THE KNIGHT MOVE CONJECTURE IS FALSE

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Abstract. The Knight Move Conjecture claims that the Khovanov homology of any knot decomposes as direct sums of some “knight move” pairs and a single “pawn move” pair. This is true for instance whenever the Lee spectral sequence from Khovanov homology to $\mathbb{Q}^2$ converges on the second page, as it does for all alternating knots and knots with unknotting number at most 2. We present a counterexample to the Knight Move Conjecture. For this knot, the Lee spectral sequence admits a nontrivial differential of bidegree $(1, 8)$.

1. Introduction

Almost 20 years ago, Khovanov \cite{Kh} introduced a categorification of the Jones polynomial, now known by the name of Khovanov homology. This is an invariant of links in $S^3$ that is strictly more powerful than the Jones polynomial \cite{Jo}, and it detects the unknot \cite{R}. Furthermore, using Khovanov homology, Rasmussen defined a concordance homomorphism $s: \mathcal{C} \to \mathbb{Z}$ from the smooth knot concordance group, and used it to give the first combinatorial proof of Milnor’s conjecture \cite{Mil}.

Given a knot $K \subset S^3$, its Khovanov homology over $\mathbb{Q}$ is a bigraded vector space over $\mathbb{Q}$, endowed with a homological grading $i \in \mathbb{Z}$ and a quantum grading $j \in 1 + 2\mathbb{Z}$. We denote this bigrading by $(i, j)$. We denote the Khovanov homology of a knot $K \subset S^3$ by

$$\text{Kh}(K) = \bigoplus_{i \in \mathbb{Z}, j \in 1+2\mathbb{Z}} \text{Kh}^{i,j}(K),$$

and its Poincaré series by $\text{Kh}(K)(t, q) = \sum_{i, j} \dim_\mathbb{Q}(\text{Kh}^{i,j}(L)) t^i q^j$.

1.1. The structure of Khovanov homology. An early conjecture about the structure of Khovanov homology \cite{BarConj} Conjecture 1], known as the Knight Move Conjecture, is due to Bar-Natan, Garoufalidis, and Khovanov. It says that it is always possible to decompose the Khovanov homology of a knot into the direct sum of elementary pieces.

Conjecture 1.1 (Knight Move Conjecture \cite{BarConj}). Given a knot $K$, its Khovanov homology over $\mathbb{Q}$ is the direct sum of a single pawn move piece

$$\mathbb{Q}\{0, s - 1\} \oplus \mathbb{Q}\{0, s + 1\},$$

where $s$ is Rasmussen’s invariant, and several knight move pieces

$$\mathbb{Q}\{i, j\} \oplus \mathbb{Q}\{i + 1, j + 4\},$$

for various $i, j \in \mathbb{Z}$.

In terms of Poincaré series, this conjecture can be rewritten as follows (see \cite{BarConj} Conjecture 5.2]):

Conjecture 1.2 (Reformulation of the Knight Move Conjecture). For any knot $K$, there is a Laurent polynomial $f_2 \in \mathbb{N}[t^{\pm 1}, q^{\pm 1}]$ so that

$$\text{Kh}(K)(t, q) = q^s(q + q^{-1}) + f_2(t, q)(1 + tq^4).$$
1.2. Lee’s deformation. In [9], Lee introduced a deformation of the (co-)chain complex yielding Khovanov homology, which in fact is a filtered differential, where the filtration level is given by the quantum degree. The zeroth page of the resulting spectral sequence is the usual Khovanov complex, with the usual differential $d_0$. Thus, the first page $E_1$ is simply Khovanov homology\(^1\). The higher differentials $d_n$ on $E_n$ have degree $(1, 4n)$. Lastly, if $K$ is a knot, the resulting spectral sequence converges to $\mathbb{Q}\{0, s-1\} \oplus \mathbb{Q}\{0, s+1\}$, where $s$ is Rasmussen’s invariant.

Using the above properties, it is immediate to check that if the Lee spectral sequence of a knot $K$ degenerates after the first page, then there must be a Knight Move decomposition of $\text{Kh}(K)$. This is true for example for all alternating knots [8], and more generally for all quasi-alternating knots [10], as well as for all knots with unknotting number not bigger than 2 [2].

