Determining mean first-passage time on a class of treelike regular fractals

Yuan Lin, Bin Wu, and Zhongzhi Zhang

School of Computer Science, Fudan University, Shanghai 200433, China and Shanghai Key Lab of Intelligent Information Processing, Fudan University, Shanghai 200433, China

(Dated: October 1, 2010)

Relatively general techniques for computing mean first-passage time (MFPT) of random walks on networks with a specific property are very useful, since a universal method for calculating MFPT on general graphs is not available because of their complexity and diversity. In this paper, we present techniques for explicitly determining the partial mean first-passage time (PMFPT), i.e., the average of MFPTs to a given target averaged over all possible starting positions, and the entire mean first-passage time (EMFPT), which is the average of MFPTs over all pairs of nodes on regular treelike fractals. We describe the processes with a family of regular fractals with treelike structure. The proposed fractals include the T fractal and the Peano basin fractal as their special cases. We provide a formula for MFPT between two directly connected nodes in general trees on the basis of which we derive an exact expression for PMFPT to the central node in the fractals. Moreover, we give a technique for calculating EMFPT, which is based on the relationship between characteristic polynomials of the fractals at different generations and avoids the computation of eigenvalues of the characteristic polynomials. Making use of the proposed methods, we obtain analytically the closed-form solutions to PMFPT and EMFPT on the fractals and show how they scale with the number of nodes. In addition, to exhibit the generality of our methods, we also apply them to the Vicsek fractals and the iterative scale-free fractal tree and recover the results previously obtained.

PACS numbers: 05.40.Fb, 61.43.Hv, 89.75.Hc, 05.60.Cd

I. INTRODUCTION

As an integral ingredient of nonlinear science, fractals have attracted an increasing attention in physics and other scientific fields [1, 2] not only because of the striking beauty intrinsic in their structure [3] but also due to the significant impact of the idea of fractals on a large variety of scientific disciplines, such as astrophysics, plasma physics, optics, economy, ecology, and so on [4]. Among various fractal categories, regular fractals constitute an important family of fractals. Frequently cited examples include the Cantor set [5], the Koch curve [6], the Sierpinski gasket [7], the Vicsek fractals [42], the T fractal [8], etc. These structures have been a focus of research objects [10]. One of the main justifications for studying regular fractals is that many problems can be exactly solvable on regular fractals [11], the explicit solutions for which provide useful insight different from that of approximate solutions for random fractals.

One of the ultimate goals of research efforts on fractals is to unveil how their underlying geometrical and structural features affect dynamical processes and critical phenomenon on them [12]. Among a variety of dynamics, due to their simplicity and wide range of applications [13], random walks play a central role in many branches of sciences and engineering and have gained a considerable attention within the scientific community [14, 15]. A key quantity related to random walks is mean first-passage time (MFPT) [16, 17] that is the expected time for the walker starting off from a source node to first reach a given target node. In addition to its role as a basic quantitative measure of the transportation efficiency, MFPT also encodes useful information of other quantities concerned with random walks [18].

Concerted efforts have been devoted to study MFPT in fractals in order to obtain the scaling of MFPT with system size [19]. For instance, MFPT associated to a given starting node has been discussed by many authors [19, 20] since it depends on the starting position and is an important issue in its own right. On the other hand, recent papers have addressed the explicit determination of partial mean first-passage time (PMFPT) on some regular fractals, defined as the average of MFPTs to a selected target node averaged over all possible starting nodes, such as the Sierpinski gasket [21, 22], the T fractal [23, 24], the iterative scale-free treelike fractal [25], as well as the hierarchical lattice fractals [26, 27]. Besides, entire mean first-passage time (EMFPT), i.e., the average of MFPTs over all pairs of nodes, has been computed for the T fractal [28], the Vicsek fractals [29], and some other regular fractals [30]. Thus far, most techniques used to calculate PMFPT or EMFPT are only applicable to specific structures; universal (even relatively general) methods for computing MFPTs are much less [17, 30].

In this paper, we develop techniques for the explicit determination of PMFPT and EMFPT on regular treelike fractals, which are also expected to be valid to other deterministic media. For PMFPT, we provide a universal formula for MFPT between two adjacent nodes in generic trees based on which one can determine PMFPT between a given node and all other nodes. For EMFPT, by considering only the relationship between characteristic polynomials at different iterations, we can find exactly
the analytical expression for EMFPT, which needs only local information of characteristic polynomials and can avoid the laborious explicit computation of the eigenvalues.

To illustrate the calculation processes, we put forward a family of treelike regular fractals, which include the famous $T$ fractal and the Peano basin fractal [31] as their particular cases. Using the presented methods, we first study the random-walk problem with a trap fixed at the central node of the fractals and obtain the PMFPT to the trap; then, we address the case that the trap is uniformly selected among all nodes and derive the rigorous solution to EMFPT between all node pairs. From the obtained analytical results, we give the scalings for both PMFPT and EMFPT and show that they both increase as a power-law function of the number of nodes, with the exponent larger than 1 but smaller than 2.

II. CONSTRUCTIONS AND PROPERTIES OF THE TREELIKE FRACTALS

Let us first introduce the model for the treelike fractals that are built in an iterative way and controlled by a positive-integer parameter $m$ (i.e., $m \geq 1$). We denote by $T_g$ ($g \geq 0$) the treelike fractals after $g$ iterations. Notice that the number of iterations also represents the generation of the treelike fractals. Initially ($g = 0$), $T_0$ is an edge connecting two nodes. For $g \geq 1$, $T_g$ is obtained from $T_{g-1}$ by performing the following operations on every edge in $T_{g-1}$ as shown in Fig. 1: replace the edge by a path of two links long, with the two end points of the path being the same end points of the original edge, then attach $m$ new nodes to the middle node of the path. Figure 2 illustrates the construction process of a particular fractal for the case of $m = 2$, showing the first several generations. Notice that there are several limiting cases of our model. When $m = 1$, it is reduced to the $T$ fractal [31]. When $m = 2$, it turns out to be the Peano basin fractal [31].

According to the construction algorithm, at each step the number of edges in the treelike fractals increases by a factor of $m + 2$. Thus, we can easily know that the total number of edges of $T_g$ is $E_g = (m + 2)^g$ and that the total number of nodes (often named network order) in $T_g$ is $N_g = E_g + 1 = (m + 2)^g + 1$. One can partition all the $N_g$ nodes into two classes: nodes having degree $m + 2$ are called internal nodes and nodes with degree of 1 are named external nodes. Since at each generation, a preexisting edge creates $m + 1$ nodes: one of which has a degree of $m + 2$, the rest $m$ nodes have a degree of 1. Thus, at generation $g_i$ ($g_i \geq 1$), the numbers of internal and external nodes generated at this generation are $(m + 2)^{g_i - 1}$ and $m(m + 2)^{g_i - 1}$, respectively.

On the other hand, it is obvious that after each iteration, the diameter (namely, the maximum of shortest distances between all pairs of nodes) of the fractals doubles. Thus, we have that the fractal dimension of the treelike fractals is $d_f = \ln(m + 2)/\ln 2$. In addition, as we will show below, for any two nodes $i$ and $j$ at current generation, the MFPT from node $i$ to node $j$ increases by a factor of $2(m + 2)$ at next generation. Hence, the random-walk dimension of the fractals is $d_w = \ln[2(m + 2)]/\ln 2 = 1 + d_f$, and their spectral dimension is $d = 2d_f/d_w = 2 \ln(m + 2)/\ln[2(m + 2)] = 2d_f/(1 + d_f)$.

It is worth mentioning that the treelike fractals can also be alternatively constructed using another method. If we define the central node (see Fig. 2 for an example) as the innermost node and the nodes farthest from the central node as the outermost nodes, then the second generating algorithm for the fractals can be described as follows: Given the generation $g$, $T_{g+1}$ can be obtained by joining $m + 2$ replicas of $T_g$ (see Fig. 3). That is to say, to obtain $T_{g+1}$ one can merge together the discrete outermost nodes of the $m + 2$ copies of $T_g$. The $m + 2$ outermost nodes belonging to separate copies merge into a single new node, which is then the innermost node in $T_{g+1}$.

After introducing the construction algorithms and the properties of the treelike fractals, in what follows we will study the random-walk dynamics on the fractal family $T_g$. First, we will investigate random walks with a single immobile trap located on the innermost node. Then, we will continue to address random walks with the trap selected randomly (i.e., distributed uniformly) among all nodes.
and time, for a walker, staring from node $i$ is presented as the trap. What we are concerned is the PMFPT pre-absorption [23]. By definition, $\langle F \rangle_g$ is also often called mean trapping time or mean time to absorption. In the proposed fractals other than the trap, PMFPT over all starting nodes distributed uniformly over nodes in the tree. We denote by $u, v$ the nodes in the innermost node, while $K$ is an outermost node.

