Structured Decompositions: Structural and Algorithmic Compositionality.

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Abstract

We introduce structured decompositions: category-theoretic generalizations of many combinatorial invariants – including tree-width, layered tree-width, co-tree-width and graph decomposition width – which have played a central role in the study of structural and algorithmic compositionality in both graph theory and parameterized complexity. Structured decompositions allow us to generalize combinatorial invariants to new settings (for example decompositions of matroids) in which they describe algorithmically useful structural compositionality. As an application of our theory we prove an algorithmic meta theorem for the \( \text{Sub}_p\text{-COMPOSITION} \) problem which, when instantiated in the category of graphs, yields compositional algorithms for NP-hard problems such as: \text{MAXIMUM BIPARTITE SUBGRAPH}, \text{MAXIMUM PLANAR SUBGRAPH} and \text{LONGEST PATH}.

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1 Introduction

Compositionality can be understood as the perspective that the meaning, structure or function of the whole is given by that of its constituent parts [56]. This idea has always played a key role in mathematics and computer science where age-old notions such as recursion and divide and conquer make sense of mathematical objects by looking at them through a compositional lens. More recently there has been a tremendous push towards the systematic study of the mathematics of compositional systems and of how and where such systems crop-up in science [28, 47, 12, 15, 42].

Thanks to these efforts, we can now make sense of how one might build graphs, Petri nets [3], chemical reaction networks [4], stock and flow diagrams [2] or epidemiological models [41] from smaller constituent parts. But how should we study real-world instances of this data? Clearly we would like to develop algorithms which are not specific only to one of these application domains. Indeed, it would be ideal to have algorithms that take the categorial and compositional structure of their inputs into account and which can be applied to any of these instances. As of yet such algorithms do not exist and this paper intends to set the groundwork from which to initiate this study.

The field of parameterized complexity guides us in this endeavor by providing us with a vast and well-established literature on how to develop compositional algorithms when the inputs are graphs or other kinds of combinatorial objects which display some sort of compositional (often recursive) structure [27, 16, 14, 31].

Here we introduce the notion of structured decompositions: categorical objects which vastly generalize tree-width – one of the most important notions in the parameterized complexity toolbox [27, 16, 14, 31] – and which allow us to initiate the study of how to develop algorithms which exploit the compositional structure of their inputs.

Structured decompositions aim to generalize the following idea. Suppose we have a morphism of undirected multi-graphs \( f : H \rightarrow G \). We think of this morphism ‘fibrationally’. For each vertex \( x \) of \( G \), there is a set \( f^{-1}(x) \) called the ‘bag’ of vertices sitting over a node \( x \). Similarly, for each edge \( e : x \rightarrow y \) of \( G \), we have a set \( f^{-1}(e) \) called the ‘bag’ of edges sitting over \( e \). For example, consider the graph morphism depicted below:
The codomain $G$ of $f$ is drawn below with the dashed edges denoting the fibers of its vertices. Although not indicated in the picture, the edges $a$ and $b$ sit over the edge $e$ and the edges $c$ and $d$ sit over $f$. In this paper we are more interested in the ‘dependent perspective’ of $f$; the fibers of this $f$ collect into a functor of the form

$$f^{-1}: \int G \rightarrow \text{FinSet}$$

where $G$ is a graph and $\int G$ is the category having an object for each vertex of $G$, an object for each edge of $G$ and a span joining each edge to its source and target vertices (this is an instance of the Grothendieck construction; we will explain in detail in Section 3). A proof of an equivalence between the fibrational perspective and the dependent perspective is given in Proposition 3.6. In the above example, the ‘dependent’ perspective is drawn as follows.

This picture shows the image of the functor on the generating objects $b_1, b_2, e$ and $f$ of $\int G$ and on its generating morphisms. In this paper, we generalize this idea by replacing sets with arbitrary objects of a category $K$.

There is a functor

$$\binom{-}{2}: \text{FinSet} \rightarrow \text{Gr_H} \quad X \mapsto \binom{X}{2} \quad \text{(i.e. the complete graph on } X\text{)}$$

This functor, called the ‘spine’ of $\text{Gr}_H$, summarizes the data necessary to define tree-width (Proposition 5.3).

The tree-width of a graph records the minimum size of a tree decomposition for that graph. Tree-width is a key notion in both graph structure theory (e.g. Robertson and Seymour’s celebrated graph minor theorem [51]) and parameterized complexity (e.g. Courcelle’s famous algorithmic meta-theorem for bounded tree-width graphs [13]).

In this paper we allow the spine to be an arbitrary functor $\Omega: K \rightarrow C$ to obtain generalized notions of width. By choosing $\Omega$ to be the functor $\Omega: \text{FinSet} \rightarrow \text{Gr}_H$ sending a set to the discrete graph on that set we obtain complemented tree-width [9, 25] (or co-tree-width [25]). Complemented tree-width has algorithmic applications when computing on dense graph classes [25].

Other than tree-width and its complemented variant, we find many further combinatorial examples of our construction. These include layered tree-width [26, 54] (a
tree-width variant which, unlike tree-width, remains bounded on planar graphs) as well as the recent notion of a graph decomposition (due to Carmesin [10] and related to coverings and fundamental groups of graphs by Diestel, Jacobs, Knappe and Kurkofka [23]). Furthermore, a more fine-grained version of Jansen, de Kroon, and Włodarczyk’s [36] $H$-tree-width is also easily seen to fit within our formalism.

We note that, except for $H$-tree-width, we recover all of these width measures we just mentioned ‘up-to FPT’ (Definition 6.2) in the sense that parametrizations by our ‘recovered’ width measures yield the same algorithmic consequences as parametrizations in the usual, non category-theoretic sense.

Given a spine $\Omega : K \to C$, a structured decomposition should decompose objects of $K$, as these decompositions are used to measure the width of objects in $C$. Therefore, we define a $K$-valued structured decomposition to be a functor of the type $d : \mathcal{J}(G) \to K$, generalizing the case when $K = \text{FinSet}$. Furthermore, we show there is a category $\mathfrak{T}(K)$ of $K$-valued structured decompositions whose morphisms are suitably defined natural transformations.

The codomain $C$ of the functor $\Omega$ may be thought of as a semantics for $K$-valued structured decompositions and its relationship with $K$-valued structured decompositions is the main ingredient for defining all of these invariants or ‘width measures’ which we just mentioned. This is done by via studying the left Kan extension given in the following diagram.

$$
\begin{array}{ccc}
K & \xrightarrow{\Omega} & C \\
\downarrow \text{const} & \Downarrow \text{Lan}_{\text{const}} \Omega & \\
\mathfrak{T}(K) & \xrightarrow{\text{const}} & C
\end{array}
$$

The intuitive idea is that all subobjects of anything in the range of $\text{Lan}_{\text{const}} \Omega$ should be understood as being ‘at most as complex’ as the objects of $\mathfrak{T}(K)$. This allows us to define (Definition 5.4) the notion of $(G,\Omega)$-width, which provides a vast generalization of the width measure induced by Carmesin’s graph decompositions [10], to objects of any category $C$ with colimits.

We show that as long as $C$ is adhesive and $\Omega : K \to C$ preserves monomorphisms, we can always relate the $(G,\Omega)$-width of any object $x \in C$ to a ‘recipe’ (i.e. a structured decomposition) which exhibits how to construct $x$ as a colimit in $C$ (Theorem 5.8). This result allows us to conveniently switch perspective between width measures (and completions) and decompositions.

Our main algorithmic application of structured decompositions has to do with the algorithmic task of determining the value of a functor $F : C \to D$ on some object $c \in C$. Whenever $F$ preserves colimits, we can easily show a result (Observation 6.4) which suggests that a ‘compositional algorithm on $c$ is itself a structured decomposition of the solution space for $F$ on $c$’.

When $F$ does not preserve colimits the situation is more complex. Nevertheless, for adhesive $C$, we are still able to prove an algorithmic metatheorem which, when instantiated in the category of graphs, provides us with compositional (but not FPT-time) algorithms for NP-hard problems such as MAXIMUM BIPARTITE SUBGRAPH, MAXIMUM PLANAR SUBGRAPH and LONGEST PATH (and more generally for maximizing – or minimizing – the size of subgraphs satisfying some given property invariant under taking subobjects).
1.1 Related work

As we already mentioned, in graph theory there has been a considerable recent drive to generalize tree decompositions in two different ways. The first is to allow more general decomposition ‘shapes’ [10, 23] (e.g. cycle or planar decompositions etc.). This is the general notion of graph decompositions introduced by Carmesin [10]. The second approach is to allow for more complex ‘bags’ in a tree decomposition; this has been done, for example, by studying the notions such as $H$-tree-width [36] or layered tree-width [54, 26]. These notions and combinations thereof can be expressed and studied by using our structured decompositions and thus this work opens exciting new research prospects in the interplay of categorial ideas with combinatorial ones.

In the realm of category theory, structured decompositions bear significant similarity to undirected wiring diagrams [55]. These provide an operadic view on a construction which is very similar to what we would call $\text{FinSet}$-valued structured decompositions. Although it is beyond the scope of this paper, investigating the connections between these two notions is an interesting direction for further study.

Finally we note that there have been two other categorifications of tree-width to date. The first is through the notion of spined categories [9, 8] which are categories $\mathcal{C}$ equipped with a so called proxy-pushout operation and a sequence $\Omega : \mathbb{N}_0 \to \mathcal{C}$ of objects called the spine. This paper originated as a generalization of spined categories to the case where the spine is allowed to be any functor $\Omega : \mathcal{K} \to \mathcal{C}$.

The other categorification of tree-width in the literature is through the notion of monoidal width introduced by Di Lavore and Sobociński [17, 18]. Monoidal width measures the width of a morphism in a monoidal category $(\mathcal{C}, \otimes)$. Instead of a spine $s : \mathbb{N} \to \mathcal{C}$, monoidal width is determined by a width function $w : A \subset \text{MorC} \to \mathbb{N}$ sending a subclass of atomic morphisms to natural numbers. There are two main differences between our approach and the approach of Di Lavore and Sobociński. Firstly, for monoidal width, the morphisms are being decomposed whereas in the the present structured decomposition approach we decompose the objects. Setting that aside, the only other difference is that the spine and width function are going in opposite directions. Indeed, a spine $s$ may be obtained from a width function $w$ by choosing $s(n) \in w^{-1}(n)$.

