THE LEVEL CROSSINGS OF RANDOM SUMS

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Abstract. Let \( \{a_j\}_{j=0}^N \) and \( \{b_j\}_{j=0}^N \) be sequences of mutually independent and identically distributed, real, normal random variables with mean zero and variances \( \{\sigma^2_a\}_{j=0}^N \) and \( \{\sigma^2_b\}_{j=0}^N \). Let \( \{f_j\}_{j=0}^N \) be a sequence of basis functions that are entire and real-valued on \( \mathbb{R} \). For studying the number of times a random sum crosses a complex level, we establish an exact formula for the expected intensity of the complex roots of \( \sum_{j=0}^N (a_j + ib_j)f_j(z) = K_1 + iK_2 \), where \( K_1 \) and \( K_2 \) are constants independent of \( z \), and apply this formula to a standard Brownian motion.

1. Introduction

An exact formula for the expected number of real roots of a random polynomial was obtained by Kac \[12\] under independent and identically distributed (i.i.d.), real, standard normal coefficients. For complex coefficients, Dunnage \[5, 6\] gave some estimates for the number of real roots. For complex roots, the expected intensity of roots was studied by Shepp and Vanderbei \[16\] for i.i.d., real, standard normal coefficients and generalized by Ibragimov and Zeitouni \[11\] for a wider class of distributions of coefficients. Relevant to these investigations is the work of Kostlan \[13\]. The expected intensity was dealt with, also, by Hammersley \[10\], Edelman and Kostlan \[7\], and Farahmand and Grigorash \[9\]. Vanderbei \[17\] generalized the work in \[16\] to random sums with entire functions that are real-valued on \( \mathbb{R} \) as basis functions. Motivated by the studies conducted by Vanderbei \[17\] and Farahmand \[8\], the present authors \[4\] obtained results on the level crossings of these random sums. The chief purpose of the present paper is to extend certain of these results.

In what follows, let \( \{a_j\}_{j=0}^N \) and \( \{b_j\}_{j=0}^N \) be sequences of mutually i.i.d., real, normal random variables defined on the complete probability space \( (\Omega, \mathcal{F}, \text{Prob}) \) with mean zero and variances \( \{\sigma^2_a\}_{j=0}^N \) and \( \{\sigma^2_b\}_{j=0}^N \). As per usual, \( \Omega \) is a set with generic elements \( \omega \), \( \mathcal{F} \) is a \( \sigma \)-field of subsets of \( \Omega \) and \( \text{Prob} \) is a probability measure on \( \mathcal{F} \). Assume all sub \( \sigma \)-fields contain all sets of measure zero. Let \( \{f_j\}_{j=0}^N \) be a sequence of analytic functions \( f_j(z) = u_j(x, y) + iv_j(x, y) \) for \( 0 \leq j \leq N \) and \( (x, y) \in \mathbb{R}^2 \) that are real-valued on \( \mathbb{R} \), so that \( f_j(z) = f_j(\overline{z}) \) for \( 0 \leq j \leq N \) and all \( z \in \mathbb{C} \). Define

\[
S(z) = \sum_{j=0}^N (a_j + ib_j)f_j(z).
\]

It is of interest to study the number of times that \( S \) crosses a complex level. If, for each compact subset \( T \) of \( \mathbb{C} \), \( N^T_K(T) \) denotes the random number of complex
roots, counted with multiplicity, in $T$ of $S$ in (1) that cross the complex level $K = K_1 + iK_2$, where $K_1$ and $K_2$ are constants independent of $z$, then from (3), with probability one, the expected intensity $h_K$ of the complex roots of 

$$S(z) = K$$

(2)

is given by

$$E(N_S^K(T)) = \int_T h_K(z) \, dz.$$  

(3)

The explicit derivation of $h_K$ constitutes the primary reason for studying the roots of (2). The main device for treating $h_K$ throughout $\mathbb{C}$ is the Rice formula, which provides a representation for the expected number of roots of certain random fields. This remarkable result can be summarized as follows (see [1, Theorem 11.2.3, Corollary 11.2.4, pp. 269-271] and [2, 3]).

