HEEGAARD DIAGRAMS CORRESPONDING TO TURAEV SURFACES

CODY ARMOND, NATHAN DRUIVENGA, AND THOMAS KINDRED

Abstract. We describe a correspondence between Turaev surfaces of link diagrams on $S^2 \subset S^3$ and special Heegaard diagrams for $S^3$ adapted to links.

1. Introduction

To construct the Turaev surface $\Sigma$ of a link diagram $D$ on $S^2 \subset S^3$, one pushes the all-A and all-B states of $D$ to opposite sides of $S^2$, connects these two states with a certain cobordism, and caps the state circles with disks. Turaev’s original construction [19] streamlined Murasugi’s proof [16], based on Kauffman’s work [12] on the Jones polynomial [11], of Tait’s longstanding conjecture on the crossing numbers of alternating links [17]. See also [18]. More recently, Turaev surfaces have provided geometric means for interpreting Khovanov and knot Floer homologies, as in [3, 5, 6, 9, 10, 14, 20].

Dasbach, Futer, Kalfagianni, Lin, and Stoltzfus showed that the Turaev surface of any connected link diagram $D$ on $S^2 \subset S^3$ is a splitting surface for $S^3$ on which $D$ forms an alternating link diagram [8]. When equipped with the type of crossing ball structure developed by Menasco [15], the projection sphere provides natural attaching circles for the two handlebodies of this splitting, completing a Heegaard diagram $(\Sigma, \alpha, \beta)$ for $S^3$. By characterizing the interplay between this Heegaard diagram and the original link diagram $D$, we obtain a correspondence between Turaev surfaces and particular Heegaard diagrams adapted to links. Figure 1 shows a typical example of such a diagram $(\Sigma, \alpha, \beta, D)$.

First, §2 defines Heegaard splittings and diagrams, link diagrams, crossing ball structures, and Turaev surfaces. Next, §3 constructs and describes the special, link-adapted Heegaard diagrams $(\Sigma, \alpha, \beta, D)$. Finally, §4 establishes the following correspondences:

**Theorem 4.1.** There is a 1-to-1 correspondence between Turaev surfaces of connected link diagrams on $S^2 \subset S^3$ and diagrams $(\Sigma, \alpha, \beta, D)$ with the following properties:

- $(\Sigma, \alpha, \beta)$ is a Heegaard diagram for $S^3$, with $\alpha \pitchfork \beta$.
- $D$ is an alternating link diagram on $\Sigma$ which cuts $\Sigma$ into disks, with $D \pitchfork \alpha$ and $D \pitchfork \beta$.
- $D \cap \alpha = D \cap \beta = \alpha \cap \beta$, none of these points being crossings of $D$.
- There is a checkerboard partition $\Sigma \setminus (\alpha \cup \beta) = \Sigma_\varnothing \cup \Sigma_K$, in which $\Sigma_\varnothing$ consists of disks disjoint from $D$, in which $D$ cuts $\Sigma_K$ into disks each of whose boundary contains at least one crossing point and at most two points of $\alpha \cap \beta$, and in which $2g(\Sigma) + |\Sigma_\varnothing| = |\alpha| + |\beta|$.

**Theorem 4.2.** There is a 1-to-1 correspondence between generalized Turaev surfaces, constructed from dual pairs of states of connected link diagrams on $S^2 \subset S^3$, and diagrams $(\Sigma, \alpha, \beta, D)$ with the properties in Theorem 4.1, except that $D$ need not alternate on $\Sigma$. 

arXiv:1408.1304v2 [math.GT] 8 Aug 2014
Figure 1. A link diagram on $S^2$, and the link-adapted Heegaard diagram $(\Sigma, \alpha, \beta, D)$ corresponding to its Turaev surface, the torus in Figure 7. As in all figures, the link is black; the crossing balls are white; the attaching circles comprising $\alpha$ and $\beta$ are red and blue, respectively; and the circles and disks from the all-A state are green, while those from the all-B state are brown.

Acknowledgements: We would like to thank Charlie Frohman, Maggy Tomova, Ryan Blair, Oliver Dasbach, Adam Lowrance, Neal Stoltzfus, and Effie Kalfagianni for helpful conversations.

2. Background

2.1. Heegaard splittings and diagrams. A Heegaard splitting of an orientable 3-manifold $M$ is a decomposition of $M$ into two handlebodies $H_\alpha$ and $H_\beta$ with common boundary. The surface $\partial H_\alpha = \partial H_\beta = \Sigma$ is called a splitting surface for $M$. In this paper, we address only the case in which $M = S^3$.

