MULTIPLICITY ONE THEOREM OVER CHARACTERISTIC 2

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Abstract. It is shown for all local fields \( F \) which are of characteristic different from 2 that any distribution on \( GL_{n+1}(F) \) which is invariant under conjugation by \( GL_n(F) \) is also invariant under transposition. In this paper we give an adaptation of the proof of this theorem to fields of characteristic 2.

1. Introduction

Let \( F \) be a local field of characteristic 2. In this paper we prove the following theorem:

**Theorem 1.1.** Any distribution on \( GL_{n+1}(F) \) invariant under conjugation by \( GL_n(F) \) is also invariant under transposition.

For non-archimedean fields of characteristic zero it is proven in [4], for archimedean fields in [8] and [3], and for fields of odd characteristic in [6]. In this paper we will give an adaptation of the proof in [6] to characteristic 2.

It is shown in [4, section 1] that Theorem 1.1 has the following corollary, already known by different methods (see [1]).

**Theorem 1.2.** Let \( \pi \) be an irreducible smooth representation of \( GL_{n+1} \), and let \( \rho \) be an irreducible smooth representation of \( GL_n \). Then

\[
\dim \text{Hom}_{GL_n}(\pi, \rho) \leq 1
\]

Let \( V \) be an \( n \)-dimensional vector space over \( F \). Let \( \tilde{G} := GL(V) \rtimes \{\pm 1\} \) be the semidirect product with the respect to the action of \( \{\pm 1\} \) on \( GL(V) \) by \( A \mapsto (A^t)^{-1} \). The group \( \tilde{G} \) acts on \( gl(V) \) by \( (g, 1). (A, v, \phi) = (gAg^{-1}, gv, (g^t)^{-1}\phi) \), and \( (g, -1). (A, v, \phi) = ((gAg^{-1})^t, g^{\phi^t}, (g^t)^{-1}v^t) \). Let \( \chi \) be the character of \( G \) defined by \( (g, \delta) \mapsto \delta \). It is shown in [2] (the same proof works verbatim) that Theorem 1.1 reduces to the following theorem:

**Theorem 1.3.** Any \( (\tilde{G}, \chi) \)-equivariant distribution on \( gl(V) \times V \times V^* \) is 0.

We will prove this theorem by induction on the dimension of \( V \), and so we will assume this theorem for all smaller \( n \). Throughout the paper, let \( \xi \) be a \( (\tilde{G}, \chi) \)-equivariant distribution on \( gl(V) \times V \times V^* \).

There are two points in the proof in [6] in which the assumption \( \text{char}(\mathbb{F}) \neq 2 \) was made use of. The first and more significant one is the proof of Proposition 4.6. The main goal of this paper is to prove this theorem over fields of characteristic 2 (Corollary 3.4). In section 2 we use the technique of Harish-chandra descent to a further extent than was used in [6], to get a stronger restriction on the support of \( \xi \),

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which will be used in the proof of Corollary 3.4 in section 3. The second usage of the
assumption \( \text{char}(\mathbb{F}) \neq 2 \) in [6] was the usage of a theorem by Rallis and Schiffman
(6 Theorem 2.9)), which is relied on the theory of the Weil representation, a theory
which is a bit different when working over a field of characteristic 2. In Appendix A
we prove a version of the theorem over a field of characteristic 2, which is sufficient
for the proof as given in [6].

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2. Reduction to the purely inseparable locus

Notation 2.1. Use \( \Delta : \mathfrak{gl}(V) \times A \times A^* \to \mathbb{F}[x] \) to denote the map which sends
\((A, v, \phi)\) to the characteristic polynomial of \( A \).

Theorem 2.2. For any point \((A, v, \phi)\) in the support of \( \xi \), the characteristic poly-
nomial of \( A \) is a power of an irreducible polynomial.

