A GENERAL FRAMEWORK FOR NONHOLONOMIC MECHANICS:
NONHOLONOMIC SYSTEMS ON LIE AFFGEBROIDS

DAVID IGLESIAS, JUAN C. MARRERO, D. MARTÍN DE DIEGO, AND DIANA SOSA

ABSTRACT. This paper presents a geometric description of Lagrangian and Hamiltonian systems on Lie affgebroids subject to affine nonholonomic constraints. We define the notion of nonholonomically constrained system, and characterize regularity conditions that guarantee that the dynamics of the system can be obtained as a suitable projection of the unconstrained dynamics. It is shown that one can define an almost aff-Poisson bracket on the constraint AV-bundle, which plays a prominent role in the description of nonholonomic dynamics. Moreover, these developments give a general description of nonholonomic systems and the unified treatment permits to study nonholonomic systems after or before reduction in the same framework. Also, it is not necessary to distinguish between linear or affine constraints and the methods are valid for explicitly time-dependent systems.

CONTENTS

I. Introduction 2
II. Lagrangian and Hamiltonian formalism on Lie affgebroids 4
II.1. Lie algebroids 4
II.2. The prolongation of a Lie algebroid over a fibration 5
II.3. Lie affgebroids 6
II.4. The Lagrangian formalism on Lie affgebroids 8
II.5. The Hamiltonian formalism 10
II.6. The Legendre transformation and the equivalence between the Lagrangian and Hamiltonian formalisms 14
III. Affinely constrained Lagrangian systems 16
IV. Solution of Lagrange-d’Alembert equations 19
IV.1. Projectors 24
IV.2. The constrained Poincaré-Cartan 2-section 27
V. Constrained Hamiltonian Systems and the nonholonomic bracket 28
V.1. Constrained Hamiltonian Systems 28
V.2. The nonholonomic bracket 35
VI. Examples 41
VI.1. Lagrangian systems with linear nonholonomic constraints on a Lie algebroid 41
VI.2. Standard affine nonholonomic Lagrangian systems 43

2000 Mathematics Subject Classification. 17B66, 37J60, 53D17, 70F25, 70H33.
Key words and phrases. Lie algebroids, Lie affgebroids, Lagrangian Mechanics, Hamiltonian Mechanics, Nonholonomic Mechanics, Lagrange-d’Alembert equations, Projectors, AV-bundles, Aff-Poisson brackets, Nonholonomic brackets.
I. INTRODUCTION

Nonholonomic constraints is one of the more fascinating and applied topics actually. First, there are many open questions related with this subject: characterization of the integrability, construction of geometric integrators, stabilization, controllability... But moreover, there is a wide range of applications of this kind of systems in engineering, robotics... (see [2, 6, 29] and references therein).

During the last years, many authors have studied in detail the geometry of nonholonomic systems. Some of them developed a geometric formalism for the most typical nonholonomic systems, the ones determined by a mechanical Lagrangian, that is,

$$L(v_q) = \frac{1}{2} g(v_q, v_q) - V(q), \quad v_q \in T_q Q,$$

where $V : Q \to \mathbb{R}$ is the potential energy on the configuration space $Q$, $g$ is a Riemannian metric on $Q$ and, additionally, the system is subjected to linear constraints on the velocities, expressed as a nonintegrable distribution on $Q$ [6, 19]. The usual formalism for these systems was the use of adequate projections of the Levi-Civita connection associated to $g$ to obtain the equations of motion of the system. Other authors preferred to work on the tangent bundle of the configuration space, which permits to introduce more general Lagrangians and different type of constraints (linear or nonlinear). Usually these systems are specified by a Lagrangian function $L : TQ \to \mathbb{R}$ and a constraint submanifold $M$ of $TQ$ [15, 18]. Moreover, also a description in terms of a nonholonomic bracket was introduced [5, 13, 36] with considerable applications to reduction. In this sense, reduction of nonholonomic systems was intensively studied using different geometric techniques. For instance, introducing modified Ereshmann connections (nonholonomic connection) [3], using a symplectic distribution on the constraint submanifold [1, 35] and distinguishing the different casistic depending on the ‘position’ of the Lie group of symmetries acting on the system and the nonholonomic distribution or in a Poisson context [22] (see [6] and references therein). Moreover, using jet bundle techniques, many authors studied the geometry of nonholonomic systems admitting an extension to explicitly time-dependent nonholonomic systems [4, 14, 16, 20, 33, 34] and ready for extension to nonholonomic field theories [37].

Today we find an scenario with a very rich theory but with an important lack: a general framework unifying the different casistic (unreduced and reduced equations, systems subjected to linear or affine constraints, time-dependent or time-independent systems...).

The aim of the present paper is cover this lack and present a geometrical framework covering the different cases. Obviously, in order to reach this objective it will be necessary to use some new and sophisticated techniques, in particular, the concept of a Lie affgebroid (an affine version of
a Lie algebroid) and appropriate versions of Lagrangian and nonholonomic mechanics in this setting.

The previous and other motivations coming from other fields (topology, algebraic geometry...) have recently caused a lot of interest in the study of Lie algebroids [21], which in our setting can be thought of as ‘generalized tangent bundles’, since it allows to consider, in a unified formalism, mechanical systems on Lie algebras, integrable distributions, tangent bundles, quotients of tangent bundles by Lie groups when an action is considered... In this sense, Lie algebroids can be used to give unified geometric descriptions of Hamiltonian and Lagrangian Mechanics. In [38], A. Weinstein introduced Lagrangian systems on a Lie algebroid $E \to M$ by means of the linear Poisson structure on the dual bundle $E^*$ and a Legendre-type map from $E$ to $E^*$. Motivated by this description, E. Martínez [23] developed a geometric formalism on Lie algebroids extending the Klein’s formalism in ordinary Lagrangian Mechanics on tangent bundles. This line of research has been followed in [17, 24]. In fact, a geometric description of Lagrangian and Hamiltonian dynamics on Lie algebroids, in terms of Lagrangian submanifolds of symplectic Lie algebroids, was given in [17]. More recently, in [7] (see also [8, 27, 28]) a comprehensive treatment of Lagrangian systems on Lie algebroids subject to linear nonholonomic constraints was developed. The proposed formalism allows us to treat in a unified way a variety of situations for time-independent Lagrangian systems subject to linear nonholonomic constraints (systems with symmetry, nonholonomic LL systems, nonholonomic LR systems,...).

On the other hand, in [9, 26] an affine version of the notion of a Lie algebroid structure was introduced. The resultant geometric object is called a Lie affgebroid structure. Lie affgebroid structures may be used to develop a time-dependent version of Lagrange and Hamilton equations on Lie algebroids (see [10, 12, 25, 26, 31]). In fact, one may obtain Lagrange and Hamilton equations on Lie affgebroids using a cosymplectic formalism [12, 25]. In the same setting of Lie affgebroids, one also may obtain the Hamilton equations using the notion of an aff-Poisson structure on an AV-bundle (see [10, 12]). An AV-bundle is an affine bundle of rank 1 modelled on the trivial vector bundle and an aff-Poisson bracket on an AV-bundle may be considered as an affine version of a Poisson bracket on a manifold (see [9]).

The aim of this paper is develop a geometric description of Lagrangian systems subject to affine constraints using the Lie affgebroid theory. The general geometric framework proposed covers the most interesting previous methods in the literature.

The paper is organized as follows. In Section II, we recall several constructions (which will be useful in the sequel) about the geometric description of Lagrangian and Hamiltonian Mechanics on Lie affgebroids and the equivalence between both formalisms in the hyperregular case. In Section III, we introduce the Lagrange-d’Alembert equations for an affine nonholonomic Lagrangian system on a Lie affgebroid. In Section IV, we discuss the existence and uniqueness of solutions for this type of systems. Moreover, we prove that in the regular case the nonholonomic dynamics can be obtained by different projections from the unconstrained dynamics. In Section V, we develop the Hamiltonian description and we discuss the equivalence between the Lagrangian and Hamiltonian formalism. We also introduce the nonholonomic bracket which gives the evolution of an observable for the nonholonomic dynamics. The nonholonomic bracket is an almost aff-Poisson bracket (an aff-Poisson bracket which doesn’t satisfy, in general, "the Jacobi identity") on the constraint AV-bundle. In Section VI, we apply the results obtained in...
In this section we recall some well-known facts concerning the geometry of Lie algebroids and Lagrangian and Hamiltonian formalism on Lie affgebroids. For this purpose, we will use a particular class of Lie affgebroids, namely, Atiyah affgebroids.

II. LAGRANGIAN AND HAMILTONIAN FORMALISM ON LIE AFFGEBROIDS

In this section we recall some well-known facts concerning the geometry of Lie algebroids and Lie affgebroids.

II.1. Lie algebroids. Let $E$ be a vector bundle of rank $n$ over the manifold $M$ of dimension $m$ and $\tau_E : E \to M$ be the vector bundle projection. Denote by $\Gamma(\tau_E)$ the $C^\infty(M)$-module of sections of $\tau_E : E \to M$. A Lie algebroid structure $([\cdot, \cdot]_E, \rho_E)$ on $E$ is a Lie bracket $[\cdot, \cdot]_E$ on the space $\Gamma(\tau_E)$ and a bundle map $\rho_E : E \to TM$, called the anchor map, such that if we also denote by $\rho^0_E : \Gamma(\tau_E) \to \mathfrak{X}(M)$ the homomorphism of $C^\infty(M)$-modules induced by the anchor map then $[X, fY]_E = f[X, Y]_E + \rho_E(X)(f)Y$, for $X, Y \in \Gamma(\tau_E)$ and $f \in C^\infty(M)$. The triple $(E, [\cdot, \cdot]_E, \rho_E)$ is called a Lie algebroid over $M$ (see [21]). In such a case, the anchor map $\rho_E : \Gamma(\tau_E) \to \mathfrak{X}(M)$ is a homomorphism between the Lie algebras $(\Gamma(\tau_E), [\cdot, \cdot]_E)$ and $(\mathfrak{X}(M), [\cdot, \cdot])$.

If $(E, [\cdot, \cdot]_E, \rho_E)$ is a Lie algebroid, one may define a cohomology operator which is called the differential of $E$, $d^E : \Gamma(\wedge^k \tau^*_E) \to \Gamma(\wedge^{k+1} \tau^*_E)$, as follows

$$
(d^E\mu)(X_0, \ldots, X_k) = \sum_{i=0}^{k} (-1)^i \rho_E(X_i)(\mu(X_0, \ldots, \hat{X}_i, \ldots, X_k)) + \sum_{i<j} (-1)^{i+j} \mu([X_i, X_j]_E, X_0, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_k),
$$

(1)

for $\mu \in \Gamma(\wedge^k \tau^*_E)$ and $X_0, \ldots, X_k \in \Gamma(\tau_E)$. Moreover, if $X \in \Gamma(\tau_E)$, one may introduce, in a natural way, the Lie derivative with respect to $X$, as the operator $\mathcal{L}_X^E : \Gamma(\wedge^k \tau^*_E) \to \Gamma(\wedge^k \tau^*_E)$ given by $\mathcal{L}_X^E = i_X \circ d^E + d^E \circ i_X$.

If $E$ is the standard Lie algebroid $TM$ then the differential $d^E = d^{TM}$ is the usual exterior differential associated with $M$. On the other hand, if $E$ is a real Lie algebra $\mathfrak{g}$ of finite dimension then $\mathfrak{g}$ is a Lie algebroid over a single point and the differential $d^\mathfrak{g}$ is the algebraic differential of the Lie algebra.

Now, suppose that $(E, [\cdot, \cdot]_E, \rho_E)$ and $(E', [\cdot, \cdot]_{E'}, \rho_{E'})$ are Lie algebroids over $M$ and $M'$, respectively, and that $F : E \to E'$ is a vector bundle morphism over the map $f : M \to M'$. Then $(F, f)$ is said to be a Lie algebroid morphism if

$$
d^F((F, f)^* \phi') = (F, f)^*(d^{E'}\phi'), \text{ for } \phi' \in \Gamma(\wedge^k(\tau_{E'})^*), \text{ and for all } k.
$$

Note that $(F, f)^* \phi'$ is the section of the vector bundle $\wedge^k E^* \to M$ defined by

$$
((F, f)^* \phi')_x(a_1, \ldots, a_k) = \phi'_{f(x)}(F(a_1), \ldots, F(a_k)),
$$
for $x \in M$ and $a_1, \ldots, a_k \in E_x$. If $(F, f)$ is a Lie algebroid morphism, $f$ is an injective immersion and $F|_{E_x} : E_x \to E'_f(x)$ is injective, for all $x \in M$, then $(E, [\cdot, \cdot]_E, \rho_E)$ is said to be a Lie subalgebroid of $(E', [\cdot, \cdot]_{E'}, \rho_{E'})$.

If we take local coordinates $(x^i)$ on $M$ and a local basis $\{e_\alpha\}$ of sections of $E$, then we have the corresponding local coordinates $(x^i, y^\alpha)$ on $E$, where $y^\alpha(a)$ is the $\alpha$-th coordinate of $a \in E$ in the given basis. Such coordinates determine local functions $\rho_\alpha^i, C^\gamma_{\alpha\beta}$ on $M$ which contain the local information of the Lie algebroid structure, and accordingly they are called the structure functions of the Lie algebroid. They are given by

$$\rho_E(e_\alpha) = \rho_\alpha^i \frac{\partial}{\partial x^i} \quad \text{and} \quad [e_\alpha, e_\beta]_E = C^\gamma_{\alpha\beta} e_\gamma,$$

and they satisfy the following relations

$$\rho_\alpha^i \frac{\partial \rho_\beta^j}{\partial x^i} - \rho_\beta^j \frac{\partial \rho_\alpha^i}{\partial x^i} = \rho_\gamma^i C^\gamma_{\alpha\beta} \quad \text{and} \quad \sum_{\text{cyclic}(\alpha, \beta, \gamma)} \left[ \rho_\alpha^i \frac{\partial C^\mu_{\beta\gamma}}{\partial x^i} + C^\mu_{\beta\gamma} C^\nu_{\alpha\mu} \right] = 0.$$

If $f \in C^\infty(M)$, we have that

$$d^E f = \frac{\partial f}{\partial x^i} \rho_\alpha^i e_\alpha,$$

where $\{e_\alpha\}$ is the dual basis of $\{e_\alpha\}$. Moreover, if $\theta \in \Gamma(\tau^*_E)$ and $\theta = \theta_\gamma e_\gamma$ it follows that

$$d^E \theta = \left( \frac{\partial \theta_\gamma}{\partial x^i} \rho_\beta^i - \frac{1}{2} \partial_\alpha C^\gamma_{\beta\gamma} \right) e_\beta \wedge e_\gamma.$$

II.2. The prolongation of a Lie algebroid over a fibration. In this section, we will recall the definition of the Lie algebroid structure on the prolongation of a Lie algebroid over a fibration (see [11, 17]).

Let $(E, [\cdot, \cdot]_E, \rho_E)$ be a Lie algebroid of rank $n$ over a manifold $M$ of dimension $m$ with vector bundle projection $\tau_E : E \to M$ and $\pi : M' \to M$ be a fibration.

We consider the subset $T^E M'$ of $E \times TM'$ and the map $\tau^*_E : T^E M' \to M'$ defined by

$$T^E M' = \{ (b, v') \in E \times TM'/ \rho_E(b) = (T\pi)(v') \}, \quad \tau^*_E(b, v') = \pi_{M'}(v'),$$

where $T\pi : TM' \to TM$ is the tangent map to $\pi$ and $\pi_{M'} : TM' \to M'$ is the canonical projection. Then, $\tau^*_E : T^E M' \to M'$ is a vector bundle over $M'$ of rank $n + \dim M' - m$ which admits a Lie algebroid structure $(\{\cdot, \cdot\}^*_E, \rho^*_E)$ characterized by

$$[\{X \circ \pi, U'\}, \{Y \circ \pi, V'\}]^*_E = \{ [X, Y]_E \circ \pi, [U', V'] \}, \quad \rho^*_E(X \circ \pi, U') = U',$$

for all $X, Y \in \Gamma(\tau_E)$ and $U', V'$ vector fields which are $\pi$-projectable to $\rho_E(X)$ and $\rho_E(Y)$, respectively. $(T^E M', [\cdot, \cdot]^*_E, \rho^*_E)$ is called the prolongation of the Lie algebroid $E$ over the fibration $\pi$ or the $E$-tangent bundle to $M'$ (for more details, see [11, 17]).

An element $Z \in T^E M'$ is said to be vertical if its projection onto the first factor is zero. Therefore, it is of the form $(0, v_{x'}^i)$, with $v_{x'}^i$ a tangent vector to $M'$ at $x'$ which is $\pi$-vertical.

Next, we consider a particular case of the above construction. Let $E$ be a Lie algebroid over a manifold $M$ with vector bundle projection $\tau_E : E \to M$ and $T^E E^*$ be the prolongation of $E$ over the projection $\tau^*_E : E^* \to M$. $T^E E^*$ is a Lie algebroid over $E^*$ and we can define a canonical section $\lambda_E$ of the vector bundle $(T^E E^*)^* \to E^*$ as follows. If $a^* \in E^*$ and $(b, v) \in T^E_{a^*} E^*$ then

$$\lambda_E(a^*)(b, v) = a^*(b). \tag{2}$$
λ_E is called the Liouville section associated with the Lie algebroid E.

Now, one may consider the nondegenerate 2-section Ω_E = −dTE∗E∗λ_E of (T^E E∗)∗ → E∗. It is clear that dTE∗E∗Ω_E = 0. In other words, Ω_E is a symplectic section. Ω_E is called the canonical symplectic section associated with the Lie algebroid E. Using the symplectic section Ω_E one may introduce a linear Poisson structure Π_E∗ on E∗, with linear Poisson bracket {·, ·}E∗ given by

\[ \{F, G\}_E = \Omega_E(\tau^E_F \cdot \tau^E_G), \quad \text{for } F, G \in C^\infty(E^*), \]

where \( \tau^E_F \) and \( \tau^E_G \) are the Hamiltonian sections associated with \( F \) and \( G \), that is, \( i_{\tau^E_F} \Omega_E = d\tau^E_F E∗F \) and \( i_{\tau^E_G} \Omega_E = d\tau^E_F E∗G \).

Suppose that (\( x^i \)) are local coordinates on an open subset \( U \) of \( M \) and that \( \{e_\alpha\} \) is a local basis of sections of the vector bundle \( \tau_E^{-1}(U) \rightarrow U \) as above. Then, \( \{\tilde{e}_\alpha, \tilde{e}_\alpha\} \) is a local basis of sections of the vector bundle \( (\tau^E_F)^{-1}((\tau^E_F)^{-1}(U)) \rightarrow (\tau^E_F)^{-1}(U) \), where \( \tau^E_F : T^E E∗ \rightarrow E∗ \) is the vector bundle projection and

\[ \tilde{e}_\alpha(a^*) = (e_\alpha(\tau^E(a^*)), \rho^{\alpha}_i \frac{\partial}{\partial x^i}|_{a^*}), \quad \tilde{e}_\alpha(a^*) = (0, \frac{\partial}{\partial y_\alpha}|_{a^*}). \]

Here, \( (x^i, y_\alpha) \) are the local coordinates on \( E∗ \) induced by the local coordinates \( (x^i) \) and the dual basis \( \{e_\alpha\} \) of \( \{e_\alpha\} \). Moreover, we have that

\[ \left[ \tilde{e}_\alpha, \tilde{e}_\beta \right]^E = C^\gamma_{\alpha \beta} \tilde{e}_\gamma, \quad \left[ \tilde{e}_\alpha, \tilde{e}_\beta \right]^E = \left[ \tilde{e}_\alpha, \tilde{e}_\beta \right]^E = 0, \quad \rho^{\alpha}_E(\tilde{e}_\alpha) = \rho^{\alpha}_i \frac{\partial}{\partial x^i}, \quad \rho^{\alpha}_E(\tilde{e}_\alpha) = \frac{\partial}{\partial y_\alpha}, \]

and

\[ \lambda_E(x^i, y_\alpha) = y_\alpha e^\alpha, \quad \Omega_E(x^i, y_\alpha) = \tilde{e}^\alpha \wedge \tilde{e}^\alpha + \frac{1}{2} C_{\alpha \beta}^\gamma y_\gamma \tilde{e}^\alpha \wedge \tilde{e}^\beta, \quad (4) \]

\[ \Pi_E = \rho^{\alpha}_i \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial y_\alpha} - \frac{1}{2} C_{\alpha \beta}^\gamma y_\gamma \frac{\partial}{\partial y_\alpha} \wedge \frac{\partial}{\partial y_\beta}. \quad (5) \]

(For more details, see [17, 24].)

II.3. Lie algebroids. Let \( \tau_A : A \rightarrow M \) be an affine bundle with associated vector bundle \( \tau_V : V \rightarrow M \). Denote by \( \tau_{A^+} : A^+ = \text{Aff}(A, \mathbb{R}) \rightarrow M \) the dual bundle whose fibre over \( x \in M \) consists of affine functions on the fibre \( A_x \). Note that this bundle has a distinguished section \( 1_A \in \Gamma(\tau_{A^+}) \) corresponding to the constant function 1 on \( A \). We also consider the bidual bundle \( \tilde{\tau}_A : \tilde{A} \rightarrow M \) whose fibre at \( x \in M \) is the vector space \( \tilde{A}_x = (A_x^+)^∗ \). Then, \( \tilde{A} \) may be identified with an affine subbundle of \( \tilde{A} \) via the inclusion \( i_A : A \rightarrow \tilde{A} \) given by \( i_A(a)(\varphi) = \varphi(a) \), which is an injective affine map whose associated linear map is denoted by \( i_V : V \rightarrow \tilde{A} \). Thus, \( V \) may be identified with a vector subbundle of \( \tilde{A} \). Using these facts, one can prove that there is a one-to-one correspondence between affine functions on \( A \) and linear functions on \( \tilde{A} \). On the other hand, there is an obvious one-to-one correspondence between affine functions on \( A \) and sections of \( A^+ \).

A Lie algebroid structure on \( A \) consists of a Lie algebra structure \( [·, ·]_V \) on the space \( \Gamma(\tau_V) \) of the sections of \( \tau_V : V \rightarrow M \), a \( \mathbb{R} \)-linear action \( D : \Gamma(\tau_A) \times \Gamma(\tau_V) \rightarrow \Gamma(\tau_V) \) of the sections of \( A \) on \( \Gamma(\tau_V) \) and an affine map \( \rho_A : A \rightarrow TM \), the anchor map, satisfying the following conditions:

- \( D_X[Y, Z]_V = [D_X Y, Z]_V + [Y, D_X Z]_V \),
- \( D_{X + Y} Z = D_X Z + [Y, Z]_V \),
\[ \bullet D_X(f\dot{Y}) = fD_X\dot{Y} + \rho_A(X)(f)\dot{Y}, \]

for \( X \in \Gamma(\tau_A), \dot{Y}, \dot{Z} \in \Gamma(\tau_V) \) and \( f \in C^\infty(M) \) (see [9, 26]).

If \((\cdot, \cdot)_V, D, \rho_A)\) is a Lie affgebroid structure on an affine bundle \( A \) then \((V, [\cdot, \cdot]_V, \rho_V)\) is a Lie algebroid, where \( \rho_V : V \to TM \) is the vector bundle map associated with the affine morphism \( \rho_A : A \to TM \).

A Lie affgebroid structure on an affine bundle \( \tau_A : A \to M \) induces a Lie algebroid structure \(([\cdot, \cdot]_A, \rho_A)\) on the bidual bundle \( \tilde{A} \) such that \( 1_A \in \Gamma(\tau_A^+) \) is a 1-cocycle in the corresponding Lie algebroid cohomology, that is, \( d^A 1_A = 0 \). Indeed, if \( X_0 \in \Gamma(\tau_A) \) then for every section \( \tilde{X} \) of \( \tilde{A} \) there exists a function \( f \in C^\infty(M) \) and a section \( \tilde{X} \in \Gamma(\tau_V) \) such that \( \tilde{X} = fX_0 + \tilde{X} \) and

\[ \rho_{\tilde{A}}(fX_0 + \tilde{X}) = f\rho_A(X_0) + \rho_V(\tilde{X}), \]
\[ [[fX_0 + \tilde{X}, gX_0 + \tilde{Y}]_A = (\rho_V(\tilde{X})(g) - \rho_V(\tilde{Y})(f) + f\rho_A(X_0)(g) - g\rho_A(X_0)(f))X_0 + [[\tilde{X}, \tilde{Y}]_V + fD_X\tilde{Y} - gD_{X_0}\tilde{Y}. \]

Conversely, let \((U, [\cdot, \cdot]_U, \rho_U)\) be a Lie algebroid over \( M \) and \( \phi : U \to \mathbb{R} \) be a 1-cocycle of \((U, [\cdot, \cdot]_U, \rho_U)\) such that \( \phi|_{U_x} \neq 0 \), for all \( x \in M \). Then, \( A = \phi^{-1}\{1\} \) is an affine bundle over \( M \) which admits a Lie affgebroid structure in such a way that \((U, [\cdot, \cdot]_U, \rho_U)\) may be identified with the bidual Lie algebroid \((\tilde{A}, [\cdot, \cdot]_A, \rho_{\tilde{A}})\) to \( A \) and, under this identification, the 1-cocycle \( 1_A : \tilde{A} \to \mathbb{R} \) is just \( \phi \). The affine bundle \( \tau_A : A \to M \) is modelled on the vector bundle \( \tau_V : V = \phi^{-1}\{0\} \to M \). In fact, if \( i_V : V \to U \) and \( i_A : A \to U \) are the canonical inclusions, then

\[ i_V \circ [\tilde{X}, \tilde{Y}]_V = [i_V \circ \tilde{X}, i_V \circ \tilde{Y}]_U, \quad i_V \circ D_X\tilde{Y} = [i_A \circ X, i_V \circ \tilde{Y}]_U, \]
\[ \rho_A(X) = \rho_U(i_A \circ X), \]

for \( \tilde{X}, \tilde{Y} \in \Gamma(\tau_V) \) and \( X \in \Gamma(\tau_A) \).

Let \( \tau_A : A \to M \) be a Lie affgebroid modelled on the Lie algebroid \( \tau_V : V \to M \). Suppose that \((x^i)\) are local coordinates on an open subset \( U \) of \( M \) and that \([e_0, e_\alpha]\) is a local basis of sections of \( \tau_A : \tilde{A} \to M \) in \( U \) which is adapted to the 1-cocycle \( 1_A \), i.e., such that \( 1_A(e_0) = 1 \) and \( 1_A(e_\alpha) = 0 \), for all \( \alpha \). Note that if \([e^0, e^\alpha]\) is the dual basis of \([e_0, e_\alpha]\) then \( e^0 = 1_A \). Denote by \((x^i, y^0, y^\alpha)\) the corresponding local coordinates on \( \tilde{A} \). Then, the local equation defining the affine subbundle \( A \) (respectively, the vector subbundle \( V \)) of \( \tilde{A} \) is \( y^0 = 1 \) (respectively, \( y^0 = 0 \)). Thus, \((x^i, y^\alpha)\) may be considered as local coordinates on \( A \) and \( V \).

The standard example of a Lie affgebroid may be constructed as follows. Let \( \tau : M \to \mathbb{R} \) be a fibration and \( \tau_{1,0} : J^1\tau \to M \) be the 1-jet bundle of local sections of \( \tau : M \to \mathbb{R} \). It is well known that \( \tau_{1,0} : J^1\tau \to M \) is an affine bundle modelled on the vector bundle \( \pi = (\pi_M)|_{J^1\tau} : V \tau \to M \), where \( V \tau \) is the vertical bundle of \( \tau : M \to \mathbb{R} \). Moreover, if \( t \) is the usual coordinate on \( \mathbb{R} \) and \( \eta \) is the closed 1-form on \( M \) given by \( \eta = \tau^*(dt) \) then we have the following identification \( J^1\tau \cong \{v \in TM/\eta(v) = 1\} \) (see, for instance, [32]). Note that \( V \tau = \{v \in TM/\eta(v) = 0\} \). Thus, the bidual bundle \( \tilde{J}^1\tau \) to the affine bundle \( \tau_{1,0} : J^1\tau \to M \) may be identified with the tangent bundle \( TM \to M \) and, under this identification, the Lie algebroid structure on \( \pi_M : TM \to M \) is the standard Lie algebroid structure and the 1-cocycle \( 1_{J^1\tau} \) on \( \pi_M : TM \to M \) is just the 1-form \( \eta \).
II.4. The Lagrangian formalism on Lie affgebroids. In this section, we will develop a geometric framework, which allows to write the Euler-Lagrange equations associated with a Lagrangian function \( L \) on a Lie affgebroid \( A \) in an intrinsic way (see \([26]\)).