**Theorem 1.** Let $Z: U \to \mathbb{R}^N$ be a random field, let $U$ be an open subset of $\mathbb{R}^N$ and let $u \in \mathbb{R}^N$ be a fixed point in the codomain. Assume the following conditions are satisfied with probability one:

(i) $Z$ is normal.

(ii) Almost surely the function $t \leadsto Z(t)$ is of class $C^1$.

(iii) For each $t \in U$, $Z(t)$ has a nondegenerate distribution—i.e., $\text{Var}(Z(t)) > 0$.

(iv) For each $u \in \mathbb{R}^N$, $\text{Prob}(\exists \ t \in U: Z(t) = u, \det(Z'(t)) = 0) = 0$.

If $N_u^Z(B)$ denotes the number of roots of $Z(t) = u$ that belong to the Borel subset $B \subset U$, then one has

$$E(N_u^Z(B)) = \int_B E(|\det(Z'(t))| \mid Z(t) = u) \, p_{Z(t)}(u) \, dt,$$

(4)

where $p_{Z(t)}(u)$ is the probability density function of $Z(t)$ at $u$. If $B$ is compact, then both sides of (4) are finite.

The function $Z$ in (4) is defined on $\mathbb{R}^N$. In our application, we need to find the real and complex roots of (2)—i.e., the real roots of $\text{Re}(S(x + iy)) = K_1$ and $\text{Im}(S(x + iy)) = K_2$ for $(x, y) \in \mathbb{R}^2$. The conditions (i)-(iv) are easy to check. Formula (4) is interesting. It shows that $h_K$, as defined by (3), can be expressed through a conditioned mean function of a quadratic form of i.i.d., real, normal random variables conditioned on certain linear combinations. This process can get very involved technically.

**Theorem 2.** Provided all the conditions imposed on $S$ in (1) and $T$ are satisfied, then for all integers $N > 1$ one has

$$h_K = \frac{1}{2\pi D_0} \exp \left( - \frac{K_1^2 Y_3 + K_2^2 Y_1 - 2K_1 K_2 Y_2}{2D_0} \right) \times \left\{ D_3 - \frac{|D_1|^2}{D_0} \left( \frac{Y_2 + Y_3}{D_0} - \frac{(K_1 Y_3 - K_2 Y_2)(K_1 (Y_2 + Y_3) - K_2 (Y_1 + Y_2))}{D_0^2} \right) \right. $$

$$- \left. \frac{|D_2|^2}{D_0} \left( \frac{Y_1 + Y_2}{D_0} - \frac{(K_1 Y_2 - K_2 Y_1)(K_1 (Y_2 + Y_3) - K_2 (Y_1 + Y_2))}{D_0^2} \right) \right.$$ 

$$+ \left. \frac{|D_1 + iD_2|^2}{D_0} \left( \frac{Y_2}{D_0} - \frac{(K_1 Y_3 - K_2 Y_2)(K_1 Y_2 - K_2 Y_1)}{D_0^3} \right) \right\},$$

where $Y_1, Y_2, Y_3$ are the roots counted with multiplicity.
where

\[ Y_1(z) = \sum_{j=0}^{N} (\sigma_{a_j}^2 u_j^2 + \sigma_{b_j}^2 v_j^2), \quad Y_2(z) = \sum_{j=0}^{N} (\sigma_{a_j}^2 - \sigma_{b_j}^2) u_j v_j, \]

\[ Y_3(z) = \sum_{j=0}^{N} (\sigma_{b_j}^2 u_j^2 + \sigma_{a_j}^2 v_j^2), \]

and

\[ D_0(z) = \sqrt{Y_1(z)Y_3(z) - Y_2^2(z)}, \quad D_1(z) = \sum_{j=0}^{N} (\sigma_{a_j}^2 u_j - i\sigma_{b_j}^2 v_j)(u_{jx} + iv_{jx}), \]

\[ D_2(z) = \sum_{j=0}^{N} (\sigma_{a_j}^2 u_j - i\sigma_{b_j}^2 v_j)(u_{jx} + iv_{jx}), \quad D_3(z) = \sum_{j=0}^{N} (\sigma_{a_j}^2 + \sigma_{b_j}^2)(u_{jx}^2 + v_{jx}^2). \]

