Low-lying meson spectrum of large $N_C$ strongly coupled lattice QCD

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Abstract

We compute the low energy mass spectrum of lattice QCD in the large $N_C$ limit. Expanding around a gauge-invariant ground state, which spontaneously breaks the discrete chiral symmetry, we derive an improved strong-coupling expansion and evaluate, for any value of $N_C$, the masses of the low-lying states in the unflavored meson spectrum. We then take the ’t Hooft limit by rescaling $g^2 N_C \to g^2$; the ’t Hooft limit is smooth and no arbitrary parameters are needed. We find, already at the fourth order of the strong coupling perturbation theory, a very good agreement between the results of our lattice computation and the known continuum values.
I. INTRODUCTION

Since the seminal work of ’t Hooft [1, 2] the large $N_C$ limit (with $g^2 N_C$ fixed) has played an increasingly important role in studying gauge theories in the continuum and on the lattice [3] and, more recently, the duality between gauge and string theories [4, 5]. Our aim is to use the ’t Hooft limit to investigate some features of the meson spectrum of strongly coupled lattice QCD. Our results exhibit a good agreement with the observed meson masses, thus implying that in this context also the ’t Hooft limit offers a very accurate method to analyze the QCD spectrum.

A quite successful approach to studying gauge theories with confining spectra is the strong-coupling expansion [6]. In the strong coupling limit, confinement is explicit, the confining string is a stable object [7], and some other qualitative features of the spectrum are easily obtained. The formulation of the strong coupling expansion requires a gauge invariant ultraviolet cutoff, which is most conveniently implemented using lattice regularization.

One of the most difficult problems of the strong-coupling approach to lattice gauge theory remains its extrapolation to the continuum limit, which usually occurs at weak coupling. In spite of this difficulty, there are strong-coupling computations that claim a high degree of success. A useful test of strong-coupling expansions in lattice gauge theories has been the study of lower dimensional models whose solution in the continuum is known even in the strong coupling regime; for the Schwinger models [8] and the two-dimensional ’t Hooft model [9, 10], the meson spectrum and the chiral condensate have been evaluated using the strong-coupling approximation and a remarkable agreement with the known exact results was found. Furthermore, it is by now well known [11] that relevant features of strongly coupled lattice gauge theories have analogues in quantum spin systems: in many instances selecting the appropriate ground state for building the strong-coupling expansion of a gauge model turns out to be equivalent to finding the ground state of a generalized quantum antiferromagnet [12]. This equivalence turns out to provide pertinent nonperturbative information about the structure of the gauge invariant states and, in many instances, greatly simplifies the computation of the chiral condensate.

Many choices of strong-coupling theory produce identical continuum physics. As it is well known, any irrelevant operator may be added to the lattice Hamiltonian with an arbitrary parameter. Even if such ad hoc terms do not modify the continuum limit their coefficients
might appear in physical quantities such as the hadron masses \[13\]. Our study evidences that the 't Hooft limit does not require any arbitrary parameter and leads naturally to unambiguous and well defined results.

We shall develop a method enabling us to extend to a general number of colors the celebrated approach to the computation of the meson spectrum in strongly coupled lattice QCD first introduced by Banks et al. in Ref. \[13\]. We shall use the Hamiltonian approach to lattice gauge theory using staggered fermions \[14\]. The staggered formalism is known to yield good results in the strong-coupling evaluation of the hadron spectrum \[13\]; in contrast, other types of lattice fermions such as domain-wall or overlap fermions at strong coupling are expected to suffer both doubling and explicit breaking of chiral symmetry \[15\]. In the staggered fermion formalism, due to fermion doubling, the number of continuum fermions with \(N_f\) lattice fermions in \(D\) space-time dimensions is \(N_f 2^{(D/2) - 1}\). If, as in this paper, one includes only one lattice fermion, the continuum limit yields the two lightest quarks (\(u,d\)). In this paper we shall determine the spectrum of the low-lying unflavored mesons.

The computations are performed using \(x = 1/g^2\) as the expansion parameter and the extrapolation to the continuum limit is carried by means of Padé approximants. As the ground state we take the one of the pertinent antiferromagnetic Ising model. The gauge invariant eigenstates of the unperturbed Hamiltonian are used to develop the perturbative expansion. The 't Hooft limit is then taken: the coupling constant is rescaled according to \(g^2 N_C \to g^2\) and then \(N_C\) is sent to infinity. As shown in \[16\], mesons for large \(N_c\) are free, stable and non-interacting and their masses have smooth limits. Zweig’s rule is exact at large \(N_C\), mixing of mesons with glue states are suppressed and mesons for large \(N_C\) are pure \(q \bar{q}\) states. The meson energies computed perturbatively in the strong-coupling expansion contain terms depending on the number of lattice links. In the thermodynamic limit these cancel against the ground state vacuum energy. Thus the results we shall present are finite, parameter independent and already in good agreement with the known physical values at the fourth order in the perturbative strong-coupling expansion.

In Sec. II we review the well known Hamiltonian formulation of lattice QCD with staggered fermions and provide a classification of the symmetries of the theory.

In Sec. III we consider the strong-coupling limit of lattice QCD and determine the chiral symmetry breaking ground state. We then evaluate its energy up to the fourth order in the strong-coupling expansion.
In Sec. IV we construct the operators creating mesons from the vacuum and then compute their energies up to the fourth order in the perturbative expansion. We show that these energies are finite and well defined in the large $N_C$ limit.

Section V is devoted to the extrapolation of the lattice results to the continuum theory. There we show that the lattice theory, properly extrapolated to the continuum via Padé approximants, already well reproduces the known experimental values for the ratios between meson masses at low orders in the strong-coupling expansion.

Section VI is devoted to some concluding remarks, while the appendices illustrate some technical aspects needed to clarify the computations reported in the paper.

II. LATTICE QCD IN THE HAMILTONIAN FORMULATION

In the Hamiltonian formulation of lattice QCD with staggered fermions, time is a continuous variable and space is discretized on a 3-dimensional cubic lattice with $M$ sites, labeled by $\vec{r} = (x, y, z)$; with $x, y$ and $z$ integers. The conventions and the notation used in our paper are briefly reviewed in Appendix A.

The lattice QCD Hamiltonian with one lattice flavor of massless quarks may be written as the sum of three contributions

$$H = H_e + \tilde{H}_q + H_m,$$

where

$$H_e = \frac{g^2}{2a} \sum_{[\vec{r}, \hat{n}]} E^a[\vec{r}, \hat{n}]^2$$

$$\tilde{H}_q = \frac{1}{2a} \sum_{[\vec{r}, \hat{n}]} \eta(\hat{n}) \Psi_A^\dagger(\vec{r} + \hat{n}) U_{AB}[\vec{r}, \hat{n}] \Psi_B(\vec{r}) + \text{H.c.} \equiv H_q + H_q^\dagger$$

$$H_m = \frac{1}{2g^2a} \sum_{[\vec{r}, \hat{n}, \hat{m}]} \left\{ Tr(U[\vec{r}, \hat{n}] U[\vec{r} + \hat{n}, \hat{m}] U^\dagger[\vec{r} + \hat{m}, \hat{n}] U^\dagger[\vec{r}, \hat{m}]) + \text{H.c.} \right\}$$

are the electric field Hamiltonian, the interaction Hamiltonian between quarks and gauge fields and the magnetic Hamiltonian, respectively. The sums $\sum_{[\vec{r}, \hat{n}]}$ are extended to the $N$ lattice links, whereas $\sum_{[\vec{r}, \hat{n}, \hat{m}]}$ is a sum over the plaquettes. $\hat{n} = \hat{x}, \hat{y}, \hat{z}$ is the unit vector in the $\vec{n}$ direction and

$$\eta(\hat{x}) = (-1)^x, \quad \eta(\hat{y}) = (-1)^y, \quad \eta(\hat{z}) = (-1)^y$$
are the Dirac $\bar{\alpha}$ matrices for staggered fermions\[17].

The gauge field $U[\vec{r}, \hat{n}]$ is associated with the link $[\vec{r}, \hat{n}]$ and it is a group element in the fundamental representation of $SU(N_C)$. Two gauge fields occupying the same link are related by

$$U[\vec{r}, \hat{n}] = U[^{1}[\vec{r} + \hat{n}, -\hat{n}].$$

(6)

The electric field operator $E^a[\vec{r}, \hat{n}]$ is defined on a link and it obeys the algebra

$$[E^a[\vec{r}, \hat{n}], E^b[\vec{r}', \hat{m}]] = i f^{abc} E^c[\vec{r}, \hat{n}]\delta([\vec{r}, \hat{n}] - [\vec{r}', \hat{m}]).$$

(7)

$$E[\vec{r}, \hat{n}] = E^a[\vec{r}, \hat{n}] T^a,$$\[12, 18\] where $T^a$, $a = 1, \ldots, N_C^2 - 1$, are the generators of the Lie algebra of $U(N_C)$. They generate the left-action of the Lie algebra on $U[\vec{r}, \hat{n}]$\[12, 18\]

$$[E^a[\vec{r}, \hat{n}], U[\vec{r}', \hat{m}]] = -T^a U[\vec{r}, \hat{n}]\delta([\vec{r}, \hat{n}] - [\vec{r}', \hat{m}])$$

(8)

$$[E^a[\vec{r}, \hat{n}], U[^{1}[\vec{r}', \hat{m}]] = U[^{1}[\vec{r}, \hat{n}] T^a\delta([\vec{r}, \hat{n}] - [\vec{r}', \hat{m}]).$$

(9)

The fermion fields $\Psi$ are defined on the lattice sites and obey the anticommutation relations

$$\{\Psi_A(\vec{r}), \Psi_B^{\dagger}(\vec{r}')\} = \delta_{AB}\delta(\vec{r} - \vec{r}')$$

(10)

$$\{\Psi_A(\vec{r}), \Psi_B(\vec{r}')\} = 0$$

In addition to gauge invariance, the lattice Hamiltonian\[11\] is invariant under the following discrete symmetries

1. Lattice translation by even integers,

$$\Psi(x, y, z) \rightarrow \Psi(x + 2l, y + 2p, z + 2q)$$

(11)

where $l, p, q$ are integers. This symmetry operation amounts to a discrete translational invariance on the lattice.

2. Lattice translation by a single link,

$$\Psi(r) \rightarrow \Psi(r + \hat{x})(-1)^y$$

$$\Psi(r) \rightarrow \Psi(r + \hat{y})(-1)^z$$

(12)

$$\Psi(r) \rightarrow \Psi(r + \hat{z})(-1)^x.$$

In momentum space the last equation can be written as

$$q \rightarrow e^{ikz\gamma_5\tau_3}q$$
which in the continuum limit, where \( k_z \) is infinitesimal, gives

\[
q \rightarrow \gamma_5 \tau_3 q;
\]

the other two transformations yield

\[
q \rightarrow \gamma_5 \tau_2 q \\
q \rightarrow \gamma_5 \tau_1 q.
\]

The invariance under translation by a single link plays then the role of a discrete chiral invariance of the theory. This symmetry is broken by an explicit mass term in the Hamiltonian. As we shall show later, the chiral symmetry on the lattice is also spontaneously broken by the vacuum, and this should generate a nonvanishing chiral condensate \( \langle \bar{\psi} \psi \rangle \).

3. Shift along a face diagonal,

\[
\begin{align*}
\Psi(r) & \rightarrow (-1)^{x+y} \Psi(r + \hat{x} + \hat{z}) \\
\Psi(r) & \rightarrow (-1)^{y+z} \Psi(r + \hat{y} + \hat{x}) \\
\Psi(r) & \rightarrow (-1)^{z+x} \Psi(r + \hat{z} + \hat{y}).
\end{align*}
\]

These transformations correspond to the discrete isospin rotations

\[
q \rightarrow \tau_2 q \\
q \rightarrow \tau_3 q \\
q \rightarrow \tau_1 q.
\]

4. Parity,

\[
\Psi(r) \rightarrow \Psi(-r).
\]

This is the reflection through the origin.

5. G-parity

\[
\Psi(r) \rightarrow \Psi^\dagger(-r).
\]

This is just complex conjugation.
III. THE STRONG-COUPLING LIMIT AND THE VACUUM STATE ENERGY

In this section we shall describe in some detail the method we propose for implementing the strong-coupling expansion for QCD with a generic number of colors $N_C$, by explicitly computing the vacuum state energy up to the fourth order in the strong coupling expansion.

In the strong-coupling expansion the electric field Hamiltonian $H_e$, Eq.(2), is the unperturbed Hamiltonian while the interaction Hamiltonian between quarks and gauge fields $H_q$, Eq.(3), and the magnetic Hamiltonian $H_m$, Eq.(4), are treated as perturbations. Furthermore, each term in $H$, Eq.(1), is gauge invariant since

$$[G^a(\vec{r}), H_i] = 0, \quad i = e, q, m.$$ (16)

In Eq.(16)

$$G^a(\vec{r}) = \sum_{\vec{i}=\vec{-\vec{r}}} \vec{n} E^a[\vec{r}, \hat{i}] + \Psi_A^\dagger(\vec{r}) T^a_{AB} \Psi_B(\vec{r})$$ (17)

are the generators of static gauge transformations and $a = 1, ..., N_C^2 - 1$; these generators obey the Lie algebra

$$[G^a(\vec{r}), G^b(\vec{r}')] = i f^{abc} G^c(\vec{r}) \delta(\vec{r} - \vec{r}').$$ (18)

The empty vacuum $|0>$ is a gauge-invariant ($G^a(\vec{r})|0> = 0$) singlet of the electric field algebra ($E^a[\vec{r}, \hat{i}]|0> = 0$) which contains no fermions ($\psi_A(\vec{r})|0> = 0$).

Due to gauge invariance, $|0>$ must also be color singlet and should be charge neutral, i.e., it should obey the equation $\sum_{\vec{r}} \rho(\vec{r})|0> = 0$, with $\rho(\vec{r})$ given by the local fermion number operator

$$\rho(\vec{r}) = \frac{1}{2} [\Psi^\dagger(\vec{r}), \Psi(\vec{r})] = \Psi^\dagger(\vec{r}) \Psi(\vec{r}) - \frac{N_C}{2}.$$ (19)

At a given site one may form a color singlet in two ways: one can leave the site unoccupied or one may put on it $N_C$ fermions with antisymmetrized singlet wave function. Fermi statistics allow at most one singlet per site and it is possible to distribute the $M/2$ singlets arbitrarily ($M$ is the total number of lattice sites). The degeneracy of the lowest eigenstate of $H_e$ is then given by

$$\frac{M!}{(M/2)!}.$$ (20)

The degeneracy may be removed by diagonalizing the first nontrivial order in perturbation theory. All matrix elements in the vector space of degenerate vacua of the first order
Hamiltonian vanish, since \(<0|H_q|0> = 0\) (as \(\int dU U_{AB}^{(1)} = 0\)). The first nonvanishing contribution comes from second order. At this order one should compute

\[ E_0^{(2)} = <0|H_m|0> + <0|\tilde{H}_q \frac{\Pi_0}{E_0 - H_e} \tilde{H}_q|0>, \]  

(20)

where

\[ <,> = \prod_{[\vec{r},\hat{n}]} \int dU[\vec{r},\hat{n}] \left\{ (,) \right\} \]

is the inner product in the full Hilbert space of the model. \(dU\) is the Haar measure on the gauge group manifold and (, ) the fermion Fock space inner product; \(\Pi_0\) is the projection operator projecting onto states orthogonal to \(|0>\). In Eq.(20) \(\Pi_0\) is ineffective since the states created when \(\tilde{H}_q\) acts on the vacuum are always orthogonal to \(|0>\).

