Two-Component Cosmological Fluids with Gravitational Instabilities

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Abstract

A survey of linearized cosmological fluid equations with a number of different matter components is made. To begin with, the one-component case is reconsidered to illustrate some important mathematical and physical points rarely discussed in the literature. The work of some previous studies of two-component systems are examined and re-analyzed to point out some deficiencies of solutions, and further solutions and physical interpretation are then presented. This leads into a general two-component model with variable velocity dispersion parameters and mass density fractions of each component. The equations, applicable to both hot dark matter (HDM) and cold dark matter (CDM) universes are solved in the long wavelength limit. This region is of interest, because some modes in this range of wavenumbers are Jeans unstable. The mixture Jeans wavenumber of the two-component system is introduced and interpreted, and the solutions are discussed, particularly in comparison to analogous solutions previously derived for plasma modes. This work is applicable to that region in the early Universe (20 < z < 140), where large scale structure formation is thought to have occurred.

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I. INTRODUCTION

The theory of structure formation in the Universe has become one of the most popular and intensely studied topics in modern cosmology. Throughout the past century there has been an accumulating volume of work on the analytic investigation of the cosmological structure formation equations. The various approaches include both fluid and kinetic theory formulations. They principally consider the gravitational interaction of components of the cosmological medium, though sometimes other forms of interaction such as magnetic fields are also included (for some standard examples see e.g. [1, 2]). The analysis of these equations has employed ever more diverse and complicated techniques and approximation schemes to model increasingly realistic physical situations. This has been comprehensively supported and now superseded by large N-body simulations. The algorithms which govern these large numerical studies have grown progressively more refined and subtle, and are now producing very accurate and realistic results, which can be directly compared with observations (e.g. [3]).

Despite the current trends in modern cosmological structure formation theory, much can still be learned from relatively simple analytic models. We consider such models, in the face of modern computing power, to analyze at a fundamental level some of the basic physical processes which cause the clustering observed in the Universe. This helps to isolate physical mechanisms difficult to discern numerically. In this paper our interest will focus on the linearized cosmological fluid equations. These equations have been used to build up the components of the cosmological density perturbation power spectrum, and must be evolved through the various stages of cosmological evolution, and over a large range of physical scales. There are several reasons for taking such an approach. The equations may be solved numerically to give detailed power spectra for the various cosmological models currently viable. The power spectra may then be used as initial data to evolve the large N-body simulations, which are ultimately compared to observations. The equations may also be used to build up a semi-quantitative picture of the evolution of the power spectrum. This can show how the various sized perturbations scale with respect to the Friedmann expansion parameter $a$ during different epochs of the Universe, and give a direct insight into some of the fundamental physical processes operating to produce structure in the Universe (see e.g. [4]).
The evolution of perturbation modes with wavelength greater than the Hubble radius may be studied through a relativistic formulation of the perturbation equations, whereas for modes with wavelengths smaller than this radius, a Newtonian formalism suffices. There is a range of physical parameters and variety of differential equations describing the evolution processes of density fluctuations in the early Universe. This involves such elements as equations of state for energy constituents, and specification of the expansion parameter $a$ by the Friedmann cosmological equations. As a consequence, there is a large wealth of literature on this subject, and most of the currently important techniques and results have been collected in some well known textbooks [1, 2, 4]. Some of the relatively complicated systems of equations have also been studied in the literature, and it is our goal to both review many of these studies, and to extend them in new directions, achieving some new unique results.

In this paper we will only study the Newtonian limit of the linearized cosmological perturbation equations, valid for density fluctuations on scales well within the Hubble radius. Our main concern is with some mathematically more complicated multi-component models, which although not usually considered in standard power spectrum analysis, have realistic and interesting physical meaning. The primary concept of the Jeans gravitational instability [5] has been investigated in a static universe for multi-component models, to reveal the more complicated structure of modes possible [6, 7]. This provides some interesting qualitative ideas about the possible mechanisms for structure formation, but the lack of an expanding background spacetime in the models leads to unrealistic solutions, exponential in form. The inclusion of cosmological expansion in the equations leads to the more realistic power law and logarithmic solutions, familiar from the standard power spectrum analysis. Previous work in this area has focused both on some particular models [8], and on a more general classification of the equations and solutions for a range of parameter values (some of them only of mathematical interest) and physical contexts [9, 10, 11]. Analytic solutions for some of the most general cases of the equations considered above, which often have significant physical interest, have not been achieved. It is our aim here to rectify this situation and investigate a system of equations modeling a two-component fluid in the matter dominated post-recombination era of an Einstein–deSitter universe. One of these components consists of baryons, and the other some form of nonrelativistic dark matter particles. Through this work we will amend what appear to be some errors in the previous general studies of Haubold
This paper makes a comprehensive study of the linear perturbation equations for cosmological fluids with gravitational instabilities with application to large scale structure formation. For a historical perspective we note that Lifshitz concluded that gravitational instability could not be responsible for the formation of structure in the Universe. The correct conclusion, that gravitational instability suffices, was pointed out by Novikov. As our work details the mathematical structure of the appropriate equations describing cosmological structure formation, we note the nice series of papers by Ratra and Peebles directed to understanding the applications of special functions to the problem of gravitational instability in cosmological models. We further note the nice series by Buchert et al concerning analytic results and their relevance to observational cosmology.

We will also make a comparison with work done in cosmological plasma physics in an Einstein-deSitter background. This is interesting due to the mathematically very similar form of fluid equations for both type of systems. This similarity is largely due to the similarity of the electromagnetic and gravitational forces. In this paper we will analyze the long wavelength region of the solutions. This corresponds to the Jeans unstable region of parameter space, and requires use of Frobenius methods of expansion of the differential equations. In a follow-up paper we will investigate the short wavelength region of the solutions, which will require a WKB approximation scheme to be developed.

In these papers we take explicitly the temperature relationship $T \sim 1/a(t)^2$, where $a(t)$ is the radius of the Universe. We give here the explanation why this is so. Following standard textbook material in Padmanabhan (as summarized in equation (3.118)) and Peebles (1993, page 179) we find that the baryons follow this relationship when

$$1 + z = 142(\Omega_b h^2/0.024)^{2/5}.$$ 

Here $\Omega_b$ has been scaled to the WMAP best fit value. Thus for redshifts below $z \sim 140$ the baryon temperature drops as $1/a^2$ down to $z \sim 20$, where the Universe reionizes (probably in a patchy fashion) and the $1/a^2$ scaling no longer holds. This $20 < z < 140$ redshift region is important, as it is where early large scale structure formation is thought to have occurred.

The paper is to be organized as follows. In Section 2 we introduce the most general cosmological density perturbation equations in the Newtonian approximation. We review and classify previous work on these equations to put our current work into context, showing
what has been achieved, what needs amendment, and where we will seek to expand current knowledge. In Section 3 we revert to the one-component equations, to illustrate some of the basic principles which will be important later in our analysis, and to reveal some apparently new results. This will enable us to begin to tackle the two-component problem in Section 4. In this section the CDM two-component model in an expanding universe will be investigated. This has ties with the previous work cited, and we will demonstrate the limitations of the existing formalism here. We present new results apparently overlooked in the work of Haubold and Mathai [11]. After this, we are ready to study the most general baryonic and dark matter equations in Section 5, where we will consider the long wavelength approximation, applicable to either HDM or CDM. This is followed by our conclusions in Section 6.

II. A CLASSIFICATION OF COSMOLOGICAL DENSITY PERTURBATION EQUATIONS

As discussed in Section 1, there are a vast spectrum of equations describing cosmological density perturbations in different physical regimes. We begin directly with the linearized Newtonian approach. The equations for an \( n \)-component system of nonrelativistic species is derived in all the standard texts. Given a density perturbation \( \delta_i \) in the \( i \)-th component of the mass density \( \rho_i \):

\[
\delta_i(r, t) = \frac{\delta \rho_i}{\rho_i}, \tag{2.1}
\]

it may be decomposed into its Fourier plane wave modes with wave vector \( \mathbf{k} \)

\[
\delta_i(r, t) = \frac{1}{(2\pi)^3} \int \delta_{k}(t) \exp(-i\mathbf{k} \cdot \mathbf{r}) d^3r. \tag{2.2}
\]

Here \( \mathbf{r} \) is the physical spatial coordinate, and \( t \) is cosmic time.

To be able to solve the equations, the implicit time dependence of the physical variables needs to be removed. We will adopt the convention that barred variables will denote comoving quantities, independent of time. Thus we define the comoving wave number \( \bar{k} = a \mathbf{k} \). Using the Eulerian equations of motion describing a perfect fluid, a set of coupled second order equations for the Fourier modes \( \delta_i(t) \) (where we now drop the subscript \( \mathbf{k} \)) are achieved:

\[
\frac{d^2 \delta_i}{dt^2} + 2\frac{\dot{a}}{a} \frac{d \delta_i}{dt} + \frac{v_i^2 \bar{k}^2}{a^2} \delta_i = 4\pi G \sum_{i=1}^{n} \rho_i \delta_i, \quad i = 1, 2, \ldots n. \tag{2.3}
\]
Overdots will denote derivatives with respect to $t$. The above equations contain the sound velocity

$$ v_i^2 = \frac{dp_i}{d\rho_i} \propto \rho_i^{\gamma_i^{-1}}. \quad (2.4) $$

The sound velocity depends on the equation of state of the medium, and is in general dependent on time. We have introduced the specific heat ratio $\gamma_i$, and assumed an equation of state of the form $p_i \propto \rho_i^{\gamma_i}$. The sound velocity implicitly assumes that the fluids involved are collisional. This means that there are considerable interactions between the particles comprising each matter component. It is in fact generally assumed that dark matter is collisionless, in which case a fluid equation is not strictly correct. The dark matter component would better be modeled by a distribution function satisfying the Vlasov equation. “Fluid-like” equations can still be derived in this case by taking velocity moments of the Vlasov equation, and identifying the velocity dispersion with the above parameter $v_i$. Thus although in the present paper we will refer to “sound velocities”, this should be taken as a generic expression for a velocity dispersion parameter. Such an approach should work fine for CDM, but may neglect an important damping term found in HDM models. The complete analogous equation to (2.3) for HDM gives a fluid equation perspective on free-streaming, the phenomenon found in HDM models of neutrino-like matter. An approximate equation has been derived for a hot neutrino-like component by Setayeshgar [20], which tends to wipe out perturbations below a certain scale (see also the lecture notes by Bertschinger [21]). In this work the exact Vlasov equation kinetic treatment was considered, and the usual Fermi-Dirac distribution function was replaced by a carefully chosen approximate form, allowing the conversion of the integro-differential equation into the following pure differential equation:

$$ \ddot{\delta}_\nu + \left( \frac{2a}{a^2} + \frac{2k\nu}{ka^2} \right) \dot{\delta}_\nu + \frac{v_\nu^2k^2}{a^2} - \delta_\nu = 4\pi G \sum_{i=1}^{n} \rho_i \delta_i. \quad (2.5) $$

Here the damping term $2k\nu$ gives rise to non-oscillating solutions heavily damped at short wavelengths. Thus an equation of the form (2.3) is not correct for HDM of a neutrino-like nature. We examine the general equations for both CDM and HDM, without specifying too carefully the exact nature of the dark component involved. This allows comparison of the results in this paper with previous work in the literature, which has also neglected this point. If some aspects of HDM models are poorly described by (2.3), the equations are still
applicable to other two-component cosmological systems such as a hydrogen-helium gas not in equilibrium, where the lighter hydrogen component has a greater sound speed.

