Analytic Bethe ansatz related to the Lie superalgebra $C(s)$

Zengo Tsuboi  
Institute of Physics, University of Tokyo,  
Komaba 3-8-1, Meguro-ku, Tokyo 153-8902, Japan

Abstract

An analytic Bethe ansatz is carried out related to the type 1 Lie superalgebra $C(s)$. We present eigenvalue formulae of transfer matrices in dressed vacuum form (DVF) labeled by Young superdiagrams with one row or one column. We also propose a DVF related to a one parameter family of finite dimensional irreducible representations. A class of transfer matrix functional relations among these formulae is discussed.

PACS numbers: 02.20.Qs, 02.20.Sv, 03.20.+i, 05.50.+q  
Keywords: Analytic Bethe Ansatz, Lie superalgebra, Dressed vacuum form, Solvable lattice model, Transfer matrix, T-system.

Journal-ref: Physica A 267 (1999) 173-208  
DOI: 10.1016/S0378-4371(98)00673-6
1 Introduction

The Analytic Bethe ansatz [1, 2] is a powerful technique. We can postulates the eigenvalues of transfer matrices in solvable lattice models associated with complicated representations of underlying algebras, which are too hard to obtain by other method. In fact they are realized systematically in the dressed vacuum form (DVF) in terms of Yangians \( Y(G) \) [3] analogue of skew-Young tableaux for \( G = A_r, B_r, C_r, D_r \) and \( G_2 \) [4, 5, 6, 7].

Recently a similar analysis has been done [8, 9, 10] for the type 1 Lie superalgebra \( G = \mathfrak{sl}(r + 1|s + 1) \) case. A set of DVFs are constructed and shown to satisfy a class of functional relations.

In this paper we will extend similar analyses to another type 1 Lie superalgebra \( G = C(s) = \mathfrak{osp}(2|2s - 2) \) case. Solvable lattice models related to Lie superalgebras attract much interest in strongly correlated electron systems. The analytic Bethe ansatz will be powerful technique to analyze such systems.

Remarkable enough, we can express [20, 21, 6] the Bethe ansatz equation (BAE) by the representation theoretical data of a Lie algebra. This is also the case with the Lie superalgebras: [22] for \( \mathfrak{sl}(r + 1|s + 1) \); [19] for \( B(r|1), B(0|s), C(s) \) and \( D(r|1) \). We assume, as our starting point, the BAE (3.1) associated with the distinguished simple root system of \( C(s) \) [11]. Then we execute an analytic Bethe ansatz axiomatically and construct various kind of DVFs systematically. Our guiding principals to construct DVFs are pole-freeness under the BAE and top term hypothesis [5, 6].

We define the functions \( T^a(u) \) (3.18) and \( T_m^{(1)}(u) \) (3.27) as summations over certain tableaux labelled by Young superdiagrams with one column and one row respectively. They will be eigenvalue formulae of transfer matrices in DVFs generated by certain top terms. These top terms carry highest weights of some irreducible representations of \( C(s) \). The function \( T^1(u) = T_1^{(1)}(u) \), the simplest example of \( T^a(u) \) and \( T_m^{(1)}(u) \), corresponds to the eigenvalue formula of a \( C(s) \) vertex model in Ref. [19]. We prove pole-freeness of \( T^a(u) \) under the BAE (3.1), an essential property in the analytic Bethe ansatz. This is based on the assumption that the BAE (3.1) is common to all the

---

1In the main text, we assume \( s \geq 3 \) (\( s \in \mathbb{Z} \)). However, many formulae in the main text are formally valid for \( C(2) \) case (see Appendix B). We note that solvable lattice models related to \( C(2) \simeq \mathfrak{sl}(1|2) \) were studied in many literatures; see for example [12, 13, 14, 15, 16, 17, 18, 19, 10].
DVF s for transfer matrices with various fusion types in the auxiliary space as long as they act on a common quantum space.

The type 1 Lie superalgebra admits one parameter families of finite dimensional representations, which are not tensor-like. In the previous paper [10], we gave the DVF related to a one parameter family of finite dimensional representations of $sl(r + 1|s + 1)$. On constructing this DVF, we deformed some DVF at which the top term carries the highest weight of a typical representation. Based on the similar idea, we consider a deformation of the function $T^{(1)}_{s-1}(u)$ and construct a DVF with a continuous parameter. This deformation is compatible with the top term hypothesis [5, 6]. We prove the pole freeness of this function. We further define DVF s $T^{(a)}_{m}(u)\ (a \in \{2, 3, \ldots, s\})$, whose top term will carry ‘fundamental weights’ $\omega_a$. We present a class of transfer matrix functional relations among DVF s. In particular, we conjecture a set of functional relations for DVF s $T^{(a)}_{m}(u)$, which have $T^{(1)}_{-1}(u)$ and $T^{(a)}_{1}(u)\ (a \in \{2, 3, \ldots, s\})$ as initial conditions. It may be viewed as a kind of the $T$-system [23] (see also [4, 5, 6, 7, 8, 9, 10, 13, 15, 24, 25, 26, 27, 28, 29, 30, 31, 32]).

The outline of this paper is given as follows. In section 2, we briefly review the Lie superalgebra $C(s)$. In section 3, we execute an analytic Bethe ansatz based upon the BAE (3.1) associated with distinguished simple root system. We prove pole-freeness of the function $T^{a}(u)$. In section 4, we present an extension of the DVF defined in the previous section. In section 5, we mention functional relations for DVF s. Section 6 is devoted to summary and discussion. Appendix A gives an example of the DVF related to $C(3)$. In Appendix B, we briefly mention a special case ($sl(1|2)$ case) of our previous results [8, 9, 10] and point out relation to $C(2)$ case. In this paper, we adopt similar notation in [5, 8, 9, 10].

2 The Lie superalgebra $C(s)$

In this section, we briefly review the Lie superalgebra $C(s)$ (see for example [11, 33, 34, 35, 36, 37]). A Lie superalgebra is a $Z_2$ graded algebra with a product, whose homogeneous elements satisfy the graded Jacobi identity.

There are several choices of simple root systems depending on the choices of Borel subalgebras. The simplest system of simple roots is the so called distinguished one. For example, the distinguished simple root system $\{\alpha_1, \ldots, \alpha_s\}$
of $C(s)$ has the following form (see Figure 1)

$$
\begin{align*}
\alpha_1 &= \epsilon - \delta_1 \\
\alpha_i &= \delta_{i-1} - \delta_i, \quad i = 2, 3, \ldots, s-1, \\
\alpha_s &= 2\delta_{s-1}
\end{align*}
$$

(2.1)

where $\epsilon; \delta_1, \ldots, \delta_{s-1}$ are the basis of the dual space of the Cartan subalgebra with the bilinear form $(\cdot | \cdot)$ such that

$$
(\epsilon | \epsilon) = \frac{1}{2}, \quad (\epsilon | \delta_j) = (\delta_i | \epsilon) = 0, \quad (\delta_i | \delta_j) = -\frac{\delta_{ij}}{2}
$$

(2.2)

$\{\alpha_i\}_{i \neq 1}$ are even roots and $\alpha_1$ is an odd root with $(\alpha_1 | \alpha_1) = 0$.

Any weight can be expressed in the following form:

$$
\Lambda = \Lambda_1 \epsilon + \sum_{j=1}^{s-1} \bar{\Lambda}_j \delta_j, \quad \Lambda_1, \bar{\Lambda}_j \in \mathbb{C}.
$$

(2.3)

We can rewrite this as follows:

$$
\Lambda = b_1 \omega_1 + b_2 \omega_2 + \cdots + b_s \omega_s,
$$

(2.4)

where $\omega_a$ is a ‘fundamental weight’

$$
\omega_a = \begin{cases}
\epsilon & \text{if } a = 1 \\
-\epsilon + \delta_1 + \delta_2 + \cdots + \delta_{a-1} & \text{if } a \in \{2, 3, \ldots, s\}
\end{cases}
$$

(2.5)

We normalized the longest simple root as $|(\alpha_s | \alpha_s)| = 2$.  

and $b_j$ is Kac-Dynkin label $^3$

$$b_j = \begin{cases} 
\Lambda_1 + \bar{\Lambda}_1 & \text{if } j = 1 \\
\bar{\Lambda}_{j-1} - \Lambda_j & \text{if } j \in \{2, 3, \ldots, s-1\} \\
\bar{\Lambda}_{s-1} & \text{if } j = s 
\end{cases} \quad (2.6)$$

An irreducible representation $V(\Lambda)$ with the highest weight $\Lambda$ is finite dimensional $[33]$ if and only if

$$b_j \in \mathbb{Z}_{\geq 0} \quad \text{for } j \neq 1. \quad (2.7)$$

Note that $b_1$ can take on complex values.

$V(\Lambda)$ is said to be typical if and only if

$$(\Lambda + \rho|\alpha) \neq 0 \quad \text{for all } \alpha \in \Delta_1^+, \quad (2.8)$$

where $\Delta_1^+ = \{\epsilon \pm \delta_j\}$; $\rho$ is the graded half sum of positive roots:

$$\rho = \sum_{i=1}^{s-1} (s-i)\delta_i - (s-1)\epsilon. \quad (2.9)$$

There is a large class of finite dimensional representations, which is not tensor-like. For example, a one parameter family of finite dimensional representations with the highest weight

$$\Lambda(c) = c\epsilon = c\omega_1, \quad (2.10)$$

is typical if

$$c \neq 0, 1, \ldots, s-2; \quad s, s+1, \ldots, 2s-2. \quad (2.11)$$

Note that the first Kac-Dynkin label of $\Lambda(c)$ takes non-integer value if the parameter $c$ is non-integral.

The dimensionality of the typical representation of $C(s)$ with the highest weight $\Lambda$ is given $[33, 37]$ as follows

$$\dim V(\Lambda) = 2^{2(s-1)} \prod_{1 \leq i \leq s-1} \frac{\bar{\Lambda}_i + s - i}{s - i} \times \prod_{1 \leq i < j \leq s-1} \frac{(\bar{\Lambda}_i - \bar{\Lambda}_j + j - i)(\bar{\Lambda}_i + \bar{\Lambda}_j + 2s - i - j)}{(j-i)(2s - i - j)}. \quad (2.12)$$

$^3$Note that Kac-Dynkin label of $\omega_a$ is $b_j = \delta_{aj}$.
As for the atypical finite dimensional representation, the dimensionality is given [37] as follows:

\[
\dim V(\Lambda) = \frac{2^{2s-3}}{(s-1)!} \prod_{1 \leq i \leq s-1} \frac{(2i)!}{(s-1-i)!(s-1+i)!} \\
\times \left( \prod_{1 \leq i \leq s-1; i \neq k} \prod_{1 \leq j < s; i,j \neq k} (x_i - x_j)(x_i + x_j) \right) \\
\times (-1)^{k-1} \sum_{j=0}^{2s-3} \sum_{l=0}^{j} (-1)^j 2^{-j} \binom{j}{l} (x_k - l) \\
\times \prod_{1 \leq i \leq s-1; i \neq k} (x_k - x_i - l)(x_k + x_i - l),
\] (2.13)

where \( x_i = \bar{\Lambda}_i + s - i \) for atypical with respect to \( \epsilon + \delta_k \); \( x_i = \bar{\Lambda}_i + s - i \) \((i \neq k)\) and \( x_k = \bar{\Lambda}_k + s - 1 - k \) for atypical with respect to \( \epsilon - \delta_k \).

### 3 Analytic Bethe ansatz

Consider the following type of the Bethe ansatz equation (cf [20, 21, 22, 19, 6]).

\[- \frac{N}{\prod_{j=1}^{N} \Phi(u_k^{(a)} - w_j^{(a)} + \frac{b_j^{(a)}}{t_a})} \Phi(u_k^{(a)} - w_j^{(a)} - \frac{b_j^{(a)}}{t_a}) = (-1)^{\deg(\alpha_a)} \prod_{b=1}^{s} \frac{Q_b(u_k^{(a)} + (\alpha_a|\alpha_b))}{Q_b(u_k^{(a)} - (\alpha_a|\alpha_b))}, \] (3.1)

\[Q_a(u) = \prod_{j=1}^{N_a} \Phi(u - u_j^{(a)}), \] (3.2)

where \( N, N_a \in \mathbb{Z}_{\geq 0}; u_j^{(a)}, w_j^{(a)} \in \mathbb{C}; a, k \in \mathbb{Z} \ (1 \leq a \leq s, 1 \leq k \leq N_a); t_1 = 2; \]
\( t_a = -2(2 \leq a \leq s - 1); t_s = -1; b_j^{(a)} \in \mathbb{Z}_{\geq 0}(2 \leq a \leq s); b_j^{(1)} \in \mathbb{C} \) and

\[
\deg(\alpha_a) = \begin{cases} 
0 & \text{for even root} \\
1 & \text{for odd root} 
\end{cases}
\] (3.3)

Here \( \Phi \) is a function, which has zero at \( u = 0 \). For example, we have

\[\Phi(u) = \frac{q^u - q^{-u}}{q - q^{-1}}, \] (3.4)
where \( q \) is generic. The parameters in the left hand side of the BAE (3.1) are the Kac-Dynkin labels

\[
(b_j^{(1)}, b_j^{(2)}, \ldots, b_j^{(s)})
\]  

(3.5)
of highest weight representations of \( C(s) \), which are related to the quantum space.

