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Brian Clarke, Yanir A. Rubinstein

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RICCI FLOW AND THE METRIC COMPLETION OF THE SPACE OF KÄHLER METRICS

By BRIAN CLARKE and YANIR A. RUBINSTEIN

Abstract. We consider the space of Kähler metrics as a Riemannian submanifold of the space of Riemannian metrics, and study the associated submanifold geometry. In particular, we show that the intrinsic and extrinsic distance functions are equivalent. We also determine the metric completion of the space of Kähler metrics, making contact with recent generalizations of the Calabi-Yau Theorem due to Dinew and Guedj-Zeriahi. As an application, we obtain a new analytic stability criterion for the existence of a Kähler-Einstein metric on a Fano manifold in terms of the Ricci flow and the distance function. We also prove that the Kähler-Ricci flow converges as soon as it converges in the metric sense.

1. Introduction. The study of the infinite-dimensional space \( \mathcal{H} \) of all Kähler metrics in a fixed Kähler class has evolved essentially independently of the study of the larger space \( \mathcal{M} \) of all Riemannian metrics on a closed, finite-dimensional base manifold \( M \). Our first purpose in this article is to draw attention to a simple connection between the two, going back to Calabi, which does not seem to be well known. Namely, we consider the space of Kähler metrics as a submanifold of the space of Riemannian metrics, and study the induced intrinsic and extrinsic geometry of \( \mathcal{H} \) under this embedding.

Our first main result is that when the space of all Riemannian metrics is equipped with the Ebin metric (often referred to as the \( L^2 \) metric), the intrinsic and extrinsic distance functions are equivalent. At the same time, the subspace of Kähler metrics is in a sense as far from being totally geodesic as possible—in fact, it shares no common geodesics with the ambient space, and geodesics in the ambient space intersect the subspace in at most two points.

Building on the equivalence result, we then determine the (metric) completion of \( \mathcal{H} \), making contact with some recent deep results in pluripotential theory, due to Dinew and Guedj-Zeriahi, that generalize the Calabi-Yau Theorem.

These results, combined with recent deep results on the Ricci flow, are then used to prove a new analytic characterization of Kähler-Einstein manifolds of positive scalar curvature in terms of the Ricci flow and the induced distance function. This result stands in clear analogy with Donaldson’s conjecture regarding “geodesic stability,” with Ricci flow paths taking the place of geodesic rays. It follows that for the Kähler-Ricci flow, convergence in the induced metric implies smooth...
convergence. This, and a related analytic condition that is also shown to be equivalent to smooth convergence, strengthen some recent results due to Phong-Song-Sturm-Weinkove.

We note that the study of “constrained” distance and geodesics also appears naturally in optimal transport in relation to the Wasserstein metric. In fact, Carlen-Gangbo [15] consider a submanifold of the space of probability measures and study its induced geometry. There are a number of analogies between their approach, as well as their results, and the ones in this article. For instance, the submanifold they study is naturally a hypersurface—a portion of a sphere—and a similar situation appears for the space of Kähler metrics.

Let $M$ be a smooth, closed (i.e., compact and without boundary) manifold, and consider the infinite-dimensional space $M$ of all smooth Riemannian metrics on $M$. The space $M$ may be endowed with a natural Riemannian structure, which we refer to as the Ebin metric [33] (cf. [27]), defined as follows,

$$g_E(h,k)|_g := \int_M \text{tr}(g^{-1}hg^{-1}k)dV_g,$$

where $g \in M$, $h,k \in T_gM$ and $T_gM \cong \Gamma(\text{Sym}^2T^*M)$, the space of smooth, symmetric $(0,2)$-tensor fields on $M$. As shown by Freed-Groisser [34], the curvature of $g_E$ is nonpositive and geodesics satisfy the equation

$$(g^{-1}g_t)_t = \frac{1}{4} \text{tr}(g^{-1}g_tg^{-1}g_t)\delta - \frac{1}{2} \text{tr}(g^{-1}g_t)g^{-1}g_t,$$

where $\delta$ denotes the Kronecker tensor. The geodesics can be computed explicitly, however the metric is incomplete, and in general not every two points can be connected by a geodesic. Nevertheless, it has been shown recently that $(M,g_E)$ is a metric space [21, 22], and a detailed description of its completion has been provided, including an explicit computation of the length-minimizing paths in it and its distance function $d_E$ [24].

Now, assume that $M$ admits a Kähler structure $(M,J,\omega)$, where $J$ denotes the complex structure, and $\omega$ denotes the Kähler form, and let $\mathcal{H} \subset M$ denote the space of all smooth Kähler metrics on $(M,J)$ whose Kähler form is cohomologous to $\omega$. Let $n$ denote the complex dimension of $M$ and $V$ denote the total volume $M$ with respect to $\omega$ (which depends only on the cohomology class of $\omega$). The space $\mathcal{H}$, by the $\partial\bar{\partial}$-lemma [36], may be parametrized by a single smooth function, the Kähler potential,

$$\mathcal{H} := \{g_\varphi : \omega_\varphi := \omega + \sqrt{-1}\partial\bar{\partial}\varphi > 0\} \subset M,$$

where $g_\varphi(\cdot,\cdot) := \omega_\varphi(\cdot,J\cdot)$, and $\varphi$ is unique up to an additive constant. The corresponding space of Kähler potentials is denoted by

$$\mathcal{H}_\omega := \{\varphi : \omega_\varphi := \omega + \sqrt{-1}\partial\bar{\partial}\varphi > 0\} \subset C^\infty(M),$$
and $\mathcal{H} \cong \mathcal{H}_\omega / \mathbb{R}$. There are several natural candidates for metrics on $\mathcal{H}$. The most widely studied is the Mabuchi metric [45],
\begin{equation}
g_M(\nu, \eta)|_\varphi := \int_M \nu \eta \omega^n_{\varphi}, \quad \nu, \eta \in T_\varphi \mathcal{H}_\omega \cong C^\infty(M),
\end{equation}
discovered independently also by Semmes [64] and Donaldson [29] (see, e.g., [56, Chapter 2] for an exposition and further references). Calabi and Chen proved that $g_M$ induces a metric space structure on $\mathcal{H}$, and that this space has nonpositive curvature in the sense of Alexandrov [12].

Similarly, one may consider metrics involving more (or fewer) derivatives. The Calabi metric is defined by
\begin{equation}
g_C(\nu, \eta)|_\varphi := \int_M \Delta \nu \Delta \eta \omega^n_{\varphi},
\end{equation}
This metric was introduced by Calabi in the 1950s in talks and in a research announcement [10, 9], however, since Calabi’s construction depends on—and in fact seems to have prompted—the Calabi-Yau Theorem (see Remark 4.1), the detailed computations leading to his results have appeared in print only in a recent article of Calamai [13]. In this metric, $\mathcal{H}$ is a section of a sphere, (i.e., has constant positive sectional curvature) of finite diameter, and any two points can be connected by a unique (explicit) smooth minimizing geodesic.

The article is organized as follows. Our first, and elementary, observation, which is undoubtedly due to Calabi, is that the metric $g_C$ on $\mathcal{H}$ is simply the metric induced by $g_E$ under the inclusion $\mathcal{H} \hookrightarrow \mathcal{M}$ (Proposition 2.1). Thus, as in the situation studied by Carlen-Gangbo, our submanifold is a portion of a sphere. In Section 3, the second fundamental form of the inclusion $\iota_\mathcal{H} : \mathcal{H} \hookrightarrow (\mathcal{M}, g_E)$ is computed, relying on results of Ebin, Freed-Groisser, and Gil-Medrano-Michor on the geometry of $(\mathcal{M}, g_E)$. It follows that no geodesic in the Calabi metric is a geodesic in the Ebin metric, and that geodesics of the Ebin metric intersect the space of Kähler metrics in at most two points (Remark 4.2). Then, we prove that the extrinsic and intrinsic distance functions $d_E$ and $d_C$, respectively, are equivalent on the space of Kähler metrics (Theorem 4.5). To do so, we use a transformation of the ambient space that makes the spherical nature of $\mathcal{H}$ self-evident and—aloguously to [15]—compare intrinsic geodesics (great circles) to extrinsic geodesics (chords). Motivated by the proof of the equivalence result, we then formulate a criterion for $d_C$-convergence which improves the criterion for $d_E$-convergence [23, Theorem 4.15] in the ambient space (Section 5). Next, we determine the completion of $\mathcal{H}$ with respect to the geometry induced by $(\mathcal{M}, g_E)$ (Theorem 5.6), building upon Theorem 4.5 and recent generalizations of the Calabi-Yau Theorem. In Section 6 we define the notion of Calabi-Ricci stability and prove, building on the description of the completion of $\mathcal{H}$, that it is equivalent to the existence of a Kähler-Einstein metric on a Fano manifold (Theorem 6.3). It follows that the Kähler-Ricci flow converges smoothly...
as soon as it \( d_C \)-converges (Corollary 6.8), strengthening a theorem of Phong et al. [52]. We also obtain an improved analytic characterization of convergence of the flow (Corollary 6.9). We conclude with some remarks and directions for future study in Section 7.

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2. The induced metric. We begin by considering the restriction of the Ebin metric \( g_e \) to the space of Kähler metrics \( \mathcal{H} \). The computations involve the Kähler-Riemannian dictionary of translating Hermitian objects written with respect to holomorphic coordinates to their Riemannian counterparts written in real coordinates. However, we include the detailed, completely elementary, computations in the proofs in this section since the exact constants are important for us later, and in order to avoid confusion between different possible conventions.

Given a Kähler metric \( g \in \mathcal{H} \subset M \) and a holomorphic coordinate chart \( z_1, \ldots, z_n \), denote by \( [g_{i\bar{j}}] \) the corresponding Hermitian matrix, \( g_{i\bar{j}} = g(\partial_{z_i}, \partial_{\bar{z}_j}) \). Denote by \( [g_{ij}] \) the matrix of coefficients of the metric \( g \) regarded as a Riemannian metric, i.e., \( g_{ij} = g(\partial_{x_i}, \partial_{x_j}) \), \( i, j \in \{1, \ldots, 2n\} \), with respect to the real coordinates \( x_1, \ldots, x_{2n} \), where \( z_i = x_i + \sqrt{-1} x_{i+n}, i = 1, \ldots, n \). If \( G = [g_{ij}] \) then \( g_{i\bar{j}} = \frac{1}{2}g_{ij} + \frac{\sqrt{-1}}{2}g_{ij+n} \). In matrix notation,

\[
[g_{ij}] = \begin{pmatrix}
G + \tilde{G} & (G - G^T)/\sqrt{-1} \\
(G^T - G)/\sqrt{-1} & G + \tilde{G}
\end{pmatrix}.
\]

(5)

For a function \( f \in C^\infty(M) \), we denote

\[
\left[ (\nabla^2 f)_{ij} \right] = \left[ \frac{\partial^2 f}{\partial x^i \partial x^j} \right] = \begin{pmatrix} A_f & B_f^T \\ B_f & C_f \end{pmatrix},
\]

then the complex Hessian is given by

\[
[ f_{ij} ] = \left[ \frac{\partial^2 f}{\partial z^i \partial \bar{z}^j} \right] = \frac{1}{4} (A_f + C_f) + \frac{\sqrt{-1}}{4} (B_f^T - B_f).
\]

(6)

We consider the map \( \iota_{\mathcal{H}_\omega} : \mathcal{H}_\omega \to M \), given as the composition

\[
\mathcal{H}_\omega \xrightarrow{g_{\omega + \sqrt{-1} \partial \bar{\partial} \phi}} \mathcal{H} \xrightarrow{\iota_{\mathcal{H}}} M.
\]

(7)

Its differential \( d\iota_{\mathcal{H}_\omega} : C^\infty(M) \to \text{Sym}^2 T^* M \) is independent of the point \( \varphi \in \mathcal{H}_\omega \).