At present (2.3) has been displayed in a quite general form, with an unspecified scale factor $a$, given by the Friedmann cosmological equation

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3} \rho + \frac{\Lambda}{3} - \frac{k_c}{a^2}.$$ (2.6)

General parameters describing the nature of the Universe in this equation are $k_c = 0, \pm 1$, the spatial curvature, and $\Lambda$, the cosmological constant. It is difficult to make much progress without first becoming more specific about the energy content of the Universe. All the studies cited previously [8, 9, 10, 11] have only examined the Einstein-deSitter, matter dominated case, with various physical components and equations of state possible. The studies [10, 11], all of which are equivalent, make the pretense to study the radiation dominated era as well, but this is incorrect for the equations presented. It was explicitly assumed that $a \propto t^{2/3}$ in the scaling of the energy densities $\rho_i = \Omega_i/(6\pi G t^2)$ and the velocity parameters, yet an allowance was made for a general Hubble expansion $H = \eta t^{-1}$. The general parameter $\eta$ can only be equal to $\frac{2}{3}$ for the equations presented to be physically correct. The fluid equations were formulated to allow for general equations of state, by writing the sound velocities such that both their magnitudes and time dependences were freely parameterized. A range of solutions were obtained for different cases of the parameters, and were generally classified by Meijer G-functions [22, 23]. We will show that the solutions found for a CDM and baryon model have been evaluated incorrectly, and will proceed to find their general exact representation. We will also proceed to investigate a more general dark matter and baryon problem than considered in any of the above. In [9], several subcases of the above mentioned studies were considered in some detail, and given a range of physical interpretations. The solutions were of a mathematically simpler nature, involving either Bessel functions or simple power law behavior. The investigations in [8] concentrated on a three-component medium, involving baryons, CDM and photons. They incorrectly used the nonrelativistic Newtonian cosmological equations to model the photon component, so that the solutions, expressed in terms of Meijer G-functions, cannot be considered as physically relevant.

Let us now make the choice of the matter dominated era of cosmological evolution in which to set (2.3), and in particular the post-recombination era, where baryons had decoupled from photons. This allows us to determine how the energy density and sound velocity scale with
respect to $a$, and consequently exhibit all explicit time dependences in the equations. We therefore introduce the comoving total background density $\bar{\rho}_0 \equiv a^3 \rho_0$, and the constant $\epsilon_i \equiv \rho_i / \rho_0$, the fraction of the total mass density contributed by species $i$. This is distinct from the in general time dependent quantity $\Omega_i(t) \equiv \rho_i / \rho_c$, where $\rho_c$ is the critical density of the Universe

$$\rho_c = \frac{3H^2}{8\pi G}. \quad (2.7)$$

We will consider a two-component fluid comprised of baryons (subscripted by B) and dark matter (subscripted by D). In the post-recombination era, the adiabatic speed of sound of species $i$ assumes the following behavior:

$$v_i^2 \propto T_i \propto a^{-2}, \quad (2.8)$$

where $T_i$ is the temperature of the component. This prompts us to define the time independent quantity $\bar{v}_i^2 \equiv a^2 v_i^2$. With these definitions, the linearized cosmological perturbation equations may be written as

$$\frac{d^2 \delta_B}{dt^2} + 2 \frac{\dot{a}}{a} \frac{d\delta_B}{dt} + \frac{\bar{v}_B^2 \bar{k}^2}{a^4} \delta_B = \frac{4\pi G \bar{\rho}_0}{a^3} (\epsilon_B \delta_B + \epsilon_D \delta_D), \quad (2.9)$$

$$\frac{d^2 \delta_D}{dt^2} + 2 \frac{\dot{a}}{a} \frac{d\delta_D}{dt} + \frac{\bar{v}_D^2 \bar{k}^2}{a^4} \delta_D = \frac{4\pi G \bar{\rho}_0}{a^3} (\epsilon_B \delta_B + \epsilon_D \delta_D). \quad (2.10)$$

The equations currently still represent a fairly general cosmological setting. The curvature parameter $k_c$ and cosmological constant $\Lambda$ have not been specified, and control the behavior of $a$ through the Friedmann equation $(2.6)$. To see how these influence the evolution of the density perturbations, we transform the dependent variable from $t$ to $a$. We also use $(2.6)$ and another cosmological dynamics equation for the acceleration of $a$:

$$\ddot{a} = \frac{4}{3} \pi G \bar{\rho}_0 \bar{a} + \frac{\Lambda}{3} a. \quad (2.11)$$

This equation is derived in conjunction with the Friedmann equation by taking the spatial components of the Einstein equation. The cosmological perturbation equations are now able to be written in a form purely dependent on $a$, and parameterized explicitly by the cosmological dynamical constants:

$$\left(\frac{8}{3} \pi G \bar{\rho}_0 + \frac{\Lambda}{3} a^3 - k_c a \right) \delta_B'' + a^{-1} \left(4\pi G \bar{\rho}_0 + \Lambda a^3 - 2k_c a \right) \delta_B'$$

$$+ \frac{\bar{v}_B^2 \bar{k}^2}{a^3} \delta_B - \frac{4\pi G \bar{\rho}_0}{a^2} (\epsilon_B \delta_B + \epsilon_D \delta_D) = 0, \quad (2.12)$$

$$\left(\frac{8}{3} \pi G \bar{\rho}_0 + \frac{\Lambda}{3} a^3 - k_c a \right) \delta_D'' + a^{-1} \left(4\pi G \bar{\rho}_0 + \Lambda a^3 - 2k_c a \right) \delta_D'$$

$$+ \frac{\bar{v}_D^2 \bar{k}^2}{a^3} \delta_D - \frac{4\pi G \bar{\rho}_0}{a^2} (\epsilon_B \delta_B + \epsilon_D \delta_D) = 0. \quad (2.13)$$
In the above, a prime denotes differentiation with respect to $a$.

A general analysis of (2.12) and (2.13) has not been attempted previously. To begin with, we can test for exactness of the equations (see [24], pp. 92, 93), to determine whether a first integral exists. In general, a second order ordinary differential equation of the form

$$A_0(x)y'' + A_1(x)y' + A_2(x)y = 0$$  \tag{2.14}$$

[arbitrary functions $A_i(x)$] is exact if

$$A_0'' - A_1' + A_2 = 0.$$  \tag{2.15}$$

Let us first consider the one-component example. This is the uncoupled case of (2.12) ($\epsilon_D = 0$, $\epsilon_B = 1$), which gives

$$A_0'' - A_1' + A_2 = \frac{\bar{v}_B^2 k^2}{a^3}.$$  \tag{2.16}$$

Thus the one-component equation is only exact for a pressureless gas $\bar{v}_B = 0$. This means that no closed form solution is possible, and approximations need to be made. We note that the pressureless one-component case has been studied extensively (e.g. [2]) for various values of the parameters.

To make progress with the perturbation equations, and also to make contact with previous work in the literature, we need to make some assumptions about $k_c$ and $\Lambda$. The $k_c \neq 0$ cases tend to be more complicated mathematically, as generally only parametric solutions can be found, where $a$ is represented by hyperbolic functions ($k_c = -1$ open universe) or trigonometric functions ($k_c = 1$ closed universe). Current observations, and the weight of theoretical tendencies in cosmology (e.g. $\Omega = 1$ as demanded by inflation) make the choice of flat universe $k_c = 0$ seem the most favorable. $\Omega$ contains a contribution from $\Lambda$ as well as matter components. The large amount of observational data now being analyzed, increasingly points to the existence of a cosmological constant comprising a major fraction of the energy density (see e.g. [25], [26], [27]), with a value of $\Omega_\Lambda \simeq 0.7$. The Einstein-deSitter ($\Lambda = 0$, $\Omega = 1$) model is generally not the model of choice anymore for detailed numerical studies in cosmology, however we do not make a claim that the solutions presented here are of an exact quantitative nature. Many other factors must also be taken into account when attempting to build up an exact, numerical model of structure formation. We wish to correct and extend some previous results, as well as perform some new semi-quantitative analysis. Our intent is to keep work analytically tractable at this stage.
We set $k_c = \Lambda = 0$ in (2.12) and (2.13). In the Einstein-deSitter model, the critical density can be written explicitly as
\[ \rho_c = \ddot{\rho}_0 = \frac{1}{6\pi G t^2}, \] (2.17)
and the relation $\epsilon_B + \epsilon_D = 1$ holds. We also introduce quantities resembling the comoving Jeans wavenumbers for each component taken separately:
\[ \bar{k}_B^2 = \frac{4\pi G \ddot{\rho}_0}{v_B^2}, \quad \bar{k}_D^2 = \frac{4\pi G \ddot{\rho}_0}{v_D^2}. \] (2.18)

The difference with the true comoving Jeans wavenumber for a one-component fluid is the inclusion of the total mass density $\ddot{\rho}_0$, rather than just the mass density of the component in question $\ddot{\rho}_i$. Equations (2.12) and (2.13) now become
\[ \delta''_B + \frac{3}{2a} \delta'_B + \frac{3}{2a^3} \left( \frac{\bar{k}}{k_B} \right)^2 \delta_B = \frac{3}{2a^2} (\epsilon_B \delta_B + \epsilon_D \delta_D), \] (2.19)
\[ \delta''_D + \frac{3}{2a} \delta'_D + \frac{3}{2a^3} \left( \frac{\bar{k}}{k_D} \right)^2 \delta_D = \frac{3}{2a^2} (\epsilon_B \delta_B + \epsilon_D \delta_D). \] (2.20)

The effects of the various physical processes are now clearly evident. The expansion of the Universe produces a damping term $3\delta'_i/(2a)$, causing the solutions to be in power law form rather than exponential. The relation of the mode wavenumber to the Jeans wavenumber is expressed as a ratio, transparently showing in which region of physical scales the mode lies. This ratio can be compared to the fractions $\epsilon_B$ and $\epsilon_D$ to decide whether gravity or pressure dominates the dynamics. The true Jeans instability scale for a two-component medium is not given by either $k_B$ or $k_D$, but by a combination of the two, as demonstrated in [6, 7]. This scale will be introduced in due course.

