Relaxation to Fractional Porous Medium Equation from Euler–Riesz System

Young-Pil Choi1 · In-Jee Jeong2

Received: 10 February 2021 / Accepted: 14 September 2021 / Published online: 1 October 2021
© The Author(s), under exclusive licence to Springer Science+Business Media, LLC, part of Springer Nature 2021

Abstract
We perform asymptotic analysis for the Euler–Riesz system posed in either $\mathbb{T}^d$ or $\mathbb{R}^d$ in the high-force regime and establish a quantified relaxation limit result from the Euler–Riesz system to the fractional porous medium equation. We provide a unified approach for asymptotic analysis regardless of the presence of pressure in the case of repulsive Riesz interactions, based on the modulated energy estimates, the Wasserstein distance of order 2, and the bounded Lipschitz distance. For the attractive Riesz interaction case, we consider the periodic domain and estimate a lower bound on the modulated internal energy to handle the modulated interaction energy.

Keywords Porous Medium Equation · Modulated energy · Relaxation limit · Asymptotic analysis · Riesz interaction

Mathematics Subject Classification 76N10 · 35Q31

1 Introduction

1.1 The Systems

In the current work, we are interested in the asymptotic analysis for the following damped Euler–Riesz system corresponding to the high-force regime:

Communicated by Pierre Degond.

In-Jee Jeong
injee_j@snu.ac.kr

Young-Pil Choi
ypchoi@yonsei.ac.kr

1 Department of Mathematics, Yonsei University, Seoul 03722, Republic of Korea
2 Department of Mathematical Sciences and RIM, Seoul National University, Seoul 08826, Republic of Korea
\[
\begin{aligned}
\partial_t \rho^{(\varepsilon)} + \nabla \cdot (\rho^{(\varepsilon)} u^{(\varepsilon)}) &= 0, \\
\partial_t (\rho^{(\varepsilon)} u^{(\varepsilon)}) + \nabla \cdot (\rho^{(\varepsilon)} u^{(\varepsilon)} \otimes u^{(\varepsilon)}) + \frac{1}{\varepsilon} c_p \nabla p(\rho^{(\varepsilon)}) &= 0, \\
-\frac{1}{\varepsilon} \rho^{(\varepsilon)} u^{(\varepsilon)} + \frac{1}{\varepsilon} c_K \rho^{(\varepsilon)} \nabla \Lambda^{\alpha-d} \rho^{(\varepsilon)},
\end{aligned}
\]  

where \(\rho^{(\varepsilon)}(t, \cdot): \Omega \to \mathbb{R}_+\) and \(u^{(\varepsilon)}(t, \cdot): \Omega \to \mathbb{R}^d\) denote the density and the velocity of the fluid, respectively. The pressure \(p(\rho^{(\varepsilon)})\) is given by the power-law \(p(\rho) = \rho^\gamma\), for some \(\gamma \geq 1\). Here, the domain is either \(\Omega = \mathbb{R}^d\) or \(\mathbb{T}^d\) and we consider the range \(0 \vee (d-2) < \alpha < d\) for the fractional Laplacian operator \(\Lambda^{\alpha-d} = (-\Delta)^{\frac{\alpha-d}{2}}\). The case \(\alpha - d = -2\) corresponds to Coulomb interaction, and we shall refer to the range \(0 \vee (d-2) < \alpha < d\) as Riesz interaction. Lastly, \(c_P \geq 0\) and \(c_K \in \mathbb{R}\) are coefficients representing the strength of the pressure and Riesz interaction force, respectively.

The system (1.1) has been recently investigated in [11, Serfaty 2020]. A rigorous derivation of the system (1.1) with \(c_P = 0\), i.e., pressureless case, from interacting particle systems by means of mean-field limits is established in Serfaty (2020) under suitable regularity assumptions on the solutions of (1.1). In Choi and Jeong, the local-in-time existence and uniqueness of classical solutions to the system (1.1) without the linear damping under suitable regularity assumptions on the initial data are established. It is clear that the existence theory developed in Choi and Jeong can be directly applied to the system (1.1). More recently, the global-in-time existence of the unique classical solution to the system (1.1) with \(c_P = 0\) and \(c_K < 0\) and its large-time behavior are studied in Choi and Jung.

Let us briefly explain how the system (1.1) behaves when \(\varepsilon\) vanishes. Define the free energy \(\mathcal{F}: L_1^+(\Omega) \to \mathbb{R}\) for the system (1.1) by

\[
\mathcal{F}(\rho^{(\varepsilon)}) := c_P \int_{\Omega} U(\rho^{(\varepsilon)}) \, dx - \frac{c_K}{2} \int_{\Omega} \rho^{(\varepsilon)} \Lambda^{\alpha-d} \rho^{(\varepsilon)} \, dx,
\]  

where \(U: L^1(\Omega) \to \mathbb{R}\) is an increasing function describing the internal energy of the density given by

\[
U(\rho) = \begin{cases} 
\rho \ln \rho & \text{if } \gamma = 1, \\
\frac{1}{\gamma - 1} \rho^\gamma & \text{if } \gamma > 1.
\end{cases}
\]

Here, \(L^1_+(\Omega)\) stands for the set of nonnegative \(L^1(\Omega)\) functions. Then, we can rewrite the momentum equations in (1.1) as

\[
\varepsilon \partial_t (\rho^{(\varepsilon)} u^{(\varepsilon)}) + \varepsilon \nabla \cdot (\rho^{(\varepsilon)} u^{(\varepsilon)} \otimes u^{(\varepsilon)}) = -\rho^{(\varepsilon)} \nabla \delta \mathcal{F}(\rho^{(\varepsilon)}) \frac{\delta \mathcal{F}(\rho^{(\varepsilon)})}{\delta \rho^{(\varepsilon)}} - \rho^{(\varepsilon)} u^{(\varepsilon)},
\]  

\(\Diamond\) Springer
where \( \frac{\delta F(\rho)}{\delta \rho} \) is the variational derivative of the free energy \( F \) with respect to \( \rho \), that is

\[
\frac{\delta F(\rho)}{\delta \rho} = c_p U'(\rho) - c_K \Lambda^{a-d} \rho.
\]

Thus, at the formal level the left hand side of (1.3) converges to zero as \( \varepsilon \to 0 \); if \( \rho^{(\varepsilon)} \to \rho \) and \( \rho^{(\varepsilon)} u^{(\varepsilon)} \to \rho u \) as \( \varepsilon \to 0 \), we deduce from (1.1) the continuity equation which has a gradient flow structure (Jordan et al. 1998; Otto 2001):

\[
\partial_t \rho + \nabla \cdot (\rho u) = 0 \quad \text{with} \quad \rho u = \rho \nabla \frac{\delta F(\rho)}{\delta \rho} = \rho (c_p U'(\rho) - c_K \Lambda^{a-d} \rho),
\]

(1.4)

which can also be rewritten as the fractional porous medium flow (Caffarelli et al. 2013):

\[
\partial_t \rho + c_K \nabla \cdot (\rho \Lambda^{a-d} \rho) = c_p \Delta \rho^\gamma.
\]

The main purpose of this work is to make the above formal derivation completely rigorous. More precisely, we will provide a unified approach for the quantitative error estimate between solutions to Eqs. (1.1) and (1.4). The high-force limit or strong relaxation limit has been studied for the damped Euler system (Coulombel and Goudon 2007; Huang et al. 2011; Junca and Rascle 2002; Luo and Zeng 2016; Marcati and Milani 1990), Euler–Poisson system (Choi 2021; Lattanzio and Tzavaras 2017), Euler system with nonlocal forces (José 2019; Carrillo et al. 2020). In the present work, we extend the previous results (Choi 2021; Lattanzio and Tzavaras 2017) to the Riesz interaction case.

1.2 Methodology

Our main strategy is based on estimates for the modulated energy, which is also often called as relative entropy. This type of method has originated in Dafermos (1979) to establish weak-strong uniqueness principle for hyperbolic systems. Later, it has been successfully applied to hydrodynamic limit problems: Ginzburg–Landau lattice model (Yau 1991), Boltzmann equation (Bouchut et al. 2000; Saint-Raymond 2009), Vlasov–Poisson system (Brenier 2000), etc.

To begin with, note that the kinetic energy \( K \) to the system (1.1) is given by

\[
K(U) := \frac{|\mathbf{m}|^2}{2 \rho} \quad \text{with} \quad U = \left( \begin{array}{c} \rho \\ \mathbf{m} = \rho \mathbf{u} \end{array} \right).
\]

Then, the modulated kinetic energy is given by

\[
\int_{\Omega} K(U|\tilde{U}) \, dx := \int_{\Omega} K(U) - K(\tilde{U}) - D_{\tilde{\rho}, \tilde{\mathbf{m}}} K(\tilde{U})(U - \tilde{U}) \, dx = \frac{1}{2} \int_{\Omega} |u - \tilde{u}|^2 \, dx,
\]
where

\[ \tilde{U} = \left( \frac{\tilde{\rho}}{\tilde{m}} \right). \]

We also introduce the modulated energy associated with the free energy defined in (1.2):

\[
\mathcal{F}(\rho | \tilde{\rho}) := \mathcal{F}(\rho) - \mathcal{F}(\tilde{\rho}) - \int_{\Omega} \frac{\delta \mathcal{F}(\tilde{\rho})}{\delta \tilde{\rho}} (\rho - \bar{\rho}) \, dx
= c_P \int_{\Omega} U(\rho) - U(\bar{\rho}) - U'(\bar{\rho})(\rho - \bar{\rho}) \, dx - \frac{c_K}{2} \int_{\Omega} (\rho - \bar{\rho}) \Lambda^{a-d}(\rho - \bar{\rho}) \, dx,
\]

where the first term on the right hand side is called \textit{modulated internal energy} and the second one is called \textit{modulated interaction energy}.

We shall divide the proof into two cases: pressureless and repulsive case \((c_P = 0\) and \(c_K < 0)\) and pressure and attractive case \((c_P > 0\) and \(c_K > 0)\).\(^1\) Let us provide some ideas of the proof, in each of these two cases.

**Pressureless case.** In the absence of pressure, the total energy does not include the internal energy \(U(\rho)\), and thus it is expected to obtain some convergence of \(\rho(\varepsilon)\) toward \(\rho\) by estimating the interaction energy. The modulated interaction energy has been employed in Duerinckx (2016), Serfaty (2020) to study the mean-field limits for Riesz-type flows. In particular, the \textit{extension representation} for the fractional Laplacian in the whole space (proposed in Caffarelli and Silvestre 2007) is used in Duerinckx 2016; Serfaty 2020). Motivated from these works, in the whole space case we estimate the modulated interaction energy to show convergence of \(\rho(\varepsilon)\) toward \(\rho\) in some negative Sobolev space. On the other hand, in the periodic domain case, it is unclear how to apply the extension method of Caffarelli and Silvestre (2007). To overcome this issue, using Fourier transform and commutator estimates, we obtain a similar type of estimate for the modulated interaction energy under an additional regularity assumption on the velocity fields \(u\).

