Group extensions and marginal series of pair of groups

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Abstract. In this article, using the concept of generalized Baer-invariant of a pair of groups, we establish some related isomorphisms between lower marginal quotient pairs of groups, which are generalized versions of some isomorphisms of Stallings. We also derive a result for the pair \((V, W, X)\) to be an ultra Hall pair for special varieties of groups. This result generalizes that of Fung in 1977, which has roots in Philip Hall’s criterion on nilpotency.

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1. Introduction

Let \( \mathcal{V} \) be a variety of groups defined by the set of words (laws) \( V \). Then for a given group \( G \) two subgroups \( V(G) \) and \( V^*(G) \) correspond to this variety are defined as follows:

\[
   V(G) = \langle v(g_1, g_2, \ldots, g_r) \mid g_i \in G, \; v \in V, \; 1 \leq i \leq r \rangle,
\]
\[
   V^*(G) = \{ a \in G \mid v(g_1, g_2, \ldots, g_i a, \ldots, g_r) = v(g_1, g_2, \ldots, g_r); \; g_j \in G, \; v \in V, \; 1 \leq i, j \leq r \},
\]

which are called the verbal and marginal subgroups of \( G \), and these are fully invariant and characteristic subgroups of \( G \) respectively; see [4, 8] for notion of variety of groups. Let \( N \) be a normal subgroup of \( G \). Then we define \( V(N, G) \) to be the subgroup of \( G \) generated by the following set:

\[
   \{ v(g_1, g_2, \ldots, g_i n, \ldots, g_r) v(g_1, g_2, \ldots, g_r)^{-1} \mid v \in V, \; g_j \in G, \; 1 \leq i, j \leq r, \; n \in N \}.
\]

This is the least normal subgroup \( T \) of \( G \) contained in \( N \) such that \( N/T \) is contained in \( V^*(G/T) \). Also \( V^*(N, G) \) is defined as \( N \cap V^*(G) \).

The following preliminary lemma gives the basic properties of these subgroups; see [4] for further information.

Lemma 1. Let \( \mathcal{V} \) be a variety of groups defined by the set of words \( V \) and \( N \) be a normal subgroup of a given group \( G \). Then
Lemma 3. Let $V$ be a set of words, $K$ and $N$ be two normal subgroups of a group $G$ such that $K$ is contained in $N$. Then

(i) $G \in V \iff V(G) = \{1\} \iff V^*(G) = G$,
(ii) $V(G/N) = V(G)/N$ and $V^*(G/N) \supseteq V^*(G)N/N$,
(iii) $N \subseteq V^*(G) \iff V(N, G) = \{1\}$,
(iv) $V(N) \subseteq V(N, G) \subseteq N \cap V(G)$. In particular, $V(G) = V(G, G)$,
(v) $V(V^*(G)) = \{1\}$ and $V^*(G/V(G)) = G/V(G)$.

The following similar lemma is straightforward.

**Lemma 2.** Let $V$ be a set of words, $K$ and $N$ be two normal subgroups of a group $G$ such that $K$ is contained in $N$. Then

(i) $V(V^*(N, G), G) = 1$, in particular $V(N, G) = 1$ if and only if $V^*(N, G) = N$,
(ii) $K \leq V^*(N, G)$ if and only if $f(K, G) = 1$,
(iii) $V(N/K, G/K) = V(N, G)K/K$.

In 1998, Ellis introduced the concept of pair of groups $(G, N)$, where $N$ is normal subgroup of a group $G$. He also established some related (co)homological and topological properties.

Let $(G, N)$ and $(H, K)$ be two pairs of groups. Then $(f, f') : (G, N) \to (H, K)$ is a homomorphism if $f : G \to H$ is homomorphism and $f(N) \subseteq K$. The series

$$N \geq N_0 \geq N_1 \geq \cdots \geq N_r \geq \cdots$$

is said to be $\mathcal{V}_G$-marginal series of $N$, or $\mathcal{V}$- marginal series of the pair $(G, N)$ if $N_i \leq G$ and $N_i/N_{i+1} \subseteq V^*(G/N_{i+1})$, for $i \geq 0$. The subgroup $N$ is said to be $\mathcal{V}_G$-nilpotent or, the pair $(G, N)$ is said to be $\mathcal{V}$-nilpotent if $N_r = 1$ for a positive integer $r$. The least such $r$ is called the $\mathcal{V}_G$-nilpotency class of $N$ or $\mathcal{V}$-nilpotency class of the pair $(G, N)$.

