Nonparametric Volatility Density Estimation

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Abstract

In this paper we consider a continuous-time stochastic volatility model. The model contains a stationary volatility process, the density of which, at a fixed instant in time, we aim to estimate. We assume that we observe the process at discrete instants in time. The sampling times will be equidistant with vanishing distance.

A Fourier-type deconvolution kernel density estimator based on the logarithm of the squared processes is proposed to estimate the volatility density. An expansion of the bias and a bound on the variance are derived.

Key words: stochastic volatility models, density estimation, kernel estimator, deconvolution, mixing

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1 Introduction

Let $S$ denote the log price process of some stock in a financial market. It is often assumed that $S$ can be modelled as the solution of a stochastic differential equation or, more general, as an Itô diffusion process. So we
assume that we can write
\[ dS_t = b_t \, dt + \sigma_t \, dW_t, \quad S_0 = 0, \]  
(1)
or, in integral form,
\[ S_t = \int_0^t b_s \, ds + \int_0^t \sigma_s \, dW_s, \]  
(2)
where \( W \) is a standard Brownian motion and the processes \( b \) and \( \sigma \) are assumed to satisfy certain regularity conditions (see Karatzas and Shreve (1991)) to have the integrals in (2) well-defined. In the financial context, the process \( \sigma \) is called the volatility process.

In this paper we model \( \sigma \) as a strictly stationary positive process satisfying a mixing condition, for example an ergodic diffusion on \([0, \infty)\), and we make the assumption that \( \sigma \) is independent of \( W \). We will assume that the one-dimensional marginal distribution of \( \sigma \) has a density with respect to the Lebesgue measure on \((0, \infty)\). This is typically the case in virtually all stochastic volatility models that are proposed in the literature, where the evolution of \( \sigma \) is modelled by a stochastic differential equation, mostly in terms of \( \sigma^2 \), or \( \log \sigma^2 \) (cf. e.g. Wiggins (1987), Heston (1993)).

For stochastic differential equations of the type
\[ dX_t = b(X_t) \, dt + a(X_t) \, dB_t, \]
with \( B_t \) equal to Brownian motion, the invariant density is up to a multiplicative constant equal to
\[ x \mapsto \frac{1}{a^2(x)} \exp \left( 2 \int_{x_0}^x \frac{b(y)}{a^2(y)} \, dy \right), \]  
(3)
where \( x_0 \) is an arbitrary element of the state space \((l, r)\), see e.g. Gillman and Skorohod (1972) or Skorokhod (1989). From formula (3) one sees that the invariant distribution of the volatility process (take \( X \) for instance equal to \( \sigma^2 \) or \( \log \sigma^2 \)) may take on many different forms, as is the case for the various models that have been proposed in the literature. This observation supports our point of view that nonparametric procedures are by all means sensible tools to get some insight in the behaviour of the volatility.

In the present paper we propose a nonparametric estimator for the volatility density. Using ideas from deconvolution theory, we will propose a procedure for the estimation of the marginal density at a fixed point. We will assume that we observe the log-asset price \( S \) at time instants \( 0, \Delta, 2\Delta, \ldots, n\Delta \), where the time gap satisfies \( \Delta = \Delta_n \to 0 \) and \( n\Delta_n \to \infty \) as \( n \to \infty \). To assess the quality of our procedure, we will study how the bias and variance of the estimator behave under these assumptions.

The remainder of the paper is organized as follows. In the next section, we give the heuristic arguments that motivate the definition of our estimator.
In Section 3 the main result concerning the asymptotic behaviour of the estimator is presented and discussed. The proof of the main theorem is given in the last two sections.

2 Construction of the estimator

To motivate the construction of the estimator, we first consider (1) without the drift term, so we assume to have

\[ dS_t = \sigma_t dW_t, \quad S_0 = 0. \]

It is assumed that we observe the process \( S \) at the discrete time instants 0, \( \Delta, 2\Delta, \ldots, n\Delta \). For \( i = 1, 2, \ldots \) we work, as in Genon-Catalot et al. (1998, 1999), with the normalized increments

\[ X^\Delta_i = \frac{1}{\sqrt{\Delta}} (S_i - S_{(i-1)\Delta}). \]

For small \( \Delta \), we have the rough approximation

\[ X^\Delta_i \approx \frac{1}{\sqrt{\Delta}} \int_{(i-1)\Delta}^{i\Delta} \sigma_t dW_t \approx \sigma_{(i-1)\Delta} \frac{1}{\sqrt{\Delta}} (W_i - W_{(i-1)\Delta}) = \sigma_{(i-1)\Delta} Z^\Delta_i, \]

where for \( i = 1, 2, \ldots \) we define

\[ Z^\Delta_i = \frac{1}{\sqrt{\Delta}} (W_i - W_{(i-1)\Delta}). \]

By the independence and stationarity of Brownian increments, the sequence \( Z_1^\Delta, Z_2^\Delta, \ldots \) is an i.i.d. sequence of standard normal random variables. Moreover, the sequence is independent of the process \( \sigma \) by assumption.

Taking the logarithm of the square of \( X^\Delta_i \) we get

\[ \log((X^\Delta_i)^2) \approx \log(\sigma_{(i-1)\Delta}^2) + \log((Z^\Delta_i)^2), \]

where the terms in the sum are independent. Assuming that the approximation is sufficiently accurate we can use this approximate convolution structure to estimate the unknown density \( f \) of \( \log(\sigma_i^2) \) from the observed \( \log((X^\Delta_i)^2) \).

