D-Instantons and Universal Hypermultiplet

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Abstract

Quantum non-perturbative geometry of the universal hypermultiplet is investigated. We consider the simple case when the D-instantons, originating from the Calabi-Yau wrapped D2-branes, preserve a $U(1) \times U(1)$ symmetry of the universal hypermultiplet moduli space. The cluster decomposition of D-instantons is proved to be valid in a curved spacetime. We find an $SL(2,\mathbb{Z})$ duality-invariant quaternionic solution to the effective NLSM metric of the universal hypermultiplet, which is governed by a modular-invariant function. This function appears to be the same function found by Green and Gutperle, and describing the modular invariant completion of the $R^4$ term by the D-instanton effects in the type-II superstring/M-theory. We argue that our solution interpolates between the perturbative (large CY volume) region and the superconformal (Landau-Ginzburg) region in the universal hypermultiplet moduli space. We also calculate a non-perturbative scalar potential in the hyper-Kähler limit, when an abelian isometry of the universal hypermultiplet moduli space is gauged in the presence of D-instantons.

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1 Introduction

Instanton corrections in compactified M-theory/superstrings are crucial for solving the fundamental problems of vacuum degeneracy and supersymmetry breaking. Some instanton corrections in the type-IIA superstring theory compactified on a Calabi-Yau (CY) threefold arise due to the Euclidean D2-branes wrapped about the CY special (supersymmetric) three-cycles \[1\]. Being BPS solutions to the Euclidean ten-dimensional (10d) supergravity equations of motion, these wrapped branes are localized in four uncompactified spacetime dimensions and thus can be identified with instantons. They are called D-instantons. The D-instanton action is essentially given by the volume of the supersymmetric 3-cycle on which a D2-brane is wrapped. The supersymmetric cycles (by definition) minimize volume in their homology class.

At the level of the Low-Energy Effective Action (LEEA), the effective field theory is given by the four-dimensional (4d), N=2 supergravity with some N=2 vector- and hyper-multiplets, whose structure is dictated by the CY cohomology, and whose moduli spaces are independent. The hypermultiplet sector of the LEEA is described by a 4d, N=2 Non-Linear Sigma-Model (NLSM) with a quaternionic metric in the NLSM target (moduli) space \[2\]. Any CY compactification gives rise to the so-called Universal Hypermultiplet (UH) in 4d, which contains a dilaton amongst its field components.

When the type-IIA supergravity 3-form has a non-vanishing CY-valued expectation value, the UH becomes electrically charged. This implies that an abelian isometry of the NLSM target (= UH moduli) space is gauged, while the UH scalar potential is non-trivial \[3, 4\].

It is of considerable interest to calculate the UH non-perturbative NLSM metric and the associated scalar potential, by including D-instanton corrections. The qualitative analysis was initiated by Witten \[5\] who showed that the D-instanton quantum corrections are given by powers of \(e^{-1/g_{\text{string}}}\), where \(g_{\text{string}}\) is the type-II superstring coupling constant \[5\]. The D-instanton induced interactions in the LEEA of ten-dimensional type-II superstrings, and a modular invariant completion of the \(R^4\)-term were found by Green and Gutperle \[6\]. These \(R^4\)-terms terms apparently arise from a one-loop calculation in eleven-dimensional M-theory \[6\]. The D-brane contributions to the \(R^4\)-couplings in any toroidal compactification of type-II superstrings, as well as their relation to the Eisenstein series (in eight and seven spacetime dimensions), were investigated by Pioline and Kiritsis \[8\]. The CY wrapped D-branes from the mathematical viewpoint were reviewed by Douglas \[9\].
Some explicit D-instanton corrections in the universal sector of the CY compactified type-II superstrings were calculated by Ooguri and Vafa \cite{10}, though in the hyper-Kähler limit when the spacetime gravity is switched off. The gravitational corrections are expected to be equally important at strong string coupling, while the UH sector is a good place for studying them. In particular, Strominger \cite{11} proved the absence of perturbative superstring corrections to the local UH metric provided that the Peccei-Quinn type isometries of the classical UH metric, described by the symmetric space $SU(2,1)/SU(2) \times U(1)$ \cite{12}, are preserved. In our earlier paper \cite{13} we proposed the procedure for a derivation of the non-perturbative UH metric in a curved spacetime. Unfortunately, no explicit quaternionic solutions, describing D-instantons, were found in ref. \cite{13}. In this paper we give such solutions by using some recent advances in differential geometry \cite{14}.

We also turn to the gauged version of the universal hypermultiplet NLSM, by gauging one of its abelian isometries preserved by D-instantons. This gives rise to the non-perturbative scalar potential whose minima determine the ‘true’ vacua in our toy model comprising the UH coupled to the single N=2 vector multiplet gauging the UH abelian isometry. As is well-known (see, e.g., ref. \cite{4}), gauging the classical UH geometry gives rise to the dilaton potential whose minima occur outside of the region where the string perturbation theory applies. However, this potential with the runaway behaviour is not protected against instanton corrections, while it is reasonable to gauge only those NLSM isometries that are not broken after the D-instanton corrections are included. Because of the brane charge quantization, the classical continuous symmetries of the UH metric are generically broken by the wrapped D2-branes and the solitonic 5-branes wrapped about the entire CY \cite{1}. However, when merely D-instantons are taken into account, a continuous abelian symmetry of the UH moduli space may survive, while it also makes actual calculations possible \cite{15}.

The paper is organized as follows: in sect. 2 we recall a few basic facts about the type-II string dilaton and the 4d NLSM it belongs to. In sect. 3 we discuss all possible deformations of this NLSM due to D-instantons under the condition of unbroken 4d, N=2 local supersymmetry, and give some explicit solutions. An $SL(2,\mathbb{Z})$ modular invariant quaternionic metric solution, governed by an order-3/2 Eisenstein series, is given in sect. 3 too. Sect. 4 is devoted to the gauged version of the UH and its scalar potential in the presence of the D-instanton corrected UH metric. Our conclusion is given in the Abstract. We made all efforts to keep our presentation as simple as possible.
2 Dilaton and NLSM

In all four-dimensional superstring theories a dilaton scalar $\varphi$ is accompanied by an axion pseudo-scalar $R$ belonging to the same scalar supermultiplet. In the (classical) supergravity approximation, their LEEA (or kinetic terms) are given by the NLSM whose structure is entirely fixed by duality: the NLSM target space is given by the two-dimensional non-compact homogeneous space $SL(2, \mathbb{R})/U(1)$. In the full ‘superstring theory’ (including branes) the continuous symmetry $SL(2, \mathbb{R}) \cong SO(2, 1) \cong SU(1, 1)$ is generically broken to its discrete subgroup $SL(2, \mathbb{Z})$ [16], whereas the local NLSM metric may receive some non-perturbative (instanton) corrections [1, 5, 10, 11, 15].

