POLYNOMIAL TRANSLATION WEINGARTEN SURFACES IN
3-DIMENSIONAL EUCLIDEAN SPACE

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Abstract. In this paper we will classify those translation surfaces in \( \mathbb{E}^3 \) involving polynomials which are Weingarten surfaces.

Mathematics Subject Classification (2000): 53A05, 53A10.

Keywords and Phrases: \( W \)-surfaces, translation surfaces, minimal surfaces, constant mean curvature (CMC), constant Gaussian curvature.

1. Preliminaries

The surfaces of constant mean curvature, \( H \)-surfaces and those of constant Gaussian curvature, \( K \)-surfaces in the Euclidean 3-dimensional space, \( \mathbb{E}^3 \), have been studied extensively. One interesting class of surfaces in \( \mathbb{E}^3 \) is that of translation surfaces, which can be parametrized, locally, as \( r(u, v) = (u, v, f(u) + g(v)) \), where \( f \) and \( g \) are smooth functions. This type of surfaces are important either because they are interesting themselves or because they furnish counterexamples for some problems (e.g. it is a known fact that a minimal surface has vanishing second Gaussian curvature but not conversely – see for details [1]). We call polynomial translation surfaces (in short, PT surfaces) those translation surfaces for which \( f \) and \( g \) are polynomials.

Scherk’s surface, obtained by H. Scherk in 1834, is the only non flat minimal surface, that can be represented as a translation surface. More precisely we have Theorem A. Let \( S \) be a translation minimal surface in 3-dimensional Euclidean space. Then \( S \) is an open part of \( \mathbb{E}^3 \) or it is congruent to the following surface

\[
z = \frac{1}{a} \log \left| \frac{\cos(ax)}{\cos(ay)} \right|, \quad a \neq 0.
\]

Other interesting results concerning translation surfaces having either constant mean curvature or constant Gaussian curvature are the following:

Theorem B. Let \( S \) be a translation surface with constant Gauss curvature \( K \) in 3-dimensional Euclidean space. Then \( S \) is congruent to a cylinder, so it is flat \( (K = 0) \) (Theorem 1 in [3]).

Theorem C. Let \( S \) be a translation surface with constant mean curvature \( H \neq 0 \) in 3-dimensional Euclidean space \( \mathbb{E}^3 \). Then \( S \) is congruent to the following surface

\[
z = \frac{\sqrt{1 + a^2}}{2H} \sqrt{1 - 4H^2x^2} + ay, \quad a \in \mathbb{R}
\]

(Theorem 2, statement (1) in [3]). Cf. also References in [3].
A surface $S$ is called a Weingarten surface if there is some (smooth) relation $W(\kappa_1, \kappa_2) = 0$ between its two principal curvatures $\kappa_1$ and $\kappa_2$, or equivalently, if there is a (smooth) relation $U(K, H) = 0$ between its mean curvature $H$ and its Gaussian curvature $K$.

In this paper we study those PT surfaces that are Weingarten surfaces. We abbreviate by a WPT surfaces.

We give the following classification theorem:

**Theorem 1.** Let $S$ be a WPT surface in 3-dimensional Euclidean space. Then

(i) $S$ is a cylinder, case in which $K = 0$;

(ii) $S$ is a paraboloid of revolution, case in which the mean curvature $H$ and the Gaussian curvature $K$ are positive everywhere and related by the relation

$$8aH^2 = \sqrt{K}(2a + \sqrt{K})^2$$

where $a$ is a positive constant.

2. **Weingarten translation surfaces**

Let $r : S \longrightarrow \mathbb{R}^3$ be an isometrical immersion of a translation surface of type

$$r(u, v) = (u, v, f(u) + g(v))$$

where $f$ and $g$ are smooth functions. The first fundamental form $I$ and the second fundamental form $II$ have particular forms, namely

$$I = (1 + f'(u)^2) du^2 + 2f'(u)g'(v) du dv + (1 + g'(v)^2) dv^2$$

$$II = \frac{1}{\sqrt{\Delta}} (f''(u) du^2 + g''(v) dv^2)$$

where $\Delta = 1 + f'(u)^2 + g'(v)^2$. Let us denote $f'$ by $\alpha$ and $g'$ by $\beta$. Hence, the mean curvature $H$ and the Gaussian curvature $K$ can be written as