\(E_0^{(2)}\) may be more conveniently computed by constructing an eigenstate of \(H_e\) and using it to evaluate the function \(f(H_e) = \frac{\Pi_0}{E_0 - H_e} \) appearing in Eq.(20). Since the vacuum state \(|0>\) is a singlet of the electric field algebra, one has

\[ E_a[\vec{r},\hat{n}]|0> = 0 \]

(21)

which, in turn, implies that

\[ H_e|0> = 0. \]

(22)

Using then Eq.(8) and Eq.(22) and putting the commutator \([H_e, U[\vec{r},\hat{n}]]|0>\) in place of \(H_e U[\vec{r},\hat{n}]|0>\), one finds

\[ H_e U[\vec{r},\hat{n}]|0> = \frac{g^2}{2a} C_2(N_C) U[\vec{r},\hat{n}]|0>, \]

(23)

where \(C_2 = (N_C^2 - 1)/2N_C\) is the Casimir operator of \(SU(N_C)\). \(U[\vec{r},\hat{n}]|0>\) is then an eigenstate of \(H_e\) with eigenvalue \(g^2 C_2(N_C)/2a\). Consequently,

\[ <0|\tilde{H}_q \frac{1}{E_0 - H_e} \tilde{H}_q|0> = -\frac{4a}{g^2 C_2} <0|H_e^2 \tilde{H}_q|0>. \]

(24)

After integration over the link variables \(U\) [see also Eqs.(C3),(C4)] one sees immediately that the first term in Eq.(20) is zero and that the second order correction to the vacuum energy is given by

\[ E_0^{(2)} = -\frac{1}{g^2 a C_2 N_C} <0| \sum_{[\vec{r},\hat{n}]} \left[ \rho(\vec{r} + \hat{n}) + \frac{N_C}{2} \right] \left[ -\rho(\vec{r}) + \frac{N_C}{2} \right]|0>. \]

(25)
The diagonalized effective Hamiltonian is, up to an additive constant, given by

\[ H_{\text{eff}} = \frac{1}{g^2 a C_2 N_C} \sum_{[\vec{r}, \hat{n}]} [\rho(\vec{r} + \hat{n}) \rho(\vec{r})]. \]  

(26)

In deriving Eq. (26) one should also take into account the fact we that gauge invariant states such as \( |0> \) should be charge neutral, i.e., \( \sum_{\vec{r}} \rho(\vec{r}) = 0 \). As it is well known, \( H_{\text{eff}} \) takes the form of the Hamiltonian of an antiferromagnet of spin \( N_C/2 \). It was shown in [15] that even for domain-wall fermions the effective Hamiltonian is that of an antiferromagnet but the fermions are massive and doubled.

Since, due to Eq. (19), \( \rho(\vec{r}) \) has only two possible eigenvalues \( \rho = \pm N_C/2 \), the lowest energy a link can have may occur only when one end has \( \rho = +N_C/2 \) and the other has \( \rho = -N_C/2 \). In the space of pure fermion states the true ground state must thus minimize \( H_{\text{eff}} \) and be fluxless.

There are two ground states of \( H_{\text{eff}} \) corresponding to those of the antiferromagnetic Ising model of spin \( N_C/2 \). In the large \( N_C \) limit the two ground states are not mixed at any finite order of perturbation theory. In fact, it would be necessary to apply \( H_q \) at least \( N_C \) times in order to transform one ground state into the other, since \( H_q \) acts as an hopping Hamiltonian destroying a quark on a site and creating it on a neighboring site. One may choose as the ground state the one in which \( \rho = +N_C/2 \) on even sites and \( \rho = -N_C/2 \) on odd sites, the other state being obtained by interchanging odd and even sites. With this choice of the ground state the sum over the lattice links \([\vec{r}, \hat{n}]\) in Eq. (25) may be easily done and, for the ground state energy at the second order in the strong-coupling perturbative expansion, one gets

\[ E_0^{(2)} = -\frac{N_C}{2g^2 a C_2} N. \]  

(27)

As evidenced in Eq. (12), the chiral symmetry on the lattice is given by the translation by a single link and takes even sites into odd sites. Having chosen one of the two ground states described above, chiral symmetry is spontaneously broken [13] in the large \( N_C \) limit. Thus, in the perturbative expansion one has to consider only diagonal matrix elements and, consequently, perturbation theory for nondegenerate states.

The ground state energy up to the fourth order in the strong coupling expansion, is given by

\[ E_0 = E_0^{(0)} + E_0^{(2)} + E_0^{(4)}, \]  

(28)

9
where

\[ E_0^{(4)} = [E_0^{(4)}]_I + [E_0^{(4)}]_{II} + [E_0^{(4)}]_{III} \]  \hspace{1cm} (29) \]

with

\[ [E_0^{(4)}]_I = \langle 0|\hat{H}_q\frac{\Pi_0}{E_0^{(0)} - H_e}\hat{H}_q\frac{\Pi_0}{E_0^{(0)} - H_e}\hat{H}_q|0 \rangle \]  \hspace{1cm} (30) \]

\[ [E_0^{(4)}]_{II} = -\langle 0|\hat{H}_q\frac{\Pi_0}{E_0^{(0)} - H_e}\hat{H}_q|0 \rangle + \langle 0|\hat{H}_q\frac{\Pi_0}{(E_0^{(0)} - H_e)^2}\hat{H}_q|0 \rangle \]  \hspace{1cm} (31) \]

\[ [E_0^{(4)}]_{III} = \langle 0|H_m\frac{\Pi_0}{E_0^{(0)} - H_m}|0 \rangle . \]  \hspace{1cm} (32) \]

Equation (31) has the same form of a second order contribution and may be evaluated following the same steps used to arrive at Eq.(27): the only difference being that, in Eq.(31), the energy denominator also appears squared so that, using Eq.(23), one gets

\[ <0|\hat{H}_q\frac{1}{(E_0^{(0)} - H_e)^2}\hat{H}_q|0 >= \frac{8a^2}{g^4C_2^2} <0|\hat{H}_q^\dagger\hat{H}_q|0 > . \]  \hspace{1cm} (33) \]

After integration over the link variables, one finds that

\[ <0|\hat{H}_q\frac{\Pi_0}{E_0^{(0)} - H_e}\hat{H}_q|0 >= <0|\hat{H}_q\frac{\Pi_0}{(E_0^{(0)} - H_e)^2}\hat{H}_q|0 >= \]

\[ -\frac{2}{g^6aC_2^3N_C^2} \left[ <0|\sum_{\vec{r}\vec{n}} n(\vec{r} + \vec{n})(n(\vec{r}) - N_C)|0 > \right]^2 = -\frac{N_C^2N_m^2}{2g^6aC_2^3}, \]  \hspace{1cm} (34) \]

where

\[ n(\vec{r}) = \rho(\vec{r}) + \frac{N_C}{2} \]  \hspace{1cm} (35) \]

yields the number of fermions at the site \( \vec{r} \).

One may now turn to the first term contributing to \( E_0^{(4)} \), Eq.(30). Using Eq.(23), one gets

\[ <0|\hat{H}_q\frac{\Pi_0}{E_0^{(0)} - H_e}\hat{H}_q\frac{\Pi_0}{E_0^{(0)} - H_e}\hat{H}_q\frac{\Pi_0}{E_0^{(0)} - H_e}\hat{H}_q|0 >= \]

\[ \frac{4a^2}{g^4C_2^2} \left( 4 <0|H_q\hat{H}_q^\dagger|0 > + 2 <0|\hat{H}_q^\dagger\hat{H}_q^\dagger|0 > \right) . \]  \hspace{1cm} (36) \]

In deriving Eq.(36) from Eq.(30), one should observe that the most external \( \Pi_0 \) is ineffective; the only task to accomplish is then to evaluate the energy denominator in the middle. For this purpose it is most convenient to rewrite \( H_q\hat{H}_q^\dagger|0 > \) and \( \hat{H}_q^\dagger\hat{H}_q^\dagger|0 > \) as linear combinations of eigenvectors of \( H_e \), and in order to do this one should first consider the action of \( H_e \) on
Taking into account Eq. (38) and Eq. (40), Eq. (36) becomes

\[
H_e U_{AB}[\vec{r}, \hat{n}] U^\dagger_{CD}[\vec{r}^2, \hat{m}] |0> = \frac{g^2}{a} \left( C_2(N_C) U_{AB}[\vec{r}, \hat{n}] U^\dagger_{CD}[\vec{r}^2, \hat{m}] 
+ \frac{1}{2N_C} U_{AB}[\vec{r}, \hat{n}] U^\dagger_{CD}[\vec{r}, \hat{n}] \delta([\vec{r}, \hat{n}] - [\vec{r}^2, \hat{m}])
- \frac{1}{2} \delta_{AD} \delta_{BC} \delta([\vec{r}, \hat{n}] - [\vec{r}^2, \hat{m}]) \right) |0> .
\]  

(37)

Keeping into account the action of \( \Pi_0 \) it is easy to see that \( U_{AB}[\vec{r}, \hat{n}] U^\dagger_{CD}[\vec{r}^2, \hat{m}] |0> \) is an eigenstate of \( f(H_e) = \Pi_0 / (E_{0}^{(0)} - H_e) \) with eigenvalue

\[
f(H_e) U_{AB}[\vec{r}, \hat{n}] U^\dagger_{CD}[\vec{r}^2, \hat{m}] |0> = -\frac{a}{g^2 C_2} U_{AB}[\vec{r}, \hat{n}] U^\dagger_{CD}[\vec{r}^2, \hat{m}] |0> .
\]  

(38)

A similar procedure shows that when \( H_e \) acts on two \( U \)'s at different links, one gets

\[
H_e U^\dagger_{AB}[\vec{r}, \hat{n}] U^\dagger_{CD}[\vec{r}^2, \hat{m}] |0> = \frac{g^2}{a} \left( C_2(N_C) U^\dagger_{AB}[\vec{r}, \hat{n}] U^\dagger_{CD}[\vec{r}^2, \hat{m}] 
- \frac{1}{N_C + 1} U^\dagger_{AB}[\vec{r}, \hat{n}] U^\dagger_{CD}[\vec{r}, \hat{n}] \delta([\vec{r}, \hat{n}] - [\vec{r}^2, \hat{m}]) \right) |0> .
\]  

(39)

In Appendix B we show that the combination \( \left( U^\dagger_{AB}[\vec{r}, \hat{n}] U^\dagger_{CD}[\vec{r}^2, \hat{m}] - U^\dagger_{AB}[\vec{r}, \hat{n}] U^\dagger_{CD}[\vec{r}, \hat{n}] \right) |0> \) is indeed an eigenstate of \( H_e \) with eigenvalue \( g^2C_2(N_C)/a \). Using this result and the fact that \( U^\dagger_{AB}[\vec{r}, \hat{n}] U^\dagger_{CD}[\vec{r}, \hat{n}] |0> \) is an eigenstate of \( H_e \) one can easily show that

\[
f(H_e) U^\dagger_{AB}[\vec{r}, \hat{n}] U^\dagger_{CD}[\vec{r}^2, \hat{m}] |0> = -\frac{a}{g^2 C_2} \left( U^\dagger_{AB}[\vec{r}, \hat{n}] U^\dagger_{CD}[\vec{r}^2, \hat{m}] 
+ \frac{1}{N_C - 2} U^\dagger_{AB}[\vec{r}, \hat{n}] U^\dagger_{CD}[\vec{r}, \hat{n}] \delta([\vec{r}, \hat{n}] - [\vec{r}^2, \hat{m}]) \right) |0> .
\]  

(40)

Taking into account Eq. (38) and Eq. (40), Eq. (39) becomes

\[
<0|\tilde{H}_q \frac{\Pi_0}{E_{0}^{(0)} - H_e} \tilde{H}_q \frac{\Pi_0}{E_{0}^{(0)} - H_e} \tilde{H}_q \frac{\Pi_0}{E_{0}^{(0)} - H_e} \tilde{H}_q |0> =
\frac{4a^3}{g^6 C_2^3} \left( -4 <0|H_q H_\dagger H_q H_\dagger |0> - 2 <0|H_q H_\dagger H_q H_\dagger |0> 
- \frac{1}{2a^2(N_C - 2)} <0|H_q H_q \sum_{[\vec{r}, \hat{n}]} \Psi_A(\vec{r}) U^\dagger_{AB}[\vec{r}, \hat{n}] \Psi_B(\vec{r} + \hat{n}) 
\Psi_C(\vec{r} U^\dagger_{CD}[\vec{r}, \hat{n}] \Psi_D(\vec{r} + \hat{n}) |0> \right) .
\]  

(41)

Integrating now over the link variables (see Eqs. (C4), (C5)), one gets

\[
<0|\tilde{H}_q \frac{\Pi_0}{E_{0}^{(0)} - H_e} \tilde{H}_q \frac{\Pi_0}{E_{0}^{(0)} - H_e} \tilde{H}_q \frac{\Pi_0}{E_{0}^{(0)} - H_e} \tilde{H}_q |0> =
\]
Thus, a plaquette acting on the vacuum $|0\rangle$:

$$-\frac{1}{g^6aC_2^2} \left[ \frac{1}{N_C^2} < 0 \right] \left( - \sum_{[\vec{r},\vec{n}] \neq [\vec{r}+\vec{m},\vec{m}]} n(\vec{r}+\vec{n})(n(\vec{r}) - N_C)(n(\vec{r}+\vec{n}+\vec{m}) - N_C) \right. \left. + \sum_{[\vec{r},\vec{n}] \neq [\vec{r}+\vec{m},\vec{m}]} n(\vec{r}+\vec{n})(n(\vec{r}) - N_C)n(\vec{r}+\vec{n}) \right) \left. \sum_{[\vec{r},\vec{n}] \neq [\vec{r}+\vec{n}-\vec{m},\vec{m}]} n(\vec{r}+\vec{n})(n(\vec{r}) - N_C)(n(\vec{r}+\vec{n}-\vec{m}) - N_C) \right) \left. + \sum_{[\vec{r},\vec{n}] \neq [\vec{r}+\vec{n},\vec{m}]} n(\vec{r}+\vec{n})(n(\vec{r}) - N_C)n(\vec{r}^+\vec{m})(n(\vec{r}) - N_C) \right) \right|0\rangle >$$

$$\left. + \frac{1}{N_C(N_C-2)} < 0 \sum_{[\vec{r},\vec{n}]} \left( -n(\vec{r}+\vec{n})(n(\vec{r}) - N_C) + n(\vec{r}+\vec{n})^2(n(\vec{r}) - N_C) - n(\vec{r}+\vec{n})(n(\vec{r}) - N_C)^2 + n(\vec{r}+\vec{n})^2(n(\vec{r}) - N_C)^2 \right) |0\rangle > = \right.$$
The integration over the link variables requires that the two plaquettes are coincident, otherwise the integral would vanish. Keeping into account that $2N$ is the number of oriented plaquettes on the lattice, one gets

$$
< 0 | H_m \frac{\Pi_0}{E_0^{(0)} - H_e} H_m | 0 > = - \frac{N}{g^6 a C_2}.
$$

(46)

Collecting all the terms (34), (42) and (46) the fourth order correction to the vacuum energy becomes

$$
E_0^{(4)} = \frac{N}{2g^6 a C_2^3} \frac{N C (10N C - 21)}{N C - 2} - \frac{N}{g^6 a C_2} \equiv E_q^{(4)} + E_m^{(4)},
$$

(47)

where $E_q^{(4)}$ is the contribution due to the quark Hamiltonian and $E_m^{(4)}$ the one due to the magnetic Hamiltonian. Note that $E_0^{(4)}$ is proportional to $N$ and thus the vacuum energy is an extensive variable. Moreover, the $N^2$ dependence of Eq.(34) is precisely canceled by Eq.(42). As a check of our computation one may set $N = 3$ in Eq.(47) and compare the result with those obtained in [13] using a completely different approach. The agreement is exact when the coefficient of the irrelevant operator introduced in [13] is set to zero [23].