The $SL(2, \mathbb{R})/U(1)$-based NLSM can be parametrized in terms of a single complex scalar,

$$A \equiv A_1 + iA_2 = R + ie^{-\varphi},$$

subject to the $SL(2, \mathbb{R})$ duality transformations

$$A \rightarrow A' = \frac{aA + b}{cA + d}, \quad \text{where} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{R}),$$

with four real parameters $(a, b, c, d)$ obeying the condition $ad - bc = 1$. The $SL(2, \mathbb{R})$ NLSM Lagrangian in the parametrization (2.1) is given by

$$\kappa^2 L(A, \bar{A}) = \frac{1}{(A - \bar{A})^2} \partial^\mu \bar{A} \partial_\mu A.$$  \hfill (2.3)

We assume that our scalars are dimensionless. The dimensional coupling constant $\kappa$ of the UH NLSM is proportional to the gravitational coupling constant. We assume that $\kappa^2 = 1$ for notational simplicity.

It is easy to check that the NLSM metric defined by eq. (2.3) is Kähler, with a Kähler potential

$$K(A, \bar{A}) = \log(A - \bar{A}).$$ \hfill (2.4)

The $SL(2, \mathbb{R})$ transformations (2.2) are generated by constant shifts of the axion (T-duality) with

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in SL(2, \mathbb{R})$$

and the S-duality transformation ($e^{-\varphi} \rightarrow e^{+\varphi}$) with

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in SL(2, \mathbb{R}).$$ \hfill (2.6)
It is worth mentioning that the Kähler potential is defined modulo Kähler gauge transformations,
\[ K(A, \bar{A}) \rightarrow K(A, \bar{A}) + f(A) + \bar{f}(\bar{A}) \, , \quad (2.7) \]
with arbitrary (locally holomorphic) functions \( f(A) \). After the field redefinition
\[ S = i\bar{A} \equiv e^{-2\phi} + i2D \, , \quad (2.8) \]
in terms of a dilaton \( \phi \) and an axion \( D \), the Kähler potential (2.4) takes the form
\[ K(S, \bar{S}) = \log(S + \bar{S}) \, . \quad (2.9) \]
This parametrization was used, for example, in refs. [1, 11, 15].

To connect the Kähler potential (2.9) with the standard (Fubuni-Study) potential used in the mathematical literature, let’s make yet another field redefinition,
\[ S = \frac{1 - z}{1 + \bar{z}} \, . \quad (2.10) \]
The new Kähler potential \( K(z, \bar{z}) \) takes the dual Fubini-Study form indeed,
\[ K(z, \bar{z}) = \log(1 - |z|^2) \, . \quad (2.11) \]
The corresponding NLSM Lagrangian is
\[ -\mathcal{L}(\phi, D) = (\partial_\mu \phi)^2 + e^{4\phi}(\partial_\mu D)^2 \, , \quad (2.12) \]
or
\[ -4\mathcal{L}(\rho, t) = \frac{1}{\rho^2} \left[ (\partial_\mu \rho)^2 + (\partial_\mu t)^2 \right] \, , \quad (2.13) \]
where we have introduced the new variables
\[ \rho = e^{-2\phi} \quad \text{and} \quad t = 2D \, . \quad (2.14) \]
The metric of the NLSM (2.13) is conformally flat, it has a negative scalar curvature and a manifest isometry, due to the \( t \)-independence of all its components.

The (complex) one-dimensional Kähler potential (2.11) has a natural (Kähler and dual Fubuni-Study type) extension to two (complex) dimensions,
\[ K(z_1, z_2, \bar{z}_1, \bar{z}_2) = \log(1 - |z_1|^2 - |z_2|^2) \, , \quad (2.15) \]
where \((z_1, z_2) \in \mathbb{C}^2\) are on equal footing inside the ball \( B^4 : |z_1|^2 + |z_2|^2 < 1 \). The Kähler potential (2.15) defines the so-called Bergmann metric (in the mathematical literature):
\[ ds^2 = \frac{dz_1d\bar{z}_1 + dz_2d\bar{z}_2}{1 - |z_1|^2 - |z_2|^2} + \frac{(\bar{z}_1dz_1 + \bar{z}_2dz_2)(z_1d\bar{z}_1 + z_2d\bar{z}_2)}{(1 - |z_1|^2 - |z_2|^2)^2} \, . \quad (2.16) \]
in the open ball $B^4$. Being equipped with the Bergmann metric, the open ball $B^4$ is equivalent to the symmetric quaternionic space $SU(2,1)/U(2)$ [17]. The relation to the UH parametrization $(\phi, D, C, \bar{C})$ used in the physical literature [1, 12, 15] is given by

$$z_1 = \frac{1 - S}{1 + S}, \quad z_2 = \frac{2C}{1 + S},$$

(2.17)

where the new complex variable $C$ can be identified with the RR-scalar of the UH, whereas another complex scalar $S$ is now given by (cf. eq. (2.8))

$$S = e^{-2\phi} + i2D + C\bar{C}.$$  

(2.18)

The two complex scalars $(S, C)$ represent all the physical scalars of the universal hypermultiplet that also has a Dirac hyperino as their fermionic superpartner. The UH metric defined by eq. (2.16) is Kähler, with a Kähler potential

$$K(S, \bar{S}, C, \bar{C}) = \log (S + \bar{S} - 2CC).$$

(2.19)

