Low regularity global well-posedness for the two-dimensional Zakharov system

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Abstract

The two-dimensional Zakharov system is shown to have a unique global solution for data without finite energy if the $L^2$-norm of the Schrödinger part is small enough. The proof uses a refined I-method originally initiated by Colliander, Keel, Staffilani, Takaoka and Tao. A polynomial growth bound for the solution is also given.

1 Introduction

Consider the Zakharov system in space dimension two:

\[
\begin{aligned}
&iu_t + \Delta u = nu, \\
&\Box n = n_{tt} - \Delta n = \Delta |u|^2, \\
&u(0, x) = u_0(x), \quad n(0, x) = n_0(x), \quad n_t(0, x) = n_1(x),
\end{aligned}
\]  

(1.1)

where $\Box$ is the Laplacian in $\mathbb{R}^2$, $u : [0, T) \times \mathbb{R}^2 \to \mathbb{C}$, $n : [0, T) \times \mathbb{R}^2 \to \mathbb{R}$.  

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The Zakharov system was introduced in [18] to describe the long wave Langmuir turbulence in a plasma. The function \( u \) represents the slowly varying envelope of the rapidly oscillating electric field, and the function \( n \) denotes the deviation of the ion density from its mean value. We assume \( u_0 \in H^s, n_0 \in H^l \) and \( n_1 \in H^{l-1} \) for some real \( s, l \).

We consider the Hamiltonian case, that is, we assume

\[
\exists v_0 \in L^2(\mathbb{R}^2, \mathbb{R}^2) : n_1 = -\nabla \cdot v_0. \tag{1.2}
\]

If \((u, n)\) is a solution of (1.1) we have in this case

\[
n_t(t) = -\nabla \cdot (v_0 - \int_0^t \nabla (n + |u|^2) ds) = -\nabla \cdot v(t),
\]

where

\[
v(t) = v_0 - \int_0^t \nabla (n + |u|^2) ds,
\]

so that

\[
v_t = -\nabla(n + |u|^2), \quad v(0) = v_0.
\]

Thus (1.1) can be written in the form

\[
\begin{aligned}
\begin{cases}
iu_t + \Delta u &= nu, \\
n_t &= -\nabla \cdot v, \\
v_t &= -\nabla n - \nabla |u|^2, \\
u(0, x) &= u_0(x), \quad n(0, x) = n_0(x), \quad v(0, x) = v_0(x).
\end{cases}
\end{aligned}
\tag{1.3}
\]

This system has two conserved quantities, namely besides mass conservation also energy conservation (cf. (2.1) and (2.2) below).

In one space dimension, the best result with minimal regularity assumptions on the data was proven by Colliander, Holmer and Tzirakis [7], who showed global well-posedness in the case \((s, l) = (0, -1/2)\), the largest \(L^2\)-based Sobolev space where local existence is known to hold.

In two space dimensions, Proposition 1.1 of [15] tells us that the Cauchy problem (1.1) with \((u_0, n_0, n_1) \in H^s \times H^l \times H^{l-1}\) is locally well posed if \( l \geq 0 \) and \( 2s - (l + 1) \geq 0 \). Therefore the lowest admissible value of \((s, l)\) is \((\frac{3}{4}, 0)\).

The main result of our paper now shows that in the Hamiltonian case these local solutions exist globally, if \( s > 3/4 \) and \( l = 0 \) , provided the datum \( u_0 \) satisfies \( ||u_0||_{L^2} < ||Q||_{L^2} \), where \( Q \) denotes the ground state of the equation \( \Delta Q - Q + |Q|^2 Q = 0 \). More precisely we prove

**Theorem 1.1.** Assume \((u_0, n_0, n_1) \in H^s \times L^2 \times \Lambda^{-1}L^2\), where \( n_1 \) fulfills (1.2), \( 1 > s > 3/4 \) and \( ||u_0||_{L^2} < ||Q||_{L^2} \). Here \( \Lambda \) denotes the operator \( \sqrt{-\Delta} \). Then the system (1.1) has a unique global solution. More precisely, for any \( T > 0 \) there exists a unique solution

\[
(u, n, \Lambda^{-1}n_t) \in X^{s, \frac{1}{2}} + [0, T] \times \tilde{X}^{0, \frac{1}{2}} + [0, T] \times \tilde{X}^{0, \frac{1}{2}} + [0, T]
\]
where the spaces $X^{s,b}$ are defined below, and $X^{0,1}_+ [0, T] := X^{0,1}_+ [0, T] + X^{0,1}_- [0, T]$. This solution satisfies
\[
(u, n, \Lambda^{-1} n_t) \in C^0 ([0, T], H^s (\mathbb{R}^2) \times L^2 (\mathbb{R}^2) \times L^2 (\mathbb{R}^2))
\]
and
\[
\| u(t) \|_{H^s} + \| n(t) \|_{L^2} + \| \Lambda^{-1} n_t(t) \|_{L^2} \lesssim (1 + T)^{-s}.
\]

Global well-posedness for $s = l + 1 \geq 3$ and small data is considered in [1]. Using the Fourier restriction norm method for finite energy solutions ($s = l + 1 = 1$) Bourgain and Colliander [6] proved local well-posedness and also global well-posedness in those cases where the energy functional controls the $H^1 \times L^2 \times \Lambda^{-1} L^2$-norm of the solution. This is the case, if $\| u_0 \|_{L^2} < \| Q \|_{L^2}$.

In order to generalize these results to global existence for data without finite energy one approach in the last years was initiated by [11], called the I-method. The main idea is to use a modified energy functional which is also defined for less regular functions and not strictly conserved. When one is able to control its growth in time explicitly, this allows to iterate a modified local existence theorem to continue the solution to any time $T$ and moreover to estimate its growth in time. This method was successfully applied by these authors to several equations which have a scaling invariance. It was used in [11] to improve Bourgain’s global well-posedness results [3],[4] for the (2+1)- and (3+1)-dimensional Schrödinger equation. Later it was applied to the (1+1)-dimensional derivative Schrödinger equation [12] and to the KdV and modified KdV equation in [9].

This method was later more refined by adding a suitable correction term to the modified energy functional in [8],[9] and [10] in order to damp out some oscillations in that functional. It was used in [14] and [19] to prove a $L^2$-concentration result for the Zakharov system and as a corollary global well-posedness for small initial data could be improved to $\frac{12}{13} < s < 1$, $l = 0$. This method is also used in our paper in order to further weaken the regularity assumptions on the data.

