Contents

Schramm-Loewner Evolution (SLE)  
G. Lawler

1

Schramm-Loewner Evolution (SLE)  
Introduction

2

Lecture 1. Scaling limits of lattice models
1. Self-avoiding walk (SAW)  
2. Loop-erased random walk  
3. Percolation  
4. Ising model  
5. Assumptions on limits  
6. Exercises for Lecture 1

3

5

9

11

12

13

13

Lecture 2. Conformal mapping and Loewner equation
1. Important results about conformal maps  
2. Half-plane capacity  
3. Loewner equation  
4. Maps generated by a curve  
5. A flow on conformal maps  
6. Doubly infinite time  
7. Distance to boundary  
8. Exercises for Lecture 2

15

17

19

21

22

23

24

25

Lecture 3. Schramm-Loewner evolution (SLE)
1. Definition  
2. Phases  
3. Dimension of the path  
4. Cardy’s formula  
5. Conformal images of SLE  
6. Exercises for Lecture 3

27

27

30

31

32

33

Lecture 4. SLE_κ in a simply connected domain D
1. Drift and locality  
2. Girsanov  
3. The restriction martingale  
4. (Brownian) boundary bubbles  
5. Brownian loop measure  
6. The measure μ_D(z,w) for κ ≤ 4  
7. Exercises for Lecture 4

35

35

36

37

38

40

41

42
# LECTURE 0. CONTENTS

| Lecture 5. Radial and two-sided radial $SLE_\kappa$ | 45 |
|---------------------------------------------------|----|
| 1. Example: SAW II | 45 |
| 2. Radial $SLE_\kappa$ | 47 |
| 3. Another definition | 49 |
| 4. Radial $SLE_\kappa$ in a smaller domain | 51 |
| 5. Two-sided radial | 51 |
| 6. Exercises for Lecture 5 | 52 |

| Lecture 6. Intersection exponents | 53 |
|----------------------------------|----|
| 1. One-sided | 53 |
| 2. Two-sided | 56 |
| 3. Nonintersecting $SLE_\kappa$ | 57 |
| 4. Radial exponent and SAW III | 58 |
| 5. Exercises for Lecture 6 | 60 |

| Tables for reference | 61 |

| Bibliography | 63 |
Schramm-Loewner Evolution
 \( (SLE) \)

G. Lawler
Schramm-Loewner Evolution (SLE)

G. Lawler

Introduction

This is the first expository set of notes on SLE I have written since publishing a book two years ago [45]. That book covers material from a year-long class, so I cannot cover everything there. However, these notes are not just a subset of those notes, because there is a slight change of perspective. The main differences are:

- I have defined SLE as a finite measure on paths that is not necessarily a probability measure. This seems more natural from the perspective of limits of lattice systems and seems to be more useful when extending SLE to non-simply connected domains. (However, I do not discuss non-simply connected domains in these notes.)
- I have made more use of the Girsanov theorem in studying corresponding martingales and local martingales.

As in [45], I will focus these notes on the continuous process SLE and will not prove any results about convergence of discrete processes to SLE. However, my first lecture will be about discrete processes — it is very hard to appreciate SLE if one does not understand what it is trying to model.

I would like to thank Michael Kozdron, Robert Masson, Hariharan Narayanan, and Xinghua Zheng for their assistance in the preparation of these notes. Figure 8 was produced by Geoffrey Grimmett.
The Schramm-Loewner evolution (SLE) is a measure on continuous curves that is a candidate for the scaling limit for discrete planar models in statistical physics. Although our lectures will focus on the continuum model, it is hard to understand SLE without knowing some of the discrete models that motivate it. In this lecture, I will introduce some of the discrete models. By assuming some kind of “conformal invariance” in the limit, we will arrive at some properties that we would like the continuum measure to satisfy.

1. Self-avoiding walk (SAW)

A self-avoiding walk (SAW) of length \( n \) in the integer lattice \( \mathbb{Z}^2 = \mathbb{Z} + i\mathbb{Z} \) is a sequence of lattice points

\[ \omega = [\omega_0, \ldots, \omega_n] \]

with \( |\omega_j - \omega_{j-1}| = 1, j = 1, \ldots, n \), and \( \omega_j \neq \omega_k \) for \( j < k \). If \( J_n \) denotes the number of SAWs of length \( n \) with \( \omega_0 = 0 \), it is well known that

\[ J_n \approx e^{\beta n}, \quad n \to \infty, \]

where \( e^{\beta} \) is the connective constant whose value is not known exactly. Here \( \approx \) means that \( \log J_n \sim \beta n \) where \( f(m) \sim g(m) \) means \( f(m)/g(m) \to 1 \). In fact, it is believed that there is an exponent, usually denoted \( \gamma \), such that

\[ J_n \propto n^{\gamma-1} e^{\beta n}, \quad n \to \infty, \]

where \( \propto \) means that each side is bounded by a constant times the other. The exponent \( \nu \) is defined roughly by saying that the typical diameter (with respect to the uniform probability measure on SAWs of length \( n \) with \( \omega_0 = 0 \)) is of order \( n^{\nu} \). The constant \( \beta \) is special to the square lattice, but the exponents \( \nu \) and \( \gamma \) are examples of lattice-independent critical exponents that should be observable in a “continuum limit”. For example, we would expect the fractal dimension of the paths in the continuum limit to be \( d = 1/\nu \).

To take a continuum limit we let \( \delta > 0 \) and

\[ \omega^{\delta} (t \delta^d) = \delta \omega(t). \]

We can think of \( \omega^{\delta} \) as a SAW on the lattice \( \delta \mathbb{Z}^2 \) parametrized so that it goes a distance of order one in time of order one. We can use linear interpolation to make \( \omega^{\delta} (t) \) a continuous curve. Consider the square in \( \mathbb{C} \)

\[ D = \{ x + iy : -1 < x < 1, -1 < y < 1 \}, \]

and let \( z = -1, w = 1 \). For each integer \( N \) we can consider a finite measure on continuous curves \( \gamma : (0, t_\gamma) \to D \) with \( \gamma(0+) = z, \gamma(t_\gamma) = w \) obtained as follows. To each SAW \( \omega \) of length \( n \) in \( \mathbb{Z}^2 \) with \( \omega_0 = -N, \omega_n = N \) and \( \omega_1, \ldots, \omega_{n-1} \in ND \)
we give measure $e^{-\beta n}$. If we identify $\omega$ with $\omega^{1/N}$ as above, this gives a measure on curves in $D$ from $z$ to $w$. The total mass of this measure is

$$Z_N(D; z, w) := \sum_{\omega: N z \to N w, \omega \subset N D} e^{-\beta|\omega|}.$$ 

It is conjectured that there is a $b$ such that as $N \to \infty$,

$$(1.1) \quad Z_N(D; z, w) \sim C(D; z, w) N^{-2b}.$$ 

Moreover, if we multiply by $N^{2b}$ and take a limit, then there is a measure $\mu_D(z, w)$ of total mass $C(D; z, w)$ supported on simple (non self-intersecting) curves from $z$ to $w$ in $D$. The dimension of these curves will be $d = 1/\nu$.

![Figure 1. Self-avoiding walk in a domain](image1)

![Figure 2. Scaling limit of SAW](image2)

Similarly, if $D$ is another domain and $z, w \in \partial D$, we can consider SAWs from $z$ to $w$ in $D$. If $\partial D$ is smooth at $z, w$, then (after taking care of the local lattice effects
— we will not worry about this here), we define the measure as above, multiply by $N^{2b}$ and take a limit. We conjecture that we get a measure $\mu_D(z, w)$ on simple curves from $z$ to $w$ in $D$. We write the measure $\mu_D(z, w)$ as

$$\mu_D(z, w) = C(D; z, w) \mu_D^\#(z, w),$$

where $\mu_D^\#(z, w)$ denotes a probability measure.

**Figure 3.** Scaling limit of SAW in a different domain

It is believed that the scaling limit satisfies some kind of “conformal invariance”. To be more precise we assume the following conformal covariance property: if $f : D \to f(D)$ is a conformal transformation and $f$ is differentiable in neighborhoods of $z, w \in \partial D$, then

$$f \circ \mu_D(z, w) = |f'(z)|^b |f'(w)|^b \mu_{f(D)}(f(z), f(w)).$$

In other words the total mass satisfies the scaling rule

$$C(D; z, w) = |f'(z)|^b |f'(w)|^b C(f(D); f(z), f(w)),$$

and the corresponding probability measures are conformally invariant:

$$f \circ \mu_D^\#(z, w) = \mu_{f(D)}^\#(f(z), f(w)).$$

Let us be a little more precise about the definition of $f \circ \mu_D^\#(z, w)$. Suppose $\gamma : (0, t_\gamma) \to D$ is a curve with $\gamma(0^+) = z, \gamma(t_\gamma^-) = w$. For ease, let us assume that $\gamma$ is simple. Then the curve $f \circ \gamma$ is the corresponding curve from $f(z)$ to $f(w)$. At the moment, we have not specified the parametrization of $f \circ \gamma$. We will consider two possibilities:

- **Ignore the parametrization.** We consider two curves equivalent if one is an (increasing) reparametrization of the other. In this case we do not need to specify how we parametrize $f \circ \gamma$.
- **Scaling by the dimension $d$.** If $\gamma$ has the parametrization as given in the limit, then the amount of time need for $f \circ \gamma$ to traverse $f(\gamma[t_1, t_2])$ is

$$\int_{t_1}^{t_2} |f'(\gamma(s))|^d \, ds.$$

In either case, if we start with the probability measure $\mu_D^\#(z, w)$, the transformation $\gamma \mapsto f \circ \gamma$ induces a probability measure which we call $f \circ \mu_D^\#(z, w)$.

**Remark 1.1.**
• Since $\mu^\#_D(z, w)$ is a conformal invariant, we can define $f \circ \mu^\#_D(z, w)$ even if $\partial D$ is not smooth at $z, w$. (For really bad conformal transformations, one needs to worry about the continuity of $f \circ \gamma$ at $f(z)$ and $f(w)$, but we do not need to consider transformations that are that bad!)

• The Riemann mapping theorem tells us that if $D, D_1$ are simply connected domains; $z, w$ distinct points in $\partial D$; $z_1, w_1$ distinct points in $\partial D_1$, then there is a one-parameter family of conformal transformations $f : D \to D_1$ with $f(z) = z_1, f(w) = w_1$. In particular, if we know the measure for one simply connected domain $D$, we know it for all simply connected domains.

• In particular, if we know $\mu^\#_H(0, \infty)$, then we know $\mu^\#_D(z, w)$ for all simply connected domains. Here $H$ denotes the upper half plane.

• The measure $\mu^\#_H(0, \infty)$ must be invariant under the dilation $z \mapsto rz$ ($r > 0$).

• Although the map $f : D \to D_1$ with $f(z) = z_1, f(w) = w_1$ is not uniquely defined the quantity $f'(z) f'(w)$ is independent of the choice.

• One can choose a unique such $f$ with $|f'(w)| = 1$ in which case the conformal covariance condition becomes

$$f \circ \mu_D(z, w) = |f'(z)|^b \mu_{f(D)}(f(z), f(w)).$$

When $D$ is a subdomain of $H$ with $H \setminus D$ bounded and $w = f(w) = \infty$, then the condition $|f'(w)| = 1$ translates to $f(w') \sim w'$ as $w' \to \infty$.

There are two more properties that we would expect the family of measures $\mu_D(z, w)$ to have. The first of these will be shared by all the examples in this section while the second will not. We just state the properties, and leave it to the reader to see why one would expect them in the limit.

• **Domain Markov property.** Consider the measure $\mu^\#_D(z, w)$ and suppose an initial segment of the curve $\gamma(0, t]$ is observed. Then the conditional distribution of the remainder of the curve given $\gamma(0, t]$ is the same as $\mu^\#_{D \setminus \gamma(0, t]}(\gamma(t), w)$.

![Figure 4. Domain Markov property](image-url)
Restriction property. Suppose $D_1 \subset D$. Then $\mu_{D_1}(z,w)$ restricted to paths that lie in $D_1$. In terms of Radon-Nikodym derivatives, this can be phrased as

$$\frac{d\mu_{D_1}(z,w)}{d\mu_D(z,w)}(\gamma) = 1 \{\gamma(0,t_\gamma) \subset D_1\}.$$ 

Figure 5. Illustrating the restriction property

We have considered the case where $z, w \in \partial D$. We could consider $z \in \partial D, w \in D$. In this case the measure is defined similarly, but (1.4) becomes

$$Z_D(z,w) \sim C(D; z,w) N^{-b} N^{-\tilde{b}},$$

where $\tilde{b}$ is a different exponent (see Lectures 5 and 6). The limiting measure $\mu_D(z,w)$ would satisfy the conformal covariance rule

$$f \circ \mu_D(z,w) = |f'(z)|^b |f'(w)|^{\tilde{b}} \mu_{f(D)}(f(z),f(w)).$$

Similarly we could consider $\mu_D(z,w)$ for $z, w \in D$.

2. Loop-erased random walk

We start with simple random walk. Let $\omega$ denote a nearest neighbor random walk from $z$ to $w$ in $D$. We no longer put in a self-avoidance constraint. We give each walk $\omega$ measure $4^{-|\omega|}$ which is the probability that the first $n$ steps of an ordinary random walk in $\mathbb{Z}^2$ starting at $z$ are $\omega$. The total mass of this measure is the probability that a simple random walk starting at $z$ immediately goes into the domain and then leaves the domain at $w$. Using the “gambler’s ruin” estimate for one-dimensional random walk, one can show that the total mass of this measure decays like $O(N^{-2})$; in fact (Exercise 1.4)

$$Z_N(D; z,w) \sim C(D; z,w) N^{-2}, \quad N \to \infty,$$

where $C(D; z,w)$ is the “excursion Poisson kernel”, $H_{\partial D}(z,w)$, defined to be the normal derivative of the Poisson kernel $H_D(\cdot, w)$ at $z$. In the notation of the previous section $b = 1$. For each realization of the walk, we produce a self-avoiding path by erasing the loops in chronological order.
Again we are looking for a continuum limit $\mu_D(z, w)$ with paths of dimension $d$ (not the same $d$ as for SAW). The limit should satisfy

- Conformal covariance
- Domain Markov property

However, we would not expect the limit to satisfy the restriction property. The reason is that the measure given to each self-avoiding walk $\omega$ by this procedure is determined by the number of ordinary random walks which produce $\omega$ after loop erasure. If we make the domain smaller, then we lose some random walks that would produce $\omega$ and hence the measure would be smaller. In terms of Radon-Nikodym derivatives, we would expect

$$\frac{d\mu_{D_1}(z, w)}{d\mu_D(z, w)} < 1.$$
3. Percolation

Suppose that every point in the triangular lattice in the upper half plane is colored black or white independently with each color having probability 1/2. A typical realization is illustrated in Figure 8 (if one ignores the bottom row).

![Figure 8. The percolation exploration process.](image)

We now put a boundary condition on the bottom row as illustrated — all black on one side of the origin and all white on the other side. For any realization of the coloring, there is a unique curve starting at the bottom row that has all white vertices on one side and all black vertices on the other side. This is called the percolation exploration process. Similarly we could start with a domain $D$ and two boundary points $z, w$; give a boundary condition of black on one of the arcs and white on the other arc; put a fine triangular lattice inside $D$; color vertices black or white independently with probability 1/2 for each; and then consider the path connecting $z$ and $w$. In the limit, one might hope for a continuous interface. In comparison to the previous examples, the total mass of the lattice measures is one; another way of saying this is that $b = 0$. We suppose that the curve is conformally invariant, and one can check that it should satisfy the domain Markov property.

The scaling limit of percolation satisfies another property called the locality property. Suppose $D_1 \subset D$ and $z, w \in \partial D \cap \partial D_1$ as in Figure 5. Suppose that only an initial segment of $\gamma$ is seen. To determine the measure of the initial segment, one only observes the value of the percolation cluster at vertices adjoining $\gamma$. Hence the measure of the path is the same whether it is considered as a curve in $D_1$ or a curve in $D$. The locality property is stronger than the restriction property which SAW satisfies. The restriction property is a similar statement that holds for the entire curve $\gamma$ but not for all initial segments of $\gamma$. 
There is another well known conformal invariant for percolation known as Cardy’s formula, named after the physicist who first predicted it\(^1\) the formula. Suppose \(D\) is a simply connected domain and the boundary is divided into four arcs, \(A_1, A_2, A_3, A_4\) in counterclockwise order. Let \(P_D(A_1, A_3)\) be the limit as the lattice spacing goes to zero of the probability that in a percolation cluster as above there is a connected cluster of white vertices connecting \(A_1\) to \(A_3\). This should be a conformal invariant. It turns out that the nicest domain to give the formula is an equilateral triangle as shown in the Figure 9.

4. Ising model

The Ising model is a simple model of a ferromagnet. We will consider the triangular lattice as in the previous section. Again we color the vertices black or white although we now think of the colors as spins. If \(x\) is a vertex, we let \(\sigma(x) = 1\) if \(x\) is colored black and \(\sigma(x) = -1\) if \(x\) is colored white. The measure on configurations is such that neighboring spins like to have the same sign.

It is easiest to define the measure for a finite collection of spins. Suppose \(D\) is a bounded domain in \(\mathbb{C}\) with two marked boundary points \(z, w\) which give us two boundary arcs. We consider a fine triangular lattice in \(D\) and fix boundary conditions +1 and −1 respectively on the two boundary arcs. Each configuration of spins is given energy

\[
\mathcal{E} = -\sum_{x \sim y} \sigma(x) \sigma(y),
\]

where \(x \sim y\) means that \(x, y\) are nearest neighbors. We then give measure \(e^{-\beta \mathcal{E}}\) to a configuration of spins. If \(\beta\) is small, then the correlations are localized and

\(^1\)The word “predicted” here means that the formula was found from a nontrivial, but not mathematically rigorous, argument. Much of the work by theoretical physicists using conformal field theory falls into this category. Although nonrigorous, the ideas are deep and involve a number of different areas of mathematics.
spins separated by a large distance are almost independent. If $\beta$ is large, there is long-range correlation. There is a critical $\beta_c$ that separates these two phases.

For each configuration of spins there is a well-defined boundary between $+1$ spins and $-1$ spins defined in exactly the same way as the percolation exploration process. At the critical $\beta_c$ it is believed that this gives an interesting fractal curve and that it should satisfy conformal covariance and the domain Markov property.

5. Assumptions on limits

Our goal is to understand the possible continuum limits for these discrete models. We will discuss the boundary to boundary case here but one can also have boundary to interior or interior to interior. (The terms “surface” and “bulk” are often used for boundary and interior.) Such a limit is a measure $\mu_D(z, w)$ on curves from $z$ to $w$ in $D$ which can be written

$$\mu_D(z, w) = C(D; z, w) \mu^\#_D(z, w),$$

where $\mu^\#_D(z, w)$ is a probability measure. The existence of $\mu_D(z, w)$ assumes smoothness of $\partial D$ near $z, w$, but the probability measure $\mu^\#_D(z, w)$ exists even without the smoothness assumption. The two basic assumptions are:

- Conformal covariance of $\mu_D(z, w)$ and conformal invariance of $\mu^\#_D(z, w)$.
- Domain Markov property.

The starting point for the Schramm-Loewner evolution is to show that if we ignore the parametrization of the curves, then there is only a one-parameter family of probability measures $\mu^\#_D(z, w)$ for simply connected domains $D$ that satisfy conformal invariance and the domain Markov property. We will construct this family. The parameter is usually denoted $\kappa > 0$. By the Riemann mapping theorem, it suffices to construct the measure for one simply connected domain and the easiest is the upper half plane $\mathbb{H}$ with boundary points $0$ and $\infty$. As we will see, there are a number of ways of parametrizing these curves.

6. Exercises for Lecture 1

Exercise 1.2. let $J_n$ denote the number of SAWs of length $n$ with $\omega_0 = 0$ in $\mathbb{Z}^2$. 

\begin{figure}
\centering
\includegraphics[width=0.5\textwidth]{ising_interface.png}
\caption{Ising interface.}
\end{figure}
(1) Show that there exists a \( \beta \) with \( 2 \leq e^\beta \leq 3 \) such that
\[
\lim_{n \to \infty} \frac{\log J_n}{n} = \beta.
\]
(2) Prove that \( 2 < e^\beta < 3 \).

**Exercise 1.3.** Suppose \( D \) is a simply connected domain and \( f : D \to D \) is a conformal transformation and \( z, w \in \partial D \). Suppose \( \partial D \) is smooth near \( z, w \) and \( f(z) = z, f(w) = w \). Show that \( f'(z) f'(w) = 1 \).

**Exercise 1.4.** Let
\[
V = V_N = \{ j + ik \in \mathbb{Z} \times i\mathbb{Z} : 1 \leq j \leq N - 1, 1 \leq k \leq N - 1 \}.
\]
Let \( S_n \) denote a simple random walk starting at \( j + ik \in V_N \) and let \( \tau = \tau_N = \min\{n : S_n \notin V_N\} \). Let
\[
H(j + ik, l + im) = \mathbf{P}^{j+ik}\{S_\tau = l + im\}.
\]
We will compute this. Let \( \partial V = \{ z \in \mathbb{Z}^2 : \text{dist}(z, V) = 1 \} \) and \( \overline{V} = V \cup \partial V \). Let \( \mathcal{A} \) denote the set of real-valued functions \( f \) on \( \overline{V} \) satisfying:

- \( f \) is discrete harmonic in \( V \) (i.e., the value of \( f \) is the average of \( f \) at its nearest neighbors).
- \( f \equiv 0 \) on \( \partial V \setminus \{N + im : m = 1, \ldots, N - 1\} \).

