On Higher Dimensional Fuzzy Spherical Branes

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Abstract

Matrix descriptions of higher dimensional spherical branes are investigated. It is known that a fuzzy $2k$-sphere is described by the coset space $SO(2k+1)/U(k)$ and has some extra dimensions. It is shown that a fuzzy $2k$-sphere is comprised of $\frac{k(k-1)}{2}$ spherical $D(2k-1)$-branes and has a fuzzy $2(k-1)$-sphere at each point. We can understand the relationship between these two viewpoints by the dielectric effect. Contraction of the algebra is also discussed.
1 Introduction

Some new features of string theory appeared since the discovery of the D-brane [1]. One of the recent interesting developments is the appreciation of noncommutative geometry. A low energy effective action of $N$ coincident D-branes is described by the $U(N)$ Yang Mills theory. In this theory, $U(N)$ adjoint scalars represent the transverse coordinates of this system [2]. Since they are given by $U(N)$ matrices, this fact suggests that the space-time probed by D-branes is related to noncommutative geometry. It is discussed that a world volume theory on D-branes in the presence of a constant NS-NS two-form background is described by a noncommutative gauge theory [3, 4, 5]. Myers showed that a low energy effective action of $N$ D-branes in the presence of a constant Ramond-Ramond background has an additional Chern-Simons term, and the minimum of the potential is given by the configuration where transverse coordinates form a fuzzy sphere [6]. The ideas of noncommutative geometry found a prominent role in string theory.

Higher dimensional fuzzy spheres have been investigated in some contexts [15, 16, 17, 18, 19, 20, 21, 22]. The authors in [21] showed that the matrix description of a fuzzy 2k-sphere is given by an $SO(2k+1)/U(k)$ coset space which is a $k(k+1)$ dimensional space. The stabilizer group of this noncommutative manifold is not $SO(2k)$ but $U(k)$. For example, a fuzzy four-sphere is described by $SO(5)/U(2)$, being a six dimensional space. In general, a higher dimensional fuzzy sphere has some extra dimensions. The existence of them is important for the quantization of a higher dimensional sphere. It is well-known that noncommutative geometry is realized by the guiding center coordinates of electrons in a constant magnetic field. The fuzzy four-sphere is naturally realized by the quantum Hall system which was constructed in [23]. The realization was first investigated in [25]. The system in [23] is composed of particles moving on a four dimensional sphere under an $SU(2)$ gauge field. It was shown that the configuration space of this system is locally $S^4 \times S^2$ which is consistent with the result in [21]. There were further analyses in [26, 27, 28, 29, 30] following these papers.

The purpose of this paper is to investigate higher dimensional spherical branes in matrix models and to elucidate the extra dimensions which appear from fuzzy spheres. The organization of this paper is as follows. A fuzzy four-sphere is considered as a example of higher dimensional fuzzy spheres in section 2. Some parts overlap with [28]. We begin with the $SO(5, 1)$ algebra. It is explained that the manifold which is described by this algebra becomes locally $S^4 \times S^2$ in a large $N$ limit. The extra two dimensional sphere is added to make a four-sphere symplectic manifold. The procedure of the Inönü-Wigner contraction gives a six dimensional noncommutative plane whose algebra is given by the Heisenberg algebra. Two out of the six dimensions come from the extra two-sphere. In section 3 we study higher dimensional fuzzy spheres. A fuzzy 2k-sphere is defined by the $SO(2k+1, 1)$ algebra. It is shown that a fuzzy 2k-sphere with a radius $\alpha^2 n(n+2k)$ has a fuzzy 2$(k-1)$-sphere with a radius $\alpha^2 n(n+2k-2)$ at each point. A fuzzy 2k-sphere is a $k(k+1)$ dimensional space including a extra $k(k-1)$ dimensional space which originates from a fuzzy 2$(k-1)$-sphere. Taking a large $N$ limit leads to a smooth manifold which is given by $S^{2k} \times S^{2(k-1)} \times \cdots \times S^2$. Contraction of the algebra gives a $k(k+1)$ dimensional noncommutative plane. In section 4 we give some comments about matrix models whose classical solutions are
fuzzy spheres. If we expand matrices around a fuzzy sphere solution, a matrix model action gives a noncommutative Yang-Mills action on it. The advantage of such formulation is that we can define the derivative and formulate matrix models in terms of field theories. The size of matrices plays a role of cut-off, and noncommutative Yang-Mills theories have no divergences. In section 5 we evaluate the value of the matrix model action for fuzzy spheres. The estimation leads to the fact that a fuzzy $2k$-sphere is comprised of $n^{k(k-1)} \times D(2k-1)$-branes in the matrix model. The dielectric effect of $D$-branes manifests the structure of a higher dimensional fuzzy sphere in an interesting way. Section 6 is devoted to summary and discussions.

2 Fuzzy four-sphere

2.1 Matrix description of fuzzy four-sphere

In this section, we briefly review the algebra of fuzzy four-sphere [16, 20, 21]. As a beginning, we examine the $SO(5, 1)$ algebra which is given by

$$[\hat{J}_{MN}, \hat{J}_{OP}] = 2 \left( \eta_{NO} \hat{J}_{MP} + \eta_{MP} \hat{J}_{NO} - \eta_{MO} \hat{J}_{NP} - \eta_{NP} \hat{J}_{MO} \right),$$

where $M, N, \ldots$ run over 0 to 5 and $\eta_{MN} = \text{diag}(-1, 1, \ldots, 1)$. These matrices are constructed as

$$\hat{J}_{0\mu} = (\Gamma_\mu \otimes 1 \otimes \cdots \otimes 1 + 1 \otimes \Gamma_\mu \otimes \cdots \otimes 1 \otimes \cdots + 1 \otimes \cdots \otimes \otimes \otimes \Gamma_\mu)_{\text{Sym}},$$

$$\hat{J}_{\mu\nu} = (\Gamma_{\mu\nu} \otimes 1 \otimes \cdots \otimes 1 + 1 \otimes \Gamma_{\mu\nu} \otimes \cdots \otimes 1 \otimes \cdots + 1 \otimes \cdots \otimes \otimes \otimes \Gamma_{\mu\nu})_{\text{Sym}},$$

where $\mu$ and $\nu$ run over 1 to 5. These are constructed from the $n$-fold symmetric tensor product of the five dimensional Gamma matrices and $\text{Sym}$ means the completely symmetrized tensor product. The size of the matrices is given by

$$N = \frac{1}{6} (n + 1)(n + 2)(n + 3).$$

We now explain that this algebra describes noncommutative geometry which becomes locally $S^4 \times S^2$ in a large $N$ limit if we regard $J_{0\mu} = G_\mu$ \footnote{Note that $\Gamma_{0\mu}$ is hermitian.} as coordinates of the four-sphere and $J_{\mu\nu} = G_{\mu\nu}$ as coordinates of the two-sphere. If we rewrite (1) using $G_\mu$ and $G_{\mu\nu}$, we have

$$[\hat{G}_\mu, \hat{G}_\nu] = 2\hat{G}_{\mu\nu},$$

$$[\hat{G}_\mu, \hat{G}_{\nu\lambda}] = 2 \left( \delta_{\mu\nu} \hat{G}_\lambda - \delta_{\mu\lambda} \hat{G}_\nu \right),$$

$$[\hat{G}_{\mu\nu}, \hat{G}_{\lambda\rho}] = 2 \left( \delta_{\nu\lambda} \hat{G}_{\mu\rho} + \delta_{\mu\rho} \hat{G}_{\nu\lambda} - \delta_{\mu\lambda} \hat{G}_{\nu\rho} - \delta_{\nu\rho} \hat{G}_{\mu\lambda} \right),$$

where $\hat{G}_{\mu\nu}$ form the $SO(5)$ algebra. The fuzzy four-sphere is constructed to satisfy the following two conditions,

$$\epsilon^{\mu\nu\rho\sigma} \hat{x}_\mu \hat{x}_\nu \hat{x}_\lambda \hat{x}_\rho = C \hat{x}_\sigma$$

and

$$\hat{x}_\mu \hat{x}_\mu = \rho^2,$$

Note that $\Gamma_{0\mu}$ is hermitian.
where $\hat{x}_\mu$ are coordinates of the fuzzy four-sphere and $\rho$ is a radius of the sphere. This sphere respects the $SO(5)$ invariance. Such $\hat{x}_\mu$ are constructed as

$$\hat{x}_\mu = \alpha \hat{G}_\mu,$$

(7)

where $\alpha$ is a dimensionful constant. If we use some formulae which are summarized in appendix A and $\rho$ is given by

$$C = (8n + 16)\alpha^3$$

(8)

and

$$\rho^2 = \alpha^2 n(n + 4).$$

(9)