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2 Notation

We follow standard category theoretic notation as in Awodey’s textbook [1] and standard graph-theoretic notation as in Diestel’s textbook [20]. Categories will be denoted in boldface – e.g. $\mathcal{C}$ – and their objects will be denoted lowercase letters $x, y, z$. The category of finite sets is denoted $\text{FinSet}$. The category of (undirected, but reflexive) graphs and homomorphisms is denoted as $\text{Gr}_H$, its subcategory which only keeps its monic arrows is denoted $\text{Gr}_M$. We denote the $n$-vertex complete graph as $K_n$. The empty graph is $K_0 = (\emptyset, \emptyset)$. 

5
3 Structured Decompositions

As we mentioned in Section 1, one should intuitively think of \( K \)-valued structured decompositions as special kinds of graphs where the edge relation is replaced by a ‘generalized relation’: i.e. a collection of spans in \( K \).

**Definition 3.1 (Structured Decomposition).** Given any graph \( G \) viewed as a functor \( G : \text{GrSch} \to \text{Set} \) where \( \text{GrSch} \) is the two object category generated by

\[
E \xrightarrow{s} V
\]

one can construct a category \( \int G \) with an object for each vertex of \( G \) and an object from each edge of \( G \) and a span joining each edge of \( G \) to each of its source and target vertices. This construction is an instance of the more general notion of Grothendieck construction. Now, fixing a base category \( K \) we define a \( K \)-valued structured decomposition of shape \( G \) (see Figure 1) as a diagram of the form

\[
d : \int G \to K.
\]

Given any vertex \( v \) in \( G \), we call the object \( dv \) in \( K \) the bag of \( d \) indexed by \( v \). Given any edge \( e = xy \) of \( G \), we call \( de \) the adhesion indexed by \( e \) and we call the span \( dx \leftarrow de \to dy \) the adhesion-span indexed by \( e \).

\[
a := \{1,2,3,4\} \quad \Rightarrow \quad b := \{3,4,10\}
\]

\[
\{1,2,3\} \quad \Rightarrow \quad c := \{1,2,3,\pi,\sqrt{2}\}
\]

\[
\{1,2\} \quad \Rightarrow \quad e := \{1,2,3/2\}
\]

\[
\{3,\pi,11\}
\]

Figure 1: A tree-shaped \( \text{FinSet} \)-valued structured decomposition \( d : \int T \to \text{FinSet} \). It consists of an underlying tree \( T \) (marked with black edges) which is then equipped with some extra information: each node \( x \) is associated to a set \( dx \) and each edge \( e = xy \) is associated to a span \( de := dx \leftarrow de \to dy \) (where we abuse notation denoting the entire span also by \( de \)).

Note that we have not drawn some of the arrows of the free dagger category since these can be computed by pullback. The reader familiar with tree decompositions is invited to skip ahead in this document to compare this figure with Figure 4.

Note that, to avoid an excess of variables, we will often abuse notation and denote by \( de \) both the adhesion and the adhesion span indexed by \( e \).
Definition 3.2. A morphism

\[(d_1 : \int G_1 \to K) \to (d_2 : \int G_2 \to K)\]

of $K$-valued structured decompositions $d_1$ and $d_2$ is a pair $(F, \eta)$ as in the following diagram where $F$ is a functor $F : \int G_1 \to \int G_2$ and $\eta$ is a natural transformation $\eta : d_1 \Rightarrow d_2 F$ as in the following diagram.

\[
\begin{array}{c}
\int G_1 \\
\downarrow F \\
\int G_2 \\
\downarrow \eta \\
K \\
\downarrow d_1 \\
\downarrow d_2 \\
\end{array}
\]

Proposition 3.3. Fixing a category $K$, $K$-valued structured decompositions (of any shape) and the morphisms between them (as in Definition 3.2) form a category $\mathcal{D}(K)$ called the category of $K$-valued structured decompositions.

Proof. The category $\mathcal{D}K$ may be realized as the slice category $\int /K$ of the Grothendieck construction functor $\int : \text{Set}^{\text{Sch}} \to \text{Cat}$ and the category $K$.

For sake of completeness, we note that the proof of Proposition 3.3 immediately implies that $\mathcal{D}$ is a functor.

Corollary 3.4. There is a functor $\mathcal{D} : \text{Cat} \to \text{Cat}$ which takes any category $K$ to the category $\mathcal{D}(K)$ and every functor $\Phi : K \to K'$ to the functor $\mathcal{D}(\Phi) : \mathcal{D}(K) \to \mathcal{D}(K')$ defined on objects as

\[\mathcal{D}(\Phi) : \left( \int G \to K \right) \mapsto \left( \int G \to K \right)\]

and on arrows as

\[
\begin{array}{c}
\int G_1 \\
\downarrow F \\
\int G_2 \\
\eta \\
\downarrow d_1 \\
K \\
\downarrow \Phi \\
\int G_1 \\
\downarrow F \\
\int G_2 \\
\Phi \eta \\
\downarrow d_2 \\
\end{array}
\]

Definition 3.5. We denote by $\mathcal{D}_m : \text{Cat} \to \text{Cat}$ the subfunctor of the structured decomposition functor $\mathcal{D}$ which takes each category $K$ to the subcategory $\mathcal{D}_m(K)$ of $\mathcal{D}K$ whose objects are only those structured decompositions consisting of diagrams of monomorphisms in $K.$

In the case when $K = \text{FinSet}$, structured decompositions are equivalent to graph homomorphisms. The bags of a $\text{FinSet}$-valued structured decomposition may be glued
together into a larger graph equipped with a homomorphism into the domain of the structured decomposition.

**Proposition 3.6.** There is an equivalence of categories

$$\mathcal{D}(\text{FinSet}) \cong \text{Gr}_H$$

between the category of FinSet-valued structured decompositions and the arrow category of $\text{Gr}_H$.

**Proof.** This is an instance of the $T$-Grothendieck construction when $T$ is the theory of graphs [45]. In more detail, a FinSet-valued structured decomposition defines a morphism of graphs

$$F : G \to \text{Span}(\text{FinSet})$$

where $\text{Span}(\text{FinSet})$ is the graph whose vertices are finite sets and whose edges are spans of finite sets. The graph morphism sends a node to its bag and each edge to its adhesion-span. This morphism of graphs may be turned into a graph morphism of type

$$\int F \to G$$

where $\int F$ is a graph whose vertices are pairs $(x, a)$ for $x$ a vertex of $G$ and $a$ is in the bag sitting over $x$. An edge of $\int F$ is a pair $(f, g)$ where $f$ is an edge of $G$ and $g$ is an element of the apex of the span-adhesion corresponding to $f$. This construction lifts to an equivalence of categories between structured decompositions and the arrow category of $\text{Gr}_H$. ■

## 4 Functorial Semantics via Kan Extensions

In the rest of this paper, other than for algorithmic applications, we will make two main uses of structured decompositions. The first will be to define so called ‘width measures’ (Section 5.4). These are invariants which intuitively capture the degree of structural and compositional simplicity of objects in a category. The second use of structured decompositions is to provide something akin to a compositional recipe which records how to construct a given object in a category as colimit (Section 5.2). In graph theory this last idea would be called a graph decomposition [10, 19, 20, 27, 16].

Colimits of diagrams $D \to C$ are the left adjoint of an adjunction

$$\begin{array}{ccc}
\text{C} & \xrightarrow{\text{colim}} & \text{C} \\
\text{const} & \downarrow & \downarrow \\
\text{D} & & \text{C}
\end{array}$$

where const sends an object $x$ to the functor which is constantly $x$. Indeed, this adjunction is often used to define $D$-shaped colimits. A structured decomposition $d : \int G \to K$ is diagram in $K$ where the shape of this diagram is given by replacing every edge of $G$ with span going into its source and target. If $K$ is already the category that we wish to interpret our structured decomposition, then the colimit of this diagram should represent its gluing. However, to study width measures we start with a spine $\Omega : K \to C$ where $K$ is the category used to define structured decompositions but $C$ is the category where
these structured decompositions are semantically interpreted. Thus to glue structured decompositions into a single object we first lift the spine $\Omega : \mathbb{K} \to \mathbb{C}$ to the functor

$$\mathfrak{T} : \mathfrak{T K} \to \mathfrak{T C}$$

and then use the colim functor to obtain objects of $\mathbb{C}$ as colimits of $\mathbb{K}$-valued structured decompositions. This can be summarized as the composite functor

$$\mathfrak{T K} \xrightarrow{\mathfrak{T} \Omega} \mathfrak{T C} \xrightarrow{\text{colim}} \mathbb{C}$$

which can be interpreted as a left Kan extension $\text{Lan}_{\text{const}} \Omega$ as in the following diagram (see Riehl’s textbook [48, Thm. 6.2.1] for the definition of a Kan extension).

Here the philosophy is that, if $\mathbb{K}$ is deemed to be in some sense simpler than the category $\mathbb{C}$ (for example $\mathbb{K}$ could be $\text{FinSet}$ and $\mathbb{C}$ could be $\text{Gr}_H$), then all subobjects in $\mathbb{C}$ of anything in the range of $\text{Lan}_{\text{const}} \Omega$ should also be considered to be ‘at most as complex’ as any object of $\mathfrak{T(K)}$.

**Proposition 4.1.** Let $\mathbb{K}$ be a small category and let $\mathbb{C}$ be a locally small category with colimits. If $\Omega : \mathbb{K} \to \mathbb{C}$ preserves colimits then we have the commutative triangles of functors

$$\begin{array}{ccc}
\mathbb{K} & \xrightarrow{\Omega} & \mathbb{C} \\
\downarrow{\text{const}} & & \downarrow{\text{Lan}_{\text{const}} \Omega} \\
\mathfrak{T(K)} & & \mathbb{C}
\end{array}$$

**Proof.** The outer square commutes because $\Omega$ preserves colimits. The inner triangles commute by the definition of Kan extension. $\blacksquare$

As we will show in Theorem 5.8 we can provide a satisfying clarification of the relationship between width measures and decompositions. The key insight is the following: by using $\mathbb{K}$-valued structured decompositions, we can delegate the task of justifying why a given $\mathbb{C}$-valued structured decomposition is ‘simple’ to the easier task of using $\Omega : \mathbb{K} \to \mathbb{C}$ to pick-out a class of objects in $\mathbb{C}$ which are simple, atomic building blocks.

