On the rate of convergence to the semi-circular law

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Abstract

Let $X = (X_{jk})$ denote a Hermitian random matrix with entries $X_{jk}$, which are independent for $1 \leq j \leq k$. We consider the rate of convergence of the empirical spectral distribution function of the matrix $X$ to the semi-circular law assuming that $E X_{jk} = 0$, $E X_{jk}^2 = 1$ and that the distributions of the matrix elements $X_{jk}$ have a uniform sub exponential decay in the sense that there exists a constant $\kappa > 0$ such that for any $1 \leq j \leq k \leq n$ and any $t \geq 1$ we have

$$\Pr\{|X_{jk}| > t\} \leq \kappa^{-1} \exp\{-t^\kappa\}.$$ 

By means of a short recursion argument it is shown that the Kolmogorov distance between the empirical spectral distribution of the Wigner matrix $W = \frac{1}{\sqrt{n}} X$ and the semicircular law is of order $O(n^{-1} \log^b n)$ with some positive constant $b > 0$.

1 Introduction

Consider a family $X = \{X_{jk}\}$, $1 \leq j \leq k \leq n$, of independent random variables defined on some probability space $(\Omega, \mathcal{F}, \Pr)$. Assume that $X_{jk} = X_{kj}$, for $1 \leq k <
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\[ j \leq n, \text{ and introduce the symmetric matrices} \]

\[ W = \frac{1}{\sqrt{n}} \begin{pmatrix} X_{11} & X_{12} & \cdots & X_{1n} \\ X_{21} & X_{22} & \cdots & X_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ X_{n1} & X_{n2} & \cdots & X_{nn} \end{pmatrix}. \]

The matrix \( W \) has a random spectrum \( \{\lambda_1, \ldots, \lambda_n\} \) and an associated spectral distribution function

\[ F_n(x) = \frac{1}{n} \text{card} \{j \leq n : \lambda_j \leq x\}, \quad x \in \mathbb{R}. \]

Averaging over the random values \( X_{ij}(\omega) \), define the expected (non-random) empirical distribution functions

\[ F_n(x) = \mathbb{E} F_n(x). \]

Let \( G(x) \) denote the semi-circular distribution function with density \( g(x) = G'(x) = \frac{1}{2\pi} \sqrt{4 - x^2} I_{[-2,2]}(x) \), where \( I_{[a,b]}(x) \) denotes an indicator function of interval \([a, b]\). We shall study the rate of convergence \( F_n(x) \) to the semi-circular law under the condition

\[ \Pr\{|X_{jk}| > t\} \leq \varkappa^{-1} \exp\{-t^\varkappa\} \text{ for some } \varkappa > 0. \]

This problem has been studied by several authors. The authors proved in [7] that the Kolmogorov distance between \( F_n(x) \) and the distribution function \( G(x) \), \( \Delta^*_n := \sup_x |F_n(x) - G(x)| = O(n^{-\frac{1}{2}}) \), Bai, [1], and Girko, [4], showed that \( \Delta_n := \sup_x |F_n(x) - G(x)| = O(n^{-\frac{1}{2}}) \). Bobkov, Götze and Tikhomirov [3] proved that \( \Delta_n \) and \( \mathbb{E}\Delta^*_n \) have order \( O(n^{-\frac{2}{3}}) \) assuming a Poincaré inequality for the distribution of the matrix elements. For the Gaussian Unitary Ensemble in [6] and for the Gaussian Orthogonal Ensemble in [11] it has been shown that \( \Delta_n = O(n^{-1}) \). Denote by \( \gamma_{n1} \leq \ldots \leq \gamma_{nn} \), the quantiles of \( G \), i.e. \( G(\gamma_{nj}) = \frac{j}{n} \). We introduce the notation

\[ l \log n := \log \log n \quad (1.1) \]

In Erdös, Yau and Yin [9] showed that for matrix elements \( X_{jk} \) which have a uniformly sub exponential decay in the sense that there exists a constant \( \varkappa > 0 \) such that for any \( 1 \leq j \leq k \leq n \) and any \( t \geq 1 \)

\[ \Pr\{|X_{jk}| \geq t\} \leq \varkappa^{-1} \exp\{-t^\varkappa\}, \]

the following result holds

\[ \Pr\left\{ \exists j : |\lambda_j - \gamma_j| \geq (\log n)^c l \log n \left[ \min\{(j, N-j+1]\right]^{-\frac{1}{4}} n^{-\frac{3}{2}} \right\} \leq C \exp\{- (\log n)^c l \log n \} \]

for large \( n \) enough. It is straightforward to check that this bound implies that with high probability

\[ \Pr\left\{ \sup_x |F_n(x) - G(x)| \leq C n^{-1} (\log n)^c l \log n \right\} \geq 1 - C \exp\{- (\log n)^c l \log n \}. \quad (1.2) \]
From the last inequality it is follows that
\[ E\Delta_{n}^{*} \leq Cn^{-1}(\log n)^{C\log n}. \]

In this paper we derive some improvement of the result (1.2) (reducing the power of logarithm) using arguments similar to [9] and provide a self-contained proof based on recursion methods developed in the papers of Götze and Tikhomirov [7], [5], [12].

For any positive constants \( \alpha > 0 \) and \( \kappa > 0 \) define the quantities
\[ l_{n,\alpha} := \log n (\log \log n)^{\alpha} \quad \text{and} \quad \beta_{n} := (l_{n,\alpha})^{\frac{1}{2}} + \frac{1}{2}. \] (1.3)

The main result of this paper is the following

**Theorem 1.1.** Let \( E X_{jk} = 0, \ E X_{jk}^{2} = 1 \). Assume that there exists a constant \( \kappa > 0 \) such that for any \( 1 \leq j \leq k \leq n \) and any \( t \geq 1 \),
\[ \Pr \{ |X_{jk}| \geq t \} \leq \kappa^{-1} \exp\{-t^{\kappa}\}. \] (1.4)

Then, for any positive \( \alpha > 0 \) there exist a positive constants \( C \) and \( c \) depending on \( \kappa \) and \( \alpha \) only such that
\[ \Pr \left\{ \sup_{x} |F_{n}(x) - G(x)| > n^{-1} \beta_{n}^{2} \right\} \leq C \exp\{-cl_{n,\alpha}\}. \] (1.5)

We apply the result of Theorem 1.1 to the investigation of the eigenvectors of the matrix \( W \). Let \( u = (u_{j1}, \ldots, u_{jn})^{T} \) be eigenvectors of the matrix \( W \) corresponding to the eigenvalues \( \lambda_{j}, j = 1, \ldots, n \). We prove the following result.

**Theorem 1.2.** Under the conditions of Theorem 1.1 for any positive \( \alpha > 0 \) there exist positive constants \( C \) and \( c \), depending on \( \kappa \) and \( \alpha \) only such that
\[ \Pr \left\{ \max_{1 \leq j, k \leq n} |u_{jk}|^{2} > \frac{\beta_{n}^{2}}{n} \right\} \leq C \exp\{-cl_{n,\alpha}\}, \] (1.6)

and
\[ \Pr \left\{ \max_{1 \leq k \leq n} \left| \sum_{\nu=1}^{k} |u_{j\nu}|^{2} - \frac{k}{n} \right| > \frac{\beta_{n}}{\sqrt{n}} \right\} \leq C \exp\{-cl_{n,\alpha}\}. \] (1.7)