Let us next observe that a fuzzy two-sphere exists at each point on the fuzzy four-sphere. We can always diagonalize a matrix $\hat{G}_\mu$ out of the five matrices. We diagonalize $\hat{x}_5 = \alpha \hat{G}_5$ as in appendix B. The $SU(2) \times SU(2)$ algebra is constructed from the $SO(4)$ algebra, which is the stabilizer group of a usual four-sphere:

$$[\hat{N}_i, \hat{N}_j] = 2i\epsilon_{ijk} \hat{N}_k, \quad [\hat{M}_i, \hat{M}_j] = 2i\epsilon_{ijk} \hat{M}_k, \quad [\hat{M}_i, \hat{N}_j] = 0,$$

(10)

These are compactly summarized as

$$\hat{N}_i = -\frac{i}{4} \eta^i_{ab} \hat{G}_{ab}, \quad \hat{M}_i = \frac{i}{4} \bar{\eta}^i_{ab} \hat{G}_{ab},$$

(12)

where $\eta^i_{ab} = \epsilon_{iabde} - \delta_{ia} \delta_{ab} + \delta_{ib} \delta_{ad}$, and $\bar{\eta}^i_{ab}$ is obtained by changing the signs in front of the second and third terms of $\eta^i_{ab}$. The Casimir of each $SU(2)$ algebra at the north pole is evaluated as

$$\hat{N}_i \hat{N}_i = 1 \frac{(n + G_5)(n + 4 + G_5)}{4} = n(n + 2),$$

$$\hat{M}_i \hat{M}_i = 1 \frac{(n - G_5)(n + 4 - G_5)}{4} = 0,$$

(13)

where we have used $G_5 = n$. Then we have a fuzzy two-sphere, which is given by the $(n + 1)$ dimensional representation of $SU(2)$, at the north pole. The radius of the fuzzy two-sphere is given by $\alpha^2 n(n + 2)$, being comparable with that of the fuzzy four-sphere. Since the fuzzy four-sphere has the $SO(5)$ symmetry, we can state that the fuzzy two-sphere, which is given by the $(n + 1)$ dimensional representation of $SU(2)$, exists at each point on the fuzzy four-sphere.

We later consider this noncommutative manifold as a classical solution of a matrix model. Since we expand the matrices $A_\mu$ around the $SO(5)$ vector $\hat{G}_\mu$, it may be natural to regard the two-sphere as an internal space. In this sense, this noncommutative space is called fuzzy four-sphere. Let us comment on the stabilizer group of this manifold. It is known that a usual four-sphere is described
as $SO(5)/SO(4)$ and the stabilizer group is $SO(4)$. The paper [21] reported that the stabilizer group of the fuzzy four-sphere is not $SO(4)$ but $U(2)$ whose generators are $\hat{M}_i$ ($i = 1, 2, 3$) and $\hat{N}_3$. This reason can be understood from the viewpoint of the deformation quantization. Since a usual four-sphere does not have the Poisson structure, it is difficult to quantize such a space. On the other hand a manifold which is described by $SO(5)/U(2)$ has the Poisson structure. Therefore we quantize $SO(5)/U(2) \cong SO(4)/SU(2)/U(1) \cong S^4 \times S^2$ instead of $SO(5)/SO(4) \cong S^4$. It is not difficult to understand that the fuzzy two-sphere plays the role of the symplectic form. This picture becomes more manifest when we consider a contraction of the algebra, which is considered in section 2.3.

The authors of [10] constructed this fuzzy four-sphere as a longitudinal five-brane in the context of BFSS matrix model. They showed that $N$ and $n$ represents the number of $D$-particles and that of longitudinal five-branes respectively. (In section 4 we regard this system as $D$-branes in IIB matrix model.) Therefore the fuzzy four-sphere is composed of $n$ overlapping longitudinal five-branes. The existence of the fuzzy two-sphere is explained via the dielectric effect found by Myers [6]. If we notice a point on the overlapping five-branes, we have $n$ $D$-particles. A collection of $n$ $D$-particles expands into the fuzzy two-sphere as in [10]. The expansion is along the extra dimensions.

Since $SO(5)/U(2)$ is isomorphic to $SO(6)/U(3)$, we may use either algebra as the starting point of the algebra which defines the fuzzy four-sphere. We have used the latter algebra for later convenience. (The Wick rotation is needed.) The stabilizer group of the $SO(5,1)$ algebra is $U(2,1)$ whose generators are $\hat{M}_i$, $\hat{N}_3$ and $\hat{G}_{\mu}$.

Let us investigate the lower dimensional brane charge in the four-branes. A two-brane charge is given by $\text{tr}[\hat{x}_\mu, \hat{x}_\nu] \propto \text{tr}(\hat{N} + \hat{M})$ and it is vanishing at each point on the four-sphere. It appears after the contraction of the algebra (see (30) or (31)).

Since we are now considering the $N \times N$ matrix, we have $N$ quanta in this system. The area occupied by the unit quantum on the four-sphere is

$$\frac{8\pi^2 \rho^4 n}{N} = \frac{16\pi^2 \rho^4 n}{(n + 1)(n + 2)(n + 3)},$$

(14)

where $8\pi^2 \rho^4/3$ is the area of the four-sphere. This is a noncommutative scale on the four-sphere. Since there are $n$ quanta on the fuzzy two-sphere, the number of the quanta on the fuzzy four-sphere is $N/n$. There are $n \sim N^{1/3}$ quanta at each point on the four-sphere which has $n^2 \sim N^{2/3}$ points.

A classical sphere is expected to be recovered in a large $N$ limit. We consider a large $N$ limit with the radius $\rho$ of the sphere fixed. In other words, it is an $\alpha \to 0$ limit with $\rho$ fixed. We define coordinates $\hat{w}_{\mu\nu}$ as

$$\hat{w}_{\mu\nu} \equiv i\alpha \hat{G}_{\mu\nu}.$$

(15)

The coordinates of the four-sphere commute each other in this limit,

$$[\hat{x}_\mu, \hat{x}_\nu] = -2i\alpha \hat{w}_{\mu\nu} \simeq O(\alpha \rho) \to 0.$$

(16)

\[\text{In this paper, we use the term "four-brane" to refer to a four dimensional object. If we treat it in IIB matrix model, it means } D3\text{-brane.}\]
The coordinates $\hat{x}_\mu$ and $\hat{w}_{\mu\nu}$ also become commuting matrices,

$$[\hat{x}_\mu, \hat{w}_{\nu\lambda}] = 0, \quad [\hat{w}_{\mu\nu}, \hat{w}_{\lambda\rho}] = 0.$$  \hspace{1cm} (17)

Thus we have obtained the classical manifold of $S^4 \times S^2$.