Show the following:

1. Show that there exist \( N - 1 \) linearly independent functions \( f_1, \ldots, f_{N-1} \in \mathcal{A} \) of the form
   \[
   f_q(j + ik) = \sinh(r_q j) \sin(s_q k), \quad q = 1, \ldots, N - 1.
   \]
2. Show that every \( f \in \mathcal{A} \) is a linear combination of \( f_1, \ldots, f_{N-1} \).
3. For any \( m = 1, \ldots, N - 1 \), find the unique \( f \in \mathcal{A} \) for which
   \[
   f(N + im) = 1, \quad f(N + im') = 0, m' \neq m.
   \]

Use this to justify \( \text{(1.2)} \) in Section 2 and show that for non-corner points \( z \),
\[
C(D; z, w) = H_{\partial D}(z, w) := \frac{d}{dn} H_D(z, w),
\]
where \( d/\text{dn} \) denotes normal derivative and \( H_D \) is the Poisson kernel.
Lecture 2

Conformal mapping and Loewner equation

1. Important results about conformal maps

Here we summarize the basic facts about conformal maps that one needs in order to use the Loewner equation effectively. A domain $D$ will be a connected subset of $\mathbb{C}$. We will call a holomorphic function $f : D \rightarrow D'$ a conformal transformation if it is one-to-one and onto. This implies that $f'(z) \neq 0$ for all $z \in D$. A domain $D$ is simply connected if $\mathbb{C} \setminus D$ is connected. We let $\mathbb{D} = \{ |z| < 1 \}$ denote the unit disk and $\mathbb{H} = \{ x + iy : y > 0 \}$ the upper half plane. The starting point is the Riemann mapping theorem.

**Theorem 2.5** (Riemann mapping theorem). If $D$ is a proper, simply connected domain in $\mathbb{C}$ and $z \in D$, then there is a unique conformal transformation $f : \mathbb{D} \rightarrow D$ with $f(0) = z, f'(0) > 0$.

In other words, there is a one-to-one correspondence between one-to-one, analytic functions $f$ on $\mathbb{D}$ with $f(0) = 0, f'(0) > 0$ and simply connected proper subdomains of $\mathbb{C}$ containing the origin. The class of such functions with $f'(0) = 1$ is denoted $\mathcal{S}$. Classical function theory devoted much time to the study of $\mathcal{S}$. The high point was the proof by de Branges of the Bieberbach conjecture.

**Theorem 2.6** (Bieberbach conjecture, de Branges theorem). If $f \in \mathcal{S}$ has power series expansion

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

then $|a_n| \leq n$ for each $n$.

Although we do not need such a deep result, we do use some facts that were developed to try to solve the conjecture.

**Theorem 2.7** (Koebe (1/4)-theorem). If $f \in \mathcal{S}$, then $(1/4) \mathbb{D} \subset f(\mathbb{D})$.

**Theorem 2.8** (Distortion theorem). If $f \in \mathcal{S}$, and $|z| \leq r < 1$, then

$$1 - r \left( \frac{1}{1 + r} \right)^3 \leq |f'(z)| \leq \frac{1 + r}{(1 - r)^3}.$$

(2.1)

The particular values $1/4$ and those in $[2.1]$ are not as important as the fact that there is some uniform bound over all $f \in \mathcal{S}$. These theorems are important for studying conformal maps even when the domains are not simply connected.
Suppose \( f : D_1 \to D_2 \) is a conformal transformation with \( f(0) = 0 \). Let \( d_j = \text{dist}(0, \partial D_j) \). If
\[
\tilde{f}(z) = \frac{f(d_1 z)}{d_1 f'(0)},
\]
then \( \tilde{f} \in S \). Therefore, \((1/4) \mathbb{D} \subset \tilde{f}(\mathbb{D}) \) which implies \( d_2 \geq \frac{|f'(0)| d_1}{4} \). By interchanging the roles of \( D_1, D_2 \), we get the corollary
\[
\frac{|f'(0)|}{4} \leq \frac{d_2}{d_1} \leq 4 |f'(0)|.
\]

There is a similar result about harmonic functions that is simple but worth emphasizing. We will state it for the gradient but there are similar results for higher derivatives. These are just corollaries of the Poisson integral formula,
\[
u(z) = \int_{\partial \mathbb{D}} u(w) H_\mathbb{D}(z, w) |dw|,
\]
where
\[
H_\mathbb{D}(z, w) = \frac{1}{2\pi} \frac{1 - |z|^2}{|z - w|^2}
\]
is the Poisson kernel in \( \mathbb{D} \) and \( |dw| \) means integration with respect to arc length.

**Proposition 2.9.** For every \( r < 1 \), there exists \( c_r < \infty \) such that the following holds for all \( |z| \leq r \).

- **(Harnack inequality)** If \( u : \mathbb{D} \to (0, \infty) \) is harmonic, then

\[
c_r^{-1} u(0) \leq u(z) \leq c_r u(0).
\]

- **(Derivative estimates)** If \( u : \mathbb{D} \to \mathbb{R} \) is harmonic, then

\[
|\nabla u(z)| \leq c_r \|u\|_\infty.
\]

In particular, there is a \( c \) such that if \( u : D \to \mathbb{R} \) is harmonic, then

\[
|\nabla u(z)| \leq \frac{c}{\text{dist}(z, \partial D)} \|u\|_\infty.
\]

The next theorem is a corollary of a stronger theorem known as the Beurling projection theorem. However, the weaker version here is what is used most often in applications (and also has discrete analogues).

**Theorem 2.10** (Beurling estimate). There is a \( c < \infty \) such that the following is true. Suppose \( \gamma : [0, 1] \to \mathbb{C} \) is a continuous curve with \( |\gamma(0)| = r < 1 = |\gamma(1)| \). Let \( B_t \) be a complex Brownian motion starting at the origin and let \( \tau = \tau_\mathbb{D} = \inf\{t : |B_t| = 1\} \). Then,
\[
P\{B[0, \tau] \cap \gamma[0, 1] = \emptyset\} \leq cr^{1/2}.
\]

**Remark 2.11.** The estimate is sharp when \( \gamma \) is a line segment from \( r \) to 1. See Exercise 2.27.
2. Half-plane capacity

If $K$ is a bounded, relatively closed subset of $\mathbb{H}$, let $D = D_K = \mathbb{H} \setminus K$ and

$$\phi_D(z) = \text{Im}(z) - \mathbb{E}^z[\text{Im}(B_{\tau_D})],$$

where $B_t$ is a standard complex Brownian motion and $\tau_D = \inf\{t : B_t \notin D\}$. Then $\phi_D$ is a positive harmonic function on $D$ that vanishes on $\partial D$ and such that

$$\phi_D(z) = \text{Im}(z) + O(|z|^{-1}), \quad z \to \infty.$$

The half-plane capacity (from infinity) of $K$ is defined by

$$\phi_D(z) = \text{Im}(z) + \frac{\text{hcap}(K)}{z} + o(1), \quad z \to \infty,$$

or, in other words,

$$\text{hcap}(K) = \lim_{z \to \infty} \text{Im}(-1/z) \mathbb{E}^z[\text{Im}(B_{\tau_D})].$$

The existence of the limit is included in the proof of the following lemma. Let $D_+ = \mathbb{D} \cap \mathbb{H}$ denote the upper half disk.

**Lemma 2.12.**

- If $r > 0$, 
  $$\text{hcap}(rK) = r^2 \text{hcap}(K), \quad \text{hcap}(r + K) = \text{hcap}(K).$$
- If $K \subset \overline{D}_+$, then
  \begin{equation}
  \text{hcap}(K) = \int_0^\pi \mathbb{E}^\theta [\text{Im}(B_{\tau_D})] \left( \frac{2}{\pi} \sin \theta \right) d\theta.
  \end{equation}

**Proof.** (sketch) The scaling rule follows immediately from the scaling rule $\phi_D(rz) = \phi_D(z)/r$, and the translation invariance by the rule $\phi_D(rz) = \phi_D(z)$. The last equality follows by taking a Brownian motion starting at $z$ and considering the hitting distribution of $\mathbb{R} \cup \overline{D}_+$, restricted to the unit circle. Then it can be shown (Exercise 2.26) that as $z \to \infty$, the hitting density of the unit circle is given by

$$\text{Im}(-1/z) \mathbb{E}^z[\text{Im}(B_{\tau_D})] = \left[ \int_0^\pi \mathbb{E}^\theta [\text{Im}(B_{\tau_D})] \left( \frac{2}{\pi} \sin \theta \right) d\theta \right] [1 + O(|z|^{-1})].$$

In other words,

$$\text{Im}(-1/z) \mathbb{E}^z[\text{Im}(B_{\tau_D})] = \left[ \int_0^\pi \mathbb{E}^\theta [\text{Im}(B_{\tau_D})] \left( \frac{2}{\pi} \sin \theta \right) d\theta \right] [1 + O(|z|^{-1})].$$

**Remark 2.13.** From the lemma one can conclude immediately that $\text{hcap}(\overline{D}_+) = 1$. One can also see this by noting that the function $f(z) = z + z^{-1}$ maps $\mathbb{H} \setminus \overline{D}_+$ conformally onto $\mathbb{H}$. If $K_1 \subset K$, then $\text{hcap}(K_1) \leq \text{hcap}(K)$. In particular, we get the estimate

$$\text{hcap}(K) \leq \text{rad}(K)^2,$$

\footnote{Actually we can only assert that it vanishes at the regular points of the boundary. We will not define regular here, but if all the connected components of $\partial D$ are larger than singletons then all points on $\partial D$ are regular.}
where \( \text{rad}(K) = \sup\{|z| : z \in K\} \). There is no corresponding bound in the opposite direction even for simply connected \( D \). In fact, one can check that there is a \( c \) such that for all \( K \)

\[
\text{hcap}(K) \leq c \text{rad}(K) \sup\{\text{Im}(z) : z \in K\}.
\]

**Remark 2.14.** The proof of (2.2) also gives an error estimate. There is a \( c < \infty \) such that for all \( K \subset \overline{D}^+ \), and \(|z| \geq 2\),

\[
|\text{Im}(-1/z)^{-1} \mathbb{E}^z[\text{Im}(B_{\tau_{D,k}})] - \text{hcap}(K)| \leq \frac{c \text{hcap}(K)}{|z|}.
\]

By scaling, this implies that for any \( K \) and any \(|z| \geq 2\text{rad}(K)\),

\[
|\text{Im}(-1/z)^{-1} \mathbb{E}^z[\text{Im}(B_{\tau_{D,K}})] - \text{hcap}(K)| \leq \frac{c \text{hcap}(K) \text{rad}(K)}{|z|}.
\]

Note that the error is of order \( \text{hcap}(K) \text{rad}(K) \) rather than \( \text{hcap}(K)^2 \). As mentioned above, \( \text{rad}(K) \) can be much larger than \( \text{hcap}(K) \).

**Remark 2.15.** In much of the literature on SLE, the half-plane capacity is called just the capacity and denoted \( \text{cap} \). However, this can lead to confusion because there are other natural definitions of capacities of sets in \( \mathbb{H} \).

If \( D = \mathbb{H} \setminus K \) is simply connected, then \( \phi_D \) is the imaginary part of a conformal transformation \( g_D : D \to \mathbb{H} \)

\[
g_D = \text{Re}[g_D] + i\phi_D.
\]

We will also write this as \( g_K \). This defines \( \text{Re}[g_D] \) up to an additive constant. We define \( g_D \) uniquely by specifying that the additive constant should be “0 at infinity”, i.e., so that \( g_D \) has the expansion

\[
g_D(z) = z + \frac{\text{hcap}(K)}{z} + O(|z|^{-2}), \quad z \to \infty.
\]

There is a error estimate similar to (2.5),

\[
|g_D(z) - \left(z - \frac{\text{hcap}(K)}{z}\right)| \leq \frac{c \text{hcap}(K) \text{rad}(K)}{|z|^2}, \quad |z| \geq 2\text{rad}(K).
\]

**Figure 1.** The conformal transformation \( g_A = g_{\mathbb{H} \setminus A} \).
3. Loewner equation

The last inequality implies a version of the Loewner equation.

**Proposition 2.16.** Suppose for each \( t > 0 \) there is a set \( \mathcal{K}_t \) as above. Suppose 
\[
\dot{a}(0) = \partial_t |\text{hcap}(\mathcal{K}_t)| \big|_{t=0+}
\]
exists and \( r_t := \text{rad}(\mathcal{K}_t) \to 0 \) as \( t \to 0^+ \). Let \( \phi_t = \phi_{\mathcal{K}_t} \). Then for fixed \( z \in \mathbb{H} \), as \( t \to 0^+ \),
\[
\phi_{\mathcal{K}_t}(z) = \text{Im}(z) + \dot{a}(0) t \text{Im}(1/z) + O(t r_t),
\]
i.e.,
\[
\partial_t [\phi_t(z)] \big|_{t=0+} = \dot{a}(0) \text{Im}(1/z).
\]

If the domains \( \mathcal{D}_t = \mathbb{H} \setminus \mathcal{K}_t \) are simply connected and \( g_t = g_{\mathcal{K}_t} \),
\[
\partial_t [g_t(z)] \big|_{t=0} = \frac{\dot{a}(0)}{z}.
\]

The last proposition is the basis for the following proposition which introduces the (chordal) Loewner differential equation. We will not give the details of the proof. One does need to prove that \( U_t \) is continuous; the Beurling estimate is a useful tool for this.

**Proposition 2.17.** Suppose \( \gamma : (0, T] \to \mathbb{H} \) is a simple curve with \( \gamma(0+) := U_0 \in \mathbb{R} \). Let \( a(t) = \text{hcap}(\gamma(0, t]) \), \( g_t = g_{\gamma(0, t]} \), and suppose that \( a \) is \( C^1 \). Then \( g_t(z) \) satisfies
\[
\dot{g}_t(z) = \frac{\dot{a}(t)}{g_t(z) - U_t}, \quad g_0(z) = z,
\]
for each \( z \in \mathbb{H} \setminus \gamma(0, T] \), this is valid for \( t \leq T \). For \( z = \gamma(s) \), this is valid for \( t < s \). The function \( t \mapsto U_t \) is continuous.

![Figure 2. The conformal transformation \( g_t \) induced by \( \gamma \).](image)

In the last proposition we started with a curve and produced a function \( U_t \). We will reverse the procedure here. Suppose \( a : [0, \infty) \to [0, \infty) \) is a strictly increasing \( C^1 \) function with \( a(0) = 0 \), and \( U : [0, \infty) \to \mathbb{R} \) is a continuous function. We will consider the (chordal) Loewner equation.

\[
\dot{g}_t(z) = \frac{\dot{a}(t)}{g_t(z) - U_t}, \quad g_0(z) = z,
\]
for each \( z \in \mathbb{C} \setminus \{0\} \), the solution of the equation above exists up to time \( T_z \in (0, \infty] \).

Using the continuity of \( U_t \), one can see that for every \( \epsilon > 0 \) there is a \( t > 0 \) such that \( T_z \geq t \) for \( |z - U_0| \geq \epsilon \). Moreover, it can be shown that for fixed \( t \), \( g_t \) is the
conformal transformation of \( H_t := \{ z \in \mathbb{H} : T_z > t \} \) onto \( \mathbb{H} \) with expansion at infinity

\[
g_t(z) = z + \frac{a(t)}{z} + O(|z|^{-2}), \quad z \to \infty.
\]

If \( f_t(z) = g_t^{-1}(z) \), then \( f_t \) is a conformal transformation of \( \mathbb{H} \) onto \( H_t \). By differentiating both sides of \( f_t(g_t(z)) = z \) with respect to \( t \), we see that \( f_t \) satisfies

\[
\dot{f}_t(z) = -\frac{\dot{a}(t) f_t'(z)}{z-U_t}, \quad f_t(0) = 0.
\]

Here, and throughout these lectures, ‘\( \cdot \)' refers to spatial derivatives.

We have made no assumptions on \( U_t \) other than continuity. Since \( U_t \) is real-valued, \( g_t(z) = g_t(z) \) and \( T_z = T_z \). Usually we consider the equation only for \( z \in \mathbb{H} \). If we write

\[
g_t(z) = u_t(z) + iv_t(z),
\]

then (2.8) becomes

\[
\dot{u}_t(z) = \frac{\dot{a}(t) (u_t(z) - U_t)}{(u_t(z) - U_t)^2 + v_t(z)^2}, \quad \dot{v}_t(z) = -\frac{\dot{a}(t) v_t(z)}{(u_t(z) - U_t)^2 + v_t(z)^2}.
\]

For fixed \( z \), we will often write

\[
Z_t = Z_t(z) = g_t(z) - U_t = X_t + iY_t,
\]

\[
X_t = X_t(z) = u_t(z) - U_t, \quad Y_t = Y_t(z) = v_t(z),
\]

in which case we can write (2.8) as

\[
\dot{u}_t(z) = \frac{\dot{a}(t) X_t}{X_t^2 + Y_t^2}, \quad \dot{v}_t(z) = \dot{Y}_t = -\frac{\dot{a}(t) Y_t}{X_t^2 + Y_t^2}.
\]

Differentiating (2.8) with respect to \( z \) gives

\[
g_t'(z) = -\frac{\dot{a}(t) g_t'(z)}{Z_t^2}.
\]

Since \( g_0'(z) = 1 \), we can solve this equation,

\[
g_t'(z) = \exp \left\{ -\int_0^t \frac{\dot{a}(s) \, ds}{Z_s^2} \right\},
\]

\[
|g_t'(z)| = \exp \left\{ -\int_0^t \text{Re} \left[ \frac{\dot{a}(s)}{Z_s^2} \right] \, ds \right\} = \exp \left\{ \int_0^t \frac{\dot{a}(s) (Y_s^2 - X_s^2) \, ds}{(X_s^2 + Y_s^2)^2} \right\}
\]

The last equation can be rewritten as

\[
\partial_t |g_t'(z)| = \dot{a}(t) |g_t'(z)| \frac{Y_t^2 - X_t^2}{(X_t^2 + Y_t^2)^2}.
\]
4. Maps generated by a curve

Any continuous function $U_t$ and $C^1$ function $a(t)$ produces the conformal maps $g_t$ and hence the domains $H_t$. It is not always true that the domains $H_t$ are obtained by slitting $\mathbb{H}$ with a curve $\gamma$. By a curve, we will mean a continuous function from an interval in $\mathbb{R}$ into $\mathbb{C}$.

**Definition 2.18.** Let

$$H_t^{\text{pion}} := \bigcup_{0 \leq s \leq t} \partial H_s$$

denote the pioneer points of $H_t$. If there is a curve $\gamma : [0, \infty) \rightarrow \mathbb{H}$ with $\gamma(0) \in \mathbb{R}$ such that

$$H_t^{\text{pion}} = \mathbb{R} \cup \gamma(0, t],$$

we say that $g_t$ is generated by the curve $\gamma$. (The term pioneer comes from the idea that a pioneer is someone who is on the frontier at some time.)

Note that $H_t$ is the unbounded component of $\mathbb{H} \setminus H_t^{\text{pion}}$. Suppose that $g_t$ is generated by a curve $\gamma$. If $\gamma$ is simple with $\gamma(0, t] \subset \mathbb{H}$, then $H_t^{\text{pion}} = \mathbb{R} \cup \gamma(0, t]$. If $\gamma$ is not simple, then it is possible for there to be points in $\mathbb{H} \setminus H_t$ that are not on $\gamma(0, t]$. Since $\partial H_t \subset \mathbb{R} \cup \gamma(0, t]$, we can see that

$$\text{hcap}(\gamma(0, t]) = \text{hcap}(\mathbb{H} \setminus H_t) = a(t).$$

In particular, for every $t, \epsilon > 0$,

$$\gamma(t, t + \epsilon) \cap H_t \neq \emptyset.$$ 

Also,

$$U_t = g_t(\gamma(t)) = \lim_{\epsilon \to 0^+} g_t(\gamma(t + \epsilon)).$$

It is sometimes difficult to tell whether or not the maps are generated by a curve. The next proposition gives a criterion.

**Proposition 2.19.** Suppose that $U_t : [0, 1] \rightarrow \mathbb{R}$ is a continuous function and $g_t$ is the solution to (2.8) with $a(t) = t$. Suppose that there exists $v : (0, 1] \rightarrow (0, 1]$ with $v(0+) = 0$ such that for all $0 \leq t \leq 1$ and all $\epsilon < 1$,

$$\int_0^t |f_t'(U_t + iy)| dy \leq v(\epsilon).$$

Then $U_t$ is generated by a curve $\gamma$.

