ATTRACTORS FOR MODEL OF POLYMER SOLUTIONS MOTION

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To Professor Rafael de la Llave

ABSTRACT. Existence of trajectory, global and pullback attractors for an incompressible non-Newtonian fluid (namely, for the mathematical model which describes a weak aqueous polymer solutions motion) in 2D and 3D bounded domains is studied in this paper. For this aim the approximating topological method is effectively combined with the theory of attractors of trajectory spaces.

1. Introduction. The motion of an incompressible fluid of constant density filling a bounded domain \( \Omega \subset \mathbb{R}^n, n = 2, 3 \), on a time interval \([0, T]\), \( T > 0 \), is described by the Cauchy momentum equation [13]:

\[
\frac{\partial v}{\partial t} + \sum_{i=1}^{n} v_i \frac{\partial v}{\partial x_i} + \text{grad } p = \text{Div } \sigma + f, \quad (x, t) \in \Omega \times (0, T),
\]

\[
div v = 0, \quad (x, t) \in \Omega \times (0, T),
\]

where \( v(x, t) = (v_1, \ldots, v_n) \) is the velocity vector of the particle at the point \( x \) at the time \( t \), \( p = p(x, t) \) is the fluid pressure at the point \( x \) at the time \( t \) and \( f = f(x, t) \) is the body force vector acting on the fluid. The symbol \( \text{Div } \sigma \) stands for the vector \( \left( \sum_{j=1}^{n} \frac{\partial \sigma_{1j}}{\partial x_j}, \sum_{j=1}^{n} \frac{\partial \sigma_{2j}}{\partial x_j}, \ldots, \sum_{j=1}^{n} \frac{\partial \sigma_{nj}}{\partial x_j} \right) \), whose coordinates are divergences of rows of the matrix \( \sigma = (\sigma_{ij}(v))_{i=1,\ldots,n} \), where \( \sigma(v) \) is the deviator of the stress tensor.

System (1)-(2) describes the motion of all kinds of fluids. However, it is incomplete. As a rule, one uses additional relations between the deviator of the stress tensor \( \sigma(v) \) and the strain velocity tensor \( \mathcal{E} = (\mathcal{E}_{ij}(v))_{j=1,\ldots,n} \), \( \mathcal{E}_{ij}(v) = \frac{1}{2} \left( \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) \).

Such relations are known as constitutive or rheological laws. Choosing a constitutive law we specify a type of fluid. Note that these laws are hypotheses and have to be verified for specific fluids by experimental data.

The rheological relation that governs the motion of viscoelastic medium is the following

\[
\sigma = 2\nu \mathcal{E} + 2\alpha \dot{\mathcal{E}},
\]

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where \( \nu > 0 \) is the viscosity of fluid and \( \kappa > 0 \) is the retardation (delay) time. This model of fluid motion describes the motion of a viscous non-Newtonian fluid that needs time to start moving under the action of force instantly applied.

As far as we know, this model was first practically considered by \([21]\). He named it the model of motion of weak aqueous polymer solutions. According to Pavlovsky, it is necessary to consider elastic properties as well as viscous ones in the case of such solutions. This is due to the fact that the stress depends both on the history of deformation and on the instantaneous value of the strain velocity tensor. The viscous properties of such a material are associated with the influence of the solvent. If the concentration of a polymer is low, this contribution is not negligible. This is confirmed by experimental research of solutions of polyethyleneoxide and polyacrylamide \([1]\) and solutions of polyacrylamide and guar gum \([2]\).

The constitutive law \((3)\) involves a time derivative \(\dot{\varepsilon}\). The first mathematical treatment of \((3)\) involved the partial derivative, in which case the law yields the Voigt model (a model of motion of linear viscoelastic non-Newtonian fluids \([19]\)). This system of equations were also proposed in \([4]\) as a regularization, for small values of the parameter \(\alpha\), of the 3D Navier-Stokes equations. Then the case of the total derivative was studied by \([20]\); however, the proofs of his results were incorrect \([15]\). A complete proof of weak solvability of \((1)-(3)\) with the total derivative was first given in \([45]\).

In the recent years rational mechanics \([29]\) has influenced scientists in the way that they have started to investigate constitutive laws independent of the observer, i.e. that do not change under the Galilean transformation:

\[
t^* = t + a, \quad x^* = x_0^* + Q(t)(t-t_0),
\]

where \(a\) is a time value, \(x_0\) is a point in a space, \(x_0^*\) is a time function with values in space, \(Q\) is a time function with values in the set of orthogonal tensors. In other words, if the original tensor function changes according to the law \((4)\), will the constitutive law be the same in different reference frames? In the case of partial and total derivatives the answer is negative. In order to answer this question positively one introduces objective derivatives \([29]\).

Example of an objective derivative of a tensor is the regularized Jaumann’s derivative \([46]\):

\[
\frac{DT(t,x)}{Dt} = \frac{dT(t,x)}{dt} + T(t,x)W_\rho(t,x) - W_\rho(t,x)T(t,x),
\]

\[
W_\rho(v)(t,x) = \int_{\mathbb{R}^n} \rho(x-y)W(t,y)dy,
\]

where \(\rho : \mathbb{R}^n \to \mathbb{R}\) is a smooth function with compact support such that \(\int_{\mathbb{R}^n} \rho(y)dy = 1\) and \(\rho(x) = \rho(y)\) for \(x\) and \(y\) with the same Euclidean norm; \(W = (W_{ij}(v))_{i,j=1,...,n}\), \(W_{ij}(v) = \frac{1}{2} \left( \frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i} \right)\) is the vorticity tensor.

Note that the constitutive law \((3)\) with the regularized Jaumann’s derivative is similar to a particular case of second grade fluids \([22]\), \([23]\). Existence and uniqueness results for weak and classical solutions of both the stationary and time-dependent problems have been established under various restrictions on the normal stress modulus and the data in \([9, 12, 26]\).

Substituting the right-hand of \((3)\) with the regularized Jaumann’s derivative for \(\sigma\) in equations \((1)-(2)\), we obtain
\[
\frac{\partial v}{\partial t} + \sum_{i=1}^{n} v_i \frac{\partial v}{\partial x_i} - \kappa \Delta v - \nu \Delta v - 2\kappa \text{Div} \left( v_k \frac{\partial \mathcal{E}(v)}{\partial x_k} \right) - 2\kappa \text{Div} \left( \mathcal{E}(v) W_{\rho}(v) - W_{\rho}(v) \mathcal{E}(v) \right) + \text{grad } p = f, \\
\text{div } v = 0.
\]

As we investigate the existence of attractors we consider the mathematical model (5)-(6) in a bounded domain \( \Omega \subset \mathbb{R}^n, n = 2, 3 \), with the boundary \( \partial \Omega \) of class \( C^2 \) on a time interval \([0, +\infty)\).

For the system (5)-(6) we consider the initial-boundary value problem with the initial condition
\[
v(x, 0) = a(x), \quad x \in \Omega,
\]
and the boundary nonslip condition
\[
v|_{\partial \Omega \times [0, +\infty]} = 0.
\]

The problem (5)–(6) is investigated in [39]–[42]. In the case of this problem neither the global solvability in the strong sense nor the uniqueness of the weak solution have been proved. Consequently, it is impossible to use classical approach to attractors.

It is well known that limiting regimes for problems which have no uniqueness theorems can be investigated on the basis of the theory of trajectory attractors. This theory was constructed by M.I. Vishik and V.V. Chepyzhov [6, 7, 8, 30]. A similar method was put forward independently by G.R. Sell [24, 25]. In particular, it turns out that an attractor can be constructed for the Navier-Stokes system in the 3D case [8] or for similar our considered problem 2D non-Newtonian fluid [34].

However, for the model of polymer solutions motion (and others viscoelastic models) we cannot always find shift-invariant trajectory spaces, so a theory attractors in which trajectory spaces were not necessarily shift-invariant was developed in [31, 47], whereas the construction used in [8] and [25] required this invariance. So why for prove of existence of attractors for this model in this paper the method which developed in [47] and which is based on approximating topological approach and on theory of attractors of trajectory spaces will be used.

Given an autonomous system with an attractor, it is only a matter of elapsed time, when the initial data gets forgotten? In nonautonomous systems the absolute times of both start and check are to be taken into account. As a consequence, there is more than one way to generalise the notion of attractor to nonautonomous systems.

One approach is to consider pullback attractors. They were first considered in [10, 14]. Initially, the theory of pullback attractors was naturally developed in the framework of processes (biparametric families of operators describing the evolution of nonautonomous systems). The infinite-dimensional setting of this theory has become quite rich both in abstract results and in applications. In particular, there are a number of results concerning pullback attractors of Newtonian fluids as well as certain non-Newtonian ones [3, 5, 33, 35, 36]. For these pullback attractors the invariant measures can be constructed [18, 37, 38]. However, typical lack of uniqueness impedes the use of processes in fluid mechanics. The notion of pullback attractor has recently been ported to trajectory spaces and constructed for the Navier-Stokes system by D.A. Vorotnikov [32].
The paper is organized as follows. In Section 2 we recall some auxiliary definitions and results and formulate main results. Section 3 is devoted to the proof of the main result on existence of weak solution. Section 4 is devoted to the proof of the main result on existence of trajectory and global attractors and in Section 5 the proof of the main result on existence of pullback attractor is described.

2. Preliminaries, notation and the main result.

2.1. Functional spaces. We use standard notation for the Lebesgue and Sobolev spaces. Let $V$ be the set of smooth divergence free functions $\Omega \to \mathbb{R}^n$, $n = 2, 3$, with compact support contained in $\Omega$. Also let $V^0$ be the closure of $V$ with respect to the norm of space $L_2(\Omega)$, $V^1$ be the closure of $V$ with respect to the norm of space $W^1_2(\Omega)$ and $V^2 = W^2_2(\Omega) \cap V^1$.

