SHARP GRADIENT ESTIMATES ON WEIGHTED MANIFOLDS WITH COMPACT BOUNDARY

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Abstract. In this paper, we prove sharp gradient estimates for positive solutions to the weighted heat equation on smooth metric measure spaces with compact boundary. As an application, we prove Liouville theorems for ancient solutions satisfying the Dirichlet boundary condition and some sharp growth restriction near infinity. Our results can be regarded as a refinement of recent results due to Kunikawa and Sakurai.

1. Introduction. In geometric analysis, gradient estimates play an important role in the studying elliptic and parabolic equations on Riemannian manifolds. In [19], Yau proved that if the Ricci curvature of manifold \((M^n, g)\) satisfies \(\text{Ric}_M \geq -(n-1)K\) for some constant \(K \geq 0\), then any positive harmonic function \(u\) on \((M^n, g)\) satisfies

\[
\lim_{B_R(x_0)} \frac{|\nabla u|}{u} \leq c_n \left( \frac{1}{R} + \sqrt{K} \right),
\]

where \(B_R(x_0)\) is a geodesic ball with radius \(R\) and center \(x_0 \in M^n\). As a consequence, any positive harmonic function on manifolds with non-negative Ricci curvature must be constant. Later, in [10], Li and Yau derived gradient estimates for parabolic equations on Riemannian manifolds with or without boundary. Moreover, they applied parabolic gradient estimates to study upper and lower bounds...
of the heat kernel, eigenvalue estimates and Betti numbers estimates on manifolds. Motivated by Li-Yau’s gradient estimate technique, in [5], Hamilton established a new gradient estimate for the heat equation. His result may allow one to compare the temperature of two different points at the same time on compact manifolds. In [16], Souplet and Zhang extended Hamilton’s estimate to a localized version. Their result enables the comparison of temperature distribution instantaneously, without any lag in time, even for noncompact manifolds.

On the other hand, there exist some interesting works regarding to Riemannian manifolds with boundary. In [2], Chen generalized Li-Yau gradient estimates to compact Riemannian manifolds with possible non-convex boundary. As an application, he gave a lower bound of the first Neumann eigenvalue in terms of geometric invariants with the boundary. In [17], Wang developed Li-Yau gradient estimates for the heat equation under Neumann boundary condition. As an application, he obtained upper and lower bounds for the heat kernel satisfying Neumann boundary conditions on compact manifolds with non-convex boundary. In [6], Hsu proved global Li-Yau gradient estimates for the conjugate heat equation on compact manifolds with boundary, whose metric evolves by the Ricci flow. In [12], Olivi proved Li-Yau gradient estimates for the heat equation, with Neumann boundary conditions, on a compact Riemannian submanifold with boundary, satisfying some integral Ricci curvature assumption.

Besides the above results related to Neumann boundary conditions, Kunikawa and Sakurai [8] recently proved Yau type gradient estimates for harmonic functions with some Dirichlet boundary condition. More precisely, they proved that

**Theorem 1.1.** Let \((M, g)\) be an \(n\)-dimensional, complete Riemannian manifold with compact boundary. Assume \(\text{Ric}_M \geq -(n-1)K\) for some constant \(K \geq 0\) and the mean curvature of \(\partial M\) satisfies \(H_{\partial M} \geq -(n-1)\sqrt{K}\). Let \(u : B_R(\partial M) \to (0, \infty)\) be a positive harmonic function with Dirichlet boundary condition (i.e., it is constant on the boundary). If the derivative \(u_\nu\) in the direction of the outward unit normal vector \(\nu\) is non-negative over \(\partial M\), then

\[
\sup_{B_R(\partial M)} \frac{|\nabla u|}{u} \leq c_n \left( \frac{1}{R} + \sqrt{K} \right).
\]

Here \(H_{\partial M}\) denotes the mean curvature of \(\partial M\) and the Dirichlet boundary condition means that \(u\) is constant on \(\partial M\).

Kunikawa and Sakurai [8] further proved local Souplet-Zhang gradient estimates for the heat equations with Dirichlet boundary condition.