$$H = \frac{(1 + \beta(v)^2) \alpha'(u) + (1 + \alpha(u)^2) \beta'(v)}{2 [1 + \alpha(u)^2 + \beta(v)^2]^{3/2}}$$

$$K = \frac{\alpha'(u)\beta'(v)}{[1 + \alpha(u)^2 + \beta(v)^2]^2}.$$ 

The existence of a Weingarten relation $U(H, K) = 0$ means that curvatures $H$ and $K$ are functionally related, and since $H$ and $K$ are differentiable functions depending on $u$ and $v$, this implies the Jacobian condition $\frac{\partial(U, K)}{\partial(u, v)} = 0$. More precisely the following condition

$$\frac{\partial H}{\partial u} \frac{\partial K}{\partial v} - \frac{\partial H}{\partial v} \frac{\partial K}{\partial u} = 0$$

needs to be satisfied. Conversely, if the above condition is satisfied, then the curvatures must be functionally related and thus, by definition, the surface must be Weingarten. The Jacobian condition characterizes W-surfaces and it is used to identify them when an explicit Weingarten relation cannot be found.
In our case, the Jacobian condition yields the following relation

\[ 8\alpha(u)\beta(v)\alpha'(u)^3\beta'(v)^2 - 8\alpha(u)\beta(v)\alpha'(u)^2\beta'(v)^3 - \\
-3\beta(v)\alpha'(u)^2\beta'(v)^2\alpha''(u) + 3\alpha(u)\alpha'(u)^2\beta'(v)\beta''(v) - \\
-2\alpha(u)^2\beta(v)\alpha'(u)\beta'(v)\alpha''(u) + 2\alpha(u)\beta(v)^2\alpha'(u)^2\beta'(v)\beta''(v) - \\
-3\beta(v)^3\alpha'(u)\beta'(v)^2\alpha''(u) + 3\alpha(u)^3\alpha'(u)^2\beta'(v)\beta''(v) - \\
-3\alpha(u)\alpha'(u)^3\beta'(v) + 3\beta(v)\beta'(v)^2\alpha''(u) + \\
+3\alpha(u)^2\beta(v)\beta'(v)^3\alpha''(u) - 3\alpha(u)\beta(v)^2\alpha'(u)^3\beta''(v) + \\
+\alpha'(u)\alpha''(u)\beta''(v) - \beta'(v)\alpha''(u)\beta''(v) + \\
+2\beta(v)^2\alpha'\alpha''(u)\beta'(v) - 2\alpha(u)^2\beta'(v)\alpha''(u)\beta''(v) + \\
+\alpha(u)^2\beta(v)^2\alpha'(u)\alpha''(u)\beta'(v) - \alpha(u)^2\beta(v)^2\beta'(v)\alpha''(u)\beta''(v) - \\
-\alpha(u)^3\beta'(v)\alpha''(u)\beta''(v) + \beta(v)^2\alpha'\alpha''(u)\beta''(v) = 0. \tag{6} \]

At this point we will consider \( \alpha \) and \( \beta \) to be polynomials of degree \( m \) and \( n \) respectively. More precisely we shall consider

\[ \alpha = a_mu^n + a_{m-1}u^{m-1} + \ldots \quad \text{and} \quad \beta = b_nv^n + b_{n-1}v^{n-1} + \ldots \]

where \( a_m \) and \( b_n \) are different from 0. Replacing \( \alpha \) and \( \beta \) in \( (6) \) we obtain a polynomial expression in \( u \) and \( v \) vanishing identically. This means that all the coefficients are 0.

Let us distinguish several cases:

**Case 1:** \( m, n \geq 2 \), i.e. \( \alpha'' \neq 0 \) and \( \beta'' \neq 0 \).

a. Suppose \( m > n (\geq 2) \)

The dominant term corresponds to \( u^{5m-2}v^{2n-3} \) and it comes from

\[ 3\alpha(u)^3\alpha'(u)^2\beta'(v)\beta''(v) - \alpha(u)^2\beta'(v)\alpha''(u)\beta''(v) \]

having the coefficient

\[ a_m^5b_n^2 \left( 3m^2n^2(n-1) - mn^2(m-1)(n-1) \right). \]

This cannot vanish since \( a_m, b_n \neq 0 \) and \( m > n \geq 2 \).