In relation to the work in [4], let us first observe that when \( \sigma_{a_j}^2 = \sigma_{b_j}^2 = \sigma^2 \) for \( 0 \leq j \leq N \)

\[ Y_1(z) = Y_3(z) = \sigma^2 B_0(z), \quad Y_2(z) = 0, \]

and

\[ D_0(z) = \sigma^2 B_0(z), \quad D_1(z) = D_2(z) = \sigma^2 B_1(z), \quad D_3(z) = 2\sigma^2 B_2(z). \]

Then

\[ |D_1(z) + iD_2(z)|^2 = |D_1(z)|^2 + |D_2(z)|^2 = 2\sigma^4 |B_1(z)|^2. \]

The following result is obtained by using these substitutions in Theorem 2, factoring and simplifying.

**Corollary 1.** If \( \sigma_{a_j}^2 = \sigma_{b_j}^2 = \sigma^2 \) for \( 0 \leq j \leq N \), then for all integers \( N > 1 \) one has

\[ h_K = \frac{1}{\pi B_0} \exp \left( -\frac{K_1^2 + K_2^2}{2\sigma^2 B_0} \right) \left\{ B_2 - \frac{|B_1|^2}{B_0} \left( 1 - \frac{K_1^2 + K_2^2}{2\sigma^2 B_0} \right) \right\}, \]

where

\[ B_0(z) = \sum_{j=0}^{N} |f_j(z)|^2, \quad B_1(z) = \sum_{j=0}^{N} f_j(z)f'_j(z), \quad B_2(z) = \sum_{j=0}^{N} |f'_j(z)|^2. \]

Then, as a consequence of Corollary 1, when \( \sigma^2 \) is set to be one, Theorem 1 in [4] is recovered. Further, the following result follows from Theorem 2.
Corollary 2. For all vectors $K$ restricted to a circle of radius $K > 0$ and all integers $N > 1$, one has
\[
h_K = \frac{1}{2\pi D_0} \exp \left( -\frac{K^2(Y_1 - Y_2 + Y_3)}{2D_0^2} \right) \times \left\{ D_3 - \frac{|D_1|^2}{D_0} \left( \frac{Y_2 + Y_3}{D_0} - \frac{K^2(Y_2 - Y_3)(Y_1 - Y_2 - 1)}{D_0^3} \right) \right. \\
- \frac{|D_2|^2}{D_0} \left( \frac{Y_1 + Y_2}{D_0} - \frac{K^2(Y_1 - Y_2)(Y_2 - Y_3 + 1)}{D_0^3} \right) \\
+ \frac{|D_1 + iD_2|^2}{D_0} \left( \frac{Y_2}{D_0} - \frac{K^2(Y_1 - Y_2)(Y_2 - Y_3)}{D_0^3} \right) \right\}.
\]

A special case of Corollary 2 follows.

Corollary 3. When $K$ is the zero vector, for all integers $N > 1$ one has
\[
h_K = \frac{D_3^2 - |D_1|^2(Y_2 + Y_3) - |D_2|^2(Y_1 + Y_2) - |D_1 + iD_2|^2 Y_2}{2\pi D_0^3}.
\]

The proof of Theorem 2 in the spirit of the method credited to Ibragimov and Zeitouni, is presented in Section 2. Finally, in relation to the works of Rezakhah and Shemehsavar and Rezakhah and Soltani, an application of Theorem 2 entailing a Brownian motion is given in Section 3.

2. THE INTENSITY FUNCTION FOR MULTIVARIATE NORMAL COEFFICIENTS

The proof of Theorem 2 starts with the decomposition
\[ S(z) = X_1 + iX_2, \]
where
\[ X_1 = \sum_{j=0}^{N} (a_j u_j - b_j v_j), \quad X_2 = \sum_{j=0}^{N} (a_j v_j + b_j u_j). \]