Before we can define the estimator, we need some more notation. Observe that the density of the ‘noise’ \( \log(Z^\Delta_i)^2 \), denoted by \( k \), is given by

\[ k(x) = \frac{1}{\sqrt{2\pi}} e^{\frac{1}{2} e^x} e^{-\frac{1}{2} e^x}. \]

The characteristic function of the density \( k \) is denoted by \( \phi_k \).

We will use a function \( w \), satisfying the following condition. For examples of such kernels see Wand (1998).

**Condition W.** Let \( w \) be a real symmetric function with real valued symmetric characteristic function \( \phi_w \) with support \([-1,1]\). Assume further
1. $\int_{-\infty}^{\infty} |w(u)| du < \infty$, $\int_{-\infty}^{\infty} w(u) du = 1$, $\int_{-\infty}^{\infty} u^2 |w(u)| du < \infty$,  
2. $\phi_w(1-t) = A t^\alpha + o(t^\alpha)$, as $t \to 0$ for some $\alpha > 0$.

Following a well-known approach in statistical deconvolution theory, we use a deconvolution kernel density estimator, see e.g. Section 6.2.4 of Wand and Jones (1995). Having the characteristic functions $\phi_k$ and $\phi_w$ at our disposal, choosing a positive bandwidth $h$, we introduce the kernel function

$$v_h(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\phi_w(s)}{\phi_k(s/h)} e^{-i s x} ds$$

and the density estimator

$$f_{nh}(x) = \frac{1}{nh} \sum_{j=1}^{n} v_h \left( x - \log((X_j^\Delta)^2) \right) .$$

One easily verifies that the function $v_h$, and therefore also the estimator $f_{nh}$, is real-valued.

### 3 Results

To derive the asymptotic behaviour of the estimator, we need a mixing condition on the process $\sigma$. For the sake of clarity, we recall the basic definitions. For a certain process $X$ let $\mathcal{F}^X_a$ be the $\sigma$-algebra of events generated by the random variables $X_t$, $a \leq t \leq b$. The mixing coefficient $\alpha(t)$ is defined by

$$\alpha(t) = \sup_{A \in \mathcal{F}^X_{-\infty}, B \in \mathcal{F}^X_{t}} |P(A \cap B) - P(A) P(B)| .$$

The process $X$ is called strongly mixing if $\alpha(t) \to 0$ as $t \to \infty$.

As we mentioned in the introduction, it is common practice to model the volatility process $V = \sigma^2$ as the stationary, ergodic solution of an SDE of the form

$$dV_t = b(V_t) \, dt + a(V_t) \, dB_t .$$

The mixing condition that we use in Theorem 3.1 below is satisfied in this setup. See for instance Corollary 2.1 of Genon-Catalot et al. (2000).

It is easily verified that for such processes it holds that $E|V_t - V_0| = O(t^{1/2})$, provided that $b \in L_1(\mu)$ and $a \in L_2(\mu)$, where $\mu$ is the invariant probability measure. Indeed we have $E|V_t - V_0| \leq E \int_0^t |b(V_s)| ds + (E \int_0^t a^2(V_s) ds)^{1/2} = t||b||_{L_1(\mu)} + \sqrt{t}||a||_{L_2(\mu)}$. Although we will not assume explicitly that $\sigma^2$ solves an SDE, the above observation motivates the following condition.

**Condition $\sigma$.** We have $E|\sigma_t^2 - \sigma_0^2| = O(t^{1/2})$ for $t \to 0$. 

The following theorem describes the asymptotic behaviour of our estimator $f_{nh}$. Note that it also covers the case where there is a drift $b_t$ present in equation (1). The condition on the drift is boundedness of $E b_t^2$. This condition is typically satisfied in realistic models for the log-returns of a stock, since $b_t$ is the local rate of return and this will be mostly bounded itself.

**Theorem 3.1.** Assume that $E b_t^2$ is bounded. Let the process $\sigma$ be strongly mixing with coefficient $\alpha(t)$ satisfying, for some $0 < q < 1$,

$$\int_0^\infty \alpha(t)^q \, dt < \infty,$$

and suppose that Condition $\sigma$ holds. Let the kernel function $w$ satisfy Condition $W$ and let the density $f$ of $\log \sigma_t^2$ be continuous, twice continuously differentiable with a bounded second derivative. Also assume that the density of $\sigma_t^2$ is bounded in a neighbourhood of zero. Suppose that $\Delta = n^{-\delta}$ for given $0 < \delta < 1$ and choose $h = \gamma \pi / \log n$, where $\gamma > 4/\delta$. Then the bias of the estimator (7) satisfies

$$E f_{nh}(x) - f(x) = \frac{1}{2} h^2 f''(x) \int u^2 w(u) \, du + o(h^2).$$

Moreover, the variance of the estimator satisfies

$$\text{Var} f_{nh}(x) = O\left(\frac{1}{n} h^{2q} e^{\pi/h}\right) + O\left(\frac{1}{nh^{1+q}\Delta}\right).$$

The proof of the theorem is deferred to the next section. We conclude the present section by a number of comments on the result.

**Remark 3.2.** The expectation of the deconvolution estimator is equal to the expectation of an ordinary kernel density estimator, as becomes clear from the proof of Lemma 4.1.

It is well-known that the variance of kernel-type deconvolution estimators heavily depends on the rate of decay to zero of $|\phi_k(t)|$ as $|t| \to \infty$. The faster the decay the larger the asymptotic variance. In other words, the smoother $k$ the harder the estimation problem. This follows for instance for i.i.d. observations from results in Fan (1991) and for stationary observations from the work of Masry (1993).

