DISPERSIONLESS AND MULTICOMPONENT BKP HIERARCHIES WITH QUANTUM TORUS SYMMETRIES

CHUANZHONG LI

Department of Mathematics, Ningbo University, Ningbo 315211, China,
Email:lichuanzhong@nbu.edu.cn

ABSTRACT. In this article, we will construct the additional perturbative quantum torus symmetry of the dispersionless BKP hierarchy basing on the $\mathcal{W}_\infty$ infinite dimensional Lie symmetry. These results show that the complete quantum torus symmetry is broken from the BKP hierarchy to its dispersionless hierarchy. Further a series of additional flows of the multicomponent BKP hierarchy will be defined and these flows constitute an $N$-folds direct product of the positive half of the quantum torus symmetries.

Mathematics Subject Classifications(2000). 37K05, 37K10, 37K20.
Keywords: multicomponent BKP hierarchy, dispersionless BKP hierarchy, additional symmetry, $\mathcal{W}_\infty$ Lie algebra, quantum torus algebra, perturbative quantum torus algebra.

CONTENTS

1. Introduction 1
2. BKP hierarchy and dispersionless BKP hierarchy 2
3. Perturbative quantum torus symmetry of the dispersionless BKP hierarchy 3
4. Multicomponent BKP hierarchy 7
5. Quantum torus symmetry of the multicomponent BKP hierarchy 7
References 10

1. INTRODUCTION

The KP hierarchy is one of the most important integrable hierarchies\cite{1} and it arises in many different fields of mathematics and physics such as the enumerative algebraic geometry, topological field and string theory. One of the most important study on the KP hierarchy is the theoretical description of the solutions of the KP hierarchy using Lie groups and Lie algebras such as in \cite{1, 2, 3}, which is closely related to the infinite dimensional Grassmann manifolds \cite{4, 5}.

In \cite{6}, Date, Jimbo, Kashiwara and Miwa extended their work on the KP hierarchy to the multicomponent KP hierarchy. In \cite{7}, Takasaki and Takebe derived a series of differential Fay identities for the multicomponent KP hierarchy from the bilinear identities and they showed that their dispersionless limits give rise to the universal Whitham hierarchy. Besides the multicomponent KP hierarchy, the extended and reduced multicomponent Toda hierarchies attract a lot of studies \cite{8, 9, 10}.

Additional symmetries have been studied in the explicit form of the additional flows of the KP hierarchy by Orlov and Shulman \cite{11}. This kind of additional flows depend on dynamical variables explicitly and constitute a centerless $W_{1+\infty}$ algebra which is closely related to the theory of matrix models\cite{12, 13} by the Virasoro constraint and string equations. As a generalization of the Virasoro algebra, the Block algebra was studied a lot in the field of Lie algebras and it was studied intensively in references\cite{14}-\cite{17}. In another paper\cite{18}, we give a novel Block type additional symmetry of the
bigraded Toda hierarchy (BTH). Later we did a series of studies on integrable systems and Block algebras such as in [19, 20, 21]. After the quantization, the Block Lie algebra becomes the so-called quantum torus Lie algebra which can be seen in several recent references as [22, 23].

It is well known that the KP hierarchy has two sub-hierarchies, i.e. the BKP hierarchy and CKP hierarchy. About the BKP hierarchy, a lot of studies on additional symmetries have been done, such as additional symmetries of the BKP hierarchy [33], dispersionless BKP hierarchy [31, 34], two-component BKP hierarchy [24] and its reduced hierarchies [20, 21], supersymmetric BKP hierarchy [25] and so on. Dispersionless integrable systems [26] are very important in the study of all kinds of nonlinear sciences in physics, particularly in the application on the topological field theory [27] and matrix models [28, 29]. In particular, dispersionless integrable systems have many typical properties such as the Lax pair, infinite conservation laws, symmetries and so on. On dispersionless integrable systems, we also did several studies such as [19, 21]. With the above preparation, we should pay our attention to the quantum torus type additional symmetry of the multi-component BKP hierarchy [30] and dispersionless BKP hierarchy [31, 32] from the points of the multi-component generalization of Lie algebras and the importance of dispersionless integrable systems.

In the next section, we firstly review the Lax equations of the BKP and dispersionless BKP hierarchies. In Section 3, under the basic Sato theory, we construct the additional symmetries of the dispersionless BKP hierarchy and the symmetries form a perturbative quantum torus Lie algebra. In Section 4, we recall the Lax equation of the multicomponent BKP hierarchy. In Section 5, we construct the additional symmetry of the multicomponent BKP hierarchy which turns out to be in an \( N \)-folds direct product of the infinite dimensional complete quantum torus Lie algebra.

