Index Defects in the Theory of Non-local Boundary Value Problems and the $\eta$-Invariant

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Abstract

The paper deals with elliptic theory of boundary value problems on manifolds whose boundary is represented as a covering space. We compute the index for a class of non-local boundary value problems on such manifolds. For a non-trivial covering, the index defect of the Atiyah–Patodi–Singer boundary value problem is computed. Poincaré duality in $K$-theory of the corresponding manifolds with singularities is obtained.

Introduction

This paper deals with boundary value problems for elliptic operators on a manifold whose boundary is the total space of a finite-sheeted covering. On such manifolds, we consider boundary value problems for operators that do not satisfy the Atiyah–Bott condition (i.e., have no well-posed classical boundary value problems). Recall that this condition does not hold, in particular, for the Hirzebruch and Dirac operators as well as some other related geometric operators.

We consider the following two classes of boundary value problems.

1. Non-local boundary value problems. Let $M$ be a smooth manifold such that the boundary $\partial M$ is a finite-sheeted covering with projection $\pi : \partial M \to X$. Then there is an isomorphism

$$C^\infty (\partial M) \overset{\beta}{\cong} C^\infty (X, \pi_1)$$

between the space $C^\infty (\partial M)$ of smooth functions on $\partial M$ and the space of sections of the vector bundle $\pi_1 \in \text{Vect} (X)$ on the base of the covering. Here $\pi_1$ is the direct image of the trivial line bundle.

For a scalar elliptic operator $D$ on $M$, the simplest non-local boundary value problem of the type considered in this paper is

$$\begin{cases} Du = f, \\ B\beta u|_{\partial M} = g. \end{cases}$$

(0.1)
Here $u$ and $f$ are functions on $M$, $g$ is a function on $X$, and the operator $B$ of boundary conditions acts also on $X$. In terms of the original manifold $M$, the boundary conditions in (0.1) are non-local, since they relate the values of functions at distinct points of $M$.

We prove a finiteness theorem and in the case of regular coverings obtain an index formula for this class of non-local boundary value problems. Without going into detail at the moment, let us mention two essential features of the theory.

First, in the proof of the index theorem we embed our manifolds in the classifying space of a finite group, while in the classical index theorem it is suffices to use embeddings in $\mathbb{R}^N$.

Second, the analogue of the Atiyah–Singer difference element for a non-local boundary value problem is an element of the $K$-group of a non-commutative $C^*$-algebra associated with the cotangent bundle and the covering. Recall that in the classical index theorem it suffices to use topological $K$-theory.

The index formula of this paper is given in a form resembling the $K$-theoretic statement of the Atiyah–Singer theorem. Local index formulae will appear elsewhere.

2. Spectral problems on manifolds with a covering. The first generalization of classical boundary value problems that is free of the Atiyah–Bott obstruction is due to Atiyah, Patodi, and Singer [1]. For a class of first-order elliptic operators, one has so-called spectral boundary value problems denoted by $(D, \Pi)$. Spectral boundary value problems enjoy the Fredholm property. However, their index is not determined by the principal symbol of $D$.

Interesting invariants arise if the boundary has the structure of a covering. Here we consider a class of elliptic operators that are lifted from the base of the covering in a neighbourhood of the boundary. In this case, the principal symbol of an elliptic operator $D$ defines an element

$$[\sigma(D)] \in K^0(\overline{T^*M^\pi})$$

in the $K$-group of the singular space $\overline{T^*M^\pi}$ obtained from the cotangent bundle $T^*M$ if we identify all points in each fiber of the covering (for details, see Section 5). The element $[\sigma(D)]$ has a topological index

$$\text{ind}_t[\sigma(D)] \in \mathbb{Q}/n\mathbb{Z},$$

where $n$ is the number of sheets. However, the analytical and the topological index coincide only for trivial coverings. For a general covering, we obtain the index defect formula

$$\text{mod } n\text{-ind } (D, \Pi) - \text{ind}_t[\sigma(D)] = \eta(D) \big|_X \otimes 1_{n-\pi_1} \in \mathbb{Q}/n\mathbb{Z}. \quad (0.2)$$

The index defect (the difference between the analytical index modulo $n$ and the topological index) is equal to the relative Atiyah–Patodi–Singer $\eta$-invariant of the restriction of $D$ to the boundary with coefficients in the flat bundle $\pi_1$. For a trivial covering, the relative $\eta$-invariant is zero, and the index defect formula becomes the index formula

$$\text{mod } n\text{-ind } (D, \Pi) = \text{ind}_t[\sigma(D)]$$

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due to Melrose and Freed [2] (see also [3, 4, 5, 6]). However, our proof is new even in this case. It is interesting to note that the main step in the proof is to realize the fractional analytic invariant

\[ \text{mod } n \text{-ind } (D, \Pi_+) - \eta(D|_X \otimes 1_{n-1}) \in \mathbb{Q}/n\mathbb{Z} \quad (0.3) \]

as the index of some non-local boundary problem of the form (0.1) (in a suitable elliptic theory with coefficients).

There is also a deeper relation between the two elliptic theories described in Subsec. 1 and 2.

3. Poincaré isomorphism and duality. We establish Poincaré isomorphisms on the singular spaces \( T^*M^\pi \) and \( M^\pi \). For the identity covering \( \pi = \text{Id}, X = \partial M \), these isomorphisms are just the well-known isomorphisms (e.g., see [7, 8, 9])

\[ K^0(T^*M) \simeq K_0(M, \partial M), \quad K^0(T^*(M \setminus \partial M)) \simeq K_0(M). \quad (0.4) \]

(For non-compact spaces, we use \( K \)-theory with compact supports.) In contrast to the smooth case, the Poincaré isomorphisms for singular spaces relate the \( K \)-groups of a commutative algebra of functions to those of a dual non-commutative algebra. They are defined on the elements as quantizations, i.e., take symbols to operators. More precisely, the analogue of the first isomorphism in (0.4) is defined in terms of the operators described in Subsec. 2, while in the second case one uses non-local problems introduced in Subsec. 1.

Let us outline the contents of the paper. The first section contains the definition of the class of non-local boundary value problems on manifolds with a covering on the boundary and a proof of the Fredholm property. The index formula is obtained in Sec. 2. By way of example, we define a non-local boundary value problem for the Hirzebruch operator on a manifold with reflecting boundary. In Sec. 3 we give the homotopy classification of non-local problems. The index defect formula (0.2) is proved in Sec. 4. This is one of the central results of the paper. Section 5 contains applications to the computation of the fractional part of the \( \eta \)-invariant. It is also shown that the invariant (0.3) can be computed by the Lefschetz formula. Poincaré isomorphisms in \( K \)-theory of the singular spaces corresponding to manifolds whose boundary bears the structure of a covering are constructed in the last two sections.

There are other interesting classes of non-local boundary value problems arising if the projection has singularities (e.g., the projection on the quotient by a non-free action of a finite group). Index theory of such boundary value problems is apparently related to index theory on orbifolds ([10, 11]). Our approach is advantageous in that if the base of the covering is smooth, then there are no additional analytic and topological difficulties related to the singularities of the covering. More general classes of non-local boundary value problems (e.g., see [12]) are beyond the scope of this paper.

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1 Non-local boundary value problems

1. Coverings and non-local operators. Let $Y$ be a finite covering over a manifold $X$ with projection $\pi : Y \rightarrow X$. The projection defines the direct image mapping

$$\pi_! : \text{Vect}(Y) \rightarrow \text{Vect}(X)$$

that takes each vector bundle $E \in \text{Vect}(Y)$ to the bundle

$$\pi_! E \in \text{Vect}(X), \quad (\pi_! E)_x = C^\infty(\pi^{-1}(x), E), \quad x \in X.$$

This clearly gives an isomorphism $\beta_E : C^\infty(Y, E) \xrightarrow{\sim} C^\infty(X, \pi_! E)$ of section spaces on $Y$ and $X$, while permits one to identify operators defined on the total space and on the base. More precisely, the direct image

$$\pi_! D = \beta_E D \beta_E^{-1} : C^\infty(X, \pi_! E) \rightarrow C^\infty(X, \pi_! E)$$

of a differential operator

$$D : C^\infty(Y, E) \rightarrow C^\infty(Y, E)$$

on $Y$ is a differential operator. However, the following example shows that the inverse image

$$\pi_! D' = \beta_E^{-1} D' \beta_E : C^\infty(Y, E) \rightarrow C^\infty(Y, E) \quad (1.1)$$

of a differential operator $D'$ on $X$ may well be a non-local operator. (It is not even pseudolocal.)

Example 1.1. For the trivial covering

$$Y = \underbrace{X \sqcup X \sqcup \ldots \sqcup X}_{n \text{ copies}} \rightarrow X$$

and the trivial bundle $E = \mathbb{C}$, we have $\pi_! E = \mathbb{C}^n$. The direct image

$$\pi_! D = \text{diag}(D|_{X_1}, \ldots, D|_{X_n}) : C^\infty(X, \mathbb{C}^n) \rightarrow C^\infty(X, \mathbb{C}^n)$$

of a differential operator on $Y$ is always a diagonal operator, and hence the inverse image of a non-diagonal operator can not be a differential operator. The off-diagonal entries produce non-local operators on $Y$, since they interchange the values of functions on different leaves of the covering.
2. Non-local boundary value problems. Let $M$ be a smooth compact manifold with boundary $\partial M$. Suppose that the boundary is a covering space over a smooth closed manifold $X$ with projection

$$\pi : \partial M \longrightarrow X.$$ 

We fix a collar neighbourhood $\partial M \times [0, 1)$ of the boundary. The normal coordinate will be denoted by $t$.

For a smooth function $u \in \mathcal{C}^\infty (M)$, let

$$j_{\partial M}^{m-1} u = \left( u|_{\partial M}, -i \frac{\partial}{\partial t} u|_{\partial M}, \ldots, (-i \frac{\partial}{\partial t})^{m-1} u|_{\partial M} \right)$$

be the restriction of its $(m-1)$st jet in the normal direction to the boundary. The operator $j_{\partial M}^{m-1}$ is continuous in the Sobolev spaces

$$j_{\partial M}^{m-1} : H^s (M) \longrightarrow \bigoplus_{k=0}^{m-1} H^{s-1/2-k} (\partial M), \quad s > m - 1/2.$$ 

Throughout the paper we assume that for vector bundles $E$ on manifolds with boundary there are given isomorphisms $p^* (E|_{\partial M}) \simeq E|_{\partial M \times [0, 1]}$ in the collar neighbourhood of the boundary, where $p : \partial M \times [0, 1] \rightarrow \partial M$ is the natural projection. In this case, the normal jet of a section of $E$ is also well defined.

**Definition 1.1.** A non-local boundary value problem for a differential operator

$$D : \mathcal{C}^\infty (M, E) \longrightarrow \mathcal{C}^\infty (M, F)$$

of order $m$ is a system of equations

$$\begin{cases} 
Du = f, & u \in H^s (M, E), f \in H^{s-m} (M, F), \\
B\beta_E j_{\partial M}^{m-1} u = g, & g \in H^\delta (X, G), 
\end{cases} \quad (1.2)$$

where the boundary condition is defined by a pseudodifferential operator

$$B : \bigoplus_{k=0}^{m-1} H^{s-1/2-k} (X, \pi_1 E|_{\partial M}) \longrightarrow H^\delta (X, G)$$

on $X$. We assume that the component

$$B_k : H^{s-1/2-k} (X, \pi_1 E|_{\partial M}) \longrightarrow H^\delta (X, G)$$

of $B$ has the order $s - 1/2 - k - \delta$.

**Remark 1.1.** One can also consider problems similar to (1.2) in which the components of the vector function $g$ belong to Sobolev spaces of different orders. The case in which all components have the same order $\delta$ is more convenient and can always be achieved by order reduction.
3. Relation to classical boundary value problems. Finiteness theorem. Note that $\pi \times 1 : \partial M \times \[0, 1\] \rightarrow X \times \[0, 1\]$. The induced isomorphism of function spaces will be denoted by 

$$\beta'_{E} : C^\infty (\partial M \times \[0, 1\], E) \rightarrow C^\infty (X \times \[0, 1\], \pi_{1}E).$$

The non-local problem $(D, B)$ can be represented in a neighbourhood of the boundary as the inverse image of the classical boundary value problem

$$\left( \begin{array}{cc} \beta'_{F} & 0 \\ 0 & 1 \end{array} \right) \circ \left( \begin{array}{c} D \\ B\beta_{E}\beta_{M}^{-1} \end{array} \right) \circ (\beta_{E}')^{-1} = \left( \begin{array}{c} \beta'_{F} D (\beta'_{E})^{-1} \\ B\beta_{X}^{-1} \end{array} \right).$$

More specifically, this is the boundary value problem

$$\left( \begin{array}{c} \beta'_{F} D (\beta'_{E})^{-1} \\ B\beta_{X}^{-1} \end{array} \right) : C^\infty (X \times \[0, 1\], \pi_{1}E) \rightarrow C^\infty (X \times \[0, 1\], \pi_{1}F) \oplus C^\infty (X, G),$$

for the differential operator $(\pi \times 1), D = \beta'_{F} D (\beta'_{E})^{-1}$ on the cylinder $X \times \[0, 1\]$. In local coordinates, this operator is represented by a diagonal matrix with elements acting on different leaves of the covering. Problem $(\pi \times 1)$ will be denoted by $(\pi \times 1), D, B$ for short.