**Proof.** Using (2.13), we can see that the limit

$$\gamma(t) := \lim_{\epsilon \to 0^+} f_t(U_t + i\epsilon),$$

exists and

$$|\gamma(t) - f_t(U_t + i\epsilon)| \leq v(\epsilon), \quad 0 < t, \epsilon \leq 1.$$ 

For fixed $\epsilon > 0$, the function $t \mapsto f_t(U_t + i\epsilon)$ is continuous and hence there is a $\delta_\epsilon$ such that

$$|f_s(U_t + i\epsilon) - f_t(U_s + i\epsilon)| \leq v(\epsilon), \quad |s - t| \leq \delta_\epsilon,$$

which implies

$$|\gamma(t) - \gamma(s)| \leq 3v(\epsilon), \quad |s - t| \leq \delta_\epsilon.$$
This shows that \( t \mapsto \gamma(t) \) is a continuous function. The definition of \( \gamma \) shows that \( \gamma(t) \in \partial H_t \) and hence \( H_t \) is contained in the unbounded component of \( \mathbb{H} \setminus \gamma(0,t) \). It is not difficult to show, in fact, that these are equal. \( \square \)

**Remark 2.20.** The uniform bound (2.13) is more than is needed to show that the limit (2.14) exists, but it is used to prove the continuity of \( \gamma \). There exist examples where the limit (2.14) exists for all \( t \) but for which \( \gamma \) is not a continuous function of \( t \).

**Remark 2.21.** The distortion theorem tells us that

\[
\int_0^{2^{-n}} |f_t'(U_t + iy)| \geq \sum_{j=0}^{\infty} 2^{-j} |f_t'(U_t + i2^{-j})|.
\]

A sufficient condition is to show that \( |f_t'(U_t + iy)| \leq \phi(y) \) where \( \phi \) satisfies

\[
\sum_{j=1}^{\infty} 2^{-j} \phi(2^{-j}) < \infty.
\]

5. A flow on conformal maps

The Loewner equation can be considered as a flow on the space of locally real conformal transformations at the origin. Suppose

\[
F(z) = \sum_{n=0}^{\infty} \frac{q_n}{n!} z^n,
\]

is a function analytic in a neighborhood \( \mathcal{N} = \mathcal{N}_F \) of the origin with \( q_1 > 0 \) and \( q_n \in \mathbb{R} \) for \( n \geq 0 \). Assume for ease that \( U_0 = 0 \). Let \( K_t = \mathbb{H} \setminus H_t \). For \( t \) sufficiently small, \( K_t \subset \mathcal{N} \) and hence we can define \( K_t^* = F(K_t) \). Let \( g_t^* = g_{K_t^*} \) and set \( \psi_t(z) = (g_t^* \circ F \circ g_t^{-1})(z) \). We can write down the differential equation for \( \psi_t \). Assume that \( a(t) = t, \dot{a}(t) = 1 \). Then the map \( g_t^* \) satisfies the equation

\[
\dot{g}_t^*(z) = \frac{\psi_t'(U_t)^2}{g_t^*(z) - U_t},
\]

where \( U_t^* = \psi_t(U_t) \). The extra term \( \psi_t'(U_t)^2 \) arises from the scaling rule \( \text{hcap}(rK) = r^2 \text{hcap}(K) \). Since \( \psi_t(z) = g_t^* \circ F \circ f_t \), the chain rule gives

\[
\dot{\psi}_t(z) = \frac{\psi_t'(U_t)^2}{\psi_t(z) - \psi_t(U_t^*)} - \frac{\psi_t'(z)}{z - U_t}.
\]

In particular, since \( U_0 = 0 \),

\[
\dot{\psi}_0(z) = (\Lambda F)(z) := \frac{F'(0)^2}{F(z) - F(0)} - \frac{F'(z)}{z}.
\]

Note that \( \Lambda F \) is analytic in \( \mathcal{N} \) with \( \Lambda F(0) = -3F''(0)/2 \). By differentiating this equation, we can find \( \dot{\psi}_0^{(k)}(n) \) for all positive integers \( k \).

**Lemma 2.22.** If

\[
F(z) = \sum_{n=0}^{\infty} q_n x^n,
\]
then
\[
\Lambda F = -\frac{3q_2}{2} + \left( \frac{q_2^2}{4q_1} - \frac{2q_3}{3} \right) z + \left( \frac{q_2 q_3}{6q_1} - \frac{5q_4}{24} - \frac{q_2^3}{8q_1^2} \right) z^2 + \cdots .
\]

**Proof.** Since \( \Lambda (rF + q_0) = r \Lambda F \), it suffices to prove the expansion for \( q_0 = 0, q_1 = 1 \), for which
\[
\Lambda F(z) = \left[ \sum_{n=1}^{\infty} \frac{q_n}{n!} z^n \right]^{-1} - \frac{1}{z} \sum_{n=0}^{\infty} \frac{q_{n+1}}{n!} z^n.
\]

We expand
\[
\left[ \sum_{n=1}^{\infty} \frac{q_n}{n!} z^n \right]^{-1} = \frac{1}{z} \left[ 1 - \left( \sum_{n=2}^{\infty} \frac{q_n}{n!} z^{n-1} \right) + \left( \sum_{n=2}^{\infty} \frac{q_n}{n!} z^{n-1} \right)^2 - \cdots \right],
\]

\[
\frac{1}{2} \left( \sum_{n=2}^{\infty} \frac{q_n}{n!} z^{n-1} \right) = \frac{q_2}{2} + \frac{q_3}{6} z + \frac{q_4}{24} z^2 + \cdots,
\]

\[
\frac{1}{z} \left( \sum_{n=2}^{\infty} \frac{q_n}{n!} z^{n-1} \right)^2 = \frac{q_2^2}{4} z + \frac{q_2 q_3}{6} z^2 + \cdots,
\]

\[
\frac{1}{z} \left( \sum_{n=2}^{\infty} \frac{q_n}{n!} z^{n-1} \right)^3 = \frac{q_2^3}{8} z^2 + \cdots,
\]

which gives
\[
\frac{1}{F(z)} = \frac{1}{z} - \frac{q_2}{2} + \left( \frac{q_2^2}{4} - \frac{q_3}{6} \right) z + \left( \frac{q_2 q_3}{6} - \frac{q_4}{24} - \frac{q_2^3}{8} \right) z^2 + \cdots .
\]

Also,
\[
\frac{1}{z} \sum_{n=0}^{\infty} \frac{q_{n+1}}{n!} z^n = \frac{1}{z} + \frac{q_2}{2} z + \frac{q_3}{6} z^2 + \cdots,
\]

giving
\[
\Lambda F(z) = -\frac{3q_2}{2} + \left( \frac{q_2^2}{4} - \frac{2q_3}{3} \right) z + \left( \frac{q_2 q_3}{6} - \frac{5q_4}{24} - \frac{q_2^3}{8} \right) z^2 + \cdots .
\]

\[
\square
\]

6. Doubly infinite time

If \( U_t : (-\infty, \infty) \to \mathbb{C} \) is a continuous function, we can consider the solution \( g_t \) of the Loewner equation
\[
(2.17) \quad \dot{g}_t(z) = \frac{a}{g_t(z) - U_t}, \quad g_0(z) = z.
\]

Let \( \tilde{g}_t(z) = g_{-t}(z), \tilde{U}_t = U_{-t} \). Then \( \tilde{g}_t, 0 \leq t < \infty \), satisfies
\[
\dot{\tilde{g}}_t(z) = -\frac{a}{\tilde{g}_t(z) - \tilde{U}_t}, \quad \tilde{g}_0(z) = z.
\]
Proposition 2.23. For each \( t \geq 0 \), \( \tilde{g}_t \) is a conformal transformation of \( \mathbb{H} \) onto a subdomain \( H_t = \tilde{g}_t(\mathbb{H}) \) with \( \text{hcap}(\mathbb{H} \setminus H_t) = \) at satisfying
\[
\tilde{g}_t(z) = z + \frac{at}{z} + O(|z|^{-2}), \quad z \to \infty.
\]

Proof. For fixed \( T > 0 \), let \( V_t = \tilde{U}_{T-t} - U_T \). If \( z \in \mathbb{H} \), then \( r_t(z) = \tilde{g}_{T-t}(z) - U_T \) satisfies
\[
\dot{r}_t(z) = \frac{a}{r_t(z) - V_t}, \quad r_0(z) = \tilde{g}_T(z) - U_T, \quad r_T(z) = z - U_T.
\]

In other words, if we let \( g_t(z) = \tilde{g}_{T-t}(\tilde{g}_T^{-1}(z + U_T)) - U_T \), then \( g_t(z) \) satisfies
\[
\dot{g}_t(z) = \frac{a}{g_t(z) - V_t}, \quad g_0(z) = z.
\]

This is the usual Loewner equation. Note that
\[
g_T(z) = \tilde{g}_0(\tilde{g}_T^{-1}(z + U_T)) - U_T = \tilde{g}_T^{-1}(z - V_T) + V_T.
\]

In particular,
\[
(2.18) \quad \tilde{g}_T'(z - V_T) = (g_T^{-1})'(z).
\]

7. Distance to boundary

Suppose \( g_t \) satisfies (2.18), \( H_t = \{ z : T_z > t \} \), and \( z \in \mathbb{H} \). Recall that \( H_t^{\text{pion}} := \bigcup_{0 \leq s < t} \partial H_s \) denotes the pioneer points of \( H_t \). If \( \gamma \) is generated by a curve \( \gamma \), then \( H_t^{\text{pion}} = \mathbb{R} \cup \gamma(0, t) \). In this section we consider
\[
\text{dist} \left[ z, H_t^{\text{pion}} \right].
\]

If the maps are generated by the curve \( \gamma \), this is the same as \( \text{dist}(z, \mathbb{R} \cup \gamma(0, t)) \). If \( t < T_z \), this is also the same as \( \text{dist}(z, \partial H_t) \). Therefore,
\[
\text{dist} \left[ z, H_t^{\text{pion}} \right] = \lim_{t \to T_z^{-}} \text{dist}(z, \partial H_t).
\]

This exact quantity is not as easy to study as a closely related quantity. For \( t < T_z \), we define
\[
(2.19) \quad Y_t = Y_{t,z} = \text{Im}[g_t(z)], \quad Y_t = Y_{t,z} = \frac{Y_t}{|g_t(z)|}
\]

Using (2.9) and (2.11) we see that
\[
(2.20) \quad \dot{Y}_t = -Y_t \frac{2 \dot{\alpha}(t) Y_t^2}{(X_t^2 + Y_t^2)^2},
\]
\[
Y_t = \text{Im}[z] \exp \left\{ -2 \int_{0}^{t} \frac{\dot{\alpha}(s) Y_s^2}{(X_s^2 + Y_s^2)^2} ds \right\}.
\]

In particular, \( Y_t \) is decreasing in \( t \), so we can define
\[
Y_{\infty,z} = Y_{T_z - z} = \exp \left\{ -2 \int_{0}^{T_z} \frac{\dot{\alpha}(s) Y_s^2}{(X_s^2 + Y_s^2)^2} ds \right\}, \quad t \geq T_z.
\]

The following lemma is an immediate corollary of the Koebe-(1/4) theorem.
Lemma 2.24. Under the assumptions above, if \( t < T_z \),

\[
\frac{\Upsilon_{t,z}}{4} \leq \text{dist}(z, H^\text{pion}_t) \leq 4 \Upsilon_{t,z}.
\]

Hence,

\[
\frac{\Upsilon_{\infty,z}}{4} \leq \text{dist}(z, H^\text{pion}_\infty) \leq 4 \Upsilon_{\infty,z}.
\]

The quantity \( \Upsilon_t \) is sometimes called the conformal radius. Note that (2.20) can be rewritten as

\[
\partial_t \Upsilon_t = -2 \dot{a}(t) \Upsilon_t \left[ \pi H(0, Z_t) \right]^2,
\]

where \( H \) denotes the Poisson kernel.

8. Exercises for Lecture 2

Exercise 2.25. Let \( D = \{ x + iy : 0 < x < \infty, 0 < y < \pi \} \) be a half-infinite rectangle. Use separation of variables, as outlined below, to find \( H_D(x + iy, iy') \) for \( x > 0, 0 < y, y' < \pi \).

1. Find all functions of the form

\[
\phi(x + iy) = \phi_1(x) \phi_2(y)
\]

that are harmonic in \( D \) and vanish on the horizontal boundaries of \( D \).

2. Find the linear combination of these functions whose boundary value is the \( \delta \)-function at \( iy' \).

Exercise 2.26. Use Exercise 2.25 and conformal invariance to justify (2.3) and (2.4).

Exercise 2.27. Let \( B_t \) be a standard complex Brownian motion starting at the origin and \( \tau = \tau_D = \inf\{ t : |B_t| = 1 \} \). Find

\[
\lim_{\epsilon \to 0} \epsilon^{-1/2} P\{ B[0, \tau] \cap [\epsilon, 1] = \emptyset \}.
\]

Exercise 2.28. Let \( D, \phi_D \) be as in the beginning of Section 2. Let \( B_t \) be a standard complex Brownian motion starting at \( z \in D \). For each \( R > 0 \), let

\[
\sigma_R = \inf\{ t : B_t \notin D \text{ or } \text{Im}[B_t] = R \}.
\]

Show that

\[
\lim_{R \to \infty} R P\{ \text{Im}[B_{\sigma_R}] = R \} = \phi_D(z).
\]

Conclude that if \( x \in \mathbb{R} \) and \( \text{dist}(x, \mathbb{H} \setminus D) > 0 \), then \( \partial_y \phi_D(x) \) is the probability that a Brownian motion started at \( x \) conditioned to stay in \( \mathbb{H} \) forever (i.e., a Brownian excursion) stays in \( D \) for all time.

Exercise 2.29. Find \( g_K \) for the following sets:

- \( K = (0, y] \)
- \( K \) is the line segment from 0 to \( e^{i\theta} \).
Exercise 2.30. Suppose $g_t$ is the solution to (2.7) with $a(t) = at$. Fix $T > 0$ and let $V_t = U_{T-t} - U_T$. Suppose $h_t$ is the solution to the reverse time Loewner equation

$$
\dot{h}_t(z) = \frac{a}{V_t - h_t(z)}, \quad 0 \leq t \leq T.
$$

Show that $h_T(z) = g_T^{-1}(z + U_T)$. 
We now return to the problem of determining possible candidates for the scaling limit of discrete systems. We will focus on $\mu^D_\#(z,w)$, and we will not worry about the parametrization. We start by considering $\mu^H_\#(0,\infty)$. If we parametrize the curve so that the half-plane capacity grows linearly, then we get conformal maps satisfying

$$\partial_t g_t(z) = \frac{a}{g_t(z) - U_t}, \quad g_0(z) = z,$$

where $U_t$ is now random. Conformal invariance and the domain Markov property translate into conditions on $U_t$. In fact, they require $U_t$ to be continuous with stationary, independent increments. It is well known that this implies that $U_t$ is a one-dimensional Brownian motion.

Since the process should be invariant under dilations of $\mathbb{H}$, we can see that $h_t(z) := r - \frac{1}{r^2} g_{r^2 t}(rz)$ should have the same distribution as $g_t(z)$. Note that if $g_t$ satisfies (3.1), then

$$\partial_t h_t(z) = \frac{a}{h_t(z) - U_t^*},$$

where $U_t^* = r^{-1} U_{r^2 t}$. If $U_t$ is a Brownian motion, then $U_t^*$ has the same distribution as $U_t$ provided that the drift is zero. If the drift is nonzero, they do not have the same distribution.

We can choose the variance of $U_t$ and we can choose the parameter $a$. A simple time change shows that, in fact, there is only one free parameter. As originally defined, the parameter $a$ was chosen to be 2 and $\kappa$ was used for the variance of the Brownian motion. Here, we choose the variance of the Brownian motion to be 1 and use $a$ as the free parameter. Choosing $a = 2/\kappa$ gives $\text{SLE}_\kappa$.

1. Definition

**Definition 3.31.** The **chordal Schramm-Loewner evolution** (from 0 to $\infty$ in $\mathbb{H}$ parametrized so that $\text{hcap}(\gamma(0,t)) = at$) with parameter $\kappa = 2/a$ is the solution of (3.1) where $U_t = -B_t$ is a standard one-dimensional Brownian motion with $B_0 = 0$. The (random) curve $\gamma$ that generates the maps $\{g_t\}$ is called the $\text{SLE}_\kappa$ curve.

It is not immediately obvious but has been proved that $\text{SLE}_\kappa$ is generated by a curve.

If $z \in \mathbb{H}$ and we write $Z_t(z) = X_t + iY_t = g_t(z) - U_t$, then (2.9) gives

$$dX_t = \frac{aX_t}{X_t^2 + Y_t^2} dt + dB_t, \quad \partial_t Y_t = -\frac{aY_t}{X_t^2 + Y_t^2} = Y_t - \frac{aX_t^2}{(X_t^2 + Y_t^2)^2}. $$

We also let

$$Y_t = Y_t(z) = \frac{Y_t}{|g_t(z)|}, \quad R_t = R_t(z) = \frac{X_t}{Y_t},$$

27
\( \Theta_t = \Theta_t(z) = \text{arg}(Z_t), \quad O_t = O_t(z) = (R_t^2 + 1) = [\sin^2 \Theta_t]^{-1}, \)

Recall that \( \Upsilon_t \) is related to the distance between \( z \) and \( \mathbb{R} \cup \gamma(0, t) \). Using (2.12), we see that
\[
\partial_t \Upsilon_t = -\Upsilon_t \frac{2a Y_t^2}{(X_t^2 + Y_t^2)^2}.
\]

Itô's formula gives
\[
d\Theta_t = \frac{(1 - 2a) X_t Y_t}{(X_t^2 + Y_t^2)^2} dt - \frac{Y_t}{X_t^2 + Y_t^2} dB_t,
\]
\[
dO_t = O_t \left[ \frac{[2r^2 + (4a - 1)r] X_t^2 + r Y_t^2}{(X_t^2 + Y_t^2)^2} dt + \frac{2r X_t}{X_t^2 + Y_t^2} dB_t \right].
\]

It is worth remembering that \( Y_t, \Upsilon_t \) are differential functions of \( t \) and the formulas for them are valid for any driving function \( U_t \). However, \( X_t, R_t, O_t \) have non-trivial quadratic variation. If we let \( N_t = \Upsilon_t - \frac{u}{a} Y_t^\frac{2a + r}{a} O_t^r \), then the product rule gives
\[
dN_t = N_t \left[ \frac{[2r^2 + (4a - 1)r - \theta] X_t^2 + [2u + r - \theta] Y_t^2}{(X_t^2 + Y_t^2)^2} dt + \frac{2r X_t}{X_t^2 + Y_t^2} dB_t \right].
\]

If the \( dt \) term is zero, this is a local martingale. Hence we get the following.

**Proposition 3.32.** If \( r \in \mathbb{R} \) and
\[
u = \nu(r) = r^2 + (2a - 1) r,
\]
Then
\[M_t = M_t(z) = \Upsilon_t^\frac{d}{a} Y_t^\frac{2a + r}{a} O_t^r,\]
is a local martingale satisfying
\[
dM_t = \frac{2r X_t}{X_t^2 + Y_t^2} M_t dB_t.
\]

**Example 3.33.** \( r = \frac{1}{2} - 2a, u = a - \frac{1}{4} \) gives the local martingale
\[M_t = \Upsilon_t^{d - 2} O_t^{\frac{1}{2} - 2a},\]
where
\[d = 1 + \frac{1}{4a} = 1 + \frac{\kappa}{8}.
\]

**Example 3.34.** \( r = -2a, u = 2a \) gives the local martingale
\[M_t = |g_t(z)|^2 O_t^{-2a}.
\]

**Example 3.35.** Let
\[r = -b = \frac{1 - 3a}{2}, \quad u = ab = \frac{b(1 - a)}{2}.
\]
This gives the martingale
\[
M_t = \Upsilon_t^{-b} Y_t^{-b} O_t^{-b} = |g_t(z)|^b Y_t^{-b} \left[ \pi H_{\mathbb{H}}(0, Z_t) \right]^b.
\]
Here \( H_{\mathbb{H}} \) denotes the Poisson kernel in \( \mathbb{H} \),
\[
H_{\mathbb{H}}(0, x + iy) = \frac{y}{\pi(x^2 + y^2)}.
\]
2. Phases

Recall that a curve is simple if it has no self-intersections.

**Theorem 3.36.** If $\kappa \leq 4$, SLE$_\kappa$ paths are simple. If $\kappa > 4$, SLE$_\kappa$ paths have self-intersections. In fact, if $\kappa > 4$ for every $s < t$ there exist $s < t_1 < t_2 < t$ with $\gamma(t_1) = \gamma(t_2)$.

**Remark 3.37.** To be more precise, the theorem states that the facts hold “with probability one”. We will feel free to leave this phrase out of statements of theorems.

We will prove a related fact and leave the proof of the theorem as an exercise (Exercise 3.40).

**Proposition 3.38.** If $\kappa \leq 4$, $\gamma(0, \infty) \subset \mathbb{H}$. If $\kappa > 4$, $\gamma(0, \infty) \cap \mathbb{R} \neq \emptyset$.

