Banding oscillations of non-Brownian particles in a rotating fluid

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Abstract. When non-Brownian particles are suspended in a fluid filling a cylindrical tube rotating about a horizontal axis, axially dependent patterns of particle density develop. In many cases, equally spaced axial bands are observed. In this work, we attempt to solve the problem of oscillations between two different banding configurations. Previous work has shown that inertial modes in the fluid are continuously excited by the particles and are responsible for the banding. We derived a one-dimensional linearized model describing the fluid–particle interaction as the source of the oscillatory phenomenon. The model invokes a mechanism of negative feedback between the wave and particle fields, which was modeled in previous work using computer simulations of multiple particles suspended in a moving fluid. It leads to the prediction of the dependence of the temporal period of the oscillations on the particle density and the fluid angular rotation frequency. These predictions were confirmed experimentally.

Online supplementary data available from stacks.iop.org/NJP/15/063036/mmedia
1. Introduction

A phenomenon of segregation into axial bands of particles in a rotating almost inviscid fluid was reported a decade ago following sets of experiments to grow dendritic crystals while they were levitated in a solution rotating at a few revolutions per second [1]. In the meantime, mixtures of fluids and granular media in a rotating horizontal tube have become a source for extensive research both experimentally and theoretically [2]. Observations show that within a short time the granular particles accumulate into periodically spaced bands along the tube, with a characteristic periodic length $\Lambda$ approximately twice the diameter $D$ of the tube. The observations indicate that the ends of the tube are situated either at the center of a band or halfway between two bands; as the length of the tube $L$ is changed, the period adjusts itself around the characteristic period so that the length is equal to an integral number of half periods [1–11]. Figure 1 shows snapshots of stationary patterns observed when the tube length, $L$, is twice the characteristic periodic length, $\Lambda$. The experiment used 3 mm polystyrene spheres suspended in water in a tube of 45 mm diameter rotating at 0.7 cycles per second. The left pair shows spaces at the ends, and the right pair shows bands at the ends.

This pattern formation has been analyzed both theoretically and experimentally by Seiden et al [9–11], who explained the phenomenon as arising from the excitation of inertial waves in the regularly rotating fluid by the disturbance by the particles. The particle bands form due to the tendency of particles sedimenting in Stokes flow to attract one another into groups, thus creating a sink that draws in additional particles and causes further clustering. The bands are synchronized spatially with the inertial wave.

Inertial waves are solutions of the Navier–Stokes equations for an incompressible fluid in a rotating frame of reference, including the Coriolis, centripetal and gravitational forces [12, 13]. These are

\[
\begin{align*}
\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} + 2 \mathbf{\Omega} \times \mathbf{u} &= \rho \mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{r}) + \rho \mathbf{g} + \mathbf{F}(t) + \nu \nabla^2 \mathbf{u}, \\
\n\\mathbf{\nabla} \cdot \mathbf{u} &= 0,
\end{align*}
\]

where $\mathbf{u}$ is the velocity field of the fluid, $\mathbf{F}$ is an applied force field and $\mathbf{g}$ is the gravity field. $\rho$ and $\nu$ are the fluid density and kinematic viscosity, respectively. Equation (1a) can be linearized by ignoring the second term on the left, which implies that the inertial forces are small compared to the Coriolis force (low Rossby number, $Ro \ll 1$). In the case where $\nu$, the kinematic viscosity, is very small compared to the Coriolis force (low Ekman number, $Ek \ll 1$), equations (1) give rise to a wave equation for the pressure field with a characteristic dispersion relation [9]. Inertial waves are transverse, and are most commonly observed in geophysical and astrophysical
Figure 1. Top (above) and front (below) views for segregation of suspended polystyrene particles in a rotating fluid into axial bands. The ends of the tube are situated either halfway between bands (left) or at the centers of bands (right).

scenarios. Rossby waves, geostrophic currents and geostrophic winds are well-known examples of inertial waves [14]. Laboratory-scale phenomena involving inertial waves are very rare.