One can describe a handlebody $H$ by identifying on its boundary $\partial H = \Sigma$ a collection of disjoint, simple closed curves $\alpha_1, \ldots, \alpha_k$, such that each $\alpha_i$ bounds a disk $\hat{\alpha}_i$ in $H$, and such that these disks together cut $H$ into a disjoint union of balls. The $\alpha_i$ are called attaching circles for $H$. Some conventions require that the $\hat{\alpha}_i$ together cut $H$ into a single ball, hence $k = g(\Sigma)$; though not requiring this, our definition does imply that $k \geq g(\Sigma)$.

A Heegaard diagram $(\Sigma, \alpha, \beta)$ combines these ideas to blueprint a 3-manifold. The diagram consists of a splitting surface $\Sigma = \partial H_\alpha = \partial H_\beta$, together with a union $\alpha = \bigcup \alpha_i$ of attaching circles for $H_\alpha$ and a union $\beta = \bigcup \beta_i$ of attaching circles for $H_\beta$. If $(\Sigma, \alpha, \beta)$ is a Heegaard diagram for $S^3$, then the circles of $\alpha$ and $\beta$ together generate $H_1(\Sigma)$. The Appendix provides an easy proof of this fact, using Seifert surfaces.
2.2. Link diagrams and crossing balls. A link diagram $D$ on a closed surface $F \subset S^3$ is the image, in general position, of an immersion of one or more circles in $F$; each arc at any crossing point is labeled with a direction normal to $F$ near that point, so that under- and over-crossings have been identified. By inserting small, mutually disjoint crossing balls $C = \bigcup C_i$ centered at the crossing points of $D$ and pushing the two intersecting arcs of each $D \cap C_i$ off $F$ to the appropriate hemisphere of $\partial C_i \setminus F$, as in Figure 2, one obtains a configuration of a link $K \subset (F \setminus C) \cup \partial C \subset S^3$. Call this a crossing ball configuration of the link $K$ corresponding to the link diagram $D$.

Conversely, given mutually disjoint crossing balls $C = \bigcup C_i$ centered at points on a closed surface $F \subset S^3$, and a link $K \subset (F \setminus C) \cup \partial C$ in which each crossing ball appears as in Figure 2, one may obtain a corresponding link diagram as follows. Consider a regular neighborhood of $F$ that contains $C$ and is parameterized by an orientation-preserving homeomorphism with $F \times [-1, 1]$ which identifies $F$ with $F \times \{0\}$. If $\pi : F \times [-1, 1] \to F$ denotes the natural projection, the link diagram corresponding to the crossing ball configuration $K \subset (F \setminus C) \cup \partial C \subset S^3$ is the projected image $\pi(K) \subset F$ with appropriate crossing labels.

In such a crossing ball configuration, each arc of $K \cap \partial C$ lies either in $F \times [-1, 0]$ or in $F \times [0, 1]$. The former arcs are called under-passes, and the latter are called over-passes. A link diagram $D$ is said to be alternating if each arc of $K \setminus C$ in a corresponding crossing ball configuration joins an under-pass with an over-pass. A link $K \subset S^3$ is alternating if it has an alternating diagram on $S^3$.

In particular, any Heegaard diagram $(\Sigma, \alpha, \beta)$ for $S^3$ provides an embedding of the closed surface $\Sigma$ in $S^3$. One may therefore superimpose a link diagram $D$ on the Heegaard diagram to obtain a new type of diagram $(\Sigma, \alpha, \beta, D)$. This new diagram describes a Heegaard splitting of $S^3$ in which the splitting surface contains a link diagram.

2.3. Turaev surfaces. Each crossing in a link diagram $D$ on a surface $F$ can be smoothed in two different ways, by inserting a crossing ball $C_i$ and replacing $D \cap C_i$ with one of the two pairs of arcs of $(\partial C_i \cap F) \setminus D$ opposite to another. Figure 3 shows the two possibilities, called the $A$-smoothing and the $B$-smoothing of the crossing. Making a choice of smoothing for each crossing in the diagram produces a disjoint union of circles on $F$, called a state of the diagram $D$. Two states of $D$ are dual if they have opposite smoothings at each crossing.
Given a link diagram $D$ on $S^2$, the two extreme states – the all-A and the all-B – are of particular interest, due in part to the bounds they give on the maximum and minimum degrees of the Jones polynomial. Kauffman’s proof [12] that these bounds are sharp for reduced, alternating diagrams provided the impetus for Murasugi [16], Thistlethwaite [18], and Turaev [19] to prove Tait’s conjecture on the crossing numbers of alternating links. Cromwell [7], Lickorish and Thistlethwaite [13] then extended these results to adequate link diagrams. Figure 4 shows the all-A and all-B states for the link diagram from Figure 1.

