Bounds on Walsh coefficients by dyadic difference and a new Koksma-Hlawka type inequality for Quasi-Monte Carlo integration

Takehito Yoshiki

Abstract

In this paper we give a new Koksma-Hlawka type inequality for Quasi-Monte Carlo (QMC) integration. QMC integration of a function \( f : [0,1)^s \rightarrow \mathbb{R} \) by a finite point set \( P \subset [0,1)^s \) is the approximation of the integral \( I(f) := \int_{[0,1)^s} f(x) \, dx \) by the average \( I_P(f) := \frac{1}{|P|} \sum_{x \in P} f(x) \). We treat a certain class of point sets \( P \) called digital nets. A Koksma-Hlawka type inequality is an inequality bounding the integration error \( \text{Err}(f;P) := |I(f) - I_P(f)| \) by a bound of the form \( |\text{Err}(f;P)| \leq C \cdot \|f\| \cdot D(P) \). We can obtain a Koksma-Hlawka type inequality by estimating bounds on \( |\hat{f}(k)| \), where \( \hat{f}(k) \) is a generalized Fourier coefficient with respect to the Walsh system. In this paper we prove bounds on Walsh coefficients \( \hat{f}(k) \) by introducing an operator called ‘dyadic difference’ \( \partial_{i,n} \). By converting dyadic differences \( \partial_{i,n} \) to derivatives \( \frac{\partial}{\partial x_i} \), we get a new bound on \( |\hat{f}(k)| \) for a function \( f \) whose mixed partial derivatives up to order \( \alpha \) in each variable are continuous. This new bound is smaller than the known bound on \( |\hat{f}(k)| \) under some condition. The new Koksma-Hlawka inequality is derived using this new bound on the Walsh coefficients.

1 Introduction

Quasi-Monte Carlo (QMC) integration of a function \( f : [0,1)^s \rightarrow \mathbb{R} \) by a finite point set \( P \subset [0,1)^s \) is the approximation of the integral \( I(f) := \int_{[0,1)^s} f(x) \, dx \) by the average \( I_P(f) := \frac{1}{|P|} \sum_{x \in P} f(x) \) (see [7], [10] and [19] for details). We want to find quadrature point sets \( P \) making the absolute value of the integration error \( |\text{Err}(f;P)| := |I(f) - I_P(f)| \) small for a set of functions \( f \). This problem is formulated as follows: We consider a function space \( H \) with a norm \( \|f\|_H \) and the worst case error \( \sup_{\|f\|_H \leq 1} |\text{Err}(f;P)| \) by a QMC rule using the point set \( P \).
(for example, see [7], [14] for details). Then, it holds that, for any \( f \in H \),

\[
|\text{Err}(f; \mathcal{P})| \leq \|f\|_H \times \sup_{\|f\|_H \leq 1} |\text{Err}(f; \mathcal{P})|.
\]

Thus, we aim to obtain quadrature point sets \( \mathcal{P} \) making the worst case error \( \sup_{\|f\|_H \leq 1} |\text{Err}(f; \mathcal{P})| \) small.

We often treat a point set \( \mathcal{P} \) called ‘digital net’ (for example, see [16]).

A digital net \( \mathcal{P} \) is defined as follows. Let \( n, m \geq 1 \) be integers with \( n \geq m \). Let \( 0 \leq h < b^m \) be an integer and \( C_1, \ldots, C_s \) be \( n \times m \) matrices over \( \mathbb{Z}_b = \mathbb{Z}/b\mathbb{Z} \). We write the \( b \)-adic expansion \( h = \sum_{j=1}^m h_j b^{-j} \) and take a vector \( h = (h_1, \ldots, h_m) \in (\mathbb{Z}_b^n)^\top \), where \( h_j \) is considered to be an element in \( \mathbb{Z}_b \). For \( 1 \leq i \leq s \), we define the vector \( (y_{h, i, 1}, \ldots, y_{h, i, n}) = h \cdot (C_i)^\top \) and a real number \( x_i(h) = \sum_{1 \leq j \leq n} y_{h, i, j} b^{-j} \in [0, 1) \), where \( y_{h, i, j} \) is considered to be an element of \( \{0, \ldots, b-1\} \subset \mathbb{Z} \). Then we define a digital net \( \mathcal{P} \) by \( \{x_0, \ldots, x_{b^m-1}\} \) where \( x_b = (x_i(h))_{1 \leq i \leq s} \). We define the dual net \( \mathcal{P}^\perp \) [16], which is essential to analyze the integration error:

\[
\mathcal{P}^\perp := \{k = (k_1, \ldots, k_s) \in \mathbb{N}_0^s \mid C_1^\top \tilde{k}_1 + \cdots + C_s^\top \tilde{k}_s = 0 \in \mathbb{Z}_b^m\},
\]

where \( \tilde{k}_i = (\kappa_{i, 1}, \ldots, \kappa_{i, n})^\top \) for \( b \)-adic expression \( k_i = \sum_{j \geq 1} \kappa_{i, j} b^{-j} \). Here \( \kappa_{i, j} \) is considered to be an element of \( \mathbb{Z}_b \). Throughout this paper, when we take a point set \( \mathcal{P} \), we assume that \( \mathcal{P} \) is a digital net with \( b = 2 \).

In the classical theory, many researchers studied the integration error of a function \( f \) with bounded variation (or function with square integrable partial derivatives up to first order in each variable) (for example, see [7], [13]). An extension to smooth periodic functions was established in [3], while a further extension to smooth (non-periodic) functions was shown in [4]. The QMC rules constructed in these papers, called higher order QMC rules, achieve (up to powers of \( \log N \)) the optimal rate of convergence. See also [5] for more background on higher order QMC rules. The purpose of this paper is to substantially improve the constants in the bounds in [4], which is crucial in problems in uncertainty quantification [6, 8]. In particular, [8] point out that the large constants from [4] cause problems in the CBC construction of interlaced polynomial lattice rules. To avoid this problem, they suggest to use much smaller constants which are more realistic. This paper provides the theoretical justification for doing so.

Dick et al. [6] introduced a smooth function space whose function \( f \) satisfies that its norm \( \|f\| \) (see below) is finite. If \( f \) is a function whose mixed partial derivatives up to order \( \alpha \) in each variable are continuous, \( f \) is contained in this space. This space has some parameters called weights \( \{\gamma_s\}_{s \in S} \subset \mathbb{R}_{>0} \) where \( S := \{1, \ldots, s\} \), which model the importance of different coordinate projections, see [20].

To state Dick’s result, we need modified dual spaces which correspond to the subset \( v \subset S \). For \( k_v \in \mathbb{N}^{|v|} \), let \( (k_v; 0) \) denote the vector whose \( j \)th component is \( k_j \) if \( j \in v \) and 0 otherwise. We define the dual space which corresponds to the subset \( v \subset S \) by \( \mathcal{P}_v^\perp := \{k_v \in \mathbb{N}^v \mid k = (k_v; 0) \in \mathcal{P}^\perp\} \) (note that none of the components in \( v \) is 0). Let \( 1 \leq r, r', q \leq \infty \) with \( 1/r + \)
of a function with large enough $\alpha$ for finding good point sets for QMC (see [10],[11]).

We write $e_{s,\alpha,\gamma,r'}(\mathcal{P})$, where

$$ e_{s,\alpha,\gamma,r'}(\mathcal{P}) = \left( \sum_{\phi \neq \emptyset \subseteq S} \left( C_\alpha^{\lceil \gamma \rceil} \sum_{\mathcal{P}_v} 2^{-\mu_\alpha(k_v)} \right)^{r'} \right)^{1/r'} .$$

This implies the following inequality of the form (1):

$$ |\text{Err}(f; \mathcal{P})| \leq \|f\|_{s,\alpha,\gamma,q,r'} e_{s,\alpha,\gamma,r'}(\mathcal{P}),$$

where

$$ \|f\|_{s,\alpha,\gamma,q,r'} := \left( \sum_{u \subseteq (1: \alpha)} \gamma_u^{-q} \sum_{v \subseteq u} \sum_{\tau_u \setminus v} \int_{[0,1]^{v \setminus u}} \left| \int_{[0,1]^{v \setminus u}} (\partial_y^{\alpha \tau_u \setminus v} f)(y) \ dy \right|^q \ dy_v \right)^{r/(r-q)} ,$$

with the obvious modifications if $q$ or $r$ is infinite. Here $(\alpha, \tau_u \setminus v, 0)$ denotes a sequence $(\nu_j)_j$ with $\nu_j = \alpha$ for $j \in v$, $\nu_j = \tau_j$ for $j \in u \setminus v$, and $\nu_j = 0$ for $j \notin u$. We write $\|f\|_{L^q} := (\int |f(x)|^q \ dx)^{1/q}$ and $f^{(n_1, \ldots, n_s)} = \partial^{n_1, \ldots, n_s} f/\partial x_1^{n_1} \ldots \partial x_s^{n_s}$.