The subcase \( n > m \geq 2 \) can be treated in similar way.

b. Suppose \( m = n \geq 2 \)

In the same manner, this case cannot occur.

**Case 2:** \( m > n = 1 \)

In this case \( \beta \) can be expressed as \( \beta(v) = av + b \), with \( a \) and \( b \) real constants, \( a \neq 0 \).

We rewrite the Jacobian condition in the following way

\[ 8a^2\alpha(u)\beta(v)\alpha'(u)^3\beta(v) - 8a^2\alpha(u)\beta(v)\alpha'(u)^2\beta'(v)^3 - \\
-2a^2\beta(v)\alpha'(u)^2\beta'(v)^2\alpha''(u) + 3a^2\beta(v)\alpha'(u)^2\beta'(v)\beta''(v) - \\
-3a^2\beta(v)^3\alpha'(u)^2\beta'(v) + 3a^2\alpha(u)^3\alpha'(u)^2\beta'(v)\beta''(v) + \\
+3a^2\beta(v)^2\alpha'(u)^2\beta'(v) - 3a^2\beta(v)^2\beta'(v)\alpha''(u) + \\
+3a^2\beta(v)\alpha''(u) + 3a^2\alpha(u)^2\beta(v)\alpha''(u) = 0. \tag{7} \]

Let us analyze the terms in \( u \) having maximum degree, namely \( u^{4m-3}v \). This comes from \(-2a\alpha(u)^2\beta(v)\alpha'(u)^2\beta''(u)\) and yields the relation \( 2a^2a_m^4n^2(m-1) = 0 \) which cannot hold in this case.

The case \( n > m = 1 \) can be treated in similar way.

**Case 3:** \( m = n = 1 \)
In this case $\alpha$ and $\beta$ can be expressed as $\alpha(u) = Au + B$ and $\beta(v) = av + b$, with $A$, $B$, $a$ and $b$ real constants, $A, a \neq 0$. The Jacobian condition becomes
\[
8\alpha(u)\beta(v)\alpha'(u)^3\beta'(v)^2 - 8\alpha(u)\beta(v)\alpha'(u)^2\beta'(v)^3 = 0.
\]
(8)

Using the same technique as above one gets $A = a$. So, the parametrization of the surface can be written (after a possible translation in $\mathbb{E}^3$) in the form
\[
r(u, v) = (u, v, a(u - u_0)^2 + a(v - v_0)^2)
\]
(9)
where $u_0, v_0 \in \mathbb{R}$. This is a paraboloid of revolution, and its curvatures $H$ and $K$ are both everywhere positive and they are related by the relation (1).

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{paraboloid_revolution.png}
\caption{The paraboloid of revolution}
\end{figure}

**Case 4: $m = 0$ (or $n = 0$)**

In this case $\alpha$ (or $\beta$) is constant and the Jacobian condition is automatically satisfied. So, the parametrization of the surface can be written in the form
\[
r(u, v) = (u, v, au + g(v))
\]
(10)
\[
r(u, v) = (u, v, f(u) + av)
\]
(11)
where $f$ and $g$ are arbitrary polynomials, $a \in \mathbb{R}$ (it can also vanishes). These two surfaces are cylinders and they are obviously flat.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{cylinders_wpt.png}
\caption{Cylinders as WPT surfaces}
\end{figure}
It is interesting to remark that in the case when \( f(u) = au + b \) (or \( g(v) = av + b \)) (with \( a \) and \( b \) real constants) and the other function is not polynomial we still obtain a cylinder, hence a flat surface. We give in the following two examples:

![Figure 3. Other cylinders](image-url)

### 3. Linear Weingarten Translation Surfaces

A surface \( S \) is said to be a linear Weingarten surface – \( LW \) surface – if its mean curvature \( H \) and Gaussian curvature \( K \) satisfy the relation

\[
2aH + bK = c
\]

on \( S \) for real numbers \( a, b \) and \( c \) (not all zero). Such a surface is said to be elliptic, hyperbolic or parabolic depending on whether the discriminant \( a^2 + bc \) is positive, negative or zero.

Let us analyze the case when \( c = 0 \).

