 APPROXIMATION OF POINTS IN THE PLANE  
BY GENERIC LATTICE ORBITS

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ABSTRACT. We give upper and lower bounds for Diophantine exponents measuring how well a point in the plane can be approximated by points in the orbit of a lattice $\Gamma \subset \text{SL}_2(\mathbb{R})$ acting linearly on $\mathbb{R}^2$. Our method gives bounds that are uniform for almost all orbits.

1. INTRODUCTION

Let $\Gamma \subset \text{SL}_2(\mathbb{R})$ be a lattice and, for each $T > 0$, let $\Gamma_T = \{ \gamma \in \Gamma : \| \gamma \| \leq T \}$ with $\| \gamma \| = \text{tr}(\gamma^t \gamma)$ the Hilbert-Schmidt norm. For any $u \in \mathbb{R}^2 \setminus \{0\}$ and $T > 0$ consider the finite orbit $\Gamma_T u$, where $\Gamma$ acts linearly on $\mathbb{R}^2$. The limiting distribution of these orbits as $T \to \infty$ was extensively studied in [7, 11, 4] and shown to be equidistributed with respect to a suitable measure (depending on $u$). In particular, for a generic point $u \in \mathbb{R}^2$, the orbit $\Gamma u$ is dense in $\mathbb{R}^2$, and, hence, any point $v \in \mathbb{R}^2$ can be approximated by orbit points (when $\Gamma$ is co-compact, all orbits are dense).

To measure how well a point $v \in \mathbb{R}^2$ can be approximated by orbit points in $\Gamma u$, in analogy to similar problems in Diophantine approximations, Laurent and Nogueira [8] defined two exponents $\mu_{\Gamma}(u, v)$ and $\hat{\mu}_{\Gamma}(u, v)$ as follows.

DEFINITION 1. The critical exponent $\mu_{\Gamma}(u, v)$ is defined as the supremum of all $\alpha > 0$ such that the set

$$\{ \gamma \in \Gamma : \| \gamma u - v \|_{\infty} < \| \gamma \|^{-\alpha} \}$$

is unbounded. The uniform critical exponent, $\hat{\mu}_{\Gamma}(u, v)$, is defined as the supremum over all $\alpha > 0$ such that $\Gamma_T u \cap B_{\frac{1}{T^\alpha}}(v) \neq \emptyset$ for all sufficiently large $T$. Here $B_{\delta}(v) = \{ u \in \mathbb{R} : \| u - v \|_{\infty} \leq \delta \}$ denotes a small norm ball with respect to the sup norm $\| v \|_{\infty} = \max\{|v_1|, |v_2|\}$ on $\mathbb{R}^2$.

We note that the critical exponents do not depend on the choice of norms. Also, notice that $\hat{\mu}(u, v) \leq \mu(u, v)$ unless $v \in \Gamma u$ (in which case $\hat{\mu}(u, v) = \infty$). Also, as noted in [8, 9], these exponents are invariant under the $\Gamma \times \Gamma$ action on $\mathbb{R}^2 \times \mathbb{R}^2$, and by ergodicity they are constant almost everywhere. We denote these constants by $\mu_{\Gamma}$ and $\hat{\mu}_{\Gamma}$, respectively.

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In [8], Laurent and Nogueira studied these exponents for $\Gamma = \text{SL}_2(\mathbb{Z})$ and gave very precise estimates depending on the Diophantine properties of the slopes of $u$ and $v$. In particular, their analysis implies that for almost all $u, v \in \mathbb{R}^2 \setminus \{0\}$ one has that $\frac{1}{3} \leq \tilde{\mu}_{\text{SL}_2(\mathbb{Z})}(u, v) \leq \mu_{\text{SL}_2(\mathbb{Z})}(u, v) \leq \frac{1}{2}$ and that $\mu_{\text{SL}_2(\mathbb{Z})}(u, v) \geq \frac{1}{3}$ holds for every target $v$ and any $u$ with a dense orbit. In particular, this implies that

$$\frac{1}{3} \leq \tilde{\mu}_{\text{SL}_2(\mathbb{Z})} \leq \mu_{\text{SL}_2(\mathbb{Z})} \leq \frac{1}{2}. \tag{1}$$

Moreover, in [9] they showed that for any lattice $\Gamma$, the upper bound $\mu_{\Gamma}(u, v) \leq \frac{1}{2}$ holds for any $u$ with a dense orbit and a.e. $v \in \mathbb{R}^2$, so that $\mu_{\Gamma} \leq \frac{1}{2}$ for any lattice.

Another approach for this problem was given in [10], where Maucourant and Weiss gave an effective version of the equidistribution result of $\Gamma$-orbits, building on effective equidistribution of unipotent flows on $\Gamma \backslash \text{SL}_2(\mathbb{R})$. In particular, their results imply the following lower bound for the critical exponents of a generic orbit: For any lattice $\Gamma$ in $\text{SL}_2(\mathbb{R})$, for almost every $u \in \mathbb{R}^2$ (respectively, for all $u \in \mathbb{R}^2$ if $\Gamma$ is cocompact), $\tilde{\mu}_{\Gamma}(u, v) \geq \frac{1-2\tau}{144}$, for all $v \in \mathbb{R}^2 \setminus \{0\}$. Moreover, for every $u \in \mathbb{R}^2$ with a dense orbit, $\mu_{\Gamma}(u, v) \geq \frac{1-2\tau}{144}$. Here

$$\tau = \tau(\Gamma) = \mathfrak{Re}\left(\sqrt{\frac{1}{4} - \lambda_1}\right) \in [0, \frac{1}{2}), \tag{2}$$

with $\lambda_1 > 0$ the first non-zero Laplace eigenvalue in $L^2(\Gamma \backslash H)$, is a parameter measuring the spectral gap for $\Gamma$ (see section 2.4 for more details).

