ON QUANTUM CORRELATIONS AND POSITIVE MAPS

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Abstract.
We present a discussion on local quantum correlations and their relations with entanglement. We prove that vanishing coefficient of quantum correlations implies separability. The new results on locally decomposable maps which we obtain in the course of proof also seem to be of independent interest.

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1. Introduction

Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be $C^*$-algebras. For simplicity, we assume that either $\mathcal{A}_1$ or $\mathcal{A}_2$ is a nuclear $C^*$-algebra. This assumption is not particularly restrictive as most $C^*$-algebras associated with physical systems have this property. Moreover, the assumption leads to a unique construction of the $C^*$-tensor product of $\mathcal{A}_1$ and $\mathcal{A}_2$. Let $\mathcal{A} = \mathcal{A}_1 \otimes \mathcal{A}_2$. We write $\mathcal{S}(\mathcal{A}) \equiv (\mathcal{S}(\mathcal{A}_1), \mathcal{S}(\mathcal{A}_2))$ for the set of all states on $\mathcal{A}_1 \otimes \mathcal{A}_2 \equiv \mathcal{A}$ ($\mathcal{A}_1$, $\mathcal{A}_2$). We define, for a state $\omega$ in $\mathcal{S}(\mathcal{A})$ the restriction maps:

\[
(r_1\omega)(A) \equiv \omega(A \otimes 1),
\]

where $A \in \mathcal{A}_1$ and

\[
(r_2\omega)(B) \equiv \omega(1 \otimes B),
\]

where $B \in \mathcal{A}_2$. Obviously, $r_i\omega$ is a state in $\mathcal{S}(\mathcal{A}_i)$, where $i = 1, 2$. Next, take a measure $\mu$ on $\mathcal{S}(\mathcal{A})$. Using the restriction maps one can define measures $\mu_i$ on $\mathcal{S}(\mathcal{A}_i)$ in the following way: for a Borel subset $F_i \subset \mathcal{S}(\mathcal{A}_i)$ we put

\[
(1.1) \quad \mu_i(F_i) = \mu(r_i^{-1}(F_i)),
\]

where $i = 1, 2$. Having measures $\mu_1$ and $\mu_2$, both originating from the given measure $\mu$ on $\mathcal{S}(\mathcal{A})$ one can define new measure $\boxtimes \mu$ on $\mathcal{S}(\mathcal{A}_1) \times \mathcal{S}(\mathcal{A}_2)$ which encodes classical correlations between two subsystems described by $\mathcal{A}_1$ and $\mathcal{A}_2$ respectively (see [5]). We first define $\boxtimes \mu$ for discrete measures $\mu^d = \sum_i \lambda_i^d \delta_{\rho_i^d}$ with $\lambda_i^d \geq 0$, $\sum_i \lambda_i^d = 1$, $\rho_i^d \in \mathcal{S}(\mathcal{A})$. $\delta_\sigma$ stands for Dirac measure. We introduce $\mu_1^d = \sum_i \lambda_i^d \delta_{r_1\rho_i^d}$ and $\mu_2^d = \sum_i \lambda_i^d \delta_{r_2\rho_i^d}$. Define

\[
(1.2) \quad \boxtimes \mu^d = \sum_i \lambda_i^d \delta_{r_1\rho_i^d} \times \delta_{r_2\rho_i^d}.
\]

Next, let us take an arbitrary measure $\mu$ in $M_\text{b}(\mathcal{S})$. Here, $M_\text{b}(\mathcal{S}) = \{ \mu : \phi = \int_\mathcal{S} \nu d\mu(\nu) \}$; i.e. the set of all Radon probability measures on $\mathcal{S}(\mathcal{A})$ with the fixed barycenter $\phi$. For the measure $\mu$, there exists net of discrete measures $\mu_k$ such that $\mu_k \rightarrow \mu$ ($^\text{*-weakly}$). Defining $\mu_k^1$ ($\mu_k^2$) analogously as $\mu_1$ ($\mu_2$) respectively, one has $\mu_k^1 \rightarrow \mu_1$ and $\mu_k^2 \rightarrow \mu_2$ where the convergence is taken in $^\text{*}$-weak topology. Then define, for each $k$, $\boxtimes \mu^k$ as in (1.2). One can verify that $\{\boxtimes \mu_k^k\}$ is convergent to a measure $\mathcal{S}(\mathcal{A}_1) \times \mathcal{S}(\mathcal{A}_2)$, so taking the weak limit we arrive to the measure $\boxtimes \mu$ on $\mathcal{S}(\mathcal{A}_1) \times \mathcal{S}(\mathcal{A}_2)$. It follows easily that $\boxtimes \mu$ does not depend on the chosen approximation procedure.

