To the theory of solvability of degenerate integro-differential equations in Banach spaces

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Abstract. The paper considers a new approach to constructing generalized solutions of degenerate integro-differential equations of convolution type in Banach spaces. The principal idea of the method proposed implies the refusal of the condition of existence of the full Jordan set for the Fredholm operator of the higher derivative with respect to the operator bundle formed by the rest of operator coefficients of the differential part and by the operator kernel of the integral component of the equation. The conditions are superimposed upon the values of the operator function specially constructed on the basis elements of the Fredholm operator kernel. Under such an approach, the differential part of the equation may include not only the higher derivative but also any combination of lower derivatives, what allows one to consider the convolution integro-differential equations from universal positions, without any special account of the structure of the operator bundle. The method proposed represents a form of generalization of the technique based on the application of Jordan sets of Fredholm operators, and, in the case of existence of the latter, the method coincides with this technique. A generalized solution is constructed in the form of a convolution of the fundamental operator function, which corresponds to the equation under investigation, and the function, which includes both the right-hand side of the equation and the initial data. The conditions, under which such a generalized solution does not contain any singular component, and the regular component converts the initial equation into an identity and satisfies the initial data will provide for the resolvability of the initial problem in the class of functions characterized by the respective smoothness. In this case, the generalized solution constructed will be classical. The theorem on the form of fundamental operator function has been proved. The abstract results have been illustrated via examples of initial-boundary value problems of applied character (from the theory electromagnetic fields, the theory of oscillations in visco-elastic media, the theory of vibrations of thermal-elastic plates).

1. Introduction
The present paper investigates the initial-value problem of the following form

\[ Bu^{(N)}(t) - \sum_{i=1}^{l} A_{m_i} u^{(m_i)}(t) - \int_{0}^{t} k(t-s)u(s)ds = f(t), \quad (1) \]

\[ u(0) = u_0, \quad u'(0) = u_1, \ldots, \quad u^{(N-1)}(0) = u_{N-1}, \quad (2) \]

where \( N \geq 2, \ 0 \leq m_1 < m_2 < \ldots < m_l < N, \ B, A_{m_1}, \ldots, A_{m_l}, \ k(t) \) are closed linear operators with dense definition domains, which act from the Banach space \( E_1 \) into the Banach space \( E_2 \), operator \( B \) is of Fredholm type \([1]\), \( f(t) \) is a sufficiently smooth function with the values in \( E_2 \).
The interest to the problems (1)-(2) has been defined by at least the two reasons. On the one hand, this is the most general problem statement in the abstract form for a large range of initial boundary value problems of applied character (see, e.g., item 4 of the present paper). On the other hand, problem (1)-(2) is solvable in class $C^N(t \geq 0; E_1)$ only in the case of satisfaction of rigid conditions for the relationships between the initial conditions (2) and the right-hand side of the equation (1), i.e. function $f(t)$. This effect was observed by many mathematicians (see the bibliography in [2] or in [7]). For example, in [5], being considered in terms of Jordan sets of the Noether operators, the relations between the initial conditions (2) and the function $f(t)$ were explicitly written for the 1-st order differential equation ($N = 1$ and $k(t) \equiv 0$). Rejection of the conditions of connectedness has led to the statement of problem (1)-(2) in class $K'_+(E_1)$ of generalized functions with the support bounded from the left, and in [6], such a solution has been constructed in the form of a sum of the singular components with a pointwise support and the regular component. Although, the issue of uniqueness of the constructed (componentwise) solution remained open. The complete solution of the issue of univalent solvability of problem (1)-(2) in class $K'_+(E_1)$ has been obtained with the aid of construction $E_N(t)$ of the fundamental operator function (analog of the fundamental solution of the differential operator [3]). In this case, it was necessary that all the operator coefficients $A_{ij} \neq 0$, $i = 0, 1, \ldots, N - 1$ in equation (1) would be nonzero, and the Fredholm operator $B$ would have a full Jordan set with respect to the operator function

$$A_{N - 1} + A_{N - 2}t + \cdots + A_0 t^{N - 1} + \frac{t^{N - 1}}{(N - 1)!} \theta(t) * k(t) \theta(t)$$

(see [4] Theor. 4, p. 76). In this case, the desired unique generalized solution may be restored in the form of convolution $\tilde{u}(t) = E_N(t) * F(t)$,

$$F(t) = f(t) \theta(t) + Bu_{N - 1} \delta(t) + Bu_{N - 2} \delta'(t) + \cdots + Bu_0 \delta^{(N - 1)}(t). \quad (3)$$

Rejection of at least some part of the conditions $A_{ij} \neq 0$, $i = 0, 1, \ldots, N - 1$ implies impossibility of constructing the Jordan set, and, as a result, this states the problem of finding an alternative for this condition. The present paper proposes and discusses such an approach.

2. Auxiliary information and the principal denotations

Let us henceforth assume that the following condition be satisfied:

A) $D(A) = \bigcup_{i=1}^l D(A_{m_i})$, $D(k) \equiv D(k(t))$ is independent of time $t$, $\overline{D(A)} = \overline{D(B)} = \overline{D(k)} = E_1$, $D(B) \subset D(A)$, $D(B) \subset D(k)$, $\overline{R(B)} = R(B)$, dim $N(B) = \dim N(B^*) = n \geq 1$.

Let us denote the basis of kernel $N(B)$ of operator $B$ by $\{ \varphi_i \} \in E_1$; the basis of kernel $N(B^*)$ of the conjugate operator $B^*$ by $\{ \psi_i \} \in E_2^*$; here $i = 1, \ldots, n$, $\{ \gamma_i \} \in E_1^*$ and $\{ \zeta_i \} \in E_2$ are the biorthogonal systems of elements corresponding to these bases, i.e. $\langle \varphi_i, \varphi_j \rangle = \langle \psi_i, \psi_j \rangle = \delta_{ij}$, $\delta_{ij}$ are Kronecker symbols. As shown in [1], under these assumptions, there exists a bounded Trenogin-Schmidt operator of the form

$$\Gamma = \left( B_0 \right)^{-1} = \left( B + \sum_{i=1}^n \langle \cdot, \gamma_i \rangle \zeta_i \right)^{-1} \in \mathcal{L}(E_2, E_1).$$

