ON SYMPLECTIC PERIODS FOR INNER FORMS OF GLₙ

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Abstract. In this paper we study the question of determining when an irreducible admissible representation of GLₙ(D) admits a symplectic model, that is when such a representation has a linear functional invariant under Spₙ(D), where D is a quaternion division algebra over a non-Archimedean local field k and Spₙ(D) is the unique non-split inner form of the symplectic group Sp₂ₙ(k). We show that if a representation has a symplectic model it is necessarily unique. For GL₂(D) we completely classify those representations which have a symplectic model. Globally, we show that if a discrete automorphic representation of GLₙ(Dₖ) has a non-zero period for Spₙ(Dₖ), then its Jacquet-Langlands lift also has a non-zero symplectic period. A somewhat striking difference between distinction question for GL₂ₙ(k), and GLₙ(D)(with respect to Sp₂ₙ(k) and Spₙ(D) resp.) is that there are supercuspidal representations of GL₂ₙ(D) which are distinguished by Spₙ(D). The paper ends by formulating a general question classifying all unitary distinguished representations of GLₙ(D), and proving a part of the local conjectures through a global conjecture.

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1. Introduction

Let G be a group and H a subgroup of G. We recall that a complex representation π of G is said to be H-distinguished if

\[ \text{Hom}_H(\pi, \mathbb{C}) \neq 0, \]

where \( \mathbb{C} \) denotes the trivial representation of H. When \( G = \text{GL}_{2n}(k) \), and \( H = \text{Sp}_{2n}(k) \), such representations of \( \text{GL}_{2n}(k) \) are said to have a symplectic model. When \( k \) is a non-Archimedean local field of characteristic 0, and \( \pi \)

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is an irreducible admissible complex representation of $GL_{2n}(k)$, this question has been extensively studied by several authors starting with the work of M. J. Heumos and S. Rallis in [4]. A rather complete classification of $Sp_{2n}(k)$-distinguished unitary representations of $GL_{2n}(k)$ is due to O. Offen and E. Sayag [11].

When $F$ is a number field, the analogous global question is framed in terms of the non-vanishing of certain periods of automorphic forms $f$ on $G(F) \backslash G(A)$, where $A$ is the ring of adèles of $F$, given by

$$\int_{H(F) \backslash H(k)} f(h) dh.$$ 

This question has been settled in [9, 10] and, in fact, Offen and Sayag treat some aspects of the local questions via global methods.

In this paper we study the irreducible admissible representations of $GL_n(D)$ which are $Sp_n(D)$-distinguished, where $Sp_n(D)$ is an inner form of $Sp_{2n}(k)$ constructed using the unique quaternion division algebra $D$ over $k$ (we will define this more precisely in Section 2). We proceed to state the main results of this paper.

**Theorem 1.1.** Let $\pi$ be an irreducible admissible representation of $GL_n(D)$. Then

$$\dim \text{Hom}_{Sp_n(D)}(\pi, \mathbb{C}) \leq 1.$$ 

The following theorem gives a partial answer to the question on distinction of a supercuspidal representation of $GL_n(D)$ by $Sp_n(D)$.

**Theorem 1.2.** Let $\pi$ be a supercuspidal representation of $GL_n(D)$ with Langlands parameter $\sigma_x = \sigma \otimes sp_r$ where $\sigma$ is an irreducible representation of the Weil group $W_k$, and $sp_r$ is the $r$-dimensional irreducible representation of $SL_2(\mathbb{C})$ part of the Weil-Deligne group $W'_k$. Then if $r$ is odd, $\pi$ is not distinguished by $Sp_n(D)$.

In section 6, we have constructed explicit examples of supercuspidal representations of $GL_n(D)$ which are distinguished by $Sp_n(D)$ for any odd $n \geq 1$, and in section 7 we prove a complete classification of discrete series representations of $GL_n(D)$ which are distinguished by $Sp_n(D)$ assuming globalization of locally distinguished representations to globally distinguished representations together with a natural global conjecture on distinction of automorphic representations of $GL_n(D)$ by $Sp_n(D)$.

Here is a global theorem which is a simple consequence of Offen and Sayag’s work.

**Theorem 1.3.** Let $D$ be a quaternion division algebra over $F$ and $D_A = D \otimes_F A$. Let $\Pi$ be an automorphic representation of $GL_n(D_A)$ which appears in the discrete spectrum of $GL_n(D_A)$ and has non-vanishing period integral on $Sp_n(D) \backslash Sp_n(D_A)$. Let $JL(\Pi)$ be the Jacquet-Langlands lift of $\Pi$. Then the representation $JL(\Pi)$ of $GL_{2n}(A_F)$ has non-vanishing period integral on $Sp_{2n}(F) \backslash Sp_{2n}(A_F)$.
We now briefly describe the organization of this paper. In Section 2, we set up notation and give definitions. In this section we define the inner forms of a symplectic group over a local field \( k \). In Section 3, we prove the uniqueness of the symplectic model for irreducible representations of \( \text{GL}_n(D) \). In section 4, we are able to completely analyze the question of distinction of subquotients of principal series representations of \( \text{GL}_2(D) \) by \( \text{Sp}_2(D) \) via Mackey theory. In Section 5, we prove that non-vanishing of symplectic period of an irreducible discrete spectrum automorphic representation of \( \text{GL}_n(D_A) \) is preserved under the Jacquet-Langlands correspondence. In this section, we partially analyze distinction problem for supercuspidal representations of \( \text{GL}_n(D) \). In Section 6, we construct examples of supercuspidal representations of \( \text{GL}_n(D) \) which are distinguished by \( \text{Sp}_n(D) \). The paper ends by formulating a general question classifying all unitary distinguished representations of \( \text{GL}_n(D) \), and proving a part of the local conjectures through a global conjecture.

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2. Notation and Definitions

Let \( k \) be a non-Archimedean local field of characteristic zero, and let \( D \) be the unique quaternion division algebra over \( k \). We denote the reduced trace and reduced norm maps on \( D \) by \( T_{D/k} \) and \( N_{D/k} \) respectively. Let \( \tau \) be the involution on \( D \) defined by \( x \mapsto \overline{x} = T_{D/k}(x) - x \).

For \( n \in \mathbb{N} \), let

\[
V_n = e_1D \oplus \ldots \oplus e_nD
\]

be a right \( D \)-vector space of dimension \( n \).

Definition 2.1. We define a Hermitian form on \( V_n \) by

1. \( (e_i, e_{n-j+1}) = \delta_{ij} \) for \( i = 1, 2, \ldots, n \);
2. \( (v, v') = \tau(v', v) \);
3. \( (vx, v'x') = \tau(x)(v, v')x' \), for \( v, v' \in V_n, x, x' \in D \).

Let \( \text{Sp}_n(D) \) be the group of isometries of the Hermitian form \( (\cdot, \cdot) \). The group \( \text{Sp}_n(D) \) is the unique non-split inner form of the group \( \text{Sp}_{2n}(k) \). Clearly \( \text{Sp}_n(D) \subset \text{GL}_n(D) \). The group \( \text{Sp}_n(D) \) can also be defined as

\[
\text{Sp}_n(D) = \{ A \in \text{GL}_n(D) | AJ^t\bar{A} = J \},
\]
where $\mathbf{A} = (\mathbf{a}_{ji})$ for $A = (a_{ij})$ and

$$J = \begin{pmatrix} 1 & 1 \\ -1 & 1 \\ & & \ddots & \ddots \\ & & & & 1 \end{pmatrix}.$$ 

For a right $D$-vector space $V$, let $GL_D(V)$ be the group of all invertible $D$-linear transformations on $V$. Similarly, let $Sp_D(V)$ be the group of all invertible $D$-linear transformations on $V$ which preserve the above defined Hermitian form on $V$. Let $\nu$ denote the character of $GL_n(D)$ which is the absolute value of the reduced norm on the group $GL_n(D)$. For any $p$-adic group $G$, let $\delta_G$ denote the modular character of $G$. We denote the trivial representation of any group by $\mathbb{C}$. For any representation $\pi$, we will denote its contragredient representation by $\hat{\pi}$.

3. Uniqueness of symplectic models

In this section we will show that for an irreducible representation $\pi$ of $GL_n(D)$, $\dim \text{Hom}_{Sp_n(D)}(\pi, \mathbb{C}) \leq 1$. This result is due to M. J. Heumos and S. Rallis [1] when $D$ is replaced by a local field $k$. Our proof is a straightforward adaptation of their methods. We first need a result from [16] which gives the realization of the contragredient representation of an irreducible representation of $GL_n(D)$.

**Theorem 3.1.** Let $D$ be the quaternion division algebra over $k$, $x \mapsto \overline{x} = T_{D/k}(x) - x$ be the canonical anti-automorphism of order 2 on $D$. Let $G = GL_n(D)$, and let $\sigma : G \to G$ be the automorphism of $G$ given by $\sigma(g) = J(g^{-1})J$, where $\bar{g} = (\bar{g}_{ij})$ and $J$ is the anti-diagonal matrix with all entries 1. Let $\pi$ be an irreducible admissible representation of $GL_n(D)$ and $\pi^\sigma$ be the representation defined by $\pi^\sigma(g) = \pi(\sigma(g))$. Then $\pi^\sigma = \hat{\pi}$, where $\hat{\pi}$ is the contragredient of $\pi$.