2. **BKP hierarchy and dispersionless BKP hierarchy**

Similarly to the general way in describing the classical the BKP hierarchy [1, 2], we will give a brief introduction of the BKP hierarchy. We denote “\(^*\)” as a formal adjoint operation defined by

\[
A^* = \sum (-1)^i \partial^i a_i
\]

for an arbitrary scalar-valued pseudo-differential operator \( A = \sum a_i \partial^i \), and \( (AB)^* = B^* A^* \) for two scalar operators \( A, B \). Basing on the definition, the Lax operator of the BKP hierarchy is as

\[
L_B = \partial + \sum_{i \geq 1} v_i \partial^{-i},
\]

such that

\[
L_B^* = -\partial L_B \partial^{-1}.
\]

The eq. (2.2) will be called the B type condition of the BKP hierarchy.

The BKP hierarchy is defined by the following Lax equations:

\[
\frac{\partial L_B}{\partial t_k} = \left[(L_B^k + 1) \right]_{B}, \quad k \in \mathbb{Z}_{+}^{\text{odd}}.
\]

The “\(^+\)” in (2.3) means the nonnegative projection about the operator “\(\partial\)” and “\(^-\)” means the negative projection. Note that the \( \partial/\partial t_1 \) flow is equivalent to the \( \partial/\partial x \) flow, therefore it is reasonable to assume \( t_1 = x \) in the next sections. The Lax operator \( L_B \) can be generated by a dressing operator \( \Phi_B = 1 + \sum_{k=1}^{\infty} \hat{\omega}_k \partial^{-k} \) in the following way

\[
L_B = \Phi_B \partial \Phi_B^{-1},
\]

where \( \Phi_B \) satisfies

\[
\Phi_B^* = \partial \Phi_B^{-1} \partial^{-1}.
\]

The dressing operator \( \Phi_B \) needs to satisfy the following Sato equations

\[
\frac{\partial \Phi_B}{\partial t_n} = -(L_B^n) \Phi_B, \quad n = 1, 3, 5, \cdots.
\]
Introduce firstly the Lax function of dispersionless BKP hierarchy \[31, 32\] as following
\[
L = k + u_1 k^{-1} + u_3 k^{-3} + \cdots + \ldots
\] (2.7)
where the coefficients \(u_1, u_3, \ldots\) of the Lax function are same as eq.(4.1). The variables \(u_j\) are functions of the real variable \(x\). The Lax function \(L\) can be written as
\[
L = e^{\text{ad}\phi(k)}, \quad \text{ad}\phi(\psi) = \{\phi, \psi\} = \frac{\partial \phi}{\partial k} \frac{\partial \psi}{\partial x} - \frac{\partial \phi}{\partial k} \frac{\partial \psi}{\partial x}.
\]

The dressing function has the following form
\[
\varphi = \sum_{n=1}^{\infty} \varphi_{2n} k^{-2n+1}.
\] (2.8)

The dressing function \(\varphi\) is unique up to adding some Laurent series about the variable \(k\) with coefficients which do not depend on \(x\). The dispersionless BKP hierarchy can be defined as following.

**Definition 2.1.** The dispersionless BKP hierarchy consists of flows given in the Lax pair by
\[
\frac{\partial L}{\partial t_n} = \{B_n, L\} = \frac{\partial B_n}{\partial k} \frac{\partial L}{\partial x} - \frac{\partial L}{\partial k} \frac{\partial B_n}{\partial x}, \quad n \in \mathbb{Z}_0^{\text{odd}},
\] (2.9)
where the functions \(B_n\) are defined by
\[
B_n := (L^n) -1, \quad n \in \mathbb{Z}_+^{\text{odd}}.
\]

The “+” here means the nonnegative projection about the variable “\(k\)” and “-” means the negative projection. The Lax equation of the dispersionless BKP hierarchy can lead to the following dispersionless Sato equations in the next proposition.

**Proposition 2.2.** \(L\) is the Lax function of the dispersionless BKP if and only if there exists a Laurent series \(\varphi\) (dressing function) which satisfies the equations
\[
\nabla_{tn, \varphi} \varphi = -(B_n)_-, \quad n \in \mathbb{Z}_0^{\text{odd}},
\] (2.10)
where
\[
\nabla_{tn, \psi} \varphi = \sum_{m=0}^{\infty} \frac{1}{(m+1)!} (\text{ad}\psi)^m \left( \frac{\partial \varphi}{\partial t_n} \right).
\]
The Laurent function \(\varphi\) is unique up to a transformation \(\varphi \mapsto H(\varphi, \psi)\), with a constant Laurent series \(\psi = \sum_{n=1}^{\infty} \psi_{2n} k^{-2n+1}\) (\(\psi_{2n}^2\) constant), where \(H(X, Y)\) is the Hausdorff series defined by
\[
\exp(\text{ad}H(\varphi, \psi)) = \exp(\text{ad}\varphi) \exp(\text{ad}\psi).
\]
The simplest nontrivial flow in the dispersionless BKP hierarchy is the \((2 + 1)\)-dimensional dispersionless BKP equation:
\[
3u_t + 15u^2 u_x - 5uu_y - 5u_x \partial^{-1} u_y - \frac{5}{3} \partial^{-1} u_{yy} = 0.
\]

3. **Perturbative quantum torus symmetry of the dispersionless BKP hierarchy**

In \[31, 34\], they construct additional symmetries of the dispersionless BKP hierarchy. In this section, we shall construct a specific kind of additional symmetries of the dispersionless BKP hierarchy and identify its nice quantum torus algebraic structure.

