Reciprocal Complementary Distance Energy of Complement of Line Graphs of Regular Graphs
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Abstract
The reciprocal complementary distance (RCD) matrix of a graph \( G \) is defined as \( \text{RCD}(G) = [r_{ij}] \), where
\[
r_{ij} = \frac{1}{1+D-d_{ij}} \quad \text{if} \quad i \neq j \quad \text{and} \quad r_{ij} = 0, \quad \text{otherwise},
\]
where \( D \) is the diameter of \( G \) and \( d_{ij} \) is the distance between the vertices \( v_i \) and \( v_j \) in \( G \). The RCD-energy of \( G \) is defined as the sum of the absolute values of the eigenvalues of the RCD-matrix. Two graphs are said to be RCD-equienergetic if they have same RCD-energy. In this paper, the RCD-energy of the complement of line graphs of certain regular graphs in terms of the order and degree is obtained and as a consequence, pairs of RCD-equienergetic graphs of same order and having different RCD-eigenvalues are constructed.

Keywords: Reciprocal complementary distance (RCD) eigenvalues; RCD-energy of a graph; RCD-equienergetic graphs.
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1. Introduction
Let \( G \) be a simple, undirected, connected graph with \( n \) vertices and \( m \) edges. Let the vertex set of \( G \) be \( V(G) = \{v_1, v_2, \ldots, v_n\} \). The adjacency matrix of a graph \( G \) is the square matrix \( A(G) = [a_{ij}] \) of order \( n \), in which \( a_{ij} = 1 \) if \( v_i \) is adjacent to \( v_j \) and \( a_{ij} = 0 \), otherwise. The eigenvalues of \( A(G) \) are the adjacency eigenvalues of \( G \), and they are labeled as \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \). Two non-isomorphic graphs are said to be adjacency cospectral or simply cospectral if they have same adjacency eigenvalues [3].

The distance between the vertices \( v_i \) and \( v_j \), denoted by \( d_{ij} \), is the length of the shortest path joining \( v_i \) and \( v_j \). The diameter of a graph \( G \), denoted by \( \text{diam}(G) \), is the maximum distance between any pair of vertices of \( G \). A graph \( G \) is said to be \( r \)-regular if all of its vertices have same degree equal to \( r \). The complement of a graph \( G \), denoted by \( \overline{G} \), is a graph with vertex set \( V(G) \) and two vertices in \( \overline{G} \) are adjacent if and only if they are not adjacent in \( G \). The line graph of \( G \), denoted by \( L(G) \) is the graph whose vertices corresponds to the edges of \( G \) and two vertices of \( L(G) \) are adjacent if and only if the corresponding edges are adjacent in \( G \). For \( k = 1, 2, \ldots \), the \( k \)-th iterated line graph of \( G \) is defined as \( L^k(G) = L(L^{k-1}(G)) \), where \( L^0(G) = G \) and \( L^1(G) = L(G) \) [5].

The line graph of a regular graph \( G \) of order \( n_0 \) and of degree \( r_0 \) is a regular graph of order \( n_1 = (n_0r_0)/2 \) and
of degree \( r_1 = 2r_0 - 2 \). Consequently the order and degree of \( L^k(G) \) are \([1, 2]\)
\[
n_k = \frac{r_{k-1}n_{k-1}}{2} \tag{1.1}
\]
and
\[
r_k = 2r_{k-1} - 2, \tag{1.2}
\]
where \( n_i \) and \( r_i \) stands for order and degree of \( L^i(G) \), \( i = 0, 1, \ldots \).

Therefore
\[
r_k = 2^kr_0 - 2^{k+1} + 2 \tag{1.3}
\]
and
\[
n_k = \frac{n_0}{2^k} \prod_{i=0}^{k-1} r_i = \frac{n_0}{2^k} \prod_{i=0}^{k-1} (2^i r_0 - 2^{i+1} + 2). \tag{1.4}
\]

The reciprocal complementary distance matrix or \( RCD \)-matrix \([6, 8]\) of a graph \( G \) is an \( n \times n \) matrix \( RCD(G) = [r_{ij}] \), where
\[
r_{ij} = \left\{ \begin{array}{ll}
\frac{1}{1 + D - d_{ij}} & \text{if } i \neq j \\
0 & \text{if } i = j,
\end{array} \right.
\]
where \( D \) is the diameter of \( G \) and \( d_{ij} \) is the distance between the vertices \( v_i \) and \( v_j \) in \( G \).