Let $k$ be a local field of characteristic different from 2, $\overline{k}$ the algebraic closure of $k$ and $M$ (resp. $\overline{M}$) denote the set of $n \times n$ matrices with coefficients in $k$ (respectively $\overline{k}$). Let $\sigma$ denote an anti-automorphism on $\overline{M}$ of order 2. We will record two lemmas from [1] below.

**Lemma 3.2** (Lemma 2.2.1 of [1]). For any $A \in GL_n(k)$, there exists a polynomial $f \in \overline{k}[t]$ such that $f(A)^2 = A$.

**Proposition 3.3** (Proposition 2.2.2 of [1]). For any $A \in GL_n(\overline{k})$, there exists $U, V \in GL_n(\overline{k})$ such that $\sigma(U) = U, \sigma(V) = V^{-1}$ and $A = UV$.

Set $A^J = J^tAJ$ for $A \in GL_n(D)$. Then $A \mapsto A^J$ is an anti-involution on $GL_n(D)$ of order 2. By Proposition 3.3 over an algebraically closed field, there exist $U, V \in GL_n(\overline{k})$, such that $V^J = V^{-1}, U^J = U$ and $A = UV$. Then $A^J = V^JU^J = V^{-1}U = V^{-1}AV^{-1}$. Since $V \in \text{Sp}_{2n}(\overline{k})$ if and only if
$V \in \text{GL}_{2n}(k)$ and $V' = V^{-1}$, $A'$ and $A$ lie in the same double cosets over algebraic closure.

The next result shows that $A$ and $A'$ lie in the same double coset of $\text{Sp}_n(D)$ in $\text{GL}_n(D)$. Let us first recall a theorem due to Kneser and Bruhat-Tits.

**Theorem 3.4.** Let $G$ be any semi-simple simply connected group over $p$-adic field $k$. Then $H^1(k, G) = 0$.

The theorem above will be used in conjunction with our modification of Lemma 2.3.3 given below.

**Proposition 3.5.** Let $D$ be a quaternion division algebra over a local field $k$ of characteristic zero. Let $A \in \text{GL}_n(D)$. Then there exist $P_1, P_2 \in \text{Sp}_n(D)$, such that $A' = P_1A'P_2$.

**Proof.** Consider the set

$$V(A) = \{(P_1, P_2) \in \text{Sp}_n(D) \times \text{Sp}_n(D) | A' = P_1A'P_2\}.$$

The assertion contained in the proposition is equivalent to saying that $V(A)$ is non-empty. Clearly $V(A)$ is an algebraic subset of $\text{Sp}_{2n}(k) \times \text{Sp}_{2n}(k)$. Note that $A \cap \text{ASp}_n(D)A^{-1}$ is the subgroup of $\text{GL}_n(D)$ which leaves the symplectic form associated with the matrix $J' = JAJ^{-1}$ invariant. Denote the group $\text{Sp}_n(D) \cap \text{ASp}_n(D)A^{-1}$ by $\text{Sp}(J, J')$. Consider the right action of $\text{Sp}(J, J')$ on $V(A)$ by

$$R(P_1, P_2) = (P_1R^{-1}, A^{-1}RAP_2).$$

Since $P_1R^{-1}A^{-1}P_2 = P_1AP_2 = A'$;

$$(P_1R^{-1}, A^{-1}RAP_2) = R(P_1, P_2) \in V(A),$$

we have,

$$R(P_1, P_2) = (P_1R^{-1}, A^{-1}RAP_2),$$

$$S(R(P_1, P_2)) = (P_1R^{-1}S^{-1}, A^{-1}SAA^{-1}RAP_2),$$

$$= (P_1R^{-1}S^{-1}, A^{-1}RSP_2)$$

for $R, S \in \text{Sp}(J, J')$ and $(P_1, P_2) \in V(A)$, verifying that we do indeed have an action. We check that this action is fixed point free. This is because if $R(P_1, P_2) = (P_1, P_2)$ for $R \in \text{Sp}(J, J')$ and $(P_1, P_2) \in V(A)$, then $P_1R^{-1} = P_1$ which gives $R = 1$.

We next check that the action is transitive. For this let $P = (P_1, P_2)$ and $Q = (Q_1, Q_2)$ be two points in $V(A)$. We need to prove that there exists $R \in \text{Sp}(J, J')$ such that $R(P) = Q$, that is, that $R(P_1, P_2) = (Q_1, Q_2)$, or equivalently that

$$(P_1R^{-1}, A^{-1}RAP_2) = (Q_1, Q_2).$$

Let $R = Q_1^{-1}P_1 \in \text{Sp}_n(D)$ then $P_1R^{-1} = Q_1$. With this choice of $R$

$$A^{-1}RAP_2 = A^{-1}Q_1^{-1}P_1AP_2 = A^{-1}Q_1^{-1}Q_1AQ_2 = Q_2.$$

In the second equality we have used the definition of $V(A)$ because of which $A' = P_1A'P_2 = Q_1A'Q_2$. Also $P_1AP_2 = Q_1A'Q_2$ gives

$$R = Q_1^{-1}P_1 = AQ_2P_2^{-1}A^{-1} \in \text{ASp}_n(D)A^{-1}.$$ 

Hence, $R \in \text{Sp}(J, J')$ which shows that the action of $\text{Sp}(J, J')$ on $V(A)$ is transitive. Therefore $V(A)$ is a right principal homogeneous space for the group $\text{Sp}(J, J')$. 

Klyachko proved that over an algebraically closed field, $\text{Sp}(J, J')$ is an extension of a product of symplectic groups by a unipotent group. Therefore, over a general field, $\text{Sp}(J, J')$ is an extension of a form of a product of symplectic groups by a unipotent group, that is, there exists an exact sequence of algebraic groups of the form

$$1 \to U \to \text{Sp}(J, J') \to S \to 1,$$

with $S$, a form of a product of symplectic groups. Therefore we get the following exact sequence of Galois cohomology sets:

$$H^1(k, U) \to H^1(k, \text{Sp}(J, J')) \to H^1(k, S).$$

It is well-known that $H^1(k, U) = 0$ for any unipotent group $U$ over a field of characteristic zero [17]. Since by Theorem 3.4, $H^1(k, S) = 0$, the exact sequence above gives $H^1(k, \text{Sp}(J, J')) = 0$. Since $V(A)$ is a principal homogeneous for $\text{Sp}(J, J')$ and $H^1(k, \text{Sp}(J, J')) = 0$, it follows that $V(A)(k) \neq \emptyset$, proving the proposition. □

We recall the following result from [13].

**Lemma 3.6.** Let $G$ be an $l$-group and $H$ be a closed subgroup of $G$ such that $G/H$ carries a $G$-invariant measure. Suppose $x \to \bar{x}$ is an anti-automorphism of $G$ which leaves $H$ invariant and acts trivially on those distributions on $G$ which are $H$ bi-invariant. Then for any smooth irreducible representation $\pi$ of $G$, $\dim \text{Hom}_H(\pi, \mathbb{C}) \cdot \dim \text{Hom}_H(\hat{\pi}, \mathbb{C}) \leq 1$.

**Corollary 3.7.** Let $G = \text{GL}_n(D)$, $H = \text{Sp}_n(D)$, and let $i$ be the anti-automorphism on $G$ given by $A \to JAJ^{-1}$. Then for any smooth irreducible representation $\pi$ of $G$, $\dim \text{Hom}_{\text{Sp}_n(D)}(\pi, \mathbb{C}) \cdot \dim \text{Hom}_{\text{Sp}_n(D)}(\hat{\pi}, \mathbb{C}) \leq 1$.

**Proof.** The hypotheses of Lemma 3.6 follow from Proposition 3.5 by standard methods in Gelfand-Kazhdan theory. Hence, the corollary is an immediate consequence of Lemma 3.6. □

We are now in a position to prove the main theorem of this section.

**Theorem 3.8.** Let $\pi$ be an irreducible admissible representation of $\text{GL}_n(D)$. Then $\dim \text{Hom}_{\text{Sp}_n(D)}(\pi, \mathbb{C}) \leq 1$.

**Proof.** Let $(\pi_1, V)$ be the representation defined by $\pi_1(g) = \pi(Jg^{-1})$. Let $\lambda \in \text{Hom}_{\text{Sp}_n(D)}(\pi_1, \mathbb{C})$. Then $\lambda(\pi_1(g)v) = \lambda(v)$ which gives $\lambda(\pi(Jg^{-1})v) = \lambda(v)$. Since $H$ is invariant under $g \to Jg^{-1}$, $\lambda(\pi(g)v) = \lambda(v)$ for $g \in H$, so $\lambda \in \text{Hom}_{\text{Sp}_n(D)}(V, \mathbb{C})$. The other inclusion follows similarly. Therefore, $\dim \text{Hom}_{\text{Sp}_n(D)}(\pi, \mathbb{C}) = \dim \text{Hom}_{\text{Sp}_n(D)}(\pi_1, \mathbb{C})$. Now the result follows from Theorem 3.1 and above corollary. □

4. **Local theory**

The aim of this section is to analyze the principal series representations of $\text{GL}_2(D)$ which have a symplectic model. This can be easily done by the usual Mackey theory which is what we do here.
4.1. Orbits and Mackey theory. Let $H$ and $P$ be two closed subgroups of a group $G$ and let $(\sigma, W)$ be a smooth representation of $P$. We assume that $G$ and $H$ are unimodular. Also, assume that $H \backslash G/P$ has only two elements, that is, the natural action of $H$ on $G/P$ has two orbits, which we will call $O_1$ and $O_2$.