It is organized as follows. We transform the system in the usual way into a first order system. Then we apply the multiplier $I$ to the Schrödinger equation only. Here for a given $N >> 1$, we define smoothing operators $I_N$:
\[
I_N f (\xi) = m_N (\xi) \hat{f} (\xi),
\]
where
\[
m_N (\xi) = \begin{cases} 
1 & |\xi| \leq N, \\
\frac{N}{|\xi|}^{1-s} & |\xi| \geq 2N,
\end{cases}
\]
and $m_N (\xi)$ is smooth, radial, nonnegative, and nonincreasing in $|\xi|$. We drop $N$ from the notation for short when there is no confusion. We remark that $I : H^s \to H^1$ is a smoothing operator in the following sense:
\[
\| u \|_{X^m_{b}} \lesssim \| Iu \|_{X^{m+1-s}_b} \lesssim N^{1-s} \| u \|_{X^m_b}.
\]
Here we used the $X^{m,b}_\varphi$ - spaces which are defined as follows: for an equation of the form $iu_t - \varphi(-i\nabla_x)u = 0$, where $\varphi$ is a measurable function, let $X^{m,b}_\varphi$ be the completion of $\mathcal{S}(\mathbb{R} \times \mathbb{R}^2)$ with respect to

$$
\|f\|_{X^{m,b}_\varphi} := \|\langle\xi\rangle^m \langle\tau\rangle^b \mathcal{F}(e^{it\varphi(-i\nabla_x)}f(x,t))\|_{L^2_{\xi\tau}} = \|\langle\xi\rangle^m \langle\tau + \varphi(\xi)\rangle^b \tilde{f}(\xi,\tau)\|_{L^2_{\xi\tau}}.
$$

For $\varphi(\xi) = \pm|\xi|$ we use the notation $X^{m,b}_\pm$ and for $\varphi(\xi) = |\xi|^2$ simply $X^{m,b}$. For a given time interval $I$ we define $\|f\|_{X^{m,b}(I)} = \inf_{\tilde{f} \equiv f} \|\tilde{f}\|_{X^{m,b}}$ and similarly $\|f\|_{X^{m,b}_\pm(I)}$.

For the modified Zakharov system, where only the Schrödinger equation is multiplied by $I$, we then prove a local existence theorem by using the precise estimates given by [15] for the standard Zakharov system in connection with an interpolation type lemma in [13]. Our aim is to extract a factor $\delta^\kappa$ with maximal $\kappa$ from the nonlinear estimates in order to give an optimal lower bound for the local existence time $\delta$ in terms of the norms of the data.

As it is typical for the $I$-method, one then has to consider in detail the modified energy functional $H(Iu, n)$ and to control its growth in time in dependence of the time interval and the parameter $N$ (cf. the definition of $I$ above). The increment of the energy has to be small for small time intervals and large $N$. The increment of $H(Iu, n)$ is not controlled directly but one replaces $H$ by adding a correction term to it leading to a functional $\tilde{H}$, such that the difference $H - \tilde{H}$ at a fixed time is small for large $N$, and, moreover, which is the main technical difficulty, the growth in time of $\tilde{H}$ can be seen to be small for small time intervals and large $N$, so that one can control also the growth of the corresponding norms of the solutions during its time evolution. This allows to iterate the local existence theorem with time steps of equal length in order to reach any given fixed time $T$. To achieve this one has to make the process uniform, which can be done if $s$ is close enough to $1$, namely $s > 3/4$.

We use the following notation: $A \lesssim B$ means there is a universal constant $c > 0$, such that $A \leq cB$, and $A \sim B$ when both $A \lesssim B$ and $B \lesssim A$. $<\xi> = (1 + |\xi|^2)^{1/2}$. $c+$ means $c + \epsilon$, while $c-$ means $c - \epsilon$, where $\epsilon > 0$ small enough.

There are several properties of the norms $X^{m,b}_\varphi$ and $X^{m,b}_\varphi[0,T]$, for which we refer to [15] and [5].

**Proposition 1.2.**  
1. If $u$ is a solution of $iu_t + \varphi(-i\nabla_x)u = 0$ with $u(0) = f$ and $\psi$ is a cutoff function in $C_0^\infty(\mathbb{R})$ with $\text{supp}\, \psi \subset (-2,2)$, $\psi \equiv 1$ on $[-1,1]$, $\psi(t) = \psi(-t)$, $\psi(t) \geq 0$, $\psi_\delta(t) := \psi(\frac{t}{\delta}), 0 < \delta \leq 1$, we have for $b > 0$:

$$
\|\psi_1 u\|_{X^{m,b}_\varphi} \lesssim \|f\|_{H^m}.
$$
2. If \( v \) is a solution of the problem \( iv_t + \varphi(-i\nabla_x)v = F, \ v(0) = 0 \), we have for \( b' + 1 \geq b \geq 0 \geq b' > -1/2 \)
\[
\| \psi_b v \|_{X_{\varphi}^{m,b}} \lesssim \delta^{1+b'-b} \| F \|_{X_{\varphi}^{m,b'}}.
\]

3. \( u \in X_{\varphi}^{s,b}(\mathbb{R} \times \mathbb{R}^2) \iff e^{-it\varphi(-i\nabla)}u(t,\cdot) \in H^b(\mathbb{R}, H^s(\mathbb{R}^2)) \).

4. For \( \frac{2}{q} = 1 - \frac{2}{r} \), \( 2 \leq r < \infty \), the following Strichartz estimate holds:
\[
\| u \|_{L^q_t L^r_x} \lesssim \| u \|_{X^0_{R_+,b}}.
\]

For the wave part we only use
\[
\| u \|_{L^\infty_t L^2_x} \lesssim \| u \|_{X^0_{R_+,b}}.
\]

5. For \( b > \frac{1}{2} \), \( X_{\varphi}^{s,b}(\mathbb{R} \times \mathbb{R}^2) \hookrightarrow C(\mathbb{R}, H^s(\mathbb{R}^2)) \), and \( X_{\varphi}^{s,b}(0,T)(\mathbb{R}^2) \hookrightarrow C((-T,T), H^s(\mathbb{R}^2)) \).

6. (cf. [16]). For \( 0 \leq b' < b < \frac{1}{2} \), or \( 0 \geq b > b' > -1/2 \), \( 0 < T < 1 \),
\[
\| u \|_{X_{\varphi}^{s,b'}[0,T]} \lesssim T^{b-b'} \| u \|_{X_{\varphi}^{s,b}[0,T]}.
\]

7. For \( s_1 \leq s_2 \), and \( b_1 \leq b_2 \), \( X_{\varphi}^{s_2,b_2}(\mathbb{R} \times \mathbb{R}^2) \hookrightarrow X_{\varphi}^{s_1,b_1}(\mathbb{R} \times \mathbb{R}^2) \).

8. If \( f,g \in X^{0,\frac{1}{2}+} \), with
\[
1_{|\xi_1| \sim N_1} \hat{f} = \hat{f}, 1_{|\xi_2| \sim N_2} \hat{g} = \hat{g},
\]
and \( N_1 \gtrsim N_2 \), then
\[
\| fg \|_{L^2_{t,x}} \leq C(N_2^\frac{1}{2},N_1^\frac{1}{2}) \| f \|_{X^{0,\frac{1}{2}+}} \| g \|_{X^{0,\frac{1}{2}+}}.
\]

Finally, we use the sharp Gagliardo-Nirenberg embedding for \( \mathbb{R}^2 \), which could be found in [17]:
\[
\frac{1}{2} \| u \|_{L^4_{t}}^4 \leq \frac{\| u \|_{L^2_{t}}^2 \| \nabla u \|_{L^2_{t}}^2}{\| Q \|_{L^2_{t}}^2}
\]
\[
\text{for } u \in H^1, \text{ where } Q \text{ denotes the ground state for the Schrödinger equation, i.e. the unique positive solution (up to translations) of } \Delta Q - Q + |Q|^2 Q = 0.
\]