The corresponding (bosonic part of) NLSM Lagrangian of the UH in terms of the scalar fields $(\phi, D, C, \bar{C})$ reads

$$-L_{FS} = (\partial_{\mu} \phi)^2 + e^{2\phi} |\partial_{\mu} C|^2 + e^{4\phi}(\partial_{\mu} D + \frac{i}{2} \bar{C} \partial_{\mu} C)^2.$$  

(2.20)

This NLSM metric is diffeomorphism-equivalent to the quaternionic Bergmann metric on $SU(2,1)/U(2)$ by our construction. At the same time, eq. (2.20) coincides with the so-called Ferrara-Sabharwal (FS) NLSM (in the physical literature) that was derived [12] by compactifying the 10d type-IIA supergravity on a CY threefold in the universal (UH) sector down to four spacetime dimensions, $\mu = 0, 1, 2, 3$. This means that we can identify our field $\phi$ with the dilaton used in refs. [12, 11]. We conclude that the FS metric (2.20) is completely determined by the duality symmetries of $SU(2,1)/SU(2) \times U(1)$ [14]. The FS metric can be trusted as long as the string coupling is not strong, $g_{\text{string}} = e^{\langle \phi \rangle} = \langle 1/\sqrt{\rho} \rangle$, i.e. for large $\rho > 0$. The variable $\rho$ has the physical meaning of the CY space volume — see eq. (2.14) and ref. [11].

### 3 D-instantons and quaternionic geometry

Quantum non-perturbative corrections generically break all the continuous $SU(2,1)$ symmetries of the UH classical NLSM down to a discrete subgroup because of charge quantization, even if local N=2 supersymmetry in 4d remains unbroken [14]. Nevertheless, if we restrict ourselves to the special situations when some of the abelian
symmetries of the UH moduli space remain unbroken, actual calculations of instanton corrections become possible. As was demonstrated by Strominger [11], there is no non-trivial quaternionic deformation of the classical FS metric within the super-string perturbation theory when the Peccei-Quinn-type symmetries (with three real parameters $(\alpha, \beta, \gamma)$),

$$
D \rightarrow D + \alpha \ , \ C \rightarrow C + \gamma - i\beta \ , \ S \rightarrow S + 2(\gamma + i\beta)C + \gamma^2 + \beta^2 ,
$$

(3.1)

remain unbroken. However, when some of these Peccei-Quinn-type symmetries (e.g., the one associated with shifts of the $C$-field) are broken, a calculation of the D-instanton contributions is possible indeed [15]. In this paper we assume that the abelian isometry associated with constant shifts of the axionic $D$-field is preserved, as well as the $U_C(1)$ duality rotations of the RR-type $C$-field,

$$
D \rightarrow D + \alpha \ , \ C \rightarrow e^{i\delta}C , \text{ where } \delta \in [0, 2\pi] .
$$

(3.2)

The isometries (3.2) can hold in the presence of D-instantons [10, 13]. Our considerations in this paper are entirely local, so that in what follows the D-instanton corrected UH metric is assumed to be quaternionic (as long as local N=2 supersymmetry is preserved) with a single $U(1)$ or $U_D(1) \times U_C(1)$ (torus) isometry. The problem now amounts to a derivation of non-trivial quaternionic deformations of the Bergmann (or FS) metric, which can be physically interpreted as the D-instanton contributions, subject to the given abelian isometries.

A generic quaternionic manifold admits three independent almost complex structures $(\tilde{J}_A)_{ab}^c$, where $A = 1, 2, 3$ and $a, b = 1, 2, 3, 4$. Unlike the hyper-Kähler manifolds, the quaternionic complex structures are not covariantly constant, i.e. they are not integrable to some complex structures, due to a non-vanishing NLSM torsion. This torsion is induced by 4d, N=2 supergravity because the quaternionic condition on the hypermultiplet NLSM target space metric is the direct consequence of local N=2 supersymmetry in four spacetime dimensions [2]. As regards real four-dimensional quaternionic manifolds (relevant to the UH), they all have Einstein-Weyl geometry of negative scalar curvature [2, 17],

$$
W_{abcd} = 0 , \quad R_{ab} = -\frac{\Lambda}{2}g_{ab} ,
$$

(3.3)

where $W_{abcd}$ is the Weyl tensor, $R_{ab}$ is the Ricci tensor of the metric $g_{ab}$, and the constant $\Lambda > 0$ is proportional to the gravitational coupling constant. The precise value of the ‘cosmological constant’ $\Lambda$ in our notation is fixed in eq. (3.13).

Since we assume that the UH quaternionic metric has at least one abelian isometry, a good starting point is the Tod theorem [18] applicable to any Einstein-Weyl metric
of a non-vanishing scalar curvature with a Killing vector $\partial_t$. According to ref. [18], there exists a parametrization $(t, \rho, \mu, \nu)$ in which such metric takes the form
\[
\mathrm{ds}^2_{\text{Tod}} = \frac{1}{\rho^2} \left\{ \frac{1}{P} (dt + \hat{\Theta})^2 + P \left[ e^u (d\mu^2 + d\nu^2) + d\rho^2 \right] \right\} ,
\] (3.4)
in terms of two potentials, $P$ and $u$, and the one-form $\hat{\Theta}$ in three dimensions $(\rho, \mu, \nu)$. The first potential $P(\rho, \mu, \nu)$ is fixed in terms of the second potential $u$ as
\[
P = \frac{3}{2\Lambda} (\rho \partial_\rho u - 2) .
\] (3.5a)
The potential $u(\rho, \mu, \nu)$ itself obeys the 3d non-linear equation
\[
(\partial_\mu^2 + \partial_\nu^2)u + \partial_\rho^2(e^u) = 0 ,
\] (3.5b)
known as the (integrable) $SU(\infty)$ or 3d Toda equation, whereas the one-form $\hat{\Theta}$ satisfies the linear differential equation
\[
- d \wedge \hat{\Theta} = (\partial_\nu P) d\mu \wedge d\rho + (\partial_\mu P) d\rho \wedge d\nu + \partial_\rho (P e^u) d\nu \wedge d\mu .
\] (3.5c)
Some comments are in order. Given an isometry of the quaternionic metric $g_{ab}$ with a Killing vector $K^a$,
\[
K^{a;b} + K^{b;a} = 0 , \quad K^2 \equiv g_{ab} K^a K^b \neq 0 ,
\] (3.6)
we can always choose some adapted coordinates, with all the metric components being independent upon one of the coordinates $(t)$, as in eq. (3.4). We can then plug the Tod Ansatz (3.4) into the Einstein-Weyl equations (3.3). It follows [18] that this precisely amounts to the equations (3.5). The proof is straightforward, e.g. by the use of Mathematica.