If the column vector
\[ \mathbf{X} = (X_1, X_2)' \]
genuinely represents a two-dimensional random field, then for $z = x + iy$ the Jacobian matrix of $(x, y) \rightarrow (X_1, X_2)$ is
\[
\nabla \mathbf{X} = \begin{pmatrix}
\sum_{j=0}^{N} (a_j u_{jx} - b_j v_{jx}) & \sum_{j=0}^{N} (a_j v_{jx} + b_j u_{jx}) \\
\sum_{j=0}^{N} (-a_j v_{jx} - b_j u_{jx}) & \sum_{j=0}^{N} (a_j u_{jx} - b_j v_{jx})
\end{pmatrix}
\]
and
\[
\det(\nabla \mathbf{X}) = \sum_{j=0}^{N} (a_j^2 + b_j^2)(u_{jx}^2 + v_{jx}^2) + \sum_{j=0}^{N} \sum_{k=0}^{N} \sum_{k \neq j} \left( (a_j a_k + b_j b_k)(u_{jx} u_{kx} + v_{jx} v_{kx}) + (a_j b_k - b_j a_k)(v_{jx} u_{kx} - u_{jx} v_{kx}) \right) 
\]
(5)
It is interesting to note that \( \det(\nabla X) \) is always nonnegative. Since \( N \) is fixed, \( T \) contains not more than a finite number of roots of

\[ X = K, \quad \text{(6)} \]

where

\[ K = (K_1, K_2)'. \quad \text{(7)} \]

Since the set of roots of (6) is of measure zero, assume \( \partial T \) does not contain any roots of (6) and \( T \) does not contain any such roots such that \( \det(\nabla X) = 0 \). Theorem (1) applies, and

\[ h_K = E(\det(\nabla X) \mid X = K)p_x,y(K'), \quad \text{(8)} \]

where \( p_x,y \) denotes the probability density of \( X \). By (6), and since \( X_1 \) and \( X_2 \) are linear forms with respect to \( a_j \) and \( b_j \) for \( 0 \leq j \leq N \), \( h_K \) is the conditional mean of a quadratic form with respect to \( a_j + ib_j \) for \( 0 \leq j \leq N \). This form can be calculated in terms of components by means of multivariate analysis.

Based on the assumption that the scalar random variables are independent and normally distributed, the multivariate random vectors

\[ a = (a_0, \ldots, a_N)', \quad b = (b_0, \ldots, b_N)' \]

are such that

\[ \text{Cov}(a, b \mid Y = K) = \begin{pmatrix} \Sigma_{aa,x} & \Sigma_{ab,x} \\ \Sigma_{ba,x} & \Sigma_{bb,x} \end{pmatrix}. \quad \text{(9)} \]

The elements can be computed using

\[ \Sigma_{ab,x} = \Sigma_{ab} - \Sigma_{a,x} \Sigma_{X,X}^{-1} \Sigma_{X,b} \quad \text{(10)} \]

and the corresponding expression

\[ \Sigma_{ab} = E((a - E(a))(b - E(b))'). \quad \text{(11)} \]

Since the distribution of \( a_j \) and \( b_j \) is central for \( 0 \leq j \leq N \), \( E(a) = 0 \) and \( E(b) = 0 \). Then clearly

\[ \Sigma_{ab} = E(ab'). \quad \text{(11)} \]

Thusly, the conditional expected values are expressed in terms of unconditional expected values and covariances.

Then, if \( E(X_1) = 0 \) and \( E(X_2) = 0 \), \( E(X) = 0 \), whence, by (11),

\[ \Sigma_{XX} = \begin{pmatrix} E(X_1 X_1) & E(X_1 X_2) \\ E(X_2 X_1) & E(X_2 X_2) \end{pmatrix} = \begin{pmatrix} Y_1 & Y_2 \\ Y_2 & Y_3 \end{pmatrix}, \quad \text{(12)} \]

which implies that

\[ \det(\Sigma_{XX}) = Y_1 Y_3 - Y_2^2 > 0, \]

if \( X_1 \) and \( X_2 \) are not strictly correlated. Thus,

\[ \Sigma_{XX}^{-1} = \frac{1}{Y_1 Y_3 - Y_2^2} \begin{pmatrix} Y_3 & -Y_2 \\ -Y_2 & Y_1 \end{pmatrix}. \quad \text{(13)} \]