We have the following two series

$$N = V_0(N, G) \geq V_1(N, G) \geq \cdots \geq V_i(N, G) \geq \cdots,$$

where $V_i(N, G) = V(N, G)$ and $V_i(N, G) = V(V_{i-1}(N, G), G)$, for $i \geq 1$, which is called the lower $\mathcal{V}$-marginal series of $(G, N)$. The upper $\mathcal{V}$-marginal series of $(G, N)$ is defined as

$$1 = V_0^+(N, G) \leq V_1^+(N, G) \leq \cdots \leq V_i^+(N, G) \leq \cdots,$$

where $V_i^+(N, G) = V^+(N, G)$ and

$$V_{i+1}^+(N, G)/V_i^+(N, G) = V^+(N/V_i^+(N, G), G/V_i^+(N, G)), \quad i \geq 1.$$

If one puts $N = G$, then he concept of $\mathcal{V}$-marginal series and $\mathcal{V}$-nilpotency of $G$ is obtained; see [2, 9]. In addition if $V = \{\gamma_2\}$, where $\gamma_2 = [x_1, x_2] = x_1^{-1}x_2^{-1}x_1x_2$, i.e. $\mathcal{V}$ is the variety of abelian groups, one obtains the usual concepts of central series and nilpotency; see [12].

We need the following technical lemma.

**Lemma 3.** Let $V$ be a variety of groups defined by the set of words $V$, $(G, N)$ be a pair of groups and let $N = N_0 \geq N_1 \geq \cdots \geq N_r \geq \cdots$ be a $\mathcal{V}$-marginal series of $(G, N)$. Then

(i) $V_i(N, G) \leq N_i$, $i \geq 0$,
(ii) If $c$ is the class of $\mathcal{V}$-nilpotency of $(G, N)$, then $N_{c-i} \leq V_i^+(N, G)$ and hence

$$V_i(N, G) \leq N_i \leq V_{c-i}^+(N, G), \quad 0 \leq i \leq c.$$

Let $G$ be an arbitrary group and $1 \to R \to F \xrightarrow{\pi} G \to 1$ be a free presentation of $G$. Then the Baer-invariant of the group $G$ with respect to the variety $\mathcal{V}$, is defined by

$$\mathcal{V} M(G) = \frac{R \cap V(F)}{V(R, F)}$$

which is abelian and independent of the choice of free presentation of $G$; see [7].

If $\mathcal{V}$ is the variety of abelian groups, then the Baer-invariant of the group $G$ will be $R \cap F'/[R, F]$, which by Hopf’s formula is the Schur multiplier $M(G)$ of the group $G$ and is isomorphic to $H_2(G)$. 

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the second homology group of $G$; see [5, 13, 14], see also [11] for $c$-nilpotent multiplier of Lie algebras.

In 1998 Ellis [1], introduced the concept of Schur multiplier of a pair of groups $(G, N)$, where $N$ is a normal subgroup of $G$, as

$$M(G, N) = \frac{R \cap [S, F]}{[R, F]}$$

in which $N \cong S/R$ for a suitable normal subgroup $S$ of $F$, i.e. $S = \pi^{-1}(N)$. The Baer-invariant of the pair $(G, N)$ with respect to the variety $\mathcal{V}$ is defined by

$$\mathcal{V}M(G, N) = \frac{R \cap V(S, F)}{V(R, F)}.$$

Clearly if $N = G$, then $M(G, G) = M(G)$ and $\mathcal{V}M(G, G) = \mathcal{V}M(G)$.

