Real-Time Visualization in Non-Isotropic Geometries

Eryk Kopczyński
Dorota Celińska-Kopczyńska
Institute of Informatics, University of Warsaw
erykk@mimuw.edu.pl
February 25, 2020

Abstract
Non-isotropic geometries are of interest to low-dimensional topologists, physicists and cosmologists. However, they are challenging to comprehend and visualize. We present novel methods of computing real-time native geodesic rendering of non-isotropic geometries. Our methods can be applied not only to visualization, but also are essential for potential applications in machine learning and video games.

1 Introduction
Non-isotropic geometries do not behave the same in all directions. They arise in Thurston’s famous geometrization conjecture [Thu82]. This conjecture generalizes the Poincaré conjecture, one of the most important conjectures in mathematics, proven by Perelman [Per06]. Every two-dimensional compact manifold...
can be given a spherical $S^2$, Euclidean, or hyperbolic geometry $H^2$; the Thurston conjecture states that every three-dimensional compact manifold can be similarly decomposed into subsets, each of which admitting one of eight geometries, called the Thurston geometries. The eight geometries include the three isotropic geometries mentioned, two product geometries ($S^2 \times E$, $H^2 \times E$), and three other geometries ($\text{Solv}$, $\text{Nil}$ and $\text{SL}$). The interest in $\text{Solv}$ and $\text{Nil}$ ranges from geometric group theorists, as they exhibit growth patterns typical to solvable and nilpotent groups [Loh17], to physicists [GVVP02], and cosmologists [Wee01].

(a) Poincaré disk  
(b) Beltrami-Klein disk  
(c) Poincaré half-plane 
(d) Horospherical

Figure 2: Binary tiling of $H^2$, in four projections.

Non-Euclidean geometries are perceived as unnatural and confusing to navigate. To combat this, for three-dimensional manifolds, we use first-person perspective: we put structures in a given geometry, and we render how a person inside the geometry would view those structures. We assume that light rays always travel along the shortest routes in our space (geodesics). Such visualizations ace in applications where finding the shortest path between two points is critical. Compare several projections of the same scene in two-dimensional hyperbolic geometry in Figure 2. Although half-plane and horocyclic projections (cd) seem easier to understand, they misperceive the straight yellow line as the shortest route between the red and blue cells instead of the shorter green line. Geodesic-based, azimuthal projections (ab) are immune to this problem.

Three-dimensional geometries, $\text{Solv}$, $\text{Nil}$, and $\text{SL}$, are even more demanding to comprehend. [Wee01] describes the $\text{Solv}$ geometry as "This is the real weirdo. [...] I don't know any good intrinsic way to understand it.". Therefore, efficient visualization becomes a fundamental tool for gaining intuition about them. From the designers’ point of view, there are two major challenges in visualizing non-isotropic geometries. (1) the geodesics in these geometries are not necessarily given by simple formulae (especially in $\text{Solv}$), (2) for given points $a$ and $b$, there can be multiple geodesics from $a$ to $b$. As a result, while there are implementations of real-time first-person view for Euclidean, spherical, hyperbolic spaces [Wee02, HHMS17], and for product spaces [Wee06], visualizations of geometries like $\text{Solv}$, $\text{Nil}$ or $\text{PSL}$ are nearly absent. [Wee06] mentioned plans for creating visualizations of these geometries, but even their implementation of $S^2 \times E$ did not handle the multiple geodesics problem correctly. In late 2019 the subject has received independent or partially independent attention of three
other teams [Mag19, NdSV19, CMST20a, CMST20b]. Older visualizations by Pierre Berger [Ber15] are static images rather than real-time, which makes them difficult to interpret.

This paper presents novel methods of real-time native geodesic rendering of first-person perspective in non-isotropic three-dimensional geometries. Our solution has major advantages over other propositions. First, our solution outreaches pure visualization, allowing for convenient hands-on activities, e.g., in video games, education or art, as well as applications in machine learning and physics simulations. Second, our primitive-based method is better suited for Virtual Reality. Our implementation is also the only one suitable for working with large-scale scenarios; other implementations would not be suitable because of numerical issues inherent to negatively curved spaces.

2 Riemannian manifolds and isotropic geometries

We will use a simplified definition of a Riemannian manifold. While less general than the commonly used definition, our definition is convenient for computations, and satisfies our needs. Intuitively, a Riemannian manifold is an $n$-dimensional subset of $\mathbb{R}^m$ which locally behaves like an $n$-dimensional Euclidean space. Let $\mathbb{R}_{m,m}$ be the set of bilinear functions from $\mathbb{R}^m \times \mathbb{R}^m$ to $\mathbb{R}$.

Definition 2.1 An $n$-dimensional (Riemannian) manifold is $M = (A, g)$, where $A \subseteq \mathbb{R}^m$ and $g : A \to \mathbb{R}_{m,m}$, such that for every $x \in A$, $g(x)$ is a there is an open neighborhood $U \subseteq \mathbb{R}^m$ of $x$ and a differentiable bijection $f : V \to U \cap A$, where $V$ is a open neighborhood of 0 in $\mathbb{R}^n$, such that $f$ is for every vector $0 \neq v \in \mathbb{R}^n$, $g(x)(Df(x), Df(x)) > 0$.

The bilinear function $g(x)$ is known as the metric tensor, and is used to measure the length of curves. Let $\gamma : [t_1, t_2] \to A$ be a curve (i.e., a continuous differentiable function). We define the length of $\gamma$ using the following formula: ($\dot{\gamma}$ is the derivative of $\gamma$)

$$l_M(\gamma) = \int_{x=0}^{t} \sqrt{g(\dot{\gamma}(x), \dot{\gamma}(x))} dx.$$

Definition 2.2 We say that manifold $M_1 = (A_1, g_1)$ are $M_2 = (A_2, g_2)$ are isometric iff there is a bijection $f : A_1 \to A_2$ such that for every curve $\gamma$, $l_{M_1}(\gamma) = l_{M_2}(f(\gamma))$.

When $M_1$ and $M_2$ are isometric, we consider them be different models of the same abstract manifold.