Slightly more formally, the set-up for Theorem 5.8 is that we have a functor $\Omega : \mathbb{K} \to \mathbb{C}$ as before which we use to define a width measure (Definition 5.4); i.e. a notion of which objects in $\mathbb{C}$ should be deemed structurally simple. Then Theorem 5.8 states that, whenever $\mathbb{C}$ is adhesive (and other mild assumptions apply), if an object $e \in \mathbb{C}$ has small width (i.e. it is a subobject of something in the range of $\text{Lan}_{\text{const}} \Omega$), then it can always be obtained as a colimit of a similarly simple (in a technical sense) $\mathbb{C}$-valued structured decomposition.
5 Capturing Combinatorial Invariants

In this section we will explore how to use structured decompositions to capture various combinatorial invariants defined via associated notions of ‘decomposition’. Section 5.1 will give a detailed explanation of how to define tree-width: a graph invariant that has played a crucial role in the development of graph structure theory (both finite [20, 51] and infinite [19, 20]) as well as parameterized complexity [27, 16, 14, 31]. In Section 5.2 we will study the relationship between invariants and decompositions defined using our theory. One important example will constitute the connection between tree-width (the invariant) and tree decompositions (the data structures). Finally, in Section 5.4 we will use our theory to capture further notions of ‘width’ and ‘decomposition’ that arise in combinatorics such as Carmesin’s ‘graph decompositions’ [10, Def. 9.3], Jansen, de Kroon and Włodarczyk’s $H$-tree-width [36, Def. 2.3] and layered tree-width (introduced independently by Shahrokhi [54] and Dujmović, Morin, and Wood [26]).

5.1 Tree-width

One can think of tree-width as a measure of similarity between the connectivity of a graph and that of trees. For example, it takes the value 0 on edgeless graphs, 1 on forests with at least one edge, 2 on cycles, and the value $n - 1$ on the $n$-vertex complete graph $K_n$.

Tree-width admits multiple cryptomorphic definitions [5, 32, 49, 14, 9] and was independently discovered many times [5, 32, 49] (for gentle introduction to tree-width we refer the reader to the thesis by Bumpus [8, Sec. 1.2]). The definition we use here (Equation (1)) highlights the point of view that tree-width is a measure of how well one can ‘approximate’ a given graph $G$ by a chordal graph (Definition 5.1).

By a well-known theorem of Dirac [24], chordal graphs (Definition 5.1 and Figure 2) can be intuitively understood as ‘inflated trees’ where each node is replaced by a clique (i.e. a complete graph) and each edge corresponds to the intersection of two neighboring cliques. Slightly more formally, chordal graphs consist of cliques together with their closure under clique-pushouts (we are adopting the convention in which the empty graph is considered a clique). The definition we give below highlights the recursive, tree-like shape of chordal graphs.

**Definition 5.1** (Dirac’s Theorem for chordal graphs [24]). A finite simple graph $H$ is chordal if one of the following conditions holds

- either $H \cong K_n$ for some $n$,
- or there there is a monic span $H_1 \leftrightarrow K_n \leftrightarrow H_2$ in $Gr_H$ with $H_1$ and $H_2$ chordal such that $H \cong H_1 + K_n + H_2$.

Finally, denoting by $\omega$ the clique number$^1$ of a graph, we can define tree-width as follows.

**Definition 5.2** (Tree-width). Tree-width is a function $tw : G \rightarrow \mathbb{N}$ from the class

---

$^1$for a graph $G$, the clique number $\omega(G)$ of $G$ is the largest $n$ such that $G$ has a $K_n$-subgraph.
of all finite simple graphs $G$ to the naturals defined as
\[ \text{tw}(G) + 1 := \min \{ \omega(H) : G \leftarrow H \text{ and } H \text{ is chordal} \}. \] (1)

We call any monomorphism $G \leftarrow H$ with $H$ chordal a chordal completion of $G$.

**Tree-width via structured decompositions** Consider the functor
\[ \left( \frac{1}{2} \right) : \text{FinSet} \rightarrow \text{Gr}_{\mathcal{H}} \]
taking each set $S$ to the complete graph $(S, \binom{S}{2})$. Proposition 5.3 shows that the image of any object in $\mathcal{D}_{\text{forest}} \text{FinSet}$ under $\text{Lan}_{\text{const}} \left( \frac{1}{2} \right)$ is a chordal graph (see Figure 2).

**Proposition 5.3.** Let $\left( \frac{1}{2} \right) : \text{FinSet} \rightarrow \text{Gr}_{\mathcal{H}}$ be the mapping taking every set to the complete graph over that set. The image under $\text{Lan}_{\text{const}} \frac{1}{2}$ of any tree-shaped structured decomposition $\tau : \bigtriangledown T \rightarrow \text{FinSet}$ in $\mathcal{D}_{\text{forest}} \text{FinSet}$ is a chordal graph; furthermore this map is a surjection on chordal graphs: every chordal graph is obtained via at least one structured decomposition of this kind.

**Proof.** Recall from Definition 3.1 that we can view the structured decomposition
\[ \tau : \bigtriangledown T \rightarrow \text{FinSet} \]
as a labelling of the vertices and edges of the graph $T$ by objects and spans of $\text{FinSet}$. Viewing these in $\text{Gr}_{\mathcal{H}}$ under the image of $\Omega$ we see a collection of spans of complete graphs such that each span is indexed by an edge of $T$ (in the terminology of Definition 3.1, we would say that each such span is an adhesion indexed by some edge). Since pushouts are associative, it is easy to see that, fixing some root node $r$ in $T$, we can compute the colimit of this diagram in $\text{Gr}_{\mathcal{H}}$ by bottom-up recursion: take successive
pushouts starting with the spans corresponding to leaf-edges of \( T \). By Definition 5.1, we see that the resulting graph is chordal, as desired.

The second part of the claim follows by structural induction on Definition 5.1. Take any chordal graph \( H \). If \( H \cong K_\eta \), then it can be seen as the structured decomposition

\[
V(K_\eta) \leftarrow_{1_{V(K_\eta)}} V(K_\eta) \rightarrow_{1_{V(K_\eta)}} V(K_\eta).
\]

Otherwise there is a span

\[
H_1 \leftarrow K_\eta \rightarrow H_2
\]

in \( \text{Gr}_{\Omega} \) with \( H_1 \) and \( H_2 \) chordal such that \( H \cong H_1 +_{K_\eta} H_2 \). By induction there are tree-shaped structured decompositions

\[
r_1 : \int T_1 \rightarrow \text{FinSet} \quad \text{and} \quad r_2 : \int T_2 \rightarrow \text{FinSet}
\]

corresponding to \( H_1 \) and \( H_2 \). Since \( H_1 \) and \( H_2 \) are chordal, there must be nodes \( v_1 \) and \( v_2 \) in the trees \( T_1 \) and \( T_2 \) corresponding to \( H_1 \) and \( H_2 \) such that the image of \( K_\eta \) in \( H_1 \) (resp. \( K_\eta \) in \( H_2 \)) is completely contained in the bag \( B_{v_1} \) indexed by \( v_1 \) in \( D_1 \) (resp. \( B_{v_2} \) for \( v_2 \) in \( D_2 \)). Thus we can construct a new tree \( T \) by joining \( T_1 \) to \( T_2 \) via a new edge \( v_1 v_2 \). Furthermore, by labeling this new edge \( v_1 v_2 \) by the span

\[
V(H_1) \cap B_{v_1} \leftarrow V(K_\eta) \rightarrow V(H_2) \cap B_{v_1},
\]

we obtain a new structured decomposition corresponding to \( H \), as desired. \( \Box \)

**Defining tree-width**

Take a natural number \( k \in \mathbb{N} \) and let \( \text{FinSet}_k \) denote the full subcategory of \( \text{FinSet} \) induced by all sets of cardinality at most \( k \). Let \( i_k : \text{FinSet}_k \hookrightarrow \text{FinSet} \) denote the inclusion functor between these categories. Then, in accordance to Proposition 5.3, the image of \( \text{Lan}_{\mathbb{C}_{\text{forest}}} \left( \int_2 \right) \circ i_k \) consists of all chordal graphs with clique number at most \( k \). In turn, by Definition 1, any graph \( G \) is a subgraph of something in the range of \( \text{Lan}_{\mathbb{C}_{\text{forest}}} \left( \int_2 \right) \circ i_k \) precisely if it has tree-width at most \( k \).

This suggests a more general perspective on width measures: instead of regarding the natural number \( k \) as the bound on the width of our graph, we can regard the category \( \text{FinSet}_k \) itself as the value of the width measure. Accordingly, our categorial generalization of width measure will have as its values not natural numbers, but certain well-behaved subcategories of the category of atoms \( \mathbb{K} \). This perspective rests on the following notion of a completion which generalizes the concept of a chordal completion familiar from graph theory [20] (to see how our definition below specializes to chordal completions, take \( G \) to be the class of forests and \( \Omega \) to be the functor \( \left( \int_2 \right) \) of Proposition 5.3).

**Definition 5.4** (Completions). Let \( \mathbb{K} \) be a small category, \( \mathbb{C} \) be a locally small category with colimits, \( \Omega : \mathbb{K} \rightarrow \mathbb{C} \) be a functor and let \( \mathcal{G} \) be a class of graphs. For any object \( x \in \mathbb{C} \), a \((\mathcal{G}, \Omega)\)-completion of \( x \) is a monomorphism of the form

\[
\delta : x \hookrightarrow \text{Lan}_\Omega(D)
\]

where \( (d : \int G \rightarrow \mathbb{K}) \in \mathbb{P}(\mathbb{K}) \) and \( G \in \mathcal{G} \).

The obvious choice would be to allow all pullback-closed full subcategories of \( \mathbb{K} \) as width measures, but we get better behavior (especially in the algorithmic applications of Section 6) if we restrict our attention to the class of pullback-absorbing subcategories defined below.
**Definition 5.5** (Pullback-absorbing). A functor \( P : A \to C \) is pullback-absorbing if, for every \( a \in A \), every \( y, z \in C \) and every monic pullback square of the following form,

\[
\begin{array}{ccc}
Pa & \rightarrow & y \\
\uparrow & & \uparrow \\
Pa \times_y z & \rightarrow & z
\end{array}
\]

there exists an object \( a_{y,z} \in A \) such that \( Pa_{y,z} = Pa \times_y z \). We say that a subcategory \( C' \) of a category \( C \) absorbs pullbacks if the inclusion function \( i : C' \to C \) is both full and pull-back absorbing.

Notice that, up to equivalence, the class of pullback-absorbing subcategories of \( \text{FinSet} \) consists precisely of the category \( \text{FinSet} \) itself, and the full subcategories \( \text{FinSet}_k \).

Thus, for tree-width, we have the picture given in Figure 3.

**Definition 5.6.** Take a functor \( \Omega : K \to C \). Consider a pullback-absorbing subcategory \( B \) of \( K \) with inclusion functor \( i : B \hookrightarrow K \). We say that an object \( x \) of \( C \) has \((G,\Omega)\)-width bounded by \( B \) if \( x \) admits at least one \((G,\Omega \circ i)\)-completion.

