\[ d = 2, \ N = 2 \]

Superconformally Covariant Operators
and
Super \( W \)-Algebras

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Abstract

We construct and classify superconformally covariant differential operators defined on \( N = 2 \) super Riemann surfaces. By contrast to the \( N = 1 \) theory, these operators give rise to partial rather than ordinary differential equations which leads to novel features for their matrix representation. The latter is applied to the derivation of \( N = 2 \) super \( W \)-algebras in terms of \( N = 2 \) superfields.

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1 Introduction

From reference [1], we recall that the Bol operator $L_n$ (acting on a compact Riemann surface with local coordinates $z$) is the conformally covariant version of the differential operator $\partial^n \equiv \left( \frac{\partial}{\partial z} \right)^n$. The simplest examples are given by

\[
\begin{align*}
L_0 &= 1 \\
L_1 &= \partial \\
L_2 &= \partial^2 + \frac{1}{2} R \\
L_3 &= \partial^3 + 2 R \partial + (\partial R) \\
L_4 &= \partial^4 + 5 R \partial^2 + 5 (\partial R) \partial + \frac{3}{2} \left( (\partial^2 R) + \frac{3}{2} R^2 \right),
\end{align*}
\]

where $R = R_{zz}(z)$ denotes a projective connection, i.e. a locally holomorphic field transforming with the Schwarzian derivative under a conformal change of coordinates $z \rightarrow z'(z)$:

\[
R'(z') = (\partial z')^{-2} [R(z) - S(z', z)] \quad \text{where} \quad S(z', z) = \partial^2 \ln \partial z' - \frac{1}{2} (\partial \ln \partial z')^2.
\]

The most general conformally covariant operator of order $n$ whose highest coefficient is normalized to one can be written as a sum of $L_n$ and some lower order operators $M^{(n)}_{w_k}$ depending linearly on some conformal fields $w_k$ (with $k = 3, \ldots, n$), e.g.

\[
\begin{align*}
M^{(n)}_{w_n} &= w_n, \\
M^{(n)}_{w_{n-1}} &= w_{n-1} \partial + \frac{1}{2} (\partial w_{n-1}) \\
M^{(n)}_{w_{n-2}} &= w_{n-2} \left[ \partial^2 - \frac{1-n}{2} R \right] + (\partial w_{n-2}) \partial + \frac{n-1}{2(2n-3)} \left[ \partial^2 - (n-2) R \right] w_{n-2}.
\end{align*}
\]

The operators $L_n$ and $M^{(n)}_{w_k}$ can both be obtained from Möbius covariant operators and they admit a matrix representation which is related to the principal embedding of $sl(2)$ into $sl(n)$ [2]. All of these covariant operators admit numerous applications to conformal and integrable models, in particular to the theory of $W$-algebras. Their $N = 1$ supersymmetric generalization has been worked out in references [1, 3] (see also [4]) and the present paper is devoted to their $N = 2$ supersymmetric generalization.

There is a new feature in $N = 2$ superspace geometry which makes this theory considerably richer and more complicated: the “square root” of the translation generator $\partial$ is not given by a single odd operator $D_1$ as in $N = 1$ supersymmetry ($D_1^2 = \partial$), but it involves two odd operators ($\{D, \bar{D}\} = \partial$). Therefore, one has to deal with partial differential equations (involving $D$ and $\bar{D}$) rather than ordinary differential equations (only involving $D_1$). Another aspect of the algebra $\{D, \bar{D}\} = \partial$ consists of the fact that it introduces a $U(1)$ symmetry into the theory: after projection from the super Riemann surface to the underlying ordinary Riemann surface, one thereby recovers $U(1)$-transformations in addition to the familiar conformal transformations. Henceforth, the operators (1)(3) acting on
U(1)-neutral fields are to be generalized to conformally covariant operators acting on U(1)-charged fields. The latter as well as the original operators (1) arise from different classes of $N = 2$ superdifferential operators. In particular, one is led to introduce operators which relate the chiral and anti-chiral subspaces of superconformal fields. It is only for these so-called ‘sandwich operators’ that we will be able to give a matrix representation. The latter provides a gauge connection field with values in the Lie superalgebra $sl(n+1|n)$ and the associated $N = 2$ super $W$-algebra can be constructed by imposing zero curvature conditions on this connection.

To keep supersymmetry manifest, all of our considerations will be carried out in superspace while the projection to ordinary space will be indicated in each case. The passage from the $N = 2$ to the $N = 1$ superfield formalism will also be discussed so as to make contact with previous work which mostly uses the latter description. Applications of our results will be mentioned in the text and in the concluding remarks.

2 Geometric framework

In this section, we recall some definitions (following and references therein) and we introduce some tools which are needed in the sequel.

2.1 Notation and basic relations

We will work on a compact $N = 2$ super Riemann surface (SRS) locally parametrized by coordinates

$$(z; \bar{z}) \equiv (z, \theta, \bar{\theta}; z, \theta, \bar{\theta} + \bar{\theta} - \theta, \bar{\theta} - \theta)$$

with $z, \bar{z}$ even and $\theta, \bar{\theta}, \theta - \bar{\theta}, \bar{\theta} - \theta$ odd. The variables are complex and related by complex conjugation (denoted by $^*$):

$$z^* = \bar{z} \ , \ (\theta^*)^* = \theta^- \ , \ (\bar{\theta}^*)^* = \bar{\theta}^- \ .$$

As indicated in (4), we will omit the plus-indices of $\theta^+$ and $\bar{\theta}^+$ to simplify the notation.

The canonical basis of the tangent space is defined by $(\partial, D, \bar{D}; \bar{\partial}, D_-, \bar{D}_-)$ with

$$\partial = \frac{\partial}{\partial z} \ , \ D = \frac{\partial}{\partial \theta} + \frac{1}{2} \bar{\theta} \partial \ , \ \bar{D} = \frac{\partial}{\partial \bar{\theta}} + \frac{1}{2} \theta \partial \ \text{and c.c.} \ .$$

The graded Lie brackets between these vector fields are given by

$$\{D, \bar{D}\} = \partial \ \text{and c.c.} \ ,$$

all others brackets being zero, in particular,

$$D^2 = 0 = \bar{D}^2 \ \text{and c.c.} \ .$$

Note that this set of equations implies $[D, \bar{D}]^2 = \partial^2$ and c.c. \ .
2.2 Superconformal transformations

By definition of the SRS, any two sets of local coordinates, say \( z \) and \( z' \), are related by a superconformal transformation, i.e. a smooth mapping satisfying the three conditions

\[
\begin{align*}
(i) & \quad z' = z'(z) \iff D_z z' = 0 = \bar{D}_z z' \\
(ii) & \quad \bar{D} \theta' = 0 = \bar{D} \theta' \\
(iii) & \quad D z' = \frac{1}{2} \theta' \bar{D} \theta' , \quad \bar{D} z' = \frac{1}{2} \bar{\theta}' D \bar{\theta}' .
\end{align*}
\]

These relations imply that \( D \) and \( \bar{D} \) separately transform into themselves,

\[
\begin{align*}
D' &= e^w D \\
\bar{D}' &= e^{\bar{w}} D \\
\partial' &= \{ D', \bar{D}' \} = e^{w+\bar{w}} \left[ \partial + (\bar{D} w) D + (D \bar{w}) \bar{D} \right] ,
\end{align*}
\]

where

\[
\begin{align*}
e^{-w} & \equiv D \theta' , \quad Dw = 0 \\
e^{-\bar{w}} & \equiv \bar{D} \bar{\theta}' , \quad \bar{D} \bar{w} = 0
\end{align*}
\]

and

\[
e^{-w-\bar{w}} = \partial z' + \frac{1}{2} \theta' \partial \theta' + \frac{1}{2} \theta' \partial \bar{\theta}' .
\]

We note that eqs. (10) imply \((D \theta')(D' \theta) = 1 = (\bar{D} \bar{\theta'})(\bar{D}' \bar{\theta})\) and that analogous equations hold in the \( \bar{z} \)-sector.

2.3 \( U(1) \) symmetry

Equation (10) determines the product of \( D \theta' \) and \( \bar{D} \bar{\theta}' \), but not their quotient which is related to the \( U(1) \) automorphism group of the \( N = 2 \) supersymmetry algebra. To introduce \( U(1) \)-transformations parametrized by a superanalytic superfield \( K \), we decompose equation (11) in a symmetric way,

\[
\begin{align*}
D \theta' &= e^{+K/2} \left( \partial z' + \frac{1}{2} \theta' \partial \theta' + \frac{1}{2} \bar{\theta}' \partial \bar{\theta}' \right)^{1/2} \\
\bar{D} \bar{\theta}' &= e^{-K/2} \left( \partial z' + \frac{1}{2} \bar{\theta}' \partial \theta' + \frac{1}{2} \theta' \partial \bar{\theta}' \right)^{1/2} ,
\end{align*}
\]

which implies \( K = \ln \left( \frac{D \theta' / \bar{D} \bar{\theta}'}{D \theta' / \bar{D} \bar{\theta}'} \right) \).
2.4 Superconformal fields

In the following, we will consider superconformal fields $C_{p,q}(z,\bar{z})$ transforming like

$$C'_{p,q} = e^{pw+q\bar{w}}C_{p,q} \quad (p, q \in \mathbb{Z}/2, \ p + q \in \mathbb{Z})$$

(13)

and having a Grassmann parity $(-)^{p+q}$. The space of these fields is denoted by $F_{p,q}$. The pair $(p, q)$ will be called the superconformal weight of $C_{p,q}$ while $1\over 2(p+q)$ and $1\over 2(p-q)$ will be referred to as its conformal and $U(1)$ weight (or charge), respectively – see eq.(17) below for the origin of this terminology. Thus, $C_{p,q}$ is neutral with respect to $U(1)$ if $p = q$.

Fields with another index structure (transforming also with $w^*,\bar{w}^*$) can be defined in an analogous way.

From the transformation properties of derivatives and fields, one concludes that the following chirality conditions are superconformally covariant:

$$DC_{p,0} = 0 \text{ or } \bar{D}C_{0,p} = 0.$$ (14)

The space of these so-called chiral and anti-chiral fields will be denoted by $F^c_{p,0}$ and $F^a_{0,p}$, respectively.

Note that the quadratic constraints $D\bar{D}C_{0,-1} = 0$ and $\bar{D}DC_{-1,0} = 0$ are also covariant.

2.5 Projection to component fields

A generic $N = 2$ superfield admits the $\theta$-expansion

$$C(z,\bar{z}) = c + \theta\psi + \bar{\theta}\bar{\eta} + \theta\bar{\theta}d + \theta^{-}[\gamma + \theta e + \bar{\theta}f + \theta\bar{\theta}\delta]$$

$$+\bar{\theta}^{-}[e + \theta g + \bar{\theta}h + \theta\bar{\theta}\tau] + \theta^{-}\bar{\theta}^{-}[m + \theta\varphi + \bar{\theta}\chi + \theta\bar{\theta}n],$$

where the component fields $c,...,n$ depend on $z$ and $\bar{z}$. Equivalently, these space-time fields can be introduced by means of projection,

$$C| = c, \quad DC| = \psi, \quad \bar{D}C| = \bar{\eta}, \quad [D,\bar{D}]C| = -2d, \quad ...,$$

where the bar denotes the projection onto the lowest component of the corresponding superfield.

