THE TRIGONOMETRIC CASIMIR CONNECTION OF A SIMPLE LIE ALGEBRA

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To Corrado De Concini, on his 60th birthday

Abstract. Let \( \mathfrak{g} \) be a complex, semisimple Lie algebra, \( G \) the corresponding simply-connected Lie group and \( H \subset G \) a maximal torus. We construct a flat connection on \( H \) with logarithmic singularities on the root hypertori and values in the Yangian \( Y(\mathfrak{g}) \) of \( \mathfrak{g} \). By analogy with the rational Casimir connection of \( \mathfrak{g} \), we conjecture that the monodromy of this trigonometric connection is described by the quantum Weyl group operators of the quantum loop algebra \( U_\hbar(\mathcal{L}\mathfrak{g}) \).

CONTENTS

1. Introduction 1
2. The trigonometric connection of a root system 4
3. The trigonometric Casimir connection 17
4. The monodromy conjecture 21
5. The trigonometric Casimir connection of \( \mathfrak{gl}_n \) 22
6. Bispectrality 30
7. The affine KZ connection 34
8. Appendix: Tits extensions of affine Weyl groups 36
Acknowledgments 43
References 43

1. Introduction

1.1. Let \( \mathfrak{g} \) be a complex, simple Lie algebra, \( \mathfrak{h} \subset \mathfrak{g} \) a Cartan subalgebra, \( \Phi \subset \mathfrak{h}^* \) the corresponding root system and \( W \) its Weyl group. For each \( \alpha \in \Phi \), let \( \mathfrak{sl}_\alpha^2 = \langle e_\alpha, f_\alpha, h_\alpha \rangle \subset \mathfrak{g} \) be the corresponding three-dimensional subalgebra and denote by

\[
\kappa_\alpha = \frac{(\alpha, \alpha)}{2} (e_\alpha f_\alpha + f_\alpha e_\alpha)
\]
its truncated Casimir operator with respect to the restriction to \( \mathfrak{sl}_2 \) of a fixed non–degenerate, ad–invariant bilinear form \((\cdot, \cdot)\) on \( \mathfrak{g} \). Let 

\[
\mathfrak{h}_{\text{reg}} = \mathfrak{h} \setminus \bigcup_{\alpha \in \Phi} \text{Ker}(\alpha)
\]

be the set of regular elements in \( \mathfrak{h} \), \( V \) a finite–dimensional \( \mathfrak{g} \)--module, and \( \nabla \) the holomorphically trivial vector bundle over \( \mathfrak{h}_{\text{reg}} \) with fibre \( V \). Recall that the Casimir connection of \( \mathfrak{g} \) is the holomorphic connection on \( \nabla \) given by

\[
\nabla_\kappa = d - \hbar \sum_{\alpha \in \Phi_+} \frac{d\alpha}{\alpha} \kappa_\alpha
\]

where \( d \) is the de Rham differential, \( \Phi_+ \subset \Phi \) is a system of positive roots and \( \hbar \) is a complex number. This connection was discovered independently by C. De Concini around 1995 (unpublished), by J. Millson and the author [MTL, TL1] and Felder et al. [FMTV], and shown to be flat for any \( \hbar \in \mathbb{C} \).

The monodromy of \( \nabla_\kappa \) gives representations of the generalised braid group \( B = \pi_1(\mathfrak{h}_{\text{reg}}/W) \) which are described by the quantum Weyl group operators of the quantum group \( U_\hbar \mathfrak{g} \), a fact which was conjectured by De Concini (unpublished) and independently in [TL1, TL2] and proved in [TL1, TL3].

1.2. Let \( P \subset \mathfrak{h}^* \) be the weight lattice of \( \mathfrak{g} \) and \( H = \text{Hom}_\mathbb{Z}(P, \mathbb{C}^*) \) the dual algebraic torus with Lie algebra \( \mathfrak{h} \) and coordinate ring given by the group algebra \( \mathbb{C}P \). We denote the function corresponding to \( \lambda \in P \) by \( e^\lambda \in \mathbb{C}[H] \).

The main goal of the present paper is to define a trigonometric version of the connection (1.1), that is a connection defined on \( H \) with the logarithmic forms \( d\alpha/\alpha \) replaced by \( d\alpha/(e^\alpha - 1) \).

As is well known from the study of Cherednik’s affine KZ (AKZ) connection (see, e.g. [Ch3]), both the flatness and \( W \)--equivariance of such a connection require that it possess a ‘tail’, that is be of the form

\[
\hat{\nabla}_\kappa = d - \hbar \sum_{\alpha \in \Phi_+} \frac{d\alpha}{e^\alpha - 1} \kappa_\alpha - A
\]

where \( A \) is a translation–invariant one–form on \( H \). The analogy with the AKZ equations further suggests that \( A \) should take values in a suitable extension of the enveloping algebra \( U \mathfrak{g} \), which is to \( U \mathfrak{g} \) what the degenerate affine Hecke algebra \( \mathcal{H}' \) is to the group algebra \( \mathbb{C}W \).

