CRITICAL BEHAVIOR FOR A SEMILINEAR PARABOLIC EQUATION WITH FORCING TERM DEPENDING OF TIME AND SPACE

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Abstract. We investigate the large-time behavior of the sign-changing solution of the inhomogeneous semilinear heat equation \( \partial_t u = \Delta u + |u|^p + t^\sigma w(x) \) in \((0, T) \times \mathbb{R}^N\), where \( N \geq 2, p > 1, \sigma > -1, \sigma \neq 0 \) and \( w \neq 0 \). The novelty of this paper lies in considering a forcing term \( (t^\sigma w(x)) \) which depends both of time and space. We show that there is an exponent \( p^*(\sigma) \) which is critical in the following sense: the solution of the above problem blows up in finite time when \( 1 < p < p^*(\sigma) \) and \( \int_{\mathbb{R}^N} w(x) \, dx > 0 \), while global solution exists for suitably small initial data and \( w \) belonging to certain Lebesgue spaces when \( p \geq p^*(\sigma) \). Our obtained results show that the forcing term induces an interesting phenomenon of discontinuity of the critical exponent \( p^*(\sigma) \). Namely, we found that \( \lim_{\sigma \to 0^-} p^*(\sigma) \neq \lim_{\sigma \to 0^+} p^*(\sigma) \). Furthermore, \( \lim_{\sigma \to 0} p^*(\sigma) \) coincides with the critical exponent of the above problem with \( \sigma = 0 \).

1. Introduction

In this paper we investigate the global existence and blow-up of sign-changing solutions of the following inhomogeneous parabolic equation

\[
\begin{align*}
\begin{cases}
\partial_t u &= \Delta u + |u|^p + t^\sigma w(x) \quad \text{in } (0, T) \times \mathbb{R}^N, \\
u(0, x) &= u_0(x) \quad \text{in } \mathbb{R}^N,
\end{cases}
\end{align*}
\]

(1.1)

where \( N \geq 2, p > 1, \sigma > -1, \sigma \neq 0 \) and \( w \neq 0 \). Namely, we identify the critical exponent for problem (1.1), which separates the nonexistence/existence of global-in-time solutions of (1.1), and show the discontinuity of this critical exponent at \( \sigma = 0 \).

In the case \( w \equiv 0 \) with a nonnegative initial data, problem (1.1) reduces to

\[
\begin{align*}
\begin{cases}
\partial_t u &= \Delta u + u^p \quad \text{in } (0, T) \times \mathbb{R}^N, \\
u(0, x) &= u_0(x) \geq 0 \quad \text{in } \mathbb{R}^N.
\end{cases}
\end{align*}
\]

(1.2)

Fujita [6] established the following results for problem (1.2):

(I) If \( 1 < p < 1 + 2/N \), then (1.2) admits no nontrivial global-in-time solutions.

(II) If \( p > 1 + 2/N \), then (1.2) possesses global-in-time solutions for some small \( u_0 \).

Later, it was shown that the borderline case \( p = 1 + 2/N \) belongs to the blow-up category (see e.g. [1, 8, 10, 13, 14]). From above results, the number

\[ p_F := 1 + \frac{2}{N} \]

(1.3)

is called the critical Fujita exponent, which separates the nonexistence/existence of global-in-time solutions of (1.2). In [14], Weissler also proved that, for the case \( p > p_F \), if \( \|u_0\|_{L^p} \) is sufficiently small with

\[ d = \frac{N(p - 1)}{2} > 1, \]

then (1.2) has global positive solutions.

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In the case \( \sigma = 0 \), problem (1.1) reduces to
\[
\begin{align*}
\partial_t u &= \Delta u + |u|^p + w(x) & \text{in} & \quad (0, T) \times \mathbb{R}^N, \\
u(0, x) &= u_0(x) & \text{in} & \quad \mathbb{R}^N.
\end{align*}
\] 
(1.4)

Problem (1.4) was investigated by Bandle et al. [2]. Namely, it was shown that

(I) If \( 1 < p < p^* \) and \( \int_{\mathbb{R}^N} w(x) \, dx > 0 \), where
\[
p^* = \begin{cases}
\infty & \text{if } N = 1, 2, \\
\frac{N}{N-2} & \text{if } N \geq 3,
\end{cases}
\]
then (1.4) has no global solutions.

(II) If \( N \geq 3 \) and \( p = p^* \), then for any \( \delta > 0 \), there exists \( \epsilon > 0 \) such that (1.4) has global solutions provided that
\[
\max\{|w(x)|, |u_0(x)|\} \leq \frac{\epsilon}{(1 + |x|^{N+\delta})}
\]
regardless of whether or not \( \int_{\mathbb{R}^N} w(x) \, dx > 0 \).

(III) If \( N \geq 3 \), \( p = p^* \), \( \int_{\mathbb{R}^N} w(x) \, dx > 0 \), \( w(x) = O(|x|^{-\sigma}) \) as \( |x| \to \infty \) for some \( \sigma > 0 \), and either \( u \geq 0 \) or
\[
\int_{|x| > R} \frac{w^-(y)}{|x-y|^{N-2}} \, dy = o(|x|^{-N+2})
\]
when \( R \) is large, then (1.4) has no global solutions. Here, \( w^- = \max\{-w, 0\} \).

