Cantor-solus and Cantor-multus Distributions

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ABSTRACT. The Cantor distribution is obtained from bitstrings; the Cantor-solus distribution (a new name) admits only strings without adjacent 1 bits. We review moments and order statistics associated with these. The Cantor-multus distribution is introduced – which instead admits only strings without isolated 1 bits – and more complicated formulas emerge.

A bitstring is solus if all of its 1s are isolated. Such strings were called Fibonacci words (more fully, words obeying the Fibonacci restriction) in [1]. We shall reserve the name Fibonacci for a different purpose, as in [2, 3].

A bitstring is multus if each of its 1s possess at least one neighboring 1. Such strings were called good sequences in [4]. Counts of solus \( n \)-bitstrings have a quadratic character, whereas counts of multus \( n \)-bitstrings have a cubic character. More on the meaning of this and on other related combinatorics will appear later.

1. Cantor Distribution

Let \( 0 < \vartheta \leq 1/2 \); for instance, we could take \( \vartheta = 1/3 \) as in the classical case. Let \( \vartheta = 1 - \vartheta \). Consider a mapping [5]

\[
F(\omega_1\omega_2\omega_3\cdots\omega_m) = \frac{\vartheta}{\vartheta} \sum_{i=1}^{m} \omega_i \vartheta^i
\]

from the set \( \Omega \) of finite bitstrings \( (m < \infty) \) to the nonnegative reals. The \( 2^m \) bitstrings in \( \Omega \) of length \( m \) are assumed to be equiprobable. Consider the generating function [1]

\[
G_n(z) = \sum_{\omega \in \Omega} F(\omega)^n z^{\mid\omega\mid}
\]

where \( \mid \omega \mid \) denotes the length of the bitstring. Clearly

\[
G_0(z) = \sum_{\omega \in \Omega} z^{\mid\omega\mid} = \sum_{m=0}^{\infty} 2^m z^m = \frac{1}{1 - 2z}.
\]
The quantity
\[
\frac{[z^m]G_n(z)}{[z^m]G_0(z)} = \frac{1}{2^m} \frac{[z^m]G_n(z)}{[z^m]G_0(z)}
\]
is the \(n\)th Cantor moment for strings of length \(m\); let \(\mu_n\) denote the limit of this as \(m \to \infty\). Denote the empty string by \(\epsilon\). From values
\[
F(\varepsilon) = 0, \quad F(0\omega) = \vartheta F(\omega), \quad F(1\omega) = \bar{\vartheta} + \vartheta F(\omega)
\]
and employing the recurrence \([6]\)
\[
\Omega = \varepsilon + \{0, 1\} \times \Omega,
\]
we have
\[
\begin{align*}
G_n(z) &= \sum_{\omega \in \Omega} \vartheta^n F(\omega)^n z^{1+|\omega|} + \sum_{\omega \in \Omega} (\bar{\vartheta} + \vartheta F(\omega))^n z^{1+|\omega|} \\
&= \vartheta^n z G_n(z) + z \sum_{i=0}^{n} \binom{n}{i} \bar{\vartheta}^{n-i} \vartheta^i G_i(z) \\
&= 2\vartheta^n z G_n(z) + z \sum_{i=0}^{n-1} \binom{n}{i} \bar{\vartheta}^{n-i} \vartheta^i G_i(z)
\end{align*}
\]
for \(n \geq 1\); thus
\[
G_n(z) = \frac{z}{1 - 2\vartheta^n} \sum_{i=0}^{n-1} \binom{n}{i} \bar{\vartheta}^{n-i} \vartheta^i G_i(z).
\]
Dividing both sides by \(G_0(z)\), we have \([5, 7, 8, 9]\)
\[
\mu_n = \frac{1}{2(1 - \vartheta^n)} \sum_{i=0}^{n-1} \binom{n}{i} \bar{\vartheta}^{n-i} \vartheta^i \mu_i
\]
because
\[
\lim_{z \to z_0} \frac{z}{1 - 2\vartheta^n z} = \frac{1}{2(1 - \vartheta^n)}
\]
and the singularity \(z_0 = 1/2\) of \(G_n(z)\) is a simple pole. In particular, when \(\vartheta = 1/3\),
\[
\mu_1 = 1/2, \quad \mu_2 = 3/8, \quad \mu_2 - \mu_1 = 1/8
\]
and, up to small periodic fluctuations \([9, 10, 11]\),
\[
\mu_n \sim C n^{-\ln(2)/\ln(3)},
\]
\[ C = \frac{1}{2 \ln(3)} \int_0^\infty \left( \prod_{k=2}^\infty \frac{1 + e^{-2x/3^k}}{2} \right) e^{-2x/3} x^{\ln(2)/\ln(3)-1} dx = 0.733874 \ldots \]
as \( n \to \infty \).