**Theorem 1.2.** Let \((M, g)\) be an \(n\)-dimensional complete Riemannian manifold with compact boundary. Assume \(\text{Ric}_M \geq -(n-1)K\) and \(H_{\partial M} \geq 0\). Let \(0 < u < A\) for some constant \(A > 0\), be a solution to the heat equation on \(Q_{R,T}(\partial M) := B_R(\partial M) \times [-T, 0]\). If \(u\) satisfies the Dirichlet boundary condition, and \(u_\nu \geq 0\) and \(\partial_t u \leq 0\) over \(\partial M \times [-T, 0]\), then

\[
\sup_{Q_{R,T}(\partial M)} \frac{|\nabla u|}{u} \leq c_n \left( \frac{1}{R} + \frac{1}{\sqrt{T}} + \sqrt{K} \right) \left(1 + \log \frac{A}{u}\right).
\]

Inspired by the Kunikawa-Sakurai work, in this paper we will study gradient estimates for weighted harmonic functions and weighted heat equation with Dirichlet boundary condition. In particular, our results improve Theorems 1.1 and 1.2 on the manifold case.
Before stating our results, we fix some notations. A smooth metric measure space (also called weighted manifold) is a triple \((M, g, e^{-f}dv)\), where \((M, g)\) is a complete \(n\)-dimensional Riemannian manifold, \(f\) is a smooth real-valued function on \(M\) and \(dv\) is the volume form with respect to the metric \(g\). Denote by \(\nabla\) and \(\text{Hess}\) the gradient and Hessian operators. The weighted Laplacian is defined by

\[
\Delta_f := \Delta - \nabla f \cdot \nabla.
\]

On \((M, g, e^{-f}dv)\), a natural generalization of the Ricci curvature is called \(m\)-Bakry-Émery Ricci curvature, defined by

\[
\text{Ric}^m_f := \text{Ric} + \text{Hess} f - \frac{\nabla f \otimes \nabla f}{m - n}, \quad n \leq m \leq \infty.
\]

When \(m = n\), we regard \(f\) to be constant and \(\text{Ric}^m_f = \text{Ric}\). When \(m = \infty\), we have \(\infty\)-Bakry-Émery Ricci curvature \(\text{Ric}^\infty_f = \text{Ric}^\infty\). For a complete manifold \(M\) with compact boundary \(\partial M\), let \(H_f := H - \nabla f \cdot \nu\) be the weighted mean curvature on \(\partial M\), where \(\nu\) is the outer unit normal vector to \(\partial M\) and the mean curvature \(H\) is defined with respect to \(\nu\). But if \(\nu\) is the inner unit normal vector to \(\partial M\), then \(H_f := H + \nabla f \cdot \nu\). We refer the reader to [9] for basic notation on smooth manifolds with boundary and to [15] for further discussions on \(H_f\) and weighted manifolds with boundary.

Let us first state gradient estimates for \(f\)-harmonic functions with Dirichlet boundary condition on \((M, g, e^{-f}dv)\), which is a mild generalization of Theorem 1.1.

**Theorem 1.3.** Let \((M, g, e^{-f}dv)\) be an \(n\)-dimensional smooth metric measure space with compact boundary. Assume \(\text{Ric}^m_f \geq -(m - 1)K\) where \(n \leq m < \infty\) and \(H_f \geq -L\) for some constants \(K, L \geq 0\). Let \(u : B_R(\partial M) \to (0, \infty)\) be a positive \(f\)-harmonic function (i.e. \(\Delta_f u = 0\)) with Dirichlet boundary condition. If the derivative \(u_\nu\) in the direction of the outward unit normal vector \(\nu\) is non-negative over \(\partial M\), then

\[
\sup_{B_{\frac{1}{2}}(\partial M)} \frac{|\nabla u|}{u} \leq c_m \left( \frac{1}{R} + L + \sqrt{K} \right).
\]

We would like to notice that we need to assume a lower bound of \(\text{Ric}^m_f\) since our present proof strongly depends on a refined Kato inequality in Lemma 2.2, which seems to be not true on weighted manifolds. When \(L = (m - 1)\sqrt{K}\) and \(f\) is constant, our result returns to Theorem 1.1. Recently, by using Brighton’s method [1], the second and the third authors have showed in [4] that the refined Kato inequality is not necessary for our gradient estimate. In fact, they proved that a gradient estimate of Brighton type is valid under only a lower bound of \(\text{Ric}_f\).

Second, we prove elliptic gradient estimates for the \(f\)-heat equation with Dirichlet boundary condition on \((M, g, e^{-f}dv)\) by improving the argument of [3], which seems to be new even for manifolds.

**Theorem 1.4.** Let \((M, g, e^{-f}dv)\) be a complete smooth metric measure space with compact boundary. Assume \(\text{Ric}_f \geq -(n - 1)K\) for some constant \(K \geq 0\) and \(H_f \geq 0\). Let \(0 < u \leq A\) for some constant \(A > 0\) be a solution to the \(f\)-heat equation \(u_t = \Delta_f u\) on \(Q_{R,T}(\partial M) := B_R(\partial M) \times [-T, 0]\). If \(u\) satisfies the Dirichlet boundary condition (i.e. \(u(\cdot, t) |_{\partial M}\) is constant for each fixed \(t \in [-T, 0]\)), \(u_\nu \geq 0\)
and $\partial_t u \leq 0$ over $\partial M \times [-T, 0]$, then
\[
\sup_{Q \subset \mathcal{T}(\partial M)} \frac{\lvert \nabla u \rvert}{u} \leq c_n \left( \frac{\sqrt{D} + 1}{R} + \frac{1}{\sqrt{T}} + \sqrt{K} \right) \sqrt{1 + \log \frac{A}{u}},
\]
where
\[
D := 1 + \log A - \log \left( \inf_{Q, R, T}(\partial M) u \right).
\]

As a consequence, we obtain the following Liouville theorems.