In [15], Zhang investigated the initial value problem
\[
\begin{align*}
\partial_t u &= \Delta u + u^p + w(x) & \text{in} & \quad (0, \infty) \times M^N, \\
u(0, x) &= u_0(x) & \text{in} & \quad M^N,
\end{align*}
\] 
(1.6)

where \( N \geq 3 \), \( M^N \) is a non-compact complete Riemannian manifold, \( \Delta \) is the Laplace-Beltrami operator, \( u_0 \geq 0 \) and \( w \geq 0 \) is a nontrivial \( L^1_{loc} \) function. He proved that
\[
p^*_M = \frac{\alpha}{\alpha - 2},
\]
where \( \alpha > 2 \) is the decay rate of the fundamental solution of \( \partial_t u = \Delta u \) in \( M^N \), is the critical Fujita exponent for problem (1.6). (See also [12].) Moreover, it was shown that if the Ricci curvature of \( M^N \) is non-negative, then \( p^*_M \) belongs to the blow-up case. Note that in the case \( M^N = \mathbb{R}^N (N \geq 3) \), one has \( p^*_M = p^* \), where \( p^* \) is given by (1.5). On the other hand, observe that \( p^* > p_F \), where \( p_F \) is the critical Fujita exponent of (1.2) given by (1.3). This means that the additional forcing term \( w = w(x) \geq 0 \), no matter how small it is, has the effect of increasing the critical exponent. A similar phenomenon was observed recently for a nonlocal-in-time nonlinear heat equation [9].

In all the above cited works, the considered inhomogeneous term depends only of space \( w(x) \). In this paper we investigate, for the first time, the parabolic equation (1.1) with the forcing term \( t^\sigma w(x) \). We show that there is an exponent \( p^*(\sigma) \) which is critical in the following sense: when \( 1 < p < p^*(\sigma) \) and \( \int_{\mathbb{R}^N} w(x) \, dx > 0 \), the solution of problem (1.1) blows up in finite time; when \( p \geq p^*(\sigma) \), the solution is global for suitably small \( u_0 \) and \( w \).

As usual, (1.1) is equivalent in the appropriate setting to
\[
u(t) = e^{t\Delta} u_0 + \int_0^t e^{(t-s)\Delta} (|u(s)|^p + s^\sigma w) \, ds, \quad 0 \leq t \leq T,
\]
(1.7)
where \( e^{t\Delta} \) is the heat semigroup on \( \mathbb{R}^N \). Namely, for \( u_0 \in C_0(\mathbb{R}^N) \) and \( w \in C_0^\infty(\mathbb{R}^N) \) with \( \alpha \in (0, 1) \), one can see that the solution \( u \) of the integral equation (1.7) satisfies (1.1) in the classical sense (see Proposition 2.1.).
Our obtained results are given by the following theorems. We discuss separately the cases \(-1 < \sigma < 0\) and \(\sigma > 0\).

**Theorem 1.1.** Let \(N \geq 2\), \(p > 1\) and \(\sigma \in (-1,0)\). Assume \(w \in C_0^\alpha(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)\) for some \(\alpha \in (0,1)\). Then the following holds.

(i) Assume

\[
1 < p < \frac{N - 2\sigma}{N - 2 - 2\sigma},
\]

and \(\int_{\mathbb{R}^N} w(x) \, dx > 0\). Then for any \(u_0 \in C_0(\mathbb{R}^N)\), the solution of (1.7) blows up in finite time.

(ii) Assume

\[
p \geq \frac{N - 2\sigma}{N - 2 - 2\sigma}.
\]

Put

\[
d = \frac{N(p - 1)}{2}, \quad k = \frac{d}{p(\sigma + 1) - \sigma}.
\]

Then for any \(u_0 \in C_0(\mathbb{R}^N) \cap L^d(\mathbb{R}^N)\) and \(w\) with \(\|u_0\|_{L^d(\mathbb{R}^N)} + \|w\|_{L^k(\mathbb{R}^N)}\) is sufficiently small, the solution \(u\) of (1.7) exists globally.

**Theorem 1.2.** Let \(N \geq 2\), \(p > 1\) and \(\sigma > 0\). Assume \(w \in C_0^\alpha(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)\) for some \(\alpha \in (0,1)\) and \(\int_{\mathbb{R}^N} w(x) \, dx > 0\). Then for any \(u_0 \in C_0(\mathbb{R}^N)\), the solution of (1.7) blows up in finite time.

**Remark 1.3.** (i) No assumption on the sign of \(u_0\) is needed in Theorems 1.1 and 1.2.

(ii) From Theorems 1.1 and 1.2, one observes that the critical exponent for (1.1) is given by

\[
p^*(\sigma) := \begin{cases} 
\frac{N - 2\sigma}{N - 2 - 2\sigma} & \text{if } -1 < \sigma < 0, \\
\infty & \text{if } \sigma > 0.
\end{cases}
\]

Observe also that when \(N \geq 3\), \(\lim_{\sigma \to 0^+} p^*(\sigma) \neq \lim_{\sigma \to 0^-} p^*(\sigma)\).

(iii) Observe that \(\lim_{\sigma \to 0^-} p^*(\sigma) = p^*\) (which is given by (1.5)) is the critical exponent for problem (1.4) and also the critical exponent for problem (1.6) in the case \(M^N = \mathbb{R}^N, N \geq 3\).

(iv) In the assertion (ii) of Theorem 1.1, one can relax the smallness assumptions for initial data \(u_0\) and the inhomogeneous term \(w(x)\) from the Lebesgue space \(L^r\) to the Lorentz space \(L^{r,\infty}\) (the weak \(L^r\) space). In fact, applying the same argument as in the proof of the assertion (ii) of Theorem 1.1 with the weak Young inequality (see e.g. [5, (G2)]), one can get the same conclusion for the case \(p > p^*(\sigma)\). Then we can consider

\[
|u_0(x)| \sim |x|^{-\frac{N}{\sigma}}, \quad |w(x)| \sim |x|^{-\frac{N}{\sigma}}
\]

for sufficiently large \(x\), which do not belong to \(L^d(\mathbb{R}^N)\) and \(L^k(\mathbb{R}^N)\), respectively. Furthermore, for the critical case \(p = p^*(\sigma)\), namely \(k = 1\), by (4.3), one can only relax the smallness assumption for initial data \(u_0\). Therefore it is still open that, for \(w\) which behaves like \(|x|^{-N}\) for sufficiently large \(x\), there exists a global-in-time solution of (1.7) or not.