We merely mention a problem involving order statistics. Let \( \xi_n \) denote the expected value of the minimum of \( n \) independent Cantor-distributed random variables. It is known that \[ \xi_n = \frac{1}{2^n - 2} \left[ \vartheta + \vartheta \sum_{i=1}^{n-1} \binom{n}{i} \xi_i \right] \]
in general. In the special case \( \vartheta = 1/3 \), it follows that
\[ \xi_1 = 1/2, \quad \xi_2 = 3/10, \quad \xi_3 = 1/5, \quad \xi_4 = 33/230, \quad \xi_5 = 5/46 \]
and, up to small periodic fluctuations \[ \xi_n \sim c n^{-\ln(3)/\ln(2)}, \]
by symmetry.

A final problem concerns the sum of all moments of the classical Cantor distribution \[ \sum_{n=0}^\infty \mu_n = -\frac{1}{3} + \frac{2}{3} \sum_{k=1}^\infty \left( \frac{2}{3} \right)^k \sum_{j=1}^{2^k-1} \frac{1}{j} \]
answering a question asked in \[ \text{[15].} \]

2. Cantor-solus Distribution
We examine here the set \( \Omega \) of finite solus bitstrings \( (m < \infty) \). Let
\[ f_k = f_{k-1} + f_{k-2}, \quad f_0 = 0, \quad f_1 = 1 \]
denote the Fibonacci numbers. The \( f_{m+2} \) bitstrings in \( \Omega \) of length \( m \) are assumed to be equiprobable. Clearly
\[ G_0(z) = \sum_{\omega \in \Omega} z^{\lvert \omega \rvert} = \sum_{m=0}^\infty f_{m+2} z^m = \frac{1 + z}{1 - z - z^2}. \]
From additional values

\[ F(1) = \vartheta, \quad F(10\omega) = \vartheta + \vartheta^2 F(\omega) \]

and employing the recurrence \[6\]

\[ \Omega = \varepsilon + 1 + \{0, 10\} \times \Omega, \]

we have

\[
G_n(z) = \vartheta^n z + \sum_{\omega \in \Omega} \vartheta^n F(\omega)^n z^{1+|\omega|} + \sum_{\omega \in \Omega} (\vartheta + \vartheta^2 F(\omega))^n z^{2+|\omega|}
\]

\[ = \vartheta^n z + \vartheta^n z G_n(z) + z^2 \sum_{i+j=n, \ i<n} \binom{n}{i, j} \vartheta^i \vartheta^{2j} G_j(z) \]

\[ = \vartheta^n z + \vartheta^n z G_n(z) + \vartheta^2 n z^2 G_n(z) + z^2 \sum_{i+j=n, \ j<n} \binom{n}{i, j} \vartheta^i \vartheta^{2j} G_j(z) \]

for \( n \geq 1 \); thus

\[
G_n(z) = \frac{1}{1 - \vartheta^n z - \vartheta^{2n} z^2} \left[ \vartheta^n z + z^2 \sum_{i+j=n, \ j<n} \binom{n}{i, j} \vartheta^i \vartheta^{2j} G_j(z) \right].
\]