Recently, Ghosh, Gorodnik, and Nevo [1, 2] studied a similar problem, in a more general setting, regarding rates of approximation of $\Gamma$-orbits on homogeneous spaces $X = G/H$ with $\Gamma$ a lattice in a semisimple group $G$ and $H$ a closed subgroup. Their approach again builds on effective equidistribution results for the $H$ action on $\Gamma \backslash G$, but using the mean ergodic theorem instead of a pointwise ergodic theorem. A striking feature of their result is that it provides in many cases optimal rates of approximations. In this note we borrow some of their ideas, as well as ideas of [5] for proving an effective mean ergodic theorem for actions of unipotent groups, and [3] relating mean ergodic theorems to shrinking target problems, to give bounds for the critical exponents of a generic orbit. Our main result is as follows:

**Theorem 2.** Let $\Gamma \subseteq \text{SL}_2(\mathbb{R})$ be a lattice and $\tau = \tau(\Gamma) \in [0, 1/2)$ given in (2).

1. For any $v \in \mathbb{R}^2 \setminus \{0\}$, for almost all $u \in \mathbb{R}^2$

$$\frac{1-2\tau}{3} \leq \tilde{\mu}_{\Gamma}(u, v) \leq \mu_{\Gamma}(u, v) \leq \frac{1}{2}.$$  

2. For almost all $u \in \mathbb{R}^2$, for any $v \in \mathbb{R}^2 \setminus \{0\}$ we have $\tilde{\mu}_{\Gamma}(u, v) \geq \frac{1-2\tau}{5}$.

**Remark 3.** When the representation of $G = \text{SL}_2(\mathbb{R})$ on $L^2(\Gamma \backslash G)$ is tempered (in particular for $\Gamma = \text{SL}_2(\mathbb{Z})$) we have that $\tau(\Gamma) = 0$ and the first part implies that $\frac{1}{3} \leq \tilde{\mu}_{\Gamma} \leq \mu_{\Gamma} \leq \frac{1}{2}$ recovering (1). This is slightly better than the bound $\tilde{\mu}_{\Gamma} \geq \frac{1}{6}$ claimed in [2] to be obtained by similar methods. For $\Gamma$ a congruence lattice, using the best known bounds on the spectral gap, we get that $\frac{143}{288} \leq \tilde{\mu}_{\Gamma} \leq \frac{1}{3}$. It is
not unlikely that in fact \( \hat{\mu}_\Gamma = \mu_\Gamma = \frac{1}{2} \) (independent of the spectral gap), however, proving this seems beyond our abilities at the moment.

**Remark 4.** We point out a subtle difference between the first part of our result, which holds for any target point but only for generic orbits, vs. the result of [8], that holds for any dense orbit, but the exponent depends on the slopes of the target point and the orbit. In particular, for \( \Gamma = \text{SL}_2(\mathbb{Z}) \), if the target point \( v \in \mathbb{R}^2 \) has an irrational slope which is a Liouville number, the results of [8, Theorem 2 (iii)] imply that \( \hat{\mu}_{\text{SL}_2(\mathbb{Z})}(u, v) \geq \frac{1}{4} \) for almost all \( u \), while we get \( \hat{\mu}_{\text{SL}_2(\mathbb{Z})}(u, v) \geq \frac{1}{2} \). On the other hand, if the slope of \( v \) is rational, then [8, Theorem 2 (ii)] implies that \( \hat{\mu}_{\text{SL}_2(\mathbb{Z})}(u, v) \geq \frac{1}{2} \) for almost all \( u \), which is best possible.

**Remark 5.** In the second part of our result, the bound for the critical exponent is weaker because we require that the orbit of a single point \( u \) will approximate every target point simultaneously. Here the analysis of [8] implies that almost all \( u \in \mathbb{R}^2 \setminus \{0\} \) satisfy \( \hat{\mu}_{\text{SL}_2(\mathbb{Z})}(u, v) \geq \frac{1}{4} \) for all \( v \in \mathbb{R}^2 \setminus \{0\} \), which is slightly better. However, our result holds for any lattice, and moreover, the method of proof generalizes to deal with the general problem of lattice action on homogeneous spaces, thus answering the question of uniformity on a co-null set of orbits raised in [1].

**Remark 6.** One can also consider the same problem for the action of lattices \( \Gamma \subseteq \text{SL}_2(\mathbb{C}) \) acting on \( \mathbb{C}^2 \). There have been a few results in this case: for \( \Gamma = \text{SL}_2(\mathbb{C}) \) with \( \mathcal{O} = \mathbb{Z}[i] \) the ring of Gaussian integers, recent results of Singhal [13] imply that \( \frac{1}{3} \leq \hat{\mu}_{\text{SL}_2(\mathcal{O})} \leq \mu_{\text{SL}_2(\mathcal{O})} \leq \frac{1}{2} \). More generally, the work of Pollicott [12] gives a lower bound for \( \hat{\mu}_\Gamma \) for any co-compact \( \Gamma \) in \( \text{SL}_2(\mathbb{C}) \). The methods of this paper could also be generalized to handle this case as well to show that \( c_\Gamma \leq \hat{\mu}_{\text{SL}_2(\Gamma)} \leq \mu_{\text{SL}_2(\Gamma)} \leq \frac{1}{2} \) for some explicit value of \( c_\Gamma \) depending on the spectral gap for \( \Gamma \).

2. Preliminaries and notation

2.1. Notation. We write \( A \ll B \) or \( A = O(B) \) to indicate that \( A \leq cB \) for some constant \( c \). If we wish to emphasize that the constant depends on some parameters we use subscripts, for example \( A \ll \epsilon B \). We also write \( A \asymp B \) to indicate that \( A \ll B \ll A \).

