BACKGROUND FIELDS IN 2+1
TOPOLOGICAL GRAVITY

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Abstract
The partition function of 2+1 Chern-Simons Witten topological gravity has an attractive physical interpretation in terms of the unbroken and broken phases of gravity. We make this physical interpretation manifest by using the background field method.

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1 Introduction

It is well known by now that 2+1 dimensional quantum gravity can be given a
gauge theoretic interpretation in terms of a Chern-Simons theory with gauge group
ISO(2, 1), SO(3, 1) or SO(2, 2), depending on whether the cosmological constant \( \lambda \)
is zero, positive or negative \([1,2]\). One of the main differences, however, is the role of
non-invertible vierbeins which are forbidden in the former but are allowed, and even
required, in the latter. The Chern-Simons interpretation of 2+1 quantum gravity,
therefore, indicating quite naturally the existence of an unbroken and broken phase
of gravity. It turns out that these two phases also have there place in the physical
interpretation of the partition function of the theory. That is, if a certain operator
has no zero modes then the quantum theory is constrained to the unbroken phase
but if this operator has zero modes then the theory is allowed to escape to the broken
phase where the resulting space-time has a Riemannian interpretation \([2]\).

The above observations are very interesting but not very manifest. One discovers
them by studying the properties of operators on manifolds with given topologies.
The question to be asked then is can we us a more physical approach to discover
the above observations? One possibility is the background field method. If the
physical interpretation of the partition function is indeed correct, one would expect
the background fields to mimic correctly this behaviour and thus make manifest this
physical interpretation.

A perhaps naive step in this direction was made in \([3]\). Here, Chern-Simons-
Witten topological gravity, with non-zero cosmological constant, was considered on
some three manifold \( M \). Using the background field method for the vierbein field
\( e_\alpha^i \) only, a one-loop effective action, \( \Gamma \), was obtained of the form:

\[ \exp i\Gamma = \det(\lambda e_\alpha^a)^{-1/2} \det(D_i \tilde{D}^i) \det(D_i (\lambda e_\alpha^a)^{-1} D_k)^{-1/2} \exp i S[e, \omega, \lambda]. \]  

(1)

Here, \( e, \omega \) and \( \lambda \) are the background vierbein, spin connection and cosmological
constant respectively. \( D_i \) and \( \tilde{D}^j \) are certain derivatives required for gauge fixing
and \( S[e, \omega, \lambda] \) is the initial action with respect to the background fields with cos-
mological constant \( \lambda \). It was then argued that the background vierbein \( e_\alpha^i \) must be
everywhere invertible because of the way it appears in the determinants of equation
(1). However, this argument may be a little naive for two reasons. Firstly, it is clear
from equation (1) that the background vierbeins in the determinants may cancel.
If this is true, it is a bizarre result in that to perform the calculation we have to
assume that \( e_\alpha^a \) is everywhere invertible but at the end of the calculation everything
cancels implying that we have really learned nothing. The problem, therefore, might
be with the particular method used. Secondly, in obtaining (1), we ignored quantum
fluctuations in the spin connection field which may not be legal.
As an attempt to overcome these difficulties, we will make a more careful analysis of one-loop effects using the background field method. We will consider the case of zero cosmological constant as well as the non-zero case. The case of zero cosmological constant turns out to be quite straightforward with our results agreeing perfectly with Witten’s observations. The case of non-zero cosmological constant is more difficult because we want to make sure that any constraints on background fields are not a consequence of the particular method used.

The outline of this paper is as follows. In section 2, we review the relevant facts of Chern-Simons-Witten gravity. We then use the background field method to evaluate an effective action and discuss constraints that appear in this approach. We then finish with a conclusion.

2 Gravity in 2+1 dimensions

In [1,2], Witten constructed a gauge theory of gravity with cosmological constant $\lambda$ and gauge group $G$ given by:

$$G = \begin{cases} 
SO(3,1) & \text{for } \lambda > 0 \\
ISO(2,1) & \text{for } \lambda = 0 \\
SO(2,2) & \text{for } \lambda < 0.
\end{cases}$$