The purpose of using multinomial coefficients here, rather than binomial coefficients as in Section 1, is simply to establish precedent for Section 3. Let \( \varphi = (1 + \sqrt{5})/2 \approx 1.6180339887... \) be the Golden mean. Dividing both sides by \( G_0(z) \), we have \[\Pi\]

\[
\mu_n = \frac{1}{1 - \vartheta^n / \varphi - \vartheta^{2n} / \varphi^2} \left[ 0 + \frac{1}{\varphi^2} \sum_{i+j=n, \ j<n} \binom{n}{i, j} \vartheta^i \vartheta^{2j} \mu_j \right]
\]

\[= \frac{1}{\varphi^2 - \vartheta^n \varphi - \vartheta^{2n}} \sum_{i+j=n, \ j<n} \binom{n}{i, j} \vartheta^i \vartheta^{2j} \mu_j \]

because

\[
\lim_{z \to z_0} \frac{\vartheta^n z}{G_0(z)} = \lim_{z \to z_0} \frac{1 - z - z^2}{1 + z} \vartheta^n z = 0
\]

and the singularity \( z_0 = 1/\varphi \) of \( G_n(z) \) is a simple pole. In particular, when \( \vartheta = 1/3 \),

\[
\mu_1 = 0.338826..., \quad \mu_2 = 0.203899..., \quad \mu_2 - \mu_1^2 = 0.089096...
\]
and, up to small periodic fluctuations,
\[ \mu_n \sim (0.616005...) n^{-\ln(\varphi)/\ln(3)} (3/4)^n, \]
as \( n \to \infty \). An integral formula in [1] for the preceding numerical coefficient involves a generating function of exponential type:
\[ M(x) = e^{-x/3} \sum_{k=0}^{\infty} \frac{\mu_k}{k!} \left( \frac{4x}{9} \right)^k, \]
namely
\[ \frac{1}{2\varphi \ln(3)} \int_{0}^{\infty} M(x)e^{-2x/3} x^{\ln(\varphi)/\ln(3)-1} dx \]
(we believe that the fifth decimal given in [1] is incorrect, perhaps a typo). Unlike the formula for \( C \) earlier, this expression depends on the sequence \( \mu_1, \mu_2, \mu_3, \ldots \) explicitly.

With regard to order statistics, it is known that [16]
\[ \xi_n = \frac{1}{1 - \vartheta \varphi^{-n} - \vartheta^2 \varphi^{-2n}} \left[ \tilde{\vartheta} \varphi^{-2n} + \vartheta \sum_{i=1}^{n-1} \binom{n}{i} \varphi^{-i} \varphi^{-2(n-i)} \xi_i \right], \]
\[ \eta_n = \frac{1}{1 - \vartheta \varphi^{-n} - \vartheta^2 \varphi^{-2n}} \left[ \tilde{\vartheta} (1 - \varphi^{-n}) + \vartheta^2 \sum_{j=1}^{n-1} \binom{n}{j} \varphi^{-2j} \varphi^{-2(n-j)} \eta_j \right] \]
in general. In the special case \( \vartheta = 1/3 \), we have, up to small periodic fluctuations,
\[ \xi_n \sim (3.31661...) n^{-\ln(3)/\ln(\varphi)}, \]
\[ 3/4 - \eta_n \sim (5.35114...) n^{-\ln(3)/\ln(\varphi)} \]
as \( n \to \infty \).