2 Local existence

The system (1.3) has the following conserved quantities:
\[
\| u(t) \|_{L^2} = \| u_0 \|_{L^2}.
\]
The system (1.1) is transformed into a first order system in $t$ under the assumption $\|u_0\|_{L^2} < \|Q\|_{L^2}$ as follows:

$$\int n|u|^2 \, dx \leq \|n\|_{L^2}\|u\|_{L^2}^2 \leq \|n\|_{L^2}\sqrt{2\|u\|_{L^2}^2 \|Q\|_{L^2}} \|\nabla u\|_{L^2} \leq \frac{\epsilon}{2}\|n\|_{L^2}^2 + \frac{1}{\epsilon} \|u\|_{L^2}^2 \|\nabla u\|_{L^2}^2.$$ 

Choosing $1 > \epsilon > \frac{\|u_0\|_{L^2}^2}{\|Q\|_{L^2}^2}$, we get

$$H(u, n, v) \leq 2\|\nabla u\|_{L^2}^2 + \|n\|_{L^2}^2 + \frac{1}{2}\|v\|_{L^2}^2,$$ 

as well as

$$(\frac{1}{2} - \frac{\epsilon}{2})\|n\|_{L^2}^2 + \frac{1}{2}\|v\|_{L^2}^2 + (1 - \frac{\|u_0\|_{L^2}^2}{\epsilon \|Q\|_{L^2}^2})\|\nabla u\|_{L^2}^2 \leq H(u, n, v),$$

so that

$$\|\nabla u\|_{L^2}^2 + \|n\|_{L^2}^2 + \|v\|_{L^2}^2 \leq c_0 H(u, n, v).$$

The system (1.1) is transformed into a first order system in $t$ as follows: with $n_\pm := n \pm i\Lambda^{-1}n_t$, i.e. $n = \frac{1}{2}(n_+ + n_-)$, $2i\Lambda^{-1}n_t = n_+ - n_-$, and $\nu_\pm = n_-$ we get

$$iu_t + \Delta u = \frac{1}{2}(n_+ + n_-)u,$$ 

$$in_{\pm t} \mp \Lambda n_\pm = \pm \Lambda(|u|^2),$$ 

$$u(0) = u_0, \quad n_{\pm}(0) = n_{\pm 0} := n_0 \pm i\Lambda^{-1}n_1.$$ 

One easily checks that the energy $H(u, n, v)$ is transformed into

$$H(u, n_+) = \|\nabla u\|_{L^2}^2 + \frac{1}{2}\|n_+\|_{L^2}^2 + \frac{1}{2}\int (n_+ + \nu_+) |u|^2 \, dx,$$

so that (cf. (2.3))

$$H(u, n_+) \lesssim \|\nabla u\|_{L^2}^2 + \|n_+\|_{L^2}^2,$$ 

and (cf. (2.5))

$$\|\nabla u\|_{L^2}^2 + \|n_+\|_{L^2}^2 \leq c_0 H(u, n_+).$$

Now, we will apply the I-method (we refer to the introduction for the definition of I). A crucial role is played by the modified energy $H(Iu, n_+)$ for the system

$$iu_t + \Delta u = \frac{1}{2}[(n_+ + n_-)u],$$ 

$$in_{\pm t} \mp \Lambda n_\pm = \pm \Lambda(|u|^2),$$ 

$Iu(0) = Iu_0$, $n_{\pm}(0) = n_{\pm 0} = (n_0 \pm i\Lambda^{-1}n_1),$
Lemma 2.1. Assume $1>s\geq 0$. Then the following estimate holds:
\[
\|n_{\pm}u\|_{X^{s,-\frac{1}{2}}} \lesssim \delta^\frac{1}{2}\|n_\pm\|_{X^{0,\pm}u} \|u\|_{X^{s,\frac{1}{2}}} .
\]

Assume $s \geq 1/2$. Then the following estimate holds:
\[
\|\Lambda(|u|^2)\|_{X^{0,-\frac{1}{2}}} \lesssim \delta^\frac{1}{2}\|u\|^2_{X^{s,\frac{1}{2}}} .
\]

Lemma 2.2. In the case $1>s\geq 0$ the following estimate holds:
\[
\|I(n_{\pm}u)\|_{X^{1,-\frac{1}{4}}} \lesssim N^0\delta^\frac{1}{2}\|n_{\pm}\|_{X^{0,\pm}u} \|I_{1,n}u\|_{X^{1,\frac{1}{4}}} .
\]

Proof. Let $\chi$ be a smooth cutoff function which equals to 1 for $|\xi| \leq N$, and equals to 0 for $|\xi| \geq 2N$. We estimate as follows:
\[
\|I(n_{\pm}u)\|_{X^{1,-\frac{1}{4}}} \leq \|I(n_{\pm}F^{-1}(\chi Fv))\|_{X^{1,-\frac{1}{4}}} + \|I(n_{\pm}F^{-1}(1-\chi)Fu)\|_{X^{1,-\frac{1}{4}}}
\]
where $v := Iu$.

The first term is estimated by Lemma 2.1 as follows:
\[
\|I(n_{\pm}u)\|_{X^{1,-\frac{1}{4}}} \lesssim N^0\|n_{\pm}F^{-1}(\chi Fv)\|_{X^{1,-\frac{1}{4}}} \lesssim N^0\delta^\frac{1}{2}\|n_{\pm}\|_{X^{0,\pm}u} \|F^{-1}(\chi Fv)\|_{X^{1,-\frac{1}{4}}}
\]
\[
\lesssim N^0\delta^\frac{1}{2}\|n_{\pm}\|_{X^{0,\pm}u} \|I_{1,n}u\|_{X^{1,\frac{1}{4}}} .
\]

Next we consider the second term. By Lemma 2.1 we have
\[
\|n_{\pm}u_1\|_{X^{1,-\frac{1}{4}}} \lesssim \delta^\frac{1}{2}\|n_\pm\|_{X^{0,\pm}u} \|u_1\|_{X^{1,\frac{1}{4}}} .
\]

This means
\[
\left| \int_{\Sigma_1} \frac{\hat{f}(\xi_1, \tau_1)\hat{u}_1(\xi_2, \tau_2)\hat{n}_3(\xi_3, \tau_3) < \xi_1 >^s}{\sigma_1 >^\frac{1}{2} < \sigma_2 >^\frac{1}{2} < \sigma >^\frac{1}{2} < \xi_2 >^s} d\xi d\tau \right| \lesssim \delta^\frac{1}{2}\|f\|L^2\|u_1\|L^2\|n_\pm\|L^2 , \quad (2.16)
\]
where $f \in L^2$, $\Sigma_3$ denotes the set $\xi_1 + \xi_2 + \xi_3 = 0$ and $\tau_1 + \tau_2 + \tau_3 = 0$, $\sigma_j = \tau_j + |\xi_j|^2$ ($j = 1, 2$) and $\sigma = \tau_3 \pm |\xi_3|$. In order to prove
\[
\|I(n_{\pm}F^{-1}((1-\chi)Fu))\|_{X^{1,-\frac{1}{4}}} \lesssim \delta^\frac{1}{2}\|n_\pm\|_{X^{0,\pm}u} \|I_{1,n}F^{-1}((1-\chi)Fu)\|_{X^{1,\frac{1}{4}}} ,
\]
Because $|\xi_2| \geq N$, we have:

