Projective modules over the fuzzy four-sphere

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Abstract

We describe how to reduce the fuzzy four-sphere algebra to a set of four independent raising and lowering oscillator operators. In terms of them we derive the projector valued operators for the fuzzy four-sphere, which are the global definition of $k$-instanton connections over this noncommutative base manifold.
1 Introduction

In ref. [1] (see also [2]) a finite projective module description of all noncommutative monopole configurations on the fuzzy two-sphere has been presented. The basic variables which describe connections and gauge fields of the monopoles are represented by globally defined projective operators, in terms of which it is possible to deduce directly the Chern class [3]-[4]-[5]-[6].

The aim of the present paper is to generalize the above construction to the case of the fuzzy four-sphere [7]-[8]-[9]-[10]. The difference from the fuzzy two-sphere case [11]-[12] is that the commutators of the coordinates do not close between them in this case. We have to generalize the algebra from five hermitian operators to fifteen operators, which define the $SO(5,1)$ algebra. Despite this difficulty we are able to generalize our method. The starting point is the simple observation that our construction, in the classical limit, is based on the Hopf fibration $S^3 \to S^2$; since it exists its generalization to the case $S^7 \to S^4$ we conclude that the fuzzy four-sphere algebra is generated by four independent oscillators. Once that the building blocks of the more complicated algebra are recognized, it is then easy to compute the projectors, which define the projective modules over the fuzzy four-sphere. The projectors are $N \times N$ matrices, having as entries the elements of the basic noncommutative coordinate algebra of the fuzzy four-sphere, where $N = \frac{(k+1)(k+2)(k+3)}{6}$ and $k$ is an integer, labelling the instanton number.

It is important to notice that a further generalization to higher dimensional fuzzy spheres of the present method is forbidden since there is no straightforward generalization of the Hopf fibration to $S^{2n}$ spheres. At the level of the fuzzy four-sphere, there are sixteen independent entries of the type $z_i z_j$ (where $(z_i, z_j)$ are the basic raising and lowering operators of the four oscillators), corresponding to the basic operators of the $SO(5,1)$ algebra and the number operator $\hat{N}$, labelling the representations of this algebra.

2 The fuzzy four-sphere

The fuzzy four-sphere (we follow mainly ref. [7]) is realized through a set of five hermitian operators that satisfy the following two conditions:

\[
\epsilon^{\mu\nu\lambda\rho} \hat{x}_\mu \hat{x}_\nu \hat{x}_\lambda \hat{x}_\rho = C \hat{x}_\sigma \\
\hat{x}_\mu \hat{x}_\mu = R^2
\] (2.1)
where \( R \) is a radius of the sphere. The two constraints respect \( SO(5) \) invariance. Both constraints are for example realized by the five-dimensional gamma matrices, which are the simplest finite-dimensional realizations of the fuzzy four-sphere, like the Pauli matrices are the simplest one for the fuzzy two-sphere.

In general, let us define finite-dimensional matrices \( \hat{\Gamma}_\mu \) as follows

\[
\hat{x}_\mu = \rho \hat{\Gamma}_\mu \quad (2.2)
\]

where \( \rho \) is a dimensional constant. The gamma matrices algebra is defined as

\[
\hat{\Gamma}_1^{(0)} = \gamma_0 \quad \hat{\Gamma}_2^{(0)} = \gamma_0 \gamma_1 \quad \hat{\Gamma}_3^{(0)} = \gamma_0 \gamma_2 \quad \hat{\Gamma}_4^{(0)} = \gamma_0 \gamma_3 \quad \hat{\Gamma}_5^{(0)} = i\gamma_0 \gamma_5. \quad (2.3)
\]

The general matrices can be built from the \( n \)-fold symmetric tensor product of \( \hat{\Gamma}_\mu^{(0)} \) algebra:

\[
\hat{\Gamma}_\mu^{(n)} = (\hat{\Gamma}_\mu^{(0)} \otimes 1 \otimes \ldots \otimes 1 + 1 \otimes \hat{\Gamma}_\mu^{(0)} \otimes \ldots \otimes 1 + \ldots \otimes 1 \otimes \hat{\Gamma}_\mu^{(0)})_{sym}. \quad (2.4)
\]

The size of these matrices depends on the parameter \( n \) as

\[
N = \frac{(n+1)(n+2)(n+3)}{6}. \quad (2.5)
\]

The same construction is valid in two dimensions, i.e. replacing \( \hat{\Gamma}_\mu^{(0)} \) with Pauli matrices \( \hat{\Gamma}_\mu^{(n)} \) become the hermitian operators of a fuzzy two-sphere, that is a \((n+1)\)-dimensional representation of \( SU(2) \).