By combining (5) and (6), we see that in local coordinates, \( d\iota_{\mathcal{H}_\omega} f \) is given by

\[
\frac{1}{2} \begin{pmatrix} A_f + C_f & B_f^T - B_f \\ B_f - B_f^T & A_f + C_f \end{pmatrix}.
\]

(8)
In fact,
\[
(9) \quad d\iota_{\mathcal{H}} = P^{1,1} \circ \nabla^2 =: \nabla^{1,1},
\]
where $P^{1,1}$ denotes the projection of a symmetric $(0,2)$-type tensor onto its $J$-invariant part, and the action of $J$ on $\text{Sym}^2 T^* M$ is given by $J \cdot h := h(J \cdot, J \cdot)$. The formula (9) holds since $J \cdot \nabla^2 f$ is represented in coordinates by
\[
\begin{pmatrix}
0 & -I \\
I & 0
\end{pmatrix}
\begin{pmatrix}
A_f & B_f^T \\
B_f & C_f
\end{pmatrix}
\begin{pmatrix}
0 & I \\
-I & 0
\end{pmatrix},
\]
and $P^{1,1} h = \frac{1}{2} (h + J \cdot h)$. In this notation, if we let $g_f$ denote the metric associated to $\omega_f$, then $g_f + \nabla^{1,1} f = g_f + \nabla^{1,1} f$.

We note that from this description of metrics in $\mathcal{H}$, we see that $\mathcal{H}$ is the intersection of a closed affine subspace (within the space of symmetric $(0,2)$-tensor fields) with $\mathcal{M}$. Indeed, if $g_0$ is the Riemannian metric associated to the reference Kähler form $\omega$, then by the above discussion any metric $g_f \in \mathcal{H}$ is given by $g_0 + \nabla^{1,1} f$. This shows, in particular, that $\mathcal{H}$ is an embedded submanifold of $\mathcal{M}$.

Using these preliminaries, we make the following observation, which we believe is due to Calabi. It does not seem to be well-known, and serves as our starting point. It shows that $(\mathcal{H}, g_C)$ is isometrically embedded in $(\mathcal{M}, g_E)$.

**Proposition 2.1.** Consider the inclusion $\iota_{\mathcal{H}} : \mathcal{H} \hookrightarrow \mathcal{M}$. Then, $\iota_{\mathcal{H}}^* g_E = 2 g_C$.

Here and in the sequel, we abuse notation by using $g_C$ to also denote the metric on $\mathcal{H}$ obtained by pushing $g_C$ forward to $\mathcal{H}$ under the first map in (7) (i.e., we also write $\iota_{\mathcal{H}}^* (g_E g_C) = 2 g_C$). The $\partial \bar{\partial}$-lemma implies that there is no loss in doing so. We also note that by abuse of notation, we often write both $\omega \in \mathcal{H}$ and $g_\omega \in \mathcal{H}$.

**Proof.** First, note that $g_C$ may be alternatively expressed as
\[
(10) \quad g_C(\nu, \eta)|_\varphi = \int_M (\sqrt{-1} \partial \bar{\partial} \nu, \sqrt{-1} \partial \bar{\partial} \eta)_{\omega_\varphi} \omega_\varphi^n/n!.
\]
To see this, recall the following algebraic identity for any $(1,1)$-forms $\beta, \gamma$ and a strictly positive $(1,1)$-form $\alpha$ ([2], [6, Lemma 2.77]),
\[
(11) \quad (\alpha, \beta)_{\alpha} (\alpha, \gamma)_{\alpha} - (\beta, \gamma)_{\alpha} = \frac{\beta \wedge \gamma \wedge \alpha^2 / (n-2)!}{\alpha^n/n!}.
\]
Since the right-hand side in this identity is exact whenever $\beta$ and $\gamma$ are, and since $(\omega_\varphi, \sqrt{-1} \partial \bar{\partial} \nu)_{\omega_\varphi} = \Delta_\varphi \nu$, equation (10) follows.

We claim that
\[
(12) \quad 2 (\sqrt{-1} \partial \bar{\partial} \nu, \sqrt{-1} \partial \bar{\partial} \eta)_{\omega_\varphi} = \text{tr}(g_\varphi^{-1} d\iota_{\mathcal{H}_\omega} \nu g_\varphi^{-1} d\iota_{\mathcal{H}_\omega} \eta),
\]
where \( d\nu_\mathcal{H} = \frac{d}{dt} \big|_{t=0} g_{\varphi + t\nu} \) is as in (9), and similarly for \( \eta \). For the proof, it is enough to verify this identity pointwise. If \( \omega_\varphi = \sqrt{-1}g_{ij}dz^i \wedge d\bar{z}^j \), then the left-hand side of (12) is

\[
g_{\varphi}^{ij} \frac{d}{dt} g_{\varphi}^{k\ell} \partial_\ell \omega_\varphi_{ij} = tr(G_\varphi^{-1}R G_\varphi^{-1} S),
\]

where \( G_{\varphi} = [(g_{\varphi})_{ij}], R = [\nu_{ij}], S = [\eta_{ij}] \). By choosing holomorphic normal coordinates at \( p \in M \), \( (g_{\varphi})_{ij}(p) = \delta_{ij} \), and by (5) we have \( (g_{\varphi})_{ij}(p) = 2\delta_{ij} \). The left-hand side of (12) equals

\[
\frac{1}{16} tr \left( (A_\nu + C_\nu + \sqrt{-1}B_\nu^T - \sqrt{-1}B_\eta)(A_\eta + C_\eta + \sqrt{-1}B_\eta^T - \sqrt{-1}B_\eta) \right),
\]

while the right-hand side equals

\[
tr \left( 2^{-1} \left( A_\nu + C_\nu B_\nu^T - B_\nu \right) \left( B_\nu - B_\eta \right) \right) 2^{-1} \left( A_\eta + C_\eta B_\eta^T - B_\eta \right),
\]

proving (12).

To conclude the proof, observe that \( \omega^n/n! = det(g_{ij})^{\wedge_n} \sqrt{-1}dz^k \wedge d\bar{z}^k \), while \( dV_g = \sqrt{det(g_{ij})^{\wedge_2} \sqrt{-1}dz^k} \). Note that if \( G = A + \sqrt{-1}B \) then \( [g_{ij}] = 2^{-1} \left( \begin{array}{cc} A & B \\ -B & A \end{array} \right) \), hence \( det(g_{ij}) = 2^{n^2} det(g_{ij})^2 \) (see, e.g., [16, Lemma 2]). Hence \( dV_g = \omega^n/n! \), and the proposition follows. \( \square \)

Since, as we recall in Section 4.3, \((\mathcal{H}, g_\mathcal{C})\) has diameter \( \pi \sqrt{V} \), it follows that \( \mathcal{H} \) is a bounded set in \((M, g_\mathcal{E})\) (of diameter at most \( \pi \sqrt{2V} \)). This also follows directly from the fact the set of all metrics of volume not greater than \( V \) in \( M \) has diameter at most \( 4\sqrt{\frac{2V}{n}} \) [24, Lemma 4.2, Theorem 4.16]. The latter is a better bound whenever \( n > 1 \), reflecting to some degree the extent to which \( \mathcal{H} \) is far from being totally geodesic, as we show in the next subsection.

As a corollary of the proof of Proposition 2.1, we record the following property of tangent vectors to \( \mathcal{H} \subset M \).

**Lemma 2.2.** For all \( h, k \in T_{g_\varphi} \mathcal{H} \),

\[
(h, k)_E = \frac{1}{4n} (tr(g_{\varphi}^{-1}h)g_{\varphi}, tr(g_{\varphi}^{-1}k)g_{\varphi})_E = \frac{1}{2} \int_M tr(g_{\varphi}^{-1}h) tr(g_{\varphi}^{-1}k) dV_{g_\varphi}.
\]

**Proof.** Let \( \nu, \eta \in T_\mathcal{H} \mathcal{H}_\omega \), and let \( \nabla^{1,1}\nu, \nabla^{1,1}\eta \in T_{g_\varphi} \mathcal{H} \subset T_{g_\varphi} M \) (recall (9)). Then by (6) and (8),

\[
\Delta_{\varphi} \nu = \frac{1}{2} tr(g_{\varphi}^{-1} \nabla^{1,1}\nu)
\]

(remembering that if \( g_{ij}(p) = \delta_{ij} \) then \( g_{ij}(p) = 2\delta_{ij} \)). Now, by the \( \partial \bar{\partial} \)-lemma, \( \nabla^{1,1} \) is an isomorphism between \( C^\infty(M)/\mathbb{R} \) and \( T_{g_\varphi} \mathcal{H} \). Hence, given \( h, k \in T_{g_\varphi} \mathcal{H} \) there
exist $\nu, \eta \in C^\infty(M)$ with $h = \nabla^{1,1} \nu$, $k = \nabla^{1,1} \eta$. So by Proposition 2.1 and (13),

\[(h, k)_E = 2(\nu, \eta)_C = 2 \int_M \Delta_\varphi \nu \Delta_\varphi \eta \frac{\omega^n_\varphi}{n!} = \frac{1}{2} \int_M \text{tr}(g_\varphi^{-1} h) \text{tr}(g_\varphi^{-1} k) dV_{g_\varphi},\]

as claimed. \[\square\]

**Remark 2.3.** Lemma 2.2 may be interpreted as saying that the angle cut out between $\mathcal{H}$ and the conformal classes is a constant depending only on the dimension. For more on this we refer to Section 7.1.

### 3. The second fundamental form

We now compute the second fundamental form $II$ of $\iota_\mathcal{H} : \mathcal{H} \hookrightarrow \mathcal{M}$. For simplicity, and since it suffices for our purposes, we state the result only in terms of the trace of $II$. It then follows that no geodesic of $(\mathcal{H}, g_C)$ is a geodesic of $(\mathcal{M}, g_E)$.

**Proposition 3.1.** The trace of the second fundamental form of the inclusion $\iota_\mathcal{H} : \mathcal{H} \hookrightarrow (\mathcal{M}, g_E)$ is given by

\[
\text{tr}(g_\varphi^{-1} II(h, k))|_{g_\varphi} = -\frac{n}{2} \text{tr}(g_\varphi^{-1} h g_\varphi^{-1} k) + \frac{1}{4} \text{tr}(g_\varphi^{-1} h) \text{tr}(g_\varphi^{-1} k) - \frac{1}{2V}(h, k)_E,
\]

where $h = \nabla^{1,1} \nu, k = \nabla^{1,1} \eta$, with $\nu, \eta \in C^\infty(M)$ constant vector fields on $\mathcal{H}$, and $\nabla^{1,1}$ defined by (9). In particular, no geodesic in $(\mathcal{H}, g_C)$ is a geodesic in $(\mathcal{M}, g_E)$.