III. RANDOM WALKS WITH A TRAP FIXED ON THE CENTRAL NODE

In this section we focus on a particular case of random walks—trapping problem—on the family of treelike fractals $T_g$ with the trap or perfect absorber placed on the central node of the regular fractals. In the process of the random walks, at each time step, the walker, starting from its current location, jumps to any of its neighbor nodes with identical probability. For the convenience of description, we label by 1 the central node of $T_g$, while all other nodes are labeled consecutively as 2, 3, ..., $N_g - 1$, and $N_g$. Let $F_i(g)$ be the MFPT, also called trapping time, for a walker, staring from node $i$ to first arrive at the trap. What we are concerned is the PMFPT presented as $\langle F \rangle_g$, which is defined as the average of $F_i(g)$ over all starting nodes distributed uniformly over nodes in the proposed fractals other than the trap. PMFPT is also often called mean trapping time or mean time to absorption [23]. By definition, $\langle F \rangle_g$ is given by

$$\langle F \rangle_g = \frac{1}{N_g - 1} \sum_{i=2}^{N_g} F_i(g).$$

The main purpose of this section is to determine explicitly $\langle F \rangle_g$ and show how $\langle F \rangle_g$ scales with network order.

A. Mean first-passage time between two adjacent nodes in general trees

Here, we provide a universal formula for MFPT between two nodes directly connected by an edge in a treelike network, which is very helpful for the following derivation of $\langle F \rangle_g$. Let us consider a general connected tree. We denote by $e = (u, v)$ an edge in the tree connecting two nodes $u$ and $v$. Obviously, if we remove the edge $(u, v)$, the tree will be divided into two small subtrees: one subtree includes node $u$ and the other contains node $v$. We use $C_{u<v}$ to denote the number of nodes in the subtree containing node $u$. Actually, $C_{u<v}$ is the number of nodes in the original tree lying closer to $u$ than to $v$, including $u$ itself. Furthermore, let $F_{uv}$ denote the MFPT from $u$ to $v$. Then, $F_{uv}$ can be written in terms of $C_{u<v}$ as

$$F_{uv} = 2C_{u<v} - 1. \quad (2)$$

Equation (2) can be easily proved inductively. If $u$ is a leaf node (namely, node with degree 1), it is obvious that $C_{u<v} = 1$ and $F_{uv} = 1$, and thus Eq. (2) holds. We proceed to consider the case that $u$ is a non-leaf node. For this case, to prove the validity of Eq. (2), we consider the tree as a rooted one with node $v$ being its root, then $v$ is the father of $u$ and $C_{u<v}$ is in fact the number of nodes in the subtree whose root is $u$. Suppose that for a non-leaf node $u$, Eq. (2) is true for all its children, the set of which is denoted by $\Omega_u$. In other words, for an arbitrary node $x$ in $\Omega_u$, the following relation holds:

$$F_{xu} = 2C_{x<u} - 1. \quad (3)$$

Then, the MFPT $F_{uv}$ from node $u$ to its father node $v$ can be calculated by

$$F_{uv} = \frac{1}{\delta} + \frac{1}{\delta} \sum_{x \in \Omega_u} (1 + F_{ux}), \quad (4)$$

where $\delta$ is the degree of node $u$.

The first term on the right-hand side (rhs) of Eq. (4) explains the case that the walker, starting off from node $u$, jumps directly to node $v$ in one single step with probability $\frac{1}{\delta}$. The second term accounts for another case that the walker first reaches a child node $x$ of node $u$ in one step, and then it takes more $F_{ux}$ steps to first hit the target node $v$. Equation (4) can be rewritten as

$$F_{uv} = \frac{1}{\delta} + \frac{1}{\delta} \sum_{x \in \Omega_u} (1 + F_{ux} + F_{uv}), \quad (5)$$

which leads to

$$F_{uv} = 1 + \sum_{x \in \Omega_u} (1 + F_{ux}) = \delta + \sum_{x \in \Omega_u} (2C_{x<u} - 1) = 2C_{u<v} - 1, \quad (6)$$

where the assumption, viz. Eq. (3), was used. Thus, we have proved Eq. (2), which is a basic characteristic for random walks on a tree and is useful for the following computation of the key quantity $\langle F \rangle_g$.

B. Evolution law for mean first-passage time in the fractals

Let $F_{ij}(g)$ be the MFPT of random walks on $T_g$, starting from node $i$, to first reach node $j$. Let $(u, v)$ be
an edge connecting two nodes \( u \) and \( v \) in \( T_g \). Next, we will give a relation between \( F_{uv}(g+1) \) and \( F_{uv}(g) \). To this end, we look upon the treelike regular fractals \( F_{uv} \) as rooted trees with node \( v \) as the root, and thus \( u \) is the father of \( u \). Note that in the evolution of the fractals, we suppose that node \( v \) is always the root. In addition, for \( T_g \) we use \( C_u(g) \) and \( E_u(g) \) to represent the numbers of nodes and edges in the subtree, whose root is node \( u \). Clearly,

\[
E_u(g) = C_u(g) - 1.
\]

On the other hand, by construction of the fractals, each edge in \( T_g \) will be replaced by \( m+2 \) new edges at iteration \( g+1 \). Then, we can easily have

\[
E_u(g+1) = (m+2)E_u(g), \tag{8}
\]

from which we can obtain the recursion relation between \( C_u(g+1) \) and \( C_u(g) \). By definition,

\[
C_u(g+1) = E_u(g+1) + 1 = (m+2)E_u(g) + 1 = (m+2)C_u(g) - m - 1. \tag{9}
\]

Now we begin to derive the relation governing \( F_{uv}(g+1) \) and \( F_{uv}(g) \). According to the general result given in Eq. (2), we have

\[
F_{uv}(g) = 2C_u(g) - 1. \tag{10}
\]

Notice that at generation \( g+1 \), \( v \) is no longer the father of \( u \) but still an ancestor of \( u \). According to the construction algorithm of the fractals, the edge in \( T_g \) connecting nodes \( u \) and \( v \) will generate \( m+1 \) new nodes at generation \( g+1 \) (see Fig. 3): one node \( x \) has a degree \( m+2 \), while the other \( m \) nodes denoted by \( x_i \) \((i = 1, 2, \ldots, m)\) have the same degree 1. After this iteration, the father of \( u \) becomes \( x \) and \( x \) is a child of \( v \). Then, for a random walker in \( T_{g+1} \), if it wants to transfer from \( u \) to \( v \), it must pass by node \( x \). Therefore,

\[
F_{uv}(g+1) = F_{ux}(g+1) + F_{xv}(g+1)
= 2C_u(g+1) + 2C_x(g+1) - 2. \tag{11}
\]

Then, the determination of \( F_{uv}(g+1) \) is reduced to finding \( C_u(g+1) \) and \( C_x(g+1) \).

Since \( C_u(g+1) \) has been determined above, we only need to find \( C_x(g+1) \). According to the structure of the fractals, it is obvious that

\[
C_x(g+1) = C_u(g+1) + m + 1 = (m+2)C_u(g), \tag{12}
\]

where the term \( m+1 \) between the two equal marks describes the number of nodes \( x \) and \( x_i \) \((i = 1, 2, \ldots, m)\). Plugging Eqs. (9) and (12) into Eq. (11), we obtain

\[
F_{uv}(g+1) = 4(m+2)C_u(g) - 2m - 4
= 2(m+2)F_{uv}(g), \tag{13}
\]

in which Eq. (10) has been made use of.

Equation (13) tells us that for any two adjacent nodes \( u \) and \( v \) in the fractals at a given generation, the MFPT from one of them to the other will increase by a factor of \( 2(m+2) \) at the next generation. Since the considered fractals have a treelike structure, for any pair of two nodes \( i \) and \( j \), adjacent or not, the MFPT between them obeys the relation:

\[
F_{ij}(g+1) = 2(m+2)F_{ij}(g), \tag{14}
\]

which is a basic property of random walks in the regular treelike fractals, dominating the evolution of MFPT between any couple of nodes. The relation described by Eq. (14) is very important using which we will derive the rigorous formula for \( \langle F \rangle_g \).