**Theorem 2.** Let \( S \) be a linear Weingarten translation surface satisfying

\[
2aH + bK = 0
\]

\( a \) and \( b \) real constants, at least one different from zero. Then

1. \( S \) is minimal (\( H = 0 \)) (Scherk’s surface) or
2. \( S \) is flat (\( K = 0 \)) (cylinders and planes).

**Proof.** Consider the parametrization (2) and the notations from the previous section. With \( H \) and \( K \) given by (3) and (4) respectively, we rewrite (13) in terms of \( \alpha \) and \( \beta \). One gets

\[
a \left( (1 + \beta(v)^2)\alpha'(u) + (1 + \alpha(u)^2)\beta'(v) \right) + \frac{b\alpha'(u)\beta'(v)}{\sqrt{\Delta}} = 0
\]

where \( \Delta = 1 + \alpha(u)^2 + \beta(v)^2 \).

Suppose there exist \( u_0 \) and \( v_0 \) such that \( \alpha'(u_0) \neq 0 \) and \( \beta'(v_0) \neq 0 \). It follows, due the smoothness of the two functions, there exist intervals \( I \supset u_0 \) and \( J \supset v_0 \) such that \( \alpha' \neq 0 \) on \( I \) and \( \beta' \neq 0 \) on \( J \). We obtain

\[
a \left( \frac{1 + \alpha(u)^2}{\alpha'(u)} + \frac{1 + \beta(v)^2}{\beta'(v)} \right) = -\frac{b}{\sqrt{\Delta}}
\]

for all \( u \in I \) and \( v \in J \). The left hand of the above expression is a sum of two terms, the first depends on \( u \) and the second depends on \( v \). Thus, after a derivation
with respect to $u$, followed by a derivation with respect to $v$, the left part of the obtained expression vanishes. In the same time, the right hand of the expression becomes $-3\alpha'(u)\beta(v)\alpha''(u)\beta'(v)\Delta^{-\frac{3}{2}}$.

Hence $b = 0$.

Since $a$ cannot be zero, it follows that $H = 0$, i.e. the surface $S$ is minimal. The remained cases are $\alpha' = 0$ and $\beta' = 0$, respectively. This means that either $f$ or $g$ is a linear function, i.e. the surface $S$ is a cylinder or a plane.

4. The second Gaussian curvature

In this section we deal with Riemannian surfaces in Euclidean space $E^3$ of which the second fundamental form $II$ is positive definite. One may associate to such surface $S$ geometrical objects measured by means of its second fundamental form, as second mean curvature $H_{II}$ and second Gaussian curvature $K_{II}$, respectively. Several results concerning $K_{II}$ have been already obtained, e.g. it is proved that every compact, convex surface in $E^3$ for which $K_{II}$ is constant is a sphere [4].

Using the formula of the second Gaussian curvature, a similar one to Brioschi’s formula for the Gaussian curvature obtained replacing the components of the first fundamental form $E$, $F$, $G$ by those of the second fundamental form $e$, $f$, $g$

$$K_{II} = \frac{1}{(|e| - f^2)^2} \begin{pmatrix} -\frac{1}{2}e_{vv} + f_{uv} - \frac{1}{2}g_{uu} & \frac{1}{2}e_u & f_v - \frac{1}{2}e_v & 0 & \frac{1}{2}e_v & \frac{1}{2}g_u \\ f_v - \frac{1}{2}g_u & e & f & 0 & \frac{1}{2}e_v & f \\ \frac{1}{2}g_v & f & g & 0 & \frac{1}{2}g_u & f \end{pmatrix}$$

one gets for the translation surfaces:

$$K_{II} = \frac{num}{4\Delta^{3/2}},$$

where

$$num = -2\alpha(u)^2\alpha'(u)^2\beta'(v) - 2\alpha'(u)\beta(v)^2\beta'(v)^2 + 2\alpha(u)^2\beta'(u)^2 + 2\alpha'(u)^2\beta(v)^2\beta'(v) + 2\alpha'(u)^2\beta'(v)^2 + 2\alpha'(u)^2\beta(v) + \alpha'(u)\beta(v)^2\beta'(v) + \alpha'(u)\beta'(v)^2\beta(v) + \alpha'(u)\beta'(v)^3\beta(v) + \alpha'(u)^3\alpha''(u)\beta'(v)$$

(14)

The general case of the translation surfaces with vanishing second Gaussian curvature in Minkowski 3-space was recently studied in [5]. An analogous result can be formulated for the Euclidean 3-space.

