The Generalized Ricci Flow for 3D Manifolds with One Killing Vector

J. Gegenberg † G. Kunstatter ♯

† Dept. of Mathematics and Statistics and Department of Physics,
University of New Brunswick
Fredericton, New Brunswick, Canada E3B 5A3

‡ Dept. of Physics and Winnipeg Institute of Theoretical Physics,
University of Winnipeg
Winnipeg, Manitoba, Canada R3B 2E9

Abstract

We consider 3D flow equations inspired by the renormalization group (RG) equations of string theory with a three dimensional target space. By modifying the flow equations to include a U(1) gauge field, and adding carefully chosen De Turck terms, we are able to extend recent 2D results of Bakas to the case of a 3D Riemannian metric with one Killing vector. In particular, we show that the RG flow with De Turck terms can be reduced to two equations: the continual Toda flow solved by Bakas, plus its linearization. We find exact solutions which flow to homogeneous but not always isotropic geometries.
1 Introduction

The Ricci flow of d-dimensional manifolds is interesting because of its relationship to the renormalization group equations of generalized 2D sigma models with d-dimensional target space. In two spacetime dimensions, Ricci flow also provides a proof of the uniformization theorem\[1\], which states that every closed orientable two dimensional manifold with handle number 0,1, or > 1 admits uniquely the constant curvature geometry with positive, zero, or negative curvatures, respectively. Bakas \[3\] has shown that the 2D Ricci flow equations in conformal gauge provide a continual analogue of the Toda field equations. Using this algebraic approach he was able to write down the general solution.

The potential importance of a 3D uniformization theorem is evident particularly in the context of (super)membrane physics and three-dimensional quantum gravity where one should be able to perform path-integral quantization via a similar procedure to that in two dimensions. Unfortunately, there is no uniformization theorem in three dimensions, only a conjecture due to W.P. Thurston. \[4\] \[5\].

Recently there has been speculation that Perelman \[6\] has overcome some roadblocks in Hamilton’s program to prove the conjecture using the ‘Ricci flow’ \[7\] \[8\]. It is therefore important to understand in detail the properties of this flow.

In the following, we follow up on a suggestion by Bakas to use his 2D results in order to analyze the flow equations for 3D manifolds with a single Killing vector. This provides a tractable midisuperspace approach which can be systematically studied in the context of the ‘stringy flow’ first considered in \[2\]. We will show that this flow reduces to the infinite dimensional generalization of the Toda equation for the conformal factor of the invariant 2D submanifold plus a linear equation for the scale factor of the extra dimension. Note that since the latter scale factor depends on the coordinates of the invariant subspace, our manifolds are not simple direct products. In addition, we will analyze two exact analytic solutions in detail and show that they have the expected behaviour.

The paper is organized as follows. Section 2 reviews 2D flow equations and Bakas’ results, Section 3 reviews the stringy flow of \[2\], but with the De Turck modification \[9\]. The De Turck modification contains a vector field $\xi_i$, and we show that if we choose this vector field as a linear combination of two vector fields, one of which is proportional to the gradient of the dilaton field,
then the dilaton can be decoupled from the flow of the remaining fields. In Section 4 we discuss the flow for a metric ansatz where one of the coordinates is in the direction of a Killing vector field, and the remaining part of the metric is in the form of a conformal 2D metric. We use the second part of the De Turck vector field to preserve this form of the metric throughout the flow. In order that the flow is self-consistent, the U(1) vector field in the stringy flow must be fixed in terms of the functions that occur in the metric tensor. The flow then reduces to two equations for the two metric degrees of freedom. One of these is the ‘continual Toda equation’ for the conformal factor of the 2D geometry orthogonal to the Killing vector field, and the other, for the component of the metric in the direction of the Killing vector field, is the linearization of the continual Toda equation. Section 5 presents specific solutions and Section 6 ends with conclusions and prospects for future work.

2 The 2D Case

We now summarize the methodology and results of Bakas since they play a crucial role in the following. The Ricci flow equations, for arbitrary 2-metric $g_{AB}$ are:

$$\frac{\partial g_{ij}}{\partial t} = -R_{ij} + \nabla_i \xi_j + \nabla_j \xi_i. \quad (1)$$

The last two terms (the so-called “De Turck” terms) incorporate the effects of all possible diffeomorphisms and can be chosen arbitrarily in order to simplify the equations and/or optimize convergence. Originally, De Turck chose the vector field

$$\xi^i := g^{jk} \left( \Gamma^i_{jk} - \Delta^i_{jk} \right), \quad (2)$$

where $\Gamma^i_{jk}$ is the Christoffel connection with respect to the Riemannian metric $g_{ij}$ and $\Delta^i_{jk}$ is a fixed ‘background connection’. The purpose was to replace the Ricci flow, which is only weakly parabolic, by an equivalent flow which is strongly parabolic.

