Analysing a prey predator model for stability with prey refuge-stage structure on predator

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**Abstract:** This paper analyses a model of prey predator with Michaelis Menten-Holling type functional response with inclusion of a constant proportion of prey refuge and predator stage structure. The analysed model consists of three nonlinear ordinary differential equations which describe the interaction between the prey and juvenile & adult predators. The system is studied for its local stability by determining the equilibrium points and obtaining the conditions for its local stability. Suitable Lyapunov function has been constructed to obtain the global asymptotic stability of the system.

**Keywords:** prey predator, functional response, prey refuge, stage structure, stability.

1. Introduction

The interaction between prey and predator species has been generously studied by many researchers in the recent years. The interaction between them has been modeled mathematically as differential equations and the qualitative analysis of ODE has been used in studying its dynamical properties. The modeling work was first initiated in 1920’s independently by two mathematicians Vito Volterra and Lotka. In real life, the prey predator interaction depends on many factors that include carrying capacity of the biological system, competition for food among the same species, response of predator to prey, harvesting individual or both the species etc.,. Amongst them the major factor is the functional response which describes the number of prey that would be consumed by a predator in unit time. Depending upon the behavior of both the species, desirable functional responses have been arrived at. The functional responses that depend solely on prey and on predator are termed as prey dependent and predator dependent respectively. Eg Holling type I, II, III, IV[2,4], ratio dependent functional responses[7,11], Beddington De-Angelis[4,6,9], Leslie Gower[1]. In ratio dependent the predator response function is F(prey/predator) in unit time. It relies on the density of prey and predator together.

A number of studies in the recent times have explicated the multifaceted physical and biological situations between predator and prey. Situations which influence the predator’s search for food followed by competition and sharing of food warrant a more suitable general prey predator theory that relies on ratio dependent model.

In many cases, the life style of different species passes across two life stages – mature and immature. The young ones cannot hunt or reproduce and hence they depend on their mature ones for their food. Therefore introduction of stage structure is inevitable in these situations and it has been studied by [2,5,8,10] for various types of functional responses. Also prey refuge plays a vital part in preventing the prey species becoming extinct, since the prey protect themselves from being attacked by predators.
Here we consider a ratio dependent Michaelis Menten – Holling type prey predator model [7]. A stage structure on predator has been introduced with prey refuge.

2. Mathematical model
At any time \( t \), let the population densities of the prey, juvenile predator and adult predator be represented by \( u(t) \), \( v_1(t) \), \( v_2(t) \) respectively. The model focused here is of the form:

\[
\begin{align*}
\dot{u} &= R u \left( 1 - \frac{u}{K} \right) - \frac{A u (1 - \lambda) v_2}{u (1 - \lambda) + K_1 v_2} \\
\dot{v}_1 &= -D v_1 - d_1 v_1 + \frac{B A u (1 - \lambda) v_2}{u (1 - \lambda) + K_1 v_2} \\
\dot{v}_2 &= D v_1 - d_2 v_2 \\
\end{align*}
\]  

(1)

where the parameters \( R, K, A, K_1, B \) are positive constants. \( R \) and \( K \) represent the intrinsic growth rate of the prey and carrying capacity of the environment respectively. \( A, K_1, B \) represents the capture rate of the predator, predators benefit from cofeeding and conversion coefficient of the predator respectively. \( D, d_1, d_2 \) represents the transition rate from juvenile predator to mature predator, mortality rate of juvenile predator and mature predator respectively. A constant proportion of the prey \( \lambda \in [0, 1) \) is supposed to take refuge and still \( (1 - \lambda)U \) of the prey are left behind for the predators. It is also assumed that only the mature predator is capable of hunting and the immature predator depend on the adults for their food.