Ordinary conformal and $U(1)$-transformations are related to the expressions

$$(w + \bar{w})| = -\ln \partial z'$$

$$(w - \bar{w})| = -\ln \frac{D\theta'}{D\theta} | = -K| \equiv -k.$$ (16)

Here and in all following space-time equations, $z'$ stands for the lowest component of the superfield $z'\equiv z'(z)$ and the fermionic contributions are never spelled out.

If $C \equiv C_{p,q}$ belongs to $F_{p,q}$, then its transformation law (13) can be projected down to ordinary space by taking into account eqs.(12)(16):

$$c'_{p,q} = (\partial z')^{-\frac{1}{2}(p+q)} e^{-\frac{1}{2}(p-q)k} c_{p,q}.$$ (17)
Obviously, $\frac{1}{2}(p+q)$ (resp. $\frac{1}{2}(p-q)$) is the conformal (resp. $U(1)$) weight of the ordinary conformal field $c_{p,q}(z,\bar{z})$.

Whereas $C_{p,q}$ transforms homogeneously under conformal and $U(1)$-transformations, the (bosonic) space-time field $[D,\bar{D}]C_{p,q}$ does not unless $p = 0 = q$. In fact, $[D,\bar{D}]C_{p,q}$ is not a superconformal field. The remedy to this problem consists of adding an appropriate term to the projection operator $[D,\bar{D}]$ which eliminates the unwanted contribution in the transformation law of $[D,\bar{D}]C_{p,q}$ - see section 3.3.

2.6 From $N=2$ to $N=1$ formalism

Since $N=2$ theories are often formulated in terms of $N=1$ superfields, we will briefly outline how the latter are extracted from $N=2$ superfields.

A generic $N=2$ superfield (15) can be rewritten in terms of the odd coordinates

$$\theta_1 = (\theta + \bar{\theta})/2 \quad , \quad \theta_2 = (\theta - \bar{\theta})/2$$

and the complex conjugate variables $\bar{\theta}_1 = (\theta_1)^*, \bar{\theta}_2 = (\theta_2)^*$ which give rise to the $N=1$ derivative

$$D_1 \equiv \frac{\partial}{\partial \theta_1} + \theta_1 \partial = D + \bar{D} \quad , \quad (D_1)^2 = \partial \quad \text{(and c.c.)} .$$

Indeed, one finds

$$C(z, \bar{z}, \theta_1, \bar{\theta}_1, \theta_2, \bar{\theta}_2) = a + \theta_2 \alpha + \bar{\theta}_2 \beta + \theta_2 \bar{\theta}_2 b ,$$

where $a, \alpha, \beta, b$ represent $N=1$ superfields depending on the variables $z, \bar{z}, \theta_1, \bar{\theta}_1$. If $C$ is even, then $a, b (\alpha, \beta)$ are even (odd).

3 Derivatives, connections and covariant operators

Along the lines of the $N=1$ theory [1], we now introduce the necessary ingredients for the construction and description of covariant operators.

3.1 Schwarzian derivative

The super Schwarzian derivative [2, 4] associated to a superconformal change of coordinates $z \rightarrow z'(z)$ is defined by

$$- S(z', z) = 2 e^{-\frac{i}{2}(w + \bar{w})} [D, \bar{D}] e^{\frac{i}{2}(w + \bar{w})}$$

$$= [D, \bar{D}](w + \bar{w}) + D(w + \bar{w})\bar{D}(w + \bar{w})$$

$$= \partial(w - \bar{w}) + (\bar{D}w)(D\bar{w})$$

$$= \frac{\partial D\theta'}{D\theta'} - \frac{\partial D\theta'}{D\theta'} + \frac{\partial \bar{\theta}'}{D\theta'} \frac{\partial \theta'}{D\theta'} .$$

It satisfies the ‘chain rule’

$$S(z'', z) = e^{-(w + \bar{w})} S(z'', z') + S(z', z) .$$

(22)
By using equations (21) and (22), we can extract the bosonic component fields of $S$:

$$S(z', z) = \partial k$$

$$[D, \bar{D}]S(z', z) = S(z', z) + \frac{1}{2} (\partial k)^2.$$  

Here, $S(z', z)$ denotes the ordinary Schwarzian derivative (2): it represents the ‘conformal part’ of $S$ while $\partial k$ represents its ‘$U(1)$ part’.

### 3.2 Projective coordinates

A superprojective (super M"obius) mapping is a superconformal change of coordinates $Z = (z, \Theta, \bar{\Theta}) \to Z'(Z)$ satisfying $S(Z', Z) = 0$. By virtue of the definition (21), this condition is equivalent to the relation

$$(D_\Theta Z')(D_{\bar{\Theta}} Z') = (cZ + d + \Theta \bar{\gamma} + \bar{\Theta} \gamma)^{-2},$$  

where $c$ and $d$ ($\gamma$ and $\bar{\gamma}$) are Grassmann even (odd) constants. The latter relation can be integrated so as to obtain the general expression of a superprojective mapping $Z \to Z'$: by taking into account the conditions $D_\Theta Z' = 0 = D_{\bar{\Theta}} Z'$ and $D_\Theta Z' = \frac{1}{2} \Theta' D_\Theta Z'$, one finds

$$\Theta'(Z) = e^{+h} \left\{ -2 \frac{\alpha Z + \beta}{cZ + d} + \frac{\Theta}{cZ + d} \left[ 1 + 2 \frac{\bar{\alpha} Z + \bar{\beta}}{cZ + d} \right] - \frac{\Theta \bar{\Theta}}{(cZ + d)^2} \gamma \right\},$$

$$\bar{\Theta}'(Z) = e^{-\bar{h}} \left\{ -2 \frac{\bar{\alpha} Z + \bar{\beta}}{cZ + d} + \frac{\bar{\Theta}}{cZ + d} \left[ 1 - 2 \frac{\alpha Z + \beta}{cZ + d} \right] + \frac{\Theta \bar{\Theta}}{(cZ + d)^2} \bar{\gamma} \right\},$$

$$Z'(Z) = \frac{aZ + b}{cZ + d} \left[ 1 + 2(\alpha \bar{\beta} + \bar{\alpha} \beta) + 8\alpha \bar{\alpha} \beta \bar{\beta} \right]$$

$$- \frac{\Theta \bar{\Theta}}{(cZ + d)^2} \left[ 1 + 2(\alpha \bar{\beta} + \bar{\alpha} \beta) \right] + \Theta \bar{\Theta} \partial \left[ \frac{\alpha Z + \beta}{cZ + d} \frac{\bar{\alpha} Z + \bar{\beta}}{cZ + d} \right].$$

Here $h, \bar{h}, a, b, (\alpha, \beta, \bar{\alpha}, \bar{\beta})$ are Grassmann even (odd) constants which satisfy the relations

$$\alpha d - \beta c = \gamma,$$

$$\bar{\alpha} d - \bar{\beta} c = \bar{\gamma},$$

$$ad - \beta c = 1, \quad 2(\alpha \bar{\beta} + \bar{\alpha} \beta) = h - \bar{h}.$$  

The lowest order component of $Z'(Z)$ represents an ordinary projective transformation: $Z' = (aZ + b)(cZ + d)^{-1}$.

A superprojective structure on a SRS is an atlas of local coordinates for which all changes of charts $Z \to Z'$ are superprojective mappings.

A quasi-primary superfield $C_{p,q}(Z, \bar{Z})$ of superconformal weight $(p, q)$ transforms according to the rule

$$C_{p,q}'(Z', \bar{Z}') = (cZ + d + \Theta \bar{\gamma} + \bar{\Theta} \gamma)^{p+q} e^{-(p-q)H} C_{p,q}(Z, \bar{Z})$$

with

$$H(Z) = h + \frac{1}{cZ + d} \left[ 2(\bar{\alpha} Z + \bar{\beta}) + \Theta \bar{\gamma} - \bar{\Theta} \gamma + \frac{1}{2} \Theta \bar{\Theta} c \right]$$

under a superprojective change of coordinates $Z \to Z'$. 

7
3.3 Projective connection

A superprojective (super Schwarzian) connection on a SRS is a collection $\mathcal{R} \equiv \mathcal{R}_{\theta \bar{\theta}}(z)$ of superfields (one for each coordinate patch) which are locally superanalytic (i.e. $D_\theta \mathcal{R} = 0 = \bar{D}_{\bar{\theta}} \mathcal{R}$) and which transform under a superconformal change of coordinates according to

$$\mathcal{R}'(z') = e^{w + \bar{w}} \left[ \mathcal{R}(z) - S(z', z) \right].$$

(28)

The existence of such connections on compact SRS’s of arbitrary genus can be proven along the lines of the $\mathcal{N} = 0$ and $\mathcal{N} = 1$ theories [1].

If $Z$ belongs to a projective atlas, then the super Schwarzian derivative of the super-conformal mapping $z \to Z$ represents a projective connection,

$$\mathcal{R}(z) = S(Z, z).$$

(29)

In fact, equation (22) then implies the transformation law (28) and it ensures that the expression (29) is inert under a projective change of coordinates $Z \to Z'$. The relation (29) establishes the equivalence between projective structures and projective connections.

The superfield $\mathcal{R}$ admits the $\theta$-expansion

$$\mathcal{R}_{\theta \bar{\theta}}(z) = \rho_z + \theta \eta_{\theta z} + \bar{\theta} \bar{\eta}_{\bar{\theta} z} + \theta \bar{\theta} \left[ -\frac{1}{2} r_{zz} \right].$$

The component fields $\rho = \mathcal{R}|_{\theta = 0}, ..., r = [D, \bar{D}] \mathcal{R}$ are locally holomorphic $(0 = \bar{\partial} \rho = ... = \bar{\partial} r)$ and the bosonic components have the transformation laws

$$\rho' = (\partial z')^{-1} \left[ \rho - \partial k \right]$$
$$r' = (\partial z')^{-2} \left[ r - S(\rho - \frac{1}{2} \partial k) \partial k \right].$$

(30)

From these equations, we conclude that the quantity $R = r - \frac{1}{2} \rho^2$ transforms like ($\mathbb{Z}$), i.e. like an ordinary projective connection while $\rho$ represents a $U(1)$-connection.

