A CANONICAL LIFT OF FROBENIUS IN MORAVA E-THEORY

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Abstract. We prove that the \( p \)th Hecke operator on the Morava E-cohomology of a space is congruent to the Frobenius mod \( p \). This is a generalization of the fact that the \( p \)th Adams operation on the complex K-theory of a space is congruent to the Frobenius mod \( p \). The proof implies that the \( p \)th Hecke operator may be used to test Rezk’s congruence criterion.

1. Introduction

The \( p \)th Adams operation on the complex K-theory of a space is congruent to the Frobenius mod \( p \). This fact plays a role in Adams and Atiyah’s proof [AA66] of the Hopf invariant one problem. It also implies the existence of a canonical operation \( \theta \) on \( K_0(X) \) satisfying

\[ \psi^p(x) = x^p + p\theta(x), \]

when \( K_0(X) \) is torsion-free. This extra structure was used by Bousfield [Bou96] to determine the \( \lambda \)-ring structure of the K-theory of an infinite loop space. There are several generalizations of the \( p \)th Adams operation in complex K-theory to Morava E-theory: the \( p \)th additive power operation, the \( p \)th Adams operation, and the \( p \)th Hecke operator. In this note, we show that the \( p \)th Hecke operator is a lift of Frobenius.

In [Rez09], Rezk studies the relationship between two algebraic structures related to power operations in Morava E-theory. One structure is a monad \( T \) on the category of \( E_0 \)-modules that is closely related to the free \( E_\infty \)-algebra functor. The other structure is a form of the Dyer-Lashof algebra for \( E \), called \( \Gamma \). Given a \( \Gamma \)-algebra \( R \), each element \( \sigma \in \Gamma \) gives rise to a linear endomorphism \( Q_\sigma \) of \( R \). He proves that a \( \Gamma \)-algebra \( R \) admits the structure of an algebra over the monad \( T \) if and only if there exists an element \( \sigma \in \Gamma \) (over a certain element \( \bar{\sigma} \in \Gamma/p \)) such that \( Q_\sigma \) is a lift of Frobenius in the following sense:

\[ Q_\sigma(r) \equiv r^p \mod pR \]

for all \( r \in R \).

We will show that \( Q_\sigma \) may be taken to be the \( p \)th Hecke operator \( T_p \) as defined by Ando in [And95 Section 3.6]. We prove this by producing a canonical element \( \sigma_{can} \in \Gamma \) lifting the Frobenius class \( \bar{\sigma} \in \Gamma/p \) [Rez09 Section 10.3] such that \( Q_{\sigma_{can}} = T_p \). This provides us with extra algebraic structure on torsion-free algebras over the monad \( T \) in the form of a canonical operation \( \theta \) satisfying

\[ T_p(r) = r^p + p\theta(r). \]

Let \( G_{E_0} \) be the formal group associated to \( E \), a Morava E-theory spectrum. The Frobenius \( \phi \) on \( E_0/p \) induces the relative Frobenius isogeny

\[ G_{E_0/p} \rightarrow \phi^* G_{E_0/p} \]
over $E_0/p$. The kernel of this isogeny is a subgroup scheme of order $p$. By a theorem of Strickland, this corresponds to an $E_0$-algebra map
\[ \bar{\sigma} : E^0(B\Sigma_p)/I \longrightarrow E_0/p, \]
where $I$ is the image of the transfer from the trivial group to $\Sigma_p$. This map further corresponds to an element in the mod $p$ Dyer-Lashof algebra $\Gamma/p$. Rezk considers the set of $E_0$-module maps $[\bar{\sigma}] \subset \text{hom}(E^0(B\Sigma_p)/I, E_0)$ lifting $\bar{\sigma}$.

**Proposition 1.1.** There is a canonical choice of lift $\sigma_{can} \in [\bar{\sigma}]$.

The construction of $\sigma_{can}$ is an application of the formula for the $K(n)$-local transfer (induction) along the surjection from $\Sigma_p$ to the trivial group [Gan06, Section 7.3].

Let $X$ be a space and let
\[ P_p/I : E^0(X) \longrightarrow E^0(B\Sigma_p)/I \otimes_{E_0} E^0(X) \]
be the $p$th additive power operation. The endomorphism $Q_{\sigma_{can}}$ of $E^0(X)$ is the composite of $P_p/I$ with $\sigma_{can} \otimes 1$.

