Minimizing Convex Functions with Integral Minimizers

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Abstract

Given a separation oracle SO for a convex function $f$ that has an integral minimizer inside a box with radius $R$, we show how to efficiently find a minimizer of $f$ using at most $O(n(n + \log(R)))$ calls to SO. When the set of minimizers of $f$ has integral extreme points, our algorithm outputs an integral minimizer of $f$. This improves upon the previously best oracle complexity of $O(n^2(n + \log(R)))$ obtained by an elegant application of simultaneous diophantine approximation due to [Grötschel, Lovász and Schrijver, Prog. Comb. Opt. 1984, Springer 1988] over thirty years ago. We conjecture that our oracle complexity is tight up to constant factors.

Our result immediately implies a strongly polynomial algorithm for the Submodular Function Minimization problem that makes at most $O(n^3)$ calls to an evaluation oracle. This improves upon the previously best $O(n^3 \log^2(n))$ oracle complexity for strongly polynomial algorithms given in [Lee, Sidford and Wong, FOCS 2015] and [Dadush, Végh and Zambelli, SODA 2018], and an exponential time algorithm with oracle complexity $O(n^3 \log(n))$ given in the former work, answering two open problems posted therein.

Our result is achieved by an application of the LLL algorithm [Lenstra, Lenstra and Lovász, Math. Ann. 1982] for the shortest lattice vector problem. We show how an approximately shortest vector of certain lattice can be used to reduce the dimension of the problem, and how the oracle complexity of such a procedure is advantageous compared with the Grötschel-Lovász-Schrijver approach that uses simultaneous diophantine approximation. Our analysis of the oracle complexity is based on a potential function that captures simultaneously the size of the search set and the density of the lattice. To achieve the $O(n^2)$ term in the oracle complexity, technical ingredients from convex geometry are applied.

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1 Introduction

In this paper, we investigate the problem of minimizing a convex function $f$ on $\mathbb{R}^n$ accessed through a separation oracle $SO$. When queried with a point $x$, the oracle returns “YES” if $x$ minimizes $f$; otherwise, the oracle returns a hyperplane that separates $x$ from the minimizer of $f$. An algorithm is said to be strongly polynomial [GLS88] for such a problem if it makes $\text{poly}(n)$ calls to $SO$, uses $\text{poly}(n)$ arithmetic operations, and the size of numbers occurring during the algorithm is polynomially bounded by $n$ and the size of the output of the separation oracle.

Designing strongly polynomial algorithms for continuous optimization problems with certain underlying combinatorial structure is a well-studied but challenging task in general. To this date, despite tremendous effort, it remains a major open question to solve linear programming (LP) in strongly polynomial time. This problem is also widely known as Smale’s 9th question [Sma98]. Despite this barrier, such algorithms are known under additional combinatorial assumptions: linear systems with at most two non-zero entries per row/column in the constraint matrix [Meg83, AC91, CM94], LPs with bounded entries in the constraint matrix [Tar86, VY96, DHNV20], and LPs with 0-1 optimal solutions [Chu12, Chu15].

For minimizing a general convex function $f$, strongly polynomial algorithms are hopeless unless the function $f$ satisfies certain combinatorial properties. In this work, we study the setting where the minimizer of $f$ is an integral point inside a box with radius $R = 2^{\text{poly}(n)}$. The integrality assumption on the minimizer is a natural one, and is general enough to encapsulate well-known problems such as submodular function minimization, where the radius $R = 1$. Prior to our work, an elegant application of simultaneous diophantine approximation due to Grötschel, Lovász and Schrijver [GLS84, GLS88] gives a strongly polynomial algorithm that finds the minimizer of $f$ using $O(n^2(n + \log(R)))$ calls to the separation oracle. We refer interested readers to [GLS88, Chapter 6] for a detailed presentation of their approach (also see Section A). The purpose of the present paper is to design a strongly polynomial algorithm with an improved number of separation oracle calls.

A closely related problem, known as the Convex Integer Minimization problem, asks to minimize a convex function $f$ over the set of integer points. Dadush [Dad12, Section 7.5] gave an algorithm for this problem that takes $n^{O(n)}$ time and exponential space. In fact, the Convex Integer Minimization problem generalizes integer linear programming (ILP) and thus cannot be solved in sub-exponential time under standard complexity assumptions. The crucial difference of this problem from our setting is that the minimizer of $f$ might not be integral, so the integrality assumption on the minimizer of $f$ is, in some sense, necessary for obtaining efficient algorithms.

The number of separation oracle calls made by an algorithm for minimizing a convex function $f$, known as the oracle complexity, plays a central role in black-box models of convex optimization. For weakly polynomial algorithms, it’s a well-known fact that $\Theta(n \log(nR/\epsilon))$ oracle calls is optimal, with $\epsilon$ being the accuracy parameter. The first exponential time algorithm that achieves such a number of oracle calls is the well-known center of gravity method discovered independently by Levin [Lev65] and New-

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1It’s easy to show that strongly polynomial algorithm doesn’t exist if $\log(R)$ is super-polynomial (see Remark 1.4). In the statement of our main result in Theorem 1.2, we do not make such an assumption.

2The original approach by Grötschel, Lovász and Schrijver was given in the context of obtaining exact solutions to LP, but it is immediately applicable to our problem. Their approach was briefly described in [GLS84] with details given in [GLS88].

3Their approach originally used the ellipsoid method which is sub-optimal in terms of oracle complexity. The oracle complexity given here uses Vaidya’s cutting plane method [Vai89].
man [New65]. As for polynomial time algorithms, an oracle complexity of this order was first achieved over thirty years ago by the method of inscribed ellipsoids [KTE88, NN89]. In contrast, the optimal oracle complexity for strongly polynomial algorithms is largely unknown to this date. This motivates the present paper to place a focus on the oracle complexity aspect of our algorithms.

1.1 Our results

To formally state our result, we first define the notion of a separation oracle. For a convex function $f$, a separation oracle can be implemented using the sub-gradient of $f$.

**Definition 1.1 (Separation oracle).** Let $f$ be a convex function on $\mathbb{R}^n$ and $K^*$ be the set of minimizers of $f$. Then a separation oracle for $f$ is one that:

(a) when queried with a minimizer $x \in K^*$, it outputs “YES”;
(b) when queried with a point $x \not\in K^*$, it outputs a vector $c$ such that $\min_{y \in K^*} c^T y \geq c^T x$.

The main result of this paper is given in Theorem 1.2. Here, we deal with convex functions whose minimizers might not be unique. In this case, we assume that the set of minimizers is the convex hull of a set of integral points. Instead of finding an arbitrary minimizer of $f$, our algorithm is able to output an integral minimizer which is crucial for our application to submodular function minimization.

**Theorem 1.2 (Main result).** Given a separation oracle $SO$ for a convex function $f$ defined on $\mathbb{R}^n$. If the set of minimizers $K^*$ of $f$ is contained in a box of radius $R$ and satisfies

(\checkmark) all extreme points of $K^*$ are integral,

then there is an algorithm that finds an integral minimizer of $f$ using $O(n(n + \log(R)))$ calls to $SO$ and $\text{poly}(n, \log(R))$ arithmetic operations, with the numbers occurring in the algorithm having bit sizes $\text{poly}(n, \log(R))$. Moreover, the assumption (\checkmark) that all extreme points of $K^*$ are integral and the $O(n \log(R))$ term in the oracle complexity are necessary.

The seemingly strong assumption (\checkmark) in Theorem 1.2 is used to guarantee that our algorithm finds an integral minimizer of $f$. To find any minimizer of $f$, one only needs the much weaker assumption that $f$ has an integral minimizer. Corollary 1.3 below follows along the lines of the proof for Theorem 1.2.

**Corollary 1.3 (Non-integral extreme points).** Given a separation oracle $SO$ for a convex function $f$ defined on $\mathbb{R}^n$. If the set of minimizers $K^*$ of $f$ is contained in a box of radius $R$ and satisfies

(\checkmark\checkmark) $K^*$ contains an integral point $x^* \in \mathbb{Z}^n$,

then there is an algorithm that finds a minimizer of $f$ using $O(n(n+\log(R)))$ calls to $SO$ and $\text{poly}(n, \log(R))$ arithmetic operations, with the numbers occurring in the algorithm having bit sizes $\text{poly}(n, \log(R))$.

The following remark justifies the last statement of Theorem 1.2 that assumption (\checkmark) is necessary for obtaining an integral minimizer of $f$ and it takes at least $\Omega(n \log(R))$ oracle calls to find one. It also implies that one cannot hope to obtain an integral minimizer of $f$ under the weaker assumption (\checkmark\checkmark) in Corollary 1.3.

\[\text{The minimizer found in Corollary 1.3 is not guaranteed to be integral.}\]
Remark 1.4 (Assumption (∗) and lower bound). If assumption (∗) does not hold, we give an example where an exponential number of calls to SO are needed to find an integral minimizer of $f$. Consider the unit cube $K = [0, 1]^n$ and let $V(K) = \{0, 1\}^n$ be the set of vertices. For each $v \in V(K)$, define the simplex $\Delta(v) = \{x \in K : \|x - v\|_1 < 0.01\}$. Randomly pick a vertex $u \in V(K)$ and consider the convex function

$$f_u(x) = \begin{cases} 0 & x \in K \setminus (\bigcup_{v \in V(K) \setminus \{u\}} \Delta(v)) \\ \infty & \text{otherwise} \end{cases}.$$  

When queried with a point $x \in \Delta(v)$ for some $v \in V(K) \setminus \{u\}$, we let SO output a separating hyperplane $H$ such that $K \cap H \subseteq \Delta(v)$; when queried with $x \notin K$, we let SO output a hyperplane that separates $x$ from $K$. Notice that $u$ is the unique integral minimizer of $f_u$, and to find $u$, one cannot do better than randomly checking vertices in $V(K)$ which takes $2^{O(n)}$ queries to SO.

We next argue that $\Omega(n \log(R))$ calls to SO is necessary in Theorem 1.2. Consider $f$ having a unique integral minimizer which is a random integral point in $B_\infty(R) \cap \mathbb{Z}^n$, where $B_\infty(R)$ is $\ell_\infty$ ball with radius $R$. In this case, one cannot hope to do better than just bisecting the search space for each call to SO and this strategy takes $\Omega(n \log(R))$ calls to SO to reduce the size of the search space to a constant factor.

The diameter $R$ in Theorem 1.2 does not need to be given, and can be found by a standard doubling trick. When $R \leq 2^{O(n)}$ (i.e. each entry of an integral minimizer of $f$ has $O(n)$ bits), the algorithm in Theorem 1.2 is a strongly polynomial algorithm with $O(n^2)$ oracle complexity. Usually, there’s a gap that depends on the dimension between the performance of weakly and strongly polynomial algorithms. For the problem of solving LPs with small constraint matrix, the strongly polynomial algorithm of Tardos [Tar86] makes $O(n^2)$ calls to a weakly polynomial LP procedure, and the strongly polynomial interior-point method of [DHNV20] takes an extra factor of $O(n^2)$ in the number of iterations as compared to the standard weakly polynomial interior-point method. In comparison to the optimal oracle complexity of $\Theta(n \log(nR/\epsilon))$ for weakly polynomial algorithms, Theorem 1.2 implies a gap of at most $O(n)$ for optimizing convex functions with integral minimizers.

Finally, we remark that to prove Theorem 1.2, we may assume that $f$ has a unique integral minimizer without loss of generality. We make such an assumption in the rest of this paper.

Remark 1.5 (Unique minimizer). Without loss of generality, we may assume that $f$ has a unique integral minimizer $x^* \in \mathbb{Z}^n$ in Theorem 1.2. To justify this statement, suppose we are minimizing a convex function $f$ with more than one minimizers that satisfy assumption (∗). Our algorithm picks an integral vector $c \in \mathbb{Z}^n$ with entries that are independent and uniform at random in $\{-\text{poly}(n, R), \ldots, \text{poly}(n, R)\}$. Whenever SO is queried at a point $x \in K^*$ and certifies that $x$ is a minimizer of $f$, our algorithm restricts the search set inside the half-space $\{y : c^T y \geq c^T x\}$. In this way, our algorithm solves the optimization problem $\max_{x \in K^*} c^T x$, which has a unique integral solution with probability $1 - 1/\text{poly}(n, R)$ since all extreme points of $K$ are integral by assumption (∗). See, for example, Lemma 4 in [KS01] for more details.

One might also view the above argument from a hardness perspective. Suppose the adversary is given the set of minimizers $K^*$ that satisfies assumption (∗). If $K^*$ contains more than one point, the adversary can randomly pick an integral vector $c$ as in the previous paragraph and construct a new separation oracle $SO'_c$ that answers a query $x$ as follows: when $x \notin K^*$, $SO'_c$ gives the same answer as SO; if $x \in K^*$ but $x$ is not a solution to $\max_{y \in K^*} c^T y$, $SO'_c$ returns the separating hyperplane $\{y : c^T y \geq c^T x\}$; when $x \in K^*$ is an optimal solution to $\max_{y \in K^*} c^T y$, $SO'_c$ returns “YES”. This creates a new instance of the problem that is at least as hard as the original instance. Moreover, with probability $1 - 1/\text{poly}(n, R)$, the new instance has a unique integral minimizer by assumption (∗).
1.2 Application to Submodular Function Minimization

Submodular function minimization (SFM) has been recognized as an important problem in the field of combinatorial optimization. Classical examples of submodular functions include graph cut functions, set coverage function, and utility functions from economics. Since the seminal work by Edmonds in 1970 [Edm70], SFM has served as a popular tool in various fields such as theoretical computer science, operations research, game theory, and machine learning. For a more comprehensive account of the rich history of SFM, we refer interested readers to the excellent surveys [McC05, Iwa08].

The formulation of SFM we consider is the standard one: we are given a submodular function $f$ defined over subsets of an $n$-element ground set. The values of $f$ are integers, and are evaluated by querying an evaluation oracle that takes time $EO$. Since the breakthrough work by Grötschel, Lovász, Schrijver [GLS81, GLS88] that the ellipsoid method can be used to construct a strongly polynomial algorithm for SFM, there has been a vast literature on obtaining better strongly polynomial algorithms (see Table 1). These include the very first combinatorial strongly polynomial algorithms constructed by Iwata, Fleischer and Fujishige [IFF01] and Schrijver [Sch00]. Very recently, a major improvement was made by Lee, Sidford and Wong [LSW15] using an improved cutting plane method. Their algorithm achieves the state-of-the-art oracle complexity of $O(n^3 \log^2(n))$ for strongly polynomial algorithms. A simplified variant of this algorithm achieving the same oracle complexity was given in [DVZ18].

The authors of [LSW15] also noted that $O(n^3 \log(n))$ oracle calls are information theoretically sufficient for SFM ([LSW15, Theorem 71]), but were unable to give an efficient algorithm achieving such an oracle complexity. They asked as open problems ([LSW15, Section 16.1]):

(a) whether one could obtain a strongly polynomial algorithm achieving the $O(n^3 \log(n))$ oracle complexity;

(b) whether one could further (even information theoretically) remove the extraneous $\log(n)$ factor from the oracle complexity.

The significance of these questions stem from their belief that $\Theta(n^3)$ is the tight oracle complexity for strongly polynomial algorithms for SFM (see [LSW15, Section 16.1] for a more detailed discussion).

We answer both these open questions affirmatively by obtaining a strongly polynomial algorithm for SFM with $O(n^3)$ oracle complexity. This brings the oracle complexity for strongly polynomial algorithms down to the natural barrier of $O(n^3)$. The following Theorem 1.6 is obtained by directly applying Theorem 1.2 to the Lovász extension $\hat{f}$ of the function $f$, together with the well-known fact that a separation oracle for $\hat{f}$ can be implemented using $n$ calls to the evaluation oracle ([LSW15, Theorem 61]). We provide details on these definitions and the proof of Theorem 1.6 in Section C.

**Theorem 1.6 (Submodular function minimization).** Given an evaluation oracle $EO$ for a submodular function $f$ defined over subsets of an $n$-element ground set, there exists a strongly polynomial algorithm that minimizes $f$ using $O(n^3)$ calls to $EO$.

Our algorithm is conceptually simpler than the algorithm given in [LSW15, DVZ18]. Moreover, while most of the previous strongly polynomial time algorithms for SFM vastly exploit different combinatorial structures of submodularity, our result is achieved via a very general algorithm and uses the structural properties of submodular functions in a minimal way.
More generally, if the oracle complexity in Theorem 1.2 can be improved to \( O(n(n^\alpha + \log(R))) \) for some positive \( \alpha < 1 \), then this would imply a strongly polynomial algorithm for SFM with oracle complexity \( O(n^{\alpha+2}) = o(n^3) \). Such a result would be a fundamental breakthrough in the study of SFM.

