Predicative proof theory of PDL and basic applications

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1 Extended abstract

Propositional dynamic logic (PDL) was derived by M. J. Fischer and R. Ladner [6], [7] from dynamic logic where it plays the role that classical propositional logic plays in classical predicate logic. Conceptually, it describes the properties of the interaction between programs (as modal operators) and propositions that are independent of the domain of computation. The semantics of PDL is based on Kripke frames and comes from that of modal logic. Corresponding sound and complete Hilbert-style formalism was proposed by K. Segerberg [19] (see also [15], [10]). Gentzen-style treatment is more involved. This is because the syntax of PDL includes starred programs $P^*$ which make finitary sequential formalism similar to that of (say) Peano Arithmetic with induction (PA) that allows no full cut-elimination. In the case of PA, however, there is a well-known Schütte-style solution in the form of infinitary (also called semiformal) sequent calculus with Carnap-style omega-rule that allows full cut elimination, provably in PA extended by transfinite induction up to Gentzen’s ordinal $\varepsilon_0$ (cf. [3], [18]). By the same token, in the case of PDL, we introduce Schütte-style semiformal one-sided sequent calculus $\text{Seq}_{\omega}^{PDL}$ whose inferences include the omega-rule with principal formulas $[P^*]A$ and prove cut-elimination theorem using transfinite induction up to Veblen’s predicative ordinal $\varphi_\omega(0)$ (that exceeds $\varepsilon_0$, see [21], [5]). The ordinal increase in question is caused by higher upper bounds on the complexity of cut formulas that may contain nested occurrences of the starred programs, as compared to plain occurrences of quantifiers in the language of PA. The omega-rule-free derivations in $\text{Seq}_{\omega}^{PDL}$ are finite and sequents deducible by these finite derivations are valid in PDL. Hence by the cutfree subformula property we conclude that any given $[P^*]$-free sequent is valid in PDL if it is deducible in $\text{Seq}_{\omega}^{PDL}$ by a finite cut- and omega-rule free derivation, which by standard methods enables better structural analysis of the validity of $[P^*]$-free sequent involved. \footnote{\textcolor{red}{cf. e.g.} Gentzen-style conclusion that any given false equation $x = y$ (in particular $0 = 1$) is not valid in PA, since obviously it has no cutfree derivation.} The latter also refers to the computational complexity of the validity problem in PDL that is known to be EXPTIME-complete (cf. [7], [17]). We show that PDL-validity of $[P^*]$-free basic conjunctive normal expansions (~*: BCNE) is solvable in polynomial space, whereas PDL-validity of dual $[P^*]$-free basic disjunctive normal expansions (~: BCNE) is not.
BDNE), whose negations express that satisfying Kripke frames encode accepting computations of polynomial-space alternating TM, is EXPTIME-complete. Thus the conjecture \textbf{EXPTIME} = \textbf{PSPACE} holds true iff \textbf{PDL}-validity of BDNE is decidable in polynomial space. We show that cutfree-derivability in Seq\textsuperscript{pdl} (and hence \textbf{PDL}-validity) of any given BDNE, \( S \), is equivalent to the validity of a suitable “transparent” quantified boolean formula \( \hat{S} \) whose size is exponential in the size of \( S \). Hence \textbf{EXPTIME} = \textbf{PSPACE} holds true if \( \hat{S} \) is equivalent with another (hypothetical) quantified boolean formula whose size is polynomial in the size of \( S \), for every \( S \in \text{BDNE} \). This may reduce the former problem to a complexity problem in quantified boolean logic. The whole proof is formalized in \textbf{PA} extended by transfinite induction along \( \phi^{\omega}(0) \) (at most) – actually in the corresponding primitive recursive weakening, \textbf{PRA}\_\phi^{\omega}(0).

### 2 More detailed exposition

#### 2.1 Hilbert-style proof system PDL

**Language \( \mathcal{L} \)**

1. Programs PRO (abbr.: \( P, Q, R, S \), possibly indexed):
   - (a) include program-variables (\(\text{PRO}_0\)) \( \pi_0, \pi_1, \ldots \) (abbr.: \( p, q, r \), possibly indexed),
   - (b) are closed under modal connectives ; and \( \cup \) and star operation \( * \).

2. Formulas FOR (abbr.: \( A, B, C, D, F, G, H, E \), etc., possibly indexed):
   - (a) include formula-variables \( \upsilon_0, \upsilon_1, \ldots \) (abbr.: \( x, y, z \), possibly indexed),
   - (b) are closed under implication \( \rightarrow \), negation \( \neg \) and modal operation \( F \leftrightarrow \langle P \rangle F \), where \( P \in \text{PRO} \). \(^2\)

**Axioms** (cf. e.g. [19], [10]):

\((D1)\) \textit{Axioms of classical propositional logic.}

\((D2)\) \( [P] (A \rightarrow B) \rightarrow ([P] A \rightarrow [P] B) \)

\((D4)\) \( [P; Q] A \leftrightarrow [P][Q] A \)

\((D5)\) \( [P \cup Q] A \leftrightarrow [P] A \land [Q] A \)

\((D7)\) \( [P^*] A \leftrightarrow A \land [P] [P^*] A \)

\((D8)\) \( [P^*] (A \rightarrow [P] A) \rightarrow (A \rightarrow [P^*] A) \)

**Inference rules:**

\((MP)\) \[
\begin{array}{c}
A \\
A \rightarrow B
\end{array}
\]

\((G)\) \[
\begin{array}{c}
[P] A
\end{array}
\]

\(^2\)Boolean constants are definable as usual e.g. by \( 1 := v_0 \rightarrow v_0 \) and \( 0 := \neg 1 \).

\(^3\)Standard axiom \((D3)\) : \( [P] (A \land B) \leftrightarrow ([P] A \land [P] B) \) follows by \((G)\) from \((D1)-(D2)\), whereas \((D6)\) : \( [\neg A] B \leftrightarrow (A \rightarrow B) \) is obsolete in the chosen \( ? \)-free language.
2.2 Semiformal sequent calculus \( \text{SEQ}^{\text{PDL}} \)

**Definition 1** The language of \( \text{SEQ}^{\text{PDL}} \) includes seq-formulas and sequents. Seq-formulas are built up from literals \( x \) and \( \neg x \) by propositional connectives \( \vee \) and \( \wedge \) and modal operations \( [P] \) and \( \langle P \rangle \) for arbitrary \( P \in \text{PRO} \). Seq-negation \( \overline{F} \) is defined recursively as follows, for any seq-formula \( F \).

1. \( \overline{x} := \neg x \), \( \overline{\overline{x}} := x \),
2. \( \overline{A \lor B} := \overline{A \land B} \), \( \overline{A \land B} := \overline{A} \lor \overline{B} \),
3. \( \langle P \rangle A := [P]A \), \( [P]A := \langle P \rangle \overline{A} \).

In the sequel we use abbreviations \( \langle P \rangle^m := \overline{\cdots \overline{\langle P \rangle}} \) and \( [P]^m := \overline{\cdots \overline{[P]}} \). For any \( \chi \in \{0, 1\} \), let \( \langle P \rangle^\chi := \left\{ \begin{array}{ll} [P] & \text{if } \chi = 1, \\ \langle P \rangle & \text{if } \chi = 0. \end{array} \right. \) For any \( \overline{P} = P_1, \ldots, P_k \) \((k \geq 0)\) and \( f : [1, k] \rightarrow \{0, 1\} \) let \( \overline{\langle P \rangle}_f := \langle P_1 \rangle_{f(1)} \cdots \langle P_k \rangle_{f(k)} \). By \( \langle \overline{\langle P \rangle} \rangle_f \) and \( \langle \overline{\langle P \rangle} \rangle_f \) we abbreviate \( \langle \overline{\langle P \rangle} \rangle_f \) for arbitrary \( f, f \equiv 0 \) and \( f \equiv 1 \), respectively.

Formulas from FOR are represented as seq-formulas recursively by \( \neg F := \overline{F} \), \( F \rightarrow G := \overline{F} \lor \overline{G} \) and, conversely, by \( F \lor G := \neg F \rightarrow G \), \( F \land G := \neg (F \rightarrow \neg G) \), \( \langle P \rangle F := \neg [P] \neg F \). Sequents (abbr.: \( \Gamma, \Delta, \Pi, \Sigma \), possibly indexed) are viewed as multisets (possibly empty) of seq-formulas. A sequent \( \Gamma = F_1 \lor \cdots \lor F_n \) is called valid if so is the corresponding disjunction \( F_1 \lor \cdots \lor F_n \). Plain complexity of a given formula and/or program in \( \mathcal{L} \) is its ordinary length (= total number of occurrences of literals and connectives \( \lor, \land, \lor, \land, * \)).

**Definition 2** Ordinal complexity \( \omega^\omega \) of formulas, programs and sequents in \( \mathcal{L} \) is defined recursively as follows, where \( \alpha \vdash \beta \) is the symmetric sum of ordinals \( \alpha \) and \( \beta \).

1. \( \omega(x) = \omega(\neg x) = \omega(P) : = 0 \).
2. \( \omega(A \lor B) = \omega(A \land B) := \max \{ \omega(A), \omega(B) \} + 1 \).
3. \( \omega([P]Q) := \omega([P]Q) + 1 \), \( \omega([P]Q) := \omega([P]Q) + 1 \).
4. \( \omega(P^*) := \sup_{m < \omega} \omega([P]^m) = \omega(P) \cdot \omega, \quad \omega([P]A) := \omega([P]A) := \omega(P) + \omega(A) + 1 \).
5. \( \omega(\Gamma) := \sum \{ \omega(A) : A \in \Gamma \} \).

