ON THE SEMIMARTINGALE PROPERTY OF DISCOUNTED ASSET-PRICE PROCESSES

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This work is dedicated to the memory of our colleague and dear friend Nicola Bruti Liberati, who died tragically on the 28th of August, 2007.

Abstract. A financial market model where agents trade using realistic combinations of buy-and-hold strategies is considered. Minimal assumptions are made on the discounted asset-price process — in particular, the semimartingale property is not assumed. Via a natural market viability assumption, namely, absence of arbitrages of the first kind, we establish that discounted asset-prices have to be semimartingales. In a slightly more specialized case, we extend the previous result in a weakened version of the Fundamental Theorem of Asset Pricing that involves strictly positive supermartingale deflators rather than Equivalent Martingale Measures.

1. Introduction

In the process of obtaining a sufficiently general version of the Fundamental Theorem of Asset Pricing (FTAP), semimartingales proved crucial in modelling discounted asset-price processes. The powerful tool of stochastic integration with respect to general predictable integrands, that semimartingales are exactly tailored for, finally lead to the culmination of the theory in [9]. The FTAP connects the economical notion of No Free Lunch with Vanishing Risk (NFLVR) with the mathematical concept of existence of an Equivalent Martingale Measure (EMM), i.e., an auxiliary probability, equivalent to the original (in the sense that they have the same impossibility events), that makes the discounted asset-price processes have some kind of martingale property. For the above approach to work one has to utilize stochastic integration using general predictable integrands, which translates to allowing for continuous-time trading in the market. Even though continuous-time trading is of vast theoretical importance, in practice it is only an ideal approximation; the only feasible way of trading is via simple, i.e., combinations of buy-and-hold, strategies.

Recently, it has been argued that existence of an EMM is not necessary for viability of the market; to this effect, see [10], [18], [10]. Even in cases where classical arbitrage opportunities are present in the market, credit constraints will not allow for arbitrages to be scaled to any desired degree. It is rather the existence of a strictly positive supermartingale deflator, a concept weaker than existence of an EMM, that allows for a consistent theory to be developed.

Our purpose in this work is to provide answers to the following questions:
(1) Why are semimartingales important in modeling discounted asset-price processes?
(2) Is there an analogous result to the FTAP that involves weaker (both economic and mathematical) conditions, and only assumes the possibility of simple trading?

A partial, but precise, answer to question (1) is already present in [9]. Roughly speaking, market viability already imposes the semimartingale property on discounted asset-price processes. In this paper, we elaborate on the previous idea, undertaking a different approach, which ultimately leads to an improved result. We also note that in [1], [3] and [15], the semimartingale property of discounted asset-price processes is obtained via the finite value of a utility maximization problem; this approach will also be revisited.

All the conditions that have appeared previously in the literature are only sufficient to ensure that discounted asset-price processes are semimartingales. Here, we shall also discuss a necessary and sufficient condition in terms of a natural market-viability notion that parallels the FTAP, under minimal initial structural assumptions on the discounted asset-price processes themselves. The weakened version of the FTAP that we shall come up with as an answer to question (2) above is a “simple, no-short-sales trading” version of Theorem 4.12 from [13].

The structure of the paper is as follows. In Section 2, we introduce the market model, simple trading under no-short-sales constraints. Then, we discuss the market viability condition of absence of arbitrages of the first kind (a weakening of condition NFLVR), as well as the concept of strictly positive supermartingale deflators. After this, our main result, Theorem 2.3, is formulated and proved, which establishes both the importance of semimartingales in financial modelling, as well as the weak version of the FTAP. Section 3 deals with remarks on, and ramifications of, Theorem 2.3. We note that, though hidden in the background, the proofs of our results depend heavily on the notion of the numéraire portfolio (also called growth-optimal, log-optimal or benchmark portfolio), as it appears in a series of works: [14], [17], [2], [12], [18], [19], [13], [8], to mention a few.

2. The Semimartingale Property of Discounted Asset-Price Process and a Version of the Fundamental Theorem of Asset Pricing

2.1. The financial market model and trading via simple, no-short-sales strategies. The random movement of \( d \in \mathbb{N} \) risky assets in the market is modelled via càdlàg, nonnegative stochastic processes \( S^i \), where \( i \in \{1, \ldots, d\} \). As is usual in the field of Mathematical Finance, we assume that all wealth processes are discounted by another special asset which is considered a “baseline”. The above process \( S = (S^i)_{i=1,\ldots,d} \) is defined on a filtered probability space \((\Omega, F, (F_t)_{t \in \mathbb{R}_+}, \mathbb{P})\), where \((F_t)_{t \in \mathbb{R}_+}\) is a filtration satisfying \( F_t \subseteq F \) for all \( t \in \mathbb{R}_+ \), as well as the usual assumptions of right-continuity and saturation by all \( \mathbb{P} \)-null sets of \( F \).

Observe that there is no a priori assumption on \( S \) being a semimartingale. This property will come as a consequence of a natural market viability assumption.

In the market described above, economic agents can trade in order to reallocate their wealth. Consider a simple predictable process \( \theta := \sum_{j=1}^n \vartheta_j \mathbb{1}_{[\tau_j-1, \tau_j]} \). Here, \( \tau_0 = 0 \), and for all \( j \in \{1, \ldots, n\} \) (where \( n \) ranges in \( \mathbb{N} \)), \( \tau_j \) is a finite stopping time and \( \vartheta_j = (\vartheta^i_j)_{i=1,\ldots,d} \) is \( F_{\tau_j-1} \)-measurable. Each \( \tau_{j-1}, j \in \{1, \ldots, n\} \), is an instance when some given economic agent may trade in the market; then, \( \vartheta^i_j \) is the number of units from the \( i \)-th risky asset that the agent will hold in the trading interval
This form of trading is called simple, as it comprises of a finite number of buy-and-hold strategies, in contrast to continuous trading where one is able to change the position in the assets in a continuous fashion. This last form of trading is only of theoretical value, since it cannot be implemented in reality, even if one ignores market frictions. Starting from initial capital $x \in \mathbb{R}_+$ and following the strategy described by the simple predictable process $\theta := \sum_{j=1}^n \vartheta_j \mathbb{1}_{[\tau_j, \tau_j]}$, the agent’s discounted wealth process is given by

$$X^{x, \theta} = x + \int_0^T \langle \theta_t, dS_t \rangle := x + \sum_{j=1}^n \langle \vartheta_j, S_{\tau_j \wedge} - S_{\tau_{j-1} \wedge} \rangle.$$  

We use “$\langle \cdot, \cdot \rangle$” throughout to denote the usual Euclidean inner product on $\mathbb{R}^d$.