In addition to convergence of \(\rho(\varepsilon)\) in some negative-order Sobolev space, we have a stronger convergence of \(\rho(\varepsilon)\) by employing the Wasserstein distance of order 2, which is defined by

\[
d_2(\mu, \nu) := \inf_{\pi \in \Pi(\mu, \nu)} \left( \int_{\Omega \times \Omega} |x - y|^2 \, \pi(dx, dy) \right)^{1/2},
\]

for \(\mu, \nu \in \mathcal{P}_2(\Omega)\), where \(\Pi(\mu, \nu)\) is the set of all probability measures on \(\Omega \times \Omega\) with first and second marginals \(\mu\) and \(\nu\) and bounded 2-moments, respectively. Here \(\mathcal{P}_2(\Omega)\) is the set of probability measures in \(\Omega\) with second moment bounded. Note that \(\mathcal{P}_2(\Omega)\) is a complete metric space endowed with the 2-Wasserstein distance. We show that the 2-Wasserstein distance between \(\rho(\varepsilon)\) and \(\rho\) can be controlled by the associated modulated kinetic energy; see Proposition 3.1. Thus, the quantitative error bound on

\(^1\) Our previous work Choi and Jeong suggests that the Cauchy problem for the system (1.1) is ill-posed for the pressureless and attractive case, i.e., \(c_K > 0\) and \(c_P = 0\).
the modulated kinetic energy also gives convergence in terms of the 2-Wasserstein distance between the densities.

In order to show the convergence of the momentum $\rho^{(\varepsilon)}u^{(\varepsilon)}$ toward $\rho u$, we use the bounded Lipschitz distance defined by

$$d_{BL}(\mu, v) := \sup_{\phi \in \mathcal{A}} \left| \int \phi(x) (\mu(dx) - v(dx)) \right|,$$

where the admissible set $\mathcal{A}$ of test functions is given by

$$\mathcal{A} := \left\{ \phi : \Omega \to \mathbb{R} : \|\phi\|_{L^\infty} \leq 1 \text{ and } \|\phi\|_{Lip} := \sup_{x \neq y} \frac{|\phi(x) - \phi(y)|}{|x - y|} \leq 1 \right\}.$$

We prove that the bounded Lipschitz distance between the momenta can be bounded by the sum of the 2-Wasserstein distance between the associated densities and the modulated kinetic energy.

**Pressure case.** With pressure, the repulsive interaction case can be easily taken into account by almost the same arguments as the above, see Sect. 1.4 (iv). In the attractive interaction case, it is observed in Carrillo et al. (2020) and Lattanzio and Tzavaras (2017) that the modulated internal energy plays a crucial role in handling the modulated Coulomb or regular interaction energy. In particular, the attractive Coulomb interaction is considered in Lattanzio and Tzavaras (2017) in the periodic domain. In this work, we extend it to cover the attractive Riesz interaction case. Presence of pressure gives convexity of the total energy, and therefore we have the strong convergence of $(\rho^{(\varepsilon)}, \rho^{(\varepsilon)}u^{(\varepsilon)})$ toward $(\rho, \rho u)$ in some $L^p$ space.

**Notation.** Let us introduce a few notations and conventions used throughout the paper. Since the total mass is conserved in time (see Lemma 2.1), without loss of generality, we assume that $\rho^{(\varepsilon)}$ is a probability density function, i.e., $\|\rho^{(\varepsilon)}(t, \cdot)\|_{L^1} = 1$ for all $t \geq 0$ and $\varepsilon > 0$. Moreover, $L^1_2(\Omega)$ represents the space of weighted integrable functions by $1 + |x|^2$ with the norm

$$\|f\|_{L^1_2} := \int_\Omega (1 + |x|^2)|f(x)| \, dx.$$

Clearly, the weighted norm is redundant in the periodic domain case. The $L^2$ based Sobolev norms are defined by $\|f\|_{\dot{H}^s} = \|\Lambda^s f\|_{L^2}$ and $\|f\|_{H^s} = \|f\|_{\dot{H}^s} + \|f\|_{L^2}$. Finally, we denote by $C$ a generic positive constant, independent of $\varepsilon$ and whose value can vary from a line to another.

### 1.3 Main Result

Now we are ready to state the main result.
Theorem 1.1  Let $T > 0$, $d \geq 1$ and $\gamma \geq 1$. Let $(\rho^{(e)}, u^{(e)})$ and $\rho$ be classical solutions to the systems (1.1) and (1.4) on the time interval $[0, T]$, respectively. Suppose that

$$\rho^{(e)}, \rho \in L^\infty(0, T; L^1_2(\Omega)) \quad \text{and} \quad u \in W^{1,\infty}((0, T) \times \Omega). \quad (1.5)$$

In the periodic domain case, we assume in addition that $u \in L^\infty(0, T; H^s(\mathbb{T}^d))$ with $s > d/2 + 1 + (d - \alpha)/2$. Then we have the following statements.

(i) pressureless and repulsive case ($c_P = 0$ and $c_K < 0$): There exists $C > 0$, which depends on $c_K$, $T$, and $\|u\|_{W^{1,\infty}}$ when $\Omega = \mathbb{R}^d$, $\|u\|_{W^{1,\infty}}$, and $\Omega = \mathbb{T}^d$, but independent of $\epsilon > 0$ such that

$$\sup_{0 \leq t \leq T} \left( d^2_2(\rho^{(e)}(t), \rho(t)) + \| (\rho^{(e)}(t) - \rho)(t, \cdot) \|^2_{H^{-\frac{d-\alpha}{2}}} \right)$$

$$+ \int_0^T d^2_{BL}( (\rho^{(e)} u^{(e)})(t), (\rho u)(t) ) \, dt$$

$$\leq C \epsilon \int_{\Omega} \rho^{(e)}_0 |u_0^{(e)} - u_0|^2 \, dx + C d^2_2(\rho_0^{(e)}, \rho_0) + C \int_{\Omega} (\rho^{(e)}_0)$$

$$- \rho_0) \Lambda^{d/\alpha} (\rho^{(e)}_0 - \rho_0) \, dx + C \epsilon^2. \quad (1.6)$$

In particular, if the right hand side of (1.6) converges to zero as $\epsilon \to 0$, then we have

$$\rho^{(e)} \to \rho \quad \text{in} \quad L^\infty(0, T; \dot{H}^{-\frac{d-\alpha}{2}}(\Omega)) \quad \text{and weakly} \quad \star \quad \text{in} \quad L^\infty(0, T; \mathcal{M}(\Omega))$$

$$\rho^{(e)} u^{(e)} \to \rho u \quad \text{weakly} \quad \star \quad \text{in} \quad L^2(0, T; \mathcal{M}(\Omega)).$$

Here, $\dot{H}^{-\frac{d-\alpha}{2}}(\Omega) = \dot{W}^{-\frac{d-\alpha}{2},2}(\Omega)$ stands for the negative-order homogeneous Sobolev space with the finite corresponding norm, and we denote by $\mathcal{M}(\Omega)$ the space of (signed) Radon measures on $\Omega$ with finite mass.

(ii) pressure and attractive case ($c_P > 0$ and $c_K > 0$): Let us assume $d \geq 2$, $\Omega = \mathbb{T}^d$, and

$$0 < \rho_{\min} \leq \rho(t, x) \leq \rho_{\max} < \infty \quad (t, x) \in [0, T] \times \mathbb{T}^d$$

for some $\rho_{\min}, \rho_{\max} \in \mathbb{R}$. Furthermore, we assume $\gamma \geq 1 + \alpha/d$ and the strength of the attractive interaction force $c_K > 0$ is small enough, dependently only on $\mathbb{T}^d, \rho_{\min}, \rho_{\max}$, and the pressure-coefficient $c_P > 0$. Then we have

$$\sup_{0 \leq t \leq T} \| (\rho^{(e)} - \rho)(t, \cdot) \|^2_{L^\gamma} + \int_0^T \| (\rho^{(e)} u^{(e)} - \rho u)(t, \cdot) \|^2_{L^1} \, dt$$

$$\leq C \epsilon \int_{\mathbb{T}^d} \rho^{(e)}_0 |u_0^{(e)} - u_0|^2 \, dx + C \int_{\mathbb{T}^d} U(\rho^{(e)}_0 |\rho_0) \, dx + C \epsilon^2, \quad (1.7)$$

for $\gamma \in [1, 2]$, where $C > 0$ depends on $c_P, c_K, T, \rho_{\min}, \rho_{\max}$, and $\|u\|_{W^{1,\infty}}$, but independent of $\epsilon > 0$. 

\(\square \) Springer
Similarly as before, if the right hand side of (1.7) converges to zero as \( \varepsilon \to 0 \), then we have
\[
\rho^{(\varepsilon)} \to \rho \quad a.e. \text{ and in } L^\infty(0, T; L^\gamma(\mathbb{T}^d)), \quad \gamma \in [1, 2],
\]
\[
\rho^{(\varepsilon)} u^{(\varepsilon)} \to \rho u \quad a.e. \text{ and in } L^2(0, T; L^1(\mathbb{T}^d))
\]
as \( \varepsilon \to 0 \).

1.4 Remarks

We give several remarks regarding the main statement above.

(i) The existence theory for the systems (1.1) and (1.4) is developed in Choi and Jeong (2021). To be more precise, the local-in-time existence and uniqueness of classical solutions for (1.1) with either \( c_P = 0 \) and \( c_K < 0 \) (pressureless and repulsive case) or \( c_P > 0 \) and \( c_K > 0 \) (pressure and attractive case) is established in Choi and Jung. It is worth noting that in the attractive Riesz interaction case without pressure, i.e., \( c_P = 0 \) and \( c_K > 0 \), linear analysis suggests that the Cauchy problem for the system (1.1) is ill-posed. For the case with pressure, the range \( \gamma \in [1, 5/3] \) is considered. For the limiting system (1.4), the local-in-time existence of the unique classical solution is obtained in Choi and Jeong (2021) when \( \gamma = 1 \). Similarly as above, the either \( c_K < 0 \) and \( c_P = 0 \) o r \( c_K > 0 \) and \( c_P > 0 \) is taken into account. As mentioned in Choi and Jeong (2021), the local well-posedness for (1.4) with \( \gamma \neq 1 \) is a challenging problem when the possible vacuum state is considered; see (Choi and Jeong 2021, Section 1.3) for more details. On the other hand, Theorem 1.1 can be obtained by using a rather weak regularity of solutions to the system (1.1), (Lattanzio and Tzavaras 2017, Definition 3.1) for instance, see also Choi (2021).

(ii) The finite second moment of \( \rho^{(\varepsilon)} \) can be easily obtained. In fact, it follows from the continuity equation of (1.1) that
\[
\frac{1}{2} \frac{d}{dt} \int_\Omega |x|^2 \rho^{(\varepsilon)} \, dx = \int_\Omega x \cdot u^{(\varepsilon)} \rho^{(\varepsilon)} \, dx.
\]

Then, applying Young’s inequality together with Grönwall’s lemma gives
\[
\int_\Omega |x|^2 \rho^{(\varepsilon)} \, dx \leq e^T \int_\Omega |x|^2 \rho_0^{(\varepsilon)} \, dx + e^T \int_0^T \int_\Omega |u^{(\varepsilon)}|^2 \rho^{(\varepsilon)} \, dx \, d\tau.
\]

Since the right hand side can be bounded under the assumption that \( \rho^{(\varepsilon)} \in L^\infty((0, T) \times \Omega) \) and \( u^{(\varepsilon)} \in L^2((0, T) \times \Omega) \) (for instance), we have the desired result. It is worth noticing that uniform-in-\( \varepsilon \) bound is not necessarily required here.

