Using the existence of t-designs to prove Erdős-Ko-Rado

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Abstract

In 1984, Wilson proved the Erdős-Ko-Rado theorem for $t$-intersecting families of $k$-subsets of an $n$-set: he showed that if $n \geq (t + 1)(k - t + 1)$ and $\mathcal{F}$ is a family of $k$-subsets of an $n$-set such that any two members of $\mathcal{F}$ have at least $t$ elements in common, then $|\mathcal{F}| \leq \binom{n-t}{k-t}$. His proof made essential use of a matrix whose origin is not obvious. In this paper we show that this matrix can be derived, in a sense, as a projection of $t$-($n, k, 1$) design.

1 Introduction

A family of sets is $t$-intersecting if every two sets in the family have at least $t$ elements in common. The Erdős-Ko-Rado theorem states that if $\mathcal{F}$ a $t$-intersecting family of sets of size $k$ chosen from a set $N$ of size $n$ and $n \geq (t + 1)(k - t + 1)$, then

$$|\mathcal{F}| \geq \binom{n-t}{k-t}.\$$

If $n > (t + 1)(k - t + 1)$, equality holds if and only if $\mathcal{F}$ consists of the $k$-subsets that contain a given set of $t$ points from $V$. The lower bound on $n$ is necessary, because the result is false when the bound fails. Subsequently Ahlswede and Khachatrian [1, 2] determined the maximal families for all $n$. The result as just stated was proved by Wilson in 1984 [7].

The goal of this paper is to motivate a key step in Wilson’s proof. He introduces a “magic matrix” with rows and columns indexed by the $k$-subsets of a $v$-set; he then determines the eigenvalues of this matrix and, given these, fairly standard machinery then leads to the proof of the EKR-bound. From private discussions with Rick Wilson, it is clear that this matrix was the result of a lot of calculation and a lot of inspiration. Our aim in this paper is to

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present a derivation which requires less effort and less brilliance. To this end, we give a simpler formulation of this matrix and show that it is equivalent to that of Wilson, using the recent proof of the existence of $t$-designs of Keevash [5].

2 The Johnson Scheme

Assume $N = \{1, \ldots, n\}$. The Johnson scheme $J(n, k)$ is a set of 01-matrices $A_0, \ldots, A_k$, with rows and columns indexed by the $k$-subsets of $N$, where $(A_r)_{\alpha, \beta} = 1$ if $|\alpha \cap \beta| = k - r$ for $r = 0, \ldots, k$. We see that $A_0 = I$. The matrices $A_1, \ldots, A_k$ are adjacency matrices of graphs $X_1, \ldots, X_k$, where $X_1$ is the so-called Johnson graph. It can be shown that two $k$-subsets are adjacent in $X_r$ if and only if they are at distance $k - r$ in the Johnson graph. The Johnson scheme is discussed in detail in [4, Chapter 6], and anything we state here without proof is treated there.

The matrices $A_r$ satisfy
\[ \sum_r A_r = J. \]

Further, there are scalars $p_{i,j}(r)$ such that, for all $i$ and $j$,
\[ A_i A_j = \sum_r p_{i,j}(r) A_r. \]

Since the product of two symmetric matrices is symmetric if and only if the matrices commute, it follows that the space of the matrices $A_r$ is a commutative matrix algebra. (To use the standard jargon, the matrices $A_0, \ldots, A_k$ form a symmetric association scheme, and their span is known as the Bose-Mesner algebra of the scheme.) All matrices that occur in Wilson’s proof lie in the Bose-Mesner algebra of the Johnson scheme.

To define his matrix, Wilson used another basis for the Bose-Mesner algebra of the Johnson scheme. Let $W_{i,j}(n)$ denote the matrix with rows indexed by the $i$-subsets of $N$, columns indexed by the $j$-subsets of $N$ and with $(\alpha, \beta)$-entry equal to 1 if $\alpha \subseteq \beta$. (So each row of $W_{i,j}(n)$ sums to $\binom{n-1}{i}$.) Let $\overline{W}_{i,j}(n)$ denote the matrix with rows indexed by the $i$-subsets of $N$, columns indexed by the $j$-subsets of $N$ and with $(\alpha, \beta)$-entry equal to 1 if $\alpha \cap \beta = \emptyset$. Now define matrices $D_0, \ldots, D_k$ by
\[ D_i = W_{i,k} \overline{W}_{i,k}^T \]

(For details concerning these matrices see Wilson’s paper [], or [4, Section 6.4]. Despite appearances, these matrices are symmetric.) The matrix $\Omega(n, k, t)$ is given by
\[ \Omega(n, k, t) = \sum_{i=0}^{k-t} (-1)^{t-1-i} \frac{\binom{k-i}{k-t}}{\binom{n-k-t+1}{k-t}} D_{k-i}. \]

The matrices $I + \Omega(n, k, t)$ form the key to Wilson’s proof of the EKR theorem. We define
\[ N_W(n, k, t) = I + \Omega(n, k, t) \]
and abbreviate $N_{W}(n,k,t)$ to $N_{W}$ where possible.