2.2 Noncommutative field theory on fuzzy four-sphere

In this subsection, a noncommutative gauge field theory on the fuzzy four-sphere is investigated using a matrix model [28]. It is known that matrix models provide the definition of noncommutative Yang-Mills, and the idea is to expand matrices around a classical solution [9, 10, 12]. We expand the matrices as follows,

$$A_\mu = \hat{x}_\mu + \alpha \rho \hat{a}_\mu = \alpha \rho \left( \frac{1}{\rho} \hat{G}_\mu + \hat{a}_\mu \right).$$  \hspace{1cm} (18)

The details of the model are explained in section 4. Note that we expanded only around $\hat{x}_\mu$. We do not consider fluctuations around $\hat{w}_{\mu\nu}$. Fields on a noncommutative sphere are expanded by noncommutative analogue of the spherical harmonics. In this case, such a noncommutative spherical harmonics is given in [20]. The bases are classified by the $SO(5)$ representations and the matrices are expanded by the irreducible representations of $SO(5)$. The irreducible representation is characterized by the Young diagram. It is labeled by the row length $(r_1, r_2)$ in this case. Note that the representations with $r_2 = 0$ correspond to a classical four-sphere. Summing up the dimensions of all irreducible representations with the condition $n \geq r_1 \geq r_2$ leads to the square of $N$ [20]. We write the noncommutative spherical harmonics as $\hat{Y}_{r_1 r_2}(\hat{x}, \hat{w})$ and expand fields as follows,

$$\hat{a}(\hat{x}, \hat{w}) = \sum_{r_1=0}^{n} \sum_{r_2} a_{r_1 r_2} \hat{Y}_{r_1 r_2}(\hat{x}, \hat{w}),$$  \hspace{1cm} (19)

If we consider a function corresponding to the above matrix,

$$a(x, w) = \sum_{r_1=0}^{n} \sum_{r_2} a_{r_1 r_2} Y_{r_1 r_2}(x, w),$$  \hspace{1cm} (20)

a product of the fields becomes noncommutative and associative. We would like to emphasize that $n$ plays a role of a cutoff parameter for the angular momentum $r_1$. $n$ is related to the size of the matrix and it gives UV and IR cut-off parameters. This means that a field theory on this noncommutative space has no divergences.

When we consider the noncommutative field theory, an adjoint action of $\hat{G}_\mu$ becomes the following derivative operator

$$ad \left( \hat{G}_\mu \right) \rightarrow -2i \left( w_{\mu\nu} \frac{\partial}{\partial x_\nu} - x_\nu \frac{\partial}{\partial w_{\mu\nu}} \right),$$  \hspace{1cm} (21)

and an adjoint action of $\hat{G}_{\mu\nu}$ becomes

$$ad \left( \hat{G}_{\mu\nu} \right) \rightarrow 2 \left( x_\mu \frac{\partial}{\partial x_\nu} - x_\nu \frac{\partial}{\partial x_\mu} - w_{\mu\lambda} \frac{\partial}{\partial w_{\lambda\nu}} + w_{\nu\lambda} \frac{\partial}{\partial w_{\lambda\mu}} \right).$$  \hspace{1cm} (22)
We next discuss the Laplacian. It is natural to consider $ad(\hat{G}_\mu)^2$ to be the Laplacian from the matrix model point of view [28]. We now investigate the spectrum of the Laplacian. It is calculated as follows,

$$\frac{1}{4}[\hat{G}_\mu, [\hat{G}_\mu, \hat{Y}_{r_1r_2}]] = (r_1(r_1 + 3) - r_2(r_2 + 1)) \hat{Y}_{r_1r_2}. \quad (23)$$

It is well-known that noncommutative geometry is realized by the guiding center coordinates of electrons in a constant magnetic field. The authors in [23] studied the quantum Hall system on a four dimensional sphere. Their system is composed of particles moving on a four dimensional sphere under an $SU(2)$ gauge field. The existence of Yang’s $SU(2)$ monopole [24] in the system makes the coordinates of the particles noncommutative. They showed that the configuration space of this system is locally $S^4 \times S^2$ where $S^2$ represents an iso-spin space. We find that the eigenvalues (23) exactly coincide with those of the Hamiltonian of a single particle on a four-sphere under the monopole background.

2.3 Contraction of fuzzy four-sphere algebra

In this subsection, we consider the Inönü-Wigner contraction of the algebra (1) or (4) in the vicinity of the north pole. By virtue of the $SO(5)$ symmetry, this discussion is without loss of generality.

$\hat{G}_\mu$ is decomposed to $\hat{G}_5 \simeq n$ and $\hat{G}_a$, and $\hat{G}_{\mu\nu}$ to $\hat{G}_{ab}$ and $\hat{G}_{a5}$ at the north pole. Indices $a, b, \ldots$ run over 1 to 4. We rescale the matrices as

$$\hat{G}_a' = \frac{1}{\sqrt{n}} \hat{G}_a, \quad \hat{G}_{ab}' = \frac{1}{\sqrt{n}} \hat{G}_{ab}, \quad \hat{G}_{a5}' = \frac{1}{\sqrt{n}} \hat{G}_{a5}. \quad (24)$$

The radius of the four-sphere in the rescaled coordinate is

$$\rho'^2 = \hat{x}'_i \hat{x}'_i = \alpha^2 n(n + 4) \frac{n}{\alpha^2 n} = 1 \rho^2 \sim \alpha^2 n. \quad (25)$$

To contract the algebra, we take $\rho' \to \infty$ (or $n \to \infty$) limit with $\alpha$ fixed. From (A.4), we have

$$\hat{G}_{a5} = \frac{1}{n} \left(4 \hat{G}_a - \hat{G}_{ab} \hat{G}_b\right). \quad (26)$$

If we use this equation, $\hat{G}_{a5}$ is written in terms of $\hat{G}_a$ and $\hat{G}_{ab}$ at the north pole. Therefore independent matrices are now $\hat{G}_a$ and $\hat{G}_{ab}$. The commutation relations of them are given by

$$[\hat{G}_a', \hat{G}_b'] = \frac{2}{\sqrt{n}} \hat{G}_{ab}',$$

$$[\hat{G}_a', \hat{G}_{bc}'] = \frac{2}{\sqrt{n}} \left(\delta_{ac} \hat{G}_b' - \delta_{ab} \hat{G}_c'\right),$$

$$[\hat{G}_{ab}', \hat{G}_{cd}'] = \frac{2}{\sqrt{n}} \left(\delta_{bc} \hat{G}_{ad}' + \delta_{ad} \hat{G}_{bc}' - \delta_{ac} \hat{G}_{bd}' - \delta_{bd} \hat{G}_{ac}'\right). \quad (27)$$

Here we investigate the algebra of the fuzzy two-sphere in the large $n$ limit. It determines the noncommutativity for the coordinates of the four-sphere. One of the three coordinates of the fuzzy two-sphere is diagonalized as follows,

$$\hat{N}_3 = \text{diag}(n, n - 2, \ldots, -n + 2, -n). \quad (28)$$
When we take the large $n$ limit, the two-brane charge no longer vanishes. It is because the contributions from the north pole and the south pole of the two-sphere do decouple. Then $\hat{N}_3$ takes the value $+n$ (or $-n$) in the large $n$ limit. Since the magnitudes of $\hat{N}_1$ and $\hat{N}_2$ are $O(1)$, the noncommutativity $\hat{G}'_{ab}$ become as follows after taking the large $n$ limit,

$$
\hat{G}'_{12} = i \sqrt{n} \mathbf{1} \quad (\text{or} \quad -i \sqrt{n} \mathbf{1}), \quad \hat{G}'_{34} = -i \sqrt{n} \mathbf{1} \quad (\text{or} \quad i \sqrt{n} \mathbf{1}), \\
\hat{G}'_{ab} = O(1) \quad ((a, b) \neq (1, 2), (3, 4)).
$$

(29)

Note that $\hat{M}'_i \simeq 0$. We have obtained the six dimensional noncommutative space whose coordinates have the following commutation relations,

$$
[\hat{G}'_1, \hat{G}'_2] = 2i \mathbf{1}, \quad [\hat{G}'_3, \hat{G}'_4] = -2i \mathbf{1}, \quad [\hat{N}'_1, \hat{N}'_2] = 2i \mathbf{1},
$$

(30)

or

$$
[\hat{G}'_1, \hat{G}'_2] = -2i \mathbf{1}, \quad [\hat{G}'_3, \hat{G}'_4] = 2i \mathbf{1}, \quad [\hat{N}'_1, \hat{N}'_2] = -2i \mathbf{1}.
$$

(31)

In the language of D-brane, a flat $D3$-brane expands along the extra two dimensional plane and effectively form six dimensional space. We easily find that the fuzzy two-sphere plays the role of the symplectic form. Due to the presence of it, the symmetry of the plane is not $SO(4)$ but $SO(2) \times SO(2)$.