Based on these bounds on the integration error, Dick constructed 'interlaced digital nets' to obtain a point set with small integration error (for example, see [3],[4]). He showed that the worst case error of this type of point set achieves the order $O(N^{-\alpha} (\log N)^{s\alpha})$ in terms of the cardinality $N$ of a point set (see [4]). This is known to be optimal up to log terms (see [18]). He also gave a component-by-component (CBC) algorithm which is another construction method to obtain point sets whose worst case error achieve the same order (see [1],[2]).

There is another algorithm to find good quadrature point sets for QMC of a function with large enough $\alpha$. This is introduced by Matsumoto, Saito and Matoba [15]. They define the Walsh Figure of Merit (WAFOM), which is defined by the discretization of the upper bound of the worst case error of the form (2). The advantage of WAFOM is that we can compute it on the computer in reasonable time. This property enables us to find a point set with small integration error by computer search. In fact, there are some algorithms for finding good point sets for QMC (see [10],[11]).

---

1The norm in [6] Definition 3.3] has been corrected in arXiv:1309.4624v3. The correct version is restated here in Eq. (4).
In this way, to find good quadrature point sets, we need an inequality of the form (1), which bounds the integration error by the product of a norm of \( f \) and a figure of merit of \( \mathcal{P} \). These types of inequalities are called Koksma-Hlawka inequalities (for example, see [13] for details). In the following, we give a new Koksma-Hlawka type inequality to bound the integration error of smooth functions better than the inequality (3) under some condition.

**Theorem 1.1.** Let \( \alpha \in \mathbb{N} \cup \{\infty\} \) such that \( \alpha \geq 2 \). We assume that a function \( f \) satisfies that its mixed partial derivatives up to order \( \alpha \) in each variable \( x_i \) are continuous on \([0,1]^s\), and \( 1 \leq p, q, q' \leq \infty \) such that \( \frac{1}{q} + \frac{1}{q'} = 1 \). Then we have

\[
|\text{Err}(f;\mathcal{P})| \leq \|f\|_{B_{\alpha,\gamma,p,q'}} \times W_{\alpha,\gamma,q}(\mathcal{P}),
\]

where

\[
W_{\alpha,\gamma,q}(\mathcal{P}) = \left( \sum_{\phi \neq v \subset S} \left( \gamma_v \sum_{k_v \in \mathcal{P}^\perp} 2^{-\mu_{\alpha}(k_v)} \right)^q \right)^{1/q},
\]

\[
\|f\|_{B_{\alpha,\gamma,p,q'}} = \left( \sum_{\phi \neq v \subset S} \left( \gamma_v^{-1} 2^{\|v\|} \sup_{\alpha_v \in \{1,\ldots,\alpha\}^v} \|f^{(\alpha_v)}\|_p \right)^q \right)^{1/q'},
\]

and where

\[
\|f^{(\alpha_v)}\|_p = \left( \int_{[0,1]^v} \left| \int_{[0,1]^{S \setminus v}} \frac{\partial^{(\alpha_v)} f}{\partial x_{\alpha_v}} dx_{S \setminus v} \right|^p dx_v \right)^{1/p},
\]

with the obvious modifications if either \( p, q \) or \( q' \) is infinite.

In this theorem we write \( \mu_{\alpha}(l_1, \ldots, l_{|v|}) = \sum_{i=1}^{|v|} \sum_{j=1}^{\min(\alpha,N_i)} (a_{i,j} + 2) \) for \( l_i = \sum_{j=1}^{N_i} 2^{a_{i,j}} \) instead of Dick’s weight function \( \mu_{\alpha}(l_1, \ldots, l_{|v|}) \).

This result yields a significant improvement of (3). This is crucial when using the bound in a CBC algorithm, since a large constant (as it appears in [6, Theorem 3.5]) may make it impractical to perform the CBC construction in practice. For instance, [8, Section 4.1] write that The resulting large values of the worst-case error bounds [ referring to the large constants in [6, Theorem 3.5]] have been found to lead to generating vectors with bad projections.

Additionally, we also include the case \( \alpha = \infty \) which has not been studied before in the context of digital nets. In [6, Theorem 3.5], the case \( \alpha = \infty \) is not included since in this case the constant \( C_{\alpha} \) appearing in (3) is infinite. Furthermore, we can define another version of WAFOM when we consider this new bound (see [10] for details).

This theorem is based on the estimation of \( \text{Err}(f;\mathcal{P}) \) by the Walsh coefficients. Dyadic Walsh coefficients are defined as follows (see [12, 17] for details).
Definition 1.2 (Walsh functions and Walsh coefficients). Let \( f: [0, 1)^s \to \mathbb{R} \) and \( k = (k_1, \ldots, k_s) \in \mathbb{N}_0^s \). We define the \( k \)-th dyadic Walsh function \( \text{wal}_k \) by

\[
\text{wal}_k(x) := \prod_{i=1}^{s} (-1)^{\left(\sum_{j \geq 1} a_{i,j} b_{i,j}\right)} ,
\]

where for \( 1 \leq i \leq s \), we write the dyadic expansion of \( k_i \) and \( x_i \) by \( k_i = \sum_{j \geq 1} a_{i,j} 2^{j-1} \), \( x_i = \sum_{j \geq 1} b_{i,j} 2^{-j} \), where for each \( i \), infinitely many digits \( b_{i,j} \) are 0.

Using Walsh functions, we define the \( k \)-th dyadic Walsh coefficient \( \hat{f}(k) \) as follows:

\[
\hat{f}(k) := \int_{[0,1)^s} f(x) \cdot \text{wal}_k(x) \, dx .
\]

We see that the integration error \( \text{Err}(f; \mathcal{P}) \) by a digital net \( \mathcal{P} \) can be represented by Walsh coefficients \( \hat{f}(k) \) as follows ([7, Chapter 15]):

\[
\text{Err}(f; \mathcal{P}) = \sum_{k \in \mathcal{P} \setminus \{0\}} \hat{f}(k) \left( = \sum_{\phi \neq v \subseteq S} \sum_{k_v \in \mathcal{P}_v} \hat{f}(k_v; 0) \right) .
\]

The proof of Theorem 1.1 is facilitated by an improved bound on the Walsh coefficients of smooth functions. We show the bounds on Walsh coefficients \( \hat{f}(k) \) as follows:

**Theorem 1.3.** We assume the same assumptions as in Theorem 1.1. Let \( \phi \neq v \subset \{1, \ldots, s\} = S \). For \( k_v \in \mathbb{N}^{|v|} \), we have

\[
|\hat{f}(k_v; 0)| \leq 2^{\frac{|v|}{p}} \cdot 2^{-\mu_r^v(k_v)} \cdot \|f(\min(\alpha, N_{k_v}))\|_p , \quad (6)
\]

where \( 1 \leq p \leq \infty \) and \( \| \cdot \|_p \) is the norm defined in Theorem 1.1. Here \( \min(\alpha, N_l) = (\min(\alpha, N_{l_1}), \ldots, \min(\alpha, N_{l_{|v|}})) \) for \( l = (l_1, \ldots, l_{|v|}) \) with dyadic expansion \( l_i = \sum_{j=1}^{N_l} 2^{a_i,j} \).

This inequality follows from the formula for the Walsh coefficients by dyadic differences, which are defined in Section 3 (see the rough sketch of proof in Section 3).

Here we compare this result with [5, Theorem 14] and its higher dimensional analogue in [6]. But our bound includes the case \( \alpha = \infty \) for the case \( b = 2 \) and we see that our bound (6) is better under some condition. For example, we assume that \( N_1 \geq \alpha \) and \( s = 1 \). Then, if we multiply our bound by \( (5/3)^{\alpha-2} \), our bound is still smaller than that bound by Dick for any \( k_1 = \sum_{j=1}^{N_1} 2^{a_1,j} \) (see [7] chapter 14).

In the following, we assume that \( k \in \mathbb{N}_0 \) has dyadic expansion with \( k = \sum_{j=1}^{N_1} 2^{a_j} \) where \( N \) is some integer and \( a_1 > \cdots > a_N \) and set \( N = 0 \), \( \{a_j\} = \phi \) for \( k = 0 \). The reminder of this paper is organized as follows. In Section 2 we give the proof of Theorem 1.1 and we give the rough sketch of the proof of Theorem 1.3 in Section 3. In Section 4 we show the proof of lemmas to complete the proof of Theorem 1.1 in Section 3.
2 Proof of Theorem 1.1

Proof. We assume that \( f \) is continuous on \([0,1]^s\) and \( \sum_{k \in \mathbb{N}_0} |\hat{f}(k)| < \infty \). In [9, Lemma 17], we have pointwise absolute convergence

\[
\| f(x) = \sum_{k \in \mathbb{N}_0} \hat{f}(k) \omega_{k}(x). \tag{7}
\]