We point out that the classical boundary value problem is defined only in a neighbourhood of the boundary, since the covering is defined only near the boundary.

**Definition 1.2.** Problem $(D, B)$ is said to be **elliptic** if $D$ is elliptic and $(\pi \times 1), D, B$ is elliptic, i.e., satisfies the Shapiro–Lopatinski boundary condition (e.g., see [13]).

The proof of the following finiteness theorem is standard.

**Theorem 1.1.** An elliptic boundary value problem $D = (D, B)$ defines a Fredholm operator.

**Proof.** Let $D^{-1}$ be the parametrix of $D$ in the interior of the manifold. Similarly, the parametrix of the classical boundary value problem on $X \times \[0, 1\]$ will be denoted by $(L, K)$. They can be pasted together globally on $M$ by the formula

$$D^{-1} = \left( \psi_{1} D^{-1} \varphi_{1} + \psi_{2} (\pi \times 1)^{1} L\varphi_{2}, \psi_{2} K \right).$$

Here

$$\varphi_{1} + \varphi_{2} = 1, \quad \psi_{j} \varphi_{j} = \varphi_{j} \quad \psi_{1} = 0 \text{ near the boundary}, \quad \psi_{2} = 0 \text{ far from the boundary.} \quad (1.4)$$

Furthermore, $\psi_{2}$ is assumed to be constant in the fiber of $\pi \times 1$. Obviously, $D^{-1}$ is a two-sided parametrix of $D$. The proof is complete. \qed
2 The index of non-local problems

In the previous section, non-local boundary value problems were represented near the boundary in terms of equivalent classical boundary value problems. Therefore, we can apply well-known topological methods (e.g., see [14]) to compute the index of non-local boundary value problems.

1. Reduction to zero-order operators. We introduce a class of operators that are non-local in a neighbourhood of the boundary. A linear operator

\[ D : C^\infty (M, E) \longrightarrow C^\infty (M, F) \]

will be called an admissible operator of order \( m \) if it can be represented modulo operators with smooth kernels as

\[ D = \psi_1 D' \varphi_1 + \psi_2 (\pi \times 1) ! D'' \varphi_2 \] (2.1)

for cutoff functions \( \varphi_1, \varphi_2, \psi_1, \psi_2 \) as in the proof of Theorem I.1, a pseudodifferential operator \( D' : C^\infty (M, E) \rightarrow C^\infty (M, F) \), and an operator

\[ D'' : C^\infty (X \times [0, 1], \pi E) \longrightarrow C^\infty (X \times [0, 1], \pi F) \]

that is a sum of a pseudodifferential operator with compactly supported kernel on \( X \times (0, 1) \) and a differential operator

\[ \sum_{k=0}^{m} D_k (t) \left(-i \frac{\partial}{\partial t}\right)^{m-k} \] (2.2)

with respect to the normal variable \( t \).

Here the \( D_k (t) \) are smooth families of pseudodifferential operators on \( X \) of order \( k \) and \( D_0 (t) \) is induced by a vector bundle isomorphism.

To this class of operators, one can extend the notion of ellipticity, the statement of non-local boundary value problems, and the finiteness theorem (cf. a similar generalization in [13] for the classical case). In particular, the symbol of an admissible operator is a pair \((\sigma_M, \sigma_X)\), where \( \sigma_M : p^* E \rightarrow p^* F \) is defined over \( M \setminus (\partial M \times [0, \varepsilon]) \) \((p : S^* M \rightarrow M \) is the natural projection) and \( \sigma_X : p_0^*(\pi E) \rightarrow p_0^*(\pi F) \) \((p_0 : S^*(X \times [0, 1]) \rightarrow X \times [0, 1]) \) is defined over \( X \times [0, 1] \). Moreover, the symbols are smooth and satisfy the compatibility condition

\[ (\pi_0)! \sigma_M|_{\partial M \times (\varepsilon, 1)} = \sigma_X, \]

where the direct image is induced by the natural projection \( \pi_0 : T^* M|_{\partial M \times (\varepsilon, 1)} \rightarrow T^*(X \times (\varepsilon, 1)) \).

Example 2.1. Let \( E \in \text{Vect} (M) \) be a vector bundle. Suppose that its direct image over \( U_{\partial M} \) is decomposed as a sum of two subbundles

\[ \pi_! E|_{U_{\partial M}} = E_+ \oplus E_-, \quad E_\pm \in \text{Vect} (X \times [0, \varepsilon)). \]
Consider the operator $D_{\pm} : C^\infty(M, E) \to C^\infty(M, E)$ given by
\[
D_{\pm} = \psi_2(\pi \times 1)^! \left[ \left( -i\frac{\partial}{\partial t} + i\Lambda_{X, E_+} \right) \oplus \left( i\frac{\partial}{\partial t} + i\Lambda_{X, E_-} \right) \right] \varphi_2 + \psi_1 i\Lambda_M \varphi_1.
\] (2.3)
(Here $\Lambda$ stands for first-order pseudodifferential operators with principal symbol $|\xi|$ on the corresponding manifolds, and the cutoff functions are chosen as before.) This formula defines an admissible elliptic operator. We equip it with the Dirichlet boundary condition
\[
P_{E_-} \beta_E u|_{\partial M} = g \in C^\infty(X, E_-),
\]
where $P_{E_-} : \pi_! E|_{\partial M} \to \pi_! E|_{\partial M}$ is a projection onto the subbundle $E_-$. Denote this boundary value problem by $\mathcal{D}_{\pm}$. By analogy with the classical case (e.g., see \cite{13}), one proves that the index of this boundary value problem is zero.

For example, let $E_+ = \pi_! E|_{\partial M}$ and $E_- = 0$. Then the operator (2.3), which will be denoted by $D_+$, is Fredholm without any boundary condition.

**Remark 2.1.** Just as in the classical elliptic theory on a closed manifold (see \cite{15}), there are two equivalent definitions of homotopy of non-local elliptic problems. First, one can say that two problems are homotopic if they can be connected by a family of non-local elliptic problems continuous in the operator norm (in some given pair of Sobolev spaces). Second, two problems are said to be homotopic if there exists a continuous homotopy of their principal symbols (preserving ellipticity). The equivalence of the two definitions is based on the smoothing of continuous homotopies and the standard norm estimates modulo compact operators, e.g., see \cite{16}.

Let $^1$ $\text{Ell}^m (M, \pi), \; m \geq 1$, be the Grothendieck group of the semigroup of homotopy classes of elliptic boundary value problems for admissible operators of order $m$ modulo boundary value problems of the form $\mathcal{D}_{\pm} \circ D_m^{m-1}$.

The group of stable homotopy classes of zero-order admissible elliptic operators is denoted by $\text{Ell}^0 (M, \pi)$. Recall that stabilization is taken modulo trivial operators. In this case, by trivial operators we mean operators induced by vector bundle isomorphisms. It should be noted that elliptic operators of order zero do not require boundary conditions, since near the boundary they are induced by vector bundle isomorphisms.

Just as in the classical theory (see \cite{14} or \cite{13}), the order of a non-local boundary value problem can be reduced to zero by stable homotopies. More precisely, the following theorem holds.

**Theorem 2.1 (order reduction).** The composition with the operator $D_+$ (with coefficients in vector bundles) induces an isomorphism
\[
\times D_+^m : \text{Ell}^0 (M, \pi) \longrightarrow \text{Ell}^m (M, \pi),
\]
\[
[D] \quad \mapsto \quad [D \circ D_+^m].
\]

$^1$Later on, by $\text{Ell}^m (M, \pi)$ we denote also the corresponding Grothendieck groups for closed manifolds and for manifolds with boundary with projection $\pi$ defined, possibly, on an open subset. Which group is meant is always clear from the context.
The proof of this result is a straightforward generalization of the corresponding proof in the classical case (see [14]) and hence is omitted. □

Remark 2.2. Let us explicitly describe order reduction, i.e., the mapping \((\times D^m_+)^{-1}\), in the important special case of boundary value problems

\[
\begin{align*}
&Du = f, \\
&P\beta_E(u|_{\partial M}) = g, \quad g \in C^\infty(X, \text{Im} P),
\end{align*}
\]

for a first-order admissible operator \(D : C^\infty(M, E) \to C^\infty(M, F)\) that admits a decomposition

\[
(\pi \times 1)_! (D|_{\partial M}) = \Gamma \left( \frac{\partial}{\partial t} + A(t) \right)
\]

in a neighbourhood of the boundary, where \(A(t)\) is a smooth operator family on \(X\) and \(\Gamma : \pi^! E|_{\partial M} \to \pi^! F|_{\partial M}\) is a vector bundle isomorphism. The boundary condition is defined by the projection \(P\) in the bundle \(\pi_!(E|_{\partial M})\). We assume for simplicity that the symbol \(a(x, \xi)\) of \(A(0)\) is symmetric and additionally satisfies \(a^* a = |\xi|^2\). We assume that \(P\) is also symmetric.

Let \(L_+ (A(0)) \in \text{Vect}(S^*X)\) be the Calderón bundle. For our first-order operator, this is the bundle over \(S^*X\) generated by eigenvectors of \(a(x, \xi)\) with positive eigenvalues.

The ellipticity condition for \((D, P)\) requires that \(P\) define an isomorphism

\[
L_+ (A(0)) \xrightarrow{P} p_0^* \text{Im} P, \quad p_0 : S^*X \to X,
\]

of subbundles. Consider the principal symbol of our operator on the boundary:

\[
\sigma \left( \frac{\partial}{\partial t} + A(0) \right) = i\tau + a(x, \xi)
\]

(here \(\tau\) is dual to \(t\)). The linear homotopy

\[
(1 - \varepsilon) (i\tau + a(x, \xi)) + \varepsilon (2P(x) - 1), \quad \varepsilon \in [0, 1],
\]

is a homotopy of elliptic symbols for \(\tau^2 + |\xi|^2 = 1\) provided that the ellipticity condition for \((D, P)\) is satisfied. Furthermore, at the end of the homotopy (for \(\varepsilon = 1\)) the symbol does not depend on the cotangent variables. Let us treat the homotopy of elliptic symbols on \(X\) as an elliptic symbol on \(X \times [0, 1]\). Then the symbol of \(D\) and the homotopy taken together define the symbol of an admissible elliptic operator of order zero on the manifold \(M\) with \([0, 1] \times \partial M\) attached.

This zero-order symbol can be transferred to \(M\) by an obvious diffeomorphism \(M \simeq M \cup_{\partial M} ([0, 1] \times \partial M)\) that is equal to identity far from the boundary. One can show (cf. [14]) that the element defined by this symbol (operator) is precisely the image of the problem \((D, P)\) under the order reduction mapping \((\times D^m_+)^{-1}\) of Theorem 2.1.
2. Admissible operators on closed manifolds. Let $\overline{U}$ be a codimension zero submanifold of some closed manifold $M$. We assume that $\overline{U}$ is a covering space

$\overline{U} \xrightarrow{\pi} \overline{Y}$

with smooth base $\overline{Y}$. Let $U$ and $Y$ be the corresponding sets of interior points (we allow $\overline{U}$ to have a boundary). Then scalar admissible operators on $M$ are by definition operators of the form

$$D = D' + \psi (\pi! D'') \varphi,$$

where $D'$ is a pseudodifferential operator on $M$, $D''$ is a pseudodifferential operator on $Y$ acting on sections of $\pi_1 \in \text{Vect}(Y)$, and the cutoff functions $\varphi$ and $\psi$ are supported in $U$.

In the non-scalar case, we consider operators acting in the spaces slightly more general than section spaces of vector bundles.

Namely, consider triples $(E, E_0, \alpha)$ defined by vector bundles

$$E \in \text{Vect}(V), \ E_0 \in \text{Vect}(Y)$$

(here we fix a neighbourhood $V \subset M$ of $M \setminus U$ such that if a point lies in $U \cap V$ then the entire fiber containing this point also lies in $U \cap V$) and a vector bundle isomorphism

$$\pi_1 \mid_{U \cap V} \overset{\alpha}{\simeq} E_0 \mid_{\pi(U \cap V)}$$

on $\pi(U \cap V)$.

Let $\text{Vect}(M, \pi)$ be the set of isomorphism classes of such triples. Here two triples $(E, E_0, \alpha), (F, F_0, \gamma)$ are isomorphic if the vector bundles are pairwise isomorphic, $E \overset{a}{\simeq} F, E_0 \overset{b}{\simeq} F_0$, and the isomorphisms are compatible: $\gamma(\pi a) = b\alpha$.

The linear space of sections corresponding to the triple $\mathcal{E} = (E, E_0, \alpha)$ is defined as

$$C^\infty(M, \mathcal{E}) = \left\{ (u, v) \mid u \in C^\infty(V, E), v \in C^\infty(Y, E_0), \alpha \beta_E (u \mid_{U \cap V}) = v \mid_{\pi(U \cap V)} \right\} \subset C^\infty(V, E) \oplus C^\infty(Y, E_0).$$

For the identity covering, $\mathcal{E}$ defines a vector bundle on $M$ obtained by clutching $E$ with $E_0$ by the transition function $\alpha$, and $C^\infty(M, \mathcal{E})$ is just the space of sections of $\mathcal{E}$.

