ON DIMENSIONS OF TANGENT CONES IN LIMIT SPACES WITH LOWER RICCI CURVATURE BOUNDS

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Abstract. We show that if $X$ is a limit of $n$-dimensional Riemannian manifolds with Ricci curvature bounded below and $\gamma$ is a limit geodesic in $X$ then along the interior of $\gamma$ same scale tangent cones $T_{\gamma(t)}X$ have the same dimension in the sense of Colding-Naber.

1. Introduction

In this paper we obtain new restrictions on tangent cones along interiors of limit geodesics in Gromov-Hausdorff limits of manifolds with lower Ricci curvature bounds.

Our main technical result is the following

Theorem 1.1. For any $H \in \mathbb{R}$ and $0 < \delta < 1/3$, there exist $r_0(n, \delta, H), \varepsilon(n, \delta, H) > 0$ such that the following holds:

Suppose that $(M^n, g)$ is a complete $n$-dimensional Riemannian manifold with $\text{Ric}_M \geq (n-1)H$ and let $\gamma : [0, 1] \to M$ be a unit speed minimizing geodesic. Then for any $t_1, t_2 \in (\delta, 1-\delta)$ with $|t_1 - t_2| < \varepsilon$ and any $r < r_0$ there are subsets $C'_i \subset B_r(\gamma(t_i))$ $(i = 1, 2)$ such that

$$\frac{\text{vol} C'_i}{\text{vol} B_r(\gamma(t_i))} \geq 0.9$$

and there exists a bijective map $f_r : C'_1 \to C'_2$ which is 4-Bilipschitz (i.e both $f_r$ and $f_r^{-1}$ are 4-Lipschitz).

Let $d \text{vol}_{i,r} = \frac{d \text{vol}}{\text{vol} B_r(\gamma(t_i))}$ $(i = 1, 2)$ be the normalized volume measures at $\gamma(t_i)$. It’s then obvious that under the assumptions of the theorem we have

$$C^{-1}(n) d \text{vol}_{2,r} \leq (f_r)_* (d \text{vol}_{1,r}) \leq C(n) d \text{vol}_{2,r}$$

for some universal constant $C(n)$. Therefore, using precompactness and a standard Arzela-Ascoli type argument Theorem 1.1 easily yields

Corollary 1.2. Let $M^n_i \to X$ where $\text{Ric}_{M_i} \geq (n-1)H$. Let $\gamma_i : [0, 1] \to X$ be a unit speed geodesic which is a limit of geodesics in $M_i$. Let $t_1 \in (0, 1).$ Then for any $t_2$ sufficiently close to $t_1$ there exist subsets $C_i$ $(i = 1, 2)$ in the unit ball around the origin $o_i$ in the same scale tangent cones $T_{\gamma_i(t)}X$ $(i = 1, 2)$ such that

$$\text{vol}_i C_i \geq 0.9$$

and there exists a map $f : C_1 \to C_2$ satisfying

(i) $f$ is bijective and $4$-Bilipschitz;

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(ii) There exists a universal constant \( C(n) \geq 1 \) such that
\[
C^{-1}(n) \, d \vol_2 \leq f_\#(d \vol_1) \leq C(n) \, d \vol_2.
\]
In particular, \( f_\#(d \vol_1) \) (\( f_\#^{-1}(d \vol_2) \)) is absolutely continuous with respect to \( \vol_2 \) (\( \vol_1 \)).

Here \( d \vol_i \) denotes the renormalized limit volume measure on \( B_1(\alpha_i) \subset T_{\gamma(t_i)} X \). (In particular, \( \vol_i(B_1(\alpha_i)) = 1 \).)

Let \( X \) be a limit of \( n \)-manifolds with Ricci curvature bounded below. Recall that a point \( p \in X \) is called \( k \)-regular if every tangent cone \( T_p X \) is isometric to \( \mathbb{R}^k \). The collection of all \( k \)-regular points is denoted by \( R_k(X) \). (When the space \( X \) in question is clear we will sometimes simply write \( R_k \).

The set of regular points of \( X \) is the union
\[
(1.2) \quad R(X) \equiv \bigcup_i R_k(X).
\]

The set of singular points \( S \) is the complement of the set of regular points. It was proved in \([CC97]\) that \( \vol(S) = 0 \) with respect to any renormalized limit volume measure \( d \vol \) on \( X \). Moreover, by \([CC00b, Theorem 4.15]\), \( \dim_{\text{Haus}} R_k \leq k \) and \( d \vol \) is absolutely continuous on \( R_k(X) \) with respect to the \( k \)-dimensional Hausdorff measure. In particular,
\[
(1.3) \quad \dim_{\text{Haus}} R_k = k \quad \text{if} \quad \vol(R_k) > 0.
\]

It was further shown in \([CN12, Theorem 1.18]\) that there exists unique integer \( k \) such that
\[
(1.4) \quad \vol(R_k) > 0.
\]

Altogether this implies that there exists unique integer \( k \) such that
\[
(1.5) \quad \vol(X \setminus R_k) = 0.
\]

Moreover, it can be shown (Theorem 1.7 below) that this \( k \) is equal to the largest integer \( m \) for which \( R_m \) is non-empty. Following Colding and Naber we will call this \( k \) the dimension of \( X \) and denote it by \( \dim X \). (Note that it is not known to be equal to the Hausdorff dimension of \( X \) in the collapsed case).

Corollary 1.2 immediately implies

**Theorem 1.3.** Under the assumptions of Corollary 1.2 the dimension of same scale tangent cones \( T_{\gamma(t_i)} X \) is constant for \( t \in (0, 1) \).

**Proof.** For \( t_2 \) sufficiently close to \( t_1 = t \) let \( C_i \subset B_1(\alpha_i) \) \((i = 1, 2)\), \( f : C_1 \to C_2 \) be provided by Corollary 1.2. Let \( k_i = \dim T_{\gamma(t_i)} X \). Suppose \( k_1 \neq k_2 \), say \( k_1 < k_2 \). By using (1.5) and Corollary 1.2 (ii) we can assume that \( C_i \subset R_k(T_{\gamma(t_i)} X) \). By above this means that \( \dim_{\text{Haus}} C_i = k_i \).

Since \( f \) is Lipschitz we have \( \dim_{\text{Haus}}(f(C_1)) \leq \dim_{\text{Haus}} C_1 = k_1 \). Since \( d \vol_2 \) is absolutely continuous with respect to the \( k_2 \)-dim Hausdorff measure on \( R_{k_2} \) and \( k_2 > k_1 \) this implies that \( \vol_2(f(C_1)) = 0 \). This is a contradiction since \( \vol_2(f(C_1)) = \vol_2(C_2) > 0 \). \( \square \)

Note that "a cusp" can exist in the limit space of manifolds with lower Ricci curvature bound, for example, a horn \([CC97, Example 8.77]\). Theorem 1.3 indicates that a "cusp" cannot occur in the interior of limit geodesics. In particular, it provides a new way to rule the trumpet \([CC00a, Example 5.5]\) and its generalizations \([CN12, Example 1.15]\). Moreover, we show that the following example cannot arise as a Gromov-Hausdorff limit of manifolds with lower Ricci bound, even through the tangent cones are Hölder (in fact, Lipschitz) continuous along the interior of geodesics. This example cannot be ruled out by previously known results.
Example 1.4. Let $Y = \{(x, y, z) \in \mathbb{R}^3 : z \geq \sqrt{x^4 + |y| - x^2}\}$.