The rest of the paper is organized as follows. In Section 2, we investigate the local existence properties for equation (1.1). The assertion (i) of Theorem 1.1, as well as Theorem 1.2 are established in Section 3. The next section is devoted to the proof of the global existence result given by the assertion (ii) of Theorem 1.1.
2. LOCAL EXISTENCE

We first introduce some notations. For any $1 \leq r \leq \infty$, we denote by $\| \cdot \|_{L^r}$ the usual norm of $L^r := L^r(\mathbb{R}^N)$. Let $C_0(\mathbb{R}^N)$ be the space of continuous functions in $\mathbb{R}^N$ vanishing at infinity. For $\alpha \in (0, 1)$, let $C_0^\alpha(\mathbb{R}^N) = C^\alpha(\mathbb{R}^N) \cap C_0(\mathbb{R}^N)$. By the letter $C$, we denote generic positive constants and they may have different values also within the same line.

Further, let us recall some well known facts about the semigroup $e^{t\Delta}$. There exists a positive constant $c_1$ such that for any $1 \leq q \leq r \leq \infty$, one has

$$\|e^{t\Delta} \varphi\|_{L^r} \leq c_1 t^{-\frac{N}{q}} \|\varphi\|_{L^q}, \quad t > 0, \quad (2.1)$$

for any $\varphi \in L^q$. In particular,

$$\|e^{t\Delta} \varphi\|_{L^q} \leq \|\varphi\|_{L^q}, \quad t > 0. \quad (2.2)$$

Furthermore, for $\varphi \in C_0(\mathbb{R}^N)$, it holds that (see e.g. [7])

$$\lim_{t \to 0^+} e^{t\Delta} \varphi(x) = \varphi(x), \quad x \in \mathbb{R}^N.$$  

Applying these estimates, we prove the following local existence result.

**Proposition 2.1.** Let $N \geq 2$, $p > 1$, $\sigma > -1$ with $\sigma \neq 0$. Assume $u_0 \in C_0(\mathbb{R}^N)$ and $w \in C_0^\alpha(\mathbb{R}^N)$ for some $\alpha \in (0, 1)$. Then the following holds.

(i) There exists $0 < T < \infty$ and a unique solution $u \in C([0, T], C_0(\mathbb{R}^N))$ of (1.7). Furthermore, the solution $u$ satisfies (1.1) in the classical sense.

(ii) The solution $u$ can be extended to a maximal interval $[0, T_{\text{max}})$, where $0 < T_{\text{max}} \leq \infty$, and if $T_{\text{max}} < \infty$, then $\lim_{t \to T_{\text{max}}} \|u(t)\|_{L^\infty} = \infty$.

(iii) If, in addition, $u_0, w \in L^r$, where $1 \leq r < \infty$, then $u \in C([0, T_{\text{max}}), C_0(\mathbb{R}^N)) \cap C([0, T_{\text{max}}), L^r)$.

**Proof.** The proof of this proposition follows from standard arguments. For the completeness of this paper, we write the details.

We first prove the assertion (i). For the uniqueness of solutions, let $T > 0$ and $u, v \in C([0, T], C_0(\mathbb{R}^N))$ be two solutions of (1.7). Since it holds that

$$|a^p - b^p| \leq p \max\{a^{p-1}, b^{p-1}\} |a - b|, \quad a, b \geq 0, \quad (2.3)$$

by (2.2), one has

$$\|u(t) - v(t)\|_{L^\infty} \leq C \int_0^t \|u(s) - v(s)\|_{L^\infty} \, ds, \quad 0 \leq t \leq T.$$  

This together with the Gronwall inequality imply that $u(t, x) = v(t, x)$ for all $t \in [0, T]$ and $x \in \mathbb{R}^N$.

For the existence of solutions, given $0 < T \leq 1$, we define the set

$$\mathcal{V} = \left\{ u \in C([0, T], C_0(\mathbb{R}^N)) : \|u\|_{L^\infty([0, T], L^\infty)} \leq 2\delta_{\infty}(u_0, w) \right\}, \quad (2.4)$$

where $\delta_{\infty}(u_0, w) = \max\{\|u_0\|_{L^\infty}, \|w\|_{L^\infty}\}$. We endow $\mathcal{V}$ with the distance generated by the norm of $C([0, T], C_0(\mathbb{R}^N))$, that is,

$$d(u, v) = \|u - v\|_{L^\infty([0, T], L^\infty)}, \quad u, v \in \mathcal{V}. \quad (2.5)$$  

Given $u \in \mathcal{V}$, let

$$(Fu)(t) := e^{t\Delta} u_0 + \int_0^t e^{(t-s)\Delta} (|u(s)|^p + s^\sigma w) \, ds, \quad 0 \leq t \leq T. \quad (2.6)$$
Since \( u_0, w \in C_0(\mathbb{R}^N) \), \( \sigma > -1 \) and \( u \in \mathcal{V} \), one can easily verify that \( Fu \in C([0,T],C_0(\mathbb{R}^N)) \). On the other hand, by (2.2), one has

\[
\|(Fu)(t)\|_{L^\infty} \leq \|e^{tA}u_0\|_{L^\infty} + \int_0^t \left\|e^{(t-s)\Delta}(u(s))^p\right\|_{L^\infty} ds + \int_0^t s^\sigma \left\|e^{(t-s)\Delta}w\right\|_{L^\infty} ds \\
\leq \|u_0\|_{L^\infty} + T\|u\|_{L^\infty((0,T),L^\infty)}^p + \frac{T^{\sigma+1}}{\sigma + 1}\|w\|_{L^\infty} \\
\leq \left(1 + \frac{T^{\sigma+1}}{\sigma + 1}\right)\delta_\infty(u_0, w) + 2^p\delta_\infty(u_0, w)^{p-1}T
\]

for all \( 0 < t \leq T \). This yields

\[
\|Fu\|_{L^\infty((0,T),L^\infty)} \leq \left(1 + \frac{T^{\sigma+1}}{\sigma + 1} + 2^p\delta_\infty(u_0, w)^{p-1}T\right)\delta_\infty(u_0, w). \tag{2.7}
\]