Definition 2.3 A geodesic is a curve $\gamma$ that is locally shortest and constant speed. For every $t$, there is an interval $(t_1, t_2)$ such that for $t_1 < u_1 < u_2 < t_2$, $\gamma$ restricted to $[u_1, u_2]$ is the shortest curve from $\gamma(u_1)$ to $\gamma(u_2)$. Furthermore, $g(\dot{\gamma}(x), \dot{\gamma}(x))$ is a constant.
Definition 2.4 A geometry is a manifold that is complete, simply connected, and locally homogeneous. A manifold \((A, g)\) is simply connected iff for every two points \(x, y \in A\), there exists a curve from \(x\) to \(y\), and every two curves from \(x\) to \(y\) are homotopic, i.e., one can be continuously deformed into the other; locally homogeneous iff for every two points \(x, y \in A\), there exist open neighborhoods \(X \ni x, Y \ni y\) such that \((X, g)\) and \((Y, g)\) are isometric; complete iff every geodesic \(\gamma: [t_1, t_2] \to A\) can be extended to \(\gamma: \mathbb{R} \to A\).

A locally homogeneous manifold is one where every point locally looks the same. A manifold is called isotropic if additionally it looks the same in every direction. The following isotropic geometries exist for every dimension \(n \geq 2\):

**Euclidean geometry** \(\mathbb{E}^n\) given by \(A = \mathbb{R}^n\) and \(g(x)(v, w) = v \cdot w\), where \(\cdot\) is the inner product.

**Spherical geometry** \(\mathbb{S}^n\) given by \(A = \{v \in \mathbb{R}^{n+1}: v \cdot v = 1\}\) and \(g(x)(v, w) = v \cdot w\). This is the surface of a sphere in \(n + 1\)-dimensional space.

**Hyperbolic geometry** \(\mathbb{H}^n\) given by \(A = \{v \in \mathbb{R}^{n+1}: v \cdot v = -1, v_{n+1} > 0\}\) and \(g(x)(v, w) = v \cdot w\), where \(\cdot\) is the Minkowski inner product: \((x_1, \ldots, x_{n+1}) \cdot (y_1, \ldots, y_{n+1}) = x_1 y_1 + x_2 y_2 + \ldots + x_n y_n - x_{n+1} y_{n+1}\).

We have described the hyperbolic geometry in the Minkowski hyperboloid model. We will need also other models [CFK+97]:

**Beltrami-Klein model** where \(A_1 = \{v \in \mathbb{R}^n: v \cdot v \leq 1\}\), is obtained from the Minkowski hyperboloid model via the map \(f(h) = (h_1/h_{n+1}, \ldots, h_n/h_{n+1})\). The metric tensor \(g\) is defined in the unique way which yields an isometry.

**Poincaré ball model** where \(A_2 = \{v \in \mathbb{R}^n: v \cdot v \leq 1\}\), is obtained from the Minkowski hyperboloid model via the map \(f(h) = (h_1/(1+h_{n+1}), \ldots, h_n/(1+h_{n+1}))\). The metric tensor \(g\) is defined in the unique way which yields an isometry.

**Half-space model** where \(A_3 = \{v \in \mathbb{R}^n: v_n > 0\}\), is obtained from the Poincaré ball model via inversion in a circle centered at \((0, \ldots, 0, -1)\). The metric tensor \(g\) is defined in the unique way which yields an isometry; we get \(g(x)(v, w) = x_n^2(v \cdot w)\).

**Horospherical model** where \(A_4 = \mathbb{R}^n\), is obtained from the half-space model via \(f(x_1, \ldots, x_n) = (x_1, \ldots, x_{n-1}, \log x_n)\). The metric tensor \(g\) is defined in the unique way which yields an isometry; we get \(g(x)(v, w) = e^{2x_n} v_1 w_1 + e^{2x_n} v_2 w_2 + \ldots + e^{2x_n} v_{n-1} w_{n-1} + v_n w_n\).

For two manifolds \(A = (A, g_A)\) and \(B = (B, g_B)\), their product manifold \(A \times B\) is \((A \times B, g)\), where, for every \(a_1, a_2 \in A, b_1, b_2 \in B\), \(g((a_1, b_1), (a_2, b_2)) = g_A(a_1, a_2) + g_B(b_1, b_2)\). Note that \(\mathbb{E}^n \times \mathbb{E}^m = \mathbb{E}^{n+m}\).
3 Tangent spaces, geodesics and parallel transport

The tangent space $T_{a}(A)$ is the set of vectors $v \in \mathbb{R}^{m}$ such that there exists a curve $\gamma : \mathbb{R} \to A$ such that $\gamma(0) = a$ and $\dot{\gamma}(0) = v$. Let $a \in A$, and $v \in T_{a}(A)$, the exponential map $\exp_{a}(v)$ is $\gamma(1)$, where $\gamma$ is the unique geodesic such that $\gamma(0) = a$ and $\frac{d}{dt}(\gamma(0)) = v$. Intuitively, $\exp_{a}(v)$ tells us where we end up if we start in the point $a$ and follow the geodesic in the direction and distance given by $v$. The inverse of $\exp_{a}$ is the inverse exponential map $\log_{a} : A \to \mathbb{R}^{m}$; it need not be a function. The inverse exponential map $\log_{a}(b)$ tells us which direction (and distance) we should go in order to reach $b$ from $a$. It may have multiple values (similar to $\log(b)$ in the complex plane).

Let $\gamma$ be a curve such that $\gamma(t_{0}) = a$ and $\gamma(t_{1}) = b$. Parallel transport lets us move tangent vectors $v \in T_{a}(A)$ to $\in T_{b}(A)$ along the curve $\gamma$ in a natural way; contrary to Euclidean space, the resulting $w \in T_{b}(A)$ may depend on the choice of $\gamma$. For example, in $\mathbb{H}^{2}$ and $S^{2}$, the sum of internal angles in a triangle is $180^\circ + \epsilon$ where $\epsilon < 0$ in $\mathbb{H}^{2}$ and $\epsilon > 0$ in $S^{2}$. As a consequence, if we walk on a loop $\gamma$ which cycles around such a triangle, we need to turn by $360^\circ - \epsilon$ angles in total, and thus the vector $v \in T_{a}(A)$ will be transported to $v' \in T_{a}(A)$, where $v'$ is $v$ rotated by angle $\epsilon$. In general, the rotation equals the area enclosed in $\gamma$ times the curvature (or integral of the curvature for non-homogeneous manifolds).