**Definition 3** \( \text{SEQ}^{\text{PDL}} \) includes the following axiom (AX) and inference rules (\( \lor \), \( \land \), \( \lor \), \( \land \), \( ; \), \( ; \), \( \ast \), \( \ast \), \text{GEN}, \text{CUT}) in classical one-sided sequent
formalism in the language $L$. In $[\ast]$ we allow $\overline{Q} = \overline{[Q]} = \emptyset$.

\[
\begin{array}{c}
(Ax) \quad x, \neg x, \Gamma \\
(V) \quad A, B, \Gamma \quad (\forall) \quad A, \Gamma \quad B, \Gamma \\
(\cup) \quad (P) A, (R) A, \Gamma \quad (P \cup R) A, \Gamma \\
(;) \quad (P; R) A, \Gamma \\
(*) \quad \langle P \\rangle A, \Gamma \\
[\ast] \quad \cdots \quad \langle \overline{Q} \rangle (P^m) A, \Gamma \\
(\ast) \quad \langle Q \rangle (P^m) A, \Gamma \\
(\wedge) \quad [Q] A, \Gamma \\
(\lor) \quad [P] A, \Gamma \\
([;]) \quad [P; R] A, \Gamma \\
([\ast]) \quad \cdots \quad \langle \overline{Q} \rangle (P^m) A, \Gamma \\
(Gen) \quad A_1, \ldots, A_n \\
\quad (P) x_1 A_1, \ldots, (P) x_n A_n, \Gamma \\
\quad (n > 0) \\
\quad \text{if } \sum_{i=1}^{n} x_i = 1. \\
(Cut) \quad C, \Gamma \quad \overline{C}, \Pi \\
\end{array}
\]

For the sake of brevity we’ll drop “seq-” when referring to seq-formulas of $\text{Seq}^{\text{pdl}}$. $\Gamma$ is called derivable in $\text{Seq}^{\text{pdl}}$ if there exists a (tree-like, possibly infinite) $\text{Seq}^{\text{pdl}}$ derivation $\delta$ with the root sequent $\Gamma$ (abbr.: $(\delta : \Gamma)$). We assume that $\text{Seq}^{\text{pdl}}$ derivations are well-founded. The simplest way to implement this assumption is to supply nodes $x$ in $\delta$ with ordinals $\text{ord}(x)$ such that ordinals of premises are always smaller than the ones of the corresponding conclusions. Having this we let $h(\delta) := \text{ord} \left( \text{root}(\delta) \right)$ and call it the height of $\delta$.

In $\text{Seq}^{\text{pdl}}$, formulas occurring in $\Gamma$ and/or $\Pi$ are called side formulas, whereas other (distinguished) ones are called principal formulas, of axioms or inference rules exposed. These axioms and inferences, in turn, are called principal with respect to their principal formulas. Principal formulas of (Cut) are also called the corresponding cut formulas. We’ll sometimes specify (Gen) as $(\text{Gen})_P$ to indicate principal program $P$ involved.

**Theorem 4 (soundness and completeness)** $\text{Seq}^{\text{pdl}}$ is sound and complete with respect to PDL. Moreover any PDL-valid sequent (in particular formula) is derivable in $\text{Seq}^{\text{pdl}}$ using ordinals $< \omega + \omega =: \omega \cdot 2$.

\footnote{We assume that all rules exposed have nonempty premises.}

\footnote{[\ast] has infinitely many premises. \Pi is called the $\omega$-rule.}
Proof. The soundness says that any sequent $\Gamma$ that is derivable in $\text{SEQ}^\text{PDL}$ is valid in Kripke-style semantics of $\text{PDL}$. It is proved by transfinite induction on $h(\varnothing)$ of well-founded $(\varnothing: \Gamma)$ involved. Actually it suffices to verify that every inference rule of $\text{SEQ}^\text{PDL}$ preserves Kripke validity, which is easy (we omit the details; see also Remark 5 below).

The completeness is proved as follows by deducing in $\text{SEQ}^\text{PDL}$ the axioms and inferences $(D1), (D2), (D4), (D5), (D7), (D8), (\text{MP}), (G)$ of $\text{PDL}$. By $\equiv$ we denote the equivalence in propositional logic.

$(D1)$ is deducible by standard method via extended axiom $(\text{Ax})^+: F, F, \Gamma$ whose finite cutfree derivation is constructed by recursion on plain complexity of $F$ (in particular we pass by $(\text{GEN})$ from $A, A$ to $[P]A, [P]A, A$).

$(D4)$ and $(D5)$ are trivial, while $(D2), (D7), (D8)$ are derivable as follows.

$(D2): [P] (A \rightarrow B) \rightarrow ([P] A \rightarrow [P] B) \equiv (\langle P \rangle (A \land [P] A) \lor [P] \langle P \rangle [P] B)$.

$(D7): [P^*] A \equiv A \land [P] [P^*] A$

$\equiv ([P^*] A \lor (A \land [P] [P^*] A)) \land ((P^*) A \lor [P] [P^*] A)$.

$(D8): [P^*] (A \rightarrow [P] A) \rightarrow (A \rightarrow [P^*] A)$

$\equiv [P^*] (A \land [P] A) \lor [P] [P^*] A$.  

$^6$Plain (finite) induction is sufficient for $[\ast]$-free derivations.
The validity of Remark 5 for \( \Pi := \langle \Pi \rangle \) and derivable (dual) \( (D3) : \langle P \rangle (A \lor B) \leftrightarrow ((P) A \lor (P) B) \) (cf. Footnote 3), e.g. like this:

\[
\begin{array}{c}
A_1, A_2, \ldots, A_n \equiv A_1 \lor A_2 \lor \cdots \lor A_n \\
\quad \equiv [P](A_1 \lor A_2 \lor \cdots \lor A_n) \quad \equiv [P](\neg (A_2 \lor \cdots \lor A_n) \rightarrow A_1) \\
\quad \equiv [P](\neg (A_2 \lor \cdots \lor A_n) \rightarrow [P]A_1) \quad \equiv [P]A_1 \lor (P)(A_2 \lor \cdots \lor A_n) \\
\quad \equiv [P]A_1 \lor (P)(A_2 \lor \cdots \lor A_n) \lor \Gamma \quad \equiv [P]A_1, (P)A_2, \ldots, (P)A_n, \Gamma
\end{array}
\]
2.3 Cut elimination procedure

2.3.1 Auxiliary sequent calculus $\text{SEQ}_{\omega}^{\text{PDL}}$

Definition 6 $\text{SEQ}_{\omega}^{\text{PDL}}$ is a modification of $\text{SEQ}_{\omega}^{\text{PDL}}$ that includes the following upgraded inferences $\langle \cup \rangle$, $[\cup]$, $\langle ; \rangle$, $[;]$.

\[
\begin{array}{c|c|c}
\langle \cup \rangle & \frac{\langle Q \rangle (P) A, \langle Q \rangle (R) A, \Gamma}{\langle Q \rangle (P \cup R) A, \Gamma} & \frac{\langle Q \rangle [P] A, \Gamma}{\langle Q \rangle [P \cup R] A, \Gamma}
\\
\langle ; \rangle & \frac{\langle Q \rangle (P; R) A, \Gamma}{\langle Q \rangle (P) A, \Gamma} & \frac{\langle Q \rangle [P; R] A, \Gamma}{\langle Q \rangle [P] A, \Gamma}
\end{array}
\]

Obviously these upgrades are still sound in $\text{PDL}$ and cut-free derivable in $\text{SEQ}_{\omega}^{\text{PDL}}$. Hence $\text{SEQ}_{\omega}^{\text{PDL}}$ and $\text{SEQ}_{\omega}^{\text{PDL}+}$ are proof theoretically equivalent. In the sequel for the sake of brevity we use old names $\langle \cup \rangle$, $[\cup]$, $\langle ; \rangle$, $[;]$ also for the corresponding upgrades.

2.3.2 Admissible refinements

Lemma 7 The following inferences are admissible in $\text{SEQ}_{\omega}^{\text{PDL}+}$ minus (Cut). Moreover, for any inversion $\langle \partial : \Delta \rangle$ involved we have $h(\partial^\omega) < h(\partial) + \omega$. In $\langle \text{GEN} \rangle$ we assume that $\vec{P} = P_1, \cdots, P_k$ ($k > 0$), $f_1, \cdots, f_n : [1, k] \rightarrow \{0, 1\}$ and $(\forall j \in [1, k]) \sum_{i=1}^{n} f_i(j) = 1$. Note that (GEN) is a particular case of $\langle \text{GEN} \rangle$. 

7
Proof. Induction on proof height and/or formula complexity. Cases (W), (C) are standard. Note that (C) with principal (GEN) is trivial, e.g.

\[ \frac{(\l_1 : A, B)}{(\varphi)A, (\varphi)A, [\varphi]B, \Gamma} \] (GEN) \quad \rightarrow \quad \frac{(\l_1^c : A, B)}{(\varphi)A, [\varphi]B, \Gamma} \] (GEN).

Case (GEN) is an obvious iteration of (GEN).

Cases (v)\_2^\circ, (\land)\_1^\circ, (\land)\_2^\circ are standard (and trivial) boolean inversions.

Case (\cup) (\[\coprod\]) analogous. We omit trivial case of principal inversion of (\cup) and show only the crucial cases of principal (GEN) (in simple form):

\[ \frac{(\l_1 : A, B, C)}{(P \cup R)A, (P \cup R)B, (P \cup R)C, \Gamma} \] (GEN) \quad \rightarrow \quad \frac{(\l_1^\circ : A, B, C)}{(P \cup R)A, (P \cup R)B, (P \cup R)C, \Gamma} \] (GEN)\_p.
Theorem 8 (Predicative cut elimination) The following is provable in $PA$ extended by transfinite induction up to Veblen-Feferman ordinal $\varphi_\omega(0) > \varepsilon_0$.  

2.3.3 Cut elimination proper

We adapt familiar predicative cut elimination techniques ([18], [4], [16], [8], [2]).
Any sequent derivable in $\text{SEQ}_{\omega_1}^{\text{pdl}}$ is derivable in $\text{SEQ}_{\omega_2}^{\text{pdl}}$ minus (Cut). Hence any PDL-valid sequent (formula) is derivable in the cut-free fraction of $\text{SEQ}_{\omega_2}^{\text{pdl}}$, and hence also in $\text{SEQ}_{\omega_1}^{\text{pdl}}$ minus (Cut).