Suppose that \( (\tau_A: A \to M, \gamma, V \to M, ([\cdot, \cdot], V, D, \rho_A)) \) is a Lie affgebroid over \( M \). Then, the bidual bundle \( \tau_A^*: A \to M \) admits a Lie algebroid structure \(([\cdot, \cdot], \rho_A)\) in such a way that the section \( 1_A \) of the dual bundle \( A^+ \) is a 1-cocycle.

Now, we consider the Lie algebroid prolongation \( (T^\gamma A, [\cdot, \cdot]_{\gamma}, \rho_{\gamma}) \) of the Lie algebroid \((\tilde{A}, [\cdot, \cdot], \rho_A)\) over the fibration \( \tau_A: A \to M \) with vector bundle projection \( \tau_A^*: T\tilde{A}A \to A \). If \((x^i)\) are local coordinates on an open subset \( U \) of \( M \) and \( \{e_0, e_{\alpha}\} \) is a local basis of sections of the vector bundle \( \tau_A^{-1}(U) \to U \) adapted to \( 1_A \), then \( \{\tilde{T}_0, \tilde{T}_\alpha, \tilde{V}_\alpha\} \) is a local basis of sections of the vector bundle \( (\tau_A^*)^{-1}(\tau_A^{-1}(U)) \to \tau_A^{-1}(U) \), where

\[
\tilde{T}_0(a) = \left(e_0(\tau_A(a)), \rho_0^i \frac{\partial}{\partial x^i}|_a\right), \quad \tilde{T}_\alpha(a) = \left(e_\alpha(\tau_A(a)), \rho_\alpha^i \frac{\partial}{\partial x^i}|_a\right), \quad \tilde{V}_\alpha(a) = \left(0, \frac{\partial}{\partial y^\alpha}|_a\right), \quad (6)
\]

\((x^i, y^\alpha)\) are the local coordinates on \( A \) induced by the local coordinates \((x^i)\) and the basis \( \{e_\alpha\} \) and \( \rho_0^i, \rho_\alpha^i \) are the components of the anchor map \( \rho_A \). We also have that

\[
\tilde{T}_0, \tilde{T}_\alpha \in C^\alpha_0, \gamma \quad \text{and} \quad \tilde{V}_\alpha \in C^\alpha_0, \gamma,
\]

\[
\tilde{T}_0, \tilde{T}_\alpha \in C^\alpha_0, \gamma \quad \text{and} \quad \tilde{V}_\alpha \in C^\alpha_0, \gamma,
\]

\[
\rho_{T^\gamma}(\tilde{T}_0) = \rho_0^i \frac{\partial}{\partial x^i}, \quad \rho_{T^\gamma}(\tilde{T}_\alpha) = \rho_\alpha^i \frac{\partial}{\partial x^i}, \quad \rho_{T^\gamma}(\tilde{V}_\alpha) = \frac{\partial}{\partial y^\alpha}, \quad (7)
\]

where \( C^\alpha_0, \gamma \) are the structure functions of the Lie bracket \([\cdot, \cdot], \gamma \) with respect to the basis \( \{e_0, e_\alpha\} \). Note that, if \( \{\tilde{T}_0, \tilde{T}_\alpha, \tilde{V}_\alpha\} \) is the dual basis of \( \{\tilde{T}_0, \tilde{T}_\alpha, \tilde{V}_\alpha\} \), then \( \tilde{T}_0 \) is globally defined and it is a 1-cocycle. We will denote by \( \phi_0 \) the 1-cocycle \( \tilde{T}_0 \). Thus, we have that

\[
\phi_0(a)(\tilde{b}, X_a) = 1_A(\tilde{b}), \quad \text{for} \quad (\tilde{b}, X_a) \in T\tilde{A}A.
\]

One may also consider the vertical endomorphism \( S: T\tilde{A}A \to T\tilde{A}A \), as a section of the vector bundle \( T\tilde{A}A \otimes (T\tilde{A}A)^* \to A \), whose local expression is (see \([26]\))

\[
S = (\tilde{T}^\alpha - y^\alpha \phi_0) \otimes \tilde{V}_\alpha. \quad (8)
\]

A section \( \xi \) of \( \tau_A^*: T\tilde{A}A \to A \) is said to be a second order differential equation (SODE) on \( A \) if \( \phi_\xi(\xi) = 1 \) and \( S \xi = 0 \). If \( \xi \in \Gamma(\tau_A^*) \) is a SODE then \( \xi = \tilde{T}_0 + y^\alpha \tilde{T}_\alpha + \xi^a \tilde{V}_a \), where \( \xi^a \) are local functions on \( A \), and

\[
\rho_{T^\gamma}(\xi) = (\rho_0^i + y^\alpha \rho_\alpha^i) \frac{\partial}{\partial x^i} + \xi^a \frac{\partial}{\partial y^a}.
\]

Now, a curve \( \gamma: I \subseteq \mathbb{R} \to A \) in \( A \) is said to be admissible if \( \rho_A \circ i_A \circ \gamma = (\tau_A \circ \gamma) \) or, equivalently, \((i_A(\gamma(t)), \gamma(t)) \in T\gamma(\gamma(t))A \), for all \( t \in I \), \( i_A: A \to \tilde{A} \) being the canonical inclusion. We will denote by \( \text{Adm}(A) \) the space of admissible curves on \( A \). Thus, if \( \gamma(t) = (x^i(t), y^\alpha(t)) \), for all \( t \in I \), then \( \gamma \) is an admissible curve if and only if

\[
\frac{dx^i}{dt} = \rho_0^i + \rho_\alpha^i y^a, \quad \text{for} \quad i \in \{1, \ldots, m\}.
\]
If we denote by $F \text{rom } (7), (8) \text{ and } (9), \text{ we obtain that}$

$$\delta L_{\text{intrinsic way}}(\Omega_{\text{assoc}}) = \{ \delta L \text{ associated with } \} \text{ the Lagrangian } \mathcal{L} \text{ as } \gamma(t) \in \mathcal{T}_{\gamma(t)}\mathcal{A} \text{ if and only if } \gamma \text{ is admissible} \text{ and } \iota_{(i,\mathcal{A}(\gamma(t)),\mathcal{L}(\gamma(t)))} \Omega_{\mathcal{L}}(\gamma(t)) = 0, \text{ for all } t.$$ 

Now, a curve $\gamma : I = (-\epsilon, \epsilon) \subseteq \mathbb{R} \rightarrow \mathcal{A}$ in $\mathcal{A}$ is a solution of the Euler-Lagrange equations associated with $\mathcal{L}$ if and only if $\gamma$ is admissible and $i_{(i,\mathcal{A}(\gamma(t)),\mathcal{L}(\gamma(t)))} \Omega_{\mathcal{L}}(\gamma(t)) = 0, \text{ for all } t.$

If $\gamma(t) = (x^i(t), y^\alpha(t))$ then $\gamma$ is a solution of the Euler-Lagrange equations if and only if

$$\frac{dx^i}{dt} = \dot{\alpha}_0 + \rho_\alpha y^\alpha, \quad \frac{dy^\alpha}{dt} = \rho_\alpha \frac{\partial L}{\partial x^\alpha} + (C_\alpha^\beta + C_\beta^\gamma y^\beta) \frac{\partial L}{\partial y^\gamma},$$

for $i \in \{1, \ldots, m\}$ and $\alpha \in \{1, \ldots, n\}.$

We denote by $\mathcal{C}(\mathcal{A}^+)$ the set of curves in $\mathcal{A}^+$, we can define the Euler-Lagrange operator $\delta \mathcal{L} : \text{Adm}(\mathcal{A}) \rightarrow \mathcal{C}(\mathcal{A}^+)$ by

$$\delta \mathcal{L}_{\gamma(t)}(\tilde{a}) = \Omega_{\mathcal{L}}(\gamma(t))(i_{\gamma(t)}(\tilde{a}), (\dot{a}, v_{\gamma(t)}) , \gamma(t)), \quad (12)$$

for $\gamma \in \text{Adm}(\mathcal{A})$ and $\tilde{a} \in \tilde{\mathcal{A}}_{\gamma(t)}(\gamma(t))$, where $v_{\gamma(t)} \in T_{\gamma(t)}\mathcal{A}$ is such that $(\tilde{a}, v_{\gamma(t)}) \in T_{\gamma(t)}\mathcal{A}$, for all $t$. It is easy to prove that the map $\delta \mathcal{L}$ doesn’t depend on the chosen tangent vector $v_{\gamma(t)}$. From (10), we deduce that its local expression is

$$\delta \mathcal{L} = \left( -\frac{d}{dt} \left( \frac{\partial L}{\partial y^\alpha} \right) + \rho_\alpha \frac{\partial L}{\partial x^\alpha} + (C_\alpha^\beta + C_\beta^\gamma y^\beta) \frac{\partial L}{\partial y^\gamma} \right) (e^\alpha - y^\alpha e^0),$$

where $\{e^0, e^\alpha\}$ is the dual basis of $\{e_0, e_\alpha\}.$ Then the Euler-Lagrange differential equations read as $\delta \mathcal{L} = 0.$

The Lagrangian $\mathcal{L}$ is regular if and only if the matrix $(W_{\alpha\beta}) = \left( \frac{\partial^2 \mathcal{L}}{\partial y^\alpha \partial y^\beta} \right)$ is regular or, in a intrinsic way, if the pair $(\Omega_\mathcal{L}, \phi_0)$ is a cosymplectic structure on $T^{\mathcal{A}\mathcal{A}}\mathcal{A}$, that is,

$$\left\{ \Omega_\mathcal{L} \wedge \ldots \wedge \Omega_\mathcal{L} \wedge \phi_0 \right\}(a) \neq 0, \quad d^{T^{\mathcal{A}\mathcal{A}}}\Omega_\mathcal{L} = 0 \quad \text{and} \quad d^{T^{\mathcal{A}\mathcal{A}}} \phi_0 = 0, \text{ for all } a \in \mathcal{A}.$$ 

Note that the first condition is equivalent to the fact that the map $b_\mathcal{L} : T^{\mathcal{A}\mathcal{A}}\mathcal{A} \rightarrow (T^{\mathcal{A}\mathcal{A}}\mathcal{A})^*$ defined by

$$b_\mathcal{L}(X) = i_X \Omega_\mathcal{L} + \phi_0(X) \phi_0,$$

is an isomorphism of vector bundles.
Now, suppose that the Lagrangian $L$ is regular and let $R_L \in \Gamma(\tau_A^*)$ be the Reeb section of the cosymplectic structure $(\Omega_L, \phi_0)$ characterized by the following conditions

$$i_{R_L}\Omega_L = 0 \text{ and } i_{R_L}\phi_0 = 1.$$  

Then, $R_L$ is the unique Lagrangian SODE associated with $L$, that is, the integral curves of the vector field $\rho_A^* R_L$ are solutions of the Euler-Lagrange equations associated with $L$. In such a case, $R_L$ is called the *Euler-Lagrange section associated with $L$* and its local expression is

$$R_L = \tilde{T}_0 + y^\alpha \tilde{T}_\alpha + W^{\alpha\beta} \left( \rho_0^i \frac{\partial L}{\partial x^i} - (\rho_0^i + y^\gamma \rho_i^\gamma) \frac{\partial^2 L}{\partial x^i \partial y^\beta} + (C^\gamma_{ij} + y^\mu C^\gamma_{ij} \frac{\partial L}{\partial y^\mu}) \right) \tilde{V}_\alpha,$$  

where $(W^{\alpha\beta})$ is the inverse matrix of $(W_{\alpha\beta})$.

### II.5. The Hamiltonian formalism

In this section, we will develop a geometric framework, which allows to write the Hamilton equations associated with a Hamiltonian section on a Lie affgebroid (see [12, 25]).

Suppose that $(\tau_\tilde{A} : \tilde{A} \to M, \tau_\tilde{V} : V \to M, (\|\cdot\|_V, D, \rho_\tilde{A}))$ is a Lie affgebroid. Now, consider the prolongation $(\tilde{\tau}_\tilde{A} V^*, [\cdot, \cdot]_\tilde{A}, \tilde{\rho}_\tilde{A} V^*)$ of the bidual Lie algebroid $(\tilde{\tau}_\tilde{A}, [\cdot, \cdot]_\tilde{A}, \rho_\tilde{A})$ over the fibration $\tau_\tilde{V} : V^* \to M$.

Let $(x^i)$ be local coordinates on an open subset $U$ of $M$ and $\{e_0, e_\alpha\}$ be a local basis of sections of the vector bundle $\tau_\tilde{A}^{-1}(U) \to U$ adapted to $\mathfrak{g}$. Let $\{e_0, e_\alpha\}$ be a local basis of sections of the vector bundle $\tau_\tilde{A}^{-1}(U) \to U$ adapted to $\mathfrak{g}$.

Denote by $(x^i, y^0, y^\alpha)$ the induced local coordinates on $\tilde{A}$ and by $(x^i, y^0, y_\alpha)$ the dual coordinates on the dual vector bundle $\tau_{\tilde{A}^*} : \tilde{A}^* \to M \to \tilde{A}$. Then, $(x^i, y_\alpha)$ are local coordinates on $V^*$ and $\{\tilde{e}_0, \tilde{e}_\alpha, \tilde{e}_\gamma\}$ is a local basis of sections of the vector bundle $\tau_{\tilde{A}^*} : T\tilde{A} V^* \to V^*$, where

$$\tilde{e}_0(\psi) = \left( e_0(\tau_\tilde{V}(\psi)), \rho_0^i \frac{\partial}{\partial x^i}|_\psi \right), \quad \tilde{e}_\alpha(\psi) = \left( e_\alpha(\tau_\tilde{V}(\psi)), \rho_\alpha^i \frac{\partial}{\partial x^i}|_\psi \right), \quad \tilde{e}_\gamma(\psi) = \left( 0, \frac{\partial}{\partial y_\gamma}|_\psi \right).$$

Using this local basis one may introduce local coordinates $(x^i, y_\alpha, z^0, z^\alpha, v_\alpha)$ on $T\tilde{A} V^*$. A direct computation proves that

$$[e_0, \tilde{e}_\beta]_{\tilde{A}^*} = C^\gamma_{0\beta} \tilde{e}_\gamma, \quad [\tilde{e}_\alpha, \tilde{e}_\beta]_{\tilde{A}^*} = C^\gamma_{\alpha\beta} \tilde{e}_\gamma,$$

$$[\tilde{e}_0, \tilde{e}_\alpha]_{\tilde{A}^*} = \tilde{e}_\alpha, \quad [\tilde{e}_\alpha, \tilde{e}_\beta]_{\tilde{A}^*} = \tilde{e}_\beta, \quad [\tilde{e}_0, \tilde{e}_\alpha]_{\tilde{A}^*} = 0,$$

$$\tilde{\rho}_\tilde{A}(\tilde{e}_0) = \rho_0^i \frac{\partial}{\partial x^i}, \quad \tilde{\rho}_\tilde{A}(\tilde{e}_\alpha) = \rho_\alpha^i \frac{\partial}{\partial x^i}, \quad \tilde{\rho}_\tilde{A}(\tilde{e}_\beta) = \frac{\partial}{\partial y_\beta},$$

for all $\alpha$ and $\beta$. Thus, if $\{e^0, e^\alpha, e^\beta\}$ is the dual basis of $\{\tilde{e}_0, \tilde{e}_\alpha, \tilde{e}_\beta\}$ then

$$dT\tilde{A}^* f = \rho_0^i \frac{\partial f}{\partial x^i} e^0 + \rho_\alpha^i \frac{\partial f}{\partial x^i} e^\alpha + \frac{\partial f}{\partial y_\alpha} e^\alpha,$$

$$dT\tilde{A}^* e^\gamma = -\frac{1}{2} C^\gamma_{0\alpha} e^0 \wedge e^\alpha - \frac{1}{2} C^\gamma_{\alpha\beta} e^\alpha \wedge e^\beta, \quad dT\tilde{A}^* e^0 = dT\tilde{A}^* e^\gamma = 0,$$

for $f \in C^\infty(V^*)$.  

Let $\mu : \mathcal{A}^+ \to V^*$ be the canonical projection given by $\mu(\varphi) = \varphi^i$, for $\varphi \in \mathcal{A}_x^+$, with $x \in M$, where $\varphi^i \in V^x_\nu$ is the linear map associated with the affine map $\varphi$ and $h : V^* \to \mathcal{A}^+$ be a Hamiltonian section of $\mu$.

Now, we consider the Lie algebroid prolongation $\mathcal{T}\mathcal{A}^+ \to \mathcal{A}^+ \to M$ with vector bundle projection $\tau^{\mathcal{T}\mathcal{A}+}_{\mathcal{A}} : \mathcal{T}\mathcal{A}^+ \to \mathcal{A}^+$ (see Section II.2). Then, we may introduce the map $T h : \mathcal{T}\mathcal{A}^+ V^* \to \mathcal{T}\mathcal{A}^+ \mathcal{A}^+$ defined by $T h(\tilde{a}, X_\alpha) = (\tilde{a}, (T_h)(X_\alpha))$, for $\tilde{a}, X_\alpha \in \mathcal{T}_\alpha\mathcal{A}^+ V^*$, with $\alpha \in V^*$. It is easy to prove that the pair $(T h, h)$ is a Lie algebroid morphism between the Lie algebroids $\tau^{\mathcal{T}\mathcal{A}+}_{\mathcal{A}} : \mathcal{T}\mathcal{A}^+ V^* \to V^*$ and $\tau^{\mathcal{T}\mathcal{A}+}_{\mathcal{A}} : \mathcal{T}\mathcal{A}^+ \mathcal{A}^+ \to \mathcal{A}^+$.

Next, denote by $\lambda_h$ and $\Omega_h$ the sections of the vector bundles $(\mathcal{T}\mathcal{A}^+ V^*)^* \to V^*$ and $\Lambda^2(\mathcal{T}\mathcal{A}^+ V^*)^* \to V^*$ given by

$$
\lambda_h = (T h, h)^*(\lambda_{\mathcal{A}}), \quad \Omega_h = (T h, h)^*(\Omega_{\mathcal{A}}),
$$

(13)

where $\lambda_{\mathcal{A}}$ and $\Omega_{\mathcal{A}}$ are the Liouville section and the canonical symplectic section, respectively, associated with the Lie algebroid $\mathcal{A}$. Note that $\Omega_h = -d\tau^{\mathcal{T}\mathcal{A}+}_{\mathcal{A}} \lambda_h$.

On the other hand, let $\eta : \mathcal{T}\mathcal{A}^+ V^* \to \mathbb{R}$ be the section of $(\mathcal{T}\mathcal{A}^+ V^*)^* \to V^*$ defined by

$$
\eta(\tilde{a}, X_\nu) = 1_{\mathcal{A}}(\tilde{a}),
$$

(14)

for $\tilde{a}, X_\nu \in \mathcal{T}_\nu\mathcal{A}^+ V^*$, with $\nu \in V^*$. Note that if $pr_1 : \mathcal{T}\mathcal{A}^+ V^* \to \mathcal{A}$ is the canonical projection on the first factor then $(pr_1, \tau^\mathcal{T}_{\mathcal{A}})$ is a morphism between the Lie algebroids $\tau^\mathcal{T}_{\mathcal{A}} : \mathcal{T}\mathcal{A}^+ V^* \to V^*$ and $\tau_{\mathcal{A}} : \mathcal{A} \to M$ and $(pr_1, \tau^\mathcal{T}_{\mathcal{A}})(1_{\mathcal{A}}) = \eta$. Thus, since $1_{\mathcal{A}}$ is a 1-cocycle of $\tau_{\mathcal{A}} : \mathcal{A} \to M$, we deduce that $\eta$ is a 1-cocycle of the Lie algebroid $\tau^\mathcal{T}_{\mathcal{A}} : \mathcal{T}\mathcal{A}^+ V^* \to V^*$.

Suppose that $h(x^i, y_\alpha) = (x^i, -H(x^i, y_\beta), y_\alpha)$ and that $\{\dot{e}^0, \dot{e}^\alpha, \dot{\bar{e}}^\alpha\}$ is the dual basis of $\{\bar{e}^0, \bar{e}_\alpha, \bar{\bar{e}}_\alpha\}$. Then $\eta = \dot{e}^0$ and, from (11), (13) and the definition of the map $T h$, it follows that

$$
\Omega_h = \dot{e}^\alpha \wedge \dot{\bar{e}}^\alpha + \frac{1}{2} C^{\alpha}_{\beta \gamma} \dot{e}^\beta \wedge \dot{\bar{e}}^\gamma \wedge \dot{\bar{e}}^0 + (\rho^\alpha \frac{\partial H}{\partial x^\beta} - C^{\alpha}_{\beta \gamma} y_\gamma) \dot{e}^\alpha \wedge \dot{\bar{e}}^0 + \frac{\partial H}{\partial y_\alpha} \dot{\bar{e}}^\alpha \wedge \dot{\bar{e}}^0.
$$

(15)

Thus, it is easy to prove that the pair $(\Omega_h, \eta)$ is a cosymplectic structure on the Lie algebroid $\tau^\mathcal{T}_{\mathcal{A}} : \mathcal{T}\mathcal{A}^+ V^* \to V^*$.

**Remark II.1.** Let $\mathcal{T}\mathcal{A}^+ V^*$ be the prolongation of the Lie algebroid $V$ over the projection $\tau^\mathcal{T}_V : V^* \to M$. Denote by $\lambda_V$ and $\Omega_V$ the Liouville section and the canonical symplectic section, respectively, of $V$ and by $(i_V, Id) : \mathcal{T}\mathcal{A}^+ V^* \to \mathcal{T}\mathcal{A}^+ V^*$ the canonical inclusion. Then, using (12), (13), (14) and the fact that $\mu \circ h = Id$, we obtain that

$$(i_V, Id)^*(\lambda_h) = \lambda_V, \quad (i_V, Id)^*(\eta) = 0.
$$

Thus, since $(i_V, Id)$ is a Lie algebroid morphism, we also deduce that

$$(i_V, Id)^*(\Omega_h) = \Omega_V.
$$

Now, let $R_h \in \Gamma(\tau^\mathcal{T}_{\mathcal{A}})$ be the Reeb section of the cosymplectic structure $(\Omega_h, \eta)$ characterized by the following conditions

$$
i_R h = 0 \text{ and } i_{R_h} \eta = 1.
$$
With respect to the basis \{\bar{e}_0, \bar{e}_\alpha, \bar{e}_\alpha\} of \(\Gamma(Z^w_A)\), \(R_h\) is locally expressed as:

\[
R_h = \bar{e}_0 + \frac{\partial H}{\partial y_\alpha} \bar{e}_\alpha - \left( C^\gamma_{\alpha\beta} y_\gamma \frac{\partial H}{\partial y_\beta} + \rho_\alpha \frac{\partial H}{\partial x^i} - C^\gamma_{\alpha 0} y_\gamma \right) \bar{e}_\alpha.
\]  

(16)

Thus, the vector field \(\rho^w_A(R_h)\) is locally given by

\[
\rho^w_A(R_h) = (\rho_0^i + \rho_\alpha \frac{\partial H}{\partial y_\alpha}) \frac{\partial}{\partial x^i} + \left( -\rho_\alpha \frac{\partial H}{\partial x^i} + y_\gamma (C^\gamma_{\alpha 0} + C^\gamma_{\beta 0} \frac{\partial H}{\partial y_\beta}) \right) \frac{\partial}{\partial y_\alpha}
\]  

(17)

and the integral curves of \(R_h\) (i.e., the integral curves of \(\rho^w_A(R_h)\)) are just the solutions of the Hamilton equations for \(h\),

\[
\frac{dx^i}{dt} = \rho_0^i + \rho_\alpha \frac{\partial H}{\partial y_\alpha}, \quad \frac{dy_\alpha}{dt} = -\rho_\alpha \frac{\partial H}{\partial x^i} + y_\gamma (C^\gamma_{\alpha 0} + C^\gamma_{\beta 0} \frac{\partial H}{\partial y_\beta}),
\]

for \(i \in \{1, \ldots, m\}\) and \(\alpha \in \{1, \ldots, n\}\).

Next, we will present an alternative approach in order to obtain the Hamilton equations. For this purpose, we will use the notion of an aff-Poisson structure on an AV-bundle which was introduced in [9] (see also [10]).

Let \(\tau_Z : Z \to M\) be an affine bundle of rank 1 modelled on the trivial vector bundle \(\tau_M \times \mathbb{R} : M \times \mathbb{R} \to M\), that is, \(\tau_Z : Z \to M\) is an AV-bundle in the terminology of [10].

Then, we have an action of \(\mathbb{R}\) on the fibres of \(Z\). This action induces a vector field \(X_Z\) on \(Z\) which is vertical with respect to the projection \(\tau_Z : Z \to M\).

On the other hand, there exists a one-to-one correspondence between the space of sections of \(\tau_Z : Z \to M\), \(\Gamma(\tau_Z)\), and the set

\[
\{F_h \in C^\infty(Z)/X_Z(F_h) = -1\}.
\]

In fact, if \(h \in \Gamma(\tau_Z)\) and \((x^i, s)\) are local fibred coordinates on \(Z\) such that \(X_Z = \frac{\partial}{\partial s}\) and \(h\) is locally defined by \(h(x^i) = (x^i, -H(x^i))\), then the function \(F_h\) on \(Z\) is locally given by

\[
F_h(x^i, s) = -H(x^i) - s,
\]  

(18)

(for more details, see [10]).

Now, an aff-Poisson structure on the AV-bundle \(\tau_Z : Z \to M\) is a bi-affine map

\[
\{\cdot, \cdot\} : \Gamma(\tau_Z) \times \Gamma(\tau_Z) \to C^\infty(M)
\]

which satisfies the following properties:

i) Skew-symmetric: \(\{h_1, h_2\} = -\{h_2, h_1\}\).

ii) Jacobi identity:

\[
\{h_1, \{h_2, h_3\}\}_V + \{h_2, \{h_3, h_1\}\}_V + \{h_3, \{h_1, h_2\}\}_V = 0,
\]

where \(\{\cdot, \cdot\}_V\) is the affine-linear part of the bi-affine bracket.

iii) If \(h \in \Gamma(\tau_Z)\) then

\[
\{h, \cdot\} : \Gamma(\tau_Z) \to C^\infty(M), \quad h' \mapsto \{h, h'\},
\]

is an affine derivation.
Condition iii) implies that, for each \( h \in \Gamma(\tau_Z) \) the linear part \( \{ h, \cdot \}_V : C^\infty(M) \to C^\infty(M) \) of the affine map \( \{ h, \cdot \} : \Gamma(\tau_Z) \to C^\infty(M) \) defines a vector field on \( M \), which is called the Hamiltonian vector field of \( h \) (see [10]).

In [10], the authors proved that there is a one-to-one correspondence between aff-Poisson brackets \( \{ \cdot, \cdot \} \) on \( \tau_Z : Z \to M \) and Poisson brackets \( \{ \cdot, \cdot \}_\Pi \) on \( Z \) which are \( X_Z \)-invariant, i.e., which are associated with Poisson 2-vectors \( \Pi \) on \( Z \) such that \( \mathcal{L}_X \Pi = 0 \). This correspondence is determined by

\[
\{ h_1, h_2 \} \circ \tau_Z = \{ F_{h_1}, F_{h_2} \}_\Pi, \text{ for } h_1, h_2 \in \Gamma(\tau_Z).
\]

Note that the function \( \{ F_{h_1}, F_{h_2} \}_\Pi \) on \( Z \) is \( \tau_Z \)-projectable, i.e., \( \mathcal{L}_X \{ F_{h_1}, F_{h_2} \}_\Pi = 0 \) (because the Poisson 2-vector \( \Pi \) is \( X_Z \)-invariant).