## 2 Proof of the main Theorem

To bound error \( \Delta_{n}^{*} \) we shall use an approach developed in our paper [7]. We shall apply a bound of the Kolmogorov distance between distribution functions via the distance between their Stieltjes transforms. We denote the Stieltjes transform of \( F_{n}(x) \) by \( m_{n}(z) \) and the Stieltjes transform of a semi-circular law by \( s(z) \). Let \( R = R(z) \) be the resolvent matrix of \( W \) given by
\[ R = (W - zI_{n})^{-1}, \]
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for all $z = u + iv$ with $v \neq 0$. Here and in what follows $I_n$ denotes the identity matrix of dimension $n$. Sometimes we shall omit the sub index in the notation of an identity matrix. It is well-known that the Stieltjes transform of a semi-circular distribution satisfies the equation

$$s^2(z) + zs(z) + 1 = 0$$

(see, for example, equality (4.20) in [7]). Furthermore, the Stieltjes transform of empirical spectral distribution function $F_n(x)$, say $m_n(z)$, is given by

$$m_n(z) = \frac{1}{n} \sum_{j=1}^{n} R_{jj} = \frac{1}{n} \text{ETr} \mathbf{R}.$$ 

(see, for instance, equality (4.3) in [7]). Introduce the matrices $W^{(j)}$, which are obtained from $W$ by deleting the $j$-th row and the $j$-th column, and the corresponding resolvent matrix $R^{(j)}$ defined by $R^{(j)} := (W^{(j)} - zI_{n-1})^{-1}$ and let $m_n^{(j)}(z) := \frac{1}{n} \text{Tr} R^{(j)}$. Consider the index sets $T_j := \{1, \ldots, n\} \setminus \{j\}$. We shall use the representation

$$R_{jj} = \frac{1}{-z + \frac{1}{n} X_{jj} - \frac{1}{n} \sum_{k,l \in T_j} X_{jk} X_{jl} R_{kl}^{(j)}}, \quad (2.1)$$

(see, for example, equality (4.6) in [7]). We may rewrite it as follows

$$R_{jj} = -\frac{1}{z + m_n(z)} + \frac{1}{z + m_n(z)} \varepsilon_j R_{jj}, \quad (2.2)$$

where $\varepsilon_j := \varepsilon_{j1} + \varepsilon_{j2} + \varepsilon_{j3} + \varepsilon_{j4}$ with

$$\varepsilon_{j1} := \frac{1}{\sqrt{n}} X_{jj}, \quad \varepsilon_{j2} := \frac{1}{n} \sum_{k \in T_j} (X_{jk}^2 - 1) R_{kk}^{(j)},$$

$$\varepsilon_{j3} := \frac{1}{n} \sum_{k \neq l \in T_j} X_{jk} X_{jl} R_{kl}^{(j)}, \quad \varepsilon_{j4} := \frac{1}{n} \left( \text{Tr} R^{(j)} - \text{Tr} R \right).$$

This relation immediately implies the following two equations

$$R_{jj} = -\frac{1}{z + m_n(z)} - \sum_{\nu=1}^{3} \frac{\varepsilon_{j\nu}}{(z + m_n(z))^2} + \sum_{\nu=1}^{3} \frac{1}{(z + m_n(z))^2} \varepsilon_{j\nu} \varepsilon_{j} R_{jj} + \frac{1}{z + m_n(z)} \varepsilon_{j4} R_{jj}, \quad (2.3)$$
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and

\[ m_n(z) = -\frac{1}{z + m_n(z)} - \frac{1}{z + m_n(z)} \sum_{j=1}^{n} \varepsilon_{j} R_{jj} \]

\[ = -\frac{1}{z + m_n(z)} - \frac{1}{z + m_n(z)} ^{2} \sum_{j=1}^{n} \sum_{\nu=1}^{n} \varepsilon_{j\nu} \]

\[ + \frac{1}{z + m_n(z)} ^{2} \sum_{j=1}^{n} \sum_{\nu=1}^{n} \varepsilon_{j\nu} \varepsilon_{j} R_{jj} + \frac{1}{z + m_n(z)} \sum_{j=1}^{n} \varepsilon_{j4} R_{jj}. \quad (2.4) \]

2.1 Large deviations I

In the following Lemmas we shall bound \( \varepsilon_{j\nu} \), for \( \nu = 1, \ldots, n \)

Lemma 2.1. Assuming the conditions of Theorem 1.1 there exists positive constants \( C \) and \( c \) depending on \( \kappa \) and \( \alpha \) such that, for any \( j = 1, \ldots, n \)

\[ \Pr\{|\varepsilon_{j1}| \geq 2l_{n, \alpha}^{1/2} n^{-1/2}\} \leq C \exp\{-cl_{n, \alpha}\} \]

Proof. The result follows immediately from the hypothesis (1.4).

Lemma 2.2. Assuming the conditions of Theorem 1.1 we have, for any \( z = u + iv \) with \( v > 0 \) and any \( j = 1, \ldots, n \), we have

\[ |\varepsilon_{j4}| \leq \frac{1}{nv}. \]

Proof. The conclusion of Lemma 2.2 follows immediately from the obvious inequality \(|\text{Tr} R - \text{Tr} R^{(j)}| \leq v^{-1} \) (see Lemma 4.1 in [7]).

Lemma 2.3. Assuming the conditions of Theorem 1.1 for all \( v > 0 \), the following inequality holds

\[ \Pr\{|\varepsilon_{j2}| > 2l_{n, \alpha}^{1/2} n^{-1/2} \geq n^{-1} \sum_{l \in T_{j}} |R_{jl}^{(j)}|^{2} \} \leq C \exp\{-cl_{n, \alpha}\} \]

for some positive constants \( c > 0 \) and \( C \), depending on \( \kappa \) and \( \alpha \) only.

Proof. We use the following inequality for sums of independent random variables. Let \( \xi_{1}, \ldots, \xi_{n} \) be independent random variables such that \( E\xi_{j} = 0 \) and \( |\xi_{j}| \leq \sigma_{j} \). Then

\[ \Pr\{|\sum_{j=1}^{n} \xi_{j}| > x\} \leq c(1 - \Phi(x/\sigma) \leq \sigma \exp\{-x^{2}/2\sigma^{2}\}, \quad (2.5) \]

where \( \Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp\{-y^{2}/2\}dy \) and \( \sigma^{2} = \sigma_{1}^{2} + \cdots + \sigma_{n}^{2} \). We put \( \eta_{l} = X_{jl}^{2} - 1 \), and define,

\[ \xi_{l} = \left( \eta_{l}\mathbb{I}\{|X_{jl}| \leq l_{n, \alpha}^{1/2}\} - E\eta_{l}\mathbb{I}\{|X_{jl}| \leq l_{n, \alpha}^{1/2}\}\right) R_{jl}^{(j)}. \]
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Note that \( E \xi_l = 0 \) and \( |\xi_l| \leq 2l_{n,a}^2|R^{(j)}_l| \). Introduce the \( \sigma \)-algebra \( \mathcal{W}^{(j)} \) generated by the random variables \( X_{kl} \) with \( k, l \in T_j \). Let \( E_j \) and \( \text{Pr}_j \) denote the conditional expectation and the conditional probability with respect to \( \mathcal{W}^{(j)} \). Note that the random variables \( X_{jl} \) and the \( \sigma \)-algebra \( \mathcal{M}(j) \) are independent. Applying inequality (2.5) with \( x := 2n^{1/2}l_{n,a}^2(n^{-1} \sum_{l \in T_j} |R^{(j)}_l|^2)^{1/2} \), we get

\[
\text{Pr}\left\{ |\sum_{l \in T_j} \xi_l| > x \right\} = E \text{Pr}_j\left\{ |\sum_{l \in T_j} \xi_l| \geq x \right\} \leq E \exp\left\{ -\frac{x^2}{\sigma^2} \right\} \leq C \exp\{-cn_{a} \}. \tag{2.6}
\]

Furthermore, note that

\[
|E_j \eta_j (|\xi_l| \leq \frac{l}{n_{a}})| \leq E_j \frac{1}{\sigma} |\eta_l|^2 \text{Pr}_j\{ |\xi_l| > l_{n,a} \} \leq E_j \frac{1}{\sigma} |\eta_l|^2 \exp\{-\frac{c}{2}l_{n,a} \} \tag{2.7}
\]