We finally perform a couple more manipulations, to cast the equations in their simplest form. We define the dimensionless parameters
\[ K_B = \frac{\bar{k}}{k_B}, \quad K_D = \frac{\bar{k}}{k_D}. \] (2.21)

Then $K_i < 1$ corresponds to the Jeans unstable region in the one-component analog of the equations, and $K_i > 1$ to the acoustic region. We also make the variable transformation $\chi = a^{-1/2}$. This gives the final form of the system of differential equations to be studied in the ensuing sections:
\[ \delta''_B + 6 \left( K_B^2 - \frac{\epsilon_B}{\chi^2} \right) \delta_B - \frac{6\epsilon_D}{\chi^2} \delta_D = 0, \] (2.22)
\[ \delta''_D + 6 \left( K_D^2 - \frac{\epsilon_D}{\chi^2} \right) \delta_D - \frac{6\epsilon_B}{\chi^2} \delta_B = 0. \] (2.23)
A prime now denotes differentiation with respect to $\chi$. These equations bear a strong resemblance to the equations of an electron-proton cosmological plasma studied in [18] [equations (4.8) and (4.9) of that paper]. As is well known from the analogy between the simple Jeans instability and Langmuir modes, this resemblance is not surprising when the mathematical similarity between the electromagnetic and gravitational forces is considered. The techniques employed in [18] will be useful in our current analysis. In this paper we will employ the Frobenius method in obtaining long wavelength solutions. In a related paper [19], some general WKB techniques are developed further than previously. Apart from facilitating some short wavelength solutions to the current problem, these techniques will also indicate further results possible in cosmological plasma physics.

Before we proceed to a general analysis of (2.22) and (2.23), we wish to digress to the simpler case of a one-component system. Surprisingly, we will derive some apparently new results, which provide a conceptually useful introduction to the ensuing analysis.

III. THE ONE-COMPONENT EQUATION REVISITED

The Einstein-deSitter one-component equation for a baryonic or dark matter fluid in the post-recombination era is a canonical example studied in all textbooks for linearized cosmological perturbation theory. It gives the familiar Jeans unstable power law solutions $\delta \propto t^{2/3}, t^{-1}$ in the limit of large scales, and acoustic oscillations in the limit of small scales. Despite this, we have not found the full exact solutions completely displayed and analyzed in any textbooks or review articles in the current literature. Although a full analysis will not bring any startling new physical revelations, the mathematical techniques required are of some interest in their relation to the physics, and as an introduction to the more complicated analysis we will require later. This section may be seen as a useful orientation to the further work carried out in the bulk of this paper.

We begin with the one-component version of (2.19), i.e. with $\epsilon_B = 1$ and $\epsilon_D = 0$ [or vice versa for (2.20)]. Analogous to the definitions of $K_B$ and $K_D$, we define the one-component comoving Jeans ratio for the fluid, which has comoving Jeans wavenumber $\bar{k}_J$, as $K_J = \bar{k}/\bar{k}_J$.

The one-component density perturbation equation then becomes

$$\delta'' + \frac{3}{2a} \delta' + \left(\frac{3}{2a^3} K_J^2 - \frac{3}{2a^2}\right) \delta = 0.$$  
(3.1)
The solution of this equation is a Bessel function of order $5/2$. A Bessel function of half odd-integer order can be recast in terms of a spherical Bessel function. To begin with, we will choose the spherical Bessel functions of the first and second kind, $j_{\nu}$ and $y_{\nu}$ respectively. The solution may be rewritten as

$$\delta(a) = c_1 a^{-1/2} j_{2} \left( \sqrt{\frac{6}{a} K_J} \right) + c_2 a^{-1/2} y_{2} \left( \sqrt{\frac{6}{a} K_J} \right),$$

with arbitrary constants of integration $c_1$ and $c_2$. For this case the fortuitous circumstance arises that the solutions may be represented in terms of elementary trigonometric functions (see e.g. [28]). The solutions as shown are exact mathematical representations, containing all the information of the modes over all scales. As is usually the case with such solutions, a simple inspection does not reveal all the physical properties of the modes in an obvious manner. For example, it is a little difficult to interpret the time dependence of the modes through the argument of the Bessel functions $\sqrt{6/aK_J}$. We will require various approximations and numerical plotting to extract more physical meaning out of the solutions.

To begin with, we seek to place the solutions into a canonical form, for easy comparison with other examples. The most useful such form comprises, to leading order, a product of a power law time factor and complex exponential factor. This approach was adopted in the studies of cosmological plasmas [16, 17, 18] for one- and two-component systems. As mentioned previously, due to the similarity between the gravitational and electromagnetic forces, the corresponding modes display many similarities.

The most useful Bessel function solutions for our purposes are the Hankel functions, due to the fact that their leading order terms contain complex exponentials. We use the spherical Hankel functions $h^{(1)}_2$ and $h^{(2)}_2$, given by the expressions

$$h^{(1)}_2(z) = \frac{1}{z} \exp \left[ i \left( z - \frac{3\pi}{2} \right) \right] \left( 1 - \frac{3}{z^2} - \frac{3}{iz} \right),$$

$$h^{(2)}_2(z) = \frac{1}{z} \exp \left[ -i \left( z - \frac{3\pi}{2} \right) \right] \left( 1 - \frac{3}{z^2} + \frac{3}{iz} \right).$$

We may make the comparison here to plasma results, where analogous series were obtained for large $z$. Contrary to here, where the series has a finite number of terms, the series for plasma modes were only asymptotic.

We write down the explicit one-component solution via Hankel functions as

$$\delta(a) = \left( 1 + \frac{a}{2K_j^2} + \frac{a^2}{4K_j^4} \right)^{1/2} \exp \left\{ \pm i \left[ \sqrt{\frac{6}{a} K_J} a^{-1/2} + \arctan \left( \frac{\sqrt{6} K_J a^{-1/2}}{2K_j^2 a^{-1} - 1} \right) \right] \right\}.$$  

(3.5)
It must be stressed that unlike the plasma solutions, this is an exact result. The modulus of the solution grows with respect to time to leading order as \( \delta \propto a \) if \( K_J \ll 1 \), or else if \( K_J \gg 1 \) the modulus is approximately constant, with a first order time correction proportional to \( a \).

There is also a complex exponential portion to the solution, which usually gives a dispersion relation. The dispersion relation may be extracted by differentiating the phase with respect to \( t \). This follows from the general fact that given an observed frequency \( \omega \), a solution of the form

\[
\delta \propto \exp \left[ \pm i \int \omega(t) \, dt \right]
\]

is expected. This is assuming, of course, that the solution oscillates—if not, some other form of real valued solution must be available. Using the matter dominated time dependence of \( a \), namely

\[
a = \left( \frac{t}{t_i} \right)^{2/3},
\]

where \( t_i \) is an arbitrary constant, we find the frequency to be

\[
\omega = \frac{\bar{v_s} k a^{-2}}{1 + \frac{1}{2} K_J^{-2} a + \frac{1}{4} K_J^{-4} a^2} \\
\approx \frac{\bar{v_s} k}{a^2} \left( 1 - \frac{1}{2} \frac{a}{K_J^2} + \frac{1}{8} \frac{a^3}{K_J^4} - \frac{1}{16} \frac{a^4}{K_J^6} + \cdots \right), \quad K_J > 1.
\]

The result has been expanded for \( K_J > 1 \), as we may suspect that due to the Jeans instability, acoustic waves only exist in this region, and thus we can only attach physical meaning to \( \omega \) for \( K_J > 1 \). This assertion will be derived rigorously in what ensues.

The result of (3.8) may come as a surprise. How does it relate to the well-known Jeans dispersion relation derived for a static spacetime

\[
\omega^2 = v_s^2 k^2 - 4\pi G \rho_0?
\]

In a cosmological setting, we may expect the dispersion relation to follow a similar form, with appropriate time factors included. For plasma modes, it was demonstrated in [18] that the dispersion relations could be written down to leading order in exactly the same form as their static spacetime counterparts in terms of physical (non-barred) variables, and then converted to comoving variables by inserting the correct time factors. Thus we may expect

\[
\omega \approx \frac{\bar{v_s} k}{a^2} \left( 1 - \frac{a}{K_J^2} \right)^{1/2}
\]
at least in the form of a binomial expansion, namely
\[ \omega \sim \frac{\bar{v}_s \bar{k}}{a^2} \left( 1 - \frac{1}{2} \frac{a}{K_J^2} - \frac{1}{8} \frac{a^2}{K_J^4} - \frac{1}{16} \frac{a^3}{K_J^6} + \cdots \right). \] (3.11)

This form for \( \omega \) may also be expected to contain some other time dependent terms, as was demonstrated for a number of plasma modes. Comparing the expansions in (3.8) and (3.11), we see in fact that they only agree to first order. This still indicates some form of Jeans instability, but the dispersion relations are quite different. This difference in behavior between the linearized gravitational modes and plasma modes may be attributed to the special role the density plays in the gravitational perturbation equations. Eq.(3.11) contains only one free parameter, the Jeans ratio \( K_J \), whereas the plasma equations contain both the sound velocity and plasma frequency, which cannot be reduced to one parameter. This implies that the relation between the gravitational source and the Friedmann equation, which fixes the background spacetime, means that the same form for the dispersion relation as found in static spacetime need not necessarily be expected in the expanding Einstein-deSitter model.

We have a general solution in terms of a modulus and complex exponential, which is exact and thus contains all the information of the problem. How do we infer the usual Jeans instability behavior from this? Let us examine plots of the solutions to gain a pictorial idea of what is happening. In Figs. 1 and 2 we see the transition from acoustic oscillations to growing and decaying modes as \( K_J \) is decreased—we are examining ever larger scales, and passing through the instability. To all fit on the same set of axes, the plots have been approximately normalized. The dependent variable \( a \) is rather arbitrary [as can be deduced from (3.7)]. An appropriate starting time \( t_i \) may be chosen to normalize \( a \) to 1 at the beginning of the chosen epoch of evolution, and the solutions may then be propagated forward in time. The tendency for the period of the acoustic oscillations to grow longer in time is evident from the plots, and the approximate constancy of the amplitude predicted previously from (3.5) is evident. In the extreme case, the oscillation period becomes so long that a perturbation cannot complete one full oscillation, and then the instability arises. This behavior can clearly be seen in Figs. 1 and 2 for the \( K_J = 2.0, 8.0 \) plots.

