Spatial flow around an obstacle of a mixture of compressible viscous fluids

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Abstract. We present results on the existence and uniqueness of a strong solution to the inhomogeneous boundary value problem for equations of motion of viscous compressible fluids mixture with three spatial variables.

1. Introduction
Stationary spatial motions of a mixture of viscous compressible fluids are described by the following system of partial differential equations.

\[
\begin{align*}
\rho_1 (\mathbf{u}^{(1)} \cdot \nabla) \mathbf{u}^{(1)} + \nabla p_1 + a \left( \mathbf{u}^{(2)} - \mathbf{u}^{(1)} \right) &= \sum_{j=1}^{2} \left[ \mu_{1j} \Delta \mathbf{u}^{(j)} + (\mu_{1j} + \lambda_{1j}) \nabla \text{div} \mathbf{u}^{(j)} \right], \\
\rho_2 (\mathbf{u}^{(2)} \cdot \nabla) \mathbf{u}^{(2)} + \nabla p_2 - a \left( \mathbf{u}^{(2)} - \mathbf{u}^{(2)} \right) &= \sum_{j=1}^{2} \left[ \mu_{2j} \Delta \mathbf{u}^{(j)} + (\mu_{2j} + \lambda_{2j}) \nabla \text{div} \mathbf{u}^{(j)} \right], \\
\text{div} \left( \rho_1 \mathbf{u}^{(1)} \right) &= 0, \quad \text{div} \left( \rho_2 \mathbf{u}^{(2)} \right) = 0.
\end{align*}
\]

Here \( \mathbf{u}^{(i)} \), \( i = 1, 2 \), is velocity field for the \( i \)-th component of the mixture, filling a certain domain of the Euclidean space \( \mathbb{R}^3 \) of points \( x = (x_1, x_2, x_3) \); \( \rho_i, i = 1, 2 \), are scalar fields densities of the components; pressures \( p_i \) for the \( i \)-th component are assumed to be available sufficiently smooth functions of the density \( \rho_i \); the summands \( \mathbf{J}^{(i)} = (-1)^{(i)} a \left( \mathbf{u}^{(2)} - \mathbf{u}^{(1)} \right) \), \( a > 0 \), characterize the intensity of the momentum exchange between the mixture components. For thermodynamic reasons, the constant viscosity coefficients \( \mu_{ij}, \lambda_{ij}, i,j = 1,2 \), satisfy the conditions providing ellipticity by Petrovsky of the differential operator

\[
L (\vec{u}^{(1)}, \vec{u}^{(2)}) = \begin{pmatrix}
\sum_{j=1}^{2} L_{1j}(\vec{u}^{(j)}) \\
\sum_{j=1}^{2} L_{2j}(\vec{u}^{(j)})
\end{pmatrix}, \quad L_{ij}(\vec{u}) = -\mu_{ij} \Delta \vec{u} - (\lambda_{ij} + \mu_{ij}) \nabla \text{div} \vec{u}.
\]

In general, the equations (1)-(3) are a very complex nonlinear system of composite type: the momentum conservation equations (1), (2) are an elliptic system with respect to the desired
functions \( \tilde{u}^{(1)} \), \( \tilde{u}^{(2)} \), and the mass conservation equations of components (3) can be interpreted as first-order equations with respect to densities \( \rho_1 \), \( \rho_2 \).

Most of the known results for Navier–Stokes equations for viscous compressible fluids and moreover for equations of mixtures of such media concerns flows in areas bounded by impenetrable walls, while the results of study of inhomogeneous boundary value problems remain fairly modest. Among the papers dealing with the last issue, we’d like to mention the papers of S. Novo [1], V. Girinon [2] which proves existence theorem in the case of boundary conditions of a special kind and the papers of R. Farwig [3], J.R. Kweon, R.B. Kellogg [4],[5] for two-dimensional domains on the hypothesis that the velocity field at the boundary of the flow region is close to a prescribed constant. Important results relating to the existence of strong solutions of inhomogeneous boundary value problems for stationary Navier–Stokes equations in case of small Reynolds and Mach numbers were obtained by P. Plotnikov, J. Sokolowski in [6].

2. Statement of the problem

In this paper, we study the problem of flow around a family of obstacles by a three-dimensional flow of a viscous compressible fluids mixture, leading to an inhomogeneous boundary value problem for the system of equations (1)-(3) in the following statement.

