Exponential Stability of Leakage Delay and Semi-Markovian Jump for Neutral-Type Neural Network

YAN GAO¹, FENGJIAO GU¹, CARLO CATTANI², AND WANQING SONG¹

¹School of Electronic and Electrical Engineering, Shanghai University of Science and Engineering, Shanghai 201620, China
²Engineering School, DEIM, University of Tuscia, 01100 Viterbo, Italy

Corresponding author: Yan Gao (gy@sues.edu.cn)

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Abstract In this paper, a suitable Lyapunov-Krasovskii functional together with the inequality analysis technique for neutral-type system, and sufficient exponential stability conditions are proposed in the form of linear matrix inequalities (LMIs). Firstly, the system model of leakage delay and semi-Markovian process for neutral-type neural network is established, where the sojourn-time follows Weibull distribution, which is time-varying instead of being constant in the transition rate. Secondly, through combinations of stability analysis method, Lyapunov-Krasovskii functional and inequality analysis technique, some sufficient conditions have been achieved to undertake the exponential stabilization of given neural network system. Moreover, time delays of the system contain leakage and mixed delays, which are more general and appropriate. Finally, in order to show the many advantages of our model, the obtained results will be compared with previous works based on methods different from semi-Markovian jump or leakage delay.

Index Terms Exponential stability, leakage delay, neutral-type neural network, semi-Markovian jump.

I. INTRODUCTION

In the past decades, neural networks have been found in extensive studies [1]–[3] where time delay is a fundamental concept. In particular, neutral-type neural network is a special type of time-delay system, whose model structure is characterized by derivative of the current state and the past state. These systems have been received considerable attention in engineering such as population ecology, distributed networks containing lossless transmission lines, heat exchangers and robots in contact with rigid environments. The system model is more general, among neutral-type neural networks, compared to traditional delayed neural networks [4]–[6]. On the other hand, many results of Markovian process in neural network systems have been studied to analyze dynamical behaviors of Markov jump neural networks [7]–[9].

However, Markovian jump neural network systems are conservative in many practical applications, since the sojourn-time of a Markovian chain only follows exponential distribution. Specifically, its sojourn-time follows Weibull distribution in semi-Markovian switching, whose transition rate is known to be time-varying instead of being constant. This indicates that semi-Markovian switching neural network has broader application prospect than Markovian switching due to its relaxed conditions on the probability distributions [10]–[12]. Therefore, studies on semi-Markovian switching neural networks are so important both from a theoretical approach and practical significance.

In [13], Liu et al. investigated stability analysis for a class of neutral-type neural networks with Markovian jumping parameters and mode-dependent mixed delays. In [14], Dai et al. discussed adaptive exponential synchronization in mean square for Markovian jumping neutral-type coupled neural networks with time-varying delays by pinning control. In [15], Saravanakumar et al. addressed stability of markovian jump generalized neural networks with interval time-varying delays. Stability analysis of neutral-type neural networks with additive time-varying delay components and leakage delay was studied in [16]. In [17], Cheng et al. considered globally asymptotic stability of a class of neutral-type neural networks with delays.

Another typical time delay, called leakage delay, has been widely found in the negative feedback terms of neural networks. Since leakage delay has a great influence on the
dynamical behaviors of neural networks, it is necessary and important to consider the leakage delay effects on the stability of systems. Therefore, the stability analysis of neural networks with leakage delay has been one of the hot topics and many research achievements have been reported. In [18], Zheng et al. considered stability analysis of stochastic fuzzy Markovian jumping neural networks with leakage delay under impulsive perturbations. In [19], Shu discussed Global exponential stability of Markovian jumping stochastic impulsive uncertain BAM neural networks with leakage, mixed time delays, and α-inverse Hölder activation functions. In [20], Liu et al. considered stability for discrete-time stochastic neural networks with both leakage and probabilistic delays. Moreover, passivity of uncertain neural networks with both leakage delay and time-varying delay is discussed in [21]. In [22], stochastic synchronization was also considered for neutral-type chaotic impulse neural networks with Markovian jumping parameters and some conditions were derived by adopting finite-time stability theorem.

In [23], state estimation for semi-Markovian switching systems with sliding mode control and mismatched uncertainties were discussed. In [24], [25], stochastic stability of Itô differential equation [26] with semi-Markovian jump parameters was considered by using Itô inequality and LMIs. In [27], the stochastic stability of semi-Markovian jump systems with mode-dependent delays was investigated, by establishing a suitable Lyapunov-Krasovskii function and LMIs approach. The stochastic synchronization for continuous-time semi-Markovian jump neural networks with time-varying delay were addressed in [28]. Compared with existing studies on semi-Markovian jump neural networks in [24], the difference of the stability analysis is considered in this paper. The method proposed in [24] only guarantees the stochastic stability of the proposed system, but the method given in this paper can ensure that the system is exponential stability. This guarantees that the resulting system will attain a fast and satisfactory response. It should be noted that, as far as we know, exponential stability of neutral-type neural network with leakage delay and semi-Markovian jump has received very little attention in the academic field.