We now perform some approximations to make contact with some better known results of the one-component problem. Let us begin with a small \( K_J^2/a \) expansion. For the spherical
Hankel solutions (3.5), we find
\[
\delta \sim \frac{a}{2K_J^2} \left[ 1 + \frac{K_J^2}{a} + O \left( \frac{K_J^4}{a^2} \right) \right] \exp \left\{ \mp i \sqrt{\frac{6}{5}} \frac{K_J}{a^{1/2}} \left[ 1 - \frac{4}{5} \frac{K_J^4}{a^2} + O \left( \frac{K_J^6}{a^3} \right) \right] \right\}. \quad (3.12)
\]

The above expansion explains what happens to acoustic oscillations when \( K_J \lesssim 1 \). In this region, the leading order factor \( K_J a^{-1/2} \) in the exponential must always lie between 1 and 0 numerically, and decreases with increasing time. This is because \( a \geq 1 \) and increases monotonically for all time. Thus the solution lies within one period of oscillation for all time, and only the growing or decaying modes may be observed. When \( K_J \) becomes larger than 1, more than one period of oscillation may be spanned by the \( K_J a^{-1/2} \) factor, and the solution will begin to develop acoustic waves. The expansion (3.12) shows \( \delta \propto a \), which only gives the familiar growing mode, discussed in all texts. The decaying mode has not been found in the current analysis, because spherical Hankel functions have been chosen to represent the Bessel function solutions. The spherical Hankel functions are linear combinations of the original \( j_2 \) and \( y_2 \) solutions, which contain both modes. The decaying mode has been “asymptotically swamped” by the growing mode in this linear combination. The decaying mode may be liberated by a direct small variable expansion of (3.2) for each spherical Bessel function. We find the \( j_2 \) component gives the decaying mode
\[
\delta \sim a^{-3/2} \left[ 1 - \frac{3}{7} \frac{K_J^2}{a} + O \left( \frac{K_J^4}{a^2} \right) \right], \quad (3.13)
\]
and the \( y_2 \) component gives the same growing mode as (3.12) in the slightly different form
\[
\delta \sim a \left[ 1 + \frac{K_J^2}{a} + O \left( \frac{K_J^4}{a^2} \right) \right] \quad (3.14)
\]
without the exponential phase factor. These solutions clearly correspond to the usual textbook modes found when pressure is ignored. They have included the pressure corrections, given as a series in the Jeans ratio, with matching time factors to all orders in the expansion.

We now turn to the large parameter expansion, from which we expect to liberate the acoustic oscillations. Eq. (3.5) is in fact already in the form of a large parameter expansion, only that it has a finite number of terms, and is consequently exact. The phase of the exponential does not seem to show the structure of the familiar Jeans dispersion relation. This was discussed above, where it was indicated that this is not necessarily to be expected. It is possible to see why this is so in a lucid fashion by applying the WKB method to the original equation (3.1). Through this method we will derive a dispersion relation displaying
very similar characteristics to the familiar textbook one, that shows the presence of the Jeans instability. A more complicated WKB approximation scheme was developed in [18] to deal with plasma modes of a more intricate form, involving larger numbers of coupled equations. This method is used in [19] to handle the two-component cosmological density perturbation equations in the short wavelength approximation. For the present simple second order equation, the standard textbook approach suffices (for a good explanation of WKB methods, see [29]).

To see the physics most clearly, we transform (3.1) to depend on \( t \). Using the variable transformation given by (3.7), the equation

\[
\ddot{\delta} + \frac{4}{3t} \dot{\delta} + \left( \frac{2}{3} K^2 J_{\ell_i}^{2/3} - \frac{2}{3} \right) \delta = 0 \tag{3.15}
\]

results. The usual Jeans dispersion relation can be directly seen in this equation. Consider the factor \( 2/(3t^2) \), arising from the gravitational source term. This term may be transformed to explicitly see the source parameters emerge. We consider the physical (time dependent) value of the total energy density:

\[
4\pi G \rho_0 = 4\pi G \rho_c = \frac{4\pi G}{6\pi G t^2} = \frac{2}{3t^2}, \tag{3.16}
\]

and the relation

\[
K^2 = \frac{3}{2} \bar{v}^2 k^2 t_i^2. \tag{3.17}
\]

Then the Jeans dispersion relation can be directly seen in (3.15):

\[
\bar{v}^2 k^2 \left( \frac{t_i}{t} \right)^{8/3} - \frac{2}{3t^2} = \bar{v}^2 k^2 - 4\pi G \rho_0. \tag{3.18}
\]

In the static spacetime case the physical variables would of course not depend on time, and the first derivative term \( 4\dot{\delta}/(3t) \) in (3.15) would not exist. This leads to the exact exponential solutions and the familiar dispersion relation given by (3.18), as originally found by Jeans.

The most straightforward way to effect a WKB approximation in the present situation is to remove the first derivative from the equation. One way to do this is by the variable change \( \chi = a^{-1/2} \). We then find the equation

\[
\frac{d^2\delta}{d\chi^2} + \left( 6K^2_{\ell_i} - \frac{6}{\chi^2} \right) \delta = 0. \tag{3.19}
\]

Now we suggestively define

\[
\tilde{\omega}(\chi) = \left( 6K^2_{\ell_i} - \frac{6}{\chi^2} \right)^{1/2}. \tag{3.20}
\]
Applying the WKB approximation to (3.19) gives the leading order solution
\[
\delta(\chi) \sim \tilde{\omega}^{-1/2} \exp \left[ \pm i \int \tilde{\omega}(\chi) d\chi \right].
\] (3.21)

A dispersion relation has indeed been derived, and is given by \( \tilde{\omega} \) as defined in (3.20). If we consider
\[
\int \tilde{\omega}(\chi) d\chi = -\int \left( \frac{2}{3} K^2 J^{2/3} \frac{t^2}{t_i^{8/3}} - \frac{2}{3t^2} \right)^{1/2} dt
\]
then a physical frequency \( \omega(t) \) can be identified by use of (3.18) and the relation \( t_i^2 = 1/(6\pi G \bar{\rho}_0) \). Thus
\[
\int \tilde{\omega}(\chi) d\chi = \int \omega(t) dt = \int (v_s^2 k^2 - 4\pi G \bar{\rho}_0)^{1/2} dt,
\] (3.22)

and the physical connection has been made. Another point to note is that the amplitude \( \tilde{\omega}(\chi)^{-1/2} \) is time independent to leading order (because \( K J \gg 1 \)). Thus the amplitude is approximately constant, as noted previously.

To make a direct comparison of (3.21) and (3.5), we now evaluate the integral in the phase of the WKB solutions. A change of integration variable from \( \chi \) back to \( a \) puts the integral into a form which has been tabulated [30], and we find
\[
\int \tilde{\omega}(\chi) d\chi \sim \left( \frac{6K^2}{a} - 6 \right)^{1/2} - \frac{\sqrt{6}}{2} \arcsin \left( 1 - \frac{2a}{K_J^2} \right).
\] (3.24)

The solution generated by the WKB method may be compared to the exact one given by (3.5). The amplitudes and phases need to be expanded for large \( K_J \):

WKB phase:
\[
\left( \frac{6K^2}{a} - 6 \right)^{1/2} - \frac{\sqrt{6}}{2} \arcsin \left( 1 - \frac{2a}{K_J^2} \right) = \sqrt{6} \frac{K_J}{a^{1/2}} \left[ 1 - \frac{\pi}{4} \frac{a^{1/2}}{K_J} + O \left( \frac{a}{K_J^2} \right) \right],
\] (3.25)

exact phase:
\[
\frac{\sqrt{6}K_J}{a^{1/2}} + \arctan \left( \frac{\sqrt{6}K_J a^{-1/2}}{2K_J^2 a^{-1} - 1} \right) = \sqrt{6} \frac{K_J}{a^{1/2}} \left[ 1 + \frac{a}{2K_J^2} + O \left( \frac{a}{K_J^2} \right) \right],
\] (3.26)

WKB amplitude:
\[
(6K^2 - 6a)^{-1/4} \frac{1}{6^{1/4}K_J^{1/2}} \left[ 1 + \frac{a}{4K_J^2} + O \left( \frac{a^2}{K_J^4} \right) \right],
\] (3.27)

exact amplitude:
\[
\left( 1 + \frac{a}{2K_J^2} + \frac{a^2}{4K_J^4} \right)^{1/2} \left[ 1 + \frac{a}{4K_J^2} + O \left( \frac{a^2}{K_J^4} \right) \right].
\] (3.28)
It can be seen that the two solutions only agree to leading order (modulo time independent constant factors). Given that the WKB method only gives a leading order solution to the problem, no more can be expected. This discussion has highlighted the difference between the expected dispersion relations of static spacetime to those derived in an expanding universe context. WKB can reproduce the same form as the static spacetime dispersion relations, but this may only agree to leading order to the true dispersion relation found in an expanding universe scenario.

IV. IMPROVEMENTS ON PREVIOUS CDM PERTURBATION RESULTS

We now return to the two-component equations and consider the case of CDM perturbations characterized by strictly zero temperature. If the velocity dispersion is considered to be an adiabatic sound velocity as given by (2.8), then \( T_i = 0 \) corresponds to \( K_D = 0 \) in (2.23). Such an approximation facilitates an exact analytic solution to the problem, which is otherwise impossible. The resulting system of equations was one of the main cases investigated in the general analysis of [11]. In this section we will point out what appears to be an error in the analysis of that paper, which leads to some markedly different solutions, derived in what ensues.

Before we proceed with this, it is pertinent to point out a general problem with taking \( K_D = 0 \) in Eqs. (2.22), (2.23). For simplicity, it is possible to neglect spacetime expansion, as the qualitative behavior will be the same. Thus the static spacetime cosmological equations studied in [7] are sufficient for this discussion. These equations are also examined in [19], and using the notation employed there we have

\[
\ddot{\delta}_D + (v_D^2 k^2 - W_D)\delta_D - W_B \delta_B = 0, \tag{4.1}
\]

\[
\ddot{\delta}_B + (v_B^2 k^2 - W_B)\delta_B - W_D \delta_D = 0, \tag{4.2}
\]

with \( W_i = 4\pi G\rho_i \). This system of equations can be reduced to a first order linear autonomous dynamical system describing a state vector

\[
x = (x_1, x_2, x_3, x_4)^T \equiv (\dot{\delta}_D, \delta_D, \dot{\delta}_B, \delta_B)^T, \tag{4.3}
\]

where \( T \) denotes the transpose of a vector. The dynamical system has a \( k \)-dependent critical point found by solving the equation \( \dot{x} = 0 \). This critical point happens to give the Jeans
wavenumber of the mixture, and is given by
\[ k^2 = k_M^2 \equiv k_B^2 + k_D^2 = \frac{W_B}{v_B^2} + \frac{W_D}{v_D^2}. \] (4.4)

Note the slight difference in this definition of \( k_i \) in comparison to \( \bar{k}_i \) defined in (2.18). It is already evident that a problem arises if we take \( v_D \to 0 \) in (4.4), and this will be physically elucidated by studying the four independent modes of the system, given by the eigenvalues of the dynamical system.