The rate of decay of $|\phi_k(t)|$ for the density $\bar{f}$ is given by Lemma 5.1 in Section 5, stating $|\phi_k(t)| \sim \sqrt{2} e^{-\frac{1}{2} \pi |t|}$, as $|t| \to \infty$. This shows that $k$ is supersmooth, cf. Fan (1991). By the similarity of the tail of this characteristic function to the tail of a Cauchy characteristic function we can expect the same order of the mean squared error as in Cauchy deconvolution.
problems, where it decreases logarithmically in \( n \), cf. Fan (1991) for results on i.i.d. observations. Note that this rate, however slow, is faster than the one for normal deconvolution. Fan (1991) also shows that we cannot expect anything better.

Remark 3.3. The choices \( \Delta = n^{-\delta} \), with \( 0 < \delta < 1 \) and \( h = \gamma \pi / \log n \), with \( \gamma > 4/\delta \) render a variance that is of order \( n^{-1+1/\gamma}(1/\log n)^{2\alpha} \) for the first term of (11) and \( n^{-1+\delta}(\log n)^{1+q} \) for the second term. Since by assumption \( \gamma > 4/\delta \) we have \( 1/\gamma < \delta/4 < \delta \) so the second term dominates the first term. The order of the variance is thus \( n^{-1+\delta}(\log n)^{1+q} \). Of course, the order of the bias is logarithmic, hence the bias dominates the variance and the mean squared error of \( f_{nh}(x) \) is also logarithmic.

Remark 3.4. Better bounds on the asymptotic variance can be obtained under stronger mixing conditions. Consider for instance uniform mixing. In this case the mixing coefficient \( \phi(t) \) is defined for \( t > 0 \) as

\[
\phi(t) = \sup_{A \in \mathcal{F}_t^- \cap B \in \mathcal{F}_t^+} |P(A|B) - P(A)|
\]

and a process is called uniform mixing if \( \phi(t) \to 0 \) for \( t \to \infty \). Obviously, uniform mixing implies strong mixing. As a matter of fact, one has the relation

\[
\alpha(t) \leq \frac{1}{2} \phi(t).
\]

See Doukhan (1994) for this inequality and many other mixing properties. If \( \sigma \) is uniform mixing with coefficient \( \phi \) satisfying \( \int_0^\infty \phi(t)^{1/2} dt < \infty \), then the variance bound is given by

\[
\text{Var} f_{nh}(x) = O\left( \frac{1}{nh} \frac{h^{2\alpha} e^{\pi/h}}{1/nh} \right) + O\left( \frac{1}{nh} \log n \right).
\]

The proof of this bound runs similarly to the strong-mixing bound. The essential difference is that in equation (11) we use Theorem 2.3.3 of Ibragimov and Linnik (1971) with \( \tau = 0 \) instead of Deo’s (1973) lemma, as in the proof of Theorem 2 in Masry (1983).

Remark 3.5. Smoothness conditions on the density at each time of the solution of a stochastic differential equation are guaranteed under Hörmander’s condition, see Theorem 2.3.3 in Nualart (1995). Recall also relation (3), which can be used to relate the smoothness of the invariant density to the smoothness of the drift and diffusion coefficients.
4 Proof of Theorem 3.1

We give the proof under the additional assumption that $b_t = 0$. The general case is an easy consequence. Let $\mathcal{F}_\sigma$ denote the sigma field generated by the process $\sigma$ and let $\tilde{f}_{nh}$ denote the estimator based on the approximating random variables $\sigma(j-1)\Delta Z^\Delta_j$, written as $\tilde{X}_j$, i.e.

$$
\tilde{f}_{nh}(x) = \frac{1}{nh} \sum_{j=1}^{n} v_h \left( \frac{x - \log(\tilde{X}^2_j)}{h} \right).
$$

The proof of (9) follows from the following two lemmas, whose proofs are given in the next section. The first one deals with the expectation of $\tilde{f}_{nh}$.

**Lemma 4.1.** We have

$$
E \tilde{f}_{nh}(x) = \frac{1}{h} \int_{-\infty}^{\infty} w \left( \frac{x - u}{h} \right) f(u) du.
$$

The second lemma estimates the expected difference between $f_{nh}$ and $\tilde{f}_{nh}$. The bound is in terms of the functions

$$
\gamma_0(h) = \frac{1}{2\pi} \int_{-1}^{1} \frac{1}{|\phi_k(s/h)|} ds
$$

and

$$
\gamma_1(h, x) = e^{\frac{1}{2}\pi/h} + \frac{1}{h} \exp \left( \frac{\pi}{2} \frac{1 + \pi/|x|}{h} \right) \log \frac{1 + \pi/|x|}{h}.
$$

**Lemma 4.2.** For $h \to 0$ and $\varepsilon$ small enough we have

$$
|E f_{nh}(x) - E \tilde{f}_{nh}(x)| = \mathcal{O} \left( \frac{1}{h^2} \gamma_0(h) \frac{\Delta^{1/4}}{\varepsilon} + \frac{1}{h} \gamma_0(h) \frac{\Delta^{1/2}}{\varepsilon^2} + \gamma_1(h, \log 2\varepsilon/h) \frac{\varepsilon}{\log 2\varepsilon} \right).
$$

Notice that the equality (14) is the same as for ordinary kernel estimators, see for instance Wand and Jones (1995). Statement (9) of the theorem then follows by combining standard arguments of kernel density estimation and Lemma 4.2. We will show that the bound in Lemma 4.2 is essentially a negative power of $n$, whereas $h^2$ is of logarithmic order. Recall that we have assumed $\delta > 4/\gamma$. It follows that $1/2\gamma < \delta/4 - 1/2\gamma$, so we can pick a $\beta \in (1/2\gamma, \delta/4 - 1/2\gamma)$ and take $\varepsilon = n^{-\beta}$. Up to factors that are logarithmic in $n$ the order of $|E f_{nh}(x) - E \tilde{f}_{nh}(x)|$ is then

$$
n^{\frac{1}{2\gamma} - \frac{1}{4}\delta + \beta} + n^{\frac{1}{2\gamma} + 2\beta - \frac{\delta}{2}} + n^{\frac{1}{2\gamma} - \beta},
$$

(17)
which is negligible to $h^2 = \gamma^2\pi^2/(\log n)^2$ for the chosen values of the parameters.

