Disproof of a conjecture on the main spectrum of generalized Bethe trees

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Abstract

An eigenvalue of the adjacency matrix of a graph is said to be main if the all-ones vector is not orthogonal to its associated eigenspace. A generalized Bethe tree with \(k\) levels is a rooted tree in which vertices at the same level have the same degree. França and Brondani [On the main spectrum of generalized Bethe trees, Linear Algebra Appl., 628 (2021) 56-71] recently conjectured that any generalized Bethe tree with \(k\) levels has exactly \(k\) main eigenvalues whenever \(k\) is even. We disprove the conjecture by constructing a family of counterexamples for even integers \(k \geq 6\).

Keywords: Generalized Bethe tree; main eigenvalue; divisor matrix; equitable partition.
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1 Introduction

Let \(G\) be a simple graph with vertex set \(\{1, \ldots, n\}\). The adjacency matrix of \(G\) is the \(n \times n\) symmetric matrix \(A = A(G) = (a_{i,j})\), where \(a_{i,j} = 1\) if \(i\) and \(j\) are adjacent; \(a_{i,j} = 0\) otherwise. We often identify a graph \(G\) with its adjacency matrix \(A\). For example, the eigenvalues and the eigenspaces of \(A\) are called the eigenvalues and the eigenspaces of \(G\). Let \(e_n\) denote the \(n\)-dimensional all-ones vector. An eigenvalue of \(G\) is called a main eigenvalue if the associated eigenspace is not orthogonal to \(e_n\). The main spectrum of \(G\) is the set of all (distinct) main eigenvalues of \(G\). The notion of main eigenvalue was introduced by Cvetković [1] and has received considerable attention since then; see [6] for a survey.

Let \(k \geq 2\) and \(d_1, d_2, \ldots, d_{k-1}\) be \(k-1\) integers of at least 2. A generalized Bethe tree \(\mathcal{B}(d_1, d_2, \ldots, d_{k-1})\) with \(k\) levels is a rooted tree in which all vertices at the level \(i\) have the degree \(d_i\) for \(i = 1, 2, \ldots, k-1\). If \(d_2 = d_3 = \cdots = d_{k-1} = d + 1\) and \(d_1 = d\), then the generalized Bethe tree \(\mathcal{B}(d_1, d_2, \ldots, d_{k-1})\) becomes an ordinary Bethe tree, and is denoted by \(\mathcal{B}_{d,k}\). If \(d_1 = d_2 = \cdots = d_{k-1} = d\) then \(\mathcal{B}(d_1, d_2, \ldots, d_{k-1})\) is denoted by \(\mathcal{Q}_{d,k}\) and is called the quasi-regular complete tree [2]. Figure 1 shows some examples of generalized Bethe trees.

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Figure 1: Generalized Bethe trees $\mathcal{B}_{2,4}, \mathcal{Q}_{3,4}$ and $\mathcal{B}(3,3,2)$.

Note that a generalized Bethe tree with two levels is a star graph, which is known to have exactly two main eigenvalues. França and Brondani extended this result for both Bethe trees and quasi-regular complete trees.

**Theorem 1** ([4]). For any $k \geq 3$ and $d \geq 2$, both $\mathcal{B}_{d,k}$ and $\mathcal{Q}_{d,k}$ have exactly $k$ main eigenvalues.

It is known that Theorem 1 cannot be extended to all generalized Bethe trees. Indeed, for any positive integer $\alpha \geq 2$, the generalized Bethe tree $\mathcal{B}(\alpha^2 - \alpha + 1, \alpha)$ has exactly two main eigenvalues while it has three levels; see [5]. França and Brondani [4] conjectured that such an inconsistency will never happen if the number of levels of the concerned generalized Bethe tree is even.

**Conjecture 1** ([4]). Let $k$ be even. Every generalized Bethe tree with $k$ levels has exactly $k$ main eigenvalues.

Note that Conjecture 1 trivially holds for $k = 2$. The main aim of this note is to show that Conjecture 1 fails for each even $k \geq 6$. Indeed, we give a family of counterexamples to Conjecture 1.

**Theorem 2.** Let $k \geq 6$ be even. Then $\mathcal{B}(5, k - 3, 5, 3, 2, \ldots, 2)$ with $k$ levels has at most $k - 1$ main eigenvalues.

The counterexample $\mathcal{B}(5, 3, 5, 3, 2)$ constructed in Theorem 2 for $k = 6$ is shown in Figure 2, where the solid circle represents the root vertex. The proof of Theorem 2 will be presented in the next section using the tool of equitable partitions. We remark that Conjecture 1 is still open for the only remaining case that $k = 4$.