This paper aims to investigate exponential stability of neutral-type neural network with leakage delay and semi-Markovian jump. By applying Lyapunov-Krasovskii functional methods and stability analysis techniques, some sufficient conditions have been presented to undertake the exponential stability for given neural network system. Compared to the neutral-type time-delay neural networks in [14] and [15], differences in time-delay, such as mixing and leakage delays, are investigated. Moreover, compared to the model of Markovian jump in [7], this paper discusses the semi-Markovian jump. Based on exponential stability theory, new exponential stability condition for neutral-type neural network with mixed, leakage delays and semi-Markovian jump will be investigated in terms of LMIs. The contributions of this paper can be summarized as follows:

1. It is the first time to build a new model for neutral-type neural network with leakage delay and semi-Markovian jump.
2. We construct a suitable Lyapunov-Krasovskii functional to achieve the new criterions for the system.
3. The Lyapunov-Krasovskii functional and inequality analysis technique are used to undertake exponential stability for the proposed model.
4. Finally, the numerical example is provided to verify the effectiveness of exponential stability scheme.

The remaining content of this paper is arranged as follows. In Section II, the model of leakage delay and semi-Markovian jump for neutral-type neural network is presented. In addition, some preliminary topics are introduced, including some definitions and assumptions. In Section III, some sufficient conditions and results of exponential stability are addressed. Section IV provides a numerical example to demonstrate the validity of the obtained results. Finally, conclusions are summarized in Section V.

II. MODELING NEUTRAL-TYPE NEURAL NETWORK WITH LEAKAGE DELAY AND SEMI-MARKOVIAN

Considering the following neutral-type neural network with mixed and leakage delays:

\[
d\begin{bmatrix} x(t) - E x(t - h(t)) \\
- \lambda x(t - \delta) + B g(x(t)) + C g(x(t - \tau(t))) \\
+ D \int_{t-\delta}^{t} g(x(s))ds \end{bmatrix} dt + C(t) w(t),
\]

where \( x(t) = [x_1(t), x_2(t), \ldots, x_n(t)]^T \) is the state vector, \( \delta \geq 0 \) is the constant leakage delay, \( \tau(t), d(t) \) and \( h(t) \) denote the discrete, distributed and neutral time-varying delays, \( g(\cdot) \) denotes the neuron activation function, \( g(x(t)) = [g_1(x_1(t)), g_2(x_2(t)), \ldots, g_n(x_n(t))]^T \). \( A = diag(a_1, a_2, \ldots, a_n) \) is a positive diagonal matrix with positive entries \( a_i > 0 \), \( C(t) = (e_{ij})_{n \times n} \) is the noisy interconnection matrix, \( B = (b_{ij})_{n \times n}, C = (c_{ij})_{n \times n}, D = (d_{ij})_{n \times n} \) is the n-dimensional Brown motion defined [29], [30] on a complete probability space \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathcal{P})\). Let \( \{r(t), t \geq 0\} \) be a right continuous semi-Markov chain on the complete probability space \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathcal{P})\). It takes value in a finite states space \( S = \{1, 2, \ldots, N\} \) with generator \( \Pi = \{\pi_{ij}(t)\}_{S \times S} \) given by

\[
P \{ r(t + \iota) = j | r(t) = i \} = \begin{cases} \pi_{ij}(t) + o(t), & \text{if } i \neq j, \\ 1 + \pi_{ij}(t) + o(t), & \text{if } i = j, \end{cases}
\]

where \( \iota > 0 \) and \( \lim_{\iota \to 0} o(t)/\iota = 0. \) \( \pi_{ij}(t) \) is the transition rate from \( i \) to \( j \) if \( i \neq j \), while \( \pi_{ii}(t) = - \sum_{j=1,j \neq i}^{S} \pi_{ij}(t) \) for each mode \( i \).
Remark 1: (1) In continuous-time Markovian process, the sojourn-time $\tau$ is subject to the exponential distribution, the transition rate $\pi_{ij}(\tau)$ is a constant, i.e.,

$$\pi_{ij}(\tau) = \frac{f_\tau(\tau)}{1 - F_\tau(\tau)} = \frac{\pi_{ij}\exp(-\tau\pi_{ij})}{1 - (1 - \exp(-\tau\pi_{ij}))} = \pi_{ij},$$

where $f_\tau(\tau)$ and $F_\tau(\tau)$ are respectively the probability density function and cumulative distribution function of the exponential distribution, and $\pi_{ij} > 0$ is the rate parameter.

(2) In continuous-time semi Markovian process, the probability distribution of sojourn-time $\tau$ can be Weibull distribution, the transition rate $\pi_{ij}(\tau)$ is time-varying and depends on the sojourn-time $\tau$. For the case of Weibull distribution, the probability density function is

$$f(\tau) = \begin{cases} \frac{\beta}{\alpha\tau^{\beta-1}} \exp\left(-\left(\frac{\tau}{\alpha}\right)^\beta\right), & \tau \geq 0, \\ 0, & \tau < 0, \end{cases}$$

where the scale parameter $\alpha > 0$ and shape parameter $\beta > 0$. The cumulative distribution function can be described by,

$$F(\tau) = \begin{cases} 1 - \exp\left(-\left(\frac{\tau}{\alpha}\right)^\beta\right), & \tau \geq 0, \\ 0, & \tau < 0, \end{cases}$$

thus, the transition rate function for Weibull distribution can be derived

$$\pi_{ij}(\tau) = \frac{f(\tau)}{1 - F(\tau)} = \frac{\beta}{\alpha\tau^{\beta-1}},$$

obviously, when the shape parameter $\beta = 1$, Weibull distribution changes into exponential distribution. In this case, the continuous time semi-Markovian process is reduced to a traditional continuous time Markovian process, and $\pi_{ij}(\tau)$ is a constant $1/\alpha$.