Assume without loss of generality that the orbit $O_1$ of $H$ through $eP$ is closed and the orbit $O_2$ is open. Let $H_1$ be the stabilizer in $H$ of the element $eP$ in $G/P$, then $H_1 = P \cap H$. Choose an element $x$ in $G$ such that the coset $xP$ lies in $O_2$. Then $H_2 = \text{Stab}_H(xP) = H \cap xPx^{-1}$. Therefore, $O_1 \simeq H/H_1$ and $O_2 \simeq H/H_2$. Using Mackey theory we obtain an exact sequence of $H$-representations:

$$0 \to \text{Ind}^H_{H_2} \sigma_2 \to \text{Ind}_P^G \sigma \mid_H \to \text{Ind}^H_{H_1} \sigma_1 \to 0,$$

where

$$\sigma_1(h) = (\delta_P/\delta_{H_1})^{1/2} \sigma(h) \text{ for } h \in H_1,$$

and

$$\sigma_2(h) = (\delta_P/\delta_{H_2})^{1/2} \sigma(h) \text{ for } h \in H_2.$$  

The question of the existence of an $H$-invariant linear form for $\pi$ can thus be addressed by studying $H$-invariant linear forms for representations of $H$ induced from its subgroups.

Now we apply the Mackey theory discussed above to the our situation for $G = \text{GL}_2(D)$, $H = \text{Sp}_2(D)$ and a parabolic subgroup $P$ of $\text{GL}_2(D)$.

Let $V$ be a 2-dimensional Hermitian right $D$-vector space with a basis $\{e_1, e_2\}$ of $V$ with $\langle e_1, e_1 \rangle = \langle e_2, e_2 \rangle = 0$ and $\langle e_1, e_2 \rangle = 1$. Let $X$ be the set of all 1-dimensional $D$-subspaces of $V$. The group $G = \text{GL}_D(V)$ acts naturally on $V$, and induces a transitive action on $X$, realizing $X$ as homogeneous space for $G$. Then the stabilizer of a line $W$ in $G$ is a parabolic subgroup $P$ of $G$, with $X \simeq G/P$. Using the above basis, $\text{GL}_D(V)$ can be identified with $\text{GL}_2(D)$. For $W = \langle e_1 \rangle$, $P$ is the parabolic subgroup consisting upper triangular matrices in $\text{GL}_2(D)$. As we have a Hermitian structure on $V$, $H = \text{Sp}_D(V) \subset \text{GL}_D(V)$.

We want to understand the space $H \backslash G/P$. This space can be seen as the orbit space of $H$ on the flag variety $X$. This action has two orbits. One of them, say $O_1$, consists of all 1-dimensional isotropic subspaces of $V$ and the other, say $O_2$ consists of all 1-dimensional anisotropic subspaces of $V$. Here, the one dimensional subspace generated by a vector $v$ is called isotropic if $\langle v, v \rangle = 0$; otherwise, it is called anisotropic. The fact that $\text{Sp}_D(V)$ acts transitively on $O_1$ and $O_2$ follows from Witt’s theorem [7] page 6, §9, together with the well known theorem that the reduced norm $N_{D/k} : D^\times \to k^\times$ is surjective, and as a result if a vector $v \in V$ is anisotropic, we can assume that in the line $\langle v \rangle = \langle v \cdot D \rangle$ generated by $v$, there exists a vector $v'$ such that $\langle v', v' \rangle = 1$.

It is easily seen that the stabilizer of the line $\langle e_1 \rangle$ in $\text{Sp}_D(V)$ is

$$P_H = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \mid a \in D^\times, b \in D, ab + b\bar{a} = 0 \right\}.$$
Now we consider the line \( \langle e_1 + e_2 \rangle \) inside \( O_2 \). To calculate the stabilizer of this line in \( \operatorname{Sp}_D(V) \), note that if an isometry of \( V \) stabilizes the line generated by \( e_1 + e_2 \), it also stabilizes its orthogonal complement which is the line generated by \( e_1 - e_2 \). Hence, the stabilizer of the line \( \langle e_1 + e_2 \rangle \) in \( \operatorname{Sp}_D(V) \) stabilizes the orthogonal decomposition of \( V \) as

\[
V = \langle e_1 + e_2 \rangle \oplus \langle e_1 - e_2 \rangle,
\]

and also acts on the vectors \( \langle e_1 + e_2 \rangle \) and \( \langle e_1 - e_2 \rangle \) by scalars. Thus the stabilizer in \( \operatorname{Sp}_D(V) \) of the line \( \langle e_1 + e_2 \rangle \) is \( D^1 \times D^1 \) sitting in a natural way in the Levi \( D^\times \times D^\times \) of the parabolic \( P \) in \( \operatorname{GL}_2(D) \). Here \( D^1 \) is the subgroup of \( D^\times \) consisting of reduced norm \( 1 \) elements in \( D^\times \).

Now consider the principal series representation \( \pi = \sigma_1 \times \sigma_2 := \operatorname{Ind}_{P}^{\operatorname{GL}_2(D)}(\sigma) \) of \( \operatorname{GL}_2(D) \), where \( \sigma = \sigma_1 \times \sigma_2 \) is an irreducible representation of \( D^\times \times D^\times \). We analyze the restriction of \( \pi \) to \( \operatorname{Sp}_2(D) \). By Mackey theory, we get the following exact sequence of \( \operatorname{Sp}_2(D) \) representations

\[
0 \rightarrow \operatorname{Ind}_{D^1 \times D^1}^{\operatorname{Sp}_2(D)}[(\sigma_1 \otimes \sigma_2) \mid_{D^1 \times D^1}] \rightarrow \pi \rightarrow \operatorname{Ind}_{P_H}^{\operatorname{Sp}_2(D)}(\nu^{1/2}[(\sigma_1 \otimes \sigma_2) \mid_{M_H}]) \rightarrow 0.
\]

(4.1)

Here \( \nu \) is the character on \( P_H \) given by

\[
\nu \left( \begin{array}{cc} a & b \\ 0 & \bar{a}^{-1} \end{array} \right) = |N_{D/\mathbb{K}}(a)|.
\]

Suppose \( \pi \) has a nonzero \( \operatorname{Sp}_2(D) \)-invariant linear form. Then one of the representations in the above exact sequence,

\[
\operatorname{ind}_{D^1 \times D^1}^{\operatorname{Sp}_2(D)}[(\sigma_1 \otimes \sigma_2) \mid_{D^1 \times D^1}] \text{ or } \operatorname{Ind}_{P_H}^{\operatorname{Sp}_2(D)}(\nu^{1/2}[(\sigma_1 \otimes \sigma_2) \mid_{M_H}]),
\]

must have an \( \operatorname{Sp}_2(D) \)-invariant form. First, consider the case when

\[
\operatorname{Hom}_{\operatorname{Sp}_2(D)}(\operatorname{Ind}_{P_H}^{\operatorname{Sp}_2(D)}(\nu^{1/2}[(\sigma_1 \otimes \sigma_2) \mid_{M_H}], \mathbb{C}) \neq 0.
\]

Since \( H/P_H \) is compact, by Frobenius reciprocity, this is equivalent to

\[
\operatorname{Hom}_{M_H}(\nu^{1/2} \langle \sigma_1 \otimes \sigma_2 \rangle, \nu^{3/2}) \neq 0.
\]

Since \( M_H = \{(d, d^{-1}) \mid d \in D^\times \} \simeq \Delta(D^\times \times D^\times) \), we have

\[
\operatorname{Hom}_{D^\times}((\sigma_1 \otimes \hat{\sigma}_2), \nu) \neq 0,
\]

and hence

\[
\operatorname{Hom}_{D^\times}(\sigma_1, \sigma_2 \otimes \nu) \neq 0,
\]

or

\[
\sigma_1 \simeq \nu \otimes \sigma_2.
\]

(4.3)

Now assume that

\[
\operatorname{Hom}_{\operatorname{Sp}_2(D)}(\operatorname{ind}_{D^1 \times D^1}^{\operatorname{Sp}_2(D)}[(\sigma_1 \otimes \sigma_2) \mid_{D^1 \times D^1}], \mathbb{C}) \neq 0.
\]

Then by Frobenius reciprocity, this is equivalent to

\[
\operatorname{Hom}_{D^1 \times D^1}((\sigma_1 \otimes \sigma_2), \mathbb{C}) \neq 0.
\]

(4.4)
Lemma 4.1. Let $(\sigma, V)$ be a finite dimensional irreducible representation of $D^\times$ with $\text{Hom}_{D^1}(V, \mathbb{C}) \neq 0$. Then $\sigma$ is one dimensional.