1.3. The correct extension turns out to be the Yangian \( Y(\mathfrak{g}) \), which is a deformation of the enveloping algebra \( U(\mathfrak{g}[t]) \) over the ring \( \mathbb{C}[t] \). Let \( \nu : \mathfrak{h} \to \mathfrak{h}^* \) be the isomorphism determined by the inner product \((\cdot, \cdot)\), set \( t_i = \nu^{-1}(\alpha_i) \), where \( \alpha_1, \ldots, \alpha_n \) are the simple roots of \( \mathfrak{g} \) relative to \( \Phi_+ \) and let \( t^i = \lambda^i \) be the dual basis of \( \mathfrak{h} \) given by the fundamental coweights. Let \( T(u)_r, u \in \mathfrak{h}, r \in \mathbb{N} \) be the Cartan loop generators of \( Y(\mathfrak{g}) \) in Drinfeld’s new realisation (see [Dr2] and [B3] for definitions). Let \( \{u^i\} \) be a basis of \( \mathfrak{h} \), \( \{u_i\} \) the dual basis of \( \mathfrak{h}^* \) and regard the differentials \( du_i \) as translation–invariant one–forms on \( H \). The main result of this paper is the following
Theorem. The $Y(\mathfrak{g})$–valued connection on $H$ given by

$$\tilde{\nabla}_\kappa = d - \hbar \sum_{\alpha \in \Phi^+} \frac{d\alpha}{e^\alpha - 1} \kappa_\alpha + du_i \left( 2T(u^i)_1 - \frac{\hbar}{2} (u^i, u^j)_1 t^j \right)$$

is flat and $W$–equivariant.

1.4. We call $\tilde{\nabla}_\kappa$ the trigonometric Casimir connection of $\mathfrak{g}$. Its monodromy defines representations of the affine braid group $\hat{B} = \pi_1(H_{\text{reg}}/W)$ on any finite–dimensional module over $Y(\mathfrak{g})$, where

$$H_{\text{reg}} = H \setminus \bigcup_{\alpha \in \Phi} \{ e^\alpha = 1 \}$$

is the set of regular elements in $H$. By analogy with the rational case, we conjecture that these representations are equivalent to the quantum Weyl group action of $\hat{B}$ on finite–dimensional modules over the quantum loop algebra $U_\hbar(\mathfrak{g}[z, z^{-1}])$.

1.5. We turn now to a detailed description of the paper.

In Section 2 we obtain a necessary and sufficient condition for a connection of the form (1.2) to be flat and equivariant under the Weyl group $W$ (Theorem 2.5 and Proposition 2.22), thus effectively determining the Lie algebras of the fundamental groups $\pi_1(H_{\text{reg}})$ and $\pi_1(H_{\text{reg}}/W)$. Consistently with Cherednik’s study of the AKZ connection, the quadratic equations giving the flatness and equivariance of $\tilde{\nabla}_\kappa$ specialise, when $\kappa_\alpha$ is replaced by the orthogonal reflection $s_\alpha \in W$ to the defining relations of the degenerate affine Hecke algebra corresponding to $W$.

In Section 3 we review Drinfeld’s two presentations of the Yangian $Y(\mathfrak{g})$ of a simple Lie algebra $\mathfrak{g}$. We then use their interplay to solve the above quadratic equations in $Y(\mathfrak{g})$, thereby obtaining the trigonometric Casimir connection of $\mathfrak{g}$ (Theorem 3.8).

In Section 4 we explain how to define monodromy representations of the affine braid group from $\tilde{\nabla}_\kappa$ and conjecture that these are described by the quantum Weyl group operators of the quantum loop algebra $U_\hbar(\mathfrak{g}[z, z^{-1}])$.

In Section 5 we define a trigonometric connection with values in the Yangian of $\mathfrak{sl}_n$ by using the interplay between its loop and RTT presentations (Theorem 5.7). We then relate it to the trigonometric Casimir connection of $\mathfrak{sl}_n$. We also check that, when computed in a tensor product of $m$ evaluation modules, it coincides with the trigonometric dynamical differential equations [TV] which are differential equations on $(\mathbb{C}^*)^n$ with values in $U_\hbar(\mathfrak{gl}_n^\otimes m)$.

In Section 6 we show that the trigonometric Casimir connection commutes with $q$KZ difference equations of Frenkel–Reshetikhin determined by the rational $R$–matrix of $Y(\mathfrak{g})$ (Theorem 6.6), a fact which was checked in [TV] for the trigonometric dynamical differential equations.

\footnote{1We follow the standard convention that in any expression involving $u_i$ and $u^i$, or $t_i$ and $t^i$, summation over $i$ is implicit.}
In Section 7, we review the definition of the degenerate affine Hecke algebra $H'$ of $W$ [Lu] and of the corresponding $H'$–valued AKZ connection [Ch3]. We then show that if $V$ is a $Y(g)$–module whose restriction to $g$ is small, that is such that $2\alpha$ is not a weight for any root $\alpha$ [Br, Re], the zero weight space $V[0]$ carries a natural action of $H'$. Moreover, the trigonometric Casimir connection with coefficients in $V[0]$ coincides with the AKZ connection with values in this $H'$–module (Theorem 7.5).

