Fully Modified Functional Principal Component Analysis for Cointegrated Functional Time Series

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Abstract

Functional principal component analysis (FPCA) has played an important role in the development of functional time series analysis. This paper investigates how FPCA can be used to analyze cointegrated functional time series and proposes a modification of FPCA as a novel statistical tool. Our modified FPCA not only provides an asymptotically more efficient estimator of the cointegrating vectors, but also leads to novel FPCA-based tests for examining some essential properties of cointegrated functional time series. We apply our methodology to two empirical examples: U.S. age-specific employment rates and earning densities.

1 Introduction

Functional principal component analysis (FPCA) has been a central tool in analysis of functional time series (FTS) whose observations are, for example, curves, probability density functions, or images. In a similar vein to PCA for multivariate data analysis, FPCA is based on eigendecomposition of the sample covariance operator and results in an effective way of dimension-reduction of functional observations that are high-dimensional in most cases. For this reason, FPCA has been widely employed in various contexts encompassing inference for the functional autoregressive (AR) model (Bosq, 2000), testing structural breaks (Berkes et al., 2009; Horváth et al., 2010; Zhang et al., 2011), FTS regression (Park and Qian, 2012), prediction/forecasting (Hyndman and Ullah, 2007; Aue et al., 2015) and long memory FTS (Li et al., 2020) to name only a few. Hörmann and Kokoszka (2012) and Shang (2014) well illustrate and summarize how FPCA is used in FTS analysis. A recent work by Chang et al. (2016) is another example employing FPCA to analyze FTS; however, it is distinct from the foregoing articles in the sense that FPCA is applied to nonstationary FTS allowing cointegration (a.k.a. cointegrated FTS) for the purpose of characterizing some essential properties of such time series. Provided that many economic and statistical time series are nonstationary but allow a long run stable relationship, their analysis paves the way for many potential applications; see e.g., Chang et al. (2020). Moreover, their methodology was recently adopted and extended by Li et al. (2022) for nonstationary fractionally integrated FTS.

The purpose of this paper is to further investigate how FPCA can be used in statistical analysis of cointegrated FTS \( \{X_t\}_{t \geq 1} \) taking values in a Hilbert space \( \mathcal{H} \). Cointegrated FTS considered in this paper is

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partially nonstationary in the sense that it can be decomposed by orthogonal projections \( P^N \) and \( P^S = I - P^N \) (where \( I \) denotes the identity map defined on \( \mathcal{H} \)) into a finite dimensional unit root process \( \{P^N X_t\}_{t \geq 1} \) and a potentially infinite dimensional stationary process \( \{P^S X_t\}_{t \geq 1} \). When such a time series is given, it is important to estimate \( P^N \) or \( P^S \) since either of them characterizes the cointegrating behavior; note that \( P^S \) is a linear transformation that eliminates the unit root component of \( \{X_t\}_{t \geq 1} \) and leaves only its stationary component, which is as the cointegrating vectors do in a multivariate cointegrated system. In fact, estimation of \( P^N \) or \( P^S \) reduces to estimation of the cointegrating vectors if \( \mathcal{H} \) is a conventional Euclidean space.

Chang et al. (2016) earlier studied how the eigenelements of the sample covariance operator can be used to construct a consistent estimator of \( P^N \) and provide some relevant asymptotic results. In this paper, we first add some novel asymptotic results, including the convergence rate and the asymptotic limit of the estimation. Our modification of the ordinary FPCA methodology that guarantees (i) a more asymptotically efficient estimator of \( P^N \) and (ii) a more convenient asymptotic limit which leads to novel statistical tests for examining hypotheses about cointegration. Our modification of the ordinary FPCA methodology is motivated by Phillips and Hansen (1990), Park (1992), and Harris (1997) that propose semiparametric modifications to obtain asymptotically efficient estimators of the cointegrating vectors in a multivariate cointegrated system.

The rest of this paper is organized as follows. Section 2 provides notation and main assumptions about cointegrated FTS. In Section 3, we discuss on the ordinary FPCA and our proposed modification of it as statistical tools to estimate \( P^N \) or \( P^S \), and provide the relevant asymptotic theory. Based on our modified FPCA, Section 4 gives statistical tests to examine various hypotheses on cointegration. We illustrate our methodology with empirical examples in Section 5. All proofs are given in the Supplementary Appendix to this article.

2 Cointegrated FTS in Hilbert space

2.1 Notation

Let \( \mathcal{H} \) denote a real separable Hilbert space with inner product \( \langle \cdot, \cdot \rangle \) and norm \( \| \cdot \| = \langle \cdot, \cdot \rangle^{1/2} \). We let \( \mathcal{L}_\mathcal{H} \) denote the space of bounded linear operators on \( \mathcal{H} \), equipped with the usual operator norm \( \|A\|_{\mathcal{L}_\mathcal{H}} = \sup_{\|x\| \leq 1} \|Ax\| \). For an operator \( A \in \mathcal{L}_\mathcal{H} \), we let \( A^* \in \mathcal{L}_\mathcal{H} \) denote the adjoint of \( A \), and let \( \text{ran} \ A \) (resp. \( \ker \ A \)) denote the range (resp. kernel) of \( A \), which are respectively defined by

\[
\text{ran} \ A = \{Ax : x \in \mathcal{H}\}, \quad \ker \ A = \{x \in \mathcal{H} : Ax = 0\}.
\]

The rank of \( A \), denoted by \( \text{rank} \ A \), is equal to \( \dim(\text{ran} \ A) \). If \( A = A^* \), then \( A \) is said to be self-adjoint. An operator \( A \in \mathcal{L}_\mathcal{H} \) is positive semidefinite if \( \langle Ax, x \rangle \geq 0 \) for any \( x \in \mathcal{H} \), and positive definite if also \( \langle Ax, x \rangle \neq 0 \) for any nonzero \( x \in \mathcal{H} \). Throughout this paper, \( x \otimes y \) for \( x, y \in \mathcal{H} \) denotes the operator \( z \mapsto \langle x, z \rangle y \) of rank one. An operator \( A \in \mathcal{L}_\mathcal{H} \) is said to be compact if there exists two orthonormal bases \( \{u_j\}_{j \geq 1} \) and \( \{v_j\}_{j \geq 1} \), and a real-valued sequence \( \{a_j\}_{j \geq 1} \) tending to zero, such that \( A = \sum_{j=1}^{\infty} a_j u_j \otimes v_j \); we may assume that \( u_j = v_j \) and \( a_1 \geq a_2 \geq \ldots \geq 0 \) if \( A \) is also self-adjoint and positive semidefinite (Bosq, 2000, p. 35). For any \( A \in \mathcal{L}_\mathcal{H} \) with a closed range, we let \( A^\dagger \in \mathcal{L}_\mathcal{H} \) denote the Moore-Penrose
inverse; see Engl and Nashed (1981). For any self-adjoint, positive semidefinite, and compact operator 
\( A \in \mathcal{L}_H \) satisfying 
\( A = \sum_{j=1}^{\infty} a_j u_j \otimes u_j \) for some \( a_1 \geq a_2 \geq \ldots \geq 0 \) with \( a_m > 0 \), we hereafter let 
\( A_m = \sum_{j=1}^{m} a_j^{-1} u_j \otimes u_j \) and be called the \( m \)-regularized inverse of \( A \). \( A_m \) is set to 0 in \( \mathcal{L}_H \) if \( m = 0 \). The 
\( m \)-regularized inverse \( A^*_m \) is understood as the partial inverse of \( A \) on the restricted domain \( \text{span} \{ u_j \}_{j=1}^{m} \).

If \( \text{rank} \ A = m \), then \( A^*_m = A^\dagger \).

Random elements taking values in \( H \) and \( \mathcal{L}_H \) (the space bounded linear operators equipped with the usual uniform operator topology) are briefly introduced in the Supplementary Appendix (see A.1). We here only review some essential notation. \( L^2_H \) denotes the space of \( H \)-valued random variables \( X \) satisfying 
\( \mathbb{E} X = 0 \) and \( \mathbb{E} \| X \|^2 < \infty \). For any \( X \in L^2_H \), its covariance operator is defined by 
\( C_X = \mathbb{E} [X \otimes X] \). An \( \mathcal{L}_H \)-valued random element is called a random bounded linear operator. If \( A \) is such a random operator and 
\( x_1, \ldots, x_n, y_1, \ldots, y_m \in H \) for some \( n \geq 1 \), then we let the distribution of 
\( (\langle Ax_1, y_1 \rangle, \ldots, \langle Ax_n, y_m \rangle)' \) be called the finite dimensional distribution of \( A \) with respect to the choice of vectors; note that if \( n = m^2 \) for 
some positive integer \( m \), \( x_j = e_j \) and \( y_k = e_k \) for all \( j, k \leq m \), and \( \{ e_j \}_{j=1}^{m^2} \) is an orthonormal basis of 
a subspace \( H_m \), then the associated finite dimensional distribution can be viewed as the distribution of \( A \) as a 
map from \( H_m \) to \( H_m \) (more precisely, the distribution of \( P_m A P_m \), where \( P_m \) is the orthogonal projection 
onto \( H_m \)). If two random bounded linear operators \( A \) and \( B \) have the same finite dimensional distribution 
regardless of \( n \) and \( x_1, \ldots, x_n, y_1, \ldots, y_m \in H \), we write \( A \) =_{\text{fd}} B. \) Moreover, if \( \{ A_j \}_{j \geq 1} \) is a sequence of 
random elements in \( \mathcal{L}_H \) and \( \| A_j - A \|_{\mathcal{L}_H} \to_p 0 \) for some \( A \) taking its value in \( \mathcal{L}_H \), we write \( A_j \to_{\mathcal{L}_H} A \).

### 2.2 Model

We let \( \{ \varepsilon_t \}_{t \in \mathbb{Z}} \) be an iid sequence in \( L^2_H \) with a positive definite covariance and let \( \{ \Phi_j \}_{j \geq 0} \) be a sequence in \( \mathcal{L}_H \) satisfying 
\( \sum_{j=0}^{\infty} j \| \Phi_j \|_{\mathcal{L}_H} < \infty \). We define \( \{ \zeta_t \}_{t \in \mathbb{Z}} \) and \( \{ \eta_t \}_{t \in \mathbb{Z}} \) as follows,

\[
\zeta_t = \sum_{j=0}^{\infty} \Phi_j \varepsilon_{t-j}, \quad \eta_t = \sum_{j=0}^{\infty} \tilde{\Phi}_j \varepsilon_{t-j}, \quad t \in \mathbb{Z},
\]

where \( \tilde{\Phi}_j = -\sum_{k=j+1}^{\infty} \Phi_k \). Under the summability condition on \( \{ \Phi_j \}_{j \geq 0} \), \( \zeta_t \) and \( \eta_t \) are convergent in \( L^2_H \), and the operators \( \Phi(1) := \sum_{j=0}^{\infty} \Phi_j \) and \( \tilde{\Phi}(1) := \sum_{j=0}^{\infty} \tilde{\Phi}_j \) are convergent in \( \mathcal{L}_H \). We consider a sequence 
\( \{ X_t \}_{t \geq 0} \) of which first differences \( \Delta X_t = X_t - X_{t-1} \) satisfy the equations \( \Delta X_t = \zeta_t \) for \( t \geq 1 \). Then \( X_t \) allows the Phillips-Solo decomposition: 
\( X_t = (X_0 - \eta_0) + \Phi(1) \sum_{s=1}^{t} \varepsilon_s + \eta_t, \) for \( t \geq 1 \); see Phillips and Solo (1992). Ignoring the initial values \( X_0 \) and \( \eta_0 \) that are unimportant in the development of our asymptotic theory, we have

\[
X_t = \Phi(1) \sum_{s=1}^{t} \varepsilon_s + \eta_t, \quad t \geq 1.
\]

Note that \( \{ X_t \}_{t \geq 1} \) is nonstationary unless \( \Phi(1) = 0 \); however it may have an element \( x \) such that \( \{ \langle X_t, x \rangle \}_{t \geq 1} \) is stationary. Such an element is called a cointegrating vector, and the collection of the cointegrating vectors, denoted by \( \mathcal{H}^S \), is called the cointegrating space (or the stationary subspace). The attractor space (or the nonstationary subspace), denoted by \( \mathcal{H}^N \), is defined by the orthogonal complement to \( \mathcal{H}^S \). Borrowing terminology from the literature on cointegrated time series in Euclidean space (see e.g., Stock and Watson, 1988; Johansen, 1995), an element of \( \mathcal{H}^N \) may be called a stochastic trend. When \( X_t \) satisfies (2.1),
\( \mathcal{H}^S \) (resp. \( \mathcal{H}^N \)) is given by \([\text{ran } \Phi(1)]^\perp\) (resp. the closure of \( \text{ran } \Phi(1) \)); see Proposition 3.3 of Beare et al. (2017). Obviously, \( \mathcal{H} = \mathcal{H}^N \oplus \mathcal{H}^S \), and the orthogonal projection onto \( \mathcal{H}^N \) is uniquely defined, which is denoted by \( P^N \). Then \( P^S = I - P^N \) is the orthogonal projection onto \( \mathcal{H}^S \). These projections decompose \( \{X_t\}_{t \geq 1} \) into the unit root and the stationary components in the sense that \( \{P^N X_t\}_{t \geq 1} \) is a unit root process which does not allow any cointegrating vector and \( \{P^S X_t\}_{t \geq 1} \) is a stationary process in \( \mathcal{H} \).

Note that \( \{X_t\}_{t \geq 1} \) has zero mean by construction, but which is not essentially required in this paper. A deterministic component such as a nonzero intercept or a linear trend may be allowed in our analysis with a proper modification, which will be discussed in Section B of the Supplementary Appendix.

For our asymptotic analysis, we apply the following additional conditions throughout.

**Assumption M.**

(i) \( \sum_{j=0}^{\infty} j \|\tilde{\Phi}_j\|_{\mathcal{L}(\mathcal{H})} < \infty \) and (ii) \( \varphi = \text{rank } \Phi(1) \in [0, \infty) \).

Assumption M-(i) is employed for mathematical proofs, which may not be restrictive in practice. Assumption M-(ii) implies that \( \dim(\mathcal{H}^N) < \infty \), which is commonly assumed in the recent literature on cointegrated FTS for statistical analysis based on eigendecomposition of the sample covariance operator, see e.g., Chang et al. (2016) and Nielsen et al. (2022).

**Remark 2.1.** Finiteness of \( \varphi \) seems to be reasonable in many empirical examples; see Chang et al. (2016, Section 5) and Nielsen et al. (2022, Section 5). Moreover, it is known that functional AR processes with unit roots (including functional ARMA processes that are considered in e.g. Klepsch et al., 2017) with compact autoregressive operators always satisfy Assumption M-(ii); see Beare and Seo (2020), Franchi and Paruolo (2020) and Seo (2022). It is quite common to assume compactness of autoregressive operators in statistical analysis of such time series, so in this case we do not lose any generality by assuming that \( \varphi \) is finite. In this regard, Assumption M-(ii) does not seem to be restrictive in practice.

The case with \( \mathcal{H}^S = \{0\} \) for each \( \varphi \) is uninteresting for our study of cointegrated FTS, and thus such a case is not considered throughout this paper. Moreover, if \( \varphi = 0 \), \( \{X_t\}_{t \geq 1} \) is stationary and thus \( \mathcal{H}^N = \{0\} \); this is also an uninteresting case except when we examine the null hypothesis of \( \varphi = 0 \) in Section 4. Thus, our asymptotic theory, to be developed in this section, concerns the case when \( \varphi \geq 1 \).

Related to a sequence \( \{X_t\}_{t \geq 1} \) satisfying Assumption M, it will be convenient to introduce additional notation. Note that the identity \( I = P^N + P^S \) implies the following operator decomposition: for \( A \in \mathcal{L}(\mathcal{H}) \),

\[
A = A^{NN} + A^{NS} + A^{SN} + A^{SS}, \quad A^{ij} = P^i AP^j, \quad i \in \{N, S\}, \quad j \in \{N, S\}.
\]

We also define

\[
\mathcal{E}^N_t = P^N \zeta_t, \quad \mathcal{E}^S_t = P^S \eta_t, \quad \mathcal{E}_t = \mathcal{E}^N_t + \mathcal{E}^S_t;
\]

note that each sequence of \( \mathcal{E}^N_t, \mathcal{E}^S_t \) and \( \mathcal{E}_t \) is stationary in \( \mathcal{H} \). We then let \( \Omega \) (resp. \( \Gamma \)) denote the long run
covariance (resp. one-sided long run covariance) operator of \( \{ \mathcal{E}_t \}_{t \in \mathbb{Z}} \), which are defined by

\[
\Omega = \mathbb{E}[\mathcal{E}_t \otimes \mathcal{E}_t] + \sum_{j=1}^{\infty} \{ \mathbb{E}[\mathcal{E}_t \otimes \mathcal{E}_{t-j}] + \mathbb{E}[\mathcal{E}_{t-j} \otimes \mathcal{E}_t] \},
\]

\[
\Gamma = \mathbb{E}[\mathcal{E}_t \otimes \mathcal{E}_t] + \sum_{j=1}^{\infty} \mathbb{E}[\mathcal{E}_t \otimes \mathcal{E}_{t-j}].
\]

Under the summability condition \( \sum_{j=0}^{\infty} j \| \Phi_j \|_{\mathcal{L}_H} < \infty \), \( \Omega \) and \( \Gamma \) are convergent in \( \mathcal{L}_H \) (see Section 3.1 of Beare et al., 2017). As in (2.2), we may decompose those operators as \( \Omega = \Omega^{NN} + \Omega^{NS} + \Omega^{SN} + \Omega^{SS} \) and \( \Gamma = \Gamma^{NN} + \Gamma^{NS} + \Gamma^{SN} + \Gamma^{SS} \).

Let \( W = \{ W(r) \}_{r \in [0,1]} \) denote a Brownian motion in \( \mathcal{H} \) whose covariance operator is given by \( \Omega \) (see e.g., Kuelbs, 1973; Chen and White, 1998). Then we let \( W^N = P^N W \) and \( W^S = P^S W \). Note that \( W^N \) and \( W^S \) are correlated unless \( \Omega^{NS} = \Omega^{SN} = 0 \).

For mathematical convenience in our asymptotic analysis, we employ the following high level assumption: below, for \( x_1, \ldots, x_k \in \mathcal{H} \), we let \( \mathcal{E}_{k,t} = ((\mathcal{E}_t, x_1), (\mathcal{E}_t, x_2), \ldots, (\mathcal{E}_t, x_k))^\prime \) and \( W_k(s) = ((W(s), x_1), (W(s), x_2), \ldots, (W(s), x_k))^\prime \).

**Assumption W.** For any \( k \geq 1 \) and \( x_1, \ldots, x_k \in \mathcal{H} \), \( T^{-1/2} \sum_{t=1}^{T} (\sum_{s=1}^{t} \mathcal{E}_{k,s}) E'_{k,t} \) converges in distribution to \( \int_0^1 W_k(s) dW_k(s)^\prime + \sum_{j=0}^{\infty} \mathbb{E}[\mathcal{E}_{k,t-j} E'_{k,t}] \).

Appropriate conditions for the above convergence result can be found in Hansen (1992); specifically, from Theorem 4.1 of the paper, we may deduce that Assumption W is satisfied if (i) \( T^{-1/2} \sum_{t=1}^{T} \mathcal{E}_{k,t} \}_{s \in [0,1]} \) converges weakly in the Skorohod topology to \( W_k \), (ii) \( \sup_{t \geq 1} \| \mathcal{E}_t \|^n < \infty \), and (iii) \( \mathcal{E}_t \}_{t \geq 1} \) is strongly mixing of size \(-ab(a - b)\) for some \( a > b > 2 \). Among these conditions, the first is in fact a consequence of Assumption M (see Lemma C.1 in the Supplementary Appendix).

3  **FPCA of cointegrated FTS and asymptotic results**

3.1  **Ordinary FPCA of cointegrated FTS**

For given functional observations \( \{ X_t \}_{t=1}^{T} \), we are interested in statistical inference on cointegration, which boils down to estimation and hypothesis testing about \( \mathcal{H}^N \) or \( \mathcal{H}^S \). We here focus on the aspect of estimation, and investigate how FPCA can be used and improved. Throughout this section, we assume that \( \varphi = \dim(\mathcal{H}^N) \) is known. Of course, this is not a realistic assumption in most applications. However, we already have a few available methods to determine \( \varphi \) from functional observations such as those in e.g. Chang et al. (2016), Nielsen et al. (2022), and Li et al. (2022). In addition to those, it will be shown in Section 4 that the asymptotic results to be given in this section lead to a novel FPCA-based testing procedure to determine \( \varphi \).