We will also use the well-known decomposition of $L_2(\Omega)$ (see [28]): $L_2(\Omega) = V^0 \oplus \nabla W^1_2(\Omega)$, where $\nabla W^1_2(\Omega) = \{ \nabla p : p \in W^1_2(\Omega) \}$ (spaces $V^0$ and $\nabla W^1_2(\Omega)$ are orthogonal in $L_2(\Omega)$).

Let $\pi : L_2(\Omega) \to V^0$ be the Leray projector. Consider the operator $A = -\pi \Delta$ defined on $D(A)$. It can be extended to a closed self-adjoint operator in $V^0$. We denote the extension by the same letter. The extended operator $A$ is positive and has a compact inverse. Hence $A$ has countably many eigenvalues $0 < \lambda_1 \leq \lambda_2 \leq \ldots$. Let $e_k$ denote associated eigenfunctions. Vector functions $e_k(k = 1, 2, \ldots)$ are smooth (depends on boundary smoothness). Consider the set

$$E_\infty = \left\{ v = \sum_{k=1}^{m} v_k e_k, \quad m \in \mathbb{N}, v_k \in \mathbb{R} \right\}$$

(here $m$ depends on $v$) and for any $\alpha \in \mathbb{R}$ define space $V^{\alpha}$ as the closure of $E_\infty$ with respect to the norm

$$\|v\|_{V^{\alpha}} = \left( \sum_{k=1}^{\infty} \lambda_k^{\alpha} |v_k|^2 \right)^{1/2}.$$

It can be shown that for $\alpha = 0, 1, 2$ the construction described above leads to the same $V^0, V^1$ and $V^2$ as introduced at the beginning.

In case $\alpha \geq 0$ we have continuous embedding $V^{\alpha} \subset W_2^0(\Omega)$ and the norm $\|\cdot\|_{V^{\alpha}}$ is equivalent to the norm of $W_2^0(\Omega)$ (see [11, 45]). Note that for $\alpha = 0, 1, 3$ we have

$$\|v\|_{V^0} = \left( \int_\Omega v^2(x)dx \right)^{1/2} \quad \|v\|_{V^1} = \left( \int_\Omega \nabla v(x) : \nabla v(x)dx \right)^{1/2} \quad \|v\|_{V^3} = \left( \int_\Omega \nabla(\Delta v(x)) : \nabla(\Delta v(x))dx \right)^{1/2}.$$

(for matrices $A = (a_{ij})$ and $B = (b_{ij})$ of order $n$ we put $A : B = \sum a_{ij}b_{ij}$). For $\alpha > \beta \geq 0$ the embedding $V^{\alpha} \subset V^{\beta}$ is compact. Let $\alpha \geq 0$ and $(V^\alpha)^*$ be the conjugate space of $V^{\alpha}$. Then the space $(V^\alpha)^*$ is isometric to $V^{-\alpha}$. We identify these spaces.

We need the following Banach spaces in order to define weak solutions

$$W_1[0, T] = \{ v : v \in L_\infty(0, T; V^1), v' \in L_\infty(0, T; V^{-1}) \}$$

with the norm $\|v\|_{W_1[0, T]} = \|v\|_{L_\infty(0, T; V^1)} + \|v'\|_{L_\infty(0, T; V^{-1})}$ and

$$W_2[0, T] = \{ v : v \in C([0, T], V^3), v' \in L_\infty(0, T; V^3) \}$$

with the norm $\|v\|_{W_2[0, T]} = \|v\|_{C([0, T], V^3)} + \|v'\|_{L_\infty(0, T; V^3)}$. 


Also let $W_1^{1,0}(\mathbb{R}^+)$ be the class of function $v : \mathbb{R}^+ \to V^1$ ($\mathbb{R}^+$ denote the non-negative half-axis of the real axis $\mathbb{R}^+$) such that the restriction of $v$ to any segment $[0, T]$ belongs to $W_1[0, T]$; likewise, let $W_2^{1,0}(\mathbb{R}^+)$ denote the class of functions $v \in C([\mathbb{R}^+, V^3])$ such that the restriction of $v$ to any segment $[0, T]$ belongs to $W_2[0, T]$. These classes are needed to defining solutions on the nonnegative semi-axis.

The following compactness theorem is very important.

Let $X_0 \subset F \subset X_1$ be Banach spaces, where the first embedding is compact and $X_0$ is reflexive; further, let $0 < T < \infty$ and $1 \leq p_i \leq \infty$ $(i = 0, 1)$. Consider the space

$$W(0, T; p_0, p_1; X_0, X_1) = \{v : v \in L_{p_0}(0, T; X_0), v' \in L_{p_1}(0, T; X_1)\}$$

(the time derivative is in the sense of distributions on $(0, T)$ with values in $X_1$); $W(0, T; p_0, p_1; X_0, X_1)$ is endowed with the norm

$$\|v\|_{W} = \|v\|_{L_{p_0}(0, T; X_0)} + \|v'\|_{L_{p_1}(0, T; X_1)}.$$ 

**Theorem 2.1.** If $p_0 < \infty$, the following embedding is compact:

$$W(0, T; p_0, p_1; X_0, X_1) \subset L_{p_0}(0, T; F).$$

If $p_0 = \infty$ and $p_1 > 1$, the following embedding is compact:

$$W(0, T; p_0, p_1; X_0, X_1) \subset C([0, T]; F).$$

The proof can be found, e.g., in [27].

### 2.2. Statement of weak solution problem.

Let the body force $f \in L_2(\Omega)$ be fixed.

**Definition 2.2.** A weak solution of the initial-boundary value problem (5)-(8) on the interval $[0, T]$ is a function $v \in W_1[0, T]$ such that for any $\varphi \in V^3$ and almost all $t \in (0, T)$ it satisfies the equality

$$\langle \frac{\partial v}{\partial t}, \varphi \rangle - \int_{\Omega} \sum_{i,j=1}^n v_i \frac{\partial \varphi_j}{\partial x_i} \, dx + \nu \int_{\Omega} \nabla v : \nabla \varphi \, dx + \kappa \int_{\Omega} \nabla \left( \frac{\partial v}{\partial t} \right) : \nabla \varphi \, dx$$

$$- \kappa \int_{\Omega} \sum_{i,j,k=1}^n v_{ik} \frac{\partial^2 \varphi_j}{\partial x_i \partial x_k} \, dx + \kappa \int_{\Omega} \sum_{i,j,k=1}^n v_{ik} \frac{\partial^2 \varphi_j}{\partial x_i \partial x_k} \, dx$$

$$+ 2\kappa \int_{\Omega} (E(v)W_{\rho}(v) - W_{\rho'}(v)E(v)) : \nabla \varphi \, dx = \int_{\Omega} f \varphi \, dx \quad (9)$$

and the initial condition

$$v(0) = a, \quad a \in V^1. \quad (10)$$

A function $v \in W_1^{1,0}(\mathbb{R}^+)$ is called a weak solution of problem (5)-(8) on $\mathbb{R}^+$ if for any $T > 0$ the function $v$ is a weak solution of the problem (5)-(8) on the interval $[0, T]$.

**Remark 1.** If $v \in W_1[0, T]$, then $v(t) \in V \subset L_4(\Omega)$ and $\partial v_i / \partial x_j \in L_2(\Omega)$ for a.a. $t \in (0, T)$. For $\varphi \in V^3 \subset W_2^3(\Omega)$ we have $\partial^2 \varphi_j / \partial x_i \partial x_k \in W_2^2(\Omega) \subset L_4(\Omega)$. Consequently, all the integrals on the left-hand side of (9) exist.

**Remark 2.** By Theorem 2.1 we have $W_1[0, T] \subset C_0([0, T]; V^0)$. Thus the initial condition (10) is sensible for functions belonging to the class $W_1[0, T]$. 
The identity (9) is derived from equations (5)-(8) in a standard way: under the
assumption that a classical solution exists, multiply equation (5) by an arbitrary
function $\varphi \in V^\alpha$ and integrate by parts certain terms; since $\varphi$ is solenoidal, the
term grad $p$ is eliminated.

The following existence theorem holds.

**Theorem 2.3.** For any $a \in V^1$, problem (5)-(8) has a solution on the semiaxis $\mathbb{R}_+$
that satisfies the inequality

$$
\|v(t)\|_{V^1} + \|v'(t)\|_{V^{-1}} \leq R_0(1 + \|a\|_{V^1}^2 e^{-\alpha t}) \text{ for a.a. } t \geq 0,
$$

where the constants $R_0 > 0$ and $\alpha > 0$ are independent of $v$.

This theorem is proved in Section 3.

### 2.3. Statement of trajectory and global attractors problem.

We first introduce some definitions and theorems concerning the trajectory and global attractors.

The existence result of a trajectory and global attractors and its property for a non-
Newtonian fluid process one can find at the book [47] and the review article [43].

Let $E$ and $E_0$ be Banach spaces, $E \subset E_0$ (the embedding is assumed to be continuous); we also assume that $E$ is reflexive. Let $L_\infty(\mathbb{R}_+; E)$ be the Banach space of essentially bounded functions on $\mathbb{R}_+$ taking values in $E$. The linear space $C(\mathbb{R}_+; E_0)$ consists of continuous function on $\mathbb{R}_+$ taking values in $E_0$. Convergence in this space is treated as uniform convergence on each interval $[0, T]$, $T > 0$.

Now we look at the **shift operators** $T(h)$: to each function $g$ this operator assigns the function $T(h)g$ such that $T(h)g(t) = g(t + h)$. For $h \geq 0$ the $T(h)$ are bounded linear operators in $L_\infty(\mathbb{R}_+; E)$ and continuous linear operators is $C(\mathbb{R}_+; E_0)$. The identity $T(h_1)T(h_2) = T(h_1 + h_2)$ is obvious, as is the fact that $T(0)$ is the identity operator; this allows us to say that the family $\{T(h) : h \geq 0\}$ is a semigroup, the so-called **translation semigroup**.