**Proof.** Let $x > 0$. Then $\gamma(0, \infty) \cap [x, \infty) \neq \emptyset$ if and only if $T_x < \infty$. We will show that $\mathbb{P}\{T_x < \infty\}$ is 1 if $\kappa > 4$ and equals 0 if $\kappa \leq 4$. Let $Z_t = Z_t(x) = g_t(x) - U_t$. Then the Loewner equations tells us that

$$dZ_t = \frac{a}{Z_t} dt + dB_t, \quad Z_0 = x. \quad (3.5)$$

This is the Bessel equation, and it is well known that the probability that $Z_t$ reaches the origin in finite time equals 1 or 0 depending on whether $a < 1/2$ or $a \geq 1/2$. □

**Remark 3.39.** If $W_t$ is a $d$-dimensional Brownian motion starting at $z \in \mathbb{R}^d \setminus \{0\}$, then $Z_t = |W_t|$ satisfies (3.5) with $a = \frac{d-1}{2}$, $x = |z|$, and $B_t$ a standard Brownian motion.

**Theorem 3.40.** If $\kappa \geq 8$, then $\gamma(0, \infty) = \mathbb{H}$, i.e., $\gamma$ is plane-filling. If $\kappa < 8$, then for each $z \in \mathbb{H}$, $\mathbb{P}\{z \in \gamma(0, \infty)\} = 0$.

**Proof.** Suppose $\kappa = \frac{2}{a} > 4$ and let $z \in \mathbb{H}$. By using the previous theorem and scaling we can see that $\mathbb{P}\{T_z < \infty\} = 1$. Suppose $\gamma(T_z) \neq z$. Then a straightforward argument (Exercise 3.37) shows that $\Theta_{T_z} \in \{0, \pi\}$, where $\Theta_t = \Theta_t(z)$. It will be convenient to make the time change

$$\tilde{X}_t = X_{\sigma(t)}, \quad \tilde{Y}_t = Y_{\sigma(t)},$$

where

$$\partial_t \sigma(t) = X_{\sigma(t)}^2 + Y_{\sigma(t)}^2,$$

$$\partial_t \tilde{Y}_t = \partial_t Y_{\sigma(t)} = \partial_t \sigma(t) Y_{\sigma(t)} \frac{-a}{X_{\sigma(t)}^2 + Y_{\sigma(t)}^2} = -a \tilde{Y}_t,$$

i.e., $\tilde{Y}_t = e^{-at}$. Under this time change $\log \tilde{Y}_t$ is a deterministic linear function. Time $T_z$ in the usual parametrization becomes time $\infty$ in the time change. Under this time change (3.3) becomes

$$d\Theta_t = (\frac{1}{2} - a) \sin(2\Theta_t) \ dt + \sin \Theta_t dB_t,$$

for a standard Brownian motion $B_t$. We have reduced the problem to a question about a one-dimensional diffusion. One can show that if $a < 1/4$ it is not the case that $\sin \Theta_t \to 0$ as $t \to \infty$. We will not prove this but let us sketch why this is true. If we do a time change, this equation looks like

$$d\tilde{\Theta}_t = (\frac{1}{2} - a) \frac{\sin(2\tilde{\Theta}_t)}{\sin^2 \tilde{\Theta}_t} \ dt + dB_t.$$
For $\tilde{\Theta}_t$ near zero this looks like
\[ d\tilde{\Theta}_t = \frac{(1 - 2a)}{\tilde{\Theta}_t} dt + d\tilde{B}_t. \]

by comparison with a Bessel equation, we see that to keep this from reaching the origin we need $1 - 2a \geq 1/2$ or $a \leq 1/4$.

This is not quite enough to prove the statement for $4 < \kappa < 8$. Since this follows from Theorem 3.41 in the next section, we will not bother to give the complete details here. \[ \square \]

If $\kappa < 8$, then $\Theta_{T_z} - \hat{\Theta}_{\infty} \in \{0, \pi\}$. Let $\phi(z) = P\{\Theta_{\infty} = \pi\}$. Scaling shows that $\phi$ depends only on the angle. Since $\phi(\tilde{\Theta}_t)$ is a martingale, we can use Itô’s formula to conclude that $\phi(\theta)$ satisfies
\[ 2(1 - 2a) \phi'(\theta) \cos \theta + \phi''(\theta) \sin \theta = 0, \quad \phi(0) = 0, \phi(\pi) = 1. \]

3. Dimension of the path
In this section we will discuss the dimension of the path $\gamma$. We will not give all the details.

**Theorem 3.41.** There exists $c_\ast$ such that if $a > 1/4$, then for all $z \in \mathbb{H}$
\[ P\{Y_{\infty}(z) \leq \delta\} \sim c_\ast G(z) \delta^{2-d}, \quad \delta \to 0, \]
where $d = 1 + \frac{1}{8a}$ and $G$ is the “Green’s function” for chordal SLE$_{2/a}$ defined by
\[ G(y(x + i)) = y^{d-2} (x^2 + 1)^{1/2 - 2a}. \]

**Proof.** By scaling it suffices to prove this result when $z = x + i$. We fix $x$ and we consider the martingale from Example 3.33
\[ M_t = Y_t^{d-2} \sin^{4a-1} \Theta_t = |y'(z)|^{2-d} G(Z_t), \]
which satisfies
\[ dM_t = \frac{(1 - 4a) X_t}{X_t^2 + Y_t^2} M_t dB_t. \]

One can check that with probability one $M_t \to 0$. Let $\tau_\delta = \inf\{t : T_t \leq \delta\}$. For fixed $\delta$, $M_{t \wedge \tau_\delta}$ is a uniformly bounded martingale, and hence the optional sampling theorem gives
\[ G(z) = M_0 = \lim_{t \to \infty} E[M_{t \wedge \tau_\delta}] = E[M_{\tau_\delta}; \tau_\delta < \infty] = \delta^{d-2} E[\sin^{4a-1} \Theta_{\tau_\delta}; \tau_\delta < \infty]. \]

With the aid of Girsanov (details of this argument are left as Exercise 3.49), we can show that there exists $0 < c_\ast < \infty$ such that
\[ E[\sin^{4a-1} \Theta_{\tau_\delta}; \tau_\delta < \infty] \sim [1/c_\ast] P\{\tau_\delta < \infty\}. \]

**Remark 3.42.** In fact it can be shown (but is more difficult) that the Hausdorff dimension of the path $\gamma(0, \infty)$ is $d = 1 + \frac{\kappa}{8}$ if $\kappa \leq 8$. 
4. Cardy’s formula

In this section, we derive a formula for $SLE$ that corresponds to Cardy’s formula for percolation as discussed in Lecture 1, Section 3. Suppose $\gamma(0, \infty)$ is an $SLE_\kappa$ curve with $\kappa > 4$. Suppose $x, y > 0$. Then with probability one $T_x, T_{-y} < \infty$. We will consider

$$\phi(x, y) = P\{T_{-y} > T_x\}.$$ 

Note that scaling implies that $\phi(x, y) = \phi(y/x)$ where $\phi(y) = \phi(1, y)$.

**Proposition 3.43** (Cardy’s formula). If $0 < a < 1/2$, then

$$\phi(y) = \Gamma(2 - 4a) \int_0^{\infty} \frac{du}{u^{2a} (1 - u)^{2a}}.$$

**Proof.** Let $X_t = g_t(1) - U_t, J_t = g_t(-y) - U_t$ and

$$Z_t = \frac{X_t}{X_t - J_t}.$$ 

Let $\tau = \inf\{t : Z_t \in \{0, 1\}\}$ and

$$\psi(r) = P\{Z_\tau = 0 \mid Z_0 = r\}.$$ 

Then $\phi(y) = \psi(\frac{1}{y + 1})$. Since

$$dX_t = \frac{X_t}{J_t} dt + dB_t, \quad \partial_t[X_t - J_t] = \frac{a}{X_t} - \frac{a}{J_t},$$

we can see that

$$dZ_t = \frac{a}{(X_t - J_t)^2} \left[ \frac{1}{Z_t} + \frac{1}{Z_t - 1} \right] dt + \frac{1}{X_t - J_t} dB_t.$$ 

We can do a random time change $\sigma$ and see that $\tilde{Z}_t := Z_{\sigma(t)}$ satisfies

$$d\tilde{Z}_t = \left[ \frac{a}{\tilde{Z}_t} - \frac{a}{1 - \tilde{Z}_t} \right] dt + d\tilde{B}_t,$$

where $\tilde{B}_t$ is a standard Brownian motion. However, $\psi(\tilde{Z}_t)$ is a martingale. Using Itô’s formula, we can see that this implies

$$\psi''(u) + 2a \left[ \frac{1}{u} - \frac{1}{1 - u} \right] \psi'(u) = 0.$$ 

Solving this equation with the boundary conditions $\psi(0) = 1, \psi(1) = 0$ gives (3.6).

To understand why this corresponds to Cardy’s crossing formula for percolation, consider the percolation exploration process as in Lecture 1, Section 3. If we again put the boundary condition of black on the negative real axis and white on the positive real axis then the event $\{T_{-y} > T_1\}$ corresponds to the event that in the (scaled) percolation cluster there is a connected component of black sites connecting $[-y, 0]$ to $[1, \infty)$. Hence the “crossing probability” for $SLE_\kappa$ is given by

$$P_{\delta}([-y, 0], [1, \infty)) = \phi(y) = \Gamma(2 - 4a) \int_0^{\infty} \frac{du}{u^{2a} (1 - u)^{2a}}.$$
In the case $\kappa = 6$, which we will see in the next lecture corresponds to percolation, this crossing probability becomes

$$P_H([-y, 0], [1, \infty)) = \frac{\Gamma(2/3)}{\Gamma(1/3)^2} \int_0^y \frac{du}{u^{2/3}(1-u)^{2/3}}.$$  

This may not appear like a very nice formula, but if we map $\mathbb{H}$ to an equilateral triangle, we get the simple formula in Figure 9.

5. Conformal images of SLE

If $D$ is a simply connected domain and $z, w$ are distinct points in $\partial D$, then the measure $\mu_D^H(z, w)$ is defined to be the image of $\mu^H_\mathbb{H}(0, \infty)$ under a conformal transformation $f : \mathbb{H} \to D$ with $f(0) = z, f(\infty) = w$. Here, and for the rest of these lectures, we are considering the probability measures $\mu^H_D$ as being defined on curves modulo reparametrization. The map $f$ is not unique; however, any other such map $f_1$ can be written as $f_1(z) = f(rz)$ for some $r > 0$. The invariance of $\text{SLE}_\kappa$ under scaling shows that $\mu^H_D(z, w)$ is well defined.

![Figure 1. The maps $g_t^*, F_t$. Note that $F_t = g_t^* \circ F \circ g_t^{-1}$.](image)

Here we will assume that

$$F(z) = \sum_{j=0}^{\infty} a_j z^j, \quad a_j \in \mathbb{R}, \quad a_1 > 0,$$

is analytic in a neighborhood of the origin. The assumptions on the $a_j$ imply that near $0$ $F$ maps $\mathbb{R}$ into $\mathbb{R}$ and for some $\epsilon > 0$, $\epsilon \mathbb{D}_+ = \{z \in \mathbb{H} : |z| < \epsilon\}$ is mapped conformally into $\mathbb{H}$.

Suppose $\gamma$ is an $\text{SLE}_\kappa$ curve and let $\tau = \inf \{t : \gamma(t) \notin \epsilon \mathbb{D}_+\}$. For $t < \tau$, we can define the curve

$$\gamma^*(t) = F(\gamma(t)).$$

We let $H_t^*$ be the unbounded component of $\mathbb{H} \setminus \gamma^*(t)$ and $g_t^*$ the unique conformal transformation of $H_t^*$ onto $\mathbb{H}$ with

$$g_t^*(z) = z + \frac{a^*(t)}{z} + \cdots.$$
We also let $F_t = g_t^* \circ F \circ g_t^{-1}$; in other words, $F_t \circ g_t = g_t^* \circ F$. Using only properties of the Loewner equation, we can see that

\[(3.7) \quad \partial_t a^*(t) = a F'_t(U_t)^2.\]

The Loewner equation tells us that

\[
\partial_t g^*_t(z) = a F'_t(U_t)^2 \left( \frac{g'_t(z)}{g_t(z)} - U_t^* \right),
\]

where $U_t^* = g_t^*(\gamma^*(t)) = F_t(U_t)$.

**Proposition 3.44.** Under a suitable time change, $\hat{U}_t = U_{\sigma(t)}^*$ satisfies

\[
d\hat{U}_t = b \frac{\Phi''(\hat{U}_t)}{\Phi'(\hat{U}_t)} dt + dW_t,
\]

where $W_t$ is a standard Brownian motion, $b = \frac{3a-1}{2} = \frac{6 - \kappa}{2\kappa}$ is the boundary scaling exponent, and $\Phi_t = F_{\sigma(t)}^{-1}$.

**Proof.** We use Itô’s formula to get

\[
dU_t^* = dF_t(U_t) = \left[ \hat{F}_t(U_t) + \frac{1}{2} F''_t(U_t) \right] dt + F'(U_t) dU_t.
\]

The term $\hat{F}_t(U_t)$ can be calculated using the Loewner equation. In fact, it is the first term in (2.16), i.e., $\hat{F}_t(U_t) = -(3a/2) F''_t(U_t)$ (the factor $a$ appears because (2.16) assumes $a = 1$). Therefore,

\[
dU_t^* = -b F''_t(U_t) dt + F'(U_t) dU_t.
\]

With an appropriate time change we can write this as

\[
d\hat{U}_t = -b \frac{F''_{\sigma(t)}(U_{\sigma(t)})}{F'_{\sigma(t)}(U_{\sigma(t)})^2} dt + dW_t.
\]

A simple calculation shows that

\[
\frac{F''_{\sigma(t)}(U_{\sigma(t)})}{F'_{\sigma(t)}(U_{\sigma(t)})^2} = \frac{\Phi''(\hat{U}_t)}{\Phi'(\hat{U}_t)}.
\]

**Remark 3.45.** The half-plane capacity is not preserved by $F$. The time change in the proof is exactly the time change needed so that $\text{hcap}(\gamma^*(0, \sigma(t))) = at$.

6. Exercises for Lecture 3

**Exercise 3.46.** Show why Proposition 3.38 implies Theorem 3.36

**Exercise 3.47.** Verify the step in the proof of Theorem 3.40.

**Exercise 3.48.** Justify (3.7).

**Exercise 3.49.** The goal is to fill in the details to Theorem 3.41. Let $M_t$, $\tau_\delta$ be as in the paragraph following the theorem and assume $z = x + i, \theta_0 = \arg(z)$.
• Show that if we weight by the local martingale $M_t$ then
  
  $$
  d\Theta_t = \frac{2a}{2} \frac{X_t Y_t}{(X_t^2 + Y_t^2)^2} dt - \frac{Y_t}{X_t^2 + Y_t^2} dW_t,
  $$
  
  where $W_t$ is a standard Brownian motion in the weighted measure.

• Show that if we do a random time change $\sigma(t)$ so that $\tilde{\Upsilon} := \Upsilon_{\sigma(t)} = e^{-2at}$, then $\tilde{\Theta}_t = \Theta_{\sigma(t)}$ satisfies

  $$
  d\tilde{\Theta}_t = 2a \cot \tilde{\Theta}_t dt + d\tilde{W}_t,
  $$

  where $\tilde{W}_t$ is a standard Brownian motion.

• Let

  $$
  e(\theta, t) = E[\sin^{4a-1} \tilde{\Theta}_t \mid \tilde{\Theta}_0 = \theta],
  $$

  where $\tilde{\Theta}_t$ satisfies (3.8). Use the Girsanov theorem to prove that

  $$
  P\{ \Upsilon_\infty \leq e^{-2at} \} = e(t, \theta_0) G(z) e^{-2at(2-d)}.
  $$

• Find a stationary density for (3.8) and use it to show that

  $$
  \lim_{t \to \infty} e(t, \theta_0) = c^*_s
  $$

  for some $c^*_s \in (0, \infty)$. Determine $c^*_s$. 
Let $\mathcal{D}$ denote the set of simply connected subdomains $D$ of $\mathbb{H}$ with $\mathbb{H} \setminus D$ bounded and $\text{dist}(0, \mathbb{H} \setminus D) > 0$. We will write $\Phi_D$ for the unique conformal transformation of $D$ onto $\mathbb{H}$ with $\Phi_D(z) = z + o(1)$ as $z \to \infty$ (we denoted this by $g_D$ earlier, but it will be convenient to use a new notation). We will consider $\text{SLE}_\kappa$ from 0 to $\infty$ in $D$ and show that there are a number of equivalent ways of defining this process. We already have one definition: the image of $\text{SLE}_\kappa$ in $\mathbb{H}$ under the conformal transformation $F_D := \Phi_D^{-1}$. We note that (see Exercise 2.28) $\Phi_D'(0) \in (0, 1]$; in fact $\Phi_D'(0)$ is the probability that a Brownian “excursion” in $\mathbb{H}$ stays in $D$.

1. Drift and locality

Proposition 3.44 can be restated in the following way. Suppose $\gamma$ is a curve and let $g_t$ be the corresponding conformal maps. Let

$$
\tau = \tau_D = \inf \{ t : \text{dist}(\gamma(t), \mathbb{H} \setminus D) = 0 \}.
$$

**Proposition 4.50.** Suppose for $t < \tau$, $g_t$ is the solution to the Loewner equation (3.1) where $U_t$ satisfies the stochastic differential equation

$$
dU_t = b \left[ \log \Phi'_t(U_t) \right] \, dt - dB_t.
$$

Here $\Phi_t = \Phi_{D_t}$ where $D_t = g_t(D)$. Then $g_t$ (and the corresponding curves $\gamma$) have the distribution of $\text{SLE}_\kappa$ in $D$ stopped at time $\tau$.

In other words, the proposition gives an equivalent definition of $\text{SLE}_\kappa$ in $D$, at least up to time $\tau_D$. The drift is nontrivial unless $b = 0$ which holds if and only if $a = 1/3, \kappa = 6$.

**Theorem 4.51** (Locality). Suppose $\gamma$ is a curve with the distribution of $\text{SLE}_6$ in the domain $D \in \mathcal{D}$. The distribution of $\gamma$ up to time $\tau_D$ is the same as that of $\text{SLE}_\kappa$ in $\mathbb{H}$ up to time $\tau_D$.

If we consider our discrete models, we can see that the only one for which we would definitely expect the locality property is the percolation exploration process. As we saw in the last lecture, the crossing probability for $\text{SLE}_6$ is the same as that predicted by Cardy for percolation.

For other values of $\kappa$, we see that $\text{SLE}_\kappa$ in $D$ is obtained (at least for small time) by putting a drift in the Brownian motion. The Girsanov theorem tells us that Brownian motion with drift is absolutely continuous with respect to Brownian motion without drift, at least for small times.
2. Girsanov

The Girsanov theorem is a very useful tool for studying the (local) martingales arising in \( SLE_\kappa \) so we will discuss what we need here. Suppose that \( M_t \) is a nonnegative continuous local martingale satisfying

\[
dM_t = J_t M_t dB_t, \quad M_0 = 1.
\]

Although \( M_t \) may not be a martingale, we can approximate \( M_t \) by a uniformly bounded martingale. Indeed, if \( \tau_n = \inf \{ t : M_t \geq n \} \) and \( M_{t,n} = M_{t \wedge \tau_n} \), then for fixed \( n \), \( M_{t,n} \) is a uniformly bounded martingale satisfying

\[
dM_{t,n} = J_t 1\{ \tau_n > t \} M_{t,n} dB_t,
\]

and for fixed \( t \), \( M_{t,n} \to M_t \) as \( n \to \infty \). For any nonnegative martingale \( N_t \) satisfying

\[
dN_t = J_t N_t dB_t,
\]

there is a measure \( Q \) defined by

\[
Q(V) = \mathbb{E}[1_V N_t],
\]

if \( V \) is \( \mathcal{F}_t \)-measurable. The Girsanov theorem states that under the measure \( Q \),

\[
W_t = B_t - \int_0^t J_s ds,
\]

is a standard Brownian motion. In other words, \( B_t \) satisfies

\[
dB_t = J_t dt + dW_t,
\]

where \( W_t \) is a Brownian motion with respect to \( Q \).

The Girsanov theorem requires that \( N_t \) be a martingale. If we only know that \( M_t \) is a local martingale satisfying (4.1), then we can still use the Girsanov theorem as long as we run the paths up to a stopping time \( \tau_n \). In this case we get the equation

\[
dM_t = J_t M_t dB_t, \quad t < \tau_n.
\]

If we weight by the (local) martingale \( M_t \), we can say that \( B_t \) satisfies

\[
dB_t = J_t dt + dW_t, \quad t < \tau_n.
\]

One can often use this equation to determine whether or not \( M_t \) is actually a martingale. Essentially what keeps the local martingale from being a martingale is the fact that some mass “goes to infinity in finite time”. One can check that this happens by time \( t \) if and only if

\[
\lim_{n \to \infty} \mathbb{E}[M_{t,n}; \tau_n < t] > 0,
\]

i.e., if and only if in the weighted measure, there is a positive probability of explosion of \( M_t \) in finite time.