We define the sets

\[
J = \{1, 2, \ldots, s, \bar{s}, \ldots, \bar{2}, \bar{1}\},
\]

\[
J_+ = \{1, \bar{1}\}, \quad J_- = \{2, 3, \ldots, s, \bar{s}, \ldots, \bar{3}, \bar{2}\}
\]

(3.6)
with the total order

\[
1 < 2 < \cdots < s < \bar{s} < \cdots < \bar{2} < \bar{1}
\]

(3.7)
and with the grading

\[
p(a) = \begin{cases} 
0 & \text{for } a \in J_+, \\
1 & \text{for } a \in J_-. 
\end{cases}
\]

(3.8)

For \( a \in J \), we define \(^4\) the function

\[
1_u = \psi_1(u) \frac{Q_1(u - \frac{1}{2})}{Q_1(u + \frac{1}{2})},
\]

\[
a_u = \psi_a(u) \frac{Q_{a-1}(u - \frac{a-1}{2})Q_a(u - \frac{a-2}{2})}{Q_{a-1}(u - \frac{a-3}{2})Q_a(u - \frac{a-2}{2})}, \quad 2 \leq a \leq s - 1,
\]

\[
s_u = \psi_s(u) \frac{Q_{s-1}(u - \frac{s-1}{2})Q_s(u - \frac{s-3}{2})}{Q_{s-1}(u - \frac{s-3}{2})Q_s(u - \frac{s-1}{2})},
\]

\[
s_{\bar{1}} = \psi_{s}(u) \frac{Q_{s-1}(u - \frac{s-1}{2})Q_s(u - \frac{s-3}{2})}{Q_{s-1}(u - \frac{s-3}{2})Q_s(u - \frac{s-1}{2})},
\]

\[
a_{\bar{1}} = \psi_a(u) \frac{Q_{a-1}(u - \frac{2a-1}{2})Q_a(u - \frac{2a-2}{2})}{Q_{a-1}(u - \frac{2a-3}{2})Q_a(u - \frac{2a-1}{2})}, \quad 2 \leq a \leq s - 1,
\]

\[
\bar{1}_u = \psi_1(u) \frac{Q_1(u - \frac{2s-3}{2})}{Q_1(u - \frac{2s-1}{2})},
\]

(3.9)

\(^4\)In this paper, we often abbreviate the spectral parameter \( u \).
where \( Q_0(u) = Q_{s+1}(u) = 1 \). In this section, we will consider the case, as an example, where Kac-Dynkin labels in (3.5) have the form \( \delta_j^{(a)} = \delta_{a1} \) \((1 \leq a \leq s)\). In this case, the vacuum part of the function \( a \), (3.9) takes the following form:

\[
\psi_1(u) = \phi(u+1)\phi(u-s+1),
\psi_a(u) = \phi(u)\phi(u-s+1) \quad 2 \leq a \leq 2,
\psi_{\bar{1}}(u) = \phi(u)\phi(u-s),
\tag{3.10}
\]

where

\[
\phi(u) = \prod_{j=1}^{N} \Phi(u - w_j^{(1)}).
\tag{3.11}
\]

The generalization to the case of arbitrary Kac-Dynkin labels (3.5) will be achieved by suitable redefinition of the function \( \psi_a(u) \), and such redefinition will not essentially influence the subsequent argument.

Under the BAE (3.1), we have \(^5\)

\[
Res_{u=-\frac{1}{2}+u_k^{(1)}} \begin{pmatrix} 1 \\ u \end{pmatrix} - \begin{pmatrix} 2 \\ u \end{pmatrix} = 0,
\tag{3.12}
Res_{u=\frac{d}{2}+u_k^{(d)}} \begin{pmatrix} d \\ u \end{pmatrix} + \begin{pmatrix} d+1 \\ u \end{pmatrix} = 0, \quad 2 \leq d \leq s-1,
\tag{3.13}
Res_{u=\frac{s}{2}+u_k^{(s)}} \begin{pmatrix} S \\ u \end{pmatrix} + \begin{pmatrix} 3 \\ u \end{pmatrix} = 0,
\tag{3.14}
Res_{u=\frac{2s-d}{2}+u_k^{(d)}} \begin{pmatrix} d \\ u \end{pmatrix} + \begin{pmatrix} d+1 \\ u \end{pmatrix} = 0, \quad 2 \leq d \leq s-1,
\tag{3.15}
Res_{u=\frac{2s-1}{2}+u_k^{(1)}} \begin{pmatrix} 1 \\ u \end{pmatrix} - \begin{pmatrix} 2 \\ u \end{pmatrix} = 0.
\tag{3.16}
\]

We shall present functions with spectral parameter \( u \in \mathbb{C} \), which are candidates of DVF of various fusion types in the auxiliary space of transfer matrices of \( U_q(C(s)^{(1)}) \) vertex model. For \( a \in \mathbb{Z}_{\geq 0} \), we define the DVF la-

\(^5\)Here \( Res_{u=a}f(u) \) denotes the residue of a function \( f(u) \) at \( u = a \).
belled by the Young superdiagram 6 with shape \((1^a)\) as a summation 7 over products of the boxes in (3.9):

\[
T^a(u) = \sum_{\{i_k\} \in B(1^a)} (-1)^{\sum_{j=1}^a p(i_j)} \prod_{i=1}^a i_k,
\]

(3.18)

where the spectral parameter \(u\) is shifted as \(u + \frac{a-1}{2}, u + \frac{a-3}{2}, \ldots, u - \frac{a-1}{2}\) from top to the bottom. \(B(1^a)\) is a set of tableaux \(\{i_k\}\) obeying the following rule (admissibility conditions)

1. For \(i_k, i_{k+1} \in J_+\),
   \[i_k \prec i_{k+1}\]
   (3.19)

2. and for \(i_k, i_{k+1} \in J\),
   \[i_k \preceq i_{k+1}\]
   (3.20)

unless
   \[i_k = \bar{s}, \quad i_{k+1} = s.\]
   (3.21)

The top term 8 [5] of the DVF (3.18) will be

\[
(-1)^{a-1} \begin{pmatrix}
1 \\
2 \\
\vdots \\
2
\end{pmatrix} a = (-1)^{a-1} \frac{Q_1(u - \frac{a}{2})Q_2(u + \frac{a-1}{2})}{Q_1(u + \frac{a}{2})Q_2(u - \frac{a-1}{2})},
\]

(3.22)

which carries \(C(s)\) weight \(\epsilon + (a - 1)\delta_1 = a\omega_1 + (a - 1)\omega_2\). We believe the DVF (3.18) is generated from this term.

Note that Kac-Dynkin label \((b_1, b_2, \ldots, b_s)\) is related to the Young superdiagram with shape \((\mu_1, \mu_2, \ldots)\) as follows

\[
\begin{align*}
b_1 &= \mu_1 + \eta_i, \\
b_{1+i} &= \eta_i - \eta_{i+1} & \text{for } i \in \{1, 2, \ldots, s-2\}, \\
b_s &= \eta_{s-1},
\end{align*}
\]

(3.17)

where \(\eta_i = \text{Max}\{\mu_i' - 1, 0\}\).

We assume \(T^0(u) = 1\).

Here we omit the vacuum part.
Table 1: The number of the terms in $T^a(u)$ for $C(3)$.  

| $\sharp B(1^a)$ | 1 | 6 | 20 | 50 | 105 | 196 | 336 |
|-----------------|---|----|-----|-----|------|------|-----|
| $a$             | 0 | 1  | 2   | 3   | 4    | 5    | 6    |

The number of the terms in $T^a(u)$ is given as (see Table 1)  

$$\sharp B(1^a) = \sum_{n=0}^{[\frac{a}{2}]} D(a, n),$$  

(3.23)

where

$$D(a, n) = \sum_{k=0}^{a-2n} \left\{ \binom{k+s-2}{k} + \binom{k+s-3}{k-1} \right\} \times \left\{ \binom{a-2n-k+s-2}{a-2n-k} + \binom{a-2n-k+s-3}{a-2n-k-1} \right\}. $$  

(3.24)

We have checked the following relation for several cases (see Table 2),

$$D(a, n) = \begin{cases} 
\dim V(\epsilon + (a-2n-1)\delta_1) & \text{if } n \in \{0, 1, \ldots, \left[\frac{a}{2}\right] - 1\}, \\
\dim V(0) & \text{if } n = \frac{a}{2}, a \in 2\mathbb{Z}_{\geq 0}, \\
\dim V(\epsilon) & \text{if } n = \frac{a-1}{2}, a \in 2\mathbb{Z}_{\geq 0} + 1.
\end{cases}$$  

(3.25)

This relation suggests a possibility that the auxiliary space $W_{(1^a)}$\footnote{It will be a module of super Yangian or quantum affine superalgebra [38, 39, 40].} of the
DVF $T^a(u)$ decomposes as a $C(s)$ module as follows:

$$W_{(1^a)} \simeq V(\epsilon + (a - 1)\delta_1) \oplus V(\epsilon + (a - 3)\delta_1) \oplus \cdots \oplus \begin{cases} V(0) & \text{if } a \in 2\mathbb{Z}_{\geq 0}, \\ V(\epsilon) & \text{if } a \in 2\mathbb{Z}_{\geq 0} + 1. \end{cases}$$ (3.26)

This relation seem to suggest a superization of Kirillov-Reshetikhin formula [41], which gives multiplicity of occurrence of the irreducible representations of Lie superalgebra in Yangian module.

For $m \in \{1, 2, \ldots, s - 1\}$, we also define the DVF labelled by the Young superdiagram with shape $(m^1)$ as follows

$$T_m^{(1)}(u) = \sum_{\{i_k\} \in B(m^1)} (-1)^{\sum_{j=1}^m p(i_j)} \begin{array}{cccc} \vdots \end{array}$$ (3.27)

where the spectral parameter $u$ is shifted as $u - \frac{m-1}{2}, u - \frac{m-3}{2}, \ldots, u + \frac{m-1}{2}$ from left to right. $B(m^1)$ is a set of tableaux $\{i_k\}$ obeying the following rule (admissibility conditions):

1. For $i_k, i_{k+1} \in J_-$,

$$i_k \prec i_{k+1}$$ (3.28)

and

$$s + j - k \geq d \quad \text{provided} \quad i_j = d \quad \text{and} \quad i_k = \bar{d},$$ (3.29)

2. for $i_k, i_{k+1} \in J$

$$i_k \preceq i_{k+1}.$$ (3.30)

The top term $^{11}$ [5] of the DVF (3.27) will be

$$\begin{array}{cccc} 1 & 1 & \cdots & 1 \\ \hline \end{array}$$ (3.31)

which carries $C(s)$ weight $m\epsilon = m\omega_1$. We believe the DVF (3.27) is generated from this term.

$^{10}$We assume $T_0^{(1)}(u) = 1.$

$^{11}$Here we omit the vacuum part.
Table 3: The number of the terms in $T_m^{(1)}(u)$ for $C(3)$.

The number $N_m^{(1)}$ of the terms in $T_m^{(1)}(u)$ for $m \in \{0, 1, 2, \ldots, s - 1\}$ is given as (see Table 3)

$$\#B(m^1) = \sum_{k=0}^{m} \left\{ \binom{2s - 2}{k} - \binom{2s - 2}{k-2} \right\} (m - k + 1) \quad (3.32)$$

We have checked the relation $\#B(m^1) = \dim V(m\omega_1)$ for several cases. This relation suggests a possibility that the auxiliary space $W_{(m^1)}$ of the DVF $T_m^{(1)}(u)$ is $W_{(m^1)} \approx V(m\epsilon)$ as a $C(s)$ module.

Note that the function $T^1(u) = T_1^{(1)}(u)$ coincides with the eigenvalue formula by the algebraic Bethe ansatz [19].

We remark that $a$ is transformed to $a$ under the following transformation:

$$u \rightarrow -(u - s + 1),$$

$$u_j^{(a)} \rightarrow -u_j^{(a)},$$

$$w_j^{(1)} \rightarrow -w_j^{(1)},$$

if $\Phi(-u) = \pm \Phi(u)$. In this case, $T^a(u)$ (3.18) and $T_m^{(1)}(u)$ (3.27) are invariant under the transformation (3.33).

Now we shall present examples of (3.18) and (3.27) for $C(3)$, $J_+ = \{1, \overline{1}\}$, $J_- = \{2, 3, \overline{3}, \overline{2}\}$ case:

$$T^1(u) = T_1^{(1)}(u) = \begin{array}{c}
\begin{bmatrix}
1 & -2 & -3 & -3 & -2 & +1
\end{bmatrix}
\end{array}$$

$$= \phi(-2 + u)\phi(1 + u) \frac{Q_1(-\frac{1}{2} + u)}{Q_1(\frac{1}{2} + u)}$$

$$- \phi(-2 + u)\phi(u) \frac{Q_1(-\frac{1}{2} + u)Q_2(1 + u)}{Q_1(\frac{1}{2} + u)Q_2(u)}$$

\[12\] As for $m \geq s (m \in \mathbb{Z})$, see section 4.
\[ T^2(u) = -\frac{1}{2} - \frac{1}{3} - \frac{1}{3} - \frac{1}{2} + \frac{1}{1} + \frac{1}{2} + \frac{2}{2} + \frac{2}{3} + \frac{2}{2} - \frac{2}{1} \]

\[ + \frac{3}{3} + \frac{3}{3} + \frac{3}{2} - \frac{3}{1} + \frac{3}{3} + \frac{3}{2} - \frac{3}{1} + \frac{2}{2} - \frac{2}{1} \]

\[ + \frac{3}{3} \]

\[ = \phi(-\frac{3}{2} + u) \phi(-\frac{1}{2} + u) \{ \]

\[ - \phi(-\frac{5}{2} + u) \phi(\frac{3}{2} + u) \frac{Q_1(-1 + u)Q_2(\frac{1}{2} + u)}{Q_1(1 + u)Q_2(-\frac{3}{2} + u)} \]

\[ - \phi(-\frac{5}{2} + u) \phi(\frac{3}{2} + u) \frac{Q_1(u)Q_2(\frac{-3}{2} + u)Q_3(\frac{1}{2} + u)}{Q_1(1 + u)Q_2(-\frac{1}{2} + u)Q_3(-\frac{3}{2} + u)} \]

\[ - \phi(-\frac{5}{2} + u) \phi(\frac{3}{2} + u) \frac{Q_1(1, u)Q_2(\frac{-3}{2} + u)Q_3(-\frac{5}{2} + u)}{Q_1(1 + u)Q_2(-\frac{5}{2} + u)Q_3(-\frac{3}{2} + u)} \]

\[ - \phi(-\frac{5}{2} + u) \phi(\frac{3}{2} + u) \frac{Q_1(-2 + u)Q_1(u)Q_2(-\frac{7}{2} + u)}{Q_1(-3 + u)Q_1(1 + u)Q_2(-\frac{5}{2} + u)} \]

\[ + \phi(-\frac{7}{2} + u) \phi(\frac{3}{2} + u) \frac{Q_1(-2 + u)Q_1(u)}{Q_1(-3 + u)Q_1(1 + u)} \]

\[ + \phi(-\frac{5}{2} + u) \phi(\frac{1}{2} + u) \frac{Q_1(-1 + u)Q_2(\frac{3}{2} + u)}{Q_1(1 + u)Q_2(-\frac{1}{2} + u)} \]

\[ + \phi(-\frac{5}{2} + u) \phi(\frac{1}{2} + u) \frac{Q_1(u)Q_2(\frac{-3}{2} + u)Q_2(\frac{7}{2} + u)Q_3(\frac{1}{2} + u)}{Q_1(1 + u)Q_2(-\frac{1}{2} + u)Q_2(\frac{5}{2} + u)Q_3(-\frac{3}{2} + u)} \]