Proof. By a theorem due to Matsushima [8], $D^1$ is the commutator subgroup of $D^\times$. Since $D^1$ is a normal subgroup of $D^\times$, $V^{D^1} \neq \{0\}$ is invariant under $D^\times$ and so by the irreducibility of $V$, $V = V^{D^1}$. Since $(\sigma, V)$ is an irreducible representation of $D^\times$, on which $D^1$ operates trivially, $(\sigma, V)$ as a representation of $D^\times/D^1$ is also irreducible. Since $D^\times/D^1$ is abelian, $\sigma$ must be one dimensional.

From the analysis above, we deduce that if the representation

$$\pi = \sigma_1 \times \sigma_2 := \text{Ind}_{P}^{GL_2(D)}(\sigma_1 \otimes \sigma_2)$$

has an $\text{Sp}_2(D)$-invariant linear form, then either

1. $\sigma_1 \simeq \sigma_2 \otimes \nu$, or
2. both $\sigma_1$ and $\sigma_2$ are 1-dimensional representations of $D^\times$, hence are of the form $\sigma_1 = \chi_1 \circ N_{D/k}$, $\sigma_2 = \chi_2 \circ N_{D/k}$ for characters $\chi_i : k^\times \to \mathbb{C}^\times$.

Further, we note that the closed orbit for the action of $\text{Sp}_2(D)$ on $P \setminus GL_2(D)$ contributes to a $\text{Sp}_2(D)$-invariant form in the first case above, whereas it is the open orbit which contributes to a $\text{Sp}_2(D)$-invariant linear form in the second case. Since the part of the representation supported on the closed orbit arises as a quotient of $\pi$, we find that in the first case $\pi$ must have a $\text{Sp}_2(D)$-invariant linear form.

If $\text{dim}(\sigma_1 \otimes \sigma_2) > 1$, then the open orbit cannot contribute to an $\text{Sp}_2(D)$-invariant linear form, and therefore we conclude that if $\text{dim}(\sigma_1 \otimes \sigma_2) > 1$, then $\pi = \sigma_1 \times \sigma_2$ has an $\text{Sp}_2(D)$-invariant form if and only if $\sigma_1 = \sigma_2 \otimes \nu$. Observe that if $\pi$ has an $\text{Sp}_2(D)$-invariant linear form, and is irreducible, then by an analogue of a theorem of Gelfand-Kazhdan [3] due to Raghuram [16], $\hat{\pi}$ too has an $\text{Sp}_2(D)$-invariant linear form. However, if $\pi = \sigma_1 \times \sigma_2$, and $\pi$ is irreducible, then $\hat{\pi} = \hat{\sigma}_1 \times \hat{\sigma}_2$, and if $\sigma_1 \simeq \sigma_2 \otimes \nu$, we get $\hat{\sigma}_1 \simeq \hat{\sigma}_2 \otimes \nu^{-1}$. This means by our analysis above that the representation $\hat{\sigma}_1 \times \hat{\sigma}_2$ of $GL_2(D)$ does not carry an $\text{Sp}_2(D)$-invariant linear form. Therefore, we conclude that if $\sigma_1 \simeq \sigma_2 \otimes \nu$, then $\pi = \sigma_1 \times \sigma_2$ must be reducible, which is one part of the following theorem of Tadic [15].

Theorem 4.2. (Tadic) Let $\sigma_1$ and $\sigma_2$ be two irreducible representations of $D^\times$. Let $\pi = \text{Ind}_{P}^{GL_2(D)}(\sigma_1 \otimes \sigma_2)$ be the corresponding principal series representation of $GL_2(D)$. Assume $\text{dim}(\sigma_1 \otimes \sigma_2) > 1$. Then $\pi$ is reducible if and only if $\sigma_1 \simeq \sigma_2 \otimes \nu^{\pm 1}$. If $\pi$ is reducible then it has length two. Assuming $\sigma_1 = \sigma_2 \otimes \nu$, we have the following non-split exact sequence:

$$0 \to \text{St}(\pi) \to \pi \to \text{Sp}(\pi) \to 0,$$

where $\text{St}(\pi)$ is a discrete series representation called a generalized Steinberg representation of $GL_2(D)$ and $\text{Sp}(\pi)$ is called a Speh representation of $GL_2(D)$. If $\text{dim}(\sigma_1 \otimes \sigma_2) = 1$, then $\pi = \chi_1 \times \chi_2$ is reducible if and only if $\sigma_1 \simeq \sigma_2 \otimes \nu^{\pm 2}$. If $\sigma_1 = \sigma_2 \otimes \nu$, $\pi$ has a one dimensional quotient, and the submodule is a twist of the Steinberg representation of $GL_2(D)$. 
In the exact sequence of $GL_2(D)$-modules

$$0 \to \text{Sp}(\sigma_1) \to \text{Ind}_P^{GL_2(D)}(\sigma_1 \nu^{-1/2} \otimes \sigma_1 \nu^{1/2}) \to \text{St}(\sigma_1) \to 0,$$

and assuming that $\dim(\sigma_1) > 1$, we know by our previous analysis that $\text{Ind}_P^{GL_2(D)}(\sigma_1 \nu^{-1/2} \otimes \sigma_1 \nu^{1/2})$ does not have an $\text{Sp}_2(D)$-invariant linear form. Therefore, from the exact sequence above, it is clear that $\text{St}(\sigma_1)$ also does not have an $\text{Sp}_2(D)$-invariant linear form.

On the other hand, we know that $\text{Ind}_P^{GL_2(D)}(\sigma_1 \nu^{1/2} \otimes \sigma_1 \nu^{-1/2})$ does have an $\text{Sp}_2(D)$-invariant linear form, and $\text{Ind}_P^{GL_2(D)}(\sigma_1 \nu^{1/2} \otimes \sigma_1 \nu^{-1/2})$ fits in the following exact sequence:

$$0 \to \text{St}(\sigma_1) \to \text{Ind}_P^{GL_2(D)}(\sigma_1 \nu^{1/2} \otimes \sigma_1 \nu^{-1/2}) \to \text{Sp}(\sigma_1) \to 0.$$ 

Since we have already concluded that $\text{St}(\sigma_1)$ does not have an $\text{Sp}_2(D)$-invariant linear form and since $\text{Ind}_P^{GL_2(D)}(\sigma_1 \nu^{1/2} \otimes \sigma_1 \nu^{-1/2})$ has a $\text{Sp}_2(D)$-invariant linear form, we conclude that $\text{Sp}(\sigma_1)$ must have an $\text{Sp}_2(D)$-invariant linear form.

Having completed the analysis of $\text{Sp}_2(D)$-invariant linear forms on representations $\pi = \sigma_1 \times \sigma_2$ with $\dim(\sigma_1 \otimes \sigma_2) > 1$, we turn our attention to the case when $\sigma_1$ and $\sigma_2$ are both one dimensional representations of $D^\times$. In this case, the part of $\pi$ supported on the open orbit, which is a submodule of $\pi$, contributes to an $\text{Sp}_2(D)$-invariant linear form. Suppose that $\sigma_1 \neq \sigma_2 \otimes \nu$, as otherwise there is an $\text{Sp}_2(D)$-invariant linear form arising from the closed orbit.

Since the part of $\pi$ supported on the open orbit, that is, $\text{ind}_{D^1 \times D^1}^{\text{Sp}_2(D)}(\sigma_1 \otimes \sigma_2)$, is a submodule of $\pi$, it is not obvious that an $\text{Sp}_2(D)$-invariant linear form on $\text{ind}_{D^1 \times D^1}^{\text{Sp}_2(D)}(\sigma_1 \otimes \sigma_2)$ will extend to an $\text{Sp}_2(D)$-invariant linear form on $\pi$. For this, as in [13], we need to ensure that

$$\text{Ext}_1^{\text{Sp}_2(D)}[\text{Ind}_P^{\text{Sp}_2(D)}(\nu^{1/2} (\sigma_1 \otimes \sigma_2))_{\chi_M}, \mathbb{C}] = 0.$$ 

For proving this, we recall the notion of the Euler-Poincaré pairing between two finite length representations of any reductive group $G$, defined by

$$\text{EP}_G[\pi_1, \pi_2] = \sum_{i=0}^{r(G)} (-1)^i \dim E\text{xt}_G^i(\pi_1, \pi_2),$$

where $r(G)$ is the split rank of $G$ which for $\text{Sp}_2(D)$ is 1. Therefore, for $\text{Sp}_2(D)$,

$$\text{EP}_{\text{Sp}_2(D)}[\pi_1, \pi_2] = \dim \text{Hom}_{\text{Sp}_2(D)}[\pi_1, \pi_2] - \dim E\text{xt}_1^{\text{Sp}_2(D)}[\pi_1, \pi_2].$$