The following scaling minimizes the parameters,

\[
U = \frac{u}{K}, V_i = \frac{K_i}{K} v_i, V_2 = \frac{K_i}{K} v_2, T = R t
\]

The system (1) becomes

\[
\begin{align*}
\dot{U} &= U (1 - U) - \frac{b U (1 - \lambda) V_2}{U (1 - \lambda) + V_2} = f_1(U, V_1, V_2) \\
\dot{V}_1 &= -D v_1 - a_1 v_1 + \frac{b c U (1 - \lambda) V_2}{U (1 - \lambda) + V_2} = f_2(U, V_1, V_2) \\
\dot{V}_2 &= D v_1 - a_2 v_2 = f_3(U, V_1, V_2) \\
\end{align*}
\]

(3)

where \( b = \frac{A}{K_i R}, c = B K_i D_i = \frac{D}{R}, a_1 = \frac{d_1}{R}, a_2 = \frac{d_2}{R} \)

3. Positivity and Boundedness
The variables \( U, V_1, V_2 \) represent the living species and hence the system of equations (3) with (4) has the domain \( \mathbb{R}^3 \). Further the functions \( f_1, f_2, f_3 \) of the system of equations (3) are Lipschitz continuous and are partially differentiable on the state space \( \mathbb{R}^3 \). Therefore
solution of system (3) is unique for any given initial values that belong to $\mathbb{R}_+^3$. Also all the solutions of system (3) with non negative initial values are uniformly bounded.

**Theorem 1:** All the solutions of the model equation (3) with (4) that start in $\mathbb{R}_+^3$ are uniformly bounded.

**Proof:** Let $(U(T), V_1(T), V_2(T))$ be any solution of model equation (3) with non negative initial conditions $(U_0, V_{10}, V_{20})$. From equation (3), we have

$$\dot{U} < U(1-U)$$

Solving this differential inequality we get

$$U(T) \leq \frac{e^t U_0}{1 - U_0 + e^t U_0}$$

Therefore, $\lim_{T \to \infty} \sup U(T) \leq 1$.

Define the function

$$\varphi(T) = U(T) + \frac{1}{c} V_1(T) + \frac{1}{c} V_2(T)$$

The time derivative of the function $\varphi(T)$ along the solution curve of the system (3) is derived as:

$$\dot{\varphi}(T) = \dot{U}(T) + \frac{1}{c} \dot{V}_1(T) + \frac{1}{c} \dot{V}_2(T)$$

Let $d = \min \{a_1, a_2 \}$

Then we have,

$$\varphi + d \varphi \leq (1+d)$$

solving the above we get,

$$\varphi(T) \leq \frac{(1+d)}{d} + \varphi(0)e^{-dT} - \frac{(1+d)}{d}e^{-dT}$$

$T \to \infty$, gives

$$\varphi(T) \leq \left[ \frac{1+d}{d} \right]$$

Hence all the solutions of the system (3) enters into the region

$$\mathcal{N} = \left\{ (U(T), V_1(T), V_2(T) \in \mathbb{R}_+^3) : U(T) \leq 1 \text{ and } \varphi(T) \leq \left[ \frac{1+d}{d} \right] \right\}$$

which proves the uniform boundedness.

**4. Equilibrium points and stability analysis of the system**

In this section, the existence of the equilibrium points of the system (3) are investigated and examined for their local stability at all possible non negative equilibrium points.

**4.1. Equilibrium Points**

System (3) has three non negative equilibrium points.

- a. The equilibrium point $P_0(0, 0, 0)$ always exists and is trivial.
- b. The prey population grows to the carrying capacity in the absence of predator. Therefore the axial equilibrium $P_1(\bar{U}, 0, 0) = P_1(1, 0, 0)$ always exists.
c. The predator population perish without prey and consequently equilibrium point no longer exists along the \( V_1V_2 \)-plane.

d. The interior equilibrium point \( P_2(\text{U}', \text{V}', \text{W}') \) is obtained by solving the following system of equations:

\[
(1 - \text{U}) - \frac{\text{bU}(1 - \lambda_1)\text{V}_2}{\text{U}(1 - \lambda) + \text{V}_2} = 0; \quad \text{D}_1\text{V}_1 - \text{a}_1\text{V}_1 + \frac{\text{bcU}(1 - \lambda_2)\text{V}_2}{\text{U}(1 - \lambda) + \text{V}_2} = 0; \quad \text{D}_1\text{V}_1 - \text{a}_2\text{V}_2 = 0
\]