Following Turaev [19], one can construct a cobordism between the all-A and all-B states as follows. Parameterize a bi-collaring of $S^2$ as in §2.2 and push the all-A and all-B states off $S^2$ to $S^2 \times \{1\}$ and $S^2 \times \{-1\}$, respectively, such that each state circle sweeps out an annulus to one side of $S^2$. Assume that these annuli are mutually disjoint, and that they are disjoint from the crossing balls $C = \bigcup C_i$ used to construct the all-A and all-B states. Gluing together these annuli and the disks of $S^2 \cap C$ produces the cobordism between the two states. (See Figure 5.) Near each crossing, the cobordism has a saddle, as in Figure 6.
Figure 6. Turaev’s cobordism between the all-A and all-B states has a saddle near each crossing, shown here with and without a crossing ball.

Figure 7. This torus is the Turaev surface of the link diagram in Figures 1 and 4, seen from the ambient space. To provide a window to the far side of the surface, one of the three disks of the all-A state is only partly shown.

Having constructed the cobordism, one caps the all-A and all-B states with mutually disjoint disks to form a closed surface $\Sigma$, called the Turaev surface of the original link diagram $D$ on $S^2$. Since $\Sigma$ contains a neighborhood of $S^2$ around each crossing point, the crossing information of $D$ on $S^2$ translates to crossing information on the Turaev surface. Thus, $D$ forms a link diagram on $\Sigma$. A crossing ball configuration corresponding to this link diagram is $K \subset (\Sigma \setminus C) \cup \partial C$, with under- and over-passes defined as in §2.2.

Observe that $D$ cuts $\Sigma$ into disks, each of which contains exactly one state disk, and that $S^2 \cap \Sigma = S^2 \cap (C \cup K) = \Sigma \cap (C \cup K)$. Note also that if $D$ is alternating on $S^2$, then $\Sigma$ is a sphere which can be isotoped to $S^2$ while fixing $D$. Figure 7 shows a less trivial example.
The construction of the Turaev surface generalizes to any pair of states $s$ and $\tilde{s}$ dual to one another. By pushing $s$ and $\tilde{s}$ to opposite sides of $S^2$ to sweep out annuli on opposite sides of $S^2$, gluing in disks near the crossings to obtain a cobordism between $s$ and $\tilde{s}$, and capping off with disks, one obtains a closed surface $\Sigma$ on which $D$ forms a link diagram \cite{DFKLS}. Call this surface $\Sigma$ the \textit{generalized Turaev surface} of the dual states $s$ and $\tilde{s}$.

3. Construction of Heegaard diagrams for Turaev surfaces

Given a connected link diagram $D$ on $S^2 \subset S^3$ and its Turaev surface $\Sigma$, this section constructs a link-adapted Heegaard diagram $(\Sigma, \alpha, \beta, D)$. Theorem \ref{thm:heegaard} then characterizes this diagram, providing one direction of the correspondence to come in Theorem \ref{thm:classification}.

Let $K \subset (S^2 \setminus C) \cup \partial C$ be a crossing ball structure corresponding to $D$, and let $H_\alpha$ and $H_\beta$ be the two components of $S^3 \setminus \Sigma$. Define $\hat{\alpha} := (S^2 \setminus (C \cup K)) \cap H_\alpha$ and $\hat{\beta} := (S^2 \setminus (C \cup K)) \cap H_\beta$ to be the two checkerboard classes of $S^2 \setminus (C \cup K)$, with $\alpha := \partial \hat{\alpha}$ and $\beta := \partial \hat{\beta}$. From this setup, three modifications will complete the construction of the diagram $(\Sigma, \alpha, \beta, D)$. During these changes, $\Sigma$, $D$, $S^2$, $C$, and $K$ will remain fixed.

First, perturb $\alpha$ and $\beta$ through the cobordism as follows, carrying along the disks of $\hat{\alpha}$ and $\hat{\beta}$. Let $X = \{x_1, \ldots, x_n\}$ consist of one point on each arc of $K \setminus C$ which joins two under-passes on $S^2$, and let $Y = \{y_1, \ldots, y_n\}$ consist of one point on each arc of $K \setminus C$ which joins two over-passes on $S^2$. Each arc of $\alpha \setminus (X \cup Y)$ runs along a circle from either the all-A state or the all-B state. Isotope $\alpha$ through the cobordism so as to push arcs of the former type to $S^2 \times (0, 1)$ and arcs of the latter type to $S^2 \times (-1, 0)$, giving $\alpha \cap C = \emptyset$ and $\alpha \cap D = X \cup Y$. Next, isotope $\beta$ in the same manner, after which $\alpha$ and $\beta$ will both be disjoint from $C$, while $\alpha$, $\beta$, and $D$ will be pairwise transverse and will intersect exclusively at triple points: $\alpha \cap \beta = \alpha \cap D = \beta \cap D = X \cup Y$.