Now we have that \( f \) is continuous by the assumption of \( f \). We show that \( f \) also satisfies the second condition. If we apply Theorem 1.3 for \( \alpha = 2 \), we have

\[
\sum_{k \in \mathbb{N}_0 \setminus \{0\}} |\hat{f}(k)| = \sum_{\phi \neq \nu \subseteq S} \sum_{k \in \mathbb{N}^v} |\hat{f}((k_0; 0))| \leq \sum_{\phi \neq \nu \subseteq S} \sum_{k \in \mathbb{N}^v} 2^{\min(2, \nu_2(k))} \|f(\min(2, \nu_2(k)))\|_p
\]

\[
\leq \sum_{\phi \neq \nu \subseteq S} \sum_{k \in \mathbb{N}^v} 2^{\min(2, \nu_2(k))} \max_{n \in \{1,2\}^r} \|f(n)\|_p \leq 2^s \max_{n' \in \{0,1,2\}^r} \|f(n')\|_{L^p} \sum_{k \in \mathbb{N}^v \setminus \{0\}} 2^{-\nu_2(k)}.
\]

Note that \( \|f\|_{L^p} := (\int |f|^p)^{1/p} \) is different from \( \|f\|_p \). Thus we have \( \sum_{k \in \mathbb{N}^v \setminus \{0\}} |\hat{f}(k)| \leq 2^s \max_{n' \in \{0,1,2\}^r} \|f(n')\|_{L^p} \sum_{k \in \mathbb{N}^v} 2^{-\nu_2(k)} \). Since \( \max_{n' \in \{0,1,2\}^r} \|f(n')\|_{L^p} < \infty \) holds by the assumption on \( f \), we have only to show the last summation is finite. We prove this in the following way:

\[
\sum_{k \in \mathbb{N}^v} 2^{-\nu_2(k)} = \left( \sum_{k \in \mathbb{N}_0} 2^{-\nu_2(k)} \right)^s = \left( 1 + \sum_{l \in \mathbb{N}_0} 2^{-l-2} + \sum_{l_1, l_2 \in \mathbb{N}_0, l_1 < l_2} \sum_{k \in \mathbb{N}_0, k < 2^l} 2^{-l_1 - l_2 - 4} \right)^s
\]

\[
= \left( \frac{3}{2} + \sum_{l_2 \in \mathbb{N}_0} l_2 2^{l_2 - l_2 - 4} \right)^s \leq \left( \frac{3}{2} + 2^{-4} \cdot \sum_{l_2 \in \mathbb{N}_0} \frac{3}{4} l_2 \right)^s < \infty.
\]

Then we can apply the formula (7) to \( f \) to get

\[
|\text{Err}(f; \mathcal{P})| = \int_{[0,1]^s} f(x) \, dx \cdot \left( \frac{1}{|P|} \sum_{x \in \mathcal{P}} f(x) \right) = \left| \frac{1}{|P|} \sum_{x \in \mathcal{P}} \hat{f}(k) \omega_{k}(x) \right|.
\]

Now we introduce the property of Walsh coefficients \( \omega_{k}(x) \) (see [7]). Let \( \mathcal{P} \) be a digital net in \([0,1]^s\) where \( |P| = 2^m \). Then we have (see [7, Lemma 4.75])

\[
\sum_{x \in \mathcal{P}} \omega_{k}(x) = \begin{cases} 2^m & \text{if } k \in \mathcal{P}^\perp, \\ 0 & \text{otherwise}. \end{cases}
\]

Using this fact, we have

\[
|\text{Err}(f; \mathcal{P})| = \left| \frac{1}{|P|} \sum_{x \in \mathcal{P}^\perp} \hat{f}(k) \right| \leq \sum_{x \in \mathcal{P}^\perp} |\hat{f}(k)| = \sum_{\phi \neq \nu \subseteq S} \sum_{k \in \mathcal{P}^\perp} |\hat{f}(k; 0)|.
\]
Let \( \alpha \in \mathbb{N} \cup \{\infty\} \) with \( \alpha \geq 2 \) and \( 1 \leq p, q, q' \leq \infty \) such that \( 1/q + 1/q' = 1 \). Applying Theorem \( 1.3 \) to \( f \), we have

\[
|\text{Err}(f; \mathcal{P})| \leq \sum_v \sum_{\mathbf{k} \in P^v} 2^{i_v} 2^v \mu''(\mathbf{k}) \|f(\min(\alpha, N\mathbf{k}))\|_p
\]

\[
\leq \sum_v \gamma^{-1} 2^{i_v} \sup_{\alpha_v \in \{1, \ldots, \alpha\}} \|f(\alpha_v)\|_p \sum_{\mathbf{k} \in P^v} \gamma^v 2^v \mu''(\mathbf{k})
\]

\[
\leq \|f\|_{\mathcal{B}_{\alpha, \gamma, p, q}^{\perp}} \times W_{\alpha, \gamma, q}(\mathcal{P}).
\]

We use Hölder’s inequality in the last inequality. \( \square \)

### 3 Proof of Theorem 2

We introduce some useful notation in the following.

#### 3.1 Some notations

To calculate bounds on Walsh coefficients, we introduce the dyadic difference \( \partial_{i,n} \) and the weight function \( \mu''_{\mathbf{u}}(\mathbf{k}) \).

**Definition 3.1** (dyadic difference). Let \( t, n, i \in \mathbb{N} \) with \( i \leq t \). For a function \( f : [0, 1]^t \to \mathbb{R} \), we define the dyadic difference \( \partial_{i,n} f \) of a function \( f(x_1, \ldots, x_t) \) by

\[
\partial_{i,n} f(x_1, \ldots, x_t) := \frac{f(x_1, \ldots, x_i \oplus 2^{-n}, \ldots, x_t) - f(x_1, \ldots, x_i, \ldots, x_t)}{2^{-n}}.
\]

Here we write \( z \oplus 2^{-n} := z + 2^{-n}(-1)^z \) for \( z \) having dyadic expansion \( z = \sum_{j=1}^{\infty} z_j 2^{-j} \), where infinitely many digits \( z_j \) are 0.

Let a vector \( \mathbf{k} = (k_1, \ldots, k_t) \in \mathbb{N}_0^t \) with \( k_i = \sum_{j=1}^{N_i} 2^{a_{i,j}} \). Let a vector \( \mathbf{u} = (u_1, \ldots, u_t) \in \mathbb{N}_0^t \). We write the composition of the dyadic differences \( \{\partial_{i,n} f\}_{\|1 \leq j \leq t \| k_j \neq 0 \| 1 \leq j \leq \min(N, u_i)} \) by \( \mathbf{d}_{\mathbf{u}} \). If \( u_i = \alpha \in \mathbb{N} \cup \{\infty\} \) for every \( i \), we write \( \mathbf{d}_{\mathbf{u}} \) instead of \( \mathbf{d}_{\mathbf{u}} \).

**Remark 3.2.** Since any two dyadic differences commute, \( \mathbf{d}_{\mathbf{u}} \) is defined independent of the order of a composition.

**Definition 3.3** (the new weight function \( \mu''_{\mathbf{u}}(\mathbf{k}) \)). Let \( t \in \mathbb{N} \) and \( (u_1, \ldots, u_t) \in (\mathbb{N}_0 \cup \{\infty\})^t \). We take a vector \( \mathbf{k} = (k_1, \ldots, k_t) \in \mathbb{N}_0^t \) with \( k_i = \sum_{j=1}^{N_i} 2^{a_{i,j}} \).

Then the weight function \( \mu''_{\mathbf{u}}(\mathbf{k}) \) of \( \mathbf{k} \) is defined by

\[
\mu''_{\mathbf{u}}(\mathbf{k}) := \sum_{i \in \{1 \leq j \leq t \| k_j \neq 0 \}} \sum_{j \leq \min(N, u_i)} (a_{i,j} + 2),
\]

where we define \( \mu''_{\mathbf{u}}(0) = 0 \). When \( u_i = \alpha \in \mathbb{N} \cup \{\infty\} \) for every \( i \), we denote \( \mu''_{\mathbf{u}} \) by \( \mu''_{\alpha} \).
We define the important two functions $\chi_n(x, y)$ and $W(k)$.

**Definition 3.4.** We define the function $\chi_n(x, y): [0, 1)^2 \rightarrow \mathbb{R}$ by

$$\chi_n(x, y) := \begin{cases} 2^n & \text{if } y \in [\min(x, x \oplus 2^{-n}), \max(x, x \oplus 2^{-n})], \\ 0 & \text{otherwise}. \end{cases}$$

Recall that $x \oplus 2^{-n}$ is defined as in Definition 3.1.

Using this, we define the 1-dimensional versions $W(k)$ inductively by

$$W(2^n_1)(y) := \int_0^1 \chi_{n+1}(x, y) \, dx,$$

$$W(2^n_1 + \cdots + 2^n_{N+1})(y) := \int_0^1 \chi_{n+1}(x, y) W(2^n_1 + \cdots + 2^n_N)(x) \, dx,$$

where $n_1 > \cdots > n_{N+1}$. We define $W(0)$ by 1.

Let $t \in \mathbb{N}$ and a vector $(k_1, \ldots, k_t) \in \mathbb{N}_0^t$. We define the $t$-dimensional versions $W(k): [0, 1)^t \rightarrow \mathbb{R}$ by

$$W(k) := \prod_{i=1}^t W(k_i).$$

**Remark 3.5.** By definition, $W(k)$ is continuous on $[0, 1)$ for any $k \in \mathbb{N}_0$.

The following important property of these functions is proven in Section 4.3.