We use $M \circ N$ to denote the Schur product of two matrices of the same order, thus

$$(M \circ N)_{i,j} = M_{i,j}N_{i,j}.$$ 

Since the set $\{0, A_0, \ldots, A_k\}$ is closed under the Schur product, it follows that the Bose-Mesner algebra of the Johnson scheme is closed under Schur product.

The pertinent properties of $N_{W}$ are summarized in the following:

**2.1 Theorem.** The matrix $N_{W}(n,k,t)$ is positive semidefinite and lies in the span of the matrices $A_{k-t+1}, \ldots, A_k$.

Wilson’s proof that $N_{W}$ is positive semidefinite is highly non-trivial; it is presented at somewhat greater length, but with no essential improvement, in [4, Chapter 8].

### 3 Projections on to matrix algebras

We use $\text{sum}(M)$ to denote the sum of the entries of a matrix $M$. We note that

$$\text{tr}(M^T N) = \text{sum}(M \circ N)$$

and so we have two expressions for the standard inner product on real matrices:

$$\langle M, N \rangle = \text{tr}(M^T N) = \text{sum}(M \circ N).$$

Relative to this inner product, the Schur idempotents $A_0, \ldots, A_k$ form an orthogonal basis for the Bose-Mesner algebra. We also observe that

$$\text{tr}(M) = \langle I, M \rangle, \quad \text{sum}(M) = \langle J, M \rangle.$$

We state a version of a result known as the clique-coclique bound. It is proved, for general association schemes, as Lemma 3.8.1 in [4].

**3.1 Lemma.** Assume $v = \binom{n}{k}$. If $M$ and $N$ are matrices in the Bose-Mesner algebra of the Johnson scheme and

(a) $M$ and $N$ are positive semidefinite, and

(b) for some constant $\gamma$ we have $M \circ N = \gamma I$,

then

$$\frac{\text{sum}(M) \text{sum}(N)}{\text{tr}(M) \text{tr}(N)} \leq v.$$ 

**3.2 Lemma.** The orthogonal projection of a positive semidefinite matrix onto a transpose-closed real matrix algebra is positive semidefinite.

**Proof.** This is a special case of Tomiyama’s theorem, see [6].
Given an orthogonal basis for the Bose-Mesner algebra, we can compute orthogonal projections of matrices onto it—if $M$ is an $\binom{n}{k} \times \binom{n}{k}$ matrix, its orthogonal projection $\Psi(M)$ is given by Gram-Schmidt:

$$\Psi(M) = \sum_i \frac{\langle M, A_i \rangle}{\langle A_i, A_i \rangle} A_i.$$ 

Note that

$$\langle M - \Psi(M), A \rangle = 0$$

for any matrix $A$ in the Bose-Mesner algebra, and taking $A$ to be $J$ and $I$ in turn yields that

$$\sum \Psi(M) = \text{tr}(M), \quad \text{tr}(\Psi(M)) = \text{tr}(M).$$

We consider an example. For any family $F$ of $k$-subsets of $N$, we denote by $N_F$ the matrix $xx^T$ where $x$ is the characteristic vector of $F$. Let $F$ be a $t$-intersecting family of $k$-subsets. Then

$$\langle A_r, N_F \rangle = \text{tr}(AN_F) = x^TA_rx,$$

which equals the number of pairs $(\alpha \beta)$ in $F \times F$ such that $|\alpha \cap \beta| = k - r$. Therefore $\langle A_r, N_F \rangle = 0$ if $r \geq k - t + 1$.

3.3 Lemma. Let $F$ be a $t$-intersecting family. Then $\Psi(N_F)$ is a positive semidefinite matrix lying in the span of $A_0, \ldots, A_{k-t}$ and

$$\text{tr}(\Psi(N_F)) = |F|, \quad \sum(\Psi(N_F)) = |F|^2.$$ 

Proof. Observe that if $A$ lies in the Bose-Mesner algebra of the Johnson scheme, then

$$0 = \langle M - \Psi(M), A \rangle = \langle M, A \rangle - \langle \Psi(M), A \rangle,$$

whence $\langle \Psi(M), A \rangle = \langle M, A \rangle$. Therefore $\langle \Psi(M), A_r \rangle = \langle M, A_r \rangle$, which proves that $\Psi(N_F)$ lies in the span of $A_0, \ldots, A_{k-t}$. The remaining two claims follow from the fact that $\Psi$ preserves trace. $\square$