3. Cantor-multus Distribution
We examine here the set \( \Omega \) of finite multus bitstrings \( (m < \infty) \). Let
\[ f_k = 2f_{k-1} - f_{k-2} + f_{k-3}, \quad f_0 = 0, \quad f_1 = f_2 = 1 \]
denote the second upper Fibonacci numbers [17]. The \( f_{m+2} \) bitstrings in \( \Omega \) of length \( m \) are assumed to be equiprobable. Clearly
\[ G_0(z) = \sum_{\omega \in \Omega} z^{\mid \omega \mid} = \sum_{m=0}^{\infty} f_{m+2} z^m = \frac{1 - z + z^2}{1 - 2z + z^2 - z^3}. \]
From additional values
\[ F(11\omega) = \vartheta + \vartheta\vartheta + \vartheta^2 F(\omega), \]
\[ F(1110\omega) = \vartheta + \vartheta\vartheta + \vartheta^2 + \vartheta^4 F(\omega) \]
and employing the recurrence
\[ \Omega = \varepsilon + 1 + \{0, 11, 1110\} \times \Omega, \]
we have
\[ G_n(z) = \vartheta^n z + \sum_{\omega \in \Omega} \vartheta^n F(\omega)^n z^{|\omega|} + \sum_{\omega \in \Omega} (\vartheta + \vartheta\vartheta + \vartheta^2 F(\omega))^n z^{|\omega|} \]
\[ + \sum_{\omega \in \Omega} (\vartheta + \vartheta\vartheta + \vartheta^2 + \vartheta^4 F(\omega))^n z^{|\omega|} \]
\[ = \vartheta^n z + \vartheta^n z G_n(z) + z^2 \sum_{i+j+k=n} \binom{n}{i,j,k} \vartheta^i (\vartheta\vartheta)^j (\vartheta^2)^k G_k(z) \]
\[ + z^4 \sum_{i+j+k+l=n, \ell<n} \binom{n}{i,j,k,\ell} \vartheta^i \vartheta^j \vartheta^{j+k} G_k(z) \]
\[ = \vartheta^n z + \vartheta^n z G_n(z) + \vartheta^{2n} z^2 G_n(z) + z^2 \sum_{i+j+k=n, \ell<n} \binom{n}{i,j,k} \vartheta^{i+j} \vartheta^{j+k} G_k(z) \]
\[ + \vartheta^{4n} z^4 G_n(z) + z^4 \sum_{i+j+k+l=n, \ell<n} \binom{n}{i,j,k,\ell} \vartheta^{i+j+k} \vartheta^{j+k+4\ell} G_k(z) \]
for \( n \geq 1 \); thus
\[ G_n(z) = \frac{1}{1 - \vartheta^n z - \vartheta^{2n} z^2 - \vartheta^{4n} z^4} \left[ \vartheta^n z + \vartheta^n z G_n(z) + z^2 \sum_{i+j+k=n, \ell<n} \binom{n}{i,j,k} \vartheta^{i+j} \vartheta^{j+k} G_k(z) \right. \]
\[ + z^4 \sum_{i+j+k+l=n, \ell<n} \binom{n}{i,j,k,\ell} \vartheta^{i+j+k} \vartheta^{j+k+4\ell} G_k(z) \]
\[ = \frac{1}{3} \left[ 2 + \left( \frac{25 + 3\sqrt{69}}{2} \right)^{1/3} + \left( \frac{25 - 3\sqrt{69}}{2} \right)^{1/3} \right] = 1.7548776662... \]
be the second upper Golden mean \[17, 18\]. Dividing both sides by \( G_0(z) \), we have

\[
\mu_n = \frac{1}{1 - \vartheta^n/\psi - \vartheta^{2n}/\psi^2 - \vartheta^{4n}/\psi^4} \left[ 0 + \frac{1}{\psi^2} \sum_{i+j+k=n, \ k<n} \binom{n}{i, j, k} \bar{\vartheta}^{i+j+k} \vartheta^{j+2k} \mu_k \right]
\]

\[
+ \frac{1}{\psi^4} \sum_{i+j+k+\ell=n, \ \ell<n} \binom{n}{i, j, k, \ell} \bar{\vartheta}^{i+j+k+2\ell} \vartheta^{j+2k+4\ell} \mu_\ell
\]

\[
= \frac{1}{\psi^4 - \vartheta^n\psi^3 - \vartheta^{2n}\psi^2 - \vartheta^{4n}} \left[ \psi^{2} \sum_{i+j+k=n, \ k<n} \binom{n}{i, j, k} \bar{\vartheta}^{i+j+k} \vartheta^{j+2k} \mu_k \right]
\]

\[
+ \sum_{i+j+k+\ell=n, \ \ell<n} \binom{n}{i, j, k, \ell} \bar{\vartheta}^{i+j+k+2\ell} \vartheta^{j+2k+4\ell} \mu_\ell
\]

because

\[
\lim_{z \to z_0} \frac{\bar{\vartheta}^n z}{G_0(z)} = \lim_{z \to z_0} \frac{1 - 2z + z^2 - z^3}{1 - z + z^2} \bar{\vartheta}^n z = 0
\]

and the singularity \( z_0 = 1/\psi \) of \( G_n(z) \) is a simple pole. In particular, when \( \vartheta = 1/3 \),

\[
\mu_1 = 0.504968..., \quad \mu_2 = 0.416013..., \quad \mu_2 - \mu_1^2 = 0.161020...
\]

but no asymptotics for \( \mu_n \) are known. Order statistics likewise remain open.