The reciprocal complementary distance matrix is an important source of structural descriptors in the quantitative structure property relationship (QSPR) model in chemistry \([6, 8]\).

The eigenvalues of \( RCD(G) \), labeled as \( \mu_1 \geq \mu_2 \geq \cdots \geq \mu_n \), are said to be the reciprocal complementary distance eigenvalues or \( RCD \)-eigenvalues of \( G \) and their collection is called \( RCD \)-spectra of \( G \). Two non-isomorphic graphs are said to be \( RCD \)-cospectral if they have same \( RCD \)-spectra.

The reciprocal complementary distance energy or \( RCD \)-energy of a graph \( G \), denoted by \( RCDE(G) \), is defined as \([11]\)
\[
RCDE(G) = \sum_{i=1}^{n} |\mu_i|. \tag{1.5}
\]

The Eq. (1.5) is defined in full analogy with the ordinary graph energy \( E(G) \), defined as \([4]\)
\[
E(G) = \sum_{i=1}^{n} |\lambda_i|,
\]
where \( \lambda_1, \lambda_2, \ldots, \lambda_n \) are the adjacency eigenvalues of \( G \). The ordinary graph energy has a relation with the total \( \pi \)-electron energy of a molecule in quantum chemistry \([9]\).

Two connected graphs \( G_1 \) and \( G_2 \) are said to be reciprocal complementary distance equienergetic or \( RCD \)-equienergetic if \( RCDE(G_1) = RCDE(G_2) \). In \([10, 11]\) \( RCD \)-equienergetic graphs are obtained. In this paper we obtain the \( RCD \)-energy of the complement of iterated line graphs of certain regular graphs and thus give another construction of \( RCD \)-equienergetic graphs having different \( RCD \)-spectra.

We need following results.

**Theorem 1.1.** \([3]\) If \( G \) is an \( r \)-regular graph, then its maximum adjacency eigenvalue is equal to \( r \).

**Theorem 1.2.** \([13]\) If \( \lambda_1, \lambda_2, \ldots, \lambda_n \) are the adjacency eigenvalues of a regular graph \( G \) of order \( n \) and of degree \( r \), then the adjacency eigenvalues of \( L(G) \) are
\[
\lambda_i + r - 2, \quad i = 1, 2, \ldots, n, \quad \text{and} \quad -2, \quad n(r - 2)/2 \quad \text{times}.
\]

**Theorem 1.3.** \([12]\) Let \( G \) be an \( r \)-regular graph of order \( n \). If \( r, \lambda_2, \ldots, \lambda_n \) are the adjacency eigenvalues of \( G \), then the adjacency eigenvalues of \( G \) are \( n - r - 1 \) and \( -\lambda_i - 1, \quad i = 2, 3, \ldots, n \).

**Theorem 1.4.** \([11]\) Let \( G \) be an \( r \)-regular graph on \( n \) vertices and \( \text{diam}(G) = 2 \). If \( r, \lambda_2, \ldots, \lambda_n \) are the adjacency eigenvalues of \( G \), then its \( RCD \)-eigenvalues are \( n - 1 - \frac{r}{2} \) and \( -1 - \frac{\lambda_i}{2}, \quad i = 2, 3, \ldots, n \).

**Lemma 1.1.** \([7]\) Let \( G \) be an \( r \)-regular graph on \( n \) vertices. If \( r \leq \frac{n-1}{2} \) then \( \text{diam} \left( L^k(G) \right) = 2, \quad k \geq 1 \).
2. **RCD-Energy**

**Theorem 2.1.** Let $G$ be a regular graph of order $n$ and degree $r \geq 4$. If $r \leq \frac{n-1}{2}$, then

$$RCDE\left(L^2(G)\right) = \frac{3nr}{2}(r-2).$$

*Proof.* Let the adjacency eigenvalues of $G$ be $r, \lambda_2, \ldots, \lambda_n$. By Theorem 1.2, the adjacency eigenvalues of $L(G)$ are

$$\begin{align*}
2r - 2, & \quad \text{and} \\
\lambda_i + r - 2, & \quad i = 2, 3, \ldots, n, \quad \text{and} \\
-2, & \quad n(r-2)/2 \times \text{times.}
\end{align*}$$