Bakas chose to work in the conformal gauge:

$$ds^2 = g_{ij} dx^i dx^j = \frac{1}{2} \exp(\Phi)(dx^2 + dy^2). \quad (3)$$

In this gauge there is no need to add De Turck terms and the flow takes the
form of a non-linear “heat equation”:
\[ \frac{\partial}{\partial t} e^\Phi = \nabla^2 \Phi. \] (4)

The Toda equations describe the integrable interactions of a collection of two dimensional fields \( \Phi_i(x, y) \) coupled via the Cartan matrix \( K_{ij} \):
\[ \sum_j K_{ij} e^{\Phi_j(x,y)} = \nabla^2 \Phi_i(x, y). \] (5)

Bakas argues that Eq.(4) is a continual analogue of the above, with the Cartan matrix replaced by the kernel:
\[ K_{ij} \rightarrow K(t, t') = \frac{\partial}{\partial t} \delta(t, t'). \] (6)

This leads to a general solution to (4) in terms of a power series around the free field expanded in path ordered exponentials. Although the resulting expression is difficult to work with explicitly, it does provide a formal complete solution to the 2D flow equations.

In the next sections we will show that a similar formal solution can also be found for three dimensional metrics with at least one Killing field.

3 3D Flow Equations

We consider here a generalization of the Ricci flow, in which, besides the metric \( g_{ij} \), there are additional fields which flow, consisting of a dilaton \( \phi \), a gauge two-form potential \( B_{ij} \) with field strength \( H_{ijk} \) and finally, a \( U(1) \) gauge field with potential 1-form \( A_i \) and corresponding field strength \( F_{ij} \) which couples as a ‘Maxwell-Chern-Simons theory’. Including De Turck terms plus gauge terms for the flow of the non-metric fields, the flow is

\[ \dot{g}_{ij} = -2 \left( R_{ij} + 2\phi_{ij} - (\epsilon_F F_i^k F_{jk} + \frac{\epsilon_H}{4} H_{ikl} H_{j}^{kl} \right) + L_\xi g_{ij}, \] (7)

\[ \dot{A}_i = - \left( e^{2\phi} \nabla_j (e^{-2\phi} F_{ij}) + \frac{\epsilon_F}{2} \eta^{jk} F_{jk} \right) + L_\xi A_i + \partial_i \Lambda, \] (8)

\[ \dot{B}_{ij} = e^{2\phi} \nabla_k (e^{-2\phi} H^{k}_{ij}) + L_\xi B_{ij} + \partial_i \Lambda_j - \partial_j \Lambda_i, \] (9)

\[ \dot{\phi} = -\chi + \Delta \phi - |\nabla \phi|^2 + \frac{\epsilon_F}{2} F^2 + \frac{\epsilon_H}{12} + L_\xi \phi. \] (10)
In the above, $L_\xi$ denotes the Lie derivative with respect to the (covariant) vector field $\xi$. The Lie derivative terms are present because we are flowing geometrical objects that are not coordinate invariant, so their time derivatives should only be determined up to arbitrary gauge transformations at each point along the flow. Similarly, the terms containing $\Lambda$ and $\Lambda_i$ correspond to arbitrary gauge transformations on the gauge fields. By choosing the gauge and coordinate transformation terms judiciously, we are able to simplify the equations considerably.

This flow is motivated by two considerations. First, as shown in [2], all of the Thurston geometries are solutions of the equations of motion of this theory for various values of the parameters $\chi, \epsilon_H, \epsilon_F, e$, as well as the other fields. In particular, the addition of the Maxwell term alone ($e = 0$) yields $S^2 \times E^1, H^2 \times E^1$ and $Sol$ as solutions. Moreover, there exists a generalized Birkhoff theorem which guarantees that these are the only solutions when $\phi = \text{constant}$ and $A \neq 0$. With $e \neq 0$, one finds that the remaining Thurston geometries $Nil$ and $SL(2,\mathbb{R})$ are also solutions. As argued in [2] it seems plausible that these are the only solutions, but to date no rigorous proof exists.