- If $|\xi_1| \leq N$: \[ \frac{m(\xi_1)|\xi_2|}{m(\xi_2)} \lesssim \left( \frac{|\xi_1|}{|\xi_2|} \right)^{1-s} \lesssim \frac{|\xi_1|^s}{|\xi_2|^s}. \]
- If $|\xi_1| \geq N$: \[ \frac{m(\xi_1)|\xi_2|}{m(\xi_2)} \lesssim \left( \frac{|\xi_1|}{|\xi_2|} \right)^{1-s} \lesssim \frac{|\xi_1|^s}{|\xi_2|^s}. \]

So (2.16) implies (2.17). Thus

\[
\|I(n_+ F^{-1}((1 - \chi)F u))\|_{X^{1, \frac{s}{2}+}} \lesssim \delta^{\frac{1}{2}} \|n_+\|_{X^{0, \frac{s}{4}+}} \|F^{-1}((1 - \chi)F u))\|_{X^{1, \frac{s}{4}+}} \\
\lesssim \delta^{\frac{1}{2}} \|n_+\|_{X^{0, \frac{s}{4}+}} \|u\|_{X^{1, \frac{s}{4}+}}.
\]

**Proposition 2.3.** Assume $1 > s \geq 1/2$ and $(u_0, n_{+0}, n_{-0}) \in H^s \times L^2 \times L^2$. Then there exists

\[ \delta \sim \frac{1}{(\|u_0\|_{H^1} + \|n_{+0}\|_{L^2} + \|n_{-0}\|_{L^2})^2 + N^{0+}}, \]

such that the system (2.11), (2.12), (2.13) has a unique local solution in the time interval $[0, \delta]$ with the property:

\[ \|u\|_{X^{1, \frac{s}{4}+}} + \|n_+\|_{X^{0, \frac{s}{4}+}} + \|n_-\|_{X^{0, \frac{s}{4}+}} \lesssim \|u_0\|_{H^1} + \|n_{+0}\|_{L^2} + \|n_{-0}\|_{L^2}. \]

This immediately implies

\[ \|u\|_{C^0([0,\delta],H^1)} + \|n_+\|_{C^0([0,\delta],L^2)} + \|n_-\|_{C^0([0,\delta],L^2)} \lesssim \|u_0\|_{H^1} + \|n_{+0}\|_{L^2} + \|n_{-0}\|_{L^2}. \]

**Proof.** We use the corresponding integral equations to define a mapping $S = (S_0, S_1)$ by

\[
S_0(Iu(t)) = Ie^{it\Delta}u_0 + \frac{1}{2} \int_0^t e^{i(t-s)\Delta} I(u(s)(n_+(s) + n_-(s))) ds \\
S_1(n_\pm(t)) = e^{it\Lambda}n_{\pm0} + i \int_0^t e^{i(t-s)\Delta} \Lambda(|u(s)|^2) ds.
\]

Combining Lemma 2.2 with the interpolation lemma of [13] we get

\[ \|\Lambda(|u|^2)\|_{X^{1-s, \frac{s}{4}+}} \lesssim \|\Lambda(|u|^2)\|_{X^{1, \frac{s}{4}+}} \lesssim \delta^{\frac{1}{2}} \|u\|_{X^{1, \frac{s}{4}+}} \]

This immediately implies

\[ \|S_1(n_\pm)\|_{X^{0, \frac{s}{4}+}} \lesssim \|n_{\pm0}\|_{L^2} + \delta^{\frac{1}{2}} \|u\|_{X^{1, \frac{s}{4}+}} \]

Using Lemma 2.2 we get

\[ \|S_0(Iu)\|_{X^{1, \frac{s}{4}+}} \lesssim \|Iu_0\|_{H^1} + \|n_{\pm0}\|_{X^{0, \frac{s}{4}+}} \|u\|_{X^{1, \frac{s}{4}+}}, \]

Choosing $\delta$ as in the statement of this proposition the standard contraction argument gives a unique fixed point of $S$, thus the claimed result. 

\[ \square \]
3 Estimates for the modified energy

In this section, let us get the control of the increment of the modified energy.

As the modified energy is

$$H(Iu, n_+)(t) = \|\nabla Iu\|_{L^2}^2 + \frac{1}{2} \|n_+\|_{L^2}^2 + \frac{1}{2} \int_{\mathbb{R}} (n_+ + \hat{n}_+) |Iu|^2 \, dx,$$

which is not conserved any more, we have to control its growth.

For functions depending on $t$ we drop $t$ from the notation here and in the following.

First of all, let us define a new quantity $\tilde{H}(u, n_+)$, which is a slight variant of $H(Iu, n_+)(t)$, and establish an almost conservation law for that quantity instead.

Definition 3.1. Let $k$ be an integer and $\Sigma_k \subset (\mathbb{R}^2)^k$ denote the space

$$\Sigma_k := \{(\xi_1, \cdots, \xi_k) \in (\mathbb{R}^2)^k : \xi_1 + \cdots + \xi_k = 0\},$$

then

$$\tilde{H}(u, n_+)(t) = - \int_{\Sigma_2} \xi_1 m_1 \cdot \xi_2 m_2 \hat{u}(\xi_1) \hat{u}(\xi_2) + \frac{1}{2} \int_{\Sigma_2} \hat{n}_+ (\xi_1) \hat{n}_+ (\xi_2) + \frac{1}{2} \int_{\Sigma_3} \sigma \hat{u}(\xi_1) \hat{u}(\xi_2) (\hat{n}_+ + \hat{n}_+)(\xi_3)$$

is called the refined energy, where $m_i = m_N(\xi_i)$, and $\sigma = \frac{|\xi_1|^2 m_1^2 - |\xi_2|^2 m_2^2}{|\xi_1|^2 - |\xi_2|^2}$.

Then we shall show the following:

Proposition 3.2. (Fixed-time difference) For $s > \frac{1}{2}$ we have

$$|H(Iu, n_+)(t) - \tilde{H}(u, n_+)(t)| \lesssim N^{-1+} \|Iu(t)\|_{H^1(\mathbb{R}^2)} \|n_+(t)\|_{L^2(\mathbb{R}^2)}.$$

(3.1)

Proposition 3.3. (Almost conservation law) For $s > \frac{1}{2}$, if $(Iu, n_+, n_-)$ is the solution to the Cauchy problem [2.11], [2.12], [2.13] on the time interval $[0, \delta]$ with initial data $(Iu_0, n_{0+}, n_{0-}) \in H^1(\mathbb{R}^2) \times L^2(\mathbb{R}^2) \times L^2(\mathbb{R}^2)$, then we have

$$|\tilde{H}(u, n_+)(\delta) - \tilde{H}(u, n_+)(0)|$$

$$\lesssim N^{-\frac{1}{4}+} \delta^{\frac{1}{2}+} \|Iu\|_{X^1_{\frac{1}{2}+}} \|n_+\|_{X^0_{\frac{1}{2}+}} + (N^{-2+} + N^{-1+} \delta^{\frac{1}{2}+}) \|Iu\|_{X^1_{\frac{1}{2}+}} \|n_+\|_{X^0_{\frac{1}{2}+}}.$$  (3.2)

The remaining part of this section is devoted to prove the above two propositions.