**Corollary 1.** Let $(M, g, e^{-f} dv)$ be a complete smooth metric measure space with compact boundary. Assume $\text{Ric} \geq 0$ and $H \geq 0$. Let $u : M \to (0, \infty)$ be a positive $f$-harmonic function with Dirichlet boundary condition. If $u_\nu \geq 0$ over $\partial M$, then $u$ is constant.

**Corollary 2.** Let $(M, g, e^{-f} dv)$ be a complete smooth metric measure space with compact boundary. Assume $\text{Ric} \geq 0$ and $H \geq 0$. Let $u$ be an ancient solution to the $f$-heat equation.

1. Suppose that $u > 0$. If $u_\nu \geq 0$ and $\partial_t u \leq 0$ on $\partial M \times (-\infty, 0]$ and
   \[u(x, t) = e^{\alpha(\rho(x) + |t|)}\]
   near infinity, then $u$ is constant.

2. If $u_\nu \geq 0$ and $\partial_t u \leq 0$ on $\partial M \times (-\infty, 0]$ and
   \[u(x, t) = o(\rho(x) + |t|)\]
   near infinity, then $u$ is constant.

Here, $\rho(x) = \rho_{\partial M}(x)$ denotes the Riemannian distance from the boundary.

**Remark 1.** We would like to emphasize that the results in Corollary 2 are better than those in Corollary 1.7 in [8]. This is because the gradient estimates in Theorem 1.4 (also for manifolds) control the growth at infinity better than those in Theorem 1.2. Indeed, our gradient estimate is sharp in both spatial and time directions due to an example that $M = (-\infty, 0]$, $f = 2x$, $u = e^{x^{-1}}$ is an ancient positive solution to the $f$-heat equation, $u_\nu \geq 0$, $\partial_t u \leq 0$ on $\partial M \times (-\infty, 0]$ and its growth near infinity is $e^{x^{-1} + |t|}$. We also would like to mention that the main differences between the proof of our sharp results and Kunikawa-Sakurai results are due to a choice of the gradient quantity in the Bochner formula in (2.3). This choice was first used by the first two authors in the manifold case [3] and leads to a sharper estimate in Theorem 1.4 as compared with Theorem 1.2.

The rest of this paper is organized as follows. In Section 2, we will introduce some basic facts on smooth metric measure space with boundary. In particular, we will give the Laplacian comparison with boundary. In Section 3, we will prove gradient estimates for $f$-harmonic functions with Dirichlet boundary condition, i.e. Theorem 1.3. In Section 3, we will prove Theorem 1.4 and Corollaries 1 and 2.

2. Preliminaries. We start to introduce some basic facts. For a smooth metric measure space $(M^n, g, e^{-f} dv)$ with boundary $\partial M$, the distance function from the boundary is defined by
\[
\rho := \rho_{\partial M} := d(\cdot, \partial M).
\]
Note that $\rho$ is smooth outside of the cut locus for the boundary (see [13]) and the completeness of $M$ does not imply the boundedness of $\rho$. The Laplacian comparison
for the distance function on smooth metric measure spaces with boundary was first proved in [18]; see also [15]. Here, we give a general statement due to [14]. We assume that $x \in B_R(\partial M)$, let $B(x,d(x,\partial M)) \subset M$ be the largest geodesic ball centered at $x$ and $z = \partial B(x,d(x,\partial M)) \cap \partial M$. Suppose that $\gamma_{z,x}(t)$ is the geodesic curve starting from $z$ and connecting $z$ and $x$. We see that $\gamma_{z,x}'(0)$ is the unit inner normal vector for $\partial M$ at $z$. Since $\gamma_{z,x}(d(x,\partial M)) = x$ and $d(x,\partial M) \leq R$, Lemma 6.1 in [14] implies the Laplacian comparison theorem for $\infty$-Bakry-Émery Ricci curvature, which will be used in our proof of Theorems 1.3 and 1.4.