We next study an action of noncommutative gauge theory on the noncommutative plane. Although we have just obtained the six dimensional noncommutative plane, the two dimensional space is an extra dimensional space and we obtain a four dimensional noncommutative gauge theory. The matrices $A_\mu$ are expanded as

$$
A_a = \hat{x}_a + \alpha \rho \hat{a}_a \\
\equiv \alpha \rho' D'_a, \\
A_5 = \alpha \rho' \phi',
$$

(32)

where we have rescaled the field as $\sqrt{n}a_\mu = a'_\mu$. $ad(D_a)$ is the covariant derivative on the flat background and is given by

$$
ad(D_a) = B_{ab} \frac{\partial}{\partial x'_b} + [a'_a, \cdot ]s,
$$

(33)

where $B_{12} = 2i$, $B_{34} = -2i$ and other components of $B_{ab}$ are zero. It must be noted that the fields have dependence not only $\hat{G}_\mu$ but also $\hat{N}_i$. An action of the noncommutative Yang-Mills is obtained from an action of the matrix model as

$$
S = -\frac{(\alpha \rho')^4}{4g^2} Tr \left( [D'_a, D'_b] [D'_a, D'_b] + 2[D'_a, \phi'][[D'_a, \phi']] \right) \\
-\frac{\alpha \rho'}{2g^2} Tr \left( \bar{\psi} \Gamma^a [D'_a, \psi] + \bar{\psi} \Gamma^5 [\phi', \psi] \right).
$$

(34)

After replacing the trace with the integral as

$$
\frac{1}{N} Tr \rightarrow \frac{1}{8 \pi \rho^4 \cdot 4 \pi \rho^2} \int d^4x' d^2y',
$$

(35)
we have the following action of the noncommutative field theory,

\[
S = -\frac{3N\alpha^4}{128\pi^2\rho^2g^2} \int d^4x' d^2y' \left( [D'_a, D'_b][D'_a, D'_b] + 2[D'_a, \phi'][D'_a, \phi'] \right) - \frac{N\alpha}{64\pi^2\rho g^2} \int d^4x' d^2y' \left( \bar{\psi} \Gamma^{a}[D'_a, \psi] + \bar{\psi} \Gamma^5[\phi', \psi] \right). \tag{36}
\]

Note that the fields propagate only on the \( x \) directions since there are no derivatives for \( y \) coordinates. After performing the integral \( d^2y' \), we obtain a four dimensional gauge theory with gauge coupling \( g^2 Y M = \frac{32\pi g^2}{3N\alpha^4} \).

3 Fuzzy 2k-sphere

3.1 Matrix description of fuzzy 2k-sphere

In this section, we study a higher dimensional fuzzy sphere \[20, 21\] and see that a lower dimensional fuzzy sphere is attached at each point on the higher dimensional fuzzy sphere as in the case of the fuzzy four-sphere.

Let us start with the following \( SO(2k+1) \) algebra \[3\]

\[
[J_{MN}, J_{OP}] = 2 \left( \eta_{NO} J_{MP} + \eta_{MP} J_{NO} - \eta_{MO} J_{NP} - \eta_{NP} J_{MO} \right), \tag{37}
\]

where \( M, N, \ldots \) run over 0 to \( 2k+1 \) and \( \eta_{MN} = diag(-1, 1, \ldots, 1) \). \( N_k \) dimensional representation of the algebra is constructed from the \( n \)-fold symmetric tensor product of the \( (2k+1) \) dimensional Gamma matrices as

\[
\hat{J}_{0\mu} = (\Gamma_\mu \otimes 1 \otimes \cdots \otimes 1 + 1 \otimes \Gamma_\mu \otimes \cdots \otimes 1 + \cdots + 1 \otimes 1 \otimes \cdots \otimes \Gamma_\mu)_{Sym},
\]

\[
\hat{J}_{\mu\nu} = (\Gamma_{\mu\nu} \otimes 1 \otimes \cdots \otimes 1 + 1 \otimes \Gamma_{\mu\nu} \otimes \cdots \otimes 1 + \cdots + 1 \otimes 1 \otimes \cdots \otimes \Gamma_{\mu\nu})_{Sym}. \tag{38}
\]

The value of \( N_k \) is summarized in appendix A and becomes \( n \frac{k(k+1)}{2} \) for a large \( n \). We rewrite \( \hat{J}_{0\mu} \) and \( \hat{J}_{\mu\nu} \) as \( \hat{G}_\mu \) and \( \hat{G}_{\mu\nu} \), and they represent coordinates of a fuzzy 2k-sphere and a fuzzy 2\((k-1)\)-sphere respectively as shown from now on. The commutation relations of them are given by

\[
\begin{align*}
\left[ \hat{G}_\mu, \hat{G}_\nu \right] &= 2 \hat{G}_{\mu\nu}, \\
\left[ \hat{G}_\mu, \hat{G}_{\nu\lambda} \right] &= 2 \left( \delta_{\mu\nu} \hat{G}_{\lambda} - \delta_{\mu\lambda} \hat{G}_{\nu} \right), \\
\left[ \hat{G}_{\mu\nu}, \hat{G}_{\lambda\rho} \right] &= 2 \left( \delta_{\nu\lambda} \hat{G}_{\mu\rho} + \delta_{\mu\rho} \hat{G}_{\nu\lambda} - \delta_{\mu\lambda} \hat{G}_{\nu\rho} - \delta_{\nu\rho} \hat{G}_{\mu\lambda} \right). \tag{39}
\end{align*}
\]

We can confirm that \( \hat{G}_\mu \) satisfy the following relation,

\[
\epsilon^{\mu_1 \cdots \mu_{2k+1}} \hat{G}_{\mu_1} \cdots \hat{G}_{\mu_{2k+1}} = C_k \hat{G}_{\mu_{2k+1}}, \tag{40}
\]

\(^3k = 1\) case is special since \( \hat{G}_{\mu\nu} \) is written by \( \hat{G}_\mu \) using the three-rank anti-symmetric tensor. In this case \( \hat{G}_\mu \) generate the \( SO(3) \) algebra. Some arguments of this section is applied for \( k = 1 \) case.
where $C_k$ is a constant which depends on $n$ and $\epsilon^{\mu_1 \cdots \mu_{2k+1}}$ is the $SO(2k+1)$ invariant tensor. This relation respects the $SO(2k+1)$ symmetry which is the isometry of a 2$k$-sphere. The radius of the fuzzy 2$k$-sphere is given by

$$\hat{x}_\mu \hat{x}_\mu = \alpha^2 \hat{G}_\mu \hat{G}_\mu = \alpha^2 n(n + 2k) \equiv \rho^2.$$ \hfill (41)