Let \( T > 0 \) be a sufficiently small constant such that

\[
\frac{T^{\sigma+1}}{\sigma + 1} + 2^p\delta_\infty(u_0, w)^{p-1}T \leq 1. \tag{2.8}
\]

Then, by (2.7), one obtains

\[
\|Fu\|_{L^\infty((0,T),L^\infty)} \leq 2\delta_\infty(u_0, w),
\]

which yields \( F(\mathcal{V}) \subset \mathcal{V} \). Furthermore, for \( u, v \in \mathcal{V} \), by (2.2), (2.3) and (2.6), one has

\[
\|(Fu)(t) - (Fv)(t)\|_{L^\infty} \leq \int_0^t \left\|e^{(t-s)\Delta}(|u(s)|^p - |v(s)|)^p\right\|_{L^\infty} ds \\
\leq \int_0^t \|u(s)|^p - |v(s)|^p\|_{L^\infty} ds \\
\leq p\delta_\infty(u_0, w)^{p-1}T\|u - v\|_{L^\infty((0,T),L^\infty)} \tag{2.9}
\]

for all \( 0 < t \leq T \). By (2.8) and (2.9), one obtains

\[
\|Fu - Fv\|_{L^\infty((0,T),L^\infty)} \leq \frac{p}{2^p}\|u - v\|_{L^\infty((0,T),L^\infty)}.
\]

Since \( 2^p > p \), under the condition (2.8), the self-mapping \( F : \mathcal{V} \to \mathcal{V} \) is a contraction. Moreover, since \( (\mathcal{V}, d) \) is a complete metric space, from the Banach contraction principle, it follows that (1.7) admits a solution \( u \in \mathcal{V} \), which is the unique solution to (1.7) in \( C([0,T],C_0(\mathbb{R}^N)) \). Furthermore, since \( \sigma > -1 \) and under the assumption on the function \( w \), applying similar arguments in regularity theorems for second order parabolic equation (see e.g. [4, Chapter 1]) to (1.7), one see that the solution \( u \) satisfies (1.1) in the classical sense, and the assertion (i) follows.

Next we prove the assertion (ii). Applying the uniqueness of solutions, we see that the solution \( u \), which is obtained above, can be extended to a maximal interval \([0,T_{\text{max}}]\), where

\[
T_{\text{max}} = \sup \left\{ t > 0 : (1.7) \text{ admits a solution in } C([0,t],C_0(\mathbb{R}^N)) \right\}.
\]

Suppose that \( T_{\text{max}} < \infty \), and there exists \( M > 0 \) such that

\[
\|u(t)\|_{L^\infty} \leq M, \quad 0 \leq t < T_{\text{max}}. \tag{2.10}
\]

Let \( t_* \) be such that \( T_{\text{max}}/2 < t_* < T_{\text{max}} \). For \( 0 < \tau < T_{\text{max}} \), we define the set

\[
\mathcal{W} = \{ v \in C([0,\tau],C_0(\mathbb{R}^N)) : \|v\|_{L^\infty((0,\tau),L^\infty)} \leq 2\delta_\infty(M, w) \},
\]

where \( \delta_\infty(M, w) = \max \{ M, \|w\|_{L^\infty} \} \). Given \( v \in \mathcal{W} \), let

\[
(Gv)(t) := e^{t\Delta}u(t_*) + \int_0^t e^{(t-s)\Delta}|v(s)|^p ds + \int_0^t (s + t_*)^\sigma e^{(t-s)\Delta}w ds, \quad 0 \leq t \leq \tau.
\]
Similarly to $\mathcal{V}$ and $Fu$, we endow $\mathcal{W}$ with the distance $d$, which is defined by (2.5), and we have $Gv \in C([0, \tau], C_0(\mathbb{R}^N))$. Furthermore, by (2.2) and (2.10), we obtain

$$
\|(Gv(t))\|_{L^\infty} \leq \|e^{\Delta u(t_s)}\|_{L^\infty} + \int_0^t \left\| e^{(t-s)\Delta} |v(s)|^p \right\|_{L^\infty} \, ds
$$

$$
+ \int_0^t (s + t_s)^\sigma \left\| e^{(t-s)\Delta} w \right\|_{L^\infty} \, ds
$$

$$
\leq \|u(t_s)\|_{L^\infty} + \tau \|v\|_p \|_{L^\infty((0, \tau), L^\infty)} + \frac{(t + t_s)^{\sigma+1} - t_s^{\sigma+1}}{\sigma+1} \|w\|_{L^\infty}
$$

$$
\leq \delta_{\infty}(M, w) + 2^p \delta_{\infty}(M, w)^p \tau + \frac{(t + t_s)^{\sigma+1} - t_s^{\sigma+1}}{\sigma+1} \delta_{\infty}(M, w)
$$

(2.11)

for all $0 \leq t \leq \tau$. On the other hand, applying the mean value theorem, we see that, for any $0 < t \leq \tau$, there exists a constant $c_{t, t_s} \in (t_s, t + t_s)$ such that

$$
\frac{(t + t_s)^{\sigma+1} - t_s^{\sigma+1}}{\sigma+1} = c_{t, t_s}^\sigma t \leq c_{t, t_s}^\sigma \tau.
$$

(2.12)

Since it holds from the definition of $t_s$ that

$$
\frac{T_{\max}}{2} < t_s < c_{t, t_s} < t + t_s < 2T_{\max},
$$

by (2.12), we have

$$
\frac{(t + t_s)^{\sigma+1} - t_s^{\sigma+1}}{\sigma+1} \leq C_{\sigma} \tau
$$

for all $0 < t \leq \tau$, where $C_{\sigma} = T_{\max}^{\sigma} \max\{2^\sigma, 2^{-\sigma}\}$. This together with (2.11) implies

$$
\|Gv\|_{L^\infty((0, \tau), L^\infty)} \leq \left(1 + 2^p \delta_{\infty}(M, w)^p \tau + C_{\sigma} \tau\right) \delta_{\infty}(M, w).
$$

