Asymptotic expansions of zeros of a partial theta function

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Abstract

The bivariate series \( \theta(q,x) := \sum_{j=0}^{\infty} q^{(j+1)/2} x^j \) defines a partial theta function. For fixed \( q \) (\(|q| < 1\)), \( \theta(q,.) \) is an entire function. We prove a property of stabilization of the coefficients of the Laurent series in \( q \) of the zeros of \( \theta \). The coefficients \( r_k \) of the stabilized series are positive integers. They are the elements of a known increasing sequence satisfying the recurrence relation \( r_k = \sum_{\nu=1}^{\infty} (-1)^{\nu-1}(2\nu + 1)r_{k-\nu(\nu+1)/2} \).

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1 Introduction

The bivariate series \( \theta(q,x) := \sum_{j=0}^{\infty} q^{(j+1)/2} x^j \) (where \((q,x) \in \mathbb{C}^2, |q| < 1\)) defines a partial theta function. For fixed \( q \), \( \theta(q,.) \) is an entire function.

Different domains in which the partial theta function finds applications are asymptotic analysis (see [2]), the Ramanujan type \( q \)-series (see [16]), the theory of (mock) modular forms (see [3]), statistical physics and combinatorics (see [15]) and also some questions concerning hyperbolic polynomials (i.e. real polynomials with all roots real, see [8], [11] and [9]). The latter are connected to a problem considered by Hardy, Petrovitch and Hutchinson, see [5], [7], [12] and [13]. Other properties of \( \theta \) are considered in [1].

In the article [10] the zeros of \( \theta \) are presented in the form \( -\xi_j = -1/q^j \Delta_j \). It is shown in [10] that \( \Delta_j \) are formal power series (FPS) of the form \( 1 + O(q) \). For \(|q| \leq 0.108\) all zeros \( -\xi_j \) are distinct (see [10]), so they depend analytically on \( q \) and the series \( \Delta_j \) converge. In the present paper we present the zeros of \( \theta \) also as Laurent series of the form \( -q^{-j} + a_j q^{\kappa_j} + o(q^{\kappa_j}) \), where \( \mathbb{Z} \ni \kappa_j > -j \).

Theorem 1. (1) The series \( \Delta_j \) is of the form \( 1 + (-1)^j q^{(j+1)/2} \Phi_j(q) \), where \( \Phi_j := 1 + O(q) \) is an FPS in \( q \). Hence \( a_j = (-1)^j \) and \( \kappa_j = j(j-1)/2 \).

(2) Represent the zeros \( -\xi_j \) in the form \( -\xi_j = -q^{-j} + (-1)^j q^{(j-1)/2}(1 + \sum_{k=1}^{\infty} g_{j,k} q^k) \). There exists an FPS of the form \( (H) : 1 + \sum_{k=1}^{\infty} r_k q^k, r_k \in \mathbb{Z} \), such that \( g_{j,k} = r_k \) for \( k = 1, \ldots, j \) and \( j \geq 2 \). The coefficients \( r_k \) satisfy the following recurrence relation (we set \( r_k := 0 \) for \( k < 0 \) and \( r_0 := 1 \)):

\[
 r_k = \sum_{\nu=1}^{\infty} (-1)^{\nu-1}(2\nu + 1)r_{k-\nu(\nu+1)/2}.
\]

Remarks 2. (1) The relation \( (1) \) implies that all numbers \( r_k \) are integer. The first 20 of them are equal to
The sequence \( \{r_k\} \) is known (see A000716 in Sloane’s database of integer sequences [14]). Set \((q)_\infty := \prod_{k=1}^{\infty} (1 - q^k)\). It is shown in [4] that \(1 + \sum_{k=1}^{\infty} r_k q^k = 1/(q)_\infty^3\), and that, using Jacobi’s triple product (see [6], p. 377), one obtains the equality

\[
(q)_\infty^3 = \sum_{j=0}^{\infty} (-1)^j (2j + 1) q^{j(j+1)/2}.
\]

With the help of this equality the authors of [4] obtain the recurrence formula [11].