**Definition 2.4.** A nonempty set

$$
\mathcal{H}^+ \subset C(\mathbb{R}_+; E_0) \cap L_\infty(\mathbb{R}_+; E)
$$

is called a trajectory space for the problem; its elements are called trajectories for the problem.

The only requirement is that the trajectory space must be nonempty (and consist of functions in $C(\mathbb{R}_+; E_0) \cap L_\infty(\mathbb{R}_+; E)$).

**Definition 2.5.** A set $P \subset C(\mathbb{R}_+; E_0) \cap L_\infty(\mathbb{R}_+; E)$ is said to be attracting (for a trajectory space $\mathcal{H}^+$) if for each subset $B$ of $\mathcal{H}^+$ which is bounded in $L_\infty(\mathbb{R}_+; E)$,

$$
\lim_{h \to 0} \sup_{u \in B} \inf_{v \in P} \|T(h)u - v\|_{C([0,M]; E_0)} = 0 \quad \forall M > 0.
$$

**Definition 2.6.** A set $P \subset C(\mathbb{R}_+; E_0) \cap L_\infty(\mathbb{R}_+; E)$ is said to be absorbing (for a trajectory space $\mathcal{H}^+$) if for each subset $B$ of $\mathcal{H}^+$ which is bounded in $L_\infty(\mathbb{R}_+; E)$, there exists $h \geq 0$ such that $T(h)B \subset P$ for all $t \geq h$.

Of course, it follows from these definitions that each absorbing set is attracting.

**Definition 2.7.** A nonempty set $P \subset C(\mathbb{R}_+; E_0) \cap L_\infty(\mathbb{R}_+; E)$ is called a trajectory semi-attractor (of the trajectory space $\mathcal{H}^+$), if the following conditions hold:

(i) $P$ is compact in $C(\mathbb{R}_+; E_0)$ and bounded in $L_\infty(\mathbb{R}_+; E)$;
(ii) the inclusion $T(t)P \subset P$ holds for all $t \geq 0$;
(iii) $P$ is an attracting set in the sense of Definition 2.5.

**Definition 2.8.** A set $U \subset C(\mathbb{R}_+; E_0) \cap L_\infty(\mathbb{R}_+; E)$ is called a trajectory attractor (of a trajectory space $\mathcal{H}^+$) if it satisfies the following conditions:

(i) $U$ is compact in $C(\mathbb{R}_+; E_0)$ and bounded in $L_\infty(\mathbb{R}_+; E)$;
(ii) $T(t)U = U$ for all $t \geq 0$;
(iii) the set $U$ is attracting in the sense of Definition 2.5.

**Definition 2.9.** A minimal trajectory attractor (of a trajectory space $\mathcal{H}^+$) is a trajectory attractor which is minimal with respect to inclusion (it lies in any other trajectory attractor of the space $\mathcal{H}^+$).

**Definition 2.10.** A set $A \subset E$ is called a global attractor (in $E_0$) of a trajectory space $\mathcal{H}^+$ if it satisfies the following conditions:

(i) $A$ is compact in $E_0$ and bounded in $E$;
(ii) for each subset $B$ of $\mathcal{H}^+$ which is bounded in $L_\infty(\mathbb{R}_+; E)$ the following conditions of attraction is satisfied:

$$
\sup_{u \in B} \inf_{y \in A} \|u(t) - y\|_{E_0} \to 0, \quad t \to \infty;
$$

(iii) $A$ is the minimal set satisfying conditions (i) and (ii) in this definition ($A$ lies in any set satisfying these two conditions).

**Remark 3.** It follows directly from definitions that if a trajectory space has a minimal trajectory attractor or a global attractor, then this attractor is unique.

**Theorem 2.11.** Suppose the trajectory space $\mathcal{H}^+$ has a trajectory semi-attractor $P$. Then $\mathcal{H}^+$ has the minimal trajectory attractor $U$ and the kernel of $\mathcal{H}^+$ is compact in $C(\mathbb{R}; E_0)$ and bounded with respect to the norm of $L_\infty(\mathbb{R}; E)$.

**Theorem 2.12.** Suppose the trajectory space $\mathcal{H}^+$ has the minimal trajectory attractor. Then the global attractor $A$ of $\mathcal{H}^+$ exists.

Now we go over to our problem (5)-(8).

**Definition 2.13.** A function $v \in W_{1,loc}^1(\mathbb{R}_+) \cap L_\infty(\mathbb{R}_+; V^1)$ is called a trajectory of problem (5)-(8) if it is a solution of this problem with some $a \in V^1$ and the following inequality holds:

$$
\|v(t)\|_{V^1} + \|v'(t)\|_{V^{-1}} \leq R_0 \left(1 + \|v\|^2_{L_\infty(\mathbb{R}_+, V^1)} e^{-\alpha t}\right) \text{ for a.a. } t \geq 0. \tag{12}
$$

The set of trajectories is called its trajectory space of the problem and is denoted by $\mathcal{H}^+$.

**Remark 4.** Weak solutions of the problem (5)-(8) are weakly continuous in $V^1$, whence $\|a\|_{V^1} = \|v(0)\|_{V^1} \leq \|v\|_{L_\infty(\mathbb{R}_+, V^1)}$. Thus inequality (12) follows from inequality (11), and by Theorem 2.3 we see that any point $a \in V^1$ is the beginning of a trajectory.

Consider a number $\delta \in (0, 1)$ and suppose that $f \in L_2(\Omega)$.

These are main results concerning the existence of attractors.

**Theorem 2.14.** The trajectory space $\mathcal{H}^+$ of problem (5)-(8) has the minimal trajectory attractor $U$. The attractor is bounded in $L_\infty(\mathbb{R}_+; V^1)$ and compact in $C(\mathbb{R}_+; V^{1-\delta})$; it attracts sets of trajectories bounded in $L_\infty(\mathbb{R}_+; V^1)$ with respect to the topology of $C(\mathbb{R}_+; V^{1-\delta})$. 
**Theorem 2.15.** The trajectory space $\mathcal{H}^+$ of problem (5)-(8) has the global trajectory attractor $\mathcal{A}$. The attractor is bounded in $V^1$ and compact in $V^{1-\delta}$; it attracts sets of trajectories bounded in $L^\infty(\mathbb{R}^+; V^1)$ with respect to the topology of $V^{1-\delta}$.

These theorems are proved in Section 4.

2.4. **Statement of pullback attractors problem.** We first introduce some definitions and theorems concerning the pullback attractors.

Let $E$ and $E_0$ be Banach spaces, $E \subset E_0$ (the embedding is assumed to be continuous); we also assume that $E$ is reflexive. For every $\tau \in \mathbb{R}$ we assign the non-empty set

$$\mathcal{H}^+ \subset \mathcal{T} := C(\mathbb{R}^+; E_0) \cap L^\text{loc}_\infty(\mathbb{R}^+; E).$$

We start with basic definitions and results of the abstract theory of trajectory pullback attractors [32, 44].

The sets $\mathcal{H}^+$ are called the *trajectory spaces* and elements thereof are called *trajectories*. The family $\mathcal{H}^+ = \{\mathcal{H}^+\}_{\tau \in \mathbb{R}}$ is called the *family of trajectory spaces*.

Fix a class of families of sets $\mathcal{D}$ over $E$ assuming that for any family $\mathcal{D} = \{D_t\} \in \mathcal{D}$ we have $D_t \neq \emptyset$ for any $t \in \mathbb{R}$. For any $\mathcal{D} = \{D_t\} \in \mathcal{D}$ we consider the family $\mathcal{H}^+(\mathcal{D}) = \{\mathcal{H}^+\}_{\tau \in \mathbb{R}}$, where

$$\mathcal{H}^+(\mathcal{D}) = \{v \in \mathcal{H}^+: v(0) \in D_\tau\}.$$

**Definition 2.16.** A family $\mathcal{P} = \{P_\theta\}$ ($P_\theta \subset \mathcal{T}$) is called pullback $\mathcal{D}$-attracting for $\mathcal{H}^+$, if for any family $\mathcal{D} \in \mathcal{D}$ and for any $\theta \in \mathbb{R}$ we have

$$\sup_{u \in \mathcal{H}^+(\mathcal{D})} \inf_{v \in P_\theta} \|T(\theta - \tau)u - v\|_{C(\mathbb{R}^+; E_0)} \to 0 \quad (\tau \to -\infty).$$

**Definition 2.17.** A family $\mathcal{P} = \{P_\theta\}$ ($P_\theta \subset \mathcal{T}$) is called pullback $\mathcal{D}$-absorbing for $\mathcal{H}^+$, if for any family $\mathcal{D} \in \mathcal{D}$ and for any $\theta \in \mathbb{R}$ it exists the number $\tau_\mathcal{D}(\theta) \leq \theta$, such that for all $\tau \leq \tau_\mathcal{D}(\theta)$ the inclusion

$$T(\theta - \tau)\mathcal{H}^+(\mathcal{D}) \subset P_\theta,$$

holds, and the function $\tau_\mathcal{D} : \mathbb{R} \to \mathbb{R}$ is nondecreasing.

**Definition 2.18.** A family $\mathcal{P} = \{P_\theta\}$ ($P_\theta \subset \mathcal{T}$) is called the $\mathcal{T}$-precompact, if

(i) $P_\theta$ is precompact in $C(\mathbb{R}^+; E_0)$ for every $\theta \in \mathbb{R}$;

(ii) for any $\theta \in \mathbb{R}$ there exists a continuous function $\varphi_\theta : \mathbb{R}^+ \to \mathbb{R}$ such that for any trajectory $v \in P_\theta$ the inequality $\|v(t)\|_E \leq \varphi_\theta(t)$ holds for all $t \in \mathbb{R}^+$.