To further simplify the picture, push the state circles through the cobordism to align with $\alpha \cup \beta$, so that each state disk becomes a component of $\Sigma \setminus (\alpha \cup \beta)$. This causes the neighborhood of each arc of $K \setminus C$ to appear as in Figure \ref{fig:splitting} possibly with red and blue reversed. Note that the state disks’ interiors remain disjoint from $D$, in fact from $S^2$.

To complete the construction, remove any attaching circles that are disjoint from $D$. Also remove the corresponding disks of $\hat{\alpha}$ and $\hat{\beta}$, and let $\alpha$, $\beta$, $\hat{\alpha}$ and $\hat{\beta}$ retain their names. Because each removed circle lies in some disk of $\Sigma \setminus D$, each removed disk is parallel to $\Sigma$.

\textbf{Lemma 3.1 (DFKLS \cite{DFKLS}).} The Turaev surface $\Sigma$ of any connected link diagram $D$ on $S^2 \subset S^3$ is a splitting surface for $S^3$.

\textbf{Proof.} Observe that $S^2 \cup C$ cuts $S^3$ into two balls, which $\Sigma$ cuts into smaller balls. Also, $S^3 \setminus (S^2 \cup C \cup \Sigma) = (H_\alpha \setminus (S^2 \cup C)) \cup (H_\beta \setminus (S^2 \cup C))$, where $H_\alpha$ and $H_\beta$ are the two components of $S^3 \setminus \Sigma$. Hence, $H_\alpha \setminus C$ and $H_\beta \setminus C$ are handlebodies, as are $H_\alpha$ and $H_\beta$. \hfill $\square$

The proof of Lemma \ref{lem:classification} implies that $(\Sigma, \alpha, \beta)$ was a Heegaard diagram for $S^3$ when $\alpha$ and $\beta$ were first defined. The fact that each removed disk of $\hat{\alpha}$ and of $\hat{\beta}$ was parallel to $\Sigma$ implies that $(\Sigma, \alpha, \beta)$ is a Heegaard diagram for $S^3$ in the finished construction as well.
Lemma 3.2 (DFKLS [8]). Any connected link diagram $D$ on $S^2 \subset S^3$ forms an alternating link diagram on its Turaev surface $\Sigma$.

Proof. Recall from §2.3 that $D$ forms a link diagram on $\Sigma$. On $S^2$, each arc $\kappa$ of $K \setminus C$ joins either two over-passes, two under-passes, or one of each. Figure 8 shows the three possible configurations of $\Sigma$ near $\kappa$, prior to the removal of attaching circles, up to reversal of $\alpha$ and $\beta$. In all three cases, the two arcs of $K \cap \partial C$ incident to $\kappa$ lie to opposite sides of $\Sigma$, so that one is an over-pass on $\Sigma$ and the other is an under-pass on $\Sigma$.  

One defines the Turaev genus $g_T(K)$ of a link $K \subset S^3$ to be the minimum genus among the Turaev surfaces of all diagrams of $K$ on $S^2$. The resulting invariant, surveyed in [4], measures how far a link is from being alternating. See also [2]. In particular, Turaev genus provides the crux of Turaev’s proof of Tait’s conjecture:

Corollary 3.3 (Turaev [19], DFKLS [8]). A link $K$ is alternating if and only if $g_T(K) = 0$. 
Given a diagram \((\Sigma, \alpha, \beta, D)\) with the properties in Theorems 3.4, 4.1, or 4.2, removing the disks of \(\Sigma_\emptyset\) from \(\Sigma\) and gluing in the disks of \(\hat{\alpha}\) and \(\hat{\beta}\) produces a sphere on which \(D\) forms a link diagram. Near each point of \(\alpha \cap \beta\), this surgery appears as shown, up to mirroring.

**Theorem 3.4.** From the Turaev surface \(\Sigma\) of a connected link diagram \(D\) on \(S^2 \subset S^3\), the construction in this section produces a diagram \((\Sigma, \alpha, \beta, D)\) with the following properties:

- \((\Sigma, \alpha, \beta)\) is a Heegaard diagram for \(S^3\), with \(\alpha \pitchfork \beta\).
- \(D\) is an alternating link diagram on \(\Sigma\) which cuts \(\Sigma\) into disks, with \(D \pitchfork \alpha\) and \(D \pitchfork \beta\).
- \(D \cap \alpha = D \cap \beta = \alpha \cap \beta\), none of these points being crossings of \(D\).
- There is a checkerboard partition \(\Sigma \setminus (\alpha \cup \beta) = \Sigma_\emptyset \cup \Sigma_K\), in which \(\Sigma_\emptyset\) consists of disks disjoint from \(D\) and constitute a checkerboard class of \(\Sigma \setminus (\alpha \cup \beta)\).