2.2. Coordinates. Let \( G = \text{SL}_2(\mathbb{R}) \) and consider the Cartan decomposition \( G = NAK \) with \( N \) unipotent, \( A \) diagonal, and \( K \) compact. We will use the following coordinates

\[
a_y = \begin{pmatrix} \sqrt{y} & 0 \\ 0 & 1/\sqrt{y} \end{pmatrix} \in A, \quad n_x = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \in N, \quad k_\theta = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix} \in K
\]

In the coordinates \( g = n_x a_y k_\theta \), the Haar measure of \( G \) is \( dg = \frac{dxdy\theta}{y^2} \).

Let \( \bar{n}_x = \begin{pmatrix} 1 \\ x \end{pmatrix} \) and let \( \bar{N} = \{ \bar{n}_x : x \in \mathbb{R} \} \). For any \( g \in G \) apart from a set of measure zero we can also write \( g = n_x a \bar{n}_x \) and the Haar measure in these coordinates is given by \( dg = \frac{dxdy\bar{x}}{y} \).
2.3. **Norms.** Fix a basis $B = \{X_1, X_2, X_3\}$ for the Lie algebra $g$ of $G$. Given a smooth test function $\psi \in C^\infty(\Gamma \backslash G)$, define the “$L^p$, order-$d$” Sobolev norm $\mathcal{S}_{p,d}(\psi)$ as

$$
\mathcal{S}_{p,d}(\psi) := \sum_{\text{ord}(\mathcal{D}) \leq d} \|\mathcal{D}\psi\|_{L^p(\Gamma \backslash G)}.
$$

Here $\mathcal{D}$ ranges over monomials in $B$ of order at most $d$ and $\mathcal{D}$ acts on $\psi$ by left differentiation (e.g., $X\psi(g) = \frac{d}{dt}(\psi(g e^{tX}))|_{t=0}$). This definition depends on the basis, however, changing the basis $B$ only distorts $\mathcal{S}_{p,d}$ by a bounded factor.

2.4. **Spectral gap.** The group $G$ acts on the upper half plane $\mathbb{H} = \{x + iy : y > 0\}$ by linear fractional transformation preserving the hyperbolic metric. The (self adjoint extension of the) hyperbolic Laplacian $\Delta = -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$ acts on $L^2(\Gamma \backslash \mathbb{H})$, and its spectrum consists of a discrete part $0 = \lambda_0 < \lambda_1 \leq \lambda_2 < \ldots$ and a continues part contained in $[\frac{1}{4}, \infty)$ and spanned by Eisenstein Series (when $\Gamma$ is non uniform). The spectral gap for $\Gamma$ is the gap between the first positive eigenvalue $\lambda_1$ and the trivial eigenvalue $\lambda_0 = 0$ and is controlled by the parameter $\tau = \tau(\Gamma) \in [0, 1/2)$, defined by $\tau(\Gamma) = \arg \left( \frac{1}{2} - \lambda_1 \right)$.

When $\Gamma$ is a congruence group, more is known about the spectral gap. We recall that a congruence subgroup of $\text{SL}_2(\mathbb{Z})$ is a subgroup containing one of the principal congruence groups $\Gamma(q) = \{g \in \text{SL}_2(\mathbb{Z}) : g \equiv I \pmod{q}\}$ (there are also congruence subgroups of co-compact lattices derived from quaternion algebras that are defined similarly). For these congruence groups, Selberg’s eigenvalue conjecture states that $\tau(\Gamma) = 0$. This is known for $\Gamma = \text{SL}_2(\mathbb{Z})$ as well as for some other congruence groups of small level $q$. The best known bound for a general congruence lattice is $\tau(\Gamma) \leq \frac{\ln}{54}$ [6]. On the other hand, there are also non congruence subgroups $\Gamma \subseteq \text{SL}_2(\mathbb{Z})$ with $\tau(\Gamma)$ arbitrarily close to $1/2$.

2.5. **Decay of matrix coefficients.** Given a lattice $\Gamma \subseteq G$, let $\mu$ denote the $G$ invariant probability measure on $\Gamma \backslash G$. The group $G$ acts on the right on the space $L^2(\Gamma \backslash G, \mu)$ via $\pi(g)\psi(x) = \psi(xg)$, and for any two functions $\psi, \varphi$ the corresponding matrix coefficient is

$$
\langle \pi(g)\psi, \varphi \rangle = \int_{\Gamma \backslash G} \psi(xg)\varphi(x) d\mu(x).
$$

For $\psi, \varphi \in L^2_0(\Gamma \backslash G)$ (the space orthogonal to the constant function) the corresponding matrix coefficients go to zero as $g \to \infty$, and the rate of decay is related to the spectral gap of $\Gamma$ as follows (see [14, Section 9.11]): For any smooth $\psi, \varphi \in L^2_0(\Gamma \backslash G) \cap C^\infty(\Gamma \backslash G)$, for all sufficiently small $\epsilon > 0$,

$$
\langle \pi(ka_jk^\prime)\psi, \varphi \rangle \ll (1 + y)^{-1/2+\epsilon} (\mathcal{S}_{2,1}(\psi)\mathcal{S}_{2,1}(\varphi))^{1/2+\epsilon} (\mathcal{S}_{2,0}(\psi)\mathcal{S}_{2,0}(\varphi))^{1/2-\epsilon},
$$

where $\tau = \tau(\Gamma)$ the spectral gap parameter given in (2).
3. PROOF OF MAIN RESULTS

For the proof, we first use the duality of the ζ action on $G/\tilde{N} \cong \mathbb{R}^2 \sim \{0\}$ and the $\tilde{N}$ action on $\Gamma \setminus G$ to reduce the problem to a shrinking target problem for a unipotent flow. Then we prove an effective mean ergodic theorem and use it to give a partial solution for the shrinking target problem. Combining these results will give the proof of Theorem 2.