The last inequality implies that

\[
\left| \sum_{l \in T_j} E_j \eta_j (|X_{jl}| \leq \frac{l}{n_{a}}) \right| R^{(j)}_l \right| \leq C \exp\{-\frac{c}{2}l_{n,a} \} \left( \frac{1}{n} \sum_{l \in T_j} |R^{(j)}_l|^2 \right)^{1/2}. \tag{2.8}
\]

The inequalities (2.6) and (2.8) together conclude the proof of Lemma 2.3. Thus the Lemma is proved.

\[\square\]

**Corollary 2.4.** Assuming the conditions of Theorem 1.1 for any \( \alpha > 0 \) there exists positive constants \( c \) and \( C \), depending on \( \kappa \) and \( \alpha \) such that for any \( z = u + iv \) with \( v > 0 \)

\[
\text{Pr}\{|\epsilon_{j2}| > \beta_{n}^2(nv)^{-1/2}(\text{Im} \ m^{(j)}_n(z))^{1/2} \} \leq C \exp\{-cn_{a} \}. \tag{2.9}
\]

**Proof.** Note that

\[
n^{-1} \sum_{l \in T_j} |R^{(j)}_l|^2 \leq n^{-1} \text{Tr} |R^{(j)}|^2 = \frac{1}{v} \text{Im} m^{(j)}_n(z), \tag{2.10}
\]

where \( |R^{(j)}|^2 = R^{(j)}R^{(j)\ast} \). The result follows now from Lemma 2.3. \[\square\]

**Lemma 2.5.** Assuming the conditions of Theorem 1.1, for any \( j = 1, \ldots, n \) and for any \( v > 0 \), the following inequality holds

\[
\text{Pr}\{|\epsilon_{j3}| > \beta_{n}^2(nv)^{-1/2}(\frac{1}{n} \sum_{k \neq l \in T_j} |R^{(j)}_k|^2)^{1/2} \} \leq C \exp\{-cn_{a} \}. \tag{2.11}
\]

**Proof.** We shall use a large deviation bound for quadratic forms which follows from results by Ledoux (see [10]).
**Proposition 2.1.** Let \( \xi_1, \ldots, \xi_n \) be independent random variables such that \( |\xi_j| \leq 1 \). Let \( a_{ij} \) denote complex numbers such that \( a_{ij} = a_{ji} \) and \( a_{jj} = 0 \). Let \( Z = \sum_{l,k=1}^{n} \xi_l \xi_k a_{lk} \). Let \( \sigma^2 = \sum_{l,k=1}^{n} |a_{lk}|^2 \). Then for every \( t > 0 \) there exists some positive constant \( c > 0 \) such that the following inequality holds

\[
\Pr\{|Z| \geq \frac{3}{2} E[Z] + t\} \leq \exp\{-\frac{ct}{\sigma}\}. \tag{2.12}
\]

**Proof.** Proposition 2.1 follows from Theorem 3.1 in [10]. \( \square \)

In order to bound \( \varepsilon_{j3} \) we use Proposition 2.1 with

\[
\xi_l = (X_{jl}\{ |X_{jl}| \leq \frac{1}{n} \} - EX_{jl}\{ |X_{jl}| \leq \frac{1}{n} \}) / 2\frac{1}{n}. \tag{2.13}
\]

Note that the random variables \( X_{jl}, l \in T_j \) and the matrix \( R^{(j)} \) are mutually independent for any fixed \( j = 1, \ldots, n \). Moreover, we have \( |\xi_l| \leq 1 \). Put \( Z := \sum_{k \neq l \in T_j} \xi_l \xi_k R^{(j)}_{kl} \).

Applying Proposition 2.1 with \( t = l_{n, \alpha}(\sum_{l \neq k \in T_j} |R^{(j)}_{kl}|^2)^{\frac{1}{2}} \), we get

\[
EPr\{ |Z| \geq l_{n, \alpha}(\sum_{l \neq k \in T_j} |R^{(j)}_{kl}|^2)^{\frac{1}{2}} \} \leq C \exp\{-cl_{n, \alpha}\}. \tag{2.14}
\]

Furthermore,

\[
\Pr\{ \exists j, l \in [1, \ldots, n]: |X_{jl}| > \frac{1}{n} \} \leq C \exp\{-cl_{n, \alpha}\} \tag{2.15}
\]

and

\[
|EX_{jl}|( |X_{jl}| \leq \frac{1}{n} ) | \leq \Pr\{\exists j, l, k \in [1, \ldots, n]: |X_{jl}| > \frac{1}{n} \} \leq C \exp\{-cl_{n, \alpha}\}. \tag{2.16}
\]

Introduce the random variables \( \hat{\xi}_l = \xi_{jl}\{ |X_{jl}| \leq \frac{1}{n} \} / 2\frac{1}{n} \) and \( \hat{Z} = \sum_{l, k \in T_j} \xi_l \xi_k R^{(j)}_{kl} \).

Note that

\[
\Pr\left\{ \sum_{l, k \in T_j} X_{jk} X_{jl} R^{(j)}_{kl} \neq \sum_{l, k \in T_j} \hat{\xi}_k \hat{\xi}_l R^{(j)}_{kl} \right\} \leq C \exp\{-cl_{n, \alpha}\}. \tag{2.17}
\]

Inequalities (2.14) - (2.17) together imply

\[
\Pr\left\{ |\varepsilon_{j3}| > \beta_n^2 n^{-\frac{1}{2}} (\frac{1}{n} \sum_{l \neq k \in T_j} |R^{(j)}_{kl}|^2)^{\frac{1}{2}} \right\} \leq C \exp\{-cl_{n, \alpha}\}. \tag{2.18}
\]

Thus Lemma 2.5 is proved. \( \square \)

**Corollary 2.6.** Under the conditions of Theorem 1.1 there exist positive constants \( c \) and \( C \) depending on \( \varkappa \) and \( \alpha \) such that for any \( z = u + iv \) with \( v > 0 \)

\[
\Pr\{ |\varepsilon_{j3}| > \beta_n^2 (nv) (-\frac{1}{2})(\text{Im} m_n^{(j)}(z))^{\frac{1}{2}} \} \leq C \exp\{-cl_{n, \alpha}\}. \tag{2.19}
\]
Proof. Note that
\begin{equation}
\frac{1}{n} \sum_{k \neq l \in T_j} |R^{(j)}_{kl}|^2 \leq \frac{1}{n} \text{Tr} |R^{(j)}|^2 = \frac{1}{v} \text{Im} m_n^{(j)}(z). \tag{2.20}
\end{equation}
The result now follows from Lemma 2.5.

To summarize these results we recall
\begin{equation}
\beta_n = (l_{n,\alpha})^{\frac{1}{2} + \frac{1}{2}}, \tag{2.21}
\end{equation}
defined previously in (1.3). Then we may write that, for \( \nu = 1, 2, 3 \)
\begin{equation}
\text{Pr}\left\{ |\varepsilon_{j\nu}| > \frac{\beta_n}{\sqrt{n}} \left( 1 + \frac{\text{Im} \frac{1}{2} m_n(z)}{\sqrt{v}} + \frac{1}{\sqrt{v \sqrt{n} v'}} \right) \right\} \leq C \exp\{-cn_{n,\alpha}\}. \tag{2.22}
\end{equation}
Denote by
\begin{equation}
\Omega_n(z) = \left\{ \omega \in \Omega : |\varepsilon_j| \leq \frac{\beta_n}{\sqrt{n}} \left( 1 + \frac{\text{Im} \frac{1}{2} m_n(z)}{\sqrt{v}} + \frac{1}{\sqrt{v \sqrt{n} v'}} \right) \right\}. \tag{2.23}
\end{equation}
Let \( v_0 = \frac{a^{\frac{3}{2}}}{n} \) with a sufficiently small positive constant \( a > 0 \). We introduce the region \( D = \{ z = u + iv \in \mathbb{C} : |u| \leq 2, v_0 < v \leq 2 \} \). Furthermore, we introduce the sequence \( z_t = u_t + v_t \) in \( D \), recursively defined via \( u_{t+1} - u_t = \frac{4}{n} \) and \( v_{t+1} - v_t = \frac{2}{n^2} \). Using a union bound, we have
\begin{equation}
\text{Pr}\{ \cap_{z_t \in D} \Omega_n(z_t) \} \geq 1 - C \exp\{-cn_{n,\alpha}\}. \tag{2.24}
\end{equation}
It is straightforward to check that
\begin{equation}
|\varepsilon_j(z) - \varepsilon_j(z')| \leq \frac{\beta_n}{v_0^2} \left( 1 + \frac{\text{Im} \frac{1}{2} m_n(z)}{\sqrt{v}} + \frac{1}{\sqrt{v \sqrt{n} v'}} \right). \tag{2.25}
\end{equation}
This immediately implies that
\begin{equation}
\text{Pr}\{ \cap_{z \in D} \Omega_n(z) \} \geq 1 - C \exp\{-cn_{n,\alpha}\}, \tag{2.26}
\end{equation}
for appropriately some chosen constant in the definition (2.23) of the event \( \Omega_n(z) \).

