NEW CONSEQUENCES OF THE RIEMANN-SIEGEL FORMULA
AND A LAW OF ASYMPTOTIC EQUALITY OF
SIGNUM-AREAS OF Z(t) FUNCTION

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Abstract. In this paper we obtain the first mean-value theorems for the function Z(t) on some disconnected sets. Next, we obtain a geometric law that controls chaotic behavior of the graph of the function Z(t). This paper is the English version of the papers [8] and [9], except of the Appendix that connects our results with the theory of Jacob’s ladders, namely new third-order formulae have been obtained.

1. Introduction

1.1. First of all we define the following collection of sequences

\[ \{t_\nu(\tau)\}, \ \nu = 1, 2, \ldots, \ \tau \in [-\pi, \pi] \]

by the equation

\[ \varsigma[t_\nu(\tau)] = \pi \nu + \tau; \ \ t_\nu(0) = t_\nu, \]

where (see [12], pp. 79, 329)

\[ Z(t) = e^{i\varphi(t)} \zeta \left( \frac{1}{2} + it \right), \]

\[ \varphi(t) = -\frac{t}{2} \ln \pi + \text{Im} \ln \Gamma \left( \frac{1}{4} + \frac{it}{2} \right). \]

If we use this collection together with the Riemann-Siegel formula (see [10], p. 60, comp. [12], p. 79)

\[ Z(t) = 2 \sum_{n \leq t} \frac{1}{\sqrt{n}} \cos \{\varphi(t) - t \ln n\} + \mathcal{O}(t^{-1/4}), \ \bar{t} = \sqrt{\frac{t}{2\pi}}, \]

then we obtain a new kind of mean-value theorems for the function Z(t) on some disconnected sets. These formulae contain new information about the distribution of positive and negative values of the function Z(t) on corresponding sets.

1.2. The main reason to introduce mentioned disconnected sets lies in the study of the internal structure of the Hardy-Littlewood estimate (1918), (see [1], p. 178)

\[ \int_T^{T+\Omega} Z(t) dt = o(\Omega) \]

for corresponding \( \Omega \).

Key words and phrases. Riemann zeta-function.
Let us remind the following about the estimate (1.4). Hardy and Littlewood have obtained the following estimate (see [1], p. 178)

\[ \int_{T}^{T+M} x(t) \, dt = O(T^{\delta}), \quad M = T^{1/4+\epsilon}, \quad \epsilon > 0, \quad \delta > 0, \]

where (see [1], p. 177)

\[ f(s) = \pi^{-s} e^{-\frac{1}{2}(s-\frac{1}{2})\pi i} \Gamma(s) \zeta(2s), \quad s = \sigma + it, \]

\[ x(t) = f\left(\frac{1}{4} + it\right). \]

Since (see [1], p. 178)

\[ x(t) = -e^{-\pi t} \Xi(2t) \frac{\pi}{4} + 4t^2, \]

and (see [12], p. 68, (4.12.2); \( \sigma = \frac{1}{4} \))

\[ Z(t) = -2\pi^{1/4} \frac{\Xi(t)}{(\frac{1}{4} + t^2) |\Gamma(\frac{1}{4} + it)|}, \]

\[ \left| \Gamma\left(\frac{1}{4} + \frac{t}{2}\right) \right| = t^{-1/4} e^{-\frac{1}{2}\pi t} \sqrt{2\pi} \left\{ 1 + O\left(\frac{1}{t}\right) \right\}, \]

then we have the expression

\[ x(t) = 2^{1/2} \pi^{-1/4} t^{-1/4} Z(2t) \left\{ 1 + O\left(\frac{1}{t}\right) \right\}. \]

Hence, we obtain from (1.5) by (1.6) the estimate

\[ \int_{T}^{T+M} Z(2t) \, dt = O(T^{1/4+\delta}), \]

and, consequently

\[ \int_{2T}^{2T+2M} Z(t) \, dt = O(T^{1/4+\delta}); \quad 2T \to \bar{T}, \quad 2M \to \bar{M} \]

\[ \int_{\bar{T}}^{\bar{T}+\bar{M}} Z(t) \, dt = O(T^{1/4+\delta}), \]

i.e. we have (1.4).