Remark 4.52. Although we will not prove the Girsanov theorem here, let us give a heuristic reason why it is true. Imagine that \( \Delta B_t := B_{t+\Delta t} - B_t \) were equal to \( \sqrt{\Delta t} \) with probability 1/2 and \( -\sqrt{\Delta t} \) with probability 1/2. Then \( M_{t+\Delta t} \) is about \((1 + J_t \sqrt{\Delta t}) M_t \) with probability 1/2 and \((1 - J_t \sqrt{\Delta t}) M_t \) with probability 1/2. If we weight by \( M_{t+\Delta t} \), then in the weighted measure \( \Delta B_t \) equals \( \sqrt{\Delta t} \) with probability \((1 + J_t \sqrt{\Delta t})/2 \) and equals \(-\sqrt{\Delta t} \) with probability \((1 - J_t \sqrt{\Delta t})/2 \). Under this measure

\[
\mathbb{E}[\Delta B_t] = J_t \Delta t.
\]
In other words, by weighting by the martingale we obtain a drift of $J_t$.

3. The restriction martingale

Let $D, \Phi, \Phi_t$ be as above. We will consider $\Phi'(U_t)^b$. In the next proposition, we use the fact that

$$\Phi'(U_t) = a \left[ \frac{\Phi''(U_t)^2}{4 \Phi'(U_t)} - \frac{2 \Phi'''(U_t)}{3} \right].$$

This is a property of the Loewner equation and can be seen as the second term in (2.16). With this, the following proposition is a straightforward Itô’s formula calculation. The Schwarzian derivative of a function $f$, $S_f$, is defined by

$$S_f(z) = \frac{f'''(z)}{f'(z)} - \frac{3}{2} \frac{f''(z)^2}{f'(z)^2},$$

and the central charge $c$ is defined by

$$c = \frac{2b(3 - 4a)}{a} = \frac{(3\kappa - 8)(6 - \kappa)}{2\kappa}.$$

**Proposition 4.53.** If $U_t$ is $SLE_{2/\kappa}$ and $D \in \mathcal{D}$, then for $t < \tau_D$,

$$d[\Phi'(U_t)^b] = \Phi'(U_t)^b \left[ \frac{ac}{12} S\Phi_t(U_t) dt - b \frac{\Phi''(U_t)}{\Phi'(U_t)} dB_t \right].$$

If we let

$$M_t = \exp \left\{ -c \int_0^t \frac{aS\Phi_t(U_t)}{12} dt \right\} \Phi'(U_t)^b,$$

then $M_t$ is a local martingale satisfying

$$dM_t = -b \frac{\Phi''(U_t)}{\Phi'(U_t)} M_t dB_t.$$

In particular, if $\kappa = 8/3$,

$$M_t = \Phi'(U_t)^{5/8},$$

is a martingale.

Then under the weighted measure $B_t$ satisfies

$$dB_t = -b \frac{\Phi''(U_t)}{\Phi'(U_t)} dt + dW_t,$$

where $W_t$ is a Brownian motion in the new measure, i.e., $U_t$ satisfies

$$dU_t = b \frac{\Phi''(U_t)}{\Phi'(U_t)} dt - dW_t,$$

**Fact.** $SLE_\kappa$ weighted by $\Phi'(U_t)^b$ gives $SLE_\kappa$ in the smaller domain.

This fact is valid for all $\kappa$ provided that $t$ is sufficiently small. For the rest of this lecture we will consider $\kappa \leq 4$ for which the curves are simple. In Proposition 4.57 we will see that

$$S\Phi_t(U_t) \leq 0,$$

so $M_t \leq 1$ for $\kappa \leq 8/3$. This implies that $M_t$ is a martingale. In fact, for all $\kappa \leq 4$, this is a martingale.
Theorem 4.54 (Restriction). If $\kappa = 8/3$ and $D \in \mathcal{D}$, then

$$P\{\gamma(0, \infty) \subset D\} = \Phi'_D(0)^{5/8}.$$ 

Moreover,

$$\frac{d\mu_D(0, \infty)}{d\mu_H(0, \infty)} = 1\{\gamma(0, \infty) \subset D\}.$$ 

Proof. We first need to observe (we omit the argument) that if $\gamma$ is a fixed curve with $\gamma(t) \to \infty$ and $\gamma(0, \infty) \subset D$, then $\Phi'_t(U_t) \to 1$. For each $\epsilon > 0$, let $\rho_\epsilon = \inf\{t : \text{dist}(\gamma(t), \mathbb{H} \setminus D) \leq \epsilon\}$.

Since $M_{t,}\epsilon = \Phi_{t \land \rho_\epsilon}(U_{t \land \rho_\epsilon})^{5/8}$ is a uniformly bounded martingale,

$$\Phi'_D(0)^{5/8} = E[M_0] = E[M_\infty; \rho_\epsilon = \infty] + E[M_{\rho_\epsilon}; \rho_\epsilon < \infty].$$

But we know that if we weight by the martingale $M_t$, the weighted paths have the distribution of $SLE_{8/3}$ in $D$ which is the same as the conformal image of $SLE_{8/3}$ in $\mathbb{H}$ under $\Phi_D^{-1}$. Since $\kappa \leq 4$, we know that these paths stay in $D$ and hence

$$\lim_{\epsilon \to 0^+} E[M_{\rho_\epsilon}; \rho_\epsilon < \infty] = 0,$$

and

$$\Phi'_D(0)^{5/8} = \lim_{\epsilon \to 0^+} E[M_\infty; \rho_\epsilon = \infty] = E[1\{\gamma(0, \infty) \subset D\}] = P\{\gamma(0, \infty) \subset D\}.$$ 

This generalizes to all $\kappa \leq 4$.

Theorem 4.55 (Restriction). If $\kappa = 2/a \leq 4$ and $D \in \mathcal{D}$, then

$$\frac{d\mu_D(0, \infty)}{d\mu_H(0, \infty)} = M_\infty = 1\{\gamma(0, \infty) \subset D\} \exp\left\{-c \int_0^\infty \frac{aS\Phi_t(U_t)}{12} dt \right\}.$$ 

For $\kappa \leq 8/3$ ($c \leq 0$), the martingale $M_t$ is uniformly bounded and the proof proceeds as the previous proof. For $8/3 < \kappa \leq 4$, the martingale is not uniformly bounded, but we can use an argument using the Girsanov theorem as above. In the next section, we study $M_\infty$.

4. (Brownian) boundary bubbles

The Brownian bubble measure $\nu_\mathbb{H}(x, x)$ is an infinite measure on curves $\gamma : (0, t_\gamma) \to \mathbb{H}$ with $\gamma(0+) = \gamma(t_\gamma-) = x$. There are a number of ways of defining it. One way is to consider Brownian motion starting at $x + \epsilon i$ and conditioned to leave $\mathbb{H}$ at $x$. Instead of thinking of this as a probability measure, we consider it as a finite measure with total mass $\pi H_\mathbb{H}(x + \epsilon i, x) = 1/\epsilon$. Here $H_\mathbb{H}$ denotes the Poisson kernel. (Recall that $H_\mathbb{H}(x + \epsilon i, x) = (\pi \epsilon)^{-1}$.) We define the Brownian bubble measure at $x$ by

$$\nu_\mathbb{H}(x) = \lim_{\epsilon \to 0^+} \pi \nu(x + \epsilon i, x) = \lim_{\epsilon \to 0^+} \frac{1}{\epsilon} \nu^\#(x + \epsilon i, x),$$

where $\nu^\#(x + \epsilon i, x)$ is the probability measure corresponding to Brownian motion starting at $x + \epsilon i$ conditioned to leave $\mathbb{H}$ at $x$. If $D$ is a subdomain of $\mathbb{H}$, then we let $\Gamma(x, D)$ be the $\nu_\mathbb{H}(x, x)$ measure of all $\gamma$ with $\gamma(0, t_\gamma) \not\in D$. If $\text{dist}(x, \mathbb{H} \setminus D) > 0$, then $\Gamma(x, D)$ is finite. We let $\Gamma(D) = \Gamma(0, D)$.

The quantity $\Gamma(x, D)$ is expressed nicely in terms of excursion measure. If $D$ is a domain and $z, w$ are distinct points in $\partial D$ at which the boundary is smooth,
the excursion Poisson kernel (sometimes called the Dirichlet to Neumann map) is defined by

$$H_{\partial D}(z, w) = \lim_{\epsilon \to 0^+} \epsilon^{-1} H_D(z + \epsilon n, w),$$

where $n$ denotes the inward unit normal at $z$. If $f : D \to D'$ is a conformal transformation and the boundary is sufficiently smooth, we have the scaling rule

$$H_{\partial D}(z, w) = |f'(z)|^2 H_{\partial D'}(f(z), f(w)).$$

Note that this is the scaling rule from Lecture 1 with $b = 1$. If $D_1 \subset D$ and the boundaries are nice, we define

$$\Gamma_D(z, D_1) = \int_{\partial D_1} H_{\partial D_1}(z, w) H_D(w, z) \, |dw|. \quad (1)$$

We could also write this as the integral over $D \cap \partial D_1$. In this notation $\Gamma(x, D) = \Gamma_H(x, D)$. The definition of $\Gamma_D(z, D_1)$ does not need smoothness of $D \cap \partial D_1$ — the same integral can be expressed in terms of Brownian excursion measure. Using the scaling rule for the (regular and excursion) Poisson kernel, we get the following conformal covariance rule

$$\Gamma_D(z, D_1) = |f'(z)|^2 \Gamma_{f(D)}(f(z), f(D_1)).$$

**Example 4.56.** Suppose $D = \mathbb{D}_+ = \{ z \in \mathbb{H} : |z| < 1 \}$. As $\epsilon \to 0$,

$$H_D(\epsilon i, e^{i\theta}) \sim \frac{2\epsilon}{\pi} \sin \theta,$$

and hence

$$\Gamma(D) = \int_0^\pi H_{\mathbb{D}_+}(0, e^{i\theta}) H_{\mathbb{H}}(e^{i\theta}, 0) = \int_0^\pi \frac{2}{\pi} \sin^2 \theta \, d\theta = 1.$$

The normalization of the bubble measure is chosen so that $\Gamma(\mathbb{D}_+) = 1$.

The next proposition (Exercise 4.68) relates $\Gamma$ to the Schwarzian derivative.

**Proposition 4.57.** Suppose $D \subset \mathbb{H}$ is a simply connected domain with $\text{dist}(0, \mathbb{H} \setminus D) > 0$. Suppose $f : D \to \mathbb{H}$ is a conformal transformation. Then,

$$\Gamma_{\mathbb{H}}(0, D) = -\frac{1}{6} S f(0),$$

where $S$ denotes Schwarzian derivative.

We can write the local martingale $M_t$ from the previous section as

$$M_t = \left[ \exp \left\{ -\int_0^t a \Gamma(U_s, g_s(D)) \, ds \right\} \right]^{-e/2} \Phi'_t(U_t)^b,$$

and if $\kappa \leq 4$,

$$M_\infty = 1\{ \gamma(0, \infty) \subset D \} [e^{-\Theta}]^{-e/2},$$

where

$$\Theta = \Theta(\gamma, D) = a \int_0^\infty \Gamma(U_t, g_t(D)) \, dt.$$

Note that $\Theta$ is a deterministic function of $\gamma$ and $D$. The factor $a$ comes from the fact that $\gamma$ has been parametrized so that $\text{hcap}[\gamma(0, t] = at$. The value of $\Theta$ does not depend on the parametrization; we can write

$$\Theta = \Theta(\gamma, D) = \int_0^\infty \Gamma(U_t, g_t(D)) \, d\text{hcap}(\gamma(0, t]).$$
The very nice feature of $\Theta$ is conformal invariance. More generally if $\gamma$ is a curve connecting boundary points in $D$ and $D_1 \subset D$ we can define $\Theta_D(\gamma, D_1)$.

**Proposition 4.58.** If $D_1 \subset D$; $z, w$ distinct points on $\partial D$; and $\gamma$ a simple curve from $z, w$ in $D$, then

$$\Theta_D(\gamma; D_1) = \Theta_{f(D)}(f \circ \gamma; f(D_1)).$$

**Remark 4.59.** Because $\Theta$ is a conformal invariant, we can see that we no longer need to assume that $\partial D$ is smooth at $z$.

**Remark 4.60.** We do not need $D$ or $D_1$ to be simply connected in order to define $\Gamma_D(z, D_1)$ or $\Theta_D(\gamma; D_1)$. However, Proposition 4.57 which relates the bubble measure to the Schwarzian derivative does assume that the domain is simply connected.

5. Brownian loop measure

A (rooted) loop in $\mathbb{C}$ is a continuous function $\gamma: [0, t_\gamma] \to \mathbb{C}$ with $\gamma(0) = \gamma(t_\gamma)$. There is a one-to-one correspondence between loops and ordered triples

$$\langle z, s, \eta \rangle,$$

where $z \in \mathbb{C}, s > 0$, and $\eta$ is a loop with $t_\eta = 1$ and $\eta(0) = \eta(1) = 0$. The correspondence is by translation and Brownian scaling,

$$t_\gamma = s, \quad \gamma(t) = z + s^{1/2} \eta(t/s), \quad 0 \leq s \leq t_\gamma.$$

An unrooted loop is an equivalence class of rooted loops under the equivalence $\tilde{\gamma} \sim \gamma$ if $\tilde{\gamma}(t) = \gamma(t + r)$ for some $r \in \mathbb{R}$ (here addition is modulo $t_\gamma$). In other words, an unrooted loop is a loop that forgets where its starting point is.

**Definition 4.61.**

- The (rooted) Brownian loop measure is the measure on loops given by putting the following measure on $(z, s, \eta)$:

$$\text{area} \times \frac{dt}{2\pi t^2} \times \text{Brownian bridge},$$

where Brownian bridge refers to the probability measure on Brownian paths in $\mathbb{R}^2$ conditioned to return to the origin at time 1.

- The (unrooted) Brownian loop measure is the measure obtained from the rooted loop measure by forgetting the roots.

**Remark 4.62.** Recall that the density for a two-dimensional Brownian motion at time $t$ is $(2\pi t)^{-1} e^{-|z|^2/2t}$. Roughly speaking, we can think of $(2\pi t)^{-1}$ as the probability that a Brownian motion starting at the origin is at the origin at time $t$. We think of the unrooted loop measure as giving measure about $(2\pi t)^{-1}$ to each unrooted loop of length $t$. The rooted loop measure then chooses a root for the loop by using the uniform distribution on $[0, t]$; this gives the extra factor of $t^{-1}$ in the definition. These heuristics can be made precise by considering random walk approximations to the loop measure.

The unrooted loop measure turns out to be conformally invariant. In fact, $\Theta_D(\gamma, D_1)$ is the measure of unrooted loops in $D$ that intersect both $\gamma$ and $D_1$. 


6. The measure $\mu_D(z, w)$ for $\kappa \leq 4$

The ideas in this lecture can be used to define

$$\mu_D(z, w) = C(D; z, w) \mu_D^\#(z, w),$$

for all simply connected domains $D$ and distinct boundary points $z, w$ at which $\partial D$ is locally analytic. We allow $w = \infty$. In this case we say $\partial D$ is locally analytic at $\infty$ if $\partial[f(D)]$ is locally analytic at $0$ where $f(z') = 1/z'$. Given any two such triples $(D, z, w), (D_1, z_1, w_1)$ there is a unique conformal transformation, which we call the canonical transformation, $f : D \to D'$ with $f(z) = z_1, f(w) = w_1$ and $|f'(w)| = 1$ with the appropriate interpretation of this if $w = \infty$ or $w_1 = \infty$. (For example, if $w = \infty, w_1 \neq \infty$, then as $z' \to \infty, |f(z') - w_1| \sim |z'|^{-1}.$) We define $C_{\mathbb{H}}(0, \infty) = 1$ and $\mu_{\mathbb{H}}(0, \infty) = \mu_{\mathbb{H}}^\#(0, \infty)$ to be the probability measure given by $SLE_\kappa$. We then define

$$\mu_D(z, w) = |f'(0)|^{-b} [f \circ \mu_{\mathbb{H}}(0, \infty)],$$

where $f : \mathbb{H} \to D$ is the canonical transformation with $f(0) = z, f(\infty) = w$. In other words,

$$C(D; z, w) = |f'(0)|^{-b}, \quad \mu_D^\#(0, \infty) = f \circ \mu_{\mathbb{H}}^\#(0, \infty).$$

**Example 4.63.** The map $f(z) = z/(1-z)$ is the canonical transformation from $\mathbb{H}$ to $\mathbb{H}$ with $f(0) = 0, f(1) = \infty$. Using this we see that $C(\mathbb{H}; 0, 1) = 1$. More generally,

$$C(\mathbb{H}; x_1, x_2) = |x_2 - x_1|^{-2b}.$$

**Example 4.64.** If $D \in D$, then $\Phi_D^{-1}$ is a canonical transformation from $\mathbb{H}$ to $D$. Therefore,

$$C(D; x, \infty) = \Phi_D^{-1}(x)^b.$$

We get the following properties.

- **Conformal covariance.** If $f : D \to f(D)$ is a conformal transformation, then

$$f \circ \mu_D(z, w) = |f'(z)|^b |f'(w)|^b \mu_{f(D)}(f(z), f(w))$$

(assuming sufficient smoothness at the boundary points).

- **Boundary perturbation.** If $D_1 \subset D, \partial D, \partial D_1$ agree near boundary points $z, w$, then

$$\frac{d\mu_{D_1}(z, w)}{d\mu_D(z, w)} (\gamma) = 1\{\gamma \subset D_1\} e^{\Theta/2},$$

where $\Theta$ is the measure of the set of Brownian loops in $D$ that intersect both $\gamma$ and $D \setminus D_1$.

- In particular, if $D_1 \subset D, \partial D, \partial D_1$ agree near boundary points $z, w$, and $f : D \to f(D)$ is a conformal transformation, then

$$\frac{d\mu_{D_1}(z, w)}{d\mu_D(z, w)} = \frac{d\mu_{f(D)}(f(z), f(w))}{d\mu_{f(D)}(f(z), f(w))}.$$

- If $D_1 \subset D, \partial D, \partial D_1$ agree near boundary points $z, w$, then the probability measure $\mu_D^\#(z, w)$ can be obtained from $\mu_D^\#(z, w)$ by “weighting paths locally” by $C(D_1; z, w)$, or equivalently by “weighting paths locally” by

$$\frac{d\mu_{D_1}(z, w)}{d\mu_D(z, w)}.$$
Remark 4.65. The last property is stated informally. A precise statement for $D = \mathbb{H}, z = 0, w = \infty$ was given in Section 3. For other $D_1, D$, we can first map $D$ to $\mathbb{H}$ and use conformal invariance.

Remark 4.66. Because the quantity
\[ \frac{d\mu_{D_1}(z,w)}{d\mu_{D}(z,w)} \]
is a conformal invariant, it is well defined even if $\partial D$ is not smooth near $z, w$ (but still assuming that $\partial D, \partial D_1$ agree in neighborhoods of $z, w$).

Remark 4.67. If $D_1, D_2 \in \mathcal{D}$ and $f : D_1 \to D_2$ is a conformal transformation with $f(\infty) = \infty$, we write $f'(\infty) = r$ if $f(z) \sim z/r$ as $z \to \infty$. In particular, if $f(z) = rz$, then $f'(\infty) = 1/r$. Using this convention, (4.2) holds for such transformations.

7. Exercises for Lecture 4

Exercise 4.68. Prove Proposition 4.57.

Exercise 4.69. Suppose $\gamma : (0, \infty) \to \mathbb{H}$ is a simple curve with $\gamma(0^+) = 0, \gamma(z) \to \infty$ as $z \to \infty$. Let $g_t$ be the corresponding conformal maps. Fix $\kappa \leq 4$. Show that if $0 < x < y$,
\[ C(\mathbb{H} \setminus \gamma(0,t); x, y) = \frac{g'_t(x)^b g'_t(y)^b}{|g_t(y) - g_t(x)|^{2\kappa}}, \]
and
\[ C(\mathbb{H} \setminus \gamma(0,\infty); x, y) = \lim_{t \to \infty} \frac{g'_t(x)^b g'_t(y)^b}{|g_t(y) - g_t(x)|^{2\kappa}}. \]

Exercise 4.70. Suppose $\kappa \leq 4$. Suppose $D$ is a bounded, simply connected domain, and $A_1, A_2$ are disjoint, closed, analytic, subarcs of $D$ larger than a single point. Define the measure
\[ \mu_D(A_1, A_2) = \int_{A_1} \int_{A_2} \mu_D(z,w) |dw| |dz|, \]
where $\mu_D(z,w)$ denotes the SLE$_\kappa$ measure and $|d\cdot|$ denotes integration with respect to arc length. We can write
\[ \mu_D(A_1, A_2) = C_D(A_1, A_2) \mu_D^\#(A_1, A_2), \]
where
\[ C_D(A_1, A_2) = \int_{A_1} \int_{A_2} C(D; z, w) |dw| |dz|, \]
is the total mass and $\mu_D^\#(A_1, A_2)$ is a probability measure. Suppose $f : D \to D_1$ is a conformal transformation with $f(A_1), f(A_2)$ being analytic arcs.