The sample covariance operator \( \hat{C} \) of \( \{ X_t \}_{t=1}^{T} \) is the random operator given by \( \hat{C} = T^{-1} \sum_{t=1}^{T} X_t \otimes X_t \). Broadly speaking, FPCA reduces to solving the following eigenvalue problem associated with \( \hat{C} \),

\[
\hat{C} \hat{v}_j = \lambda_j \hat{v}_j, \quad j = 1, 2, \ldots, \tag{3.1}
\]

where \( \lambda_1 \geq \ldots \geq \lambda_T > 0 = \lambda_{T+1} = \lambda_{T+2} = \ldots \) and \( \{ \hat{v}_j \}_{j \geq 1} \) constitutes an orthonormal basis of \( \mathcal{H} \).
From the set of the estimated eigenvectors \( \{ \hat{v}_j \}_{j \geq 1} \), we construct our preliminary estimators of \( P^N \) and \( P^S \) as follows,

\[
\hat{P}^N_{\phi} = \sum_{j=1}^{\varphi} \hat{v}_j \otimes \hat{v}_j, \quad \hat{P}^S_{\phi} = I - \hat{P}^N_{\phi}, \tag{3.2}
\]

which are the orthogonal projections onto \( \text{span}\{ \hat{v}_j \}_{j=1}^{\varphi} \) and its orthogonal complement, respectively. The estimators of \( H^N \) and \( H^S \) are then given by \( \hat{H}^N_{\phi} = \text{ran} \, \hat{P}^N_{\phi} \) and \( \hat{H}^S_{\phi} = \text{ran} \, \hat{P}^S_{\phi} \). It is worth noting that \( \hat{P}^N_{\phi} \) and \( \hat{P}^S_{\phi} \) are constructed from only a finite set of the estimated eigenvectors rather than the entire set, so there is no theoretical difficulties in computing those. Note also that \( \varphi \) as a subscript in (3.2) denotes the number of eigenvectors used to construct \( \hat{P}^N_{\phi} \); even if keeping such a subscript may increase notational complexity, it will be eventually helpful to avoid potential confusion when we need to consider \( \hat{P}^N_{\varphi_0} \), the projection onto \( \text{span}\{ \hat{v}_j \}_{j=1}^{\varphi_0} \), for a hypothesized value \( \varphi_0 \) in Section 4.

To investigate the asymptotic properties of the preliminary estimators, we may establish the limiting behavior of either of \( \hat{P}^N_{\varphi} - P^N \) or \( \hat{P}^S_{\varphi} - P^S \) (one is simply given by the negative of the other). The following theorem deals with the former case: we hereafter write \( \int A(r) \) to denote \( \int_0^1 A(r)dr \) for any operator- or vector-valued function \( A \) defined on \([0, 1] \).

**Theorem 3.1.** Suppose that Assumptions M and W hold with \( \varphi \geq 1 \). Then

\[
T(\hat{P}^N_{\varphi} - P^N) \overset{\text{fdd}}{\rightarrow} \mathcal{L} \quad F + F^*,
\]

where \( F = \int \left( W^N(r) \otimes W^N(r) \right)^{\dagger} \left( \int dW^S(r) \otimes W^N(r) + \Gamma^{NS} \right) \).

The asymptotic properties of \( \hat{P}^N_{\varphi} \) given in Theorem 3.1 are enough to show consistency of \( \hat{P}^N_{\varphi} \) in the sense that \( \hat{P}^N_{\varphi} \rightarrow_{\mathcal{L}} P^N \). However, Theorem 3.1 does not facilitate a direct asymptotic inference since \( F \) depends on nuisance parameters. We especially focus on \( \Gamma^{NS} \) and \( \Omega^{NS} \); as shown in Theorem 3.1, \( \Gamma^{NS} \neq 0 \) makes the limiting random operator \( F + F^* \) not centered at zero, and \( \Omega^{NS} \neq 0 \) makes \( W^N \) and \( W^S \) correlated. Therefore, the asymptotic limit of any statistic based on \( \hat{P}^N_{\varphi} \) or \( \hat{P}^S_{\varphi} \) is, if it exists, also dependent on \( \Gamma^{NS} \) and \( \Omega^{NS} \) in general. As will be shown throughout this paper, elimination of such dependence is the most important step to our modified FPCA and statistical tests based on it.

**Remark 3.1.** Chang et al. (2016, Proposition 2.1) earlier established that \( \| \hat{P}^N_{\varphi} - P^N \|_{\mathcal{L}\mathcal{H}} = O_p(T^{-1}) \), and hence consistency of \( \hat{P}^N_{\varphi} \) in the norm of \( \mathcal{L}\mathcal{H} \) is what we already know. It should be noted that Theorem 3.1 extends their result by providing a more detailed limiting behavior of the considered estimator. More importantly, the convergence result given by the theorem becomes an essential input to the modified FPCA methodology that will be discussed in the next section.

### 3.2 Modified FPCA for cointegrated FTS

In this section, we propose a modification of the ordinary FPCA, which not only provides an asymptotically more efficient estimator of \( P^N \) (or \( P^S \)) but establishes the foundation for our statistical tests that will be developed in Section 4. Our modified FPCA may be viewed as an adjustment of the ordinary FPCA for cointegrated FTS rather than an alternative methodology; we actually take full advantage of the asymptotic properties of the preliminary estimators given in Section 3.1.
To introduce our methodology, we first define

\[ Z_{\varphi,t} = \hat{P}_{N\varphi} \Delta X_t + \hat{P}_{S\varphi} X_t, \quad t = 1, \ldots, T, \tag{3.3} \]

where \( \hat{P}_{N\varphi} \) and \( \hat{P}_{S\varphi} \) are the preliminary estimators obtained from the ordinary FPCA in Section 3. Then the sample counterparts of \( \Omega \) and \( \Gamma \) given in (2.4) and (2.5) are defined by

\[ \hat{\Omega}_{\varphi} = \frac{1}{T} \sum_{t=1}^{T} Z_{\varphi,t} \otimes Z_{\varphi,t} + \frac{1}{T} \sum_{s=1}^{T-1} k \left( \frac{s}{h} \right) \sum_{t=s+1}^{T} \{ Z_{\varphi,t} \otimes Z_{\varphi,t-s} + Z_{\varphi,t-s} \otimes Z_{\varphi,t} \}, \tag{3.4} \]

\[ \hat{\Gamma}_{\varphi} = \frac{1}{T} \sum_{t=1}^{T} Z_{\varphi,t} \otimes Z_{\varphi,t} + \frac{1}{T} \sum_{s=1}^{T-1} k \left( \frac{s}{h} \right) \sum_{t=s+1}^{T} Z_{\varphi,t} \otimes Z_{\varphi,t-s}, \tag{3.5} \]

where \( k(\cdot) \) (resp. \( h \)) is the kernel (resp. the bandwidth) satisfying the following conditions:

**Assumption K.** (i) \( k(0) = 1 \), \( k(u) = 0 \) if \( u > \kappa \) with some \( \kappa > 0 \), \( k \) is continuous on \( [0, \kappa] \), and (ii) \( h \to \infty \) and \( h/T \to 0 \) as \( T \to \infty \).

The above conditions are provided by Horváth et al. (2013) for estimation of the long run covariance of a stationary FTS, and, under these conditions, Rice and Shang (2017) provide a detailed discussion on the optimal choice of \( h \) for various kernel functions.

From the identity \( I = \hat{P}_{N\varphi} + \hat{P}_{S\varphi} \), we may decompose \( \hat{\Omega}_{\varphi} \) and \( \hat{\Gamma}_{\varphi} \) as in (2.2), i.e.,

\[ \hat{\Omega}_{\varphi} = \hat{\Omega}_{NN\varphi} + \hat{\Omega}_{NS\varphi} + \hat{\Omega}_{SN\varphi} + \hat{\Omega}_{SS\varphi}, \quad \hat{\Gamma}_{\varphi} = \hat{\Gamma}_{NN\varphi} + \hat{\Gamma}_{NS\varphi} + \hat{\Gamma}_{SN\varphi} + \hat{\Gamma}_{SS\varphi}. \tag{3.6} \]

We then define a modified variable \( X_{\varphi,t} \) as follows,

\[ X_{\varphi,t} = X_t - \hat{\Omega}_{SN\varphi} \left( \hat{\Omega}_{NN\varphi} \right)^{\dagger} \hat{P}_{N\varphi} \Delta X_t, \quad t = 1, \ldots, T. \tag{3.7} \]

The \( \varphi \)-regularized inverse \( \hat{\Omega}_{NN\varphi}^{\dagger} \) in (3.7) can easily be computed from eigendecomposition of \( \hat{\Omega}_{NN\varphi} \) once we know \( \varphi \). Let \( \hat{C}_{\varphi} \) be the sample covariance operator of \( \{ X_{\varphi,t} \}_{t=1}^{T} \), i.e.,

\[ \hat{C}_{\varphi} = \frac{1}{T} \sum_{t=1}^{T} X_{\varphi,t} \otimes X_{\varphi,t}. \tag{3.8} \]

Roughly speaking, replacing \( \hat{C} \) with \( \hat{C}_{\varphi} \) in the eigenvalue problem (3.1) has the effect of transforming \( W^S \), appearing in the expression of \( F \) in Theorem 3.1, into another Brownian motion that is independent of \( W^N \); however, this does not resolve the issue that the center of the limiting operator depends on nuisance parameters. We therefore need a further adjustment, which is achieved by considering the following modified eigenvalue problem:

\[ \left( \hat{C}_{\varphi} - \hat{\Gamma}_{\varphi} \right) \hat{w}_j = \hat{\mu}_j \hat{w}_j, \quad j = 1, 2, \ldots, \tag{3.9} \]

where

\[ \hat{\Gamma}_{\varphi} = \hat{\Gamma}_{NS\varphi} - \hat{\Gamma}_{NN\varphi} \left( \hat{\Omega}_{NN\varphi} \right)^{\dagger} \hat{\Omega}_{NS\varphi}. \tag{3.10} \]
Note that the operator $\hat{C}_\varphi - \hat{Y}_\varphi - \hat{Y}'_\varphi$ is self-adjoint, so (3.9) does not produce any complex eigenvalues. From the set of the estimated eigenvectors $\{\hat{w}_j\}_{j=1}^\varphi$, we construct our proposed estimators of $P^N$ and $P^S$ as follows:

$$
\hat{\Pi}^N_\varphi = \sum_{j=1}^\varphi \hat{w}_j \otimes \hat{w}_j, \quad \hat{\Pi}^S_\varphi = I - \hat{\Pi}^N_\varphi.
$$

(3.11)

Note that $\hat{\Pi}^N_\varphi$ and $\hat{\Pi}^S_\varphi$ are simply obtained by replacing $\hat{v}_j$ with $\hat{w}_j$ in (3.2). The asymptotic properties of $\hat{\Pi}^N_\varphi$ or $\hat{\Pi}^S_\varphi$ are described by the following theorem.

**Theorem 3.2.** Suppose that Assumptions M, W and K hold with $\varphi \geq 1$. Then

$$
T(\hat{\Pi}^N_\varphi - P^N) \rightarrow_{\mathcal{L}_H} G + G^*,
$$

where $G = \text{fd}_{\text{odd}} \left( \int W^N(r) \otimes W^N(r) \right)^\dagger \left( \int dW^{S,N}(r) \otimes W^N(r) \right)$, $W^{S,N}(r) = W^S(r) - \Omega^{SN}(\Omega^{NS})^\dagger W^N(r)$, and $W^{S,N}$ is independent of $W^N$.

The asymptotic limit of the proposed estimator $\hat{\Pi}^N_\varphi$ is of a more convenient form than that of the preliminary estimator based on the ordinary FPCA. First note that the limiting operator $G + G^*$ is now centered at zero, whereas $F + F^*$ in Theorem 3.1 is not so in general due to $\Gamma^{NS}$. In addition, $G$ is characterized by two independent Brownian motions while $F$ is not so in general due to $\Omega^{NS}$. These properties of $\hat{\Pi}^N_\varphi$ not only help us develop statistical tests for examining various hypotheses about cointegration in Section 4, but also make $\hat{\Pi}^N_\varphi$ asymptotically more efficient than $\hat{P}^N_\varphi$ in a certain sense, see Remark 3.2.

**Remark 3.2.** For any $k, x = (x_1, \ldots, x_k) \in \mathcal{H}^k$ and $y = (y_1, \ldots, y_k) \in \mathcal{H}^k$, let $\hat{P}(k, x, y) = \langle \langle TP^N \hat{P}^S x_1, y_1 \rangle, \ldots, \langle TP^N \hat{P}^S x_k, y_k \rangle \rangle$ and $\hat{\Pi}(k, x, y) = \langle \langle TP^N \hat{\Pi}^S x_1, y_1 \rangle, \ldots, \langle TP^N \hat{\Pi}^S x_k, y_k \rangle \rangle$. From similar arguments used in Saikkonen (1991, Theorem 3.1) and Harris (1997, Section 2), the following can be shown: for any arbitrary choice of $k, x, y,$ and $\Theta \subset \mathbb{R}^k$ that is convex and symmetric around the origin,

$$
\lim_{T \to \infty} \text{Prob.} \left\{ \hat{\Pi}(k, x, y) \in \Theta \right\} = \lim_{T \to \infty} \text{Prob.} \left\{ \hat{P}(k, x, y) \in \Theta \right\},
$$

and the equality does not hold in general unless $\Omega^{NS} = \Gamma^{NS} = 0$; a more detailed discussion is given at the end of Section C.1.2 in the Supplementary Appendix. This implies that the asymptotic distribution of $\hat{\Pi}(k, x, y)$ is more concentrated at zero than that of $\hat{P}(k, x, y)$. A similar result can be shown for $P^S \hat{\Pi}^N_\varphi$ and $P^S \hat{P}^N_\varphi$. In this sense, we say that $\hat{\Pi}^N_\varphi$ is more asymptotically efficient than $\hat{P}^N_\varphi$.

### 3.3 A special case: PCA for cointegrated time series by Harris (1997)

We consider a special case when $\dim(\mathcal{H}) < \infty$ and the minimum eigenvalue of $E[\mathcal{E}_t \otimes \mathcal{E}_t]$ is strictly positive. Even if our modified FPCA can be applied without any adjustment in this case, we here provide a way to obtain an estimator whose asymptotic properties are equal to those of $\hat{\Pi}^N_\varphi$ given in Theorem 3.2. Define

$$
\bar{X}_{\varphi,t} = X_t - \hat{\Omega}^{SN}_\varphi \left( \hat{\Omega}^{NN}_\varphi \right)^\dagger \hat{P}^N_\varphi \Delta X_t - \hat{P}^S_\varphi \hat{C}_\varphi Z_{\varphi,t}, \quad t = 1, \ldots, T,
$$

8
where \( \hat{C}_{\varphi,Z} = T^{-1} \sum_{t=1}^{T} Z_{\varphi,t} \otimes Z_{\varphi,t} \). Let \( \tilde{C}_{\varphi} \) be the sample covariance of \( \{ \tilde{X}_{\varphi,t} \}_{t=1}^{T} \) and consider the following eigenvalue problem,

\[
\tilde{C}_{\varphi} \tilde{w}_j = \tilde{\mu}_j \tilde{w}_j, \quad j = 1, 2, \ldots, \dim(\mathcal{H}).
\] (3.12)

We then define \( \tilde{\Pi}^N_{\varphi} = \sum_{j=1}^{\varphi} \tilde{w}_j \otimes \tilde{w}_j \) and \( \tilde{\Pi}^S_{\varphi} = I - \tilde{\Pi}^N_{\varphi} \). In fact, (3.12) is a simple adaptation of the eigenvalue problem proposed in Harris (1997) for multivariate cointegrated systems. Based on the asymptotic results given in Harris (1997), the following can be shown.

**Proposition 3.1.** Suppose that Assumptions M, W and K hold with \( \varphi \geq 1 \), \( \dim(\mathcal{H}) < \infty \), and the minimum eigenvalue of \( \mathbb{E}[\mathcal{E}_t \otimes \mathcal{E}_t] \) is strictly positive. Then

\[
T(\tilde{\Pi}^N_{\varphi} - P^N) \to_{\mathcal{L}_H} G + G^*,
\]

where \( G \) is given in Theorem 3.2.

However, the asymptotic result given in Proposition 3.1 essentially requires \( \hat{C}^{-1}_{\varphi,Z} \) to converge in probability to the inverse of \( \mathbb{E}[\mathcal{E}_t \otimes \mathcal{E}_t] \) (see the proof of Theorem 2 in Harris, 1997). In an infinite dimensional setting, \( \mathbb{E}[\mathcal{E}_t \otimes \mathcal{E}_t] \) does not allow its inverse as an element of \( \mathcal{L}_H \) and, moreover, \( \hat{C}^{-1}_{\varphi,Z} \) does not converge to any bounded linear operator since the inverse of the minimum eigenvalue of \( \mathbb{E}[\mathcal{E}_t \otimes \mathcal{E}_t] \) diverges to infinity. This is the reason why Proposition 3.1, unlike Theorem 3.2, is not applicable in an infinite dimensional setting.

## 4 Statistical tests based on the modified FPCA

We develop statistical tests to examine various hypotheses about \( \mathcal{H}^N \) or \( \mathcal{H}^S \) based on the asymptotic results given in Section 3.2. As in the previous sections, we focus on the case without deterministic terms; the discussion is extended to allow a deterministic component in Section B of the Supplementary Appendix.

### 4.1 FPCA-based test for the dimension of \( \mathcal{H}^N \)

In the previous sections, we need prior knowledge of \( \varphi = \dim(\mathcal{H}^N) \). We here provide a novel FPCA-based test that can be applied sequentially to determine \( \varphi \); as will be shown, the proposed test is similar to KPSS-type tests for examining stationarity or cointegration in a finite dimensional Euclidean space setting.

Consider the following null and alternative hypotheses,

\[
H_0 : \dim(\mathcal{H}^N) = \varphi_0 \quad \text{against} \quad H_1 : \dim(\mathcal{H}^N) > \varphi_0,
\] (4.1)

for \( \varphi_0 \geq 0 \). The null hypothesis in (4.1) can be a pre-specified hypothesis of interest, or it can be reached by a sequential procedure starting from \( \varphi_0 = 0 \) to estimate the true dimension. It is worth mentioning that stationarity of \( \{ X_t \}_{t \geq 1} \), i.e., \( \varphi_0 = 0 \), can be examined by the test to be developed. This means that our test can be used as an alternative to the existing tests of stationarity of FTS proposed by e.g. Horváth et al. (2014), Kokoszka and Young (2016), and Aue and Van Delft (2020). It should also be noted that our selection of hypotheses in (4.1) is the opposite of that used for the existing tests proposed by Chang et al. (2016) and Nielsen et al. (2022) in the sense that the alternative hypothesis in those tests is set to \( H_1 : \dim(\mathcal{H}^N) < \varphi_0 \).
Due to this difference, our test has its own advantage especially when it is sequentially applied to estimate \( \varphi \); this will be more detailed in Remark 4.2.

For each \( \varphi_0 \), we define \( \hat{P}^N_{\varphi_0} \) and \( \hat{P}^S_{\varphi_0} \), as in (3.2) from the eigenvalue problem (3.1); if \( \varphi_0 = 0 \), \( \hat{P}^N_{\varphi_0} \) is set to zero. Given \( \hat{P}^N_{\varphi_0} \) and \( \hat{P}^S_{\varphi_0} \), define \( Z_{\varphi_0,t}, \hat{\Omega}_{\varphi_0}, \hat{\Gamma}_{\varphi_0}, \hat{\Upsilon}_{\varphi_0}, X_{\varphi_0,t}, \) and \( \hat{C}_{\varphi_0} \) as in Section 3; see (3.3)-(3.8) and (3.10).

We expect from Theorem 3.1 that \( \{ \hat{P}^S_{\varphi_0}X_t \}_{t \geq 1} \) will behave as a stationary process if \( \varphi_0 = \varphi \) (since \( \| \hat{P}^S_{\varphi_0} - P^S \|_{\mathcal{L}_2} = O_p(T^{-1}) \)), and the eigenvectors \( \{ \hat{v}_j \}_{j=\varphi_0+1}^{\varphi} \) are asymptotically included in \( \mathcal{H}_N \) if \( \varphi_0 < \varphi \). In the latter case, \( \{ \langle X_t, \hat{v}_{\varphi_0+1} \rangle, \ldots, \langle X_t, \hat{v}_{\varphi} \rangle \} \) is expected to behave as a unit root process, hence \( \{ \hat{P}^S_{\varphi_0}X_t \}_{t \geq 1} \) will not be stationary. Based on this idea, it may be reasonable to ask if we can distinguish the correct hypothesis from incorrect ones specifying \( \varphi_0 < \varphi \) by examining stationarity of \( \{ \hat{P}^S_{\varphi_0}X_t \}_{t \geq 1} \).