Introduce a recurrent family of bounded operators:

$$L_0 = I, \quad L_{-1} = L_{-2} = \cdots = L_{-(N - 1)} \equiv 0, \quad L_k = A_{m_1} \Gamma L_{k-(N-m_1)} + \cdots + A_{m_2} \Gamma L_{k-(N-m_2)} + A_{m_1} \Gamma L_{k-(N-m_1)}, \quad k \geq 1.$$
The property of pseudo-commutation is proved by induction with respect to \( k \):

\[
L_k = L_{k-(N-m_1)}A_{m_1}\Gamma + \cdots + L_{k-(N-m_2)}A_{m_2}\Gamma + L_{k-(N-m_1)}A_{m_1}\Gamma.
\]

With aid of this family it is possible to construct the following operator function of bounded operators

\[
\mathcal{U}_N(t) = \sum_{k=0}^{\infty} L_k t^{k+N-1} \frac{1}{(k+N-1)!}.
\]

**Remark 1.** For the operator function \( \mathcal{U}_N(t) \) introduced valid is another (integral) representation via the operator resolvent

\[
\tilde{\mathcal{R}}_N(\lambda; A_{m_1}\Gamma, \ldots, A_{m_2}\Gamma, A_{m_1}\Gamma) = \left( \lambda^N I - \lambda^{m_1} A_{m_1}\Gamma - \cdots - \lambda^{m_2} A_{m_2}\Gamma - \lambda^{m_1} A_{m_1}\Gamma \right)^{-1} = \sum_{k=0}^{\infty} L_k \lambda^k \frac{1}{\lambda^{k+N}}
\]

in the form of contour integral

\[
\mathcal{U}_N(t) = \frac{1}{2\pi i} \oint_{C_\rho} \tilde{\mathcal{R}}_N(\lambda; A_{m_1}\Gamma, \ldots, A_{m_2}\Gamma, A_{m_1}\Gamma)e^{\lambda t} d\lambda.
\]

Here \( C_\rho \) is the circumference of the complex plane with the center in the origin and with radius \( \rho > R \), where

\[
\frac{\|A_{m_1}\Gamma\|}{R^{N-m_1}} + \cdots + \frac{\|A_{m_2}\Gamma\|}{R^{N-m_2}} + \frac{\|A_{m_1}\Gamma\|}{R^{N-m_1}} < 1.
\]

By immediate verification it is possible to make sure that the following statement is valid.

**Lemma 1.** The operator function \( \mathcal{U}_N(t) \theta(t) \) is fundamental for the differential operator

\[
I\delta^{(N)}(t) - \sum_{i=1}^{l} A_{m_i}\Gamma \delta^{(m_i)}(t).
\]

Let us denote by \( \mathcal{R}(t) \) the resolvent of kernel \( k(t)\Gamma\theta(t) * \mathcal{U}_N(t)\theta(t) \), hence the following statement is valid.

**Lemma 2.** The generalized operator function

\[
\tilde{\mathcal{E}}_N(t) = \mathcal{U}_N(t)\theta(t) * \left( I\delta(t) + \mathcal{R}(t)\theta(t) \right)
\]

is fundamental for the following integro-differential operator

\[
\tilde{L}_N(\delta(t)) = I\delta^{(N)}(t) - \sum_{i=1}^{l} A_{m_i}\Gamma \delta^{(m_i)}(t) - k(t)\Gamma\theta(t),
\]

furthermore, it is obvious that \( k(t)\Gamma\theta(t) * \tilde{\mathcal{E}}_N(t) \equiv \mathcal{R}(t) \).

**Proof.** According to the definition of the fundamental operator function \[2\] let us verify the validity of the following equalities:

\[
\tilde{L}_N(\delta(t)) * \tilde{\mathcal{E}}_N(t) = \tilde{\mathcal{E}}_N(t) * \tilde{L}_N(\delta(t)) = I_2 \delta(t),
\]

where \( I_2 \) is a unit operator of space \( E_2 \).
Indeed,

\[ \tilde{L}_N(\delta(t)) * \tilde{E}_N(t) = \left( I \delta(t) - k(t) \Gamma \theta(t) * \mathcal{U}_N(t) \theta(t) \right) * \left( I \delta(t) + \mathcal{R}(t) \theta(t) \right) = I_2 \delta(t). \]

On the other hand, the following other representation is valid for \( \tilde{E}_N(t) \):

\[ \tilde{E}_N(t) = \mathcal{U}_N(t) \theta(t) * \left( I \delta(t) + \mathcal{R}(t) \theta(t) \right) = \mathcal{U}_N(t) \theta(t) + k(t) \Gamma \theta(t) * \mathcal{U}_N(t) \theta(t) = \left[ I \delta(t) + \mathcal{U}_N(t) \theta(t) \right] * k(t) \Gamma \theta(t) \]

therefore

\[ \tilde{E}_N(t) * \tilde{L}_N(\delta(t)) = \left[ I \delta(t) + \mathcal{U}_N(t) \theta(t) \right] * \left( I \delta(t) + \mathcal{R}(t) \theta(t) \right) * k(t) \Gamma \theta(t) - \mathcal{U}_N(t) \theta(t) * k(t) \Gamma \theta(t) - \mathcal{R}(t) \theta(t) * k(t) \Gamma \theta(t) = I_2 \delta(t). \]

Let \( G(t) = \frac{d^N}{dt^N} \tilde{E}_N(t) \). Let us construct the operator function (a regulatory one)

\[ \mathcal{M}(t) = G(t) - I \delta(t) = \mathcal{R}(t) \theta(t) + \mathcal{U}_N^{(N)}(t) \theta(t) * \left( I \delta(t) + \mathcal{R}(t) \theta(t) \right) \]

and introduce the projector in space \( E_2 \)

\[ Q = \sum_{i=1}^{n} Q_i = \sum_{i=1}^{n} \langle \cdot, \psi_i \rangle z. \]

From now on, by \( \frac{d^N}{dt^N} \) we will denote differentiation in the generalized sense, and via \( \mathcal{U}_N^{(N)}(t) \) – classical differentiation.