**Theorem 2.1** ([14]). Let $(M^n,g,e^{-f}dv)$ be a complete smooth metric measure space with boundary $\partial M$ (not necessary to be compact). Assume that $\text{Ric}_f \geq -K$ and $H_f \geq -L$ for some constants $K \geq 0$ and $L \in \mathbb{R}$. Then

$$\Delta_f \rho(x) \leq KR + L$$

for all $x \in B_R(\partial M)$.

Next, we recall the so-called weighted Reilly formula, we refer the reader to [7, 4] for further discussion.

**Proposition 1.** For all $\varphi \in C^\infty(M)$,

$$\left(\frac{d\varphi}{\rho}\right)_\nu = 2\varphi(\Delta f \varphi - \Delta_{\partial M,f}(\varphi |_{\partial M}) - \varphi H_f) + 2g_{\partial M}(\nabla_{\partial M} \varphi |_{\partial M}, \nabla_{\partial M} \varphi) + 2(\nabla_{\partial M} \varphi)_{\partial M}.\varphi_{\partial M}.$$

For the $f$-harmonic function, we have a useful inequality, which was proved in Theorem 2.2 of [11].

**Lemma 2.2** (see [11]). Let $(M,g,e^{-f}dv)$ be a smooth metric measure space with $\text{Ric}^m_f \geq -(m-1)K$. If $u$ is a positive $f$-harmonic function and $m \geq n$, then $\phi := |\nabla \log u|$ satisfies

$$\Delta_f \phi \geq -(m-1)K \phi - 2\frac{m-2}{m-1} \frac{\langle \nabla \phi, \nabla u \rangle}{u} + \frac{\phi^2}{n-1}.$$

For the $f$-heat equation, we have another useful inequality, which was established in [3] for the manifold case.

**Lemma 2.3.** Under the same assumption as in Theorem 1.4, let $w = |\nabla h|^2$, where $h = \sqrt{1 + \log \left(\frac{A}{u}\right)}$. For any $(x,t) \in Q_{R,T}(\partial M)$,

$$\Delta_f w - w_t \geq -2(n-1)K w + 2 \left(2h - \frac{1}{h}\right) \langle \nabla w, \nabla h \rangle + 2 \left(2 + \frac{1}{h^2}\right) w^2. \quad (2.1)$$

**Proof.** Let $u(x,t)$ be a solution to the $f$-heat equation and $0 < u \leq A$ for some constant $A$ in $Q_{R,T}(\partial M)$. Let $B = Ae$ and

$$h = \sqrt{\log \left(\frac{B}{u}\right)} = \sqrt{1 + \log \left(\frac{A}{u}\right)} \geq 1.$$

Then

$$u = Be^{-h^2} \quad \text{and} \quad \log u = \log B - h^2.$$

By the $f$-heat equation, we directly compute that

$$h_t = \Delta_f h + |\nabla h|^2 \left(\frac{1}{h} - 2h\right). \quad (2.2)$$
On the other hand, by the Bochner formula and $\text{Ric}_f \geq -(n-1)K$, we deduce that
\[
\Delta f w - w_t = 2|\nabla h|^2 + 2\text{Ric}_f(\nabla h, \nabla h) + 2\langle \nabla \Delta h, \nabla h \rangle - w_t
\geq -2(n-1)Kw + 2\langle \nabla \Delta h, \nabla h \rangle - w_t, \tag{2.3}
\]
By the equality (2.2), we obtain
\[
\Delta f w - w_t \geq -2(n-1)Kw + 2 \langle \nabla (h_t + |\nabla h|^2 \left(2h - \frac{1}{h}\right)), \nabla h \rangle - w_t
\geq -2(n-1)Kw + 2 \langle \nabla (h_t), \nabla h \rangle + 2 \left(2h - \frac{1}{h}\right) \langle \nabla \left(|\nabla h|^2\right), \nabla h \rangle
+ 2|\nabla h|^2 \langle \nabla \left(2h - \frac{1}{h}\right), \nabla h \rangle - w_t. \tag{2.4}
\]
Observe that
\[
2\langle \nabla (h_t), \nabla h \rangle = (|\nabla h|^2)_t = w_t,
\]
and
\[
\nabla \left(2h - \frac{1}{h}\right) = 2\nabla h + \frac{\nabla h}{h^2} = \left(2 + \frac{1}{h^2}\right) \nabla h.
\]
Hence, (2.4) implies the desired inequality. \hfill \Box

3. Gradient estimates for harmonic functions. In this section we will follow an argument of [8] to prove Theorem 1.3.

Proof of Theorem 1.3. Define $G := (R^2 - \rho^2)\phi$, where $\phi$ is as in Lemma 2.2. Assume $G$ obtains its maximal at $x_1 \in B_R(\partial M)$.