Direct evaluation shows that

\[ \Sigma_{aa} = E(a_j a_k) = \delta_{jk} \sigma^2_{a_j} \quad \text{(14)} \]

and

\[ \Sigma_{bb} = E(b_j b_k) = \delta_{jk} \sigma^2_{b_j}, \quad \text{(15)} \]
where
\[ \delta_{jk} = \begin{cases} 
1 & \text{if } j = k, \\
0 & \text{if } j \neq k. 
\end{cases} \]

Further, notice that
\[ \Sigma_{ab} = E(a_j b_k) = 0 \] (16)
and
\[ \Sigma_{ba} = 0. \] (17)

Next, since \( E(a_j X_1) = \sigma^2_{a_j} u_j \) and \( E(a_j X_2) = \sigma^2_{a_j} v_j \) for \( 0 \leq j \leq N \),
\[ \Sigma_{aX} = (\sigma^2_{a_j} u_j \quad \sigma^2_{a_j} v_j), \] (18)
whence
\[ \Sigma_{Xa} = \begin{pmatrix} \sigma^2_{a_k} u_k \\ \sigma^2_{a_k} v_k \end{pmatrix}. \] (19)

Analogously, \( E(b_j X_1) = -\sigma^2_{b_j} v_j \) and \( E(b_j X_2) = \sigma^2_{b_j} u_j \) for \( 0 \leq j \leq N \). Then
\[ \Sigma_{bX} = (-\sigma^2_{b_j} v_j \quad \sigma^2_{b_j} u_j), \] (20)
whence
\[ \Sigma_{Xb} = \begin{pmatrix} -\sigma^2_{b_k} v_k \\ \sigma^2_{b_k} u_k \end{pmatrix}. \] (21)
for \( 0 \leq k \leq N \).

Then, from (10), (13), (14), (18) and (19), for the \( j \)th row and \( k \)th column
\[ \Sigma_{aa,X} = \delta_{jk} \sigma^2_{a_j} - \frac{\sigma^2_{a_j} \sigma^2_{a_k}}{Y_1 Y_3 - Y^2_2} (Y_1 v_j v_k - Y_2 (u_j v_k + v_j u_k) + Y_3 u_j u_k). \] (22)
Likewise, from (10), (13), (15), (20) and (21)
\[ \Sigma_{bb,X} = \delta_{jk} \sigma^2_{b_j} - \frac{\sigma^2_{b_j} \sigma^2_{b_k}}{Y_1 Y_3 - Y^2_2} (Y_1 u_j u_k + Y_2 (u_j v_k + v_j u_k) + Y_3 v_j v_k). \] (23)

From (10), (13), (16), (18) and (21)
\[ \Sigma_{ab,X} = -\frac{\sigma^2_{a_j} \sigma^2_{a_k}}{Y_1 Y_3 - Y^2_2} (Y_1 v_j u_k - Y_2 (u_j v_k - v_j u_k) - Y_3 u_j v_k). \] (24)
From (10), (13), (17), (19) and (20)
\[ \Sigma_{ba,X} = -\frac{\sigma^2_{a_j} \sigma^2_{a_k}}{Y_1 Y_3 - Y^2_2} (Y_1 u_j v_k - Y_2 (u_j u_k - v_j v_k) - Y_3 v_j u_k). \] (25)

The mean function in (8) is then found by applications of
\[ E(a \mid X = K) = E(a) + \Sigma_{aX} \Sigma^{-1}_{XX} (K - E(X)), \]
which, for the aforesaid reasons, reduces to
\[ E(a \mid X = K) = \Sigma_{aX} \Sigma^{-1}_{XX} K. \] (26)

From (7), (13), (18) and (20)
\[ E(a_j \mid X = K) = \frac{\sigma^2_{a_j}}{Y_1 Y_3 - Y^2_2} ((K_1 Y_3 - K_2 Y_2) u_j - (K_1 Y_2 - K_2 Y_1) v_j). \] (27)
From (7), (13), (20) and (26)
\[ E(b_j | X = K) = - \frac{\sigma^2_{b_j}}{Y_1 Y_3 - Y_2^2} ((K_1 Y_3 - K_2 Y_2)v_j + (K_1 Y_2 - K_2 Y_1)u_j). \]  