2. NEW ASYMPTOTIC FORMULAE FOR THE FUNCTION \( Z(t) \)

2.1. Let

\[ G_1(x) = G_1(x; T, H) = \]

\[ = \bigcup_{T \leq t_{2T} \leq T+H} \{ t : t_{2T}(-x) < t < t_{2T}(x) \}, \quad 0 < x \leq \frac{\pi}{2}, \]

(2.1)

\[ G_2(y) = G_2(y; T, H) = \]

\[ = \bigcup_{T \leq t_{2y} \leq T+H} \{ t : t_{2y+1}(-y) < t < t_{2y+1}(y) \}, \quad 0 < y \leq \frac{\pi}{2}, \]

where

\[ H = T^{1/6+2\epsilon}; \quad T^{1/6} \psi^2 \ln^5 \bar{T} \to T^{1/6+\epsilon}. \]

The following theorem holds true.
Theorem 1.

\[
\int_{G_1(x)} Z(t) dt = \frac{2}{\pi} H \sin x + O(x T^{1/6+\epsilon}),
\]
\[
\int_{G_2(y)} Z(t) dt = -\frac{2}{\pi} H \sin y + O(y T^{1/6+\epsilon}),
\]
\[x, y \in (0, \frac{\pi}{2}]\]

(2.3)

Remark 1. By the formula (2.3) we have expressed the internal structure of the Hardy-Littlewood integral

\[
\int_{T}^{T+H} Z(t) dt,
\]
i. e. the decomposition of this integral into its parts.

Remark 2. We will assume that

(2.4) \quad G_1(x), G_2(y) \subset [T, T + H]
since we may put

\[G_1(x) \cap [T, T + H] = \bar{G}_1(x) \rightarrow G_1(x), \ldots\]

Remark 3. The existence of the odd zero of the function

\[Z(t), \quad t \in [T, T + T^{1/6+2\epsilon}]\]

follows directly from our formulae (2.3).

2.2. Since (see (1.1))

\[
\vartheta[t_2 \nu(x)] - \vartheta[t_2 \nu(-x)] = 2x,
\]
\[
\vartheta[t_2 \nu+1(y)] - \vartheta[t_2 \nu+1(-y)] = 2y,
\]
we obtain (see [1], p. 102; [3], (42))

(2.5)

\[
t_{2 \nu}(x) - t_{2 \nu}(-x) = \frac{4x}{\ln \frac{2\pi}{T}} + O\left(\frac{x H}{T \ln^2 T}\right),
\]
\[
t_{2 \nu+1}(y) - t_{2 \nu+1}(-y) = \frac{4y}{\ln \frac{2\pi}{T}} + O\left(\frac{y H}{T \ln^2 T}\right).
\]

(2.6)

Next we have (see (2.1), (2.6) and [4], (23))

(2.7)

\[m\{G_1(x)\} = \frac{x}{\pi} H + O(x), \quad m\{G_2(y)\} = \frac{y}{\pi} H + O(y),\]

where \(m\{G_1(x)\}, m\{G_2(y)\}\) stand for measures of the corresponding sets. Following (2.3) we obtain

Corollary 1.

\[
\frac{1}{m\{G_1(x)\}} \int_{G_1(x)} Z(t) dt \sim \frac{2}{x} \sin x,
\]
\[
\frac{1}{m\{G_2(y)\}} \int_{G_2(y)} Z(t) dt \sim -\frac{2}{y} \sin y.
\]

(2.8)
Since (see (1.1), (2.1))

\[ G_1(x) \cap G_2(y) = \emptyset; \quad t_{2\nu} \left( \frac{\pi}{2} \right) = t_{2\nu+1} \left( -\frac{\pi}{2} \right), \]

we obtain from (2.3) the following

**Corollary 2.**

\[
\int_{G_1(x) \cup G_2(y)} Z(t) dt = \left\{ \begin{array}{ll}
\frac{2}{\pi} (\sin x - \sin y) H + O\{(x + y)T^{1/6+\epsilon}\} & \text{, } x \neq y \\
O(xT^{1/6+\epsilon}) & \text{, } x = y.
\end{array} \right.
\]

**Remark 4.** Since (see (2.4))

\[ m\{G_1(\pi/2)\} + m\{G_2(\pi/2)\} = H, \]

we have (see (2.2), (2.9))

\[
\int_{G_1(\pi/2) \cup G_2(\pi/2)} Z(t) dt = \int_T^{T+H} Z(t) dt = o(H),
\]

comp. (1.3). Consequently, the estimate of the Hardy-Littlewood type is direct connects of the asymptotic formulae (2.3).

### 3. Law of asymptotic equality of signum-areas of $Z(t)$ function

3.1. Let us point out the chaotic behavior of the graph of function $Z(t)$ at $t \to \infty$, (comp., for example, the graph of $Z(t)$ in the neighborhood of the first Lehmer pair of the zeroes, [2], pp. 296, 297). In this direction we obtain a new law that controls this chaotic behavior.

