AUTOMIZERS AS EXTENDED REFLECTION GROUPS

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1. Introduction

Broué, Malle and Michel have shown that the automizer of an abelian Sylow $p$-subgroup $P$ in a finite simple Chevalley group is an irreducible complex reflection group (for $p$ not too small and different from the defining characteristic) [BrMaMi, BrMi].

The aim is this note is to show that a suitable version of this property holds for general finite groups.

We give a simple direct proof, building on the Lehrer-Springer theory [LeSp], that the property above holds for simply connected simple algebraic groups $G$, provided $p$ is not a torsion prime (Proposition 4.1): the automizer $E = N_G(P)/C_G(P)$ is a reflection group on $\Omega_1(P)$, the largest elementary abelian subgroup of $P$.

On the other hand, we show that the presence of $p$-torsion in the Schur multiplier of a finite group $G$ prevents the subgroup of $E$ generated by reflections from being irreducible (Proposition 3.5).

This suggests considering covering groups of finite simple groups or equivalently finite simple groups $G$ such that $H^2(G, F_p) = 0$. We also need to allow $p'$-automorphisms and we now look for a description of the automizer as an extension of an irreducible reflection group $W$ by a subgroup of $N_{GL(\Omega_1(P))}(W)/W$.

We actually need a slight generalization: $\Omega_1(P)$ should be viewed in some cases as a vector space over a larger finite field (for example in the case of $\text{PSL}_2(F_{p^n})$) and we need to allow field automorphisms.

As an example, the automizer of an 11-Sylow subgroup in the Monster is the 2-dimensional complex reflection group $G_{16}$.

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2. Notation and definitions

Let $p$ be a prime. Given $P$ an abelian group, we denote by $\Omega_1(P)$ the subgroup of $P$ of elements of order 1 or $p$, i.e., the largest elementary abelian $p$-subgroup of $P$.

Let $V$ be a free module of finite rank over a commutative algebra $K$. A reflection is an element $s \in \text{GL}_K(V)$ of finite order such that $V/\ker(s - 1)$ is a free $K$-module of rank 1 (note that we do not require $s^2 = 1$). A finite subgroup of $\text{GL}_K(V)$ is a reflection group if it is generated by reflections.

3. Main result and remarks

Let $p$ be a prime and $H$ a simple group such that the $p$-part of the Schur multiplier of $H$ is trivial, i.e. $H^2(H, F_p) = 0$. Assume $H$ has an abelian Sylow $p$-subgroup $P$. Let $\tilde{H} \leq \text{Aut}(H)$
be a finite group containing $H$ and such that $\tilde{H}/H$ is a Hall $p'$-subgroup of $\text{Out}(H)$. Let $E = N_{\tilde{H}}(P)/C_{\tilde{H}}(P)$.

**Theorem 3.1.** There is

- a finite field $K$
- an $\mathbb{F}_p$-subspace $V$ of $\Omega_1(P)$ and an isomorphism of $\mathbb{F}_p$-vector spaces $K \otimes_{\mathbb{F}_p} V \cong \Omega_1(P)$
- a subgroup $N$ of $\text{GL}_K(\Omega_1(P))$ and
- a subgroup $\Gamma$ of $\text{Aut}(K)$

such that $E = N \rtimes \Gamma$, as subgroups of $\text{Aut}(\Omega_1(P))$, and such that the normal subgroup $W$ of $N$ generated by reflections acts irreducibly on $\Omega_1(P)$.

The theorem will be proven in \[\text{[4]}\]

**Remark 3.2.** Gorenstein and Lyons have shown that $N_{\tilde{H}}(P)/C_{\tilde{H}}(P)$ acts irreducibly on $\Omega_1(P)$ viewed as a vector space over $\mathbb{F}_p$ and, as a consequence, $P$ is homocyclic \[\text{[GoLy] (12.1)}\].

Note nevertheless that the subgroup of $N_{\tilde{H}}(P)/C_{\tilde{H}}(P)$ generated by reflections might not be irreducible in its action on $\Omega_1(P)$: this happens for example in the case $H = \mathfrak{A}_{2p}$, $p > 3$.

We can take $K = \mathbb{F}_p$ in Theorem 3.1 except for

- $\text{PSL}_d(p^n)$, $n > 1$: $K = \mathbb{F}_{p^n}$
- $J_1$ and $2G_2(q)$, $p = 2$: $K = \mathbb{F}_8$.