(28)

Then, from (9), (22)–(25) and (27),
\[
E(a_j a_k | X = K) = E(a_j | X = K)E(a_k | X = K) + \operatorname{Cov}(a_j, a_k | X = K) \\
= \frac{\sigma^2_{a_j} \sigma^2_{a_k}}{(Y_1 Y_3 - Y_2^2)^2} ((K_1 Y_3 - K_2 Y_2)^2 u_j u_k + (K_1 Y_2 - K_2 Y_1)^2 v_j v_k \\
- (K_1 Y_3 - K_2 Y_2)(K_1 Y_2 - K_2 Y_1)(u_j v_k + v_j u_k) + \delta_{jk} \sigma^2_{a_j}) \\
- \frac{\sigma^2_{a_j} \sigma^2_{a_k}}{Y_1 Y_3 - Y_2^2} (Y_2 u_j u_k + Y_3 v_j v_k - Y_2(u_j v_k + v_j u_k)).
\]

(29)

From (9), (22)–(25) and (28)
\[
E(b_j b_k | X = K) = E(b_j | X = K)E(b_k | X = K) + \operatorname{Cov}(b_j, b_k | X = K) \\
= \frac{\sigma^2_{b_j} \sigma^2_{b_k}}{(Y_1 Y_3 - Y_2^2)^2} ((K_1 Y_2 - K_2 Y_1)^2 u_j u_k + (K_1 Y_3 - K_2 Y_2)^2 v_j v_k \\
+ (K_1 Y_2 - K_2 Y_1)(K_1 Y_3 - K_2 Y_2)(u_j v_k + v_j u_k)) + \delta_{jk} \sigma^2_{b_j} \\
- \frac{\sigma^2_{b_j} \sigma^2_{b_k}}{Y_1 Y_3 - Y_2^2} (Y_3 u_j u_k + Y_3 v_j v_k + Y_2(u_j v_k + v_j u_k)).
\]

(30)

From (9), (22)–(25), (27) and (28)
\[
E(a_j b_k | X = K) = E(a_j | X = K)E(b_k | X = K) + \operatorname{Cov}(a_j, b_k | X = K) \\
= \frac{\sigma^2_{a_j} \sigma^2_{b_k}}{(Y_1 Y_3 - Y_2^2)^2} ((K_1 Y_2 - K_2 Y_1)^2 v_j u_k - (K_1 Y_3 - K_2 Y_2)^2 u_j v_k \\
- (K_1 Y_3 - K_2 Y_2)(K_1 Y_2 - K_2 Y_1)(u_j v_k - v_j u_k)) \\
- \frac{\sigma^2_{a_j} \sigma^2_{b_k}}{Y_1 Y_3 - Y_2^2} (Y_1 v_j u_k - Y_2(u_j u_k - v_j v_k) - Y_3 u_j v_k)
\]

and
\[
E(b_j a_k | X = K) = E(b_j | X = K)E(a_k | X = K) + \operatorname{Cov}(b_j, a_k | X = K) \\
= \frac{\sigma^2_{b_j} \sigma^2_{a_k}}{(Y_1 Y_3 - Y_2^2)^2} ((K_1 Y_2 - K_2 Y_1)^2 v_j u_k - (K_1 Y_3 - K_2 Y_2)^2 v_j u_k \\
- (K_1 Y_3 - K_2 Y_2)(K_1 Y_2 - K_2 Y_1)(u_j v_k - v_j u_k)) \\
- \frac{\sigma^2_{b_j} \sigma^2_{a_k}}{Y_1 Y_3 - Y_2^2} (Y_1 v_j v_k - Y_2(u_j u_k - v_j v_k) - Y_3 v_j u_k).
\]

