Gravitational lensing by spinning and radially moving lenses

M. Sereno$^a,b$

$^a$Dipartimento di Scienze Fisiche, Università degli Studi di Napoli “Federico II”, Compl. Univ. Monte S. Angelo, via Cinthia, 80126, Napoli, Italia

$^b$Istituto Nazionale di Fisica Nucleare, Sez. Napoli, Compl. Univ. Monte S. Angelo, Edificio G, 80126, Napoli, Italia

Abstract

The effect of currents of mass on bending of light rays is considered in the weak field regime. Following Fermat’s principle and the standard theory of gravitational lensing, we derive the gravito-magnetic correction to time delay function and deflection angle caused by a geometrically-thin lens. The cases of both rotating and shifting deflectors are discussed.

Key words: Gravitational Lensing

PACS: 4.20.Cv, 4.25.Nx, 95.30.sf, 98.62.Sb

1 Introduction

Gravitational lensing has become a powerful tool for observational astrophysics and cosmology. The bending of light has been widely investigated in the framework of general relativity but some topics, such as the dynamical properties of the deflectors, are usually neglected. Peculiar and intrinsic motions of the lenses are expected to be small second order effects. However, gravity induced by moving matter is related to the dragging of inertial frames and the effects of mass currents on the propagation of light signals deserve attention from the theoretical point of view. Furthermore, since the impressive development of technological capabilities, it is not a far hypothesis to obtain observational evidences in a next future.

Email address: sereno@na.infn.it (M. Sereno).
Lensing of light rays by stars with angular momentum has been addressed by several authors with very different approaches. Epstein & Shapiro [4] performed a calculation based on the post-newtonian expansion. Ibáñez and coworkers [6,7] resolved the motion equation for two spinning point-like particles, when the spin and the mass of one of the particles were zero, by expanding the Kerr metric in a power series of gravitational constant $G$. Dymnikova [3] studied the time delay of signal in a gravitational field of a rotating body by integrating the null geodesics of the Kerr metric. Glicenstein [5] applied an argument based on Fermat’s principle to the Lense-Thirring metric to study the lowest order effects of rotation of the deflector. The listed results give a deep insight on some peculiar aspects of spinning lenses but are very difficult to generalize. On the other hand, Capozziello et al. [2] discussed the gravito-magnetic correction to the deflection angle caused by a point-like, shifting lens in weak field regime and slow motion approximation.

In this paper, we show that usual, basic assumptions of gravitational lensing theory allow to consider effects of currents of mass in a very general way. In the weak field approximation, we derive the gravito-magnetic correction to the time delay function and the deflection angle caused by a geometrically-thin lens. Our results, based on Fermat’s principle, apply to almost all observed gravitational lensing systems and contribute to fill the gap between the weak field approximation and a full theory of gravitational lensing enlarged to any order of approximation.

2 Gravito-magnetic correction

We follow the standard hypotheses of gravitational lensing as summarized in the monographs by Schneider et al. [9] and Petters et al. [8]. The gravitational lens is localized in a very small region of the sky and its lensing effect is weak. The deflector changes its position slowly with respect to the coordinate system, i.e. the matter velocity is much less than $c$, the speed of light; matter stresses are also small (the pressure is much smaller than the energy density times $c^2$). In this weak field regime and slow motion approximation, space-time is nearly flat near the lens. Up to leading order in $c^{-3}$, the metric is

$$ds^2 \simeq \left(1 + 2\frac{\phi}{c^2}\right)c^2dt^2 - 8cdt\frac{V dx}{c^3} - \left(1 - 2\frac{\phi}{c^2}\right)dx^2;$$

(1)

$\phi$ is the Newtonian potential,

$$\phi(t, x) \simeq \int_{\Re^3} \frac{\rho(t, x')}{|x - x'|}d^3x';$$

(2)
\( \mathbf{V} \) is a vector potential taking into account the gravito-magnetic field produced by mass currents,

\[
\mathbf{V}(t, \mathbf{x}) \simeq \int_{\mathbb{R}^3} \frac{(\rho \mathbf{v})(t, \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} d^3\mathbf{x}',
\]

where \( \mathbf{v} \) is the velocity field of the mass elements of the deflector as measured in a coordinate system defined such that one of its spatial coordinate axes follows the line from the observer to the source, and the other two spatial axes lie in the plane of the sky. In Eqs. (2,3), we have neglected the retardation (Schneider et al. [9]). The line element in Eq.(1) satisfies the weak-field condition once \( |\phi| \ll c^2 \).

