Black Holes in Matrix Theory

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We review recent progress in understanding black hole structure and dynamics via matrix theory. Talk by the second author, presented at Strings ’97 (Amsterdam, June 16-20, 1997).

1. Introduction

Matrix theory appears to capture a remarkable amount of the nonperturbative structure of string/M-theory. At generic values of the moduli, most of the localized states of the latter are black holes. One would like to see how matrix theory encodes their basic properties – geometry, dynamics of test particles, thermodynamics, decay via the Hawking process, etc. First steps in this direction have recently been taken; our purpose here is to summarize the results (with a few extensions), as well as to argue for a general picture of black hole structure.

We begin in section 2 with a discussion of the geometry we are trying to reproduce – classical eleven-dimensional supergravity. The focus is on five-dimensional (near)BPS black holes (for a review, see [6]), as these are best understood from the standpoint of string theory. We then point out a few properties of supergravitons as probes of the geometry, and indicate how matrix theory reproduces various asymptotic features of the metric.

Adapting a D-brane calculation of Maldacena [7], we reproduce in section 3 the near-extremal black hole entropy as the entropy of the noncritical string theory that describes matrix theory on $T^5$ \[8\]. Rephrasing the calculation in natural M-theory variables renders the interpretation of many formulae transparent. The basic theme will be that black hole thermodynamics is generalized supersymmetric Yang-Mills statistical mechanics, in the microcanonical ensemble \[9\].

Section 4 presents some speculations on the dynamics of black holes in matrix theory, summarizes our results and lists a few directions for further research.

2. Black holes in supergravity

The class of black holes and black strings we will consider arises in 11d supergravity on $\mathbb{R}^{5,1} \times T^5$ \[10\]. Take the coordinates of $\mathbb{R}^{5,1}$ to be $(x^0, ..., x^4; x^{10})$; sometimes we write $x^1, ..., x^4$ in spherical coordinates $r, \Omega$. The coordinates of $T^5$ will be $(x^5, ..., x^9)$. A black string stretched along $x_{10}$ may be constructed as a bound state of fivebranes, membranes, and gravitons of the 11d theory in the configuration

\[
\begin{bmatrix}
  6 & 7 & 8 & 9 & 10 \\
  \cdot & \cdot & \cdot & \cdot & p_{10} \\
  5 & \cdot & \cdot & \cdot & 10 
\end{bmatrix}
\]

Compactification of $x_{10}$ on a circle produces a black hole in five-dimensional spacetime. The generic (nonextremal) metric for such a configuration is

\[
ds^2 = H_2^{1/3} H_5^{2/3} \times \left[ H_2^{-1} H_5^{-1} (-H_0^{-1} h dx_0^2 + H_0 dx_{10}^2) + H_2^{-1} dx_5^2 + H_5^{-1} (dx_6^2 + ... + dx_9^2) + h^{-1} dr^2 + r^2 d\Omega_3^2 \right],
\]

where $H_i(r)$, $i = 0, 2, 5$, and $h(r)$ are harmonic functions

\[
H_i = 1 + \frac{r_i^2}{r^2}, \quad r_i^2 = r_g \sinh^2 \alpha_i
\]

Various tori.

\[\]
and \( \Delta = H_2 H_5 [\partial_u \partial_v - (H_0 - 1) \partial_t^2] \)

\[
\begin{align*}
\Delta &= H_2 H_5 [\partial_u \partial_v - (H_0 - 1) \partial_t^2] \\
&+ (\partial_t^2 + \partial_5^2) \\
&+ H_2 \partial_5^2 + H_5 (\partial_t^2 + \partial_5^2) \\
&+ (\text{connection terms}).
\end{align*}
\]

(5)

For a probe of the causal structure of the black string, we want a massless particle, therefore we consider waves independent of the internal coordinates on the \( T^5 \). One can further choose special polarizations of supergraviton (e.g. the 567 component of \( A_{MNP} \)) such that the connection term vanishes. Wavepackets of this sort behave as if they are scalars, hence travel light cones of the 11d metric – precisely the probes we want.