Using this correspondence we will prove the following result.

**Theorem II.2.** [12] Let \( \tau_A : A \to M \) be a Lie affgebroid modelled on the vector bundle \( \tau_V : V \to M \). Denote by \( \tau_{A^+} : A^+ \to M \) (resp., \( \tau_V^* : V^* \to M \)) the dual vector bundle to \( A \) (resp., to \( V \)) and by \( \mu : A^+ \to V^* \) the canonical projection. Then:

i) \( \mu : A^+ \to V^* \) is an AV-bundle which admits an aff-Poisson structure.

ii) If \( h : V^* \to A^+ \) is a Hamiltonian section (that is, \( h \in \Gamma(\mu) \)) then the Hamiltonian vector field of \( h \) with respect to the aff-Poisson structure is a vector field on \( V^* \) whose integral curves are just the solutions of the Hamilton equations for \( h \).

**Proof.** i) It is clear that \( \mu : A^+ \to V^* \) is an AV-bundle. In fact, if \( a^+ \in A^+_x \), with \( x \in M \), and \( t \in \mathbb{R} \) then

\[
a^+ + t = a^+ + t1_A(x).
\]

Thus, the \( \mu \)-vertical vector field \( X_{A^+} \) on \( A^+ \) is just the vertical lift \( 1_A^V \) of the section \( 1_A \in \Gamma(\tau_{A^+}) \). Moreover, one may consider the Lie algebroid \( \tilde{\tau}_A : \tilde{A} = (A^+_*)^* \to M \) and the corresponding linear Poisson 2-vector \( \Pi_{A^+} \) on \( A^+ \). Then, using the fact that \( 1_A \) is a 1-cocycle of \( \tau_A : A = (A^+_*)^* \to M \), it follows that the Poisson 2-vector \( \Pi_{A^+} \) is \( X_{A^+} \)-invariant. Therefore, \( \Pi_{A^+} \) induces an aff-Poisson structure \( \{ \cdot, \cdot \} \) on \( \mu : A^+ \to V^* \) which is characterized by the condition

\[
\{ h_1, h_2 \} \circ \mu = \{ F_{h_1}, F_{h_2} \}_\Pi_{A^+}, \text{ for } h_1, h_2 \in \Gamma(\mu).
\]

One may also prove this first part of the theorem using the relation between special Lie affgebroid structures on an affine bundle \( A' \) and aff-Poisson structures on the AV-bundle \( AV((A')^\delta) \) (see Theorem 23 in [10]).

ii) From (18) and (19), we deduce that the linear map \( \{ h, \cdot \}_V : C^\infty(V^*) \to C^\infty(V^*) \) associated with the affine map \( \{ h, \cdot \} : \Gamma(\mu) \to C^\infty(V^*) \) (that is, the Hamiltonian vector field of \( h \)) is given by

\[
\{ h, \cdot \}_V(\phi) \circ \mu = \{ F_h, \phi \circ \mu \}_\Pi_{A^+}, \text{ for } \phi \in C^\infty(V^*).
\]

Now, suppose that the local expression of \( h \) is

\[
h(x^i, y_\alpha) = (x^i, -H(x^j, y_\beta), y_\alpha).
\]

On the other hand, using (5), we have that

\[
\Pi_{A^+} = \rho_i^j \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial y_0} + \rho_i^j \frac{\partial}{\partial x^j} \wedge \frac{\partial}{\partial y_0} - C^0_{\alpha\beta} \frac{\partial}{\partial y_\alpha} \wedge \frac{\partial}{\partial y_\beta} - \frac{1}{2} C^i_{\alpha\beta} \frac{\partial}{\partial y_\alpha} \wedge \frac{\partial}{\partial y_\beta}.
\]
Thus, from (20), (21) and (22), we conclude that the Hamiltonian vector field of $h$ is locally given by
\[
(\rho^0 + \rho^\alpha \frac{\partial H}{\partial y^\alpha}) \frac{\partial}{\partial x^i} + \left( -\rho^\alpha \frac{\partial H}{\partial y^\alpha} + y^\gamma (C^\gamma_{0\alpha} + C^\gamma_{\beta\alpha} \frac{\partial H}{\partial y^\beta}) \right) \frac{\partial}{\partial y^\alpha},
\]
which proves our result (see (17)).

II.6. The Legendre transformation and the equivalence between the Lagrangian and Hamiltonian formalisms. Let $L : \mathcal{A} \to \mathbb{R}$ be a Lagrangian function and $\Theta_L \in \Gamma(\tau_{\mathcal{A}}^*)$ be the Poincaré-Cartan 1-section associated with $L$. We introduce the extended Legendre transformation associated with $L$ as the smooth map $Leg_L : \mathcal{A} \to \mathcal{A}^+$ defined by $Leg_L(a)(b) = \Theta_L(a)(z)$, for $a, b \in \mathcal{A}_x$, where $z \in T^* \mathcal{A}$ is such that $pr_1(z) = i_\mathcal{A}(b)$, $pr_1 : T^* \mathcal{A} \to \mathcal{A}$ being the restriction to $T^* \mathcal{A}$ of the first canonical projection $pr_1 : \mathcal{A} \times T \mathcal{A} \to \mathcal{A}$. The map $Leg_L$ is well-defined and its local expression in fibred coordinates on $\mathcal{A}$ and $\mathcal{A}^+$ is
\[
Leg_L(x^i, y^\alpha) = (x^i, L - \frac{\partial L}{\partial y^\alpha} y^\alpha, \frac{\partial L}{\partial y^\alpha}).
\]
Thus, we can define the Legendre transformation associated with $L$, $leg_L : \mathcal{A} \to V^*$, by $leg_L = \mu \circ Leg_L$. From (23) and since $\mu(x^i, y_0, y_\alpha) = (x^i, y_\alpha)$, we have that
\[
leg_L(x^i, y^\alpha) = (x^i, \frac{\partial L}{\partial y^\alpha}).
\]
The maps $Leg_L$ and $leg_L$ induce the maps $T Leg_L : T^* \mathcal{A} \to T^* \mathcal{A}^+$ and $T leg_L : T^* \mathcal{A} \to T^* V^*$ defined by
\[
(T Leg_L)(\tilde{b}, X_\alpha) = (\tilde{b}, (T_x Leg_L)(X_\alpha)), \quad (T leg_L)(\tilde{b}, X_\alpha) = (\tilde{b}, (T_x leg_L)(X_\alpha)),
\]
for $a \in \mathcal{A}$ and $(\tilde{b}, X_\alpha) \in T^*_a \mathcal{A}$.

Now, let $\{\tilde{T}_0, \tilde{T}_0, \tilde{V}_\alpha\}$ (respectively, $\{\check{e}_0, \check{e}_\alpha, \check{e}_0, \check{e}_\alpha\}$) be a local basis of $\Gamma(\tau_{\mathcal{A}}^*)$ as in Section 11.3 (respectively, of $\Gamma(\tau_{\mathcal{A}}^{*+})$ as in Section 11.2) and denote by $(x^i, y^\alpha; z^0, z^\alpha, v^\alpha)$ (respectively, $(x^i, y_0, y_\alpha; z^0, z^\alpha, v_\alpha)$) the corresponding local coordinates on $T^* \mathcal{A}$ (respectively, $T^* \mathcal{A}^+$). In addition, suppose that $(x^i, y_\alpha; z^0, z^\alpha, v_\alpha)$ are local coordinates on $T^* \mathcal{A}^+$ as in Section 11.3.

Then, from (23), (24) and (25), we deduce that the local expression of the maps $T Leg_L$ and $T leg_L$ is
\[
T Leg_L(x^i, y^\alpha; z^0, z^\alpha, v^\alpha) = (x^i, L - \frac{\partial L}{\partial y^\alpha} y^\alpha, \frac{\partial L}{\partial y^\alpha}; z^0, z^\alpha, v^\alpha \rho^\beta_0 \frac{\partial L}{\partial x^i} - \frac{\partial^2 L}{\partial x^i \partial y^\alpha} y^\alpha - \rho^\alpha \frac{\partial^2 L}{\partial x^i \partial y^\beta} y^\beta - \rho^\beta \frac{\partial^2 L}{\partial x^i \partial y^\beta} y^\beta)
+ z^\alpha \rho^\alpha \frac{\partial^2 L}{\partial x^i \partial y^\beta} (\frac{\partial}{\partial x^i} \frac{\partial^2 L}{\partial x^i \partial y^\beta} - \frac{\partial^2 L}{\partial x^i \partial y^\beta} y^\beta)\]
\[
T leg_L(x^i, y^\alpha; z^0, z^\alpha, v^\alpha) = (x^i, \frac{\partial L}{\partial y^\alpha}; z^0, z^\alpha, v^\alpha \rho^\beta_0 \frac{\partial L}{\partial x^i} + z^\alpha \rho^\beta \frac{\partial^2 L}{\partial x^i \partial y^\beta} y^\beta).
\]
Thus, using (41), (40), (20) and (26), we can prove the following result.
From (24), it follows

\[ (T \text{Leg}_L, \text{Leg}_L)^* (\lambda_A) = \Theta_L, \quad (T \text{Leg}_L, \text{Leg}_L)^* (\Omega_A) = \Omega_L. \]  

(28)

From (24), it follows

**Proposition II.4.** The Lagrangian \( L \) is regular if and only if the Legendre transformation \( \text{leg}_L : A \to V^* \) is a local diffeomorphism.

Next, we will assume that \( L \) is hyperregular, that is, \( \text{leg}_L \) is a global diffeomorphism. Then, from (25) and Theorem II.3 we conclude that the pair \( (T \text{Leg}_L, \text{Leg}_L) \) is a Lie algebroid isomorphism.

Moreover, we can consider the Hamiltonian section \( h : V^* \to A^* \) defined by

\[ h = \text{Leg}_L \circ \text{leg}_L^{-1}, \]  

(29)

the corresponding cosymplectic structure \( (\Omega_h, \eta) \) on the Lie algebroid \( T\bar{\Delta}V^* \to V^* \) and the Hamiltonian section \( R_h \in \Gamma(\tau_{\bar{\Delta}}V^*) \).

Using (13), (27), (28), (29) and Theorem II.3 we deduce that

**Theorem II.5.** If the Lagrangian \( L \) is hyperregular then the Euler-Lagrange section \( R_L \) associated with \( L \) and the Hamiltonian section \( R_h \) associated with \( h \) satisfy the following relation

\[ R_h \circ \text{leg}_L = T\text{Leg}_L \circ R_L. \]

Moreover, if \( \gamma : I \to A \) is a solution of the Euler-Lagrange equations associated with \( L \), then \( \text{leg}_L \circ \gamma : I \to V^* \) is a solution of the Hamilton equations associated with \( h \) and, conversely, if \( \bar{\gamma} : I \to V^* \) is a solution of the Hamilton equations for \( h \) then \( \gamma = \text{leg}_L^{-1} \circ \bar{\gamma} \) is a solution of the Euler-Lagrange equations for \( L \).

Note that if the local expression of the inverse of the Legendre transformation \( \text{leg}_L : A \to V^* \) is

\[ \text{leg}_L^{-1}(x^i, y_\beta) = (x^i, y^\alpha(x^i, y_\beta)), \]

then the Hamiltonian section \( h \) is locally given by

\[ h(x^i, y_\alpha) = (x^i, -H(x^j, y_\beta), y_\alpha), \]

where the function \( H \) is

\[ H(x^i, y_\alpha) = y_\beta y^\alpha(x^i, y_\alpha) - L(x^i, y^\beta(x^i, y_\alpha)). \]

Thus,

\[ \frac{\partial H}{\partial x^i} |_{(x^i, y_\alpha)} = - \frac{\partial L}{\partial x^i} |_{(x^i, y^\alpha(x^i, y_\beta))}, \quad \frac{\partial H}{\partial y_\beta} |_{(x^i, y_\alpha)} = y^\beta(x^i, y_\alpha). \]  

(30)

Therefore,

\[ \text{leg}_L^{-1}(x^i, y_\alpha) = (x^i, \frac{\partial H}{\partial y_\alpha}), \]  

(31)
III. Affinely constrained Lagrangian systems

We start with a free Lagrangian system on a Lie algebroid $\mathcal{A}$ of rank $n$. Now, we plug in some nonholonomic affine constraints described by an affine subbundle $\mathcal{B}$ of rank $n-r$ of the bundle $\mathcal{A}$ of admissible directions, that is, we have an affine bundle $\tau_\mathcal{B}: \mathcal{B} \to M$ with associated vector bundle $\tau_\mathcal{B}: \mathcal{B} \to M$ and the corresponding inclusions $i_\mathcal{B}: \mathcal{B} \hookrightarrow \mathcal{A}$ and $i_\mathcal{B}: \mathcal{B} \hookrightarrow \mathcal{V}$, $\tau_\mathcal{V}: \mathcal{V} \to M$ being the vector bundle associated with the affine bundle $\tau_\mathcal{A}: \mathcal{A} \to M$. If we impose to the admissible solution curves $\gamma(t)$ the condition to stay on the manifold $\mathcal{B}$, we arrive at the equations $\delta L_{\gamma(t)} = \lambda(t)$ and $\gamma(t) \in \mathcal{B}$, where the constraint force $\lambda(t) \in \mathcal{A}^+_{\tau_\mathcal{A}(\gamma(t))}$ is to be determined. In the standard case ($\mathcal{A} = J^1\tau$, $\tau: M \to \mathbb{R}$ being a fibration), $\lambda$ takes values in the annihilator of the constraint submanifold $\mathcal{B}$. Therefore, in the case of a general Lie algebroid, the natural \textit{Lagrange-d’Alembert equations} one should pose are

$$\delta L_{\gamma(t)} \in \mathcal{B}^\circ_{\tau_\mathcal{B}(\gamma(t))} \text{ and } \gamma(t) \in \mathcal{B},$$

where $\mathcal{B}^\circ = \{ \varphi \in \mathcal{A}^+/\varphi|_{\mathcal{B}} \equiv 0 \}$ is the affine annihilator of $\mathcal{B}$ which is a vector subbundle of $\mathcal{A}^+$ with rank equal to $r$.

In more explicit terms, we look for curves $\gamma(t) \in \mathcal{A}$ such that

- $\gamma \in \text{Adm}(\mathcal{A})$,
- $\gamma(t) \in \mathcal{B}_{\tau_\mathcal{A}(\gamma(t))}$,
- There exists $\lambda(t) \in \mathcal{B}^\circ_{\tau_\mathcal{A}(\gamma(t))}$ such that $\delta L_{\gamma(t)} = \lambda(t)$.

If $\gamma(t)$ is one of such curves, then $(\gamma(t), \dot{\gamma}(t))$ is a curve in $T\mathcal{A}$ with $i_{(\gamma(t), \dot{\gamma}(t))}\phi_0 = 1$. Moreover, since $\gamma(t)$ is in $\mathcal{B}$, we have that $\dot{\gamma}(t)$ is tangent to $\mathcal{B}$, that is, $(\gamma(t), \dot{\gamma}(t)) \in T\mathcal{B}$ with $i_{(\gamma(t), \dot{\gamma}(t))}\phi_0 = 1$. Note that, since $\mathcal{B}$ is an affine subbundle of $\mathcal{A}$, the Lie algebroid prolongation $\tau_{\mathcal{A}}^\mathcal{B}: T\mathcal{A} \to \mathcal{B}$ of the Lie algebroid $\mathcal{A}$ over the fibration $\tau_{\mathcal{B}}: \mathcal{B} \to M$ is well-defined. Under some regularity conditions (to be made precise later on), we may assume that the above admissible curves are integral curves of a section $X$ which we will assume that it is a SODE section taking values in $T\mathcal{B}$. Then, from (12), we deduce that

$$(i_X\Omega_L)(b) \in \mathcal{B}^\circ_v, \text{ for all } b \in \mathcal{B},$$

where $\mathcal{B}^\circ_v$ is the vector bundle over $\mathcal{B}$ whose fibre at the point $b \in \mathcal{B}$ is

$$\mathcal{B}^\circ_v = \{ \alpha \in (T\mathcal{B})^* / \alpha(b', v_b) = 0, \forall (b', v_b) \in T\mathcal{B} \text{ such that } b' \in \mathcal{B}_{\tau_\mathcal{A}(b)} \}. $$

Based on the previous arguments, we may reformulate geometrically our problem as the search for such a SODE $X$ (defined at least on a neighborhood of $\mathcal{B}$) satisfying $i_{X(b)}\Omega_L(b) \in \mathcal{B}^\circ_v$, $i_{X(b)}\phi_0(b) = 1$ and $X(b) \in T\mathcal{B}$, at every point $b \in \mathcal{B}$.

Now, we will prove the following result.

\textbf{Proposition III.1.} \textit{If $S$ is the vertical endomorphism then the vector bundles over $\mathcal{B}$, $\mathcal{B}^\circ \to \mathcal{B}$ and $S^*(T\mathcal{B})^\circ \to \mathcal{B}$, are equal. In other words,}

$$\mathcal{B}^\circ_v = S^*(T\mathcal{B})^\circ, \text{ for all } b \in \mathcal{B}.$$

\textbf{Proof.} First, we will see that the map

$$S^*: (T\mathcal{B})^\circ \to S^*(T\mathcal{B})^\circ$$
is a linear isomorphism.
In fact, suppose that $\alpha \in (T^\mathcal{A}_b\mathcal{B})^\circ$ and $S^*\alpha = 0$. Then, we have that
$$\alpha(0, v_b) = 0, \text{ for all } v_b \in T_b\mathcal{A} \text{ such that } (T_b\tau_\mathcal{A})(v_b) = 0.$$ 
Moreover, if $(\tilde{a}, u_b) \in T^\mathcal{A}_b\mathcal{B}$, then, since $T_b\tau_\mathcal{A} : T_b\mathcal{B} \to T_{\tau_\mathcal{A}(b)}M$ is a linear epimorphism, it follows that there exists $v_b \in T_b\mathcal{B}$ satisfying
$$(T_b\tau_\mathcal{A})(v_b) = \rho^\mathcal{A}_b(\tilde{a}).$$ 
Thus, using that $\alpha \in (T^\mathcal{A}_b\mathcal{B})^\circ$, we deduce that
$$\alpha(\tilde{a}, u_b) = \alpha(\tilde{a}, v_b) + \alpha(0, u_b - v_b) = 0.$$ 
Therefore, $\alpha = 0$.
This implies that $S^* : (T^\mathcal{A}_b\mathcal{B})^\circ \to S^*(T^\mathcal{A}_b\mathcal{B})^\circ$ is a linear isomorphism and
$$\dim S^*(T^\mathcal{A}_b\mathcal{B})^\circ = r.$$ 
On the other hand, since $T_b\tau_\mathcal{A} : T_b\mathcal{B} \to T_{\tau_\mathcal{A}(b)}M$ is a linear epimorphism, we obtain that
$$\dim \tilde{\mathcal{B}}_b^\circ = r = \dim S^*(T^\mathcal{A}_b\mathcal{B})^\circ.$$ 
Finally, we will prove that $S^*(T^\mathcal{A}_b\mathcal{B})^\circ \subseteq \tilde{\mathcal{B}}_b^\circ$.
In fact, if $\alpha \in (T^\mathcal{A}_b\mathcal{B})^\circ$, $(b', v_b) \in T^\mathcal{A}_b\mathcal{B}$ and $b' \in B_{\tau_\mathcal{B}(b)}$, it follows that
$$S(b', v_b) = (0, v'_b), \text{ with } v'_b \in T_b\mathcal{B}.$$ 
Consequently,
$$(S^*\alpha)(b', v_b) = 0.$$ 
\hfill \Box

Proposition III.1 suggests us to introduce the following definition.

**Definition III.2.** A nonholonomically constrained Lagrangian system on a Lie affgebroid $\mathcal{A}$ is a pair $(L, B)$, where $L : \mathcal{A} \to \mathbb{R}$ is a $C^\infty$-Lagrangian function, and $i_B : B \hookrightarrow \mathcal{A}$ is a smooth affine subbundle of $\mathcal{A}$, called the constraint affine subbundle. By a solution of the nonholonomically constrained Lagrangian system $(L, B)$ we mean a section $X \in \Gamma(\tau^\mathcal{A}_B)$ which is a SODE on $B$ and satisfies the Lagrange-d’Alembert equations
$$\begin{align*}
(i_X\Omega_L)|_B &= \Gamma(\tau^\mathcal{A}_{S^*(T^\mathcal{A}_B)^\circ}), \\
(i_X\phi_0)|_B &= 1, \\
X|_B &= \Gamma(\tau^\mathcal{A}_B),
\end{align*}$$
where $\tau^\mathcal{A}_{S^*(T^\mathcal{A}_B)^\circ}$ is the vector bundle projection of $S^*(T^\mathcal{A}_B)^\circ \to \mathcal{B}$.

With a slight abuse of language, we will interchangeably refer to a solution of the constrained Lagrangian system as a section or the collection of its corresponding integral curves.
Remark III.3. We want to stress that a solution of the Lagrange-d’Alembert equations needs to be defined only over $\mathcal{B}$, but for practical purposes we consider it extended to $\mathcal{A}$ (or just to a neighborhood of $\mathcal{B}$ in $\mathcal{A}$). We will not make any notational distinction between a solution on $\mathcal{B}$ and any of its extensions. Solutions which coincide on $\mathcal{B}$ will be considered as equal. In accordance with this convention, by a SODE on $\mathcal{B}$ we mean a section of $T^*\mathcal{B}$ which is the restriction to $\mathcal{B}$ of some SODE defined in a neighborhood of $\mathcal{B}$. 

Now, we will analyze the form of the Lagrange-d’Alembert equations in local coordinates. Let $(x^i)$ be local coordinates on an open subset $U$ of $M$ and $\{e_0, e_\alpha\}$ be a local basis of sections of the vector bundle $T^*\mathcal{A} \rightarrow U$ adapted to $1_\mathcal{A}$. Denote by $(x^i, y^\alpha, \gamma^\alpha)$ the corresponding local coordinates on $\tilde{\mathcal{A}}$ and suppose that $\{\mu^\alpha_a e_\alpha\}$ is a local basis of the vector bundle $U^\mathcal{B} \rightarrow M$ and that $e_\alpha^\mathcal{B}$ is a local section of the affine bundle $\mathcal{B} \rightarrow M$ such that $e_0 - e_\alpha^\mathcal{B} = a_\alpha^\mathcal{B} e_\alpha$. Then, using that an element $e_0(x) + y^\alpha e_\alpha(x)$ of $\mathcal{A}_1$ belongs to $\mathcal{B}_x$ if and only if $(e_0(x) + y^\alpha e_\alpha(x)) - e_0^\mathcal{B}(x) \in (\mathcal{U}_\mathcal{B})_x$, we deduce that the local equations defining the constrained subbundle $\mathcal{B}$ as an affine subbundle of $\mathcal{A}$ are

$$\Psi^\alpha = \mu^\alpha_0 + \mu^\alpha_a y^\alpha = 0, \quad \text{for all } a \in \{1, \ldots, r\},$$

where $\mu^\alpha_0 = \mu^\alpha_a e^a_0$. Observe that $d(T^A\mathcal{B}) \in \Gamma((\tau^\mathcal{A}_1)^*)$ verifies that

$$(dT^A\mathcal{B})(b)(\tilde{a}, X_b) = 0, \quad \text{for all } (\tilde{a}, X_b) \in T^\mathcal{A}_1 \mathcal{B},$$

and, therefore, $(dT^A\mathcal{B})(b) \in (T^\mathcal{A}_1 \mathcal{B})^\circ$, for every $b \in \mathcal{B}$ and $a \in \{1, \ldots, r\}$. On the other hand, using (11) and (17), we deduce that

$$dT^A\mathcal{B} = \rho^i_0 \left( \frac{\partial \mu^\alpha_0}{\partial x^i} + y^\beta \frac{\partial \mu^\alpha_\beta}{\partial x^i} \right) \phi_0 + \rho^i_a \left( \frac{\partial \mu^\alpha_0}{\partial x^i} + y^\beta \frac{\partial \mu^\alpha_\beta}{\partial x^i} \right) \tilde{T}^\alpha + \mu^\alpha_a \tilde{V}^\alpha,$$

$\{\phi_0, \tilde{T}^\alpha, \tilde{V}^\alpha\}$ being the dual basis of $\{\tilde{T}_0, \tilde{T}_\alpha, \tilde{V}_\alpha\}$. Thus, since the matrix $(\mu^\alpha_a)$ has maximum rank $r$, it follows that the sections $\{dT^A\mathcal{B}\}_{a=1,\ldots,r}$ are linearly independent and, using that $\text{rank}(T^A\mathcal{B})^\circ = r$, we deduce that $\{dT^A\mathcal{B}(b)\}_{a=1,\ldots,r}$ is a local basis of sections of $(T^A\mathcal{B})^\circ \rightarrow \mathcal{B}$.

Therefore, using (33) and the fact that $S^* : (T^A\mathcal{B})^\circ \rightarrow S^*(T^A\mathcal{B})^\circ$ is an isomorphism, for all $b \in \mathcal{B}$, we obtain that $\{\mu^\alpha_a \theta^\alpha\}_{a=1,\ldots,r}$ is a local basis of sections of $S^*(T^A\mathcal{B})^\circ \rightarrow \mathcal{B}$, where $\theta^\alpha = \tilde{T}^\alpha - y^\alpha \phi_0$.

Consequently, the constrained motion equations (32) can be written as

$$\begin{cases}
\dot{T}_0 \Omega_L = \lambda_\alpha \mu^\alpha_a \theta^\alpha, \\
\dot{T}_\alpha \phi_0 = 1, \\
(dT^A\mathcal{B})(X) = 0,
\end{cases} \quad (33)$$

along the points of $\mathcal{B}$, where $\lambda_\alpha$ are some Lagrange multipliers to be determined.

Now, a section $X \in \Gamma(T^*\mathcal{A})$ is of the form $X = z^0 \tilde{T}_0 + z^\alpha \tilde{T}_\alpha + v^\alpha \tilde{V}_\alpha$. If we assume that the Lagrangian $L$ is regular, then the first two equations in (33) imply that any solution $X$ has to be a SODE, that is, $X = \tilde{T}_0 + y^\alpha \tilde{T}_\alpha + x^\alpha \tilde{V}_\alpha$ and, then, the first equation in (33) becomes (see (11))

$$X^\alpha \frac{\partial^2 L}{\partial y^\alpha \partial y^\beta} + (\rho^\alpha_0 + y^\alpha \rho^\alpha_0) \frac{\partial^2 L}{\partial x^i \partial y^\beta} - \rho^\beta \frac{\partial L}{\partial x^i} - (\gamma^\alpha_0 + y^\alpha \gamma^\alpha_0) \frac{\partial L}{\partial y^\beta} + \lambda_\alpha \mu^\alpha_0 = 0.$$
Thus, if $R_L$ is the Euler-Lagrange section associated with $L$, $X = R_L - W^{\alpha \beta} \lambda_\alpha \mu_\beta \bar{V}_\alpha$ and then, the third equation in (33) implies that

$$(d^{T^A \Phi})(R_L) + \lambda_b (d^{T^A \Phi})(-W^{\alpha \beta} \mu_\beta \bar{V}_\alpha) = 0, \text{ for all } a \in \{1, \ldots, r\}.$$ 

As a consequence, we get that there exists a unique solution of the Lagrange-d’Alembert equations if and only if the matrix $C^{ab}(h) = (-W^{\alpha \beta} \mu_\beta \mu_\alpha)(h)$ is regular, for all $b \in B$. 