It is not difficult to realize that these matrices satisfy the two conditions defining the fuzzy four-sphere \((2.1)\):

\[
\hat{\Gamma}_\mu^{(n)} \hat{\Gamma}_\nu^{(n)} = n(n+4) \\
e^{\mu\nu\lambda\rho} \hat{\Gamma}_\mu^{(n)} \hat{\Gamma}_\nu^{(n)} \hat{\Gamma}_\lambda^{(n)} \hat{\Gamma}_\rho^{(n)} = e^{\mu\nu\lambda\rho} \hat{\Gamma}_\mu^{(n)} \hat{\Gamma}_\nu^{(n)} = (8n+16) \hat{\Gamma}_\sigma^{(n)} \quad (2.6)
\]

where \( \hat{\Gamma}_\mu^{(n)} = \frac{1}{2}[\hat{\Gamma}_\mu^{(n)}, \hat{\Gamma}_\nu^{(n)}] \).

The two parameters \( C \) and \( R^2 \) can be computed as:

\[
C = (8n+16)\rho^3 \\
R^2 = n(n+4)\rho^2. \quad (2.7)
\]
In the large \( n \)-limit, fixing \( R^2 \), one recovers the usual classical four-sphere, since then \( C \) goes to zero as \( \frac{1}{n^2} \). From the second constraint we can recover the following useful relation:

\[
\hat{\Gamma}^{(n)}_{\mu\nu} = -\frac{1}{2(n+2)}\epsilon^{\mu\nu\lambda\rho\sigma} \hat{\Gamma}^{(n)}_{\lambda\rho} \hat{\Gamma}^{(n)}_{\sigma} = -\frac{1}{2(n+2)}\epsilon^{\mu\nu\lambda\rho\sigma} \hat{\Gamma}^{(n)}_{\lambda} \hat{\Gamma}^{(n)}_{\rho} \hat{\Gamma}^{(n)}_{\sigma}.
\] (2.8)

For completeness let us give a whole set of relations, obtained from the fundamental ones (2.6) and (2.8):

\[
\hat{\Gamma}^{(n)}_{\mu\nu} \hat{\Gamma}^{(n)}_{\nu} = 4\hat{\Gamma}^{(n)}_{\mu}
\]

\[
\hat{\Gamma}^{(n)}_{\mu\nu} \hat{\Gamma}_{\nu\mu} = 4n(n+4)
\]

\[
\hat{\Gamma}^{(n)}_{\mu\nu} \hat{\Gamma}^{(n)}_{\nu\lambda} = n(n+4)\delta_{\mu\lambda} + \hat{\Gamma}^{(n)}_{\mu} \hat{\Gamma}^{(n)}_{\lambda} - 2\hat{\Gamma}^{(n)}_{\lambda} \hat{\Gamma}^{(n)}_{\mu}.
\] (2.9)

Unfortunately, the basic coordinate operators \( \hat{x}_\mu \) do not close between them but we need to enlarge the algebra to the commutators:

\[
\hat{\Gamma}^{(n)}_{\mu\nu} = \frac{1}{2}[\hat{\Gamma}^{(n)}_{\mu} , \hat{\Gamma}^{(n)}_{\nu}]
\]

\[
[\hat{\Gamma}^{(n)}_{\mu} , \hat{\Gamma}^{(n)}_{\nu\lambda}] = 2(\delta_{\mu\nu} \hat{\Gamma}^{(n)}_{\lambda} - \delta_{\mu\lambda} \hat{\Gamma}^{(n)}_{\nu})
\]

\[
[\hat{\Gamma}^{(n)}_{\mu\nu} , \hat{\Gamma}^{(n)}_{\nu\lambda}] = 2(\delta_{\nu\lambda} \hat{\Gamma}^{(n)}_{\mu\nu} + \delta_{\mu\rho} \hat{\Gamma}^{(n)}_{\nu\lambda} - \delta_{\mu\lambda} \hat{\Gamma}^{(n)}_{\nu\rho} - \delta_{\nu\rho} \hat{\Gamma}^{(n)}_{\mu\lambda}).
\] (2.10)

These form the \( SO(5, 1) \) algebra.