Based on the discussions in the section above, we consider the following semi-Markovian jumping neutral-type neural network systems with mixed and leakage delays:

$$\begin{align*}
\dot{x}(t) &= -A(x(t))x(t) - h(t) \\
&= [-A_i x_i(t) - A_j x_j(t)] + C_i x_i(t) + C_j x_j(t) + D_i x_i(t) + D_j x_j(t) + E_i x_i(t) + E_j x_j(t),
\end{align*}$$

Thus, system (2) can be rewritten as the following:

$$\begin{align*}
\dot{x}(t) &= -A(x(t))x(t) - h(t) \\
&= [-A_i x_i(t) - A_j x_j(t)] + C_i x_i(t) + C_j x_j(t) + D_i x_i(t) + D_j x_j(t) + E_i x_i(t) + E_j x_j(t),
\end{align*}$$

Assumption 1: The neuron activation function $g_i(s)$ satisfy

$$l_i^+ \leq g_i(\xi_1) - g_i(\xi_2) \xi_1 - \xi_2 \leq l_i^-,$$

where $\xi_1, \xi_2 \in R$, $\xi_1 \neq \xi_2$.

Definition 1: The dynamics (2) are said to be exponentially mean-square stable, if for all finite initial functions $\phi(t)$ on $(-\infty, 0)$, all initial conditions $x_0 \in R^n$, and $r_0 \in N$, there exist constants $\alpha > 0$ and $\beta > 0$, such that

$$E \{||x(t)||^2 | x_0, r_0\} \leq e^{-\alpha t} ||\phi||^2, \ \forall t \geq 0,$$

where $||\phi||^2 \triangleq \sup_{x \in [-x, 0]} \{||\phi(s)||^2 \cdot ||\phi(s)||^2\}$.

III. DERIVATION OF EXPONENTIAL STABILITY CONDITION

The infinitesimal operator is defined as follows:

$$L V(x(t), t, r(t)) = \lim_{\Delta \to 0} \frac{1}{\Delta} \left[ E\{V(x(t + \Delta), \tau(t + \Delta))| x(t), r(t)\} - V(x(t), r(t)) \right],$$

where $\Delta$ is a small positive number.

Theorem 1: For given scalars $h, h, \tilde{h}, \tau, \tilde{\tau}, \mu, d, \tilde{d}$, the Markovian jumping neutral-type neural network is exponentially stable if there exist positive scalar $\gamma$, positive definite symmetric matrices $P_i, Q_i, R_i, S_i, \tilde{Q}, \tilde{R}, \tilde{S}, S_0$, and diagonal matrices $Z_0 > 0, Z_2 > 0, H_i$ and positive scalar $\epsilon$, such that

$$\begin{align*}
\begin{bmatrix}
\sum_{j=0}^{N} (i) + \sum_{j=0}^{N} (i) \\
\epsilon T_1 \\
T_2 \\
T_2 \\
\end{bmatrix} &< 0,
\end{align*}$$

where

$$\begin{align*}
\sum_{j=1}^{N} \lambda_{ij}(h) I_j &\leq \tilde{Q}, \\
\sum_{j=1}^{N} \lambda_{ij}(h) R_j &\leq \tilde{R}, \\
\sum_{j=1}^{N} \lambda_{ij}(h) S_j &\leq \tilde{S},
\end{align*}$$

and

$$\begin{align*}
\sum_{j=1}^{N} \lambda_{ij}(h) I_j &\leq \tilde{Q}, \\
\sum_{j=1}^{N} \lambda_{ij}(h) R_j &\leq \tilde{R}, \\
\sum_{j=1}^{N} \lambda_{ij}(h) S_j &\leq \tilde{S},
\end{align*}$$