Proof. Assume that a polynomial \( f \) of degree \( n \) has two coprime components
\( f = f_1 f_2 \). By the localization principle (see [6] Theorem 2.4]), it is enough for
us to show that for any such polynomial \( f \), the fiber of \( \Delta \) above \( f \) has no non-
zero \((\tilde{G}, \chi)\)-equivariant distributions. Let \( \zeta \) be such a distribution on \( \Delta^{-1}(f) \). Let
\( d_1 = \deg f_1, d_2 = \deg f_2 \). Denote by \( \Lambda \) the space of all pairs of subspaces \( V_1, V_2 \) of \( V \)
such that \( \dim(V_1) = d_1, \dim(V_2) = d_2, V = V_1 \oplus V_2 \). \( \Lambda \) has a natural action of \( G \) on it,
which extends to an action of \( \tilde{G} \) by the involution \( (V_1, V_2) \mapsto (V_2^*, V_1^*) \) with
respect to the quadratic form \( v \mapsto v^t v \). These two actions are (both) transitive. There is a
\( \tilde{G} \)-equivariant map \( \Delta^{-1}(f) \to \Lambda \), given by taking the (unique) pair of \( A \)-invariant
subspaces of \( V \) on which \( A \) acts with characteristic polynomials \( f_1 \) and \( f_2 \). The fiber
of this map above \( (V_1, V_2) \) is a closed subspace of \( (\mathfrak{gl}(V_1) \times V_1 \times V_1^*) \times (\mathfrak{gl}(V_2) \times V_2 \times V_2^*) \),
and the stabilizer of this point is equal to \( \tilde{G}(V_1) \times \tilde{G}(V_2) \). By the localization
principle (see [6] Theorem 2.4]) and induction hypothesis, there are no non-zero
\((\tilde{G}(V_1) \times \tilde{G}(V_2), \chi)\)-equivariant distributions on \((\mathfrak{gl}(V_1) \times V_1 \times V_1^*) \times (\mathfrak{gl}(V_2) \times V_2 \times V_2^*) \),
and so it follows from Frobenius descent (see [6] Theorem 2.7) that \( \zeta = 0 \) too.

Theorem 2.3. For any point \((A, v, \phi)\) in the support of \( \xi \), the irreducible factor in the
characteristic polynomial of \( A \) is purely inseparable.

Proof. By the localization principle (see [6] Theorem 2.4]) it is enough to consider
a \((\tilde{G}, \chi)\)-equivariant distribution \( \zeta \) on \( F \times V \times V^* \), where \( F \) is the fiber of \( \Delta \) over
some \( f^m \), \( f \) being irreducible and not purely separable, and show that \( \zeta = 0 \). We
have \( f(x) = g(x^2) \) for \( g(x) \) irreducible and separable. By assumption, \( \deg g > 1 \).
For any \( A \in F \), the characteristic polynomial of \( B := A^{2^k} \) is equal to \( g(x)^{2^km} \), as
it is the polynomial whose roots are \( 2^k \) powers of the roots of \( g(x^2)^m \). All of its
irreducible factors are separable, and so $B$ has a well defined Jordan decomposition $B = B_s + B_n$. Moreover, the map $h : A \mapsto B_s$ is continuous on $F$. Note that since $B_s$ is expressible as a polynomial in $B$ (and thus as a polynomial in $A$), it commutes with $A$. Use Frobenius descent (see \cite[Theorem 2.7]{Frobenius}) with respect to $h$ - the stabilizer of a point is isomorphic to $GL_{2k_m}(E)$, where $E := \mathbb{F}[x]/g(x)$. The fiber of $h$ above a point is isomorphic to a closed subspace of $gl_{2k_m}(E)$, and so by induction hypothesis applied to the Frobenius descent of $\zeta$, we get that $\zeta = 0$.  

3. Vanishing of linear invariants

The following proposition is proved in \cite[Lemma 7.2]{Vanishing} over a field of characteristic 0, and the proof there applies verbatim over arbitrary characteristic.

