sqrt{(\theta_t)^2}$, $g = \exp(\theta_t \tau_3)$.

We now compute

$$
< \Psi_{g,t}, \Psi_{g',t} > = \frac{1}{\sin(\theta) \sin(\theta')} \sum_{n=0}^{\infty} e^{-\frac{t}{2}(n + 1)^2 - 1} \sin((n + 1)\bar{\theta}) \sin((n + 1)\theta')
$$

$$
= \frac{e^{t/2}}{\sin(\theta) \sin(\theta')} \sum_{n=1}^{\infty} e^{-\frac{t}{2}n^2} \sin(n\bar{\theta}) \sin(n\theta')
$$

$$
= -\frac{e^{t/2}}{4 \sin(\theta) \sin(\theta')} \sum_{n=1}^{\infty} e^{-\frac{t}{2}n^2} \times
$$
where in the last step we have noticed that the term \( n = 0 \) vanishes. Let now \( x_n := \sqrt{t}n \), \( \Delta x := x_{n+1} - x_n = \sqrt{t} \), then (6.11) can be written in the form

\[
< \Psi_{g,t}, \Psi_{g',t} > = \frac{e^{t/2}}{4 \sqrt{t} \sin(\tilde{\theta}) \sin(\tilde{\theta}') \sum_{n \in \mathbb{Z} - \{0\}} e^{-\frac{1}{2}n^2} [\exp(ix_n \tilde{\theta} + \tilde{\theta}') - \exp(ix_n \tilde{\theta} - \tilde{\theta}')] \]

(6.12)

which suggests to replace the sum by a Riemann integral for small \( t \). It would be literally a Riemann sum if it was not for the explicit \( t \)-dependence of the integrand. Thus, the following expression is only an approximation to (6.12) which becomes exact as \( t \to 0 \)

\[
i^t(g, g') = \frac{e^{t/2}}{4 \sqrt{t} \sin(\tilde{\theta}) \sin(\tilde{\theta}') \int_{-\infty}^{\infty} dx \ e^{-\frac{1}{2}x^2} [\exp(i x_n \tilde{\theta} + \tilde{\theta}') - \exp(i x_n \tilde{\theta} - \tilde{\theta}')] \]

(6.13)

where we have used a Cauchy integral formula. The overlap integral thus is approximated by

\[
\tilde{I}(g, g', t) = \frac{|i^t(g, g')|^2}{i^t(g, g') i^t(g', g')} = \frac{|\exp(-\frac{(\tilde{\theta}+\tilde{\theta}')^2}{2t}) - \exp(-\frac{(\tilde{\theta}-\tilde{\theta}')^2}{2t})|^2}{[\exp(-\frac{(\tilde{\theta}+\tilde{\theta}')^2}{2t}) - \exp(-\frac{(\tilde{\theta}-\tilde{\theta}')^2}{2t})][\exp(-\frac{(\tilde{\theta}+\tilde{\theta}')^2}{2t}) - \exp(-\frac{(\tilde{\theta}-\tilde{\theta}')^2}{2t})]} \]

(6.14)

or when decomposing \( \theta = \phi + is \), \( \theta' = \phi' + is' \)

\[
\tilde{I}(g, g', t) = \frac{|\exp(-\frac{(|\phi'|-i|s'-s'|)^2}{2t}) - \exp(-\frac{(|\phi'-\phi|+|s+s'|)^2}{2t})|^2}{[\exp(-2\frac{\phi^2}{t}) - \exp(2\frac{s'^2}{t})][\exp(-2\frac{\phi'^2}{t}) - \exp(2\frac{s^2}{t})]} \]

(6.15)

We now multiply numerator and denominator of (6.13) with \( \exp([-s^2-(s')^2+\phi^2+(\phi')^2]/t) \) and obtain

\[
\tilde{I}(g, g', t) = \frac{|\exp(-\frac{(|\phi'|-is'+s')^2}{2t}) - \exp(-\frac{(|\phi'-\phi|+is')^2}{2t})|^2}{4 \sinh(\frac{(\phi^2+s^2)}{t}) \sinh(\frac{(\phi')^2+(s')^2}{t})} \]

\[
= \frac{\cosh(2\frac{\phi'\phi'+ss'}{t}) - \cos(2\frac{\phi'\phi'+ss'}{t})}{2 \sinh(\frac{(\phi^2+s^2)}{t}) \sinh(\frac{(\phi')^2+(s')^2}{t})} \]

(6.16)

Now, for \( \theta, \theta' \neq 0 \), in the limit \( t \to 0 \) we have (since \( \cos \) is a bounded function and \( ss', \phi\phi' \geq 0 \))

\[
\tilde{I}(g, g', t) \to \exp(-\frac{[\phi - \phi']^2 + [s - s']^2}{t}) \]

(6.17)

which is indeed rapidly vanishing as \( t \to 0 \) unless \( \theta = \theta' \) in which case it equals unity as it should.

If either of \( \theta, \theta' \) vanishes, say \( s' = \phi' = \theta' = 0 \) then expression (6.16) is of the type \( 0/0 \) and we can evaluate it, provided the limit exists, by picking up the leading order terms
of numerator and denominator by Cauchy’s formula. It turns out to be sufficient to keep the terms of second order in $s', \phi'$. The numerator becomes

$$\frac{1}{2}[(2\frac{\phi' + ss'}{t})^2 + (2\frac{\phi' - \phi's}{t})^2] + O((t')^3) = 2\frac{(\phi'^2 + s^2)((\phi')^2 + (s')^2)}{t^2} + O((t')^3)$$

while the denominator becomes

$$2\sinh\left(\frac{s^2 + \phi^2}{t}\right)(\frac{\phi'}{t})^2 + (s')^2 + ..$$

where the dots denote terms of at least fourth order in $s', \phi'$. Thus, (5.10) has the well-defined limit

$$\tilde{I}(g, g' = 1, t) = \frac{s^2 + \phi^2}{t\sinh\left(\frac{s^2 + \phi^2}{t}\right)}$$

which is again exponentially damped as $t \to 0$ unless $s = \phi = 0$ in which case it equals unity as it should.

To conclude, the overlap integral $I^t(g, g') \to_{t \to 0} \tilde{I}(g, g', t)$ is exponentially damped unless $T^\Phi = (T')^\Phi$.

Next, we should also show that the normalized coherent state itself, in both the configuration and the momentum representation, is peaked. These and other issues will be much more systematically analyzed for general graphs in [40] by using the Poisson summation formula.

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