**Lemma 3.6.** Let $t \in \mathbb{N}$ and a vector $(k_1, \ldots, k_t) \in \mathbb{N}_0^t$. We have that

$$W(k)(x) \geq 0$$

on $[0, 1)^t$ and, for $1 \leq p \leq \infty$, we have

$$\|W(k)\|_{L^p} \leq 2^{-(1 - \frac{1}{p})t}.$$  

### 3.2 Proof of Theorem 1.3

In this subsection, we show bounds on the Walsh coefficients $\hat{f}(k)$ admitting the following Lemmas 3.7 and 3.8, which we prove in the next two sections.

When we analyze Walsh coefficients, it is suitable to use dyadic differences. In fact, the $k$-th Walsh coefficient $\hat{f}(k)$ can be represented by $\partial_{i,a} f(k)$ as follows:

$$\hat{f}(k) = (-1) \cdot 2^{-a_{i,j} - 2} \partial_{i,a_{i,j}+1}(f)(k),$$

where $k = (k_1, \ldots, k_s)$ and $k_i = \sum_{j=1}^{N_i} 2^{a_{i,j} - 0}$ satisfying $a_{i,1} > \cdots > a_{i,N_i}$ for each $i$ satisfying $k_i \neq 0$. We set $N_i = 0$ for $i$ satisfying $k_i = 0$. Applying the above formula (8) repeatedly, we have the following formula:

**Lemma 3.7.** Let $s \in \mathbb{N}$, $f \in L^1([0, 1]^s)$, $k = (k_1, \ldots, k_s) \in \mathbb{N}_0^s$ and $u = (u_1, \ldots, u_s) \in (\mathbb{N}_0 \cup \{\infty\})^s$.

Then we have $d_{k,u} f \in L^1([0, 1]^s)$ and we have

$$\hat{f}(k) = (-1)^{\sum_{i=1}^s \min(N_i, u_i)} 2^{-\nu_u(k)} d_{k,u} f(k).$$

8
This formula (9) means that dyadic differences connect the $k$-th Walsh coefficient $f(k)$ to the weight function $\mu_0(x)$ for $f \in L^1([0,1]_x)$.

Dyadic differences $\partial_{i,n}f$ are similar to derivatives $\frac{\partial f}{\partial x_i}$. In the following formula, we can replace $\partial_{i,n}f$ with $\frac{\partial f}{\partial x_i}$ using a slight modification.

To state the result, we denote the symbols used in the statements. Let $t \in \mathbb{N}$ and $k = (k_1, \ldots, k_t) \in \mathbb{N}_0^t$ with $k_i = \sum_{j=1}^{N_i} 2^{a_{i,j}}$. Let $u = (u_1, \ldots, u_t) \in (\mathbb{N}_0 \cup \{\infty\})^t$. We define

$$k_{i,:}^* := \sum_{j > u_i} 2^{a_{i,j}}, \quad k_{i,:}^\leq := \sum_{j \leq u_i} 2^{a_{i,j}}, \quad i \text{ if } k_i \neq 0,$$

$$k_{i,:}^* := 0, \quad k_{i,:}^\leq := 0, \quad i \text{ if } k_i = 0,$$

$$k_u^* := (k_{1,:}^*, \ldots, k_{t,:}^*), \quad k_u^\leq := (k_{1,:}^\leq, \ldots, k_{t,:}^\leq),$$

$$\min(u, N_k) := (\min(u_1, N_1), \ldots, \min(u_t, N_t)).$$

We have the following formula.

**Lemma 3.8.** Let $s \in \mathbb{N}$ and $u = (u_1, \ldots, u_s) \in (\mathbb{N}_0 \cup \{\infty\})^s$. We assume that a function $f$ satisfies that its mixed partial derivatives up to order $u_i$ in each variable $x_i$ are continuous on $[0,1]^s$. For any vector $k = (k_1, \ldots, k_s) \in \mathbb{N}_0^s$, we have

$$\hat{f}(k) = (-1)^{\sum_{i=1}^{s} \min(u_i, N_i)} 2^{-\mu_0(k)} \int_{[0,1]^s} f^{(\min(u, N_k))}(x) \cdot W(k_u^*) (x) \cdot \text{wal}_{k_u^*} (x) \, dx.$$

Then we can get the following bound on $|\hat{f}(k)|$:

**Lemma 3.9.** Let $\phi \neq v \in \{1, \ldots, s\} = S$ and $k_\phi \in \mathbb{N}^{[v]}$. Under the same assumptions as in Lemma 3.8, we have

$$|\hat{f}((k_\phi; 0))| \leq 2^\phi \cdot 2^{-\mu_0(k_\phi)} \cdot \|f^{(\min(u, N_{k_\phi}))}\|_p < \infty,$$

where $1 \leq p \leq \infty$ and $\| \cdot \|_p$ is the norm defined in Theorem 7.7.

**Proof.** For $(k_\phi; 0) \in \mathbb{N}_0^s$, it holds that $W((k_\phi; 0)_u^\leq) = W(k_u^\leq) = \text{wal}_{k_u^\leq}$. And by the definition of Walsh functions, we see $|\text{wal}_{k_u^\leq}| = 1$. Then, by the above Lemma, we have

$$|\hat{f}((k_\phi; 0))| \leq 2^{-\mu_0(k_\phi)} \int_{[0,1]^s} f^{(\min(u, N_{k_\phi}; 0))}(x) \cdot W(k_u^\leq)(x) \cdot \text{wal}_{k_u^\leq}(x) \, dx$$

$$\leq 2^{-\mu_0(k_\phi)} \int_{[0,1]^s} \int_{[0,1]^{(S \setminus \{\phi\})}} f^{(\min(u, N_{k_\phi}; 0))}(x) \, dx_{S \setminus \{\phi\}} \cdot |W(k_u^\leq)(x)| \, dx_{\{\phi\}}$$

$$\leq 2^{-\mu_0(k_\phi)} \|f^{(\min(u, N_{k_\phi}))}\|_p \cdot \|W(k_u^\leq)\|_{L^\frac{p}{p-1}} \leq 2^{-\mu_0(k_\phi)} 2^{\frac{\mu_0}{p}} \|f^{(\min(u, N_{k_\phi}))}\|_p,$$

where we used Hölder’s inequality in the third inequality and Lemma 3.6 in the last inequality. 

\[ \square \]
In particular, when \( u_i = \alpha \in \mathbb{N} \cup \{\infty\} \) for every \( i \), we have Theorem 1.3.

In the following three sections, we will prove the lemmas which we used in this section. From now, we write the composition of maps \( \phi_1 \circ \cdots \circ \phi_n \) by \( \Pi_{i=1}^n \phi_i \).

# 4 Proof of Lemmas

## 4.1 Proof of Lemma 3.7

**Proof.** We show the 1-dimensional version here. In the case \( s > 1 \), we obtain the result by using the same method in the case \( s = 1 \).

We easily obtain the first statement as follows. Since \( \partial_{1, a_j + 1} \) is the sum of \( f(x_1 \oplus 2^{-a_j-1}) \in L^1([0, 1]) \) and \( f \in L^1([0, 1]) \), we have \( \partial_{1, a_j + 1} f \in L^1([0, 1]^s) \).

By repeating this argument, we have \( d_{k_1, u_1}(f) \in L^1([0, 1]) \).

We show the second statement inductively. We show the case \( u_1 = 1 \): \( f(k_1) = (-1) \cdot 2^{-a_j-2} \partial_{1, a_j + 1}(f)(k_1) \),

(10)

where \( k_1 = \sum_{j=1}^N 2^{a_j} \). By changing variables \( x \mapsto x_1 \oplus 2^{-a_j-1} \), we have

\[
\int_0^1 f(x_1 \oplus 2^{-a_j-1}) \cdot \text{wal}_{k_1}(x_1) \, dx_1 = \sum_{c=0}^{2^{a_j-1}-1} \left( \int_{2^{-a_j-1}, 2c} f(x_1 + 2^{-a_j-1}) \cdot \text{wal}_{k_1}(x_1) \, dx_1 ight. \\
+ \int_{2^{-a_j-1}, 2c+1} f(x_1 - 2^{-a_j-1}) \cdot \text{wal}_{k_1}(x_1) \, dx_1 \\
\left. + \int_{2^{-a_j-1}, 2c+1} f(x_1) \cdot \text{wal}_{k_1}(x_1 - 2^{-a_j-1}) \, dx_1 \\
+ \int_{2^{-a_j-1}, 2c+1} f(x_1) \cdot \text{wal}_{k_1}(x_1 + 2^{-a_j-1}) \, dx_1 \right)
\]

\[
= \int_0^1 f(x_1) \cdot \text{wal}_{k_1}(x_1 \oplus 2^{-a_j-1}) \, dx_1 = \int_0^1 f(x_1) \cdot \text{wal}_{k_1}(x_1) \cdot \text{wal}_{k_1}(2^{-a_j-1}) \, dx_1
\]

\[
= - \int_0^1 f(x_1) \cdot \text{wal}_{k_1}(x_1) \, dx_1,
\]

where the last two identities follows from the definition of Walsh functions. Using this calculation, we obtain

\[
\partial_{1, a_j + 1}(f)(k_1) = -2 \cdot 2^{a_j+1} \int_0^1 f(x_1) \cdot \text{wal}_{k_1}(x_1) \, dx_1 = (-1) \cdot 2^{a_j+2} \hat{f}(k_1).
\]
Using (10) inductively, we obtain

\[
\hat{d}_{k_1 u_1} f(k_1) = -2^{a_1+1} \left( \prod_{j=2}^{\min(u_1,N)} \partial_{1,a_j+1}(f)(k_1) \right) = (-1)^{2^{a_1+1} \sum_{j=1}^{\min(u_1,N)} \prod_{j=3}^{\min(u_1,N)} \partial_{1,a_j+1}(f)}(k_1) = \cdots = (-1)^{\min(u_1,N)} \cdot 2^{\mu_u(k_1)} f(k_1).
\]

\[\square\]

4.2 Proof of Lemma 3.8

4.2.1 Important properties of dyadic differences \(\partial_{i,n}\)

In order to prove Lemma 3.8, we show some properties of dyadic differences \(\partial_{i,n}\). We define the following symbols.