3 Large deviations II

In this Section we obtain bounds for large deviation probabilities of the sum of \( \varepsilon_j \). We start with
\begin{equation}
\delta_{n1} = \frac{1}{n} \sum_{j=1}^{n} \varepsilon_{j1}. \tag{3.1}
\end{equation}
Lemma 3.1. There exist constants $c$ and $C$ such that
\[
\Pr\{|\delta_{n1}| > n^{-1} \beta_n\} \leq C \exp\{-cl_{n,\alpha}\}.
\] \hfill (3.1)

Proof. We repeat the proof of Lemma 2.1. Consider the truncated random variables
\[
\hat{X}_{jj} = X_{jj} I\{|X_{jj}| \leq l_{n,\alpha}^{\frac{1}{2}}\}.
\] \hfill (3.2)

By assumption (1.4),
\[
\Pr\{|X_{jj}| > l_{n,\alpha}^{\frac{1}{2}}\} \leq C \exp\{-cl_{n,\alpha}\}.
\] \hfill (3.3)

Moreover,
\[
|E \hat{X}_{jj}| \leq C \exp\{-cl_{n,\alpha}\}.
\] \hfill (3.4)

We define
\[
\tilde{X}_{jj} = \hat{X}_{jj} - E \hat{X}_{jj}
\] \hfill (3.5)

and consider the sum
\[
\tilde{\delta}_{n1} := \frac{1}{n\sqrt{n}} \sum_{j=1}^{n} \tilde{X}_{jj}.
\] \hfill (3.6)

Since
\[
|\tilde{X}_{jj}| \leq 2l_{n,\alpha}^{\frac{1}{2}},
\]
we have
\[
\Pr\{|\tilde{\delta}_{n1}| > n^{-1} l_{n,\alpha}^{\frac{1}{2}}\} \leq C \exp\{-cl_{n,\alpha}\}.
\] \hfill (3.7)

Note that
\[
|\tilde{\delta}_{n1} - \delta_{n1}| \leq \frac{1}{n} \sum_{j=1}^{n} |E \hat{X}_{jj}| \leq \exp\{-cl_{n,\alpha}\}.
\] \hfill (3.8)

This inequality and inequality (3.7) together imply
\[
\Pr\{|\delta_{n1}| > n^{-1} l_{n,\alpha}^{\frac{1}{2}}\} \leq C \exp\{-cl_{n,\alpha}\}.
\] \hfill (3.9)

Thus, Lemma 3.1 is proved.

Consider now the quantity
\[
\delta_{n2} := \frac{1}{n^2} \sum_{j=1}^{n} \sum_{l \in \mathcal{T}_j} (X_{jl}^2 - 1) R_{il}^{(j)}.
\] \hfill (3.10)

We prove the following Lemma

Lemma 3.2. Assume that there exists a constant $C$ such that for any $j = 1, \ldots, n$ and any $l \in \mathcal{T}_j$
\[
|R_{il}^{(j)}| \leq C.
\] \hfill (3.11)

Then there exist constants $c$ and $C$, depending on $\nu$ and $\alpha$ such that
\[
\Pr\{|\delta_{n2}| \leq n^{-1} \beta_n^2\} \leq C \exp\{-cl_{n,\alpha}\}.
\] \hfill (3.12)
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Proof. Introduce the truncated random variables

\[ \xi_{jl} = \hat{X}_{jl}^2 - \mathbb{E}\hat{X}_{jl}^2, \]  

where \( \hat{X}_{jl} = X_j\mathbb{I}\{|X_j| \leq \frac{l}{\sqrt{n}}\}. \) It is straightforward to check that

\[ 0 \leq 1 - E\hat{X}_{jl}^2 \leq C \exp\{-c l n, \alpha\}. \]  

We shall need the following quantities as well

\[ \hat{\delta}_n^2 = \frac{1}{n^2} \sum_{j=1}^{n} \sum_{l \in \mathcal{T}_j} (\hat{X}_{jl}^2 - 1) R_{ll}^{(j)} \quad \text{and} \quad \tilde{\delta}_n^2 = \frac{1}{n^2} \sum_{j=1}^{n} \sum_{l \in \mathcal{T}_j} \xi_{jl} R_{ll}^{(j)}. \]  

By assumption (1.4),

\[ \Pr\{\delta_n^2 \neq \hat{\delta}_n^2\} \leq C \exp\{-c l n, \alpha\}. \]  

Let

\[ \zeta_j := \frac{1}{\sqrt{n}} \sum_{l \in \mathcal{T}_j} \xi_{jl} R_{ll}^{(j)}. \]  

Then

\[ \hat{\delta}_n^2 = \frac{1}{n^2} \sum_{j=1}^{n} \zeta_j. \]  

Note that the sequence \( \hat{\delta}_n^2 \) is a martingale with respect to the \( \sigma \)-algebras \( \mathcal{M}_j \). In fact,

\[ \mathbb{E}\{\zeta_j | \mathcal{M}_{j-1}\} = \mathbb{E}\{\mathbb{E}\{\zeta_j | \mathcal{M}^{(j)}\} | \mathcal{M}_{j-1}\} = 0. \]  

In order to use large deviation bounds for \( \hat{\delta}_n^2 \) we replace the differences \( \zeta_j \) by truncated random variables. We put

\[ \hat{\zeta}_j = \zeta_j \mathbb{I}\{|\zeta_j| \leq \frac{l}{\sqrt{n}}\mathbb{I}\left\{\frac{1}{n} \sum_{l \in \mathcal{T}_j} |R_{ll}^{(j)}|^2 \right\}^{\frac{1}{2}}\}. \]  

Since \( \zeta_j \) is a sum of independent bounded random variables with mean zero, we have

\[ \Pr\{|\zeta_j| > \frac{1}{n} \sum_{l \in \mathcal{T}_j} |R_{ll}^{(j)}|^2 \} \leq C \exp\{-c l n, \alpha\}. \]  

This implies that

\[ \Pr\{\sum_{j=1}^{n} \zeta_j \neq \sum_{j=1}^{n} \hat{\zeta}_j\} \leq C \exp\{-c l n, \alpha\}. \]  

Furthermore, introduce the random variables

\[ \tilde{\zeta}_j = \hat{\zeta}_j - \mathbb{E}\{\hat{\zeta}_j | \mathcal{M}_{j-1}\}. \]
The rate of convergence to the semi-circular law

Using the Cauchy-Schwartz inequality and the boundedness of the random variables \( \xi_{jl} R^{(j)}_{ll} \) we may show that

\[
|E\{\zeta_j | M_{j-1}\}| \leq C \exp\{-cl_{n,\alpha}\}. \tag{3.24}
\]