Using a superprojective connection, we can define the second order operator

$$\mathcal{D}_{p, q} = p \bar{D} D - q D \bar{D} + pq \mathcal{R}$$

(31)

which projects out an ordinary conformal field from the superconformal field $C_{p, q}$:

$$\left( \mathcal{D}_{p, q} C_{p, q} \right)' = e^{-\frac{1}{2}(p-q)} (\partial z')^{-\frac{1}{2}(p+q+2)} \left( \mathcal{D}_{p, q} C_{p, q} \right).$$

3.4 Affine connection, covariant derivative and Miura transformation

A superaffine connection on a SRS is a collection $B \equiv B_{\theta}(z), \bar{B} \equiv \bar{B}_{\bar{\theta}}(z)$ of superfields (one for each coordinate patch) which are locally superanalytic, which satisfy the chirality conditions $DB = 0 = \bar{D}\bar{B}$ and which transform under a superconformal change of coordinates
The only compact SRS’s which admit a globally defined affine connection are those of genus one \([1]\). Nevertheless, these quantities can always be introduced locally and they represent extremely useful computational tools for deriving globally well-defined results.

Using an affine connection, we can introduce \textit{supercovariant derivatives}

\[
\nabla_{p,q} = D - qB : \ F_{p.q} \rightarrow F_{p+1,q} \\
\tilde{\nabla}_{p,q} = \tilde{D} - p\tilde{B} : \ F_{p,q} \rightarrow F_{p,q+1} .
\]

Products of covariant derivatives are defined by taking into account the weights, e.g.

\[
\nabla_\nabla_\nabla C_{p,q} = \nabla_{p+1,q+1} \nabla_{p+1,q} \nabla_{p,q} C_{p,q} .
\]

From the very definitions, it then follows that

\[
\nabla^2 = 0 = \nabla^2 .
\]

Locally, we can express \(B\) and \(\tilde{B}\) as

\[
B = DQ \quad , \quad \tilde{B} = \tilde{D}Q
\]

with

\[
Q = \ln \left( D\Theta \tilde{D}\Theta \right) = \ln \left( \partial Z + \frac{1}{2} \Theta \partial \tilde{\Theta} + \frac{1}{2} \tilde{\Theta} \partial \Theta \right) ,
\]

where \(\Theta\) and \(\tilde{\Theta}\) belong to an atlas of superprojective coordinates \(Z\). Then, we have the operatorial relations

\[
\nabla_{p,q} = (\tilde{D}\Theta)^q \cdot D \cdot (\tilde{D}\Theta)^{-q} \\
\tilde{\nabla}_{p,q} = (D\Theta)^p \cdot \tilde{D} \cdot (D\Theta)^{-p} .
\]

Affine and projective connections are related by the \textit{super Miura transformation}

\[
R = D\tilde{B} - \tilde{D}B - B\tilde{B} \\
= [D, \tilde{D}]Q - (DQ)(\tilde{D}Q) .
\]

This formula follows from the expression of the projective connection in terms of the Schwarzian derivative (i.e equation (29)) or, equivalently, by comparing the expressions (52) and (54) below for the basic conformally covariant operator \(L^\text{sym}_1\).
From eq. (38) we can determine some explicit expressions for the component fields $[D, \bar{D}] R|_r = r = R + \frac{1}{2} \rho^2$ and $R| = \rho$. In fact, by taking into account eq. (36), one recovers the ordinary Miura transformation for the projective connection $R$ and an explicit expression for $\rho$:

$$R = \partial^2 \ln \partial Z| - \frac{1}{2} (\partial \ln \partial Z)^2$$

$$\rho = ([D, \bar{D}] \ln \partial Z)| .$$

Furthermore, from equations (35) and (36), we conclude that

$$b \equiv DB| = \frac{1}{2} (\partial \ln \partial Z| - \rho)$$

$$\bar{b} \equiv D\bar{B}| = \frac{1}{2} (\partial \ln \partial Z| + \rho) .$$

From these expressions, we obtain the conformally and $U(1)$-covariant space-time derivative

$$\{\nabla, \bar{\nabla}\} C_{p,q} = \left[ \tilde{\partial}_{p,q} - \frac{p+q}{2} (\partial \ln \partial Z)| \right] c_{p,q} \quad \text{with} \quad \tilde{\partial}_{p,q} \equiv \partial - \frac{p-q}{2} \rho .$$

Here, $\tilde{\partial}_{p,q}$ denotes the $U(1)$-covariant derivative associated to the $U(1)$-connection $\rho$ (cf. eq. (30)) and the local expression $\partial \ln \partial Z|$ transforms like an affine connection with respect to a conformal change of coordinates $z \to z'(z)$ [1].

### 3.5 Determination of conformally covariant operators

We are interested in superconformally covariant differential operators which are globally defined on compact SRS’s of any genus. In order to construct these quantities, we start from an operator $L$ which is a polynomial in the covariant derivatives $\nabla, \bar{\nabla}$ and require that it only depends on the affine connections $B, \bar{B}$ through the particular combination $\mathcal{R} = DB - \bar{D}B - B\bar{B}$ (which is globally defined): using a variational argument [8, 2], one imposes $\delta L = 0$ while varying $B, \bar{B}$ subject to the condition that $\mathcal{R}$ is fixed:

$$0 = \delta \mathcal{R} = D\delta \bar{B} - \bar{D}\delta B - \delta B\bar{B} - B\delta \bar{B} .$$

The latter relation can be rewritten as

$$\nabla \delta \bar{B} = \bar{\nabla} \delta B ,$$

and the nilpotency of $\nabla, \bar{\nabla}$ then yields

$$\bar{\nabla} \nabla \delta \bar{B} = 0 = \nabla \bar{\nabla} \delta B ,$$

while the chirality properties of $B, \bar{B}$ imply

$$\nabla \delta B = 0 = \bar{\nabla} \delta \bar{B} .$$
There are two basic classes of covariant differential operators: besides the super Bol operators (which only depend on a projective structure or equivalently on a projective connection), one can introduce operators which also depend linearly on a conformal field. These two classes will be determined in sections 4 and 5, respectively, and at the end of section 5, we will show that the most general covariant operator can be written as a sum of these.

4 Super Bol operators

We successively discuss the cases where the differential operator \( L \) contains an even and odd number of derivatives \( \nabla, \bar{\nabla} \). All operators discussed in this section are supposed to be normalized in the sense that the coefficients of the highest order derivatives are constant.

4.1 Even number of derivatives

The general form of a normalized covariant operator involving an even number of derivatives is

\[
L_{n}^{\text{even}} = \alpha (\nabla \bar{\nabla})^{n} + \beta (\bar{\nabla} \nabla)^{n} \quad (n \in \mathbb{N}^*, \ (\alpha, \beta) \neq (0, 0)) \tag{45}
\]

and such that

\[
L_{n}^{\text{even}} : \mathcal{F}_{p,q} \longrightarrow \mathcal{F}_{p+n,q+n} \quad \text{for some} \ (p, q) . \tag{46}
\]

Thus, the operator does not modify the \( U(1) \) charge of the fields. In order to simplify the notation, we have not spelled out the dependence of \( L \) on \( p \) and \( q \).

Imposing \( \delta L_{n}^{\text{even}} = 0 \) and using eqs.(42)-(44), one finds that \( p, q \) and \( n \) are related: the general solution reads

\[
L_{n}^{\text{even}} = q (\nabla \bar{\nabla})^{n} - p (\bar{\nabla} \nabla)^{n} \quad \text{with} \quad p + q = -n \tag{47}
\]

Here and in the following, the constant overall factor of \( L \) has been chosen in a convenient way. For \( n = 1 \) and \( n = 2 \), we have the explicit expressions

\[
L_{1}^{\text{even}} = qD\bar{D} - pD\bar{D} - pqR \quad \text{with} \quad p + q = -1
\]

\[
= \frac{p+q}{2} L_{1}^{\text{sym}} - \frac{p-q}{2} \left( \partial - \frac{p-q}{2} R \right) \tag{48}
\]

\[
L_{2}^{\text{even}} = \frac{p+q}{2} L_{2}^{\text{sym}} - \frac{p-q}{2} \left( \partial^{2} - R[D,\bar{D}] - 2 \frac{p-q}{2} R \partial + (D\bar{R})D - (D\bar{R})\bar{D} \right.
\]

\[
\left. - \frac{p-q}{2} \partial R - pq\bar{R}^{2} \right) \quad \text{with} \quad p + q = -2 ,
\]

where \( L_{1}^{\text{sym}} \) and \( L_{2}^{\text{sym}} \) denote the symmetric solution \((p=q)\), see eq.(42) below. Thus, the covariant operator \( L_{n}^{\text{even}} \) has a conformal part (a multiple of \( L_{n}^{\text{sym}} \)) and a \( U(1) \) part, each of which is proportional to the corresponding weight.
Projection to components is achieved by virtue of the operator \( \mathcal{D}_{p,q} \) introduced in eq. (31):

\[
( \mathcal{D}_{p+n,q+n} \mathcal{L}^{\text{even}}_{n} \mathcal{C}_{p,q} ) = pq L_{n+1}^{p,q} .
\]

(49)

Here, \( L_{n}^{p,q} \) denotes the generalization of the usual Bol operator \( L_{n} \) (depending on the ordinary projective connection \( R = r - \frac{1}{2} \rho^{2} \)) to charged conformal fields: it amounts to replacing the ordinary partial derivative (acting on \( \partial \)) and the simplest examples are

\[
\tilde{\mathcal{L}}^{\text{sym}}_{n} = 0)
\]

Obviously, the general solution (47) involves two different classes of solutions: the symmetric solution \((p = q)\) for which the operator acts on neutral fields and the asymmetric solutions \((p \neq q)\) among which we find as particular cases the chiral \((p = 0)\) and anti-chiral \((q = 0)\) solutions.

4.1.1 Symmetric solution

The symmetric solution \( \mathcal{L}^{\text{sym}}_{n} : \mathcal{F}_{-\frac{n}{2},-\frac{n}{2}} \to \mathcal{F}_{-\frac{n}{2},\frac{n}{2}} \) has the general form

\[
\mathcal{L}^{\text{sym}}_{n} = (\nabla \nabla)^{n} - (\overline{\nabla} \nabla)^{n}
\]

\[
= \bar{\partial}^{n-1} [D, \bar{D}] + \frac{n}{2} \mathcal{R} \bar{\partial}^{n-1} + ...
\]