The eigenvalues give the structure of the modes (see [7] or [19] for the details). They are found to be of the general form
\[
\begin{align*}
\lambda_1 &= -\lambda_2 = \frac{1}{\sqrt{2}} \sqrt{f + \sqrt{f^2 + 4g}}, \\
\lambda_3 &= -\lambda_4 = \frac{1}{\sqrt{2}} \sqrt{f - \sqrt{f^2 + 4g}}.
\end{align*}
\] (4.5)

For the case of CDM currently under consideration, where \( v_D = 0 \), the \( k \)-dependent functions \( f \) and \( g \) are given by
\[
\begin{align*}
f(k) &= W_B + W_D - k^2 v_B^2, \quad (4.6) \\
g(k) &= k^2 W_D^2 v_B^2. \quad (4.7)
\end{align*}
\]

It was previously found that the eigenvalues \( \lambda_1 \) and \( \lambda_2 \) described the Jeans unstable modes, whereas \( \lambda_3 \) and \( \lambda_4 \) described acoustic oscillations at all wavenumbers. An examination of \( \lambda_1 \) in the current context will show this not to be the case for CDM. If we define (analogously to the one-component scenario)
\[ K_J = \frac{v_B^2 k^2}{W_B + W_D}, \] (4.8)

\( \lambda_1 \) may be written as follows:
\[ \lambda_1 = \frac{1}{\sqrt{2}} (W_B + W_D)^{1/2} \left\{ 1 - K_J^2 + \left[ (1 - K_J^2)^2 + 4\epsilon_D K_J^2 \right]^{1/2} \right\}^{1/2}. \] (4.9)

This does not equal zero for \( K_J = 1 \), and it is straightforward to show that it has no zeros for all \( k \neq 0 \). Thus a CDM perturbation would collapse for all scales, clearly a physical impossibility.

If we examine Eqs.(4.1), (4.2), it can be seen that with the removal of the pressure term \( v_D k^2 \), there is no mechanism to counter the remaining gravitational source terms \( -W_D \delta_D \) and \( -W_B \delta_B \), whose sign indicate an attractive forcing, initiating gravitational collapse. This is the case no matter how small the fraction of dark matter compared to baryons, thus no
amount of baryonic pressure support can prevent a collapse at any scale. This physically absurd situation is the root of the problem of taking $v_D \to 0$ in the fluid models. A physically correct equation must include some sort of velocity dispersion term, even if the matter is totally collisionless and an adiabatic speed of sound cannot be defined.

With these thoughts in mind, we must view the current section as more of a mathematical digression, than a physically realistic model. It nevertheless serves a purpose. We can make contact with the work of [11], and uncover some interesting mathematical properties associated with these cosmological perturbation equations in general. We find mathematical subtleties overlooked in [11], which also indicate the nature of the general solutions to follow in the next section. They allow an interesting comparison with cosmological plasma modes discussed in [18], where the limit $T_D \to 0$ is valid. The special nature of gravity, and the extra complications it entails are revealed by this comparison.

We now proceed to obtain a solution of Eqs. (2.22), (2.23) (with $K_D = 0$) in the long wavelength (small $k$) limit. A short wavelength solution is trivially obtained by setting $K_D = 0$ everywhere in the results presented in [19]. To align ourselves with earlier notation, and to stress the fact that there is only one Jeans related scale now occurring, we will rename $K_B$ to $K_J$. Rather than directly reducing Eqs. (2.22), (2.23) into a single equation, we attempt a solution by the Frobenius method. This is useful as a precursor to the general solution derived in the following section in a similar manner.

To begin with, we assume an arbitrary expansion of the solutions in the form

$$
\delta_B(\rho, \chi) = \chi^\rho \sum_{n=0}^{\infty} a_n \chi^n, \quad (4.10)
$$

$$
\delta_D(\rho, \chi) = \chi^\rho \sum_{n=0}^{\infty} b_n \chi^n. \quad (4.11)
$$

Here $\rho$ is an arbitrary exponent to be determined, and $a_n$ and $b_n$ are series coefficients also to be determined by the Frobenius method. We substitute these series into (2.22) and (2.23) to obtain a set of algebraic relations between the undetermined coefficients. Then all coefficients of like power of $\chi$ are collected and equated to zero.

Arising out of this procedure are a pair of indicial equations for $\rho$, with one of $a_0$ or $b_0$ remaining an arbitrary constant:

$$
\rho(\rho - 1)a_0 - 6\epsilon_B a_0 - 6\epsilon_D b_0 = 0, \quad (4.12)
$$

$$
\rho(\rho - 1)b_0 - 6\epsilon_D b_0 - 6\epsilon_B a_0 = 0. \quad (4.13)
$$
A set of recursion relations also arise for all higher order coefficients:

\[ 6K_J^2 a_n + [(\rho + n + 2)(\rho + n + 1) - 6\epsilon_B]a_{n+2} - 6\epsilonDb_{n+2} = 0, \]  
(4.14)

\[ (\rho + n + 2)(\rho + n + 1) - 6\epsilon_Db_{n+2} - 6\epsilonBa_{n+2} = 0. \]  
(4.15)

In retaining the arbitrary constant \(a_0\) or \(b_0\), it is assumed that all odd indexed terms vanish from propagation of the initial values \(a_1 = b_1 = 0\) through the recursion relations.

If we solve (4.12) and (4.13) for \(\rho\), we find four possible values:

\[ \rho = 0, 1, 3, -2. \]  
(4.16)

A comparison with the analogous case for cosmological plasma modes immediately shows a difference in the nature of the exponents. In the present case the exponents are exactly determined integers, whereas for plasmas the exponents depended on the plasma frequency. When the one-component solutions were discussed in the previous section, an analogous difference was observed between the spherical Bessel function solutions of the gravitational perturbation modes, and the general order Bessel function solutions of the plasma modes. Whereas in the one-component study the solutions were simplified by this property of gravity, in the present case they are in fact complicated. The plasma solutions were representable in terms of \(2F_3\) generalized hypergeometric functions, but a similar representation is not well-defined here. This is because the exponents differ by integers—a fact which necessitates a modification of the basic Frobenius method. This modification is borne out in the solutions by the fact that parameters appearing in the denominator of generalized hypergeometric function expansions cannot differ by integers. In such situations the generalized hypergeometric functions are not definable, and one must resort to classifying solutions of the equation by Meijer G-functions.

To apply the Frobenius method to indices differing by integers, the recursion relations must first be solved for general \(\rho\). This is achieved by employing (4.12), (4.13), and writing the system of differential equations as

\[ \mathbf{L} \begin{bmatrix} \delta_B(\rho, \chi) \\ \delta_D(\rho, \chi) \end{bmatrix} = \begin{bmatrix} \rho(\rho - 1)a_0 - 6\epsilon_Ba_0 - 6\epsilon_Db_0 \\ \rho(\rho - 1)b_0 - 6\epsilon_Db_0 - 6\epsilon_Ba_0 \end{bmatrix} \chi^{\rho - 2}, \]  
(4.17)

where the operator \(\mathbf{L}\) is defined by

\[ \mathbf{L} = \begin{bmatrix} \frac{\partial^2}{\partial \chi^2} + 6 \left( K_J^2 - \frac{\epsilon_B}{\chi^2} \right) - \frac{6\epsilon_D}{\chi^2} \\ - \frac{6\epsilon_B}{\chi^2} \end{bmatrix}. \]  
(4.18)
At present there is no relation between $a_0$ and $b_0$, but the recursion relations will provide one. A lengthy algebraic exercise is needed to solve the recursion relations. The result is

$$a_{2n} = a_0 \left( \frac{1}{2} \rho - \frac{1}{2} \nu_D + \frac{3}{4} \right)_n \left( \frac{1}{2} \rho + \frac{1}{2} \nu_D + \frac{3}{4} \right)_n \left( -\frac{3}{2} K^2 \right)_n,$$  

$$b_{2n} = \frac{6a_0 \epsilon_B}{\rho(\rho - 1) - 6\epsilon_D} \left( \frac{1}{2} \rho - \frac{1}{2} \nu_D - \frac{1}{4} \right)_n \left( \frac{1}{2} \rho + \frac{1}{2} \nu_D - \frac{1}{4} \right)_n \left( -\frac{3}{2} K^2 \right)_n. \quad (4.20)$$

In the above, the parameter $\nu_D$ has been introduced as a shorthand:

$$\nu_D \equiv \sqrt{\frac{1}{4} + 6\epsilon_D}, \quad (4.21)$$

and the notation $()_n$ is the Pochhammer symbol. It is clear that the coefficients as derived will not exist for $\rho = 1, -2$. This is the basis for requiring a modification to the straightforward method of substituting in the four calculated values of $\rho$ (4.16) into (4.19) and (4.20) to generate four independent solutions.

To proceed, we extract from (4.19) and (4.20) the relation

$$b_0 = \frac{6a_0 \epsilon_B}{\rho(\rho - 1) - 6\epsilon_D}, \quad (4.22)$$

which allows (4.17) to be rewritten as

$$L \begin{bmatrix} \delta_B(\rho, \chi) \\ \delta_D(\rho, \chi) \end{bmatrix} = a_0 \begin{bmatrix} \frac{1}{\epsilon_B} \rho(\rho - 1)(\rho + 2)(\rho - 3) \\ \rho(\rho - 1) - 6\epsilon_D \end{bmatrix} \chi^{\rho-2}. \quad (4.23)$$

For $\rho = 0, 3$, direct substitution is permissible to obtain two independent solutions, which turn out to be generalized hypergeometric $2F_3$ functions, analogously to the plasma results. For $\rho = 1, -2$, we take advantage of the fact that $a_0$ is arbitrary, and set it respectively equal to $\rho - 1$ and $\rho + 2$. After differentiation with respect to $\rho$ and evaluation at the respective points $\rho = 1$ and $\rho = -2$, we obtain the result

$$L \left[ \frac{\partial}{\partial \rho} \begin{bmatrix} \delta_B(\rho, \chi) \\ \delta_D(\rho, \chi) \end{bmatrix} \right] \bigg|_{\rho=1,-2} = 0. \quad (4.24)$$

This gives two more solutions.