For the purpose of illustration, we focus on the time series \( \{ \langle X_t, \hat{v}_{\varphi_0+1} \rangle \}_{t \geq 1} \), which is expected to behave as a stationary process under \( H_0 \) or a unit root process under \( H_1 \). To examine stationarity of \( \{ \langle X_t, \hat{v}_{\varphi_0+1} \rangle \}_{t \geq 1} \), we consider the following test statistic,

\[
\frac{1}{T^2} \sum_{t=1}^{T} \sum_{s=1}^{t} \langle X_s, \hat{v}_{\varphi_0+1} \rangle^2 / \text{LRV}(\langle X_t, \hat{v}_{\varphi_0+1} \rangle),
\]

where \( \text{LRV}(\langle X_t, \hat{v}_{\varphi_0+1} \rangle) = \frac{1}{T} \sum_{s=-T+1}^{T-1} k(|s|/h) \sum_{t=|s|+1}^{T} \langle X_t, \hat{v}_{\varphi_0+1} \rangle \langle X_{t-|s|}, \hat{v}_{\varphi_0+1} \rangle \) and \( k(\cdot) \) and \( h \) satisfy Assumption K. Note that for each \( T \) the test statistic is given by the ratio of the sum of squared partial sums of the univariate time series \( \{ \langle X_t, \hat{v}_{\varphi_0+1} \rangle \}_{t=1}^{T} \) to its sample long-run variance \( \text{LRV}(\langle X_t, \hat{v}_{\varphi_0+1} \rangle) \). This is similar to the well-known KPSS test for examining the null hypothesis of stationarity; however, it should be noted that (4.2) is not identical to the original LM statistic proposed by Kwiatkowski et al. (1992) unless \( \{ \langle X_t, \hat{v}_{\varphi_0+1} \rangle \}_{t=1}^{T} \) is demeaned (see Appendix of their paper). If \( \Omega^{NS} = \Gamma^{NS} = 0 \), the asymptotic null distribution of (4.2) does not depend on any nuisance parameters and is simply given by a functional of two independent standard Brownian motions. We thus may asymptotically evaluate the plausibility of the null hypothesis in (4.1) relative to the alternative once quantiles of the limiting distribution, which can be easily simulated by standard methods, are obtained. In general cases where \( \Omega^{NS} = \Gamma^{NS} = 0 \) is not satisfied, the limiting distribution of (4.2) depends on both of \( \Omega^{NS} \) and \( \Gamma^{NS} \), which are unknown in practice (see Theorem 4.1 and its proof given in Section C.1 of the Supplementary Appendix). As expected from Theorem 3.1, this issue is linked closely to the fact that the asymptotic limit of \( \hat{P}^S_{\varphi_0} \) depends on \( \Omega^{NS} \) and \( \Gamma^{NS} \).

The assumption that \( \Omega^{NS} = \Gamma^{NS} = 0 \) is too restrictive in modeling cointegrated FTS, hence a naive use of (4.2) in practice to examine (4.1) should be limited to very special circumstances. However, using the asymptotic results developed for our modified FPCA in Section 3, the test statistic (4.2) can be modified to have an asymptotic null distribution that is free of nuisance parameters in general cases. Consider the modified eigenvalue problem given by

\[
\left( \hat{C}_{\varphi_0} - \hat{\Upsilon}_{\varphi_0} - \hat{\Upsilon}_{\varphi_0}^* \right) \hat{w}_j = \hat{\mu}_j \hat{w}_j, \quad j = 1, 2, \ldots.
\]

(4.3)
We then replace $X_t$ and $\hat{v}_{\varphi_0+1}$ in (4.2) with $X_{\varphi_0,t}$ and $\hat{w}_{\varphi_0+1}$, respectively, and obtain the following statistic:

$$
\frac{1}{T^2} \sum_{t=1}^{T} \sum_{s=1}^{t} \langle X_{\varphi_0,s}, \hat{w}_{\varphi_0+1} \rangle^2 / \text{LRV}(\langle X_{\varphi_0,t}, \hat{w}_{\varphi_0+1} \rangle).
$$

There is a big gain from this simple replacement: different from (4.2), the asymptotic null distribution of (4.4) does not depend on any nuisance parameters and is given by a functional of two independent standard Brownian motions without requiring $\Omega^{NS} = \Gamma^{NS} = 0$; this is, of course, closely related to the asymptotic result given by Theorem 3.2. We thus may asymptotically assess the plausibility of the null hypothesis relative to the alternative in an obvious way.

The asymptotic test based on (4.4) in fact corresponds to a special case of the general version of our FPCA-based test, which we want to detail here. Let $\hat{\Pi}^K_{\varphi_0}$ be the projection onto the span of $\{\hat{w}_j\}_{j=1}^K$ for any arbitrary finite integer $K$ in $(\varphi_0, \dim(\mathcal{H})]$ (note that $K = \dim(\mathcal{H})$ is possible only when $\dim(\mathcal{H}) < \infty$); that is, $\hat{\Pi}^K_{\varphi_0} = \sum_{j=1}^{K} \hat{w}_j \otimes \hat{w}_j$. Define

$$
z_{\varphi_0,t} = \langle (X_{\varphi_0,t}, \hat{w}_{\varphi_0+1}), \ldots, (X_{\varphi_0,t}, \hat{w}_K) \rangle.
$$

Once $\{\hat{w}_j\}_{j=1}^K$ are given, $z_{\varphi_0,t}$ can easily be computed. Note that $z_{\varphi_0,t}$ may be understood as $\hat{\Pi}^K_{\varphi_0} \hat{\Pi}^S_{\varphi_0} X_{\varphi_0,t}$, i.e., the projected image of possibly infinite dimensional element $\hat{\Pi}^S_{\varphi_0} X_{\varphi_0,t}$ on $\text{span}(\{\hat{w}_j\}_{j=1}^K)$ of dimension $K$; this $K$-dimensional time series of $z_{\varphi_0,t}$ is all we need to construct our test statistic. It may be helpful to outline how the sequence $\{z_{\varphi_0,t}\}_{t \geq 1}$ behaves under $H_0$ and $H_1$ prior to defining the test statistic. From the asymptotic results derived in Section 3, the following can be shown (Lemma C.3): for any $\varphi_0 \leq \varphi$,

$$
\hat{w}_j \to_p w_j \in \mathcal{H}^N, \quad j = 1, \ldots, \varphi,
$$

$$
\hat{w}_j \to_p w_j \in \mathcal{H}^S, \quad j = \varphi + 1, \ldots, K.
$$

From the above results, we expect that the subvector $\langle (X_{\varphi_0,t}, \hat{w}_{\varphi_0+1}), \ldots, (X_{\varphi_0,t}, \hat{w}_\varphi) \rangle$ of $z_{\varphi_0,t}$ will behave as a unit root process while the remaining subvector will behave as a stationary process. Only when $\varphi_0 = \varphi$, $\{z_{\varphi_0,t}\}_{t \geq 1}$ will behave as a stationary process. Based on this idea, our test statistic is constructed to examine stationarity of $\{z_{\varphi_0,t}\}_{t \geq 1}$ as follows:

$$
\hat{Q}(K, \varphi_0) = \frac{1}{T^2} \sum_{t=1}^{T} \left( \sum_{s=1}^{t} z_{\varphi_0,s} \right)' \text{LRV}(z_{\varphi_0,t})^{-1} \left( \sum_{s=1}^{t} z_{\varphi_0,s} \right),
$$

where

$$
\text{LRV}(z_{\varphi_0,t}) = \frac{1}{T} \sum_{t=1}^{T} z_{\varphi_0,t} z_{\varphi_0,t}' + \frac{1}{T} \sum_{s=1}^{t-1} k \left( \frac{s}{T} \right) \sum_{t=s+1}^{T} \{z_{\varphi_0,t} z_{\varphi_0,t-s} + z_{\varphi_0,t-s} z_{\varphi_0,t}' \}.
$$

The test statistic is similarly constructed as other KPSS-type statistics developed for multivariate cointegrated systems; see e.g. Shin (1994), Choi and Ahn (1995), and Harris (1997). Computation of our test statistic is easy once the projected time series $\{z_{\varphi_0,t}\}_{t=1}^T$ is obtained from the modified eigenvalue problem (4.3). If $K = \varphi_0 + 1$, the test statistic (4.6) becomes identical to (4.4).
We hereafter let B and W denote independent $\varphi_0$-dimensional and $(K - \varphi_0)$-dimensional standard Brownian motions, respectively. The asymptotic properties of our test statistic are described by these independent Brownian motions as follows.

**Theorem 4.1.** Suppose that Assumptions M, W and K hold, and K is a finite integer in $(\varphi_0, \dim(\mathcal{H})]$. Under $H_0$ of (4.1),

$$\hat{Q}(K, \varphi_0) \to_d \int V(r)^t V(r),$$

where $V(r) = W(r) - \int dW(s) B(s)^t \left( \int B(s) B(s)^t \right)^{-1} \int_0^r B(s)$, and the second term is regarded as zero if $\varphi_0 = 0$. Under $H_1$ of (4.1), $\hat{Q}(K, \varphi_0) \to P \infty$.

At least to some extent, our test may be viewed as an extension of the PCA-based test proposed by Harris (1997), which extends the KPSS test for univariate/multivariate cointegrated time series. In such a finite dimensional setting, the test statistic (4.6) can be reconstructed by the outputs from the PCA described in Section 3.3 and $K$ can be set to $\dim(\mathcal{H})$; this makes our test become identical to that given by Harris (1997).

It should be noted that the asymptotic null distribution given in Theorem 4.1 depends on $K$. This is a new aspect that arises from the fact that the proposed test examines stationarity of potentially infinite dimensional time series \{\(\hat{\Pi}_{\varphi_0}^t X_{\varphi_0,t}\)\}_{t \geq 1} by projecting it onto a $K$-dimensional subspace as in Horváth et al. (2014). As a consequence, critical values for the test statistic depend on both $K$ and $\varphi_0$; however, for any fixed $K$ and $\varphi_0$, those can easily be simulated by standard methods since the limiting distribution of the test statistic is simply given by a functional of two independent standard Brownian motions. One can refer to Table 1 of Harris (1997) reporting critical values for a few different choices of $K$ and $\varphi_0$, with the caution that the table reports critical values depending on the dimension of time series and the cointegration rank, which correspond to $K$ and $K - \varphi_0$ in this paper, respectively. Even if Theorem 4.1 holds for any arbitrary finite integer $K$ in $(\varphi_0, \dim(\mathcal{H})]$, it may be better, in practice, to set $K$ to be only slightly greater than $\varphi_0$, such as $K = \varphi_0 + 1$ or $\varphi_0 + 2$; our simulation results support that a large value of $K$ tends to yield worse finite sample properties of the test; see Section 4.3 and also the discussion given by Nielsen et al. (2022, Section 3.5) in a similar context.

**Remark 4.1.** To determine $\varphi$, we may apply our test for $\varphi_0 = 0, 1, \ldots$ sequentially. Let $\hat{\varphi}$ denote the value under $H_0$ that is not rejected for the first time at a fixed significance level $\alpha$. We deduce from Theorem 4.1 that $\mbox{Prob.} \{ \hat{\varphi} = \varphi \} \to 1 - \alpha$ and $\mbox{Prob.} \{ \hat{\varphi} < \varphi \} \to 0$. Note that, even if the sequential procedure requires multiple applications of the proposed test, the correct asymptotic size, $\alpha$, is guaranteed without any adjustments; this property is shared by the existing sequential procedures developed in a Euclidean/Hilbert space setting (see e.g., Johansen, 1995; Nyblom and Harvey, 2000; Chang et al., 2016; Nielsen et al., 2022). If the significance level is chosen such that $\alpha \to 0$ as $T \to \infty$ then, $\mbox{Prob.} \{ \hat{\varphi} = \varphi \} \to 1$.

**Remark 4.2.** In Chang et al. (2016) and Nielsen et al. (2022), $\varphi$ is determined by testing the following hypotheses: for a pre-specified positive integer $\varphi_{\max}$ and $\varphi_0 = \varphi_{\max}, \varphi_{\max} - 1, \ldots, 1$,

$$H_0 : \dim(\mathcal{H}^N) = \varphi_0 \text{ against } H_1 : \dim(\mathcal{H}^N) < \varphi_0.$$
sequentially until $H_0$ is not rejected for the first time. Note that we need a prior information on an upper bound of $\varphi$, i.e., $\varphi_{\text{max}}$, for this type of procedure. However, in an infinite dimensional setting, there is no natural upper bound of $\dim(\mathcal{H}^N)$. On the other hand, our sequential procedure described in Remark 4.1 does not require such a prior information; the procedure first examines the null hypothesis $H_0: \dim(\mathcal{H}^N) = 0$, and 0 is the minimal possible value of $\dim(\mathcal{H}^N)$.

\textbf{Remark 4.3.} If $\varphi \geq 1$ is known in advance, $\varphi$ can also be estimated by the eigenvalue ratio criterion proposed by Li et al. (2022) under their assumptions. The estimator is originally developed to estimate the dimension of the dominant subspace of nonstationary fractionally integrated functional time series $\{X_t\}_{t \geq 1}$. If the highest fractional order is equal to one in their paper, the dominant subspace is equal to $\mathcal{H}^N$.

4.2 Applications: tests of hypotheses about cointegration

Practitioners may be interested in testing various hypotheses about $\mathcal{H}^N$ or $\mathcal{H}^S$. For example, we may want to test if a specific element $x_0$ is included in $\mathcal{H}^N$ or the span of a specified set of vectors contains $\mathcal{H}^N$. More generally, we here consider testing the following hypotheses: for a specified subspace $\mathcal{M}$,

\begin{align*}
H_0: \mathcal{M} \subset \mathcal{H}^N, & \quad \text{or equivalently, } \mathcal{M}^\perp \supset \mathcal{H}^S & \text{against } H_1: H_0 \text{ is not true,} \quad (4.8) \\
H_0: \mathcal{M} \supset \mathcal{H}^N, & \quad \text{or equivalently, } \mathcal{M}^\perp \subset \mathcal{H}^S & \text{against } H_1: H_0 \text{ is not true.} \quad (4.9)
\end{align*}

Let $P^\mathcal{M}$ denote the projection onto $\mathcal{M}$ and assume that $\varphi$ is known; in practice, we may apply any statistical method discussed in Remarks 4.1–4.3 to determine $\varphi$ in advance. In this case, (4.8) and (4.9) can be tested by examining the dimension of the attractor space associated with the residuals $\{(I - P^\mathcal{M})X_t\}_{t \geq 1}$.

Consider first testing (4.8). If $\mathcal{M} \subset \mathcal{H}^N$, then the attractor subspace associated with $\{(I - P^\mathcal{M})X_t\}_{t \geq 1}$ is $(\varphi - \dim(\mathcal{M}))$–dimensional, and this can be examined using the test developed in Section 4. To be more specific, let $\hat{Q}_1(K)$ denote our FPCA-based test statistic (4.6) computed from $\{(I - P^\mathcal{M})X_t\}_{t=1}^T$ for $\varphi_0 = \varphi - \dim(\mathcal{M})$ and $K > \varphi_0$. We then deduce from (4.1) that

\begin{align*}
\hat{Q}_1(K) & \rightarrow_d \int V(r)'V(r) \quad \text{under } H_0 \text{ of (4.8)}, \\
\hat{Q}_1(K) & \rightarrow_p \infty \quad \text{under } H_1 \text{ of (4.8)},
\end{align*}

where $V(r)$ is defined as in Theorem 4.1 for $\varphi_0 = \varphi - \dim(\mathcal{M})$ and $K > \varphi_0$.

On the other hand, in the case where $\mathcal{M} \supset \mathcal{H}^N$, $\{(I - P^\mathcal{M})X_t\}_{t \geq 1}$ becomes stationary. Therefore, if we let $\hat{Q}_2(K)$ be our FPCA-based test statistic (4.6) computed from $\{(I - P^\mathcal{M})X_t\}_{t=1}^T$ for $\varphi_0 = 0$ and $K > 0$, then the following is deduced from Theorem 3.1:

\begin{align*}
\hat{Q}_2(K) & \rightarrow_d \int W_K(r)'W_K(r) \quad \text{under } H_0 \text{ of (4.9)}, \\
\hat{Q}_2(K) & \rightarrow_p \infty \quad \text{under } H_1 \text{ of (4.9)},
\end{align*}

where $W_K$ is $K$–dimensional standard Brownian motion. Given that what we need to test (4.9) is actually a statistical test for examining stationarity of functional time series, we may employ other existing stationarity tests such as those in Horváth et al. (2014) and Aue and Van Delft (2020).
4.3 Monte Carlo Simulations

We investigate the finite sample performances of our tests by Monte Carlo study. For all simulation experiments, the number of replications is 2000, and the nominal size is 5%.

Simulation setups

Let \( \{ f_j \}_{j \geq 1} \) be the Fourier basis of \( L^2[0,1] \), the Hilbert space of square integrable functions on \([0,1]\) equipped with inner product \( \langle f,g \rangle = \int f(u)g(u)du \) for \( f, g \in L^2[0,1] \). For each \( \varphi (\leq 5, \text{in our simulation experiments}) \), we let \( E_t^N \) and \( E_t^S \) be generated by the following stationary functional AR(1) models:

\[
E_t^N = \sum_{j=1}^{\varphi} \alpha_j \langle g_j^N, E_{t-1} \rangle g_j^N + P^N \varepsilon_t, \quad E_t^S = \sum_{j=1}^{10} \beta_j \langle g_j^S, E_{t-1} \rangle g_j^S + P^S \varepsilon_t, \tag{4.10}
\]

where \( \{ g_j^N \}_{j=1}^{\varphi} \) (resp. \( \{ g_j^S \}_{j=1}^{10} \)) are randomly drawn from \( \{ f_1, \ldots, f_6 \} \) (resp. \( \{ f_7, \ldots, f_{18} \} \)) without replacement, and \( \varepsilon_t \) is given as follows: for standard normal random variables \( \{ \theta_{j,t} \}_{j \geq 1} \) that are independent across \( j \) and \( t \), \( \varepsilon_t = \sum_{j=1}^{80} \theta_{j,t}(0.95)^{j-1} f_j \). We then construct \( X_t \) from the relationships \( P^N \Delta X_t = E_t^N \) and \( P^S X_t = E_t^S \) for \( t \geq 1 \). Note that \( \mathcal{H}^N \) is given by the span of \( \{ g_j^N \}_{j=1}^{\varphi} \), which is not fixed across different realizations of the DGP; moreover, an orthonormal basis \( \{ g_j^N \}_{j=1}^{\varphi} \) of \( \mathcal{H}^N \) is selected only from a set of smoother Fourier basis functions \( \{ f_j \}_{j=1}^{6} \). The former is to avoid potential effects caused by the particular shapes of \( \mathcal{H}^N \) as in Nielsen et al. (2022), and the latter is not to make estimation of \( \mathcal{H}^N \) with small samples too difficult: as expected from the results given by Nielsen et al. (2022), the finite sample performances of tests for the dimension of \( \mathcal{H}^N \) may become poorer as \( \mathcal{H}^N \) includes less smoother functions. We also let \( \{ \alpha_j \}_{j=1}^{\varphi} \) and \( \{ \beta_j \}_{j=1}^{10} \) be randomly chosen as follows: for some \( \beta_{\text{min}} \) and \( \beta_{\text{max}} \),

\[
\alpha_j \sim U[-0.5, 0.5], \quad \beta_j \sim U[\beta_{\text{min}}, \beta_{\text{max}}].
\]

It is known that persistence of the stationary component (\( \{ E_t^S \}_{t \in \mathbb{Z}} \) in this paper) has a significant effect on finite sample properties of KPSS-type tests (see e.g., Kwiatkowski et al., 1992; Nyblom and Harvey, 2000). We thus will investigate finite sample properties of our tests under a lower persistence scheme (\( \beta_{\text{min}} = 0, \beta_{\text{max}} = 0.5 \)) and a higher persistence scheme (\( \beta_{\text{min}} = 0.5, \beta_{\text{max}} = 0.7 \)). It may be more common in practice to have a nonzero intercept or a linear time trend in (2.1), and Section B.2 of the Supplementary Appendix provides a relevant extension of Theorem 4.1 to accommodate such a case. We here consider the former case and add an intercept \( \mu_1 \), which is also randomly chosen, to each realization of the DGP (some simulation results for the case with a linear trend are reported in Table A4 of the Supplementary Appendix); specifically, \( \mu_1 = \sum_{j=1}^{4} \tilde{\theta}_j p_j / \sqrt{\sum_{j=1}^{4} \tilde{\theta}_j^2}, \{ \tilde{\theta}_j \}_{j=1}^{4} \) are independent standard normal random variables, and \( p_j \) is \((j-1)\)-th order Legendre polynomial defined on \([0,1]\). Finally, functional observations used to compute the test statistic are constructed by smoothing \( X_t \) observed on 200 regularly spaced points of \([0,1]\) using 20 quadratic B-spline basis functions (the choice of basis functions has minimal effect in our simulation setting).