A wavepacket \( \psi \) with \( \partial_u \psi \gg \partial_v \psi \) \((p_+ \gg p_- = E_{L/C})\) comoves with the wave bound to the black string and is therefore near-BPS. In a sense, it adiabatically approaches and crosses the horizon. The wave equation \( \Delta \psi = 0 \) reduces to

\[
\begin{align*}
n^2(r) \omega_{\text{eff}}^2 \psi &= \tilde{\partial}^2 \psi, \\
n^2(r) &= H_2 H_5.
\end{align*}
\]

(6)

where \( \tilde{\partial}^2 = \partial_1^2 + \ldots + \partial_5^2 \), \( \omega_{\text{eff}}^2 \psi \approx \partial_u \partial_v \psi \), and \( n(r) = H_2 H_5 \). Thus the effective dynamics of wavepackets is equivalent to geometric optics with the spatially dependent index of refraction \( n(r) \). Since the index increases as the radius decreases, null geodesics are focussed onto the string (gravitational lensing).

In the next section we will exhibit a matrix theory configuration for near-extremal black strings/holes of the type (3). The effective dynamics maps onto an equivalent D-brane system (5+1d SYM), although the interpretation is rather different. One of the most important distinctions is that, whereas the D-brane moduli space describes slowly moving massive particles, the matrix model moduli space describes the transverse motion of massless particles (the supergravitons) – thus giving direct information about the causal structure.

Calculations in this D-brane system [12,13] reproduce all \( 1/r^2 \) terms in the near-extremal geometry. Curved space geometry is a low-energy approximation in matrix theory. The dynamics of the probe-black hole system in \( \mathbb{R}^{5,1} \) is described by matrix Higgs fields in the D-brane SYM on \( T^5 \)

\[
\Phi = \begin{bmatrix} \tilde{X}_{BH} & \tilde{Y} \\ \tilde{Y}^\dagger & \tilde{X}_{\text{probe}} \end{bmatrix},
\]

(7)

where the vector components run over \((x_1, ..., x_4)\). The \( \tilde{Y} \) variables are massive, schematically \( V_{\text{eff}} = [\Phi^2, \Phi^2]^2 \sim |\tilde{X}_{BH} - \tilde{X}_{\text{probe}}|^2 |\tilde{Y}|^2 \equiv r^2 |\tilde{Y}|^2 \). Integrating out \( \tilde{Y} \) yields an effective action that is an expansion in powers of derivatives of the light fields \( X \) and inverse powers of \( r \). All the \( (1/r^2) \) terms in the asymptotic metric come from integrating out \( \tilde{Y} \) at one loop [12,13]. Thus, in matrix theory \( n^2(r) \sim 1 + \frac{4}{r^2} + \frac{4}{r^4} \) is an optical index – it is generated by the spatially dependent vacuum polarization whose effect is a spatially varying dielectric function seen by the light degrees of freedom. However, this correspondence makes it clear that the light-cone structure of spacetime is indeed only a low-energy approximation in matrix theory. This structure is defined by the trajectories of massless particles, such as the probe wavepackets considered above. A supergraviton probe will only follow the classical black hole light cones in this (moduli space) approximation, i.e. when the \( Y \) variables are sufficiently

\[2\text{The interaction with the gravitational wave } H_2 \text{ bound to the string is none other than the (}\tilde{t}_{\text{ probe}} = \tilde{r} \text{ of the wave})^2 \text{/r}^7 \text{ Coulomb interaction of matrix theory} \text{ [12], the } r^{-7} \text{ becomes } r^{-2} \text{ when smeared over the internal } T^5 .\]
heavy that they may be consistently integrated out. Near the black hole (the ‘stretched horizon’ in black hole physics [15]; the ‘stadium region’ in D-brane terminology [14]), this approximation breaks down. Thus in the Schwarzschild-type coordinate frame intrinsic to the matrix description, the classical causal structure and classical notions of information propagation only make sense sufficiently far from the black hole (or string in the present case). Near the hole, spacetime literally becomes nonabelian, and the light cones are not meaningful (at least in this coordinate frame).