The measure $\boxtimes \mu$ leads to the concept of degree of local (quantum) correlations for $\phi \in \mathcal{S}(\mathcal{A})$, $a_1 \in \mathcal{A}_1$, $a_2 \in \mathcal{A}_2$, which is defined as

\[d(\phi, a_1, a_2) = \inf_{\mu \in M_\text{b}(\mathcal{S}(\mathcal{A}))} |\phi(a_1 \otimes a_2)|
\]

\[\quad = -\left(\int \xi d(\boxtimes \mu)(\xi))(a_1 \otimes a_2)\right).
\]

Recently, we have studied relations between the coefficient of quantum correlations and entanglement (cf [5]). R. Werner has kindly pointed out that the proof of the statement saying that $d(\phi; a, b) = 0$ for all $a \in \mathcal{A}_1$, $b \in \mathcal{A}_2$ and a state $\phi$ on $\mathcal{A}$ implies separability of $\phi$ contains a gap (see Proposition 5.3 in [5]). The aim of this letter is to give the proof of the properly amended statement (Theorem 4.3, Section 4). To this end we also give a generalization of Størmer theory of locally decomposable maps (see Section 3) which seems to be of independent interest. All definitions and notations used here are taken from [5].
2. Local separability 1.

Assume \( d(\phi; a, b) = 0 \) for all \( a \in A_1, b \in A_2 \) and for a state \( \phi \) on \( A \). Then as
\[
\mu \mapsto \left( \int \phi(d(\xi\mu)(\xi))(a_1 \otimes a_2) \right)
\]
is \("\)-weak continuous, there exists a measure \( \mu \in M_\phi(S) \)
(Radon probability measures on \( S(A_1 \otimes A_2) \) with barycenter \( \phi \)) such that
\[
(2.1) \quad \phi(a \otimes b) = \int_{S(A_1) \times S(A_2)} \xi d(\xi\mu)(a \otimes b).
\]
Using the Riemann approximation property of the classical measure one has
\[
(2.2) \quad \phi(a \otimes b) = \lim_i \sum_i \lambda_i (a, b) \xi_i^{(1)}(a) \xi_i^{(2)}(b),
\]
where \( \lambda_i (a, b) \) are non-negative numbers, depending on \( a \) and \( b \), \( \sum_i \lambda_i (a, b) = 1 \) and
states \( \xi_i^{(1)} \) (\( \xi_i^{(2)} \)) are defined on \( A_1 \) (on \( A_2 \) respectively) and depend on the chosen
element \( a \otimes b \).

**Definition 2.1.** Let a state \( \phi \) on \( A_1 \otimes A_2 \) have a representation of the form \(2.1\)
with the measure \( \mu \) depending on the chosen element \( a \otimes b \). Such state will be called
locally separable.

In other words, one can say that if the coefficient of quantum correlations for
a state \( \phi \) vanishes on \( a \otimes b \) then the state \( \phi \) is locally separable. Now we wish
to examine the property of local separability. Let us begin with a particular case:
assume that \( a \) is a normal element of \( A_1 \) while \( b \) is arbitrary one in \( A_2 \). Let
\( \phi \in S(A_1 \otimes A_2) \). We observe that
\[
(2.3) \quad \phi(a \otimes b) = \phi|_{A_1^0 \otimes A_2^0}(a \otimes b),
\]
where \( \phi|_{A_1^0 \otimes A_2^0} \) is the restriction of \( \phi \) to the subalgebra \( A_1^0 \otimes A_2^0 \subset A_1 \otimes A_2 \). Here,
\( A_1^0 \) is the abelian \( C^* \)-algebra generated by \( a \) and \( 1 \) (\( a \) was normal!) while \( A_2^0 \)
is the algebra, in general non-commutative, generated by \( b \) and \( 1 \). But in such case,
each state in \( S(A_1^0 \otimes A_2^0) \) is a separable one. Moreover, \( \phi \) has the decomposition
depending on \( a \) and \( b \). However, we wish to stress: the assumption of normality
for \( a \) was crucial. Namely, taking an arbitrary \( a \) and \( b \), the condition of vanishing
of coefficient \( d \) implies the uniformity of decomposition with respect to hermitian
and antihermitian part of \( a \) in \( a \otimes b \). In that context it is worth adding that by the
genuine separability we understand decomposition of type \(2.1\) which is uniform
with respect to elements of algebra \( A \).