The final appendix, Section 8 contains a discussion of the Tits extensions of affine Weyl groups which is needed for Section 4.

2. THE TRIGONOMETRIC CONNECTION OF A ROOT SYSTEM

2.1. General form. Let $E$ be a Euclidean vector space, $\Phi \subset E^*$ a reduced, crystallographic root system. Let $Q^\vee \subset E$ be the lattice generated by the co-roots $\alpha^\vee$, $\alpha \in \Phi$ and $P \subset E^*$ the dual weight lattice. Let $H = \text{Hom}_\mathbb{Z}(P, \mathbb{C}^*)$ be the complex algebraic torus with Lie algebra $\mathfrak{h} = \text{Hom}_\mathbb{Z}(P, \mathbb{C})$ and coordinate ring given by the group algebra $\mathbb{C}P$. We denote the function corresponding to $\lambda \in P$ by $e^\lambda \in \mathbb{C}[H]$ and set

$$H_{\text{reg}} = H \setminus \bigcup_{\alpha \in \Phi} \{e^\alpha = 1\} \tag{2.1}$$

Let $A$ be an algebra endowed with the following data:

- a set of elements $\{t_\alpha\}_{\alpha \in \Phi} \subset A$ such that $t_{-\alpha} = t_\alpha$
- a linear map $\tau : \mathfrak{h} \to A$

Consider the $A$–valued connection on $H_{\text{reg}}$ given by

$$\nabla = d - \sum_{\alpha \in \Phi^+} \frac{d\alpha}{e^\alpha - 1} t_\alpha - du_i \tau(u^i) \tag{2.2}$$

where $\Phi^+ \subset \Phi$ is a chosen system of positive roots, $\{u_i\}$ and $\{u^i\}$ are dual bases of $\mathfrak{h}^*$ and $\mathfrak{h}$ respectively, the differentials $du_i$ are regarded as translation–invariant one–forms on $H$ and the summation over $i$ is implicit.

2.2. Positive roots. The form of the connection (2.2) depends upon the choice of the system of positive roots $\Phi^+ \subset \Phi$. Let however $\Phi'_+ \subset \Phi$ be another such system, then

**Proposition.** The connection (2.2) may be rewritten as

$$\nabla = d - \sum_{\alpha \in \Phi'_+} \frac{d\alpha}{e^\alpha - 1} t_\alpha - du_i \tau'(u^i)$$

where $\tau' : \mathfrak{h} \to A$ is given by

$$\tau'(v) = \tau(v) - \sum_{\alpha \in \Phi^+_+ \cap \Phi'_-} \alpha(v) t_\alpha \tag{2.3}$$
Proof. Write the second summand in (2.2) as
\[
\sum_{\alpha \in \Phi_+ \cap \Phi'_+} \frac{d\alpha}{e^{\alpha} - 1} t_{\alpha} - \sum_{\alpha \in \Phi_- \cap \Phi'_+} \frac{d\alpha}{e^{-\alpha} - 1} t_{-\alpha}
\]
where $\Phi_- = -\Phi_+$. Since
\[
\frac{1}{1 - e^{-\alpha}} = \frac{e^\alpha}{e^\alpha - 1} = \frac{1}{e^\alpha - 1} + 1
\]
and $t_{-\alpha} = t_\alpha$, the above is equal to
\[
\sum_{\alpha \in \Phi'_+} \frac{d\alpha}{e^\alpha - 1} t_{\alpha} + \sum_{\alpha \in \Phi_- \cap \Phi'_+} d\alpha t_{\alpha}
\]
which yields the required result since $\alpha = u_i(\alpha^i)$.

Definition. If $W$ is the Weyl group of $\Phi$ and $w \in W$ the unique element such that $\Phi'_+ = w\Phi_+$, we denote $\tau'$ by $\tau_w$. Thus,
\[
\tau_w(v) = \tau(v) - \sum_{\alpha \in \Phi_+ \cap w\Phi_-} \alpha(v) t_{\alpha}
\]

2.3. Delta form. Choose $\Phi'_+ = \Phi_-$ in Proposition 2.2. Comparing the corresponding expressions for $\nabla$ shows that it may be more invariantly rewritten as
\[
\nabla = d - \frac{1}{2} \sum_{\alpha \in \Phi} \frac{d\alpha}{e^\alpha - 1} t_{\alpha} - du_i \delta(u^i)
\]
where $\delta : h \to A$ is given by
\[
\delta(v) = \tau(v) - \frac{1}{2} \sum_{\alpha \in \Phi_+} \alpha(v) t_{\alpha}
\]
Alternatively, substituting (2.6) into (2.2) yields
\[
\nabla = d - \frac{1}{2} \sum_{\alpha \in \Phi_+} \frac{e^\alpha + 1}{e^\alpha - 1} d\alpha t_{\alpha} - du_i \delta(u^i)
\]
We shall occasionally refer to (2.2) and (2.7) as the $\tau$ and $\delta$-forms of the connection $\nabla$ respectively. Note that the latter does not depend upon the choice of $\Phi_+ \subset \Phi$.