From (33), we deduce that the differential equations for the integral curves of the vector field $\rho^T_A(X)$ are the Lagrange-d’Alembert differential equations, which read

$$\begin{align*}
\frac{dx^i}{dt} &= \rho_0^i + \rho_A^i y^A, \\
\frac{d}{dt} \left( \frac{\partial L}{\partial y^A} \right) - \rho_A^i \frac{\partial L}{\partial x^i} + (C^0_\alpha + C^\gamma_\alpha y^\gamma) \frac{\partial L}{\partial y^\gamma} &= -\lambda_\alpha \mu_\alpha, \\
\mu_0^\gamma + \mu_\alpha^\gamma y^\alpha &= 0,
\end{align*}$$

with $i \in \{1, \ldots, m\}$, $\alpha \in \{1, \ldots, n\}$ and $a \in \{1, \ldots, r\}$.

**Remark III.4.** In the above discussion, we obtained a complete set of differential equations that determine the dynamics. Now, we will analyze the form of the Lagrange-d’Alembert equations in terms of adapted coordinates to $B$. Consider local coordinates $(x^i)$ on a open set $U$ of $M$ and the local basis $\{\phi_A\}_{A=1,\ldots,n-r}$ of sections of $U_B$. Complete it to a local basis of sections $\{e_0, \phi_A, \phi_a\}$ of the vector bundle $\tau^{-1}_A(U) \to U$ adapted to $1_A$. Then, in coordinates $(x^i, y^0, y^A, y^a)$ adapted to this basis, the equations defining the constrained subbundle $B$ as an affine subbundle of $A$ (respectively, as an affine subbundle of $\tilde{A}$) are $y^a = 0$ (respectively, $y^0 = 1$ and $y^a = 0$). Thus, the Lagrange-d’Alembert differential equations read as

$$\begin{align*}
\frac{dx^i}{dt} &= \rho_0^i + \rho_A^i y^A, \\
\frac{d}{dt} \left( \frac{\partial L}{\partial y^A} \right) - \rho_A^i \frac{\partial L}{\partial x^i} + (C^0_\alpha + C^\gamma_\alpha y^\gamma) \frac{\partial L}{\partial y^\gamma} &= 0, \\
y^a &= 0.
\end{align*}$$

\diamond

**IV. Solution of Lagrange-d’Alembert Equations**

In this section, we will perform a precise global analysis of the existence and uniqueness of the solution of Lagrange-d’Alembert equations. In what follows, we will assume that the Lagrangian $L$ is regular at least in a neighborhood of $B$.

**Definition IV.1.** A constrained Lagrangian system $(L, B)$ is said to be regular if the Lagrange-d’Alembert equations have a unique solution.
In order to characterize geometrically these nonholonomic systems which are regular, we define the vector subbundle $F \subset T^\mathcal{A}_\mathcal{B} \to \mathcal{B}$ whose fibre at point $b \in \mathcal{B}$ is $F_b = b^{-1}(S^*(T^\mathcal{A}_\mathcal{B}))^\circ$, where $b_L: T^\mathcal{A} \to (T^\mathcal{A})^*$ is the vector bundle isomorphism defined by

$$b_L(X) = i_X \Omega_L + \phi_0(X)\phi_0, \text{ for all } X \in T^\mathcal{A}.$$

More explicitly,

$$F_b = \{ X \in T^\mathcal{A}/ \text{exists } \chi_\alpha \text{ s.t. } b_L(X) = \chi_\alpha \mu_\alpha^{a}\partial_a |_b \}.$$

From the definition, it is clear that the rank of $F$ is $\text{rank}(F) = \text{rank}(T^\mathcal{A}_\mathcal{B})^\circ = \text{rank}(T^\mathcal{A})^\circ - \text{rank}(T^\mathcal{A}_\mathcal{B})^\circ = r$. If we consider the sections $Z_a \in \Gamma(\mathcal{A}_\mathcal{B})$ such that $b_L(Z_a) = \mu_\alpha^{a}\partial_a$, with $a \in \{1, \ldots, r\}$, then $\{Z_a\}$ is a local basis of sections of $F$. Moreover, if $R_L$ is the Euler-Lagrange section associated with $L$, we have that

$$(i_{Z_a} \Omega_L)(R_L) + \phi_0(Z_a)\phi_0(R_L) = \mu_\alpha^{a}\partial_a(R_L) = 0$$

which implies that $\phi_0(Z_a) = 0$. Therefore, $Z_a$ is completely characterized by the conditions

$$iZ_a \Omega_L = \mu_\alpha^{a}\partial_a \text{ and } iZ_a \phi_0 = 0.$$

In addition, using these two conditions, we conclude that the local expression of $Z_a$ is the following

$$Z_a = -W^{\alpha\beta}\mu_\beta^{a}\tilde{V}_\alpha,$$

where $(W^{\alpha\beta})$ is the inverse of $(W_{\alpha\beta})$. Thus, the matrix defined in (34) is just $C^{ab} = (dT^\mathcal{A}_{\mathcal{B}})(Z_b) = -W^{\alpha\beta}\mu_\beta^{a}\mu_\alpha^{b}$ and we will denote it by $C = (C^{ab})$.

A second important geometric object is the vector subbundle $G \subset T^\mathcal{A}_\mathcal{B} \to \mathcal{B}$ whose annihilator at a point $b \in \mathcal{B}$ is $S^*(T^\mathcal{A}_\mathcal{B})^\circ$. We can consider the subbundle $G^\perp \subset T^\mathcal{A}_\mathcal{B} \to \mathcal{B}$, the orthogonal to $G$ with respect to the cosymplectic structure $(\Omega_L, \phi_0)$, which is given by

$$G^\perp_b = \{ X \in T^\mathcal{A}/ (i_X \Omega_L + \phi_0(X)\phi_0)|_{G_b} = 0 \}, \text{ for } b \in \mathcal{B}.$$ 

Note that $G^\perp_b = b^{-1}(G^\circ_b)$ is $F_b$, for all $b \in \mathcal{B}$. Moreover, we obtain

**Proposition IV.2.** $G_b$ is coisotropic in $(T^\mathcal{A}_\mathcal{B}, \Omega_L(b), \phi_0(b))$, for all $b \in \mathcal{B}$. In other words, $G^\perp_b \subseteq G_b$, for $b \in \mathcal{B}$. 

**Proof.** In fact, using (37) and since $F$ (respectively, $G^\circ$) is locally generated by $\{Z_a\}_{a=1, \ldots, r}$ (respectively, $\{S^*(dT^\mathcal{A}_{\mathcal{B}})(\tilde{\Psi})\}_{a=1, \ldots, r}$), we deduce that

$$G^\perp_b = F_b \subseteq G_b, \text{ for all } b \in \mathcal{B}. \quad \square$$

Now, we introduce the section $\Lambda_L$ of the vector bundle $\Lambda^2(T^\mathcal{A}) \to \mathcal{A}$ defined by

$$\Lambda_L(\alpha, \beta) = \Omega_L(b^{-1}(\alpha), b^{-1}(\beta)), \quad \text{(38)}$$

for $\alpha, \beta \in (T^\mathcal{A})^\circ$. $\Lambda_L$ is the (algebraic) Poisson structure (on the vector bundle $T^\mathcal{A}$) associated with the cosymplectic structure $(\Omega_L, \phi_0)$. We will denote by $\sharp_{\Lambda_L}: (T^\mathcal{A})^* \to T^\mathcal{A}$ the vector bundle morphism given by

$$\sharp_{\Lambda_L}(\alpha) = i_\alpha \Lambda_L, \text{ for } \alpha \in (T^\mathcal{A})^\circ.$$
Note that
\[ \phi_0(\sharp_{\Lambda_L}(\alpha)) = \Omega_L(\flat^{-1}(\alpha), R_L) = 0. \]

On the other hand, it is clear that
\[ \alpha = \flat_L(\flat^{-1}(\alpha)) = i_{\flat_L}(\alpha) \Omega_L + \phi_0(\flat^{-1}(\alpha)) \phi_0 \]
and, thus
\[ \alpha(R_L) = \phi_0(\flat^{-1}(\alpha)). \quad (39) \]

Moreover, from (38), we have that
\[ \text{g}_{\Lambda_L}(\alpha) = -i_{\flat_L}(\beta) \Omega_L(\flat^{-1}(\alpha)) = -\beta(\flat^{-1}(\alpha)) + \phi_0(\flat^{-1}(\beta)) \phi_0(\flat^{-1}(\alpha)) \]
which implies that (see (39))
\[ \beta(\sharp_{\Lambda_L}(\alpha)) = \beta(\flat^{-1}(\alpha) + \alpha(R_L) R_L). \]

Therefore, we have proved that
\[ \sharp_{\Lambda_L}(\alpha) = \beta(\flat^{-1}(\alpha) + \alpha(R_L) R_L), \quad \text{for } \alpha \in (T^\mathcal{A} L)^*. \quad (40) \]

Next, consider the vector subbundle \( G_{\Lambda_L} \subset T^\mathcal{A} L |_{\mathcal{B}} \rightarrow \mathcal{B} \), the orthogonal to \( G \) with respect to the Poisson structure \( (\Lambda_L) \), which is given by
\[ G^\perp_{\Lambda_L} = \sharp_{\Lambda_L}(G^\perp), \quad \text{for } b \in \mathcal{B}. \]

From (40) and since \( S^\ast(dT^\mathcal{A} L \Psi^a)(R_L) = 0 \), we deduce that
\[ \sharp_{\Lambda_L}(S^\ast(dT^\mathcal{A} L \Psi^a) |_{\mathcal{B}}) = -Z_a. \]

Consequently, we obtain that
\[ G^\perp_{\Lambda_L} = G^\perp = F_b, \quad \text{for } b \in \mathcal{B}. \]

Now, we will consider the vector subbundle \( H \subset T^\mathcal{A} L |_{\mathcal{B}} \rightarrow \mathcal{B} \) whose fibre at point \( b \in \mathcal{B} \) is \( H_b = T^\mathcal{A} L |_{\mathcal{B}} \cap G_b \). We will denote, as above, by \( H^\perp \) the orthogonal to \( H \) with respect to the cosymplectic structure \( (\Omega_L, \phi_0) \) and by \( H^\perp_{\Lambda_L} \) the orthogonal to \( H \) with respect to the Poisson structure \( \Lambda_L \).

Using (38) and (40), we have that \( i_S \Omega_L = 0 \), that is,
\[ \Omega_L(SX, Y) + \Omega_L(X, SY) = 0, \quad \text{for all } X, Y \in \Gamma(T^\mathcal{A} L), \]
or, equivalently,
\[ i_S \Omega_L = -S^\ast(i_X \Omega_L), \quad \text{for all } X \in \Gamma(T^\mathcal{A} L). \quad (41) \]

Next, denote by \( \text{grad} \Psi^a \) the gradient section corresponding to the function \( \Psi^a \) with respect to the cosymplectic structure \( (\Omega_L, \phi_0) \), that is, \( \text{grad} \Psi^a = \flat^{-1}(dT^\mathcal{A} L \Psi^a) \). Then, using (41) and the fact that \( S^\ast \phi_0 = 0 \), we have that \( i_{(\text{grad} \Psi^a)_{|\mathcal{B}} \Omega_L = -i_{Z_a} \Omega_L \) and \( \phi_0(S \text{grad} \Psi^a) = \phi_0(Z_a) = 0 \) which implies that
\[ (S \text{grad} \Psi^a)_{|\mathcal{B}} = -Z_a, \quad \text{for all } a \in \{1, \ldots, r\}. \quad (42) \]
Since the sections \( \{d^{T,A}\Psi^a\} \) are linearly independent, it follows that the sections \( \{\text{grad } \Psi^a\} \) also are linearly independent. Thus, using (12), we conclude that the sections \( \{(\text{grad } \Psi^a)|_{B}, Z_a\} \) are linearly independent. On the other hand, \( H^\circ = (T \hat{A} B)^\circ + G^\circ \) (43) and, therefore, the space of sections of \( H \) is locally generated by \( \{(\text{grad } \Psi^a)|_{B}, Z_a\} \). This implies that \( \{(\text{grad } \Psi^a)|_{B}, Z_a\} \) is a local basis of sections of \( H^\perp \) and \( \text{rank}(H^\perp) = \text{corank}(H) = 2r \).

Now, using (43), we deduce that
\[
H^\perp \Lambda_L = (T \hat{A} B)^\perp \Lambda_L + F.
\]
On the other hand, we introduce the Hamiltonian section \( X_{\Psi^a}^{\Lambda_L} \) associated with the function \( \Psi^a \) with respect to \( \Lambda_L \) which is defined by
\[
X_{\Psi^a}^{\Lambda_L} = -\sharp_{\Lambda_L} (d^{T,A} \Psi^a), \text{ for } a \in \{1, \ldots, r\}.
\]
From (40), we have that
\[
\flat_L X_{\Psi^a}^{\Lambda_L} = d^{T,A} \Psi^a - (d^{T,A} \Psi^a)(R_L)\phi_0.
\]
Thus, using that the sections \( \{d^{T,A} \Psi^a, \phi_0\} \) are independent, it follows that the sections \( \{X_{\Psi^a}^{\Lambda_L}\} \) are also independent. Moreover, from (45), we obtain that
\[
X_{\Psi^a}^{\Lambda_L} = \text{grad } \Psi^a - (d^{T,A} \Psi^a)(R_L)R_L
\]
which implies that
\[
S X_{\Psi^a}^{\Lambda_L} = -Z_a, \text{ for } a \in \{1, \ldots, r\}.
\]
Therefore, \( \{X_{\Psi^a}^{\Lambda_L}, Z_a\} \) is a local basis of sections of \( H^\perp \Lambda_L \) and \( \text{rank}(H^\perp \Lambda_L) = \text{corank}(H) = 2r \).

The relation among these objects is described by the next result.

**Theorem IV.3.** The following properties are equivalent:

1. The constrained Lagrangian system \((L, B)\) is regular,
2. the matrices \( C = (C^{ab}) \) are non-singular,
3. \( T_0 \hat{A} B \cap F_b = \{0\}, \text{ for all } b \in B, \)
4. \( H_b \cap H_b^\perp = \{0\}, \text{ for all } b \in B, \)
5. \( H_b \cap H_b^\perp \Lambda_L = \{0\}, \text{ for all } b \in B. \)

**Proof.** [(1) \( \iff \) (2)] This result was proved in Section III.

[(2) \( \iff \) (3)] (\( \Rightarrow \)) Suppose that \( C \) is non-singular and let be \( X \in T_0 \hat{A} B \cap F_b \). Thus, \( X = \sum_{a=1}^{r} \lambda_a Z_a(b) \) and \( (d^{T,A} \Psi^a)(b)(X) = 0, \text{ for all } a \in \{1, \ldots, r\} \), which implies that
\[
\sum_{c=1}^{r} \lambda_c (d^{T,A} \Psi^a)(b)(Z_c(b)) = \sum_{c=1}^{r} \lambda_c C^{ac} = 0.
\]
Therefore, we deduce that \( \lambda_c = 0, \text{ for all } c, \) and consequently \( X = 0 \).

(\( \Leftarrow \)) Conversely, take an arbitrary linear combination of columns of \( C \) at some point \( b \) such that
\[
\sum_{c=1}^{r} \lambda_c C^{ac}(b) = 0, \text{ for all } a \in \{1, \ldots, r\}.
\]
Thus, \( \sum_{c=1}^{r} \lambda_c Z_c(b) \in T_b^0 \mathcal{B} \) which implies that \( \sum_{c=1}^{r} \lambda_c Z_c(b) = 0 \), and hence \( \lambda_c = 0 \), for all \( c \in \{1, \ldots, r\} \).

\[(3) \Leftrightarrow (4) \] \( \Rightarrow \) Let \( b \in \mathcal{B} \), then we have that \( T_b^0 \mathcal{B} \cap F_b = T_b^0 \mathcal{B} \cap G_b^1 = \{0\} \) and \( T_b^0 \mathcal{A} = T_b^0 \mathcal{B} \oplus G_b^1 \) (note that \( \dim T_b^0 \mathcal{B} + \dim G_b^1 = \dim T_b^0 \mathcal{A} \)). Hence, from Proposition IV.2 we obtain that

\[ G_b = (T_b^0 \mathcal{B} \cap G_b) \oplus G_b^1 = H_b \oplus G_b^1. \]

If \( X \in H_b \cap H_b^1 \) then, from the above decomposition, we also have that \( X \in G_b^1 \) and, thus, \( X = 0 \).

\( \Leftarrow \) Conversely, let \( b \in \mathcal{B} \) and take an element \( X \in T_b^0 \mathcal{B} \cap F_b = T_b^0 \mathcal{B} \cap G_b^1 \subset T_b^0 \mathcal{B} \cap G_b = H_b \).

Since, \( \Omega_L(b)(X, Y) = 0 \) and \( \phi_0(b)(X) = 0 \), for all \( Y \in H_b \), we conclude that \( X \in H_b^1 \) and, therefore, \( X = 0 \).

\[(4) \Leftrightarrow (5) \] \( \Rightarrow \) Let \( b \in \mathcal{B} \), then we have that \( H_b \cap H_b^1 = \{0\} \) and, as a consequence, the restriction \( (\Omega_L^{H_b}, \phi_0^{H_b}) \) to \( H_b \) of the cosymplectic structure \( (\Omega_L, \phi_0) \) is a cosymplectic structure on \( H_b \). In other words, the map

\[ \flat^{H_b}_L : H_b \to H_b^*, \quad X \in H_b \mapsto (i_X \Omega_L(b) + \phi_0(b)(X)\phi_0(b))|_{H_b} \in H_b^*, \]

is a linear isomorphism. Thus, we can consider the Reeb vector \( R^H(b) \) of the cosymplectic vector space \( (H_b, \Omega_L^{H_b}, \phi_0^{H_b}) \) which is characterized by the following conditions

\[ i_{R^H(b)}\Omega_L^{H_b} = 0 \quad \text{and} \quad i_{R^H(b)}\phi_0^{H_b} = 1. \quad (46) \]

Now, we will prove that \( H_b \cap \sharp^L_{\Lambda_L}(H_b^o) = \{0\} \).

Suppose that \( \alpha \in H_b^o \) and \( \sharp^L_{\Lambda_L}(\alpha) \in H_b \). Then, using (40) and the fact that \( \phi_0(b)(\sharp^L_{\Lambda_L}(\alpha)) = 0 \), we deduce that

\[ i_{\sharp^L_{\Lambda_L}(\alpha)}\Omega_L(b) = -\alpha + \alpha(R_L(b))\phi_0(b). \quad (47) \]

Therefore, from (40) and since \( \alpha \in H_b^o \) and \( \sharp^L_{\Lambda_L}(\alpha) \in H_b \), we have that

\[ 0 = (i_{\sharp^L_{\Lambda_L}(\alpha)}\Omega_L(b))(R^H(b)) = \alpha(R_L(b)) \]

which implies that (see (47))

\[ \flat^{H_b}_L(\sharp^L_{\Lambda_L}(\alpha)) = 0. \]

Consequently, \( \sharp^L_{\Lambda_L}(\alpha) = 0 \).

\( \Leftarrow \) Let \( b \in \mathcal{B} \) and take an element \( X \in H_b \cap H_b^1 \). Then

\[ (i_X \Omega_L(b) + \phi_0(b)(X)\phi_0(b))|_{H_b} = 0. \]

Thus, using that \( X \in H_b \), it follows that

\[ \phi_0(b)(X) = 0 \quad \text{and} \quad \alpha = i_X \Omega_L(b) \in H_b^o. \]

Therefore, \( \alpha(R_L(b)) = 0 \) and \( \alpha = \beta_L(X) \). This implies that (see (40))

\[ X = -\sharp^L_{\Lambda_L}(\alpha) \in \sharp^L_{\Lambda_L}(H_b^o) = H_b^\perp_{\Lambda_L}. \]

Consequently, \( X = 0 \).

\[ \square \]

**Proposition IV.4.** Conditions (3), (4) and (5) in Theorem IV.3 are equivalent, respectively, to
Proof. The equivalence of (3) and (3') follows by computing the dimension of the corresponding spaces. The ranks of $T\tilde{A}$, $T\tilde{A}B$ and $F$ are

\[
\begin{align*}
\text{rank}(T\tilde{A}) &= 2 \text{rank}(A) + 1, \\
\text{rank}(T\tilde{A}B) &= \text{rank}(A) + \text{rank}(B) + 1, \\
\text{rank}(F) &= \text{rank}(B^c) = \text{rank}(A) - \text{rank}(B). \\
\end{align*}
\]

Thus, $\text{rank}(T\tilde{A}) = \text{rank}(T\tilde{A}B) + \text{rank}(F)$, and the result follows. The equivalence between (4) and (4') is obvious, since we are assuming that the free Lagrangian is regular, i.e., $(Ω_L, φ_0)$ is a cosymplectic structure on $T\tilde{A}$. Finally, the equivalence of (5) and (5') is also obvious because we have that $\text{rank}(H^{1, −λ_1}) = \text{corank}(H)$.

IV.1. Projectors. Now, we can express the constrained dynamical section in terms of the free dynamical section by projecting to the adequate space, either $T\tilde{A}B$ or $H$, according to each of the above decompositions of $T\tilde{A}|_{I|}$. Of course, both procedures give the same result.

IV.1.1. Projection to $T\tilde{A}B$. Assuming that the constrained Lagrangian system is regular, we have a direct sum decomposition

\[T\tilde{A}|_{I|} = T\tilde{A}B \oplus F,\quad \text{for all } b \in B,\]

where we recall that the subbundle $F \subset T\tilde{A}$ is defined by $F = b^{-1}_L(S^*(T\tilde{A}B)^c)$. We will denote by $P$ and $Q$ the complementary projectors defined by this decomposition, that is,

\[P_b : T\tilde{A}|_{I|} \to T\tilde{A}B \quad \text{and} \quad Q_b : T\tilde{A}|_{I|} \to F, \quad \text{for all } b \in B.\]

Theorem IV.5. Let $(L, B)$ be a regular constrained Lagrangian system and $R_L$ be the solution of the free dynamics, i.e., $i_{R_L}Ω_L = 0$ and $i_{R_L}φ_0 = 1$. Then, the solution of the constrained dynamics is the SODE $R_{nh}$ obtained by projection $R_{nh} = P(R_L|_{I|})$.

Proof. Indeed, if we write $R_{nh}(b) = R_L(b) - Q_b(R_L(b))$, for $b \in B$, then we have

\[i_{R_{nh}(b)}Ω_L(b) = i_{R_L(b)}Ω_L(b) - i_{Q_b(R_L(b))}Ω_L(b) = -i_{Q_b(R_L(b))}Ω_L(b)\]

which is an element of $S^*(T\tilde{A}B)^c$, because $Q_b(R_L(b)) \in F$ and $φ_0(Q_b(R_L(b))) = 0$. Moreover, using this last fact and since $R_L$ is a SODE and $S(Q(R_L)) = 0$, we have that $R_{nh}$ is also a SODE.

Let $(x^i)$ be local coordinates on an open subset $U$ of $M$ and $\{e_0, e_α\}$ be a local basis of sections of the vector bundle $T\tilde{A}|_{I|} \to U$ adapted to $L$. Denote by $(x^i, y^h, y^α)$ the corresponding local coordinates on $\tilde{A}$ and suppose that the equations defining the constrained subbundle $B$ as an affine subbundle of $A$ are

\[Ψ^a = μ^a_0 + μ^a_α y^α = 0, \quad \text{for } a \in \{1, \ldots, r\}.\]
Then, the local expression of the projector over $\mathcal{T}^\mathcal{A}\mathcal{B}$ is

$$P_b = Id - \sum_{1 \leq a,c \leq r} C_{ac}(b)Z_c(b) \otimes (d^{\mathcal{T}^\mathcal{A}\mathcal{B}}\Psi^a)(b),$$

for all $b \in \mathcal{B}$, $(C_{ab})$ being the inverse matrix of $(\mathcal{C}^{ab})$.

Hence, if the constrained Lagrangian system is regular, the solution of the constrained dynamics is

$$R_{nh} = R_L|_\mathcal{B} - \sum_{1 \leq a,c \leq r} C_{ac}\rho^{\mathcal{T}^\mathcal{A}\mathcal{B}}(R_L)(\Psi^c)Z_c|_\mathcal{B}.$$  

From the regularity of the local matrices $\mathcal{C}$ we deduce that $(P,Q)$ may be extended (in many ways) to an open neighborhood of $\mathcal{B}$. Therefore, $R_{nh}$ may also be extended to an open neighborhood of $\mathcal{B}$. This fact will be used in the following proposition.

**Proposition IV.6.** Let $(L, \mathcal{B})$ be a regular constrained Lagrangian system, $\Theta_L$ be the Poincaré-Cartan 1-section and $\mathcal{L}^{\mathcal{T}^\mathcal{A}\mathcal{A}}$ be the Lie derivative on $\mathcal{T}^\mathcal{A}\mathcal{A}$.

i) If $R_{nh}$ is the solution of the constrained dynamics

$$\mathcal{L}^{\mathcal{T}^\mathcal{A}\mathcal{A}}R_{nh} = d^{\mathcal{T}^\mathcal{A}\mathcal{A}}L - \mathcal{L}^{\mathcal{T}^\mathcal{A}\mathcal{A}}Q(R_L)|\mathcal{B},$$

ii) We have that

$$\mathcal{L}^{\mathcal{T}^\mathcal{A}\mathcal{A}}(\Theta_L)|\mathcal{B} \in \Gamma(t_{S^*(\mathcal{T}^\mathcal{A}\mathcal{B})}).$$

**Proof.** (i) It follows since $R_{nh} = P(R_L) = R_L - Q(R_L)$ and $\mathcal{L}^{\mathcal{T}^\mathcal{A}\mathcal{A}}R_{nh} = d^{\mathcal{T}^\mathcal{A}\mathcal{A}}L$.

(ii) Since $Q(R_L) = \sum_{a=1}^r \Lambda_a Z_a$, with $\Lambda_a = C_{ac}\rho^{\mathcal{T}^\mathcal{A}\mathcal{B}}(R_L)(\Psi^c)$, and using (10) and (37), we deduce that

$$\mathcal{L}^{\mathcal{T}^\mathcal{A}\mathcal{A}}R_{nh} = i\sum_{a=1}^r \Lambda_a d^{\mathcal{T}^\mathcal{A}\mathcal{A}}\Theta_L + d^{\mathcal{T}^\mathcal{A}\mathcal{A}}i\sum_{a=1}^r \Lambda_a Z_a \Theta_L = -\sum_{a=1}^r \Lambda_a i Z_a \Omega_L = -\Lambda_a \mu_a^a \theta^a.$$

Thus, $(\mathcal{L}^{\mathcal{T}^\mathcal{A}\mathcal{A}}R_{nh})|\mathcal{B} \in \Gamma(t_{S^*(\mathcal{T}^\mathcal{A}\mathcal{B})}).$ 

**IV.1.2. Cosymplectic Projection to $H$.** We have seen that the regularity condition for the constrained system $(L, \mathcal{B})$ can be equivalently expressed by requiring that the subbundle $H$ is a cosymplectic subbundle of $(\mathcal{T}^\mathcal{A}\mathcal{A}, \Omega_L, \phi_0)$. Thus, the restriction $(\Omega^H, \phi^H_0)$ to $H$ is a cosymplectic structure on $H$. Therefore, there exists a unique solution on $H$ of the equations

$$i_X \Omega_L^H = 0 \text{ and } i_X \phi^H_0 = 1.$$

(48)

**Proposition IV.7.** The solution of equations (48) is precisely the solution of the constrained dynamics $R_{nh}$.

**Proof.** Since $R_{nh}$ is the solution of the constrained dynamics, then $R_{nh}(b) \in \mathcal{T}^\mathcal{A}\mathcal{B}_b$ for all $b \in \mathcal{B}$. Moreover, using that $R_{nh}$ is a SODE, we obtain that $R_{nh}(b) \in G_b$. Then, $R_{nh}(b) \in H_b$ and it is obvious that it verifies (48). Note that $i_{R_{nh}} \Omega_L(b) \in S^*(\mathcal{T}^\mathcal{A}\mathcal{B})^o \subset G_b \subset H_b^o$, for all $b \in \mathcal{B}$.