For each element \( z_i \in E_2, i = 1, \ldots, n \) we introduce the denotation \( l_k(\varphi_i) = \mathcal{M}^{(k)}(0) z_i \), furthermore, \( \langle l_k(\varphi_i), \psi_j \rangle = 0, j = 1, \ldots, n, k = 0, 1, \ldots, p_i - 1, \) but not all the numbers \( \langle l_p(\varphi_i), \psi_j \rangle, j = 1, \ldots, n \) are equal to zero. Since for \( m_i \leq N - 2, \mathcal{M}^{(k)}(0) = 0, k = 0, 1, \ldots, N - m_i - 2 \) and \( \mathcal{M}^{(N-m_i-1)}(0) = A_{m_i} \Gamma \), we have \( p_i \geq 1 \).

Remark 2. The conditions given above imply solvability (see [1]) of equations \( B_0(\varphi_i^{(k+1)} = \mathcal{M}^{(k)}(0) z_i, k = 0, 1, \ldots, p_i - 1, \) in this case, the partial solution, which has the form \( \varphi_i^{(k+1)} = \Gamma \mathcal{M}^{(k)}(0) z_i \), may be called a formal \((k + 1)\)-adjoined element to \( \varphi_i \). The set of elements \( \{ \varphi_i^{(k)} \}, i = 1, \ldots, n, k = 1, \ldots, p_i \), in which \( \varphi_i^{(k+1)} = \varphi_i \), when \( \mathcal{M}^{(k)}(0) z_i = 0, \) may be called a set of formally adjoined elements, and the set \( \{ \varphi_i^{(k)} \}, k = 1, \ldots, p_i \) — the chain of formally adjoined elements; parameter \( p_i \) may be called the length of the chain of formally adjoined elements. Obviously, the set \( \{ \varphi_i^{(k)} \}, i = 1, \ldots, n, k = 1, \ldots, p_i \) constructed like that won’t – in the general case – represent an independent system of elements.

Next, let us enumerate the basis elements of kernel \( \{ \varphi_i \} \in N(B) \) in the order of growth of the parameters \( p_i \), i.e. \( 1 \leq p_1 \leq p_2 \leq \ldots \leq p_n \). The set of formally adjoined elements \( \{ \varphi_i^{(k)} \} \)
will be called full, when satisfied is condition det \(\|\langle l_{p_i}(\varphi_i), \psi_j\rangle\| \neq 0\), in this case, the basis of the kernel of the conjugate operator \(\{\psi_i\} \in N(B^*)\) may be reconstructed so (see [1]) that the following equalities \(\langle l_{p_i}(\varphi_i), \psi_j\rangle = \delta_{ij}\) be satisfied. So, let us henceforth consider the set \(\{\varphi_i^{(k)}\}\) as complete, and basis \(\{\psi_i\}\) reconstructed.

So, let us assume that the following condition is satisfied:

\[B)\]

\[\langle l_k(\varphi_i), \psi_j\rangle = \begin{cases} 0, & k = 0, 1, \ldots, p_i - 1, \quad i, j = 1, \ldots, n, \\ \delta_{ij}, & k = p_i, \end{cases} \]

furthermore,

\[\langle l_{p_i+k}(\varphi_i), \psi_j\rangle = 0, \quad i = 1, \ldots, n - 1, \quad j = 1, \ldots, n, \quad k = 1, \ldots, p_n - p_i.\]

**Remark 3.** In the case of the “full” differential operator (i.e. \(A_i \neq 0, i = 0, 1, \ldots, N - 1\)), condition \(B)\) in equation (1) transforms into the condition of existence of a full Jordan set from the publication [10]. Condition \(B)\) was used in papers [4] and [11] right in this sense.

**Remark 4.** Due to the assumption that \(1 \leq p_1 \leq p_2 \leq \cdots \leq p_n\) let us group the basis elements of kernel \(N(B)\) as follows: the first group will include \(l_1\) initial elements of the basis having minimum lengths of the chains of formally adjoined elements \(p_1\); the second group will include the following \(l_2\) elements of the basis having the lengths of the chains \(p_2 > p_1\), etc. the third group will join the last \(l_n\) elements of the basis with the maximum lengths of the chains equal to \(p_n\). Therefore, there will be \(m\) blocks formed and \(l_1 + l_2 + \ldots + l_m = n\), for the projector \(Q\) in this case we obtain an expansion of the form

\[Q = \sum_{i=1}^{n} Q_i = \sum_{j=1}^{m} \tilde{Q}_j, \quad \text{here} \quad \tilde{Q}_j = \sum_{i=l_1+l_2+\ldots+l_{j-1}+1}^{l_1+l_2+\ldots+l_j} Q_i,\]

furthermore, due to condition \(B)\)

\[\tilde{Q}_j \mathcal{M}^{(k)}(0) \tilde{Q}_i = \begin{cases} \delta_{ij} \tilde{Q}_i, & k = p_i, \\ 0, & k < p_i. \end{cases} \]  

(4)

Obviously, the lengths of the chains of elements \(\varphi_i\); which have appeared in the same \(j\)-th block, are similar, and so, we shall denote them by \(p_j\).

Let us denote by \(\mathcal{M}_1(t)\) the resolvent of kernel

\[-\sum_{i=1}^{n} Q_i \mathcal{M}^{(p_i+1)}(t) \theta(t) = -\sum_{j=1}^{m} \tilde{Q}_j \mathcal{M}^{(p_j+1)}(t) \theta(t),\]

hence, due to its construction \(Q \delta(t) * \mathcal{M}_1(t) = \mathcal{M}_1(t)\), and, according to the rule of differentiation of the convolution of generalized functions (see [3]), the following property of pseudocommutation

\[\tilde{E}_N(t) * \left( I \delta(t) + \mathcal{M}_1(t) \theta(t) \right) * \sum_{j=1}^{m} \tilde{Q}_j \delta^{(p_j+1)}(t) \ast G(t) = \]

\[G(t) * \left( I \delta(t) + \mathcal{M}_1(t) \theta(t) \right) * \sum_{j=1}^{m} \tilde{Q}_j \delta^{(p_j+1)}(t) \ast \tilde{E}_N(t)\]

is valid.