The wealth process $X^{x, \theta}$ of (2.1) is càdlàg and adapted, but could in principle become negative. In real markets, some economic agents, for instance pension funds, face several institution-based constraints when trading. The most important constraint is prevention of having negative positions in the assets; we plainly call this no-short-sales constraints. In order to ensure that no short sales are allowed in the risky assets, which also include the baseline asset used for discounting, we define $\mathcal{X}_s(x)$ to be the set of all wealth processes $X^{x, \theta}$ given by (2.1), where $\theta = \sum_{j=1}^n \vartheta_j \mathbb{1}_{[\tau_{j-1}, \tau_j]}$ is simple and predictable and such that $\vartheta_i \geq 0$ and $\langle \vartheta_j, S_{\tau_{j-1}} \rangle \leq X^{x, \theta}_{\tau_{j-1}}$ hold for all $i \in \{1, \ldots, d\}$ and $j \in \{1, \ldots, n\}$. (The subscript “s” in $\mathcal{X}_s(x)$ is a mnemonic for “simple”; the same is true for all subsequent definitions where this subscript appears.) Note that the previous no-short-sales constraints, coupled with the nonnegativity of $S^i$, $i \in \{1, \ldots, d\}$, imply the stronger $\vartheta_i \geq 0$ for all $i = 1, \ldots, d$ and $\langle \theta, S_- \rangle \leq X^{-, \theta}$. (The subscript “-” is used to denote the left-continuous version of a càdlàg process.) It is clear that $\mathcal{X}_s(x)$ is a convex set for all $x \in \mathbb{R}_+$. Observe also that $\mathcal{X}_s(x) = x\mathcal{X}_s(1)$ for all $x \in \mathbb{R}_+ \setminus \{0\}$. Finally, define $\mathcal{X}_s := \bigcup_{x \in \mathbb{R}_+} \mathcal{X}_s(x)$.

2.2. Market viability. We now aim at defining the essential “no-free-lunch” concept to be used in our discussion. For $T \in \mathbb{R}_+$, an $\mathcal{F}_T$-measurable random variable $\xi$ will be called an arbitrage of the first kind on $[0, T]$ if $\mathbb{P}[\xi \geq 0] = 1$, $\mathbb{P}[\xi > 0] > 0$, and for all $x > 0$ there exists $X \in \mathcal{X}_s(x)$, which may depend on $x$, such that $\mathbb{P}[X_T \geq \xi] = 1$. If, in a market where only simple, no-short-sales trading is allowed, there are no arbitrages of the first kind on any interval $[0, T]$, $T \in \mathbb{R}_+$ we shall say that condition NA1$\xi$ holds. It is straightforward to check that condition NA1$\xi$ is weaker than condition NFLVR (appropriately stated for simple, no-short-sales trading). The next result describes an equivalent reformulation of condition NA1$\xi$ in terms of boundedness in probability of the set of outcomes of wealth processes, which is essentially condition “No Unbounded Profit with Bounded Risk” of [13] for all finite time-horizons in our setting of simple, no-short-sales trading.

**Proposition 2.1.** Condition NA1$\xi$ holds if and only if, for all $T \in \mathbb{R}_+$, the set $\{X_T \mid X \in \mathcal{X}_s(1)\}$ is bounded in probability, i.e., $\lim_{\ell \to \infty} \sup_{X \in \mathcal{X}_s(1)} \mathbb{P}[X_T > \ell] = 0$ holds for all $T \in \mathbb{R}_+$.

**Proof.** Using the fact that $\mathcal{X}_s(x) = x\mathcal{X}_s(1)$ for all $x > 0$, it is straightforward to check that if an arbitrage of the first kind exists on $[0, T]$ for some $T \in \mathbb{R}_+$ then $\{X_T \mid X \in \mathcal{X}_s(1)\}$ is not bounded in probability. Conversely, assume the existence of $T \in \mathbb{R}_+$ such that $\{X_T \mid X \in \mathcal{X}_s(1)\}$ is not bounded in probability. As $\{X_T \mid X \in \mathcal{X}_s(1)\}$ is further convex, Lemma 2.3 of [5] implies the existence of $\Omega_\alpha \in \mathcal{F}_T$ with $\mathbb{P}[\Omega_\alpha] > 0$ such that, for all $n \in \mathbb{N}$, there exists $X^n \in \mathcal{X}_s(1)$ with
Proof of Theorem 2.3, statement (1).

(i) ⇒ (ii). Define the set of dyadic rational numbers $\mathbb{D} := \{m/2^k \mid k \in \mathbb{N}, m \in \mathbb{N}\}$, which is dense in $\mathbb{R}_+$. Further, for $k \in \mathbb{N}$, define the set of trading times $T^k := \{m/2^k \mid m \in \mathbb{N}, 0 \leq m \leq k2^k\}$. Before stating our main Theorem 2.3, recall that $X^0 \in \mathcal{X}_1$ (1) holds on $[0, T]$, which finishes the proof. □

Remark 2.2. The constant wealth process $X \equiv 1$ belongs to $\mathcal{X}_1(1)$. Then, Proposition 2.1 implies that condition NA1 is also equivalent to the requirement that the set $\{X_T \mid X \in \mathcal{X}_1(1)\}$ is bounded in probability for all finite stopping times $T$. 2.3. Strictly positive supermartingale deflators. Define the set $\mathcal{Y}_s$ of strictly positive supermartingale deflators for simple, no-short-sales trading to consist of all càdlàg processes $Y$ such that $\mathbb{P}[Y_0 = 1, \text{ and } Y_t > 0 \forall t \in \mathbb{R}_+] = 1$, and $YX$ is a supermartingale for all $X \in \mathcal{X}_s$. Note that existence of a strictly positive supermartingale deflator is a condition closely related, but strictly weaker, to existence of equivalent (super)martingale probability measures.

2.4. The main result. Condition NA1, existence of strictly positive supermartingale deflators and the semimartingale property of $S$ are immensely tied to each other, as will be revealed below.