The Chern-Simons term on some three manifold $M$ takes the form:

$$S = \frac{1}{2} \int_M A \wedge dA + \frac{2}{3} A \wedge A \wedge A,$$

where $A$ is a Lie algebra valued one form, and $<>$ denotes the invariant quadratic form. Denoting tangent space indices by $i, j, k$ and “Lorentz” indices by $a, b, c$, we expand the gauge field as:

$$A_i = e_i^a P_a + \omega_i^a J_a,$$

where, depending on the sign of $\lambda$, $P_a$ and $J_a$ are the momentum and angular momentum generators of the anti-de Sitter, Poincare’ or de-Sitter groups. $e_i^a$ and $\omega_i^a$ are the vierbein and spin connection fields on $M$ respectively. Substituting (4) into (3), we arrive at:

$$S = \frac{1}{2} \int_M \epsilon^{ijk}(\epsilon_{ia}(\partial_j \omega^a_k - \partial_k \omega^a_j) + \epsilon_{abc} e_i^a \omega^b_j \omega^c_k + \frac{1}{3} \lambda \epsilon_{abc} e_i^a e_j^b e_k^c).$$

Varying equation (5) with respect to $e$ and $\omega$ gives their respective equations of motion:

$$\partial_i \omega^a_j - \partial_j \omega^a_i + \epsilon^{abc} (\omega_{ib} \omega^c_j + \lambda e_ib e_j^c) = 0,$$
Provided that \( \det(e^a_i) \neq 0 \), the classical equivalence between the gauge theory and general relativity is seen once one solves \( \omega \) in terms of \( e \) via equation (7). When substituted back into equation (5) we find the Einstein lagrangian with a cosmological constant \( \lambda \) \([2,4]\).

Let us now consider the partition function of \( S \) on some closed manifold \( M \). Looking at equation (5) we see that its path integral in \( e^a_i \) will become particularly simple when \( \lambda = 0 \). We therefore discuss this case since we expect to be able to calculate the full partition function. Following Witten \([2]\), we therefore want to evaluate:

\[
Z(M) = \int DeD\omega... \exp i \left( \frac{1}{2} \int_M e^{ijk} e_{ia} (\partial_j \omega_k^a - \partial_k \omega_j^a + [\omega_j, \omega_k]^a) + i S_{gf} \right),
\]

where \( S_{gf} \) is the gauge fixing + ghost term given by:

\[
S_{gf} = \int_M (uD_i e^i + vD_i \omega^i + \bar{f} D_i \bar{D}^i f + \bar{g} D_i \bar{D}^i g).
\]

Here \( f \) and \( g \) are respectively ghosts for the rotations and translations in ISO(2,1), \( \bar{f} \) and \( \bar{g} \) are the corresponding antighosts. \( u \) and \( v \) are the Lagrange multipliers which enforce the gauge conditions:

\[
D_i \omega^i = D_i e^i = 0,
\]

where \( D_i = D_i^{\text{grav}} + [\omega(\alpha)_i, \cdot] \), \( D_i^{\text{grav}} \) being the gravitational covariant derivative with \( \omega(\alpha) \) being a flat connection, and finally \( \bar{D}_i = D_i^{\text{grav}} + [\omega_i, \cdot] \) is the full covariant derivative in the combined gravitational and gauge fields. On imposing the gauge conditions (10), one picks a metric \( g_{ij} \) on \( M \) but this metric is unrelated to the vierbein which we are regarding as a gauge field. Witten then distinguishes two cases for the evaluation of \( Z(M) \).

In case (1), the topology of the manifold \( M \) is such that the moduli space of flat \( SO(2,1) \) connections \( \mathcal{N} \) consists of finitely many points. In this case, the partition function is given by:

\[
Z(M) = \sum_{\alpha} \frac{(\det \Delta)^2}{|\det L|},
\]

where one is summing over a set of gauge equivalence classes of flat \( SO(2,1) \) connections labelled by \( \alpha \), \( L \) is the bosonic kinetic operator appearing in equation (8), and \( \Delta = D_i \bar{D}^i |_{\omega = \omega_\alpha} \).
In case (2), the topology of $M$ is such that the moduli space $\mathcal{M}$ does not consist of finitely many points but has positive dimension. In this case the operator $L$ has zero modes with the partition function given by:

$$Z(M) = \int_\Gamma \frac{(\text{det} \Delta)^2}{|\text{det}' L|}, \quad (12)$$

where $\Gamma$ is the space of classical solutions of the classical field equations on $M$, and $\text{det}'$ represents the product of the non-zero eigenvalues.

Let us now consider possible divergences in $Z(M)$ together with their physical interpretations [2]. If one formulates the quantum theory on some manifold $M$ that is impossible classically, then the partition function $Z(M)$ corresponds to case (1). Here, the quantum theory never escapes the quantum regime of Planckian distances with the partition function being perfectly finite since it is being dominated by distances of order the Planck scale as there are no zero modes. That is, case (1) resembles the unbroken phase of gravity.