(iii) One can slightly relax the assumptions (1.5) on solutions in the whole space case. To be more specific, instead of (1.5), under the assumption that \( \partial_t u, \nabla u \in \)
$L^\infty((0, T) \times \mathbb{R}^d)$, the error estimate (1.6) can be replaced by

\[
\sup_{0 \leq t \leq T} \| ( \rho^{(e)} - \rho ) ( t, \cdot ) \|_{L^{\infty} \left( (0, T) \times \mathbb{R}^d \right)}^2 + \int_0^T \int_{\mathbb{R}^d} \rho^{(e)} ( t, x ) ( u^{(e)} - u ) ( t, x )^2 \, dx \, dt \\
\leq C \varepsilon \int_{\mathbb{R}^d} \rho_0^{(e)} | u_0^{(e)} - u_0 |^2 \, dx + C \int_{\mathbb{R}^d} ( \rho_0^{(e)} - \rho_0 ) \Lambda^{\alpha - d} \left( \rho_0^{(e)} - \rho_0 \right) \, dx \\
+ C \varepsilon^3 \left( \int_{\mathbb{R}^d} \rho_0^{(e)} | u_0^{(e)} |^2 \, dx + \frac{1}{\varepsilon} \int_{\mathbb{R}^d} \rho_0^{(e)} \Lambda^{\alpha - d} \rho_0^{(e)} \, dx \right) + C \varepsilon^2,
\]

where $C > 0$ is independent of $\varepsilon > 0$. Thus, we also conclude

\[
\rho^{(e)} \to \rho \quad \text{in} \quad L^\infty (0, T; \dot{H}^{-\frac{d-\alpha}{2}}),
\]

as $\varepsilon \to 0$. We refer to Remark 2.12 for a detailed discussion.

(iv) The result of Theorem 1.1 (i) can be naturally extended to the pressure and repulsive case without any further difficulties since the free energy $F(\rho_0^{(e)})$ is always nonnegative. Indeed, if $(\rho^{(e)})^\gamma, \rho^\gamma \in L^\infty (0, T; L^1(\Omega))$ uniformly in $\varepsilon$, then we have

\[
\sup_{0 \leq t \leq T} \| ( \rho^{(e)} - \rho ) ( t, \cdot ) \|_{L^2}^2 + \int_0^T \| ( \rho^{(e)} u^{(e)} - \rho u ) ( t, \cdot ) \|_{L^2}^2 \, dt \\
\leq C \varepsilon \int_{\Omega} \rho_0^{(e)} | u_0^{(e)} - u_0 |^2 \, dx + C \int_{\Omega} \mathcal{U}(\rho_0^{(e)} \rho_0) \, dx + C \varepsilon^2,
\]

for $\gamma \in [0, 2]$, where $C > 0$ is independent of $\varepsilon > 0$. One may find the detailed discussion in Remark 3.4.

(v) In a recent work Hung et al., the modulated interaction energy estimates with general interaction potential are discussed. Note that the inverse of fractional Laplacian can be given as $K \ast \rho$ with $K = |x|^{-\alpha}$, e.g., the term $\Lambda^{\alpha - d} \rho$ can be rewritten as $K \ast \rho$ with $\alpha \in (d - 2, d)$. In Hung et al., the sub-Coulombic case, which means $\alpha \in (0, d - 2)$, is dealt with. This would provide the estimate of modulated interaction energy in Lemma 2.7 for the case $\alpha \in (0, d - 2)$ under suitable assumptions on the solution $u$, and thus our relaxation limit results in Theorem 1.1 (i) hold for any $\alpha \in (0, d)$.

The rest of this paper is organized as follows. In Sect. 2, we provide the modulated kinetic, internal, and interaction energy estimates. Section 3 is devoted to proving Theorem (1.1).

2 Modulated Energy Estimates

The goal of this section is to establish modulated energy estimates for the system (1.1). Before we proceed, let us begin with some standard energy estimates.
Lemma 2.1 Let $T > 0$. Let $(\rho^{(e)}, u^{(e)})$ be a classical solution to the system (1.1) on the time interval $[0, T]$. Then, we have

$$
\frac{d}{dt} \int_\Omega \rho^{(e)} \, dx = 0, \quad \frac{d}{dt} \int_\Omega \rho^{(e)} u^{(e)} \, dx = -\frac{1}{\varepsilon} \int_\Omega \rho^{(e)} u^{(e)} \, dx,
$$

and

$$
\frac{d}{dt} \left( \frac{1}{2} \int_\Omega \rho^{(e)} |u^{(e)}|^2 \, dx + \frac{1}{\varepsilon} \mathcal{F}(\rho^{(e)}) \right) + \frac{1}{\varepsilon} \int_\Omega \rho^{(e)} |u^{(e)}|^2 \, dx = 0.
$$

**Proof** The first two assertions are clear. For the third one, a direct computation gives

$$
\frac{1}{2} \frac{d}{dt} \int_\Omega \rho^{(e)} |u^{(e)}|^2 \, dx + \frac{1}{\varepsilon} \int_\Omega \rho^{(e)} |u^{(e)}|^2 \, dx = -\frac{1}{\varepsilon} \int_\Omega \rho^{(e)} u^{(e)} \cdot \nabla \frac{\delta \mathcal{F}(\rho^{(e)})}{\delta \rho^{(e)}} \, dx.
$$

We also find

$$
\frac{d}{dt} \mathcal{F}(\rho^{(e)}) = \int_\Omega \frac{\delta \mathcal{F}(\rho^{(e)})}{\delta \rho^{(e)}} \partial_t \rho^{(e)} \, dx = \int_\Omega \rho^{(e)} u^{(e)} \cdot \nabla \frac{\delta \mathcal{F}(\rho^{(e)})}{\delta \rho^{(e)}} \, dx.
$$

Combining those two estimates concludes the desired result. $\square$

The main purpose of this section is to prove the following proposition.

**Proposition 2.2** Let $T > 0$. Let $(\rho^{(e)}, u^{(e)})$ and $\rho$ be classical solutions on the time interval $[0, T]$ to the systems (1.1) and (1.4), respectively. Suppose that $u \in W^{1,\infty}((0, T) \times \Omega)$ and $\rho^{(e)} \in L^\infty(0, T; L^1(\Omega))$. Then, we have

$$
\frac{d}{dt} \left( \frac{1}{2} \int_\Omega \rho^{(e)} |u^{(e)}|^2 \, dx + \frac{1}{\varepsilon} \int_\Omega \mathcal{F}(\rho^{(e)}|\rho) \, dx \right) + \frac{1}{2\varepsilon} \int_\Omega \rho^{(e)} |u^{(e)}|^2 \, dx
\leq \frac{c\rho C}{\varepsilon} (\gamma - 1) \int_\Omega U(\rho^{(e)}|\rho) \, dx + \frac{C\varepsilon}{\rho} \int_\Omega (\rho^{(e)} - \rho) \Lambda^\sigma d (\rho^{(e)} - \rho) \, dx + C\varepsilon,
$$

(2.1)

where $C > 0$ is independent of $\varepsilon > 0$.

**Remark 2.3** The integral version of the modulated energy estimate can be also obtained by using the weak energy inequality, see (Lattanzio and Tzavaras 2017, Definition 3.1) for instance.

**Remark 2.4** In addition, if we assume $\partial_t u, \nabla u \in L^\infty((0, T) \times \Omega)$ instead of $u \in W^{1,\infty}((0, T) \times \Omega)$, then we have an additional term, the kinetic energy term, on the right hand side of (2.1). See Remark 2.12 for details.
2.1 Modulated Internal Energy

We first estimate the modulated internal energy:

\[
\int_{\Omega} U(\rho^{(e)}|\rho) \, dx = \int_{\Omega} U(\rho^{(e)}) - U(\rho) - U'(\rho)(\rho^{(e)} - \rho) \, dx \\
= \left\{ \begin{array}{ll}
\int_{\Omega} \rho^{(e)} \ln(\rho^{(e)}) - \rho \ln \rho + (\rho - \rho^{(e)})(1 + \ln \rho) \, dx & \text{if } \gamma = 1, \\
\frac{1}{\gamma - 1} \int_{\Omega} (\rho^{(e)})^\gamma - \rho^\gamma + \gamma(\rho - \rho^{(e)})\rho^{\gamma-1} \, dx & \text{if } \gamma > 1.
\end{array} \right.
\]

By Taylor’s theorem, we can easily have the following lemma.

**Lemma 2.5** (Lower bounds on the modulated internal energy) *Let \( \gamma \geq 1 \). For any \( \rho^{(e)}, \rho \in (0, \infty) \), we have*

\[
U(\rho^{(e)}|\rho) \geq \frac{\gamma}{2} \min\{ (\rho^{(e)})^{\gamma-2}, \rho^{\gamma-2} \} |\rho^{(e)} - \rho|^2.
\]

**Lemma 2.6** (Temporal derivative of the modulated internal energy) *Let \( T > 0 \). Let \((\rho^{(e)}, u^{(e)})\) and \(\rho\) be classical solutions to the systems (1.1) and (1.4) on the time interval \([0, T]\), respectively. Then, we have*

\[
\frac{d}{dt} \int_{\Omega} U(\rho^{(e)}|\rho) \, dx = \int_{\Omega} \rho^{(e)}(u^{(e)} - u) \cdot \nabla(U'\rho^{(e)} - U'\rho) \, dx \\
- (\gamma - 1) \int_{\Omega} U(\rho^{(e)}|\rho) \nabla \cdot u \, dx. \tag{2.2}
\]

**Proof** We consider two cases: \( \gamma > 1 \) and \( \gamma = 1 \). For \( \gamma > 1 \), we observe that

\[
U(\rho) = \frac{1}{\gamma - 1} \rho^\gamma, \quad \rho U'(\rho) = \gamma U(\rho), \quad \text{and} \quad \rho U''(\rho) = (\gamma - 1)U'(\rho).
\]

Then, we estimate

\[
\frac{d}{dt} \int_{\Omega} U(\rho^{(e)}|\rho) \, dx = \int_{\Omega} (U'(\rho^{(e)}) - U'(\rho)) \partial_t \rho^{(e)} - U''(\rho)(\partial_t \rho)(\rho^{(e)} - \rho) \, dx \\
= \int_{\Omega} \nabla(U'(\rho^{(e)}) - U'(\rho)) \cdot \rho^{(e)} u^{(e)} \, dx \\
+ \int_{\Omega} U''(\rho)(\nabla \cdot (\rho u))(\rho^{(e)} - \rho) \, dx \\
= \int_{\Omega} \rho^{(e)}(u^{(e)} - u) \cdot \nabla(U'(\rho^{(e)}) - U'(\rho)) \, dx \\
+ \int_{\Omega} \rho^{(e)} u \cdot \nabla(U'(\rho^{(e)}) - U'(\rho)) \, dx \\
+ \int_{\Omega} U''(\rho)(\nabla \cdot (\rho u))(\rho^{(e)} - \rho) \, dx
\]
\[ =: I + I_1 + I_2. \quad (2.3) \]

Here,
\[
I_1 = - \int_{\Omega} (\nabla \rho^{(e)} \cdot u + \rho^{(e)} \nabla \cdot u)(U'(\rho^{(e)}) - U'(\rho)) \, dx
\]
\[
= - \int_{\Omega} \left( U'(\rho^{(e)})\nabla \rho^{(e)} - U'(\rho)\nabla \rho + U'(\rho)(\nabla \rho - \nabla \rho^{(e)}) \right) \cdot u \, dx
\]
\[
= - \int_{\Omega} \left( \nabla (U(\rho^{(e)}) - U(\rho)) + U'(\rho)\nabla (\rho - \rho^{(e)}) \right) \cdot u \, dx
\]
\[
= - \int_{\Omega} \left( \gamma (U(\rho^{(e)}) - U(\rho)) + U'(\rho)(\rho - \rho^{(e)}) \right) \nabla \cdot u \, dx
\]

and
\[
I_2 = \int_{\Omega} U''(\rho)(\nabla \rho \cdot u + \rho \nabla \cdot u)(\rho^{(e)} - \rho) \, dx
\]
\[
= - \int_{\Omega} (\nabla U'(\rho))(\rho - \rho^{(e)}) \cdot u \, dx - (\gamma - 1) \int_{\Omega} U'(\rho)(\rho - \rho^{(e)}) \nabla \cdot u \, dx.
\]