(51)

and the simplest examples are

\[
\mathcal{L}^{\text{sym}}_{1} = [D, \bar{D}] + \frac{1}{2} \mathcal{R}
\]

\[
\mathcal{L}^{\text{sym}}_{2} = \bar{\partial}[D, \bar{D}] + \mathcal{R} \partial - (D \mathcal{R}) \bar{D} - (\bar{D} \mathcal{R}) D + (\partial \mathcal{R})
\]

\[
\mathcal{L}^{\text{sym}}_{3} = \partial^{2}[D, \bar{D}] + \frac{3}{2} \mathcal{R} \partial^{2} - 3(D \mathcal{R}) \partial \bar{D} - 3(\bar{D} \mathcal{R}) \partial D + 3(\partial \mathcal{R}) \partial
\]

\[
+ \left( \frac{1}{2}[D, \bar{D}] \mathcal{R} - \frac{1}{4} \mathcal{R}^{2} \right) [D, \bar{D}] - \left( 2 \partial D \mathcal{R} + \frac{1}{4} D \mathcal{R}^{2} \right) \bar{D} - \left( 2 \partial \bar{D} \mathcal{R} - \frac{1}{4} \bar{D} \mathcal{R}^{2} \right) D
\]

\[
+ \frac{3}{2} \left( \partial^{2} \mathcal{R} + \frac{1}{2} \mathcal{R}[D, \bar{D}] \mathcal{R} - \frac{1}{4} \mathcal{R}^{3} - 2(D \mathcal{R})(\bar{D} \mathcal{R}) \right)
\]

\[
\mathcal{L}^{\text{sym}}_{4} = \partial^{3}[D, \bar{D}] + 2 \mathcal{R} \partial^{3} - 6(\bar{D} \mathcal{R}) \partial^{2} D - 6(D \mathcal{R}) \partial^{2} \bar{D} + 6 \partial^{2} \mathcal{R} \partial^{2}
\]

\[
+ \left( 2([D, \bar{D}] \mathcal{R}) - \mathcal{R}^{2} \right) \partial [D, \bar{D}] - \left( 8 \partial \bar{D} \mathcal{R} - 2 \mathcal{R} \bar{D} \mathcal{R} \right) \partial D - \left( 8 \partial D \mathcal{R} + 2 \mathcal{R} D \mathcal{R} \right) \partial \bar{D}
\]

\[
+ (6 \partial^{2} \mathcal{R} + 4 \mathcal{R}[D, \bar{D}] \mathcal{R} - 15 D \mathcal{R} D \mathcal{R} - 2 \mathcal{R}^{3}) \partial + (\partial[D, \bar{D}] \mathcal{R} - \mathcal{R} \partial \mathcal{R}) [D, \bar{D}]
\]

\[
- \left( 3 \partial^{2} \bar{D} \mathcal{R} - \mathcal{R} \partial D \mathcal{R} - \frac{3}{2} D \mathcal{R} \partial \mathcal{R} + \frac{9}{2} \bar{D} \mathcal{R} [D, \bar{D}] \mathcal{R} - 2 \mathcal{R}^{2} D \mathcal{R} \right) D
\]

\[
- \left( 3 \partial^{2} D \mathcal{R} + \mathcal{R} \partial D \mathcal{R} + \frac{3}{2} D \mathcal{R} \partial \mathcal{R} + \frac{9}{2} D \mathcal{R} [D, \bar{D}] \mathcal{R} - 2 \mathcal{R}^{2} D \mathcal{R} \right) \bar{D}
\]

\[
+ 2 \mathcal{R} \partial [D, \bar{D}] \mathcal{R} - 10 D \mathcal{R} \partial \bar{D} \mathcal{R} + 10 \bar{D} \mathcal{R} \partial D \mathcal{R} + 4 \partial \mathcal{R} [D, \bar{D}] \mathcal{R} - 4 \mathcal{R}^{2} \partial \mathcal{R} + 2 \partial^{3} \mathcal{R},
\]
where the derivatives only act on the field to their immediate right. As indicated above, the result for \( \mathcal{L}_1^{\text{sym}} \) is nothing but the Miura transformation (38). The operator \( \mathcal{L}_n^{\text{sym}} \) has the hermiticity property

\[
\int dz d\theta d\bar{\theta} f \mathcal{L}_n^{\text{sym}} g = (-1)^{n+1} \int dz d\theta d\bar{\theta} (\mathcal{L}_n^{\text{sym}} f) g \quad \text{for } f, g \in \mathcal{F}_{-\frac{n}{2},-\frac{n}{2}}.
\]

The operators (52) play the role of Poisson (Hamiltonian) operators in the description of superintegrable models - see section 7.

### 4.1.2 Chiral and anti-chiral solutions

The anti-chiral \((q = 0)\) and chiral \((p = 0)\) solutions read

\[
\mathcal{L}_n^{\text{anti}} = (\nabla \nabla)^n = \tilde{D}(\nabla \nabla)^{n-1} D : \mathcal{F}_{-n,0} \rightarrow \mathcal{F}_{0,n} \quad (53)
\]

\[
\mathcal{L}_n^{\text{chir}} = (\nabla \nabla)^n = D(\nabla \nabla)^{n-1} \tilde{D} : \mathcal{F}_{0,-n} \rightarrow \mathcal{F}_{c,n,0}.
\]

For these operators, the target spaces are the anti-chiral and chiral subspaces, respectively, since \( \tilde{D}\mathcal{L}_n^{\text{anti}} = 0 \) and \( D\mathcal{L}_n^{\text{chir}} = 0 \). Both classes of operators are related by hermitian conjugation:

\[
\int dz d\theta d\bar{\theta} f \mathcal{L}_n^{\text{chir}} g = \int dz d\theta d\bar{\theta} (\mathcal{L}_n^{\text{anti}} f) g \quad \text{for } f \in \mathcal{F}_{-n,0}, g \in \mathcal{F}_{0,-n}.
\]

Differential operators of this type have been introduced in reference [9]: they occur as Lax operators in the \( N = 2 \) supersymmetric KdV and KP hierarchies [10, 9, 11] - see section 5.

Since \( q \) and \( p \) vanish for \( \mathcal{L}_n^{\text{anti}} \) and \( \mathcal{L}_n^{\text{chir}} \), respectively, the r.h.s. of eq.(49) vanishes for both of these operators: we will discuss their relation with ordinary Bol operators in the next section.

### 4.2 Sandwich operators

Before considering operators involving an odd number of derivatives, it is worthwhile to have another look at the chiral and anti-chiral solutions.

The anti-chiral solution (53) has the form

\[
\mathcal{L}_n^{\text{anti}} C_{-n,0} = \tilde{D}(\nabla \nabla)^{n-1} DC_{-n,0} = \tilde{D}\mathcal{K}_{n-1} DC_{-n,0}, \quad (54)
\]

where \( \mathcal{K}_n \) will be referred to as ‘sandwich operator’ for obvious reasons. Since \( DC_{-n,0} = \Phi_{-n+1,0} \) represents a chiral field, we have obtained a new conformally covariant operator

\[
\tilde{D}\mathcal{K}_{n-1} = \tilde{D}(\nabla \nabla)^{n-1} : \mathcal{F}_{-n+1,0} \rightarrow \mathcal{F}_{0,n}.
\]

Although \( \mathcal{K}_{n-1} \) formally coincides with \( \mathcal{L}_n^{\text{chir}} = (\nabla \nabla)^{n-1} \), it does not represent the same operator, because it does not act on the same space. We have the operatorial relations

\[
\tilde{D}\mathcal{K}_1 = \tilde{D}[D\tilde{D} + \mathcal{R}] = \tilde{D}[\partial + \mathcal{R}]
\]

\[
\tilde{D}\mathcal{K}_2 = \tilde{D}[\partial^2 + 3\mathcal{R}\partial + (D\tilde{D}\mathcal{R}) + 2(D\tilde{D}\mathcal{R}) + 2\mathcal{R}^2].
\]
Note that the operator $K_n$ can be written in different forms since it is only defined as ‘sandwiched’ between $\bar{D}$ and $D$.

The projection to component field expressions is simply done by applying $D$ to $\bar{D}K_n$:

$$ (D\bar{D}K_n C) | = L_{n+1}L^0 C \quad \text{for} \quad C \in \mathcal{F}_{-n,0}^c . \quad (57) $$

Here, $C | = c$ and $L_{p,q}^n$ denotes the generalization of the Bol operator $L_n$ to charged conformal fields – see eq. (50). This shows that $\mathcal{L}_n^{\text{anti}}$ also represents a supersymmetric generalization of the Bol operators, though the latter have to be projected out according to the procedure (57) rather than (49).

Of course, one can apply the same line of reasoning to the chiral solution for which

$$ \mathcal{L}_{n}^{\text{chir}} C_{0,-n} = D(\bar{\nabla}\nabla)^{n-1} \bar{D}C_{0,-n} \equiv D\bar{K}_{-n-1} \bar{D}C_{0,-n} \quad (58) $$

and

$$ D\bar{K}_{n-1} : \mathcal{F}_{0,-n+1}^a \rightarrow \mathcal{F}_{n,0}^c . \quad (59) $$

### 4.3 Odd number of derivatives

The general form of a normalized covariant operator involving an odd number of derivatives is

$$ \mathcal{L}_{n}^{\text{odd}} C_{0,-n} = \bar{\nabla}(\nabla\bar{\nabla})^n \quad (60) $$

or

$$ \mathcal{L}_{n}^{\text{odd}} C_{p,0} = \nabla(\nabla\bar{\nabla})^n \quad \text{with} \quad n = 0, 1, 2, .. $$

acting on $\mathcal{F}_{p,q}$. These operators modify the $U(1)$ charge of the fields.

For $n = 0$, the condition of covariance $\delta \mathcal{L} = 0$ leads to the explicit results

$$ \mathcal{L}_{0}^{\text{odd}} C_{0,q} = \bar{\nabla}C_{0,q} = \bar{D}C_{0,q} \quad (61) $$

$$ \mathcal{L}_{0}^{\text{odd}} C_{p,0} = \nabla C_{p,0} = DC_{p,0} . $$

For $n \geq 1$, we find

$$ \mathcal{L}_{n}^{\text{odd}} = \bar{D}(\nabla\bar{\nabla})^n : \{ C \in \mathcal{F}_{-n,0} / \mathcal{L}_{n}^{\text{anti}} C = 0 \} \rightarrow \mathcal{F}_{0,n+1}^a , \quad (62) $$

the simplest examples being

$$ \mathcal{L}_{1}^{\text{odd}} C_{-1,0} = \bar{D}[\partial + \mathcal{R}]C_{-1,0} \quad (63) $$

$$ \mathcal{L}_{2}^{\text{odd}} C_{-2,0} = \bar{D}[\partial^2 + 3R\partial - (D\mathcal{R})D + (\bar{D}D\mathcal{R}) + 2(DD\mathcal{R}) + 2\mathcal{R}^2]C_{-2,0} $$

with

$$ 0 = \mathcal{L}_{1}^{\text{anti}} C_{-1,0} = \bar{D}DC_{-1,0} \quad (64) $$

$$ 0 = \mathcal{L}_{2}^{\text{anti}} C_{-2,0} = \bar{D}[\partial + \mathcal{R}]DC_{-2,0} . $$
If $C_{-n,0}$ represents a chiral field, it automatically satisfies $L_n^{anti}C_{-n,0} = 0$, henceforth

$$F_{-n,0}^c \subset \{ C \in F_{-n,0} / L_n^{anti}C = 0 \}.$$  

By comparing equations (55) and (62), we conclude that $L_n^{odd} = \bar{D}((\nabla\nabla)^n)$ is an extension (in the functional analytic sense [12]) of the operator $\bar{D}K_n = \bar{D}((\nabla\nabla)^n)$: both operators differ by their domain of definition and act in the same way on the smaller domain, i.e. on $F_{-n,0}^c$. The Bol operator $L_{n+1}^{-1}0$ acting on $c = C|$ is extracted in the same manner from $L_n^{odd}C$ and $\bar{D}K_nC$, i.e. according to eq.(77).