### 5.2 Completions yield decompositions

The algorithmic and structural usefulness of tree-width stems in many cases from the associated notion of tree-decompositions [20, 27, 16]. In this section we will generalize the notion of a tree decomposition by showing how they relate to the completions of Definition 5.4. In Section 5.4 we will give further examples of combinatorial invariants such as layered tree-width [54, 26] or graph decomposition width [10, 23] which are special cases of our notion of \((G,\Omega)\)-width.
**Background on tree decompositions**  In Section 5.1 we gave a definition of $(\mathcal{G}, \Omega)$-width via the notion of $(\mathcal{G}, \Omega)$-completions. This was in direct analogy to the definition of tree-width of a graph $G$ (Equation (1)) as the minimum clique-number over all chordal supergraphs $H$ of $G$ (also known as chordal completions [20]). This definition highlights the perspective that a graph $G$ should be understood as a structurally simple graph because it is contained in a graph $H$ which is itself structurally simple in a very concrete way: $H$ is built by recursively gluing (i.e. taking pushouts of) small constituent cliques.

There is, however, another equivalent definition of tree-width in terms of so-called tree decompositions (Definition 5.7; see also Figure 4). In contrast to chordal completions, tree decompositions do not delegate the task of highlighting the recursive structure of a graph $G$ to those supergraphs of $G$ which display a particularly clear recursive structure (i.e. the chordal supergraphs of $G$). Instead, tree decompositions explicitly record the tree-like shape of $G$.

A graph $G$.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{tree_decomposition.png}
\caption{An example of a tree decomposition of a graph $G$. Note that, to ease legibility, we have only recorded the vertices that appear in each bag rather than drawing the entire subgraph contained in that bag. For didactic purposes, we invite the reader to compare this figure with Figure 1 for a visual understanding of the connection between tree decompositions and structured decompositions.}
\end{figure}

**Definition 5.7.** The pair $(T, B)$ is a tree decomposition of a graph $G$ if $B := (B_t)_{t \in V(T)}$ is a sequence of subgraphs of $G$ indexed by the nodes of the tree $T$ such that:

- **(T1)** for every edge $xy$ of $G$, there is a node $t \in V(T)$ such that $\{x, y\} \subseteq B_t$,
- **(T2)** for every $x \in V(G)$, the set $\{t \in V(T) : x \in B_t\}$ induces a non-empty connected subgraph in $T$. 

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The elements of \( B \) are called the \textit{bags} of the decomposition and the sets of the form \( B_t \cap B_{t'} \) where \( t't' \) is an edge in \( T \) are called the \textit{adhesion sets} of the decomposition.

We invite the reader to compare the terminology of bags and adhesion sets above with our terminology in Definition 3.1: a tree-decomposition of a graph \( G \) is nothing more than an object of \( \mathcal{D}(\text{Gr}_H) \) whose colimit is \( G \).

### 5.3 Completions vs decompositions

It is well-known [20] (and not too hard to show) that tree decompositions and chordal completions are two equivalent perspectives on the same notion: given a tree decomposition \((T, B)\) of a graph \( G \), we can find a chordal supergraph \( H \) of \( G \) by filling-in any missing edges of the bag \( B_t \) for each \( t \in T \). Conversely, given a chordal completion \( G \hookrightarrow H \) of \( G \), we can recover a decomposition \((T, B)\) for \( G \) by letting the bags be the restriction of the image of \( G \) to the maximal cliques of \( H \).

This correspondence has a neat statement (and indeed, generalization) in our categorical formalism.

**Theorem 5.8.** Denote by \( \mathcal{D}_{\text{sd}} \mathcal{K} \) the subcategory of \( \mathcal{sd} \mathcal{K} \) consisting of diagrams whose morphisms are monomorphisms. Let \( \Omega : \mathcal{K} \to \mathcal{C} \) be a functor to an adhesive category \( \mathcal{C} \) with colimits of diagrams of monomorphisms. Let \( c \in \text{Ob}(\mathcal{C}) \) and consider any \( \Omega \)-completion

\[
\delta : c \leftrightarrow \text{Lan}_{\mathcal{D}_{\text{m}}} \Omega(d : \int G \to \mathcal{K}).
\]

Then there exists a \( \mathcal{C} \)-valued structured decomposition (which is also of shape \( G \))

\[
d_c : \int G \to \mathcal{C}
\]

and an arrow

\[(1_G, \eta_b) : d_c \to \mathcal{D}_{\text{m}} \Omega d \text{ in } \mathcal{D}_{\text{m}}(\mathcal{C})\]

as in the following diagram

\[
\begin{array}{ccc}
\int G & \xrightarrow{1_G} & \int G \\
\downarrow d_c & & \downarrow d \\
C & \xleftarrow{\eta_b} & \mathcal{D}_{\text{m}} \Omega d
\end{array}
\]

such that \((1_G, \eta_b)\) is mapped to \( \delta \) by the colimit functor; i.e. \( \text{colim} d_c = c \) and

\[
\text{colim} : (d_c : (1_G, \eta_b) \to \mathcal{D}_{\text{m}} \Omega d) \mapsto \left( \text{colim} d_c \xrightarrow{\delta} \text{colim} \right).
\]

Rather than proving Theorem 5.8 straight away, we will explain why this theorem allows us to recover the relationship we just mentioned between tree-width, chordal completions and tree-decompositions. To do this, take \( \mathcal{K} = \text{FinSet}, \mathcal{C} = \text{Gr}_H \) and \( \Omega = (\cdot)^2 \) to be the functor described in Proposition 5.3 which takes each set \( S \) to the complete graph \((S, (\cdot)^2_S)\). Proposition 5.3 tells us that, for every chordal completion \( \delta : X \to H \), we can
find an appropriate tree \( T_H \) and \( \text{FinSet} \)-valued decomposition \( d : \int T_H \to \text{FinSet} \) such that \( H = \text{Lan}_{\mathcal{D}_{\leq m}}(\delta)(d : \int T_H \to \mathcal{K}) \) and hence we can rewrite \( \delta \) as

\[
\delta : x \leftarrow \text{Lan}_{\mathcal{D}_{\leq m}}(\delta)(d : \int T_H \to \mathcal{K}).
\]

Then Theorem 5.8 states that we can find a \( \text{Gr}_H \)-valued structured decomposition

\[
d_X : \int T_H \to \text{Gr}_H
\]

of the same shape as \( d \) and an arrow

\[
(1_{T_H}, \eta_\delta) : d_x \to \mathcal{D}_{\leq m} \Omega d
\]

(which is an arrow in \( \mathcal{D}_{\leq m}(C) \)) such that \( \text{colim}(1_{T_H}, \eta_\delta) = \delta \).

**Observation 5.9.** In other words, when instantiated over graphs (according to the set-up of Proposition 5.3), Theorem 5.8 states that we can lift any chordal completion \( \delta : X \to H \) to a morphism \( d_X \to d \) between two structured decompositions which respectively give rise to the graphs \( X \) and \( H \) as colimits and furthermore this allows us to relate any completion of \( X \) to a genuine decomposition of \( X \).

Since it is not hard to show that, when thinking of tree-shaped decompositions, the structured decomposition \( d_X \) is a tree decomposition of \( X \), we see that Theorem 5.8 is precisely the generalization of the relationship between width-measures and decompositions that we set-out to find. All that remains to do now is to prove it. To that end, we first need the following Lemma.

**Lemma 5.10.** Let \( C \) be an adhesive category with colimits of all diagrams whose arrows are monomorphisms. For any arrow \( \delta : x \to y \) in \( C \) and any diagram \( d_y : J \to C \) whose colimit is \( y \) we can obtain a diagram \( d_x : J \to C \) whose colimit is \( x \) by point-wise pullbacks of \( \delta \) and the arrows of the colimit cocone over \( d_y \). Furthermore this assembles into a functor

\[
\text{subMon} : C^{\text{op}} \to \text{Set}
\]

\[
\text{subMon} : e \mapsto \{d \mid \text{colim} d = e \text{ and } d \text{ is a diagram with monic arrows in } C\}
\]

\[
\text{subMon} : (f : x \to y) \mapsto (f' : (d_y \in \text{subMon} y) \mapsto (d_x \in \text{subMon} x)).
\]

**Proof.** We will prove this by induction on the number of objects of \( J \). Notice that the claim is trivially true if \( J \) has only one object. So now suppose \( J \) has more than object and let \( q : J' \to J \) be a full subcategory of \( J \) with exactly one object fewer than \( J \). Then
we have the following commutative diagram which we will explain in what follows.

The arrow $\delta$ is any given morphism in $C$. By taking pointwise pullbacks of the colimit cocone $\Lambda_y$ over $d_y$, we obtain a diagram $d_x$ and a cocone $\Lambda_x$ over it. Our goal is to prove that this cocone is a colimit cocone.

The bottom face of the diagram is a pushout which is stating that we can obtain the colimit of $d_y$ via a pushout of the colimit of $qd_y$, and the remaining arrows of $d_x$.

The front and right faces are obtained by pullback (by construction). Now our goal is to prove that the back face is a pullback as well since then the fact that the morphisms of $d_y$ are monic (and the fact that the pushout of monos are monos in adhesive category) implies that the whole cube is a Van Kampen cube and the top face is a pushout. This will in turn imply that $x$ is the colimit of the diagram $d_x$, as desired.

Thus, to conclude the proof, observe that the long rectangular face formed by pasting together the back and left faces is a pullback. Thus, by the two pullback lemma, this implies that the back face is also a pullback (since the left face is).

By specializing Lemma 5.10 we obtain the following result which also implies a proof of Theorem 5.8.

**Corollary 5.11.** Let $C$ be an adhesive category with colimits of all diagrams whose arrows are monomorphisms. For any arrow $\delta : x \to y$ in $C$ and any structured decomposition whose arrows are monic $d_y : J \to C$ (i.e. $d_y \in \mathcal{D}_{mC}$ whose colimit is $y$) we can obtain a diagram $d_x : J \to C$ whose colimit is $x$ by point-wise pullbacks of $\delta$ and the arrows of the colimit cocone over $d_y$. Furthermore this allows us to define the following functor

$$\text{Decomp} : C^{op} \to \text{Set}$$

$$\text{Decomp} : c \mapsto \{d \in \mathcal{D}_{mC} \mid \text{colimd} = c \text{ and } d \text{ is a diagram with monic arrows in } C\}$$

$$\text{Decomp} : (f : x \to y) \mapsto \{f' : (d_y \in \text{Decomp}(y)) \mapsto (d_x \in \text{Decomp}(x))\}.$$

**Proof.** The category of structured decompositions is a subcategory of the category of diagrams in $C$. Thus $\text{Decomp}$ is evidently a subfunctor of $\text{subMon}$. ■

**Proof of Theorem 5.8.** Apply Corollary 5.11 to any given completion. ■

### 5.4 Instances of $(\mathcal{G}, \Omega)$-width in combinatorics

Our notion of $(\mathcal{G}, \Omega)$-width unifies various recently introduced notions of graph width measures (and structured decompositions, as we shall see in Section 5.2) which play
significant roles in graph structure theory [10] and parameterized complexity [36]. Here we will mention some of these connections.