\section{Exact solution to partial mean first-passage time in \( T_g \)}

Having obtained the evolution rule of MFPT for random walks in the fractals, we now determine the mean time to absorption averaged over all non-trap nodes in \( T_g \). To attain this goal, we represent the set of nodes in \( T_g \) as \( \Lambda_g \), and denote the set of those nodes created at generation \( g \) by \( \Lambda_g \). Obviously, \( \Lambda_g = \Lambda_{g-1} + \Lambda_g \). For the convenience of description, we define the following two quantities for \( n \leq g \):

\[
F_n^{\text{tot}}(g) = \sum_{i \in \Lambda_n} F_i(g), \tag{15}
\]

and

\[
\bar{F}_n^{\text{tot}}(g) = \sum_{i \in \Lambda_n} F_i(g). \tag{16}
\]

Thus, we have

\[
F_g^{\text{tot}}(g) = 2(m+2)F_{g-1}^{\text{tot}}(g) + F_g^{\text{tot}}(g), \tag{17}
\]

where we have used Eq. (14). Thus, to obtain \( F_g^{\text{tot}}(g) \), we should first determine the quantity \( F_g^{\text{tot}}(g) \).
To find $\bar{F}_g^{\text{tot}}(g)$, we separate set $\bar{\Lambda}_g$ of the nodes created at generation $g$ into two subsets $\bar{\Lambda}_g^{\text{int}}$ and $\bar{\Lambda}_g^{\text{ext}}$, such that $\bar{\Lambda}_g = \bar{\Lambda}_g^{\text{ext}} \cup \bar{\Lambda}_g^{\text{int}}$, where $\bar{\Lambda}_g^{\text{int}}$ is the set of internal nodes and $\bar{\Lambda}_g^{\text{ext}}$ is the set of external nodes. As shown in Sec. II the cardinalities (the cardinality of a set is the number of nodes in the set) of the two subsets, denoted by $|\bar{\Lambda}_g^{\text{ext}}|$ and $|\bar{\Lambda}_g^{\text{int}}|$, are $|\bar{\Lambda}_g^{\text{int}}| = (m+2)^g - 1$ and $|\bar{\Lambda}_g^{\text{ext}}| = m(m+2)^g - 1$, respectively. Clearly, the two variables satisfy the following relation:

$$|\bar{\Lambda}_g^{\text{ext}}| = m|\bar{\Lambda}_g^{\text{int}}|. \quad (18)$$

Then, the quantity $\bar{F}_g^{\text{tot}}(g)$ can be rewritten as

$$\bar{F}_g^{\text{tot}}(g) = \sum_{i \in \bar{\Lambda}_g^{\text{int}}} F_i(g) + \sum_{i \in \bar{\Lambda}_g^{\text{ext}}} F_i(g), \quad (19)$$

which shows that to determine $\bar{F}_g^{\text{tot}}(g)$, one may alternatively find the two quantities on the rhs of the equal mark. We begin from determining the first sum term $\sum_{i \in \bar{\Lambda}_g^{\text{int}}} F_i(g)$.

Since at a given generation $g_i$ ($g_i \geq 2$), any internal node has a degree of $m+2$ and two internal nodes in $\bar{\Lambda}_g^{\text{int}}$ are not directly connected to each other, we have that any of the $E_{g_i}$ edges in network $T_{g_i}$ must have one (and only one) endpoint belonging to $\bar{\Lambda}_g^{\text{int}}$. In addition, by construction any internal node in $\bar{\Lambda}_g^{\text{int}}$ at generation $g$ that generated the internal node, which means that any internal node in $\bar{\Lambda}_g^{\text{int}}$ must have one (and only one) neighbor belonging to $\bar{\Lambda}_g^{\text{int}}$. Furthermore, we can know that at generation $g_i+1$, the $m+2$ neighbors of any internal node in $\bar{\Lambda}_g^{\text{int}}$ are all internal nodes in $\bar{\Lambda}_g^{\text{int}}$, and for any two internal nodes in $\bar{\Lambda}_g^{\text{int}}$, their neighbors are completely different. From this information, we can obtain a recursion relation for $\sum_{i \in \bar{\Lambda}_g^{\text{int}}} F_i(g)$.

Consider the random walks on the fractals $T_g$. For an internal node $i_{g-1}^{\text{int}}$ in $\bar{\Lambda}_g^{\text{int}}$, and all its $m+2$ neighbors $i_{g-1}^{\text{ext}}(x = 1, 2, \ldots, m+2)$ belonging to $\bar{\Lambda}_g^{\text{int}}$, their trapping times obey the following relation:

$$F_{i_{g-1}^{\text{int}}}(g) = 1 + \frac{1}{m+2} \sum_{x=1}^{m+2} F_{i_{g-1}^{\text{ext}}(x)}(g). \quad (20)$$

Summing Eq. (20) over all internal nodes in $\bar{\Lambda}_g^{\text{int}}$, we obtain

$$\sum_{i \in \bar{\Lambda}_g^{\text{int}}} F_i(g) = \left|\bar{\Lambda}_g^{\text{int}}\right| + \frac{1}{m+2} \sum_{i \in \bar{\Lambda}_g^{\text{int}}} F_i(g), \quad (21)$$

which may be rewritten as

$$\sum_{i \in \bar{\Lambda}_g^{\text{int}}} F_i(g) = (m+2) \sum_{i \in \bar{\Lambda}_g^{\text{int}}} F_i(g) - (m+2)|\bar{\Lambda}_g^{\text{int}}| \quad = 2(m+2)^2 \sum_{i \in \bar{\Lambda}_g^{\text{int}}} F_i(g-1) - (m+2)^g - 1. \quad (22)$$

Using $\sum_{i \in \bar{\Lambda}_g^{\text{int}}} F_i(2) = (2m+3)(m+2)$, Eq. (22) is resolved by induction,

$$\sum_{i \in \bar{\Lambda}_g^{\text{int}}} F_i(g) = \frac{(m+2)^g - 2}{3+2m} [2 + m + 2^g(m+1)(m+2)^g]. \quad (23)$$

After obtaining the first sum $\sum_{i \in \bar{\Lambda}_g^{\text{int}}} F_i(g)$ in Eq. (19), we proceed to determine the second sum $\sum_{i \in \bar{\Lambda}_g^{\text{ext}}} F_i(g)$. A random walker starting from a node $i_{\text{ext}}$ belonging to $\bar{\Lambda}_g^{\text{ext}}$ must be on its only neighbor $i_{\text{int}}$—an internal node in $\bar{\Lambda}_g^{\text{int}}$. Thus,

$$F_{i_{\text{ext}}}(g) = F_{i_{\text{int}}}(g) + 1. \quad (24)$$

Summing Eq. (24) over all external nodes in $\bar{\Lambda}_g^{\text{ext}}$ gives rise to

$$\sum_{i \in \bar{\Lambda}_g^{\text{ext}}} F_i(g) = m \sum_{i \in \bar{\Lambda}_g^{\text{ext}}} F_i(g) + |\bar{\Lambda}_g^{\text{ext}}|, \quad (25)$$

where we have considered Eq. (18). Substituting Eq. (23) for $\sum_{i \in \bar{\Lambda}_g^{\text{int}}} F_i(g)$ and $|\bar{\Lambda}_g^{\text{ext}}| = m(m+2)^g - 1$ into Eq. (25), we have

$$\sum_{i \in \bar{\Lambda}_g^{\text{ext}}} F_i(g) = \frac{m(2+m)^g - 2}{3+2m} [2 + m + 2^g(m+1)(2+m)^g] + m(2+m)^g - 1. \quad (26)$$

With the results obtained for $\sum_{i \in \bar{\Lambda}_g^{\text{int}}} F_i(g)$ and $\sum_{i \in \bar{\Lambda}_g^{\text{ext}}} F_i(g)$, we can determine $\bar{F}_g^{\text{tot}}(g)$. Substituting the two sums given in Eqs. (23) and (26) for $\bar{F}_g^{\text{tot}}(g)$ into Eq. (17) yields to

$$\bar{F}_g^{\text{tot}}(g) = 2(m+2)\bar{F}_g^{\text{tot}}(1) + m(m+2)^g\left(\frac{2^g m + 2^g m + 2^g}{3 + 2m}\right) \quad \left[2 + m + 2^g(m+1)(2+m)^g\right]. \quad (27)$$

Considering the initial condition $\bar{F}_g^{\text{tot}}(1) = m+2$, Eq. (27) is resolved by induction to yield

$$\bar{F}_g^{\text{tot}}(g) = \frac{(2+m)^g}{2(2m^2 + 7m + 6)} \left[2^{g+1}(m+1)(2+m)^g + (2m^2 + 5m + 3)2^g - 4m^2 - 8m - 2\right]. \quad (28)$$

Plugging the last expression into Eq. (11), we arrive at the explicit formula for the PMFPT on the studied treelike regular fractals,

$$\langle F \rangle_g = \frac{1}{N_g - 1} \sum_{i=2}^{N_g} \bar{F}_i(g) = \frac{1}{N_g - 1} \bar{F}_g^{\text{tot}}(g) = \frac{1}{2(2m^2 + 7m + 6)} \left[2^{g+1}(m+1)(2+m)^g + (2m^2 + 5m + 3)2^g - 4m^2 - 8m - 2\right]. \quad (29)$$
When $m = 1$, Eq. \((29)\) recovers the result previously obtained in Refs. \([23]\) and \([24]\), confirming that the solution given by Eq. \((29)\) is right.