(2) The functions \(M(q) := (q)_\infty^3 = 1 - 3q + 5q^3 - \cdots\) and \((q)_\infty = 1 - q - q^2 + \cdots\) are positive valued for \(q \in (-1, 1)\) and flat at \pm 1 (all this follows from \((q)_\infty := \prod_{k=1}^{\infty} (1 - q^k))\). They are decreasing on \([0, 1)\) because every factor \(1 - q^k\) is positive valued and decreasing. Their derivatives at 0 are negative, therefore they attain their maximal values (which are > 1) at some point in \((-1, 0)\) (the same for both of them). Their second derivatives at negative points close to 0 are negative, so they have inflection points in \((-1, 0)\). At 0 the function \(M\) has an inflection point. The function \((q)_\infty\) has an inflection point in \((0, 1)\) (because it is flat at 1 and \(((q)_\infty)'_q)_0 < 0\).

The above remarks imply the following proposition:

**Proposition 3.** (1) The integer sequences \(\{r_k\} \ (k \geq 0)\), \(\{r_{k+1} - r_k\} \ (k \geq 0)\) and \(\{r_{k+2} - 2r_{k+1} + r_k\} \ (k \geq 1)\) are increasing.

(2) The radius of convergence of the Taylor series \((H)\) of Theorem [3] equals 1.

**Proof of Proposition 3** Set \(R(q) := \sum_{j=0}^{\infty} q^j, \ 1/(q)_\infty^3 := R^2 W, \) i. e. \(W = (\prod_{k=2}^{\infty} R(q)^k)^3\). The numbers \(r_k\) are the coefficients of the series \(1/(q)_\infty^3\). Denote by \(s_k, t_k\) and \(u_k\) the coefficients of the series \(R^2W, RW\) and \(W\). The equality \(R = 1 + qR\) implies

\[
r_{k+1} = r_k + s_{k+1}, \quad s_{k+1} = s_k + t_{k+1} \quad \text{and} \quad t_{k+1} = t_k + u_{k+1}.
\]

As \(r_k > 0, s_k > 0, t_k > 0\) and \(u_k > 0\) for \(k \geq 2\), this proves statement (1). The radius of convergence of the right-hand side of (2) is 1 (and \((q)_\infty^3 > 0\) for \(q \in [0, 1)\)), therefore this is the case of the series \((H)\) as well. This is statement (2).

\[\square\]

**Proposition 4.** The function \(M\) is convex on \((0, 1)\).

The proposition is proved in Section [3].

**Remark 5.** The first 10 coefficients of the FPS \(\Delta_1, \Delta_2, \Delta_3\) are listed below:

| \(r_k\) | \(s_k\) | \(t_k\) | \(u_k\) |
|---|---|---|---|
| 1 | -1 | -1 | -2 |
| 1 | 0 | 0 | 0 |
| 1 | 0 | 0 | 0 |

The reason why part (2) of the theorem does not hold true for \(j = 1\) is explained in Remark [7]. It would be interesting to (dis)prove that for \(k \geq 1\) the sequence \(\{r_{k+1}/r_k\}\) is decreasing.

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2 Proof of Theorem 1

Prove part (1) of the theorem. Consider the condition \(\theta(q, x) = 0\). If instead of \(-\xi_j\) one substitutes just \(-q^{-j}\) for \(x\) in \(\theta\), then the negative powers of \(q\) cancel in \(\theta(q, -q^{-j})\). Indeed, denote by \(\lambda_s\) the degree of the \(s\th\) monomial of the Laurent series \(\theta(q, -q^{-j})\) (i.e. the degree of the monomial \(q^{s(s+1)/2}x^s|_{x=-q^{-j}}\)). Hence \(\lambda_s = (s^2 + s)/2 - js\).