3.1. Reduction to a shrinking target problem. To define our shrinking targets, fix $\nu = (\nu_1, \nu_2) \in \mathbb{R}^2$ and assume that $\nu_1 \nu_2 \neq 0$. For small $\delta \in (0, 1/2)$, consider the set $\mathcal{A}_\delta(\nu) \subseteq G$ given by

$$\mathcal{A}_\delta(\nu) = \{n_x a_y \tilde{n}_{x'} : |x'| \leq 1/2, |\frac{1}{\sqrt{\nu_1}} - 1| \leq \frac{\delta}{|\nu_2|}, |\frac{x}{\sqrt{\nu_1}} - 1| \leq \frac{\delta}{|\nu_1|}\}.$$

Note that, for any $g \in \mathcal{A}_\delta(\nu)$, we have that $\|g(\nu) - \nu\|_\infty \leq \delta$ and, moreover, if $\|g(\nu) - \nu\|_\infty \leq \delta$, then $g \tilde{n}_k \in \mathcal{A}_\delta(\nu)$ for some $k \in \mathbb{Z}$. For each of the sets $\mathcal{A}_\delta(\nu) \subseteq G$ we define the corresponding set $\mathcal{B}_\delta(\nu) \subseteq \Gamma \setminus G$ by

$$\mathcal{B}_\delta(\nu) = \{g \in \mathcal{A}_\delta(\nu)\}.$$

The following lemma shows that these shrinking targets $\mathcal{B}_\delta(\nu)$ are stable under small perturbation in $\nu$.

**Lemma 7.** If $\nu, \tilde{\nu} \in \mathbb{R}^2$ are off the axes and satisfy that $\|\nu - \tilde{\nu}\| < \delta$, then $\mathcal{B}_\delta(\nu) \subseteq \mathcal{B}_{\delta}(\tilde{\nu})$.

**Proof.** Let $\Gamma g \in \mathcal{B}_\delta(\nu)$. Then there is some $\gamma \in \Gamma$ with $\gamma g \in \mathcal{A}_\delta(\nu)$ and, hence, $\|\gamma g(\nu) - \nu\| \leq \delta$. But then $\|\gamma g(\nu) - \nu\| \leq 2\delta$ and, hence, $\gamma g \tilde{n}_k \in \mathcal{A}_{\delta}(\tilde{\nu})$ for some $k \in \mathbb{Z}$. Now, on one hand, $\gamma g = n_x a_y \tilde{n}_{x'}$ with $|x'| \leq 1/2$ and, on the other hand, $\gamma g = n_x a_y \tilde{n}_{x'}$ with $|x' - k| < 1/2$, implying that $x = \tilde{x}, y = \tilde{y}, x' = \tilde{x}'$ and $k = 0$, and, hence, $\gamma g \in \mathcal{A}_{\delta}(\tilde{\nu})$ and $\Gamma g \in \mathcal{B}_{\delta}(\tilde{\nu})$, as claimed.

The shrinking target problem is then to determine how fast can targets $\mathcal{B}_{\delta_k}(\nu)$ shrink so that the finite orbits

$$\Theta_k(x) = \{x \tilde{n}_i : |i| \leq k\},$$

keeps hitting them. The following lemma connects this shrinking target problem to the critical exponents (cf. [2, Proposition 3.2]).

**Lemma 8.** Fix $g \in G$ and let $u = g(\nu) \in \mathbb{R}^2$ and $x = \Gamma g \in \Gamma \setminus G$. For any $\alpha < \eta$

1. If $\{\gamma \in \Gamma : \|\gamma u - v\| \leq \|\gamma\|^{-\eta}\}$ is unbounded then $\{k : x \tilde{n}_k \in \mathcal{B}_{1/k^\alpha}(\nu)\}$ is unbounded.
2. If $\Theta_T(x) \cap \mathcal{B}_{1/T^\alpha}(\nu) \neq \emptyset$ for all sufficiently large $T$, then $\Gamma T u \cap B_{1/T^\alpha}(\nu) \neq \emptyset$ for all sufficiently large $T$.

**Proof.** Assume that $\gamma_i \in \Gamma$ has $\|\gamma_i\| \to \infty$ and satisfies $\|\gamma_i u - v\| \leq \|\gamma_i\|^{-\eta}$. Let $\delta_i = \|\gamma_i\|^{-\eta}$. Then for each $i \in \mathbb{N}$, there is $k_i \in \mathbb{Z}$ such that $\gamma_i g \tilde{n}_{k_i} \in \mathcal{A}_{\delta_i}(\nu)$ and, hence, $x \tilde{n}_{k_i} \in \mathcal{B}_{\delta_i}(\nu)$. Moreover, since $\gamma_i g \tilde{n}_{k_i} \in \mathcal{A}_{\delta_i}(\nu)$ and $\mathcal{A}_{\delta_i}(\nu)$ is contained in a compact set depending only on $\nu$, comparing norms we see that $\|\gamma_i\|^{-\eta} \leq \|g\|^{-\eta}$.

Journal of Modern Dynamics

Volume 11, 2017, 143–153
\(B_{c/k_i^\eta}(v)\). Now, for any \(\alpha < \eta\), we have that \(\frac{c}{k_i^\eta} \leq \frac{1}{k_i^\tau}\) for \(k_i\) sufficiently large, and so, from some point, \(x\bar{n}_k \in B_{1/k_i^\alpha}(v)\) and, indeed, the set \(\{k : x\bar{n}_k \in B_{1/k_i^\alpha}(v)\}\) is unbounded.