It is worth mentioning that after the conformal rescaling
\[
g_{ab} \rightarrow \rho^2 g_{ab}
\] (3.7)
a generic Einstein-Weyl metric of the form (3.4) becomes Kähler with the vanishing scalar curvature [19]. After this conformal rescaling the Tod Ansatz (3.4) precisely takes the form of the standard (LeBrun) Ansatz for scalar-flat Kähler metrics [20],
\[
\mathrm{ds}^2_{\text{LeBrun}} = \frac{1}{P} (dt + \hat{\Theta})^2 + P \left[ e^u (d\mu^2 + d\nu^2) + d\rho^2 \right] ,
\] (3.8)
whose potential $u$ still satisfies the 3d Toda equation (3.5b), whereas the potential $P$ is a solution to
\[
(\partial_\mu^2 + \partial_\nu^2)P + \partial_\rho^2(e^u P) = 0 .
\] (3.9)
This equation is nothing but the integrability condition for eq. (3.5c) that holds too.

According to LeBrun [20], a scalar-flat Kähler metric is \textit{hyper}-Kähler if and only if

\[ P \propto \partial_{\rho} u \quad . \]  

(3.10)

Given eq. (3.10), the LeBrun Ansatz reduces to the Boyer-Finley Ansatz [21] for a four-dimensional hyper-Kähler metric with a rotational isometry [21], or to the Gibbons-Hawking Ansatz [22] in the case of a translational (or tri-holomorphic) isometry that essentially implies \( u = 0 \) in addition. Both Ansätze are well known in general relativity (see, e.g., ref. [23] for a review). In particular, exact solutions to the Boyer-Finley Ansatz are governed by the same 3d Toda equation, whereas exact solutions to the Gibbons-Hawking Ansatz \[ ds_{\text{GH}}^2 = \frac{1}{P}(dt + \hat{\Theta})^2 + P(d\mu^2 + dv^2 + d\rho^2) \]  

(3.11)

are governed by the \textit{linear} equations, \( (\partial_{\mu}^2 + \partial_{\nu}^2 + \partial_{\rho}^2) P = 0 \) and \( \vec{\nabla} P + \vec{\nabla} \times \vec{\Theta} = 0 \), whose solutions are given by harmonic functions. Given another commuting isometry, each of such \( U(1) \times U(1) \)-invariant hyper-Kähler metrics is described by a harmonic function depending upon two variables, like in ref. [10].

The hyper-Kähler geometry arises in the limit when the spacetime N=2 supergravity decouples, because \textit{any} 4d NLSM with rigid N=2 supersymmetry has a hyper-Kähler metric [24]. The existence of such approximation is dependent upon the validity of eq. (3.10). Otherwise, the hyper-Kähler limit may not exist. Given a \( U(1) \times U(1) \) isometry of a hyper-Kähler metric, the existence of a translational (i.e. tri-holomorphic) isometry does not pose a problem, since there always exists a linear combination of two commuting abelian isometries that is tri-holomorphic [25]. Some explicit examples of the correspondence between four-dimensional hyper-Kähler and quaternionic metrics were derived in ref. [26] from harmonic superspace.

We are now in a position to rewrite the classical UH metric (2.20) into the Tod form (3.4) by using \textit{the same} coordinates as in eq. (2.20). We find

\[ P = 1 \ , \ e^a = \rho \ , \ \text{and} \ d \wedge \hat{\Theta} = d\nu \wedge d\mu \quad , \]  

(3.12)

which are all agree with eqs. (3.5a), (3.5b) and (3.5c), respectively. Eq. (3.5a) also implies that

\[ \Lambda = 3 \quad . \]  

(3.13)

The classical UH metric does not have a hyper-Kähler limit because \( \partial_{\rho} u = 1/\rho \) is not proportional to \( P = 1 \), so that eq. (3.10) is not valid. This conclusion is confirmed by direct checking that \( \rho^2 ds_{\text{FS}}^2 \) is Kähler and scalar-flat, but it is \textit{not} Ricci-flat,
and, hence, it is not hyper-Kähler. This result does not seem to be very surprising after taking into account the fact that the dilaton supermultiplet is dual to the supergravity multiplet under the mirror symmetry \[11\]. We can now identify the \(\rho\) and \(t\) coordinates in eq. (2.14) with the \(\rho\) and \(t\) coordinates here, as well as set up
\[ C = \mu + i\nu . \] (3.14)

The classical UH story is now complete. The non-perturbative UH metrics (with instanton corrections) are governed by non-separable solutions to the \(SU(\infty)\) Toda equation (3.5b) with \(P \neq 1\) \[13\], and they are very difficult to find \[13, 15\].

However, we didn’t take advantage of the second (linearly independent) abelian isometry of the UH metric yet! Given two abelian isometries commuting with each other, as in eq. (3.2), one can write down another Ansatz for the UH metric in adapted coordinates where both isometries are manifest (i.e. in terms of some potentials depending upon \(two\) coordinates only), and then impose the Einstein-Weyl conditions (3.3). Surprisingly enough, this programm was successfully accomplished in the mathematical literature only recently by Calderbank and Petersen \[14\].

