COMPARISON BETWEEN SPIN AND ROTATION PROPERTIES OF LORENTZ EINSTEIN AND REFLECTION SYMMETRIC TRANSFORMATIONS

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Abstract

We have shown that reflection symmetric transformation is Lorentz invariant; it is also associative. We have also shown that reflection symmetric sum of vectors has a spin-like term comparable to the spin of Dirac electron. As a consequence of reflection symmetry we have found that the sum is bounded. This corresponds to Einstein’s postulate.

Key words: Reflection symmetry, Lorentz invariance, Spin, Associativity.

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1. Introduction

We have defined a reflection symmetry sum \( \hat{+} \) ( + with a cap ^) of vectors \( A \) and \( B \) as [1].

\[
A +^ B = \frac{A + B + iAxB}{1 + A.B}
\]  

(1)

\( A' \) will be called a reciprocal of \( A \) if \( A'.A = 1 \)

With the help of an arbitrarily chosen vector \( G \), we define reciprocals of \( A \) as

\[
A'_\pm = \frac{G \pm iAxG}{A.G}
\]  

(2)

We now have the symmetry relation

\[
A'_+ + B'_- = A +^ B
\]  

(3)

We intend to study the relation of reflection symmetric sum to Lorentz invariance and also its rotational (spin) property.

2. Pauli Quaternion

We construct a 4-dimensional vector [2] and follow the convention adopted by Kyrala [3] to write it as a sum of a scalar and a Cartesian vector

\[
A = A_0 + A
\]
with the help of basis vectors \( \sigma_0, \sigma_x, \sigma_y, \sigma_z \) (they are slightly different from those of Kyrala [3]) we write it as
\[
A = \sigma_0 A_0 + \sigma_x A_x + \sigma_y A_y + \sigma_z A_z = \sigma_0 A_0 + \sigma A
\] (5)
where \( A_0 \) is a number and \( A_x, A_y, A_z \) are the components of the Cartesian vector \( A \).

As a consequence of linearity we have
\[
kA = \sigma_0 kA_0 + \sigma kA
\] (6)
where \( k \) is a constant.

From \( A = A_0 + A \) we construct
\[
A^* = A_0 - A
\] (7)
and define \( A^2 = AA^* \) where, so that
\[
A^2 = (\sigma_0 A_0 + \sigma_x A_x + \sigma_y A_y + \sigma_z A_z) (\sigma_0 A_0 - \sigma_x A_x - \sigma_y A_y - \sigma_z A_z)
\] (8)
or
\[
A^2 = \sigma_0^2 A_0^2 - \sigma_x^2 A_x^2 - \sigma_y^2 A_y^2 - \sigma_z^2 A_z^2
\] (9)
\[
\begin{align*}
&= (\sigma_x \sigma_0 - \sigma_0 \sigma_x) A_x A_x + (\sigma_y \sigma_0 - \sigma_0 \sigma_y) A_y A_y + (\sigma_z \sigma_0 - \sigma_0 \sigma_z) A_z A_z \\
&- (\sigma_x \sigma_y + \sigma_y \sigma_x) A_x A_y - (\sigma_x \sigma_z + \sigma_z \sigma_x) A_x A_z - (\sigma_y \sigma_z + \sigma_z \sigma_y) A_y A_z
\end{align*}
\]
We require
\[
A^2 = A_0^2 - A.A = A_0^2 - A^2
\] (10)
This will be fulfilled if
\[
\begin{align*}
\sigma_0^2 &= \sigma_x^2 = \sigma_y^2 = \sigma_z^2 = 1 \\
\sigma_x \sigma_0 - \sigma_0 \sigma_x &= \sigma_y \sigma_0 - \sigma_0 \sigma_y = \sigma_z \sigma_0 - \sigma_0 \sigma_z = 0 \\
\sigma_x \sigma_y + \sigma_y \sigma_x &= \sigma_x \sigma_z + \sigma_z \sigma_x = \sigma_y \sigma_z + \sigma_z \sigma_y = 0
\end{align*}
\] (11) to (13)
Therefore
\[
\sigma_0 = \pm 1
\] (14)
we shall choose
\[
\sigma_0 = +1
\] (15)
We require that multiplication of these 4-vectors form a group so that
\[
AB = D = \sigma_0 D_0 + \sigma D = \sigma_0 D_0 + \sigma_x D_x + \sigma_y D_y + \sigma_z D_z
\] (16)
where
\[
AB = (\sigma_0 A_0 + \sigma A)(\sigma_0 B_0 + \sigma B)
\] (17)
Comparison between (16) and (17) using (11) through (15) shows that we have to have
\[
D_0 = A_0 B_0 + A.B
\] (18)
\[
\sigma D = A_0 \sigma B + B_0 \sigma A + \sigma_x \sigma_y A_y B_x + \sigma_y \sigma_x A_x B_y + \sigma_x \sigma_z A_z B_x + \sigma_z \sigma_x A_x B_z
\] (19)
Group requirement will be fulfilled if we set
\[
\begin{align*}
\sigma_x \sigma_y &= -\sigma_y \sigma_x = \beta \sigma_z \\
-\sigma_x \sigma_z &= \sigma_z \sigma_x = \beta \sigma_y \\
\sigma_y \sigma_z &= -\sigma_z \sigma_y = \beta \sigma_x
\end{align*}
\] (20)
where \( \beta \) is a number.
Associativity