4.4 Comments

4.4.1 Super Bol operators in projective coordinates

The Bol operator $L_n$ is most directly obtained from the Möbius covariant operator $\partial^n_Z$ (acting on quasi-primary fields of appropriate weight) by going over from the projective coordinates $Z$ to a generic system of conformal coordinates $z$ by a conformal transformation (see [1] and references therein). Proceeding in the same way in the $N = 2$ supersymmetric case, we start from the differential operator $\partial^n_Z^{-1}[D_{\Theta}, \bar{D}_{\Theta}] = (D_{\Theta}D_{\Theta})^{-1} - (D_{\Theta}D_{\Theta})^{-1}$ which transforms homogenously under a superprojective change of coordinates $Z \rightarrow Z'$ when acting on quasi-primary fields of appropriate weight (see eq.(77)):

$$\left( \partial^n_Z^{-1}[D_{\Theta}, \bar{D}_{\Theta}] \mathcal{C}_{-\frac{n}{2}, -\frac{n}{2}} \right) = (cZ + d + \Theta\bar{\gamma} + \bar{\Theta}\gamma)^n \partial^n_Z^{-1}[D_{\Theta}, \bar{D}_{\Theta}] \mathcal{C}_{-\frac{n}{2}, -\frac{n}{2}}.$$  \hspace{1cm} (65)

By passing over from the projective coordinates $Z$ to generic conformal coordinates $z$ by a superconformal transformation, the operator $\partial^n_Z^{-1}[D_{\Theta}, \bar{D}_{\Theta}]$ becomes the super Bol operator $L_n^{sym}$: we have

$$\partial^n_Z^{-1}[D_{\Theta}, \bar{D}_{\Theta}] \mathcal{C}_{-\frac{n}{2}, -\frac{n}{2}} \equiv (D\Theta \bar{D}\Theta)^{-\frac{n}{2}} L_n^{sym} \mathcal{C}_{-\frac{n}{2}, -\frac{n}{2}},$$  \hspace{1cm} (66)

where $\mathcal{C}$ and $C$ are related by

$$\mathcal{C}_{-\frac{n}{2}, -\frac{n}{2}}(Z, \bar{Z}) \equiv (D\Theta \bar{D}\Theta)^{\frac{n}{2}} \mathcal{C}_{-\frac{n}{2}, -\frac{n}{2}}(z, \bar{z}).$$  \hspace{1cm} (67)

In operatorial form, equation (66) reads

$$L_n^{sym} = (D\Theta \bar{D}\Theta)^{\frac{n}{2}} \cdot \partial^n_Z^{-1}[D_{\Theta}, \bar{D}_{\Theta}] \cdot (D\Theta \bar{D}\Theta)^{\frac{n}{2}},$$  \hspace{1cm} (68)

which means that $L_n^{sym}$ represents the superconformally covariant version of the differential operator $\partial^n[\bar{D}, \bar{D}]$ acting on $U(1)$-neutral fields.

The same line of arguments applies to the sandwich operator $K_n$ acting on chiral fields. In fact, $DC_{-n,0} = 0$ is equivalent to $D_{\Theta} \mathcal{C}_{-n,0} = 0$ where $\mathcal{C}_{-n,0} \equiv (D\Theta)^{n} C_{-n,0}$ and for such fields one has

$$D_{\Theta} \partial^n_Z \mathcal{C}_{-n,0} = (\bar{D}\Theta)^{(n+1)} \bar{D}K_n C_{-n,0}.$$  \hspace{1cm} (69)
4.4.2 Recursion and factorization relations

The super Bol operators can be determined by using a recursion relation [1], e.g. if \( \psi \in F_{-\frac{1}{2}, -\frac{1}{2}} \) is assumed to be Grassmann even, then

\[
L_{\mathrm{sym}}^1 \psi = 0 \implies L_{\mathrm{sym}}^n \psi^n = 0 \, .
\] (70)

Thus, one can determine the explicit form of \( L_{\mathrm{sym}}^2 \) by applying \( \partial[D, \bar{D}] \) to \( \psi^2 \) while using \( L_{\mathrm{sym}}^1 \psi = 0 \) (i.e. \( [D, \bar{D}] \psi = -\frac{1}{2} \mathcal{R} \psi \)) and so on for \( L_{\mathrm{sym}}^3, \ldots \). The proof of (70) consists of going over to superprojective coordinates.

The operators \( \bar{D}K_n \) can be determined in the same way: if \( \psi \in F_{c, -1, 0} \) is Grassmann even, then

\[
\bar{D}K_1 \psi = 0 \implies \bar{D}K_n \psi^n = 0 \, .
\] (71)

If the super Bol operators are defined by virtue of M"obius covariant operators as in equation (68), their expressions in terms of covariant derivatives (eq.(51)) represent factorization formulae. The latter may then be proven by going over to projective coordinates for which \( \nabla \) and \( \bar{\nabla} \) reduce to \( D_{\Theta} \) and \( D_{\bar{\Theta}} \), respectively [1].

4.4.3 From \( N = 2 \) to \( N = 1 \) operators

An \( N = 2 \) superfield \( F \) can be projected to an \( N = 1 \) superfield \( F|_{N=1} \) by evaluating it at \( \theta = \bar{\theta}, \theta^- = \bar{\theta}^- \). By virtue of this projection, one finds that the quantity \( \mathcal{R}_1 = -\frac{1}{2} (D - \bar{D}) \mathcal{R} \big|_{N=1} \) transforms like an \( N = 1 \) superprojective connection, i.e. according to \( \mathcal{R}'_1 = e^{3W}(\mathcal{R}_1 - S_1) \) where \( W \equiv w|_{N=1} = \bar{w}|_{N=1} \) and where \( S_1 \) denotes the \( N = 1 \) super Schwarzian derivative.

Since sandwich operators act on chiral fields \( C \), we have \( \bar{D}C = (D + \bar{D})C \equiv D_1 C \) where \( D_1 \) is the basic \( N = 1 \) derivative introduced in section 2.6. For the simplest sandwich operator, we thus obtain the expression

\[
(\bar{D}K_1 C)\big|_{N=1} = \left( [\partial\bar{D} + (\bar{D}\mathcal{R}) + \mathcal{R}\bar{D}]C \right)\big|_{N=1}
\]

\[= \left( [D_1^3 + \mathcal{R}_1]C + \frac{1}{2}(D_1\mathcal{R}) + \mathcal{R}D_1]C \right)\big|_{N=1} ,\]

which represents a \( U(1) \)-covariant version of the \( N = 1 \) super Bol operator \( D_1^3 + \mathcal{R}_1 \).

5 Covariant operators involving conformal fields

In this section, we introduce covariant operators \( \mathcal{M}^{(n)}_{W_{p,q}} \) which depend on a projective connection \( \mathcal{R} \) and, in a linear way, on a superconformal field \( W_{p,q} \). We restrict our attention to the case of sandwich operators since these are the only ones for which we will be able to give a matrix representation.

Thus, our goal is to construct a covariant operator which acts between the same spaces as \( \bar{D}K_n \) (cf.eq.(53)):

\[
\bar{D}\mathcal{M}^{(n)}_{W_{p,q}} : F_{c,-n,0} \longrightarrow F^a_{0,n+1} ,
\]
To do so, we proceed as before and first write $M^{(n)}_{W_{p,q}}$ as a polynomial in the nilpotent operators $\nabla$ and $\bar{\nabla}$. It turns out that a covariant operator can only be obtained in the case of neutral superfields $W_{k,k}$ of conformal weight $k \in \mathbb{N}$ with $1 \leq k \leq n$: the corresponding operator reads

$$M^{(n)}_{W_{k,k}} = \sum_{l=0}^{n-k} \left\{ a^{(n)}_{kl} \left[ (\nabla \bar{\nabla})^l W_{k,k} \right] + b^{(n)}_{kl} \left[ (\bar{\nabla} \nabla)^l W_{k,k} \right] \right\} (\nabla \bar{\nabla})^{n-k-l} \quad (72)$$

with constant coefficients $a^{(n)}_{kl}$ and $b^{(n)}_{kl}$. The latter are determined by imposing the condition $\delta \bar{D} M^{(n)}_{W_{k,k}} = 0$ and by using eqs. (12)-(14). This leads to the following result (which is unique up to a global factor):

$$a^{(n)}_{kl} = \binom{n-k}{l} \binom{k+l}{l} / (2k+l) \binom{l}{l} \quad , \quad b^{(n)}_{kl} = \binom{n-k}{l} \binom{k+l-1}{l} / (2k+l) \binom{l}{l} \quad (73)$$

for $l = 1, \ldots, n-k$ and $a^{(n)}_{k0} + b^{(n)}_{k0} = 1$. For instance,

$$M^{(n)}_{W_{n,n}} = W_{n,n}$$

$$M^{(n)}_{W_{n-1,n-1}} = W_{n-1,n-1} \bar{\partial} + \frac{n-1}{2n-1} (D W_{n-1,n-1}) \bar{D} + \frac{n}{2n-1} (D \bar{D} W_{n-1,n-1})$$

$$+ \frac{n^2}{2n-1} \mathcal{R} W_{n-1,n-1} \quad , \quad (74)$$

where we recall that sandwich operators can be cast into different forms since they are only defined as sandwiched between $\bar{D}$ and $D$. Component field results are recovered by applying the derivative $D$ and subsequently projecting to the lowest component:

$$(D \bar{D} M^{(n)}_{W_{k,k}} C_{n,0}) = \left[ M^{(n+1)}_{w_k} - \frac{n+k+1}{2k(2k+1)} \bar{M}^{(n+1)}_{v_{k+1}} \right] C_{n,0} \quad (75)$$

with $w_k = W_{k,k}$ and $v_{k+1} = (D_{k,k} W_{k,k})$ where $D_{k,k}$ denotes the second order operator (31). Here, $\bar{M}^{(n)}_{w_k}$ is the $\mathcal{U}(1)$-covariant generalization of the operator $M^{(n)}_{w_k}$ mentioned in the introduction - see eq. (3).

The most general normalized superconformally covariant sandwich operator of order $n$ has the form

$${\mathcal{L}^{(n)}} : \mathcal{F}^{c}_{-n,0} \rightarrow \mathcal{F}^{n}_{0,n+1}$$

$$C_{n,0} \mapsto \mathcal{L}^{(n)} C_{n,0} = \bar{D} \left[ \partial^n + a^{(n)}_1 \partial^{n-1} + a^{(n)}_2 \partial^{n-2} + \ldots + a^{(n)}_n \right] C_{n,0} \quad (76)$$

with even coefficient functions $a^{(n)}_k(z)$ transforming in an appropriate way under superconformal changes of coordinates - see references [2, 1] and [3] for the $N = 0$ and $N = 1$. 