This family is called $\mathcal{T}$-compact, if in addition $P_\theta$ is closed (and thus, compact) in $C(\mathbb{R}^+; E_0)$ for any $\theta \in \mathbb{R}$.

**Definition 2.19.** A family $\mathcal{P}$ consisting of nonempty subsets of $\mathcal{T}$ called a *trajectory pullback $\mathcal{D}$-semiattractor* for $\mathcal{H}^+$, if

(i) $\mathcal{P}$ is $\mathcal{T}$-compact;

(ii) $T(h)\mathcal{P} \subset \mathcal{P}$ for any $h \geq 0$;

(iii) $\mathcal{P}$ is pullback $\mathcal{D}$-attracting.

**Definition 2.20.** A trajectory pullback $\mathcal{D}$-semiattractor $\mathcal{P}$ for $\mathcal{H}^+$ is called a *trajectory pullback $\mathcal{D}$-attractor* for $\mathcal{H}^+$, if $T(h)\mathcal{P} = \mathcal{P}$ for any $h \geq 0$.

**Definition 2.21.** A trajectory pullback $\mathcal{D}$-attractor $\mathcal{U} = \{U_\theta\}$ ($U_\theta \subset \mathcal{T}$) for $\mathcal{H}^+$ is called *minimal*, if it is contained in any trajectory pullback $\mathcal{D}$-attractor $\mathcal{P} = \{P_\theta\}$.
Definition 2.22. A family $A = \{A_\theta\} \subset E$ is called a global pullback $\mathcal{D}$-attractor for $H^+$, if

(i) $A_\theta$ is compact in $E_0$ and bounded in $E$ for all $\theta \in \mathbb{R}$;
(ii) for any $D \in \mathcal{D}$ and $\theta \in \mathbb{R}$ the pullback attraction

$$\sup_{v \in H_0^1(D)} \inf_{a \in A_{\theta}} \|v(\theta - \tau) - a\|_{E_0} \to 0 \quad (\tau \to -\infty)$$

holds;

(iii) $A$ is contained in any family $A' = \{A'_{\theta}\} (A'_{\theta} \subset E)$ satisfying (i) and (ii).

Remark 5. The minimal trajectory pullback attractor is unique, and so is the global pullback attractor [32].

Theorem 2.23. Suppose that $H^+$ admits a $\mathcal{T}$-precompact pullback $\mathcal{D}$-absorbing family $P$, and let $\overline{P}$ denote the closure of $P$ with respect to the topology of $C(\mathbb{R}_+; E_0)$. Then there exists a minimal trajectory pullback $\mathcal{D}$-attractor $U \subset P$.

Theorem 2.24. Suppose that $H^+$ has a trajectory pullback $\mathcal{D}$-semiattractor $P$. Then it also has the minimal trajectory pullback $\mathcal{D}$-attractor $U \subset P$.

Theorem 2.25. Let $U = \{U_\theta\}$ be the minimal trajectory pullback $\mathcal{D}$-attractor for $H^+$. Then the family $A = \{A_\theta\}$, where $A_\theta = \{u(0): u \in U_\theta\} \subset E$, is the global pullback $\mathcal{D}$-attractor for $H^+$.

We use this approach to consider pullback attractors of the problem (5)-(8).

In this section we consider equalities (5) and (6) on $\Omega \times (\tau, +\infty)$ with the initial condition

$$v(x, \tau) = a(x), \quad x \in \Omega,$$  

and the boundary condition

$$v|_{\partial\Omega \times [\tau, +\infty]} = 0.$$  

We assume that the body force $f$ in the equation (5) belongs to $L^2_{\text{loc}}(\mathbb{R}; V^0)$ and verifies

$$\int_{-\infty}^{t} e^{\alpha \xi} \|f(\xi)||_{V^0}^2 d\xi < \infty$$

for all $t \in \mathbb{R}$ (where $\alpha > 0$).

Fix $\delta \in (0, 1]$. For the introduction of the class $\mathcal{T}$ we use $E = V^1$ and $E_0 = V^{1-\delta}$.

Theorem 2.26. Suppose that $f \in L^2_{\text{loc}}(\mathbb{R}, V^0)$ satisfies (15). Then the family of trajectory spaces $H^+$ has a minimal trajectory pullback $\mathcal{D}$-attractor $U$ and a global pullback $\mathcal{D}$-attractor $A = U(0)$.

The proof of this theorem will be given below in the section 5.

3. Solvability of the initial-boundary value problem (5)-(8). To prove Theorem 2.3 we study an auxiliary problem.

3.1. Auxiliary problem. Find a function $v \in W_2[0, T]$ on $[0, T]$ ($v \in W_2^{\text{loc}}(\mathbb{R}_+)$ on $\mathbb{R}_+$) satisfying for every $\varphi \in V^3$ and a.a. $t \in (0, T)$ the identity
\[ \int_{\Omega} \frac{\partial v}{\partial t} \varphi dx - \int_{\Omega} \sum_{i,j=1}^{n} v_{ij} \frac{\partial \varphi_j}{\partial x_i} dx + \nu \int_{\Omega} \nabla v : \nabla \varphi dx + \kappa \int_{\Omega} \frac{\partial v}{\partial t} : \nabla \varphi dx + \varepsilon \int_{\Omega} \nabla (\Delta \varphi) dx - \kappa \int_{\Omega} \sum_{i,j,k=1}^{n} v_{kj} \frac{\partial^2 \varphi_j}{\partial x_i \partial x_k} dx + 2\kappa \int_{\Omega} (E(v)W_{\rho}(v) - W'_{\rho}(v)E(v)) : \nabla \varphi dx = \int_{\Omega} f \varphi dx, \quad (16) \]

and the initial condition
\[ v(0) = b, \quad (17) \]

where \( \varepsilon > 0, \alpha = \nu/(k_0^2 + \kappa), k_0 \) and \( b \) are constants.

Note that (16) differs from (9) by presence of the term \( \varepsilon e^{-\alpha t} \int_{\Omega} \nabla (\Delta v') : \nabla (\Delta \varphi) dx \).

The identity (16) transforms into (9) as \( \varepsilon \to 0 \).

In what follows we consider operator equations. Consider the following operators:

\[ I : V^3 \to V^{-3}, \langle Iv, \varphi \rangle = \int_{\Omega} v \varphi dx, \quad v, \varphi \in V^3; \]
\[ A : V^1 \to V^{-1}, \langle Av, \varphi \rangle = \int_{\Omega} \nabla v : \nabla \varphi dx, \quad v, \varphi \in V^1; \]
\[ N : V^3 \to V^{-3}, \langle Nv, \varphi \rangle = \int_{\Omega} \nabla (\Delta v) : \nabla (\Delta \varphi) dx, \quad v, \varphi \in V^3; \]
\[ B_1 : L_4(\Omega) \to V^{-1}, \langle B_1(v), \varphi \rangle = \int_{\Omega} \sum_{i,j=1}^{n} v_{ij} \frac{\partial \varphi_j}{\partial x_i} dx, \quad v \in L_4(\Omega), \varphi \in V^1; \]
\[ B_2 : V^1 \to V^{-3}, \langle B_2(v), \varphi \rangle = \int_{\Omega} \sum_{i,j,k=1}^{n} v_{kj} \frac{\partial^2 \varphi_j}{\partial x_i \partial x_k} dx, \quad v \in V^1, \varphi \in V^3; \]
\[ B_3 : V^1 \to V^{-3}, \langle B_3(v), \varphi \rangle = \int_{\Omega} \sum_{i,j,k=1}^{n} v_{kj} \frac{\partial^2 \varphi_j}{\partial x_i \partial x_k} dx, \quad v \in V^1, \varphi \in V^3; \]
\[ D : V^1 \to V^{-3}, \langle D(v), \varphi \rangle = \int_{\Omega} (E W_{\rho} - W_{\rho} E)(v) : \nabla \varphi dx, \quad v \in V^1, \varphi \in V^3. \]

It will be convenient to have a notation for the exponential function. By definition, for any \( \beta \in \mathbb{R} \) put
\[ e^{\beta}(t) = e^{\beta t}. \]

Since \( \varphi \in V^3 \) is arbitrary in (16) this identity is equivalent to the following operator equation
\[ Iv' - B_1(v) + \nu Av + \varepsilon e^{-\alpha t} Nv' + \kappa Av' - \kappa B_2(v) - \kappa B_3(v) + 2\kappa D(v) = f. \quad (18) \]

**Definition 3.1.** A function \( v \in W^[2](0,T) \) is called a solution of (18) on \([0,T]\) if it yields a true equality in \( L_1(0,T; V^{-3}) \) when substituted into (18). A function
$v \in W^{1,\infty}_2(\mathbb{R}_+)$ is called a solution of (18) on $\mathbb{R}_+$ if it is a solution of (18) on each finite segment $[0, T]$.

We also define the following operators:

$L : W_2^2[0, T] \to L_\infty(0, T; V^{-3}) \times V^3$, \hspace{1em} $L(v) = ((I + \varepsilon e_{-\alpha} N + \kappa A)v', v|_{t=0})$

$K : W_2^2[0, T] \to L_\infty(0, T; V^{-3}) \times V^3$, \hspace{1em} $K(v) = (\nu Av - B_1(v) - \kappa B_2(v) - \kappa B_3(v) + 2\kappa D(v), 0)$.

The problem of finding a solution of equation (18) satisfying the initial condition (17) is equivalent to the problem of finding a solution for the following operator equation:

$L(v) + K(v) = (f, b)$.

### 3.2. Properties of operators.

**Lemma 3.2.** The following properties hold.

1) The operator $L : W_2^2[0, T] \to L_2(0, T; V^{-3}) \times V^3$ is invertible and the inverse operator is continuous.

2) The operator $K : W_2^2[0, T] \to L_2(0, T; V^{-3}) \times V^3$ is compact.

