Probability, Markov Chain, and their applications

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Abstract. Probability is a branch of mathematics that focuses on the possibility of an event or the possibility of a position. Probability is significant to the field of politics since it could help people analyze and predict election results. Specifically, the Markov chain plays an important role in this because it focuses on transition of states. With election, reelection, retirement, and other elements involved, the Markov chain is helpful for predicting election results.

1. Introduction
Probability is a measure of the likelihood of an event to occur. The probability of an event is in the range of number 0 and 1. The higher the probability of an event is, the more likely the event will happen.

During the 17th century, European aristocrats were enjoying a certain gambling game of dice. Since he constantly lost money, he turned to Blaise Pascal, a renowned mathematician, for help, in order to resolve the problem of losing money mathematically. Pascal and another mathematician, Pierre de Fermat studied the problem together and contributed to the emergence of the academic field of probability [1].

In the field of probability, many tools are used to calculate probability and make predictions; specifically, the theory of Markov chain is utilized to predict certain electoral outcomes. In order to make predictions about a politician’s career, a Markov chain and related tools must be created based on the scenario and probabilities. Markov chain is a mathematical system that experiences transitions from one state to another according to certain probabilistic rules. The defining characteristic of a Markov chain is that no matter how the process arrived at its present state, the possible future states are fixed [2].

A Markov chain can demonstrate scenarios involving probabilities more clearly since it shows transition between states. By using the theory of Markov chain, more and more results of real-life elections can be predicted [3].

2. Characteristics of Markov chain
In the early 20th century, Andrei Andreyevich Markov, a Russian Mathematician founded a new branch of probability theory by applying mathematics to poetry. When studying the text of Alexander Pushkin’s novel in verse Eugene Onegin, Markov spent hours sifting through patterns of vowels and consonants. On January 23, 1913, he sent his findings to the Imperial Academy of Sciences in St. Petersburg. Although his analysis did not change the comprehension or appreciation of Pushkin’s poem, the technique he developed-now known as Markov chain—extended the theory of probability in a new direction. [4]

The Markov chain is a type of Markov process, a process for which predictions can be made regarding future outcomes based solely on its present state and—most importantly—such predictions are just as good as the ones that could be made knowing the process's full history. In a Markov chain, the probability of each event solely depends on the state from the previous event. [5-6]
Markov chain can be used to solve probabilistic problems. In a Markov chain, probabilities of events can be calculated as long as events are viewed as states that can transition to other states or themselves.

3. Real-life application of Markov chain

3.1. Weather prediction
Markov chains can be used to predict the weather. Assume that the probability of a sunny day followed by a rainy day is $\frac{1}{10}$, the probability of a sunny day followed by a sunny day is $\frac{9}{10}$, the probability of a rainy day followed by a sunny day is $\frac{2}{5}$, and the probability of a rainy day followed by a rainy day is $\frac{3}{5}$. This scenario can be demonstrated in a transition diagram:

Figure 1. Transition diagram of weather prediction

In order to calculate the probability of the next state, the transition matrix is needed:

$$
P = \begin{pmatrix}
S & R \\
\frac{9}{10} & \frac{1}{10} \\
\frac{2}{5} & \frac{3}{5}
\end{pmatrix}
$$

Figure 2. Transition matrix of weather prediction

3.2. The voting problem
A politician named AP is running for the US Congress. If AP has never been elected, then the probability that he will be elected is $\frac{1}{2}$. If AP loses the ballot, he can run for the US Congress again in the following election, two years later. If AP has already been elected and is currently in office then their probability of being reelected is $\frac{9}{10}$. If AP loses the ballot, he/she retires from politics, meaning that he/she cannot run for the US Congress again.

3.3. Key terms explained:
- Transition diagram: a diagram that clearly shows the transition between states with probabilities of each state transitioning to each other.
- Transition matrix: a matrix that shows transition between states as a table.
- Sample space: set of all possible outcomes or results of an experiment
- Random variable: given an experiment with sample space $S$, a random variable is a function $X$ from $S$ to $R$ ($R$ stands for all real numbers)
- Support: the set of all possible values of $X$
- Absorbing Markov chain: a Markov chain that has at least one absorbing state
- Transient state: a state in an absorbing Markov chain that is not absorbing
- Absorbing state: a state in a Markov chain that once this state is reached, it is impossible to leave it
- Prediction involving Markov chain: a transition diagram where all possible transitions of states are demonstrated can be drawn as followed:
Figure 3. transition diagram of voting problem