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supersymmetric theories, respectively. It can be expressed as a sum of the previously introduced operators,

$$L^{(n)} = \bar{D} \left[ K_n + \sum_{k=2}^{n} M^{(n)}_{W_k,k} \right] = \bar{D} \left[ \partial^n + \frac{n(n+1)}{2} \mathcal{R} \partial^{n-1} + \ldots \right],$$

(77)

which means that it can be parametrized in terms of a projective connection $\mathcal{R}$ and $n-1$ neutral superconformal fields $W_{2,2}, \ldots, W_{n,n}$. The relation between these fields and the coefficients $a_k^{(n)}$ is invertible and given by differential polynomials.

Operators of the form (76) with $C_{-n,0} = D\Phi$ occur as Lax operators, e.g. $L^{(1)}$ and $L^{(2)}$ give rise to the $N=2$ super KdV and Boussinesq equations, respectively [9, 11].

6 Matrix representation of covariant operators

In this section, we discuss the matrix representation of the sandwich operators (77) while starting with the particular case of the super Bol operator

$$\bar{D}K_n = \bar{D}(\nabla \bar{\nabla})^n = \bar{D}[D-nB][\bar{D}+\bar{B}][D-(n-1)B][\bar{D}+2\bar{B}] \cdots [D-B][\bar{D}+n\bar{B}].$$

(78)

The two conformally covariant differential equations

$$Df = 0, \quad \bar{D}K_n f = 0 \quad (f \in \mathcal{F}_{-n,0})$$

(79)

are equivalent to two systems of first-order differential equations which can be cast into matrix form:

$$Q^{fac}_n \vec{f} = \vec{0}, \quad \bar{Q}^{fac}_n \vec{f} = \vec{0}.$$  

(80)

Here, $\vec{f} = (f_1, f_2, \ldots, f_{2n}, f)^t$ and $Q^{fac}_n, \bar{Q}^{fac}_n$ denote the $(2n+1) \times (2n+1)$ matrix operators

$$Q^{fac}_n = -\mu - D1 - BH, \quad \bar{Q}^{fac}_n = -\bar{\mu} - \bar{D}1 - \bar{B}\bar{H},$$

(81)

where $\mu_-, \bar{\mu}_-, H, \bar{H}$ belong to the superprincipal embedding $sl(2|1)_{\text{pal}} \subset sl(n+1|n)$ of the Lie superalgebra $sl(2|1)$ into the Lie superalgebra $sl(n+1|n)$ - see eqs. (A.3) of appendix A. We note that the operators (81) can be rewritten in the form

$$Q^{fac}_n = D1 - A^{fac}_g, \quad \bar{Q}^{fac}_n = \bar{D}1 - A^{fac}_{\bar{g}},$$

(82)

where $A^{fac}_g$ and $A^{fac}_{\bar{g}}$ are to be interpreted as components of a $sl(2|1)_{\text{pal}}$-valued connection - a viewpoint which will be further developed in the next section.

$Q^{fac}_n$ and $\bar{Q}^{fac}_n$ describe the factorized form (78) of the super Bol operator. One obtains an equivalent matrix representation for this operator by conjugation with a group element $N \in SL(2|1)_{\text{pal}} \subset SL(n+1|n)$,

$$Q_n = \hat{N}^{-1} \cdot Q^{fac}_n \cdot N, \quad \bar{Q}_n = \hat{N}^{-1} \cdot \bar{Q}^{fac}_n \cdot N,$$

(83)
where \( N \) is an upper triangular matrix with entries 1 on the diagonal \([3, 3]\). (The systems of equations \((80)\) then become \( Q_n \vec{f}' = \vec{0}, \bar{Q}_n \vec{f}' = \vec{0} \) with \( \vec{f}' \equiv N^{-1} \vec{f} = (f'_1, ..., f'_{2n}, f) \), i.e. the component \( f \) of \( \vec{f} \) is not modified, which explains why \( Q_n \) and \( \bar{Q}_n \) still represent the scalar equations \((79)\).) The hatted group element \( \hat{N} = \exp \hat{M} \) follows from \( N = \exp M \) by changing the sign of the anticommuting part of the Lie superalgebra element \( M \). By using the gauge freedom \((83)\), we can find a form of \( Q_n \) and \( \bar{Q}_n \) which makes the dependence on the superprojective structure manifest:

\[
Q_n = -\mu_- + D1 - \frac{1}{2} [R\hat{\mu}_+ + (D\hat{R})E_{++}] \quad \text{(84)}
\]

\[
\bar{Q}_n = -\bar{\mu}_- + \bar{D}1 + \frac{1}{2} [R\mu_+ + (\bar{D}\hat{R})E_{++}] .
\]

This matrix representation of \( \bar{D}K_n \) in which \( \mu_\pm, \bar{\mu}_\pm \) and \( E_{++} \equiv -\{\mu_+, \bar{\mu}_+\} \) belong to the Lie superalgebra \( sl(2|1)_{pal} \) exhibits most clearly the underlying algebraic structure which is due to the covariance with respect to super Möbius transformations. By unravelling the matrix equations \( Q_n \vec{f}' = \vec{0}, \bar{Q}_n \vec{f}' = \vec{0} \), one recovers our results \((56)\) for the operator \( \bar{D}K_n \).

The matrix representation \((84)\) can be extended to the sandwich operators involving superconformal fields that we discussed in section 5:

\[
Q_n = -\mu_- + D1 - \frac{1}{2} \sum_{i=1}^{n} [i\mathcal{V}_{i,i}\hat{\mu}_+ + (D\mathcal{V}_{i,i})E_{++}] E_{++}^{i-1} \quad \text{(85)}
\]

\[
\bar{Q}_n = -\bar{\mu}_- + \bar{D}1 + \frac{1}{2} \sum_{i=1}^{n} [i\mathcal{V}_{i,i}\mu_+ + (\bar{D}\mathcal{V}_{i,i})E_{++}] E_{++}^{i-1} .
\]

Here, \( \mathcal{V}_{1,1} \equiv \mathcal{R} \) and \( \mathcal{V}_{i,i} \) is a superconformal field for \( i \geq 2 \). The \( \mathcal{V}_{i,i} \) (\( i = 1, ..., n \)) can be expressed in terms of the projective connection \( \mathcal{R} \) and the conformal fields \( \mathcal{W}_{k,k} \) (\( k = 2, ..., n \)) appearing in the expression \((77)\) by means of invertible relations involving differential polynomials. Henceforth, the expressions \((85)\) provide a matrix representation for the most general sandwich operator. Thanks to this result, explicit expressions for covariant operators can be derived in a straightforward way.

If the sandwich operators are parametrized in the form \((76)\), their matrix realization reads

\[
Q_{n}^{\text{hor}} = -\mu_- + D1 \quad \text{(86)}
\]

\[
\bar{Q}_{n}^{\text{hor}} = -\bar{\mu}_- + \bar{D}1 + \sum_{i=1}^{n} \left[ \lambda_i a_i^{(n)} E_{1,2i} + \mu_i (\bar{D}a_i^{(n)}) E_{1,2i+1} \right] ,
\]

where \( E_{ij} \) are the matrices with entries \( (E_{ij})_{kl} = \delta_{ik}\delta_{jl} \) and

\[
\lambda_i = \frac{(n-i)!}{n! (i-1)!}, \quad \mu_i = \frac{(n-i)!}{n! i!} .
\]

This gauge choice is referred to as the horizontal gauge \([13]\) while \((84,85)\) is known as the \( osp(1|2) \) highest weight gauge. In fact, as we will now verify, the combination \( Q_n + \bar{Q}_n \) leads
to the matrix representation of $N = 1$ superconformally covariant differential operators in terms of highest weight generators of $osp(1|2)_{\text{pal}} \subset sl(2|1)_{\text{pal}}$ (see appendix A for the algebraic details).

By adding the expressions (85), we find

$$Q_n + \bar{Q}_n = -\kappa_- + (D + \bar{D})1 + \frac{1}{2} \sum_{i=1}^{n} \left[ i \mathcal{V}_{i,i} M_{2i-1} - (D - \bar{D}) \mathcal{V}_{i,i} M_{2i} \right].$$

Here, $\kappa_- \equiv \mu_- + \bar{\mu}_- \in osp(1|2)_{\text{pal}}$ and $M_k \equiv (\mu_+ - \bar{\mu}_+)^k$ are highest weight generators of $osp(1|2)_{\text{pal}}$ while $D + \bar{D} = D_1$ denotes the basic $N = 1$ derivative. By projecting from $N = 2$ to $N = 1$ superfields along the lines of section 4.4.3, we get

$$Q_n^{(N=1)} \equiv (Q_n + \bar{Q}_n)|_{N=1} = -\kappa_- + D_1 1 + \sum_{k=1}^{2n} V_{k+1} M_k$$

with

$$V_{2i} = \frac{i}{2} \mathcal{V}_{i,i}|_{N=1}, \quad V_{2i+1} = -\frac{1}{2} (D - \bar{D}) \mathcal{V}_{i,i}|_{N=1} \quad (i = 1, \ldots, n).$$

The expression (87) coincides with the $N = 1$ matrix representation found in references [3, 16] up to the fact that it involves a different (though equivalent) realization of the superprincipal embedding of $osp(1|2)_{\text{pal}}$ in $sl(n+1|n)$. The same realization can be obtained by choosing $Q_n$ and $\bar{Q}_n$ along the lines of reference [13]:

$$Q'_n = -\bar{\mu}_+ + D 1 + \frac{1}{2} \sum_{i=1}^{n} \left[ (D \mathcal{V}_{i,i}) \bar{E}_{i, -}^i - \mathcal{V}_{i,i}[\bar{\mu}_+, \bar{E}_{i, -}^i] \right] \quad (88)$$

$$\bar{Q}'_n = +\bar{\mu}_+ + \bar{D} 1 - \frac{1}{2} \sum_{i=1}^{n} \left[ (\bar{D} \mathcal{V}_{i,i}) E_{i, -}^i + \mathcal{V}_{i,i}[\bar{\mu}_+, \bar{E}_{i, -}^i] \right].$$

Here, the tilde denotes the transposed matrix and the common structure of the results (88) and (88) becomes apparent if one takes into account the identities

$$[\mu_-, E_{++}^i] = -i \bar{\mu}_+ E_{++}^{i-1}, \quad [\mu_-, E_{++}^i] = -i \mu_+ E_{++}^{i-1}.$$ (89)

7 Ward identities and $W$-algebras from a zero curvature condition

The highest weight matrix representation (87) of covariant sandwich operators has the form

$$Q_n = D 1 - A_{\theta}, \quad \bar{Q}_n = \bar{D} 1 - A_{\bar{\theta}},$$

where $A_{\theta}$ and $A_{\bar{\theta}}$ can be interpreted as the spinorial components of a connection with values in the Lie superalgebra $sl(2|1)_{\text{pal}} \subset sl(n+1|n)$. The spinorial components can
be supplemented with spatial components \(A_z\) and \(A_\bar{z}\) represented by generic elements of \(sl(n+1|n)\). Altogether these fields represent the fundamental variables for the superspace formulation of a \((2,0)\)-supersymmetric gauge theory based on graded Lie algebras. (For the general framework of such theories, we refer to the appendix of [16].)