Let

\[
\begin{align*}
G_1^+(x) &= \{ t : t \in G_1(x), Z(t) > 0 \}, \\
G_1^-(x) &= \{ t : t \in G_1(x), Z(t) < 0 \}, \\
G_2^+(x) &= \{ t : t \in G_2(x), Z(t) > 0 \}, \\
G_2^-(x) &= \{ t : t \in G_2(x), Z(t) < 0 \}, \\
G_3(x) &= \{ t : t \in G_1(x), Z(t) = 0 \}, \\
G_4(x) &= \{ t : t \in G_2(x), Z(t) = 0 \}.
\end{align*}
\]

(3.1)

Of course,

\[ m\{G_3(x)\} = m\{G_4(x)\} = 0. \]

The following Theorem holds true.

**Theorem 2.**

\[
\int_{G_1^+(x) \cup G_2^+(x)} Z(t) dt \sim -\int_{G_1^-(x) \cup G_2^-(x)} Z(t) dt,
\]

\[ T \to \infty, \quad x \in \left( 0, \frac{\pi}{2} \right]. \]

Let

\[
D^+(x) = \{(t, u) : t \in G_1^+(x) \cup G_2^+(x), \ 0 < u \leq Z(t)\},
\]

(3.2)

\[
D^-(x) = \{(t, u) : t \in G_1^-(x) \cup G_2^-(x), \ Z(t) \leq u < 0\}. 
\]

(3.3)
Remark 5. The asymptotic equality (3.2) expresses the following geometric law
(3.4) \[ m\{D^+(x)\} \sim m\{D^-(x)\}, \quad T \to \infty, \quad x \in \left(0, \frac{\pi}{2}\right), \]
where \( D^+(x), D^-(x) \) (see (3.1)) are the first mentioned signum-sets.

Remark 6. It is just the geometric law (3.4) that controls the chaotic behavior of the graph of function \( Z(t) \).

4. Lemma 1

The following lemma holds true.

Lemma 1.

\[ \sum_{T \leq t_\nu(\tau) \leq T + H} Z[t_\nu(\tau)] = O(T^{1/6+\epsilon}), \quad \tau \in [-\pi, \pi]. \] (4.1)

Proof. Let us remind the formulae (see [3], (42); [4], (23))

\[ t_{\nu+1} - t_\nu = \frac{2\pi}{\ln \frac{T}{2\pi}} + O\left(\frac{H}{T \ln^2 T}\right), \]

\[ \sum_{T \leq t_\nu \leq T + H} 1 = \frac{1}{2\pi} H \ln \frac{T}{2\pi} + O(1). \] (4.2)

By the same way (comp. [11], p. 102; [3], (40) – (42)) we obtain (see (1.1)) the formulae

\[ t_{\nu+1}(\tau) - t_\nu(\tau) = \frac{2\pi}{\ln \frac{T}{2\pi}} + O\left(\frac{H}{T \ln^2 T}\right), \]

\[ \sum_{T \leq t_\nu(\tau) \leq T + H} 1 = \frac{1}{2\pi} H \ln \frac{T}{2\pi} + O(1). \] (4.3)

Next, from (1.3) we have

\[ Z(t) = 2 \sum_{n < P_0} \frac{1}{\sqrt{n}} \cos\{\vartheta - t \ln n\} + O(T^{-1/4}), \quad P_0 = \sqrt{\frac{T}{2\pi}}, \]

\[ t \in [T, T + H] \]

and, consequently (see (1.3))

\[ Z[t_\nu(\tau)] = 2(-1)^\nu \cos \tau + \]

\[ + 2(-1)^\nu \sum_{2 \leq n < P_0} \frac{1}{\sqrt{n}} \cos\{\tau - t_\nu(\tau) \ln n\} + O(T^{-1/4}). \] (4.4)