Define the (first) bankruptcy time of $X \in \mathcal{X}_s$ to be $\xi^X := \inf\{t \in \mathbb{R}_+ \mid X_{t-} = 0 \text{ or } X_t = 0\}$. We shall say that $X \in \mathcal{X}_s$ cannot revive from bankruptcy if $X_t = 0$ holds for all $t \geq \xi^X$ on $\{\xi^X < \infty\}$. As $S^i \in \mathcal{X}_s$ for $i \in \{1, \ldots, d\}$, the previous definitions apply in particular to each $S^i$, $i \in \{1, \ldots, d\}$.

Before stating our main Theorem 2.3 recall that $S^i$, $i \in \{1, \ldots, d\}$, is an exponential semimartingale if there exists a semimartingale $R^i$ with $R^i_0 = 0$, such that $S^i = S^i_0 \mathcal{E}(R^i)$ where $\mathcal{E}$ denotes the stochastic exponential operator.

Theorem 2.3. Let $S = (S^i)_{i=1,\ldots,d}$ be an adapted, càdlàg stochastic process such that $S^i$ is non-negative for all $i \in \{1, \ldots, d\}$. Consider the following four statements:

(i) Condition NA1 holds in the market.
(ii) $\mathcal{Y}_s \neq \emptyset$.
(iii) $S$ is a semimartingale, and $S^i$ cannot revive from bankruptcy for all $i \in \{1, \ldots, d\}$.
(iv) For all $i \in \{1, \ldots, d\}$, $S^i$ is an exponential semimartingale.

Then, we have the following:

(1) It holds that (i) ⇔ (ii) ⇒ (iii), as well as (iv) ⇒ (i).
(2) Assume further that $S^i_{\xi^i_{S^i}} > 0$ holds on $\{\xi^S < \infty\}$ for all $i \in \{1, \ldots, d\}$. Then, we have the equivalences (i) ⇔ (ii) ⇔ (iii) ⇔ (iv).

2.5. Proof of Theorem 2.3, statement (1).

(i) ⇒ (ii). Define the set of dyadic rational numbers $\mathbb{D} := \{m/2^k \mid k \in \mathbb{N}, m \in \mathbb{N}\}$, which is dense in $\mathbb{R}_+$. Further, for $k \in \mathbb{N}$, define the set of trading times $T^k := \{m/2^k \mid m \in \mathbb{N}, 0 \leq m \leq k2^k\}$. For all $n \in \mathbb{N}$ set $A^n = \mathbb{I}_{\{X^n_\xi > n\}} \cap \Omega_u \in \mathcal{F}_T$. Then, set $A := \cap_{n \in \mathbb{N}} A^n \in \mathcal{F}_T$ and $\xi := \mathbb{I}_A$. It is clear that $\xi$ is $\mathcal{F}_T$-measurable and that $\mathbb{P}[\xi > 0] = 1$. Furthermore, since $A \subseteq \Omega_u$ and

$$\mathbb{P}[\Omega_u \setminus A] = \mathbb{P}\left( \bigcup_{n \in \mathbb{N}} (\Omega_u \setminus A^n) \right) \leq \sum_{n \in \mathbb{N}} \mathbb{P}[\Omega_u \setminus A^n] = \sum_{n \in \mathbb{N}} \mathbb{P}\{X^n_\xi \leq n\} \cap \Omega_u \leq \sum_{n \in \mathbb{N}} \frac{\mathbb{P}[\Omega_u]}{2^{n+1}} = \frac{\mathbb{P}[\Omega_u]}{2},$$

we obtain $\mathbb{P}[A] > 0$, i.e., $\mathbb{P}[\xi > 0] > 0$. For all $n \in \mathbb{N}$ set $X^n := (1/n) \tilde{X}^n_\xi$, and observe that $X^n \in \mathcal{X}_1(1/n)$ and $\xi = \mathbb{I}_A \leq \mathbb{I}_{A^n} \leq X^n_\xi$ hold for all $n \in \mathbb{N}$. It follows that $\xi$ is and arbitrage of the first kind on $[0, T]$, which finishes the proof. □
Then, $T^k \subset T^{k'}$ for $k < k'$ and $\bigcup_{k \in \mathbb{N}} T^k = \mathbb{D}$. In what follows, $\mathcal{X}_s^k(1)$ denotes the subset of $\mathcal{X}_s(1)$ consisting of wealth processes where trading only may happen at times in $T^k$. We now state and prove an intermediate result that will help to establish implication (i) $\Rightarrow$ (ii) of Theorem 2.3.

**Lemma 2.4.** Under condition NA1, and for each $k \in \mathbb{N}$, there exists a wealth process $\tilde{X}^k \in \mathcal{X}_s^k(1)$ with $\mathbb{P}[\tilde{X}^k_1 > 0] = 1$ for all $t \in T^k$ such that, by defining $\tilde{Y}^k := 1/\tilde{X}^k$, $\mathbb{E}[\tilde{Y}^k X_t \mid \mathcal{F}_s] \leq \tilde{Y}^k X_s$ holds for all $X \in \mathcal{X}_s^k(1)$, where $T^k \ni s \leq t \in T^k$.

**Proof.** The existence of such “numéraire portfolio” $\tilde{X}^k$ essentially follows from Theorem 4.12 of [13]. However, we shall explain in detail below how one can obtain the validity of Lemma 2.4 following the idea used to prove Theorem 4.12 of [13] in this simpler setting, rather than using the latter heavy result. Throughout the proof we keep $k \in \mathbb{N}$ fixed, and we set $T^k_{++} := T^k \setminus \{0\}$.