We now show that a fuzzy 2$(k-1)$-sphere exists at each point on a fuzzy 2$k$-sphere and coordinates of the fuzzy 2$(k-1)$-sphere are given by $\hat{G}_{ab}$ $(a,b = 1, \ldots, 2k)$. We consider the algebra at the north pole of the fuzzy 2$k$-sphere, $\hat{G}_{2k+1} \simeq n$. $\hat{G}_{ab}$ are generators of the $SO(2k)$ rotation around the north pole. The lower dimensional fuzzy sphere is embedded in the $SO(2k)$ algebra. If we regard the vector coordinates of the 2$(k-1)$-sphere as $i\hat{G}_{\alpha 2k}$, $i\hat{G}_{\alpha 2k}$ and $\hat{G}_{\alpha \beta}$ $(\alpha, \beta = 1, \ldots, 2k-1)$ form $SO(2k-1, 1)$ algebra. This algebra indeed describes a fuzzy 2$(k-1)$-sphere. $\hat{G}_{a2k+1}$ depend on $\hat{G}_a$ and $\hat{G}_{ab}$ since they are written as

$$\hat{G}_{a2k+1} = \frac{1}{n} (2k\hat{G}_a - \hat{G}_{ab}\hat{G}_b) ,$$ \hfill (42)

where we used the formula (A.4). Since the fuzzy 2$k$-sphere has the $SO(2k+1)$ symmetry, the fuzzy 2$(k-1)$-sphere exists at each point on the fuzzy 2$k$-sphere.

It can be proved that the fuzzy 2$(k-1)$-sphere on the fuzzy 2$k$-sphere satisfies the relations which are analogous to (11) and (14). To prove them, we use the following identifications

$$\hat{G}_\alpha^{(2k)} = \hat{G}_{\alpha \beta}^{(2(k-1))} , \quad i\hat{G}_\alpha^{(2k)} = \hat{G}_\alpha^{(2(k-1))}.$$ \hfill (43)

We begin with the relation (10) for the fuzzy 2$k$-sphere, and set $\hat{G}_{2k+1} = n$,

$$C_k n = \epsilon^{\mu_1 \cdots \mu_{2k}} \hat{G}_{\mu_1}^{(2k)} \cdots \hat{G}_{\mu_{2k}}^{(2k)} = \epsilon^{\mu_1 \cdots \mu_{2k}} \hat{G}_{\mu_1 \mu_2}^{(2k)} \cdots \hat{G}_{\mu_{2k-1} \mu_{2k}}^{(2k)} = 2k \epsilon^{\mu_1 \cdots \mu_{2k-1}} \hat{G}_{\mu_1 \mu_2}^{(2k)} \cdots \hat{G}_{\mu_{2k-1} \mu_{2k}}^{(2k)} = -i2k \epsilon^{\mu_1 \cdots \mu_{2k-1}} \hat{G}_{\mu_1 \mu_2}^{(2(k-1))} \cdots \hat{G}_{\mu_{2k-1} \mu_{2k}}^{(2(k-1))} = -i2k \epsilon^{\mu_1 \cdots \mu_{2k-1}} \hat{G}_{\mu_{2k-1} \mu_{2k}}^{(2(k-1))} \cdots \hat{G}_{\mu_1}^{(2(k-1))}.$$ \hfill (44)

The identifications (13) are used from the third line to the forth line. We have arrived at the following equation,

$$\epsilon^{\mu_1 \cdots \mu_{2k-1}} \hat{G}_{\mu_1}^{(2(k-1))} \cdots \hat{G}_{\mu_{2k-1}}^{(2(k-1))} = \frac{i}{2k} \eta C_k.$$ \hfill (45)

This expression is equivalent to the following expression (see appendix A),

$$\epsilon^{\mu_1 \cdots \mu_{2k-2}} \hat{G}_{\mu_1} \cdots \hat{G}_{\mu_{2k-2}} = C_{k-1} \hat{G}_{\mu_{2k-1}}.$$ \hfill (46)

Then we have obtained the relation (46) for the fuzzy 2$(k-1)$-sphere from the relation (10) for the fuzzy 2$k$-sphere using the identifications (13). Let us now calculate the radius of the fuzzy 2$(k-1)$-sphere using the relations for the fuzzy 2$k$-sphere. From the formulae given in appendix A we can calculate as

$$2kn(n + 2k) = \hat{G}_{\mu \nu}^{(2k)} \hat{G}_{\nu}^{(2k)}.$$
This shows that a fuzzy 2-sphere and a two-sphere. The area of the unit quantum on a 2-sphere is given by 
\[ G_g(2k)G_g(2k) = 2\hat{G}_{2k+1}^2 + 2\hat{G}_{2k+1}^2 \hat{G}_{2k+1} \cdot \hat{G}_{2k+1} \]  
(47)

Then we have

\[ \hat{G}_{ab}^2 G_{ba}^2 = (2k - 2)n(n + 2k) + 2\hat{G}_{2k+1}^2 \hat{G}_{2k+1} \]
\[ = 2kn(n + 2k - 2), \]  
(48)

where we used \( \hat{G}_{2k+1}^2 = n. \) \( \hat{G}_{ab}^2 G_{ba}^2 \) is also calculated as

\[ \hat{G}_{ab}^2 G_{ba}^2 = \hat{G}_{ab}^2 G_{ba}^2 = \hat{G}_{ab}^2 G_{ba}^2 = (2(2k - 2)) = 2k(2k - 2) \hat{G}_{ab}^2 G_{ba}^2 \]
\[ = 2k(2k - 2) \hat{G}_{ab}^2 G_{ba}^2 \]
(49)

From the second line to the third line, we have used the following equation which is derived from \( (A.2) \) and \( (A.3) \),

\[ \hat{G}_{(2(k-1))} = (2k - 2) \hat{G}_{(2(k-1))} \hat{G}_{(2(k-1))} \]
\[ = (2k - 2) \hat{G}_{(2(k-1))} \hat{G}_{(2(k-1))} \]
(50)

Therefore we find from \( (A.48) \) and \( (A.49) \) that the fuzzy 2(k - 1)-sphere has a radius \( \alpha^2 n(n + 2k - 2) \). The size of the extra dimensions is also given by \( \rho \).

Eigenvalues of the matrices represent the positions of quanta in the noncommutative spacetime, and we have \( N_k \sim n^{k(k+1)/2} \) quanta in this system. For the \( k = 3 \) case, we have \( N_3 \sim n^6 \) quanta. Since a fuzzy four-sphere exists at each point on a six-sphere, there are \( n^3 \) points on the six-sphere and \( N_2 \sim n^3 \) points on the fuzzy four-sphere. Since the fuzzy four-sphere has a fuzzy two-sphere at each point, \( n^3 \) can be divided into \( n^2 \cdot n \). For the \( k = 4 \) case, \( N_4 \sim n^{10} = n^4 \cdot n^3 \cdot n^2 \cdot n \) where each factor represents the number of the quanta on an eight-sphere, a six-sphere, a four-sphere and a two-sphere. The area of the unit quantum on a 2k-sphere is given by

\[ \frac{2\pi^{k-\frac{1}{2}}}{\Gamma(k+\frac{1}{2})}\rho^{2k} N_{k-1} \sim \frac{n^{2k}}{n^k} \sim \alpha^2 n^k, \]
(51)

where \( \frac{2\pi^{k-\frac{1}{2}}}{\Gamma(k+\frac{1}{2})}\rho^{2k} \) is the area of a 2k-sphere. This is a noncommutative scale on this system.