The above discussion has defined an algorithm for finding small $k$ expansions of the solutions of the cosmological density perturbation equations. This algorithm is readily implemented into a symbolic manipulation computer code. We will write such a code to find
the solutions to the general equations in the next section. The solutions for the CDM case under consideration may now be written down. The need to differentiate with respect to $\rho$ for the $\rho = 1, -2$ cases result in solutions involving digamma functions $\psi$ and logarithmic terms—quite a complication to the $2F_3$ functions derived for the plasma perturbations in [18]. To display the modes in their simplest form, linear combinations of the modes derived directly by the above algorithm need to be taken, and the use of various mathematical identities involving digamma functions employed. The final set of four CDM modes are given as follows:

\[
\delta_{B1}(\chi) = \frac{c_1}{\epsilon_B} \sum_{n=0}^{\infty} \frac{(\frac{3}{4} - \frac{3}{2} \nu_D)_n (\frac{3}{4} + \frac{1}{2} \nu_D)_n}{(-\frac{1}{2})_n (\frac{1}{2})_n (2)_n n!} (-\frac{3}{2} K_f^2 \chi^2)^n
\]

\[
= \frac{c_1}{\epsilon_B} 2F_3 \left( \frac{3}{4} - \frac{3}{2} \nu_D, \frac{3}{4} + \frac{1}{2} \nu_D; -\frac{1}{2}, \frac{1}{2}, 2; -\frac{3}{2} K_f^2 \chi^2 \right)
\]  

(4.25)

\[
\delta_{D1}(\chi) = \frac{c_1}{\epsilon_D} \sum_{n=0}^{\infty} \frac{(-\frac{1}{4} - \frac{3}{2} \nu_D)_n (-\frac{1}{4} + \frac{1}{2} \nu_D)_n}{(-\frac{1}{2})_n (\frac{1}{2})_n (2)_n n!} (-\frac{3}{2} K_f^2 \chi^2)^n
\]

\[
= \frac{c_1}{\epsilon_D} 2F_3 \left( -\frac{1}{4} - \frac{3}{2} \nu_D, -\frac{1}{4} + \frac{1}{2} \nu_D; -\frac{1}{2}, \frac{1}{2}, 2; -\frac{3}{2} K_f^2 \chi^2 \right)
\]  

(4.26)

\[
\delta_{B2}(\chi) = \frac{c_2}{\epsilon_B} \sum_{n=0}^{\infty} \frac{(\frac{5}{4} - \frac{3}{2} \nu_D)_n (\frac{5}{4} + \frac{1}{2} \nu_D)_n}{(-\frac{3}{2})_n (\frac{3}{2})_n (2)_n n!} (-\frac{3}{2} K_f^2 \chi^2)^n
\]

\[
\times \left[ \psi \left( \frac{5}{4} - \frac{3}{2} \nu_D + n \right) + \psi \left( \frac{5}{4} + \frac{1}{2} \nu_D + n \right) - \psi(n) - \psi(n + 1) - \psi(n + \frac{3}{2}) - \psi(n + \frac{5}{2}) + \log \chi^2 \right]
\]  

(4.27)

\[
\delta_{D2}(\chi) = -\frac{c_2}{\epsilon_D} \sum_{n=0}^{\infty} \frac{(\frac{1}{4} - \frac{3}{2} \nu_D)_n (\frac{1}{4} + \frac{1}{2} \nu_D)_n}{(-\frac{1}{2})_n (\frac{1}{2})_n (2)_n n!} (-\frac{3}{2} K_f^2 \chi^2)^n
\]

\[
\times \left[ \psi \left( \frac{1}{4} - \frac{3}{2} \nu_D + n \right) + \psi \left( \frac{1}{4} + \frac{1}{2} \nu_D + n \right) - \psi(n) - \psi(n + 1) - \psi(n + \frac{3}{2}) - \psi(n + \frac{5}{2}) + \log \chi^2 \right]
\]  

(4.28)

\[
\delta_{B3}(\chi) = c_3 \sum_{n=0}^{\infty} \frac{(\frac{5}{4} - \frac{3}{2} \nu_D)_n (\frac{5}{4} + \frac{1}{2} \nu_D)_n}{(-\frac{3}{2})_n (\frac{3}{2})_n (2)_n n!} (-\frac{3}{2} K_f^2 \chi^2)^n
\]

(4.29)

\[
\delta_{D3}(\chi) = c_3 \sum_{n=0}^{\infty} \frac{(\frac{1}{4} - \frac{3}{2} \nu_D)_n (\frac{1}{4} + \frac{1}{2} \nu_D)_n}{(-\frac{1}{2})_n (\frac{1}{2})_n (2)_n n!} (-\frac{3}{2} K_f^2 \chi^2)^n
\]  

(4.30)

\[
\delta_{B4}(\chi) = c_4 \left( \frac{3}{2} K_f^2 \chi^2 \right)^{\frac{1}{2}} + 2c_4 \sum_{n=0}^{\infty} \frac{(\frac{3}{4} - \frac{3}{2} \nu_D)_n (\frac{3}{4} + \frac{1}{2} \nu_D)_n}{(-\frac{1}{2})_n (\frac{1}{2})_n (2)_n n!} (-\frac{3}{2} K_f^2 \chi^2)^n
\]

\[
\times \left[ \psi \left( \frac{3}{4} - \frac{3}{2} \nu_D + n \right) + \psi \left( \frac{3}{4} + \frac{1}{2} \nu_D + n \right) - \psi(n + 2) - \psi(n + 1) - \psi(n + \frac{1}{2}) - \psi(n + \frac{1}{2}) + \log \chi^2 \right]
\]  

(4.31)

\[
\delta_{D4}(\chi) = c_4 \left( \frac{3}{2} K_f^2 \chi^2 \right)^{\frac{1}{2}} - 2c_4 \sum_{n=0}^{\infty} \frac{(-\frac{1}{4} - \frac{3}{2} \nu_D)_n (-\frac{1}{4} + \frac{1}{2} \nu_D)_n}{(-\frac{1}{2})_n (\frac{1}{2})_n (2)_n n!} (-\frac{3}{2} K_f^2 \chi^2)^n
\]

\[
\times \left[ \psi \left( -\frac{1}{4} - \frac{3}{2} \nu_D + n \right) + \psi \left( -\frac{1}{4} + \frac{1}{2} \nu_D + n \right) - \psi(n + 2) - \psi(n + 1) - \psi(n + \frac{1}{2}) - \psi(n + \frac{1}{2}) + \log \chi^2 \right]
\]  

(4.32)
The same solutions may be obtained by considering Meijer G-function solutions to the original differential equations. The procedure for determining solutions to generalized hypergeometric-like equations which contain parameters differing by integers is discussed in detail by Luke [23], pp.138-143. The study outlined a method for developing Meijer G-function solutions from the equations, which involved the differentiation of generalized hypergeometric functions with respect to their parameters—a procedure analogous to the differentiation of $\rho$ indices in the above. The analysis involved is lengthy and tedious, but leads to the solutions obtained above. We refrain from a physical interpretation of the above gravitational modes for now, and take that up in the next section when we discuss the more general solutions. The one obvious difference will be the fact that no Jeans instability is apparent for any of the above modes, with two modes always collapsing and two modes always acoustic, whereas for the general modes a Jeans instability will be apparent.

We now compare the solutions obtained to those of [10, 11]. In these studies general solutions were written as

$$\phi_1 \equiv t^{-\alpha} \delta_B = c_1 G_1 + c_2 G_2 + c_3 G_3 + c_4 G_4.$$  \hspace{1cm} (4.33)

Here $c_i$ are constants, and the functions $G_i$ denote Meijer G-functions

$$G_h = G_{2,4}^{m,n} \left( x \bigg| \begin{array}{c} a_1^*, a_2^* + 1 \\ b_h^*, b_1^* \ldots \# \ldots b_4^* \end{array} \right), \quad h = 1, 2, 3, 4.$$  \hspace{1cm} (4.34)

The notation # signifies that $b_h$ is to be omitted in its usual place. The G-function is defined so that $0 \leq m \leq 4$ and $0 \leq n \leq 2$. The parameters $a_i^*, b_j^*$ depend on $\epsilon_B, \epsilon_D$, the adiabatic index $\gamma_B$, and the exponent $\eta$ of $t$ in the Hubble expansion parameter (which we pointed out earlier must be equal to $\frac{2}{3}$ for the equations as formulated to be physically correct—even though [10, 11] used a greater range of values). The time parameter $x = \frac{2}{3} K^2 \chi^2$ in our notation. A particular set of solutions is given by $m$ and $n$ being given specific values. In general, $m = 1$, $n = 2$ will give such a set of solutions for small $x$ in the above example. The G-functions can then normally be expressed in terms of $2F_3$ functions (for example the plasma solutions), but in the particular case under discussion, since some of the $b_i^*$ differ by integers, this is not possible. It is this point that Haubold and Mathai missed in [11].

Let us get down to specifics to illustrate the point. Under the general classification scheme, the CDM case under consideration corresponds to $\eta = \frac{2}{3}$, $\gamma_i = \frac{5}{3}$, in the notation of
This implies that the parameters take the following values:

\[ a_1^*, a_2^* = -1 \pm \frac{1}{2} \nu D, \]
\[ b_1^*, b_2^* = \pm \frac{1}{4}, \quad b_3^*, b_4^* = \pm \frac{5}{4}. \]

To evaluate the G-functions in this special case, where some parameters differ by integers, the integral representation of the G-functions was considered in [11]:

\[ G_1 = \frac{1}{2\pi i} \int_L \frac{\Gamma(\frac{1}{4} + s)\Gamma(-a_1^* - s)\Gamma(-a_2^* - s)}{\Gamma(\frac{5}{4} - s)\Gamma(-\frac{1}{4} - s)\Gamma(\frac{9}{4} - s)} x^{-s} ds. \]  
(4.35)

Using the fact that

\[ \frac{\Gamma(\frac{1}{4} + s)}{\Gamma(-\frac{1}{4} - s)} - \frac{\Gamma(\frac{5}{4} + s)}{\Gamma(\frac{9}{4} - s)}, \]  
(4.36)

it was found that

\[ G_1 = -x^{5/4} \frac{\Gamma(-a_1^* + \frac{5}{4})\Gamma(-a_2^* + \frac{5}{4})}{\Gamma(\frac{5}{4})\Gamma(2)\Gamma(\frac{9}{4})} \, _2F_3 \left( -a_1^* + \frac{5}{4}, -a_2^* + \frac{5}{4}, \frac{5}{2}, \frac{7}{2}; -x \right), \]  
(4.37)

which agrees with \( \delta_{B3} \) in (4.29). Performing the same analysis as above for \( G_3 \) however, it is found that \( G_1 \) and \( G_3 \) are exactly the same function. It appears as though this was never checked in [11]. The same applies to \( G_2 \) and \( G_4 \):

\[ G_2 = G_4 = -x^{-1/4} \frac{\Gamma(-a_1^* - \frac{1}{4})\Gamma(-a_2^* - \frac{1}{4})}{\Gamma(\frac{1}{2})\Gamma(2)\Gamma(-\frac{1}{2})} \times \, _2F_3 \left( -a_1^* - \frac{1}{4}, -a_2^* - \frac{1}{4}; \frac{1}{2}, 2, \frac{-1}{2}; -x \right), \]  
(4.38)

which agrees with \( \delta_{B1} \) in (4.25). Thus the solutions degenerate by employing this method, and two of the solutions, namely \( \delta_{B2} \) and \( \delta_{B4} \) which contain logarithmic terms, are totally missed. The correct way to evaluate the G-functions when some parameters differ by integers is given in Luke [23], pp.143-147. A careful study of this rather complicated procedure will show that the logarithmic solutions \( \delta_{B2} \) and \( \delta_{B4} \) are found in this way.