The following auxiliary operator-convolution equalities are also satisfied.
Lemma 3. Let conditions A) and B) be satisfied, hence

\[
(I\delta(t) + M_1(t)\theta(t)) \ast \sum_{j=1}^{m} \tilde{Q}_j \delta^{(p_j+1)}(t) \ast M(t)\theta(t) \ast Q\delta(t) = Q\delta(t); \tag{5}
\]

\[
Q\delta(t) + \left( I\delta(t) + M_1(t)\theta(t) \right) \ast \sum_{j=1}^{m} \tilde{Q}_j \delta^{(p_j+1)}(t) \ast \left\{ Q\delta(t) + G(t) \ast (I - Q)\delta(t) \right\} =
\]

\[
\left( I\delta(t) + M_1(t)\theta(t) \right) \ast \sum_{j=1}^{m} \tilde{Q}_j \delta^{(p_j+1)}(t) \ast G(t); \tag{6}
\]

\[
F_k(t) = Q\delta(t) \ast \left[ M^{(p_k+1)}(t)\theta(t) + \frac{d^{p_k+1}}{dt^{p_k+1}} \left( M(t)\theta(t) \right) \ast Q\delta(t) \ast M_1(t)\theta(t) \right] \ast \tilde{Q}_k \delta(t) \equiv 0, \quad k = 1, \ldots, m. \tag{7}
\]

Proof. Due to condition B) and in accordance with equality (4) we have

\[
\sum_{j=1}^{m} \tilde{Q}_j \delta^{(p_j+1)}(t) \ast M(t)\theta(t) \ast Q\delta(t) = \sum_{j=1}^{m} \left( \tilde{Q}_j \delta(t) + \tilde{Q}_j M^{(p_j+1)}(t)\theta(t) \ast Q\delta(t) \right) =
\]

\[
Q\delta(t) + \sum_{j=1}^{m} \tilde{Q}_j M^{(p_j+1)}(t)\theta(t) \ast Q\delta(t),
\]

whence the validity of equality (5) follows. Indeed,

\[
\left( I\delta(t) + M_1(t)\theta(t) \right) \ast \sum_{j=1}^{m} \tilde{Q}_j \delta^{(p_j+1)}(t) \ast M(t)\theta(t) \ast Q\delta(t) =
\]

\[
\left( I\delta(t) + M_1(t)\theta(t) \right) \ast \left( Q\delta(t) + \sum_{j=1}^{m} \tilde{Q}_j M^{(p_j+1)}(t)\theta(t) \ast Q\delta(t) \right) =
\]

\[
Q\delta(t) + M_1(t)\theta(t) \ast Q\delta(t) - M_1(t)\theta(t) \ast Q\delta(t) = Q\delta(t).
\]

In turn, from relation (5) we obtain the following chain of identities

\[
Q\delta(t) + \left( I\delta(t) + M_1(t)\theta(t) \right) \ast \sum_{j=1}^{m} \tilde{Q}_j \delta^{(p_j+1)}(t) \ast \left\{ I\delta(t) - G(t) \right\} \ast Q\delta(t) =
\]

\[
Q\delta(t) - Q\delta(t) \equiv 0,
\]

which proves the validity of equality (6).

Let us now prove the validity for the equality (7). According to the rule of differentiation of generalized functions, we can find

\[
F_k(t) = Q\delta(t) \ast \left[ M^{(p_k+1)}(t)\theta(t) + \frac{d^{p_k+1}}{dt^{p_k+1}} \left( M(t)\theta(t) \right) \ast Q\delta(t) \ast M_1(t)\theta(t) \right] \ast \tilde{Q}_k \delta(t) =
\]
\[
Q \mathcal{M}^{(p_k+1)}(t) \theta(t) - \sum_{j=1}^{m} Q \delta(t) * \left( \mathcal{M}^{(p_k+1)}(t) \theta(t) + \mathcal{M}^{(p_k)}(0) \delta(t) + \mathcal{M}^{(p_k-1)}(0) \delta'(t) + \ldots + \mathcal{M}(0) \delta^{(p_k)}(t) \right) * \tilde{Q}_j \mathcal{M}^{(p_j+1)}(t) \theta(t) * \left( I \delta(t) + \mathcal{M}_1(t) \theta(t) \right) \right] * \tilde{Q}_k \delta(t).
\]

Hence due to relations (4) we have:

\[
\mathcal{F}_k(t) = \left[ Q \mathcal{M}^{(p_k+1)}(t) \theta(t) * \left( I \delta(t) + \mathcal{M}_1(t) \theta(t) \right) - \sum_{i=1}^{k} \frac{d^{p_k-p_i}}{d^{p_k-p_i}} \left( \tilde{Q}_i \mathcal{M}^{(p_i+1)}(t) \theta(t) * \left( I \delta(t) + \mathcal{M}_1(t) \theta(t) \right) \right) * \tilde{Q}_i \delta(t) \right] * \tilde{Q}_k \delta(t).
\]

After computing all the generalized derivatives \( \frac{d^{p_k-p_i}}{d^{p_k-p_i}} \left( \tilde{Q}_i \mathcal{M}^{(p_i+1)}(t) \theta(t) \right) \) and collecting similar terms we obtain

\[
\mathcal{F}_k(t) = \sum_{j=k+1}^{m} \tilde{Q}_j \mathcal{M}^{(p_k+1)}(t) \theta(t) * \left( I \delta(t) + \mathcal{M}_1(t) \theta(t) \right) * \tilde{Q}_k \delta(t) =
\]

\[
\sum_{j=k+1}^{m} \tilde{Q}_j \frac{t^{p_j-p_k}}{(p_j-p_k)!} \theta(t) * \frac{d^{p_j-p_k}}{d^{p_j-p_k}} \left( \tilde{Q}_j \mathcal{M}^{(p_k+1)}(t) \theta(t) \right) * \left( I \delta(t) + \mathcal{M}_1(t) \theta(t) \right) * \tilde{Q}_j \delta(t) =
\]

\[
\sum_{j=k+1}^{m} \tilde{Q}_j \frac{t^{p_j-p_k}}{(p_j-p_k)!} \theta(t) * \tilde{Q}_j \mathcal{M}^{(p_j+1)}(t) \theta(t) * \left( I \delta(t) + \mathcal{M}_1(t) \theta(t) \right) * \tilde{Q}_j \delta(t).
\]

According to relations (4), from this representation we finally obtain the following:

\[
\mathcal{F}_k(t) = \sum_{j=k+1}^{m} \tilde{Q}_j \frac{t^{p_j-p_k}}{(p_j-p_k)!} \theta(t) * \left[ \tilde{Q}_j \mathcal{M}^{(p_j+1)}(t) \theta(t) * \left( I \delta(t) + \mathcal{M}_1(t) \theta(t) \right) \right] * \tilde{Q}_k \delta(t) \equiv 0.
\]
3. Theorem on the form of the fundamental operator function

The following statement is valid.