IV. THE LOW-MASS MESON STATES

In the strong coupling expansion the lowest-lying states in the meson spectrum are those consisting of a quark and an antiquark at opposite ends of a single link. If the quark is at $(\vec{r} + \hat{n})$ and the antiquark at $\vec{r}$ a basis for such states is given by

$$
| \vec{r}, \hat{n} > = \Psi_t^j(\vec{r} + \hat{n}) U[\vec{r}, \hat{n}] \Psi(\vec{r}) | 0 >.
$$

(48)

For a given meson, the wave function may be determined through the following steps. One may first take the quark bilinear in the continuum with the desired transformation properties and appropriate continuum quantum numbers and then write it in point-separated lattice form using the discrete symmetries of the theory. Only after fixing the pertinent lattice quantum numbers one may apply the bilinear to the vacuum.

In the following we shall be interested only in the low-lying unflavored mesons: $\pi_0, \rho, \omega, b_1, a_1, f_2, f_0$. In the continuum theory the wave functions for these mesons are given by

$$
| \pi_0 > \sim i \bar{\Psi} \gamma_5 \frac{1}{2} \gamma_3 \Psi | 0 >
$$

(49)
\[ |\omega > \sim \Psi^\dagger \alpha_x \Psi |0 > \]  
\[ |\rho > \sim \Psi^\dagger \alpha_x \frac{1}{2} \tau_3 \Psi |0 > \]  
\[ |b_1 > \sim i \Psi^\dagger \gamma_5 \partial_2 \tau_3 \Psi |0 > \]  
\[ |a_1 > \sim i \Psi^\dagger (\alpha_x \partial_y - \alpha_y \partial_x) \tau_3 \Psi |0 > \]  
\[ |f_2 > \sim i \Psi^\dagger (\alpha_x \partial_z + \alpha_z \partial_x - 2\alpha_y \partial_y) \Psi |0 > \]  
\[ |f_0 > \sim i \nabla \Psi |0 > . \]  

The choice done is as in \[13\] and it is based on the quantum numbers labeling the mesonic states. For example, for \( \pi_0 \) one needs a pseudoscalar with nontrivial isospin; thus \( i \nabla \gamma_5 \tau_3 \frac{1}{2} \Psi \) is a pertinent choice of the wave function. The choice for the components of the vector mesons (\( \omega, \rho, b_1, a_1 \)) or of the spin-2 meson \( (f_2) \) is made by observing that these are the only components of these mesons that on the lattice have the standard form \( \mathbf{(13)} \).

The lattice form of these operators may be obtained by applying the staggered fermion formalism to derive operators with appropriate lattice quantum numbers. The lattice wave functions at zero momentum are then given by

\[ |\pi_0 > = \frac{i}{\sqrt{N_{C,M}}} \left[ \sum_\vec{r} (-1)^z \Psi_A^\dagger (\vec{r} + \hat{z}) U_{AB}[\vec{r}, \hat{z}] \Psi_B(\vec{r}) - h.c. \right] |0 > \]  
\[ |\omega > = \frac{i}{\sqrt{N_{C,M}}} \left[ \sum_\vec{r} (-1)^y \Psi_A^\dagger (\vec{r} + \hat{z}) U_{AB}[\vec{r}, \hat{z}] \Psi_B(\vec{r}) - h.c. \right] |0 > \]  
\[ |\rho > = \frac{1}{\sqrt{N_{C,M}}} \left[ \sum_\vec{r} (-1)^y \Psi_A^\dagger (\vec{r} + \hat{y}) U_{AB}[\vec{r}, \hat{y}] \Psi_B(\vec{r}) + h.c. \right] |0 > \]  
\[ |b_1 > = \frac{i}{\sqrt{2N_{C,M}}} \left[ \sum_\vec{r} (-1)^z \Psi_A^\dagger (\vec{r} + \hat{z}) U_{AB}[\vec{r}, \hat{z}] \Psi_B(\vec{r}) \right. \]  
\[ + \left. (-1)^{x+y+z} \Psi_A^\dagger (\vec{r} + \hat{x}) U_{AB}[\vec{r}, \hat{x}] \Psi_B(\vec{r}) \right] |0 > \]  
\[ |a_1 > = \frac{i}{\sqrt{2N_{C,M}}} \left[ \sum_\vec{r} [(-1)^y \Psi_A^\dagger (\vec{r} + \hat{y}) U_{AB}[\vec{r}, \hat{y}] \Psi_B(\vec{r}) \right. \]  
\[ + \left. (1)^{x+y+z} \Psi_A^\dagger (\vec{r} + \hat{x}) U_{AB}[\vec{r}, \hat{x}] \Psi_B(\vec{r}) \right] |0 > \]  
\[ |f_2 > = \frac{1}{\sqrt{2N_{C,M}}} \left[ \sum_\vec{r} [(-1)^y \Psi_A^\dagger (\vec{r} + \hat{z}) U_{AB}[\vec{r}, \hat{z}] \Psi_B(\vec{r}) \right. \]  
\[ + \left. (1)^{x+y+z} \Psi_A^\dagger (\vec{r} + \hat{x}) U_{AB}[\vec{r}, \hat{x}] \Psi_B(\vec{r}) \right] + h.c. \right] |0 > \]  
\[ |f_0 > = \frac{1}{\sqrt{3N_{C,M}}} \left[ \sum_\vec{r} \eta(\hat{n}) \Psi_A^\dagger (\vec{r} + \hat{n}) U_{AB}[\vec{r}, \hat{n}] \Psi_B(\vec{r}) + h.c. \right] |0 > . \]  

The normalizations are fixed in the standard way by integrating over the link variables.

All the mesons are degenerate at the lowest order and their energy is given by

\[ E_M^{(0)} = \langle \mathcal{M}|H_c|\mathcal{M} \rangle = \frac{g^2 C_2}{2a} , \]  

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as it can be easily seen using Eq. (23) and integrating over the link variables using Eq. (C4). Since a static limit (fixed and large $a$) is used -and thus the states cannot propagate in this approximation- all the single-link mesons have the same mass regardless of the character (e.g., $s$-wave or $p$-wave) of their continuum wave functions. As we shall see, the fourth order computation will cure this unphysical effect of the static lattice approximation.

The meson energy has been computed up to the fourth order in the perturbative expansion using a method similar to the one used in the evaluation of the vacuum energy. If one evaluates the meson energy up to the fourth order, one gets

\[ E_M = \frac{g^2 C_2}{2a} + E_M^{(2)} + E_M^{(4)}, \]

where

\[ E_M^{(2)} = <\mathcal{M}|\tilde{H}_q\frac{\Pi_M}{E_M^{(0)} - H_e}\tilde{H}_q|\mathcal{M} > \]  

and

\[ E_M^{(4)} = - <\mathcal{M}|\tilde{H}_q\frac{\Pi_M}{E_M^{(0)} - H_e}\tilde{H}_q|\mathcal{M} > <\mathcal{M}|\tilde{H}_q\frac{\Pi_M}{(E_M^{(0)} - H_e)^2}\tilde{H}_q|\mathcal{M} > \]

\[ + <\mathcal{M}|H_m\frac{\Pi_M}{E_M^{(0)} - H_e}H_m|\mathcal{M} > . \]

$\Pi_M$ is the projection operator projecting onto states orthogonal to those states that have the unperturbed energy of a generic single link meson $|\mathcal{M} >$. $|\mathcal{M} >$ is the meson operator.

A. Second order

The projection operator $\Pi_M$ does not affect the second order in the strong coupling expansion since the states created by $\mathcal{M}$ and $\tilde{H}_q$ acting on the vacuum $|0 >$ are orthogonal to the meson state. Thus, the matrix elements to be computed are

\[ E_M^{(2)} = <(M + M^\dagger)|(H_q + H_q^\dagger)\frac{1}{E_M^{(0)} - H_e}(H_q + H_q^\dagger)|(M + M^\dagger) >, \]

where $|\mathcal{M} >$ is one of the meson operator $|53\rangle$ and $\mathcal{M} \equiv M + M^\dagger$. In Eq. (68) the terms containing a different number of $U$ and $U^\dagger$ vanish when integrated over the link variables.
The remaining terms may be conveniently grouped as

\[
E_M^{(2)} = 2 \left[ \langle M | H_q \tilde{\Lambda}_M(H_e) H^\dagger_q | M^\dagger \rangle + \langle M | H^\dagger_q \tilde{\Lambda}_M(H_e) H_q | M^\dagger \rangle + \langle M | H^\dagger_q \tilde{\Lambda}_M(H_e) H^\dagger_q | M \rangle \right],
\]

where

\[
\tilde{\Lambda}_M(H_e) = \frac{1}{E_M^{(0)} - H_e}.
\]

All the one-link meson states \([56]-[62]\) consist of linear combinations of directed links on the lattice with appropriate phases, which are responsible for the differences in the matrix elements of the various mesons \([56]-[62]\).

To elucidate the method used to compute these matrix elements, it is sufficient to consider a generic meson of the form

\[
\sum_{\vec{r}} S(\vec{r}) \Psi^\dagger_A(\vec{r}, \hat{n}) U_{AB}[\vec{r}, \hat{n}] \Psi_B(\vec{r}) |0\rangle;
\]

In Eq.\([70]\) \(S(\vec{r})\) is one of the phases of the meson operators \([56]-[62]\). In order to compute

\[
(II)_1 = \langle M | H_q \tilde{\Lambda}_M(H_e) H^\dagger_q | M^\dagger \rangle \quad (71)
\]

\[
(II)_2 = \langle M | H^\dagger_q \tilde{\Lambda}_M(H_e) H_q | M^\dagger \rangle \quad (72)
\]

\[
(II)_3 = \langle M | H^\dagger_q \tilde{\Lambda}_M(H_e) H^\dagger_q | M \rangle \quad (73)
\]

one should construct suitable eigenstates of the unperturbed Hamiltonian \(H_e\) in order to evaluate the function of \(H_e, \tilde{\Lambda}_M(H_e)\). Using the eigenvectors found in the previous section, it is quite easy to obtain

\[
\tilde{\Lambda}_M(H_e) U_{AB}[\vec{r}, \hat{n}] U_{CD}[\vec{r}^\prime, \hat{m}] |0\rangle = \frac{2a}{g^2 C_2} \left( -U^\dagger_{AB}[\vec{r}, \hat{n}] U_{CD}[\vec{r}^\prime, \hat{m}] - \frac{2}{N_C - 3} U^\dagger_{AB}[\vec{r}, \hat{n}] U_{CD}[\vec{r}, \hat{m}] \right) |0\rangle \quad (74)
\]

and

\[
\tilde{\Lambda}_M(H_e) U_{AB}[\vec{r}, \hat{n}] U_{CD}[\vec{r}^\prime, \hat{m}] |0\rangle = \frac{2a}{g^2 C_2} \left( -U_{AB}[\vec{r}, \hat{n}] U_{CD}[\vec{r}^\prime, \hat{m}] + \frac{2}{N_C + 1} U_{AB}[\vec{r}, \hat{n}] U_{CD}[\vec{r}, \hat{m}] \delta \left( [\vec{r}, \hat{n}] - [\vec{r}^\prime, \hat{m}] \right) \right) + \frac{2N_C}{N_C + 1} \delta_{AD} \delta_{BC} \delta \left( [\vec{r}, \hat{n}] - [\vec{r}^\prime, \hat{m}] \right) |0\rangle \quad (75).
\]

Plugging Eq.\([74]\) and Eq.\([75]\) in Eqs.\([71]-[73]\), one gets

\[
(II)_1 = - \frac{2a}{g^2 C_2} < M | H_q H^\dagger_q | M^\dagger >
\]

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One now needs the results of Eq. (C4) and Eq. (C5) to integrate over the link variable \( \vec r \). As an example, consider the case of the \( S \) selecting a given meson.

As an example, consider the case of the \( \rho \) meson. In terms of the fermion number operator \( n(\vec r) \) one has

\[
(II)_1 = - \frac{2a}{g^2 C_2} < M|H_q^\dagger H_q|M >
\]
\[
+ \frac{4}{g^2 C_2 (N_C^2 + 1)} < M|H_q^\dagger H_q| \sum_{[\vec r, \hat{n}]} \eta(\hat{n}) \Psi_A^\dagger(\vec r) U_{AB}[\vec r, \hat{n}] \Psi_B(\vec r) < M|H_q^\dagger H_q|\sum_{[\vec r, \hat{n}]} \eta(\hat{n}) \Psi_A^\dagger(\vec r + \hat{n}) \Psi_B(\vec r) >.
\]

One now needs the results of Eq. (C4) and Eq. (C5) to integrate over the link variable \( U \) and to choose the particular \( S(\vec r) \) selecting a given meson.

