Reconstructing the Potential of the Generalized Heat Equation

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Received: 7 February 2022 / Accepted: 14 April 2022 / Published online: 10 May 2022
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Abstract
We reconstruct the potential \( q(x) \) for the generalized heat equation of the form
\[
    u_t - b(x)u_{xx} - a(x)u_x - q(x)u = 0
\]
under general conditions, and by means of a Carleman’s estimate.

Keywords  Inverse problem · Carleman’s estimate · Generalized heat operator

1 Introduction

The inverse problems for partial differential equations is of fundamental importance in the theory of ill-posed problems and its applications to mathematics physics. The general idea of these problems is to determine (uniquely or nonuniquely) the unknown coffecients of differential equations. A basic type of equations is the parabolic type of equations. Parabolic equations have been under immense amount of research. It’s almost impossible to mention literature related to this type of equation because it captures the interests of researches from different disciplines in mathematics, physics, and engineering. One of the most important problems related to partial differential equations related to mathematical physics is the inverse problem of determining one of the unknown cofficients in the equation, provided that the input data are known. This problem is called “coefficient identification problem”. If the missing term is the source, then it is called “source identification problem”. The source identification problem has been intensively studied in literature. The motivation of this problem is that the coefficients in partial differential equations usually describe some physical properties for the medium and the system under investigation, so in many cases they are unknown. Equations of parabolic type arise in the study of heat conduction, thermoelasticity, fluid mechanics, and electromagnetic theory. PDE with singular potential arising in combustion theory and quantum mechanics. Cannon and Zachmann [1] were the first to consider the source identification problem for the heat equation. The identification problem

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has been studied by Cannon [2, 3], and Cannon and Lin [4], Isakov, Prilepko, Susuki, and Rundell investigated the restoration problem for parabolic equations with different conditions. Rundell [5, 6] and Isakov [7] obtained uniqueness theorems for determining $q = q(x, u)$ in the semilinear equation

$$u_t - \Delta u + q(x, u) = 0$$

given results of all lateral boundary measurements, where $a_u$ is bounded or nonnegative. Prilepko and Kostin [8, 9] investigated generalized solutions of the problem with integral overdetermination. Susuki [10, 11] and Pierce [12] used the Gelfand and Leviton technique in converting the problem to an inverse Sturm–Liouville problem. Boumenir and Tuan [13] recovered the potential $q(x) \in L^1(0, \pi)$ for the equation

$$u_t - u_{xx} - q(x)u = 0$$

from $u(0, t)$ when $q$ is sufficiently small, and using the conditions

$$u(x, 0) = f(x), \quad u(0, t) = hu_x(0, t), \quad u(\pi, t) = -Hu_x(\pi, t)$$

where $h$ is the convective heat transfer coefficient. Korolev et al. [14] investigated the inverse problem of determining the expected premium $f = \mu(x) - r$ in the equation

$$u_t + \frac{\sigma^2}{2} x^2 u_{xx} + \mu x u_x - ru = 0,$$

which boils down to the Black–Scholes option pricing model. The equation

$$u_t - \Delta u - \frac{\lambda}{|x|^2} u = 0$$

has been investigated by Baras–Goldstein. It is worth noting that we only mentioned the notable research with classic results. For a thorough treatment of the subject, one can consult the monograph of Klibanov [15], and Bukhgeim [16] for a careful analysis of uniqueness theorems for coefficients of some classical PDEs of mathematical physics.

In this paper, and following [16], we investigate the inverse problem of the generalized heat equation

$$u_t - b(x)u_{xx} - a(x)u_x - q(x)u = 0, \quad (x, t) \in \Omega \subset \mathbb{R}^2$$  \hspace{1cm} (1.1)

where $a \in L^\infty(\Omega), b \in C^2(\overline{\Omega}),$ and $q \in C(\overline{\Omega}),$ and the coefficient $q(x)$ is the heat medium property of generating heat source or sink and describes the heat exchange of the material with its surrounding, and $q(x)u$ is a heat source. Note here that if

$$b(x) = 1, \quad a(x) = \frac{2}{x}, \quad q(x) = \frac{-n(n + 1)}{x^2},$$

then Eq. (1.1) reduces to the three-dimensional heat equation
where

\[ F(x, y, z) = u(r, t)P_n(\cos \phi), \]

for the Legendre polynomials \( P_n \). Moreover, if \( b(x) = 1 \), and \( a(x) = q(x) = 0 \), then Eq. (1.1) reduces to the classical heat equation.