**Proof.** Our cut elimination operator $\partial \hookrightarrow \mathcal{E}(\partial)$ satisfying $\deg(\mathcal{E}(\partial)) = 0$ is defined for any derivation $\partial$ in $\text{SEQ}_{\omega_1}^{\text{pdl}} + \text{minus (Cut)}$ by simultaneous transfinite recursion on $h(\partial)$ and ordinal cut-degree $\deg(\partial)$.

$$\deg(\partial) := \max\{0, \sup\{\sigma(C) + 1 : C \text { occurs as cut formula in } \partial\}\}$$

Namely, for any inference rule $(R) \neq \text{(Cut)}$ with

$$(\partial : \Gamma) = \frac{(\partial_1 : \Gamma_1)}{\Gamma} \quad \text{(R)}$$
$$\text{or} \quad (\partial : \Gamma) = \frac{(\partial : \Gamma)}{\Gamma} \frac{(\partial_1 : \Gamma_1) \quad (\partial_2 : \Gamma_2)}{(R)}$$

we respectively let

$$(\mathcal{E}(\partial) : \Gamma) = \frac{(\mathcal{E}(\partial_1) : \Gamma_1)}{\Gamma} \quad \text{(R)}$$
$$\text{or} \quad (\mathcal{E}(\partial) : \Gamma) = \frac{(\mathcal{E}(\partial_1) : \Gamma_1) \quad (\mathcal{E}(\partial_2) : \Gamma_2)}{\Gamma} \quad \text{(R)}$$
$$\text{or} \quad (\mathcal{E}(\partial) : \Gamma) = \frac{(\mathcal{E}(\partial_m) : \Gamma_m \quad \{m \geq 0\})}{\Gamma} \quad [\ast].$$

Otherwise, if $(R) = \text{(Cut)}$ with

$$(\partial : \Gamma \cup \Pi) = \frac{(\partial_1 : \Gamma \cup \Pi)}{\Gamma \cup \Pi} \quad \text{(Cut)}$$

then we stipulate

$$(\mathcal{E}(\partial) : \Gamma \cup \Pi) = \left[\mathcal{E} \left(\mathcal{R} \left(\frac{(\mathcal{E}(\partial_1) : \Gamma \cup \Pi)}{\Gamma \cup \Pi} \quad \text{(Cut)}\right)\right)\right] : \Gamma \cup \Pi$$

with respect to a suitable cut reduction operation $\partial \hookrightarrow \mathcal{R}(\partial)$ such that

$$\deg(\mathcal{R}(\partial)) < \deg(\partial) \text{ if } \deg(\partial_1) = \deg(\partial_2) = 0,$$

which makes $\mathcal{E}(\partial), \deg(\mathcal{E}(\partial)) = 0$, definable by induction on $\deg(\partial)$ and $h(\partial)$.

Now $\mathcal{R}(\partial)$ is defined for any

$$(\partial : \Gamma \cup \Pi) = \frac{(\partial_1 : \Gamma \cup \Pi)}{\Gamma \cup \Pi} \quad \text{(Cut)}$$
by following double induction on ordinal complexity of $C$ and $\max\{h(\partial_1), h(\partial_2)\}$, provided that $\deg(\partial_1) = \deg(\partial_2) = 0$.

1. Case $C = L$ and $\overline{C} = \overline{L}$ for $L \in \{x, \neg x\}$. This case is standard. Namely, we observe that $L$ is principal left-hand side cut formula only if $\langle \partial_1 : L, \Gamma \rangle$ with $\Gamma = \overline{L}, \Gamma'$. But then $\langle \partial_2 : \overline{L}, \Pi \rangle$ infers $\Gamma \cup \Pi = \overline{L}, \Gamma', \Pi$ by derivable weakening (W). Thus following Mints-style graphic presentation, we can construct $R(\partial)$ by (1) setting $L := \Pi$ for every non-principal predecessor of the left-hand cut formula $L$ while ascending $\partial_1 : L, \Gamma$ either up to its disappearance as a side formula of (GEN) – then we are done – or else up to its principal occurrence in (AX) $\overline{L}, L, \Gamma''$, which yields sequent $\overline{L}, \Pi, \Gamma''$ instead, followed in the latter case by (2) setting $\overline{L} := \overline{L}, \Gamma''$ for every non-principal predecessor of the right-hand cut formula $\overline{L}$ while ascending $\partial_2 : \overline{L}, \Pi$ up to its disappearance as a side formula of (GEN) or else up to any occurrence in a leaf. In both cases we eventually arrive at a correct derivation of $\Gamma \Rightarrow \Pi$, since in the case of principal occurrence of $\overline{L}$ in a modified axiom (AX) $\overline{L}, L, \Pi'$ we obtain another axiom (AX) $\overline{L}, \Gamma'', L, \Pi'$.

2. Case $C = A \lor B$ and $\overline{C} = \overline{A} \land B$. Use derivable inversions $(\lor)^\circ, (\land)^\circ, (\forall)^\circ$.

3. Case $C = \langle \overline{Q} \rangle P \cup R \Rightarrow A$ and $\overline{C} = [\overline{Q}] [P \cup R] \overline{A}$. Analogous reduction to (CUT)'s on $\langle \overline{Q} \rangle P$ and $\langle \overline{Q} \rangle R \Rightarrow A$ by derivable inversions $(\cup)^\circ, (\cup)\circ, (\forall)^\circ$.

4. Case $C = \langle \overline{Q} \rangle P \cup R \Rightarrow A$ and $\overline{C} = [\overline{Q}] [P \cup R] \overline{A}$. Immediate reduction to (CUT) on $\langle \overline{Q} \rangle P \Rightarrow A$ by derivable inversions $(\land)^\circ, (\land)^\circ$.

5. Case $C = \langle \overline{Q} \rangle F$ and $\overline{C} = [\overline{Q}] F \overline{\Gamma}$ where $\overline{Q} = Q_1 \ldots Q_n (n > 0)$ and $(\forall j \in [1, n]) (Q_j = p_j$ or $Q_j = P'_j)$, while $F \neq \langle \overline{Q} \rangle F'$. The reduction is either trivial, if $\partial_1 = (\text{AX})^\uparrow$, or else defined hereditarily with respect to left-hand side non-principal subcases like

$$
\partial_1 : \begin{array}{c}
(\partial_1 : C, \Gamma') \\
C, \Gamma
\end{array}
$$

with $\partial_2 : [p] \overline{A}, \Pi$, when we let

$$
\begin{array}{c}
\begin{array}{c}
(\partial_1 : C, \Gamma') \\
C, \Gamma
\end{array}
\end{array}
$$

$$
R(\partial) := \begin{array}{c}
(\partial_1 : C, \Gamma') \\
C, \Gamma
\end{array}
$$

or analogous non-principal subcases $\partial_1 : (\partial_1^{(m)} : C, \Gamma^{(m)})$...$(\forall m \geq 0)$.
as well as the following principal subcases 5 (a), 5 (b), 5 (c).

5 (a). $C = \left\langle \overrightarrow{Q} \right\rangle F = \left\langle \overrightarrow{Q} \right\rangle (P^*) A$
and
\[
\partial_1 : \frac{\left\langle \partial_1 : \left\langle \overrightarrow{Q} \right\rangle (P^*) A, \left\langle \overrightarrow{Q} \right\rangle (P^*) A, \Gamma \right\rangle}{\left\langle \overrightarrow{Q} \right\rangle (P^*) A, \Gamma} \text{ with } \partial_2 : \left[ \overrightarrow{Q} \right\rangle (P^*) A, \Pi. \text{ Let}
\]
\[
\mathcal{R}(\partial) :=
\]
\[
\left(\mathcal{R}(\partial^{\prime}) : \left\langle \overrightarrow{Q} \right\rangle (P^*) A, \Gamma \cup \Pi \right) \quad \left(\partial_2^{[\cdot]} : \left[ \overrightarrow{Q} \right\rangle (P^*) A, \Pi \right) \quad \text{(Cut)}
\]
\[
\partial^{\prime} : \frac{\left\langle \partial^{\prime}_1 : \left\langle \overrightarrow{Q} \right\rangle (P^*) A, \left\langle \overrightarrow{Q} \right\rangle (P^*) A, \Gamma \right\rangle}{\left\langle \overrightarrow{Q} \right\rangle (P^*) A, \Gamma \cup \Pi} \text{ (Cut)}
\]
\[
\text{where}
\]

5 (b). $C = \left\langle \overrightarrow{Q} \right\rangle F = \left\langle p \right\rangle A$
and
\[
\partial_1 : \frac{\left\langle \partial_1^{\prime} : A, \overrightarrow{B}, D \right\rangle}{\left\langle p \right\rangle A, \left\langle p \right\rangle \overrightarrow{B}, \left\langle p \right\rangle D, \Gamma} \quad \text{(Gen)} \quad \text{with } \partial_2 : \left[ \overrightarrow{p} \right\rangle A, \Pi. \text{ Then let}
\]
\[
\mathcal{R}(\partial) := \cdots \left(\partial_m^{\prime} : \left\langle p \right\rangle \overrightarrow{B}, \left\langle p \right\rangle D, \Gamma \cup \Pi \right) \cdots (\forall m \geq 0) \quad [\ast] \quad \text{where}
\]
\[
\partial_m^{\prime} :=
\]
\[
\left(\partial_1^{\prime} : A, \overrightarrow{B}, D \right) \quad \left\langle p \right\rangle A, \left\langle p \right\rangle \overrightarrow{B}, \left\langle p \right\rangle D, \Gamma \cup \Pi \quad \text{(Gen)} \quad \left(\partial_2^{[\cdot]} : \left[ \overrightarrow{p} \right\rangle A, \Pi \right) \quad \text{(Cut)}
\]
\[
\left(\overrightarrow{p} \right\rangle A, \left\langle p \right\rangle \overrightarrow{B}, \left\langle p \right\rangle D, \Gamma \cup \Pi \quad \text{(Cut)}
\]
\[
\left(\overrightarrow{p} \right\rangle \overrightarrow{B}, \left\langle p \right\rangle D, \Gamma \cup \Pi \quad [\ast]
\]

5 (c). $C = \left\langle \overrightarrow{Q} \right\rangle F = \left\langle p \right\rangle A$
and
\[
\partial_1 : \frac{\left\langle \partial_1^{\prime} : A, \overrightarrow{B}, D \right\rangle}{\left\langle p \right\rangle A, \left\langle p \right\rangle \overrightarrow{B}, \left\langle p \right\rangle D, \Gamma} \quad \text{(Gen)} \quad \text{with } \partial_2 : \left[ \overrightarrow{p} \right\rangle A, \Pi. \text{ Then we let}
\]
\[
\mathcal{R}(\partial) := \left(\mathcal{R}(\partial^{\prime}, \partial_2) : \left\langle p \right\rangle \overrightarrow{B}, \left\langle p \right\rangle D, \Pi \right) \quad \text{(W)}
\]
where \( \mathcal{R}'(\partial_1, \partial_2) \) is defined by induction on \( h(\partial_2) \) – either trivially, if \( \partial_2 = (AX)^\top \), or hereditarily, in the non-principal subcases, while in the principal subcases

\[
\partial_2 : \frac{\left( \partial_1' : A, \overrightarrow{A}, \Pi' \right)}{[p] A, (p) \overrightarrow{A}, \Pi'} \quad \text{and} \quad \partial_2 : \frac{\left( \partial_2' : \overrightarrow{A}, \Pi' \right)}{[p] A, \Pi}
\]

we respectively let

\[
\begin{align*}
\mathcal{R}'(\partial_1, \partial_2) := & \quad \frac{\left( \partial_1' : A, \overrightarrow{B}, \Pi' \right)}{[p] A, (p) \overrightarrow{B}, \Pi'} \quad \text{(GEN)} \quad \text{and} \quad \frac{\left( \partial_2' : \overrightarrow{A}, \Pi' \right)}{[p] A, \Pi} \quad \text{(GEN)} \\
& \quad \frac{\left( \partial_2' : \overrightarrow{A}, \Pi' \right)}{[p] A, \Pi} \quad \text{(GEN)}
\end{align*}
\]

