A counterexample to a geometric Hales-Jewett type conjecture

Vytautas Gruslys*

October 13, 2014

Abstract

Pór and Wood conjectured that for all $k, l \geq 2$ there exists $n \geq 2$ with the following property: whenever $n$ points, no $l + 1$ of which are collinear, are chosen in the plane and each of them is assigned one of $k$ colours, then there must be a line (that is, a maximal set of collinear points) all of whose points have the same colour. The conjecture is easily seen to be true for $l = 2$ (by the pigeonhole principle) and in the case $k = 2$ it is an immediate corollary of the Motzkin-Rabin theorem. In this note we show that the conjecture is false for $k, l \geq 3$.

1 Introduction

Given a finite set $S$ in the plane, we will use the term line to denote any maximal set of collinear points of $S$. Pór and Wood posed the following conjecture.

Conjecture 1 (Pór and Wood [4]). For all integers $k \geq 1$ and $l \geq 2$, there is an integer $n$ such that for every finite set $S$ of size $|S| \geq n$ in the plane $\mathbb{R}^2$, if each point of $S$ is assigned one of $k$ colours, then

- $S$ contains $l + 1$ collinear points, or
- $S$ contains a monochromatic line.

*Department of Pure Mathematics and Mathematical Statistics, Centre for Mathematical Sciences, University of Cambridge, Wilberforce Road, Cambridge, CB3 0WB, United Kingdom; e-mail: v.gruslys@dpmms.cam.ac.uk
The motivation for this conjecture comes from the Hales-Jewett theorem. By a combinatorial line in the grid \([l]^n \subset \mathbb{R}^n\) (where \([l]\) stands for the set \(\{1, 2, \ldots, l\}\)) we mean a set of the form
\[
\{(x_1, \ldots, x_n) \in [l]^n : x_i = x_j \text{ for all } i, j \in I\}
\]
for fixed \(I \subset [n], I \neq \emptyset\) and fixed \(x_i, i \in [n] \setminus I\). Now the Hales-Jewett theorem can be stated as follows.

**Theorem 2** (Hales and Jewett [1]). For all integers \(k, l \geq 1\), there is an integer \(n\) such that whenever each of the points in \([l]^n \subset \mathbb{R}^n\) is given one of \(k\) colours, there is a monochromatic combinatorial line.

Conjecture 1 is a natural geometric version of this theorem, where the lines are not necessarily parallel to a fixed set of axes, and the ambient set can be any set without many collinear points.

For \(l = 2\) the result is trivial: we may take \(n = k + 1\) and by the pigeonhole principle there is a line containing two points of the same colour. The case \(k = 2\) is a special case of the Motzkin-Rabin theorem that was proved in [3]. In this paper we demonstrate by a counterexample that the conjecture is false in the next smallest case \(k = l = 3\), and hence it is false whenever \(k, l \geq 3\).

**Theorem 3.** For any \(n \geq 2\), there is a set \(S \subset \mathbb{R}^2\) of size \(n\) satisfying:

- no four points of \(S\) are collinear, and
- the points of \(S\) can be coloured using three colours in such a way that no line is monochromatic.

## 2 Proof of Theorem 3

We start by noting that it is sufficient to find a set with the required properties in the projective plane \(\mathbb{R}P^2\). Indeed, given a finite set \(S \subset \mathbb{R}P^2\), one can choose a line \(l \subset \mathbb{R}P^2\) that does not meet \(S\) and apply a projective transformation that sends \(l\) to the line at infinity. The image of \(S\) under this transformation is contained in the affine plane \(\mathbb{R}^2\) while the collinearity relations of the original set \(S\) are preserved.

Our counterexample is a finite subset of the irreducible cubic curve \(y^2 = x^3 - x^2\). More specifically, we will use a subset of the set of its non-singular points \(\Gamma = \{(x, y) \in \mathbb{R}^2 : y^2 = x^3 - x^2, x \neq 0\} \cup \{O\} \subset \mathbb{R}P^2\) where \(O\) is a point at infinity that is contained in all lines parallel to the \(y\)-axis and in
the line at infinity. By the Bézout theorem, \( \Gamma \) does not contain a set of four collinear points. Moreover, it is a well known fact in algebraic geometry that \( \Gamma \) forms an abelian group with the property that distinct points \( P, Q, R \in \Gamma \) are collinear if and only if \( P + Q + R = 0 \), and that \( \Gamma \) is isomorphic to the circle group \( \mathbb{R}/\mathbb{Z} \) (see [2], p. 19–20).

In fact, any choice of an elliptic curve whose group is isomorphic to \( \mathbb{R}/\mathbb{Z} \) would do. However, we choose this particular cubic curve (which is not an elliptic curve as it contains a singular point \((0,0)\)) because it admits a simple explicit group isomorphism \( \phi : \mathbb{R}/\mathbb{Z} \to \Gamma \), given by

\[
\phi(x) = \begin{cases} 
(cot(\pi x)^2 + 1, cot(\pi x)(cot(\pi x)^2 + 1)) & \text{if } x \neq 0, \\
O & \text{if } x = 0.
\end{cases}
\]

This enables us to give a self-contained proof of the theorem without referring to any results from algebraic geometry. However, the reader familiar with elliptic curves can skip the proof of the following proposition.