This gives
\[
I_1 + I_2 = - \int_{\Omega} \nabla (U(\rho^{(e)}|\rho)) \cdot u \, dx - \gamma \int_{\Omega} U(\rho^{(e)}|\rho) \nabla \cdot u \, dx
\]
\[
= (1 - \gamma) \int_{\Omega} U(\rho^{(e)}|\rho) \nabla \cdot u \, dx.
\]

Combining this with (2.3) asserts (2.2) for \( \gamma > 1 \). In case \( \gamma = 1 \), we easily find
\[
I_1 = \int_{\Omega} u \cdot \rho^{(e)} \nabla (\ln \rho^{(e)} - \ln \rho) \, dx = \int_{\Omega} u \cdot (\nabla \rho^{(e)} - \rho^{(e)} \rho \nabla \rho) \, dx
\]

and
\[
I_2 = \int_{\Omega} \nabla \cdot (\rho u) \left( \frac{\rho^{(e)}}{\rho} - 1 \right) \, dx = - \int_{\Omega} \rho u \cdot \left( \frac{\nabla \rho^{(e)} \rho - \rho^{(e)} \nabla \rho}{\rho^2} \right) \, dx = -I_1.
\]

Thus, we also have the estimate (2.2) when \( \gamma = 1 \). \( \square \)

### 2.2 Modulated Interaction Energy Estimate

In this part, we discuss the temporal derivative of the modulated interaction energy. More specifically, we provide the following lemma.
Lemma 2.7 (Temporal derivative of the modulated interaction energy) Let $T > 0$. Let $(\rho^{(e)}, u^{(e)})$ and $\rho$ be classical solutions to the systems (1.1) and (1.4) on the time interval $[0, T]$, respectively. Then, we have

$$-\frac{c_K}{2} \frac{d}{dt} \int_{\Omega} (\rho^{(e)} - \rho) \Lambda^{a-d}(\rho^{(e)} - \rho) \, dx \leq -c_K \int_{\Omega} (\rho^{(e)} u^{(e)} - u) \cdot \nabla \Lambda^{a-d}(\rho^{(e)} - \rho) \, dx + C|c_K| \int_{\Omega} (\rho^{(e)} - \rho) \Lambda^{a-d}(\rho^{(e)} - \rho) \, dx$$

for all $t \in [0, T]$, where $C > 0$ depends on $\|\nabla u\|_{L^\infty}$ when $\Omega = \mathbb{R}^d$ and $\|u\|_{H^s}$ otherwise, but independent of $\varepsilon > 0$.

We present the details of the proof of the above lemma by dividing into two cases: $\Omega = \mathbb{T}^d$ or $\Omega = \mathbb{R}^d$.

2.2.1 Whole Space Domain Case

We first notice that the Riesz interaction can be rewritten as

$$\Lambda^{a-d} \rho = K \ast \rho,$$

where the kernel $K$ is given by

$$K(x) = \frac{c_{\alpha, d}}{|x|^{\alpha}}$$

for some constant $c_{\alpha, d} > 0$. We then extend it to $\mathbb{R}^d \times \mathbb{R}$ via

$$\int_{\mathbb{R}^d} K((x, \xi) - (y, 0)) \rho(y) \, dy =: (K \ast (\rho \otimes \delta_0))(x, \xi),$$

where we denote

$$K(x, \xi) := \frac{c_{\alpha, d}}{|(x, \xi)|^{\alpha}}$$

see Caffarelli and Silvestre (2007) for the detailed discussion on the extension problems for the fractional Laplacian. We also refer to Petrache and Serfaty (2017) for the periodic domain case. Then, we find that the extended interaction force satisfies

$$-\nabla_{(x, \xi)} \cdot (|\xi|^{\alpha} \nabla_{(x, \xi)} K \ast (\rho \otimes \delta_0)) = \rho (x) \otimes \delta_0(\xi) \quad \text{on} \quad \mathbb{R}^d \times \mathbb{R} \quad (2.5)$$

with $\xi := \alpha + 1 - d \in (-1, 1)$ in the sense of distributions.

In the following lemma, motivated from Duerinckx (2016), Petrache and Serfaty (2017) and Serfaty (2020), we show that the modulated interaction energy can be expressed in terms of the kernel $K$. 

\(\square\) Springer
Lemma 2.8  The modulated potential energy can be rewritten as

\[
\int_{\mathbb{R}^d} \left( (\rho^{(e)} - \rho) \Lambda^{\alpha-d} (\rho^{(e)} - \rho) \right) \, dx \\
= \int_{\mathbb{R}^d \times \mathbb{R}} |\xi|^\xi |\nabla_{(x,\xi)} K \star ((\rho^{(e)} - \rho) \otimes \delta_0) (x, \xi)|^2 \, dx \, d\xi.
\]

**Proof**  By (2.4) and (2.5), we find

\[
\int_{\mathbb{R}^d} \left( (\rho^{(e)} - \rho) \Lambda^{\alpha-d} (\rho^{(e)} - \rho) \right) \, dx \\
= \int_{\mathbb{R}^d \times \mathbb{R} \times \mathbb{R}} (\rho^{(e)} - \rho)(x) \otimes \delta_0(\xi) K((x, \xi) - (y, 0))(\rho^{(e)} - \rho)(y) \, dx \, dy \, d\xi \\
= \int_{\mathbb{R}^d \times \mathbb{R} \times \mathbb{R}} (\rho^{(e)} - \rho)(x) \otimes \delta_0(\xi) (K \star ((\rho^{(e)} - \rho) \otimes \delta_0))(x, \xi) \, dx \, d\xi \\
= - \int_{\mathbb{R}^d \times \mathbb{R} \times \mathbb{R}} \nabla_{(x,\xi)} \cdot \left( |\xi|^\xi \nabla_{(x,\xi)} K \star ((\rho^{(e)} - \rho) \otimes \delta_0)(x, \xi) \right) \\
(K \star ((\rho^{(e)} - \rho) \otimes \delta_0))(x, \xi) \, dx \, d\xi \\
= \int_{\mathbb{R}^d \times \mathbb{R} \times \mathbb{R}} |\xi|^\xi |\nabla_{(x,\xi)} K \star ((\rho^{(e)} - \rho) \otimes \delta_0)(x, \xi)|^2 \, dx \, d\xi.
\]

\(\Box\)

In the lemma below, we show the estimate of the temporal derivative of the modulated interaction energy in case \(\Omega = \mathbb{R}^d\).

Lemma 2.9  Let \(T > 0\). Let \((\rho^{(e)}, u^{(e)})\) and \(\rho\) be classical solutions to the systems (1.1) and (1.4) on the time interval \([0, T]\), respectively. Then, we have

\[
- \frac{c_K}{2} \frac{d}{dt} \int_{\mathbb{R}^d} \left( \rho^{(e)} - \rho \right) \Lambda^{\alpha-d} \left( \rho^{(e)} - \rho \right) \, dx \\
\leq -c_K \int_{\mathbb{R}^d} \rho^{(e)} (u^{(e)} - u) \cdot \nabla \Lambda^{\alpha-d} \left( \rho^{(e)} - \rho \right) \, dx \\
+ C |c_K| \int_{\mathbb{R}^d} \left( \rho^{(e)} - \rho \right) \Lambda^{\alpha-d} \left( \rho^{(e)} - \rho \right) \, dx
\]

for all \(t \in [0, T]\), where \(C > 0\) depends on \(\|\nabla u\|_{L^\infty}\), but independent of \(\varepsilon > 0\).

**Proof**  By Lemma 2.8, we estimate

\[
- \frac{c_K}{2} \frac{d}{dt} \int_{\mathbb{R}^d} \left( K \star (\rho^{(e)} - \rho) \right) (\rho^{(e)} - \rho) \, dx \\
= -c_K \int_{\mathbb{R}^d} (\rho^{(e)} u^{(e)} - \rho u) \cdot \nabla K \star (\rho^{(e)} - \rho) \, dx
\]
\[\begin{align*}
&= -c_K \int_{\mathbb{R}^d} \rho^{(e)}(u^{(e)} - u) \cdot \nabla K \ast (\rho^{(e)} - \rho) \, dx \\
&\quad + c_K \int_{\mathbb{R}^d} (\rho - \rho^{(e)}) u \cdot \nabla K \ast (\rho^{(e)} - \rho) \, dx \\
&=: J + J_1.
\end{align*}\]

For notational simplicity, in the rest of this proof, we set \( \mathcal{R}^{(e)}(x, \xi) := (K \ast (\rho^{(e)} - \rho) \otimes \delta_0)(x, \xi) \). Then, for \( J_1 \) we obtain

\[ J_1 = c_K \int_{\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}} (\rho - \rho^{(e)})(x) \otimes \delta_0(\xi)(u(x), 0) \cdot \nabla_{(x, \xi)} \mathcal{R}^{(e)}(x, \xi)) \, dx \, dy \, d\xi \]

\[ = c_K \int_{\mathbb{R}^d \times \mathbb{R}} (\rho - \rho^{(e)})(x) \otimes \delta_0(\xi)(u(x), 0) \cdot \nabla_{(x, \xi)} \mathcal{R}^{(e)}(x, \xi) \, dx \, d\xi \]

\[ = -c_K \int_{\mathbb{R}^d \times \mathbb{R}} \nabla_{(x, \xi)} \cdot (|\xi|^\xi \nabla_{(x, \xi)} \mathcal{R}^{(e)}(x, \xi))(u(x), 0) \cdot \nabla_{(x, \xi)} \mathcal{R}^{(e)}(x, \xi) \, dx \, d\xi \]

\[ = c_K \int_{\mathbb{R}^d \times \mathbb{R}} |\xi|^\xi \nabla_{(x, \xi)} \mathcal{R}^{(e)}(x, \xi) \otimes \nabla_{(x, \xi)} \mathcal{R}^{(e)}(x, \xi) : \nabla_{(x, \xi)} (u(x), 0) \, dx \, d\xi \]

\[ =: J_{11} + J_{12}, \]

where \( J_{12} \) can be easily controlled by

\[ |c_K| \| \nabla u \|_{L^\infty} \int_{\mathbb{R}^d \times \mathbb{R}} |\xi|^\xi |\nabla_{(x, \xi)} \mathcal{R}^{(e)}(x, \xi)|^2 \, dx \, d\xi. \]

For \( J_{11} \), by the integration by parts, we get

\[ J_{11} = -c_K \int_{\mathbb{R}^d \times \mathbb{R}} |\xi|^\xi \nabla_{(x, \xi)} |\nabla_{(x, \xi)} \mathcal{R}^{(e)}(x, \xi)|^2 \cdot (u(x), 0) \, dx \, d\xi \]

\[ = c_K \int_{\mathbb{R}^d \times \mathbb{R}} |\nabla_{(x, \xi)} \mathcal{R}^{(e)}(x, \xi)|^2 \nabla_{(x, \xi)} \cdot ((u(x), 0)|\xi|^\xi) \, dx \, d\xi \]

\[ \leq |c_K| \| \nabla u \|_{L^\infty} \int_{\mathbb{R}^d \times \mathbb{R}} |\xi|^\xi |\nabla_{(x, \xi)} \mathcal{R}^{(e)}(x, \xi)|^2 \, dx \, d\xi, \]

where we used

\[ \nabla_{(x, \xi)} \cdot ((u(x), 0)|\xi|^\xi) = |\xi|^\xi (\nabla \cdot u)(x). \]

Thus, we obtain

\[ J_1 \leq 2|c_K| \| \nabla u \|_{L^\infty} \int_{\mathbb{R}^d \times \mathbb{R}} |\xi|^\xi |\nabla_{(x, \xi)} K \ast ((\rho^{(e)} - \rho) \otimes \delta_0)(x, \xi)|^2 \, dx \, d\xi. \]
This together with Lemma 2.8 concludes the desired result. □

2.2.2 Periodic Domain Case

In this part, we take into account the periodic domain case. It is worth noticing that
the method based on the extension representation for the fractional Laplacian would
not be applicable to this case.