The space $C^\infty(M, \mathcal{E})$ is generated by the subspaces

$$C^\infty_0(V, E), C^\infty_0(Y, E_0) \subset C^\infty(M, \mathcal{E})$$

of compactly supported sections. More precisely, the first embedding takes $u$ to the pair $(u, \beta_E u \mid_{U \cap V})$, where the tilde stands for the extension of a function by zero at the points where the function was not originally defined. Similarly, the second embedding takes $v$ to the pair $(\beta_E^{-1} v \mid_{\pi(U \cap V)}, v)$. Now non-local operators acting in spaces $C^\infty(M, \mathcal{E})$ can readily be defined by analogy with the scalar case. Namely, an admissible operator of order $m$ is an operator

$$D : C^\infty(M, \mathcal{E}) \longrightarrow C^\infty(M, \mathcal{F})$$

on $\mathcal{E}$.
that is equal, modulo operators with smooth kernel, to

\[ D = D_1 \varphi_1 + D_2 \varphi_2, \]  

(2.4)

where

\[ D_1 : C_0^\infty (V, E) \to C_0^\infty (V, F), \quad D_2 : C_0^\infty (Y, E_0) \to C_0^\infty (Y, F_0) \]

are \( m \)th-order pseudodifferential operators with compactly supported kernels. Here we assume that the cutoff function \( \varphi_1 \) is zero in some neighbourhood of \( M \setminus V \) and \( \varphi_2 \) is zero in a neighbourhood of \( M \setminus U \).

The **symbol of an admissible operator** is a pair \( (\sigma_M, \sigma_Y) \) of usual elliptic symbols

\[ \sigma_M : p_M^* E|_{M \setminus U} \to p_M^* F|_{M \setminus U}, \quad \sigma_Y : p_Y^* E_0|_{\overline{Y}} \to p_Y^* F_0|_{\overline{Y}}, \]

where \( p_M : S^* M \to M \) and \( p_Y : S^* Y \to Y \), are compatible in the sense that

\[ \gamma((\pi_0)!) \sigma_M|_{\partial M} \alpha^{-1} = \sigma_Y|_{\partial \overline{Y}}. \]

Let \( \text{Ell}^k (M, \pi) \) be the group of stable homotopy classes of admissible elliptic operators of order \( k \) on \( M \), modulo elliptic operators with principal symbols independent of the cotangent variables.

**Remark 2.3.** On manifolds with boundary, one can also consider a similar class of elliptic operators and boundary value problems. More precisely, let \( M \) be a manifold with boundary, with a projection \( \pi \) defined on a closed subset \( \overline{U} \subset M \) as above. We assume that \( \overline{U} \) is a codimension zero submanifold in the interior \( M \setminus \partial M \) and is the Cartesian product \( [0, \varepsilon) \times \overline{U}_0 \) in some collar neighbourhood of the boundary for some codimension zero submanifold \( \overline{U}_0 \) in \( \partial M \). Then on \( M \) we consider operators similar to (2.4), where both \( D_1 \) and \( D_2 \) are of order \( m \) and are differential operators with respect to the normal variables in neighbourhoods of the boundaries of the corresponding manifolds (see (2.2)). Such operators are considered in the spaces \( C^\infty (M, \mathcal{E}) \). One considers boundary value problems of the form

\[ (D, Bj) : C^\infty (M, \mathcal{E}) \to C^\infty (M, \mathcal{F}) \oplus C^\infty (\partial M, \mathcal{G}), \]

where \( \mathcal{E}, \mathcal{F} \in \text{Vect}(M, \pi), \mathcal{G} \in \text{Vect}(\partial M, \pi|_{\partial M}) \), \( j \) is the jet operator of order \( m \), \( j : C^\infty (M, \mathcal{E}) \to C^\infty (\partial M, \mathcal{E}^m|_{\partial M}) \), and the boundary conditions are defined by an admissible operator \( B \) on the boundary. One can readily extend all results of this section, including the definition of trivial problems \( D_\pm \), the group of stable homotopy classes of boundary value problems, and order reduction, to this class of boundary value problems.

**Remark 2.4.** Let \( M \) be a manifold with covering \( \pi \) on the boundary. In Subsec. 1, we defined the group \( \text{Ell}^m (M, \pi) \) generated by elliptic non-local problems for the usual operators. At the same time, the projection \( \pi \times 1 : \partial M \times [0, 1) \to X \times [0, 1) \) is defined in a collar neighbourhood of the boundary, and one can consider the corresponding group \( \text{Ell}^m (M, \pi \times 1) \) generated by non-local problems for admissible operators in the sense of Remark 2.3. It turns out that these two groups are isomorphic under the natural mapping

\[ \text{Ell}^m (M, \pi) \to \text{Ell}^m (M, \pi \times 1). \]

This essentially follows from the isomorphism \( \text{Vect}(M) \simeq \text{Vect}(M, \pi \times 1) \).
3. Reduction to a closed manifold. We return to the problem of computing the index of non-local operators on a manifold $M$ with a covering $\pi$ defined on $\partial M$. Consider an embedding $f : M \to M'$ in a closed manifold of the same dimension as $M$ (for example, $M'$ can be the double $2M = M \cup_{\partial M} M$). Just as in the classical case [15], $f$ induces the direct image mapping

$$ f_!: \text{Ell}^0(M, \pi) \to \text{Ell}^0(M', \pi \times 1), $$

where $\pi \times 1$ is the extension of $\pi$ to $\partial M \times [-1, 1] \subset M'$. This mapping takes the symbol $\sigma(D) = (\sigma_M, \sigma_X)$ of an elliptic operator $D : C^\infty(M, E) \to C^\infty(M, C^k)$ to the symbol on $M'$ that coincides on $M$ with the original symbol and is the identity $\text{id} : C^k \to C^k$ on the complement $M' \setminus M$. The extended symbol is defined on the bundle obtained by clutching $E$ with $C^k$ using the isomorphism $\sigma_X|_X$ and maps this bundle to the bundle $C^k$ over the ambient closed manifold $M'$.

**Lemma 2.1.** The mapping $f_!: \text{Ell}^0(M, \pi) \to \text{Ell}^0(M', \pi \times 1)$ is well defined and is index preserving.

**Proof.** This is a restatement of the well-known excision property of the index. The proof is standard, and hence we omit it altogether. $\square$

4. Embedding in a universal space. In the index theorems of the present paper, we assume that the following condition is satisfied.

**Assumption 2.1.** The covering $\pi$ is regular and there is a free action of a finite group $G$ on the submanifold $\overline{U}$ such that $\pi$ is the projection onto the quotient.

Let $(M, \pi)$ and $(M', \pi')$ be two pairs (both manifolds are assumed to be closed) and let $U$ and $U'$ be the domains of $\pi$ and $\pi'$, respectively.

**Definition 2.1.** We say that $f$ is an embedding of $(M, \pi)$ in $(M', \pi')$ if there is an embedding $f : M \to M'$, $f(\overline{U}) \subset \overline{U}'$, that is equivariant on the domain of $\pi$.

Denote by $\pi_N : EG_N \to BG_N$ the $N$-universal bundle for $G$. We assume that $EG_N$ and $BG_N$ are closed manifolds. There is an explicit construction for such a model (e.g., see [17]). For example, consider the embedding

$$ G \subset S_{|G|} \subset U(|G|) $$

in the unitary group. (Here $|G|$ is the order of $G$.) Consider the bundle $V_{k,|G|} \to V_{k,|G|}/G$, where $V_{k,n}$ is the Stiefel manifold of $n$-frames in $C^k$. For sufficiently large $k$, this bundle is $N$-universal.

\footnote{An arbitrary operator $D' : C^\infty(M, E) \to C^\infty(M, F)$ is reduced to this form by adding the identity operator in the sections of the complementary bundle to $F$.}
Proposition 2.1. For \((M, \pi)\) satisfying Assumption 2.1, there exists an embedding in \((EG_N, \pi_N)\) provided that \(N\) is sufficiently large.

Proof. By \(N\)-universality of \(\pi_N\), there exists an equivariant mapping \(\overline{U} \to EG_N\). We can assume that this mapping is a smooth embedding. This can be achieved by a small deformation provided that the dimension of \(EG_N\) is sufficiently large.

This embedding can be extended to a smooth mapping \(M \to EG_N\) owing to the \(N\)-connectedness of \(EG_N\). Finally, a small deformation outside a neighbourhood of \(U\) makes it a global embedding. \(\square\)

5. The Euler operator on the disc. Consider the Neumann problem

\[
\begin{align*}
(d + \delta) u &= f, \\
(\ast u)|_{S^{n-1}} &= g,
\end{align*}
\]

for the Euler operator in the unit disc \(D^n \subseteq \mathbb{R}^n\) with the Euclidean metric. Here \(g \in \Lambda^{e+n}(S^{n-1})\). This boundary value problem is elliptic, and Hodge theory shows that the cokernel is trivial and the one-dimensional kernel consists of constant functions.

The same is true for the homogeneous boundary value problem, which we rewrite in the operator form

\[
D_{dR} = d + \delta : \Lambda_0^e(D^n) \to \Lambda_o^o(D^n).
\]

(Here \(\Lambda_0^0(D^n)\) is the space of forms satisfying the homogeneous boundary condition.) This operator is \(O(n)\)-equivariant with respect to the natural action of the orthogonal group on \(D^n\).

6. Embeddings and the index of elliptic operators. Let \(f : (M, \pi) \to (M', \pi')\) be an embedding of positive codimension. We choose a Riemannian metric on \(M'\) that is \(G\)-invariant over \(\overline{U}' \subset M'\). Denote the normal bundle to \(M\) by \(N M\). Then a closed tubular neighbourhood \(W\) of \(M\) in \(M'\) is diffeomorphic to the unit ball subbundle \(D M \subseteq N M\). Additionally, we can assume this diffeomorphism to be \(G\)-equivariant over \(U \subset U'\).

Consider an admissible elliptic operator

\[
D : C^\infty(M, E) \to C^\infty(M, F).
\]

We define a boundary value problem on \(D M\) as the exterior tensor product of \(D\) by a family of boundary value problems for the Euler operator in the fibers. The definition of this product is the same as in [15].

More precisely, the exterior tensor product gives the operator

\[
\mathcal{D} = \begin{pmatrix}
\tilde{D} \otimes 1_{\Lambda^e} & -1_F \otimes \tilde{D}_{dR}^* \\
1_E \otimes \tilde{D}_{dR} & \tilde{D}^* \otimes 1_{\Lambda^o}
\end{pmatrix},
\]

where the pullback of \(D\) to the bundle \(D M\) with coefficients in even forms on the fibers is denoted by

\[
\tilde{D} \otimes 1_{\Lambda^e} : C^\infty(D M, p^*E \otimes \Lambda_0^e(D M)) \to C^\infty(D M, p^*F \otimes \Lambda_0^o(D M)).
\]
Here $\tilde{D}^* \otimes 1_{A^o}$ is the pullback of the adjoint operator with coefficients in odd forms. The family of Neumann problems for the Euler operator $\tilde{D}^M_{dR}$ with coefficients in the triple $\mathcal{E} = (E, E_0, \alpha)$ is denoted by

$$1_E \otimes \tilde{D}_{dR} : C^\infty_{\alpha \otimes 1} (DM, p^* E \otimes \Lambda^o_0 (DM)) \to C^\infty_{\alpha \otimes 1} (DM, p^* E \otimes \Lambda^o (DM)).$$

The off-diagonal entries of $\mathcal{D}$ commute with entries on the diagonal by construction. As in ordinary Atiyah–Singer theory, this leads to the following result.

**Lemma 2.2.** One has $\text{ind} D = \text{ind} \mathcal{D}$.  

The proof is similar to [15]. □

Thus an elliptic operator on the submanifold $M \subset M'$ induces an elliptic boundary value problem with the same index on the tubular neighbourhood $DM \simeq W \subset M'$. Further, we can apply the order reduction procedure to this problem (see Remark 2.2) and extend the resulting zero-order operator from $W$ to the entire manifold $M'$ as in Subsec. 2.

Summarizing, we see that the embedding $f$ of $(M, \pi)$ in $(M', \pi')$ induces the direct image mapping

$$f_! : \text{Ell}^1 (M, \pi) \to \text{Ell}^0 (M', \pi'),$$

which preserves the index.

**Remark 2.5.** A straightforward computation shows that the linear homotopy of order reduction for boundary value problems (defined in Remark 2.2) which extends the symbol $\sigma (d + \delta)$ from $T^* \mathbb{D}^n$ to $T^* \mathbb{R}^n$ as an invertible element outside a compact set defines an element of the equivariant $K$-group equal to the element

$$f_! (1) \in K_{O(n)} (T^* \mathbb{R}^n), \quad j : pt \to \mathbb{R}^n,$$

which is used in the standard proof of the Atiyah–Singer theorem.