**Theorem 1.** If conditions A) and B) are satisfied, then the integro-differential operator \( L_N(\delta(t)) = B \delta^{(N)}(t) - \sum_{i=1}^{l} A_m \delta^{(m_i)}(t) - k(t) \theta(t) \) has the following fundamental operator function in the class of distributions with the support bounded from the left \( K'_+(E_1) \)

\[
E_N(t) = \Gamma \delta(t) * \tilde{E}_N(t) \ast \left[ I \delta(t) - \left( I \delta(t) + M_1(t) \theta(t) \right) \ast \sum_{j=1}^{m} \tilde{Q}_j \delta^{(p_j+1)}(t) \ast G(t) \right] \tag{8}
\]

or (due to the property of pseudocommutation)

\[
E_N(t) = \Gamma \delta(t) \ast \left[ I \delta(t) - G(t) \ast \left( I \delta(t) + M_1(t) \theta(t) \right) \ast \sum_{j=1}^{m} \tilde{Q}_j \delta^{(p_j+1)}(t) \right] \ast \tilde{E}_N(t). \tag{9}
\]

**Proof.** In accordance with the definition of the fundamental operator function [2], let us show that the following two operator-convolution equalities are satisfied.

\[
E_N(t) \ast L_N(\delta(t)) \ast u(t) = u(t), \quad \forall u(t) \in K'_+(E_1), \tag{10}
\]

\[
L_N(\delta(t)) \ast E_N(t) \ast v(t) = v(t), \quad \forall v(t) \in K'_+(E_2). \tag{11}
\]

To the end of proving the identity (10) let us use representation (9) for \( E_N(t) \)

\[
E_N(t) \ast L_N(\delta(t)) = E_N(t) \ast L_N(\delta(t)) \ast \Gamma \tilde{B} \delta(t) = E_N(t) \ast \left( \tilde{L}_N(\delta(t)) - Q \delta^{(N)}(t) \right) \ast \tilde{B} \delta(t) =
\]

\[
\Gamma \delta(t) \ast \left[ I \delta(t) - G(t) \ast \left( I \delta(t) + M_1(t) \theta(t) \right) \ast \sum_{j=1}^{m} \tilde{Q}_j \delta^{(p_j+1)}(t) \right] \ast \left[ I \delta(t) - G(t) \ast Q \delta(t) \right] \ast \tilde{B} \delta(t).
\]

Having proved the equality

\[
\mathcal{F}(t) = \left[ I \delta(t) - G(t) \ast \left( I \delta(t) + M_1(t) \theta(t) \right) \ast \sum_{j=1}^{m} \tilde{Q}_j \delta^{(p_j+1)}(t) \right] \ast \left[ I \delta(t) - G(t) \ast Q \delta(t) \right] = I \delta(t),
\]

we obtain the desired statement. Indeed,

\[
\mathcal{F}(t) = I \delta(t) - G(t) \ast Q \delta(t) - G(t) \ast \left( I \delta(t) + M_1(t) \theta(t) \right) \ast \sum_{j=1}^{m} \tilde{Q}_j \delta^{(p_j+1)}(t) \ast \left[ I \delta(t) - G(t) \right] \ast Q \delta(t),
\]

Here we have used the properties of the projectors in the following manner

\[
\sum_{j=1}^{m} \tilde{Q}_j \delta^{(p_j+1)}(t) \ast \left[ I \delta(t) - G(t) \ast Q \delta(t) \right] = \sum_{j=1}^{m} \tilde{Q}_j \delta^{(p_j+1)}(t) \ast Q \delta(t) \ast \left[ I \delta(t) - G(t) \ast Q \delta(t) \right] =
\]

\[
\sum_{j=1}^{m} \tilde{Q}_j \delta^{(p_j+1)}(t) \ast \left[ Q^2 \delta(t) - Q \delta(t) \ast G(t) \ast Q \delta(t) \right] =
\]

\[
\sum_{j=1}^{m} \tilde{Q}_j \delta^{(p_j+1)}(t) \ast Q \delta(t) \ast \left[ I \delta(t) - G(t) \right] \ast Q \delta(t) = \sum_{j=1}^{m} \tilde{Q}_j \delta^{(p_j+1)}(t) \ast \left[ I \delta(t) - G(t) \right] \ast Q \delta(t).
\]
Due to identity (5) we have
\[ F(t) = I\delta(t) - G(t) * Q\delta(t) + G(t) * Q\delta(t) = I\delta(t). \]

Therefore, the validity of equality (10) has been completely proved.

To the end of proving the validity of identity (11) let us use representation (8) for \( E_N(t) \):
\[ L_N(\delta(t)) * E_N(t) = (\tilde{L}_N(\delta(t)) - Q\delta^{(N)}(t)) * \tilde{E}_N(t) \]
\[ = \left[ I\delta(t) - \left( I\delta(t) + M_1(t)\theta(t) \right) * \sum_{j=1}^{m} \tilde{Q}_j \delta^{(p_j+1)}(t) * G(t) \right] \]
\[ = I\delta(t) - \left[ Q\delta(t) \left( I\delta(t) + M_1(t)\theta(t) \right) * \sum_{j=1}^{m} \tilde{Q}_j \delta^{(p_j+1)}(t) \right] * G(t). \]

In order to complete the proof for the equality (11) it remains only to make sure that
\[ N(t) = Q\delta(t) + \left( I\delta(t) - Q\delta(t) * G(t) \right) * \left( I\delta(t) + M_1(t)\theta(t) \right) * \sum_{j=1}^{m} \tilde{Q}_j \delta^{(p_j+1)}(t) \equiv 0. \]