The proof can be found in [16], [39] and [45].

### 3.3. A priori estimate.

Consider the family of auxiliary problems depending on the parameter $\lambda \in [0, 1]$:

$$
(I + \varepsilon e_{-\lambda \alpha} N + \kappa A)v' + \lambda \nu Av - B_1(v) - \kappa B_2(v) - \kappa B_3(v) + 2\kappa D(v) = \lambda f, \hspace{1em} (19)
$$

$$
v(0) = \lambda b.
$$

The definition of a solution has the same sense for (19) as for (18).

We will use necessary estimates from [17]. Let $v \in W_2^2[0, T]$ be a solution of (19) on $[0, T]$ for certain $\lambda \in [0, 1]$. Apply both sides (16) with some $\lambda$ to $v(t)$ and observe that

$$
\langle v'(t), v(t) \rangle = \frac{1}{2} \frac{d}{dt} \|v(t)\|_{V,0}^2,
$$

$$
\langle e^{-\lambda \alpha t} N v'(t), v(t) \rangle = e^{-\lambda \alpha t} \frac{1}{2} \frac{d}{dt} \|v(t)\|_{V,3}^2,
$$

$$
\langle Av'(t), v(t) \rangle = \frac{1}{2} \frac{d}{dt} \|v(t)\|_{V,1}^2.
$$

Moreover, it is known [39] that $\langle B_i(v(t)), v(t) \rangle = 0$ ($i = 1, 2, 3$) and $\langle D(v(t)), v(t) \rangle = 0$. Thus we obtain:

$$
\int_0^T \frac{1}{2} \frac{d}{dt} \|v(t)\|_{V,0}^2 + \varepsilon \frac{d}{dt} \|v(t)\|_{V,3} + \kappa \frac{d}{dt} \|v(t)\|_{V,1} + \lambda \nu \|v(t)\|_{V,1}^2 = \int_\Omega f \cdot v(t) \, dt. \hspace{1em} (20)
$$

Now we get from (20) a dissipative estimate with a decaying exponential. We estimate the right-hand side of the latter equation using the Cauchy inequality:

$$
\int_\Omega f \cdot v(t) \, dt \leq \|f\|_{V^{-1}} \|v(t)\|_{V,1} \leq \frac{1}{2\nu} \|f\|_{V^{-1}}^2 + \frac{\nu}{2} \|v(t)\|_{V,1}^2.
$$

Combining this with (20), we get

$$
\frac{d}{dt} \|v(t)\|_{V,0}^2 + \kappa \frac{d}{dt} \|v(t)\|_{V,3} + \varepsilon e^{-\lambda \alpha t} \frac{d}{dt} \|v(t)\|_{V,1} + \lambda \nu \|v(t)\|_{V,1}^2 \leq \frac{\lambda}{\nu} \|f\|_{V^{-1}}^2. \hspace{1em} (21)
$$
Consider an auxiliary norm on $V^1$ defined by the formula $\|u\|^2 = \|u\|^2_{V_0} + \varepsilon \|u\|^2_{V_1}$. This norm is equivalent to $\| \cdot \|_{V^1}$. We have:

$$\frac{d}{dt}\|v(t)\|^2_{V_0} + \varepsilon \frac{d}{dt}\|v(t)\|^2_{V_1} = \frac{d}{dt}\|v(t)\|^2; \quad \nu\|v(t)\|^2_{V_1} \geq \frac{\nu}{k_0 + \varepsilon}\|v(t)\|^2 = \alpha\|v(t)\|^2,$$

where $\alpha = \frac{\nu}{k_0 + \varepsilon}$ and $k_0$ is a constant which does not depend on $v$.

Thus it follows from (21) that

$$\frac{d}{dt}\|v(t)\|^2 + \varepsilon e^{-\lambda \alpha t} \frac{d}{dt}\|v(t)\|^2_{V_3} + \lambda \alpha \|v(t)\|^2 \leq \frac{\lambda}{\nu}\|f\|^2_{V^{-1}}.$$

Substitute $v(t) = \tilde{v}(t) \exp(-\lambda \alpha t/2)$ in the first and third terms in the left-hand side of the last inequality. We get

$$-\lambda \alpha e^{-\lambda \alpha t}\|v(t)\|^2 + e^{-\lambda \alpha t} \frac{d}{dt}\|v(t)\|^2 + \varepsilon e^{-\lambda \alpha t} \frac{d}{dt}\|v(t)\|^2_{V_3} + \lambda \alpha e^{-\lambda \alpha t}\|v(t)\|^2 \leq \frac{\lambda}{\nu}\|f\|^2_{V^{-1}}.$$

Multiplying both sides by $\exp(\lambda \alpha t)$, we obtain

$$\frac{d}{dt} (\|\tilde{v}(t)\|^2 + \varepsilon \|v(t)\|^2_{V_3}) \leq \frac{\lambda}{\nu}\|f\|^2_{V^{-1}} e^{\lambda \alpha t}.$$

Integrating the last inequality, we have

$$\|\tilde{v}(t)\|^2 + \varepsilon \|v(t)\|^2_{V_3} \leq \|v(0)\|^2 + \varepsilon \|v(0)\|^2_{V_3} + \frac{1}{\alpha \nu}\|f\|^2_{V^{-1}} (e^{\lambda \alpha t} - 1),$$

for all $t$ (this is true both for $\lambda > 0$ and for $\lambda = 0$). Now multiply both parts of the last inequality by $\exp(-\lambda \alpha t)$, whence we obtain

$$\|v(t)\|^2 + \varepsilon e^{-\lambda \alpha t}\|v(t)\|^2_{V_3} \leq \left(\|v(0)\|^2 + \varepsilon \|v(0)\|^2_{V_3}\right) e^{-\lambda \alpha t}.$$

Since norms $\| \cdot \|$ and $\| \cdot \|_{V^1}$ are equivalent, it follows from the last equality that

$$\|v(t)\|^2_{V_1} + \varepsilon e^{-\alpha t}\|v(t)\|^2_{V_3} \leq C \left(1 + \left(\|v(0)\|^2_{V_1} + \varepsilon \|v(0)\|^2_{V_3}\right) e^{-\lambda \alpha t}\right)$$

with a constant $C$ independent of $\lambda$, $\varepsilon$, and $v$.

Using (19) it is possible to estimate the derivative $v'$ in terms of $v$. Combining the estimate obtained in this way with (23), we obtain

$$\|v(t)\|_{V^1} + \sqrt{\varepsilon} e^{-\alpha t/2}\|v(t)\|_{V^3} + \|v'(t)\|_{V^{-1}} + \varepsilon e^{-\alpha t}\|v'(t)\|_{V^3} \leq R_1 \left(1 + \left(\|v(0)\|^2_{V_1} + \varepsilon \|v(0)\|^2_{V_3}\right) e^{-\lambda \alpha t}\right).$$

with a constant $R_1$ that does not depend on $\varepsilon$, $\lambda$, and $v$.

3.4. Existence of solutions. Now we state the main existence theorem for the auxiliary problem.

**Theorem 3.3.** For any $b \in V^3$ problem (18), (17) has a solution on the semiaxis $\mathbb{R}_+$. Any solution of this problem satisfies

$$\|v(t)\|_{V^1} + \sqrt{\varepsilon} e^{-\alpha t/2}\|v(t)\|_{V^3} + \|v'(t)\|_{V^{-1}} + \varepsilon e^{-\alpha t}\|v'(t)\|_{V^3} \leq R_1 \left(1 + \left(\|v(0)\|^2_{V_1} + \varepsilon \|v(0)\|^2_{V_3}\right) e^{-\lambda \alpha t}\right)$$

a. e. on $\mathbb{R}_+$ with a constant $R_1$ independent of $\varepsilon$, $\lambda$, and $v$. 

The proof of Theorem 3.3. involves two steps. First we prove the solvability on a finite segment $[0, T]$ with an arbitrary $T > 0$ and then we prove that there exists a solution on $\mathbb{R}_+$.  

**Step I.** Let $T > 0$. Let us prove that problem (18), (17) has a solution on $[0, T]$. Consider the following family of equations dependent on $\lambda \in [0, 1].$

$$Lv + \lambda K(v) = \lambda(f, b).$$

Equation (26) with $\lambda = 1$ corresponds to (18), (17). Note that it follows from (24) that solutions of (26) (if they exist) satisfy the following a priori estimate:

$$\|v\|_{C([0, T]; \mathbb{V}^3)} + \varepsilon e^{-\alpha T} \|v'\|_{L^\infty(0, T; \mathbb{V}^3)} \leq C,$$

where $C$ does not depend on $\lambda$ (but generally speaking, it can depend on other parameters of the equation). Indeed, it follows from (24) that for a.a. $t \in [0, T]$ the norms $\|v(t)\|_{\mathbb{V}^3}$ and $e^{-\alpha T} \|v'(t)\|_{\mathbb{V}^3}$ do not exceed

$$R_1(1 + (\lambda^2 \|a\|_{\mathbb{V}^3}^2 + \varepsilon \lambda^2 \|a\|_{\mathbb{V}^3}^2))e^{-\lambda \alpha t} \leq R_1(1 + (\|a\|_{\mathbb{V}^3}^2 + \varepsilon \|a\|_{\mathbb{V}^3}^2)),$$

and the right-hand part of the last inequality does not depend on $t$ and $\lambda$. Also it follows from (27) that solutions of (26) satisfy

$$\|v\|_{W_2[0, T]} \leq R,$$

where $R$ does not depend on $\lambda$.