In those cases, $V = \mathbb{F}_p$ and $P = \Omega_1(P) = K$.

Note that the theorem is trivial when $P$ is cyclic: one takes $K = \mathbb{F}_p$ and $N = E = W \subset \mathbb{F}_p^\times$.

Using the classification of finite simple groups, we deduce a statement about finite groups with abelian Sylow $p$-subgroups.

**Corollary 3.3.** Let $G$ be a finite group with an abelian Sylow $p$-subgroup $P$. Let $H = O^p(G/O_p(G))$.

Assume the $p$-part of the Schur multiplier of $H$ is trivial. Then, there is a finite group $X$ containing $H$ as a normal subgroup of $p'$-index and

- a product $K$ of finite field extensions of $\mathbb{F}_p$
- an $\mathbb{F}_p$-subspace $V$ of $\Omega_1(P)$ and an isomorphism of $\mathbb{F}_p$-vector spaces $K \otimes_{\mathbb{F}_p} V \cong \Omega_1(P)$
- a subgroup $N$ of $\text{GL}_K(\Omega_1(P))$ and
- a subgroup $\Gamma$ of $\text{Aut}(K)$

such that $N_X(P)/C_X(P) = N \rtimes \Gamma$, as subgroups of $\text{Aut}(\Omega_1(P))$, and such that denoting by $W$ the normal subgroup of $N$ generated by reflections, we have $\Omega_1(P)^W = 1$.

**Proof.** The case where $H$ is simple is Theorem 3.1. In general, the classification of finite simple groups shows that there are finite simple groups $H_1, \ldots, H_r$ such that $H = H_1 \times \cdots \times H_r$ (cf. eg \[\text{[FoHa] \text{§}5]}\]). Note that $O_p(H) = 1$, i.e., there is no non-trivial $p$-group as a direct factor of $H$, since $H^2(H, \mathbb{F}_p) = 0$. Now, we take $X = X_1 \times \cdots \times X_r$, where the $X_i$ are associated with $H_i$. We put $K = K_1 \times \cdots \times K_r$, etc.

Following \[\text{[GoLy] Proof of (12.1)}\], we give now the list of possible finite simple groups $H$ and primes $p$ such that Sylow $p$-subgroups of $H$ are abelian non-cyclic and the $p$-part of the
Schur multiplier of \( H \) is trivial. In the first case, instead of providing the group \( H \), we provide a group \( G \) such that \( H \leq G/O_{p'}(G) \leq \text{Aut}(H) \) and \( p \nmid [G/O_{p'}(G) : H] \).

- \( G = G^F \) where \( G \) is a simply connected simple algebraic group and \( F \) is an endomorphism of \( G \), a power of which is a Frobenius endomorphism defining a rational structure over a finite field with \( q \) elements, \( p \nmid q \) and \( p \) is not a torsion prime for \( G \)
- \( H = \mathfrak{A}_n \) and \( n < p^2 \)
- \( H = \text{PSL}_2(p^n) \)
- \( H = 2G_2(q) \), \( p = 2 \)
- \( H \) is sporadic

Assume \( K = \mathbb{F}_p \). We have \( V = \Omega_1(P) \) and \( \Gamma = 1 \). Furthermore, \( N = E \subset N_{\text{GL}(P)}(W) \). So, in this case, the theorem is equivalent to the statement that \( W \) acts irreducibly on \( P \). As a consequence, in order to show that the theorem holds, it is enough to prove the statement with \( H \) replaced by a group \( G \) as above.

**Remark 3.4.** The finite simple groups with an abelian Sylow \( p \)-subgroup such that the \( p \)-part of the Schur multiplier is non-trivial are the following (cf [AT1]):

- \( H = M_{22}, ON, \mathfrak{A}_6, \mathfrak{A}_7 \) and \( p = 3 \)
- \( H = \text{PSL}_2(q) \), \( q \equiv 3, 5 \pmod{8} \) and \( p = 2 \)
- \( H = \text{PSL}_3(q) \) and \( 3|q - 1 \) or \( H = \text{PSU}_3(q) \) and \( 3|q + 1 \) (here \( p = 3 \))

Note that the automizer of a Sylow 3-subgroup \( P \) in \( \text{Aut}(ON) = ON \cdot 2 \) does not contain any reflection (when \( P \) is viewed as a vector space over \( \mathbb{F}_3 \)). That automizer is not a subgroup of \( \text{GL}_2(9) \cdot 2 \) (extension by the Frobenius).