7. **The Index theorem.** Let $f$ be an embedding of $(M, \pi)$ in the universal space defined in Proposition 2.1. For the universal space $EG_N$, the projection $\pi_N$ is defined globally. Therefore, the direct image of a non-local operator can be treated as a usual elliptic operator on the base $BG_N$; i.e., we have a natural mapping

$$(\pi_N)_! : \text{Ell} (EG_N, \pi_N) \to \text{Ell} (BG_N) \simeq K (T^* BG_N).$$

**Theorem 2.2.** For a pair $(M, \pi)$ satisfying Assumption 2.1, the diagram

$$\begin{array}{ccc}
\text{Ell}^1 (M, \pi) & \xrightarrow{f_!} & \text{Ell}^0 (EG_N, \pi_N) \\
\text{ind} \downarrow & & \downarrow (\pi_N)_! \\
\mathbb{Z} & \xleftarrow{\text{ind}_t} & K (T^* BG_N)
\end{array}$$

commutes. Here $\text{ind}_t$ is the usual topological index on a closed manifold.
Proof. Indeed, we have
\[ \text{ind}D = \text{ind} f; [D] = \text{ind} (\pi_N); f; [D] = \text{ind}_t ((\pi_N); f; [D]). \]
The first equality here follows from the invariance of the index for embeddings, the second from the fact that \((\pi_N)\); does not change the operator, and the last equality is just the Atiyah–Singer formula on \(BG_N\).

\[ \square \]

8. Example. Manifolds with reflecting boundary \[18\]. Let \(M\) be a \(4k\)-dimensional compact oriented Riemannian manifold with boundary \(\partial M\). Suppose that \(\partial M\) is equipped with an orientation-reversing smooth involution \(G\) without fixed points. The involution defines a free action of the group \(\mathbb{Z}_2\) and the corresponding double covering \(\pi: \partial M \to \partial M/\mathbb{Z}_2\). Consider the Hirzebruch operator \[19\]
\[ d + d^* : \Lambda^+ (M) \to \Lambda^- (M). \]

In a neighbourhood of the boundary, let us take a metric lifted from \([0, 1] \times \partial M/\mathbb{Z}_2\). Then the Hirzebruch operator can be decomposed near the boundary as (see \[11\])
\[ \frac{\partial}{\partial t} + A \]
(up to a bundle isomorphism), where \(A\) is an elliptic self-adjoint operator on the boundary and is given by the formula
\[ A : \Lambda^* (\partial M) \to \Lambda^* (\partial M), \quad A\omega = (-1)^{k+p} (d \ast - \varepsilon \ast d) \omega; \]
here for an even degree form \(\omega \in \Lambda^{2p} (\partial M)\) we set \(\varepsilon = 1\), and \(\varepsilon = -1\) otherwise. Since \(G\) reverses the orientation, it follows that \(A\) and \(G^*\) anticommute:
\[ G^* A = -AG^*. \]

It is known that the Hirzebruch operator has no well-posed classical boundary conditions. However, it admits the non-local boundary value problem
\[ \begin{cases} (d + d^*) \omega = f, \\ \frac{(1+G^*)}{2} \omega|_{\partial M} = g, \quad g \in \Lambda^* (\partial M)^{\mathbb{Z}_2} \simeq \Lambda^* (\partial M/\mathbb{Z}_2) \end{cases} \tag{2.5} \]
on the manifold with reflecting boundary. Here \(\Lambda^* (\partial M)^{\mathbb{Z}_2}\) is the subspace of \(G\)-invariant forms on the boundary.

**Proposition 2.2.** The non-local boundary value problem \[2.5\] is elliptic.

**Proof.** Consider an arbitrary point \(x \in \partial M/\mathbb{Z}_2\). An explicit computation shows that near this point the equivalent classical boundary value problem is
\[ \begin{cases} \left( \frac{\partial}{\partial t} + A \right) \omega_1 = f_1, \quad \left( \frac{\partial}{\partial t} - A \right) \omega_2 = f_2, \\ \omega_1|_{\partial M/\mathbb{Z}_2} + \omega_2|_{\partial M/\mathbb{Z}_2} = g. \end{cases} \]
It is elliptic (satisfies the Shapiro–Lopatinskii condition), since the symbol of the operator of boundary conditions defines an isomorphism

\[ \text{Im} \sigma (\Pi_+) (x, \xi) \oplus \text{Im} \sigma (\Pi_-) (x, \xi) \cong \Lambda^* (\partial M)_x \]

at an arbitrary point \((x, \xi) \in S^*(\partial M/\mathbb{Z}_2)\), where

\[ \Pi_+ = \frac{A + |A|}{2|A|} \]

is the non-negative spectral projection of \(A\) and \(\Pi_- = 1 - \Pi_+\) is the negative projection. The ellipticity of the classical boundary value problem proves the desired statement. \(\square\)

**Proposition 2.3.** One has

\[ \text{ind} (d + d^*, (1 + G^*)) = \text{sign} M, \]

where \(\text{sign} M\) is the signature of \(M\).

**Proof.** The symbol of \((2.5)\) coincides with that of the composition of the spectral Atiyah–Patodi–Singer boundary value problem

\[
\begin{cases}
(d + d^*) \omega = f \\
\Pi_+ \omega|_{\partial M} = \omega', \quad \omega' \in \text{Im} \Pi_+ \subset \Lambda^* (\partial M),
\end{cases}
\]

and the Fredholm operator

\[ (1 + G^*) : \text{Im} \Pi_+ \longrightarrow \Lambda^* (\partial M)^2. \ \ (2.6) \]

Let us compute both indices.

1) For the index of the spectral boundary value problem, one has \[\text{ind} (d + d^*, \Pi_+) = \text{sign} M - \frac{\dim \ker A}{2}. \]

In addition, by the Hodge–de Rham theory we obtain \(\dim \ker A = \dim H^* (\partial M)\).

2) On the other hand, one can readily verify that the operator in Eq. \((2.6)\) is surjective and its kernel coincides with the space of \(G\)-antiinvariant harmonic forms. The Hodge operator \(\ast\) interchanges the antiinvariant and invariant subspaces. Thus we obtain

\[ \dim \ker (1 + G^*)|_{\text{Im} \Pi_+ (A)} = \frac{\dim \ker A}{2}. \]

Adding the index of the spectral problem to the index of \((1 + G^*)\), we obtain

\[ \text{ind} (d + d^*, (1 + G^*)) = \text{sign} M - \frac{\dim \ker A}{2} + \frac{\dim \ker A}{2} = \text{sign} M. \]

The proof of the theorem is complete. \(\square\)
3 The homotopy classification of non-local operators

Let us cut $M$ into two parts

$$M' = M \setminus \{\partial M \times [0, 1]\} \simeq \partial M \times [0, 1].$$

Then the symbol $\sigma(D)$ of an admissible elliptic operator $D$ of order zero is naturally represented as a pair of usual symbols

$$\sigma(D)|_{M'} \text{ and } (\pi \times 1)_* \sigma(D)|_{\partial M \times [0, 1]}.$$ (3.1)

Both symbols define difference elements

$$[\sigma_M] \in K(T^*M'), \quad [\sigma_X] \in K(T^*(X \times (0, 1))).$$

Here and in what follows, we use $K$-groups with compact supports. In the latter case, the elliptic symbol $\sigma_X$ of order zero is invertible over $X \times \{0\}$ (this follows from ellipticity and the decomposition in Eq. (2.2)) and hence defines element in the above-mentioned $K$-group with compact supports.

However, it is impossible to define an element of a single topological $K$-group; indeed, the manifolds $T^*M'$ and $T^*(X \times (0, 1])$ cannot be glued together, for their boundaries are not diffeomorphic. Nonetheless, we can glue the algebras of functions on these spaces instead of the original manifolds.

1. The $C^*$-algebra of a manifold with a covering on the boundary. To each space, we assign an algebra of continuous functions vanishing at infinity:

$$C_0(T^*M'), \quad C_0(T^*(X \times (0, 1]), \text{End}p^*\pi_1).$$

More precisely, on the space $T^*(X \times (0, 1])$ we consider functions ranging in the set of endomorphisms of the bundle $\pi_1 \in \text{Vect}(X)$, where $p : T^*(X \times (0, 1]) \to X$ is the natural projection. In the direct sum of these algebras, consider the subalgebra determined by the compatibility condition

$$A_{T^*M,\pi} = \left\{(u, v) \mid u \in C_0(T^*M'), v \in C_0(T^*(X \times (0, 1]), \text{End}p^*\pi_1), \beta u|_{\partial M}, \beta^{-1} = v|_{t=1}\right\}.$$ (3.2)

Here $t$ is the coordinate on $(0, 1]$.

For the trivial covering $\partial M \to \partial M = X$, this algebra is just the commutative algebra of continuous functions on $T^*(M \setminus \partial M)$ vanishing at infinity. Let us also mention that this algebra can be also viewed as the groupoid $C^*$-algebra [20] of the equivalence relation defined by $\pi (x \sim y$ if either $x = y$ or $x, y \in \partial M$ and $\pi(x) = \pi(y)$). For a trivial covering, this algebra was used in [6].

2. The difference construction. Let us define the difference construction for non-local operators. This will be a mapping

$$\chi : \text{Ell}^0(M, \pi) \to K_0(A_{T^*M,\pi})$$ (3.3)
into the $K_0$ group of the $C^*$-algebra $\mathcal{A}_{T^*M,\pi}$. To this end, we take an elliptic operator

$$D : C^\infty(M, E) \longrightarrow C^\infty(M, \mathbb{C}^k),$$

fix some embeddings of $E$ and $\mathbb{C}^k$ in trivial bundles of sufficiently large dimension, and denote by $P_E$ and $P_{C^k}$ the projections that define the corresponding subbundles:

$$E \simeq \text{Im } P_E \subset \mathbb{C}^N \oplus 0, \quad \mathbb{C}^k \simeq \text{Im } P_{C^k} \subset 0 \oplus \mathbb{C}^L.$$

We denote the direct images of these projections near the boundary by $P_{\pi E}$ and $P_{\pi C^k}$. The difference element of $D$ is, by definition, the difference

$$\chi [D] = [P_1 \oplus P_2] - [P_{C^k} \oplus P_{\pi C^k}] \in K_0 (\mathcal{A}_{T^*M,\pi}),$$

where the projection $P_1$ over $M'$ is given by

$$P_1 = \begin{cases} P_E \cos^2 |\xi| + P_{C^k} \sin^2 |\xi| + (\sigma^{-1}_M (x, \xi) P_{C^k} + \sigma_M (x, \xi) P_E) \sin |\xi| \cos |\xi|, & |\xi| \leq \pi/2, \\
\frac{1}{2} (\tilde{\sigma}^{-1} (x', \xi) P_{\pi C^k} + \tilde{\sigma} (x', \xi) P_{\pi E}) \sin 2 |\xi|, & |\xi| > \pi/2. \end{cases}$$

(3.4)

(We assume that the principal symbol is zero-order homogeneous in $\xi$.) The projection $P_2$ over $X \times [0, 1]$ is defined by the formula

$$P_2 = \begin{cases} P_{\pi E} \cos^2 |\xi| + P_{\pi C^k} \sin^2 |\xi| + \frac{1}{2} (\tilde{\sigma}^{-1} (x', \xi) P_{\pi C^k} + \tilde{\sigma} (x', \xi) P_{\pi E}) \sin 2 |\xi|, & |\xi| \leq \pi/2, \\
\frac{1}{2} (\tilde{\sigma}^{-1} (x', 0) P_{\pi C^k} + \tilde{\sigma} (x', 0) P_{\pi E}) \sin 2 \varphi, & |\xi| > \pi/2. \end{cases}$$

where the first case is used for $x' \in X \times [1/2, 1], |\xi| \leq \pi/2$, the second for $x' \in X \times [0, 1/2], |\xi| < \pi t$, and the third otherwise. Here we write

$$\varphi = |\xi| + \pi/2 (1 - 2t), \quad \tilde{\sigma} (x', \xi) = \sigma_X (x', \xi)$$

for brevity. Geometrically, these projections define a subbundle that coincides with $E \subset \mathbb{C}^{N L}$ over the zero section (for $\xi=0$); coincides with the orthogonal bundle $\mathbb{C}^k \subset \mathbb{C}^{N L}$ for $|\xi| \geq \pi/2$; and is obtained by the rotation of the first bundle towards the second bundle with the use of $\sigma (D)$ at the intermediate points. (The symbol is treated as an isomorphism of the two bundles.) By construction, $P_1$ and $P_{C^k}$ coincide outside a compact set in $T^*M'$, and $P_2$ and $P_{\pi C^k}$ coincide outside a compact set in $T^* (X \times (0, 1])$. Therefore, the difference $[P_1 \oplus P_2] - [P_{C^k} \oplus P_{\pi C^k}]$ is indeed in $K_0 (\mathcal{A}_{T^*M,\pi})$.