**Graph decompositions** It is easy to see that, in the case of simple, undirected graphs, the above definition coincides with Carmesin’s notion of graph decomposition width [10, Def. 9.3] which generalizes tree decompositions (see Definition 5.7) and tree-width to graph decompositions of arbitrary shapes. To keep the presentation self-contained, we include Carmesin’s notion as Definition 5.12 below.

**Definition 5.12** ([10]). A graph-decomposition consists of a bipartite graph \((B, S)\) with bipartition classes \(B\) and \(S\), where the elements of \(B\) are referred to as ‘bags-nodes’ and the elements of \(S\) are referred to as ‘separating-nodes’. This bipartite graph is referred to as the ‘decomposition graph’. For each node \(x\) of the decomposition graph, there is a graph \(G_x\) associated to \(x\). Moreover for every edge \(e\) of the decomposition graph from a separating-node \(s\) to a bag-node \(b\), there is a map \(\iota_e\) that maps the associated graph \(G_s\) to a subgraph of the associated graph \(G_b\). We refer to \(G_s\) with \(s \in S\) as a local separator and to \(G_b\) with \(b \in B\) as a bag.

Note that our definition of \((G, \Omega)\)-width immediately extends Carmesin’s Definition 5.12 to directed graphs as well since the category \(\text{Rel}\) of reflexive, directed multi-graphs has all colimits. This is an interesting direction for further study.

**Complemented tree-width** Another combinatorial instance of \((G, \Omega)\)-width comes from the notion of complemented tree-width. This is the map \(\text{tw} : G \mapsto \text{tw}(\overline{G})\) taking each graph \(G\) to the tree-width of its complement. Complemented tree-width has useful applications in algorithmics [25] since it allows for recursive algorithms on graph classes that are both dense and incomparable to bounded tree-width classes. Using the theory of spined categories [9, Sec. 5.2], one can already provide a category-theoretic characterization of complemented treewidth. Now, through the use of \((G, \Omega)\)-width, not only can we define complemented tree-width, but also complemented versions of Carmesin’s graph decompositions [10]. This is direct consequence of the fact that the category \(\text{Gr}_{\text{Ref}}\) of graphs and reflexive homomorphisms [9, Def. 5.6] (Definition 5.13 below) is isomorphic to \(\text{Gr}_{\text{H}}\) [9, Prop. 5.10] and hence has all colimits\(^2\).

**Definition 5.13** ([9]). A reflexive graph homomorphism is a vertex map \(f : V(G) \rightarrow V(H)\) from the vertices of a graph \(G\) to those of a graph \(H\) such that, if \(f(x)f(y)\) is an edge in \(H\), then \(xy\) is an edge in \(G\) as well. There is a category \(\text{Gr}_{\text{Ref}}\) whose objects are the same as those of \(\text{Gr}_{\text{H}}\) and whose arrows are reflexive graph homomorphisms.

**Layered tree-width** We can replace the category \(\text{FinSet}\) with other elementary topoi to obtain different width measures. Many of the measures obtained this way have already been investigated in the combinatorics literature. For example, layered tree-width is a variant of tree-width introduced independently by Shahrokhi [54] and Dujmović, Morin, and Wood [26] which, in contrast to tree-width [20], remains bounded on planar graphs. Here we show choosing the category \(K\) as the category \(\text{FinSet}^n\), which consists of finite copresheaves on the category \(\mathbb{N}\) with finite support (i.e. functors

\(^2\)NB this is not the same as the category of reflexive graphs and homomorphisms!
Formally $\mathbb{N}_\subseteq \to \text{FinSet}$ satisfying $F[X] \not\cong \perp$ for only finitely many objects $X$ of $\mathbb{N}_\subseteq$ gives rise to a structured decomposition that has layered tree-width as its associated width measure.

**Definition 5.14** ([54, 26] stated as in [7]). Given a finite graph $G$, a *layering* of $G$ is a finite sequence of vertex subsets $\mathcal{L} := (L_1, L_2, \ldots)$ partitioning the vertex-set of $G$ such that whenever we are given a pair of vertices $(x_i, x_j) \in L_i \times L_j$, if $x_i x_j$ is an edge in $G$, then $|i - j| \leq 1$. A *layered tree decomposition* of a graph $G$ is a pair $(\mathcal{L}, (T, B))$ where $\mathcal{L}$ is a layering of $G$ and $(T, B)$ is a tree decomposition of $G$. The *width* of a layered tree decomposition is the maximum number of vertices shared by any bag in the decomposition with any layer; i.e., $\text{width}(T, B) := \max_{L \in \mathcal{L}} \max_{B \in T} |B \cap L|$. The layered tree-width $\text{ltw}(G)$ is the minimum width attained by any layered tree decomposition of $G$.

Again, as we did above for FinSet*, we let $\text{Gr}_H^*$ denote the full subcategory of $\text{Gr}_H^{\mathbb{N}_\subseteq}$ consisting of sequences with finite support. Notice that we can lift the functor $(\gamma)$ of Proposition 5.3 to a functor $\mathcal{K}^* : \text{FinSet}^* \to \text{Gr}_H^*$ by setting

$$
\mathcal{K}^* : \text{FinSet}^* \to \text{Gr}_H^*,
$$

$$
\mathcal{K}^* : (F : \mathbb{N}_\subseteq \to \text{FinSet}) \mapsto (\left\lceil \frac{1}{2}\right\rceil \circ F : \mathbb{N}_\subseteq \to \text{Gr}_H).
$$

Clearly this allows us to obtain a width measure $\text{Gr}_H^*$. We can use this to induce a width measure on $\text{Gr}_H$ itself by defining a suitable functor $\otimes : \text{Gr}_H^* \to \text{Gr}_H$ which maps any sequence $F : \mathbb{N}_\subseteq \to \text{Gr}_H$ of graphs to the graph obtained by making each graph in the sequence complete$^3$ to the graph following it in the sequence. Formally $\otimes$ is defined as follows (where $+$ and $\coprod$ denote coproducts)

$$
\otimes : \text{Gr}_H^* \to \text{Gr}_H
$$

$$
\otimes : F \mapsto \left(\coprod_{n \in \mathbb{N}} V(F(n)), \coprod_{n \in \mathbb{N}} E(F(n)) + \coprod_{n \in \mathbb{N}} \{xy : x \in V(F(n)), y \in V(F(n+1))\}\right)
$$

$$
\otimes : \left(\coprod_{n \in \mathbb{N}} f_n : F \to G\right) \mapsto \left(\coprod_{n \in \mathbb{N}} f_n : \otimes F \to \otimes G\right).
$$

**Proposition 5.15.** Given the full subcategory $t_k : \text{FinSet}_{k}^* \to \text{FinSet}^*$, any graph $G$ has (forest, $\otimes \mathcal{K}^*$)-width at most $\text{FinSet}_{k}^*$ if and only if $G$ has layer tree-width at most $k$.

**Proof.** Since it can be shown that $\otimes$ preserves colimits, and since this is also true for $(\gamma)$ (and hence for $\mathcal{K}^*$), we can deduce the commutativity of the following diagram.

$$
\begin{array}{cccc}
\text{FinSet}* & \xrightarrow{\otimes \mathcal{K}^*} & \text{Gr}_H^* & \xrightarrow{\otimes} & \text{Gr}_H \\
\text{colim} & & \text{colim} & & \text{colim} \\
\text{FinSet}* & \xrightarrow{\otimes k^*} & \text{Gr}_H^* & \xrightarrow{\otimes} & \text{Gr}_H
\end{array}
$$

$^3$Recall that we say that a graph $G$ is obtained by making two graphs $G_1$ and $G_2$ complete to one another if $G$ is isomorphic the graph obtained as follows: starting from the disjoint union of $G_1$ and $G_2$, add an edge from every vertex of $G_1$ to every vertex of $G_2$.
Now consider any graph $G$ and take any (forest, $\mathcal{K}$)-completion $\delta : G \leftrightarrow H$ of $G$ where

$$H := \text{Lan}(\mathcal{K}^*) (d : \int T \to \text{FinSet}^*)$$

is given from some monic decomposition $d$. By Theorem 5.8 there exists a $\text{Gr}_H$-valued structured decomposition $d_G : \int T \to \text{Gr}_H$ of the same shape as $d$ and a morphism

$$(1_{T}, \eta_\delta) : d_G \to \mathfrak{D}(\mathfrak{S}) \mathfrak{D}(\mathcal{K}^*)(d)$$

in $\mathfrak{D}(\text{Gr}_H)$ such that colim$(1_{T}, \eta_\delta) = \delta$.

For clarity, we explicitly point out that the structured decomposition $d_G$ yields a tree decomposition $(T, (B_t)_{t \in V T})$ of $G$ where $B_t := (d_G)_t$ for each node $t \in V T$.

Now notice that under the colim map the decomposition $d \in \mathfrak{D}_m(\text{FinSet}^*)$ gets mapped to the sequence of sets

$$\mathcal{L} := (\text{colim}(d) : \mathbb{N}_m \to \text{FinSet})$$

Furthermore, $\mathcal{L}$ is a layering of $H$ and hence, since $G$ is a subgraph of $H$, this induces a layering on $G$ as well. It thus follows from the definition of morphisms of structured decompositions, that max${\{|B_t \cap L| : t \in T, L \in \mathcal{L}\}} \leq k$. Thus we have shown that $G$ has layer tree-width at most $k$ whenever it has (forest, $\mathcal{K}$)-width at most $\text{FinSet}_k$.