We continue to show how to represent PMFPT as a function of network order $N_g$, with the aim to obtain the scaling between these two quantities. Recalling $N_g = (m+2)^2 + 1$, we have $g = \log_{m+2}(N_g - 1)$. Hence, Eq. \((29)\) can be recast as

$$
(F)_g = \frac{1}{2(2m^2 + 7m + 6)}[(2m+1)(N_g-1)^{1+\ln 2/\ln(m+2)}
+ (2m^2 + 5m + 3)(N_g-1)^{2\ln 2/\ln(m+2)}
- 4m^2 - 8m - 2].
$$

(30)

For systems with large order, i.e., $N_g \to \infty$,

$$(F)_g \sim (N_g)^{1+\ln 2/\ln(m+2)} = (N_g)^{2/d}. 
$$

(31)

This confirms the lower bound for PMFPT provided in for general graphs \([27]\). Thus, for the whole family of the treelike fractals, in the large $g$ limit, the PMFPT grows as a power-law function of the network order with the exponent, represented by $\theta(m) = 1 + \ln 2/\ln(m+2)$, being a decreasing function of $m$. When $m$ grows from 1 to infinite, the exponent $\theta(m)$ drops from $1 + \ln 2/\ln 3$ and approaches to 1, indicating that the PMFPT grows superlinearly with network order. This also means that the efficiency of the trapping process depends on parameter $m$: the larger the value of $m$, the more efficient the trapping process.

Before closing this section, we stress that although we only focus on a special family of fractals, above computational method and process are also applicable to the trapping problem on other self-similar trees. For example, we have used this method to compute the PMFPT to a target hub node in an iterative scale-free tree first introduced in \([32]\) and studied in detail in \([33]\) and to reproduce the result on PMFPT previously reported \([25]\).

IV. RANDOM WALKS WITH THE TRAP DISTRIBUTED UNIFORMLY ON THE FRACTALS

In the Sec. III C, we have studied the PMFPT from a node to a trap fixed on the central node of treelike fractals averaged over all possible starting points, and showed that the PMFPT scales algebraically with network order. In this section, we will investigate random walks in $T_g$ with the trap uniformly distributed among all nodes of the fractals. As will be shown, the EMFPT $\langle H \rangle_g$, which is the average of MFPTs over all pairs of nodes, exhibits the same scaling as that of $\langle F \rangle_g$.

A. Mean first-passage times and effective resistance

The quantity EMFPT $\langle H \rangle_g$ concerned in this case that trap is randomly selected from all nodes involves a double average: the first one is over all the starting nodes to a given trap node and the second one is the average of the first one over all trap nodes. By definition, $\langle H \rangle_g$ is given by

$$
\langle H \rangle_g = \frac{1}{N_g(N_g-1)} \sum_{i=1}^{N_g} \sum_{j=1, j\neq i}^{N_g} F_{ij}(g).
$$

(32)

In general, one can apply the method for pseudoinverse of the Laplacian matrix \([34]\) to calculate numerically but exactly $\langle H \rangle_g$. However, this approach makes heavy demands on time and computational resources and is thus not tractable for large networks. To overcome these shortcomings, we will resort to the theory between electrical networks and random walks to derive $\langle H \rangle_g$, providing a closed-form expression for this related quantity.

To obtain the solution to $\langle H \rangle_g$, we view $T_g$ as resistor networks \([35]\) by considering all edges of $T_g$ to be unit resistors. Let $R_{ij}(g)$ be the effective resistance between two nodes $i$ and $j$ in the electrical networks obtained from $T_g$. Then, according to the connection between MFPTs and effective resistance \([36, 37]\), we have

$$
F_{ij}(g) + F_{ji}(g) = 2 E_g R_{ij}(g),
$$

(33)

using which Eq. \((32)\) can be recast as

$$
\langle H \rangle_g = \frac{E_g}{N_g(N_g-1)} \sum_{i=1}^{N_g} \sum_{j=1, j\neq i}^{N_g} R_{ij}(g)
= \frac{1}{N_g} \sum_{i=1}^{N_g} \sum_{j=1, j\neq i}^{N_g} R_{ij}(g),
$$

(34)

where the sum of effective resistance between all pairs of nodes is the so-called Kirchhoff index \([38]\), which we denote by $R_{\text{tot}}(g)$. Using the previously obtained results \([39, 40]\), $R_{\text{tot}}(g)$ can be rewritten as

$$
R_{\text{tot}}(g) = \sum_{i=1}^{N_g} \sum_{j=1, j\neq i}^{N_g} R_{ij}(g) = 2 N_g \sum_{i=2}^{N_g} \frac{1}{\lambda_i(g)},
$$

(35)

where $\lambda_i(g)$ $(i = 2, \ldots, N_g)$ are all the nonzero eigenvalues of the Laplacian matrix corresponding to $T_g$. Note that Eq. \((35)\) only holds for a tree. Then, the EMFPT $\langle H \rangle_g$ can be expressed as

$$
\langle H \rangle_g = 2 \sum_{i=2}^{N_g} \frac{1}{\lambda_i(g)}.
$$

(36)

We use $L_g$ to represent the Laplacian matrix for $T_g$. The entries $L_{ij}(g)$ of $L_g$ are defined as follows: the off-diagonal element $L_{ij}(g)$ is $-1$ if the pair of nodes $i$ and $j$ are directly linked to each other, otherwise $L_{ij}(g)$ equals 0, while the diagonal entry $L_{ii}(g)$ is equal to the degree of node $i$. After reducing the computation of $\langle H \rangle_g$ to determining the sum of the reciprocal of all nonzero eigenvalues of $L_g$, the next step is to evaluate this sum.
B. Using eigenvalues of the Laplacian matrix to determine entire mean first-passage time

Let $P_g(\lambda)$ express the characteristic polynomial of matrix $L_g$, i.e.,

$$P_g(\lambda) = \det (L_g - \lambda I_g),$$  \hspace{0.5cm} (37)

where $I_g$ is an $N_g \times N_g$ identity matrix. As mentioned above, one of the main goals is to find the sum of the reciprocal of all nonzero eigenvalues of $L_g$, namely, all nonzero roots of polynomial $P_g(\lambda)$.

In order to determine this sum, we denote $Q_g$ as a $(N_g - 1) \times (N_g - 1)$ sub-matrix of $(L_g - \lambda I_g)$, which is obtained by removing from $(L_g - \lambda I_g)$ the row and column corresponding to an outermost node, e.g., node $K$ in Fig. 3. Also, we define $R_g$ as a sub-matrix of $(L_g - \lambda I_g)$ with order $(N_g - 2) \times (N_g - 2)$, obtained by removing from $(L_g - \lambda I_g)$ two rows and columns corresponding to two arbitrary outermost nodes in two different copies, $T^{(1)}_{g-1} (i = 1,2, \ldots, m+1, m+2)$, of $T_{g-1}$ that are constituents of $T_g$ (see Fig. 3). On the other hand, we denote $Q_g(\lambda)$ and $R_g(\lambda)$ as the determinants of $Q_g$ and $R_g$, respectively. Then, the three quantities $P_g(\lambda)$, $Q_g(\lambda)$, and $R_g(\lambda)$ satisfy the following recursion relations:

$$P_{g+1}(\lambda) = \begin{vmatrix} m + 2 - \lambda & -e_q & -e_q \ldots -e_q \\ -e_q & Q_g & O \ldots O \\ \vdots & \vdots & \vdots & \vdots \\ -e_q & O & O \ldots Q_g \end{vmatrix},$$  \hspace{0.5cm} (38)

$$Q_{g+1}(\lambda) = \begin{vmatrix} m + 2 - \lambda & -e_r & -e_r \ldots -e_r \\ -e_r & R_g & O \ldots O \\ \vdots & \vdots & \vdots & \vdots \\ -e_r & O & O \ldots Q_g \end{vmatrix},$$  \hspace{0.5cm} (39)

and

$$R_{g+1}(\lambda) = \begin{vmatrix} m + 2 - \lambda & -e_r & -e_r \ldots -e_q \\ -e_r & R_g & O \ldots O \\ \vdots & \vdots & \vdots & \vdots \\ -e_r & O & O \ldots Q_g \end{vmatrix}. $$  \hspace{0.5cm} (40)