In 1976 Leedham-Green and McKay [7], introduced the concept of the generalized Baer-invariant of a pair of groups, the generalized Baer-invariant of the pair $(G, N)$ with respect to the variety $\mathcal{V}$ is defined by

$$\mathcal{W}\mathcal{V}M(G, N) = \frac{R \cap W(S, F)W(F)}{V(R, F)W(F)}.$$

which is also abelian and independent of the choice of the free presentation of $G$. Similar to the Baer-invariant of the pair, the generalized Baer-invariant of the pair $(G, N)$, where $G \in \mathcal{W}$, with respect to the variety $\mathcal{V}$ is defined by

$$\mathcal{W}\mathcal{V}M(G, N) = \frac{(R \cap V(S, F))W(F)}{V(R, F)W(F)}.$$

If one puts $\mathcal{W}$ variety of all groups, then $W(F) = \{1\}$. Thus $\mathcal{W}\mathcal{V}M(G) = \mathcal{V}M(G)$ and $\mathcal{W}\mathcal{V}M(G, N) = \mathcal{V}M(G, N)$; see [7, 10].

In Section 2 we get a generalized version of the well-known 5-term exact sequence of homology groups and then obtain some isomorphisms between lower marginal factors of pairs of groups, under special conditions. In Section 3, we study $\mathcal{V}$-nilpotency of the pair $(G, N)$ and then derive a result which has roots in the Philip Hall’s criterion on nilpotency.

2. Homological methods and generalized Baer-invariant of pair of groups

In this section using the concept of generalized Baer-invariant of a pair of groups, we obtain a generalization of well-known 5-term exact sequence and then we establish some isomorphisms which are wide generalization of some results of Stallings [15]. The following main result generalizes [9, Theorem 3.2] extensively; see also [5].

**Theorem 4.** Let $\mathcal{V}$ and $\mathcal{W}$ be varieties of groups defined by the set of laws $V$ and $W$, respectively, and $E \in \mathcal{W}$. If $1 \to N \xrightarrow{\pi} E \xrightarrow{\pi'} G \to 1$ is a group extension and $L$ is a normal subgroup of $E$ such that $1 \to N \xrightarrow{i} L \xrightarrow{\pi'} M \to 1$ is a group extension which $i$ is the inclusion map, then the following sequence is exact:

$$\mathcal{W}\mathcal{V}M(E, L) \xrightarrow{\psi} \mathcal{W}\mathcal{V}M(G, M) \xrightarrow{\varphi} \frac{N}{V(N, E)} \xrightarrow{\alpha} \frac{L}{V(L, E)} \xrightarrow{\pi'} \frac{M}{V(M, G)} \to 1.$$
Proof. We define the following maps
\[ \pi' : \begin{array}{ccc} L & \rightarrow & M \\ V(L,E) & \rightarrow & V(M,G) \\ xV(L,E) & \rightarrow & \pi(x)V(M,G) \end{array}, \quad \sigma : \begin{array}{ccc} N & \rightarrow & L \\ V(N,E) & \rightarrow & V(L,E) \\ nV(N,E) & \rightarrow & nV(L,E). \end{array} \]
Clearly, \( \pi' \) is an epimorphism with the kernel \( \frac{V(N,E)}{V(L,E)} \). The image and the kernel of \( \sigma \) are \( \frac{V(N,E)}{V(L,E)} \) and \( \frac{V(N,E)}{V(L,E)} \), respectively. So, the exactness at \( L \) and \( M \) follows immediately. Now, let \( 1 \rightarrow R \rightarrow F \rightarrow E \rightarrow 1 \) be a free presentation of \( E \) and \( L \equiv T \oplus R \) for a normal subgroup \( T \) of the free group \( F \). Then \( \pi \circ \pi_1 : F \rightarrow G \) is a free presentation of \( G \). Put ker \( \pi \circ \pi_1 = S \). Therefore, \( S \) is the inverse image of \( N \) under \( \pi_1 \). Hence, \( R \subseteq S \subseteq T, N \subseteq S/R \) and \( M \subseteq T/S \). Also,
\[ W(V(E,L) = \frac{(R \cap V(T,F))W(F)}{V(R,F)W(F)}, \quad W(V(G,M) = \frac{(S \cap V(T,F))W(F)}{V(S,F)W(F)}. \]
Now, we define the maps
\[ \varphi : W(V(M,G,M) \rightarrow \frac{N}{V(N,E)}, \quad \psi : W(V(M,E,L) \rightarrow W(V(M,G,M), \]
\[ xV(S,F)W(F) \rightarrow \pi_1(x)V(N,E) \quad xV(R,F)W(F) \rightarrow \pi(x)V(S,F)W(F). \]
It can be easily checked that the image of \( \varphi \) is \( \frac{NV(N,E)}{V(N,E)} \) which is the same as the kernel of \( \sigma \). Also, the kernel of \( \varphi \) is \( \frac{(R \cap V(T,F))V(S,F)W(F)}{(V(S,F)W(F)} \) which is the same as the image of \( \psi \). Thus, the sequence is exact and the proof is completed. \( \Box \)

The above lemma has the following important corollary, which generalizes [15, Theorem 2.1].