We need to specify the kernel function \( k(\cdot) \), the bandwidth \( h \), and a positive integer \( K \) satisfying \( K > \varphi_0 \) to implement our test. In our simulation experiments, \( k(\cdot) \) is set to the Parzen kernel, and two different values of \( h, T^{1/3} \) and \( T^{2/5} \), are employed to see the effect of the bandwidth parameter on the finite sample
performances of our tests. Moreover, our simulation results reported in this section are obtained by setting $K = \varphi_0 + 1$, which is the smallest possible choice of $K$. We found some simulation evidence supporting that this choice of $K$ tends to result in better finite sample properties; to see this, one can compare the results reported in Table 1 and those in Tables A2 and A3 (in the Supplementary Appendix).

**Simulation results**

We investigate the finite sample performance of the proposed test for the dimension of $H^N$. In our simulation experiments, finite sample powers are computed when $\varphi = \varphi_0 + 1$; as $\varphi$ gets farther away from $\varphi_0$, our test tends to exhibit a better finite sample power as is expected (see Table A1 in the Supplementary Appendix with Table 1). Table 1 summarizes the simulation results under the two different persistence schemes.

Under the lower persistence scheme ($\beta_{\text{min}} = 0$, $\beta_{\text{max}} = 0.5$), for all considered values of $\varphi$ and $h$ the test has excellent size control with a reasonably good finite sample power. On the other hand, it displays over-rejection under the higher persistence scheme ($\beta_{\text{min}} = 0.5$, $\beta_{\text{max}} = 0.7$). Our simulation results evidently show that (i) such an over-rejection tends to disappear as $T$ gets larger and/or $h$ gets bigger, and (ii) choosing a bigger bandwidth with fixed $T$ tends to lower finite sample power. To summarize, employing a bigger bandwidth helps us avoid potential over-rejection, which can happen when $\{E_t^S\}_{t \in \mathbb{Z}}$ is persistent, at the expense of power. This trade-off between correct size and power in finite samples seems to be commonly observed for other KPSS-type tests; see e.g. Kwiatkowski et al. (1992) and Nyblom and Harvey (2000).

We also investigate finite sample properties of the tests for examining hypotheses about $H^N$ or $H^S$. Among many potentially interesting hypotheses, we consider the following: for a specific vector $x_0 \in \mathcal{H}$ and an orthonormal set $\{x_j\}_{j=1}^{\varphi} \subset \mathcal{H}$,

$$H_0 : x_0 \in \mathcal{H}^N \quad \text{against} \quad H_1 : x_0 \notin \mathcal{H}^N,$$

(4.11)

$$H_0 : \text{span}(\{x_j\}_{j=1}^{\varphi}) = \mathcal{H}^N \quad \text{against} \quad H_1 : \text{span}(\{x_j\}_{j=1}^{\varphi}) \neq \mathcal{H}^N.$$

(4.12)

Note that $\text{span}(\{g_j^N\}_{j=1}^{\varphi}) = \mathcal{H}^N$ and $g_1^S \in \mathcal{H}^S$ in each realization of the DGP (see (4.10)). In our simulation experiments for (4.11), finite sample sizes and powers are computed by setting

$$x_0 = g_1^N + \frac{\gamma}{T} g_1^S, \quad \gamma = 0, 20, 40, 60.\$$

Clearly, $x_0 = g_1^N \in \mathcal{H}^N$ if $\gamma = 0$. On the other hand, $x_0$ deviates slightly from $\mathcal{H}^N$ in the direction of $g_1^S$ when $\gamma > 0$, but such a deviation gets smaller as $T$ increases. In our experiments for (4.12), finite sample sizes and powers are computed by setting

$$x_1 = g_1^N + \frac{\gamma}{T} g_1^S, \quad x_j = g_j^N \text{ for } j = 2, \ldots, \varphi, \quad \gamma = 0, 20, 40, 60.\$$

(4.13)

Obviously, $\gamma = 0$ implies that $\text{span}(\{x_j\}_{j=1}^{\varphi}) = \mathcal{H}^N$ in (4.13) while $\gamma > 0$ makes $\text{span}(\{x_j\}_{j=1}^{\varphi})$ deviates slightly from $\mathcal{H}^N$ in the direction of $g_1^S$. The simulation results for our tests of the hypotheses (4.11) and (4.12) for a few different values of $\varphi$ are reported in Tables A5 and A6 in the Supplementary Appendix; since the proposed tests are essentially simple modifications of our test for the dimension of $\mathcal{H}^N$, they seem to have similar finite sample properties to those we observed in Table 1. We can also observe a trade-off
Table 1: Rejection frequencies (%) of the test for (4.1), \( K = \varphi_0 + 1 \)

| \( T \) | \( \varphi_0 = 0 \) | 1 | 2 | 3 | 4 |
|--------|----------------|---|---|---|---|
|        | size (\( \varphi = \varphi_0 \)) |   |   |   |   |
| 250    | 4.6            | 4.4| 4.1| 4.3| 4.5|
| 500    | 4.6            | 5.2| 5.1| 4.8| 4.5|
|        | power (\( \varphi = \varphi_0 + 1 \)) |   |   |   |   |
| 250    | 97.1           | 89.0| 79.9| 73.9| 69.5|
| 500    | 99.3           | 96.5| 94.6| 92.5| 90.9|

| \( T \) | \( \varphi_0 = 0 \) | 1 | 2 | 3 | 4 |
|--------|----------------|---|---|---|---|
|        | size (\( \varphi = \varphi_0 \)) |   |   |   |   |
| 250    | 10.9           | 10.6| 12.0| 11.2| 14.2|
| 500    | 8.0            | 8.9 | 9.4 | 9.2 | 11.6|
|        | power (\( \varphi = \varphi_0 + 1 \)) |   |   |   |   |
| 250    | 97.1           | 89.2| 80.1| 74.0| 69.7|
| 500    | 99.3           | 96.5| 94.6| 92.6| 91.0|

5 Empirical illustrations

We revisit two empirical applications considered, respectively, by Nielsen et al. (2022, Section 5.1) and Chang et al. (2016, Section 5.1) with extended time spans. In this section, we discuss the first empirical example in detail; the second example is discussed in Section E of the Supplementary Appendix, which is due to that some preliminaries are required related to mathematical/statistical issues in dealing with density-valued observations in a Hilbert space setting (see e.g., Egozcue et al., 2006; Delicado, 2011; Hron et al., 2016; Petersen and Müller, 2016; Kokoszka et al., 2019; Seo and Beare, 2019; Zhang et al., 2021).

Specifically, we here consider the time series of U.S. age-specific employment rates for the working age (15-64) population, observed monthly from January 1986 to Dec 2019. The raw survey data is available from the Current Population Survey (CPS) at https://www.ipums.org/. For age \( a \in [15, 64] \), the age-specific employment rate at time \( t \), denoted by \( Z_{a,t} \), is computed as in Nielsen et al. (2022, Section 5.1). Such employment rates take values in \([0, 1]\) by construction, hence we take the logit transformation \( \varphi(Z_{a,t}) \) as suggested by Nielsen et al. (2022), and then obtain functional observations \( X_t(u) \) for \( u \in [15, 64] \) by smoothing \( \varphi(Z_{a,t}) \) over \( a \) using 18 quadratic B-spline functions. In Figure 1 we plot the functional observations and univariate time series \{⟨X_t, x⟩\}_{t \in \mathbb{Z}}\) for some choices of \( x \), which may help us explore characteristics of the FTS. Specifically we consider \( x = x_1, x_2 \) and \( x_3 \) defined as follows:

\[
x_1(u) = \frac{1}{25} \mathbb{1} \{15 \leq u < 40\}, \quad x_2(u) = \frac{1}{24} \mathbb{1} \{40 \leq u \leq 64\}, \quad x_3 = x_2 - x_1.
\]

Clearly, ⟨\( X_t, x_1 \)⟩ (resp. ⟨\( X_t, x_2 \)⟩) computes the average (logit) employment rate of individuals aged less than 40 years (resp. no less than 40 years), and ⟨\( X_t, x_3 \)⟩ computes their difference. Provided that the two
Table 2: Test results for the dimension of $\mathcal{H}^N$ - age-specific employment rates

| $\phi_0$ | 0    | 1    | 2    |
|----------|------|------|------|
| Test statistic | 0.3175∗∗ | 0.1195 | 0.0665 |

Notes: $T = 408$. We use * and ** to denote rejection at 5% and 1% significance level, respectively; the approximate critical values for 95% (resp. 99%) are given by 0.15, 0.12, and 0.10 (resp. 0.22, 0.18, and 0.15) sequentially.

time series given in Figure 1-(b) seem to be nonstationary, we expect that there exists at least one stochastic trend; however, it is not possible to conclude from the plots that there are multiple stochastic trends since a single stochastic trend, say $w$, can make both time series of $\langle X_t, x_1 \rangle$ and $\langle X_t, x_2 \rangle$ nonstationary if $\langle w, x_1 \rangle$ and $\langle w, x_2 \rangle$ are nonzero. It can also be seen from Figure 1-(c) that for some choice of $x$ the time series of $\langle X_t, x \rangle$ may exhibit a lower persistence than those of $x_1$ and $x_2$; this may be suggestive of the possible existence of cointegrating vectors.

When a cointegrated FTS is given, it is important to estimate $\mathcal{H}^N$. We already know from our earlier results that $\mathcal{H}^N$ can be estimated by the span of the first eigenvectors computed from the proposed eigenvalue problem (3.9); however in practice $\phi$ is unknown. We thus apply our FPCA-based sequential procedure to determine $\phi$. As in Nielsen et al. (2022), the model with a linear trend (see Section B in the Supplementary Appendix) is considered for this empirical example. Table 2 reports the test results when $k(\cdot)$, $h$, and $K$ are set to the Parzen kernel, $T^{2/5}$, and $\phi_0 + 1$, respectively. The proposed sequential procedure concludes that the dimension of $\mathcal{H}^N$ is 1 both at 5% and 1% significance levels, hence $\mathcal{H}^N$ can be estimated by the span of the first eigenvector $\tilde{w}_1$ obtained from our modified FPCA. Figure 2 shows the estimated eigenvector and the time series $\{\langle \tilde{U}_t, \tilde{w}_1 \rangle \}_{t=1}^T$. Note that the time series of $\langle X_t, x \rangle$ for any $x \in \mathcal{H}^N$ is a unit root process with a linear deterministic trend, hence the time series of $\langle \tilde{U}_t, \tilde{w}_1 \rangle$ is expected to behave as a unit root process, which is as shown in Figure 2-(b).

In practice, one may also be interested in testing various hypotheses about cointegration such as (4.8) and (4.9). These hypotheses can be examined by the tests given in Section 4.2 once $\phi (=\dim(\mathcal{H}^N))$ is estimated. For illustrative purposes, we consider the following hypotheses,

$H_0 : x \in \mathcal{H}^N$ against $H_1 : x \notin \mathcal{H}^N$,
$H_0 : x \in \mathcal{H}^S$ against $H_1 : x \notin \mathcal{H}^S$.

Figure 1: Age group characteristics

(a) observations ($X_t$) (b) two age groups ($\langle X_t, x_1 \rangle & \langle X_t, x_2 \rangle$) (c) difference ($\langle X_t, x_3 \rangle$)
Figure 2: Estimated orthonormal basis of $\mathcal{H}^N$ - age-specific employment rates

![Figure 2](image)

Table 3: Test results for $H_0: x \in \mathcal{H}^N$ or $\mathcal{H}^S$ - age-specific employment rates

| $x$ | $x_1$ | $x_2$ | $x_3$ |
|-----|-------|-------|-------|
| Test of $H_0: x \in \mathcal{H}^N$ | 0.3128** | 0.3215** | 0.3178** |
| Test of $H_0: x \in \mathcal{H}^S$ | 0.3221** | 0.3133** | 0.1960* |

Notes: $T = 408$. We use * and ** to denote rejection at 5% and 1% significance level, respectively; the approximate critical value for 95% (resp. 99%) is 0.15 (resp. 0.22), in each case.

where $x = x_1, x_2$ or $x_3$ given in (5.1). The test results are reported in Table 3 under the assumption that $\varphi = 1$ as we earlier concluded from our sequential procedure. The null hypothesis that $x \in \mathcal{H}^N$ is rejected in every case at 1% significance level, which means that each of the considered functions is not entirely included in $\mathcal{H}^N$ but given by a linear combination of nonzero elements of $\mathcal{H}^N$ and $\mathcal{H}^S$; these results are, at least to some extent, expected from that $\mathcal{H}^N$ is estimated by $\text{span}\{\tilde{w}_1\}$ and the considered functions have quite different shapes from that of $\tilde{w}_1$; see Figure 2-(a). Moreover, the null hypothesis that $x \in \mathcal{H}^S$ is rejected at 1% significance level for $x = x_1$ or $x_2$, and it is rejected at 5% (but not 1%) significance level for $x = x_3$. These test results would lead us to similarly conclude that each of the considered functions is not entirely included in $\mathcal{H}^S$, but now it is worth noting that the null hypothesis for $x = x_3$ is rejected with less confidence; that is, the data supports that $x_3$ is relatively closer to $\mathcal{H}^S$ than $x_1$ and $x_2$. This result was earlier conjectured from the plots given in Figure 1, and we here note that such a conjecture has been, at least to some extent, examined by our proposed FPCA-based test.

In this example, we found a strong evidence that there exists at least one stochastic trend in the time series of age-specific employment rates. It may be of interest to investigate whether this time series is cointegrated with some economic variables exhibiting a unit root-type nonstationarity. This can be further explored in the future by developing a cointegrating regression model involving functional observations.

6 Conclusion

Nonstationarity caused by unit roots and the property of cointegration have been widely studied in a multivariate time series setting while it was not until recently that those were studied in the literature on FTS analysis. We investigate how a widely used statistical tool, FPCA, can be used in analysis of cointegrated FTS, and then propose a modification of FPCA for statistical inference on the cointegrating behavior. This paper develops a novel test for the dimension of the attractor space, which may be understood as an extension of some existing KPSS-type tests to examine stationarity or the cointegration rank of a cointegrated time series taking values in a finite dimensional Euclidean space. Our test is free of nuisance parameters and it can be
used to test various hypotheses on cointegration. We illustrate our methodology with two empirical examples. The presence of nonstationarity caused by unit roots and cointegration in FTS seems to be empirically relevant, so further exploration beyond this paper will be in demand.

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Supplementary Appendix

A Preliminaries

A.1 Random elements in \( \mathcal{H} \) and \( \mathcal{L}_\mathcal{H} \)

Let \((S, \mathcal{F}, \mathbb{P})\) denote the underlying probability triple. An \( \mathcal{H} \)-valued random variable \( X \) is defined as a measurable map from \( S \) to \( \mathcal{H} \), where \( \mathcal{H} \) is understood to be equipped with the usual Borel \( \sigma \)-field. An \( \mathcal{H} \)-valued random variable \( X \) is said to be integrable (resp. square integrable) if \( E\|X\| < \infty \) (resp. \( E\|X\|^2 < \infty \)). If \( X \) is integrable, there exists a unique element \( EX \in \mathcal{H} \) satisfying \( E \langle X, x \rangle = \langle EX, x \rangle \) for any \( x \in \mathcal{H} \). The element \( EX \) is called the expectation of \( X \). Let \( L^2_\mathcal{H} \) denote the space of \( \mathcal{H} \)-valued random variables \( X \) (identifying random elements that are equal almost surely) that satisfy \( \mathbb{E}X = 0 \) and \( \mathbb{E}\|X\|^2 < \infty \). For any \( X \in L^2_\mathcal{H} \), we may define its covariance operator as \( C_X = \mathbb{E}[X \otimes X] \), which is guaranteed to be self-adjoint, positive semidefinite and compact.

Now let \( A \) be a map from \( S \) to \( \mathcal{L}_\mathcal{H} \) such that \( \langle Ax, y \rangle \) is Borel measurable for all \( x, y \in \mathcal{H} \). Such a map \( A \) is called a random bounded linear operator; see Skorohod (1983). For such operators \( A \) and \( B \), we write \( A =_{fdd} B \) if \( \langle (Ax_1, y_1), \ldots, (Ax_k, y_k) \rangle = \langle (Bx_1, y_1), \ldots, (Bx_k, y_k) \rangle \) for every \( k \geq 1 \) and \( x_1, \ldots, x_k, y_1, \ldots, y_k \in \mathcal{H} \). For any sequence \( \{A_j\}_{j \geq 1} \) in \( \mathcal{L}_\mathcal{H} \), we write \( A_j \to_{\mathcal{L}_\mathcal{H}} A \), if \( \|A_j - A\|_{\mathcal{L}_\mathcal{H}} \to 0 \). It will be convenient to define two other modes of convergence of random bounded linear operators for our proofs of the main results. First, we write \( A_j \to_w A \), if for all \( x, y \in \mathcal{H} \),

\[
|\langle A_j x, y \rangle - \langle Ax, y \rangle| \to_p 0.
\]

Moreover, we write \( A_j \to_{wd} A \) if, for all \( k \) and \( x_1, \ldots, x_k, y_1, \ldots, y_k \in \mathcal{H} \),

\[
\lim_{j \to \infty} \mathbb{E} f(\langle A_j x_1, y_1 \rangle, \ldots, \langle A_j x_k, y_k \rangle) = \mathbb{E} f(\langle Ax_1, y_1 \rangle, \ldots, \langle Ax_k, y_k \rangle)
\]

for any bounded continuous function \( f \). When \( A_j \to_w A \) (resp. \( A_j \to_{wd} A \)), \( A_j \) is said to converge weakly (resp. weakly in distribution) to \( A \); these modes of convergence are introduced in detail by Skorohod (1983). Note that \( A_j \to_{wd} A \) means that the finite dimensional distribution of \( A_j \) converges to that of \( A \). Furthermore, it can be shown that

\[
A_j \to_{\mathcal{L}_\mathcal{H}} A \quad \Rightarrow \quad A_j \to_w A \quad \Rightarrow \quad A_j \to_{wd} A.
\]

A.2 Convergence of random bounded linear operators

We collect some useful results on convergence of random bounded linear operators.

Lemma A.1. Let \( \{A_j\}_{j \geq 1} \) and \( \{B_j\}_{j \geq 1} \) be sequences of random bounded linear operators. Then the following hold.

(i) If \( A_j \to_{wd} A \), then \( \sup_j \|A_j\| = O_p(1) \).

(ii) If \( A_j \to_{\mathcal{L}_\mathcal{H}} A \) and \( B_j \to_{\mathcal{L}_\mathcal{H}} B \), then \( A_j B_j \to_{\mathcal{L}_\mathcal{H}} AB \).

(iii) If \( A_j \to_{\mathcal{L}_\mathcal{H}} A \), then \( A_j^* \to_{\mathcal{L}_\mathcal{H}} A^* \).

Proof. (i) is given in Skorohod (1983); see, the proof of Theorem 3 in Ch 1.

Note that \( \|A_j B_j - AB\|_{\mathcal{L}_\mathcal{H}} \leq \|A_j - A\|_{\mathcal{L}_\mathcal{H}}\|B_j\|_{\mathcal{L}_\mathcal{H}} + \|A\|_{\mathcal{L}_\mathcal{H}}\|B_j - B\|_{\mathcal{L}_\mathcal{H}} \). Using (i), we may deduce that the right hand side converges to zero, hence (ii) is established.

(iii) immediately follows from the fact that \( \|A_j - A\|_{\mathcal{L}_\mathcal{H}} = \|A_j^* - A^*\|_{\mathcal{L}_\mathcal{H}} \). \( \square \)
Lemma A.2. Let \( \{ A_j \}_{j \geq 1} \) be a random sequence in \( \mathcal{L}_h \) and for \( j \geq 1 \) \( \{ \Psi_{A_j}(\mathcal{U}) \} \) be their characteristic functionals that are given by \( \Psi_{A_j}(\mathcal{U}) = \mathbb{E} \exp(i \text{tr}(A_j \mathcal{U})) \), where \( \mathcal{U} \) is any operator of the form

\[
\mathcal{U} = \sum_{i=1}^{k} x_i \otimes y_i, \quad x_i, y_i \in \mathcal{H}, \quad i = 1, \ldots, k, \quad k = 1, 2, \ldots \tag{A.1}
\]

Then the following hold.

(i) \( \Psi_{A_j}(\mathcal{U}) \to \Psi_A(\mathcal{U}) \) for all \( \mathcal{U} \) of the form (A.1) if and only if \( A_j \to_{\text{wd}} A \).

(ii) (Cramér-Wold device) \( A_j \to_{\text{wd}} A \) if and only if \( \sum_{i=1}^{k} \langle A_j y_i, x_i \rangle \to_d \sum_{i=1}^{k} \langle Ay_i, x_i \rangle \) for any \( k, x_1, \ldots, x_k \in \mathcal{H} \) and \( y_1, \ldots, y_k \in \mathcal{H} \).