**Proposition 1.2.** For any space $X$, the following operations on $E^0(X)$ are equal:
\[ Q_{\sigma_{can}} = (\sigma_{can} \otimes 1)(P_p/I) = T_p. \]

This has the following immediate consequence:

**Corollary 1.3.** Let $X$ be a space such that $E^0(X)$ is torsion-free. There exists a canonical operation
\[ \theta : E^0(X) \longrightarrow E^0(X) \]
such that, for all $x \in E^0(X)$,
\[ T_p(x) = x^p + p\theta(x). \]

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2. Tools

Let $E$ be a height $n$ Morava $E$-theory spectrum at the prime $p$. We will make use of several tools that let us access $E$-cohomology. We summarize them in this section.

For the remainder of this paper, let $E(X) = E^0(X)$ for any space $X$. We will also write $E$ for the coefficients $E^0$ unless we state otherwise.

**Character theory** Hopkins, Kuhn, and Ravenel introduce character theory for $E(BG)$ in [HKR00]. They construct the rationalized Drinfeld ring $C_0$ and introduce a ring of generalized class functions taking values in $C_0$:
\[ Cl_n(G, C_0) = \{ C_0\text{-valued functions on conjugacy classes of map from } \mathbb{Z}_p^n \text{ to } G\}. \]

They construct a map
\[ E(BG) \longrightarrow Cl_n(G, C_0) \]
and show that it induces an isomorphism after the domain has been base-changed to $C_0$ [HKR00 Theorem C]. When $n = 1$, this is a $p$-adic version of the classical character map from representation theory.

**Good groups** A finite group $G$ is good if the character map
\[ E(BG) \longrightarrow Cl_n(G, C_0) \]
is injective. Hopkins, Kuhn, and Ravenel show that $\Sigma_{p^k}$ is good for all $k$ [HKR00, Theorem 7.3].

**Transfer maps** It follows from a result of Greenlees and Sadofsky [GS96] that there are transfer maps in $E$-cohomology along all maps of finite groups. In [HKR00, Section 7.3], Ganter studies the case of the transfer from $G$ to the trivial group and shows that there is a simple formula for the transfer on the level of class functions. Let

$$\text{Tr}_{C_0} : C_1(G, C_0) \to C_0$$

be given by the formula $f \mapsto \frac{1}{|G|} \sum_{[\alpha]} f([\alpha])$, where the sum runs over conjugacy classes of maps $\alpha : \mathbb{Z}_p^n \to G$. Ganter shows that there is a commutative diagram

$$
\begin{array}{ccc}
E(BG) & \xrightarrow{\text{Tr}_E} & E \\
\downarrow & & \downarrow \\
C_1_n(G) & \xrightarrow{\text{Tr}_{C_0}} & C_0,
\end{array}
$$

in which the vertical maps are the character map.

**Subgroups of formal groups** Let $\mathbb{G}_E = \text{Spf}(E(\text{BS}^1))$ be the formal group associated to the spectrum $E$. In [Str98], Strickland produces a canonical isomorphism

$$\text{Spf}(E(\Sigma p^k)/I) \cong \text{Sub}_{p^k}(\mathbb{G}_E),$$

where $I$ is the image of the transfer along $\Sigma_{p^k-1} \subset \Sigma p^k$ and $\text{Sub}_{p^k}(\mathbb{G}_E)$ is the scheme that classifies subgroup schemes of order $p^k$ in $\mathbb{G}_E$. We will only need the case $k = 1$.

**The Frobenius class** The relative Frobenius is a degree $p$ isogeny of formal groups

$$\mathbb{G}_{E/p} \xrightarrow{\phi^*} \mathbb{G}_{E/p},$$

where $\phi : E/p \to E/p$ is the Frobenius. The kernel of the map is a subgroup scheme of order $p$. Using Strickland’s result, there is a canonical map of $E$-algebras

$$\hat{\sigma} : E(\Sigma p)/I \to E/p$$

picking out the kernel. In [Rez09, Section 10.3], Rezk describes this map in terms of a coordinate and considers the set of $E$-module maps $[\hat{\sigma}] \subset \text{hom}(E(\Sigma p), E)$ that lift $\hat{\sigma}$.

**Power operations** In [GH04], Goerss, Hopkins, and Miller prove that the spectrum $E$ admits the structure of an $E_\infty$-ring spectrum in an essentially unique way. This implies a theory of power operations. These are natural multiplicative non-additive maps

$$P_m : E(X) \to E(B\Sigma_m) \otimes E(X)$$

for all $m > 0$. For $m = p^k$, they can be simplified to obtain interesting ring maps by further passing to the quotient

$$P_{p^k}/I : E(X) \to E(B\Sigma_{p^k}) \otimes E(X) \to E(B\Sigma_{p^k})/I \otimes E(X),$$

where $I$ is the transfer ideal that appeared above.