The resulting linearized wave equation for the fluid velocity field was solved for the boundary conditions of zero radial velocity on the tube walls and zero axial velocity at the ends. The dominant mode that was stationary in the laboratory frame of reference and satisfied the boundary conditions had wavelength $\Lambda = 1.97D$, independent of the speed of rotation, which agreed with the experimental evidence for a variety of tube diameters and lengths [9]. Note that zero axial velocity of the fluid at the ends of the tube occurs either at a maximum or a minimum of the wave amplitude. When non-Brownian particles fill a small fraction of the tube volume, and settle due to gravity, in the rotating frame of reference they provide a periodic stimulus of frequency $\Omega_1$ that excites the inertial waves at the frequency of the rotation. This stimulus is provided via the fluid viscosity, which couples the fluid velocity and the particle motion. Since the tube rotation frequency and the wave frequency are identical, the rate of rotation does not affect the inertial wave pattern and the banding phenomenon. However, no effects are observed at very low frequencies, when the particles slide on the tube wall and are not lifted by the fluid, nor at high frequencies where they centrifuge out to the walls [9–11].

It was observed in many of the experiments that the particle bands, which are linked to the wave amplitude, oscillate between two allowed states at a frequency much lower than that of the rotation (see the supplementary data for a video, available online from stacks.iop.org/NJP/15/063036/mmedia). These allowed states arise because of the two possibilities for the wave amplitude at the ends of the tube that satisfy the boundary conditions. A similar effect, reported by the authors as traveling waves, was seen by Breu et al [15]. The purpose of this work is to provide a physical understanding of these oscillations. The oscillations are dominant when the time for viscously limited fall of a particle across the tube diameter is of the order of the tube rotation period. This suggests that oscillations should only exist for a limited range of viscosities. Experiments verified the suggested model, particularly observation of the induced flow pattern using microscopic tracer particles imaged with a light sheet [11].

As stated before, the theoretical mechanism involving inertial waves establishes the relationship between the specific modes of the waves and the tube dimensions when the Rossby and Ekman numbers, which relate nonlinear and viscous forces respectively to the Coriolis
forces, are much less than unity. On the other hand, the viscosity cannot be zero because it is required to couple the particle motion and the wave field.

It appears both experimentally and theoretically that two degenerate modes with the same band spacing are obtained with either a band center or a space center at each end of the tube (figure 1). At lower rotation rates, periodic oscillations between the two degenerate modes were reported [9, 15], but these have not previously been explained theoretically. In figure 2, time-sequence snapshots of the breakdown and oscillation process are demonstrated (see supplementary data, available from stacks.iop.org/NJP/15/063036/mmedia).

Based on the physical model already demonstrated in the explanation for the banding phenomenon [11], we derive a simple model to describe the mechanism behind these oscillations.

2. A dynamic model

A first step in developing an analytical model will be to make the simplifying assumption that the fluid–particle interaction can be described as a function of the axial dimension (z-axis) only, instead of attempting a more exact three-dimensional N-body model. We consider the suspended particles to be the source of a time-dependent periodic disturbance to the otherwise unperturbed bounded fluid, with frequency equal to that of the rotating tube. Using cylindrical coordinates (r, φ, z) in this derivation where z is along the cylinder axis, we define scalar wave fields representing the fluid inertial wave (FIW) pressure and the particle density, averaged over (r, φ).
Considering the complexity of the system, we must make some simplifications in searching for an analytical model for the phenomenon described:

1. We focus on the inviscid, linear regime (small Ekman and Rossby numbers).
2. We take the wave fields to be averaged over the plane $z = \text{constant}$ and look at their interactions with the particles as a function of $z$.
3. We assume that the fields change at a rate that is small compared to the rotation rate $\Omega$. This is consistent with the observations.

Following this approach, the FIW pressure field will take the form of $W(t, z) = W_0(t)e^{ikz}$, where $W_0(t)$ is the FIW pressure amplitude and $k$ is the wave number. In the same manner, we define the particle scalar density field to be $P(t, z) = \bar{P} + P_0(t)e^{ikz}$, where $\bar{P}$ is the average particle density and $P_0(t)$ is the particle density disturbance amplitude. We further define the particle flux field, averaged over the $r$–$\phi$ plane, to be $\vec{j}(t, z)$. This flux averages parts that have opposite signs depending on the values of $r$ and $\phi$.

