LIMITING ABSORPTION PRINCIPLE ON MANIFOLDS HAVING ENDS WITH VARIOUS MEASURE GROWTH RATE LIMITS

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Abstract. The purpose of this paper is to study the property of the resolvent of the Laplace-Beltrami operator on a noncompact complete Riemannian manifold with various ends each of which has a different limit of the growth rate of the Riemannian measure at infinity, in particular, focusing on the limiting absorption principle. As a result, we will obtain the absolute continuity of the Laplace-Beltrami operator.

1. Introduction

The Laplace-Beltrami operator $\Delta_g$ on a noncompact complete Riemannian manifold $(M, g)$ is essentially self-adjoint on $C_0^\infty(M)$, and its self-adjoint extension to $L^2(M, v_g)$ has been studied by several authors from various points of view. The purpose of this paper is to prove the limiting absorption principle and the absolute continuity of the Laplace-Beltrami operator on a Riemannian manifold with a combination of a plurality of ends, each of which has a different limit of the growth rate of the Riemannian measure at infinity. A Riemannian manifold with a plurality of ends has been studied by several authors (see Mazzeo-Melrose [21], Mazzeo [20], Perry [24], Bouclet [3] and so on), but they assumed that the curvatures of all ends converged to a common constant at infinity; such an assumption does not produce any difference between “one” and “more-than-one” with respect to the number of ends. This paper studies a Riemannian manifold with a combination of different geometries for each end.

It is important to note that the spectral structure of the Laplace-Beltrami operator determines the asymptotic behavior of a solution of the wave equation or of a time-dependent Schrödinger equation as time tends to infinity. For example, if $u$ is a vector of the absolutely continuous subspace of the Laplace-Beltrami operator, then the “wave function” $e^{i\Delta t}u$ decays locally as $t \to \pm \infty$; on the other hand, if $u$ is an element of the pure point subspace of the Laplace-Beltrami operator, the wave function $e^{i\Delta t}u$ remains localized for all time $t$.

To state our results, we shall first introduce some terminology and notations. Let $(M, g)$ be an $n$-dimensional connected noncompact complete Riemannian manifold and $V$ be an (possibly unbounded) open subset of $M$. We shall say that $E := M \setminus V$ is an end with radial coordinates if and only if $V$ has a compact connected $C^\infty$-boundary $\partial V$ such that the outward normal exponential map $\exp_{\partial V}$ :
$N^+(\partial V) \to M\setminus V$ induces a diffeomorphism, where $N^+(\partial V) := \{ w \in T(\partial V) \mid w$ is outward normal to $\partial V \}$. We assume that there exists a relatively compact open subset $U$ of $M$ such that $M\setminus U$ consists of a finite number of ends $E_1, E_2, \cdots, E_m$ with radial coordinates:

$$M\setminus U = E_1 \cup E_2 \cup \cdots \cup E_m$$

(disjoint union),

where $m \geq 1$ is an integer. We denote $r(x) := \text{dist}_g(U, x)$ for $x \in M\setminus U$. For convenience, we shall introduce the following terminologies: we shall say that an end $E$ with radial coordinates satisfies the condition “MC($\frac{a}{r}, \frac{b}{r}; \delta$)”, if there exist constants $a > 0, b > 0, \delta \in (0, 1)$ such that

$$
\begin{align*}
\nabla \! dr & \geq \left\{ \frac{a}{r} + O(r^{-1-\delta}) \right\} (g - dr \otimes dr) \quad \text{on } E, \\
\Delta_g r & = \frac{b}{r} + O(r^{-1-\delta}) \quad \text{on } E;
\end{align*}
$$

(1)

also, we shall say that an end $E$ satisfies the condition “MC($\alpha; \beta; \delta$)”, if there exist constants $\alpha > 0, \beta > 0, \delta \in (0, 1)$ such that

$$
\begin{align*}
\nabla \! dr & \geq \{ \alpha + O(r^{-1-\delta}) \} (g - dr \otimes dr) \quad \text{on } E, \\
\Delta_g r & = \beta + O(r^{-1-\delta}) \quad \text{on } E;
\end{align*}
$$

(2)

“MC” stands for “mean curvature”; note that $\Delta_g r$ is the mean curvatures of each level-hypersurface of the function $r$, and that $\Delta_g r$ expresses the growth rate of the Riemannian measure $v_g$ on the level-hypersurfaces of $r$. We assume that

(3) \quad $E_j$ satisfies $\text{MC}(\frac{a_j}{r}, \frac{b_j}{r}; \delta_j)$ \quad for $1 \leq j \leq m_0$;

(4) \quad $E_j$ satisfies $\text{MC}(\alpha_j; \beta_j; \delta_j)$ \quad for $m_0 + 1 \leq j \leq m$,

where $a_1, a_2, \cdots, a_{m_0}, b_1, b_2, \cdots, b_{m_0}, \alpha_{m_0+1}, \alpha_{m_0+2}, \cdots, \alpha_m$ are positive constants;

$$0 = \beta_1 = \beta_2 = \cdots = \beta_{m_0} < \beta_{m_0+1} \leq \beta_{m_0+2} \leq \cdots \leq \beta_m$$

are real constants; $\delta_1, \delta_2, \cdots, \delta_m$ are constant in $(0, 1)$; 0 “$\leq$” $m_0 \leq m$ is an integer.

**Remark 1.1.** If $m_0 = 0$, we mean that every end satisfies (4), and that there is no end satisfying (3), and hence, \{a_1, a_2, \cdots, a_{m_0}\} = \{b_1, b_2, \cdots, b_{m_0}\} = \{\delta_1, \delta_2, \cdots, \delta_{m_0}\} = \emptyset.

In order to state our theorems, we need to introduce the weighted $L^2$-spaces; let $v_g$ be the Riemannian measure of $(M, g)$. For any $s \in \mathbb{R}$, let $L_s^2(M, v_g)$ denote the Hilbert space of all complex-valued measurable functions $f$ such that $|(1 + r)^s f|$ is square integrable on $M$; the inner product and norm will be denoted as follows:

$$
(f, g)_{L_s^2(M, v_g)} = \int_M \{1 + r(x)\}^{2s} f(x) g(x) \overline{dv_g}(x),
$$

$$
\|f\|_{L_s^2(M, v_g)} = \sqrt{(f, f)_{L_s^2(M, v_g)}},
$$

where recall that $r = \text{dist}_g(U, *)$. Let $R(z)$ denote the resolvent $(\Delta_g - z)^{-1}$ of $-\Delta_g$ for $z \in \rho(-\Delta_g)$, where $\rho(-\Delta_g)$ stands for the resolvent set of $-\Delta_g$. For $t, t' \in \mathbb{R}$, let $\mathcal{B}(t, t')$ denote the space of all bounded linear operators $T : L_t^2(M, v_g) \to L_{t'}^2(M, v_g)$
with $\text{Dom}(T) = L^2_v(M, v_g)$; $\mathbb{B}(t, t')$ is a Banach space with the operator norm $\|*\|_{\mathbb{B}(t, t')}$. We set
\[ \delta := \min \{ \delta_j \mid 1 \leq j \leq m \} \]
and denote
\[ I := \left( \frac{(\beta_1)^2}{4}, \infty \right) - \left\{ \frac{(\beta_j)^2}{4} \mid j = 2, 3, \ldots, m \right\}; \]
\[ \Pi_+ := \left\{ z = x + iy \in \mathbb{C} \mid x > \frac{(\beta_1)^2}{4}, y \geq 0 \right\} - \left\{ \frac{(\beta_j)^2}{4} \mid j = 2, 3, \ldots, m \right\}; \]
\[ \Pi_- := \left\{ z = x + iy \in \mathbb{C} \mid x > \frac{(\beta_1)^2}{4}, y \leq 0 \right\} - \left\{ \frac{(\beta_j)^2}{4} \mid j = 2, 3, \ldots, m \right\}. \]
Now, we shall state our main result.

**Theorem 1.1 (Principle of limiting absorption).** Let $(M, g)$ be an $n$-dimensional connected complete Riemannian manifold. Assume that there exists a relatively compact open subset $U$ of $M$ such that $M \setminus U$ consists of a disjoint union of finitely many ends $E_1, E_2, \ldots, E_m$ with radial coordinates, where $m \geq 1$ is an integer. Assume that the conditions (3) and (4) holds, where $a_1, a_2, \ldots, a_m, b_1, b_2, \ldots, b_m, a_{m+1}, a_{m+2}, \ldots, a_m$ are positive constant;
\[ 0 = \beta_1 = \beta_2 = \cdots = \beta_m \leq \beta_{m+1} \leq \beta_{m+2} \leq \cdots \leq \beta_m \]
are real constants; $\delta_1, \delta_2, \ldots, \delta_m$ are constant in $(0, 1)$; $0 \leq \delta$ $m_0 \leq m$ is an integer. Note that, if $m_0 = 0$, we follows the convention stated in Remark 1.1. Let $s$ and $s'$ be real numbers satisfying
\[ 0 < s' < s < \min \left\{ a_{\min}, \frac{1}{2} \right\}; \quad s' + s \leq \delta = \min \{ \delta_1, \cdots, \delta_m \}, \]
where
\[ a_{\min} := \begin{cases} \min \{ a_1, \cdots, a_m \} & \text{if } \beta_1 = 0; \\ \infty & \text{if } \beta_1 > 0. \end{cases} \]
Then, in the Banach space $\mathbb{B} \left( \frac{1}{2} + s, -\frac{1}{2} - s' \right)$, we have the limit
\[ R(\lambda \pm i0) := \lim_{\epsilon \downarrow 0} R(\lambda \pm i \epsilon) \quad \text{for } \lambda \in I \quad \text{(double sign in same order)}. \]
Moreover, the convergence (5) is uniform on any compact subset of $I$, and hence, $R(z)$ is continuous on $\Pi_+$ and $\Pi_-$ with respect to the operator norm $\|*\|_{\mathbb{B} \left( \frac{1}{2} + s, -\frac{1}{2} - s' \right)}$, by considering $R(\lambda + i0)$ on $\Pi_+ \cap (0, \infty)$, and $R(\lambda - i0)$ on $\Pi_- \cap (0, \infty)$, respectively.

In order to state the next theorem, we shall recall some terminology from the spectral theory: let $H$ be a self-adjoint operator on a Hilbert space $(X, \langle , \rangle_X)$, and $E(\Lambda)$ ($\Lambda \in \mathcal{B}$) denote the spectral measure of $H$ on $X$, where $\mathcal{B}$ stands for the set of all Borel sets of $\mathbb{R}$; “spectral measure” is also called “spectral decomposition” or “spectral projection”. Any $u \in X$ defines the measure $m_u$ on $\mathbb{R}$, by $m_u(\Lambda) := \langle E(\Lambda)u, u \rangle_X$ for $\Lambda \in \mathcal{B}$. Let $|*|$ denote the Lebesgue measure on $\mathbb{R}$. The property of $m_u$ classifies vectors in $X$ as follows: set
\[ X_{pp} := \text{the closure of the linear hull of eigenvectors of } H; \]
\[ X_c := \{ u \in X \mid m_u(\{a\}) = 0 \text{ for any } a \in \mathbb{R} \}; \]
\[ X_{ac} := \{ u \in X_c \mid m_u \text{ is absolutely continuous with respect to } |*| \}; \]
then, $X$ is decomposed into the direct sum of three closed linear subspaces:

$$\tag{6} X = X_{pp} \oplus X_{ac} \oplus X_{sc},$$

where $X_{pp}$, $X_{ac}$, and $X_{sc}$ are orthogonal to each other, and reduce $H$; thus, corresponding to the decomposition (6), $H$ is decomposed into the direct sum of three self-adjoint operators:

$$H = H_{pp} \oplus H_{ac} \oplus H_{sc}.$$  

$X_{pp}$, $X_{ac}$, and $X_{sc}$ are called the pure point subspace of $H$, continuous subspace of $H$, absolutely continuous subspace of $H$, and singular continuous subspace of $H$, respectively. If $H = H_{ac}$ (i.e., $X = X_{ac}$), then $H$ is said to be absolutely continuous. Also, if $H = H_{sc}$ (i.e., $X = X_{sc}$), $H$ is said to be singular continuous. Moreover, if $H$ is absolutely continuous on $\mathbb{R}$, then $H|_{E(J)X}$ is absolutely continuous, then $H$ is said to be absolutely continuous on $J$. (For the details mentioned above, see [14] Chapter 10, or [26] VII.2).

**Theorem 1.2.** Assume that $(M,g)$ satisfies the assumptions in Theorem 1.1. Then, $\sigma_{\text{ess}}(-\Delta_g) = \left(\frac{(n-1)^2}{4}, \infty\right)$, and $-\Delta_g$ is absolutely continuous on $\left(\frac{(n-1)^2}{4}, \infty\right)$, where $\sigma_{\text{ess}}(-\Delta_g)$ stands for the essential spectrum of $-\Delta_g$. In particular, $-\Delta_g$ has no singular continuous spectrum.

**Corollary 1.1.** Assume that $(M,g)$ satisfies the assumptions in Theorem 1.1. If $m_0 \geq 1$, then $-\Delta_g$ is absolutely continuous on $(0, \infty)$ and 0 is not an eigenvalue of $-\Delta_g$.

Note that the decay order (2) is fairly sharp; indeed, there exists a rotationally symmetric manifold $(\mathbb{R}^n, g := dr^2 + f(r)^2 g_{S^{n-1}(1)})$ which satisfies $\Delta_gr = (n - 1)(1 + O(r^{-1}))$ as $r \to \infty$; $\sigma_{\text{ess}}(-\Delta_g) = \left[\frac{(n-1)^2}{4}, \infty\right)$; $\sigma_{\text{pp}}(-\Delta_g) \cap \left(\frac{(n-1)^2}{4}, \infty\right) = \left(\frac{(n-1)^2}{4}, 1\right)$ [see 17].

Now, we shall recall some earlier works and compare them to the theorems above. First, recall that Xavier proved the following:

**Theorem 1.3** (Xavier [29]). Let $(M,g)$ be an Hadamard manifold and $r$ be the distance function to some fixed point of $M$. Assume that the function $f = (r^2+1)^{1/2}$ satisfies the following conditions (i) and (ii):

(i) $\Delta_g f \leq C_1$,
(ii) $(\Delta_g)^2 f \leq C_2 f^{-3},$

where $C_1$ and $C_2$ are positive constants. Then, $-\Delta_g$ is absolutely continuous on $(\alpha, \infty)$, where $4\alpha = 6C_1 + C_2$.

The nature of Theorem 1.3 seems to be more analytic than geometric; note that the term $(\Delta_g)^2 f$ in (ii) contains the derivatives of the curvature tensor. Theorem 1.1 and 1.2 seem to be more geometric than Xavier’s in the sense that Theorem 1.1 and 1.2 do not require any estimates of derivatives of the curvature tensor.

After that, Donnelly proved the following by using the Mourre theory:

**Theorem 1.4** (Donnelly [6]). Assume that a complete Riemannian manifold $(M,g)$ admits a proper $C^2$-exhaustion function $b$ satisfying the following:

(i) $c_1 r \leq b \leq c_2 r$ for some positive constants $c_1$ and $c_2$;
(ii) $|\nabla b - 1| \leq c b^{-\varepsilon}$;
(iii) $|\nabla db - b^{-1}(g - db \otimes db)| \leq c b^{-1-\varepsilon}$;
(iv) $|(b^2)_{kks}| + |(b^2)_{skk}| \leq c b^{-\varepsilon},$

where $c$ and $\varepsilon$ are positive constants; $r$ denotes the distance function to a fixed point of $M$; indices stand for the components of the covariant differential; the repeated indices expresses contraction. Then $-\Delta g$ is absolutely continuous.

Since the function $b$ is not a distance function, Theorem 1.4 seems to be general in that sense. But, the author has a feeling that the condition (iv) of Theorem 1.4 is somehow technical.

In this paper, we will modify the arguments used in a classical method for the Schrödinger operator in Euclidean space; see, Eidus [7, 8], Ikebe and Saito [13], and Mochizuki and Uchiyama [23] and so on. We will dare to use the distance function explicitly.