To compute the exponential function and parallel transport in non-isotropic manifolds we will be using the Christoffel symbols. Assume $n = m$ (otherwise use another model which has this property). Let $g^{ij}$ be the matrix of coefficients of $g$, $g_{ij}$ be the inverse of this matrix, and $\partial_{i} = \frac{\partial}{\partial x^{i}}$. The Christoffel symbols are given by $\Gamma_{ij}^{k} = \frac{1}{2} \sum_{m} g^{km}(\delta_{i}g_{mj} + \delta_{j}g_{im} - \delta_{m}g_{ij})$. Parallel transport is given by the following system of differential equations: $v(t_{0}) = v$ and

$$\dot{v}^{k} = - \sum_{i} \sum_{j} v^{i} \dot{\gamma}^{j} \Gamma_{ij}^{k}. \quad (1)$$

The curve $\gamma$ is a geodesic iff the above hold for $\dot{\gamma}$, i.e.,

$$\dot{\gamma}^{k} = - \sum_{i} \sum_{j} \dot{\gamma}^{i} \dot{\gamma}^{j} \Gamma_{ij}^{k}. \quad (2)$$

4 Tessellations

A tessellation of a manifold is its tiling using a compact shape (called tile or cell) with no overlaps or gaps. In general, tessellations may use multiple shapes; the tessellations in our paper will always use just one. The simplest tessellation of the hyperbolic plane is the binary tiling [B74]. In the horospherical model, the shape is given as $S = [0, 1] \times [0, \log 2]$. By translating $S$ with the isometry $f_{k}(x, y) = (x + k, y)$ for $n \in \mathbb{Z}$ we tessellate the horoannulus $S = \mathbb{R} \times [0, \log 2]$; by translating in two dimensions using the isometry $f_{k,l}(x, y) = (2^{-l}x + k, y + 5$
We tessellate the whole hyperbolic plane. The binary tiling is shown in Figure 2. The binary tiling has a structure similar to that of the infinite binary tree.

The binary tiling can be seen as a combinatorial graph that can be generated lazily. Every tile has pointers to its five neighbors (left, right, up-left, up-right, down); these pointers are initially null pointers, and point to specific tiles once the relevant tile is known. We start with a single root tile, and generate new tiles as required, using simple rules:

- if we are asked about an unknown up-left or up-right neighbor of \( X \), we create a new tile \( Y \). We connect \( X \) to \( Y \) (setting \( Y \) as the up neighbor of \( X \), and \( X \) as the down neighbor of \( Y \)).

- the left neighbor of \( X \) is the up-left neighbor of the down neighbor of \( X \) (if \( X \) is the up-right neighbor of its down neighbor), or the up-right neighbor of the left neighbor of the down neighbor of \( X \) (if \( X \) is the up-left neighbor). The right neighbor can be found similarly.

- if we are asked about an unknown down neighbor of \( X \), we create a new tile, and we arbitrarily assign \( X \) as one of the upper neighbors of \( Y \).

Similar, but somewhat more complicated rules can be also used to generate regular tilings of the hyperbolic plane [Mar14]. Representing points in \( \mathbb{H}^2 \) with a pointer to the tile they are in, and Minkowski coordinates relative to that tile, lets us avoid numerical issues that appear when representing faraway points using only model coordinates.

5 Homogeneous coordinates

In computer graphics, we commonly represent \( n \)-dimensional Euclidean space using homogeneous coordinates \((x_1, \ldots, x_n, x_{n+1})\), where \( x_{n+1} = 1 \). This lets one represent both translations and rotations as matrix multiplications. The same property also holds in the spherical and Minkowski hyperboloid coordinates. Rotations around the homogeneous origin \( h_0 = (0, \ldots, 0, 1) \) are described by the same matrices in all three geometries. Translation by \( x \) along the first axis does not change the coordinates except \( x_1 \) and \( x_{n+1} \), while \( x_1 \) and \( x_{n+1} \) are affected in the following way: \[
\begin{pmatrix}
x'_1 \\
x'_{n+1}
\end{pmatrix} = M \begin{pmatrix} x_1 \\
x_{n+1}
\end{pmatrix},
\]
where the matrix \( M \) is of form \[
\begin{pmatrix} x_0 & x \\
0 & 1
\end{pmatrix}
\] in Euclidean geometry, \[
\begin{pmatrix} \cos x & \sin x \\
-\sin x & \cos x
\end{pmatrix}
\] in spherical geometry, and \[
\begin{pmatrix} \cosh x & \sinh x \\
\sinh x & \cosh x
\end{pmatrix}
\] in hyperbolic geometry. These formulas make visualizations of isotropic geometries, including the camera movement, a straightforward generalization of the Euclidean methods [PG92].

- According to our experience, people starting their work with hyperbolic rendering usually tend to use the Poincaré model, since that model is commonly used in courses while the
In isotropic geometries to move and rotate the world seems easier, than to move and rotate the camera. In non-isotropic geometries we can still move our point of vision by moving the world, and to make this work we will also use coordinate systems where translations are represented by matrix multiplications. However, we can no longer rotate the world (rotations are no longer isometries), so we will represent the camera orientation as a triple of vectors (top, right, front directions, denoted $d_1, d_2, d_3 \in T_c(A)$ where $c$ is the camera position), or equivalently a view matrix $V$ such that $Ve_i = d_i$ and $Ve_4 = e_4$, where $e_i$ is the $i$-th unit vector. To find out the screen coordinates of an object located at $x$, we apply the perspective projection to $(TV)^{-1} \log h_0(Tx)$, where $T$ is the translation matrix which moves the current camera position $c$ to the homogeneous origin $h_0$. The camera can be rotated in the standard way. To move the camera $d$ units forward, we compute the geodesic $\gamma$ such that $\gamma(0) = c$ and $\dot{\gamma}(0) = d_3$. The new camera position will be $\gamma(d)$, and the new camera orientation $(d_i)$ is computed by parallel transport of respective $(d_i)$ along the geodesic $\gamma$ (this will keep the front vector $d_3$ always point forward as we traverse our geodesic; for vectors $d_1$ and $d_2$, using parallel transport ensures that the camera is not weirdly rotated as we travel).