(2.13)

Let $\tau > 0$ be a sufficiently small constant such that

$$
2^p \delta_{\infty}(M, w)^p \tau + C_{\sigma} \tau \leq 1.
$$

(2.14)

Then, by (2.13), we obtain

$$
\|Gv\|_{L^\infty((0, \tau), L^\infty)} \leq 2\delta_{\infty}(M, w),
$$

which yields $G(\mathcal{W}) \subset \mathcal{W}$. Furthermore, similarly to (2.9) with (2.14), one can see that under the condition (2.14), the self-mapping $G : \mathcal{W} \to \mathcal{W}$ is a contraction. Applying the Banach contraction principle, we see that there exists a unique function $v \in \mathcal{W}$ satisfying

$$
v(t) = e^{t\Delta} u(t_s) + \int_0^t e^{(t-s)\Delta} |v(s)|^p \, ds + \int_0^t (s + t_s)^\sigma e^{(t-s)\Delta} w \, ds, \quad 0 \leq t \leq \tau.
$$

For $\max\{T_{\max}/2, T_{\max} - \tau\} < \tilde{t} < T_{\max}$, let

$$
\tilde{u}(t) = \begin{cases} u(t) & \text{if } 0 \leq t \leq \tilde{t}, \\
v(t - \tilde{t}) & \text{if } \tilde{t} \leq t \leq \tilde{t} + \tau. \end{cases}
$$

Then we observe that $\tilde{u} \in C([0, \tilde{t} + \tau], C_0(\mathbb{R}^N))$ is a solution to (1.7) and $\tilde{t} + \tau > T_{\max}$, which contradicts the definition of $T_{\max}$. Hence, we see that if $T_{\max} < \infty$, then $\lim_{t \to T_{\max}} \|u(t)\|_{L^\infty} = \infty$, and the assertion (ii) follows.

Finally we prove the assertion (iii). Instead of the functional space $\mathcal{V}$ given by (2.4), we define the set

$$
\mathcal{V}_r = \{ u \in C([0, T], C_0(\mathbb{R}^N)) \cap C([0, T], L^r) : \|u\|_{L^\infty((0, T), L^\infty)} \leq 2\delta_{\infty}(u_0, w), \quad \|u\|_{L^\infty((0, T), L^r)} \leq 2\delta_r(u_0, w) \},
$$
where \( \delta_r(u_0, w) = \max \{ \|u_0\|_{L^r}, \|w\|_{L^r} \} \). We endow \( \mathcal{V}_r \) with the distance
\[
d_r(u, v) = \|u - v\|_{L^\infty(0,T),L^r} + \|u - v\|_{L^\infty(0,T),L^r}, \quad u, v \in \mathcal{V}_r.
\]
Since it holds that \( \|u(t)|p|\|_{L^r} \leq \|u(t)|p-1\|_{L^r} |u(t)|_{L^r} \), applying same argument as in the proof of the assertion (i), we obtain a unique solution \( u \in \mathcal{V}_r \), and we see that \( u \in C([0,T_{\max}), C_0(\mathbb{R}^N)) \cap C([0,T_{\max}), L^r) \). Thus the assertion (iii) follows. \( \square \)

3. Blow-up of solutions

In order to prove the blow-up results given by Theorems 1.1 and 1.2, we use the well-known rescaled test function method (see \([11]\)).