Note that the presence of \( p \)-torsion in the Schur multiplier is an obstruction to the irreducibility of the subgroup of the automizer generated by reflections on \( \Omega_1(P) \), viewed as a vector space over \( \mathbb{F}_p \).

**Proposition 3.5.** Let \( G \) be a finite group with an abelian Sylow \( p \)-subgroup \( P \). Let \( E = N_G(P)/C_G(P) \) and let \( W \) be the subgroup of \( E \) generated by reflections on \( \Omega_1(P) \), viewed as an \( \mathbb{F}_p \)-vector space. Assume \( p > 2 \).

If \( H^2(G, \mathbb{F}_p) \neq 0 \), then \( \Omega_1(P)^W \neq 0 \).

**Proof.** Let \( V = \Omega_1(P)^* \). We have \( H^2(G, \mathbb{F}_p) \simeq H^2(N_G(P), \mathbb{F}_p) \simeq H^2(P, \mathbb{F}_p)^E \). On the other hand, we have an isomorphism of \( \mathbb{F}_p \)-modules \( H^2(P, \mathbb{F}_p) \to V \oplus \Lambda^2(V) \), so \( H^2(G, \mathbb{F}_p) \simeq V^E \oplus \Lambda^2(V)^E \subset V^W \oplus \Lambda^2(V)^W \). By Solomon’s Theorem [So], we have \( \Lambda^2(V)^W \simeq \Lambda^2(V^W) \). The result follows.

**Remark 3.6.** Let \( W \) be a reflection group on a complex vector space \( L \), with minimal field of definition \( K \). The subgroup of the outer automorphism group of \( W \) of elements fixing the set of reflections has always a decomposition as a semi-direct product \( (N_{\text{GL}(L)}(W)/W) \times \text{Gal}(K/\mathbb{Q}) \) as shown by Marin and Michel [MaMi].

**Remark 3.7.** It would be interesting to investigate if there is a version of Theorem 3.1 for non-principal blocks with abelian defect groups.

In a work in progress, we study automizers of maximal elementary abelian \( p \)-subgroups in covering groups of simple groups.
We run through the list of groups $H$ (or $G$) as described above.

4.1. **Chevalley groups.** Let $G$ be a connected and simply connected reductive algebraic group over an algebraic closure $k$ of a finite field and endowed with an endomorphism $F$, a power of which is a Frobenius endomorphism. Let $G = G^F$. Assume $p$ is invertible in $k$ and $p$ is not a torsion prime for $G$.

4.1.1. **Abelian $p$-subgroups.** Since $p$ is not a torsion prime for $G$, every abelian $p$-subgroup $Q$ of $G$ is contained in an $F$-stable maximal torus $T$ of $G$ and $L = C_G(Q)$ is a Levi subgroup ([SpSt Corollary 5.10 and Theorem 5.8] and [GeHi Proposition 2.1]). Furthermore, $N_G(Q) = N_G(Q)C_G(Q)$ [SpSt Corollary 5.10], hence $N_G(Q)/C_G(Q) = N_G(Q)/C_G(Q)$.

Let $W = N_G(T)/T$, $X = \text{Hom}(T, G_m)$ and $Y = \text{Hom}(G_m, T)$. If $G$ is simple, then the action of $W$ on $C \otimes \mathbb{Z} X$ is irreducible.

We have a canonical map $N_W(Q) \to N_G(Q)/T$. Since $L \subset N_G(Q) \subset N_G(L)$, we obtain an isomorphism

$$N_W(Q)/C_W(Q) \xrightarrow{\sim} N_G(Q)/C_G(Q).$$

Given $L$ an abelian group, we denote by $L_{p^\infty}$ the subgroup of $p$-elements of $L$. Let $\mu = k^\times$. We have an isomorphism

$$T_{p^\infty} \xrightarrow{\sim} \text{Hom}(X, \mu_{p^\infty}), \ t \mapsto (\chi \mapsto \chi(t)).$$

This provides an isomorphism

$$T_{p^\infty} \cong Y \otimes \mathbb{Z} \mu_{p^\infty}.$$

These isomorphisms are equivariant for the actions of $W$ and $F$.