(iii) If \( \Psi_A((x \otimes y)) \) is continuous in \( \alpha \) for all \( x, y \in \mathcal{H} \), then there exists a sequence of random operators \( \{ \tilde{A}_j \}_{j \geq 1} \) and \( \tilde{A} \) such that (i) \( \Psi_{\tilde{A}_j}(\cdot) = \Psi_A(\cdot) \), (ii) \( \Psi_{\tilde{A}}(\cdot) = \Psi_A(\cdot) \), and \( A_j \to_w \tilde{A} \).

Proof. The results can be found in or deduced from Chapter 3.3 of Skorohod (1983).

\[ \square \]

B Inclusion of deterministic terms

We now extend the main results given in Sections 3 and 4 to allow deterministic terms that may be included in the time series of interest. The proofs of the results in this section will be given later in Section C.

In particular, we in this section consider the following unobserved component models:

\[
\text{Model D1} : \quad X_{c,t} = \mu_1 + X_t, \tag{B.1}
\]

\[
\text{Model D2} : \quad X_{\ell,t} = \mu_1 + \mu_2 t + X_t, \tag{B.2}
\]

where \( X_t \) is a cointegrated time series considered in Section 3. We define the functional residual \( \bar{U}_t \) (resp. \( \bar{\tilde{U}}_t \)) for Model D1 (resp. Model D2) as follows: for \( t = 1, \ldots, T \),

\[
\bar{U}_t = X_{c,t} - \frac{1}{T} \sum_{i=1}^{T} X_{c,t}, \quad \bar{\tilde{U}}_t = X_{\ell,t} - \frac{1}{T} \sum_{i=1}^{T} X_{\ell,t} - \left( t - \frac{T+1}{2} \right) \frac{\sum_{i=1}^{T} (t - \frac{T+1}{2}) X_{\ell,t}}{\sum_{i=1}^{T} t^2};
\]

see Kokoszka and Young (2016). The sample covariance operators of \( \{ \bar{U}_t \}_{t=1}^{T} \) and \( \{ \bar{\tilde{U}}_t \}_{t=1}^{T} \) are given by \( \bar{\mathcal{C}} = \frac{1}{T} \sum_{t=1}^{T} \bar{U}_t \otimes \bar{U}_t \) and \( \bar{\mathcal{C}} = \frac{1}{T} \sum_{t=1}^{T} \bar{\tilde{U}}_t \otimes \bar{\tilde{U}}_t \), respectively.

B.1 Extension of the results given in Section 3

Consider the eigenvalue problems given by

\[
\bar{\mathcal{C}} \pi_j = \bar{\lambda}_j \pi_j, \quad j = 1, 2, \ldots, \tag{B.3}
\]

\[
\bar{\mathcal{C}} \tilde{\pi}_j = \bar{\lambda}_j \tilde{\pi}_j, \quad j = 1, 2, \ldots. \tag{B.4}
\]

We similarly obtain our preliminary estimator \( \bar{\mathcal{P}}_\varphi^N \) (resp. \( \bar{\mathcal{P}}_\varphi^N \)) from (B.3) (resp. (B.4)) as in (3.2), and let \( \mathcal{P}_\varphi^S = I - \mathcal{P}_\varphi^N \) and \( \bar{\mathcal{P}}_\varphi^S = I - \bar{\mathcal{P}}_\varphi^N \). As will be shown in Theorem B.1, the asymptotic limits of these preliminary estimators depend on \( \Omega^{NS} \) and \( \Gamma^{NS} \). Define for \( t = 1, \ldots, T \),

\[
\bar{Z}_{\varphi,t} = \bar{\mathcal{P}}_\varphi^N \Delta \bar{U}_t + \mathcal{P}_\varphi^S \bar{U}_t,
\]

\[
\bar{\tilde{Z}}_{\varphi,t} = \bar{\mathcal{P}}_\varphi^N \Delta \bar{\tilde{U}}_t + \bar{\mathcal{P}}_\varphi^S \bar{\tilde{U}}_t.
\]
The sample long run covariance (resp. one-sided long run covariance) of \( \{\bar{Z}_{\varphi,t}\}_{t=1}^T \) is similarly defined as in (3.4) (resp. (3.5)), and denoted by \( \bar{\Omega}_\varphi \) (resp. \( \bar{\Gamma}_\varphi \)). Such operators are also defined for \( \{\bar{Z}_{\varphi,t}\}_{t=1}^T \) and denoted by \( \Omega_\varphi \) and \( \Gamma_\varphi \) respectively. As in (2.2) and (3.6), we consider the following operator decompositions: for \( i \in \{N,S\} \) and \( j \in \{N,S\} \),

\[
\begin{align*}
\bar{\Omega}^i_j &= \bar{P}^i \phi, \quad \bar{\Gamma}^i_j = \phi \bar{P}^i, \\
\bar{\Omega}^i_j &= \bar{P}^i \phi, \quad \bar{\Gamma}^i_j = \phi \bar{P}^i,
\end{align*}
\]

For \( t = 1, \ldots, T \), define

\[
\begin{align*}
\bar{U}_{\varphi,t} &= U_t - \bar{\Omega}^N_N \left( \bar{\Omega}^N_N \right) \bar{P}^N_N \Delta U_t, \\
\bar{U}_{\varphi,t} &= U_t - \bar{\Omega}^N_N \left( \bar{\Omega}^N_N \right) \bar{P}^N_N \Delta U_t.
\end{align*}
\]

As in (3.8) and (3.10), we let

\[
\begin{align*}
\bar{\mathbf{U}}_\varphi &= T^{-1} \sum_{t=1}^T U_{\varphi,t} \otimes U_{\varphi,t}, \\
\bar{\mathbf{T}}_\varphi &= \bar{\Gamma}^{NS} - \bar{\Omega}^{NS} \left( \bar{\Omega}^{NS} \right) \bar{\Omega}^{NS}, \\
\bar{\mathbf{C}}_\varphi &= T^{-1} \sum_{t=1}^T \bar{U}_{\varphi,t} \otimes \bar{U}_{\varphi,t}, \\
\bar{\mathbf{T}}_\varphi &= \bar{\Gamma}^{NS} - \bar{\Omega}^{NS} \left( \bar{\Omega}^{NS} \right) \bar{\Omega}^{NS},
\end{align*}
\]

and consider the following modified eigenvalue problems that are parallel to (3.9),

\[
\begin{align*}
\left( \bar{\mathbf{C}}_\varphi - \bar{\mathbf{T}}_\varphi - \bar{\mathbf{T}}_\varphi^* \right) \bar{w}_j &= \bar{\mu}_j \bar{w}_j, \quad j = 1, 2, \ldots, \\
\left( \bar{\mathbf{C}}_\varphi - \bar{\mathbf{T}}_\varphi - \bar{\mathbf{T}}_\varphi^* \right) \bar{w}_j &= \bar{\mu}_j \bar{w}_j, \quad j = 1, 2, \ldots.
\end{align*}
\]

We then construct \( \bar{\Pi}_\varphi^N, \bar{\Pi}_\varphi^S, \bar{\Pi}_\varphi^N \), and \( \bar{\Pi}_\varphi^S \) as in (3.11).

To describe the asymptotic properties of \( \bar{\mathbf{P}}_\varphi^N, \bar{\mathbf{P}}_\varphi^S, \bar{\mathbf{P}}_\varphi^N \), and \( \bar{\mathbf{P}}_\varphi^S \), we hereafter let \( \bar{W}_N = W_N - \int W_N(s) \) and \( \bar{\tilde{W}}_N = W_N - \int W_N(s) + (6r - 4) \int \bar{W}_N(s) + (6 - 12r) \int \bar{\tilde{W}}_N(s) \). Moreover, we let \( \mathbf{F}, \bar{\mathbf{F}}, \mathbf{G}, \) and \( \bar{\mathbf{G}} \) be random bounded linear operators satisfying that

\[
\begin{align*}
\mathbf{F} &= \text{fdd} \left( \int \bar{W}_N(r) \otimes \bar{W}_N(r) \right)^* \left( \int dW_N(r) \otimes \bar{W}_N(r) + \Gamma^{NS} \right), \\
\bar{\mathbf{F}} &= \text{fdd} \left( \int \tilde{W}_N(r) \otimes \tilde{W}_N(r) \right)^* \left( \int dW_N(r) \otimes \tilde{W}_N(r) + \Gamma^{NS} \right), \\
\mathbf{G} &= \text{fdd} \left( \int \bar{W}_N(r) \otimes \bar{W}_N(r) \right)^* \left( \int dW_N(r) \otimes \bar{W}_N(r) \right), \\
\bar{\mathbf{G}} &= \text{fdd} \left( \int \tilde{W}_N(r) \otimes \tilde{W}_N(r) \right)^* \left( \int dW_N(r) \otimes \tilde{W}_N(r) \right),
\end{align*}
\]

where, as in Theorem 3.2, \( W^{S,N} = W^S - \Omega^{SN} \left( \Omega^{SN} \right)^* W^N \) and \( W^{S,N} \) is independent of \( W^N \). The asymptotic properties of the estimators are given as follows.
Theorem B.1. Suppose that Assumptions M, W and K hold with \( \varphi \geq 1 \).

\[
T(\mathcal{P}^N_{\varphi} - P^N) \rightarrow_{\mathcal{L}_W} F + F^* \quad \text{and} \quad T(\mathcal{P}^N_{\varphi} - P^N) \rightarrow_{\mathcal{L}_W} G + G^* \quad \text{if Model D1 is true},
\]
\[
T(\mathcal{P}^N_{\varphi} - P^N) \rightarrow_{\mathcal{L}_W} \mathcal{F} + \mathcal{F}^* \quad \text{and} \quad T(\mathcal{P}^N_{\varphi} - P^N) \rightarrow_{\mathcal{L}_W} \mathcal{G} + \mathcal{G}^* \quad \text{if Model D2 is true}.
\]

Moreover, as in Remark 3.2, it can be shown without difficulty that \( \mathcal{P}^N_{\varphi} \) (resp. \( \mathcal{P}^N_{\varphi} \)) is more asymptotically efficient than \( \mathcal{P}^N_{\varphi} \) (resp. \( \mathcal{P}^N_{\varphi} \)); see Remark 3.2 and our detailed discussion to be given in Section C.1.2.

B.2 Extension of the results given in Section 4

For any hypothesized value \( \varphi_0 \), we similarly construct \( \mathcal{P}^N_{\varphi_0} \) and \( \mathcal{P}^N_{\varphi_0} \) from (B.3) and \( \mathcal{P}^N_{\varphi_0} \) and \( \mathcal{P}^N_{\varphi_0} \) from (B.4). Define \( Z_{\varphi_0,t}, \Omega_{\varphi_0}, \Gamma_{\varphi_0}, \Upsilon_{\varphi_0}, \Upsilon_{\varphi,t}, \) and \( \Upsilon_{\varphi,t} \) for Model D1, and also define \( Z_{\varphi_0,t}^{*}, \Omega_{\varphi_0}, \Gamma_{\varphi_0}, \Upsilon_{\varphi_0}, U_{\varphi,t}, \) and \( \Upsilon_{\varphi,t} \) for Model D2 as in Section B. We then consider the following modified eigenvalue problems,

\[
\begin{align*}
\left( \mathcal{C}_{\varphi_0} - \mathcal{Z}_{\varphi_0} - \mathcal{Z}_{\varphi_0}^* \right) w_j &= \mu_j w_j, \quad j = 1, 2, \ldots, \quad (B.5) \\
\left( \mathcal{C}_{\varphi_0} - \mathcal{Z}_{\varphi_0} - \mathcal{Z}_{\varphi_0}^* \right) w_j &= \mu_j w_j, \quad j = 1, 2, \ldots. \quad (B.6)
\end{align*}
\]

Similar to (4.5), we define the following: for \( K > \varphi_0 \),

\[
\begin{align*}
\tilde{z}_{\varphi,t} &= (\langle U_{\varphi,t}, w_{\varphi_0+1} \rangle, \ldots, \langle U_{\varphi,t}, w_K \rangle)', \\
\tilde{z}_{\varphi,t} &= (\langle U_{\varphi,t}, w_{\varphi_0+1} \rangle, \ldots, \langle U_{\varphi,t}, w_K \rangle)'.
\end{align*}
\]

To examine the null and alternative hypotheses given in (4.1), we consider the following statistics for Model D1 and Model D2 respectively:

\[
\begin{align*}
\mathcal{Q}(K, \varphi_0) &= \frac{1}{T^2} \sum_{t=1}^{T} \left( \sum_{s=1}^{t} \tilde{z}_{\varphi,t} \right)' \mathcal{LRV}(\tilde{z}_{\varphi,t})^{-1} \left( \sum_{s=1}^{t} \tilde{z}_{\varphi,t} \right), \\
\bar{Q}(K, \varphi_0) &= \frac{1}{T^2} \sum_{t=1}^{T} \left( \sum_{s=1}^{t} \tilde{z}_{\varphi,t} \right)' \mathcal{LRV}(\tilde{z}_{\varphi,t})^{-1} \left( \sum_{s=1}^{t} \tilde{z}_{\varphi,t} \right) .
\end{align*}
\]

To describe the asymptotic properties of \( \mathcal{Q}(K, \varphi_0) \) and \( \bar{Q}(K, \varphi_0) \), we let \( \mathcal{B}(r) = \mathcal{B}(r) - \int \mathcal{B}(s) \), \( \bar{B}(r) = \mathcal{B}(r) + (6r - 4) \int \mathcal{B}(s) + (6 - 12r) \int s \mathcal{B}(s), \) \( \tilde{W}(r) = \mathcal{W}(r) - r \mathcal{W}(1), \) and \( \bar{W}(r) = \mathcal{W}(r) + (2r - 3r^2) \mathcal{W}(1) + (6r^2 - 6r) \int \mathcal{W}(s), \) where \( \mathcal{B} \) and \( \mathcal{W} \) the independent Brownian motions that are defined for the case with no deterministic terms. The asymptotic properties of the statistics are given follows:

Theorem B.2. Suppose that Assumptions M, W and K hold, and \( K \) is a finite integer in \( (\varphi_0, \dim(\mathcal{H})) \). Under \( H_0 \),

\[
\begin{align*}
\mathcal{Q}(K, \varphi_0) &\rightarrow_{d} \int \mathcal{V}(r)' \mathcal{V}(r) \quad \text{if Model D1 is true}, \\
\bar{Q}(K, \varphi_0) &\rightarrow_{d} \int \bar{V}(r)' \bar{V}(r) \quad \text{if Model D2 is true},
\end{align*}
\]

where

\[
\begin{align*}
\mathcal{V}(r) &= \mathcal{W}(r) - \int d\mathcal{W}(s) \mathcal{B}(s)' \left( \int \mathcal{B}(s) \mathcal{B}(s)' \right)^{-1} \int_0^r \mathcal{B}(s), \\
\bar{V}(r) &= \bar{W}(r) - \int d\mathcal{W}(s) \mathcal{B}(s)' \left( \int \mathcal{B}(s) \mathcal{B}(s)' \right)^{-1} \int_0^r \mathcal{B}(s),
\end{align*}
\]
and the second term in each expression of $\bar{V}$ and $\bar{V}$ is regarded as zero if $\varphi_0 = 0$. Under $H_1$ and each of the models, the relevant statistic diverges in probability to infinity.

Tables 2 and 3 of Harris (1997) report critical values for some choices of $K$ and $\varphi_0$.

C Mathematical Proofs

Let $\{A_j\}_{j \geq 1}$ be a sequence in the space equipped with norm $\|\cdot\|_B$ and satisfy that $\|A_j - A\|_B = O_p(n)$. As is common in the literature, we sometimes, for convenience, write this as $A_j = A + O_p(n)$. We similarly write $A_j = A + o_p(n)$ to denote $\|A_j - A\|_B = o_p(n)$.

C.1 Mathematical proofs for the results in Section 3

C.1.1 Preliminary results

We provide useful lemmas.

Lemma C.1. Under Assumptions $M$ for every $k \geq 1$ and $x_1, \ldots, x_k \in \mathcal{H}$,

$$\frac{1}{\sqrt{T}} \sum_{t=1}^{\lfloor Ts \rfloor} \mathcal{E}_{k,t} \to_d W_k(s) \quad (C.1)$$

in the Skorohod space $D[0, 1]^k$, where $\mathcal{E}_{k,t} = (\langle E_{t,1}, x_1 \rangle, \ldots, \langle E_{t,k}, x_k \rangle)'$ and $W_k(s) = ((W(s), x_1), \ldots, (W(s), x_k))'$.

Proof. Under the summability conditions $\sum_{j=1}^{\infty} j \|\Phi_j\|_{L_H} < \infty$ and $\sum_{j=1}^{\infty} j \|\Phi_j\|_{L_H} < \infty$, $\{E_{nit}\}_{t \in \mathbb{Z}}$ and $\{E_{nt}^S\}_{t \in \mathbb{Z}}$ are, respectively, so-called $L^2$-m-approximable (Hörmann and Kokoszka, 2010, Proposition 2.1), and then $\{\mathcal{E}_t\}_{t \in \mathbb{Z}}$ is also $L^2$-m-approximable (Hörmann and Kokoszka, 2010, Lemma 2.1). Then the desired weak convergence result follows from Theorem 1.1. of Berkes et al. (2013).

Lemma C.2. Under Assumptions $M$ and $K$, $\hat{\Omega}_\varphi$ in (3.4) and $\hat{\Gamma}_\varphi$ in (3.5) are consistent, i.e., $\hat{\Omega}_\varphi \to_{L_H} \Omega$ and $\hat{\Gamma}_\varphi \to_{L_H} \Gamma$. Moreover, let $\tilde{\Omega}_\varphi$, $\tilde{\Gamma}_\varphi$, $\tilde{\Omega}_\varphi$, and $\tilde{\Gamma}_\varphi$ be defined as in Section B. Under the same assumptions, $\tilde{\Omega}_\varphi$ and $\tilde{\Gamma}_\varphi$ (resp. $\hat{\Omega}_\varphi$ and $\hat{\Gamma}_\varphi$) are consistent in the same sense if Model D1 given by (B.1) (resp. Model D2 given by (B.2)) is true.

Proof. We know from Theorem 3.1 that $\hat{P}_\varphi^N = P^N + O_p(T^{-1})$ and $\hat{P}_\varphi^S = P^S + O_p(T^{-1})$, hence

$\hat{\Omega}_\varphi = \hat{\Omega}_\varphi^{N,N} + \hat{\Omega}_\varphi^{N,S} + \hat{\Omega}_\varphi^{S,N} + \hat{\Omega}_\varphi^{S,S} = \hat{\Omega}_{0,\varphi} + O_p(T^{-1})$,

where $\hat{\Omega}_{0,\varphi} = P^N \hat{\Omega}_\varphi P^N + P^N \hat{\Omega}_\varphi P^S + P^S \hat{\Omega}_\varphi P^N + P^S \hat{\Omega}_\varphi P^S$. Note that $\hat{\Omega}_{0,\varphi}$ is the sample long-run covariance of $\{\mathcal{E}_t\}_{t=1}^T$, where $\mathcal{E}_t = \mathcal{E}_t^N + \mathcal{E}_t^S$, $\mathcal{E}_t^N = \sum_{j=0}^{\infty} P^N \Phi_j \mathcal{E}_{t-j}$, and $\mathcal{E}_t^S = \sum_{j=0}^{\infty} P^S \Phi_j \mathcal{E}_{t-j}$. We know from our proof of Lemma C.1 that the summability conditions $\sum_{j=1}^{\infty} j \|\Phi_j\|_{L_H} < \infty$ and $\sum_{j=1}^{\infty} j \|\Phi_j\|_{L_H} < \infty$ imply that $\{\mathcal{E}_t\}_{t \in \mathbb{Z}}$ is $L^2$-m-approximable. We then apply Theorem 2 of Horváth et al. (2013) to obtain $\hat{\Omega}_{0,\varphi} \to_{L_H} \Omega$, which in turn establishes $\hat{\Omega}_\varphi \to_{L_H} \hat{\Omega}$. To show $\hat{\Gamma}_\varphi \to_{L_H} \Gamma$, we note that $\tilde{\Gamma}_\varphi = \tilde{\Gamma}_{0,\varphi} + O_p(T^{-1})$ where $\tilde{\Gamma}_{0,\varphi} = P^N \tilde{\Gamma}_\varphi P^N + P^N \tilde{\Gamma}_\varphi P^S + P^S \tilde{\Gamma}_\varphi P^N + P^S \tilde{\Gamma}_\varphi P^S$. Then $\tilde{\Gamma}_\varphi \to_{L_H} \Gamma$ may be deduced from the proof of Theorem 2 in Horváth et al. (2013).