Apply $L^{-1}$ to both sides of (26) and write the equation thus obtained in the form

$$v - \lambda L^{-1}((f, a) - K(v)) = 0.$$

The mapping $\Phi(\lambda, v) = \lambda L^{-1}((f, a) - K(v))$ is continuous with respect to $(\lambda, v)$, so it is a deformation between vector fields $\Phi_1 v = v - L^{-1}((f, a) - K(v))$ and $\Phi_0 v = v$. It can be proved that $\Phi(\lambda, v)$, regarded as a function of $v$, is uniformly continuous with respect to $\lambda$ and that $\Phi(\lambda, v)$ is compact with respect to $(\lambda, v)$. Moreover, it follows from (28) that $\Phi(\lambda, v)$ does not vanish on the boundary of the ball $B_{R+1}$. Hence, $\Phi(\lambda, v)$ is a homotopy between $\Phi_0 v$ and $\Phi_1 v$ on $B_{R+1}$.

Since the deformation $\Phi(\lambda, v)$ is nondegenerate on the boundary of $B_{R+1}$, the Leray–Schauder degree of the compact vector fields $\Phi_1 v$ and $\Phi_0 v$ on $B_{R+1}$ is well defined. By the homotopic invariance of the Leray–Schauder degree we have

$$\deg_{LS}(I - L^{-1}((f, a) - K(\cdot)), B_{R+1}, 0) = \deg_{LS}(I, B_{R+1}, 0) = 1.$$

Since the field $I - L^{-1}((f, a) - K(\cdot))$ has non-zero degree, there exists a solution $v \in W_2[0, T]$ of the operator equation

$$v - L^{-1}((f, a) - K(v)) = 0.$$

This equation is equivalent to equation (26) with $\lambda = 1$, and the latter equation is in turn equivalent to problem (18), (17). We have thus proved that the auxiliary problem (18), (17) has a solution on $[0, T]$.

**Step II.** Let $v_m$ be a solution of problem (18), (17) on $[0, m]$ ($m = 1, 2, \ldots$). Consider the extension of the functions $v_m$ to $\mathbb{R}_+$ defined by the formula

$$\tilde{v}_m(t) = \begin{cases} v(t), & 0 \leq t \leq m, \\ v(m), & t \geq m. \end{cases}$$

It is obvious that the functions $\tilde{v}_m$ belong to $W_2^{loc}(\mathbb{R}_+)$. 

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Suppose that $0 < \delta < 1$. Take an arbitrary $T > 0$. All but finitely many terms of the sequence $\{\hat{v}_m\}$ are solutions of (18), (17) on $[0, T]$. Since functions $\hat{v}_m$ take the same value $b$ at $0$, by Step I of Theorem 3.3 it follows that they satisfy the estimate
\[
\|\hat{v}_m\|_{L_\infty(0,T;V^1)} + \|\hat{v}_m\|_{L_\infty(0,T;V^3)} + \|\hat{v}_m\|_{L_\infty(0,T;V^{-1})} \leq C(\delta, T),
\]
where $C(\epsilon, T)$ does not depend on $m$.

Thus the sequence $\{\hat{v}_m\}$ is bounded in $L_\infty(0,T;V^1)$ and the sequence of derivatives $\{\hat{v}'_m\}$ is bounded with respect to the norm of $L_\infty(0,T;V^{-1})$. By Theorem 2.1 we have that the sequence $\{\hat{v}_m\}$ is precompact in $C([0,T];V^{1-\delta})$. Since this is true for arbitrary $T$, the sequence is precompact in $C(\mathbb{R}_+;V^{1-\delta})$.

Thus the sequence $\{\hat{v}_m\}$ has a subsequence $\{\hat{v}_{m_k}\}$ that converges to some function $v_*$ in the space $C(\mathbb{R}_+,V^{1-\delta})$. It can be proved [39, 43] that this limit function is the sought for solution of problem (18), (17) on $\mathbb{R}_+$.

**Proof of Theorem 2.3.** Since $V^3$ is dense in $V^1$, there exists a sequence $\{b_m\}$ in $V^3$ such that $\|b_m - a\|_{V^1} \to 0$. Suppose the sequence $\{\epsilon_m\}$ tends to zero and
\[
\epsilon_m \|b_m\|_{V^3}^2 \leq 1. \tag{29}
\]
One can put
\[
\epsilon_m = \frac{1}{m \max \{\|b_m\|_{V^3}^2, 1\}}
\]
to obtain such a sequence.

Substitute $\epsilon_m$ for $\epsilon$ in (18) and consider the initial condition
\[
v_m(0) = b_m
\]
for this equation. By Theorem 3.3 this initial-boundary value problem has a solution $v_m$ on $\mathbb{R}_+$. Inequalities (25) and (29) yield the following estimate:
\[
\|v_m(t)\|_{V^1} + \|v'_m(t)\|_{V^{-1}} + \epsilon_m e^{-\alpha t} \|v'_m(t)\|_{V^3} \leq R_1 \left(1 + \left(\|a_m\|_{V^1}^2 + 1\right) e^{-\alpha t}\right), \tag{30}
\]
a.e. on $\mathbb{R}_+$. More precisely, for each $m$ the last inequality holds for all $t \in \mathbb{R}_+ \setminus Q_m$, where $Q_m$ is a set of zero measure. Hence for any $t \in \mathbb{R}_+ \setminus Q$, where $Q = \bigcup_m Q_m$ is a set of zero measure, inequality (30) holds for all $m$.

Suppose that $0 < \delta \leq 1$. According to (30) we have that for any $T > 0$ the sequence $\{v_m\}$ is bounded in $L_\infty(0,T;V^1)$ and the sequence $\{v'_m\}$ is bounded in $L_\infty(0,T;V^{-1})$. By Theorem 2.1 it follows that the sequence $\{v_m\}$ is compact in $C([0,T];V^{1-\delta})$. Since $T$ is arbitrary, it follows that the latter sequence is precompact in $C(\mathbb{R}_+,V^{1-\delta})$ and thus has a subsequence $\{v_{mk}\}$ converging in $C(\mathbb{R}_+,V^{1-\delta})$ to a function $v_*$. It can be proved in [39] and [43] that $v_*$ is a solution of problem (18), (17).

Now we get (11). Discarding certain nonnegative terms in the left-hand side of (30) we obtain
\[
\|v_{mk}(t)\|_{V^1} \leq R_1 \left(1 + \left(\|a_m\|_{V^1}^2 + 1\right) e^{-\alpha t}\right). \tag{31}
\]
Given $k$, this inequality holds for any $t$ to a subset of $\mathbb{R}_+$ of full measure that does not depend on $k$. Take such a $t$. First observe that $v_{mk}(t) \to v_*(t)$ in $V^{1-\delta}$, since the convergence in $C(\mathbb{R}_+,V^{1-\delta})$ implies pointwise convergence. Further, it follows from (31) that the sequence $\{v_{mk}(t)\}$ is bounded in $V^1$. Consequently, it has a subsequence $\tilde{v}_\mu(t)$ that converges to $v_*(t)$ weakly in $V^1$. Therefore,
\[
\|v_*(t)\|_{V^1} \leq \liminf\|\tilde{v}_\mu(t)\|_{V^1} \leq R_1 \left(1 + \left(\|a\|_{V^1}^2 + 1\right) e^{-\alpha t}\right) \quad (\mu \to \infty).
\]
Thus for a. a. $t \in \mathbb{R}_+$ we have
\[ ||v_h(t)||_{V^1} \leq R_1 \left( 1 + (||a||_{L^1}^2 + 1) e^{-\alpha t} \right). \]
Moreover, one can use (18) and estimate $v'_* \text{ in terms of } v$. Combining such an estimate with (32), we get (11).

4. Trajectory and global attractors for the initial-boundary value problem (5)-(8). In this subsection we fix a number $\delta \in (0,1)$.

Consider $E = V^1$ and $E_0 = V^{1-\delta}$ as the pair of Banach spaces needed to introduce a trajectory space. This choice is justified by the fact that $V^1$ is reflexive and is continuously embedded in $V^{1-\delta}$.

By Remark 4 the trajectory space introduced by Definition 2.13 is nonempty. Thus it suffices to check the inclusion
\[ \mathcal{H}^+ \subset C(\mathbb{R}_+; E_0) \cap L_\infty(\mathbb{R}_+; E). \]
so as to make sure that the trajectory space is well defined.

The inclusion $\mathcal{H}^+ \subset L_\infty(\mathbb{R}_+; E)$ directly follows from the definition of the trajectory space. We use Theorem 2.1 in order to prove that the trajectories are continuous. Consider three spaces $V^1 \subset V^{1-\delta} \subset V^{-1}$. Let $v$ be an arbitrary trajectory. It follows from (12) that for any segment $[0,T]$ we have $v \in L_\infty(0,T; V^1)$ and $v' \in L_\infty(0,T; V^{-1})$. Hence by Theorem 2.1 we obtain that $v$ belongs to $C([0,T]; V^{1-\delta})$. This is true for any $T$, so $v \in C(\mathbb{R}_+; V^{1-\delta})$.

Let $\tilde{R} > 4R_0$. Consider the set
\[ \tilde{P} = \{ v \in C(\mathbb{R}_+; V^{1-\delta}) \cap L_\infty(\mathbb{R}_+; V^1); v' \in L_\infty(\mathbb{R}_+; V^{-1}), \]
\[ ||v||_{L_\infty(\mathbb{R}_+; V^1)} + ||v'||_{L_\infty(\mathbb{R}_+; V^{-1})} \leq \tilde{R} \} \]

Let us establish several properties of this set.

**Lemma 4.1.** The set $\tilde{P}$ is bounded in $L_\infty(\mathbb{R}_+; V^1)$, compact in $C(\mathbb{R}_+; V^{1-\delta})$, and the following inclusion holds:
\[ T(h)\tilde{P} \subset \tilde{P} \quad (h \geq 0). \]

**Proof.** It follows from the definition of $\tilde{P}$ that it is bounded in $L_\infty(\mathbb{R}_+; V^1)$.

It is not hard to prove that the set $\tilde{P}$ is precompact in $C(\mathbb{R}_+; V^{1-\delta})$. Indeed, take $T > 0$. It follows easily from the construction that $\tilde{P}$ is bounded in $L_\infty(0,T; V^1)$ and the set $\{ v'; v \in \tilde{P} \}$ is bounded in $L_\infty(0,T; V^{-1})$. By Theorem 2.1 the set $\tilde{P}$ is precompact in $C([0,T]; V^{1-\delta})$. Since $T$ is arbitrary, $\tilde{P}$ is precompact in $C(\mathbb{R}_+; V^{1-\delta})$.