To this end, firstly we define the following dispersionless function \(\Gamma\) and the dispersionless Orlov-Shulman’s function \(M\) as
\[
\Gamma = x + \sum_{i \in \mathbb{Z}_0^{\text{odd}}} it_i \lambda^{i-1}, \quad M = e^{\text{ad}\varphi}(\Gamma) = \sum_{n=0}^{\infty} (2n + 1) t_{2n+1} L^{2n} + \sum_{n=0}^{\infty} v_{2n+2} L^{-2n-2}.
\] (3.1)
The dispersionless Lax function $L$ and the dispersionless Orlov-Shulman’s function $M$ satisfy the following canonical relation

$$\{L, M\} = 1.$$ 

Then basing on a quantum parameter $q$, the additional flows for the time variables $t_{m,n}, t^*_{m,n}$ are defined respectively as follows

$$\nabla_{t_{m,2n+1}, \varphi} = -(M^m L^{2n+1})_-, \nabla_{t^*_{m,n}, \varphi} = -(e^{mM} (q^nL - q^{-nL}))_-, \; m, n \in \mathbb{N}, \quad (3.2)$$

or equivalently rewritten as

$$\frac{\partial L}{\partial t_{m,2n+1}} = -\{(M^m L^{2n+1})_-, L\}, \quad \frac{\partial M}{\partial t_{m,2n+1}} = -\{(M^m L^{2n+1})_-, M\}, \quad (3.3)$$

$$\frac{\partial L}{\partial t^*_{m,n}} = -\{(e^{mM} (q^nL - q^{-nL}))_-, L\}, \quad \frac{\partial M}{\partial t^*_{m,n}} = -\{(e^{mM} (q^nL - q^{-nL}))_-, M\}. \quad (3.4)$$

Further one can also derive

$$\partial_{t^*_{l,k}} (e^{mM} q^{nL}) = \{- (e^{lM} (q^{kL} - q^{-kL}))_-, e^{mM} q^{nL}\}. \quad (3.5)$$

One can find the functions’ set $\{M^m L^n, \; m, n \geq 0, \; 1 \leq \alpha \leq N\}$ has an isomorphism with the operators’ set $\{z^n \partial_z^m, \; m, n \geq 0, \; 1 \leq \beta \leq N\}$ as

$$M^m L^n \mapsto z^n \partial_z^m, \quad (3.6)$$

with the following commutator

$$[M^m L^n, M^k L^l] = C^{(mn)(kl)}_{ab} M^a L^b. \quad (3.7)$$

One can find the functions’ set $\{e^{mM} q^{nL}, \; m, n \geq 0, \; 1 \leq \alpha \leq N\}$ has an isomorphism with the operators’ set $\{q^n z^m \partial_z, \; m, n \geq 0, \; 1 \leq \beta \leq N\}$ as

$$e^{mM} q^{nL} \mapsto q^n z^m \partial_z, \quad (3.8)$$

with the following commutator

$$[q^n z^m \partial_z, q^l z^k \partial_z] = (q^{ml} - q^{nk}) q^{(n+l)} z^{(m+k)} \partial_z. \quad (3.9)$$

The additional flows $\frac{\partial}{\partial t_{m,n}}$ will be proved later to commute with the flows $\frac{\partial}{\partial t_k}$, i.e. $[\frac{\partial}{\partial t_{m,n}}, \frac{\partial}{\partial t_k}] = 0$, but they do not commute with each other. They can form a $W_\infty$ infinite dimensional Lie algebra. This further leads to the commutativity of the additional flows $\frac{\partial}{\partial t^*_{m,n}}$ with the flows $\frac{\partial}{\partial t_k}$ and the additional flows $\frac{\partial}{\partial t^*_{m,n}}$ themselves constitute a perturbative quantum torus algebra which will be proved later.

**Proposition 3.1.** The additional flows $\partial_{t_{l,k}}$ are symmetries of the dispersionless BKP hierarchy, i.e. they commute with all $\partial_{t_n}$ flows of the dispersionless BKP hierarchy.
Proof. According the action of $\partial_{t_{l,k}}$ and $\partial_{t_n}$ on the dressing function $L$, then

$$\left[\partial_{t_{m,l}}, \partial_{t_n}\right]L$$

$$= \partial_{t_{m,l}}\partial_{t_n}e^{ad\varphi}(k) - \partial_{t_n}\partial_{t_{m,l}}e^{ad\varphi}(k)$$

$$= \partial_{t_{m,l}}\left\{\nabla_{t_n,\varphi}e^{ad\varphi}(k)\right\} - \partial_{t_n}\left\{\nabla_{t_{m,l},\varphi}e^{ad\varphi}(k)\right\}$$

$$= \partial_{t_{m,l}}\left\{-\{B_n\}_-, L\right\} - \partial_{t_n}\left\{\left(M^m L^l\right)_+, L\right\}$$

$$= \{-\left(M^m L^l\right)_+, B_n\}_-, L\} + \{-B_n\}_-, \left\{\left(M^m L^l\right)_+, L\right\} - \{-B_n\}_-, \left(M^m L^l\right)_+, L\} - \{B_n\}_-, \left\{\left(M^m L^l\right)_+, L\right\}$$

$$= \{-L, \left\{B_n\}_-, \left(M^m L^l\right)_+\} + \left\{\left(M^m L^l\right)_+, B_n\}_+ - \{-B_n\}_-, \left(M^m L^l\right)_+\}$$

$$= 0.$$

Therefore the proposition holds. \qed

With the help of this proposition, we can derive the following theorem.