Obviously \( \mathcal{R} \) reduces the cut degree of \( \partial \). That is, in each case 1–5 we have \( \deg(\mathcal{R}(\partial)) < \deg(\partial) < \omega^\omega \), provided that both \( \partial_1 \) and \( \partial_2 \) involved are cut-free. Moreover, it’s readily seen that nodes in \( \mathcal{R}(\partial) \) can be augmented with ordinals such that

\[ h(\partial) < h(\partial_1) + h(\partial_2) + \omega < h(\partial) \cdot 2 + \omega. \]

Having this one can define ordinal assignments also for (slightly modified) cut-free derivations \( \mathcal{E}(\partial) \) such that for any \( \partial \) with \( \deg(\partial) < \omega^\omega \) it holds

\[ h(\mathcal{E}(\partial)) < \varphi(\alpha, h(\partial)) \]

which for \( \deg(\partial) < \omega^\omega \) and \( h(\partial) < \omega \cdot 2 \) (cf. Theorem 4) yields

\[ h(\mathcal{E}(\partial)) < \sup_{n < \omega} \varphi(n, \omega \cdot 2) = \varphi(\omega, 0) = \varphi_\omega(0) \]

(see Appendix A for a detailed presentation). It is readily seen that the entire proof is formalizable in \( \text{PA}_{\omega + (0)} \), i.e. \( \text{PA} \) extended by schema of transfinite induction along (canonical primitive recursive representation of) ordinal \( \varphi_\omega(0) \).

\[ \square \]

**Corollary 9** Let \( \Gamma \) be any sequent that does not contain occurrences \([P^*] \) and suppose that \( \Gamma \) is derivable in \( \text{SEQ}_{\omega}^{\text{PDL}} \). Then \( \Gamma \) is derivable in a subsystem of \( \text{SEQ}_{\omega}^{\text{PDL}} \) according to ordinal notations used in \( \mathcal{R} \).

\[ \varphi_\omega(0) = D(\omega^{\Omega+\omega}) \]
SEQ$_{\omega}$, called SEQ$_{1}^{\text{PDL}}$, that does not contain inferences $[*]$ and/or (Cut). Note that every derivation in SEQ$_{1}^{\text{PDL}}$ is finite. Consequently, any given $[P^*]$-free seq-formula is valid in PDL iff it is derivable in SEQ$_{1}^{\text{PDL}}$.

Proof. This is obvious by the subformula property of cutfree derivations.

Remark 10 Here and below we argue in PA$\varphi_{\omega}(0)$ that is a proper extension of PA, as $\varphi_{\omega}(0) > \varepsilon_{0}$. Actually by standard arguments the whole proof is formalizable in the corresponding primitive recursive weakening, PRA$\varphi_{\omega}(0)$.

2.4 Herbrand-style conclusions

Let $L_0$ be the star-free sublanguage of $L$. Denote by SEQ$_{0}^{\text{PDL}}$ the $L_0$-subsystem of SEQ$_{1}^{\text{PDL}}$.

Theorem 11 Let $\Sigma = \langle P^* \rangle A, \Pi \in L_0$. Suppose that $\Sigma$ is derivable in SEQ$_{\omega}^{\text{PDL}}$. Then there exists a $k \geq 0$ such that $\hat{\Sigma}_k := A, \langle p \rangle A, \cdots, \langle p \rangle^k A, \Pi$ is derivable in SEQ$_{0}^{\text{PDL}}$.

Proof. The nontrivial implication $\text{SEQ}_{\omega}^{\text{PDL}} \vdash \Sigma \Rightarrow \text{SEQ}_{0}^{\text{PDL}} \vdash \hat{\Sigma}_k$ follows by standard arguments from the cut elimination theorem by induction on the height of the corresponding finite cutfree proof $\vartheta$ of $\Sigma$ in SEQ$_{\omega}^{\text{PDL}}$. Since no $[P^*]$ occurs in $\Sigma$, no $\langle P^* \rangle A$ can be principal formula in (GEN). Thus the only crucial case is when some $\langle P^* \rangle A$ is principal formula in

\[
\begin{array}{c}
\langle * \rangle \langle P^* \rangle A, \langle P^* \rangle^m A, \Sigma \\
\hline
\langle* \rangle A, \Sigma
\end{array}
\]

which by the induction hypothesis yields $k$ such that $\langle P^* \rangle^m A, \hat{\Sigma}_k$ is derivable in SEQ$_{0}^{\text{PDL}}$. By (C) or (W) this yields the derivability of $\hat{\Sigma}_{k'}$ for $k' := \max (k, m)$.

Remark 12 By the same token, for any $[P^*]$-free seq-formula $F$, one can successively replace all subformulas $\langle P^* \rangle A$ by appropriate disjunctions $\bigvee_{i=0}^{k} \langle P \rangle^i A$ such that $F$ is PDL-valid iff the resulting expansion $\hat{F}$ is derivable in SEQ$_{0}^{\text{PDL}}$.

2.4.1 PSPACE refinement

Denote by $L_{00}$ a sublanguage of $L_0$ over atomic programs PRO$_0$ and let $L_0$ be purely propositional fraction of $L$. Note that program operations “;” and “[;” are definable in $L_{00}$ via $(P;Q)A := (P)(Q)A, (P \cup Q)A := (P)A \lor (Q)A$ and $[P \cup Q]A := [P]A \land [Q]A$. Let SEQ$_{00}^{\text{PDL}}$ be the following $L_{00}$-restriction of SEQ$_{0}^{\text{PDL}}$ (that proves the same $L_{00}$-sequents as SEQ$_{0}^{\text{PDL}}$).
Remark 13 Theorems 4 and 8 confirm that PDL is a conservative extension of classical propositional logic that is formalized by the (Gen)-free subsystem of Seq\textsuperscript{pdl}_0. Thus any L\textsubscript{0}-formula A is derivable in Seq\textsuperscript{pdl}_0 if it is valid in propositional logic, and hence, by contraposition, Seq\textsuperscript{pdl}_0 \nvdash A iff \nexists A (\therefore \neg A is satisfiable).

Lemma 14 (p-inversion) Suppose that \[ \Delta = (\text{Ax}) x, \neg x, \Gamma \]

\begin{align*}
\frac{(\lor)}{A, B, \Gamma} & \quad \frac{(\land)}{A, \Gamma} \quad \frac{B, \Gamma}{A \land B, \Gamma} \\
\frac{(\text{Gen})}{A_1, \ldots, A_n, \Gamma} & \quad \frac{(p)\chi_x A_1, \ldots, (p)\chi_x A_n, \Gamma}{(n > 0)} \\
& \quad \text{if } \sum_{i=1}^{n} \chi_i = 1.
\end{align*}

Then so is either \( \Delta \) or \( A_i, B_1, \ldots, B_k \), for some \( i \in [1, j] \), without increasing the height of the former derivation.

Proof. By straightforward induction on the derivation height. In the crucial principal case we have

\begin{align*}
\frac{(\text{Gen})}{A_i, \Delta} & \quad \frac{(\text{Ax})}{(p)A_1, \ldots, (p)A_k, (p)B_1, \ldots, (p)B_l} \\
& \quad \text{where } 0 < i \leq k \text{ and } \Delta \subseteq B_1, \ldots, B_i, \text{ which by derivable (W) yields the required derivability of } A_i, B_1, \ldots, B_l. \quad \blacksquare
\end{align*}

Theorem 15 The derivability in Seq\textsuperscript{pdl}_0 is a PSPACE problem. \[ \footnote{Apparently this result is well-known in the context of multimodal version of K.} \]

Proof. For the sake of brevity we consider L\textsubscript{0} formulas containing at most one atomic program \( p = \pi_0 \). Furthermore, we refine the notion of Seq\textsuperscript{pdl}_0 derivability by asserting that a sequent \( \Delta \not\vdash (\text{Ax}) \) is the conclusion of a rule (R) if one of the following priority conditions 1–3 is satisfied.

1. (R) = (\lor).
2. (R) = (\land) and no disjunction \( A \lor B \) occurs as formula in \( \Delta \); thus \( \Delta \) is not a conclusion of any (\lor).
3. No disjunction \( A \lor B \) or conjunction \( A \land B \) occurs as formula in \( \Delta \). Thus \( \Delta \) is not a conclusion of any (\lor) or (\land), i.e. \( \Delta = (p)\xi F_1, \ldots, (p)\xi F_n \) for \( \sum_{i=1}^{n} \xi_i \geq 1 \). In this case we stipulate that \( \Delta \) is the conclusion of (R) if one of the following two conditions holds:
(a) \( \sum_{i=1}^{n} \xi_i = 1 \) and \( F_1, \cdots, F_n \) is the premise of \((R) = (\text{Gen})\).

(b) \( \sum_{i=1}^{n} \xi_i > 1 \) and there exists \( j \in [1, n] \) with \( \xi_j = 1 \) such that either \( \Delta(j) := F_j \cup \{ F_l \in \Delta : \xi_l = 0 \} \) or \( \Delta(-j) := \Delta \setminus \{ F_j \} \) is the premise of \((R)\). (Note that we have \((R) = (\text{Gen})\) and \((R) = (\text{W})\) in the former and in the latter case, respectively.)

Having this we consider derivations in the refined \( \text{Seq}^{pdl}_{00} \) as at most binary-branching trees \( \partial \) whose nodes are labeled with sequents of \( \mathcal{L}_{00} \). Actually, for any given \( \mathcal{L}_{00} \)-sequent \( \Sigma \) it will suffice to fix one distinguished proof search tree \( \partial_0 \) with root sequent \( \Sigma \) that is defined by bottom-up recursion while applying the conditions 1–3 in a chosen order as long as possible. It is readily seen by inversions in Lemmata 7, 14 that \( \Sigma \) is derivable in \( \text{Seq}^{pdl}_{00} \) iff \( \partial_0 \) proves \( \Sigma \), i.e. every maximal path in \( \partial_0 \) is locally correct with respect to 1–3. Moreover, by the obvious subformula property we conclude that the depth, \( d(\partial_0) \), and maximum sequent length, \( \max \{|\Delta| : \Delta \in \partial_0\} \), of \( \partial_0 \) are both proportional to \( |\Sigma| \). Hence every maximal path in \( \partial_0 \) can be encoded by a \( \mathcal{L}_{01} \)-string of the length proportional to \( |\Sigma| \) whose local correctness is verifiable by TM in \( O(|\Sigma|) \) space. The corresponding universal verification runs by counting all maximal paths successively, still in \( O(|\Sigma|) \) space, which completes the proof.