**Hecke operators** In [And95, Section 3.6], Ando produces operations

$$T_{p^k} : E(X) \to E(X)$$

by combining the structure of power operations, Strickland’s result, and ideas from character theory. Let $T = (\mathbb{Q}_p/\mathbb{Z}_p)^\times$, let $H \subset T$ be a finite subgroup, and let $D_\infty$ be the Drinfeld ring.
at infinite level so that \( \text{Spf}(D_{\infty}) = \text{Level}(\mathcal{T}, G_E) \) and \( \mathbb{Q} \otimes D_{\infty} = C_0 \). Ando constructs an Adams operation depending on \( H \) as the composite
\[
\psi^H: E(X) \xrightarrow{p^l/I_p} E(B\Sigma_p)/I \otimes_E E(X) \xrightarrow{H \otimes 1} D_{\infty} \otimes_E E(X).
\]
He then defines the \( p^k \)th Hecke operator
\[
T_{p^k} = \sum_{H \subset T \mid |H| = p^k} \psi^H
\]
and shows that this lands in \( E(X) \).

3. A canonical representative of the Frobenius class

We construct a canonical representative of the set \([\bar{\sigma}]\). The construction is an elementary application of several of the tools presented in the previous section.

We specialize the transfers of the previous section to \( G = \Sigma_p \). Let
\[
\text{Tr}_E: E(B\Sigma_p) \rightarrow E
\]
be the transfer from \( \Sigma_p \) to the trivial group and let
\[
\text{Tr}_{C_0}: \text{Cl}_n(\Sigma_p, C_0) \rightarrow C_0
\]
be the transfer in class functions from \( \Sigma_p \) to the trivial group. This is given by the formula
\[
\text{Tr}_{C_0}(f) = \frac{1}{p!} \sum_{[\alpha]} f([\alpha]).
\]
Recall that \( T = (\mathbb{Q}_p/\mathbb{Z}_p)^n \) and let \( \text{Sub}_p(T) \) be the set of subgroups of order \( p \) in \( T \).

**Lemma 3.1.** [Mar] Section 4.3.6] The restriction map along \( \mathbb{Z}/p \subseteq \Sigma_p \) induces an isomorphism
\[
E(B\Sigma_p) \cong E(B\mathbb{Z}/p)^{\text{Aut}(\mathbb{Z}/p)}.
\]
After a choice of coordinate \( x \),
\[
E(B\Sigma_p) \cong E[y]/(yf(y)),
\]
where the degree of \( f(y) \) is
\[
|\text{Sub}_p(T)| = \frac{p^n - 1}{p - 1} = \sum_{i=0}^{n-1} p^i,
\]
\( f(0) = p \), and \( y \) maps to \( x^{p-1} \) in \( E(B\mathbb{Z}/p) \cong E[x]/[p](x) \).

**Lemma 3.2.** [Qui71] Proposition 4.2] After choosing a coordinate, there is an isomorphism
\[
E(B\Sigma_p)/I \cong E[y]/(f(y)),
\]
and the ring is free of rank \( |\text{Sub}_p(T)| \) as an \( E \)-module.

After choosing a coordinate, the restriction map \( E(B\Sigma_p) \rightarrow E \) sends \( y \) to 0 and the map
\[
E(B\Sigma_p) \rightarrow E(B\Sigma_p)/I
\]
is the quotient by the ideal generated by \( f(y) \).

**Lemma 3.3.** The index of the \( E \)-module \( E(B\Sigma_p) \) inside \( E \times E(B\Sigma_p)/I \) is \( p \).
Proof. This can be seen using the coordinate. There is a basis of \( E(B\Sigma_p) \) given by the set \( \{ 1, y, \ldots, y^m \} \), where \( m = |\text{Sub}_p(T)| \), and a basis of \( E \times E(B\Sigma_p)/I \) given by 
\[
\{(1,0), (0,1), (0,y), \ldots, (0,y^{m-1})\}.
\]
By Lemma 3.1 the image of the elements \( \{1, y, \ldots, y^{m-1}, p - f(y)\} \) in \( E(B\Sigma_p) \) is the set 
\[
\{(1,1), (0,y), \ldots, (0,y^{m-1}), (0,p)\}
\]
in \( E \times E(B\Sigma_p)/I \). The image of \( y^m \) is in the span of these elements and the submodule generated by these elements has index \( p \).

\[\Box\]

Lemma 3.4. \([\text{Rez09}, \text{Section 10.3}]\) In terms of a coordinate, the Frobenius class \( \bar{\sigma}: E(B\Sigma_p)/I \rightarrow E/p \) is the quotient by the ideal \((y)\).