4. **Bitsums**

We turn to a more fundamental topic: given a set \( \Omega \) of finite bitstrings, what can be said about the bitsum \( S_n \) of a random \( \omega \in \Omega \) of length \( n \)? If \( \Omega \) is unconstrained, i.e., if all \( 2^n \) strings are included in the sample, then

\[
\mathbb{E}(S_n) = n/2, \quad \mathbb{V}(S_n) = n/4
\]

because a sum of \( n \) independent Bernoulli(1/2) variables is Binomial(\( n, 1/2 \)). Expressed differently, the average density of 1s in a random unconstrained string is 1/2, with a corresponding variance 1/4.

Let us impose constraints. If \( \Omega \) consists of solus bitstrings, then the total bitsum \( a_n \) of all \( \omega \in \Omega \) of length \( n \) has generating function \[19, 20\]

\[
\sum_{n=0}^\infty a_n z^n = \frac{z}{(1 - z - z^2)^2} = z + 2z^2 + 5z^3 + 10z^4 + 20z^5 + \cdots
\]
and the total bitsum squared \( b_n \) has generating function

\[
\sum_{n=0}^{\infty} b_n z^n = \frac{z(1 - z + z^2)}{(1 - z - z^2)^3} = z + 2z^2 + 7z^3 + 16z^4 + 38z^5 + \cdots ;
\]

hence \( c_n = f_{n+2}b_n - a_n^2 \) has generating function

\[
\sum_{n=0}^{\infty} c_n z^n = \frac{z(1 - z)}{(1 + z)^2(1 - 3z + z^2)^2} = z + 2z^2 + 10z^3 + 28z^4 + 94z^5 + \cdots
\]

where \( f_n \) is as in Section 2. Standard techniques [6] give asymptotics

\[
\lim_{n \to \infty} \frac{\mathbb{E}(S_n)}{n} = \lim_{n \to \infty} \frac{a_n}{nf_{n+2}} = \frac{5 - \sqrt{5}}{10} = 0.2763932022\ldots,
\]

\[
\lim_{n \to \infty} \frac{\mathbb{V}(S_n)}{n} = \lim_{n \to \infty} \frac{c_n}{nf_{n+2}^2} = \frac{1}{5\sqrt{5}} = 0.0894427190\ldots
\]

for the average density of 1s in a random solus string and corresponding variance.

If instead \( \Omega \) consists of multus bitstrings, then the total bitsum \( a_n \) of all \( \omega \in \Omega \) of length \( n \) has generating function [21]

\[
\sum_{n=0}^{\infty} a_n z^n = \frac{z^2(2 - z)}{(1 - 2z + z^2 - z^3)^2} = 2z^2 + 7z^3 + 16z^4 + 34z^5 + \cdots
\]

and the total bitsum squared \( b_n \) has generating function

\[
\sum_{n=0}^{\infty} b_n z^n = \frac{z^2(4 - 7z + 4z^2 + 3z^3 - z^4)}{(1 - 2z + z^2 - z^3)^3} = 4z^2 + 17z^3 + 46z^4 + 116z^5 + \cdots ;
\]

hence \( c_n = f_{n+2}b_n - a_n^2 \) has generating function

\[
\sum_{n=0}^{\infty} c_n z^n = \frac{z^2(4 - 9z + 9z^2 - 9z^3 - 6z^4 + z^5 - 6z^6 + z^8)}{(1 - z + 2z^2 - z^3)^2(1 - 2z - 3z^2 - z^3)^2} = 4z^2 + 19z^3 + 66z^4 + 236z^5 + \cdots
\]

where \( f_n \) is as in Section 3. We obtain asymptotics

\[
\lim_{n \to \infty} \frac{\mathbb{E}(S_n)}{n} = \lim_{n \to \infty} \frac{a_n}{nf_{n+2}} = \frac{1}{3} \left[ 2 - \left( \frac{23 + 3\sqrt{69}}{1058} \right)^{1/3} + \left( \frac{-23 + 3\sqrt{69}}{1058} \right)^{1/3} \right] = 0.5885044113\ldots,
\]
for the average density of 1s in a random multus string and corresponding variance. Unsurprisingly $0.588 > 1/2 > 0.276$ and $0.281 > 1/4 > 0.089$; a clumping of 1s forces a higher density than a separating of 1s.