Remark 6. The first \(j\) degrees \(\lambda_s\) decrease from \(\lambda_0 = 0\) to \(\lambda_{j-1} = -j(j-1)/2\). Starting from \(\lambda_j\), the degrees increase. One has

\[
\begin{align*}
(A) : & \quad \lambda_\nu = \lambda_{2j-1-\nu} \quad \text{for} \quad \nu = 0, \ldots, j - 1. \\
(B) : & \quad \lambda_{s+1} - \lambda_s = s + 1 - j \quad \text{(for} \quad s \geq j - 1 \text{ this gives} \quad 0, 1, 3, 6, 10, \ldots) .
\end{align*}
\]

The expansion of \(\theta(q, -q^{-j})\) contains the monomials \((-1)^\nu q^{\lambda_s}\) and \((-1)^{2j-1-\nu} q^{\lambda_s}\) which cancel. When one substitutes \(-q^{-j} + a_j q^{\kappa_j} + o(q^{\kappa_j})\) for \(x\) in \(\theta(q, x)\), one gets

\[
\Psi_s := q^{s(s+1)/2}x^s|_{x=-q^{-j}+a_j q^{\kappa_j}+o(q^{\kappa_j})} = (-1)^s q^{\lambda_s} + (-1)^{s-1} s a_j q^{\mu_s} + o(q^{\mu_s}) ,
\]

where \(\mu_s = s(s+1)/2 + (s - 1) - j + \kappa_j\). The lowest value of \(\mu_s\) is attained for and only for \(s = j - 1\) and \(s = j\). (One has \(\mu_{j-1} = \mu_j = -j(j-3)/2 + \kappa_j\).) In the Taylor series expansion of \(\Psi_{j-1}\) and \(\Psi_j\) the sum of the coefficients of the two monomials with this power of \(q\) equals

\[
S := (-1)^j (j - 1)a_j + (-1)^{j-1}ja_j = (-1)^{j-1}a_j .
\]

The quantity \(S_1 := Sq^{\mu_j}\) must cancel with other monomials or with just another monomial from the expansion of \(\theta(q, -\xi_j)\). This must be just one monomial, and it has to be \(q^{\lambda_{2j}}\). Indeed, any monomial \(T\) in the expansion of \(\Psi_s\) for \(s \neq j - 1\) and \(s \neq j\) which is not \(q^{\lambda_s}\), has a degree higher than \(\mu_j\). The same holds true for the monomials of \(\Psi_{j-1}\) and \(\Psi_j\) which are different from \(\pm q^{\lambda_j}\) and \((-1)^{s-1} sa_j q^{\mu_j}\), \(s = j - 1\) or \(j\). On the other hand, the monomials \(q^{\lambda_s}\) cancel for \(s = 0, \ldots, 2j - 1\). Hence \(S_1\) can cancel with (a) monomial(s) of degree at least \(\lambda_{2j}\). The only monomial of degree \(\lambda_{2j}\) is \(q^{\lambda_{2j}}\) (where \(\lambda_{2j} = (4j^2 + 2j)/2 - 2j^2 = j\), and \(S_1\) must cancel with it. Indeed, otherwise no monomial \(T\) cancels with it either, i.e. the quantity \(\theta(q, \xi_j)\) is not identically equal to 0 which is a contradiction. Hence \((-1)^j a_j + 1 = 0\), i.e. \(a_j = (-1)^j\), and

\[-j(j-3)/2 + \kappa_j = j \quad \text{hence} \quad \kappa_j = j(j-1)/2 .
\]

Thus the zero \(\xi_j = -q^{-j}/\Delta_j\) is of the form

\[-q^{-j} + (-1)^j q^{j(j-1)/2} + o(q^{j(j-1)/2}) = (-q^{-j})(1 - (-1)^j q^{j(j+1)/2} + o(q^{j(j+1)/2}))
\]

which means that \(\Delta_j = 1 + (-1)^j q^{j(j+1)/2} + o(q^{j(j+1)/2})\). This proves part (1).

We prove part (2) for \(j \geq 4\). For \(j = 2\) and \(j = 3\) its proof is contained in Remark 7 below. Recall that \(-\xi_j\) is represented in the form \(-q^{-j} + (-1)^j (q^{\kappa_j} + g_{j,1} q^{\kappa_j+1} + g_{j,2} q^{\kappa_j+2} + o(q^{\kappa_j+2}))\). When one computes the expansion of \(\Psi_s\) in powers of \(q\), one applies the formula of the Newton binomial to the Laurent series of \(-\xi_j\):

(i) The term containing the first power of \(q\) obtained in the expansion of \(\Psi_s\) is \((-1)^s q^{\lambda_s}\).