## 2 Proof of Theorem 2

Let $\Pi = \{C_1, C_2, \ldots, C_k\}$ be a partition of the vertex set $V(G)$. We say that $\Pi$ is an *equitable partition* of $G$ if for all $i, j \in \{1, 2, \ldots, k\}$, there is a constant $b_{ij}$ such that every vertex in $C_i$ has exactly $b_{ij}$ neighbors in $V_j$. The *divisor* (or *quotient graph*) of $G$ with respect to $\Pi$, denoted by $G/\Pi$, is the directed multigraph with $k$ vertices $C_1, C_2, \ldots, C_k$ and $b_{ij}$ arcs from $C_i$ to $C_j$. The $k \times k$ matrix $(b_{ij})$ is called the *divisor matrix* of $\Pi$, denoted by $A(G/\Pi)$. The *characteristic matrix* of $\Pi$, denoted by $C$, is the $n \times k$ matrix with the
characteristic vectors of the cells $C_i$’s as its columns. Note that $C^TC$ is the $k \times k$ diagonal matrix $\text{diag} (|C_1|,|C_2|,\ldots,|C_k|)$, which is clearly invertible as each $|C_i|$ is nonzero.

The following lemma collects some basic results on divisors.

**Lemma 1** ([3]). Let $\Pi = \{C_1,C_2,\ldots,C_k\}$ be an equitable partition of the graph $G$, with characteristic matrix $C$. Let $A = A(G)$ and $B = A(G/\Pi)$. Then

(i) $AC = CB$;
(ii) $\det(xI - B)$ divides $\det(xI - A)$ and hence each eigenvalue of $B$ is an eigenvalue of $A$;
(iii) Each main eigenvalue of $A$ is an eigenvalue of $B$.

**Lemma 2.** Using the notations of Lemma 1, and letting $D = C^TC = \text{diag} (|C_1|,|C_2|,\ldots,|C_k|)$, we have $DBD^{-1} = B^T$.

**Proof.** Let $b'_{ij}$ and $b_{ij}$ denote the $(i,j)$-th entry of $DBD^{-1}$ and $B$, respectively. Noting that $D^{-1} = \text{diag} \left( \frac{1}{|C_1|}, \frac{1}{|C_2|}, \ldots, \frac{1}{|C_k|} \right)$, we see that

$$b'_{ij} = \frac{b_{ij}|C_i|}{|C_j|}.$$ 

As $\Pi$ is an equitable partition, we see that $b_{ij}|C_i|$ equals $b_{ji}|C_j|$ since both count the number of edges with one end in $C_i$ and the other in $C_j$. Thus $b'_{ij} = b_{ji}$ and the lemma follows. \qed

**Corollary 1.** Using the notations of Lemma 1, and letting $\lambda$ be an eigenvalue of $B$ (or equivalently, $B^T$), the following two statements are equivalent:

(i) $B$ has an eigenvector associated with $\lambda$ which is not orthogonal to the vector $(|C_1|,\ldots,|C_k|)^T$.
(ii) $B^T$ has an eigenvector associated with $\lambda$ which is not orthogonal to $e_k$.

**Proof.** Let $\xi$ be a vector such that $B\xi = \lambda\xi$ and $(|C_1|,\ldots,|C_k|)\xi \neq 0$. Write $D = C^TC$. Then by Lemma 2, we see that $DB = B^TD$ and hence $B^TD\xi = DB\xi = \lambda D\xi$. Noting that $e_k^TD = (|C_1|,\ldots,|C_k|)$, we have $e_k^T(D\xi) = (|C_1|,\ldots,|C_k|)\xi \neq 0$. This means that $D\xi$ is an eigenvector of $B^T$ associated with $\lambda$ which is not orthogonal to $e_k$. Thus, (i) implies (ii).
Conversely, let \( \eta \) be a vector such that \( B^T\eta = \lambda \eta \) and \( e_k^T\eta \neq 0 \). Similarly, \( BD^{-1} = D^{-1}B^T \) and hence \( D^{-1}\eta \) is an eigenvector of \( B \) associated with \( \lambda \). Moreover, we have \((|C_1|, \ldots, |C_k|)D^{-1}\eta = e_k^T\eta \neq 0 \). Thus (ii) implies (i) and the proof is complete. \( \square \)

The following definition was first introduced by Teranishi [7].

**Definition 1** ([7]). Let \( \Pi = \{C_1, C_2, \ldots, C_k\} \) be an equitable partition of the graph \( G \). An eigenvalue \( \lambda \) of the divisor matrix \( A(G/\Pi) \) is called a *main eigenvalue* of \( G/\Pi \) if the associated eigenspace of \((A(G/\Pi))^T \) is not orthogonal to the all-ones vector \( e_k \).

**Remark 1.** The original definition of main eigenvalues of divisors given in [7] requires both conditions as described in Corollary 1. But as these two conditions are indeed equivalent, the current simplified definition is essentially equivalent to the original one.

We need the following key lemma due to Teranishi [7], which refines the last two assertions of Lemma 1. We give a simpler proof for the convenience of readers.

**Lemma 3** ([7]). Let \( G \) be a graph with an equitable partition \( \Pi = \{C_1, \ldots, C_k\} \). Then \( G \) and \( G/\Pi \) have the same main spectrum.