6 Solv geometry

Our implementation of the Solv geometry is based on BS07. This geometry is $(\mathbb{R}^3, g)$, where $g^{11}(x, y, z) = \exp z$, $g^{22}(x, y, z) = \exp -z$, $g^{33}(x, y, z) = 1$, and $g^{ij} = 0$ for $i \neq j$. The isometry taking $(x, y, z)$ to $(0, 0, 0)$ is given by $m(x', y', z') = (x - e^z x', y - e^{-z} y', z - z')$; as explained in Section 5, we add the fourth homogeneous coordinate, always equal 1, so that this isometry can be represented as matrix multiplication. (This fourth coordinate will be ignored below.) Note that the plane $y = 0$ is the hyperbolic plane in the horospherical model, and the plane $x = 0$ is also the hyperbolic plane in the horospherical model, but where the coordinate $z$ is reversed. The plane $z = 0$ is Euclidean.

To understand the geodesics in Solv, consider special cases. What is the shortest curve from $(0, 0, 0)$ to $(M, 0, 0)$ (where $M$ is large)? Both points belong to the hyperbolic plane $y = 0$, thus the geodesic will act just like in this hyperbolic plane (we can easily see that we cannot obtain a shorter curve by changing $y$). We already know how geodesics work in the horospherical coordinates model: the obvious curve $\gamma(t) = (t, 0, 0)$ is not the shortest (and thus not a geodesic), because its length is $M$, and we get a shorter curve by moving first to $(0, 0, -\log M)$, then to $(M, 0, -\log M)$, then to $(M, 0, 0)$. The total length of this curve is $\log M + 1 + \log M = 2\log M + 1$; the actual geodesic is obtained from the $\mathbb{H}^2$ geodesic by adding the extra coordinate $y = 0$. Its construction is similar to that of the polyline constructed above; in particular, its length is also $\Theta(\log M)$.

Minkowski hyperboloid model is ignored. Later, they learn about the Minkowski hyperboloid model and find out that it is easier to understand by analogy to spherical geometry and also due to its much better numerical properties.
Similarly we can find the shortest curve from \((0,0,0)\) to \((0,M,0)\); however, since in the hyperbolic plane \(x = 0\), the coordinate \(z\) is reversed, our polyline will first move to \((0,0,\log M)\). We can find the geodesics from \((0,0,0)\) to any point \((x,0,z)\) or \((0,y,z)\) by adding a zero coordinate to the respective geodesic in \(\mathbb{H}^2\).

The situation is more difficult for points \((x,y,z)\) where \(x,y \neq 0\). In particular, let us try to find the shortest curve from \((0,0,0)\) to \((M,M,0)\). While the curve \(\gamma(t) = (t,t,0)\) is a geodesic of length \(M\sqrt{2}\), it is not the globally shortest one. There is a polyline of length \(4\log M+2\), which goes through the following points: \((0,0,0)\) \(\to\) \((0,0,-\log M)\) \(\to\) \((M,0,-\log M)\) \(\to\) \((M,0,\log M)\) \(\to\) \((M,M,\log M)\) \(\to\) \((M,M,0)\). There is also another polyline of the same length, going through the points: \((0,0,0)\) \(\to\) \((0,0,\log M)\) \(\to\) \((0,M,\log M)\) \(\to\) \((0,M,-\log M)\) \(\to\) \((M,M,-\log M)\) \(\to\) \((M,M,0)\). It can be seen that the actual geodesic will again be of similar nature to one of these polylines: we have to temporarily increase the \(z\) coordinate in order to traverse the large difference in \(y\) coordinate quickly, and also to temporarily decrease the \(z\) coordinate to traverse the large difference in \(x\). However, these two movements can be done in any order, yielding two distinct geodesics (by symmetry, of the same length), one of which starts almost precisely upwards (for large values of \(M\)), and the other starts almost precisely downwards. For points \((M_1,M_2,0)\) where \(|M_1| \neq |M_2|\), one of these geodesics will be shorter.

To determine the actual geodesics, we need to solve the geodesics equations \(^2\). This has been done in \cite{BS07}; however, the result obtained is in terms of integrals of elliptic functions, and it is not clear how to compute it efficiently. Therefore, we determine the exponential function \(\exp_0\) by solving the geodesic equation \(^2\) numerically. We use the midpoint method with 100 steps. We have experimentally verified that this yields sufficient precision in our applications.

Figure 3a shows the graphs of several geodesics from \((0,0,0)\) to \((x,y,z)\) and their lengths: the geodesics are three-dimensional, color is used to visualize the third dimension. Other than the geodesics strictly embedded in hyperbolic planes \((x(t) = C\) or \(y(t) = C)\), \(x(t)\) and \(y(t)\) are strictly monotonous with time, while \(z(t)\) is periodic: it increases to the top level \(z_1\), afterwards it decreases to \(z_2\), then back to \(z(t)\). The derivatives \(\dot{x}(t)\) and \(\dot{y}(t)\) are functions of \(z(t)\) (\(|\dot{x}(t)|\) is bigger when \(z(t)\) is small, and smaller when \(z(t)\) is big).

To render \textbf{Solv} we need to find \(\log_M(a)\). We use an iterative method similar to the Newton method. In the \(n\)-th iteration, we compute \(\exp_0(t_n)\) and \(\exp_0(t_n + \epsilon \epsilon_i)\) for \(i = 1, 2, 3\) (we use \(\epsilon = 10^{-6}\)). This lets us find an affine function \(f\) which agrees with \(\exp_0\) in the four testing points; the next \(t_{n+1}\) will be such that \(f(t_{n+1}) = a\). For the interesting values of \(a\), we have verified experimentally that this method quickly and successfully finds \(t\) such that \(\exp_0(t) = a\), if we start with \(t_0 = 0\) and limit the step size to 0.1.