**Remark 3.1.** This element of the $K$-group can be equivalently defined by different expressions (cf. [21]).

**Theorem 3.1.** The difference construction (3.3) is a well-defined group isomorphism.
Proof. The mapping $\chi$ preserves the equivalence relations in $\Ell^0 (M, \pi)$ and $K_0 (A_{T^* M, \pi})$. Indeed, under an operator homotopy the symbols vary continuously. Therefore, the corresponding projections $P_{1,2}$ are joined by a continuous homotopy. Furthermore, $\chi [D]$ is independent of the choice of an embedding in a trivial bundle, since all such embeddings are homotopic, and for a trivial $D$ (i.e., one induced by a vector bundle isomorphism) $\chi [D]$ is equal to zero. This shows that $\chi$ is well defined. The proof that this mapping is one-to-one presents no essential difficulties and is left to the reader.

3. **Index theorem for families.** Later on in Section 5, we use a families index formula. Let us briefly state the corresponding results.

Let $P$ be a compact space. Denote by $\Ell_P (M, \pi)$ the group of stable homotopy classes of elliptic families on $M$ parametrized by $P$.

**Theorem 3.2** (the index of families of non-local operators). Let $(M, \pi)$ satisfy Assumption 2.1 Then for an embedding $f : M \to EG_N$ the direct image mapping $f_!$ for families is well defined and the following diagram commutes:

$$
\begin{array}{ccc}
\Ell_P (M, \pi) & \xrightarrow{f_!} & \Ell_P (EG_N, \pi_N) \\
\text{ind} & & \downarrow^{(\pi_N)_!} \\
K^0 (P) & \xleftarrow{\text{ind}_!} & K^0 (P \times T^* BG_N) & \xrightarrow{\text{ind}} & \Ell_P (BG_N).
\end{array}
$$

The proof is similar to that of Theorem 2.2 in the previous section (cf. [22]) and therefore is omitted.

Let us finally note that the difference construction can also be defined in this case as a mapping

$$
\chi_P : \Ell_P (M, \pi) \longrightarrow K_0 (C (P, A_{T^* M, \pi})),
$$

where $C (P, A_{T^* M, \pi})$ is the algebra of continuous functions on $P$ ranging in the $C^*$-algebra $A_{T^* M, \pi}$.

4 **A homotopy invariant for manifolds with covering on the boundary**

1. **The class of operators.** On a manifold $M$ with covering $\pi$ on the boundary, we consider elliptic differential operators

$$
D : C^\infty (M, E) \longrightarrow C^\infty (M, F)
$$

that are lifted from the base of the covering in a neighbourhood of the boundary. Technically, we suppose that the following condition is satisfied.
Assumption 4.1. The restrictions of the bundles $E$ and $F$ to the boundary are lifted from the base of the covering; moreover, we fix some isomorphisms

$$E|_{\partial M} \simeq \pi^* E_0, \quad F|_{\partial M} \simeq \pi^* F_0, \quad E_0, F_0 \in \text{Vect}(X),$$

and for some operator $D_0 : C^\infty(X \times [0, 1), E_0) \to C^\infty(X \times [0, 1), F_0)$ on the cylinder with base $X$ the direct image of $D$ in a collar neighbourhood of the boundary satisfies the commutative diagram

$$C^\infty(X \times [0, 1), \pi_1 E) \xrightarrow{(\pi \times 1)_!, D} C^\infty(X \times [0, 1), \pi_1 F) \quad (4.1)$$

Here $D_0 \otimes 1$ stands for the operator $D_0$ with coefficients in the flat bundle $\pi_1$ (e.g., see [23]).

We also suppose that $D$ is first-order operator and the following assumption is satisfied.

Assumption 4.2. In the neighbourhood $X \times [0, \epsilon)$ of the boundary, the operator has the form

$$D_0|_{X \times [0, \epsilon)} = \Gamma \left( \frac{\partial}{\partial t} + A_0 \right)$$

for a bundle isomorphism $\Gamma$, where $A_0$ is an elliptic self-adjoint first-order operator on $X$. This operator is called the tangential operator of $D_0$.

If $D$ satisfies Assumptions 4.1 and 4.2, then near the boundary it has the form

$$D = \frac{\partial}{\partial t} + \pi_! (A_0 \otimes 1)$$

up to a vector bundle isomorphism. For brevity, the self-adjoint operator $\pi_! (A_0 \otimes 1)$ will be denoted by $A$.

2. The homotopy invariant. For an operator $D$ satisfying Assumptions 4.1 and 4.2 consider the spectral Atiyah–Patodi–Singer boundary value problem $\Pi$

$$\begin{cases} 
Du &= f, \\
\Pi_+ u|_{\partial M} &= g, \quad g \in \text{Im} \Pi_+ \subset H^{s-1/2}(\partial M, E),
\end{cases}$$

where $\Pi_+ = (A + |A|)/2 |A|$ is the non-negative spectral projection of the self-adjoint operator $A$. (If $A$ is not invertible, then in this formula one should replace $A$ by $A + \epsilon$ for some $\epsilon$ less then the absolute value of the greatest negative eigenvalue of $A$.) The spectral problem is always Fredholm. However, its index $\text{ind} (D, \Pi_+)$ is not invariant under homotopies of $D$ and is not determined by its principal symbol. Here by definition a continuous homotopy of $D$ is a continuous homotopy in the interior of $M$ that can be covered in a neighbourhood of the boundary by a homotopy of the diagram (4.1) and a continuous homotopy of tangential operators.
Proposition 4.1. The sum
\[ \widetilde{\text{ind}} D \overset{\text{def}}{=} \text{mod } n \cdot (\text{ind } (D, \Pi_{+}) + \eta (A) - n \eta (A_0)) \in \mathbb{R}/n \mathbb{Z}, \] (4.2)
is a homotopy invariant of \( D \). Here \( n \) is the number of sheets of the covering and \( \eta (A) \) and \( \eta (A_0) \) are the spectral Atiyah–Patodi–Singer \( \eta \)-invariants of the tangential operators \( A \) and \( A_0 \).

Proof. Consider the non-reduced invariant
\[ \text{ind } (D, \Pi_{+}) + \eta (A) - n \eta (A_0). \] (4.3)
The results of [23] imply that for a smooth operator family \( D_t \) this expression is a piecewise smooth function of the parameter \( t \) (the corresponding families of tangential operators are denoted by \( A_t \) and \( A_{0,t} \)).

1) We claim that (4.3) is a piecewise constant function. Indeed, the derivative of the \( \eta \)-invariant
\[ \frac{d}{dt} \eta (A_t) \]
with respect to \( t \) is local, i.e., is equal to an integral over the manifold of an expression determined by the complete symbol of the tangential family \( A_t \). However, the complete symbols of \( A_t \) and \( A_{0,t} \) coincide locally by Assumption 4.1. Thus we have
\[ \frac{d}{dt} \eta (A_t) = n \frac{d}{dt} \eta (A_{0,t}). \]
Therefore, (4.3) is a piecewise constant function.

2) Let us show that the jumps of this function are multiples of the number of sheets of the covering. Indeed, for a homotopy \( D_t \) the index, as well as the \( \eta \)-invariants, changes by the spectral flow of the corresponding families of tangential operators. Hence
\[ [\text{ind } (D_t, \Pi_{+}, t) + \eta (A_t) - n \eta (A_{0,t})]_{t=0}^{1} = (-1 + 1) \text{ sf } (A_t)_{t\in[0,1]} - n \text{ sf } (A_{0,t})_{t\in[0,1]} \in n \mathbb{Z}, \]
as desired. \( \square \)

Remark 4.1. For a trivial covering, our invariant is none other than the mod \( n \)-index
\[ \text{mod } n \cdot \text{ind } (D, \Pi_+) \in \mathbb{Z}_n \subset \mathbb{R}/n \mathbb{Z} \]
of Freed-Melrose [2]. On the other hand, the fractional part of the invariant (4.2) is the so-called relative Atiyah–Patodi–Singer \( \eta \)-invariant [24] [23]
\[ \{ \eta (A_0 \otimes 1_{\pi!1}) - n \eta (A_0) \} \in \mathbb{R}/\mathbb{Z} \]
of \( A_0 \) with coefficients in the flat bundle \( \pi!1 \in \text{Vect } (X) \).
The invariant \( \widetilde{\text{ind}} \) has an interesting interpretation as an obstruction. Namely, suppose that \( M \) is the total space of a covering \( \widetilde{\pi} \) with base \( Y \) that induces the covering \( \pi \) over the boundary
\[
\begin{align*}
\partial M & \subset M \\
\pi & \downarrow \quad \downarrow \widetilde{\pi} \\
X & \subset Y.
\end{align*}
\]

**Proposition 4.2.** If a differential operator \( D : C^\infty (M, E) \to C^\infty (M, F) \) is the pullback of an elliptic operator \( D_0 \) on \( Y \), then
\[
\widetilde{\text{ind}} D = 0.
\]

**Proof.** According to the Atiyah–Patodi–Singer formula (see [1]), the sum
\[
\text{ind} (D, \Pi_+) + \eta (A)
\]
is equal to the integral over the manifold of a local expression defined by the complete symbol of \( D \). Since \( D \) and \( D_0 \) coincide locally, one has
\[
\text{ind} (D, \Pi_+) + \eta (A) = n (\text{ind} (D_0, \Pi_{+,0}) + \eta (A_0)).
\]
We obtain the desired formula by transposing the term \( n \eta (A_0) \) to the left-hand side. \( \square \)

### 5 The index defect formula

The aim of this section is to find a topological formula for the invariant \( \widetilde{\text{ind}} \).

**1. The difference construction.** The pair \((M, \pi)\) defines the singular space
\[
\overline{M}^\pi = M / \{x \sim x', \text{ if } x, x' \in \partial M \text{ and } \pi (x) = \pi (x') \},
\]
obtained by identification of points in the fibers of \( \pi \) (see Figure 1 in the case of a trivial covering). Likewise, the boundary of the non-compact manifold \( T^*M \) is a covering over the product \( T^*X \times \mathbb{R} \), and the corresponding singular space will be denoted by \( \overline{T^*M}^\pi \).

Consider an elliptic operator \( D \) satisfying Assumption 4.1. The diagram (4.1) implies that the principal symbol defines a \( K \)-theory element
\[
[\sigma (D)] \in K (\overline{T^*M}^\pi).
\]
Thus we have a homomorphism
\[
\chi : \text{Ell} (\overline{M}^\pi) \to K (\overline{T^*M}^\pi), \quad \chi [D] = [\sigma (D)].
\]
Here \( \text{Ell} (\overline{M}^\pi) \) is the Grothendieck group of homotopy classes of elliptic operators \( D \) on \( M \) that satisfy Assumptions 4.1 and 4.2.

The topological formula for the invariant \( \widetilde{\text{ind}} \) uses the Poincaré pairing on the manifold \( \overline{T^*M}^\pi \) with singularities. Let us define this pairing.
2. A pairing in $K$-theory of a singular manifold. By analogy with the algebra $A_{T^*M,\pi}$ of the cotangent bundle, one can define an algebra for $M$ itself:

$$A_{M,\pi} = \left\{ (u,v) \mid u \in C_0(M'), v \in C_0(X \times (0,1], \text{End}\pi_1) \right\}.$$

**Lemma 5.1.** The group $K_0(A_{M,\pi})$ is isomorphic to the group of stable homotopy classes of triples

$$(E,F,\sigma), \quad E,F \in \text{Vect}(M), \quad \sigma : \pi_1 E|_{\partial M} \longrightarrow \pi_1 F|_{\partial M}.$$

Here $\sigma$ is a bundle isomorphism, and trivial triples are those with $\sigma$ induced by an isomorphism over $M$.

**Proof.** Note that this lemma is similar to Theorem 3.1, which can be also considered as giving a realization of the group $K_0(A_{T^*M,\pi})$ in topological terms. Along the same lines, a triple $(E,\mathbb{C}^k,\sigma)$ defines the element

$$[P_E \oplus P_2] - [P_{\mathbb{C}^k} \oplus P_{\pi_1\mathbb{C}^k}] \in K_0(A_{M,\pi}),$$

where the projection $P_2$ over $X \times [0,1]$ is

$$P_2 = P_{\pi_1E} \cos^2 \varphi + P_{\pi_1\mathbb{C}^k} \sin^2 \varphi + P_{\pi_1\mathbb{C}^k} \sigma(x) P_{\pi_1E} \sin 2\varphi, \quad \varphi = \frac{\pi}{2} (1 - t).$$

Here $P_E, P_{\mathbb{C}^k}$ are projections on subbundles isomorphic to $E$ and $F$. We also suppose that the subbundles are orthogonal to each other.
The proof of the fact that this mapping induces an isomorphism with the group $K_0(A_{M,\pi})$ is similar to the previous proof and is omitted.