The main result of ref. \[14\] is the theorem that any four-dimensional quaternionic metric (of a non-vanishing scalar curvature) with two linearly independent Killing vectors can be written down in the from
\[ ds_{CP}^2 = \frac{4\rho^2(F_{\rho}^2 + F_{\eta}^2) - F^2}{4F} \left(\frac{d\rho^2 + d\eta^2}{\rho^2}\right) + \frac{[(F - 2\rho F_{\rho})\hat{\alpha} - 2\rho F_{\eta}\hat{\beta}]^2 + [-2\rho F_{\eta}\hat{\alpha} + (F + 2\rho F_{\rho})\hat{\beta}]^2}{F^2[4\rho^2(F_{\rho}^2 + F_{\eta}^2) - F^2]} , \] (3.15)
in some local coordinates \((\rho, \eta, \theta, \psi)\) inside an open region of the half-space \(\rho > 0\), where \(\partial_{\theta}\) and \(\partial_{\psi}\) are the two Killing vectors, the one-forms \(\hat{\alpha}\) and \(\hat{\beta}\) are given by
\[ \hat{\alpha} = \sqrt{\rho}d\theta \quad \text{and} \quad \hat{\beta} = \frac{d\psi + \eta d\theta}{\sqrt{\rho}} , \] (3.16)
and, most importantly, the whole metric (3.15) is governed by the function \(F(\rho, \eta)\) that is a solution to the \(linear\) differential equation
\[ \left(\partial_{\rho}^2 + \partial_{\eta}^2\right)F = \frac{3}{4\rho^2}F . \] (3.17)

Some comments are in order.

First, it is fairly straightforward (e.g., by using Mathematica) to verify that the \textit{Calderbank-Petersen (CP) Ansatz} (3.15) does satisfy the fundamental Einstein-Weyl
equations (3.3) under the conditions (3.16) and (3.17), so that the metric (3.15) is quaternionic indeed. Moreover, this calculation is also useful to verify that the metric (3.17) is of negative scalar curvature provided that

$$4\rho^2(F_\rho^2 + F_\eta^2) > F^2 > 0 .$$

(3.18)

Second, the field redefinition

$$G = F\sqrt{\rho}$$

(3.19)

allows one to rewrite the CP Ansatz (3.15) to the form

$$-ds^2 = G^{-2}\left\{\frac{1}{P}(d\psi + \hat{\Theta})^2 + Pd\gamma^2\right\} ,$$

(3.20)

where

$$P = 1 - \frac{GG_\rho}{\rho(G_\rho^2 + G_\eta^2)} , \quad \hat{\Theta} = \left(\frac{GG_\eta}{G_\rho^2 + G_\eta^2} - \eta\right)d\theta ,$$

(3.21)

and

$$d\gamma^2 \equiv \rho^2d\theta^2 + (G_\rho^2 + G_\eta^2)(d\rho^2 + d\eta^2) .$$

(3.22)

The Ansatz (3.20) is similar to the Tod Ansatz (3.4), while it allows us to identify the $G$ function (3.19) with the Tod coordinate $\rho$ in eq. (3.4). Plugging eq. (3.19) into eq. (3.17) yields the linear differential equation on $G(\rho, \eta)$:

$$\left(\partial^2_\rho + \partial^2_\eta\right) G = \frac{1}{\rho}\partial_\rho G .$$

(3.23)

Unfortunately, eq. (3.22) does not seem to imply a direct relation between the Toda potential $u$ in eq. (3.4) and a function $F$ in eq. (3.17) since yet another reparametrization is needed to put the Ansatz (3.20) into the Tod form (3.4).

Third, and, perhaps, most importantly, the linear equation (3.17) means that $F$ is a local eigenfunction, with eigenvalue $3/4$, of the two-dimensional Laplace-Beltrami operator on the hyperbolic plane $\mathcal{H}^2$ with the metric

$$ds^2_{\mathcal{H}^2} = \frac{1}{\rho^2}(d\rho^2 + d\eta^2) .$$

(3.24)

Unlike the non-linear Toda equation (3.5b), the linearity of eq. (3.17) allows a superposition of any two solutions to form yet another solution. In physical terms, this amounts to the cluster decomposition of D-instantons. The validity of such decomposition is not obvious in a curved spacetime.

Though we cannot identify a dilaton in the full moduli space of the UH (the NLSM of the UH has general coordinate invariance in its target space), we can do it in the
perturbative region where the string coupling is weak, i.e. at large \( \rho \to +\infty \), which implies
\[
G \propto \rho^2 \to +\infty, \quad P \to \text{const.} \quad \text{and} \quad F \to \rho^{3/2},
\] (3.25)
where we have used eqs. (3.17), (3.19) and (3.21).

A simple solution to eqs. (3.17) and (3.23) outside of the perturbative region is given by
\[
G = 1 \quad \text{and} \quad F_1 = \frac{1}{\sqrt{\rho}}.
\] (3.26)
This solution looks like an ‘instanton’ solution but it implies \( 4\rho^2(F_\rho^2 + F_\eta^2) - F^2 = 0 \) that is incompatible with eq. (3.18). The ‘multi-instanton’ solutions do exist [14]. However, first, we have to impose some more physical requirements on them.

First, we expect the exact (non-perturbative) UH moduli space metric to be complete inside some four-dimensional ball \( B^4 \) [27] — by analogy with the exact Seiberg-Witten-type solutions in the non-perturbative N=2 supersymmetric gauge field theories (see, e.g., ref. [23] for a review).

Second, the full UH metric should also respect the known topological ‘boundary conditions’: in the perturbative region it should reduce to the standard (Bergmann or Ferrara-Sabharwal) classical metric up to diffeomorphisms, while it should also possess the UV fixed point (or a conformal infinity [28]) at some point of \( B^4 \) outside of the perturbative domain where one expects the N=2 superconformal field theory (or Landau-Ginzburg) description to be valid [4].

Third, because the D-instantons (i.e. the D2-branes wrapped about the special 3-cycles of CY) are supposed to be the origin of non-perturbative corrections to the UH metric, we should expect the dependence of this metric upon the RR-type \( \eta \)-variable to be periodic. Indeed, given an Euclidean D2-brane wrapped \( m \)-times around \( S^3 \) in CY, it couples to the RR expectation value on \( S^3 \) and thus produces a factor of \( \exp(2\pi im\eta) \) — cf. ref. [10]. We should, therefore, search for a solution to eq. (3.17) in the form of a D-instanton sum that is periodic in \( \eta \) [13].

Fourth, a discrete \( SL(2, \mathbb{Z}) \) duality symmetry is supposed to be the exact symmetry of the full type-II ‘superstring’ theory (including branes). Hence, it must be a symmetry of our UH effective metric solution.