For the second statement, assume that for all \(T \geq T_0\) there is \(|k| \leq T\) with \(x\bar{n}_k \in \mathcal{B}_{T^{-\eta}}(v)\). Then there is \(\gamma_k \in \Gamma\) with \(\gamma_k g \bar{n}\bar{k} \in \mathcal{A}_{T^{-\eta}}(v)\), hence, \(\|\gamma_k u - v\| \leq T^{-\eta}\). Also, as before, since \(\gamma_k g \bar{n}\bar{k} \in \mathcal{A}_{T^{-\eta}}(v)\) comparing norms we get that \(\|\gamma_k\| \leq \|\bar{n}\bar{k}\| = k\) so there is \(c > 0\) (depending on \(g, v\)) such that \(\|\gamma_k\| \leq c T\). Setting \(\tilde{T} = cT\), and \(\tilde{T}_0 = cT_0\), assuming that \(\tilde{T}_0\) is sufficiently large so that \((\tilde{T}_0/c)^{-\eta} \leq \tilde{T}_0^{-\alpha}\), we get that for all \(\tilde{T} \geq \tilde{T}_0\) there is \(\gamma \in \Gamma\) with \(\|\gamma u - v\| \leq (T/c)^{-\eta} \leq T^{-\alpha}\) so \(\gamma u \in \Gamma \tilde{T} \cap B_{1/\tilde{T}_0}(v)\).

3.2. **Solution of the shrinking target problem.** In this section we prove the following result, giving a partial solution to the shrinking target problem.

**Theorem 9.** Fix \(v \in \mathbb{R}^2\) with \(v_1 v_2 \neq 0\) and let \(\mathcal{B}_\delta(v)\) be as above. Then

1. If \(\eta > 1/2\) then for almost all \(x \in \Gamma \setminus G\) the set \(\{k \in \mathbb{Z} : x\bar{n}_k \in \mathcal{B}_{k^{-\eta}}(v)\}\) is bounded.
2. If \(0 < \eta < \frac{1-\tau}{2}\), then for almost all \(x \in \Gamma \setminus G\) there is \(T_0 > 0\) such that, for all \(k \geq T_0\), we have that \(\Theta_k(x) \cap \mathcal{B}_k^{-\eta}(v) \neq \emptyset\).
3. If \(0 < \eta < \frac{1-\tau}{5}\), then for any compact set \(\Omega \subseteq \{v \in \mathbb{R}^2 : v_1 v_2 \neq 0\}\), for almost all \(x \in \Gamma \setminus G\) there is \(T_0 > 0\) (depending on \(x\) and \(\Omega\)) such that, for all \(k \geq T_0\), we have that \(\Theta_k(x) \cap \mathcal{B}_k^{-\eta}(v) \neq \emptyset\) for all \(v \in \Omega\).

**Remark 10.** This is a partial result because, even in the optimal setting when \(\tau = 0\), we do not say anything when \(\frac{1}{3} < \eta < \frac{1}{2}\). It is reasonable that the correct range for (2) is in fact \(0 < \eta < 1/2\) but we are not able to show this here. We note that for similar shrinking target problems, when the shrinking targets are spherical (i.e., right-\(K\) invariant), by a similar argument one can establish (2) with an upper bound for \(\eta\) that is the same as the bound coming from Borel-Cantelli in (1). In fact this is shown for unipotent flows on more general homogeneous spaces [5]. We also note that the exponent in (3) is even smaller because we require a much stronger form of approximation, that is, that a single orbit \(\Theta_k(x)\) approximates simultaneously all target points in \(\Omega\).

Our main tool for the proof will be an effective mean ergodic theorem for the unipotent flow \(u_t = \bar{n}_t\) on \(\Gamma \setminus G\) (we use the notation \(u_t\) to indicate that the same results holds for any unipotent flow). For any \(T > 0\), let \(\beta_T\) denote the averaging operator on \(C_c^\infty(\Gamma \setminus G)\) given by

\[
\beta_T(\varphi)(x) = \frac{1}{2T + 1} \sum_{|k| \leq T} \varphi(xu_k).
\]

Since the unipotent flow is ergodic, the mean ergodic theorem implies that \(\|\beta_T(\varphi) - \int_{\Gamma \setminus G} \varphi d\mu\|_2 \rightarrow 0\) as \(T \rightarrow \infty\) for any \(\varphi \in L^2(\Gamma \setminus G)\). Using the decay of matrix coefficients we show the following effective result.
**Proposition 11.** Let \( \tau = \tau(\Gamma) \) measure the spectral gap for \( \Gamma \). Then for any smooth \( \phi \in C_c^{\infty}(\Gamma \backslash G) \), for all sufficiently small \( \epsilon > 0 \), we have

\[
\| \beta_T(\phi) - \int_{\Gamma \backslash G} \phi d\mu \|_2 \ll \epsilon \frac{\mathcal{J}^{1+\epsilon}_{2,1}(\phi) \mathcal{J}^{1-\epsilon}_{2,0}(\phi)}{T^{1-2\epsilon}}.
\]

**Proof.** Let \( \phi_0 = \phi - \int_{\Gamma \backslash G} \phi d\mu \). Then \( \phi_0 \in L^2_0(\Gamma \backslash G) \) and \( \beta_T(\phi_0) = \beta_T(\phi) - \int_{\Gamma \backslash G} \phi d\mu \). Now expand

\[
\| \beta_T(\phi_0) \|_2^2 = \langle \beta_T\phi_0, \beta_T\phi_0 \rangle
\]

Making a change of index summation and changing the order of summation, we get

\[
\| \beta_T(\phi_0) \|_2^2 = \frac{1}{(1+2T)^2} \sum_{|l| \leq T} \sum_{|k| \leq T} \langle \pi(u_k)\phi_0, \pi(u_l)\phi_0 \rangle
\]