**Corollary 1.7.** Let \( 0 < \alpha < 1 \) be some constant. Under the assumptions of Theorem 1.2, if there exists an algorithm that finds an integral minimizer of \( f \) using \( O(n(n^\alpha + \log(R))) \) calls to SO and \( \text{poly}(n, \log(R)) \) arithmetic operations, with the numbers occurring in the algorithm having bit sizes \( \text{poly}(n, \log(R)) \), then there is a strongly polynomial algorithm for SFM using \( O(n^{\alpha+2}) \) calls to an evaluation oracle.

| Authors                  | Year  | Oracle Complexity | Remarks           |
|--------------------------|-------|-------------------|-------------------|
| Grötschel, Lovász, Schrijver [GLS81, GLS88] | 1981,88 | \( O(n^5) \) [McC05] | first strongly    |
| Schrijver [Sch00]        | 2000  | \( O(n^5) \)     | first comb. strongly |
| Iwata, Fleischer, Fujishige [IFF01] | 2000  | \( O(n^4 \log(n)) \) | first comb. strongly |
| Fleischer, Iwata [FI03]  | 2000  | \( O(n^4) \)     |                   |
| Iwata [Iwa03]            | 2002  | \( O(n^6 \log(n)) \) |                   |
| Vygen [Vyg03]            | 2003  | \( O(n^4) \)     |                   |
| Orlin [Orl09]            | 2007  | \( O(n^4) \)     |                   |
| Iwata, Orlin [IO09]      | 2009  | \( O(n^5 \log(n)) \) |                   |
| Lee, Sidford, Wong [LSW15] | 2015  | \( O(n^2 \log^2(n)) \) | current best strongly |
| Lee, Sidford, Wong [LSW15] | 2015  | \( O(n^4 \log(n)) \) | exponential time |
| Dadush, Végh, Zambelli [DVZ18] | 2018  | \( O(n^2 \log^2(n)) \) | current best strongly |
| This paper               | 2020  | \( O(n^4) \)     |                   |

Table 1: Strongly polynomial algorithms for submodular function minimization. The oracle complexity measures the number of calls to the evaluation oracle EO. In the case where a paper is published in both conference and journal, the year we provide is the earliest one.

### 1.3 Discussion of Lower Bound

Remark 1.4 shows that the \( O(n \log(R)) \) term in the oracle complexity in Theorem 1.2 is necessary. It’s a natural question whether the \( O(n^2) \) term is also required. We conjecture that this is the case and provide a few reasons in the following to justify our belief.

**Conjecture 1.8** (Lower bound). *Given a separation oracle SO for a convex function \( f \) defined on \( \mathbb{R}^n \). If the set of minimizers \( K^* \) of \( f \) is a subset of \([0, 1]^n\) that satisfies

\[ (*) \text{ all extreme points of } K^* \text{ are integral,} \]

then any strongly polynomial time algorithm for finding an integral minimizer of \( f \) makes at least \( \Omega(n^2) \) calls to SO.*

As we have seen in Section 1.2, any improvement (in fact, even by a logarithmic factor) on the \( O(n^2) \) term in the oracle complexity in Theorem 1.2 will lead to a strongly polynomial algorithm with \( o(n^3) \) oracle complexity for SFM. Such a result, if possible, will be a major breakthrough in the study of strongly polynomial algorithms for SFM that contradicts a commonly conjectured lower bound. Another evidence
of Conjecture 1.8 arises from the gap between strongly and weakly polynomial algorithms discussed in Section 1.1. A gap of $\Omega(n)$ seems intrinsically natural. One further reason to believe in a lower bound of $\Omega(n^2)$ comes from the proof of Theorem 1.2 itself: it takes $O(n)$ oracle calls to shrink the volume of the search region by a factor of $2^{-O(n)}$ which is needed to reduce the dimension of the problem by one; as one need to reduce the problem by $(n-1)$ overall, it is expectable that $\Omega(n^2)$ oracle calls are necessary.

1.4 Our Techniques

In this subsection, we give an overview of our techniques for proving Theorem 1.2. As noted in Remark 1.5, we can assume wlog that $f$ has a unique integral minimizer $x^*$. For simplicity, we further assume in the subsequent discussions that $x^*$ lies on the set of vertices of the unit cube $\{0, 1\}^n$, which does not change the problem inherently.

On a high level, our algorithm maintains a convex search set $K$ that contains the integral minimizer $x^*$ of $f$, and iteratively shrinks $K$ using the cutting plane method; as the volume of $K$ becomes small enough, our algorithm finds a hyperplane $P$ that contains all the integral points in $K$ and recurse on the lower-dimensional search set $K \cap P$. The assumption that $x^*$ is integral guarantees that $x^* \in K \cap P$. Such an idea is natural and was previously used in [GLS84, GLS88] to handle rational polytopes that are not full-dimensional and in [LSW15] to argue that $O(n^3 \log(n))$ oracle calls is information theoretically sufficient for SFM. The main technical difficulties in efficiently implementing such an idea are two-fold:

(a) we need to efficiently find the hyperplane $P$ that contains $K \cap \mathbb{Z}^n$;
(b) we need to carefully control the amount $\text{vol}(K)$ is shrinked so that progress is not lost.

The second difficulty is key to achieving a small oracle complexity and deserves some further explanation. To see why shrinking $K$ arbitrarily might result in a loss of progress, it’s instructive to consider the following toy example: suppose an algorithm starts with the unit cube $K = [0, 1]^n$ and $x^*$ lies on the face $K_1 = \{x : x_1 = 0\}$; suppose the algorithm obtains, in its $i$th call to SO, the separating hyperplane $H_i = \{x : x_1 \leq 2^{-i}\}$. After $T$ calls to SO, the algorithm obtains the refined search set $K \cap H_T$ with volume $2^{-T}$. However, when the algorithm reduces the dimension and recurse on the face $K_1$, the $(n-1)$-dimensional volume of the search set again becomes 1, and the progress made by the algorithm in shrinking the volume of $K$ is entirely lost. In contrast, the correct algorithm can reduce the dimension after only one call to SO when it’s already clear that $x^* \in K_1$.

1.4.1 The Grötschel-Lovász-Schrijver Approach: Previous $O(n^3)$ Oracle Complexity

For the moment, let’s take $K$ to be an ellipsoid. Such an ellipsoid can be obtained by Vaidya’s volumetric center cutting plane method\(^5\) [Vai89] (see Theorem 2.12). One natural idea in finding the hyperplane comes from the following geometric intuition: when the ellipsoid $K$ is “flat” enough in one direction, then all its integral points lie on a hyperplane $P$. To find such a hyperplane $P$, Grötschel, Lovász and Schrijver [GLS84, GLS88] gave an elegant application of simultaneous diophantine approximation. We briefly explain the main ideas behind this application, and leave a more detailed discussion to Section A.\(^5\)

\(^5\)Perhaps a more natural candidate is the ellipsoid method developed in [YN76, Sho77, Kha80]. This method, however, shrinks the volume of $K$ by a factor of $O(n)$ slower than Vaidya’s method. In fact, the Grötschel-Lovász-Schrijver approach [GLS84] originally used the ellipsoid method which results in an oracle complexity of $O(n^4)$.
We also refer interested readers to [GLS88, Chapter 6] for a more comprehensive presentation of their approach and its implications to finding exact LP solutions.

For simplicity, we assume $K$ is centered at 0. Let $a$ be its shortest axis vector whose Euclidean length is denoted as $\mu_{\text{min}}$. Approximating the vector $a$ using the simultaneous diophantine approximation algorithm by Lenstra, Lenstra and Lovász [LLL82], one obtains an integral vector $v \in \mathbb{Z}^n$ and a positive integer $q \in \mathbb{Z}$ such that

$$\|qa - v\|_\infty < 1/3n \quad \text{and} \quad 0 < q < 2^{2n^2}.$$  

This implies that for any integral point $x \in K \cap \mathbb{Z}^n$,

$$|v^\top x| \leq |qa^\top x| + \frac{1}{3n} \cdot \|x\|_1 \leq q \cdot \mu_{\text{min}} + 1/3.$$

When $\mu_{\text{min}} < 2^{-3n^2}$, the integral inner product $v_i^\top x$ has to be 0 and therefore all integral points in $K$ lie on the hyperplane $P = \{x : v^\top x = 0\}$. An efficient algorithm immediately follows: we first run the cutting plane method until the shortest axis has length $\mu_{\text{min}} \approx 2^{-3n^2}$, then apply the above procedure to find the hyperplane $P$ on which we recurse.

To analyze the oracle complexity of this algorithm, one naturally uses $\text{vol}(K)$ as the potential function. An amortized analysis using such a volume potential previously appeared, for example, in [DVZ20] for finding maximum support solutions in the linear conic feasibility problem. Roughly speaking, each cutting plane step (corresponding to one oracle call) decreases $\text{vol}(K)$ by a constant factor; each dimension reduction step increases $\text{vol}(K)$ by roughly $1/\mu_{\text{min}} \approx 2^{3n^2}$. As there are $n - 1$ dimension reduction steps before the problem becomes trivial, the total number of oracle calls is thus $O(n^3)$.

One might wonder if the oracle complexity upper bound can be improved using a better analysis. However, there is some fundamental issue in getting such an improvement. In particular, the upper bound of $2^{\Theta(n^2)}$ on $q$ in simultaneous diophantine approximation corresponds to the $2^{\Theta(n)}$-approximation factor of the Shortest Vector Problem in lattices, first obtained by Lenstra, Lenstra and Lovász [LLL82] (see Theorem 2.11). Despite forty years of effort, this approximation factor was not essentially improved for polynomial time algorithms.

### 1.4.2 Lattices to the Rescue: Getting $O(n^2 \log(n))$ Oracle Complexity

To bypass the previous bottleneck, we give a novel application of the LLL algorithm [LLL82] above that does not rely on simultaneous diophantine approximation. We show how the approximately shortest vector of certain lattice could be used to find the hyperplane for dimension reduction, and how the oracle complexity of such a procedure improves by a factor of $O(n/\log(n))$ over the Grötschel-Lovász-Schrijver approach. Our analysis is based on a novel potential function that captures simultaneously the volume of the search set $K$ and the density of the lattice. We believe this result, while having an extraneous factor of $\log(n)$, is elegant and interesting in its own right. The details for this algorithm and its analysis are given in Section 4.

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6In fact, an approximation factor of $2^{\Theta(n)}$ would be a huge breakthrough in lattice algorithms, and might result in the breaking of cryptosystems that rely on the inapproximability of the Shortest Vector Problem.
**Finding the hyperplane.** Again we assume that $K = \{x : x^T Ax \leq 1\}$ is an ellipsoid centered at 0. We first show a different procedure for finding the hyperplane $P$ that contains all integral points in $K$. Let $x \in K \cap \mathbb{Z}^n$ be an arbitrary integral point. For any vector $v$, we have

$$|v^T x| \leq \|v\|_{A^{-1}} \cdot \|x\|_A \leq \|v\|_{A^{-1}}.$$ 

As long as $\|v\|_{A^{-1}} < 1/2$ and $v^T x$ is an integer, we can conclude that $v^T x = 0$ and this implies that all integral points in $K$ lie on the hyperplane $P = \{v : v^T x = 0\}$.

One might attempt to guarantee that $v^T x$ is integral by choosing $v$ to be an integer vector. However, this idea has a fundamental flaw: as the algorithm reduces the dimension by restricting on a subspace $W$, the set of integral points on $W$ might become much sparser. As such, one needs $\text{vol}(K)$ to be very small to guarantee that $\|v\|_{A^{-1}} < 1/2$ and this results in a very large number of oracle calls.

To avoid this issue, we take $v = \Pi_W(z) \neq 0$ as the projection of some integral point $z \in \mathbb{Z}^n$ on $W$, where $W$ is the subspace on which $K$ lies. Since $z - v \in W^\perp$, we have $v^T x = z^T x$ and this guarantees that $v^T x$ is integral. For the more general case where $K$ is not centered at 0, a simple rounding procedure computes the desired hyperplane. We postpone the details of this construction to Lemma 3.1.

How do we find a vector $v \in \Pi_W(\mathbb{Z}^n) \setminus \{0\}$ that satisfies $\|v\|_{A^{-1}} < 1/2$? This is where lattices come into play. In particular, since $\Lambda = \Pi_W(\mathbb{Z}^n)$ forms a lattice, we can use the LLL algorithm [LLL82] to find an approximately shortest non-zero lattice vector under the norm $\|\cdot\|_{A^{-1}}$. If the shortest non-zero vector has $A^{-1}$-norm at most $2^{-n}$, then the LLL algorithm finds a vector $v$ that satisfies $\|v\|_{A^{-1}} < 1/2$.

**The algorithm.** This new approach of finding the hyperplane immediately leads to the following algorithm: we run the cutting plane method for $O(n \log(n))$ oracle calls\(^7\) to decrease the volume of the ellipsoid $K = E(x_0, A)$; then we run the LLL algorithm to find a vector $v$ for reducing the dimension. If $\|v\|_{A^{-1}} \geq 1/2$, then we continue to run the cutting plane method; otherwise, we use the above procedure to find a hyperplane $P$ containing all integral points in $K$, update\(^8\) the ellipsoid $K$ to be $K \cap P$ and recurse.

**The analysis of $O(n^2 \log(n))$ oracle complexity.** To analyze such an algorithm, one might attempt to use $\text{vol}(K)$ as the potential function as in the Grötschel-Lovász-Schrijver approach. However, one quickly realizes that $\text{vol}(K \cap P)/\text{vol}(P)$ can be as large as $\|v\|_2^2/\|v\|_{A^{-1}}$. While it’s expectable that $\|v\|_{A^{-1}} \geq n^{-O(n)}$ as we are running the LLL algorithm frequently to check for a short lattice vector, one has no control over $\|v\|_2$ and it can be as large as $n^{O(n^2)}$ in general.

Key to our analysis is the novel potential function $\Phi = \text{vol}(K)/\rho(\Lambda)$ that measures simultaneously the volume of $K$ and the density $\rho(\Lambda) = 1/\det(\Lambda)$ of the lattice $\Lambda$. Intuitively, the denser the lattice, the smaller the $A^{-1}$-norm of the shortest vector. While $\text{vol}(K)$ increases by $\|v\|_2/\|v\|_{A^{-1}}$ after the dimension reduction, we show in Lemma 3.2 that the density of the lattice would also increase by a factor of $\|v\|_2$. The increase in the density of the lattice thus elegantly cancels out the increase in $\text{vol}(K)$, leading to an overall increase in the potential of at most $n^{O(n)}$. It follows that the total increase in the potential over all $n - 1$ dimension reduction steps is at most $n^{O(n^2)}$. Note that each cutting plane step still decreases the potential function by a constant factor since the lattice is unchanged. Therefore, the total number of oracle calls is at most $O(n^2 \log(n))$.

---

\(^7\)We need to run the cutting plane method for $O(n \log(n))$ steps because of the $O(n \log(n))$ factor in Theorem 2.12. This factor comes from the ellipsoidal approximation of a polytope and cannot be avoided in Vaidya’s algorithm [Vai89].

\(^8\)For simplicity, we are assuming here that the hyperplane $P$ passes through the center of $K$; see Section 4 for the update rule in the general case.
The argument above ignores a slight technical issue: while we can guarantee that \( \|v\|_{A^{-1}} \) doesn’t become smaller than \( n^{-O(n)} \) after cutting plane steps by checking the length of the shortest lattice vector using the LLL algorithm, it’s not immediately clear why \( \|v\|_{A^{-1}} \) cannot be too small after a sequence of dimension reduction steps. To this end, we note that the new ellipsoid obtained by the intersection of ellipsoid \( E(A) = \{ x : x^\top Ax \leq 1 \} \) and a hyperplane \( P \) essentially corresponds to taking the Schur complement of \( P \) of the matrix \( A^{-1} \) (see Section 2.2 for definitions). We then prove in Lemma 3.4 that the LLL-reduced basis under the \( A^{-1} \)-norm is inherently preserved under lattice projection and taking the Schur complement of the matrix \( A^{-1} \). As an immediate consequence, the length of the shortest non-zero lattice vector will not decrease after each dimension reduction step. This completes the argument for the \( O(n^2 \log(n)) \) oracle complexity. More details on the algorithm and its analysis can be found in Section 4.