(ii) The next ones are of the form \((-1)^{s-1} s q^{\lambda_s+j+\kappa_j}\) and \((-1)^{s-1} s g_{j,0} q^{\lambda_s+j+\kappa_j+\nu}, \nu = 1, \ldots, j + \kappa_j - 1\).
(iii) To compute the higher powers of \( q \) one has to take into account the monomial \((-1)^{s-2}(s(s-1)/2)g^{\lambda_j+2j+2\kappa_j}\) and then other monomials in which participate two or more of the factors \( q^{s_j}, g_j, 1, q^{\kappa_j+1}, g_j, 2q^{\kappa_j+2} \) etc.

(iv) Consider the sum \( \Psi_{j-1-l} + \Psi_{j+l}, \ l = 0, 1, \ldots, j-1 \). Its terms mentioned in (i) cancel. Its terms mentioned in (ii) equal \((-1)^{j-l-1}(2l+1)q^\lambda_{j+l+j+\kappa_j}\) and \((-1)^{j-l-1}(2l+1)g_j, q^{\lambda_j+2j+2\kappa_j+\nu}, \ \nu = 1, \ldots, j + \kappa_j - 1\).

(v) The lowest power of \( q \) in the expansion of \( \Psi_{2j+r}, r = 0, 1, \ldots, \) is \( q^{\lambda_{2j+r}} \), where \( \lambda_{2j+r} = jr + j + r(r+1)/2 \geq j \).

The following matrix illustrates the case \( j = 4 \). We present the quantity \(-\xi_4\) in the form

\[
-\xi_4 = -q^{-4} + aq^6 + bq^7 + cq^8 + dq^9 + hq^{10} + uq^{11} + \cdots
\]

with \( b = g_{j,1}, c = g_{j,2}, \ldots \) (In fact, we know that \( a = 1 \), but for the moment we prefer to keep \( a \) as unknown quantity.) The first column indicates the power of \( q \). The next columns show the coefficients of the corresponding powers of \( q \) in the expansions of \( \Psi_s \) for \( s = 0, \ldots, 9 \). This means, in particular, that

\[
\Psi_4 = -q^{-6} + 3aq^4 + 3bq^5 + 3cq^6 + 3dq^7 + 3hq^8 + 3uq^9 + \cdots
\]

The entries \( \pm 1 \) of the matrix are the coefficients of \((-1)^s q^{\lambda_s}\).

\[
\begin{array}{cccccccc}
9 & c & -2h & 3u & -4u & 5h & -6c & -1 \\
8 & b & -2d & 3h & -4h & 5d & -6b & \\
7 & a & -2c & 3d & -4d & 5c & -6a & \\
6 & -2b & 3c & -4c & 5b & \\
5 & -2a & 3b & -4b & 5a & \\
4 & & 3a & -4a & 1 & \\
1 to 3 & 0 & 1 & -1 & \\
-1 & -1 & \\
-2 & - & \\
-3 & -1 & 1 & \\
-4 & - & \\
-5 & 1 & -1 & \\
-6 & & -1 & 1 & \\
\end{array}
\]

As \( \Psi_8 = q^4 + o(q^9) \) and \( \Psi_9 = -q^9 + o(q^9) \), one obtains the following system of equations from which one finds the quantities \( a, \ldots, h \) (one writes the conditions that the sums of the coefficients of \( q^s \) are 0, \( s = 9, 8, \ldots, 4 \); the coefficient of \( q^5 \), for instance, equals \(-2a + 3b - 4b + 5a = -b + 3a\):

\[
\begin{align*}
-u + 3h & = 0 \\
-6 & = 0 \\
-1 & = 0 \\
-c & + 3b = 0 \\
-h & + 3d - 5b = 0 \\
-b & + 3a = 0 \\
-d & + 3c - 5a = 0 \\
-a & + 1 = 0
\end{align*}
\]