The Mourre theory ([22, 2]) is a powerful tool for studying of the Schrödinger operator on the Euclidean space, and its nature is quite abstract. Note that Froese and Hislop [9] studied spectral properties of second-order elliptic operators on non-compact manifolds by using the Mourre theory from “analytic” points of view (they required the estimates of some derivatives of the curvature tensor); Golénia and Moroianu [11] proved the limiting absorption principle under a bounded condition on the second derivative of the metric on conformally cusp manifolds by modifying the Mourre theory; see also Froese-Hislop-Perry [10] for a hyperbolic manifold, and Guillopé [12] and so on.

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2. UNITARILY EQUIVALENT OPERATOR $L$ AND RADIATION CONDITION

In this section, we shall define the unitarily equivalent operator $L$, and introduce the radiation condition for $L$.

First, we shall list the notation used in the sequel:

$$E_j(s, t) := \{x \in E_j \mid s < r(x) < t\};$$
$$E_j(s, \infty) := \{x \in E_j \mid s < r(x)\};$$
$$S_j(t) := \{x \in E_j \mid r(x) = t\};$$
$$E(s, t) := \{x \in M \mid s < r(x) < t\} = \bigcup_{j=1}^m E_j(s, t);$$
$$E(s, \infty) := \{x \in M \mid s < r(x)\} = \bigcup_{j=1}^m E_j(s, \infty);$$
$$S(t) := \{x \in M \mid r(x) = t\} = \bigcup_{j=1}^m S_j(t);$$
$$U(R) := U \cup \{x \in M \mid 0 \leq r(x) < R\},$$

where $R > 0; 0 \leq s < t$.

We take a real-valued $C^\infty$ function $w$ on $M$ so that

$$w(x) = \frac{\beta_j}{2} r(x) \quad \text{for} \quad x \in E_j(1, \infty) \quad (1 \leq j \leq m),$$

where recall $r(x) = \text{dist}_g(U, x)$ for $x \in M \setminus U$. We shall introduce a new measure $\mu$ on $M$ and the operator $L$ as follows:

$$\mu := e^{-2w} \nu_g; \quad L := e^w \circ \Delta_g \circ e^{-w}.$$
Then, a direct computation shows that
\[(8) \quad Lf = \Delta_g f - 2(\nabla w, \nabla f) - Vf; \quad V := \Delta_g w - |\nabla w|^2.\]

Since the multiplication operator \(e^w : L^2(M, v_g) \ni h \mapsto e^w h \in L^2(M, \mu)\) is unitary, \(L\) with \(\text{Dom}(L) = \{ u \in L^2(M, \mu) \mid Lu \in L^2(M, \mu) \}\) is a nonnegative self-adjoint operator on \(L^2(M, \mu)\). Note that assumptions (3) and (4), together with (1) and (2), imply that \(\Delta_g r \to \beta_j\) as \(r \to \infty\) on \(E_j\), and hence,
\[(9) \quad V(x) = \frac{\beta_j}{2} \Delta_g r(x) - \left(\frac{\beta_j}{2}\right)^2 \to \frac{(\beta_j)^2}{4} \quad \text{as} \quad x \in E_j \quad \text{and} \quad r(x) \to \infty.\]

In particular, \(V\) is bounded on \(M\).

Let \(A\) denote the induced Riemannian measures on each level hypersurface \(S(t)\) for \(t \geq 0\), and set
\[A_w := e^{-2w} A.\]

For \(\Omega \subset M\) and \(s \in \mathbb{R}\), let \(L^2_s(\Omega, \mu)\) denote the Hilbert space of all complex-valued measurable functions \(f\) such that \(|(1 + r)^s f|\) is square integrable over \(\Omega\).

Now, we shall consider the equation
\[(10) \quad -Lu - z u = f\]
for some suitably chosen \(z \in \mathbb{C}\) and \(f \in L^2(M, \mu)\). If \(z \in \rho(-L)\), this equation has a unique solution \(u \in L^2(M, \mu)\) for \(f \in L^2(M, \mu)\), where \(\rho(-L)\) is the resolvent set of \(-L\). In order to extend this uniqueness theorem for \(z \in \sigma_c(-L)\), we have to consider the operator \(-L\) in a wider class and introduce the boundary condition at infinity for our manifolds, where \(\sigma_c(-L)\) stands for the continuous spectrum of \(-L\); we choose smooth functions \(p_+ : M \times \Pi_+ \to \mathbb{C}\) and \(p_- : M \times \Pi_- \to \mathbb{C}\) so that, for each \(1 \leq j \leq m,
\[(11) \quad p_{\pm}(x, z) = \mp i \sqrt{z - \frac{(\beta_j)^2}{4}} r(x) + \alpha_j \log r(x) \quad \text{for} \quad (x, z) \in E_j(r_0, \infty) \times \Pi_{\pm},\]
where \(r_0 \geq 1\) is a constant; \(\alpha_j\) is the function defined by
\[(12) \quad \alpha_j(x) := \begin{cases} \frac{b_j}{2} & \text{if} \quad x \in E_j(r_0, \infty) \quad (1 \leq j \leq m_0), \\ 0 & \text{if} \quad x \in E_j(r_0, \infty) \quad (m_0 + 1 \leq j \leq m). \end{cases}\]

Note that the square root in (11) is the principal value, that is, the analytic extension of \(\sqrt{x}\) for \(x > 0\). Note also that we are following the usual “double sign in same order” convention” in (11). This convention will be used in the sequel of this paper. We shall consider the following condition:

**Definition 2.1 (radiation condition).** We shall say that a solution \(u\) of the equation (10) with \(z \in \Pi_{\pm}\) satisfies the **radiation condition** if there exists constants \(s'\) and \(s\) such that
\[(13) \quad 0 < s' \leq s < 1; \quad s' + s \leq 1;\]
\[(14) \quad u \in L^2_{\frac{1}{2} - s'}(M, \mu), \quad \partial_{\nu} u + (\partial_{\nu} p_{\pm}) u \in L^2_{\frac{1}{2} + s}(M, \mu).\]

A solution \(u\) of (10) satisfying the radiation condition will be called an **outgoing solution** or **incoming solution**, if \(z \in \Pi_+\) or \(z \in \Pi_-\), respectively.
3. Energy integral

This section will be devoted to proofs of Proposition 3.1 and 3.2 below, which express energy integrals of a solution of (10).

First, we extend the Riemannian metric $g = \langle \ast, \ast \rangle$ to the complex bilinear form for complex tangent vectors: $\langle u_1 + iv_1, u_2 + iv_2 \rangle = \langle u_1, u_2 \rangle - \langle v_1, v_2 \rangle + i \langle \{ u_1, v_2 \} + \{ v_1, u_2 \} \rangle$ for $u_1, u_2, v_1, v_2 \in T_xM \ (x \in M)$; we also denote $|u + iv|^2 = \langle u, u \rangle + \langle v, v \rangle$ for $u, v \in T_xM \ (x \in M)$.

For the sake of convenience of readers, we mention two lemmas below, which will immediately follow from the standard Green’s formula and divergence theorem, respectively:

**Lemma 3.1** (Green’s formula). Let $\Omega$ be a relatively compact open subset of $M$ with $C^\infty$-boundary $\partial \Omega$, and $u$ and $v$ be $C^\infty$-functions on $M$. Then, we have

$$\int_\Omega (Lu)v \, d\mu = \int_{\partial \Omega} \langle \nabla u, \overline{\nu} \rangle v \, dA_w - \int_{\Omega} \langle \nabla u, \nabla v \rangle \, d\mu - \int_{\Omega} Vuv \, d\mu,$$

where $\overline{\nu}$ stands for the outward unit normal vector field along $\partial \Omega$.

**Lemma 3.2** (divergence theorem). Let $\Omega$ be a relatively compact open subset of $M$ with $C^\infty$-boundary $\partial \Omega$, and $X$ be a $C^\infty$-vector field on $M$. Then, we have

$$\int_\Omega (\text{div } X) \, d\mu = \int_{\partial \Omega} \langle X, \overline{\nu} \rangle \, dA_w + 2 \int_{\partial \Omega} \langle X, \nabla u \rangle \, d\mu.$$

**Proposition 3.1.** Let $\varphi(r)$ be a nonnegative function of $r \geq 0$ and $u \in H^2_{\text{loc}}(M)$ be a solution of the equation (10). Let $\Omega$ be a relatively compact open subset of $M$ with $C^\infty$-boundary $\partial \Omega$. Then, we have

$$- \int_{\partial \Omega} \varphi \langle \nabla u, \overline{\nu} \rangle \, dA_w - z \int_{\Omega} \varphi |u|^2 \, d\mu$$

$$= \int_{\Omega} \{ \varphi \langle f\overline{\nu} - |\nabla u|^2 - V|u|^2 \rangle - \varphi' \langle \partial_r u, \overline{\nu} \rangle \} \, d\mu.$$

**Proof.** Multiplying the equation (10) by $\varphi \overline{\nu}$ and integrating it over $\Omega$ with respect to the measure $\mu$, we obtain the desired equation by Lemma 3.1. \qed

The following proposition will play an important role in our arguments:

**Proposition 3.2.** Let $\varphi = \varphi(r)$ be a real-valued function of $r \in [0, \infty)$ satisfying $\varphi(r) \geq 0$ for $r > 0$, and $u \in H^2_{\text{loc}}(M)$ be a solution of (10) satisfying the radiation condition. Then, for any $R > r_0$, we have

$$\int_{S(R)} \varphi |\text{Im } \partial_r p_\pm |u|^2 \, dA_w + |\text{Im } z| \int_{U(R)} \varphi |u|^2 \, d\mu$$

$$\leq \int_{S(R)} \varphi |\text{Im } (\overline{\partial_r u} \mp \partial_r p_\pm u) | \, dA_w + \int_{U(R)} \{ \varphi |\text{Im } (f\overline{\nu})| + |\varphi' \text{Im } (\overline{\partial_r u})| \} \, d\mu,$$

where we set $(\partial_r \mp \partial_r p_\pm u) := \partial_r u \mp (\partial_r p_\pm u)$ for simplicity. Here, note that

$$\text{Im } \partial_r p_\pm = \mp \text{Re } \sqrt{z - \frac{\beta_j^2}{4}} \quad \text{on each } E_j(r_0, \infty).$$

**Proof.** Applying Proposition 3.1 for $U(R)$, we obtain

$$\int_{S(R)} \varphi |\partial_r p_\pm |u|^2 \, dA_w - z \int_{U(R)} \varphi |u|^2 \, d\mu$$
Moreover, assume that there exist a constant \( s \)-valued function of \( r \)

Lemma 4.1

Thus, signs of \( \text{Im} (\Omega, \mu \phi) \)

(15) \[
\int_{S(R)} \left( \text{Im} (\partial_r, \partial_r p \pm) \right) \varphi |u|^2 dA_w + \int_{U(R)} \left\{ \varphi (f \overline{\varphi} - |\nabla u|^2 - V |u|^2) - \varphi' (\partial_r u \overline{\varphi}) \right\} d\mu.
\]

Taking the imaginary part of both sides of this equation, we get

\[
(15) \quad \int_{S(R)} \left( \text{Im} (\partial_r, \partial_r p \pm) \right) \varphi |u|^2 dA_w - \int_{U(R)} \varphi |u|^2 d\mu
\]

\[
= \int_{S(R)} \varphi \left( \overline{\varphi} (\partial_r, \partial_r p \pm) u \right) dA_w + \int_{U(R)} \left\{ \varphi \text{ Im} (f \overline{\varphi}) - \varphi' \text{ Im} ((\partial_r u \overline{\varphi}) \right\} d\mu.
\]

Note that, for a general \( z' \in \mathbb{C} \),

\[
\text{Re} \sqrt{z'} \begin{cases} 
> 0 & \text{if } \text{Im } z' > 0, \\
> 0 & \text{if } z' \in (0, \infty), \\
= 0 & \text{if } z' \in (-\infty, 0], 
\end{cases}
\]

where recall that we take the principal value as our square root. Hence, by (11) and (12), we see that,

\[
(16) \quad \text{if } z \in \mathbb{H}_+, \text{ Im } z \geq 0 \text{ and } \text{Im } \partial_r p_+ = -\text{Re} \sqrt{z - \frac{(\beta_j)^2}{4}} \leq 0 \text{ on } E_j(r_0, \infty);
\]

\[
(17) \quad \text{if } z \in \mathbb{H}_-, \text{ Im } z \leq 0 \text{ and } \text{Im } \partial_r p_- = \text{Re} \sqrt{z - \frac{(\beta_j)^2}{4}} \geq 0 \text{ on } E_j(r_0, \infty).
\]

Thus, signs of \( \text{Im } \partial_r p_+ \) and \( \text{Im } z \) are different; moreover, \( \text{Im } |u|^2 \geq 0 \). Hence, Proposition 3.2 follows from (15), (16), and (17).

\[ \square \]

In the sequel, we will simply write \( \text{Im } (\partial_r u \overline{\varphi}) := \text{Im } ((\partial_r u \overline{\varphi}) \) and so on.

4. A PRIORI ESTIMATE OF \( |\nabla u| \)

We shall introduce an operator \( L_{\text{loc}} \) by \( \text{Dom} (L_{\text{loc}}) = H^2_{\text{loc}} (M) \) and \( L_{\text{loc}} u := \Delta u - 2 (\nabla w, \nabla u) - V u \) for \( u \in H^2_{\text{loc}} (M) \). Then, the following holds:

Lemma 4.1 (local a priori estimate). Let \( \Omega \) be a domain of \( M \) and \( \varphi \) be a real-valued function of \( r \geq 0 \). Assume that supp \( \varphi \) is compact; \( \varphi |_{\partial \Omega} = 0 ; | \varphi | \leq 1 \).

Moreover, assume that there exist a constant \( s \in \mathbb{R} \) and a function \( u \) such that \( u \in H^2_{\text{loc}} (\Omega) \cap L^2_{\text{loc}} (\Omega, \mu) \) and \( L_{\text{loc}} u \in L^2 (\Omega, \mu) \). Then, for any \( \varepsilon \in (0, 1) \), we obtain

\[
(1 - \varepsilon) \int_{\Omega} \varphi^2 (1 + r)^{2s} |\nabla u|^2 d\mu \\
\leq \frac{\varepsilon}{2} \int_{\Omega} \varphi^2 (1 + r)^{2s} |L_{\text{loc}} u|^2 d\mu + \tilde{c}_0 \int_{\text{supp } \varphi} (1 + r)^{2s} |u|^2 d\mu,
\]

where

\[
\tilde{c}_0 := \frac{1}{2 \varepsilon} + \max_M |V| + \frac{1}{\varepsilon} \max_M \left| \varphi' + s(1 + r)^{-1} \varphi \right|^2.
\]

Proof. Since \( \varphi^2 (1 + r)^{2s} \nabla u = \nabla \{ \varphi^2 (1 + r)^{2s} u \} - 2u \varphi (1 + r)^{2s} \{ \varphi' + s(1 + r)^{-1} \varphi \} \nabla r \),

Lemma 3.1 implies that

\[
\int_{\Omega} \varphi^2 (1 + r)^{2s} |\nabla u|^2 d\mu \\
= \int_{\Omega} \nabla \{ \varphi^2 (1 + r)^{2s} u \} \cdot \nabla \overline{\varphi} d\mu - 2 \int_{\Omega} \varphi (1 + r)^{2s} \{ \varphi' + s(1 + r)^{-1} \varphi \} u \partial_r \varphi d\mu
\]
\[
= -\int (L_{\text{loc}})\varphi^2(1 + r)^{2s} u\, d\mu - \int \varphi^2(1 + r)^{2s} V|u|^2\, d\mu
\]
\[
- 2\int_{\Omega} \varphi(1 + r)^{2s}\{\varphi' + s(1 + r)^{-1}\varphi\} u\, \partial_r \pi\, d\mu
\]
\[
\leq \frac{\varepsilon}{2}\int_{\Omega} \varphi^2(1 + r)^{2s}|L_{\text{loc}} u|^2\, d\mu + \frac{1}{2\varepsilon}\int_{\Omega} \varphi^2(1 + r)^{2s}|u|^2\, d\mu
\]
\[
+ \max_M |V| \cdot \int_{\Omega} \varphi^2(1 + r)^{2s}|u|^2\, d\mu + \varepsilon \int_{\Omega} \varphi^2(1 + r)^{2s} |\partial_r u|^2\, d\mu
\]
\[
+ \frac{1}{\varepsilon} \max_M \left| \varphi' + s(1 + r)^{-1}\varphi \right|^2 \cdot \int_{\text{supp} \varphi}(1 + r)^{2s}|u|^2\, d\mu.
\]

Now, Lemma 4.1 follows from this inequality and the assumption $|\varphi| \leq 1$. 