□
On the other hand, we have a direct sum decomposition

\[ T^b_A b = H_b \oplus H_b^\perp, \]

for all \( b \in B \). We will denote by \( P \) and \( Q \) the complementary projectors defined by this decomposition, that is,

\[ P_b : T^b_A b \rightarrow H_b \quad \text{and} \quad Q_b : T^b_A b \rightarrow H_b^\perp, \]

for all \( b \in B \).

Then, we have the following result.

**Theorem IV.8.** Let \((L, B)\) be a regular constrained Lagrangian system and \( R_L \) be the solution of the free dynamics, i.e., \( i_{R_L} \Omega_L = 0 \) and \( i_{R_L} \phi_0 = 1 \). Then, the solution of the constrained dynamics is the SODE \( R_{nh} \) obtained by projection \( R_{nh} = P(R_L|B) \).

**Proof.** If \( b \in B \) and \( X \in T^b_A b \), then, using that \( Q_b(R_L(b)) \in H_b^\perp \), it follows that

\[
\left( i_{P_b(R_L(b))} \Omega_L(b) + \phi_0^H(b)(P_b(R_L(b)))\phi_0^H(b) \right)(P_b(X)) \\
= - \left( i_{Q_b(R_L(b))} \Omega_L(b) + \phi_0(b)(Q_b(R_L(b)))\phi_0^H(b) \right)(P_b(X)) + \phi_0^H(b)(P_b(X)) \\
= \phi_0^H(b)(P_b(X)).
\]

Thus, we deduce that

\[
i_{P(R_L|B)} \Omega_L^H + \phi_0^H(P(R_L|B))\phi_0^H = \phi_0^H.
\]

In particular, from Proposition IV.7, we obtain that

\[
1 = \phi_0^H(R_{nh}) = \phi_0^H(P(R_L|B)).
\]

Therefore, using (49), we conclude that

\[
i_{P(R_L|B)} \Omega_L^H = 0.
\]

Consequently, from (50) and (51), it follows that

\[ R_{nh} = P(R_L|B). \]

Next, we will denote by \( \{\cdot, \cdot\}_L \) the Poisson bracket on \( A \) induced by the algebraic Poisson structure \( \Lambda_L \) given by

\[
\{\Psi, \Psi'\}_L = -(d^{T^A A}\Psi')(X^L_{\Psi'}) = \Omega_L(X^L_{\Psi'}, X^L_{\Psi'}),
\]

for \( \Psi, \Psi' \in C^\infty(A) \).

On the other hand, we introduce the Hamiltonian section \( X^L_{\Psi_a} \in \Gamma(T^*_A) \) associated with the function \( \Psi_a \) with respect to the cosymplectic structure \((\Omega_L, \phi_0)\) which is defined by

\[
X^L_{\Psi_a} = b^{-1}_L(d^{T^A A}\Psi_a) - (d^{T^A A}\Psi_a)(R_L)R_L, \quad \text{for} \quad a \in \{1, \ldots, r\},
\]

where \( b_L : T^A A \rightarrow (T^A A)^* \) is the vector bundle isomorphism defined in (30). Note that, from (40) and (44), we obtain that

\[ X^L_{\Psi_a} = X^L_{\Psi_a}. \]
Then, using that \( \{(\text{grad } \Psi^a)|_B, Z_a = -S(\text{grad } \Psi^a)|_B\}_{a=1,...,r} \) is a local basis of sections of \( H^\perp \), we deduce that the local expression of the projector \( P \) is
\[
P = \text{Id} - \sum_{1 \leq a,b,c,d \leq r} C_{a b c d} \{\Psi^b, \Psi^d\}_L Z_a \otimes S^*(dT^\Lambda A \Psi^c) + \sum_{1 \leq a,b \leq r} C_{a b} X^{\Lambda^c}_{\Psi^c} \otimes S^*(dT^\Lambda A \Psi^b) - \sum_{1 \leq a,b \leq r} C_{a b} Z_a \otimes (dT^\Lambda A \Psi^b),
\]
along the points of \( B \).

IV.1.3. Poisson Projection to \( H \). Assuming that the constrained Lagrangian system is regular, we have a direct sum decomposition
\[
T^\Lambda_b A = H_b \oplus H^\perp_b, \quad \text{for all } b \in B,
\]
where we recall that \( H^\perp_b = L^\perp \oplus H^\Lambda_b \). We will denote by \( \tilde{P} \) and \( \tilde{Q} \) the complementary projectors defined by this decomposition, that is,
\[
\tilde{P}_b : T^\Lambda_b A \to H_b \quad \text{and} \quad \tilde{Q}_b : T^\Lambda_b A \to H^\perp_b, \quad \text{for all } b \in B.
\]

**Theorem IV.9.** Let \((L, B)\) be a regular constrained Lagrangian system and \( R_L \) be the solution of the free dynamics. Then, the solution of the constrained dynamics is the SODE \( R_{n h} \) obtained by projection \( R_{n h} = \tilde{P}(R_L|_B) \).

*Proof.* Suppose that \( b \in B \). Since \( Q_b(R_L(b)) = Q_b(R_L(b)) \in F_b = G^\perp_b \subset G^\Lambda_b \subset H^\perp_b \), \( \tilde{P}_b(R_L(b)) \in H_b \) and \( R_L(b) = \tilde{P}_b(R_L(b)) + Q_b(R_L(b)) \), we conclude that
\[
\begin{align*}
P_b(R_L(b)) &= \tilde{P}_b(R_L(b)) \quad \Rightarrow \quad R_{n h}(b),
\end{align*}
\]
Since also \( \{Z_a, X^{\Lambda^c}_{\Psi^c}\}_{a=1,...,r} \) is a local basis of \( H^\perp_b \), we obtain that the local expression of the projector \( \tilde{P} \) is
\[
\tilde{P} = \text{Id} - \sum_{1 \leq a,b,c,d \leq r} C_{a b c d} \{\Psi^b, \Psi^d\}_L Z_a \otimes S^*(dT^\Lambda A \Psi^c) + \sum_{1 \leq a,b \leq r} C_{a b} X^{\Lambda^c}_{\Psi^c} \otimes S^*(dT^\Lambda A \Psi^b) - \sum_{1 \leq a,b \leq r} C_{a b} Z_a \otimes (dT^\Lambda A \Psi^b),
\]
along the points of \( B \).

IV.2. The constrained Poincaré-Cartan 2-section. Let \((L, B)\) be a regular constrained Lagrangian system on a Lie algebroid \( A \) of rank \( n \) with regular Lagrangian function \( L : A \to \mathbb{R} \) and with constraint subbundle \( B \) of corank \( r \). The equations for the Lagrange-d’Alembert section \( R_{n h} = P(R_L|_B) \) can be entirely written in terms of objects in the Lie algebroid prolongation \((\tau^A_M : T^A B \to B, [,]_A^M, \rho^A_M)\) of the Lie algebroid \((\tau_A^A : \tilde{A} \to M, [,]_A^\Lambda, \rho^A_A)\) over the fibration \( \tau_B : B \to M \). In order to do this, for every point \( b \in B \), we define
\[
\omega(b) = \Omega_L(b) - (i_{Q_b(R_L(b))}\Omega_L(b)) \wedge \phi_0(b).
\]
ω is a section of the vector bundle $\wedge^2(\tilde{T}^\ast\mathcal{A})|_B \to \mathcal{B}$. We also have that $\phi_0(b) \wedge \omega^n(b) \neq 0$, for all $b \in \mathcal{B}$. Thus, there exists a unique section $X$ of $\tilde{T}^\ast\mathcal{A}|_B \to \mathcal{B}$ such that

$$i_X \omega = 0 \quad \text{and} \quad i_X \phi_0 = 1. \quad (52)$$

Note that $X$ is the solution of the constrained dynamics. In fact, since $\phi_0(P(R_{L|B})) = 1$, it follows that

$$i_{P(R_{L|B})} \omega = i_{P(R_{L|B})} \Omega_L + i_{Q(R_{L|B})} \Omega_L - (i_{Q(R_{L|B})} \Omega_L)(P(R_{L|B}))\phi_0|_B$$

$$= -i_{Q(R_{L|B})} \Omega_L)(P(R_{L|B}))\phi_0|_B.$$

Thus, using that $Q(R_{L|B})$ is a section of $F \to \mathcal{B}$ and the fact that $P(R_{L|B})$ is a SODE, we conclude that $i_{P(R_{L|B})} \omega = 0$ which proves that $X = P(R_{L|B}) = R_{nh}$.

Next, we get the following.

**Theorem IV.10.** If $\tilde{\omega}$ and $\tilde{\phi}_0$ are the restrictions of $\omega$ and $\phi_0$ to the vector subbundle $\tilde{T}^\ast\mathcal{B}$ of $\tilde{T}^\ast\mathcal{A}$, then the solution $R_{nh} = P(R_{L|B})$ of the constrained dynamics verifies the equations

$$i_X \tilde{\omega} = 0 \quad \text{and} \quad i_X \tilde{\phi}_0 = 1. \quad (53)$$

Moreover, the unique SODE $X$ on $\tilde{T}^\ast\mathcal{B}$ satisfying $[53]$ is just $R_{nh} = P(R_{L|B})$.

**Proof.** Since the section $R_{nh} = P(R_{L|B})$ satisfies $[52]$, then it also verifies $[53]$. Now, let $X$ be a SODE on $\tilde{T}^\ast\mathcal{B}$ such that $i_X \tilde{\omega} = 0$ and $i_X \tilde{\phi}_0 = 1$. Then, we have that

$$(i_X \omega)(P(Y)) = 0, \quad \text{for all} \ Y \text{ section of } \tilde{T}^\ast\mathcal{A}|_B \to \mathcal{B}. \quad (54)$$

On the other hand, using that $S(Q(Y)) = 0$, $\phi_0(Q(Y)) = 0$ and the fact that $X$ is a SODE, we obtain

$$(i_X \omega)(Q(Y)) = -(i_{Q(Y)} \Omega_L)(X) - (i_{Q(R_{L|B})} \Omega_L)(X)\phi_0(Q(Y)) + (i_{Q(R_{L|B})} \Omega_L)(Q(Y)) = 0, \quad (55)$$

for all $Y$ section of $\tilde{T}^\ast\mathcal{A}|_B \to \mathcal{B}$. Finally, from $[54]$ and $[55]$, we conclude that $i_X \omega = 0$ which implies that $X = R_{nh}$. \[\square\]

**Definition IV.11.** The 2-section $\tilde{\omega}$ is said to be the constrained Poincaré-Cartan 2-section.

**Remark IV.12.** Note that $(\tilde{\omega}, \tilde{\phi}_0)$ is not a cosymplectic structure on $\tilde{T}^\ast\mathcal{B}$ so that it may be another solution of the equations

$$i_X \tilde{\omega} = 0 \quad \text{and} \quad i_X \tilde{\phi}_0 = 1. \quad \diamond$$

**V. Constrained Hamiltonian Systems and the nonholonomic bracket**

**V.1. Constrained Hamiltonian Systems.** We now pass to the Hamiltonian description of the nonholonomic system on a Lie affgebroid. The equivalence between the Lagrangian and the Hamiltonian description of an unconstrained system was discussed in Section 11.1. We now consider a nonholonomic system, described by a hyperregular Lagrangian $L$ and a constrained affine subbundle $\tau_B : \mathcal{B} \to M$ of the bundle $\tau_A : \mathcal{A} \to M$. Denote by $\bar{\mathcal{B}}$ the image of $\mathcal{B}$ under the Legendre transformation, which is a submanifold of $V^\ast$. If $\mathcal{B}$ is again locally defined
by \( r \) independent functions \( \Psi^a \), then the constraint functions on \( V^* \) describing \( B \) become
\[
\psi^a = \Psi^a \circ \text{leg}_L^{-1},
\]
i.e.,
\[
\psi^a(x^j, y_a) = \Psi^a(x^j, \partial H / \partial y_a),
\]
where the local expression of the Hamiltonian section is
\[
h(x^i, y_a) = (x^i, -H(x^j, y_j), y_a).
\]
A curve \( \gamma : I \to V^* \) is a solution of the equations of motion for the nonholonomic Hamiltonian system \((h, B)\) if \( \text{leg}_L^{-1} \circ \gamma : I \to A \) is a solution of the Lagrange-d’Alambert equations for the system \((L, B)\). Thus, using (30), (31) and (35), we deduce that
\[
\gamma : I \to V^*, \quad t \mapsto \gamma(t) = (x^i(t), y_a(t)),
\]
is a solution of the equations of motion if
\[
\left\{
\begin{array}{l}
\frac{dx_i}{dt} = \rho^i_0 + \rho^i_a \frac{\partial H}{\partial y_a}, \\
\frac{dy_a}{dt} + \rho^a_0 \frac{\partial H}{\partial x^i} + (C^\gamma_{a0} + C^\gamma_{a\beta} \frac{\partial H}{\partial y_\beta}) y_\gamma = -\tilde{\lambda}_a H^{a\beta} \frac{\partial \psi^a}{\partial y_\beta}, \\
\psi^a(x^j, y_a) = 0,
\end{array}
\right.
\]
with \( i \in \{1, \ldots, m\}, \alpha \in \{1, \ldots, n\} \) and \( a \in \{1, \ldots, r\} \) and where \( \tilde{\lambda}_i \) are the Lagrange multipliers and \( H^{a\beta} \) are the components of the inverse matrix of the regular matrix \((H_{a\beta}) = (\partial^2 H / \partial y_a \partial y_\beta)\).

An intrinsic description of the equations of motion is obtained as follows. Consider the sub-bundle \( \bar{G} \subset T^A V^* |_{\bar{B}} \to \bar{B} \) such that \( \bar{G}_{\bar{b}} = (T\text{leg}_L)(G_b) \), with \( \bar{b} = \text{leg}_L(b) \) and \( b \in B \). It is easy to prove that \( \bar{G}_{\bar{b}} \) is locally generated by the independent sections
\[
\hat{\mu}^a = \frac{\partial \psi^a}{\partial y_a} H^{a\beta}(\bar{e}^\beta - \left. \frac{\partial H}{\partial y_\beta} \right|_{\bar{e}^0}), \quad \text{for all} \quad a \in \{1, \ldots, r\}.
\]

Note that, from (27) and (56), it follows that
\[
(T\text{leg}_L, \text{leg}_L)^*(\hat{\mu}^a) = \mu^a a^\alpha, \quad \text{for all} \quad a.
\]

Moreover, it is clear that the restriction \( \tau_{\bar{G}} : \bar{B} \to M \) to \( \bar{B} \) of the vector bundle projection \( \tau^*_V : V^* \to M \) is a fibration. Thus, we can consider the Lie algebroid \( \tau^*_A : T^A B \to \bar{B} \) and the Hamilton equations of motion of the nonholonomic system can then be rewritten in intrinsic form as
\[
\left\{
\begin{array}{l}
(i_X \Omega_h)_{|\bar{B}} \in \Gamma(\tau_{\bar{G}}^*), \\
(i_X \eta)_{|\bar{B}} = 1, \\
X_{|\bar{B}} \in \Gamma(\tau_{\bar{G}}^{|\bar{B}}),
\end{array}
\right.
\]
\((\Omega_h, \eta) \) being the cosymplectic structure on \( T^A V^* \) defined in (13) and (14) and \( \tau_{\bar{G}} \) being the vector bundle projection of \( \bar{G}^0 \to \bar{B} \). It is said that the constrained Hamiltonian system \((h, B)\) is regular if the Hamilton equations have a unique solution, which we will denote by \( \bar{R}_{nm} \).

Now, denote by \( b_h : T^A V^* \to (T^A V^*)^* \) the vector bundle isomorphism given by
\[
b_h(X) = i_X \Omega_h + \eta(X) \eta, \quad \text{for} \quad X \in T^A V^*.
\]
and by $\Lambda_h$ the (algebraic) Poisson structure on $T^A V^*$ defined by

$$\Lambda_h(\alpha, \beta) = \Omega_h(b^{-1}_h(\alpha), b^{-1}_h(\beta)), \text{ for } \alpha, \beta \in (T^A V^*)^*.$$  

We also can transport the vector subbundle $F \subset T^A A_\mathcal{B} \rightarrow \mathcal{B}$ and obtain a subbundle $\bar{F} \subset T^A V^*_\mathcal{B} \rightarrow \mathcal{B}$ such that $\bar{F}_b = (T legL)(F_b)$, with $b = legL(b)$ and $b \in \mathcal{B}$. Moreover, it satisfies that

$$\bar{F}_b = b^{-1}_h(G^b_\mathcal{B}) = \bar{G}^b_\mathcal{B} = \bar{G}^{i,\Lambda_h}$$

where $\mathcal{G}^\perp$ (respectively, $\mathcal{G}^{i,\Lambda_h}$) denotes the orthogonal to $\mathcal{G}$ with respect to the cosymplectic (respectively, Poisson) structure $(\Omega_h, \eta)$ (respectively, $\Lambda_h$). Notice that $\bar{F}$ is locally generated by the sections $Z_a = X^\Lambda_h_{ab} \in \Gamma(T^*_\mathcal{A})$ characterized by the conditions

$$i_{Z_a} \Omega_h = \bar{\mu}^a \text{ and } i_{Z_a} \eta = 0,$$

for all $a \in \{1, \ldots, r\}$. Of course, $Z_a$ and $\bar{Z}_a$ are $(T legL, legL)$-related. Moreover, its local expression is the following

$$Z_a = -\mathcal{H}^{\alpha\beta} \frac{\partial \psi^a}{\partial y_3} \bar{e}_\alpha. \tag{59}$$

We will denote by $(\bar{C}^{ab})$ the matrix which elements are

$$\bar{C}^{ab} = (dT^A V^* \psi^a)(\bar{Z}_b) = -\bar{\mu}^b(X^\Lambda_h) = -\mathcal{H}^{\alpha\beta} \frac{\partial \psi^a}{\partial y_3} \frac{\partial \psi^b}{\partial y_3}, \tag{60}$$

for all $a, b \in \{1, \ldots, r\}$, where $\psi^a_{\Lambda_h}$ is the Hamiltonian section of $\psi^a$ with respect to the (algebraic) Poisson structure $\Lambda_h$, that is, $X^\Lambda_h = -\bar{Z}_b (dT^A V^* \psi^a)$, $\bar{Z}_b : (T^A V^*)^* \rightarrow T^A V^*$ being the vector bundle morphism given by $\bar{Z}_b(\alpha) = i_a \Lambda_h$, for $\alpha \in (T^A V^*)^*$. It is easy to check that $\bar{C}^{ab} \circ legL = C^{ab}$.

In a similar way that in the Lagrangian side, we can consider the subbundle $\bar{H} \subset T^A V^*_\mathcal{B} \rightarrow \mathcal{B}$ such that $\bar{H}_b = T^A B_{\mathcal{B}} \cap \bar{G}_b = (T legL)(H_b)$, with $b = legL(b)$ and $b \in \mathcal{B}$, and its orthogonal $\bar{H}^\perp$ (respectively, $\bar{H}^{i,\Lambda_h}$) with respect to the cosymplectic (respectively, Poisson) structure $(\Omega_h, \eta)$ (respectively, $\Lambda_h$).

Using Theorem IV.3 and the fact that the Lagrangian $L$ is hyperregular, we deduce the next result.

**Theorem V.1.** The following properties are equivalent:

1. The constrained Lagrangian system $(L, \mathcal{B})$ is regular,
2. the constrained Hamiltonian system $(h, \mathcal{B})$ is regular,
3. the matrices $\bar{C} = (\bar{C}^{ab})$ are non-singular,
4. $\mathcal{T}^A B_{\mathcal{B}} \cap \bar{F}_b = \{0\}$, for all $b \in \mathcal{B}$,
5. $\bar{H}_{\bar{b}} \cap \bar{H}^\perp = \{0\}$, for all $\bar{b} \in \mathcal{B}$,
6. $\bar{H}_{\bar{b}} \cap \bar{H}^{i,\Lambda_h} = \{0\}$, for all $\bar{b} \in \mathcal{B}$.

Assuming the regularity of the constrained system, we have the corresponding direct sum decompositions of $T^A V^*_\mathcal{B}$ (similar to the Lagrangian case). But we will only develop the Poisson decomposition. Since the condition (6) of the above theorem, we have the decomposition

$$T^A V^* = \bar{H}_b \oplus \bar{H}^{i,\Lambda_h}, \text{ for all } \bar{b} \in \mathcal{B},$$
and, hence, one can define two complementary projectors

\[ \tilde{P}_b : T_b \tilde{A}^* \to \tilde{H}_b \quad \text{and} \quad \tilde{Q}_b : T_b \tilde{A}^* \to \tilde{H}_b^{\perp} \lambda_h, \quad \text{for all} \quad b \in \mathcal{B}. \]

Moreover, \((\tilde{P}, \tilde{Q})\) is \((\text{leg}_L, \text{leg}_L)\)-related with \((\hat{P}, \hat{Q})\). Then, the section \(\tilde{P}(R_{\hat{h}/\mathcal{B}})\) is the unique solution of the constrained Hamilton equations \((58)\), where \(R_{\hat{h}}\) is the solution of the Hamilton equations for the free dynamics, that is, \(R_{\hat{h}_{\text{free}}} = \tilde{P}(R_{\hat{h}/\mathcal{B}})\).

Now, we are going to construct a decomposition of \(T \tilde{A}^+\), which allows us to obtain the solution of the constrained Hamiltonian system as the projection by \(T \mu\) of a certain section. Given a Hamiltonian section \(h : V^* \to \mathcal{A}^+\), one can construct an affine function \(F_h : \mathcal{A}^+ \to \mathbb{R}\) with respect to the AV-bundle \(\mu : \mathcal{A}^+ \to V^*\) as follows. For each \(\varphi_x \in \mathcal{A}^+_x\), with \(x \in M\), exists a unique \(F_h(\varphi_x) \in \mathbb{R}\) such that

\[ h(\mu(\varphi_x)) - \varphi_x = F_h(\varphi_x)1_\mathcal{A}(x). \quad (61) \]

The function \(F_h : \mathcal{A}^+ \to \mathbb{R}\) is locally given by

\[ F_h(x^i, y_0, y_\alpha) = -H(x^i, y_\alpha) - y_0. \quad (62) \]

Note that this function was introduced in another way in Section \[\text{II.5}\]. Using \((2), (13), (14)\) and \((61)\), it is easy to prove that

\[ \lambda_{\tilde{A}} = (T \mu, \mu)^* \lambda_h - F_h(T \mu, \mu)^* \eta, \]

where \(\lambda_{\tilde{A}}\) is the Liouville section associated with the Lie algebroid \(\tilde{A}\) and \((T \mu, \mu)\) is the Lie algebroid epimorphism between the Lie algebroids \(\tau_{\tilde{A}}^+: T \tilde{A}^+ \to \mathcal{A}^+\) and \(\tau_{\tilde{A}}^* : T \tilde{A}^* \to V^*\) defined by \(T \mu(\tilde{a}, X_{\varphi_x}) = (\tilde{a}, (T_{\varphi_x}\mu)(X_{\varphi_x}))\), for \((\tilde{a}, X_{\varphi_x}) \in T_{\varphi_x}\tilde{A}^+\). As consequence, we have that

\[ \Omega_{\tilde{A}} = (T \mu, \mu)^* \Omega_h + \theta_h \wedge (T \mu, \mu)^* \eta, \quad (63) \]

where \(\theta_h = dT_{\tilde{A}}^+ F_h\) and \(\Omega_{\tilde{A}}\) is the canonical symplectic section associated with \(\tilde{A}\). We will denote by \(E \in \Gamma(\tau_{\tilde{A}}^+)\) the section given by

\[ E(\varphi_x) = (0(x), -1^V_{\mathcal{A}}(\varphi_x)) \in T_{\varphi_x}\tilde{A}^+, \quad \text{for all} \quad \varphi_x \in \mathcal{A}^+_x, \quad (64) \]

where \(1^V_{\mathcal{A}} \in \mathcal{X}(\mathcal{A}^+)\) is the vertical lift of the section \(1_\mathcal{A} \in \Gamma(\tau_{\mathcal{A}^+})\). Using \((62)\), we obtain that

\[ \ker T_{\varphi_x}\mu = E(\varphi_x), \quad \theta_h(\varphi_x)(E(\varphi_x)) = 1, \quad (65) \]

for all \(\varphi_x \in \mathcal{A}^+_x\), with \(x \in M\).

Now, we introduce the vector subbundle \((T \tilde{A}^V)^H \subset T \tilde{A}^+ \to \mathcal{A}^+\) whose fibre at point \(\varphi_x \in \mathcal{A}^+_x\) is

\[ (T_{\mu(\varphi_x)}V)^H_{\varphi_x} = \{ X_{\varphi_x} \in T_{\varphi_x}\tilde{A}^+/\theta_h(\varphi_x)(X_{\varphi_x}) = 0 \}. \]

It is not difficult to prove that \((T_{\varphi_x}\mu)(T_{\mu(\varphi_x)}V)^H_{\varphi_x} : (T_{\mu(\varphi_x)}V)^H_{\varphi_x} \to T_{\varphi_x}\tilde{A}^+\) is a linear isomorphism, for all \(\varphi_x \in \mathcal{A}^+_x\). Then, we deduce that

\[ T_{\varphi_x}\mathcal{A}^+ = (T_{\mu(\varphi_x)}V)^H_{\varphi_x} \oplus E(\varphi_x), \quad (66) \]
for all $\varphi_x \in A_x^+$, $x \in M$. We will denote by $H_{\varphi_x} : T_{\mu(\varphi_x)}^A V^* \to (T_{\mu(\varphi_x)}^A V^*)^H$ the inverse map of

$$(T_{\varphi_x}, \mu)((T_{\mu(\varphi_x)}^A V^*)^H)_{\varphi_x} : (T_{\mu(\varphi_x)}^A V^*)^H \to T_{\mu(\varphi_x)}^A V^*$$. So, if $X \in T_{\varphi_x}^A A^+$ it follows that

$$X = (T_{\varphi_x} \mu)(X)_{\varphi_x}^H + \theta_h(\varphi_x)(X_{\varphi_x})E(\varphi_x). \quad (67)$$

On the other hand, consider the section $\tilde{\eta} \in \Gamma((\tau_x^A)^*)$ defined by

$$\tilde{\eta}(\varphi_x)(\tilde{a}, X_{\varphi_x}) = 1_{\tilde{A}}(\tilde{a}),$$

for $\varphi_x \in A_x^+$ and $(\tilde{a}, X_{\varphi_x}) \in T_{\varphi_x}^A A^+$. A direct computation, using (\ref{eq:4}) and (\ref{eq:62}), proves that

$$\tilde{\eta}(X_{\varphi_x}^A) = -1,$$

where $X_{\varphi_x}^A \in \Gamma((\tau_x^A)^*)$ is the Hamiltonian section of $F_h$ with respect to the symplectic structure $\Omega_{\tilde{A}}$. Thus, since $(T_{\mu}^A, \mu)^* \eta = \tilde{\eta}$, we deduce that

$$\eta(\mu(\varphi_x))(X_{\varphi_x}^A(\varphi_x)) = 0.$$

Therefore, from (\ref{eq:63}), we obtain that

$$(i_{(T_{\varphi_x} \mu)(X_{\varphi_x}^A(\varphi_x)} \Omega_h(\mu(\varphi_x)))(T_{\varphi_x} \mu)(Z_{\varphi_x})) = 0, \text{ for } Z_{\varphi_x} \in T_{\varphi_x}^A A^+. \quad (68)$$

This implies that

$$(T_{\varphi_x} \mu)(X_{\varphi_x}^A(\varphi_x)) = -R_h(\mu(\varphi_x)), \text{ for } \varphi_x \in A_x^+. \quad (69)$$

Now, if $\alpha \in \Gamma((\tau_x^A)^*)$, then the Hamiltonian section $X_{\alpha}^{A_h} \in \Gamma(\tau_x^A)$ of $\alpha$ with respect to the (algebraic) Poisson structure $\Lambda_h$ is characterized by the following conditions

$$i_{X_{\alpha}^{A_h}} \Omega_h = \alpha - \alpha(R_h) \eta \quad \text{and} \quad i_{X_{\alpha}^{A_h}} \eta = 0.$$

On the other hand, denote by $\Lambda_{\tilde{A}} \in \Gamma(\wedge^2(\tau_x^A))$ the (algebraic) Poisson structure associated with the symplectic structure $\Omega_{\tilde{A}}$. In other words,

$$\Lambda_{\tilde{A}}(\tilde{a}, \tilde{\beta}) = \Omega_{\tilde{A}}(\tilde{\sigma}^{-1}(\tilde{a}), \tilde{\sigma}^{-1}(\tilde{\beta})), \text{ for } \tilde{a}, \tilde{\beta} \in \Gamma((\tau_x^A)^*),$$

and $\tilde{\sigma} : T_{\tilde{A}}^A A^+ \to (T_{\tilde{A}}^A A^+)^*$ being the vector bundle isomorphism induced by $\Omega_{\tilde{A}}$. Note that if $\tilde{a} \in \Gamma((\tau_x^A)^*)$ then

$$X_{\tilde{a}}^{\Lambda_{\tilde{A}}} = -_{\tilde{a}} \Lambda_{\tilde{A}}(\tilde{a}).$$

\textbf{Proposition V.2.} The prolongation of $\mu$

$$T_{\mu} : T{\tilde{A}} A^+ \to T{\tilde{A}} V^*$$

is a Poisson morphism over $\mu$, that is, if $\alpha \in \Gamma((\tau_x^A)^*)$ then

$$T_{\mu} \circ X_{\alpha}^{\Lambda_{\tilde{A}}} = X_{\alpha}^{A_{\tilde{A}}} \circ \mu. \quad (69)$$

\textbf{Proof.} If $\varphi_x \in A_x^+$ then a direct computation, using (\ref{eq:3}) and (\ref{eq:13}), proves that

$$\eta(\mu(\varphi_x))(X_{\varphi_x}^{\Lambda_{\tilde{A}}}) = 0.$$

Thus, from (\ref{eq:63}) and (\ref{eq:68}), it follows that

$$(i_{(T_{\varphi_x} \mu)(X_{\varphi_x}^{\Lambda_{\tilde{A}}}(\varphi_x)} \Omega_h(\mu(\varphi_x)))(T_{\varphi_x} \mu)(Z_{\varphi_x})) = 0.$$
Next, we consider the submanifold \( B \). From Proposition V.2, it follows that
\[
\{ \alpha(\mu)(\phi(\varphi_x))(\mu(\varphi_x))((\mu(\varphi_x))(Z_{\phi_x})) = \alpha(\mu(\varphi_x))((\mu(\varphi_x))(R_h(\mu(\varphi_x))))(\mu(\varphi_x))((\mu(\varphi_x))(Z_{\phi_x})),
\]
for \( Z_{\phi_x} \in T_{\phi_x}^A \). This implies that
\[
i(T_{\phi_x}(\mu))^{\alpha(\nu)(\phi(\varphi_x))}(\alpha(\mu))(\mu(\varphi_x)) = \alpha(\mu(\varphi_x))((\mu(\varphi_x))(R_h(\mu(\varphi_x))))(\mu(\varphi_x))\).
\]
Therefore, we deduce that (69) holds.