Lemma 2.10 Let $T > 0$. Let $(\rho^{(\varepsilon)}, u^{(\varepsilon)})$ and $\rho$ be classical solutions to the systems (1.1) and (1.4) on the time interval $[0, T]$, respectively. Then, we have

$$-rac{cK}{2} \frac{d}{dt} \int_{\mathbb{T}^d} (\rho^{(\varepsilon)} - \rho) \Lambda^{\alpha-d} (\rho^{(\varepsilon)} - \rho) \, dx$$

$$\leq -cK \int_{\mathbb{T}^d} \rho^{(\varepsilon)} (u^{(\varepsilon)} - u) \cdot \nabla \Lambda^{\alpha-d} (\rho^{(\varepsilon)} - \rho) \, dx$$

$$+ C |cK| \int_{\mathbb{T}^d} (\rho^{(\varepsilon)} - \rho) \Lambda^{\alpha-d} (\rho^{(\varepsilon)} - \rho) \, dx$$

for some $C > 0$ independent of $\varepsilon > 0$, which depends on $s$ ($s > d/2 + 1 + (d - \alpha)/2$) and $\|u\|_{H^s}$.

Remark 2.11 Compared to the whole space case discussed in Lemma 2.9, we need a
better regularity of solutions $u$.

Proof of Lemma 2.10 Proceeding as in the proof of Lemma 2.9, it suffices to obtain the bound

$$\int_{\mathbb{T}^d} (\rho - \rho^{(\varepsilon)}) u \cdot \nabla \Lambda^{\alpha-d} (\rho^{(\varepsilon)} - \rho) \, dx \leq C \int_{\mathbb{T}^d} (\rho^{(\varepsilon)} - \rho) \Lambda^{\alpha-d} (\rho^{(\varepsilon)} - \rho) \, dx.$$  

For the simplicity of notation, let us write $g = \rho^{(\varepsilon)} - \rho$ and $b = -(\alpha - d)/2$. Then, we compute

$$\int_{\mathbb{T}^d} g u \cdot \nabla \Lambda^{-2b} g \, dx = - \int_{\mathbb{T}^d} [\Lambda^{-b} \nabla \cdot (ug)]$$

$$- (u \cdot \nabla) \Lambda^{-b} g \Lambda^{-b} g \, dx - \int_{\mathbb{T}^d} (u \cdot \nabla) \Lambda^{-b} g \Lambda^{-b} g \, dx$$

$$\leq C \|\Lambda^{-b} \nabla \cdot (ug)\|_{L^2} - (u \cdot \nabla) \Lambda^{-b} g \|_{L^2} \|\Lambda^{-b} g\|_{L^2}$$

$$+ C \|\nabla u\|_{L^\infty} \|\Lambda^{-b} g\|_{L^2}^2.$$
Note that \(\int_{\mathbb{T}^d} g \, dx = 0\) implies \(\hat{g}(0) = 0\). Similarly \(\hat{u}(0) = 0\). Hence, in the above summation we may assume that \(\eta \neq 0\) and \(\xi - \eta \neq 0\). Moreover, when \(\xi = 0\),

\[
|\widehat{H}(0)| = \left| \sum_{\eta \in \mathbb{Z}^d} \eta |\eta|^{-b} \cdot \hat{u}(-\eta) \hat{g}(\eta) \right| \leq C \|\nabla u\|_{L^2} \|\Lambda^{-b} g\|_{L^2} \tag{2.6}
\]

by Hölder’s inequality. Now assuming \(\xi \neq 0\), we have \(|\xi|, |\eta| \gtrsim 1\) and estimate

\[
|\widehat{H}(\xi)| \leq C \sum_{\eta \in \mathbb{Z}^d} |\xi - \eta| (|\xi|^{-b} + |\eta|^{-b}) |\hat{u}(\xi - \eta)| |\hat{g}(\eta)| \\
\leq C \sum_{\eta \in \mathbb{Z}^d} |\xi - \eta| (|\xi - \eta|^{-b} + 1) |\hat{u}(\xi - \eta)| |\eta|^{-b} |\hat{g}(\eta)|.
\]

In the second inequality, we have used the simple inequality

\[
|\xi|^{-b} \leq C |\eta|^{-b} |\xi - \eta|^b
\]

for \(\xi, \eta \in \mathbb{Z}^d\) with \(\xi \neq 0, \eta \neq 0, \eta - \xi \neq 0\). Therefore, with Young’s convolution inequality,

\[
\|\widehat{H}\|_{\ell^2(\xi \neq 0)} \leq C \|\xi\|_{\ell^1} \|\xi|^{-b}\|_{\ell^2} \|\hat{g}(\xi)\|_{\ell^2} \leq C \|u\|_{H^s} \|\Lambda^{-b} g\|_{L^2} \tag{2.7}
\]

where \(s > d/2 + 1 + b\). Together with (2.6) and (2.7), we obtain

\[
\|H\|_{L^2} = \|\Lambda^{-b} \nabla \cdot (u g) - (u \cdot \nabla) \Lambda^{-b} g\|_{L^2} \leq C \|\Lambda^{-b} g\|_{L^2}
\]

with some \(C = C(\|u\|_{H^s}) > 0\). This completes the proof. \(\square\)

### 2.3 Proof of Proposition 2.2

In this subsection, we provide the details of the proof of Proposition 2.2. Let us first rewrite Eq. (1.4) as

\[
\begin{aligned}
\partial_t \rho + \nabla \cdot (\rho u) &= 0, \\
\partial_t (\rho u) + \nabla \cdot (\rho u \otimes u) &= -\frac{1}{\varepsilon} \rho u - \frac{1}{\varepsilon} \rho \nabla \frac{\delta \mathcal{F}(\rho)}{\delta \rho} + \rho e,
\end{aligned}
\]
where \( e \) is given by \( e = \partial_t u + (u \cdot \nabla) u \). Then, straightforward computations yield

\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} \rho^{(e)} |u^{(e)} - u|^2 \, dx + \frac{1}{\varepsilon} \int_{\Omega} \rho^{(e)} |u^{(e)} - u|^2 \, dx
= - \int_{\Omega} \rho^{(e)} (u^{(e)} - u) \otimes (u^{(e)} - u) : \nabla u \, dx
- \frac{1}{\varepsilon} \int_{\Omega} \rho^{(e)} (u^{(e)} - u) \cdot \nabla \left( \frac{\delta \mathcal{F}(\rho^{(e)})}{\delta \rho^{(e)}} - \frac{\delta \mathcal{F}(\rho)}{\delta \rho} \right) \, dx
- \int_{\Omega} \rho^{(e)} (u^{(e)} - u) \cdot e \, dx.
\]

Here, the first term on the right hand side can be easily bounded from above by

\[
C \| \nabla u \|_{L^\infty} \int_{\Omega} \rho^{(e)} |u^{(e)} - u|^2 \, dx.
\]

For the second term, we write

\[
- \int_{\Omega} \rho^{(e)} (u^{(e)} - u) \cdot \nabla \left( \frac{\delta \mathcal{F}(\rho^{(e)})}{\delta \rho^{(e)}} - \frac{\delta \mathcal{F}(\rho)}{\delta \rho} \right) \, dx
= -c_P \int_{\Omega} \rho^{(e)} (u^{(e)} - u) \cdot \nabla (\mathcal{U}'(\rho^{(e)}) - \mathcal{U}'(\rho)) \, dx
+ c_K \int_{\Omega} \rho^{(e)} (u^{(e)} - u) \cdot \nabla \Lambda^{\alpha-d} (\rho^{(e)} - \rho) \, dx
=: I + J.
\]

On the other hand, it follows from Lemmas 2.6 and 2.7 that

\[
I = -c_P \frac{d}{dt} \int_{\Omega} \mathcal{U}(\rho^{(e)}|\rho) \, dx + c_P (\gamma - 1) \int_{\Omega} \mathcal{U}(\rho^{(e)}|\rho) \nabla \cdot u \, dx
\leq -c_P \frac{d}{dt} \int_{\Omega} \mathcal{U}(\rho^{(e)}|\rho) \, dx + c_P (\gamma - 1) \| \nabla \cdot u \|_{L^\infty} \int_{\Omega} \mathcal{U}(\rho^{(e)}|\rho) \, dx
\]

and

\[
J \leq c_K \frac{d}{dt} \int_{\Omega} (\rho^{(e)} - \rho) \Lambda^{\alpha-d} (\rho^{(e)} - \rho) \, dx
+ C |c_K| \int_{\Omega} (\rho^{(e)} - \rho) \Lambda^{\alpha-d} (\rho^{(e)} - \rho) \, dx,
\]

where \( C > 0 \) is independent of \( \varepsilon > 0 \). Here, we used the fact that \( \mathcal{U}(\rho^{(e)}|\rho) \geq 0 \).

Finally, the third term can be estimated as

\[
\int_{\Omega} \rho^{(e)} (u^{(e)} - u) \cdot e \, dx \leq \| e \|_{L^\infty} \int_{\Omega} \rho^{(e)} |u^{(e)} - u| \, dx.
\]
where \( C > 0 \) depends only on \( \| e \|_{L^\infty} \) and \( \| \rho^{(e)} \|_{L^\infty(0, T; L^1)} \). Here, we used the assumption \( u \in W^{1, \infty}(0, T) \times \Omega \), which implies that \( e \in L^\infty((0, T) \times \Omega) \).

We now combine all of the above estimates to have

\[
\frac{d}{dt} \left( \frac{1}{2} \int_\Omega \rho^{(e)} |u^{(e)} - u|^2 \, dx + \frac{c_P}{\varepsilon} \int_\Omega \mathcal{U}(\rho^{(e)} |\rho|) \, dx - \frac{c_K}{2\varepsilon} \int_\Omega (\rho^{(e)} - \rho) \Lambda^{a-d} (\rho^{(e)} - \rho) \, dx \right) + \frac{1}{2\varepsilon} \int_\Omega \rho^{(e)} |u^{(e)} - u|^2 \, dx \\
\leq \frac{c_P C}{\varepsilon} (\gamma - 1) \int_\Omega \mathcal{U}(\rho^{(e)} |\rho|) \, dx + \frac{C |c_K|}{\varepsilon} \int_\Omega (\rho^{(e)} - \rho) \Lambda^{a-d} (\rho^{(e)} - \rho) \, dx + C \varepsilon,
\]

where \( C > 0 \) depends only on \( \| u \|_{W^{1, \infty}} \). This completes the proof.