To prove the bound (10) we use the two lemmas below, which are proved in the next section. First consider the variance of $\hat{f}_{nh}(x)$.

**Lemma 4.3.** We have, for $h \to 0$,

$$\text{Var} \hat{f}_{nh}(x) = O\left(\frac{1}{n} h^{2\alpha} e^{\pi/h}\right) + O\left(\frac{1}{nh^{1+q}\Delta}\right).$$

The next lemma estimates $\text{Var} (f_{nh}(x) - \hat{f}_{nh}(x))$.

**Lemma 4.4.** We have, for $h \to 0$ and $\varepsilon > 0$ small enough,

$$\text{Var} (f_{nh}(x) - \hat{f}_{nh}(x)) = O\left(\frac{1}{nh^4} \gamma_0(h) \frac{\Delta^{1/2}}{\varepsilon^2} + \frac{1}{n} \gamma_1(h, \varepsilon) \frac{\log 2\varepsilon/h}{\log 2\varepsilon} \frac{\varepsilon}{\log 2\varepsilon^2}\right).$$

The proof of (10) is finished as soon as we show that the estimate in Lemma 4.4 is of lower order than the one in Lemma 4.3. Up to terms that are logarithmic in $n$, the bound in Lemma 4.3 is of order $n^{\delta-1}$. Choosing again $\varepsilon = n^{-\beta}$, up to logarithmic factors, the order of $\text{Var} (f_{nh}(x) - \hat{f}_{nh}(x))$ is

$$n^{-1+1/\gamma - 2\beta} + n^{-1+1/\gamma - \beta} + n^{-1+2\beta+4\beta\delta} + n^{-1+\delta-\beta(1-q)}.$$  (21)

Recall our assumption $\delta \gamma > 4$. If we pick $\beta$ less than $\frac{1}{4}\delta(1-q)$, then all these terms are indeed of lower order than $n^{\delta-1}$.

## 5 Technical lemmas

### 5.1 Analytic properties

We need expansions and order estimates for the functions $\phi_k$, the kernel $v_h$, as defined in (9), $\gamma_0$, as defined in (13) and the function $\gamma_1$, as defined in (16). These are collected in the lemmas of this subsection.

**Lemma 5.1.** For $|t| \to \infty$ we have

$$|\phi_k(t)| = \sqrt{2} e^{-\frac{1}{2}t^2} (1 + O(\frac{1}{|t|})).$$

**Proof.** The characteristic function of $k$ is given by

$$\phi_k(t) = \frac{1}{\sqrt{\pi}} 2^i t \Gamma(\frac{1}{2} + it).$$

The result follows by applying the Stirling formula for the complex gamma function, cf. Abramowitz and Stegun (1964) Chapter 6.
Lemma 5.2. We have the following order estimate for the $L^2$ norm of $v_h$.
For $h \to 0$
\[
\|v_h\|_2 = O(h^{1+\alpha} e^{\pi/2h}).
\]

Proof. By Parseval’s identity
\[
\|v_h\|_2^2 = \frac{1}{2\pi} \int_{-1}^{1} \left|\frac{\phi_w(s)}{\phi_k(s/h)}\right|^2 ds.
\]
The integral on the right-hand side is bounded by
\[
\frac{1}{2} \int_{-1}^{1} \left|\frac{\phi_w(s)}{\phi_k(s/h)}\right|^2 e^{\pi|s/h|} ds + \int_{-1}^{1} \left|\frac{\phi_w(s)}{\phi_k(s/h)}\right|^2 \frac{1}{\phi_k(s/h)} - \frac{1}{2} e^{\pi|s/h|} ds.
\]
The first term in (24) can be rewritten as
\[
e^{\pi/h} h^{1+2\alpha} \int_{0}^{1/h} \left|\frac{\phi_w(1 - hv)}{(hv)^\alpha}\right|^2 v^{2\alpha} e^{-\pi v} dv \sim e^{\pi/h} h^{1+2\alpha} A^2 \int_{0}^{\infty} v^{2\alpha} e^{-\pi v} dv,
\]
by the dominated convergence theorem. We can rewrite the second term in (24) as
\[
2h^{1+2\alpha} e^{\pi/h} \int_{0}^{1/h} \left|\frac{\phi_w(1 - hv)}{(hv)^\alpha}\right|^2 \left|\frac{2e^{-\pi(1/h-v)}}{\phi_k(1/h-v)} - 1\right| v^{2\alpha} e^{-\pi v} dv,
\]
which is of order $O(h^{1+2\alpha} e^{\pi/h})$ by the dominated convergence theorem. We have used the fact that both the functions $\phi_w(1 - u)/u^\alpha$ and $\left|(2\exp(-\pi u)/|\phi_k(u)|^2 - 1\right|$ are bounded and that the second function is of order $O(1/u)$ as $u$ tends to infinity. This shows that the second term (24) is negligible with respect to the first one.

Lemma 5.3. For $h \to 0$ we have
\[
\gamma_0(h) = O\left(h^{1+\alpha} e^{\pi/2h}\right).
\]

Proof. The proof is similar to that of Lemma 5.2.