**Proof of the assertion (i) of Theorem 1.1.** We argue by contradiction. Suppose that \( T_{\max} = \infty \), i.e. \( u \in C([0, \infty), C_0(\mathbb{R}^N)) \) is a global solution of (1.7). We need to introduce two cut-off functions. Let \( \xi, \eta \in C^\infty([0, \infty)) \) satisfy
\[
0 \leq \xi \leq 1; \; \xi \equiv 1 \text{ in } [0, 1]; \; \xi \equiv 0 \text{ in } [2, \infty) \tag{3.1}
\]
and
\[
\eta \geq 0, \; \eta \neq 0, \; \text{supp}(\eta) \subset (0, 1). \tag{3.2}
\]
For sufficiently large positive constant \( T \), we put
\[
\varphi_T(t, x) = \eta_T(t) \mu_T(x), \quad (t, x) \in [0, T] \times \mathbb{R}^N,
\]
where
\[
\eta_T(t) = \eta \left( \frac{t}{T} \right)^{p-1}, \quad \mu_T(x) = \xi \left( \frac{|x|^2}{T} \right)^{\frac{2p}{p-1}}. \tag{3.3}
\]
By (3.1) and (3.3), it can be easily seen that
\[
|\Delta \mu_T(x)| \leq \frac{C}{T} \xi \left( \frac{|x|^2}{T} \right)^{\frac{2p}{p-1}}, \quad x \in \mathbb{R}^N. \tag{3.4}
\]
Since the solution \( u \) of (1.7) satisfies (1.1) in the classical sense, multiplying (1.1) by \( \varphi = \varphi_T \), and integrating by parts over \( (0,T) \times \mathbb{R}^N \), we obtain
\[
\int_0^T \int_{\mathbb{R}^N} |u|^p \varphi_T \, dx \, dt + \int_0^T \int_{\mathbb{R}^N} t^{\sigma} w(x) \varphi_T \, dx \, dt + \int_{\mathbb{R}^N} u_0(x) \varphi_T(0, x) \, dx
\]
\[
= - \int_0^T \int_{\mathbb{R}^N} u \Delta \varphi_T \, dx \, dt - \int_0^T \int_{\mathbb{R}^N} u \partial_t \varphi_T \, dx \, dt.
\]
On the other hand, by (3.2), it holds that
\[
\int_{\mathbb{R}^N} u_0(x) \varphi_T(0, x) \, dx = \eta_T(0) \int_{\mathbb{R}^N} u_0(x) \mu_T(x) \, dx = 0.
\]
Therefore, we deduce that
\[
\int_0^T \int_{\mathbb{R}^N} |u|^p \varphi_T \, dx \, dt + \int_0^T \int_{\mathbb{R}^N} t^{\sigma} w(x) \varphi_T \, dx \, dt
\]
\[
\leq \int_0^T \int_{\mathbb{R}^N} |u| |\Delta \varphi_T| \, dx \, dt + \int_0^T \int_{\mathbb{R}^N} |u| |\partial_t \varphi_T| \, dx \, dt. \tag{3.5}
\]
We claim that
\[
\int_0^T \int_{\mathbb{R}^N} t^{\sigma} w(x) \varphi_T \, dx \, dt \geq C T^{\sigma+1} \int_{\mathbb{R}^N} w(x) \, dx. \tag{3.6}
\]
Indeed, we have
\[
\int_0^T \int_{\mathbb{R}^N} t^\sigma w(x) \varphi_T \, dx \, dt = \left( \int_0^T t^\eta \left( \frac{t}{T} \right)^{\frac{p}{p-1}} \, dt \right) \left( \int_{\mathbb{R}^N} w(x) \xi \left( \frac{|x|^2}{T} \right)^{\frac{2p}{p-1}} \, dx \right). \tag{3.7}
\]
From the conditions imposed on the function \( w \), and by the dominated convergence theorem, we obtain
\[
\lim_{T \to \infty} \int_{\mathbb{R}^N} w(x) \xi \left( \frac{|x|^2}{T} \right)^{\frac{2p}{p-1}} \, dx = \int_{\mathbb{R}^N} w(x) \, dx > 0.
\]
This implies that, for a sufficiently large \( T > 0 \), we have
\[
\int_{\mathbb{R}^N} w(x) \xi \left( \frac{|x|^2}{T} \right)^{\frac{2p}{p-1}} \, dx \geq \frac{1}{2} \int_{\mathbb{R}^N} w(x) \, dx. \tag{3.8}
\]
On the other hand, we have
\[
\int_0^T t^\sigma \eta \left( \frac{t}{T} \right)^{\frac{p}{p-1}} \, dt = T^\sigma + 1 \int_0^1 s^\sigma \eta(s)^{\frac{p}{p-1}} \, ds. \tag{3.9}
\]
Using (3.2), (3.7), (3.8) and (3.9), (3.6) follows.
Next, applying the \( \varepsilon \)-Young inequality with \( \varepsilon = \frac{1}{T} \), we obtain
\[
\int_0^T \int_{\mathbb{R}^N} |u| |\Delta \varphi_T| \, dx \, dt \leq \frac{1}{2} \int_0^T \int_{\mathbb{R}^N} |u|^p \varphi_T \, dx \, dt + CI_1(T) \tag{3.10}
\]
and
\[
\int_0^T \int_{\mathbb{R}^N} |u| |\partial_t \varphi_T| \, dx \, dt \leq \frac{1}{2} \int_0^T \int_{\mathbb{R}^N} |u|^p \varphi_T \, dx \, dt + CI_2(T), \tag{3.11}
\]
where
\[
I_1(T) := \int_0^T \int_{\mathbb{R}^N} \varphi_T^{-1} |\Delta \varphi_T| \frac{p}{p-1} \, dx \, dt, \quad I_2(T) := \int_0^T \int_{\mathbb{R}^N} \varphi_T^{-1} |\partial_t \varphi_T| \frac{p}{p-1} \, dx \, dt.
\]
By the definition of \( \varphi_T \) with (3.3) and (3.4), we obtain
\[
I_1(T) = \left( \int_0^T \eta_T(t) \, dt \right) \left( \int_{\mathbb{R}^N} \mu_T(x)^{\frac{1}{p-1}} |\Delta \mu_T(x)| \frac{p}{p-1} \, dx \right)
\leq CT \int_{\mathbb{R}^N} \mu_T(x)^{\frac{1}{p-1}} |\Delta \mu_T(x)| \frac{p}{p-1} \, dx \leq CT^{1+\frac{N}{p}-\frac{p}{p-1}} \int_{1<|y|<\sqrt{T}} 1 \, dy,
\]
which yields
\[
I_1(T) \leq CT^{1+\frac{N}{p}-\frac{p}{p-1}}. \tag{3.12}
\]
Similarly, we have
\[
I_2(T) = \left( \int_0^T \eta_T(t)^{\frac{1}{p-1}} |\eta_T(t)|^{\frac{p}{p-1}} \, dt \right) \left( \int_{\mathbb{R}^N} \mu_T(x) \, dx \right). \tag{3.13}
\]
On the other hand, we have
\[
\int_{\mathbb{R}^N} \mu_T(x) \, dx = T^\frac{N}{p} \int_{1<|y|<\sqrt{T}} \xi(|y|^2)^{\frac{2p}{p-1}} \, dy = CT^\frac{N}{p} \tag{3.14}
\]
and
\[
\int_0^T \eta_T(t)^{\frac{1}{p-1}} |\eta_T(t)|^{\frac{p}{p-1}} \, dt = \lambda \frac{p}{p-1} T^{1-\frac{p}{p-1}} \int_0^1 |\eta'(s)|^{\frac{p}{p-1}} \, ds. \tag{3.15}
\]
Therefore, using (3.13), (3.14) and (3.15), we obtain
\[
I_2(T) \leq CT^{1+\frac{N}{p}-\frac{p}{p-1}}. \tag{3.16}
\]
Hence, combining (3.5), (3.6), (3.10), (3.11), (3.12) and (3.16), we see that
\[ T^{\sigma+1} \int_{\mathbb{R}^N} w(x) \, dx \leq I_1(T) + I_2(T) \leq CT^{1+\frac{N}{2}} \frac{\rho}{p-1}, \]
which yields
\[ \int_{\mathbb{R}^N} w(x) \, dx \leq CT^{\frac{N}{p} - \sigma - \frac{p}{p-1}}. \]  
(3.17)