Proof of Proposition 3.2. Since

$$H(Iu, n_+) = - \int_{\Sigma_2} \xi_1 m_1 \cdot \xi_2 m_2 \hat{u}(\xi_1) \hat{u}(\xi_2) + \frac{1}{2} \int_{\Sigma_2} \hat{n}_+ (\xi_1) \hat{n}_+ (\xi_2) + \frac{1}{2} \int_{\Sigma_3} m_1 m_2 (\xi_1) \hat{u}(\xi_2) (\hat{n}_+ + \hat{n}_+)(\xi_3),$$
Lemma 3.4. Under the above assumption, we have
\[
\tilde{H}(u, n_+) = -\int_{\Sigma_2} \xi_1 m_1 \cdot \xi_2 m_2 \hat{u}(\xi_1) \hat{u}(\xi_2) + \frac{1}{2} \int_{\Sigma_2} \hat{n}_+(\xi_1) \hat{n}_+(\xi_2) + \frac{1}{2} \int_{\Sigma_3} \sigma \hat{u}(\xi_1) \hat{u}(\xi_2) (\hat{n}_+ + \hat{n}_+)(\xi_3),
\]
we have
\[
H(Iu, n_+) - \tilde{H}(u, n_+) = \frac{1}{2} \int_{\Sigma_3} (m_1 m_2 - \sigma) \hat{u}(\xi_1) \hat{u}(\xi_2) (\hat{n}_+ + \hat{n}_+)(\xi_3).
\]

(3.3)

We use a dyadic decomposition with \( N_i \leq |\xi_i| \leq 2N_i \). As the complex conjugates will play no role here, we can suppose \( N_1 \geq N_2 \).

If \( N_2 \leq N_1 \ll N \), then by the definition of \( m_N \) and \( \sigma \), the integral vanishes. Hence, we suppose \( N_1 \gg N \).

Therefore, it remains to prove under these assumptions
\[
I = \left| \int_{\Sigma_3} (m_1 m_2 - \sigma) \hat{u}(\xi_1) \hat{u}(\xi_2) (\hat{n}_+ + \hat{n}_+)(\xi_3) \right| \lesssim N^{-1+} N_i^{0-} ||Iu||_{H^1} ||n_+||_{L^2}.
\]

(3.4)

Lemma 3.4. Under the above assumption, \( \sigma \) is bounded.

Proof. Case 1. \( N_2 \ll N \ll N_1 \).

\[
|\sigma| = \frac{|\xi_1|^2 m_1^2 - |\xi_2|^2}{|\xi_1|^2 - |\xi_2|^2} \sim \frac{|\xi_1|^2 m_1^2 - |\xi_2|^2|}{|\xi_1|^2} \lesssim m_1^2 + \frac{N_2^2}{N_1^2} \lesssim 1.
\]

(3.5)

Case 2. \( N \ll N_2 \ll N_1 \).

\[
|\sigma| = \frac{|\xi_1|^2 m_1^2 - |\xi_2|^2 m_2^2}{|\xi_1|^2 - |\xi_2|^2} = \frac{f(|\xi_1|) - f(|\xi_2|)}{|\xi_1|^2 - |\xi_2|^2},
\]

(3.6)

where \( f(r) = r^2 m_N(r)^2 \) and \( r \gtrsim N \). Thus \( |f'(r)| \lesssim N^{2(1-s)} r^{2s-1} \), which is monotone increasing w.r.t. \( r \gtrsim N \), because \( s > \frac{1}{2} \). Hence,
\[
|\sigma| \lesssim \frac{N^{2(1-s)} |\xi_1|^{2s-1} (|\xi_1| - |\xi_2|)}{(|\xi_1| + |\xi_2|)(|\xi_1| - |\xi_2|)} \lesssim \frac{N^{2(1-s)} |\xi_1|^{2s-1}}{|\xi_1|} = (\frac{N}{|\xi_1|})^{2(1-s)} \lesssim 1.
\]

(3.7)

This lemma implies \( |m_1 m_2 - \sigma| \lesssim 1 \), and for \( s > 1/2 \) we get
\[
I \lesssim \frac{1}{m(N_1) m(N_2)} ||Iu||_{L^2} ||Iu||_{L^\infty} ||n_+||_{L^2} \lesssim \left( \frac{N_1}{N} \right)^{1-s} \left( \left( \frac{N_2}{N} \right)^{1-s} + 1 \right) \frac{1}{N_1} ||Iu||_{H^1} ||n_+||_{L^2} \lesssim N^{-1+} N_i^{0-} ||Iu||_{H^1} ||n_+||_{L^2}.
\]

(3.8)
Here and in the following we abuse notation and denote

\[ m(N_i) = \inf_{|\xi| \sim N_i} m(\xi_i) \sim \sup_{|\xi| \sim N_i} m(\xi_i). \]

This completes the proof of Proposition 3.2.

**Proof of Proposition 3.3** By system (2.11), (2.12)

\[
\frac{d}{dt} \tilde{H}(u, n_+)(t) = -\sum_{\Sigma_2} \xi_1 m_1 \cdot \xi_2 m_2 \hat{u}_t(\xi_1) \hat{u}_t(\xi_2) - \sum_{\Sigma_2} \xi_1 m_1 \cdot \xi_2 m_2 \hat{\hat{u}}_t(\xi_1) \hat{\hat{u}}_t(\xi_2) + \frac{1}{2} \sum_{\Sigma_2} \hat{n}_+ t(\xi_1) \hat{n}_+(\xi_2) + \frac{1}{2} \sum_{\Sigma_2} \hat{n}_+ t(\xi_1) \hat{n}_+(\xi_2) + \frac{1}{2} \sum_{\Sigma_2} \sigma \hat{u}_t(\xi_1) \hat{u}_t(\xi_2) + \frac{1}{2} \sum_{\Sigma_2} \sigma \hat{\hat{u}}_t(\xi_1) \hat{\hat{u}}_t(\xi_2) + \frac{1}{2} \sum_{\Sigma_2} \sigma \hat{u}_t(\xi_1) \hat{n}_+(\xi_2) + \hat{n}_+(\xi_3) + \frac{1}{2} \sum_{\Sigma_3} \sigma \hat{\hat{u}}_t(\xi_1) \hat{\hat{u}}_t(\xi_2) + \hat{\hat{n}}_+(\xi_3) + \frac{1}{2} \sum_{\Sigma_3} \sigma \hat{u}_t(\xi_1) \hat{\hat{u}}_t(\xi_2) + \hat{\hat{n}}_+(\xi_3) + \frac{1}{2} \sum_{\Sigma_3} \sigma \hat{\hat{u}}_t(\xi_1) \hat{u}_t(\xi_2) + \hat{n}_+(\xi_3) + \frac{1}{2} \sum_{\Sigma_3} \sigma \hat{u}_t(\xi_1) \hat{u}_t(\xi_2) + \hat{n}_+(\xi_3)
\]

where \( \xi_{ij} = \xi_i + \xi_j \), \( m_{ij} = m_N(\xi_i + \xi_j) \), and the expression for \( \sigma \) is given above.