Then $T_{(x,0,0)}Y = \{(y, z) \in \mathbb{R}^2 : z \geq \frac{|y|}{2}\} \times \mathbb{R}$ for $x \neq 0$ and $T_{(0,0,0)}Y = \mathbb{R}_+ \times \mathbb{R}$. Let $X$ be the double of $Y$ along its boundary. Then all points not on the $x$-axis are in $\mathbb{R}^3$ and along the $x$-axis we have that for $x \neq 0$ $T_{(x,0,0)}X = \text{(double of } \{(y, z) \in \mathbb{R}^2 : z \geq \frac{|y|}{2}\}) \times \mathbb{R}$ (i.e. it’s a cone $\times \mathbb{R}$) degenerating to $T_{(0,0,0)}X = \mathbb{R}_+ \times \mathbb{R}$.

So $\dim T_{(0,0,0)}X = 2$ but $\dim T_{(x,0,0)}X = 3$ for $x \neq 0$. Lastly, any segment of the geodesic $\gamma(t) = (t, 0, 0)$ is unique shortest between its end points and hence it’s a limit geodesic if $Y$ is a limit of manifolds with $\text{Ric} \geq -(n - 1)H$. Hence Theorem 1.3 is applicable to $\gamma$ and therefore $X$ is not a limit of $n$-manifolds with $\text{Ric} \geq -(n - 1)H$.

Note that one can further smooth out the metric on $X$ along $\partial Y \\setminus \{x\text{-axis}\}$ to obtain a space $X_1$ with similar properties but which in addition is a smooth Riemannian manifold away from the $x$-axis. In particular $X_1$ is non-branching.

Next we want to mention several semicontinuity results about the Colding-Naber dimension which further suggest that this notion is a natural one.

Let $\mathcal{M}^n$ be the space of Gromov-Hausdorff limits of manifolds with $\text{Ric} \geq -(n - 1)$. Recall the following notions from [CC97]

**Definition 1.5.** Let $X \in \mathcal{M}^n$.

- $\mathcal{WE}_k(X) = \{x \in X \text{ such that some tangent cone } T_xX \text{ splits off isometrically as } \mathbb{R}^k \times Y\}$.
- $\mathcal{E}_k(X) = \{x \in X \text{ such that every tangent cone } T_xX \text{ splits off isometrically as } \mathbb{R}^k \times Y\}$.
- $(\mathcal{WE}_k)_c(X) = \{x \in X \text{ such that there exist } 0 < r \leq 1, Y \text{ and } q \in \mathbb{R}^k \times Y \text{ such that } d_{G-H}(B_r(x), B_{r^2}(q)) < \varepsilon r\}.$
By [CC97, Lemma 2.5] there exists $\varepsilon(n) > 0$ such that if $p \in (\mathcal{WE}_k)^n(X)$ for some $\varepsilon \leq \varepsilon(n)$ then $\text{vol } B_r(p) \cap \mathcal{E}_k > 0$ for all sufficiently small $r$.

Suppose $(X_i, p_i) \in \mathcal{M}^n$, $(X_i, p_i) \to (X, p)$ and $p \in \mathcal{R}_k(X)$. Then $p \in (\mathcal{WE}_k)^n(X_i)$ which obviously implies that $p_i \in (\mathcal{WE}_k)^n(X_i)$ for all large $i$ as well. By above this implies that $\text{vol } \mathcal{E}_k(X_i) > 0$ for all large $i$.

This together with $(1.5)$ yields the following result of Honda proved in [Hon13b, Prop 3.78] using very different tools.

**Theorem 1.6.** [Hon13b, Prop 3.78] Let $X_i \in \mathcal{M}^n$ and $\dim X_i = k$. Let $(X_i, p_i) \xrightarrow{\gamma} (X, p)$. Then $\dim X \leq k$. In other words, the dimension function is lower semicontinuous on $\mathcal{M}^n$ with respect to the Gromov-Hausdorff topology.

This theorem, applied to the convergence $(\frac{1}{r}X, p) \xrightarrow{\gamma} (T_pX, o) = (\mathbb{R}^{\varepsilon}, 0)$ for $p \in \mathcal{R}_k$, immediately gives the following result which also directly follows from [Hon13a, Prop 3.1] and $(1.5)$.

**Theorem 1.7.** Let $X \in \mathcal{M}^n$. Then $\dim X$ is equal to the largest $k$ for which $\mathcal{R}_k(X) \neq \emptyset$.

Another immediate consequence of Theorem 1.6 is the following

**Corollary 1.8.** [Hon13b, Prop 3.78] Let $X \in \mathcal{M}^n$. Then for any $x \in X$ and any tangent cone $T_xX$ it holds that

$$\text{dim } T_xX \leq \dim X.$$ 

It is obvious from $(1.3)$ that for any $X \in \mathcal{M}^n$ we have $\dim X \leq \dim_{\text{Haus}} X$. However, as was mentioned earlier, the following natural question remains open.

**Question 1.9.** Let $X \in \mathcal{M}^n$. Is it true that $\dim X = \dim_{\text{Haus}} X$?

### 1.1. Idea of the proof of Theorem 1.1

Let $\gamma : [0, 1] \to \mathcal{M}^n$ be a unit speed shortest geodesic in an $n$-manifold with $\text{Ric} \geq (n - 1)H$. In [CN12] Colding and Naber constructed a parabolic approximation $h_\varepsilon$ to $d(\cdot, p)$ given as the solution of the heat equation with initial conditions given by $d(\cdot, p)$, appropriately cut off near the end points of $\gamma$ and outside a large ball containing $\gamma$. They showed that $h_\varepsilon$ provides a good approximation to $d^r = d(\cdot, p)$ on an $r$-neighborhood of $\gamma|_{[\delta, 1-\delta]}$. In particular, they showed that

$$\int_0^{\varepsilon} \left( \int_{B_r(\gamma(t))} |\text{Hess}_{\gamma(t)}| \right)^2 dt \leq c(n, \delta, H)$$

for all $r \leq r_0(n, \delta, H)$. They used this to show that for any $t \in (\delta, 1 - \delta)$ most points in $B_r(\gamma(t))$ remain $r$-close to $\gamma$ under the reverse gradient flow of $d^r$ for a definite time $s \leq \varepsilon = \varepsilon(n, \delta, k)$. In section 3 we show that the same holds true for the reverse gradient flow $\phi_\varepsilon$ of $h_{\varepsilon}$. Next, the standard weak type 1-1 inequality for maximum function applied to the inequality $(1.6)$ implies that

$$\int_0^{\varepsilon} \left( \int_{B_r(\gamma(t))} (\text{Mx } |\text{Hess}_{\gamma(t)}| \right)^2 dt \leq c(n, \delta, H)$$

as well. This implies that for every $x$ in a subset $\mathcal{C}^r(\gamma(t))$ in $B_r(\gamma(t))$ of almost full measure the integral $\int_0^1 \text{Mx } |\text{Hess}_{\gamma(t)}| \phi_\varepsilon(x) ds$ is small (see estimate $(4.4)$). Using a small modification of a lemma from [KW11] this implies that for any such point $x$ and any $0 < r_1 \leq r$ most points in $B_{r_1}(x)$ remain $r_1$-close to $\phi_\varepsilon(x)$ for all $s \leq \varepsilon$ under the flow $\phi_\varepsilon$. This then easily implies that $\phi_\varepsilon$ is $B$-Vitali Kaporvitch and Nan Li. 4
1.2. Acknowledgements. We are very grateful to Aaron Naber for helpful conversations and to Shouhei Honda for bringing to our attention results of [Hon13a] and [Hon13b].