2.4. Root subsystems. For a subset $\Psi \subset \Phi$ and subring $R \subset \mathbb{R}$, let $\langle \Psi \rangle_R \subset E^*$ be the $R$-span of $\Psi$.

Definition. A root subsystem of $\Phi$ is a subset $\Psi \subset \Phi$ such that $\langle \Psi \rangle_{\mathbb{Z}} \cap \Phi = \Psi$. $\Psi$ is complete if $\langle \Psi \rangle_{\mathbb{R}} \cap \Phi = \Psi$. If $\Psi \subset \Phi$ is a root subsystem, we set $\Psi_+ = \Psi \cap \Phi_+$. 
Remark. According to the above definition, the short roots of the root system $B_2$ (resp. $G_2$) are not a root subsystem, but the long ones constitute a root subsystem of type $A_1 \times A_1$ (resp. $A_2$) which is not complete. Another root subsystem of $\Phi = G_2$ is given by $\{ \pm \alpha, \pm \beta \}$ where $\alpha, \beta$ are two orthogonal roots (necessarily of different lengths).

2.5. Integrability. The following is the main result of this section.

Theorem. 
(1) The connection $\nabla$ is flat if, and only if the following relations hold
- For any rank 2 root subsystem $\Psi \subset \Phi$ and $\alpha \in \Psi$,
  \[ [t_\alpha, \sum_{\beta \in \Psi_+} t_\beta] = 0 \]  
  
- For any $u, v \in \mathfrak{h}$,
  \[ [\tau(u), \tau(v)] = 0 \]  
- For any $\alpha \in \Phi_+, w \in W$ such that $w^{-1} \alpha$ is a simple root and $u \in \mathfrak{h}$ such that $\alpha(u) = 0$,
  \[ [t_\alpha, \tau_w(u)] = 0 \]  
  
(2) Modulo the relations (tt), the relations (tτ) are equivalent to
  \[ [t_\alpha, \delta(v)] = 0 \]  
for any $\alpha \in \Phi$ and $v \in \mathfrak{h}$ such that $\alpha(v) = 0$, where $\delta : \mathfrak{h} \to A$ is given by (2.6).

The proof of Theorem 2.5 occupies the paragraphs 2.7–2.19.

2.6. We spell out below the relations (tt) in the case when $\Phi$ is of rank 2. For $\Psi = \Phi$, they read
  \[ [t_\alpha, \sum_{\beta \in \Phi_+} t_\beta] = 0 \text{ for any } \alpha \in \Phi \]  
  
In particular, if $\Phi = \{ \pm \alpha, \pm \beta \}$ is of type $A_1 \times A_1$ then
  \[ [t_\alpha, t_\beta] = 0 \]  
  
For $\Phi = B_2$, the long roots $\{ \pm \beta_1, \pm \beta_2 \}$ form an $A_1 \times A_1$ subsystem so that
  \[ [t_{\beta_1}, t_{\beta_2}] = 0 \]  
  
For $\Phi = G_2$, there are two types of root subsystems: that formed by the $A_2$ configuration of long roots $\{ \pm \beta_1, \pm \beta_2, \pm \beta_3 \}$, leading to
  \[ [t_{\beta_i}, t_{\beta_j} + t_{\beta_k}] = 0 \]  
  
and the $A_1$ configurations $\{ \pm \beta, \pm \gamma \}$ formed by a long root and an orthogonal short one, leading to
  \[ [t_\beta, t_\gamma] = 0 \]
Combining relations \(Φ\), \((A_2 \subset G_2)\) and \((A_1 \times A_1 \subset G_2)\) yields in particular the following relations

\[
[t_β, t_{γ'} + t_{γ''}] = 0 \tag{2.8}
\]

where \(β \in G_2\) is long and \(γ', γ''\) are the short positive roots which are not orthogonal to \(β\).

**Remark.** If \(Φ\) is not simply–laced, the relations \((tt)\) are stronger than those yielding the flatness of the rational connection

\[
∇ = d - \sum_{α ∈ Φ_+} \frac{dα}{α} t_α
\]

Indeed, the latter involve two dimensional subspaces of \(h^*\) spanned by elements of \(Φ \ [K]\) and therefore only those rank 2 subsystems of \(Φ\) which are complete. The relevance of additional relations corresponding to non–complete subsystems was first pointed out in the closely related context of the Yang–Baxter equations by Cherednik \[Ch3, §6.1\].