The flow domain is a domain \( \Omega = B \setminus S \in \text{Euclidean space} \mathbb{R}^3 \) of points \( x = (x_1, x_2, x_3) \), where \( B \) is a bounded domain with boundary \( \Sigma = \partial B \) of class \( C^3 \), that contains inside a compact set \( S \) with the sufficiently smooth boundary \( \partial S \).

Let \( \tilde{U}^{(j)} \), \( j = 1, 2 \), be some given vector fields of class \( C^3(\mathbb{R}^3) \), vanishing in a neighborhood of \( S \). On the boundary \( \Sigma \) of \( B \), distinguish the inflow segments: \( \Sigma_{in} = \{ x \in \Sigma : \tilde{U}^{(j)} \cdot \tilde{n} < 0 \} \), \( j = 1, 2 \), and the outflow segments: \( \Sigma_{out} = \{ x \in \Sigma : \tilde{U}^{(j)} \cdot \tilde{n} > 0 \} \), \( j = 1, 2 \), where \( \tilde{n} \) is the outer normal to \( \Sigma \).

We want to find the velocity fields \( \tilde{u}^{(1)} \), \( \tilde{u}^{(2)} \) and density fields \( \rho_1 \), \( \rho_2 \) of the mixture components, which satisfy the equations (1)-(3) (presented in dimensionless form)

\[
\begin{align*}
\sum_{j=1}^{2} L_{ij}(\tilde{u}^{(j)}) + \text{Re} \rho_i \tilde{u}^{(j)} \cdot \tilde{\nabla} \tilde{u}^{(j)} + \frac{\text{Re}}{\text{Ma}^2} \nabla \rho_i + (-1)^{(i)}a \left( \tilde{u}^{(2)} - \tilde{u}^{(1)} \right) &= 0 \quad \text{in } \Omega, \\
\text{div}(\rho_i \tilde{u}^{(i)}) &= 0 \quad \text{in } \Omega, \quad i = 1, 2, \\
L_{ij}(\tilde{u}^{(j)}) &= -\mu_{ij} \Delta \tilde{u}^{(j)} - (\mu_{ij} + \lambda_{ij}) \nabla \text{div} \tilde{u}^{(j)}, \quad i, j = 1, 2,
\end{align*}
\]

and boundary conditions

\[
\begin{align*}
\tilde{u}^{(j)} &= \tilde{U}^{(j)} \quad \text{on } \Sigma, \\
\tilde{u}^{(j)} &= 0 \quad \text{on } \partial \Sigma, \\
\rho_j &= \rho^0_j \quad \text{on } \Sigma_{in}^j, \quad j = 1, 2,
\end{align*}
\]

where \( \rho^0_j \), \( j = 1, 2 \), are some given positive constants; \( \text{Re} \) and \( \text{Ma} \) denote the Reynolds and Mach numbers, respectively.

We assume fulfilled

**Condition 1.** The sets \( \Gamma^j = d\Sigma_{in}^j \cap (\Sigma \setminus \Sigma_{in}^j) \), \( j = 1, 2 \), (the characteristic parts of the surface) are one-dimensional closed manifolds such that \( \Sigma = \Sigma_{in}^j \cup \Gamma^j \cup \Sigma_{out}^j \) and, moreover, \( \int \tilde{U}^{(j)} \cdot \tilde{n} \text{d}s = 0, \quad j = 1, 2; \tilde{U}^{(j)} \cdot \nabla (\tilde{U}^{(j)} \cdot \tilde{n}) > C > 0 \) on \( \Gamma^j, \quad j = 1, 2 \), where \( C > 0 \) is some constant.

Since in this paper we are talking about the problem of flow around a family of obstacles, we consider the following class of deformations of the flow region. Choose a vector field \( \bar{T} \in C^2(\mathbb{R}^3) \), equal to zero in a neighborhood of the boundary \( \Sigma \), and consider the mapping \( y = \bar{T}_\varepsilon(x) = x + \varepsilon \bar{T}(x) \), that defines the perturbation of the obstacle \( S \). For small \( \varepsilon \) the mapping
$x \mapsto \tilde{T}_x(x)$ is a diffeomorphism of the flow domain $\Omega$ onto $\Omega_\varepsilon = B \setminus S_\varepsilon$, where $S_\varepsilon = \tilde{T}_x(S)$ is the perturbed streamlined obstacle.