\textbf{Remark 16} Arguing along more familiar lines we can turn \( \partial_0 \) into a Boolean circuit with (binary) AND, OR and (unary) ID gates, where ID \((x) := x \) for \( x \in \{0, 1\} \), such that AND, OR and ID correspond to the above conditions 2, 3 (b) and 1 and/or 3 (a), respectively. The corresponding truth evaluations \( \text{val}(-) \) are defined as usual via \( \text{val}(\Delta) := 1 \) (true) iff \( \Delta = (\text{Ax}) \), for every leaf \( \Delta \). Then \( \text{val}(\Sigma) = 1 \) iff \( \partial_0 \) proves \( \Sigma \), as required.

\textbf{2.4.2 Special cases}

Recall that by (a particular case of) Theorem 11, for any \( \Sigma = \langle p^* \rangle A, \Pi \) with \( A \in \mathcal{L}_{00}, \Pi \in \mathcal{L}_{0} \) the following holds. Suppose that \( \Sigma \) is derivable in \( \text{Seq}_{00}^{pdl} \). Then there exists a \( k \geq 0 \) such that \( \hat{\Sigma}_k := A, \langle p \rangle A, \cdots, \langle p \rangle^k A, \Pi \) is derivable in \( \text{Seq}_{00}^{pdl} \). It turns out that in some cases it’s possible to estimate the minimum \( k \) and hence the size of \( \hat{\Sigma}_k \).

\textbf{Definition 17} Let \( p = \pi_0 \) be fixed. Call basic conjunctive normal form (abbr.: BCNF) any \( \mathcal{L}_{00} \)-formula \( \bigwedge_{i=1}^{m} \left( B_i \lor \langle p \rangle C_i \lor \bigvee_{j=1}^{n_i} \langle p \rangle D_{i,j} \right) \) for \( m > 0, n_i \geq 0 \) and \( B_i, C_i, D_{i,j} \in \mathcal{L}_{0} \cup \{ \emptyset \} \). Formulas \( \langle p^* \rangle A \lor Z \) for \( A \in \text{BCNF} \) and \( Z \in \mathcal{L}_{0} \) are called basic conjunctive normal expressions (abbr.: BCNE).

\textsuperscript{9}This proof is dual to familiar proof of polynomial space solvability of QSAT (cf. e.g. \cite{14}).
Theorem 18  Let $A = \bigwedge_{i=1}^{m} \left( B_i \lor (p)C_i \lor \bigvee_{j=1}^{n_i} [p]D_{i,j} \right) \in \text{BCNF}$ and for any $k \geq 0$ and $\Pi \in L_0$ let $\hat{A}_k := A, \langle p \rangle A, \cdots, \langle p \rangle^k A, \hat{A}_0 = A$, and $\hat{\Sigma}_k := \hat{A}_k, \Pi$. If any $\hat{\Sigma}_k$ is derivable in $\text{SEQ}^{\text{FDL}}_{00}$ then so is $\hat{\Sigma}_{n+1}$ too, where $n = \sum_{i=1}^{m} n_i$.

Proof. For $i \in [1, m]$ let $\Delta_i := \{[p] D_{i,j} : 1 \leq j \leq n_i \}$. So Lemma 14 yields

\[ \vdash \hat{\Sigma}_0 \iff \vdash A, \Pi \iff \bigwedge_{i=1}^{m} \vdash B_i, (p)C_i, \Delta_i, \Pi \iff \]

"For every $i \in [1, m]$, either $\vdash B_i, \Pi$ or there is $j \in [1, n_i]$ such that $\vdash C_i, D_{i,j}$"

where "$\vdash$" stands for "$\text{SEQ}^{\text{FDL}}_{00} \vdash$", and hence

$\not\vdash \hat{\Sigma}_0 \iff \not\vdash A, \Pi \iff$

"There is $i \in [1, m]$ such that $\not\vdash B_i, \Pi$ and for every $j \in [1, n_i]$, $\not\vdash C_i, D_{i,j}$".

By the same token, for any $s \geq 0$ we let $\langle p \rangle \hat{A}_s := \langle p \rangle (A \lor \langle p \rangle A \lor \cdots \lor \langle p \rangle^s A)$ and arrive at

\[ \vdash \hat{\Sigma}_{s+1} \iff \vdash \hat{A}_{s+1}, \Pi \iff \vdash A, \langle p \rangle \hat{A}_s, \Pi \]

\[ \iff \bigwedge_{i=1}^{m} \vdash B_i, \langle p \rangle C_i, \Delta_i, \langle p \rangle \hat{A}_s, \Pi \iff \]

"For every $i \in [1, m]$, either $\vdash B_i, \Pi$ or there is $j \in [1, n_i]$ such that $\vdash \hat{A}_s, C_i, D_{i,j}$"

which yields

$\not\vdash \hat{\Sigma}_{s+1} \iff \not\vdash \hat{A}_{s+1}, \Pi \iff$

"There is $i \in [1, m]$ such that $\not\vdash B_i, \Pi$ and for every $j \in [1, n_i]$, $\not\vdash \hat{A}_s, C_i, D_{i,j}$".

Thus for any $k \geq 0$, the assertion $\not\vdash \hat{\Sigma}_k$ is equivalent to the existence of a labeled rooted refutation tree $T_k$ of the height $k + 1$ such that the following conditions 1–3 hold, where sequents $\ell(x)$ are the labels of nodes $x \in T_k$ ($\rho$ being the root).

1. $\ell(\rho) = \Pi$.

2. $\not\ell(\ell(x))$ holds for every leaf $x \in T_k$.
3. For any inner node $x \in T_k$ there exists $i \in [1, m]$ such that $x$ has $m_i + 1$
ordered children: $x_0$ (the son) with label $\ell(x_0) = B_i, \ell(x)$ and $x_1, \cdots, x_m,$
(the daughters) labeled $\ell(x_j) = C_i, D_i,j$, respectively; moreover $x_j$ ($j \geq 0$)
is a leaf iff it is either a son or else a daughter of the depth $k + 1$.

Such $T_k$ is easily obtained by straightforward geometric interpretation of our
translations of the conditions $\not\proves\Sigma_0$ and $\not\proves\Sigma_{n+1}$. Moreover, condition 2 above is
equivalent to

$2^\ast$. $\not\proves\ell(x)$ holds for every node $x \in T_k$,

since daughters are subsequents of their sons. That is, every inner node
$x \in T_k$ with label $\ell(x)$ has an upper neighbor (son) $x_0$ with label $B_i, \ell(x)$ being
a leaf in $T_k$, which by condition 2 yields $\not\proves B_i, \ell(x)$, and hence also $\not\proves \ell(x)$, as
required, by converting the admissible weakness rule of $\text{SEQ}^\text{pdl}$.

Now if $k \leq n + 1$ then $\hat{\Sigma}_k \subseteq \hat{\Sigma}_{n+1}$, and hence $\proves \hat{\Sigma}_k$ implies $\proves \hat{\Sigma}_{n+1}$. Furthermore, assuming $\not\proves \hat{\Sigma}_{n+1}$ we’ll infer $(\forall s > n) \not\proves \hat{\Sigma}_s$ and conclude by contraposition
that $(\exists k) \not\proves \hat{\Sigma}_k$ implies (in fact is equivalent to) $\proves \hat{\Sigma}_{n+1}$, as required. So assume $\not\proves \hat{\Sigma}_{n+1}$. We prove by induction on $s > n$ the existence of the refutation trees $T_s$, and hence $\not\proves \hat{\Sigma}_n$. Basis case $s = n + 1$ holds by the assumption. To pass
from $T_s$ to $T_{s+1}$ we argue as follows. Let $x \in T_s$ be any leaf-daughter and
$\theta = (\rho, y_1, \cdots, y_s = x)$ the corresponding maximal path, in $T_s$. Since $\theta$ contains
at most $n < s$ different labels $\ell(y_i) = C_i, D_i,j$ ($i \in [1, m], j \in [1, n_i]$), there exist a (say, minimal) pair $0 < r < t < s$ such that $\ell(y_r) = \ell(y_t)$. Let $T_{(s, x, r, t)}$ be a tree that arises from $T_s$ by substituting its subtree rooted in $y_r$ for that rooted in $y_t$. Note that $T_{(s, x, r, t)}$ is higher than $T_s$ -- so let $T_{(s+1, x)}$ be a subtree of $T_{(s, x, r, t)}$ consisting of the nodes of heights $\leq s + 1$. Proceeding this way successively with respect to all leaf-daughters $x \in T_s$ while keeping in mind condition 2*, we eventually obtain a refutation tree $T_{s+1}$ of the height $s + 1$, as required. ■

By Remark 10 and Theorem 11, the following are provable in $\text{PRA}_{\omega_1}(0)$.

**Corollary 19** Let $A \in \text{BCNF}$, $n$ and $\Pi$ be as above. Then $\Sigma := (p^\ast)A, \Pi$ is
derivable in $\text{SEQ}^\text{pdl}$ iff $\hat{\Sigma}_{n+1} := \hat{A}_{n+1}, \Pi$ is derivable in $\text{SEQ}^\text{pdl}$.

**Corollary 20** Let $S \in \text{BCNE}$. Problem $\text{PDL} \proves S$, i.e. $\text{PDL}$-validity of $S$, is
solvable by a deterministic TM in $O \left( |S|^2 \right)$ space.

**Proof.** For $A$ as above we have $n < |A|$, and hence $|\hat{A}_{n+1}| = O \left( |A|^2 \right)$. This
yields $|\hat{A}_{n+1}, Z| = O \left( |A|^2 + |Z| \right) = O \left( |S|^2 \right)$. Now by Theorem 4 followed by
Theorems 11, 18 we have

\[
\text{PDL} \proves S \iff \text{SEQ}^\text{pdl} \proves S \iff \text{SEQ}^\text{pdl} \proves \hat{A}_{n+1}, Z
\]

while problem $\text{SEQ}^\text{pdl} \proves \hat{A}_{n+1}, Z$ is solvable in $O \left( |\hat{A}_{n+1}, Z| \right) = O \left( |S|^2 \right)$ space.

■

Now consider (dual) basic disjunctive normal forms.
Definition 21 Call basic disjunctive normal form (abbr.: BDNF) any \( \mathcal{L}_{00} \)-formula \( F \lor \bigvee_{i=1}^{s} (F_i \land [p] G_i) \lor \bigvee_{j=1}^{t} (F_j \land (p) H_j) \) for \( s,t > 0 \) and \( F, F_i, G_i, H_j \in \mathcal{L}_0 \cup \{\emptyset\} \). Formulas \( \langle p^* \rangle A \lor Z \) for \( A \in \text{BDNF} \) and \( Z \in \mathcal{L}_0 \) are called basic disjunctive normal expressions (abbr.: BDNE).