2.1 The time delay function

We assume that, during the time light rays take to traverse the lens, the potentials in Eqs. (2,3) vary negligibly little. Then, the lens can be treated as stationary. Under this hypothesis, the space-time described by Eq. (1) can be interpreted as a flat one with an effective index of refraction, \( n \), given by [9]

\[
n \simeq 1 - 2\frac{\phi}{c^2} + 4\frac{c^3}{c^3} \mathbf{V} \cdot \mathbf{e},
\]

where \( \mathbf{e} \) is the unit tangent vector of a light ray. A light signal emitted at the source will arrive after

\[
\Delta T = \frac{1}{c} \int_{\mathbf{p}} n \, ds = \frac{1}{c} \int_{\mathbf{p}} \left( 1 - 2\frac{\phi}{c^2} + 4\frac{c^3}{c^3} \mathbf{V} \cdot \mathbf{e} \right) \, ds,
\]

with the integral performed along the light trajectory \( \mathbf{p} \). The time delay of the path \( \mathbf{p} \) relative to the unlensed ray \( \mathbf{p}_0 \) is

\[
\Delta T = \frac{1}{c} \left( \int_{\mathbf{p}} n \, ds - \int_{\mathbf{p}_0} n \, ds \right).
\]

Equation (6) can be expressed as a sum of geometrical and potential time delays

\[
\Delta T = \Delta T_{\text{geom}} + \Delta T_{\text{pot}},
\]
with
\[ c \Delta T_{geom} = \int_{p} ds - \int_{p_0} ds, \]  
(7)
due to the extra path length relative to the unperturbed ray, and
\[ c \Delta T_{pot} = -\int_{p} \left( 2 \frac{\phi}{c^2} - \frac{4}{c^3} \mathbf{V} \cdot \mathbf{e} \right) \ ds, \]  
(8)
due to the retardation of the deflected ray caused by the gravitational field of the lens.

In usual lensing phenomena, the deflection angle of a light ray is very small. Furthermore, we treat the gravitational lens as thin. In almost all astrophysical configurations, the extent of the lens in the direction of the incoming ray is small compared to the distances between lens and observer, \( D_d \), and lens and source, \( D_{ds} \), so that the maximal deviation of the actual ray from a light path propagating through unperturbed space-time is small compared to the length scale on which the gravitational field changes. In this picture, the deflection occurs essentially in a small region near the deflector. The actual ray path can be approximated by combining its incoming and outgoing asymptotes. This trajectory is a piecewise-smooth null geodesic curve consisting of two null geodesics of the unperturbed space-time, one from the source to the deflector and one from the deflector to the observer. It is useful to employ the spatial orthogonal coordinates \( (\xi_1, \xi_2, l) \). The \( l \)-axis is along the incoming light ray direction \( \mathbf{e}_{in} \); the \( \xi \)-axes lie in the plane of the sky. The lens plane corresponds to \( l = 0 \). With these assumptions, the geometrical time delay is [9,8]
\[ c \Delta T_{geom} \simeq \frac{1}{2} \frac{D_d D_s}{D_{ds}} \left| \frac{\xi}{D_d} - \frac{\eta}{D_s} \right|^2, \]  
(9)
where \( D_s \) is the distance from the observer to the source and \( \eta \) is the bidimensional vector position of the source from the optical axis in the source plane.

The actual ray light is deflected, but if the deflection angle is small, it can be approximated as a straight line in the neighbourhood of the lens. This corresponds to the Born approximation, which allows integrating along the unperturbed ray \( \mathbf{e}_{in} \). We get
\[ c \Delta T_{pot} \simeq -\frac{4G}{c^2} \int_{\mathbb{R}^2} d^2 \xi' \Sigma(\xi') \left( 1 - 2 \frac{(\mathbf{v} \cdot \mathbf{e}_{in}) t(\xi')}{c} \right) \ln \left| \frac{\xi - \xi'}{\xi_0} \right| + \text{const.} \]  
(10)
In Eq. (10), $\Sigma$ is the surface mass density of the deflector,
\[
\Sigma(\xi) \equiv \int \rho(\xi, l) \, dl; \quad (11)
\]
\[
\langle v \cdot e_m \rangle l(\xi) \equiv \int \frac{\rho(\xi, l) \cdot e_m}{\Sigma(\xi)} \, dl. \quad (12)
\]