We consider the boundary value problem (4) - (6) in the perturbed domain $\Omega_\varepsilon$ and its solution is denoted by $(\vec{u}_\varepsilon^{(i)}, \rho_\varepsilon)$, $i = 1, 2$, i.e.

$$
\sum_{j=1}^{2} L_{ij}(\vec{u}_\varepsilon^{(j)}) + \text{Re} \rho_\varepsilon (\vec{u}_\varepsilon^{(i)} \cdot \nabla) \vec{u}_\varepsilon^{(i)} + \frac{\text{Re}}{\text{Ma}^2} \nabla p_\varepsilon (\rho_\varepsilon) + \vec{f}^{(i)} = 0 \text{ in } \Omega_\varepsilon,
$$

$$
div(\rho_\varepsilon \vec{u}_\varepsilon^{(i)}) = 0 \text{ in } \Omega_\varepsilon, \; i = 1, 2,
$$

$$
\vec{g}^{(i)} = \vec{U}^{(i)} \text{ on } \Sigma, \quad \vec{u}_\varepsilon^{(j)} = 0 \text{ on } \partial S_\varepsilon, \; \rho_\varepsilon = \rho_j^0 \text{ on } \Sigma^j_{in}, \; j = 1, 2,
$$

Problem (7) can be conveniently reduced to a boundary value problem in the unperturbed domain $\Omega$ for a one-parameter family of differential equations with perturbed coefficients. To this end, we introduce the functions $\vec{u}^{(i)}$ and $\rho_i$, $i = 1, 2$, that are defined in $\Omega$ by the formulas:

$$
\vec{u}^{(i)}(x; \varepsilon) = \mathbf{N}(x) \vec{u}_\varepsilon^{(i)}(x + \varepsilon \tilde{T}(x)), \; \rho_i(x; \varepsilon) = \rho_\varepsilon(x + \varepsilon \tilde{T}(x)), \; x \in \Omega, \; i = 1, 2,
$$

where $\mathbf{N}(x) = (\det \mathbf{M}(x))^{1/2}$, $\mathbf{M}(x) = \mathbf{I} + \varepsilon \mathbf{D}\tilde{T}(x)$, $\mathbf{D}\tilde{T}(x) = \left\{ \frac{\partial \tilde{T}_i(x)}{\partial x_j} \right\}$ is the Jacobi matrix of the map $x \mapsto \tilde{T}(x)$.

In result of this transformation, (7) is reduced to the problem

$$
\sum_{j=1}^{2} \mu_{ij} \Delta \vec{u}^{(j)} - \nabla q_i = \sum_{j=1}^{2} \mu_{ij} \mathcal{A}(\vec{u}^{(j)}; \mathbf{N}) + \text{Re} \mathcal{B}(\rho_i, \vec{u}^{(i)}, \vec{u}^{(i)}; \mathbf{N}) + (-1)^i \mathcal{S}(\vec{u}^{(2)} - \vec{u}^{(1)}; \mathbf{N}) \text{ in } \Omega,
$$

$$
div \vec{u}^{(i)} = \sum_{j=1}^{2} g\sigma_{ij} p_j - \sum_{j=1}^{2} g\gamma_{ij} q_j \text{ in } \Omega,
$$

$$
\vec{g}^{(i)} \cdot \nabla \rho_i + \rho_i \sum_{j=1}^{2} g\sigma_{ij} p_j = \rho_i \sum_{j=1}^{2} g\gamma_{ij} q_j \text{ in } \Omega,
$$

$$
\vec{u}^{(i)} = \vec{U}^{(i)} \text{ on } \Sigma, \; \vec{u}^{(j)} = 0 \text{ on } \partial S, \; \rho_i = \rho_j^0 \text{ on } \Sigma^j_{in}, \; i = 1, 2.
$$