We began with \( (37) \) or \( (39) \) for the fuzzy 2k-sphere algebra. On the other hand, we may use the \( SO(2k + 1) \) algebra as the definition of the fuzzy 2k-sphere algebra thanks to the isomorphic relation between \( SO(2k + 2)/U(k + 1) \) and \( SO(2k + 1)/U(k) \). This relation also helps us to guess the relationship between the fuzzy 2k-sphere and the fuzzy 2(k - 1)-sphere as follows. If we start from the coset space \( SO(2k + 1)/U(k) \), it is written as

\[ SO(2k + 1)/U(k) = SO(2k + 1)/SO(2k) \times SO(2k - 1)/U(k - 1). \]
(52)

This shows that a fuzzy 2(k - 1)-sphere exists at each point on a 2k-sphere.

We can show that lower dimensional brane charges in the fuzzy 2k-branes vanish. Let us first consider the \( k = 3 \) case. A two-brane charge \( tr[\hat{x}_\mu, \hat{x}_\nu] \) vanishes because of the cyclicity of the
Trace for finite matrices. As for a four-brane charge, we consider at the north pole for simplicity, and use the identification \( \hat{G}_{\alpha\beta}^{(6)} = G^{(4)}_{\alpha\beta} \), \( i\hat{G}^{(6)}_{\alpha} = G^{(4)}_{\alpha} \).

\[
G^{(6)}_{\alpha\beta} = G^{(4)}_{\alpha\beta}, \quad iG^{(6)}_{\alpha} = G^{(4)}_{\alpha} .
\]  

(53)

A four-brane charge is given by a completely anti-symmetric product of four matrices as

\[
Tr\hat{x}_a\hat{x}_b\hat{x}_c\hat{x}_d = \alpha^4 Tr\hat{G}_a\hat{G}_b\hat{G}_c\hat{G}_d.
\]

When the indices do not include 6, it becomes as

\[
\epsilon^{abcd}Tr[\hat{G}_a^{(6)}, \hat{G}_b^{(6)}][\hat{G}_c^{(6)}, \hat{G}_d^{(6)}] = \epsilon^{abcd}Tr[\hat{G}_a^{(4)}, \hat{G}_b^{(4)}][\hat{G}_c^{(4)}, \hat{G}_d^{(4)}] = Tr\hat{G}_e^{(4)} = 0.
\]

(54)

We have used the identification from the first line to the second line. When the indices include 6, we can calculate as

\[
\epsilon^{abcde}Tr[\hat{G}_a^{(6)}, \hat{G}_b^{(6)}][\hat{G}_c^{(6)}, \hat{G}_d^{(6)}] = -2i\epsilon^{abcde}Tr[\hat{G}_a^{(4)}, \hat{G}_b^{(4)}][\hat{G}_c^{(4)}, \hat{G}_d^{(4)}] = -2i\epsilon^{abcde}Tr\hat{G}_a^{(4)}\hat{G}_b^{(4)}\hat{G}_c^{(4)}\hat{G}_d^{(4)} \propto Tr[\hat{G}_a^{(4)}, \hat{G}_b^{(4)}] = 0.
\]

(55)

Therefore the four-brane charge vanishes. We have shown that lower dimensional brane charges vanish at each point on the fuzzy six-sphere. They do not vanish after the contraction of the algebra (see the next subsection). Results for the \( k = 4 \) case are analogous.

Noncommutative field theories on the fuzzy spheres can be defined through matrix models by the same way as in section 2.2. Fields are expanded by the noncommutative spherical harmonics. It is characterized by the irreducible representations of \( SO(2k + 1) \) which are represented by the Young diagram of the low length \( (r_1, r_2, \ldots, r_k) \) where \( n \geq r_1 \geq r_2 \geq \cdots \geq r_k \), and summing up the dimensions of all irreducible representations leads to the square of \( N_k \). Coordinates which correspond to the representation with \( r_k(k \geq 3) \neq 0 \) are written in terms of \( \hat{G}_a \) and \( \hat{G}_{ab} \) at each point on the fuzzy 2\( k \)-sphere. Then the fuzzy 2\( k \)-sphere is a \( k(k + 1) \) dimensional space and it has an extra \( k(k - 1) \) dimensional space which forms the fuzzy 2\( k(k - 1) \)-sphere.

Before finishing this subsection, let us comment on a large \( N \) limit which gives a smooth manifold. A classical manifold is recovered by a large \( N \) limit with the radius \( \rho \) fixed. In the limit, the coordinates \( \hat{x}_\mu \) and \( \hat{w}_{\mu\nu} \) become commuting matrices. Since the coordinates \( \hat{w}_{\mu\nu} \) originate from the lower dimensional sphere, we obtain the classical manifold \( S^{2k} \times S^{2(k-1)} \times \cdots \times S^2 \).

### 3.2 Contraction of fuzzy 2\( k \)-sphere algebra

The contraction of the fuzzy 2\( k \)-sphere algebra is considered in this subsection. It is shown that there appear a \( k(k + 1) \) dimensional noncommutative plane including \( k(k - 1) \) extra dimensions which originate from the lower dimensional fuzzy-sphere. We first consider the \( k = 3 \) case. We rescale the generators as \( \hat{G}'_a = \frac{1}{\sqrt{n}}\hat{G}_a \) and \( \hat{G}'_{ab} = \frac{1}{\sqrt{n}}\hat{G}_{ab} \) \( (a, b = 1, \ldots, 6) \) for later convenience.
The commutation relations between $\hat{G}_a'$ become

$$[\hat{G}_a', \hat{G}_b'] = \frac{2}{\sqrt{n}} \hat{G}_{ab}.'$$

The $SO(6)$ generators $\hat{G}_{ab}$ are identified with the generators of the fuzzy four-sphere algebra as in (53). $\hat{G}_5^{(4)}$ is diagonalized as

$$\hat{G}_5^{(4)} = \text{diag}(n, n-2, \ldots, -n+2, -n).$$

We consider the region in the vicinity of the north pole of the four-sphere, $\hat{G}_5^{(4)} \approx n$ and take a large $n$ limit to obtain an infinitely extended plane. We can also diagonalize $\hat{N}_3^{(4)} = -\frac{i}{4} \hat{g}_{ab} \hat{G}_{ab}$ which is defined in (12). In the region, the generators become as $\hat{G}_a'^{(4)} \approx O(1)$, $\hat{N}_i^{(4)} \approx O(1)$ and $\hat{N}_3^{(4)} \approx \sqrt{n}$. Referring to the discussions in section 2.3, we obtain the following twelve dimensional noncommutative plane:

$$[\hat{G}_1', \hat{G}_2'] = \frac{2}{\sqrt{n}} \hat{G}_{12}^{(4)} = 2i \mathbf{1}, \quad [\hat{G}_3', \hat{G}_4'] = \frac{2}{\sqrt{n}} \hat{G}_{34}^{(4)} = -2i \mathbf{1},$$

$$[\hat{G}_5', \hat{G}_6'] = -\frac{2i}{\sqrt{n}} \hat{G}_5^{(4)} = -2i \mathbf{1},$$

$$[\hat{G}_{16}', \hat{G}_{26}'] = -[\hat{G}_1^{(4)'}, \hat{G}_2^{(4)'}] = -2i \mathbf{1}, \quad [\hat{G}_{36}', \hat{G}_{46}'] = -[\hat{G}_3^{(4)'}, \hat{G}_4^{(4)'}] = 2i \mathbf{1},$$

$$[\hat{G}_{13}', \hat{G}_{23}'] = [\hat{G}_1^{(4)'}, \hat{G}_2^{(4)'}] = -[\hat{N}_1^{(4)'}, \hat{N}_2^{(4)'}] = -2i \mathbf{1}.$$

Other commutation relations are zero. The forth and fifth commutation relations come from the fuzzy four-sphere and the last one from the fuzzy two-sphere. If we apply the same procedure to the $k = 4$ case, we obtain a twenty dimensional noncommutative plane with twelve extra dimensions. We find that the lower dimensional sphere gives the symplectic structure and the symmetry of the noncommutative plane is $SO(2)^k$.