As an example, consider the case of the \( \rho \) meson. In terms of the fermion number operator \( n(\vec r) \) one has

\[
[\langle II \rangle_1]_\rho = - \frac{1}{2g^2 a C_2 N_C} < 0 \left[ \sum_{[\vec r, \hat{n}, \hat{\bar{n}}]} n(\vec r + \hat{n})(n(\vec r) - N_C)(n(\vec r + \hat{\bar{n}}) - N_C) \right.
\]
\[
- \sum_{[\vec r, \hat{\bar{n}}]} n(\vec r + \hat{\bar{n}})(n(\vec r) - N_C)(n(\vec r + \hat{n}) - N_C) + \left. \sum_{[\vec r, \hat{\bar{n}}]} n(\vec r + \hat{n})(n(\vec r) - N_C) n(\vec r + \hat{\bar{n}} - \hat{n})(n(\vec r + \hat{\bar{n}}) - N_C) \right]
\]
\[
+ \sum_{[\vec r, \hat{\bar{n}}]} n(\vec r + \hat{n})(n(\vec r) - N_C) n(\vec r + \hat{\bar{n}})(n(\vec r + \hat{n}) - N_C) + \frac{2N_C}{C_2 - 3} \sum_{[\vec r]} \left( -n(\vec r + \hat{n})(n(\vec r) - N_C) - n(\vec r + \hat{n})(n(\vec r) - N_C)^2 \right).
\]
\[ + n(\tilde{r} + \tilde{y})^2 (n(\tilde{r}) - N_C) + n(\tilde{r} + \tilde{y})^2 (n(\tilde{r}) - N_C)^2 \] \]
\[ = \frac{1}{2g^2aC_2} \left( 3 - \frac{2(N_C - 1)}{N_C - 3} - \frac{N_C}{4N} \right). \quad (79) \]

\[ [(II)_2]_\rho = -\frac{1}{2g^2aC_2N_C^2} < 0 \left\{ \begin{array}{c}
- \sum_{\tilde{r}, [\tilde{r}, \tilde{y}] \neq [\tilde{r} - \tilde{m}, \tilde{m}]} n(\tilde{r} + \tilde{y})(n(\tilde{r}) - N_C)(n(\tilde{r} + \tilde{y} + \tilde{m}) - N_C) \\
+ \sum_{\tilde{r}, [\tilde{r}, \tilde{y}] \neq [\tilde{r} - \tilde{m}, \tilde{m}]} n(\tilde{r} + \tilde{y})(n(\tilde{r}) - N_C)n(\tilde{r} - \tilde{m}) \\
+ \sum_{\tilde{r}, [\tilde{r}, \tilde{y}] \neq [\tilde{R}, \tilde{y}]} (\tilde{r} + \tilde{y})(n(\tilde{r}) - N_C)n(\tilde{R} + \tilde{y})(n(\tilde{R}) - N_C) \\
+ \sum_{\tilde{r}, [\tilde{r}, \tilde{y}] \neq [\tilde{R}, \tilde{y}]} \left( \frac{N_C^2}{N_C^2 - 1} \left[ \sum_{\tilde{r}} \left[ n(\tilde{r} + \tilde{y})^2 (n(\tilde{r}) - N_C)^2 - n(\tilde{r} + \tilde{y})(n(\tilde{r}) - N_C)^2 \right] \\
- \frac{2N_C}{N_C^2 - 1} \left[ \sum_{\tilde{r}} \left[ n(\tilde{r} + \tilde{y})^2 (n(\tilde{r}) - N_C)^2 - n(\tilde{r} + \tilde{y})(n(\tilde{r}) - N_C) \right] \right] \\
- \frac{N_C}{N_C^2 - 1} \sum_{\tilde{r}} \left[ n(\tilde{r} + \tilde{y})^2 (n(\tilde{r}) - N_C) + n(\tilde{r} + \tilde{y})(n(\tilde{r}) - N_C)^2 \right] \right) \\
+ \sum_{\tilde{r}, [\tilde{r}, \tilde{y}] \neq [\tilde{R}, \tilde{y}]} (\tilde{r} + \tilde{y})(n(\tilde{r}) - N_C)n(\tilde{R} + \tilde{y})(n(\tilde{R}) - N_C) \right\} \left\{ n(\tilde{r} + \tilde{y}) \right\} \] \]
\[ = \frac{1}{2g^2aC_2} \left( 3 - \frac{N_C}{4N} \right). \quad (80) \]

\[ [(II)_3]_\rho = \frac{1}{2g^2aC_2N_C^2} < 0 \left\{ \sum_{\tilde{r}} n(\tilde{r} + \tilde{y})(n(\tilde{r}) - N_C)n(\tilde{r} - \tilde{y}) \\
- \sum_{\tilde{r}} n(\tilde{r} + \tilde{y})(n(\tilde{r}) - N_C)n(\tilde{r} + 2\tilde{y}) - N_C) \right\} \left\{ n(\tilde{r} + \tilde{y}) \right\} \]
\[ = -\frac{1}{2g^2aC_2}. \quad (81) \]

Adding up the three terms, one finds
\[ E^{(2)}_\rho = \frac{1}{g^2aC_2} \left[ 5 - \frac{2(N_C - 1)}{N_C - 3} - \frac{N_C}{2N} \right] . \quad (82) \]

A similar procedure yields the following results for the second order correction to the meson energies
\[ E^{(2)}_{\pi_0} = \frac{1}{g^2aC_2} \left[ 5 - \frac{2(N_C - 1)}{N_C - 3} - \frac{N_C}{2N} \right] . \quad (83) \]
where the $N$-dependent terms cancel against the vacuum energy, as should be since the masses are intensive quantities.

After rescaling the coupling constant according to the ’t Hooft prescription (large $N_C$ with $g^2N_C$ fixed)

$$ g^2N_C \to g^2 $$

one finds that the large $N_C$ limit makes Eqs. $(82)-(88)$ finite.

Our results have been checked by deriving the second order meson masses for $N_C$ generic with the graphical procedure of Ref. [13]. In this case also there is complete agreement.

**B. A comment on irrelevant operators**

Since the strong-coupling limit of QCD is not universal, adding an irrelevant operator to the Hamiltonian, leads to the same physical predictions in the continuum limit. This allows one to introduce arbitrary parameters, the coefficients of these irrelevant operators, which are then fixed by fitting the experimental data. Our analysis shows that in the ’t Hooft limit ($N_C \to \infty$, $g^2N_C$ fixed) the meson masses can be made independent of arbitrary parameters and that results in agreement with experiments can be obtained without introducing irrelevant operators.

In the celebrated computation of the hadron spectrum by Banks et al. [13] the lattice Hamiltonian was indeed modified by the addition of an irrelevant operator given by

$$ W = A \sum_{\vec{r},\hat{n}} \left( \rho(\vec{r})\rho(\vec{r} + \hat{n}) + \frac{N_C^2}{4} \right) $$

(89)
where $A$ is a dimensionless irrelevant parameter. The new term was chosen according to three demands. First, it must remove the degeneracies at zeroth order so that nondegenerate perturbation theory can be used. Second, it must preserve the symmetries of the original Hamiltonian. The vacuum state of the modified theory must again break the chiral symmetry spontaneously. Third, the added term should have no effect on the continuum limit of the lattice theory, so it should be an irrelevant operator. $W$ is a four-fermion operator and, when it is written in terms of the continuum variables with the conventional units, it depends on $Ag^2a^2$ so that in the continuum limit ($a \to 0$) it vanishes faster than $a^2$.

The irrelevant operator (89) was introduced by Banks et al. [13] mainly because in this way the meson masses are well defined even for $N_C = 3$ at the second order in the strong coupling expansion. The corrections (82)-(88) are in fact divergent for $N_C = 3$. This problem was avoided by introducing $W$ in the unperturbed Hamiltonian. The meson masses then depend on the arbitrary parameter $A$. For $N_C = 3$ the diverging contribution, evidenced in Eqs. (82)-(88), comes from a term in which the meson operators and the quark Hamiltonian are on the same link. In [13] this term varies as $A^{-1}$ and if $A$ is set to zero it yields a diverging contribution. It is not difficult to verify that, if the irrelevant operator is not introduced, up to the fourth order in the strong coupling expansion, there are divergences of the type $1/(N_C - 1)$, $1/(N_C - 2)$, $1/(N_C - 3)$ and $1/(N_C - 4)$; at the next order there are divergences up to $1/(N_C - 5)$; and so on. In the infinite $N_C$ limit these divergences are avoided. Our analysis thus shows that in the large $N_C$ approach to lattice QCD there is no need for an irrelevant operator; in fact, with the ’t Hooft prescription, the limit $N_C \to \infty$ yields series expansions for the meson masses which are free of divergences and thus well defined.

The constant $A$ in Eq. (89) was fixed in [13] by requiring that the $q\bar{q}$ state is less massive than a nucleon-antinucleon state in the static limit. As we will show below, in the ’t Hooft limit however, the baryon masses are zero at zeroth order in the strong coupling expansion and acquire a mass proportional to $N_C$ only at second order. Thus, baryons may be consistently regarded as QCD solitons [16] and the unperturbed mass of a bound state $n\bar{n}$ in the large $N_C$ limit vanishes at the lowest order in the perturbative expansion. A $q\bar{q}$ state is then not degenerate with a $n\bar{n}$ state and for this reason also there is no need of an irrelevant operator.

Nucleon masses should be determined using an antisymmetric operator creating $N_C$
quarks at the same lattice site when acting on the vacuum state. The normalized nucleon state is
\[ |n> = \frac{1}{N_C} \sqrt{\frac{2}{M}} \sum_{\vec{r}} e^{A_1 A_2 \cdots A_{N_C}} \Psi_{A_1}^\dagger (\vec{r}) \Psi_{A_2}^\dagger (\vec{r}) \cdots \Psi_{A_{N_C}}^\dagger (\vec{r}) |0> \] (90)

At zeroth order in the strong-coupling expansion, the baryon is massless, since the creation operator \(|n>\) does not contain any color flux and thus \(H_e|n> = 0\). This is in agreement with the requirement that the nucleon mass, being the mass of a soliton, should vary as the inverse of the coupling constant. At the second order in the strong coupling expansion baryons already acquire mass given by

\[ E_n^{(2)} = n|\tilde{H}_q^\dagger \Pi_n \tilde{H}_q |n> = \frac{1}{g^2 a C_2} \left[ -\frac{N_C}{2} N + N_C \right] \]

\[ m_n = E_n^{(2)} - E_0^{(2)} = \frac{1}{g^2 a C_2} N = \frac{1}{g^2 a C_2} N_C \] (91)

which, after rescaling the coupling constant according to the ’t Hooft prescription \(g^2 N_C \to g^2\), varies as \(N_C\) in the large \(N_C\) limit. \(1/N_C\) is the “coupling constant”; thus the baryon mass again varies as the inverse of the coupling constant as a soliton mass should do [16].

C. Fourth order

The fourth order corrections to the meson energies are given by the matrix elements (65), (66) and (67), where the projection operator \(\Pi_M\) eliminates the states proportional to \(|M>\). Again one can construct eigenstates of the unperturbed Hamiltonian to evaluate the function of \(H_e\) in Eqs. (65), (66) and (67).

First consider Eq. (65). The non vanishing terms in Eq. (65) can be grouped as follows

\[ <M|H_q^\dagger \lambda_M H_q \lambda_M H_q^\dagger \lambda_M H_q |M> = 2 \left[ <M|H_q^\dagger \lambda_M H_q \lambda_M H_q^\dagger \lambda_M H_q |M> + <M|H_q^\dagger \lambda_M H_q \lambda_M H_q^\dagger \lambda_M H_q |M> \right] \]

\[ + 4 \left[ <M|H_q \lambda_M H_q^\dagger \lambda_M H_q \lambda_M H_q^\dagger \lambda_M H_q |M> + <M|H_q \lambda_M H_q^\dagger \lambda_M H_q \lambda_M H_q^\dagger \lambda_M H_q |M> \right] \]

\[ + <M|H_q^\dagger \lambda_M H_q \lambda_M H_q^\dagger \lambda_M H_q |M> \] , (92)

where

\[ \lambda_M (H_e) = \frac{\Pi_M}{E_M^{(0)} - H_e} \]
To compute these matrix elements, first note that the two external projection operators do not affect the calculations. The projection operator in the middle, instead, does not allow patterns in which there are fermion operators creating and destroying the same quark at the same lattice site. For example, a term of the form

$$\ldots \frac{\Pi_M}{E_M^{(0)} - H_e} \frac{1}{4a^2} \sum_{\vec{r}, \vec{n}} \Psi_A^\dagger(\vec{r}) U_{AB}[\vec{r}, \vec{n}] \Psi_B(\vec{r} + \vec{n}) \Psi_C^\dagger(\vec{r} + \vec{n}) U_{CD}[\vec{r}, \vec{n}] \Psi_D(\vec{r}) |M>$$

is eliminated by the projection operator since it gives rise to a state with the same energy of a single link meson.

In order to illustrate the method used in the computation of the meson energy, one should focus the attention on the first two terms in Eq.\((92)\) and evaluate them for the generic meson \((70)\). Note that in the first of the two terms the projection operators are irrelevant. Let us define

$$\langle A \rangle = <M|H_q^\dagger \Lambda_M H_q^\dagger \Lambda_M H_q \Lambda_M H_q|M> \quad (93)$$

Using Eq.\((74)\), \((93)\) may be rewritten as

$$\langle A \rangle = \frac{4a^2}{g^2 C_2^2} \left[ <M|H_q^\dagger H_q \Lambda_M H_q|M> + \frac{1}{(N_C - 3)^2} \sum_{\vec{r}, [\vec{r}, \vec{n}]} S(\vec{r}) \Psi_A^\dagger(\vec{r}) U_{AB}[\vec{r}, \vec{n}] \Psi_B(\vec{r} + \vec{n}) \Psi_C^\dagger(\vec{r} + \vec{n}) U_{CD}[\vec{r}, \vec{n}] \Psi_D(\vec{r}) |0> \right]$$

Making use of Eq.\((3)\), Eq.\((22)\), Eq.\((22)\) and after constructing a suitable eigenstate of \(H_e\), one finds

$$\Lambda_M(H_e) U_{AB}[\vec{r}, \vec{n}] U_{CD}[\vec{r}, \vec{n}] U_{EF}[\vec{r}, \vec{n}] |0>$$

$$= - \frac{a}{g^2 C_2} \left[ U_{AB}[\vec{r}, \vec{n}] U_{CD}[\vec{r}, \vec{n}] U_{EF}[\vec{r}, \vec{n}] + \frac{1}{N_C - 2} \left( U_{AB}[\vec{r}, \vec{n}] U_{CD}[\vec{r}, \vec{n}] U_{EF}[\vec{r}, \vec{n}] \right) \right]$$