"
where
\[
\sum_{(i)} = \begin{bmatrix}
\sum_1 (i) \\
\sum_2 (i) \\
\sum_3 (i) \\
\sum_4 (i) \\
\sum_5 (i)
\end{bmatrix} \Psi (i) \Phi (i) \Psi^T (i) \Phi^T (i),
\]
\[
\Psi^T (i) = \begin{bmatrix}
\Lambda^T (i) \\
\Lambda^T (i) \\
\Lambda^T (i) \\
\Lambda^T (i) \\
\Lambda^T (i)
\end{bmatrix},
\]
\[
\Lambda (i) = \begin{bmatrix}
\lambda_1 (i) \\
\lambda_2 (i) \\
\lambda_3 (i) \\
\lambda_4 (i) \\
\lambda_5 (i) \\
\lambda_6 (i)
\end{bmatrix},
\]
\[
l_1 (i) = \begin{bmatrix}
-P (i) A (i) \\
-H (i) C (i) \\
-P (i) A (i) \\
-H (i) C (i) \\
-P (i) A (i) \\
-H (i) C (i)
\end{bmatrix},
\]
\[
l_2 (i) = \begin{bmatrix}
P (i) D (i) \\
0 \\
P (i) D (i)
\end{bmatrix},
\]
\[
l_3 (i) = \begin{bmatrix}
P (i) B (i) \\
H (i) C (i) \\
P (i) B (i) \\
H (i) C (i)
\end{bmatrix},
\]
\[
l_4 (i) = \begin{bmatrix}
P (i) C (i) \\
0 \\
P (i) C (i)
\end{bmatrix},
\]
\[
l_5 (i) = \begin{bmatrix}
P (i) E (i) \\
0 \\
P (i) E (i)
\end{bmatrix},
\]
\[
\Phi (i) = \text{diag} (\Phi_2 (i), \Phi_3 (i), \Phi_4 (i), \Phi_5 (i), \Phi_6 (i)),
\]
\[
\Phi_2 (i) = -P (i) Q_1^1 P (i), \quad \Phi_3 (i) = -\phi_1^{-1} P (i) S_1^{1} P (i),
\]
\[
\Phi_4 (i) = -\tau (i) P (i) S_2^1 P (i), \quad \Phi_5 (i) = -\phi_1^{-1} P (i) K_1 P (i),
\]
\[
\Phi_6 (i) = -h^{-1} P (i) K_2 P (i),
\]
\[
\sum_0 (i) = \text{diag} \{\sum_{01} (i), 0, 0, 0, 0, 0\},
\]
\[
\sum_1 (i) = \sum_{11} (i) + \sum_{12} (i) + \sum_{13} (i),
\]
\[
\sum_{11} (i) = \sum_{i=1}^{2} \sum R_i + \sum Q_i + R (i) - L_1 Z_1 + Q (i) + S (i) - L_3 Z_1 + \sum_{j=1}^{N} \lambda_j (h) P (j) - Z_1^T L_4^T + \tau \hat{R} + h \hat{Q} + \delta S \text{M}_1,
\]
\[
\sum_{12} (i) = \sum_{i=1}^{2} M_1^T \sum_{i=1}^{2} L_1 Z_1 + L_3 Z_2 - M_1^T E_1 M_1,
\]
\[
\sum_{13} (i) = \sum_{i=1}^{2} M_1^T ((1 - \dot{\tau}) R_3 (i) + L_1 Z_2 + L_3 Z_2 + Z_2^T L_4^T + Z_2^T L_4^T) M_2 - M_2^T R_1 M_3 - d^{-1} M_1^T K_3 M_1 + M_2^T L_2 Z_2 M_{10} - \phi_1^{-1} (M_5 - M_7^T) K_1 (M_5 - M_7) - \phi_0^{-1} (M_5 - M_6^T) K_1 (M_5 - M_6) - R_2 M_4 - (1 - h) M_5^T Q_5 M_3 - M_6^T Q_1 M_5 - M_7^T Q_2 M_7 - (1 - h) M_5^T Q_3 M_3 - M_6^T Q_1 M_5 - M_7^T Q_2 M_7 + M_1^T L_2 Z_1 M_{10} - \phi_1^{-1} (M_2 - M_4^T) S_1 (M_2 - M_4) + M_1^T Z_2 M_{10} + M_1^T Z_2 M_{10} + M_1^T Z_2 M_10 + M_1^T Z_2 M_10 + M_1^T Z_2 M_10 + M_1^T Z_2 M_10,
\]
\[
\sum_{01} (i) = \phi_1^{-1} (M_2 - M_4^T) S_2 (M_2 - M_4) - \phi_1^{-1} (M_2 - M_4^T) S_2 (M_2 - M_4) - \phi_1^{-1} (M_1 - M_3^T) S_2 (M_1 - M_3) - \phi_1^{-1} (M_5 - M_7^T) K_2 (M_5 - M_7).
\]

\[
V_1 (x (t), t, r (t)) = \sum_{i=1}^{N} V_i (x_i (t), t, r (t)),
\]

\[
V_2 (x (t), t, r (t)) = \int_{t-h}^{t} x^T (s) S x (s) ds + \int_{t-h}^{t} \dot{x}^T (s) \hat{R} x (s) ds + \int_{t-h}^{t} \dot{x}^T (s) \hat{Q} x (s) ds + \int_{t-h}^{t} \dot{x}^T (s) Q x (s) ds + \int_{t-h}^{t} \dot{x}^T (s) Q x (s) ds,
\]

\[
V_3 (x (t), t, r (t)) = \int_{t-h}^{t} x^T (s) \hat{S} x (s) ds + \int_{t-h}^{t} \dot{x}^T (s) \hat{S} x (s) ds + \int_{t-h}^{t} \dot{x}^T (s) \hat{S} x (s) ds + \int_{t-h}^{t} \dot{x}^T (s) \hat{S} x (s) ds + \int_{t-h}^{t} \dot{x}^T (s) \hat{S} x (s) ds,
\]