2.7. Let \(Q ⊂ P\) be the root lattice generated by \(Φ\) and \(T = \text{Hom}_Z(Q, C^*)\) the corresponding complex algebraic torus of adjoint type. \(T\) has coordinate ring \(CQ\) and is birationally isomorphic to the standard torus \((C^*)^n\) by sending \(p ∈ T\) to the point with coordinates \(z_i = e^{-α_i}(p)\), where \(α_i\) varies over the simple roots of \(Φ\) relative to \(Φ_+\).

2.8. The torus \(T\) is a quotient of \(H\) and the form of \(∇\) shows that it may be regarded as a connection on the trivial vector bundle with fibre \(A\) over \(T\). As such, \(∇\) has singularities on the codimension one subtori

\[T_α = \{e^α = 1\} \subset T\]

where \(α ∈ Φ\). Given a subset \(Ψ ⊂ Φ\), we shall be interested in the connectedness of the intersection \(\bigcap_{α ∈ Ψ} T_α\). Let \(⟨Ψ⟩_Z ⊂ Q\) be the \(Z\)-span of \(Ψ\) and set \[DCP\ §3.1\]

\[⟨Ψ⟩_Z = \{γ ∈ Q| mγ ∈ ⟨Ψ⟩_Z \text{ for some } m ∈ Z^*\}\]

Since

\[C[\bigcap_{α ∈ Ψ} T_α] = C Q/⟨Ψ⟩_Z ≅ C Q/⟨Ψ⟩_Z ⊗ C (⟨Ψ⟩_Z/⟨Ψ⟩_Z)\]

the connected components of \(\bigcap_{α ∈ Ψ} T_α\) are tori labelled by the characters of the finite abelian group \(⟨Ψ⟩_Z/⟨Ψ⟩_Z\). In particular, if \(Ψ = \{α\}\), we see that each \(T_α\) is connected since \(α\) is indivisible in \(Q\).

2.9. The necessity of relations \((tt)–(tτ)\) follows from the computation of the residues of the curvature \(Ω\) of \(∇\) to be carried out in \[2.10–2.14\].
Specifically, write $\nabla = d - A$. Since $dA = 0$, $\Omega$ is equal to $A \wedge A = \Omega_1 + \Omega_2 + \Omega_3$, where

$$\Omega_1 = \frac{1}{2} \sum_{\alpha, \beta} \frac{d\alpha}{e^\alpha - 1} \wedge \frac{d\beta}{e^\beta - 1} [t_\alpha, t_\beta]$$

(2.9)

$$\Omega_2 = \sum_{\alpha, i} \frac{d\alpha}{e^\alpha - 1} \wedge du_i [t_\alpha, \tau(u^i)]$$

(2.10)

$$\Omega_3 = \frac{1}{2} \sum_{i,j} du_i \wedge du_j [\tau(u^i), \tau(u^j)]$$

(2.11)

2.10. Let $\alpha \in \Phi$ and denote the inclusion $T_\alpha \hookrightarrow T$ by $\iota_\alpha$. Then

$$\text{res}_{T_\alpha} \Omega_1 = \iota_\alpha^* \sum_{\beta \neq \alpha} \frac{d\beta}{e^\beta - 1} [t_\alpha, t_\beta]$$

$$\text{res}_{T_\alpha} \Omega_2 = \iota_\alpha^* du_i [t_\alpha, \tau(u^i)]$$

and $\text{res}_{T_\alpha} \Omega_3 = 0$ since $\Omega_3$ is regular on $T_\alpha$.

2.11. Let $\Psi \subset \Phi$ be a rank 2 root subsystem and set

$$T_{\Psi} = \bigcap_{\beta \in \Psi} T_\beta$$

By §2.8, $T_{\Psi}$ is a codimension two subtorus of $T$ with group of components $\text{Hom}(\langle \Psi \rangle_{\mathbb{Z}}/\langle \Psi \rangle_{\mathbb{Z}}, \mathbb{C}^*)$.

Since $\langle \Psi \rangle_{\mathbb{Z}}/\langle \Psi \rangle_{\mathbb{Z}}$ is cyclic (of order 1, 2 or 3 depending on the type of $\Psi$ and $\langle \Psi \rangle_{\mathbb{Z}} \cap \Phi$), there exists a character $\chi$ of $\langle \Psi \rangle_{\mathbb{Z}}/\langle \Psi \rangle_{\mathbb{Z}}$ with trivial kernel. It follows that the corresponding component $T^\chi_{\Psi}$ of $T_{\Psi}$ is contained in some $T_\gamma$ if, and only if, $\gamma \in \Psi$.

Together with §2.10 this implies that for any $\alpha \in \Psi$,

$$\text{res}_{T^\chi_{\Psi}} \text{res}_{T_\alpha} \Omega = [t_\alpha, \sum_{\beta \in \Psi_+} t_\beta]$$

thus showing the necessity of (12).