Taking into account Eq.\((32)\), Eq.\((34)\) becomes

$$\langle A \rangle =$$
\[
\frac{4a^3}{g^6C_2} \left\{ - <M^\dagger | H_q^\dagger H_q H_q | M> \right. \\
\left. - \frac{1}{N_C - 2} \left[ <M^\dagger | H_q^\dagger H_q H_q \frac{1}{4a^2} \sum_{[\vec{r}, \hat{n}]} \Psi_G^\dagger (\vec{r} + \hat{n}) U_{GH}[\vec{r}, \hat{n}] \Psi_H(\vec{r}) \Psi_I (\vec{r} + \hat{n}) U_{IL}[\vec{r}, \hat{n}] \Psi_L(\vec{r}) | M> \right. \\
\left. + <M^\dagger | H_q^\dagger H_q H_q \frac{1}{2a} \sum_{[\vec{r}, \hat{n}]} \eta(\hat{n}) \Psi_G^\dagger (\vec{r} + \hat{n}) U_{GH}[\vec{r}, \hat{n}] \Psi_H(\vec{r}) H_q \sum_{\vec{r}} S(\vec{r}) \Psi_M^\dagger (\vec{r} + \hat{n}) U_{MN}[\vec{r}, \hat{n}] \Psi_N(\vec{r}) | 0 > \right. \\
\left. + <M^\dagger | H_q^\dagger H_q H_q \frac{1}{2a} \sum_{[\vec{r}, \hat{n}]} \eta(\hat{n}) \Psi_G^\dagger (\vec{r} + \hat{n}) U_{IL}[\vec{r}, \hat{n}] \Psi_L(\vec{r}) \sum_{\vec{r}} S(\vec{r}) \Psi_M^\dagger (\vec{r} + \hat{n}) U_{MN}[\vec{r}, \hat{n}] \Psi_N(\vec{r}) | 0 > \right. \\
\left. \frac{6}{(N_C - 4)(N_C - 2)} <M^\dagger | H_q^\dagger H_q \frac{1}{4a^2} \sum_{[\vec{r}, \hat{n}]} \Psi_C^\dagger (\vec{r} + \hat{n}) U_{GH}[\vec{r}, \hat{n}] \Psi_H(\vec{r}) \Psi_I (\vec{r} + \hat{n}) U_{IL}[\vec{r}, \hat{n}] \Psi_L(\vec{r}) \right. \\
\left. + \frac{8a^2(N_C - 1)}{g^6C_2(N_C - 3)(N_C - 2)} \left[ <M^\dagger | H_q^\dagger H_q \sum_{[\vec{r}, \hat{n}]} \eta(\hat{n}) \Psi_I^\dagger (\vec{r} + \hat{n}) U_{IL}[\vec{r}, \hat{n}] \Psi_L(\vec{r}) \sum_{\vec{r}} S(\vec{r}) \Psi_M^\dagger (\vec{r} + \hat{n}) U_{MN}[\vec{r}, \hat{n}] \Psi_N(\vec{r}) | 0 > \right. \\
\left. + \frac{1}{a(N_C - 4)} <M^\dagger | H_q^\dagger \Psi_M^\dagger (\vec{r} + \hat{n}) U_{MN}[\vec{r}, \hat{n}] \Psi_N(\vec{r}) | 0 > \right. \\
\left. - \frac{4a(N_C - 1)}{g^6C_2(N_C - 3)^2(N_C - 2)} \left[ <0 | \sum_{\vec{r}} S^*(\vec{r}) \Psi_A^\dagger (\vec{r}) U_A^\dagger [\vec{r}, \hat{n}] \Psi_B(\vec{r} + \hat{n}) \right. \\
\left. \Psi_C^\dagger (\vec{r}) U_{CD}[\vec{r}, \hat{n}] \Psi_D(\vec{r} + \hat{n}) H_q \Psi_D^\dagger (\vec{r} + \hat{m}) U_{IL}[\vec{r}, \hat{m}] \Psi_L(\vec{r}) \right. \\
\left. \sum_{\vec{r}} S(\vec{r}) \Psi_M^\dagger (\vec{r} + \hat{m}) U_{MN}[\vec{r}, \hat{m}] \Psi_N(\vec{r}) | 0 > \right. \\
\left. + \frac{1}{a(N_C - 4)} <0 | \sum_{\vec{r}} S^*(\vec{r}) \Psi_A^\dagger (\vec{r}) U_A^\dagger [\vec{r}, \hat{n}] \Psi_B(\vec{r} + \hat{n}) \right. \\
\left. \Psi_C^\dagger (\vec{r}) U_{CD}[\vec{r}, \hat{n}] \Psi_D(\vec{r} + \hat{n}) H_q \Psi_D^\dagger (\vec{r} + \hat{m}) U_{GH}[\vec{r}, \hat{m}] \Psi_H(\vec{r}) \right. \\
\left. \sum_{[\vec{r}, \hat{m}]} \eta(\hat{m}) \Psi_I^\dagger (\vec{r} + \hat{m}) U_{IL}[\vec{r}, \hat{m}] \Psi_L(\vec{r}) \sum_{\vec{r}} S(\vec{r}) \Psi_M^\dagger (\vec{r} + \hat{m}) U_{MN}[\vec{r}, \hat{m}] \Psi_N(\vec{r}) | 0 > \right. \right\}. \tag{96}
\]

The matrix elements in Eq. (96) are evaluated by means of the integrals over the gauge group elements given in Appendix C.

Group integration yields a nonvanishing result only if each link exhibits a combination of matrices from which a color singlet may be formed. Then, for each matrix element one should compute all the possible integrals obtained by putting three \(UU^\dagger\) pairs on a different link [and then using Eq. (C4) or two of them on the same link [Eqs. (C4) and (C5)], or all
the three on the same link [Eq. (C6)]. For $\rho$ this term gives

$$[(A)]_\rho = \frac{1}{4g^6aC_2^3} \left\{ -86 + \left( \frac{17}{2} N_C \right) N - \frac{N_C^2}{4} N^2 + \frac{4(N_C - 1)(10N_C - 49)}{(N_C - 2)(N_C - 4)} - \frac{5N_C(N_C - 1)}{2(N_C - 2)} N \right\}.$$  

Now, one defines

$$\text{(B)} = < M | H_q^\dagger \Lambda_M H_q^\dagger | M \dagger > .$$  

Combining Eq. (73) and the action of $\Pi_M$, Eq. (88) becomes

$$\text{(B)} = \frac{4a^2}{g^4C_2^2} < M | H_q^\dagger H_q^\dagger \Lambda_M H_q | M \dagger >$$  

$$+ \frac{1}{16a^4} < M | \sum_{[\vec{r},\vec{n}]} \Psi_A^\dagger(\vec{r}) U_{AB}[\vec{r},\vec{n}] \Psi_B(\vec{r} + \vec{n}) \Psi_C^\dagger(\vec{r}) U_{CD}[\vec{r},\vec{n}] \Psi_D(\vec{r} + \vec{n})$$  

$$\Lambda_M \Psi_E^\dagger(\vec{r} + \vec{n}) U_{EF}[\vec{r},\vec{n}] \Psi_F(\vec{r}) \Psi_G^\dagger(\vec{r} + \vec{n}) U_{GH}[\vec{r},\vec{n}] \Psi_H(\vec{r}) | M \dagger > .$$  

Using again Eq. (8), Eq. (9) and Eq. (22), one finds

$$\Lambda_M(H_e) U_{AB}[\vec{r},\vec{n}] U_{CD}[\vec{r}^\dagger,\vec{m}] U_{EF}^\dagger[\vec{r}^\dagger,\vec{l}] | 0 > - \frac{a}{g^2C_2^2} U_{AB}[\vec{r},\vec{n}] U_{CD}[\vec{r}^\dagger,\vec{m}] U_{EF}^\dagger[\vec{r}^\dagger,\vec{l}] | 0 >$$  

and

$$\Lambda_M(H_e) U_{AB}[\vec{r},\vec{n}] U_{CD}[\vec{r}^\dagger,\vec{m}] U_{EF}^\dagger[\vec{r}^\dagger,\vec{l}] | 0 > - \frac{2aN_C}{g^2(N_C + 1)(N_C - 2)} U_{AB}[\vec{r},\vec{n}] U_{CD}[\vec{r}^\dagger,\vec{m}] U_{EF}^\dagger[\vec{r}^\dagger,\vec{l}] | 0 > .$$  

Taking into account Eq. (100) and Eq. (101), one has

$$\text{(B)} = - \frac{4a^3}{g^6C_2^3} < M | H_q^\dagger H_q^\dagger \Pi_M H_q | M \dagger >$$  

$$+ \left( \frac{N_C - 1}{16a^4(N_C - 2)} \right) < M | \sum_{[\vec{r},\vec{n}]} \Psi_A^\dagger(\vec{r}) U_{AB}[\vec{r},\vec{n}] \Psi_B(\vec{r} + \vec{n}) \Psi_C^\dagger(\vec{r}) U_{CD}[\vec{r},\vec{n}] \Psi_D(\vec{r} + \vec{n})$$  

$$\Pi_M \Psi_E^\dagger(\vec{r} + \vec{n}) U_{EF}[\vec{r},\vec{n}] \Psi_F(\vec{r}) \Psi_G^\dagger(\vec{r} + \vec{n}) U_{GH}[\vec{r},\vec{n}] \Psi_H(\vec{r}) | M \dagger > .$$  

The two different combinations of link variables allowed for the first term in Eq. (102) are those where the gauge fields of the meson operators $M$ and $M^\dagger$ are defined on the same link. Then, the two possibilities may be represented by

$$U_{AB}[\vec{r},\vec{n}] U_{CD}^\dagger[\vec{r}^\dagger,\vec{m}] U_{EF}^\dagger[\vec{r}^\dagger,\vec{l}] U_{GH}[\vec{r}^\dagger,\vec{m}] U_{IL}[\vec{r},\vec{n}] U_{MN}^\dagger[\vec{r},\vec{n}]$$  

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Using Eq. (74), Eq. (75), and the result (82), the contribution of Eq. (66) to $N$ must be defined on the same link, term in Eq. (102) there is only one possibility, i.e., the gauge fields of the meson operators must be defined on the same link,

$$U_{AB}[\vec{r}, \hat{n}] U_{CD}^{\dagger}[\vec{r'}, \hat{m}] U_{EF}[\vec{r'}, \hat{l}] U_{GH}[\vec{r'}, \hat{m}] U_{IL}[\vec{r'}, \hat{l}] U_{MN}^{\dagger}[\vec{r}, \hat{n}]$$

where the link variables in this expression have the same ordering they have in Eq. (102). Since the gauge variables of the quark Hamiltonian are on the same link, for the second term in Eq. (102) there is only one possibility, i.e., the gauge fields of the meson operators must be defined on the same link,

$$U_{AB}[\vec{r}, \hat{n}] U_{CD}^{\dagger}[\vec{r'}, \hat{m}] U_{EF}[\vec{r'}, \hat{m}] U_{GH}[\vec{r'}, \hat{m}] U_{IL}[\vec{r'}, \hat{m}] U_{MN}^{\dagger}[\vec{r}, \hat{n}]$$

After the integration over the gauge fields one gets for $\rho$

$$[(B) \rho] = \frac{1}{4g^6aC^2} \left[ -72 - 12N_C + \frac{12(N_C - 1)^2}{N_C - 2} + N \left( 8N_C + \frac{N_C^2}{2} - \frac{N_C(N_C - 1)^2}{2(N_C - 2)} \right) - \frac{N_C^2}{4} N^2 \right] .$$

Analogously, the other terms of Eq. (92) yield the following contributions for $\rho$:

$$< M|H_q^\dagger \Lambda_M H_q \Lambda_M H_q^\dagger \Lambda_M H_q^\dagger|M^>| = \frac{1}{4g^6aC^2} \left[ 37 - 12N_C + N \left( 4N_C + \frac{N_C^2}{2} \right) \right]$$

$$- \frac{N_C^2}{8} N^2 - \frac{(N_C - 1)^3}{(N_C - 2)(N_C - 3)^2} \left( -12(N_C - 1) + \frac{N_C}{2}(N_C - 1)N \right)$$

$$< M|H_q^\dagger \Lambda_M H_q \Lambda_M H_q^\dagger \Lambda_M H_q^\dagger|M^>| = \frac{1}{4g^6aC^2} \left[ -37 - 12N_C + N \left( 4N_C + \frac{N_C^2}{2} \right) \right]$$

$$- \frac{N_C^2}{8} N^2 - \frac{(N_C - 1)^3}{(N_C - 2)(N_C - 3)^2} \left( -12(N_C - 1) + \frac{N_C}{2}(N_C - 1)N \right)$$

$$< M|H_q^\dagger \Lambda_M H_q \Lambda_M H_q^\dagger \Lambda_M H_q^\dagger|M^>| = \frac{1}{4g^6aC^2} \left[ -34 - 12N_C + \frac{12(N_C - 1)^2}{N_C - 2} \right]$$

$$+ N \left( 4N_C + \frac{N_C^2}{2} - \frac{N_C(N_C - 1)^2}{2(N_C - 3)(N_C - 2)} \right) - \frac{N_C^2}{8} N^2$$

$$< M|H_q^\dagger \Lambda_M H_q \Lambda_M H_q^\dagger \Lambda_M H_q^\dagger|M^>| = \frac{1}{4g^6aC^2} \left[ 16 + 4N_C - \frac{4(N_C - 1)^3}{(N_C - 3)(N_C - 2)} - N_CN \right]$$

$$< M|H_q^\dagger \Lambda_M H_q \Lambda_M H_q^\dagger \Lambda_M H_q^\dagger|M^>| = \frac{1}{4g^6aC^2} \left[ 16 + 4N_C - \frac{4(N_C - 1)^2}{N_C - 2} - N_CN \right] .$$

Using Eq. (14), Eq. (75), and the result (82), the contribution of Eq. (66) to $E_M^{(4)}$ for $\rho$ may be easily computed

$$\left( < M|H_q^\dagger \Pi_M M^< M|H_q \Pi_M (E_M^{(0)} - H_e)^2 H_q^\dagger M^> \right)$$

$$= \frac{2}{g^6aC^2} \left[ \frac{3N_C - 13}{N_C - 3} - \frac{N_C}{2}N \right] \left[ \frac{N_C^2 - 18N_C + 37}{(N_C - 3)^2} + \frac{N_C}{2}N \right] .$$

Collecting all these contributions, one finds that the $N$-dependent terms cancel against the quark Hamiltonian contribution to $E_0^{(4)}$ [see Eq. (47)] and the remaining terms are $N$-
independent. This is a very good check of our computation since the meson mass should be an intensive quantity.

Adding up all the $N$-independent terms the contribution due to the quark Hamiltonian is given by

$$E_{\rho}^{(4)} - E_{q0}^{(4)} = \frac{31180 - 48674 N_C + 29051 N_C^2 - 8183 N_C^3 + 1069 N_C^4 - 51 N_C^5}{2g^6 a C_2^3(N_C - 4)(N_C - 3)^3(N_C - 2)}$$  \hspace{1cm} (105)$$

where $E_{q0}^{(4)}$ is the contribution to the fourth order vacuum energy due to the sole quark Hamiltonian \textit{(17)}. The results for the other mesons are listed below (see also Appendix D for the results for each matrix element):

$$E_{\pi}^{(4)} - E_{q0}^{(4)} = \frac{29452 - 45650 N_C + 29963 N_C^2 - 7471 N_C^3 + 949 N_C^4 - 43 N_C^5}{2g^6 a C_2^3(N_C - 4)(N_C - 3)^3(N_C - 2)}$$  \hspace{1cm} (106)$$

$$E_{\omega}^{(4)} - E_{q0}^{(4)} = \frac{29452 - 45650 N_C + 26963 N_C^2 - 7471 N_C^3 + 949 N_C^4 - 43 N_C^5}{2g^6 a C_2^3(N_C - 4)(N_C - 3)^3(N_C - 2)}$$  \hspace{1cm} (107)$$

$$E_{b_1}^{(4)} - E_{q0}^{(4)} = \frac{36172 - 56386 N_C + 33803 N_C^2 - 9671 N_C^3 + 1309 N_C^4 - 67 N_C^5}{2g^6 a C_2^3(N_C - 4)(N_C - 3)^3(N_C - 2)}$$  \hspace{1cm} (108)$$

$$E_{a_1}^{(4)} - E_{q0}^{(4)} = \frac{124972 - 205002 N_C + 130895 N_C^2 - 40619 N_C^3 + 6121 N_C^4 - 359 N_C^5}{2g^6 a C_2^3(N_C - 4)(N_C - 3)^3(N_C - 2)}$$  \hspace{1cm} (109)$$

$$E_{f_2}^{(4)} - E_{q0}^{(4)} = \frac{29452 - 45650 N_C + 26963 N_C^2 - 7471 N_C^3 + 949 N_C^4 - 43 N_C^5}{2g^6 a C_2^3(N_C - 4)(N_C - 3)^3(N_C - 2)}$$  \hspace{1cm} (110)$$

$$E_{f_0}^{(4)} - E_{q0}^{(4)} = \frac{76396 - 127034 N_C + 83111 N_C^2 - 26803 N_C^3 + 4273 N_C^4 - 271 N_C^5}{2g^6 a C_2^3(N_C - 4)(N_C - 3)^3(N_C - 2)}$$  \hspace{1cm} (111)$$