2. Preliminaries

In this section we will list most of the technical tools needed for the proof of Theorem 1.1.

2.1. Segment inequality. Throughout the rest of the paper unless indicated otherwise we will assume that all manifolds $M^n$ involved are $n$-dimensional complete Riemannian satisfying

\[ \text{Ric}_{M^n} \geq -(n-1). \]

We will need the following result of Cheeger and Colding

**Theorem 2.1** (Segment inequality). [CC96, Theorem 2.11] Given $n$ and $r_0 > 0$ there exists $c = c(n, r_0)$ such that the following holds.

Let $F : M^n \to \mathbb{R}^+$ be a nonnegative measurable function. Then for any $r \leq r_0$ and $A, B \subset B_r(p)$ it holds

\[ \int_{A \times B} \int_0^{e^{dt(x,y)}} F(y_{x,y}(u)) \, du \, dvol_x \, dvol_y \leq c \cdot r \cdot (vol A + vol B) \int_{B_{2r}(p)} F(z) \, dvol_z, \]

where $y_{x,z}$ denotes a minimal geodesic from $z_1$ to $z_2$.

2.2. Generalized Abresch-Gromoll Inequality. Let $\gamma : [0,L] \to M$ be a minimizing unit speed geodesic with $\gamma(0) = p, \gamma(L) = q$ where $L = d(p,q)$. To simplify notations and exposition from now on we will assume that $L = 1$. Let $d^- = d(\cdot, p), d^+ = d(\cdot, q)$, and let $e = d^+ + d^- - d(p,q)$ be the excess function.

The following result is a direct consequence of [CN12, Theorem 2.8] and, as was observed in [CN12], using the fact that $|\nabla e| \leq 2$ it immediately implies the Abresch-Gromoll estimate [AG90].

**Theorem 2.2** (Generalized Abresch-Gromoll Inequality). [CN12, Theorem 2.8] There exist $c(n, \delta), r_0(n, \delta) > 0$ such that for any $0 < \delta < t < 1 - \delta < 1, 0 < r < r_0$ it holds

\[ \int_{B_r(\gamma(t))} e \leq c(n, \delta) r^2. \]

2.3. Parabolic approximation for distance functions. Fix $\delta > 0$ and let $h^\pm_\delta$ be parabolic approximations to $d^\pm$ constructed in [CN12]. They are given by the solutions to the heat equations

\[ \frac{d}{dt} h^\pm_\delta = \Delta h^\pm_\delta, \quad h^\pm_\delta(x) = \lambda(x) \cdot d^\pm(x) \]

for appropriately constructed cutoff function $\lambda$. We will need the following properties of $h_\delta$ established in [CN12].

**Lemma 2.3.** [CN12, Lemma 2.10] There exists $c(n, \delta)$ such that

(2.1) \[ \Delta h^\pm_\delta \leq c(n, \delta). \]

**Theorem 2.4.** [CN12, Theorem 2.19] There exist $c(n, \delta), r_0(n, \delta) > 0$ such that for all $r_1 \leq r_0$ there exists $r \in [\frac{r_1}{2}, 2r_1]$ such that the following properties are satisfied

(i) $|h^\pm_\delta - d^\pm(x)| \leq c r^2$ for any $x \in B_2(p) \setminus (B_\delta(p) \cup B_\delta(q))$ with $e(x) \leq r^2$

(ii) $\int_{B_r(\gamma(t))} \|\nabla h^\pm_\delta\|^2 - 1 \leq c r$

(iii) $\int_{B_r(\gamma(t))} \|\nabla h^\pm_\delta\|^2 - 1 \leq c r^2$

(iv) $\int_{B_r(\gamma(t))} |\text{Hess}_{h^\pm_\delta}|^2 \leq c$. 

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2.4. First Variation formula. We will need the following lemma (cf. [CN12, Lemma 3.4]).

Lemma 2.5. Let $X$ be a smooth vector field on $M$ and let $\sigma_1(t), \sigma_2(t)$ be smooth curves. Let $p = \sigma_1(0), q = \sigma_2(0)$. Then

$$\left| \frac{d^*}{dt} d(\sigma_1(t), \sigma_2(t)) \big|_{t=0} \right| \leq |X(p) - \sigma_1'(0)| + |X(q) - \sigma_2'(0)| + \int_{\gamma_{pq}} |\nabla X|,$$

where $\gamma_{pq} : [0,d(p,q)] \to M$ is a shortest geodesic from $p$ to $q$. Here $|\nabla X|$ means the norm of the full covariant derivative of $X$ i.e. norm of the map $v \mapsto \nabla_v X$. In particular, if $h : M \to \mathbb{R}$ is smooth and $X = \nabla h$, then

$$\left| \frac{d^*}{dt} d(\sigma_1(t), \sigma_2(t)) \big|_{t=0} \right| \leq |X(p) - \sigma_1'(0)| + |X(q) - \sigma_2'(0)| + \int_{\gamma_{pq}} |\text{Hess}_h|.$$

Proof. The lemma easily follows from the first variation formula for distance functions and the triangle inequality. ~\hfill \Box

2.5. Maximum function. Let $f : M \to \mathbb{R}$ be a nonnegative function. Consider the maximum function $M_x f(p) = \sup_{\rho \leq r} \int_{B_\rho(p)} f$ for $\rho \in (0,4]$. We’ll set $M_x f = M x_1 f$.

The following lemma is well-known [Ste93, p. 12].

Lemma 2.6 (Weak type 1-1 inequality). Suppose $(M^n, g)$ has $\text{Ric} \geq -(n-1)$ and let $f : M \to \mathbb{R}$ be a nonnegative function. Then the following holds.

(i) If $f \in L^\alpha(M)$ with $\alpha \geq 1$ then $M_x f$ is finite almost everywhere.

(ii) If $f \in L^1(M)$ then $\text{vol}(\{x \in M : M_x f(x) > c\}) \leq \frac{C(n)}{c} \int_M f$ for any $c > 0$.