The purpose of the work is to restore the potential \( q(x) \) for the generalized heat equation under general conditions and by means of Carleman’s estimates. In Sect. 2, we present some results that will help us in obtaining the main result. In Sect. 3, we state and prove the uniqueness theorem of determining the potential of the BVP.

2 Auxiliary Results

We assume \( \Omega \subset \mathbb{R}^2 \) is an open set of real variables \( x, t \) that is to be determined later. Consider the generalized heat operator

\[ P = \partial_t - b(x)\partial^2 - a(x)\partial - q(x) \tag{2.1} \]

where \( \partial = \partial/\partial x \), and \( q(x) \) is the potential, \( b \) is the diffusion or dispersion coefficient, \( a \) is the convection coefficient, and \( a \) is the potential. We always assume the medium is isotropic and homogeneous. We introduce the weight \( L^2 \)-norm

\[ \|u\|_s^2 = \int_{\Omega} e^{2s\varphi(x, t)} |u(x, t)|^2 dx dt, \tag{2.2} \]

where \( s \geq 0 \), and \( \varphi \in C^\infty(\overline{\Omega}) \). We begin with two lemmas.

Lemma 1 Let \( \varphi \in C^2[-T, T] \) with the weight \( L^2 \)-norm defined in (2.2). Define the integral operator

\[ (\mathcal{L}u)(t) = \int_0^t u(\tau) d\tau. \]

If \(-\varphi'(t)\text{sgn}(t) \geq 0 \) for all \( t \in [-T, T] \), where

\[ \text{sgn}(t) = \begin{cases} 1 : & t \geq 0 \\ -1 : & t < 0, \end{cases} \]

then

\[ \|\mathcal{L}u\|_s^2 \leq T^2\|u\|_s^2. \]
Proof It suffices to prove it for the case $\varphi \in C^2[0, T]$, then we can write $L_2(-T,T)$ as the direct sum

$$L_2(-T,0) \oplus L_2(0, T),$$

and applying the result for each one of the spaces with the substituting $\tilde{t} = -t$ in the first space. Let

$$w = e^{s\varphi} L u.$$

Then $w$ can be written as

$$w = \int_0^t e^{s\varphi(\tau)} u(\tau) d\tau = \int_0^t e^{s\varphi(\tau)} \cdot e^{-s\varphi(\tau)} \cdot e^{s\varphi(\tau)} u(\tau) d\tau.$$ 

Let $v = e^{s\varphi} u$. Since $\varphi$ increases,

$$\|w(t)\| \leq (L|v|(t)).$$

hence

$$\|w\|^2 \leq T^2\|v\|^2.$$ 

The result for $L_2(0, T)$ follows when rewriting the last inequality in terms of $u$.  

The next proposition proves an important type of Carleman’s a priori estimate that will be used in our reconstruction of the potential.

**Proposition 2** Let $\Omega \subset \mathbb{R}^2$ be an open set, and let $q, a \in L_\infty(\Omega)$, $b \in C^3(\overline{\Omega})$, and $\varphi \in C^\infty(\overline{\Omega})$, and suppose $\varphi \in C^\infty(\overline{\Omega})$, such that

$$\varphi_x^2 \geq \epsilon$$

$$2\varphi_{xx}b^2 + \varphi_x b_x b \geq \epsilon$$

for a small $\epsilon > 0$ and for all $(x,t) \in \overline{\Omega}$. If $P$ is the generalized heat operator in (2.1) such that $Pu = 0$, then there exists $s_0 > 0$ and $c > 0$ such that for every $s \geq s_0$ we have

$$s^3\|u\|_s^2 + s\|\partial u\|_s^2 \leq c\|Pu\|_s^2$$

for all $u \in C_0^\infty(\Omega)$.