Equation \textit{(17)} yields the magnetic contribution to $E_{M}^{(4)}$ and it is at the second order in the strong coupling expansion. This term gives the same result for all the mesons and reads

$$(E_{M}^{(2)})_{\text{magnetic}} = -\frac{N}{g^6 a C_2^2} + \frac{2}{g^6 a(2N_C^2 - 1)(N_C^2 - 1)} + \frac{2}{g^6 a(2N_C - 3)(N_C^2 - 1)}$$  \hspace{1cm} (112)$$

Again the $N$-dependent term in Eq.\textit{(112)} vanishes in the difference between Eq.\textit{(112)} and the magnetic contribution to the fourth order correction to the vacuum energy $E_{m0}^{(4)}$, Eq.\textit{(17)}. As for the second order, the fourth order correction to the meson energy is finite and well defined after rescaling the coupling constant $g^2 N_C \rightarrow g^2$ and taking the 't Hooft limit (large $N_C$ with $g^2 N_C$ fixed).
D. The meson spectrum

The lattice excitation masses are given by subtracting the energy of the ground state from the energies of the excitations

\[ m_M = E_M - E_0 \]

Using the results of Eqs. (82)-(88) and of Eqs. (105)-(112), one gets in the large \( N_C \) limit

\[
\begin{align*}
m_{\pi_0} &= g^2 \left( \frac{1}{4} + 6\epsilon^2 - 171\epsilon^4 \right) \\
m_\rho &= g^2 \left( \frac{1}{4} + 6\epsilon^2 - 203\epsilon^4 \right) \\
m_\omega &= g^2 \left( \frac{1}{4} + 6\epsilon^2 - 171\epsilon^4 \right) \\
m_{b_1} &= g^2 \left( \frac{1}{4} + 10\epsilon^2 - 267\epsilon^4 \right) \\
m_{a_1} &= g^2 \left( \frac{1}{4} + 14\epsilon^2 - 1435\epsilon^4 \right) \\
m_{f_2} &= g^2 \left( \frac{1}{4} + 14\epsilon^2 - 875\epsilon^4 \right) \\
m_{f_0} &= g^2 \left( \frac{1}{4} + 18\epsilon^2 - 1083\epsilon^4 \right)
\end{align*}
\]

where \( \epsilon = 1/g^2 \) and with \( g \) we indicate the rescaled coupling constant \( g^2 N_C \rightarrow g^2 \). Equations (113)-(119) provide the value of the meson masses up to the fourth order in the strong coupling expansion.

V. LATTICE VS. CONTINUUM

The series given in the preceding section are derived for large \( g^2 \). Since from renormalization group arguments for an asymptotically free theory \( g^2 = -c/\ln a \) for small \( a \), the series for the meson masses are valid only for large lattice spacings. To compare the results of the strong coupling expansion with the continuum theory one needs some method of continuing the series to the region in which \( g^2 = 0 \), i.e., \( \epsilon = \infty \). To make this extrapolation possible, it is customary to make use of Padé approximants \([20]\), which allows one to extrapolate a series expansion beyond the convergence radius. For this purpose one should consider the mass ratios, expand them as power series in \( y = \epsilon^2 \) and then use [1, 1] Padé approximants.
by writing the mass ratios in the form

\[ P_1^1 = \frac{1 + ay}{1 + by} \]

where \(a\) and \(b\) are determined by expanding to order \(y^2\) and equating coefficients. In the continuum limit this ratios yields \(a/b\).

The results obtained with this method for the mass ratios of the single link mesons are listed in Table 1. For each mass ratio considered here, the \([1, 1]\) Padé approximant exists for

| Mass ratios | Our results | Banks et al. | Experimental values |
|-------------|-------------|--------------|---------------------|
| \(\frac{m_\pi}{m_{b_1}}\) | 0.75 | 0.86 | 0.11 |
| \(\frac{m_\omega}{m_{b_1}}\) | 0.75 | 0.86 | 0.63 |
| \(\frac{m_\rho}{m_{b_1}}\) | 0.71 | 0.81 | 0.62 |
| \(\frac{m_b}{m_{f_0}}\) | 0.82 | 1.03 | 0.90 |
| \(\frac{m_{b_1}}{m_{a_1}}\) | 0.95 | 0.93 | 0.98 |
| \(\frac{m_{b_1}}{m_{f_2}}\) | 0.92 | 1.00 | 0.97 |

positive values for \(a\) and \(b\). Therefore, the extrapolation from \(y = 0\) to \(y = \infty\) is singularity free in our approximation.

We considered the ratios between meson masses involving the \(b_1\) meson. For these ratios the results are in very good agreement with the well known experimental values except for the pion mass. This foreseeable failure is due to the lack of full chiral symmetry in the theory for large lattice spacing. The \(\pi-\rho\) splitting is so tiny because of the lack of significant spin-spin forces in the first four orders of strong-coupling perturbation theory. Magnetic field effects, loops of flux, are just not important through this order. However, at sixth and eighth order such effects are important and should provide improved results.

The mass of \(f_0\) seems to be quite large but in agreement with the experimental value of the meson \(f_0(1370)\).
Our results show that the large $N_C$ expansion provides a very good and systematic theoretical setting to evaluate physical quantities of phenomenological interest.

VI. CONCLUDING REMARKS

In this paper we used the strong-coupling expansion of lattice QCD to compute -for large $N_C$- the low-lying unflavored meson spectrum. Our large $N_C$ Hamiltonian approach with staggered fermions evidences that the possible ground states of strongly coupled lattice QCD are those of a spin $N_C/2$ antiferromagnetic Ising model. Choosing one of the two ground states amounts then to the spontaneous breaking of the discrete chiral symmetry corresponding to translations by a lattice site. As a consequence a nonvanishing chiral condensate should arise; it would then be interesting to compute -within our formalism- the chiral condensate on the lattice and then compare it with the results of numerical simulations.

Mesons are created by operators that, acting on the vacuum, create $q\bar{q}$ states with the desired quantum numbers. Their energy is computed up to the fourth order in the strong coupling expansion. The meson masses are obtained by subtracting the vacuum state energy from the energies of the excitations. After rescaling the coupling constant $g^2 N_C \to g^2$, according to the 't Hooft prescription, and taking the $N_C \to \infty$ limit one finds that the series for the meson masses are well defined. Since it is expected that the continuum limit occurs without any phase transitions, the series are analytically continued by using Padé approximants.

With the exception of the pion, which turns out to be degenerate in mass with $\omega$, the results we obtained are, already at the fourth order, in good agreement with observed values. Higher orders of the strong coupling expansion should evidence the splitting between the masses of $\pi_0$ and $\omega$ since the effects of the magnetic field would be more pronounced.

The strong-coupling limit of QCD is not universal, thus adding an irrelevant operator to the Hamiltonian leads to the same physical predictions in the continuum limit. This allows one to introduce arbitrary parameters, the coefficients of these irrelevant operators, which are then fixed by fitting the experimental data. We have shown that, in the 't Hooft limit, the meson masses can be computed without introducing irrelevant operators and consequently without arbitrary parameters. Irrelevant operators are usually introduced \[13\], in fact, mainly because in this way the meson masses are well defined for $N_C = 3$. Our large $N_C$
approach does not need any irrelevant operator; it yields series expansions for the meson masses which are free of divergences and thus well defined.

We determined the ratios between two single link mesons even if these ratios are expected to be less reliable than the ratios involving meson masses relative to the nucleon mass. The reason is that all the mesons are degenerate at zeroth order so small differences of second and fourth Taylor series coefficients control the [1, 1] Padé approximants, and a considerable amount of information in the series is lost. The good agreement between our results and the experimental values for the ratios between meson masses shows that the large \( N_C \) limit is very effective in the strong-coupling region also. It should be interesting to extend our procedure to provide a large \( N_C \) evaluation of the ratios between the meson masses and the nucleon mass.

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APPENDIX A: THE STAGGERED FERMIONS FORMALISM

In this appendix the well known formalism for staggered fermions originally developed by Kogut and Susskind \[21\] is reviewed. Other formulations exist in literature \[22\] but in this paper we use the notation and the conventions of refs. \[13, 17\].

In this approach the two nonstrange quark fields \( u \) and \( d \) are represented by a single-component lattice field \( \Psi(\vec{r}) \). In a three-dimensional cubic lattice with sites labeled by triplets of indices \( \vec{r} = (x, y, z) \) one may define the lattice derivative operator

\[
\nabla_i \psi(\vec{r}) = \frac{1}{2a} \left[ \psi(\vec{r} + \hat{i}) - \psi(\vec{r} - \hat{i}) \right].
\]

(A1)

With the definition (A1) the massless Dirac equation

\[
\dot{\psi} = -\vec{\alpha} \cdot \nabla \psi
\]

becomes, on the lattice,

\[
\dot{\psi}(\vec{r}) = -\frac{1}{2a} \sum_{\vec{n}} \vec{\alpha} \cdot \hat{n} [\psi(\vec{r} + \hat{n}) - \psi(\vec{r} - \hat{n})].
\]

(A2)
FIG. 1: Labeling of lattice sites

In Eq. (A2) \( \vec{\alpha} \) is the vector of Dirac matrices in the representation

\[
\gamma_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix}
\] (A3)

The dispersion law for Eq. (A2) is given by

\[
\omega = (\alpha_x \sin l_x + \alpha_y \sin l_y + \alpha_z \sin l_z)/a
\]

and the low-energy spectrum is found at the eight corners of a cube in \( k \) space.

It is customary to remove the degeneracy by reducing the degrees of freedom. One may consider a one-component field, \( \phi(r) \), per lattice site which will represent a component of the Dirac field. The lattice is subdivided into four sublattices to place the four components of a conventional Dirac field and the corners of a unit cube are labeled as shown in Fig. 1. By further subdividing the lattice in sublattices there are two complete and independent Dirac fields which exhaust the low-frequency spectrum of the Dirac equation. The lattice is now divided into a sublattice of sites for which \( y \) is even and another of sites for which \( y \) is odd. The fields are conveniently relabeled as follows

\[
\psi_i = f_i \quad (y = \text{even})
\]

\[
\psi_i = g_i \quad (y = \text{odd}).
\]

Writing Eq. (A2) in Fourier transformed variables, one finds for \( f \) and \( g \)

\[
\omega a \dot{f} = (\alpha_z \sin k_z + \alpha_x \sin k_x)f + (\alpha_y \sin k_y)g \quad \text{(A4)}
\]

\[
\omega a \dot{g} = (\alpha_z \sin k_z + \alpha_x \sin k_x)g + (\alpha_y \sin k_y)f. \quad \text{(A5)}
\]
There are two combinations of \( f \) and \( g \) which are conventional Dirac fields for long wave-lengths

\[
u_i = f_i + g_i \quad (i = 1, 2, 3, 4)
\]

(A6)

and

\[
d_1 = f_2 - g_2 \\
d_2 = -(f_1 - g_1) \\
d_3 = -(f_4 - g_4) \\
d_4 = f_3 - g_3
\]

(A7)

This is easily seen by summing up Eqs. (A4) and (A5). One finds for the component of the \( u \) field in the representation (A3)

\[
\omega \dot{u}_1(\vec{k}) = \sin k_x u_4 - i \sin k_y u_4 + \sin k_z u_3 \\
\omega \dot{u}_2(\vec{k}) = \sin k_x u_3 + i \sin k_y u_3 - \sin k_z u_4 \\
\omega \dot{u}_3(\vec{k}) = \sin k_x u_2 - i \sin k_y u_2 + \sin k_z u_1 \\
\omega \dot{u}_4(\vec{k}) = \sin k_x u_1 + i \sin k_y u_1 - \sin k_z u_2
\]

(A8)

and identical equations for the \( d \)'s. If one now considers those normal modes with \( \vec{k} \) small and set \( \vec{k} = a\vec{K} \), Eq. (A8), for small \( \vec{k} \), takes the form

\[
\omega \dot{u}_1(\vec{K}) = K_x u_4 - i K_y u_4 + K_z u_3 \\
\omega \dot{u}_2(\vec{K}) = K_x u_3 + i K_y u_3 - K_z u_4 \\
\omega \dot{u}_3(\vec{K}) = K_x u_2 - i K_y u_2 + K_z u_1 \\
\omega \dot{u}_4(\vec{K}) = K_x u_1 + i K_y u_1 - K_z u_2
\]

(A9)

and the same with \( u \leftrightarrow d \). Equations (A9) are the normal Dirac equations for \( u \) and \( d \) in the representation (A3).

One may now turn to the free massless Dirac Hamiltonian in the continuum

\[
H = -i(u^\dagger \alpha_i \partial_i u + d^\dagger \alpha_i \partial_i d)
\]

and write it in the lattice form by means of the definitions (A6) and (A7) for the \( u \) and \( d \) fields in the representation (A3). In terms of the one component field \( \phi(\vec{r}) \) one gets

\[
H = -\frac{i}{2a} \sum_{\vec{r}} \left[ (\phi^\dagger(\vec{r})\phi(\vec{r} + \hat{z}) - h.c.)(-1)^{x+y} 
\right]
\]

32
+ (\phi^\dagger(\vec{r})\phi(\vec{r} + \hat{x}) - \text{h.c.}) \\
- i(\phi^\dagger(\vec{r})\phi(\vec{r} + \hat{y}) + \text{h.c.})(-1)^{x+y} \right]. 
\tag{A10} 

Equation (A10) may be written in a more symmetric form by defining the following functions on the lattice:

\begin{align*}
D(x, y) &= \frac{1}{2} \left[ (-1)^x + (-1)^y + (-1)^{x+y+1} + 1 \right] \\
A(n) &= \frac{1}{\sqrt{2}} \left[ i^n - \frac{1}{2} + (-1)^{n-1/2} \right]. 
\tag{A11} \\
A(y) A(y + 1) &= (-1)^y. 
\tag{A12} 
\end{align*}

On the lattice sites $D$ and $A$ are either 1 or -1 and therefore

\[ D^2 = A^2 = 1. \]

Furthermore, $D$ and $A$ satisfy the relations

\begin{align*}
D(y, x)D(y, x + 1) &= (-1)^y \\
D(y + 1, x + 1)D(y, x) &= (-1)^{x+y+1} \\
A(y)A(y + 1) &= (-1)^y. 
\tag{A13} 
\end{align*}

Defining the field

\[ \Psi(\vec{r}) = (-1)^y(i)^{x+z}A(y)D(x, z)\phi(\vec{r}) \]

and using the previous definitions for $A$ and $D$ the lattice Hamiltonian becomes

\[ H_q = \frac{1}{2} \sum_{\vec{r}} \left[ \Psi^\dagger(\vec{r} + \hat{x})\Psi(\vec{r})(-1)^x \right. \\
\left. \Psi^\dagger(\vec{r} + \hat{y})\Psi(\vec{r})(-1)^y \right. \\
\left. \Psi^\dagger(\vec{r} + \hat{z})\Psi(\vec{r})(-1)^y + \text{h.c.} \right], 
\tag{A15} 
\]

which is the quark Hamiltonian used in our paper.