Hence, (see [4.3], (4.4))

\[ \sum_{T \leq t_\nu(\tau) \leq T + H} Z[t_\nu(\tau)] = \]

\[ = 2 \cos \tau \sum_{2 \leq n < P_0} \frac{1}{\sqrt{n}} \sum_{T \leq t_\nu(\tau) \leq T + H} (-1)^\nu \cos\{t_\nu(\tau) \ln n\} + \]

\[ + 2 \sin \tau \sum_{2 \leq n < P_0} \frac{1}{\sqrt{n}} \sum_{T \leq t_\nu(\tau) \leq T + H} (-1)^\nu \cos\{t_\nu(\tau) \ln n\} + O(\ln T) = \]

\[ = 2w_1 \cos \tau + 2w_2 \sin \tau + O(\ln T). \] (4.5)
By the method of the papers [3], [7] we obtain the estimate
\[ w_1 = \mathcal{O}(T^{1/6+\epsilon}). \]
Next, instead of [3], (54) we have by [5], (66)
\[ \sum_{T \leq T_1(\tau)} \leq t_\nu \leq T + H \]
\[ \sum_{T \leq t_\nu(\tau) \leq T + H} (-1)^\nu \sin\{t_\nu \ln n\} = \]
\[ = \frac{1}{2}(-1)^{N+\nu} \sin(\omega N + \varphi) + \frac{1}{2}(-1)^{N+\nu} \tan \frac{\omega}{2} \cos \varphi - \]
\[ - \frac{1}{2}(-1)^{N+\nu} \tan \frac{\omega}{2} \cos(\omega N + \varphi) + O\left(\frac{H^3 \ln n}{T}\right), \]
and instead of [7], (42) we have
\[ \cos \varphi - \cos(\omega N + \varphi) = 2 \sin \left(\frac{\omega}{2} + \varphi\right) = \]
\[ = -2 \sin(Nx(n)) \sin(T \ln n). \]
Then, by the method of the papers [3], [7] we obtain the estimate
\[ w_2 = \mathcal{O}(T^{1/6+\epsilon}). \]
Hence, the estimate (4.1) follows from (4.5), (4.6), (4.7). □

5. **Lemma 2**

Next, the following lemma holds true.

**Lemma 2.**
\[ \sum_{T \leq t_\nu(\tau) \leq T + H} (-1)^\nu Z[t_\nu(\tau)] = \]
\[ = \frac{1}{\pi} \ln H \frac{T}{2\pi} \cos \tau + O(T^{1/6} \ln T), \quad \tau \in [-\pi, \pi]. \]

**Remark 7.** The formula (5.1) is the asymptotic formula for
\[ H = T^{1/6+2\epsilon}, \quad \tau \in [-\pi, \pi], \quad \tau \neq -\frac{\pi}{2}, \frac{\pi}{2}. \]

**Proof.** We have the following formula from [12]
\[ \sum_{T \leq t_\nu(\tau) \leq T + H} (-1)^\nu Z[t_\nu(\tau)] = \frac{1}{\pi} \ln H \frac{T}{2\pi} \cos \tau + \]
\[ + 2 \cos \tau \sum_{2 \leq n < P_0} \frac{1}{\sqrt{n}} \sum_{T \leq t_\nu(\tau) \leq T + H} \cos\{t_\nu(\tau) \ln n\} + \]
\[ + 2 \sin \tau \sum_{2 \leq n < P_0} \frac{1}{\sqrt{n}} \sum_{T \leq t_\nu(\tau) \leq T + H} \sin\{t_\nu(\tau) \ln n\} + O(\ln T) = \]
\[ = \frac{1}{\pi} \ln H \frac{T}{2\pi} \cos \tau + 2w_3 \cos \tau + 2w_4 \sin \tau + O(\ln T) = \]
\[ = \frac{1}{\pi} \ln H \frac{T}{2\pi} \cos \tau + \mathcal{O}(T^{1/6} \ln T), \]
where the estimates for the sums \( w_3, w_4 \) were obtained by the methods of papers [4], [6]. □
6. Proof of Theorem 1

First of all, we have the following formulae (see (4.1), (5.1))

\[
\sum_{T \leq t_{2\nu} \leq T + H} Z[t_{2\nu}(\tau)] = \frac{1}{2\pi} H \ln \frac{T}{2\pi} \cos \tau + O(T^{1/6+\varepsilon}),
\]

(6.1)

\[
\sum_{T \leq t_{2\nu+1} \leq T + H} Z[t_{2\nu+1}(\tau)] = -\frac{1}{2\pi} H \ln \frac{T}{2\pi} \cos \tau + O(T^{1/6+\varepsilon}).
\]

Since (see [11], p. 100)

\[
\vartheta'(t) = \frac{1}{2} \ln \frac{t}{2\pi} + O\left(\frac{1}{t}\right),
\]

we obtain from (1.1) that

\[
\int_{t_{2\nu}(-x)}^{t_{2\nu}(x)} Z(t) dt = O(HT^{-5/6}),
\]

where

\[
(6.3)
\]

\[
Z(t) = O(t^{1/6} \ln t), \quad t \to \infty.
\]

(6.4)

Finally, by integration (see (6.2)) of the first formula in (6.1) (after the transformation (6.4)) we obtain the first formula in (2.3). The second formula in (2.3) can be obtained by the similar way.