\[ + \phi(-\frac{5}{2} + u) \phi(\frac{1}{2} + u) \frac{Q_1(u)Q_2(\frac{-3}{2} + u)Q_2(\frac{7}{2} + u)Q_3(\frac{1}{2} + u)}{Q_1(1 + u)Q_2(-\frac{1}{2} + u)Q_2(\frac{5}{2} + u)Q_3(-\frac{3}{2} + u)} \]

\[ = \phi(-\frac{3}{2} + u) \phi(-\frac{1}{2} + u) \}

\[ (3.34) \]
\[
T_2^{(1)}(u) = \left\{
\begin{array}{c}
+ \phi(-\frac{5}{2} + u)\phi(\frac{1}{2} + u) - Q_1(u)Q_2(-\frac{3}{2} + u)Q_2(\frac{3}{2} + u)Q_3(-\frac{7}{2} + u) \\
+ \phi(-\frac{5}{2} + u)\phi(\frac{1}{2} + u) Q_1(-2 + u)Q_1(u)Q_2(-\frac{7}{2} + u)Q_2(\frac{3}{2} + u) \\
+ \phi(-\frac{7}{2} + u)\phi(\frac{1}{2} + u) \frac{Q_1(-2 + u)Q_1(u)Q_2(\frac{3}{2} + u)}{Q_1(-3 + u)Q_1(1 + u)Q_2(-\frac{5}{2} + u)Q_2(\frac{1}{2} + u)} \\
+ \phi(-\frac{5}{2} + u)\phi(\frac{1}{2} + u) \frac{Q_1(-2 + u)Q_1(u)Q_2(\frac{3}{2} + u)}{Q_1(-3 + u)Q_1(1 + u)Q_2(\frac{3}{2} + u)} \\
+ \phi(-\frac{5}{2} + u)\phi(\frac{1}{2} + u) \frac{Q_2(-\frac{3}{2} + u)Q_2(-\frac{1}{2} + u)Q_3(-\frac{7}{2} + u)Q_3(\frac{3}{2} + u)}{Q_2(-\frac{5}{2} + u)Q_2(\frac{1}{2} + u)Q_3(-\frac{3}{2} + u)Q_3(-\frac{1}{2} + u)} \\
+ \phi(-\frac{5}{2} + u)\phi(\frac{1}{2} + u) \frac{Q_1(-2 + u)Q_2(-\frac{7}{2} + u)Q_2(-\frac{1}{2} + u)Q_3(\frac{3}{2} + u)}{Q_1(-3 + u)Q_2(-\frac{5}{2} + u)Q_2(\frac{1}{2} + u)Q_3(-\frac{1}{2} + u)} \\
+ \phi(-\frac{7}{2} + u)\phi(\frac{1}{2} + u) \frac{Q_1(-2 + u)Q_2(-\frac{1}{2} + u)Q_3(\frac{3}{2} + u)}{Q_1(-3 + u)Q_2(\frac{1}{2} + u)Q_3(-\frac{1}{2} + u)} \\
+ \phi(-\frac{5}{2} + u)\phi(\frac{1}{2} + u) \frac{Q_2(-\frac{1}{2} + u)Q_3(-\frac{7}{2} + u)Q_3(-\frac{5}{2} + u)}{Q_2(-\frac{5}{2} + u)Q_3(-\frac{3}{2} + u)Q_3(-\frac{1}{2} + u)} \\
+ \phi(-\frac{5}{2} + u)\phi(\frac{1}{2} + u) \frac{Q_1(-2 + u)Q_2(-\frac{7}{2} + u)Q_2(-\frac{1}{2} + u)Q_3(-\frac{5}{2} + u)}{Q_1(-3 + u)Q_2(-\frac{5}{2} + u)Q_2(-\frac{3}{2} + u)Q_3(-\frac{1}{2} + u)} \\
+ \phi(-\frac{7}{2} + u)\phi(\frac{1}{2} + u) \frac{Q_1(-2 + u)Q_2(-\frac{1}{2} + u)Q_3(-\frac{5}{2} + u)}{Q_1(-3 + u)Q_2(-\frac{5}{2} + u)Q_3(-\frac{1}{2} + u)} \\
+ \phi(-\frac{5}{2} + u)\phi(\frac{1}{2} + u) \frac{Q_1(-1 + u)Q_2(-\frac{7}{2} + u)}{Q_1(-3 + u)Q_2(-\frac{3}{2} + u)} \\
+ \phi(-\frac{7}{2} + u)\phi(\frac{1}{2} + u) \frac{Q_1(-1 + u)Q_2(-\frac{5}{2} + u)}{Q_1(-3 + u)Q_2(-\frac{3}{2} + u)} \\
+ \phi(-\frac{5}{2} + u)\phi(\frac{1}{2} + u) \frac{Q_3(-\frac{5}{2} + u)Q_3(\frac{1}{2} + u)}{Q_3(-\frac{3}{2} + u)Q_3(-\frac{1}{2} + u)} \right\},
\end{array}\right.
(3.35)
\]
\begin{align*}
&= \phi(-\frac{5}{2} + u)\phi(\frac{1}{2} + u)\{\phi(-\frac{3}{2} + u)\phi(\frac{3}{2} + u)\frac{Q_1(-1 + u)Q_2(\frac{3}{2} + u)}{Q_1(1 + u)Q_2(\frac{1}{2} + u)} \\
&- \phi(-\frac{3}{2} + u)\phi(\frac{1}{2} + u)\frac{Q_1(-1 + u)Q_2(\frac{3}{2} + u)}{Q_1(1 + u)Q_2(\frac{1}{2} + u)} \\
&- \phi(-\frac{3}{2} + u)\phi(\frac{1}{2} + u)\frac{Q_1(-1 + u)Q_2(-\frac{1}{2} + u)Q_3(\frac{3}{2} + u)}{Q_1(u)Q_2(\frac{1}{2} + u)Q_3(-\frac{1}{2} + u)} \\
&- \phi(-\frac{3}{2} + u)\phi(\frac{1}{2} + u)\frac{Q_1(-1 + u)Q_2(-\frac{1}{2} + u)Q_3(-\frac{5}{2} + u)}{Q_1(u)Q_2(-\frac{1}{2} + u)Q_3(-\frac{5}{2} + u)} \\
&- \phi(-\frac{3}{2} + u)\phi(\frac{1}{2} + u)\frac{Q_1(-1 + u)^2Q_2(-\frac{5}{2} + u)}{Q_1(-2 + u)Q_1(u)} \\
&+ \phi(-\frac{5}{2} + u)\phi(\frac{1}{2} + u)\frac{Q_1(-1 + u)Q_3(\frac{3}{2} + u)}{Q_1(u)Q_3(-\frac{1}{2} + u)} \\
&+ \phi(-\frac{3}{2} + u)\phi(\frac{1}{2} + u)\frac{Q_1(-1 + u)Q_3(\frac{3}{2} + u)}{Q_1(u)Q_3(-\frac{1}{2} + u)} \\
&+ \phi(-\frac{3}{2} + u)\phi(\frac{1}{2} + u)\frac{Q_1(-1 + u)^2Q_2(-\frac{5}{2} + u)Q_2(\frac{1}{2} + u)}{Q_1(-2 + u)Q_1(u)Q_2(-\frac{3}{2} + u)Q_2(-\frac{1}{2} + u)} \\
&+ \phi(-\frac{5}{2} + u)\phi(\frac{1}{2} + u)\frac{Q_1(-1 + u)^2Q_2(\frac{1}{2} + u)}{Q_1(-2 + u)Q_1(u)Q_2(-\frac{1}{2} + u)} \\
&+ \phi(-\frac{3}{2} + u)\phi(\frac{1}{2} + u)\frac{Q_1(-1 + u)Q_2(-\frac{5}{2} + u)Q_3(\frac{1}{2} + u)}{Q_1(-2 + u)Q_2(-\frac{1}{2} + u)Q_3(-\frac{3}{2} + u)} \\
&- \phi(-\frac{5}{2} + u)\phi(\frac{1}{2} + u)\frac{Q_1(-1 + u)Q_2(-\frac{3}{2} + u)Q_3(-\frac{7}{2} + u)}{Q_1(-2 + u)Q_2(-\frac{3}{2} + u)Q_3(-\frac{3}{2} + u)} \\
&- \phi(-\frac{5}{2} + u)\phi(\frac{1}{2} + u)\frac{Q_1(-1 + u)Q_2(-\frac{7}{2} + u)}{Q_1(-3 + u)Q_2(-\frac{5}{2} + u)}
\end{align*}
\[
+ \phi(-\frac{7}{2} + u)\phi(-\frac{1}{2} + u)\frac{Q_1(-1 + u)}{Q_1(-3 + u)}.
\]

(3.36)

Thanks to Theorem 3.1 and Theorem 3.2 (see later), these DVFs are free of poles under the following BAE:

\[
\phi(u)_{k+\frac{1}{2}} = Q_2(u)_{k-\frac{1}{2}} = Q_3(u)_{k+\frac{1}{2}} = Q_3(u)_{k-\frac{1}{2}}.
\]

(3.37)

We note that DVFs have so called Bethe-strap structures [5, 7], which bear resemblance to weight space diagrams. For example, Bethe-strap structures of (3.34), (3.35) and (3.36) are given in Figure 2, Figure 3 and Figure 4 respectively. We also note that the Bethe-stap resembles crystal graph [43, 44] as was pointed out in [5] for simple Lie algebras cases.

Remark: Recently we have found curious terms in many Bethe-straps. They may be called pseudo-top terms, which have the following properties (cf. Figure 2, Figure 3, Figure 4):

1. They carry Lie (super) algebras weights, which are lower than the one for the top term.
2. They send out arrows but does not suck in arrows.

We also found pseudo-bottom terms, which have the following properties:

1. They carry Lie (super) algebras weights, which are higher than the one for the bottom term.
2. They suck in arrows but does not send out arrows.

We found a pseudo-top term in the Bethe-strap of \( T_{2}(u) \) (As for \( T_{m}^{(a)}(u) \), see section 5.) for \( C(4) \). Pseudo-top term and pseudo-bottom term are not peculiar for Bethe-straps related to Lie superalgebras. Actually, we found such terms even for \( sl_{2} \) case: the DVF \( T_{2}(u)T_{1}(u + k) \) has a pseudo-top term
Figure 2: The Bethe-strap structure of the DVF $T^{(1)}_a(u)$ (3.34) for the Lie superalgebra $C(3)$: The pair $(a, b)$ denotes the common pole $u^{(a)}_k + b$ of the pair of the tableaux connected by the arrow. This common pole vanishes under the BAE (3.37). The leftmost (resp. rightmost) tableau corresponds to the ‘highest weight vector’ (resp. ‘lowest weight vector’), which is called the \textit{top term} (resp. \textit{bottom term}). Such a correspondence between certain term in the DVF and a highest weight (to be more precise, a kind of Drinfel’d polynomial) may be called \textit{top term hypothesis} [5, 6].

if $k = 1$; a pseudo-bottom term if $k = -1$ (see [5] for the definition of $T_m(u)$).

In spite of existence of these curious terms, auxiliary spaces of DVFs in this paper are supposed to correspond to some \textit{irreducible} representations as long as the Bethe straps are \textit{connected} in the whole (cf [5, 6, 42]). In fact we have confirmed, for several cases, that the Bethe straps of the DVFs $T^a(u)$ and $T^{(a)}_m(u)$ are connected in the whole. We hope to discuss about these curious terms in detail elsewhere.

In general, we have the following Theorems, which are essential in the analytic Bethe ansatz.

\textbf{Theorem 3.1} For $a \in Z_{\geq 0}$, DVF $T^a(u)$ (3.18) is free of poles under the condition that the BAE (3.1) is valid.\textsuperscript{13}

\textbf{Theorem 3.2} For $m \in \{0, 1, 2, \ldots, s - 1\}$, DVF $T^{(1)}_m(u)$ (3.27) is free of poles under the condition that the BAE (3.1) is valid.

We can prove Theorem 3.2 by using the following lemmas.

\textsuperscript{13}We assume that $u^{(b)}_i - u^{(b)}_j \neq (\alpha_i|\alpha_b)$ for any $i, j \in \{1, 2, \ldots, N_b\}$ ($i \neq j$) and $b \in \{1, 2, \ldots, s\}$ in BAE (3.1). If this assumption does not hold, we will need separate consideration. This assumption may require detailed analysis of the BAE (3.1), which is beyond the scope of this paper. We also note that similar assumptions are implicitly assumed in many literature concerning analytic Bethe ansatz.
Figure 3: The Bethe-strap structure of the DVF $T^2(u)$ (3.35) for the Lie superalgebra $C(3)$: The topmost (resp. bottommost) tableau corresponds to the 'highest weight vector ' (resp. 'lowest weight vector '), which is called the top term (resp. bottom term ).
Figure 4: The Bethe-strap structure of the DVF $T_{2}^{(1)}(u) \ (3.36)$ for the Lie superalgebra $C(3)$: The topmost (resp. bottommost) tableau corresponds to the ‘highest weight vector ’ (resp. ‘lowest weight vector ’), which is called the top term (resp. bottom term).
Lemma 3.3 (1) For \( b \in \{2, 3, \ldots, s-1\} \),

\[
\text{Res}_{v = u_k^{(b)}} = \frac{1}{2} \left( b \begin{array}{c} b+1 \\ b+s-b \end{array} + b \begin{array}{c} b \end{array} \right) = 0,
\]

\[
\text{Res}_{v = u_k^{(b)}} = \frac{1}{2} \left( b \begin{array}{c} b+1 \\ b+s-b \end{array} + b+1 \begin{array}{c} b \end{array} \right) = 0
\]

under the BAE (3.1).

(2) \[ \begin{array}{c} b \\ b+1 \end{array} \begin{array}{c} b+s-b \end{array} + b \begin{array}{c} b \end{array} \begin{array}{c} b+s-b \end{array} + b+1 \begin{array}{c} b \end{array} \begin{array}{c} b+s-b \end{array} \]

is free of color \( b \) \((b \in \{2, 3, \ldots, s-1\}) \) pole under the BAE (3.1).

The following lemma is \( C(s) \) version of lemma 3.3.4. in [5] for \( C_s \).

Lemma 3.4 For \( b \in \{2, 3, \ldots, s-1\} \), let

\[
\begin{array}{c} \xi \\ b \end{array} \begin{array}{c} \eta \end{array} \begin{array}{c} b+1 \end{array} \begin{array}{c} \zeta \end{array} \quad \begin{array}{c} \xi \\ b+1 \end{array} \begin{array}{c} \eta \end{array} \begin{array}{c} b \end{array} \begin{array}{c} \zeta \end{array}
\]

be the terms that appear in \( T_{(1)}^a(u) \) (3.27). Here \( \xi, \eta \) and \( \zeta \) do not contain \( b, b+1, b+1 \) or \( b \). In this case, the length of \( \eta \) is less than \( s - b \).