This realization permits one to define a product
$$K_0(T^*M) \times K_0(A_{M,\pi}) \longrightarrow K_0(A_{T^*M,\pi})$$
using the construction of a symbol with coefficients in a vector bundle. More precisely, for elements 

$$[\sigma] \in K(T^*M), \quad [E, F, \sigma'] \in K_0(A_{M,\pi}),$$

consider the symbol

$$\sigma \otimes 1_E \oplus \sigma^{-1} \otimes 1_F$$

on $M$. The direct image of its restriction to the boundary can be written as

$$\pi_! (\sigma \otimes 1_E \oplus \sigma^{-1} \otimes 1_F|_{\partial M}) = \pi_! \sigma \otimes 1_{\pi_! E|_{\partial M}} \oplus \pi_! \sigma^{-1} \otimes 1_{\pi_! F|_{\partial M}} \cong (\pi_! \sigma \oplus \pi_! \sigma^{-1}) \otimes 1_{\pi_! E|_{\partial M}}.$$

The latter isomorphism is induced by a vector bundle isomorphism

$$\pi_! E|_{\partial M} \simeq \pi_! F|_{\partial M}.$$

Now $(\pi_! \sigma \oplus \pi_! \sigma^{-1}) \otimes 1_{\pi_! E|_{\partial M}}$ is trivially homotopic to the identity. The homotopy is

$$\begin{pmatrix} \pi_! \sigma & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos \tau & \sin \tau \\ -\sin \tau & \cos \tau \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \pi_! \sigma^{-1} \end{pmatrix} \begin{pmatrix} \cos \tau & -\sin \tau \\ \sin \tau & \cos \tau \end{pmatrix}, \quad \tau \in [0, \pi/2].$$

Thus we have extended the symbol (5.1) to a non-local elliptic symbol on $M$. Now the desired product is defined as the difference construction of the latter symbol, which we denote by

$$[\sigma] \times [E, F, \sigma'] \in K_0(A_{T^*M,\pi}).$$

Using the homotopy classification $K_0(A_{T^*M,\pi}) \simeq \text{Ell}(M, \pi)$, we can apply the index mapping to this product and define the pairing of groups as the composition

$$\langle , \rangle : K^0(T^*M) \times K_0(A_{M,\pi}) \longrightarrow K_0(A_{T^*M,\pi}) \longrightarrow \mathbb{Z}.$$ 

This pairing is an analogue of Poincaré duality on the singular manifold $T^*M\overline{\sigma}$ (see Section 8). Let us note that for a regular covering the index can be computed topologically (by the index theorem) and hence the pairing is also topologically computable.

3. The element of $K$-theory with coefficients defined by a manifold with a covering on the boundary. We start from a universal example. Denote the half-infinite cylinder $EG_N \times [0, +\infty)$ by $M_N$, and the projective limit of the groups $K_0(A_{M_N,\pi_N})$ as $N \to \infty$ by $K_0(A_{M,\pi})$. Let us show that the universal bundle $\gamma = \pi_11 \in \text{Vect}(BG)$, where $1 \in \text{Vect}(EG)$, defines an element

$$[\gamma] \in K_0(A_{M,\pi}, \mathbb{Q}/n\mathbb{Z}), \quad n = |G|.$$  

(5.2)
To this end, note that the universal bundle gives the element
\[ [\gamma] - n \in K^0(BG). \]

We use the following lemma to show that this difference defines the desired element.

**Lemma 5.2.** There is an isomorphism \( \tilde{K}^0(BG) \simeq K^1(BG, \mathbb{Q}/n\mathbb{Z}) \), which is defined as the coboundary mapping \( \partial \) in the exact sequence
\[
\rightarrow K^1(BG_N) \otimes \mathbb{Q} \rightarrow K^1(BG_N, \mathbb{Q}/n\mathbb{Z}) \overset{\partial}{\rightarrow} K^0(BG_N) \overset{\times n}{\rightarrow} K^0(BG_N) \otimes \mathbb{Q}
\]
induced by the inclusion of the coefficient groups \( n\mathbb{Z} \subset \mathbb{Q} \).

**Proof.** Let us rewrite the sequence
\[
\rightarrow K^1(BG_N) \otimes \mathbb{Q} \rightarrow K^1(BG_N, \mathbb{Q}/n\mathbb{Z}) \rightarrow K^0(BG_N) \overset{\times n}{\rightarrow} K^0(BG_N) \otimes \mathbb{Q}
\]
as the short exact sequence
\[
0 \rightarrow K^1(BG_N) \otimes \mathbb{Q}/n\mathbb{Z} \rightarrow K^1(BG_N, \mathbb{Q}/n\mathbb{Z}) \rightarrow \text{Tor}K^0(BG_N) \rightarrow 0.
\]
Then we obtain the following sequence of projective limits as \( N \to \infty \) (this sequence may not be exact):
\[
0 \rightarrow \lim_{\leftarrow} K^1(BG_N) \otimes \mathbb{Q}/n\mathbb{Z} \rightarrow K^1(BG, \mathbb{Q}/n\mathbb{Z}) \rightarrow \lim_{\leftarrow} \text{Tor}K^0(BG_N) \rightarrow 0. \quad (5.3)
\]

As \( N \) increases, the sequence \( \tilde{K}^*(BG_N) \) has the following property (e.g., see [21, 23]). For an arbitrary \( N \), there exists an \( L > 0 \) such that the range of the mapping
\[
\tilde{K}^*(BG_{N+L}) \rightarrow \tilde{K}^*(BG_N)
\]
is in the torsion subgroup. Using this, we obtain the following expressions for the limits:
\[
\lim_{\leftarrow} K^1(BG_N) \otimes \mathbb{Q}/n\mathbb{Z} = 0, \quad \lim_{\leftarrow} \text{Tor}K^0(BG_N) = \lim_{\leftarrow} \tilde{K}^0(BG_N) = \tilde{K}^0(BG).
\]

One can also use (5.4) to prove the exactness of (5.3). The proof is based on commutative diagrams of the form
\[
\begin{array}{ccc}
0 \rightarrow K^1(BG_N) \otimes \mathbb{Q}/n\mathbb{Z} & \rightarrow & K^1(BG_N, \mathbb{Q}/n\mathbb{Z}) \rightarrow \text{Tor}K^0(BG_N) \rightarrow 0 \\
0 \uparrow & & \uparrow & & \uparrow \leftarrow \text{Tor}K^0(BG_N) \rightarrow 0, \text{Tor}K^0(BG_{N+L}) \rightarrow 0, \text{Tor}K^0(BG_{N+L}) \rightarrow 0,
\end{array}
\]
where \( L \) is chosen as in (5.4).

\[3\text{Here and below, the } K\text{-groups of classifying spaces are defined as the projective limits over their finite-dimensional approximations.}\]
Thus we obtain the desired isomorphism
\[ \lim_{\leftarrow} K^1 (BG_N, \mathbb{Q}/n\mathbb{Z}) \cong \lim_{\leftarrow} \tilde{K}^0 (BG). \]
□

Finally, the desired element (5.2) is obtained from the isomorphism
\[ K^{*+1} (BG_N, \mathbb{Q}/n\mathbb{Z}) \cong K_*(A_{M_N, \pi}, \mathbb{Q}/n\mathbb{Z}) \]
induced by the inclusion of the ideal \( C_0(BG_N \times (0,1), \text{End}(\pi_N); 1) \). Thus
\[ K^1 (BG, \mathbb{Q}/n\mathbb{Z}) \cong K_0(A_{M_\infty}, \mathbb{Q}/n\mathbb{Z}), \]
and \([\gamma]\) can be viewed as an element of both groups.

Suppose that we are now given a pair \((M, \pi)\). In the remaining part of the section, we assume that \(\pi\) is a principal \(G\)-covering for a finite group \(G\). There exists a mapping \(f : M \to M_N\) that takes the boundary to the base \(EG_N \times \{0\}\) and is equivariant on the boundary (see Proposition 2.1). The inverse image of \([\gamma]\) is denoted by
\[ [\tilde{\pi}] \overset{\text{def}}{=} f^* [\gamma] \in K_0(A_{M, \pi}, \mathbb{Q}/n\mathbb{Z}). \] (5.6)
This element does not depend on the choice of \(f\), since a map into the universal space is unique up to homotopy. Let us obtain a geometric realization of this element. We do this in two steps.

4. A geometric realization of \(K_0(A_{M, \pi}, \mathbb{Q}/n\mathbb{Z})\). The group \(\mathbb{Q}/n\mathbb{Z}\) is the direct limit of the finite groups
\[ \mathbb{Z}_{nN} \subset \mathbb{Q}/n\mathbb{Z}, \quad x \mapsto x/N. \]
Therefore, the \(K\)-group with coefficients in \(\mathbb{Q}/n\mathbb{Z}\) is defined as the direct limit
\[ K_0(A_{M, \pi}, \mathbb{Q}/n\mathbb{Z}) = \lim_{\rightarrow} K_0(A_{M, \pi}, \mathbb{Z}_{nN}). \] (5.7)

Further, the elements of the groups with finite coefficients can be constructed by using the following proposition (cf. [24]).

Proposition 5.1. A triple \((E, F, \sigma)\), where
\[ E \in \text{Vect} (M), \quad F \in \text{Vect} (X), \quad \pi_! (E|_{\partial M}) \overset{\sigma}{\cong} kF, \] (5.8)
and \(\sigma\) is an isomorphism on \(X\), defines an element in \(K_0(A_{M, \pi}, \mathbb{Z}_k)\).

Proof. By analogy with the topological case (e.g., see [24]), the theory with coefficients in \(\mathbb{Z}_k\) is defined in terms of the Moore space \(M_k\) of this group by the formula
\[ K_0(A_{M, \pi}, \mathbb{Z}_k) = K_0 \left( \tilde{C}_0 (M_k, A_{M, \pi}) \right), \] (5.9)
where $\tilde{C}_0(M_k, A_{M,\pi})$ is the algebra of $A_{M,\pi}$-valued functions on the Moore space vanishing at the fixed point.

One can readily generalize Lemma 5.1 to the case of families. More precisely, the same method shows that the group $K_0(\tilde{C}_0(M_k, A_{M,\pi}))$ is isomorphic to the group of stable homotopy classes of triples $(E', F', \sigma')$, where $E', F' \in \text{Vect}(M \times M_k)$ and the isomorphism

$$\sigma' : \pi_1(E'|_{\partial M}) \longrightarrow \pi_1(F'|_{\partial M})$$

is defined over $X \times M_k$.

Let $\varepsilon$ be the line bundle over the Moore space representing the generator $[\varepsilon] - 1 \in \tilde{K}(M_k) \simeq \mathbb{Z}_k$. (Further information about the Moore spaces can be found, e.g., in [24, 26].) Let us also fix a trivialization $\rho : k\varepsilon \to \mathbb{C}^k$.

To the triple $(E, F, \sigma)$ in (5.8), we assign the element

$$[E \otimes \varepsilon, E, \sigma'] \in K_0(\tilde{C}_0(M_k, A_{M,\pi})),$$

where the isomorphism $\sigma'$ is defined as the composition (see [26])

$$\pi_1(E|_{\partial M}) \otimes \varepsilon \xrightarrow{\sigma_\otimes} kF \otimes \varepsilon \simeq F \otimes k\varepsilon \xrightarrow{\otimes 1} F \otimes \mathbb{C}^k \simeq kF \xrightarrow{1} \pi_1(E|_{\partial M}).$$

(5.10)

□

5. A geometric realization of $[\tilde{\pi}1]$ (see (5.1)). For sufficiently large $N$, consider a triple $(N, 1, \alpha)$, where

$$N \in \text{Vect}(M), \quad 1 \in \text{Vect}(X),$$

are trivial vector bundles of the corresponding dimensions and $\pi_1N \simeq \mathbb{C}^nN$ is some trivialization. By Proposition 5.1, this triple defines an element

$$[N, 1, \alpha] \in K_0(A_{M,\pi}, \mathbb{Q}/n\mathbb{Z}).$$

Now, using the diagram (5.5), the reader can verify that this element coincides with $[\tilde{\pi}1]$ if the number $N$ and the trivialization are chosen as follows.

Suppose that the range of the classifying mapping $f : M \longrightarrow M_\infty$ is contained in the skeleton $M_{N'}$, then for $N'$ there exists an $L'$ such that property (5.4) is valid. Now we can choose an $N$ such that the restriction of the direct sum $N\gamma$ of the universal bundle to $BG_{N'+L'}$ is trivial with some trivialization

$$N\gamma \simeq \mathbb{C}^{nN}.$$ 

Finally, over $M$ we choose the induced trivialization

$$\alpha = f^*\alpha'.$$

6. The index defect theorem.
Theorem 5.1. Let \((M, \pi)\) be a manifold with a covering on the boundary corresponding to a free action of a finite group \(G\). Then the diagram

\[
\begin{array}{ccc}
\text{Ell}(M) & \xrightarrow{\chi} & K(T^*M) \\
\sim \text{ind} & & \sim \\
\mathbb{R}/n\mathbb{Z} & \xleftarrow{\langle \cdot, [\pi^1] \rangle}
\end{array}
\]

commutes. Here \(\langle \cdot, \rangle\) is the Poincaré pairing with coefficients,

\[
\langle \cdot, \rangle : K(T^*M) \times K_0(A_{M,\pi}, \mathbb{Q}/n\mathbb{Z}) \to K_0(A_{T^*M,\pi}, \mathbb{Q}/n\mathbb{Z}) \xrightarrow{\text{ind}} \mathbb{Q}/n\mathbb{Z}. \tag{5.11}
\]

Remark 5.1. Theorem 5.1 expresses \(\widetilde{\text{ind}} \text{D} \) in topological terms via the principal symbol. Indeed, by (5.7) and (5.9), the index mapping in (5.11) can be expressed topologically using the index theorem for families (Theorem 3.2).