The transition matrix is shown below in order to predict and calculate future states:

![Transition Matrix Image]

Figure 4. transition matrix of voting problem

“N”, “E”, “IO”, and “R” represents 4 states in this scenario. “N” stands for “never have been elected”. “E” stands for “elected”. “IO” stands for “currently in office”. “R” stands for “retired”. If AP has never been elected, there’s a probability of \( \frac{1}{2} \) that he will be elected and a probability of \( \frac{1}{2} \) that he will not be elected, which means that he/she can try again in the next election. If AP is elected, the probability of AP is “in office” is 1. Later on, if AP tries to get reelected, there’s a \( \frac{9}{10} \) chance that he/she will be reelected and a \( \frac{1}{10} \) chance of failing, which means that AP retires. It is observed that “R” is absorbing, which means that if the state of “R” is reached, it is impossible to leave it. In this case, if AP retires, he/she will stay retired meaning that he cannot run for US Congress anymore.

4. Other Applications of probability
Probability also play important roles in other fields of study and real-life situations.

For example, probability can be applied to medical situations. Assume that a test for covid-19 is created, and 8% people have covid-19. The sensitivity of the test is 0.9, which means that if a person has covid-19, the probability of the person being tested positive is 0.9. The specificity of the test is 0.9, which means that if a person does not have covid-19, the probability of the person being tested negative is 0.9.

Assume a person is tested positive for covid-19, the probability of the person actually having covid-19 can be calculated due to Bayes’ Theorem:

\[
P(D|T_+) = \frac{P(D \cap T_+)}{P(T_+)} = \frac{P(T_+|D) \times P(D)}{P(T_+|D) \cup P(T_+ \cap H)} = \frac{P(T_+|D) \times P(D)}{P(T_+|D) \times P(D) + P(T_+|H) \times P(H)}
\]

\[
= \frac{0.9 \times 0.08}{0.9 \times 0.08 + 0.1 \times 0.92} \approx 43.9\%
\]

According to the probability of the person actually has covid-19 knowing that the person is tested positive, many people will be misdiagnosed, and people’s health will be endangered.

However, if a person is tested positive for the second time, the probability of the person actually having covid-19 will be higher:
\[ P(D|T_+) = \frac{P(T_+|D) \times P(D)}{P(T_+|D) \times P(D) + P(T_+|H) \times P(H)} = \frac{0.9 \times 0.439}{0.9 \times 0.439 + 0.1 \times 0.561} \approx 87.6\% \]

It is quite normal for the test for certain disease to be only effective sometimes. In this case, it would be better for people to get tested twice than once due to the greatly increased probability. Even though the accuracy of the test cannot be improved, by taking the test more than once, people can obtain more accurate results. Furthermore, it is more beneficial because chances of being misdiagnosed is lower, and people who actually don’t have covid-19 don’t need to spend their money on nothing and risk their lives making contact to doctors and covid-19 patients in hospitals.

Moreover, probability has great impact on sports other than gambling. For example, in football, European clubs that are qualified for the Champions League need to draw to be put into 8 separate groups. Later on, when the group stage ends, 16 teams that rank top 2 in their groups need to draw again to find out who are they playing in the next round. Specifically, if Bayern Munich draws first, the probability of them playing Chelsea is 1/15. When there’s 4 teams left, including Olympique Lyonnais, Juventus, Manchester City, and Real Madrid, the probability of Juventus playing Olympique Lyonnais is 1/3. Also, probability has an important application on a specific aspect of the game of football: penalty. Normally, each football team has one penalty taker who would take almost all the penalties the team gets. Different penalty takers have different penalty-taking techniques and preferred spots of the goal. By analyzing a player’s penalties taken in the past, the coaching staff and the goalkeeper of the opposing team could figure out which direction the ball is more likely to go, thus helping the goalkeeper to go to the right direction to make the save and help the team to win.

5. Conclusion
Although in daily lives, the future cannot be accurately predicted, and things that have not happened cannot be foreseen, probability is still an effective tool of dealing with uncertainty. From predicting the weather to gambling, from manufacturing to sports, probability is everywhere in our lives, not just in textbooks, and it makes our lives much more convenient and easier. Hopefully, in the future, the human kind could further develop probability and better apply it.

Acknowledgement
I would like to extend my sincere thanks to Professor Pierre Clare and teaching assistant Bohang Zong. I also wish to thank Eva Chen for her assistance.

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