In Eqs. (38)-(40), the superscript $\top$ of a vector represents transpose and $e_q$ ($e_r$) is a vector of order $N_g - 1$ ($N_g - 2$) with only the first entry being 1 and all other $N_g - 2$ ($N_g - 3$) entries equaling 0, i.e.,

$$e_q = (1, 0, 0, 0, \ldots, 0, 0) \quad \text{and} \quad e_r = (1, 0, 0, 0, \ldots, 0, 0).$$  \hspace{0.5cm} (41) \hspace{0.5cm} (42)

In Appendix A using the elementary matrix operations we show that $P_{g+1}(\lambda)$, $Q_{g+1}(\lambda)$, and $R_{g+1}(\lambda)$ evolve as:

$$P_{g+1}(\lambda) = (m+2)[Q_g(\lambda)]^{m+1}P_g(\lambda) + (m+1)\lambda [Q_g(\lambda)]^{m+2},$$  \hspace{0.5cm} (43)

$$Q_{g+1}(\lambda) = [Q_g(\lambda)]^{m+2} + (m+1)\lambda R_g(\lambda)[Q_g(\lambda)]^{m+1} + (m+1)R_g(\lambda)[Q_g(\lambda)]^m P_g(\lambda)$$  \hspace{0.5cm} (44)

and

$$R_{g+1}(\lambda) = 2R_g(\lambda)[Q_g(\lambda)]^{m+1} + (m+1)\lambda[R_g(\lambda)]^2[Q_g(\lambda)]^m + m[R_g(\lambda)]^2 [Q_g(\lambda)]^{m-1} P_g(\lambda).$$  \hspace{0.5cm} (45)

After obtaining the recursion relations for $P_g(\lambda)$, $Q_g(\lambda)$, and $R_g(\lambda)$, next we will compute the sum of the reciprocal of the non-zero roots of $P_g(\lambda)$. Since $P_g(\lambda)$ has one and only one root equal to zero, say $\lambda_1(g) = 0$, to find this sum, we define a new polynomial $P'_g(\lambda)$ as

$$P'_g(\lambda) = \frac{1}{\lambda} P_g(\lambda).$$  \hspace{0.5cm} (46)

Obviously, we have

$$\sum_{i=2}^{N_g} \frac{1}{\lambda_i(g)} = \sum_{i=1}^{N_g-1} \frac{1}{\lambda'_i(g)},$$  \hspace{0.5cm} (47)

where $\lambda'_1(g), \lambda'_2(g), \ldots, \lambda'_{N_g-1}(g)$ are the $N_g - 1$ roots of polynomial $P'_g(\lambda)$. Then, we reduce the problem to finding the sum on the rhs of Eq. (47).

Note that one can also express the polynomial $P'_g(\lambda)$ in the form of $P'_g(\lambda) = \sum_{j=0}^{N_g-1} p'_g(j)\lambda^j$ [here $p'_g(j)$ is the coefficient of term $\lambda^j$ with degree $j$], such that

$$\sum_{j=0}^{N_g-1} p'_g(j)\lambda^j = p'_g(N_g - 1) \prod_{i=1}^{N_g-1} [\lambda - \lambda'_i(g)].$$  \hspace{0.5cm} (48)

Comparing the coefficients of both sides of Eq. (48), we have

$$\sum_{i=1}^{N_g-1} \frac{1}{\lambda'_i(g)} = -\frac{p'_g(1)}{p'_g(0)}.$$  \hspace{0.5cm} (49)

Thus, we turn our aim into determining the two coefficients $p'_g(0)$ and $p'_g(1)$.

From Eqs. (43)-(45) we can easily get the recursive equations,

$$P'_{g+1}(\lambda) = (m+2)[Q_g(\lambda)]^{m+1}P'_g(\lambda) + (m+1)[Q_g(\lambda)]^{m+2},$$  \hspace{0.5cm} (50)

$$Q'_{g+1}(\lambda) = [Q_g(\lambda)]^{m+2} + (m+1)\lambda R_g(\lambda)[Q_g(\lambda)]^{m+1} + (m+1)R_g(\lambda)[Q_g(\lambda)]^m P'_g(\lambda),$$  \hspace{0.5cm} (51)

and

$$R'_{g+1}(\lambda) = 2R_g(\lambda)[Q_g(\lambda)]^{m+1} + (m+1)\lambda[R_g(\lambda)]^2[Q_g(\lambda)]^m + m[R_g(\lambda)]^2 [Q_g(\lambda)]^{m-1} P'_g(\lambda).$$  \hspace{0.5cm} (52)
Based on these relations we can find the values for \( p'_g(0) \) and \( p'_g(1) \).

We first evaluate \( p'_g(0) \). For this purpose, let \( q_0(0) \) and \( r_g(0) \) be the constant terms of \( Q_g(\lambda) \) and \( R_g(\lambda) \), respectively. According to Eqs. (50)-(52), the quantities \( p'_g(0), q_0(0), \) and \( r_g(0) \) can be represented recursively as follows:

\[
p_{g+1}(0) = (m + 2)[q_g(0)]^{m+1}p_g(0) + (m + 1)[q_g(0)]^{m+2},
\]

\[
q_{g+1}(0) = [q_g(0)]^{m+2},
\]

and

\[
r_{g+1}(0) = 2r_g(0)q_g^{m+1}(0).
\]

Using the initial conditions \( p'_1(0) = -m - 3, q_1(0) = 1 \) and \( r_1(0) = 2 \), Eqs. (53)-(55) can be solved to yield

\[
p'_g(0) = -(m + 2)^g - 1,
\]

\[
q_g(0) = 1,
\]

and

\[
r_g(0) = 2^g.
\]

With these obtained results, we go on to compute \( p'_g(1) \). To this end, let \( q_g(1) \) denote the coefficient of term \( \lambda \), i.e., term with degree 1, of polynomial \( Q_g(\lambda) \). Then, using Eqs. (50)-(52), we have

\[
p'_{g+1}(1) = (m + 2)[q_g(0)]^{m+1}p'_g(1) + (m + 2)(m + 1)[q_g(0)]^{m+1}q_g(1) + (m + 2)(m + 1)[q_g(0)]^{m+1}q_g(1)
\]

and

\[
q_{g+1}(1) = (m + 2)[q_g(0)]^{m+1}q_g(1) + (m + 1)r_g(0)[q_g(0)]^{m+1} + (m + 1)r_g(0)[q_g(0)]^{m+1}p'_g(0).
\]

Substituting Eqs. (56)-(58) into Eqs. (50) and (51), we obtain the explicit expressions for \( p'_g(1) \) and \( q_g(1) \) as

\[
p'_g(1) = \frac{(m + 2)^g - 1}{2m + 3} - 2\frac{m + 2)^g + 2(m + 1)}{2m + 2 - m - 2},
\]

\[
q_g(1) = -(m + 2)^g + 1 - 2^g + 2^g(m + 2).
\]

Thus far, we have obtained the intermediate quantities, we now be in position to determine EMFPT \( \langle H \rangle_g \). Inserting Eqs. (59) and (61) into Eq. (49), from which we can arrive at the explicit formula of the EMFPT \( \langle H \rangle_g \) for random walks in \( T_g \) as

\[
\langle H \rangle_g = 2 \sum_{i=1}^{N_g-1} \frac{1}{\lambda_i(g)} = \frac{2(m + 2)^g - 1}{(m + 2)^g + 2m + 3}
\]

\[
[2^g(m + 2)^g - 2(2^g - 1)(m + 2)^g + 2(m + 1) + 2^g - 1](m + 2)^g + (m + 2)^2 - m - 2].
\]

We have checked this rigorous result given by Eq. (60) against that obtained via numerical calculations by using the method of pseudoinverse matrix [34], which fully agree with each other.

Similar to \( \langle F \rangle_g \), we can rewrite EMFPT \( \langle H \rangle_g \) as a function of network order \( N_g \) as

\[
\langle H \rangle_g = \frac{2(N_g - 1)}{(2m^2 + 7m + 6)N_g}
\]

\[
[(m^2 + 2m + 1)(N_g - 1)^{1+2ln2/ln(m+2)} + (2m + 3)(N_g - 1) + m^2 + 3m + 2].
\]

Therefore, in the thermodynamic limit \( (N_g \to \infty) \), we have

\[
\langle H \rangle_g \sim (N_g)^{1+2ln2/ln(m+2)} = (N_g)^{2/d},
\]

showing that the EMFPT \( \langle H \rangle_g \) grows as a power-law function of the network order \( N_g \). It should be mentioned that Eq. (65) is in complete agreement with the general result for fractals given by Eq. (9) in [27].