While the method above always finds a geodesic, it might not find the shortest geodesic. E.g., for \(a = (x,0,0)\) it will find \(t = a\) which is of length \(x \sqrt{2}\), while the shortest geodesic is of length \(\Theta(\log x)\). For our applications finding the shortest geodesic is the most important. The problem is for \(a = (x,y,z)\) where both \(|x|\) and \(|y|\) are large. In this case, the shortest geodesic has a structure
similar to the shortest paths described earlier in this section: we move through to a point \( b \) close to either \( p_1 = (x,0,\frac{z}{2}) \) or \( p_2 = (0,y,\frac{z}{2}) \). To find the actual shortest geodesic, we need to find the point \( b \), of form \( b = p_i + (x,-x,z) \) (since this is enough to find an arbitrary point on the geodesic, we find one in the intersection of the geodesic with the hyperplane \( \{p_i + (x,-x,z) : x,z \in \mathbb{R}\} \)). We set \( b_0 = p_i \) for \( i = 1, 2 \), and then iteratively minimize the sum of geodesic distance from 0 to \( b_n \) and from \( b \) to \( a \); this can be done by computing \( f(b) = |\log_b(0)| + |\log_b(I_ba)| \) where \( I_b \) is an isometry that takes \( b \) to 0. In both cases we compute \( \log_b \) for a point that has only one coordinate distant from 0, and for such points the Newton method described above works. We minimize \( f(b) \) by approximating first-order and second-order derivatives of \( f \), and finding the minimum of the obtained quadratic function. Once \( b \) is found, \( \log_0 a \) can be computed as \( t' = \frac{\log_0(b)+\log_0(I_ba)}{\log_0(b)} \log_0 b \); to combat the precision issues, we find the actual \( t \) by the Newton method, starting the iteration from \( t' \).

The methods described above are too computationally expensive for real-time visualization. We solve this by constructing a \( D \times D \times D \) table of precomputed values, and then use interpolation. Such interpolation can be performed efficiently on GPU hardware (in GLSL, the table is loaded as a texture). Since \( x, y, z \) are unbounded, we will actually precompute a function \( g \) such that \( \log_0(x,y,z) = k^{-1}(g(i_x(x),i_y(y),i_z(z))) \), where \( i_x(x),i_y(y),i_z(z) \in [0,1] \). We only consider \( x, y, z \geq 0 \) (we can use symmetry to compute \( \log_0 \) for negative arguments). For \( i_x(x) \) we map the represent the point \((x,0)\) in the horocyclic coordinates to the Poincaré disk model; in the Poincaré disk model, the horocycle is mapped to a circle (see Figure 2), and \( i_x(x) \) is the angular coordinate on that (semi-)circle, scaled to \([0,1]\). Function \( i_y \) works in the same way. Our \( i_z(z) \) is the Poincaré disk coordinate of the horocyclic point \((0,z)\). The function \( k \) maps points in \( \mathbb{R}^3 \) to \([0,1]^3\); this is necessary for technical reasons, since the GPU expects the coordinates in textures to be \([0,1]\). Our function \( k(x,y,z) \) considers \((x,y,z)\) as azimuthal equidistant coordinates of a point in \( \mathbb{H}^3 \), and returns its coordinates in the Poincaré ball model. Our choices of \( i_x, i_z \) and \( k \) ensure that the function \( g(x,y,z) \) will be linear when \( x \) or \( y \) are close to 1, and thus the interpolation will yield good approximate results for large values of \( x, y, z \).

Figure 3b presents \textsc{Solv} in a Poincaré-ball like model. The colored planes are surfaces of constant \( z \). We graph \( k(k_0(x,y,z)) \) for the \( k_0 \) computed using the method above. The cuts \( x = 0 \) and \( y = 0 \) are hyperbolic planes in the Poincaré disk model, as shown in Figure 3c. The surfaces of constant \( z \) are mapped to torus-like shapes in this projection; the tori for \( z > 0 \) and \( z < 0 \) are interlocking. Since this model is azimuthal, it corresponds to what the user positioned in the center using a first-person perspective visualization perceives: surfaces of constant \( z \) are perceived as interlocking tori (Figure 1a).

The binary tiling we have used for the hyperbolic plane (Figure 2) generalizes straightforwardly to \textbf{Solv}. We will build the honeycomb in levels, where \( i \)-th level has the \( z \) coordinate in range \((i - \frac{1}{2}) \log 2, (i + \frac{1}{2}) \log 2 \)). On the level 0, our tessellation projects to the tessellation of the plane \( z = 0 \) by squares.
of side length \( l \). Tessellation on level \( i \) projects to rectangles of dimensions \( 1/2^i \times 2^i \). This way, we have subdivided Solv into isometric cube-like shapes. This tessellation can be implementing on a computer by representing every cell as a pair of its projections to the hyperbolic planes \((x, z)\) and \((y, z)\); these projections are cells in the respective binary tilings on these planes. (This tessellation is not the only choice – we can also construct a honeycomb based on the Asonov cat mapping torus, which has an advantage of admitting quotient spaces with finitely many cells.)

Our method for computing \( \log_0 \) returns a single value. Therefore, the visualization based only on the direction computed using this method does not give the whole picture. However, it turns out that the effects of this issue are in fact minor. Multiple-valued \( \log_0 \) exist only for a small region of visible space (points \((x, y, z)\) where both \(|x|\) and \(|y|\) are greater than \( \pi \), and \( z \) is small), and the geodesics which reach \( a \) after multiple oscillations in \( z \) tend to stretch objects in unrecognizable ways, and to be hidden by other objects. Thus, our basic implementation does not handle this issue. The issue can be solved by using ray-based methods, using two-valued \( \log_0 \) which includes both candidate geodesics computed (based on \( p_1 \) and \( p_2 \)), or by special handling of the difficult case. Since \( \log_0(x, y, z) \) computes the shortest geodesic, it is not a continuous function in the points where the two geodesics computed from \( p_1 \) and \( p_2 \) are different paths of the same length (this happens at points where \(|x| = |y| > \pi \) and \( z = 0 \)); special care must be taken when rendering triangles which cross the non-continuous region.