**Theorem 3.2.** The additional flows $\partial_{t_{l,k}}$ are symmetries of the dispersionless BKP hierarchy, i.e. they commute with all $\partial_{t_n}$ flows of the dispersionless BKP hierarchy.

**Proof.** According to the action of $\partial_{t_{l,k}}$ and $\partial_{t_n}$ on the dispersionless dressing function $L$, we can rewrite the quantum torus flow $\partial_{t_{l,k}}$ in terms of a combination of $\partial_{p,s}$ flows

$$\partial_{t_{l,k}} L = \{-L, \left\{B_n\}_-, \left(M^m L^l\right)_+\} + \left\{\left(M^m L^l\right)_+, B_n\}_+ - \{-B_n\}_-, \left(M^m L^l\right)_+\}$$

$$= \sum_{p,s=0}^{\infty} \frac{lp(k log q)^{2s+1}M^p L^{2s+1}}{p!(2s+1)!} \partial_{p,2s+1} L,$$

which further leads to

$$\left[\partial_{t_{l,k}}, \partial_{t_n}\right]L = \left\{\sum_{p,s=0}^{\infty} \frac{lp(k log q)^{2s+1}M^p L^{2s+1}}{p!(2s+1)!} \partial_{p,2s+1}, \partial_{t_n}\right\}L$$

$$= \sum_{p,s=0}^{\infty} \frac{lp(k log q)^{2s+1}}{p!(2s+1)!} \left[\partial_{p,2s+1}, \partial_{t_n}\right]L$$

$$= 0.$$

Therefore the theorem holds. \qed

Because

$$[z^s \partial^p, z^b \partial^\alpha] = \sum_{\alpha\beta} C^{(ps)(ab)}_{\alpha\beta} z^\beta \partial^\alpha,$$

and

$$[q^n z^m \partial_k, q^l z^{\kappa} \partial_s] = (q^{m-l} - q^{n-k})q^{(n+l)}z^{(m+k)}\partial_k,$$

therefore we can derive the following identity

$$3.10$$
Now it is time to identify the algebraic structure of the additional $\partial_{t_{i,k}}^*$ flows of the dispersionless BKP hierarchy in the following theorem.

**Theorem 3.3.** The additional flows $\partial_{t_{i,k}}^*$ of the dispersionless BKP hierarchy form a perturbative quantum torus algebra, i.e.,

$$[\partial_{t_{n,m}}^*, \partial_{t_{i,k}}^*]L = [(q^{mL} - q^{-nk})\partial_{n+i,m-k}^* - (q^{mL} - q^{-nk})\partial_{n+i,m+k}^*]L - \{A_{nmkl}, L\},$$

(3.11)

where

$$A_{nmkl} = (q^{mL} - q^{-nk} + q^{-ml} - q^{-nk})e^{(n+l)M}(q^{(-m-k)L} - q^{(-m+k)L}).$$

**Proof.** Using the Jacobi identity, we can derive the following computation which will finish the proof of this theorem

$$[\partial_{t_{n,m,\beta}}^*, \partial_{t_{i,k}}^*]L = \partial_{t_{n,m}}^*\{-(e^M(q^{kL} - q^{-kL})_L), L\} - \partial_{t_{i,k}}^*\{-(e^M(q^{mL} - q^{-ml})_L), L\}$$

$$+\{(e^M(q^{kL} - q^{-kL})_L), e^M(q^{mL} - q^{-ml})_L, L\}$$

$$+\{(e^M(q^{mL} - q^{-ml})_L), (e^M(q^{kL} - q^{-kL})_L, L\}$$

$$+\{(e^M(q^{mL} - q^{-ml})_L, e^M(q^{kL} - q^{-kL})_L, L\}$$

$$+\{(e^M(q^{mL} - q^{-ml})_L, e^M(q^{kL} - q^{-kL})_L, L\}$$

$$+\{(e^M(q^{mL} - q^{-ml})_L, e^M(q^{kL} - q^{-kL})_L, L\}$$

$$= \{(e^M(q^{mL} - q^{-ml})_L, e^M(q^{kL} - q^{-kL})_L, L\}$$

Comparing the above proposition with [23] in which the complete quantum torus symmetry of the BKP hierarchy was constructed, the following remark should be noted.