Passing to the limit as \( T \to \infty \) in (3.17) with (1.8), we obtain
\[ \int_{\mathbb{R}^N} w(x) \, dx \leq 0, \]
which contradicts the fact that \( \int_{\mathbb{R}^N} w(x) \, dx > 0 \). This completes the proof of the assertion (i) of Theorem 1.1.

\[ \square \]

Proof of Theorem 1.2. As previously, suppose that \( u \in C([0, \infty), C_0(\mathbb{R}^N)) \) is a global solution of (1.7). For sufficiently large positive constants \( T \) and \( R \), we put
\[ \psi_{T,R}(t,x) = \eta_T(t) \mu_R(x), \quad (t,x) \in [0,T] \times \mathbb{R}^N, \]
where \( \eta_T \) is given by (3.3),
\[ \mu_R(x) = \xi \left( \frac{|x|^2}{R^2} \right)^{\frac{2p}{p-1}}, \quad x \in \mathbb{R}^N \]
and \( \xi \in C^\infty([0,\infty)) \) is a cut-off function satisfying (3.1). Replacing \( \varphi_T \) with \( \psi_{T,R} \) and applying same arguments as in the proof of the assertion (i) of Theorem 1.1, we obtain
\[ \int_{\mathbb{R}^N} w(x) \, dx \leq C \left( T^{-\sigma} R^{N-\frac{2p}{p-1}} + T^{-\frac{p}{p-1}-\sigma} R^N \right). \]  
(3.18)

Fixing \( R \) and passing to the limit as \( T \to \infty \) in (3.18), since \( \sigma > 0 \), we obtain
\[ \int_{\mathbb{R}^N} w(x) \, dx \leq 0, \]
which contradicts the fact that \( \int_{\mathbb{R}^N} w(x) \, dx > 0 \), and the proof of Theorem 1.2 is complete. \( \square \)

4. Global existence

The following lemma will be used in the proof of the global existence part of Theorem 1.1.

Lemma 4.1. Let \( N \geq 2 \) and \( -1 < \sigma < 0 \). Assume (1.9). Then
\[ 2\sigma p^2 - (N + 2\sigma - 2)p + N < 0. \]  
(4.1)