\section*{APPENDIX B: CONSTRUCTION OF AN EIGENSTATE OF $H_e$}

In this Appendix we explicit the construction of eigenstates of the unperturbed Hamiltonian $H_e$ in order to evaluate the function of $H_e$ in the perturbative expansion. Using Eq.(3), Eq.(9) and Eq.(22), one finds

\[ [H_e, U^\dagger_{AB}[\vec{r}, \hat{n}]U^\dagger_{CD}[\vec{r}, \hat{m}]]|0> = \frac{g^2}{a} \left( C_2 U^\dagger_{AB}[\vec{r}, \hat{n}]U^\dagger_{CD}[\vec{r}, \hat{m}] \right] 
\]
Using Eq. (B1), one finds

\[
[H_e, U_{AB}^\dagger(\vec{r}, \hat{n}) U_{CD}^\dagger(\vec{r}, \hat{n})]|0 > = g^2 \frac{(N_C - 1)(N_C - 2)}{2aN_C} U_{AB}^\dagger(\vec{r}, \hat{n}) U_{CD}^\dagger(\vec{r}, \hat{n})|0 > .
\]  \hspace{1cm} (B2)

One may now look for an eigenstate of \( H_e \) with eigenvalue \( g^2 C_2/a \)

\[
H_e \left( U_{AB}^\dagger(\vec{r}, \hat{n}) U_{CD}^\dagger(\vec{r}, \hat{m}) + a U_{AB}^\dagger(\vec{r}, \hat{n}) U_{CD}^\dagger(\vec{r}, \hat{n}) \right) |0 > =
\]

\[
\frac{g^2}{a} C_2 \left( U_{AB}^\dagger(\vec{r}, \hat{n}) U_{CD}^\dagger(\vec{r}, \hat{m}) + a U_{AB}^\dagger(\vec{r}, \hat{n}) U_{CD}^\dagger(\vec{r}, \hat{n}) \right) |0 >
\]  \hspace{1cm} (B3)

Taking into account Eq. (B1) and Eq. (B2) one gets \( a = -1 \) and then the pertinent eigenstate is

\[
H_e \left( U_{AB}^\dagger(\vec{r}, \hat{n}) U_{CD}^\dagger(\vec{r}, \hat{m}) - U_{AB}^\dagger(\vec{r}, \hat{n}) U_{CD}^\dagger(\vec{r}, \hat{n}) \right) |0 > =
\]

\[
\frac{g^2}{a} C_2 \left( U_{AB}^\dagger(\vec{r}, \hat{n}) U_{CD}^\dagger(\vec{r}, \hat{m}) - U_{AB}^\dagger(\vec{r}, \hat{n}) U_{CD}^\dagger(\vec{r}, \hat{n}) \right) |0 >
\]  \hspace{1cm} (B4)

Using Eq. (B4), one may evaluate the function of \( H_e \) appearing in the perturbative expansion

\[
f(H_e) = \frac{\Pi_0}{E_0^{(0)} - H_e}.
\]

From Eq. (B2), one has

\[
f(H_e) U_{AB}^\dagger(\vec{r}, \hat{n}) U_{CD}^\dagger(\vec{r}, \hat{n})|0 > = - \frac{2aN_C}{g^2(N_C + 1)(N_C - 2)} U_{AB}^\dagger(\vec{r}, \hat{n}) U_{CD}^\dagger(\vec{r}, \hat{n})|0 > .
\]  \hspace{1cm} (B5)

Using Eq. (B5), one gets from Eq. (B4)

\[
f(H_e) U_{AB}^\dagger(\vec{r}, \hat{n}) U_{CD}^\dagger(\vec{r}, \hat{m})|0 > = - \frac{a}{g^2 C_2} \left( U_{AB}^\dagger(\vec{r}, \hat{n}) U_{CD}^\dagger(\vec{r}, \hat{m})
\right.
\]

\[
+ \frac{1}{N_C - 2} U_{AB}^\dagger(\vec{r}, \hat{n}) U_{CD}^\dagger(\vec{r}, \hat{n}) \delta([\vec{r}, \hat{n}] - [\vec{r}, \hat{m}]) \right) |0 > .
\]  \hspace{1cm} (B6)

**APPENDIX C: INTEGRATION OVER SU(N_C)**

In this appendix a table of the integrals over the group elements of \( SU(N_C) \) needed in the paper is provided.

It is well known that a basic ingredient to formulate \( QCD \) on a lattice is to define the measure of integration over the gauge degrees of freedom. Unlike the continuum gauge fields, the lattice gauge fields are \( SU(N_C) \) matrices with elements bounded in the range [0, 1];
Wilson \cite{7} proposed an invariant group measure, the Haar measure, for the integration over the group elements. The integral is defined so that, for any elements $g_1$ and $g_2$ of the group, one has

$$\int dU f(U) = \int dU f(Ug_1) = \int dU f(g_2U), \quad (C1)$$

with $f(U)$ a generic function over the group. When used in nonperturbative studies of gauge theory, the definition (C1) avoids the problem of introducing a gauge fixing, since the field variables are compact. The measure is normalized as

$$\int dU = 1. \quad (C2)$$

The strong coupling expansion for an $SU(N_C)$ gauge theory depends on the following identities for integration over link matrices \cite{19}

$$\int dUU_{ab} = \int dUU_{ab}^\dagger = 0 \quad (C3)$$

$$\int dUU_{ab}U_{cd}^\dagger = \frac{1}{N_C} \delta_{ad} \delta_{bc} \quad (C4)$$

$$\int dUU_{ab}U_{cd}^\dagger U_{ef}^\dagger U_{gh}^\dagger = \frac{1}{N_C(N_C^2-1)} \left( \delta_{ad} \delta_{be} \delta_{ef} \delta_{gh} + \delta_{ah} \delta_{bg} \delta_{cf} \delta_{de} \right) - \frac{1}{N_C(N_C^2-1)} (\delta_{ad} \delta_{be} \delta_{ef} \delta_{fg} + \delta_{ah} \delta_{bg} \delta_{ed} \delta_{fg}) \quad (C5)$$

One also needs the group integral over six elements, which occurs at the fourth order in the strong coupling expansion of the mass spectrum. The pertinent integral is \cite{8}

$$\int dUU_{ab}U_{cd}^\dagger U_{ef}^\dagger U_{gh}^\dagger U_{il}^\dagger = \frac{N_C^2-2}{N_C(N_C^2-1)(N_C^2-4)} \left( \delta_{ad} \delta_{eh} \delta_{in} \delta_{be} \delta_{fg} \delta_{lm} + \delta_{ah} \delta_{di} \delta_{en} \delta_{bc} \delta_{f} \delta_{gl} \right) + \delta_{ah} \delta_{de} \delta_{in} \delta_{bg} \delta_{cf} \delta_{lm} + \delta_{an} \delta_{de} \delta_{eh} \delta_{bm} \delta_{cl} \delta_{fg} + \delta_{an} \delta_{de} \delta_{hi} \delta_{bm} \delta_{cf} \delta_{gl} + \delta_{ah} \delta_{de} \delta_{en} \delta_{bg} \delta_{cl} \delta_{fm} \right)$$

$$+ \frac{1}{N_C(N_C^2-1)(N_C^2-4)} \left( \delta_{an} \delta_{de} \delta_{hi} \delta_{bm} \delta_{cf} \delta_{fg} + \delta_{ah} \delta_{de} \delta_{in} \delta_{bg} \delta_{cf} \delta_{gl} + \delta_{ah} \delta_{de} \delta_{in} \delta_{bm} \delta_{cf} \delta_{gl} + \delta_{an} \delta_{de} \delta_{hi} \delta_{bm} \delta_{cl} \delta_{fg} + \delta_{an} \delta_{de} \delta_{hi} \delta_{bm} \delta_{cl} \delta_{fg} \right)$$

$$+ \frac{2}{N_C(N_C^2-1)(N_C^2-4)} \left( \delta_{an} \delta_{de} \delta_{hi} \delta_{bm} \delta_{cf} \delta_{fg} + \delta_{ah} \delta_{de} \delta_{in} \delta_{bg} \delta_{cf} \delta_{fg} + \delta_{ah} \delta_{de} \delta_{in} \delta_{bm} \delta_{cf} \delta_{gl} + \delta_{ah} \delta_{de} \delta_{in} \delta_{bg} \delta_{cl} \delta_{fm} \right). \quad (C6)$$
APPENDIX D: MATRIX ELEMENTS

In this appendix the matrix elements useful for the computation of the \( \pi, \omega, b_1, a_1, f_2 \) and \( f_0 \) energies at the fourth order in the strong-coupling expansion are reported.

1. \( \pi \)

\[
< M | H_q^\dagger \Lambda_M H_q^\dagger \Lambda_M H_q \Lambda_M H_q | M > = \frac{1}{4g^6aC^3_2} \left\{ -76 + \left( \frac{17}{2} N_C \right) N - \frac{N_C^2}{4} N^2 - \frac{(N_C - 1)}{N_C - 2} \left[ \frac{5}{2} N_C N - 58 \right] \right. \\
- \frac{18(N_C - 1)}{N_C - 4} - \frac{4(N_C - 1)^2}{N_C - 3} \left[ \frac{(N_C N - 22)}{N_C - 2} + \frac{6}{N_C - 4} \right] \\
- \left. \frac{4(N_C - 1)^2}{(N_C - 3)^2} \left[ \frac{N_C/2N + 2N_C - 14}{N_C - 2} + \frac{6}{N_C - 4} \right] \right\} 
\]

(D1)

\[
< M | H_q^\dagger \Lambda_M H_q^\dagger \Lambda_M H_q \Lambda_M H_q^\dagger | M > = \frac{1}{4g^6aC^3_2} \left[ -72 - 12N_C \right] \\
+ \frac{12(N_C - 1)^2}{N_C - 2} + N \left( 8N_C + \frac{N_C^2}{2} - \frac{N_C(N_C - 1)^2}{2(N_C - 2)} - \frac{N_C^2}{4} N^2 \right) 
\]

(D2)

\[
< M | H_q \Lambda_M H_q^\dagger \Lambda_M H_q \Lambda_M H_q^\dagger | M > = \frac{1}{4g^6aC^3_2} \left[ -33 - 12N_C - \frac{N_C^2}{8} N^2 \right] \\
+ N \left( 4N_C + \frac{N_C^2}{2} - \frac{(N_C - 1)^4}{(N_C - 2)(N_C - 3)^2} \left( -12 + \frac{N_C}{2} N \right) \right) 
\]

(D3)

\[
< M | H_q^\dagger \Lambda_M H_q \Lambda_M H_q^\dagger \Lambda_M H_q | M > = \frac{1}{4g^6aC^3_2} \left[ -38 - 12N_C \right] \\
+ \frac{12(N_C - 1)^2}{N_C - 2} + N \left( 4N_C + \frac{N_C^2}{2} - \frac{N_C(N_C - 1)^2}{2(N_C - 2)} - \frac{N_C^2}{8} N^2 \right) 
\]

(D4)

\[
< M | H_q \Lambda_M H_q^\dagger \Lambda_M H_q \Lambda_M H_q | M > = \frac{1}{4g^6aC^3_2} \left[ -32 - 12N_C + \frac{12(N_C - 1)^3}{(N_C - 3)(N_C - 2)} \right] \\
+ N \left( 4N_C + \frac{N_C^2}{2} - \frac{N_C(N_C - 1)^3}{2(N_C - 3)(N_C - 2)} - \frac{N_C^2}{8} N^2 \right) 
\]

(D5)

\[
< M | H_q \Lambda_M H_q^\dagger \Lambda_M H_q \Lambda_M H_q^\dagger | M > = \frac{1}{4g^6aC^3_2} \left[ 14 + 4N_C - N_C N - \frac{4(N_C - 1)^3}{(N_C - 3)(N_C - 2)} \right] \\
+ 4N_C - \frac{4(N_C - 1)^2}{N_C - 2} - N_C N 
\]

(D6)
\[
\begin{align*}
\left< M | H_q - E^{(0)}_M H_q | M \right>& \left< M | H_q - E^{(0)}_M H_q | M \right> \\
= & \left( \frac{\Pi_M}{E^{(0)}_M - H_e} H_q M \right) \left( \frac{\Pi_M}{E^{(0)}_M - H_e} H_q M \right) \\
= & \frac{2}{g^6C_2^3} \left[ \frac{3N_C - 13}{N_C - 3} - \frac{N_C}{2} N \right] \left\{ -\frac{N_C^2 - 18N_C + 37}{(N_C - 3)^2} + \frac{N_C}{2} N \right\} \pi
\end{align*}
\]

(2. \( \omega \))

\[
\left< M | H_q^\dagger \Lambda_M H_q^\dagger \Lambda_M H_q \Lambda_M H_q | M \right> = \frac{1}{4g^6aC_2^3} \left\{ -96 + \left( \frac{17}{2} N_C \right) N - \frac{N_C^2}{4} N^2 - \frac{(N_C - 1)}{2} N_C N - 58 \right\} \\
- \frac{18(N_C - 1)}{N_C - 4} \left[ 4(N_C - 1)^2 \left( \frac{(N_C N - 22)}{N_C - 2} + \frac{6}{N_C - 4} \right) \right]
\]

(9)

\[
\left< M | H_q^\dagger \Lambda_M H_q^\dagger \Lambda_M H_q | M \right> = \frac{1}{4g^6aC_2^3} \left\{ -72 - 12N_C + \frac{12(N_C - 1)^2}{N_C - 2} \right. \\
+ \left. \frac{N_C^2}{8} N^2 - \frac{(N_C - 1)^4}{N_C - 2}(N_C - 3)^2 \left( -12 + \frac{N_C}{2} N \right) \right\}
\]

(10)

\[
\left< M | H_q^\dagger \Lambda_M H_q^\dagger \Lambda_M H_q | M \right> = \frac{1}{4g^6aC_2^3} \left\{ -41 - 12N_C + N \left( 4N_C + \frac{N_C^2}{2} \right) \right\}
\]

(11)

\[
\left< M | H_q^\dagger \Lambda_M H_q^\dagger \Lambda_M H_q^\dagger | M \right> = \frac{1}{4g^6aC_2^3} \left\{ -42 - 12N_C + \frac{12(N_C - 1)^2}{N_C - 2} \right. \\
+ \left. \frac{N_C^2}{8} N^2 \right\}
\]

(12)

\[
\left< M | H_q^\dagger \Lambda_M H_q^\dagger | M \right> = \frac{1}{4g^6aC_2^3} \left\{ -36 - 12N_C + \frac{12(N_C - 1)}{N_C - 3}(N_C - 2) \right. \\
+ \left. \frac{N_C^2}{8} N^2 \right\}
\]

(13)

\[
\left< M | H_q^\dagger | M \right> = \frac{1}{4g^6aC_2^3} \left\{ 22 + 4N_C - N_C N - \frac{4(N_C - 1)^3}{(N_C - 3)(N_C - 2)} \right\}
\]

(14)

\[
\left< M | H_q^\dagger | M \right> = \frac{1}{4g^6aC_2^3} \left\{ 24 + 4N_C - \frac{4(N_C - 1)^2}{N_C - 2} - N_C N \right\}
\]