Within the \(N = 1\) superfield formalism, it has been shown [16] that classical super-\(W\)-algebras can be constructed in the following way: in the \(z\)-sector, one considers the highest weight gauge for the spinorial components of the connection (i.e. eqs. (53) in the present case), all entries of the connection components \(A_z, A_\bar{z}, A_\theta, \ldots\) are supposed to be smooth superfields and zero curvature conditions are imposed on the connection. We will now apply this procedure to the \(N = 2\) theory.

The integrability conditions for the system of differential equations \(\bar{0} = Q_n \bar{f} = \tilde{Q}_n \bar{f} = (\bar{\partial} - A_\bar{z}) \bar{f}\) are the zero curvature conditions

\[
0 = DA_\theta - \hat{A}_\theta A_\theta, \quad 0 = \bar{D}A_\bar{\theta} - \hat{A}_{\bar{\theta}} A_{\bar{\theta}}
\]

\[
A_z = DA_\bar{\theta} + \bar{D}A_\theta - \hat{A}_{\bar{\theta}} A_\theta - \hat{A}_\theta A_{\bar{\theta}}
\]

\(0 = \bar{D}A_{\bar{z}} - \bar{\partial}A_{\bar{\theta}} - A_{\bar{\theta}} A_{\bar{z}} + \hat{A}_{\bar{\theta}} A_\theta\)

\(0 = \bar{D}A_{\bar{z}} - \bar{\partial}A_{\bar{\theta}} - A_{\bar{\theta}} A_{\bar{z}} + \hat{A}_{\bar{\theta}} A_\theta\),

(91)

where the hat again denotes the automorphism of the Lie superalgebra which reverses the signs of all odd elements. As may be verified explicitly, the expressions (85) (involving smooth superfields \(\mathcal{V}_{\alpha,\beta}\)) verify the constraints of the first line. The second line is simply a redefinition constraint which determines the component \(A_z\) in terms of the variables \(A_{\theta}\) and \(A_{\bar{\theta}}\). Finally, \(A_{\theta}\) and \(A_{\bar{\theta}}\) having the form (83), the last two conditions represent constraints for the entries of the matrix \(A_{\bar{z}}\). Actually, most of these constraints are algebraic equations which allow to express the entries of \(A_{\bar{z}}\) in terms of other fields while the other ones represent partial differential equations for the independent fields. These differential equations are the Ward identities for the super-\(W_{n+1}\)-algebra associated to \(sl(n+1|n)\).

In the simplest case \((n = 1)\), the matrix \(A_{\bar{z}}\) only contains one independent superfield \(H\) satisfying the differential equation

\[
\mathcal{L}_2^{sym} H = \bar{\partial} \mathcal{R},
\]

(92)

where \(\mathcal{L}_2^{sym}\) denotes the symmetric super Bol operator (52). This relation is nothing but the \((2,0)\) superconformal Ward identity [6], the superfield \(H \equiv H_{\bar{z}}\) being interpreted as the Beltrami superfield (in the so-called restricted geometry) and the superprojective connection \(\mathcal{R}\) as the stress-energy tensor. It expresses the superdiffeomorphism invariance of the generating functional \(Z_c[H]\) in superconformal field theory.

For \(n = 2\), \(A_{\bar{z}}\) contains two independent superfields \(H \equiv H_{\bar{z}}\) and \(G \equiv G_{\bar{z}}\) which are associated to \(\mathcal{R}\) and \(\mathcal{V} \equiv \mathcal{V}_{2,2}\), respectively. These variables satisfy the system of differential equations

\[
\bar{\partial} \mathcal{R} = \mathcal{L}_2^{sym} H + [2\mathcal{V}\partial - (D\mathcal{V})D - (D\mathcal{V})\bar{D} + 2(\partial\mathcal{V})] G
\]

\[\bar{\partial} \mathcal{V} = \frac{-1}{16} \left[ \mathcal{L}_4^{sym} + 4\mathcal{M}_4^{(4) sym} \right] G + [2\mathcal{V}\partial - (D\mathcal{V})D - (D\mathcal{V})\bar{D} + (\partial\mathcal{V})] H,\]

(93)
where $\mathcal{M}_V^{(4)\text{sym}}$ is the superconformally covariant operator \cite{footnoted}
\begin{align*}
\mathcal{M}_V^{(4)\text{sym}} &= 5V\partial[D, \bar{D}] + 5D\partial\bar{D} - 5\bar{D}\partial D + \frac{5}{2}\partial V[D, \bar{D}] + \frac{3}{2}[D, \bar{D}]V + 7RV)\partial \\
&\quad + (3\partial DV + RDV - 9DRV)\bar{D} - (3\partial D\bar{V} - \bar{R}D\bar{V} + 9\bar{R}DV)D \\
&\quad + \partial[D, \bar{D}]V + 3RD\partial V - 5\bar{D}RDV - 5D\bar{R}DV + 8\partial RV .
\end{align*}
The relations (93) generalize the Ward identity (92) which is recovered for $V = 0 = G$. They represent the Ward identities associated to the $N = 2$ super $W_3$-algebra and they are manifestly covariant since all differential operators occurring on their r.h.s. are superconformally covariant when acting on the given fields.

As a matter of fact, the Ward identities (93) are equivalent to the Poisson bracket algebra (commutation relations)
\begin{align*}
[R_2, R_1] &= (L_2^{\text{sym}})_{1} \delta^{(3)}(z_2, z_1) \\
[R_2, V_1] &= (2V\partial - (D\bar{V})D - (DV)\bar{D} + (\partial V))D \delta^{(3)}(z_2, z_1) \\
[V_2, V_1] &= -\frac{1}{16} \left( L_4^{\text{sym}} + 4\mathcal{M}_V^{(4)\text{sym}} \right)_{1} \delta^{(3)}(z_2, z_1) ,
\end{align*}
for which we used the notation $R_i \equiv R(z_i)$, $V_i \equiv V(z_i)$ and the Dirac distribution $\delta^{(3)}$ defined by eq.(B.3). By spelling out all expressions on the r.h.s. of eqs.(94), one finds that this result exactly coincides with the one found in reference \cite{17} by applying quite different methods (see also \cite{9, 18} as well as \cite{11} for a general formula). The introduction of covariant operators not only allows us to cast the results in a compact form, it also helps to exhibit the conformal symmetry which plays a crucial role in the formulation of integrable models. In fact, the covariant operators occurring in eqs.(93, 94) yield the second Hamiltonian structure for the $N = 2$ super Boussinesq equation \cite{17, 18, 19}. We note that the algebra (94) can be rewritten in terms of $N = 1$ superfields \cite{18} in which case all $N = 2$ covariant operators reduce to $N = 1$ covariant ones (see \cite{3} for the latter operators and the corresponding algebra).

By integrating the Ward identities (93), one can deduce the operator product expansions (OPE’s) of the spin 1 and spin 2 supercurrents $R$ and $V$. To do so, one proceeds along the lines of reference \cite{16} and defines $R$ and $V$ as derivatives of a generating functional $Z_c = Z_c [H, G]$,
\begin{align*}
R &= - \frac{\delta Z_c}{\delta H} \bigg|_{H=0, G=0} , \quad V = - \frac{\delta Z_c}{\delta G} \bigg|_{H=0, G=0} .
\end{align*}
By using eqs.(B.1) and (B.4) of appendix B, one obtains the (singular parts of the) OPE’s of the classical $N = 2$ super $W_3$-algebra \cite{19}, in terms of the notation $R_i \equiv R(z_i)$ and

\footnotetext{1}{This operator is an example of a covariant operator involving a conformal superfield and generalizing the symmetric Bol operators of section 4.1.1. These operators can be determined in a systematic way along the lines of section 5.}
\( \mathcal{V}_i \equiv \mathcal{V}(z_i) \), we have

\[
\begin{align*}
\mathcal{R}_2 \mathcal{R}_1 &= \frac{2}{z_{12}^2} + \left[ \frac{\bar{\theta}_{12} \theta_{12}}{z_{12}^2} + \frac{\bar{\theta}_{12} D - \theta_{12} D - \bar{\theta}_{12} \theta_{12} \partial}{z_{12}} \right] \mathcal{R}_1 \\
\mathcal{R}_2 \mathcal{V}_1 &= \left[ \frac{2 \bar{\theta}_{12} \theta_{12}}{z_{12}^2} + \frac{\bar{\theta}_{12} D - \theta_{12} D - \bar{\theta}_{12} \theta_{12} \partial}{z_{12}} \right] \mathcal{V}_1 \\
-16 \mathcal{V}_2 &= \frac{12}{z_{12}^3} + \frac{12 \bar{\theta}_{12} \theta_{12} \mathcal{R}_1}{z_{12}^3} + \frac{12 (\bar{\theta}_{12} D - \theta_{12} D - \bar{\theta}_{12} \theta_{12} \partial) \mathcal{R}_1}{z_{12}^3} \\
&\quad + \frac{2 (20 \mathcal{V}_1 + 2[D, \bar{D}] \mathcal{R}_1 - \mathcal{R}_1^2)}{z_{12}^2} - (\bar{\theta}_{12} D + \theta_{12} D)(20 \mathcal{V}_1 + 8[D, \bar{D}] \mathcal{R}_1 - \mathcal{R}_1^2) \\
&\quad + \bar{\theta}_{12} (6[D, \bar{D}] \mathcal{V}_1 + 6 \partial^2 \mathcal{R}_1 + U_1) - \partial (20 \mathcal{V}_1 + 2[D, \bar{D}] \mathcal{R}_1 - \mathcal{R}_1^2) \\
&\quad + \bar{\theta}_{12} (12 \partial D \mathcal{V}_1 + 3 \partial^2 \mathcal{D} \mathcal{R}_1 + \psi_1) + \bar{\theta}_{12} (12 \partial D \mathcal{V}_1 + 3 \partial^2 \mathcal{D} \mathcal{R}_1 + \psi_1) \\
&\quad - \frac{1}{4} \bar{\theta}_{12} \theta_{12} \frac{16 \partial D [D, \bar{D}] \mathcal{V}_1 + 8 \partial^3 \mathcal{R}_1 + \partial [D, \bar{D}] \mathcal{R}_1^2 + 2 (\partial U_1 - D \psi_1 + D \psi_1)}{z_{12}}.
\end{align*}
\]

which expressions involve the composite supercurrents \( \mathcal{R}^2 \) and \( U, \bar{\psi}, \psi \) defined by

\[
\begin{align*}
U &= 28 \mathcal{R} \mathcal{V} + 4 \mathcal{R} [D, \bar{D}] \mathcal{R} - 15 D \mathcal{D} \mathcal{D} \mathcal{R} - 2 \mathcal{R}^3 \\
\bar{\psi} &= -4 \mathcal{R} \bar{D} \mathcal{V} + 36 \mathcal{D} \mathcal{R} \mathcal{V} - \mathcal{R} \partial \bar{D} \mathcal{R} - \frac{3}{2} \partial \mathcal{R} \bar{D} \mathcal{R} + \frac{9}{2} [D, \bar{D}] \mathcal{R} \mathcal{D} \mathcal{R} - 2 \mathcal{R}^2 \bar{D} \mathcal{R} \\
\psi &= 4 \mathcal{R} D \mathcal{V} - 36 \mathcal{D} \mathcal{R} \mathcal{V} - \mathcal{R} \partial D \mathcal{R} - \frac{3}{2} \partial \mathcal{R} D \mathcal{R} - \frac{9}{2} [D, \bar{D}] \mathcal{R} D \mathcal{R} + 2 \mathcal{R}^2 D \mathcal{R}.
\end{align*}
\]

In conclusion, we note that one can explicitly derive the Poisson brackets \( \{ \tau_4 \} \) from the OPE’s \( \{ \tau_6 \} \) by applying the supersymmetric version of Cauchy’s theorem (appendix B) \( \{ \tau_9 \} \).