For a general knot, a corollary of Lee’s spectral sequence is that we have a decomposition of Khovanov homology into a pawn move and several, possibly “longer” knight moves, of the form

$$\mathbb{Q}\{i, j\} \oplus \mathbb{Q}\{i+1, j+4n\}.$$ 

In other words, for any knot $K$, there is a family of two variable Laurent polynomials $f_{2l} \in \mathbb{N}[t^{\pm 1}, q^{\pm 1}]$ for $l \geq 1$, so that

$$\text{Kh}(K)(t, q) = q^{s}(q + q^{-1}) + \sum_{l \geq 1} f_{2l}(t, q)(1 + tq^{4l}).$$

The Knight Move Conjecture is equivalent to saying that $f_{2l}$ can be set to 0 for all $l \geq 2$.

In this note, we present a counterexample to the Knight Move Conjecture. The example that we give has a non-trivial differential $d_2$.

2. The Counterexample

Our counterexample is the knot $K$ illustrated in Figure 1. It is obtained from an 8-crossing diagram of the unknot by doing a full positive twist along 6 strands. The resulting diagram has 38 crossings.

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\(^1\)In some of the literature, for example in [12], what we call the $E_n$ page of the Lee spectral sequence is denoted $E_{n+1}$, and our differential $d_n$ is their $d_{n+1}$.
Theorem 2.1. The knot $K$ in Figure 1 does not satisfy the Knight Move Conjecture. Moreover, the second Lee differential $d_2$ of bidegree $(1, 8)$ is non-vanishing.

Proof. The Khovanov homology of $K$ is computed using the program “JavaKh-v2”, an update by Scott Morrison of Jeremy Green’s original program, both of which are available on the Knot Atlas [1]. The result is shown in Table 1. The entry $tq$ (marked in red) is non-empty. If the Knight Move Conjecture were true, this should be matched by a non-zero entry in either $q^{-3}$ or $t^2q^5$. However, these are both empty.

Regarding the Lee spectral sequence, the entry $tq$ cannot be cancelled by a $d_1$ differential, because both the entries $q^{-3}$ and $t^2q^5$ are empty. It follows that it must be cancelled by a higher differential, which is necessarily $d_2$, since there is no room for non-trivial maps of bidegree $(1, 4n)$ for $n \geq 3$, as one can easily check from Table 1.

In fact, one can determine the whole structure of the Lee spectral sequence for $K$ using the program “UniversalKh” of Scott Morrison [1, 11]. It turns out that the $d_1$ differential (the knight move) cancels most of the terms in Khovanov homology, leaving only four copies of $\mathbb{Q}$ on the $E_2$ page, in bidegrees $(0, -1), (0, 1), (1, 1)$ and $(2, 9)$. The last two are cancelled by the $d_2$ differential, and the first two survive to the $E_\infty$ page. Rasmussen’s invariant for this knot is $s = 0$.

Remark 2.2. We came across the knot $K$ while studying the generalized crossing changes introduced by Cochran and Tweedy in [4]. The full twist shown in Figure 1 is an example of a generalized negative crossing. The resulting knot $K$ is slice in the blown-up ball $B^4 \# \overline{CP}^2$, but it is not slice in $B^4$, because its Alexander polynomial

$$\Delta_K(t) = -3t^{-1} + 7 - 3t$$

does not satisfy the Fox-Milnor criterion.

Table 1. The Khovanov homology of the knot $K$. The homological grading $i$ is on the horizontal axis, and the quantum grading $j$ on the vertical axis. The red box marks an entry that cannot be cancelled by a $d_1$ differential.
Remark 2.3. It is easy to see that the knot $K$ can be unknotted by three crossing changes. Since the Knight Move Conjecture holds for knots of unknotting number at most two [2], it follows that $K$ has unknotting number 3.

We end with an open problem.

Question 2.4. Given any $n \geq 3$, does there exist a knot for which the $d_n$ differential in the Lee spectral sequence is nonzero?

In view of the work of Alishahi and Dowlin [2], if for a knot $K$ we have $d_n \neq 0$, then $K$ needs to have unknotting number at least $2n - 1$.

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