We study the PT surfaces with vanishing second Gaussian curvature, $K_{II} = 0$. It turns out that we need to find those polynomials $\alpha$ and $\beta$ of degree $m$, respectively $n$, so that $num = 0$ in (14).

At this point, using the same technique as in Section 2 the only convenient case is obtained for $m \geq n = 0$, or $n \geq m = 0$ namely for $\beta$ constant or $\alpha$ constant. We retrieve the Case 4. from Preliminaries, the surfaces given by the parametrization: (10) or (11).

Theorem 3. The only PT surfaces with vanishing second Gaussian curvature are cylindrical surfaces, which are also WPT surfaces.
Until now we studied PT surfaces. Inspired from the example given by Blair in [1] and mentioned in Preliminaries, we analyze other types of translation surfaces, involving power functions, i.e.
\[
\alpha = au^p \text{ and } \beta = bv^q \text{ with } a, b \in \mathbb{R}, \ a, b \neq 0 \text{ and } p, q \in \mathbb{Q}.
\]
The condition for vanishing second Gaussian curvature becomes:
\[
a(3p-1)u^pv + a^2b(3q-1)u^{2p+1}v^q + a^3(-p-1)u^{3p}v + b(3q-1)uv^q + ab^2(3p-1)u^pv^{2q+1} + b^3(-q-1)uv^{3q} = 0
\]
Again, using the same technique proposed in Section 2, the only possible case is for degrees \( p = q = \frac{1}{2} \), which implies the additional condition for coefficients, \( a = -b \).
We get the surface \( S \) given by the parametrization
\[
r(u, v) = \left( u, v, c(u^{\frac{1}{2}} - v^{\frac{1}{2}}) \right), c \in \mathbb{R}.
\]
We remark that, up to the multiplication factor \( c \), the example given by Blair is the only one translation surface of this type with vanishing second Gaussian curvature.

5. BACK TO THE JACOBIAN CONDITION

In this section we rewrite the Jacobian condition (11). Substituting \( \alpha \) and \( \beta \) with the corresponding power functions, one gets:
\[
ab^2pq(-q + q^2 + pq - pq^2)u^{p+1}v^{2q} + a^3b^2pq(-2q - pq + 2q^2 + pq^2)u^{3p+1}v^{2q} +
\]
\[
+a^5b^2pq(-q - 2pq + q^2 + 2pq^2)u^{5p+1}v^{2q} + ab^4pq(-q + pq - 2q^2 + 2pq^2)u^{p+1}v^{4q} +
\]
\[
+a^3b^4pq(-q - pq - 2q^2 - 4pq^2)u^{3p+1}v^{4q} +
\]
\[
+a^2b^5pq(p - p^2 - pq + p^2q)u^{2p}v^{5q+1} + a^2b^3pq(2p - 2p^2 - pq - p^2q)u^{2p}v^{3q+1} +
\]
\[
+a^2b^5pq(p - p^2 + 2pq - 2p^2q)u^{2p}v^{5q+1} + abpq(p + 2p^2 - pq - 2p^2q)u^{4p}v^{q+1} +
\]
\[
+a^4b^3pq(p + 2p^2 + pq + 4p^2q)u^{4p}v^{3q+1} = 0.
\]
It is easy to see that for \( p = 0 \) and any \( q \in \mathbb{Q} \) or for \( q = 0 \) and any \( p \in \mathbb{Q} \) the identity (17) is obviously true. So, if \( \alpha = \text{const} \) or \( \beta = \text{const} \), namely \( f = au + b \) or \( g = cv + d \), with \( a, b, c, d \in \mathbb{R} \), then we retrieve the same parametrization (10) or (11) like in Case 4, from Section 2.

The other interesting case is \( p = q = 1, \alpha = au \) and \( \beta = bv \), retrieving Case 3, from Section 2, the parametrization of the surface is given by (9).

Acknowledgement. The first author was supported by grant ID_308/2007-2010, ANCS, Romania. The second author was partially supported by grant CEEX – ET n. 5883/2006-2008, ANCS, Romania.
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