**Proposition 3.1.** For any point $(A, v, \phi)$ in the support of $\xi$, we have $<v, \phi >= 0$.

**Definition 3.2.** Let $\mu \in \mathbb{F}$. Define $\rho_\mu$ as the following $GL(V)$-equivariant automorphism on $gl(V) \times V \times V^*$:

$$ (A, v, \phi) \mapsto (A + \mu v \otimes \phi, v, \phi). $$

**Theorem 3.3.** For any point $(A, v, \phi)$ in the support of $\xi$, we have $<Av, \phi >= 0$.

**Proof.** By applying Theorem \cite[2.3]{Vanishing} to $\rho_\mu(\xi)$ for some $\mu \in \mathbb{F}$, we get that the characteristic polynomial of $A + \mu v \otimes \phi$ must be a power of an irreducible purely inseparable polynomial too. Denote the characteristic polynomial of $A + \mu v \otimes \phi$ by $\sum_{i=0}^n c_i(A + \mu v \otimes \phi)x^{n-i}$.

**Case 1.** $n$ is odd

The characteristic polynomial of $A + \mu v \otimes \phi$ is of the form $(x + \lambda_\mu)^n$. Since $\lambda_\mu = c_1(A + \mu v \otimes \phi) = c_1(A) - \mu <v, \phi >= c_1(A) = \lambda_0$, we get that $\lambda_\mu$ (and so also the characteristic polynomial of $A + \mu v \otimes \phi$) is independent of $\mu$. Thus we get that $c_2(A) = c_2(A + \mu v \otimes \phi) = c_2(A) - \mu(<Av, \phi > + c_1(A) <v, \phi >) = c_2(A) - \mu <Av, \phi >$, and so $<Av, \phi >= 0$.

**Case 2.** $n$ is divisible by 4

In this case $n = \binom{n}{2} = 0$ in $\mathbb{F}$. The characteristic polynomial of $A + \mu v \otimes \phi$ is always of the form $(x^{2k} + \lambda)^\frac{n}{2}$ for some $\lambda$ dependent on $\mu$. By maybe changing $\lambda$, we can assume that $2^k$ is the maximal power of 2 that divides $n$. In particular our polynomial is a polynomial in $x^4$, and so we have $c_2(A + \mu v \otimes \phi) = 0$ for all $\mu$. However,

$$ c_2(A + \mu v \otimes \phi) = c_2(A) - \mu(<Av, \phi > + c_1(A) <v, \phi >) = c_2(A) - \mu <Av, \phi >. $$

Thus we must have $<Av, \phi >= 0$.

**Case 3.** $n = 2 \mod 4$ and $n > 2$

The irreducible factor of the characteristic polynomial is either linear or quadratic. Thus the characteristic polynomial must be either $(x + \lambda)^n$ or $(x^2 + \lambda)^{n/2}$. Allowing $\lambda$ to be a square, we assume it is of the second form. So $c_2$ is equal $(n/2)\lambda = \lambda$. Let $\lambda_\mu$ be such that the characteristic polynomial of $A + \mu v \otimes \phi$ is $(x^2 + \lambda_\mu)^{n/2}$. We have

$$ \lambda_\mu = c_2(A + \mu v \otimes \phi) = c_2(A) - \mu <Av, \phi >. $$
\[(c_2(A) - \mu(\langle A, \phi \rangle))^{n/2} = \lambda^{n/2}_\mu = c_n(A + \mu v \otimes \phi) = c_n(A) - \mu(\ldots)\]

The right hand side is linear in \(\mu\) while the left hand side is polynomial of degree \(n/2\) unless \(\langle A, \phi \rangle = 0\). Thus assuming \(n > 2\) we are done.