The system is triangular and one readily finds that

\[
(a, b, c, d, h, u) = (1, 3, 9, 22, 51, 107).
\]
Now we describe what the analogs of the above matrix and the above system of equations are when \( j \) is arbitrary. The analogs of the lines of the powers 4 and 9 of the matrix are the lines of the powers \( j \) and \( 2j + 1 \). Their rightmost indicated entries equal \((-1)^{2j} = 1\) and \((-1)^{2j+1} = -1\). The columns of \( \Psi_{j-3}, \ldots, \Psi_{j+2} \) are given in the next matrix \((j \text{ is presumed greater than } 4)\):

\[
\begin{array}{ccccccccc}
  j + 5 & (j - 3)c & -(j - 2)h & (j - 1)u & -ju & (j + 1)h & -(j + 2)c  \\
  j + 4 & (j - 3)b & -(j - 2)d & (j - 1)h & -jh & (j + 1)d & -(j + 2)b  \\
  j + 3 & (j - 3)a & -(j - 2)c & (j - 1)d & -jd & (j + 1)c & -(j + 2)a  \\
  j + 2 & -(j - 2)b & (j - 1)c & -(j - 1)d & -jc & (j + 1)b  \\
  j + 1 & -(j - 2)a & (j - 1)b & -jb & (j + 1)a  \\
     j & -ja & (j - 1)a  \\
\end{array}
\]

\[
\begin{array}{ccccccccc}
  1 to j - 1 &  &  &  &  &  &  &  &  & -1  \\
  0 & 1 &  &  &  &  &  &  &  & \lambda_{j+2}  \\
  &  &  \lambda_{j+2} - 1  \\
  &  &  \lambda_{j+1}  \\
  &  & \lambda_j  \\
\end{array}
\]

\[
\begin{array}{ccccccccc}
  \lambda_{j+2} & (-1)^{j-3} &  &  &  &  & (\lambda_{j+2})^+ & (\lambda_{j+2})^-  \\
  \lambda_{j+1} & (-1)^{j-2} &  &  &  &  & (\lambda_{j+1})^+ & (\lambda_{j+1})^-  \\
  \lambda_j & (\lambda_j)^+ & (\lambda_j)^- & (\lambda_j)^+ & (\lambda_j)^-  \\
\end{array}
\]

\[
\begin{array}{cccccccc}
  \Psi_0 & \Psi_{j-3} & \Psi_{j-2} & \Psi_{j-1} & \Psi_j & \Psi_{j+1} & \Psi_{j+2} & \Psi_{2j-1} & \Psi_{2j}  \\
\end{array}
\]

If one writes then the analog \((E)\) of system \([3]\), only its first equation will be slightly different: \(-u + 3h - 5c = 0\). Hence

\[
(a, b, c, d, h, u) = (1, 3, 9, 22, 51, 108).
\]

Denote by \((E_k)\) the equation of system \((E)\) expressing the fact that the sum of the coefficients of \(q^k\) must be 0. (In system \([3]\) we have written explicitly equations \((E_0) - (E_4)\).) Fix \(j_0 \in \mathbb{N}\).

It is easy to see that for \( j \geq j_0 \) the equations \((E_j) - (E_{j+j_0})\) do not depend on \( j \). Their form is \([1]\) (this follows from statement (B) of Remark \([6]\). Therefore the values of the first \( j_0 + 1 \) quantities \( a, b, \ldots \), do not depend on \( j \) for \( j \geq j_0 \). Part (2) is proved.

**Remark 7.** Part (2) of the theorem does not hold true for \( j = 1 \) for the following reason.

Consider the matrices of coefficients of the Laurent series \( \Psi_j \). Recall that \( \Psi_s = (-1)^s + (-1)^{s-1}sa_jq^\mu_s + \cdots \) with \( \mu_s \geq j(j+1)/2 \) for \( s \geq 2j - 1 \). For \( j > 3 \) the inequality \( j(j+1)/2 > 2j \) holds true which means that the columns of \( \Psi_{2j-1} \) and \( \Psi_{2j} \) (considered only for degrees of \( q \) ranging from \(-j(j-1)/2\) to \(2j\)) contain as only nonzero entries \( \pm 1 \) in the rows corresponding respectively to degree 0 and \( j \). (The columns of \( \Psi_\nu \) for \( \nu > 2j \) contain only zeros in these rows.)