It is interesting to note that a linear combination of the solutions \( \delta_{B2} \) and \( \delta_{B4} \) are in fact found in [11] in the large \( x \) limit. In this case the function \( G_{2,4}^{4,1} \) was evaluated from the contour integral representation to achieve some analogous series to (4.27) and (4.31). Although the solutions (4.26)–(4.32) are valid for all \( x \), such a representation does not seem very useful for \( x \) large, as the solutions are comprised of an infinite ascending series in \( x \). Thus very many terms would be required to represent the solutions accurately in this limit.
through $\delta_{B2}$ and $\delta_{B4}$. In [19] we develop a much better method for evaluating the solutions for large $x$ using a WKB approximation scheme.

We have now thoroughly investigated the CDM two-component model using the $K_D = 0$ limit, and tidied up previous work in this area. As discussed earlier, although this work is of dubious physical relevance, it has been an interesting mathematical investigation, and has allowed comparisons with the previous work in cosmological perturbation theory and cosmological plasma physics. We now turn to the more general two-component model, which has a firmer physical basis.

V. THE GENERAL TWO-COMPONENT SOLUTIONS

We finally investigate the most general set of equations, those with both $K_B$ and $K_D$ nonzero. The equations thus posed can model both CDM and HDM, though the discussion initiated earlier about the lack of a free streaming damping term for neutrino-like HDM should be heeded.

The algorithm needed to solve the equations was developed in the previous section for $K_D = 0$, and can be used here. Thus if (4.10) and (4.11) are substituted into (2.22) and (2.23), we obtain the same indicial equations (4.12) and (4.13) as previously, and the following set of coupled recursion relations:

$$
(\rho + n + 2)(\rho + n + 1)(\rho + n + 4)(\rho + n - 1)a_{n+2} + 6K_B^2(\rho + n + \frac{3}{2} - \nu_B)(\rho + n + \frac{3}{2} + \nu_B)a_n + 36\epsilon_D K_D^2 b_n = 0, \quad (5.1)
$$

$$
(\rho + n + 2)(\rho + n + 1)(\rho + n + 4)(\rho + n - 1)b_{n+2} + 6K_D^2(\rho + n + \frac{3}{2} - \nu_B)(\rho + n + \frac{3}{2} + \nu_B)b_n + 36\epsilon_B K_B^2 a_n = 0. \quad (5.2)
$$

Here $\nu_B^2 = \frac{1}{4} + 6\epsilon_B$ in analogy with the previous definition of $\nu_D$. A closed solution for this set of recursion relations cannot be obtained for all $n$, so that solutions to (5.1) and (5.2) can only be generated iteratively, to whatever order desired. Lacking the ability to generate an infinite series representation for the solutions means that they cannot be classified by known analytic functions. To handle the complicated algebra involved in finding successive terms iteratively, we have developed a symbolic computation code using the functional programming language Mathematica. The algorithm described in the previous section can be used to generate a solution up to a certain power in $\chi$. 

26
The coefficients increase in complexity very quickly for increasing \( n \). Although the code can generate solutions up to arbitrary order, we find it sufficient to present only the first two orders for each solution here. We express the results in terms of the original parameters \( K_B \) and \( K_D \), rather than in terms of \( \nu_B \) and \( \nu_D \), as no simplification is gained in using the latter. The solutions, corresponding to \( \rho = 0, 1, 3, -2 \) respectively, are:

\[
\begin{align*}
\delta_{B1}(\chi) &= 1 + \frac{3}{2}(K_B^2 - 3\epsilon_D K_B^2 - 3\epsilon_B K_B^2)\chi^2 + O(\chi^4), \\
\delta_{D1}(\chi) &= -\frac{\epsilon_B}{\epsilon_D} \left[ 1 + \frac{3}{2}(K_B^2 - 3\epsilon_D K_B^2 - 3\epsilon_B K_B^2)\chi^2 + O(\chi^4) \right], \\
\delta_{B2}(\chi) &= \chi + \frac{6}{25} \left( 6K_B^2 - 31\epsilon_D K_B^2 - 31\epsilon_B K_B^2 + 5\frac{\epsilon_B}{\epsilon_D} K_D^2 \right) \chi^3 \\
&\quad + \frac{6}{5} \epsilon_B (K_D^2 - K_B^2) \chi^3 \log \chi + O(\chi^5), \\
\delta_{D2}(\chi) &= -\frac{\epsilon_B}{\epsilon_D} \left[ \chi - \frac{1}{25} (31\epsilon_D K_B^2 + 31\epsilon_B K_B^2 - K_D^2) \chi^3 \\
&\quad - \frac{6}{5} \epsilon_D (K_D^2 - K_B^2) \chi^3 \log \chi + O(\chi^5) \right], \\
\delta_{B3}(\chi) &= \chi^3 + \frac{3}{70} (3\epsilon_D K_B^2 - 3\epsilon_D K_D^2 - 10K_B^2) \chi^5 + O(\chi^7), \\
\delta_{D3}(\chi) &= \chi^3 + \frac{3}{70} (3\epsilon_B K_D^2 - 3\epsilon_B K_B^2 - 10K_D^2) \chi^5 + O(\chi^7), \\
\delta_{B4}(\chi) &= \chi^{-2} + \left( K_B^2 - 5\frac{\epsilon_B}{\epsilon_D} K_D^2 + 5\epsilon_D K_B^2 - 5\epsilon_D K_D^2 \right) \\
&\quad + 6\epsilon_D (K_B^2 - K_D^2) \log \chi + O(\chi^2), \\
\delta_{D4}(\chi) &= \chi^{-2} + (6K_B^2 - 5\epsilon_B K_B^2 + 5\epsilon_B K_D^2) \\
&\quad + 6\epsilon_D (K_D^2 - K_B^2) \log \chi + O(\chi^2).
\end{align*}
\]

It remains now to make a physical interpretation of these solutions. This is most usefully achieved by constructing a table comparing different properties of the modes, and comparing the solutions to the corresponding cosmological plasma modes. The results are summarized in Table II, which uses notation defined in [18]. Both the \( K_D = 0 \) modes of the previous section and the modes of this current section are included in each category under the table. The plasma modes \( y_1 \ldots y_4 \) of \[18\] Eq.(4.16) correspond to the \( K_D = 0 \) modes \[4.25]–\[4.32\]. The current gravitational modes \[5.3]–\[5.10\] (that is \( \delta_1 \ldots \delta_4 \)) correspond to the more general expansions Eqs.(4.20), (4.21), (4.29)–(4.32) of \[18\].

In Table II, the power of \( \chi \) gives the corresponding exponent \( \rho \). The parameter \( \nu \equiv \sqrt{\chi^2 - P_i^2 - P_e^2} \) found in the plasma modes depends on the plasma frequencies of the electron
TABLE I: A comparison of gravitational and plasma linear perturbation modes

| Gravitational Modes | Plasma Modes |
|---------------------|--------------|
| $\delta_1 \sim \chi^0 \sim t^0$ | $y_1 \sim \eta^0 \sim t^0$ |
| $\frac{\delta_B}{\delta_D} \sim -\frac{\epsilon_D}{\epsilon_B}$ | $\frac{n_e}{n_i} \sim 1$ |
| lower $2F_3$ parameters: $-\frac{1}{2}, \frac{1}{2}, 2$ | $\frac{1}{3}, \frac{3}{4} - \frac{1}{4}\nu, \frac{3}{4} + \frac{1}{4}\nu$ |
| acoustic mode | ion-sound mode |

| $\delta_2 \sim \chi \sim a^{-1/2} \sim t^{-1/3}$ | $y_2 \sim \eta^{-1} \sim a^{-1/2} \sim t^{-1/3}$ |
| $\frac{\delta_B}{\delta_D} \sim -\frac{\epsilon_D}{\epsilon_B}$ | $\frac{n_e}{n_i} \sim 1$ |
| logarithmic solutions | parameters do not correspond |
| acoustic mode | ion-sound mode |

| $\delta_3 \sim \chi^3 \sim a^{-3/2} \sim t^{-1}$ | $y_3 \sim \eta^{-1/2-\nu} \sim a^{-1/4-(1/2)\nu} \sim t^{-1/6-(1/3)\nu}$ |
| $\frac{\delta_B}{\delta_D} \sim 1$ | $\frac{n_e}{n_i} \sim -\frac{P^2}{P^i}$ |
| lower $2F_3$ parameters: $-\frac{5}{2}, \frac{7}{2}, 2$ | $1 + \nu, \frac{3}{4} + \frac{1}{2}\nu, \frac{5}{4} + \frac{1}{2}\nu$ |
| collapsing mode | Langmuir mode |

| $\delta_4 \sim \chi^{-2} \sim a \sim t^{2/3}$ | $y_4 \sim \eta^{-1/2+\nu} \sim a^{-1/4+(1/2)\nu} \sim t^{-1/3+(1/3)\nu}$ |
| $\frac{\delta_B}{\delta_D} \sim 1$ | $\frac{n_e}{n_i} \sim -\frac{P^2}{P^i}$ |
| logarithmic solutions | parameters do not correspond |
| collapsing mode | Langmuir mode |

and ion components. For the gravitational modes, the signs of $P_i$ and $P_e$ must be reversed since (as opposed to the electromagnetic force) gravity is always attractive. In addition $P_i^2 + P_e^2$ corresponds to $\epsilon_B + \epsilon_D = 1$ (once again the special nature of gravity in cosmology is apparent). This implies that the general parameter $\nu$ in the plasma modes should be replaced with $\frac{5}{2}$ for the gravitational modes—the reason why many of the parameters in the $2F_3$ and Meijer G-functions were pure rational numbers not depending on physical constants. Notice that the ratio of the amplitudes of baryonic/dark matter modes and electron/ion modes differ. This is because the couplings in the differential equations are different. For the gravitational modes the couplings involve terms such as $\epsilon_D \delta_D$ and $\epsilon_B \delta_B$, whereas for the plasma modes the couplings involve terms such as $P_i^2 \bar{n}_e$ and $P_e^2 \bar{n}_i$.

We have indicated the corresponding collapsing and acoustic modes for the gravitational
density perturbations. It is difficult to show this rigorously for the series solutions as presented. We can make comparisons to the one-component results, and identify the leading order powers of the expansion parameter $a$. This yields the classification as stated. We can also make an analogy to the ion-sound modes of plasma physics, which are of a similar nature to acoustic oscillations. They show a collective behavior of both components oscillating approximately in phase.