To the best of our knowledge, no solutions with all the above-mentioned features are known. Nevertheless, it is known how to construct exact ‘multi-centre’ solutions to eq. (3.17) with a finite instanton number \( m > 1 \) [29, 30]. These solutions were originally found by using the quaternionic-Kähler quotients of the \( 4(m-1) \)-dimensional quaternionic projective space \( HP^{m-1} \) by an \( (m-2) \)-torus (i.e. by an
(m − 2)-dimensional family of commuting Killing vectors) with m > 1 \cite{21, 30}. In the case relevant for our investigation, the quaternionic projective plane \( HP^{m-1} \) should be replaced by the non-compact quaternionic hyperboloid, \( H^{m-1} \rightarrow H^{p-1,q} \), with \((p, q) = (m - 1, 1)\).

In terms of the CP description \cite{14} of the four-dimensional quaternionic metrics with torus isometry, governed by eq. (3.17), the ‘multi-instanton’ metric solutions are described by the following simple solution to the linear equation (3.17) on \( \mathcal{H}^2 \):

\[
F_m(\rho, \eta) = \sum_{k=1}^{m} \frac{\sqrt{a_k^2 \rho^2 + (a_k \eta - b_k)^2}}{\sqrt{\rho}} \tag{3.27}
\]

with some real moduli \((a_k, b_k)\). Since the superposition principle applies, it is easy to check that eq. (3.27) is a solution to eq. (3.17) indeed. Each additive contribution to the right-hand-side of eq. (3.27) is just a simple generalization of the ‘basic’ solution (3.26) corresponding to \( a = 0 \) and \( b = 1 \). When the hyperbolic plane \( \mathcal{H}^2 \) is mapped onto an open disc \( D \), the ‘positions’ of instantons are given by the marked points on the boundary of this disc where the torus action has its fixed points. The ‘twistors’ (in the terminology of ref. \cite{14}) \( \{a_k, b_k\}_{k=1}^{m} \) form the 2\(m\)-dimensional vector space where the three-dimensional \( SL(2, \mathbb{R}) \) duality group naturally acts. In addition, the solutions \( F \) are merely defined modulo an overall real factor. Hence, the total (real) dimension of the D-instanton moduli space \( \mathcal{M}_m \) (of a finite instanton number \( m \)) is given by \cite{14}

\[
\dim \mathcal{M}_m = 2m - \dim SL(2, \mathbb{R}) - 1 = 2m - 4 \tag{3.28}
\]

As is clear now, the whole moduli space (for all \( m \)) is infinite dimensional, in agreement with the LeBrun theorem \cite{31}.

Many explicit examples of the four-dimensional quaternionic metrics in the case of \( m = 2 \) and \( m = 3 \) can be found, e.g., in ref. \cite{14}. These examples include, in particular, the quaternionic-Kähler extensions of generic four-dimensional hyper-Kähler metrics with two centers and \( U(1) \times U(1) \) isometry (like Taub-NUT and Eguchi-Hanson) \cite{26}. In the case of \( m = 2 \) one finds only non-interesting (hyperbolic) metrics that have nothing to do with instantons or monopoles. The most general solution in the case of \( m = 3 \) reads \cite{14}

\[
F_3(\rho, \eta) = \frac{1}{\sqrt{\rho}} \left( \frac{(b + c/q)\sqrt{\rho^2 + (\eta + q)^2}}{\sqrt{\rho}} + \frac{(b - c/q)\sqrt{\rho^2 + (\eta - q)^2}}{\sqrt{\rho}} \right), \tag{3.29}
\]

where \((b, c)\) are two real (non-negative) moduli and \( q^2 = \pm 1 \). Amongst the solutions (3.29) there are the ones that do possess the physically important features, such as \cite{22, 23, 33, 35, 14}.

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• completeness in certain domains,
• negative scalar curvature,
• existence of the UV fixed point (or a conformal infinity),
• smoothness, i.e. no unremovable singularities,
• equivalence to the classical metric at some special values of the moduli.

The holographic principle may also apply on the conformal boundary \[36\]. We have, therefore, good reasons to expect all these features to be valid for higher \( m \) too.

The periodicity condition with respect to the RR-type coordinate \( \eta \),

\[
F(\rho, \eta) = F(\rho, \eta + 1), \quad (3.30)
\]
is non-trivial since it cannot be true for any finite value of \( m \). However, eq. (3.30) may be satisfied by the infinite D-instanton sum, \( \text{viz.} \)

\[
F(\rho, \eta) = \frac{1}{\sqrt{\rho}} + \sum_{n=-\infty}^{+\infty} |a_n| \frac{\sqrt{\rho^2 + (\eta + n)^2}}{\sqrt{\rho}} = \frac{1}{\sqrt{\rho}} + \sum_{n=-\infty}^{+\infty} |a_n| \sqrt{\frac{\rho + (\eta + n)^2}{\rho}} \quad (3.31)
\]
whose moduli \( \{a_n\} \) are supposed to guarantee convergence of the infinite series. We conclude that \textit{all} D-instanton (winding) numbers have to be present in the non-perturbative corrections to the UH metric.

In fact, the full UH solution to eq. (3.17), which describes all D-instanton corrections, should also respect the non-perturbative \( SL(2, \mathbb{Z}) \) duality (sect. 2),

\[
\tau \rightarrow \frac{a\tau + b}{c\tau + d}, \quad (a, b, c, d) \in \mathbb{Z}, \quad ad - bc = 1, \quad (3.32)
\]
where we have introduced the complex coordinate \( \tau \),

\[
\tau = \tau_1 + i\tau_2 \equiv \eta + i\rho. \quad (3.33)
\]
It is, therefore, natural to search for a non-holomorphic solution \( F(\tau, \bar{\tau}) \) amongst the modular functions \( f_s(\tau, \bar{\tau}) \) of order \( s \), which are defined by the Eisenstein series \[32\]

\[
f_s(\tau, \bar{\tau}) = \sum_{(p,n)\neq(0,0)} \left( \frac{\tau_2}{|p + n\tau|^2} \right)^s = \sum_{(p,n)\neq(0,0)} \frac{\rho^s}{[p^2 + n^2(\eta^2 + \rho^2) + 2np\eta]^s}, \quad (3.34)
\]
and obey the eigenvalue equation

\[
\tau_2^2 (\partial^2_{\tau_1} + \partial^2_{\tau_2}) f_s(\tau, \bar{\tau}) = s(s-1)f_s(\tau, \bar{\tau}). \quad (3.35)
\]
Equation (3.17) exactly coincides with eq. (3.35) provided that \( s(s-1) = 3/4 \) or
\[
s = 3/2 \quad .
\]
(3.36)
Thus we conclude that the UH solution to eq. (3.17) is proportional to \( f_{3/2}(\tau, \bar{\tau}) \),
\[
F(\tau, \bar{\tau}) \propto f_{3/2}(\tau, \bar{\tau}) = \sum_{(p, n) \neq (0, 0)} \frac{\tau_{2}^{3/2}}{|p + n\tau|^3} .
\]
(3.37)