Proof. Assume (1.9). Then
\[ N \geq s^* := 2\sigma + \frac{2p}{p-1}. \]
Consider the function
\[ \varrho(s) = 2\sigma p^2 - (s + 2\sigma - 2)p + s, \quad s \geq s^*. \]
Since it follows from \( p > 1 \) that \( \varrho \) is a decreasing function, we obtain \( \varrho(s) \leq \varrho(s^*) \) for \( s \geq s^* \). On the other hand, we have \( \varrho(s^*) = 2\sigma(p-1)^2 < 0 \). Therefore we see that \( \varrho(s) < 0 \) for \( s \geq s^* \), and taking \( s = N \) in this inequality, (4.1) follows. \( \square \)
Proof of the assertion (ii) of Theorem 1.1. The proof is inspired by that of [3, Theorem 1.1]. By (1.9) and (4.1), we can take a positive constant $q$ satisfying
\[
\frac{2}{N} \max \left\{ \frac{1}{p(p-1)}, \sigma + \frac{1}{p-1} \right\} < \frac{1}{q} < \min \left\{ \frac{2}{N(p-1)}, \frac{1}{p} \right\}.
\] (4.2)
Furthermore, it follows that
\[
q > d > k \geq 1,
\] (4.3)
where $d$ and $k$ are given by (1.10). Let
\[
\beta = \frac{1}{p-1} - \frac{N}{2q}.
\] (4.4)
Then we verify easily that
\[
\beta > 0, \quad \beta p < 1
\] and
\[
\beta - \frac{N}{2} \left( \frac{1}{d} - \frac{1}{q} \right) = \beta (1-p) + 1 - \frac{N}{2q} (p-1) = \beta - \frac{N}{2} \left( \frac{1}{k} - \frac{1}{q} \right) + \sigma + 1 = 0.
\] (4.5)
Let $\delta$ be a sufficiently small positive constant. We define the set
\[
\Xi = \left\{ u \in L^\infty((0, \infty), L^q(\mathbb{R}^N)) : \sup_{t>0} t^\beta \|u(t)\|_{L^q} \leq \delta \right\}.
\]
We endow $\Xi$ with the distance
\[
d(u, v) = \sup_{t>0} t^\beta \|u(t) - v(t)\|_{L^q}, \quad u, v \in \Xi.
\]
The proof is inspired by that of [3, Theorem 1.1]. By (4.2) and (4.4), we have
\[
\|e^{t\Delta} u_0\|_{L^q} \leq c_1 t^{-\frac{N}{2q} \left( \frac{1}{d} - \frac{1}{q} \right)} \|u_0\|_{L^d} = c_1 t^{-\beta} \|u_0\|_{L^d}, \quad t > 0,
\] (4.7)
where $c_1$ is the constant given in (2.1). Furthermore, since it follows form (4.2) that $q > p$, by (1.9) and (4.5), we obtain
\[
\int_0^t \left\| e^{(t-s)\Delta} |u(s)|^p \right\|_{L^q} ds \leq c_1 \int_0^t (t-s)^{-\frac{N}{2q}(p-1)} \|u(s)|^p\|_{L^q} ds
\]
\[
\leq c_1 \delta p \int_0^t s^{-\beta p} (t-s)^{-\frac{N}{2q}(p-1)} ds
\]
\[
= c_1 \delta p t^{-\frac{N}{2q}(p-1) + 1 - \beta p} B \left( 1 - \beta p, 1 + \frac{N}{2q}(p-1) \right)
\]
\[
= c_1 C \delta p t^{-\beta}, \quad t > 0,
\] (4.8)
where $B$ denotes the beta function. We note that by (4.2) and (4.4), $B \left( 1 - \beta p, 1 + \frac{N}{2q}(p-1) \right)$ is well-defined. Similarly, it holds that
\[
\int_0^t s^\sigma \left\| e^{(t-s)\Delta} w \right\|_{L^q} ds \leq c_1 \|w\|_{L^k} \int_0^t s^\sigma (t-s)^{-\frac{N}{2q} \left( \frac{1}{d} - \frac{1}{q} \right)} ds
\]
\[
= c_1 t^{-\frac{N}{2q} \left( \frac{1}{d} - \frac{1}{q} \right) + 1 + \sigma} B \left( \sigma + 1, 1 + \frac{N}{2} \left( \frac{1}{k} - \frac{1}{q} \right) \right) \|w\|_{L^k}
\]
\[
= c_1 C t^{-\beta} \|w\|_{L^k}, \quad t > 0.
\] (4.9)
We note again that by $\sigma > -1$ and (4.2), $B \left( \sigma + 1, 1 - \frac{N}{2} \left( \frac{1}{k} - \frac{1}{q} \right) \right)$ is well-defined. Then, by (4.6), (4.7), (4.8) and (4.9), we have
\[ t^\beta \| (Su)(t) \|_{L^q} \leq C_* \left( \|u_0\|_{L^d} + \delta^p + \|w\|_{L^k} \right), \quad t > 0, \]
where $C_* > 0$ is a constant, independent of $\delta$. Therefore, we can choose a sufficiently small positive constant $\delta$ satisfying
\[ 0 < \delta \leq \left( \frac{1}{2C_*} \right)^{\frac{1}{p-1}}, \]
and if the initial data $u_0$ and the inhomogeneous term $w$ satisfy
\[ \|u_0\|_{L^d} + \|w\|_{L^k} \leq \frac{\delta}{2C_*}, \]
we get
\[ C_* \left( \|u_0\|_{L^d} + \delta^p + \|w\|_{L^k} \right) \leq \delta. \]
This yields $S(\Xi) \subset \Xi$. Furthermore, assuming $\|u_0\|_{L^d} + \|w\|_{L^k}$ and $\delta$ small enough if necessary, and applying similar arguments as above, we see that the self-mapping $S : \Xi \to \Xi$ is a contraction, so it admits a fixed point $u \in L^\infty((0, \infty), L^q)$, which solves (1.7). We claim that
\[ u \in C([0, \infty), C_0(\mathbb{R}^N)). \quad (4.10) \]
In order to prove our claim, we first show that for $T > 0$ small enough, $u \in C([0, T], C_0(\mathbb{R}^N))$. For any $T > 0$ (small enough), we observe that the above argument yields uniqueness in
\[ \Xi_T = \left\{ u \in L^\infty((0, T), L^q) : \sup_{0 < t < T} t^\beta \|u(t)\|_{L^q} \leq \delta \right\}. \]
Let $\tilde{u}$ be the local solution to (1.7) obtained by Proposition 2.1. Since it follows from (4.3) that $u_0, w \in L^q$, by Proposition 2.1 (iii) we have $\tilde{u} \in C([0, T_{\text{max}}), C_0(\mathbb{R}^N)) \cap C([0, T_{\text{max}}), L^q)$. Then, by the boundedness of $\|\tilde{u}(t)\|_{L^q}$, for a sufficiently small $T > 0$, we see that $\sup_{0 < t < T} t^\beta \|\tilde{u}(t)\|_{L^q} \leq \delta$. Hence, by the uniqueness of solutions, we deduce that $u = \tilde{u}$ in $[0, T]$, so that
\[ u \in C([0, T], C_0(\mathbb{R}^N)). \quad (4.11) \]
Next, applying a bootstrap argument, we show that $u \in C([T, \infty), C_0(\mathbb{R}^N))$. Indeed, for $t > T$, it holds that
\[ u(t) - e^{t\Delta} u_0 - \int_0^t s^\sigma e^{(t-s)\Delta} w ds = \int_0^T e^{(t-s)\Delta} |u(s)|^p ds + \int_T^t e^{(t-s)\Delta} |u(s)|^p ds \]
\[ := J_1(t) + J_2(t). \]
Since $u \in C([0, T], C_0(\mathbb{R}^N))$, we can easily show that $J_1 \in C([T, \infty), C_0(\mathbb{R}^N))$. Furthermore, by the above calculations used to construct the fixed point, we have $J_1 \in C([T, \infty), L^q)$. On the other hand, by (4.2), we see that $q > N(p - 1)/2$, and we can take a constant $r \in (q, \infty]$ such that
\[ \frac{N}{2} \left( \frac{p}{q} - \frac{1}{r} \right) < 1. \]
Since $u \in L^\infty((0, \infty), L^q)$, for $\tilde{T} > T$, we know that $|u|^p \in L^\infty((T, \tilde{T}), L^{\frac{q}{p}})$, and it easily follows that $J_2 \in C([T, \tilde{T}], L^r)$. By the arbitrariness of $\tilde{T}$, it holds $J_2 \in C([T, \infty), L^r)$. Since the terms $e^{t\Delta} u_0, \int_0^t s^\sigma e^{(t-s)\Delta} w ds$ and $J_1$ belong to $C([T, \infty), C_0(\mathbb{R}^N)) \cap C([T, \infty), L^q)$, we deduce that $u \in C([T, \infty), L^r)$. Iterating this process a finite number of times, we obtain
\[ u \in C([T, \infty), C_0(\mathbb{R}^N)). \quad (4.12) \]
Hence, (4.10) follows from (4.11) and (4.12), and the proof of the assertion (ii) of Theorem 1.1 is complete. \qed
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