Case 1: If $x_1 \in B_R(\partial M) \setminus \partial M$, we may assume that $x_1 \notin \text{Cut}(\partial M)$ by the Calabi’s argument. At $x_1$, we have
\[
\nabla G = 0 \quad \text{and} \quad \Delta f G \leq 0.
\]
Consequently,
\[
\frac{\nabla \rho^2}{R^2 - \rho^2} = \frac{\nabla \phi}{\phi}, \quad \frac{\Delta f \rho^2}{R^2 - \rho^2} + \frac{\Delta f \phi}{\phi} - 2\langle \nabla \rho^2, \nabla \phi \rangle \leq 0.
\]
Hence
\[
\frac{\Delta f \rho}{\phi} - \frac{\Delta f \rho^2}{R^2 - \rho^2} - 2 \frac{|\nabla \rho|^2}{(R^2 - \rho^2)^2} \leq 0. \tag{3.1}
\]
Since $\text{Ric}^n_\rho \geq -(m-1)K$ (and hence $\text{Ric}_f \geq -(m-1)K$), by the Laplacian comparison of Theorem 2.1, we have
\[
\Delta f \rho^2 = 2|\nabla \rho|^2 + 2\rho \Delta f \rho \leq 2 + 2\rho((m-1)KR + L)
\leq 2 + 2LR + 2(m-1)KR^2, \tag{3.2}
\]
where we used $|\nabla \rho| = 1$ and $\rho \leq R$. Since $|\nabla \rho^2| = 4\rho^2|\nabla \rho|^2 = 4\rho^2$, Lemma 2.2 together with (3.1) and (3.2) implies
\[
0 \geq \frac{\Delta f \rho}{\phi} - 2 + 2LR + 2(m-1)KR^2 - 8\rho^2 \frac{R^2 - \rho^2}{(R^2 - \rho^2)^2} \geq -(m-1)K - 2\frac{(m-2)}{m-1} \frac{|\nabla \phi, \nabla u|}{\phi u} + \frac{\phi^2}{m-1} - 2 + 2LR + 2(m-1)KR^2 - 8\rho^2 \frac{R^2 - \rho^2}{(R^2 - \rho^2)^2}. \tag{3.3}
\]
at \( x_1 \). By the definition of \( G \) and
\[
\frac{\langle \nabla \phi, \nabla u \rangle}{\phi u} = \frac{2\rho(\nabla \phi, \nabla u)}{(R^2 - \rho^2)u} \leq \frac{2\rho\phi}{R^2 - \rho^2},
\]
(3.3) can be written as
\[
0 \geq \frac{G^2}{m-1} - \frac{4(m-2)}{m-1}\rho G - [2 + 2LR + 2(m-1)KR^2](R^2 - \rho^2) - 8\rho^2
- (m-1)K(R^2 - \rho^2)^2
\geq \frac{G^2}{m-1} - \frac{4(m-2)}{m-1}RG - [2 + 2LR + 2(m-1)KR^2]R^2 - 8R^2 - (m-1)KR^4
\geq \frac{G^2}{2(m-1)} - [(c_m + 2LR)]R^2 - 3(m-1)KR^4
\]
at \( x_1 \), where we used the Cauchy-Schwarz inequality. This gives
\[
G(x_1) \leq c_m \left( \sqrt{1 + LR}R + \sqrt{KR^2} \right).
\]
By the definition of \( G \), we conclude
\[
\frac{3R^2}{4} \sup_{B_{R/2}(\partial M)} \frac{\|\nabla u\|}{u} \leq c_m \left( \sqrt{1 + LR}R + \sqrt{KR^2} \right)
\]
and the desired result follows.

**Case 2:** If \( x_1 \in \partial M \), the proof follows by the argument of [8]. We include it for the completeness. Indeed, at \( x_1 \), \( G \nu \geq 0, \phi \nu \equiv \frac{G \nu}{\rho} \geq 0 \) and hence \( (\phi^2) \nu \geq 0 \). Since \( u \) satisfies the Dirichlet boundary condition and \( u \nu \geq 0 \), then \( |\nabla u| = u \nu \). We note that \( \phi = |\nabla u| \) and \( \log u \) also satisfies Dirichlet boundary condition. So Proposition 1 gives
\[
0 \leq (\phi^2) \nu = (|\nabla \log u|^2) \nu = 2(\log u \nu)(\Delta_f \log u - (\log u) \nu H_f)
= 2 \frac{u \nu}{u} \left( -\frac{|\nabla u|^2}{u^2} - \frac{u \nu}{u} H_f \right) = 2 \left( \frac{u \nu}{u^2} \right)^2 \left( -\frac{u \nu}{u} - H_f \right),
\]
which implies
\[
\phi(x_1) = \frac{u \nu}{u} \leq -H_f \leq L.
\]
Hence \( G(x_1) = R^2 \phi(x_1) \leq LR^2 \). This gives
\[
\frac{3R^2}{4} \sup_{B_{R/2}(\partial M)} \frac{|\nabla u|}{u} \leq LR^2.
\]
The proof is complete.