**Proof.** For the following formula we refer to [33, p. 19], or [34, p. 335] (or [35, p. 189] with a different sign convention).

**Lemma 3.2.** The Levi-Civita connection of $(\mathcal{M}, g_E)$ is given by

\[
\nabla^g_E h|_g = \frac{1}{2} \nabla h g^{-1} k - \frac{1}{2} k g^{-1} h - \frac{1}{4} \text{tr}(g^{-1} h g^{-1} k) g + \frac{1}{4} \text{tr}(g^{-1} h) k + \frac{1}{4} \text{tr}(g^{-1} k) h,
\]

for constant vector fields $h, k \in T\mathcal{M}$.

Next, we compute the Levi-Civita connection of $(\mathcal{H}_\omega, g_C)$. We first compute this on the level of Kähler potentials, and then translate to the level of metrics. Let $\nu, \eta, \psi \in T_\varphi \mathcal{H}_\omega \cong C^\infty(M)$ be constant vector fields. Since

\[
\frac{d}{ds} \bigg|_{s=0} \Delta_\varphi + s \psi \nu = - (\sqrt{-1} \partial \bar{\partial} \psi, \sqrt{-1} \partial \bar{\partial} \nu) \omega_\varphi,
\]

it follows that

\[
\psi(\nu, \eta) = \int_M \left( \Delta_\varphi \nu \Delta_\varphi \eta \Delta_\varphi \psi - (\sqrt{-1} \partial \bar{\partial} \psi, \sqrt{-1} \partial \bar{\partial} \nu) \omega_\varphi \Delta_\varphi \eta \nu \right) \omega_\varphi^n \frac{1}{n!} - (\sqrt{-1} \partial \bar{\partial} \psi, \sqrt{-1} \partial \bar{\partial} \eta) \omega_\varphi \Delta_\varphi \nu \omega_\varphi^n \frac{1}{n!}.
\]
The Koszul formula (see, e.g., [56, (11)]) then gives that
\[ \Delta \varphi (\nabla^g_c \eta | \varphi) \]
is equal, up to a constant, to
\[ -(\sqrt{-1} \bar{\partial} \bar{\partial} \eta, \sqrt{-1} \bar{\partial} \bar{\partial} \nu)_{\omega_\varphi} + \frac{1}{2} \Delta \varphi \eta \Delta \varphi \nu. \]

By (11) it follows that
\[ \Delta \varphi (\nabla^g_c \eta | \varphi) = \frac{1}{2} \Delta \varphi \eta \Delta \varphi \nu + \frac{1}{2} V^{-1} \int_M (\Delta \varphi \eta \Delta \varphi \nu) \frac{\omega_\varphi^n}{n!} \]
\[ - (\sqrt{-1} \bar{\partial} \bar{\partial} \eta, \sqrt{-1} \bar{\partial} \bar{\partial} \nu)_{\omega_\varphi}. \]

On the level of Kähler forms the tangent vector is expressed as
\[ \sqrt{-1} \bar{\partial} \bar{\partial} \Delta \varphi^{-1} \left( \frac{1}{2} \Delta \varphi \eta \Delta \varphi \nu + \frac{1}{2} V^{-1} \int_M (\Delta \varphi \eta \Delta \varphi \nu) \frac{\omega_\varphi^n}{n!} - (\sqrt{-1} \bar{\partial} \bar{\partial} \eta, \sqrt{-1} \bar{\partial} \bar{\partial} \nu)_{\omega_\varphi} \right). \]

The corresponding tangent vector in \( T_{g_\varphi} M \) is given by \( \nabla^1,1 \), \( (\nabla^g_c \eta | \varphi) \) (recall (9)).

Let \( h = \nabla^1,1 \eta \) and \( k = \nabla^1,1 \nu \) be elements of \( T_{g_\varphi} M \). Slightly abusing notation, we have \( \nabla^g_c k |_{g_\varphi} = \nabla^1,1 (\nabla^g_c \eta | \varphi) \). By Proposition 2.1, (12), and (13), on the level of metrics then,
\[ \frac{1}{2} \text{tr} \left( g_\varphi^{-1} \nabla^g_c k \right) = \frac{1}{8} \text{tr} \left( g_\varphi^{-1} h \right) \text{tr} \left( g_\varphi^{-1} k \right) + \frac{1}{4} V^{-1} (h, k)_E - \frac{1}{2} \text{tr} \left( g_\varphi^{-1} h g_\varphi^{-1} k \right). \]

Since, by Proposition 2.1,
\[ II(h, k) = \nabla^g_c k - \nabla^g_c k, \]
the claimed formula follows (note \( \text{tr}(g_\varphi^{-1} g_\varphi) = 2n \)).

Now, Lemma 2.2 implies that whenever \( h, k \in T_{g_\varphi} \mathcal{H} \),
\[ (II(h, k), g_\varphi)_E = \int_M \text{tr} \left( g_\varphi^{-1} II(h, k) \right) dV_{g_\varphi} = - \frac{n}{2} (h, k)_E. \]

Hence, for any nonzero vector \( h \in T_{g_\varphi} \mathcal{H} \) we have \( II(h, h) \neq 0 \), and this completes the proof of the Proposition.

Alternatively, the last conclusion may be proved by examining the explicit expressions for the exponential map of \( g_E \). In fact, we will see below (Remark 4.2) that geodesics of \( \mathcal{M} \) intersect \( \mathcal{H} \) in at most two points.

4. **Intrinsic and extrinsic distance on the space of Kähler metrics.** By Proposition 2.1, when \( \mathcal{H} \) is considered as a submanifold of \( (\mathcal{M}, g_E) \), its induced metric precisely coincides with twice the Calabi metric. Comparing the distance
between Kähler metrics measured with respect to these two $L^2$ metrics then corresponds to comparing the extrinsic (Ebin) distance, and the intrinsic (Calabi) distance. Let $d_E, d_C$ denote the distance functions of $(\mathcal{M}, g_E)$ and $(\mathcal{H}, g_C)$, respectively. The main result of this section (Theorem 4.5) shows that these two distance functions are equivalent.

The proof of this fact uses the Calabi-Yau Theorem and the associated diffeomorphism $\mathcal{H} \cong \mathcal{V}$, where $\mathcal{V}$ is the space of all smooth volume forms on $M$ with total volume $V = \text{Vol}(M, \omega)$. On $\mathcal{V}$, an explicit expression for $d_C$ can be obtained, and we use this to show that $d_C$ is equivalent to the metric induced from the Ebin metric on the ambient space $\tilde{\mathcal{V}}$ of all smooth volume forms on $M$. We then translate this result back to $\mathcal{M}$ by using the natural submersion $\mathcal{M} \to \tilde{\mathcal{V}}$ and the product structure it induces on $\mathcal{M}$.

4.1. The space of volume forms as a submanifold of $\mathcal{M}$. Our references for this subsection are Ebin [33] and Freed-Groisser [34] (see also [21, Section 2.5.3]).

Consider the space $\tilde{\mathcal{V}}$ of all smooth volume forms on $M$. At any point $\mu$, the tangent space to $\tilde{\mathcal{V}}$ is canonically isomorphic to $\Omega^{2n}(M)$, the space of smooth $2n$-forms on $M$. On the other hand, for each fixed $\mu \in \tilde{\mathcal{V}}$, consider the smooth submanifold [33, Lemma 8.8]

$$
\mathcal{M}_\mu := \{ g \in \mathcal{M} : dV_g = \mu \} \subset \mathcal{M}.
$$

Since the map $i_\mu : \mathcal{M}_\mu \times \tilde{\mathcal{V}} \to \mathcal{M}$, $(g, \nu) \mapsto (\nu/\mu)^{2/n}g$ (sending $(g, \nu)$ to the unique metric conformal to $g$ with volume form $\nu$) is a diffeomorphism, the space $\mathcal{M}$ inherits the structure of a product manifold. Define $\pi : \mathcal{M} \to \tilde{\mathcal{V}}$ by $\pi(g) := dV_g$. It is surjective and its differential is

$$
d\pi|_g h = \frac{1}{2} \text{tr} (g^{-1} h) dV_g, \quad g \in \mathcal{M}, \ h \in T_g \mathcal{M}.
$$

When $\mathcal{M}$ is equipped with the metric $g_E$ and $\tilde{\mathcal{V}}$ with the metric $\frac{4}{2n} g_{\tilde{\mathcal{V}}}$, where

$$
g_{\tilde{\mathcal{V}}} (\alpha, \beta) := \int_M \frac{\alpha \wedge \beta}{\mu} \mu, \quad \alpha, \beta \in T_{\mu} \tilde{\mathcal{V}} \cong \Omega^{2n}(M),
$$

the map $\pi$ becomes a Riemannian submersion whose vertical fibers are of the form $\mathcal{M}_\mu$, with vertical tangent spaces $T_g^v \mathcal{M} = \{ h \in T_g \mathcal{M} : \text{tr}(g^{-1} h) = 0 \}$ and horizontal tangent spaces $T_g^h \mathcal{M} = C^\infty(M) \cdot g = \{ h \in T_g \mathcal{M} : h = \frac{1}{2n} \text{tr}(g^{-1} h)g \}$. The leaf of the horizontal distribution through $g \in \mathcal{M}$ is precisely the conformal class $\mathcal{P}g$ of $g$, where $\mathcal{P} := \{ F \in C^\infty(M) : F > 0 \}$ and $\mathcal{P} \cong \tilde{\mathcal{V}}$. 
4.2. The space of fixed-volume volume forms. Consider the submanifold

\[ \mathcal{V} := \left\{ \mu \text{ is a smooth volume form on } M : \int_M \mu = V \right\} \]

of \( \tilde{\mathcal{V}} \), and the map \( \mathcal{H} \ni g \rightarrow \iota^*_{\mathcal{V}} V^\omega = \omega^n/n! \in \mathcal{V} \). By the maximum principle (and the \( \partial\bar{\partial} \)-lemma), it is injective [11]. Let \( g_{\mathcal{V}} \) denote the metric on \( \mathcal{V} \) induced from the inclusion \( \mathcal{V} \hookrightarrow (\tilde{\mathcal{V}}, g_{\tilde{\mathcal{V}}}) \),

\[ g_{\mathcal{V}}(\alpha, \beta)|_\mu = \int_M \frac{\alpha \beta}{\mu^2} \mu, \quad \alpha, \beta \in T_\mu \mathcal{V} \cong \Omega_{0}^{2n}(M), \]

where \( \Omega_{0}^{2n}(M) \) denotes the space of smooth 2n-forms on \( M \) that integrate to zero. Since \( d\iota^*_{\mathcal{V}} V^\omega_{\mathcal{V}}|_{g_{\mathcal{V}}} \nabla_{1,1}^V = \Delta_{\omega^V} \omega^n/n! \), it follows that \( \iota^*_{\mathcal{H}, \mathcal{V}} g_{\mathcal{V}} = g_C \). The Calabi-Yau theorem [76] states that \( \iota_{\mathcal{H}, \mathcal{V}} \) is also surjective, and hence that \( (\mathcal{H}, g_C) \) is isometric to \( (\mathcal{V}, g_{\mathcal{V}}) \).