We can prove Theorem 3.2 by a similar idea used in the proof of Theorem 3.3.1. in [5]. So we prove only Theorem 3.1 from now on.

Proof of Theorem 3.1. For simplicity, we assume that the vacuum parts are formally trivial, that is, the left hand side of the BAE (3.1) is constantly \(-1\). We prove that \( T^a(u) \) (3.18) is free of color \( b \) pole, namely, \( \text{Res}_{u = u_k^{(b)} + \ldots} T^a(u) = 0 \) for any \( b \in \{1, 2, \ldots, s\} \) under the condition that the BAE (3.1) is valid. The function \( c_{(b)} \) (3.9) with \( c \in J \) has color \( b \) poles only at \( c = b, b+1, b+1 \) or \( b \) for \( b \in \{1, 2, \ldots, s-1\} \); at \( c = s \) or \( c = \overline{s} \) for \( b = s \), so we shall trace only \( b, b+1, b+1 \) or \( b \) for \( b \in \{1, 2, \ldots, s-1\} \); \( s \) or \( \overline{s} \) for \( b = s \). Denote \( S_k \) the partial sum of \( T^a(u) \), which contains \( k \) boxes among \( b, b+1, b+1 \) or \( b \) for \( 1 \leq b \leq s-1 \); \( s \) or \( \overline{s} \) for \( b = s \). Apparently, \( S_0 \) does not have color \( b \) pole.

Now we examine \( S_1 \) which is a summation of the tableaux (with sign) of the form

\[
\begin{array}{c} \xi \\ \eta \end{array} \begin{array}{c} \zeta \end{array}
\]
where \( \xi \) and \( \zeta \) are columns with total length \( a - 1 \) and they do not involve \( Q_b \). \( \eta \) is \( b, b + 1, b + 1 \) or \( \bar{b} \) for \( 1 \leq b \leq s - 1 \); \( s \) or \( \bar{s} \) for \( b = s \).

Thanks to the relations (3.12)-(3.16), \( S_1 \) is free of color \( b \) pole under the BAE (3.1). Hereafter we consider \( S_k \) \( (k \geq 2) \).

- The case \( b = 1 : S_k \) \( (k \geq 2) \) is a summation of the tableaux (with sign) of the form

\[
\begin{array}{cccc}
D_{11} & D_{11} & D_{12} & D_{12} \\
\eta & \eta & \eta & \eta \\
D_{21} & D_{22} & D_{21} & D_{22} \\
\end{array}
= \left( \begin{array}{cc}
D_{11} - D_{12} \\
D_{21} - D_{22}
\end{array} \right) \eta
\]

(3.43)

where \( \eta \) is a column with length \( a - k \), which does not contain \( 1, 2 \) and \( 2 \) and \( \bar{D}_{11} \) is a column 14 of the form:

\[
\begin{array}{cc}
1 & v \\
2 & v - 1 \\
\vdots & \\
2 & v - k_1 + 1
\end{array}
= \frac{Q_1(v + \frac{1}{2} - k_1)Q_2(v)}{Q_1(v + \frac{1}{2})Q_2(v - k_1 + 1)},
\]

(3.44)

\( D_{12} \) is a column of the form:

\[
\begin{array}{cc}
2 & v \\
2 & v - 1 \\
\vdots & \\
2 & v - k_1 + 1
\end{array}
= \frac{Q_1(v + \frac{1}{2} - k_1)Q_2(v + 1)}{Q_1(v + \frac{1}{2})Q_2(v - k_1 + 1)},
\]

(3.45)

where \( v = u + h_1 \); \( h_1 \) is some shift parameter; \( \bar{D}_{21} \) is a column 15 of the form:

\[
\begin{array}{cc}
2 & w \\
\vdots & \\
2 & w - k_2 + 2 \\
\bar{2} & w - k_2 + 1
\end{array}
= \frac{Q_1(w - \frac{2s-3}{2})Q_2(w - \frac{2s-4}{2} - k_2)}{Q_1(w - \frac{2s-3}{2} - k_2)Q_2(w - \frac{2s-2}{2})},
\]

(3.46)

14We assume that \( \bar{D}_{11} = \begin{array}{c}
1
\end{array} \) if \( k_1 = 1 \).
15We assume that \( \bar{D}_{21} = \begin{array}{c}
\bar{1}
\end{array} \) if \( k_2 = 1 \).
$D_{22}$ is a column of the form:

$$
\begin{array}{c}
2 \\
\vdots \\
2 \\
2 \\
\end{array}
\begin{array}{c}
w \\
-w-k_2+2 \\
w-k_2+1 \\
\end{array}
= \frac{Q_1(w - \frac{2s-3}{2})Q_2(w - \frac{2s-2}{2} - k_2)}{Q_1(w - \frac{2s-3}{2} - k_2)Q_2(w - \frac{2s-2}{2})},
\end{array}
\quad (3.47)
$$

where $w = u + h_2$: $h_2$ is some shift parameter; $k = k_1 + k_2$.\(^{16}\) Obviously, the color $b = 1$ residues at $v = -\frac{1}{2} + u_j^{(1)}$ in (3.44) and (3.45) cancel each other under the BAE (3.1). And the color $b = 1$ residues at $w = \frac{2s-3}{2} + k_2 + u_j^{(1)}$ in (3.46) and (3.47) cancel each other under the BAE (3.1). Thus $S_k$ does not have color 1 pole under the BAE (3.1).

- The case $2 \leq b \leq s - 1$: $S_k(k \geq 2)$ is a summation of the tableaux (with sign) of the form

$$
\begin{array}{c}
k_1 \\
\sum_{n_1=0}^{k_1} \\
k_2 \\
\sum_{n_2=0}^{k_2} \\
\end{array}
\begin{array}{c}
E_{1n_1} \\
\eta \\
E_{2n_2} \\
\zeta \\
\end{array}
= \left( \sum_{n_1=0}^{k_1} E_{1n_1} \right) \left( \sum_{n_2=0}^{k_2} E_{2n_2} \right) \times \xi \times \eta \times \zeta,
\end{array}
\quad (3.48)
$$

where $\xi$, $\eta$ and $\zeta$ are columns with total length $a - k$, which do not

\(^{16}\)Here we discussed the case for $k_1 \geq 1$ and $k_2 \geq 1$; the case for $k_1 = 0$ or $k_2 = 0$ can be treated similarly.
contain $b$, $b+1$, $b+1$ and $E_{1m}$ is a column \(^{17}\) of the form:

\[
\begin{array}{c|c}
  b & v \\
  \vdots & \vdots \\
  b & v-n_1+1 \\
  b+1 & v-n_1 \\
  \vdots & \vdots \\
  b+1 & v-k_1+1 \\
\end{array}
\]

\[
= \frac{Q_{b-1}(v - \frac{b-3}{2} - n_1)Q_b(v - \frac{b-4}{2})}{Q_{b-1}(v - \frac{b-3}{2})Q_b(v - \frac{b-4}{2} - n_1)}
\]

\[\text{(3.49)}\]

\[
\times \frac{Q_b(v - \frac{b-2}{2} - k_1)Q_{b+1}(v - \frac{b-3}{2} - n_1)}{Q_b(v - \frac{b-2}{2} - n_1)Q_{b+1}(v - \frac{b-3}{2} - k_1)}
\]

for \(2 \leq b \leq s - 2;\)

\[
\begin{array}{c|c}
  s-1 & v \\
  \vdots & \vdots \\
  s-1 & v-n_1+1 \\
  s & v-n_1 \\
  \vdots & \vdots \\
  s & v-k_1+1 \\
\end{array}
\]

\[
= \frac{Q_{s-2}(v - \frac{s-4}{2} - n_1)Q_{s-1}(v - \frac{s-5}{2})}{Q_{s-2}(v - \frac{s-4}{2})Q_{s-1}(v - \frac{s-5}{2} - n_1)}
\]

\[\text{(3.50)}\]

\[
\times \frac{Q_{s-1}(v - \frac{s-3}{2} - k_1)Q_s(v - \frac{s-5}{2} - n_1)Q_s(v - \frac{s-3}{2} - n_1)}{Q_{s-1}(v - \frac{s-3}{2} - n_1)Q_s(v - \frac{s-3}{2} - k_1)Q_s(v - \frac{s-5}{2} - k_1)}
\]

for \(^{18}\) \(b = s - 1; v = u + h_1; h_1\) is some shift parameter and $E_{2n_2}$ is a column

\[^{17}\text{We assume that } E_{10} = \begin{array}{c|c}
  b+1 & v \\
  \vdots & \vdots \\
  b+1 & v-k_1+1 \\
\end{array} \text{ and } E_{1k_1} = \begin{array}{c|c}
  b & v \\
  \vdots & \vdots \\
  b & v-k_1+1 \\
\end{array}.\]

\[^{18}\text{We need not take care of the sequence of } \begin{array}{c|c}
  s \end{array} \text{ since this does not contain } Q_{s-1}.\]
19 of the form:

\[
\begin{array}{c|c|c|c|c}
   & b+1 & w \\
  \vdots & \vdots & \vdots \\
   b+1 & w-n_2+1 & w-n_2 \\
   b & w-n_2 & w-k_2+1 \\
   \vdots & \vdots & \vdots \\
   b & w-k_2+1 \\
\end{array}
\]

\[
= \frac{Q_{b-1}(w - \frac{2s-b-1}{2} - n_2)Q_b(w - \frac{2s-b}{2} - k_2)}{Q_{b-1}(w - \frac{2s-b-1}{2} - k_2)Q_b(w - \frac{2s-b}{2} - n_2)} (3.51)
\]

\[
\times \frac{Q_b(w - \frac{2s-b-2}{2})Q_{b+1}(w - \frac{2s-b-1}{2} - n_2)}{Q_b(w - \frac{2s-b-2}{2} - n_2)Q_{b+1}(w - \frac{2s-b-1}{2})}
\]

for \(2 \leq b \leq s-2;\)

\[
\begin{array}{c|c|c|c|c}
   & s & w \\
  \vdots & \vdots & \vdots \\
   s & w-n_2+1 \\
   \vdots & \vdots & \vdots \\
   s & w-k_2+1 \\
\end{array}
\]

\[
= \frac{Q_{s-2}(w - \frac{s}{2} - n_2)Q_{s-1}(w - \frac{s+1}{2} - k_2)}{Q_{s-2}(w - \frac{s}{2} - k_2)Q_{s-1}(w - \frac{s+1}{2} - n_2)} (3.52)
\]

\[
\times \frac{Q_{s-1}(w - \frac{s-1}{2})Q_s(w - \frac{s+1}{2} - n_2)Q_s(w - \frac{s-1}{2} - n_2)}{Q_{s-1}(w - \frac{s-1}{2} - n_2)Q_s(w - \frac{s-1}{2})Q_s(w - \frac{s+1}{2})}
\]

for \(b = s-1; w = u + h_2; h_2\) is some shift parameter; \(k = k_1 + k_2.\)

For \(2 \leq b \leq s-1,\) \(E_{1n_1}\) has color \(b\) poles at \(u = -h_1 + \frac{b-2}{2} + n_1 + u_p^{(b)}\)
and \(u = -h_1 + \frac{b-4}{2} + n_1 + u_p^{(b)}\) for \(1 \leq n_1 \leq k_1 - 1;\) at \(u = -h_1 + \frac{b-2}{2} + u_p^{(b)}\) for \(n_1 = 0;\) at \(u = -h_1 + \frac{b-4}{2} + k_1 + u_p^{(b)}\) for \(n_1 = k_1.\) The Color \(b\) residues at \(u = -h_1 + \frac{b-2}{2} + n_1 + u_p^{(b)}\) in \(E_{1n_1}\) and \(E_{1n_1+1}\) cancel each other under the BAE (3.1). Thus, under the BAE (3.1), \(\sum_{n_1=0}^{k_1} E_{1n_1}\) is free of color \(b\) poles (see Figure 5).

\(E_{2n_2}\) has color \(b\) poles at \(u = -h_2 + \frac{2s-b}{2} + n_2 + u_p^{(b)}\) and \(u = -h_2 + \frac{2s-b-2}{2} + n_2 + u_p^{(b)}\) for \(1 \leq n_2 \leq k_2 - 1;\) at \(u = -h_2 + \frac{2s-b}{2} + u_p^{(b)}\) for \(n_2 = 0\)

19 We assume that \(E_{2b} = \begin{array}{c|c|c|c|c}
   & b & v \\
  \vdots & \vdots & \vdots \\
   b & v-k_2+1 & v \\
\end{array}\) and \(E_{2k_2} = \begin{array}{c|c|c|c|c}
   & b+1 & v \\
  \vdots & \vdots & \vdots \\
   b+1 & v-k_2+1 & v \\
\end{array}.

24
Figure 5: Partial Bethe-strap structure of $E_{1n}$ for color $b$ poles: The number $n$ on the arrow denotes the common color $b$ pole $-h_1 + \frac{b-2}{2} + n + u^{(b)}_p$ of the pair of the tableaux connected by the arrow. This common pole vanishes under the BAE (3.1).

; at $u = -h_2 + \frac{2s-b-2}{2} + k_2 + u^{(b)}_p$ for $n_2 = k_2$. The color $b$ residues at $u = -h_2 + \frac{2s-b}{2} + n_2 + u^{(b)}_p$ in $E_{2n_2}$ and $E_{2,n_2+1}$ cancel each other under the BAE (3.1). Thus, under the BAE (3.1), $\sum_{n_2=0}^{k_2} E_2,n_2$ is free of color $b$ poles, so is $S_k$.