For the cases with deterministic terms, the desired results are deduced from Theorem 2 of Horváth et al. (2013) (when Model D1 is true), Theorem 5.3 of Kokoszka and Young (2016) (when Model D2 is true), and our previous proof for the case without deterministic terms. 

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C.1.2 Proofs of the main results

Proof of Theorem 3.1. We note the identity

\[ \hat{P}_\varphi^N - P^N = P^S \hat{P}_\varphi^N - P^N \hat{P}_\varphi^S. \]  

(C.2)

Since \( \hat{P}_\varphi^N \) is the projection onto the first \( \varphi \) leading eigenvectors, \( P^N \hat{C} \hat{P}_\varphi^S = P^N \hat{C} P^N \hat{P}_\varphi^S + P^N \hat{C} P^S \hat{P}_\varphi^S = P^N \hat{C}(I - \sum_{j=1}^{\infty} \lambda_j \delta_j \otimes \delta_j) = P^N \hat{P}_\varphi^N \Lambda, \) where \( \Lambda = \sum_{j=1}^{\infty} \lambda_j \delta_j \otimes \delta_j. \) We thus find that

\[ TP^N \hat{P}_\varphi^S = \left( T^{-1} P^N \hat{C} P^N \right)^\dagger P^N \hat{C} P^S + \left( T^{-1} P^N \hat{C} P^N \right)^\dagger P^N \hat{P}_\varphi^S \Lambda, \]  

(C.3)

where \( \left( T^{-1} P^N \hat{C} P^N \right)^\dagger \) is well defined since \( T^{-1} P^N \hat{C} P^N \) is a finite rank operator, hence has a closed range. We first deduce from the weak convergence result given in (C.1) and the continuous mapping theorem that the following holds

\[ \langle T^{-1} P^N \hat{C} P^N x, y \rangle = \frac{1}{T^2} \sum_{t=1}^{T} \langle P^N X_t, x \rangle \langle P^N X_t, y \rangle \rightarrow_d \int \langle W^N(r), x \rangle \langle W^N(r), y \rangle, \]  

jointly.

This result implies that

\[ T^{-1} P^N \hat{C} P^N \rightarrow_{wd} \int W^N(r) \otimes W^N(r), \]  

(C.4)

due to the Cramér-Wold device (Lemma A.2-(ii)). Let \( X_{k,t}^S = (\langle P^S X_t, x_1 \rangle, \ldots, \langle P^S X_t, x_k \rangle) \), \( X_{k,t}^N = (\langle P^N X_t, y_1 \rangle, \ldots, \langle P^N X_t, y_k \rangle) \). We deduce from the convergegence result given by Assumption W that

\[ \frac{1}{T} \sum_{t=1}^{T} X_{k,t}^S X_{k,t}^N \rightarrow_d \int_0^1 dW_k^S(s)W_k^N(s) + \sum_{j=0}^{\infty} \mathbb{E}[X_{k,t}^S X_{k,t}^N], \]

where \( X_{k,t}^S = (\langle X_{k,t}^S, x_1 \rangle, \ldots, \langle X_{k,t}^S, x_k \rangle) \), \( X_{k,t}^N = (\langle X_{k,t}^N, y_1 \rangle, \ldots, \langle X_{k,t}^N, y_k \rangle) \), \( W_k^S(s) = (\langle W^S(s), x_1 \rangle, \ldots, \langle W^S(s), x_k \rangle) \), \( W_k^N(s) = (\langle W^N(s), y_1 \rangle, \ldots, \langle W^N(s), y_k \rangle) \). Moreover, for any \( x, y \in \mathcal{H} \), \( \langle P^N \hat{C} P^S x, y \rangle = \langle P^N \hat{C} P^N X_t, y \rangle \), from which we find that \( \sum_{j=1}^{k} \langle P^N \hat{C} P^S x, y \rangle = \text{tr}(T^{-1} \sum_{t=1}^{T} X_{k,t}^S X_{k,t}^N) \). From the continuous mapping theorem, we have

\[ \sum_{j=1}^{k} \langle P^N \hat{C} P^S x, y \rangle \rightarrow_d \sum_{j=1}^{k} \left( \int \langle dW^S(r), x \rangle \langle W^N(r), y \rangle + \langle \Gamma^N X_t, x \rangle \right). \]  

(C.5)

From (C.5) and the Cramér-Wold device (Lemma A.2-(ii)), we deduce that

\[ P^N \hat{C} P^S \rightarrow_{wd} \int dW^S(r) \otimes W^N(r) + \Gamma^NS. \]  

(C.6)

It will be shown that the convergence results given by (C.4) and (C.6) can be strengthened as follows:

Claim 1: \( T^{-1} P^N \hat{C} P^N \rightarrow_{L_2} A_1 = \text{fdd} \int W^N(r) \otimes W^N(r). \)

Claim 2: \( P^N \hat{C} P^S \rightarrow_{L_2} A_2 = \text{fdd} \int dW^S(r) \otimes W^N(r) + \Gamma^NS. \)

Almost surely, the eigenvalues of \( A_1 \) are distinct and each eigenvalue is associated with one-dimensional eigenspace.

We then know from Claim 1 and Lemma 4.3 of Bosq (2000) that the eigenvectors of \( T^{-1} P^N \hat{C} P^N \) converge to those of \( A_1 \). From similar arguments to the proofs of Proposition 3.2 and Theorem 3.3 of Chang et al. (2016), we find that
\( P^N \hat{P}^S = O_p(T^{-1}) \), \( P^S \hat{P}^S - P^S = O_p(T^{-1}) \) and \( \hat{\lambda}_{\varphi+k} = O_p(1) \) for \( k \geq 1 \). Given these results, the following can be additionally proved.

**Claim 3:** \( TP^S \hat{P}^N = -T(P^N \hat{P}^S)^* + O_p(T^{-1}) \).

Since \( P^N \hat{P}^S = O_p(T^{-1}) \), (C.3) can be written as \( TP^N \hat{P}^S = - \left( T^{-1} P^N \hat{C} P^N \right)^\dagger P^N \hat{C} P^S + O_p(T^{-1}) \). From this equation, Claim 3 and (C.2), we find that

\[
T(\hat{P}^N - P^N) = \left( T^{-1} P^N \hat{C} P^N \right)^\dagger P^N \hat{C} P^S + P^S \hat{C} P^N \left( T^{-1} P^N \hat{C} P^N \right)^\dagger + O_p(T^{-1}).
\]

We know from Claims 1-2 and Lemma A.1 that our proof becomes complete if we show

\[
\left( T^{-1} P^N \hat{C} P^N \right)^\dagger \rightarrow_{\mathcal{L}_H} A_1^\dagger. \tag{C.7}
\]

Note that both \( T^{-1} P^N \hat{C} P^N \) and \( A_1 \) are self-adjoint positive definite operators of rank \( \varphi \), almost surely. Consider the spectral representations of \( T^{-1} P^N \hat{C} P^N \) and \( A_1 \) as follows:

\[
T^{-1} P^N \hat{C} P^N = \sum_{j=1}^\varphi \gamma_j \hat{u}_j \otimes \hat{u}_j, \quad A_1 = \sum_{j=1}^\varphi \gamma_j u_j \otimes u_j.
\]

\( T^{-1} P^N \hat{C} P^N \rightarrow_{\mathcal{L}_H} A_1 \) implies that the eigenvalues of \( T^{-1} P^N \hat{C} P^N \) converge to those of \( A_1 \) (Bosq, 2000, Lemma 4.2), i.e.,

\[
\sup_{1 \leq j \leq \varphi} |\hat{\gamma}_j - \gamma_j| \rightarrow_p 0. \tag{C.8}
\]

Note that \( \gamma_j > 0 \) and the associated eigenspace is one-dimensional for each \( j = 1, \ldots, \varphi \) almost surely. It is deduced from Lemma 4.3 of Bosq (2000) that the eigenvectors satisfy

\[
\sup_{1 \leq j \leq \varphi} \|\hat{u}_j - \text{sgn}(\langle \hat{u}_j, u_j \rangle) u_j\| \rightarrow_p 0. \tag{C.9}
\]

Let \( \bar{u}_j = \text{sgn}(\langle \hat{u}_j, u_j \rangle) u_j \). Then it can be shown that \( \sup_{\|x\| \leq 1} \| (T^{-1} P^N \hat{C} P^N)^\dagger x - A_1^\dagger x \| \) is bounded above by

\[
\sum_{j=1}^\varphi |\gamma_j^{-1} - \hat{\gamma}_j^{-1}| + 2 \sum_{j=1}^\varphi \hat{\gamma}_j^{-1} \|\hat{u}_j - \bar{u}_j\|, \tag{C.10}
\]

We then deduce from (C.8) and (C.9) that (C.10) converges in probability to zero as desired, so (C.7) holds.

**Proofs of Claims 1 & 2:** We only provide our proof of Claim 2; the arguments to be given can be applied to prove Claim 1 with a minor modification. To simplify expressions, we let \( \hat{A} = P^N \hat{C} P^S \) and \( \hat{A}^* = P^S \hat{C} P^N \). From (C.6) and Lemma A.2-(iii), we may assume that \( \hat{A} \rightarrow_w A_2 = \lim \int dW(x) \otimes W_N(x) + \Gamma \) and \( \hat{A}^* \rightarrow_w A_2^* \). Let \( \{u_j\}_{j=1}^\varphi \) (resp. \( \{u_j\}_{j=\varphi+1}^\infty \)) denote an orthonormal basis of \( \mathcal{H}^N \) (resp. \( \mathcal{H}^S \)). Since \( \hat{A} \rightarrow_w A_2 \), ran \( \hat{A} \subset \mathcal{H}^N \) and ran \( A_2 \subset \mathcal{H}^N \), we have for any \( x \in \mathcal{H} \)

\[
\| (\hat{A} - A_2) x \|^2 = \sum_{j=1}^\varphi |\langle (\hat{A} - A_2) x, u_j \rangle|^2 \rightarrow 0, \tag{C.11}
\]

where the equality follows from Parseval’s identity. From the properties of operator norm, we have

\[
\| \hat{A} - A_2 \|_{\mathcal{L}_H}^2 = \| \hat{A} \hat{A}^* - A_2 \hat{A}^* - \hat{A} A_2^* + A_2 A_2^* \|_{\mathcal{L}_H}. \tag{C.12}
\]
Note that \( \hat{A}A^* - A_2\hat{A}^* - \hat{A}A_2^* + A_2A_2^* \) is self-adjoint, positive semidefinite and has finite rank. Moreover, its operator norm is bounded above by the trace norm (see equation (1.55) in Bosq, 2000), which is in turn bounded by \( \sum_{j=1}^{\infty} |\langle \hat{A}A^* - A_2\hat{A}^* - \hat{A}A_2^* + A_2A_2^* \rangle u_j, u_j \rangle |. \) Since \( \text{ran}(\hat{A}A^* - A_2\hat{A}^* - \hat{A}A_2^* + A_2A_2^*) \) is orthogonal to \( H^S \), this upper bound is simplified to

\[
\sum_{j=1}^{\infty} |\langle \hat{A}A^* - A_2\hat{A}^* - \hat{A}A_2^* + A_2A_2^* \rangle u_j, u_j \rangle |.
\]

We note that \( |\langle \hat{A}A^* u_j, u_j \rangle - \langle A_2A_2^* u_j, u_j \rangle| \leq |\langle A^* u_j, (\hat{A}^* - A_2^*) u_j \rangle| + |\langle \hat{A}^* - A_2^* \rangle u_j, u_j \rangle |, \) hence

\[
|\langle \hat{A}A^* u_j, u_j \rangle - \langle A_2A_2^* u_j, u_j \rangle| \leq \| \hat{A}^* \| \| (\hat{A}^* - A_2^*) \| u_j \| + \| A_2^* \| \| (\hat{A}^* - A_2^*) \| u_j \| = o_p(1),
\]

where the last equality follows from the fact that (C.11) and \( \sup_p \| \hat{A}^* \| = O_p(1) \) (Lemma A.1-(i)). We therefore conclude that

\[
\hat{A}A^* \rightarrow_w A_2A_2^*.
\]

From parallel arguments,

\[
A_2\hat{A}^* \rightarrow_w A_2A_2^* , \quad \hat{A}A_2^* \rightarrow_w A_2A_2^*.
\]

Equations (C.14) and (C.15) imply that (C.13) is \( o_p(1) \), which in turn implies that (C.12) is \( o_p(1) \).

**Proof of Claim 3:** Note that \( \hat{P}_\varphi^N = P^S\hat{P}_\varphi^N + P^N\hat{P}_\varphi^N \) and \( \hat{P}_\varphi^N = -T\hat{P}_\varphi^N P^N\hat{P}_\varphi^N \) by construction, we thus have \( T\hat{P}_\varphi^N P^N\hat{P}_\varphi^N = -T\hat{P}_\varphi^N P^N\hat{P}_\varphi^N \). From this equation and the identity \( I = \hat{P}_\varphi^N + \hat{P}_\varphi^N \), the following can be shown:

\[
TP^S\hat{P}_\varphi^N - T\hat{P}_\varphi^N P^N\hat{P}_\varphi^N = -T(P^N\hat{P}_\varphi^N)^* + T(\hat{P}_\varphi^N P^N\hat{P}_\varphi^N)^*.
\]

Since \( P^N\hat{P}_\varphi^N = O_p(T^{-1}) \) and \( P^N\hat{P}_\varphi^N - P^S = O_p(T^{-1}) \), we find that \( \| T\hat{P}_\varphi^N P^N\hat{P}_\varphi^N \|_{L_H} = \| T(I - \hat{P}_\varphi^N) P^N\hat{P}_\varphi^N \|_{L_H} = O_p(T^{-1}) \) and \( \| T\hat{P}_\varphi^N P^N\hat{P}_\varphi^N \|_{L_H} = \| T\hat{P}_\varphi^N P^N\hat{P}_\varphi^N \|_{L_H} = O_p(T^{-1}) \). We therefore conclude that

\[
TP^S\hat{P}_\varphi^N = -T(P^N\hat{P}_\varphi^N)^* = O_p(T^{-1}).
\]

**Proof of Theorem 3.2.**

We first show that the desired result can be obtained from the following eigenvalue problem:

\[
\left( \hat{C}_\varphi - \hat{\Upsilon}_\varphi \right) \hat{w}_j = \hat{\mu}_j \hat{w}_j, \quad j = 1, 2, \ldots.
\]

It can be shown that \( P^N(\hat{C}_\varphi - \hat{\Upsilon}_\varphi)P^S \) and \( P^S(\hat{C}_\varphi - \hat{\Upsilon}_\varphi)P^S \) weakly converge in distribution to some elements in \( L_H \), then Lemma A.1-(i) implies that each of \( \| P^N(\hat{C}_\varphi - \hat{\Upsilon}_\varphi)P^S \|_{L_H} \) and \( \| P^S(\hat{C}_\varphi - \hat{\Upsilon}_\varphi)P^S \|_{L_H} \) is \( O_p(1) \). We know from Lemma C.2 that \( \hat{\Upsilon}_\varphi = O_p(1) \), and also deduce from our construction of \( X_{\varphi,t} \) that \( T^{-1}P^N\hat{C}_\varphi P^N = T^{-1}P^N\hat{C}_\varphi + O_p(T^{-1}) \). Combining all these results and arguments used in our proof of Theorem 3.1, we have

\[
T^{-1}(\hat{C}_\varphi - \hat{\Upsilon}_\varphi) = T^{-1}P^N\hat{C}_\varphi P^N + O_p(T^{-1}) \rightarrow_{L_H} A_1 = \text{fdd} \int W^N(r) \otimes W^N(r).
\]

Then from similar arguments to the proofs of Proposition 3.2 and Theorem 3.3 of Chang et al. (2016), we find that \( P^N\hat{\Pi}_\varphi^S = O_p(T^{-1}) \), \( P^S\hat{\Pi}_\varphi^S - P^S = O_p(T^{-1}) \) and \( \hat{\mu}_{\varphi} + k = O_p(1) \) for \( k \geq 1 \). Since \( P^N(\hat{C}_\varphi - \hat{\Upsilon}_\varphi)\hat{\Pi}_\varphi^S = P^N\hat{\Pi}_\varphi^S \hat{M} \) for \( \hat{M} = \sum_{j=1}^{\infty} \hat{\mu}_j \hat{w}_j \otimes \hat{w}_j \) and \( \hat{\Pi}_\varphi^S = P^N\hat{\Pi}_\varphi^S + P^S\hat{\Pi}_\varphi^S \),

\[
P^N(\hat{C}_\varphi - \hat{\Upsilon}_\varphi)P^N\hat{\Pi}_\varphi^S + P^N(\hat{C}_\varphi - \hat{\Upsilon}_\varphi)P^S\hat{\Pi}_\varphi^S = P^N\hat{\Pi}_\varphi^S \hat{M}.
\]
We know from \( \hat{\Pi}^S P^N = O_p(T^{-1}) \) and \( \hat{\Pi}^* P^N = O_p(T^{-1}) \) that \( P^N \hat{\Pi}^S M = O_p(T^{-1}) \) and \( P^N \hat{\Pi}^* P^N = O_p(T^{-1}) \). This in turn implies that

\[
T P^N \hat{\Pi}^S = - \left( T^{-1} P^N \hat{\Pi}^* P^N \right) \dagger \left( P^N \hat{\Pi}^* P^N - P^N \hat{\Pi}^* P^N \right) + O_p(T^{-1}), \tag{C.18}
\]

where note that \( (T^{-1} P^N \hat{\Pi}^* P^N) \dagger \) is well define since \( T^{-1} P^N \hat{\Pi}^* P^N \) is a finite rank operator, so has a closed range. Using similar results to (C.2) and Claim 3 in our proof of Theorem 3.1, we find that

\[
T(\hat{\Pi}^N - P^N) = \left( P^S \hat{\Pi}^* P^N - P^S \hat{\Pi}^* P^N \right) \left( T^{-1} P^N \hat{\Pi}^* P^N \right) \dagger \\
+ \left( T^{-1} P^N \hat{\Pi}^* P^N \right) \dagger \left( P^N \hat{\Pi}^* P^N - P^N \hat{\Pi}^* P^N \right) + O_p(T^{-1}). \tag{C.19}
\]

From (C.17) and nearly identical arguments to those used to derive (C.7),

\[
\left( T^{-1} P^N \hat{\Pi}^* P^N \right) \dagger \to L_\mathcal{H} A_1^\dagger. \tag{C.20}
\]

We will derive the limit of \( \hat{A} = P^N \hat{\Pi}^* P^S - P^N \hat{\Pi}^* P^S \). First, it may be deduced from Lemma C.2 that

\[
\hat{\Gamma}^{NN} \to L_\mathcal{H} \Gamma^{NN}, \quad \hat{\Gamma}^{NS} \to L_\mathcal{H} \Gamma^{NS}, \quad \hat{\Gamma}^{SS} \to L_\mathcal{H} \Gamma^{SS}. \tag{C.21}
\]

Moreover, from the fact that \( \text{rank} \hat{\Omega}^{NN} = \text{rank} \Omega^{NN} = \varphi \) almost surely, \( B \mapsto B^{-1} \) is a continuous map for every positive definite matrix \( B \in \mathbb{R}^{\varphi \times \varphi} \), and \( \hat{\Omega}^{NN} \to L_\mathcal{H} \Omega^{NN} \) (Lemma C.2) and this may be understood as a convergence in probability in \( \mathbb{R}^{\varphi \times \varphi} \), we find that

\[
\hat{\Omega}^{NN} \to L_\mathcal{H} \left( \Omega^{NN} \right)^\dagger. \tag{C.22}
\]

Combining (C.21) and (C.22), we have, for any \( k \geq 1 \), \( x_1, \ldots, x_k \in \mathcal{H} \) and \( y_1, \ldots, y_k \in \mathcal{H} \), \( \langle P^N \hat{\Pi}^* P^S x_j, y_j \rangle \to_p \langle \Upsilon x_j, y_j \rangle \), where \( \Upsilon = \Gamma^{NS} - \Gamma^{NN} \left( \Omega^{NN} \right)^\dagger \Omega^{NS} \). We then note that

\[
\langle \hat{A} x_j, y_j \rangle = \frac{1}{T} \sum_{t=1}^T \langle P^S X_t - \Omega^{SN} \left( \Omega^{NN} \right)^\dagger P^N \Delta X_t, x_j \rangle \langle P^N X_t, y_j \rangle \langle \Upsilon x_j, y_j \rangle + O_p(1),
\]

where the first term converges in distribution to \( \int \langle dW^{SN}(r), x_j \rangle \langle W^N(r), y_j \rangle + \langle \Upsilon x_j, y_j \rangle \); see e.g. the proofs of Theorems 1 and 2 of Harris (1997). Using this result and nearly identical arguments used to derive (C.6), it is quite obvious to establish that \( \hat{A} \to_{wd} \int dW^{SN}(r) \otimes W^N(r) \). Furthermore, from similar arguments to those used to prove Claim 2 in our proof of Theorem 3.1, we find that

\[
\hat{A} \to_{L_\mathcal{H}} A_2 = \text{id}_\mathcal{H} \int dW^{SN}(r) \otimes W^N(r). \tag{C.23}
\]

Combining (C.19), (C.20), (C.23), and Lemma A.1-(ii) and (iii), we obtain the desired result.