\[
V_4 (x (t), t, r (t)) = \int_{t-h}^{t} \dot{x}^T (s) K_1 x (s) ds + \int_{t-h}^{t} \dot{x}^T (s) K_1 x (s) ds + \int_{t-h}^{t} \dot{x}^T (s) K_1 x (s) ds + \int_{t-h}^{t} \dot{x}^T (s) K_1 x (s) ds + \int_{t-h}^{t} \dot{x}^T (s) K_1 x (s) ds,
\]

\[
V_5 (x (t), t, r (t)) = \int_{t-h}^{t} \dot{x}^T (s) K_2 \dot{x} (s) ds + \int_{t-h}^{t} \dot{x}^T (s) K_2 \dot{x} (s) ds + \int_{t-h}^{t} \dot{x}^T (s) K_2 \dot{x} (s) ds + \int_{t-h}^{t} \dot{x}^T (s) K_2 \dot{x} (s) ds + \int_{t-h}^{t} \dot{x}^T (s) K_2 \dot{x} (s) ds,
\]

\[
V_6 (x (t), t, r (t)) = \int_{t-h}^{t} g^T (x (s)) K_3 g (x (s)) ds + \int_{t-h}^{t} g^T (x (s)) K_3 g (x (s)) ds + \int_{t-h}^{t} g^T (x (s)) K_3 g (x (s)) ds + \int_{t-h}^{t} g^T (x (s)) K_3 g (x (s)) ds + \int_{t-h}^{t} g^T (x (s)) K_3 g (x (s)) ds,
\]

\[
\Lambda_1 (t) = P (i) \begin{bmatrix}
\Lambda_2 (i) \\
C_1 (i) \\
\Lambda_2 (i)
\end{bmatrix},
\]

\[
\Lambda_2 (i) = [A (i) 0 0 0 0 0 0 0 0 0 0 E (i) B (i) C (i) D (i)].
\]
According to the definition, from (6), we have

\[ \mathcal{L}V_1(x(t), t, r(t)) = 2x^T(t)P(t)\dot{x}(t) + x^T(t)\sum_{j=1}^{N} \lambda_{ij}(\eta)P(j)x(t) \]

\[ = 2x^T(t)P(t)\Lambda_2(t)\dot{x}(t) + x^T(t)\sum_{j=1}^{N} \lambda_{ij}(\eta)P(j)x(t) \]

\[ = \xi^T(t)[M_1^T P(t) A(i) + \Lambda_2(t) P(i) M_1 + M_1^T \sum_{j=1}^{N} \lambda_{ij}(\eta)P(j)]M_1^{1/2}(t), \quad (12) \]

\[ \mathcal{L}V_2(x(t), t, r(t)) \leq x^T(t)(S(i) + R(i) + R_1 + R_2 + Q(i) + Q_1 + Q_2) + x^T(t)\sum_{j=1}^{N} \lambda_{ij}(\eta)R(j)x(t)ds \]

\[ + \int_{t-h}^{t} x^T(s)\sum_{j=1}^{N} \lambda_{ij}(\eta)Q(j)x(s)ds \]

\[ + \int_{t-h}^{t} x^T(s)\sum_{j=1}^{N} \lambda_{ij}(\eta)S(j)x(s)ds \]

\[ \leq \xi^T(t)\sum_{j=1}^{N} \lambda_{ij}(\eta)R(j)x(t) \]

\[ + \xi^T(t)\sum_{j=1}^{N} \lambda_{ij}(\eta)Q(j)x(t) \]

\[ + \xi^T(t)\sum_{j=1}^{N} \lambda_{ij}(\eta)S(j)x(t) \]

\[ \leq \xi^T(t)\sum_{j=1}^{N} \lambda_{ij}(\eta)R(j)x(t) \]

\[ + \xi^T(t)\sum_{j=1}^{N} \lambda_{ij}(\eta)Q(j)x(t) \]

\[ + \xi^T(t)\sum_{j=1}^{N} \lambda_{ij}(\eta)S(j)x(t) \]

\[ \leq \xi^T(t)\sum_{j=1}^{N} \lambda_{ij}(\eta)R(j)x(t) \]

\[ + \xi^T(t)\sum_{j=1}^{N} \lambda_{ij}(\eta)Q(j)x(t) \]

\[ + \xi^T(t)\sum_{j=1}^{N} \lambda_{ij}(\eta)S(j)x(t) \]

\[ \leq \xi^T(t)\sum_{j=1}^{N} \lambda_{ij}(\eta)R(j)x(t) \]

\[ + \xi^T(t)\sum_{j=1}^{N} \lambda_{ij}(\eta)Q(j)x(t) \]

\[ + \xi^T(t)\sum_{j=1}^{N} \lambda_{ij}(\eta)S(j)x(t) \]

\[ \leq \xi^T(t)\sum_{j=1}^{N} \lambda_{ij}(\eta)R(j)x(t) \]

\[ + \xi^T(t)\sum_{j=1}^{N} \lambda_{ij}(\eta)Q(j)x(t) \]

\[ + \xi^T(t)\sum_{j=1}^{N} \lambda_{ij}(\eta)S(j)x(t) \]

\[ \leq \xi^T(t)\sum_{j=1}^{N} \lambda_{ij}(\eta)R(j)x(t) \]

\[ + \xi^T(t)\sum_{j=1}^{N} \lambda_{ij}(\eta)Q(j)x(t) \]