The modular-invariant function \( f_{3/2}(\tau, \bar{\tau}) \) is precisely the same function that describes multi-instanton contributions to the \( R^4 \) terms in the ten-dimensional type-II superstrings [3], which are due to instantons of discrete energy \( p \) and discrete charge \( n \). The solution (3.37) is simply related to the \( K_1 \) Bessel function [6],
\[
F(\rho, \eta) \propto f_{3/2}(\tau, \bar{\tau}) = 2\zeta(3)\rho^{3/2} + \frac{2\pi^2}{3} \rho^{-1/2} + 8\pi\rho^{1/2} \sum_{m \neq 0}^{m \neq 0, n \geq 1} \frac{m}{n} |e^{2\pi imn\eta}| K_1(2\pi |mn|\rho) ,
\]
(3.38)
where \( \zeta(3) = \sum_{m>0}(1/m)^3 \). The asymptotical expansion of the function (3.38) in the perturbative (large \( \rho \)) region is given by [6]
\[
f_{3/2}(\tau, \bar{\tau}) = 2\zeta(3)\rho^{3/2} + \frac{2\pi^2}{3} \rho^{-1/2} + 4\pi^{3/2} \sum_{m,n \geq 1} \left( \frac{m}{n^3} \right)^{1/2} \left[ e^{2\pi imn(\eta+i\rho)} + e^{-2\pi imn(\eta-i\rho)} \right] \times
\]
\[
\times \left[ 1 + \sum_{k=1}^{\infty} \Gamma(k-1/2) \frac{1}{\Gamma(-k-1/2) (4\pi mn\rho)^k} \right] ,
\]
(3.39)
while it is just the sum of tree level, one-loop, and instanton contributions indeed. The one-loop correction has no local meaning since it can be removed by a NLSM field redefinition [11]. Since the function (3.39) is periodic in \( \eta \), it should be possible to rewrite it into the form (3.31).

The exact UH quaternionic metric governed by the solution (3.37) via eq. (3.15) does not depend upon details of the CY moduli space, as it may have been expected in the universal sector of CY compactification. The discovered relation to the earlier results [1, 6, 7, 8, 10] about D-instantons is important for justifying the consistency of our approach.

It is also worth mentioning that to be quaternionic does not automatically mean to be Kähler. Though the classical UH metric is quaternionic-Kähler (sect. 2), this does not apply to the instanton-corrected quaternionic metrics discussed in this section. This observation also implies that our results for the UH coupled to N=2 supergravity cannot be rewritten to the N=1 supergravity form without truncation, since N=1 local supersymmetry in 4d requires the NLSM metric to be Kähler.
4 The instanton-induced scalar potential

When the UH is electrically charged, its scalar potential becomes non-trivial (we do not consider any magnetic charges here). This happens because of gauging of an abelian isometry in the UH moduli space. Abelian gauging in the classical UH target space was discussed in great detail in ref. [4] – see also refs. [38, 39, 40]. Since the D-instantons are supposed to preserve the $U(1) \times U(1)$ isometry of the classical UH moduli space, it is quite natural to gauge a $U(1)$ part of it in the presence of the D-instanton quantum corrections, in order to generate a non-perturbative UH scalar potential. The fixed points (zeroes) of the UH scalar potential determine new vacua in type-II string theory. Gauging an abelian isometry introduces an extra N=2 vector gauge multiplet into our model of the UH. In 4d, N=2 supergravity it is the quaternionic NLSM metric and its Killing vector that fully determine the corresponding scalar potential. Unfortunately, the literature about the hypermultiplet scalar potentials in N=2 supergravity is rather confusing (or very complicated, at least), so we begin with the case of 4d, rigid N=2 supersymmetry that is well understood [41, 42].

Any hyper-Kähler N=2 NLSM in 4d can be obtained from its counterpart in 6d by dimensional reduction. No scalar potential for hypermultiplets is possible in 6d. Hence, a non-trivial scalar potential can only be generated via a Scherk-Schwarz-type mechanism of dimensional reduction with a non-trivial dependence upon extra spacetime coordinates, like [41, 42]

$$\partial_4 \phi^a = K^a(\phi), \quad \text{and} \quad \partial_5 \phi^a = 0,$$

where $K^a(\phi)$ is a Killing vector in the NLSM target space with a hyper-Kähler metric $g_{ab}(\phi)$ parametrized by four real scalars $\phi^a$ and $a = 1, 2, 3, 4$. The Scherk-Schwarz procedure is consistent with rigid N=2 supersymmetry if and only if the Killing vector $K^a(\phi)$ represents a translational (or tri-holomorphic) isometry, while there is always one such isometry in the case of an $U(1) \times U(1)$ symmetric hyper-Kähler metric. Upon the dimensional reduction (4.1) down to 4d, the 6d NLSM kinetic terms produce the scalar potential

$$V(\phi) = \frac{1}{2} g_{ab} K^a K^b \equiv \frac{1}{2} K^2$$

that is just given by half of the Killing vector squared.