Lemma 5.4. The functions $v_h$ are bounded and Lipschitz. More precisely, for all $x$ we have $|v_h(x)| \leq \gamma_0(h)$ and for all $x$ and $u$
\[
|v_h(x + u) - v_h(x)| \leq \gamma_0(h) |u|.
\]

Proof. The bound for $|v_h(x)|$ is obvious. To prove (26) write
\[
|v_h(x + u) - v_h(x)| \leq \frac{1}{2\pi} \int_{-1}^{1} \left|\frac{\phi_w(s)}{\phi_k(s/h)}\right| e^{-isu} - 1 |ds \leq \gamma_0(h) |u|.
\]
Lemma 5.5. For $x \to \infty$ we have the following estimate on the behavior of $v_h$. For some positive constant $D$ it holds that

$$|v_h(x)| \leq \frac{D}{|x|} \gamma_1(h, x) \text{ as } |x| \to \infty,$$

(27)

and

$$\gamma_1(h, x) = O\left(\frac{|\log h|}{h} e^{\frac{1}{2} \pi (1 + \pi/|x|)/h}\right) \text{ as } h \to 0. \quad (28)$$

Proof. By a bound in the proof of the Riemann Lebesgue lemma on page 402 of Hewitt and Stromberg (1965) we have, with $y = \pi/x$,

$$|v_h(x)| = \frac{1}{2\pi} \left| \int_{-\infty}^{\infty} \frac{\phi_w(s)}{\phi_k(s/h)} e^{-i s x} ds \right| \leq \frac{1}{2\pi} \int_{-\infty}^{\infty} \left| \frac{\phi_w(s)}{\phi_k(s/h)} - \frac{\phi_w(s + y)}{\phi_k((s + y)/h)} \right| ds$$

(29)

$$\leq \frac{1}{2\pi} \int_{-\infty}^{\infty} \phi_w(s) \left| \frac{1}{\phi_k(s/h)} - \frac{1}{\phi_k((s + y)/h)} \right| ds \quad \text{and, as } |x| \to \infty,$$

(30)

First we need a bound on the integral (29). Since it follows from Conditions W that $\phi_w$ is Lipschitz (the proof is similar to that of (26)), with Lipschitz constant $C_1$ say, we have

$$\int_{-\infty}^{\infty} \left| \frac{\phi_w(s) - \phi_w(s + y)}{\phi_k(s/h)} \right| ds \leq C_1 \int_{-1}^{1} \frac{1}{|\phi_k(s/h)|} ds |y| \sim C_1 \sqrt{2} e^{\frac{1}{2} \pi/h} |y|,$$

by Lemma 5.1. To bound the integral (30) we need an estimate on the behaviour of $|\phi_k'|/|\phi_k|^2$. Recall the expression (22) for $\phi_k$. Hence, with $\Psi = \Gamma'/\Gamma$ the digamma function,

$$|\phi_k'(t)| = \frac{1}{\sqrt{\pi}} \left| i \log 2 e^{it \log 2} \Gamma'(\frac{1}{2} + it) + i e^{it \log 2} \Gamma'(\frac{1}{2} + it) \right| \leq \frac{1}{\sqrt{\pi}} \left( \log 2 |\Gamma'(\frac{1}{2} + it)| + |\Gamma'(\frac{1}{2} + it)| \right)$$

and, as $|t| \to \infty$,

$$\frac{|\phi_k'(t)|}{|\phi_k(t)|^2} \leq \sqrt{\pi} \frac{1}{|\Gamma'(\frac{1}{2} + it)|} \left( \log 2 + |\Psi(\frac{1}{2} + it)| \right) \leq 4 \sqrt{\pi} \log(|t|) e^{\frac{1}{2} \pi |t|}, \quad (31)$$
by Lemma 5.1 and by the expansion $|\Psi(z)| \sim \log z$ for $z \to \infty$, $|\text{Arg} z| < \pi$, cf. Abramowitz and Stegun (1964), Chapter 6. We now turn back to the integral (30) and write

$$
\int_{-\infty}^{\infty} \phi_w(s + y) \left| \frac{1}{\phi_k(s/h)} - \frac{1}{\phi_k((s + y)/h)} \right| ds
$$

$$
= \int_{-1}^{1} \phi_w(s) \left| \frac{1}{\phi_k((s - y)/h)} - \frac{1}{\phi_k(s/h)} \right| ds
$$

$$
\leq \frac{2}{h} \sup_{(\pi+1)|y|/h \leq 1 + |y|/h} \frac{|\phi_k'(s)| |y|}{\phi_k(s)^2}
$$

$$
\leq \frac{2}{h} \sup_{(\pi+1)|y|/h \leq 1 + |y|/h} 4\sqrt{\pi} \log(|y|) e^{\pi |y|/h} |y|
$$

$$
= 8 \sqrt{\pi} \log((1 + |y|)/h) e^{\pi (1 + |y|)/h} |y|
$$

in view of (31). This completes the proof. □

5.2 Proof of lemmas 4.1-4.4

We start with the proof of Lemma 4.1. Recall that $F_{\sigma}$ is the $\sigma$-algebra generated by the process $\sigma$.