**Remark 3.4.** The classical quantum torus structure

$$[\partial_{t_{n,m}}^*, \partial_{t_{i,k}}^*] = (q^{mL} - q^{-nk})\partial_{n+i,m-k}^* - (q^{mL} - q^{-nk})\partial_{n+i,m+k}^*$$

(3.12)

is broken from the BKP hierarchy to its dispersionless hierarchy of which there exists one more perturbative term $\{A_{nmkl}, L\}$. 
4. Multicomponent BKP hierarchy

For an $N$-component BKP hierarchy, there are $N$ infinite families of time variables $t_{\alpha,n}, \alpha = 1, \ldots, N, n = 1, 3, 5, 7, \ldots$. The coefficients $A, u_1, u_2, \ldots$ of the Lax operator

$$\mathcal{L}_B = A\partial + u_1 \partial^{-1} + u_2 \partial^{-2} + \ldots$$  \hspace{1cm} (4.1)

are $N \times N$ matrices and $A = \text{diag}(a_1, a_2, \ldots, a_N)$. There are another $N$ pseudo-differential operators $R_1, \ldots, R_N$ in the form of

$$R_\alpha = E_\alpha + u_{\alpha,1} \partial^{-1} + u_{\alpha,2} \partial^{-2} + \ldots,$$

where $E_\alpha$ is the $N \times N$ matrix with “1” on the $(\alpha, \alpha)$-component and zero elsewhere, and $u_{\alpha,1}, u_{\alpha,2}, \ldots$ are also $N \times N$ matrices. The operators $\mathcal{L}_B, R_1, \ldots, R_N$ satisfy the following conditions:

$$\mathcal{L}_B R_\alpha = R_\alpha \mathcal{L}_B, \quad R_\alpha R_\beta = \delta_{\alpha\beta} R_\alpha, \quad \sum_{\alpha=1}^N R_\alpha = E.$$

Let $*$ denote a formal adjoint operation defined by $p^* = \sum (-1)^i \partial^i p_i^T$ for an arbitrary matrix-valued pseudo-differential operator $p = \sum p_i^T \partial^i$, and $(AB)^* = B^* A^*$ for two matrix-valued pseudo-differential operators $A, B$. Here $\mathcal{L}_B$ must satisfy

$$\mathcal{L}_B^* = -\partial \mathcal{L}_B \partial^{-1}. \hspace{1cm} (4.2)$$

The Lax equations of the multicomponent BKP hierarchy are:

$$\frac{\partial \mathcal{L}_B}{\partial t_n} = [B_{\alpha,n}, \mathcal{L}_B], \quad \frac{\partial R_\beta}{\partial t_n} = [B_{\alpha,n}, R_\beta], \quad B_{\alpha,n} := (\mathcal{L}_B^n R_\alpha)_+, \quad n \in \mathbb{Z}^{\text{odd}}.$$

The operator $\partial$ now is equal to $a_1^{-1} \partial_{\xi_1} + \ldots + a_N^{-1} \partial_{\xi_N}$. In fact the Lax operators $\mathcal{L}_B$ and $R_\alpha$ can have the following dressing structures

$$\mathcal{L}_B = \Phi A \partial \Phi^{-1}, \quad R_\alpha = \Phi E_\alpha \Phi^{-1}.$$

Then the dressing operator $\Phi$ needs to satisfy

$$\Phi^* = \partial \Phi^{-1} \partial^{-1}. \hspace{1cm} (4.3)$$

We call the eq. (4.2) the B type condition of the $N$-component BKP hierarchy. We can get the operators $B_{\alpha,n}, R_j$ satisfy the following B type condition

$$B_{\alpha,n}^* = -\partial B_{\alpha,n} \partial^{-1}, \quad R_j^* = \partial R_j \partial^{-1}. \hspace{1cm} (4.4)$$

and the dressing operator $\Phi$ satisfies the following Sato equations

$$\frac{\partial \Phi}{\partial t_n} = -(\mathcal{L}_B^n R_\alpha)_- \Phi.$$

5. Quantum torus symmetry of the multicomponent BKP hierarchy

To construct the additional quantum torus symmetry of the multicomponent BKP hierarchy, firstly we define the operator $\Gamma_B$ and the Orlov-Shulman’s operator $\mathcal{M}_B$ as

$$\Gamma_B = \sum_{i \in \mathbb{Z}^{\text{odd}}} \sum_{j=1}^N i t_i^j A^{-1} E_{jj} \partial^{-1}, \quad \mathcal{M}_B = \Phi \Gamma_B \Phi^{-1}. \hspace{1cm} (5.1)$$

The Lax operator $\mathcal{L}_B$ and the Orlov-Shulman’s operator $\mathcal{M}_B$ satisfy the following matrix canonical relation

$$[\mathcal{L}_B, \mathcal{M}_B] = E.$$

Given an operator $\mathcal{L}_B$, the dressing operators $\Phi$ are determined uniquely up to a multiplication to the right by operators with constant coefficients.
We denote \( t = (t_1, t_3, t_5, \ldots) \) and introduce the wave function as
\[
w_B(t; z) = \Phi e^{\xi_B(t; z)},
\]
where the matrix function \( \xi_B \) is defined as \( \xi_B(t; z) = \sum_{k \in \mathbb{Z}^+} \sum_{j=1}^{N} t_j^k E_{j} z^k \). It is easy to see
\[
\partial_i e^{xz} = z^i e^{xz}, \quad i \in \mathbb{Z}
\]
and
\[
L_B w_B(t; z) = zw_B(t; z), \quad \partial w_B \partial t_{2n+1} = \left( L_{2n+1}^R j \right) w_B.
\]