Here, we may use a martingale bound due to Bentkus, \[2\], Theorem 1.1. It provides the following result. Let \( M_0 = \{\emptyset, \Omega\} \subset M_1 \subset \cdots \subset M_n \subset \mathbb{R} \) be a family of \( \sigma \)-algebras of a measurable space \( \{\Omega, \mathbb{R}\} \). Let \( M_n = \xi_1 + \cdots + \xi_n \) be a martingale with bounded differences \( \xi_j = M_j - M_{j-1} \) such that

\[
Pr\{|\xi_j| \leq b_j\} = 1 \quad \text{for} \quad j = 1, \ldots, n.
\]

Then, for \( x > \sqrt{8} \)

\[
Pr\{|M_n| \geq x\} \leq c(1 - \Phi(\frac{x}{\sigma})) = \int_{-\infty}^{\infty} \varphi(t)dt, \quad \varphi(t) = \frac{1}{\sqrt{2\pi}} \exp\{-\frac{t^2}{2}\}. \tag{3.25}
\]

with some numerical constant \( c > 0 \) and \( \sigma^2 = b_1^2 + \cdots + b_n^2 \). Note that for \( t > C \)

\[
1 - \Phi(t) \leq \frac{1}{C} \varphi(t).
\]

Thus, this leads to the inequality

\[
Pr\{|M_n| \geq x\} \leq \exp\{-\frac{x^2}{2\sigma^2}\}, \tag{3.26}
\]

which we shall use to bound \( \tilde{\delta}_{n2} \). By assumption \((3.11)\) and definition of \( \tilde{\zeta}_j \), we may take \( \beta_j = \frac{\beta}{\sqrt{n,\alpha}} \). We get

\[
Pr\{|\tilde{\delta}_{n2}| > n^{-1} \beta^2_n\} \leq C \exp\{-cl_{n,\alpha}\} \tag{3.27}
\]

Inequalities \((3.22)\)–\((3.27)\) together conclude the proof of Lemma 3.2. \( \square \)

Let

\[
\delta_{n3} := \frac{1}{n^2} \sum_{j=1}^{n} \sum_{l \neq k \in T_j} X_{jl} X_{jk} R^{(j)}_{lk} \tag{3.28}
\]

**Lemma 3.3.** Assume that there exists a constant \( B > 0 \) such that for any \( j = 1, \ldots, n \)

\[
\text{Im} \ m_n^{(j)}(z) \leq B. \tag{3.29}
\]

Then there exist constants \( c \) and \( C \), depending on \( \kappa \) and \( \alpha \) such that

\[
Pr\{|\delta_{n3}| > \frac{\beta^2_n}{n \sqrt{v}}\} \leq C \exp\{-cl_{n,\alpha}\} \tag{3.30}
\]
The rate of convergence to the semi-circular law

Proof. The proof of this Lemma is similar to the proof of Lemma 3.2. We introduce the random variables

$$\eta_j = \frac{1}{\sqrt{n}} \sum_{l \neq k \in \mathcal{T}_j} X_{jk} X_{jl} R_{lk}^{(j)}$$  \hspace{1cm} (3.31)$$

and note that the sequence

$$M_n = \sum_{j=1}^{n} \eta_j$$  \hspace{1cm} (3.32)$$
is martingale with respect to the $\sigma$–algebras $\mathfrak{M}_j$. Then we apply the martingale bound of Bentkus twice replacing $\eta_j$ by truncated random variables. Thus the Lemma is proved. \hfill \square

Finally, we shall bound

$$\delta_{n4} := \frac{1}{n^2} \sum_{j=1}^{n} (\text{Tr} R - \text{Tr} R^{(j)}) R_{jj}.$$  \hspace{1cm} (3.33)$$

Lemma 3.4. For any $z = u + iv$ with $v > 0$ the following inequality

$$|\delta_{n4}| \leq \frac{1}{nv} \text{Im} m_n(z)$$  \hspace{1cm} (3.34)$$
holds.

Proof. By formula (5.4) in [7], we have

$$(\text{Tr} R - \text{Tr} R^{(j)}) R_{jj} = (1 + \frac{1}{n} \sum_{l,k \in \mathcal{T}_j} X_{jl} X_{jk} (R^{(j)})_{lk}^2) R_{jj}^2 = \frac{d}{dz} R_{jj}.$$ \hspace{1cm} (3.35)$$

From here it follows that

$$\frac{1}{n^2} \sum_{j=1}^{n} (\text{Tr} R - \text{Tr} R^{(j)}) R_{jj} = \frac{1}{n^2} \frac{d}{dz} \text{Tr} R = \frac{1}{n^2} \text{Tr} R^2.$$ \hspace{1cm} (3.36)$$

Finally, we note that

$$|\frac{1}{n^2} \text{Tr} R^2| \leq \frac{1}{nv} \text{Im} m_n(z).$$ \hspace{1cm} (3.37)$$

The last inequality concludes the proof. Thus, Lemma 3.4 is proved. \hfill \square

3.1 Stieltjes transforms

We shall derive auxiliary bounds for the difference between the Stieltjes transforms $m_n(z)$ of the empirical spectral measure of the matrix $X$ and the Stieltjes transform $s(z)$ of the semi-circular law. Introduce the additional notations

$$\delta_n := \delta_{n1} + \delta_{n2} + \delta_{n3}, \quad \hat{\delta}_n := \delta_{n4}, \quad \bar{\delta}_n := \frac{1}{n} \sum_{v=1}^{3} \sum_{j=1}^{n} \varepsilon_{jv} \varepsilon_j R_{jj}.$$ \hspace{1cm} (3.38)$$
Recall that \( s(z) \) satisfies the equation
\[
s(z) = \frac{1}{z + s(z)}.
\] (3.39)

Introduce \( g_n(z) := m_n(z) - s(z) \). The representation (2.4) and equality (3.39) together imply
\[
g_n(z) = \frac{g_n(z)}{(z + s(z))(z + m_n(z))} - \frac{\delta_n}{(z + m_n(z))^2} + \frac{\tilde{\delta}_n}{z + m_n(z)} + \frac{\overline{\delta}_n}{(z + m_n(z))^2}. \tag{3.40}
\]

This equality yields
\[
|g_n(z)| \leq \frac{|\delta_n| + |\overline{\delta}_n|}{|z + m_n(z)||z + s(z) + m_n(z)|} + \frac{|\tilde{\delta}_n|}{|z + s(z) + m_n(z)|}. \tag{3.41}
\]

For any \( z \in \mathcal{D} \) introduce the events
\[
\hat{\Omega}_n(z) := \{ \omega \in \Omega : |\delta_n| \leq \frac{\beta_n}{n\sqrt{v}} \}, \quad \overline{\Omega}_n(z) := \{ \omega \in \Omega : |\hat{\delta}_n| \leq \frac{C\text{Im} m_n(z)}{nv} \}, \tag{3.42}
\]
\[
\overline{\Omega}_n(z) := \{ \omega \in \Omega : |\overline{\delta}_n| \leq \left( \frac{\beta_n^2\text{Im} m_n(z)}{n^2v} + \frac{\beta_n^2}{nv^2} \right) \frac{1}{n} \sum_{j=1}^{n} |R_{jj}|^2 \}. \tag{3.43}
\]

Put \( \Omega_n^* := \hat{\Omega}_n(z) \cup \overline{\Omega}_n(z) \cup \overline{\Omega}_n(z) \). By Lemmas 3.1–3.4, we have
\[
\Pr\{\hat{\Omega}_n(z)\} \geq 1 - C \exp\{-c l_n,\alpha\}. \tag{3.44}
\]

By Lemma 3.4,
\[
\Pr\{\overline{\Omega}_n(z)\} = 1. \tag{3.45}
\]

Note that
\[
|\varepsilon_{j\nu}\varepsilon_{j4}| \leq \frac{1}{2}(|\varepsilon_{j\nu}|^2 + |\varepsilon_{j4}|^2). \tag{3.46}
\]