Integrating with respect to \( t \) on \([0, \delta]\), we have

\[
\tilde{H}(u, n_+)(\delta) - \tilde{H}(u, n_+)(0) = -\frac{i}{2} \int_0^\delta \sum_{\Sigma_3} (1 - \sigma)|\xi_3| \hat{\hat{u}}(\xi_1) \hat{\hat{u}}(\xi_2) + \frac{i}{2} \int_0^\delta \sum_{\Sigma_3} (1 - \sigma)|\xi_3| \hat{\hat{u}}(\xi_1) \hat{\hat{u}}(\xi_2) + \frac{i}{2} \sum_{\Sigma_3} \frac{\xi_2^2 (m_{23}^2 - m_{22}^2)}{\xi_{23}^2 - \xi_{22}^2} \hat{\hat{u}}(\xi_1) \hat{\hat{u}}(\xi_2) \hat{\hat{n}}(\xi_3). \]

Because the complex conjugates play no role here, there are two kinds of terms we have to deal with:

\[
II = \int_0^\delta \sum_{\Sigma_3} (1 - \sigma)|\xi_3| \hat{\hat{u}}(\xi_1) \hat{\hat{u}}(\xi_2) \hat{\hat{n}}(\xi_3),
\]
and

\[ III = \int_{0}^{\delta} \int_{\mathbb{R}^4} \frac{|\xi_{2}|^2(m_{23}^2 - m_{2}^2)}{|\xi_{23}|^2 - |\xi_{2}|^2} \tilde{u}(\xi_{1})\tilde{u}(\xi_{2})\hat{n}_{+}(\xi_{3})\hat{n}_{+}(\xi_{4}) \, d\xi_{1} d\xi_{2} d\xi_{3} d\xi_{4}. \]  

(3.12)

First we prove

\[ II \lesssim N^{-\frac{1}{2}+}N_{1}^{-\frac{1}{2}-}\|Iu\|_{X^{1+\frac{1}{2}-}}^{2}\|n_{+}\|_{X^{0,\frac{1}{2}+}}, \]

where we can assume as above \( N_{2} \leq N_{1} \), \( N_{1} \gtrsim N \), and \( N_{3} \lesssim N_{1} \).

As \(|1 - \sigma| \lesssim 1\),

\[ II \lesssim N_{3}\frac{1}{m(N_{1})m(N_{2})}\|Iu_{1}Iu_{2}\|_{L^{2}_{x}}\|n_{+}\|_{L^{2}_{x}} \]

\[ \lesssim N_{3}\left(\frac{N_{1}}{N}\right)^{1-s}\left(\frac{N_{2}}{N}\right)^{1-s} + 1\right)\left(\frac{N_{2}}{N_{1}}\right)^{1-s}\|Iu_{1}\|_{X_{+}^{0,\frac{1}{2}+}}\|Iu_{2}\|_{X_{+}^{0,\frac{1}{2}+}}\|n_{+}\|_{X_{+}^{0,\frac{1}{2}+}} \]

\[ \lesssim N_{3}\left(\frac{N_{1}}{N}\right)^{1-s}\left(\frac{N_{2}}{N}\right)^{1-s} + 1\right)\left(\frac{N_{2}}{N_{1}}\right)^{1-s} \frac{1}{N_{1} < N_{2} >} \delta^{\frac{1}{2}}\|Iu\|_{X^{1,\frac{1}{2}+}}\|n_{+}\|_{X^{0,\frac{1}{2}+}} \]

\[ \lesssim N^{-\frac{1}{2}+}N_{1}^{-\frac{1}{2}-}\|Iu\|_{X^{1,\frac{1}{2}+}}\|n_{+}\|_{X^{0,\frac{1}{2}+}}. \]  

(3.14)

Next we prove

\[ III \lesssim (N^{-2+} + N^{-1+}\delta^{\frac{1}{2}+})\|Iu\|_{X_{+}^{0,\frac{1}{2}+}}\|n_{+}\|_{X_{+}^{0,\frac{1}{2}+}}. \]  

(3.15)

If both \( N_{2} \) and \( N_{3} \ll N \), then \( m_{23}^2 - m_{2}^2 = 0 \), which is trivial. Thus we suppose \( N_{2} \) or \( N_{3} \gtrsim N \).

Case 1. \( N_{2} \ll N_{3} \), and \( N_{3} \gtrsim N \).

\[ \|\xi_{2}|^2(m_{23}^2 - m_{2}^2)| \lesssim \frac{|\xi_{2}|^2}{|\xi_{23}|^2 - |\xi_{2}|^2} \lesssim \frac{|\xi_{2}|^2}{|\xi_{3}|^2}. \]  

(3.16)

Since \( N_{2} \ll N_{3} \) and \( \xi_{1} + \xi_{2} + \xi_{3} + \xi_{4} = 0 \), \( N_{4} \lesssim \max\{N_{1}, N_{3}\} \), and \( N_{\max} \lesssim \max\{N_{1}, N_{3}\} \).

Subcase 1.1. \( N_{1} \ll N \).

So \( N_{\max} \approx N_{3} \).

\[ III \lesssim \frac{N_{2}^{2}}{N_{3}^{2}m(N_{1})m(N_{2})}\|Iu_{1}\|_{L^{\infty}_{x}}\|Iu_{2}\|_{L^{2}_{x}}\|n_{+}\|_{L^{2}_{x}}\|n_{+}\|_{X_{+}^{0,\frac{1}{2}+}} \]

\[ \lesssim \frac{N_{2}^{2}}{N_{3}^{2}}\left(\frac{N_{2}}{N}\right)^{1-s} + 1\right)\|Iu_{1}\|_{L^{\infty}_{x}H^{1}_{x}}\|Iu_{2}\|_{X_{+}^{0,\frac{1}{2}+}}\|n_{+}\|_{H^{0,\frac{1}{2}+}}\|n_{+}\|_{X_{+}^{0,\frac{1}{2}+}} \]

\[ \lesssim \frac{N_{2}^{2}}{N_{3}^{2}}\left(\frac{N_{2}}{N}\right)^{1-s} + 1\right)\frac{1}{< N_{2} >}N_{3}^{-\frac{1}{2}+}\|Iu\|_{X^{1,\frac{1}{2}+}}\|n_{+}\|_{X_{+}^{0,\frac{1}{2}+}} \]

\[ \lesssim N^{-\frac{1}{2}+}N_{\max}^{-\frac{1}{2}-}\|Iu\|_{X^{1,\frac{1}{2}+}}\|n_{+}\|_{X_{+}^{0,\frac{1}{2}+}}. \]  

(3.17)

Subcase 1.2. \( N_{1} \gtrsim N \).
III \lesssim \frac{N_2^2}{N_3^3 m(N_1)m(N_2)}\Vert Iu_1 \Vert_{L_t^2 L_x^{\infty}} \Vert Iu_2 \Vert_{L_t^2 L_x^{\infty}} \Vert n_+ + 3 \Vert_{L_t^{-\infty} L_x^2} \Vert n_+ + 4 \Vert_{L_t^4 L_x^2} \approx \frac{N_2^2}{N_3^3 \left( \frac{N_1}{N} \right)^{1-s} \left( \frac{N_2}{N} \right)^{1-s} + 1} \Vert Iu_1 \Vert_{X^{0, \frac{1}{2}+}} \Vert Iu_2 \Vert_{X^{0, \frac{1}{2}+}} \Vert n_+ + 3 \Vert_{L_t^{4} H_x^1} \Vert n_+ + 4 \Vert_{X^{0, \frac{1}{2}+}}.