**Definition 4.1.** Let \(x = (x_1, \ldots, x_s) \in [0,1)^s\) and \(i, n\) be positive integers. Then we define

\[
\begin{align*}
\hat{w}_{i,n}(x) &= \text{wal}_{2^{a_i}}(x_i), \\
\hat{w}_{i,n}(f)(x) &= \hat{w}_{i,n}(x) \cdot \hat{\partial}_{i,n}(f)(x).
\end{align*}
\]

**Remark 4.2.** Let \(k = (k_1, \ldots, k_s) \in N_0^s\) with \(k_i = \sum_{j=1}^{N_i} 2^{a_{i,j}}\). Then we have

\[
\hat{w}_{k}(x) = \prod_{i \in \{1 \leq g \leq s \mid k_g \neq 0\}} \prod_{j=1}^{N_i} \hat{w}_{i,a_{i,j}+1}(x).
\]

We see that \(\hat{w}_{j,m}(x)\) and \(\hat{\partial}_{i,n}\) commute.

**Lemma 4.3.** When \((i, n) \neq (j, m) \in N^2\), we have the following identity:

\[
\hat{w}_{j,m}(x) \cdot \hat{\partial}_{i,n}(f)(x) = (\hat{\partial}_{i,n}(f \cdot \hat{w}_{j,m}))(x).
\]

**Proof.** We omit the proof here since it is easy. \(\square\)

We first prove the following property.

**Lemma 4.4.** Let \(f \in L^1([0,1)^s)\), \(T \subset \{1, \ldots, s\}\) and \(p_i \leq q_i\) be positive integers for each \(i\). Then we have \(\prod_{i \in T} \prod_{j=p_i}^{q_i} w_{i,a_{i,j}+1} f \in L^1([0,1)^s)\) for \(\{a_{i,j}\}_{i \in T, p_i \leq j \leq q_i} \subset N_0\).

**Proof.** Since the above lemma, we have that \(\prod_{i \in T} \prod_{j=p_i}^{q_i} w_{i,a_{i,j}+1} f \) equals \(\prod_{i \in T} \prod_{j=p_i}^{q_i} w_{i,a_{i,j}+1} \prod_{i \in T} \prod_{j=p_i}^{q_i} \partial_{i,a_{i,j}+1} f\). By the definition of Walsh functions, we see \(|w_{i,a_{i,j}+1}| = 1\). Since \(\prod_{j=p_i}^{q_i} \partial_{i,a_{i,j}+1} f\) is the sum of the function in \(L^1([0,1)^s)\), the result follows. \(\square\)
4.2.2 Proof of Lemma 3.8

The following Lemma is the key to prove Lemma 3.8 which connects \( w \partial_{i,n} (f) \) with the derivative \( \frac{\partial f}{\partial x_i} \).

**Lemma 4.5.** Let \( n, s, i \in \mathbb{N} \) satisfy \( s \geq i \). Let \( f : [0, 1]^s \to \mathbb{R} \) as a function of the \( i \)th component \( x_i \), satisfy

\[
f \in C^1 \left( [2^{-n+1}c, 2^{-n+1}(c+1)] \right), \quad c = 0, \ldots, 2^{n-1} - 1.
\]

Then for any \((x_1, \ldots, x_s) \in [0, 1]^s\), we have

\[
w \partial_{i,n} (f)(x_1, \ldots, x_s) = \int_0^1 \frac{\partial f}{\partial x_i} (x_1, \ldots, x_{i-1}, y, x_i+1, \ldots, x_s) \cdot \chi_n(x_i, y) \, dy,
\]

where we define

\[
\frac{\partial f}{\partial x_i} := \frac{\partial f}{\partial x_i} \quad \text{on } x_i \in [2^{-n+1}c, 2^{-n+1}(c+1)], \quad c = 0, \ldots, 2^{n-1} - 1.
\]

**Proof.** Let \( c' \in \mathbb{N}_0 \) satisfying \( x_i \in [2^{-n}c', 2^{-n}(c'+1)) \). We consider two cases: \( c' = 2c \) and \( c' = 2c + 1 \) for some integer \( c \). We only calculate the case \( c = 2c' \) since the other case can be calculated by the same way. In this case, by the calculation \( w_{i,n}(x_i) = 1 \) and the assumption (11), we have

\[
w \partial_{i,n} (f)(x) = \partial_{i,n} (f)(x) = 2^n \cdot (f(x_1, \ldots, x_i+2^{-n}, \ldots, x_s) - f(x_1, \ldots, x_s))
\]

\[
= \int_{x_i}^{x_i+2^{-n}} 2^n \cdot \frac{\partial f}{\partial x_i} (x_1, \ldots, x_{i-1}, y, x_i+1, \ldots, x_s) \, dy
\]

\[
= \int_0^1 \frac{\partial f}{\partial x_i} (x_1, \ldots, x_{i-1}, y, x_i+1, \ldots, x_s) \cdot \chi_n(x_i, y) \, dy.
\]

The last equality follows from \([x_i, x_i+2^{-n}] = [\min(x_i, x_i+2^{-n}), \max(x_i, x_i+2^{-n})]\) and the definition of \( \chi_n(x, y) \). \( \square \)

**Proof. Lemma 3.8**

We prove the case \( s = 1 \). Let \( k_1 = \sum_{j=1}^N 2^{a_j} \). Then we have

\[
\left( \prod_{j=1}^{\min(u_1,N)} \partial_{a_j+1} \right)(f)(k) = \int_0^1 \left( \prod_{j=1}^{\min(u_1,N)} \partial_{a_j+1} \right)(f) \cdot \left( \prod_{j=1}^N w_{a_j+1} \right) \, dx_1
\]

\[
= \int_0^1 \left( \prod_{j=1}^{\min(u_1,N)} w_{a_j+1} \right) (f) \cdot \prod_{j>\min(u_1,N)} w_{a_j+1} \, dx_1.
\]

We use Lemma 4.1 in the second equality. Using the assumption of \( f \) and the definition of \( w_{a_j+1} \), we have that

\[
f \cdot \prod_{j>\min(u_1,N)} w_{a_j+1} \in C^{a_1} \left( [2^{-a_\min(u_1,N)+1} c, 2^{-a_\min(u_1,N)+1}(c+1)] \right),
\]

12
and we have
\[
\frac{\partial}{\partial x_1} \left( f \cdot \prod_{j > \min(u_1, N)} w_{a_j + 1} \right) = \frac{\partial f}{\partial x_1} \prod_{j > \min(u_1, N)} w_{a_j + 1},
\]
on $x_1 \in [2^{-a_{\min(u_1, N)}+1} c, 2^{-a_{\min(u_1, N)}+1}(c+1)]$ with $0 \leq c \leq 2^{a_{\min(u_1, N)}+1} - 1$.

Let $1 \leq n, 0 \leq c' < 2^n$ be integers. By the definition of $w \partial_{1,n}$, we have that, if $g \in C^1([c2^{-n}, (c+1)2^{-n}])$, it holds that $w \partial_{1,n} g \in C^1([c2^{-n}, (c+1)2^{-n}])$ and $\frac{\partial}{\partial x_1} (w \partial_{1,n} g) = w \partial_{1,n} \left( \frac{\partial}{\partial x_1} g \right)$ on $[c2^{-n}, (c+1)2^{-n}]$.