Equations (41) and (65) indicate that \( \langle F \rangle_g \) and \( \langle H \rangle_g \) in the treelike regular fractals show a similar behavior, both of which grow approximately as a power-law function of network order \( N_g \) with the exponent \( \theta(m) = 2/d = 1 + \ln 2/\ln(m+2) \) being a decreasing function of \( m \) and lying between 1 and 2, implying that they both increase superlinearly with network order. The sameness of scalings for \( \langle F \rangle_g \) and \( \langle H \rangle_g \) in \( T_g \) is in comparison with the results obtained for the deterministic recursive tree [41], where the PMFPT to a central node (a node with highest degree) varies linearly with network order, while the EMFPT averaged over all node pairs grows with network order \( N \) as \( N \ln N \), a scaling larger than the linear one for PMFPT.

Finally, we show the universality of our method for computing EMFPT in self-similar treelike fractals. Using a process similar to that above, we determine the EMFPT in the Vicsek fractals, and recover the exact solution presented in one of our previous papers [29]. For details, see Appendix B.

V. CONCLUSIONS

In this paper, we have presented techniques to determine the explicit solutions to PMFPT and EMFPT in regular treelike fractals. First, we provided a formula
for the MFPT between two adjacent nodes in any connected tree, using which one can find PMFPT to a given target node. We then proposed a method for computing EMFPT, i.e., average of MFPTs over all node pairs, which is based on the relationship of characteristic polynomials of the fractals at different iterations. It only needs partial knowledge of the polynomials, but avoids the laborious computation of eigenvalues that some other methods need. Thus, it is a simple and elegant method. We have used in detail our techniques to a family of treelike regular fractals, which include some well-known fractals, e.g., the T fractal and the Peano basin fractal, as its peculiar cases. We also applied the methods to other treelike fractals to demonstrate their universality.

Acknowledgments

We would like to thank Xin Li for assistance. This work was supported by the National Natural Science Foundation of China under Grant Nos. 60704044 and 61074119, and the Shanghai Leading Academic Discipline Project No.B114. Y. L also acknowledges the Hui-Chun Chin and Tsung-Dao Lee Chinese Undergraduate Research Endowment.

Appendix A: Derivation of recursion relations for relevant polynomials

In this appendix, we give the detailed process for the derivation of recursive relations for polynomials $P_g(\lambda)$, $Q_g(\lambda)$, and $R_g(\lambda)$. We first derive in detail the evolution equation for $P_g(\lambda)$, while the evolution relations for $Q_g(\lambda)$, and $R_g(\lambda)$ can be obtained analogously.

Note that the vector $(m+2-\lambda,-e_q,-e_q,\ldots,-e_q)$ of the first row of the matrix on the rhs of Eq. (38) can be looked upon as the sum (i.e., a linear combination with all scalars being 1) of the following $m+3$ vectors: $(m+2-\lambda,0_q,0_q,\ldots,0_q)$, $(0,-e_q,0_q,\ldots,0_q)$, and $(0,0_q,-e_q,\ldots,0_q)$, $\ldots$, $(0,0_q,0_q,\ldots,-e_q)$, where $0_q$ is a zero vector with order $N_q-1$. For all these $m+3$ vectors, only one entry is non-zero $(m+2-\lambda$ or $-1)$, while the $N_q-1$ entries are zeros. According to the properties of determinants, we can rewrite $P_{g+1}(\lambda)$ as a linear function of the first row on the rhs of Eq. (38) when the remaining rows are held fixed. Concretely, $P_{g+1}(\lambda)$ can be expressed as

\[
P_{g+1}(\lambda) = \begin{vmatrix}
m+2-\lambda & 0_q & 0_q & \cdots & 0_q \\
-e_q^T & Q_g & 0 & \cdots & 0 \\
-e_q^T & 0 & Q_g & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
-e_q^T & 0 & 0 & \cdots & Q_g \\
0 & 0 & e_q & \cdots & 0_q \\
-e_q^T & Q_g & 0 & \cdots & 0 \\
-e_q^T & 0 & Q_g & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
-e_q^T & 0 & 0 & \cdots & Q_g
\end{vmatrix} + \begin{vmatrix}
0 & -e_q & 0_q & \cdots & 0_q \\
-e_q^T & Q_g & 0 & \cdots & 0 \\
-e_q^T & 0 & Q_g & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
-e_q^T & 0 & 0 & \cdots & Q_g \\
0 & 0 & 0_q & \cdots & -e_q \\
-e_q^T & Q_g & 0 & \cdots & 0 \\
-e_q^T & 0 & Q_g & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
-e_q^T & 0 & 0 & \cdots & Q_g
\end{vmatrix} + \cdots + \begin{vmatrix}
0 & 0 & 0_q & \cdots & -e_q \\
-e_q^T & Q_g & 0 & \cdots & 0 \\
-e_q^T & 0 & Q_g & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
-e_q^T & 0 & 0 & \cdots & Q_g
\end{vmatrix}.
\] (A1)

It is easy to see that the last $m+2$ determinants on the rhs of Eq. (A1) are equal to one another. We use $P^{(1)}_{g+1}(\lambda)$ to denote the first determinant and use $P^{(2)}_{g+1}(\lambda)$ to represent any of the last $m+2$ determinants. Thus,

\[
P_{g+1}(\lambda) = P^{(1)}_{g+1}(\lambda) + (m + 2)P^{(2)}_{g+1}(\lambda).
\] (A2)

Below we will calculate these two quantities $P^{(1)}_{g+1}(\lambda)$ and $P^{(2)}_{g+1}(\lambda)$.

For $P^{(1)}_{g+1}(\lambda)$, it is obvious to have

\[
P^{(1)}_{g+1}(\lambda) = (m+2-\lambda)[Q_g(\lambda)]^{m+2}.
\] (A3)

With regard to $P^{(2)}_{g+1}(\lambda)$, after some transformations of the matrix corresponding to the determinant, we have

\[
P^{(2)}_{g+1}(\lambda) = [Q_g(\lambda)]^{m+1}
\begin{vmatrix}
0 & -e_q^T \\
-e_q^T & Q_g
\end{vmatrix}
\begin{vmatrix}
1 - \lambda & -e_q^T \\
-e_q^T & Q_g
\end{vmatrix}
\begin{vmatrix}
\lambda - 1 & 0_q \\
-e_q^T & Q_g
\end{vmatrix}
=[Q_g(\lambda)]^{m+1}[[L_g + (\lambda - 1)Q_g]]
=[Q_g(\lambda)]^{m+1}[P_g(\lambda) + (\lambda - 1)Q_g(\lambda)].
\] (A4)
Plugging Eqs. (A3) and (A4) into Eq. (A2), we obtain

\[ P_{g+1}(\lambda) = (m + 2 - \lambda) |Q_g(\lambda)|^{m+2} + (m + 2) |Q_g(\lambda)|^{m+1} [P_g(\lambda) + (\lambda - 1) Q_g(\lambda)] \]

\[ = (m + 2)|Q_g(\lambda)|^{m+1} P_g(\lambda) + (m + 1) \lambda |Q_g(\lambda)|^m \]

which is exactly Eq. (13) in the main text.

In a similar way, we can easily derive the recursion relations for \( Q_{g+1}(\lambda) \) and \( R_{g+1}(\lambda) \) expressed in Eqs. (44) and (45). The concrete derivation processes for \( Q_{g+1}(\lambda) \) and \( R_{g+1}(\lambda) \) are as follows:

\[ Q_{g+1}(\lambda) = |Q_g(\lambda)|^{m+1} \begin{bmatrix} m + 2 - \lambda & -e_r \cr -e_r^T & R_g \end{bmatrix} + (m + 1) R_g(\lambda) |Q_g(\lambda)|^m \begin{bmatrix} 0 & -e_q \cr -e_q^T & Q_g \end{bmatrix} \]

\[ = |Q_g(\lambda)|^{m+1} \begin{bmatrix} m + 1 & 0_r \cr -e_r^T & R_g \end{bmatrix} + \begin{bmatrix} 1 - \lambda & -e_r \cr -e_r^T & R_g \end{bmatrix} \]

\[ + (m + 1) R_g(\lambda) |Q_g(\lambda)|^m \begin{bmatrix} 0 & -e_q \cr -e_q^T & Q_g \end{bmatrix} \]

\[ = |Q_g(\lambda)|^{m+1} \begin{bmatrix} m + 1 & 0_r \cr -e_r^T & R_g \end{bmatrix} + R_g(\lambda) \begin{bmatrix} 1 - \lambda & -e_r \cr -e_r^T & R_g \end{bmatrix} \]

\[ + R_g(\lambda) \begin{bmatrix} 0 & -e_q \cr -e_q^T & Q_g \end{bmatrix} \]

\[ = |Q_g(\lambda)|^m \left( 2 R_g(\lambda) |Q_g| + (m + \lambda) R_g(\lambda) |R_g| \right) + m R_g(\lambda) |Q_g(\lambda)|^{m-1} [P_g(\lambda) + (\lambda - 1) Q_g(\lambda)] \]

\[ = |Q_g(\lambda)|^m \left( 2 R_g(\lambda) |Q_g| + (m + \lambda) R_g(\lambda) |Q_g(\lambda)|^2 \right) + m R_g(\lambda) |Q_g(\lambda)|^{m-1} [P_g(\lambda) + (\lambda - 1) Q_g(\lambda)] \]

\[ = 2 R_g(\lambda) |Q_g(\lambda)|^{m+1} + (m + 1) \lambda |R_g(\lambda)|^2 |Q_g(\lambda)|^m + m R_g(\lambda) |Q_g(\lambda)|^{m-1} P_g(\lambda), \]

where \( 0_r \) is a zero vector with order \( N_g - 2 \).