- Convince yourself that the integral makes sense (i.e., there is no trouble integrating a “measure-valued” function).
- Show that $0 < C_D(A_1, A_2) < \infty$.
- In the cases of self-avoiding walk, loop-erased walk, and Ising interface, describe what $\mu_D(A_1, A_2)$ represents in terms of limits of discrete models.
• Show that if $\kappa = 2$, then
  
  $$f \circ \mu_D(A_1, A_2) = \mu_D(f(A_1), f(A_2)).$$

  In particular,

  (4.4)  
  
  $$f \circ \mu^\#_D(A_1, A_2) = \mu^\#_D(f(A_1), f(A_2)).$$

• Show that (4.4) does not necessarily hold if $\kappa \neq 2$. 
LECTURE 5

Radial and two-sided radial $SLE_k$

1. Example: SAW II

We return to the example of the self-avoiding walk (SAW) from Lecture 1. We let $z$ be a boundary point and $w$ be an interior point and consider the set of SAWs from $z$ to $w$. We again give the measure $e^{-\beta|\omega|}$ to each walk where $\beta$ is the critical value and we let $Z_N(D; z, w)$ be the partition function as before. It is conjectured that

$$Z_N(D; z, w) \sim C(D; z, w) N^{-b} N^{-\tilde{b}},$$

where now we have a (one-sided) interior scaling exponent $\tilde{b}$. If we multiply by $N^{b+\tilde{b}}$ and take a limit, we expect to have a measure on simple paths from $z$ to $w$ (or $w$ to $z$). The model for the continuum limit of this is called (one-sided) radial $SLE$.

We can also consider a case with two boundary points, $z, y$ and one interior point $w$. We look at the set of all SAWs $\omega$ from $z$ to $y$ that go through $w$. Equivalently, we can look at the set of pairs of SAWs $(\omega, \omega')$ where $\omega$ goes from $w$ to $z$; $\omega'$ gives from $w$ to $y$; and $\omega \cap \omega' = \{w\}$. In this case, we expect the partition function to scale like

$$Z_N(D; z, y; w) \sim C(D; z, y; w) N^{-2b} N^{-\tilde{b}},$$
where $\hat{b}$ is a two-sided interior scaling exponent. By comparison to $Z_N(D; z, y)$, we can see that $N^{-\hat{b}}$ should be comparable to the probability that a SAW from $z$ to $y$ goes through $w$ which we can conjecture to be about $N^d/N^2$ where $d$ is the fractal dimension of the paths. We therefore get

$$\hat{b} = 2 - d.$$ 

Since this relation holds, we do not adopt the notation $\hat{b}$ but rather just refer to the exponent as $2 - d$.

Consider the marginal measure on $\omega$. Then for any $\omega$ from $w$ to $z$ the measure is

$$Z_N(D \setminus \omega; y, w).$$

In other words we can first choose $\omega$ using the one-sided measure but then we weight this distribution by the measure of walks $\omega'$ from $w$ to $y$ that avoid $\omega$. We will be able to look at the scaling limit of the measure on $\omega$ in two different ways: chordal SLE from $z$ to $y$ conditioned to go through $w$ or radial SLE from $z$ to $w$ weighted by the total mass of paths from $w$ to $y$ in $D \setminus \omega$. (Both of these interpretations must be considered in some kind of limit.) If we fix $\omega$ and consider the probability measure on $\omega'$ obtained by conditioning, then this will be the same as the probability measure given by the normalized measure on $D \setminus \omega$. In the scaling limit, the probability measure on $\omega'$ given $\omega$ will be $\mu_{D \setminus \omega}^\#(w, y)$ (note that $w$ is a boundary point of $D \setminus \omega$).

![Discrete approximation of two-sided radial](image)

**Figure 2.** Discrete approximation of two-sided radial

Note that

$$\frac{Z_N(D; z, y; w)}{Z_N(D; z, w) Z_N(D; y, w)} \asymp N^{-(\hat{b} - 2\tilde{b})}.$$ 

The left-hand side represents the probability that two SAWs starting at the origin do not intersect. If we recall from Lecture 1 that we expect this probability to
decay like $|\omega|^{-(\gamma-1)}$, then we see that we would expect this quantity to decay like $N^{-d(\gamma-1)}$. This gives the following relation between the exponents

$$d(\gamma - 1) = \hat{b} - 2\hat{b} = 2 - d - 2\hat{b}, \quad \gamma = 2\nu(1 - \hat{b}).$$

One can also consider the case for two interior points $w, w'$ for which we would expect the scaling

$$Z_N(D; w, w') \sim C(D; w, w') N^{-2\hat{b}}.$$ 

The corresponding probability measures are called whole plane SLE. We will not consider this case in these lectures.

2. Radial $SLE_\kappa$

Radial $SLE_\kappa$ is a measure

$$\mu_D(z, w) = C_D(z, w) \mu_D^\#(z, w), \quad z \in \partial D, w \in D$$
on paths $\gamma : (0, t_\gamma] \to D$ with $\gamma(0^+) = z$ and $\gamma(t_\gamma) = w$. We will consider paths modulo reparametrization, but we could also consider the paths with parametrization. It satisfies the following properties:

- **Domain Markov property** (for $\mu_D^\#(z, w)$). Given $\gamma(0, t]$, the distribution of the remainder of the path is the same as $\mu_D^\#_{\gamma(0, t]}(\gamma(t), w)$

- **Conformal covariance**. If $f : D \to f(D)$ is a conformal transformation, then

\begin{equation}
C(D; z, w) = |f'(z)|^b |f'(w)|^{\hat{b}} C(f(D); f(z), f(w)),
\end{equation}

$$f \circ \mu_D^\#(z, w) = \mu_{f(D)}^\#(f(z), f(w)).$$

- **Conformal invariance for boundary perturbations** (for $\kappa \leq 4$). If $w \in D_1 \subset D; \ z \in \partial D; \text{ and } \partial D_1, \partial D \text{ agree near } z; \text{ then } \mu_{D_1}(z, w)$ is absolutely continuous with respect to $\mu_D(z, w)$ and the Radon-Nikodym derivative

\begin{equation}
\frac{d\mu_{D_1}(z, w)}{d\mu_D(z, w)}
\end{equation}
is a conformal invariant.

**Remark 5.71.** Here we have introduced the interior scaling exponent

$$\hat{b} = \frac{\kappa - 2}{4} b.
$$

We discuss below why this value is chosen — essentially, it is the value that makes the quantity on the right hand side of (5.3) a local martingale. The total mass $C(D; z, w)$ is nonzero and finite provided that $\partial D$ is locally analytic at $z$. One does not need smoothness of the boundary at $z$ to define the probability measures $\mu_D^\#(z, w)$ or the Radon-Nikodym derivative $\mu_D^\#_{\gamma(0, t]}(\gamma(t), w)$.

**Remark 5.72.** We will normalize so that $C(\mathbb{H}; 0, i) = 1$. Once we have done this, the scaling rule (5.1) determines $C(D; z, w)$ for all simply connected domains (assuming we have determined $\hat{b}$). The domain Markov property and conformal invariance will determine $\mu_{\mathbb{H}}^\#(0, i)$ and from this we get $\mu_D^\#(z, w)$ for all simply connected $D$. 
Example 5.73. We compute $C(\mathbb{H}; 0, z)$. Let $x > 0$ and

$$f(z) = \frac{z(1 + x^2)}{x(x - z)}, \quad f'(z) = -\frac{1 + x^2}{(z - x)^2}, \quad f(i) = -\frac{1}{x} + i,$$

which is a conformal transformation of $\mathbb{H}$ onto $\mathbb{H}$ with $f(0) = 0$. Note that

$$|f'(i)| = 1, \quad |f'(0)|^{-1} = \frac{x^2}{x^2 + 1} = \pi H_\mathbb{H}\left(-\frac{1}{x} + i, 0\right),$$

where $H_\mathbb{H}$ denotes the Poisson kernel. Hence we get

$$C(\mathbb{H}; 0, y(x + i)) = y^{-b} \left[\pi H_\mathbb{H}(x + i, 0)\right]^b = y^{-b} \left[\pi H_\mathbb{H}(y(x + i), 0)\right]^b.$$

Example 5.74. We choose (somewhat arbitrarily)

$$C(\mathbb{H}; \infty, i) = 1.$$

The map $F(z) = rz$ satisfies $F'(\infty) = 1/r$, so we get

$$C(\mathbb{H}; \infty, iy) = C(\mathbb{H}; \infty, iy) = y^{b-b},$$

If $w \in D \subset D$, then $\Phi'_D(\infty) = 1$ and

$$C(D; \infty, w) = |\Phi'_D(w)|^b C(\mathbb{H}; \infty, \Phi_D(w)).$$

In particular, if $\gamma$ is a path and $t < T_w$, then

$$C(\mathbb{H} \setminus \gamma(0, t]; \infty, w) = |g'_t(w)|^b Y_t^\gamma - \bar{b} = Y_t^{-b} Y_t^b.$$

If $g_t$ is chordal $SLE_\kappa$ from 0 to $\infty$, then (see (3.4)),

$$M_t = |g'_t(z)|^b C(\mathbb{H}; U_t, g_t(z)) = |g'_t(z)|^b C(\mathbb{H}; 0, Z_t) = Y_t^{-b} \left[\pi H_\mathbb{H}(Z_t, 0)\right]^b,$$

is a local martingale for $t \leq T_\gamma$ satisfying

$$dM_t = (1 - 3a) \frac{X_t}{X_t^2 + Y_t^2} M_t dB_t.$$

We can consider this property as the defining property for the exponent $\bar{b}$. If we use the Girsanov theorem and weight the paths by $M_t$ then under the weighted measure $U_t = -B_t$ satisfies

$$dU_t = (3a - 1) \frac{X_t}{X_t^2 + Y_t^2} dt - dW_t,$$

where $W_t$ is a Brownian motion in the new measure. This leads to one definition for the probability measure $\mu_{\mathbb{H}}(0, w)$.

Definition 5.75. Let $w \in \mathbb{H}$ and suppose $g_t$ is the solution to the Loewner equation (3.1) where $U_t$ satisfies

$$dU_t = (3a - 1) \frac{X_t}{X_t^2 + Y_t^2} dt - dW_t,$$

where $X_t = X_t(w), Y_t = Y_t(w)$. Then the corresponding curve $\gamma(t), 0 < t < T_w$ is radial $SLE_\kappa$ from 0 to $w$ stopped at time $T_w$. 


Note that for radial $SLE_\kappa$,
\[
dX_t = \frac{(1-2a)X_t}{X_t^2 + Y_t^2} dt + dW_t, \\
d\Theta_t = \frac{aX_tY_t}{(X_t^2 + Y_t^2)^2} dt - \frac{Y_t}{X_t^2 + Y_t^2} dW_t.
\]

**Proposition 5.76.**
\begin{itemize}
  \item If $\kappa \leq 4$, then $T_w < \infty$ and $\gamma(T_w-) = w$.
  \item If $4 < \kappa < 8$, then $T_w < \infty$ and $\gamma(T_w-) \neq w$.
  \item If $\kappa \geq 8$, then $T_w = \infty$.
\end{itemize}

**Proof.** We consider a new parametrization for which the conformal radius $\Upsilon_t$ decreases deterministically. Assume for ease that $w = x + i$ and let $\rho(t)$ be defined by $Y_{\rho(t)} = e^{-2at}$. In other words,
\[
\dot{\rho}(t) = \frac{(X_{\rho(t)}^2 + Y_{\rho(t)}^2)^2}{Y_{\rho(t)}^2}.
\]
This is valid up to the time that $Y_{\rho(t)} = 0$. Recall that $\Upsilon_t \asymp \text{dist}[w, \gamma(0,t] \cup \mathbb{R}]$. Then $\Theta_t = \Theta_{\rho(t)}$ satisfies
\begin{equation}
(5.4) \quad d\Theta_t = a \cot \Theta_t dt + dW_t,
\end{equation}
where $\tilde{W}_t$ is a standard Brownian motion. By comparison with a Bessel process we see that if $a \geq 1/2$, then $\sin \Theta_t > 0$ for all $t$, but if $a < 1/2$, then $\sin \Theta_t$ reaches zero in finite time. \hfill \Box

**Remark 5.77.**
\begin{itemize}
  \item Our definition shows immediately that radial $SLE_\kappa$ up to time $T_w$ is absolutely continuous with respect to chordal $SLE_\kappa$. (To be more accurate, we need to stop the process slightly before $T_w$).
  \item If $\kappa = 2/a = 6$, then radial $SLE_\kappa$ has the same distribution as chordal $SLE_\kappa$ up to time $T_w$. This is another version of the locality property for $SLE_6$.
  \item For $\kappa > 4$, this definition only defines radial $SLE_\kappa$ up to time $T_w$. In order to define the measure $\mu_\#(0, i)$ we can consider $\tilde{g}_t$, a conformal transformation of $\mathbb{H} \setminus \gamma(0,t]$ onto $\mathbb{H}$ with $\tilde{g}_t(i) = i$. This is the basis of the original definition which we describe in the next section. There the definition is done for $\mu_{\#}^w(z, 0)$.
\end{itemize}

3. Another definition

Here we give an alternative definition of radial $SLE_\kappa$ in simply connected domains. We will define $\mu_{\#}^w(z, 0)$ for $z \in \partial \mathbb{D}$. Given $\gamma(0,t]$, let $\hat{g}_t$ denote the unique conformal transformation of $\mathbb{D} \setminus \gamma(0,t]$ onto $\mathbb{D}$ with $\hat{g}_t(0) = 0, \hat{g}_t'(0) > 0$. If we parametrize the curve so that $\log \hat{g}_t'(0) = t$, then $\hat{g}_t$ satisfies
\[
\partial_t \hat{g}_t(z) = \hat{g}_t(z) \frac{e^{2U_t} + \hat{g}_t(z)}{e^{2U_t} - \hat{g}_t(z)}, \quad \hat{g}_0(z) = z,
\]
where $\hat{g}_t(\gamma(t)) = e^{2U_t}$. If we choose $2U_t = B_{st}$, where $B_t$ is a standard Brownian motion, then $\gamma$ has the distribution of $\mu_{\#}^w(z, 0)$. $\mu_{\#}^w(z, w)$ for other simply connected
domains is defined by conformal invariance. The equivalence of the definitions of radial \( SLE_\kappa \) can be checked in a straightforward way by studying the image of \( SLE \) under conformal transformations. In studying the equation above it is often useful to consider the logarithm: suppose \( \tilde{g}_t(e^{2i\theta}) = e^{2i\tilde{h}_t(\theta)} \). (Since \( \tilde{g}_t \) vanishes at the origin, there are technical issues in taking the logarithm. If we stay in a simply connected subdomain of \( \mathbb{D} \) that avoids the origin, e.g., a neighborhood of a boundary point, there is no problem.) Then \( h_t \) satisfies

\[
\partial_t \tilde{h}_t(\theta) = \frac{1}{2} \cot \left( h_t(\theta) - \frac{1}{2} B_{\kappa t} \right).
\]

If \( h_t = \tilde{h}_{4t/\kappa} \), then this becomes

\[
\partial_t h_t(\theta) = a \cot \left( h_t(\theta) - \tilde{W}_t \right),
\]

where \( \tilde{W}_t \) is a standard Brownian motion. In other words \( \Psi_t := h_t(\theta) - \tilde{W}_t \) satisfies

\[
d\Psi_t = a \cot \Psi_t \, dt + dW_t,
\]

where \( W_t = -\tilde{W}_t \) is a standard Brownian motion. Note that this is the same equation as (5.4). If we let \( g_t = \tilde{g}_{4t/\kappa} \), then

\[
|g_t'(e^{2i\theta})| = \exp \left\{ -\int_0^t \frac{a \, ds}{\sin^2 \Psi_s} \right\}.
\]

in this parametrization \( g_t'(0) = e^{4t/\kappa} = e^{2at} \).

4. Radial \( SLE_\kappa \) in a smaller domain

The computation starts getting a little messy, but we can also do radial \( SLE \) in a smaller domain.

**Proposition 5.78.** Let \( D \in \mathcal{D} \) and \( w \in D \). Let

\[
K_t = C(D_t; U_t, g_t(z)) = \Phi_t(U_t)^b |\Phi_t(g_t(z))|^b C(\mathbb{H}; 0, L_t),
\]

where \( L_t = \Phi_t(g_t(z)) - \Phi_t(U_t) \). Then,

\[
M_t = \exp \left\{ -\int_0^t \frac{ac}{12} \Im \Phi_t(U_s) \, ds \right\} K_t,
\]

is a local martingale.

If \( \kappa \leq 4 \), then (4.3) holds for radial \( SLE_\kappa \). In particular, for \( \kappa = 8 / 3 \), radial \( SLE_\kappa \) satisfies the restriction property.

**Proposition 5.79.** Suppose \( w \in D_1 \subset D \) where \( D_1, D \) are simply connected. Suppose \( z \in \partial D, \partial D \) is smooth near \( z \); and \( D_1, D \) agree near \( z \). Then if \( \gamma \) has the distribution of \( \mu_{f(z)}^{w}(z, w) \), then the probability that \( \gamma(t) \subset D_1 \) is \( |F'(z)|^{5/8} |F'(w)|^{5/18} \) where \( F \) is the conformal transformation of \( D_1 \) onto \( D \) fixing \( z, w \).
5. Two-sided radial

Here we let $\kappa < 8$ and introduce a process that can be called either two-sided radial $SLE_\kappa$ or chordal $SLE_\kappa$ conditioned to go through a point. This is a measure

$$\mu_D(z_1, z_2; w) = C(D; z_1, z_2; w) \mu_D^\#(z_1, z_2; w),$$

where $D$ is a domain; $z_1, z_2$ are distinct points in $\partial D$; and $w \in D$. We define

$$C(\mathbb{H}; 0, \infty; w) = G(w),$$

where $G(y(x+i)) = y^{d-2}(x^2 + 1)^{1/2-a}$ is the Green’s function for chordal $SLE_\kappa$ as introduced in Lecture 3. For other simply connected domains, we define

$$C(D; z_1, z_2; w) = |f'(0)|^{-b} |f'(w')|^{-b} G(w') = |f'(0)|^{-b} |f'(w')|^{d-2} G(w').$$

where $f: \mathbb{H} \to D$ is the canonical transformation with $f(0) = z_1, f(\infty) = z_2$ and $w' = f^{-1}(w)$. It satisfies the following conformal covariance relations.

- **Conformal covariance.** If $f: D \to f(D)$ is a conformal transformation, then

\begin{equation}
(5.5) \quad C(D; z_1, z_2; w) = |f'(z_1)|^b |f'(z_2)|^b |f'(w)|^{2-d} C(f(D); f(z_1), f(z_2); f(w)),
\end{equation}

$$f \circ \mu_D^\#(z_1, z_2; w) = \mu_D^\#(f(z_1), f(z_2); f(w)).$$

- **Conformal invariance for boundary perturbations** (for $\kappa \leq 4$). If $w \in D_1 \subset D$; $z \in \partial D_1$; and $\partial D_1, \partial D$ agree near $z_1, z_2$; then $\mu_D(z_1, z_2; w)$ is absolutely continuous with respect to $\mu_D(z_1, z_2; w)$ and the Radon-Nikodym derivative

\begin{equation}
(5.6) \quad \frac{d\mu_D(z_1, z_2; w)}{d\mu_D(z_1, z_2; w)}
\end{equation}

is a conformal invariant.

There are two different ways to think of the probability measure $\mu_D^\#(z_1, z_2; w)$. We illustrate this with $D = \mathbb{H}, z_1 = 0, z_2 = \infty$.

5.1. Chordal weighted by $G$

Recall that if $z \in \mathbb{H}$ and $M_t = M_t(z) = |g_t'(z)|^{2-d} G(Z_t)$, then $M_t$ is a local martingale satisfying

$$dM_t = (1 - 4a) \frac{X_t}{X_t^2 + Y_t^2} M_t dB_t.$$ 

Hence if we weight the paths by the martingale, then

$$dB_t = (1 - 4a) \frac{X_t}{X_t^2 + Y_t^2} dt + dB_t,$$

where $B_t$ denotes a Brownian motion in the new measure. In other words,

$$dX_t = (1 - 3a) \frac{X_t}{X_t^2 + Y_t^2} dt + d\tilde{B}_t,$$

$$d\Theta_t = \frac{2a X_t Y_t}{(X_t^2 + Y_t^2)^2} dt - \frac{Y_t}{X_t^2 + Y_t^2} d\tilde{B}_t.$$ 

If we do the time reparametrization as in the previous section, we can write

\begin{equation}
(5.7) \quad d\tilde{\Theta}_t = 2a \cot \tilde{\Theta}_t dt + d\tilde{W}_t,
\end{equation}

where $\tilde{W}_t$ is a standard Brownian motion (in the weighted measure).
5.2. Radial weighted by radial weights

We will now show that we can consider two-sided radial SLE\(_κ\) from 0 to \(w\) in \(\mathbb{H}\) as (one-sided) radial SLE\(_κ\) weighted by \(C(\mathbb{H} \setminus \gamma(0, t]; \infty, w)\). For ease, let us assume \(w = i\). Let us start with radial SLE\(_κ\) parametrized by \(\Upsilon_t\) and recall that

\[
C(\mathbb{H} \setminus \gamma(0, \rho(t)]; \infty, \rho(t)) = \Upsilon_{\rho(t)}^{-h} Y_{\rho(t)}^h = e^{-2ab \tilde{Y}_t^h}.
\]

Note that

\[
\partial_t \tilde{Y}_t = -a \csc^2 \tilde{\Theta}_t \tilde{Y}_t,
\]

i.e.,

\[
J_t := \tilde{Y}_t^h = \exp \left\{ -ab \int_0^t \frac{ds}{\sin^2 \tilde{\Theta}_s} \right\}.
\]

Recall that for radial SLE\(_κ\), \(\tilde{\Theta}_t\) satisfies \((5.4)\). Let us now assume that there is a function \(\phi(\theta)\) such that

\[
\phi(\theta) = \lim_{t \to \infty} e^{\lambda t} E[J_t \mid \tilde{\Theta}_0 = \theta].
\]

Then \(e^M \phi(\tilde{\Theta}_t) J_t\) is a martingale. Using Itô’s formula, we get the equation

\[
\phi''(\theta) + 2a \phi'(\theta) \cot \theta + (2\lambda - 2ab \csc^2 \theta) \phi(\theta) = 0,
\]

which gives \(\phi(\theta) = e^{3at/2} \sin^a \theta\). Therefore, \(M_t = e^{3at/2} \sin^a \tilde{\Theta}_t\) satisfies

\[
dM_t = a \cot \tilde{\Theta}_t d\tilde{W}_t,
\]

If we do Girsanov and weight by \(M_t\), then \(\tilde{\Theta}_t\) satisfies

\[
d\tilde{\Theta}_t = 2a \cot \tilde{\Theta}_t dt + d\tilde{W}_t,
\]

where \(\tilde{W}_t\) is a standard Brownian motion in the new measure. Note that this is the same as \((5.7)\).