Thus the system to which \((a, b, \ldots)\) is solution is completely defined (and according to one and the same rule) by the columns of the matrix containing the coefficients of \( \Psi_0, \ldots, \Psi_{2j} \).

For \( j = 2 \) and \( j = 3 \) one has to check directly that the system is of the same form, i. e.

\[
\begin{align*}
-a + 1 &= 0, \\
-b + 3a &= 0, \\
-c + 3b &= 0, \\
-a + 1 &= 0, \\
-b + 3a &= 0, \\
-c + 3b &= 0, \\
-d + 3c - 5a &= 0.
\end{align*}
\]

For \( j = 1 \) the system becomes \(-a + 1 = 0, -b + 2a = 0\) and the value of \( b \) is not 3, but 2.

### 3 Proof of Proposition \([4]\)

Set \( M = e^{3L} \), where \( 3L := \ln M = 3 \sum_{k=1}^{\infty} \ln(1 - q^k) \). Hence \( M'' = (9(L')^2 + 3L'')e^{3L} \) and it suffices to show that \( 3(L')^2 + L'' > 0 \) on \((0, 1)\). Using Taylor series at 0 of the logarithm one
gets \( L = - \sum_{j=1}^{\infty} (1/j) \sum_{k=1}^{\infty} q^{jk} = - \sum_{j=1}^{\infty} \frac{q^j}{j(1-q^j)} = - \sum_{j=1}^{\infty} (-1/j + 1/j(1-q^j)), \) so

\[
L' = - \sum_{j=1}^{\infty} q^{j-1}/(1-q^j)^2 \quad \text{and} \quad L'' = U + 2V \quad \text{, where}
\]

\[
U := - \sum_{j=2}^{\infty} (j-1)q^{j-2}/(1-q^j)^2 \quad \text{,} \quad V := - \sum_{j=1}^{\infty} jq^{2j-2}/(1-q^j)^3.
\]

In the expansion of 3(\(L'\))^2 all terms are of the form \(q^s/(1-q^k)(1-q^l)^2\). For \(s\) fixed denote by \(q^sS_s\) the sum of all these terms, i.e. \(3(\rho')^2 = \sum_{s=0}^{\infty} q^sS_s\). One has

\[
S_s = \begin{cases} 
6 \sum_{\nu=1}^{s/2} 1/(1-q^{2\nu})^2(1-q^{s+2-\nu})^2 + 3/(1-q^{s/2+1})^4 & \text{for } s \text{ even} \\
6 \sum_{\nu=1}^{(s+1)/2} 1/(1-q^{s+2-\nu})^2 & \text{for } s \text{ odd}.
\end{cases}
\]

On the other hand, \(L'' = - \sum_{s=0}^{\infty} q^sT_s\), where \(T_s := (s+1)/(1-q^{s+2})^2 + 2(s+1)q^s/(1-q^{s+1})^3\). To prove that \(3(\rho')^2 + L'' > 0\) it suffices to show that for \(s = 0, 1, \ldots\) one has \(S_s > T_s\) for \(q \in (0, 1)\).

Observe that \(T_s < 3(s+1)/(1-q^{s+1})^3\). One has \(1/(1-q^{s+2})^2(1-q^{s+2-\nu})^2 > 1/(1-q^{s+1})^3, 1/(1-q^{s+2+1})^4 > 1/(1-q^{s+1})^3\) and \(1/(1-q^2)^2(1-q^{s+2-\nu})^2 > 1/(1-q^{s+1})^3\) for the values of the indices \(s\) and \(\nu\) as indicated above. Hence \(S_s > 3(s+1)/(1-q^{s+1})^3 > T_s\) from which the proposition follows.

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