Case (2) corresponds to formulating the quantum theory on some manifold $M$ with a topology that is allowed classically. In this case, the classical space-time can be arbitrarily big due to the scale invariance of the classical equations of motion. The partition function $Z(M)$ will then diverge. This divergence being an infrared divergence which corresponds to the classical region of macroscopic space-times. This clearly representing the broken phase of the theory.

Our next task is to try and confirm the above observations using the background field method.

### 3 The background fields

For $\lambda = 0$, the full partition function of the corresponding gauge theory is precisely the Ray-Singer analytic torsion which is a topological invariant. This in turn has an attractive physical interpretation in terms of the unbroken and broken phases of gravity. It is precisely this physical interpretation we would like to reproduce in terms of the background fields $(e^a_i, \omega^a_i)$.

For the time being we will consider the case $\lambda \neq 0$ and apply the background field method to evaluate the one-loop contribution to the effective action of the background fields. The case of zero cosmological constant is simply obtained by setting $\lambda = 0$. Our basic object of interest is a functional of background fields which, in principle, contains all the information about the quantised fields. Our starting point is equation (5). We use the background field method for the vierbein and spin connection fields to write:

$$e^a_i \rightarrow e^a_i + \tilde{e}^a_i, \quad \omega^a_i \rightarrow \omega^a_i + \tilde{\omega}^a_i, \quad (13)$$
where we are expanding around a non-zero classical solution \((e_i^a, \omega_i^a)\), and \((\tilde{e}_i^a, \tilde{\omega}_i^a)\) are the quantum fluctuations. At one-loop we require that part of the action, \(S_2\), which is at most quadratic in the quantum fields \((\tilde{e}_i^a, \tilde{\omega}_i^a)\). Thus, making the substitution of (13) into (5), we obtain:

\[
S_2 = \frac{1}{2} \int_M e^{ijk}(e_i^a F_{ajk}(\omega + \tilde{\omega}) + \tilde{e}_i^a F_{ajk}(\omega + \tilde{\omega}) + \frac{1}{3} \lambda \epsilon_{abc} e_i^a e_j^b e_k^c + \lambda \epsilon_{abc} \tilde{e}_i^a e_j^b \tilde{e}_k^c + \lambda \epsilon_{abc} \tilde{e}_i^a \tilde{e}_j^b \tilde{e}_k^c),
\]

where \(F_{ajk}\) is given by:

\[
F_{ajk}(\omega) = \partial_j \omega_{ka} - \partial_k \omega_{ja} + \epsilon_{abc} \omega_j^b \omega_k^c.
\]

Our next task is to integrate out the quantum fields \((\tilde{e}_i^a, \tilde{\omega}_i^a)\). Before we can do this, gauge fixing is required. The one-loop effective action, \(\Gamma(e, \omega, \lambda)\), for the background fields is then given by:

\[
\exp i\Gamma = \int D\tilde{\omega} D\tilde{e} \ldots \exp i(S_2 + S_{gf}),
\]

where \(S_{gf}\) is the gauge + ghost term now given by:

\[
S_{gf} = \int_M (uD_i \tilde{e}^i + vD_i \tilde{\omega}^i + fD_i \tilde{D}^i f + gD_i \tilde{D}^i g).
\]

As in equation (9), \(f\) and \(g\) are respectively ghosts for the \(\tilde{e}\) and \(\tilde{\omega}\) fields, \(\tilde{f}\) and \(\tilde{g}\) are the corresponding antighosts. \(u\) and \(v\) are Lagrange multipliers which enforce the gauge conditions:

\[
D_i \tilde{e}^i = D_i \tilde{\omega}^i = 0,
\]

where now, \(D_i = D_i^{grav} \{[\omega_i, \text{ ]}\}, D_i^{grav}\) being the gravitational covariant derivative, \(\omega_i\) being the background field connection, which we will be assuming obeys the background field equations, and now \(\tilde{D}_i = D_i\).

What we would like to do is expand \(\tilde{e}\) in terms of eigenfunctions of some operator but since the background field \(e\) is a classical object, we cannot regard it as an operator and so doing this will give us little information regarding the background field \(e_i^a\). We will therefore proceed by using formal manipulations of the path integrals.