(iii) If $f \in L^\alpha(M)$ with $\alpha > 1$ then $M_x f \in L^\alpha(M)$ and $\|M_x f\|_{\alpha} \leq C(n, \alpha)\|f\|_{\alpha}$.

This lemma easily generalizes to functions defined on subsets as follows:

Corollary 2.7. Let $\text{Ric}_{M^n} \geq -(n-1)$ and $f : M \to \mathbb{R}^+$ be measurable. Let $A \subset M$ be measurable such that $f \in L^\alpha(U_\rho(A))$ where $\alpha > 1$. Here $U_\rho(A)$ denotes the $\rho$-neighborhood of $A$.

Then

$$\|M_x f\|_{L^\alpha(A)} \leq C(n, \alpha)\|f\|_{L^\alpha(U_\rho(A))}.$$

Proof. Let $\tilde{f} = f \cdot \chi_{U_\rho(A)}$. Obviously, $M_x f(x) = M_x \tilde{f}(x)$ for any $x \in A$. The result follows by applying Lemma 2.6 (iii) to $\tilde{f}$. ~\hfill \Box

3. Gradient flow of the parabolic approximation

Let $\psi_s$ be the reverse gradient flow of $h = h^{-1}_t$ (i.e. the gradient flow of $-h^{-1}_t$) and let $\psi_s$ be the reverse gradient flow of $d'$. We first want to show that for most points $x \in B_{r_t}(y(t))$ we have that $\psi_s(x) \in B_{2r}(\gamma(t-s))$ for all $t \in (\delta, 1-\delta)$ and $s \in [0, \epsilon]$ for some uniform $\epsilon = \epsilon(n, \delta)$.

Note that this (and more) is already known for $\psi_s$ by [CN12]. Following Colding-Naber we use the following

Definition 3.1. For $0 < s < t < 1$ define the set $\mathcal{A}_s(t) \equiv \{ z \in B_{r_t}(\gamma(t)) : \psi_s(z) \in B_{2r}(\gamma(t-u)) \forall 0 \leq u \leq s \}$. Similarly, we define $\mathcal{B}_s(t) \equiv \{ z \in B_{r_t}(\gamma(t)) : \phi_s(z) \in B_{2r}(\gamma(t-u)) \forall 0 \leq u \leq s \}$. An important technical tool used to prove the main results of [CN12] is the following
Proposition 3.2. [CN12, Proposition 3.6] There exist \( r_0(n, \delta) \) and \( \epsilon_0(n, \delta) \) such that if \( t \in (\delta, 1 - \delta) \) and \( \epsilon \leq \epsilon_0 \) then \( \forall r \leq r_0 \) as in Theorem 2.4 we have
\[
\frac{1}{2} \leq \frac{\text{vol}(\mathcal{A}_t^\epsilon(r))}{\text{vol}(B_t(y(t)))}.
\]

Unlike Colding-Naber we prefer to work with the gradient flow of the parabolic approximation \( h \) rather than the gradient flows of \( d^h \), because the gradient flow of \( h \) provides better distance distortion estimates since in that case the two terms outside the integral in Lemma 2.5 vanish and the resulting inequality scales better in the estimates involving maximum function (see Lemma 4.2 below). Therefore, our first order of business is to establish the following lemma which says that Proposition 3.2 holds for the gradient flow of \( -h \) as well:

Lemma 3.3. There exists \( r_1(n, \delta) \) and \( \epsilon_1(n, \delta) \) such that if \( \delta < t - \epsilon < 1 - \delta \) with \( \epsilon \leq \epsilon_1 \) then \( \forall r \leq r_1 \) we have
\[
\frac{1}{2} \leq \frac{\text{vol}(\mathcal{A}_t^{\epsilon}(r))}{\text{vol}(B_t(y(t)))}.
\]

The proof of Proposition 3.2 uses bootstrapping in \( \epsilon, r \) starting with infinitesimally small (depending on \( M! \)) \( r \) (cf. Lemma 4.2 below) for which the claim easily follows from Bochner’s formula applied to \( d^\gamma \) along \( \gamma \). We don’t utilize bootstarping in \( r \) and instead use that the result has already been established for the gradient flow of \( -d^\gamma \).

Proof. Of course, we only need to prove the second inequality as the first one holds by Proposition 3.2 for some \( r_0(n, \delta), \epsilon_0(n, \delta) > 0 \). By possibly making \( r_0 \) smaller we can ensure that it satisfies Theorem 2.4.

Let \( 0 < \epsilon < \epsilon_0 \) be small (how small it will be chosen later). Let
\[
S_t \equiv \left\{ 0 \leq s < t - \delta : \frac{1}{2} < \frac{\text{vol}(B_t^\epsilon(r))}{\text{vol}(B_t(y(t)))} \right\}.
\]

We wish to show that \( S_t \) contains \([0, \epsilon]\) for some uniform \( \epsilon = \epsilon(n) \). Obviously \( S_t \) is open in \([0, \epsilon]\) so it’s enough to show that it’s also closed. To establish this it’s enough to show that if \( \epsilon' \leq \epsilon \) and \([0, \epsilon'] \subset S_t \) then \( \epsilon' \in S_t \).

For any \( 0 < s < t \) we define \( \bar{\mathcal{A}}_t^\epsilon \) to be the characteristic function of the set \( \mathcal{A}_t^\epsilon(r) \times \mathcal{B}_t(y) \).

The same argument as in [CN12] shows that
\[
\int_{B_t(y(t)) \times B_t(y(t))} \bar{\mathcal{A}}_t^\epsilon(x, y) \left( \int_{\gamma_{t, x} \times \gamma_{t, y}} |\text{Hess}_h| \right) d\text{vol}_x \ d\text{vol}_y \leq C(n, \delta) r \left( \frac{\text{vol}(B_t(y(t - s)))}{\text{vol}(B_t(y(t)))} \right)^2 \int_{B_t(y(t - s))} |\text{Hess}_h|.
\]

Indeed, we have
\[
\int_{B_t(y(t)) \times B_t(y(t))} \bar{\mathcal{A}}_t^\epsilon(x, y) \left( \int_{\gamma_{t, x} \times \gamma_{t, y}} |\text{Hess}_h| \right) d\text{vol}_x \ d\text{vol}_y = \int_{\mathcal{A}_t^\epsilon(r) \times \mathcal{B}_t(y)} \left( \int_{\gamma_{t, x} \times \gamma_{t, y}} |\text{Hess}_h| \right) d\text{vol}_x \ d\text{vol}_y \leq C(n, \delta) \int_{\psi_t(\mathcal{A}_t^\epsilon(r)) \times \phi_t(\mathcal{B}_t(y))} \left( \int_{\gamma_{t, x}, \gamma_{t, y}} |\text{Hess}_h| \right) d\text{vol}_x \ d\text{vol}_y,
\]
where the last inequality follows from the fact that $\Delta h \leq c(n, \delta)$ by Lemma 2.3 and hence the Jacobian of $\phi_s$ satisfies
\begin{equation}
J_{\phi_s} \geq e^{C(n, \delta) s}.
\end{equation}