Now we modify \( \text{Tr}_{C_0} \) to construct a map \( \sigma_{\text{can}}: E(B\Sigma_p)/I \rightarrow E \).

By Ganter’s result \([\text{Gan06}, \text{Section 7.3}]\) and the fact that \( \Sigma_p \) is good, the restriction of \( \text{Tr}_{C_0} \) to \( E(B\Sigma_p) \) is equal to \( \text{Tr}_E \). It makes sense to restrict \( \text{Tr}_{C_0} \) to 
\[
E \times E(B\Sigma_p)/I \subset \text{Cl}_n(\Sigma_p, C_0).
\]
Lemma 3.3 implies that this lands in \( \frac{1}{p^2} E \). Thus we see that the target of the map 
\[
p! \text{Tr}_{C_0} \mid_{E \times E(B\Sigma_p)/I}/I
\]
can be taken to be \( E \). We may further restrict this map to the subring \( E(B\Sigma_p)/I \) to get 
\[
p! \text{Tr}_{C_0} \mid_{E(B\Sigma_p)/I} : E(B\Sigma_p)/I \rightarrow E.
\]
From the formula for \( \text{Tr}_{C_0} \), for \( e \in E \subset E(B\Sigma_p)/I \), we have 
\[
p! \text{Tr}_{C_0} \mid_{E(B\Sigma_p)/I}(e) = |\text{Sub}_p(T)|e.
\]
Note that \( |\text{Sub}_p(T)| \) is congruent to 1 mod \( p \) (and therefore a \( p \)-adic unit). We set 
\[
\sigma_{\text{can}} = p! \text{Tr}_{C_0} \mid_{E(B\Sigma_p)/I}.
\]

Remark 3.5. One may also normalize \( \sigma_{\text{can}} \) by dividing by \( |\text{Sub}_p(T)| \) so that \( e \) is sent to \( e \).

We now show that \( \sigma_{\text{can}} \) fits in the diagram
\[
\begin{array}{c}
\text{E} \\
\sigma_{\text{can}}
\end{array}
\]
\[
\begin{array}{c}
E(B\Sigma_p)/I \\
\text{E}/p \uparrow
\end{array}
\]
where \( \bar{\sigma} \) picks out the kernel of the relative Frobenius.

Proposition 3.6. The map 
\[
\sigma_{\text{can}}: E(B\Sigma_p)/I \rightarrow E
\]
is a representative of Rezk’s Frobenius class.
**Proof.** We may be explicit. Choose a coordinate so that the quotient map
\[ q: E(B\Sigma_p) \longrightarrow E(B\Sigma_p)/I \]
is given by
\[ q: E[y]/(yf(y)) \longrightarrow E[y]/(f(y)). \]
We must show that
\[ E(B\Sigma_p)/I \xrightarrow{\sigma_{can}} E \mod p \xrightarrow{} E/p \]
is the quotient by the ideal \((y) \subset E(B\Sigma_p)/I\).

There is a basis of \(E(B\Sigma_p)\) (as an \(E\)-module) given by \(\{1, y, \ldots, y^m\}\), where \(m = |\text{Sub}_p(T)|\). We will be careful to refer to the image of \(y^i\) in \(E(B\Sigma_p)/I\) as \(q(y^i)\). For the basis elements of the form \(y^i\), where \(i \neq 0\), the restriction map \(E(B\Sigma_p) \rightarrow E\) sends \(y^i\) to 0. Thus
\[ \text{Tr}_E(y^i) = \text{Tr}_{C_0}|_{E(B\Sigma_p)/I}(q(y^i)) \in E. \]
Now the definition of \(\sigma_{can}\) implies that \(\sigma_{can}(q(y^i))\) is divisible by \(p\). So
\[ \sigma_{can}(q(y^i)) \equiv 0 \mod p. \]
It is left to show that, for \(e\) in the image of \(E \rightarrow E(B\Sigma_p)/I\),
\[ \sigma_{can}(e) \equiv e \mod p. \]
We have already seen that
\[ p! \text{Tr}_{C_0}|_{E(B\Sigma_p)/I}(e) = |\text{Sub}_p(T)|e. \]
The result follows from the fact that \(|\text{Sub}_p(T)| \equiv 1 \mod p. \]

4. The Hecke operator congruence

We show that the \(p\)th additive power operation composed with \(\sigma_{can}\) is the \(p\)th Hecke operator. This implies that the Hecke operator satisfies a certain congruence.