Now let us show that $\tilde{P}$ is closed and therefore compact in $C(\mathbb{R}_+; V^{1-\delta})$. Suppose that the sequence $\{ v_m \} \subset \tilde{P}$ converges to $v_0$ in $C(\mathbb{R}_+; V^{1-\delta})$. This sequence is bounded in $L_\infty(\mathbb{R}_+; V^1)$, so it converges to its limit function $*$-weakly in $L_\infty(\mathbb{R}_+; V^1)$. Moreover, the sequence of derivatives $\{ v'_m \}$ converges to $v'_0$ in the sense of distributions and also $*$-weakly in $L_\infty(\mathbb{R}_+; V^{-1})$, since it is bounded in $L_\infty(\mathbb{R}_+; V^{-1})$. Therefore
\[ ||v_0||_{L_\infty(\mathbb{R}_+; V^1)} + ||v'_0||_{L_\infty(\mathbb{R}_+; V^{-1})} \leq \lim\inf_{m \to \infty} ||v_m||_{L_\infty(\mathbb{R}_+; V^1)} + \lim\inf_{m \to \infty} ||v'_m||_{L_\infty(\mathbb{R}_+; V^{-1})} \]
\[ \leq \lim\left( ||v_m||_{L_\infty(\mathbb{R}_+; V^1)} + ||v'_m||_{L_\infty(\mathbb{R}_+; V^{-1})} \right) \leq \tilde{R} \quad (m \to \infty). \]
This proves that $\tilde{P}$ contains the limit function $v_0$. So $\tilde{P}$ is closed.

Finally, let us prove the inclusion (33). Take $h \geq 0$. For any $v \in \tilde{P}$ we have
\[ ||T(h)v||_{L_\infty(\mathbb{R}_+; V^1)} + ||T(h)v'||_{L_\infty(\mathbb{R}_+; V^{-1})} \leq ||v||_{L_\infty(\mathbb{R}_+; V^1)} + ||v'||_{L_\infty(\mathbb{R}_+; V^{-1})} \leq \tilde{R}, \]
whence \( T(h)v \in \hat{P} \).

**Proof of Theorem 2.14.** Let us prove that \( \hat{P} \) is a semi-attractor of \( \mathcal{H}^+ \). By Lemma 4.1 we have that \( \hat{P} \) satisfies conditions (i) and (ii) of Definition 2.7. Let us prove that \( \hat{P} \) is absorbing. Consider an arbitrary set \( B \subset \mathcal{H}^+ \) bounded in \( L_\infty(\mathbb{R}_+; V^1) \); to be definite, assume that \( \|v\|_{L_\infty(\mathbb{R}_+; V^1)} \leq R \) for any \( v \in B \). Take \( h_0 \geq 0 \) such that \( R^2 e^{-\alpha h_0} \leq 1 \). Let \( v \) be an arbitrary function belonging to \( B \). Since \( v \) satisfies inequality (12), for all \( h \geq h_0 \) we have

\[
\|T(h)v(t)\|_{V^1} + \|T(h)v'(t)\|_{V^{-1}} = \|v(t + h)\|_{V^1} + \|v'(t + h)\|_{V^{-1}} \\
\leq R_0(1 + R^2 e^{-\alpha(t + h)}) \leq R_0(1 + R^2 e^{-\alpha h_0}) \leq 2R_0.
\]

Hence \( \|T(h)v\|_{L_\infty(\mathbb{R}_+; V^1)} \leq 2R_0, \|T(h)v'\|_{L_\infty(\mathbb{R}_+; V^{-1})} \leq 2R_0 \), and therefore

\[
\|T(h)v\|_{L_\infty(\mathbb{R}_+; V^1)} + \|T(h)v'\|_{L_\infty(\mathbb{R}_+; V^{-1})} \leq 4R_0 \leq \hat{R}.
\]

Thus \( T(h)v \in \hat{P} \). Since \( v \) is arbitrary, we have \( T(h)B \subset \hat{P} \) for all \( h \geq h_0 \). Consequently \( \hat{P} \) is absorbing.

We have proved that \( \hat{P} \) is a semi-attractor of \( \mathcal{H}^+ \). By Theorem 2.11 the trajectory space \( \mathcal{H}^+ \) has the minimal trajectory attractor. □

**Proof of Theorem 2.15.** According to Theorem 2.12, the global attractor of a trajectory space exists if the trajectory space has the minimal trajectory attractor. Theorem 2.14 implies that the trajectory space \( \mathcal{H}^+ \) satisfies this requirement. □

5. Pullback attractors for the initial-boundary value problem (5)-(6), (13)-(14). At the pullback attractor trajectory space theory the trajectory is usually defined on the semiaxis \( \mathbb{R}_+ = [0, +\infty) \). Therefore, along with (5)-(6), (13)-(14) we consider the auxiliary problem

\[
\frac{\partial v}{\partial t} - \nu \Delta v + \sum_{i=1}^{n} v_i \frac{\partial v}{\partial x_i} - \kappa \frac{\partial \Delta v}{\partial t} - 2\kappa \text{Div} \left( \sum_{i=1}^{n} v_i \frac{\partial \mathcal{E}(v)}{\partial x_i} \right) - 2\kappa \text{Div} (\mathcal{E}(v)W_p(v) - W'_p(v)\mathcal{E}(v)) + \text{grad} p = F, \quad (x, t) \in \Omega \times (0, +\infty); \quad \text{div} v = 0, \quad (x, t) \in \Omega \times (0, +\infty); \quad \left| v \right|_{\partial \Omega \times (0, +\infty)} = 0, \quad v \bigg|_{t=0} = a. \tag{37}
\]

Here, the value of the right part \( F \) is not specified in advance. If \( F(x, t) = f(x, t + \tau) \), then the problem (34)-(37) is obtained from the problem (5)-(6), (13)-(14) by linear change of the independent variable \( t \), which is transforming \( \tau \) into 0.

Similarly to Section 2.2 problem (34)-(37) has analogous existence theorem.

**Theorem 5.1.** For any \( F \in L^2_{\text{loc}}(\mathbb{R}_+; V^0) \) and \( a \in V^1 \) the problem (34)-(37) has a weak solution on the half-axis \( \mathbb{R}_+ \), which satisfies an inequality

\[
\|v(t)\|_{V^1} \leq e^{-\alpha t} \left( \frac{k^2}{\kappa} + 1 \right) \|a\|_{V^0} + \frac{1}{\nu^2} \int_{0}^{t} e^{\alpha \xi} \|F(\xi)\|_{V^{-1}}^2 d\xi \tag{38}
\]

for almost all \( t > 0 \). Moreover, for any weak solution \( v \) on \( \mathbb{R}_+ \) of the problem (34)-(37) we have that the derivative \( v' \in L^2_{\text{loc}}(\mathbb{R}_+; V^{-1}) \) and the inequality

\[
\|v'(t)\|_{V^{-1}} \leq C(\|F(t)\|_{V^{-1}} + \|v(t)\|_{V^1} + \|v(t)\|_{V^0}^2) \tag{39}
\]
holds for almost all $t > 0$ with the constance $C$ which does not depend on $t$, $v$ and $f$.

**Remark 6.** Estimates (38) and (39) are obtained similarly to Section 3.3 with addition that $F$ depends on $t$. It means, when we integrate the inequality (22), we get

$$
\|\bar{v}(t)\|^2 + \|v(t)\|_{V,3}^2 \leq \|v(0)\|^2 + \varepsilon \|v(0)\|_{V,3}^2 + \frac{1}{\nu} \int_0^t e^{\lambda \alpha t} \|F(\xi)\|_{V,-3}^2 d\xi.
$$

Rest arguments remain to be the same.

We assume that the density of the external force $f$ in the equation (5) belongs to the space $L^{loc}_2(\mathbb{R}; V^0)$ and satisfies the condition (15) for all $t \in \mathbb{R}$.

In this subsection we fix a number $\delta \in (0, 1)$. Consider $E = V^1$ and $E_0 = V^{1-\delta}$ as the pair of Banach spaces needed to introduce the class $\mathcal{T}$.

Let $\tau \in \mathbb{R}$. As the trajectory space $\mathcal{H}_+^\tau$ of the problem (5)-(6), (13)-(14) we consider the set of a weak solution $v$ of the problem (34)-(37) with the right part $F = T(\tau)f$ and some initial condition $a \in V^1$ (each for every $v$), which is satisfying the estimate

$$
\|v(t)\|_{V,1}^2 \leq e^{-\alpha t} \left( \left( \frac{k^2}{\alpha} + 1 \right) \|v(0)\|_{V,1}^2 + \frac{1}{\nu} \int_0^t e^{\alpha \xi} \|f(\xi + \tau)\|_{V,-3}^2 d\xi \right). \tag{40}
$$

These trajectory spaces form a family of trajectory spaces $H^+ = \{ \mathcal{H}_+^\tau \}$.

**Remark 7.** Note that for spaces $\mathcal{H}_+^\tau$ the embedding $\mathcal{H}_+^\tau \subset \mathcal{T}$ holds. In fact, from the inequality (40) and condition (15) the uniform with respect to $t$ estimate of the trajectory $v \in \mathcal{H}_+^\tau$ on an arbitrary interval $[0, T]$ follows

$$
\|v(t)\|_{V,1}^2 \leq \left( \left( \frac{k^2}{\alpha} + 1 \right) \|v(0)\|_{V,1}^2 + \frac{1}{\nu} e^{-\alpha t} \int_{-\infty}^{T+\tau} e^{\alpha \xi} \|f(s)\|_{V,-3}^2 ds \right) < \infty,
$$

where $v \in L_\infty(0, T; V^1)$. By virtue of the arbitrary of $T$ we get $v \in L^{loc}_\infty(\mathbb{R}_+; V^1)$.