We are finally left with the question of how the Jeans scale enters into the solutions. In Section 4 the mixture wavenumber $k_M$, Eq. (4.4) was briefly introduced as being the only physically meaningful scale for instabilities in a two-component fluid. To make this quantity dimensionless, it would be appropriate to make the definition

$$K^2_M = \frac{k^2}{W_B/v_B^2 + W_D/v_D^2}. \quad (5.11)$$

This quantity is only of relevance to a static spacetime scenario. To place it in the context of the expanding Universe, the substitutions

$$W_B \rightarrow \frac{6 \epsilon_B}{\chi^2}, \quad v_B^2 k^2 \rightarrow 6 K_B^2,$$

$$W_D \rightarrow \frac{6 \epsilon_D}{\chi^2}, \quad v_D^2 k^2 \rightarrow 6 K_D^2$$

are required. Then $K_M$ takes on the revised definition

$$K^2_M = \frac{\chi^2}{\epsilon_B/K_B^2 + \epsilon_D/K_D^2} = \frac{\chi^2}{\chi_c^2}. \quad (5.12)$$

We have introduced the quantity $\chi_c(k)$, which can be thought of as a critical time. For $\chi > \chi_c$, $K_M > 1$ and acoustic oscillations would only be expected to exist for all modes. For $\chi < \chi_c$, $K_M < 1$ and two of the modes become unstable and undergo gravitational collapse. Since the precise magnitude of the scale factor $a$ is not determined by cosmology [see Eq. (3.7), which contains an arbitrary initial time $t_i$], we may arbitrarily assign an initial time $a_0 = 1$, so that $\chi_0 = 1$ and decreases with increasing time. Then we may interpret the Jeans instability in two ways by considering the critical time $\chi_c$. Initially we may study all $k$-dependent modes at a particular time $\chi$, where a subset will be unstable for values of $k$ for which $\chi_c(k) > \chi$ (we stress that $\chi_c$ is a function of $k$). We may then consider what occurs as the modes evolve through time from this particular instant. The critical time $\chi_c$ is fixed for any particular mode, so that a subset of modes that were originally acoustic will become unstable as $\chi \rightarrow \chi_c^+$ (those modes corresponding to the solutions $\delta_3$ and $\delta_4$ in
Here $\chi/\chi_c$ in terms of increasing powers of $v$ evolves. The physical wavenumber $k$ is of course dependent on time, thus the dependence of the instability on a time $\chi_c$ shows the inextricable link between the wavenumber and time.

It is illuminating at this stage to refer back to the one-component modes discussed in Section 3. For the one-component case, solutions were found in terms of the combination $KJa^{-1/2}$. With the identification $\chi = a^{-1/2}$, the quantity $\chi_c$ is seen to be the two-component analog of $K_J$.

It would be useful to convert the expansions (5.3)–(5.10) to depend on $K_M$ or $\chi_c$, to see how the Jeans scale enters. This is achieved by the following relations:

$$K_B^2 = \frac{1}{\chi_c^2}(\epsilon_B + \epsilon_D V^2),$$
$$K_D^2 = \frac{1}{\chi_c^2}(\epsilon_D + \frac{\epsilon_B}{V^2}).$$

(5.13)

Here $V = v_B/v_D$ is the ratio of sound velocities. We then find that the expansions are all in terms of increasing powers of $\chi/\chi_c$, with coefficients in terms of $\epsilon_B, \epsilon_D$ and $V$:

$$\delta_{B1}(\chi) = 1 + \frac{3}{2} \left( \frac{\epsilon_B + \epsilon_D V^2 - 6\epsilon_B \epsilon_D - 3\epsilon_D V^2 - 3\epsilon_B^2 V^2}{\chi_c^2} \right) \frac{\chi^2}{\chi_c^2} + O \left( \frac{\chi^4}{\chi_c^4} \right),$$
$$\delta_{D1}(\chi) = -\frac{\epsilon_B}{\epsilon_D} \left[ 1 + \frac{3}{2} \left( \frac{\epsilon_D + \frac{\epsilon_B}{V^2} - 6\epsilon_B \epsilon_D - 3\epsilon_D V^2 - 3\epsilon_B^2 V^2}{\chi_c^2} \right) \frac{\chi^2}{\chi_c^2} + O \left( \frac{\chi^4}{\chi_c^4} \right) \right] \frac{\chi^2}{\chi_c^2} + O \left( \frac{\chi^4}{\chi_c^4} \right),$$

(5.14)

(5.15)

$$\delta_{B2}(\chi) = \chi \left[ 1 + \frac{1}{25} \left( 5 + \epsilon_D + \frac{6\epsilon_B}{V^2} + \frac{5\epsilon_B^2}{\epsilon_D V^2} - 31 \left( \epsilon_D V + \frac{\epsilon_B}{V} \right)^2 \right) \frac{\chi^2}{\chi_c^2} + \frac{6\epsilon_B}{5 \epsilon_D} \left( \epsilon_D - \epsilon_B - \epsilon_D V^2 + \frac{\epsilon_B}{V^2} \right) \frac{\chi^2}{\chi_c^2} \log \chi + O \left( \frac{\chi^4}{\chi_c^4} \right) \right],$$
$$\delta_{D2}(\chi) = -\frac{\epsilon_B}{\epsilon_D} \chi \left[ 1 - \frac{1}{25} \left( \epsilon_D - \frac{\epsilon_B}{V^2} + 31 \left( \epsilon_D V + \frac{\epsilon_B}{V} \right)^2 \right) \frac{\chi^2}{\chi_c^2} \right] - \frac{6\epsilon_B}{5 \epsilon_D} \left( \epsilon_D - \epsilon_B - \epsilon_D V^2 + \frac{\epsilon_B}{V^2} \right) \frac{\chi^2}{\chi_c^2} \log \chi + O \left( \frac{\chi^4}{\chi_c^4} \right),$$

(5.16)

(5.17)

$$\delta_{B3}(\chi) = \chi^3 \left[ 1 + \frac{3}{70} \left[ 3\epsilon_D \left( \epsilon_B - \epsilon_D + \epsilon_D V^2 - \frac{\epsilon_B}{V^2} \right) \right] \frac{\chi^2}{\chi_c^2} \right] - 10 \left( \epsilon_B + \epsilon_D V^2 \right) \frac{\chi^2}{\chi_c^2} + O \left( \frac{\chi^4}{\chi_c^4} \right),$$
$$\delta_{D3}(\chi) = \chi^3 \left[ 1 + \frac{3}{70} \left[ 3\epsilon_B \left( \epsilon_D - \epsilon_B - \epsilon_D V^2 + \frac{\epsilon_B}{V^2} \right) \right] \frac{\chi^2}{\chi_c^2} \right] - 10 \left( \epsilon_D + \frac{\epsilon_B}{V^2} \right) \frac{\chi^2}{\chi_c^2} + O \left( \frac{\chi^4}{\chi_c^4} \right),$$

(5.18)

(5.19)
\[ \delta_{B4}(\chi) = \chi^{-2} \left\{ 1 + \left[ -4\epsilon_B + \epsilon_D V^2 + 5\epsilon_B \epsilon_D + 5\epsilon_D^2 (V^2 - 1) \right. \right. \\
- 5 \left( \frac{\epsilon_B}{\epsilon_D} + \epsilon_D \right) \frac{\chi^2}{\chi_c^2} \left. \right. \\
+ 6\epsilon_D \left( \epsilon_B - \epsilon_D + \epsilon_D V^2 - \frac{\epsilon_B}{V^2} \right) \frac{\chi^2}{\chi_c^2} \log \chi + O\left( \frac{\chi^4}{\chi_c^4} \right) \right\}, \quad (5.20) \]

\[ \delta_{D4}(\chi) = \chi^{-2} \left\{ 1 + \left[ 6\epsilon_D - 5\epsilon_B^2 + 5\epsilon_B \epsilon_D (1 - V^2) + \frac{5\epsilon_B}{V^2} \right] \frac{\chi^2}{\chi_c^2} \log \chi + O\left( \frac{\chi^4}{\chi_c^4} \right) \right\}, \quad (5.21) \]

This is a convenient parameterization of the solutions. The scale of the modes are chosen by \( \chi_c \), the nature of the matter involved is determined by \( V \), and the proportions are determined by \( \epsilon_B \) and \( \epsilon_D \). A complete solution to the problem has thus been achieved up to whatever order desired.

VI. CONCLUSIONS AND FURTHER WORK

A method for determining the small \( k \) solutions of a general two-component cosmological density perturbation model has been expounded in this paper. We have only displayed the solutions to first order, but it is possible derive them up to any order by the method in principle. We have explored the mathematical properties and peculiarities of density perturbations influenced by gravitational interaction, particularly contrasting them to plasma modes, and correcting a number of previous misconceptions in the literature. The expanding Universe introduces new features not predictable from simple static spacetime considerations. In particular, totally new structures to the dispersion relations are found, even in the one-component example. We have shown how the mixture Jeans wavenumber enters the solutions, and clarified its role in an expanding universe context.

More work is required to investigate the solutions around the critical scale defined by \( k_M \). Although the expansions as derived in this paper are applicable to this region, they are not particularly useful, as many terms in the equations need to be retained when the expansion parameter \( \chi/\chi_c \) is of \( O(1) \). It is unclear how an analytical investigation of this region could proceed at present. We have performed some preliminary studies which involved producing a large number of terms in the expansions \((5.3) - (5.10)\) using the Mathematica program described, and then substituting in numerical values for the various physical parameters to obtain numerical coefficients with an ascending series in \( \chi \). At present the plots of these
expansions over a range of values of $\chi$ do not yield reliable results—it is possible that many more terms than are practically calculable will be required, and a very high order of numerical precision will have to be maintained. Other methods of analyzing the modes in this interesting region probably need to be investigated.

Of ultimate interest is exploring how these type of modes contribute to the power spectrum. More physical effects may need to be introduced, such as a cosmological constant, or the addition of more matter components. To determine the actual density contrast at a given scale $1/k$, the Fourier modes of the density contrast as derived in this paper would also need to be integrated over the whole range $0 < k < 1/k$. It would be of considerable interest to compare the power spectra calculated by such a method with the well-known power spectra of the various cosmological models in existence today.

In concluding, we remark that a similar analysis could be carried out in the post-recombination region $140 < z < 1150$, where now the baryons follow the $T \sim 1/a$ relationship. The differential equations in Section II will now be different, as will be their solutions; but we expect that the ensuing analysis would yield qualitatively similar results but quantitatively different scaling. This would be a useful future study.

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FIG. 1: The transition of the decaying one-component modes as $K_J$ varies through the Jeans instability
FIG. 2: The transition of the growing one-component modes as $K_J$ varies through the Jeans instability