To summarize, we saw that: (i) when \( \tilde{\mathcal{V}} \) is considered as a subspace of \( \mathcal{M} \) via the map \( i_{\mu}(g, \cdot) \) (with some arbitrary choices of \( g \) and \( \mu \)), its natural \( L^2 \) metric \( g_{\tilde{\mathcal{V}}} \) coincides with the one induced from \( (\mathcal{M}, \frac{\omega}{2} g_E) \); (ii) by the Calabi-Yau theorem, Calabi’s metric \( g_C \) on \( \mathcal{H} \) induces a metric on \( \mathcal{V} \); and (iii) the latter metric coincides with the metric induced on \( \mathcal{V} \) from the inclusion \( \mathcal{V} \hookrightarrow (\tilde{\mathcal{V}}, g_{\tilde{\mathcal{V}}}) \).

Hence, \( \mathcal{V} \) inherits two distance functions—the intrinsic distance \( d_{\mathcal{V}} \) from \( (\mathcal{V}, g_{\mathcal{V}}) \) and the extrinsic distance \( d_{\tilde{\mathcal{V}}} \) induced from \( (\tilde{\mathcal{V}}, g_{\tilde{\mathcal{V}}}) \). In the next subsection we compute \( d_{\mathcal{V}} \), and in Section 4.4 we prove it is equivalent to \( d_{\tilde{\mathcal{V}}} \). Since

\[ \iota_{\mathcal{H}, \mathcal{V}}^* d_{\mathcal{V}} = d_C, \]

this will allow us in Section 4.5 to estimate \( d_C \) from above in terms of \( d_{\tilde{\mathcal{V}}} \).

Remark 4.1. Using only that \( (\mathcal{H}, g_C) \) is isometrically embedded in \( (\mathcal{V}, g_{\mathcal{V}}) \), the ensuing inequality \( \iota_{\mathcal{H}, \mathcal{V}}^* d_{\mathcal{V}} \leq d_C \) would not suffice to prove our main result. The isomorphism \( \iota_{\mathcal{H}, \mathcal{V}}^{-1} : \mathcal{V} \rightarrow \mathcal{H} \) allows us to compute in \( (\mathcal{V}, g_{\mathcal{V}}) \) and then translate back to \( \mathcal{H} \). In this sense, the Calabi-Yau isomorphism serves as a change of variable for the equation

\[ (\Delta_{\varphi^\mathcal{V}}) \varphi^\mathcal{V} = \frac{\omega^n}{\omega^\mathcal{V}} \left( 2\Delta_{\varphi^\mathcal{V}} \varphi^\mathcal{V} - 2|\sqrt{-1} \partial \bar{\partial} \varphi^\mathcal{V}|^2 + \frac{C^2 \omega^n}{V} \right). \]

In fact, it seems that being able to compute \( d_C \) from (15) was one of Calabi’s original motivations for his well-known conjecture, and the reason why his results summarized in [10] remained unpublished until recently.

4.3. Geodesics in Calabi’s metric. Next, we recall the equation for Calabi’s geodesics on \( \mathcal{V} \) [10, 13]. For completeness, and in order to fix conventions, we derive the equation directly, in a slightly different and more economical manner than in [13].
Let $\mu \in \mathcal{V}$. The energy of a path $\{\mu(t) = F(t)\mu\} \subset \mathcal{V}$ is given by

$$
\int_{[0,1] \times M} \left( \frac{\mathcal{L}(t)}{\mu} \right)^2 \mu \wedge dt = \int_{[0,1] \times M} \frac{F_t^2}{F} \mu \wedge dt,
$$

where $F = F(t,z)$, and subscripts denote differentiation. Taking the first variation of the energy with respect to variations $\varphi(t,s,z)$ fixing the endpoints, we obtain that for a geodesic $F(t)$,

$$
0 = \int_{[0,1] \times M} \frac{2F_t \mathcal{L}_t F - F_t^2 \mathcal{L}_s F_s}{F^2} \mu \wedge dt
$$

$$
= \int_{[0,1] \times M} \left( -2(\log F)_{tt} - ((\log F)_t)^2 \right) F_s \mu \wedge dt.
$$

The expression in parentheses is orthogonal to $\Omega^2_0(M)$, i.e., constant, and by integrating against $\mu(t)$ is seen to equal $\frac{1}{4} |\mu_t(t)|_{\mathcal{V}}^2$. When $F(t) > 0$, the equation for geodesics of constant speed $C$ is therefore

$$
F_t^2 - 2F_{tt} F - \frac{C^2}{V} F^2 = 0,
$$

where $C^2 = |\mu_t(t)|^2_{\mathcal{V}}$, and when $F(t) > 0$ this simplifies to $(\sqrt{F})_{tt} + \frac{C^2}{4V} \sqrt{F} = 0$. The unit-speed geodesic connecting $F_\mu$ and $G_\mu$ thus satisfies

$$
\sqrt{F(t)} = \frac{\sin \left( \frac{1}{2} (T-t)/\sqrt{V} \right)}{\sin \left( \frac{1}{2} T/\sqrt{V} \right)} \sqrt{F} + \frac{\sin \left( \frac{1}{2} t/\sqrt{V} \right)}{\sin \left( \frac{1}{2} T/\sqrt{V} \right)} \sqrt{G},
$$

where $T$ is the length of the geodesic. Noting that

$$
F_t|_{t=T} \mu = \frac{G_\mu}{\sqrt{V}} \cot \left( \frac{1}{2} T/\sqrt{V} \right) - \frac{\sqrt{FG_\mu}}{\sqrt{V} \sin \left( \frac{1}{2} T/\sqrt{V} \right)}
$$

must have unit length yields that $T = 2\sqrt{V} \cos^{-1} \left( \frac{1}{V} \int \sqrt{FG_\mu} \right)$. In fact, geodesics minimize length in $(\mathcal{V}, g_\mathcal{V})$ [10], [13, Lemma 6.3], and so

$$
d_\mathcal{V}(\mu_1, \mu_2) = 2\sqrt{V} \cos^{-1} \left( \frac{1}{V} \int_M \sqrt{\frac{\mu_1}{\mu_0} \frac{\mu_2}{\mu_0}} \mu_0 \right),
$$

where $\mu_0 \in \mathcal{V}$ is any fixed volume form.

**4.4. Intrinsic and extrinsic distance on the space of fixed-volume volume forms.** We turn to proving the equivalence of the intrinsic and extrinsic distance functions on the space of fixed-volume volume forms. The results of this subsection (excepting Remark 4.2) hold on a general Riemannian (and not necessarily Kähler) manifold.
Before stating the result we make some remarks. The metric $g_{\tilde{V}}$ is only a weak Riemannian metric, but it nevertheless induces a metric space structure [22, Corollary 11]; we denote the distance function by $d_\Phi$. On the other hand, the submanifold $\tilde{V}$ is not totally geodesic in $(\tilde{V},g_{\tilde{V}})$ (nor, equivalently, is its inclusion in $(\mathcal{M},g_E)$). In fact, no geodesic of the latter is a geodesic of the former; this follows from the explicit formula for geodesics of $g_{\tilde{V}}$,

$$\mu(t) = \left(1 + \frac{t}{2\mu} \right)^2 \mu, \quad \alpha \in T_\mu \tilde{V},$$

which shows that if $\mu(0) = \alpha \in T_\mu \tilde{V}$ then $\mu(t) = \alpha + \frac{t}{2} (\frac{\alpha}{\mu})^2 \mu \notin \Omega^2_0(M)$ for any $t \neq 0$. Hence, geodesics in $\tilde{V}$ intersect $\tilde{V}$ tangentially in at most one point, and all other intersections are transverse. In particular, this implies that Vol$(M,\mu(t))$ is not constant. Additionally, by (19), Vol$(M,\mu(t))$ is quadratic in $t$. By transversality then, Vol$(M,\mu(t)) = V$ for exactly one positive value of $t$. Thus, a geodesic of $\tilde{V}$ that intersects $\tilde{V}$ does so in exactly two points.

**Remark 4.2.** We make a slight digression to observe that, similarly, geodesics of $(\mathcal{M},g_E)$ intersect $\mathcal{H}$ in at most two points. Indeed, let $\{g(t)\}$ be a geodesic of $(\mathcal{M},g_E)$ with $g := g(0)$ and $h := g_t(0) \in T_g \mathcal{H}$. Denote by $\mu(t) := \pi(g(t))$ the volume form induced by $g(t)$, and $h_0 := h - \frac{1}{2\pi} \text{tr}(g^{-1}h)$. Then by [34, Theorem 2.3], [35, Theorem 3.2], $\mu(t) = \left(\left(1 + \frac{t}{4} \text{tr}(g^{-1}h)\right)^2 + \frac{n}{2} \text{tr}(\text{tr}(g^{-1}h^2) t^2)\right) \mu(0)$. From this, the variation in the total volume of $\mu(t)$ is

$$\frac{d}{dt} \int_M \mu(t) = \frac{1}{2} \int_M \left(\text{tr}(g^{-1}h) + \frac{t}{4} \text{tr}(g^{-1}h)^2 + \frac{n}{2} \text{tr}(\text{tr}(g^{-1}h^2) t^2)\right) \mu(0).$$

As $h \in T_g \mathcal{H}$, the first term vanishes, implying that if $\mu(t)$ is tangent to $\mathcal{H}$ for some $t \neq 0$, then $\int_M (\text{tr}(g^{-1}h)^2 + 2n \text{tr}(\text{tr}(g^{-1}h^2))) \mu(0) = 0$. But this gives that $h = 0$, proving $\{g(t)\}$ intersects $\mathcal{H}$ tangentially in at most one point. Since Vol$(M,g(0))$ is quadratic in $t$, $\{g(t)\}$ intersects $\mathcal{H}$ in at most two distinct points (the second point where Vol$(M,g(t)) = V$ might not be Kähler).

At this point, motivated by [13, 15], making what amounts to a change of coordinates on $\tilde{V}$ allows for a clearer picture of the geometry of $\mathcal{V} \subset \tilde{V}$. So fix any $\mu_0 \in \mathcal{V}$, and consider the map $\Phi : \tilde{V} \to \mathcal{P}$ defined by $\Phi(\mu) := 2\sqrt{\mu/\mu_0}$, which is seen to be a diffeomorphism. By (19), a path $\mu(t) = F(t)\mu_0$ is a geodesic of $\mathcal{V}$ if and only if $(\sqrt{F(t)})_{tt} = 0$, that is, if and only if $\Phi(\mu(t))_{tt} = 0$. Thus, in the coordinates defined by $\Phi$, $\tilde{V}$ is manifestly flat. Furthermore, we see that $\Phi$ is an isometry (both in the Riemannian sense and in the sense of metric spaces) between $(\tilde{V},g_{\tilde{V}})$ and its image in $L^2(M,\mu_0)$, since $g_{\tilde{V}}(\alpha,\alpha) = ||d\Phi(\mu)\alpha||_{L^2(M,\mu_0)}^2$.