Here $g = g(x; \mathbf{N}) = \sqrt{\det \mathbf{N}(x)}$; the linear operators $\mathcal{A}$, $\mathcal{S}$ and nonlinear map $\mathcal{B}$ are defined by the formulas

$$
\mathcal{A}(\vec{u}; \mathbf{N}) = \Delta \vec{u} - \left( \mathbf{N}^T \right)^{-1} \text{div} \left( g^{-1} \mathbf{N} \mathbf{N}^T \nabla \left( \mathbf{N}^{-1} \vec{u} \right) \right),
$$

$$
\mathcal{B}(\rho, \vec{u}, \vec{w}; \mathbf{N}) = \rho \left( \mathbf{N}^T \right)^{-1} \left( \vec{u} \nabla \left( \mathbf{N}^{-1} \vec{w} \right) \right), \quad \mathcal{S}(\vec{u}; \mathbf{N}) = g \cdot a \left( \mathbf{N}^T \right)^{-1} \mathbf{N}^{-1} \vec{u};
$$

$$
q_i = -\sum_{j=1}^{2} g^{-1}(\mu_{ij} + \lambda_{ij}) \text{div}\vec{u}^{(j)} + \frac{\text{Re}}{\text{Ma}^2} p_i(\rho_i), \; i = 1, 2, \text{ are effective viscous pressures; } \gamma_{ij} \text{ are the entries of the inverse matrix } \chi^{-1}, \text{ of } \chi \text{ whose entries are } \mu_{ij} + \lambda_{ij}, i, j = 1, 2; \text{ and } \sigma_{ij} = \frac{\text{Re}}{\text{Ma}^2} \gamma_{ij}.
$$

The solution of (8) is constructed in the form of some perturbation of a specially chosen sufficiently smooth flow $\vec{u}_*^{(i)}$, $\rho_*^i$, $q_*^i$, $\varphi_*$, $q_i = q_*^i + \pi_i + \Lambda \cdot p_i(\rho_*^i) + \sum_{j=1}^{2} (\mu_{ij} + \lambda_{ij}) m_j$, i.e.

$$
\vec{u}^{(i)} = \vec{u}_*^{(i)} + \vec{u}^{(i)}, \; \rho_i = \rho_*^i + \varphi_*, \; q_i = q_*^i + \pi_i + \Lambda \cdot p_i(\rho_*^i) + \sum_{j=1}^{2} (\mu_{ij} + \lambda_{ij}) m_j,
$$

where $\Lambda$ is the second invariant of the perturbation $\chi$, $\pi = \frac{1}{2} \text{tr} \chi$ and $\varphi = \frac{1}{2} \text{tr} \chi - \frac{1}{2} \text{tr} \chi^2$.
functions with the finite norm \( l^1 \) is defined as the space of measurable functions with the finite norm by the method of real interpolation \[8\] between \( L^W \) where
\[
\sigma
\]

Theorem 2. Here the constant parameters \( \mu \) and the given boundary values \( p_i^0; \Phi_i[\overline{\theta}], \Psi_i[\overline{\theta}] \) are the functions of the components of \( \overline{\theta} = (\overline{v}^{(1)}, \overline{v}^{(2)}; \pi_1, \pi_2, \varphi_1, \varphi_2) \), the expressions for which are given in \[7\].

3. The Main Result

Theorem 2. Then the numbers \( \sigma^* > 1 \) and \( \tau^* \in (0, 1) \) can be found so that, if the matrix \( \mathbf{N} \) is chosen so as to satisfy the condition \( |\mathbf{I} - \mathbf{N}|_{C^2(\Omega)} \leq \tau^2, \tau \in (0, \tau^*) \), \( p_j(\rho_j) \in C^3(0, \infty), j = 1, 2, \) and the parameters of the problem are such that \( \frac{1}{\Lambda} \leq \tau^2, \Re \leq \tau^2, \alpha \leq \tau^2, |\tau_{12}| \leq \tau, |\tau_{12}| \leq \tau, \tau_{ii} \geq \sigma^*, i = 1, 2, \tau \in (0, \tau^*) \), then (9) has a unique solution \( \overline{\theta} = (\overline{v}^{(1)}, \overline{v}^{(2)}; \pi_1, \pi_2, \varphi_1, \varphi_2), m_i, \zeta_i^{(j)}, i, j = 1, 2, \) such that \( \overline{\theta} \in V_{s,r}^* \times X^{s,r}, m_i \in R, \zeta_i^{(j)} \in X^{s,r}, \) where the exponents \( s \in (0, 1) \) and \( r \in (1, \infty) \) satisfy the conditions \( \frac{1}{2} < s < 1, s \cdot r > 3, (1 - s) \cdot r > 3, 2s - \frac{3}{r} < 1 \). In addition, \( \overline{\theta} \) belongs to the ball \( B_r \) of radius \( r \) centered at the origin of \( V_{s,r}^* \times X^{s,r}. \) Here \( V_{s,r}^* \) and \( X^{s,r} \) denote the spaces