With the above preparation, it is time to construct additional symmetries for the multicomponent BKP hierarchy in the next part. It is easy to get that the operator \( M_B \) satisfy
\[
[L_B, M_B] = 1, \quad M_B w_B(z) = \partial_z w_B(z);
\]
\[
\frac{\partial M_B}{\partial t_k} = \left[ (L_k^R j)_{+,}, M_B \right], \quad k \in \mathbb{Z}^+\text{odd}.
\]

Given any pair of integers \((m, n)\) with \( m, n \geq 0 \), we will introduce the following matrix-valued operator \( B_{mnj} \)
\[
B_{mnj} = \mathcal{M}_B^m L_B^n R_j - (-1)^n R_j L_B^{n-1} \mathcal{M}_B^m L_B.
\]

For any matrix operator \( B_{mnj} \) in \((5.5)\), one has
\[
\frac{\partial B_{mnj}}{\partial t_k} = \left[ (L_k^R j)_{+,}, B_{mnj} \right], \quad k \in \mathbb{Z}^+\text{odd}.
\]

To prove that the \( B_{mnj} \) satisfies the B type condition, we need the following lemma.

**Lemma 5.1.** The matrix operator \( \mathcal{M}_B \) satisfies the following identity,
\[
\mathcal{M}_B^* = \partial \mathcal{L}_B^{-1} \mathcal{M}_B \mathcal{L}_B \partial^{-1}.
\]

**Proof.** Using identities as
\[
\Phi^* = \partial \Phi^{-1} \partial^{-1}, \quad \Gamma_B^* = \Gamma_B;
\]
the following calculations
\[
\mathcal{M}_B^* = \Phi^{-1} \Gamma_B \Phi^* = \partial \Phi^{-1} \Gamma_B \partial \Phi^{-1} \partial^{-1} = \partial \Phi \partial^{-1} \Phi^{-1} \mathcal{M}_B \Phi \partial \Phi^{-1} \partial^{-1},
\]
will lead to the lemma. \( \square \)

Basing on the Lemma 5.1 above, it is easy to check that the matrix-valued operator \( B_{mnj} \) satisfy the B type condition, namely
\[
B_{mnj}^* = -\partial B_{mnj} \partial^{-1}.
\]

Now we will denote the matrix operator \( D_{mnj} \) as
\[
D_{mnj} := e^{m \mathcal{M}_B^p q^n \mathcal{L}_B} R_j - L_B^{-1} R_j q^{-n} \mathcal{L}_B e^{m \mathcal{M}_B \mathcal{L}_B},
\]
which further leads to
\[
D_{mnj} = \sum_{p,s=0}^{\infty} \frac{m^p (n \log q)^s (\mathcal{M}_B^p \mathcal{L}_B^s R_j - (-1)^s R_j \mathcal{L}_B^{s-1} \mathcal{M}_B^p \mathcal{L}_B)}{p! s!} = \sum_{p,s=0}^{\infty} \frac{m^p (n \log q)^s B_{psj}}{p! s!}.
\]
Using the eq. \((5.7)\), the following calculation will lead to the B type anti-symmetry property of \(D_{mnj}\) as
\[
D_{mnj}^* = \left( \sum_{p,s=0}^\infty \frac{m^p (n \log q)^s B_{psj}}{p!s!} \right)^* \\
= -\left( \sum_{p,s=0}^\infty \frac{m^p (n \log q)^s \partial B_{psj} \partial^{-1}}{p!s!} \right) \\
= -\partial \left( \sum_{p,s=0}^\infty \frac{m^p (n \log q)^s B_{psj}}{p!s!} \right) \partial^{-1} \\
= -\partial D_{mnj} \partial^{-1}.
\]
Therefore we get the following important B type condition, i.e. the matrix operator \(D_{mnj}\) satisfies
\[
D_{mnj}^* = -\partial D_{mnj} \partial^{-1}. \quad (5.10)
\]
Then basing on a quantum parameter \(q\), the additional flows for the time variable \(t_m,n, t^*_{m,n}\) are defined as follows
\[
\frac{\partial \Phi}{\partial t^j_{m,n}} = -(B_{mnj}) - \Phi, \quad \frac{\partial \Phi}{\partial t^*_{j_{m,n}}} = -(D_{mnj}) - \Phi, \quad (5.11)
\]
or equivalently rewritten as
\[
\frac{\partial L_B}{\partial t^j_{m,n}} = -[(B_{mnj}) - , L_B], \quad \frac{\partial M_B}{\partial t^*_{j_{m,n}}} = -[(D_{mnj}) - , M_B]. \quad (5.12)
\]
Generally, one can also derive
\[
\partial_{t^*_l,k} (D_{mnj}) = [-(D_{lk}) - , D_{mnj}]. \quad (5.13)
\]
The way in the construction of the matrix-valued operator \(D_{mnj}\) depends on the reduction condition in the eq. \((4.2)\) on the generators of the additional flows.
This further leads to the commutativity of the additional flow \(\frac{\partial}{\partial t^j_{m,n}}\) with the flow \(\frac{\partial}{\partial t^k}\) in the following theorem.