4 Fuzzy sphere as classical solution of matrix model

We consider matrix model realizations of the fuzzy spheres in this section. Original matrix models [7, 8] do not have static higher dimensional fuzzy spheres as classical solutions. Let us consider IIB matrix model [8] as an example,

$$S = -\frac{1}{g^2} \text{Tr} \left( \frac{1}{4} [A_\mu, A_\nu] [A_\mu, A_\nu] + \frac{1}{2} \bar{\psi} \Gamma^\mu [A_\mu, \psi] \right).$$

Supersymmetries of this model are given by

$$\delta^{(1)} \psi = \frac{i}{2} [A_\mu, A_\nu] \Gamma^{\mu\nu} \epsilon, \quad \delta^{(1)} A_\mu = i \epsilon \Gamma^\mu \psi,$$

$$\delta^{(2)} \psi = \xi, \quad \delta^{(2)} A_\mu = 0,$$

and these form $\mathcal{N} = 2$ supersymmetry algebra.

In the matrix model, eigenvalues of bosonic variables are interpreted as space-time coordinates [11]. Space-time coordinates are represented by matrices and noncommutative geometry
is expected to appear. A flat noncommutative space which is given by the following is a formal classical solution of this model in a large \( N \) limit,

\[
[x_\mu, x_\nu] = iC_{\mu\nu}1
\]

and this solution preserves the \( \mathcal{N} = 1 \) supersymmetry. We can embed a noncommutative manifold whose isometry is a subgroup of \( SO(10) \) in the matrix model. The fuzzy spheres satisfy this condition. Since the fuzzy \( 2k \)-spheres satisfy the following equation

\[
[G_{\nu}, [G_\mu, G_{\nu}]] = 2[G_\nu, \hat{G}_\mu, \hat{G}_\nu] = -16\hat{G}_\mu = -\frac{16}{C_k}c^{\mu_1...\mu_{2k}\mu}\hat{G}_{\mu_1}...\hat{G}_{\mu_{2k}},
\]

the fuzzy \( 2k \)-sphere becomes a classical solution of the matrix model by adding a \((2k+1)\)-rank Chern-Simons term or a mass term. A Chern-Simons term \(^4\) appeared in a low energy effective action of \( D \)-branes under a R-R background \([6]\). A matrix model with a mass term is investigated in \([14]\). See also \([31]\) in which a massive matrix model is analyzed from the viewpoint of a supermatrix model. At present it is not clear whether the matrix model receives such deformations. The issue should be discussed in the future publications. The embedding of the curved space-time in matrix models is also discussed in \([13, 14, 28, 34, 35, 36, 37]\).

Although the fuzzy sphere solutions do not preserve the supersymmetry, it is shown to be recovered locally on the sphere. (It can be easily checked that \([30]\) and \([38]\) preserve the \( \mathcal{N} = 1 \) supersymmetry.) As for the fuzzy two-sphere, we can construct matrix models in which the fuzzy two-sphere is a supersymmetric classical solution \([13, 32]\). In these deformed models, the supersymmetry transformations are deformed such that the fuzzy two-sphere is supersymmetric and form a \( \mathcal{N} = 2 \) algebra in spite of the deformation.

The advantage of the matrix model construction of noncommutative gauge theories is that the gauge symmetry is manifest. If we expand \( A_\mu \) around a classical background as

\[
A_\mu = \hat{x}_\mu + \hat{a}_\mu,
\]

the fluctuation \( \hat{a}_\mu \) behaves as a gauge field on a noncommutative background \( \hat{x}_\mu \). The gauge symmetry in noncommutative gauge theories originates from the unitary symmetry of the matrix model, and \( A_\mu \) is a gauge covariant quantity. By considering such a expansion, the relationship between matrix models and field theories becomes more manifest. One can introduce a natural ultraviolet cut-off which is given by the size of matrix into noncommutative Yang-Mills theories, and letting it large in a way leads to usual field theories. Fluctuations of different backgrounds in the matrix model are described by different gauge theories. From this point of view, \( A_\mu \) are clearly background independent \([33]\).

5 Dielectric effect and higher dimensional fuzzy sphere

In this section, we show that \( n^{\frac{k(k-1)}{2}} \) spherical \( D(2k-1) \)-branes give a fuzzy \( 2k \)-sphere in the matrix model. The existence of lower dimensional fuzzy spheres is understood from the dielectric

\(^4\)Note that \( C_k(k \neq 2) \) depend on \( n \) while \( C_2 \) does not. The equations of motion determine the dimension of the fuzzy sphere for \( k \neq 2 \).
effect. The first point is established by calculating the value of the action \( S \) for fuzzy spheres. This calculation is analogous to the calculation for flat D-branes in \cite{38}. We identify \( g^2 \) as \( g_s l_s^4 \) in this calculation, where \( g_s \) and \( l_s \) are the string coupling constant and the string length scale respectively \cite{8}. It seems reasonable to suppose that the string length scale is related to the noncommutative scale on the fuzzy sphere. (See, for example, \cite{39} on this point.) From (51) we can find the following relation,

\[
\rho^{-2k} n^k \sim \alpha^{-2k} n^k.
\]  

(64)

The value of the action for a fuzzy sphere is estimated as follows,

\[
S = -\frac{1}{4g_s l_s^4} Tr[A_\mu, A_\nu][A_\mu, A_\nu]
\]

\[
= -\frac{\alpha^4}{g_s l_s^4} Tr\hat{G}_{\mu\nu}\hat{G}_{\mu\nu}
\]

\[
= \frac{\alpha^4}{g_s l_s^4} N_k 2kn(n + 2k)
\]

\[
\sim \frac{1}{g_s l_s^{2k}} \rho^{-2k} N_{k-1}.
\]  

(65)

where \( g_s l_s^{2k} \) is the tension of a \( D(2k - 1) \)-brane up to a numerical coefficient. \( \sim \) means that we considered a large \( n \). This is the energy of \( N_{k-1} \) spherical \( D(2k-1) \)-branes, where \( N_{k-1} \sim n^{\frac{(k+1)}{2}} \).

We can conclude that a fuzzy 2k-sphere is comprised of \( N_{k-1} \) spherical \( D(2k-1) \)-branes. On the other hand, we showed that a fuzzy 2k-sphere has a fuzzy \( 2(k-1) \)-sphere at each point. The consistency of these two viewpoints becomes clear by considering the dielectric effect. We may say that \( N_{k-1} \) D-instantons are present at each point on the \( D(2k-1) \)-branes since we are considering \( N_{k-1} \) \( D(2k-1) \)-branes. Since \( D \)-branes expand into a fuzzy sphere under a constant anti-symmetric tensor background due to the dielectric effect \cite{6}, the \( N_{k-1} \) D-instantons expand into a fuzzy \( 2(k-1) \)-sphere. It must be noted that coordinates of the fuzzy \( 2(k-1) \)-sphere are embedded in extra dimensions. It is known that matrix descriptions of D-branes correspond to \( D \)-branes under NS-NS or R-R form backgrounds. D-instantons under a constant R-R three-form field strength background expand to a fuzzy two-sphere. In general, D-instantons under a R-R \( (2k+1) \)-form field strength background expand to a fuzzy \( 2k \)-sphere. A fuzzy \( 2(k-1) \)-sphere appears at each point on the fuzzy \( 2k \)-sphere since a constant \( (2k+1) \)-form contains lower dimensional forms. This section can be summarized in the following sentences: A fuzzy \( 2k \)-sphere is given by a bound state of \( N_{k-1} \) \( D(2k-1) \)-branes and \( N_k \) D-instantons. The dielectric effect occurs at each point on the \( D(2k-1) \)-branes, and this effect makes extra spherical spaces.