Similar inequality holds for $\psi_s$ by Bishop-Gromov volume comparison. Since $\psi_s(\mathcal{A}_i^s(r), \phi_s(B_i^s(r)) \subseteq B_{\mathcal{A}_i^s(\gamma(t-s))}$ by definition, by the segment inequality (Theorem 2.1) we have
\begin{equation}
\int_{\phi_s(\mathcal{A}_i^s(r), \phi_s(B_i^s(r))} \left( \int_{\gamma_{\phi_s}} |\text{Hess}_h| \right) d\vol_x \ d\vol_y \\
\leq C(n, \delta) r \left[ \vol(\psi_s(\mathcal{A}_i^s(r))) + \vol(\phi_s(B_i^s(r))) \right] \int_{B_s(\gamma(t-s))} |\text{Hess}_h| \\
\leq C(n, \delta) r \vol(B_s(\gamma(t-s))) \int_{B_s(\gamma(t-s))} |\text{Hess}_h| \\
= C(n, \delta) r \vol(B_s(\gamma(t-s)))^2 \int_{B_s(\gamma(t-s))} |\text{Hess}_h|.
\end{equation}

where the last inequality follows by Bishop-Gromov. Thus,
\begin{equation}
\int_{B_s(\gamma(t-s))} \tilde{c}_i(x, y) \left( \int_{Y_{\phi_s(\gamma(t-s))}} |\text{Hess}_h| \right) d\vol_x \ d\vol_y \\
\leq C(n, \delta) r \int_{B_s(\gamma(t-s))} |\text{Hess}_h|.
\end{equation}

Dividing by $\vol(B_s(\gamma(t)))^2$ we get (3.2). By [CN12, Cor 3.7] we have that
\begin{equation}
C^{-1} \leq \frac{\vol(B_s(\gamma(t-s)))}{\vol(B_s(\gamma(t)))} \leq C.
\end{equation}

for some universal $C = C(n, \delta)$ and therefore
\begin{equation}
\int_{B_s(\gamma(t-s))} \tilde{c}_i(x, y) \left( \int_{Y_{\phi_s(\gamma(t-s))}} |\text{Hess}_h| \right) d\vol_x \ d\vol_y \\
\leq C(n, \delta) r \int_{B_s(\gamma(t-s))} |\text{Hess}_h|.
\end{equation}

Let
\begin{equation}
\tilde{I}_c = \int_{B_s(\gamma(t-s))} \tilde{c}_i(x, y) \left( \int_{Y_{\phi_s(\gamma(t-s))}} |\text{Hess}_h| \right) d\vol_x \ d\vol_y = \int_0^c \int_{B_s(\gamma(t-s))} \tilde{c}_i(x, y) \left( \int_{Y_{\phi_s(\gamma(t-s))}} |\text{Hess}_h| \right) d\vol_x \ d\vol_y \ ds.
\end{equation}

Then by (3.8) and Theorem 2.4 we have that
\begin{equation}
\tilde{I}_c = \int_0^c \int_{B_s(\gamma(t-s))} \tilde{c}_i(x, y) \left( \int_{Y_{\phi_s(\gamma(t-s))}} |\text{Hess}_h| \right) d\vol_x \ d\vol_y \ ds \\
\leq C(n, \delta) r \int_0^c \int_{B_s(\gamma(t-s))} |\text{Hess}_h| d\vol_y \ ds \\
\leq C(n, \delta) r \sqrt{c^2} \left( \int_0^c \int_{B_s(\gamma(t-s))} |\text{Hess}_h|^2 d\vol_x \ ds \right)^{1/2} \\
\leq C(n, \delta) r \sqrt{c^2}.
\end{equation}
Let
\[ T^r_\eta \equiv \left\{ x \in B_r(y(t)) \colon x \in \mathcal{A}_\epsilon(r) \text{ and } \int_{\|x\|B_r(y(t))}^{\gamma} \int_0^{\gamma} \bar{c}^\prime_s(x, y) \left( \int_{\gamma_\epsilon(x, y, \theta)} |\text{Hess}_s| \right) ds \leq \eta^{-1} h \right\}, \]
and for \( x \in T^r_\eta \) let us define
\[ (3.11) \quad \bar{T}^r_\eta(x) \equiv \left\{ y \in B_r(y(t)) : \int_0^{\gamma} \bar{c}^\prime_s(x, y) \left( \int_{\gamma_\epsilon(x, y, \theta)} |\text{Hess}_s| \right) ds \leq \eta^{-2} h \right\}. \]

Here \( \eta = \eta(n, \delta, d(p, q)) > 0 \) is small and chosen first. Then \( \epsilon \) is chosen later depending on \( \eta \). By [CN12, (3.37)] we can assume that
\[ (3.12) \quad \frac{\text{vol}(\mathcal{A}_\epsilon(r))}{\text{vol}(B_r(y(t)))} \geq 1 - C(n, \delta)\eta. \]

Therefore, by construction we have that
\[ (3.13) \quad \frac{\text{vol}(\bar{T}^r_\eta)}{\text{vol}(B_r(y(t)))} \geq 1 - C(n, \delta)\eta, \]

and hence
\[ (3.14) \quad \frac{\text{vol}(\bar{T}^r_\eta(x))}{\text{vol}(B_r(y(t)))} \geq 1 - C(n, \delta)\eta, \quad \forall x \in T^r_\eta. \]

We choose \( \eta \) so that \( C(n, \delta)\eta \ll 1 \) in (3.13) and (3.14).

Let \( x \in \bar{T}^r_\eta \cap \bar{T}^r_\eta^{100} \) (this intersection is non-empty for small \( \eta = \eta(n) \) by Bishop-Gromov) and let \( y \in \bar{T}^r_\eta(x) \). We will fix \( \eta > 0 \) satisfying the above conditions from now on. We claim that then \( y \in \bar{B}_r^\epsilon(r) \).

Indeed \( d(\phi_s(x), y(t-s)) \leq r/50 \) for all \( s \leq \epsilon' \) since \( x \in \bar{T}^r_\eta^{100} \subset \mathcal{A}_\epsilon(r/100) \). So by the triangle inequality it’s enough to show that \( d(\phi_s(x), \phi_s(y)) \leq 1.1r \) for any \( s \leq \epsilon' \).

Let \( S = \{ s \leq \epsilon' : y \in \bar{B}_r^\epsilon(r) \} \). This set is obviously open and connected in \([0, \epsilon'] \). We claim that \( S = [0, \epsilon'] \). Let \( \bar{\epsilon} = \sup\{ s : s \in S \} \).

Note that for any \( 0 < s < \bar{\epsilon} \) we have that \( \bar{c}^\prime_s(x, y) = 1 \). Therefore, by (3.10) and (3.11) for any \( 0 < s < \bar{\epsilon} \) we have
\[ (3.15) \quad \int_0^{\epsilon'} \left( \int_{\gamma_\epsilon(x, \theta)} |\text{Hess}_s| \right) ds \leq \eta^{-2} \bar{T}^r_\eta \leq \frac{C(n, \delta)}{\eta^2} \sqrt{\epsilon^2 r} \leq 0.001r \]
if \( \epsilon = \epsilon(\eta) \) is chosen small enough.