Adding the geometrical contribution, Eq. (9), and the potential term, Eq. (10), we get the time delay of a kinematically possible ray with impact parameter $\xi$ in the lens plane, relative to the unlensed one for a single lens plane. The time delay function is
\[
c\Delta T = \frac{1}{2} \frac{D_d D_s}{D_{ds}} \left| \frac{\xi}{D_d} - \frac{\eta}{D_s} \right|^2 - \psi(\xi) + \text{const.}, \quad (13)
\]
where $\psi$ is the deflection potential up to the order $v/c$,
\[
\psi(\xi) \equiv \frac{4G}{c^2} \int d^2 \xi' \Sigma(\xi') \left( 1 - 2 \frac{(v \cdot e_m)_l(\xi')}{c} \right) \ln \frac{|\xi - \xi'|}{\xi_0}. \quad (14)
\]

Hereafter, we will neglect the const. in Eq. (13), since it has no physical significance [9]. We remind that the time delay function is not an observable, but the time delay between two actual rays can be measured.

Since the potential time delay is a local effect which arises when a ray traverses the neighbourhood of the lens, the time delay function can be easily generalized to a cosmological context. In an homogeneous and isotropic background perturbed by an isolated lens, $\Delta T_{\text{pot}}$ has to be redshifted; $\Delta T_{\text{geom}}$, except for a cosmological factor $(1 + z_d)$ with $z_d$ the redshift of the deflector, takes the same form of Eq. (9) once we consider the distances as angular diameter distances [9,8]. The time delay measured at the observer is
\[
c\Delta T = (1 + z_d) \left\{ \frac{1}{2} \frac{D_d D_s}{D_{ds}} \left| \frac{\xi}{D_d} - \frac{\eta}{D_s} \right|^2 - \psi(\xi) \right\}. \quad (15)
\]

The velocity $v$ is the peculiar velocity with respect to the coordinate system.
In a cosmological context, the recession velocity of the deflector does not contribute to the gravito-magnetic correction.
2.2 The lens equation

To derive the lens equation, we can apply Fermat’s principle to the bending of light caused by an isolated perturbation (Schneider et al. [9] and Petters et al. [8]). Fermat’s principle states that a light ray from a source to an observer follows a trajectory, from among all kinematically possible paths, that is a stationary value of the arrival path. In terms of a variational formulation,

$$\delta \int_p n \, ds = 0,$$

with the integral performed along the light trajectory. Actual light rays, given the source position, are characterized by critical points of $\Delta T(\xi)$, i.e. $\Delta T(\xi)$ is stationary with respect to variations of $\xi$. The lens equation is then obtained calculating

$$\nabla_\xi \Delta T(\xi) = 0; \quad (16)$$

we get

$$\eta = \frac{D_s}{D_d} \xi - D_d \alpha(\xi); \quad (17)$$

$\alpha \equiv \nabla_\xi \psi$ is the deflection angle, i.e. the difference of the initial and final ray direction. It is, to the order $c^{-3}$,

$$\alpha(\xi) \equiv \frac{4G}{c^2} \int_{\mathbb{R}^2} d^2\xi' \Sigma(\xi') \left( 1 - 2 \frac{\langle \mathbf{v} \cdot \mathbf{e}_m \rangle_l(\xi')}{c} \right) \frac{\xi - \xi'}{|\xi - \xi'|^2}. \quad (18)$$

In the thin lens approximation, the only components of the velocities parallel to the line of sight enter the equations of gravitational lensing. A change in position of the deflector orthogonal to the line of sight can be noticeable in a variation of the luminosity of the source but does not affect the individual light rays, i.e. does not contribute to the gravito-magnetic correction.