We now consider the eigenvalue problem (3.9). On top of the previous proof, it is easy to show that \( \hat{\Upsilon}^* = O_p(1) \) and \( P^N \hat{\Upsilon}^* P^S = O_p(T^{-2}) \), so (C.18) still holds for \( \hat{\Pi}^S \) obtained from (3.9). The rest of proof is almost identical to our proof for the eigenvalue problem (C.16).

Independence of \( W^{SN} \) and \( W^N \) is deduced from the fact that \( \mathbb{E}[W^{SN} \otimes W^N] = 0 \).

**Proof of Proposition 3.1.** Since \( \dim(\mathcal{H}) < \infty \) and the minimum eigenvalue of \( \mathbb{E}[\mathcal{E} \otimes \mathcal{E}] \) is strictly positive, we may deduce from the proof of Theorem 2 in Harris (1997) and arguments similar to those used in Theorems 3.1 and
Suppose that Assumptions M, W and K hold.

Lemma C.3. \( T^{-1}\tilde{C}_\varphi = T^{-1}P^N\tilde{C}_\varphi P^N + O_p(T^{-1}) \rightarrow_{L^2} A_1 = \int W^N(r) \otimes W^N(r), \) \( P^N\tilde{C}_\varphi P^S \rightarrow_{L^2} A_2 = \int dW^S(r) \otimes W^N(r). \) (C.24) (C.25)

As in our proof of Theorem 3.2, we have \( P^N\tilde{\Pi}_\varphi^S = O_p(T^{-1}), P^S\tilde{\Pi}_\varphi^S - P^S = O_p(T^{-1}) \) and \( \tilde{\lambda}_{\varphi+k} = O_p(1) \) for \( k \geq 1. \) From similar arguments to those used to derive (C.19), we have

\[
T(\tilde{\Pi}_\varphi^N - P^N) = (T^{-1}P^N\tilde{C}_\varphi P^N)^\dagger P^N\tilde{C}_\varphi P^S + P^S\tilde{C}_\varphi P^N (T^{-1}P^N\tilde{C}_\varphi P^N)^\dagger + O_p(T^{-1}).
\] (C.26)

Then the desired result is deduced from (C.24)-(C.26).

\[ \square \]

A detailed discussion on Remark 3.2. We here consider the simple case when \( k = 1. \) From Theorem 3.1 of Saikkonen (1991), it can be shown that

\[
\lim_{T \rightarrow \infty} \text{Prob.} \{ |T\langle P^N\tilde{\Pi}_\varphi^S x, y \rangle| < \delta \} \geq \lim_{T \rightarrow \infty} \text{Prob.} \{ |T\langle P^N\tilde{\Pi}_\varphi^S x, y \rangle| < \delta \}. \] (C.27)

If \( \Omega^{NS} = \Gamma^{NS} = 0, \) then \( T\langle P^N\tilde{\Pi}_\varphi^S \rangle \) and \( T\langle P^N\tilde{\Pi}_\varphi^S \rangle \) have the same limiting distribution. Moreover, if \( y \in H^S \) both sides of (C.27) are equal to zero regardless of the limiting behaviors of \( P^N\tilde{\Pi}_\varphi^S \) and \( P^N\tilde{\Pi}_\varphi^S. \) Therefore the inequality given in (C.27) is not always strict. However, if \( \Omega^{NS} = \Gamma^{NS} = 0 \) is not true and our choice of \( x, y \) and \( \delta \) makes the left hand side of (C.27) positive, then the inequality is strict; see Theorem 3.1 of Saikkonen (1991). Moreover, the left hand side can be made positive by choosing \( x, y \) and \( \delta \) appropriately. For example, suppose that \( x \in H^S \setminus \{0\} \) and \( y \in H^N \setminus \{0\}. \) As shown in our proof of Theorem 3.1, we may assume that \( \int W^N(r) \otimes W^N(r) = \sum_{j=1}^\varphi \gamma_j u_j \otimes u_j, \) \( \{\gamma_j\}_{j=1}^\varphi \) are all positive, and \( \text{span}(\{u_j\}_{j=1}^\varphi) = H^N. \) Then from Theorem 3.2, we have \( T\langle P^N\tilde{\Pi}_\varphi^S x, y \rangle \rightarrow_d \sum_{j=1}^\varphi \gamma_j \int (dW^S(r), x) \langle u_j, W^N(r) \rangle \langle u_j, y \rangle. \) That is, \( T\langle P^N\tilde{\Pi}_\varphi^S x, y \rangle \) converges to a nondegenerate distribution centered at \( 0 \in \mathbb{R}. \) Therefore, the left hand side of (C.27) becomes positive for any strictly positive \( \delta. \) A similar result can be obtained for \( T^{-1}P^N\tilde{\Pi}_\varphi^S \) with only a minor modification of the above arguments.

C.2 Mathematical proofs for the results in Section 4

C.2.1 Preliminary results

Lemma C.3. Suppose that Assumptions M, W and K hold.

(i) For any \( \varphi_0 \leq \varphi \) and \( K > \varphi, \) \( \hat{w}_j \) given in (4.3) satisfies

\[
\|\hat{w}_j - \text{sgn}(\langle \hat{w}_j, w_j \rangle) w_j\|_p \rightarrow_0 0, \quad j = 1, \ldots, K,
\] (C.28)

where \( \{w_j\}_{j=1}^\varphi \) (resp. \( \{w_j\}_{j=\varphi+1}^K \)) is an orthonormal set of \( H^N \) (resp. \( H^S \)).

(ii) Then (C.28) still holds if we replace \( \hat{w}_j \) with \( \overline{w}_j \) (resp. \( \hat{w}_j \)) given in (B.5) (resp. (B.6)) when Model D1 given by (B.1) (resp. Model D2 given by (B.2)) is true.

Proof. We first show (i). The proof is trivial when \( \varphi = 0, \) so we hereafter assume \( \varphi \geq 1. \) Note that

\[
X_{\varphi_0,t}/\sqrt{T} = X_t/\sqrt{T} - \tilde{\alpha}_{\varphi_0}^SN (\tilde{\alpha}_{\varphi_0}^SN)_{\hat{\varphi}_0}^\dagger \tilde{\alpha}_{\varphi_0}^SN \Delta X_t/\sqrt{T}.
\] (C.29)

It can be shown that \( \tilde{\alpha}_{\varphi_0}^SN \) converges to a well defined limit if \( \varphi_0 \leq \varphi. \) Thus the second term on the right hand side of (C.29) is \( O_p(T^{-1/2}) \) and this implies that \( T^{-1}\tilde{C}_\varphi = T^{-1}C + O_p(T^{-1}). \) Moreover, we deduce
from Lemma C.2 that \( \hat{T}_{\varphi_0} = O_p(1) \). From similar arguments to our proof of Theorem 3.2, we have

\[
T^{-1}(\hat{C}_{\varphi_0} - \hat{T}_{\varphi_0} - \hat{Y}^*_0) \to_{\mathcal{L}_N} A_1 \ = \text{fdd} \int W^N(r) \otimes W^N(r).
\]  

We note that the eigenvalues of the limiting operator are distinct and each eigenvalue is associated with one-dimensional eigenspace, almost surely. Then Lemma 4.3 of Bosq (2000) and (C.30) jointly imply that the eigenvectors of \( T^{-1}(\hat{C}_{\varphi} - \hat{T}_{\varphi} - \hat{Y}^*_\varphi) \) converge to those of \( \int W^N(r) \otimes W^N(r) \). Note also that \( \int W^N(r) \otimes W^N(r) \) is a positive definite operator of rank \( \varphi = \dim(\mathcal{H}_N) \) almost surely. This implies that the eigenvectors of \( T^{-1}(\hat{C}_{\varphi} - \hat{T}_{\varphi} - \hat{Y}^*_\varphi) \) corresponding to the largest \( \varphi \) eigenvalues converge to an orthonormal basis of \( \mathcal{H}_N \). We now let \( \tilde{Q}_{\varphi} = \sum_{j=1}^{\varphi} \tilde{w}_j \otimes \tilde{w}_j \) and let \( \tilde{Q}^S = I - \tilde{Q}_{\varphi} \). Then from (C.30) and similar arguments used in the proof of Proposition 3.2 in Chang et al. (2016), it can be shown that \( \|\tilde{Q}_{\varphi} - P_N\|_{\mathcal{L}_N} = O_p(T^{-1}) \) and \( \|\tilde{Q}^S - P^S\|_{\mathcal{L}_N} = O_p(T^{-1}) \). Then from a nearly identical argument used in the proof of Theorem 3.3 in Chang et al. (2016), it can be shown that

\[
\|\tilde{w}_j - \text{sgn}(\langle \tilde{w}_j, w_j \rangle)w_j\| \to_p 0, \quad j = \varphi + 1, \ldots,
\]

where \( w_{\varphi + 1}, \ldots \) are the eigenvectors of \( E[\mathcal{E}_S \otimes \mathcal{E}_S] \). This completes our proof of (i).

We now show (ii). From our proof of Theorem B.1, we may similarly deduce that \( T^{-1}(\hat{C}_{\varphi_0} - \hat{T}_{\varphi_0} - \hat{Y}^*_0) \to_{\mathcal{L}_N} \tilde{A}_1 \ = \text{fdd} \int \tilde{W}^N(r) \otimes \tilde{W}^N(r) \). Then the rest of proof is similar to that for the case without deterministic terms.

\[\square\]

### C.2.2 Proofs of the main results

**Proof of Theorem 4.1.** For notational convenience, we let \( Y_{\varphi_0,t} = \sum_{s=1}^t X_{\varphi_0,t}. \) Then \( \sum_{s=1}^t z_{\varphi_0,s} = (\langle Y_{\varphi_0,t}, \hat{w}_{\varphi_0+1} \rangle, \ldots, \langle Y_{\varphi_0,t}, \hat{w}_0 \rangle)' \). We first consider the case when \( \varphi_0 \geq 1 \). Note that

\[
T^{-1/2}\langle Y_{\varphi_0,t}, \hat{w}_j \rangle = \langle T^{-1/2}Y_{\varphi_0,t}, P^S \hat{\Pi}^S_{\varphi_0} \hat{w}_j \rangle + \langle T^{-3/2}Y_{\varphi_0,t}, TP^N \hat{\Pi}^S_{\varphi_0} \hat{w}_{\varphi_0+1} \rangle
\]

for \( j = \varphi_0 + 1, \ldots, K \). From Theorem 3.1 and Lemma C.3-(i),

\[
P^S \hat{\Pi}^S_{\varphi_0} \to_{\mathcal{L}_N} P^S,
\]

\[
\|\hat{w}_j - \text{sgn}(\langle \hat{w}_j, w_j \rangle)w_j\| \to_p 0, \quad j = \varphi_0 + 1, \ldots, K,
\]

where \( \{w_j\}_{j=\varphi_0+1}^K \) is an orthonormal set included in \( \mathcal{H}_S \). Moreover, we may deduce the following from our proof of Theorem 3.2: for any \( v \in \mathcal{H}_S \) and \( w \in \mathcal{H}_N \),

\[
\left< \left< T^{-1/2} Y_{\varphi_0,t}, v \right>, \left< T^{-3/2} Y_{\varphi_0,t}, w \right> \right> \to_d \left< \left< W^{S,N}(r), v \right>, \left< \int_0^r W^N(s), w \right> \right>.
\]

From (C.32) and (C.33), we conclude that for any \( v \in \mathcal{H} \),

\[
\left< v, P^S \hat{\Pi}^S_{\varphi_0} \hat{w}_j \right> \to_p - \text{sgn}(\langle \hat{w}_j, w_j \rangle) \left< v, w_j \right>.
\]

From Theorem 3.2 and (C.33), we may deduce that for any \( v \in \mathcal{H}_N \) and \( j = \varphi_0 + 1, \ldots, K \),

\[
\left< v, TP^N \hat{\Pi}^S_{\varphi_0} \hat{w}_j \right> \to_d - \text{sgn}(\langle \hat{w}_j, w_j \rangle) \left< v, A w_j \right> = d - \text{sgn}(\langle \hat{w}_j, w_j \rangle) \left< A^* v, w_j \right>.
\]
where \( A^* = \text{Id} \left( \int W^N (r) \otimes dW^{S,N} (r) \right) \left( \int W^N (r) \otimes W^N (r) \right)^\dagger \). Combining (C.31), (C.34), (C.35), (C.36), and the Cramér-Wold device, we obtain the following convergence result:

\[
T^{-1/2} \langle Y_{\varphi_0, t}, \tilde{w}_j \rangle \rightarrow_d \left( W^{S,N} (r) - A^* \int_0^r W^N (s), w_j \right), \text{ jointly for } j = \varphi_0 + 1, \ldots, K, \tag{C.37}
\]

where we used the property that the limiting distribution does not depend on \( \text{sgn}(\langle w_j, w_j \rangle) \). We let

\[
B^N (r) = (\langle W^N (r), w_1 \rangle, \ldots, \langle W^N (r), w_{\varphi_0} \rangle)^T, \quad B^{S,N} (r) = (\langle W^{S,N} (r), w_{\varphi_0 + 1} \rangle, \ldots, \langle W^{S,N} (r), w_K \rangle)^T.
\]

Due to isomorphism between \( n \)-dimensional subspace of \( \mathcal{H} \) and \( n \)-dimensional Euclidean space for any finite integer \( n \), the joint limiting distribution in (C.37) may be understood as

\[
V^S (r) = B^{S,N} (r) - \int dB^{S,N} (s) B(s)^\dagger \left( \int B(s) B(s)^\dagger \right)^{-1} \int_0^r B(s), \tag{C.38}
\]

where we used the distributional identity given by \( B^N (s)^\dagger \left( \int B(s) B^N (s) \right)^{-1} \int_0^r B^N (s) = d B(s)^\dagger \left( \int B(s) B(s)^\dagger \right)^{-1} \int_0^r B(s) \). We next show that LRV \((z_{\varphi_0, t})\) converges to the covariance matrix of \( B^{S,N} \). For time series \( \{V_t\}_{t=1}^T \) and \( \{W_t\}_{t=1}^T \), we let \( \mathcal{G}(V_t, W_t) \) denote the operator given by

\[
\mathcal{G}(V_t, W_t) = \frac{1}{T} \sum_{t=1}^T V_t \otimes W_t + \frac{1}{T} \sum_{s=1}^{T-1} k \left( \frac{s}{T} \right) \sum_{t=s+1}^T \{V_t \otimes W_{t-s} + W_{t-s} \otimes V_t \}. \tag{C.39}
\]

We will show that \( \| \mathcal{G}(\hat{\Pi}^{S}_\varphi X_{\varphi_0, t}, \hat{\Pi}^{S}_\varphi X_{\varphi_0, t}) - \Omega^{S,N} \| = o_p(1) \), which is a stronger result implying that LRV \((z_{\varphi_0, t})\) converges to the covariance matrix of \( B^{S,N} \). Note that \( \mathcal{G}(\hat{\Pi}^{S}_\varphi X_{\varphi_0, t}, \hat{\Pi}^{S}_\varphi X_{\varphi_0, t}) \) is equal to

\[
\mathcal{G}(\hat{\Pi}^{S}_\varphi X_{\varphi_0, t}, \hat{\Pi}^{S}_\varphi X_{\varphi_0, t}) = \mathcal{G}(\hat{\Pi}^{S}_\varphi X_{\varphi_0, t}, \hat{\Pi}^{S}_\varphi \hat{\Omega}^{S,N} (\hat{\Omega}_\varphi^{S,N} |_{\varphi_0}) \hat{P}^{N}_\varphi \Delta X_t) - \mathcal{G}(\hat{\Pi}^{S}_\varphi \hat{\Omega}^{S,N} (\hat{\Omega}_\varphi^{S,N} |_{\varphi_0}), \hat{P}^{N}_\varphi \Delta X_t, \hat{\Pi}^{S}_\varphi X_{\varphi_0, t}) + \mathcal{G}(\hat{\Pi}^{S}_\varphi \hat{\Omega}^{S,N} (\hat{\Omega}_\varphi^{S,N} |_{\varphi_0}), \hat{P}^{N}_\varphi \Delta X_t, \hat{\Pi}^{S}_\varphi \hat{\Omega}^{S,N} (\hat{\Omega}_\varphi^{S,N} |_{\varphi_0}) \hat{P}^{N}_\varphi \Delta X_t).
\]

From Lemma C.2, Theorems 3.1 and B.1, and (C.22), we find that \( \mathcal{G}(\hat{\Pi}^{S}_\varphi X_{\varphi_0, t}, \hat{\Pi}^{S}_\varphi X_{\varphi_0, t}) \rightarrow_{\mathcal{L}_2} \Omega^{S,N} \) and each of the other terms converges to \( \Omega^{S,N} (\hat{\Omega}_\varphi^{S,N}) \Omega^{S,N} \) in the same sense. This implies that \( \mathcal{G}(\hat{\Pi}^{S}_\varphi X_{\varphi_0, t}, \hat{\Pi}^{S}_\varphi X_{\varphi_0, t}) \rightarrow_{\mathcal{L}_2} \Omega^{S,N} \). From this result and (C.38), we find that under \( H_0 \),

\[
LRV(z_{\varphi_0, t})^{-1/2} V^S (r) \rightarrow_d V (r). \tag{C.40}
\]

From (C.37), (C.38), (C.40) and the continuous mapping theorem, we may conclude that \( \hat{Q}(K, \varphi_0) \rightarrow_d \int V (r) / V (r) \), which establishes the desired result under \( H_0 \).

Under \( H_1 \), Lemma C.3-(i) implies that \( \hat{\Pi}^{N}_\varphi \) converges to a projection onto a strict subspace of \( \mathcal{H}^{N} \). Therefore for some \( j \), we have \( \hat{\Pi}^{N}_\varphi \tilde{w}_j \rightarrow_p w \in \mathcal{H}^{N} \). We then deduce from (C.34) that for such \( j \) \( T^{-3/2} \langle Y_{\varphi_0, t}, P^S \hat{\Pi}^{S}_\varphi \tilde{w}_j \rangle \) converges to a functional of \( W^N \), so \( T^{-3/2} \langle Y_{\varphi_0, t}, \tilde{w}_{\varphi_0+1} \rangle, \ldots, \langle Y_{\varphi_0, t}, \tilde{w}_K \rangle \) converges in distribution to a nondegenerate limit. It is also deduced from the unnumbered equation between (A.10) and (A.11) of Phillips (1991) that \( (\kappa h T)^{-1} LRV(z_{\varphi_0, t}) \) converges in probability to a nonzero limit. Combining all these results, \( \hat{Q}(K, \varphi_0) \rightarrow_p \infty \) is deduced.

It remains to prove the case when \( \varphi_0 = 0 \). In this case, the second term in (C.31) is equal to zero, \( A^* = 0 \) in (C.37), \( W^N = 0 \) and \( W^{S,N} \) is understood as \( W^S \). Then the rest of the proof is similar to that for the case when \( \varphi_0 \geq 1 \).
C.3 Mathematical proofs of the results given in Section B

Proof of Theorem B.1. From Lemma 3 of Nielsen et al. (2022) and our proof for the case without deterministic terms, we may deduce that $T^{-1} \mathcal{C} = T^{-1}P_N \mathcal{C}P_N + O_p(T^{-1})$ and $T^{-1} \tilde{\mathcal{C}} = T^{-1}P_N \tilde{\mathcal{C}}P_N + O_p(T^{-1})$, and thus

$$T^{-1} \mathcal{C} \xrightarrow{\text{fdd}} \mathcal{A}_1 = \int \mathbb{W}^N(r) \otimes \mathbb{W}^N(r), \quad T^{-1} \tilde{\mathcal{C}} \xrightarrow{\text{fdd}} \tilde{\mathcal{A}}_1 = \int \tilde{\mathbb{W}}^N(r) \otimes \tilde{\mathbb{W}}^N(r);$$

moreover, $P_N \Phi^S_N = O_p(T^{-1})$, $P_S \Phi^S_N - P_S = O_p(T^{-1})$, $P_N \tilde{\Phi}^S_N = O_p(T^{-1})$, $P_S \tilde{\Phi}^S_N - P_S = O_p(T^{-1})$, and $\tilde{\lambda}_\varphi + k = O_p(1)$ and $\bar{\lambda}_\varphi + k = O_p(1)$ for $k \geq 1$. With a standard modification to allow deterministic terms from our proof of Theorem 3.1, we deduce that (i) $P_N \mathcal{C}P_S \xrightarrow{\text{fdd}} \mathcal{A}_2 = \int dW^S(r) \otimes \mathbb{W}^N(r) + \Gamma_{NS}$ and (ii) $P_N \tilde{\mathcal{C}}P_S \xrightarrow{\text{fdd}} \mathcal{A}_2 = \int dW^S(r) \otimes \tilde{\mathbb{W}}^N(r) + \Gamma_{NS}$. From these results and our proof of Theorem 3.1, the limits of $T(P_N^* - P_N)$ and $T(\tilde{P}_N^* - P_N)$ are obtained as desired.