2.12. Let $\mathbf{T} \cong \mathbb{C}^n$ be the partial compactification determined by the embedding $T \hookrightarrow (\mathbb{C}^*)^n$ given by sending $p \in T$ to the point with coordinates $z_i = e^{-\alpha_i(p)}$. We wish to determine the residues of $\Omega$ on the divisors

$$T_i = \{z_i = 0\} \subset \mathbf{T}$$

To this end, we first rewrite $\nabla$ in the coordinates $z_i$. Choosing $u_i = \alpha_i$ as basis of $\mathfrak{h}^*$, so that the dual basis $\{u^i\}$ of $\mathfrak{h}$ is given by the fundamental coweights $\{\lambda^\vee_i\}$ yields $du_i = -dz_i/z_i$ and

$$du_i \tau(u^i) = -\frac{d z_i}{z_i} \tau(\lambda^\vee_i)$$
Further, if $\alpha = \sum_i m^i_\alpha \alpha_i$ is a positive root, then $e^\alpha = \prod_i z_i^{-m^i_\alpha}$ so that
\[
\frac{d\alpha}{e^\alpha - 1} = \frac{e^{-\alpha}}{1 - e^{-\alpha}} d\alpha = -\sum_i m^i_\alpha \frac{\prod_j z_j^{-m^j_\alpha}}{1 - \prod_j z_j^{-m^j_\alpha}} dz_i
\]
which is a regular on each $T_i$. It follows that $\text{res}_{T_i} \Omega_1 = 0$ and
\[
\text{res}_{T_j \cap T_i} \Omega_2 = s_i^* \sum_\alpha \frac{d\alpha}{e^\alpha - 1} [t_\alpha, \tau(\lambda^j_\vee)]
\]
\[
\text{res}_{T_j \cap T_i} \Omega_3 = s_i^* \sum_{j \neq i} \frac{dz_j}{z_j} [\tau(\lambda^j_\vee), \tau(\lambda^i_\vee)]
\]
where $s_i$ is the inclusion $T_i \hookrightarrow T$.

2.13. Thus, for any $j \neq i$,
\[
\text{res}_{T_j \cap T_i} \Omega = [\tau(\lambda^j_\vee), \tau(\lambda^i_\vee)]
\]
which shows the necessity of the relations (2.7).

2.14. Let now $\alpha = \sum_i m^i_\alpha \alpha_i$ be a positive root and let $T_\alpha = \{ \prod_i z_i^{-m^i_\alpha} = 1 \}$ be the closure of $T_\alpha$ in $T$. The intersection
\[
T_{\alpha,i} = T_\alpha \cap T_i \subset T
\]
is clearly nonempty if, and only if $\alpha(\lambda^i_\vee) = m^i_\alpha = 0$. When that is the case, $T_{\alpha,i}$ is connected and contained in no other $T_\beta$, $\beta \neq j$ or $T_j$ for $j \neq i$.

It follows that whenever $\alpha(\lambda^j_\vee) = 0$,
\[
\text{res}_{T_{\alpha,i}} \text{res}_{T_\alpha} \Omega = [t_\alpha, \tau(\lambda^i_\vee)]
\]
thus showing the necessity of (2.7) for $\alpha$ simple and $w = 1$. The general case follows by repeating the computations of the last two subsections in the compactification of $T$ corresponding to a different basis $\Delta$ of simple roots and using the alternative form of the connection $\nabla$ given by Proposition 2.2.

2.15. We next turn to the sufficiency of the relations (2.7)–(2.11). This may be proved by embedding $T$ in the toric variety corresponding to the fan determined by the chambers of $\Phi$ in $E$ and using a general integrability criterion of E. Looijenga as in \[Lo, \S 1–2\]. We prefer a more direct approach which will occupy \[2.16–2.19\].

\[\text{Note however that line 2 of the statement of Corollary 1.3 in } [Lo] \text{ should read ”for every irreducible component } I \text{ of a codimension two intersection”, the words in bold are missing in } [Lo].\]
2.16. Since the relations (17) imply that $\Omega_3 = 0$, we need to show that the relations (11)–(17) imply that $\Omega_1 + \Omega_2 = 0$. To this end, we rewrite $\Omega_2$ in a different form below and, in §2.17, rewrite $\Omega_1$.

**Lemma.** Modulo the relations (17), the curvature term

$$\Omega_2^\alpha = \sum_i \frac{d\alpha}{e^{\alpha} - 1} \wedge du_i \left[ t_\alpha, \tau(u^i) \right]$$

corresponding to $\alpha \in \Phi_+$ is equal to

$$\sum_{\beta \in \Phi_+ \cap w\Phi_-} \frac{d\alpha}{e^{\alpha} - 1} \wedge d\beta \left[ t_\alpha, t_\beta \right]$$

for any $w \in W$ such that $w^{-1} \alpha$ is a simple root.