Of course, the $1/r^2$ terms in the metric are all required by Gauss’ law – they represent the energy of gauge fluxes carrying the BPS or Noether charges ($Q_0, Q_2, Q_5, E$) at infinity. It is more a relief than a triumph only for matrix theory to reproduce them. Subleading terms are allowed to depend sensitively on nonuniversal details – the wavefunction of the black hole and probe degrees of freedom, form factors for their scattering, etc. It may be that the precise black hole geometry is reproduced only in the large N limit, with all these effects taken into account. In other words, we need to understand how the Einstein equations come out of matrix theory. It is encouraging that the behavior of supergraviton probes [16] has the same form that one would expect to get from the abelianized moduli space approximation to matrix theory – the classical geometry acts as an ‘optical medium’ whose optical index bends the trajectories of probes.

3. Matrix black hole thermodynamics

The entropy: To discuss thermodynamic properties, we will postulate 1) that there exists some theory (perhaps of noncritical strings [19]) whose large $N$ dynamics formulates matrix theory on $T^5$; and 2) this theory has a regime in which it is well-approximated by 5+1d Yang-Mills with 16 supersymmetries. One can think of this auxiliary theory heuristically as the T-dual of the original matrix theory on all five circles of $T^5$:  

\[
\begin{align*}
\text{graviton} & \sim D0 \quad \longrightarrow \quad D5 \\
\text{long. fivebrane} & \sim D4 \quad \longrightarrow \quad D1 \\
\text{long. membrane} & \sim \text{IIA string} \quad \longrightarrow \quad \text{momentum} .
\end{align*}
\]

Thus the three charges correspond, respectively, to the rank $N$ of the gauge group, the instanton number, and the field momentum in the SYM theory. This maps the system to a well-studied type IIB D-brane system (for a review, see [10]). Finite $N$ in matrix theory is supposed to be related to compactification of the longitudinal [9] or light-front [14] coordinate on a circle of radius $R$. In matrix theory, branes’ wrapping/momenta are SYM fluxes:  

\[
\begin{align*}
q_i &= \int \text{tr} F_{0i} = \left(\text{graviton KK charge along } x_i\right) \\
m_{ij} &= \int \text{tr} F_{ij} = \left(\text{membrane wrapping } x_i x_j \text{ cycle}\right) \\
m_{+i} &= \int T_{0i} = \left(\text{longitudinal membrane wrapping } x_i x_{10}\right) \\
f_i &= \int (F \wedge F)_i = \int \epsilon_{ijklm} F_{jk} F_{lm} \\
&= \left(\text{longitudinal fivebrane along } x_i x_k x_l x_m x_{10}\right) .
\end{align*}
\]

Note that the instanton is a solitonic string in 5+1d, and $f_i$ is its wrapping along the $x_i$ cycle. The configuration [9] corresponds to exciting $m_{+5}$ and $f_5$ in the SYM theory.

Horowitz, Maldacena, and Strominger [7] proposed an identification of the parameters of the classical geometry with ‘brane charges’:

\[
\begin{align*}
N_{0,0} \frac{\ell_p}{R} &= \frac{VR_5 R}{4 \ell_p^8} r_9^2 e^{\pm 2 \alpha_0} \equiv E_{0,0} \\
N_{2,2} \frac{R_5 R}{\ell_p^2} &= \frac{VR_5 R}{4 \ell_p^8} r_9^2 e^{\pm 2 \alpha_2} \equiv E_{2,2} \\
N_{5,5} \frac{VR}{\ell_p^5} &= \frac{VR_5 R}{4 \ell_p^8} r_9^2 e^{\pm 2 \alpha_5} \equiv E_{5,5} ,
\end{align*}
\]

where $V = R_6 R_7 R_8 R_9$ is the volume of the $T^4$ spanned by the fivebrane, and $E_i$, $E_i$ are the contributions of these ‘constituents’ to the ADM energy  

\[
E_{\text{ADM}} = \sum_{i=0,2,5} \left( E_i + E_i \right) .
\]

We will argue below that the identifications [10] are somewhat misleading, nevertheless they will serve our purpose for the moment.
In the classical supergravity solution, one requires a balance of the pressures and tensions exerted by the various branes on the internal $T^5$ in order to have a nonsingular horizon at extremality; in terms of the quantities $E_0, E_\ell, E_2, E_3, E_5$.

$$E_0E_\ell = E_2E_2 = E_3E_5.$$  

We propose to interpret this as an equipartition of the ‘invariant masses’ among the different BPS charges, since the first of these is the invariant mass of a 1d gas, and the others are U-dual to charges, as such a quantity. Note that $\text{mass of a 1d gas}$, and the others are U-dual to $\text{charges}$, since the first of these is the invariant mass of the ‘invariant masses’ among the different BPS charges.