In addition, in view of Theorem 5.1 we have $v' \in L^{loc}_\infty(\mathbb{R}_+; V^{-1})$. So by Theorem 2.1 applied in the case of triple spaces $V^1 \subset V^{1-\delta} \subset V^{-1}$, we get $v \in C(\mathbb{R}_+; V^{1-\delta})$.

The inclusion $\mathcal{H}_+^\tau \subset \mathcal{T}$ is proved.

**Theorem 5.2.** For every $a \in V^1$, there exists a trajectory $v \in \mathcal{H}_+^\tau (\tau \in \mathbb{R})$, which satisfies the initial condition $v(0) = a$.

**Proof.** Theorem is a direct consequence of existence Theorem 5.1.

We describe the class $\mathcal{D}$ of attracting sets families. Let $\mathcal{R}$ denotes the set of functions $r: \mathbb{R} \rightarrow \mathbb{R}_+$, such that the function $\tau \rightarrow e^{\alpha \tau}(r(\tau))^2$ increases and

$$
\lim_{\tau \rightarrow -\infty} e^{\alpha \tau}(r(\tau))^2 = 0.
$$

The class $\mathcal{D}$ consists of families of $\mathcal{D} = \{ D_\tau \} (D_\tau \subset V^1)$ for which there exist functions $r_\mathcal{D} \in \mathcal{R}$, such that $w \in D_\tau$ and $|w|_{1} \leq r_\mathcal{D}(\tau)$ for all $\tau \in \mathbb{R}$.

**Proof of Theorem 2.26.** We construct a family of sets $\mathcal{P} = \{ P_\theta \} (P_\theta \subset \mathcal{T})$ which is a $\mathcal{T}$-precompact and pullback-absorbing. Then the theorem follows from Theorems 2.23 and 2.25.
Let a set $P_\theta$ consists of functions $v \in T$ which satisfy inequalities

\[
\|v(t)\|_{V^1}^2 \leq e^{-\alpha t} \left( \frac{k_0^2}{\nu} + 1 + \frac{1}{\nu \kappa} \int_{-\infty}^t e^{\alpha s} \|f(s + \theta)\|_{V^{3-\delta}}^2 \, ds \right),
\]

(41)

\[
\|v'(t)\|_{V^{-1}} \leq C(\|f(t + \theta)\|_{V^{3-\delta}} + \|v(t)\|_{V^1} + \|v(t)\|_T^2)
\]

(42)

(here $C$ is a constant from the inequality (39)).

Now we show that the family $\mathcal{P} = \{P_\theta\}$ is $T$-precompact. Condition (ii) of the definition 2.18 is performed with the function

\[
\varphi_\theta(t) = e^{-\alpha t/2} \left( \frac{k_0^2}{\nu} + 1 + \frac{1}{\nu \kappa} \int_{-\infty}^t e^{\alpha s} \|f(s + \theta)\|_{V^{3-\delta}}^2 \, ds \right)^{1/2}.
\]

The function $\varphi_\theta$ is bounded from the condition (15)

\[
\int_{-\infty}^t e^{\alpha s} \|f(s + \theta)\|_{V^{3-\delta}}^2 \, ds = e^{-\alpha \theta} \int_{-\infty}^{t+\theta} e^{\alpha \xi} \|f(\xi)\|_{V^{3-\delta}}^2 \, d\xi < \infty.
\]

Verify the condition (ii) of Definition 2.18. Let $[0, T]$ be an arbitrary interval. Put $R_T \equiv \max_{t \in [0, T]} \varphi_\theta(t)$. Then for every trajectory $v \in P_\theta$ we have $\|v\|_{L_\infty(0, T; V^1)} \leq R_T$, and from the inequality (42) we get

\[
\|v'(t)\|_{L_2(0, T; V^{3-\delta})} \leq C(\|f\|_{L_2(T; V^{3-\delta})} + \sqrt{T} R_T + \sqrt{T} R_T^2).
\]

Thus, the set $P_\theta$ is bounded in the norm of space $L_\infty(0, T; V^1)$ and the set $\{v' \mid v \in P_\theta\}$ is bounded in the norm $L_2(0, T; V^{3-\delta})$. By Theorem 2.1 applied for the case of spaces $V^1 \subset V^{1-\delta} \subset V^{-1}$ we get that the set $P_\theta$ is precompact in $C([0, T]; V^{1-\delta})$. By virtue of arbitrary of $T$ we have that $P_\theta$ is precompact in $C(\mathbb{R}_+; V^{1-\delta})$, as it is required.

We have shown that the family $\mathcal{P}$ is $T$-precompact.

Now we show that the family $\mathcal{P}$ satisfies conditions of Definition 2.17. Let $\mathcal{D} = \{D_\tau\} \in \mathcal{D}$. Take a number $\theta$ and show that there is $\tau_{D}(\theta) \leq \theta$ that for $\tau \leq \tau_{D}(\theta)$ the inclusion

\[
T(\theta - \tau) \mathcal{H}_+^1(\mathcal{D}) \subset P_\theta
\]

(43)

holds and the function $\tau_{D}$ increases.

By the definition of the class $\mathcal{D}$ for the family $\mathcal{D}$ there is a function $r_D : \mathbb{R} \to \mathbb{R}_+$, such that for $w \in D_\tau$ we have the estimate $\|w\|_1 \leq r_D(\tau)$, and that the function $\chi_{\mathcal{D}}(\tau) = e^{\alpha \tau} (r_D(\tau))^2$ increases and tends to 0 when $\tau \to -\infty$. By the monotony of the function $\chi_{\mathcal{D}}$ the inverse function $\chi_{\mathcal{D}}^{-1}$ is increasing.

Consider the inequality

\[
\chi(\tau) \leq e^{\alpha \theta}.
\]

(44)

Due to the properties of $\chi_{\mathcal{D}}$ it holds either on the whole axis or on the line $(-\infty, \chi^{-1}(e^{\alpha \theta})]$. In the first case put $\tau_{D}(\theta) = \theta$ and in the second case let $\tau_{D}(\theta) = \min \{ \chi^{-1}(e^{\alpha \theta}), \theta \}$. Clearly, in any case the function $\tau_{D}$ increases, satisfies $\tau_{D}(\theta) \leq \theta$ and for $\tau \leq \tau_{D}(\theta)$ the inequality (44) holds or, equivalently,

\[
e^{-\alpha (\theta - \tau)} (r_D(\tau))^2 \leq 1 \quad (\tau \leq \tau_{D}(\theta)).
\]

(45)

To prove the embedding (43) for $\tau \leq \tau_{D}(\theta)$, we take the trajectory $v \in \mathcal{H}_+^1$, such that $v(0) \in D_\tau$, and show that $T(\theta - \tau)v \in P_\theta$. 

Show that for the function $T(\theta - \tau)v$ the estimate (41) holds. With the estimate (40) and inequality (45) we obtain:

$$
\|T(\theta - \tau)v(t)\|_{V^{-1}} \leq ||v(t+\theta - \tau)||_{V^{1}}^2
\leq e^{-\alpha(\tau+\theta-\tau)}\left(\frac{k_0^2}{\nu} + 1\right)||v(0)||_{V^{1}}^2 + \frac{1}{\nu \kappa} \int_{0}^{\tau+\theta-\tau} e^{\alpha \xi} ||f(\xi + \tau)||_{V^{-1}}^2 d\xi
\leq e^{-\alpha(t)}\left(\frac{k_0^2}{\nu} + 1\right)e^{-\alpha(\theta-\tau)}(r_\mathcal{D}(\tau))^2 + \frac{1}{\nu \kappa} \int_{-\infty}^{\tau+\theta-\tau} e^{\alpha \xi+\alpha \tau-\alpha \theta} ||f(\xi + \tau)||_{V^{-1}}^2 d\xi
\leq e^{-\alpha(t)}\left(\frac{k_0^2}{\nu} + 1 + \frac{1}{\nu \kappa} \int_{-\infty}^{t} e^{\alpha s} ||f(s + \theta)||_{V^{-1}}^2 d\tau\right).
$$

The estimate (41) for the function $T(\theta - \tau)v$ is proved.

Show that for the function $T(\theta - \tau)v$ the estimate (42) holds. Since $v$ is a weak solution of the problem (34)–(37) with the function $F = T(\tau)f$, by Theorem 5.1 for almost all $t > 0$ the inequality

$$
||v(t)||_{V^{-1}} \leq C(||f(t+\tau)||_{V^{-3}} + ||v(t)||_{V^{1}} + ||v(t)||_{V^{1}})
$$

holds. Consequently, we have

$$
\|T(\theta - \tau)v(t)\|_{V^{-1}} = ||v(t+\theta - \tau)||_{V^{-1}}
\leq C(||f(t+\theta)||_{V^{-3}} + ||v(t+\theta - \tau)||_{V^{1}} + ||v(t+\theta - \tau)||_{V^{1}})
= C(||f(t+\theta)||_{V^{-3}} + \|T(\theta - \tau)v(t)||_{V^{1}} + \|T(\theta - \tau)v(t)||_{V^{1}}).
$$

Thus, we have $T(\theta - \tau)v \in P_\theta$ and the embedding (43) is proved.

We have proved that a family of $\mathcal{P}$ is $\mathcal{T}$-precompact and pullback $\mathcal{D}$-absorbing, as it is required. According to Theorems 2.23 and 2.25 we finish our proof of Theorem 2.26.

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