**Theorem 5.2.** The additional flows of \(\partial_{t^*_l,k}\) are symmetries of the multicomponent BKP hierarchy, i.e. they commute with all \(\partial_{t^*_l}\) flows of the multicomponent BKP hierarchy.

**Proof.** The proof is similar as the KP hierarchy by using the Theorem 6.2 in [23], i.e. the additional flows of \(\partial_{t^*_l,k}\) can commute with all \(\partial_{t^*_l}\) flows of the multicomponent BKP hierarchy. The detail will be omitted here. \(\square\)

Comparing with the additional symmetry of the single-component BKP hierarchy in [33], the additional flows \(\partial_{t^*_l,k}\) of the multicomponent BKP hierarchy form the following \(N\)-folds direct product of the \(W_\infty\) algebra as following
\[
[\partial_{t^*_p,a}, \partial_{t^*_n,b}] L_B = \delta_{rc} \sum_{\alpha\beta} C^{(ps)(ab)}_{\alpha\beta} \partial_{t^*_a,\beta} L_B.
\]
Now it is time to identity the algebraic structure of the additional \(t^*_l,k\) flows of the multicomponent BKP hierarchy.
Theorem 5.3. The additional flows $\partial_{t^r_{n,m}}$ of the multicomponent BKP hierarchy form the $\bigotimes^N QT_+$ algebra (an $N$-folds direct product of the positive half of the quantum torus algebra $QT$), i.e.,

$$[\partial_{t^r_{n,m}}, \partial_{t^j_{l,k}}] = \delta_{rs}(q^{ml} - q^{nk})\partial_{t^r_{n+1,m+k}}, \quad n, m, l, k \geq 0, \quad 1 \leq r, j \leq N. \quad (5.14)$$

Proof. One can also prove this theorem as following by rewriting the quantum torus flow in terms of a combination of $\partial_{t^r_{n,n}}$ flows

$$[\partial_{t^r_{n,m}}, \partial_{t^j_{l,k}}] \mathcal{L}_B$$

$$= \sum_{p,s=0}^{\infty} \frac{n^p(m \log q)^s}{p!s!} \partial_{t^r_{p,s}} \sum_{a,b=0}^{\infty} \frac{l^a(k \log q)^b}{a!b!} \partial_{t^j_{a,b}} \mathcal{L}_B$$

$$= \sum_{p,s=0}^{\infty} \sum_{a,b=0}^{\infty} \frac{n^p(m \log q)^s l^a(k \log q)^b}{p!s!a!b!} \partial_{t^r_{p,s}} \partial_{t^j_{a,b}} \mathcal{L}_B$$

$$= \sum_{p,s=0}^{\infty} \sum_{a,b=0}^{\infty} \frac{n^p(m \log q)^s l^a(k \log q)^b}{p!s!a!b!} \sum_{a, \beta} C^{(ps)(ab)}_{a \beta} \delta_{rj} \delta_{r, a, \beta} \mathcal{L}_B$$

$$= (q^{ml} - q^{nk}) \sum_{a, \beta=0}^{\infty} \frac{(m + l)^a((m + k) \log q)^\beta}{a!\beta!} \delta_{rj} \partial_{t^r_{n+1,m+k}} \mathcal{L}_B$$

$$= (q^{ml} - q^{nk}) \delta_{rj} \partial_{t^r_{n+1,m+k}} \mathcal{L}_B.$$  

From these above, we can find that the quantum torus symmetry can be generalized from the BKP hierarchy to the multicomponent BKP hierarchy by the multi-folds product.

Acknowledgements: This work is supported by the National Natural Science Foundation of China under Grant No. 11571192 and the K. C. Wong Magna Fund in Ningbo University.

References

[1] E. Date, M. Kashiwara, M. Jimbo, and T. Miwa, Transformation groups for soliton equations, Nonlinear integrable systems-classical theory and quantum theory (Kyoto, 1981), World Sci. Publishing, Singapore, 1983, 39-119.

[2] L. Dickey, Soliton equations and hamiltonian systems, World Scientific, Singapore, 1991.

[3] V. Kac, J. van de Leur, The N-component KP hierarchy and representation theory, in: A.S. Fokas, V.E. Zakharov (Eds.), Important developments in soliton theory, Springer-Verlag, Berlin, Heidelberg, 1993.

[4] M. Sato, Soliton equations as dynamical systems on infinite dimensional Grassmann manifolds, RIMS Kokyuroku 30 (1981).

[5] M. Sato, The KP hierarchy and infinite-dimensional Grassmann manifolds, in Theta functions–Bowdoin 1987, Part I, Proc. Sympos. Pure Math., 49, Part 1, Amer. Math. Soc., Providence, RI, (1989), 51-66.