Given a four-dimensional hyper-Kähler metric with a triholomorphic isometry, we are in a position to use the Gibbons-Hawking Ansatz (3.11) where this isometry is manifest, with $K^a = (1, 0, 0, 0)$ and $\phi^a = (t, \mu, \nu, \rho)$. Equation (4.2) now implies

$$V = \frac{g_{tt}}{2} = \frac{1}{2} P^{-1},$$

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where $P(\vec{X})$ is a harmonic function of $\vec{X} = (\mu, \nu, \rho)$. For example, the (Gibbons-Hawking) multi-centre hyper-Kähler metrics are described by the harmonic function

$$P(\vec{X}) = \sum_{k=1}^{m} \frac{1}{|\vec{X} - \vec{X}_k|},$$

(4.4)

where the moduli $\vec{X}_i$ denote locations of the centers. The corresponding scalar potential (4.3) is non-negative, while its absolute minima occur precisely at the fixed points where the harmonic function (4.4) diverges. Since $V = 0$ at these points, N=2 supersymmetry remains unbroken in all of these vacua. It is worth mentioning that the vacua are independent upon the NLSM parametrization used (the fixed points are mapped into themselves under the NLSM reparametrizations).

In the case of the UH with 4d, local N=2 supersymmetry (i.e. coupled to N=2 supergravity), we have to deal with a quaternionic NLSM metric having a gauged abelian isometry. First, the Gibbons-Hawking Ansatz (3.11) is to be replaced by the Tod Ansatz (3.4) \([3]\). Second, we have also take into account the presence of the abelian gauge N=2 vector multiplet whose complex scalar component ($a$) is going to enter the NLSM scalar potential too. As was demonstrated in refs. \([39, 40]\), the scalar potential in the gauged N=2 supergravity appears to be a very natural generalization of the scalar potential (4.2) in the absence of N=2 supergravity,

$$V = \frac{1}{\text{Im}[^\tau(a)]} \frac{1}{2} K^2 = \frac{1}{\text{Im}[^\tau(a)]} \frac{P^{-1}}{2\rho^2},$$

(4.5)

where $\tau(a)$ is the function governing the kinetic terms of the N=2 vector multiplet. We have used eq. (3.4) in the second equation (4.5).

The standard way of deriving the scalar potential in the gauged N=2 supergravity uses the local N=2 supersymmetry transformation laws of the fermionic fields (gauginos, hyperinos and gravitini) \([13, 18]\). The contributions of gauginos and hyperinos are always positive, whereas the contribution of gravitini is negative. The recent results of ref. \([39, 40]\) imply that the negative (gravitini) contribution to the scalar potential cancels against the positive contributions due to the matter fermions (gaugino and hyperino) in the gauged N=2 supergravity. This is not the case in the gauged N=1 supergravity theories \([14]\). Hence, it may not be possible to rewrite an N=2 gauged supergravity theory in the N=1 locally supersymmetric form without truncations \(\text{cf. our remarks at the end of sect. 3)}\).

Because of unitarity of the N=2 supergravity theory, effectively describing the unitary CY-compactified theory of type-II superstrings, there should be no ghosts in
the \( N=2 \) vector multiplet sector too, so that we should have

\[ \text{Im}[\tau(a)] > 0. \tag{4.6} \]

Being interested in the vacua of the effective \( N=2 \) supergravity theory, which are determined by the minima of its scalar potential (4.5), we do not need to know the function \( \tau(a) \) explicitly – eq. (4.6) is enough.

In the classical approximation for the UH metric, eq. (3.12) tells us that \( P = \text{const.} \). This immediately gives rise to the run-away behaviour of the potential (4.5) with its absolute minimum at \( \rho = \infty \), in agreement with refs. [4, 40]. This run-away solution is, of course, physically unacceptable because it implies the infinite CY volume i.e. a decompactification, as well as the ‘infinite’ string coupling. One may hope that the use of the full (non-perturbative) UH metric may improve the scalar potential behaviour, because the D-instanton corrections imply that \( P \neq \text{const.} \) (see sect. 3).

Unfortunately, finding an exact potential in this case amounts to solving the (non-linear, partial differential) Toda equation (3.5b), since the \( P \)-function is governed by the Toda potential via eq. (3.5a). Though the 3d Toda equation is known to be integrable, it is notoriously difficult to find its explicit (non-separable) solutions.

When \( N=2 \) supergravity is switched off (after gauging), we can take a solution for the \( P \)-function in the hyper-Kähler limit [10],

\[
P(\rho, \eta) = \frac{1}{4\pi} \sum_{n=-\infty}^{+\infty} \frac{1}{\sqrt{\rho^2/g_{\text{string}}^2 + (\eta + n)^2}} - \frac{1}{|n|} + \text{const.}
\]

\[
\begin{align*}
P(\rho, \eta) &= \frac{1}{4\pi} \log \left( \frac{1}{\rho^2} \right) + \sum_{m \neq 0} \frac{1}{2\pi} e^{2\pi i m \eta} K_0 \left( 2\pi \frac{|m\rho|}{g_{\text{string}}} \right),
\end{align*}
\]

where \( K_0 \) is the modified Bessel function, and we have re-introduced the dependence upon the string coupling constant \( g_{\text{string}} \) for reader’s convenience. The conjectured \( U(1) \times U(1) \) symmetry of the UH metric in the form (3.11) and the Poisson resummation formula were used in deriving eq. (4.7) — see ref. [10]. In the perturbative region (large \( \rho \)) the asymptotic expansion of the Bessel function in eq. (4.7) yields the infinite D-instanton sum [10]

\[
P(\rho, \eta) = \frac{1}{4\pi} \log \left( \frac{1}{\rho^2} \right) + \sum_{m=1}^{\infty} \exp \left( -\frac{2\pi |m\rho|}{g_{\text{string}}} \right) \cos(2\pi m \eta)
\]

\[
\times \sum_{n=0}^{\infty} \frac{\Gamma(n + \frac{1}{2})}{\sqrt{\pi n!}} \left( \frac{g_{\text{string}}}{4\pi |m\rho|} \right)^{n + \frac{1}{2}}.
\]

The \( \exp (-1/g_{\text{string}}) \) type dependence of the solution (4.8) agrees with the general expectations [3] so that eq. (4.8) describes the D-instantons indeed.
We conclude that the non-perturbative vacua of our toy model for the electrically charged UH in the presence of D-instantons are given by poles of the $P$-function defined by eqs. (3.4) and (3.5). In the hyper-Kähler limit, the vacua are given by the fixed points of the D-instanton function (4.7).

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