**Remark V.3.** From Proposition [V.2] it follows that
\[
T_{\phi_x}(\mu) \circ \sharp_{\lambda_A}(T_{\phi_x}(A^+)) \circ (T_{\phi_x}(\mu))^* = \sharp_{\lambda_A}(T_{\phi_x}(A^+)), \text{ for } \varphi_x \in A^+.
\]

Next, we consider the submanifold \( B^+ \) of \( A^+ \) defined by \( B^+ = \mu^{-1}(\bar{B}) \) and the vector subbundle \( H^+ \subset T^A_{A^+} \rightarrow B^+ \) given by
\[
H^+_{\phi_x} = (T_{\phi_x}(\mu))^{-1}(\tilde{H}(\mu(\varphi_x))) \cap E(\varphi_x), \text{ for all } \varphi_x \in B^+.
\]
where, we denote \( (H(\mu(\varphi_x)))^H_{\phi_x} \) the orthogonal to \( H^+_{\phi_x} \) with respect to the symplectic structure \( \Omega_{\bar{A}}(\varphi_x) \) of \( T_{\phi_x}^A \), for all \( \varphi_x \in B^+ \). In other words,
\[
(H^+_{\phi_x})^\perp_{\phi_x} = \mathbb{R}^{-1}((H^+_{\phi_x})^\circ) = \sharp_{\lambda_A}((H^+_{\phi_x})^\circ).
\]
Using (71), we have that
\[
(H^+_{\phi_x})^\circ = (T_{\phi_x}(\mu))^*((H(\mu(\varphi_x)))^\circ)
\]
and, from (70) and (72), it follows that
\[
(T_{\phi_x}(\mu))((H^+_{\phi_x})^\perp_{\phi_x}) = (\tilde{H}(\mu(\varphi_x)))^\perp_{\phi_x}, \text{ for all } \varphi_x \in B^+.
\]
In addition, using that \( \{\mu, d\bar{A}^V, \psi^a\} \) is a local basis of \( H^\circ \), we obtain that \( \{((T(\mu))^*\mu, (T(\mu))^*d\bar{A}^V, \psi^a)\} \) is a local basis of \( (H^+)^\circ \). This implies that \( \{X_{\phi_x}^{\bar{A}}(T(\mu))^*\mu, X_{\phi_x}^{\bar{A}}(T(\mu))^*d\bar{A}^V, \psi^a\} \) is a local basis of \( (H^+)^\perp_{\phi_x} \).

Note that, from Proposition [V.2] we have that
\[
(T_{\phi_x}(\mu))X_{\phi_x}^{\bar{A}}(T(\mu))^*\mu = \Xi_{\phi_x}(\mu(\varphi_x)),
\]
\[
(T_{\phi_x}(\mu))X_{\phi_x}^{\bar{A}}(T(\mu))^*(d\bar{A}^V, \psi^a) = \Xi_{\phi_x}(\mu(\varphi_x)).
\]
Then, we will prove the following result.

**Theorem V.4.** The following conditions are equivalent:

1. The constrained Hamiltonian system \((h, \bar{B})\) is regular;
2. \( H^+_{\phi_x} \cap (H^+_{\phi_x})^\perp_{\phi_x} = \{0\} \), for all \( \varphi_x \in B^+ \), with \( x \in M \).

**Proof.** (1) \(\Rightarrow\) (2) Suppose that the constrained Hamiltonian system is regular. Then, since \( B^+ = \mu^{-1}(\bar{B}) \) and using Theorem [V.1] we have that \( \tilde{H}(\mu(\varphi_x)) \cap (\tilde{H}(\mu(\varphi_x)))^\perp_{\phi_x} = \{0\} \), for all \( \varphi_x \in B^+ \). If \( X \in H^+_{\phi_x} \cap (H^+_{\phi_x})^\perp_{\phi_x} \) then, from (71), \( X = (T_{\phi_x}(\mu))X \in H_{\mu(\varphi_x)} \). Moreover, using (73) we deduce that \( \tilde{X} \in (H_{\mu(\varphi_x)})^\perp_{\phi_x} \) and, therefore, \( \tilde{X} = 0 \). Thus, \( X \in E(\varphi_x) \). Therefore, from (63) and (65), \( i_X \Lambda_A(\varphi_x) = \lambda(X(\mu(\mu))^\ast\eta)(\varphi_x) \in (H^+_{\phi_x})^\circ \), with \( \lambda \in \mathbb{R} \). However,


\((T \mu, \mu)\) \(\not\in (H')^\circ\). In fact, there exists \(\tilde{R}_{nh}(\varphi_x) \in H^+_{\varphi_x}\) such that \((T_{\varphi_x, \mu})(\tilde{R}_{nh}(\varphi_x)) = \tilde{R}_{nh}(\mu(\varphi_x))\) and, thus, \(((T \mu, \mu)^*\eta)(\varphi_x) = \eta(\mu(\varphi_x))\). Consequently, \(\lambda = 0\) and \(X = 0\).

(2) \(\Rightarrow\) (1) Suppose that \(H^+_{\varphi_x} \cap (H^+_{\varphi_x})^{1, \Lambda_h} = \{0\}\), for all \(\varphi_x \in B_+^x\) and take \(\tilde{X} \in \tilde{H}_{\mu(\varphi_x)} \cap \tilde{H}_{\mu(\varphi_x)}^{1, \Lambda_h}\).

Using (73), there exists \(X \in (H^+_{\varphi_x})^{1, \Lambda_h}\) such that \((T_{\varphi_x, \mu})(X) = \tilde{X}\) and, since \(\tilde{X} \in \tilde{H}_{\mu(\varphi_x)}\), we have that \(X \in H^+_{\varphi_x}\). Thus, \(X = 0\), which implies that \(X = 0\). Finally, using Theorem V.4, we conclude the result.

Assuming that the constrained system is regular, the bundle \(H^+ \to B^+\) is a symplectic sub-bundle of the symplectic bundle \((T^\Lambda A^+|_{B^+} \to B^+), (\Omega^\Lambda, \Lambda)|_{B^+}\) and we have a direct sum decomposition

\[
T_{\varphi_x, \mu}^\Lambda A^+ = H^+_{\varphi_x} \oplus (H^+_{\varphi_x})^{1, \Lambda_h}, \quad \text{for all} \quad \varphi_x \in B_+^x, \quad \text{with} \quad x \in M.
\]

Let us denote by \(P^+\) and \(Q^+\) the complementary projectors defined by this decomposition, that is,

\[
P_{\varphi_x}^+ : T_{\varphi_x, \mu}^\Lambda A^+ \to H^+_{\varphi_x} \quad \text{and} \quad Q_{\varphi_x}^+ : T_{\varphi_x, \mu}^\Lambda A^+ \to (H^+_{\varphi_x})^{1, \Lambda_h}, \quad \text{for all} \quad \varphi_x \in B_+^x.
\]

**Theorem V.5.** Let \((\hbar, B)\) be a regular constrained Hamiltonian system. Then, the solution of the constrained dynamics is the section \(\tilde{R}_{nh}\) obtained as follows \(\tilde{R}_{nh} \circ \mu = -T\mu(P^+(X^\Lambda_{\hbar}))\).

**Proof.** If \(\varphi_x \in B_+^x\), then, from (71) and since the map \(T_{\varphi_x, \mu}^\Lambda A^+ \to T_{\mu(\varphi_x)}^\Lambda V^*\) is a linear epimorphism, it follows that

\[
(T_{\varphi_x, \mu})(H^+_{\varphi_x}) = H_{\mu(\varphi_x)}.
\]

Thus, if \(Z_{\varphi_x} \in T_{\varphi_x}^\Lambda A^+\) then, using (73) and (75), we deduce that

\[
(T_{\varphi_x, \mu})(P_{\varphi_x}^+(Z_{\varphi_x})) \in \tilde{H}_{\mu(\varphi_x)}, \quad (T_{\varphi_x, \mu})(Q_{\varphi_x}^+(Z_{\varphi_x})) \in (\tilde{H}_{\mu(\varphi_x)})^{1, \Lambda_h}
\]

and it is clear that

\[
(T_{\varphi_x, \mu})(Z_{\varphi_x}) = (T_{\varphi_x, \mu})(P_{\varphi_x}^+(Z_{\varphi_x})) + (T_{\varphi_x, \mu})(Q_{\varphi_x}^+(Z_{\varphi_x})).
\]

Therefore, we have proved that

\[
(T_{\varphi_x, \mu})(P_{\varphi_x}^+(Z_{\varphi_x})) = \tilde{P}_{\mu(\varphi_x)}((T_{\varphi_x, \mu})(Z_{\varphi_x})),
\]

\[
(T_{\varphi_x, \mu})(Q_{\varphi_x}^+(Z_{\varphi_x})) = \tilde{Q}_{\mu(\varphi_x)}((T_{\varphi_x, \mu})(Z_{\varphi_x})),
\]

i.e.,

\[
T_{\varphi_x, \mu} \circ P_{\varphi_x}^+ = \tilde{P}_{\mu(\varphi_x)} \circ T_{\varphi_x, \mu} \quad \text{and} \quad T_{\varphi_x, \mu} \circ Q_{\varphi_x}^+ = \tilde{Q}_{\mu(\varphi_x)} \circ T_{\varphi_x, \mu}.
\]

Finally, from (68) and (76), we conclude that

\[
\tilde{R}_{nh}(\mu(\varphi_x)) = \tilde{P}_{\mu(\varphi_x)}(\tilde{R}_{h}(\mu(\varphi_x))) = -\tilde{P}_{\mu(\varphi_x)}(T_{\varphi_x, \mu}(X^\Lambda_{\hbar}(\varphi_x))) = -T_{\varphi_x, \mu}(P_{\varphi_x}^+(X^\Lambda_{\hbar}(\varphi_x))).
\]

Next, we will denote by \(\{\cdot, \cdot\}_{\hbar}\) the Poisson bracket on \(V^*\) induced by the algebraic Poisson structure \(\Lambda_h\) given by

\[
\{\psi, \psi'\}_\hbar = -(d^T T^* \psi')(X^\Lambda_{\hbar}) = \Omega_h(X^\Lambda_{\hbar}, X^\Lambda_{\hbar}),
\]
for $\psi, \psi' \in C^\infty(V^*)$.

Then, using (60), (71), (74) and the fact that $\bar{\mu}^a(\bar{Z}_h) = 0$ (see (57) and (59)), we deduce that the local expression of the projector $\mathcal{P}^+$ is

$$
\mathcal{P}^+ = \text{Id} - (\tilde{C}_{ab} \circ \mu)(\tilde{C}_{cd} \circ \mu)(\{\psi^b, \psi^d\} + \mu)X^{\Omega_{\bar{\mu}}(X_{\psi^b})} \otimes (T\mu, \mu)^* \tilde{\mu}^c
$$

$$
- (\tilde{C}_{ab} \circ \mu)X^{\Omega_{T\mu, \mu}(X_{\psi^b})} \otimes (T\mu, \mu)^* (dT^{\bar{A}V^*} \psi^b)
$$

$$
+ (\tilde{C}_{ab} \circ \mu)X^{\Omega_{T\mu, \mu}(dT^{\bar{A}V^*} \psi^b)} \otimes (T\mu, \mu)^* \tilde{\mu}^b,
$$

(78)

along the points of $\mathcal{B}^+$.

Now, we will denote by $\tilde{h}_h$ the section of $(T^{\bar{A}V^*})^*|_{\mathcal{B}} \to \mathcal{B}$ given by

$$
\tilde{h}_h(\tilde{b}) = -(i_{\tilde{R}_{nh}} \tilde{\Omega}_h)(\tilde{b}), \text{ for all } \tilde{b} \in \mathcal{B}.
$$

Then, using (63), (76), Theorem V.5 and the facts that $\text{Im}(\mathcal{P}^+) \subseteq H^+$, $\text{Im}(\mathcal{Q}^+) \subseteq (H^+)_{\bar{\mu}^\perp}$, $\tilde{R}_{nh} \in \tilde{H}$ and $i_{\tilde{R}_{nh}} = 1$, it is not difficult to prove that

$$
\tilde{h}_h(\mu(\varphi_x))(\tilde{Q}_{\mu(\varphi_x)}(T_{\varphi_x}\mu(X_{\varphi_x}))) = -\tilde{h}_h(\varphi_x)(\mathcal{Q}^+_{\varphi_x}(X_{\varphi_x})) = 0,
$$

(79)

for all $X_{\varphi_x} \in T_{\varphi_x}^A\mathcal{A}^+$ and $\varphi_x \in \mathcal{B}^+$, with $x \in \mathcal{M}$. Note that $\eta(\mu(\varphi_x)) \circ T_{\varphi_x}\mu \circ \mathcal{Q}^+_{\varphi_x} = \eta(\mu(\varphi_x)) \circ \tilde{Q}_{\mu(\varphi_x)} \circ T_{\varphi_x}\mu = 0$. We also recall that $\theta_h = d\bar{T}^\mathcal{A}L_h$.

Thus, using (67), (76) and (79), we obtain that

$$
\mathcal{P}^+_{\varphi_x}(X_{\varphi_x}) = (\hat{\mathcal{P}}_{\mu(\varphi_x)}(T_{\varphi_x}\mu(X_{\varphi_x})))_{\bar{\varphi}_x} + \tilde{h}_h(\varphi_x)(X_{\varphi_x})
$$

$$
+ \tilde{h}_h(\mu(\varphi_x))(\tilde{Q}_{\mu(\varphi_x)}(T_{\varphi_x}\mu(X_{\varphi_x})))_{\bar{\varphi}_x} \bigg|_{\mathcal{B}^+} E(\varphi_x),
$$

(80)

$$
\mathcal{Q}^+_{\varphi_x}(X_{\varphi_x}) = (\tilde{Q}_{\mu(\varphi_x)}(T_{\varphi_x}\mu(X_{\varphi_x})))_{\bar{\varphi}_x} - \tilde{h}_h(\mu(\varphi_x))(\tilde{Q}_{\mu(\varphi_x)}(T_{\varphi_x}\mu(X_{\varphi_x})))_{\bar{\varphi}_x} E(\varphi_x),
$$

for all $X_{\varphi_x} \in T_{\varphi_x}^A\mathcal{A}^+$, with $\varphi_x \in \mathcal{B}^+$ and $x \in \mathcal{M}$.

V.2. The nonholonomic bracket. We consider a regular nonholonomic system on a Lie affgebroid $\mathcal{A}$ described by a hyperregular Lagrangian function $L: \mathcal{A} \to \mathcal{R}$ and a constraint affine subbundle $\tau_{\mathcal{B}}: \mathcal{B} \to \mathcal{M}$ of the bundle $\tau_{\mathcal{A}}: \mathcal{A} \to \mathcal{M}$. We will denote by $(h, \mathcal{B})$ the corresponding regular constrained Hamiltonian system, by $\tilde{h}$ the restriction to $\mathcal{B}$ of $h$, by $\mathcal{B}^+$ the submanifold of $\mathcal{A}^+$ given by $\mathcal{B}^+ = \mu^+(\mathcal{B})$ and by the corresponding complementary projectors, for $\varphi_x \in \mathcal{B}^+$.

We have that $\mathcal{B}^+$ is the total space of an AV-bundle over $\tilde{\mathcal{B}}$. In fact, the affine bundle projection is the restriction $\mu_{\mathcal{B}^+}: \mathcal{B}^+ \to \mathcal{B}$ to $\mathcal{B}^+$ of the canonical projection $\mu: \mathcal{A}^+ \to V^*$.

Note that the restriction $\tau_{\mathcal{B}^+}: \mathcal{B}^+ \to \mathcal{M}$ of the projection $\tau_{\mathcal{A}^+}: \mathcal{A}^+ \to \mathcal{M}$ is a fibration. Thus, one may consider the prolongation $T\bar{A}\mathcal{B}^+$ of the Lie algebroid $\mathcal{A}$ over $\tau_{\mathcal{B}^+}: \mathcal{B}^+ \to \mathcal{M}$. $T\bar{A}\mathcal{B}^+$ is a Lie algebroid over $\mathcal{B}^+$.

Now, suppose that

$$
\tilde{h}', \tilde{h}'' : \tilde{\mathcal{B}} \to \mathcal{B}^+ \in \Gamma(\mu_{\mathcal{B}^+})
$$
are two sections of the AV-bundle $\mu_{B^+} : B^+ \to B$ and that

$$h', h'' : V^* \to A^+ \in \Gamma(\mu)$$

are arbitrary extensions to $V^*$ of $\bar{h}'$ and $\bar{h}''$, respectively.

We will denote by

$$F_{h'}, F_{h''} : A^+ \to \mathbb{R}$$

the affine functions associated with the sections $h'$ and $h''$, respectively, and by $X^{\Omega}_{\bar{h}', A}$ and $X^{\Omega}_{\bar{h}'', A}$ the corresponding Hamiltonian sections. Then, we have that

$$h'(\mu(\varphi_x)) - \varphi_x = X^\Omega_{\bar{h}', A}(x), \quad h''(\mu(\varphi_x)) - \varphi_x = X^\Omega_{\bar{h}'', A}(x),$$

$$i_{X^{\Omega}_{\bar{h}', A}} = d^{\Omega}A^+ F_{h'}, \quad i_{X^{\Omega}_{\bar{h}'', A}} = d^{\Omega}A^+ F_{h''},$$

for $\varphi_x \in A^+_x$.

Moreover, we will prove the following results.

**Lemma V.6.** The real function on $B^+$

$$\Omega_{\mathcal{A}}(\mathcal{P}^+(X^{\Omega}_{\bar{h}', B^+}), \mathcal{P}^+(X^{\Omega}_{\bar{h}'', B^+}))$$

does not depend on the chosen extensions $h'$ and $h''$ of $\bar{h}'$ and $\bar{h}''$, respectively.

**Proof.** If $\varphi_x \in B^+_x$ then $Q^+_{\varphi_x}(X^{\Omega}_{\bar{h}', A}(\varphi_x)) \in (H^+_{\varphi_x})^{\perp \Omega \bar{A}}$ and, thus,

$$\Omega_{\mathcal{A}}(\varphi_x)(Q^+_{\varphi_x}(X^{\Omega}_{\bar{h}', A}(\varphi_x)), P^+_{\varphi_x}(X^{\Omega}_{\bar{h}', A}(\varphi_x))) = 0$$

which implies that

$$\Omega_{\mathcal{A}}(\mathcal{P}^+(X^{\Omega}_{\bar{h}', B^+}), \mathcal{P}^+(X^{\Omega}_{\bar{h}'', B^+})) = \Omega_{\mathcal{A}}(X^{\Omega}_{\bar{h}', B^+}, \mathcal{P}^+(X^{\Omega}_{\bar{h}'', B^+})). \quad (81)$$

Now, let $h'_1 : V^* \to A^+$ be another extension of $\bar{h}' : B \to B^+$. Then, it is clear that

$$(F_{h'} - F_{h'_1})_{|B^+} = 0.$$ 

Therefore, since $P^+_{\varphi_x}(X^{\Omega}_{\bar{h}', A}(\varphi_x)) \in (H^+_{\varphi_x})^{\perp \Omega \bar{A} B^+}$, we obtain that

$$\Omega_{\mathcal{A}}(\varphi_x)(X^{\Omega}_{\bar{h}', A}(\varphi_x) - X^{\Omega}_{\bar{h}', A}(\varphi_x), P^+_{\varphi_x}(X^{\Omega}_{\bar{h}', A}(\varphi_x))) = \rho_{\mathcal{A}}(\mathcal{P}^+(X^{\Omega}_{\bar{h}', A}(\varphi_x)))(F_{h'} - F_{h'_1}) = 0.$$

This proves that

$$\Omega_{\mathcal{A}}(\mathcal{P}^+(X^{\Omega}_{\bar{h}', B^+}), \mathcal{P}^+(X^{\Omega}_{\bar{h}'', B^+})) = \Omega_{\mathcal{A}}(\mathcal{P}^+(X^{\Omega}_{\bar{h}', B^+}), \mathcal{P}^+(X^{\Omega}_{\bar{h}'', B^+})).$$

Consequently, using that $\Omega_{\mathcal{A}}$ is skew-symmetric, we deduce the result. \hfill \Box

**Lemma V.7.** The real function on $B^+$

$$\Omega_{\mathcal{A}}(\mathcal{P}^+(X^{\Omega}_{\bar{h}', B^+}), \mathcal{P}^+(X^{\Omega}_{\bar{h}'', B^+}))$$

is basic with respect to the affine bundle projection $\mu_{B^+} : B^+ \to B$. 

Proof. Denote by \( \theta_{h'} \) the section of \((T\tilde{A})^+ \to A^+\) defined by
\[
\theta_{h'} = d^{T\tilde{A}}A^+ F_{h'}.
\]
Then, from (31), it follows that
\[
\Omega_\tilde{A}^+(p^+(X_{F_{h'}|B^+}^\Omega_{\tilde{A}}), p^+(X_{F_{h'}|B^+}^\Omega_{\tilde{A}})) = \theta_{h'} \mid_{B^+} (X_{F_{h'}|B^+}^\Omega_{\tilde{A}}) - \theta_{h'} \mid_{B^+} (Q^+(X_{F_{h'}|B^+}^\Omega_{\tilde{A}})).
\]
Moreover, if \( \Pi_{A^+} \) is the linear Poisson structure on \( A^+ \) induced by the Lie algebroid \( \tilde{A} \), we have that \( \theta_{h'}(X_{F_{h'}|B^+}^\Omega_{\tilde{A}}) = \{F_{h'}, F_{h'}\} \Pi_{A^+} \). In addition, the vertical bundle of the fibration \( \mu : A^+ \to V^* \) is generated by the vertical lift \( 1^V_A \) of the section \( 1_A \in \Gamma(\tau_{A^+}) \). Since \( 1_A \) is a 1-cocycle of \( \tilde{A} \), we deduce that \( 1^V_A \) is an infinitesimal Poisson automorphism of \( \Pi_{A^+} \) and
\[
1^V_A(\{F_{h'}, F_{h'}\} \Pi_{A^+}) = \{1^V_A(F_{h'}), F_{h'}\} \Pi_{A^+} + \{F_{h'}, 1^V_A(F_{h'})\} \Pi_{A^+} = -\{1, F_{h'}\} \Pi_{A^+} - \{F_{h'}, 1\} \Pi_{A^+} = 0.
\]
This implies that the function \( \theta_{h'}(X_{F_{h'}|B^+}^\Omega_{\tilde{A}}) \) is basic with respect to the fibration \( \mu : A^+ \to V^* \). Thus, the function \( \theta_{h'}(X_{F_{h'}|B^+}^\Omega_{\tilde{A}}) \) basic with respect to the fibration \( \mu_{B^+} : B^+ \to \tilde{B} \).

On the other hand, from (30), we deduce that
\[
\theta_{h'}(\varphi_x)(Q_{F_{x}^\Omega_{\tilde{A}}})(X_{F_{h'}|B^+}^\Omega_{\tilde{A}}(\varphi_x))) = \theta_{h'}(\varphi_x)((\tilde{Q}_{\mu(\varphi_x)})(T_{\varphi_x, \mu}(X_{F_{h'}|B^+}^\Omega_{\tilde{A}}(\varphi_x)))) + \tilde{h}(\mu(\varphi_x))(\tilde{Q}_{\mu(\varphi_x)})(\mu(\varphi_x)))\theta_{h'}(\varphi_x)(E(\varphi_x)),
\]
for \( \varphi_x \in B^+_x \).