Applying this argument to the function $f \cdot \prod_{j > \min(u_1, N)} w_{a_j + 1}$ inductively, we have, for $x_1 \in [2^{-a_{2-1}} c, 2^{-a_{2-1}}(c+1)]$,
\[
\frac{\partial}{\partial x_1} \left( \prod_{j = 2}^{\min(u_1, N)} w_{a_j + 1} \right) (f \cdot \prod_{j > \min(u_1, N)} w_{a_j + 1})
= \left( \prod_{j = 2}^{\min(u_1, N)} w_{a_j + 1} \right) \left( \prod_{j > \min(u_1, N)} w_{a_j + 1} \right),
\]
where $0 \leq c \leq 2^{a_{2-1}} - 1$. Since $2^{-a_{2-1}} \geq 2^{-a_1}$, we can apply Lemma 4.5 to this function. Thus we continue the computation of $\left( \prod_{j = 1}^{\min(u_1, N)} \partial_{a_j + 1} \right) (f)(k_1)$ as follows
\[
\left( \prod_{j = 1}^{\min(u_1, N)} \partial_{a_j + 1} \right) (f)(k_1) = \int_0^1 w_{a_1 + 1} \circ \left( \prod_{j = 2}^{\min(u_1, N)} w_{a_j + 1} \right) (f \cdot \prod_{j > \min(u_1, N)} w_{a_j + 1}) \, dx_1
= \int_0^1 \int_0^1 \chi_{a_1 + 1}(x_1, y) \left( \prod_{j = 2}^{\min(u_1, N)} w_{a_j + 1} \right) \left( \frac{\partial f}{\partial x_1} \cdot \prod_{j > \min(u_1, N)} w_{a_j + 1} \right)(y) \, dy \, dx_1.
\]
We have $\frac{\partial f}{\partial x_j} \cdot \prod_{j > \min(u_1, N)} w_{a_j + 1} \in L^1([0, 1])$ since $|w_{a_1 + 1}| = 1$ and the assumption of $f$. If we apply Lemma 4.3 to this function and consider the fact $0 \leq \chi_{a_1 + 1}(x_1, y) \leq 2^{a_1 + 1}$, we see that the integrand belongs to $L^1([0, 1])$. Thus we can use Fubini’s Theorem as follows.
\[
\left( \prod_{j = 1}^{\min(u_1, N)} \partial_{a_j + 1} \right) (f)(k)
= \int_0^1 \int_0^1 \chi_{a_1 + 1}(x_1, y) \, dx_1 \left( \prod_{j = 2}^{\min(u_1, N)} w_{a_j + 1} \right) \left( \frac{\partial f}{\partial x_1} \cdot \prod_{j > \min(u_1, N)} w_{a_j + 1} \right)(y) \, dy
= \int_0^1 \left( \prod_{j = 2}^{\min(u_1, N)} w_{a_1 + 1} \right) \left( \frac{\partial f}{\partial x_1} \cdot \prod_{j > \min(u_1, N)} w_{a_j + 1} \right)(x_1) \cdot W(2^{a_1})(x_1) \, dx_1.
\]
By repeating the argument we have

\[
\left( \prod_{j=1}^{\min(u_1,N)} \partial_{a_j+1} \right) (f)(k_1)
\]

\[
= \int_0^1 \int_0^1 \chi_{a_2+1}(x_1,y) \cdot W(2^{a_1})(x_1) \, dx_1 \left( \prod_{j=3}^{\min(u_1,N)} w \partial_{a_j+1} \right) (\partial^2 f / \partial x_1^2 \cdot \prod_{j>\min(u_1,N)} w_{a_j+1})(y) \, dy
\]

\[
= \int_0^1 \left( \prod_{j=3}^{\min(u_1,N)} w \partial_{a_j+1} \right) (\partial^2 f / \partial x_1^2 \cdot \prod_{j>\min(u_1,N)} w_{i,a_j+1})(x_1) \cdot W(2^{a_1} + 2^{a_2})(x_1) \, dx_1
\]

\[
= \cdots
\]

\[
= \int_0^1 \left( \partial^{\min(u_1,N)} f / \partial x_1^{\min(u_1,N)} \cdot \prod_{j>\min(u_1,N)} w_{a_j+1} \right) \cdot W(k_{1,\leq})(x_1) \, dx_1,
\]

thus we have the result for \( s = 1 \). By calculating in the same manner as in the case \( s = 1 \), we have the result for the case \( s \geq 1 \). \( \square \)

In fact, we can determine the sign of \( \hat{f}(k) \) in the special case.

**Corollary 4.6.** We assume that \( f \in C^\infty([0,1]^s) \). We use the same symbol \( N_i \) as in Theorem 1.3. Then, we have \( \hat{f}(k) \cdot (-1)^{\sum_{i=1}^s N_i} \geq 0 \).

**Proof.** By Lemma 3.6, we have \( W(k)(x) \geq 0 \). By combining this fact and the assumption that \( f \in C^\infty([0,1]^s) \), we have \( f^{(N_1,\ldots,N_s)}(x) \cdot W(k)(x) \geq 0 \). Thus, by the above lemma with \( u_i = \infty \) we have

\[
\hat{f}(k) \cdot (-1)^{\sum_{i=1}^s N_i} = 2^{-\mu_c(k)} \int_{[0,1]^s} f^{(N_1,\ldots,N_s)}(x) \cdot W(k)(x) \, dx \geq 0.
\]

\( \square \)

### 4.3 Proof of Lemma 3.6

#### 4.3.1 Important properties of \( \chi_n(x,y) \) and \( W(k) \)

We show the important properties of \( \chi_n(x,y) \) and \( W(k) \) in this subsection. See Definition 3.4 for the definitions of \( \chi_n(x,y) \) and \( W(k) \).

We see that \( \chi_n(x,y)/2^n \) is a characteristic function of some region in \( [0,1)^2 \).

**Lemma 4.7.** Let \( n \in \mathbb{N} \) and \( c \in \mathbb{N}_0 \) satisfying \( c < 2^{-1} \).

1. Let \( x, y \in [0,2^{-n+1}) \), then we have

\[
\chi_n(x + c2^{-n+1}, y + c2^{-n+1}) = \chi_n(x, y).
\]
2. Let \( x \in [e^{2^{-n+1}}, (c+1)2^{-n+1}] \) and \( y \not\in [e^{2^{-n+1}}, (c+1)2^{-n+1}] \). Then we have \( \chi_n(x, y) = 0 \).

And let \( y \in [e^{2^{-n+1}}, (c+1)2^{-n+1}] \) and \( x \not\in [e^{2^{-n+1}}, (c+1)2^{-n+1}] \). Then we have \( \chi_n(x, y) = 0 \).

**Proof.**

1. We have \((x + c2^{-n+1}) \oplus 2^{-n} = (x \oplus 2^{-n}) + c2^{-n+1}\). Thus, the result follows from the fact that

\[
y \in \left[ \min(x, x \oplus 2^{-n}), \max(x, x \oplus 2^{-n}) \right] \iff y + c2^{-n+1} \in \left[ \min(x + c2^{-n+1}, (x + c2^{-n+1}) \oplus 2^{-n}), \max(x + c2^{-n+1}, (x + c2^{-n+1}) \oplus 2^{-n}) \right].
\]

2. We prove \( \chi_n(x, y) = 0 \) in the case \( x \in [e^{2^{-n+1}}, (c+1)2^{-n+1}] \) and \( y \not\in [e^{2^{-n+1}}, (c+1)2^{-n+1}] \). Let \( x \in [d2^{-n}, (d+1)2^{-n}] \) for \( d \in \mathbb{N}_0 \) where \( d = 2c \) or \( 2c+1 \). When \( d = 2c \), it holds that \( c2^{-n+1} \leq x < x \oplus 2^{-n} = x + 2^{-n} < (c+1)2^{-n+1} \). In the case \( d = 2c+1 \), it holds that \( c2^{-n+1} \leq x \oplus 2^{-n} = x - 2^{-n} < x < (c+1)2^{-n+1} \). So we have

\[
\left[ \min(x, x \oplus 2^{-n}), \max(x, x \oplus 2^{-n}) \right] \subset [e^{2^{-n+1}}, (c+1)2^{-n+1}].
\]

So, if \( y \not\in [e^{2^{-n+1}}, (c+1)2^{-n+1}] \), we have \( y \not\in \left[ \min(x, x \oplus 2^{-n}), \max(x, x \oplus 2^{-n}) \right] \). Then we obtain \( \chi_n(x, y) = 0 \).

For the case \( x \not\in [e^{2^{-n+1}}, (c+1)2^{-n+1}] \) and \( y \in [e^{2^{-n+1}}, (c+1)2^{-n+1}] \), there is some integer \( e \) such that \( x \in [e2^{-n+1}, (e+1)2^{-n+1}] \) and \( y \not\in [e2^{-n+1}, (e+1)2^{-n+1}] \). Thus the result follows from the above argument.

\(\square\)

The function \( \chi_n(x, y) \) is defined by using a characteristic function of \( y \). In Lemma 4.8, we rewrite \( \chi_n(x, y) \) by using a characteristic function of \( x \).