Let us first practice with the case \(\lambda = 0\). Here, the one-loop effective action becomes the full effective action since we have only linear terms in \(e\). Performing the \(\tilde{e}, v, f\) and \(g\) integrals, gives:

\[
\exp i\Gamma(e, \omega, \lambda = 0) = \int D\tilde{\omega} D u \delta(\frac{1}{2} \epsilon^{ijk} F_{ajk}(\omega + \tilde{\omega}) + D^i u_a)
\]
\[
\times \delta(D^i \tilde{\omega}_i^a) \det(\Delta) \exp i \frac{1}{2} \int_M \epsilon^{ijk} e_i^a F_{ajk}(\omega + \tilde{\omega}).
\] (19)

In order to evaluate the \(\tilde{\omega}\) integral, we borrow the method of reference [2]. We start by demanding that the background field \(\omega_i^a\) obeys the classical equation of motion given by equation (6). This means that:

\[
F_{aij}(\omega) = 0.
\] (20)

In order that \(F_{aij}(\omega + \tilde{\omega}) = 0\), which follows from equation (19), we find to lowest order in \(\tilde{\omega}\) that:

\[
\frac{1}{2} \epsilon^{ijk} F_{ajk}(\omega + \tilde{\omega}) = \epsilon^{ijk} D_{(\omega)j} \tilde{\omega}_{ak},
\] (21)

where \(D_{(\omega)j}\) is the covariant derivative with respect to the background flat connection \(\omega^a_i\). Assuming that \(D_{(\omega)j}\) has no zero modes, we can change variables in the path integral from \(\tilde{\omega}\) to \(\epsilon^{ijk} D_{(\omega)j} \tilde{\omega}_{jk}\) to obtain:

\[
\exp i \Gamma = \int D\tilde{\omega} \frac{(\det \Delta)^2}{\det D_{(\omega)j}} \exp i \int_M e_i^a D^i u_a.
\] (22)

Finally, performing the \(u\) integral gives, assuming that \(D^i\) has no zero modes:

\[
\exp i \Gamma = \frac{(\det \Delta)^2}{\det D_{(\omega)j} \cdot \det D^i} \delta(e_i^a).
\] (23)

We see that the background field \(e_i^a\) is manifestly constrained to be zero. That is, when the operators \(D_{(\omega)j}\) and \(D^i\) have no zero modes, the theory is constrained to the unbroken phase\(^2\).

Let us now consider the opposite situation in which the operators \(D_{(\omega)j}\) and \(D^i\) have zero modes. In this case it is clear what we have to do. Starting from equation (19), we will, as in reference [2], not integrate over these zero modes which we label by \(\tilde{e}_0\). Thus equation (19) becomes:

\[
\exp i \Gamma = \int D\tilde{e}_0 D\tilde{u} \frac{(\det \Delta)^2}{\det D_{(\omega)j} \cdot \det D^i} \exp i \int_M e_i^a D^i u_a,
\] (24)

where \(\det'\) represents the product of non-zero eigenvalues. Performing the \(u\) integral, noting that we cannot use the substitution \(u_a \rightarrow D^i u_a\) in the path integral which was used in the case when \(D\) has no zero modes, gives:

\[
\exp i \Gamma = \int D\tilde{e}_0 \frac{(\det \Delta)^2}{\det' D_{(\omega)j}} \delta(D^i e_i^a).
\] (25)

\(^2\)Note that the operator \(L\) in equation (11) is given by \(L = \begin{pmatrix} \epsilon^{ijk} D_{(\omega)j} & D^i \\ -D^k & 0 \end{pmatrix}\).
Clearly, the background field $e^a_i$ is now constrained to:

$$D^i e^a_i = 0,$$

that is, equation (26) has non-zero solutions. What does this equation mean? When gravity is discussed in the first order formalism, it is always possible to define the affine connection by postulating that the vierbein is covariantly constant [5], that is:

$$D_i e^a_j = \partial_i e^a_j - \Gamma^k_{ji} e^a_k + \epsilon^{abc} \omega^b_i e^c_j = 0.$$  

(27)

From this equation we can deduce the standard metric connection of general relativity. Equation (26) is saying much the same thing when we recognise that (26) becomes:

$$g^{ij} D_i e^a_j = 0,$$

(28)

where $g_{ij}$ is the gauge fixing metric which is also consistent with equation (27). What all this is saying is that the non-zero background field $e^a_i$ is a solution to Einstein’s equations and thus corresponds to the broken phase of gravity. This of course confirming the observations of equation (12).