This phase is well-defined at fixed energy (microcanonical ensemble).

The Virasoro constraints on the instanton string determine its excitation level:

$$n_{L,R} = \alpha' \ell_S [E^2_{LC} - (p \pm \mathcal{W})^2] .$$  

Plugging in and using $T_{\text{eff}} = (2\pi\alpha')^{-1} = (4\pi^2/g_{YM}^2 N) = \frac{\sqrt{R_{L}R_{R}}}{2\pi N}$, one finds the well-known answer

$$S_{BH} = 2\pi(\sqrt{N} + \sqrt{N}) = 2\pi\sqrt{N}(\sqrt{N_2} + \sqrt{N_3}) \times (\sqrt{N_5} + \sqrt{N_5}) .$$  

In the $R, N \to \infty$ limit, a graviton with any nonzero transverse velocity can be boosted into the forward direction (since the transverse momentum acts effectively as a mass with respect to longitudinal boosts). If matrix theory is Lorentz covariant, in this limit there may be a sense in which all nonextremalities are effectively turned on. Finally, the fact that we get the right answer independent of the moduli of $T^5$ suggests that the above picture captures the thermodynamic properties of the noncritical string of matrix theory on $T^5$.

There are a few simple generalizations of the above calculation. First, one can turn on other BPS charges. The general extremal entropy is

$$S_{BH} = 2\pi \left[ (Nf_1 + \frac{1}{2}(m \wedge m)) \times (m_1 - q^2 m_1) / N \right]^{1/2} .$$  

Halyo has considered the (near)extremal black hole obtained from transverse rather than longitudinal gravitons and membranes (the second term rather than the first in the last factor). This should correspond to turning off the momentum charge carried by the Hagedorn string, replacing its contribution to the energy by that of the transverse zero- and two-brane charges that are turned on instead. Second, the considerations

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4 With $N/R^2$ fixed in order to have a finite entropy per unit length in the limit.

5 Note, however, that when $R > r_p$ an instability develops, whereby the black string will become a black hole threaded by an extremal string. This suggests we keep $R, N$ finite of order the size of the black hole in order to capture the relevant physics.
of included Hagedorn strings with angular momentum; it is straightforward to translate the results to the context of matrix theory to obtain the entropy of spinning matrix black holes. Third, nothing in the entropy calculation required \( P_r | W \); this simply corresponds to choosing orthogonally intersecting membranes and fivebranes. Allowing \( P_r \) to make an angle \( \zeta \) with respect to \( W \) means that there is a component of the membrane charge along the fivebrane – the branes intersect at angles. Repeating the above exercise, one finds e.g.

\[
n_{L,R} = \alpha'_{\text{eff}}[E_{L,C}^2 - (P_5 + W_5)^2 - \mathcal{P}_5^2]
\]

\[
= \left( \frac{\alpha'_{\text{eff}}}{\ell_p^2} \right) \left[ \frac{V R R_5}{4 f_5^8} \right] \left( \frac{r^2}{g} \right)^2
\]

\[
\times \left[ \cos^2 \zeta \, \text{ch}^2(\alpha_2 + \alpha_3) + \sin^2 \zeta \, \text{ch}^2(\alpha_2 + \alpha_3) \right].
\]

In the extremal limit \( \alpha_2, \alpha_3 \to \infty \) the entropy agrees with known results \([20]\); the nonextremal entropy has not been computed in supergravity, and it would be interesting to see if it matches \([21]\).

**Interpreting the perturbation from extremality:** It was claimed above that the identification \([10]\) is somewhat misleading. To see this, let us consider the excitations of the instanton string. Its transverse oscillators (in Green-Schwarz formalism) consist of \( X^i, S^a_L, S^b_R \), where \( \alpha, \beta \) are spinor indices of the SO(4) transverse to the \( T^5 \) (the R-symmetry of the 5+1d SYM), and \( i, a, b \) are vectors and spinors of the SO(4) little group in 5+1d. The center of mass mode of the string has the quantum numbers of the vector multiplet (the string we are considering is the T-dual of the one in \([4]\)), and one can interpret the oscillator modes as coupling to fluctuations of the SYM theory. Therefore local oscillations along the Hagedorn string are fluctuations of the Hagedorn gas, equivalent to all types of local fluctuations of SYM. This picture may carry over largely unmodified to the full noncritical interacting matrix string theory of \([13,15]\); there, the SYM ‘particles’ are microscopic noncritical strings, and the Hagedorn string is simply a macroscopic string cut from the same cloth. Dumping a lot of energy into the system forces it into the Hagedorn phase, and the soft excitations of the Hagedorn string indeed couple to all quantum numbers of the theory. This explains why \([13,15]\) are valid over all of the parameter space of the 5+1d matrix theory.