### 1.4.3 Implicitly Maintaining Ellipsoid: Towards \( O(n^2) \) Oracle Complexity

The extraneous \( \log(n) \) factor in the oracle complexity of the previous algorithm comes from maintaining the ellipsoid: Vaidya’s cutting plane method actually maintains a polytope \( K \), and an ellipsoid \( E \) enclosing the polytope \( K \) only serves as an upper bound on \( \text{vol}(K) \). It’s well-known that the smallest enclosing ellipsoid of \( K \) has volume larger than \( \text{vol}(K) \) by a factor of \( n^{O(n)} \). This factor, over the \( n - 1 \) dimension reductions steps, becomes \( n^{O(n^2)} \) and leads to the extra log\( (n) \) factor in the oracle complexity.

To avoid this extra logarithmic factor, we maintain a polytope \( K \) formed by the separating hyperplanes directly, and use the approximate center of gravity method by Bertsimas and Vempala [BV02] (see Theorem 2.14) to shrink the volume of \( K \) by a constant factor with each oracle call. Such an approach lead to two new challenges:

(a) How do we find the hyperplane that contains all integral points in \( K \)?

(b) How do we upper bound the oracle complexity?

Motivated by the algorithm in Section 1.4.2, it’s natural to construct the ellipsoid \( E(\text{Cov}(K)^{-1}) = \{ x : x^\top \text{Cov}(K)^{-1} x \leq 1 \} \), where Cov\( (K) \) is the covariance matrix of the uniform distribution over \( K \). Of course, we cannot compute \( \text{Cov}(K) \) exactly, but a sufficiently good approximation can be obtained by sampling from \( K \) [BV02]. The following sandwiching condition is well-known when \( K \) is centered at 0:

\[
E(\text{Cov}(K)^{-1}) \subseteq K \subseteq (n + 1) \cdot E(\text{Cov}(K)^{-1}).
\]  

Thus whenever the vector \( \|v\|_{A^{-1}} < 1/2(n + 1) \), the approach in Section 1.4.2 can be used to construct a hyperplane \( P \) that contains all integral points in the ellipsoid \( (n + 1) \cdot E(\text{Cov}(K)^{-1}) \), and in particular, all integral points in \( K \).

One might attempt to take \( K \cap P \) as the polytope after the dimension reduction and recursively apply the approach from the previous paragraph. This idea, however, still has a gap from achieving \( O(n^2) \) oracle complexity. To analyze the oracle complexity, the natural candidate is the potential function \( \text{vol}(K)/\rho(\Lambda) \) from the previous analysis. Using standard results from convex geometry, one can show that after reducing the dimension, the potential increases by roughly \( 1/\|v\|_{\text{Cov}(K)} \). If we can prove that \( \|v\|_{\text{Cov}(K)} \geq 2^{-O(n)} \), this would imply an oracle complexity of \( O(n^2) \). Unfortunately, \( \text{Cov}(K \cap P) \) might be very different from \( \text{Cov}(K) \) and this might produce very short vectors in \( \text{Cov}(K) \)-norm after a sequence of consecutive dimension reduction steps.
1.4.4 Reusing the Covariance Matrix: Achieving $O(n^2)$ Oracle Complexity

To get around the aforementioned issue, we propose a novel approach to reuse the same covariance matrix for maintaining the ellipsoid throughout a sequence of dimension reduction steps. To be more precise, let the current polytope $K$ be obtained by the cutting plane method and suppose we find a hyperplane $P$ that contains all integral points in $K$. Instead of updating polytope $K$ to $K \cap P$ and ellipsoid $E(\text{Cov}(K)^{-1})$ to $E(\text{Cov}(K \cap P)^{-1})$, we update $E(\text{Cov}(K)^{-1})$ to $E(\text{Cov}(K)^{-1}) \cap P_0$, where hyperplane $P_0$ is the translation of $P$ that passes through the origin. The intention here is to exploit the fact that LLL-reduced basis is preserved under lattice projection and taking the Schur complement of the matrix defining the norm (see Lemma 3.4).

To make sure that the new ellipsoid $E(\text{Cov}(K)^{-1}) \cap P_0$ still approximates the new polytope, we update $K$ to be $2K \cap P$, where we again assume that $K$ is centered at 0. This guarantees that the volume of $K$ increases by at most $2^{O(n)}$ and the new ellipsoid and polytope satisfies the new sandwiching condition

$$E(\text{Cov}(K)^{-1}) \cap P_0 \subseteq 2K \cap P \subseteq (2n + 1) \cdot E(\text{Cov}(K)^{-1}) \cap P_0.$$ (2)

We note that the new sandwiching condition in (2) has a factor of 2 off on the RHS from the old sandwiching condition in (2) due to scaling up the polytope $K$. In a sequence of at most $n$ dimension reduction steps, this factor of 2 accumulates to a factor of at most $2^n$. Thus the ellipsoid $E$ and the polytope $K$ always satisfies $E \subseteq K \subseteq 2^n \cdot E$. In this case, to find a hyperplane that contains all integral points in $K$, we need the more restrictive condition that $\|v\|_{\text{Cov}(K)} < 2^{-2n}$. As the LLL algorithm already has an approximation ratio of $2^{O(n)}$, this extra exponential factor translates to only a constant factor in the oracle complexity. Combining everything above gives an algorithm that achieves $O(n^2)$ oracle complexity. We leave the details of this algorithm and its analysis to Section 5.

1.5 Further Related Works

Strongly polynomial algorithms for combinatorial optimization problems. There is an enormous body of literature on designing strongly polynomial algorithms for combinatorial optimization problems. We do not aim present a comprehensive overview here but only highlight some of the landmark results.

Among classical work, Grötschel, Lovász and Schrijver [GLS81] used the ellipsoid method to give strongly polynomial algorithms for many combinatorial optimization problems. More applications can be found in the excellent textbook [GLS88]. Tardos [Tar86] gave a strongly polynomial algorithm for combinatorial LPs where the constraint matrix has small bit size. This is an extension, for example, of the strongly polynomial result of the minimum cost flow problem [Tar85, GT89, Orl93]. A strengthening of Tardos’ result to handle real input was given by Vavasis and Ye [VY96] using a “layered-step” interior point method and a recent improvement was obtained by Dadush et al. [DHNV20]. The framework by Frank and Tardos [FT87] can be used to turn many weakly polynomial algorithms into strongly polynomial algorithms.

Recently, strongly polynomial algorithms were discovered for the Markov decision problem with a fixed discount rate [Ye05, Ye11], minimum-cost flow problems with separable convex objectives [Végh12], generalized flow maximization [Végh17, OV20] and computing market equilibriums for linear exchange markets [GV19]. We refer to [GV19] for more references on strongly polynomial algorithms for related market problems.
Cutting plane methods. Cutting plane methods optimize a convex function $f$ by maintaining a convex set $K$ that contains the minimizer of $f$, which gets refined iteratively using the separating hyperplanes returned by the separation oracle. Since the breakthrough result of [YN76, Sho77, Kha80] on the Ellipsoid method, various different cutting plane methods have been proposed [KTE88, NN89, Vai89, AV95, BV02, LSW15]. The current fastest cutting plane method is due to [JLSW20].

Cutting plane methods are very general optimization protocols that use minimal information of the convex function to be optimized. For more specific problems, structured optimization methods such as interior point methods are sometimes more desirable (e.g. for linear programming and empirical risk minimization [CLS19, LSZ19, JSWZ20]).

2 Preliminaries

2.1 Notations

For any positive integer $n \in \mathbb{Z}_{\geq 1}$, we use $[n]$ to denote the set $\{1, \cdots, n\}$. Given a real number $a \in \mathbb{R}$, the floor of $a$, denoted as $\lfloor a \rfloor$, is the largest integer that is at most $a$. Define the closest integer to $a$, denoted as $\lceil a \rceil$, to be $\lceil a \rceil := [a + 1/2]$.

For any $i \in [n]$, we denote $e_i$ the $i$th standard orthonormal basis vector of $\mathbb{R}^n$. $S^{n-1}$ will be used to denote the unit Euclidean sphere in $\mathbb{R}^n$. We use $B_p(R)$ to denote the $\ell_p$-ball of radius $R$ in $\mathbb{R}^n$ and $B_p = B_p(1)$ the unit $\ell_p$-ball. For any set of vectors $V \subseteq \mathbb{R}^n$, we use $\text{span}\{V\}$ to denote the linear span of vectors in $V$. Throughout, a subspace $W$ is a subspace of $\mathbb{R}^n$ with $0 \in W$; an affine subspace $W$ is a translation of a subspace of $\mathbb{R}^n$ (and thus might not pass through the origin). Given a subspace $W$, we denote $W^\perp$ the orthogonal complement of $W$ and $\Pi_W(\cdot)$ the orthogonal projection onto the subspace $W$. Given a PSD matrix $A \in \mathbb{R}^{n \times n}$ and a subspace $V \subseteq \mathbb{R}^n$, we say $A$ has full rank on $V$ if $\text{rank}(A) = \text{dim}(V)$ and the eigenvectors corresponding to non-zero eigenvalues of $A$ form an orthogonal basis of $V$.

Given a subspace $V \subseteq \mathbb{R}^n$ and a PSD matrix $A \in \mathbb{R}^{n \times n}$ that has full rank on $V$, the function $\langle \cdot, \cdot \rangle_A$ given by $\langle x, y \rangle_A = x^T A y$ defines an inner product on $V$. The inner product $\langle \cdot, \cdot \rangle_A$ induces a norm on $V$, i.e. $\|x\|_A = \sqrt{\langle x, x \rangle_A}$ for any $x \in V$, which we call the $A$-norm. In this work, the matrix $A$ in the previous definitions often comes from an ellipsoid $E(x_0, A) := \{x : (x - x_0)^T A (x - x_0) \leq 1\}$. Usually, $A$ needs to be full-rank for this expression of ellipsoid to be rigorous. For convenience, we abuse notations and use such an expression for the case where $A$ might not have full-rank. In this case, let $W_A$ be the subspace spanned by eigenvectors corresponding to non-zero eigenvalues of $A$. Then $E(x_0, A)$ is used to denote the ellipsoid given by $E(x_0, A) := \{x \in x_0 + W_A : (x - x_0)^T A (x - x_0) \leq 1\}$. In particular, let $k = \text{rank}(A)$, then $E(x_0, A)$ lies in a $k$-dimensional affine subspace. Denoting $A^{-1}$ the Moore-Penrose inverse of $A$, the $k$ eigenvectors corresponding to non-zero eigenvalues of $A^{-1}$ give the $k$ axes of ellipsoid $E(x_0, A)$. When the ellipsoid is centered at $0$, we use the short-hand notation $E(A)$ to denote $E(0, A)$.
2.2 Schur Complement

**Definition 2.1** (Schur complement). Given a symmetric matrix $M \in \mathbb{R}^{n \times n}$. Fix some integer $k \in [n]$, let $W = \text{span}\{e_1, \cdots, e_k\}$ be the subspace corresponding to the first $k$ coordinates. We write $M$ as

$$M = \begin{pmatrix} A & B^\top \\ B & D \end{pmatrix},$$

where $A \in \mathbb{R}^{k \times k}$ and $D \in \mathbb{R}^{(n-k) \times (n-k)}$ are symmetric matrices. If the block $A$ is invertible, then the Schur complement of the subspace $W$ of the matrix $M$ is defined as the $(n-k) \times (n-k)$ matrix $\text{SC}(M, W) := D - BA^{-1}B^\top$.

The following property of Schur complement is well-known.

**Fact 2.2** (Property of Schur complement). Let $W = \text{span}\{e_1, \cdots, e_k\}$. Given a PSD matrix $M \in \mathbb{R}^{n \times n}$ whose top-left $k \times k$ block submatrix is invertible and any vector $v \in W^\perp$. Let $v_{[k+1:n]} \in \mathbb{R}^{n-k}$ be the vector formed by the last $n-k$ coordinates of $v$. Then we have

$$\min_{u \in W} (v + u)^\top M(v + u) = v_{[k+1:n]}^\top \cdot \text{SC}(M, W) \cdot v_{[k+1:n]}.$$

Definition 2.1 and Fact 2.2 naturally generalizes to an arbitrary subspace $W \subseteq \mathbb{R}^n$. In this case, we first rotate the space so that $W$ corresponds to the first $k$ coordinates, where $k = \dim(W)$, and then express the matrix $M$ in this new coordinate system. The Schur complement of the subspace $W$ of the matrix $M$, denoted as $\text{SC}(M, W)$, then follows from Definition 2.1 in this new coordinate system.

2.3 Lattices

Given a set of linearly independent vectors $b_1, \cdots, b_k \in \mathbb{R}^n$, denote $\Lambda(b_1, \cdots, b_k) = \{\sum_{i=1}^{k} \lambda_i b_i : \lambda_i \in \mathbb{Z}\}$ the lattice generated by $b_1, \cdots, b_k$. Here, $k$ is called the rank of the lattice. A lattice is said to have full-rank if $k = n$. Any set of $k$ linearly independent vectors that generates the lattice $\Lambda = \Lambda(b_1, \cdots, b_k)$ under integer linear combinations is called a basis of $\Lambda$. In particular, the set $\{b_1, \cdots, b_k\}$ is a basis of $\Lambda$. Different basis of a full-rank lattice are related by unimodular matrices, which are integer matrices with determinant $\pm 1$.

Given a basis $B \in \mathbb{R}^{n \times k}$, the fundamental parallelepiped of $\Lambda = \Lambda(B)$ is the polytope $\mathcal{P}(B) := \{\sum_{i=1}^{k} \lambda_i b_i : \lambda_i \in [0, 1), \forall i \in [k]\}$. The determinant of the lattice, denoted as $\det(\Lambda)$, is defined to be the volume of the fundamental parallelepiped, which is independent of the basis.

**Definition 2.3** (Dual lattice). Given a lattice $\Lambda \subseteq \mathbb{R}^n$, the dual lattice $\Lambda^*$ is the set of all vectors $x \in \text{span}\{\Lambda\}$ such that $\langle x, y \rangle \in \mathbb{Z}$ for all $y \in \Lambda$.

The following standard fact gives a basis for $\Lambda^*$ from a basis of $\Lambda$.

**Fact 2.4** (Basis of dual lattice). Let $B$ be a basis of lattice $\Lambda$. Then $B(B^\top B)^{-1}$ is a basis of the dual lattice $\Lambda^*$.

---

9By definition, $\text{SC}(M, W)$ is a matrix $M'$ that operates in the subspace $W^\perp$. For convenience, we abuse the definition in certain occasions where we extend $M'$ to the entire $\mathbb{R}^n$ with $M'w = 0$ for any $w \in W$. 13
2.3.1 Minkowski’s First Theorem

Minkowski’s first theorem asserts the existence of a non-zero lattice point in a symmetric convex set with large enough volume.

**Theorem 2.5** (Minkowski’s first theorem, [Min53]). Let \( \Lambda \subset \mathbb{R}^n \) be a full-rank lattice and \( K \subset \mathbb{R}^n \) be a symmetric convex set with \( \text{vol}(K) > 2^n \text{det}(\Lambda) \). Then \( K \cap (\Lambda \setminus \{0\}) \neq \emptyset \).

An important consequence of Minkowski’s theorem is the following theorem.

**Theorem 2.6** (Consequence of Minkowski’s first theorem). Given a full-rank lattice \( \Lambda \subseteq \mathbb{R}^n \), one has \( \lambda_1(\Lambda) \leq \sqrt{n} \cdot \text{det}(\Lambda)^{1/n} \) where \( \lambda_1(\Lambda) \) is the length of the shortest non-zero vector in \( \Lambda \) under the Euclidean norm.

2.3.2 Hermite Normal Form

**Definition 2.7** (Hermite normal form). Given an integer matrix \( B \in \mathbb{Z}^{m \times n} \) with \( m \leq n \), we say that \( B \) is in Hermite normal form if

(a) \( B = [L, 0] \) where \( L \) is a lower triangular matrix,

(b) \( B_{i,j} \geq 0 \) for all \( i, j \in [n] \), and

(c) each diagonal entry \( B_{i,i} \) is the unique maximum entry for row \( i \).

The following fundamental result on Hermite normal form is well-known. We refer interested readers to Chapter 4-5 of the excellent book [Sch98] for more details on Hermite normal form.

**Fact 2.8** (Hermite normal form). Given any integer matrix \( B \in \mathbb{Z}^{m \times n} \) with full row rank, there exists a unimodular matrix \( U \in \mathbb{Z}^{n \times n} \) such that \( BU \) is in Hermite normal form. Each entry of the matrix \( L \) and the unimodular matrix \( U \) is upper bounded by the greatest sub-determinant of \( B \). Moreover, the Hermite normal form of every matrix is unique.