First of all, it is straightforward to check that condition NA1 implies that each $X \in \mathcal{X}_s$, and in particular also each $S^i$, $i \in \{1, \ldots, d\}$, cannot revive from bankruptcy. This implies that we can consider an alternative “multiplicative” characterization of wealth processes in $\mathcal{X}_s(1)$, as we now describe. Consider a process $\pi = (\pi_t)_{t \in T^k_{++}}$ such that, for all $t \in T^k_{++}$, $\pi_t \equiv (\pi_t^i)_{i \in \{1, \ldots, d\}}$ is $\mathcal{F}_{t-1/2k}$-measurable and takes values in the $d$-dimensional simplex $\Delta^d := \{z | z_i \geq 0 \text{ for } i = 1, \ldots, d, \text{ and } \sum_{i=1}^d z_i \leq 1\}$. Define $X_0^\pi := 1$ and, for all $t \in T^k_{++}$, $X_t^{\pi}(z) := \prod_{t \in T^k_{++} \ni u \leq t} (1 + \langle \pi_u, \Delta R^k_u \rangle)$, where, for $u \in T^k_{++}$, $\Delta R^k_u = (\Delta R^k_{ui})_{i \in \{1, \ldots, d\}}$ is such that $\Delta R^k_{ui} = (S^i_{u+1/2k} - S^i_{u-1/2k} - 1) I_{\{S^i_{u-1/2k} > 0\}}$ for $i \in \{1, \ldots, d\}$. Then, define a simple predictable $d$-dimensional process $\theta$ as follows: for $i \in \{1, \ldots, d\}$ and $u \in [t-1/2k, t]$, where $t \in T^k_{++}$, set $\theta^i_t = (\pi^i_t X^{\pi}_{t-1/2k}/S^i_{t-1/2k}) I_{\{S^i_{t-1/2k} > 0\}}$; otherwise, set $\theta^i_t = 0$. It is then straightforward to check that $X^{1,\theta}$, in the notation of (2.1), is an element of $\mathcal{X}_s^k(1)$, as well as that $X^{1,\theta}_t = X_t^{\pi}$ holds for all $t \in T^k$. We have then established that $\pi$ generates a wealth process in $\mathcal{X}_s^k(1)$. We claim that every wealth process of $\mathcal{X}_s^k(1)$ can be generated this way. Indeed, starting with any predictable $d$-dimensional process $\theta$ such that $X^{1,\theta}$, in the notation of (2.1), is an element of $\mathcal{X}_s^k(1)$, we define $\pi^i_t = (\theta^i_t S^i_{t-1/2k}/X^{1,\theta}_{t-1/2k}) I_{\{X^{1,\theta}_{t-1/2k} > 0\}}$ for $i \in \{1, \ldots, d\}$ and $t \in T^k_{++}$. Then, $\pi = (\pi_t)_{t \in T^k_{++}}$ is $\mathcal{F}_{t-1/2k}$-measurable for $t \in T^k_{++}$, and $\pi$ generates $X^{1,\theta}$ in the way described previously — in particular, $X^{1,\theta}_t = X_t^{\pi}$ holds for all $t \in T^k$. (In establishing the claims above it is important that all wealth processes of $\mathcal{X}_s$ cannot revive from bankruptcy.)

Continuing, since $\Delta^d$ is a compact subset of $\mathbb{R}^d$, for all $t \in T^k$ there exists a $\mathcal{F}_{t-1/2k}$-measurable $\rho_t = (\rho^i_t)_{i \in \{1, \ldots, d\}}$ such that, for all $\mathcal{F}_{t-1/2k}$-measurable and $\Delta^d$-valued $\pi_t = (\pi^i_t)_{i \in \{1, \ldots, d\}}$, we have

$$
\mathbb{E}\left[\frac{1 + \langle \pi_t, \Delta R^k_t \rangle}{1 + \langle \rho_t, \Delta R^k_t \rangle} \mid \mathcal{F}_{t-1/2k}\right] \leq 1
$$

(It is exactly the existence of such $\rho_t$ can be seen as a stripped-down version of Theorem 4.12 in [13]; in effect, $\rho_t$ is the optimal proportions of wealth connected with the log-utility maximization problem, modulo technicalities arising when the value of the log-utility maximization problem has infinite value.) Setting $\tilde{X}^k$ to be the wealth process in $\mathcal{X}_s^k(1)$ generated by $\rho$ as described in the previous paragraph, the result of Lemma 2.4 is immediate. \qed
We proceed with the proof of implication \((i) \Rightarrow (ii)\) of Theorem 2.24 using the notation from the statement of Lemma 2.4. For all \(k \in \mathbb{N}\), \(\bar{Y}^k\) satisfies \(\bar{Y}^k_0 = 1\) and is a positive supermartingale when sampled from times in \(T^k\), since \(1 \in \mathcal{X}_s^k\). Therefore, for any \(t \in \mathbb{D}\), the convex hull of the set \(\{\bar{Y}^k_t \mid k \in \mathbb{N}\}\) is bounded in probability. We also claim that, under condition NA1\(s\), for any \(t \in \mathbb{R}_+\), the convex hull of the set \(\{\bar{Y}^k_t \mid k \in \mathbb{N}\}\) is bounded away from zero in probability. Indeed, for any collection \((\alpha^k)_{k \in \mathbb{N}}\) such that \(\alpha^k \geq 0\) for all \(k \in \mathbb{N}\), having all but a finite number of \(\alpha^k\)'s non-zero and satisfying \(\sum_{k=1}^{\infty} \alpha^k = 1\), we have

\[
\frac{1}{\sum_{k=1}^{\infty} \alpha^k Y^k} \leq \sum_{k=1}^{\infty} \frac{\alpha^k}{Y^k} = \sum_{k=1}^{\infty} \alpha^k \bar{X}^k \in \mathcal{X}_s(1).
\]

Since, by Proposition 2.21, \(\{X_t \mid X \in \mathcal{X}_s(1)\}\) is bounded in probability for all \(t \in \mathbb{R}_+\), the previous fact proves that the convex hull of the set \(\{\bar{Y}^k_t \mid k \in \mathbb{N}\}\) is bounded away from zero in probability.

Now, using Lemma A1.1 of [9], one can proceed as in the proof of Lemma 5.2(a) in [11] to infer the existence of a sequence \((\bar{Y}^k)_{k \in \mathbb{N}}\) and some process \((\bar{Y}_t)_{t \in \mathbb{D}}\) such that, for all \(k \in \mathbb{N}\), \(\bar{Y}^k\) is a convex combination of \(\bar{Y}^k, \bar{Y}^{k+1}, \ldots\), and \(\mathbb{P}[\lim_{k \to \infty} \bar{Y}^k_t = Y_t, \forall t \in \mathbb{D}] = 1\). The discussion of the preceding paragraph ensures that \(\mathbb{P}[0 < \bar{Y}_t < \infty, \forall t \in \mathbb{D}] = 1\).