The same approach can be used for other manifolds similar to Solv. Such manifolds can be obtaining by changing \( g^{11}(x, y, z) = \exp(a_1 z) \) and \( g^{22}(x, y, z) = \exp(a_2 z) \). For Solv we have \( a_1 = 1, a_2 = -1 \), for \( E^3 \) we have \( a_1 = a_2 = 0 \), for \( \mathbb{H}^3 \) we have \( a_1 = a_2 = 1 \), for \( \mathbb{H}^2 \times E \) we have \( a_1 = 1, a_2 = 0 \). Thus, we can obtain a non-isotropic variant of hyperbolic space by taking for example \( a_1 = \log 2, a_2 = \log 3 \), or a less symmetric variant of Solv by taking \( a_1 = \log 2, a_2 = -3 \log 3 \). Our methods generalize to such manifolds. Our methods also generalize to \( n \)-dimensional versions of Solv for \( n > 3 \), defined by \( g^i(x_1, \ldots, x_n) = \exp(a_i x_n) \).
for \( i = 1, \ldots, n - 1 \) and \( g^{nm}(x_1, \ldots, x_n) = 1 \), \( g^{ij} = 0 \) for \( i \neq j \). As long as the sequence \( (a_i) \) contains only two different non-zero values, we can compute \( \log g(x) \) using a three-dimensional precomputed texture by rotating \( x \) so that it lies in a three-dimensional subspace. For the product geometry \( \mathbb{H}^2 \times \mathbb{R} \) (and similarly \( \mathbb{S}^2 \)), taking the product of homogeneous representations of \( \mathbb{H}^2 \) and \( \mathbb{R} \) yields a five-dimensional homogeneous representation, which is difficult to work with in OpenGL. However, there is also a three-dimensional coordinate system which has the desired property: for \( h \in \mathbb{H}^2 \) and \( x \in \mathbb{R} \), we represent \((h, e)\) as \( h \cdot \exp f x \in \mathbb{R}^3 \).

7 Nil geometry

Our implementation of Nil geometry is based on the paper [PM03]. The translations of Nil are given by the formula \( T(x, y, z)(a, b, c) = (x, y, z) \ast (a, b, c) = (a + x, b + y, c + x b + z) \), and thus can be represented as matrices when we add the fourth homogeneous coordinate. The metric \( g \) is given by \( g(0)(v, w) \) is the inner product of \( v \) and \( w \), and for other points \( a, g(a) \) is uniquely defined by the fact that \( T(x, y, z) \) is an isometry. Note that \( (r, 0, 0) \ast (0, r, 0) \ast (-r, 0, 0) \ast (0, 0, -r) = (0, 0, -r^2) \); thus, when we move \( r \) steps in the direction of increasing \( x \), increasing \( y \), decreasing \( x \) and decreasing \( y \), we end up \( r^2 \) units below the starting point. This aspect makes Nil similar to Penrose staircases.

Formulas for Christoffel coefficients and geodesics in Nil are computed in [PM03]. Let \( t = (c \cos \alpha, c \sin \alpha, w) \). Then, in the general case, \( \exp_0(t) = \left( \frac{\sqrt{2}}{w} \sin(w t + \alpha) - \sin \alpha, -2 \frac{\sqrt{2}}{w} \cos(w t + \alpha) - \cos \alpha \right), w t + \frac{e^2}{2w} t - \frac{e^2}{2w} (\sin(2w t + 2\alpha) - \sin(2\alpha)) + \frac{e^2}{2w^2} (\sin(w t + 2\alpha) - \sin(2\alpha) - \sin(w t)) \right) \). In the special case \( c = 0 \) we have \( \exp_0(t) = (0, 0, wt) \) and in the case \( w = 0 \) we have \( \exp_0(t) = (ct \cos \alpha, ct \sin \alpha, \frac{\sqrt{2}}{2} c \cos \alpha \sin \alpha t^2) \). To find \( t \) such that \( \exp_0(t) = (x, y, z) \), note that, for the given value of \( w \), we can compute \( c \) and \( \alpha \) which will give the correct \( x \) and \( y \), and then compute \( z(w) \) based on \( w, c(w) \) and \( \alpha(w) \). The obtained \( z(w) \), as a function of \( w \in (-2\pi, 2\pi) \), is monotonic, and thus we can find the correct \( w \) using the bisection method. (Other solutions exist where \( |w| > 2\pi \), but these represent longer geodesics, and similar to Solv are less important in visualization.)

The obvious honeycomb for Nil has a cell for every point \((x, y, z) \in \mathbb{Z}^3 \). The cells adjacent to \((x, y, z) \) are given by \((x, y, z) \ast \pm e_i \). The cells are not cubes - note that after four moves \( e_1, -e_2, -e_1, -e_2 \) we end up below the original cell. The side faces of cell 0 are given by \( \pm \frac{1}{2} e_i \ast \pm e_j \ast \pm e_3 \), where \( k \in \left( -\frac{1}{2}, \frac{1}{2} \right) \), \( j \in \{1, 2\}, i \in \{1, 2\}, j = 3 - i \). The top and bottom face consists of four triangles given by \( m e_i \ast \pm e_j \ast \pm e_3 \), where \( m \in \left( -\frac{1}{2}, \frac{1}{2} \right), |k| < |m|, i \in \{1, 2\}, j = 3 - i \), and four vertical walls connecting the four triangles. This construction makes the side faces of a cell similar to that of a cube, while the top and bottom faces have little Penrose staircases on them. While these top and bottom faces are not flat, such a honeycomb is good for visualization, as it shows the basic Penrose staircase-like nature of Nil, as well as its rotational symmetry in the XY plane.
8 PSL geometry

The Nil geometry has been obtained from the Euclidean plane by adding a third coordinate behaving in a twisted way: whenever we go along a curve $\gamma$ which is locally flat (i.e., the $z$ coordinate of $T(\dot{\gamma}(t))$, where $T$ is the isometry which takes $\gamma(t)$ to 0) and projects to an Euclidean loop $\gamma'$, we end up directly above where we started, with the displacement equal to the area inside the curve $\gamma'$ in the Euclidean plane. A similar process can be also applied to $S^2$ and $H^2$. As mentioned in Section 3, in these geometries the parallel transport along a curve $\gamma'$ results in rotating vectors by angle equal to the area inside $\gamma'$. Thus, the obtained space is the space of rotations of $S^2$ or $H^2$. In case of $S^2$, this space of rotations is the elliptic 3-space, i.e., the quotient space of $S^3$ where we identify $(x,y,z,w)$ and $(-x,-y,-z,-w)$ (this fact is frequently used in computer graphics, where unit quaternions are used for representing rotations of the three-dimensional space).