Hermite normal form of a matrix \( B \) can be computed efficiently.

**Theorem 2.9** (Computation of Hermite normal form, [SL96]). Given any integer matrix \( B \in \mathbb{Z}^{m \times n} \) with full row rank, there exists an algorithm that computes the unimodular matrix \( U \in \mathbb{Z}^{n \times n} \) such that \( BU \) is in Hermite normal form using \( \tilde{O}(m^{\omega} n \log(\|B\|_\infty)) \) arithmetic operations, where \( \omega \) is the exponent of matrix multiplication and \( \|B\|_\infty = \max_{i,j} |B_{i,j}| \).

2.3.3 Gram-Schmidt Orthogonalization

Given a set of linearly independent vectors \( b_1, \cdots, b_k \in \mathbb{R}^n \) and a PSD matrix \( A \in \mathbb{R}^{n \times n} \) that has full-rank on the subspace \( \text{span}\{b_1, \cdots, b_k\} \), the Gram-Schmidt orthogonalization procedure computes a basis \( \{b_1^*, \cdots, b_k^*\} \) of \( \text{span}\{b_1, \cdots, b_k\} \) that is orthogonal with respect to the inner product \( \langle \cdot, \cdot \rangle_A \), and the corresponding coefficients \( \mu_{i,j} \) such that \( b_j^* = b_j - \sum_{i=1}^{j-1} \mu_{i,j} b_i^* \). The procedure is shown in Algorithm 1.
2.3.4 Lenstra-Lenstra-Lovasz Algorithm

Given a lattice $\Lambda$ and a PSD matrix $A$ that has full rank on $\text{span}\{\Lambda\}$, the Lenstra-Lenstra-Lovasz (LLL) algorithm [LLL82] finds a good approximation to the shortest non-zero lattice vector under $A$-norm\(^{10}\), whose length is denoted as $\lambda_1(\Lambda, A)$. In particular, the LLL algorithm finds an LLL-reduced basis defined as follows.

**Definition 2.10** (LLL-reduced basis, [LLL82]). Let $B \in \mathbb{R}^{n \times k}$ be a lattice basis and $A \in \mathbb{R}^{n \times n}$ be a PSD matrix that has full-rank on the column space of $B$. Let $b_1^*, \ldots, b_k^*$ be the vectors and coefficients from the Gram-Schmidt orthogonalization with respect to $\langle \cdot, \cdot \rangle_A$. The basis $B$ is called LLL-reduced under $A$-norm if the following is satisfied:

(a) Coefficient reduced: $|\mu_{i,j}| \leq 1/2$ for all $1 \leq i < j \leq n$,

(b) Lovász condition: $\|b_i^*\|^2_A \leq 2 \|b_{i+1}^*\|^2_A$ for $i = 1, \ldots, n - 1$.

**Theorem 2.11** (LLL algorithm, [LLL82]). Given a basis $b_1, \ldots, b_k \in \mathbb{Z}^n$ for lattice $\Lambda$ and a PSD matrix $A \in \mathbb{Z}^{n \times n}$ that has full rank on $\text{span}\{\Lambda\}$. Let $D \in \mathbb{R}$ be such that $\|b_i\|^2_A \leq D$ for any $i \in [k]$. Then there exists an algorithm $\text{LLL}(\Lambda, A)$ that finds a basis $b'_1, \ldots, b'_k$ for $\Lambda$ that is LLL-reduced under $A$-norm using $O(n^4 \log(D))$ arithmetic operations, and the integers occuring in the algorithm have bit sizes at most $O(n \log(D))$. Moreover, the first basis vector $b'_1$ satisfies that

$$\|b'_1\|^2_A \leq 2^{n-1} \cdot \min_{i \in [k]} \left\| (b'_i)^* \right\|_A^2 \leq 2^{n-1} \cdot \lambda_1^2(\Lambda, A),$$

where $(b'_1)^*, \ldots, (b'_k)^*$ is the Gram-Schmidt orthogonalization of $b'_1, \ldots, b'_k$ with respect to $\langle \cdot, \cdot \rangle_A$.

2.4 Cutting Plane Methods

Cutting plane methods optimize a convex function $f$ by maintaining a convex set $K$ that contains the minimizer of $f$, which gets refined iteratively using the separating hyperplanes returned by the separation oracle. The following result of [JLSW20] is the current fastest cutting plane method, which is built upon Vaidya’s volumetric center cutting plane method [Vai89].

\(^{10}\)Equivalently, one could think of finding an approximately shortest vector under the Euclidean norm in the lattice $A^{1/2}\Lambda$. 

---

Algorithm 1

\begin{verbatim}
Algorithm 1

1: procedure GS($b_1, \ldots, b_k \in \mathbb{R}^n, A \in \mathbb{R}^{n \times n}$) \quad $\triangleright A$ is PSD and has full rank on $\text{span}\{b_1, \ldots, b_k\}$
2: \hspace{30pt} $b_1^* \leftarrow b_1$
3: \hspace{30pt} for $j = 2, \ldots, k$ do
4: \hspace{60pt} $b_j^* \leftarrow b_j - \sum_{i=1}^{j-1} \mu_{i,j} b_i^*$ with $\mu_{i,j} = \frac{\langle b_j, b_i^* \rangle_A}{\|b_i^*\|_A^2}$
5: \hspace{30pt} end for
6: \hspace{30pt} return $b_1^*, \ldots, b_k^*$
7: end procedure
\end{verbatim}
Theorem 2.12 (Faster implementation of Vaidya’s method, Theorem 4.1 of [JLSW20]). Given a separation oracle \( SO \) for a convex function \( f \) defined on \( \mathbb{R}^n \) with minimizer \( x^* \in \mathcal{B}(x_0, R) \) and a parameter \( 0 < \varepsilon < 1 \). There exists a cutting plane method \( \text{CuttingPlane}(SO, \varepsilon, \mathcal{B}(x_0, R)) \) that uses \( O(n \log(n/\varepsilon)) \) calls to \( SO \) and an extra \( O(n^3 \log(n/\varepsilon)) \) arithmetic operations to output an ellipsoid \( E \) with center \( x_E \) containing the optimal solution \( x^* \) such that

\[
\text{vol}(E) \leq \varepsilon^n \cdot \text{vol}(\mathcal{B}(x_0, R)) \quad \text{and} \quad \left( \frac{\varepsilon}{n} \right)^O(n) \cdot \mathcal{B}(0, R) \subseteq -x_E + E \subseteq \left( \frac{n}{\varepsilon} \right)^O(n) \cdot \mathcal{B}(0, R).
\]

Remark 2.13. In fact, the cutting plane method can start from any ellipsoid \( E_0 \supseteq K \) with center \( x_0 \). In this case, \( \text{CuttingPlane}(SO, \varepsilon, E_0) \) uses \( O(n \log(n/\varepsilon)) \) calls to \( SO \) and an extra \( O(n^3 \log(n/\varepsilon)) \) arithmetic operations to output an ellipsoid \( E \) with center \( x_E \) that satisfies

\[
\text{vol}(E) \leq \varepsilon^n \cdot \text{vol}(E_0) \quad \text{and} \quad \left( \frac{\varepsilon}{n} \right)^O(n) \cdot (-x_0 +E_0) \subseteq -x_E + E \subseteq \left( \frac{n}{\varepsilon} \right)^O(n) \cdot (-x_0 + E_0).
\]

To achieve our \( O(n(n + \log(R))) \) oracle complexity result, we need the following cutting plane method which is an efficient implementation of the center of gravity method.

Theorem 2.14 (Approximate center of gravity method, [BV02]). Given a separation oracle \( SO \) for a convex function \( f \) defined on \( \mathbb{R}^n \). Given a polytope \( K \subseteq \mathbb{R}^n \) with \( m \) constraints that contains the minimizer \( x^* \) of \( f \), a number of iterations \( T \), and an error parameter \( \varepsilon > 0 \), there exists a cutting plane method \( \text{RandomWalkCG}(SO, K, T, \varepsilon) \) that uses \( T \) calls to \( SO \) and an extra \( O((m+T)\text{poly}(n, 1/\varepsilon)) \) arithmetic operations to output a polytope \( K' \) with \( m+T \) constraints, an approximate centroid \( x' \) of \( K' \), and an approximate covariance matrix \( \Sigma_{K'} \) of \( K' \) such that the following hold with high probability:

\[ \begin{align*}
(a) & \quad x^* \in K' \quad \text{and} \quad K' \text{ is the intersection of } K \text{ with } T \text{ hyperplanes output by } SO, \\
(b) & \quad \text{vol}(K') \leq (2/3)^T \cdot \text{vol}(K), \\
(c) & \quad \|x' - \text{cg}(K')\|_{\text{Cov}(K')}^{-1} < \varepsilon, \\
(d) & \quad (1 - \varepsilon) \cdot \text{Cov}(K') \leq \Sigma_{K'} \leq (1 + \varepsilon) \cdot \text{Cov}(K').
\end{align*} \]

2.5 Convex Geometry

A function \( g : \mathbb{R}^n \to \mathbb{R}_+ \) is log-concave if its support \( \text{supp}(g) \) is convex and \( \log(g) \) is concave on \( \text{supp}(g) \). An integrable function \( g : \mathbb{R}^n \to \mathbb{R}_+ \) is a density function, if \( \int_{\mathbb{R}^n} g(x)dx = 1 \). The centroid of a density function \( g : \mathbb{R}^n \to \mathbb{R}_+ \) is defined as \( \text{cg}(g) = \int_{\mathbb{R}^n} g(x)xdx \); the covariance matrix of the density function \( g \) is defined as \( \text{Cov}(g) = \int_{\mathbb{R}^n} g(x)(x - \text{cg}(g))(x - \text{cg}(g))^T dx \). A density function \( g : \mathbb{R}^n \to \mathbb{R}_+ \) is isotropic, if its centroid is 0 and its covariance matrix is the identity matrix, i.e. \( \text{cg}(g) = 0 \) and \( \text{Cov}(g) = I \).

A typical example of a log-concave distribution is the uniform distribution over a convex body \( K \subseteq \mathbb{R}^n \). We shall use \( \mathbb{E}_{x \sim K} \cdot \cdot \cdot \) to denote the expectation where the random sample \( x \) is drawn from the uniform distribution over \( K \). Given a convex body \( K \) in \( \mathbb{R}^n \), its volume is denoted as \( |K| \). The centroid (resp. covariance matrix) of \( K \), denoted as \( \text{cg}(K) \) (resp. \( \text{Cov}(K) \)), is defined to be the centroid (resp. covariance matrix) of the uniform distribution over \( K \). A convex body \( K \) is said to be isotropic if the uniform density over it is isotropic. Any convex body can be put into its isotropic position via an affine transformation.
Definition 2.15 (Cross-sectional volume). Let $K$ be a convex body in $\mathbb{R}^n$ and $\nu \in \mathbb{S}^{n-1}$ be a unit vector. Define the cross-sectional volume function $g_{K,\nu} : \mathbb{R} \to \mathbb{R}_+$ to be the $(n-1)$-volume $g_{K,\nu}(t) := |K \cap (\nu^\perp + t\nu)|$.

See e.g. [BGVV14] for a proof the following well-known result.

Theorem 2.16 (Brunn’s principle). Let $K$ be a convex body in $\mathbb{R}^n$ and $\nu \in \mathbb{S}^{n-1}$ be a unit vector. Then, the cross-sectional volume function $g_{K,\nu}$ is log-concave on its support.

Theorem 2.17 (Property of log-concave density, Theorem 5.14 of [LV07]). Let $g : \mathbb{R} \to \mathbb{R}_+$ be an isotropic log-concave density function.

(a) $c \leq g(0) \leq C$, for some universal constant $c, C > 0$,

(b) For any $x \in \mathbb{R}$, we have $g(x) < C'$ for some universal constant $C' > 0$.

An immediate corollary of Theorem 2.16 and 2.17 is the following.

Corollary 2.18 (Almost largest cross-section). Let $K$ be a convex body in $\mathbb{R}^n$ with centroid at 0. Then for any unit vector $\nu \in \mathbb{S}^{n-1}$ and any $t \in \mathbb{R}$, we have $g_{K,\nu}(t) \leq C \cdot g_{K,\nu}(0)$, where $C > 0$ is some universal constant.

The following result on the volume of the intersection of ellipsoid with a hyperplane appeared in [LS90].

Theorem 2.19 (Volume of the intersection of ellipsoid with a hyperplane, [LS90]). Let $E = \{x \in \mathbb{R}^n : x^\top Ax \leq 1\}$ be a full-rank ellipsoid, and $H = \{x \in \mathbb{R}^n : \nu^\top x = b\}$ be a hyperplane where $\nu \neq 0$. Then the intersection $E \cap H$ is an ellipsoid and that

$$\frac{\text{vol}(E \cap H)}{\text{vol}(E)} \leq \frac{\|\nu\|_2}{\|\nu\|_{A^{-1}}} \cdot \frac{\Gamma(n/2 + 1)}{\Gamma((n + 1)/2) \cdot \sqrt{\pi}}.$$

with equality above if and only if $b = 0$.

More generally, the following theorem on the intersection of convex body with a hyperplane passing through its centroid was proved in [Hen80].

Theorem 2.20 (Cross-section through the centroid, [Hen80]). Let $K$ be a convex body in $\mathbb{R}^n$ with centroid at 0. Then there exist universal constants $c, C > 0$ such that for any unit vector $\nu \in \mathbb{S}^{n-1}$, we have

$$\frac{c}{\|\nu\|_{\text{Cov}(K)}} \leq \frac{g_{K,\nu}(0)}{|K|} \leq \frac{C}{\|\nu\|_{\text{Cov}(K)}}.$$

We also need the following result from [KLS95].

Theorem 2.21 (Ellipsoidal approximation of convex body, [KLS95]). Let $K$ be an isotropic convex body in $\mathbb{R}^n$. Then,

$$\sqrt{\frac{n + 1}{n}} \cdot B_2 \subseteq K \subseteq \sqrt{n(n + 1)} \cdot B_2,$$

where $B_2$ is the unit Euclidean ball in $\mathbb{R}^n$. 

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We have since

We use these convex geometry tools mainly for improving the oracle complexity of our algorithm to $K$. This implies that Lemma 3.1 (Dimension reduction that preserves integral points) In this section, we prove a few technical lemmas which are key to our result.

3 Technical Lemmas

In this section, we prove a few technical lemmas which are key to our result.

3.1 Dimension Reduction that Preserves Integral Points

**Lemma 3.1** (Dimension reduction that preserves integral points). *Given an affine subspace $W = x_0 + W_0$, where $W_0$ is a subspace of $\mathbb{R}^n$ and $x_0 \in \mathbb{R}^n$ is some fixed point, and an ellipsoid $E(x_0, A)$ that has full rank on $W$. Given a vector $v \in \Pi_{W_0}(\mathbb{Z}^n) \setminus \{0\}$ with $\|v\|_{A^{-1}} < 1/2$, then there exists a hyperplane $P \not\subset W$ such that $E \cap \mathbb{Z}^n \subseteq P \cap W$.***

*Proof.* We may assume that $E \cap \mathbb{Z}^n \neq \emptyset$ as otherwise there’s nothing to prove. Clearly we have $E \cap \mathbb{Z}^n \subseteq W$ since $E \subseteq W$. It therefore suffices to find a hyperplane $P$ such that $E \cap \mathbb{Z}^n \subseteq P$. Let $z \in \mathbb{Z}^n$ be such that $v = \Pi_{W_0}(z)$. Consider the hyperplane given by

$$P = \{x : v^T x = (v - z)^T x_0 + [z^T x_0]\}.$$

Since $v \in W_0 \setminus \{0\}$ and $W_0$ is a translation of $W$, we have $P \not\subset W$. For any integral vectors $x_1, x_2 \in E \cap \mathbb{Z}^n$, we have

$$|v^T(x_1 - x_2)| \leq \|v\|_{A^{-1}} \cdot \|x_1 - x_2\|_A$$

$$< \frac{1}{2} \cdot (\|x_1 - x_0\|_A + \|x_2 - x_0\|_A) \leq 1.$$
Since \( x_1, x_2 \in W \cap \mathbb{Z}^n \), we have \( x_1 - x_2 \in W_0 \cap \mathbb{Z}^n \). As \( v = \Pi_{W_0}(z) \) where \( z \in \mathbb{Z}^n \), we have \( v^\top(x_1 - x_2) \in \mathbb{Z} \).