Also apparent is the fact that $\mathcal{V}$ is a section of a sphere. Indeed, if $\mu \in \mathcal{V}$, then we have $||\Phi(\mu)||_{L^2(M,\mu_0)} = 2\sqrt{V}$, so $\Phi(\mathcal{V})$ is precisely the intersection of the sphere
of radius $2\sqrt{V}$ in $L^2(M,\mu_0)$ with $\mathcal{P}$. Either from this description, or by using the Cauchy-Schwarz inequality to see that the argument of $\cos^{-1}$ in (18) is strictly between 0 and 1 if $\mu_1 \neq \mu_2$, one sees that great circles on this spherical section have length strictly less than $\pi \sqrt{V}$, since $d_{\mathcal{V}}(\mu_1,\mu_2) < \pi \sqrt{V}$ for any $\mu_1,\mu_2 \in \mathcal{V}$. Thus, an arc of a great circle connecting two boundary points of $\Phi(\mathcal{V})$ is at most a quarter-circle.

Now, as in [15], we can see that geodesics in $\mathcal{V}$ are projections of chordal geodesics in $\tilde{\mathcal{V}}$. Indeed, let $\mu,\nu \in \mathcal{V}$ be given, and let $\mu(t)$, for $t \in [0,1]$, be the unique geodesic of $\tilde{\mathcal{V}}$ connecting them. Then by the above discussion, $\Phi(\mu(t))$ is simply the line segment (chord) between $\Phi(\mu)$ and $\Phi(\nu)$. By elementary geometry, we know that the geodesic (i.e., arc of a great circle) between $\mu$ and $\nu$ on $\Phi(\mathcal{V})$ is the projection of $\Phi(\mu(t))$ onto $\Phi(\mathcal{V})$, which is explicitly given by $\sqrt{V/v(t)}\Phi(\mu(t))$, where $v(t) = \int_M \mu(t)$. This arc is length-minimizing in $L^2(M,\mu_0)$, and since $\Phi$ is an isometry, its length equals $d_{\mathcal{V}}(\mu,\nu)$.

Using that $\Phi$ is an isometry, we also get the following formula for $d_{\tilde{\mathcal{V}}}$.

**Lemma 4.3.** Let $\mu_1,\mu_2 \in \tilde{\mathcal{V}}$ be given. Then we have

$$d_{\tilde{\mathcal{V}}}(\mu_1,\mu_2) = \|\Phi(\mu_2) - \Phi(\mu_1)\|_{L^2(M,\mu_0)} = 2 \left( \int_M \left( \sqrt{\frac{\mu_2}{\mu_0}} - \sqrt{\frac{\mu_1}{\mu_0}} \right)^2 \mu_0 \right)^{1/2}.$$

**Proposition 4.4.** The intrinsic and extrinsic metrics $d_{\mathcal{V}}$ and $d_{\tilde{\mathcal{V}}}$, respectively, are equivalent on $\mathcal{V}$. More specifically,

$$d_{\tilde{\mathcal{V}}} \leq d_{\mathcal{V}} < \frac{\pi}{2\sqrt{2}} d_{\tilde{\mathcal{V}}},$$

and these bounds are optimal.

**Proof.** We first give an essentially algebraic proof of equivalence which yields a suboptimal bound. The optimal bound is given by a geometric argument.

So suppose we are given $\mu_1,\mu_2 \in \mathcal{V}$. As noted above, $d_{\mathcal{V}}(\mu_1,\mu_2) < \pi \sqrt{V}$. Hence, by convexity of $x - \frac{\pi}{2} \sin x$ on $[0,\pi/2]$,

$$\frac{1}{2\sqrt{V}} d_{\mathcal{V}}(\mu_1,\mu_2) < \frac{\pi}{2} \sin \left( \frac{1}{2} d_{\mathcal{V}}(\mu_1,\mu_2)/\sqrt{V} \right)$$

$$= \frac{\pi}{2} \left( 1 - \cos^2 \left( \frac{1}{2} d_{\mathcal{V}}(\mu_1,\mu_2)/\sqrt{V} \right) \right)^{1/2}$$

$$= \frac{\pi}{2} \left( 1 - \left( \frac{1}{V} \int_M \sqrt{\frac{\mu_1}{\mu_0} \frac{\mu_2}{\mu_0}} \right)^2 \right)^{1/2}.$$
Since $1 - x^2 \leq 2(1 - x)$ for all $x \in \mathbb{R}$, we can further estimate

$$d_{\mathcal{V}}(\mu_1, \mu_2) < \pi \sqrt{2V} \left(1 - \frac{1}{V} \int_M \sqrt{\frac{\mu_1 \mu_2}{\mu_0 \mu_0}}\right)^{1/2}.$$ 

Since $\int \mu_i = V$, the result follows from Lemma 4.3.

Now we make use of the geometric discussion preceding the proposition. An arc of a great circle diverges more from a straight line the longer it is. As already noted, great circles between boundary points of $\Phi(V)$ can be at most quarter-circles. But such an arc has length equal to $\pi/(2\sqrt{2})$ times that of the chord between the boundary points. Since $\Phi$ is an isometry, we can thus deduce that

$$d_{\mathcal{V}} \leq \frac{\pi}{2\sqrt{2}} d_{\tilde{\mathcal{V}}}.$$  \hspace{1cm} (20)

Furthermore, since the bound $d_{\mathcal{V}}(\mu, \nu) < \pi \sqrt{V}$ given above is optimal, the factor in (20) is optimal. \hfill $\square$

4.5. **Intrinsic and extrinsic distance on the space of Kähler metrics.** We are now in a position to prove our first main result.

**Theorem 4.5.** The intrinsic and extrinsic metrics $d_C$ and $d_E$, respectively, are equivalent on $\mathcal{H}$. More specifically,

$$\frac{1}{\sqrt{2}} d_E \leq d_C < \frac{\pi \sqrt{n}}{4} d_E.$$ 

**Proof.** Recall from Section 4.1 that for each $g \in \mathcal{M}$, $d_\pi|_g$ is an isometry between the horizontal tangent space $T^h_g \mathcal{M} \cong T_g \tilde{\mathcal{V}}$ at $g$ and the tangent space at $\pi(g) = dV_g \in \tilde{\mathcal{V}}$. In particular, if we denote by $L_E$ the $g_E$-length of a path, this implies that $L_E(\{g(t)\}) \geq L_{\tilde{\mathcal{V}}}(\{\pi(g(t))\})$ for any path $\{g(t)\}$ in $\mathcal{M}$, with equality if and only if $g(t)$ is horizontal. (Note that “$L_{\tilde{\mathcal{V}}}$” on the right side of the inequality

Figure 1. A schematic representation of the submanifold geometry of $\mathcal{H}$, an isometrically embedded section of a sphere in the ambient space $\mathcal{M}$, which is nonpositively curved.
stands for the length with respect to $\iota^*_{\nabla^*_{\mathcal{M}g}}$. In particular, by Section 4.2,

$$d_{\tilde{V}}(\mu_1, \mu_2) \leq \sqrt{\frac{n}{2}} d_E(\mathcal{M}_{\mu_1}, \mathcal{M}_{\mu_2}). \quad (21)$$

Let $\omega_1$ and $\omega_2$ denote cohomologous Kähler forms with volume forms $\mu_1$ and $\mu_2$. Combining (15), Proposition 4.4, and (21), we have

$$d_C(g_{\omega_1}, g_{\omega_2}) = d_{\tilde{V}}(\mu_1, \mu_2) \leq \frac{\pi \sqrt{n}}{2 \sqrt{2}} d_{\tilde{V}}(\mu_1, \mu_2) \leq \frac{\pi \sqrt{n}}{4} d_E(\mathcal{M}_{\mu_1}, \mathcal{M}_{\mu_2}) \leq \frac{\pi \sqrt{n}}{4} d_E(g_{\omega_1}, g_{\omega_2}),$$

which is the required upper bound on $d_C$. This concludes the proof, since the lower bound follows from Proposition 2.1. \qed

5. The completion of $\mathcal{H}$. In this section, we use the equivalence of $d_{\tilde{V}}$ and $d_V$ to first determine the completion of $(\mathcal{V}, d_{\tilde{V}})$ on a general Riemannian manifold (by “completion” we will always mean the metric completion). From this, we obtain a simple criterion for the convergence of metrics in $\mathcal{H}$ with respect to $d_C$. By using recent deep results from pluripotential theory, this then gives a description of the completion of $(\mathcal{H}, d_C)$. It can be viewed as giving a geometric description of a subset of the class of functions $\mathcal{E}(M, \omega)$ (to be defined below).

The completions of $\tilde{V}$ and $V$ can be quickly obtained using the map $\Phi$ defined in the last section. First, though, we recall several elementary facts from functional analysis. Let $(X, \mu)$ be a measure space with $\mu(X) < \infty$.

**Definition 5.1.** A collection $F$ of measurable functions on $X$ is called uniformly integrable if, for each $\epsilon > 0$, there exists $t \geq 0$ such that for all $f \in F$,

$$\int_{\{x \in X : |f(x)| \geq t\}} |f(x)| \, d\mu(x) < \epsilon.$$  

**Lemma 5.2.** (Vitali’s Convergence Theorem; [55, Theorem 8.5.14], [40, (13.38)]) A sequence $\{f_k\}$ in $L^p(X, \mu)$ converges to $f \in L^p(X, \mu)$ if and only if $f_k$ converges to $f$ in measure and $\{\|f_k\|^p : k \in \mathbb{N}\}$ is uniformly integrable.

The next lemma is a simple consequence of Vitali’s Convergence Theorem, but we include its short proof for completeness.

**Lemma 5.3.** A sequence of nonnegative functions $\{f_k\}$ converges to $f$ in $L^2(X, \mu)$ if and only if $f_k^2$ converges to $f^2$ in $L^1(X, \mu)$.

**Proof.** Suppose $f_k \to f$ in $L^2(X, \mu)$. Then by Lemma 5.2, $f_k \to f$ in measure, and $\{|f_k|^2\}$ is uniformly integrable. Now, consider the sequence $f_k^2$. Clearly $f_k^2 \to f^2$ in measure. Also, $\{|f_k|^2\}$ is uniformly integrable, because it is equal to the set $\{|f_k|^2\}$. So by Lemma 5.2, $f_k^2$ converges to $f^2$ in $L^1(X, \mu)$. The converse direction...
follows in precisely the same way (and is also where nonnegativity of the functions is required).

With these preliminaries, we can determine the completions of $\mathcal{V}$ and $\tilde{\mathcal{V}}$.