Proof We set

$$v(x, t) = e^{s\varphi(x,t)} u(x, t).$$

Introduce the new operator
\[ L = e^{wP}e^{-wq}. \]

Since \[ \|Pu\|_s = \|Lv\|_s, \] we can express the estimate in terms of \( L \) which can be represented as the sum of the symmetric operator and the skew-symmetric operator \( L_+ \) and \( L_- \) respectively, in which \( L_\pm = (L \pm L^*)/2 \), where \( L^* \) is the adjoint operator of \( L \) which can be determined from the inequality

\[(Lv_1, v_2) = (v_1, L^*v_2)\]

for any \( v_1, v_2 \in C^\infty_0(\Omega) \). It can be readily seen that

\[ 2Re(L_+v, L_-v) = ([L_+, L_-]v, v), \]

where the commutator \([S, T] = S \cdot T - TS\), from which we deduce that

\[ \|Pu\|^2_s \geq ([L_+, L_-]v, v). \quad (2.6) \]

Since

\[ Lu = e^{wq}(\partial_t - b(x)\partial^2 - a(x)\partial - q(x))e^{-wq}u, \]

doing all computations gives

\[ L = -b\partial^2 + (2sb\varphi_x - a)\partial + \partial_t - s^2b\varphi_x^2 + s(a\varphi_x - \varphi_t + b\varphi_{xx}) - q. \]

So, finding the adjoint operator \( L^* \) taking into account that \((\partial)^* = -\partial\), we obtain

\[ L_+ = -b\partial^2 - s^2b\varphi_x^2 + s(a\varphi_x - \varphi_t) + a_x - q, \]

\[ L_- = \partial_t + (2sb\varphi_x - a)\partial + sb\varphi_{xx}. \]

So

\[ [L_+, L_-] = [-b\partial^2, \partial_t] + [-b\partial^2, 2sb\varphi_x\partial] + [-b\partial^2, -a\partial] + [-b\partial^2, bs\varphi_{xx}] \]

\[ + [as\varphi_x, \partial_t] + [as\varphi_x, 2sb\varphi_x\partial] + [as\varphi_x, -a\partial] + [as\varphi_x, bs\varphi_{xx}] \]

\[ + [-bs^2\varphi_x^2, \partial_t] + [-bs^2\varphi_x^2, 2sb\varphi_x\partial] + [-bs^2\varphi_x^2, -a\partial] + [-bs^2\varphi_x^2, bs\varphi_{xx}] \]

\[ + [-s\varphi_x, \partial_t] + [-s\varphi_x, 2sb\varphi_x\partial] + [-s\varphi_x, -a\partial] + [-s\varphi_x, bs\varphi_{xx}] \]

\[ + [-q, \partial_t] + [-q, 2sb\varphi_x\partial] + [-q, -a\partial] + [-q, bs\varphi_{xx}] \]

\[ + [a_x, \partial_t] + [a_x, 2sb\varphi_x\partial] + [a_x, -a\partial] + [a_x, bs\varphi_{xx}]. \]

Performing all commutators, we obtain the following
where

\[ A_2 = -4sb^2\varphi_{xx} - s(b^2)\varphi_x - ab_x - 2ba_x, \quad A_1 = O(s), \quad A_0 = 2s^3b\varphi_x(b\varphi^2_x) + O(s). \]

From (2.3) and (2.4) we have

\[ b\varphi_x(b\varphi^2_x)_x \geq \epsilon^2 > 0 \]

then for sufficiently large \( s_0 \) we have

\[ (A_0v, v) \geq s^3\epsilon^2\|v\|^2. \]

Moreover, we can use integration by parts to get

\[ (A_2\partial^2 v, v) = -(A_2\partial v, \partial v) - (A_{2x}\partial v, v) \]

Again, from (2.4),

\[ 4\varphi_{xx}b^2 + \varphi_x^2bb_x > \epsilon, \]

and using \( \alpha \)-inequality

\[ 2xy \leq ax^2 + a^{-1}y^2 \]

for \( (A_{2x}\partial v, v) \) and \( (A_1\partial v, v) \) we obtain

\[ ([L_+, L_-]v, v) \geq s^3\rho\|v\|^2 + s\rho\|\partial v\|^2 \]

for some positive \( \rho < \epsilon^2 \) and \( s \geq s_0 \) for a sufficiently large \( s_0 \). Restoring the function \( u \) from \( v \), noting that

\[ \|v\| = \|u\|_s, \]

and taking into account (2.6), the result follows.

\[ \square \]