It then follows that \( v^\top x_1 = v^\top x_2 \). Finally, we note that for any integral vector \( x_1 \in E \cap \mathbb{Z}^n \), we have

\[
|z^\top(x_1 - x_0)| = |v^\top(x_1 - x_0)| \leq \|v\|_{\Lambda^{-1}} \cdot \|x_1 - x_0\|_\Lambda < 1/2.
\]

Since \( z^\top x_1 \in \mathbb{Z} \), we have \( z^\top x_1 = [z^\top x_0] \). Therefore, we have

\[
v^\top x_1 = [z^\top x_0] + (v - z)^\top x_1
= [z^\top x_0] + (v - z)^\top x_0,
\]

where the last equality is because \( v - z \in W_0^\perp \) and \( x_1 - x_0 \in W_0 \). This finishes the proof of the lemma. \( \square \)

### 3.2 Lattice Projection

**Lemma 3.2** (Lattice projection). Let \( \Lambda \subseteq \mathbb{R}^n \) be a rank-\( k \) lattice, and \( v_1, \ldots, v_k \in \mathbb{R}^n \) be a basis of \( \Lambda \). Then the projection of \( \Lambda \) onto the hyperplane \( P = \{ x \in \mathbb{R}^n : v_i^\top x = 0 \} \) is a rank-(\( k - 1 \)) lattice \( \Lambda' \) with a basis given by \( \Pi_P(v_2), \ldots, \Pi_P(v_k) \), where \( \Pi_P(\cdot) \) denote the orthogonal projection onto the subspace \( P \). Moreover, we have

\[
\frac{\det(\Lambda')}{\det(\Lambda)} = \frac{1}{\|v_1\|_2}.
\]

**Proof.** We first prove that \( \Lambda' \) is a rank-(\( k - 1 \)) lattice with a basis given by \( \Pi_P(v_2), \ldots, \Pi_P(v_k) \). Note that the vectors \( \Pi_P(v_2), \ldots, \Pi_P(v_k) \) are all non-zero and linearly independent, as otherwise there exist linear dependencies for the lattice basis \( v_1, \ldots, v_k \). Thus the set \( \Lambda'' := \{ \sum_{i=2}^k \lambda_i \cdot \Pi_P(v_i) : \lambda_i \in \mathbb{Z} \} \) is a rank-(\( k - 1 \)) lattice. It’s not hard to verify that the set \( \Lambda' = \Lambda'' \).

To prove the second part of the lemma. We let \( v_1^*, \ldots, v_k^* \) be the Gram-Schmidt orthogonalization of the lattice basis \( v_1, \ldots, v_k \) in the given order. In particular, we have \( v_1^* = v_1 \) and that

\[
\det(\Lambda) = \prod_{i=1}^k \|v_i^*\|_2.
\]

Now consider the projections \( \Pi_P(v_2), \ldots, \Pi_P(v_k) \). For convenience, denote \( V_i = \text{span}\{v_1, \ldots, v_i\} \) for each \( i \in [k], V_i' = \text{span}\{\Pi_P(v_2), \ldots, \Pi_P(v_i)\} \) and \( V_i \setminus V_1 = V_i \cap V_1^\perp \) for each \( i \in \{2, \ldots, k\} \). We first observe that \( V_i' = V_i \setminus V_1 \) for each \( i \in \{2, \ldots, k\} \). Notice that

\[
v_2^* = \Pi_{V_2 \setminus V_1}(v_2) = \Pi_P(v_2),
\]

and for each \( i > 2 \), we have

\[
v_i^* = \Pi_{V_i \setminus V_{i-1}}(v_i) = \Pi_{V_i' \setminus V_{i-1}'}(v_i) = \Pi_{V_i' \setminus V_{i-1}'}(\Pi_P(v_i)),
\]

where the first equality is the definition of Gram-Schmidt orthogonalization, and the second and third equality follows because subspace \( V_i \supset V_{i-1} \supset V_1 \) and \( v_i - \Pi_P(v_i) \in V_1 \) which is orthogonal to \( V_i' \setminus V_{i-1}' \). It thus follows that \( v_2^*, \ldots, v_k^* \) is the Gram-Schmidt orthogonalization of the basis \( \Pi_P(v_2), \ldots, \Pi_P(v_k) \) of \( \Lambda' \). Therefore, we have

\[
\det(\Lambda') = \prod_{i=2}^k \|v_i^*\|_2.
\]

The lemma then follows immediately from (3) and (4). \( \square \)
3.3 Shortest Vector in $A$-Norm

**Lemma 3.3** (Shortest vector in $A$-norm). Let $\Lambda$ be a rank-$k$ lattice on a $k$-dimensional subspace $W \subseteq \mathbb{R}^n$ and $A \in \mathbb{R}^{n \times n}$ be a PSD matrix that has full rank on $W$. Then there exists a vector $v \in \Lambda$ such that

$$
\|v\|_A \leq \sqrt{k} \cdot \det(A^{1/2})^{1/k} \cdot \det(\Lambda)^{1/k},
$$

where $\det(A^{1/2})$ is the product of non-zero eigenvalues of $A^{1/2}$.

**Proof.** For notational convenience, we reparametrize the subspace $W$ by an orthonormal basis of it (geometrically, we rotate the space so that $W$ becomes the first $k$ dimensions). Abusing the notation, we use $A \in \mathbb{R}^{k \times k}$ to denote the matrix $A$ after the reparametrization. Let $B \in \mathbb{R}^{k \times k}$ be the basis of $\Lambda$ after reparametrization. It immediately follows that $\det(\Lambda) = \det(B)$.

Consider the rank-$k$ lattice $A^{1/2}B\mathbb{Z}^k$. By Minkowski’s first theorem (Theorem 2.6), we have

$$
\lambda_1(A^{1/2}B\mathbb{Z}^k) \leq \sqrt{k} \cdot \det(A^{1/2}B)^{1/k}.
$$

Let $A^{1/2}Bz$ for some $z \in \mathbb{Z}^k$ be the shortest vector of this lattice, we define $v = Bz \in \Lambda$. It follows that

$$
\|v\|_A = \sqrt{v^TAv} = \|A^{1/2}Bz\|_2 \leq \sqrt{k} \cdot \det(A^{1/2})^{1/k} \cdot \det(\Lambda)^{1/k}.
$$

This finishes the proof of the lemma. □

3.4 Dimension Reduction Preserves LLL-Reduced Basis

**Lemma 3.4** (Dimension reduction preserves LLL-reduced basis). Given a full-rank lattice $\Lambda \subseteq \mathbb{R}^n$ and a full-rank matrix $A \in \mathbb{R}^{n \times n}$. Let $b_1, \cdots, b_n$ be an LLL-reduced basis of $\Lambda$ under $A$-norm and subspace $V = \text{span}\{b_1, \cdots, b_k\}$. Then $\Pi_{V^\perp}(b_{k+1}), \cdots, \Pi_{V^\perp}(b_n)$ is an LLL-reduced basis of lattice $\Pi_{V^\perp}(\Lambda)$ under $A^\prime$-norm, where $A^\prime = \text{SC}(A, V)$ is the Schur complement of the subspace $V$ of the matrix $A$.

Moreover, let $b_1^*, \cdots, b_n^*$ be the Gram-Schmidt orthogonalization of $b_1, \cdots, b_n$ under the inner product $\langle \cdot, \cdot \rangle_A$ and $(b_{k+1}^*)^*, \cdots, (b_n^*)^*$ the Gram-Schmidt orthogonalization of $\Pi_{V^\perp}(b_{k+1}), \cdots, \Pi_{V^\perp}(b_n)$ under the inner product $\langle \cdot, \cdot \rangle_{A^\prime}$. Then for any $k + 1 \leq i \leq n$, we have

$$
\|b_i^*\|_{A^\prime} = \|(b_i^*)^*\|_{A^\prime}.
$$

In particular, this implies that

$$
\min_{i \in [k+1, n]} \|(b_i^*)^*\|_{A^\prime} \geq \min_{i \in [n]} \|b_i^*\|_A.
$$

**Remark 3.5.** We note that the RHS above is the lower bound on $\lambda_1(\Lambda, A)$ used in the performance guarantee of the LLL algorithm (see Theorem 2.11).

**Proof.** For simplicity, we denote $\Pi_{V^\perp}(b_{k+1}), \cdots, \Pi_{V^\perp}(b_n)$ as $b_{k+1}', \cdots, b_n'$. Let $V_i = \text{span}\{b_1, \cdots, b_i\}$ for each $i \in [n]$ and $V_i' = \text{span}\{b_{k+1}', \cdots, b_i'\}$ for each $i \in [k + 1 : n]$. In particular, we have $V_k = V$ and $V + V_i' = V_i$ for each $i \in [k + 1 : n]$. Let $\mu_{i,j} = \frac{(b_i, b_j)_A}{\|b_i^*\|_A}$ for all $1 \leq i < j \leq n$ be the Gram-Schmidt coefficient for $b_i^*, \cdots, b_n^*$, and $\mu_{i,i} = \frac{(b_i, (b_i^*)^*)_A}{\|b_i^*\|_A^2}$ for all $k + 1 \leq i < j \leq n$ be the Gram-Schmidt coefficient for $(b_{k+1}')^*, \cdots, (b_n')^*$. Recall the definition of an LLL-reduced basis: the basis $b_1, \cdots, b_n$ is LLL-reduced under $A$-norm if
• (Coefficient reduced): $|\mu_{ij}| \leq 1/2$ for any $1 \leq i < j \leq n$,

• (Lovász condition): $\|b_i^*\|_A^2 \leq 2 \|b_{i+1}^*\|_A^2$.

We first verify the Lovász condition for $b_{k+1}', \ldots, b_n'$ by proving the following claim.

**Claim 3.6** (Lovász condition). For any $i \in [k+1:n]$, we have $\|b_i^*\|_A = \|(b_i')^*\|_{A'}$ and $b_i^* - (b_i')^* \in V$. 

**Proof of Claim 3.6.** By the Gram-Schmidt normalization, $b_i^* \in b_i - V_{i-1} = b_i' - V_{i-1}' - V$ is the vector that minimizes $\|b_i^*\|_A$ over the affine subspace $b_i - V_{i-1}$, and $(b_i')^* \in b_i' - V_{i-1}'$ is the vector that minimizes $\|(b_i')^*\|_{A'} = \min_{u \in V} \|(b_i')^* - u\|_A$, over the affine subspace $b_i' - V_{i-1}'$. We note that the above two minimization problems are essentially equivalent:

$$\min_{b \in b_i - V_{i-1} - V} \|b\|_A = \min_{b' \in b_i' - V_{i-1}' - V} \min_{u \in V} \|b' - u\|_A.$$ 

This implies that $\|b_i^*\|_A = \|(b_i')^*\|_{A'}$ and that $b_i^* - (b_i')^* \in V$. This proves Claim 3.6. \hfill $\square$

Next we verify the coefficient reduced condition for $b_{k+1}', \ldots, b_n'$ by showing that $\mu_{ij} = \mu_{ij}'$.

**Claim 3.7** (Coefficient reduced). For any $k+1 \leq i < j \leq n$, we have $\mu_{ij} = \mu_{ij}'$.

**Proof of Claim 3.7.** Consider any $k+1 < j \leq n$. We have

$$b_j^* = b_j - \sum_{i=1}^{j-1} \mu_{i,j} b_i^* \in b_j' - \sum_{i=k+1}^{j-1} \mu_{i,j} (b_i')^* + V,$$

where the last step uses $b_i^* - (b_i')^* \in V$ from Claim 3.6 and the fact that $V = \text{span}\{b_1^*, \ldots, b_k^*\}$. Note that the coefficients $\mu_{i,j}$ satisfy that

$$\|b_j^*\|_A = \min_{\mu_{i,j}} \min_{v \in V} \left\| b_j' - \sum_{i=k+1}^{j-1} \mu_{i,j} (b_i')^* + v \right\|_A.$$ 

We also have

$$(b_j')^* = b_j' - \sum_{i=k+1}^{j-1} \mu_{i,j}' (b_i')^*,$$

and the coefficients $\mu_{i,j}'$ satisfy that

$$\|(b_j')^*\|_{A'} = \min_{\mu_{i,j}'} \min_{u \in V} \left\| b_j' - \sum_{i=k+1}^{j-1} \mu_{i,j}' (b_i')^* + u \right\|_A.$$ 

By Claim 3.6 and the uniqueness of the Gram-Schmidt coefficients, the set of coefficients $\mu_{ij}'$ is the same as the set of coefficients $\mu_{ij}$, for any $k+1 \leq i < j$. This proves Claim 3.7. \hfill $\square$

Combining Claim 3.6 and 3.7, we have $b_{k+1}', \ldots, b_n'$ is an LLL-reduced basis of the projected lattice $\Pi_{V^\perp}(A)$. The “moreover” part of the lemma is an immediate consequence of Claim 3.6. This completes the proof of the lemma. \hfill $\square$
4 The Basic Algorithm: Almost Quadratic Oracle Complexity

In this section, we present an efficient algorithm that achieves an oracle complexity of \( O(n(\log(n) + \log(R))) \). While having an extra \( \log(n) \) factor in the oracle complexity, this algorithm is conceptually simpler and its analysis contains some of the key ideas behind our main result in Theorem 1.2. We will present our main algorithm for Theorem 1.2 in Section 5.

**Theorem 4.1 (Basic Algorithm).** Given a separation oracle \( SO \) for a convex function \( f \) defined on \( \mathbb{R}^n \). If the set of minimizers \( K^* \subseteq B_2(R) \) of \( f \) satisfies

\[
(\ast) \text{ all extreme points of } K^* \text{ are integral,}
\]

then there is an algorithm that finds an integral minimizer of \( f \) using \( O(n(\log(n) + \log(R))) \) calls to \( SO \) and \( O(n^7(n + \log(R))) \) arithmetic operations, with the numbers occurring in the algorithm having bit sizes \( \text{poly}(n, \log(R)) \).

4.1 The Basic Algorithm

We may assume we know a radius \( R \) such that \( \log(R) \in [\log(\|x^*\|_2), 2\log(\|x^*\|_2)] \), which can obtained by the standard doubling trick. Our algorithm maintains an affine subspace \( W \), an ellipsoid \( E \subseteq W \) containing the integral minimizer \( x^* \) of \( f \), and a lattice \( \Lambda \) for dimension reduction. In the beginning, the affine subspace \( W = \mathbb{R}^n \), ellipsoid \( E = B_2(R) \) and lattice \( \Lambda = \mathbb{Z}^n \). In each iteration of the algorithm (i.e. each while loop), the algorithm uses the LLL algorithm [LLL82] to find a basis vector \( v \in \Lambda \setminus \{0\} \) with small \( A^{-1} \)-norm. If the vector \( v \) doesn’t satisfy \( \|v\|_{A^{-1}} < 1/2 \), then the algorithm runs the cutting plane method inside the affine subspace \( W \) as in Theorem 2.12 to obtain a new ellipsoid \( E' \) with smaller volume. The new ellipsoid \( E' \) is used to replace the old ellipsoid \( E \) and the iteration ends.

If, on the other hand, the vector \( v \in \Lambda \) satisfies that \( \|v\|_{A^{-1}} < 1/2 \), then the algorithm recurses on the lower-dimensional affine subspace \( W \cap P \), where hyperplane \( P = \{x : v^T x = (v - z)^T x_0 + [z^T x_0]\} \) for some integer vector \( z \in \mathbb{Z}^n \) such that \( v = \Pi_{W_0}(z) \) and \( W_0 = -x_0 + W \) is the translation of \( W \) that passes through the origin. In particular, one can find such a vector \( z \in \mathbb{Z}^n \) by solving the closest vector problem \( \min_{z \in \mathbb{Z}^n} \|z - v\|_{P_{W_0}} \), where \( P_{W_0} \) is the projection matrix onto the subspace \( W_0 \).

Now we specify more details of the recursion. Let hyperplane \( P' = \{x : v^T x = 0\} \) (resp. \( P_0 = \{x : v^T x = 0\} \)) be the translation of \( P \) that passes through the center of \( E \) (resp. the origin). Let ellipsoid \( E' = \{x : (x - x_0')^T A'(x - x_0') \leq 1\} \) be a translation of \( E \cap P \) s.t. \( E \cap P \subseteq E' \). The recursion is then applied on the affine subspace \( W \cap P \) with ellipsoid \( E' \) and lattice \( \Pi_{P_0}(\Lambda) \). We remark that as \( v \in \Lambda \setminus \{0\} \) is a basis vector, \( \Pi_{P_0}(\Lambda) \) is a lattice with rank reduced by 1.

When the dimension of the affine subspace \( W \) becomes 1, we find an integral minimizer of \( f \) on the segment \( E \) directly using binary search. A formal description of the algorithm can be found in Algorithm 2.