**Corollary 4.1** (global a priori estimate). Assume that there exist a constant $s \in \mathbb{R}$ and a function $u$ such that $u \in \text{Dom}(L_{\text{loc}}) \cap L^2_s(M, \mu)$ and $L_{\text{loc}} u \in L^2_s(M, \mu)$. Then, for any $\varepsilon \in (0, \frac{1}{2})$, there exists a constant $\hat{c}(\varepsilon) > 0$ such that
\[
\left\| \nabla u \right\|_{L^2_s(M, \mu)}^2 \leq \varepsilon \|L_{\text{loc}} u\|^2_{L^2_s(M, \mu)} + \hat{c}(\varepsilon)\|u\|^2_{L^2_s(M, \mu)},
\]
where $\hat{c}(\varepsilon) := \frac{1}{1 - 2\varepsilon} \{ \frac{1}{s} + \frac{1}{2s}(1 + |s|)^2 + \max_M |V| \}$. 

**Proof.** For $t > 0$, set
\[
h_t(r) := \begin{cases} 1 & \text{if } r \leq t, \\
-r + t + 1 & \text{if } t \leq r \leq t + 1, \\
0 & \text{if } t + 1 \leq r
\end{cases}
\]
and put $\Omega = M$ and $\varphi(r) = h_t(r)$ in Lemma 4.1. Then,
\[
(1 - 2\varepsilon) \int_M h_t^2(1 + r)^{2s} \left| \nabla u \right|^2\, d\mu
\]
\[
\leq \varepsilon \int_M h_t^2(1 + r)^{2s}|L_{\text{loc}} u|^2\, d\mu + c_t \int_{U(t + 1)}(1 + r)^{2s}|u|^2\, d\mu,
\]
where $c_t := \frac{1}{4|s|} + \max_M |V| + \frac{1}{2s} \max_M |h_t'| + s(1 + r)^{-1}|h_t|^2$. Since $|h_t'| \leq 1$ and $|s|(1 + r)^{-1}|h_t| \leq |s|$, we have $c_t \leq \frac{1}{4|s|} + \frac{1}{2s}(1 + |s|)^2 + \max_M |V|$. Letting $t \to \infty$, we get the desired inequality. 

5. **Estimate of $|\nabla u + u\nabla p_\pm|$**

The purpose of this section is to prove Proposition 5.2 below, which shows a decay estimate of $|\nabla u + u\nabla p_\pm|$ of a solution $u$ of (10); decay assumptions (3) and (4) (see also (1) and (2)) will be systematically used; in this sense, this is the most important section.

To prove Proposition 5.2, we first prove a preparative proposition. Let $\eta : M \times \mathbb{H}_\pm \to \mathbb{C}$ be a complex-valued $C^\infty$-function and consider a function
\[
v(x, z) := e^{\eta(x, z)} u(x),
\]
where $u$ is a solution of the equation (10). In view of (8) and (10), direct computations show the following:
\[
\Delta_g v + 2\langle \nabla \eta + \nabla w, \nabla v \rangle - qv = e^\eta f;
\]
\[
q = q(x, z) := z - \Delta_g \eta + \langle 2\nabla w + \nabla \eta, \nabla \eta \rangle - V.
\]
The following Proposition 5.1 will serve the estimate of $|\nabla u + u\nabla p_\pm|$ on $M$ (see Proposition 5.2):

**Proposition 5.1.** Let $\eta : M \times \Pi_\pm \to \mathbb{C}$ and $\psi : M \times \Pi_\pm \to \mathbb{R}$ be $C^\infty$-functions and $X$ be a “real” $C^\infty$-vector field on $M$. Let $u$ be a solution of the equation (10) and set $v = e^{\psi}u$. Let $\Omega \subset M$ be a relatively compact open subset with $C^\infty$-boundary $\partial \Omega$. Then, we have

$$\int_{\partial \Omega} \psi \left\{ \text{Re} \left( X, \nabla \nabla \right) \left\langle \nabla v, \overline{\nabla w} \right\rangle - \frac{1}{2} |\nabla v|^2 \left\langle X, \overline{\nabla} \right\rangle \right\} dA_w$$

$$= \int_{\Omega} \left( \langle X, \nabla w \rangle \psi - \frac{1}{2} \langle X, \nabla \psi \rangle - \frac{1}{2} \psi \text{div} X \right) |\nabla v|^2 d\mu$$

$$+ \int_{\Omega} \left\{ \text{Re} \left( \nabla \psi + 2\psi \nabla \eta, \nabla v \right) \left\langle X, \nabla \nabla \right\rangle + \psi \text{Re} \left( \nabla \nabla v, \nabla \nabla \right) \right\} d\mu$$

$$- \int_{\Omega} \psi \text{Re} \left( qv + e^{\psi}f \right) \left\langle X, \nabla \nabla \right\rangle d\mu.$$  

**Proof.** Direct calculations show the following:

$$-(\Delta_\eta v) \left\langle X, \nabla \nabla \right\rangle = - \text{div} \left( \langle X, \nabla \nabla \rangle \nabla v \right) + \langle \nabla \nabla v, X, \nabla \nabla \rangle + \langle \nabla \eta \nabla v \rangle \langle X, \nabla \nabla \rangle;$$

$$2\text{Re} \langle \nabla \eta \rangle \langle X, \nabla v \rangle = \text{div} \left( \langle \nabla v, \nabla \nabla \rangle X \right) - \langle \nabla v, \nabla \nabla \rangle \text{div} X.$$

Combining these equation makes

$$- \text{Re} \left( \Delta_\eta v \right) \langle X, \nabla \nabla \rangle$$

$$= - \text{Re} \text{div} \left( \langle X, \nabla \nabla \rangle \nabla v \right) + \text{Re} \langle \nabla \nabla v, X, \nabla \nabla \rangle + \frac{1}{2} \text{div} \left( |\nabla v|^2 X \right) - \frac{1}{2} |\nabla v|^2 \text{div} X.$$

Therefore, multiplying the equation (18) by $\psi \langle X, \nabla \nabla \rangle$ and taking its real part, we obtain

$$- \text{Re} \text{div} \left( \psi \langle X, \nabla \nabla \rangle \nabla v \right) + \text{Re} \langle X, \nabla \nabla \rangle \langle \nabla v, \nabla \psi \rangle + \psi \text{Re} \langle \nabla \nabla v, X, \nabla \nabla \rangle$$

$$+ \frac{1}{2} \{ \text{div} \left( \psi |\nabla v|^2 X \right) - |\nabla v|^2 \langle X, \nabla \psi \rangle \} - \frac{1}{2} \psi |\nabla v|^2 \text{div} X$$

$$+ \psi \text{Re} \left\{ \left( 2\langle \nabla \eta + \nabla w, \nabla \psi \rangle - qv \right) \langle X, \nabla \nabla \rangle \right\} = \psi \text{Re} e^{\psi}f \langle X, \nabla \nabla \rangle,$$

where note that $\psi$ is real-valued. Integrating this equation on $\Omega$ with respect to the measure $\mu$ and applying Lemma 3.2 to the first and fourth term above make

$$\int_{\partial \Omega} \psi \left\{ \text{Re} \left( X, \nabla \nabla \right) \left\langle \nabla v, \overline{\nabla w} \right\rangle - \frac{1}{2} |\nabla v|^2 \left\langle X, \overline{\nabla} \right\rangle \right\} dA_w$$

$$+ 2 \int_{\Omega} \psi \left\{ -\text{Re} \left( X, \nabla \nabla \right) \langle \nabla v, \nabla w \rangle + \frac{1}{2} |\nabla v|^2 \langle X, \nabla w \rangle \right\} d\mu$$

$$- \frac{1}{2} \int_{\Omega} |\nabla v|^2 \left\{ \langle X, \nabla \psi \rangle + \psi \text{div} X \right\} d\mu + \int_{\Omega} \psi \text{Re} \left( \nabla \nabla v, X, \nabla \nabla \right) d\mu$$

$$+ \int_{\Omega} \text{Re} \left( X, \nabla \nabla \right) \left\{ \langle \nabla v, \nabla \psi \rangle + 2\psi \langle \nabla \eta + \nabla w, \nabla \psi \rangle - \psi qv - \psi e^{\psi}f \right\} d\mu = 0.$$  

Since the term $2 \int_{\Omega} \psi \text{Re} \langle X, \nabla \nabla \rangle \langle \nabla v, \nabla w \rangle d\mu$ appears twice with different signs on the left hand side of the equation above, we see that Proposition 5.1 follows from this equation.  

In the sequel, let $K_+$ and $K_-$ be any fixed compact subsets in $\Pi_+$ and $\Pi_-$, respectively. Then, we have the following:

**Lemma 5.1.** If we set $\eta = p_\pm$ and if $z \in K_\pm$, then the function $q$ defined by (19) has the following asymptotic property on $M$:

$$q = O \left( r^{-1 - \delta} \right)$$

uniformly for $z \in K_\pm$ as $r \to \infty$.

Here, recall that $\delta = \min \{ \delta_j \mid 1 \leq j \leq m \}$.

**Proof.** First, we shall consider the case that $x \in E_j(r_0, \infty)$ $(1 \leq j \leq m_0)$ and $r(x) \to \infty$. Then, $w = 0$, $\eta = p_\pm = \mp i\sqrt{z} r + \frac{b_j}{2r} \log r$, and hence,

$$q = z - \Delta_\eta \eta + \langle 2\nabla w + \nabla \eta, \nabla \eta \rangle - V = z - \Delta_\eta \eta + \langle \nabla \eta, \nabla \eta \rangle$$

$$= \pm i \sqrt{z} \left\{ \Delta_\eta r - \frac{b_j}{r} \right\} - \frac{b_j}{2r} \Delta_\eta r + \frac{b_j}{2r^2} + \frac{(b_j)^2}{4r^2}.$$

The condition (3) is $\Delta_\eta r = \frac{b_j}{r} + O \left( r^{-1 - \delta} \right)$ on $E_j$ $(1 \leq j \leq m_0)$, and hence, (20) implies the desired result for ends, $E_1, \cdots, E_{m_0}$.

Next, we shall consider the case that $x \in E_j(r_0, \infty)$ $(m_0 + 1 \leq j \leq m)$ and $r(x) \to \infty$. Then, $w = \frac{\beta_j}{2r} \; r$; $V = \frac{\beta_j}{2} \Delta_\eta r - \frac{(\beta_j)^2}{4}$; $\eta = p_\pm = \mp i\sqrt{z - \frac{(\beta_j)^2}{4}} r$, and hence,

$$q = z - \Delta_\eta \eta + \langle 2\nabla w + \nabla \eta, \nabla \eta \rangle - V$$

$$= z \pm \frac{\beta_j}{2} \Delta_\eta r + \frac{(\beta_j)^2}{4}$$

Since the condition (4) is $\Delta_\eta r = \beta_j + O \left( r^{-1 - \delta} \right)$ on $E_j$ $(m_0 + 1 \leq j \leq m)$, we see that (21) implies the desired result for ends, $E_{m_0+1}, \cdots, E_m$. \hfill \Box

The purpose of this section is to prove the following:

**Proposition 5.2.** Let $s$, $s'$, and $R$ be positive real numbers, and $z$ be a complex number satisfying

$$s < a_{\min} = \min \{ a_j \mid 1 \leq j \leq m_0 \}, \ s + s' \leq \delta, \ 2 \leq R, \ z \in K_\pm.$$

Assume that $f \in L^2_{z_+} (M, \mu)$ and that $u$ is a solution of (10) satisfying the radiation condition. Then, we have

$$\| \nabla u + u \nabla p_\pm \|^2_{L^2_{z_+} (E_j(R+1, \infty), \mu)}$$

$$\leq \hat{c}_4(s, a_{\min}, R, K_\pm) \cdot \left\{ \| u \|^2_{L^2_{z_+} (E_j(R-1, \infty), \mu)} + \| f \|^2_{L^2_{z_+} (E_j(R-1, \infty), \mu)} \right\},$$

where $\hat{c}_4(s, a_{\min}, R, K_\pm)$ is a constant depending only on $s$, $a_{\min}$, $R$, and $K_\pm$. \hfill 11
Proof. Let $\varphi_R(r)$ be the function of $r \geq 0$ defined by

$$
\varphi_R(r) := \begin{cases} 
0 & \text{if } r \leq R, \\
- \left( r - R \right) & \text{if } R \leq r \leq R + 1, \\
1 & \text{if } R + 1 \leq r.
\end{cases}
$$

In Proposition 5.1, we shall substitute

$$
\eta(x, z) = p_\pm(x, z); \ \psi = \varphi_R(r) r^{2s} \exp(-2\text{Re } p_\pm); \ \Omega = E_j(R, t); \ X = \nabla r,
$$

where $t > R + 1$ is a constant; recall that $p_\pm$ is defined by (11). For simplicity, we denote $Y := e^{-\eta}v = \nabla u + u\nabla p_\pm$. Then, $e^{-2\text{Re } \eta} = e^{-\eta} - |e^{-\eta}|^2$; $\nabla \psi = |e^{-\eta}|^2 r^{2s} \{ \varphi'_R + (2s r^{-1} - 2 \text{Re } \partial_r \eta) \varphi_R \} \nabla r$. Hence, each term, appeared in Proposition 5.1, is calculated as follows:

\begin{align*}
(22) \quad & \psi \left\{ \text{Re } \langle \nabla r, \nabla \bar{v} \rangle \langle \nabla v, \nabla r \rangle - \frac{1}{2} |\nabla v|^2 \langle \nabla r, \nabla r \rangle \right\} \\
& = \varphi_R(r) r^{2s} |\eta|^2 \left\{ \text{Re } \langle \nabla r, e^\eta Y \rangle \langle e^\eta Y, \nabla r \rangle - \frac{1}{2} |e^\eta Y|^2 \right\} \\
& = \varphi_R(r) r^{2s} \left\{ |\langle Y, \nabla r \rangle|^2 - \frac{1}{2} |Y|^2 \right\}; \\
(23) \quad & \left\{ \langle \nabla r, \nabla w \rangle \psi - \frac{1}{2} \langle \nabla r, \nabla \psi \rangle - \frac{1}{2} \psi \Delta_g r \right\} |\nabla v|^2 \\
& = \left\{ \partial_r \psi \varphi_R r^{2s} |\eta|^2 - \frac{1}{2} \partial_r \psi - \frac{1}{2} \varphi_R r^{2s} |\eta|^2 \Delta_g r \right\} |e^\eta Y|^2 \\
& = r^{2s} |Y|^2 \left\{ \varphi_R \left( \partial_r w - sr^{-1} + \text{Re } \partial_r p_\pm - \frac{1}{2} \Delta_g r \right) - \frac{1}{2} \varphi'_R \right\}; \\
(24) \quad & \text{Re } \langle \nabla \psi + 2 \psi \nabla \eta, \nabla v \rangle \langle \nabla r, \nabla r \rangle \\
& = \text{Re } \langle \partial_r \psi \nabla r + 2 \psi \partial_r \eta \nabla r, e^\eta Y \rangle \langle \nabla r, e^\eta \overline{Y} \rangle \\
& = \text{Re } \left\{ \partial_r \psi + 2 \psi \partial_r \eta \right\} |e^\eta Y|^2 \\
& = r^{2s} \text{Re } \left\{ \varphi'_R + (2sr^{-1} - 2 \text{Re } \partial_r \eta) \varphi_R + 2 \varphi_R \partial_r \eta \right\} |\langle Y, Y \rangle|^2 \\
& = r^{2s} \left\{ \varphi'_R + 2sr^{-1} \varphi_R \right\} |\langle Y, Y \rangle|^2; \\
(25) \quad & \psi \text{Re } \langle \nabla v, \nabla r \rangle \overline{\eta} = \varphi_R r^{2s} |\eta|^2 \text{Re } \langle e^\eta Y, \nabla r, e^\eta \overline{Y} \rangle \\
& = \varphi_R r^{2s} \text{Re } \langle \nabla \psi \rangle \langle Y, Y \rangle; \\
(26) \quad & - \psi \text{Re } (qu + e^\eta f) \langle \nabla r, \nabla r \rangle = - \varphi_R r^{2s} |\eta|^2 \text{Re } (qe^\eta u + e^\eta f) \langle \nabla r, e^\eta \overline{Y} \rangle \\
& = - \varphi_R r^{2s} \text{Re } \langle qu + f \rangle \langle \nabla r, Y \rangle.
\end{align*}

Also, we have

\begin{align*}
(27) \quad & \varphi'_R = \begin{cases} 
1 & \text{on } E_j(R, R + 1), \\
0 & \text{on } M \setminus E_j(R, R + 1).
\end{cases}
\end{align*}

Substituting (22)–(27) into the equation of Proposition 5.1, we obtain

\begin{align*}
(28) \quad & \int_{S(t)} r^{2s} \left\{ |\langle Y, \nabla r \rangle|^2 - \frac{1}{2} |Y|^2 \right\} dA \\
& - \int_{E_j(R, R + 1)} r^{2s} \left\{ |\langle Y, \nabla r \rangle|^2 - \frac{1}{2} |Y|^2 \right\} d\mu
\end{align*}
\[
+ \int_{E_j(R,t)} r^{2s} \varphi^R \text{Re} (qu + f) (\nabla Y, \nabla r) \, d\mu
\]

\[
= \int_{E_j(R,t)} r^{2s} \varphi^R \left\{ (\partial_r w - sr^{-1} + \text{Re} \partial_r p_\pm - \frac{1}{2} \Delta_g r) |Y|^2 + 2sr^{-1}|(Y, \nabla r)|^2 + \text{Re} (\nabla dr)(Y, \nabla Y) \right\} \, d\mu;
\]

note that \( \varphi^R(R) = 0 \), and hence, the boundary integral on \( S_j(R) \) vanishes.