Proof. The proof of the theorem is essentially analytic in nature. The main idea is to reduce the analytic invariant \(\widetilde{\text{ind}} \text{D} \) to the index with values in \(\mathbb{Q}/n\mathbb{Z}\). Thus, we start by defining the corresponding operators.

1. First, we define elliptic theory \(\text{Ell}(M, \pi, \mathbb{Q}/n\mathbb{Z})\) with coefficients \(\mathbb{Q}/n\mathbb{Z}\). The definition can be given using the direct limit

\[
\text{Ell}(M, \pi, \mathbb{Q}/n\mathbb{Z}) = \lim_{\rightarrow} \text{Ell}(M, \pi, \mathbb{Z}_{nN}), \quad \mathbb{Z}_{nN} \subset \mathbb{Z}_{nNM} \subset \mathbb{Q}/n\mathbb{Z},
\]

of theories with finite coefficients. More precisely, elliptic theory with coefficients in \(\mathbb{Z}_k\) is defined by families of non-local elliptic operators of order one parametrized by the Moore space \(\mathbb{M}_k\) of \(\mathbb{Z}_k\):

\[
\text{Ell}(M, \pi, \mathbb{Z}_k) = \text{Ell}_{\mathbb{M}_k}(M, \pi).
\]

For elliptic theory with coefficients, we refer the reader to [26].

2. Consider the mapping

\[
\text{Ell}(M) \xrightarrow{\Phi} \text{Ell}(M, \pi, \mathbb{Q}/n\mathbb{Z}) \tag{5.12}
\]

that takes an operator \(D\) to the family

\[
D^* \oplus (D \otimes 1_\varepsilon) : C^\infty(M, F \oplus E \otimes \varepsilon) \to C^\infty(M, E \oplus F \otimes \varepsilon)
\]

of first-order elliptic operators on \(M\) parametrized by \(\mathbb{M}_{nN}\) (the number \(N\) will be chosen below). Here \(D^*\) is the adjoint operator, and the operator family obtained by twisting \(D\) with the bundle \(\varepsilon\) is denoted by \(D \otimes 1_\varepsilon\). Consider the direct sum of \(N\) copies of this family. It turns out that if \(N\) is sufficiently large, then this family admits an elliptic boundary condition. Indeed, for \(N\) sufficiently large there exists a trivialization

\[
N\pi_1 \cong \mathbb{C}^n, \tag{5.13}
\]

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since $\pi_1$ is flat. Hence on the base of the covering we have the vector bundle isomorphism

$$\pi_1 (N \, E|_{\partial M}) \simeq \pi_1 N \otimes E_0 \overset{\alpha \otimes 1}{\to} \mathbb{C}^{nN} \otimes E_0$$

and the similar isomorphism

$$\pi_1 (N \, E|_{\partial M}) \otimes \varepsilon \simeq \pi_1 N \otimes E_0 \otimes \varepsilon \overset{\alpha \otimes 1}{\to} \mathbb{C}^{nN} \otimes E_0 \otimes \varepsilon \simeq nN \varepsilon \otimes E_0 \overset{\rho \otimes 1}{\to} \mathbb{C}^{nN} \otimes E_0.$$  

We denote the induced isomorphisms on sections by

$$B_1 = \alpha \otimes 1 : C^\infty (X, \pi_1 (N \, E|_{\partial M})) \to C^\infty (X, \mathbb{C}^{nN} \otimes E_0),$$

$$B_2 = \rho \otimes 1 (\alpha \otimes 1) : C^\infty (X, \pi_1 (N \, E|_{\partial M}) \otimes \varepsilon) \to C^\infty (X, \mathbb{C}^{nN} \otimes E_0).$$

Now we can define the family of non-local boundary value problems (in the sense of Section 1)

$$\left\{ \begin{array}{c} ND^* u = f_1, \\
N (D \otimes 1_\varepsilon) v = f_2, \\
B_1 \beta E u|_{\partial M} + B_2 \beta E v|_{\partial M} = g, \quad g \in C^\infty (X, \mathbb{C}^{nN} \otimes E_0), \end{array} \right. \quad (5.14)$$

which consists of elliptic elements. We define the mapping (5.12) as follows: it takes $D$ to the family of non-local problems (5.14). Note that $\Phi$ depends on the choice of the trivialization (5.13).

3. There is a natural index mapping

$$\text{ind} : \text{Ell} (M, \pi, \mathbb{Q}/n\mathbb{Z}) \to \mathbb{Q}/n\mathbb{Z}$$

that takes an element $[D]$ represented by a family $D$ of elliptic operators parametrized by $\mathbb{M}_{nN}$ to the (reduced) index of the family

$$\text{ind} [D] \overset{\text{def}}{=} \text{ind} D \in \tilde{K} (\mathbb{M}_{nN}) \simeq \mathbb{Z}_{nN} \subset \mathbb{Q}/n\mathbb{Z}.$$

**Lemma 5.3.** The diagram

$$\begin{array}{ccc}
\text{Ell} (\overline{M}^\sigma) & \overset{\Phi}{\to} & \text{Ell} (M, \pi, \mathbb{Q}/n\mathbb{Z}) \\
\text{ind} \downarrow & & \text{ind} \downarrow \\
\mathbb{R}/n\mathbb{Z} & & \mathbb{R}/n\mathbb{Z},
\end{array}$$

where $N$ and the trivialization $\alpha$ in (5.13) are chosen as in Subsec. 5, commutes.

**Proof of the lemma.** The boundary value problem (5.14) is linearly homotopic to the problem

$$\left\{ \begin{array}{c} ND^* u = f_1, \\
N (D \otimes 1_\varepsilon) v = f_2, \\
B_1 \beta E u|_{\partial M} + B_2 \beta E v|_{\partial M} = g, \quad g \in C^\infty (X, \mathbb{C}^{nN} \otimes E_0), \end{array} \right.$$
within the class of elliptic problems. The last formula shows that the index of \( \Phi [D] \) is equal to the sum of the index of the family of spectral problems for \( ND^* \) and \( N (D \otimes 1_\varepsilon) \) and the index of the operator family

\[
N \im \Pi_- (A) \oplus N \im \Pi_+ (A) \otimes \varepsilon \stackrel{B_1 + B_2}{\longrightarrow} C^\infty (X, C^{nN} \otimes E_0)
\]

on the boundary. Note that we specify the self-adjoint operators in the notation of spectral projections. Let us compute the index of the former family on \( X \).

1) There is a decomposition

\[
C^\infty (X, C^{nN} \otimes E_0) \simeq nN \im \Pi_-(A_0) \oplus nN \varepsilon \otimes \im \Pi_+(A_0)
\]

of the target space for the family (5.15). This decomposition is defined as

\[
nN \im \Pi_-(A_0) \oplus nN \varepsilon \otimes \im \Pi_+(A_0) \rightarrow C^\infty (X, C^{nN} \otimes E_0).
\]

Using this isomorphism, we represent the index of (5.15) in the form

\[
= \text{ind} \left( N \im \Pi_+ (\pi_A) \stackrel{\Pi_+(A_0)}{\longrightarrow} nN \im \Pi_+(A_0) \right) ([\varepsilon] - 1) \in \tilde{K} (\mathbb{M}_{nN}).
\]

Finally, we rewrite the index by pushing forward the space \( \im \Pi_+ (A) \) to the base of the covering:

\[
= \text{ind} \left( N \im \Pi_+ (\pi_A) \stackrel{\Pi_+(A_0)}{\longrightarrow} nN \im \Pi_+(A_0) \right) ([\varepsilon] - 1).
\]

The index of the elliptic operator (not a family!) in the last formula can be expressed by the Atiyah–Patodi–Singer formula [23]

\[
\text{ind} \left( N \im \Pi_+ (\pi_A) \stackrel{\Pi_+(A_0)}{\longrightarrow} nN \im \Pi_+(A_0) \right) = N \eta (A) - nN \eta (A_0) + \langle [\sigma (A_0)] , [\pi_1] \rangle,
\]

where the brackets \( \langle , \rangle \) denote the pairing

\[
\langle , , \rangle : K^1 (T^*X) \times K^1 (X, \mathbb{Q}) \longrightarrow \mathbb{Q}
\]

of the difference element \( [\sigma (A_0)] \in K^1 (T^*X) \) of an elliptic self-adjoint operator \( A_0 \) with the element \( [\pi_1] \in K^1 (X, \mathbb{Q}) \) defined by the trivialized flat bundle \( N \pi_1 \) (more about this formula can be found in the book [24]).

2) It turns out that for our choice of the trivialization (5.13) the last term in (5.16) is equal to zero. Indeed, consider the classifying mapping \( f : X \rightarrow BG_{\gamma} \). We can evaluate (5.17) on the classifying space:

\[
\langle [\sigma (A_0)] , [\pi_1] \rangle = \langle f_! [\sigma (A_0)] , [\gamma] \rangle , \quad [\pi_1] = f^* [\gamma] \in K^1 (X, \mathbb{Q}),
\]

where \( [\gamma] \in K^1 (BG_{\gamma}) \otimes \mathbb{Q} \) is the element defined by the trivialized flat bundle \( N \gamma \). The inclusion \( BG_{N'} \subset BG_{N'+L'} \) induces the commutative diagram

\[
\begin{array}{ccc}
K^1 (T^*BG_{N'}) & \times & K^1 (BG_{N'}) \otimes \mathbb{Q} \\
\downarrow & & \uparrow \\
K^1 (T^*BG_{N'+L'}) & \times & K^1 (BG_{N'+L'}) \otimes \mathbb{Q}
\end{array}
\]

30
Using this diagram and (5.4), one can prove the triviality of the pairing (5.18) by a diagram chase argument.

Thus we have reduced $\text{ind}\Phi [D]$ to the desired form

$$\text{ind}\Phi [D] = \widehat{\text{ind}} [D].$$

4. To complete the proof of the theorem, it suffices to show that the value of the Poincaré pairing $\langle [\sigma(D)], [\pi_1]\rangle$ coincides with the index of $\Phi(D)$.

We denote the product by $[\pi_1]$ by $\varphi$:

$$\varphi : K (T^*M^\pi) \longrightarrow K_0 (A_{T^*M,\pi}, \mathbb{Q}/n\mathbb{Z}).$$

**Lemma 5.4.** Under the assumptions of Lemma 5.3, the diagram

$$\begin{array}{ccc}
\text{Ell} (M^\pi) & \xrightarrow{\Phi} & \text{Ell} (M, \pi, \mathbb{Q}/n\mathbb{Z}) \\
\chi \downarrow & & \downarrow \chi' \\
K (T^*M^\pi) & \xrightarrow{\varphi} & K (A_{T^*M,\pi}, \mathbb{Q}/n\mathbb{Z}),
\end{array}$$

where $\chi'$ is induced by the difference constructions for families (see Subsec. 3.3).

**Proof.** Substituting the definitions of $[\sigma(D)]$ and $[\pi_1]$ (according to Subsecs. 4.4 and 4.5) into Eq. (5.1), defining the product, one can show that the desired product $\varphi[\sigma(D)]$ is determined by the family of elliptic symbols that are equal to

$$N\sigma(D) \otimes 1_{\varepsilon} \oplus N\sigma(D)^{-1} \otimes 1$$

far from the boundary. The direct image of the restriction of this symbol to the boundary is equal to

$$N\pi_1 (\sigma(D) \otimes 1_{\varepsilon} \oplus \sigma(D)^{-1} \otimes 1) = (\sigma(D_0) \otimes 1_{N\varepsilon \oplus \pi_1} \oplus \sigma(D_0)^{-1} \otimes 1_{N\pi_1}) \simeq (5.21)$$

In the last equality, we use the isomorphisms $N\pi_1 \simeq \mathbb{C}^n\mathbb{N}, \quad \mathbb{C}^n\varepsilon \simeq \mathbb{C}^n\mathbb{N}$. The symbol is extended to a neighbourhood of the boundary using the homotopy of the direct sum $\sigma(D_0) \oplus \sigma(D_0)^{-1}$ to the identity.

It remains to prove that the difference elements for the principal symbol of the family of boundary value problems (5.14) and the symbol defined by (5.20), (5.21) coincide. Indeed, this equality is obvious far from the boundary, since the only difference here is in the components $\sigma(D^*)$ and $\sigma(D)^{-1}$. These components are joined by the standard homotopy

$$\sigma(D^*) [\sigma(D) \sigma(D^*)]^{-s}, \quad s \in [0, 1].$$

The reader can also prove the equality near the boundary using the formulae for order reduction given in Remark 2.2.

By combining Lemmata 5.4 and 5.3 we complete the proof of the theorem. □
6 Applications

1. Theorem 5.1 enables one to express the fractional part of the $\eta$-invariant in the following situation.

Let $M$ be an even-dimensional spin manifold with boundary represented as the total space of a covering such that the spin structure on the boundary is the pullback of a spin structure on the base. Let us also fix an $E \in \text{Vect}(M)$ that is also pulled back from the base near the boundary: $E|_{\partial M} \simeq \pi^*E_0$. We choose a metric on $M$ that is a product metric induced by a metric on the base near the boundary. Finally, we choose a similar connection in $E$.