Let us now consider the case of $\lambda \neq 0$. Looking back at equation (14), we see that we now have a quadratic and a linear term in the quantum field $\tilde{\omega}^a_i$. Integrating over this field will straight away imply non-trivial constraints on the background field $e^a_i$ since we will have factors like $(\det e^a_i)^{-1/2}$ and $(e^a_i)^{-1}$, which would imply that the background fields must be everywhere invertible on $M$ for the one-loop partition function to make sense. Of course, this may not be true. It is quite possible that the determinants may cancel as discussed in the introduction. Thus by demanding that the background field be everywhere invertible might just be a facet of the particular method used. For this reason we will follow a different route in evaluating (19). We will do this by assuming that the partition function (19) is still dominated by flat connections as for the case $\lambda = 0$. This certainly seems reasonable for very small $\lambda$ which in any case must be true for the one-loop partition function to be a good approximation to the full theory. Starting from equation (19), we will firstly perform the $\tilde{\omega}$ integral together with the $v$, $f$ and $q$ integrals. Assuming then that the one-loop partition function is dominated by flat connections, we can use equation (21). Assuming that $D_\omega$ has no zero modes leads to:

$$\exp i\Gamma = \int D\tilde{\omega} Du \frac{(\det \Delta)^2}{|\det D_\omega|} \delta(e^a_i + \tilde{e}^a_i) \times \exp i \int_M \epsilon^{ijk} \left( \frac{1}{3} \lambda \epsilon_{abc} e^a_i e^b_j e^c_k + \lambda \epsilon_{abc} \tilde{e}^a_i e^b_j e^c_k + \lambda \epsilon_{abc} \tilde{e}^a_i e^b_j \tilde{e}^c_k \right) + u_a D^i \tilde{e}^a_i.$$  

(29)
It is now quite clear that we may perform the $\tilde{e}^a_i$ integral, and also performing the $u$ integral as in the $\lambda = 0$ case, gives:

$$\exp i\Gamma = \frac{(\det \Delta)^2}{|\det D_{(\omega)j}| |\det D^i|} \delta(e^a_i) \exp i \int_M \epsilon^{ijk} \epsilon_{abc} \frac{1}{3} \lambda e^a_i e^b_j e^c_k. \quad (30)$$

Again, we see that the background field is constrained to the unbroken phase as with the $\lambda = 0$ case.

Consider now the case when $D_{(\omega)}^i = D^i$ has zero modes. Since we are now performing the $\tilde{\omega}^a_i$ integral first, we cannot follow the corresponding situation as in $\lambda = 0$ since there we considered the operator $D_{(\omega)j}$ in terms of eigenfunctions of $\tilde{e}^a_i$. So it would not make sense in this case. We therefore expand $\tilde{\omega}^a_i$ in terms of eigenfunctions of $D$, leaving un-integrated these zero modes $\tilde{\omega}_0$, we find on performing the $u$ integral:

$$\exp i\Gamma = \int D\tilde{\omega}_0 \frac{(\det \Delta)^2}{|\det D|} \delta(D^i e^a_i) \exp i \int_M \frac{1}{3} \lambda \epsilon^{ijk} \epsilon_{abc} e^a_i e^b_j e^c_k, \quad (31)$$

where the same conclusions follow as for the $\lambda = 0$ case.

## 4 Conclusion

The point of this paper as been to try and reproduce Witten’s physical interpretation of the partition function of 2+1 topological gravity in terms of background fields. The case $\lambda = 0$ was quite straightforward. Here, the background field $e^a_i$ was shown to mimic correctly the unbroken and broken phases of gravity. The case $\lambda \neq 0$ was a little more subtle. If one attempted to integrate over the quantum field $\tilde{e}^a_i$ first, then one must assume that the background field $e^a_i$ is everywhere invertible on $M$ to make sense of the expression. As discussed in the introduction, this may be a little naive and at the end of the calculation, all constraints may cancel implying that we really cannot make sense of this procedure. To overcome these difficulties, we chose in this case to integrate over $\tilde{\omega}^a_i$ first where now we must think of expanding the quantum field $\tilde{\omega}^a_i$ in terms of eigenfunctions of $D$. We then assumed that the one-loop partition function is dominated by flat connections as with the $\lambda = 0$ case. We note that this is different from assuming that the one-loop partition function is dominated by classical solutions given by equation (6). The above assumption certainly makes sense if $\lambda$ is very small, which as to be true for the one-loop partition function to be a good representation of the full theory. We then followed a similar procedure as for the $\lambda = 0$ case and obtained similar results. This of course should be no surprise since we expect to recover the $\lambda = 0$ results in the limit $\lambda \to 0$. 

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Along some what different lines, 2+1 topological gravity was investigated by introducing a space-time “metric” $G_{ij}$ as a collective coordinate [6]. It was shown that $G_{ij}$ also mimiced correctly the two phases of quantum gravity giving encouraging evidence that $G_{ij}$ is indeed some space-time metric. Thus, whether working with background vierbein and spin connection fields or a space-time “metric” $G_{ij}$, we find that we are able to make the broken and unbroken phases manifest. It would be interesting to understand the relationship between these two approaches in more detail.

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