Corollary 5. Let \( G \) be a group with two normal subgroups \( K \) and \( N \) such that \( K \subseteq N \). Then the following sequence is exact:
\[ W(V(M,G,N) \rightarrow W(V(M,G/K,N/K) \rightarrow \frac{K}{V(K,G)} \rightarrow \frac{N}{V(N,G)} \rightarrow \frac{N}{V(N,G)K} \rightarrow 1. \]

By using Corollary 5, we have the following theorem, which generalizes [5, Theorem 7.9.1]; see also [15, Theorem 3.4].

Theorem 6. Let \( (f, f) : (G, N) \rightarrow (H, K) \) be a homomorphism, where \( G, H \in \mathcal{W} \). Suppose \( f \) induces isomorphisms \( f_0 : G/N \rightarrow H/K \) and \( f_1 : N/V(N,G) \rightarrow K/V(K,H) \), and that \( f_* : W(V(M,G,N) \rightarrow W(V(M,H,K) \) is an epimorphism. Then \( f \) induces isomorphisms
\[ (f_n, f_n) : (G/V_n(N,G), N/V_n(N,G)) \rightarrow (H/V_n(K,H), K/V_n(K,H)), \quad \forall n \geq 0. \]

Proof. Let us define \( P_n = V_n(N,G) \) and \( Q_n = V_n(K,H) \). We proceed by induction. For \( n = 0 \), the assertion is trivial. For \( n = 1 \), consider the following diagram:
\[ \begin{array}{ccc} 1 & \rightarrow & N/V(N,G) \rightarrow G/V(N,G) \rightarrow G/N \rightarrow 1 \\ & \downarrow f_1 & \downarrow f_1 & \downarrow f_0 \\ 1 & \rightarrow & K/V(K,H) \rightarrow H/V(K,H) \rightarrow H/K \rightarrow 1. \end{array} \]
By the hypothesis, \( f_1 \) and \( f_0 \) are isomorphism. Hence, \( f_1 \) is an isomorphism. Assume that \( n \geq 2 \). By considering Corollary 5, we can conclude the following diagram:
\[ \begin{array}{ccc} W(V(M,G,N) \rightarrow W(V(M/P_n-1,G/P_n-1, N/P_n-1) \rightarrow P_n/P_n-1 \rightarrow N/V(N,G) \rightarrow N/V(N,G)P_n-1 \rightarrow 1 \\ \downarrow \alpha_1 \downarrow \alpha_2 \downarrow \alpha_3 \downarrow \alpha_4 \downarrow \alpha_5 \\ W(V(M,H,K) \rightarrow W(V(M/Q_n-1,G/Q_n-1, K/Q_n-1) \rightarrow Q_n-1/Q_n \rightarrow K/V(K,H) \rightarrow K/V(K,H)Q_n-1 \rightarrow 1. \end{array} \]
Note that the naturality of the map \( f \) induces homomorphisms \( \alpha_i, i = 1, 2, \ldots, 5 \) such that the above diagram is commutative. By hypothesis, \( \alpha_1 \) is an epimorphism, \( \alpha_4 \) and \( \alpha_5 \) are isomorphisms. Also, by considering the induction hypothesis and definition of the Baer-invariant of the
pair of groups, $\alpha_2$ is an isomorphism. Hence, by the well-known five lemma, $\alpha_3$ is an isomorphism. Now, consider the following diagram:

\[
\begin{array}{cccccc}
1 & \longrightarrow & P_{n-1}/P_n & \longrightarrow & N/P_n & \longrightarrow & N/P_{n-1} & \longrightarrow & 1 \\
\bigg\downarrow_{\alpha_3} & & \bigg\downarrow_{f_n/1} & & \bigg\downarrow_{f_n-1/1} & & \\
1 & \longrightarrow & Q_{n-1}/Q_n & \longrightarrow & K/Q_n & \longrightarrow & K/Q_{n-1} & \longrightarrow & 1.
\end{array}
\]