By inequality (2.22), we have, for \( \nu = 1, 2, 3 \),
\[
\Pr\left\{|\varepsilon_{j\nu}|^2 > \frac{\beta_n^2}{n}(1 + \frac{\text{Im} m_n(z)}{v} + \frac{1}{nv^2})\right\} \leq C \exp\{-c l_n,\alpha\} \tag{3.47}
\]
and
\[
\Pr\{|\varepsilon_{j4}|^2 \leq \frac{1}{n^2v^2}\} = 1. \tag{3.48}
\]

Similarly as in (2.26) we may show that
\[
\Pr\{\cap_{z \in \mathcal{D}} \Omega_n^*(z) \cap \Omega_n\} \geq 1 - C \exp\{-c l_n,\alpha\}. \tag{3.49}
\]

Let
\[
\Omega_n^* := \cap_{z \in \mathcal{D}} \Omega_n^*(z) \cap \Omega_n. \tag{3.50}
\]

We now prove now some auxiliary Lemmas.
Lemma 3.5. Let $z = u + iv \in \mathcal{D}$ and $\omega \in \Omega_n^*$. Assume that

$$|g_n(z)| \leq \frac{1}{2}. \quad (3.51)$$

Then the following bound holds

$$|g_n(z)| \leq \frac{C\beta_n^2}{nv} + \frac{C\beta_n^2}{n^2v^2\sqrt{\gamma + v}}.$$  \hspace{1cm} (3.52)

Proof. First we note that the inequality $|g_n(z)| \leq \frac{1}{2}$ implies

$$|z + m_n(z)| \geq |z + s(z)| - |g_n(z)| \geq \frac{1}{2}. \quad (3.52)$$

Moreover,

$$\text{Im} m_n^{(j)}(z) \leq |m_n^{(j)}(z)| \leq |m_n(z)| + \frac{1}{nv} \leq |s(z)| + |g_n(z)| + \frac{1}{nv} \leq C. \quad (3.53)$$

Furthermore, we obviously obtain

$$|z + s_n(z) + s(z)| \geq \text{Im} z + \text{Im} m_n(z) + \text{Im} s(z) \geq \text{Im} (z + s(z)) \geq \text{Im}\{\sqrt{z^2 - 4}\}. \quad (3.54)$$

For $z \in \mathcal{D}$ we get $\text{Re}(z^2 - 4) \leq 0$ and $\frac{\pi}{2} \leq \arg(z^2 - 4) \leq \frac{3\pi}{2}$. Therefore,

$$\text{Im}\{\sqrt{z^2 - 4}\} \geq \frac{1}{\sqrt{2}}|z^2 - 4|^{\frac{1}{2}} \geq \frac{1}{4}\sqrt{\gamma + v}, \quad (3.55)$$

where $\gamma = 2 - |u|$. Inequality (3.41) implies that for $\omega \in \Omega_n^*$

$$|g_n(z)| \leq \frac{\beta_n}{n\sqrt{v}|z + m_n(z)||z + s(z) + m_n(z)|} + \frac{C\text{Im} m_n(z)}{nv|z + s(z) + m_n(z)|} + \frac{\beta_n^2}{nv|z + m_n(z)||z + s(z) + m_n(z)|} \left(\text{Im} m_n(z) + \frac{1}{nv}\right) \frac{1}{n} \sum_{j=1}^{n}|R_{jj}|^2. \quad (3.56)$$

Furthermore, equation (2.2), inequality (3.52) and the definition of $\Omega_n^*$ in (2.23) together imply that, for $\omega \in \Omega_n^*$ and $z \in \mathcal{D}$

$$|R_{jj}| \leq \frac{2}{|z + s_n(z)|}. \quad (3.57)$$

Equation (3.56) and inequality (3.57) together imply

$$|g_n(z)| \leq \frac{C\beta_n^2}{nv} \left(1 + \frac{1}{nv}\sqrt{\gamma + v}\right). \quad (3.58)$$

This inequality completes the proof of lemma. \hfill \Box
Put now $v_0' := v_0'(z) = \frac{v_0}{\sqrt{\gamma}}$, where $\gamma := 2 - |u|$ and $z = u + iv$. Denote $\hat{D} := \{z \in \mathcal{D} : v \geq v_0\}$.

**Corollary 3.6.** Assume that for $\omega \in \Omega_n^*$ and $z = u + iv \in \hat{D}$

$$|g_n(z)| \leq \frac{1}{2}.$$  

Then

$$|g_n(z)| \leq \frac{1}{100}.$$  

**Proof.** Note that for $v \geq v_0'$

$$\frac{C\beta^2_n}{nv} + \frac{C\beta^2_n}{n^2v^2\sqrt{\gamma + v}} \leq \frac{1}{100}$$  \hspace{1cm} (3.59)

Thus, the Corollary is proved.  

**Corollary 3.7.** Let $z = u + iv \in \mathcal{D}$. Assume that

$$|z + g_n(z)| > \frac{1}{2}.$$  

Then for any $\omega \in \Omega_n$, the following bound holds

$$|g_n(z)| \leq \frac{C\beta^2_n}{nv} + \frac{C\beta^2_n}{n^2v^2\sqrt{\gamma + v}}.$$  

**Proof.** In the proof of Lemma 3.5 we have only used condition (3.51) of Lemma 3.5 to prove inequality (3.60). This proves the Corollary.

Assume that $N_0$ is sufficiently large number such that for any $n \geq N_0$ and for any $v \in \mathcal{D}$ the right hand side of inequality (3.58) is smaller then $\frac{1}{100}$. In the what follows we shall assume that $n \geq N_0$ is fixed. The following lemma plays a crucial role in our proof. It is similar to Lemma 3.4 in [S].

**Lemma 3.8.** Assume that for some $z = u + iv \in \mathcal{D}$ with $v \geq v_0$ condition (3.51) holds. Then it holds for $z' = u + iv' \in \mathcal{D}$ with $v \geq v' \geq v - n^{-8}$.

**Proof.** First of all note that

$$|m_n(z) - m_n(z')| = \frac{1}{n}(v - v')|\text{Tr} R(z)R(z')| \leq \frac{v - v'}{vv'} \leq \frac{C}{n^4} \leq \frac{1}{100}$$  \hspace{1cm} (3.61)

and

$$|s(z) - s(z')| \leq \frac{1}{100}$$  \hspace{1cm} (3.62)

By Corollary 3.7 we have

$$|g_n(z)| \leq \frac{1}{100}.$$  \hspace{1cm} (3.63)
The rate of convergence to the semi-circular law

All these inequalities together imply

\[ |g_n(z')| \leq \frac{3}{100} < \frac{1}{2}. \]  \hspace{2cm} (3.64)

Thus, the Lemma is proved.

**Proposition 3.1.** There exists positive constants \( C, c \), depending on \( \alpha \) and \( \kappa \) only such that

\[ \Pr \left\{ |g_n(z)| > \frac{\beta_n^2 (\text{Im} m_n + \frac{1}{nv})}{n \sqrt{v} \sqrt{\gamma + v}} \right\} \leq C \exp \{-cl_{n,\alpha}\}. \]  \hspace{2cm} (3.65)

for all \( z \in D \)

**Proof.** Note that for \( v = 2 \) we have, for any \( \omega \in \Omega_n^* \),

\[ |z + m_n(z)| \geq \text{Im}(z + m_n(z)) \geq 2 \geq \frac{1}{2}. \]  \hspace{2cm} (3.66)

By Lemma 3.5, we obtain inequality (3.65) for \( v = 2 \). By Lemma 3.8, this inequality holds for any \( v \) with \( v_0 \leq v \leq 2 \) as well. Thus Proposition 3.1 is proved.

\[ \square \]