4.2 Analysis of the Basic Algorithm

Theorem 4.1 immediately follows from the four lemmas below: Lemma 4.2 shows the correctness of Algorithm 2, Lemma 5.3 analyzes the number of oracle calls, Lemma 4.4 shows the numbers occuring
in the algorithm have polynomial bit sizes, and Lemma 4.5 bounds the number of arithmetic operations. The proofs of Lemma 4.4 and 4.5 are not central to our analysis and are thus postponed to Appendix B.

**Lemma 4.2** (Correctness of the basic algorithm). Assuming the conditions in Theorem 4.1, Algorithm 2 finds an integral minimizer \( x^* \in \mathbb{Z}^n \cap E \).

**Proof.** As we assumed that \( f \) has a unique integral minimizer \( x^* \in \mathbb{Z}^n \), we prove that Algorithm 2 finds \( x^* \). Note that in the beginning of each iteration, \( E \subseteq W \) and \( \Lambda \subseteq W_0 \), where \( W_0 \) is the translation of \( W \) that passes through the origin. We first argue that the lattice \( \Lambda \) is in fact the orthogonal projection of \( \mathbb{Z}^n \) onto the subspace \( W_0 \), i.e. \( \Lambda = \Pi_{W_0}(\mathbb{Z}^n) \). This is required for Lemma 3.1 to be applicable. Clearly \( \Lambda = \Pi_{W_0}(Z) \) holds in the beginning of the algorithm since \( \Lambda = \mathbb{Z}^n \) and \( W = \mathbb{R}^n \). Notice that the cutting plane method in Line 7 keeps \( \Lambda \) and \( W \) the same. Each time we reduce the dimension in Line 10-15, we have

\[
\Pi_{W_0 \cap P_0}(\mathbb{Z}^n) = \Pi_{W_0 \cap P_0}(\Pi_{W_0}(\mathbb{Z}^n)) = \Pi_{W_0 \cap P_0}(\Lambda),
\]

where the first equality follows because \( W_0 \cap P_0 \) is a subspace of \( W_0 \). Since \( \Pi_{P_0}(\Lambda) = \Pi_{W_0 \cap P_0}(\Lambda) \) as \( v \in W_0 \), this shows that the invariant \( \Lambda = \Pi_{W_0}(\mathbb{Z}^n) \) holds throughout the algorithm.

Now we prove that \( x^* \in E \) holds throughout the algorithm. We prove this by induction. Note that \( x^* \in E \) holds in the beginning of the algorithm by the assumption that \( x^* \in B_2(R) \). Assume that \( x^* \in E \) in the beginning of an iteration. If \( \|v\|_{A^{-1}} \geq 1/2 \) and we run the cutting plane method to obtain ellipsoid \( E' \), we have \( x^* \in E' \) by Theorem 2.12 and thus \( x^* \in E \) holds in the beginning of the next iteration. If on the other hand that \( \|v\|_{A^{-1}} < 1/2 \), since \( v \in \Lambda \setminus \{0\} = \Pi_{W_0}(\mathbb{Z}^n) \setminus \{0\} \), Lemma 3.1 gives \( E \cap \mathbb{Z}^n \subseteq P \cap W \).
and therefore \( E \cap \mathbb{Z}^n \subseteq P \cap E \). In particular, the integral minimizer \( x^* \in E \cap P \). Since the new ellipsoid \( E' \supseteq E \cap P \), we thus have \( x^* \in E' \). This proves that \( x^* \in E \) holds throughout the algorithm. \( \square \)

**Lemma 4.3 (Oracle complexity of the basic algorithm).** Assuming the conditions in Theorem 4.1, Algorithm 2 made at most \( O(n \log(n) + \log(R)) \) calls to the separation oracle \( SO \).

**Proof.** We note that the oracle is only called when the cutting plane method is invoked in Line 7, and each run of the cutting plane method makes \( O(n \log(n)) \) calls to \( SO \) according to Theorem 2.12. To upper bound the total number of runs of the cutting plane method, we consider the potential function \( \Phi = \log(\text{vol}(E) \cdot \det(\Lambda)) \). In the beginning, \( \Phi = \log(\text{vol}(B_2(R)) \cdot \det(I)) = O(n \log(R)) \). Each time the cutting plane method is called in Line 7 of Algorithm 2, we have from Theorem 2.12 that the volume of \( E \) decreases by a factor of at least \((1/e)^n = 2^{n \log(n)}\), and thereby the potential function decreases by at least \( n \log(n) \) additively.

Each time the dimension of \( W \) is reduced, denote \( \Lambda' = \Pi_{P_b}(\Lambda) \) the new lattice. It follows from Theorem 2.19 and 3.2 that

\[
\frac{\text{vol}(E') \cdot \det(\Lambda')}{\text{vol}(E) \cdot \det(\Lambda)} \leq O(n) \cdot \frac{1}{\|v\|_{A^{-1}}}. \tag{5}
\]

We shall argue that \( \|v\|_{A^{-1}} \geq n^{-O(n)} \) by considering the previous iteration. There are two cases to consider: (1) the previous iteration runs the cutting plane method in Line 7, and (2) the previous iteration reduces the dimension as in Line 10-15.

In case (1), we denote \( \widetilde{v} \) the vector used to construct the hyperplanes, and \( \tilde{A} \) the matrix in the expression of the ellipsoid \( \widetilde{E} \). Note that \( \|\widetilde{v}\|_{\tilde{A}^{-1}} \geq 1/2 \) since the previous iteration is a cutting plane iteration. By Theorem 2.12, we have \( n^{-O(n)} \cdot \widetilde{E} \subseteq E \subseteq n^{O(n)} \cdot \widetilde{E} \). If \( \|v\|_{A^{-1}} < n^{-O(n)} \) for some sufficiently large constant, then we have \( \|v\|_{\tilde{A}^{-1}} < n^{-O(n)} \cdot n^{O(n)} \leq n^{-n} \). This shows that the vector \( \tilde{A}^{-1/2} \tilde{v} \) is shorter than the vector \( A^{-1/2} v \) by a factor of at least \( n^n/2 \), thus contradicting the property that the LLL algorithm is \( 2^{n/2} \)-approximation. We thus have that \( \|v\|_{A^{-1}} \geq n^{-O(n)} \) in case (1).

Now we consider case (2). In this case, the current iteration belongs to a sequence of consecutive dimension reduction iterations. Consider the first dimension reduction iteration in this sequence and denote \( \widehat{v} \) the vector used to construct the hyperplanes, \( \hat{A} \) the matrix in the expression of the ellipsoid \( \hat{E} \). We already showed in case (1) that \( \|\widehat{v}\|_{\hat{A}^{-1}} \geq n^{-O(n)} \). By the guarantee of the LLL algorithm, the shortest Gram-Schmidt orthogonalization vector has length at least \( n^{-O(n)} \) in that iteration. It then follows from Lemma 3.4 that the shortest Gram-Schmidt orthogonalization vector in the current iteration has length at least \( n^{-O(n)} \). This proves that \( \|v\|_{A^{-1}} \geq n^{-O(n)} \).

Together with (5), we have

\[
\frac{\text{vol}(E') \cdot \det(\Lambda')}{\text{vol}(E) \cdot \det(\Lambda)} \leq n^{O(n)}.
\]

This shows that the potential increases by at most \( O(n \log(n)) \) whenever the dimension is reduced.

Finally we note that whenever the potential becomes smaller than \( -n^2 \), Lemma 3.3 shows the existence of a vector \( \overline{v} \in \Lambda \setminus \{0\} \) with \( \|\overline{v}\|_{\Lambda^{-1}} \leq 2^{-n} \), and thus the LLL algorithm in Line 5 would find a vector \( v \in \Lambda \) with \( \|v\|_{\Lambda^{-1}} \leq 2^{-n/2} \). It follows that such an iteration will not invoke the cutting plane method. As there are at most \( n \) dimension reduction iterations, the total amount of increase in the potential
function is at most $O(n^2 \log(n) + n \log(R))$, and thus the algorithm runs the cutting plane method at most $O(n + \log(R)/\log(n))$ times. Since each run of the cutting plane method makes $O(n \log(n))$ calls to SO, the total number of calls to SO in Algorithm 2 is thus $O(n(n \log(n) + \log(R)))$. □

**Lemma 4.4** (Polynomial bit sizes of numbers occurring in the basic algorithm). *Assuming the conditions in Theorem 4.1, the numbers occurring in Algorithm 2 have bit sizes poly($n, \log(R)$).

**Lemma 4.5** (Number of operations of the basic algorithm). *Assuming the conditions in Theorem 4.1, the number of arithmetic operations needed by Algorithm 2 is at most $O(n^3(n + \log(R)))$.

See Appendix B for the proofs of Lemma 4.4 and 4.5.
5  The Main Algorithm: Achieving Quadratic Oracle Complexity

In this section, we prove Theorem 1.2 by giving a polynomial time algorithm that achieves $O(n(n + \log(R)))$ oracle complexity. For convenience, we restate Theorem 1.2 below.

**Theorem 1.2** (Main result). Given a separation oracle $SO$ for a convex function $f$ defined on $\mathbb{R}^n$. If the set of minimizers $K^*$ of $f$ is contained in a box of radius $R$ and satisfies

\[ (*) \text{ all extreme points of } K^* \text{ are integral,} \]

then there is an algorithm that finds an integral minimizer of $f$ using $O(n(n + \log(R)))$ calls to $SO$ and $\text{poly}(n, \log(R))$ arithmetic operations, with the numbers occurring in the algorithm having bit sizes $\text{poly}(n, \log(R))$. Moreover, the assumption $(*)$ that all extreme points of $K^*$ are integral and the $O(n \log(R))$ term in the oracle complexity are necessary.

5.1  The Main Algorithm

Our main algorithm is given formally in Algorithm 3. It is in spirit similar to the basic algorithm in Section 4, but with a few key differences which we highlight below. In particular, instead of keeping track of an ellipsoidal search set, we maintain a polytope $K$ that is formed by the cutting planes directly. An ellipsoid $E(x_{\text{in}}, A)$ is only maintained implicitly and serves as an approximation to $K$. In fact, the matrix $A$ is guaranteed to be a $2^{O(n)}$-approximation of $\text{Cov}(K)^{-1}$. One might be tempted to approximate $\text{Cov}(K)^{-1}$ much better by sampling from $K$ in every iteration; this, however, suffers from the technical issue of producing much shorter vectors after a sequence of dimension reduction steps.

As in Algorithm 2, we run the LLL algorithm to find a short non-zero lattice vector $v$ under the $A^{-1}$-norm. The criterion for performing a cutting plane step is now taken as $\|v\|_{A^{-1}} \geq 2^{-\Theta(n)}$, which corresponds to the quality of matrix $A$ as an approximation to $\text{Cov}(K)^{-1}$. If $\|v\|_{A^{-1}} \geq 2^{-\Theta(n)}$, then we run the approximate center of gravity method (Theorem 2.14) for one step, update $x_{\text{in}}$ to be an approximate centroid of $K$, and the matrix $A$ to be a $(1 \pm \epsilon)$-approximation to $\text{Cov}(K)^{-1}$.

If, on the other hand, that $\|v\|_{A^{-1}} < 2^{-\Theta(n)}$, then we again use Lemma 3.1 to find a hyperplane $P$ that contains all integral points in $K$. We note that one cannot simply update the polytope $K$ to be $K \cap P$ in this case. The issue here is that $\text{Cov}(K \cap P)$ might be different from $\text{Cov}(K)$ by a factor of $\text{poly}(n)$ on the subspace $P$, which might result in a much shorter vector after a sequence of dimension reduction steps. Instead, we enlarge the polytope $K$ from point $x_{\text{in}}$ by a factor of 2 to obtain a scaled-up polytope $x_{\text{in}} + 2(-x_{\text{in}} + K)$. The polytope $K$ is then updated to be the intersection of this scaled-up polytope with $P$. Such a procedure is illustrated in Figure 1.

After obtaining the updated polytope, which we denote as $K'$, we update $A$ to be the matrix $A'$ defining the ellipsoid $E(A) \cap P_0$, where $P_0$ is a translation of $P$ that passes through the origin. This method of updating the matrix $A$ is key to our analysis: in particular, Lemma 3.4 guarantees that the shortest non-zero lattice vector in $A$-norm will not become much shorter after dimension reduction. We will further show that $A'$ continues to approximate $\text{Cov}(K')^{-1}$, with the approximation factor worse by a factor of 2. This factor of 2, over all $n - 1$ dimension reduction steps, accumulates to a factor of at most $2^n$, and thus we get the guarantee that $A$ approximates $\text{Cov}(K)^{-1}$ within a factor of $2^{O(n)}$ as promised earlier. As we scaled-up $K$ by a factor of 2, the inner center $x_{\text{in}}$ can be updated to be any point on $P \cap K$. 

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Algorithm 3

1: procedure MAIN(SO, R)  
   R can be obtained by doubling trick
2:   Affine subspace $W \leftarrow \mathbb{R}^n$, lattice $\Lambda \leftarrow \mathbb{Z}^n$
3:   Polytope $K \leftarrow B_\infty(R)$, matrix $A \leftarrow \text{Cov}(K)^{-1}$
4:   Approximate centroid $x_K \leftarrow 0$, inner center $x_{in} \leftarrow 0$  
   $x_{in} + E(A)/2 \subseteq K \subseteq x_K + 2n2^n \cdot E(A)$
5:   $T \leftarrow 1, \varepsilon \leftarrow 0.01$  
   Parameters in Theorem 2.14
6:   while dim($W$) > 1 do
7:     $v \leftarrow \text{LLL}(\Lambda, A^{-1})$  
     $v \in \Lambda \setminus \{0\}$
8:     if $\|v\|_{A^{-1}} \geq \frac{1}{10n2^n}$ then
9:       $(K', x_{K'}, \Sigma_K, \cdot) \leftarrow \text{RANDOMWALKCG}(SO, K, T, \varepsilon)$ as in Theorem 2.14
10:      $K \leftarrow K'$, $A \leftarrow \Sigma_K^{-1}$, $x_K \leftarrow x_{K'}$, $x_{in} \leftarrow x_{in}$
11:   else
12:     Find $z \in \mathbb{Z}^n$ such that $v = \Pi_{W_0}(z)$  
     Subspace $W_0 = -x_k + W$
13:     Construct $P \leftarrow \{y : v^T y = (v - z)^T x_K + [z^T x_K]\}$
14:     Pick any point $x_{in}' \in P \cap K$
15:     $W \leftarrow W \cap P$, $K \leftarrow (x_{in} + 2(-x_{in} + K)) \cap P$, $x_{in} \leftarrow x_{in}'$  
     Dimension reduction
16:     Obtain approximate centroid $x_K$ of $K$ as in Theorem 2.14
17:     Construct hyperplane $P_0 \leftarrow \{y : v^T y = 0\}$
18:     $\Lambda \leftarrow \Pi_{P_0}(\Lambda)$  
     Lattice projection
19:     Let $E'(A') = E(A) \cap P_0$, $A \leftarrow A'$  
     Restricting $A$ to subspace $W_0 \cap P_0$
20:   end if
21: end while
22: Find integral minimizer $x^* \in \mathbb{Z}^n \cap E$
23: Return $x^*$
24: end procedure

Figure 1: Update rule for the polytope $K$. We enlarge $K$ from $x_{in}$ by a factor of 2, and then take the intersection of this scaled-up polytope $x_{in} + 2 \cdot (-x_{in} + K)$ with $P$.  

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5.2 Proof of Main Result

By Remark 1.5, we assume without loss of generality that $x^* \in \mathbb{Z}^n$ is the unique minimizer of $f$. The proofs of the statements that Algorithm 3 uses $\text{poly}(n, \log(||x^*||_2))$ arithmetic operations and that the numbers occurring in the algorithm have bit sizes $\text{poly}(n, \log(||x^*||_2))$ are very similar to the proofs of Lemma 4.4 and 4.5, and are thus omitted. We only prove the correctness and oracle complexity of our main algorithm.

**Lemma 5.1** (Correctness of the main algorithm). Assuming the conditions in Theorem 1.2, Algorithm 3 finds an integral minimizer of the function $f$.

**Proof.** As in the proof of Lemma 4.2, we have $\Lambda = \Pi_{W_0}(\mathbb{Z}^n)$. Since RANDOMWALKCG in Line 9 preserves the minimizer of $f$, we only need to prove that the dimension reduction step in Line 15 preserves the minimizer of $f$. In the following, we show the stronger statement that each dimension reduction step taken by Algorithm 3 in Line 15 preserves all integral points in $K$.