\[ + \xi^T(t)\sum_{j=1}^{N} \lambda_{ij}(\eta)S(j)x(t) \]

\[ \leq \xi^T(t)\sum_{j=1}^{N} \lambda_{ij}(\eta)R(j)x(t) \]

\[ + \xi^T(t)\sum_{j=1}^{N} \lambda_{ij}(\eta)Q(j)x(t) \]

\[ + \xi^T(t)\sum_{j=1}^{N} \lambda_{ij}(\eta)S(j)x(t) \]

\[ \leq \xi^T(t)\sum_{j=1}^{N} \lambda_{ij}(\eta)R(j)x(t) \]

\[ + \xi^T(t)\sum_{j=1}^{N} \lambda_{ij}(\eta)Q(j)x(t) \]

\[ + \xi^T(t)\sum_{j=1}^{N} \lambda_{ij}(\eta)S(j)x(t) \]

\[ \leq \xi^T(t)\sum_{j=1}^{N} \lambda_{ij}(\eta)R(j)x(t) \]

\[ + \xi^T(t)\sum_{j=1}^{N} \lambda_{ij}(\eta)Q(j)x(t) \]

\[ + \xi^T(t)\sum_{j=1}^{N} \lambda_{ij}(\eta)S(j)x(t) \]

\[ \leq \xi^T(t)\sum_{j=1}^{N} \lambda_{ij}(\eta)R(j)x(t) \]

\[ + \xi^T(t)\sum_{j=1}^{N} \lambda_{ij}(\eta)Q(j)x(t) \]

\[ + \xi^T(t)\sum_{j=1}^{N} \lambda_{ij}(\eta)S(j)x(t) \]

\[ \leq \xi^T(t)\sum_{j=1}^{N} \lambda_{ij}(\eta)R(j)x(t) \]

\[ + \xi^T(t)\sum_{j=1}^{N} \lambda_{ij}(\eta)Q(j)x(t) \]

\[ + \xi^T(t)\sum_{j=1}^{N} \lambda_{ij}(\eta)S(j)x(t) \]

\[ \leq \xi^T(t)\sum_{j=1}^{N} \lambda_{ij}(\eta)R(j)x(t) \]

\[ + \xi^T(t)\sum_{j=1}^{N} \lambda_{ij}(\eta)Q(j)x(t) \]

\[ + \xi^T(t)\sum_{j=1}^{N} \lambda_{ij}(\eta)S(j)x(t) \]

\[ \leq \xi^T(t)\sum_{j=1}^{N} \lambda_{ij}(\eta)R(j)x(t) \]

\[ + \xi^T(t)\sum_{j=1}^{N} \lambda_{ij}(\eta)Q(j)x(t) \]

\[ + \xi^T(t)\sum_{j=1}^{N} \lambda_{ij}(\eta)S(j)x(t) \]

\[ \leq \xi^T(t)\sum_{j=1}^{N} \lambda_{ij}(\eta)R(j)x(t) \]

\[ + \xi^T(t)\sum_{j=1}^{N} \lambda_{ij}(\eta)Q(j)x(t) \]

\[ + \xi^T(t)\sum_{j=1}^{N} \lambda_{ij}(\eta)S(j)x(t) \]

\[ \leq \xi^T(t)\sum_{j=1}^{N} \lambda_{ij}(\eta)R(j)x(t) \]

\[ + \xi^T(t)\sum_{j=1}^{N} \lambda_{ij}(\eta)Q(j)x(t) \]

\[ + \xi^T(t)\sum_{j=1}^{N} \lambda_{ij}(\eta)S(j)x(t) \]

\[ \leq \xi^T(t)\sum_{j=1}^{N} \lambda_{ij}(\eta)R(j)x(t) \]

\[ + \xi^T(t)\sum_{j=1}^{N} \lambda_{ij}(\eta)Q(j)x(t) \]

\[ + \xi^T(t)\sum_{j=1}^{N} \lambda_{ij}(\eta)S(j)x(t) \]
\begin{align*}
K\left(\int_{t-d}^{t} g(x(t))ds\right) \\
\leq dT(t)M_{10}^{T}KM_{10}\hat{x}(t) - d^{-1}\hat{x}(t)M_{12}^{T}KM_{12}\hat{x}(t).
\end{align*}

(17)

For the function \(g(s)\), from (4), we have

\[
[g_i(x_i(t)) - l^+_i x_i(t)] [g_j(x_j(t)) - l^+_j x_j(t)] \leq 0,
\]

\[
[g_i(x_i(t) - (t)) - l^+_i x_i(t) - (t))] \times [g_j(x_j(t) - (t)) - l^+_j x_j(t) - (t)))] \leq 0.
\]

For convenience, we denote the matrices:

\[
L_1 = \text{diag}([1, l^+_1, l^+_2, \ldots, l^+_n]),
\]

\[
L_2 = \text{diag}([1, l^+_1, l^+_2, l^+_2, \ldots, l^+_n + l^+_n]).
\]