The two maps in question are the composite
\[ E(X) \xrightarrow{T_p/I} E(B\Sigma_p)/I \otimes_E E(X) \xrightarrow{\sigma_{can} \otimes 1} E(X) \]
and the Hecke operator \(T_p\) described in Section 2.

**Proposition 4.1.** The \(p\)th additive power operation composed with the canonical representative of the Frobenius class is equal to the \(p\)th Hecke operator:
\[ (\sigma_{can} \otimes 1)(P_p/I) = T_p. \]

**Proof.** This follows in a straight-forward way from the definitions. Unwrapping the definition of the character map, the map \(\sigma_{can}\) is the sum of a collection of maps
\[ E(B\Sigma_p)/I \longrightarrow C_0, \]
one for each subgroup of order \(p\) in \(T\). These are the maps induced by the canonical isomorphism
\[ C_0 \otimes \text{Sub}_p(G_E) \cong \text{Sub}_p(T). \]
In other words, they classify the subgroups of order \(p\) in \(T. \)
Since $\sigma_{\text{can}} \in \overline{\sigma}$, the following diagram commutes
\[
\begin{array}{cccccc}
E(X) & \longrightarrow & E(B\Sigma_2) & \otimes & E(X) & \longrightarrow & E(B\Sigma_2)/I \otimes_E E(X) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
E(X) & \longrightarrow & E(X)/p
\end{array}
\]
and this implies that
\[(\sigma_{\text{can}} \otimes 1)(P_{/I})(x) \equiv x^p \mod p.
\]

**Corollary 4.2.** For $x \in E(X)$, there is a congruence
\[T_p(x) \equiv x^p \mod p.
\]

Let $X$ be a space with the property that $E(X)$ is torsion-free. The corollary above implies the existence of a canonical function
\[\theta: E(X) \longrightarrow E(X)
\]
such that
\[T_p(x) = x^p + p\theta(x).
\]

**Example 4.3.** When $n = 1$, $G_E$ is a height 1 formal group,
\[E(B\Sigma_2)/I
\]
is a rank one $E$-module, and $\sigma_{\text{can}}$ is an $E$-algebra isomorphism. The composite
\[E(X) \xrightarrow{\sigma_{\text{can}} \otimes 1} E(X)
\]
is the $p$th unstable Adams operation. In this situation, the function $\theta$ is understood by work of Bousfield [Bou96].

**Example 4.4.** At arbitrary height, we may consider the effect of $T_p$ on $z \in \mathbb{Z}_p \subset E$. Since $T_p$ is a sum of ring maps
\[T_p(z) = |\text{Sub}_p(T)|z.
\]
This is congruent to $z^p \mod p$.

**Example 4.5.** At height 2 and the prime 2, Rezk constructed an $E$-theory associated to a certain elliptic curve [Rezi]. He calculated $P_2/I$, when $X = \ast$. He found that, after choosing a particular coordinate $x$,
\[E(B\Sigma_2)/I \cong \mathbb{Z}_2[u_1][x]/(x^3 - u_1x - 2)
\]
and
\[P_2/I: \mathbb{Z}_2[u_1] \longrightarrow \mathbb{Z}_2[u_1][x]/(x^3 - u_1x - 2)
\]
sends $u_1 \mapsto u_1^2 + 3x - u_1x^2$. In [Drin74 Section 4B], Drinfeld explains how to compute the ring that corepresents $\mathbb{Z}/2 \times \mathbb{Z}/2$-level structures. Note that in the ring
\[\mathbb{Z}_2[u_1][y, z]/(y^3 - u_1y - 2),
\]
y is a root of $z^3 - u_1z - 2$ and
\[
\frac{z^3 - u_1z - 2}{z - y} = z^2 + yz + y^2 - u_1.
\]
Drinfeld’s construction gives
\[D_1 = \Gamma \text{Level}(\mathbb{Z}/2 \times \mathbb{Z}/2, G_E) \cong \mathbb{Z}_2[u_1][y, z]/(y^3 - u_1y - 2, z^2 + yz + y^2 - u_1).\]
The point of this construction is that $x^3 - u_1 x - 2$ factors into linear terms over this ring. In fact,

$$x^3 - u_1 x - 2 = (x - y)(x - z)(x + y + z).$$

The three maps $E(B\Sigma_2)/I \to D_1 \subset C_0$ that show up in the character map are given by sending $x$ to these roots. A calculation shows that

$$\sigma_{can}(x) = 0$$

and that

$$T_p(u_1) = (\sigma_{can} \otimes 1)(P_2/I)(u_1) = u_1^2.$$