- The case $b = s$: $S_k (k \geq 2)$ is a summation of the tableaux (with sign) of the form

$$g(l, n) = \frac{Q_{s-1}(v - \frac{s-3}{2} - l)Q_{s-1}(v - \frac{s-1}{2} - l - 2n)}{Q_{s-1}(v - \frac{s-3}{2})Q_{s-1}(v - \frac{s-1}{2} - k)}$$

(3.53)
Table 4: The conditions for occurrence of cancellation of poles and zeros in \( g(l, n) \) (3.53): \( g(l, n) \) has a pole at \( u = y - h + u_p^{(s)} \) and a zero at \( u = x - h + u_p^{(s)} \), where \( y \) is a number listed in the first column and \( x \) is a number listed in the first row. The content of the table indicates when these pole and zero cancel each other, i.e. the condition for occurrence of \( x = y \) for \( 0 < n < \left\lfloor \frac{k}{2} \right\rfloor \), \( 0 < l < k - 2n \).

\[
\frac{Q_s(v - \frac{s-5}{2})Q_s(v - \frac{s-3}{2})Q_s(v - \frac{s-1}{2} - k)}{Q_s(v - \frac{s-5}{2} - l)Q_s(v - \frac{s-1}{2} - l)Q_s(v - \frac{s-3}{2} - l - 2n)}
\times \frac{Q_s(v - \frac{s+1}{2} - k)}{Q_s(v - \frac{s+1}{2} - l - 2n)} \times \Xi \times \zeta,
\]

where \( \Xi \) and \( \zeta \) are columns with total length \( a - k \), which do not contain \( s \) and \( s \), \( v = u + h_1 \); \( h_1 \) is some shift parameter. \( g(l, n) \) has color \( s \) poles, some of which are canceled by zeros in the numerator (see Table 4), and remaining poles are canceled by the functions ‘around’ \( g(l, n) \) under the BAE (3.1) (see Figure 6 and Figure 7). Hereafter we consider the case \( k \geq 3 \). The case \( k = 2 \) can be treated similarly.

Denote the sets of \( l, n \in \mathbb{Z} \) (0 ≤ \( n \) ≤ \( \left\lfloor \frac{k}{2} \right\rfloor \), 0 ≤ \( l \) ≤ \( k - 2n \)) satisfying the conditions in Table 4 as alphabet. Set \( \mathcal{G} = \{ l, n \in \mathbb{Z} | n = 0, 1 \leq l \leq k - 1 \} \).

(1) For \( A \cap B \cap C \cap D \cap E \cap F \cap G = \{ l, n \in \mathbb{Z} | 2 \leq l \leq k - 2n - 2, 1 \leq n \leq \left\lfloor \frac{k-1}{2} \right\rfloor - 1 \} \), \( g(l, n) \) has color \( s \) poles\(^{20} \) at \( u = -h_1 + \frac{s-5}{2} + l + u_p^{(s)} \), \( u = -h_1 + \frac{s-1}{2} + l + 2n + u_p^{(s)} \), \( u = -h_1 + \frac{s+1}{2} + l + 2n + u_p^{(s)} \) and \( u = -h_1 + \frac{s-3}{2} + l + 2n + u_p^{(s)} \).

\(^{20}\)We assume that these poles at \( u = -h_1 + \frac{s-5}{2} + l + u_i^{(s)} \), \( u = -h_1 + \frac{s+1}{2} + l + 2n + u_j^{(s)} \),

\( u = -h_1 + \frac{s-1}{2} + l + u_p^{(s)} \) and \( u = -h_1 + \frac{s-3}{2} + l + 2n + u_p^{(s)} \) are not coincide each other for any \( i, j, p, q \in \{ 1, 2, \ldots, N_s \} \). If some of these poles coincide, we will need separate consideration. For example, if poles at \( u = -h_1 + \frac{s+1}{2} + l + 2n + u_j^{(s)} \) and \( u = -h_1 + \frac{s-1}{2} + l + u_p^{(s)} \) coincide, i.e. \( u_p^{(s)} = u_j^{(s)} + 2n + 1 \), we have to consider not
and these poles are canceled by \( g(l - 2, n + 1) \), \( g(l, n + 1) \), \( g(l + 2, n - 1) \) and \( g(l, n - 1) \) respectively under the BAE (3.1).

(2) For \( A \cap D = D = \{ l, n \in \mathbb{Z} | l = n = 0 \} \), \( g(l, n) \) has color \( s \) poles at 
\[ u = -h_1 + \frac{s+1}{2} + u_p^{(s)} \] 
and these poles are canceled by \( g(0, 1) \) and \( g(1, 0) \) respectively under the BAE (3.1).

(3) For \( B \cap G = \{ l, n \in \mathbb{Z} | l = 1, n = 0 \} \), \( g(l, n) \) has color \( s \) poles at 
\[ u = -h_1 + \frac{s+3}{2} + u_p^{(s)} \] 
and these poles are canceled by \( g(1, 1) \), \( g(2, 0) \) and \( g(0, 0) \) respectively under the BAE (3.1).

(4) For \( \bar{B} \cap G \cap \bar{E} = \{ l, n \in \mathbb{Z} | 2 \leq l \leq k - 2, n = 0 \} \), \( g(l, n) \) has color \( s \) poles at 
\[ u = -h_1 + \frac{s-5}{2} + l + u_p^{(s)} \] 
and these poles are canceled by \( g(0, 1) \) and \( g(1, 0) \) respectively under the BAE (3.1).

(5) For \( G \cap E = \{ l, n \in \mathbb{Z} | l = k - 1, n = 0 \} \), \( g(l, n) \) has color \( s \) poles at 
\[ u = -h_1 + \frac{s-7}{2} + k + u_p^{(s)} \] 
and these poles are canceled by \( g(k - 3, 1) \), \( g(k, 0) \) and \( g(k - 2, 0) \) respectively under the BAE (3.1).

(6) For \( (A \cap F \cap \{ k : \text{even} \}) \cup (A \cap E \cap \{ k : \text{odd} \}) = \{ l, n \in \mathbb{Z} | l = 0, n = \lfloor \frac{k}{2} \rfloor \} \), 
\( g(l, n) \) has color \( s \) poles at 
\[ u = -h_1 + \frac{s-1}{2} + u_p^{(s)} \] 
and these poles are canceled by \( g(2, \lfloor \frac{k}{2} \rfloor - 1) \) and \( g(0, \lfloor \frac{k}{2} \rfloor - 1) \) respectively under the BAE (3.1).

(7) For \( (A \cap \bar{D} \cap \bar{F} \cap \{ k : \text{even} \}) \cup (A \cap \bar{D} \cap \bar{E} \cap \{ k : \text{odd} \}) = \{ l, n \in \mathbb{Z} | l = 0, 1 \leq n \leq \lfloor \frac{k}{2} \rfloor - 1 \} \), 
\( g(l, n) \) has color \( s \) poles at 
\[ u = -h_1 + \frac{s+1}{2} + 2n + u_p^{(s)} \] 
and these poles are canceled by \( g(0, n + 1) \), \( g(2, n - 1) \) and \( g(0, n - 1) \) respectively under the BAE (3.1).

(8) For \( (B \cap E \cap \{ k : \text{even} \}) \cup (B \cap F \cap \{ k : \text{odd} \}) = \{ l, n \in \mathbb{Z} | l = 1, n = \lceil \frac{k-1}{2} \rceil \} \), 
\( g(l, n) \) has color \( s \) poles at 
\[ u = -h_1 + \frac{s+1}{2} + u_p^{(s)} \] 
and these poles are canceled by \( g(3, \lceil \frac{k-1}{2} \rceil - 1) \) and \( g(1, \lceil \frac{k-1}{2} \rceil - 1) \) respectively under the BAE (3.1).

(9) For \( (B \cap E \cap G \cap \{ k : \text{even} \}) \cup (B \cap F \cap G \cap \{ k : \text{odd} \}) = \{ l, n \in \mathbb{Z} | l = 1, 1 \leq n \leq \lceil \frac{k-1}{2} \rceil - 1 \} \), 
\( g(l, n) \) has color \( s \) poles at 
\[ u = -h_1 + \frac{s+3}{2} + 2n + u_p^{(s)} \]

only ‘nearest functions’ \( g(l, n + 1) \) and \( g(l + 2, n - 1) \) but also ‘next nearest function’ 
\( g(l, n): g(l, n) + g(l, n + 1) + g(l + 2, n - 1) + g(l + 2, n) \) is free of color \( s \) pole at 
\[ u = -h_1 + \frac{s+1}{2} + l + 2n + u_p^{(s)} = -h_1 + \frac{s+1}{2} + l + u_p^{(s)} \] under the BAE (3.1). Other cases (including (2)-(12)) can be treated by a similar idea.
u = -h_1 + \frac{s+1}{2} + u_p(s) and u = -h_1 + \frac{s-1}{2} + 2n + u_p(s), and these poles are canceled by \(g(1, n+1), g(3, n-1)\) and \(g(1, n-1)\) respectively under the BAE (3.1).

(10) For \((\bar{A} \cap \bar{C} \cap F \cap \{k: \text{ even}\}) \cup (\bar{B} \cap \bar{C} \cap F \cap \{k: \text{ odd}\}) = \{l, n \in \mathbf{Z}| l = -2n + k, 1 \leq n \leq \lfloor \frac{k}{2} \rfloor - 1\}, g(l, n)\) has color \(s\) poles at \(u = -h_1 + \frac{s-5}{2} + k - 2n + u_p(s), u = -h_1 + \frac{s-1}{2} + k - 2n + u_p(s)\) and \(u = -h_1 + \frac{s-3}{2} + k + u_p(s)\), and these poles are canceled by \(g(k - 2n - 2, n + 1), g(k - 2n + 2, n - 1)\) and \(g(k - 2n, n - 1)\) respectively under the BAE (3.1).

(11) For \((\bar{B} \cap E \cap \bar{G} \cap \{k: \text{ even}\}) \cup (\bar{A} \cap E \cap \bar{G} \cap \{k: \text{ odd}\}) = \{l, n \in \mathbf{Z}| l = -2n + k - 1, 1 \leq n \leq \lfloor \frac{k+1}{2} \rfloor - 1\}, g(l, n)\) has color \(s\) poles at \(u = -h_1 + \frac{s+5}{2} + k - 2n + u_p(s), u = -h_1 + \frac{s-3}{2} + k - 2n + u_p(s)\) and \(u = -h_1 + \frac{s-5}{2} + k + u_p(s)\), and these poles are canceled by \(g(k - 2n - 3, n + 1), g(k - 2n + 1, n - 1)\) and \(g(k - 2n - 1, n - 1)\) respectively under the BAE (3.1).

(12) For \(C \cap F = C = \{l, n \in \mathbf{Z}| l = k, n = 0\}, g(l, n)\) has color \(s\) poles at \(u = -h_1 + \frac{s-5}{2} + k + u_p(s)\) and \(u = -h_1 + \frac{s-3}{2} + k + u_p(s)\), and these poles are canceled by \(g(k - 2, 1)\) and \(g(k - 1, 0)\) respectively under the BAE (3.1).

Thus, under the BAE (3.1), \(\sum_{n=0}^{\lfloor \frac{k}{2} \rfloor} \sum_{l=0}^{k-2n} g(l, n)\) is free of color \(s\) poles, so is \(S_k\).

4 An extension of the DVF \(T_m^{(1)}(u)\)

In the previous section, we have given the DVF \(T_m^{(1)}(u)\) (3.27) only for \(m \in \mathbf{Z}(1 \leq m \leq s - 1)\). In this section, we shall extend \(T_m^{(1)}(u)\) for \(m \in \mathbf{C}\). Hereafter, we will consider the case where the quantum space is formally trivial. In this case, the vacuum part of the function \(\bar{\mathfrak{M}}_k\) (3.9) is constantly 1.

We assume \(T_m^{(1)}(u)\) for \(m \geq s - 1 (m \in \mathbf{Z})\) is given by using a deformation of \(T_{s-1}^{(1)}(u)\):

\[
\mathcal{T}_m(u) = \frac{Q_1(u - \frac{m}{2})}{Q_1(u + \frac{m}{2} - s + 1)} \times T_{s-1}^{(1)}(u + \frac{m - s + 1}{2}). \tag{4.54}
\]

For \(m = s, s + 1, \ldots, 2s - 2,\)

\[
T_m^{(1)}(u) = \mathcal{T}_m(u) - T_{2s-2-m}^{(1)}(u) \tag{4.55}
\]
Figure 6: Partial Bethe-strap structure of $g(l,n)$ for color $s$ poles ($k = 7$: odd case): Each term of $g(l,n)$ is marked as some symbol: (1) $\triangleright$, (2) $\oplus$, (3) $\ominus$, (4) $\otimes$, (5) $\otimes$, (6) $\odot$, (7) $\llcorner$, (8) $\ast$, (9) $\ast$, (10) $\odot$, (11) $\bullet$, (12) $\diamondsuit$. The terms connected by the arrow have a common pole, and this common pole vanishes under the BAE (3.1).
Figure 7: Partial Bethe-strap structure of $g(l, n)$ for color $s$ poles ($k = 6$: even case): Each term of $g(l, n)$ is marked as some symbol: (1) $\triangleright$, (2) $\oplus$, (3) $\ominus$, (4) $\otimes$, (5) $\odot$, (6) $\odot$, (7) $\triangleleft$, (8) $\star$, (9) $\ast$, (10) $\odot$, (11) $\bullet$, (12) $\diamond$. The terms connected by the arrow have a common pole, and this common pole vanishes under the BAE (3.1).
and for \( m = 2s - 1, 2s, \ldots \),
\[
T_{m}^{(1)}(u) = T_{m}(u). \tag{4.56}
\]

This deformation is compatible with the top term hypothesis \([5, 6]\). In fact the top term of \( T_{m}^{(1)}(u) \) will be \( \frac{Q_{1}(u - \frac{m}{2})}{Q_{1}(u + \frac{m}{2})} \), which carries \( C(s) \) weight \( m\omega_{1} \). We have confirmed, for several cases, the fact that this top term generates \( T_{m}^{(1)}(u) \) (cf. Figure 2, Figure 4). Moreover, we have also checked, for several cases, that the number of the term in \( T_{m}^{(1)}(u) \) coincides with \( \text{dim}V(m\omega_{1}) \) (cf. Table 2, Table 3). Furthermore \( T_{m}^{(1)}(u) \) is free of pole under the BAE (3.1) (see later). We remark that the right hand side of (4.54) is given as a summation of the tableaux of the form
\[
(-1)^{s-1} \sum_{k=1}^{p(i_{k})} \prod_{i=1}^{s-1} \begin{array}{l}
1 \\
m-s+1 \\
\vdots \\
1 \\
\vdots \\
s-1 \\
i_{s-1}
\end{array}
\tag{4.57}
\]
where \( \{i_{k}\} \in B((s-1)^{1}) \); the spectral parameter \( u \) is shifted as \( u - \frac{m-1}{2}, u - \frac{m-3}{2}, \ldots, u + \frac{m-1}{2} \) from the left to the right. In all these tableaux, the left side is always occupied by the same number 1. This circumstance resembles the \( sl(1|s-1) \) case \([10]\).