Solving this we get

\[
\text{U'} = 1 - \frac{(1 - \lambda_1)}{\text{D}_1\text{c}} \left[ \text{D}_1\text{bc} - (\text{D}_1 + \text{a}_1)\text{a}_2 \right]; \quad \text{V}_2' = (1 - \lambda_2) \left[ \frac{\text{D}_1\text{bc}}{(\text{D}_1 + \text{a}_1)(\text{a}_2)} - 1 \right] \text{U'}; \quad \text{V}_1' = \frac{\text{a}_2}{\text{D}_1}\text{V}_2'
\]

4.2. Local stability analysis:

To observe the local dynamical behavior of the model near the equilibrium points, the variational matrix of the system (3) at each of these equilibrium points are found. The eigenvalues of the resulting variational matrix are then obtained.

At the equilibrium points \( P_i \), the variational matrix of system (3) is given by

\[
J(P_i) = \begin{bmatrix}
\frac{\partial f_1}{\partial \text{U}} & \frac{\partial f_1}{\partial \text{V}_1} & \frac{\partial f_1}{\partial \text{V}_2} \\
\frac{\partial f_2}{\partial \text{U}} & \frac{\partial f_2}{\partial \text{V}_1} & \frac{\partial f_2}{\partial \text{V}_2} \\
\frac{\partial f_3}{\partial \text{U}} & \frac{\partial f_3}{\partial \text{V}_1} & \frac{\partial f_3}{\partial \text{V}_2}
\end{bmatrix}
\]

**Proposition 1**: The equilibrium point \( P_0(0, 0, 0) \) is a saddle node.

**Proof**: At the equilibrium point \( P_0 \), the variational matrix of the system (3) is

\[
J(P_0) = \begin{bmatrix}
1 & 0 & 0 \\
0 & -(\text{D}_1 + \text{a}_1) & 0 \\
0 & \text{D}_1 & -\text{a}_2
\end{bmatrix}
\]

The eigenvalues are 1, \(-(\text{D}_1 + \text{a}_1)\), \(-\text{a}_2\). One of the eigenvalues is positive and the other two eigenvalues are negative. Hence the trivial equilibrium \( P_0 \) is a saddle point.

**Theorem 2**: The equilibrium point \( P_1(1, 0, 0) \) of system (3) is locally asymptotically stable if \((\text{D}_1 + \text{a}_1)\text{a}_2 > \text{bcD}_1\).

**Proof**: At the equilibrium point \( P_1 \), the variational matrix of the system (3) is

\[
J(P_1) = \begin{bmatrix}
-1 & 0 & -\text{b} \\
0 & -(\text{D}_1 + \text{a}_1) & \text{bc} \\
0 & \text{D}_1 & -\text{a}_2
\end{bmatrix}
\]

The eigenvalues are \(-1, -(\text{D}_1 + \text{a}_1 + \text{a}_2)\) and \(\text{bcD}_1 - (\text{D}_1 + \text{a}_1)\text{a}_2\). Obviously the equilibrium point \( P_1 \) is locally asymptotically stable provided \((\text{D}_1 + \text{a}_1)\text{a}_2 > \text{bcD}_1\). Hence the theorem.
Theorem 3: The interior equilibrium point $P_2(U^*, V_1^*, V_2^*)$ is locally asymptotically stable under the conditions $(D_1 + a_1) a_2 > \frac{b c U^* (1 - \lambda)^2 D_1}{U(1 - \lambda) + V_1^*}$ and $H_1 H_2 - H_3 > 0$.