**Proof.** We have already established the first three properties. Let \(\Sigma_\emptyset\) consist of the interiors of all adjusted state disks whose boundary contains at least one point of \(\alpha \cap \beta\), i.e. those whose boundary still lies in \(\alpha \cup \beta\) after the removal of the attaching circles disjoint from \(D\). These state disks are disjoint from \(D\) and constitute a checkerboard class of \(\Sigma \setminus (\alpha \cup \beta)\). See Figure 9

Let \(\Sigma_K\) denote the other checkerboard class of \(\Sigma \setminus (\alpha \cup \beta)\). Each component of \(\Sigma_K \setminus D\) is also a component of \((\Sigma \setminus D) \setminus (\alpha \cup \beta)\), and each attaching circle intersects \(D\); therefore, \(D\) cuts \(\Sigma_K\) into disks. Further, each arc of \(K \setminus C\) contains at most one point of \(\alpha \cap \beta\), and each arc of \((\alpha \cup \beta) \setminus D\) is parallel through \(\Sigma\) to \(D\); consequently, the boundary of each disk of \(\Sigma_K \setminus D\) contains at least one crossing point and at most one arc of \((\alpha \cup \beta) \setminus D\), hence at most two points of \(\alpha \cap \beta\).

Finally, to see that \(2g(\Sigma) + |\Sigma_\emptyset| = |\alpha| + |\beta|\), consider Euler characteristic in light of the observation that removing the disks of \(\Sigma_\emptyset\) from \(\Sigma\) and gluing in the disks of \(\hat{\alpha}\) and \(\hat{\beta}\) yields a sphere isotopic to \(S^2\). Near each point of \(\alpha \cap \beta\), this surgery appears as in Figure 9. \(\square\)
4. Correspondence between Heegaard diagrams and Turaev surfaces

4.1. Main correspondence. From the Turaev surface of a connected link diagram on \( S^2 \subset S^3 \), we have constructed a link-adapted Heegaard diagram \((\Sigma, \alpha, \beta, D)\) with several properties. We will now see that any such diagram corresponds to the Turaev surface of some link diagram on the sphere.

**Theorem 4.1.** There is a 1-to-1 correspondence between Turaev surfaces of connected link diagrams on \( S^2 \subset S^3 \) and diagrams \((\Sigma, \alpha, \beta, D)\) with the following properties:

- \((\Sigma, \alpha, \beta)\) is a Heegaard diagram for \( S^4 \), with \( \alpha \cap \beta \).
- \( D \) is an alternating link diagram on \( \Sigma \) which cuts \( \Sigma \) into disks, with \( D \cap \alpha \) and \( D \cap \beta \).
- \( D \cap \alpha = D \cap \beta = \alpha \cap \beta \), none of these points being crossings of \( D \).
- There is a checkerboard partition \( \Sigma \setminus (\alpha \cup \beta) = \Sigma_{\varnothing} \cup \Sigma_K \), in which \( \Sigma_{\varnothing} \) consists of disks disjoint from \( D \), in which \( D \) cuts \( \Sigma_K \) into disks each of whose boundary contains at least one crossing point and at most two points of \( \alpha \cap \beta \), and in which \( 2g(\Sigma) + |\Sigma_{\varnothing}| = |\alpha| + |\beta| \).

**Proof.** Theorem 3.4 provides one direction of this correspondence. It remains to prove the converse.

Assume that the diagram \((\Sigma, \alpha, \beta, D)\) is as described. Remove the disks of \( \Sigma_{\varnothing} \) from \( \Sigma \) and glue the disks of \( \alpha \) and \( \beta \) to obtain a closed surface. (See Figure 9) Because \( D \) is connected and \( 2g(\Sigma) + |\Sigma_{\varnothing}| = |\alpha| + |\beta| \), this surface is a sphere – call it \( S^2 \). Moreover, \( D \), being disjoint from \( \Sigma_{\varnothing} \) and having its crossing points in \( \Sigma_K \), forms a link diagram on \( S^2 \).

We claim, up to isotopy, that \( \Sigma \) is the Turaev surface of the link diagram \( D \) on \( S^2 \).

The property that \( D \) cuts \( \Sigma_K \) into disks implies that \( D \) intersects each attaching circle, cutting \( \alpha \) and \( \beta \) into arcs. Because the boundary of each disk of \( \Sigma_K \setminus D \) contains at most two points of \( \alpha \cap \beta \), each of these arcs is parallel through one of these disks to \( D \). The property that the boundary of each disk of \( \Sigma_K \setminus D \) contains at least one crossing point then implies that there is at most one point of \( \alpha \cap \beta \) on \( D \) between any two adjacent crossings.