(32)
Then, from \([39]\) and \([30]\),

\[
E(a_j a_k + b_j b_k \mid X = K) = \frac{1}{(Y_1 Y_3 - Y_2^2)^2}((K_1 Y_3 - K_2 Y_2)^2(\sigma_{a_j}^2 \sigma_{a_k}^2 u_j u_k
+ \sigma_{b_j}^2 \sigma_{b_k}^2 v_j v_k) + (K_1 Y_2 - K_2 Y_1)^2(\sigma_{a_j}^2 \sigma_{a_k}^2 v_j v_k + \sigma_{b_j}^2 \sigma_{b_k}^2 u_j u_k)
- (K_1 Y_3 - K_2 Y_2)(K_1 Y_2 - K_2 Y_1)(u_j v_k + v_j u_k)(\sigma_{a_j}^2 \sigma_{a_k}^2 - \sigma_{b_j}^2 \sigma_{b_k}^2)) \tag{33}
\]

From \([31]\) and \([32]\)

\[
E(a_j b_k - b_j a_k \mid X = K) = \frac{1}{(Y_1 Y_3 - Y_2^2)^2}((K_1 Y_2 - K_2 Y_1)^2(\sigma_{a_j}^2 \sigma_{b_k}^2 v_j u_k
- \sigma_{a_j}^2 \sigma_{b_k}^2 u_j v_k) - (K_1 Y_3 - K_2 Y_2)^2(\sigma_{a_j}^2 \sigma_{b_k}^2 v_j u_k - \sigma_{b_j}^2 \sigma_{b_k}^2 v_j u_k)
- (K_1 Y_3 - K_2 Y_2)(K_1 Y_2 - K_2 Y_1)(u_j v_k - v_j u_k)(\sigma_{a_j}^2 \sigma_{b_k}^2 - \sigma_{b_j}^2 \sigma_{a_k}^2)) \tag{34}
\]

Altogether, in view of \([5]\), \([33]\) and \([34]\), after all the necessary simplifications,

\[
E(\det(\nabla X) \mid X = K) = \sum_{j=0}^{N} (\sigma_{a_j}^2 + \sigma_{b_j}^2)(u_j^2 + v_j^2)
- \left( \frac{Y_1 + Y_2}{Y_1 Y_3 - Y_2^2} \right) \left( \frac{(K_1 Y_2 - K_2 Y_1)^2 + (K_1 Y_3 - K_2 Y_2)(K_1 Y_2 - K_2 Y_1)}{(Y_1 Y_3 - Y_2^2)^2} \right)
\times \left| \sum_{j=0}^{N} (\sigma_{b_j}^2 u_j - i \sigma_{a_j}^2 v_j)(u_j x + iv_j) \right|^2
- \left( \frac{Y_2 + Y_3}{Y_1 Y_3 - Y_2^2} \right) \left( \frac{(K_1 Y_3 - K_2 Y_2)^2 + (K_1 Y_3 - K_2 Y_2)(K_1 Y_2 - K_2 Y_1)}{(Y_1 Y_3 - Y_2^2)^2} \right)
\times \left| \sum_{j=0}^{N} (\sigma_{a_j}^2 u_j - i \sigma_{b_j}^2 v_j)(u_j x + iv_j) \right|^2
+ \left( \frac{Y_2}{Y_1 Y_3 - Y_2^2} \right) \left( \frac{(K_1 Y_3 - K_2 Y_2)(K_1 Y_2 - K_2 Y_1)}{(Y_1 Y_3 - Y_2^2)^2} \right)
\times \left| \sum_{j=0}^{N} ((\sigma_{a_j}^2 u_j - i \sigma_{b_j}^2 v_j) + i(\sigma_{b_j}^2 u_j - i \sigma_{a_j}^2 v_j))(u_j x + iv_j) \right|^2.
\]
Since $X_1$ and $X_2$ are random variables distributed according to the normal law, their joint density is

$$p_{x,y}(K') = \frac{1}{2\pi \sqrt{Y_1 Y_3 - Y_2^2}} \exp\left(-\frac{K_1^2 Y_3 - 2K_1 K_2 Y_2 + K_2^2 Y_1}{2(Y_1 Y_3 - Y_2^2)}\right)$$

(36)

Finally, in accordance with (8), (35) and (36), the required result is proved.