\lesssim \frac{N_2^2}{N_3^3 \left( \frac{N_1}{N} \right)^{1-s} \left( \frac{N_2}{N} \right)^{1-s} + 1} \frac{1}{N_1 < N_2} \frac{1}{N_3^{0+}} \Vert Iu \Vert_{X^{1, \frac{1}{2}+}} \Vert n_+ \Vert_{X^{0, \frac{1}{2}+}} \Vert n_+ \Vert_{X^{0, \frac{1}{2}+}} \Vert n_+ \Vert_{X^{0, \frac{1}{2}+}}.

(3.18)

Case 2. \( N_3 < N_2, N_2 \gtrsim N \).

So

\frac{\vert \xi_2 \vert^2 (m_{23}^2 - m_2^2)}{\vert \xi_2 \vert^2 - \vert \xi_2 \vert^2} = \frac{\vert \xi_2 \vert^2 (m_N (\xi_2 + \xi_3)^2 - m_N (\xi_2)^2)}{(\vert \xi_2 \vert^2 + \vert \xi_3 \vert) (\vert \xi_3 \vert - \vert \xi_2 \vert)} \lesssim \frac{\vert \xi_2 \vert^2 N^{2(1-s)} \vert \xi_2 \vert^2 - \vert \xi_2 \vert}{\vert \xi_2 \vert^2 \vert \xi_3 \vert - \vert \xi_2 \vert} \lesssim m_2^2,

(3.19)

and \( N_4 \lesssim \max\{N_1, N_2\}, N_{\text{max}} \lesssim \max\{N_1, N_2\} \).

Subcase 2.1. \( N_1 << N \).

So \( N_{\text{max}} \sim N_2 \).

As above

III \lesssim \frac{1}{m(N_2)} \frac{m(N_1)m(N_2)}{m(N_1)} \Vert Iu_1 \Vert_{L_t^2 L_x^{\infty}} \Vert Iu_2 \Vert_{L_t^2 L_x^{\infty}} \Vert n_+ + 3 \Vert_{L_t^{2} L_x^{2}} \Vert n_+ + 4 \Vert_{L_t^{4} L_x^{2}} \approx \frac{1}{N_2} N_3^{0+} \Vert Iu \Vert_{X^{1, \frac{1}{2}+}} \Vert n_+ \Vert_{X^{0, \frac{1}{2}+}} \Vert n_+ \Vert_{X^{0, \frac{1}{2}+}} \Vert n_+ \Vert_{X^{0, \frac{1}{2}+}} \Vert n_+ \Vert_{X^{0, \frac{1}{2}+}}.

(3.20)

Subcase 2.2. \( N_1 \sim N \).

As in subcase 1.2,

III \lesssim \frac{1}{m(N_2)} \frac{m(N_1)m(N_2)}{m(N_1)} \Vert Iu_1 \Vert_{L_t^{2} L_x^{\infty}} \Vert Iu_2 \Vert_{L_t^{2} L_x^{\infty}} \Vert n_+ + 3 \Vert_{L_t^{-\infty} L_x^2} \Vert n_+ + 4 \Vert_{L_t^{4} L_x^{2}} \approx \frac{N_1^{1-s} N_2^{1-s}}{N_1} \frac{1}{N_2} N_3^{0+} \Vert Iu \Vert_{X^{1, \frac{1}{2}+}} \Vert n_+ \Vert_{X^{0, \frac{1}{2}+}} \Vert n_+ \Vert_{X^{0, \frac{1}{2}+}} \Vert n_+ \Vert_{X^{0, \frac{1}{2}+}} \Vert n_+ \Vert_{X^{0, \frac{1}{2}+}} \Vert n_+ \Vert_{X^{0, \frac{1}{2}+}} \Vert n_+ \Vert_{X^{0, \frac{1}{2}+}}.

(3.21)

Case 3 \( N_2 \sim N_3 \gtrsim N \).

Hence \( N_4 \lesssim \max\{N_1, N_2, N_3\} \sim \max\{N_1, N_2\} \), and \( N_{\text{max}} \lesssim \max\{N_1, N_2\} \).
We have
\[ |\xi_2|^{2}(m_{23}^2 - m_{2}^2) = |\xi_2|^{2}m_{23}^2 - |\xi_2|^2m_{2}^2 - m_{23}^2 \lesssim |\xi_2|^{2}m_{23}^2 - |\xi_2|^2m_{2}^2 + m_{23}^2. \]

By the proof of Lemma 3.3, the above expression is bounded.

Subcase 3.1. $N_1 \gtrsim N$.

\[ III \lesssim \frac{1}{m(N_1)m(N_2)} \|Iu_1\|_{L_{t,x}^{\infty}} \|Iu_2\|_{L_{t,x}^{2}L_{x}^{\infty}} \|n_{+}\|_{L_{t,x}^{2}L_{x}^{\infty}} \|n_{-}\|_{L_{t,x}^{2}L_{x}^{\infty}} + 4N \|u\|_{L_{t,x}^{2}L_{x}^{\infty}} \]
\[ \lesssim \left( \frac{N_1}{N} \right)^{1-s} \left( \frac{N_2}{N} \right)^{1-s} \frac{1}{N_1 N_2} \|Iu_1\|_{X_{\gamma,0}^{1,\frac{1}{2}}} \|n_{+}\|_{X_{\gamma,0}^{1,\frac{1}{2}}} \|Iu_2\|_{X_{\gamma,0}^{1,\frac{1}{2}}} \|n_{-}\|_{X_{\gamma,0}^{1,\frac{1}{2}}} \]
\[ \lesssim N^{-1 + N_{\max}^{-1}} \|Iu\|_{X_{\gamma,0}^{1,\frac{1}{2}}} \|n_{+}\|_{X_{\gamma,0}^{1,\frac{1}{2}}} \|n_{-}\|_{X_{\gamma,0}^{1,\frac{1}{2}}} \].

Subcase 3.2. $N_1 \lesssim N$.