We now see why the identification \([10]\) is not the full story. The Hagedorn gas contains fluctuations of all possible quantum numbers \([4]\), not just those which have an expectation value such as \( m_{45} \) and \( f_5 \). Rather, the energy above extremality is equipartitioned into all types of branes/antibranes according to their energy cost per quantum. What were called \( N_i, Q_i, i = 0, 2, 5 \) in \([4]\), are simply a characterization of the state of excitation of the Hagedorn string. They were actually determined by the ADM charges \( E_{\text{ADM}}, Q_0, Q_2, Q_5 \), and the two ‘pressure balance’ conditions \([12,14]\).

**4. Discussion and speculations**

**The infalling probe:** We have seen that matrix theory encodes the key features of 5+1d black holes – their leading asymptotic geometry and their density of states. Whether the rest of the structure is present is tantamount to an understanding of how general relativity appears as the effective theory. It is at least encouraging that, in supergravity, the motion of probes in the black hole background can be recast in the form of geometric optics. In matrix theory, this is just what one gets from the vacuum polarization effects of the heavy matrix degrees of freedom. The issue is whether these effects reproduce the right ‘optical index’.

A legitimate question to ask of matrix theory concerns the fate of an infalling probe. In the exterior static coordinates intrinsic to the infinite momentum frame of matrix theory\([5]\) when

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\( ^6 \)The membrane/fivebrane pressure balance can be motivated by extremizing the entropy of the Hagedorn string \([10]\); the remaining pressure balance will require an interpretation within matrix theory of the factor of \( N_i \) appearing in \([12]\). Perhaps one can get both pressure balance conditions by turning on other charges as in the discussion after \([10]\).

\( ^7 \)This is because the optical index blows up near the horizon, causing the probe wave to infinitely slow down, as one expects in static coordinates. The divergence is fake, however, merely reflecting the failure of the approximation made in integrating out off-diagonal matrix elements which are becoming light there.

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the probe reaches the ‘stretched horizon’ of the black hole, it dissolves into the Hagedorn gas of the black hole. In terms of matrices, the evolution has the schematic form

$$\begin{bmatrix} \text{BH} & 0 \\ 0 & \text{probe} \end{bmatrix} \rightarrow \begin{bmatrix} \text{BH}' & 0 \\ 0 & \text{probe}' \end{bmatrix}, \quad (18)$$

where $\text{BH}'$ denotes an excited black hole. To describe the proper motion of the probe one would like to at least approximately diagonalize the probe’s collective field theory. This would give an approximate evolution looking like

$$\begin{bmatrix} \text{BH} & 0 \\ 0 & \text{probe} \end{bmatrix} \rightarrow \begin{bmatrix} \text{BH}' & 0 \\ 0 & \text{probe}' \end{bmatrix}, \quad (19)$$

at least until one gets close to the singularity. Such a rediagonalization should involve passing to infalling coordinates, for instance by performing a sequence of boosts to keep the probe in its instantaneous rest frame. In general relativity, the coordinate transformation that results has the form

$$u \rightarrow U \sim \frac{1}{R} e^{au},$$
$$v \rightarrow V \sim v - a(b^2 r),$$
$$r \rightarrow W \sim e^{-ru} \cdot \left(\frac{r}{2}\right). \quad (20)$$