4.1.2. **Abelian Sylow $p$-subgroups.** Assume now $P = Q$ is a abelian Sylow $p$-subgroup of $G$. Let $V = Y \otimes \mathbb{F}_p$. We have $V^F \cong \Omega_1(P)$.

**Proposition 4.1.** The group $N_W(P)/C_W(P)$ is a reflection group on $\Omega_1(P)$. If $G$ is simple, then this reflection group is irreducible.

**Proof.** Note that $N_W(P)/C_W(P)$ is a $p'$-group, since $P$ is an abelian Sylow $p$-subgroup of $G$ and $N_W(P)/C_W(P) \cong N_G(P)/C_G(P)$. So, the canonical map is an isomorphism

$$N_W(P)/C_W(P) \xrightarrow{\sim} N_W(\Omega_1(P))/C_W(\Omega_1(P)).$$

The proposition follows now from the next lemma by Lehrer-Springer theory [LeSp] extended to positive characteristic [Rou].

**Lemma 4.2.** We have $\dim V^F \geq \dim V_w^F$ for all $w \in W$.

**Proof.** Let $w \in N_G(T)$. By Lang’s Lemma, there is $x \in G$ such that $w = x^{-1}F(x)$. Given $t \in T$, we have $F(xtx^{-1}) = xwF(t)x^{-1}$. So, $xTtx^{-1}$ is $F$-stable and the isomorphism

$$T \cong xTtx^{-1} \quad t \mapsto xtx^{-1}$$

transfers the action of $wF$ on the left to the action of $F$ on the right. So,

$$V_w^F \cong (Y(xTtx^{-1}) \otimes \mathbb{F}_p)^F \cong \Omega_1((xTtx^{-1})^F).$$

The rank of that elementary abelian $p$-subgroup of $G$ is at most the rank of $P$ and we are done.
4.2. Alternating groups. Let $G = S_n$, $n > 7$. Put $n = pr + s$ with $0 \leq s \leq p - 1$ and $r < p$. We have $P \simeq (\mathbb{Z}/p)^r$. We put $K = F_p$, $N = W = F_p^\times \wr \mathfrak{S}_r$.

Remark 4.3. Note that when $n = 5$ and $p = 2$ or $n = 6, 7$ and $p = 3$, the $p$-part of the Schur multiplier is not trivial but the description above is still valid. Note though that when $n = 6$ and $p = 3$, then $G$ contains $S_6$ as a subgroup of index 2. We have $K = F_3$, $W = F_8 \times \mathfrak{S}_2$, $N = E$, $W$ is a Weyl group of type $B_2$ and $[N : W] = 2$.

4.3. PSL$_2$. Assume $H = \text{PSL}_2(K)$ for a finite field $K$ of characteristic $p$. We have $W = N = K^\times$ and $\Gamma = \text{Gal}(K/F_p)$.

4.4. $2G_2(q)$. Assume $H = 2G_2(q)$ and $p = 2$. We have $K = F_8$, $W = N = K^\times$ and $\Gamma = \text{Gal}(K/F_2)$.

4.5. Sporadic groups. We refer to [BrMaRou] for the diagrams for complex reflection groups. For sporadic groups, we have $P = \Omega_1(P)$.

| $H$ | $K$ | $\dim_k(P)$ | $W$ | $N/W$ | $\Gamma$ | diagram of $W$ |
|-----|-----|-------------|-----|-------|---------|----------------|
| $J_1$ | $F_8$ | 1 | $F_8^\times$ | 1 | $\text{Gal}(F_8/F_2)$ | (7) |
| $M_{11}, M_{23}, HS, 2$ | $F_3$ | 2 | $B_2$ | 2 | 1 | \text{Diagram} |
| $J_2, 2, Su_2, 2$ | $F_5$ | 2 | $G_2$ | 2 | 1 | \text{Diagram} |
| $He, 2, F_{122}, 2, F_{123}, F_{124}$ | $F_5$ | 2 | $G_8$ | 1 | 1 | \text{Diagram} |
| $Co, 1$ | $F_7$ | 2 | $G_{31}$ | 1 | 1 | \text{Diagram} |
| $Th, BM$ | $F_7$ | 2 | $G_{31}$ | 1 | 1 | \text{Diagram} |
| $M$ | $F_{11}$ | 2 | $G_{16}$ | 1 | 1 | \text{Diagram} |

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