Due to the properties of the projectors we have
\[ \left( I\delta(t) - Q\delta(t) * G(t) \right) * \left( I\delta(t) + M_1(t)\theta(t) \right) * \sum_{j=1}^{m} \tilde{Q}_j \delta^{(p_j+1)}(t) = \]
\[ \left( I\delta(t) - Q\delta(t) * G(t) \right) * \left( I\delta(t) + M_1(t)\theta(t) \right) * Q\delta(t) * \sum_{j=1}^{m} \tilde{Q}_j \delta^{(p_j+1)}(t) = \]
\[ \left( I\delta(t) - Q\delta(t) * G(t) \right) * \left( Q^2 \delta(t) + Q\delta(t) * M_1(t)\theta(t) * Q\delta(t) \right) * \sum_{j=1}^{m} \tilde{Q}_j \delta^{(p_j+1)}(t) = \]
\[ \left( I\delta(t) - Q\delta(t) * G(t) \right) * Q\delta(t) * \left( I\delta(t) + M_1(t)\theta(t) \right) * Q\delta(t) * \sum_{j=1}^{m} \tilde{Q}_j \delta^{(p_j+1)}(t) = \]
\[ \left( Q^2 \delta(t) - Q\delta(t) * G(t) * Q\delta(t) \right) * \left( I\delta(t) + M_1(t)\theta(t) \right) * \sum_{j=1}^{m} \tilde{Q}_j \delta^{(p_j+1)}(t) = \]
\[ Q\delta(t) * \left( I\delta(t) - G(t) \right) * Q\delta(t) * \left( I\delta(t) + M_1(t)\theta(t) \right) * \sum_{j=1}^{m} \tilde{Q}_j \delta^{(p_j+1)}(t) = \]
\[ -Q\delta(t) * M(t)\theta(t) * Q\delta(t) * \left( I\delta(t) + M_1(t)\theta(t) \right) * \sum_{j=1}^{m} \tilde{Q}_j \delta^{(p_j+1)}(t) = \]
\[ -\sum_{j=1}^{m} \frac{d^{p_j+1}}{dt^{p_j+1}} \left( Q M(t) \tilde{Q}_j \theta(t) \right) - \sum_{j=1}^{m} Q M(t) Q \theta(t) * M_1(t)\theta(t) * \tilde{Q}_j \delta^{(p_j+1)}(t). \]
Meanwhile, due to relations (4) and the rules of differentiation of generalized functions, we have

\[
\sum_{j=1}^{m} \frac{d^{p_j+1}}{dt^{p_j+1}} \left( Q \mathcal{M}(t) \dot{Q}_j \delta(t) \right) = \sum_{j=1}^{m} \left( \dot{Q}_j \delta(t) + Q \mathcal{M}^{(p_j+1)}(t) \ddot{Q}_j \delta(t) \right) = Q \delta(t) + \sum_{j=1}^{m} Q \mathcal{M}^{(p_j+1)}(t) \theta(t) * \ddot{Q}_j \delta(t).
\]

Therefore, under the denotations of Lemma 3 (see relation (7))

\[
\mathcal{N}(t) = Q \delta(t) - Q \delta(t) - \sum_{j=1}^{m} \left( \mathcal{M}^{(p_j+1)}(t) \theta(t) \right) + \frac{d^{p_j+1}}{dt^{p_j+1}} \left( \mathcal{M}(t) \theta(t) \right) * Q \delta(t) * \mathcal{M}_1(t) \theta(t) \right) * \ddot{Q}_j \delta = - \sum_{j=1}^{m} \mathcal{F}_j(t) \equiv 0.
\]

**Remark 5.** With the use of the method of proving Theorem 1 given in the present paper it is possible to obtain more compact proofs of the principal statements for the publications [2], [4] and [11].

Due to relation (6) of Lemma 3 we have obtained the following (alternative) form of the record for \( \mathcal{E}_N(t) \). And this is the ground for the following statement.

**Theorem 2.** If conditions A) and B) are satisfied, then the integro-differential operator \( \mathcal{L}_N(\delta(t)) = B \delta^{(N)}(t) - \sum_{i=1}^{l} A_{m_i} \delta^{(m_i)}(t) - k(t) \theta(t) \) has the following fundamental operator function in the class of distributions with the support \( \mathcal{K}_+^t(E_1) \) bounded from the left:

\[
\mathcal{E}_N(t) = \Gamma \delta(t) * \mathcal{E}_N(t) * \left[ (I - Q) \delta(t) - \left( I \delta(t) + \mathcal{M}_1(t) \theta(t) \right) * \sum_{j=1}^{m} \dot{Q}_j \delta^{(p_j+1)}(t) * \{ Q \delta(t) + G(t) * (I - Q) \delta(t) \} \right].
\]

As noted above, with the aid of the fundamental operator function it is possible to conduct complete investigation of the initial-value problem (1)-(2). In terms of generalized functions, problem (1)-(2) may be rewritten in the form

\[
\mathcal{L}_N(\delta(t)) * \tilde{u}(t) = F(t), \text{ where } F(t) \text{ in (3)}.
\]

Hence the generalized function

\[
\tilde{u}(t) = \mathcal{E}_N(t) * F(t) \in \mathcal{K}_+^t(E_1)
\]

represents a solution of this equation (see (11)), furthermore, due to (10), it is unique in class \( \mathcal{K}_+^t(E_1) \).

By analyzing the structure of the solution \( \tilde{u}(t) = \mathcal{E}_N(t) * F(t) \in \mathcal{K}_+^t(E_1) \), it is possible to obtain the statement of solvability of the initial-value problem (1)-(2) in class \( \mathcal{C}^N(t \geq 0, E_1) \). In particular, if operator \( B \) does not have any adjoined elements (see [1]) with respect to operator \( A_{m_i} \) (following after the \( B \) nonzero operator coefficient in equation (1)), then all the
\[ p_i = N - m_l - 1, \ i = 1, \ldots, n \] and the formula for the fundamental operator function assumes the form:

\[ \mathcal{E}_N(t) = \Gamma \delta(t) \star \left[ \tilde{\mathcal{E}}_N(t) - G(t) \star \left( I \delta(t) + \mathcal{M}_1(t) \theta(t) \right) \star Q \left( \frac{t^{m_l - 1}}{(m_l - 1)!} \theta(t) \star G(t) \right) \right]. \]

In this case, the generalized solution \( \tilde{u}(t) = \mathcal{E}_N(t) \star F(t) \in \mathcal{R}_+^l(E_1) \) turns out to be a regular generalized function, which transforms equation (1) into an identity. Having required the satisfaction of the initial conditions, we can obtain the statement of solvability of the initial-value problem (1)-(2) in class \( C^N(t \geq 0, E_1) \).