\[
X^{s,r} = W_{s,r}^*(\Omega) \cap W^{1,2}(\Omega), \quad V_{s,r}^* = W^{s+1,r}(\Omega) \cap W^{2,2}(\Omega),
\]

where \( W^{l,p}(\Omega) \) is the standard Sobolev space, consisting of measurable functions in \( \Omega \), with generalized derivatives in \( \Omega \) up to the order \( l \) and summable to the power \( p \). For real \( s \in (0, 1) \) and \( r \in (1, \infty) \), the function Sobolev space \( W_{s,r}(\Omega) \) is obtained by the method of real interpolation \[8\] between \( L^r(\Omega) \) and \( W^{l,p}(\Omega) \) and consists of measurable functions with the finite norm

\[
\|u\|_{W_{s,r}(\Omega)} = \|u\|_{L^r(\Omega)} + |u|_{s,r,\Omega}, \quad |u|_{s,r,\Omega} = \int_{\Omega} \left| x - y \right|^{-3} \frac{|u(x) - u(y)|}{|x - y|^s} \, dx \, dy.
\]

In the general case, the Sobolev space \( W_{s,r}^{l+s,r}(\Omega) \), \( 0 < s < 1, 1 \leq r < \infty \), with \( l \geq 0 \) an integer, is defined as the space of measurable functions with the finite norm

\[
\|u\|_{W_{s,r}^{l+s,r}(\Omega)} = \|u\|_{W_{s,r}^l(\Omega)} + \sup_{|\alpha|=l} \|D^\alpha u\|_{W_{s,r}(\Omega)}.
\]
The vector quantity $\vec{F}$ belonging to the direct product of spaces $W_1 \times W_2 \times W_3$ should be understood in the sense that $\vec{F}$ is composed of three entries (vector or scalar) separated by semicolons $\vec{F} = (F_1; F_2; F_3)$, and, in addition, $F_1 \in W_1$, $F_2 \in W_2$ and $F_3 \in W_3$. If $W_1 = W_2 = W_3 = W$, then we write $\vec{F} \in W$ and separate the entries of the vector by commas.

The volume of the article allows to describe only the scheme of the proof of this theorem.

Let us characterize the steps of the proof of the existence of solutions to the problem (9).

Fix a ball $B_{\tau}$ of radius $\tau \in (0, 1)$ and choose an arbitrary matrix $N$ such that $\|N - I\|_{C^2(\Omega)} \leq \tau^2$, and then take an arbitrary element $\vec{\theta} = (\vec{v}^{(1)}, \vec{v}^{(2)}, \pi_1, \pi_2, \varphi_1, \varphi_2) \in B_{\tau}$ and put

$$\vec{u}^{(i)} = \vec{u}_*^{(i)} + \vec{v}^{(i)}; \quad \rho_i = \rho_i^* + \varphi_i; \quad q_i = q_i^* + \pi_i + \Lambda \cdot p_i(\rho_i^*) + \sum_{j=1}^{2}(\mu_{ij} + \lambda_{ij})m_j.$$  

Consider the following boundary value problem:

Find a field $\vec{\theta}_1 = (\vec{v}_1^{(1)}, \vec{v}_1^{(2)}, \pi_1, \pi_2, \varphi_1, \varphi_2)$ such that

$$\vec{u}^{(i)} \cdot \nabla \varphi_i^1 + \tau_i\pi_i^1 = \Psi_i[\vec{\theta}] + g \ m_i \rho_i \ \text{in} \ \Omega; \quad \varphi_i^1 = 0 \ \text{on} \ \Sigma_{in}, \ i = 1, 2, \ (10)$$

$$\sum_{j=1}^{2} \mu_{ij} \triangle \vec{v}_1^{(j)} - \nabla \pi_i^1 = \sum_{j=1}^{2} \mu_{ij} A(\vec{v}_j^{(j)}) + \text{Re}B(\rho_i, \vec{u}^{(i)}, \vec{u}^{(i)}) + \ (-1)^i \ S(\vec{v}^{(2)} - \vec{u}^{(1)}) \ \text{in} \ \Omega, \ (11)$$

$$\text{div} \vec{v}_1^{(i)} = \Pi \left( \sum_{j=1}^{2} \frac{1}{\rho_i} \tau_{ij} \varphi_j^1 - g \Phi_i[\vec{\theta}] - gm_i \right) \ \text{in} \ \Omega, \ (12)$$

$$\vec{u}_1^{(i)} = 0 \ \text{on} \ \partial \Omega; \quad \Pi \pi_i^1 = \pi_i^1, \ i = 1, 2. \ (13)$$