To show that \( d(\phi, \cdot) = 0 \) can imply separability, we will use another property
of entangled states. Namely, one of the intriguing features of non-separable states
is their complicated behaviour under transformations by positive maps. To be
more precise, one is interested in inspection of the functional \( \phi \circ \alpha \otimes id_2(\cdot) \), where
\( \phi \) is a state on \( A = A_1 \otimes A_2 \), \( \alpha : A_1 \to A_3 \) is a linear, unital positive map
while \( id_2 \) is the identity map on \( A_2 \). To proceed with answering this question we
need a description of locally decomposable maps and a modification of definition
of coefficient of quantum correlations which will be given in the next sections.

3. Locally decomposable maps

This section is a fairly straightforward generalization of the Størmer concept of
local decomposibility; see Definition 7.1 as well as Lemma 7.2 and Theorem 7.4 in
[8].
Definition 3.1. Let $\alpha$ be a linear positive map of a $C^*$-algebra $A$ into $B(H)$, $H$ being a Hilbert space. The map $\alpha$ is locally decomposable if for each normal state $\phi(\cdot) = Tr\rho(\cdot)$ on $B(H)$ there exists a Hilbert space $H_\rho$, and a linear map $V_\rho$ of $H_\rho$ into $H_0 = \langle B(H)\rangle^{1/2} > cl$ with property $||V_\rho|| \leq M$ for all $\rho$ and a $C^*$-homomorphism $\pi_\rho$ of $A$ into $B(H_\rho)$ such that
\[ V_\rho \pi_\rho(a) V_\rho^* \rho^{1/2} = \alpha(a) \rho^{1/2}, \]
for all $a \in A$.

We will need

Lemma 3.2. Let $A$ be a $C^*$-algebra, $H$ a Hilbert space, and $\alpha$ a positive unital linear map of $A$ into $B(H)$. If $\rho$ is a density matrix on $H$ defining a normal state $\phi$ on $B(H)$ then there is an $*$-representation $\pi$ of $A$ as $C^*$-algebra on a Hilbert space $H_\pi$, a vector $\Omega_\pi \in H_\pi$ cyclic under $\pi(A)$, and a bounded linear map $V$ of the set $\pi(A) \Omega_\pi; a \in A, a = a^*$ is dense into $H_\rho = \langle \alpha(a) \rho^{1/2}; a \in A\rangle > cl$ such that
\[ V \pi(a) V^* \rho^{1/2} = \alpha(a) \rho^{1/2}, \]
for each self-adjoint $a \in A$.

Proof. Let $\omega(\cdot) = Tr\rho(\cdot)$ denote by $\pi_\omega$ the $*$-representation of $A$ induced by $\omega$ on $H_\omega$, and let $\Omega$ be a cyclic vector for $\pi_\omega(A)$ in $H_\omega$ such that $\omega(\cdot) = (\Omega, \pi_\omega(\cdot) \Omega)$. For selfadjoint $a \in A$, define $V \pi_\omega(a) \Omega = \alpha(a) \rho^{1/2}$. The set $\{ \pi_\omega(a) \Omega; a = a^*, a \in A\}$ is a real linear subspace of $H_\omega$ whose complexification is dense in $H_\omega$. If $\pi_\omega(a) \Omega = 0$ then
\[ 0 = (\pi_\omega(a^2) \Omega, \Omega) = \omega(a^2) = Tr\rho(a^2) \geq Tr\rho(\alpha(a))^2 \geq 0. \]
Hence $\alpha(a) \rho^{1/2} = 0$. It follows that $V$ is well defined and linear. Note that
\[ V \pi_\omega(1) \Omega = V \Omega = \alpha(1) \rho^{1/2}, \]
and that
\[ (V^* \rho^{1/2}, \pi_\omega(a) \Omega) = (\rho^{1/2}, V \pi_\omega(a) \Omega) = (\rho^{1/2}, \alpha(a) \rho^{1/2}) = \omega(a) = (\Omega, \pi_\omega(a) \Omega), \]
for any self-adjoint $a \in A$. Thus $V^* \rho^{1/2} = \Omega$ and $V \pi_\omega(a) V^* \rho^{1/2} = \alpha(a) \rho^{1/2}$ for each self-adjoint $a \in A$. Moreover
\[ ||\alpha(a) \rho^{1/2}||^2 = (\alpha(a)^2 \rho^{1/2}, \rho^{1/2}) \leq (\alpha(a^2) \rho^{1/2}, \rho^{1/2}) = \omega(a^2) = ||\pi_\omega(a) \Omega||^2, \]
so that $||V|| \leq 1$ and with the identification, $\pi = \pi_\omega$, $\Omega = \Omega_\pi$, the proof is complete. \hfill $\square$