**Lemma 4.8.** Let \( y \in [0, 1) \) and \( c, n \in \mathbb{N}_0 \) satisfy \( y \in [2^{-n}c, 2^{-n}(c+1)] \).

If \( c = 2c' \) for some integer \( c' \), we have

\[
\chi_n(x, y) = \begin{cases} 2^n & \text{if } x \in [2^{-n}c, y] \cup [2^{-n}(c+1), y + 2^{-n}], \\ 0 & \text{otherwise}. \end{cases}
\]

And if \( c = 2c' + 1 \) for some integer \( c' \), we have

\[
\chi_n(x, y) = \begin{cases} 2^n & \text{if } x \in [y - 2^{-n}, 2^{-n}c) \cup [y, 2^{-n}(c+1)), \\ 0 & \text{otherwise}. \end{cases}
\]

**Proof.** We only prove the case \( c = 2c' \) here since the case \( c = 2c' + 1 \) follows from the same argument. In this case, by Item 2 of Lemma 4.7, we have that for all \( y \in [2^{-n+1}c', 2^{-n+1}(c' + 1)] \),

\[
\chi_n(x, y) = 0, \quad x \not\in [2^{-n+1}c', 2^{-n+1}(c' + 1)]. \tag{12}
\]
Let \( x \in [0, 1) \) and \( d \in \mathbb{N}_0 \) satisfy \( x \in [2^{-n}d, 2^{-n}(d + 1)) \). We calculate \( \chi_n(x, y) \) in the three cases: \( d \notin \{2c', 2c' + 1\}, d = 2c' \) and \( d = 2c' + 1 \).

We consider the case \( d \notin \{2c', 2c' + 1\} \). By condition (12), we have

\[
\chi_n(x, y) = 0, \quad x \in [2^{-n}d, 2^{-n}(d + 1)).
\]

In the case \( d = 2c' \), since \( x + 2^{-n} = x + 2^{-n} \), we have

\[
\chi_n(x, y) = 2^n \iff \left\{ \begin{array}{ll} x \leq y \leq x + 2^{-n}, \\ 2^{-n+1}c' \leq x < 2^{-n}(2c' + 1), \end{array} \right.
\]

Since \( 2^{-n+1}c' \leq y \leq 2^{-n}(2c' + 1) \), we have

\[
\chi_n(x, y) = \left\{ \begin{array}{ll} 2^n & \text{if } x \in [2^{-n+1}c', y], \\ 0 & \text{if } x \in (y, 2^{-n}(2c' + 1)). \end{array} \right.
\]

When \( d = 2c' + 1 \), by a similar argument to the case \( d = 2c' \), we have

\[
\chi_n(x, y) = \left\{ \begin{array}{ll} 2^n & \text{if } x \in [2^{-n}(2c' + 1), y + 2^{-n}], \\ 0 & \text{if } x \in (y + 2^{-n}, 2^{-n}(2c' + 2)). \end{array} \right.
\]

By combining these cases, we have the result. \( \square \)

In the last lemma, we show the period of a function \( W(k) \).

**Lemma 4.9.** We have that \( W(k) \) is a periodic function with period \( 2^{-a_N} \) for \( k = \sum_{i=1}^{N} 2^{a_i} \).

**Proof.** We proceed by induction on \( N \). We prove the result for \( k = 2^{a_1} \). Let \( c \in \mathbb{N}_0 \) satisfying \( c < 2^{a_1} \). We have that for \( y \in [0, 2^{-a_1}) \),

\[
W(2^{a_1})(y + c2^{-a_1}) = \int_{0}^{1} \chi_{a_1+1}(x, y + c2^{-a_1}) \, dx
\]

\[
= \int_{c2^{-a_1}}^{(c+1)2^{-a_1}} \chi_{a_1+1}(x, y + c2^{-a_1}) \, dx = \int_{0}^{2^{-a_1}} \chi_{a_1+1}(z + c2^{-a_1}, y + c2^{-a_1}) \, dz
\]

\[
= \int_{0}^{2^{-a_1}} \chi_{a_1+1}(z, y) \, dz = \int_{0}^{1} \chi_{a_1+1}(z, y) \, dz = W(2^{a_1})(y).
\]

The second and fifth equalities follow from Item 2 of Lemma 4.7, the forth equality follows from Item 1 of Lemma 4.7 and the change of variable \( x = z + c2^{-a_1} \) in the third equality.

Now we assume that the lemma holds for the case \( k_N = \sum_{i=1}^{N} 2^{a_i} \). We prove the result for the case \( k = \sum_{i=1}^{N+1} 2^{a_i} \) satisfying \( a_{N+1} < a_N \). By the induction assumption, we have that \( W(k_N)(z + 2^{-a_N}) = W(k_N)(z) \) for \( z \in [0, 2^{-a_N}) \) and an integer \( d \) satisfying \( 0 \leq d < 2^{a_N} \). Let \( c \in \mathbb{N}_0 \) satisfying \( c < 2^{a_N+1} \). Then we
Thus we have that for $y \in [0, 2^{-a_{N+1}})$,
\[
W(k)(y + c2^{-a_{N+1}}) = W(k_N + 2^{a_{N+1}})(y + c2^{-a_{N+1}})
\]
\[
= \int_0^1 \chi_{a_{N+1}+1}(x, y + c2^{-a_{N+1}}) \cdot W(k_N)(x) \, dx
\]
\[
= \int_{c2^{-a_{N+1}}}^{(c+1)2^{-a_{N+1}}} \chi_{a_{N+1}+1}(x, y + c2^{-a_{N+1}}) \cdot W(k_N)(x) \, dx
\]
\[
= \int_0^{2^{-a_{N+1}}} \chi_{a_{N+1}+1}(z + c2^{-a_{N+1}}, y + c2^{-a_{N+1}}) \cdot W(k_N)(z + c2^{-a_{N+1}}) \, dz.
\]
The third equality follows from Item 2 of Lemma 4.7 and the change of variables $x = z + c2^{-a_{N+1}}$ in the last equality. By the induction assumption, we have $W(k_N)(z + c2^{-a_{N+1}}) = W(k_N)(z)$ for $z \in [0, 2^{-a_{N+1}})$. Thus we obtain
\[
W(k)(y + c2^{-a_{N+1}}) = \int_0^{2^{-a_{N+1}}} \chi_{a_{N+1}+1}(z + c2^{-a_{N+1}}, y + c2^{-a_{N+1}}) \cdot W(k_N)(z) \, dz.
\]
Then we continue the computation as follows:
\[
W(k)(y + c2^{-a_{N+1}}) = \int_0^{2^{-a_{N+1}}} \chi_{a_{N+1}+1}(z, y) \cdot W(k_N)(z) \, dz
\]
\[
= \int_0^1 \chi_{a_{N+1}+1}(z, y) \cdot W(k_N)(z) \, dz = W(k)(y).
\]
The first equality follows from Item 1 of Lemma 4.7 and the second equality follows from Item 2 of Lemma 4.7.

4.3.2 Proof of Lemma 3.6

We prove Lemma 3.6 using the results in the above subsection.

Proof. Lemma 3.6

Since $\chi_n(x, y) \geq 0$, we see that $W(k)(x) \geq 0$ on $[0, 1]^d$ by induction. We omit the details. We use this property of $W(k)(x) \geq 0$ to prove $\|W(k)\|_{L^p} \leq 2^{(1-\frac{1}{p})t}$.

Using H"older’s inequality, we have
\[
\|W(k)\|_{L^p}^p = \int_{[0,1]^d} |W(k)(x)| \cdot |W(k)(x)|^{p-1} \, dx \leq \|W(k)\|_{L^1} \|W(k)\|_{L^{p-1}}^{p-1}.
\]
Thus we have
\[
\|W(k)\|_{L^p} \leq \|W(k)\|_{L^1}^{\frac{1}{p}} \|W(k)\|_{L^{p-1}}^{1-\frac{1}{p}}.
\]
Then, if we have $\|W(k)\|_{L^1} = 1$ and $\|W(k)\|_{L^{p-1}} \leq 2^t$ for $k \in N_0^t$, we have
\[
\|W(k)\|_{L^p} \leq \|W(k)\|_{L^1}^{\frac{1}{p}} \|W(k)\|_{L^{p-1}}^{1-\frac{1}{p}} \leq 2^{(1-\frac{1}{p})t}.
\]
We prove \( \|W(k)\|_{L^1} = 1 \) and \( \|W(k)\|_{L^\infty} \leq 2 \) for any \( k \in \mathbb{N}_0 \) to complete the proof.

When \( k = 0 \), we have \( \|W(0)\|_{L^1} = 1 \) and \( \|W(0)\|_{L^\infty} = 1 \leq 2 \) since \( W(0) = 1 \).