Let \(\mathbb{D} \ni s \leq t \in \mathbb{D}\). Then, \(s \in T^k\) and \(t \in T^k\) for all large enough \(k \in \mathbb{N}\). According to the conditional version of Fatou’s Lemma, for all \(X \in \bigcup_{k=1}^{\infty} \mathcal{X}_s^k\) we have that

\[
E[\bar{Y}_t X_t \mid \mathcal{F}_s] \leq \liminf_{k \to \infty} E[\bar{Y}^k_t X_t \mid \mathcal{F}_s] \leq \liminf_{k \to \infty} \bar{Y}^k_s X_s = \bar{Y}_s X_s.
\]

It follows that \((\bar{Y}_t X_t)_{t \in \mathbb{D}}\) is a supermartingale for all \(X \in \bigcup_{k=1}^{\infty} \mathcal{X}_s^k\). (Observe here that we sample the process \(\bar{Y}X\) only at times contained in \(\mathbb{D}\).) In particular, \((\bar{Y}_t)_{t \in \mathbb{D}}\) is a supermartingale.

For any \(t \in \mathbb{R}_+\) define \(Y_t := \lim_{s \downarrow t, s \in \mathbb{D}} \bar{Y}_s\) — the limit is taken in the \(\mathbb{P}\)-a.s. sense, and exists in view of the supermartingale property of \((\bar{Y}_t)_{t \in \mathbb{D}}\). It is straightforward that \(Y\) is a càdlàg process; it is also adapted because \((\mathcal{F}_t)_{t \in \mathbb{R}_+}\) is right-continuous. Now, for \(t \in \mathbb{R}_+\), let \(T \in \mathbb{D}\) be such that \(T > t\); a combination of the right-continuity of both \(Y\) and the filtration \((\mathcal{F}_t)_{t \in \mathbb{R}_+}\), the supermartingale property of \((\bar{Y}_t)_{t \in \mathbb{D}}\), and Lévy’s martingale convergence Theorem, give \(E[\bar{Y}_T \mid \mathcal{F}_t] \leq Y_t\). Since \(\mathbb{P}[\bar{Y}_T > 0] = 1\), we obtain \(\mathbb{P}[Y_t > 0] = 1\). Right-continuity of the filtration \((\mathcal{F}_t)_{t \in \mathbb{R}_+}\), coupled with (2.2), imply that \(E[Y_t X_t \mid \mathcal{F}_s] \leq Y_s X_s\) for all \(\mathbb{R}_+ \ni s \leq t \in \mathbb{R}_+\) and \(X \in \bigcup_{k=1}^{\infty} \mathcal{X}_s^k\). In particular, \(Y\) is a càdlàg nonnegative supermartingale; since \(\mathbb{P}[Y_t > 0] = 1\) holds for all \(t \in \mathbb{R}_+\), we conclude that \(\mathbb{P}[Y_t > 0, \forall t \in \mathbb{R}_+] = 1\).

Of course, \(1 \in \mathcal{X}_s^k\) and \(S^i \in \mathcal{X}_s^k\) hold for all \(k \in \mathbb{N}\) and \(i \in \{1, \ldots, d\}\). It follows that \(Y\) is a supermartingale, as well as that \(YS^i\) is a supermartingale for all \(i \in \{1, \ldots, d\}\). In particular, \(Y\) and \(YS = (YS^i)_{i \in \{1, \ldots, d\}}\) are semimartingales. Consider any \(X^{x, \theta}\) in the notation of (2.1). Using the integration-by-parts formula, we obtain

\[
Y X^{x, \theta} = x + \int_0^t \left( X^{x, \theta}_t - \langle \theta_t, S_t^- \rangle \right) \, dY_t + \int_0^t \langle \theta_t, d(Y_t S_t) \rangle.
\]

If \(X^{x, \theta} \in \mathcal{Y}_s(x)\), we have \(X^{x, \theta}_t - \langle \theta_t, S_t^- \rangle \geq 0\), as well as \(\theta^i \geq 0\) for \(i \in \{1, \ldots, d\}\). Then, the supermartingale property of \(Y\) and \(YS^i\), \(i \in \{1, \ldots, d\}\), gives that \(Y X^{x, \theta}\) is a supermartingale. Therefore, \(Y \in \mathcal{Y}_s\), i.e., \(\mathcal{Y}_s \neq \emptyset\).

\(\Box\)
(ii) ⇒ (i). Let \( Y \in \mathcal{Y}_s \), and fix \( T \in \mathbb{R}_+ \). Then, \( \sup_{X \in \mathcal{X}_s(1)} \mathbb{E}[Y_T X_T] \leq 1 \). In particular, the set \( \{ Y_T X_T \mid X \in \mathcal{X}_s(1) \} \) is bounded in probability. Since \( \mathbb{P}[Y_T > 0] = 1 \), the set \( \{ X_T \mid X \in \mathcal{X}_s(1) \} \) is bounded in probability as well. An invocation of Proposition 2.1 finishes the argument. □

(ii) ⇒ (iii). Let \( Y \in \mathcal{Y}_s \). Since \( S^i \in \mathcal{X}_s \), \( YS^i \) is a supermartingale, thus a semimartingale, for all \( i \in \{1, \ldots, d\} \). Also, the fact that \( Y > 0 \) and Itô’s formula give that \( 1/Y \) is a semimartingale. Therefore, \( S^i = (1/Y)(YS^i) \) is a semimartingale for all \( i \in \{1, \ldots, d\} \). Furthermore, since \( YS^i \) is a nonnegative supermartingale, we have \( Y_tS^i_t = 0 \) for all \( t \geq \zeta^{S^i} \) on \( \{ \zeta^{S^i} < \infty \} \), for \( i \in \{1, \ldots, d\} \).

Now, using \( Y > 0 \) again, we obtain that \( S^i_t = 0 \) holds for all \( t \geq \zeta^{S^i} \) on \( \{ \zeta^{S^i} < \infty \} \). In other words, each \( S^i \), \( i \in \{1, \ldots, d\} \), cannot revive after bankruptcy.

(iv) ⇒ (i). Since \( S \) is a semimartingale, we can consider continuous-time trading. For \( x \in \mathbb{R}_+ \), let \( \mathcal{X}(x) \) be the set of all wealth processes \( X^{x,\theta} := x + \int_0^\tau \langle \theta_t, dS_t \rangle \), where \( \theta \) is \( d \)-dimensional, predictable and \( S \)-integrable, “\( \int_0^\tau \langle \theta_t, dS_t \rangle \)” denotes a vector stochastic integral, \( X^{x,\theta} \geq 0 \) and \( 0 \leq \langle \theta, S_- \rangle \leq X^{x,\theta} \). (Observe that the qualifying subscript “\( - \)” denoting simple trading has been dropped in the definition of \( \mathcal{X}_s(x) \), since we are considering continuous-time trading.) Of course, \( \mathcal{X}_s(x) \subseteq \mathcal{X}(x) \). We shall show in the next paragraph that \( \{ X_T \mid X \in \mathcal{X}(1) \} \) is bounded in probability for all \( T \in \mathbb{R}_+ \), therefore establishing condition NA1s, in view of Proposition 2.1.