**Remark 2.12** If we only assume \( \partial_t u, \nabla u \in L^\infty((0, T) \times \Omega) \), then we modify the estimate (2.8) as

\[
\frac{1}{4\varepsilon} \int_\Omega \rho^{(e)} |u^{(e)} - u|^2 \, dx + C \varepsilon \int_\Omega \rho^{(e)} |u^{(e)}|^2 \, dx + C \varepsilon
\]

for \( \varepsilon > 0 \) small enough, where \( C > 0 \) depends on \( \| \rho^{(e)} \|_{L^\infty(0, T; L^1)}, \| \partial_t u \|_{L^\infty}, \) and \( \| \nabla u \|_{L^\infty} \). This yields

\[
\frac{d}{dt} \left( \frac{1}{2} \int_\Omega \rho^{(e)} |u^{(e)} - u|^2 \, dx + \frac{c_P}{\varepsilon} \int_\Omega \mathcal{U}(\rho^{(e)} |\rho|) \, dx - \frac{c_K}{2\varepsilon} \int_\Omega (\rho^{(e)} - \rho) \Lambda^{a-d} (\rho^{(e)} - \rho) \, dx \right) + \frac{1}{2\varepsilon} \int_\Omega \rho^{(e)} |u^{(e)} - u|^2 \, dx \\
\leq \frac{c_P C}{\varepsilon} (\gamma - 1) \int_\Omega \mathcal{U}(\rho^{(e)} |\rho|) \, dx + \frac{C |c_K|}{\varepsilon} \int_\Omega (\rho^{(e)} - \rho) \Lambda^{a-d} (\rho^{(e)} - \rho) \, dx \\
+ C \varepsilon \int_\Omega \rho^{(e)} |u^{(e)}|^2 \, dx + C \varepsilon,
\]

where \( C > 0 \) depends only on \( \| \partial_t u \|_{L^\infty} \) and \( \| \nabla u \|_{L^\infty} \). Thus, in this case, the kinetic energy \( \int_\Omega \rho^{(e)} |u^{(e)}|^2 \, dx \) appears in Proposition 2.2. On the other hand, it can be controlled by the total energy estimate in Lemma 2.1 that

\[
\int_0^t \int_\Omega \rho^{(e)} |u^{(e)}|^2 \, dx \, dt \leq \varepsilon \left( \int_\Omega \rho^{(e)}_0 |u^{(e)}_0|^2 \, dx + \frac{1}{\varepsilon} \int_\Omega \rho^{(e)}_0 \Lambda^{a-d} \rho^{(e)}_0 \, dx \right).
\]
We then combine this with (2.9) to conclude
\[
\frac{d}{dt} \left( \frac{1}{2} \int_{\Omega} \rho^{(\varepsilon)} |u^{(\varepsilon)} - u|^2 \, dx + \frac{c_P}{\varepsilon} \int_{\Omega} U(\rho^{(\varepsilon)} | \rho) \, dx - \frac{c_K}{2\varepsilon} \int_{\Omega} (\rho^{(\varepsilon)} - \rho) \Lambda^{\alpha-d} (\rho^{(\varepsilon)} - \rho) \, dx \right) \\
+ \frac{1}{2\varepsilon} \int_{\Omega} \rho^{(\varepsilon)} |u^{(\varepsilon)} - u|^2 \, dx \\
\leq \frac{c_P C}{\varepsilon} (\gamma - 1) \int_{\Omega} U(\rho^{(\varepsilon)} | \rho) \, dx + \frac{C |c_K|}{\varepsilon} \int_{\Omega} (\rho^{(\varepsilon)} - \rho) \Lambda^{\alpha-d} (\rho^{(\varepsilon)} - \rho) \, dx \\
+ C \varepsilon^2 \left( \int_{\Omega} \rho_0^{(\varepsilon)} |u_0^{(\varepsilon)}|^2 \, dx + \frac{1}{\varepsilon} \int_{\Omega} \rho_0^{(\varepsilon)} \Lambda^{\alpha-d} \rho_0^{(\varepsilon)} \, dx \right) + C \varepsilon,
\]
where \( C > 0 \) is independent of \( \varepsilon > 0 \). Now, we integrate it over \([0, t]\) and use Lemma 2.1 to have
\[
\frac{1}{2} \int_{\Omega} \rho^{(\varepsilon)} |u^{(\varepsilon)} - u|^2 \, dx + \frac{1}{2\varepsilon} \int_{\Omega} \int_{0}^{t} (\rho^{(\varepsilon)} - \rho) \Lambda^{\alpha-d} (\rho^{(\varepsilon)} - \rho) \, dx \, d\tau \\
\leq \frac{1}{2} \int_{\Omega} \rho_0^{(\varepsilon)} |u_0^{(\varepsilon)} - u_0|^2 \, dx + \frac{1}{2\varepsilon} \int_{\Omega} \int_{0}^{t} (\rho_0^{(\varepsilon)} - \rho_0) \Lambda^{\alpha-d} (\rho_0^{(\varepsilon)} - \rho_0) \, dx \, d\tau \\
+ \frac{C}{\varepsilon} \int_{0}^{t} \int_{\Omega} (\rho^{(\varepsilon)} - \rho) \Lambda^{\alpha-d} (\rho^{(\varepsilon)} - \rho) \, dx \, d\tau + C \varepsilon.
\]
In particular, this implies

3 Proof of Theorem 1.1

3.1 Pressureless and Repulsive Case

In this part, we provide the details of the proof for Theorem 1.1 when \( c_P = 0 \) and \( c_K < 0 \). For simplicity, without loss of generality, we set \( c_K = -1 \). In this case, it follows from (2.2) that
\[
\frac{d}{dt} \left( \frac{1}{2} \int_{\Omega} \rho^{(\varepsilon)} |u^{(\varepsilon)} - u|^2 \, dx + \frac{1}{2\varepsilon} \int_{\Omega} (\rho^{(\varepsilon)} - \rho) \Lambda^{\alpha-d} (\rho^{(\varepsilon)} - \rho) \, dx \right) \\
+ \frac{1}{2\varepsilon} \int_{\Omega} \rho^{(\varepsilon)} |u^{(\varepsilon)} - u|^2 \, dx \\
\leq \frac{C}{\varepsilon} \int_{\Omega} (\rho^{(\varepsilon)} - \rho) \Lambda^{\alpha-d} (\rho^{(\varepsilon)} - \rho) \, dx + C \varepsilon,
\]
We then combine this with (3.1) to yield
\[
\int \Omega (\rho) |u - u_0|^2 dx + \frac{1}{\varepsilon} \int_0^T \int \Omega (\rho) \Lambda^{\alpha-d} (\rho - \rho_0) dx dt \\
\leq C \int \Omega (\rho_0) |u_0 - u_0|^2 dx + \frac{1}{\varepsilon} \int_0^T \int \Omega (\rho) \Lambda^{\alpha-d} (\rho - \rho_0) dx dt + C \varepsilon.
\]
where $C > 0$ is independent of $\varepsilon > 0$. On the other hand, due to the symmetry of the operator $\Lambda^{\frac{\alpha-d}{2}}$, we find
\[
\int \Omega (\rho) \Lambda^{\alpha-d} (\rho - \rho_0) dx = \int \Omega |\Lambda^{\frac{\alpha-d}{2}} (\rho - \rho_0)|^2 dx = \| (\rho - \rho_0)(\tau, \cdot) \|_{L^{\frac{\alpha-d}{2}}}^2.
\]
Then, for almost any $t \in (0, T)$

$$\frac{1}{2} \frac{d}{dt} d^2_2(\mu, \nu) = \iint_{\Omega \times \Omega} (x - y) \cdot (\xi(x) - \eta(y)) \pi(dx, dy)$$

where $\pi$ is the optimal coupling between $\mu$ and $\nu$.

**Lemma 3.2** Let $T > 0$. Let $(\rho^{(e)}, u^{(e)})$ and $\rho$ be classical solutions to the systems (1.1) and (1.4) on the time interval $[0, T]$, respectively. Then, we have

$$d^2_2(\rho^{(e)}(t), \rho(t)) \leq C \exp(C \|\nabla u\|_{L^\infty})$$

$$\left( d^2_2(\rho^{(e)}, \rho_0) + \int_0^t \int_\Omega \rho^{(e)}|u^{(e)} - u|^2 \, dx \, d\tau \right)^{1/2}$$

(3.3)

for $0 \leq t \leq T$, where $C > 0$ depends only on $T$.

**Proof** By Proposition 3.1, we find

$$\frac{1}{2} \frac{d}{dt} d^2_2(\rho^{(e)}, \rho) = \iint_{\Omega \times \Omega} (x - y) \cdot (u^{(e)}(x) - u(y)) \pi(dx, dy)$$

$$\leq d_2(\rho^{(e)}, \rho) \left( \iint_{\Omega \times \Omega} |u^{(e)}(x) - u(y)|^2 \pi(dx, dy) \right)^{1/2}.$$ 

(3.4)

On the other hand, we get

$$\iint_{\Omega \times \Omega} |u^{(e)}(x) - u(y)|^2 \pi(dx, dy)$$

$$\leq 2 \int_\Omega |u^{(e)}(x) - u(x)|^2 \rho^{(e)}(x) \, dx + 2\|\nabla u\|_{L^\infty}^2 d^2_2(\rho^{(e)}, \rho),$$

where we used the fact $\pi$ is the optimal coupling between $\rho^{(e)}$ and $\rho$. This together with (3.4) yields

$$\frac{d}{dt} d_2(\rho^{(e)}, \rho) \leq C \left( \int_\Omega |u^{(e)}(x) - u(x)|^2 \rho^{(e)}(x) \, dx \right)^{1/2} + C \|\nabla u\|_{L^\infty} d_2(\rho^{(e)}, \rho).$$

Applying the Grönwall’s lemma to the above asserts

$$d^2_2(\rho^{(e)}(t), \rho(t)) \leq C \exp(C \|\nabla u\|_{L^\infty}) \left( d^2_2(\rho^{(e)}, \rho_0) + \int_0^t \int_\Omega \rho^{(e)}|u^{(e)} - u|^2 \, dx \, d\tau \right)^{1/2},$$

where $C > 0$ depends only on $T$.

**Remark 3.3** Lemma 3.2 requires rather strong regularities of solutions to the systems (1.1) and (1.4). To be more specific, as stated in Proposition 3.1, the corresponding velocity fields $u$ and $u^{(e)}$ should be locally Lipschitz. However, this assumption can be relaxed by employing a probabilistic representation formula for continuity equations, see (José 2021, 2020; Choi 2021; Figalli and Kang 2019) for detailed discussion.
For the quantitative bound on the second term on the left hand side of (1.6), we obtain that for any $\phi \in (L^\infty \cap Lip)(\Omega)$,

$$
\int_{\Omega} \phi(\rho^{(e)} u^{(e)} - \rho u) \, dx \\
= \int_{\Omega} \phi(\rho^{(e)} - \rho) u \, dx + \int_{\Omega} \phi \rho^{(e)} (u^{(e)} - u) \, dx \\
\leq \|\phi u\|_{L^\infty \cap Lip} \, d_{BL}(\rho^{(e)}, \rho) + \|\phi\|_{L^\infty} \|\rho^{(e)}\|_{L^1}^{1/2} \left( \int_{\Omega} \rho^{(e)} |u^{(e)} - u|^2 \, dx \right)^{1/2} \\
\leq C \, d_2(\rho^{(e)}, \rho) + C \left( \int_{\Omega} \rho^{(e)} |u^{(e)} - u|^2 \, dx \right)^{1/2}
$$

due to $d_{BL}(\rho^{(e)}, \rho) \leq d_2(\rho^{(e)}, \rho)$. This together with Lemma 3.2 implies

$$
\begin{align*}
\int_{0}^{T} d_{BL}^2((\rho^{(e)} u^{(e)})(t), (\rho u)(t)) \, dt &\leq C \, d_2^2(\rho^{(e)}_0, \rho_0) \\
+ C \, \int_{0}^{T} \int_{\Omega} \rho^{(e)} |u^{(e)} - u|^2 \, dx \, dt,
\end{align*}
$$

and subsequently,

$$
\begin{align*}
\int_{0}^{T} d_{BL}^2((\rho^{(e)} u^{(e)})(t), (\rho u)(t)) \, dt &\leq C \, d_2^2(\rho^{(e)}_0, \rho_0) \\
+ C \, \int_{0}^{T} \int_{\Omega} \rho^{(e)} |u^{(e)} - u|^2 \, dx \, dt,
\end{align*}
$$

(3.5)

where $C > 0$ is independent of $\varepsilon > 0$.