Thus $N_{\max} \lesssim N_2$.

\[ III \lesssim \frac{1}{m(N_1)m(N_2)} \|Iu_1\|_{L_{t,x}^{\infty}} \|Iu_2\|_{L_{t,x}^{2}L_{x}^{\infty}} \|n_{+}\|_{L_{t,x}^{2}L_{x}^{\infty}} \|n_{-}\|_{L_{t,x}^{2}L_{x}^{\infty}} + 4N \|u\|_{L_{t,x}^{2}L_{x}^{\infty}} \]
\[ \lesssim \left( \frac{N_2}{N} \right)^{1-s} \|Iu_1\|_{X_{\gamma,0}^{1,\frac{1}{2}}} \|n_{+}\|_{X_{\gamma,0}^{1,\frac{1}{2}}} \|Iu_2\|_{X_{\gamma,0}^{1,\frac{1}{2}}} \|n_{-}\|_{X_{\gamma,0}^{1,\frac{1}{2}}} \]
\[ \lesssim (N_2)^{s} N_{\max}^{-1} \|Iu\|_{X_{\gamma,0}^{1,\frac{1}{2}}} \|n_{+}\|_{X_{\gamma,0}^{1,\frac{1}{2}}} \|n_{-}\|_{X_{\gamma,0}^{1,\frac{1}{2}}} \]
\[ \lesssim (N_2)^{-1 + N_{\max}^{-1}} \|Iu\|_{X_{\gamma,0}^{1,\frac{1}{2}}} \|n_{+}\|_{X_{\gamma,0}^{1,\frac{1}{2}}} \|n_{-}\|_{X_{\gamma,0}^{1,\frac{1}{2}}} \].

4 Proof of Theorem [1.1]

Proof. The data satisfy the estimate

\[ \|Iu_0\|_{H^1} \leq cN^{1-s}\|u_0\|_{H^s}. \]

We use our local existence theorem on $[0, \delta]$, where

\[ \delta \sim \frac{1}{(\|Iu_0\|_{H^1} + \|n_{+}\|_{L^2} + \|n_{-}\|_{L^2})^2 + N^{0+}}. \]

and conclude

\[ \|Iu\|_{X_{\gamma,0}^{1,\frac{1}{2}}[0, \delta]} + \|n_{+}\|_{X_{\gamma,0}^{1,\frac{1}{2}}[0, \delta]} + \|n_{-}\|_{X_{\gamma,0}^{1,\frac{1}{2}}[0, \delta]} \leq c(\|Iu_0\|_{H^1} + \|n_{+}\|_{L^2} + \|n_{-}\|_{L^2}) \leq c_2N^{1-s}. \]
From (2.9) we get

\[ H(Iu_0, n_{+}(\delta)) \leq c_0(\|Iu_0\|_{H^1}^2 + \|n_{+}\|_{L^2}^2) \leq c_0 N^{2(1-s)}, \]

and from (2.10)

\[ \|\Lambda Iu_0\|_{L^2}^2 + \|n_{+}\|_{L^2}^2 + \|n_{-}\|_{L^2}^2 \leq c_0 N^{2(1-s)}, \quad \|Iu_0\|_{L^2} \leq \|u_0\|_{L^2} =: M \]

with \( \tilde{c} = \tilde{c}(\varepsilon) \). Thus the constant in (3.1) depends only on \( \varepsilon \) and \( M \), i.e. \( c_2 = c_2(\varepsilon, M) \).

In order to reapply the local existence result with time intervals of equal length we need a uniform bound of the solution at time \( t = \delta \) and \( t = 2\delta \) etc. which follows from a uniform control over the energy by (2.10). The increment of the energy is controlled by Proposition 3.2 and Proposition 3.3 as follows:

\[
|H(Iu(\delta), n_{+}(\delta)) - H(Iu_0, n_{+}(\delta))| \\
\leq |H(Iu(\delta), n_{+}(\delta)) - \tilde{H}(u(\delta), n_{+}(\delta))| \\
+ |\tilde{H}(u(\delta), n_{+}(\delta)) - \tilde{H}(u_0, n_{+}(\delta))| + |\tilde{H}(u_0, n_{+}(\delta)) - H(Iu_0, n_{+}(\delta))| \\
\leq c[N^{-1+\|Iu(\delta)\|_{H^1}^2\|n_{+}(\delta)\|_{L^2} + N^{-\frac{1}{2}+\delta\frac{1}{2}+\|n_{+}\|_{X^{0,\frac{1}{2}+[0,\delta]}^1}}^2 + \|Iu\|_{X^{1,\frac{1}{2}+[0,\delta]}^1}}^2 + N^{-1+\|Iu_0\|_{H^1}^2\|n_{+}(\delta)\|_{L^2}^2}].
\]

Using (4.1) and the definition of \( \delta \) we arrive at

\[
|H(Iu(\delta), n_{+}(\delta)) - H(Iu_0, n_{+}(\delta))| \\
\leq c_3(N^{-1+\|Iu(\delta)\|_{H^1}^2\|n_{+}(\delta)\|_{L^2} + N^{-\frac{1}{2}+\delta\frac{1}{2}+\|n_{+}\|_{X^{0,\frac{1}{2}+[0,\delta]}^1}}^2 + \|Iu\|_{X^{1,\frac{1}{2}+[0,\delta]}^1}}^2 + N^{-1+\|Iu_0\|_{H^1}^2\|n_{+}(\delta)\|_{L^2}^2}) + N^{-1+\|Iu_0\|_{H^1}^2\|n_{+}(\delta)\|_{L^2}^2}N^{4(1-s)},
\]

where \( c_3 = c_3(\varepsilon, M) \). This is easily seen to be bounded by \( \varepsilon N^{2(1-s)} \) (for large \( N \)).

The number of iteration steps to reach the given time \( T \) is \( \frac{T}{\delta} \sim TN^{2(1-s)}+ \). This means that in order to give a uniform bound of the energy of the iterated solutions, namely by \( 2\varepsilon N^{2(1-s)} \), from the last inequality the following condition has to be fulfilled:

\[
c_3TN^{2(1-s)}+(N^{-1+\|Iu(\delta)\|_{H^1}^2\|n_{+}(\delta)\|_{L^2} + N^{-\frac{1}{2}+\delta\frac{1}{2}+\|n_{+}\|_{X^{0,\frac{1}{2}+[0,\delta]}^1}}^2 + \|Iu\|_{X^{1,\frac{1}{2}+[0,\delta]}^1}}^2 + N^{-1+\|Iu_0\|_{H^1}^2\|n_{+}(\delta)\|_{L^2}^2})N^{4(1-s)}) < \varepsilon N^{2(1-s)}
\]

where \( c_3 = c_3(2\varepsilon, 2M) \) (recall here that the initial energy is bounded by \( \varepsilon N^{2(1-s)} \)).

One easily checks that this can be fulfilled by choosing \( N \sim T^\frac{1}{2s-\frac{3}{2}}+ \gg 1 \) provided \( s > 3/4 \). So here is the point where the decisive bound on \( s \) appears.

A uniform bound of the energy implies by (2.5) uniform control of

\[
\|\Lambda Iu(t)\|_{L^2} + \|n(t)\|_{L^2} + \|\Lambda^{-1}n(t)\|_{L^2} \leq cN^{1-s}.
\]
Moreover \( \|u(t)\|_{L^2} \leq \|u(t)\|_{L^2} = \|u_0\|_{L^2} \), thus
\[
\|u(t)\|_{H^s} + \|n(t)\|_{L^2} + \|\Lambda^{-1} n_t(t)\|_{L^2} \leq cN^{1-s}.
\]

This implies
\[
\sup_{0 \leq t \leq T} (\|u(t)\|_{H^s} + \|n(t)\|_{L^2} + \|\Lambda^{-1} n_t(t)\|_{L^2}) \leq c(1 + T)^{\frac{1-s}{2s}}.
\]

\[\square\]

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