7. Proof of Theorem 2

The following holds true (see (2.2), (2.9), (3.1))

\[
\int_{G_1(x) \cup G_2(x)} Z(t) dt = \int_{G_1^+(x) \cup G_1^-(x) \cup G_2^+(x) \cup G_2^-(x)} Z(t) dt =
\]

(7.1)

\[
= \int_{G_1^+(x) \cup G_1^+(x)} Z(t) dt + \int_{G_1^-(x) \cup G_2^-(x)} Z(t) dt = O(xT^{1/6+\varepsilon}) = o(xH).
\]

Since (see (2.3))

\[
\int_{G_1(x)} Z(t) dt > \left(\frac{2}{\pi} \sin x - \varepsilon\right) H = A(x, \varepsilon) H, \quad 0 < \varepsilon < \frac{1}{\pi} \sin x
\]
and
\[
\int_{G_1(x)} Z(t) dt = \int_{G_1^+(x) \cup G_1^-(x)} Z(t) dt \leq \int_{G_1^+(x) \cup G_1^+(x)} Z(t) dt \leq \int_{G_1^-(x) \cup G_1^-(x)} Z(t) dt,
\]
we have the following inequality
\[
(7.2) \quad \int_{G_1^+(x) \cup G_1^+(x)} Z(t) dt > A(x, \epsilon) H
\]
and, by the similar way, we have
\[
(7.3) \quad -\int_{G_1^-(x) \cup G_1^-(x)} Z(t) dt > B(x, \epsilon) H.
\]
Hence, the equality (see (7.1))
\[
\int_{G_1^+(x) \cup G_1^+(x)} Z(t) dt = \int_{G_1^-(x) \cup G_1^-(x)} Z(t) dt + o(H), \quad x \in \left(0, \frac{\pi}{2}\right)
\]
is the asymptotic equality (3.2) by (7.2) and (7.3).

**Appendix A. Jacob’s ladders and new third-order formulae corresponding to (2.8) and (3.2)**

In our paper [13], (9.2), (9.5) we have proved the following lemma: if
\[
\varphi_1\{[\hat{T}, \hat{T} + U]\} = [T, T + U],
\]
then for every Lebesgue-integrable function
\[
f(x), \quad x \in [T, T + U]
\]
we have
\[
(A.1) \quad \int_T^{T+U} f[\varphi_1(t)] \hat{Z}^2(t) dt = \int_T^{T+U} f(x) dx,
\]
where
\[
\hat{Z}^2(t) = \frac{Z^2(t)}{\left(1 + \mathcal{O}\left(\frac{\ln \ln t}{\ln t}\right)\right)} \ln t = \omega(t) Z^2(t);
\]
\[
(A.2) \quad \omega(t) = \frac{1}{\left(1 + \mathcal{O}\left(\frac{\ln \ln t}{\ln t}\right)\right)} \ln t = \frac{1}{\ln t} \left(1 + \mathcal{O}\left(\frac{\ln \ln t}{\ln t}\right)\right),
\]
and \(\varphi_1(t)\) is a fixed Jacob’s ladder. Consequently we have (see (A.1), (A.2))
\[
(A.3) \quad \int_T^{T+U} \omega(t) f[\varphi_1(t)] Z^2(t) dt = \int_T^{T+U} f(x) dx, \quad U \in \left(0, \frac{T}{\ln T}\right).
\]
Now, we obtain from (2.8), (3.2) by (A.3) the following third-order formulae

\[
\frac{1}{m\{G_1(x)\}} \int_{G_1(x)} \omega(t)Z[\varphi_1(t)]Z^2(t)dt \sim \frac{2\sin x}{x},
\]

\[
(2.8')
\]

\[
\frac{1}{m\{G_2(y)\}} \int_{G_2(y)} \omega(t)Z[\varphi_1(t)]Z^2(t)dt \sim -\frac{2\sin y}{y},
\]

and

\[
\int_{G_1^+(x)\cup G_1^-(x)} \omega(t)Z[\varphi_1(t)]Z^2(t)dt \sim
\]

\[
(3.2')
\]

\[
\sim -\int_{G_1^+(x)\cup G_1^-(x)} \omega(t)Z[\varphi_1(t)]Z^2(t)dt,
\]

\[T \to \infty.\]

I would like to thank Michal Demetrian for helping me with the electronic version of this work.

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