In the case of $H^2$, the situation is similar, but now, following [DESS09], the rotations (i.e., isometries of $H^2$ which keep orientation) can be represented as the unit split-quaternions $\{ (x,y,z,w) : z^2 + w^2 - x^2 - y^2 = 1 \}$; $(x,y,z,w)$ represents the isometry of $H^2$ given by the same formulas as quaternions representing the isometries of $S^2$:

$$
\begin{pmatrix}
+ x^2 - y^2 - z^2 + w^2 & -2(xy + zw) & 2(xz - yw) \\
-2(xy - zw) & -x^2 + y^2 - z^2 + w^2 & -2(yz + xw) \\
2(xz + yw) & -2(yz - xw) & -x^2 - y^2 + z^2 + w^2
\end{pmatrix}.
$$

Again, the points $(x,y,z,w)$ and $(-x,-y,-z,-w)$ are identified. This space is called $PSL(2,\mathbb{R})$; if we do not identify the opposite points, we get $SL(2,\mathbb{R})$. This parametrization lets us represent translations as matrices:

$$
T(x,y,z,w) = \begin{pmatrix}
w & -z & y & x \\
z & w & -x & y \\
y & -x & w & z \\
x & y & -z & w
\end{pmatrix}.
$$

We can parametrize $SL(2,\mathbb{R})$ using polar coordinates: $(r, \theta, \phi)$ corresponds to $(\sinh(r) \cos(\theta - \phi), \sinh(r) \sin(\theta - \phi), \cosh(r) \sin(\phi), \cosh(r) \cos(\phi))$. This space is not simply connected (the loop $(0,0, \cos \phi, \sin \phi)$ for $\phi \in [0, 2\pi]$ cannot be contracted), thus the actual Thurston geometry is the universal cover of $SL(2,\mathbb{R})$, obtained by considering the points with polar coordinates $(r, \theta, k\pi + \phi)$ to be separate points for $k \in \mathbb{Z}$.

The formulas for $\exp_0$ have been found in [DESS09]. Let $\sin_c(\alpha) = \sin(\sqrt{c} \alpha)/\sqrt{c}$ for $c > 0$, $\sin_c(\alpha) = \sin(\sqrt{-c} \alpha)/\sqrt{-c}$ for $c < 0$, and $\sin_0(\alpha) = 0$; $\tan_c(\alpha)$ is

\footnotetext{2}{The order of coordinates have been changed from [DESS09] to match our conventions for other geometries.}
defined similarly. We compute \( \exp_0(x, y, z) \) as follows:

\[
\begin{align*}
    z &= s \sin \alpha \\
    x &= s \cos \alpha \sin \beta \\
    y &= s \cos \alpha \cos \beta \\
    c &= \cos(2\alpha) \\
    r &= \mathrm{arsinh}(\cos \alpha \cdot \sin \psi(s)) \\
    \tan(\theta) &= \sin \alpha \cdot \tan \psi(s) \\
    \phi &= 2 \sin \alpha s + \theta(s)
\end{align*}
\]

We use the first three formulas to convert \((x, y, z)\) to spherical coordinates; then we compute \(c\) from the fourth formula; then we use the next three to find the polar coordinates of \(\exp(x, y, z)\) in \(SL\); then we convert the polar coordinates to our model coordinates. For \(c > 0\), \(\theta\) should be chosen so that the closest integer to \(\theta/\pi\) should be the same as the closest integer to \(cs\).

We compute \(\log_0(a)\) as follows. Convert \(a\) to the polar coordinates \((r, \theta, \phi)\). Without loss of generality assume \(\phi \geq 0\). Knowing \(\alpha\) and \(r\), we can compute \(s\) using the formula \(\sin c(s) = \sinh(r)/\cos \alpha\), \(\beta - \theta\) as a function of \(\alpha\) and \(s\), and \(\phi\) as a function of \(\alpha\) and \(s\). First, we use bisection to find \(\alpha \in [0, \pi/2]\) for which \(\phi = \phi(\alpha, s_0(\alpha, r))\), where \(s_0\) is the smallest \(s > 0\) such that \(\sin c(s) = \sinh(r)/\cos \alpha\). We may find such an \(\alpha\) or not; in the latter case, for \(\alpha > \alpha_0 > \pi/4\) \(s_0(\alpha, r)\) does not exist because \(\sqrt{-c} \sinh(r)/\cos \alpha > 1\). In this case, we again use bisection to find \(\alpha \in [-\pi/4, \alpha_0]\) for which \(\phi = \phi(\alpha, s_1(\alpha, r))\), where \(s_1\) is the next possible \(s\) such that \(\sin c(s) = \sinh(r)/\cos \alpha\).

Our construction of a honeycomb in \(PSL\) is analogous to the construction of a honeycomb in \(Nil\), but based on a regular tessellation of the hyperbolic plane, instead of a square tessellation of the Euclidean plane.

9 Ray-based method and comparisons

We have also implemented ray-based rendering for all the geometries except \(PSL\) and \(S^3\). To render each pixel, we send a ray in the direction depending on the pixel coordinates and camera orientation, and starting in the current camera position, and find out where does it hit a wall. Given a honeycomb in our manifold, we use coordinates relative to the cell \(C\) the ray is currently in, and we need to find out which face \(F\) of \(C\) our ray hits; if the cell \(C'\) on the other side of \(F\) is filled, the ray ends (and we color the pixel appropriately), otherwise we compute the coordinates relative to \(C'\) and continue tracing the ray.

In the case of isotropic and product manifolds, it is straightforward to find formulas for the distance we need to travel in order to hit a plane \(F\). In \(Solv\), we proceed by making small steps of length \(u\), and compute the new coordinates after each such step (using the midpoint method); if it turns out that after \(u\) units we are already in another cell, we use bisection to find the collision point.
up to precision $\epsilon$: we halve $u$ and repeat as long as $u > \epsilon$. We have shown experimentally that $u = 0.02$ and $\epsilon = 0.001$ yields enough precision to be not readily distinguishable from more accurate computations. The similar method can be used in Nil, except that we can use precise geodesic formulas instead of the midpoint method, and therefore a larger maximum step value of $u = 0.1$ works.\footnote{PSL is not implemented at the current time, but it can be rendered in the same way.}

Our experiments show that primitive-based and ray-based methods yield the same output (modulo multiple geodesics), which shows that the approximations we have used when computed $\log_2$ for Solv indeed do not destroy the visualization effect. Our implementation of a primitive-based renderer for $\mathbb{S}^2 \times \mathbb{E}$ improves on \cite{Wec06} by rendering the multiple geodesics correctly; this is done by finding out the triangles which are close to the current camera position or its antipodal point, and subdividing them. This process is quite involved, and a similar process for totally correct primitive-based rendering of non-product, non-isotropic geometries would be even more complicated. However, the multiple geodesics are not an issue in $\mathbb{S}^3$, where $\exp(a) = \exp(b)$ happens only when $a$ is the camera position or its antipodal point, or when $a$ and $b$ point in the same direction; therefore, $\mathbb{S}^3$ can be rendered very well using primitive-based methods.