**Case 4.** \(n = 2\)

We can use the localization principle (see [6, Theorem 2.4]) with respect to the map \((A, v, \phi) \mapsto \langle A, \phi \rangle\), to be left with proving that if \(\xi\) is supported on \(\{(A, v, \phi) | \langle A, \phi \rangle = m\}\) (for some \(m \neq 0\)) then \(\xi = 0\). We already know that for any point \((A, v, \phi)\) in the support of \(\xi\), we must have \(\text{tr}A = 0\), and that \(\langle v, \phi \rangle = 0\). Recalling that by the assumption \(\langle A, \phi \rangle = m\) we necessarily have \(v \neq 0\), we can write explicitly \(A = \begin{pmatrix} a & b \\ c & a \end{pmatrix}, v = \begin{pmatrix} x \\ y \end{pmatrix}, \phi = (ty \ tx)\). This yields \(\langle A, \phi \rangle = t(cx^2 + by^2)\). Let \(\sigma := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, -1 \in \tilde{G}\). It acts by

\[
(\begin{pmatrix} a & b \\ c & a \end{pmatrix}, (x, y), (ty \ tx)) \mapsto (\begin{pmatrix} a & b \\ c & a \end{pmatrix}, (tx, ty), (y, x)).
\]

Use Frobenius descent (see [6, Theorem 2.7]) with respect to the \(\tilde{G}\)-equivariant map

\[
\left\{\left(\begin{pmatrix} a & b \\ c & a \end{pmatrix}, (x, y), (ty \ tx) \right) \mid (ty \ tx) \neq 0\right\} \to \left\{\begin{pmatrix} x \\ y \end{pmatrix}, (ty \ tx) \right\} \mid (ty \ tx) \neq 0\right\}
\]

given in the above coordinates. It is easy to see that the action on the target is transitive. The stabilizer of the point \(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, (0, 1)\) inside \(G\) is the unimodular subgroup \(N := \left\{\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}\right\}\). The stabilizer inside \(\tilde{G}\) is \(\tilde{N} := N \rtimes \{1, \sigma\}\), and it is indeed also unimodular. The fiber above \(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, (0, 1)\) is \(\left\{\begin{pmatrix} a & b \\ m & a \end{pmatrix}\right\}\), on which \(\sigma\) acts trivially. Thus, since our distribution is \(\chi\)-equivariant (and so \(\sigma\)-anti-invariant), it must be 0.

\[\square\]

The following Corollary of the previous theorem appears in [6, Proposition 4.6] and is proved there in a way which fails over a field of characteristic 2.

**Corollary 3.4.** For any point \((A, v, \phi)\) in the support of \(\xi\) and any \(k \geq 0\), we have \(\langle A^kv, \phi \rangle = 0\).

**Proof.** Let \(\Delta : \mathfrak{gl}(V) \times V \times V^* \to \mathbb{F}[x]\) be the characteristic polynomial map. Recall the automorphism \(\rho_g : \Delta^{-1}(f) \to \Delta^{-1}(f)\) defined for every \(g\) coprime to \(f\) by \(\rho_g((A, v, \phi)) := (A, g(A)v, g(A^*)\phi)\). By using the localization principle (see [6, Theorem 2.4]), we can reduce the claim to a distribution on a single fiber of \(\Delta\) over a polynomial \(f\). Then for any \(g\) coprime to \(f\), we can apply \(\rho_g\), then extend the distribution back to \(\mathfrak{gl}(V) \times V \times V^*\) and apply Theorem 3.1 to get that \(\langle g(A)v, g(A^*)\phi \rangle = 0\) and Theorem 3.3 to get that \(\langle Ag(A)v, g(A^*)\phi \rangle = 0\). Since this is true for a Zariski dense set of polynomials \(g\), it is true for all polynomials. Thus we get that for any \(g\) we have \(\langle g(A)^2v, \phi \rangle = \langle g(A), g(A^*)\phi \rangle = 0\) and \(\langle Ag(A)^2v, \phi \rangle = \langle Ag(A), g(A^*)\phi \rangle = 0\). In particular for any \(k \geq 0\) we can take \(g(x) = x^k\) to get that \(\langle A^{2k}v, \phi \rangle = 0\) and \(\langle A^{2k+1}v, \phi \rangle = 0\). \(\square\)
Once this Corollary is proven, the rest of the proof of Theorem 1.3 given in [6] applies almost verbatim also over a field of characteristic 2. The only point in which there is a difference is the usage of [6, Theorem 2.9] of which the proof relies on the theory of the Weil representation. This theory is a bit different over characteristic 2. We give in Appendix A the necessary adaptations, and prove Theorem A.1 which plays the same role as [6, Theorem 2.9].