For all \( i \in \{1, \ldots, d\} \), write \( S^i = S^i_0 \mathcal{E}(R^i) \), where \( R^i \) is a semimartingale with \( R^i_0 = 0 \). Let \( R := (R^i)_{i=1,\ldots,d} \). It is straightforward to see that \( \mathcal{X}(1) \) coincides with the class of all processes of the form \( \mathcal{E}(\int_0^\tau \langle \pi_t, dR_t \rangle) \), where \( \pi \) is predictable and take values in the \( d \)-dimensional simplex \( \Delta^d := \{ z = (z^i)_{i=1,...,d} \in \mathbb{R}^d \mid z^i \geq 0 \text{ for } i = 1, \ldots, d, \text{ and } \sum_{i=1}^d z^i \leq 1 \} \). Since, for all \( T \in \mathbb{R}_+ \),

\[
\log \left( \mathcal{E} \left( \int_0^T \langle \pi_t, dR_t \rangle \right) \right) \leq \int_0^T \langle \pi_t, dR_t \rangle
\]

holds for all \( \Delta^d \)-valued and predictable \( \pi \), it suffices to show the boundedness in probability of the class of all \( \int_0^T \langle \pi_t, dR_t \rangle \), where \( \pi \) ranges in all \( \Delta^d \)-valued and predictable processes. Write \( R = B + M \), where \( B \) is a process of finite variation and \( M \) is a local martingale with \( |\Delta M^i| \leq 1 \), \( i \in \{1, \ldots, d\} \). Then, \( \int_0^T |\langle \pi_t, dB_t \rangle| \leq \sum_{i=1}^d |\int_0^T |\pi^i_t dM^i_t| < \infty \). This establishes the boundedness in probability of the class of all \( \int_0^T \langle \pi_t, dB_t \rangle \), where \( \pi \) ranges in all \( \Delta^d \)-valued and predictable processes. We have to show that the same holds for the class of all \( \int_0^T \langle \pi_t, dM_t \rangle \), where \( \pi \) is \( \Delta^d \)-valued and predictable. For \( k \in \mathbb{N} \), let \( \tau^k := \inf\{ t \in \mathbb{R}_+ \mid \sum_{i=1}^d (M^i, M^i)^{\tau^k}_t \geq k \} \wedge T \). Note that \( [(M^i, M^i)^{\tau^k}]^{\tau^k} = (M^i, M^i)^{\tau^k} + |\Delta M^i|^2 \leq k + 1 \) holds for all \( i \in \{1, \ldots, d\} \). Therefore, using the notation \( \|\eta\|_{L^2} := \sqrt{\mathbb{E}[\eta^2]} \) for a random variable \( \eta \), we obtain

\[
\left\| \int_0^\tau \langle \pi_t, dM_t \rangle \right\|_{L^2} \leq \sum_{i=1}^d \left\| \int_0^\tau \pi^i_t dM^i_t \right\|_{L^2} \leq \sum_{i=1}^d \left\| \sqrt{[M^i, M^i]}^{\tau^k} \right\|_{L^2} \leq d \sqrt{k + 1}
\]

Fix \( \epsilon > 0 \). Let \( k = k(\epsilon) \) be such that \( \mathbb{P}[\tau^k < T] < \epsilon/2 \), and also let \( \ell := d \sqrt{2(k + 1)/\epsilon} \). Then,

\[
\mathbb{P} \left[ \int_0^T \langle \pi_t, dM_t \rangle > \ell \right] \leq \mathbb{P} \left[ \tau^k < T \right] + \mathbb{P} \left[ \int_0^\tau \langle \pi_t, dM_t \rangle > \ell \right] \leq \frac{\epsilon}{2} + \left( \frac{\left\| \int_0^\tau \langle \pi_t, dM_t \rangle \right\|_{L^2}}{\ell} \right)^2 \leq \epsilon.
\]
The last estimate is uniform over all $\Delta^d$-valued and predictable $\pi$. We have, therefore, established the boundedness in probability of the class of all $\int_0^T \langle \pi_t, dM_t \rangle$, where $\pi$ ranges in all $\Delta^d$-valued and predictable processes. This completes the proof. □

2.6. **Proof of Theorem 2.3, statement (2).** In view of statement (1) of Theorem 2.3, we only need to show the validity of $(iii) \iff (iv)$ under the extra assumption of statement (2). This equivalence is really Proposition 2.2 in [7], but we present the few details for completeness.

For the implication $(iii) \Rightarrow (iv)$, simply define $R^i := \int_0^T (1/S^i_t) dS^i_t$ for $i \in \{1, \ldots, d\}$. The latter process is a well-defined semimartingale because, for each $i \in \{1, \ldots, d\}$, $S^i$ is a semimartingale, $S^i_{-}$ is locally bounded away from zero on the stochastic interval $[0, \zeta_{S^i}]$, and $S = 0$ on $[\zeta_{S^i}, \infty[$.

Now, for $(iv) \Rightarrow (iii)$, it is clear that $S$ is a semimartingale. Furthermore, for all $i \in \{1, \ldots, d\}$, $S^i$ cannot revive from bankruptcy; this follows because stochastic exponentials stay at zero once they hit zero. □

3. **ON AND BEYOND THE MAIN RESULT**

3.1. **Comparison with the result of Delbaen and Schachermayer.** Theorem 7.2 of the seminal paper [9] establishes the semimartingale property of $S$ under condition NFL VR for simple admissible strategies, coupled with a local boundedness assumption on $S$ (always together with the càdlàg property and adaptedness). The assumptions of Theorem 2.3 are different than the ones in [9]. Condition NA1$_k$ (valid for simple, no-short-sales trading) is weaker than NFL VR for simple admissible strategies. Furthermore, local boundedness from above is not required in our context, but we do require that each $S^i$, $i \in \{1, \ldots, d\}$, is nonnegative. In fact, as we shall argue in §3.2 below, nonnegativity of each $S^i$, $i \in \{1, \ldots, d\}$, can be weakened by local boundedness from below, indeed making Theorem 2.3 a generalization of Theorem 7.2 in [9]. Note that if the components of $S$ are unbounded both above and below, not even condition NFL VR is enough to ensure the semimartingale property of $S$; see Example 7.5 in [9].