$(2.1)$

Since $L(G)$ is a regular graph of order $nr/2$ and of degree $2r - 2$, by Theorem 1.2 and Eq. (2.1), the adjacency eigenvalues of $L^2(G)$ are

$$\begin{align*}
4r - 6, & \quad \text{and} \\
\lambda_i + 3r - 6, & \quad i = 2, 3, \ldots, n, \quad \text{and} \\
2r - 6, & \quad n(r-2)/2, \quad \text{and} \\
-2, & \quad nr(r-2)/2 \times \text{times.}
\end{align*}$$

$(2.2)$

From Theorem 1.3 and Eq. (2.2), the adjacency eigenvalues of $L^2(G)$ are

$$\begin{align*}
(nr(r-1)/2 - 4r + 5, & \quad \text{and} \\
-\lambda_i - 3r + 5, & \quad i = 2, 3, \ldots, n, \quad \text{and} \\
-2r + 5, & \quad n(r-2)/2, \quad \text{and} \\
1, & \quad nr(r-2)/2 \times \text{times.}
\end{align*}$$

$(2.3)$

The graph $L^2(G)$ is a regular graph of order $nr(r-1)/2$ and of degree $(nr(r-1)/2 - 4r + 5$. Since $r \leq \frac{n-1}{2}$, by Lemma 1.1, $diam\left(L^2(G)\right) = 2$. Therefore by Theorem 1.4 and Eq. (2.3), the RCD-eigenvalues of $L^2(G)$ are

$$\begin{align*}
(nr^2 - nr + 8r - 14)/4, & \quad \text{and} \\
(\lambda_i + 3r - 7)/2, & \quad i = 2, 3, \ldots, n, \quad \text{and} \\
(2r - 7)/2, & \quad n(r-2)/2, \quad \text{and} \\
-(3/2), & \quad nr(r-2)/2 \times \text{times.}
\end{align*}$$

$(2.4)$

All adjacency eigenvalues of a regular graph of degree $r$ satisfy the condition $-r \leq \lambda_i \leq r$ [3]. If $r \geq 4$, then $(nr^2 - nr + 8r - 14) \geq 0$, $\lambda_i + 3r - 7 \geq 0$ and $2r - 7 \geq 0$. Therefore by Eq. (2.4),

$$RCDE\left(L^2(G)\right) = \frac{nr^2 - nr + 8r - 14}{4} + \sum_{i=2}^{n} \frac{(\lambda_i + 3r - 7)}{2}
+ \left(\frac{2r - 7}{2}\right) \frac{n(r-2)}{2} + \frac{3}{2} \left|\frac{nr(r-2)}{2}\right|
= \frac{3nr}{2}(r-2) \quad \text{since} \quad \sum_{i=2}^{n} \lambda_i = -r.$$  \[\square\]
Corollary 2.1. Let \( G \) be a regular graph of order \( n_0 \) and of degree \( r_0 \geq 4 \). Let \( n_k \) and \( r_k \) be the order and degree respectively of the \( k \)-th iterated line graph \( L^k(G) \), \( k \geq 2 \). If \( r_0 \leq \frac{n_0 - 1}{2} \), then

\[
\text{RCDE} \left( L^k(G) \right) = \frac{3n_k - 2r_k - 2}{2} (r_k - 2).
\]

Proof. If \( r_0 \leq \frac{n_0 - 1}{2} \), then by Eqs. (1.1) and (1.2), we have

\[
r_1 = 2r_0 - 2 \leq n_0 - 3 \leq \frac{1}{2} \left( \frac{n_0 r_0}{2} - 1 \right) = \frac{n_1 - 1}{2}.
\]

Hence

\[
r_{k-2} \leq \frac{n_{k-2} - 1}{2}.
\]

Therefore by Theorem 2.1,

\[
\text{RCDE} \left( L^k(G) \right) = \text{RCDE} \left( L^{k-2}(G) \right) = \frac{3n_k - 2r_k - 2}{2} (r_k - 2).
\]

\( \square \)

Corollary 2.2. Let \( G \) be a regular graph of order \( n_0 \) and of degree \( r_0 \geq 4 \). Let \( n_k \) and \( r_k \) be the order and degree respectively of the \( k \)-th iterated line graph \( L^k(G) \), \( k \geq 2 \). If \( r_0 \leq \frac{n_0 - 1}{2} \), then

\[
\text{RCDE} \left( L^k(G) \right) = \frac{3n_k - 2r_k - 2}{2} \prod_{i=0}^{k-2} (2^i r_0 - 2^{i+1} + 2).
\]