We start by proving that $K$ is sandwiched between certain scales of the ellipsoid $E(A)$.

**Claim 5.2** (Sandwiching condition). In any iteration of Algorithm 3, we have

$$x_{in} + E(A)/2 \subseteq K \subseteq x_K + 2n \cdot 2^n \cdot E(A). \quad (6)$$

Moreover, if the previous iteration runs RANDOMWALKCG in Line 9, then we have

$$x_K + E(A)/2 \subseteq K \subseteq x_K + 2n \cdot E(A). \quad (7)$$

**Proof.** We first prove the second part of the statement. If the previous iteration runs RANDOMWALKCG in Line 9, then it follows from Theorem 2.14 that the current iteration satisfies $(1 - \epsilon) \cdot \text{Cov}(K) \preceq A^{-1} \preceq (1 + \epsilon) \cdot \text{Cov}(K)$. An immediate application of Theorem 2.21 gives the sandwiching condition $\text{cg}(K) + \frac{1}{\sqrt{1+\epsilon}} \cdot E(A) \subseteq K \subseteq \text{cg}(K) + \frac{1}{\sqrt{1-\epsilon}} \cdot (n + 1) \cdot E(A)$. By Theorem 2.14, the point $x_K$ satisfies that $||x_K - \text{cg}(K)||_A < \epsilon = 0.01$, from which (7) follows.

To prove (6), we let the current iteration be $t_0 + t$, where $t_0$ is the last iteration (before the current iteration) which runs RANDOMWALKCG in Line 9. We prove via induction that in iteration $t_0 + i$, we have

$$x_{in} + E(A)/2 \subseteq K \subseteq x_{out} + 2n \cdot 2^{i-1} \cdot E(A), \quad (8)$$

for some point $x_{out} \in \mathbb{R}^n$. Then, (6) is an immediate consequence of (8) by observing that $x_{out} + 2n \cdot 2^{i-1} \cdot E(A) \subseteq x_K + 2n \cdot 2^i \cdot E(A)$ since $x_K \in x_{out} + 2n \cdot 2^{i-1} \cdot E(A)$.

It follows from (7) that (8) holds for $i = 1$ with $x_{in} = x_{out} = x_K$. For the induction step, we assume (8) holds for some $1 \leq i < t$ and show in the following that it holds for $i + 1$. To make our notations explicit, we use superscript $(i)$ (resp. superscript $(i+1)$) for the corresponding notations in iteration $t + i$ (resp. iteration $t + i + 1$), e.g. $K^{(i)}$ and $A^{(i)}$ (resp. $K^{(i+1)}$ and $A^{(i+1)}$). By our induction hypothesis in (8), we have

$$x_{in}^{(i)} + E(A^{(i)})/2 \subseteq K^{(i)} \subseteq x_{out}^{(i)} + 2n \cdot 2^{i-1} \cdot E(A^{(i)}).$$

Recall from Line 15 that $K^{(i+1)} = P^{(i)} \cap (x_{in}^{(i)} + 2(-x_{in}^{(i)} + K^{(i)}))$, and from Line 19 that $E(A^{(i+1)}) = E(A^{(i)}) \cap P^{(i)}$. 

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We first prove the RHS of (8). Since $K^{(i)} \subseteq x^{(i)}_{\text{out}} + 2n \cdot 2^{i-1} \cdot E(A^{(i)})$, it's geometrically clear that

$$x^{(i)}_{\text{in}} + 2(-x^{(i)}_{\text{in}} + K^{(i)}) \subseteq x^{(i+1/2)}_{\text{out}} + 2n \cdot 2^i \cdot E(A^{(i)}),$$

where $x^{(i+1/2)}_{\text{out}} = 2x^{(i)}_{\text{out}} - x^{(i)}_{\text{in}}$. Taking $x^{(i+1)}_{\text{out}}$ to be the center of $P^{(i)} \cap (x^{(i+1/2)}_{\text{out}} + 2n \cdot 2^i \cdot E(A^{(i)}))$, we have the containment $K^{(i+1)} \subseteq x^{(i+1)}_{\text{out}} + 2n \cdot 2^i \cdot E(A^{(i+1)})$, since the cross-section through the center of an ellipsoid is the largest among all parallel cross-sections. See Figure 2 below for an illustration of this outer containment. This proves the RHS of (8).

![Figure 2: Illustration of the outer containment (RHS of (8)). Denote $E^{(i)}_{\text{out}}$ the ellipsoid $x^{(i)}_{\text{out}} + 2n \cdot 2^{i-1} \cdot E(A^{(i)})$ and $E^{(i+1/2)}_{\text{out}}$ the ellipsoid $x^{(i+1/2)}_{\text{out}} + 2n \cdot 2^i \cdot E(A^{(i)})$. The ellipsoid $E^{(i+1)}_{\text{out}}$ is given by $E^{(i+1/2)}_{\text{out}} \cap P$ and is lower-dimensional.](image)

To prove the LHS of (8), we take any $x^{(i+1)}_{\text{in}} \in P^{(i)} \cap K^{(i)}$. This intersection is non-empty as we shall see that the induction hypothesis implies that $P^{(i)}$ contains all integral points in $K^{(i)}$. In particular, the cone formed by connecting $x^{(i+1)}_{\text{in}}$ and $x^{(i)}_{\text{in}} + E(A^{(i+1)})/2$ lies inside $K$ due to convexity. Therefore, the scaled-up cone formed by connecting $2x^{(i+1)}_{\text{in}} - x^{(i)}_{\text{in}}$ and $x^{(i)}_{\text{in}} + E(A^{(i+1)})$ is contained inside the scaled-up convex body $x^{(i)}_{\text{in}} + 2(-x^{(i)}_{\text{in}} + K^{(i)})$. This implies that $x^{(i+1)}_{\text{in}} + E(A^{(i+1)})/2 \subseteq P^{(i)} \cap (x^{(i)}_{\text{in}} + 2(-x^{(i)}_{\text{in}} + K^{(i)})) = K^{(i+1)}$, which proves the LHS of (8). See Figure 3 below for an illustration of this inner containment. This finishes the proof of the claim.

Now we proceed to show that each dimension reduction iteration preserves all integral points in $K$. By the RHS of Claim 5.2, we have $K \cap \mathbb{Z}^n \subseteq (x_K + 2n \cdot 2^n \cdot E(A)) \cap \mathbb{Z}^n$. Since $\|v\|_{A^{-1}} < \frac{1}{10n \cdot 2^n}$ is satisfied in a dimension reduction iteration, Lemma 3.1 shows that all integral points in $x_K + 2n \cdot 2^n \cdot E(A)$ lie on the hyperplane given by $P = \{y : v^T y = (v^T x_K + [v^T x_K]).$ Thus we have $K \cap \mathbb{Z}^n \subseteq K \cap P$. Finally, we note that $K \cap P \subseteq (x_{\text{in}} + 2(-x_{\text{in}} + K)) \cap P$ by convexity of $K$. This implies that $K \cap \mathbb{Z}^n \subseteq (x_{\text{in}} + 2(-x_{\text{in}} + K)) \cap P$ and finishes the proof of the lemma.

□
Figure 3: Illustration of the inner containment (LHS of (8)). The three points, from left to right, are $2x_{in}^{(i+1)} - x_{in}^{(i)}$, $x_{in}^{(i)}$, and $x_{in}^{(i)}$. Denote $E_{in}^{(i)}$ the ellipsoid $x_{in}^{(i)} + E(A^{(i)})/2$ and $E_{in}^{(i+1)}$ the ellipsoid $x_{in}^{(i+1)} + E(A^{(i+1)})/2$. The ellipsoid $E_{in}^{(i+1)}$ is lower-dimensional.

Lemma 5.3 (Oracle complexity of the main algorithm). **Assuming the conditions in Theorem 1.2, Algorithm 3 made at most $O(n(n + \log(R)))$ calls to the separation oracle $SO$.**

**Proof.** We may assume, by union bound, that all the high probability events in Theorem 2.14 happen. We note that the oracle is only called when $R/a.sc/n.sc/d.sc/o.sc/m.scW/a.sc/l.sc/k.scCG$ is invoked in Line 9, and each run of $R/a.sc/n.sc/d.sc/o.sc/m.scW/a.sc/l.sc/k.scCG$ makes $T = 1$ call to $SO$ according to Theorem 2.14. To upper bound the total number of runs of $R/a.sc/n.sc/d.sc/o.sc/m.scW/a.sc/l.sc/k.scCG$, we consider the potential function $\Phi = \log(|K| \cdot \det(\Lambda))$.

In the beginning, $\Phi = \log(|B_{\infty}(R)| \cdot \det(I)) = n \log(R)$. Each time RANDOMWALKCG is called in Line 9, we have from Theorem 2.14 that the volume of $E$ decreases by at least a factor of 1.5, and thereby the potential function decreases by at least $\Omega(1)$ additively.

Each time the dimension of $W$ is reduced, denote $\Lambda' = \Pi_{P_i}(\Lambda)$ the new lattice. Let $K' = (x_{in} + 2(-x_{in} + K)) \cap P$ be the new convex body. $K'$ is a cross-section of $K$ (orthogonal to $v$) that is scaled up by a factor of 2, and thus by Corollary 2.18 and Lemma 2.20, we have

$$\frac{|K'|}{|K|} \leq O(2^n) \cdot \frac{\|v\|_2}{\|v\|_{\text{Cov}(K)}}.$$

By Claim 5.2, we have $x_{in} + E(A)/2 \subseteq K \subseteq x_K + 2n \cdot 2^n \cdot E(A)$. Theorem 2.21 gives a sandwiching condition in terms of the covariance matrix: $x_K + E(\text{Cov}(K)^{-1}) \subseteq K \subseteq x_K + (n + 1) \cdot E(\text{Cov}(K)^{-1})$. In particular, the containment $x_{in} + E(A)/2 \subseteq x_K + (n + 1) \cdot E(\text{Cov}(K)^{-1})$ implies that $A^{-1} \leq 4(n + 1)^2 \cdot \text{Cov}(K)$, and therefore, $\|v\|_{\text{Cov}(K)} \geq \frac{1}{2(n+1)} \cdot \|v\|_{A^{-1}}$. By Lemma 3.2, we have

$$\frac{\det(\Lambda')}{\det(\Lambda)} = \frac{1}{\|v\|_2^2}.$$
It then follows that

$$\frac{|K'| \cdot \det(\Lambda')}{|K| \cdot \det(\Lambda)} \leq 2^{O(n)} \cdot \frac{1}{\|v\|_{A^{-1}}}. \quad (9)$$

We shall argue that $\|v\|_{A^{-1}} \geq 2^{-O(n)}$ by considering the previous iteration. If the previous iteration runs RANDOMWALKCG, then Lemma 2.22 implies that $\|v\|_{A^{-1}} \geq 2^{-O(n)}$, as otherwise $v$ would have been a short enough vector in the previous iteration. It thus remains to consider the case where the previous iteration reduces the dimension. Let the current iteration be $t_0 + t$, where $t_0$ is the last iteration (before the current iteration) where RANDOMWALKCG was invoked. We shall use subscript $(t_0 + i)$ to denote the corresponding notations in iteration $t_0 + i$. Define subspace $V := W_0^{(t_0+1)} \setminus W_0^{(t_0+t)}$. It follows from Line 19 that $A^{(t_0+t)} = SC(A^{(t_0+1)}, V)$ is the Schur complement of the subspace $V$ of the matrix $A^{(t_0+1)}$. Then by Lemma 3.4, the $A$-norm of the shortest Gram-Schmidt vector of the LLL-reduced basis in iteration $t_0 + t$ is at least that in iteration $t_0 + 1$, which is at least $2^{-O(n)}$ by the previous argument. As the shortest Gram-Schmidt vector of any basis gives a lower bound on the shortest vector in the lattice, this proves that $\|v\|_{A^{-1}} \geq 2^{-O(n)}$.

Together with (9), we have

$$\frac{|K'| \cdot \det(\Lambda')}{|K| \cdot \det(\Lambda)} \leq 2^{O(n)}.$$  

This shows that the potential increases by at most $O(n)$ whenever the dimension is reduced.

Finally we note that whenever the potential becomes smaller than $-10n^2$, Lemma 3.3 shows the existence of a vector $v \in \Lambda \setminus \{0\}$ with $\|v\|_{A^{-1}} \leq 2^{-10n}$, and thus such an iteration will not invoke RANDOMWALKCG. As there are at most $n$ dimension reduction iterations, the total amount of increase in the potential function is at most $O(n^2)$, and thus the algorithm runs RANDOMWALKCG at most $O(n(n + \log(R)))$ times. Since each run of the cutting plane method makes $T = 1$ call to SO, the total number of calls to SO by Algorithm 3 is thus $O(n(n + \log(R)))$. This finishes the proof of the lemma.  \( \square \)
A The Grötschel-Lovász-Schrijver Approach

In this section, we give an algorithm with $O(n^2(n + \log(R)))$ oracle complexity and $\text{poly}(n, \log(R))$ runtime using simultaneous diophantine approximation. This elegant application is due to Grötschel, Lovász and Schrijver [GLS84, GLS88]. The proof we give here is partially taken from the excellent textbook [GLS88].

**Theorem A.1** (Grötschel, Lovász and Schrijver [GLS84, GLS88]). Given a separation oracle $SO$ for a convex function $f$ defined on $\mathbb{R}^n$. If the set of minimizers $K^* \subseteq B_2(R)$ of $f$ satisfies

$$(\star) \text{ all extreme points of } K^* \text{ are integral},$$

then there is an algorithm that finds an integral minimizer of $f$ using $O(n^2(n + \log(R)))$ calls to $SO$ and $\text{poly}(n, \log(R))$ arithmetic operations, with the numbers occurring in the algorithm having bit sizes $\text{poly}(n, \log(R))$.

By Remark 1.5, we can assume that $f$ has a unique integral minimizer $x^* \in \mathbb{Z}^n$. The following result on simultaneous diophantine approximation is given in [LLL82].

**Theorem A.2** (Simultaneous diophantine approximation, [LLL82]). Given a positive integer $N$ and rational numbers $a_1, \cdots, a_n$, there exists a polynomial time algorithm that finds integers $\mathbf{v}_1, \cdots, \mathbf{v}_n$ and $q$ such that $1 \leq q \leq 2^{2n^2} N^n$ and

$$|qa_i - \mathbf{v}_i| < \frac{1}{N} \quad \text{for} \quad i = 1, \cdots, n.$$  

Moreover, the numbers occurring in the course of the computation have sizes polynomially bounded in the input size.

Here the factor of $2^{2n^2}$ in the upper bound on $q$ corresponds to the approximation factor of $2^{\Theta(n)}$ of the LLL algorithm. Using the above Theorem A.2, we sketch a proof of Theorem A.1.

**Proof of Theorem A.1 (Sketch).** Using the cutting plane method in Theorem 2.12, we can maintain an ellipsoid $E(x_0, A)$ that contains the minimizer $x^*$. Let $\mu_{\text{min}}$ be the smallest non-zero eigenvalue of the matrix $A^{-1/2}$. In particular, $\mu_{\text{min}}$ is the Euclidean length of the shortest non-trivial axis of $E(x_0, A)$. We show that if $\mu_{\text{min}} < 2^{-3n(n+\log(R))}$, then one can efficiently compute a hyperplane $P$ that contains all the integral points in $E(x_0, A)$.

Let $a$ be the unit eigenvector that corresponds to $\mu_{\text{min}}$ (in general, one can only guarantee $\|a\|_2 \in [1, 2]$ using elementary arithmetic operations, but this difference is immaterial to us). Applying the simultaneous diophantine approximation algorithm in Theorem A.2 to the vector $(a, a^\top x_0)$ for some positive integer $N$ which we specify later, we obtain an integer vector $(\mathbf{v}, b) (\mathbf{v} \in \mathbb{Z}^n, b \in \mathbb{Z})$ and a positive integer $0 < q < 2^{2n^2} N^n$ such that

$$\|qa - \mathbf{v}\|_\infty < \frac{1}{N} \quad \text{and} \quad |qa^\top x_0 - b| < \frac{1}{N}.$$

For any integer point $x \in E(x_0, A) \cap \mathbb{Z}^n$, we thus have

$$|\mathbf{v}^\top x - b| \leq |qa^\top x - qa^\top x_0| + \frac{1}{N} \cdot (\|x\|_1 + 1) \leq q\mu_{\text{min}} + \frac{nR + 1}{N},$$

By Remark 1.5, we can assume that $f$ has a unique integral minimizer $x^* \in \mathbb{Z}^n$. The following result on simultaneous diophantine approximation is given in [LLL82].
where the first inequality uses triangular inequality and the second inequality uses $\|x\|_\infty \leq R$. Taking $N = 10(nR + 1)$ above, we get
\[
|v^T x - b| \leq q\mu_{\min} + 1/10.
\]

Note that $q \leq 2^{n^2} N^n \leq 2^{2n(n+\log(R))}$. Thus when $\mu_{\min} < 2^{-3n(n+\log(R))}$, we have $|v^T x - b| = 0$ since the quantity is an integer. We thus obtain a hyperplane $P$ that contains all the integral points in $E(x_0, A)$.