Problem 22 Let \( S \in \text{BDNE} \). Is problem \( \text{PDL} \vdash S \) solvable by a TM in \(|S|\)-polynomial space?

2.4.3 More on BDNE

PDL-satisfiability problem for certain statements \( \text{ACCEPTS}_{M,x} = \langle p^* \rangle V \land W \) for \( V \in \text{BCNF}, W \in \mathcal{L}_0 \) – expressing that satisfying Kripke frames encode accepting computations of polynomial-space alternating TM – is known to be \( \text{EXPTIME} \)-complete (cf. \cite{1} and \cite{10}; Theorem 8.5, et al; see also \cite{20}). Hence so is also the PDL-validity problem for the corresponding negations \( S := \text{ACCEPTS}_{M,x} = \langle p^* \rangle A \lor Z \in \text{BDNE} \). So the affirmative solution to Problem 22 would infer \( \text{EXPTIME} = \text{PSPACE} \) (and vice versa, since general PDL-validity is EXPTIME-complete). That is, problem \( \text{EXPTIME} = \text{PSPACE} \) reduces to a particular case of Problem 22 for \( S := \overline{\text{ACCEPTS}_{M,x}} \) (see Appendix B for precise definition).

Now let \( S = \langle p^* \rangle A \lor Z \in \text{BDNE} \) for \( A = F \lor \bigvee_{i=1}^{s} (F_i \land [p] G_i) \lor \bigvee_{j=1}^{t} (F_j \land (p) H_j) \in \text{BDNF} \) and \( Z \in \mathcal{L}_0 \). We wish to present the assertion \( \text{PDL} \vdash S \) in a suitable “transparent” quantified boolean form. To this end, by DeMorgan laws, we first convert \( A \) to \( R = \bigwedge_{\xi \in \Xi} R_\xi \in \text{BCNF} \), where \( R_\xi = B_\xi \lor (p) C_\xi \lor \bigvee_{j \in J_\xi} [p] D_{\xi,j} \) for \( \Xi := \{\xi = (\xi(1), \cdots, \xi(s + t))\} \) with \( \xi(k) \in \{1,2\}, 1 \leq k \leq s + t \), while

\[
B_\xi := F \lor \bigvee \{F_k : 1 \leq k \leq s + t \land \xi(k) = 1\}, \\
C_\xi := \bigvee \{H_{k-s} : s < k \leq t \land \xi(k) = 2\}, \\
D_{\xi,j} := G_j \text{ for } j \in J_\xi := \{k : 1 \leq k \leq s \land \xi(i) = 2\}.
\]

Clearly \( \text{PDL} \vdash A \Leftrightarrow R \) (also by PDL-equivalence \( \langle p \rangle H \lor (p) H' \Leftrightarrow \langle p \rangle (H \lor H') \)). Note that \(|\Xi| = 2^{s+t}\) and \(|R_\xi| < |A|\), for every \( \xi \in \Xi \).

By the cut-elimination theorem, \( \text{PDL} \vdash S \) is equivalent to \( \text{SEQ}_{\omega}^{\text{PDL}} \vdash \langle p^* \rangle R, Z \), which by Theorem 18 is equivalent to \( \text{SEQ}_{00}^{\text{PDL}} \vdash \widehat{R}_{n+1}, Z \), where

\[
\widehat{R}_{n+1} = R, (p) R, \cdots, (p)^{n+1} R
\]

for \( n := \sum_{\xi \in \Xi} |J_\xi| < s \cdot |\Xi| = s2^{s+t} \). Arguing as in the proof of Theorem 18 we get

\[
\text{PDL} \vdash S \Leftrightarrow \text{SEQ}_{00}^{\text{PDL}} \vdash \widehat{R}_{n+1}, Z \Leftrightarrow f(s2^t + 1, Z) = 1
\]

where \( f \) is a boolean-valued binary function that is defined for every \( i \geq 0 \) and propositional formula \( X \) by the following recursive clauses 1–2, where “\( \vdash_\emptyset Y \)” stands for plain boolean validity of propositional formula \( Y \).
1. \[ f(0, X) = 1 \iff \bigwedge_{\xi \in \Xi} \left( \vdash_\emptyset (B_\xi \lor X) \text{ or } \bigvee_{j \in \xi} \vdash_\emptyset (C_\xi \lor D_{\xi,j}) \right) \]

2. \[ f(t + 1, X) = 1 \iff \bigwedge_{\xi \in \Xi} \left( \vdash_\emptyset (B_\xi \lor X) \text{ or } \bigvee_{j \in J_\xi} f(t, C_\xi \lor D_{\xi,j}) = 1 \right) \]

Note that every "\( \vdash_\emptyset Y \)" involved is expressible in quantified boolean logic as \( \forall x_1 \cdots \forall x_q Y \), where \( \{x_1, \ldots, x_q\} \) is the set of propositional variables occurring in \( Y \). Having this, by recursion on \( t \) with respect to clauses 1–2 we obtain a desired “transparent” quantified boolean formula \( \hat{S} \) such that

\[
PDL \vdash S \iff f(s2^{t+1} + 1, Z) = 1 \iff QBL \vdash \hat{S}
\]

(QBL being the canonical proof system for quantified boolean logic).

**Remark 23** The size of \( \hat{S} \) is exponential in that of \( S \), \( ^{10} \) whereas quantified boolean validity (and/or satisfiability) is known to be \( \text{PSPACE} \)-complete (cf. e.g. [14]). Hence \( \text{EXPTIME} = \text{PSPACE} \) holds if \( \hat{S} \) is equivalent with another quantified boolean formula whose size is polynomial in the size of \( S \). Moreover, this holds true of \( S := \overline{\text{Accepts}}_{M,x} \) (see above and Appendix B).

### 2.5 Conclusion

Soundness and completeness together with full cut elimination [Theorems 4, 8] in semiformal (infinite) sequent calculus \( \text{SEQ}^\text{pdl}_\omega \) shows that Hilbert-Bernays-style proof system \( \text{PDL} \) is a conservative extension of formal (finite) cutfree sequent calculi \( \text{SEQ}^\text{pdl}_0 \varsubsetneq \text{SEQ}^\text{pdl}_1 \varsubsetneq \text{SEQ}^\text{pdl}_0 \). This is in contrast to analogous polynomial BCNE case, see Corollary 20.
**SEQ**{$^0_{00}$} :=

| Rule | Premises | Conclusions |
|------|----------|-------------|
| (Ax) | $x, \neg x, \Gamma$ |
| (V)  | $A, B, \Gamma$ | $A \lor B, \Gamma$ |
| (L)  | $A, \Gamma$ | $B, \Gamma$ |
| (Gen) | $A_1, \ldots, A_n$ | $(p)_\chi A_1, \ldots, (p)_\chi A_n, \Gamma$ (n > 0) if $\sum_{i=1}^n \chi_i = 1$ |

**SEQ**{$^0_{01}$} :=

| Rule | Premises | Conclusions |
|------|----------|-------------|
| (Ax) | $x, \neg x, \Gamma$ |
| (V)  | $A, B, \Gamma$ | $A \lor B, \Gamma$ |
| (L)  | $A, \Gamma$ | $B, \Gamma$ |
| (U)  | $(P)A, (R)A, \Gamma$ | $(P \cup R)A, \Gamma$ |
| [U]  | $[P]A, \Gamma$ | $[R]A, \Gamma$ |
| (;)  | $(P|R)A, \Gamma$ | $[P; R]A, \Gamma$ |
| (Gen) | $A_1, \ldots, A_n$ | $(P)_\chi A_1, \ldots, (P)_\chi A_n, \Gamma$ (n > 0) if $\sum_{i=1}^n \chi_i = 1$ |

**SEQ**{$^0_{10}$} :=

| Rule | Premises | Conclusions |
|------|----------|-------------|
| (Ax) | $x, \neg x, \Gamma$ |
| (V)  | $A, B, \Gamma$ | $A \lor B, \Gamma$ |
| (L)  | $A, \Gamma$ | $B, \Gamma$ |
| (S)  | $(p)^m A, (p^*) A, \Gamma$ | $(p^*) A, \Gamma$ (m ≥ 0) |
| (Gen) | $A_1, \ldots, A_n$ | $(p)_\chi A_1, \ldots, (p)_\chi A_n, \Gamma$ (n > 0) if $\sum_{i=1}^n \chi_i = 1$ |
\[
\text{SEQ}^\text{pdl}_1 :=
\]

\[
\begin{array}{ll}
& (\text{AX}) \quad x, \neg x, \Gamma \\
& (\lor) \quad A, B, \Gamma \\
& \quad \frac{A \lor B, \Gamma}{A, \Gamma} \\
& (\land) \quad A, B, \Gamma \\
& \quad \frac{A \land B, \Gamma}{A, \Gamma} \\
& (\cup) \quad \Gamma \\
& \quad \frac{(P)\Gamma A, (R)\Gamma A, \Gamma}{(P \cup R)\Gamma A, \Gamma} \\
& (\vdash) \quad \Gamma \\
& \quad \frac{(P)(R)\Gamma A, \Gamma}{(P; R)\Gamma A, \Gamma} \\
& (\ast) \quad (Q)\Gamma A, (Q^\ast)\Gamma A, \Gamma \\
& \quad \frac{(Q^\ast)\Gamma A, \Gamma}{(P^\ast)\Gamma A, \Gamma} \\
& (\text{GEN}) \quad \Gamma \\
& \quad \frac{A_1, \ldots, A_n, \Gamma}{(P)_{X_1} A_1, \ldots, (P)_{X_n} A_n, \Gamma} \\
& \quad \Gamma \\
& \quad \Gamma \\
& \quad \Gamma \\
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& \quad \Gamma \\
\end{array}
\]

It is readily seen that \text{SEQ}^\text{pdl}_0 and \text{SEQ}^\text{pdl}_0 (resp. \text{SEQ}^\text{pdl}_1 and \text{SEQ}^\text{pdl}_1) have the same provable sequents modulo basic interpretation of \text{FOR}_0 (\text{FOR}_1) within \text{FOR}_0 (\text{FOR}_1).

As usual in proof theory, our cutfree sequent calculi provide useful help in the verification of \text{PDL}-(un)provability of concrete formulas of simple shapes. Concerning general computational complexities the following holds.

1. The derivability (provability) in \text{SEQ}^\text{pdl}_0 is a PSPACE problem. The case of \text{SEQ}^\text{pdl}_1 is less clear as our interpretation of program connectives does not preserve polynomial size.