By a well known theorem, $\text{EP}_G[\pi_1, \pi_2] = 0$ if $\pi_1$ is a (not necessarily irreducible) principal series representation of $G$. Therefore, we find that

$$\text{EP}_{\text{Sp}_2(D)}[\text{Ind}_P^{\text{Sp}_2(D)}(\nu^{1/2} (\sigma_1 \otimes \sigma_2)), \mathbb{C}] = 0,$$

and so

$$\dim \text{Hom}_{\text{Sp}_2(D)}[\text{Ind}_P^{\text{Sp}_2(D)}(\nu^{1/2} (\sigma_1 \otimes \sigma_2)), \mathbb{C}] = \dim \text{Ext}_1^{\text{Sp}_2(D)}[\text{Ind}_P^{\text{Sp}_2(D)}(\nu^{1/2} (\sigma_1 \otimes \sigma_2)), \mathbb{C}]$$

Since we are assuming that $\sigma_1 \neq \sigma_2 \otimes \nu$,

$$\dim \text{Hom}_{\text{Sp}_2(D)}[\text{Ind}_P^{\text{Sp}_2(D)}(\nu^{1/2} (\sigma_1 \otimes \sigma_2)), \mathbb{C}] = 0.$$
Therefore we conclude that
\[ \text{Ext}^1_{\text{Sp}_2(D)}[\text{Ind}^{\text{Sp}_2(D)}_{\text{Sp}_2(D)}(\sigma_1 \otimes \sigma_2), \mathbb{C}] = 0. \]

As a result, we now have proved that if \( \sigma_1 \) and \( \sigma_2 \) are one dimensional representations of \( D^\times \), with \( \sigma_1 \neq \sigma_2 \otimes \nu \), then \( \pi = \sigma_1 \times \sigma_2 \) has a \( \text{Sp}_2(D) \)-invariant linear form.

We have proved most of the following theorem, which we will now complete.

**Theorem 4.3.** The only subquotients of a principal series representation \( \pi = \sigma_1 \times \sigma_2 := \text{Ind}^{\text{GL}_2(D)}_P(\sigma_1 \otimes \sigma_2) \) of \( \text{GL}_2(D) \) which have a \( \text{Sp}_2(D) \)-invariant linear form are the following.

1. When \( \dim(\sigma_1 \otimes \sigma_2) > 1 \), the unique irreducible quotient of the principal series representation \( \text{Ind}^{\text{GL}_2(D)}_P(\sigma \nu^{1/2} \otimes \sigma \nu^{-1/2}) \) denoted by \( \text{Sp}(\sigma) \).
2. When \( \dim(\sigma_1) = \dim(\sigma_2) = 1 \), any of the irreducible principal series representations \( \text{Ind}^{\text{GL}_2(D)}_P(\sigma_1 \otimes \sigma_2) \), whenever \( \sigma_1 \neq \sigma_2 \otimes \nu^{\pm 2} \).
3. When \( \dim(\sigma_1) = \dim(\sigma_2) = 1 \), and \( \sigma_1 = \sigma_2 \otimes \nu^2 \), the principal series representation \( \text{Ind}^{\text{GL}_2(D)}_P(\sigma_1 \nu \otimes \sigma_1 \nu^{-1}) \) fits in the following exact sequence:
   \[ 0 \to \text{St} \otimes \chi \to \text{Ind}^{\text{GL}_2(D)}_P(\chi \nu \otimes \chi \nu^{-1}) \to \mathbb{C}_\chi \to 0, \]
   where \( \mathbb{C}_\chi \) is the one dimensional representation of \( \text{GL}_2(D) \) on which \( \text{GL}_2(D) \) operates by the character \( \chi \circ N_{D/k} \), \( N_{D/k} \) is the reduced norm map and \( \text{St} \) is the Steinberg representation of \( \text{GL}_2(D) \). The only subquotient of \( \text{Ind}^{\text{GL}_2(D)}_P(\chi \nu \otimes \chi \nu^{-1}) \) having \( \text{Sp}_2(D) \)-invariant linear form is \( \mathbb{C}_\chi \).

**Proof.** The only part of this theorem not shown by the arguments above is that
\[ \text{Hom}_{\text{Sp}_2(D)}[\text{St}, \mathbb{C}] = 0, \]
where \( \text{St} \) is the Steinberg representation of \( \text{GL}_2(D) \), an irreducible admissible representation of \( \text{GL}_2(D) \) fitting in the exact sequence
\[ 0 \to \text{St} \to \text{Ind}^{\text{GL}_2(D)}_P(\nu \otimes \nu^{-1}) \to \mathbb{C} \to 0. \]
Applying \( \text{Hom}_{\text{Sp}_2(D)}[\cdot, \mathbb{C}] \) to this exact sequence, we have:
\[ 0 \to \text{Hom}_{\text{Sp}_2(D)}[\mathbb{C}, \mathbb{C}] \to \text{Hom}_{\text{Sp}_2(D)}[\text{Ind}^{\text{GL}_2(D)}_P(\nu \otimes \nu^{-1}), \mathbb{C}] \to \text{Hom}_{\text{Sp}_2(D)}[\text{St}, \mathbb{C}] \to \text{Ext}^1_{\text{Sp}_2(D)}[\mathbb{C}, \mathbb{C}] \to \cdots. \]
However, it is easy to see that \( \text{Ext}^1_{\text{Sp}_2(D)}[\mathbb{C}, \mathbb{C}] = 0 \). Therefore, we have a short exact sequence
\[ 0 \to \mathbb{C} \to \text{Hom}_{\text{Sp}_2(D)}[\text{Ind}^{\text{GL}_2(D)}_P(\nu \otimes \nu^{-1}), \mathbb{C}] \to \text{Hom}_{\text{Sp}_2(D)}[\text{St}, \mathbb{C}] \to 0. \]
Hence, if \( \text{Hom}_{\text{Sp}_2(D)}[\text{St}, \mathbb{C}] \neq 0 \), \( \dim \text{Hom}_{\text{Sp}_2(D)}[\text{Ind}^{\text{GL}_2(D)}_P(\nu \otimes \nu^{-1}), \mathbb{C}] \geq 2 \). However, by the analysis with Mackey theory done above, we know that \( \dim \text{Hom}_{\text{Sp}_2(D)}[\text{Ind}^{\text{GL}_2(D)}_P(\nu \otimes \nu^{-1}), \mathbb{C}] = 1 \). Thus we have proved that
\[ \text{Hom}_{\text{Sp}_2(D)}[\text{St}, \mathbb{C}] = 0. \]
Remark 4.4. As an important corollary of the theorem above, note that the irreducible principal series representation \( \pi = \chi_1 \times \chi_2 := \text{Ind}^{GL_2(D)}_P(\chi_1 \otimes \chi_2) \) for characters \( \chi_1 \) and \( \chi_2 \) of \( D^\times \) which arise from the characters \( \chi_1 \) and \( \chi_2 \) of \( k^\times \) via the reduced norm map of \( D^\times \) to \( k^\times \), with \( \chi_1 \chi_2^{-1} \neq \nu \pm 2 \), the representation \( \pi \) is distinguished by \( \text{Sp}_2(D) \). However \( JL(\pi) \), a representation of \( GL_4(k) \) is the irreducible principal series representation \( JL(\pi) = \text{Ind}^{GL_4(k)}_P(\chi_1 \text{St}_2 \otimes \chi_2 \text{St}_2) \) where \( \text{St}_2 \) denote the Steinberg representation of \( GL_4(k) \). Since \( JL(\pi) \) is a generic representation of \( GL_4(k) \), it is not distinguished by \( \text{Sp}_4(k) \). Thus Jacquet-Langlands correspondence for representations of \( GL_2(D) \) to \( GL_4(k) \) does not always preserve distinction.

5. Global theory

Let \( F \) be a number field and \( D \) be a quaternion division algebra over \( F \). For each place \( v \) of \( F \), let \( F_v \) be the completion of \( F \) at \( v \). We can define \( GL_n(D) \) and \( \text{Sp}_n(D) \) as in the local case in the Section 2.

Let \( \mathbb{A} \) be the ring of adèles of \( F \). Let \( D_\mathbb{A} = D \otimes_F F_\mathbb{A} \) and \( D_\mathbb{A} = D \otimes_F \mathbb{A} \). Then we can consider topological groups \( GL_n(D_\mathbb{A}) \), \( \text{Sp}_n(D_\mathbb{A}) \), \( GL_n(D_v) \), \( \text{Sp}_n(D_v) \), \( GL_n(\mathbb{A}_F) \), \( \text{Sp}_n(\mathbb{A}_F) \). For an automorphic representation \( \Pi \) of \( GL_n(D_\mathbb{A}) \), we denote by \( JL(\Pi) \), its Jacquet-Langlands lift to \( GL_{2n}(\mathbb{A}_F) \).

In this section, we will prove that the non-vanishing symplectic period of a discrete automorphic representation is taken to a non-vanishing period by the Jacquet-Langlands correspondence. In [10], Offen studied the symplectic periods on the discrete automorphic representations of \( GL_{2n}(\mathbb{A}_F) \). For an automorphic form \( f \) in the discrete spectrum of \( GL_{2n}(\mathbb{A}_F) \), consider the period integral

\[
\int_{\text{Sp}_{2n}(F) \backslash \text{Sp}_{2n}(\mathbb{A}_F)} f(h) dh.
\]

We say that an irreducible, discrete automorphic representation \( \Pi \) of \( GL_{2n}(\mathbb{A}_F) \) is \( \text{Sp}_{2n}(\mathbb{A}_F) \)-distinguished if the above period integral is not identically zero on the space of \( \Pi \). We now recall a result from [12] that we will use in this section.