**Theorem 5.4.** Fix $\mu_0 \in \tilde{\mathcal{V}}$. Let $\tilde{\mathcal{V}}_0$ denote the space of all nonnegative, measurable sections of $\bigwedge^{2n} T^* M$. That is, $\tilde{\mathcal{V}}_0$ consists of all tensor fields represented in local coordinates by $f \, dx_1 \wedge \cdots \wedge dx_{2n}$, where $f \geq 0$ is a measurable, locally defined function; or globally by $F \mu_0$, where $F \geq 0$ is a measurable, globally defined function. Then the metric completion of $(\tilde{\mathcal{V}}, d_{\tilde{\mathcal{V}}})$ is given by

$$
(\tilde{\mathcal{V}}, d_{\tilde{\mathcal{V}}}) \cong \left\{ \mu \in \tilde{\mathcal{V}}_0 : \frac{\mu}{\mu_0} \in L^1(M, \mu_0) \right\},
$$

i.e., the $L^1$ completion of $\tilde{\mathcal{V}}$. (Here, as usual, we identify elements that agree up to a $\mu$-nullset.) We also have

$$
(\mathcal{V}, d_{\mathcal{V}}) \cong \left\{ \mu \in (\tilde{\mathcal{V}}, d_{\tilde{\mathcal{V}}}) : \int_M \mu = V \right\}.
$$

Given an element $\mu$ and a sequence $\{\mu_k\}$ in $(\tilde{\mathcal{V}}, d_{\tilde{\mathcal{V}}})$ (resp. $(\mathcal{V}, d_{\mathcal{V}})$), $\{\mu_k\}$ converges to $\mu$ if and only if $\int_M |\mu - \mu_k| \to 0$.

**Proof.** We will prove the results of the theorem for $(\tilde{\mathcal{V}}, d_{\tilde{\mathcal{V}}})$; the results for $(\mathcal{V}, d_{\mathcal{V}})$ then follow directly from Proposition 4.4.

The completion of $\Phi(\mathcal{V})$ is given by

$$
\Phi(\mathcal{V})^{L^2(M, \mu_0)} = \left\{ F \in L^2(M, \mu_0) : F \geq 0 \mu_0\text{-a.e.} \right\}.
$$

Thus, the completion of $(\tilde{\mathcal{V}}, d_{\tilde{\mathcal{V}}})$ can be isometrically identified with the image of this set under the map $\Phi^{-1}$, where we formally extend $\Phi$ and $\Phi^{-1}$ by the same algebraic formulas to nonnegative forms and functions, respectively.

Now, from Lemma 5.3, and because $\Phi(\mu)^2 = 4\mu/\mu_0$ for all $\mu \in \tilde{\mathcal{V}}$, it follows that the completion of and convergence in $(\tilde{\mathcal{V}}, d_{\tilde{\mathcal{V}}})$ are those of $L^1(M, \mu_0)$, where we identify $2n$-forms with functions via $\Phi$. The statements of the theorem follow.

Using this, and the isometry between $(\mathcal{V}, d_{\mathcal{V}})$ and $(\mathcal{H}, d_{C})$, we get the following corollary—a very simple criterion for convergence with respect to the Calabi metric.

**Corollary 5.5.** A sequence $\{g_k\} \subset \mathcal{H}$ converges to $g \in \mathcal{H}$ with respect to $d_C$ if and only if $d_{V_{g_k}} \to d_{V_g}$ in the $L^1$ sense; i.e.,

$$
\int_M |dV_g - dV_{g_k}| \to 0.
$$
Note that this convergence result improves upon that in the ambient space $(\mathcal{M}, d_E)$ as given in [23, Theorem 4.15]. In that result, it is required in addition that the sequence $\{g_k\} \subset \mathcal{M}$ converges to $g \in \mathcal{M}$ in measure (when this is defined in a suitable sense). This extra assumption is essential in the ambient space—where there are, in contrast to $\mathcal{H}$, many metrics inducing the same volume form.

Since $(\mathcal{V}, d_{\mathcal{V}})$ and $(\mathcal{H}, d_C)$ are isometric metric spaces, Theorem 5.4 determines the completion of $\mathcal{H}$ with respect to $d_C$. However, at the moment this is only abstractly, and not on the level of metrics. To describe $(\mathcal{H}, d_C)$ in terms of metrics, it is necessary to appeal to generalizations of the Calabi-Yau Theorem that give results about the domain and image of the Monge-Ampère operator, as we now briefly elaborate.

Let

$$\text{PSH}(M, \omega) := \{ \varphi \in L^1(M, \omega^n) : \omega + \sqrt{-1} \partial \bar{\partial} \varphi \geq 0, \, \varphi \text{ is upper semi-continuous} \}.$$ 

For $\varphi \in \text{PSH}(M, \omega) \cap C^2(M)$ let $\text{MA}(\varphi) := (\omega + \sqrt{-1} \partial \bar{\partial} \varphi)^n$ denote the Monge-Ampère operator. Much work has gone into understanding what the largest subset of $\text{PSH}(M, \omega)$ is to which $\text{MA}$ can be extended in a meaningful way. Bedford and Taylor were able to define $\text{MA}$ on $\text{PSH}(M, \omega) \cap L^\infty(M)$, and showed that thus defined, it is continuous under decreasing sequences [4, 5]. Recently, Guedj-Zeriahi showed that $\text{MA}$ can be further extended to

$$\mathcal{E}(M, \omega) := \left\{ \varphi \in \text{PSH}(M, \omega) : \lim_{j \to \infty} \int_{\{\varphi \leq -j\}} (\omega + \sqrt{-1} \partial \bar{\partial} \text{max}\{\varphi, -j\})^n = 0 \right\},$$

maintaining continuity under decreasing sequences in $\text{PSH}(M, \omega) \cap L^\infty(M)$ [38]. We note that this recent development builds upon the work of many authors, and we refer to [28, 38] for a historical overview and references.

The class $\mathcal{E}(M, \omega)$ is also important since, by other recent results, a generalized version of the Calabi-Yau Theorem holds for it. To state these results we recall that a pluripolar set is by definition a subset $A \subset M$ for which there exists a function $\varphi \in \text{PSH}(M, \omega)$ such that $A \subset \{\varphi = -\infty\}$. Guedj-Zeriahi proved that if $\mu$ is a nonnegative Borel measure on $M$ that vanishes on all pluripolar sets, then there exists $\varphi \in \mathcal{E}(M, \omega)$ satisfying $\omega^n_\varphi = \mu$, and Dinew showed that such a $\varphi$ is unique up to a constant within $\mathcal{E}(M, \omega)$ [38, 28].

Returning to our previous discussion, we obtain the following description of the completion of $(\mathcal{H}, d_C)$. Fix a smooth volume form $\mu_0 \in \mathcal{V}$.

**THEOREM 5.6.** The metric completion of $(\mathcal{H}, d_C)$ is given by

$$\overline{(\mathcal{H}, d_C)} \cong \{ \varphi \in \mathcal{E}(M, \omega) : \omega^n_\varphi \text{ is absolutely continuous with respect to } \mu_0 \text{ and } \omega^n_\varphi/\mu_0 \in L^1(M, \mu_0) \},$$

and is a strict subset of $\mathcal{E}(M, \omega)$. 

Proof. Let $\nu \in (V, d_V)$ represent an element of the completion. According to Corollary 5.5, $\nu/\mu_0 \in L^1(M, \mu_0)$. In particular, $\nu$ is absolutely continuous with respect to $\mu_0$, and so any $\mu_0$-nullset is a $\nu$-nullset. (Here, we regard both $\mu_0$ and $\nu$ as Borel measures.) Since locally in $\mathbb{C}^n$, pluripolar sets are of Lebesgue measure zero (and hence are contained in a Borel nullset [63, 11.11(d)]), it follows that pluripolar sets are $\mu_0$-nullsets [7, Section 3.1].

Thus, by Theorem 5.4 and the aforementioned results of Dinew and Guedj-Zeriahi, it follows that we have an isomorphism

$$\overline{(H, d_C)} \cong \{\omega_\varphi \in \mathcal{E}(M, \omega) : \sup \varphi = 0, \omega_\varphi^n/\mu_0 \in L^1(M, \mu_0)\}.$$ 

The inclusion

$$\overline{(H, d_C)} \subset \{\omega_\varphi : \varphi \in \mathcal{E}(M, \omega)\}$$

is strict, since measures that charge $\mu_0$-nullsets that are not pluripolar are still in the image of $\mathcal{E}(M, \omega)$ under MA (such examples exist, cf. [41, Section 3], [37, Section 5]).

We remark that it would be interesting to understand the regularity properties of the subclass $\overline{(H_\omega, d_C)} \subset \mathcal{E}(M, \omega)$.

Remark 5.7. The simpler geodesic completion of $(H, d_C)$ can also be computed. From Section 4.3, any unit-speed geodesic emanating from $g \in H$ will satisfy

$$dV_{g(t)} = dV_g \left(G \sqrt{V} \sin \left(\frac{1}{2} t/\sqrt{V}\right) + \cos \left(\frac{1}{2} t/\sqrt{V}\right)\right)^2$$

for some $G \in C^\infty(M)$ with $\int_M G dV_g = 0$ and $\int_M G^2 dV_g = 1$. Thus, the geodesic completion can be identified with metrics whose volume form is smooth, nonnegative and of mass $V$. This is because $G$ changes sign and so for some maximal time $T \in (0, \pi \sqrt{V})$, the term in parentheses above will vanish. It then follows by the work of Kołodziej [41, 42, 43] that there exists a unique $\varphi \in \text{PSH}(M, \omega) \cap L^\infty(M)$ that, moreover, is Hölder continuous, such that $\omega_\varphi^n = n! dV_g(T)$. In general $\varphi$ will not be $C^2$ (but see [8] for some additional regularity statements).

6. Ricci flow, distance, and stability. An interesting problem is to understand the relation of the Ricci flow to the geometry of $(M, d_E)$. On the other hand, a problem in Kähler geometry, often referred to as the Yau-Tian-Donaldson conjecture, is to characterize the existence of Kähler-Einstein metrics in terms of some algebraic or analytic notions of “stability” (the specification of the appropriate notion being part of the problem). Our purpose in this section is twofold. First, we define an analytic stability notion and prove that it gives a new characterization of
Kähler-Einstein Fano manifolds. Second, we derive new conditions under which the Kähler-Ricci flow converges.

6.1. Calabi-Ricci stability and existence of Kähler-Einstein metrics. In the context of the Yau-Tian-Donaldson conjecture, a number of algebraic notions of stability have been introduced, starting with Tian’s notion of K-stability [73], subsequently refined by Donaldson [30] and others. At present it is still an open problem to show that such algebraic notions imply the existence of a Kähler-Einstein metric, although much progress has been made (see, e.g., the surveys [54, 70]).

The first analytic stability criterion for the existence of a Kähler-Einstein metric on a Fano manifold was obtained by Tian [73] in terms of the properness of the Mabuchi K-energy [44], and this has later been extended to other energy functionals [57, 68]. Another, conjectural, notion of stability is that of “geodesic stability,” due to Donaldson [29].

**Conjecture 6.1.** (See [29]) The following are equivalent:

(i) There exists no constant scalar curvature metric in $\mathcal{H}$.

(ii) There exists a geodesic ray in $(\mathcal{H}_\omega, g_M)$ along which the derivative of the K-energy is negative.

(iii) There exists a geodesic ray as in (ii) starting at any point in $\mathcal{H}$.

For the remainder of this section, we assume $(M, J, \omega)$ is a Fano manifold (i.e., its first Chern class $c_1(M)$ is positive), with $[\omega] = c_1(M)$. In stark contrast to the geodesic flow of $\mathcal{M}$, which instantly leaves $\mathcal{H}$ (Remark 4.2), Hamilton’s Ricci flow on $\mathcal{M}$ (which we will always assume is volume normalized), defined by

\[
\frac{\partial \omega(t)}{\partial t} = -\text{Ric} \omega (t) + \omega (t), \quad \omega (0) = \omega \in \mathcal{H},
\]

preserves $\mathcal{H}$ [39] and exists for all $t > 0$ [14].