**Proof of Lemma 4.1.** Write

$$
E \left( \tilde{f}_{nh}(x) | F_{\sigma} \right) = \frac{1}{nh} \sum_{t=1}^{n} E \left( v_h \left( \frac{x - \log \sigma_{(t-1)\Delta}^2 - \log(Z_{\Delta}^2)}{h} \right) | F_{\sigma} \right)
$$

$$
= \frac{1}{nh} \sum_{t=1}^{n} \frac{1}{2\pi} \int_{-\infty}^{\infty} \phi_w(s) e^{-is(x - \log \sigma_{(t-1)\Delta}^2 - \log(Z_{\Delta}^2))/h} ds
$$

$$
= \frac{1}{nh} \sum_{t=1}^{n} \frac{1}{2\pi} \int_{-\infty}^{\infty} \phi_w(s) e^{-isx/h} e^{is \log \sigma_{(t-1)\Delta}^2/h} \phi_k(s/h) ds
$$

$$
= \frac{1}{nh} \sum_{t=1}^{n} \frac{1}{2\pi} \int_{-\infty}^{\infty} \phi_w(s) e^{-is(x - \log \sigma_{(t-1)\Delta})/h} ds
$$

$$
= \frac{1}{nh} \sum_{t=1}^{n} w \left( \frac{x - \log \sigma_{(t-1)\Delta}}{h} \right).
$$

By taking expectation the result follows. □

For the proof of Lemma 4.2 we need a few properties of the process $\sigma$, valid under Condition $\sigma$. Since $(x - y)^2 \leq |x^2 - y^2|$ for $x, y \geq 0$, it holds that $E (\sigma_t - \sigma_0)^2 = O(t^{1/2})$ for $t \to 0$. Consequently, there exists a constant $C > 0$ such that

$$
E (X_{t}^\Delta - \sigma_0 Z_{\Delta}^2)^2 \leq C \Delta^{1/2} \text{ for } \Delta \to 0,
$$

(32)
since \( \text{E} \left( X_1^\Delta - \sigma_0 Z_1^\Delta \right)^2 = \frac{1}{\Delta} \text{E} \int_0^\Delta (\sigma_t - \sigma_0)^2 \, dt \). Moreover, Condition \( \sigma \) implies that

\[
\text{E} \left| \frac{1}{\Delta} \int_0^\Delta \sigma_t^2 \, dt - \sigma_0^2 \right| = O(\Delta^{1/2}) \text{ for } \Delta \to 0. \tag{33}
\]

**Proof of Lemma 4.2.** Writing

\[
W_j = v_h \left( \frac{x - \log((X_j^\Delta)^2)}{h} \right) - v_h \left( \frac{x - \log(\tilde{X}_j^2)}{h} \right),
\]

so that \( f_{nh}(x) - \tilde{f}_{nh}(x) = \frac{1}{nh} \sum_{j=1}^n W_j \), we have

\[
|E f_{nh}(x) - E \tilde{f}_{nh}(x)| \leq \frac{1}{h} E|W_j| \leq \frac{1}{h} E|W_j| I_{|X_1^\Delta| \geq \epsilon} \text{ and } |\tilde{X}_1| \geq \epsilon \tag{34}
\]

\[
= \frac{1}{h} E|W_j| I_{|X_1^\Delta| \leq \epsilon} \text{ or } |\tilde{X}_1| \leq \epsilon \tag{35}
\]

\[
+ \frac{1}{h} E|W_j| I_{|X_1^\Delta| \leq \epsilon} \text{ or } |\tilde{X}_1| \leq \epsilon \tag{36}
\]

By Lemma 5.4 and (32) the term (34) can be bounded by

\[
\frac{2}{h^2} \gamma_0(h) E \left| \log((X_1^\Delta) - \log(\tilde{X}_1) \right| I_{|X_1^\Delta| \geq \epsilon} \text{ and } |\tilde{X}_1| \geq \epsilon \leq \frac{2}{h^2} \gamma_0(h) \sqrt{C} \Delta^{1/4}/\epsilon.
\]

In the same way the term (35) can be bounded by

\[
\frac{2}{h} \gamma_0(h) P(|X_1^\Delta - \tilde{X}_1| \geq \epsilon) \leq \frac{2}{h} \gamma_0(h) C \Delta^{1/2} \epsilon^2.
\]

Since the absolute value of both arguments of \( v_h \) below are eventually larger than \( |\log 2\epsilon|/h \), by Lemma 5.5 the term (36) can be bounded by

\[
\frac{1}{h} \gamma_1(h, |\log 2\epsilon|/h) \frac{1}{|\log 2\epsilon|/h} P(|\tilde{X}_1| \leq 2\epsilon) \leq C_2 \gamma_1(h, |\log 2\epsilon|/h) \frac{\epsilon}{|\log 2\epsilon|},
\]

for some constant \( C_2 \). Here we used the fact that the density of \( \tilde{X}_1 \) is bounded which follows from the assumption that \( \sigma_0^2 \) has a bounded density in a neighbourhood of zero. \( \square \)

**Proof of Lemma 4.3.** Consider the decomposition

\[
\text{Var} (\tilde{f}_{nh}(x)) = \text{Var} (E (\tilde{f}_{nh}(x)|\mathcal{F}_\sigma)) + E (\text{Var} (\tilde{f}_{nh}(x)|\mathcal{F}_\sigma)). \tag{37}
\]
By the proof of Lemma 4.1, the conditional expectation \( E(\tilde{f}_{nh}(x)|\mathcal{F}_\sigma) \) is equal to a kernel estimator of the density of \( \log \sigma^2_t \). By Theorem 3 of Masry (1983), we can bound its variance by

\[
\frac{20(1+o(1))}{nh^{1+q\Delta}} f(x)^{1-q} \left( \int_{-\infty}^{\infty} |w(u)|^{2/(1-q)} du \right)^{1-q} \int_0^\infty \alpha(\tau)^q d\tau = O\left( \frac{1}{nh^{1+q\Delta}} \right).
\]

Given the process \( \sigma \) the random variables \( \log \tilde{X}_t^2 \) are independent, so we can bound the second term in (37) by

\[
\frac{1}{n^2h^2} \sum_{t=1}^n \text{Var} \left( v_h \left( \frac{x - \log \tilde{X}_t^2}{h} \right) \right) \leq \frac{1}{nh^2} \text{E} \left( v_h \left( \frac{x - \log \tilde{X}_t^2}{h} \right) \right)^2 \leq \frac{1}{nh^2} \gamma_0(h)^2,
\]

by Lemma 5.4. The result follows by an application of Lemma 5.3. \( \square \)

**Proof of Lemma 4.4.** Note that for different \( i, j \), conditional on the process \( \sigma \), the pairs \( X_i^\Delta, \tilde{X}_i \) and \( X_j^\Delta, \tilde{X}_j \) are independent. Hence the conditional covariances of functions of these pairs vanish.