Theorem 5.1. Let \( F \) be a number field and let \( \Pi = \otimes_v \Pi_v \) be an irreducible automorphic representation of \( GL_{2n}(\mathbb{A}_F) \) in the discrete spectrum. Then the following are equivalent:

1. \( \Pi \) is \( \text{Sp}_{2n}(\mathbb{A}_F) \)-distinguished,
2. \( \Pi_v \) is \( \text{Sp}_{2n}(F_v) \)-distinguished for all places \( v \) of \( F \),
3. \( \Pi_{v_0} \) is \( \text{Sp}_{2n}(F_{v_0}) \)-distinguished for some finite place \( v_0 \) of \( F \),

Jacquet and Rallis have shown in [5], that the symplectic period vanishes for a cuspidal automorphic representation of \( GL_{2n}(\mathbb{A}_F) \), that is

\[
\int_{\text{Sp}_{2n}(F) \backslash \text{Sp}_{2n}(\mathbb{A}_F)} f(h) dh = 0.
\]

In the next theorem, in the spirit of Jacquet-Rallis result mentioned above, we prove that those cuspidal automorphic representations \( \Pi \) of \( GL_n(D_\mathbb{A}) \) for which
JL(Π) is a cuspidal automorphic representation of GL\(_{2n}(\mathbb{A}_F)\), have vanishing symplectic periods.

**Theorem 5.2.** Suppose that Π is a cuspidal automorphic representation of GL\(_n(D)\) whose Jacquet-Langlands lift JL(Π) to GL\(_{2n}(\mathbb{A}_F)\) is cuspidal then the symplectic period integrals of Π vanish identically.

**Proof.** Assume if possible that Π has a non-zero symplectic period. Then Π\(_v\) has a non-zero symplectic period for all places \(v\) of \(F\). The representations JL(Π) and Π are the same at all places \(v\) of \(F\) where \(D\) splits and therefore by Theorem 3.2.2 of [4], Π\(_v\) is not generic for any \(v\) where \(D\) splits. Since a cuspidal automorphic representation of GL\(_{2n}(\mathbb{A}_F)\) is globally generic, the local representations Π\(_v\) are locally generic for all \(v\), which gives a contradiction.  □

**Theorem 5.3.** If Π is an automorphic representation of GL\(_n(D)\) which appears in the discrete spectrum, and is distinguished by Sp\(_n(D)\) then JL(Π), which is an automorphic representation of GL\(_{2n}(\mathbb{A}_F)\), is globally distinguished by Sp\(_{2n}(\mathbb{A}_F)\).

**Proof.** If Π is Sp\(_n(D)\)-distinguished, then it is locally distinguished at all places \(v\) of \(F\). Also we know that \(D\) splits at almost all places of \(F\) so Π\(_{\overline{v}}\) = JL(Π)\(_{\overline{v}}\) at almost all places of \(F\). By Theorem 5.1 global distinction of Jacquet-Langland lift JL(Π) is a consequence of local distinction at any place \(v\) of \(F\) which we know. □

**Remark 5.4.** If Π is a global automorphic representations of GL\(_2(D)\) which is distinguished by Sp\(_2(D)\) with a local component Π\(_v\) = \(\chi_1 \times \chi_2\), a representation of GL\(_2(D_v)\) for characters \(\chi_1, \chi_2 : D_v^\times \rightarrow \mathbb{C}^\times\), then JL(Π), an automorphic representation of GL\(_4(\mathbb{A}_F)\), must be distinguished by Sp\(_4(\mathbb{A}_F)\) by Theorem 5.3. Since JL(Π\(_v\)) = \(\chi_1 \circ \text{St} \times \chi_2 \circ \text{St}\) as a representation of GL\(_4(k_v)\), this seems to be in contradiction to the fact that JL(Π) is globally distinguished by Sp\(_4(\mathbb{A}_F)\). The source of this apparent contradiction is the fact that in this case, JL(Π\(_v\)) = \(\chi_1 \times \chi_2\) as a representation of GL\(_4(k_v)\), as follows from the work of Badulescu.

A supercuspidal representation of GL\(_{2n}(k)\) is not distinguished by Sp\(_{2n}(k)\).

The situation in the case of GL\(_n(D)\) is different, that is, it may happen that a supercuspidal representation of GL\(_n(D)\) is distinguished by Sp\(_n(D)\). We have an example of distinguished supercuspidal representations due to Dipendra Prasad in the next section. The following theorem gives a partial answer to the question on distinction of a supercuspidal representation of GL\(_n(D)\) by Sp\(_n(D)\).

**Theorem 5.5.** Let \(\pi_v\) be a supercuspidal representation of GL\(_n(D_v)\) with Langlands parameter \(\sigma_{\pi_v} = \sigma \otimes \text{sp}_r\) where \(\sigma\) is an irreducible representation of the Weil-group \(W_L\), and \(\text{sp}_r\) is the \(r\)-dimensional irreducible representation of SL\(_2(\mathbb{C})\) part of the Weil-Deligne group \(W'_L\). Then if \(r\) is odd, \(\pi_v\) is not distinguished by Sp\(_n(D_v)\).
Proof. Assuming $r$ is odd, we prove that $\pi_v$ is not distinguished by $\text{Sp}_{n}(D_v)$. Using a theorem of [15], we globalize $\pi_v$ to be globally distinguished automorphic representation $\Pi$ of $\text{GL}_n(D_v)$ where $D$ is a global division algebra over a number field $F$ such that $F_v = k$, and $D \otimes F_v = D_v$.

Using the Jacquet-Langlands correspondence of Badulescu, we get an automorphic representation $\text{JL}(\Pi)$ of $\text{GL}_2(A_F)$ which is locally distinguished by $\text{Sp}_{2n}(A_F)$ at all places $w$ of $F$ where $D$ splits. By a theorem of Offen-Sayag, $\text{JL}(\Pi)$ is globally distinguished by $\text{Sp}_{2n}(A_F)$. By work of Badulescu, $\text{JL}(\Pi)_v$ is one of the following

1. $\text{JL}(\Pi)_v = \text{JL}(\Pi_v)$, a discrete series representation, or
2. $\text{JL}(\Pi)_v$ = a Speh representation with Langlands parameter $\sigma \otimes (\nu^{(r-1)/2} \oplus \nu^{(r-3)/2} \oplus \cdots \oplus \nu^{-r(1)/2})$.

The first choice being a discrete series representation, in particular generic, is never distinguished by $\text{Sp}_{2n}(A_F)$.

Remark 5.6. The only place we used supercuspidality of the representation $\pi_v$ of $\text{GL}_n(D_v)$ with Langlands parameter $\sigma_{\pi_v} = \sigma \otimes \text{sp}_r$, where $\sigma$ is an irreducible representation of the Weil-group $W_k$, and $\text{sp}_r$ is the $r$-dimensional irreducible representation of the $\text{SL}_2(\mathbb{C})$ part of the Weil-Deligne group $W'_k$ is in the globalization theorem of [15]. If we grant ourselves such a globalization theorem for discrete series too, then we have the same conclusion as in the theorem.

The theorem below together with local analysis done in Section 4 completes the distinction problem for $\text{GL}_2(D)$.

**Theorem 5.7.** No discrete series representation of $\text{GL}_2(D_v)$ is distinguished by $\text{Sp}_{2}(D_v)$.

Proof. By our local analysis, we know this already for those discrete series representations of $\text{GL}_2(D_v)$ which are not supercuspidal. By the previous theorem, we also know that no supercuspidal representation of $\text{GL}_2(D_v)$ is distinguished by $\text{Sp}_{2}(D_v)$ as long as its Langlands parameter is not of the form $\sigma_{\pi} = \sigma \otimes \text{sp}_r$ where $r = 2, 4$. But by the work of Badulescu (cf. Proposition 7.2 below), such Langlands parameter correspond to non-supercuspidal discrete series representations of $\text{GL}_2(D_v)$, completing the proof of theorem.

6. **Explicit examples of supercuspidals with symplectic period**

In this section we construct examples of supercuspidal representations of $\text{GL}_n(D)$ which are distinguished by $\text{Sp}_{n}(D)$ for any odd $n \geq 1$.

Recall that $\mathcal{O}_D$ is the maximal compact subring of $D$ with $\pi_D$ a uniformizing parameter of $\mathcal{O}_D$, and $\mathcal{O}_D/\langle \pi_D \mathcal{O}_D \rangle \simeq \mathbb{F}_{q^2}$ where $\mathbb{F}_q$ is the residue field of $k$. The anti-automorphism $x \rightarrow \bar{x}$ of $D$ preserve $\mathcal{O}_D$ and acts as the Galois involution of $\mathbb{F}_{q^2}$ over $\mathbb{F}_q$. 