Appendix A. The Weil representation over characteristic 2

In this section we prove the following theorem, due to [7] in the case where \( \text{char} \mathbb{F} \neq 2 \):

**Theorem A.1.** Let \( \mathbb{F} \) be a local field with characteristic 2. Let \( V \) be a finite dimensional linear space over \( \mathbb{F} \), and let \( V^* \) be its dual space. Define \( Z := \{(v, \phi) \in V \times V^* | < v, \phi > = 0 \} \). Then for every distribution \( \xi \) on \( V \times V^* \) such that both \( \xi \) and its Fourier transform \( \mathcal{F}(\xi) \) are supported on \( Z \), we have that \( \xi \) must be "abs-homogeneous" of degree \( \dim V \). That is, for any \( t \in \mathbb{F} \) and \( \Phi \in \mathcal{S}(V \times V^*) \), we have:

\[
|(t\xi)(\Phi)| = |t|^\dim V \cdot |\xi(\Phi)|.
\]

Let us recall the facts which we will need about the Weil representation in characteristic 2 and the pseudo-symplectyc group (see [9] and [5]).

**Definition A.2.** Let \( \mathbb{F}, V \) as above. We define \( X := V \oplus V^* \). Define a bilinear form \( B \) on \( X \oplus X^* \cong V \oplus V^* \oplus V \oplus V^* \) by

\[
B((u_1, u'_1, v_1, v'_1), (u_2, u'_2, v_2, v'_2)) = < u_1, v'_2 > + < v_2, u'_1 >.
\]

Let \( Q \) be the quadratic form on \( X \oplus X^* \) defined by \( B \), and denote by \( \mathcal{Q}(X \oplus X^*) \) the linear space of all quadratic forms on \( X \oplus X^* \).

Define the pseudo-symplectic group

\[
PSp := \{(\sigma, f) \in O(Q) \times \mathcal{Q}(X \oplus X^*) \mid \forall w, w' \in X \oplus X^*,
B(w, w') + B(\sigma w, \sigma w') = f(w + w') + f(w) + f(w') \}
\]

with group law \((\sigma_1, f_1)(\sigma_2, f_2) = (\sigma_1 \sigma_2, f_1 \sigma_2 + f_2)\).

**Proposition A.3.** We have an embedding \( j : SL_2(\mathbb{F}) \to PSp \) defined by

\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} aI & bI \\ cI & dI \end{pmatrix}, f
\]

Where \( \sigma \) above is defined in coordinates \( V \oplus V^* \oplus V \oplus V^* \), \( I \) being the identity matrix of \( V \oplus V^* \), and \( f \) is defined by \( f(u, u', v, v') = ac < u, u' > + bd < v, v' > + bc(< u, v' > + < v, u'>). \)
Proof. Indeed:

\[ B(\sigma(u_1, u_1', v_1, v_1'), \sigma(u_2, u_2', v_2, v_2')) + B((u_1, u_1', v_1, v_1'), (u_2, u_2', v_2, v_2')) = \]

\[ = B((au_1 + bv_1, au_1' + bv_1', cu_1 + dv_1, cu_1' + dv_1'), (au_2 + bv_2, au_2' + bv_2', cu_2 + dv_2, cu_2' + dv_2')) + \]

\[ + B((u_1, u_1', v_1, v_1'), (u_2, u_2', v_2, v_2')) = \]

\[ = ac(\sigma(u_1, u_1') + < u_2, u_1' >) + bd(< v_1, v_2' > + < v_2, v_1' >) + \]

\[ + (ad + 1)(< u_1, v_2' > + < v_2, u_1' >) = \]

\[ = ac(\sigma(u_1, u_1') + < u_2, u_1' >) + bd(< v_1, v_2' > + < v_2, v_1' >) + \]

\[ + f((u_1, u_1', v_1, v_1') + (u_2, u_2', v_2, v_2')) + f((u_1, u_1', v_1, v_1')) = \]