For shifting lenses, $\langle \mathbf{v} \cdot \mathbf{e}_m \rangle_l(\xi) = v_l$, the gravito-magnetic correction reduces to a multiplicative factor to the zero order expressions. The deflection angle and the related quantities, such as the optical depth, up to order $v/c$ are derived from the zero-order expressions just by a product by $1 - 2v_l/c$. For deflector moving towards the observer and far away from the source ($v_l > 0$), the optical depth decreases; for receding lenses ($v_l < 0$), the deflection angle increases.
The velocity field of the deflector can break the axial symmetry of a spherical distribution of mass with respect to the lensing effect. In general, a slowly moving deflector, with surface mass density \( \Sigma^{\text{SLMO}} \), has the same lensing effect of a really static lens with

\[
\Sigma^{\text{STAT}}(\xi) = \Sigma^{\text{SLMO}}(\xi) \left( 1 - 2 \frac{\langle \mathbf{v} \cdot \mathbf{e}_n \rangle_l(\xi)}{c} \right). \tag{19}
\]

As an illustrative case, let us consider a rotating lens, a classical example of dragging of inertial frames. To simplify the calculations, we will take the angular momentum of the deflector along the \( \xi_2 \)-axis and a constant angular velocity \( \omega \). In this configuration, the component of the velocity \( \mathbf{v} \) orthogonal to the lens plane is constant along the line of sight. Introducing polar coordinates \((\theta, \xi)\) in the lens plane, we get

\[
\langle \mathbf{v} \cdot \mathbf{e}_n \rangle_l = -\omega \xi \cos \theta;
\]

for a general distribution of mass \( \Sigma \), the deflection angles in Eq. (18) reduces to

\[
\begin{align*}
\alpha_1(\theta, \xi) &= \frac{4G}{c^2} \int_0^\infty \xi' d\xi' \int_0^{2\pi} \left[ 1 + 2 \frac{\omega}{c} \xi' \cos \theta' \right] \frac{\xi \cos \theta - \xi' \cos \theta'}{\xi^2 + \xi'^2 - 2\xi \xi' \cos(\theta' - \theta)} \Sigma(\theta', \xi') d\theta' ; \tag{20} \\
\alpha_2(\theta, \xi) &= \frac{4G}{c^2} \int_0^\infty \xi' d\xi' \int_0^{2\pi} \left[ 1 + 2 \frac{\omega}{c} \xi' \cos \theta' \right] \frac{\xi \sin \theta - \xi' \sin \theta'}{\xi^2 + \xi'^2 - 2\xi \xi' \cos(\theta' - \theta)} \Sigma(\theta', \xi') d\theta'. \tag{21} 
\end{align*}
\]

For an axially-symmetric surface mass density, \( \Sigma = \Sigma(|\xi|) \), a not negligible value of \( \omega/c \) breaks the symmetry of the system and the full vector equation must be used.

3 Conclusions

We have considered the effects of gravity induced by currents of mass by calculating the gravito-magnetic corrections to the time delay function and the deflection angle for a single lens plane in the weak field regime. Fermat’s principle has been applied under the usual assumptions of small deflection angles and geometrically thin lenses to give simple formulae for a general mass distribution of the deflector. For the point-like lens, our expression of the deflection angle agrees with what found in Capozziello et al. [2]. Our analysis applies to almost all observed gravitational lensing phenomena, both on galactic and extra-galactic scales.
We remark that a full analysis of higher order corrections to the lensing theory makes possible a comparison between the predictions of general relativity and extended relativistic theory of gravity where corrections depend on the interaction scale [1]. An analysis to the lowest order might hide such differences.

In a forthcoming paper, we will discuss some specific lens models and the perspective of precise observations to detect the gravito-magnetic effect.

References

[1] S. Calchi Novati, S. Capozziello, G. Lambiase, Grav. Cosm. 6 (2000), 173.
[2] S. Capozziello, G. Lambiase, G. Papini, G. Scarpetta, Phys. Lett. A 254 (1999), 11.
[3] I. Dymnikova, Relativity in Celestial Mechanics and Astrometry, eds. J. Kovalevsky, A. Brumberg (1986), 411.
[4] R. Epstein, I. Shapiro, Phys. Rev. D 22 (1980), 2947.
[5] J.F. Glicenstein, Astron. Astrophys. 343 (1999), 1025.
[6] J. Ibáñez, Astron. Astrophys. 124 (1983), 175.
[7] J. Ibáñez, J. Martín, Phys. Rev. D 26 (1982), 384.
[8] A.O. Petters, H. Levine, J. Wambsganss, Singularity Theory and Gravitational Lensing, (Birkhäuser, Boston, 2001).
[9] P. Schneider, J. Ehlers, E.E. Falco, Gravitational Lenses, (Springer, Berlin, 1992).