3 Uniqueness Theorem

Let \( P \) be the generalized heat operator (2.1), and consider the boundary value problem

\[ Pu = 0, \quad x \geq 0, \quad t \in \mathbb{R}. \]  

\[ u(x,0) = f(x), \quad u_x(0, t) = 0, \]  

\[ u(0, t) = g(t) \quad t \in (-T, T). \]
We assume that \( q \in C(\Omega) \) for some \( \Omega \subset \mathbb{R}^2 \) to be determined later. Our goal is to restore the potential \( q(x) \) by presenting a uniqueness theorem for finding \( q(x) \) if other coefficients are given. We assume the problem has two solutions \( u_1, q_1 \) and \( u_2, q_2 \) and let \( u = u_1 - u_2 \) and \( q = q_1 - q_2 \). Then the equation can be written as

\[
\partial_t u - b(\partial^2 u) - a(\partial u) - q_1(x) = u_2(x, t)q(x).
\]

Let

\[
h(x, t) = u_2(x, t) \cdot q(x).
\]

Observe that

\[
\partial_t \left( \frac{h}{u_2} \right) = 0,
\]

hence

\[
u_2(x, t)h_t - (\partial_t u_2) h = 0.
\]

If \( u_2(x, t) \neq 0 \), then

\[
h_t - \frac{(\partial_t u_2) h}{u_2} = 0.
\]

Define

\[
Q = \partial_t - b(x, t),
\]

where

\[
b(x, t) = \frac{(\partial_t u_2) h}{u_2}.
\]

Then the equation takes the form

\[
QPu = Qh = 0.
\]

Hence,

\[
PQu = [P, Q]u
\]

where \([P, Q]\) is the commutator of \( P \) and \( Q \). By letting \( v = Qu \) our problem takes the following Cauchy form

\[
Pv = [P, Q]u = [P, Q]Q^{-1}v,
\]

\[
v(0, t) = v_1(0, t) = 0.
\]
\[ K(x, t, \tau) = \frac{u_2(x, t)}{u_2(x, \tau)}, \]

it follows that

\[ u(x, t) = Q^{-1}v = \int_0^t K(x, t, \tau)v(x, \tau)d\tau. \] (3.6)

Writing (3.4) in terms of \( v \) gives

\[ v_t - b(x)v_{xx} - a(x)v_x = q_1(x)v + [P, Q]Q^{-1}v. \] (3.7)

Calculating the commutator of \( P \) and \( Q \) gives

\[ [P, Q] = [\partial_t - b\partial^2 - a\partial - q_1, \partial_t - b] = C_1\partial + C_0, \] (3.8)

where

\[ C_1 = (b^2)_x, \quad C_0 = bb_{xx} + ab_x - b_t. \]

Substituting (3.8) in (3.7), the right hand side of the equation becomes

\[ q_1(x)v + \int_0^t (C_1K_x + C_0)v + C_1Kv_x)\] (3.9)

So, (3.7) implies

\[ |v_t - b(x)v_{xx} - a(x)v_x| \leq c(|v(x, t)| + |\mathcal{L}(v)(x, t)| + |\mathcal{L}(v_x)(x, t)|) \] (3.10)

Let us assume the weight function decreases with increasing distance from a point of the space to the support; i.e. \(-\varphi'(t) \geq 0\). If we assume \( f(x) \neq 0 \), then continuity of \( u_2(x, t) \) implies that \( u_2(x, t) \neq 0 \) in a small neighbourhood of a segment \([0, \sigma]\) at which \( f(x) \neq 0 \). Choose

\[ \varphi(x, t) = \exp(\lambda\psi(x, t)) \]

where

\[ \psi(x, t) = (x/r)^2 - (t/\rho)^2 \]

with \( \rho \) sufficiently small, and set

\[ \Omega = \{x, t : \varphi(x, t) > 0, \ x > 0\}. \]

Then we have

\[ u_2(x, t) \neq 0 \ \forall(x, t) \in \Omega. \]

Let \( \Omega_e = \Omega \cap \{\varphi > 0\} \), and define a smooth function
Choose \( w = \chi v \). Then

\[
    w(0, t) = w_x(0, t) = 0.
\]