With \( W_j \) as in the proof of Lemma 4.2 we have

\[
\text{Var} \left( f_{nh}(x) - \tilde{f}_{nh}(x) \right) = \frac{1}{nh^2} \text{Var} W_1 + \frac{1}{n^2h^2} \sum_{i \neq j} \text{Cov} \left( \text{E} (W_i|\mathcal{F}_\sigma), \text{E} (W_j|\mathcal{F}_\sigma) \right).
\]

Let us first derive a bound on \( \text{Var} W_1 \). We have \( \text{Var} W_1 \leq \text{E} W_1^2 \), which can be split up in three terms

\[
\frac{1}{h} \text{E} W_j^2 I_{[|X_i^\Delta| \geq \varepsilon \text{ and } |\tilde{X}_i| \geq \varepsilon]}
\]

\[
+ \frac{1}{h} \text{E} W_j^2 I_{[|X_i^\Delta| \leq \varepsilon \text{ or } |\tilde{X}_i| \leq \varepsilon] I_{||X_i^\Delta - \tilde{X}_i| \geq \varepsilon]}
\]

\[
+ \frac{1}{h} \text{E} W_j^2 I_{[|X_i^\Delta| \leq \varepsilon \text{ or } |\tilde{X}_i| \leq \varepsilon] I_{||X_i^\Delta - \tilde{X}_i| < \varepsilon]}.
\]

By (32) and Lemma 5.4, the term (39) can be bounded by \( \frac{2}{h} \gamma_0(h)^2 C \frac{\Delta^{1/2}}{\varepsilon^2} \). Again by (32) and Lemma 5.4, the term (40) can be bounded by

\[
4\gamma_0(h)^2 P(\|X_1^\Delta - \tilde{X}_1\| \geq \varepsilon) \leq 4\gamma_0(h)^2 C \frac{\Delta^{1/2}}{\varepsilon^2}.
\]

Since the absolute value of both arguments of \( v_h \) below are eventually larger than \( |\log 2\varepsilon|/h \), by Lemma 5.5 the term (41) can be bounded by

\[
\frac{\gamma_1(h, |\log 2\varepsilon|/h)^2}{(|\log 2\varepsilon|/h)^2} P(\|\tilde{X}_1\| \leq 2\varepsilon) \leq C_2 h^2 \gamma_1(h, |\log 2\varepsilon|/h)^2 \frac{\varepsilon}{|\log 2\varepsilon|^2},
\]

13
for some constant $C_2$, where we use again, as in the proof of Lemma 4.2, that the density of $\tilde{X}_1$ is bounded.

We get

$$E W_1^2 = O\left(\frac{1}{h^2} \gamma_0(h)^2 C \frac{\Delta^{1/2}}{\varepsilon^2} + h^2 \gamma_1(h, |\log 2\varepsilon|/h)^2 \frac{\varepsilon}{|\log 2\varepsilon|^2}\right),$$

which gives the first order bound (41).

Next we concentrate on the sum of covariances in (38). Define

$$\tilde{\sigma}_i = \frac{1}{\Delta} \int_{(i-1)\Delta}^{i\Delta} \sigma_t^2 dt. \quad (43)$$

Note that given $F_{\sigma}$, $X_{\Delta i}$ is $N(0, \tilde{\sigma}_i)$ distributed and $\tilde{X}_i$ is $N(0, \sigma_i^2 (i-1)\Delta)$. As in the proof of Lemma 4.1 it follows that

$$E \left(W_i | F_{\sigma}\right) = w\left(\frac{x - \log \tilde{\sigma}_i}{h}\right) - w\left(\frac{x - \log \sigma_{(i-1)\Delta}}{h}\right).$$

We follow the line of arguments in the proof of Theorem 3 in Masry (1983). The stationarity of $W_j$ implies that also the conditional expectations $\tilde{W}_j := E (W_j | F_{\sigma})$ are stationary. Hence we have

$$\sum_{i \neq j} \text{Cov} (\tilde{W}_i, \tilde{W}_j) = 2 \sum_{k=1}^{n-1} (n-k) \text{Cov} (\tilde{W}_0, \tilde{W}_k).$$

Now note that the process $\tilde{W}_j$ is strongly mixing with a mixing coefficient $\tilde{\alpha}(k) \leq \alpha((k-1)\Delta), k = 1, 2, \ldots$, where $\alpha$ is the coefficient of the process $\sigma$. By a lemma of Deo (1973) for strongly mixing processes it follows that for all $\tau > 0$

$$|\text{Cov} (\tilde{W}_0, \tilde{W}_k)| \leq 10\alpha((k-1)\Delta)^{\tau/(2+\tau)} \left(E |\tilde{W}_1|^{2+\tau}\right)^{2/(2+\tau)}. \quad (44)$$