Writing \( u_l = k a_l k' \), with \( y \geq 1 \) and \( k, k' \in K \), and comparing Hilbert-Schmidt norms we see that \( 2 + t^2 = y + y^{-1} \). Using the decay of matrix coefficients (3), we can bound

\[
| \langle \pi(u_k)\phi_0, \phi_0 \rangle | \ll \epsilon |k|^{2^{\tau-1 + 2\epsilon}} \mathcal{J}^{1+2\epsilon}_{2,1}(\phi_0) \mathcal{J}^{1-2\epsilon}_{2,0}(\phi_0),
\]

and, hence,

\[
\| \phi_0 \|_2^2 \ll \epsilon \frac{\mathcal{J}^{1+2\epsilon}_{2,1}(\phi_0) \mathcal{J}^{1-2\epsilon}_{2,0}(\phi_0)}{2T + 1} \left( 1 + \sum_{k=1}^{2T} \frac{1}{|k|^{1-2\tau-2\epsilon}} \right) \ll \frac{\mathcal{J}^{1+2\epsilon}_{2,1}(\phi_0) \mathcal{J}^{1-2\epsilon}_{2,0}(\phi_0)}{T^{1-2\tau-2\epsilon}}.
\]

Finally, from orthogonality, \( \mathcal{J}_{2,0}(\phi_0) \leq \mathcal{J}_{2,0}(\phi) \) and, since, for any derivative \( \mathcal{D}\phi_0 = \mathcal{D}\phi \), we also have \( \mathcal{J}_{2,1}(\phi_0) \leq \mathcal{J}_{2,1}(\phi) \), concluding the proof. \( \Box \)

Using the effective mean ergodic theorem as a variance estimate, we can estimate the measure of points whose orbit miss the small set \( \mathcal{B} \). Explicitly, we show

**Proposition 12.** Let \( \mathcal{E}_{T,\delta} = \{ x \in \Gamma \backslash G : \Sigma_T(x) \cap \mathcal{B}(\nu) = \emptyset \} \). Then, for all sufficiently small \( \epsilon > 0 \),

\[
\mu(\mathcal{E}_{T,\delta}) \ll \nu, \epsilon \frac{1}{T^{1-2\tau+\epsilon}}.
\]
Proof: Let $\rho \in C_c^\infty(\mathbb{R})$ be non-negative, supported in $(-1/2, 1/2)$ with mean one. Define a function $f_\delta \in C_c^\infty(G)$ by

$$f_\delta(n_x a_y \tilde{n}_x') = \rho \left(\frac{x - \sqrt{T}v_1}{\delta}\right) \rho \left(\frac{y - 1/2v_2}{\delta}\right) \rho(x'),$$

(this defines $f_\delta$ only on $NA\tilde{N}$, but, since $f_\delta(g) = 0$ for $y$ sufficiently large and sufficiently small, it extends smoothly by zero to all of $G$). Let $F_\delta \in C_c^\infty(\Gamma \setminus G)$ be the corresponding $\Gamma$-invariant function,

$$F_\delta(\gamma g) = \sum_{\gamma \in \Gamma} f_\delta(\gamma g).$$

Clearly $f_\delta$ is supported on $\mathcal{A}_\delta(v)$ and $F_\delta$ is supported on $\mathcal{B}_\delta(v)$. Moreover, since $\mathcal{A}_\delta(v) \subseteq \mathcal{A}_{1/2}(v)$ is contained in some fixed compact set, there is some $C > 0$ (depending only on $\nu$) such that $\mathcal{A}_\delta$ is contained in a union of $C$ fundamental domains for $\Gamma \setminus G$. Consequently, we also have that $\mu(F_\delta) = \nu(\mathcal{A}_\delta) = \delta^2$ and

$$\mathcal{J}_{2,0}(F_\delta) \precsim \nu \left(\int_G |f_\delta|^2 d\mu\right)^{1/2} = \delta.$$

Since derivatives of $\tilde{f}_\delta$ are of the form $Df_\delta = \frac{1}{2} \tilde{f}_\delta$ with $\tilde{f}_\delta$ supported on $\mathcal{A}_\delta(v)$ with $\int_G \tilde{f}_\delta d\mu = \nu \delta^2$, we also have that $\mathcal{J}_{2,1}(F_\delta) = 1$. With these estimates, Proposition 11 implies that

$$\|\beta_T(F_\delta) - \mu(F_\delta)\| \precsim \nu(\mathcal{A}_\delta) \% T^2 \delta^{-1-\epsilon}.$$ 

On the other hand, since $\beta_T(F_\delta)(x) = 0$ for all $x \in \mathcal{C}_{T,\delta}$, we can bound from below

$$\|\beta_T(F_\delta) - \mu(F_\delta)\|^2 \gtrsim \int_{\mathcal{C}_{T,\delta}} |\beta_T(F_\delta) - \mu(F_\delta)|^2 d\mu = \nu(\mathcal{C}_{T,\delta}) \delta^4$$

from which the result follows. \hfill $\Box$

We now go back to the shrinking target problem and give the

**Proof of Theorem 9.** First, noting that $\mu(\mathcal{B}_\delta(v)) \precsim \delta^2$ if we take $\delta_k = k^{-\eta}$ with $\eta > 1/2$, the series

$$\sum_k \mu(\mathcal{B}_{\delta_k}(v)) \precsim \nu \sum_k \frac{1}{k^{2\eta}} < \infty,$$

converges and, hence, by the easy half of the Borrel-Cantelli lemma, for almost all $x \in \Gamma \setminus G$, we have that $\{k : x \in \mathcal{B}_{\delta_k}(v)\}$ is bounded.