Now, we recall (see Lemma 7.3 in [8]): If $A : A \rightarrow B(H)$ is unital, positive map then
\[ (3.1) \quad \alpha(a^*a + aa^*) \geq \alpha(a^*) \alpha(a) + \alpha(a) \alpha(a^*), \]
for all $a \in A$. Lemma 3.2 and the inequality (3.1) lead to

Theorem 3.3. Every unital positive linear map of a $C^*$-algebra $A$ into $B(H)$ is locally decomposable.

Proof. Let $\rho$, $\omega$ and $\pi_\omega$ be as in Lemma 3.2. Define $\pi_\rho$ in terms of the right kernel as a $*$-anti-homomorphism (i.e. $<a, b > = \omega(ab^*)$, $I_\omega = \{a; <a, a> = 0\}$,
\[ \pi'_\omega(u) = \omega + L_\omega \] of \( \mathcal{A} \) on the Hilbert space \( \mathcal{H}'_\omega \) and let \( \tilde{\pi}_\omega = \pi_\omega \oplus \pi'_\omega \). Let \( \mathcal{H} \) be the Hilbert space \( \mathcal{H}_\omega \oplus \mathcal{H}'_\omega \) with the inner product

\[ (z + z', y + y') = 1/2[(z,y) + (z',y')], \]

where \( y, z \in \mathcal{H}_\omega \), \( y', z' \in \mathcal{H}'_\omega \). \( \tilde{\pi}_\omega \) is a \( * \)-homomorphism of \( \mathcal{A} \) into \( \mathcal{B}(\mathcal{H}) \). With \( \Omega \) and \( \Omega' \) the vacuum vectors of \( \omega \) for \( \pi_\omega \) and \( \pi'_\omega \), respectively, let \( \Omega = \Omega \oplus \Omega' \). Define a map \( V' \) of the linear submanifold \( \tilde{\pi}_\omega(\mathcal{A})\tilde{\Omega} \) of \( \mathcal{H} \) into \( < \alpha(\mathcal{A})g^{1/2} >^d \) by

\[ V'\tilde{\pi}_\omega(a)\tilde{\Omega} = \alpha(a)g^{1/2}, \]

for each \( a \in \mathcal{A} \). Note that if \( \tilde{\pi}_\omega(a)\tilde{\Omega} = 0 \) then \( \pi_\omega(a)\Omega = 0 = \pi'_\omega(a)\Omega' \). Thus

\[ \pi_\omega(a^*a)\pi_\omega(a)\Omega = \pi_\omega(a^*a)\Omega = 0 = \pi'_\omega(a^*)\pi'_\omega(a)\Omega' = \pi'_\omega(aa^*)\Omega', \]

so that \( \omega(aa^*) = 0 = \omega(a^*a) \). Thus by 3.1

\[ 0 = (\alpha(a^*a) + \alpha(aa^*))g^{1/2}, g^{1/2} \geq (\alpha(a^*)\alpha(a) + \alpha(a)\alpha(a^*))g^{1/2}, g^{1/2} \geq 0. \]

Hence \( \alpha(a)g^{1/2} = 0 \). Consequently, \( V' \) is well defined and linear. Moreover,

\[ ||V'|| = \sup\{||\alpha(a)g^{1/2}|| : \|\tilde{\pi}_\omega(a)\tilde{\Omega}\| = 1\} = \sup\{||\alpha(a)g^{1/2}|| : \|\pi_\omega(a)\Omega \oplus \pi'_\omega(a)\Omega'\| = 1\} \]

\[ = \sup\{||\alpha(a)g^{1/2}|| : ((\alpha(a^*) + \alpha(aa^*))g^{1/2}, g^{1/2} = 2\}. \]

By 3.1 if \( (\alpha(a^*a + aa^*))g^{1/2}, g^{1/2} = 2 \) then \( (\alpha(a^*)\alpha(a) + \alpha(a)\alpha(a^*))g^{1/2}, g^{1/2} \leq 2 \). Hence \( ||\alpha(a)g^{1/2}||^2 \leq 2 \). Consequently \( ||V'|| \leq 2^{1/2} \).