Theorem 2.2. Let \( G \) be a cubic graph of order \( n \geq 7 \). Then

\[
\text{RCDE} \left( L(G) \right) = \frac{3n + \text{E}(G)}{2}.
\]

Proof. Let the adjacency eigenvalues of \( G \) be \( 3, \lambda_2, \ldots, \lambda_n \). From Theorem 1.2, the adjacency eigenvalues of \( L(G) \) are

\[
\left\{ \begin{array}{c}
4, \quad \text{and} \\
\lambda_i + 1, \quad i = 2, 3, \ldots, n, \quad \text{and} \\
-2, \quad n/2 \times 
\end{array} \right\
\]

(2.5)

From Theorem 1.3 and the Eq. (2.5), the adjacency eigenvalues of \( L(G) \) are

\[
\left\{ \begin{array}{c}
(3n/2) - 5, \quad \text{and} \\
-\lambda_i - 2, \quad i = 2, 3, \ldots, n, \quad \text{and} \\
1, \quad n/2 \times 
\end{array} \right\
\]

(2.6)

Since \( G \) is a cubic graph on \( n \geq 7 \) vertices, \( 3 \leq \frac{n-1}{2} \). Therefore by Lemma 1.1, \( \text{diam} \left( L(G) \right) = 2 \).

Therefore by Theorem 1.4 and Eq. (2.6), the \( RCD \)-eigenvalues of \( L(G) \) are

\[
\left\{ \begin{array}{c}
(3n + 6)/4, \quad \text{and} \\
\frac{\lambda_i}{2}, \quad i = 2, 3, \ldots, n, \quad \text{and} \\
(-3/2), \quad n/2 \times 
\end{array} \right\
\]

(2.7)

Therefore

\[
\text{RCDE} \left( L(G) \right) = \left\lfloor \frac{3n + 6}{4} \right\rfloor + \sum_{i=2}^{n} \left\lfloor \frac{\lambda_i}{2} \right\rfloor + \left\lfloor \frac{3n}{2} \right\rfloor = \frac{3n}{4} + \frac{3}{2} + \frac{1}{2} (\text{E}(G) - 3) + \frac{3n}{4} = \frac{3n + \text{E}(G)}{2}.
\]

\( \square \)
3. RCD-Equienergetic graphs

If \( G_1 \) and \( G_2 \) are two regular graphs of same order and of same degree, then by Eq. (1.3) and (1.4) for any \( k \geq 1 \), \( L^k(G_1) \) and \( L^k(G_2) \) are also regular graphs of the same order and have the same number of edges. Hence \( L^k(G_1) \) and \( L^k(G_2) \) are regular graphs of the same order and have the same number of edges.

**Proposition 3.1.** Let \( G_1 \) and \( G_2 \) be regular graphs of same order \( n \) and of same degree \( r \). If \( r \leq \frac{n-1}{2} \), then \( k \geq 1 \), \( L^k(G_1) \) and \( L^k(G_2) \) are RCD-cospectral if and only if \( G_1 \) and \( G_2 \) are cospectral.

**Proof.** If \( G_1 \) and \( G_2 \) are regular cospectral graphs then applying Theorem 1.2 repeatedly we get that \( L^k(G_1) \) and \( L^k(G_2) \) are cospectral for \( k \geq 1 \). Therefore by Theorem 1.3, \( L^k(G_1) \) and \( L^k(G_2) \) are cospectral. Since \( r \leq \frac{n-1}{2} \), by Lemma 1.1, \( \text{diam}(L^k(G_1)) = 2 \) and \( \text{diam}(L^k(G_2)) = 2 \). Therefore by Theorem 1.4, \( L^k(G_1) \) and \( L^k(G_2) \) are RCD-cospectral.

Conversely, let \( L^k(G_1) \) and \( L^k(G_2) \) are RCD-cospectral. Suppose \( G_1 \) and \( G_2 \) are not cospectral. Then by Theorem 1.2, \( L^k(G_1) \) and \( L^k(G_2) \) are not cospectral for \( k \geq 1 \). Hence by Theorem 1.3, \( L^k(G_1) \) and \( L^k(G_2) \) are not cospectral. Now, by using Theorem 1.4, \( L^k(G_1) \) and \( L^k(G_2) \) are not RCD-cospectral, which is a contradiction. Hence \( G_1 \) and \( G_2 \) are cospectral. \( \square \)

**Theorem 3.1.** Let \( G_1 \) and \( G_2 \) be regular, not cospectral graphs of same order \( n \) and of same degree \( r \). If \( r \leq \frac{n-1}{2} \), then \( L^2(G_1) \) and \( L^2(G_2) \) form a pair of not RCD-cospectral, RCD-equierenergetic graphs of equal order and of equal number of edges.