2. The derivability (provability) in \text{SEQ}^\text{pdl}_1 is EXPTIME-complete and in fact so is the derivability in \text{SEQ}^\text{pdl}_0, too.

3. The latter is characteristic also for a subclass BDNE (: “basic disjunctive normal expressions”) of \text{FOR}_1, whereas \text{SEQ}^\text{pdl}_0-derivability of dual BCNE expressions (: “basic conjunctive normal expressions”) turns out to be decidable in polynomial space. Moreover, \text{SEQ}^\text{pdl}_0-derivability (and hence \text{PDL}-provability) of BDNE is equivalent to \text{QBL}-validity of corresponding “transparent” quantified boolean formulas of exponential length.

Proofs of our claims use transfinite induction on predicative ordinal \( \varphi_\omega(0) \).

It is not clear yet whether conservative extension results (see above) are provable in Peano Arithmetic.

\textbf{Relevant papers} [13] formalized \text{PDL} as proof system \text{CSPDL} with \( \omega \)-rule for \( \ast \) that is based on \textit{hypersequents} (more precisely: \textit{zoom tree hypersequents}), rather than sequents. Sequent calculi proper are not exposed there. Consequently, conservative extension corollaries and complexity connections are not mentioned, either. It might appear, however, that the hypersequents were
chosen to allow “deep inferences”, i.e. transformations applied to subformulas of those explicitly shown (along the lines of Herbrand version [13] of Gentzen’s sequent calculus). Cut-elimination theorem is claimed for CSPDL but proof thereof is informal. Apparently it should follow by Schütte-style predicative pattern adapted to the hypersequents, instead of plain sequents. However no ordinal bounds on the height transformation is given and hence proof theoretic strength of the required logic formalism remains unclear. According to well-known predicative cut elimination in the presence of \( \omega \)-rule(s), one would expect it to be the same as in the present paper, provided that ordinal complexity of cut formulas with nested occurrences of starred programs has the same natural upper bound \( \omega^\omega \) (which does not explicitly follow from [13]: Definition 3.1).

[11] considered finite sequent calculus GPDL in an extended language with mixed formulas (possibly including atomic programs) and contextual sequents (whose antecedents and/or succedents might include program terms). It is claimed that all but analytic cuts in special form can be eliminated from GPDL derivations. There is no discussion of possible conservative extensions and/or computational complexity connections.

[9] presented a different approach in form of an optimal tableau-based EXPTIME algorithm for deciding satisfiability for PDL with converse (CPDL) without the use of analytic cuts. In order to decide the satisfiability of a given input formula \( \phi \) the algorithm builds a suitable directed graph \( G \) and checks the applicability of one of the four attached rules Rule 1, ..., Rule 4. There is no obvious translation into plain sequent calculus formalism.

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3 Appendix A: Ordinal assignments

3.1 Ordinal arithmetic

We use basic properties 1–8 of Veblen’s ordinals (abbr.: α, β, γ, δ) (\[21\], \[5\], \[16\]).

1. Basic relation $<$ is linearly ordered.

2. Symmetric sum is associative and commutative.
3. $0 < 1 = \omega^0$, $\omega = \omega^1$, $\omega^\beta = \varphi(0, \beta)$.

4. $\alpha + 0 = \alpha$, $\alpha < \beta \rightarrow \alpha + \gamma < \beta + \gamma + \delta$.

5. $\alpha < \beta \rightarrow \varphi(\alpha, \gamma) < \varphi(\beta, \gamma) \land \varphi(\gamma, \alpha) < \varphi(\gamma, \beta)$.

6. $\alpha \leq \beta < \varphi(\gamma, \delta) \rightarrow \alpha + \beta < \varphi(\gamma, \delta)$.

7. $\alpha < \beta \land \gamma < \varphi(\beta, \delta) \rightarrow \varphi(\alpha, \gamma) < \varphi(\beta, \delta) \land \varphi(\alpha, \varphi(\beta, \delta)) = \varphi(\beta, \delta)$.

8. $\alpha \leq \omega^\alpha$, $0 < \alpha \rightarrow \omega^\varphi(\alpha, \beta) = \varphi(\alpha, \beta)$.

$\varphi(\alpha, \beta)$ is also denoted by $\varphi_a(\beta)$. Note that $\varepsilon_0 = \varphi_1(0) < \varphi_\omega(0) < \Gamma_0$.

In the rest of this chapter we freely use these properties without explicit references.

### 3.2 Cut elimination $\partial \mapsto E(\partial)$

For the sake of brevity we’ll slightly refine our inductive definition of $E(\partial)$. To this end we upgrade $R$ to $R^+$:

$$(\partial : \Gamma \cup \Pi) = (R^+(\rho, \alpha, \partial : \Gamma) \leftarrow R^+(\rho, \alpha, \partial : \Gamma)) \quad (\text{Cut})$$

That is, for any $\rho > 0$, $\alpha$ and $\partial$ with $\deg(\partial) < \rho$ we define $(R^+(\rho, \alpha, \partial : \Gamma))$ such that $\deg(R^+(\rho, \alpha, \partial : \Gamma)) < \rho$ and $h(R^+(\rho, \alpha, \partial : \Gamma)) < \varphi(\alpha, h(\partial))$. Then for any $\partial$ with cuts we let

$$E(\partial) := R^+(1, \alpha, \partial), \text{ where } \alpha := \min\{\beta : \deg(\partial) < \omega^\beta\}$$

and conclude that $\deg(E(\partial)) = 0$ and $h(E(\partial)) < \varphi(\alpha, h(\partial))$.

Now $R^+(\rho, \alpha, \partial)$ is defined for any $\partial$ with $\deg(\partial) < \rho + \omega^\alpha$ as follows by double induction on $\alpha$ and $h(\partial)$. Let $(R)$ be the lowermost inference in $\partial$. If $(R)$ is not a (Cut) on $C$ with $\rho(C) + 1 \geq \rho$ then $R^+(\rho, \alpha, \partial)$ arises from $\partial$ by substituting $R^+(\rho, \alpha, \partial_i)$ for the lowermost subdeductions $\partial_i$ (recall that $h(\partial_i) < h(\partial)$). Otherwise, we have

$$(\partial : \Gamma \cup \Pi) = (\partial_1 : C, \Gamma) \text{ or } (\partial_2 : C, \Pi) \quad (\text{Cut})$$

where $\rho \leq \rho(C) + 1 \leq \deg(\partial) < \rho + \omega^\alpha$. Let

$$(\hat{\partial} : \Gamma \cup \Pi) := (R^+(\rho, \alpha, \partial_1 : C, \Gamma) \text{ or } \rho(C) + 1 \leq \deg(\partial) < \rho + 1)$$

and consider two cases.

**Case** $\alpha = 0$. Let $R^+(\rho, \alpha, \partial) = R^+(\rho, 0, \partial) := R(\hat{\partial})$. Recall that

$$\deg(R(\hat{\partial})) < \deg(\hat{\partial}) = \rho + 1$$

for the sake of brevity we’ll slightly refine our inductive definition of $E(\partial)$. To this end we upgrade $R$ to $R^+$:

$$(\partial : \Gamma \cup \Pi) = (R^+(\rho, \alpha, \partial : \Gamma) \leftarrow R^+(\rho, \alpha, \partial : \Gamma)) \quad (\text{Cut})$$

That is, for any $\rho > 0$, $\alpha$ and $\partial$ with $\deg(\partial) < \rho$ we define $(R^+(\rho, \alpha, \partial : \Gamma))$ such that $\deg(R^+(\rho, \alpha, \partial : \Gamma)) < \rho$ and $h(R^+(\rho, \alpha, \partial : \Gamma)) < \varphi(\alpha, h(\partial))$. Then for any $\partial$ with cuts we let

$$E(\partial) := R^+(1, \alpha, \partial), \text{ where } \alpha := \min\{\beta : \deg(\partial) < \omega^\beta\}$$

and conclude that $\deg(E(\partial)) = 0$ and $h(E(\partial)) < \varphi(\alpha, h(\partial))$.

Now $R^+(\rho, \alpha, \partial)$ is defined for any $\partial$ with $\deg(\partial) < \rho + \omega^\alpha$ as follows by double induction on $\alpha$ and $h(\partial)$. Let $(R)$ be the lowermost inference in $\partial$. If $(R)$ is not a (Cut) on $C$ with $\rho(C) + 1 \geq \rho$ then $R^+(\rho, \alpha, \partial)$ arises from $\partial$ by substituting $R^+(\rho, \alpha, \partial_i)$ for the lowermost subdeductions $\partial_i$ (recall that $h(\partial_i) < h(\partial)$). Otherwise, we have

$$(\partial : \Gamma \cup \Pi) = (\partial_1 : C, \Gamma) \text{ or } (\partial_2 : C, \Pi) \quad (\text{Cut})$$

where $\rho \leq \rho(C) + 1 \leq \deg(\partial) < \rho + \omega^\alpha$. Let

$$(\hat{\partial} : \Gamma \cup \Pi) := (R^+(\rho, \alpha, \partial_1 : C, \Gamma) \text{ or } \rho(C) + 1 \leq \deg(\partial) < \rho + 1)$$

and consider two cases.

**Case** $\alpha = 0$. Let $R^+(\rho, \alpha, \partial) = R^+(\rho, 0, \partial) := R(\hat{\partial})$. Recall that

$$\deg(R(\hat{\partial})) < \deg(\hat{\partial}) = \rho + 1$$
and hence $\deg(\mathcal{R}^+(\rho, \alpha, \partial)) = \deg(\mathcal{R}(\hat{\partial})) < \rho$. On the other hand

$$h\left(\mathcal{R}(\hat{\partial})\right) < h(\mathcal{R}^+(\rho, \alpha, \partial_1)) + h(\mathcal{R}^+(\rho, \alpha, \partial_2)) + \omega$$

$$\leq \omega h(\mathcal{R}^+(\rho, \alpha, \partial_1)) + \omega h(\mathcal{R}^+(\rho, \alpha, \partial_2)) + \omega$$

$$< \omega h(\hat{\partial}) = \varphi\left(0, h\left(\hat{\partial}\right)\right)$$

which yields $h(\mathcal{R}^+(\rho, \alpha, \partial)) = h\left(\mathcal{R}(\hat{\partial})\right) < \varphi\left(0, h\left(\hat{\partial}\right)\right)$, as desired.