The first of these undoes the exponential redshift of static coordinates near the horizon, the second is a large shift of light-front time, and the last rewrites the radial variable as a kind of ‘tortoise’ coordinate. The black hole interior has $U < 0$, $W < 0$. But this is an analytic continuation of the IMF description; $u$ is the conjugate variable to $p_+ \sim N$, so the meaning of continuing past $u = \infty$ is unclear. Similarly, $W < 0$ may involve a continuation to complex eigenvalues of $\vec{X}_{\text{probe}}$. Even so, it seems clear that the sequence of boosts required to keep the probe in its rest frame involves boosts which mix light-front time and $\vec{X}_{\text{probe}}$; therefore, the probe proper time is a noncommutative (matrix) variable, and the evolution equation of the probe with respect to its proper time involves a moduli space approximation in this time coordinate. One might imagine that the classical singularity of the black hole is simply a reflection of the breakdown of this moduli space approximation in the time direction. It is also interesting to note that the boosts involved are matrix transformations; it is possible that observables appropriate to asymptotic observers and observables measured in the infalling frame will not commute, leading to a form of black hole complementarity.

**Directions for further research:** It is important to expand the lexicon of translations between matrix theoretic and gravitational quantities, especially the gross geometrical features of the region near the black hole. It should be possible to calculate the properties of Hawking radiation in the matrix theory approach. Near extremality, the Hawking temperature and the temperature of the Hagedorn gas coincide. One can relate the Hawking temperature to the temperature of the Hagedorn string arbitrarily far from extremality:

$$\beta_H = \beta_{str}(\alpha'_{\text{eff}})^{1/2} E_{\text{LC}} \cdot (21)$$

It would be nice to understand the factor of proportionality. There are also similar relations which need a proper explanation, for instance the formula

$$\langle n_{\text{L}n_{\text{R}}} \rangle^{1/2} = \frac{N}{R} E_{\text{LC}} r_g^2 \quad (22)$$

which relates the Hagedorn string’s excitation to the gravitational radius of the black hole. Larsen has related $n_L - n_R$ to the area of the inner horizon. How are these reflected in the dynamics of probes?

Perhaps the main lesson to be drawn from the above analysis is that black hole thermodynamics becomes conventional statistical mechanics in matrix theory. Thus supergravity gives a whole host of predictions – for the $(2,0)$ field theory governing matrix theory on $T^4$, for $\mathcal{N}=4$ SYM on $T^3$, and so on. The thermodynamic properties of these higher-dimensional black holes (c.f. [11]) are currently not understood in string theory; there is no weak coupling limit where the horizon is nonsingular near extremality. Nevertheless, one

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8In principle, $r_g^2 \sim \langle X^2 \rangle$; the excitations of $\vec{X}_{\text{L}}$ are R-R fluctuations of the Hagedorn string and are determined by equipartition. This is related to the fact that absorption of higher angular momenta proceeds via the worldsheet fermions.
might hope to directly study the nonextremal black hole via matrix theory. The challenge is to find the relevant degrees of freedom and to understand their behavior.

If we are successful, we should be able to explain the black hole correspondence principle \[21\]. We saw that the moduli of the internal torus act as chemical potentials, altering the balance of degrees of freedom in the equilibrium SYM ensemble. At small \( R \), membrane/antimembrane excitations dominate; at small \( V \), fivebrane/antifivebrane excitations are more prominent. Shrinking any circle of the torus to sub-Planckian size, one will recover the matrix description of black holes as bound states of perturbative strings and D-branes. It is implicit in \[22\] that the level density is \( o(\exp[M]) \) in the perturbative string regime. The black hole level density is \( o(\exp[M^{2/3}]) \), so there will have to be a substantial crossover in the SYM level density as a function of \( R \). The correspondence principle is a statement that there are no phase transitions in this crossover region, so that the spectra of black holes and of string states match smoothly onto one another.

Perhaps the simplest example of this phenomenon occurs in matrix 1+1d SYM on \( S^1 \). The flux \( \oint T_{01} \) (a momentum mode of the SYM) represents a longitudinally wrapped membrane. At large radius and far from extremality, the SYM statistical mechanics describes a black string; at small radius, an excited fundamental IIA string emerges, stretched across the longitudinal direction. This situation is currently under investigation \[23\].

Acknowledgements: E.M. thanks the Aspen Center for Physics for hospitality during the preparation of the manuscript. This work is supported in part by funds provided by the DOE under grant No. DE-FG02-90ER-40560.

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47. Y. Imamura, \texttt{hep-th/9703077}.
48. M. Li and E. Martinec, work in progress.