Proof: The variational matrix of the system (3) at the equilibrium point $P_2$ is

$$J(P_2) = \begin{bmatrix}
1 - 2U^* - \frac{b V_2^* (1 - \lambda)}{U^* (1 - \lambda) + V_2^*} & 0 & -\frac{b U^* (1 - \lambda)^2}{U^* (1 - \lambda) + V_2^*} \\
\frac{bc V_2^* (1 - \lambda)}{U^* (1 - \lambda) + V_2^*} & -(D_1 + a_1) & \frac{bc U^* (1 - \lambda)^2}{U^* (1 - \lambda) + V_2^*} \\
0 & D_1 & -a_2
\end{bmatrix}$$

For the above variational matrix, the characteristic equation is $\omega^3 + H_1 \omega^2 + H_2 \omega + H_3 = 0$ where $H_i = -(h_{11} + h_{23} + h_{33})$

$h_{11} = h_{11} h_{22} + h_{13} h_{33} + h_{23} h_{32} - h_{33} h_{32}$

$h_{12} = h_{11} (h_{23} h_{32} - h_{22} h_{33}) - h_{13} h_{23} h_{32}$

$h_{13} = \frac{b c U^* (1 - \lambda)^2}{U^* (1 - \lambda) + V_2^*}$; $h_{21} = \frac{b c V_2^* (1 - \lambda)}{U^* (1 - \lambda) + V_2^*}$

$h_{22} = -(D_1 + a_1)$; $h_{23} = \frac{b c U^* (1 - \lambda)^2}{U^* (1 - \lambda) + V_2^*}$; $h_{31} = D_1$; $h_{32} = -a_2$

By Routh Hurwitz criterion system (3) is locally asymptotically stable if $H_i > 0$ for $i=1,3$ and $H_1 H_2 - H_3 > 0$. Hence the theorem.

4.3. Global stability:
By constructing appropriate Lyapunov function the global stability of the system (3) is obtained near the equilibrium points $P_1$ and $P_2$.

If Theorem 4: the equilibrium point $P_1$ is locally asymptotically stable with the following condition

$$\frac{(D_1 + a_1) a_2}{Dc} > \frac{b (1 - \lambda) \tilde{U}}{U(1 - \lambda) + V_2}$$

then it is a globally asymptotically stable equilibrium point.

Proof: Consider the following function about point $P_1(\tilde{U}, 0, 0)$

$$L(U, V_1, V_2) = n_1 \left[ U - \tilde{U} - \frac{U}{\tilde{U}} \ln \frac{U}{\tilde{U}} \right] + n_2 V_1 + n_3 V_2$$

where $n_1$, $n_2$, and $n_3$ are positive constants which are to be found. It is obvious that $L$ is a positive definite function.
\[ \dot{L} = n_1 \left( \frac{U - \bar{U}}{U} \right) \dot{U} + n_2 \dot{V}_1 + n_3 \dot{V}_2 \]
\[ = -n_1 (U - \bar{U})^2 - \frac{U b(1 - \lambda) V_2}{U (1 - \lambda) + V_2} (n_1 - n_2 c) - \left[ n_3 (D_1 + a_i) - n_3 D_1 \right] V_1 - \left[ n_3 a_2 - \frac{b(1 - \lambda) \bar{U}}{U (1 - \lambda) + V_2} \right] V_2 \]

Take \( n_1 = 1, \ n_2 = \frac{1}{c}, \ n_3 = \frac{(D_1 + a_i)}{Dc} \)

\[ \dot{L} = -(U - \bar{U})^2 - \frac{(D_1 + a_i) a_2}{Dc} - \frac{b(1 - \lambda) \bar{U}}{U (1 - \lambda) + V_2} V_2 \]

Now, if the condition (5) stated in the theorem holds then we get \( \dot{L} < 0 \). Hence the function defined by \( L \) is Lyapunov. Therefore, \( P_2 \) is globally asymptotically stable in the int \( R^3_+ \).