Interestingly, and in contrast to [9], the proof of Theorem 2.3 provided here does not use the deep Bichteler-Dellacherie theorem on the characterization of semimartingales as “good integrators” (see [4], [20], where one starts by defining semimartingales as good integrators and obtains the classical definition as a by-product). Actually, and in view of Proposition 2.1, statement (2) of Theorem 2.3 can be seen as a “multiplicative” counterpart of the Bichteler-Dellacherie theorem. Its proof exploits two simple facts: (a) positive supermartingales are semimartingales, which follows directly from the Doob-Meyer decomposition theorem; and (b) reciprocals of strictly positive supermartingales are semimartingales, which is a consequence of Itô’s formula. Crucial in the proof is also the concept of the numéraire portfolio.

3.2. **The semimartingale property of $S$ when each $S^i$, $i \in \{1, \ldots, d\}$, is locally bounded from below.** As mentioned previously, implication $(i) \Rightarrow (iii)$ actually holds even when each $S^i$, $i \in \{1, \ldots, d\}$, is locally bounded from below, which we shall establish now. We still, of course, assume that each $S^i$, $i \in \{1, \ldots, d\}$, is adapted and càdlàg. Since “no-short-sales” strategies have ambiguous meaning when asset prices can become negative, we need to make some changes in the class of admissible wealth processes. For $x \in \mathbb{R}_+$, let $\mathcal{X}'_d(x)$ denote the class of all wealth processes...
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\(X^{x,\theta}\) using simple trading as in \(2.1\) that satisfy \(X^{x,\theta} \geq 0\). Further, set \(\mathcal{X}'_s = \bigcup_{x \in \mathbb{R}_+} \mathcal{X}'(x)\). Define condition \(\text{NAI}'_s\) for the class \(\mathcal{X}'_s\) in the obvious manner, replacing “\(\mathcal{X}_s\)” with “\(\mathcal{X}'_s\)” throughout in \(2.2\). Assume then that condition \(\text{NAI}'_s\) holds. To show that \(S\) is a semimartingale, it is enough to show that \((S^i)_{i \in \mathbb{N}}\) is a semimartingale for each \(i \in \mathbb{N}\), where \((r^k)_{k \in \mathbb{N}}\) is a localizing sequence such that \(S^i \geq -k\) on \([0, r^k]\) for all \(i \in \{1, \ldots, d\}\) and \(k \in \mathbb{N}\). In other words, we might as well assume that \(S^i \geq -k\) for all \(i \in \{1, \ldots, d\}\). Define \(\tilde{S}:=(\tilde{S}^i)_{i \in \{1,\ldots,d\}}\); then, \(\tilde{S}\) is nonnegative for all \(i \in \{1, \ldots, d\}\). Let \(\tilde{S} = (\tilde{S}^i)_{i \in \{1,\ldots,d\}}\). If \(\tilde{X}\) is (in self-explanatory notation) the collection of all wealth processes resulting from simple, no-short-sales strategies investing in \(\tilde{S}\), it is straightforward that \(\tilde{X} \subseteq \mathcal{X}'_s\). Therefore, \(\text{NAI}_s\) holds for simple, no-short-sales strategies investing in \(\tilde{S}\); using implication \((i) \Rightarrow (iii)\) in statement \((1)\) of Theorem \(2.3\), we obtain the semimartingale property of \(\tilde{S}\). The latter is of course equivalent to \(S\) being a semimartingale.

One might wonder why we do not simply ask from the outset that each \(S^i, i \in \{1, \ldots, d\}\), is locally bounded from below, since it certainly contains the case where each \(S^i, i \in \{1, \ldots, d\}\), is nonnegative. The reason is that by restricting trading to using only no-short-sales strategies (which we can do when each \(S^i, i \in \{1, \ldots, d\}\), is nonnegative) enables us to be as general as possible in extracting the semimartingale property of \(S\) from the \(\text{NAI}_s\) condition. Consider, for example, the discounted asset-price process given by \(S = a\mathbb{I}_{[0,1]} + b\mathbb{I}_{[1,\infty]}\), where \(a > 0\) and \(b \in \mathbb{R}_+\) with \(a \neq b\). This is a really elementary example of a nonnegative semimartingale. Now, if we allow for any form of simple trading, as long as it keeps the wealth processes nonnegative, it is clear that condition \(\text{NAI}'_s\) will fail (since it is known that at time \(t = 1\) there will be a jump of size \((b-a) \in \mathbb{R} \setminus \{0\}\) in the discounted asset-price process). On the other hand, if we only allow for no-short-sales strategies, \(\text{NAI}_s\) will hold — this is easy to see directly using Proposition \(2.1\) since \(X_T \leq |b-a|/a\) for all \(T \geq 1\) and \(X \in \mathcal{X}_s(1)\). Therefore, we can conclude that \(S\) is a semimartingale using implication \((i) \Rightarrow (iii)\) in statement \((1)\) of Theorem \(2.3\). (Of course, one might argue that there is no need to invoke Theorem \(2.3\) for the simple example here. The point is that allowing for all nonnegative wealth processes results in a rather weak sufficient criterion for the semimartingale property of \(S\).)

3.3. The semimartingale property of \(S\) via bounded indirect utility. There has been previous work in the literature obtaining the semimartingale property of \(S\) using the finiteness of the value function of a utility maximization problem via use of only simple strategies — see, for instance, \([1], [3], [15]\). In all cases, there has been an assumption of local boundedness (or even continuity) on \(S\). We shall offer a result in the same spirit, dropping the local boundedness requirement. We shall assume either that discounted asset-price processes are nonnegative and only no-short-sales simple strategies are considered (which allows for a sharp result), or that discounted asset-price processes are locally bounded from below. In the latter case, Proposition \(3.1\) that follows is a direct generalization of the corresponding result in \([1]\), where the authors consider locally bounded (both above and below) discounted asset-price processes. In the statement of Proposition \(3.1\) below, we use the notation \(\mathcal{X}'_s(x)\) introduced previously in \(\S 3.2\).