The link diagram \( D \) cuts \( S^2 \) into disks admitting a checkerboard partition. Because \( S^2 \) appears near each point of \( \alpha \cap \beta \) as in Figure 9, one of the checkerboard classes contains \( \hat{\alpha} \), and the other contains \( \hat{\beta} \). Yet, some disks of \( S^2 \setminus D \) may be entirely contained in \( \Sigma_K \), hence disjoint from \( \alpha \) and \( \beta \). Construct an attaching circle in the interior of each such disk, and incorporate it into either \( \alpha \) or \( \beta \) according to the checkerboard pattern, letting \( \alpha \) and \( \beta \) retain their names. Span each new circle of \( \alpha \) by a new disk of \( \hat{\alpha} \) on the same side of \( \Sigma \) as the other disks of \( \hat{\alpha} \), and similarly span each new circle of \( \beta \) by a new disk of \( \hat{\beta} \).

The components of \( \Sigma \setminus (\alpha \cup \beta) \) still admit a checkerboard partition, \( \Sigma \setminus (\alpha \cup \beta) = \Sigma_{\varnothing} \cup \Sigma_K \), in which \( \Sigma_{\varnothing} \) consists of disks disjoint from \( D \), though \( D \) no longer need cut \( \Sigma_K \) into disks. The preceding modification of \( \alpha, \beta, \hat{\alpha}, \text{ and } \hat{\beta} \) corresponds to an isotopy of \( S^2 \), which again may be obtained from \( \Sigma \) by removing the disks of \( \Sigma_{\varnothing} \) and gluing in the disks of \( \hat{\alpha} \) and \( \hat{\beta} \).

Let \( K \subset (\Sigma \setminus C) \cup \partial C \) be a crossing ball configuration corresponding to the link diagram \( D \) on \( \Sigma \), with \( C \cap \alpha = \varnothing = C \cap \beta \). Note that \( K \subset (S^2 \setminus C) \cup \partial C \) is also a crossing ball configuration corresponding to the link diagram \( D \) on \( S^2 \).
Currently $\Sigma$ and $S^2$ are non-transverse, even away from $C$, as both $\Sigma$ and $S^2$ contain $\Sigma_K$. Rectify this by perturbing $S^2$ as follows, fixing $\Sigma$, $\alpha$, $\beta$, $\alpha^*$, $\beta^*$, $D$, $\Sigma_\varnothing$, $\Sigma_K$, $C$, and $K$ in the process. (We initially constructed $S^2$ by gluing together $\alpha^*$, $\beta^*$, and $\Sigma_K$, but now we are pushing $S^2$ off of them.) Each disk of $S^2 \setminus (C \cup K)$ currently contains a disk of either $\alpha^*$ or $\beta^*$; push the disk of $S^2 \setminus (C \cup K)$ off $\Sigma$ in the corresponding direction, while fixing its boundary, which lies in $\Sigma \cap (K \cup C)$. This isotopy makes $S^2$ disjoint from $\alpha$ and $\beta$, except at the points of $\alpha \cap \beta$. In fact, this isotopy gives $S^2 \cap \Sigma = S^2 \cap (C \cup K) = \Sigma \cap (C \cup K)$, as was the case in §2.3 (Recall Figure 6).

Because $D$ is alternating on $\Sigma$, the disks of $\Sigma \setminus (C \cup K)$ admit a checkerboard partition – the boundaries of the disks in the two classes are the all-A and all-B state circles for the link diagram $D$ on $\Sigma$. Further, each of these state circles on $\Sigma$ encloses precisely one disk of $\Sigma_\varnothing$. Color green each disk of $\Sigma_\varnothing$ enclosed by a circle from the all-A state, and color brown each disk of $\Sigma_\varnothing$ enclosed by a circle from the all-B state. Near each arc of $K \setminus C$, $\Sigma$ now appears as in Figure 8 (possibly with red and blue reversed). As a final adjustment, slightly perturb the green and brown disks so that they become disjoint from $\alpha$, $\beta$, and $D$.

Removing the green and brown disks from $\Sigma$ leaves a cobordism between their boundaries. Cutting this cobordism along $\Sigma$ is therefore the Turaev surface of the link diagram $D$.

4.2. Generalization to arbitrary dual states. As noted at the end of §2.3, the construction of the Turaev surface from the all-A and all-B states of a link diagram $D$ on $S^2$ generalizes to any pair of states of $D$ which are dual to one another, having opposite smoothings at each crossing. The correspondence developed in §3 and §4.1 between link-adapted Heegaard diagrams $(\Sigma, \alpha, \beta, D)$ and Turaev surfaces extends to these generalized Turaev surfaces, the only difference being that $D$ no longer need alternate on $\Sigma$.