6. Exercises for Lecture 5

**Exercise 5.80.** Redo Section 1 for loop-erased walk.

**Exercise 5.81.** Verify \((5.5)\).
LECTURE 6

Intersection exponents

In this lecture we will compute the intersection exponents for $SLE_\kappa$, $\kappa \leq 4$. The particular case of $\kappa = \frac{8}{3}$ gives the Brownian motion intersection exponents. We fix $a = \frac{2}{\kappa} \geq 1/2$ and constants depend on $a$.

1. One-sided

We assume that $g_t$ satisfies (3.1) where $U_t = -B_t$ is a standard Brownian motion. If $z \in \mathbb{H}$ we write $Z_t = Z_{t,z} = g_t(z) - U_t$ and we write $Z_t = X_t + iY_t$. Recall that

$$dX_t = \frac{aX_t}{X_t^2 + Y_t^2} dt + dB_t, \quad dY_t = -\frac{aY_t}{X_t^2 + Y_t^2} dt.$$

(6.1)

$$d|g_t'(z)| = |g_t'(z)| \frac{a(Y_t^2 - X_t^2)}{(X_t^2 + Y_t^2)^2} dt.$$

We will consider the case $z = x \in \mathbb{R} \setminus \{0\}, t \leq T_x$, for which $g_t'(x) \in (0,1]$,

$$dX_t = \frac{a}{X_t} dt + dB_t, \quad \partial_t g_t(x) = -\frac{a g_t'(x)}{X_t^2}, \quad g_t'(x) = \exp \left\{ -a \int_0^t ds \frac{X_s}{X_s^2} \right\}.$$

(6.2)

Define

$$\lambda_0 = \lambda_0(a) = -\frac{(2a-1)^2}{8a},$$

(6.3)

$$q(\lambda) = q_+(\lambda) = q(\lambda; a) = \frac{(1 - 2a) + \sqrt{(2a - 1)^2 + 8a\lambda}}{2}, \quad \lambda \geq \lambda_0,$$

$$q_-(\lambda) = q_-(\lambda; a) = \frac{(1 - 2a) - \sqrt{(2a - 1)^2 + 8a\lambda}}{2}, \quad \lambda \geq \lambda_0.$$

Note that $q = q_\pm(\lambda)$ is a solution to the quadratic equation

$$q^2 + (2a - 1)q - 2a\lambda = 0,$$

(6.4)

and that

$$q(\lambda_1) + q(\lambda_2) = q \left( \lambda_1 + \lambda_2 + \frac{q(\lambda_1)q(\lambda_2)}{a} \right).$$

(6.5)

**Proposition 6.82.** Suppose $x > 0$, $\lambda \geq \lambda_0$, $q = q(\lambda)$. Then $M_t := X_t^q g_t'(x)^\lambda$ is a martingale satisfying

$$dM_t = M_t \frac{q}{X_t} dB_t.$$

(6.6)

If $q = q_-(\lambda)$, the same holds with “local martingale” replacing “martingale”.

53
Proof. Itô’s formula gives
\[ dX_t^q = X_t^q \left[ a + \frac{1}{2} q(q - 1) \frac{dt}{X_t^q} \right] + \frac{q}{X_t^q} dB_t, \]
and hence the product rule and \((6.2)\) give
\[ dM_t = M_t \left[ a + \frac{1}{2} q(q - 1) - a \lambda \frac{dt}{X_t^q} \right] + \frac{q}{X_t^q} dB_t. \]
By \(6.4\), we see that \(M_t\) is a local martingale. If we apply Girsanov and weight the paths by \(M_t\), the paths \(B_t\) satisfy
\[ dB_t = \frac{q}{X_t} dt + dW_t, \]
where \(W\) is a standard Brownian motion in the new weighting. Hence \(X_t\) satisfies
\[(6.7) \quad dX_t = \frac{a + q}{X_t} dt + dW_t. \]
If \(\lambda \geq \lambda_0\) then \(q \geq (1/2) - a\) and hence \(a + q \geq 1/2\). Therefore, the weighted paths never reach the origin. Also, since the weighted paths follow a Bessel process, one can use the criterion from Section 2 to see that \(M_t\) is actually a martingale. \(\square\)

If we consider the pair of processes \(U_t = -B_t\) and \(K_t = g_t(x)\), then the pair of processes \((U_t, K_t)\) satisfy the simple system
\[(6.8) \quad dU_t = \frac{q}{U_t - K_t} dt - dW_t, \quad \partial_t K_t = \frac{a}{K_t - U_t}. \]
This is an example of an \(SLE(\kappa, \rho)\) process where \(\rho = q\kappa\). (The parameter \(\rho\) is the coefficient of the drift of \(U_t\) if the process is parametrized so that \(\text{heap}(\gamma(0, t)) = 2t\). This is probably not the best way to parametrize these processes.)

**Example 6.83.** Let \(x > 0\), \(\lambda = b\), \(q = q(\lambda) = a\). Note that \(C(\mathbb{H} \setminus \gamma(0, t); x, \infty) = g_t'(x)^b\). Then \((6.7)\) becomes
\[ dX_t = \frac{2a}{X_t} dt + dW_t, \]
and \((6.8)\) becomes
\[(6.9) \quad dU_t = \frac{a}{U_t - K_t} dt - dW_t, \quad \partial_t K_t = \frac{a}{K_t - U_t}. \]
This is the \(SLE(\kappa, \rho)\) process with \(\rho = a\kappa = 2\). It is also the measure for *two-sided chordal \(SLE_{\kappa}\)* from \((0, x)\) to \(\infty\) in \(\mathbb{H}\). We can let \(x \to 0\) and obtain *two-sided chordal \(SLE_{\kappa}\)* from \(0\) to \(\infty\) in \(\mathbb{H}\). If we are only interested in the distribution of \(X_t = K_t - U_t\) we can replace \((6.9)\) with
\[ dU_t = \frac{a}{U_t - K_t} dt - \sqrt{r(t)} dW_t^1, \quad dK_t = \frac{a}{K_t - U_t} dt - \sqrt{1 - r(t)} dW_t^2, \]
where \(0 \leq r(t) \leq 1\) and \(W^1, W^2\) are independent standard Brownian motions. The value of \(r(t)\) determines the best ratio of growth rates of the two paths.

**Example 6.84.** Consider \(\mu_{12}^\#(0, x) = x^{-2b} \mu_{12}^\#(0, x)\). We can obtain the probability measure \(\mu_{12}^\#(0, x)\) from \(SLE_{\kappa}\) from \(0\) to \(\infty\) by weighting by \(X_t^{-2b}\), i.e., by choosing \(q_\kappa = -2b\). Then \((6.7)\) becomes
\[ dX_t = \frac{1 - 2a}{X_t} dt + dW_t, \]
and (6.8) becomes
\[ dU_t = -\frac{2b}{U_t - K_t} dt - dW_t, \quad \partial_t K_t = \frac{a}{K_t - U_t}. \]
This is \(\text{SLE}(\kappa, \rho)\) with
\[ \rho = -2b\kappa = \kappa - 6. \]
In this case the local martingale \(M_t\) is not a martingale. This can be seen by noting that the amount of time (in hcap from infinity) to go from 0 to \(x\) is finite, i.e., there is explosion in finite time.

**Proposition 6.85.** If \(\lambda \geq \lambda_0\) and \(q = q(\lambda)\), there is a constant \(c\) such that for all \(t, x > 0\),
\[ \mathbb{E}[g_{L^2}(x)^{\lambda}] = \mathbb{E}[g_{L}(1)^{\lambda}] \sim ct^{-q/2}. \]

**Proof.** (Sketch) The equality holds by scaling so we may assume \(x = 1\). This is a fact about Bessel processes; we are computing \(\mathbb{E}[M_t] = \mathbb{E}[M_t X_t^{-q}] = M_0 \mathbb{E}[X_t^{-q}]\),
where \(\mathbb{E}\) denotes expectations in the new measure \(\tilde{\mathbb{P}}\). In the \(\tilde{\mathbb{P}}\) measure, \(X_t\) satisfies \((6.7)\), and hence has a limiting distribution proportional to \(r^{2(a+q)} e^{-r^2/2}\). The constant \(c\) is the expectation of \(X^{-q}\) with respect to this limit measure. \(\square\)

**Remark 6.86.** We will interpret \(q\) as an “intersection exponent” or “crossing exponent” for \(\text{SLE}_\kappa\). Suppose \(L > 0\) is large (we will take asymptotics as \(L \to \infty\)) and \(\gamma\) is an \(\text{SLE}_\kappa\) curve from 0 to \(\infty\). Let us consider (see Exercise 4.69)
\[ C(\mathbb{H} \setminus \gamma(0, \infty); 1, L) = \lim_{t \to \infty} \frac{g'_L(1)^b g'_L(L)^b}{g_L(L) - g_L(1)^{2b}}. \]
Since the diameter of \(\gamma(0, t]\) is of order \(\sqrt{t}\), we can expect that
\[ C(\mathbb{H} \setminus \gamma(0, \infty); 1, L) \propto \frac{g'_L(1)^b g'_L(L)^b}{(g_L(L) - g_L(1))^{2b}} \leq L^{-2b} g'_L(1)^b, \]
\[ \mathbb{E}[C(\mathbb{H} \setminus \gamma(0, \infty); 1, L)^r] \propto \mathbb{E} \left[ \frac{g'_L(1)^{br} g'_L(L)^{br}}{(g_L(L) - g_L(1))^{2br}} \right] \leq L^{-2br} \mathbb{E}[g'_L(1)^{br}]. \]
Or, in other words,
\[ (6.10) \quad \mathbb{E}[C(\mathbb{H} \setminus \gamma(0, \infty); 1, L)^r] \propto L^{-q(br) - 2b(r-1)}. \]
2. Two-sided

We will now do a similar result for two processes.

**Lemma 6.87.** Suppose $y < 0 < x$, $X_t = g_t(x) - U_t$, $\tilde{X}_t = U_t - g_t(y)$. Let $\lambda_1, \lambda_2 \geq \lambda_0$ and let $q = q(\lambda_1), \tilde{q} = q(\lambda_2)$, $r = \tilde{q}q/a$. Then

$$M_t = X_t^q \tilde{X}_t^{\tilde{q}} (X_t + \tilde{X}_t)^r g'_t(x)^{\lambda_1} g'_t(y)^{\lambda_2}$$

is a martingale.

**Proof.** Recalling that

$$dX_t = \frac{a}{X_t} dt + dB_t, \quad d\tilde{X}_t = \frac{q}{X_t} dt - dB_t,$$

we can use Itô’s formula and the product rule to show that

$$dM_t = M_t \left[ K_t dt + \left( \frac{q}{X_t} - \frac{\tilde{q}}{\tilde{X}_t} \right) dB_t \right],$$

where

$$K_t = \frac{qa + \frac{1}{2}q(q-1) - a\lambda_1}{X_t^2} + \frac{\tilde{q}a + \frac{1}{2}\tilde{q}(\tilde{q}-1) - a\lambda_2}{\tilde{X}_t^2} + \frac{ar - \tilde{q}q}{X_t \tilde{X}_t} = 0,$$

for our choice of $q, \tilde{q}, r$. If we weight the paths by $M_t$, we get

$$dB_t = \left( \frac{q}{X_t} - \frac{\tilde{q}}{\tilde{X}_t} \right) dt + dW_t.$$

Therefore,

$$dX_t = \left( \frac{a + q}{X_t} - \frac{\tilde{q}}{\tilde{X}_t} \right) dt + dW_t,$$

$$d\tilde{X}_t = \left( \frac{a + \tilde{q}}{X_t} - \frac{q}{\tilde{X}_t} \right) dt - dW_t.$$

By examining these coupled Bessel-like processes we can see that $M_t$ is a martingale. □

This lemma and (6.5) go a long way to proving an important estimate. For large $t$, $X_t \approx t^{1/2}$, $\tilde{X}_t \approx t^{1/2}$. Since $E[M_t] = M_0$, we get

$$E[g'_t(x)^{\lambda_1} g'_t(y)^{\lambda_2}] \approx t^{-q(\lambda_1, \lambda_2)/2},$$

where

$$q(\lambda_1, \lambda_2) = q(\lambda_1) + q(\lambda_2) + \frac{q(\lambda_1) q(\lambda_2)}{a}.$$ 

In fact, one can show that for each $x > 0$ there is a $c_x > 0$ such that

$$\lim_{t \to \infty} t^{q(\lambda_1, \lambda_2)/2} E[g'_t(x)^{\lambda_1} g'_t(-1)^{\lambda_2}] = c_x.$$

We will now define the chordal crossing exponents $\tilde{\xi} = \tilde{\xi}_\kappa$

- $\tilde{\xi}(\lambda) = \lambda$,

- $\tilde{\xi}(\lambda_1, \lambda_2) = \lambda_1 + \lambda_2 + q(\lambda_1) q(\lambda_2)$.

Note that

$$\tilde{\xi}(\lambda_1, \lambda_2) = q^{-1} \left( q(\tilde{\xi}(\lambda_1)) + q(\tilde{\xi}(\lambda_2)) \right) = q^{-1} (q(\lambda_1) + q(\lambda_2)).$$
More generally we define
\[
\tilde{\xi}(\lambda_1, \lambda_2, \ldots, \lambda_n) = q^{-1} (q(\lambda_1) + \cdots + q(\lambda_n)).
\]

Clearly \(\tilde{\xi}\) is a symmetric function and one can check that it satisfies the “cascade relation”
\[
\tilde{\xi}(\lambda_1, \ldots, \lambda_n) = \tilde{\xi}(\tilde{\xi}(\lambda_1, \ldots, \lambda_k), \tilde{\xi}(\lambda_{k+1}, \ldots, \lambda_n)).
\]
If \(\tilde{\xi}_n = \tilde{\xi}(b, \ldots, b)\), then
\[
\tilde{\xi}_{n+1} = \tilde{\xi}(b, \tilde{\xi}_n) = \tilde{\xi}_n + a + \sqrt{(2a - 1)^2 + 8a\tilde{\xi}_n},
\]
which yields by induction
\[
\tilde{\xi}_n = an^2 + (2a - 1)n.
\]

**Example 6.88.** The case \(\kappa = 8/3\) gives the (chordal or half-plane) Brownian intersection exponents. Here we have
\[
q(\beta) = -\frac{1}{4} + \frac{1}{4} \sqrt{1 + 24\beta},
\]
\[
\tilde{\xi}(\beta, \lambda) = \beta + \lambda + q(\beta) \left[ \sqrt{1 + 24\lambda} - 1 \right].
\]
In particular, \(q(1) = 1, q(2) = 3/2\) and hence
\[
\tilde{\xi}(1, \lambda) = 1 + \lambda + \frac{4}{3} q(\lambda) = \lambda + \frac{2}{3} + \frac{1}{3} \sqrt{1 + 24\lambda},
\]
\[
\tilde{\xi}(2, \lambda) = 2 + \lambda + 2q(\lambda) = \lambda + \frac{3}{2} + \frac{1}{2} \sqrt{1 + 24\lambda}.
\]

### 3. Nonintersecting SLE\(\kappa\)

We will assume \(\kappa \leq 4\) and \(k\) is a positive integer. Suppose \(D\) is a simply connected domain and \(z_1, \ldots, z_k, w_1, \ldots, w_k\) are distinct points on the boundary of \(D\). We will assume that \(\partial D\) is locally analytic near all of these boundary points. We define \(SLE_D(z_1, \ldots, z_k; w_1, \ldots, w_k)\) which is a measure on \(k\)-tuples of curves \(\tilde{\gamma} = (\gamma^1, \ldots, \gamma^k)\) with \(\gamma^j\) connecting \(z_j\) to \(w_j\). The measure is defined inductively on \(k\).

- If \(k = 1\), \(\mu_D(z_1, w_1)\) is \(SLE_\kappa\) from \(z_1\) to \(w_1\) in \(D\).
- Suppose \(SLE_D(z_1, \ldots, z_{k-1}; w_1, \ldots, w_{k-1})\) has been defined. This is a measure on \((k - 1)\)-tuples \(\tilde{\gamma} = (\gamma^1, \ldots, \gamma^{k-1})\). For each realization of \(\tilde{\gamma}\), let \(D_{\tilde{\gamma}}\) denote the connected component of \(D \setminus \tilde{\gamma}\) that contains \(z_k\) and \(w_k\) on the boundary. Then for the marginal on \(\tilde{\gamma}\) we define
  \[
  \frac{d\mu_D(z_1, \ldots, z_k; w_1, \ldots, w_k)}{d\mu_D(z_1, \ldots, z_{k-1}; w_1, \ldots, w_{k-1})}(\tilde{\gamma}) = C(D_{\tilde{\gamma}}; z_k, w_k).
  \]
If \(z_k, w_k\) are not on the boundaries of the same connected component we set \(C(D_{\tilde{\gamma}}; z_k, w_k) = 0\).
- The conditional probability measure on \(z_k, w_k\) given \(\hat{\gamma}\) is that of \(\mu_{D_{\hat{\gamma}}}(z_k, w_k)\).
While the definition above makes it appear that the definition depends on the order that we specify the pairs \((z_1, w_1), \ldots, (z_k, w_k)\), we can give an equivalent definition that is independent. Let \(Y = Y_D(z_1, \ldots, z_k, w_1, \ldots, w_k)\) denote the Radon-Nikodym derivative of \(\mu_D(z_1, \ldots, z_k; w_1, \ldots, w_k)\) with respect to the product measure

\[
\mu_D(z_1, w_1) \times \cdots \times \mu_D(z_k, w_k).
\]

Then if \(\gamma = (\gamma^1, \ldots, \gamma^k)\),

\[
Y(\gamma) = 1 \{ \gamma^j \cap \gamma^n = \emptyset, 1 \leq j < n \leq k \} \exp \frac{-\Theta c}{2}
\]

where \(\Theta\) denotes the Brownian loop measure of the set of loops that intersect at least two of the curves. We will not give the proof which is essentially an application of the ideas from Lecture 4.

The quantity

\[
\frac{C(D; z_1, z_2, \ldots, z_k; w_1, w_2, \ldots, w_k)}{C(D; z_1, w_1) \cdot \cdots \cdot C(D; z_k, w_k)}
\]

is a conformal invariant. For \(k = 2\), it is given by \(\phi(L)\) defined in (6.10).