**Proposition 4.** Let \( x, y \) and \( z \) be distinct elements of \( \mathbb{R}/\mathbb{Z} \). Then the points \( \phi(x), \phi(y) \) and \( \phi(z) \) are collinear if and only if \( x + y + z = 0 \). Moreover, \( \phi : \mathbb{R}/\mathbb{Z} \to \Gamma \) is a well defined bijection.

**Proof.** The fact that \( \phi \) is a well defined bijection follows from the basic properties of the cotangent function. To prove the equivalence of the geometric and algebraic relations, we will use the identity

\[
\cot(x + y) = \frac{\cot(x)\cot(y) - 1}{\cot(x) + \cot(y)},
\]

which holds whenever \( x + y \neq 0 \). Given a real number \( r \notin \mathbb{Z} \), define \( c_r = \cot(\pi r) \).

If one of \( x, y, z \in \mathbb{R}/\mathbb{Z} \) is 0 (say, \( x = 0 \)) then \( \phi(z) \) is collinear with \( \phi(x) = O \) and \( \phi(y) \) if and only if \( \phi(z) \) is the reflection of \( \phi(y) \) in the \( x \)-axis, that is, \( z = -y \). Similarly, if two of the numbers (say, \( x \) and \( y \)) sum to 0, then the three points are collinear if and only if \( \phi(z) = O \), that is, \( z = 0 \). Now we can assume that \( x, y, z \) are all non-zero and that no two of them sum to 0. Then the points \( \phi(x), \phi(y) \) and \( \phi(z) \) are collinear if and only if

\[
\frac{c_z(c_y^2 + 1) - c_x(c_x^2 + 1)}{(c_y^2 + 1) - (c_x^2 + 1)} = \frac{c_x(c_x^2 + 1) - c_y(c_y^2 + 1)}{(c_x^2 + 1) - (c_y^2 + 1)},
\]

which after rearrangement becomes

\[
c_z = \frac{c_x c_y - 1}{c_x + c_y}.
\]
Notice that \( z = -x - y \) is a solution by (1), and it is unique in \( \mathbb{R}/\mathbb{Z} \) as \( \cot \) is injective on \((0, \pi)\).

Now we are ready to finish the proof of the theorem.

**Proof of Theorem 3.** As noted before, it is enough to construct a set \( S' \subset \mathbb{RP}^2 \) with the two required properties, and take a projective transformation that maps \( S' \) into \( \mathbb{R}^2 \).

For the set \( S' \) (see Fig. 1 and 2) we will take \( S' = \{ \phi(i/n) : i = 0, \ldots, n-1 \} \). Notice that by Proposition 4 there are no four collinear points in \( S' \). Indeed, if \( \phi(x), \phi(y), \phi(z) \) and \( \phi(w) \) were distinct and collinear, then \( z = -x - y = w \) in \( \mathbb{R}/\mathbb{Z} \), giving a contradiction. Colour \( \phi(i/n) \)

- red if \( 0 \leq i < \frac{n}{3} \),
- green if \( \frac{n}{3} \leq i < \frac{2n}{3} \),
- blue if \( \frac{2n}{3} \leq i < n \).

![Figure 1: The set \( S' \) with \( n = 16 \). The sixteenth point is at infinity, and has red colour. The framed section is shown in smaller scale in Fig. 2.](image)
Suppose for contradiction that there is a monochromatic line $l$. It must pass through two distinct points $\phi(i/n)$ and $\phi(j/n)$, $0 \leq i, j < n$. There is an integer $0 \leq k < n$ such that $k \equiv -i - j \pmod{n}$, possibly $k = i$ or $k = j$. Then $i/n + j/n + k/n = 0$ in $\mathbb{R}/\mathbb{Z}$, and so by Proposition 4 either $\phi(i/n), \phi(j/n)$ and $\phi(k/n)$ are distinct colinear points, or $\phi(k/n)$ coincides with one of the other two points. In either case $l$ passes through all of these points, and hence they have the same colour.

Now write $i/n = x + \alpha$, $j/n = x + \beta$ and $k/n = x + \gamma$, where $x \in \{0, \frac{1}{3}, \frac{2}{3}\}$ and $\alpha, \beta, \gamma \in [0, \frac{1}{3})$. Considered as real numbers, $3x$ and $i/n + j/n + k/n = 3x + \alpha + \beta + \gamma$ are integers, so $\alpha + \beta + \gamma$ is also an integer. But $0 \leq \alpha, \beta, \gamma < \frac{1}{3}$, so this is only possible if $\alpha = \beta = \gamma = 0$. In particular, $i/n = j/n$, contradicting the assumption that $\phi(i/n) \neq \phi(j/n)$.

This finishes the proof.

\begin{flushright}
\Box
\end{flushright}

References

[1] A.W. Hales and R.I. Jewett, Regularity and positional games, Trans. Amer. Math. Soc. 106 (1963), 222–229.

[2] D. Husemoller, Elliptic curves (2nd ed., Springer, 2004).

[3] T.S. Motzkin, Nonmixed connecting lines, Notices Amer. Math. Soc. 14 (1967), 837.

[4] A. Pór and D.R. Wood, On visibility and blockers, J. Computational Geometry 1 (2010), 29–40.