**Theorem 4.2.** There is a 1-to-1 correspondence between generalized Turaev surfaces of connected link diagrams on $S^2 \subset S^3$, and diagrams $(\Sigma, \alpha, \beta, D)$ with the following properties:

- $(\Sigma, \alpha, \beta)$ is a Heegaard diagram for $S^3$, with $\alpha \cap \beta$.
- $D$ is a link diagram on $\Sigma$ which cuts $\Sigma$ into disks, with $D \cap \alpha$ and $D \cap \beta$.
- $D \cap \alpha = D \cap \beta = \alpha \cap \beta$, none of these points being crossings of $D$.
- There is a checkerboard partition $\Sigma \setminus (\alpha \cup \beta) = \Sigma_\varnothing \cup \Sigma_K$, in which $\Sigma_\varnothing$ consists of disks disjoint from $D$, in which $D$ cuts $\Sigma_K$ into disks each of whose boundary contains at least one crossing point and at most two points of $\alpha \cap \beta$, and in which $2g(\Sigma) + |\Sigma_\varnothing| = |\alpha| + |\beta|$.

**Proof.** Given the generalized Turaev surface $\Sigma$ for dual states $s$ and $\tilde{s}$ of a connected link diagram $D$ on $S^2 \subset S^3$, reverse some collection of the crossings of $D$ to obtain a new link diagram $D'$ for which $s$ and $\tilde{s}$ are the all-A and all-B states. Construct the corresponding diagram $(\Sigma, \alpha, \beta, D')$ as in §3. Finally, switch back the reversed crossings of $D'$ to obtain the required diagram $(\Sigma, \alpha, \beta, D)$.
Conversely, suppose that $(\Sigma, \alpha, \beta, D)$ is as described. The proof of Theorem 4.1 extends almost verbatim. The only concern, as $D$ need not alternate on $\Sigma$, is whether or not the disks of $\Sigma \setminus D$ admit a checkerboard partition; it suffices to show that they do.

The condition that $D \cap \Sigma = \emptyset$ implies that one endpoint of each arc of $(\alpha \cup \beta) \setminus D$ appears as in Figure 9, and the other appears as the mirror image. Thus, each attaching circle intersects $D$ in an even number of points. The fact that the attaching circles generate $H_1(\Sigma)$ then implies that any simple closed curve on $\Sigma$ in general position with respect to $D$ must intersect $D$ in an even number of points, and hence that the disks of $\Sigma \setminus D$ admit a checkerboard partition.

4.3. Conclusion. Up to isotopy, each link diagram on $S^2 \subset S^3$ has a unique Turaev surface. Theorem 4.1 thus establishes – via Turaev surfaces – a 1-to-1 correspondence between link diagrams on $S^2 \subset S^3$ and alternating, link-adapted Heegaard diagrams $(\Sigma, \alpha, \beta, D)$.

Similarly, Theorem 4.2 establishes – via generalized Turaev surfaces constructed from dual states – a 2-to-1 correspondence between states of link diagrams on $S^2 \subset S^3$ and link-adapted Heegaard diagrams $(\Sigma, \alpha, \beta, D)$ for $S^3$ in which $D$ need not alternate on $\Sigma$.

5. Appendix

Let $(\Sigma, \alpha, \beta)$ be a Heegaard diagram for $S^3$, and let $\gamma \subset \Sigma$ be an oriented, simple closed curve. The following construction yields an expression for $[\gamma] \in H_1(\Sigma)$ in terms of the homology classes of the oriented attaching circles, proving that the latter generate $H_1(\Sigma)$.

Because $H_1(S^3)$ is trivial, $\gamma$ bounds a Seifert surface $S \subset S^3$, on which $\gamma$ induces an orientation. Fixing $\gamma$, isotope $S$ so that its interior intersects $\Sigma$ transversally – along simple closed curves and along arcs with endpoints on $\gamma$.