**4 Proof of Theorem 1.1**

To conclude the proof of Theorem 1.1 we modify the bound of the Kolmogorov distance of the spectral distribution functions via Stieltjes transforms obtained in [7] Lemma 2.1. Given \( \varepsilon > 0 \) introduce the interval \( \mathbb{I}_\varepsilon = [-2 + \varepsilon, 2 - \varepsilon] \) and \( \mathbb{I}'_\varepsilon = [-2 + \frac{1}{2}\varepsilon, 2 - \frac{1}{2}\varepsilon] \). For any \( x \in \mathbb{I}_\varepsilon \) define \( \gamma = \gamma(x) := 2 - |x| \). For any distribution function \( F \) denote by \( S_F(z) \) its Stieltjes transform,

\[ S_F(z) = \int_{-\infty}^{\infty} \frac{1}{x - z} dF(x). \]

**Proposition 4.1.** Let \( v > 0 \) and \( a \) and \( \varepsilon > 0 \) be positive numbers such that

\[ \alpha = \frac{1}{\pi} \int_{|u| \leq a} \frac{1}{u^2 + 1} du = \frac{3}{4}, \]  \hspace{2cm} (4.1)

and

\[ 2va \leq \varepsilon \sqrt{\gamma}. \]  \hspace{2cm} (4.2)

If \( G \) denotes the distribution function of the standard semi-circular law, and \( F \) is any distribution function, there exists some absolute constants \( C_1, C_2 \) and \( C_3 \) such that

\[ \Delta(F, G) := \sup_x |F(x) - G(x)| \]

\[ \leq \sup_{x \in \mathbb{I}'_\varepsilon} \left| \text{Im} \int_{-\infty}^{\infty} (S_F(u + i \frac{v}{\sqrt{\gamma}}) - S_G(u + i \frac{v}{\sqrt{\gamma}})) du \right| + C_2v + C_3\varepsilon^{\frac{3}{2}}. \]  \hspace{2cm} (4.3)
Proof. The proof of Proposition 4.1 is straightforward adaptation of the proof of Lemma 2.1 from [7]. We include it here for the sake of completeness. Note that

$$\sup_x |F(x) - G(x)| \leq \sup_{x \in J_e^\varepsilon} |F(x) - G(x)| + G(-2 + \varepsilon),$$

(4.4)

and

$$G(-2 + \varepsilon) \leq C\varepsilon^{\frac{3}{2}}.$$  

(4.5)

Let \( x \in J_e^\varepsilon \). Then according to condition (4.2) \( x + \frac{v_0}{\sqrt{\gamma}} \in J_e^\varepsilon \). Denote by \( v' = \frac{v}{\sqrt{\gamma}} \). For any \( x \in J_e^\varepsilon \)

$$\left| \frac{1}{\pi} \text{Im} \left( \int_{-\infty}^{x} (S_F(u + iv') - S_G(u + iv'))du \right) \right|$$

$$\geq \frac{1}{\pi} \text{Im} \left( \int_{-\infty}^{x} (S_F(u + iv') - S_G(u + iv'))du \right)$$

$$= \frac{1}{\pi} \left[ \int_{-\infty}^{\infty} \frac{v'd(F(y) - G(y))}{(y - u)^2 + v'^2} \right] du$$

$$= \frac{1}{\pi} \int_{-\infty}^{x} \left[ \int_{-\infty}^{\infty} \frac{2v'(y - u)(F(y) - G(y))dy}{((y - u)^2 + v'^2)^2} \right]$$

$$= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{(F(y) - G(y)) \left[ \int_{-\infty}^{x} \frac{2v'(y - u)du}{((y - u)^2 + v'^2)^2} \right]}{y^2 + 1}. \quad \text{(4.6)}$$

Since \( F \) is non decreasing, we have

$$\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{(F(x - v'y) - G(x - v'y))dy}{y^2 + 1}$$

$$\geq \alpha(F(x - v'a) - G(x - v'a)) - \frac{1}{\pi} \int_{|y| \leq a} |G(x - v'y) - G(x - v'a)|dy$$

$$\geq \alpha(F(x - v'a) - G(x - v'a)) - \frac{1}{v'\pi} \int_{|y| \leq v'a} |G(x - y) - G(x - v'a)|dy. \quad \text{(4.7)}$$

Denote by \( \Delta_e(F, G) = \sup_{x \in J_e^\varepsilon} |F(x) - G(x)| \). Let \( x_n \in J_e^\varepsilon \) such that \( F(x_n) - G(x_n) \to \Delta_e(F, G) \). Then \( x_n = x_n + v'a \in J_e^\varepsilon \). We have

$$\sup_{x \in J_e^\varepsilon} \left| \text{Im} \left( \int_{-\infty}^{x} (S_F(u + iv') - S_G(u + iv'))du \right) \right| \geq \alpha(F(x_n) - G(x_n))$$

$$- \frac{1}{\pi v} \sup_{x \in J_e^\varepsilon} \frac{1}{\sqrt{\gamma}} \int_{|y| < 2v'a} |G(x + y) - G(x)|dy - (1 - \alpha)\Delta_e(F, G). \quad \text{(4.8)}$$
The rate of convergence to the semi-circular law

Note that
\[ \frac{1}{\pi v} \sup_{x \in J_\varepsilon} \frac{1}{\sqrt{\gamma}} \int_{|y| < 2v} |G(x + y) - G(x)| dy \]
\[ \leq \frac{1}{\pi v} \sup_{x \in J_\varepsilon} \frac{1}{\sqrt{\gamma}} \sqrt{4 - x^2} \leq C_v. \] (4.9)

Inequalities (4.4), (4.8) and (4.9) together imply
\[ \sup_{x \in J_\varepsilon} \left| \int_{-\infty}^{x} (S_F(u + iv') - S_G(u + iv')) du \right| \geq (2\alpha - 1)\Delta_\varepsilon(F, G) - C_v - C\varepsilon^2. \] (4.10)

Similar arguments may be used for the sequence \( x_n \in J_\varepsilon \) such \( F(x_n) - G(x_n) \to -\Delta_\varepsilon(F, G) \). This completes the proof.

**Corollary 4.1.** Under the conditions of Proposition 4.1, for any \( V > v \), the following inequality holds
\[ \sup_{x \in J_\varepsilon} \left| \int_{-\infty}^{x} (S_F(u + iv') - S_G(u + iv')) du \right| \leq \int_{-\infty}^{L} |S_F(u + iV) - S_G(u + iV)| du 
+ \sup_{x \in J_\varepsilon} \left| \int_{\varepsilon}^{V} (S_F(x + iu) - S_G(x + iu)) du \right|. \] (4.11)

**Proof.** Put \( z = u + iv' \). \( v' \leq 2 \). Since the functions of \( S_F(z) \) and \( S_G(z) \) are analytic in the upper half-plane, it is enough to use Cauchy’s theorem. We can write
\[ \int_{-\infty}^{\infty} \text{Im}(S_F(z) - S_G(z)) du = \lim_{L \to \infty} \int_{-\infty}^{x} (S_F(u + iv') - S_G(u + iv')) du. \] (4.12)

Since \( v' = \frac{v}{\sqrt{\gamma}} \leq \frac{\varepsilon}{2a} \), we may assume without loss of generality that \( v' \leq 2 \). By Cauchy’s integral formula, we have
\[ \int_{-\infty}^{L} (S_F(z) - S_G(z)) du = \int_{-\infty}^{L} (S_F(u + iV) - S_G(u + iV)) du 
+ \int_{\varepsilon}^{V} (S_F(-L + iu) - S_G(-L + iu)) du 
- \int_{\varepsilon}^{V} (S_F(x + iu) - S_G(x + iu)) du. \] (4.13)

Denote by \( \xi(\eta) \) a random variable with distribution function \( F(x)(G(x)) \). Then we have
\[ |S_F(-L + iv')| = \left| \mathbb{E}_{\xi \sim L - iv'} \frac{1}{\xi + L - iv'} \right| \leq v'^{-1} \Pr\{|\xi| > L/2\} + \frac{2}{L}. \] (4.14)
The rate of convergence to the semi-circular law

Similarly,

$$|S_G(-L + iv^')| \leq v'^{-1} \Pr\{|\eta| > L/2\} + \frac{2}{L}. \quad (4.15)$$

These inequalities imply that

$$\left| \int_{v'}^V (S_F(-L + iu) - S_G(-L + iu))du \right| \to 0 \quad \text{as} \quad L \to \infty, \quad (4.16)$$

which completes the proof.