4. **Gradient estimates for heat equations.** To prove Theorem 1.4, we introduce a smooth cut-off function as in [8]. This cut-off function is originally used in [10] (see also [16]).

**Lemma 4.1.** There exists a smooth cut-off function \( \psi = \psi(x,t) \) supported in \( Q_{R,T}(\partial M) \) such that

(i) \( \psi = \psi(\rho_{\partial M}(x,t)) \equiv \psi(\rho,\tau); \psi(\rho,\tau) = 1 \text{ in } Q_{\rho \tau} \partial M, \quad 0 \leq \psi \leq 1 \).

(ii) \( \psi \) is decreasing as a radial function in the spatial variables, and \( \frac{\partial \psi}{\partial \rho} = 0 \) in \( Q_{R/2, T}(\partial M) \).

(iii) \( |\frac{\partial \psi}{\partial \tau}| \leq C \), \( |\frac{\partial \psi}{\partial \tau}| \leq \frac{C \psi}{\tau} \) and \( |\frac{\partial^2 \psi}{\partial \tau^2}| \leq \frac{C \psi}{\tau^2} \), \( 0 < \varepsilon < 1 \).
Using the cut-off function, we have

**Lemma 4.2.** If \( \Phi := \psi (\Delta_f w - w_t) + w \Delta_f \psi - w \psi_t - 2 \frac{|\nabla \psi|^2}{\psi} w \), then

\[
\psi w^2 \leq c \left( \frac{D^2 + 1}{R^4} + \frac{1}{T^2} + K^2 \right) + \frac{\Phi}{4},
\]

where

\[
D := 1 + \log A - \log \left( \inf_{Q_n T(\partial M)} u \right).
\]

Here \( c \) denotes a constant depending only on \( n \) whose value may change from line to line in the following.

**Proof.** Plugging (2.1) into \( \Phi \),

\[
\Phi \geq -2(n-1)K \psi w - 2 \left( 2h - \frac{1}{h} \right) \langle \nabla h, \nabla \psi \rangle w
\]

\[
+ 2 \left( 2 + \frac{1}{h^2} \right) \psi w^2 + w \Delta_f \psi - w \psi_t - 2 \frac{|\nabla \psi|^2}{\psi} w.
\]

This is equivalent to

\[
4\psi w^2 \leq \frac{2h^2}{1 + 2h^2} 2(n-1)K \psi w - \frac{4h (1 - 2h^2)}{1 + 2h^2} \langle \nabla h, \nabla \psi \rangle w
\]

\[
- \frac{2h^2}{1 + 2h^2} w \Delta_f \psi + \frac{2h^2}{1 + 2h^2} w \psi_t + \frac{4h^2}{1 + 2h^2} \frac{|\nabla \psi|^2}{\psi} w + \frac{2h^2}{1 + 2h^2} \Phi.
\]

Since \( 0 < \frac{2h^2}{1 + 2h^2} \leq 1 \) and \( 0 < \frac{2}{1 + 2h^2} \leq 2 \), we get

\[
4\psi w^2 \leq 2(n-1)K \psi w - \frac{4h (1 - 2h^2)}{1 + 2h^2} \langle \nabla h, \nabla \psi \rangle w
\]

\[
- \frac{2h^2}{1 + 2h^2} w \Delta_f \psi + w \psi_t + \frac{2|\nabla \psi|^2}{\psi} w + \frac{2h^2}{1 + 2h^2} \Phi.
\]

In the following we will estimate each term of the right hand side of (4.1). Since \( \text{Ric}_f \geq -(n-1)K \) and \( H_f \geq 0 \), by the Laplacian comparison in Theorem 2.1,

\[
\Delta_f \rho \leq (n-1)KR.
\]

Using this and Lemma 4.1, we have

\[
- \frac{2h^2}{1 + 2h^2} w \Delta_f \psi = - \frac{2h^2}{1 + 2h^2} w \left( \psi \Delta_f \rho + \psi_{rr} |\nabla \rho|^2 \right)
\]

\[
\leq \frac{2h^2}{1 + 2h^2} w \left( |\psi_r| (n-1)KR + |\psi_{rr}| \right)
\]

\[
\leq \psi^{1/2} w \frac{|\psi_{rr}|}{\psi^{1/2}} + (n-1)KR \psi^{1/2} w \frac{|\psi_r|}{\psi^{1/2}}
\]

\[
\leq \frac{3}{5} \psi w^2 + c \left[ \frac{|\psi_{rr}|}{\psi^{1/2}} \right] + \left( (n-1)KR \frac{|\psi_r|}{\psi^{1/2}} \right)^2
\]

\[
\leq \frac{3}{5} \psi w^2 + \frac{c}{R^4} + cK^2.
\]