**Case** $\alpha > 0$. Thus $\omega^a = \omega^{a_1} + \cdots + \omega^{a_n}$ for $\alpha > a_1 \geq \cdots \geq a_n$ (by Cantor’s normal form). In this case we apply inductive hypotheses successively for $a_1, \ldots, a_n$ and let

$$\mathcal{R}^+(\rho, \alpha, \partial) := \mathcal{R}^+(\rho, a_1, \mathcal{R}^+(\rho, a_2, \mathcal{R}^+(\cdots, \mathcal{R}^+(\rho_{n-1}, a_n, \partial))))$$

where $\rho_0 := \rho$ and $(\forall i > 0) \rho_{i+1} := \rho_i + \omega^{a_{i+1}}$. Then $\deg(\mathcal{R}^+(\rho, \alpha, \partial)) < \rho$ and $h(\mathcal{R}^+(\rho, \alpha, \partial)) < \varphi(\alpha_1, \varphi(\alpha_2, \varphi(\cdots, \varphi(\alpha_n, h(\partial)))))) < \varphi(\alpha, h(\partial))$, as desired.

### 3.3 Formalization

We fix a chosen “canonical” primitive recursive ordinal representation

$$\mathcal{O} = \langle 0, 1, \omega, <, +, \omega^(-), \varphi(-, -) \rangle$$

(also known as *system of ordinal notations*) in the language of $\text{PA}$ that is supposed to be well-ordered by $<$ up to $\varphi_\omega(0)$ (at least). To formalize the latter assumption we extend standard formalism of $\text{PA}$ by the transfinite induction axiom (schema) for arbitrary arithmetical formulas, $\text{TI}O(\varphi_\omega(0))$. The extended proof system is abbreviated by $\text{PA}_{\varphi_\omega(0)}$. Derivations $\hat{\partial}$ used in the proofs are interpreted as primitive recursive trees whose nodes $x$ are labeled with sequents and ordinals $\text{ord}(x) < \varphi_\omega(0)$. Having this it is easy to formalize in $\text{PA}_{\varphi_\omega(0)}$ the whole cut elimination proof; note that the operators $\hat{\mathcal{R}}$, $\hat{\mathcal{R}}^+$ and $\hat{\mathcal{E}}$ involved are constructively defined and $\text{TI}O(\varphi_\omega(0))$ is used in the corresponding termination-and-correctness proofs only. Actually we can restrict $\text{TI}O(\varphi_\omega(0))$ to primitive recursive induction formulas thus reducing $\text{PA}_{\varphi_\omega(0)}$ to $\text{PRA}_{\varphi_\omega(0)}$.

### 4 Appendix B: Formula Accepts

#### 4.1 Semantics

Consider a given polynomial-space-bounded $k$-tape alternating Turing machine $M$ on a given input $x$ of length $n$ with blanks over $M$’s input alphabet; $\vdash$ and $\dashv$ are the left and right endmarkers, respectively. Formula $\text{Accepts}_{M,x}$ involves

\[\text{This is a recollection of } [10]: 8.2.\]
the single atomic program `Next`, atomic propositions `Symbol^a_i` and `State^q_i` for each symbol `a` in `M`’s tape alphabet, `q` a state of `M`’s finite control, and `0 ≤ i ≤ n`, and an atomic proposition `Accept`. Then `Accepts_{M,x}` has the property that any satisfying Kripke frame encodes an accepting computation of `M` on `x`. In any such Kripke frame, states `u` represent configurations of `M` occurring in the computation tree of `M` on input `x = x_1, · · · , x_n`; the truth values of `Symbol^a_i` and `State^q_i` at state `u` give the tape contents, current state, and tape head position in the configuration corresponding to `u`. The truth value of `Accept` will be 1 iff the computation beginning in state `u` is an accepting computation according to the rules of alternating Turing machine acceptance. Then `M` accepts `x` iff `Accepts_{M,x}` is satisfiable. `Accepts_{M,x}` is EXPTIME-complete (cf. [10]: Theorem 8.5) and hence so is the negation `¬Accepts_{M,x}`.

4.2 Formal definition

Let `Γ` be `M`’s tape alphabet and `Q` the set of states; there is a distinguished start-state `s ∈ Q` and left/right annotations `ℓ, r ∈ Q`. Let `U ⊆ Q` and `E ⊆ Q` be the sets of universal and existential states, respectively. Thus `U ∪ E = Q` and `U ∩ E = ∅`. For each pair `(q, a) ∈ Q × Γ` let `Δ(q, a)` be the set of all triples describing a possible action when scanning `a` in state `q`. Working in `L` we let

```
Accept_{M,x} := \text{Acc} \land \text{Start} \land [\text{Next}^*](\text{Config} \land \text{Move} \land \text{Acceptance})
```

where `Acc(\text{ept})`, `State_{\va{l}^\wedge}` and `Symbol_{\va{l}^\wedge}` ∈ `VAR`, `Next` ∈ `PRO` while `Start`, `Config`, `Move` and `Acceptance` are defined as follows.

1. **Config** :=

   `State^0_0 \land \bigwedge_{1 ≤ i ≤ n} Symbol^x_i \land \bigwedge_{n+1 ≤ i ≤ n+1} Symbol^\square_i`.

2. **Config** :=

   `\bigwedge_{0 ≤ i ≤ n+1} \bigvee a \in Γ` (`Symbol^a_i \land \bigwedge_{a \neq b \in Γ} Symbol^b_i`) \land `Symbol^0_0 \land Symbol^n_{n+1}` \land

   `\bigvee_{0 ≤ i ≤ n+1} \bigwedge_{0 ≤ i ≤ n+1} q \in Q \cup \{ℓ, r\}` (`\bigwedge_{q \neq p \in Q \cup \{ℓ, r\}} State^q_i \land \bigwedge_{p \in Q \cup \{ℓ, r\}} State^p_i`) \land

   `\bigwedge_{0 ≤ i ≤ n} \bigwedge_{q \in Q \cup \{ℓ\}} (State^q_i \lor State^q_{i+1}) \land \bigwedge_{0 ≤ i ≤ n+1} \bigwedge_{q \in Q \cup \{r\}} (State^q_i \lor State^q_{i-1})`.

3. **Move** :=

   `\bigwedge_{0 ≤ i ≤ n+1} (State^q_i \lor State^q_{i+1}) \land \bigwedge_{a \in Γ} (Symbol^a_i \lor [\text{Next}] Symbol^a_i)` \land

   `\bigwedge_{0 ≤ i ≤ n+1} \bigwedge_{(p, b, d) \in Δ(q, a)} (Symbol^b_{i+d} \land State^p_{i+d}) \land`.

4. **Acceptance** :=

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\[
\left( \bigwedge_{0 \leq i \leq n+1} \text{STATE}^i \lor \left( \left( \text{ACC} \lor [\text{NEXT}] \neg \text{ACC} \right) \land \left( \neg \text{ACC} \lor [\text{NEXT}] \text{ACC} \right) \right) \right) \land \\
\left( \bigwedge_{0 \leq i \leq n+1} \text{STATE}^i \lor \left( \left( \text{ACC} \lor [\text{NEXT}] \neg \text{ACC} \right) \land \left( \neg \text{ACC} \lor [\text{NEXT}] \text{ACC} \right) \right) \right) \\
\text{Hence}
\]

\[
\overline{\text{ACCEPTS}}_{M,x} = \text{ACC} \lor \text{START} \lor (\text{NEXT}^*) \lor (\text{CONFIG} \lor \text{MOVE} \lor \text{ACCEPTANCE})
\]

is equivalent to \( \langle p^* \rangle A \lor Z \) for \( p = \text{NEXT}, Z = \text{ACC} \lor \text{START} \in \mathcal{L}_{00} \) and

\[
A = F_0 \lor (F_1 \land [p] G_1) \lor (F_2 \land [p] G_2) \lor \bigvee_{\alpha \in R} (F_\alpha \land [p] G_\alpha) \lor (F_3 \land [p] G_3) \\
\lor (F_4 \land [p] G_4) \lor \bigvee_{\beta \in T} (F_\beta \land [p] G_\beta) \lor \bigvee_{\gamma \in S} (F_\gamma \land [p] G_\gamma) \in \text{BDNF}
\]

where:

\( R = \{ \alpha = (i, a, q, (p, b, d)) \in [n+1] \times \Gamma \times Q \times \Delta(q, a) \}, \)

\( T = \{ \beta = (i, a) \in [n+1] \times \Gamma \}, \)

\( S = \{ \gamma = (i, a, q) \in [n+1] \times \Gamma \times Q \}, \)

\( F_0 = \bigvee_{0 \leq i \leq n+1} \bigwedge_{\ell \in \Gamma} \left( \text{Symbol}^i_\ell \lor \bigvee_{a \neq \ell} \text{Symbol}^i_a \right) \lor \text{Symbol}^i_0 \lor \text{Symbol}^i_{n+1} \)

\( \lor \bigvee_{0 \leq i \leq n+1} \bigwedge_{q \in Q} \left( \text{STATE}^i_q \lor \bigvee_{q \neq p \in Q \cup \{l, r\}} \text{STATE}^i_p \right) \lor \\
\lor \bigvee_{0 \leq i \leq n+1} \bigwedge_{q \in Q \cup \{l, r\}} \left( \text{STATE}^i_q \land \text{STATE}^i_{q+1} \right) \lor \bigvee_{0 \leq i \leq n+1} \bigwedge_{q \in Q \cup \{l, r\}} \left( \text{STATE}^i_q \land \text{STATE}^i_{q-1} \right) \\
F_1 = \bigwedge_{0 \leq i \leq n+1} \text{STATE}^i \land \text{ACC}, G_1 = \overline{\text{ACC}}, \\
F_2 = \bigwedge_{0 \leq i \leq n+1} \text{STATE}^i \land \text{ACC}, G_2 = \text{ACC}, \\
F_3 = \bigwedge_{0 \leq i \leq n+1} \text{STATE}^i \land \text{ACC}, G_3 = \text{ACC}, \\
F_4 = \bigwedge_{0 \leq i \leq n+1} \text{STATE}^i \land \text{ACC}, G_4 = \overline{\text{ACC}}, \\
F_\alpha = \text{Symbol}^i_\alpha \land \text{STATE}^i_\alpha, G_\alpha = \overline{\text{Symbol}^i_\alpha} \lor \text{STATE}^i_{\alpha+d}, \\
F_\beta = \text{STATE}^i_\beta \land \text{Symbol}^i_\beta, G_\beta = \overline{\text{Symbol}^i_\beta}, \\
F_\gamma = \text{Symbol}^i_\gamma \land \text{STATE}^i_\gamma, G_\gamma = \bigvee_{(p, b, d) \in \Delta(q, a)} (\text{Symbol}^i_\gamma \lor \text{STATE}^i_{\gamma+d}) \\
\text{Note that } |\langle p^* \rangle A \lor Z| \text{ is at most quadratic in } |\overline{\text{ACCEPTS}}_{M,x}|.\]