Combining the results of Proposition 4.1 and Corollary 4.1, we get

**Corollary 4.2.** Under the conditions of Proposition 4.1 the following inequality holds

$$\Delta(F, G) \leq 7C_1 \int_{-\infty}^{\infty} |S_F(u + iV) - S_G(u + iV)|du + C_2v + C_3 \varepsilon^{2} \quad (4.17)$$

where $v' = \frac{v}{\sqrt{\gamma}}$ with $\gamma = 2 - |x|$.

We shall now apply the result of Corollary 4.2 to the empirical spectral distribution function $F_n(x)$ of the random matrix $X$. At first we bound the integral over the line $V = 2$. Note that in this case we have $|z + m_n(z)| \geq 1$. Moreover, $\text{Im} m_n^{(j)}(z) \leq \frac{1}{V} \leq \frac{1}{2}$.

We may now apply the results of the previous Lemmas regarding large deviation probabilities. This implies the following bound for $g_n(z)$ for all $z = u + iV$ with $u \in \mathbb{R}$.

$$|g_n(z)| \leq \frac{\beta_n}{n \sqrt{V}|z + m_n(z)||z + s(z) + m_n(z)|} \left(1 + \frac{\beta_n \text{Im} m_n(z)}{\sqrt{V}} + \frac{\beta_n}{nV \sqrt{V}} \right)$$

$$+ \frac{C \text{Im} m_n(z)}{nV|z + s(z) + m_n(z)|}. \quad (4.18)$$

Note that for $V = 2$

$$|z + m_n(z)||z + m_n(z) + s(z)| \geq \begin{cases} 4 & \text{for} \quad |u| \leq 2, \\ \frac{1}{4}|z|^2 & \text{for} |u| > 2. \end{cases} \quad (4.19)$$

We may rewrite the bound (4.18) as follows

$$|g_n(z)| \leq \frac{C \beta_n^2}{n(|z|^2 + 1)} + \frac{C \text{Im} m_n(z)}{nV}. \quad (4.20)$$

Note that for any distribution function $F(x)$ we have

$$\int_{-\infty}^{\infty} \text{Im} s_F(u + iv)du \leq \pi \quad (4.21)$$
From here it follows that, for $V = 2$

$$\int_{|u| \geq n} |m_n(z) - s(z)| du \leq \frac{C}{n} \quad (4.22)$$

Denote $\mathcal{D}_n := \{z = u + 2i : |u| \leq n\}$ and

$$\Omega_n := \{\cap_{z \in \mathcal{D}_n} \{\omega \in \Omega : |g_n(z)| \leq \frac{C\beta^2_n}{n(|z|^2 + 1)}\} \cap \Omega_n^\ast\}$$

Using a union bound, we may show that

$$\Pr\{\Omega_n\} \geq 1 - C \exp\{-c\ell_n,\alpha\}. \quad (4.23)$$

It is straightforward to check that for $\omega \in \Omega_n$

$$\int_{-\infty}^{\infty} |m_n(z) - s(z)| du \leq \frac{C}{n} \quad (4.24)$$

We put $\varepsilon = n^{-\frac{2}{3}}$ and $v_0 = \frac{\beta^2_n}{n}$. To conclude the proof we need to consider the "vertical" integrals for $z = x + iv'$ with $x \in \mathbb{J}'$, $v' = \frac{\beta_n}{\sqrt{\gamma}}$ and $\gamma = 2 - |x|$. It is enough to consider one of these integrals only. For example

$$\int_{v'}^{2v_0} \frac{1}{n^2v^2\sqrt{\gamma} + v} dv \leq \frac{1}{n^2v_0^2\sqrt{\gamma}} \leq \frac{1}{n^2v_0} \leq \frac{\beta^2_n}{n}. \quad (4.25)$$

Finally, we get for any $\omega \in \Omega_n$

$$\Delta(F_n, G) = \sup_{x} |F_n(x) - G(x)| \leq \frac{\beta^2_n}{n}. \quad (4.26)$$

Thus Theorem 1.1 is proved.

5 Proof of Theorem 1.2

We may express the diagonal entries of the resolvent matrix $R$ as follows

$$R_{jj} = \sum_{k=1}^{n} \frac{1}{\lambda_k - z}|u_{jk}|^2. \quad (5.1)$$

Consider the distribution function, say $F_{nj}(x)$, of the probability distribution of the eigenvalues $\lambda_k$

$$F_{nj}(x) = \sum_{k=1}^{n} |u_{jk}|^2 I\{\lambda_k \leq x\}. \quad (5.2)$$
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Then we have
\[ R_{jj} = R_{jj}(z) = \int_{-\infty}^{\infty} \frac{1}{x-z} dF_{nj}(x). \]  

which means that \( R_{jj} \) is the Stieltjes transform of the distribution \( F_{nj}(x) \). Note that, for any \( \lambda > 0 \)
\[ \max_{1 \leq k \leq n} |u_{jk}|^2 \leq \sup_x (F_{nj}(x + \lambda) - F_{nj}(x)) =: Q_{nj}(\lambda). \]  

On the other hand, it is easy to check that
\[ Q_{nj}(\lambda) \leq 2 \sup_u \lambda \text{Im} R_{jj}(u + i\lambda). \]  

By relations (2.23) and (2.26), we obtain for any \( v \geq v_0 \) with \( v_0 = \frac{c_0}{n} \) with a sufficiently small constant \( c \),
\[ \Pr\{ \left| \frac{\varepsilon_j}{|z + m_n(z)|} \right| \leq \frac{1}{2} \} \leq \exp\{-cl_{n,\alpha}\}. \]  

Furthermore, the representation (2.2) and inequality (5.6) together imply, for \( v \geq v_0 \),
\[ \text{Im} R_{jj} \leq |R_{jj}| \leq C_1 \]  

with some positive constant \( C_1 > 0 \) depending on \( \kappa, \alpha \). This implies that
\[ \Pr\{ \max_{1 \leq k \leq n} |u_{jk}|^2 \leq \frac{\beta_n}{n} \} \leq C\exp\{-cl_{n,\alpha}\}. \]  

By a union bound we arrive at the inequality (1.6). To prove inequality (1.7), we consider the quantity
\[ r_j := R_{jj} - s(z). \]  

Using equalities (2.2) and (3.39), we get
\[ r_j = -s(z)g_n(z) + \frac{\varepsilon_j}{z + m_n(z)} R_{jj}. \]  

By inequalities (3.65) and (2.26), we have
\[ |r_j| \leq \frac{c\beta_n}{\sqrt{nv}}. \]  

From here it follows that
\[ \sup_{x \in \mathbb{R}} \int_{-\infty}^{\infty} |r_j(x + iv)| dv \leq \frac{C}{\sqrt{n}}. \]  

Similar to (1.24) we get
\[ \int_{-\infty}^{\infty} |r_j(x + iV)| dx \leq \frac{C}{\sqrt{n}}. \]
Applying Corollary 4.2, we get
\[
\Pr\left\{ \sup_x |F_{nj}(x) - G(x)| \leq \frac{\beta_n}{\sqrt{n}} \right\} \geq 1 - C \exp\{-cl_{n,\alpha}\}. \tag{5.14}
\]
Using now that
\[
\Pr\left\{ \sup_x |F_n(x) - G(x)| \leq \frac{\beta_n^2}{n} \right\} \geq 1 - C \exp\{-cl_{n,\alpha}\}, \tag{5.15}
\]
we get
\[
\Pr\left\{ \sup_x |F_{nj}(x) - G(x)| \leq \frac{\beta_n}{\sqrt{n}} \right\} \geq 1 - C \exp\{-cl_{n,\alpha}\}. \tag{5.16}
\]
Thus, Theorem 1.2 is proved.

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