**Appendix B: Using recurrence relations for relevant polynomials to determine EMFPT of Vicsek fractals**

In order to demonstrate the universality of our method for calculating EMFPT of self-similar tree-like fractals, here we apply our proposed method to compute the EMFPT for the Vicsek fractals [42, 43], which is one of the most important and frequently studied regular fractal classes.

The Vicsek fractals are built in an iterative way [42, 43] controlled by the two parameters \( f \) and \( g \). Denote by \( \mathbb{V}_{f,g} \) (\( f \geq 2, g \geq 1 \)) the Vicsek fractals after \( g \) iterations (generations). The initial construction (\( g = 1 \)) is a starlike cluster composed of \( f + 1 \) nodes arranged in a crosswise pattern with \( f \) peripheral nodes connected to a central node. This corresponds to \( \mathbb{V}_{f,1} \). Hereafter, we call those nodes farthest from the central node at any iteration as outmost nodes. For \( g \geq 2 \), \( \mathbb{V}_{f,g} \) is obtained from \( \mathbb{V}_{f,g-1} \).

To obtain \( \mathbb{V}_{f,g} \), we generate \( f \) identical copies of \( \mathbb{V}_{f,g-1} \) (denoted by \( \mathbb{V}_{f,g-1}^{(i)} \), \( i = 1, 2, \ldots, f \)) and arrange them around the periphery of the original \( \mathbb{V}_{f,g-1} \) (denoted by \( \mathbb{V}_{f,g-1}^{(0)} \)), then we add \( f \) new edges, each of them connects an outmost node in one of the \( f \) corner copy structures and that of the original central structure as shown in Fig. 24.

After introducing the Vicsek fractals, in what follows, we will apply our method to determine exactly the EMFPT between two nodes over the whole Vicsek fractal family. Note that in the case without confusion, we will use the same notations as that used above for the tree-like fractals \( T_g \). It should be mentioned that since the number of recurrence relations for related polynomials increases with parameter \( f \), below we only provide the calculation detail of the EMFPT for a particular case of
In order to obtain the sum of the reciprocal of all nonzero
roots of $V_g(\lambda)$, we denote
$$V_g^1(\lambda) = \frac{1}{\lambda} \left[ V_g^2(\lambda) + 3 V_g^3(\lambda) \right].$$

For the convenience of description, we define $V_g(\lambda)$ as the characteristic polynomial of the Laplacian matrix $L_g$
corresponding to $V_{3g}$, i.e.,
$$V_g(\lambda) = \text{det}(L_g - \lambda I_g).$$

In order to obtain the sum of the reciprocal of all nonzero
eigenvalues of $L_g$, viz., all nonzero roots of $V_g(\lambda)$, we denote $V_g^1(\lambda)$, $V_g^2(\lambda)$, and $V_g^3(\lambda)$ as determinants of sub-matrices of $(L_g - \lambda I_g)$, where $V_g^i(\lambda)$ ($i = 1, 2, 3$) is obtained by deleting from $(L_g - \lambda I_g)$ $i$
rows and columns corresponding to $i$ outmost nodes that belong to $i$
different copy structures forming $V_{3g}$, i.e.,
$$V_{3g-1}, V_{3g-2}, \text{ or } V_{3g-3}.$$

The polynomials $V_g(\lambda)$, $V_g^1(\lambda)$, $V_g^2(\lambda)$, and $V_g^3(\lambda)$ obey the following recursive relations:

$$V_{g+1}(\lambda) = [V_g^2(\lambda) + 3V_g^3(\lambda)] [V_g(\lambda)]^3 + [9V_g^1(\lambda)]^2 + 3V_g^1(\lambda) [V_g^2(\lambda)]^2 + 4[V_g^1(\lambda)]^3 V_g(\lambda), \quad (B2)$$

and

$$V_{g+1}^1(\lambda) = [V_g^2(\lambda) + 3V_g^3(\lambda)] [V_g(\lambda)]^3 + [5V_g^1(\lambda)]^2 + 7V_g^1(\lambda) V_g^2(\lambda) + V_g^1(\lambda) V_g^3(\lambda)] [V_g(\lambda)]^2 + [5V_g^1(\lambda)]^3 + 5V_g^1(\lambda) V_g^2(\lambda) V_g(\lambda) + [V_g^1(\lambda)]^4, \quad (B3)$$

$$V_{g+1}^2(\lambda) = [V_g^2(\lambda) + 3V_g^3(\lambda)] [V_g(\lambda)]^3 + [4V_g^1(\lambda)]^3 + 9[V_g^1(\lambda)]^2 + 4V_g^1(\lambda) [V_g^2(\lambda)]^2 + [V_g^1(\lambda)]^2 V_g^3(\lambda)] V_g(\lambda) + 2[V_g^1(\lambda)]^4 + 3V_g^1(\lambda) [V_g^3(\lambda)] V_g(\lambda), \quad (B4)$$

Using above recursion relations one can compute the
sum of the reciprocal of the non-zero roots of $V_g(\lambda)$. To
this end, we define another polynomial $V_g^i(\lambda)$ as

$$V_g^i(\lambda) = \frac{1}{\lambda} V_g(\lambda).$$

Obviously, the sum of roots of $V_g^i(\lambda)$ is equal to the sum of
non-zero roots of $V_g(\lambda)$. According to the above-obtained
results, we have

$$\langle H \rangle_g = -\frac{\nu(1)}{\nu(0)}, \quad (B7)$$

where $\nu(0)$ is the constant term of $V_g(\lambda)$ and $\nu(1)$ is the
coefficient of term $\lambda$ of $V_g(\lambda)$. Then, the problem
of determining $\langle H \rangle_g$ is reduced to finding the quantities
$\nu(0)$ and $\nu(1)$.

From Eqs. ($B2$)-($B5$) we can easily get the following
four recursive equations:

$$V_{g+1}^1(\lambda) = \lambda^3 [V_g^2(\lambda)]^5 + \lambda^2 [6V_g^1(\lambda) + 3V_g^2(\lambda) + V_g^3(\lambda)] [V_g^2(\lambda)]^3 + \lambda [9V_g^1(\lambda) [V_g^2(\lambda)]^2 + 3V_g^1(\lambda) V_g^2(\lambda)] [V_g^3(\lambda)]^2 + 4[V_g^2(\lambda)]^3 V_g(\lambda), \quad (B8)$$

and

$$V_{g+1}^2(\lambda) = \lambda^2 [V_g^1(\lambda) + V_g^2(\lambda)]^5 [V_g^3(\lambda)]^3 + \lambda [5V_g^1(\lambda)]^2 + 5V_g^1(\lambda) V_g^2(\lambda)] V_g^3(\lambda) + 9[V_g^1(\lambda)]^2 V_g^2(\lambda) V_g^3(\lambda) + 2[V_g^1(\lambda)]^4 + 3[V_g^1(\lambda)]^3 V_g^2(\lambda), \quad (B9)$$

and

$$V_{g+1}^3(\lambda) = \lambda [V_g^1(\lambda) + V_g^2(\lambda)]^3 V_g^3(\lambda) + 3[V_g^1(\lambda)]^4 + 9[V_g^1(\lambda)]^3 V_g^2(\lambda) + 6[V_g^1(\lambda)]^2 V_g^3(\lambda)]^2 + [V_g^1(\lambda)]^3 V_g^3(\lambda). \quad (B10)$$

Based on these relations we can find the values for $v_g(0)$
and $v_g(1)$.