For \( m = s - 1, s, \ldots, 2s - 2 \), every term in \( T_{2s-m-2}^{(1)}(u) \) coincides \(^{21}\) with a term in \( T_{m}(u) \). This fact can be verified as follows. Noting the relation,
\[
\prod_{i=1}^{s-1} \begin{array}{l}
1 \\
m-s+1 \\
\vdots \\
1 \\
\vdots \\
s-1 \\
i_{s-1}
\end{array} = 1, \tag{4.59}
\]
we find
\[
(-1)^{s-1} \sum_{k=1}^{p(i_{k})} \prod_{i=1}^{s-1} \begin{array}{l}
1 \\
m-s+1 \\
\vdots \\
1 \\
\vdots \\
s-1 \\
i_{s-1}
\end{array} \times \begin{array}{l}
1 \\
2s-2-m \\
\vdots \\
1 \\
\vdots \\
m-s+1
\end{array} = 1
\tag{4.58}
\]
\(^{21}\)We may trace this fact back to the observation (cf \([5]\)) that the DVF has the following form
\[
\sum \frac{Q_{a_{1}}(u + \xi_{1}) \cdots Q_{a_{k}}(u + \xi_{k})}{Q_{a_{1}}(u + \eta_{1}) \cdots Q_{a_{k}}(u + \eta_{k})} \tag{4.58}
\]
if one neglect the signs originated from the grading (3.8) since the transfer matrix is defined as a super-trace of a monodromy matrix.
\[
(-1)^\sum_{k=1}^{2s-2-m} p(i_k) \times \frac{\prod_{i_1} \cdots \prod_{i_{2s-2-m}}}{\prod_{i_s-2}}
\]  \hspace{2cm} (4.60)

where the spectral parameter \( u \) is shifted as \( u - \frac{m-1}{2}, u - \frac{m+1}{2}, \ldots, u + \frac{m-1}{2} \) from the left to the right on the left hand side of (4.60); \( u - \frac{2s-m-3}{2}, u - \frac{2s-m-5}{2}, \ldots, u + \frac{2s-m-3}{2} \) from the left to the right on the right hand side of (4.60). Apparently, the left hand side of (4.60) coincides with a term in \( T_m(u) \) (see (4.57)) if \( \{i_j\} \in B((2s - m - 2)^1) \).

In the DVF \( T_{m}^{(1)}(u) \), we assume \( m \in \mathbb{Z}_{\geq 0} \). However, in view of the fact that one can construct finite dimensional module whose first Kac-Dynkin label is complex number, \( T_{m}^{(1)}(u) \) will be also valid for \( m \in \mathbb{C} \) by ‘analytic continuation’. Namely, we assume that the DVF whose top term carries \( C(s) \) weight \( \omega_1 \) \( (c \in \mathbb{C}; c \neq 0, 1, \ldots, s - 2, s, s + 1, \ldots, 2s - 2) \) is given by the right hand side of the (4.54) for \( m = c \in \mathbb{C} \). \(^{22}\)

Then we find the following theorem, which is a generalization of Theorem 3.2.

**Theorem 4.1** For any \( c \in \mathbb{C} \), the DVF \( T_{c}^{(1)}(u) \) is free of poles under the condition that the BAE (3.1) is valid.

**Proof.** Thanks to the Theorem 3.2, \( T_{s-1}^{(1)}(u + \frac{c-s+1}{2}) \) and \( T_{2s-2-m}^{(1)}(u) \) \( (m \in \{s, s+1, \ldots, 2s-2\}) \) are free of poles under the BAE (3.1). Then we have only to show that the function \( T_{c}(u) \), i.e. \( (4.54) \) for \( m = c \in \mathbb{C} \), is free of pole at \( u = u_{k}^{(1)} - \frac{c}{2} - s - 1 \) \( (k = 1, \ldots, N_1) \). We will show that

\[
T_{s-1}^{(1)}(u + \frac{c-s+1}{2}) = \sum_{\{i_k\} \in B((s-1)^1)} (-1)^{\sum_{k=1}^{s-1} p(i_k)} \times \frac{\prod_{i_1} \cdots \prod_{i_{s-1}}}{\prod_{i_s-2}}
\]  \hspace{2cm} (4.61)

is divisible by \( Q_{1}(u + \frac{c}{2} - s + 1) \). In the set \( \{a: a \in \mathbb{J}, \xi \in \mathbb{C} \} \), only \( \prod_{i_{s-1} + \frac{c-1}{2} - s} \in \mathbb{J}_{u + \frac{c}{2} - s} \) and \( \prod_{i_{s-1} + \frac{c-1}{2} - s} \) have \( Q_{1}(u + \frac{c}{2} - s + 1) \) in their numerators. So we have only to show that every term in (4.61) contains at least one of them. Then all we have to do is to show that \( i_1 = 2 \) or \( i_{s-1} = 2 \) in (4.61) since the argument of \( i_j \) \( (j = 1, 2, \ldots, s-1) \)

\(^{22}\)Note that this weight corresponds to typical representation.

\(^{23}\)See Appendix A for an example of \( T_{c}^{(1)}(u) \) for \( c \in \mathbb{C} \).
in (4.61) becomes \( u + \frac{c+3}{2} - s \) (resp. \( u + \frac{c-1}{2} \)) only when it’s subscript is \( j = 1 \) (resp. \( j = s - 1 \)).

Now we assume \( i_1 \neq 1, 2 \) and \( i_{s-1} \neq \bar{1}, \bar{2}, \) which will lead contradiction. From the admissibility conditions (3.28)-(3.30), there is at least one \( d \in J \) such that both \( d \) and \( \bar{d} \) appear in the row since \( i_j \in \{3, 4, \ldots, \bar{s}, \ldots, \bar{3}\} \) and the length of the row is \( s - 1 \). If \( d_{\text{min}} \) is minimum of such \( d \), then we find that the tableaux in right hand side of (4.61) have the following form:

\[
\begin{array}{c|c|c|c|c}
\xi & d_{\text{min}} & \eta & d_{\text{min}} & \zeta \\
\end{array}
\]  

(4.62)

where \( \xi \) contains only the elements in \( \{3, 4, \ldots, d_{\text{min}} - 1\} \); \( \zeta \) contains only the elements in \( \{d_{\text{min}} - 1, d_{\text{min}} - 2, \ldots, \bar{3}\} \); \( \beta \) and \( \bar{\beta} \) do not appear simultaneously in \( \xi \) and \( \zeta \). Then the following inequality is valid:

\[
|\xi| + |\zeta| \leq d_{\text{min}} - 3,
\]  

(4.63)

where \( |\xi| \) and \( |\zeta| \) are the length of \( \xi \) and \( \zeta \) respectively. On the other hand, from the admissibility condition (3.28)-(3.30), we have

\[
d_{\text{min}} \leq s - (|\eta| + 1),
\]  

(4.64)

where \( |\eta| \) is the length of \( \eta \). These inequalities lead contradiction:

\[
s - 1 = |\xi| + 2 + |\eta| + |\zeta| \leq (d_{\text{min}} - 3) + 2 + (s - d_{\text{min}} - 1) = s - 2.
\]  

(4.65)

In the proof the Theorem 3.2, we need not make use of the factor \( Q_1(u + \xi - s + 1) \) to prove the fact that \( T^{(1)}_{s-1}(u + \frac{s+1}{2}) \) does not have a color 1 pole under the BAE (3.1). So division by \( Q_1(u + \xi - s + 1) \) does not influence the proof of the pole-freeness of \( T^{(1)}_{s-1}(u + \frac{s+1}{2}) \) under the BAE (3.1). Therefore the function \( T_c(u) \) is free of poles under the BAE (3.1), so is \( T^{(1)}_c(u) \).

To construct a transfer matrix whose eigenvalue formula is given by \( T^{(1)}_c(u) \), one may be able to use the \( R \) matrix which is constructed by tensor product graph method [40].

Using the function \( T_{a-2}(u) \) (4.54), we may define the DVF \( T^{(a)}_1(u) \) \((\in \{2, 3, \ldots, s\})\) whose top term will carry ‘fundamental weight’ \( \omega_a \)

\[
T^{(a)}_1(u) = T_{a-2}(u) - T^{(1)}_{a-2}(u) \quad a \in \{2, 3, \ldots, s\}.
\]  

(4.66)
We remark that the right hand side of (4.54) for \( m = a - 2 \) \((a \in \{2, 3, \ldots s\})\) is given as a summation of the tableaux of the form

\[
(-1)^{s-1} \sum_{k=1}^{s-1} \rho(i_k) \times \begin{array}{c}
\begin{array}{c}
i_1 \cdots i_{s-1} \\[s-1]\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
[1] \cdots [1] \\[s+1-a]\end{array}
\end{array}
\]

(4.67)

where \( \{i_k\} \in B((s-1)^1) \); the spectral parameter \( u \) is shifted as \( u - \frac{2s-a-1}{2}, u - \frac{2s-a-3}{2}, \ldots, u + \frac{2s-a-1}{2} \) from the left to the right. For \( a = 2, 3, \ldots, s \), every term in \( T_{a-2}^{(1)}(u) \) (in the right hand side of (4.66)) coincides with a term in \( T_{a-2}(u) \). This fact can be verified as follows. Noting the relation,

\[
\sum_{j=1}^{a-2} \rho(j_k) \times \begin{array}{c}
\begin{array}{c}
j_1 \cdots j_{a-2} \\[a-2]\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
[1] \cdots [1] \\[s+1-a]\end{array}
\end{array}
\]

(4.69)

we find

\[
(-1)^{a-2} \sum_{k=1}^{a-2} \rho(j_k) \times \begin{array}{c}
\begin{array}{c}
j_1 \cdots j_{a-2} \\[a-2]\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
[1] \cdots [1] \\[s+1-a]\end{array}
\end{array}
\]

(4.70)

where the spectral parameter \( u \) is shifted as \( u - \frac{2s-a-1}{2}, u - \frac{2s-a-3}{2}, \ldots, u + \frac{a-3}{2}, u - \frac{a-5}{2}, \ldots, u + \frac{a-3}{2} \) from the left to the right on the left hand side of (4.69); \( u - \frac{a-3}{2}, u - \frac{a-5}{2}, \ldots, u + \frac{a-3}{2} \) from the left to the right on the tableau in the right hand side of (4.69). Apparently, the left hand side of (4.69) coincides with a term in \( T_{a-2}(u) \) if \( \{j_k\} \in B((a-2)^1) \).

The top term of \( T_1^{(a)}(u) \) (4.66) will be \((-1)^{a-1} \frac{Q_a(u - \frac{1}{ta})}{Q_a(u + \frac{1}{ta})}\), which carries \( C(s) \) weight \( \omega_a \). In fact \( T_1^{(a)}(u) \) contains a term of the form:

\[
(-1)^{a-1} \times \begin{array}{c}
\begin{array}{c}
1 \cdots 1 \\[s-a]\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
2 3 \cdots a \\[a-1]\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
[1] \cdots [1] \\[s+1-a]\end{array}
\end{array}
\]

(4.70)

\[
= (-1)^{a-1} \frac{Q_a(u - \frac{1}{ta})}{Q_a(u + \frac{1}{ta})},
\]

34
where the spectral parameter $u$ is shifted as $u - \frac{2s-a-1}{2}, u - \frac{2s-a-3}{2}, \ldots, u + \frac{2s-a-1}{2}$ from the left to the right on the left hand side of (4.70). Apparently, the left hand side of (4.70) coincides with a term in $T_{a-2}(u)$; dose not formally coincide with any term in $T_{a-2}^{(1)}(u)$. Moreover, we have checked, for several cases, that the number of the terms in $T_{1}^{(a)}(u)$ (4.66) coincides with $\dim V(\omega_{a})$ (cf. Table 2, Table 5).

Owing to the Theorem 4.1, we have:

**Theorem 4.2** For any $a \in \{2, 3, \ldots, s\}$, $T_{1}^{(a)}(u)$ (4.66) is free of poles under the condition that the BAE (3.1) is valid.

### 5 On functional relations among DVFs

Now we briefly mention the functional relations among DVFs. From the admissibility conditions (3.19)-(3.21) and (3.28)-(3.30), the following relation holds.

$$T_{1}^{(1)}(u - \frac{1}{2})T_{1}^{(1)}(u + \frac{1}{2}) = T_{1}^{(2)}(u) + T_{2}(u).$$

(5.71)

This functional relation may be related to specialization of the Hirota bilinear difference equation [45]. Note however that there is another functional relation, which comes from a one parameter family of finite dimensional representations. For instance, $T_{c}(u)$ satisfies

$$T_{c}(u - \frac{d}{2})T_{c}(u + \frac{d}{2}) = T_{c-d}(u)T_{c+d}(u),$$

(5.72)

where $c, d \in \mathbb{C}$. As special cases, this reduces to the following relations:

$$
(T_{m}^{(1)}(u - \frac{1}{2}) + T_{1}^{(m+1)}(u))(T_{m+1}^{(1)}(u) + T_{1}^{(m+2)}(u)) = (T_{m+1}^{(1)}(u) + T_{1}^{(m+1)}(u))(T_{m+1}^{(1)}(u) + T_{1}^{(m+3)}(u)) \quad m \in \{1, 2, \ldots, s-3\},
$$

$$
(T_{s-2}^{(1)}(u - \frac{1}{2}) + T_{1}^{(s)}(u - \frac{1}{2}))(T_{s-2}^{(1)}(u + \frac{1}{2}) + T_{1}^{(s)}(u + \frac{1}{2}))
$$

(5.74)

24 This functional relation is not complete one. In fact, we do not know general expressions of this type of functional relations for $T_{m}^{(1)}(u)$ ($m \in \{2, 3, \ldots, 2s-1\}$) since we are lack of DVFs labelled by Young superdiagrams with shape $(n^{a})$ for $m, a \geq 2$. We left this point for future study.
Let $T_m^{(a)}(u) \ (m \in \mathbb{Z}_{\geq 0}, \ a \in \{2, 3, \ldots, s\})$ be the DVF $^{25}$ whose top term $^{26}$ carries $C(s)$ weight $m \omega_a$ ($a \in \{2, 3, \ldots, s\}$). As for specific values of $(m,a)$, we have already given the expression of $T_m^{(a)}(u)$. In general, we conjecture that $T_m^{(a)}(u)$ is given as a solution of the following set of functional relations:

For $C(3); \ m \in \mathbb{Z}_{\geq 1}$,

$$
T_m^{(1)}(u) = (T_{m-3}^{(1)}(u) + T_1^{(s-1)}(u))T_{s-1}^{(1)}(u),
$$

$$
T_s^{(1)}(u - \frac{1}{2})T_{s-1}^{(1)}(u + \frac{1}{2}) = (T_{s-2}^{(1)}(u) + T_1^{(s)}(u))(T_s^{(1)}(u) + T_{s-2}^{(1)}(u)),
$$

$$
(T_s^{(1)}(u - \frac{1}{2}) + T_{s-2}^{(1)}(u - \frac{1}{2}))(T_s^{(1)}(u + \frac{1}{2}) + T_{s-2}^{(1)}(u + \frac{1}{2})) = T_{s-1}^{(1)}(u)(T_{s+1}^{(1)}(u) + T_{s-3}^{(1)}(u)),
$$

$$
(T_m^{(1)}(u - \frac{1}{2}) + T_{2s-m-2}^{(1)}(u - \frac{1}{2}))(T_m^{(1)}(u + \frac{1}{2}) + T_{2s-m-2}^{(1)}(u + \frac{1}{2})) = (T_{m-1}^{(1)}(u) + T_{m-1}^{(1)}(u))T_{m+1}^{(1)}(u) + T_{2s-m-3}^{(1)}(u)
$$

$$
m \in \{s + 1, s + 2, \ldots, 2s - 3\},
$$

$$
(T_{2s-2}^{(1)}(u - \frac{1}{2}) + 1)(T_{2s-2}^{(1)}(u + \frac{1}{2}) + 1) = T_{2s-1}^{(1)}(u)(T_{2s-3}^{(1)}(u) + T_1^{(1)}(u)),
$$

$$
T_{2s-1}^{(1)}(u - \frac{1}{2})T_{2s-1}^{(1)}(u + \frac{1}{2}) = (T_{2s-2}^{(1)}(u) + 1)T_{2s}^{(1)}(u),
$$

$$
T_m^{(1)}(u - \frac{1}{2})T_m^{(1)}(u + \frac{1}{2}) = T_{m-1}^{(1)}(u)T_{m+1}^{(1)}(u) \ m \in \{2s, 2s + 1, \ldots\}.
$$

$^{25}$We assume $T_0^{(a)}(u) = 1$.