*Proof.* Let \( A = A(G) \), \( B = A(G/\Pi) \) and \( C \) be the characteristic matrix of \( \Pi \). By Lemma 1(i), we have

\[
AC = CB. \tag{1}
\]

Let \( \lambda \) be a main eigenvalue of \( G \). Then there exists a vector \( \xi \in \mathbb{R}^n \) such that \( A\xi = \lambda \xi \) and \( e_k^T\xi \neq 0 \). Taking transpose for both sides of (1) and noting that \( A \) is symmetric, we have \( C^TA = B^TC^T \) and hence \( B^TC^T\xi = C^TA\xi = \lambda C^T\xi \). Write \( \eta = C^T\xi \). Then \( B^T\eta = \lambda \eta \).

Moreover, as \( Ce_k = e_n \) and \( e_n^T\xi \neq 0 \), we have

\[
e_k^T\eta = e_k^T(C^T\xi) = (Ce_k)^T\xi = e_n^T\xi \neq 0.
\]

This indicates that \( \lambda \) is a main eigenvalue of \( G/\Pi \).

Conversely, let \( \lambda \) be a main eigenvalue of \( G/\Pi \). By Corollary 1 there exists a \( \eta \in \mathbb{R}^k \) such that \( B\eta = \lambda \eta \) and \((|C_1|, \ldots, |C_k|)^T\eta \neq 0 \). Write \( \xi = C\eta \). It follows from (1) that

\[
A\xi = AC\eta = CB\eta = \lambda C\eta = \lambda \xi.
\]

Moreover, as \( e_n^TC = (|C_1|, \ldots, |C_k|) \), we have

\[
e_n^T\xi = e_n^TC\eta = (|C_1|, \ldots, |C_k|)\eta \neq 0.
\]

This proves that \( \lambda \) is a main eigenvalue of \( G \). The proof is complete. \( \square \)

Now we are ready to prove Theorem 2. Let \( G_k = B(5, k - 3, 5, 3, 2, 2, \ldots, 2) \), where \( k \) is even and \( k \geq 6 \). Let \( \Pi_k = \{C_1, C_2, \ldots, C_k\} \) be a partition \( V(G_k) \), where \( C_i \) collects all vertices at the \( \hat{i} \)-th level of the generalized Bethe tree \( G_k \). Clearly, \( \Pi_k \) is an equitable partition and the corresponding divisor matrix \( B \) is the following tridiagonal matrix:
Let $\xi = (1, -2, -1, 2(k - 3), -4(k - 4), 4(k - 5), -4(k - 6), \ldots, -8, 4)^T$. Note that $k$ is even and the last $k - 4$ entries of $\xi$ have alternating signs. Direct calculation shows that

$$
(2I + B^T)\xi = \begin{pmatrix}
2 & 1 & 0 & 0 & 0 & \ldots & 0 \\
5 & 2 & 1 & 0 & 0 & \ldots & 0 \\
0 & 2 & 1 & 0 & 0 & \ldots & 0 \\
0 & 0 & 2 & 1 & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & 0 & 0 & \ldots & 2(k - 3) \\
0 & 0 & 0 & 0 & 0 & \ldots & -4(k - 4) \\
0 & 0 & 0 & 0 & 0 & \ldots & 4(k - 5) \\
0 & 0 & 0 & 0 & 0 & \ldots & -4(k - 6) \\
0 & 0 & 0 & 0 & 0 & \ldots & 8 \\
0 & 0 & 0 & 0 & 0 & \ldots & 4 \\
\end{pmatrix}
\begin{pmatrix}
0 \\
1 \\
-2 \\
-1 \\
2(k - 3) \\
-4(k - 4) \\
4(k - 5) \\
-8 \\
4 \\
\end{pmatrix}
= \begin{pmatrix}
0 \\
0 \\
0 \\
0 \\
\end{pmatrix}
$$

that is, $B^T\xi = -2\xi$. This indicates that $\lambda = -2$ is an eigenvalue of $B^T$. Moreover, as the sum of the last $k - 4$ entries of $\xi$ is $-2(k - 4)$, we see that

$$
e_k^T\xi = 1 - 2 - 1 + 2(k - 3) - 2(k - 4) = 0. \quad (2)
$$

Note that $G_k/\Pi_k$ has exactly $k$ vertices. We claim that the divisor $G_k/\Pi_k$ has at most $k - 1$ main eigenvalues. Otherwise all the $k$ eigenvalues of $G_k/\Pi_k$ must be simple and main. Consider the eigenvalue $\lambda = -2$. Then the associated eigenspace is one-dimensional and is spanned by $\xi$. Thus, by [2], we see that $\lambda = -2$ is not a main eigenvalue, a contradiction.

By Lemma [3] we know that $G_k$ and $G_k/\Pi_k$ have the same number of main eigenvalues. Thus, $G_k$ has at most $k - 1$ main eigenvalues. This completes the proof of Theorem [2].

**Declaration of competing interest**

There is no conflict of interest.

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