Now we substitute \( w \) into the Carleman’s estimate (2.5), we get

\[
    s \int_{\Omega} e^{2s\phi}(|v(x, t)|^2 + |v_x(x, t)|^2)dxdt \leq c \int_{\Omega} e^{2s\phi} P(v)\chi^2 dxdt + \int_{\Omega \setminus \Omega} e^{2s\phi} |P(\chi v)|^2 dxdt
\]

Using (3.10) and Cauchy–Schwarz inequality, then using Lemma 1, we obtain

\[
    \int_{\Omega} e^{2s\phi} P(v)^2 dxdt \leq c \int_{\Omega} e^{2s\phi} (|v|^2 + |v_x|^2) dxdt.
\]

From (3.11) and (3.12) we obtain

\[
    (s - c^2) \int_{\Omega} e^{2s\phi} (|v(x, t)|^2 + |v_x(x, t)|^2)dxdt \leq c \int_{\Omega \setminus \Omega} e^{2s\phi} |P(\chi v)|^2 dxdt.
\]

Choose \( s_0 \) such that \( 1 - \frac{c}{s_0} \geq 1/2 \), then

\[
    s - c^2 \geq s/2.
\]

Moreover, from the definition of \( \phi \) we have

\[
    \min_{\Omega} \phi = \max_{\Omega \setminus \Omega} \phi = \epsilon.
\]

Hence (3.13) becomes

\[
    s \int_{\Omega} (|v|^2 + |v_x|^2)dxdt \leq c \int_{\Omega \setminus \Omega} |P(\chi v)|^2 dxdt
\]

Dividing by \( s \) then taking \( s \to \infty \) gives

\[
    \int_{\Omega} (|v|^2 + |v_x|^2)dxdt = 0
\]

Letting \( \epsilon \to 0 \), we conclude that \( v = 0 \) in \( \Omega \), which implies that \( u = 0 \) in \( \Omega \). Since

\[
    Pu = 0 \Rightarrow h = u_2 \cdot q(x)
\]

and \( u_2 \neq 0 \), then \( q(x) = 0 \), therefore
for all \( x \geq 0 \). Thus, we have the following result.

**Theorem 3** Let \( u(x, t), \ x \geq 0 \), be a solution to the boundary value problem (3.1–3.3), where \( a \in L^\infty((0, \infty)) \), \( b \in C^3([0, \infty)) \), and \( q \in C([0, \infty)) \). If \( f(x) \neq 0 \) for all \( x \geq 0 \), then the potential \( q(x) \) is uniquely determined by \( u(0, t) = g(t), t \in (-T, T) \).

### 4 Concluding Remarks

The paper proposed an inverse problem of recovering the potential coefficient of the generalized heat equation given the information about the solution and an initial observation. We established the uniqueness of the solution of the inverse problem of recovering the potential coefficient if the solution \( u(x, t) \) and an initial observation \( u(x, 0) \) are given. Knowing that the solution of (3.1–3.3) represents temperature and the potential represents the heat medium property of generating heat source or sink, the result implies that the potential can be uniquely recovered from the temperature with an initial observation, so that one potential leads to the known solution. The paper also shows that the problem can be reduced to a Cauchy problem for integro-differential equation, whose solution can be obtained numerically. This is especially interesting and important in applied mathematics as there are several useful techniques and methods that we can apply to obtain such numerical solutions, and many physical models are associated with evolutionary integrodifferential equations [17].

Many physical and engineering processes are described by parabolic and hyperbolic equations. Due to their importance and various application to applied sciences, coefficient identification inverse problems for parabolic and hyperbolic differential equations of mathematical physics start to play a major role in several areas of physics and engineering, which explains why they are receiving increasing attention in literature. It has been found recently that many problems in science and engineering, applied to fluid mechanics, thermodynamics, electromagnetic theory, beam theory of elasticity, scattering theory, radiology, acoustics, geophysics, seismology, and much more topics and areas are closely related to some type of an inverse problem [18].

**Acknowledgements** The author would like to acknowledge the support of Prince Sultan University, Saudi Arabia for paying the Article Processing Charges (APC) of this publication. The author would like to thank Prince Sultan University for their support.

**Funding** Not applicable.

**Data Availability Statement** Not applicable.

**Declarations**

**Conflict of interest** No potential competing interest was reported by the author.
Ethics Approval and Consent to Participate  Hereby I confirm that article is not under consideration in other journals.

Consent for Publication  Not Applicable.

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