By the above discussion, $\alpha_3$ is an isomorphism and by induction hypothesis, $f_{n-1}|$ is an isomorphism. Therefore, $f_n|/1$ is an isomorphism. Finally, by the following diagram:

\[
\begin{array}{cccccc}
1 & \longrightarrow & N/P_n & \longrightarrow & G/P_n & \longrightarrow & G/N & \longrightarrow & 1 \\
\bigg\downarrow_{f_n/1} & & \bigg\downarrow_{f_n} & & \bigg\downarrow_{f_1} & & \\
1 & \longrightarrow & K/Q_n & \longrightarrow & H/Q_n & \longrightarrow & H/K & \longrightarrow & 1
\end{array}
\]

and in the same way, $f_n$ is an isomorphism.

Now we obtain the following corollary, which generalizes [15, Corollary 3.5] and [9, Corollary 3.4].

**Corollary 7.** Let $(f, f') : (G, N) \rightarrow (H, K)$ be a homomorphism which satisfies the hypotheses of Theorem 6. Suppose further that $(G, N)$ and $(H, K)$ are $\mathcal{V}$-nilpotent. Then $(f, f')$ is an isomorphism.

**Proof.** There exists $n \geq 0$ such that $V_n(N, G) = \{1\}$ and $V_n(K, H) = \{1\}$. So, the assertion follows from Theorem 6.

As a final result we have the following theorem, which is of interest in its own account.

**Theorem 8.** Let $(f, f') : (G, N) \rightarrow (H, K)$ be an epimorphism of pairs of groups, where $G, H \in \mathcal{W}$. Let $(G, N)$ be a $\mathcal{V}$-nilpotent pair. If $\ker f \subseteq V(N, G)$ and $\mathcal{W}V M(H, K)$ is trivial, then $(f, f')$ is an isomorphism.

**Proof.** Put $M = \ker f$. Then $N/\ker(N, G) \cong N / V(K, H)$, $G \cong H / K$ and $V_n(N, G)M \cong V_n(K, H)$ for all $n \geq 0$. Now, the result follows from Corollary 7.

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3. Ultra Hall pair

The concept of a Schur pair was first introduced by Philip Hall [3] in 1940. Then in 1976, Hulse and Lennox [6] studied more properties of this pair and introduced the notion of an ultra Schur pair, a persistent pair and an ultra persistent pair. In 1977, Fung introduced the notion of a Hall pair as the following.

**Definition 9.** Let $\mathcal{X}$ be a class of groups and $\mathcal{V}$ be a variety of groups. If for every group $G$ and normal $\mathcal{V}$-nilpotent subgroup $N$ of $G$, $G/N \in \mathcal{X}$ implies that $G \in \mathcal{X}$, then the pair $(\mathcal{V}, \mathcal{X})$ is said to be a Hall pair.

In the special case if $\mathcal{V}$ is the variety of abelian groups and $\mathcal{X}$ is the class of nilpotent groups, we observe that this notion has roots in the well-known nilpotency criterion of Philip Hall; see [12, Theorem 5.2.10].

Let $F_\infty$ be the free group with the set of free generators $\{x_1, x_2, x_3, \ldots\}$. The outer commutator words (henceforth o.c. words) are defined inductively as follows. The word $x_i$ is an o.c. word of weight one. If $U = U(x_1, \ldots, x_m)$ and $V = V(x_{m+1}, \ldots, x_{m+n})$ are o.c. words of weight $m$ and $n$, respectively, then

$$W(x_1, \ldots, x_{m+n}) = [U(x_1, \ldots, x_m), V(x_{m+1}, \ldots, x_{m+n})].$$

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the commutator of $U$ and $V$, is an o.c. word of weight $m + n$. Let $V = V(x_1, \ldots, x_n)$ and $W = W(x_1, \ldots, x_n)$ be two arbitrary words. Then $V o W$, the composite of $V$ and $W$, is defined as $V o W = V(y_1, \ldots, y_m)$, where $y_i = W(x(i-1)n+1, \ldots, x_i n)$, $1 \leq i \leq m$. In the sequel, $V . W$ is the variety of groups defined by the word $V o W$.