Next, for the lower bound, for a fixed target point $v$, assume that $\delta_k = k^{-\eta}$ with $0 < \eta < \frac{1-2\gamma}{2}$ and let $\mathcal{C}_v \subseteq \Gamma \setminus G$ denote the set of all points such that, for any $T \in \mathbb{N}$, there is $k \geq T$ with $B_{\delta_k}(v) \cap \mathcal{B}_{\delta_k}(v) = \emptyset$, that is,

$$\mathcal{C}_v = \bigcap_{T \in \mathbb{N}} \bigcup_{k \geq T} \mathcal{C}_k,$$

with $\mathcal{C}_k = \mathcal{C}_{k,\delta_k}$ the set of points, $x$, for which $B_{\delta_k}(x) \cap \mathcal{B}_{\delta_k}(v) = \emptyset$. Now consider the sets

$$\mathcal{E}_k = \{x \in \Gamma \setminus G : \Omega_k(x) \cap \mathcal{B}_{\delta_k}(v) = \emptyset\},$$

\textbf{Journal of Modern Dynamics} \textbf{Volume 11, 2017, 143-153}
and note that, since the orbits $\mathcal{O}_k(x)$ are increasing sets and the targets $\mathcal{B}_{\delta_k}(v)$ are decreasing, we have that

$$\bigcup_{l=k}^{2k} \mathcal{C}_l = \{ x \in \Gamma \setminus G : \exists l \in [k, 2k], \mathcal{O}_I(x) \cap \mathcal{B}_{\delta_k}(v) = \emptyset \} \subseteq \mathcal{C}_k.$$

We thus have that

$$\mathcal{C}_v = \bigcap_{T \in \mathbb{N}} \bigcup_{l \geq \log(T)} \bigcup_{k=2^l}^{2^{l+1}} \mathcal{C}_k \subseteq \bigcup_{T \in \mathbb{N}} \bigcup_{l \geq \log(T)} \mathcal{C}_{2^l}.$$

By Proposition 12 we can estimate

$$\mu(\mathcal{C}_{2^l}) \ll \frac{1}{2^{l(1-2\tau-3\eta)}}$$

and, hence,

$$\mu\left( \bigcup_{l \geq \log(T)} \mathcal{C}_{2^l} \right) \leq \sum_{l \geq \log T} \mu(\mathcal{C}_{2^l}) \ll \epsilon \sum_{l \geq \log T} \frac{1}{2^{l(1-2\tau-3\eta)}}.$$

Now, from our assumption $1 - 2\tau - 3\eta > 0$ and taking $\epsilon = \frac{1-2\tau-3\eta}{2}$ we can estimate

$$\mu\left( \bigcup_{l \geq \log(T)} \mathcal{C}_{2^l} \right) \ll \sum_{l \geq \log(T)} 2^{-\epsilon} \ll T^{-\epsilon},$$

implying that $\mu(\mathcal{C}_v) = 0$.

Finally, for the uniform bound, let $\delta_k = k^{-a}$ with $0 < \eta < \alpha < \frac{1-2\tau}{5}$ and for each $k$ let $\{v_{k,l}\}_{l=1}^{m_k} \subseteq \Omega$ be $\delta_k$-dense in $\Omega$ (so that $m_k \asymp \Omega \delta_k^{-2}$). Moreover, by choosing our points to have rational coordinates with dyadic denominators, we ensure that the set $\{v_{k,l}\}_{l=1}^{m_k}$ contains all points $v_{k,j}$ with $k' \leq k$ and $j \leq m_k$. Now, let $\mathcal{C}_\Omega = \bigcap_{T \in \mathbb{N}} \bigcup_{k \geq T} m_k \mathcal{C}_{k,i},$ with $\mathcal{C}_{k,i} = \{ x \in \Gamma \setminus G : \mathcal{O}_k(x) \cap \mathcal{B}_{\delta_k}(v_{k,i}) = \emptyset \}$. As before, we have

$$\mathcal{C}_\Omega \subseteq \bigcap_{T \in \mathbb{N}} \bigcup_{l \geq \log(T)} \bigcup_{i=1}^{m_{2^l+1}} \mathcal{C}_{2^l,i},$$

with

$$\mathcal{C}_{k,i} = \{ x \in \Gamma \setminus G : \mathcal{O}_k(x) \cap \mathcal{B}_{\delta_k}(v_{k,i}) = \emptyset \}.$$
Then, for all sufficiently large $k$, $Ω_k(x) \cap B_δ_k(ν_{k,i}) \neq ∅$. Since $∥v - ν_{k,i}∥ ≤ δ_k$, we have that, for $k$ sufficiently large, $B_δ_k(ν_{k,i}) ⊆ B_δ_k(ν) ⊆ B_δ_κ(ν)$, implying that $Ω_k(x) \cap B_δ_κ(ν) \neq ∅$ as well.

3.3. Conclusion. Now combining the shrinking target results in Theorem 9 with Lemma 8, we get estimates on the critical exponents.

Proof of Theorem 2. Let $0 \neq ν ∈ ℝ^2$. Since the orbit $νΓ$ can not be contained in the set $\{ν ∈ ℝ^2 : v_1v_2 = 0\}$ and the critical exponents $μ(u, ν) = μ(u, γ ν)$ and $\hat{μ}(u, ν) = \hat{μ}(u, γ ν)$, we may assume that $v_1v_2 \neq 0$. For any $δ ∈ (0, 1/2)$, let $A_δ(ν)$ and $B_δ(ν)$ be as above.