## 8 Concluding remarks

Interestingly enough, the zero curvature conditions provide a link between two different classes of \( N = 2 \) superconformally covariant operators: by starting from the matrix representation of sandwich operators, one obtains Ward identities which involve symmetric covariant operators.

Concerning the matrix representation of covariant operators which are not of sandwich type, we remark the following: if one disregards the covariance properties and considers \( \mathcal{L}_n^{sym} \) as sandwiched between \( D \) and \( D \), it has the form \( \{ \tau_6 \} \) and equation \( \{ \tau_9 \} \) with properly chosen coefficients \( a_i^{(n)} \) then provides a matrix representation. However, this result is neither satisfactory nor complete, since it does not properly take into account conformal symmetry and since the scalar operator \( \mathcal{L}_n^{sym} \) can only be recovered from this matrix representation up to contributions \( \mathcal{L}_a \) satisfying \( D \mathcal{L}_a D = 0 \).
The expression $M^{(n)}_{W}C$ may be viewed as the result of a bilinear and covariant map $J$, i.e. $M^{(n)}_{W}C = J(W,C)$ which is known as the super Gordan transvectant [1]-[3]. Such bilinear as well as trilinear covariant operators [3] can be constructed along the lines of section 3.5. These operators occur in particular in the Poisson brackets of super $W$-algebras [3].

There is a remarkable relationship between the matrix representation of conformally covariant operators and the general formula determining the singular vectors of the Virasoro algebra - see the second of references [2]. This relationship generalizes to the $N = 1$ supersymmetric theory [3] and our results concerning $N = 2$ covariant operators should also allow to draw close parallels to the recent study of singular vectors of the $N = 2$ superconformal algebra [20].

Apart from the Gelfand-Dickey derivation of (super) $W$-algebras and the zero curvature construction that we discussed here, various other approaches to these algebras have been considered in the literature - see for instance [21, 22]. The tools developed in the present work should provide the appropriate ingredients for generalizing these constructions to the $N = 2$ theory.

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Appendix A : Non-standard matrix realization of $sl(n+1|n)$

$sl(n+1|n)$ is the graded Lie algebra of $(2n+1) \times (2n+1)$ matrices with vanishing supertrace. It admits a purely fermionic system of simple roots.

We consider a non-standard matrix realization of $sl(n+1|n)$ which is referred to as the diagonal representation $[3, 16]$. It consists of assigning a $\mathbb{Z}_2$-grading $i+j \pmod{2}$ to a matrix element $M_{ij}$, defining the supertrace by the alternating sum

$$\text{str } M = \sum_{i=1}^{2n+1} (-)^{i+1} M_{ii}$$

and the graded commutator by

$$[M, N]_{ik} = \sum_{j=1}^{2n+1} \left( M_{ij} N_{jk} - (-1)^{(i+j)(j+k)} N_{ij} M_{jk} \right) .$$

In the Serre-Chevalley basis, the basic commutation relations of $sl(n+1|n)$ read

$$[h_i, h_j] = 0 , \quad [h_i, e_j] = +a_{ij} e_j , \quad [e_i, f_j] = \delta_{ij} h_j , \quad [h_i, f_j] = -a_{ij} f_j$$

for $i, j \in \{1, \ldots, 2n\}$.

Here, the $h_i$ belong to the Cartan subalgebra, the $e_i$ denote the fermionic simple roots, $f_i$ the associated negative roots and the $a_{ij}$ are the elements of the Cartan matrix:

$$a_{ij} = \delta_{i+1,j} - \delta_{i,j+1} \quad (A.1)$$

In the diagonal representation, the Serre-Chevalley generators can be represented by the matrices

$$h_i = E_{i,i} + E_{i+1,i+1}$$
$$e_i = E_{i,i+1}$$
$$f_i = E_{i+1,i} \quad (A.2)$$

where $E_{ij}$ denotes the $(2n+1) \times (2n+1)$ matrix with entries $(E_{ij})_{kl} = \delta_{ik}\delta_{jl}$.

The superprincipal embedding of $sl(2|1)$ in $sl(n+1|n)$ is denoted by $sl(2|1)_{\text{pal}} \subset sl(n+1|n)$ and defined as follows $[15]$

$$H = \sum_{i=1}^{n} (n-i+1) h_{2i-1} , \quad \bar{H} = -\sum_{i=1}^{n} i h_{2i}$$
$$\mu_+ = \sum_{i=1}^{n} e_{2i-1} , \quad \bar{\mu}_+ = -\sum_{i=1}^{n} e_{2i} \quad (A.3)$$
$$\mu_- = \sum_{i=1}^{n} (n-i+1) f_{2i-1} , \quad \bar{\mu}_- = \sum_{i=1}^{n} i f_{2i} .$$
Here, $H, \bar{H}$ represent the Cartan generators of $sl(2|1)$, $\mu_+, \bar{\mu}_+$ the fermionic simple roots and $\mu_-, \bar{\mu}_-$ the associated negative roots. The non-vanishing commutators of these generators read

$$[H, \bar{\mu}_\pm] = \pm \bar{\mu}_\pm , \quad [\bar{H}, \mu_\pm] = \pm \mu_\pm$$

$$\{\mu_+, \mu_-\} = H , \quad \{\bar{\mu}_+, \bar{\mu}_-\} = \bar{H}$$

$$\{\mu_+, \bar{\mu}_+\} \equiv -E_{++} , \quad \{\mu_-, \bar{\mu}_-\} \equiv -E_{--} \quad \text{(A.4)}$$

where $E_{++}$ and $E_{--}$ correspond to the bosonic roots of $sl(2|1)$.

The Lie superalgebra $sl(2|1)_{\text{pal}}$ contains the superprincipal embedding of $osp(1|2)$ into $sl(n + 1|n)$, i.e. $osp(1|2)_{\text{pal}} \subset sl(2|1)_{\text{pal}} \subset sl(n + 1|n)$: this embedding is defined by the combinations

$$J_0 = H + \bar{H} , \quad \kappa_+ = \mu_+ + \bar{\mu}_+ , \quad \kappa_- = \mu_- + \bar{\mu}_-$$

$$J_+ = E_{++} , \quad J_- = -E_{--} \quad \text{(A.5)}$$

which satisfy the commutation relations

$$[J_0, \kappa_\pm] = \pm \kappa_\pm , \quad \{\kappa_+, \kappa_-\} = J_0 , \quad \{\kappa_\pm, \kappa_\pm\} = \pm 2J_\pm \quad \text{(A.6)}$$

Here, $J_0$ is the Cartan generator and $\kappa_+$ the fermionic simple root. The highest weight generators $M_k$ of $osp(1|2)_{\text{pal}}$ are defined by the relations

$$[J_0, M_k] = k M_k , \quad [\kappa_+, M_k] = 0 \quad (k = 1, 2, ..., 2n) \quad \text{(A.7)}$$

which are solved by

$$M_k = M_1^k \quad \text{with} \quad M_1 = \mu_+ - \bar{\mu}_+ \quad \text{(A.8)}$$

### Appendix B: Some distributional relations

In this appendix, we gather some distributional relations which hold in $d = 2, N = 2$ superspace and we indicate the correspondence rules between supercurrent OPE’s and commutation relations.

Let $(z_i, \bar{z}_i) \equiv (z_i, \theta_i, \bar{\theta}_i; \bar{z}_i, \theta^-_i, \bar{\theta}^-_i)$ with $i = 1, 2$ denote the local coordinates of two points on a $N = 2$ SRS. We then define the relative coordinates by

$$\theta_{12} = \theta_1 - \theta_2 \quad , \quad \bar{\theta}_{12} = \bar{\theta}_1 - \bar{\theta}_2 \quad , \quad z_{12} = z_1 - z_2 - \frac{1}{2}(\theta_1 \bar{\theta}_2 + \bar{\theta}_1 \theta_2) \quad \text{(B.1)}$$

For a superanalytic superfield $F$ (i.e. a field depending only on $z$), one has the following Cauchy formulae $^{23,3}$ ($n \in \mathbb{N}$):

$$\int d^3z_1 \left(\theta_{12}\theta_{12}, \theta_{12}, \bar{\theta}_{12}, 1\right) \frac{F(z_1)}{z_{12}^{n+1}} = \left(1 - \bar{D}, D, \frac{1}{2}[D, \bar{D}]\right) \frac{\partial^n F(z_2)}{n!} \quad \text{(B.2)}$$
Here, $d^3z$ denotes the measure $(2\pi i)^{-1}dzd\theta d\bar{\theta}$ with respect to which the Dirac distribution takes the form
\[
\delta^{(3)}(z_1, z_2) \equiv \delta(z_1 - z_2)(\bar{\theta}_1 - \bar{\theta}_2)(\theta_1 - \theta_2)
\]
(B.3)

The integration of Ward identities is performed by means of the Dirac distribution defined with respect to the measure $d^4z = (2\pi i)^{-1}d\bar{z}dzd\theta d\bar{\theta}$:
\[
\delta^{(4)}(z_1, z_2) \equiv \delta(z_1 - z_2)\delta(\bar{z}_1 - \bar{z}_2)(\bar{\theta}_1 - \bar{\theta}_2)(\theta_1 - \theta_2) = \partial_{\bar{z}_1}^{\bar{z}_2} \delta_{\bar{z}z_1} \theta_{z_2}
\]
(B.4)

A general OPE of two supercurrents $A(z_2)$ and $B(z_1)$ can be written in the form
\[
A(z_2)B(z_1) = \sum_{n \in \mathbb{N}} \frac{a_n(z_1) + \bar{\theta}_{12} \alpha_n(z_1) + \theta_{12} \beta_n(z_1) + \bar{\theta}_{12} \beta_{12} b_n(z_1)}{z_{21}^{n+1}} + \text{regular terms}
\]
(B.5)
and by virtue of the relations (B.2), this OPE is equivalent to the commutation relation
\[
[A(z_2), B(z_1)] = \sum_{n \in \mathbb{N}} \frac{1}{n!} \left( \frac{a_n(z)}{2} [D, \bar{D}] + \alpha_n(z)D - \beta_n(z)\bar{D} + b_n(z) \right) \bigg|_{z=z_1} \partial_z^n \delta^{(3)}(z_2, z_1)
\]
(B.6)
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