First, the particle density $P(t, z)$ is conserved as the total number of particles in the system is fixed. Thus, the continuity equation for the particle field is

$$\nabla \cdot \vec{j} (t, z) + \frac{\partial P(t, z)}{\partial t} = 0. \quad (2)$$

Now we assume that the particle field is pumped by the periodic gravity field in the rotating frame of reference. The mechanism is basically that the sedimenting particles lose energy to the FIW pressure field by viscous (Stokes) forces, thereby transferring energy from one to the other. The strength of this interaction is represented by a constant $\alpha$ that clearly depends on fluid viscosity, size and shape of the particles, and their buoyancy. For example, if the buoyancy is reversed (as in the case of bubbles) the bands and spaces interchange [9]. Since both fields oscillate at frequency $\Omega$ in the rotating frame of reference, $P(t, z)$ is synchronized with $W(t, z)$. In addition, we introduce a damping coefficient $\beta$ that represents the viscous damping of an excited wave in the absence of particles. These assumptions suggest the following relation:

$$\frac{dW(t, z)}{dt} = \alpha P(t, z) - \beta W(t, z). \quad (3)$$

The value of $\beta$ can be derived by comparing the first and last terms of equation (1a) for a wave-like disturbance, which shows that $\beta = v/\Lambda^2 = \Omega R^2 Ek/\Lambda^2 \approx \Omega Ek/16 \ll 1$, where $Ek$ is the Ekman number.

In previous work by Seiden et al [10, 11], including both simulations and experimental results, the simulation showed how multiple suspended particles interact with the inertial wave field. The interaction manifests itself in the repulsion of the particles by the FIW field maxima as a result of the pressure gradient, thus giving

$$-\vec{j} = \gamma P(t, z) \nabla W(t, z), \quad (4)$$

where $\gamma$ represents the interaction between the FIW pressure field and the particle field. This multiparticle–fluid interaction is obviously complicated but depends on the particle properties and is expected to be inversely proportional to the viscosity of the fluid. This is because the particle flux $j$ gets smaller as we increase the viscosity of the fluid. This effect is similar to that which occurs in the Chladni plate experiment where sand particles migrate to the nodes of the transverse waves of the vibrating plate.
Using the definitions for \( W(t,z) \) and \( P(t,z) \) with equation (4), we obtain

\[
-j = i k \gamma \left( \bar{P} + P_0(t) \right) e^{ikz} W_0(t) e^{ikz}
\]

\[
= i k \gamma \bar{P} W_0(t) e^{ikz} + i k \gamma P_0(t) W_0(t) e^{ikz}.
\]

(5)

Linearizing by neglecting the higher harmonic \( e^{i2kz} \), we approximate the particle flux field to be

\[
-j \simeq i k \gamma \bar{P} W_0(t) e^{ikz}.
\]

(6)

Now, substituting (6) in the continuity equation (2) results in

\[
\frac{dP_0(t)}{dt} = -k^2 \gamma \bar{P} W_0(t).
\]

(7)

Differentiating (3) with respect to time and using (7) and (3) again we thus obtain

\[
\frac{d^2W_0(t)}{dt^2} e^{ikz} + \left( \alpha \gamma k^2 \bar{P} - \beta^2 \right) W_0(t) e^{ikz} + \beta \alpha \bar{P} + \beta \alpha P_0(t) e^{ikz} = 0.
\]

(8)

The term \( \beta \alpha \) is proportional to the square of the viscosity and therefore can be neglected. Defining \( \bar{P} = N_0/L \), where \( N_0 \) is the number of particles and \( L \) is the tube length, gives

\[
\frac{d^2W_0(t)}{dt^2} + \left( \alpha \gamma k^2 \frac{N_0}{L} - \beta^2 \right) W_0(t) = 0,
\]

(9)

which is a harmonic oscillator equation for the FIW pressure amplitude with a frequency of \( \sqrt{\frac{\alpha \gamma k^2 N_0}{L} - \beta^2} \).