The second motivation comes from string theory. In particular, the RG flow for a non-linear sigma model with a 4D Kaluza-Klein target space resembles the flow above, with the $A_i$ potential originating as the twist potential of the 4D Kaluza-Klein metric. The details of this are being investigated elsewhere [10].

We choose $\xi_i = k_i + 2\nabla_i \phi$ and let $\Lambda = -2A_j \phi^{ij}$ and $\Lambda_i = 2B_{ji} \phi^{ij}$, where $k_i$ is as yet arbitrary. With these choices the dilaton is completely eliminated from the flow equations for the metric and gauge fields:

$$
\dot{g}_{ij} = -2 \left( R_{ij} - (\epsilon_F F_{i}^{k} F_{jk} + \frac{\epsilon_H}{4} H_{ikl} H_{j}^{kl}) \right) + L_k g_{ij},
$$

$$
\dot{A}_i = - \left( \nabla_j F_{ij}^{\ell} + \frac{\epsilon_F}{2} \eta^{jk} F_{jk} \right) + L_k A_i + \partial_i \lambda,
$$

$$
\dot{B}_{ij} = \nabla_k H_{ij}^k + L_k B_{ij} + \partial_i \lambda_j - \partial_j \lambda_i,
$$

$$
\dot{\phi} = -\chi + \Delta \phi - 2|\nabla \phi|^2 + \frac{\epsilon_F}{2} F^2 + \frac{\epsilon_H}{6} H^2 + L_k \phi.
$$

The arbitrary vector $k^i$ and gauge parameters $\lambda, \lambda_i$ indicate that we are still free to add further De Turck and gauge terms to the equations. We will use this freedom later to simplify the equations that result from a particular ansatz.
4 A Particular Case with One Killing Vector Field

Henceforth we set $B_{ij}$ identically equal to 0, which is consistent with the flow equations. We also consider the case $e = 0$ (no Chern-Simons term). We assume the metric to have a single Killing vector and to be manifestly hypersurface orthogonal (i.e. diagonal):

$$ds^2 = e^\Phi(dx^2 + dy^2) + e^\sigma dw^2,$$

We also choose the following ansatz for the vector potential:

$$A_i = [e^\sigma/2, 0, 0].$$

Consistency of the above ansatz requires that the flow equations preserve the diagonal nature of the metric. It turns out that this can be accomplished by choosing the vector field $k_i$ as

$$k_i = -\frac{1}{2}\partial_i \sigma,$$

With these choices the flow equations simplify to:

$$\dot{g}_{xx} = e^\Phi \dot{\Phi} = \nabla^2 \Phi + \frac{1}{2}(1 + \epsilon_F)(\partial_x \sigma)^2,$$

$$\dot{g}_{yy} = e^\Phi \dot{\Phi} = \nabla^2 \Phi + \frac{1}{2}(1 + \epsilon_F)(\partial_y \sigma)^2,$$

$$\dot{g}_{xy} = 0 = \frac{1}{2}(1 + \epsilon_F)\partial_x \sigma \partial_y \sigma,$$

$$\dot{A}_x = \dot{A}_y = 0,$$

$$\dot{A}_w = \epsilon_f \nabla^2 e^\sigma/2 + (1 + \epsilon_F)(\partial \sigma)^2.$$

In the above, $\nabla^2$ denotes the flat space Laplacian.

We now fix $\epsilon_F = -1$, in which case the flow boils down to two simple partial differential equations. The first is

$$\partial_t e^\Phi = \nabla^2 \Phi,$$
which is the ‘continual Toda eqn’ a la [3]. The other flow is the linearization of the continual Toda flow:

\[ e^\Phi \partial_t e^{-\sigma/2} = -\nabla^2 e^{-\sigma/2}. \tag{21} \]

To the best of our knowledge, there is no ‘simpler flow’ constructed from the Ricci-De Turck flow alone, without other fields, which can self-consistently flow the metric preserving the manifestly static form. Note that for any given solution \( \Phi \) of (20), there exists a corresponding solution for \( \sigma \):

\[ e^{-\sigma/2} = \Phi(x, y, -t) + \chi(x, y), \tag{22} \]

where \( \chi(x, y) \) is any harmonic function on the \( x, y \) subspace, i.e. satisfying: \( \nabla^2 \chi = 0 \).