We finally combine (3.3), (3.5), (3.2), and Remark 2.12 to assert

$$
\begin{align*}
\sup_{0 \leq t \leq T} \left( d_2^2(\rho^{(e)}(t), \rho(t)) + \|\rho^{(e)} - \rho\|(t, \cdot)\|_{H^{-d+\alpha/2}}^2 \right) \\
+ \int_{0}^{T} d_{BL}^2((\rho^{(e)} u^{(e)})(t), (\rho u)(t)) \, dt \\
\leq C \varepsilon \int_{\Omega} \rho^{(e)}_0 |u^{(e)}_0 - u_0|^2 \, dx + C \, d_2^2(\rho^{(e)}_0, \rho_0) \\
+ C \int_{\Omega} (\rho^{(e)}_0 - \rho_0) \Lambda^{-d}(\rho^{(e)}_0 - \rho_0) \, dx + C \varepsilon^2,
\end{align*}
$$

where $C > 0$ is independent of $\varepsilon > 0$. This completes the proof.
**Remark 3.4** If the initial free energy $\mathcal{F}(\rho_0^{(e)})$ is bounded uniformly in $\varepsilon$, then it follows from Lemma 2.1 that $\rho^{(e)} \in L^\infty(0, T; L^1(\Omega))$ uniformly in $\varepsilon$. Together with this uniform estimate, assuming $\rho^\gamma \in L^\infty(0, T; L^1(\Omega))$ enables us to estimate

$$
\|\rho - \rho^{(e)}\|_{L^\gamma} \leq \left( \int_{\Omega} \left( \frac{\gamma}{2} \min\{\rho^{\gamma-2}, (\rho^{(e)})^{\gamma-2}\} \right)^{\frac{2}{\gamma}} \right)^{\frac{\gamma}{2}} \leq C \left( \int_{\Omega} \frac{\gamma}{2} \min\{\rho^{\gamma-2}, (\rho^{(e)})^{\gamma-2}\}\|\rho - \rho^{(e)}\|_{L^2}^2 \right)^{\frac{\gamma}{2}},
$$

for $\gamma \in [1, 2]$, and thus

$$
\|\rho - \rho^{(e)}\|_{L^\gamma}^2 \leq C \int_{\Omega} \frac{\gamma}{2} \min\{\rho^{\gamma-2}, (\rho^{(e)})^{\gamma-2}\}|\rho - \rho^{(e)}|^2 \, dx \leq C \int_{\Omega} \mathcal{U}(\rho^{(e)}|\rho) \, dx
$$

due to Lemma 2.5, for some $C > 0$ independent of $\varepsilon > 0$. Furthermore, we find

$$
\int_{\Omega} |\rho^{(e)} u^{(e)} - \rho u| \, dx \leq \int_{\Omega} |\rho^{(e)} - \rho||u| \, dx + \int_{\Omega} \rho^{(e)}|u^{(e)} - u| \, dx \leq \|u\|_{L^\gamma} \|\rho^{(e)} - \rho\|_{L^\gamma} + \|\rho^{(e)}\|_{L^1}^{1/2} \left( \int_{\Omega} \rho^{(e)}|u^{(e)} - u|^2 \, dx \right)^{1/2},
$$

where $\gamma_*$ is the Hölder’s conjugate of $\gamma$, i.e., $\gamma_* = \gamma/(\gamma - 1)$.

Hence, we have

$$
\|(\rho^{(e)} u^{(e)} - \rho u)(t, \cdot)\|_{L^1}^2 \leq C \int_{\Omega} \mathcal{U}(\rho^{(e)}|\rho) \, dx + C \int_{\Omega} \rho^{(e)}|u^{(e)} - u|^2 \, dx,
$$

where $C > 0$ is independent of $\varepsilon > 0$, and this combined with Proposition 2.2 yields

$$
\sup_{0 \leq t \leq T} \|(\rho^{(e)} - \rho)(t, \cdot)\|_{L^\gamma}^2 + \int_0^T \|(\rho^{(e)} u^{(e)} - \rho u)(t, \cdot)\|_{L^1}^2 \, dt \leq C \varepsilon \int_{\Omega} \rho_0^{(e)}|u_0^{(e)} - u_0|^2 \, dx + C \int_{\Omega} \mathcal{U}(\rho_0^{(e)}|\rho_0) \, dx + C \varepsilon^2,
$$

for $\gamma \in [0, 2]$, where $C > 0$ is independent of $\varepsilon > 0$. 
3.2 Pressure and Attractive Case

We first recall Hardy–Littlewood–Sobolev inequality.

**Lemma 3.5** (Elliott 1983) For all $f \in L^p(\Omega)$, $g \in L^q(\Omega)$, $1 < p, q < \infty$, $d - 2 < \alpha < d$ and $\frac{1}{p} + \frac{1}{q} + \frac{\alpha}{d} = 2$, it holds

$$\left| \int_{\Omega} f \Lambda^{\alpha-d} g \, dx \right| \leq C \| f \|_{L^p} \| g \|_{L^q},$$

where $C = C(\alpha, d, p, q) > 0$.

Next, let us present a lower bound estimate on the modulated interaction energy. Since the proof can be found in (Lattanzio and Tzavaras 2017, Lemma 3.4, Remark 3.5), we omit it here.

**Lemma 3.6** Let $d \geq 2$. Suppose that $\rho$ and $\bar{\rho}$ satisfy

$$0 \leq \bar{\rho}(t, x) \quad \text{and} \quad 0 < \rho_{\text{min}} \leq \rho(t, x) \leq \rho_{\text{max}} < \infty \quad (t, x) \in [0, T] \times \mathbb{R}^d$$

for some $\rho_{\text{min}}, \rho_{\text{max}} \in \mathbb{R}$. Furthermore, we assume $\gamma > \frac{2d}{2d - \alpha}$. Then, if $\gamma \geq 2$, then we have

$$\mathcal{U}(\rho | \bar{\rho}) \geq C |\rho - \bar{\rho}|^2,$$

where $C > 0$ depends only on $\rho_{\text{min}}$ and $\rho_{\text{max}}$. On the other hand, if $2 > \gamma \geq \frac{2d}{2d - \alpha}$, then there exist positive constants $R_0 = R_0(\rho_{\text{min}}, \rho_{\text{max}})$, $C_1 = C_1(R_0, \rho_{\text{min}}, \rho_{\text{max}})$, and $C_2 = C_2(R_0, \rho_{\text{min}}, \rho_{\text{max}})$ such that

$$\mathcal{U}(\rho | \bar{\rho}) \geq \begin{cases} C_1 |\rho - \bar{\rho}|^2 & \text{for } 0 \leq \bar{\rho} \leq R_0, \\ C_2 |\rho - \bar{\rho}|^{\gamma} & \text{for } \bar{\rho} > R_0. \end{cases} \tag{3.6}$$

**Remark 3.7** More precisely, (3.6) holds for $\gamma > 1$ (see Lattanzio and Tzavaras 2017, Lemma 3.4); however, we modified the statement for our case.

We now provide the details on the proof of Theorem 1.1 in pressure and attractive case.

**Proof of Theorem 1.1 (ii)** By Lemma 3.5, we obtain

$$\left| \frac{cK}{2\varepsilon} \int_{\Omega} (\rho^{(e)} - \rho) \Lambda^{\alpha-d}(\rho^{(e)} - \rho) \, dx \right| \leq \frac{CcK}{\varepsilon} \| \rho^{(e)} - \rho \|_{L^\theta}^2,$$

where $\theta$ is given by

$$\theta = \frac{2d}{2d - \alpha} \in \left( \frac{2d}{d+2}, 2 \right).$$
We first easily observe from Lemma 3.6 that for \( \gamma \geq 2 \)

\[
\| \rho^{(e)} - \rho \|_{L^\theta}^2 \leq C \| \rho^{(e)} - \rho \|_{L^2}^2 \leq C \int_{\mathbb{T}^d} U(\rho^{(e)}|\rho) \, dx,
\]

where \( C > 0 \) is independent of \( \epsilon > 0 \). On the other hand, if \( d \geq 2 \) and \( \gamma \in [\theta, 2) \), then we estimate

\[
\| \rho^{(e)} - \rho \|_{L^\theta(T^d)}^2 \leq C \| \rho^{(e)} - \rho \|_{L^2(T^d \cap \{\rho \leq R_0\})}^2 + C \| \rho^{(e)} - \rho \|_{L^\theta(T^d \cap \{\rho > R_0\})}^2.
\]

Here, the first term on the right hand side can be bounded as

\[
\| \rho^{(e)} - \rho \|_{L^\theta(T^d \cap \{\rho \leq R_0\})}^2 \leq C \| \rho^{(e)} - \rho \|_{L^2(T^d \cap \{\rho \leq R_0\})}^2,
\]

due to the monotonicity of \( L^p(T^d) \) norm. For the second term, for \( \theta' = \frac{2}{3} \gamma \), we get

\[
\| \rho^{(e)} - \rho \|_{L^{\theta'}(T^d \cap \{\rho > R_0\})}^2 \leq C \| \rho^{(e)} - \rho \|_{L^{1\theta'}(T^d \cap \{\rho > R_0\})}^{2(1-\beta)} \| \rho^{(e)} - \rho \|_{L^\gamma(T^d \cap \{\rho > R_0\})}^{2\beta},
\]

for \( \beta > 0 \) satisfying \( \frac{1}{\theta'} = \frac{\beta}{\gamma} + 1 - \beta \). Here, we used the fact that \( \gamma \geq 1 + \alpha/d \) implies

\[
1 < \frac{2d}{2d - \alpha} = \theta \leq \theta' = \frac{2}{3 - \gamma} < \gamma
\]

for \( d \geq 2 \). Note that

\[
2\beta = 2 \frac{\gamma}{\gamma - 1} \frac{1 - \theta'}{\theta'} = 2 \frac{\gamma}{\gamma - 1} \frac{1 - \gamma}{2} = \gamma,
\]

and thus

\[
\| \rho^{(e)} - \rho \|_{L^{\theta'}(T^d \cap \{\rho > R_0\})}^2 \leq C \| \rho^{(e)} - \rho \|_{L^\gamma(T^d \cap \{\rho > R_0\})}^2,
\]

where \( C > 0 \) is independent of \( \epsilon > 0 \). Combining all of the above estimates, we have

\[
\| \rho^{(e)} - \rho \|_{L^\theta(T^d)}^2 \leq C \| \rho^{(e)} - \rho \|_{L^\theta(T^d \cap \{\rho \leq R_0\})}^2 + C \| \rho^{(e)} - \rho \|_{L^\gamma(T^d \cap \{\rho > R_0\})}^2.
\]

This together with Lemma 3.6 yields

\[
\| \rho^{(e)} - \rho \|_{L^\theta}^2 \leq C \int_{\mathbb{T}^d} U(\rho^{(e)}|\rho) \, dx.
\]

for \( \gamma \in [\theta, 2) \).
Hence, we have

\[
\left| \frac{c_K}{2\varepsilon} \int_{\mathbb{T}^d} (\rho^{(e)} - \rho) \Lambda^{\alpha-d} (\rho^{(e)} - \rho) \, dx \right| \leq \frac{C_0 c_K}{\varepsilon} \int_{\mathbb{T}^d} \mathcal{U}(\rho^{(e)} | \rho) \, dx,
\]

where \( C_0 > 0 \) depends on \( \rho_{\text{min}} \) and \( \rho_{\text{max}} \), but independent of \( \varepsilon > 0 \).