For the converse direction, suppose $G$ admits the following layer-tree decomposition of width at most $k$:

$$(F : \mathbb{N}_m \to \text{FinSet}^*, d_G : \int T \to \text{Gr}_H)$$

This gives rise to an object of $d_G^* \in \mathfrak{D} \text{Gr}_H^*$ by inducing a tree decomposition on each layer; on objects $d_G^*$ is given as follows (to reduce clutter, we omit its definition on arrows):

$$d_G^* : \int T \to \text{Gr}_H^*$$

$$d_G^* : t \in \int T \to \bigsqcup_{n \in \mathbb{N}} d_G(t)[F n]$$

(where, as usual in graph theory, the square bracket notation in $D_G(t)[F n]$ denotes taking the induced subgraph on the vertices in $F n$).

Now recall that, since the forgetful vertex-set functor $V : \text{Gr}_H \to \text{FinSet}$ preserves colimits, we can conclude that the following diagram commutes.

$$\begin{array}{ccc}
\mathfrak{D} \text{FinSet}^* & \xleftarrow{\mathfrak{D} V^*} & \mathfrak{D} \text{Gr}_H^* \\
\downarrow \text{colim} & & \downarrow \text{colim} \\
\text{FinSet}^* & \xleftarrow{V^*} & \text{Gr}_H^*
\end{array}$$

(3)

Now, to conclude the proof, recall that there is an adjunction $\mathcal{K} \dashv V$ and that this adjunction lifts to an adjunction $\mathcal{K}^* \dashv V^*$. Consequently, by the properties of this adjunction, we find that there is a monomorphism

$$d_G^* \leftrightarrow \mathfrak{D}_m(\mathcal{K}^*) \mathfrak{D}_m(V^*) d_G^*$$

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which one can easily show gives rise to the desired monomorphism

\[ G \xleftarrow{\text{const}} \text{Lan}(\mathcal{E}\mathcal{K})(\mathcal{D}(V^*)d^*_G). \]

\[ \square \]

**\( \mathcal{H} \)-tree-width**  Another notion that has been particularly impactful in parameterized algorithms is the notion of **\( \mathcal{H} \)-tree-width** which was introduced by Jansen, de Kroon and Włodarczyk \cite[Def. 2.3]{36}. This width measure allows leaf-bags\(^4\) of tree decompositions to be arbitrarily large, as long as they belong to a fixed graph class **\( \mathcal{H} \)**. Whenever the class **\( \mathcal{H} \)** is closed under taking subgraphs, we obtain a finer-grained (and in some senses better behaved) version of **\( \mathcal{H} \)-tree-width** as an instance of \((G,\Omega)\)-width.

We briefly sketch the idea behind this construction. Let \( K_w \) denote the full subcategory of \( \text{Gr}_{\mathcal{H}} \) consisting of all graphs in the class **\( \mathcal{H} \)**, together with all complete graphs \( \mathcal{E} \) with at most some fixed constant \( w \in \mathbb{N} \). Let \( \Omega : K_w \rightarrow \text{Gr}_{\mathcal{H}} \) be the obvious inclusion and consider any tree-shaped object \( d_H : \int T \rightarrow K_w \) of \( \mathcal{E} \). If \( \mathcal{H} \) is closed under taking subgraphs, then whenever we have a mono \( G \xleftarrow{\text{const}} \text{Lan}_{\mathcal{E}}(\Omega(d_H)) \), we can conclude that \( G \) admits a tree decomposition \((T,B)\) where the width of this decomposition is the size of the largest bag \( B \) such that \( B \notin \mathcal{H} \). Clearly whenever a graph has **\( \mathcal{H} \)-tree-width** at most \( w \), it will have \((\text{forest},\Omega)\)-width at most \( K_w \). The converse, however, does not follow: our notion does not force all of the bags in **\( \mathcal{H} \)** which have width larger than \( w \) to be leaves. We conjecture that, with slightly more effort, one can proceed along similar lines of thinking use structured decompositions to define similar classes whenever \( \mathcal{H} \) is merely closed under taking induced subgraphs.\(^5\)

**Matroids**  Heunen and Patta showed that there is a category \( \text{Matr} \) of matroids and strong maps \cite{33} and a right-adjoint functor \( C : \text{FinSet} \rightarrow \text{Matr} \) called the cofree functor \cite[Theorem 2.9]{33}. Although \( \text{Matr} \) does not have all pushouts, all pushouts under cofree matroids exist \cite[Prop. 3.6]{33}. This means that we can define the \((G,C)\)-width of matroids for various choices of decomposition shapes \( G \). Comparing the resulting width measure to Hliněný and Whittle’s *matroid tree-width* \cite{34} is a promising avenue for further study.

Finally, although we will not treat this in the present paper, we note that further interesting applications for structured decompositions include Petri nets, chemical reaction networks and other kinds of ‘typed graphs’.

6 Algorithms

As we have seen so far, structured decompositions are ways of organizing data, defining width measures (Sections 5.1 and 5.4) and giving compositional formulae for computing objects of a category as colimits (Section 5.2). Here we study another aspect of structured decompositions; namely their algorithmic applications.

\(^4\)i.e. bags indexed by leaves of the tree decomposition

\(^5\)A graph \( G \) is an induced subgraph of a graph \( H \) if there is a vertex-subset \( S \subseteq V(H) \) of \( H \) such that \( G \cong H - S \). Note that, if a class is closed under taking subgraphs, then it is also closed under taking induced subgraphs. The converse, however, does not hold.
| Decomposition          | Functor $\Omega$                              | Shape class $\mathfrak{G}$ | Width measure           |
|------------------------|----------------------------------------------|----------------------------|-------------------------|
| tree decomp.           | $\langle \_ \rangle : \text{FinSet} \rightarrow \text{Gr}_{\mathfrak{T}}$ (complete graphs) | trees                      | tree-width              |
| complemented tree decomp. | $\langle \_ \rangle : \text{FinSet} \rightarrow \text{Gr}_{\mathfrak{T}}$ (discrete graphs) | trees                      | complemented tree-width |
| $G$-shaped graph decomp.       | $\langle \_ \rangle : \text{FinSet} \rightarrow \text{Gr}_{\mathfrak{T}}$ (complete graphs) | $G$                        | graph $G$-decomposition width |
| layered tree decomp.     | $\mathfrak{A}^\star : \text{FinSet} \rightarrow \text{Gr}_{\mathfrak{T}}$ | trees                      | layered tree-width      |
| $H$-tree decomp.         | $\langle \_ \rangle \times : \text{FinSet} \times H \rightarrow \text{Gr}_{\mathfrak{T}}$ where $i : H \hookrightarrow \text{Gr}_{\mathfrak{T}}$ for full $H \subseteq \text{Gr}_{\mathfrak{T}}$ and $\langle \_ \rangle$ as above | trees                      | $H$-tree-width           |

Table 1: Summary of the combinatorial invariants we described using structured decompositions in Sections 5.1 and 5.4.

### 6.1 Algorithmic preliminaries

In what follows, we work with recursive categories: a straightforward, if tedious, formalization of what it means for certain operations on a category to be computable. We refer the reader to the appendix for the formal definition of a strict recursive category (Definition A.1; furthermore, all recursive co/limits are defined analogously to the recursive pullbacks of Definition A.1.

**Definition 6.1.** Take a strict recursive category $\mathcal{K}$ with recursive pullbacks, a strict recursive category $\mathcal{C}$ with (not necessarily recursive) colimits equipped with a recursive functor $\Omega : \mathcal{K} \rightarrow \mathcal{C}$ and a computable function $| \cdot | : \text{Ob}(\mathcal{C}) \rightarrow \mathbb{N}$, and a class of graphs $\mathfrak{G}$.

Consider a decision problem $P$ defined on the objects of the category $\mathcal{C}$. We say that the decision problem $P$ is **fixed-parameter tractable parameterized by $(\mathfrak{G}, \Omega)$-width** if for every object $x$ of $\mathcal{C}$ there exists a pullback-absorbing subcategory $\mathcal{B}_x$ of $\mathcal{K}$ such that

1. $x$ has $(\mathfrak{G}, \Omega)$-width at most $\mathcal{B}_x$, and
2. there is a constant $c$ such that the problem $P$ can be solved in time $O(|y|^c)$ on all objects $y$ of $(\mathfrak{G}, \Omega)$-width at most $\mathcal{B}_x$.

Note that, the definition above states that, whenever a problem is fixed-parameter tractable parameterized by $(\mathfrak{G}, \Omega)$-width, then this means that the problem is polynomial-time solvable (in the classical sense) on classes of bounded $(\mathfrak{G}, \Omega)$-width. For instance, when specialized to the case of tree-width, this definition is stating that a problem is in FPT parameterized by tree-width if there is a constant $c$ and an algorithm whose worst-case running time is polynomial on any given class of graphs of bounded tree-width.

Using the notion of fixed-parameter tractability for a width measure, we can now state what it means to **recover** an ordinary (graph-theoretic) width measure as an instance of our categorial construction.

**Definition 6.2 (FPT-recovery).** Consider a graph-theoretic width measure $\nu$, a
class of graphs $\mathcal{G}$ and a functor $\Omega$ whose codomain is the category $\text{Gr}_H$. We say that $(\mathcal{G}, \Omega)$-width FPT-recovers $w$ if a decision problem $P$ is fixed-parameter tractable parameterized by $w$ (in the usual sense) precisely if it is fixed-parameter tractable parameterized by $(\mathcal{G}, \Omega)$-width (in the sense of Definition 6.1).

**Observation 6.3.** The tree-width (resp. complemented tree-width, layered tree-width and graph decomposition width) instance of $(\mathcal{G}, \Omega)$-width presented in Section 5.4 FPT-recovers tree-width (resp. complemented tree-width, layered tree-width and graph decomposition width).

### 6.2 Algorithmic applications

Throughout this section, we are given a functor $\mathcal{F} : \mathcal{A} \to \mathcal{B}$ between recursive categories $\mathcal{A}$ and $\mathcal{B}$. We think of the functor $\mathcal{F}$ as encoding a computational problem (or its solution space). Our task is to determine the value of $\mathcal{F}$ on a pushout, given the values of $\mathcal{F}$ on the objects of the associated span. This is the $\mathcal{F}$-COMPOSITION problem given below.

| **F-COMPOSITION** |
|-------------------|
| **Input:** A functor $\mathcal{F} : \mathcal{C} \to \mathcal{D}$, a span $L \leftarrow M \rightarrow R$ in $\mathcal{C}$ and the objects $\mathcal{F}(L), \mathcal{F}(M)$ and $\mathcal{F}(R)$. |
| **Task:** Determine $\mathcal{F}(L +_M R)$. |

The goal of this section is twofold. First of all we want to investigate how to lift algorithms for $\mathcal{F}$-COMPOSITION to the problem of determining $\mathcal{F}$ on objects admitting structured decompositions of any shape (and not just those of shape $K_2$, which is what the $\mathcal{F}$-COMPOSITION problem asks). Our second goal is to provide concrete algorithms for various instances of the $\mathcal{F}$-COMPOSITION problem.