We also require the operation to be associative. This will be guaranteed if the \( \sigma \)'s are associative i.e.

\[
(\sigma_i \sigma_j) \sigma_k = \sigma_i (\sigma_j \sigma_k)
\]

for

\[
i = 0, x, y, z; \quad j = 0, x, y, z \quad k = 0, x, y, z
\]

Then we have

\[
(AB)C = A(BC)
\]

To determine \( \beta \)

Choose relation (20)

\[
\sigma_x \sigma_y = \beta \sigma_z
\]

Multiplying both sides and using (11)

\[
(\sigma_x \sigma_y)(\sigma_x \sigma_y) = \beta^2 \sigma_z^2 = \beta^2
\]

Using (20) and (21) the above relation becomes

\[-(\sigma_x \sigma_y \sigma_y \sigma_x) = -\sigma_x \sigma_y^2 \sigma_x = -\sigma_x^2 \sigma_y = -1 = \beta^2
\]

Or

\[
\beta^2 = -1 \quad \text{or} \quad \beta = \pm i
\]

We shall choose \( \beta = i \)

With this (19) becomes

\[
\sigma \cdot D = \sigma \cdot A_0 \cdot B + \sigma \cdot B_0 \cdot A
\]

\[
+ i \sigma_z (A_x B_y - A_y B_x) - i \sigma_y (A_x B_z - A_z B_x) + i \sigma_x (A_x B_z - A_z B_y)
\]

Or

\[
\sigma \cdot D = \sigma \cdot A_0 \cdot B + \sigma \cdot B_0 \cdot A + i \sigma \cdot (A \cdot B)
\]

Or

\[
D = A_0 \cdot B + B_0 \cdot A + i (A \cdot B)
\]

3. Conjugate, Norm and Inverse

\( A^* \), the conjugate of \( A \), is constructed from \( A \) by changing the sign of the Cartesian part of \( A \).

\( A^2 \), the square Norm of \( A \), is

\[
A^2 = AA^* = A^* A = A_0^2 - A^2
\]

\( A^{-1} \), the inverse of \( A \), is

\[
A^{-1} = \frac{A^*}{A^2}
\]

so that

\[
A^{-1}A = AA^{-1} = 1
\]

and the unit vector is 1

Let

\[
f(\sigma_x, \sigma_y) = \sigma_x + \sigma_y + \sigma_z \sigma_y
\]