We shall write

\[
Y = (Y, \nabla r) \nabla r \perp Y^\perp, \quad \text{where } \nabla r \perp Y^\perp.
\]

In the following, we shall bound the integrand of the right hand side of (28) from below.

First, let us consider the case that \( 1 \leq j \leq m_0 \); then, on \( E_j \),

\[
w \equiv 0; \quad p_\pm = \mp \text{i} \sqrt{z - r^2} \, \log r(x);
\]

\[
\text{Re} \partial_r p_\pm = \pm \text{Im} \sqrt{z - r^2} \, \frac{b_j}{2r} \quad \text{for } z \in I_\pm;
\]

\[
\Delta_g r = \frac{b_j}{r} + O \left( r^{-1-\delta} \right); \quad \text{Re} (\nabla dr)(Y, \nabla Y) \geq \left( \frac{\delta_j}{r} + O \left( r^{-1-\delta} \right) \right) |Y^\perp|^2.
\]

Hence, in this case, the integrand of the right hand side of (28) is bounded from below as follows:

\[
(29)
\]

\[
r^{2s} \varphi^R \left\{ (\partial_r w - \frac{s}{r} + \text{Re} \partial_r p_\pm - \frac{1}{2} \Delta_g r) |Y|^2 + 2\frac{s}{r} |(Y, \nabla r)|^2 + \text{Re} (\nabla dr)(Y, \nabla Y) \right\}
\]

\[
\geq r^{2s} \varphi^R \left\{ \left( \frac{s}{r} - O \left( r^{-1-\delta} \right) \right) |(Y, \nabla r)|^2 + \left( \frac{\delta_j - \frac{r}{2}}{r} - O \left( r^{-1-\delta} \right) \right) |Y^\perp|^2 \right\}
\]

\[
\geq \frac{1}{2} r^{2s-1} \varphi^R \cdot \min \{ s, a_j - s \} \cdot |Y|^2 \quad \text{for } x \in E_j(R_j', \infty) \quad (1 \leq j \leq m_0),
\]

where \( R_j' > 0 \) is a constant depending only on the geometry of \( E_j \) \( (1 \leq j \leq m_0) \).

Next, consider the case that \( m_0 + 1 \leq j \leq m \); then, on \( E_j \),

\[
\partial_r w = \frac{\beta_j}{2}; \quad p_\pm = \mp \text{i} \sqrt{z - \frac{(\beta_j)^2}{4}} \, r, \quad \text{Re} \partial_r p_\pm = \pm \text{Im} \sqrt{z - \frac{(\beta_j)^2}{4}} \geq 0 \quad \text{for } z \in I_\pm;
\]

\[
\Delta_g r = \beta_j + O \left( r^{-1-\delta} \right); \quad \text{Re} (\nabla dr)(Y, \nabla Y) \geq \left\{ \alpha_i + O \left( r^{-1-\delta} \right) \right\} |Y^\perp|^2.
\]

Hence, in this case, the integrand of the right hand side of (28) is bounded from below as follows:

\[
(30)
\]

\[
r^{2s} \varphi^R \left\{ (\partial_r w - \frac{s}{r} + \text{Re} \partial_r p_\pm - \frac{1}{2} \Delta_g r) |Y|^2 + 2\frac{s}{r} |(Y, \nabla r)|^2 + \text{Re} (\nabla dr)(Y, \nabla Y) \right\}
\]

\[
\geq r^{2s} \varphi^R \left\{ \left( \frac{\beta_j}{2} - \frac{s}{r} \right) \pm \text{Im} \sqrt{z - \frac{(\beta_j)^2}{4} - \frac{\beta_j}{2} - O \left( r^{-1-\delta} \right)} |Y|^2
\]

\[
+ 2\frac{s}{r} |(Y, \nabla r)|^2 + (\alpha_i - O \left( r^{-1-\delta} \right)) |Y^\perp|^2 \right\}
\]

\[
\geq \frac{1}{2} r^{2s-1} \varphi^R |Y|^2 \quad \text{for } x \in E_j(R_j, \infty) \quad (m_0 + 1 \leq j \leq m),
\]

13
where \(R_j > 0\) is a constant depending only on the geometry of \(E_j\). Thus, (29) and (30) imply that
\[
(31) \quad r^{2s} \varphi_R \left\{ \left( \partial_r w - \frac{s}{r} + \text{Re} \partial_r p_\pm - \frac{1}{2} \Delta_g r \right) |Y|^2 + 2 \frac{r^2}{r} |(Y, \nabla r)|^2 + \text{Re}(\nabla dr)(Y, Y) \right\} \geq \hat{c}_1(s) \cdot r^{2s-1} \varphi_R |Y|^2
\]
for any \(x \in M\) with \(r(x) \geq R_3 > 0\), where \(R_3\) is a constant depending only on the geometry of \(M\) and \(\hat{c}_1(s) := \min \{s, 2s - s\}\).

Now, we shall bound the left hand side of (28) from above. We begin with the third term of the left hand side of (28). As we have seen in Lemma 5.1, there exists a constant \(\bar{c} > 0\) such that \(|q| \leq \bar{c} r^{-1-\delta}\) for \(r(x) \geq 1\). Hence, by Schwarz’s inequality, we have
\[
|q||u||Y| \leq \bar{c} r^{-1-\delta} |u||Y| \leq \frac{\bar{c}^2}{c_1(s)} r^{-1-2\delta} |u|^2 + \frac{\hat{c}_1(s)}{4} r^{-1} |Y|^2;
\]
\[
|f||Y| \leq \frac{r}{c_1(s)} |f|^2 + \frac{\hat{c}_1(s)}{4} r^{-1} |Y|^2.
\]
From these inequalities, we have
\[
(32) \quad r^{2s} \varphi_R \text{Re} (qu + f)(Y, \nabla r) \leq r^{2s} \varphi_R \left\{ \frac{\hat{c}_1(s)}{2} r^{-1} |Y|^2 + \hat{c}_2(s) r^{-1-2\delta} |u|^2 + \hat{c}_2(s) r |f|^2 \right\},
\]
where \(\hat{c}_2(s) := \max \{\bar{c}^2, 1\}\).

Next, we bound the second term of the left hand side of (28). Since \(Y = e^{-\eta} \nabla v = \nabla u + u \nabla p_\pm\), we have
\[
(33) \quad \frac{1}{2} \int_{E_j(R, R+1)} r^{2s} |Y|^2 \, d\mu \leq \int_{E_j(R, R+1)} r^{2s} \left\{ |\nabla u|^2 + |\nabla p_\pm|^2 |u|^2 \right\} \, d\mu.
\]
In Lemma 4.1, we set
\[
\varphi(r) = \begin{cases} r - R + 1 & \text{if } R - 1 \leq r \leq R, \\ 1 & \text{if } R \leq r \leq R + 1, \\ -r + R + 2 & \text{if } R + 1 \leq r \leq R + 2, \\ 0 & \text{otherwise}, \end{cases}
\]
and substitute \(\Omega = E_j, \varepsilon = \frac{1}{2}\), and \(-Lu = zu + f\); then, we obtain
\[
(34) \quad \int_{E_j(R, R+1)} r^{2s} |\nabla u|^2 \, d\mu \leq \left\{ |z|^2 + 10 + 2 \max_M |V| \right\} \int_{E_j(R-1, R+2)} r^{2s} |u|^2 \, d\mu + \int_{E_j(R-1, R+2)} r^{2s} |f|^2 \, d\mu,
\]
where we have used the assumptions, \(R \geq 2\) and \(0 < s \leq 1/2\). Combining (33) and (34) yields
\[
(35) \quad \frac{1}{2} \int_{E_j(R, R+1)} r^{2s} |Y|^2 \, d\mu \leq \tilde{c}_3 \int_{E_j(R-1, R+2)} r^{1-2s} |u|^2 \, d\mu + \int_{E_j(R-1, R+2)} r^{1+2s} |f|^2 \, d\mu,
\]
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where we have again used the assumption, \( R \geq 2 \), and set
\[
\hat{c}_3 := \left\{ 10 + \lvert z \rvert^2 + 2 \max_M \lvert V \rvert + \max_{E_j(R, R + 1)} \lvert p_\pm \rvert^2 \right\} (R + 2)^{1 + 2s + 2s'}.
\]
Putting together (28), (31), (32), and (35), we obtain
\[
\frac{1}{2} \hat{c}_1(s) \int_{E_j(R, R + 1)} \int_{E_j(R, R + 1, t)} r^{2s-1} \lvert Y \rvert^2 d\mu.
\]
\[
\leq \hat{c}_2(s) \int_{E_j(R, R + 1)} \int_{E_j(R, R + 1)} r^{-1 + 2s - 2\delta} \lvert u \rvert^2 d\mu + \hat{c}_2(s) \int_{E_j(R, R + 1, t)} \int_{E_j(R, R + 1)} r^{1 + 2s} \lvert f \rvert^2 d\mu
\]
\[
+ \int_{S_j(t)} r^{2s} \lvert \langle Y, \nabla r \rangle \rvert^2 dA_w + \int_{S_j(t)} r^{2s} \lvert \langle Y, \nabla r \rangle \rvert^2 dA_w
\]
\[
\leq \left\{ (\hat{c}_2(s) + \hat{c}_3) \int_{E_j(R, R - 1, R + 2)} + \hat{c}_2(s) \int_{E_j(R, R + 2, t)} \right\} \int_{E_j(R, R + 1)} \int_{E_j(R, R + 1, t)} r^{-1 - 2s'} \lvert u \rvert^2 d\mu
\]
\[
+ \left\{ (\hat{c}_2(s) + 1) \int_{E_j(R, R - 1, R + 2)} + \hat{c}_2(s) \int_{E_j(R, R + 2, t)} \right\} \int_{E_j(R, R + 1)} \int_{E_j(R, R + 1, t)} r^{1 + 2s} \lvert f \rvert^2 d\mu
\]
\[
+ \int_{S_j(t)} r^{2s} \lvert \langle Y, \nabla r \rangle \rvert^2 dA_w,
\]
where we have used the assumption \( s + s' \leq \delta \).

Since \( u \) satisfies the radiation condition (see (14)), \( \langle Y, \nabla r \rangle = (\partial_r + \partial_r p_\pm) u \in L^2_{\frac{1}{2} + s}(M, \mu) \), and hence, there exists a divergent sequence \( \{ t_i \} \rightarrow 1 \) of positive numbers such that \( \lim_{t \rightarrow \infty} \int_{S_j(t)} r^{2s} \lvert \langle Y, \nabla r \rangle \rvert^2 dA_w = 0 \). Hence, substituting \( t = t_i \) in (36) and letting \( i \rightarrow \infty \), we obtain
\[
\frac{\hat{c}_1(s)}{2} \int_{E_j(R + 1, \infty)} \int_{E_j(R + 1, t)} r^{2s-1} \lvert \nabla u + u \nabla p_\pm \rvert^2 d\mu
\]
\[
\leq (\hat{c}_2(s) + \hat{c}_3) \int_{E_j(R, R - 1, \infty)} \int_{E_j(R, R + 1, \infty)} r^{-1 - 2s'} \lvert u \rvert^2 d\mu + (\hat{c}_2(s) + 1) \int_{E_j(R, R + 1, \infty)} \int_{E_j(R, R + 1, \infty)} r^{1 + 2s} \lvert f \rvert^2 d\mu.
\]
Proposition 5.2 follows from this inequality. \( \square \)

6. Decay estimate for the case \( \text{Re } z < \frac{(\beta_j)^2}{4} \)

This section studies the decay estimate of a solution \( u \) of (10) on an end \( E_j \) satisfying \( \text{MC}(\alpha_j; \beta_j; \delta_j) \) in the case that \( \text{Re } z < \frac{(\beta_j)^2}{4} \).

First, let us recall (9), that is, on an end \( E_j \) satisfying \( \text{MC}(\alpha_j; \beta_j; \delta_j) \), the real valued function \( V = \frac{\delta_j}{2}(\Delta_g r - \frac{\beta_j}{2}) \) converges to the positive constant \( \frac{(\beta_j)^2}{4} \) at infinity. An a priori decay estimate of solutions of (10) will be obtained by taking the real part of the integral in Proposition 3.1:

**Proposition 6.1.** Let \( (N, g_N) \) be an \( n \)-dimensional Riemannian manifold with compact connected \( C^\infty \)-boundary \( \partial N \). Assume that the inward normal exponential map \( \exp_{\partial N}^+ : N^+(\partial N) \rightarrow N \) induces a diffeomorphism, where \( N^+(\partial N) := \{ \text{inward normal vectors to } \partial N \} \). Let \( \beta > 0 \) be a constant and set \( r(*) := \text{dist}_g(\partial N, *) \); \( \tilde{V} := \frac{\delta_j}{2}(\Delta_g r - \frac{\beta_j}{2}) ; N(t, \infty) := \{ x \in N \mid r(x) > t \} \) for \( t > 0 \); \( \mu := e^{-\beta v_{g_N}} \), where \( v_{g_N} \) stands for the Riemannian measure of \( (N, g_N) \). Let \( s \) and \( s' \) be constants satisfying \( 0 < s' \leq s < 1, s' + s \leq 1 \), and \( u \) be a solution of the equation \( -\tilde{L} u - z u = f \) on
Then, by (37), we have

\[ \Delta_{g_N} u - \beta \partial_t u - \tilde{V}u. \]

Assume that there exist positive constants \( \theta, \varepsilon, \) and \( R_1 \) such that

\[ 2\theta > \beta \varepsilon; \quad \text{Re} z \leq \frac{\beta^2}{4} - \theta; \quad \Delta_{g_N} r \geq \beta - \varepsilon \]
on \( N(R_1, \infty). \)

Then, for any \( \ell \geq 2 \) and \( R \geq \max\{R_1, 2\} \), we obtain

\[ \|u\|_{L^2_{\tilde{z}, \infty}(N(R, \infty), \mu)} \leq \frac{\hat{c}_0(\theta, \beta, \varepsilon)}{(\ell - 1)R} \left\{ \|u\|_{L^2_{\tilde{z}, \infty}(N(R, \infty), \mu)}^2 + \|f\|_{L^2_{\tilde{z}, \infty}(N(R, \infty), \mu)}^2 \right\}, \]

where \( \hat{c}_0(\theta, \beta, \varepsilon) := \frac{9}{2\beta - \varepsilon}. \)