By the Young inequality, we also have

\[
- \frac{4h (1 - 2h^2)}{1 + 2h^2} \langle \nabla h, \nabla \psi \rangle w
\]
\[
\begin{align*}
\leq & 4h \left| 1 - \frac{2h^2}{1 + 2h^2} \right| |\nabla \psi| |\nabla h| w 
\leq 4h |\nabla \psi| w^{3/2} \\
= & 4h |\nabla \psi| \psi^{-3/4} (\psi w^2)^{3/4} \leq \frac{3}{5} \psi w^2 + ch^4 \frac{|\nabla \psi|^4}{\psi^3} \leq \frac{3}{5} \psi w^2 + cD^2 R^4,
\end{align*}
\]
where
\[
D := 1 + \log A - \log \left( \inf_{Q_{R,T}(\partial M)} u \right).
\]

By the Cauchy-Schwarz inequality, it is not hard to see that the following estimates hold. We first estimate that
\[
2(n - 1) K \psi w \leq \frac{3}{5} \psi w^2 + 4cK^2;
\]
for \( w |\psi| \), we estimate that
\[
w |\psi| = \psi^{1/2} w \frac{|\psi|}{\psi^{1/2}} \leq \frac{3}{5} \left( \psi^{1/2} w \right)^2 + c \left( \frac{|\psi|}{\psi^{1/2}} \right)^2 \leq \frac{3}{5} \psi w^2 + \frac{c}{T^2};
\]
we also estimate that
\[
2\frac{|\nabla \psi|^2}{\psi} w = 2 \left( |\nabla \psi|^2 \psi^{-3/2} \right) \left( \psi^{1/2} w \right) \leq \frac{3}{5} \psi w^2 + c \frac{|\nabla \psi|^4}{\psi^3} \leq \frac{3}{5} \psi w^2 + \frac{c}{R^4}.
\]
We substitute (4.2)-(4.6) into the right hand side of (4.1), and get that
\[
\psi w^2 \leq c \left( \frac{D^2 + 1}{R^4} + \frac{1}{T^2} + K^2 \right) + \frac{2h^2}{2(1 + 2h^2)} \Phi.
\]
and the result follows. \( \square \)

Now we are ready to prove Theorem 1.4 by improving the argument of [8].

**Proof of Theorem 1.4.** We may assume \( u \) is non-constant in \( Q_{R/2,T/2} \). If \( u \) is constant, it is trivial. Hence we let \( \psi w \) be a positive function at a point in \( Q_{R/2,T/3}(\partial M) \) and achieves its maximal value at \( (x_1, t_1) \) in \( Q_{R/2,T/3}(\partial M) \). We first claim \( x_1 \notin \partial M \). Indeed, by contradiction, assume that \( x_1 \in \partial M \). Then at \( (x_1, t_1) \), \( (\psi w)_\nu \geq 0 \), consequently
\[
\psi w + \psi w_\nu = \psi w_\nu \geq 0;
\]
in particular, \( w_\nu \geq 0 \). Since \( w = |\nabla \sqrt{\log \left( \frac{B}{w} \right)}|^2 \), and \( h = \sqrt{\log \left( \frac{B}{w} \right)} \) also satisfies the Dirichlet boundary condition, Proposition 1 implies
\[
0 \leq w_\nu = (|\nabla h|^2)_\nu = 2h_\nu (\Delta f - h_\nu H_f).
\]
Since \( u \) satisfies the Dirichlet boundary condition, then \( |\nabla u| = u_\nu \). Hence,
\[
h_\nu = -\frac{u_\nu}{2u \sqrt{\log \left( \frac{B}{w} \right)}} = -\frac{|\nabla u|}{2u \sqrt{\log \left( \frac{B}{w} \right)}} = -w_\nu.
\]
We now compute

\[ \Delta_f h = \text{div} \left( -\frac{\nabla u}{2 u \sqrt{\log \left( \frac{B}{u} \right)}} \right) - \langle \nabla f, \nabla h \rangle \]

\[ = -\frac{\Delta u}{2 u \sqrt{\log \left( \frac{B}{u} \right)}} - \frac{1}{2} \left\langle \nabla u, \nabla \left( \frac{1}{u \sqrt{\log \left( \frac{B}{u} \right)}} \right) \right\rangle + \left\langle \nabla f, -\frac{\nabla u}{2 u \sqrt{\log \left( \frac{B}{u} \right)}} \right\rangle \]

\[ = -\frac{\Delta f u}{2u \sqrt{\log \left( \frac{B}{u} \right)}} - \frac{1}{2} \left( \frac{|\nabla u|^2}{u^2 \sqrt{\log \left( \frac{B}{u} \right)}} + \frac{1}{2} \frac{u^2}{u^2 \log \left( \frac{B}{u} \right)} \right) \]

\[ = -\frac{u_t}{2uh} + \left( \frac{2h^2 - 1}{h} \right) w. \]