(15)
\[
< M | H_q \Lambda_M H_q^\dagger \Lambda_M H_q \Lambda_M H_q | M > = \frac{1}{4g^6aC_2^3} \left\{ -76 + \left( \frac{17}{2}N_C \right) N - \frac{N_C^2}{4}N^2 - \frac{(N_C - 1)^2}{N_C - 2} \left[ \frac{5}{2}N_C N - 58 \right] \right. \\
- \frac{18(N_C - 1)N}{N_C - 4} - \frac{4(N_C - 1)^2}{N_C - 3} \left[ \frac{(N_C N - 22)}{N_C - 2} + \frac{6}{N_C - 4} \right] \\
- \frac{4(N_C - 1)^2}{(N_C - 3)^2} \left[ \frac{N_C/2N + 2N_C - 14}{N_C - 2} + \frac{6}{N_C - 4} \right] \left\} \right. \\
\] 

\[
\frac{1}{4g^6aC_2^3} \left( -72 - 12N_C + \frac{12(N_C - 1)^2}{N_C - 2} \right) \\
N \left( 8N_C + \frac{N_C^2}{2} - \frac{N_C(N_C - 1)^2}{2(N_C - 2)} \right) - \frac{N_C^2}{4}N^2 
\]

\[
< M | H_q \Lambda_M H_q^\dagger \Lambda_M H_q \Lambda_M H_q | M > = \frac{1}{4g^6aC_2^3} \left\{ -33 - 12N_C + \left( 4N_C + \frac{N_C^2}{2} \right) \right. \\
- \frac{N_C^2}{8}N^2 - \frac{(N_C - 1)^4}{(N_C - 2)(N_C - 3)^2} \left( -12 + \frac{N_C}{2}N \right) \left\} \right. \\
\] 

\[
\frac{1}{4g^6aC_2^3} \left( -38 - 12N_C + \frac{12(N_C - 1)^2}{N_C - 2} \right) \\
N \left( 4N_C + \frac{N_C^2}{2} - \frac{N_C(N_C - 1)^2}{2(N_C - 2)} \right) - \frac{N_C^2}{8}N^2 
\]

\[
< M | H_q \Lambda_M H_q^\dagger \Lambda_M H_q \Lambda_M H_q | M > = \frac{1}{4g^6aC_2^3} \left\{ -32 - 12N_C + \frac{12(N_C - 1)^3}{(N_C - 3)(N_C - 2)} \right. \\
+ N \left( 4N_C + \frac{N_C^2}{2} - \frac{N_C(N_C - 1)^3}{2(N_C - 3)(N_C - 2)} \right) - \frac{N_C^2}{8}N^2 \left\} \right. \\
\] 

\[
\frac{1}{4g^6aC_2^3} \left( -14 - 4N_C + N_CN + \frac{4(N_C - 1)^3}{(N_C - 3)(N_C - 2)} \right) 
\]

\[
< M | H_q \Lambda_M H_q^\dagger \Lambda_M H_q \Lambda_M H_q^\dagger | M > = \frac{1}{4g^6aC_2^3} \left[ -12 - 4N_C + \frac{4(N_C - 1)^2}{N_C - 2} + N_CN \right] 
\]
\[
\frac{\Pi_M}{E^{(0)}_M - H_e} \frac{E^{(0)}_M - H_e}{(E^{(0)}_M - H_e)^2} H_q |M > < M | H_q \frac{\Pi_M}{E^{(0)}_M - H_e} H_q |M >
\]
\[
= \frac{2}{g^6 a C_2^3} \left[ \frac{5 N_C - 19 - N_C}{N_C - 3} - \frac{N_C}{2} N \right] \left[ \frac{3 N_C^2 - 30 N_C + 55}{(N_C - 3)^2} + \frac{N_C}{2} N \right]
\]

4. \(a_1\)
\[
< M^\dagger | H_q^\dagger \Lambda M H_q^\dagger \Lambda M H_q \Lambda M H_q |M >= \frac{1}{4g^6 a C_2^3} \left\{ -91 + \left( \frac{17}{2} N_C \right) N - \frac{N_C^2}{4} N^2 - \frac{(N_C - 1)}{N_C - 2} \left[ \frac{5}{2} N_C N - 58 \right] \right. \]
\[
- \frac{18(N_C - 1)}{N_C - 4} \left( \frac{N_C}{2} - \frac{5}{N_C - 4} \right) \left[ \frac{(N_C N - 22)}{N_C - 2} + \frac{6}{N_C - 4} \right] \]
\[
- \frac{4(N_C - 1)^2}{(N_C - 3)^2} \left[ \frac{N_C/2N + 2N_C - 14}{N_C - 2} + \frac{6}{N_C - 4} \right] \}
\]

\[
< M | H_q^\dagger \Lambda M H_q^\dagger \Lambda M H_q \Lambda M H_q |M^\dagger >= \frac{1}{4g^6 a C_2^3} \left[ -72 - 12N_C + \frac{12(N_C - 1)^2}{N_C - 2} \right]
\]
\[
+ N \left( 8N_C + \frac{N_C^2}{2} - \frac{N_C(N_C - 1)^2}{2(N_C - 2)} \right) - \frac{3}{16} \frac{N_C^2}{N^2}
\]

\[
< M | H_q^\dagger \Lambda M H_q^\dagger \Lambda M H_q \Lambda M H_q |M^\dagger >= \frac{1}{4g^6 a C_2^3} \left[ -39 - 12N_C + 4N_C + \frac{N_C^2}{2} \right]
\]
\[
- \frac{N_C^2}{8} N^2 - \frac{(N_C - 1)^4}{(N_C - 2)(N_C - 3)^2} \left( -12 + \frac{N_C}{2} N \right)
\]

\[
< M | H_q^\dagger \Lambda M H_q^\dagger \Lambda M H_q \Lambda M H_q |M^\dagger >= \frac{1}{4g^6 a C_2^3} \left[ -41 - 12N_C + \frac{12(N_C - 1)^2}{N_C - 2} \right]
\]
\[
+ N \left( 4N_C + \frac{N_C^2}{2} - \frac{N_C(N_C - 1)^2}{2(N_C - 2)} \right) - \frac{3}{16} \frac{N_C^2}{N^2}
\]

\[
< M | H_q \Lambda M H_q^\dagger \Lambda M H_q^\dagger \Lambda M H_q |M^\dagger >= \frac{1}{4g^6 a C_2^3} \left[ -35 - 12N_C + \frac{12(N_C - 1)^3}{(N_C - 3)(N_C - 2)} \right]
\]
\[
+ N \left( 4N_C + \frac{N_C^2}{2} - \frac{N_C(N_C - 1)^3}{2(N_C - 3)(N_C - 2)} \right) - \frac{3}{16} \frac{N_C^2}{N^2}
\]

\[
< M | H_q^\dagger \Lambda M H_q^\dagger \Lambda M H_q^\dagger \Lambda M H_q |M > = \frac{1}{4g^6 a C_2^3} [-3]
\]

\[
< M | H_q^\dagger \Lambda M H_q^\dagger \Lambda M H_q^\dagger \Lambda M H_q |M > = \frac{1}{4g^6 a C_2^3} [-4]
\]

\[
\left( < M | H_q \frac{\Pi_M}{E^{(0)}_M - H_e} H_q |M > < M | H_q \frac{\Pi_M}{E^{(0)}_M - H_e} H_q |M > \right)_a^1
\]
\[
= \frac{2}{g^6 a C_2^3} \left[ \frac{5N_C - 25 - N_C}{N_C - 3} - \frac{N_C}{2} N \right] \left[ \frac{5N_C^2 - 42N_C + 73}{(N_C - 3)^2} + \frac{N_C}{2} N \right]
\]

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\[ 5. \ f_2 \]

\[ \langle M^\dagger | H_q^{\dagger} \Lambda_M H_q^{\dagger} \Lambda_M H_q \Lambda_M H_q | M \rangle = \]

\[ \frac{1}{4g^6aC_2^3} \left\{ -86 + \left( \frac{17}{2} \right) N_c - \frac{N_c^2}{4} N^2 - \frac{(N_c - 1)}{N_c - 2} \left[ \frac{5}{2} N_c N - 58 \right] \right. \]

\[ - \frac{18(N_c - 1)}{N_c - 4} - \frac{4(N_c - 1)^2}{N_c - 3} \left[ (N_c N - 22) + \frac{6}{N_c - 4} \right] \]

\[ - \frac{4(N_c - 1)^2}{(N_c - 3)^2} \left[ \frac{N_c/2N + 2N_c - 14}{N_c - 2} + \frac{6}{N_c - 4} \right] \} \]  

(D33)

\[ \langle M | H_q^{\dagger} \Lambda_M H_q^{\dagger} \Lambda_M H_q \Lambda_M H_q | M^\dagger \rangle = \frac{1}{4g^6aC_2^3} \left[ -72 - 12N_c + \frac{12(N_c - 1)^2}{N_c - 2} \right] \]

+ \( N \left( 8N_c + \frac{N_c^2}{2} - \frac{N_c(N_c - 1)^2}{2(N_c - 2)} \right) - \frac{N_c^2}{4} N^2 \]  

(D34)

\[ \langle M | H_q \Lambda_M H_q^{\dagger} \Lambda_M H_q \Lambda_M H_q | M^\dagger \rangle = \frac{1}{4g^6aC_2^3} \left[ -37 - 12N_c + N \left( 4N_c + \frac{N_c^2}{2} \right) \right] \]

\[ - \frac{N_c^2}{8} N^2 - \frac{(N_c - 1)^4}{(N_c - 2)(N_c - 3)^2} \left( -12 + \frac{N_c}{2} N \right) \]  

(D35)

\[ \langle M | H_q^{\dagger} \Lambda_M H_q \Lambda_M H_q^{\dagger} \Lambda_M H_q | M^\dagger \rangle = \frac{1}{4g^6aC_2^3} \left[ -40 - 12N_c + \frac{12(N_c - 1)^2}{N_c - 2} \right] \]

+ \( N \left( 4N_c + \frac{N_c^2}{2} - \frac{N_c(N_c - 1)^2}{2(N_c - 2)} \right) - \frac{N_c^2}{8} N^2 \]  

(D36)

\[ \langle M | H_q \Lambda_M H_q^{\dagger} \Lambda_M H_q \Lambda_M H_q | M^\dagger \rangle = \frac{1}{4g^6aC_2^3} \left[ -34 - 12N_c + \frac{12(N_c - 1)^3}{(N_c - 3)(N_c - 2)} \right] \]

+ \( N \left( 4N_c + \frac{N_c^2}{2} - \frac{N_c(N_c - 1)^3}{2(N_c - 3)(N_c - 2)} \right) - \frac{N_c^2}{8} N^2 \]  

(D37)

\[ \langle M | H_q \Lambda_M H_q^{\dagger} \Lambda_M H_q \Lambda_M H_q | M \rangle = \frac{1}{4g^6aC_2^3} \left[ -18 - 4N_c + N_c N + \frac{4(N_c - 1)^3}{(N_c - 3)(N_c - 2)} \right] \]  

(D38)

\[ \langle M | H_q \Lambda_M H_q^{\dagger} \Lambda_M H_q^{\dagger} \Lambda_M H_q | M \rangle = \frac{1}{4g^6aC_2^3} \left[ -18 - 4N_c + \frac{4(N_c - 1)^2}{N_c - 2} + N_c N \right] \]  

(D39)

\[
\left( \langle M | H_q E_M^{(0)} - H_e \rangle \frac{\Pi_M}{-H_e} H_q | M \rangle > \langle M | H_q \frac{\Pi_M}{E_M^{(0)} - H_e} H_q | M \rangle \right) \\
= \frac{2}{g^6aC_2^3} \left[ \frac{7N_c - 25}{N_c - 3} - \frac{N_c}{2} N \right] \left[ -5N_c^2 - 42N_c + 73 + \frac{N_c}{2} N \right] \]

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6. $f_0$

$$< M^i | H^i_q \Lambda_M H^i_q \Lambda_M H_q \Lambda_M H_q | M > =$$

$$\frac{1}{4g^6 a C_2^3} \left\{ -181 + \left( \frac{25}{2} N_C \right) N - \frac{N_C^2}{4} N^2 - \frac{(N_C - 1)}{N_C - 2} \left[ \frac{5}{2} N_C N - 90 \right] \right.$$  

$$- \frac{18(N_C - 1)}{N_C - 4} - \frac{4(N_C - 1)^2}{N_C - 3} \left[ \frac{(N_C N - 30)}{N_C - 2} + \frac{6}{N_C - 4} \right]\}

$$< M | H^i_q \Lambda_M H^i_q \Lambda_M H_q \Lambda_M H_q | M^\dagger > = \frac{1}{4g^6 a C_2^3} \left[ -72 - 12N_C + \frac{12(N_C - 1)^2}{N_C - 2} \right.$$  

$$+ N \left( 8N_C + \frac{N_C^2}{2} - \frac{N_C(N_C - 1)^2}{2(N_C - 2)} \right) - \frac{N_C^2}{4} N^2 \right]$$

$$< M | H_q \Lambda_M H^i_q \Lambda_M H_q \Lambda_M H_q | M^\dagger > = \frac{1}{4g^6 a C_2^3} \left[ -70 - 12N_C + N \left( 5N_C + \frac{N_C^2}{2} \right) \right.$$  

$$- \frac{N_C^2}{8} N^2 - \frac{(N_C - 1)^4}{(N_C - 2)(N_C - 3)^2} \left( -12 + \frac{N_C}{2} N \right) \right]$$

$$< M | H^i_q \Lambda_M H_q \Lambda_M H^i_q \Lambda_M H_q | M^\dagger > = \frac{1}{4g^6 a C_2^3} \left[ -72 - 12N_C + \frac{12(N_C - 1)^2}{N_C - 2} \right.$$  

$$+ N \left( 5N_C + \frac{N_C^2}{2} - \frac{N_C(N_C - 1)^2}{2(N_C - 2)} \right) - \frac{N_C^2}{8} N^2 \right]$$

$$< M | H_q \Lambda_M H^i_q \Lambda_M H_q \Lambda_M H^i_q | M^\dagger > = \frac{1}{4g^6 a C_2^3} \left[ -66 - 12N_C + \frac{12(N_C - 1)^3}{(N_C - 3)(N_C - 2)} \right.$$  

$$+ N \left( 5N_C + \frac{N_C^2}{2} - \frac{N_C(N_C - 1)^3}{2(N_C - 3)(N_C - 2)} \right) - \frac{N_C^2}{8} N^2 \right]$$

$$< M | H_q \Lambda_M H^i_q \Lambda_M H_q \Lambda_M H^i_q | M > = \frac{1}{4g^6 a C_2^3} \left[ -66 - 6N_C + 3N_C N + \frac{6(N_C - 1)^3}{(N_C - 3)(N_C - 2)} \right]$$

$$< M | H_q \Lambda_M H^i_q \Lambda_M H_q \Lambda_M H^i_q | M > = \frac{1}{4g^6 a C_2^3} \left[ -72 - 6N_C + \frac{6(N_C - 1)^2}{N_C - 2} + 3N_C N \right]$$

$$\left( < M | H_q \frac{\Pi \Lambda \Lambda M}{E_M^{(0)} - H_e} H_q | M > < M | H_q \frac{\Pi \Lambda \Lambda M}{(E_M^{(0)} - H_e)^2} H_q | M > \right)$$

$$= \frac{2}{g^6 a C_2^3} \left[ 9N_C - 29 \right] - \frac{N_C}{2} N \right] \left[ -7N_C^2 - 54N_C + 91 \right] + \frac{N_C}{2} N \right]$$

$$f_0$$
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[23] There should be a minor misprint in the final equation for $\omega_0^{(4)}$ in [13]. In fact if one adds up all the contributions from each graph the result is slightly different and coincides with Eq. [17] for $N_C = 3$. 

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