Now, using (31) and (34), one proves that
\[
\theta_{h'}(\varphi_x)(E(\varphi_x)) = -1^V_A(\varphi_x)(F_{h'}) = -\frac{d}{dt}_{\mid_{t=0}} (F_{h'}(\varphi_x) - t), \text{ for } \varphi_x \in A^+_x. \tag{82}
\]
Furthermore, proceeding as in Section V.1 (see (68)), we have that there exists a section \( R_{h''} \in \Gamma(\tau_{A^+}) \) such that
\[
(T_{\varphi_x, \mu}(X_{F_{h'}|B^+}^\Omega_{\tilde{A}}(\varphi_x))) = -R_{h''}(\mu(\varphi_x)), \text{ for all } \varphi_x \in A^+_x.
\]

Therefore,
\[
\theta_{h'}(\varphi_x)(Q_{F_{x}^\Omega_{\tilde{A}}})(X_{F_{h'}|B^+}^\Omega_{\tilde{A}}(\varphi_x))) = -\theta_{h'}(\varphi_x)((\tilde{Q}_{\mu(\varphi_x)})(R_{h''}(\mu(\varphi_x)))) + \tilde{h}(\mu(\varphi_x))(\tilde{Q}_{\mu(\varphi_x)})(R_{h''}(\mu(\varphi_x))), \text{ for } \varphi_x \in B^+_x.
\]
Consequently, it only remains to prove that the function \( \theta_{h'}(\tilde{Q}(R_{h''}))^{H} : B^+ \to \mathbb{R} \) defined by
\[
\theta_{h'}(\tilde{Q}(R_{h''}))^{H}(\varphi_x) = \theta_{h'}(\varphi_x)((\tilde{Q}_{\mu(\varphi_x)})(R_{h''}(\mu(\varphi_x))))^{H}_{\varphi_x}
\]
is \( \mu_{B^+} \)-projectable or, equivalently, that
\[
\mathcal{L}^{T\tilde{A}A^+}_{E[B^+]}(\theta_{h'}(\tilde{Q}(R_{h''}))^{H}) = 0.
\]
In fact, we will see that if \( Z \in \Gamma(\tau_{A^+}) \) and \( Z^{H} \) is the section of \( T\tilde{A}A^+ \) given by \( Z^{H}(\varphi_x) = (Z(\mu(\varphi_x)))^{H}_{\varphi_x} \), then
\[
\mathcal{L}^{T\tilde{A}A^+}_{E}[Z^{H}] = 0. \tag{83}
\]
Note that, from (82), it follows that
\[ \mathcal{L}_E^T\hat{A}^+ \theta_{h'} = d^{T\hat{A}^+} (\mathcal{L}_E^{T\hat{A}^+} F_{h'}) = d^{T\hat{A}^+} (\theta_{h'}(E)) = 0. \]
Thus,
\[ \mathcal{L}_E^T\hat{A}^+ (\theta_{h'}(Z^H)) = \theta_{h'}([E, Z^H]_{\hat{A}}^{T\hat{A}^+}). \]
Now, since \( d^{T\hat{A}^+} \theta_h = 0 \) and using that \( \theta_h(E) = 1 \) and the fact that \( \theta_h(Z^H) = 0 \), we obtain that
\[ \theta_h([E, Z^H]_{\hat{A}}^{T\hat{A}^+}) = 0. \tag{84} \]
In addition,
\[ T\mu \circ E = 0, \quad T\mu \circ Z^H = Z \circ \mu \]
and the pair \((T\mu, \mu)\) is a morphism between the Lie algebroids \( T\hat{A}^+, \hat{A} \rightarrow A^+ \) and \( T\hat{A}V^* \rightarrow V^* \).
This implies that
\[ T\mu \circ [E, Z^H]_{\hat{A}}^{T\hat{A}^+} = 0. \tag{85} \]
Therefore, from (67), (84) and (85), we deduce that \([E, Z^H]_{\hat{A}}^{T\hat{A}^+} = 0 \) and (83) holds.

From Lemmas \textbf{V.6} and \textbf{V.7}, we have that there exists a real function \( \{\bar{h}', \bar{h}''\}_{nh} \in C^\infty(\bar{B}) \) which is characterized by the following condition
\[ \{\bar{h}', \bar{h}''\}_{nh} \circ \mu_{B^+} = \Omega_{\bar{A}}(\mathcal{P}^+(X^\Omega_{\bar{A}}_{B^+, \infty}), \mathcal{P}^+(X^\Omega_{\bar{A}}_{B^+, \infty})). \tag{86} \]
This function is called the nonholonomic bracket of the sections \( \bar{h}' \) and \( \bar{h}'' \) and the resultant map
\[ \{\cdot, \cdot\}_{nh} : \Gamma(\mu_{B^+}) \times \Gamma(\mu_{B^+}) \rightarrow C^\infty(\bar{B}) \]
is called the nonholonomic bracket associated with the regular constrained Hamiltonian system \((h, \bar{B})\).

**Theorem V.8.** a) The nonholonomic bracket \( \{\cdot, \cdot\}_{nh} : \Gamma(\mu_{B^+}) \times \Gamma(\mu_{B^+}) \rightarrow C^\infty(\bar{B}) \) associated with the system \((h, \bar{B})\) is an almost-aff-Poisson bracket on the AV-bundle \( \mu_{B^+} : B^+ \rightarrow \bar{B} \), that is,
\begin{enumerate}
  \item \( \{\cdot, \cdot\}_{nh} \) is a bi-affine map,
  \item \( \{\cdot, \cdot\}_{nh} \) is skew-symmetric and
  \item If \( \bar{h}' \in \Gamma(\mu_{B^+}) \) then
    \[ \{\bar{h}', \cdot\}_{nh} : \Gamma(\mu_{B^+}) \rightarrow C^\infty(\bar{B}), \quad \bar{h}' \mapsto \{\bar{h}', \cdot\}_{nh}, \]
    is an affine derivation.
\end{enumerate}
b) If \( \bar{f} \in C^\infty(\bar{B}) \) is an observable and \( \{\bar{h}, \cdot\}^a_{nh} : C^\infty(\bar{B}) \rightarrow C^\infty(\bar{B}) \) is the linear map associated with the affine map \( \{\bar{h}, \cdot\}_{nh} : \Gamma(\mu_{B^+}) \rightarrow C^\infty(\bar{B}) \), then
\[ \bar{f} = \{\bar{h}, \bar{f}\}^a_{nh} \]
where \( \bar{f} \) is the evolution of \( \bar{f} \) along the solutions of the constrained Hamilton equations.
Proof. a) It is clear that $\{\cdot,\cdot\} _{nh}$ is skew-symmetric.

Next, we will prove that $\{\cdot,\cdot\} _{nh}$ is a bi-affine map. Suppose that $\tilde{h}'$ and $\tilde{h}''$ are sections of the $\ AVB : B^+ \to B$ and that $\tilde{f}'$ and $\tilde{f}''$ are real functions on $\B$. If $h', h'' \in \Gamma(\mu)$ are arbitrary extensions of $\tilde{h}'$ and $\tilde{h}''$, respectively, and $f', f'' \in C^\infty(\V^*)$ are arbitrary extensions of $\tilde{f}'$ and $\tilde{f}''$, respectively, then we can consider the real functions on $\B^+$ given by

$$
\Omega_{\mathcal{A}}(X_{\mathcal{F}_{h'}}, \mathcal{P}^+(X_{\mathcal{F}_{f'}})|_{B^+}) = -\Omega_{\mathcal{A}}(X_{\mathcal{F}_{h''}}, \mathcal{P}^+(X_{\mathcal{F}_{f'''}})|_{B^+}),
$$

$$
\Omega_{\mathcal{A}}(X_{\mathcal{F}_{h''}}, \mathcal{P}^+(X_{\mathcal{F}_{f''}})|_{B^+}) = -\Omega_{\mathcal{A}}(X_{\mathcal{F}_{h'''}}, \mathcal{P}^+(X_{\mathcal{F}_{f''''}})|_{B^+}),
$$

Now, proceeding as in the proof of Lemma V.6 we deduce that these functions don’t depend on the chosen extensions $h', h''$ and $f', f''$ of $\tilde{h}', \tilde{h}''$ and $\tilde{f}', \tilde{f}''$, respectively.

Thus, we can introduce the linear maps

$$
\{\cdot,\cdot\} _{nh}^a : C^\infty(\B) \to C^\infty(\B^+), \quad \tilde{f}' \mapsto \{\cdot,\tilde{f}'\} _{nh}^a = \Omega_{\mathcal{A}}(X_{\mathcal{F}_{h'}}, \mathcal{P}^+(X_{\mathcal{F}_{f'}})|_{B^+}),
$$

$$
\{\cdot,\tilde{h}''\} _{nh}^l : C^\infty(\B) \to C^\infty(\B^+), \quad \tilde{f}' \mapsto \{\tilde{f}',\cdot\} _{nh}^l = \Omega_{\mathcal{A}}(X_{\mathcal{F}_{h''}}, \mathcal{P}^+(X_{\mathcal{F}_{f''}})|_{B^+})
$$

and the bilinear map

$$
\{\cdot,\cdot\} _{nh}^u : C^\infty(\B) \times C^\infty(\B) \to C^\infty(\B^+) \quad (\tilde{f}', \tilde{f}'') \mapsto \{\tilde{f}',\tilde{f}''\} _{nh}^u = \Omega_{\mathcal{A}}(X_{\mathcal{F}_{h'}}, \mathcal{P}^+(X_{\mathcal{F}_{f'}})|_{B^+}).
$$

Using that

$$
F_{h''+f''} = F_{h''} + (f'' \circ \mu),
$$

we obtain that

$$
\{\tilde{h}', \tilde{h}'' + \tilde{f}''\} _{nh} \circ \mu_{B^+} = \{\tilde{h}', \tilde{h}''\} _{nh} \circ \mu_{B^+} + \{\tilde{h}', \tilde{f}''\} _{nh}^a.
$$

This implies that there exists $\{\tilde{h}', \tilde{f}''\} _{nh}^a \in C^\infty(\B)$ such that

$$
\{\tilde{h}', \tilde{f}''\} _{nh}^a \circ \mu_{B^+} = \{\tilde{h}', \tilde{f}''\} _{nh}^a.
$$

and, in addition,

$$
\{\tilde{h}', \tilde{h}'' + \tilde{f}''\} _{nh} = \{\tilde{h}', \tilde{h}''\} _{nh} + \{\tilde{h}', \tilde{f}''\} _{nh}^a. \quad (87)
$$

Since $\{\cdot,\tilde{h}''\} _{nh} = -\{\tilde{h}'',\cdot\} _{nh}$, we have that there exists $\{\tilde{f}', \tilde{h}''\} _{nh}^l \in C^\infty(\B)$ such that

$$
\{\tilde{f}', \tilde{h}''\} _{nh}^l \circ \mu_{B^+} = \{\tilde{f}', \tilde{h}''\} _{nh}^l
$$

and

$$
\{\tilde{h}' + \tilde{f}', \tilde{h}''\} _{nh} = \{\tilde{h}', \tilde{h}''\} _{nh} + \{\tilde{f}', \tilde{h}''\} _{nh}^l. \quad (88)
$$

On the other hand, from (87) and (88), it follows that

$$
\{\tilde{h}' + \tilde{f}', \tilde{h}'' + \tilde{f}''\} _{nh} \circ \mu_{B^+} = \{\tilde{h}'', \tilde{f}''\} _{nh} + \{\tilde{h}', \tilde{f}''\} _{nh}^a + \{\tilde{f}', \tilde{h}''\} _{nh}^l \circ \mu_{B^+} + \{\tilde{f}', \tilde{f}''\} _{nh}^u.
$$
Therefore, there exists \( \{ \tilde{f}', \tilde{f}'' \}_n \in C^\infty(\mathcal{B}) \) such that

\[
\{ \tilde{f}', \tilde{f}'' \}_n \circ \mu_{\mathcal{B}^+} = \{ \tilde{f}', \tilde{f}'' \}_n
\]

and

\[
\{ \tilde{h}', \tilde{f}' \}_n + \{ \tilde{h}'', \tilde{f}'' \}_n = \{ \tilde{h}', \tilde{h}'' \}_n + \{ \tilde{h}', \tilde{f}'' \}_n + \{ \tilde{f}', \tilde{h}'' \}_n + \{ \tilde{f}', \tilde{f}'' \}_n.
\]

This proves that the map \( \{ \cdot, \cdot \}_n \) is bi-affine.

Moreover, the linear map associated with the affine map \( \{ \tilde{h}', \cdot \}_n \) is just the map \( \{ \tilde{h}', \cdot \}_n : C^\infty(\mathcal{B}) \rightarrow C^\infty(\mathcal{B}) \) and we have that

\[
\{ \tilde{h}', \tilde{f} \}_n \circ \mu_{\mathcal{B}^+} = -d\mathcal{T}^A\mathcal{A}^+ (f \circ \mu)|_{\mathcal{B}^+} (\mathcal{P}^+ (X_{\mathcal{F}_{\mu}^+}))(\mathcal{P}^+ (X_{\mathcal{F}_{\mu}^+})(\varphi_x)).
\]

In addition, proceeding as in Section 6.1 (see (68)), we deduce that there exists a section \( R_{h'} \in \Gamma(\mathcal{T}) \) such that

\[
(T_{\varphi, \mu})(X_{\mathcal{F}_{\mu}^+}(\varphi_x)) = -R_{h'}(\mu_{\mathcal{B}^+}(\varphi_x)), \text{ for } \varphi_x \in \mathcal{B}^+.
\]

and, using (70), we obtain that

\[
(T_{\varphi, \mu})(\mathcal{P}^+ (X_{\mathcal{F}_{\mu}^+}(\varphi_x))) = -\mathcal{P}^+ (R_{h'}(\mu_{\mathcal{B}^+}(\varphi_x)))(\mathcal{P}^+ (X_{\mathcal{F}_{\mu}^+})(\varphi_x)).
\]

Therefore, from (89), (91) and since \( \mathcal{T}^A\mathcal{B} \) is a Lie subalgebroid of \( \mathcal{T}^A\mathcal{V}^* \), we conclude that

\[
\{ \tilde{h}', \tilde{f} \}_n = \frac{\mathcal{T}}{\mathcal{A}} (\mathcal{P}(R_{h'}))(f).
\]

Consequently, \( \{ \tilde{h}', \cdot \}_n \) is a vector field on \( \mathcal{B} \) and \( \{ \tilde{h}', \cdot \}_n \) is an affine derivation.

b) It follows using (72) and the fact that the solution of the constrained Hamilton equations is the section \( \mathcal{P}(R_{h'}|_{\mathcal{B}}) \).

\[ \square \]

Now, let \( \tilde{h}', \tilde{h}'' \) be two sections of the AV-bundle \( \mu_{\mathcal{B}^+} : \mathcal{B}^+ \rightarrow \mathcal{B} \) and \( h', h'' \in \Gamma(\mu) \) be two arbitrary extensions to \( \mathcal{V}^* \) of \( \tilde{h}' \) and \( \tilde{h}'' \), respectively. Then, we may consider the nonholonomic bracket \( \{ \tilde{h}', \tilde{h}'' \}_n \) of the sections \( \tilde{h}' \) and \( \tilde{h}'' \) and the aff-Poisson bracket \( \{ h', h'' \} \) of the sections \( h' \) and \( h'' \) defined in (111). Using (11), (112), (113), (114), (115), and (116), we obtain that \( \{ \tilde{h}', \tilde{h}'' \}_n \) and \( \{ h', h'' \} \) are locally related by the following condition

\[
\{ \tilde{h}', \tilde{h}'' \}_n = \{ h', h'' \}_V - \mathcal{C}_{ab} \mathcal{C}_{cd} (\psi^b, \psi^d) h \tilde{\mu}^c (R_{h'}) \tilde{\mu}^a (R_{h'}) - \mathcal{C}_{ab} (\psi^b, h'' \psi^d) \tilde{\mu}^c (R_{h'}) + \mathcal{C}_{ab} (\psi^b, h' \psi^d) \tilde{\mu}^c (R_{h'}) \big|_{\mathcal{B}^+}
\]

where \( \{ h', h'' \}_V : C^\infty(\mathcal{V}^*) \rightarrow C^\infty(\mathcal{V}^*) \) is the linear map associated with the affine map \( \{ h', h'' \} : \Gamma(\mu) \rightarrow C^\infty(\mathcal{V}^*) \) (see (20)). Thus, if the local expressions of the sections \( h' \) and \( h'' \) are...
\( h'(x^i, y_\alpha) = (x^i, -H'(x^i, y_\beta), y_\alpha) \) and \( h''(x^i, y_\alpha) = (x^i, -H''(x^i, y_\beta), y_\alpha) \), then, using (15), (16), (18), (19), (20), (22), (57) and (77), we deduce that
\[
\{\bar{h}', \bar{h}''\}_{nh} = \rho_0^i \frac{\partial (H' - H'')}{\partial x^i} + \rho_0^i \left( \frac{\partial H'}{\partial y_\alpha} - \frac{\partial H''}{\partial y_\alpha} \right) + C^\alpha_\alpha y_\gamma \frac{\partial (H' - H'')}{\partial y_\alpha} - C^\alpha_\alpha y_\gamma \frac{\partial H'}{\partial y_\alpha} \frac{\partial H''}{\partial y_\gamma}
\]
\[
+ \tilde{c}_{ab} \tilde{c}_{cd} \left[ \rho_0^a \left( \frac{\partial \psi_b}{\partial y_\alpha} \frac{\partial \psi^d}{\partial y_\alpha} - \frac{\partial \psi_b}{\partial y_\alpha} \frac{\partial \psi^d}{\partial y_\beta} \right) + \frac{\partial \psi_c}{\partial y_\gamma} C^\alpha_\alpha y_\gamma \frac{\partial (H' - H'')}{\partial y_\gamma} - C^\alpha_\alpha y_\gamma \frac{\partial H''}{\partial y_\gamma} \right]
\]
\[
+ \tilde{c}_{ab} \left[ \rho_0^a \frac{\partial \psi^b}{\partial x^i} - \rho_0^a \left( \frac{\partial \psi^b}{\partial y_\alpha} \frac{\partial H'}{\partial x^i} - \frac{\partial \psi^b}{\partial y_\alpha} \frac{\partial H''}{\partial x^i} \right) + \frac{\partial \psi^c}{\partial y_\gamma} C^\alpha_\alpha y_\gamma \frac{\partial (H' - H'')}{\partial y_\gamma} - C^\alpha_\alpha y_\gamma \frac{\partial H''}{\partial y_\gamma} \right]
\]
\[
+ \tilde{c}_{ab} \left[ \rho_0^a \frac{\partial \psi^b}{\partial x^i} - \rho_0^a \left( \frac{\partial \psi^b}{\partial y_\alpha} \frac{\partial H'}{\partial x^i} - \frac{\partial \psi^b}{\partial y_\alpha} \frac{\partial H''}{\partial x^i} \right) + \frac{\partial \psi^c}{\partial y_\gamma} C^\alpha_\alpha y_\gamma \frac{\partial (H' - H'')}{\partial y_\gamma} - C^\alpha_\alpha y_\gamma \frac{\partial H''}{\partial y_\gamma} \right]
\]

VI. EXAMPLES

VI.1. Lagrangian systems with linear nonholonomic constraints on a Lie algebroid.

In this section, we will discuss the particular case of Lagrangian systems with linear nonholonomic constraints on a Lie algebroid (see [7]).

First of all, we will recall an standard construction which will be useful in the sequel.

Let \( \tau_E : E \to N \) be a Lie algebroid over a manifold \( N \) with Lie algebroid structure \([\cdot, \cdot]_E, \rho_E\).

Then, the vector bundle \( \tilde{\tau}_E : E \times \mathbb{R} \to N \) admits a natural Lie algebroid structure \([\cdot, \cdot]_{E \times \mathbb{R}}, \rho_{E \times \mathbb{R}}\) given by
\[
[(X, f), (Y, g)]_{E \times \mathbb{R}} = ([X, Y]_E, \rho_E(X)(g) - \rho_E(Y)(f)),
\]
\[
\rho_{E \times \mathbb{R}}(X, f) = \rho_E(X),
\]
for \((X, f), (Y, g) \in \Gamma(\tau_E) \times C^\infty(N)\).

Next, suppose that \( \tau_V : V \to M \) is a Lie algebroid over a manifold \( M \) with Lie algebroid structure \([\cdot, \cdot]_V, \rho_V\). Then, the affine bundle \( \tau_A = \tau_V : A = V \to M \) is a Lie affine bundle over \( M \). In fact, the bidual bundle \( \tau_A^* : \tilde{\mathcal{A}} \to M \) may be identified with the Lie algebroid \( \tilde{\tau}_V : V \times \mathbb{R} \to M \). In addition, it is easy to prove that the prolongation \( \mathcal{T}^A \mathcal{A} \) of \( \tilde{\mathcal{A}} \) over the fibration \( \tau_A : A \to M \) is isomorphic to the Lie algebroid \( \tilde{\tau}_V^* : \mathcal{T}^V V \times \mathbb{R} \to V \).

We will denote by \( S : TVV \to TVV \) the vertical endomorphism on \( TVV \) (see [17] [23]) and by \( E_0 \) the section \((0, 1)\) of \( \tilde{\tau}_V^* : TVV \times \mathbb{R} \to V \).

Now, let \( L : V \to \mathbb{R} \) be a Lagrangian function on \( V \) and \( E_L = \Delta(L) - L \in C^\infty(V) \) be the Lagrangian energy (here, \( \Delta \) is the Liouville vector field of \( V \)). Then, the dual bundle to \( \tau_A^* : \mathcal{T}^A \mathcal{A} \to A \) is isomorphic to the vector bundle \((\tilde{\tau}_V^*)^* : (TVV)^* \times \mathbb{R} \to V \) and, under this identification, the 1-cocycle \( \bar{\phi}_0 \) is the section \((0, 1)\) and
\[
\Omega_L = \omega_L + (0, 1) \wedge dTVV E_L,
\]
where \( \omega_L : V \to \wedge^2(TVV)^* \) is the Poincaré-Cartan 2-section associated with the Lagrangian \( L \) on \( V \) ([17] [23])

Next, suppose that \( \tau_U : U \to M \) is a vector subbundle of \( \tau_V : V \to M \). Then, \( \tau_B = \tau_U : B = U \to M \) is an affine subbundle of \( \tau_A : A \to M \) and we can consider the affine constrained Lagrangian system \((L, B)\) on \( A \). Moreover, if \( \tilde{X} \) is a section of \( \tau_A^* = \tilde{\tau}_V^* : \mathcal{T}^A \mathcal{A} \cong TVV \times \mathbb{R} \to \)}
V then, from [13], it follows that \( \dot{X} \) is a solution of the Lagrange-d’Alembert equations for the affine nonholonomic system \((L, B)\) if and only if

\[
Y = Y_U = (\dot{X} - E_0)|_U \text{ is a SODE on } U,
\]

\[
(i_Y \omega_L - d(T^V \nu E_L))|_U \in \Gamma(\tau^V \nu (\tau^V U^+)),
\]

\[
Y_U \in \Gamma(\tau^V U^+).
\]

(94) are just the Lagrange-d’Alembert equations for the linear nonholonomic system \((L, U)\) in the terminology of [12].

Thus, the affine constrained Lagrangian system \((L, B)\) on \(A\) is regular if and only if the linear nonholonomic system \((L, U)\) on \(V\) is regular in the sense of [12].

Therefore, using the results in Section [12] of this paper, one directly deduces some results, for linear nonholonomic systems on Lie algebroids, which were obtained in [12] (see Section 3 in [12]).

Now, assume that the Lagrangian function \(L : A = V \to \mathbb{R}\) is hyperregular (that is, the Legendre transformation \(leg_L : A = V \to V^+\) is a global diffeomorphism) and that the affine constrained Lagrangian system \((L, B)\) is regular. Then, the space \(A^+\) may be identified with the product manifold \(V^+ \times \mathbb{R}\) and, under this identification, we have that:

(i) The vector bundle projection \(\tau_{A^+} : A^+ \to M\) is the map \(\tau^+_V : V^+ \times \mathbb{R} \to M\) given by

\[
\tau^+_V(\alpha_x, t) = \tau^+_V(\alpha_x) = x, \text{ for } (\alpha_x, t) \in V^+ \times \mathbb{R} \text{ and } x \in M.
\]

(ii) The fibration \(\mu : A^+ \to V^+\) is the canonical projection \(pr_1 : V^+ \times \mathbb{R} \to V^+\) on the first factor and, thus, the spaces \(\Gamma(\mu)\) and \(C^\infty(V^+)\) are isomorphic.

(iii) The Hamiltonian section \(h = Leg_L \circ leg_L^{-1} \in \Gamma(\mu)\) is the Hamiltonian energy \(H \in C^\infty(V^+)\) of the free system, that is, \(H = E_L \circ leg_L^{-1} : V^+ \to \mathbb{R}\).

(iv) The canonical aff-Poisson bracket on the trivial AV-bundle \(pr_1 : V^+ \times \mathbb{R} \to V^+\) is just the linear Poisson bracket on \(V^+\) induced by the Lie algebroid structure on \(V\).

Next, let \(\bar{B} = \bar{U}\) be the Hamiltonian constrained submanifold of \(V^+\), that is, \(\bar{B} = \bar{U} = leg_L(U)\). Then, the constrained AV-bundle \(\mu_{B^+} : B^+ \to \bar{B}\) may be identified with the trivial AV-bundle \(pr_1 : \bar{U} \times \mathbb{R} \to \bar{U}\). Therefore, the nonholonomic bracket associated with the nonholonomic system \((L, B)\) is an almost-Poisson bracket on \(\bar{U}\), i.e., a \(\mathbb{R}\)-bilinear map

\[
\{\cdot, \cdot\}_{nh} : C^\infty(\bar{U}) \times C^\infty(\bar{U}) \to C^\infty(\bar{U})
\]

which is skew-symmetric and a derivation in each argument with respect to the standard product of functions \(\{\cdot, \cdot\}_{nh}\) doesn’t satisfy, in general, the Jacobi identity).

Finally, since the restriction of the Legendre transformation to \(U\)

\[
leg_U = leg_L|_U : U \to \bar{U}
\]

is a global diffeomorphism, one may consider the corresponding nonholonomic bracket on \(U\), which we also denote by \(\{\cdot, \cdot\}_{nh}\), defined by

\[
\{f, g\}_{nh} = \{f \circ leg_U^{-1}, g \circ leg_U^{-1}\}_{nh} \circ leg_U, \text{ for } f, g \in C^\infty(U).
\]

This bracket was introduced in [12] and its properties were discussed in this paper (see Section 3.5 in [12]).
VI.2. Standard affine nonholonomic Lagrangian systems. Let $\tau : M \to \mathbb{R}$ be a fibration and $A = J^1\tau$ be the 1-jet bundle of local sections of $\tau : M \to \mathbb{R}$. Then, as we know (see Section 11.3, the affine bundle $\tau_A = \tau_{1,0} : A = J^1\tau \to M$ is a Lie affgebroid modelled on the Lie algebroid $\tau_V = (\pi_M)|_{V_T} : V = V\tau \to M$. Moreover, the bidual bundle $\tilde{A}$ may be identified with the standard Lie algebroid $\pi_M : TM \to M$ and, under this identification, the 1-cocycle $1_A$ of $\tilde{A}$ is just the closed 1-form $\tau^*(dt)$.

Note that if $(t, q^i)$ are local coordinates on $M$ which are adapted to the fibration $\tau$ then

$$\{e_0 = \frac{\partial}{\partial t}, e_\alpha = \frac{\partial}{\partial q^\alpha}\}$$

is a local basis of sections of the Lie algebroid $\tau_\tilde{A} = \pi_M : \tilde{A} = TM \to M$ such that $1_A(e_0) = 1$ and $1_A(e_\alpha) = 0$. Furthermore, we have that

$$\rho_A(e_0) = \frac{\partial}{\partial t}, \quad \rho_A(e_\alpha) = \frac{\partial}{\partial q^\alpha}, \quad \{e_0, e_\alpha\} = [e_\alpha, e_\beta] = 0,$$

(95)

for all $\alpha$ and $\beta$.

On the other hand, it is easy to prove that the prolongation $\tau_\tilde{A}^\tau : T\tilde{A} \to A$ of $\tilde{A}$ over the fibration $\tau_A : A \to M$ is isomorphic to the standard Lie algebroid $\pi_J : T(J^1\tau) \to J^1\tau$. Thus, the space $\Gamma(\tau_\tilde{A}^\tau)$ may be identified with the set of vector fields on $J^1\tau$.

Now, let $L : J^1\tau \to \mathbb{R}$ be a regular Lagrangian function. If $\tau_1 = \tau \circ \tau_{1,0} : J^1\tau \to \mathbb{R}$ is the canonical projection then, under the above identifications, the 1-cocycle $\rho_0$ of the Lie algebroid $\tau_\tilde{A}^\tau : T\tilde{A} \to A$ is the closed 1-form $\eta = \tau_1^*(dt)$ on $J^1\tau$. Moreover, the Poincaré-Cartan 2-section $\Omega_L$ associated with $L$ is just the Poincaré-Cartan 2-form on $J^1\tau$.

Next, suppose that $\tau_B : B \to M$ is an affine subbundle of $\tau_A = \tau_{1,0} : A = J^1\tau \to M$. Then, we can consider the affine constrained Lagrangian system $(L, B)$.