[6] E. Date, M. Jimbo, M. Kashiwara and T. Miwa, Transformation groups for soliton equations III, J. Phys. Soc. Japan 50 (1981), 3806–3812.

[7] K. Takasaki and T. Takebe, Universal Whitham hierarchy, dispersionless Hirota equations and multicomponent KP hierarchy, Physica D 235 (2007), 109-125.

[8] C. Álvarez Fernández, U. Fidalgo Prieto, and M. Mañas 2010 Multiple orthogonal polynomials of mixed type: Gauss-Borel factorization and the multi-component 2D Toda hierarchy, Advances in Mathematics, 227(2011), 1451-1525.

[9] C. Z. Li, J. S. He, The extended multi-component Toda hierarchy, Math. Phys. Analysis and Geometry. 17(2014), 377-407.

[10] C. Z. Li, J. S. He, The extended $Z_N$-Toda hierarchy, Theor. Math. Phys. 185(2015), 1614-1635.

[11] A. Yu. Orlov, E. I. Schulman, Additional symmetries of integrable equations and conformal algebra reprensentaion, Lett. Math. Phys. 12(1986), 171-179.

[12] R. Dijkgraaf, E. Witten, Mean field theory, topological field theory, and multimatrix models, Nucl. Phys. B 342 (1990), 486-522.
[13] M. Douglas, Strings in less than one dimension and the generalized KdV hierarchy, Phys. Lett. B 238 (1990), 176-180.

[14] R. Block, On torsion-free abelian groups and Lie algebras, Proc. Amer. Math. Soc., 9(1958), 613-620.

[15] J. M. Osborn, K. Zhao, Infinite-dimensional Lie algebras of generalized Block type, Proc. Amer. Math. Soc., 127(1999), 1641-1650.

[16] X. Xu, Generalizations of Block algebras, Manuscripta Math., 100(1999), 489-518.

[17] J. M. Osborn, K. Zhao, Infinite-dimensional Lie algebras of generalized Block type, Proc. Amer. Math. Soc., 127(1999), 1641-1650.

[18] X. Xu, Generalizations of Block algebras, Manuscripta Math., 100(1999), 489-518.

[19] C. Z. Li, J. S. He, Dispersionless bigraded Toda hierarchy and its additional symmetry, Reviews in Mathematical Physics, 24(2012), 1230003.

[20] C. Z. Li, J. S. He, Block algebra in two-component BKP and D type Drinfeld-Sokolov hierarchies, J. Math. Phys. 54(2013), 113501.

[21] C. Z. Li, J. S. He, Y. C. Su, Block (or Hamiltonian) Lie symmetry of dispersionless D type Drinfeld-Sokolov hierarchy, Commun. Theor. Phys. 61(2014), 431-435.

[22] T. Nakatsu, K. Takasaki, Melting Crystal, Quantum Torus and Toda Hierarchy, Commun. Math. Phys. 285(2009), 445-468.

[23] C. Z. Li, J. S. He, Quantum Torus symmetry of the KP, KdV and BKP hierarchies, Lett. Math. Phys. 104(2014), 1407-1423.

[24] S. Q. Liu, C. Z. Wu, Y. Zhang, On the Drinfeld-Sokolov hierarchies of D type, Intern. Math. Res. Notices 2011(2011),1952-1996.

[25] C. Z. Li, J. S. He, Supersymmetric BKP systems and their symmetries, Nuclear Physics B 896(2015), 716-737.

[26] K. Takasaki, T. Takebe, Integrable hierarchy and dispersionless limit, Review in Mathematical Physics, 7(1995),743-808.

[27] K. Takasaki, Dispersionless Toda hierarchy and two-dimensional string theory, Commun. Math. Phys. 170(1995), 101-116.

[28] Y. Kodama, V. U. Pierce, Combinatorics of dispersionless integrable systems and universality in random matrix theory, Commun. Math. Phys. 292(2009), 529-568.

[29] T. Eguchi, S. K. Yang, The topological $CP^1$ model and the large-N matrix integral. Modern Phys. Lett. A 9(1994), 2893-2902.

[30] V. Kac, J. van de Leur, The geometry of spinors and the multicomponent BKP and DKP hierarchies, CRM Proc. Lect. Notes, Providence, American Mathematical Society, 14(1998), 159-202.

[31] K. Takasaki, Quasi-Classical Limit of BKP Hierarchy and W-Infinity Symmetries, Lett. Math. Phys. 28(1993), 177-185.

[32] L. V. Bogdanov, B. G. Konopelchenko, On dispersionless BKP hierarchy and its reductions, J. Nonlinear Math. Phys., 12(2005), 64-75.

[33] M. H. Tu, On the BKP hierarchy: additional symmetries, Fay identity and Adler-Shiota-van Moerbeke formula, Lett. Math. Phys. 81(2007), 93-105.

[34] Y. T. Chen, M. H. Tu, A note on the dispersionless BKP hierarchy, J. Phys. A: Math. Gen. 39(2006), 7641-7655.