**Proof.** If \( G_1 \) and \( G_2 \) are regular, not cospectral graphs of same order \( n \), same degree \( r \geq 4 \), and \( r \leq \frac{n-1}{2} \), then by Proposition 3.1, \( L^2(G_1) \) and \( L^2(G_2) \) form a pair of not RCD-cospectral graphs of same order and same size. And by Theorem 2.1, \( RCD(E(G_1)) = \frac{3n}{2}r - 2 = RCD(E(G_2)) \), which implies that \( L^2(G_1) \) and \( L^2(G_2) \) form a pair RCD-equierenergetic graphs. \( \square \)

**Theorem 3.2.** Let \( G_1 \) and \( G_2 \) be regular, not cospectral graphs of same order \( n \) and of same degree \( r \). If \( r \leq \frac{n-1}{2} \), then for \( k \geq 2 \), \( L^k(G_1) \) and \( L^k(G_2) \) form a pair of not RCD-cospectral, RCD-equierenergetic graphs of equal order and of equal number of edges.

**Proof.** Since \( L^k(G_1) = L^2(L^{k-2}(G_1)) \) and \( L^k(G_2) = L^2(L^{k-2}(G_2)) \), the result follows from Theorem 3.1. \( \square \)

**Proposition 3.2.** Let \( G_1 \) and \( G_2 \) be cubic graphs of order \( n \geq 7 \), such that \( E(G_1) = E(G_2) \). Then

\[
RCD(E(G_1)) = RCD(E(G_2)).
\]

**Proof.** The result follows from Theorem 2.2 as \( E(G_1) = E(G_2) \). \( \square \)

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**References**

[1] Buckley, F.: *Iterated line graphs*. Congr. Numer. 33, 390–394 (1981).

[2] Buckley, F.: *The size of iterated line graphs*. Graph Theory Notes of New York. 25, 33–36 (1993).

[3] Cvetković, D., Rowlinson, P., Simić, S.: Introduction to the Theory of Graph Spectra. Cambridge University Press. Cambridge (2010).

[4] Gutman, I.: *The energy of a graph*. Ber. Math. Stat. Sekt. Forschungsz. Graz. 103, 1–22 (1978).

[5] Harary, F.: Graph Theory. Addison-Wesley Publishing Co., Reading (1969).
[6] Ivanciuc, O., Ivanciuc, T., Balaban, A. T.: *The complementary distance matrix, a new molecular graph metric*. ACH-Models Chem. **137**, 57–82 (2000).

[7] Indulal, G.: *D-spectrum and D-energy of complements of iterated line graphs of regular graphs*. J. Alg. Stru. Appl. **4**, 51–56 (2017). https://doi.org/10.29252/asta.4.1.51

[8] Jenežić, D., Miličević, A., Nikolić, S., Trinajstić, N.: *Graph Theoretical Matrices in Chemistry*. University of Kragujevac. Kragujevac (2007). https://doi.org/10.1021/ci700278s

[9] Li, X., Shi, Y., Gutman, I.: *Graph Energy*. Springer. New York (2012). https://doi.org/10.1007/978-1-4614-4220-2

[10] Ramane, H. S., Gudodagi, G. A.: *Reciprocal complementary equienergetic graphs*. Asian-European J. Math. **9**, ID: 1650084, pages 15 (2016). https://doi.org/10.1142/S1793557116500844

[11] Ramane, H. S., Yalnaik, A. S.: *Reciprocal complementary distance spectra and reciprocal complementary distance energy of line graphs of regular graphs*. El. J. Graph Theory Appl. **3**, 228–236 (2015). http://dx.doi.org/10.5614/ejgta.2015.3.2.10

[12] Sachs, H.: *Über selbstkomplementare Graphen*. Publ. Math. Debrecen. **9**, 270–288 (1962).

[13] Sachs, H.: *Über Teiler, Faktoren und charakteristische Polynome von Graphen, Teil II*. Wiss. Z. TH Ilmenau. **13**, 405–412 (1967).

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