The constants $m_i$ (as operators on $\vec{\theta}$) are defined by the formulas

$$\vec{m} = (m_1, m_2)^T; \quad \vec{m} = (kI - A)^{-1} \vec{f}; \quad A = \{a_{ij}\}_{i,j=1}^{2}, \quad \vec{f} = (f_1, f_2)^T, \ (14)$$

$$k = \int_{\Omega} gdx, \ a_{ij} = \frac{1}{\rho_i} \int_{\Omega} g \cdot \rho_j \cdot \Psi_i[j] \ dx, \ f_i = \frac{1}{\rho_i} \sum_{j=1}^{2} \int_{\Omega} (\varphi_j^{(j)} \cdot \Psi_j[j] - g \Phi_i[j])dx,$$

where $\varphi_i^{(j)}$ is a solution to the boundary value problem

$$-\text{div}(\vec{u}^{(i)} \cdot \varphi_j^{(i)}) + \tau_{ij} \varphi_j^{(i)} = \tau_{ij} g \ \text{in} \ \Omega, \ \varphi_j^{(i)} = 0 \ \text{on} \ \Sigma_{out}, \ i, j = 1, 2. \ (15)$$

Based on a number of known results on linear Stokes systems and transport equations [6], we show that (10)-(15) provides a complete system of equations and boundary conditions for the unique determination of the vector $\vec{\theta}_1$, and so the problem defines a mapping $W : \vec{\theta} \rightarrow \vec{\theta}_1$. Each fixed point of $W$ gives a solution to (9).

Based on the solution of (10)-(15) theorems [7], it is proved that for Reynolds numbers sufficiently small, the mapping $W$ takes the ball $B_{\tau}$ into itself.

Note also that the mapping $W$ is sequentially weakly continuous. Suppose that a sequence $\vec{\theta}_n = (\vec{v}_n^{(1)}, \vec{v}_n^{(2)}, \pi_n^1, \pi_n^2, \varphi_1^n, \varphi_2^n) \in B_{\tau}$ converges weakly in $V^{s,r} \times X^{s,r}$. Since $V^{s,r} \times X^{s,r}$ is reflexive, the limit $\vec{\theta}$ of this sequence also belongs to $V^{s,r} \times X^{s,r}$ and, moreover, $\vec{\theta} \in B_{\tau}$, since the ball $B_{\tau}$ is closed and compact. Since the elements $\vec{\theta}_{1,n} = W(\vec{\theta}_n)$ lie in $B_{\tau} \in V^{s,r} \times X^{s,r}$, there exist a sequence $\{\vec{\theta}_{1,n_k}\} \subset \{\vec{\theta}_{1,n}\}$ and $\vec{\theta}_1 \in B_{\tau}$ such that $\vec{\theta}_{1,n_k} \rightarrow \vec{\theta}_1$ weakly in $V^{s,r} \times X^{s,r}$.
At this stage, the structure of the solutions of (9), corresponding to different matrices, does not matter, and so the results below hold for an arbitrary $N$. In the equations of the form (10)-(15), connecting the vector functions $\vec{w}$, $\vec{q}$, $\vec{v}$, $\vec{q}^0$, $\vec{v}^0$, and so on, the subsequences $\{\vec{w}_{i,n}\} \subset \{\vec{w}^{(j)}\}$ converge weakly in $X^{s,r}$, in particular, $\nabla\vec{w}_{i,n} \rightarrow \nabla\vec{w}^{(j)}$ weakly in $L^2(\Omega)$. By the compactness of the embeddings $X^{s,r} \subset C(\Omega)$ and $V^{s,r} \times X^{s,r} \subset C(\Omega)$, we have $\vec{\theta}_n \rightarrow \vec{\theta}$ in $C(\Omega)$, $\vec{\theta}_{1,n_k} \rightarrow \vec{\theta}_1$ in $C(\Omega)$ and $\zeta^{(j)}_{i,n_k} \rightarrow \zeta^{(j)}_i$ in $C(\Omega)$.