Let us first evaluate $v_g(0)$. To this end, we use $v_g^1(0)$,
$v_g^2(0)$, and $v_g^3(0)$ to denote separately the constant terms
of $V_g^1(\lambda)$, $V_g^2(\lambda)$ and $V_g^3(\lambda)$. Making use of Eqs. ($B5$)-
($B11$), we can derive the following important relations:

$$v_{g+1}^1(0) = 4[v_g^1(0)]^3 v_g^3(0), \quad (B12)$$

FIG. 5: The first several iterative processes of a particular
Vicsek fractal $V_{4,i}$. The open circles express the starting
structure $V_{4,1}$. The outmost nodes that
\[ v_{g+1}(0) = |v_g(0)|^4, \quad \text{(B13)} \]
\[ v_{g+1}^2(0) = 2|v_g^1(0)|^4 + 3|v_g^0(0)|^3v_g^2(0), \quad \text{(B14)} \]
and
\[ v_{g+1}^3(0) = 3|v_g^1(0)|^4 + 9|v_g^0(0)|^3v_g^2(0) + 6|v_g^0(0)|^2v_g^2(0)v_g^2(0) + |v_g^0(0)|^3v_g^3(0) \quad \text{(B15)} \]

Considering \( v_1^1(0) = -4 \), \( v_1^0(0) = 1 \), \( v_1^1(0) = 2 \), and \( v_1^2(0) = 3 \), Eqs. \( \text{(B12)-(B15)} \) are resolved to obtain
\[ v_g^0(0) = -4^g, \quad \text{(B16)} \]
\[ v_g^1(0) = 1, \quad \text{(B17)} \]
\[ v_g^2(0) = 3^g - 1, \quad \text{(B18)} \]
and
\[ v_g^3(0) = \frac{3}{4}(3^g - 1)^2. \quad \text{(B19)} \]

With these obtained results, we continue to calculate \( v_g(1) \). To achieve this goal, we define \( v_g^1(1) \) as the coefficient of term \( \lambda \) of polynomial \( V_g^1(\lambda) \). According to Eqs. \( \text{(B8)-(B11)} \), the quantities \( v_g^1(1) \) and \( v_g^2(1) \) can be represented recursively as follows:
\[ v_{g+1}^0(1) = 4|v_g^1(0)|^3v_g^1(0) + 12|v_g^1(0)|^2v_g^2(0)v_g^1(1) + 9|v_g^0(0)|^2v_g^2(0)v_g^2(0) \quad \text{(B20)} \]

and
\[ v_{g+1}^1(1) = 4|v_g^1(0)|^3v_g^1(1) + 5|v_g^1(0)|^3v_g^2(0) + 5|v_g^0(0)|^3v_g^2(0)v_g^1(0). \quad \text{(B21)} \]

Plugging Eqs. \( \text{(B10)-(B11)} \) into Eqs. \( \text{(B20)} \) and \( \text{(B21)} \), and using the initial conditions \( v_1^1(1) = 9 \) and \( v_1^2(1) = -5 \), we obtain the exact solutions to \( v_g^0(1) \) and \( v_g^2(1) \),
\[ v_g^0(1) = \frac{2^{2g-3} - 11}{11}(21 \times 12^g - 11 \times 4^g - 10) \quad \text{(B22)} \]
and
\[ v_g^1(1) = -5 \times 2^{2g-3}(3^g - 1). \quad \text{(B23)} \]

After obtaining all the intermediate quantities, we can determine the EMFPT of the fractal \( V_{3,g} \). Inserting Eqs. \( \text{(B10)} \) and \( \text{(B22)} \) into Eq. \( \text{(B7)} \), we obtain the explicit formula for the EMFPT \( \langle H \rangle_g \) in \( V_{3,g} \) as
\[ \langle H \rangle_g = \frac{1}{44}(21 \times 12^g - 11 \times 4^g - 10), \quad \text{(B24)} \]
which is completely consistent with the result previously reported in \( \text{(22)} \).

[1] S. Havlin and D. ben-Avraham, Adv. Phys. 36, 695 (1987).
[2] D. ben-Avraham and S. Havlin, Diffusion and Reactions in Fractals and Disordered Media (Cambridge University Press, Cambridge, 2000).
[3] B. Mandelbrot, The Fractal Geometry of Nature (Freeman, San Francisco, 1982).
[4] J. Aguirre, R. L. Viana, and M. A. F. Sanjuán, Rev. Mod. Phys. 81, 333 (2009).
[5] G. Cantor, Math. Ann. 21, 545 (1981).
[6] H. von Koch, Acta Math. 30, 145 (1906).
[7] W. Sierpinski, Compt. Rendus Acad. Sci. Paris 160, 302 (1915).
[8] T. Vicsek J. Phys. A 16, L647 (1983).
[9] S. Havlin and H. Weissman, J. Phys. A 19, L1021 (1986).
[10] K. J. Falconer, Fractal Geometry: Mathematical Foundations and Applications (Wiley, Chichester, 2003).
[11] W. A. Schwalm, M. K. Schwalm, and M. Giona, Phys. Rev. E 55, 6741 (1997).
[12] S. N. Dorogovtsev, A. V. Goltsev, and J. F. F. Mendes, Rev. Mod. Phys. 80, 1275 (2008).
[13] G. H. Weiss, Aspects and Applications of the Random Walk (North Holland, Amsterdam, 1994).
[14] R. Metzler and J. Klafter, Phys. Rep. 339, 1 (2000).
[15] R Burioni and D Cassi, J. Phys. A 38, R45 (2005).
[16] S. Redner, A Guide to First-Passage Processes (Cambridge University Press, Cambridge, 2001).
[17] J. D. Noh and H. Rieger, Phys. Rev. Lett. 92, 118701 (2004).
[18] S. Condamin, O. Bénichou, V. Tejedor, R. Voituriez, and J. Klafter, Nature (London) 450, 77 (2007).
[19] S. Condamin, O. Bénichou, and M. Moreau, Phys. Rev. Lett. 95, 260601 (2005).
[20] S. Condamin, O. Bénichou, and M. Moreau, Phys. Rev. E 75, 021111 (2007).
[21] J. J. Kozak and V. Balakrishnan, Phys. Rev. E 65, 021105 (2002).
[22] J. J. Kozak and V. Balakrishnan, Int. J. Bifurcation Chaos Appl. Sci. Eng. 12, 2379 (2002).
[23] E. Agliari, Phys. Rev. E 77, 011128 (2008).
[24] C. P. Haynes and A. P. Roberts, Phys. Rev. E 78, 041101 (2008).
[25] Z. Z. Zhang, W. L. Xie, S. G. Zhou, S. Y. Gao, and J. H. Guan, EPL 88, 10001 (2009).
[26] Z. Z. Zhang, W. L. Xie, S. G. Zhou, M. Li, and J. H. Guan, Phys. Rev. E 80, 061111 (2009).
[27] V. Tejedor, O. Bénichou, and R. Voituriez, Phys. Rev. E 80, 065104(R) (2009).
[28] Z. Z. Zhang, Y. Lin, S. G. Zhou, B. Wu, and J. H. Guan, New J. Phys. 11, 103043 (2009).
[29] Z. Z. Zhang, B. Wu, H. J. Zhang, S. G. Zhou, J. H. Guan, and Z. G. Wang, Phys. Rev. E 81, 031118 (2010).
[30] F. Comellas and A. Miralles, Phys. Rev. E 81, 061103 (2010).
[31] S. De Bartolo, F. Dell’Accio, and M. Veltri, Phys. Rev. E 79, 026108 (2009).
[32] C. Song, S. Havlin, H. A. Makse, Nat. Phys. 2, 275 (2006).
[33] H. D. Rozenfeld, S. Havlin, and D. ben-Avraham, New J. Phys. 9, 175 (2007).
[34] C. Rao and S. Mitra, Generalized Inverse of Matrices and Its Applications (Wiley, New York, 1971).
[35] P. G. Doyle and J. L. Snell, Random Walks and Electric Networks (The Mathematical Association of America, Oberlin, OH, 1984).
[36] A. K. Chandra, P. Raghavan, W. L. Ruzzo, and R. Smolensky, Proceedings of the 21st Annual ACM Symposium on the Theory of Computing (ACM Press, New York, 1989), pp. 574-586.
[37] P. Tetali, J. Theor. Probab. 4, 101 (1991).
[38] D. Bonchev, A. T. Balaban, X. Liu, and D. J. Klein, Int. J. Quantum Chem. 50, 1 (1994).
[39] I. Gutman and B. Mohar, J. Chem. Inf. Comput. Sci. 36, 982 (1996).
[40] H.-Y. Zhu, D. J. Klein, and I. Lukovits, J. Chem. Inf. Comput. Sci. 36, 420 (1996).
[41] Z. Z. Zhang, Y. Qi, S. G. Zhou, S. Y. Gao, and J. H. Guan, Phys. Rev. E 81, 016114 (2010).
[42] T. Vicsek J. Phys. A 16, L647 (1983).
[43] A. Blumen, Ch. von Ferber, A. Jurjiu, and Th. Koslowski, Macromolecules 37, 638 (2004).