**Proof.** Let $w \in W$ be such that $w^{-1} \alpha$ is a simple root $\alpha_i$. By (2.5),

$$\Omega_2^\alpha = \sum_j \frac{d\alpha}{e^{\alpha} - 1} \wedge du_j \left[ t_\alpha, \tau_w(u^j) \right] + \sum_{\beta \in \Phi_+ \cap w\Phi_-} \beta(u^j) \left[ t_\alpha, t_\beta \right]$$

Choosing $u_j = w\alpha_j$ yields a commutator $\left[ t_\alpha, \tau_w(u^j) \right] = \left[ t_\alpha, \tau_w(w\lambda^j) \right]$ which is zero for all $j \neq i$ by (17). Since $d\alpha \wedge w\alpha_i = 0$, this yields

$$\Omega_2^\alpha = \sum_j \frac{d\alpha}{e^{\alpha} - 1} \wedge du_j \left[ t_\alpha, \sum_{\beta \in \Phi_+ \cap w\Phi_-} \beta(u^j) t_\beta \right] = \sum_{\beta \in \Phi_+ \cap w\Phi_-} \frac{d\alpha}{e^{\alpha} - 1} \wedge d\beta \left[ t_\alpha, t_\beta \right]$$

since $\beta = \beta(u^i)u_i$.

2.17. For any $a \in Q$, let $\eta_a$ be the meromorphic one–form on $T$ given by

$$\eta_a = \frac{da}{e^{\alpha} - 1}$$

**Lemma.** Let $\Psi \subset \Phi$ be a rank 2 root subsystem and consider the curvature term

$$\Omega_1^\Psi = \frac{1}{2} \sum_{\alpha, \beta \in \Psi_+} \eta_\alpha \wedge \eta_\beta \left[ t_\alpha, t_\beta \right]$$

Assume that the relations (17) hold. Then, if $\Psi$ is of type $A_1 \times A_1$,

$$\Omega_1^\Psi = 0 \hspace{1cm} (2.12)$$

If $\Psi$ is of type $A_2$ with $\Psi_+ = \{ \alpha, \beta, \alpha + \beta \}$

$$\Omega_1^\Psi = -\eta_{\alpha + \beta} \wedge d\beta \left[ t_{\alpha + \beta}, t_\beta \right] \hspace{1cm} (2.13)$$

If $\Psi$ is of type $B_2$ with $\Psi_+ = \{ \alpha, \beta, \alpha \pm \beta \}$

$$\Omega_1^\Psi = -\eta_{\alpha + \beta} \wedge d\beta \left[ t_{\alpha + \beta}, t_\beta \right] - \eta_\alpha \wedge d(\alpha - \beta) \left[ t_\alpha, t_{\alpha - \beta} \right] \hspace{1cm} (2.14)$$
If $\Psi$ is of type $G_2$, with $\Psi_+ = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2, \alpha_1 + 2\alpha_2, \alpha_1 + 3\alpha_2, 2\alpha_1 + 3\alpha_2\}$

$$\Omega_1^\Psi = -\eta_{\alpha_1+\alpha_2} \wedge da_1 \left[ t_{\alpha_1+\alpha_2}, t_{\alpha_1} \right]$$

(2.15)

$$- \eta_{\alpha_1+2\alpha_2} \wedge (da_2 \left[ t_{\alpha_1+2\alpha_2}, t_{\alpha_2} \right] + d(\alpha_1 + 3\alpha_2) \left[ t_{\alpha_1+3\alpha_2}, t_{\alpha_1+3\alpha_2} \right])$$

$$- \eta_{\alpha_1+3\alpha_2} \wedge da_2 \left[ t_{\alpha_1+3\alpha_2}, t_{\alpha_2} \right]$$

$$- \eta_{2\alpha_1+3\alpha_2} \wedge (da_1 \left[ t_{2\alpha_1+3\alpha_2}, t_{\alpha_1} \right] + d(\alpha_1 + \alpha_2) \left[ t_{2\alpha_1+3\alpha_2}, t_{\alpha_1+3\alpha_2} \right])$$

Proof. If $\Psi$ is of type $A_1 \times A_1$, the result follows from $(A_1 \times A_1)$. For other types, we shall need the following easily verified identity. For $a, b \in Q$, set

$$\eta_{a,b} = \frac{da \wedge db}{e^{a+b} - 1}$$

Then,

$$\eta_a \wedge \eta_b = \eta_a \wedge \eta_{a+b} + \eta_{a+b} \wedge \eta_b + \eta_{a,b}$$

(2.16)

We shall apply (2.16) to $\Omega_1^\Psi$ repeatedly, specifically to terms of the form $\eta_a \wedge \eta_b$ with $\alpha + \beta \in \Phi_+$, until no such terms are left.