We extend \( V' \) by continuity to all of the subspace \( \tilde{\mathcal{H}}_0 = \langle \tilde{\pi}_\omega(\mathcal{A})\tilde{\Omega} \rangle \) and call the extension \( V' \) of \( \mathcal{H}_0 \) equals \( V' \) and \( V \) restricted to orthocomplement of \( \tilde{\mathcal{H}}_0 \) is equal to 0. Then \( ||V'|| \leq 2^{1/2} \). Moreover, repeating the corresponding argument given in the proof of Lemma 3.2 one can show \( (V')^*g^{1/2} = \tilde{\Omega} \) and this completes the proof.

4. Local separability 2.

Having the notion of locally decomposable maps one might be tempted to study local PPT (positive partial transposition) property, now without any restriction with respect to dimension. One can also study relations between local separability and locally decomposable maps. To proceed with these questions one should evaluate functionals and study the coefficient \( d(\cdot) \) on an arbitrary positive element of \( \mathcal{A} \). To this end we propose

**Definition 4.1.** Let \( \phi \) be a state on \( \mathcal{A} = \mathcal{A}_1 \otimes \mathcal{A}_2 \) and \( A \) be an element in \( \mathcal{A} \). The general coefficient of quantum correlations \( d_0(\cdot) \) for \( \phi \) and \( A \) is defined as

\[ d_0(\phi, A) = \inf_{\mu \in M_\mu(S)} \| \int_S \xi d\mu(\xi)(A) - \int_{S_1 \times \hat{S}_2} \xi d\mathbb{E}\mu(\xi)(A) \|. \]

To clarify this definition we recall that, by definition, \( \mu_1 \) and \( \mu_2 \) are probability measures on \( \mathcal{S}(\mathcal{A}_1) \) and \( \mathcal{S}(\mathcal{A}_2) \), respectively (they are basic ingredients of the definition of \( \mathbb{E}\mu \); see Introduction or 4). Consequently, \( \mathbb{E}\mu \) is a probability measure on \( \mathcal{S}(\mathcal{A}_1) \times \mathcal{S}(\mathcal{A}_2) \). However, as \( \mathcal{S}(\mathcal{A}_1) \times \mathcal{S}(\mathcal{A}_2) \subset S \) is a measurable
subset of $\mathcal{S}$ one can consider $\mathbb{E}\mu$ as a probability measure on $\mathcal{S}$ supported by $\mathcal{S}(\mathcal{A}_1) \times \mathcal{S}(\mathcal{A}_2)$. To summarize, $\int_{\mathcal{S}(\mathcal{A}_1) \times \mathcal{S}(\mathcal{A}_2)} \xi d(\mathbb{E}\mu)(\xi)$ is a well defined element of $\mathcal{S}(\mathcal{A})$. Therefore $\int_{\mathcal{S}(\mathcal{A}_1) \times \mathcal{S}(\mathcal{A}_2)} \xi d(\mathbb{E}\mu)(\xi)(A) = \sum_i \int_{\mathcal{S}(\mathcal{A}_1) \times \mathcal{S}(\mathcal{A}_2)} \xi d(\mathbb{E}\mu)(\xi)(a_i \otimes b_i)$ is also well defined ($A = \sum_i a_i \otimes b_i$ is a general element of $\mathcal{A}$).

Obviously, the just given definition of $d_0(\cdot)$ is equivalent to that given for $d(\cdot)$ (cf [8]) if one restrict oneself to simple tensors! Moreover, it is worth noting that, in measure terms, separability of $\mathbb{E}\mu$ is equivalent to $\mathbb{E}\mu \in M_{\phi}(\mathcal{S})$ (cf [1]).

Let us consider a state $\phi$ on $\mathcal{A}$ such that $d_0(\phi, A) = 0$ for some fixed $A \in \mathcal{A} = \mathcal{A}_1 \otimes \mathcal{A}_2$, where $\mathcal{A}_1, \mathcal{A}_2$ are finite dimensional $\text{C}^*$-algebras. This is the most important case considered within Quantum Information Theory. The general case needs more complicated arguments based on approximation procedures and it will be not considered here. We also assume that $A \geq 0$ and we suppose that the measure $\mu$ appearing in the condition $d_0(\phi, A) = 0$ is finitely supported. This involves no loss of generality, as there exist (finite) optimal decompositions (cf [5]). Then, there are states $\{\phi_{1,i}^A\} \subset \mathcal{S}(\mathcal{A}_1)$, and $\{\phi_{2,i}^A\} \subset \mathcal{S}(\mathcal{A}_2)$ and non-negative numbers $\lambda_i(A)$, $\sum_i \lambda_i(A) = 1$ such that:

$$\phi(A) = \phi(\sum_{kl} a^*_k a_l \otimes b^*_l b_k) = \sum_i \sum_{kl} \lambda_i(A) \phi_{1,i}^A(a^*_k a_l) \phi_{2,i}^A(b^*_l b_k).$$