Then, for \(Z_j = \text{diag}(z_1, z_2, \ldots, z_n) \geq 0, j = 1, 2, \) we obtain

\[
0 \leq -2\sum_{i=1}^{n} z_1 [g_i(x_i(t)) - l^+_i x_i(t)] [g_i(x_i(t)) - l^+_i x_i(t)] \\
= -2\hat{x}(t)^T M_{10}^{T} Z_{11} M_{10}\hat{x}(t) + 2\hat{x}(t)^T M_{1}^{T} L_{2} Z_{11} M_{10}\hat{x}(t) \\
- 2\hat{x}(t)^T M_{1}^{T} L_{1} Z_{1} M_{10}\hat{x}(t).
\]

(18)

\[
0 \leq -2\sum_{i=1}^{n} z_2 [g_i(x_i(t) - (t)) - l^+_i x_i(t) - (t))] \\
\times [g_i(x_i(t) - (t)) - l^+_i x_i(t) - (t)))] \\
= -2\hat{x}(t)^T M_{1}^{T} Z_{12} M_{12}\hat{x}(t) + 2\hat{x}(t)^T M_{1}^{T} L_{2} Z_{22} M_{12}\hat{x}(t) \\
- 2\hat{x}(t)^T M_{1}^{T} L_{1} Z_{1} M_{12}\hat{x}(t),
\]

(19)

from (12) to (19), we can get that

\[
\mathcal{L}V(x(t), t, r(t)) \\
= \sum_{i=1}^{5} \mathcal{L}V_{i}(x(t), t, r(t)) \\
\leq \hat{x}(t)^T M_{1}^{T} Z_{1} M_{10}\hat{x}(t) - \hat{x}(t)^T M_{1}^{T} L_{1} Z_{1} M_{10}\hat{x}(t) \\
- \hat{x}(t)^T M_{1}^{T} L_{2} Z_{11} M_{10}\hat{x}(t) + \hat{x}(t)^T M_{1}^{T} L_{1} Z_{1} M_{10}\hat{x}(t).
\]

(20)

Applying Dynkin’s formula, we can have

\[
E[x^{2\alpha t} V(x(t), t, r(t))] = E[V(x_0, r_0)] \\
= E\int_{0}^{r(t)} x^{2\alpha u} \left(2\alpha V(x(u), u, r_0) + \mathcal{L}V(x(u), u, r_0)\right)du,
\]

(21)

where \(\alpha > 0\) is a parameter to be determined.

\[
\int_{0}^{t} x^{2\alpha u} \left(2\alpha V(x(u), u, r_0) + \mathcal{L}V(x(u), u, r_0)\right)du \leq \Theta(r_1, t),
\]

where

\[
\Theta(r_1, t) \\
= (2\alpha \lambda_{\max}(P(t)) - \lambda_{\min}(\sum_{i=1}^{5} S_i)) \int_{0}^{t} x^{2\alpha u} \|x(u)\|^2 du \\
+ (2\alpha \lambda_{\min}(S_2)) \int_{u}^{t} x^{2\alpha u} \int_{u}^{t} \|\hat{x}(s)\|^2 ds du.
\]

Choose the suitable \(\alpha > 0\), such that

\[
2\alpha \lambda_{\max}(P(t)) - \lambda_{\min}(\sum_{i=1}^{5} S_i) + 2\alpha \mu_1 x^{2\alpha t} + 2\alpha d\mu_5 x^{2\alpha t} + 2\alpha d \delta x^{2\alpha t} < 0,
\]

then, we have

\[
E\left[x^{2\alpha t} V(x(t), t, r(t))\right] - E\left[V(x_0, 0)\right] \leq \chi E\left[\left\|\phi\right\|^2\right],
\]

which yields

\[
\lambda_{\min}(P(t)) E\left[\|x(t)\|^2\right] \leq E\left[V(x(t), t, r(t))\right] \leq x^{-2\alpha t} \tilde{\chi} E\left[\left\|\phi\right\|^2\right],
\]

where

\[
\tilde{\chi} = 2\alpha (\mu_1 x^{2\alpha t} + \mu_2 x^{2\alpha t} + \mu_5 x^{2\alpha t}) + \chi,
\]

therefore, we can receive

\[
E\left[\|x(t)\|_2\right] \leq \sqrt{\tilde{\chi}} \frac{x^{-2\alpha t} E\left[\left\|\phi\right\|^2\right]},
\]

(22)
thus, according to Definition 1, system (2) is exponential stability.

Remark 2: Theorem 1, provide some novel delay-dependent conditions guaranteeing exponential stability of the proposed neutral-type neural network system. To the best of our knowledge, there have not been any exponential stability conditions for neutral-type neural networks with leakage delay and semi-Markovian jump has been addressed first time in this paper.

Remark 3: As we known, the leakage delays are unavoidable and their occurrence causes instability or oscillation. The previous studies cannot be applied to neutral-type neural networks with leakage delay due to the existence of the term $\delta$ in those systems. In addition, semi-Markovian jump was not taken into account in those models. Therefore, Theorem 1 considers not only the different time delays but also semi-Markovian jump.