Between all the possible representations of this algebra, we can always choose to diagonalize a matrix \( \hat{\Gamma}_\mu \) out of the five matrices, for example \( \hat{x}_5 = \rho \hat{\Gamma}_5 \), whose eigenvalue \( \Gamma_5 \) runs between \( n, n - 2, ..., -n + 2, -n \).

The matrices \( \hat{\Gamma}^{(n)}_{\mu\nu} \), with \( \mu, \nu = 1, ... 4 \) form an \( SO(4) \) algebra, which is a subalgebra of the \( SO(5) \) algebra:

\[
[\hat{N}_i, \hat{N}_j] = i\epsilon_{ijk} \hat{N}_k
\]

\[
[\hat{M}_i, \hat{M}_j] = i\epsilon_{ijk} \hat{M}_k
\]

\[
[\hat{N}_i, \hat{M}_j] = 0
\] (2.11)

where

\[
\hat{N}_1 = -\frac{i}{4}(\hat{\Gamma}_{23} - \hat{\Gamma}_{14}) \quad \hat{M}_1 = -\frac{i}{4}(\hat{\Gamma}_{23} + \hat{\Gamma}_{14})
\]
\[ \hat{N}_2 = \frac{i}{4}(\hat{\Gamma}_{13} + \hat{\Gamma}_{24}) \quad \hat{M}_2 = \frac{i}{4}(\hat{\Gamma}_{13} - \hat{\Gamma}_{24}) \]
\[ \hat{N}_3 = -\frac{i}{4}(\hat{\Gamma}_{12} - \hat{\Gamma}_{34}) \quad \hat{M}_1 = -\frac{i}{4}(\hat{\Gamma}_{12} + \hat{\Gamma}_{34}). \]  
(2.12)

\[ \hat{\Gamma}_{ab} \text{ is rewritten as:} \]
\[ \hat{\Gamma}_{23} = 2 \frac{i}{4}(\hat{\mathbf{N}}_1 + \hat{\mathbf{M}}_1) \quad \hat{\Gamma}_{14} = -2 \frac{i}{4}(\hat{\mathbf{N}}_1 - \hat{\mathbf{M}}_1) \]
\[ \hat{\Gamma}_{13} = -2 \frac{i}{4}(\hat{\mathbf{N}}_2 + \hat{\mathbf{M}}_2) \quad \hat{\Gamma}_{24} = -2 \frac{i}{4}(\hat{\mathbf{N}}_2 - \hat{\mathbf{M}}_2) \]
\[ \hat{\Gamma}_{12} = 2 \frac{i}{4}(\hat{\mathbf{N}}_3 + \hat{\mathbf{M}}_3) \quad \hat{\Gamma}_{34} = -2 \frac{i}{4}(\hat{\mathbf{N}}_3 - \hat{\mathbf{M}}_3). \]  
(2.13)

The Casimir of each SU(2) algebra depends on \( n \) and on the eigenvalue \( \Gamma_5 \):
\[ \hat{\mathbf{N}}_i \hat{\mathbf{N}}_i = \frac{1}{16}(n + \Gamma_5)(n + 4 + \Gamma_5) \]
\[ \hat{\mathbf{M}}_i \hat{\mathbf{M}}_i = \frac{1}{16}(n - \Gamma_5)(n + 4 - \Gamma_5). \]  
(2.14)

Therefore matrices \( \hat{\mathbf{N}}_i \) and \( \hat{\mathbf{M}}_i \) are realized by \( \frac{n + \Gamma_5 + 2}{2} \) and \( \frac{n - \Gamma_5 + 2}{2} \) dimensional representations of SU(2) respectively.