Next, recall that by [CN12, Lemma 2.20(3)] we have that for any \( x \in B_r(y(t)) \),
\[ (3.16) \quad \int_0^{\epsilon'} (\nabla h(\phi_s(x)) - \nabla \bar{h}(\phi_s(x))) ds \leq c(n, \delta) \sqrt{\bar{\epsilon}} (\sqrt{\epsilon} + r) \]

Further, by Theorem 2.2 we know that
\[ (3.17) \quad \int_{B_r(y(t))} \epsilon \leq c(n, \delta) r^2. \]

Therefore, without losing generality by making the sets \( \bar{T}^r_\eta \) slightly smaller we can assume that for any \( x \in \bar{T}^r_\eta \) we have
\[ (3.18) \quad \epsilon(x) \leq \eta^{-1} r^2. \]
Thus, for all $x \in \tilde{T}_r^\eta$ we have

\begin{equation}
\int_0^\infty |\nabla h(\psi_u(x) - \nabla \delta \cdot (\eta^{-1/2} r + r) < 0.001 r
\end{equation}

if $\varepsilon = \varepsilon(n, \delta, \eta)$ is small enough. Therefore, by Lemma 2.5 and using (3.15) and (3.19) we get that

\begin{equation}
d(\psi_s(x), \phi_s(y)) \leq 0.002 r + d(x, y) < 1.1 r.
\end{equation}

By the triangle inequality,

\begin{equation}
d(\phi_s(x), \gamma(t-s)) \leq r/50 + 1.1 r \leq 1.5 r < 2 r.
\end{equation}

By continuity the same holds for \bar{s} and hence \bar{s} \in S. Thus $S$ is both open and closed in $[0, \varepsilon']$ and therefore $S = [0, \varepsilon']$. Unwinding this further we see that this means that $\tilde{T}_r^\eta(x) \subset B_{s'}(r)$. Therefore, by (3.14)

\begin{equation}
\frac{1}{2} \leq \frac{\text{vol}(B_{s'}(r))}{\text{vol}(B_r(\gamma(t)))}
\end{equation}

when $\eta$ was chosen small enough so that $C(n, \delta) \eta < 1/2$. Hence $\varepsilon' \in S$. Therefore, $S$ is both open and closed and $\varepsilon' = \varepsilon$.

\begin{remark}
It follows from the proof above that the constant $\varepsilon_1$ can be chosen explicitly depending on the constant $c$ in Theorem 2.4 and $\varepsilon_0$ in Proposition 3.2.
\end{remark}

The proof of Lemma 3.3 shows that $\tilde{T}_r^\eta(x) \subset B_{s'}(r)$ for appropriately chosen $\varepsilon$ depending on $\eta$. In view of (3.13) this means that the conclusion can be strengthened as follows (cf. [CN12]).

\begin{lemma}
For every $\eta \leq \eta_0(n, \delta)$ and $r \leq r_0(n, \delta)$ that there exists $\varepsilon = \varepsilon(n, \eta, \delta)$ such that the set $B_{s'}(r) \equiv \{ z \in B_r(\gamma(t)) : \phi_s(z) \in B_{2s}(\gamma(t-s)) \} \forall 0 \leq s \leq \varepsilon$ satisfies

\[ \frac{\text{vol}(B_{s'}(r))}{\text{vol}(B_r(\gamma(t)))} \geq 1 - C(n, \delta) \eta. \]

When $\eta$ is sufficiently small this means that most points in $B_r(\gamma(t))$ remain close to the geodesic $\gamma$ under the flow $\phi_s$. Also, because of the above lemma, (3.4) and (3.7) we have the following

\begin{lemma}
Let $F : M \to \mathbb{R}$ be a nonnegative measurable function. Then

\[ \int_{B_{s'}(r)} F(\phi_s(x)) d\text{vol}_x \leq C(n, \delta, \eta) \int_{B_{2s}(\gamma(t-s))} F(\bar{s}) d\text{vol}_{\bar{s}} \]

for any $s \leq \varepsilon$ as in Lemma 3.5.
\end{lemma}

\begin{remark}
It is obvious that all results of this section concerning the flow of $-h_r^\eta$ are also true for the flow of $-h_r^\eta$.
\end{remark}

4. Bilipschitz control.

The goal of this section is to prove the following somewhat stronger version of Theorem 1.1.
\textbf{Theorem 4.1.} Given $H \in \mathbb{R}$, $0 < \delta < 1/3, 0 < \eta < 1$ there exist $r_0(n, \delta, H, \eta), \epsilon(n, \delta, H, \eta) > 0$ such that the following holds:

Suppose $(M^n, g)$ is complete with $\text{Ric}_{M^n} \geq (n-1)H$ and let $\gamma : [0, 1] \to M^n$ be a unit speed minimizing geodesic. Then for any $t_1, t_2 \in (\delta, 1-\delta)$ with $|t_1 - t_2| < \epsilon$ and any $r < r_0$ there are subsets $C_1' \subset B_r(\gamma(t_i)) (i = 1, 2)$ such that

$$\frac{\text{vol} C_i'}{\text{vol} B_r(\gamma(t_i))} \geq 1 - \eta$$

and there exists a 4-Bilipschitz map $f_i : C_1' \to C_2'$.

As before, to simplify notation we will assume that $H = 1$ and $\text{Ric}_{M^n} \geq -(n-1)$.

Corollary 2.7 essentially means that all estimates involving integrals of $|\text{Hess}_{h_1}|$ from the previous section remain true for $M_{x_p} [\text{Hess}_h]$. In particular, for $r_0$ as in Theorem 2.4 and any $r \leq r_0/10$ we have

$$\int_0^{1-\delta} \left( \int_{B_r(\gamma(t))} |\text{Hess}_{h(t)}|^2 \right) dt \leq \frac{C}{\delta} .$$

By Corollary 2.7 this implies that

$$\int_0^{1-\delta} \left( \int_{B_r(\gamma(t))} |\text{Hess}_h|^2 \right) dt \leq \frac{C}{\delta} .$$

where $h = h_{16r}^\delta$. It is clear that all results from the previous section work for this $h$ as well as $h_{16r}^\delta$. Therefore, everywhere in the previous section where we used (4.1) we could have used (4.2) instead. Indeed, we have for any $2\delta < t < 1 - 2\delta$,

$$\int_0^t \left( \int_{B_r(\gamma(t+s))} |\text{Hess}_h|^2 \right) ds \leq \sqrt{t} \int_0^t \left( \int_{B_r(\gamma(t+s))} |\text{Hess}_h|^2 \right) ds \leq C(n, \delta) \sqrt{t} .$$

In view of Lemma 3.6 this implies that

$$\int_0^t \left( \int_{B_r(\gamma(t+s))} |\text{Hess}_h(\phi_s(x))|^2 \right) ds \leq C(n, \delta, \eta) \sqrt{t} .$$