In the equations of the form (10)-(15), connecting the vector functions $\vec{\theta}_{i,n_k}$ and $\vec{\theta}_{1,n_k}$, by what was said, we can pass to the limit as $n_k \rightarrow \infty$, which gives the relation $\vec{\theta}_1 = W(\vec{\theta})$. Thus, every weakly convergent subsequence $\{\vec{\theta}_{1,n_k}\}$ of the sequence $\{\vec{\theta}_1\}$ has as the limit the unique element $\vec{\theta}_1 \in B_r$. Consequently, the entire sequence $\{\vec{\theta}_1\}$ converges weakly to $\vec{\theta}_1$ which proves the sequential weak continuity of $W$. The Tikhonov fixed point theorem [6] imply the existence of an element $\vec{\theta} \in B_r$, such that $W(\vec{\theta}) = \vec{\theta}$. The fixed point $\vec{\theta} = W(\vec{\theta}) \in V^{s,r} \times X^{s,r}$ is a solution to problem (9).

To prove the uniqueness of the solution of problem (9) we need to study, first of all, the dependence of these solutions on the matrix $N$, is completely defined by the deformation of the flow domain.

For the difference $\vec{q}_0 - \vec{q}_1$, \[ \vec{q}_i = (\vec{q}_{i1}^{(1)}, \vec{q}_{i2}^{(2)}; \pi_1^{i}, \pi_2^{i}, \psi_1^{i}, \psi_2^{i}, \xi_1^{(1)}, \xi_2^{(1)}, \zeta_1^{(2)}, \zeta_2^{(2)}; m_i^1, m_i^2), i = 0, 1, \]
of the solutions of (9), corresponding to different matrices $N_0$ and $N_1$, we introduce the notation \[ \vec{w}^{(j)} = \vec{w}_0^{(j)} - \vec{w}_1^{(j)}, \quad \omega_j = \pi_j^0 - \pi_j^1, \quad \psi_j = \varphi_j^0 - \varphi_j^1, \quad n_j = m_j^0 - m_j^1, \]
\[ \xi_j^{(k)} = \xi_{j1}^{(k)} - \xi_{j1}^{(k)}, \quad k, j = 1, 2 \]
At this stage, the structure of $N$ does not matter, and so the results below hold for an arbitrary smooth matrix-valued function $N(x)$, $x \in \Omega$.

It follows from (9) that the vector-function \[ \vec{q}_0 - \vec{q}_1 = (\vec{w}^{(1)}, \vec{w}^{(2)}; \omega_1, \omega_2, \psi_1, \psi_2, \xi_1^{(1)}, \xi_2^{(1)}, \zeta_1^{(2)}, \zeta_2^{(2)}; m_1^1, m_2^1) \]
is the solution of the linear problem:

\[
\sum_{j=1}^{2} \mu_{ij} \Delta \vec{w}^{(j)} - \nabla \omega_i = \sum_{j=1}^{2} \mu_{ij} A_0(\vec{w}^{(j)}) + Re\mathcal{L}(\psi_i, \vec{w}^{(i)}) + D_i + (-1)^i(\mathcal{E} + S_0(\vec{w}^{(2)} - \vec{w}^{(1)})) \quad \text{in} \quad \Omega, \tag{16}
\]
\[
div \vec{w}^{(i)} = \sum_{j=1}^{2} \alpha_{ij} \psi_j + \sum_{j=1}^{2} \beta_{ij} \omega_j + \gamma_{i} n_i + \delta_i d \quad \text{in} \quad \Omega, \tag{17}
\]
\[
\vec{u}^{(i)}_0 \cdot \nabla \psi_i + \tau_i \psi_i = -\vec{w}^{(i)} \cdot \nabla \psi_i^1 + \sum_{j=1}^{2} \alpha_{ij} \psi_j + \sum_{j=1}^{2} \beta_{ij} \omega_j + \gamma_i n_i + \delta_i d \quad \text{in} \quad \Omega, \tag{18}
\]
\[
-div(\vec{u}^{(i)}_0) \xi_j^{(i)} + \tau_{i} \xi_j^{(i)} = div(\vec{w}^{(i)} \xi_j^{(i)}) + \tau_{ij} d \quad \text{in} \quad \Omega, \tag{19}
\]
\[
\vec{w}^{(i)} = 0 \quad \text{on} \quad \Omega, \quad \psi_i = 0 \quad \text{on} \quad \Sigma_{in}, \quad \omega_i = \Pi \omega_i, \quad \xi_j^{(i)} = 0 \quad \text{on} \quad \Sigma_{out}, \tag{20}
\]
\[
n_i = \sum_{k=1}^{2} \chi_{ik} \int_{\Omega} \left[ \delta_k d + \sum_{j=1}^{2} (\beta_{kj} \psi_j + \gamma_{kj} \omega_j + \tau_{kj} \xi_j^{(k)}) \right] dx, \quad i, j = 1, 2. \tag{21}
\]
In the representation of these equations, we use the notation