**Theorem 10 (cf. [2, Theorem 3]).** Let $V$ be variety of groups defined by an o.c. word $V$ of weight at least two and let $W$ be a variety of groups defined by a single word $W$. Then the assumption that $(V, \mathcal{X})$ is a Hall pair always implies that $(V . W, \mathcal{X})$ is also a Hall pair.

In the following we state the definition of ultra Hall pair and derive a result which is a generalization of [2, Theorem 3].

**Definition 11.** Let $\mathcal{X}$ be a class of groups and $V$ be a variety of groups defined by the set of words $V$. If for every normal subgroups $K$ and $N$ of a given group $G$ such that $K$ is $\mathcal{V}_N$-nilpotent, the assumption $G / V(K, N) \in \mathcal{X}$ implies that $G \in \mathcal{X}$, then $(V, \mathcal{X})$ is called an ultra Hall pair.

The following lemma will be useful for the proof of our results; see [2, Lemma 2.6].

**Lemma 12.** Let $V$ and $W$ be two words of distinct variables and $U = [V, W]$. Then for every normal subgroup $N$ of a given group $G$, the following statements hold

(i) $U(N, G) = [V(N, G), W(G)] [W(N, G), V(G)]$,

(ii) If $V$ is an o.c. word, then $V o W(N, G) = V(W(N, G), W(G))$.

The following easy lemma is useful in the next result.

**Lemma 13.** Let $V$ be an o.c. word of weight at least two. Then for every normal subgroup $N$ of a given group $G$, $V(N, G) \leq [N, G]$.

**Proof.** Let $c$ be the weight of $V$. For $c = 2$, $V = y_2$, then $V(N, G) = [N, G]$. Let the result holds for o.c. words of weight less than $c$. Then $V = [V_1, V_2]$, where $V_1$ and $V_2$ are o.c. words of weight less than $c$. By Lemma 12 (i)

$$V(N, G) = [V_1(N, G), V_2(G)] [V_2(N, G), V_1(G)]$$

$$\leq [[N, G], V_2(G)] [N, G], V_1(G)]$$

$$\leq [N, G].$$

The following theorem gives a necessary and sufficient condition for a normal subgroup $N$ of a group $G$ to be $\mathcal{V}_G$-nilpotent, where $\mathcal{V}$ is the variety of groups defined by the word $V o W$.

**Theorem 14.** Let $V$ and $W$ be two words of distinct variables such that $V$ is an o.c. word of weight at least two. Then for any normal subgroup $N$ of a group $G$, the subgroup $N$ is $\mathcal{V}_G$-nilpotent if and only if $W(N, G)$ is $\mathcal{V}_{W(G)}$-nilpotent.

**Proof.** Let $W(N, G)$ be $\mathcal{V}_{W(G)}$-nilpotent. By considering $U = V o W$, since $V$ is an o.c. word, then $U(N, G) = V o W(N, G) = V(W(N, G), W(G))$. Using induction on $k$, we prove that for any positive integer $k$, $U_k(N, G) \leq V_k(W(N, G), W(G))$. The result is true for $k = 1$. Suppose that for $k = i$ the statement holds. Then

$$U_{i+1}(N, G) = U(U_i(N, G), G) = V(W(U_i(N, G), W(G))$$

$$\leq V(W(V_i(W(N, G), W(G)), W(G)))$$

$$\leq V(V_i(W(N, G), W(G)), W(G))$$

$$= V_{i+1}(W(N, G), W(G)).$$
where (1) follows from Lemma 1(iv). Since $W(N, G)$ is $\mathcal{W}(G)$-nilpotent, then
\[ V_\ell(W(N, G), W(G)) = 1 \]
for a positive integer $r$. Thus $U_\ell(N, G) = 1$, which implies that $N$ is $\mathcal{X}(G)$-nilpotent.