Corollary 1. If in the conditions of Theorem 1 \( N = 4, l = 2, m_l = 2, m_1 = 0, p_i = 1 \), then the initial-value problem (1)-(2) is solvable in class \( C^4(t \geq 0, E_1) \) if and only if the following conditions are satisfied

\[ Q(A_2u_2 + A_0u_0 + f(0)) = 0, \quad Q(A_2u_3 + A_0u_1 + k(0)u_0 + f'(0)) = 0. \]

Corollary 2. If in the conditions of Theorem 1 \( N = 3, l = 1, m_l = 1, p_i = 1 \), then the initial-value problem (1)-(2) is solvable in class \( C^3(t \geq 0, E_1) \) if and only if the following conditions are satisfied

\[ Q(A_1u_1 + f(0)) = 0, \quad Q(A_1u_2 + k(0)u_0 + f'(0)) = 0. \]

Corollary 3. If in the conditions of Theorem 1 \( N \geq 2, l = 1, m_1 = 0, p_i = N - 1 \), then the initial-value problem (1)-(2) is solvable in class \( C^N(t \geq 0, E_1) \) if and only if the following conditions are satisfied

\[ Q \left( A_0u_4 + f^{(i)}(0) + k(0)u_{i-1} + k'(0)u_{i-2} + \ldots + k^{(i-2)}(0)u_1 + k^{(i-1)}(0)u_0, \psi_j \right) = 0, \]

\[ i = 0, 1, \ldots, N - 1. \]

4. Applications

Let us investigate the following initial boundary value problems with the aid of Theorem 1.

Consider the Cauchy-Dirichlet problem for the generalized potential of electric field [8]

\[ \frac{\partial^2}{\partial t^2} \left( \frac{\partial^2}{\partial \bar{x}^2} + \alpha \right) (\Delta u - \beta u) + \gamma \frac{\partial^2}{\partial \bar{x}^2} \Delta u + \alpha \gamma \frac{\partial^2}{\partial x^2} + \frac{t}{\int_0^t g(t-s) \Delta^2 u(s, \bar{x}) ds = div f(t, \bar{x})} \]

\[ u^{(i)} \Bigg|_{t=0} = u_i(\bar{x}), \quad i = 0, 1, 2, 3, \quad \bar{x} \in \Omega; \quad u \Bigg|_{x \in \partial \Omega} = 0, \quad t \geq 0. \]

Here \( g(t), f(t, \bar{x}) \) are the given functions, \( u = u(t, \bar{x}) \) is the desired (sought) function, \( \bar{x} \in \Omega \subset \mathbb{R}^m \) is a bounded domain with an infinitely smooth boundary \( \partial \Omega \), \( \Delta \) is the Laplace operator, \( u = u(t, \bar{x}) \) is defined on the cylinder \( R_+ \times \Omega, \beta \in \sigma(\Delta) \). Here, in the denotations of Theorem 1, \( N = 4, l = 2, m_l = 2, m_1 = 0 \).
In the case of the Cauchy-Dirichlet problem (12)-(13), the Banach spaces and the operators are given as follows:

\[ E_1 \equiv \left\{ v(\bar{x}) \in W^{k+4}_2(\Omega) : v|_{\partial\Omega} = 0, \right\}, \quad E_2 \equiv W^{k}_2(\Omega), \]

\[ B = \Delta - \beta, \quad A_2 = -(\alpha + \gamma)\Delta + \alpha\beta, \quad A_0 = -\alpha\gamma \frac{\partial^2}{\partial x_N^2}, \]

where \( W^{k+4}_2(\Omega) \) and \( W^{k}_2(\Omega) \) are Sobolev’s spaces; let \( \varphi_i(\bar{x}), \ i = 1, \ldots, n \) is the orthonormalized basis of the space of solutions of the homogeneous problem: \( \beta\varphi_i = \Delta\varphi_i, \ \varphi_i|_{\partial\Omega} = 0 \), i.e. operator \( B \) is Fredholm, and due to the relations \( \langle A_2\varphi_i, \varphi_j \rangle = \gamma\beta\delta_{ij} \) it does not have any \( A_2 \)-adjointed elements. Consequently, according to Corollary 1, we can make the following statement.

**Theorem 3.** The Cauchy-Dirichlet problem (12)-(13) is unequivocally solvable in class \( C^4(t \geq 0, E_1) \) if and only if

\[ \left\langle \gamma\beta u_2(\bar{x}) + \alpha\gamma \frac{\partial^2 u_1(\bar{x})}{\partial x_N^2} - \text{div} f'_1(0, \bar{x}) + g(0)\beta^2 u_0(\bar{x}), \varphi_i(\bar{x}) \right\rangle = 0, \]

\[ \left\langle \gamma\beta u_2(\bar{x}) + \alpha\gamma \frac{\partial^2 u_0(\bar{x})}{\partial x_N^2} - \text{div} f(0, \bar{x}), \varphi_i(\bar{x}) \right\rangle = 0, \ i = 1, \ldots, n. \]

The followings 2 examples are considered similarly.

Consider the Cauchy-Dirichlet problem for the equation from the theory of vibrations of thermoelastic plates [9] (when \( k = 0 \))

\[ \frac{\partial^3}{\partial t^3}(\Delta u - \beta u) + \gamma \frac{\partial}{\partial t} \Delta^2 u + \int_0^t g(t-s)\Delta^3 u(s, \bar{x}) ds = f(t, \bar{x}), \quad \left. u^{(i)}_t \right|_{t=0} = u_i(\bar{x}), \ i = 0, 1, 2, \ \bar{x} \in \Omega; \quad u|_{\bar{x} \in \partial\Omega} = 0, \ t \geq 0, \]  

(14)  

(15)

Here, likewise above, \( g(t), \ f(t, \bar{x}) \) are the given functions, \( u = u(t, \bar{x}) \) is the desired (sought) function, \( \bar{x} \in \Omega \subset R^n \) is the bounded domain with an infinitely smooth boundary \( \partial\Omega \), \( \Delta \) is the Laplace operator, \( u = u(t, \bar{x}) \) is defined the cylinder \( R_+ \times \Omega, \ \beta \in \sigma(\Delta) \). Furthermore, in the denotations of Theorem 1, we imply \( N = 3, l = 1, m_l = 1 \).