6 Summary and Discussions

In this paper, we studied matrix descriptions of higher dimensional spherical branes. A higher dimensional sphere which is described by the coset space \( SO(2k+1)/SO(2k) \) can not be quantized since it does not admit a symplectic structure. Instead of considering this space, we consider the coset space \( SO(2k+1)/U(k) \). It is a Kähler space and a symplectic structure is
given by a Kähler form \[30\]. We showed that there is a fuzzy \(2(k-1)\)-sphere at each point on the fuzzy \(2k\)-sphere. The fuzzy \(2k\)-sphere is a \(k(k+1)\) dimensional space including \(k(k-1)\) extra dimensions. The coordinates of the fuzzy \(2k\)-sphere and the fuzzy \(2(k-1)\)-sphere which exists at each point on the fuzzy \(2k\)-sphere are given by \(\hat{G}_\mu\) and \(\hat{G}_{\mu\nu}\) respectively. The lower dimensional spheres serve the symplectic structure. This fact becomes more obvious after the contraction of the algebra as discussed in section \[23\]. It is well known that the quantum Hall system is an example of noncommutative geometry. As is discussed in \[23\], the quantum Hall system on a four dimensional sphere is constructed by considering a system of particles under a \(SU(2)\) gauge field. The existence of the gauge field background produces the noncommutativity, and the same observation applies to the fuzzy sphere system. The existence of the lower dimensional fuzzy sphere produces the noncommutativity on the higher dimensional sphere.

The values of the matrix model for fuzzy spheres are computed in section \[5\]. The computation shows that the fuzzy \(2k\)-sphere system is composed of \(N_{k-1}\) \(D(2k-1)\)-branes where \(N_{k-1}\) corresponds to the dimension of the fuzzy \(2(k-1)\)-sphere. The fuzzy \(2k\)-sphere is formed from \(N_k\) \(D\)-instantons under a constant R-R \((2k+1)\)-form field strength background. This is the so called Myers effect. Since the \((2k+1)\)-form includes a \((2k-1)\)-form, the fuzzy \(2(k-1)\)-sphere appears on the fuzzy \(2k\)-sphere. This is also due to the Myers effect. The appearance of the fuzzy \(2(k-1)\)-sphere is closely related to the overlap of \(N_{k-1}\) \(D(2k-1)\)-branes. The Myers effect arises in such an interesting way for higher dimensional spheres.

It is known that a matrix model around a noncommutative background gives the definition of a noncommutative Yang-Mills theory on the background. A noncommutative Yang-Mills on a fuzzy sphere is obtained through a matrix model. Since we take account of fluctuations only around \(\hat{G}_\mu\), we would obtain a \(2k\) dimensional noncommutative Yang-Mills. The extra \(k(k-1)\) dimensional space plays the same role as the gauge field background in the quantum Hall effect.

The advantage of compact noncommutative manifolds is that one can construct them in terms of finite size matrices while a solution which represents a noncommutative plane cannot be represented by finite size matrices. From the viewpoint of noncommutative field theories, \(N\) plays the role of the cutoff parameter. The noncommutative field theories are completely finite as long as \(N\) is finite. We showed that a gauge theory on a noncommutative plane were reproduced from a gauge theory on a fuzzy sphere by taking a large \(N\) limit. The existence of the symplectic form breaks the full Lorentz invariance of the plane and we obtain a noncommutative plane which is the Heisenberg algebra.
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A Some formulae

In this appendix, we summarize several formulae involving $\hat{G}_\mu$ and $\hat{G}_{\mu\nu}$ in diverse dimensions. $\hat{G}_\mu$ and $\hat{G}_{\mu\nu}$ are generators of $SO(2k+1,1)$, and they are constructed from the $n$-fold symmetrized tensor product of the Gamma matrices. The dimensions $N_k$ of the representation are given by

$$N_1 = n + 1, \quad N_2 = \frac{1}{6}(n + 1)(n + 2)(n + 3),$$
$$N_3 = \frac{1}{360}(n + 1)(n + 2)(n + 3)^2(n + 4)(n + 5),$$
$$N_4 = \frac{1}{302400}(n + 1)(n + 2)(n + 3)^2(n + 4)^2(n + 5)^2(n + 6)(n + 7).$$  \hspace{1cm} (A.1)

These are calculated in [20]. We have the following invariants

$$\hat{G}_\mu \hat{G}_\mu = n(n + 2k)$$  \hspace{1cm} (A.2)

and

$$\hat{G}_{\mu\nu} \hat{G}_{\nu\mu} = 2k(n + 2k).$$  \hspace{1cm} (A.3)

The following relations are also satisfied

$$\hat{G}_{\mu\nu} \hat{G}_{\nu\lambda} = 2k \hat{G}_\mu$$  \hspace{1cm} (A.4)

and

$$\hat{G}_{\mu\nu} \hat{G}_{\nu\lambda} = n(n + 2k)\delta_{\mu\lambda} + (k - 1)\hat{G}_\mu \hat{G}_\lambda - k\hat{G}_\lambda \hat{G}_\mu.$$  \hspace{1cm} (A.5)

$G_\mu$ satisfy the following relation

$$\epsilon^{\mu_1 \cdots \mu_{2k+1}} \hat{G}_{\mu_1} \cdots \hat{G}_{\mu_{2k}} = C_k \hat{G}_{\mu_{2k+1}}$$  \hspace{1cm} (A.6)

where $\epsilon^{\mu_1 \mu_2 \cdots \mu_{2k} \mu_{2k+1}}$ is the $SO(2k+1)$ invariant tensor. $C_k$ is a constant which depends on $n$,

$$C_1 = 2i, \quad C_2 = 8(n + 2), \quad C_3 = -48i(n + 2)(n + 4), \quad C_4 = -384(n + 2)(n + 4)(n + 6).$$  \hspace{1cm} (A.7)

The details of this calculation are given in [31]. By multiplying the equation (A.6) by $\hat{G}_{\mu_{2k+1}}$, we have

$$\epsilon^{\mu_1 \cdots \mu_{2k} \mu_{2k+1}} \hat{G}_{\mu_1} \cdots \hat{G}_{\mu_{2k}} \hat{G}_{\mu_{2k+1}} = C_k n(n + 2k)$$
$$= \frac{i}{2(k + 1)} n C_{k + 1},$$  \hspace{1cm} (A.8)

where we have used the following relation which is found from (A.7),

$$C_k = -i2k(n + 2k - 2)C_{k - 1}.$$  \hspace{1cm} (A.9)
B Notations of Gamma matrices

We summarize an explicit representation of Gamma matrices in odd dimensions $d = 2k + 1$. Higher dimensional ones are composed of lower dimensional ones as

\[
\Gamma_\mu = \begin{pmatrix} 0 & -i\gamma_\mu \\ i\gamma_\mu & 0 \end{pmatrix} = \gamma_\mu \otimes \sigma_2 \quad (\mu = 1, \ldots, 2k - 1),
\]

\[
\Gamma_{d-1} = \begin{pmatrix} 0 & 1_{2k-1} \\ 1_{2k-1} & 0 \end{pmatrix} = 1_{2k-1} \otimes \sigma_1,
\]

\[
\Gamma_d = \begin{pmatrix} 1_{2k-1} & 0 \\ 0 & -1_{2k-1} \end{pmatrix} = 1_{2k-1} \otimes \sigma_3,
\]

where $\sigma_\mu$ is the Pauli matrices. They satisfy the Clifford algebra,

\[
\{\Gamma_\mu, \Gamma_\nu\} = 2\delta_{\mu\nu} \quad (\mu, \nu = 1, \ldots, d = 2k + 1).
\]

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