Then as a consequence of (16)
The product $AB$ involves quantities of type $f$ above. Therefore,

$$(AB)^* = B^*A^* \quad \text{(35)}$$

$$(AB)^{-1} = B^{-1}A^{-1} \quad \text{(36)}$$

$$(AB)^2 = (AB)(AB)^* = ABB^*A^* = A(B^2)A^* = A^2B^2 \quad \text{(37)}$$

4. Lorentz Invariant Reflection Symmetric Transformation

Let

$$L = ct + X \quad \text{(38)}$$

So that, by (30)

$$L^2 = LL^* = (ct)^2 - X^2 \quad \text{(39)}$$

and

$$R = g - g\sigma.V/c \quad \text{(40)}$$

where

$$g = \frac{1}{\sqrt{1 - (V/c)^2}} \quad \text{(41)}$$

so that

$$R^2 = R_0^2 - R^2 = 1 \quad \text{(42)}$$

and

$$RL = L' = ct' + X' \quad \text{(43)}$$

With

$$t' = gt - gX.V/c^2 \quad \text{(44)}$$

and

$$X' = g(X - tV) - ig \frac{V \times X}{c} \quad \text{(45)}$$

Using (37) and (42)

$$(L')^2 = (RL)^2 = L^2 \quad \text{(46)}$$

Therefore, the transformation $L \rightarrow L'$, (38) through (46), is Lorentz invariant.

5. Comparison between the Rotational Properties of Lorentz–Einstein Transformation and Reflection Symmetry Transformation

We define Lorentz-Einstein product of 4-vectors $L$ and $R$ defined by (38) and (39) as

$$R \times L = cg\{t - X.V/c^2\} + \left\{X + [(g - 1)\frac{X.V}{V^2} - gt]V\right\}. \quad \text{(47)}$$

Observing that in the limit $c \rightarrow 0$

$$\frac{1}{g} = \frac{1}{\sqrt{1 - (V/c)^2}} \xrightarrow{c \rightarrow 0} i \frac{|V|}{c} \quad \text{(48)}$$

We find in the limit $c \rightarrow 0$

$$-\frac{c}{g |X||V|} \xrightarrow{c \rightarrow 0} \cos \theta + i \frac{1}{|X|} \left\{\frac{(X.V)V}{V^2} - X\right\} = \cos \theta + im \sin \theta \quad \text{(49)}$$
where \( \theta \) is the angle between \( \mathbf{X} \) and \( \mathbf{V} \)

Similarly using (43)-(45)

\[
\frac{-c \mathbf{RL}}{g \frac{V}{X}} \cos \theta + i n \sin \theta \quad (50)
\]

where

\[
m = \frac{(X \cdot V) V / V^2 - X}{(X \cdot V) V / V^2 - X} \quad \text{and} \quad n = \frac{V \times X}{|V \times X|} \quad (51)
\]

An important difference between (49) and (50) is that \( m \) and \( n \) are orthogonal so that \( m \cdot n = 0 \)

### 6. Matrix Representation and Spin

For a comparison with the spin of Dirac electron theory, it is worthwhile representing reflection symmetric transformation in matrix formalism.

Pauli matrices [4] have properties (7) and (13). Therefore, it is possible to represented \( \sigma_z, \sigma_y, \sigma_x \) and \( \sigma_0 \) by 2x2 Pauli matrices and a unit matrix

\[
\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad \text{and} \quad \sigma_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad (52)
\]

We now construct the matrix \( \mathbf{A} \)

\[
\mathbf{A} = \sigma_0 A_0 + \sigma_x A_x + \sigma_y A_y + \sigma_z A_z \quad (53)
\]

Using

\[
\sigma_0 A_0 = \begin{bmatrix} A_0 & 0 \\ 0 & A_0 \end{bmatrix}, \quad \sigma_x A_x = \begin{bmatrix} 0 & A_x \\ A_x & 0 \end{bmatrix}, \quad \sigma_y A_y = \begin{bmatrix} 0 & -i A_y \\ i A_y & 0 \end{bmatrix}, \quad \sigma_z = \begin{bmatrix} 0 & A_z \\ A_z & 0 \end{bmatrix} \quad (54)
\]

we have

\[
\mathbf{A} = \sigma_0 A_0 + \sigma_\mathbf{A} = \begin{bmatrix} A_0 + A_z & A_x - i A_y \\ A_x + i A_y & A_0 - A_z \end{bmatrix} \quad (55)
\]

Constructing matrices of type (55) using \( \mathbf{L} \) and \( \mathbf{R} \) of (38) and (40) and multiplying we get \( \mathbf{L}' = \mathbf{RL} \)

The product, \( \mathbf{L}' \), is a 2x2 matrix. The cross product term of \( \mathbf{L}' \), corresponding to cross product term of (45), corresponds to the spin term of Dirac electron. Spin term of Dirac electron has its origin in the product of Pauli matrices [5].