Recall also that we have defined $\text{Sp}_n(D)$ to be the subgroup of $\text{GL}_n(D)$ by:

$$\text{Sp}_n(D) = \{ A \in \text{GL}_n(D) | AJ^t \bar{A} = J \},$$

where $^t \bar{A} = (\bar{a}_{ji})$ for $A = (a_{ij})$ and

$$J = \begin{pmatrix} 1 & 1 \\ & 1 \\ & & 1 \end{pmatrix}.$$

It follows that $\text{Sp}_n(O_D) \subset \text{GL}_n(O_D)$, and taking the reduction of these compact groups modulo $\pi_D$, we have:

$$U_n(F_q) \hookrightarrow \text{GL}_n(F_{q^2}),$$

where $U_n$ is defined using the Hermitian form

$$J = \begin{pmatrix} 1 & 1 \\ & 1 \\ & & 1 \end{pmatrix}.$$

**Proposition 6.1.** Let $\pi_{00}$ be an irreducible cuspidal representation of $\text{GL}_n(F_q)$, $n$ an odd integer, and $\pi_0 = \text{BC}(\pi_{00})$ be the base change of $\pi_{00}$ to $\text{GL}_n(F_{q^2})$. Using the reduction mod $\pi_D : \text{GL}_n(O_D) \rightarrow \text{GL}_n(F_{q^2})$, we can lift $\pi_0$ to an irreducible representation of $\text{GL}_n(O_D)$ to be denoted by $\pi_0$ again. Let $\chi$ be a character of $k^\times$ which matches with the central character of $\pi_0$ on $O_k^\times$. Then

$$\pi = \text{ind}_{k^\times \text{GL}_n(O_D)}^{k^\times \text{GL}_n(D)}(\chi \cdot \pi_0)$$

is an irreducible supercuspidal representation of $\text{GL}_n(D)$ which is distinguished by $\text{Sp}_n(D)$.

**Proof.** The fact that $\pi$ is an irreducible supercuspidal representation of $\text{GL}_n(D)$ is a well-known fact about compact induction valid in a great generality once we have checked that $\pi_0 = \text{BC}(\pi_{00})$ is a cuspidal representation. This assertion on $\text{GL}_n(F_{q^2})$ follows from the fact that $n$ is odd in which case we have a diagram of fields:
In particular,
\[ \text{Gal}(\mathbb{F}_{q^{2n}}/\mathbb{F}_q) = \text{Gal}(\mathbb{F}_{q^n}/\mathbb{F}_q) \times \text{Gal}(\mathbb{F}_{q^2}/\mathbb{F}_q). \]

Thus given a character \( \chi_{00} : \mathbb{F}_{q^{2n}}^\times \rightarrow \mathbb{C}^\times \) whose Galois conjugate are distinct (and which gives rise to the cuspidal representation \( \pi_{00} \) of \( \text{GL}_n(\mathbb{F}_q) \)), the character \( \chi_0 : \mathbb{F}_{q^{2n}}^\times \rightarrow \mathbb{C}^\times \) obtained from \( \chi_{00} \) using the norm map: \( \mathbb{F}_{q^{2n}}^\times \rightarrow \mathbb{F}_{q^n}^\times \), has exactly \( n \) distinct Galois conjugates, therefore \( \chi_0 \) gives rise to a cuspidal representation \( \pi_0 \) of \( \text{GL}_n(\mathbb{F}_{q^2}) \) which is the base change of the representation \( \pi_{00} \) of \( \text{GL}_n(\mathbb{F}_q) \).

The distinction of \( \pi \) by \( \text{Sp}_n(D) \) follows from the earlier observation that reduction mod \( \pi_D \) of the inclusion \( \text{Sp}_n(O_D) \subset \text{GL}_n(O_D) \) is
\[ U_n(\mathbb{F}_q) \hookrightarrow \text{GL}_n(\mathbb{F}_{q^2}), \]

together with the well-known fact, Theorem 2 of [14], that irreducible representations of \( \text{GL}_n(\mathbb{F}_{q^2}) \) which are base change from \( \text{GL}_n(\mathbb{F}_q) \) are distinguished by \( U_n(\mathbb{F}_q) \).

\[ \square \]

Remark 6.2.  
(1) The Langlands parameter of the irreducible representation \( \pi = \text{ind}_{k^* \text{GL}_n(O_D)}^{\text{GL}_n(D)}(\pi_0) \) is of the form \( \sigma = \sigma_0 \otimes \text{sp}_2 \) where \( \sigma_0 \) is the Langlands parameter of the supercuspidal representation of \( \text{GL}_n(k) \) compactly induced from the representation \( \chi \cdot \pi_{00} \) of \( k^* \text{GL}_n(O_k) \), and \( \text{sp}_2 \) is the 2-dimensional natural representation of the \( \text{SL}_2(\mathbb{C}) \) part of the Weil-Deligne group \( W'_k = W_k \times \text{SL}_2(\mathbb{C}) \) of \( k \).

(2) If, on the other hand, the cuspidal representation \( \pi_0 \) of \( \text{GL}_n(\mathbb{F}_{q^2}) \) is not obtained by base change from \( \text{GL}_n(\mathbb{F}_q) \) then the Langlands parameter of such a \( \pi \) is that of the cuspidal representation of \( \text{GL}_{2n}(k) \) which is obtained by compact induction of the representation of \( k^* \text{GL}_{2n}(O_k) \) which is \( \chi \) on \( k^* \), and on \( \text{GL}_{2n}(O_k) \) it corresponds to a representation of \( \text{GL}_{2n}(\mathbb{F}_q) \) which is the automorphic induction of the representation \( \pi_{00} \) of \( \text{GL}_n(\mathbb{F}_{q^2}) \) (and which is cuspidal since we are assuming that the representation \( \pi_0 \) of \( \text{GL}_n(\mathbb{F}_{q^2}) \) is not a base change for \( \text{GL}_n(\mathbb{F}_q) \)).

7. Conjectures on distinction

The following conjectures have been proposed by Dipendra Prasad.

(1) An irreducible discrete series representation \( \pi \) of \( \text{GL}_n(D_v) \) is distinguished by \( \text{Sp}_n(D_v) \) if and only if \( \pi \) is supercuspidal and the Langlands parameter \( \sigma_{\pi} \) of \( \pi \) is of the form \( \sigma_{\pi} = \tau \otimes \text{sp}_r \) where \( \tau \) is irreducible and \( \text{sp}_r \) is the \( r \)-dimensional natural representation of the \( \text{SL}_2(\mathbb{C}) \) part of the Weil-Deligne group \( W'_k = W_k \times \text{SL}_2(\mathbb{C}) \) of \( k \) for \( r \) even. By Proposition 7.2 below, this is the case if and only if \( r = 2 \), and \( n \) is odd. (This is thus exactly the case in which we constructed in the last section a supercuspidal representation of \( \text{GL}_n(D_v) \) which is distinguished by \( \text{Sp}_n(D_v) \).)

(2) We follow the notation of Offen-Sayag, Theorem 1 of [11], to recall that the unitary representations of \( \text{GL}_{2k}(F_v) \) which are distinguished
by $\text{Sp}_{2n}(F_v)$ are of the form

\[ \sigma_1 \times \cdots \times \sigma_t \times \tau_{t+1} \times \cdots \times \tau_{t+s}, \]

where $\sigma_i$ are the Speh representations $U(\delta_i, 2m_i)$ for discrete series representations $\delta_i$ of $GL_{r_i}(F_v)$, and $\tau_i$ are complementary series representations $\pi(U(\delta_i, 2m_i), \alpha_i)$ with $|\alpha_i| < 1/2$. We suggest that unitary representations of $GL_n(D_v)$ distinguished by $\text{Sp}_n(D_v)$ are exactly those representations of $GL_n(D_v)$ which are of the form

\[ \pi = \sigma_1 \times \cdots \times \sigma_t \times \tau_{t+1} \times \cdots \times \tau_{t+s} \times \mu_{t+s+1} \times \cdots \times \mu_{t+s+r}, \]

where

(a) The parameter $\sigma_\pi$ of $\pi$ is relevant for $GL_n(D_v)$, that is, all irreducible subrepresentations of $\sigma_\pi$ have even dimension.

(b) $\sigma_i$ and $\tau_i$ are as in the theorem of Offen-Sayag recalled above.

(c) $\mu_i$ are supercuspidal representations of $GL_{m_i}(D_v)$ as in Part (1) of the conjecture.

(3) A global automorphic representation of $GL_n(D_A)$ is distinguished by $\text{Sp}_n(D_A)$ if and only $JL(\pi)$ as an automorphic representation of $GL_{2n}(\mathbb{A}_F)$ (which is same as $\pi$ at places of $F$ where $D$ splits) is distinguished by $\text{Sp}_{2n}(\mathbb{A}_F)$.

Proposition 7.1. The global conjecture in part 3 above implies the local conjecture in part 1.

Proof. To prove the Proposition, note that a discrete series representation $\pi$ of $GL_n(D_v)$ with parameter $\tau \otimes \text{sp}_r$ with $r$ odd is not distinguished by $\text{Sp}_n(D_v)$ as follows from Theorem 5.5 and the remark 5.6 following it (which assumes validity of the globalization theorem of [15] for discrete series representations).