For \( \Gamma_5 = n \) at the north pole, the Casimir of \( \hat{\mathbf{N}}_i \) and \( \hat{\mathbf{M}}_i \) are given by:
\[ \hat{\mathbf{N}}_i \hat{\mathbf{N}}_i = \frac{n(n + 2)}{4} \]
\[ \hat{\mathbf{M}}_i \hat{\mathbf{M}}_i = 0. \]  
(2.15)

A fuzzy two-sphere appears at the north pole and it is given by the \( (n + 1) \)-dimensional representation of SU(2). By using the SO(5) symmetry we can attach a fuzzy two-sphere to every point on the fuzzy four-sphere. We can regard this two-sphere as the (spin) internal two-dimensional space.

It is not difficult to introduce actions on a fuzzy four-sphere, for example by introducing the following matrix model:
\[ S = -\frac{1}{g^2} Tr \left\{ \frac{1}{4}[A_\mu, A_\nu][A_\mu, A_\nu] + \frac{2\lambda}{5(n + 2)} \epsilon^{\mu
u\lambda\rho\sigma} A_\mu A_\nu A_\lambda A_\rho A_\sigma \right\} \]
\[ + 8(1 - \lambda) \rho^2 A_\mu A_\mu \]  
(2.16)

which contains the fuzzy four sphere as a classical solution:
\[ A_\mu = \hat{x}_\mu = \rho \hat{f}^{(n)}_\mu. \] (2.17)

By expanding matrices around the classical solution corresponding to the fuzzy four-sphere, we obtain an action of noncommutative gauge theory.

### 3 Hopf fibration and projectors for the fuzzy four-sphere

Instead of searching for instanton solutions on the fuzzy four-sphere by exploring the classical solutions of the action (2.16), as we did for the fuzzy sphere [13], we appeal to another method which is simpler and is based on the Hopf principal fibration from \( S^7 \) to \( S^4 \).

In two dimensions noncommutative monopoles over the fuzzy two-sphere were derived by introducing projector operators \( P_n \) which characterize the non trivial bundles over the fuzzy two-sphere. The canonical connection associated with the projector \( P_n(x) \) has curvature given by

\[ \nabla^2 = P_n dP_n dP_n. \] (3.1)

The projector operator is made in terms of the Hopf fibration from \( S^3 \) to \( S^2 \) as follows:

\[
\begin{align*}
x_1 &= \frac{\rho}{2}(z_0 \overline{z}_1 + z_1 \overline{z}_0) \quad \quad |z_0|^2 + |z_1|^2 = 1 \\
x_2 &= \frac{\rho}{2}(z_0 \overline{z}_1 - z_1 \overline{z}_0) \quad \quad x_1^2 + x_2^2 + x_3^2 = \frac{\rho^2}{4} \\
x_3 &= \frac{\rho}{2}(z_0 \overline{z}_0 - z_1 \overline{z}_1).
\end{align*}
\] (3.2)

In this classical setting, \( z_0 \) and \( z_1 \) are two complex numbers constrained to be \( S^3 \) and the Hopf fibration produces the real coordinates \( x_i \) which are instead constrained to be \( S^2 \).

This representation is particularly useful for the noncommutative case since generalizing the complex coordinates to a couple of oscillators:

\[ [z_i, \overline{z}_j] = \delta_{ij} \quad [z_i, z_j] = 0 \] (3.3)

produces for the real coordinates \( x_i \), promoted to operators, the more complex algebra of the fuzzy two-sphere:
\[ [\hat{x}_i, \hat{x}_j] = i\rho \epsilon_{ijk} \hat{x}_k. \] (3.4)

Introducing the number operator \( \hat{N} = z_0 \bar{z}_0 + \bar{z}_1 z_1 \) (3.5)

the parameter \( \rho \) is given in terms of \( n \), the eigenvalue of \( \hat{N} \), and \( R \) the radius of the sphere:

\[
\sum_i (\hat{x}_i)^2 = R^2 \\
\rho = \frac{2R}{\sqrt{n(n+2)}}. \tag{3.6}
\]

Here is the idea of the present letter. Since the Hopf fibration \( S^3 \to S^2 \) is generalizable to the case \( S^7 \to S^4 \), is it possible to represent the complicated algebra of the fuzzy four-sphere in terms of four independent oscillators? The answer is yes, and it is also possible to generalize the construction that led us to compute the projectors \( P_n \) for the fuzzy two-sphere to the case of fuzzy four-sphere.