Proposition 6.1. The Dirac operator $D_M$ on $M$ with coefficients in $E$ satisfies the assumptions of Theorem 5.1 and the fractional part of the $\eta$-invariant is equal to

$$\{\eta(D_X)\} = \frac{1}{n} \left( \int_M \hat{A}(M) \, \text{ch}_E - \left( [\sigma(D_M)], [\pi_1] \right) \right) \in \mathbb{R}/\mathbb{Z},$$

where $D_X$ is the self-adjoint Dirac operator on $X$ with coefficients in $E_0$.

Proof. The formula follows from Theorem 5.1 if we decompose the index of the spectral problem using the Atiyah–Patodi–Singer formula

$$\text{ind} (D_M, \Pi_+) = \int_M \hat{A}(M) \, \text{ch}_E - \eta(D_{\partial M}).$$

2. The invariant $\widetilde{\text{ind}}$ can be effectively computed via Lefschetz theory. Suppose that $\pi$ is regular, i.e., the boundary is a principal $G$-bundle for a finite group $G$. Let $D$ be a $G$-invariant elliptic differential operator of order one on $M$. For $g \in G$, let $L(D, g) \in \mathbb{C}$ be the usual contribution to the Lefschetz formula (see [28]) of the fixed point set of the diffeomorphism $g : M \to M$.

Proposition 6.2. One has

$$\widetilde{\text{ind}} D \equiv - \sum_{g \neq e} L(D, g) \pmod{n}. \quad (6.1)$$

Proof. Consider the equivariant index $\text{ind}_g (D, \Pi_+)$ of the Atiyah–Patodi–Singer problem and the equivariant $\eta$-function (see [28]) of the tangential operator $A$ on the boundary.

Denote by $(D, \Pi_+)^G$ and $A^G$ the restrictions of the corresponding operators to the subspaces of $G$-invariant sections. Clearly, $A^G$ is isomorphic to $A_0$ on $X$. On the other hand, one can express the usual invariants in terms of their equivariant counterparts:

$$\text{ind} (D, \Pi_+)^G = \frac{1}{|G|} \sum_{g \in G} \text{ind}_g (D, \Pi_+) \quad \eta (A^G) = \frac{1}{|G|} \sum_{g \in G} \eta (A, g).$$
These expression follow from elementary character theory. Using them, we write

\[ \tilde{\text{ind}} D = \text{ind}_e (D, \Pi_+) - \sum_{g \neq e} \eta (A, g) . \]

Let us substitute the expression for the \( \eta \)-invariant given by the equivariant Atiyah–Patodi–Singer formula (see [28])

\[ -\eta (A, g) = \text{ind}_g (D, \Pi_+) - L (D, g) \]

into this formula. This gives the desired congruence (6.1):

\[ \tilde{\text{ind}} D = |G| \text{ind} (D, \Pi_+)^G - \sum_{g \neq e} L (D, g) . \]

\[ \square \]

### 7 Poincaré isomorphisms

1. **A closed smooth manifold.** It is well known (see [29, 3] or the monograph [30]) that elliptic operators of order zero on a compact closed manifold define elements in \( K \)-theory:

\[ [\sigma (D)] \in K^* (T^* M) , \quad [D] \in K^* (C (M)) \equiv K^*_e (M) . \]

The latter group is the analytic \( K \)-homology group, and the grading is odd for self-adjoint operators and even otherwise. The first element is the difference element of the operator. To define the second element, we recall that an elliptic operator \( D \) of order zero is a Fredholm operator

\[ D : L^2 (M, E) \longrightarrow L^2 (M, F) , \]

where both \( L^2 \)-spaces are modules over \( C (M) \) (the module structure is given by the pointwise product of functions). In addition, \( D \) commutes with the module structure up to compact operators. Thus, for a self-adjoint \( D \) (of course in this case the bundles coincide) the pair \( (L^2 (M, E), D) \) is an element

\[ [D] \in K^1 (C (M)) . \]

For a nonself-adjoint \( D \), we consider a self-adjoint matrix operator

\[ T = \begin{pmatrix} 0 & D^* \\ D & 0 \end{pmatrix} \]

in the naturally \( \mathbb{Z}_2 \)-graded \( C (M) \)-module \( L^2 (M, E) \oplus L^2 (M, F) \). The operator \( T \) is odd with respect to the grading. Hence it defines a \( K \)-theory element, denoted by

\[ [D] \in K^0 (C (M)) . \]
2. Manifold with boundary. On the other hand, elliptic operators of order one on a manifold with non-empty boundary define similar elements

\[ [\sigma(D)] \in K^i(T^*M), \quad [D] \in K_i(M\setminus\partial M). \]

The former is the Atiyah–Singer difference element, and the latter is defined as follows. Consider an embedding

\[ M \subset \tilde{M} \]

of \( M \) in some closed manifold \( \tilde{M} \) of the same dimension (e.g., the double \( 2M \)). Let \( \tilde{D} \) be an arbitrary extension of \( D \) to \( \tilde{M} \). On \( \tilde{M} \), we consider the zero-order operator

\[ \tilde{F} = \left(1 + \tilde{D}^*\tilde{D}\right)^{-1/2} \tilde{D}. \]

We define the restriction of this operator to \( M \) as the bounded operator

\[ F = i^*\tilde{F}i_* : L^2(M, E) \rightarrow L^2(M, F), \tag{7.2} \]

where \( i_* : L^2(M) \rightarrow L^2(\tilde{M}) \) is the extension by zero and \( i^* : L^2(\tilde{M}) \rightarrow L^2(M) \) is the restriction operator.

For a symmetric \( D \), we find that \( F \) satisfies

\[ F - F^* \in \mathcal{K}, \quad f(F^2 - 1) \in \mathcal{K}, \quad [F, f] \in \mathcal{K} \]

for functions \( f \in C_0(M\setminus\partial M) \) vanishing on the boundary. Here \( \mathcal{K} \) is the ideal of compact operators. These relations show that \( F \) defines an element of \( K^1(C_0(M\setminus\partial M)) \) (see [31]).

If \( D \) is nonself-adjoint, then one considers the matrix as in Eq. (7.1). This defines an element in \( K^0(C_0(M\setminus\partial M)) \).

The mappings

\[ K^* (T^*(M\setminus\partial M)) \rightarrow K^* (M), \]

\[ K^* (T^*M) \rightarrow K^* (M\setminus\partial M), \]

\[ [\sigma(D)] \rightarrow [D], \]

which take symbols to operators, define Poincaré isomorphisms on a smooth manifold \( M \) with boundary (e.g., see [8]). The top mapping is defined in terms of elliptic operators of order zero on \( M \) that are induced by vector bundle isomorphisms near the boundary.

3. Manifolds with singularities. An elliptic non-local zero-order operator \( D \) defines elements

\[ [\sigma(D)] \in K^*_+(A_{T^*M,\pi}), [D] \in K^* \left(C_0(\overline{M}^\pi)\right) \simeq K^*_+(\overline{M}^\pi). \]

The first is the difference element defined in Section 2. To define the second element, we note that a non-local elliptic operator \( D \) of order zero does not almost commute with the entire algebra \( C(M) \) but only with the functions pulled back from the quotient space \( \overline{M}^\pi \). This leads to a smaller algebra.
On the other hand, the operators of Sections 4 and 5 define similar elements

\[ [\sigma(D)] \in K^* (T^* M^\pi), [D] \in K_* (A_{M, \pi}). \]

In this case, the corresponding operators (7.2), on the contrary, almost commute with functions \( C_0 (M \setminus \partial M) \) as well as with the elements of the algebra \( A_{M, \pi} \).

**Theorem 7.1.** For an arbitrary manifold with a covering on the boundary \((M, \pi)\), the following Poincaré isomorphisms are valid:

\[
\begin{align*}
K_* (A_{T^* M, \pi}) & \rightarrow K_* (\overline{M}^\pi), \\
K_* (T^* M^\pi) & \rightarrow K_* (A_{M, \pi}), \\
[\sigma(D)] & \mapsto [D].
\end{align*}
\]

**Proof.** 1) Let us prove the latter isomorphism. Consider the ideal

\[ I = C_0 (T^* (X \times (0, 1)), \text{End} p^* \pi_1) \subset A_{T^* M, \pi} \]

with the quotient \( A_{T^* M, \pi} / I \simeq C_0 (T^* M) \). The long exact sequence of the pair can be written as

\[
\rightarrow K (T^* X) \xrightarrow{\partial} K_0 (A_{T^* M, \pi}) \rightarrow K (T^* M) \rightarrow K^1 (T^* X) \rightarrow \ldots \quad (7.3)
\]

Here we have taken into account the isomorphism \( K_* (C_0 (Y, \text{End} G)) \simeq K_* (C_0 (Y)) \simeq K_* (Y) \) for a vector bundle \( G \in \text{Vect} (Y) \).

Consider the commutative diagram

\[
\begin{array}{cccccc}
\rightarrow & K^0 (T^* X) & \rightarrow & K^0 (A_{T^* M, \pi}) & \rightarrow & K^0 (T^* M) & \rightarrow & K^1 (T^* X) & \rightarrow & \ldots \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
\rightarrow & K_0 (X) & \rightarrow & K_0 (\overline{M}^\pi) & \rightarrow & K_0 (M, \partial M) & \rightarrow & K_1 (X) & \rightarrow & \ldots
\end{array}
\]

Here the lower sequence is the exact sequence of the pair \( X \subset \overline{M}^\pi \) in \( K \)-homology. The vertical mappings of the diagram (except for the second one) are isomorphisms (see [32] [3]). Thus, using the 5-lemma, we find that the middle mapping

\[ K_* (A_{T^* M, \pi}) \rightarrow K_* (\overline{M}^\pi) \]

is also an isomorphism.

2) In the second case, the proof follows the same scheme, but one uses the diagram

\[
\begin{array}{cccccc}
\leftarrow & K^1 (T^* X) & \leftarrow & K^0 (\overline{T^* M}^\pi) & \leftarrow & K^0 (T^* (M \setminus \partial M)) & \leftarrow & K^0 (T^* X) & \leftarrow & \ldots \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
\leftarrow & K_1 (X) & \leftarrow & K_0 (A_{M, \pi}) & \leftarrow & K_0 (M) & \leftarrow & K_0 (X) & \leftarrow & \ldots
\end{array}
\]

The upper row corresponds to the pair \( \mathbb{R} \times T^* X \subset \overline{T^* M}^\pi \).

The proof of the theorem is complete. \qed
8 Poincaré duality

An analogue of the pairing for the groups $K^0(T^*\overline{M})$ and $K_0(A_{M,\pi})$ in Section 5 is also valid for the odd groups. The definition is left to the reader.

**Theorem 8.1.** On a manifold $M$ with covering $\pi$ on the boundary, the pairings

$$K^i(T^*\overline{M}) \times K_i(A_{M,\pi}) \longrightarrow \mathbb{Z}, \quad i = 1, 2,$$

(8.1)

are non-degenerate on the free parts of the groups.

**Proof.** Fixing the first argument of the pairing, we obtain a mapping

$$K^i(T^*\overline{M}) \otimes \mathbb{Q} \longrightarrow K'_i(A_{M,\pi}),$$

where for brevity we write $G' = \text{Hom}(G, \mathbb{Q})$. This mapping is part of the commutative diagram

$$
\begin{array}{c}
K^1(T^*X) \otimes \mathbb{Q} & \leftarrow & K^0(T^*\overline{M}) \otimes \mathbb{Q} & \leftarrow & K^0(T^*(M\setminus\partial M)) \otimes \mathbb{Q} & \leftarrow & K^0(T^*X) \otimes \mathbb{Q} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
K^i(X) & \leftarrow & K'_i(A_{M,\pi}) & \leftarrow & K^0(M) & \leftarrow & K^0(X) \\
\end{array}
$$

Here the vertical mappings, except for the second one, are isomorphisms (by virtue of Poincaré duality on a closed manifold and on a manifold with boundary). Thus, by the 5-lemma, the second mapping is an isomorphism. Hence the pairing (8.1) is non-degenerate in the second variable.

The non-degeneracy with respect to the first argument can be proved in a similar way.

\[\Box\]

By way of example, consider $M$ with a $\text{spin}^c$-structure that on the boundary is induced by a $\text{spin}^c$-structure on the base $X$ of the covering $\pi$. Then the group $K^*(T^*\overline{M})$ is a free $K^{*+n}(\overline{M})$-module with one generator (where $n = \dim M$); as a generator one can take the difference construction

$$[\sigma(D)] \in K^n(T^*\overline{M})$$

of the principal symbol of the Dirac operator on $M$ (this can be proved by analogy with the usual case of closed manifolds; e.g., see [33]). Consequently, one can define the Poincaré duality pairing

$$K^{*+n}(\overline{M}) \times K_*(A_{M,\pi}) \longrightarrow \mathbb{Z}$$

as the composition with $K^{*+n}(\overline{M}) \rightarrow K(T^*\overline{M})$. The above theorem shows that this pairing is non-degenerate on the free parts of the groups.
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