This estimate together with Proposition 2.2 yields

\[
\frac{1}{2} \int_{\mathbb{T}^d} \rho^{(e)} | u^{(e)} - u |^2 \, dx + \frac{1}{\varepsilon} (c_P - C_0 c_K) \int_{\mathbb{T}^d} \mathcal{U}(\rho^{(e)} | \rho) \, dx \\
+ \frac{1}{2\varepsilon} \int_0^t \int_{\mathbb{T}^d} \rho^{(e)} | u^{(e)} - u |^2 \, dx \, d\tau \\
\leq \frac{1}{2} \int_{\mathbb{T}^d} \rho^{(e)} | u^{(e)} - u |^2 \, dx + \frac{c_P}{\varepsilon} \int_{\mathbb{T}^d} \mathcal{U}(\rho^{(e)} | \rho) \, dx \\
- \frac{c_K}{2\varepsilon} \int_{\mathbb{T}^d} (\rho^{(e)} - \rho) \Lambda^{\alpha-d} (\rho^{(e)} - \rho) \, dx \\
+ \frac{1}{2\varepsilon} \int_0^t \int_{\mathbb{T}^d} \rho^{(e)} | u^{(e)} - u |^2 \, dx \, d\tau \\
\leq \frac{1}{2} \int_{\mathbb{T}^d} \rho_0^{(e)} | u_0^{(e)} - u_0 |^2 \, dx + \frac{c_P}{\varepsilon} \int_{\mathbb{T}^d} \mathcal{U}(\rho_0^{(e)} | \rho_0) \, dx \\
- \frac{c_K}{2\varepsilon} \int_{\mathbb{T}^d} (\rho_0^{(e)} - \rho_0) \Lambda^{\alpha-d} (\rho_0^{(e)} - \rho_0) \, dx \\
+ \frac{C}{\varepsilon} \int_0^t \int_{\mathbb{T}^d} \mathcal{U}(\rho^{(e)} | \rho) \, dx \, d\tau + C\varepsilon,
\]

where we chose \( c_K > 0 \) small enough so that \( c_P > C_0 c_K \). We then apply Grönwall’s lemma to have

\[
\frac{1}{2} \int_{\mathbb{T}^d} \rho^{(e)} | u^{(e)} - u |^2 \, dx + \frac{1}{\varepsilon} (c_P - C_0 c_K) \int_{\mathbb{T}^d} \mathcal{U}(\rho^{(e)} | \rho) \, dx \\
+ \frac{1}{2\varepsilon} \int_0^t \int_{\mathbb{T}^d} \rho^{(e)} | u^{(e)} - u |^2 \, dx \, d\tau \\
\leq \frac{1}{2} \int_{\mathbb{T}^d} \rho_0^{(e)} | u_0^{(e)} - u_0 |^2 \, dx + \frac{c_P}{\varepsilon} \int_{\mathbb{T}^d} \mathcal{U}(\rho_0^{(e)} | \rho_0) \, dx + C\varepsilon.
\]

On the other hand, we use Lemma 3.6 to obtain that for \( 2 > \gamma > \frac{2d}{2d-\alpha} \)

\[
\int_{\mathbb{T}^d} | \rho^{(e)} u^{(e)} - \rho u | \, dx \\
\leq \int_{\mathbb{T}^d \cap \{ \rho^{(e)} \leq R_0 \}} | \rho^{(e)} - \rho | | u | \, dx \\
+ \int_{\mathbb{T}^d \cap \{ \rho^{(e)} > R_0 \}} | \rho^{(e)} - \rho | | u | \, dx + \int_{\mathbb{T}^d} \rho^{(e)} | u^{(e)} - u | \, dx
\]
\[
\leq C \| \rho^{(e)} - \rho \|_{L^2(\mathbb{T}^d \cap \{ \rho^{(e)} \leq R_0 \})} + C \| \rho^{(e)} - \rho \|_{L^2(\mathbb{T}^d \cap \{ \rho^{(e)} > R_0 \})} \\
+ \| \rho^{(e)} \|_{L^1}^{1/2} \left( \int_{\mathbb{T}^d} \rho^{(e)} |u^{(e)} - u|^2 \, dx \right)^{1/2},
\]

where \( C > 0 \) depends on \( \| u \|_{L^2 \cap L^\gamma} \). For \( \gamma \geq 2 \), we simply find

\[
\int_{\mathbb{T}^d} |\rho^{(e)} u^{(e)} - \rho u| \, dx \leq \| u \|_{L^2} \| \rho^{(e)} - \rho \|_{L^2} + \left( \int_{\mathbb{T}^d} \rho^{(e)} |u^{(e)} - u|^2 \, dx \right)^{1/2}.
\]

In conclusion, we have

\[
\int_0^t \| (\rho^{(e)} u^{(e)} - \rho u)(\tau, \cdot) \|_{L^1}^2 \, d\tau \leq C \int_0^t \int_{\Omega} \mathcal{U}(\rho^{(e)} |\rho|) \, dx \, d\tau \\
+ C \int_0^t \int_{\Omega} \rho^{(e)} |u^{(e)} - u|^2 \, dx \, d\tau.
\]

We finally use this, (3.7), and Lemma 3.6 to complete the proof. \( \square \)

Acknowledgements The authors would like to thank the anonymous reviewers for the invaluable comments which significantly improved the quality and readability of this article. YPC has been supported by NRF grant (No. 2017R1C1B2012918) and Yonsei University Research Fund of 2019-22-0212 and 2020-22-0505. IJJ has been supported by the New Faculty Startup Fund from Seoul National University, the Science Fellowship of POSCO TJ Park Foundation, and the National Research Foundation of Korea grant (No. 2019R1F1A1058486).

References

Ambrosio, L., Gigli, N., Savaré, G.: Gradient Flows in Metric Spaces and in the Space of Probability Measures, 2nd edn. Lectures in Mathematics. ETH Zürich, Birkhäuser Verlag, Basel (2008)
Bouchut, F., Golse, F., Pulvirenti, M.: Kinetic Equations and Asymptotic Theory. Series in Applied Mathematics (Paris), vol. 4, Gauthier-Villars, Éditions Scientifiques et Médicales Elsevier, Paris, Edited and with a foreword by Benoît Perthame and Laurent Desvillettes (2000)
Brenier, Y.: Convergence of the Vlasov–Poisson system to the incompressible Euler equations. Commun. Partial Differ. Equ. 25(3–4), 737–754 (2000)
Caffarelli, L., Silvestre, L.: An extension problem related to the fractional Laplacian. Commun. Partial Differ. Equ. 32(7–9), 1245–1260 (2007)
Caffarelli, L., Soria, F., Vázquez, J.L.: Regularity of solutions of the fractional porous medium flow. JEMS 15(5), 1701–1746 (2013)
Carrillo, J.A., Choi, Y.P.: Quantitative error estimates for the large friction limit of Vlasov equation with nonlocal forces. Ann. Inst. H. Poincaré Anal. Non Linéaire 37(4), 925–954 (2020)
Carrillo, J.A., Choi, Y.P.: Mean-field limits: from particle descriptions to macroscopic equations. Arch. Ration. Mech. Anal. 241(3), 1529–1573 (2021)
Carrillo, J.A., Choi, Y.-P., Tse, O.: Convergence to equilibrium in Wasserstein distance for damped Euler equations with interaction forces. Commun. Math. Phys. 365(1), 329–361 (2019)
Carrillo, J.A., Peng, Y., Wróblewska-Kamińska, A.: Relative entropy method for the relaxation limit of hydrodynamic models. Netw. Heterog. Med. 15(3), 369–387 (2020)
Choi, Y.-P.: Large friction limit of pressureless Euler equations with nonlocal forces. J. Differ. Equ. 299, 196–228 (2021)
Choi, Y.-P., Jeong, I.-J.: On well-posedness and singularity formation for the Euler–Riesz system, preprint
Choi, Y.-P., Jeong, I.-J.: Classical solutions for fractional porous medium flow. Nonlinear Anal. 210 (2021), Paper No. 112393, 13
Choi, Y.-P., Jung, J.: The pressureless damped Euler–Riesz equations (preprint)
Coulombel, J.-F., Goudon, T.: The strong relaxation limit of the multidimensional isothermal Euler equations. Trans. Am. Math. Soc. 359(2), 637–648 (2007)
Dafermos, C.M.: The second law of thermodynamics and stability. Arch. Ration. Mech. Anal. 70(2), 167–179 (1979)
Duerinckx, M.: Mean-field limits for some Riesz interaction gradient flows. SIAM J. Math. Anal. 48(3), 2269–2300 (2016)
Figalli, A., Kang, M.-J.: A rigorous derivation from the kinetic Cucker–Smale model to the pressureless Euler system with nonlocal alignment. Anal. PDE 12(3), 843–866 (2019)
Huang, F., Pan, R., Wang, Z.: $L^1$ convergence to the Barenblatt solution for compressible Euler equations with damping. Arch. Ration. Mech. Anal. 200(2), 665–689 (2011)
Jordan, R., Kinderlehrer, D., Otto, F.: The variational formulation of the Fokker–Planck equation. SIAM J. Math. Anal. 29(1), 1–17 (1998)
Junca, S., Rascle, M.: Strong relaxation of the isothermal Euler system to the heat equation. Z. Angew. Math. Phys. 53(2), 239–264 (2002)
Lattanzio, C., Tzavaras, A.E.: From gas dynamics with large friction to gradient flows describing diffusion theories. Commun. Partial Differ. Equ. 42(2), 261–290 (2017)
Lieb, E.H.: Sharp constants in the Hardy–Littlewood–Sobolev and related inequalities. Ann. Math. (2) 118(2), 349–374 (1983)
Luo, T., Zeng, H.: Global existence of smooth solutions and convergence to Barenblatt solutions for the physical vacuum free boundary problem of compressible Euler equations with damping. Commun. Pure Appl. Math. 69(7), 1354–1396 (2016)
Marcati, P., Milani, A.: The one-dimensional Darcy’s law as the limit of a compressible Euler flow. J. Differ. Equ. 84(1), 129–147 (1990)
Otto, F.: The geometry of dissipative evolution equations: the porous medium equation. Commun. Partial Differ. Equ. 26(1–2), 101–174 (2001)
Petrache, M., Serfaty, S.: Next order asymptotics and renormalized energy for Riesz interactions. J. Inst. Math. Jussieu 16(3), 501–569 (2017)
Hung, N.Q., Matthew, R., Sylvia, S.: Mean-field limits of Riesz-type singular flows with possible multiplicative transport noise (preprint)
Saint-Raymond, L.: Hydrodynamic Limits of the Boltzmann Equation. Lecture Notes in Mathematics, vol. 1971. Springer, Berlin (2009)
Serfaty, S.: Mean field limit for Coulomb-type flows. Duke Math. J. 169(15), 2887–2935 (2020)
Villani, C.: Optimal transport, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 338. Springer, Berlin (2009) Old and new
Yau, H.-T.: Relative entropy and hydrodynamics of Ginzburg–Landau models. Lett. Math. Phys. 22(1), 63–80 (1991)

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.