Given the above procedure, an algorithm follows naturally: we run the cutting plane method for an integer multiple of $\Theta(n \log(n))$ steps until $\mu_{\min} < 2^{-3n(n+\log(R))}$, at which point we use the above procedure to find a hyperplane that contains all the integral points in $E(x_0, A)$ and recurse on the lower dimensional affine subspace.

To prove that this algorithm has an oracle complexity of $O(n^2(n + \log(R)))$, we consider the potential function $\Phi = \log(\text{vol}(E))$, where $E$ is the ellipsoid maintained by the cutting plane method in Theorem 2.12. Roughly, every $\Theta(n \log(n))$ steps of the cutting plane method decreases the potential function by an additive factor of $n \log(n)$, and every dimension reduction iteration increases the potential function by approximately $\log(1/\mu_{\min}) \approx 3n(n + \log(R))$. Thus over all $n - 1$ dimension reduction iterations (before the dimension of the problem becomes 1), the total increase in the potential is $O(n^2(n + \log(R)))$.

Note also that whenever the potential function becomes smaller than $-\Theta(n^2(n + \log(R)))$, we have $\text{vol}(E) \leq 2^{-\Theta(n^2(n+\log(R)))}$ and thus $\mu_{\min} < 2^{-3n(n+\log(R))}$ is always satisfied. In such an occasion, the algorithm doesn’t need to run the cutting plane method further. Thus the total number of cutting plane steps, and therefore the oracle complexity, is $O(n^2(n + \log(R)))$. This proves the theorem. \hfill \Box

## B Missing Proofs in Section 4

**Lemma 4.4** (Polynomial bit sizes of numbers occurring in the basic algorithm). *Assuming the conditions in Theorem 4.1, the numbers occurring in Algorithm 2 have bit sizes poly($n, \log(R)$).

**Proof.** Since $-O(n^2) \leq \log(\text{vol}(E) \cdot \det(\Lambda)) \leq O(n \log(R))$ from the proof of the previous lemma, the volume of the ellipsoid satisfies $2^{-O(n^2)} \leq \text{vol}(E) \leq R^{O(n)}$ in each iteration of the algorithm. Let $\lambda(\Lambda^{-1})$ be any non-zero eigenvalue of $\Lambda^{-1}$. Since $E \subseteq B_2(R)$, we derive from the volume bound that $2^{-O(n(n+\log(R)))} < \lambda(\Lambda^{-1}) < R$. This implies that each entry of the matrix $\Lambda^{-1}$ defining the ellipsoid $E$ requires at most $O(n(n + \log(R)))$ bits.

We yet need to show how to obtain the new lattice given by the lattice projection in Line 15. It’s not clear that projecting a lattice $O(n)$ times using Lemma 3.2 results in a lattice basis with polynomial bit sizes. As argued in the proof of Lemma 4.2, the lattice $\Lambda$ in each iteration of the algorithm is exactly $\Pi_{W_0}(\mathbb{Z}^n)$, where $W_0$ is the subspace on which the lattice lies in. We thus avoid lattice projection and directly compute a basis for the lattice $\Lambda = \Pi_{W_0}(\mathbb{Z}^n)$. We show how to do this in the following.

Let $v_k \in \Lambda_k \setminus \{0\}$ be the vector $v$ in the algorithm that is used for dimension reduction in Line 10-15 when $\dim(W_0) = n - k + 1$, where $k \in \{1, \cdots, n-1\}$ and $\Lambda_k$ is the corresponding lattice. Since $n^{-O(n)} < \|v_k\|_\infty < 1/2$ as argued in the proof of Lemma 5.3 and that $2^{-O(n(n+\log(R)))} < \lambda(\Lambda^{-1}) < R$, we have $2^{-O(n(n+\log(R)))} < \|v_k\|^2 < 2^{O(n(n+\log(R)))}$. Further denote $\text{span}\{\Lambda_k\}$ as $W_k$, where we recall

\[\text{We need to run the cutting plane method in batches of } \Theta(n \log(n)) \text{ steps because of the } n \log(n) \text{ factor in Theorem 2.12.}\]
that $\Lambda_k = \Pi_{W_k}(\mathbb{Z}^n)$. We first show how to find integral vectors $z_k \in \mathbb{Z}^n$ such that $\Pi_{W_k}(z_k) = v_k$ and $\|z_k\|^2 < 2^{O(n(n+\log(R)))}$. This procedure is given in Algorithm 4.

### Algorithm 4

1: procedure FindPreImage($v_1, \ldots, v_k, \Lambda_1, \ldots, \Lambda_k$)  
2: $z_k \leftarrow v_k$  
3: for $j = k-1, k-2, \ldots, 1$ do  
4: Find $|\alpha_j| \leq 1/2$ such that $z_k + \alpha_j v_j \in \Lambda_j$  
5: $z_k \leftarrow z_k + \alpha_j v_j$  
6: end for  
7: Return $z_k$  
8: end procedure

To justify the existence of $|\alpha_j| \leq 1/2$ in Line 4 of Algorithm 4, we fix any $j \in [k+1]$ and denote the vector $z_k$ prior to Line 4 as $z_k^{j+1} \in \Lambda_{j+1}$. Recall that $\Lambda_{j+1}$ is obtained from $\Lambda_j$ by projecting onto the hyperplane $\{x : v_j^T x = 0\}$. We can thus find some $\beta_j \in \mathbb{R}$ such that $z_k^{j+1} + \beta_j v_j \in \Lambda_j$. Since $v_j \in \Lambda_j$, we have $z_k^{j+1} + (\beta_j - [\beta_j]) v_j \in \Lambda_j$ and thus $\alpha_j$ can be taken as $\beta_j - [\beta_j]$. Note that $z_k = v_k + \sum_{j=1}^{k-1} \alpha_j v_j$. Since $\|v_j\|^2 < 2^{O(n(n+\log(R)))}$ for each $j \in [k-1]$, we have $\|z_k\|^2 < 2^{O(n(n+\log(R)))}$.

Now we are ready to show how to compute a basis for $\Lambda_{k+1} = \Pi_{W_{k+1}}(\mathbb{Z}^n)$. Consider matrix $Z \in \mathbb{Z}^{k \times n}$ whose $i$th row is $z_i^T$. Since $W_{j+1} = W_j \cap \{x : v_j^T x = 0\} = W_j \cap \{x : z_j^T x = 0\}$, for any $j \in [k]$, we have that $W_{k+1}$ is the null space of $Z$. The procedure to compute a basis for $\Pi_{W_{k+1}}(\mathbb{Z}^n)$ is given in Algorithm 5.

### Algorithm 5

1: procedure FindBasis($Z \in \mathbb{Z}^{k \times n}$)  
2: Compute Hermite normal form $ZU = [B, 0]$ $\quad \triangleright U \in \mathbb{Z}^{n \times n}$ is unitary, $B \in \mathbb{Z}^{k \times k}$ has full rank  
3: Let matrix $V \in \mathbb{Z}^{n \times (n-k)}$ be the last $n-k$ columns of $U$ $\quad \triangleright V$ is the basis of $\Lambda_{k+1}^*$ $\triangleright$ Fact 2.4  
4: Return $V(V^T V)^{-1}$  
5: end procedure

The main observation in Algorithm 5 is the following claim.

**Claim B.1** (Dual lattice of $\Lambda_{k+1}$ and its basis). The dual lattice of $\Lambda_{k+1}$, denoted as $\Lambda_{k+1}^*$, is given by $W_{k+1} \cap \mathbb{Z}^n$. Moreover, the matrix $V \in \mathbb{Z}^{n \times (n-k)}$ in Algorithm 5 is a basis of $\Lambda_{k+1}^*$.

**Proof.** We first prove that $W_{k+1} \cap \mathbb{Z}^n$ is the dual lattice of $\Lambda_{k+1}$. Denote $P_{k+1}$ the projection matrix onto the subspace $W_{k+1}$. Note that any vector $u \in \Lambda_{k+1}$ can be expressed as $P_{k+1} z_u$ for some integral vector $z_u \in \mathbb{Z}^n$. For any vector $u' \in W_{k+1} \cap \mathbb{Z}^n$, we have

$$\langle u', u \rangle = \langle P_{k+1} u', z_u \rangle = \langle u', z_u \rangle \in \mathbb{Z}.$$

This implies that $W_{k+1} \cap \mathbb{Z}^n \subseteq \Lambda_{k+1}^*$. To prove the other direction, we consider any vector $u' \in \Lambda_{k+1}^*$. Note that $\langle u', u \rangle = \langle u', z_u \rangle \in \mathbb{Z}$ for any $u \in \Lambda_{k+1}$ and thus for any $z_u \in \mathbb{Z}^n$. It follows that $u' \in \mathbb{Z}^n$. This proves that $\Lambda_{k+1}^* = W_{k+1} \cap \mathbb{Z}^n$.
We next prove that $V \in \mathbb{Z}^{n \times (n-k)}$ is a basis of $\Lambda_{k+1}^*$. Let $V_i$ be the $i$th column of $V$. Since $ZV_i = 0$, we have $V_i \in W_{k+1}$. It follows that $V_i \in \Lambda_{k+1}^*$ since $V_i \in \mathbb{Z}^n$. Thus we have $V \mathbb{Z}^{n-k} \subseteq \Lambda_{k+1}^*$. Now consider any vector $u \in \Lambda_{k+1}^* = W_{k+1} \cap \mathbb{Z}^n$. Since $U$ is a unitary matrix, we have $u = Uz_u$ for some integral vector $z_u \in \mathbb{Z}^n$. Thus we have

$$0 = Zu = ZUz_u = [B, 0] \cdot z_u.$$  

Since $Z$ has full row rank, the matrix $B$ is lower-diagonal with non-zero diagonal entries. This implies that the first $k$ coordinates of $z_u$ are all 0. Therefore, $u = Uz_u$ is an integer combination of the last $n-k$ columns of $U$, which are exactly the columns of matrix $V$. This finishes the proof of the claim. \qed

It immediately follows from Claim B.1 and Fact 2.4 that $V(V^TV)^{-1}$ returned by the algorithm is a basis of the lattice $\Lambda_{k+1}$. We now upper bound the bit size of this basis. Since $\|z_k\|_2 < 2^{O(n(n + \log(R)))}$, the matrix $Z$ has bit size $O(n(n + \log(R)))$ per entry. It follows that the matrix $U$ (and thus $V$) has bit size $O(n^2(n + \log(R)))$ per entry. Since entries of the inverse $(V^TV)^{-1}$ can be expressed by the ratio between sub-determinants of the matrix $V^TV$, the basis $V(V^TV)^{-1}$ has bit size at most $O(n^3(n + \log(R)))$ per entry. This proves that we can find a basis of lattice $\Lambda$ with poly$(n, \log(R))$ bit size in each iteration of the algorithm.

It follows that all numbers occurring in the algorithm have polynomial bit sizes. This finishes the proof of the lemma. \qed

**Lemma 4.5** (Number of operations of the basic algorithm). *Assuming the conditions in Theorem 4.1, the number of arithmetic operations needed by Algorithm 2 is at most $O(n^8(n + \log(R)))$.*

**Proof.** The most costly step of the algorithm in terms of the number of operations is the LLL algorithm in Line 5. As seen from the proof of Lemma 5.3, the total number of calls to the LLL algorithm is $O(n)$. In each call to the LLL algorithm, the proof of Lemma 4.4 shows that the basis of the lattice $\Lambda$ has bit size $O(n^3(n + \log(R)))$ per entry. It then follows from Theorem 2.11 that each call to the LLL algorithm takes $O(n^3(n + \log(R)))$ arithmetic operations. This finishes the proof of the lemma. \qed

### C Submodular Function Minimization

In this section, we do not seek to give a comprehensive introduction to submodular functions, but only provide the necessary definitions and properties that are needed for the proof of Theorem 1.6. We refer interested readers to the famous textbook by Schrijver [Sch03] or the extensive survey by McCormick [McC05] for more details on submodular functions.

#### C.1 Preliminaries

Throughout this section, we use $[n] = \{1, \cdots, n\}$ to denote the ground set and let $f : 2^{[n]} \to \mathbb{Z}$ be a set function defined on subsets of $[n]$. For a subset $S \subseteq [n]$ and an element $i \in [n]$, we define $S + i := S \cup \{i\}$. A set function $f$ is submodular if it satisfies the following property of diminishing marginal differences:

**Definition C.1** (Submodularity). *A function $f : 2^{[n]} \to \mathbb{Z}$ is submodular if $f(T+i) - f(T) \leq f(S+i) - f(S)$, for any subsets $S \subseteq T \subseteq [n]$ and $i \in [n] \setminus T.$*
Throughout this section, the set function \( f \) we work with is assumed to be submodular even when it is not stated explicitly. We may assume without loss of generality that \( f(\emptyset) = 0 \) by replacing \( f(S) \) by \( f(S) - f(\emptyset) \). We assume that \( f \) is accessed by an evaluation oracle, and use EO to denote the time to compute \( f(S) \) for a subset \( S \). Our algorithm for SFM is based on a standard convex relaxation of a submodular function, known as the Lovász extension [GLS88].

**Definition C.2 (Lovász extension).** The Lovász extension \( \hat{f} : [0, 1]^n \rightarrow \mathbb{R} \) of a submodular function \( f \) is defined as

\[
\hat{f}(x) = \mathbb{E}_{t \sim [0,1]}[f(\{i : x_i \geq t\})],
\]

where \( t \sim [0,1] \) is drawn uniformly at random from \([0, 1]\).

The Lovász extension \( \hat{f} \) of a submodular function \( f \) has many desirable properties. In particular, \( \hat{f} \) is a convex relaxation of \( f \) and it can be evaluated efficiently.

**Theorem C.3 (Properties of Lovász extension).** Let \( f : 2^n \rightarrow \mathbb{Z} \) be a submodular function and \( \hat{f} \) be its Lovász extension. Then,

(a) \( \hat{f} \) is convex and \( \min_{x \in [0,1]^n} \hat{f}(x) = \min_{S \subseteq [n]} f(S) \);

(b) \( f(S) = \hat{f}(I_S) \) for any subset \( S \subseteq [n] \), where \( I_S \) is the indicator vector for \( S \);

(c) Suppose \( x_1 \geq \cdots \geq x_n \geq x_{n+1} := 0 \), then \( \hat{f}(x) = \sum_{i=1}^n (f([i]) - f([i-1]))x_i \);

(d) The set of minimizers of \( \hat{f} \) is the convex hull of the set of minimizers of \( f \).

Next we address the question of implementing the separation oracle (as in Definition 1.1) using the evaluation oracle of \( f \).

**Theorem C.4 (Separation oracle implementation for Lovász extension, Theorem 61 of [LSW15]).** Let \( f : 2^n \rightarrow \mathbb{Z} \) be a submodular function and \( \hat{f} \) be its Lovász extension. Then a separation oracle for \( \hat{f} \) can be implemented in time \( O(n \cdot EO + n^2) \).

### C.2 Proof of Theorem 1.6

Before presenting the proof, we restate Theorem 1.6 for convenience.

**Theorem 1.6 (Submodular function minimization).** Given an evaluation oracle EO for a submodular function \( f \) defined over subsets of an \( n \)-element ground set, there exists a strongly polynomial algorithm that minimizes \( f \) using \( O(n^3) \) calls to EO.

**Proof.** We apply Theorem 1.2 to the Lovász extension \( \hat{f} \) of the submodular function \( f \) with \( R = 1 \). By part (a) and (d) of Theorem C.3, \( \hat{f} \) is a convex function that satisfies the assumption \( (\star) \) in Theorem 1.2. Thus Theorem 1.2 gives a strongly polynomial algorithm for finding an integral minimizer of \( \hat{f} \) that makes \( O(n^2) \) calls to a separation oracle of \( \hat{f} \). This integral minimizer also gives a minimizer of \( f \). Since a separation oracle for \( \hat{f} \) can be implemented using \( O(n) \) calls to EO by Theorem C.4, the total number of calls to the evaluation oracle is thus \( O(n^3) \). This proves the theorem. \( \square \)
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