We note that the composition problem has been solved in the case when $\mathcal{F}$ computes shortest paths [43]. Extending this result to more general structured decompositions will be left to future work.

In Section 6.3 we show that, whenever $\mathcal{F}$ is colimit-preserving, we can lift any algorithm for the $\mathcal{F}$-COMPOSITION problem to an algorithm running on objects admitting structured decompositions. Although this section will provide some valuable insights and although the ensuing algorithms already provide benefits compared to non-compositional algorithms, we conjecture that our algorithmic lifting procedure can be still greatly improved (see Section 7 for a more thorough discussion). Indeed this opens fascinating new directions for future research.

On the other hand in Section 6.4 we will consider some cases when $\mathcal{F}$ is not pushout preserving. In particular we will provide an algorithm for the $\text{Sub}_P$-Composition problem for any functor $P : \mathcal{C}' \to \mathcal{C}$ which is pullback-absorbing (Definition 5.5). This result yields compositional (but not FPT-time) algorithms for many optimization problems including $\text{MAXIMUM BIPARTITE SUBGRAPH}$, $\text{MAXIMUM PLANAR SUBGRAPH}$ and $\text{LONGEST PATH}$ which are intractable in general [29].

### 6.3 A naïve algorithmic lifting theorem

Suppose $\mathcal{C}$ and $\mathcal{D}$ are categories with pullbacks and $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ is a functor which preserves colimits. Then, we can lift any algorithm $A$ for solving the $\mathcal{F}$-COMPOSITION
problem to the problem of determining the value of $F$ on any object $X$ whenever we are given a structured decomposition of $X$. To see this, suppose $C$ has colimits and $x$ is an object in $C$ such that there is a tame $C$-valued structured decomposition (of any shape) $d_x \in \mathcal{D}_m(C)$ such that $\text{colim}d_x = x$. We can recursively compute this colimit as a sequence of pushouts and hence we can simply determine $F_x$ recursively by invoking multiple calls of $A$ for each pushouts in this sequence. This means that the algorithm we just described is nothing more than using $A$ to traverse the structured decomposition of the solutions space of $F$. We summarise this below.

### Observation 6.4 (Algorithms as structured decompositions of solution spaces)

Suppose $C$ and $D$ are categories with pullbacks and colimits and $F : C \to D$ is a functor which preserves colimits. Suppose further that we are given an object $x \in C$ and a $C$-valued structured decomposition $d_x : \int G \to C$ such that $\text{colim}d_x = x$. If $A$ is an algorithm for the $F$-composition problem, then we can use $A$ to compute $F_x = \text{colim} \mathcal{D}(F)d_x$ as in the following diagram.

![Diagram](image)

6.4 The $P$-subobject problem

In this section we investigate the $P$-subobject problem which asks to determine the poset of all subobjects of the form $P_a \leftarrow x$ of a fixed object $x \in C$ where $P : A \to C$ is a pullback-absorbing functor. The associated composition task gives rise to the $\text{Sub}_P$-composition problem.

As an example, we first consider the functor

$P_{\text{ph}} : \text{DisjointPaths} \to \text{Gr}_H$

given by the inclusion of the full subcategory of $\text{Gr}_H$ induced by disjoint unions of paths. Notice that if $f : P_{\text{ph}}X \leftarrow Y$ is a disjoint union of paths that maps monically into a graph $Y$, then the intersection of $f(P_{\text{ph}}X)$ with any subgraph $g : Z \leftarrow Y$ of $Y$ (i.e. the pullback of $f$ and $g$) is again a disjoint union of paths. Consequently, the functor $P_{\text{ph}}$ absorbs pullbacks.

Since any subgraph of a planar graph is still planar and any subgraph of a bipartite graph is still bipartite, we can define similar pullback-absorbing functors

$P_{\text{pl}} : \text{Planar} \to \text{Gr}_H$ and $P_{\text{bp}} : \text{Bipartite} \to \text{Gr}_H$.

Since $P_{\text{ph}}, P_{\text{pl}}$ and $P_{\text{bp}}$ are pullback-absorbing, the following algorithm for $\text{Sub}_P$-composition of pullback-absorbing functors immediately yields algorithms for LONGEST PATH (see Figure 5), MAXIMUM PLANAR SUBGRAPH and MAXIMUM BIPARTITE SUBGRAPH, which are intractable in general [29]. Mutatis mutandis, the same works for any subgraph-closed graph property and indeed for subobject closed properties for other kinds of ‘typed graphs’ such as Petri nets or chemical reaction networks [40].

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Figure 5: The graph above is obtained as a pushout $K_3 + K_3$ of two triangles over a single vertex. A longest path in this pushout is the 4-edge path $p := a_1c_1a_2c_2b_2$. It is constructed as a pushout of two 2-edge paths $a_1c_1b_1$ and $a_2c_2b_2$ which lie respectively on the left-hand- and right-hand-side triangle.

Figure 6: Diagrammatic summary of the steps of the Compose algorithm. Solid arrows are given as part of the input, dashed arrows are a result of the construction and the dotted arrow is the unique pushout arrow. Given objects $(\ell' : PL' \rightarrow L) \in \text{Sub}_pL$ and $(r' : PR' \rightarrow R) \in \text{Sub}_pR$, a candidate for an object of $\text{Sub}_pL +_M R$ is found via the pushout $PL' +_M PR'$. (Note that the fact that $C$ is adhesive ensures that pushouts of monic spans yield monic pushout cocones.)

**Lemma 6.5.** Let $C$ be an skeletal adhesive category that is strict recursive with recursive pullbacks and pushouts. Let $P : A \rightarrow C$ be a pullback-absorbing functor, let $L \xleftarrow{\ell} M \xrightarrow{r} R$ be a monic span in $C$. If we are given oracle for

1. computing pullbacks and pushouts in $C$ and
2. deciding whether any given object in $C$ is in the range of $P$, then there algorithm for the $\text{Sub}_p$-Composition problem running in time

$$O((|\text{Ob(Sub}_pL||\text{Ob(Sub}_pR)|))$$.

**Proof.** First we will describe the algorithm below (summarized diagrammatically in Figure 6) and then we prove correctness and the running time bound.

1. **procedure** COMPOSE($\ell : M \leftrightarrow L, r : M \leftrightarrow R, \text{Sub}_pL, \text{Sub}_pR$)
2: \( S \leftarrow \{ \} \quad \triangleright \text{set of objects of } \text{Sub}_p(L +_M R); \text{initially empty.} \\
3: \text{for } (\ell' : PL' \to L) \in \text{Sub}_p L \text{ do} \\
4: \quad \text{for } (r' : PR' \to R) \in \text{Sub}_p R \text{ do} \\
5: \quad \text{compute the pullback } L'' := PL' \times_M M \\
6: \quad \text{compute the pullback } R'' := PR' \times_R M \\
7: \quad \text{compute the pullback } M' := PL' \times_M PR' \\
8: \quad \text{if the pushout } L'' +_M R'' \text{ is a } P\text{-subobject of } L +_M R, \text{ then} \\
9: \quad \text{add } L'' +_M R'' \text{ to } S \\
10: \quad \text{return } S \cup \text{Ob(Sub}_p L) \cup \text{Ob(Sub}_p R). \\

Since lines 5 through 9 of the algorithm above are calls to the oracle, the running time analysis is trivial. Proving the correctness of the algorithm amounts to showing that every object \((q : PQ \to L +_M R) \in \text{Sub}_p(L +_M R)\) is in \(S \cup \text{Ob(Sub}_p L) \cup \text{Ob(Sub}_p R)\) (since the fact that \(P\) is pullback-absorbing together with the check of Line 8 of the algorithm implies that the set returned by the algorithm above is clearly a subset of the objects of \(\text{Sub}_p(L +_M R)\)). To that end, take any such \((q : PQ \to L +_M R) \in \text{Sub}_p(L +_M R)\), and consider Diagram (4) below, obtained by computing five pullbacks.

Note that, since the bottom face of Diagram (4) is a pushout square, we have \(f_r \ell = f_r r\). Thus, since \(L''\) and \(R''\) are pullbacks of \(q\) with \(f_r \ell\) and \(f_r r\) respectively (by the two-pullback Lemma [1]), we have \(L'' = M' = R''\), \(\ell_q = \ell^q = r^q\), and \(m_q = m_r\): this is easily seen from the following diagram.
This means that we can rewrite Diagram (4) as the following commutative cube.

Since since C is adhesive, pushouts squares along monomorphisms are Van Kampen squares [40, Defs. 2 and 5]. In particular this means that the bottom face of the cube in Diagram (5) is a Van Kampen square. This fact, combined with the fact that all four side-faces of the cube are pullbacks (by construction for the front faces and by what we just showed for the back faces) implies, by the property of Van Kampen squares [40, Def. 2] that the top face of the cube is a pushout. In other words we just showed that \( PQ = L_q +_{M'} R_q \). As desired, this means that \( q : PQ \to L +_{M'} R \) is found by the Compose algorithm.

Clearly, by keeping track of a running maximum or minimum, the algorithm of Lemma 5.10 can be extended to the maximization and minimization variants of \( \text{Sub}_p \)-composition.

**Corollary 6.6.** Let \( C \) be an skeletal adhesive category that is strict recursive with recursive pullbacks and pushouts. Let \( L \xleftarrow{f} M \xrightarrow{g} R \) be a monic span in \( C \). Furthermore, suppose we are given a function \( f : \text{Ob}(A) \to \mathbb{Q} \). If we are given oracle for

1. computing pullbacks and pushouts in \( C \) and
2. deciding whether any given object in \( C \) is in the range of \( P \) and
3. evaluating \( f \),

then there algorithm for the maximization and minimization variants of the \( \text{Sub}_p \)-Composition problem running in time

\[ \Theta(\text{\text{Ob(Sub}_p L)} + \text{\text{Ob(Sub}_p R))} \). \]
7 Discussion

Compositionality has just as much to do with composing pieces to make a whole as it is does with decomposing the whole into constituent pieces. This is easily seen by comparing the perspectives on compositionality that have originated from three different branches of mathematics: applied category theory, graph theory and parameterized complexity theory.

The first has placed a great emphasis on developing a general, object-agnostic theory (e.g. structured [15] and decorated [28] cospans) which details how to go about composing many different kinds of mathematical objects along given interfaces [28, 47, 12, 15, 42, 3, 4, 41, 2].