**Proof.** First, note the following:

(37) \[ \tilde{V} - \text{Re} z \geq \frac{\beta}{2}(\beta - \varepsilon - \frac{\beta}{2}) - \frac{\beta^2}{4} > \theta - \frac{\beta \varepsilon}{2} > 0 \quad \text{on } N(R_1, \infty). \]

Let \( h = h(r) \) be any real valued function of \( r \in (R_1, \infty) \) with compact support in \( (R_1, \infty) \). We shall set \( \varphi = h^2 \) and \( \Omega = E \) in Proposition 3.1 and take its real part. Then, by (37), we have

\[ (\theta - \frac{\beta \varepsilon}{2}) \int_N |h|^2 d\mu \leq \int_N (\tilde{V} - \text{Re} z)|hu|^2 d\mu \]

\[ = \int_N \left\{ h^2 \text{Re} f\varpi - h^2|\nabla u|^2 - 2h\text{Re}(\partial_t \varpi) \right\} d\mu \]

\[ \leq \int_N \left\{ (h')^2|u|^2 + h^2\text{Re} f\varpi \right\} d\mu. \]

For any constants \( \ell \geq 2, R \geq 2 \) and \( t \) satisfying \( R_1 < R < \ell R < t - 1 \), set

\[ h(r) = \begin{cases} 
0 & \text{if } r \leq R, \\
\frac{(\ell - 1)R(r - R)}{(\ell - 1)R^2} & \text{if } R \leq r \leq \ell R, \\
1 & \text{if } \ell R \leq r \leq t - 1, \\
-r + t & \text{if } t - 1 \leq r \leq t, \\
0 & \text{if } t \leq r, 
\end{cases} \]

and substitute this function \( h \) into (38). Then, we have

\[ (\theta - \frac{\beta \varepsilon}{2}) \int_{N(tR, t-1)} |u|^2 d\mu \]

\[ \leq \frac{1}{(\ell - 1)^2 R^2} \int_{N(R, t)} |u|^2 d\mu + \int_{N(t-1, t)} |u|^2 d\mu + \int_{N(R, t)} |uf| d\mu. \]

We shall multiply both sides of (39) by \((1 + t)^{-2 - 2s'}\) and integrate it with respect to \( t \) over \([\ell R + 1, \infty)\); for convenience, we set \( F(r) := \int_{S(r)} |u|^2 e^{-2s'} dA, \) where \( A \) stands for the Riemannian measures induced on each level hypersurface \( S(r) = \{ x \in N \mid \text{dist} (\partial N, x) = r \} \). Then, as for the integral of the left hand side of (39), we obtain, by Fubini’s theorem,

\[ \int_{\ell R + 1}^{\infty} (1 + t)^{-2 - 2s'} dt \int_{N(tR, t-1)} |u|^2 d\mu = \int_{\ell R + 1}^{\infty} (1 + t)^{-2 - 2s'} dt \int_{tR}^{t-1} F(r) dr. \]
where, in the first inequality, we have used Fubini’s theorem; in the second inequality, we have

\[ \int_{N(\ell R, \infty)} (1 + r) \cdot (1 + r)^{-1 - 2s'} |u|^2 d\mu \]

As for the integral of the second term on the right hand side of (39), we have

\[ \int_{N(\ell R, \infty)} (1 + r)^{-1 - 2s'} |u|^2 d\mu \]

As for the integral of the first term on the right hand side of (39), we have

\[ \int_{N(\ell R, \infty)} (1 + t)^{-2 - 2s'} dt \]

As for the integral of the second term on the right hand side of (39), we have

\[ \int_{N(\ell R, \infty)} (1 + r)^{-1 - 2s'} |u|^2 d\mu \]

As for the integral of the third term on the right hand side of (39), by Fubini’s theorem, we have

\[ \int_{\ell R + 1}^{\infty} (1 + t)^{-2 - 2s'} dt \int_{N(\ell R, \infty)} |u|^2 d\mu = \int_{\ell R + 1}^{\infty} (1 + t)^{-2 - 2s'} dt \int_{\ell R + 1}^{\infty} F(r) dr \]

where, in the first inequality, we have used Fubini’s theorem; in the second inequality, we have used the fact that \( \int_{r}^{r+1} (1 + t)^{-2 - 2s'} dt \leq (1 + r)^{-2 - 2s'} \int_{r}^{r+1} dt = (1 + r)^{-2 - 2s'} \).

As for the integral of the third term on the right hand side of (39), by Fubini’s theorem, we have

\[ \int_{\ell R + 1}^{\infty} (1 + t)^{-2 - 2s'} dt \int_{N(\ell R, \infty)} |u|^2 d\mu = \int_{\ell R + 1}^{\infty} (1 + t)^{-2 - 2s'} dt \int_{\ell R + 1}^{\infty} F(r) dr \]

\[ = \int_{R}^{\infty} \tilde{F}(r) dr \int_{\ell R + 1}^{\infty} (1 + t)^{-2 - 2s'} dt + \int_{R}^{\infty} \tilde{F}(r) dr \int_{r}^{\infty} (1 + t)^{-2 - 2s'} dt \]

\[ = \frac{1}{(1 + 2s')(\ell R + 2)^{1 + 2s'}} \int_{R}^{\ell R + 1} \tilde{F}(r) dr + \frac{1}{1 + 2s'} \int_{\ell R + 1}^{\infty} (1 + r)^{-1 - 2s'} \tilde{F}(r) dr \]
\[
= \frac{1}{1 + 2s'} \left\{ \frac{1}{(\ell R + 2)^{1+2s'}} \int_{N(R,\ell R+1)} |uf| \, d\mu + \int_{N(\ell R+1,\infty)} (1 + r)^{-1-2s'} |uf| \, d\mu \right\},
\]

where we have set \( \bar{F}(r) := \int_{S(r)} |uf|e^{-2 \rho \sigma} \, dA \). Since \( s' \leq s \), we have

\[
\int_{N(R,\ell R+1)} |uf| \, d\mu \leq \frac{1}{2} \int_{N(\ell R+1,\infty)} \left\{ (1 + r)^{-1-2s'} |u|^2 + (1 + r)^{1+2s'} |f|^2 \right\} \, d\mu.
\]

Also, we have

\[
\int_{N(\ell R+1,\infty)} (1 + r)^{-1-2s'} |uf| \, d\mu = \int_{N(\ell R+1,\infty)} (1 + r)^{(-1-s'-s)+(s-s')} |uf| \, d\mu
\]

\[
\leq \frac{1}{2(2 + \ell R)^{1+s+s'}} \int_{N(\ell R+1,\infty)} (1 + r)^{s-s'} |uf| \, d\mu
\]

\[
\leq \frac{1}{2(2 + \ell R)^{1+s+s'}} \int_{N(\ell R+1,\infty)} \left\{ (1 + r)^{-1-2s'} |u|^2 + (1 + r)^{1+2s'} |f|^2 \right\} \, d\mu.
\]

Therefore, since \( s' \leq s \), by combining (43), (44), and (45), we obtain

\[
\int_{\ell R+1}^{\infty} (1 + t)^{-2-2s'} \, dt \int_{N(\ell R,t)} |uf| \, d\mu
\]

\[
\leq \frac{1}{2(2 + \ell R)^{1+s+s'}} \int_{N(\ell R,\infty)} \left\{ (1 + r)^{-1-2s'} |u|^2 + (1 + r)^{1+2s'} |f|^2 \right\} \, d\mu
\]

\[
\leq \frac{1}{2(\ell - 1)R} \int_{N(R,\infty)} \left\{ (1 + r)^{-1-2s'} |u|^2 + (1 + r)^{1+2s'} |f|^2 \right\} \, d\mu.
\]

Now, putting together (40), (41), (42) and (46), we obtain

\[
\frac{1}{3} \left( \frac{\beta \rho}{2} - \frac{\beta \rho}{2} \right) \int_{N(t R, R)} (1 + r)^{-1-2s'} |u|^2 \, d\mu
\]

\[
\leq \frac{3}{2(\ell - 1)R} \int_{N(R,\infty)} (1 + r)^{-1-2s'} |u|^2 \, d\mu + \frac{1}{2(\ell - 1)R} \int_{N(\ell R,\infty)} (1 + r)^{1+2s'} |f|^2 \, d\mu.
\]

Proposition 6.1 follows from this inequality.

\[\square\]

7. Decay estimate for the case that \( \Re z > \frac{(\beta_j)^2}{4} \)

In this section, we will show an a-priori-decay-estimate of an outgoing or incoming solution of the equation (10) on the end \( E_j \) in the case that \( \Re z > \frac{(\beta_j)^2}{4} \). This will be accomplished by combining Proposition 5.2 and Lemma 7.1 below; lemma 7.1 will be proved by taking the imaginary part of the integral in Proposition 3.1.

**Proposition 7.1.** Let \( 2 \leq R, z \in K_{\pm}, \) and \( f \in L^2_{\frac{1}{2} + s}(M, \mu), \) and \( u \) be a solution of (10) satisfying the radiation condition. Assume that

\[
S < a_{\min}; \quad s + s' \leq \delta; \quad \Re z > \frac{(\beta_j)^2}{4}.
\]

Then, for any \( R \) and \( R_1 \) satisfying \( \max\{1, r_0\} \leq R_1 < R_1 + 1 < R, \) we obtain

\[\Re \sqrt{z - \frac{(\beta_j)^2}{4} \|u\|_{L^2_{\frac{1}{2} - s'}(E_j(R,\infty), \mu)}^2}.\]
\[
\leq \tilde{c}_3 (1 + R)^{-2s'} \left\{ \| u \|^2_{L^2_{\frac{3}{4} + s}(E_j(R_1 - 1, \infty), \mu)} + \| f \|^2_{L^2_{\frac{1}{4} + s}(E_j(R_1 - 1, \infty), \mu)} \right\},
\]
where \( \tilde{c}_3 = \tilde{c}_3(s, s', a_{\min}, R_1, K_{\pm}) \) is a constant depending only on \( s, s', a_{\min}, R_1 \), and \( K_{\pm} \).

As is mentioned above, Proposition 7.1 immediately follows from the following Lemma 7.1 and Proposition 5.2, and hence, it suffices to prove the following:

**Lemma 7.1.** Let \( f \in L^2_{\frac{3}{4} + s}(M, \mu), z \in \Pi_{\pm}, \) and \( \text{Re} \, z > \frac{(\beta_j)^2}{4} \). Assume that \( u \) is an outgoing or incoming solution of (10). Then, for any \( R \) and \( R_1 \) satisfying \( \max\{1, r_0\} \leq R_1 < R_1 + 1 < R \), we obtain

\[
\text{Re} \left\{ -\frac{(\beta_j)^2}{4} \right\} \| u \|^2_{L^2_{\frac{3}{4} + s}(E_j(R_\infty), \mu)} \leq \frac{(1 + R)^{-2s'}}{2s'} \left\{ \| (\partial_r + \partial_r p_{\pm}) u \|^2_{L^2_{\frac{1}{4} + s}(E_j(R_\infty), \mu)} + \| f \|^2_{L^2_{\frac{1}{4} + s}(E_j(R_1 - 1, \infty), \mu)} \right\} + \tilde{c}_4 \| u \|^2_{L^2_{\frac{1}{4} + s}(E_j(R_1 - 1, \infty), \mu)}
\]

where \( \tilde{c}_4 := 64 (R_1)^2 \left\{ |z|^2 + 1 + \max_M |V| \right\} \).

**Proof.** Let \( t \) be a constant satisfying \( t > R_1 + 1 \). In proposition 3.1, we shall set \( \Omega = E_j(R_1, t) \) and

\[
\varphi(r) = \begin{cases} 
0 & \text{if } R_1 < r, \\
r - R_1 & \text{if } R_1 \leq r \leq R_1 + 1, \\
1 & \text{if } R_1 + 1 < r,
\end{cases}
\]

and take the imaginary part. Then, as in the proof of Proposition 3.2, we obtain

\[
\int_{S_{j}(t)} \text{Im} \, (\partial_r p_{\pm}) |u|^2 \, dA_w - \text{Im} \, z \int_{E_j(R_1, t)} \varphi |u|^2 \, d\mu
\]

\[
= \int_{S_{j}(t)} \text{Im} \, \varphi (\partial_r + \partial_r p_{\pm}) u \, dA_w + \int_{E_j(R_1, t)} \varphi \text{Im} \, f \, d\mu - \int_{E_j(R_1, t)} \text{Im} \, (\partial_r u) \varphi \, d\mu.
\]

Here, note that

\[
\text{Im} \, z \geq 0, \quad \text{Im} \, \partial_r p_{+} = -\text{Re} \left\{ \sqrt{z - \frac{(\beta_j)^2}{4}} \right\} < 0 \quad \text{if } z \in \Pi_{+};
\]

\[
\text{Im} \, z \leq 0, \quad \text{Im} \, \partial_r p_{-} = \text{Re} \left\{ \sqrt{z - \frac{(\beta_j)^2}{4}} \right\} > 0 \quad \text{if } z \in \Pi_{-};
\]

(48) and (49) follow from the assumption \( \text{Re} \, z > \frac{(\beta_j)^2}{4} \). Thus, signs of real numbers, \( \text{Im} \, z \) and \( \text{Im} \, \partial_r p_{\pm} \), are different in both cases. Hence, by (47), (48), and (49), we obtain

\[
\text{Re} \left\{ \sqrt{z - \frac{(\beta_j)^2}{4}} \right\} \int_{S_{j}(t)} |u|^2 \, dA_w \leq \text{Re} \left\{ \sqrt{z - \frac{(\beta_j)^2}{4}} \right\} \int_{S_{j}(t)} |u|^2 \, dA_w + |\text{Im} \, z| \int_{E_j(R_1, t)} \varphi |u|^2 \, d\mu
\]
right hand side of (50), we have, by Schwarz’s inequality,

\[ R, \]

it over \( t \). We shall multiply both sides of the inequality (50) by \((1 + R)^{-1 - 2s'}\) and integrate it over \([R, \infty)\) with respect to \( t \). Then, as for the integral of the first term on the right hand side of (50), we have, by Schwarz’s inequality,

\[
(51) \quad \int_R^\infty (1 + t)^{-1 - 2s'} \, dt \int_{S_j(t)} |u| (\partial_r + \partial_r p_{\pm}) u \, dA_w
\]

\[
\leq (1 + R)^{-s' - s} \int_R^\infty dt \int_{S_j(t)} (1 + t)^{-1 + s - s'} |u| (\partial_r + \partial_r p_{\pm}) u \, dA_w
\]

\[
\leq \frac{(1 + R)^{-s' - s}}{2} \left\{ \int_{E_j(R, \infty)} (1 + r)^{-1 + 2s'} |(\partial_r + \partial_r p_{\pm}) u|^2 \, d\mu + \int_{E_j(R, \infty)} (1 + r)^{-1 - 2s'} |u|^2 \, d\mu \right\}.
\]

We shall set \( \tilde{F}(r) := \int_{S_j(t)} |u| \, d\mu \) and use Fubini’s theorem for the integral of the second term on the right hand side of (50). Then, we have

\[
(52) \quad \int_R^\infty (1 + t)^{-1 - 2s'} \, dt \int_{E_j(R, t)} |u| f \, d\mu = \int_R^\infty (1 + t)^{-1 - 2s'} \, dt \int_{R_1}^t \tilde{F}(r) \, dr
\]

\[
= \int_{R_1}^R \tilde{F}(r) \, dr \int_R^\infty (1 + t)^{-1 - 2s'} \, dt + \int_R^\infty \tilde{F}(r) \, dr \int_r^\infty (1 + t)^{-1 - 2s'} \, dt
\]

\[
= \frac{(1 + R)^{-2s'}}{2s'} \int_{E_j(R_1, R)} |u| f \, d\mu + \frac{1}{2s'} \int_{E_j(R, \infty)} (1 + r)^{-2s'} |u| f \, d\mu
\]

\[
\leq \frac{(1 + R)^{-2s'}}{2s'} \int_{E_j(R_1, \infty)} |u| f \, d\mu
\]

\[
\leq \frac{(1 + R)^{-2s'}}{4s'} \left\{ \int_{E_j(R_1, \infty)} (1 + r)^{1 + 2s} |f|^2 \, d\mu + \int_{E_j(R_1, \infty)} (1 + r)^{-1 - 2s'} |u|^2 \, d\mu \right\},
\]

where, in the last line, we have used Schwarz’s inequality and the assumption \( s' \leq s \).