If we can show that the Bose-Mesner algebra of $J(n, k)$ contains a matrix $L$ such that:

(a) $L$ is positive semidefinite,

(b) $\Psi(N_F) \circ L = \gamma I$ for some $\gamma$,

(c) $\sum(L)/\text{tr}(L) = \binom{n}{t}/\binom{n-t}{k-t},$

then Lemma 3.1 implies that

$$|F| \leq \binom{n-t}{k-t}.$$ 

The key to Wilson’s proof was to demonstrate that, provided

$$n \leq (t + 1)(k - t + 1),$$

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the matrix $I + \Omega(n, k, t)$ satisfies these conditions.

Recall that a $t$-$(n, k, \lambda)$-design is a collection of subsets of size $k$ from an $n$-set such that any any subset of $t$ points from $V$ lies in exactly $\lambda$ blocks (aka $k$-sets). If $\lambda = 1$, we call the design a Steiner system. The construction of Steiner systems for large $t$ is something of a mystery (to which we shall return), but projective and affine planes of finite order provide examples with $t = 2$ and Möbius planes give examples with $t = 3$.

3.4 Lemma. Let $\mathcal{D}$ be a $t$-$(n, k, 1)$ design. Then $\Psi(N_{\mathcal{D}})$ is a positive semidefinite matrix lying in the span of $A_{k-t+1}, \ldots, A_t$ and

$$\text{tr}(\Psi(N_{\mathcal{D}})) = |\mathcal{D}|, \quad \sum(\Psi(N_{\mathcal{D}})) = |\mathcal{D}|.$$  

3.5 Lemma. If a $t$-$(n, k, 1)$-design exists, then a $t$-intersecting family of $k$-subsets of a set of size $v$ has size at most $\binom{n-t}{k-t}$.

We can compute $\Psi(N_{\mathcal{D}})$ explicitly. If $\lambda_i$ denotes the number of blocks of $\mathcal{D}$ that contain a given set of $i$ points and $0 \leq i \leq t$, then

$$\lambda_i = \binom{n-i}{k-i} \binom{n}{k} - \binom{n-t}{k-t}.$$

If we define

$$\gamma_s = \sum_{i=s}^{t} (-1)^{i-s} \binom{i}{s} \binom{k}{i} (\lambda_i - 1),$$

then from Exercise 1 in Chapter 8 of [4], we find that

$$\Psi(N_{\mathcal{D}}) = \sum_{s=0}^{t} \gamma_s \binom{n-k}{k-s} A_{k-s}. $$

For $n, k, t$, we will denote by $M(n, k, t)$ the following:

$$M(n, k, t) = \sum_{s=0}^{t} \frac{\gamma_s}{\binom{n-k}{k-s}} A_{k-s}. $$

If there exists a $t$-$(n, k, 1)$-design $\mathcal{D}$ exists, then $\Psi(N_{\mathcal{D}}) = M(n, k, t)$. Observe that the matrix $M(n, k, t)$ is always well-defined, whether or not the design exists. We use Keevash’s result [5] on the existence of $t$-designs to show that this projection is equal to Wilson’s matrix. For a second proof of Keevash’s result, see [3].

The following theorem is a restatement of Theorem 1.4 of [5] applied to $G = K^t_n$, in the language of block designs instead of hypergraphs.

3.6 Theorem. (Keevash) For fixed $k$ and $t$, there exists $N$ such that for $n > N$, if $\binom{k-i}{t-i}$ divides $\binom{n-i}{i}$ for $i = 0, \ldots, t-1$, then there exists a $t$-$(n, k, 1)$ block design.
We are now able to prove the following.

**3.7 Theorem.** For any $n \geq k \geq t$, we have that $M(n, k, t) = \Omega(n, k, t) + I$.

**Proof.** Fix $k$ and $t$. Let

$$f_r(n) = \theta_r(M(n, k, t))$$

and

$$g_r(n) = \theta_r(\Omega(n, k, t)) + 1.$$  

If there exists a $t$-$(n, k, 1)$-design $D$ exists, then $f_r(n) = g_r(n)$ for $r = 0, \ldots, t$. By Theorem 3.6, we have that $f_r(n)$ and $g_r(n)$ are equal for infinitely many $n$. Consider $h_r(n) = f_r(n) - g_r(n)$. We see that $h_r(n)$ is a rational function whose numerator $p(n)$ is a polynomial. Since $p(n) = 0$ infinitely often, we have that $p(n) = 0$ and so $h_r(n) = 0$. We thus have that $f_r(n) = g_r(n)$ for all $n$. This shows that $M(n, k, t) = \Omega(n, k, t) + I$ for all $n$. \qed

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