The second, graph-theoretical approach has instead focussed on studying a myriad of subtly different ways in which one may go about decomposing graphs and other combinatorial objects into small parts [5, 32, 49, 14, 31, 19, 46, 37, 6, 35, 52, 39, 10, 26, 54, 36]. This approach has focused on understanding the structural properties which can be inferred by the existence of decompositions of ‘small width’ [20, 49, 50, 51, 19] as well as characterizing obstructions to such well-behaved decompositions [22, 21, 30].

The third perspective is that of parameterized complexity. It shines yet another light on combinatorial objects that admit decompositions into simple pieces by observing that many computational problems are tractable whenever their inputs are (recursively) decomposable into small constituent parts [27, 16, 14, 31]. Indeed this is often the case even when the computational problems in question are NP-hard in general [27, 16, 14, 31].

Here we promote the unification of these three perspectives: we should understand how objects in arbitrary categories – not just combinatorial ones – can be composed, decomposed and studied algorithmically. The theory of structured decompositions introduced here is a first step towards developing a framework capable of reunifying the three perspectives.

The story of structured decompositions is built around a functor $\Omega : K \to C$ from a category with pullbacks to a category with colimits. The objects of $K$ are the simple, atomic building blocks from which we generate the category $\mathcal{D}(K)$ of $K$-valued structured decompositions. The functor $\Omega : K \to C$ provides, albeit indirectly, the semantics which accompanies the abstract compositional syntax given by structured decompositions themselves, summarized in the diagram below.

\[
\begin{array}{ccc}
\mathcal{K} & \xrightarrow{\text{const}} & \mathcal{D}(\mathcal{K}) & \xrightarrow{\text{Lan}_{\text{const}} \Omega} & C \\
\end{array}
\]

In combinatorics width measures are always accompanied by appropriate decompositions which witness the degree of simplicity purported by the width measures themselves. In Theorem 5.8 we show that this relationship is retained in our abstract setting as long as the category whose objects we are studying is sufficiently well-behaved. Indeed, as an application we use structured decompositions to provide a unified generalization of many width measures in combinatorics such as tree-width, graph decomposition width [10], layered tree-width [54, 26], $H$-tree-width [36] and even new width measures on other kinds of objects such as matroids.
From the algorithmic perspective, we showed (Observation 6.4) that, if \( \Omega : K \to C \) is a colimit-preserving functor encoding a computational problem, then the lines between structure and algorithms begin to blur. Indeed Observation 6.4 shows that, for such problems, algorithms computing on a decomposition of an object are the same thing as structured decompositions of the solution space of the problem on that object.

Our main algorithmic result, however, does not concern a colimit preserving functor. It has to do with the \( \mathcal{P} \)-subobject problem. Here we provide an algorithmic metatheorem for finding maximal or minimal subobjects of the form \( P_a \leftrightarrow x \) whenever \( P \) is a pullback-absorbing functor. Examples of the kinds of problems this encapsulates are **LONGEST PATH**, **MAXIMUM PLANAR SUBGRAPH** and **MAXIMUM BIPARTITE SUBGRAPH** which are all intractable in general [29].

### 7.1 Connection to spined categories

Recall (from [9]) that a spined category consists of a triple \((C, \Omega, \mathfrak{P})\) where

- \( C \) is a category,
- \( \Omega : \mathbb{N}_\infty \to \text{Ob} C \) is a functor called the spine of \( C \),
- \( \mathfrak{P} \) is an operator called the proxy pushout, which that assigns to each diagram of the form

  \[
  G \xleftarrow{h} \Omega_n \xrightarrow{h} H
  \]

  in \( C \) a distinguished cocone

  \[
  G \xrightarrow{\mathfrak{P}(g, h)} \mathfrak{P}(g, h)_n \xleftarrow{\mathfrak{P}(g, h)_n} H,
  \]

subject to two conditions which we omit here. Roughly, the proxy pushout operation axiomatizes a property that pushout squares in a category \( C \) maintain in the subcategory \( C_{\text{mon}} \) of \( C \). Similar ‘proxy notions’ have recently found applications in the theory of asymmetric lenses [44].

Spined categories provide a simple, minimal setting for the study of well-behaved abstract analogues of tree decompositions. The main result regarding spined categories \( C \) is the existence of a distinguished functor from \( C \) to the poset of natural numbers (the so-called triangulation functor) satisfying a certain maximality property. The triangulation functor of the spined category of graphs (with graphs as objects, graph monomorphisms as arrows, the pushout of graphs as the proxy pushout, and the sequence of cliques as the spine) coincides with tree-width: this provides a purely category-theoretic construction of a width measure.

Notice that the spine is always an \( \mathbb{N} \)-indexed sequence of objects, subject to some reasonable technical requirements. While many interesting categories of combinatorial objects admit such sequences, more infinitary settings would benefit from “longer” (say ordinal or cardinal-indexed) spine analogues, while other combinatorial settings of interest lack a unique, linear choice of spine. Examples of the latter sort include the category of layered graphs (Section 5.4) or the category of planar graphs, as pointed out by Lavrov [38].

Our attempt to replace the \( \mathbb{N} \)-indexed spine with a diagram of more general shape, and hence to recover more width measures (including e.g. layered treewidth above) as
triangulation functors, revealed formidable technical difficulties: resolving these ultimately required (and led us to) the more intricate theory of structured decompositions presented above.

7.2 Further questions

Recall that a $G$-shaped structured decomposition valued in some category $C$ is just a diagram $d : \int G \to C$ whose shape is given by the Grothendieck construction applied to a graph $G$. In principle there is nothing stopping us from letting $G$ be another kind of presheaf and indeed, one could for instance study structured decompositions whose shapes are given not by graphs, but simplicial complexes or other combinatorial objects. This is a fascinating direction of future research which might shine light on new tools in graph theory and algorithmics.

In terms of further questions on the category-theoretic side, at least two other main lines of inquiry deserve explicit study. The first of these is the relationship between FinSet-valued structured decompositions and Spivak’s undirected wiring diagrams [55]. The second is to further investigate the uses of the morphisms of structured decompositions, which may be seen as decompositions of morphisms in a category.

In graph-theory, one natural direction for further work is to characterize what obstacles prevent general objects from admitting simple (i.e. low width) structured decompositions. When it comes to graphs and tree-shaped structured decompositions, there is a well-established theory of obstacles to having low tree-width, including notions such as brambles [53], $k$-blocks [11], tangles [50] and abstract separation systems [21]. It is a fascinating, but highly non-trivial research direction to lift these ideas to the more general, category-theoretic setting of structured decompositions.

Other research directions stem from the algorithmic applications of structured decompositions. Our naïve approach to lifting algorithms (Observation 6.4) already provides a running time improvement compared to algorithms which are not compositional. However, the generic algorithm provided by Observation 6.4 will often be excessively slow compared to state of the art specific algorithms studied in parameterized complexity. When solving an NP-hard problem on a graph $G$ using a tree-decomposition $(T, \mathcal{V})$ (i.e. a FinSet-valued structured decomposition of $G$), the usual goal is to obtain running times of the form $c^{(\text{max}_{i \in [|T|]} |\mathcal{V}[i]|)}|T|$ for some small constant $c$ [27, 16]. In contrast, the running time of the algorithm given by Observation 6.4 is easily seen to be at least $\mathcal{O}(c^{(|G|/2)})$.

We aspire to obtain a less naïve lifting theorem, one which yields algorithms with running times comparable to the parameterized complexity state of the art. In particular, turning our compositional algorithm for SubP-composition (Lemma 6.5) into an FPT-time algorithm will require new ideas, and will constitute a major avenue of future work.

A Strict Recursive Categories

**Definition A.1.** A strict recursive category consists of the following data:

- A computable function $\text{obj} : \mathbb{N} \to \{0, 1\}$,
- A computable function $\text{Hom} : \mathbb{N}^3 \to \{0, 1\}$.
• a computable function \( \text{id} : \mathbb{N} \rightarrow \mathbb{N} \),
• a computable function \( \cdot : \mathbb{N}^2 \rightarrow \mathbb{N} \)

subject to the requirements that, for all \( x, y, x', y', f \in \mathbb{N} \), the following conditions are met:

1. if \( \text{Hom}(x, y, f) = 1 \) then 
   \( \text{obj}(x) = 1 \) and 
   \( \text{obj}(y) = 1 \) (definedness of arrows),
2. if \( \text{Hom}(x, y, f) = \text{Hom}(x', y', f) = 1 \) then \( x = x' \) and \( y = y' \) (uniqueness of domain and codomain),
3. if \( \text{obj}(x) = 1 \) then \( \text{Hom}(x, x, \text{id}(x)) = 1 \) (existence of identity arrows),
4. if \( \text{Hom}(x, y, f) = 1 \) and \( \text{Hom}(y, z, g) = 1 \), then \( \text{Hom}(x, z, g \cdot f) = 1 \) (composition),
5. if \( \text{Hom}(x, y, f) = 1 \), then \( \text{id}(y) \cdot f = f \) and \( f \cdot \text{id}(x) = f \) (identity),
6. the set \( \{ f \mid \text{Hom}(x, y, f) = 1 \} \) is finite (finiteness of homsets).

We say that a strict recursive category has recursive pullbacks if there are computable functions \( P : \mathbb{N}^2 \rightarrow \mathbb{N}, l : \mathbb{N}^2 \rightarrow \mathbb{N}, r : \mathbb{N}^2 \rightarrow \mathbb{N} \) and \( u : \mathbb{N}^4 \rightarrow \mathbb{N} \) so that for any \( x, y, z, f, g \in \mathbb{N} \) with \( \text{Hom}(x, z, f) = 1 \) and \( \text{Hom}(y, z, g) = 1 \), all of the following hold:

1. \( \text{Hom}(P(f, g), x, l(f, g)) = 1 \) (existence of left leg),
2. \( \text{Hom}(P(f, g), y, r(f, g)) = 1 \) (existence of right leg),
3. \( f \cdot l(f, g) = g \cdot r(f, g) \) (pullback cone condition),
4. for any other \( Q \in \mathbb{N} \) and \( f', g' \in \mathbb{N} \) having both \( \text{Hom}(Q, x, f') = 1 \) and \( \text{Hom}(Q, y, g') = 1 \), if \( f' \cdot f = g \cdot g' \), then we have \( l(f, g) \cdot u(f, g, f', g') = f' \) and \( r(f, g) \cdot u(f, g, f', g') = g' \) (pullback test-objects).

As customary in category theory, we denote \( P(f, g) \) as \( x \times_y y \).

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