Moreover, as we did in our proof of Theorem 3.2, it can be shown that (i) $T^{-1}(\mathcal{C}_\varphi - \mathcal{C}_\varphi) = T^{-1}P_N \mathcal{C}P_N + O_p(T^{-1}) \Rightarrow_{\text{fdd}} \mathcal{A}_1 = \int \mathbb{W}^N(r) \otimes \mathbb{W}^N(r)$, (ii) $T^{-1}(\tilde{\mathcal{C}}_\varphi - \tilde{\mathcal{C}}_\varphi) = T^{-1}P_N \tilde{\mathcal{C}}P_N + O_p(T^{-1}) \Rightarrow_{\text{fdd}} \tilde{\mathcal{A}}_1 = \int \tilde{\mathbb{W}}^N(r) \otimes \mathbb{W}^N(r)$, and (iii) $P_N \Pi_\varphi = O_p(T^{-1})$, $P_S \Phi^S_N - P_S = O_p(T^{-1})$, $P_N \tilde{\Phi}^N = O_p(T^{-1})$, $P_S \tilde{\Phi}^N - P_S = O_p(T^{-1})$, and $\bar{\theta}_\varphi + k = O_p(1)$ and $\bar{\theta}_\varphi + k = O_p(1)$ for $k \geq 1$. The rest of proof is similar to that of Theorem 3.2 concerning with the case without deterministic terms; that is, it can be shown that (i) $T(P_N^* - P_N) = (P_S \mathcal{C}_\varphi P_N - P_S \mathcal{C}_\varphi P_N^*)(T^{-1} P_N \mathcal{C}P_N^N) + (T^{-1} P_N \mathcal{C}P_N^N)(P^N \mathcal{C}_\varphi P_S - P^N \mathcal{C}_\varphi P_S) + O_p(T^{-1})$ and (ii) $(T^{-1} P_N \mathcal{C}_\varphi P_N^N) \Rightarrow_{\text{fdd}} \mathcal{A}_1$ and $P_N \mathcal{C}_\varphi P_S = P^N \mathcal{C}_\varphi P_S \Rightarrow_{\text{fdd}} \mathcal{A}_2 = \int dW^S(r) \otimes \mathbb{W}^N(r) + \Gamma_{NS}$. From these, the desired result for $P_N^*$ is obtained. A similar result can be easily obtained for $\tilde{P}_N^*$, and hence the details are omitted.

Proof of Theorem B.2. From Theorem B.1, Lemma C.3-(ii) and a slight modification of the arguments used in our proof of Theorem 4.1, the desired results may be easily deduced.

D Additional simulation results

Table A1: Supplementary results to Table 1

| $\beta_j \sim U[0,0.5], \ h = T^{1/3}$ | $\beta_j \sim U[0,0.5], \ h = T^{2/5}$ |
|-------------------------------------|-------------------------------------|
| $T$ | $\varphi_0=0$ | 1 | 2 | 3 | 4 | $T$ | $\varphi_0=0$ | 1 | 2 | 3 | 4 |
|-------------------------------------|-------------------------------------|
|-------------------------------------|-------------------------------------|
| 250 | 99.1 | 95.4 | 87.7 | 81.9 | 77.7 | 250 | 97.8 | 85.5 | 71.8 | 62.4 | 55.9 |
| 500 | 100.0 | 98.8 | 97.5 | 96.1 | 94.7 | 500 | 99.5 | 95.2 | 89.7 | 85.9 | 82.1 |
| -----------------------------------|-------------------------------------|
| $\beta_j \sim U[0.5,0.7], \ h = T^{1/3}$ | $\beta_j \sim U[0.5,0.7], \ h = T^{2/5}$ |
| $\varphi_0=0$ | 1 | 2 | 3 | 4 | $\varphi_0=0$ | 1 | 2 | 3 | 4 |
| --------------------------------------|--------------------------------------|
| 250 | 99.2 | 95.4 | 87.5 | 82.1 | 77.9 | 250 | 97.9 | 85.5 | 72.0 | 62.5 | 56.2 |
| 500 | 100.0 | 98.8 | 97.5 | 96.2 | 94.8 | 500 | 99.5 | 95.2 | 89.7 | 85.8 | 82.3 |
Table A2: Simulation results for (4.1), $K = \varphi_0 + 2$

|          | $\varphi_0=0$ | 1 | 2 | 3 | 4  |
|----------|---------------|---|---|---|----|
| $T$      |               |   |   |   |    |
| 250      | 4.6           | 4.8| 5.6| 4.9| 4.6|
| 500      | 5.6           | 4.8| 5.2| 4.6| 4.6|

Table A3: Simulation results for (4.1), $K = \varphi_0 + 3$

|          | $\varphi_0=0$ | 1 | 2 | 3 | 4  |
|----------|---------------|---|---|---|----|
| $T$      |               |   |   |   |    |
| 250      | 4.0           | 5.2| 6.0| 4.9| 4.3|
| 500      | 4.0           | 5.4| 5.0| 4.2| 4.6|

|          | $\varphi_0=0$ | 1 | 2 | 3 | 4  |
|----------|---------------|---|---|---|----|
| $T$      |               |   |   |   |    |
| 250      | 13.6          | 13.6| 14.2| 13.0| 15.0|
| 500      | 10.7          | 10.3| 10.0| 10.1| 12.0|

|          | $\varphi_0=0$ | 1 | 2 | 3 | 4  |
|----------|---------------|---|---|---|----|
| $T$      |               |   |   |   |    |
| 250      | 93.4          | 73.3| 58.3| 50.4| 47.1|
| 500      | 97.5          | 88.4| 78.3| 76.4| 72.4|

|          | $\varphi_0=0$ | 1 | 2 | 3 | 4  |
|----------|---------------|---|---|---|----|
| $T$      |               |   |   |   |    |
| 250      | 87.6          | 55.0| 37.6| 31.1| 29.4|
| 500      | 93.2          | 72.3| 56.0| 51.2| 48.2|
### Table A4: Simulation results for (4.1), $K = \varphi_0 + 1$ (trend-adjusted)

|   | $\varphi_0 = 0$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| $T$ | size $(\varphi = \varphi_0)$ | | | | |
| 250 | 4.3 | 5.1 | 4.8 | 5.8 | 5.6 |
| 500 | 4.4 | 4.4 | 5.1 | 5.2 | 4.7 |
| power $(\varphi = \varphi_0 + 1)$ | | | | | |
| 250 | 98.6 | 95.5 | 89.3 | 82.4 | 76.8 |
| 500 | 99.9 | 99.4 | 98.5 | 97.2 | 95.2 |

|   | $\varphi_0 = 0$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| $T$ | size $(\varphi = \varphi_0)$ | | | | |
| 250 | 4.0 | 5.0 | 4.9 | 5.3 | 5.1 |
| 500 | 4.0 | 4.4 | 5.0 | 4.8 | 4.0 |
| power $(\varphi = \varphi_0 + 1)$ | | | | | |
| 250 | 94.7 | 85.3 | 74.4 | 62.1 | 54.1 |
| 500 | 98.7 | 95.7 | 91.6 | 85.9 | 82.8 |

### Notes: For each realization of the DGP of Model D2, $\mu_1 = \sum_{j=1}^4 \hat{\theta}_j p_j / \sqrt{\sum_{j=1}^4 \hat{\theta}_j^2}$, $\{\hat{\theta}_j\}_{j=1}^4$ are independent standard normal random variables, and $p_j$ is $(j-1)$-th order Legendre polynomial defined on $[0, 1]$. $\mu_2$ is generated in a similar manner.

### Table A5: Rejection frequencies (%) of the test for (4.11)

|   | $\varphi = 1$ | 2 | 3 | 4 |
|---|---|---|---|---|
| $T$ | $\gamma / T$ | 250 | 0 | 4.6 | 4.8 | 5.1 | 5.2 |
| 500 | 0.08 | 26.2 | 19.9 | 15.8 | 14.3 |
| 500 | 0.04 | 14.1 | 9.9 | 10.6 | 8.8 |
| 500 | 0.16 | 61.1 | 50.7 | 41.2 | 38.6 |
| 500 | 0.08 | 42.9 | 33.0 | 28.2 | 27.6 |
| 500 | 0.24 | 81.1 | 72.1 | 62. | 59.6 |
| 500 | 0.12 | 67.7 | 56.3 | 49.8 | 47.8 |

|   | $\varphi = 1$ | 2 | 3 | 4 |
|---|---|---|---|---|
| $T$ | $\gamma / T$ | 250 | 0 | 4.3 | 4.4 | 4.2 | 4.0 |
| 500 | 0.08 | 25.8 | 20.6 | 16.8 | 14.6 |
| 500 | 0.04 | 13.6 | 10.7 | 11.4 | 10.0 |
| 500 | 0.16 | 60.0 | 50.0 | 40.4 | 38.4 |
| 500 | 0.08 | 42.1 | 34.6 | 32.0 | 30.6 |
| 500 | 0.24 | 79.0 | 68.0 | 57.5 | 54.3 |
| 500 | 0.12 | 66.7 | 57.6 | 53.4 | 51.1 |

### Notes: For each realization of the DGP of Model D2, $\mu_1 = \sum_{j=1}^4 \hat{\theta}_j p_j / \sqrt{\sum_{j=1}^4 \hat{\theta}_j^2}$, $\{\hat{\theta}_j\}_{j=1}^4$ are independent standard normal random variables, and $p_j$ is $(j-1)$-th order Legendre polynomial defined on $[0, 1]$. $\mu_2$ is generated in a similar manner.
Table A6: Rejection frequencies (%) of the test for (4.12)

|         | (a) $\beta_j \sim U[0, 0.5]$, $h = T^{1/3}$ | (b) $\beta_j \sim U[0, 0.5]$, $h = T^{2/5}$ |
|---------|--------------------------------------------|--------------------------------------------|
| $T$     | $\gamma/T$ | $\varphi = 1$ | 2 | 3 | 4 | $\gamma/T$ | $\varphi = 1$ | 2 | 3 | 4 |
| 250     | 0 | 4.6 | 4.4 | 4.3 | 4.8 | 250 | 0 | 4.4 | 4.4 | 4.2 | 4.3 |
| 500     | 0.08 | 4.8 | 4.8 | 4.8 | 4.6 | 500 | 0.08 | 4.8 | 4.6 | 4.6 | 4.8 |
| 250     | 0.04 | 24.2 | 25.4 | 27.6 | 30.9 | 500 | 0.04 | 24.0 | 24.9 | 26.6 | 29.5 |
| 250     | 0.16 | 14.7 | 14.4 | 16.0 | 18.2 | 500 | 0.16 | 14.6 | 14.2 | 15.6 | 17.8 |
| 250     | 0.08 | 60.7 | 61.3 | 64.2 | 67.0 | 500 | 0.08 | 58.8 | 59.8 | 61.9 | 64.9 |
| 250     | 0.24 | 81.1 | 80.0 | 82.7 | 83.8 | 500 | 0.24 | 79.0 | 77.9 | 79.8 | 80.7 |
| 500     | 0.12 | 66.2 | 68.8 | 71.5 | 72.3 | 500 | 0.12 | 64.9 | 68.0 | 70.5 | 71.1 |

|         | (c) $\beta_j \sim U[0.5, 0.7]$, $h = T^{1/3}$ | (d) $\beta_j \sim U[0.5, 0.7]$, $h = T^{2/5}$ |
|---------|--------------------------------------------|--------------------------------------------|
| $T$     | $\gamma/T$ | $\varphi = 1$ | 2 | 3 | 4 | $\gamma/T$ | $\varphi = 1$ | 2 | 3 | 4 |
| 250     | 0 | 12.8 | 12.9 | 13.4 | 14.0 | 250 | 0 | 9.4 | 9.0 | 10.2 | 9.6 |
| 500     | 0 | 8.7 | 9.8 | 9.4 | 11.0 | 500 | 0 | 6.7 | 7.4 | 7.2 | 7.6 |
| 250     | 0.08 | 38.8 | 41.0 | 40.6 | 43.0 | 250 | 0.04 | 35.0 | 36.2 | 36.0 | 37.6 |
| 500     | 0.04 | 26.2 | 29.2 | 28.6 | 31.1 | 500 | 0.08 | 23.9 | 25.4 | 25.2 | 27.0 |
| 250     | 0.16 | 68.7 | 69.3 | 70.1 | 72.5 | 250 | 0.16 | 64.8 | 65.2 | 65.5 | 67.4 |
| 500     | 0.08 | 53.8 | 59.1 | 57.9 | 59.2 | 500 | 0.08 | 51.3 | 55.4 | 54.6 | 55.1 |
| 250     | 0.24 | 85.0 | 83.6 | 85.2 | 86.3 | 250 | 0.24 | 81.0 | 79.7 | 80.6 | 81.6 |
| 500     | 0.12 | 74.3 | 76.3 | 78.9 | 78.4 | 500 | 0.12 | 71.9 | 74.1 | 76.3 | 75.0 |

E  Empirical example: cross-sectional earning densities

We consider a monthly sequence of U.S. earning densities running from January 1989 to December 2019; this empirical example is similar to that given by Chang et al. (2016, Section 5.1). The raw individual weekly earning data can be obtained from the CPS at https://ipums.org. Since individual earnings in each month are reported in current dollars, they are all adjusted to January 2000 prices using monthly consumer price index data obtained from the Federal Reserve Economic Database at https://fred.stlouisfed.org. Reported weekly earnings are censored from above, with the threshold for censoring switching partway through the sample: the top-coded nominal earning is 1923 before January 1998, and 2885 afterward. In addition, the raw dataset contains many abnormally small nominal earnings, such as $0.01 a week. In the subsequent analysis, weekly earnings below the 3th percentile and above the 97th percentile are excluded to avoid potential effects of those abnormal earnings. Each earning density is estimated from the remaining individual earnings as in Seo and Beare (2019) by applying local likelihood density estimation proposed in Loader (1996); see Section E.1 for more details. One may treat such density-valued observations as random elements of the familiar Hilbert space $L^2$ of square integrable functions to apply our methodology; however, as pointed out by Petersen and Müller (2016), this is not in general advisable since the set of probability densities is not a linear subspace in this case (see also Delicado, 2011; Hron et al., 2016; Kokoszka et al., 2019; Zhang et al., 2021). Therefore, we first embed the estimated densities into a Hilbert space via the transformation approach proposed in Egozcue et al. (2006). Let $S \subset \mathbb{R}$ be the support of the probability density functions and let $|S|$ be the length of $S$; in this empirical example $S$ is set to $[75.36, 1823.94]$ where the left endpoint (resp. the right endpoint) corresponds to the minimal (resp. the maximal) individual earning over the time span. We then define

$$f_t(u) = \psi(X_t) = \log X_t(u) - \frac{1}{|S|} \int \log X_t(u) du, \quad u \in S. \quad (E.1)$$
Figure A1: Estimated earning densities and transformed densities

(a) earning densities
(b) transformed densities

Table A7: Test results for the dimension of $H^N$ - earning densities

| Test statistic | 0   | 1   | 2     |
|---------------|-----|-----|-------|
| $\varphi_0$   | 0.4398** | 0.2734** | 0.0715 |

Notes: $T = 372$. We use * and ** to denote rejection at 5% and 1% significance level, respectively; the approximate critical values for 95% (resp. 99%) are given by 0.15, 0.12, and 0.10 (resp. 0.22, 0.18, and 0.15) sequentially.

Then the transformed series $\{f_t\}_{t \geq 1}$ are regarded as a time series taking values in $L_0^2[0,1]$, the collection of all $x \in L^2[0,1]$ satisfying $\int x(u)du = 0$, which is a Hilbert space. The transformation $\psi$ given above turns out to be an isomorphism between a Hilbert space of probability density functions (called a Bayes Hilbert space) and $L_0^2[0,1]$, and its inverse is given by $\psi^{-1}(f) = \exp(f)/\int \exp(f(u))du$ for any $f \in L_0^2[0,1]$; see Egozcue et al. (2006). For our purpose to illustrate our modified FPCA methodology, we hereafter only concern with the transformed series $\{f_t\}_{t \geq 1}$; the results obtained in this section can be naturally converted into those for the original density-valued time series $\{X_t\}_{T=1}^T$ using the inverse transformation $\psi^{-1}$ under certain mathematical conditions; see Seo and Beare (2019) for a more detailed discussion. Figure A1 reports the estimated earning densities and their transformations as our functional observations.

Given that the functional observations are constructed from weekly earnings which tend to increase over time, it may be reasonable to consider the model with a linear trend as in Section 5. We apply the sequential procedure based on our proposed test to determine $\varphi$. We use 18 quadratic B-spline functions for the representation of the functional observations, and $k(\cdot)$, $h$, and $K$ are set in the same way. Table A7 reports the test results. Our sequential procedure concludes that the dimension of $H^N$ is 2 at 1% significant level. Given this result, $H^N$ is estimated by the span of the first two eigenvectors $\{\tilde{w}_j\}_{j=1}^2$, which are obtained from our modified FPCA. If we let $\tilde{f}_j$ denote the residual from detrending $f_t$, it is then expected that the time series of $\langle \tilde{f}_t, \tilde{w}_j \rangle$ behaves as a unit root process for $j = 1$ and 2. These are shown in Figure A2. Once the dimension of $H^N$ is estimated, then we may examine various hypotheses about cointegration using the tests developed in Section 4.2.

We in this example found an evidence that the time series of earning densities exhibits stochastic trends. As in the example given in Section 5, it may be interesting to investigate if this time series of earning densities is cointegrated with other economic time series, and this can certainly be further explored in the future study.

E.1 Estimation of densities of earning

To obtain densities of earning, we employ the local likelihood method by Loader (1996, 2006). Suppose that $X$ is the density of interest which is supported on $S$, and there are $n$ individual earnings available for estimation of $X$. Given survey responses $a_1, \ldots, a_n$ with design weights $w_1, \ldots, w_n$ such that $\sum_{i=1}^n w_i = n$, the weighted log-likelihood is
Figure A2: Eigenvectors from the modified FPCA & their characteristics - earning densities

![Figure A2](image)

The eigenvectors $\tilde{w}_1$ and $\tilde{w}_2$ are shown along with their time series. The characteristics of the earning densities $\langle \tilde{f}_t, \tilde{w}_1 \rangle$ and $\langle \tilde{f}_t, \tilde{w}_2 \rangle$ are also illustrated.

given by $L(X) = \sum_{i=1}^n w_i \log(X(a_i)) - n \left( \int X(u) du - 1 \right)$. Under some smoothness assumptions, we can consider a localized version of $L(X)$, and $\log X(a)$ can be locally approximated by a polynomial function, as follows.

$$L_p(X)(a) = \sum_{i=1}^n w_i \mathcal{W}\left(\frac{a_i - a}{b}\right) Q(a_i - a; \varpi) - n \int \mathcal{W}\left(\frac{u - a}{b}\right) \exp(Q(u - a; \varpi)) du,$$

(E.2)

where $\mathcal{W}(\cdot)$ is a suitable kernel function, $b$ is a nearest neighborhood bandwidth ensuring that a fixed percent of the data is included in the local neighborhood of $a$, and $Q(u - a; \varpi)$ is the $q$-th order polynomial in $u - a$ with coefficients $\varpi = (\varpi_0, \ldots, \varpi_q)$. Let $(\hat{\varpi}_0, \ldots, \hat{\varpi}_q)$ be the maximizer of (E.2). The local likelihood log-density estimate is then given by $\hat{\log} X(a) = \hat{\varpi}_0$ and therefore $\hat{X}(a) = \exp(\hat{\varpi}_0)$. The procedure is repeated for a fine grid of points, and then $\hat{X}$ may be obtained from an interpolation method described in (Loader, 2006, Chapter 12). Following this procedure, each earning density is obtained on the common support $S = [75.35, 1823.94]$ in our empirical application in Section E: to be more specific, we set $\mathcal{W}(u) = (1 - |u|^3)^3 1\{|u| < 1\}$, $Q(u - a; \varpi) = \varpi_0 + \varpi_1(u - a) + \varpi_2(u - a)^2$, and choose $b$ so that 40\% of the data is contained in the local neighborhood of $a$. The employed kernel function is called the tricube kernel and used in Loader (1996).

References for Supplementary Appendix

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