IV. SIMULATION CASE

In order to evaluate the proposed exponential stability result, we consider system (2) with two subsystems. The parameters of the discussed system (2) are listed as follows:

$$A_1 = \begin{bmatrix} 1.1 & 0 \\ 0 & 2 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1.6 & 0 \\ 0 & 1.8 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} -0.3 & 0 \\ -0.25 & 0.2 \end{bmatrix}, \quad B_2 = \begin{bmatrix} -0.22 & 0 \\ -0.43 & 0.25 \end{bmatrix},$$

$$C_1 = \begin{bmatrix} 1.04 & 0.23 \\ -0.43 & -1.06 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 0.22 & 0.13 \\ -0.14 & 0.16 \end{bmatrix},$$

$$D_1 = \begin{bmatrix} 0.12 & -0.09 \\ 0.18 & 0.42 \end{bmatrix}, \quad D_2 = \begin{bmatrix} 0.32 & -0.49 \\ 0.18 & 0.22 \end{bmatrix},$$

$$E_1 = \begin{bmatrix} 0.1 & 0 \\ 0.5 & -0.1 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 0.2 & 0 \\ 0.6 & -0.2 \end{bmatrix},$$

$$C_{11} = \begin{bmatrix} 1.23 & 3.33 \\ 0.23 & -1.18 \end{bmatrix}, \quad C_{12} = \begin{bmatrix} -0.34 & 2.21 \\ 0.31 & 0.30 \end{bmatrix},$$

$$E_{11} = \begin{bmatrix} 1.3 & 1.32 \\ 0.52 & 2.16 \end{bmatrix}, \quad E_{12} = \begin{bmatrix} 1.82 & 1.18 \\ 0.18 & 2.23 \end{bmatrix},$$

$$E_{11}(t) = 0.01I, \quad E_{12}(t) = 0.01I, \quad \tau(t) = 1.26 + 0.8\sin t,$$

$$l_{11}^1 = l_{12}^1 = -0.01, \quad l_{11}^2 = l_{12}^2 = 0.02, \quad g(x) = \tanh(x),$$

$$h(t) = 1.78 + 0.01\sin t, \quad d(t) = 1.36 + 0.7\sin t.$$

The transition rate of semi Markovian switching systems in this mode are given as follows

$$\pi_{11}(t) \in (-3.05, -2.95), \quad \pi_{12}(t) \in (2.95, 3.05),$$

$$\pi_{21}(t) \in (1.95, 2.05), \quad \pi_{22}(t) \in (-2.05, -1.95),$$

$$\pi_{11} = -3, \quad \pi_{12} = 3, \quad \lambda_{11} = \lambda_{12} = 0.02,$$

$$\pi_{21} = 2, \quad \pi_{22} = -2, \quad \lambda_{21} = \lambda_{22} = 0.08.$$

Solving the matrix inequality by the LMI Toolbox of Matlab, we can receive

$$J_1 = \begin{bmatrix} 0.7274 & 0.2012 \\ -0.3209 & 0.4562 \end{bmatrix}, \quad J_2 = \begin{bmatrix} 0.3193 & 1.0761 \\ -0.2282 & 0.8562 \end{bmatrix}.$$
the model of Markovian jump in [7], this paper discussed the semi-Markovian jump.

In Figures 2 and 3, we depict the state trajectory of system (3) with the initial condition above. The values of $x_1(t)$ and $x_2(t)$ in Figures 2 and 3 clearly show that the system (3) has exponential stability, which illustrate the effectiveness of the proposed design methods and the theoretical results in this paper.

V. CONCLUSION

Concluding, the problem of exponential stabilization has been investigated for neural networks of neutral type with semi-Markovian switching parameters and leakage delay. Based on inequality analysis method, the developed exponential stability theorem and Lyapunov-Krasovskii functional theory, a new criterion has been achieved to undertake the exponential stabilization for neutral-type neural networks with semi-Markovian jump and leakage delay. In addition, a numerical example has been represented to demonstrate the effectiveness of the theoretical analysis and the potential of the stabilization criteria obtained in the paper. In the future research, we may consider uncertain semi-Markovian jump instead of Markovian jump in our model, and our results may be expanded and applied to synchronization problems and other synthesis problems for neural networks with time-varying delays.

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YAN GAO received the Ph.D. degree in control theory and control engineering from Donghua University, in 2013. She is currently a Teacher with the Shanghai University of Engineering Science. Her main research interests include the stability, the synchronization, and the control of neural networks and complex networks.

FENGJIAO GU is currently pursuing the master’s degree with the Department of Electronic and Electrical Engineering, Shanghai University of Science Engineering, China. Her main research interests include neural networks theory and complex networks.

CARLO CATTANI is a Professor of mathematical physics with the Engineering School, Tuscia University, Italy, an Adjunct Professor with Ton Duc Thang University, HCMC, Vietnam, and an Honorary Professor with BSP University, Ufa, Russia. He has authored more than 200 articles. His research interests include wavelets, fractals, fractional and stochastic equations, nonlinear waves, nonlinear dynamical systems, computational and numerical methods, and data mining.

WANQING SONG received the B.Sc. degree from the Inner Mongolia University of Science and Technology, in 1983, the M.Sc. degree from the University of Science and Technology Beijing, in 1990, and the Ph.D. degree from Donghua University, in 2010. He is a Professor with the Shanghai University of Engineering Science. His main research interests include condition monitor and fault diagnosis.

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