To check that this is a morphism of groups, one needs to check that given \( g_1, g_2 \in SL_2(\mathbb{F}) \) which map into \( (\sigma_1, f_1), (\sigma_2, f_2) \), the quadratic form associated to \( \sigma_1 \sigma_2 \) is \( f_1 \sigma_2 + f_2 \). Indeed

\[ (f_1 \sigma_2 + f_2)(u, u', v, v') = \]

\[ = (a_1 c_1 < a_2 u + b_2 v, a_2 u' + b_2 v' > + b_1 d_1 < c_2 u + d_2 v, c_2 u' + d_2 v'> + \]

\[ + b_1 c_1 < a_2 u + b_2 v, c_2 u' + d_2 v'> + c_2 u + d_2 v, a_2 u' + b_2 v' >) + \]

\[ + a_2 c_2 < u, u' > + b_2 d_2 < v, v'> + b_2 c_2 < u, v'> + < v, u' >) = \]

\[ = (a_1 c_1 a_2^2 + b_1 d_1 c_2^2 + a_2 c_2) < u, u' > + (a_1 c_1 b_2^2 + b_1 d_1 d_2^2 + b_2 d_2) < v, v' > + \]

\[ + (a_1 c_1 b_2 a_2 b_2 + b_1 d_1 c_2 b_2 + b_1 c_1 d_1 c_2 + b_1 c_1 a_2 d_1 + a_2 c_2 b_1 c_1) < u, u' > + \]

\[ + (a_1 c_1 b_2 a_2 b_2 + b_1 d_1 c_2 b_2 + a_1 d_1 b_2 d_2 + b_1 c_1 b_2 d_2)(< u, v' > + < v, u' >) = \]

\[ = (a_1 c_1 a_2 b_2 + a_1 c_1 a_2 + d_1 c_2) < u, u' > + (a_1 b_2 + b_1 d_1)(c_1 b_2 + d_1 d_2) < v, v' > + \]

\[ + (a_1 b_2 + b_1 d_1)(c_1 a_2 + d_1 c_2) < u, v' > + < v, u' >) \]

which is indeed the quadratic form associated to \( \sigma_1 \sigma_2 \). \( \square \)

Theorem A.4 (see [9] and [5]). Fix an additive character \( \psi \) of \( \mathbb{F} \). There is a projective representation \( \rho \) of \( PSp \) on \( S(X) \), such that for any \( a, b \in PSp \), we have \( \rho(ab)^{-1} \rho(a) \rho(b) = \pm 1 \). We also have the explicit formulas, with \( x = (v, v') \in X \):

\[ \left( \left( \rho j \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \right) \Phi \right) (x) = |t|^\dim V \Phi(tx), \]

\[ \left( \left( \rho j \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \right) \Phi \right) (x) = \psi(u < v, v'>)\Phi(x), \]

\[ \left( \left( \rho j \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right) \Phi \right) (x) = \mathcal{F}(\Phi)(x). \]

In the last equation \( \mathcal{F} \) denotes the Fourier transform on \( X \) with respect to the symmetric bilinear non-degenerate form \( \langle (u, u'), (v, v') \rangle := < u, v' > + < v, u' > \).
Proof of Theorem A.1. The projective action of $PSp$ on $S(V \oplus V^*)$ gives a projective representation of $PSp$ on $S^*(V \oplus V^*)$. Looking in the formulas for this action, we see that if $\xi \in S^*(V \oplus V^*)$ is as in the formulation of the theorem, we have

$$\rho_j \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} (\xi) = \pm \xi$$

and also

$$\rho_j \begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix} (\xi) = \rho_j \left( \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right) (\xi) = \pm \xi.$$

Since elements of the form $\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix}$ generate $SL_2(\mathbb{F})$, we get that $\xi$ is $SL_2(\mathbb{F}) \pm$-invariant. In particular $\rho_j \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} (\xi) = \pm \xi$. So we get that

$$|t|^{-\dim V} \cdot |(t\xi)(\Phi)| = |\xi(\Phi)|$$

as desired. \qed

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