As for the integration of the third term on the right hand side of (50), we have

\[
(53) \quad \int_R^\infty (1 + t)^{-1 - 2s'} \, dt \int_{E_j(R_1, R_1 + 1)} |(\partial_r u)_w| \, d\mu
\]

\[
\leq \frac{(1 + R)^{-2s'}}{4s'} \left\{ \int_{E_j(R_1, R_1 + 1)} |\partial_r u|^2 \, d\mu + \int_{E_j(R_1, R_1 + 1)} |u|^2 \, d\mu \right\}.
\]
Now, we shall apply Lemma 4.1 to the first term on the right hand side of (53); set \( \Omega = E_j, \varepsilon = \frac{1}{2}, \) and

\[
\varphi(r) = \begin{cases} 
0 & \text{if } 0 \leq r \leq R_1 - 1, \\
_r - R_1 + 1 & \text{if } R_1 - 1 \leq r \leq R_1, \\
1 & \text{if } R_1 \leq r \leq R_1 + 1, \\
-r + R_1 + 2 & \text{if } R_1 + 1 \leq r \leq R_1 + 2, \\
0 & \text{if } R_1 + 2 \leq r.
\end{cases}
\]

Then, in view of \(-L_{loc} u = z u + f,\) we obtain

\[
\int_{E_j(R_1 - 1, R_1 + 1)} \left| \nabla u \right|^2 d\mu \\
\leq \int_{E_j(R_1 - 1, R_1 + 2)} (1 + r)^{1+2s'} \left| f \right|^2 d\mu \\
+ (3 + R_1)^{1+2s'} \left\{ \left| z \right|^2 + \tilde{c}_5 \right\} \int_{E_j(R_1 - 1, R_1 + 2)} (1 + r)^{-1-2s'} \left| u \right|^2 d\mu.
\]

Here, \( \tilde{c}_5 := \frac{1}{2} + 2 \max M |V| + 4(1 + s) \leq 7 + 2 \max M |V| \) by the fact \( 0 < s < 1/2. \) Hence, (53) and (54) imply that

\[
\int_{E_j(R_1 - 1, R_1 + 1)} \left| \partial_r u \right| d\mu \\
\leq \frac{(1 + R)^{-2s'}}{4s'} \left\{ \int_{E_j(R_1 - 1, R_1 + 2)} (1 + r)^{1+2s'} \left| f \right|^2 d\mu \\
+ \tilde{c}_6 \int_{E_j(R_1 - 1, R_1 + 2)} (1 + r)^{-1-2s'} \left| u \right|^2 d\mu \right\},
\]

where \( \tilde{c}_6 := (3 + R_1)^{1+2s'} \left\{ \left| z \right|^2 + \tilde{c}_5 + 1 \right\} \leq (3 + R_1)^2 \left\{ \left| z \right|^2 + 8 + 2 \max M |V| \right\}. \)

Thus, putting together (50), (51), (52), and (55), we obtain

\[
\text{Re} \sqrt{z - \frac{\beta_j^2}{4}} \int_{E_j(R_1, \infty)} (1 + r)^{-1-2s'} \left| u \right|^2 d\mu \\
\leq \frac{(1 + R)^{-s'-s}}{2} \int_{E_j(R_1, \infty)} (1 + r)^{-1+2s} \left| \partial_r + \partial_r p_+ \right| u \left| ^2 d\mu \\
+ \frac{(1 + R)^{-2s'}}{2s'} \int_{E_j(R_1 - 1, \infty)} (1 + r)^{1+2s'} \left| f \right|^2 d\mu \\
+ \tilde{c}_7(R) \int_{E_j(R_1 - 1, \infty)} (1 + r)^{-1-2s'} \left| u \right|^2 d\mu,
\]

where \( \tilde{c}_7(R) := \frac{(1+R)^{-s'-s}}{2} + (1 + \tilde{c}_6) \frac{(1+R)^{-2s'}}{4s'}. \) Since \( s' \leq s \) and \( 0 < s' \leq \frac{1}{2}, \) we see that

\[
\frac{1 + R)^{-s'-s}}{2} \leq \frac{(1 + R)^{-2s'}}{4s'};
\]

\[
2s'(1 + R)^{2s'} \cdot \tilde{c}_7(R) \leq 1 + \frac{(3 + R_1)^{1+2s'}}{2} \left\{ \left| z \right|^2 + 8 + \max M |V| \right\}.
\]

Lemma 7.1 follows from (56), (57), and (58). \( \square \)
8. THE PROOF OF MAIN THEOREMS

In this section, we shall prove main theorems after preparing some lemmas. In the following, we shall fix $s', s \in \mathbb{R}$ satisfying

$$0 < s' < s < \min \left\{ a_{\min}, \frac{1}{2} \right\}; \quad s' + s \leq \delta,$$

and use these numbers in the definition of the radiation condition (13) and (14). We denote by $R_{-L}(z)$ the resolvent $(-L - z)^{-1}$ of $-L$ for $z \in \text{resolv}(-L)$, where \text{resolv}(-L) stands for the resolvent set of $-L$. Moreover, we denote by $\mathbb{B}_{\mu} \left( \frac{1}{2} + s, -\frac{1}{2} - s' \right)$ the space of all bounded linear operators $T$ from $L^2_{-\frac{1}{2} + s}(M, \mu)$ into $L^2_{-\frac{1}{2} - s'}(M, \mu)$ with $\text{Dom}(T) = L^2_{-\frac{1}{2} + s}(M, \mu)$. $\mathbb{B}_{\mu} \left( \frac{1}{2} + s, -\frac{1}{2} - s' \right)$ is a Banach space by the operator norm $\| * \|_{\mathbb{B}_{\mu}(\frac{1}{2} + s, -\frac{1}{2} - s')}$.

We shall begin with the following:

**Proposition 8.1.** Let $z \in \mathbb{C} \setminus \mathbb{R}$. Then, (i) and (ii) below hold:

(i) Let $f \in L^2(M, \mu)$. Assume that $u$ is an outgoing or incoming solution of (10), i.e., $u \in L^2_{-\frac{1}{2} - s'}(M, \mu)$ and $\partial_r u + (\partial_r p_{\pm}) u \in L^2_{-\frac{1}{2} + s}(M, \mu)$. Then, $u \in L^2(M, \mu)$; in particular, $|\text{Im } z| \| u \|_{L^2(M, \mu)} \leq \| f \|_{L^2(M, \mu)}$, and $u$ coincides with the $L^2(M, \mu)$-solution $R_{-L}(z)f$.

(ii) Let $f \in L^2(M, \mu)$ for $\gamma \in (0, 1]$. Assume that $u$ is an outgoing (or incoming) solution of (10). Then, $u \in L^2(M, \mu)$ and

$$|\text{Im } z| \| u \|_{L^2_{\gamma}(M, \mu)} \leq (1 + \gamma) \| f \|_{L^2_{\gamma}(M, \mu)} + 10\gamma \left\{ 1 + |z| + \max_{M} \sqrt{|V|} \right\} \| u \|_{L^2(M, \mu)}.$$

**Proof.** (i) We may assume that $u$ is non-trivial. Setting $\varphi = 1$ in Proposition 3.2, we obtain

$$|\text{Im } z| \int_{U(R)} |u|^2 \, d\mu \leq \int_{S(R)} \left| (\partial_r + \partial_r p_{\pm}) u \right| |u| \, dA_w + \int_{U(R)} |u| \| f \| \, d\mu.$$

Here, in view of $s' \leq s$, we obtain $|(\partial_r + \partial_r p_{\pm}) u| |u| \leq \frac{1}{2} \left\{ r^{2s} |(\partial_r + \partial_r p_{\pm}) u|^2 + r^{-2s'} |u|^2 \right\}$; also, $\| u \|_{L^2(U(R), \mu)} \| f \|_{L^2(U(R), \mu)} \leq \| u \|_{L^2(U(R), \mu)} \| f \|_{L^2(U(R), \mu)}$. Hence, we obtain

$$\left(60\right) \quad |\text{Im } z| \| u \|_{L^2(U(R), \mu)} \leq \frac{1}{2} \| u \|_{L^2(U(R), \mu)}^{-1} \int_{S(R)} \left\{ r^{2s} |(\partial_r + \partial_r p_{\pm}) u|^2 + r^{-2s'} |u|^2 \right\} \, dA_w + \| f \|_{L^2(U(R), \mu)}.$$

Since $u$ satisfies (14), we see that $\liminf_{R \to \infty} \int_{S(R)} \left\{ r^{-2s'} |u|^2 + r^{2s} |(\partial_r + \partial_r p_{\pm}) u|^2 \right\} \, dA_w = 0$. Therefore, substituting an appropriate divergent sequence $\{ R_{\ell} \}_{\ell=1}^{\infty}$ of positive numbers into (60), we get our desired result.

(ii) The proof of (ii) will be completed through two steps. First, we shall show that $u \in L^2_{\gamma}(M, \mu)$. Note that $-Lu = f + zu \in L^2(M, \mu)$ by (i), which implies that $|\nabla u| \in L^2(M, \mu)$ by Corollary 4.1. Hence, $|(\partial_r + \partial_r p_{\pm}) u| \in L^2(M, \mu)$ by (11). Therefore, we see that

$$\liminf_{R \to \infty} \int_{S(R)} (1 + r) \gamma |(\partial_r + \partial_r p_{\pm}) u| |u| \, dA_w = 0 \quad \text{for } \gamma \in (0, 1].$$
Now, for $\gamma \in (0, 1]$, set $\varphi(r) = (1 + r)^\gamma$ in Proposition 3.2. Then,

\[
\text{(62)} \quad \left| \text{Im } z \right| \|u\|^2_{L^2_\infty(U(R), \mu)} \\
\leq \int_{S(R)} (1 + r)^\gamma |(\partial_r + \partial_r p_\pm) u| \left| u \right| \, dA_w \\
+ \int_{U(R)} \left\{ \gamma(1 + r)^{\gamma - 1} |\partial_r u| \left| u \right| + (1 + r)^\gamma |f| \left| u \right| \right\} \, d\mu.
\]

Here, $\int_{U(R)} (1+r)^{\gamma-1} |\partial_r u| \left| u \right| \, d\mu \leq \|\partial_r u\|_{L^{2-\gamma}(U(R), \mu)} \cdot \|u\|_{L^2_\infty(U(R), \mu)}$ and $\int_{U(R)} (1+r)^\gamma |f| \left| u \right| \, d\mu \leq \|f\|_{L^2_\infty(U(R), \mu)} \cdot \|u\|_{L^2_\infty(U(R), \mu)}$. Therefore, bearing (61) in mind, substituting an appropriate divergent sequence $\{R_i\}_{i=1}^\infty$ into (62), and letting $i \to \infty$, we obtain

\[
\left| \text{Im } z \right| \|u\|^2_{L^2_\infty(M, \mu)} \leq \gamma \|\partial_r u\|_{L^{2-\gamma}(M, \mu)} + \|f\|_{L^2_\infty(M, \mu)} < \infty,
\]

where, the last inequality follows from $-1 + \frac{\gamma}{2} \leq 0$ and $|\nabla u| \in L^2(M, \mu)$. Thus, $u \in L^2_\infty(M, \mu)$.

Next, we shall prove that $u \in L^2_\infty(M, \mu)$. Note that $-Lu = f + zu \in L^2_\infty(M, \mu)$ by the fact, $u \in L^2_\infty(M, \mu)$, as is shown above. Hence, Corollary 4.1 implies that

\[
|(|\partial_r + \partial_r p_\pm) u| \in L^2_\infty(M, \mu),
\]

which, together with $0 < \gamma \leq 1$ and $u \in L^2_\infty(M, \mu)$, implies that

\[
\text{(63)} \quad \liminf_{R \to \infty} \int_{S(R)} (1 + r)^{2\gamma} |(\partial_r + \partial_r p_\pm) u| \left| u \right| \, dA_w = 0 \quad \text{for } 0 < \gamma \leq 1.
\]

Now, we shall set $\varphi(r) = (1 + r)^{2\gamma}$ in Proposition 3.2 and repeat the same arguments as above: then, we obtain

\[
\text{(64)} \quad \left| \text{Im } z \right| \|u\|^2_{L^2_\infty(U(R), \mu)} \leq \|u\|^2_{L^2_\infty(U(R), \mu)} \int_{S(R)} (1 + r)^{2\gamma} |(\partial_r + \partial_r p_\pm) u| \left| u \right| \, dA_w \\
+ 2\gamma \|\partial_r u\|_{L^{2+\gamma}(U(R), \mu)} + \|f\|_{L^2_\infty(U(R), \mu)}.
\]

Bearing (63) in mind, substituting an appropriate divergent sequence $\{R_i\}_{i=1}^\infty$ into (64), and letting $i \to \infty$, we obtain

\[
\text{(65)} \quad \left| \text{Im } z \right| \|u\|^2_{L^2_\infty(M, \mu)} \leq 2\gamma \|\partial_r u\|^2_{L^{2+\gamma}(M, \mu)} + \|f\|_{L^2_\infty(M, \mu)}.
\]

Again, since $-1 + \gamma \leq 0$, Corollary 4.1 implies that

\[
\|\nabla u\|^2_{L^{2+\gamma}(M, \mu)} \leq \frac{1}{2} \|f + zu\|^2_{L^{2+\gamma}(M, \mu)} + \tilde{c}_8 \|u\|^2_{L^{2+\gamma}(M, \mu)} \\
\leq \frac{1}{2} \|f\|^2_{L^2(M, \mu)} + \left( \tilde{c}_8 + \frac{|z|}{2} \right) \|u\|^2_{L^2(M, \mu)},
\]

where $\tilde{c}_8 := \sqrt{\tilde{c} \left( \frac{1}{4} \right)} \leq \frac{3}{2} + \frac{3}{2} \max_M \sqrt{|V|}$. Combining (65) and (66), we obtain

\[
\left| \text{Im } z \right| \|u\|^2_{L^2_\infty(M, \mu)} \\
\leq \gamma \|f\|^2_{L^2(M, \mu)} + 10\gamma \left\{ 1 + |z| + \max_{M} \sqrt{|V|} \right\} \|u\|^2_{L^2(M, \mu)} + \|f\|_{L^2_\infty(M, \mu)}.
\]

Assertion (ii) follows from this inequality. \hfill \Box

Proposition 8.1 implies the following two corollaries:
Corollary 8.1. Assume that \( z \in \Pi_\pm \setminus (0, \infty) \). Then, for any \( f \in L^2_{\frac{1}{2} + s}(M, \mu) \), the equation (10) has a unique outgoing or incoming solution \( u \); moreover, \( u \) coincides with the \( L^2(M, \mu) \)-solution \( R_{-\lambda}(z)f \) and belongs to \( L^2_{\frac{1}{2} + s}(M, \mu) \).

Proof. Let \( f \in L^2_{\frac{1}{2} + s}(M, \mu) \) and consider \( L^2(M, \mu) \)-solution \( R_{-\lambda}(z)f \). By Proposition 8.1, it suffices to prove that \( R_{-\lambda}(z)f \) satisfies the radiation condition. Then, \( R_{-\lambda}(z)f \) satisfies \( -L(R_{-\lambda}(z)f) = z(R_{-\lambda}(z)f) + f \in L^2(M, \mu) \), and hence, Corollary 4.1 implies that \( |\nabla (R_{-\lambda}(z)f)| \in L^2(M, \mu) \). In view of the facts, \( 0 < s \leq \frac{1}{2} \) and \( \sup_{x \in M} |\partial_x p_{\pm}(x, z)| < \infty \) for each fixed \( z \), we see that \( \partial_x (R_{-\lambda}(z)f) + (\partial_x p_{\pm}(z, *)) (R_{-\lambda}(z)f) \in L^2(M, \mu) \subset L^2_{\frac{1}{2} + s}(M, \mu) \). Thus, \( R_{-\lambda}(z)f \) satisfies the condition (14), and hence, \( R_{-\lambda}(z)f \in L^2_{\frac{1}{2} + s}(M, \mu) \) by Proposition 8.1.