This means that for most points $x \in B^t_r$ the integral $\int_{0}^{t} |\text{Hess}_h(\phi_s(x))| ds$ is bounded. More precisely, given any $0 < \nu < 1$ let

$$B^t_{x}(r, \nu) \equiv \left\{ x \in B^t_{x}(r) : \int_{0}^{t} |\text{Hess}_h(\phi_s(x))| ds \leq \frac{C(n, \delta, \eta) \sqrt{t}}{\nu} \right\} .$$

Then

$$\frac{\text{vol} B^t_{x}(r, \nu)}{\text{vol} B^t_{x}(r)} > 1 - \nu .$$

We will need the following slight modification of Lemma 3.7 from [KW11]
Lemma 4.2. Given $c > 0$, there exists (explicit) $C = C(n, \lambda)$ such that the following holds. Suppose $(M^n, g)$ has $\text{Ric}_{M^n} \geq -(n-1)$ and $X'$ is a vector field with compact support, which depends on time but is piecewise constant in time. Let $c(t)$ be the integral curve of $X'$ with $c(0) = p_0 \in M^n$ and assume that $\text{div} X' \geq -\lambda$ on $B_{10}(c(t))$ for all $t \in [0, 1]$.

Let $\phi_t$ be the flow of $X'$. Define the distortion function $d_t(t)(p, q)$ of the flow on scale $r$ by the formula
\begin{equation}
  d_t(t)(p, q) := \min \{ r, \max_{0 \leq t < T} d(p, q) - d(\phi_t(p), \phi_t(q)) \}.
\end{equation}

Put $\varepsilon = \int_0^1 \text{Mx}_1(||\nabla X'||)(c(t)) \, dt$. Then for any $r \leq 1/10$ we have
\begin{equation}
  \int_{B_r(p_0) \setminus B_r(p_0)} d_t(1)(p, q) \, d \text{vol}_p \, d \text{vol}_q \leq Cr \cdot \varepsilon
\end{equation}
and there exists $B_r(p_0)' \subset B_r(p_0)$ such that
\begin{equation}
  \frac{\text{vol}(B_r(p_0)')}{\text{vol}(B_r(p_0))} \geq 1 - C\varepsilon
\end{equation}
and $\phi_t(B_r(p_0)') \subset B_{2r}(c(t))$ for all $t \in [0, 1]$.

This lemma immediately implies

Corollary 4.3. Under the assumptions of Lemma 4.2 for every $r \leq 1/10$ there exists $B_r(p_0)'' \subset B_r(p_0)$ such that
\begin{equation}
  \frac{\text{vol}(B_r(p_0)'')}{\text{vol}(B_r(p_0))} \geq 1 - C\varepsilon
\end{equation}
\begin{equation}
  \phi_t(B_r(p_0)''') \subset B_{2r}(c(t)) \text{ for all } t \in [0, 1]
\end{equation}
and
\begin{equation}
  \forall p, q \in B_r(p_0)'', \quad d_t(1)(p, q) \leq Cr \cdot \varepsilon.
\end{equation}

In [KW11] Lemma 4.2 is stated for divergence free vector fields. We want to apply it to $X = -\nabla h$ which is not divergence free. However, recall that it does satisfy $\text{div} X \geq -\lambda$ and hence the Jacobian of its flow map satisfies
\begin{equation}
  J_{\phi_t} \geq e^{-\lambda s}
\end{equation}
pointwise.

The proof given in [KW11] goes through verbatim with a straightforward change in one place using (4.10) instead of the flow of $X$ being volume-preserving. We include the proof here for reader’s convenience.

Proof of Lemma 4.2. We prove the statement for a constant in time vector field $X'$. The general case is completely analogous except for additional notational problems.

Notice that all estimates are trivial if $\varepsilon \geq \frac{\lambda}{2}$. Therefore it suffices to prove the statement with a universal constant $C(n, \lambda)$ for all $\varepsilon \leq \varepsilon_0(n, \lambda)$. We put $\varepsilon_0 = \frac{\lambda}{2n}$ and determine $C \geq 2$ in the process. We again proceed by induction on the size of $r$.

Notice that the differential of $\phi_t$ at $c(0)$ is Bilipschitz with Bilipschitz constant
\begin{equation}
  \frac{\int_0^t ||\nabla X'||(c(t)) \, dt}{1 + 2\varepsilon}.
\end{equation}

Thus the Lemma holds for very small $r$. 

\[\varepsilon\]
Suppose the result holds for some $r/10 \leq 1/100$. It suffices to prove that it then holds for $r$. By induction assumption we know that for any $t$ there exists $B_{r/10}(c(t)) \subset B_{r/10}(c(t))$ such that for any $s \in [-t, 1-t]$ we have

\[ \text{vol}(B_{r/10}(c(t))) \geq (1-C\varepsilon) \text{vol}(B_{r/10}(c(t))) \geq \frac{1}{2} \text{vol}(B_{r/10}(c(t))) \]  

and

\[ \phi_s(B_{r/10}(c(t))) \subset B_{r/5}(c(t+s)), \]

where we used $\varepsilon \leq \frac{1}{7C}$ in the inequality. This easily implies that $\text{vol}(B_{r/10}(c(t)))$ are comparable for all $t$. More precisely, for any $t_1, t_2 \in [0, 1]$ we have that

\[ \frac{1}{c_0} \text{vol } B_{r/10}(c(t_1)) \leq \text{vol } B_{r/10}(c(t_2)) \leq C_0 \text{vol } B_{r/10}(c(t_1)) \]

with a computable universal $C_0 = C_0(n)$. Put

\[ h(s) = \int_{B_{r/10}(c(0)) \times B_r(c(0))} d\mathcal{t}_s(p, q) \ d \text{vol}_p \ d \text{vol}_q, \]

\[ U_s \ := \ \{(p, q) \in B_{r/10}(c(0)) \times B_r(c(0)) \mid d\mathcal{t}_s(p, q) < r\}, \]

\[ \phi_s(U_s) \ := \ \{(\phi_s(p), \phi_s(q)) \mid (p, q) \in U_s\}, \text{ and} \]

\[ d\mathcal{t}_s'(p, q) \ := \ \limsup_{h \rightarrow 0} \frac{d\mathcal{t}_s((p+h)(p, q) - d\mathcal{t}_s(p, q)).}{h}. \]

As $d\mathcal{t}_s(t) \leq r$ is monotonously increasing, we deduce that if $d\mathcal{t}_s(p, q) = r$, then $d\mathcal{t}_s'(p, q) = 0$. Since $d\mathcal{t}_s(p + h)(p, q) \leq d\mathcal{t}_s(p, q) + d\mathcal{t}_s(h)(\phi_s(p), \phi_s(q))$ and $\phi_s$ satisfies $J_{\phi_s} \geq e^{-\lambda t}$ it follows

\[ h'(s) \leq \int_{U_s} d\mathcal{t}_s'(\phi_1(x), \phi_2(y)) \leq e^{\lambda t} \int_{\phi_s(U_s)} d\mathcal{t}_s'(0)(p, q) \]

\[ \leq e^{\lambda t} \frac{4 \text{vol } B_{r/10}(c(0))}{\text{vol } B_{r/10}(c(0))} \int_{B_{r/10}(c(0))} d\mathcal{t}_s'(0)(p, q), \]

where we used that $\phi_s(B_{r/10}(p_0)) \subset \phi_s(U_s) \subset B_{3\varepsilon}(c(s))^2$. We would like to point out that using $J_{\phi_s} \geq e^{-\lambda t}$ instead of $J_{\phi_s} = 1$ (which is true for flows of harmonic maps) in the above inequality is the only place where the proof of Lemma 4.2 differs from the proof of [KW11, Lemma 3.7].