4. Radial exponent and SAW III

Let us consider radial SLE\(_\kappa\) in \(\mathbb{D}\) from 1 to 0 parametrized as in Lecture 5, Section 3 with \(g'_t(0) = e^{2at}\). Let \(\theta \in (0, \pi)\). We will consider \(E[|g'_t(e^{2i\theta})|^\lambda]\). As seen in that section, this is an estimate about a one-dimensional diffusion. Indeed,

\[
|g'_t(e^{2i\theta})|^\lambda = \exp \left\{ -a\lambda \int_0^t \frac{dt}{\sin^2 \Psi_t} \right\},
\]

where \(\Psi_t\) satisfies

\[
d\Psi_t = a \cot \Psi_t dt + dW_t.
\]

Proposition 6.89. If \(a \geq 1/4\) and \(\lambda > \lambda_0\), then there exists \(c\) such that

\[
E[|g'_t(e^{2i\theta})|^\lambda] \sim c e^{-2a\beta t} \sin^r \theta = c g'_t(0)^{-\beta} \sin^q \theta,
\]

where

\[
q = q(\lambda) = \frac{(1 - 2a) + \sqrt{(2a - 1)^2 + 8a\lambda}}{2}, \quad \beta = \beta(\lambda) = \frac{\lambda}{2} + \frac{q}{4a}.
\]

Proof. (Sketch) If

\[
r^2 + r(2a - 1) - 2a\lambda = 0, \quad k = a\lambda + \frac{r}{2},
\]

then \(M_t = |g'_t(e^{2i\theta})|^\lambda \sin^r \Psi_t e^{kt}\), is a local martingale satisfying

\[
dM_t = r \cot \Psi_t M_t dW_t.
\]

If \(r\) is chosen as in (6.12), then using Girsanov we can show that \(M_t\) is actually a martingale and hence

\[
\sin^r \theta = e^{kt} E \left[ |g'_t(e^{2i\theta})|^\lambda \sin^r \Psi_t \right].
\]

We then proceed as in Proposition 6.85.
Example 6.90. We have already seen an example of this exponent. Let \( \lambda = b \). Then \( r = a, \beta = 3a/4 \). If \( C \) denotes the radial \( SLE \) total mass, then

\[
C(\mathbb{D} \setminus \gamma(0,t); e^{2i\theta},0) = g'_t(0)^{b} |g'_t(e^{2i\theta})|^b.
\]

Hence we can interpret this as

\[
E \left[ C(\mathbb{D} \setminus \gamma(0,t); e^{2i\theta},0) \right] \sim c \sin^b \theta g'_t(0)^{\tilde{b} - 2\pi/b}.
\]

Example 6.91. (SAW III) Here we will show how to predict a critical exponent for self-avoiding walk. Define the exponent \( u \) by saying that the probability that two self-avoiding walks of \( n \) steps starting at the origin avoid each other decays like \( n^{-u} \). The typical distance of a SAW of length \( n \) is \( n^{1/d} \) where \( d = 4/3 \) is the fractal dimension of the paths. Hence, we can rephrase the definition of \( u \) as saying that the probability that two SAWs go distance \( R \) without an intersection decays like \( R^{-ud} \). Given the conjectured relation between SAWs and \( SLE_{8/3} \), we can guess that if \( \gamma^1, \gamma^2 \) are independent radial \( SLE_{8/3} \) in \( \mathbb{D} \) going to 0 starting at \( \pm 1, -1 \), respectively, then the probability that \( \gamma^1 \) reaches the ball of radius \( \delta \) about the origin without hitting the path of \( \gamma^2 \) decays like \( \delta^{ud} \). The restriction property tells us that for a fixed realization of \( \gamma^1 \) (with corresponding conformal maps \( g_t \)), the probability that \( \gamma^2 \) avoids \( \gamma^1(0,t) \) is given by

\[
g'_t(0)^b |g'_t(-1)|^b,
\]

with \( \tilde{b} = 5/48, b = 5/8 \). Also, the Koebe-(1/4) theorem tells us that the distance from the origin to \( \gamma^1(0,t) \) is comparable to \( g'_t(0) \). Our proposition tells us that

\[
E[|g'_t(-1)|^b] \propto g'_t(0)^{-\beta}, \quad \beta = \beta(b) = \frac{3a}{4} = \frac{9}{16}.
\]

Hence the probability of no intersection is given by

\[
E[g'_t(0)^{\tilde{b}} |g'_t(-1)|^b] \sim g'_t(0)^{-11/24}.
\]

This gives \( ud = 11/24, u = 11/32 \) and matches the prediction first given by Nienhuis. The exponent \( u \) is the same as \( \gamma - 1 \) described in Lecture 1, Section 1.

The radial exponent \( \xi \) is defined by

\[
\xi(b, \lambda) = \tilde{b} + \beta(\lambda) = \tilde{b} + \frac{\lambda}{2} + \frac{g(\lambda)}{4a}.
\]

More generally, we define

\[
\xi(b, \lambda_1, \ldots, \lambda_k) = \xi(b, \xi(\lambda_1, \ldots, \lambda_k)).
\]

Note that the chordal exponent \( \tilde{\xi} \) appears in the definition of \( \xi \). The exponent \( \xi(\lambda_1, \lambda_2) \) is defined so that the following holds:

\[
\xi(b, \lambda, \lambda_2) = \xi(b, \tilde{\xi}(\lambda, \lambda_2)) = \xi(\tilde{\xi}(b, \lambda_1), \lambda_2).
\]

Example 6.92. If \( \kappa = 8/3 \), then \( \tilde{\xi}(5/8, 1/8) = 1, g(1/8) = 1/4, \xi(5/8, 5/8) = 2, q(5/8) = 3/4, \]

\[
\xi(5/8, \lambda) = \frac{5}{48} + \frac{\lambda}{2} + \frac{1}{3} g(\lambda) = \frac{1}{48} + \frac{\lambda}{2} + \frac{1}{12} \sqrt{1 + 24\lambda}.
\]

One can check that

\[
\xi(1, \lambda) = \frac{1}{8} + \frac{\lambda}{2} + \frac{1}{8} \sqrt{1 + 24\lambda}.
\]
\[ \xi(2, \lambda) = \frac{11}{24} + \frac{\lambda}{2} + \frac{5}{24} \sqrt{1 + 24\lambda}. \]

These are the one-sided and two-sided Brownian intersection exponents. The values \( \xi(1, 0) = 1/4, \xi(2, 0) = 2/3 \) are the one-sided and two-sided disconnection exponents.

5. Exercises for Lecture 6

Exercise 6.93. Find the constant \( c \) in Proposition 6.85.

Exercise 6.94. Complete the details in the proof of Proposition 6.89. Note that in the proposition, \( r \) was chosen to be a particular root of the quadratic equation in (6.13). Show that if we choose \( r \) to be the other root of that equation, then \( M_t \) is not a martingale (even though it is a local martingale). Express \( c \) in terms of invariant densities for diffusions of the form (6.11).

Exercise 6.95. In Proposition 6.89, note that \( \beta > \lambda/2 \) if \( \lambda > 0 \) and \( \beta \sim \lambda/2 \) as \( \lambda \to \infty \). Show how these facts can be deduced from the Beurling estimate.

Exercise 6.96. Fix \( \kappa \leq 4 \) and positive integer \( n \). Let \( 0 < y_1 < \cdots < y_n < \pi \) and let \( R_L = [0, L] \times [0, \pi] \). Let \( z_j = y_j i, w_j = w_j L = L + y_j i \). Show that as \( L \to \infty \),
\[ C(R_L; z_1, \ldots, z_n; w_1, \ldots, w_n) \asymp e^{-L \tilde{\xi}_n}, \]
where \( \tilde{\xi}_n \) is as defined in Section 2.

Exercise 6.97. For \( \kappa = 2 \), it can be shown using an identity of Fomin that
\[ C(R_L; z_1, \ldots, z_n; w_1, \ldots, w_n) = \det [H_{\partial R_L}(z_j, w_k)]. \]
Use this identity to give another proof that
\[ \tilde{\xi}_n = \frac{n^2 + n}{2}, \quad \kappa = 2. \]
### Table 1. Parameters for SLE

| $\kappa$ | variance in driving function | $\kappa > 0$ |
|----------|-----------------------------|--------------|
| $a$      | rate of capacity growth     | $\frac{2}{\kappa}$ |
| $d$      | dimension of paths          | $1 + \frac{\kappa}{8} = 1 + \frac{4}{a}$, $\kappa \leq 8$ |
| $b$      | boundary scaling exponent   | $\frac{3a-1}{2} = \frac{6-\kappa}{2\kappa}$ |
| $\hat{b}$ | one-sided interior scaling exponent | $\frac{1-a}{2a} b = \frac{\kappa-2}{4} b$ |
| $\hat{\hat{b}}$ | two-sided interior scaling exponent | $2 - d = 1 - \frac{\kappa}{8}$ |
| $c$      | central charge              | $\frac{2b(3-4a)}{a} = \frac{(4-4a)(3a-1)}{a} = \frac{(3\kappa-8)(6-\kappa)}{2\kappa}$ |

### Table 2. Some discrete models

|                      | $\kappa$ | $a$ | $b$ | $\hat{b}$ | $c$ | $d$ |
|----------------------|----------|-----|-----|-----------|-----|-----|
| loop-erased walk     | 2        | 1   | 1   | 0         | $-2$ | $\frac{5}{4}$ |
| self-avoiding walk   | $\frac{8}{3}$ | $\frac{3}{4}$ | $\frac{5}{8}$ | $\frac{5}{48}$ | 0 | $\frac{4}{3}$ |
| Ising interface      | 3        | $\frac{2}{3}$ | $\frac{1}{2}$ | $\frac{1}{8}$ | $\frac{1}{2}$ | $\frac{11}{8}$ |
| harmonic explorer    | 4        | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{8}$ | 1 | $\frac{3}{2}$ |
| free field interface |          |     |     |           |     |     |
| percolation interface| 6        | $\frac{1}{3}$ | 0   | 0         | 0   | $\frac{7}{4}$ |
| uniform spanning tree| 8        | $\frac{1}{4}$ | $-\frac{1}{8}$ | $-\frac{3}{16}$ | $-2$ | 2 |
[1] M. Bauer, D. Bernard (2002), $SLE_\kappa$ growth processes and conformal field theories, Phys. Lett. B 543, 135–138.
[2] M. Bauer, D. Bernard (2003), Conformal field theories of stochastic Loewner evolutions. Comm. Math. Phys. 239, 493–521.
[3] M. Bauer, D. Bernard (2003), $SLE$ martingales and the Virasoro algebra. Phys. Lett. B 557, 309–316.
[4] M. Bauer, D. Bernard (2004), Conformal transformations and the $SLE$ partition function martingale. Ann. Henri Poincaré 5, 289–326.
[5] M. Bauer, D. Bernard (2004), CFTs of $SLEs$: the radial case, Phys. Lett. B 583, 324–330.
[6] M. Bauer, D Bernard (2006), 2D growth processes: $SLE$ and Loewner chains. Phys. Rep. 432, 115–221.
[7] M. Bauer, D. Bernard, D., J. Houdayer (2005), Dipolar stochastic Loewner evolutions. J. Stat. Mech. Theory Exp., P03001.
[8] M. Bauer, D. Bernard, K. Kytölä, Multiple Schramm-Loewner evolutions and statistical mechanics martingales. J. Stat. Phys. 120, 1125–1163.
[9] R. Bauer, R. Friedrich (2004), Stochastic Loewner evolution in multiply connected domains. C. R. Math. Acad. Sci. Paris 339, 579–584.
[10] R. Bauer, R. Friedrich (2006), On radial stochastic Loewner evolution in multiply connected domains. J. Funct. Anal. 237, 565–588.
[11] V. Beffara (2004), Hausdorff dimension for $SLE_6$, Ann. Probab 32, 2606–2629.
[12] V. Beffara, The dimension of the $SLE$ curves, to appear in Ann. Probab.
[13] A. Belavin, A. Polyakov, A. Zamolodchikov (1984), Infinite conformal symmetry of critical fluctuations in two dimensions, J. Stat. Phys. 34, 763–774.
[14] A. Belavin, A. Polyakov, A. Zamolodchikov (1984), Infinite conformal symmetry in two-dimensional quantum field theory. Nuclear Phys. B 241, 333–380.
[15] D. Bernard (2005), Conformal field theories in random domains and stochastic Loewner evolutions. XIVth International Congress on Mathematical Physics, World Sci. Publ., 535–549.
[16] E. Bettelheim, I. Gruzberg, A. Ludwig, P. Wiegmann, (2005), Stochastic Loewner evolution for conformal field theories with Lie group symmetries. Phys. Rev. Lett. 95, 251601.
[17] E. Bogomolny, R. Dubertrand, C. Schmit (2007), SLE description of the nodal lines of random wavefunctions. J. Phys. A 40, 381–395.
[18] F. Camia, C. Newman (2006). Two-dimensional critical percolation: the full scaling limit. Comm. Math. Phys. 268, 1–38.
[19] J. Cardy (1984), Conformal invariance and surface critical behavior, Nucl. Phys. B 240 (FS12), 514–532.
[20] J. Cardy (1992), Critical percolation in finite geometries, J. Phys. A 25, L201–L206.
[21] J. Cardy (2003), Stochastic Loewner evolution and Dyson’s circular ensembles, J. Phys. A 36, L379–L386.
[22] J. Cardy (2004), $SLE$ for theoretical physicists (2005), Ann. Physics 318, 81–118.
[23] J. Cardy (2007), ADE and $SLE$. J. Phys. A 40, 1427–1438.
[24] B. Doyon, V. Riva,J. Cardy (2006), Identification of the stress-energy tensor through conformal restriction in $SLE$ and related processes. Comm. Math. Phys. 268, 687–716.
[25] J. Dubédat (2003), $SLE$ and triangles, Electron. Comm. Probab. 8, 28–42.
[26] J. Dubédat (2004), Critical percolation in annuli and $SLE_6$, Comm.Math. Phys. 245, 627–637.
[28] J. Dubédat (2005), SLE(κ, ρ) martingales and duality, Ann. Probab. 33, 223–243.
[29] J. Dubédat, Euler integrals for commuting SLEs (2006), J. Stat. Phys. 123, 1183–1218.
[30] J. Dubédat, Excursion decompositions for SLE and Watts’ crossing formula (2006), Probab. Theory Related Fields 134, 453–488.
[31] B. Duplantier (1998), Random walks and quantum gravity in two dimensions, Phys. Rev. Lett. 81, 5489–5492.
[32] R. Friedrich, J. Kalkkinen (2004), On conformal field theory and stochastic Loewner evolution. Nuclear Phys. B 687, 279–302.
[33] R. Friedrich, W. Werner (2002), Conformal fields, restriction properties, degenerate representations and SLE, C. R. Acad. Sci. Paris Ser. I Math 335, 947–952.
[34] R. Friedrich, W. Werner (2003), Conformal restriction, highest-weight representations and SLE, Comm. Math. Phys. 243, 105–122.
[35] I. Gruzberg (2006), Stochastic geometry of critical curves, Schramm-Loewner evolutions and conformal field theory. J. Phys. A 39, 12601–12655.
[36] I. Gruzberg, and L. Kadanoff (2004), The Loewner equation: maps and shapes, J. Stat. Phys, 114, 1183-1198.
[37] W. Kager, B. Nienhuis (2004), A guide to stochastic Loewner evolution and its applications. J. Statist. Phys. 115, 1149–1229.
[38] W. Kager, B. Nienhuis, L. Kadanoff (2004), Exact solutions for Loewner evolutions. J. Statist. Phys. 115, 805–822.
[39] N.-G. Kang (2007), Boundary behavior of SLE. J. Amer. Math. Soc. 20, 185–210.
[40] T. Kennedy (2003), Monte Carlo tests of SLE predictions for 2D self-avoiding walks, Phys. Rev. Lett. 88, 130601.
[41] T. Kennedy (2004), Conformal invariance and stochastic Loewner evolution predictions for the 2D self-avoiding walk–Monte Carlo tests, J. Statist. Phys. 114, 51–78.
[42] M. Kozdron, G. Lawler (2007). The configuration measure on mutually avoiding SLE paths, in Universality and Renormalization: From Stochastic Evolution to Renormalization of Quantum Fields, I. Binder, D. Kreimer, ed., Amer. Math. Soc., 199–224.
[43] K. Kytölä, On conformal field theory of SLE(κ, ρ) (2006), J. Stat. Phys. 123, 1169–1181.
[44] K. Kytölä, A. Kemppainen (2006), Antti SLE local martingales, reversibility and duality, J. Phys. A 39, L657–L666.
[45] G. Lawler (2005). Conformally Invariant Processes in the Plane, American Mathematical Society.
[46] G. Lawler (2004), An introduction to the stochastic Loewner evolution, in Random Walks and Geometry, V. Kaimanovich, ed., de Gruyter.
[47] G. Lawler (2004), Conformal invariant processes in the plane, in School and Conference on Probability Theory, ICTP Lecture Notes 17, G. Lawler, ed., 305–351.
[48] G. Lawler (2006), The Laplacian-b-random walk and the Schramm-Loewner evolution, Illinois J. Math. 50, no. 1–4, 701–746.
[49] G. Lawler, J. Lind (2007), Two-sided SLE8/3 and the infinite self-avoiding polygon, in Universality and Renormalization: From Stochastic Evolution to Renormalization of Quantum Fields, I. Binder, D. Kreimer, ed., Amer. Math. Soc., 249–280.
[50] G. Lawler, O. Schramm, W. Werner (2001), Values of Brownian intersection exponents I: Half-plane exponents, Acta Math. 187, 237–273.
[51] G. Lawler, O. Schramm, W. Werner (2001), Values of Brownian intersection exponents II: Plane exponents, Acta Math. 187, 273–308.
[52] G. Lawler, O. Schramm, W. Werner (2002), Values of Brownian intersection exponents III: Two-sided exponents, Ann. Inst. Henri Poincaré PR 38, 109-123.
[53] G. Lawler, O. Schramm, W. Werner (2002), Analyticity of intersection exponents for planar Brownian motion, Acta Math. 189, 179–201.
[54] G. Lawler, O. Schramm, W. Werner (2001), The dimension of the planar Brownian frontier is 4/3, Math. Res. Lett. 8, 401–411.
[55] G. Lawler, O. Schramm, W. Werner (2002), Sharp estimates for Brownian non-intersection probabilities, in In and out of equilibrium. Probability with a Physics flavour, V. Sidoravicius, ed., Birkhäuser.
[56] G. Lawler, O. Schramm, W. Werner (2004), Conformal invariance of planar loop-erased random walks and uniform spanning trees, Annals of Probab. 32, 939–995.
57. G. Lawler, O. Schramm, W. Werner (2004), On the scaling limit of planar self-avoiding walk, in Fractal Geometry and Applications: A Jubilee of Benoit Mandelbrot, Vol. II., M. Lapidus, M. van Frankenhuijsen, ed., Amer. Math. Soc., 339–364.

58. G. Lawler, O. Schramm, W. Werner (2003), Conformal restriction: the chordal case, J. Amer. Math. Soc. 16, 917–955.

59. G. Lawler, J. A. Trujillo Ferreras (2007), Random walk loop soup, Trans. Amer. Math. Soc. 359, 767–787.

60. G. Lawler, W. Werner (1999), Intersection exponents for planar Brownian motion, Annals of Probability 27, 1601–1642.

61. G. Lawler, W. Werner (2000), Universality for conformally invariant intersection exponents, J. Europ. Math. Soc. 2, 291-328.

62. G. Lawler, W. Werner (2004), The Brownian loop soup, Probab. Theory Related Fields 128, 565–588.

63. F. Lesage, J. Rasmussen (2004), SLE-type growth processes and the Yang-Lee singularity. J. Math. Phys. 45, 3040–3048

64. D. Marshall, S. Rohde (2005), The Loewner differential equation and slit mappings. J. Amer. Math. Soc. 18, 763–778.

65. S. Moghimi-Araghi, M. Rajabpour, S. Rouhani (2004), Logarithmic conformal null vectors and SLE, Phys. Lett. B 600, 297–301.

66. S. Moghimi-Araghi, M. Rajabpour, S. Rouhani, S. (2006), SLE(κ, ρ) and boundary Coulomb gas, Nuclear Phys. B 740, 348–357.

67. V. Riva, J. Cardy (2006), Holomorphic parafermions in the Potts model and stochastic Loewner evolution. J. Stat. Mech. Theory Exp., P12001,

68. O. Schramm (2000), Scaling limits of loop-erased random walks and uniform spanning trees, Israel J. Math. 118, 221–288.

69. O. Schramm (2001), A percolation formula, Electron. Comm. Probab. 6, 115–120.

70. O. Schramm (2007), Conformally invariant scaling limits: an overview and a collection of problems, Proc. Int. Congress of Mathematicians, Vol. I, Eur. Math. Soc., 513–543.

71. O. Schramm, S. Sheffield (2005), Harmonic explorer and its convergence to SLE4. Ann. Probab. 33, 2127–2148.

72. O. Schramm, D. Wilson (2005), SLE coordinate changes. New York J. Math. 11, 659–669.

73. S. Smirnov (2001), Critical percolation in the plane: Conformal invariance, Cardy’s formula, scaling limits, C. R. Acad. Sci. Paris S. I Math. 333 no. 3, 239–244.

74. W. Werner (2001), Critical exponents, conformal invariance and planar Brownian motion, in Proceedings of the 4th ECM Barcelona 2000, Prog. Math, 202, Birkhäuser.

75. W. Werner (2003), SLEs as boundaries of clusters of Brownian loops. C. R. Math. Acad. Sci. Paris 337, 481–486.

76. W. Werner (2004), Random planar curves and Schramm-Loewner evolutions, Ecole d’Eté de Probabilités de Saint-Flour XXXII - 2002, Lecture Notes in Mathematics 1840, Springer-Verlag, 113–195.

77. W. Werner (2004), Girsanov’s transformation for SLE(κ, ρ) processes, intersection exponents and hiding exponents. Ann. Fac. Sci. Toulouse Math. (6) 13, 121–147.

78. W. Werner (2006), Conformal restriction properties, Proc. Int. Congress of Mathematicians, Vol. III, Eur. Math. Soc., 741–762.

79. D. Zhan (2004), Stochastic Loewner evolution in doubly connected domains. Probab. Theory Related Fields 129, 340–380.

80. D. Zhan (2006), Some properties of annulus SLE, Electron. J. Probab. 11, 1069–1093