A famous example of an infinite aperiodic solus bitstring is the Fibonacci word [2, 3], which is the limit obtained recursively starting with 0 and satisfying substitution rules $0 \rightarrow 01$, $1 \rightarrow 0$. The density of 1s in this word is $1 - 1/\varphi \approx 0.382$ [22], which exceeds the average 0.276 but falls well within the one-sigma upper limit $0.276 + \sqrt{0.089} = 0.575$. We wonder if an analogously simple construction might give an infinite aperiodic multus bitstring with known density.

5. Longest Bitruns

We turn to a different topic: given a set $\Omega$ of finite bitstrings, what can be said about the duration $R_{n,1}$ of the longest run of 1s in a random $\omega \in \Omega$ of length $n$? If $\Omega$ is unconstrained, then [6]

$$E(R_{n,1}) = \frac{1}{2n} \sum_{k=1}^{\infty} \left( \frac{1}{1-2z} - \frac{1 - z^k}{1 - 2z + z^{k+1}} \right),$$

the Taylor expansion of the numerator series is [23]

$$z + 4z^2 + 11z^3 + 27z^4 + 62z^5 + 138z^6 + 300z^7 + 643z^8 + 1363z^9 + 2866z^{10} + \cdots$$

and, up to small periodic fluctuations [24, 25],

$$E(R_{n,1}) \sim \frac{\ln(n)}{\ln(2)} - \left( \frac{3}{2} - \frac{\gamma}{\ln(2)} \right)$$

as $n \rightarrow \infty$. Of course, identical results hold for $R_{n,0}$, the duration of the longest run of 0s in $\omega$.

If $\Omega$ consists of solus bitstrings, then it makes little sense to talk about 1-runs. For 0-runs, over all $\omega \in \Omega$, we have

$$E(R_{n,0}) = \frac{1}{f_{n+2}} \sum_{k=1}^{\infty} \left( \frac{1 + z}{1 - z - z^2} - \frac{1 + z - z^k - z^{k+1}}{1 - z - z^2 + z^{k+1}} \right)$$
and the Taylor expansion of the numerator series is

\[ z + 4z^2 + 9z^3 + 18z^4 + 34z^5 + 62z^6 + 110z^7 + 192z^8 + 331z^9 + 565z^{10} + \cdots \]

where \( f_n \) is as in Section 2.

If instead \( \Omega \) consists of multus bitstrings, then we can talk both about 1-runs:

\[
E(R_{n,1}) = \frac{1}{f_{n+2}} [z^n] \left\{ -\frac{z}{(1-z)(1-z+z^2)} + \sum_{k=1}^{\infty} \left( \frac{1+z^2}{1-2z+z^2-z^3} - \frac{1+z^2-z^{k-1}-z^k}{1-2z+z^2-z^3+z^{k+1}} \right) z \right\},
\]

\[
\text{num} = 2z^2 + 7z^3 + 16z^4 + 32z^5 + 62z^6 + 118z^7 + 221z^8 + 409z^9 + 751z^{10} + \cdots
\]

and 0-runs:

\[
E(R_{n,0}) = \frac{1}{f_{n+2}} [z^n] \sum_{k=1}^{\infty} \left( \frac{1+z^2}{1-2z+z^2-z^3} - \frac{1+z^2-z^{k-1}+z^k-2z^{k+1}}{1-2z+z^2-z^3+z^{k+2}} \right) z,
\]

\[
\text{num} = z + 2z^2 + 5z^3 + 11z^4 + 23z^5 + 45z^6 + 87z^7 + 165z^8 + 309z^9 + 573z^{10} + \cdots
\]

where \( f_n \) is as in Section 3. Proof: the number of multus bitstrings with no runs of \( k \) 1s has generating function

\[
\frac{1+z^2-z^{k-1}-z^k}{1-2z+z^2-z^3+z^{k+1}} z \quad \text{if } k > 1; \quad \frac{z}{1-z} \quad \text{if } k = 1;
\]

we conclude by use of the summation identity

\[
\sum_{j=0}^{\infty} j \cdot h_j(z) = \sum_{k=0}^{\infty} (\sum_{i=0}^{\infty} h_i(z) - \sum_{i=0}^{k} h_i(z)).
\]

Study of runs of \( k \) 0s proceeds analogously. The solus and multus results here are new, as far as is known. Asymptotics would be good to see someday.

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