\[ A_k(\vec{w}) = A(\vec{w}; \mathbf{N}_k), \quad B_k(\rho, \vec{u}, \vec{w}) = B(\rho, \vec{u}, \vec{w}; \mathbf{N}_k), \quad S_k(\vec{w}) = S(\vec{w}; \mathbf{N}_k), \quad k = 0, 1, \]

\[ \mathcal{L}(\psi_i, \vec{w}^{(i)}) = B_0(\psi_i, \vec{u}^{(i)}, \vec{w}^{(i)}_0) + B_0(\rho_1, \vec{w}^{(i)}, \vec{w}^{(i)}_0) + B_0(\rho_1, \vec{u}^{(i)}, \vec{w}^{(i)}_0), \]

\[ \mathcal{E} = S_0(\vec{u}^{(2)}_1 - \vec{u}^{(1)}_1) - S_1(\vec{u}^{(2)}_1 - \vec{u}^{(1)}_1), \]

\[ D_i = \sum_{j=1}^{2} \mu_{ij} \left( A_0(\vec{u}^{(j)}_1) - A_1(\vec{u}^{(j)}_1) \right) + \text{Re} \left( B_0(\rho_1, \vec{u}^{(i)}_1, \vec{u}^{(i)}_1) - B_1(\rho_1, \vec{u}^{(i)}_1, \vec{w}^{(i)}_1) \right), \]

\[ d = g_0 - g_1, \quad g_k = \sqrt{\det \mathbf{N}_k}. \]

The symbols \( \alpha_{ik} \) denote the entries of \((k(\mathbf{N}_0) - A(\mathbf{N}_0, \vec{\theta}_0))^{-1}. \) The coefficients \( \alpha_{ij}, \beta_{ij}, \gamma_i, \delta_i, \) \( \hat{\alpha}_{ij}, \hat{\beta}_{ij}, \hat{\gamma}_i, \hat{\delta}_i, \) \( \tilde{\alpha}_{ij}, \tilde{\beta}_{ij}, \tilde{\gamma}_{ij}, \) and \( \tilde{\delta}_i, i, j = 1, 2, \) on the right-hand sides of (17), (18) and (21) depend on the solutions \( \vec{q}_0 \) and \( \vec{q}_1 \) and are considered as known. The expressions for these coefficients are very cumbersome. Thus, we do not present them here.

Problem (16)-(21) for the difference \( \vec{q}_0 - \vec{q}_1 \) has the peculiarity that the results [6] on the transport equations are inapplicable to the equations (18) that are included in this problem because the summand \( \vec{w}^{(i)} \cdot \nabla \varphi^j_1 \) does not satisfy the necessary smoothness conditions. However, from the result on existence of solutions of (9) it follows that the terms \( \vec{w}^{(i)} \cdot \nabla \varphi^j_1 \) belong to the such space [9], that it is possible to consider the difference \( \vec{q}_0 - \vec{q}_1 \) as a very weak solution of (16)-(21). The results [9] on existence and uniqueness of strong and weak solutions of problem conjugate to the problem (16)-(21), enable us to derive some estimates for the norms of the differences \( \vec{w}^{(i)}(\omega_i, \psi_i, \xi^{(j)}_i) \) and \( u_i \) through norms of \( \mathcal{E}, D_1, D_2, d. \) From these estimates follows the uniqueness of a solution to (9). This completes the proof of Theorem 2.

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