The Hopf fibration \( S^7 \to S^4 \) is made by four complex coordinates \( z_i \) constrained to be \( S^7 \), that are mixed together to give five real coordinates \( x_i \) constrained to be \( S^4 \):

\[
\begin{align*}
x_1 &= \rho(a_1 + \bar{a}_1) \\
x_2 &= i\rho(a_1 - \bar{a}_1) \\
x_3 &= \rho(a_2 + \bar{a}_2) \\
x_4 &= i\rho(a_2 - \bar{a}_2) \\
x_5 &= \rho(z_0\bar{z}_0 + z_1\bar{z}_1 - z_2\bar{z}_2 - z_3\bar{z}_3) \\
a_1 &= z_0\bar{z}_2 + z_3\bar{z}_1 \\
a_2 &= z_0\bar{z}_3 - z_2\bar{z}_1 \\
\sum_i x_i^2 &= \rho^2 \\
\sum_i |z_i|^2 &= 1. \tag{3.7}
\end{align*}
\]

By promoting the complex coordinates \( z_i \) to four oscillators as follows:

\[
[z_i, \bar{z}_j] = \delta_{ij} \quad [z_i, z_j] = 0 \tag{3.8}
\]

the corresponding algebra for \( \hat{x}_i \) is the fuzzy four-sphere algebra. For example the \( SU(2) \times SU(2) \) subalgebra made by \( \hat{N}_i \) and \( \hat{M}_i \) can be easily represented in terms of \( z_i \) as:
\[ \hat{N}_3 = \frac{1}{2}(z_3 \bar{z}_3 - z_2 \bar{z}_2) \quad \hat{M}_3 = \frac{1}{2}(z_0 \bar{z}_0 - z_1 \bar{z}_1) \]
\[ \hat{N}_+ = \hat{N}_1 + i \hat{N}_2 = z_2 \bar{z}_3 \quad \hat{M}_+ = \hat{M}_1 + i \hat{M}_2 = z_1 \bar{z}_0 \]
\[ \hat{N}_- = \hat{N}_1 - i \hat{N}_2 = z_3 \bar{z}_2 \quad \hat{M}_- = \hat{M}_1 - i \hat{M}_2 = z_0 \bar{z}_1. \] (3.9)

The parameter \( n \) is simply the eigenvalue of the number operator \( \hat{N} \):

\[ \hat{N} = \bar{z}_0 z_0 + \bar{z}_1 z_1 + \bar{z}_2 z_2 + \bar{z}_3 z_3 \quad \hat{N} \to n. \] (3.10)

In terms of \( \hat{N} \), the Casimir for \( \hat{x}_i^2 \) is:

\[ \sum_i \hat{x}_i^2 = \rho^2 \hat{N}(\hat{N} + 4) = R^2. \] (3.11)

To construct the \( k \)-instanton projectors \( P_k(x) \) for the fuzzy four-sphere, let us consider the following vectors:

\[ |\psi_k > = N_k \left( \begin{array}{c} (z_0)^k \\
\ldots \\
\sqrt{\frac{k!}{i_1!i_2!i_3!(k-i_1-i_2-i_3)!}} z_0^{k-i_1-i_2-i_3} z_1^i z_2^j z_3^k \\
\ldots \\
(z_1)^k \end{array} \right) \] (3.12)

where

\[ 0 \leq i_1 \leq k \quad 0 \leq i_2 \leq k - i_1 \quad 0 \leq i_3 \leq k - i_1 - i_2. \] (3.13)

Fixing \( i_1 \) and \( i_2 \), the index \( i_3 \) takes \( k - i_1 - i_2 + 1 \) values. Fixing \( i_1 \), the indices \( i_2 \) and \( i_3 \) take \( \frac{(k-i_1+1)(k-i_1+2)}{2} \) values.

In total the number of entries of the vector \( |\psi_k > \) is given by summing over \( i_1 \) as follows:

\[ N_k = \frac{(k+1)(k+2)(k+3)}{6} \] (3.14)

which is equal to the number \( N \) of the size of the matrix \( \hat{\Gamma}_\mu^{(k)} \).