Plugging these identities into the above inequality, we obtain

\[ 0 \leq -2w \left( -\frac{u_t}{2uh} + \left( \frac{2h^2 - 1}{h} \right) w \right) \leq -2w \left( \left( \frac{2h^2 - 1}{h} \right) w + H_f \right) \]

where we used \( u_t \leq 0 \) in the last inequality. This implies

\[ \left( \frac{2h^2 - 1}{h} \right) w \leq -H_f \leq 0. \]

Since \( h \geq 1 \), \( w = 0 \) at \((x_1, t_1)\). This means \( \psi w \equiv 0 \) on \( Q_{R,T}(\partial M) \). This is a contradiction.

We have shown that \( x_1 \notin \partial M \). By the standard argument of Calabi, we may assume that \( x_1 \notin \partial M \cup \text{Cut}(\partial M) \). Now at \((x_1, t_1)\), Lemma 4.2 gives

\[ w^2 \leq c \left( \frac{D^2 + 1}{R^4} + \frac{1}{T^2} + K^2 \right) + \frac{h^2}{2(1 + 2h^2)^2} \Phi. \quad (4.7) \]

On the other hand, since \((x_1, t_1)\) is a maximal point, we have

\[ \nabla (\psi w) = 0, \quad \Delta_f (\psi w) \leq 0 \quad \text{and} \quad (\psi w)_t \geq 0 \]

at \((x_1, t_1)\). Thus, at \((x_1, t_1)\),

\[ 0 \geq \Delta_f (\psi w) - (\psi w)_t = \psi (\Delta_f w - w_t) + w (\Delta_f \psi - \psi_t) + 2 \langle \nabla w, \nabla \psi \rangle, \]

that is, \( \Phi(x_1, t_1) \leq 0 \). Hence, (4.7) implies

\[ (\psi w)(x, t) \leq (\psi w)(x_1, t_1) \leq c \left( \frac{D + 1}{R^2} + \frac{1}{T} + K \right) \]

for all \((x, t) \in Q_{R,T}(\partial M)\). Since \( \psi \equiv 1 \) in \( Q_{R/2,T/2}(\partial M) \), by the definition of \( w \), we get

\[ \frac{|\nabla u|}{u} \leq 2c \left( \frac{\sqrt{D} + 1}{R} + \frac{1}{\sqrt{T}} + \sqrt{K} \right) \sqrt{1 + \log \frac{A}{u}}. \]

The proof is complete. \( \square \)

Corollary 1 follows by letting \( R \to \infty \). So we only give a proof of Corollary 2.
Proof of Corollary 2. Now $K = L = 0$ and $u_t = \Delta_t u$. Let $v = u + 1$ an then $v_t = \Delta_t v$. Moreover, $u$ and $v$ have the same growth at infinity. Hence, without loss of generality, we may assume $u \geq 1$. Fix $(x_0, t_0)$. We apply Theorem 1.4 to $Q_{R, R}(\partial M) = B_{R}(\partial M) \times [t_0 - R, t_0]$ and get

\[
\frac{|\nabla u|}{u}(x_0, t_0) \leq c_m \left( \sqrt{1 + \log \frac{A}{R}} + \frac{1}{\sqrt{R}} \right) \sqrt{1 + \log \frac{A}{u(x_0, t_0)}}
\]

\[
\leq c_m \left( \sqrt{o(R + |R|)} \frac{R}{R} + \frac{1}{\sqrt{R}} \right) \sqrt{o(R + |R|) - \log(u(x_0, t_0))}.
\]

Letting $R \to \infty$, we get $|\nabla u(x_0, t_0)| = 0$ and $u$ is constant because $(x_0, t_0)$ is arbitrary. The proof of (1) is complete.

The proof of (2) is similar as in [16], and we omit the details. \(\Box\)

Remark 2. In Theorem 4.2 of [18], the authors proved that $\rho$ is finite if some extra conditions on $K$ and $L$ are given. We point out that our gradient estimates and Liouville theorems work well in this case. In fact, if $\rho$ is finite, then all boundary balls $B_{\partial M}(R)$ are the same if $R$ is large. The proof still work without any change. Therefore, we can let $R \to \infty$ even when $\rho$ is finite.

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