3. A FURTHER RAMIFICATION

If $\{A_j\}_{j=0}^\infty$ and $\{B_j\}_{j=0}^\infty$ are sequences of i.i.d., real, normal random variables for which the respective increments $A_j - A_{j-1}$ and $B_j - B_{j-1}$ are independent for $j \geq 0$ and $A_{-1} = B_{-1} = 0$ by convention, then the increments

$$\Delta_j = (A_j - A_{j-1}) + i(B_j - B_{j-1})$$

are independent, real, normal random variables with mean zero and finite variances $\sigma_j^2$ such that $A_j + iB_j = \Delta_0 + \cdots + \Delta_j$ for $j \geq 0$. Then $\{A_j + iB_j\}_{j=0}^\infty$ can be interpreted as a sequence of successive observations of a Brownian motion. More precisely, $A_j + iB_j = W(t_j)$ for $j \geq 0$, where $t_0 < t_1 < \ldots$ and $\{W(t)\}_{t=0}^\infty$ is the standard Brownian motion. It is plain that $\text{Var}(\Delta_j)$ is the distance between the successive times $t_{j-1}$ and $t_j$ for $j \geq 0$. Thus, the sum in (2) assumes the form

$$S(z) = \sum_{j=0}^N (A_j + iB_j)f_j(z) = \sum_{k=0}^N F_k(z) \Delta_k,$$

(37)

where

$$F_k(z) = \sum_{j=k}^N u_j(x, y) + i \sum_{j=k}^N v_j(x, y)$$

(38)

for $0 \leq k \leq N$ and $(x, y) \in \mathbb{R}^2$. In fact, $\{F_k\}_{k=0}^N$ is a sequence of analytic functions that are real-valued on $\mathbb{R}$. Hence, $F_k(z) = F_k(\overline{z})$ for $0 \leq k \leq N$ and all $z \in \mathbb{C}$. Regard that the covariance matrix of $\Delta_k$ is given by

$$\Gamma_k = \begin{pmatrix} \sigma_{a_k}^2 & 0 \\ 0 & \sigma_{b_k}^2 \end{pmatrix}$$

for $0 \leq k \leq N$. Then, from Theorem 2 the following result is attained.

**Theorem 3.** Provided all the conditions imposed on $S$ in (37) and (38) and $T$ are satisfied, then for all integers $N > 1$ the formula for $h_K$ in Theorem 2 now holds for

$$D_0(z) = \sqrt{Y_1(z)Y_3(z) - Y_2^2(z)},$$
where

\[
Y_1(z) = \sum_{k=0}^{N} \left( \sigma_{a_k}^2 \left( \sum_{j=k}^{N} u_j \right)^2 + \sigma_{b_k}^2 \left( \sum_{j=k}^{N} v_j \right)^2 \right),
\]

\[
Y_2(z) = \sum_{k=0}^{N} \left( \sigma_{a_k}^2 - \sigma_{b_k}^2 \right) \sum_{j=k}^{N} u_j \sum_{j=k}^{N} v_j,
\]

\[
Y_3(z) = \sum_{k=0}^{N} \left( \sigma_{b_k}^2 \left( \sum_{j=k}^{N} u_j \right)^2 + \sigma_{a_k}^2 \left( \sum_{j=k}^{N} v_j \right)^2 \right),
\]

and

\[
D_1(z) = \sum_{k=0}^{N} \left( \sigma_{a_k}^2 \sum_{j=k}^{N} u_j - i \sigma_{a_k}^2 \sum_{j=k}^{N} v_j \right) \left( \sum_{j=k}^{N} u_{jz} + i \sum_{j=k}^{N} v_{jz} \right),
\]

\[
D_2(z) = \sum_{k=0}^{N} \left( \sigma_{b_k}^2 \sum_{j=k}^{N} u_j - i \sigma_{a_k}^2 \sum_{j=k}^{N} v_j \right) \left( \sum_{j=k}^{N} u_{jz} + i \sum_{j=k}^{N} v_{jz} \right),
\]

\[
D_3(z) = \sum_{k=0}^{N} (\sigma_{a_k}^2 + \sigma_{b_k}^2) \left( \sum_{j=k}^{N} u_{jz} \right)^2 + \left( \sum_{j=k}^{N} v_{jz} \right)^2.
\]

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