We introduce the following notion of analytic stability.

**Definition 6.2.** We say that $(M, J)$ is Calabi-Ricci unstable (or CR-unstable) if there exists a Ricci flow trajectory that diverges in $(\mathcal{H}, d_C)$. Otherwise, we say $(M, J)$ is CR-stable.

Note that the derivative of the K-energy is negative along the Ricci flow (22). Also, as will follow from the results below, the definition of CR-instability can be equivalently phrased in terms of the existence of a Ricci flow of infinite $d_C$-length. Thus, this definition stands in precise analogy to Donaldson’s geodesic stability, with Ricci flow paths and $d_C$-distance taking the place of $g_M$-geodesic rays and $d_M$-distance.

Definition 6.2 is motivated by the following result, that similarly stands in clear analogy to Donaldson’s conjecture.
THEOREM 6.3. A Fano manifold \((M, J)\) is CR-stable if and only if it admits a Kähler-Einstein metric. Moreover, if it is CR-unstable then any Ricci flow trajectory diverges in \((\mathcal{H}, \mathcal{d}_C)\).

Proof. First, assume that a Kähler-Einstein metric exists. Then, a theorem of Perelman and work of Chen-Tian, Tian-Zhu and Phong-Song-Sturm-Weinkove implies that any Ricci flow (22) will converge to a Kähler-Einstein metric exponentially fast in any \(C^k\) norm [18, 74, 52]. In particular, the metrics along the flow are uniformly equivalent, and \(|\frac{\partial \omega}{\partial t}|_{\omega(t)} < C_1 \frac{\partial \omega}{\partial t}|_{\omega} < C_1 e^{-C_2 t}\), for some \(C_1, C_2 > 0\) independent of \(t\). Hence the \(\mathcal{d}_C\)-length of \(\{\omega(t)\}_{t \geq 0}\) is finite and the flow converges in \(\mathcal{d}_C\).

Now, assume that \((M, J)\) admits no Kähler-Einstein metric. The flow (22) induces the Kähler-Ricci flow
\[
\omega_n^\phi(t) = \omega_n e^{f_\phi(t) + \psi(t)}, \quad \phi(0) = c_0,
\]
on \(\mathcal{H}_\omega\), where \(\sqrt{-1} \partial \bar{\partial} f_\phi = \text{Ric} \omega - \omega\) and \(\frac{1}{V} \int_M e^{f_\phi} \omega^n = 1\). The initial condition \(c_0\) is a certain constant uniquely determined by \(\omega\), fixed once and for all [18, 50].

Recall that the multiplier ideal sheaf associated to a function \(\phi \in \text{PSH}(M, \omega)\) is defined as the sheaf \(I(\phi)\) defined for each open set \(U \subset M\) by local sections
\[
I(\phi)(U) = \{h \in \mathcal{O}_M(U) : |h|^2 e^{-\phi} \in L^1_{\text{loc}}(M, \omega^n)\}.
\]
Such a sheaf is called proper if it is neither zero nor the structure sheaf \(\mathcal{O}_M\), and is called a Nadel sheaf whenever there exists \(\epsilon > 0\) such that \((1 + \epsilon) \phi \in \text{PSH}(M, \omega)\).

We recall the following result, describing the limiting behavior of the Kähler-Ricci flow in terms of a Nadel multiplier ideal sheaf.

THEOREM 6.4. [59, Theorem 1.3] Let \((M, J)\) be a Fano manifold not admitting a Kähler-Einstein metric. Let \(\gamma \in (n/(n+1), 1)\) and let \(\omega \in \mathcal{H}\). Then there exists a subsequence \(\{\phi(t_j)\}_{j \geq 1}\) of solutions of (23) with \(\lim_{j \to \infty} t_j = \infty\), such that \(\phi(t_j) - \frac{1}{V} \int_M \phi(t_j) \omega^n\) converges in the \(L^1(M, \omega^n)\)-topology to \(\phi_\infty \in \text{PSH}(M, \omega)\) and \(I(\gamma \phi_\infty)\) is a proper Nadel multiplier ideal sheaf.

Now, fix some \(\gamma \in (n/(n+1), 1)\), and let \(I(\gamma \phi_\omega)\) be the Nadel sheaf constructed by Theorem 6.4. This sheaf cuts out a subscheme in \(M\) whose support, which we denote by \(S\), is a nonempty subvariety of positive codimension [47].

The following lemma is an analogue for the Ricci flow of a well-known fact for the continuity method [71, 48].

LEMMA 6.5. Let \(K \subset M \setminus S\) be a compact set, and let \(\{\phi(t_j)\}_{j \geq 1}\) be as in Theorem 6.4. Then
\[
\lim_{j \to \infty} \int_K \omega_n^\phi(t_j) = 0.
\]
Proof. Given the Sobolev inequality along the Ricci flow [77, 78, 79] and Perelman’s estimates for the Kähler-Ricci flow [65, 74], the proof follows in the same way as the corresponding result for the continuity method [48, Proposition 4.1], and so we only outline the proof for completeness.

By (23) and Perelman’s estimate (see [18, 50, 65], or [59, Theorem 2.1(i)])

\[ |\dot{\varphi}(t)| < C, \]

it suffices to estimate \( \int_K e^{-\varphi(t_j)}\omega^n \). Fix \( \gamma \in (n/(n+1), 1) \). From [59, (16), (21)]

\[ \lim_{j \to \infty} \int_M e^{-\gamma(\varphi(t_j) - \sup \varphi(t_j))} = \infty, \]

It follows then from [71, Theorem 3.1] that

\[ \int_K e^{-\gamma(\varphi(t_j) - \sup \varphi(t_j))}\omega^n < C \]

for some uniform constant \( C \) depending on \( K \) (here and in the statement it might be necessary to take a subsequence, but we omit this from the notation). By (26),

\[ \int_K e^{-\varphi(t_j)}\omega^n \leq Ce^{-\gamma \sup \varphi(t_j) - (1-\gamma)\inf \varphi(t_j)}. \]

To conclude it suffices to use the Harnack inequality \( -\inf \varphi(t) \leq n \sup \varphi(t) + C \).
Note that in [59, (15)] the Harnack inequality \( -\frac{1}{V} \int_M \varphi(t)\omega^n_{\varphi(t)} \leq n \sup \varphi(t) + C \) is proved, and that as in [71, 72] (cf. [66]) one can then deduce from it the previous inequality. Alternatively, the former follows from the latter via a Green’s function estimate (cf. [50, p. 626], [59, p. 5847]). Indeed, \( -\inf \varphi(t) \leq -\frac{1}{V} \int_M \varphi(t)\omega^n_{\varphi(t)} + nA_t \), where \( -A_t \) is the minimum of the Green function of \((M, \omega(t))\), normalized to have average zero (see [59, p. 5845]). As shown by Bando-Mabuchi [3, (3.4)], the heat kernel estimate of Cheng-Li [20, (2.9)] implies such an estimate as soon as one has uniform Poincaré and Sobolev inequalities (assume \( n > 1 \), as \( n = 1 \) is treated in [59, p. 5847]), and these indeed hold, by the results of Perelman, Ye and Zhang [59, Theorem 2.1]. Here we note (as pointed out to us by V. Tosatti) that the classical “weighted” Poincaré inequality [59, Lemma 2.3] implies the usual Poincaré inequality by a straightforward argument using the Cauchy-Schwarz inequality.

It follows from this lemma that \( \varphi_\infty \notin E(M, \omega) \), and hence by Theorem 5.6 that the limit point \( \omega_{\varphi_\infty} \) of the Ricci flow is not in \( (\mathcal{H}, d_C) \). Thus, the Ricci flow with initial condition \( \omega(0) = \omega \) does not converge with respect to \( d_C \), and so \((M, J)\) is CR-unstable. Since \( \omega \in \mathcal{H} \) was arbitrary, this concludes the proof.

Figure 2 illustrates the possible types of Kähler-Ricci flow trajectories on CR-stable and CR-unstable complex manifolds.
Figure 2. The figure schematically depicts various possibilities for a Kähler-Ricci flow trajectory in $H_\omega$. In the case that $(M,J)$ does not admit a Kähler-Einstein metric, then by Theorem 6.3, $(M,J)$ is CR-unstable. In this case, the theorem rules out the solid and dotted paths, which converge to points in the completion and have infinite and finite length, respectively. The only allowed type is the divergent (dashed) path.

Remark 6.6. As pointed out to us by S. Boucksom, as far as the proof of Theorem 6.3 is concerned, Lemma 6.5 may be substituted with the following result of Guedj-Zeriahi [38, Corollary 1.8]: any element $\phi \in \mathcal{E}(M,\omega)$ has zero Lelong number at every point of $M$. Therefore, by Skoda’s theorem [67], $\int_M e^{-p\phi} \omega^n$ is finite for each $p > 0$. It follows then that $\varphi_\infty \notin \mathcal{E}(M,\omega)$. Note, however, that Perelman’s estimates are still needed in order to prove Theorem 6.4 in [59]. We preferred Lemma 6.5 since it, in passing, gives more precise information about the volume concentration on the singularity set of $\varphi_\infty$ (i.e., it shows that all the mass is concentrated there).

6.2. Analytic criteria for the convergence of the Kähler-Ricci flow. The goal of this subsection is to derive new analytic criteria for the convergence of the Kähler-Ricci flow.

First, we show that $d_C$-convergence of the flow implies an a priori $C^0$ estimate. Such an estimate does not follow from Kołodziej’s deep results [41, 42] that require slightly stronger control on the volume form than $L^1$.

**Theorem 6.7.** Assume that the Kähler-Ricci flow trajectory (23) $d_C$-converges, i.e., assume that $\{\omega^n_{\varphi(t)}\}_{t \geq 0}$ converges in $L^1$ (see Corollary 5.5). Then there exists a constant $C > 0$ independent of $t$ such that $||\varphi(t)||_{C^0(M)} < C$.

**Proof.** The proof can be extracted from the proofs of Theorem 6.3 and [59, Theorem 1.3], but we summarize it below for the reader’s convenience.

By [59, (15), (16), (20)], the following a priori estimates hold:

$$\frac{1}{V} \int_M -\varphi(t) \omega^n_{\varphi(t)} \leq \frac{n}{V} \int_M \varphi(t) \omega^n,$$

$$\sup \varphi(t) \leq \frac{1}{V} \int_M \varphi(t) \omega^n + C,$$
\[-\inf \varphi(t) \leq \frac{C}{V} \int_M -\varphi(t) \omega^n \varphi(t),\]

Let \( \gamma \in (n/(n+1), 1) \). These inequalities, combined with Perelman’s estimate (24), imply that a uniform bound on \( \int_M e^{-\gamma(\varphi(t)) - \frac{1}{\gamma} \int_M \varphi(t) \omega^n} \omega^n \) leads to a uniform estimate \( \|\varphi(t)\|_{C^0(M)} < C \).