Corollary 8.2 (decay estimate for \( |\text{Im} \ z| > 0 \)). Let \( z \in \Pi_\pm \setminus (0, \infty) \), \( f \in L^2_{\frac{1}{2} + s}(M, \mu) \), and \( u \) be the corresponding outgoing or incoming solution of (10). Then, for any \( R > 1 \),

\[
|\text{Im} \ z|^2 \int_{E(R, \infty)} (1 + r)^{-1 - 2s'} |u|^2 \, d\mu \leq (\hat{c}_{11})^2 (1 + R)^{-2 - 2s - 2s'} \int_M (1 + r)^{1 + 2s} |f|^2 \, d\mu,
\]

where \( \hat{c}_{11} := 10 \left\{ 1 + \frac{1}{|\text{Im} \ z|} \left( 1 + |z| + \max_M |V| \right) \right\} \).

Proof. Corollary 8.1 and Proposition 8.1 imply that

\[
|\text{Im} \ z| \|u\|_{L^2_{\frac{1}{2} + s}(M, \mu)} \leq 10 \left\{ \frac{1 + |z| + \max_M \sqrt{|V|}}{|\text{Im} \ z|} \right\} \|f\|_{L^2(M, \mu)} \leq \hat{c}_{11} \|f\|_{L^2_{\frac{1}{2} + s}(M, \mu)},
\]

where \( \hat{c}_{11} = 10 \left\{ 1 + \frac{1}{|\text{Im} \ z|} \left( 1 + |z| + \max_M \sqrt{|V|} \right) \right\} \). Therefore, we obtain

\[
|\text{Im} \ z|^2 (1 + R)^{2 + 2s + 2s'} \int_{E(R, \infty)} (1 + r)^{-1 - 2s'} |u|^2 \, d\mu \leq |\text{Im} \ z|^2 \int_M (1 + r)^{1 + 2s} |u|^2 \, d\mu \leq (\hat{c}_{11})^2 \int_M (1 + r)^{1 + 2s} |f|^2 \, d\mu \quad \text{(by (67))}.
\]

Lemma 8.2 follows from this inequality.

Next, we shall show the following uniqueness theorem:

Lemma 8.1 (uniqueness). Assume that \( z = \lambda \in \Pi_\pm \setminus (0, \infty) \). Then, outgoing or incoming solution \( u \) of (10), if it exists, is uniquely determined by \( z \) and \( f \).

Proof. First, note that \( z = \lambda > \frac{(\beta_1)^2}{4} \) by the definition of \( \Pi_\pm \). Let \( u_1 \) and \( u_2 \) be two solutions of the same equation (10). Then, \( u := u_1 - u_2 \) is a solution to the eigenvalue equation

\[
-Lu - \lambda u = 0; \quad \lambda > \frac{(\beta_1)^2}{4}.
\]
Since \( \lambda \in \mathbb{R} \), we may assume that \( u \) is real-valued by considering the real and imaginary part of \( u \). Hence, setting \( \varphi \equiv 1 \) and \( f = 0 \) in Proposition 3.2, we obtain, for \( R \geq r_0 \),

\[
\int_{S(R)} |\text{Im} \partial_r p_\pm | |u| dA_w \leq \int_{S(R)} |(\partial_r + \partial_r p_\pm) u| |u| dA_w.
\]

Therefore, in view of (11) and (12), we get, for \( R \geq r_0 \),

\[
(69) \quad \sqrt{\lambda - \frac{(\beta_j)^2}{4}} \int_{S_1(R)} |u|^2 dA_w \leq \int_{S_1(R)} |(\partial_r + \partial_r p_\pm) u| |u| dA_w.
\]

Since \((\partial_r + \partial_r p_\pm) u \in L^2_{\frac{1}{2} + s}(M, \mu)\) and \( u \in L^2_{\frac{1}{2} + s'}(M, \mu) \) by (14), we see that

\[
\int_M (1 + r)^{s-s'-1} |(\partial_r + \partial_r p_\pm) u| |u| d\mu \leq \frac{1}{2} \int_M \{(1 + r)^{1-2s'}|u|^2 + (1 + r)^{1-2s'} |u|^2 \} d\mu < \infty.
\]

Hence, multiplying both sides of (69) by \((1 + R)^{s-s'-1}\) and integrating it over \([r_0, \infty)\) with respect to \( R \), we obtain

\[
\sqrt{\lambda - \frac{(\beta_j)^2}{4}} \int_{E_{(r_0, \infty)}} (1 + R)^{s-s'-1} |u|^2 d\mu \leq \int_{E_{(r_0, \infty)}} (1 + R)^{s-s'-1} |(\partial_r + \partial_r p_\pm) u| |u| d\mu < \infty.
\]

Thus, \( Lu = -\lambda u \in L^2_{\frac{1}{2}+s'}(E_1, \mu) \); hence, \(|\nabla u| \in L^2_{\frac{1}{2}+s'}(E_1, \mu)\) by the same arguments of the proof of Corollary 4.1. Therefore, we obtain

\[
(70) \quad \liminf_{R \to \infty} R^{s-s'} \int_{S_1(R)} \{|\partial_r u|^2 + |u|^2 \} d\mu = 0,
\]

where \( s-s' > 0 \) (see (59)).

Now, by reconsidering the arguments developed in [17] and [18], it is not hard to see that, if (68) holds on \( E_1 \), then (70), together with (1) or (2) with \( E = E_1 \), implies that \( u \equiv 0 \) on \( E_1 \) (in both cases, (1) and (2)), and hence, \( u \equiv 0 \) on \( M \) by the unique continuation theorem; for details and other interesting facts, see [19]. \( \square \)

**Lemma 8.2 (precompactness).** Let \( \{ z_k \}_{k=1}^\infty \subset K_+ \) be a sequence, and \( \{ f_k \}_{k=1}^\infty \) be a bounded sequence in \( L^2_{\frac{1}{2}+s}(M, \mu) \). Assume that \( \{ u_k \}_{k=1}^\infty \) is the sequence of the corresponding outgoing or incoming solution to the equation (10). If \( \{ u_k \}_{k=1}^\infty \) is bounded in \( L^2_{\frac{1}{2}+s'}(M, \mu) \), then \( \{ u_k \}_{k=1}^\infty \) is precompact in \( L^2_{\frac{1}{2}+s'}(M, \mu) \).

**Proof.** First, recall that \( K_+ \) and \( K_- \) are any fixed compact subsets in \( \Pi_+ \) and \( \Pi_- \), respectively. Therefore,

\[
(71) \quad \min \left\{ \left| z - \frac{(\beta_j)^2}{4} \right| \mid z \in K_\pm, \ j = 1, \cdots, m \right\} > 0.
\]

For \( \varepsilon > 0 \), let \( \text{Rect}_\pm \left( \frac{(\beta_j)^2}{4}, \varepsilon \right) \) be a rectangle around \( \frac{(\beta_j)^2}{4} \) defined by

\[
\text{Rect}_\pm \left( \frac{(\beta_j)^2}{4}, \varepsilon \right) := \left\{ x + iy \in \Pi_\pm \mid \max \left\{ |y|, \left| \frac{(\beta_j)^2}{4} - x \right| \right\} \leq \varepsilon \right\}.
\]
Then, (71) implies that there exists a constant $\varepsilon_0 = \varepsilon_0(K_\pm, c_1, \ldots, c_m) > 0$ such that, $\text{Rect}_+ \left( \frac{(\beta_j)^2}{4}, \varepsilon_0 \right) \cap K_+ = \emptyset$ for any $j = 1, \ldots, m$, and $\text{Rect}_- \left( \frac{(\beta_j)^2}{4}, \varepsilon_0 \right) \cap K_- = \emptyset$ for any $j = 1, \ldots, m$. Thus, for each fixed $j \in \{1, \ldots, m\}$, if $z \in K_\pm$, then one of the following three cases occurs: (a) $\text{Re} z \leq \frac{(\beta_j)^2}{4} - \varepsilon_0$; (b) $|\text{Im} z| \geq \varepsilon_0$; (c) $\text{Re} z \geq \frac{(\beta_j)^2}{4} + \varepsilon_0$. For the case (a), we shall apply Proposition 6.1; for the case (b), we shall apply Corollary 8.2; for the case (c), we shall apply Proposition 7.1. Then, for any $R > \max\{1, r_0\}$, we obtain

$$
(72) \quad \|u_k\|_{L^2_{\frac{2}{2+s}}(E_j(R, \infty), \mu)} \leq \tilde{c}_{12} \cdot (1 + R)^{-s} \left\{ \|f_k\|_{L^2_{\frac{2}{2+s}}(M, \mu)} + \|u_k\|_{L^2_{\frac{2}{2-s}}(M, \mu)} \right\},
$$

where $\tilde{c}_{12} > 0$ is a constant independent of $\{u_k\}_{i=1}^\infty$ and $\{f_k\}_{i=1}^\infty$. Since (72) holds for any $j \in \{1, \ldots, m\}$, by summing up for every $j \in \{1, \ldots, m\}$, we obtain, for any $R > \max\{1, r_0\}$,

$$
(73) \quad \|u_k\|_{L^2_{\frac{2}{2-s}}(E(R, \infty), \mu)} \leq \tilde{c}_{14} \cdot (1 + R)^{-s} \left\{ \|f_k\|_{L^2_{\frac{2}{2+s}}(M, \mu)} + \|u_k\|_{L^2_{\frac{2}{2-s}}(M, \mu)} \right\},
$$

where $\tilde{c}_{14} > 0$ is a constant independent of $\{u_k\}_{i=1}^\infty$ and $\{f_k\}_{i=1}^\infty$. Note that the right hand side of (73) tends to zero as $R \to \infty$ uniformly with respect to $k$.

On the other hand, since $-Lu_k = z_k u_k + f_k$, we see that $\{-Lu_k\}_{k=1}^\infty$ is a bounded sequence in $L^2_{\frac{2}{2-s'}}(M, \mu)$, and hence, by Corollary 4.1, $\{\nabla u_k\}_{k=1}^\infty$ is also a bounded sequence in $L^2_{\frac{2}{2-s'}}(M, \mu)$. Therefore, the Rellich’s lemma implies that

$$
(74) \quad \{u_k\}_{U(R)}^\infty_{k=1} \text{ is precompact in } L^2_{\frac{2}{2-s'}}(U(R), \mu), \quad \text{ for any } R > 0.
$$

Thus, by (73) and (74), we see that $\{u_k\}_{k=1}^\infty$ is precompact in $L^2_{\frac{2}{2-s}}(M, \mu)$. \hfill \Box

**Lemma 8.3 (preservation of radiation condition).** Let $\{z_k\}_{k=1}^\infty \subset K_{\pm}$ and $\{f_k\}_{k=1}^\infty \subset L^2_{\frac{2}{2+s}}(M, \mu)$ be sequences. Assume that

$$
\lim_{k \to \infty} z_k = z_\infty; \quad f_k \to f_\infty \quad (k \to \infty) \quad \text{ weakly in } L^2_{\frac{2}{2+s}}(M, \mu)
$$

and that $u_k$ is the outgoing or incoming solution of the equation (10) with $z = z_k$ and $f = f_k$. Assume that

$$
\lim_{k \to \infty} \|u_k - u_\infty\|_{L^2_{\frac{2}{2-s'}}(M, \mu)} = 0.
$$

Then, $u_\infty$ is the outgoing or incoming solution of (10) with $z = z_\infty$ and $f = f_\infty$.

*Proof.* By taking the limit of the equation

$$
(75) \quad -Lu_k - z_k u_k = f_k,
$$

we have

$$
(76) \quad -Lu_\infty - z_\infty u_\infty = f_\infty
$$

in the sense of distribution. But, the elliptic regularity theorem implies that $u_\infty \in H^2_{\text{loc}}(M)$, and hence, the equation (76) holds in the sense of $L^2_{\text{loc}}(M)$. On the other hand, since any weakly convergent sequence is bounded by the principle of uniform boundedness, $\|f_k\|_{L^2_{\frac{2}{2+s}}(M, \mu)}$ is uniformly bounded with respect to $k$. Therefore, Proposition 5.2 implies that $\|\nabla u_k + u_k \nabla p_{\pm}(z_k, \ast)\|_{L^2_{\frac{2}{2+s}}(E(3, \infty), \mu)}^2$ is also uniformly
bounded with respect to \( k \). Hence, by taking a subsequence if necessary, we may assume that

\[
\nabla u_k + u_k \nabla p_\pm(z_k, \cdot) \to X \quad (k \to \infty)
\]

weakly in \( L^2_{\frac{1}{2}+s}(E(3, \infty), \mu) \).

Since \( \|f_k\|_{L^2_{\frac{1}{2}+s}(\mu)} \) and \( \|u_k\|_{L^2_{\frac{1}{2}+s}(\mu)} \) are bounded uniformly with respect to \( k \), the equation (75) implies that, for any \( R > 0 \), there exists a constant \( c_0(R) \) such that \( \|L u_k\|_{L^2(U(R), \mu)} + \|u_k\|_{L^2(U(R), \mu)} \leq c_0(R) \), where \( c_0(R) \) is independent of \( k \). Therefore \( \|u_k\|_{H^2(U(R))} \) is uniformly bounded with respect to \( k \). Hence, the Rellich's lemma implies that \( \{u_k|_{U(R)}\}_{k=1}^\infty \) is precompact in \( H^1(U(R)) \), and hence, our assumption, \( \lim_{k \to \infty} \|u_k - u_\infty\|_{L^2_{\frac{1}{2}+s}(\mu)} = 0 \), implies that

\[
\lim_{k \to \infty} \|u_k - u_\infty\|_{W^{1,1}(U(R), \mu)} = 0 \quad \text{for any } R > 0.
\]

In view of (77) and (78), we see that \( \exists \) limit \( \pm \uvec{\nabla}u_\infty + \uvec{\nabla}p_\pm(z_\infty, \cdot) \), and hence, \( u_\infty \) satisfies the condition (14). Thus, we have proved Lemma 8.5.

Lemma 8.4 (boundedness of solutions). For \((z, f) \in K_+ \times L^2_{\frac{1}{2}+s}(M, \mu)\), let \( u(z, f) \) denote the corresponding outgoing or incoming solution of (10). Then, there exists a constant \( c(K_+) > 0 \), depending only on \( K_+ \), such that

\[
\|u(z, f)\|_{L^2_{\frac{1}{2}+s}(\mu)} \leq c(K_+) \|f\|_{L^2_{\frac{1}{2}+s}(\mu)}.
\]

Proof. We shall prove this lemma by contradiction. If we deny the conclusion, there exist sequences \( \{z_k\}_{k=1}^\infty \subset K_+ \) and \( \{f_k\}_{k=1}^\infty \subset L^2_{\frac{1}{2}+s}(M, \mu) \) such that

\[
\|u_k\|_{L^2_{\frac{1}{2}+s}(\mu)} \equiv 1; \quad \lim_{k \to \infty} \|f_k\|_{L^2_{\frac{1}{2}+s}(\mu)} = 0; \quad \lim_{k \to \infty} z_k = z_\infty \in K_+,
\]

where \( u_k := u(z_k, f_k) \) and we have used the fact that \( K_+ \) is compact. By Lemma 8.2, by taking a subsequence if necessary, we may assume that there exists \( u_\infty \in L^2_{\frac{1}{2}+s}(\mu) \) such that \( \lim_{k \to \infty} \|u_k - u_\infty\|_{L^2_{\frac{1}{2}+s}(\mu)} = 0 \). Then, by taking the limit of the equation \(- Lu_k - z_ku_k = f_k\), we obtain

\[
-L u_\infty - z_\infty u_\infty = 0
\]

in the sense of distribution. But, the elliptic regularity theorem implies that \( u \in C^\infty(M) \) and (79) holds in the sense of \( C^\infty(M) \). Hence, if \( \Im z_\infty \neq 0 \), Corollary 8.1 and Proposition 8.1 (i) imply that \( u_\infty \equiv 0 \); if \( \Im z_\infty = 0 \), Lemma 8.3 and Lemma 8.1 imply that \( u_\infty \equiv 0 \). This contradicts the fact that \( \|u_\infty\|_{L^2_{\frac{1}{2}+s}(\mu)} = \lim_{k \to \infty} \|u_k\|_{L^2_{\frac{1}{2}+s}(\mu)} = 1 \). Thus, we have proved Lemma 8.4.