$^{26}$In this case, the top term is supposed to be proportional to $\frac{Q_a(u - \frac{1}{2})}{Q_a(u + \frac{1}{2})}$. 

36
For $C(s)$; $s \geq 4$; $m \in \mathbb{Z}_{\geq 1}$,

\begin{align*}
T^{(1)}_{-m}(u - \frac{1}{2})T^{(1)}_{-m}(u + \frac{1}{2}) &= \begin{cases} 
T^{(1)}_{-2}(u)(T^{(2)}_{m}(u) + 1) & \text{for } m = 1 \\
T^{(1)}_{-m+1}(u)T^{(1)}_{-m-1}(u) & \text{for } m \in \mathbb{Z}_{\geq 2},
\end{cases} 
(5.85)
T^{(2)}_{m}(u - \frac{1}{2})T^{(2)}_{m}(u + \frac{1}{2}) = T^{(2)}_{m+1}(u)T^{(2)}_{m-1}(u) + T^{(1)}_{m}(u)T^{(3)}_{m}(u),
(5.86)
T^{(a)}_{m}(u - \frac{1}{2})T^{(a)}_{m}(u + \frac{1}{2}) = T^{(a)}_{m+1}(u)T^{(a)}_{m-1}(u) + T^{(a-1)}_{m}(u)T^{(a+1)}_{m}(u) \\
&\quad 3 \leq a \leq s - 2,
(5.87)
T^{(s-1)}_{2m}(u - \frac{1}{2})T^{(s-1)}_{2m}(u + \frac{1}{2}) = T^{(s-1)}_{2m+1}(u)T^{(s-1)}_{2m-1}(u) \\
&\quad + T^{(s-2)}_{2m}(u)T^{(s)}_{m}(u - \frac{1}{2})T^{(s)}_{m}(u + \frac{1}{2}),
(5.88)
T^{(s-1)}_{2m-1}(u - \frac{1}{2})T^{(s-1)}_{2m-1}(u + \frac{1}{2}) = T^{(s-1)}_{2m}(u)T^{(s-1)}_{2m-2}(u) \\
&\quad + T^{(s-2)}_{2m-1}(u)T^{(s)}_{m-1}(u)T^{(s)}_{m}(u),
(5.89)
T^{(s)}_{m}(u - 1)T^{(s)}_{m}(u + 1) = T^{(s)}_{m+1}(u)T^{(s)}_{m-1}(u) + T^{(s-1)}_{m}(u).
(5.90)
\end{align*}

Remark: Apparently the solutions of these functional relations are not polynomials of $T^{(1)}_{-1}, T^{(2)}_{1}, \ldots, T^{(s)}_{1}$ but rational functions. Nevertheless we have confirmed, for several cases, the fact that the solutions have the form (4.58) if we express $T^{(1)}_{-1}, T^{(2)}_{1}, \ldots, T^{(s)}_{1}$ by using the $Q_{a}$-functions. We also note that these functional relations have determinant or pfaffian solutions whose matrix elements are $T^{(1)}_{-1}, T^{(2)}_{1}, \ldots, T^{(s)}_{1}$ if one change the boundary conditions for $m = 1$ in (5.81) and (5.85) to $T^{(1)}_{-1}(u - \frac{1}{2})T^{(1)}_{-1}(u + \frac{1}{2}) = T^{(1)}_{-2}(u)$.

As an example, we give the number of the terms in $T^{(a)}_{m}(u)$ for $C(3)$ for several cases (see Table 3 and Table 5). In general, we conjecture that the number of the terms in $T^{(a)}_{m}(u)$ is given as follows:

\begin{equation}
N^{(a)}_{m} = \begin{cases} 
\dim V(m\omega_1) & \text{if } a = 1, \\
\sum_{\{k_j\} \in K_{(a,m)}} \dim V(-k_1\omega_1 + k_2\omega_2 + \cdots + k_a\omega_a) & \text{if } a \in \{2, 3, \ldots, s - 1\}, \\
\dim V(m\omega_s) & \text{if } a = s,
\end{cases}
(5.91)
\end{equation}

where $K_{(a,m)} = \{\{k_j\} \mid k_1 + k_2 + \cdots + k_a \leq m, k_j \equiv m\delta_{ja} \pmod{2}, k_j \in \mathbb{Z}_{\geq 0}\}$. For example, for $C(3)$ case, we have (cf. Tables 2, 3, 5).
Table 5: The number $N_m^{(a)}$ of the terms in $T_m^{(a)}(u)$ ($a = 2, 3$) for $C(3)$.

| $m$ | 1   | 2   | 3   | 4   | 5   |
|-----|-----|-----|-----|-----|-----|
| $N_m^{(2)}$ | 15  | 65  | 175 | 385 | 735 |
| $N_m^{(3)}$ | 10  | 35  | 84  | 165 | 286 |

\[
\begin{align*}
N_l^{(1)} &= \dim V(l\omega_1) \quad l \in \mathbb{R}, \\
N_l^{(2)} &= \dim V(\omega_2), \\
N_2^{(2)} &= \dim V(-2\omega_1) + \dim V(2\omega_2), \\
N_3^{(2)} &= \dim V(-2\omega_1 + \omega_2) + \dim V(3\omega_2), \\
N_4^{(2)} &= \dim V(-4\omega_1) + \dim V(-2\omega_1 + 2\omega_2) + \dim V(4\omega_2), \\
N_5^{(2)} &= \dim V(-4\omega_1 + \omega_2) + \dim V(-2\omega_1 + 3\omega_2) + \dim V(5\omega_2), \\
&\vdots \\
N_m^{(3)} &= \dim V(m\omega_3) \quad m \in \mathbb{Z}_{\geq 0}.
\end{align*}
\]

These relations seem to suggest decompositions of the auxiliary spaces similar to (3.26).

6 Summary and discussion

In the present paper, we have executed an analytic Bethe ansatz based on the Bethe ansatz equation (3.1) with distinguished simple root system of the type 1 Lie superalgebra $C(s)$. Eigenvalue formulae of transfer matrices in DVF’s are proposed not only for tensor-like representations but also for a one parameter family of finite dimensional representations. The key is the top term hypothesis and pole-freeness under the BAE. Pole-freeness of the DVF was proven. Functional relations have been discussed for the DVF’s. To the author’s knowledge, this paper is the first systematic treatment of the analytic Bethe ansatz related to the Lie superalgebra $C(s)$.

The Lie superalgebras or their quantum analogues are not straightforward extension of their non-super counterparts. They have several inequivalent sets of simple root systems. In view of this fact, we discussed [9] relations
among sets of the Bethe ansatz equations for any simple root systems in relation to the Weyl supergroup for $sl(r+1|s+1)$ case. A similar argument will be also valid for $C(s)$ case.

It will be an interesting problem to extend similar analysis for the type 2 Lie superalgebras $B(r|s)$ and $D(r|s)$. In this case, there is no parameter family of finite dimensional representations. To construct DVFs related to spinorial representations of the type 2 Lie superalgebras will be rather cumbersome problem.

Analytic Bethe ansatz attracts our interest not only in the context of solvable lattice models but also representation theory in mathematics. DVFs in analytic Bethe ansatz may be viewed \[5, 6\] as characters of representations of Yangians. A remarkable coincidence between currents of deformed Virasoro algebras and DVFs was pointed out in Ref. \[46\]. We hope that this paper not only provides us practical formulae in statistical physics but also contributes to the development of such interplay between mathematics and physics.

**Acknowledgment**

The author would like to thank Prof. A. Kuniba for discussions. This work is supported by a Grant-in-Aid for JSPS Research Fellows from the Ministry of Education, Science and Culture of Japan.

**Appendix A An example of the DVF**

In this section \[27\], we present an example of the DVF $T_c^{(1)}(u)$ ($c \in \mathbb{C}; c \neq 0, 1, 3, 4$) and the Theorem 4.1 for $C(3); J_+ = \{1, \bar{1}\}; J_- = \{2, 3, 3, \bar{2}\}$ case:

$$\begin{align*}
T_c^{(1)}(u) &= \frac{Q_1(\frac{-c}{2} + u)}{Q_1(\frac{c}{2} - 2 + u)} T_2^{(1)}(u + \frac{c - 2}{2}) \\
&= \frac{Q_1(\frac{-c}{2} + u)}{Q_1(\frac{c}{2} + u)} + \frac{Q_1(\frac{c}{2} + u)}{Q_1(\frac{-c}{2} + u)} + \frac{Q_1(\frac{-c}{2} + u)Q_1(\frac{-4+c}{2} + u)}{Q_1(\frac{-c}{2} + u)Q_1(\frac{-2+c}{2} + u)} \\
&- \frac{Q_1(\frac{-c}{2} + u)Q_2(\frac{-6+c}{2} + u)}{Q_1(\frac{-c}{2} + u)Q_2(\frac{-4+c}{2} + u)} \\
&= \frac{Q_1(\frac{-c}{2} + u)Q_1(\frac{-4+c}{2} + u)Q_2(\frac{-7+c}{2} + u)}{Q_1(\frac{-c}{2} + u)Q_1(\frac{-2+c}{2} + u)Q_2(\frac{-5+c}{2} + u)}
\end{align*}$$

\[27\]We assume that the vacuum part of the DVF is formally trivial.
\[ Q_1\left(\frac{-c}{2} + u\right)Q_1\left(\frac{-1+c}{2} + u\right)Q_2\left(\frac{-1+c}{2} + u\right) \\
+ \frac{Q_1\left(\frac{-c}{2} + u\right)Q_1\left(\frac{-\frac{7}{2}+c}{2} + u\right)Q_2\left(\frac{-\frac{7}{2}+c}{2} + u\right)Q_2\left(\frac{-3+c}{2} + u\right)}{Q_1\left(\frac{-c}{2} + u\right)Q_1\left(\frac{-\frac{5}{2}+c}{2} + u\right)Q_2\left(\frac{-\frac{5}{2}+c}{2} + u\right)Q_2\left(\frac{-3+c}{2} + u\right)} \\
- \frac{Q_1\left(\frac{-c}{2} + u\right)Q_2\left(\frac{1+c}{2} + u\right)}{Q_1\left(\frac{-c}{2} + u\right)Q_2\left(\frac{-1+c}{2} + u\right)Q_1\left(\frac{-3+c}{2} + u\right)Q_3\left(\frac{-5+c}{2} + u\right)} \\
- \frac{Q_1\left(\frac{-c}{2} + u\right)Q_2\left(\frac{-\frac{5}{2}+c}{2} + u\right)Q_3\left(\frac{-9+c}{2} + u\right)}{Q_1\left(\frac{-c}{2} + u\right)Q_2\left(\frac{-\frac{3}{2}+c}{2} + u\right)Q_3\left(\frac{-7+c}{2} + u\right)} \\
- \frac{Q_1\left(\frac{-c}{2} + u\right)Q_2\left(\frac{-\frac{7}{2}+c}{2} + u\right)Q_3\left(\frac{-7+c}{2} + u\right)}{Q_1\left(\frac{-c}{2} + u\right)Q_2\left(\frac{-\frac{5}{2}+c}{2} + u\right)Q_3\left(\frac{-3+c}{2} + u\right)} \\
+ \frac{Q_1\left(\frac{-c}{2} + u\right)Q_2\left(\frac{-\frac{7}{2}+c}{2} + u\right)Q_3\left(\frac{-7+c}{2} + u\right)}{Q_1\left(\frac{-c}{2} + u\right)Q_2\left(\frac{-\frac{5}{2}+c}{2} + u\right)Q_3\left(\frac{-3+c}{2} + u\right)} \\
+ \frac{Q_1\left(\frac{-c}{2} + u\right)Q_2\left(\frac{-\frac{5}{2}+c}{2} + u\right)Q_3\left(\frac{-\frac{7}{2}+c}{2} + u\right)}{Q_1\left(\frac{-c}{2} + u\right)Q_2\left(\frac{-\frac{3}{2}+c}{2} + u\right)Q_3\left(\frac{-3+c}{2} + u\right)} \\
- \frac{Q_1\left(\frac{-c}{2} + u\right)Q_2\left(\frac{1+c}{2} + u\right)Q_3\left(\frac{1+c}{2} + u\right)}{Q_1\left(\frac{-c}{2} + u\right)Q_2\left(\frac{-\frac{1}{2}+c}{2} + u\right)Q_3\left(\frac{-3+c}{2} + u\right)} \\
- \frac{Q_1\left(\frac{-c}{2} + u\right)Q_3\left(\frac{1+c}{2} + u\right)}{Q_1\left(\frac{-c}{2} + u\right)Q_2\left(\frac{-\frac{1}{2}+c}{2} + u\right)Q_3\left(\frac{-3+c}{2} + u\right)}, \tag{A.1}
\]

where \( T_3^{(1)}(u) \) is given in (3.36). The first term in the right hand side of (A.1) is the top term, which is related to the highest weight \( c\omega_1 \). Although the DVF (A.1) depends on a continuous parameter \( c \), thanks to Theorem 4.1, (A.1) is free of pole under the BAE (3.37).