\section{Discussion and Applications}

\textbf{Comparison with ray marching.} Competitive attempts are based on ray-based algorithms: for every pixel, they trace the ray (geodesic), and color the pixel depending on the object that the ray hit. Our method is primitive-based: we represent our objects as triangles, and we compute the screen position for every vertex.

On the one hand, our method is more challenging. It is easier to trace geodesics than to find a geodesic which hits the given point in a non-isotropic space. On the other hand, our methods outperform ray marching in rendering shapes that are generated in a more complex way, such as 3D models. This makes them more applicable for gaming and scientific visualizations. Moreover, Virtual Reality relies on displaying separate images for both eyes. When we see an object at a specific point in our single eye vision, this means that the object is on a line; our brain then finds out where the lines defined by left eye and the right eye cross. This process, together with raytracing, works in isotropic geometries, with only the minor disadvantage of incorrect depth perception. The world is perceived as stretched Klein/gnomonic projection, which makes $\mathbb{H}^3$ look bounded while $\mathbb{S}^3$ looks unbounded. Non-isotropic ray-based VR will not work correctly, as the rays perceived by both eyes do not cross. We could do non-isotropic primitive-based VR by finding out the direction and distance to every object and then using an Euclidean renderer to render it in the right spot for both eyes.
To validate our approximations we have also implemented ray-based rendering for **Solv** and **Nil**. Our experiments show that primitive-based and ray-based methods have no major visible differences in the placement of walls.

**Differences in motivation.** Implementations by low-dimensional topologists, mostly aimed at visualizing the compact manifolds, and depicting local effects in the geometries, such as holonomy or lensing effects. Practical applications were not the focal point. Our motivation is different: we want to work with large-scale structures that are not necessarily periodic. Our methods combat problems resulting from *exponential growth* of negatively curved spaces, where large-scale computations are susceptible to floating-point errors [SDSGR18]. Our visualizations are based on tessellations, which are constructed precisely without using floating point arithmetics, and thus circumvent these problems. Tessellations are also used to build landmarks that can be used to navigate our spaces, and are important by themselves in our applications in data analysis (e.g., the self-organizing maps [Rit99, OR01]) and in gaming (level design).

**Applications.** In machine learning, a common approach is to embed our data into a manifold, in such a way that the relationships between the points correspond to the relationships between our data. While Euclidean geometry is used most commonly, non-Euclidean geometries have recently proven useful: hyperbolic geometry [PKS+12] for hierarchical data and spherical [WHPD14, LWY+17] and product [GSGR19] geometries for other data. We suppose our methods should facilitate working with Thurston geometries in data analysis; this will be a direction of our future work.

Other than the scientific purposes, the visualization of non-isotropic geometries has potential applications in video games or art. Many popular (mostly independent) video games experiment with spaces that work differently from our Euclidean world. This includes spaces with weird topology (Portal, Antichamber, Manifold Garden), interactions between 2D and 3D (Perspective, Fez, Monument Valley), non-Euclidean geometry (HyperRogue), extra dimensions (Miegakure). Similar experimentation also happens in art. Such games and art are interesting not only for mathematicians and physicists wanting to understand these spaces intuitively, but also for casual players curious to challenge their perception of the world. Non-isotropic geometries are especially relevant here because of their easily observable weirdness. **Nil**, a reminiscent of Penrose’s staircases, and M. C. Escher’s artworks, should be promising for game design.

**Acknowledgments.** We would like to thank the HyperRogue community, in particular to Kaida Tong and MagmaMcFry for sparking our interest in the **Solv** geometry and discussions. We would also like to thank the organizers of the ICERM Illustrating Geometry and Topology Workshop, partially funded by the Alfred P. Sloan Foundation award G-2019-11406 and supported by a Simons Foundation Targeted Grant to Institutes, for inviting us; this workshop has been a big inspiration for our work.
A Our implementation

The methods described in this paper have been implemented as a part of HyperRogue’s non-Euclidean visualization engine, RogueViz.

HyperRogue can be downloaded from http://roguetemple.com/z/hyper/ and its source code is available under GPL at https://github.com/zenorogue/hyperrogue/. The following files are the most relevant for this paper:

- nonisotropic.cpp – implementation of nonisotropic geometries
- raycaster.cpp – implementation of ray-based rendering
- hyperpoint.cpp – basic geometry routines for all geometries
- devmods/solv-table.cpp – producing the geodesic tables for Solv and its variants

In HyperRogue 11.2x (the current version as of writing of this), press Ctrl+T while in the start menu to try a racing game in Thurston geometries. Currently, raycasting is used by default (in several geometries); press ’o’ then ’9’ (3D configuration) then ’A’ (configure raycasting) to disable it. Racing mode can be also turned off (in ’o’) for a more random environment. Also in 3D configuration the sight range can be increased. (Note that, especially outside of the racing mode, the sight range may appear low – however, the number of cells rendered in this range is quite high because of exponential expansion.)

Some visualization videos made using our engine:

- https://youtu.be/C8HoCf_hkn8 – a simple structure in Solv geometry.
- https://youtu.be/2LotRqzidM – a longer video in Solv. This video uses an older version of our renderer; some of its details are different.
- https://youtu.be/YmFDd49WsrY – a Penrose triangle in Nil geometry.
- https://youtu.be/HeFyuVs-TTs – our honeycomb in Nil geometry.
- https://youtu.be/2ePY7Do5WvA – a structure in PSL geometry.
- https://youtu.be/_5l8v6Gn2eE – a structure in $S^2 \times \mathbb{E}$ geometry.

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