We are now ready to prove the following:

Theorem 8.1 (principle of limiting absorption). Let \( s \) and \( s' \) be constants satisfying (59). Then, in the Banach space \( B_\mu\left(\frac{1}{2} + s, -\frac{1}{2} - s'\right) \), we have the limit

\[
R_{-L}(\lambda + i 0) = \lim_{\varepsilon \downarrow 0} R_{-L}(\lambda + i \varepsilon), \quad \lambda \in I.
\]

Moreover, this convergence (80) is uniform on any compact subset of \( I \), and \( R_{-L}(z) \) is continuous on \( \Pi_+ \) and \( \Pi_- \) with respect to the operator norm \( \| \cdot \|_{B_\mu(\frac{1}{2} + s, -\frac{1}{2} - s')} \) by considering \( R_{-L}(\lambda + i 0) \) and \( R_{-L}(\lambda - i 0) \) on \( \Pi_+ \cap (0, \infty) \) and \( \Pi_- \cap (0, \infty) \), respectively.
Proof. Let $K$ be any compact subset of $I$, and we will show that there exists an operator $R_{-L}(\lambda + i0)$ in $B_{\mu}(\frac{1}{2} + s, -\frac{1}{2} - s')$ such that
\[\lim \sup_{\tau \to 0} \| R_{-L}(\lambda + i0) - R_{-L}(\lambda + i\tau) \|_{B_{\mu}(\frac{1}{2} + s, -\frac{1}{2} - s')} = 0\]
by contradiction. If we assume the contrary, there exist constant $\varepsilon_0 > 0$, sequences \(\{\tau_k^1\}_{k=1}^\infty, \{\tau_k^2\}_{k=1}^\infty\) of positive numbers, \(\{f_k\}_{k=1}^\infty \subset L^2_{\frac{1}{2} + s}(M, \mu)\), and \(\{\lambda_k\}_{k=1}^\infty \subset K\) such that
\[0 < \tau_k^1, \tau_k^2 < 1/k ; \quad \| f_k \|_{L^2_{\frac{1}{2} + s}(M, \mu)} = 1 ; \quad (81)\]
\[\| u(\lambda_k + i\tau_k^1, f_k) - u(\lambda_k + i\tau_k^2, f_k) \|_{L^2_{\frac{1}{2} - s'}(M, \mu)} \geq \varepsilon_0, \quad (82)\]
where we have used the notation in Lemma 8.4. By taking a subsequence if necessary, we may assume that
\[\lim_{k \to \infty} \lambda_k = \lambda_\infty ; \quad f_k \to f_\infty (k \to \infty) \quad \text{weakly in } L^2_{\frac{1}{2} + s}(M, \mu) ; \quad (83)\]
moreover, by Lemma 8.4 and Lemma 8.2,
\[\lim_{k \to \infty} \| u(\lambda_k + i\tau_k^1, f_k) - u_\infty \|_{L^2_{\frac{1}{2} - s'}(M, \mu)} = 0 ; \quad (84)\]
\[\lim_{k \to \infty} \| u(\lambda_k + i\tau_k^2, f_k) - u'_\infty \|_{L^2_{\frac{1}{2} - s'}(M, \mu)} = 0. \quad (85)\]
Then, Lemma 8.3 implies that $u_\infty$ and $u'_\infty$ are outgoing solutions of the same equation (10) with $z = \lambda_\infty$ and $f = f_\infty$. However, Lemma 8.1 implies that $u_\infty = u'_\infty$, which contradicts (82), (83), and (84). Thus we have proved (81).

The proof of the existence of an operator $R_{-L}(\lambda - i0) \in B_{\mu}(\frac{1}{2} + s, -\frac{1}{2} - s')$ satisfying $\lim_{\tau \to 0} \| R_{-L}(\lambda - i0) - R_{-L}(\lambda - i\tau) \|_{B_{\mu}(\frac{1}{2} + s, -\frac{1}{2} - s')} = 0$ is quite the same.

Corollary 8.3 (absolutely continuity). Let $(M, g)$ be a Riemannian manifold as in Theorem 1.1. Then, $-L$ is absolutely continuous on $\left(\frac{(b_j)^2}{4}, \infty\right)$ and has no singular continuous spectrum.

Proof. First, note that the essential spectrum of $-L$ is equal to $\left[\frac{(b_j)^2}{4}, \infty\right)$ (see [15], [16]), because $\Delta_g r \to \beta_j$ as $r \to \infty$ on $E_j$ for $j = 1, 2, \ldots, m$.

It is now a standard fact that the limiting absorption principle implies the absolute continuity; see [28, Theorem XIII.19 and Theorem XIII.20]. Hence, we obtain the absolute continuity of $-L$ on $I = \left(\frac{(b_j)^2}{4}, \infty\right) - \left\{\frac{(b_j)^2}{4} \mid j = 2, \ldots, m\right\}$ by Theorem 8.1. Moreover, Lemma 8.1 implies that $\frac{(b_j)^2}{4} \notin \sigma_{pp}(-L)$ for $j = 2, \ldots, m$ satisfying $\beta_j > \beta_1$. Here, $\sigma_{pp}(-L)$ stands for the point spectrum of $-L$. Hence, $-L$ is absolutely continuous on $\left(\frac{(b_j)^2}{4}, \infty\right)$, and has no singular continuous spectrum.

Corollary 8.4. Let $(M, g)$ be a Riemannian manifold as in Theorem 8.1. When $M$ has at least one end satisfying $MC(\frac{1}{2}, \frac{b_j}{2}, \frac{\delta}{4})$ for some $a > 0$, $b > 0$, and $\delta > 0$, then $-L$ is absolutely continuous on $(0, \infty)$ and $0 \notin \sigma_{pp}(-L)$.

Since the multiplication operator $e^w : L^2_a(M, v_M) \to L^2_a(M, \mu)$ is unitary for any $\alpha \in \mathbb{R}$, operators, $-L$ and $-\Delta_g$, are unitarily equivalent. Hence, Theorem 8.1, Corollary 8.3, and Corollary 8.4 imply Theorem 1.1, Theorem 1.2, and Corollary 1.1, respectively.
9. Further Discussions

Corollary 9.1 below means that ends into which a “wave function” \(e^{it\Delta_g}u\) will recede as \(t \to \pm \infty\) are not high energy ends \(\cup \{E_k \mid \beta_k \geq \beta_j, \ k = 2, \cdots, m\}\) but low energy ends \(\cup \{E_k \mid \beta_k < \beta_j, \ k = 1, \cdots, j - 1\}\), when \(u \in E_{-\Delta_g}(I_j)L^2(M,v_g)\) and \(I_j = \left((\frac{\beta_1}{4})^2, (\frac{\beta_1}{4})^2\right)\).

**Corollary 9.1.** Let \((M,g)\) be an n-dimensional Riemannian manifold as in Theorem 1.1, and \(E_{-\Delta_g}(\Lambda)\) \((\Lambda \in \mathcal{B})\) denotes the spectral resolution of \(-\Delta_g\) on \(L^2(M,v_g)\). Let \(j \in \{2, \cdots, m\}\) be an integer satisfying \(\beta_1 < \beta_j\), and set \(I_j := \left((\frac{\beta_1}{4}), (\frac{\beta_1}{4})^2\right)\). Then, for any \(u \in E_{-\Delta_g}(I_j)L^2(M,v_g)\), we obtain

\[
\lim_{t \to \pm \infty} \int_{\cup \{E_k \mid \beta_k \geq \beta_j, \ k = 2, \cdots, m\}} |e^{it\Delta_g}u|^2 dv_g = 0.
\]

**Proof.** For any constant \(\varepsilon > 0\) satisfying \((\frac{\beta_1}{4})^2 - \varepsilon < (\frac{\beta_1}{4})^2\), we denote \(I_j(\varepsilon) := \left((\frac{\beta_1}{4})^2, (\frac{\beta_1}{4})^2 - \varepsilon\right)\). Then, \(E_{-\Delta_g}(I_j(\varepsilon))\) strongly converges to \(E_{-\Delta_g}(I_j)\) as \(\varepsilon \to 0\). Hence, it suffices to consider the case that \(u = E_{-\Delta_g}(I_j(\varepsilon))u\). Theorem 1.2 implies that \(u\) is an element in the absolutely continuous subspace of \(-\Delta_g\); hence, \((E_{-\Delta_g}(\Lambda)u, v)(\Lambda \in \mathcal{B})\) is an absolutely continuous signed measure, for any \(v \in L^2(M,v_g)\). Thus, there exists a function \(f(\lambda) \in L^1(\mathbb{R})\) such that \((E_{-\Delta_g}(\lambda)u, v)_{L^2(M,v_g)} = \int_{\lambda} f(\lambda) d\lambda\) for any \(\Lambda \in \mathcal{B}\). Therefore, for each integer \(k \geq 0\), we obtain

\[
(85) \quad (e^{it\Delta_g}(\Delta_g)^k(1 - \Delta_g)u, v)_{L^2(M,v_g)} = \int_{I_j(\varepsilon)} e^{it\lambda} \chi_k(1 + \lambda) d(E_{-\Delta_g}(\lambda)u, v)
= \int_{I_j(\varepsilon)} e^{it\lambda} \chi_k(1 + \lambda)f(\lambda) d\lambda;
\]

Riemann–Lebesgue lemma implies that the last term of (85) converge to zero as \(t \to \pm\). Thus, \(e^{it\Delta_g}(\Delta_g)^k(1 - \Delta_g)u\) weakly converges to zero as \(t \to \pm \infty\) in \(L^2(M,v_g)\) for each \(k \geq 0\). Set

\[
\chi_{U(R)}(x) := \begin{cases} 
1 & \text{if } x \in U(R), \\
0 & \text{if } x \in M \setminus U(R).
\end{cases}
\]

Then, the Rellich’s lemma implies that \(\chi_{U(R)}(1 - \Delta_g)^{-1}\) is a compact operator on \(L^2(M,v_g)\). Therefore, \(\chi_{U(R)}(\Delta_g)^k e^{it\Delta_g}u = \chi_{U(R)}(1 - \Delta_g)^{-1} e^{it\Delta_g}(\Delta_g)^k(1 - \Delta_g)u\) strongly converges to zero as \(t \to \pm \infty\) in \(L^2(M,v_g)\), that is,

\[
\lim_{t \to \pm \infty} \int_{U(R)} |(\Delta_g)^k e^{it\Delta_g}u|^2 dv_g = 0
\]

for any nonnegative integer \(k\) and \(R > 0\). Thus, the Sobolev embedding theorem implies that, for any integer \(k \geq 0\) and \(R > 0\),

\[
(86) \quad e^{it\Delta_g}u \to 0 \quad \text{as } \quad t \to \pm \infty \quad \text{on } U(R) \quad \text{in the sense of } C^k\text{-topology}.
\]

Now, we shall prove

\[
(87) \quad \lim_{t \to \infty} \int_{\cup \{E_k \mid \beta_k \geq \beta_j, \ k = 2, \cdots, m\}} |e^{it\Delta_g}u|^2 dv_g = 0
\]
by contradiction. If we assume contrary, there exist $j_1 \in \{2, \ldots, m\}$ satisfying $\beta_{j_1} \geq \beta_j$, a positive constant $a_{j_1}$, and a divergent sequence $\{t_k\}_{k=1}^\infty$ of positive real numbers such that

$$
(88) \quad \lim_{t \to \infty} \int_{E_{j_1}} |e^{it_j \Delta_g u}|^2 \, dv_g \geq a_{j_1} > 0.
$$

Then, since

$$
\inf \left\{ \left\| \nabla h \right\|^2_{L^2(E_{j_1}(R, R))} \bigg| 0 \neq h \in C^\infty_0 (E_{j_1}(R, \infty)) \right\} \to \frac{\beta_{j_1}^2}{4} \quad \text{as } R \to \infty,
$$

we see that (86) and (88) yield

$$
(89) \quad \lim_{t \to \infty} \| (\Delta_g)^k e^{it \Delta_g u} \|^2_{L^2(M, v_g)} = \sum_{j=1}^m \lim_{t \to \infty} \| (\Delta_g)^k e^{it_j \Delta_g u} \|^2_{L^2(E_j, v_g)}
$$

$$
\geq \lim_{t \to \infty} \| (\Delta_g)^k e^{it \Delta_g u} \|^2_{L^2(E_{j_1}, v_g)} \geq \left( \frac{\beta_{j_1}}{4} \right)^4 a_{j_1}.
$$

On the other hand, since $u = E(I_j(\varepsilon)) u$, we have, for any $t \in \mathbb{R}$,

$$
(90) \quad \| (\Delta_g)^k e^{it \Delta_g u} \|^2_{L^2(M, v_g)} = \int_{I_j(\varepsilon)} \lambda^{2k} d \| E_{-\Delta_g} (\lambda) u \|^2
$$

$$
\leq \left( \frac{\beta_j^2}{4} - \varepsilon \right)^2 k \int_{I_j(\varepsilon)} d \| E_{-\Delta_g} (\lambda) u \|^2 = \left( \frac{\beta_j^2}{4} - \varepsilon \right)^2 \| u \|^2_{L^2(M, v_g)}.
$$

From (89) and (90), we obtain $\theta^{2k} \| u \|^2_{L^2(M, v_g)} \geq a_{j_1}$ for all integer $k \geq 0$, where $\theta := ((\beta_j^2 - 4\varepsilon)/\beta_j^2)$; since $\theta \in (0, 1)$, letting $k \to \infty$, we obtain $a_{j_1} = 0$, which contradicts (88). This completes the proof of (87). The proof of the case, $t \to -\infty$, is quite the same.

In view of Proposition 6.1, we see that the following holds:

**Theorem 9.1.** Let $(M, g)$ be an $n$-dimensional connected complete noncompact Riemannian manifold and $U$ be a relatively compact open subset of $(M, g)$. Assume that $M \setminus U$ consists of the disjoint union of ends, $E_1, \ldots, E_m$, with radial coordinates, where $m \geq 2$. Assume that there exist positive constants, $\gamma$ and $\delta$, such that

- $E_j$ satisfies MC $\left( \frac{a_j}{r}, \frac{b_j}{r}, \delta \right)$ for $1 \leq j \leq m_0$;
- $E_j$ satisfies MC $\left( \alpha_j, \beta_j, \delta \right)$ for $m_0 + 1 \leq j \leq m_1$;
- $\lim_{t \to \infty} \inf \{ \Delta_g r(x) \mid x \in E_{m_1+1}(t, \infty) \cup \cdots \cup E_m(t, \infty) \} \geq \gamma > \frac{(\beta_j)^2}{4},$

where

$$
0 = \beta_1 = \beta_2 = \cdots = \beta_{m_0} < \beta_{m_0+1} \leq \beta_{m_0+2} \leq \cdots \leq \beta_{m_1}
$$

are real constants; $0 \leq m_0 \leq m_1$ and $1 \leq m_1 \leq m - 1$ are integer; $a_j, b_j, \alpha_j, \beta_j$ are all positive constants. Then, in the Banach space $B(\frac{1}{2} + s, -\frac{3}{2} - s')$, we have the limit

$$
R(\lambda \pm i0) = \lim_{\varepsilon \to 0} R(\lambda \pm i\varepsilon), \quad \lambda \in i', \quad \varepsilon = \left( \frac{(\beta_1)^2}{4} \right) - \left\{ \frac{(\beta_k)^2}{4} \mid 1 \leq k \leq m_1 \right\},
$$

where $i'$ is the subset of the complex plane $i' \subseteq \mathbb{C}$.
where \( s \) and \( s' \) are constants satisfying (59). Moreover, this convergence is uniform on any compact subset of \( I' \) and \(-\Delta_g\) is absolutely continuous on \((\frac{\lambda_1^2}{4}, \gamma)\).

**Proof.** Reconsidering the arguments in Section 8, we see that the precompactness lemma plays an important role there; it follows from decay estimates of solutions of (10) on every ends. Hence, when \( \lambda \in I' \), we shall apply Proposition 6.1 for ends \( E_{m+1}, \ldots, E_m \); for the rest of ends, we shall apply the arguments in Section 8. Then, we obtain the precompactness lemma. The rest of the proof holds good, and we obtain Theorem 9.1. \( \square \)

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