Hence, supposing that \( \{\|\varphi(t)\|_{C^0(M)}\}_{t \in [0, \infty)} \) is unbounded, it follows by [59] that one can find a subsequence \( \{\varphi(t_j)\}_{j \geq 1} \) as in Theorem 6.4, which by Lemma 6.5 satisfies \( \lim_{j \to \infty} \varphi(t_j) = \varphi_\infty \in \text{PSH}(M, \omega) \setminus \mathcal{E}(M, \omega) \). Thus, one can conclude as in the proof of Theorem 6.3. □

As is well-known, once the crucial \( C^0 \) estimate is established for the flow, higher derivative estimates then follow [76, 14, 18, 74, 50, 49], and one obtains smooth convergence up to automorphisms [18, 50, 74]. On the other hand, a theorem of Phong et al. [52] shows that the latter convergence implies exponential convergence of the original flow. To summarize, we have the following statement that shows that the very weak notion of \( d_C \)-convergence (Corollary 5.5) implies such strong convergence.

**COROLLARY 6.8.** If \( \{\omega^n_{\varphi(t)}\}_{t \geq 0} \) converges in \( L^1 \), i.e., if the Kähler-Ricci flow trajectory (22) \( d_C \)-converges in \( (\mathcal{H}, d_C) \), then it converges smoothly (exponentially fast).

In the ambient space \( \mathcal{M} \), it seems to be difficult to find (nontrivial) settings under which \( d_E \)-convergence implies a more synthetic-geometric notion of convergence, such as Cheeger-Gromov, Lipschitz, or even Gromov-Hausdorff. Indeed, examples show that \( d_E \)-convergence is too weak to control the geometry in any way (for further discussion, see [23, Section 4.3], [24, Sections 1 and 5] and cf. [1]), and one can use Theorem 4.5 and Corollary 5.5 to construct examples that show this is also the case for \( \mathcal{H} \subset \mathcal{M} \). The previous corollary thus provides a rather striking instance of such a setting.

Let \( s(t) := t\varphi(t) \text{Ric} \omega(t) \) denote the scalar curvature along the flow. We also record the following weaker version of Corollary 6.8.

**COROLLARY 6.9.** The Kähler-Ricci flow (22) converges smoothly if and only if
\[
\|s - n\|_{L^1(\mathbb{R}^+, L^2(M, \omega(t)))} < \infty, \tag{27}
\]
i.e., if and only if its trajectory has finite \( d_C \)-length.

This improves a result of Phong et al. [52], where \( \|s - n\|_{L^1(\mathbb{R}^+, C^0(M))} < \infty \) is assumed instead.

**Remark 6.10.** The problem of finding conditions for the convergence of the Kähler-Ricci flow has been studied by many authors. Many of these results involve assuming that the curvature tensor is uniformly bounded along the flow, combined
with some further analytic and algebraic conditions: Phong et al. assumed uniform curvature bounds, the vanishing of the Futaki invariant and a certain stability condition on the complex structure [51, 53]; Székelyhidi [69] assumed that the K-energy is bounded from below and the manifold is K-polystable (together with the curvature bounds), and Tosatti [75] then replaced the K-energy bound by assuming asymptotic Chow semistability. Other results include: convergence when the K-energy is bounded below and the first eigenvalue of the Laplacian on \( T^{1,0}_{M} \) is uniformly positive [52] (cf. [46, 80]), and convergence when the evolving volume forms satisfy \( \omega(t)^n \geq C\omega^n \) [49]. The list above is by no means exhaustive and we refer to these articles for further references.

7. Some remarks and further study. We end with some remarks and indicate some possible directions for future study.

7.1. Angles between \( \mathcal{H} \) and conformal classes. Lemma 2.2 is, geometrically, a statement that \( \mathcal{H} \) intersects conformal classes in \( \mathcal{M} \) at a constant angle. For \( g \in \mathcal{H} \), we consider the conformal class \( \mathcal{P}g \), where we recall that \( \mathcal{P} \) denotes the group of smooth positive functions on \( M \), acting on \( \mathcal{M} \) by pointwise multiplication. Tangent vectors to \( \mathcal{P}g \) are of the form \( \rho g \) for \( \rho \in C^\infty(M) \).

Let \( h \in T_g \mathcal{H} \) and \( k \in T_g(\mathcal{P}g) \) with \( |h|_E = |k|_E = 1 \) be given. Denote the pure trace part of \( h \) by \( h_T := \frac{1}{n} \text{tr}(g^{-1}h)g \), and the traceless part of \( h \) by \( h_0 := h - h_T \). On the one hand, the decomposition \( h = h_0 + h_T \) is orthogonal, so we have \( (h, h)_E = |h|_E^2 = |h_0|_E^2 + |h_T|_E^2 \). On the other hand, by Lemma 2.2, \( |h|_E^2 = n|h_0|_E^2 \), or \( |h_T|_E = n^{1/2} \). We then have \( (h, k)_E = (h_T, k)_E \leq |h_T|_E |k|_E = n^{1/2} \), with equality if and only if \( k = \sqrt{n}h_T \). Hence, the angle between \( \mathcal{H} \) and \( \mathcal{P}g \) is \( \cos^{-1}(n^{-1/2}) \), independently of \( g \in \mathcal{H} \). In particular, in the case \( n = 1 \), the angle is 0—reflecting the fact that \( \mathcal{H} \) is contained within a single conformal class. For \( n = 2 \), the angle is \( \pi/4 \), and the tangent space to the space of Kähler metrics lies “halfway” between those of \( \mathcal{P}g \) and \( \mathcal{M}_{\mu} \), and as \( n \) grows the solution of the Calabi-Yau equation diverges more and more from a conformal transformation.

Note also that given the geometric description of \( \mathcal{V} \subset \hat{\mathcal{V}} \) in Section 4.4, the space \( (\mathcal{V}, d_{\nu}) \) can be readily shown to have constant positive curvature, both in the sense of the Riemannian sectional curvature, and well as in the synthetic sense of Alexandrov.

To summarize, \( \mathcal{H} \) is a section of a sphere isometrically embedded in \( \mathcal{M} \), and each conformal class \( \mathcal{P}g \) is an incomplete, isometrically embedded Euclidean domain. Since each of these flat spaces intersects the spherical section at the same angle, we expect the former to all meet in the completion of \( (\mathcal{M}, d_E) \)—as indeed they do, at the point represented by the zero tensor (this follows from [25, Proposition 4.1], which implies that \( d_E(\lambda g_1, \lambda g_2) \to 0 \) as \( \lambda \to 0 \) for any \( g_1, g_2 \in \mathcal{M} \).

7.2. Other metrics on \( \mathcal{H} \) and on \( \mathcal{M} \). As mentioned in the introduction, one may consider different metrics on \( \mathcal{H}_\omega \). Currently of greatest interest, perhaps,
is the Mabuchi metric $g_M$ (3), in part due to its intimate relation to several important problems in Kähler geometry concerning the existence of canonical metrics [17, 19, 29, 45]. Geodesics of $g_M$ are solutions of a homogeneous complex Monge-Ampère equation (HCMA). At present it is not known how to construct the exponential map of $g_M$, or equivalently how to solve the Cauchy problem for the HCMA. This problem seems quite difficult due to issues of ill-posedness, and it seems plausible that most directions will not exponentiate to geodesics in $\mathcal{H}$. We refer to [60, 61, 62] for more precise statements. Even the more standard Dirichlet problem of constructing a geodesic between two given metrics is not completely understood, although progress has been made by Chen and Tian toward a partial regularity theory [19], and this has been used by Chen to study the geodesic distance induced by $g_M$ [17].

Thus, while geodesics in $g_C$ are not completely explicit, they are still considerably simpler to understand, and it would be of interest to compare the Mabuchi geometry to that of Calabi. Since the length of smooth minimizing geodesics in $(\mathcal{H}_\omega, g_M)$ need not be uniformly bounded (e.g., a one-parameter family of automorphisms induces a geodesic line), one may only ask whether the Mabuchi distance dominates the Calabi distance. An analogous problem would be to compare the Donaldson metric [32], which is an analogue of the Mabuchi metric on $V$, to the metric $g_V$ induced from $(\mathcal{M}, g_E)$. Another metric on $\mathcal{H}$ can be defined by the $L^2$ norm of the gradient (for which the Kähler-Ricci flow is a gradient flow) and again, it would be interesting to compare it to $g_C$ and $g_M$. Similarly, one could also consider metrics involving more derivatives than $d_C$, and these can be induced by metrics on $\mathcal{M}$. Stronger metrics might lead to notions of convergence that could be of use in various geometric settings, where the rather weak notion of convergence associated with $g_E$ is often insufficient (cf. [23]). We refer to our follow-up paper [26] for some initial progress.

**Remark 7.1.** We were informed by Eugenio Calabi that he considered the gradient metric several decades ago. We found it not difficult to compute its geodesic equation and curvature (and we were informed by John Lott that he has carried out these computations out as well), however it seems to be an interesting open problem to solve the geodesic equation (except when $n = 1$ and then the metric is manifestly flat: $\langle \nu, \eta \rangle |_\varphi = \int \sqrt{-1} \partial \bar{\partial} \nu \wedge \bar{\partial} \eta$ with the right hand side independent of $\varphi$; cf. [13] for a different proof of this fact) and compute the corresponding metric completion.

**7.3. Other submanifolds of metrics.** As noted in Section 2, the Calabi metric, intrinsically defined in terms of Kähler potentials, is obtained by restriction of the $L^2$ metric on $\mathcal{M}$. One possible application of this is that one can define natural metrics on other submanifolds of $\mathcal{M}$, e.g., spaces of almost-Kähler metrics, where the $\partial \bar{\partial}$-lemma is absent. The question of whether the induced geometry can
be understood successfully is then essentially equivalent to whether a version of the
Calabi-Yau Theorem exists in those settings, itself a topic of current research [31].

7.4. Kählerian and Riemannian extensions of CR-stability. Definition
6.2 can be extended to an arbitrary polarized Kähler manifold, for instance, by
considering generalizations of the Kähler-Ricci flow whose stationary points are
constant scalar curvature or extremal metrics (see [59, Section 3] and [58, Sec-
tions 7–9]). A natural question is whether a result corresponding to Theorem 6.3
holds in these more general cases. We also remark that multiplier ideal sheaves
can also be constructed for the Ricci iteration, a discrete version of the Kähler-
Ricci flow introduced in [58], for which a similar, and likely equivalent, notion of
stability may be defined.

On the other hand, by the equivalence of $d_C$ and $d_E$ (Theorem 4.5) and the fact
that the (volume normalized) Ricci flow preserves $\mathcal{H}$, the notion of CR-stability
for a Fano manifold is a purely Riemannian one, i.e., it can be stated in terms of
$\left(\mathcal{M}, d_E \right)$. Thus this notion can be extended naturally to any Riemannian manifold.
It is then an interesting problem whether Theorem 6.3 has a suitable analogue for
the Ricci flow and Einstein metrics in this more general setting. Other questions
naturally arise as well: for instance, can the singularity type of a Ricci flow be
detected from the metric space geometry of $\left(\mathcal{M}, d_E \right)$?

DEPARTMENT OF MATHEMATICS, STANFORD UNIVERSITY, STANFORD, CA 94305
E-mail: brian.clarke@uni-muenster.de, yanir@member.ams.org

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