5 Exact Solutions of the Flow

We first examine a non-trivial flow, namely the sausage solution of Bakas [3] (also called the Rossineau flow in the mathematical literature [8]). This is an exact solution of the continual Toda equation of the form:

\[ e^\Phi = \frac{2 \sinh [2\gamma t]}{\gamma (\cosh [2\gamma t] + \cosh (2y))}. \tag{23} \]

In this case:

\[ e^{-\sigma/2} = \ln \left| \frac{2 \sinh [2\gamma t]}{\gamma (\cosh [2\gamma t] + \cosh (2y))} \right|, \tag{24} \]

where we have eliminated an imaginary term from \( e^{-\sigma/2} \) by using the freedom to shift by a harmonic function.

In the limit as \( t \to \infty \), \( e^\Phi \to 2/\gamma \), and \( e^\sigma \to \ln |2/\gamma|^{-2} \), so that the Ricci tensor goes to zero and in this limit, the geometry is flat. On the other hand, in the limit \( t \to 0^+ \), \( e^\Phi \to 2t/\cosh^2 y \). In this limit, we find that the Ricci scalar \( R \sim \frac{1}{t} \). So, if we flow the highly curved non-homogeneous metric with initial value at \( t = \epsilon > 0 \)

\[ ds^2 = \left( \ln \frac{2\epsilon}{\cosh^2 y} \right)^{-2} dw^2 + \frac{2\epsilon}{\cosh^2 y} \left( dx^2 + dy^2 \right), \tag{25} \]

we end up at \( t \to \infty \) with the flat metric. This is consistent with Thurston’s conjecture.
The second type of solution is of the Liouville type. We set

\[ e^{\Phi(x,y;t)} = T(t)e^{\psi(x,y)} . \]  

(26)

Now for \( t \geq 0 \), we find that

\[
T(t) = \beta t, \\
\nabla^2 \psi - \beta e^{\psi} = 0,
\]

(27)

where \( \beta \) is a separation constant. The second of the above equations is the Liouville equation, so the two dimensional part of the metric, \( e^{\Phi}(dx^2 + dy^2) \) has constant negative curvature (for \( t \geq 0 \)).

Again we choose \( e^{-\sigma/2} = \Phi \), so that

\[ e^{\sigma} = [\log \beta t + \psi(x,y)]^{-2} . \]

(28)

The separation constant \( \log \beta \) can be absorbed into \( \psi \) without loss of generality.

Hence the metric is

\[ ds^2 = [\log t + \psi(x,y)]^{-2} dw^2 + \beta t e^{\psi(x,y)}(dx^2 + dy^2) . \]

(29)

The quantity \( \psi(x,y) \) is a solution of the Liouville equation.

If \( t \geq 0 \), then the flow starts from some highly curved non-homogeneous metric near \( t = 0 \). As \( t \to \infty \), we have

\[
R_{AB} \sim -\frac{1}{2t} g_{AB}, \\
R_{ww} \sim 0,
\]

(30)

with \( A,B,\ldots = x,y \). Hence, the geometry is asymptotically that of the homogeneous, but anisotropic geometry \( H^2 \times E^1 \).

Thus the flow is consistent with the Thurston conjecture.

6 Conclusions

We have shown that the modified Ricci flow equations Eq(9) for 3D metrics with at least one Killing vector can be integrated in precisely the same manner as the 2D equations, at least for the special case \( \epsilon_F = -1 \). In addition
to extending this analysis to other values of the parameters in the action, and hence to other topologies, it is interesting to speculate whether these techniques could work for more general 3D metrics.

Consider, without loss of generality, a diagonal metric

$$ds^2 = e^{\Phi_1(x;t)}(dx^1)^2 + e^{\Phi_2(x;t)}(dx^2)^2 + e^{\Phi_3(x;t)}(dx^3)^2,$$

(31)

where the functions $\Phi_i(x; t)$ depend on all 3 coordinates $x^i$. The resulting bare Ricci flow is again not manifestly elliptic, and the equations have non-trivial off-diagonal terms on the RHS that make direct integration difficult. Since in three dimensions any metric can be made diagonal with a suitable coordinate transformation, it is reasonable to assume that there exists a modified flow that ensures that diagonal metrics evolve into diagonal metrics. We have as yet not succeeded in constructing this modified flow, but if it did exist, it is possible that the resulting three flow equations for each of the three scale factors would take a form similar to what we have found above, albeit with non-trivial coupling. It may therefore provide a basis for solving the 3D flow equations in a more general setting.

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