Now, let $N$ be $\mathcal{X}(G)$-nilpotent. By induction we will prove that
\[ V_k(W(N, G), W(G)) \leq U_{\left\lceil \frac{k+1}{2} \right\rceil}(N, G), \quad (2) \]
for any positive integer $k$, where $\left\lceil \frac{k+1}{2} \right\rceil$ is the integer part of $\frac{k+1}{2}$. Clearly the result holds for $k = 1$. Suppose the statement holds for every $i$, where $i \leq k$. Then
\[ V_k(W(N, G), W(G)) = V(V_{k-1}(W(N, G), W(G)), W(G)). \]
Since $V$ is an o.c. word, by the above lemma the right hand of equality is contained in $[V_{k-1}(W(N, G), W(G)), W(G)]$. This subgroup is contained in
\[ V(V_{k-1}(W(N, G), W(G)), G) \leq W(U_{\left\lceil \frac{k}{2} \right\rceil}(N, G), G), \]
by Lemma 1(v). So
\[ V_{k+1}(W(N, G), W(G)) = V(V_k(W(N, G), W(G)), W(G)) \]
\[ \leq V(W(U_{\left\lceil \frac{k+1}{2} \right\rceil}(N, G), G), W(G)) \]
\[ = U_{\left\lceil \frac{k+1}{2} \right\rceil}(N, G) \]
\[ = U_{\left\lceil \frac{k}{2} \right\rceil}(N, G). \]
Thus for every positive integer $k$, (2) holds. As $N$ is $\mathcal{X}(G)$-nilpotent, $U_\ell(N, G) = 1$ for a positive integer $r$. So, $V_{2r-1}(W(N, G), W(G)) = 1$, i.e. $W(N, G)$ is $\mathcal{W}(G)$-nilpotent.

The following result generalizes [2, Theorem 3].

**Theorem 15.** Let $\mathcal{V}$ and $\mathcal{W}$ be two varieties of groups as in the above theorem. Then the assumption that $(\mathcal{V}, \mathcal{X})$ is an ultra Hall pair, implies that $(\mathcal{V}, \mathcal{X})$ is also an ultra Hall pair.

**Proof.** Let $K$ and $N$ be two normal subgroups of $G$ such that $K$ is $\mathcal{X}_N$-nilpotent, where $\mathcal{X} = \mathcal{V}, \mathcal{W}$, and $G/U(K, N) \in \mathcal{X}$. So, $G/V(W(K, N), W(N)) \in \mathcal{X}$. By the above theorem, $W(K, N)$ is $\mathcal{W}(W(N))$-nilpotent. Since $(\mathcal{V}, \mathcal{X})$ is an ultra Hall pair, then $G \in \mathcal{X}$.

If one puts $K = N$, then the result that of Fung yields. The following result generalizes [9, Theorem 2.4].

**Theorem 16.** Let $\mathcal{V}$ be a variety of groups and $N$ be a $\mathcal{V}(G)$-nilpotent subgroup of $G$. If $K$ is nontrivial normal subgroup of $G$, contained in $N$, then $K \cap V^*(N, G) \neq 1$.

**Proof.** Let the $\mathcal{V}_G$-nilpotency class of $N$ be $c$. Then by Lemma 3(ii), $V_c^*(N, G) = N$. So, there exists a least integer $i$ such that $K \cap V_i^*(N, G) \neq 1$. Clearly
\[ V(K \cap V_i^*(N, G), G) \leq K \cap V(V_i^*(N, G), G). \]
On the other hand by Lemma 1(iv) and Lemma 2,
\[ V \left( \frac{V_i^*(N, G)}{V_{i-1}^*(N, G)}, G \right) = V \left( V^* \left( \frac{N}{V_{i-1}^*(N, G)}, G \right), G \right) \]
\[ = 1_{G/V_{i-1}^*(N, G)}. \]
Therefore, $V(K \cap V_i^*(N, G), G) \leq K \cap V_{i-1}^*(N, G) = 1$. Hence,
\[ K \cap V_i^*(N, G) \leq K \cap V^*(N, G), \]
our required result. \(\square\)
If one puts $N = G$ and considers $\mathcal{V}$ as the variety of abelian groups, then the well-known result of Philip Hall is obtained; see [8, Theorem 31.26].

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