The coefficients \( \alpha \) and \( \beta \) are proportional to the viscosity and \( \gamma \) is inversely proportional to the viscosity, so we expect \( \alpha \gamma \) in the expression for the frequency to be independent of viscosity. The \( \beta^2 \) term, as expected, reduces the frequency slightly and can be neglected in the case of an inviscid fluid; however, if the viscosity is large the oscillations become over-damped. This results in a time period of oscillation that is proportional to the inverse square root of the density of particles \( N_0/L \). From dimensional considerations, we assume a relation to the only time-like variable which is the rotation frequency, \( \Omega \), thus giving

\[
T_0 \propto \sqrt{\frac{L}{\Omega^2 N_0}}.
\]

(10)

3. Experimental verification

We conducted a series of experiments to confirm the dependence of the time period on the number of particles \( N_0 \) and tube length \( L \). The experiments were performed using a glass tube of internal diameter 45 mm, rotated about a horizontal axis by a variable speed motor. The length of the sample within the tube could be adjusted by means of a Teflon plug. The experiments were carried out using polystyrene spheres 3 mm in diameter whose total number could be changed. The quantitative experiments were carried out using water at 20 °C although qualitative experiments were carried out using glycerol–water mixtures. In figure 3, the apparatus is shown schematically.

In figure 4, we present the experimental data with a fit to \( \Omega^2 N_0 \). The range of \( N_0 \) was 70–285 polystyrene spheres and oscillations were observed at frequencies \( \Omega \) in the range of...
0.6–0.8 Hz. The rest of the parameters (tube length, tube radius, size of particles, fluid viscosity, etc) were kept at a constant value.

The fit to the experimental data gave us the relation

\[ T_0 \propto (\Omega^2 N_0)^{-0.475 \pm 0.024}, \]  

(11)

which is in good agreement with the prediction by the model presented.

The dependence on the tube length was evaluated using the same apparatus with a change of the tube length \( L \) and rotation frequency \( \Omega \) while keeping the number of particles \( N_0 \) at a constant value of 285 polystyrene balls. The rest of the parameters (tube radius, size of particles, fluid viscosity, etc) were also kept at a constant value. In figure 5 we present the experimental data with a fit to \( \Omega^{-2} L \). The range of \( L \) was 80–320 mm and oscillations were observed at frequencies \( \Omega \) in the range of 0.6–0.75 Hz.

The fit to the experimental data gave us the relation

\[ T_0 \propto (L / \Omega^2)^{0.462 \pm 0.033}, \]  

(12)

which is in good agreement with the prediction by the model presented.

Figure 3. The apparatus. Rotation frequency \( \Omega \), tube length \( L \) and diameter \( D \).

Figure 4. Logarithmic plot of the time period, \( T \), versus \( \Omega^2 N_0 \) for the oscillations between two states where \( L = 163 \) mm, \( R = 22.5 \) mm for water (\( \sim 1 \) cP).
Figure 5. Logarithmic plot of the time period, $T$, versus $\Omega^{-2}L$ for the oscillations between two states where $N_0 = 285$ polystyrene balls, $R = 22.5$ mm for water ($\sim 1$ cP).

Following the work done by Seiden et al [10], qualitative observations were made with several values of viscosity greater than that of water. We observed that as the viscosity was increased we needed to increase $\Omega$ in order to induce the oscillations in the otherwise stationary banding pattern. When the viscosity was greater than 2.5 cP, the oscillations were damped out, indicating that the oscillation phenomenon occurs in a limited range of viscosity as our model suggests.

To conclude, we have devised a model introducing fluid–particle interaction to address the problem of oscillations between two allowed states of axial bands in a rotating fluid. We predicted that the time period of the oscillations relates to the rotation frequency of the fluid and to the number of particles per unit length, and confirmed it experimentally. Although the model relates only to average values of the fields across the $r-\phi$ plane, it appears to provide a physically intuitive model for these intriguing oscillations.

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