We prove the case \( k = \sum_{i=1}^{N} 2^a_i \) by induction on \( N \). In the case \( k = 2^a_1 \), we have

\[
\|W(2^a_1)\|_{L^1} = \int_0^1 \int_0^1 \chi_{a_1+1}(x, y) \, dy \, dx = \int_0^1 \int_{\min(x, 2^{-a_1-1})}^{\max(x, 2^{-a_1-1})} 2^{a_1+1} \, dy \, dx = 1.
\]

The first equality follows from \( \chi_{a_1+1}(x, y) \geq 0 \). We prove \( \|W(2^a_1)\|_{L^\infty} = 2 \) to complete the case \( k = 2^a_1 \). By Lemma 4.8 we have

\[
\|W(2^a_1)\|_{L^\infty} = \sup_{y \in [0, 1)} |W(2^a_1)(y)| = \sup_{y \in [0, 2^{-a_1})} |W(2^a_1)(y)|.
\]

Since \( W(2^a_1)(y) = \int_0^1 \chi_{a_1+1}(x, y) \, dx \) and \( \chi_{a_1+1}(x, y) \geq 0 \), we have

\[
\|W(2^a_1)\|_{L^\infty} = \sup_{y \in [0, 2^{-a_1})} \int_0^1 \chi_{a_1+1}(x, y) \, dx = \max \left( \sup_{y \in [0, 2^{-a_1})} \int_0^1 \chi_{a_1+1}(x, y) \, dx, \sup_{y \in [2^{-a_1-1}, 2^{-a_1})} \int_0^1 \chi_{a_1+1}(x, y) \, dx \right).
\]

We calculate the supremum on \([0, 2^{-a_1-1})\) and \([2^{-a_1-1}, 2^{-a_1})\) separately. We assume that \( y \in [0, 2^{-a_1-1}) \). By Lemma 4.8 we have

\[
\int_0^1 \chi_{a_1+1}(x, y) \, dx = \int_0^y 2^{a_1+1} \, dx + \int_{2^{-a_1-1}}^{2^{-a_1-1}+y} 2^{a_1+1} \, dx = 2^{a_1+2}y.
\]

Hence we obtain

\[
\sup_{y \in [0, 2^{-a_1-1})} \int_0^1 \chi_{a_1+1}(x, y) \, dx = \sup_{y \in [0, 2^{-a_1-1})} 2^{a_1+2}y = 2.
\]

By the same argument we get the same result in the case \( y \in [2^{-a_1-1}, 2^{-a_1}) \).

We omit the details. Therefore we have \( \|W(2^a_1)\|_{L^\infty} = 2 \).

Now we assume that for any \( k_N = \sum_{i=1}^{N} 2^a_i \), we have that \( \|W(k_N)\|_{L^1} = 1 \) and \( \|W(k_N)\|_{L^\infty} \leq 2 \). Let \( k = k_N + 2^{a_{N+1}} \) satisfying \( a_N > a_{N+1} \). We prove that for any \( k = k_N + 2^{a_{N+1}} \), \( \|W(k)\|_{L^1} = 1 \) and \( \|W(k)\|_{L^\infty} \leq 2 \). By the fact that \( W(k_N + 2^{a_{N+1}}) \geq 0 \) and Fubini’s Theorem, we have

\[
\|W(k_N + 2^{a_{N+1}})\|_{L^1} = \int_0^1 \int_0^1 \chi_{a_N+1}(x, y)W(k_N)(x) \, dx \, dy = \int_0^1 W(k_N)(x) \, dx.
\]

By the fact \( W(k_N) \geq 0 \) and the assumption on \( k_N \), we obtain \( \|W(k_N + 2^{a_{N+1}})\|_{L^1} = \|W(k_N)\|_{L^1} = 1 \).
We prove $\|W(k)\|_{L^\infty} \leq 2$ as follows. By the fact $W(k_N + 2^{a_{N+1}}) \geq 0$ and Lemma 4.9, we have

$$\|W(k)\|_{L^\infty} = \sup_{y \in [0,2^{-a_{N+1}}]} W(k_N + 2^{a_{N+1}})$$

$$= \max \left( \sup_{y \in [0,2^{-a_{N+1}}-1]} \int_0^1 \chi_{a_{N+1}+1}(x,y) W(k_N)(x) \, dx, \right.$$  

$$\left. \sup_{y \in [2^{-a_{N+1}}-1,2^{-a_{N+1}}]} \int_0^1 \chi_{a_{N+1}+1}(x,y) W(k_N)(x) \, dx \right).$$

We can calculate the supremum as in the above argument. Then, we obtain

$$\|W(k)\|_{L^\infty} \leq \int_0^{2^{-a_{N+1}}} 2^{a_{N+1}+1} W(k_N)(x) \, dx = 2^{a_{N+1}+1} \sum_{i=0}^{2^{a_{N-1}+1}-1} \int_{2^{-a_N}}^{(i+1)2^{-a_N}} W(k_N)(x) \, dx$$

$$= 2^{a_{N+1}+1} \sum_{i=0}^{2^{a_{N-1}-1}} \int_0^{2^{-a_N}} W(k_N)(x) \, dx = 2^{a_{N+1}+1} \cdot \int_0^{2^{-a_N}} W(k_N)(x) \, dx.$$  

The second equality follows from Lemma 4.9. Thus we have

$$\|W(k)\|_{L^\infty} \leq 2^{a_{N+1}} \cdot \int_0^{2^{-a_N}} W(k_N)(x) \, dx$$

$$= 2 \sum_{i=0}^{2^{a_{N-1}-1}} \int_{2^{-a_N}}^{(i+1)2^{-a_N}} W(k_N)(x) \, dx = 2 \int_0^1 W(k_N)(x) \, dx.$$  

The first equality follows from Lemma 4.9. By the assumption on $k_N$, it follows that $\|W(k_N)\|_{L^1} = 1$, and hence we obtain $\|W(k)\|_{L^\infty} \leq 2 \cdot \|W(k_N)\|_{L^1} = 2$.  

### Acknowledgement

The author would like to thank Prof. Makoto Matsumoto and Prof. Josef Dick for helpful discussions and comments. The works of T. Y. was supported by the Program for Leading Graduate Schools, MEXT, Japan.

### References

[1] J. Baldeaux, J. Dick, J. Greslehner, and F. Pillichshammer, Construction algorithms for higher order polynomial lattice rules. J. Complexity, 27 (2011), 281-299.

[2] J. Baldeaux, J. Dick, G. Leobacher, D. Nuyens, and F. Pillichshammer, Efficient calculation of the worst-case error and (fast) component-by-component construction of higher order polynomial lattice rules. Numer. Algorithms, 59 (2012), 403-431.
[3] J. Dick, Explicit constructions of quasi-Monte Carlo rules for the numerical integration of high-dimensional periodic functions. SIAM J. Numer. Anal. 45 (2007), 2141-2176.

[4] J. Dick, Walsh spaces containing smooth functions and quasi-Monte Carlo rules of arbitrary high order. SIAM J. Numer. Anal. 46 (2008), 1519-1553.

[5] J. Dick. On quasi-Monte Carlo rules achieving higher order convergence. In: P. L’Ecuyer, A.B. Owen (eds.), Monte Carlo and Quasi-Monte Carlo Methods 2008, Springer, (2009), 73-96.

[6] J. Dick, F. Y. Kuo, Q. T. Le Gia, D. Nuyens, and Ch. Schwab, Higher order QMC Petrov-Galerkin discretization for affine parametric operator equations with random field inputs. SIAM J. Numer. Anal. 52 (2014), 2676-2702.

[7] J. Dick and F. Pillichshammer, Digital Nets and Sequences. Discrepancy Theory and Quasi-Monte Carlo Integration. Cambridge University Press, Cambridge, 2010.

[8] R. N. Gantner and Ch. Schwab Computational Higher Order Quasi-Monte Carlo Integration. Submitted, 2014. Available at http://www.sam.math.ethz.ch/sam_reports/reports_final/reports2014/2014-25.pdf.

[9] T. Goda, K. Suzuki, and T Yoshiki. The $b$-adic baker’s transformation for quasi-Monte Carlo integration using digital nets. arXiv:1312.5850v2.

[10] S. Harase. Quasi-Monte Carlo point sets with small t-values and WAFOM. arXiv:1406.1967v3.

[11] S. Harase and R. Ohori. A search for extensible low-WAFOM point sets. arXiv:1309.7828v2.

[12] N. J. Fine. On the Walsh functions. Trans. Amer. Math. Soc., 65 (1949), 372-414.

[13] L. Kuipers and H. Niederreiter. Uniform Distribution of Sequences. John Wiley, New York, 1974. Reprint, Dover Publications, Mineola, NY, 2006.

[14] F. J. Hickernell. A generalized discrepancy and quadrature error bound. Math. Comp., 67 (1998), 299-322.

[15] M. Matsumoto, M. Saito, and K. Matoba, A computable figure of merit for Quasi-Monte Carlo point sets. Math. Comp., 83 (2014), 1233-1250.

[16] H. Niederreiter, Random Number Generation and Quasi-Monte Carlo Methods. CBMS-NSF, Philadelphia, Pennsylvania, 1992.

[17] F. Schipp, W. R. Wade, and P. Simon. Walsh series. An introduction to dyadic harmonic analysis. With the collaboration of J. Pál. Adam Hilger Ltd., Bristol, 1990.
[18] I. F. Sharygin, A lower estimate for the error of quadrature formulas for certain classes of functions. Zh. Vychisl. Mat. i Mat. Fiz., 3 (1963), 370-376.

[19] I. H. Sloan and S. Joe. Lattice Methods for Multiple Integration. Clarendon Press, Oxford, 1994.

[20] I. H. Sloan and H. Wozniakowski, When are quasi-Monte Carlo algorithms efficient for high-dimensional integrals? J. Complexity 14 (1998), 1-33.