First, to show that for almost all $u ∈ ℝ^2$, we have $μ(u, ν) ≤ 1/2$, fix some $η > 1/2$ and let $U ⊆ ℝ^2$ denote the set of all $u ∈ ℝ^2$ such that there is a sequence $γ_k$ with $∥γ_k u - ν∥ ≤ ∥γ_k∥_1$. For each $u ∈ U$, let $g_u = \begin{pmatrix} u_{11} & u_{12} \\ 0 & u_{22} \end{pmatrix}$ and let $Ω ⊆ G$ be defined by $Ω = \{g_u ν_k : u ∈ U, |x| ≤ 1/2\}$. Let $η > α > 1/2$. Then by the first part of Lemma 8, for any $g ∈ Ω$, the set $\{k : xν_k ∈ B_δ_κ(ν)\}$ is unbounded, and hence $\{g : g ∈ U\} ⊆ \{x ∈ Γ \cap G : \{k : xν_k ∈ B_δ_κ(ν)\} \text{ is unbounded}\}$.

By the first part of Theorem 9, the set on the right has measure zero. We thus get that the set $\{g : g ∈ Ω\} ⊆ Γ \cap G$ is a null set. Then the set $Ω ⊆ G$ and, hence, $U ⊆ ℝ^2$ must also have measure zero. This shows that for almost all $u ∈ ℝ^2$, the set $\{γ : ∥γ u - ν∥ ≤ ∥γ∥_1\}$ is bounded, so $μ(u, ν) ≤ η$ for almost all $u ∈ ℝ^2$. Since this holds for any $η > 1/2$, we get the upper bound $μ(u, ν) ≤ 1/2$ for almost all $u ∈ ℝ^2$.

Next, to show that for almost all $u ∈ ℝ^2$, we have $\hat{μ}(u, ν) ≥ \frac{1-2τ}{3}$, fix some $α < \frac{1-2τ}{3}$. Let $α < η < \frac{1-2τ}{3}$, let $δ_k = k^{-η}$, and let $C ⊆ Γ \cap G$ be the set of all points $x ∈ Γ \cap G$ such that, for every $T ∈ ℕ$, there is a $k ≥ T$ with $Ω_k(x) \cap B_δ_k(ν) = ∅$. Let $C_0 = \{g ∈ G : Γ g ∈ C\}$. Then, for any $g ∈ G \setminus C_0$ and any $u = g(1)$, by the second part of Lemma 8, $Γ_T u \cap B_{1/τ}(ν) \neq ∅$ for all sufficiently large $T$. By the second part of Theorem 9, we have that $μ(C) = 0$ and, hence, $C_0$ is a null set and the set $\{g(1) : g ∈ G \setminus C_0\}$ is set of full measure. This shows that for almost all $u ∈ ℝ^2$, we have that $Γ_T u \cap B_{1/τ}(ν) \neq ∅$ for all sufficiently large $T$, and hence $\hat{μ}(u, ν) ≥ α$. Since this holds for any $α < \frac{1-2τ}{3}$, we get that $\hat{μ}(u, ν) ≥ \frac{1-2τ}{3}$ for almost all $u ∈ ℝ^2$.

Finally, for the uniform bound, write $\{ν ∈ ℝ^2 : v_1v_2 \neq 0\} = ∪_i Ω_i$ as a union of countably many compact sets (with each $Ω_i$ bounded away from the axis). Using the third part of Theorem 9 with the same argument as above shows that for each $i ∈ ℕ$, for almost all $u ∈ ℝ^2$, we have that $\hat{μ}(u, ν) ≥ \frac{1-2τ}{3}$ for all $ν ∈ Ω_i$. Since these are countably many conditions, we have the same result for all $ν ∈ ℝ^2$ with $v_1v_2 \neq 0$ and, since $\hat{μ}(u, ν) = \hat{μ}(u, γ ν)$, this holds for all $ν ∈ ℝ^2 \setminus \{0\}$.

REFERENCES

[1] A. Ghosh, A. Gorodnik and A. Nevo, Best possible rates of distribution of dense lattice orbits in homogeneous spaces, *J. Reine Angew. Math.*, to appear.
[2] A. Ghosh, A. Gorodnik and A. Nevo, Diophantine approximation exponents on homogeneous varieties, in Recent Trends in Ergodic Theory and Dynamical Systems, Contemp. Math., 631, Amer. Math. Soc., Providence, RI, 2015, 181–200.
[3] A. Ghosh and D. Kelmer, Shrinking targets for semisimple groups, arXiv:1512.05848, 2015.
[4] A. Gorodnik and B. Weiss, Distribution of lattice orbits on homogeneous varieties, Geom. Funct. Anal., 17 (2007), 58–115.
[5] D. Kelmer, Shrinking targets for discrete time flows on hyperbolic manifolds, preprint.
[6] H. Kim and P. Sarnak, Refined estimates towards the Ramanujan and Selberg conjectures, J. Amer. Math. Soc., 16 (2003), 139–183.
[7] F. Ledrappier, Distribution des orbites des réseaux sur le plan réel, C. R. Acad. Sci. Paris Sér. I Math., 329 (1999), 61–64.
[8] M. Laurent and A. Nogueira, Approximation to points in the plane by SL(2, Z)-orbits, J. Lond. Math. Soc. (2), 85 (2012), 409–429.
[9] M. Laurent and A. Nogueira, Inhomogeneous approximation with coprime integers and lattice orbits, Acta Arith., 154 (2012), 413–427.
[10] F. Maucourant and B. Weiss, Lattice actions on the plane revisited, Geom. Dedicata, 157 (2012), 1–21.
[11] A. Nogueira, Orbit distribution on R^2 under the natural action of SL(2, Z), Indag. Math. (N.S.), 13 (2002), 103–124.
[12] M. Pollicott, Rates of convergence for linear actions of cocompact lattices on the complex plane, Integers, 11B (2011), Paper No. A12, 7pp.
[13] L. Singhal, Diophantine exponents for standard linear actions of SL_2 over discrete rings in C, Acta Arith., 177 (2017), 53–73.
[14] A. Venkatesh, Sparse equidistribution problems, period bounds and subconvexity, Ann. of Math. (2), 172 (2010), 989–1094.

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