**Theorem 5:** Let the equilibrium point \( P_2 \) exists in the positive quadrant and is locally asymptotically stable in the int \( R^3_+ \), then \( P_2 \) is globally asymptotically stable under the following conditions.

\[ c_{12}^2 < c_{12} c_{22}; \quad c_{13}^2 < c_{13} c_{33}; \quad c_{23}^2 < c_{23} c_{33} \] (6)

where,

\[ c_{11} = \frac{1 - b(1 - \lambda) V_2^*}{SS}, \quad c_{22} = \frac{\left( D_1 + a_i \right)}{V_1}; \quad c_{13} = \frac{a_2}{V_2}; \quad c_{12} = \frac{bc(1 - \lambda) V_2^*}{V_1 SS}; \quad c_{23} = \frac{bc U(1 - \lambda)}{V_1 S} - \frac{bc U^*(1 - \lambda) V_2^*}{V_1 SS} \]

with \( S = U(1 - \lambda) + V_2 \) and \( S_1 = U^*(1 - \lambda) + V_2^* \)

**Proof:** At the equilibrium point \( P_2 \), define the positive definite function \( L_1 \) as

\[ L_1(U, V_1, V_2) = \left[ \frac{U - U^* - U^* \ln \frac{U}{U^*}}{U^*} \right] + \left[ \frac{V_1 - V_1^* - V_1^* \ln \frac{V_1}{V_1^*}}{V_1^*} \right] + \left[ \frac{V_2 - V_2^* - V_2^* \ln \frac{V_2}{V_2^*}}{V_2^*} \right] \]

Obviously \( L_1 : R^1 \rightarrow R \) such that \( L_1(P_2) = 0 \) and \( L_1(U, V_1, V_2) > 0 \) for all \((U, V_1, V_2) \in R^3_+ \)
and \((U, V_1, V_2) \neq P_2 \). Hence \( L_1 \) is a positive definite function.

Now,

\[ \dot{L}_1 = \left[ \frac{U - U^*}{U} \right] \dot{U} + \left[ \frac{V_1 - V_1^*}{V_1^*} \right] \dot{V}_1 + \left[ \frac{V_2 - V_2^*}{V_2^*} \right] \dot{V}_2 \]
\[ = -\left[ \frac{1 - b(1 - \lambda) V_2^*}{SS} \right] (U - U^*)^2 - \frac{b(1 - \lambda)}{S} (U - U^*)(V_2 - V_2^*) + \frac{bc(1 - \lambda) V_2^*}{V_1} (U - U^*)(V_1 - V_1^*) \]
\[ - \left( \frac{D_1 + a_i}{V_1} \right) (V_1 - V_1^*)^2 + \frac{bc U(1 - \lambda)}{SV_1} + \frac{D_1}{V_2} - \frac{bc(1 - \lambda) U^* V_2^*}{V_1} \left( V_1 - V_1^* \right) (V_2 - V_2^*) - \frac{a_2}{V_2} (V_2 - V_2^*)^2 \]

From the above we get,
\[
\dot{L}_1 = -\frac{1}{2} c_{11} x_1^2 + c_{13} x_1 x_3 - \frac{1}{2} c_{33} x_3^2 - \frac{1}{2} c_{11} x_1^2 + c_{12} x_1 x_2 - \frac{1}{2} c_{22} x_2^2 - \frac{1}{2} c_{22} x_2^2 + c_{23} x_2 x_3 - \frac{1}{2} c_{33} x_3^2
\]

where \( x_1 = (U - U^*); x_2 = (V_1 - V_1^*); x_3 = (V_2 - V_2^*) \)

By using the conditions (6), we obtain that

\[
\dot{L}_1 < -\frac{1}{2} \left[ \sqrt{c_{11} x_1} - \sqrt{c_{33} x_3} \right]^2 - \frac{1}{2} \left[ \sqrt{c_{11} x_1} - \sqrt{c_{22} x_2} \right]^2 - \frac{1}{2} \left[ \sqrt{c_{22} x_2} - \sqrt{c_{33} x_3} \right]^2
\]

Hence \( \dot{L}_1 < 0 \) under the conditions (6) and therefore \( L_1 \) is a Lyapunov function. Therefore, \( P_2 \) is globally asymptotically stable in the int \( R^3 \).

5. Conclusion
In this paper a prey predator model with predator stage structure is considered. Also prey refuge is incorporated in this model. The equilibrium points have been found and analysed for their local stability. The interior equilibrium point is found to be locally asymptotically stable under certain conditions. The conditions for global asymptotic stability have also been obtained.

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