### 7. Normalized Vector and Addition of Velocities

The vector \( \mathbf{W} \) will be called normalized for velocity when its scalar part is \( c \); i.e. \( W_0 = c \)

In this case its Cartesian part will be called a velocity.

Let

\[
\mathbf{U} = U_0 + \mathbf{U} \quad (56)
\]

Consider the product

\[
\mathbf{RU} = \mathbf{W}' = \frac{R_0 U_0 + R \mathbf{U}}{c} \mathbf{W} \quad (57)
\]

\[
\mathbf{W} = c + \sigma_\mathbf{W} \quad (58)
\]

We set \( U_0 = c \). \( \mathbf{R} \) is as defined by (38). \( \mathbf{U} \) and \( \mathbf{V} \) are velocities and
\[
RU = W' = \frac{1 - V.U/c^2}{\sqrt{1 - (V/c)^2}} W
\]

\[
W = W_0 + \sigma W = c + \sigma \frac{-V + U - iVxU}{1 - \frac{V.U}{c^2}}
\]

We introduce the symbol \( \hat{+} \) and write

\[
W = (-V) \hat{+} U = \frac{-V + U - iVxU}{1 - \frac{V.U}{c^2}}
\]

As a consequence of (29), (48) is associative

\[
\{(-V) + U\} \hat{+} W = (-V) + \{U + W\}
\]

8. Comparison with Lorentz-Einstein Transformation

The magnitude

The magnitude is the same in both the cases, reflection symmetric, \((\hat{+})\) and L-E, \((\hat{\sim})\).

\[
|V + U| = |V \hat{+} U|
\]

Invariance of Limiting Velocity under Reflection Symmetric Addition

We observe that

\[
A \hat{+} G \xrightarrow{G \to \infty} A' = \frac{G + iAxG}{A.G}
\]

Let \(W\) be

\[
W = A \hat{+} B
\]

Because of associativity

\[
W \hat{+} G = A \hat{+} (B \hat{+} G)
\]

Going to the limit \(G \to \infty\)

\[
W' = A \hat{+} B'
\]

By (2)

\[
W.W' = 1
\]

Therefore

\[
(A \hat{+} B).(A \hat{+} B') = 1 \text{ if } B.B' = 1
\]

If

\[
B = B'
\]

corresponding to the choice

\[
G = V \text{ and } |V| = 1
\]

Then (72) becomes

\[
W.W = (A \hat{+} B).(A \hat{+} B) = 1 \text{ if } B.B = 1
\]

We may replace all the quantities \(A, B\) and \(W\) by

\[
A \to a/c, B \to b/c \text{ and } W \to w/c
\]
Then (72) gives

$$w \cdot w = c^2 \quad \text{if} \quad b \cdot b = c^2$$  \hspace{1cm} (74)

$a$, $b$ and $w$ have the dimension of $c$ and $A$, $B$ and $W$ are dimensionless. If $c$ is a velocity, the magnitude of the sum of velocities $a$ and $b$ is also $c$. This is the reflection symmetric analogue of Einstein’s postulate.

9. Conclusion

A non-commutative sum with reflection property (3) has been written. It is Lorentz invariant (46) and associative (22, 29). It has a spin-like property comparable to that of Dirac electron (sections 6). Lorentz transformation also has this spin-like property (49) but is hidden and becomes manifest in the limit $c \to 0$. Relation (74) corresponds to Einstein’s postulate. We have found it as a consequence of reflection symmetry.

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