In the case of the Cauchy-Dirichlet problem (14)-(15), the Banach spaces and the operators will be given as follows:

\[ E_1 \equiv \left\{ v(\bar{x}) \in W^{k+6}_2(\Omega) : v|_{\partial\Omega} = 0, \right\}, \quad E_2 \equiv W^{k}_2(\Omega), \quad B = \Delta - \beta, \quad A_1 = -\gamma\Delta^2, \]

where \( W^{k+6}_2(\Omega) \) and \( W^{k}_2(\Omega) \) are Sobolev’s subspaces; let \( \varphi_i(\bar{x}), \ i = 1, \ldots, n \) be an orthonormalized basis of the space of solutions for the homogeneous problem: \( \beta\varphi_i = \Delta\varphi_i, \ \varphi_i|_{\partial\Omega} = 0 \), i.e. operator \( B \) is Fredholm, and due to relations \( \langle A_1\varphi_i, \varphi_j \rangle = -\gamma\beta^2\delta_{ij} \) it does not have any \( A_1 \)-adjointed elements. Consequently, according to Corollary 2, we obtain the following statement.

**Theorem 4.** The Cauchy-Dirichlet problem (14)-(15) is univalently solvable in class \( C^3(t \geq 0, E_1) \) if and only if

\[ \left\langle \gamma\beta^2 u_2(\bar{x}) - f'_1(0, \bar{x}) + g(0)\beta^3 u_0(\bar{x}), \varphi_i(\bar{x}) \right\rangle = 0, \]
\[ \langle \gamma \beta^2 w_1(\bar{x}) - f(0, \bar{x}), \varphi_i(\bar{x}) \rangle = 0, \quad i = 1, \ldots, n. \]

Consider the Cauchy-Dirichlet problem from the theory of oscillations in visco-elastic media [12]

\[ (\lambda - \Delta) u_{tt} - (\mu - \Delta) u - \int_0^t g(t - \tau) \Delta^2 u(\tau, \bar{x}) d\tau = f(t, \bar{x}), \]

\[ u \big|_{t=0} = u_0(\bar{x}), \quad u_t \big|_{t=0} = u_1(\bar{x}), \quad \bar{x} \in \Omega; \quad u \big|_{\bar{x} \in \partial \Omega} = 0, \quad t \geq 0, \]

(16) (17)

Here, likewise above, \( g(t) \), \( f(t, \bar{x}) \) are the given functions, \( u = u(t, \bar{x}) \) is the desired (sought) function, \( \bar{x} \in \Omega \subset \mathbb{R}^m \) is the bounded domain with an infinitely smooth boundary \( \partial \Omega \), \( \Delta \) is the Laplace operator, \( u = u(t, \bar{x}) \) is defined the cylinder \( \mathbb{R}_+ \times \Omega \), \( \beta \in \sigma(\Delta) \). Furthermore, in the denotations of Theorem 1, we imply \( N = 2, \ l = 1, \ m_l = 0 \).

In the case of the Cauchy-Dirichlet problem (16)-(17), the Banach spaces and the operators will be given as follows:

\[ E_1 \equiv \left\{ v(\bar{x}) \in W^2_2(\Omega) : v|_{\partial \Omega} = 0 \right\}; \ E_2 \equiv W_2(\Omega), \ B = \lambda - \Delta, \ A_0 = (\mu - \Delta) \lambda \in \sigma(\Delta), \]

where \( W^2_2(\Omega) \) and \( W_2(\Omega) \) are Sobolev’s subspaces; let \( \varphi_i(\bar{x}), \ i = 1, \ldots, n \) be an orthonormalized basis of the space of solutions for the homogeneous problem: \( \lambda \varphi_i = \Delta \varphi_i, \ \varphi_i|_{\bar{x} \in \partial \Omega} = 0, \) i.e. operator \( B \) is Fredholm, and due to relations \( \langle A_0 \varphi_i, \varphi_j \rangle = (\mu - \lambda) \delta_{ij} \) it does not have any \( A_0 \)-adjoined elements. Consequently, according to Corollary 3, we obtain the following statement.

**Theorem 5.** The Cauchy-Dirichlet problem (16)-(17) is univalently solvable in class \( C^2(t \geq 0, E_1) \) if and only if

\[ \langle (\mu - \lambda) u_1(\bar{x}) + f_1'(0, \bar{x}) + g(0) \lambda^2 u_0(\bar{x}), \varphi_i(\bar{x}) \rangle = 0, \]

\[ \langle (\mu - \lambda) u_0(\bar{x}) + f(0, \bar{x}), \varphi_i(\bar{x}) \rangle = 0, \quad i = 1, \ldots, n, \]

5. Conclusion

The method described in the present paper allows one to solve the problems of unequivocal solvability of degenerate initial-value problems from the universal positions, furthermore, both in the classes of distributions and in the spaces of functions characterized by some finite smoothness. In such cases, there are no additional constraints superimposed upon the structure of the operator bundle of the equation in the form of subordination of one of its parts to another. The abstract results obtained by the proposed method may be applied in investigations of various mathematical models, in which the current status of the process is indirectly influenced by the total prehistory of the observations.

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References

[1] Weinberg M M and Trenogin V A 1969 Theory of Branching Solutions of Nonlinear Equations (Moscow: Nauka)

[2] Sidorov N, Loginov B, Sinitsyn A and Falaleev M 2002 Lyapunov-Schmidt Methods in Nonlinear Analysis and Applications (Dordrecht: Kluwer Academic Publishers)

[3] Vladimirov V S 1979 Generalized Functions in Mathematical Physics (Moscow: Nauka)

[4] Falaleev M V and Orlov S S 2011 Russian Math. 10 68–79

[5] Sidorov N A and Romanova O A 1983 Differential equations B 19 1516–26

[6] Sidorov N A and Falaleev M V 1987 Differential equations B 23 726-8

[7] Falaleev M V and Orlov S S 2012 Proc. of the Institute of Mathematics and Mechanics of the Ural Branch of the Russian Academy of Sciences B 18 (4) 286–97

[8] Sveshnikov A G, Alshin A B, Korpusov M O and Pletner Yu D 2007 Linear and Nonlinear Sobolev Type Equations (Moscow: Nauka)

[9] Rivera J E and Fatori L H 1998 Math. Meth. Appl. Sci. 21 797–21

[10] Loginov B V and Rusak Y B 1978 Direct and Inverse Problems for Partial Differential Equations and their Applications (Tashkent: FAN) 133–48

[11] Falaleev M V and Orlov S S 2011 The Bulletin of Irkutsk State University. Series Mathematics 4(1) 118–34

[12] Cavalcanti M M, Domingos Cavalcanti V N and Ferreira J 2001 Math. Meth. Appl. Sci. 24 1043–53