Now, we are in position to analyse $\phi \circ \alpha \otimes id_2$ for a state $\phi$ on $\mathcal{A}$ having $d_0(\phi, A) = 0$ for all $A \in \mathcal{A}$. Here, $\alpha$ is an arbitrary linear unital positive map on $\mathcal{A}_1$; $\alpha : \mathcal{A}_1 \to \mathcal{A}_1$. Moreover, we put $A \geq 0$ and again observe that

$$\phi \circ \alpha \otimes id_2(A) = \sum_i \sum_{kl} \lambda_i(A) \phi_{1,i}^A(\alpha(a^*_k a_l)) \phi_{2,i}^A(b^*_l b_k)$$

(4.2) $$= \sum_i \sum_{kl} \lambda_i(A) \phi_{1,i}^A(V_{\phi,i,A}^\ast \pi_{\phi,i,A}(a^*_k a_l)V_{\phi,i,A}) \phi_{2,i}^A(b^*_l b_k),$$

where $\pi_{\phi,i,A}(\cdot)$ is a $\text{C}^*$-morphism (cf Section 3).

Our first remark on [4,2] is that any $\text{C}^*$-morphism is, in fact, a sum of $\ast$-morphism and $\ast$-antimorphism (cf [9] or [24]). The second observation says that $\{a^*_k a_l\}_{kl}$ and $\{b^*_l b_k\}_{kl}$ are positive semidefined matrices with $\mathcal{A}_1$ ($\mathcal{A}_2$)-valued entries (cf [9]). Taking states $\varphi^1$ and $\varphi^2$ on $\mathcal{A}_1$ and $\mathcal{A}_2$ respectively, one gets positive semidefined matrices $\{\varphi^1(a^*_k a_l)\}_{kl}$ and $\{\varphi^2(b^*_l b_k)\}_{kl}$ with entries in $\mathbb{C}$. The next remark is that the Hadamard product of positive semidefined matrices is a positive semidefined matrix (cf [1]). Finally, we recall that the transposition of a positive semidefined matrix with complex valued entries is again positive semidefined. Taking all that into account one gets:

**Lemma 4.2.** Assume that the antimorphism in the decomposition of $\pi_{\phi,i,A}$ is composed of a $\ast$-morphism with transposition. Then, for any positive $A \in \mathcal{A}$ ($\phi \circ \alpha \otimes id_2)(A)$ is positive. Hence, provided that the assumption of this Lemma is satisfied, a state $\phi$ with $d_0(\phi, A) = 0$ for any $A \in \mathcal{A}$ is the separable one.

We have used the fact that only separable states are invariant (globally) with respect to “partially positive maps” (see [10], [7], [9] and [6]). It is well known
that any antimorphism can be represented as the composition of morphism and transposition (transposition is an antimorphism of order two, while the composition of two antimorphisms leads to a morphism). Thus, the assumption of Lemma 4.2 is always satisfied. As a conclusion one has that the condition $d_0(\phi, A) = 0$ for any $A \in \mathcal{A}$ is the sufficient condition for separability of $\phi$. Hence, we got

**Theorem 4.3.** Assume $\mathcal{A}$ is the tensor product of finite dimensional $C^*$-algebras $\mathcal{A}_1$ and $\mathcal{A}_2$. Then, a state $\phi$ is separable if and only if $d(\phi; A) = 0$ for any $A \in \mathcal{A}$.

**Proof.** We have just proved, Lemma 4.2, that $d_0(\phi; A) = 0$ for all $A \in \mathcal{A}$ implies separability of $\phi$. Conversely, the definition of separability implies that the coefficient $d_0$ is equal to zero (cf [5]). This completes the proof. □

We want to close this section with an obvious remark that having a state $\phi$ with $d_0(\phi, A) = 0$ for any $A \in \mathcal{A}$, the positivity of partial transformation is the sufficient condition for separability.

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