The normalization condition for these vectors fixes the function \( \hat{N}_k \) to be dependent only on the number operator \( \hat{N} \):
The corresponding $k$-instanton connection 1-form can be computed in terms of the vector $|\psi_k>$:

$$A^\nabla_k = <\psi_k|d|\psi_k>.$$  \hspace{1cm} (3.16)

The projector for the $k$-instanton on the fuzzy four sphere is defined to be:

$$P_k = |\psi_k><\psi_k| \quad P_k^2 = P_k \quad P_k^\dagger = P_k.$$  \hspace{1cm} (3.17)

In the product of the two ket and bra vectors, the sixteen combinations of oscillators can be written in terms of the algebra of the fuzzy sphere and the number operator, which is equal to its eigenvalue $n$ on a definite representation. Therefore $P_k$ is a matrix having as entries the basic operator algebra of the theory.

The trace of the projector $P_k$ is always positive definite since

$$TrP_k = \frac{(n+k+1)(n+k+2)(n+k+3)}{(n+1)(n+2)(n+3)} Tr1 = \frac{(n+k+1)(n+k+2)(n+k+3)}{6} \frac{(n+1)(n+2)(n+3)}{6} \frac{(k+1)(k+2)(k+3)}{6} = Tr1_P$$  \hspace{1cm} (3.18)

where $1_P$ is the identity projector.

To construct the $k$-anti-instanton solution it is enough to take the adjoint of the vector $|\psi_k>$. Consider the $\frac{(k+1)(k+2)(k+3)}{6}$-dimensional vectors:

$$|\psi_{-k}>= N_k \left( \begin{array}{c} \sqrt{1}^{k} \prod_{i_1,i_2,i_3,(k-i_1-i_2-i_3)!=0}^{k!} z_{1}^{i_1-1} z_{2}^{i_2-1} z_{3}^{i_3-1} \\
\cdots\\n(\bar{z}_1)^k 
\end{array} \right).$$  \hspace{1cm} (3.19)
The normalization condition for these vectors fixes the function $N_k$ to be dependent only on the number operator $\hat{N}$:

$$<\psi_{-k}|\psi_{-k}> = 1 \quad N_k = N_k(\hat{N}) = \frac{1}{\sqrt{\prod_{i=0}^{k-1}(\hat{N} + i + 4 - k)}} = \frac{1}{\sqrt{\prod_{i=0}^{k-1}(n + i + 4 - k)}}. \quad (3.20)$$

The corresponding projector for anti-instantons is

$$P_{-k} = |\psi_{-k}><\psi_{-k}| \quad k < n + 4. \quad (3.21)$$

The trace of this projector

$$TrP_{-k} = \frac{(n - k + 1)(n - k + 2)(n - k + 3)}{(n + 1)(n + 2)(n + 3)}Tr1 = \frac{(n - k + 1)(n - k + 2)(n - k + 3)}{6} \quad (3.22)$$

is positive definite if and only if the following bound is respected

$$k < n + 1. \quad (3.23)$$

For the special cases $k = n + 1, k = n + 2, k = n + 3$ $P_{-k}$ is simply the null projector.

The projector above, defining the $k$-projective moduli of the non commutative gauge theory over the fuzzy four-sphere can be used to compute the following Chern class

$$c_k = Tr(\gamma_5 P_k dP_k dP_k dP_k dP_k) \quad (3.24)$$

as we did in [1] for the fuzzy two-sphere.

4 Conclusions

In this paper we have made use of an alternative description to the instanton connections and gauge fields in terms of globally defined projectors, a method which is suitable for generalization like noncommutative geometry.
In particular we have shown how to derive exact expressions for the noncommutative $k$-instantons on the fuzzy four-sphere. The projectors defined here are finite-dimensional matrices as in the two dimensional case, and can be used to compute the corresponding topological charge (Chern class for the vector bundle) in four dimensions. We believe that this intrinsic description of connections and gauge fields in global terms will help in extending the methods of noncommutative geometry to the case of the fuzzy four-sphere.

A possible generalization of this paper would be to study the flat limit of the fuzzy four-sphere, which should define the noncommutative $k$-instantons on the noncommutative four plane. This could represent an alternative method with respect to the papers [14]-[15]-[16]-[17]-[18]-[19]-[20], based on the Hopf fibration $S^7 \rightarrow S^4$ and the flat limit.

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