By the monotonicity of the mixing coefficient $\alpha$ we get

$$\left|\frac{1}{n^2 h^2} \sum_{i \neq j} \text{Cov} (\tilde{W}_i, \tilde{W}_j)\right| \leq \frac{10}{nh^2} \left(E |\tilde{W}_1|^{2+\tau}\right)^{2/(2+\tau)} \sum_{k=1}^{n-1} (1 - \frac{k}{n}) \alpha((k-1)\Delta)^{\tau/(2+\tau)}$$

$$\leq \frac{10}{nh^2} \alpha(0)^{\tau/(2+\tau)} + \frac{1}{\Delta} \int_0^\infty \alpha(t)^{\tau/(2+\tau)} dt \left(E |\tilde{W}_1|^{2+\tau}\right)^{2/(2+\tau)}.$$
Next we derive a bound on $E|\tilde{W}_1|^{2+\tau}$. Fix $\kappa \in (0, 1]$. We have

$$E|\tilde{W}_1|^{2+\tau} = E \left| w \left( \frac{x - \log(\tilde{\sigma}_1)}{h} \right) - w \left( \frac{x - \log(\sigma_0^2)}{h} \right) \right|^{2+\tau}$$

$$\times I_{[\sigma_1 \geq \varepsilon \text{ and } \sigma_0^2 \geq \varepsilon]}$$

$$+ E \left| w \left( \frac{x - \log(\tilde{\sigma}_1)}{h} \right) - w \left( \frac{x - \log(\sigma_0^2)}{h} \right) \right|^{2+\tau}$$

$$\times I_{[\sigma_1 \leq \varepsilon \text{ or } \sigma_0^2 \leq \varepsilon]} I_{[|\tilde{\sigma}_1^\kappa - \sigma_0^{2\kappa}| \geq \varepsilon]}$$

$$+ E \left| w \left( \frac{x - \log(\tilde{\sigma}_1)}{h} \right) - w \left( \frac{x - \log(\sigma_0^2)}{h} \right) \right|^{2+\tau}$$

$$\times I_{[\sigma_1 \leq \varepsilon \text{ or } \sigma_0^2 \leq \varepsilon]} I_{[|\tilde{\sigma}_1^\kappa - \sigma_0^{2\kappa}| < \varepsilon]}.$$  

Note that by Condition W and Fourier inversion $w$ is Lipschitz with constant $1/\pi$ and bounded by $1/\pi$. Hence the term (43) can be bounded by $E |\tilde{\sigma}_1^\kappa - \sigma_0^{2\kappa}|^{2+\tau}/(\kappa \varepsilon h)^{2+\tau}$. The term (46) can be bounded by

$$P(|\tilde{\sigma}_1^\kappa - \sigma_0^{2\kappa}| \geq \varepsilon) \leq \frac{1}{\varepsilon^{2+\tau}} E |\tilde{\sigma}_1^\kappa - \sigma_0^{2\kappa}|^{2+\tau}.$$  

Likewise, the term (47) can be bounded by

$$E \left| w \left( \frac{x - \log(\tilde{\sigma}_1)}{h} \right) - w \left( \frac{x - \log(\sigma_0^2)}{h} \right) \right|^{2+\tau} I_{[\sigma_1 \leq \varepsilon/(1+\varepsilon^{1-\kappa})]} I_{[\sigma_0^2 \leq \varepsilon/(1+\varepsilon^{1-\kappa})]} I_{[\tilde{\sigma}_1^\kappa - \sigma_0^{2\kappa}| \geq \varepsilon]} I_{[|\tilde{\sigma}_1^\kappa - \sigma_0^{2\kappa}| < \varepsilon]},$$  

which is bounded by $P(\sigma_0^2 \leq 2\varepsilon) = O(\varepsilon)$ since $\sigma_0^2$ was assumed to have a bounded density in a neighbourhood of zero.

With $\tau = 2q/(1-q)$ and $\kappa = \frac{1}{2+\tau} = \frac{1-2}{2}$ we have with an application of the basic inequality $|u^\kappa - v^\kappa| \leq |u - v|^\kappa$ for $u, v \geq 0$ and $\kappa \in (0, 1]$ in the second equality below and from condition $\sigma$ and its consequence (33) in the fourth equality

$$\left| \frac{1}{n^2 h^2} \sum_{i \neq j} \text{Cov} (\tilde{W}_1, \tilde{W}_j) \right|$$

$$= \frac{1}{n h^2 \Delta} O \left( \frac{1}{h^{2+\tau}} \frac{1}{\varepsilon^{2+\tau}} E |\tilde{\sigma}_1^\kappa - \sigma_0^{2\kappa}|^{2+\tau} + \varepsilon \right)^{2/(2+\tau)}$$

$$= \frac{1}{n h^2 \Delta} O \left( \frac{1}{h^{2+\tau}} \frac{1}{\varepsilon^{2+\tau}} E |\tilde{\sigma}_1 - \sigma_0^2|^{(2+\tau)} + \varepsilon \right)^{2/(2+\tau)}$$

$$= \frac{1}{n h^2 \Delta} O \left( \frac{(E |\tilde{\sigma}_1 - \sigma_0^2|)^{2/(2+\tau)}}{h^2 \varepsilon^2} + \varepsilon^{2/(2+\tau)} \right)$$

$$= \frac{1}{nh^2 \Delta} O \left( \frac{\Delta^{2/(2+\tau)}}{h^2 \varepsilon^2} + \varepsilon^{2/(2+\tau)} \right)$$

$$= \frac{1}{nh^2 \Delta} O \left( \frac{\Delta^{1/(2+\tau)}}{h^2 \varepsilon^2} + \varepsilon^{1-q} \right),$$  

which gives the second order bound (20).  

□
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