**Proposition 3.1.** Let \(S = (S^i)_{i=1,\ldots,d}\) be such that \(S^i\) is adapted and càdlàg process for \(i \in \{1, \ldots, d\}\). Also, let \(U: \mathbb{R}_+ \mapsto \mathbb{R} \cup \{-\infty\}\) be a nondecreasing function with \(U > -\infty\) on \([0, \infty]\) and \(U(\infty) = \infty\). Fix some \(x > 0\). Finally, let \(T\) be a finite stopping time. Assume that either:
• each $S^i$, $i \in \{1, \ldots, d\}$, is nonnegative and $\sup_{X \in \mathcal{X}^i(x)} \mathbb{E}[U(X_T)] < \infty$, or
• each $S^i$, $i \in \{1, \ldots, d\}$, is locally bounded from below and $\sup_{X \in \mathcal{X}^i(x)} \mathbb{E}[U(X_T)] < \infty$.

Then, the process $(S_{T\wedge t})_{t \in \mathbb{R}_+}$ is a semimartingale.

Proof. Start by assuming that each $S^i$, $i \in \{1, \ldots, d\}$ is nonnegative and that $\sup_{X \in \mathcal{X}^i(x)} \mathbb{E}[U(X_T)] < \infty$. Since we only care about the semimartingale property of $(S_{T\wedge t})_{t \in \mathbb{R}_+}$, assume without loss of generality that $S_t = S_{T\wedge t}$ for all $t \in \mathbb{R}_+$. Suppose that condition NA1$_s$ fails. According to Proposition 2.2 and Remark 2.2, there exists a sequence $(\tilde{X}^n)_{n \in \mathbb{N}}$ of elements in $\mathcal{X}^i(x)$ and $p > 0$ such that $\mathbb{P}[\tilde{X}^n > 2n] \geq p$ for all $n \in \mathbb{N}$. For all $n \in \mathbb{N}$, let $X^n := (x + \tilde{X}^n)/2 \in \mathcal{X}^i(x)$. Then, $\sup_{X \in \mathcal{X}^i(x)} \mathbb{E}[U(X_T)] \geq \lim_{n \to \infty} \mathbb{E}[U(X^n_T)] \geq (1 - p)U(x/2) + p \lim_{n \to \infty} U(n) = \infty$. This is a contradiction to $\sup_{X \in \mathcal{X}^i(x)} \mathbb{E}[U(X_T)] < \infty$. We conclude that $(S_{T\wedge t})_{t \in \mathbb{R}_+}$ is a semimartingale using implication $(i) \Rightarrow (iii)$ in statement (1) of Theorem 2.3.

Under the assumption that each $S^i$, $i \in \{1, \ldots, d\}$ is locally bounded from below and that $\sup_{X \in \mathcal{X}^i(x)} \mathbb{E}[U(X_T)] < \infty$, the proof is exactly the same as the one in the preceding paragraph, provided that one replaces “$\mathcal{X}^i$” with “$\mathcal{X}^i_n$” throughout, and uses the fact that condition NA1$_s$ for the class $\mathcal{X}^i_n$ implies the semimartingale property for $S$, as was discussed in (3.2).

3.4. On the implication $(iii) \Rightarrow (i)$ in Theorem 2.3 If we do not require the additional assumption on $S$ in statement (2) of Theorem 2.3, implication $(iii) \Rightarrow (i)$ might fail. We present below a counterexample where this happens.

On $(\Omega, \mathcal{F}, \mathbb{P})$, let $W$ be a standard, one-dimensional Brownian motion (with respect to its own natural filtration—we have not defined $(\mathcal{F}_t)_{t \in \mathbb{R}_+}$ yet). Define the process $\xi$ via $\xi_t := \exp(-t/4 + W_t)$ for $t \in \mathbb{R}_+$. Since $\lim_{t \to -\infty} W_0/t = 0$, $\mathbb{P}$-a.s., it is straightforward to check that $\xi_\infty := \lim_{t \to -\infty} \xi_t = 0$, and actually that $\int_0^\infty \xi_t \, dt < \infty$, both holding $\mathbb{P}$-a.s. Write $\xi = A + M$ for the Doob-Meyer decomposition of the continuous submartingale $\xi$ under its natural filtration, where $A = (1/4) \int_0^\infty \xi_t \, dt$ and $M = \int_0^\infty \xi_t \, dW_t$. Due to $\int_0^\infty \xi_t \, dt < \infty$, we have $A_\infty < \infty$ and $[M, M]_\infty = \int_0^\infty |\xi_t|^2 \, dt < \infty$, where $[M, M]$ is the quadratic variation process of $M$. In the terminology of [9], $\xi$ is a semimartingale up to infinity. If we define $S$ via $S_t = \xi_t/(1-t)$ for $t \in [0,1[$ and $S_t = 0$ for $t \in [1,\infty[$, then $S$ is a nonnegative semimartingale. Define $(\mathcal{F}_t)_{t \in \mathbb{R}_+}$ to be the augmentation of the natural filtration of $S$. Observe that $\zeta^S = 1$ and $S_{\xi_S^=} = 0$; the condition of statement (2) of Theorem 2.3 is not satisfied. In order to establish that NA1$_s$ fails, and in view of Proposition 2.1, it is sufficient to show that $\{X_1 \mid X \in \mathcal{X}(1)\}$ is not bounded in probability. Using continuous-time trading, define a wealth process $\tilde{X}$ for $t \in [0,1[, \tilde{X}_0 = 1$ and the dynamics $d\tilde{X}_t/\tilde{X}_t = (1/4)(dS_t/S_t)$ for $t \in [0,1[$. Then, $\tilde{X}_t = \exp((1/16)(t/(1-t)) + (1/4)W_t/(1-t))$ for $t \in [0,1[$, which implies that $\mathbb{P}[\lim_{t \uparrow 1} \tilde{X}_t = \infty] = 1$, where “$t \uparrow 1$” means that $t$ strictly increases to 1. Here, the percentage of investment is $1/4 \in [0,1]$, i.e, $\tilde{X}$ is the result of a no-short-sales strategy. One can then find an approximating sequence $(X^k)_{k \in \mathbb{N}}$ such that $X^k \in \mathcal{X}(1)$ for all $k \in \mathbb{N}$, as well as $\mathbb{P}[|X^k_t - \tilde{X}_{t-1/k}| < 1] > 1 - 1/k$. (Approximation results of this sort are discussed in greater generality in [21].) Then, $(X^k)_{k \in \mathbb{N}}$ is not bounded in probability; therefore, NA1$_s$ fails. Of course, in this example we also have $(iii) \Rightarrow (iv)$ of Theorem 2.3 failing.
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