If $p$ is not in the cut locus of $q$ and $\gamma_{pq} : [0, 1] \rightarrow M$ is a minimal geodesic between $p$ and $q$, then by Lemma 2.5

\[ d\mathcal{t}_s'(0)(p, q) \leq d(p, q) + \int_0^1 ||\nabla X||(\gamma_{pq}(t)) \ dt. \]

Combining the last two inequalities with the segment inequality we deduce

\[ h'(s) \leq C_1(n, \lambda) r \int_{B_{r/10}(c(s))} ||\nabla X|| \]

\[ \leq C_1(n, \lambda) r \max ||\nabla X||(c(s)). \]

Note that the choice of the constant $C_1(n, \lambda)$ can be made explicit and independent of the induction assumption. We deduce $h(1) \leq C_1(n, \lambda) r e$ and thus the subset

\[ B_r(p_0) := \{p \in B_r(p_0) \mid \int_{B_{r/10}(p_0)} d\mathcal{t}_s(1)(p, q) \ d \text{vol}_q \leq r/2\} \]
Bishop-Gromov, we can find expected, in the statement of the theorem can be replaced by any constant $\beta > 0$. The proof of Theorem 4.1 can be easily adapted to show that the Bilipschitz constant 4 in the definition of \( \mathcal{B}_2(r) \) is 4-Lipschitz on \( d \) and \( C \) This completes the induction step with \( (4.24) \).

This means that we can apply Corollary 4.3 at all such points.

\[
(4.21) \quad \text{vol}(B_r(p_0)) \geq (1 - 2C_1(n, \lambda)\varepsilon) \text{vol}(B_r(p_0)).
\]

It is elementary to check that

\[
(4.22) \quad \phi_t(B_r(p_0)) \subset B_{2r}(c(t)) \text{ for all } t \in [0, 1].
\]

Then arguing as before we estimate that

\[
(4.23) \quad \int_{B_r(p_0) \times B_r(p_0)} dt_t(1)(p, q) d\text{vol}_p d\text{vol}_q \leq C_2(n, \lambda) \cdot r \cdot \varepsilon.
\]

Using \( dt_t(1) \leq r \) and the volume estimate \( (4.21) \) this gives

\[
(4.24) \quad \int_{B_r(p_0)} dt_t(1)(p, q) d\text{vol}_p d\text{vol}_q \leq C_2 \cdot r \cdot \varepsilon + 2\varepsilon C_1 \varepsilon =: C_3 \varepsilon r.
\]

This completes the induction step with \( C(n, \lambda) = C_3 \) and \( \varepsilon_0 = \frac{\varepsilon}{2C_3} \). In order to remove the restriction \( \varepsilon \leq \varepsilon_0 \) one can just increase \( C(n, \lambda) \) by the factor 4, as indicated at the beginning.

\[\square\]

Remark 4.4. It’s obvious from the proof that Lemma 4.2 and Corollary 4.3 remain valid for \( r \leq \rho/10 \) if we change \( M_{x_1} \) to \( M_{x_\rho} \) in the assumptions.

We can now finish the proof of Theorem 4.1 by establishing the following

Lemma 4.5. Fix a small \( R < r_0/10 \) and let \( \nu = \eta \ll 1 \) and let \( \varepsilon \leq \frac{C(n, \lambda)\sqrt{\nu}}{\nu} \) satisfies Lemma 3.5. Then for any \( s \leq \varepsilon \) the map \( \phi_s(\partial B_r, v) \) is 4-Bilipschitz onto its image.

Proof. Since \( \varepsilon \leq \frac{C(n, \lambda)\sqrt{\nu}}{\nu} \) we have \( \frac{C(n, \lambda)\sqrt{\nu}}{\nu} \leq \nu \) and therefore by the definition of \( \mathcal{B}_r(\partial B_r, v) \) for all \( x \in \mathcal{B}_r(\partial B_r, v) \) it holds

\[
(4.25) \quad \int_0^\infty M_{x_\rho}|\text{Hess}_h(\phi_s(x))| ds \leq \nu.
\]

This means that we can apply Corollary 4.3 at all such points.

Let \( x, y \in \mathcal{B}_r(\partial B_r, v) \). By Bishop-Gromov we can assume that \( B_r(x)'' \cap B_r(y)'' \neq \emptyset \) provided \( v = v(n) \) is chosen small enough. Pick any \( z \in B_r(x)'' \cap B_r(y)'' \neq \emptyset \). Then by Corollary 4.3 \( \phi_t(z) \in B_{2r}(\phi_t(x)) \cap B_{2r}(\phi_t(y)) \) i.e \( d(\phi_t(x), \phi_t(z)) \leq 2r \) and \( d(\phi_t(y), \phi_t(z)) \leq 2r \) and therefore \( d(\phi_t(x), \phi_t(y)) \leq 4r = 2.4d(x, y) \). This shows that \( \phi_t \) is 4-Lipschitz on \( \mathcal{B}_r(\partial B_r, v) \).

To get that it’s Bilipschitz let \( r_1 = 0.99d(x, y) \) and \( r_2 = 0.02d(x, y) \). As before, using Bishop-Gromov, we can find \( p \in B_{r_1}(x)'' \cap B_{r_2}(y)'' \) and \( q \in B_{r_1}(x)'' \cap \partial B_{r_2}(y)'' \). This implies that \( d(\phi_t(p), \phi_t(q)) \leq 2r_2 = 0.04d(x, y) \), \( d(\phi_t(q), \phi_t(x)) \leq 2r_1/100 \leq 0.02d(x, y) \) and \( |d(p, q) - d(\phi_t(p), \phi_t(q))| \leq 0.01r_1 \leq 0.01d(x, y) \). By the triangle inequality this yields

\[
(4.26) \quad d(\phi_t(x), \phi_t(y)) \geq d(p, q) - 0.04d(x, y) - 0.02d(x, y) - 0.01d(x, y) \\
\geq 0.9d(x, y)
\]

which finally proves Lemma 4.5. \[\square\]

Remark 4.6. It’s easy to see form the proof that the constant 2 in Lemma 4.2 and in the definition of \( \mathcal{B}_r(\partial B_r) \) (Definition 3.1) can be replaced by any constant \( > 1 \). With that change the proof of Theorem 4.1 can be easily adapted to show that the Bilipschitz constant 4 in in the statement of the theorem can be replaced by any constant \( \beta > 1 \). However, as to be expected, \( \varepsilon(\lambda) \to 0 \) as \( \beta \to 1 \).
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