Given a component $S_{\alpha,i}$ of $S \cap H_\alpha$, one may obtain an expression for $[\partial S_{\alpha,i}] \in H_1(\Sigma)$ in terms of the $[\alpha_j]$ by surgering $S_{\alpha,i}$ along successive outermost disks of $\alpha \setminus S_{\alpha,i}$ until $\partial S_{\alpha,i}$ lies entirely in the punctured sphere $\Sigma \setminus \alpha$, at which point the expression is evident. An analogous procedure expresses the homology class of each component of $S \cap H_\beta$ in terms of the $[\beta_j]$. Summing over all components of $S \setminus \Sigma$ gives the desired expression for $[\gamma] \in H_1(\Sigma)$:

$$[\gamma] = [\partial S] = \sum_{\text{Components } S_{\alpha,i} \text{ of } S \cap H_\alpha} [\partial S_{\alpha,i}] + \sum_{\text{Components } S_{\beta,i} \text{ of } S \cap H_\beta} [\partial S_{\beta,i}] = \sum_{i,j} a_{i,j} [\alpha_j] + \sum_{i,j} b_{i,j} [\beta_j]$$

Conversely, if $(\Sigma, \alpha, \beta)$ is a Heegaard diagram for a 3-manifold $M$ with nontrivial first homology, then the oriented attaching circles do not generate $H_1(\Sigma)$, since inclusion $\Sigma \hookrightarrow M$ induces a surjective map $H_1(\Sigma) \to H_1(M)$, whose kernel contains all the $[\alpha_j]$ and $[\beta_j]$. 

References

[1] Y. Bae, H.R. Morton, The spread and extreme terms of Jones polynomials, J. Knot Theory Ramifications 12 (2003), 359-373.

[2] C.L.J. Balm, Topics in knot theory: On generalized crossing changes and the additivity of the Turaev genus, Thesis (Ph.D.) – Michigan State University (2013).
[3] A. Champanerkar, I. Kofman, *Spanning trees and Khovanov homology*, Proc. Amer. Math. Soc. 137 (2009), no. 6, 2157-2167.

[4] A. Champanerkar, I. Kofman, *A survey on the Turaev genus of knots*, arXiv:1406.1945 [math]-preprint.

[5] A. Champanerkar, I. Kofman, N. Stoltzfus, *Graphs on surfaces and Khovanov homology*, Algebr. and Geom. Topol. 7 (2007), 1531-1540.

[6] A. Champanerkar, I. Kofman, N. Stoltzfus, *Quasi-tree expansion for the Bollobás-Riordan-Tutte polynomial*, Bull. Lond. Math. Soc. 43 (2011), no. 5, 972-984.

[7] P.R. Cromwell, *Homogeneous links*, J. London Math. Soc. (2) 39 (1989), no. 3, 535-552.

[8] O.T. Dasbach, D. Futer, E. Kalfagianni, X.-S. Lin, N. Stoltzfus, *The Jones polynomial and graphs on surfaces*, J. Combin. Theory Ser. B 98 (2008), no. 2, 384-399.

[9] O.T. Dasbach, A. Lowrance, *Turaev genus, knot signature, and the knot homology concordance invariants*, Proc. Amer. Math. Soc. 139 (2011), no. 7, 2631-2645.

[10] O.T. Dasbach, A. Lowrance, *A Turaev surface approach to Khovanov homology*, arXiv:1107.2344v2.

[11] V.F.R. Jones, *A polynomial invariant for knots via Von Neumann algebras*, Bull. Amer. math. Soc. (N.S.) 12 (1985), no. 1, 103-111.

[12] L.H. Kauffman, *State models and the Jones polynomial*, Topology 26 (1987), no. 3, 395-407.

[13] W.B.R. Lickorish and M.B. Thistlethwaite, *Some links with non-trivial polynomials and their crossing numbers*, Comment. Math. Helv. 63 (1988), no. 4, 527-539.

[14] A. Lowrance, *On knot Floer width and Turaev genus*, Algebr. Geom. Topol. 8 (2008), no. 2, 1141-1162.

[15] W. Menasco, *Closed incompressible surfaces in alternating knot and link complements*, Topology 23 (1984), no. 1, 37-44.

[16] K. Murasugi, *Jones polynomials and classical conjectures in knot theory*, Topology 26 (1987), no. 2, 187-194.

[17] P.G. Tait, *On Knots I, II, and III*, Scientific papers 1 (1898), 273-347.

[18] M.B. Thistlethwaite, *A spanning tree expansion of the Jones polynomial*, Topology 26 (1987), no. 3, 297-309.

[19] V.G. Turaev, *A simple proof of the Murasugi and Kauffman theorems on alternating links*, Enseign. Math. (2) 33 (1987), no. 3-4, 203-225.

[20] S. Wehrli, *A spanning tree model for Khovanov homology*, J. Knot Theory Ramifications 17 (2008), no. 12, 1561-1574.

**Department of Mathematics, University of Iowa, Iowa City, IA 52242-1419, USA**

*E-mail address: cody-armond@uiowa.edu*

**Department of Mathematics, University of Iowa, Iowa City, IA 52242-1419, USA**

*E-mail address: nathan-druivenga@uiowa.edu*

**Department of Mathematics, University of Iowa, Iowa City, IA 52242-1419, USA**

*E-mail address: thomas-kindred@uiowa.edu*