**Appendix B** \( C(2) \simeq sl(1|2) \) case

We mainly considered the DVFs related to \( C(s) \) (\( s \geq 3 \)) in the main text. Many of the formulae in the main text are also valid for \( C(2) \) case.\(^{28}\) For

\(^{28}\) In this section, we assume that the vacuum part of the DVF is formally trivial.
Figure 8: Dynkin diagram for the Lie superalgebra $C(2) \simeq sl(1|2)$ corresponding to the distinguished simple root system: white circle denotes even root; grey (a cross) circle denotes odd root $\alpha$ with $\langle \alpha|\alpha \rangle = 0$.

$C(2)$; $m \in \mathbb{Z}_{\geq 1}$, the relations (5.81)-(5.90) take the following form:

$$T_{m}^{(1)}(u) - \frac{1}{2} T_{m}^{(1)}(u + \frac{1}{2} ) = \begin{cases} T_{m}^{(1)}(u)(T_{m+1}^{(2)}(u) + 1) & \text{for } m = 1, \\ T_{m+1}^{(1)}(u)T_{m}^{(1)}(u) & \text{for } m \in \mathbb{Z}_{\geq 2}, \end{cases}$$

(B.1)

$$T_{m}^{(2)}(u - 1)T_{m}^{(2)}(u + 1) = T_{m+1}^{(2)}(u)T_{m-1}^{(2)}(u) + T_{m-2}^{(1)}(u).$$

(B.2)

Now we briefly mention special cases of the results [8, 9, 10] concerning $sl(1|2)$ and point out relation to $C(2)$ case. The distinguished simple root system of $sl(1|2)$ is $\{\epsilon^* - \delta_1^*, \delta_1^* - \delta_2^*\}$ (see Figure 8), where $\langle \epsilon^*|\epsilon^* \rangle = 1$, $\langle \epsilon^*|\delta_j^* \rangle = 0$, $\langle \delta_j^*|\delta_j^* \rangle = -\delta_{ij}$, $\epsilon^* - \delta_1^* - \delta_2^* = 0$. Let $F_c^{(1)}(u)$ and $F_m^{(2)}(u)$ be the DVFs whose top term carry $sl(1|2)$ weights $c\epsilon^*$ and $-m\delta_2^*$ respectively. By using the functions:

$$1 = \frac{Q_1(u - 1)}{Q_1(u + 1)}, \quad 2 = \frac{Q_1(u - 1)Q_2(u + 2)}{Q_1(u + 1)Q_2(u)},$$

(B.3)

and

$$-1 = \frac{Q_1(u)}{Q_1(u - 2)}, \quad -2 = \frac{Q_1(u)Q_2(u - 3)}{Q_1(u - 2)Q_2(u - 1)},$$

(B.4)

$29$We assume that these top terms are given respectively as follows: $\frac{Q_1(u - \frac{t_1}{t_1})}{Q_1(u + \frac{t_2}{t_1})}$,

$(-1)^m \frac{Q_2(u - \frac{m}{t_2})}{Q_2(u + \frac{m}{t_1})}$, where $t_1 = 1, t_2 = -1$ (cf. $C(2)$ case: $t_1 = 2, t_2 = -1$).

$30$Kac-Dynkin labels $\{b_1, b_2\}$ of these weights are $\{c, 0\}$ and $\{0, m\}$ respectively.
they are given as follows:

\[ F_0^{(1)}(u) = F_0^{(2)}(u) = 1, \]
\[ F_1^{(1)}(u) = \begin{array}{c}
1 \\
2 \\
3
\end{array} \] (B.5)

\[ F_c^{(1)}(u) = \frac{Q_1(u-c)}{Q_1(u+c-4)} F_2^{(1)}(u+c-2) \]
\[ = \frac{Q_1(u-c)}{Q_1(u+c+2)} F_{-1}^{(1)}(u+c+1) \quad c \neq 0, 1 (c \in \mathbb{C}), \] (B.7)

where

\[ F_2^{(1)}(u) = \begin{array}{cccc}
1 & 1 & -1 & 2 \\
-1 & 3 & 2 & 3
\end{array} \]
\[ F_{-1}^{(1)}(u) = \begin{array}{cccc}
-3 & -2 & -3 & -1 \\
-2 & -1 & -1 & -1
\end{array} \] (B.9)

Here the spectral parameter \( u \) is shifted as \( u - 1, u + 1 \) from the left to the right.

\[ F_m^{(2)}(u) = \sum_{\{i_k\}} (-1)^{\sum_{k=1}^{m} p(i_k)} \begin{array}{c}
i_1 \\
i_1 \\
\vdots \\
i_m
\end{array} \text{ for } m \in \mathbb{Z}_{\geq 1}, \] (B.10)

where the spectral parameter \( u \) is shifted as \( u + m - 1, u + m - 3, \ldots, u - m + 1 \) from the top to the bottom; \( p(-1) = 0, p(-2) = p(-3) = 1 \). Summation is taken over the tableaux \( \{i_j\} \ (i_j \in \{-1, -2, -3\}) \) with the condition: \( i_j \preceq i_{j+1}, \) and \( i_k \prec i_{k+1} \) if \( i_{k+1} = -1 \), where we assume \(-3 \prec -2 \prec -1\). Due to the isomorphism \( C(2) \simeq sl(1|2) \), the following relation will hold:

\[ T_m^{(1)}(u) = F_{\frac{m}{2}}^{(1)}(u), \quad T_m^{(2)}(u) = F_{m}^{(2)}(u). \] (B.11)

Noting this relation, we can verify the functional relation (B.1), i.e.

\[ F_{-\frac{m}{2}}^{(1)}(u - \frac{1}{2}) F_{-\frac{m}{2}}^{(1)}(u + \frac{1}{2}) = \begin{cases}
F_{-1}^{(1)}(u)(F_1^{(2)}(u) + 1) & \text{for } m = 1, \\
F_{-m+1}^{(1)}(u)F_{-m+1}^{(1)}(u) & \text{for } m \in \mathbb{Z}_{\geq 2}.
\end{cases} \] (B.12)
The DVF labelled by the (dotted) Young superdiagram \(^{31}\) with shape \((m^a)\) is given as follows:

\[
\mathcal{F}_m^a(u) = \det_{1 \leq i, j \leq a} \left( \mathcal{F}_{m+i-j}^1(u + a - i - j + 1) \right).
\]  

(B.13)

This satisfies the following functional relation:

\[
\mathcal{F}_m^a(u - 1)\mathcal{F}_m^a(u + 1) = \mathcal{F}_{m-1}^a(u)\mathcal{F}_m^a(u) + \mathcal{F}_{m-1}^a(u)\mathcal{F}_{m+1}^a(u).
\]  

(B.14)

Noting the following relations (B.15) and (B.16),

\[
\mathcal{F}_m^a(u) = 0 \quad \text{if} \quad m \in \mathbb{Z}_{\geq 3} \quad \text{and} \quad a \in \mathbb{Z}_{\geq 2},
\]  

(B.15)

\[
\mathcal{F}_2^a(u) = \mathcal{F}_{a+1}^1(u) \quad \text{for} \quad a \in \mathbb{Z}_{\geq 1},
\]  

(B.16)

we find (B.14) reduces to the following set of functional relations:

\[
\mathcal{F}_1^a(u - 1)\mathcal{F}_1^a(u + 1) = \mathcal{F}_2^1(u) + \mathcal{F}_1^a(u)\mathcal{F}_1^{a+1}(u) \quad a \in \mathbb{Z}_{\geq 1},
\]  

(B.17)

\[
\mathcal{F}_2^a(u - 1)\mathcal{F}_2^a(u + 1) = \mathcal{F}_1^2(u)\mathcal{F}_3^1(u) + \mathcal{F}_3^1(u),
\]  

(B.18)

\[
\mathcal{F}_m^a(u - 1)\mathcal{F}_m^a(u + 1) = \mathcal{F}_{m-1}^a(u)\mathcal{F}_{m+1}^a(u) \quad m \in \mathbb{Z}_{\geq 3}.
\]  

(B.19)

This can be rewritten as

\[
F_{-m}^{(1)}(u - 1)F_{-m}^{(1)}(u + 1) = \begin{cases} 
F_{-m}^{(1)}(u)(F_{1}^{(2)}(u) + 1) & \text{for} \quad m = 1, \\
F_{m+1}^{(1)}(u)F_{-m-1}^{(1)}(u) & \text{for} \quad m \in \mathbb{Z}_{\geq 2},
\end{cases}
\]  

(B.20)

\[
F_{m}^{(2)}(u - 1)F_{m}^{(2)}(u + 1) = F_{m+1}^{(2)}(u)F_{m-1}^{(2)}(u) + F_{m}^{(1)}(u) \quad m \in \mathbb{Z}_{\geq 1},
\]  

(B.21)

where \(F_{-m}^{(1)}(u) = \mathcal{F}_2^m(u) = \mathcal{F}_{m+1}^1(u)\) and \(F_{m}^{(2)}(u) = \mathcal{F}_1^m(u)\). Noting the relations (B.11), we find (B.21) is equivalent to (B.2).

\(^{31}\)(Dotted) Young superdiagram \(\mu = (\mu_1, \mu_2, \ldots)\) is related to Kac-Dynkin label \((b_1, b_2)\) of \(sl(1|2)\) as follows: \(b_1 = -\xi_1 - \mu_2, b_2 = \mu_1 - \mu_2\), where \(\xi_1 = \text{Max}(\mu_1 - 2, 0)\).
References

[1] N. Yu. Reshetikhin, Sov. Phys. JETP 57 (1983) 691.
[2] N. Yu. Reshetikhin, Lett. Math. Phys. 14 (1987) 235.
[3] V. G. Drinfel’d, Sov. Math. Dokl 36 (1988) 212.
[4] V. V. Bazhanov, N. Yu. Reshetikhin, J. Phys. A Math. Gen. 23 (1990) 1477.
[5] A. Kuniba, J. Suzuki, Commun. Math. Phys. 173 (1995) 225.
[6] A. Kuniba, Y. Ohta, J. Suzuki, J. Phys. A: Math. Gen. 28 (1995) 6211.
[7] J. Suzuki, Phys. Lett. A195 (1994) 190.
[8] Z. Tsuboi, J. Phys. A: Math. Gen. 30 (1997) 7975.
[9] Z. Tsuboi, Physica A 252 (1998) 565.
[10] Z. Tsuboi, J. Phys. A: Math. Gen. 31 (1998) 5485.
[11] V. Kac, Adv. Math. 26 (1977) 8.
[12] T. Deguchi, A. Fujii, K. Ito, Phys. Lett. B 238 (1990) 242.
[13] Z. Maassarani, J. Phys. A: Math. Gen. 28 (1995) 1305.
[14] P. B. Ramos, M. J. Martins, Nucl. Phys. B 474 (1996) 678.
[15] M. P. Pfannmüller, H. Frahm, Nucl. Phys. B 479 (1996) 575.
[16] V. V. Bazhanov, A. G. Shadrikov, Theor. Math. Phys. 73 (1988) 1302.
[17] F. H. L. Essler, V. E. Korepin, Phys. Rev. B 46 (1992) 9147.
[18] A. Foerster, M. Karowski, Nucl. Phys. B 396 (1993) 611.
[19] M. J. Martins, P. B. Ramos, Nucl. Phys. B 500 (1997) 579.
[20] N. Yu. Reshetikhin, P. B. Wiegmann, Phys. Lett. B189 (1987) 125.
[21] E. I. Ogievetsky, P. B. Wiegmann, Phys. Lett. B168 (1986) 360.

[22] P. P. Kulish, J. Sov. Math. 35 (1986) 2648.

[23] A. Kuniba, T. Nakanishi, J. Suzuki, Int. J. Mod. Phys. A9 (1994) 5215.

[24] I. Krichever, O. Lipan, P. Wiegmann, A. Zabrodin, Commun. Math. Phys. 188 (1997) 267.

[25] A. Kuniba, J. Phys. A: Math. Gen. 27 (1994) L113.

[26] A. Klümper, P. Pearce, Physica A183 (1992) 304.

[27] A. Kuniba, J. Suzuki, J. Phys. A: Math. Gen. 28 (1995) 711.

[28] A. Kuniba, T. Nakanishi, J. Suzuki, Int. J. Mod. Phys. A9 (1994) 5267.

[29] A. Kuniba, S. Nakamura, R. Hirota, J. Phys. A: Math. Gen. 29 (1996) 1759.

[30] Z. Tsuboi, A. Kuniba, J. Phys. A Math. Gen. 29 (1996) 7785.

[31] Z. Tsuboi, J. Phys. Soc. Jpn. 66 (1997) 3391.

[32] G. Jüttner, A. Klümper, J. Suzuki, Nucl. Phys. B 512 (1998) 581.

[33] V. Kac, Lecture Notes in Mathematics 676 (1978) 597.

[34] A. B. Balantekin, I. Bars, J. Math. Phys. 22 (1981) 1149.

[35] R. J. Farmer, P. D. Jarvis, J. Phys. A: Math. Gen. 17 (1984) 2365.

[36] M. Gourdin, J. Math. Phys. 27 (1986) 2832.

[37] J. Van der Jeugt, J. Math. Phys. 37 (1996) 4176.

[38] H. Yamane, Publ. RIMS, Kyoto Univ. 30 (1994) 15.

[39] H. Yamane, Preprint q-alg/9603015.

[40] G. W. Delius, M. D. Gould, J. R. Links, Y. Z. Zhang, Int. J. Mod. Phys. A 10 (1995) 3259.
[41] A. N. Kirillov, N. Yu. Reshetikhin, J. Sov. Math. 52 (1990) 3156.
[42] A. Kuniba, Preprint, in Japanese, 1998.
[43] M. Kashiwara, T. Nakashima, J. Algebra 165 (1994) 295.
[44] T. Nakashima, Commun. Math. Phys. 154 (1993) 215.
[45] R. Hirota, J. Phys. Soc. Jpn. 50 (1981) 3787.
[46] E. Frenkel, N. Reshetikhin, Commun. Math. Phys. 178 (1996) 237.