Now we prove that a non-cuspidal discrete series representation $\pi$ of $GL_n(D_v)$ with parameter $\tau \otimes \text{sp}_r$ with $r$ even are not distinguished by $\text{Sp}_n(D_v)$. Again we will grant ourselves an automorphic representation $\pi$ of $GL_n(D_v)$ which is globally distinguished by $\text{Sp}_n(D_v)$. By the Jacquet-Langlands transfer, we get a representation $JL(\pi)$ of $GL_{2n}(\mathbb{A}_F)$ which is distinguished by $\text{Sp}_{2n}(\mathbb{A}_F)$, and therefore by the theorem of Offen-Sayag $JL(\pi)$ is in the residual spectrum with the Moeclglin-Waldspurger type, $JL(\pi) = \Sigma \otimes \text{sp}_d$, where $\Sigma$ is a cuspidal automorphic representation of $GL_n(\mathbb{A}_F)$ for some integer $r$, and $d$ is a certain even integer; here the notation $\Sigma \otimes \text{sp}_d$ is supposed to denote a certain Speh representation. The only option for $d$ in our case is $d = r$, and $\Sigma_v = \tau$. By Proposition 7.3 below, we get a contradiction to $\pi$ being a non-cuspidal discrete series representation of $GL_n(D_v)$.

Finally we prove that if we have a cuspidal representation $\pi$ of $GL_n(D_v)$ with parameter $\tau \otimes \text{sp}_r$ with $r$ even, so $r = 2$, and $\dim \tau = n$ odd, then $\pi$ is distinguished by $\text{Sp}_n(D_v)$.

Construct an automorphic representation of $GL_n(\mathbb{A}_F)$ whose local component at the place $v$ of $F$ has Langlands parameter $\tau$ with $\dim \tau = n$. Since $\tau$ is an irreducible representation of the Weil group, we are considering supercuspidal representation of $GL_n(F_v)$, and therefore globalization is possible.
We moreover assume in this globalization that the global automorphic representation of $GL_n(A_F)$ is supercuspidal at all places of $F$ where $D$ is not split. By Moeglin-Waldspurger, this gives an automorphic representation say $\Pi$ of $GL_n(A_F)$ in the residual spectrum, which by the theorems of Offen and Sayag is distinguished by $Sp_{2n}(A_F)$. By the work of Badulescu, $\Pi$ can be lifted to $GL_n(D_A)$, which by our global conjecture (3) above is globally distinguished by $Sp_2(D_A)$, and therefore locally distinguished at every place of $F$. It remains to make sure that in this Jaquet-Langlands transfer from $GL_2(A_F)$ to $GL_n(D_A)$, the local representation obtained for $GL_n(D_v)$ is the cuspidal representation $\pi$ with parameter $\tau \otimes sp_2$; this is forced on us when $\pi$ is cuspidal by lemma 7.4 below. (The representation $\pi$ could have changed to its Zelevinsky involution, but $\pi$ being cuspidal remains invariant under the Zelevinsky involution.)

The following proposition is due to Deligne-Kazhdan-Vigneras [2], Theorem B.2.b.1, as well as Badulescu, proposition 3.7 of [1].

**Proposition 7.2.** A discrete series representation of $GL_n(D_v)$, where $D_v$ is an arbitrary division algebra over the local field $F_v$, with parameter $\tau \otimes sp_r$ is a cuspidal representation of $GL_n(D_v)$ if and only if $(r, n) = 1$.

In the following proposition, we refer to Badulescu [1] for the notion of a $d$-compatible representation of $GL_{nd}(F_v)$.

**Proposition 7.3.** Let $D_v$ be a division algebra over a local field $F_v$ of dimension $d^2$. The map $|LJ|$ from $d$-compatible irreducible admissible unitary representations of $GL_{nd}(F_v)$ to irreducible unitary representations of $GL_n(D_v)$ takes a Speh representation associated to a cuspidal representation on $GL_{nd}(F_v)$ to either a cuspidal representation on $GL_n(D_v)$, or to a Speh representation, i.e., the image under $|LJ|$ of a Speh representation associated to a cuspidal representation on $GL_{nd}(F_v)$ is never a non-cuspidal discrete series representation on $GL_n(D_v)$.

**Proof.** The proof follows from the fact that $|LJ|$ commutes with the Zelevinsky involution, and that the Zelevinsky involution of a discrete series representation is itself if and only if the discrete series representation is supercuspidal. (We apply this latter fact on $GL_n(D_v)$.)

We also had occasion to use the following lemma.

**Lemma 7.4.** The map $|LJ|$ from $d$-compatible irreducible admissible unitary representations of $GL_{nd}(F_v)$ to irreducible unitary representations of $GL_n(D_v)$ has fibers of cardinality one or two over a discrete series representation of $GL_n(D_v)$, and if of cardinality two, the two elements in the fiber are Zelevinsky involution of each other, and the image consists of a cuspidal representation of $GL_n(D_v)$.

**Proof.** Assume that we are considering the fibers of the map $|LJ|$ over a discrete series representation of $GL_n(D_v)$ with Langlands parameter $\tau \otimes sp_r$. All
the representations in the fiber are contained in the principal series representation
\[ \tau \nu^{(r-1)/2} \times \tau \nu^{(r-3)/2} \times \cdots \times \tau \nu^{-(r-1)/2}. \]

It is well-known that there are exactly two irreducible unitary representations among sub-quotients of this principal series, one of which is the Langlands quotient which is a Speh module, and the other the discrete series representation with parameter \( \tau \otimes \text{sp}_r \), proving the lemma. \( \square \)

\section*{References}

[1] Alexandru Ioan Badulescu (With an appendix by Neven Grbac). Global Jacquet-Langlands correspondence, multiplicity one and classification of automorphic representations. \textit{Invent. Math.}, 172(2):383–438, 2008.

[2] P. Deligne, D. Kazhdan, M. -F. Vignéras, \textit{Représentations des algèbres centrales simples p-adiques}, in Représentations des groupes réductifs sur un corps local, Travaux en Cours, Hermann, Paris, 1984, pp. 331–117.

[3] I. M. Gelfand and D. A. Kazhdan. \textit{Representations of \text{GL}(n, K) in Lie Groups and their Representations 2}, Akadémiai Kiado, Budapest, 1974.

[4] Michael J. Heumos and Stephen Rallis. Symplectic-Whittaker models for \( \text{GL}_n \). \textit{Pacific J. Math.}, 146(2):247–279, 1990.

[5] Harvé Jacquet and Stephen Rallis. Symplectic periods. \textit{J. Reine Angew. Math.}, 423:175–197, 1992.

[6] A. A. Klyachko. Models for complex representations of groups \( \text{GL}(n, q) \). \textit{Mat. Sb. (N.S.)}, 120(162):371–386, 1983.

[7] C. Moeglin, M-F Vigneras, J.-L. Waldspurger, \textit{Correspondances de Howe sur un corps p-adique}, Lecture Notes in Mathematics. \textbf{1291}, Springer Verlag, 1987.

[8] T. Nakayama and Y. Matsushima. Über die multiplikative Gruppe einer \( p \)-adischen Divisionsalgebra. \textit{Proceedings of the Imperial Academy of Japan}, vol. 19, 1943.

[9] Omer Offen. Distinguished residual spectrum. \textit{Duke Math. J.}, 134(2):313–357, 2006.

[10] Omer Offen. On sympletic periods of discrete spectrum of \( \text{GL}_{2n} \). \textit{Israel J. Math.}, 154:253–298, 2006.

[11] Omer Offen and Eitan Sayag. On unitary representations of \( \text{GL}_{2n} \) distinguished by the symplectic group. \textit{J. Number Theory}, 125(2):344-355, 2007.

[12] Omer Offen and Eitan Sayag. Uniqueness and disjointness of Klyachko models. \textit{J. of Functional analysis}, 254:2846-2865, 2008.

[13] Dipendra Prasad. Trilinear forms for representations of \( \text{GL}(2) \) and local \( \epsilon \)-factors. \textit{Compositio Mathematica}, 75:1–46, 1990.

[14] Dipendra Prasad. Distinguished representations for quadratic extensions. \textit{Compositio Mathematica}, 119(3):343–354, 1999.

[15] Dipendra Prasad and R. Schulze-Pillot. Generalised form of a conjecture of Jacquet and a local consequence. \textit{J. Reine Angew. Math.}, 616: 219–236, 2008.

[16] A. Raghuram. \textit{Some topics in algebraic groups: Representation theory of \( \text{GL}_2(\mathfrak{D}) \) where \( \mathfrak{D} \) is a division algebra over a nonarchemedian local fields}, thesis, Tata Institute of Fundamental Research, University of Mumbai, 1999.

[17] J. P. Serre. Cohomologie Galoisienne. \textit{2nd ed., Lecture Notes in Mathematics}, vol. 5, Springer-Verlag, 1964.

[18] Marko Tadić. Induced representations of \( \text{GL}(n, A) \) for \( p \)-adic division algebras \( A \). \textit{J. Reine Angew. Math.}, 405:48–77, 1990.

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