In addition, we have that a vector field $X$ on $J^1\tau$ is a solution of the Lagrange-d’Alembert equations if and only if

$$(i_X \Omega_L)|_B \in \Gamma_{S^*\tau(T\mathbb{R}^n)}, \quad (i_X \eta)|_B = 1, \quad X|_B \in \mathcal{X}(B).$$

(96)

These two equations were considered in [14] (see also [16]). Note that the first two equations imply that $X$ is a SODE on $B$.

From [16], it follows that the affine nonholonomic Lagrangian system $(L, B)$ on $A$ is regular if and only if it is regular in the sense of [14]. Thus, using the results in Section IV, one directly deduces some results, for standard affine nonholonomic Lagrangian systems which were obtained in [14] (see also [16]).

On the other hand, we have that the space $A^+$ may be identified with the cotangent bundle $T^*M$ to $M$ and, under this identification, the fibration $\mu : A^+ \to V^*$ is just the dual map $i_{\nu_\tau^*} : T^*M \to V^*\tau$ of the canonical inclusion $i_{\nu_\tau} : V^* \to TM$.

Next, assume that the Lagrangian function $L : A = J^1\tau \to \mathbb{R}$ is hyperregular and that the affine constrained Lagrangian system $(L, B)$ is regular. Then, we can consider the Hamiltonian constrained submanifold $\tilde{B} = leg_L(B)$ of $V^*\tau$ and the constrained AV-bundle $\mu_{B^+} : B^+ = (i_{\nu_\tau}^{-1})(\tilde{B}) \to B$.

Now, denote by $h : V^*\tau \to T^*M$ the Hamiltonian section induced by the hyperregular Lagrangian $L$ ($h = Leg_L \circ leg_L^L$) and suppose that $h'$ and $\tilde{h}''$ are two sections of the AV-bundle $\mu_{B^+} : B^+ \to B$ and that $h', \tilde{h}'' : V^*\tau \to T^*M \in \Gamma(i_{\nu_\tau})$ are arbitrary extensions of $\tilde{h}'$ and $\tilde{h}''$, respectively. Then, we will obtain the local expression of the nonholonomic bracket of $\tilde{h}'$ and
and the local equations defining $\tilde{B}$. For this purpose, we take canonical local coordinates $(t, q^\alpha, p_\alpha)$ (respectively, $(t, q^\alpha, p_\alpha)$) on $V^*\tau$ (respectively, $T^*M$) such that

$$
\bar{h}(t, q^\alpha, p_\alpha) = (t, q^\alpha, -H(t, q^\beta, p_\beta), p_\alpha),
$$

$$
\bar{h}'(t, q^\alpha, p_\alpha) = (t, q^\alpha, -H'(t, q^\beta, p_\beta), p_\alpha),
$$

$$
\bar{h}''(t, q^\alpha, p_\alpha) = (t, q^\alpha, -H''(t, q^\beta, p_\beta), p_\alpha),
$$

and the local equations defining $\tilde{B}$ as a submanifold of $V^*\tau$ are

$$
\psi^a(t, q^\alpha, p_\alpha) = 0, \quad a \in \{1, \ldots, r\}.
$$

Then, using the general local expression of the nonholonomic bracket (see Section V.2) and (95), we deduce that

$$
\{\bar{h}', \bar{h}''\}_\text{nh} = \left\{ \frac{\partial H'}{\partial t} - \frac{\partial H''}{\partial t} + \frac{\partial H'}{\partial q^a} \frac{\partial H''}{\partial p_a} - \frac{\partial H'}{\partial q^b} \frac{\partial H''}{\partial p_b} \right\} + C_{ab} C_{cd} \left( \frac{\partial \psi^b}{\partial q^d} \frac{\partial \psi^d}{\partial p_a} - \frac{\partial \psi^b}{\partial q^e} \frac{\partial \psi^e}{\partial p_a} \right) \frac{\partial H''}{\partial p_c} + C_{ab} \left( \frac{\partial \psi^b}{\partial q^a} \frac{\partial H''}{\partial p_a} - \frac{\partial \psi^b}{\partial q^e} \frac{\partial H''}{\partial p_e} \right) \frac{\partial H''}{\partial p_c}
$$

Finally, we remark that the linear-linear part of the bi-affine map $\{\cdot, \cdot\}_\text{nh} : \Gamma(\mu_{g+}) \times \Gamma(\mu_{g+}) \to C^\infty(\tilde{B})$ is an almost-Poisson bracket on $\tilde{B}$. This bracket was considered in [4].

VI.3. A homogeneous rolling ball without sliding on a rotating table with time-dependent angular velocity. [3, 4, 20, 29] Consider the vector bundle $\tau_A : \tilde{A} \to M$ where $\tilde{A} = T\mathbb{R}^3 \times \mathbb{R}^3$, $M = \mathbb{R}^3$ and $\tau_A$ is the composition of the projection over the first factor with the canonical projection $\tau_{\mathbb{R}^3} : T\mathbb{R}^3 \to \mathbb{R}^3$. Denote by $(t, x, y)$ the coordinates on $M$. A global basis of sections of $\tau_A$ can be construct as follows

$$
\left\{ e_0 = \left( \frac{\partial}{\partial t}, 0 \right), e_1 = \left( \frac{\partial}{\partial x}, 0 \right), e_2 = \left( \frac{\partial}{\partial y}, 0 \right), e_3 = (0, u_1), e_4 = (0, u_2), e_5 = (0, u_3) \right\},
$$

where $u_1, u_2, u_3 : \mathbb{R}^3 \to \mathbb{R}^3$ are the constant maps $u_1(t, x, y) = (1, 0, 0)$, $u_2(t, x, y) = (0, 1, 0)$ and $u_3(t, x, y) = (0, 0, 1)$. Consider the projection $\rho_A : T\mathbb{R}^3 \times \mathbb{R}^3 \to T\mathbb{R}^3$ over the first factor and the Lie bracket on the space of sections $\Gamma(\tau_A)$ where the only non-zero Lie brackets are

$$
[e_4, e_3]_A = e_5, \quad [e_5, e_4]_A = e_3 \quad \text{and} \quad [e_3, e_5]_A = e_4.
$$

Thus, $(\cdot, \cdot)_A, \rho_A)$ induces a Lie algebroid structure on $\tilde{A} = T\mathbb{R}^3 \times \mathbb{R}^3$. Denote by $(t, x, y; \dot{t}, \dot{x}, \dot{y}, \omega_x, \omega_y, \omega_z)$ the coordinates on $\tilde{A}$ induced by $\{e_0, e_1, e_2, e_3, e_4, e_5\}$. Moreover, $\phi : T\mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ given by $\phi(t, x, y; \dot{t}, \dot{x}, \dot{y}, \omega_x, \omega_y, \omega_z) = \dot{t}$ is a 1-cocycle in the corresponding Lie algebroid cohomology and, then, it induces a Lie affgebroid structure over $\tilde{A} = \phi^{-1}\{1\} \subseteq \mathbb{R} \times T\mathbb{R}^3$. Note that the Lie affgebroid structure on $\tilde{A} = \mathbb{R} \times T\mathbb{R}^2 \times \mathbb{R}^3$ is a special type of Lie affgebroid structure called Atiyah affgebroid structure (see Section 9.3.1 in [12] for a general construction). Moreover, the affine bundle $\tau_A : \mathbb{R} \times T\mathbb{R}^2 \times \mathbb{R}^3 \to \mathbb{R}^3$ is modelled
on the vector bundle $\tau_V : V = \phi^{-1}\{0\} \equiv \mathbb{R} \times T\mathbb{R}^2 \times \mathbb{R}^3 \rightarrow \mathbb{R}^3$. Thus, $(t, x, y; \dot{x}, \dot{y}, \omega_x, \omega_y, \omega_z)$ may be considered as local coordinates on $A$ and $V$.

In this case, from (6), we have that a basis of sections of $T^\mathcal{A}A$ is

$$\left\{ \tilde{T}_0 = (e_0, \frac{\partial}{\partial t}), \tilde{T}_1 = (e_1, \frac{\partial}{\partial x}), \tilde{T}_2 = (e_2, \frac{\partial}{\partial y}), \tilde{T}_3 = (e_3, 0), \tilde{T}_4 = (e_4, 0), \tilde{T}_5 = (e_5, 0), \right\}$$

$$\tilde{V}_1 = (0, \frac{\partial}{\partial \tilde{x}}), \tilde{V}_2 = (0, \frac{\partial}{\partial \tilde{y}}), \tilde{V}_3 = (0, \frac{\partial}{\partial w_x}), \tilde{V}_4 = (0, \frac{\partial}{\partial w_y}), \tilde{V}_5 = (0, \frac{\partial}{\partial w_z}) \right\}$$

Consider now the following mechanical problem. A (homogeneous) sphere of radius $r > 0$, mass $m$ and inertia about any axis $k^2$, rolls without sliding on a horizontal table which rotates with time-dependent angular velocity $\Omega(t)$ about a vertical axis through one of its points. Apart from the constant gravitational force, no other external forces are assumed to act on the sphere. Therefore, the Lagrangian of the system corresponds to the kinetic energy. Moreover, observe that the kinetic energy may be expressed as a Lagrangian $L : \mathcal{A} \rightarrow \mathbb{R}$:

$$L(t, x, y; \dot{x}, \dot{y}, \omega_x, \omega_y, \omega_z) = \frac{1}{2}(m\dot{x}^2 + m\dot{y}^2 + k^2(\omega_x^2 + \omega_y^2 + \omega_z^2))$$

where $(\omega_x, \omega_y, \omega_z)$ are the components of the angular velocity of the sphere (see [4], for more details).

After some straightforward calculations using (10), we deduce that the Poincaré-Cartan sections associated with $L$ are given by:

$$\Theta_L = -L\phi_0 + m\dot{x}\tilde{T}^1 + m\dot{y}\tilde{T}^2 + k^2\omega_x\tilde{T}^3 + k^2\omega_y\tilde{T}^4 + k^2\omega_z\tilde{T}^5,$$

$$\Omega_L = d^\mathcal{A}A_L \wedge \phi_0 + m\tilde{V}^1 \wedge \tilde{V}^3 + \tilde{V}^2 \wedge \tilde{V}^4 + k^2\omega_x\tilde{T}^5 \wedge \tilde{V}^3 + k^2\omega_y\tilde{T}^5 \wedge \tilde{V}^4 + k^2\omega_z\tilde{T}^5 \wedge \tilde{V}^3$$

Since the ball is rolling without sliding on a rotating table then the system is subjected to the affine constraints:

$$\Psi^1 = \Omega(t)y + \dot{x} - rw_y,$$

$$\Psi^2 = -\Omega(t)x + \dot{y} + r\omega_x,$$

which define an affine subbundle $B$ of $\mathcal{A}$. Then, we have that

$$d^\mathcal{A}A\Psi^1 = \Omega'(t)y\phi_0 + \Omega(t)\tilde{T}^2 + \tilde{V}^1 - r\tilde{V}^4,$$

$$d^\mathcal{A}A\Psi^2 = -\Omega'(t)x\phi_0 - \Omega(t)\tilde{T}^1 + \tilde{V}^2 + r\tilde{V}^3.$$
and then, the matrix $C$ is
\[
C = \begin{pmatrix}
-\frac{1}{m} - \frac{r^2}{k^2} & 0 \\
0 & -\frac{1}{m} - \frac{r^2}{k^2}
\end{pmatrix}.
\]

The projector $P$ over $\mathcal{T}\mathcal{A}\mathcal{B}$ is
\[
P = \text{Id} + \frac{mk^2}{k^2 + mr^2} Z_1 \otimes d\mathcal{T}\mathcal{A}\mathcal{B} + \frac{mk^2}{k^2 + mr^2} Z_2 \otimes d\mathcal{T}\mathcal{A}\mathcal{B}.
\]

Since the solution of the unconstrained dynamics is
\[
R_L = \dot{T}_0 + \dot{x}\dot{T}_1 + \dot{y}\dot{T}_2 + \omega_x \dot{T}_3 + \omega_y \dot{T}_4 + \omega_z \dot{T}_5,
\]
then the solution of the nonholonomic problem will be
\[
R_{nh} = P(R_L) = \dot{T}_0 + \dot{x}\dot{T}_1 + \dot{y}\dot{T}_2 + \omega_x \dot{T}_3 + \omega_y \dot{T}_4 + \omega_z \dot{T}_5 + \frac{mk^2}{k^2 + mr^2} (\Omega'(t)y + \Omega(t)y)Z_1 - \frac{mk^2}{k^2 + mr^2} (\Omega'(t)x + \Omega(t)x)Z_2.
\]

Thus,
\[
\rho_{\mathcal{A}}^{\mathcal{T}}(R_{nh}) = \frac{\partial}{\partial t} + \dot{x} \frac{\partial}{\partial x} + \dot{y} \frac{\partial}{\partial y} - \frac{k^2}{k^2 + mr^2} (\Omega'(t)y + \Omega(t)y) \frac{\partial}{\partial x} + \frac{k^2}{k^2 + mr^2} (\Omega'(t)x + \Omega(t)x) \frac{\partial}{\partial y} + \frac{mr}{k^2 + mr^2} (\Omega'(t)x + \Omega(t)x) \frac{\partial}{\partial \omega_x} + \frac{mr}{k^2 + mr^2} (\Omega'(t)y + \Omega(t)y) \frac{\partial}{\partial \omega_y}
\]

and, therefore, the equations of motion of the nonholonomic problem are
\[
\begin{align*}
\ddot{x} &= -\frac{k^2}{k^2 + mr^2} (\Omega'(t)y + \Omega(t)y), \\
\ddot{y} &= \frac{k^2}{k^2 + mr^2} (\Omega'(t)x + \Omega(t)x), \\
\dot{\omega}_x &= \frac{mr}{k^2 + mr^2} (\Omega'(t)x + \Omega(t)x), \\
\dot{\omega}_y &= \frac{mr}{k^2 + mr^2} (\Omega'(t)y + \Omega(t)y), \\
\dot{\omega}_z &= 0.
\end{align*}
\]

Now, we take the coordinates $(t, x, y; p_t, p_x, p_y, u_x, u_y, u_z)$ on $\mathcal{A}^+$ and the corresponding coordinates $(t, x, y; p_x, p_y, u_x, u_y, u_z)$ on $V^+$. Then, we obtain that the extended Legendre transformation $\text{Leg}_L : \mathcal{A} \to \mathcal{A}^+$ and the Legendre transformation $\text{leg}_L : \mathcal{A} \to V^+$ associated with $L$ are given by
\[
\text{Leg}_L(t, x, y; \dot{x}, \dot{y}, \omega_x, \omega_y, \omega_z) = (t, x, y; -L, m\dot{x}, m\dot{y}, k^2\omega_x, k^2\omega_y, k^2\omega_z),
\]
\[
\text{leg}_L(t, x, y; \dot{x}, \dot{y}, \omega_x, \omega_y, \omega_z) = (t, x, y; m\dot{x}, m\dot{y}, k^2\omega_x, k^2\omega_y, k^2\omega_z).
\]
In this particular example, the expression of the nonholonomic bracket is given by
\[ \psi^1 = \Omega(t)y + \frac{1}{m}p_x - \frac{r}{k^2}u_x, \]
\[ \psi^2 = -\Omega(t)x + \frac{1}{m}p_y + \frac{r}{k^2}u_x. \]
Moreover, the induced Hamiltonian section \( h = Leg_L \circ \gamma^{-1} : V^* \to A^+ \) is given by
\[ h(t, x, y; p_x, y, u_x, u_y, u_z) = (t, x, y; -H, p_x, y, u_x, u_y, u_z), \]
where the function \( H \) is
\[ H(t, x, y; p_x, y, u_x, u_y, u_z) = \frac{1}{2}\left(\frac{1}{m}(p_x^2 + p_y^2) + \frac{1}{k^2}(u_x^2 + u_y^2 + u_z^2)\right). \]
A curve \( \gamma : I \to V^*, t \mapsto (t, x(t), y(t); p_x(t), p_y(t), u_x(t), u_y(t), u_z(t)) \), is a solution of the equations of motion for the nonholonomic Hamiltonian system \( (h, \mathcal{B}) \) if it satisfies the equations
\[
\begin{align*}
\dot{p}_x &= -\lambda_1, \\
\dot{p}_y &= -\lambda_2, \\
\dot{u}_x &= -r\lambda_2, \\
\dot{u}_y &= r\lambda_1, \\
\dot{u}_z &= 0,
\end{align*}
\]
and the constraints \( \psi^1 = 0 \) and \( \psi^2 = 0 \).

In this particular example, the expression of the nonholonomic bracket is given by
\[
\{\hat{h}', \hat{h}'\}_nh = \frac{\partial(H' - H'')}{\partial t} + \left(\frac{\partial H'}{\partial x} \frac{\partial H''}{\partial p_x} - \frac{\partial H'}{\partial p_x} \frac{\partial H''}{\partial x}\right) + \left(\frac{\partial H'}{\partial y} \frac{\partial H''}{\partial p_y} - \frac{\partial H'}{\partial p_y} \frac{\partial H''}{\partial y}\right) \\
- u_x \left(\frac{\partial H'}{\partial u_x} - \frac{\partial H'}{\partial u_x} \frac{\partial H''}{\partial u_x}, \frac{\partial H''}{\partial u_x} \right) - u_y \left(\frac{\partial H'}{\partial u_y} - \frac{\partial H'}{\partial u_x} \frac{\partial H''}{\partial u_x}, \frac{\partial H''}{\partial u_y} \right) - u_z \left(\frac{\partial H'}{\partial u_z} - \frac{\partial H'}{\partial u_x} \frac{\partial H''}{\partial u_x}, \frac{\partial H''}{\partial u_z} \right) \\
- \frac{k^2 m}{k^2 + r^2 m} \left[\Omega'(t)y - \frac{1}{m}\frac{\partial H'''}{\partial x} + \Omega(t)\frac{\partial H'''}{\partial p_y} + \frac{r}{k^2}(u_x^2\frac{\partial H'''}{\partial u_x} - u_x\frac{\partial H'''}{\partial u_y}) + \frac{\partial (H' - H)}{\partial p_x} + \frac{\partial (H' - H)}{\partial u_x}\right] \\
- \frac{k^2 m}{k^2 + r^2 m} \left[-\Omega'(t)x - \Omega(t)\frac{\partial H'''}{\partial p_x} + \frac{1}{m}\frac{\partial H'''}{\partial y} - \frac{r}{k^2}(u_y^2\frac{\partial H'''}{\partial u_x} - u_y\frac{\partial H'''}{\partial u_y}) + \frac{\partial (H' - H)}{\partial p_y} + \frac{\partial (H' - H)}{\partial u_y}\right] \\
+ \frac{k^2 m}{k^2 + r^2 m} \left[\Omega'(t)y - \frac{1}{m}\frac{\partial H'''}{\partial x} + \Omega(t)\frac{\partial H'''}{\partial p_y} + \frac{r}{k^2}(u_x^2\frac{\partial H'''}{\partial u_x} - u_x\frac{\partial H'''}{\partial u_y}) + \frac{\partial (H' - H)}{\partial p_x} + \frac{\partial (H' - H)}{\partial u_x}\right] \\
+ \frac{k^2 m}{k^2 + r^2 m} \left[-\Omega'(t)x - \Omega(t)\frac{\partial H'''}{\partial p_x} + \frac{1}{m}\frac{\partial H'''}{\partial y} - \frac{r}{k^2}(u_y^2\frac{\partial H'''}{\partial u_x} - u_y\frac{\partial H'''}{\partial u_y}) + \frac{\partial (H' - H)}{\partial p_y} + \frac{\partial (H' - H)}{\partial u_y}\right],
\]
for sections \( \hat{h}' \) and \( \hat{h}'' \) with associated functions \( H' \) and \( H'' \), respectively.

VII. Conclusions and Future Work

We have developed a general geometrical setting for nonholonomic mechanical systems in the context of Lie affgebroids. We list the main results obtained in this paper:

- The notion of regularity of a nonholonomic mechanical system with affine constraints on a Lie affgebroid was elucidated and characterized in geometrical terms.
In the regular case, the solution of the nonholonomic problem was obtained by project-
ing the unconstrained one using several decompositions of the prolongation of the
Lie affgebroid along the affine constraint subbundle.
The hamiltonian formalism for nonholonomic systems is completely analyzed and a
nonholonomic bracket was defined.
Several examples were discussed showing the versatility of this geometric framework.

In a forthcoming paper we will study the reduction of the Lie affgebroid nonholonomic dyna-

mics under symmetry and we will obtain a Lie affgebroid version of the momentum equation
introduced in [7] for Lie algebroids.

Other goal we have proposed is to develop a geometric formalism for vakonomic Mechanics and
optimal control theory on Lie affgebroids.

In addition, we will explore the construction of geometric integrators for mechanical systems
on Lie affgebroids and, in particular, for nonholonomic systems.

Acknowledgments
This work has been partially supported by MEC (Spain) Grants MTM 2006-03322, MTM
2004-7832, project “Ingenio Mathematica” (i-MATH) No. CSD 2006-00032 (Consolider-Ingenio
2010) and S-0505/ESP/0158 of the CAM. D. Iglesias wants to thank MEC for a Research
Contract “Juan de la Cierva”.

References
[1] L. Bates and J. Šniatycki: Nonholonomic reduction, Rep. Math. Phys. 32 (1), 99–115 (1992).
[2] A. M. Bloch: Nonholonomic mechanics and control. Interdisciplinary Applied Mathematics, 24. Systems
and Control. Springer-Verlag, New York, 2003
[3] A. M. Bloch, P.S. Krishnaprasad, J. E. Marsden and R. M. Murray: Nonholonomic mechanical systems
with symmetry, Arch. Rational Mech. Anal. 136, 21–99 (1996).
[4] F. Cantrijn, M. de León, J.C. Marrero and D. Martín de Diego: On almost-Poisson structures in
nonholonomic mechanics. II The time-dependent framework, Nonlinearity 13, 1379–1409 (2000).
[5] F. Cantrijn, M. de León and D. Martín de Diego: On almost-Poisson structures in nonholonomic
mechanics, Nonlinearity 12, 721–737 (1999).
[6] J. Cortés: Geometric, control and numerical aspects of nonholonomic systems, Lecture Notes in Math-
ematics, vol. 1793, Springer-Verlag, 2002.
[7] J. Cortés, M. de León, J.C. Marrero and E. Martínez: Nonholonomic Lagrangian systems on Lie alge-
broids, Preprint, 2005
[8] J. Cortés and E. Martínez: Mechanical control systems on Lie algebroids, IMA J. Math. Control. Inform.
21, 457–492 (2004).
[9] J. Grabowski, K. Grabowska, P. Urbanski: Lie brackets on affine bundles, Ann. Glob. Anal. Geom., 24,
101–130 (2003).
[10] K. Grabowska, J. Grabowski, P. Urbański: AV-differential geometry: Poisson and Jacobi structures, J.
Geom. Phys. 52, 398–446 (2004).
[11] P.J. Higgins, K. Mackenzie: Algebraic constructions in the category of Lie algebroids, J. Algebra, 129,
194–230 (1990).
[12] D. Iglesias, J.C. Marrero, E. Padrón and D. Sosa, Lagrangian submanifolds and dynamics on Lie affge-
broids, Rep. Math. Phys. 38, 385–436 (2006).
[13] W.S. Koon and J.E. Marsden: Poisson reduction of nonholonomic mechanical systems with symmetry,
Rep. Math. Phys. 42 (1/2), 101–134 (1998).
[14] M. de León, J.C. Marrero and D. Martín de Diego: Non-holonomic Lagrangian systems in jet manifolds. J. Phys. A 30 no. 4, 1167–1190 (1997).
[15] M. de León, J.C. Marrero and D. Martín de Diego: Mechanical systems with nonlinear constraints, International Journal of Theoretical Physics 36 (4), 979–995 (1997).
[16] M. de León, J.C. Marrero and D. Martín de Diego: Time-dependent mechanical systems with non-linear constraints, Proc. Conf. on Differential Geometry (Budapest, July 27-30 1996) (Dordrecht: Kluwer), 221–234 (1999).
[17] M. de León, J.C. Marrero, E. Martínez: Lagrangian submanifolds and dynamics on Lie algebroids, J. Phys. A.: Math. Gen. 38, R241–R308 (2005).
[18] M. de León and D. Martín de Diego: On the geometry of non-holonomic Lagrangian systems, J. Math. Phys. 37 (7), 3389–3414 (1996).
[19] A.D. Lewis: Affine connections and distributions with applications to nonholonomic mechanics, Rep. Math. Phys. 42, 135–164 (1998).
[20] A.D. Lewis and R.M. Murray: Variational principles for constrained systems: theory and experiment, International Journal of Nonlinear Mechanics 30 (6), 793–815 (1995).
[21] K. Mackenzie: General theory of Lie groupoids and Lie algebroids. London Mathematical Society Lecture Note Series, 213, Cambridge University Press, Cambridge, 2005.
[22] ChM. Marle: Reduction of constrained mechanical systems and stability of relative equilibria, Comm. Math. Phys. 174, 295–318 (1995).
[23] E. Martínez: Lagrangian mechanics on Lie algebroids Acta Appl. Math. 67, 295–320 (2001).
[24] E. Martínez: Geometric Formulation of Mechanics on Lie algebroids, Proceedings of the VIII Fall Workshop on Geometry and Physics (Medina del Campo, 1999). Publicaciones de la RSME, vol. 2, 209–222 (2001).
[25] E. Martínez: Lie algebroids, Some Generalizations and Applications, Proceedings of the XI Fall Workshop on Geometry and Physics (Oviedo, 2002). Publicaciones de la RSME, vol. 6, 103–117.
[26] E. Martínez, T. Mestdag, W. Sarlet: Lie algebroid structures and Lagrangian systems on affine bundles, J. Geom. and Phys. 44, 70–95 (2002).
[27] T. Mestdag: Lagrangian reduction by stages for non-holonomic systems in a Lie algebroid framework, J. Phys. A: Math. Gen. 38, 10157–10179 (2005).
[28] T. Mestdag and B. Langerock B: A Lie algebroid framework for nonholonomic systems, J. Phys. A: Math. Gen. 38, 1097–1111 (2005).
[29] J. Neimark J and N. Fufaev: Dynamics on Nonholonomic systems, Translation of Mathematics Monographs, 33, AMS, Providence, RI (1972).
[30] W. Sarlet, F. Cantrijn and D.J. Saunders: A geometrical framework for the study of nonholonomic Lagrangian systems, J. Phys. A: Math. and Gen. 28, 3253–3268 (1995).
[31] W. Sarlet, T. Mestdag, E. Martínez: Lie algebroid structures on a class of affine bundles, J. Math. Phys. 43, 5654–5674 (2002).
[32] D.J. Saunders: The geometry of jet bundles, London Math. Soc., Lecture Note Series, 142 Cambridge Univ. Press, (1989).
[33] D.J. Saunders, W. Sarlet and F. Cantrijn: A geometrical framework for the study of nonholonomic Lagrangian systems: II, J. Phys. A: Math. and Gen. 29, 4265–4274 (1996).
[34] D.J. Saunders, F. Cantrijn and W. Sarlet: Regularity aspects and Hamiltonization of non-holonomic systems, J. Phys. A: Math. and Gen. 32, 6869–6890 (1999).
[35] J. Śniatycki: Nonholonomic Noether Theorem and reduction of symmetries. Rep. Math. Phys. 42, 5–23 (1998).
[36] A.J. Van der Schaft and V.M. Maschke: On the Hamiltonian formulation of non-holonomic mechanical systems. Rep. Math. Phys. 34, 225–233 (1994).
[37] J. Vankerschaver, F. Cantrijn, M. de León and D. Martín de Diego: Geometric aspects of nonholonomic field theories. Rep. Math. Phys. 56 no. 3, 387–411 (2005).
[38] A. Weinstein: Lagrangian Mechanics and groupoids, In Mechanics day (Waterloo, ON, 1992), Fields Institute Communications 7, 207–231 (1996).