For $\Psi_+ = \{\alpha, \beta, \alpha + \beta\}$ of type $A_2$, this yields

$$\Omega_1^\Psi = \eta_{\alpha+\beta} \wedge \eta_\alpha \left[ t_{\alpha+\beta} + t_\beta, t_\alpha \right] + \eta_{\alpha+\beta} \wedge \eta_\beta \left[ t_{\alpha+\beta} + t_\alpha, t_\beta \right] + \eta_\alpha \wedge \eta_\beta \left[ t_\alpha, t\beta \right]$$

By (2.15), the first two commutators are 0 and the third is equal to $[t_\alpha, t_\beta] = [t_\alpha + t_\beta, t_\beta] = -[t_{\alpha+\beta}, t_\beta]$. This yields the required answer since

$$\eta_{a,b} = \eta_{a+b} \wedge db = -\eta_{a+b} \wedge da$$

(2.17)

To keep track of the repeated applications of (2.16) for $\Psi$ of type $B_2, G_2$, we proceed as follows. Recall that the height of $\alpha = \sum_i m^i_\alpha \alpha_i \in \Phi_+$ is defined by $ht(\alpha) = \sum_i m^i_\alpha$. Arrange pairs of distinct roots $(\alpha, \beta)$ on consecutive rows according to the value of $ht(\alpha) + ht(\beta)$: each $(\alpha, \beta)$ stands for a term $\eta_\alpha \wedge \eta_\beta$. From each pair $(\alpha, \beta)$ such that $\alpha + \beta \in \Phi_+$ draw an arrow to $(\alpha, \alpha + \beta)$ and $(\alpha + \beta, \beta)$ to signify that (2.16) has been applied with $a = \alpha$ and $b = \beta$.

For $\Psi_+ = \{\alpha, \beta, \alpha \pm \beta\}$ of type $B_2$ with simple roots $\alpha_1 = \alpha - \beta, \alpha_2 = \beta$, the corresponding graph is

\[
\begin{align*}
(\alpha - \beta, \beta) & \quad \rightarrow \\
(\alpha - \beta, \alpha) & \quad \rightarrow \\
(\alpha - \beta, \alpha + \beta) & \quad \rightarrow \\
(\alpha, \alpha + \beta) & \\
(\alpha, \beta) & \quad \rightarrow \\
(\alpha + \beta, \beta) & 
\end{align*}
\]
This yields an \( \eta_a \land \eta_b \) component of \( \Omega_1^\Psi \) equal to
\[
\eta_{a-\beta} \land \eta_a [t_{a-\beta}, t_\alpha + t_\beta] + \eta_{a-\beta} \land \eta_{a+\beta} [t_{a-\beta}, t_\alpha + t_\beta] + \eta_{a+\beta} \land \eta_\beta [t_{a+\beta} + t_\alpha + t_{a-\beta}, t_\beta] + \eta_\alpha \land \eta_{a+\beta} ([t_\alpha, t_{a+\beta} + t_\beta] + [t_{a-\beta}, t_\beta])
\]
The second commutator is equal to zero by \( A_1 \times A_1 \subset B_2 \), the first by \( \Phi \) and \( A_1 \times A_1 \subset B_2 \) and the third by \( \Phi \). By \( \Phi \), the coefficient of \( \eta_a \land \eta_{a+\beta} \) is equal to \( -[t_{a+\beta}, t_\beta] + [t_{a-\beta}, t_\alpha] = [t_{a-\beta}, t_\alpha + t_\beta] = 0 \).
\( \Omega_1^\Psi \) is therefore equal to its \( \eta_{a,b} \) component, namely
\[
\eta_{a-\beta, \beta} [t_{a-\beta}, t_\beta] + \eta_{a, \beta} [t_\alpha + t_{a-\beta}, t_\beta]
\]
which yields the required answer since, by \( \Phi \)
\[
[t_\alpha + t_{a-\beta}, t_\beta] = -[t_{a+\beta}, t_\beta]
\]
and by \( \Phi \) and \( A_1 \times A_1 \subset B_2 \)
\[
[t_{a-\beta}, t_\beta] = -[t_{a-\beta}, t_\alpha + t_\beta] = -[t_{a-\beta}, t_\alpha]
\]
while, as previously noted
\[
\eta_{a,b} = \eta_{a+b} \land db = -\eta_{a+b} \land da
\]
Assume now that \( \Psi \) is of type \( G_2 \) and has simple roots \( \alpha_1, \alpha_2 \), with \( \alpha_1 \) long. The sets of long and short positive roots are, respectively
\[
\Psi_+ = \{\alpha_1, 2\alpha_1 + 3\alpha_2, \alpha_1 + 3\alpha_2\} \quad \text{and} \quad \Psi_+^s = \{\alpha_1 + \alpha_2, \alpha_1 + 2\alpha_2, \alpha_2\}
\]
and the pairs \( (\beta, \gamma) \) of orthogonal positive roots are
\[
(\alpha_1, \alpha_1 + 2\alpha_2), \quad (2\alpha_1 + 3\alpha_2, \alpha_2), \quad (\alpha_1 + 3\alpha_2, \alpha_1 + \alpha_2)
\]
The corresponding graph reads
\[
\begin{align*}
(\alpha_1, \alpha_2) & \quad \longrightarrow \quad (\alpha_1 + \alpha_2, \alpha_2) \\
(\alpha_1, \alpha_1 + \alpha_2) & \quad \longrightarrow \quad (\alpha_1 + \alpha_2, \alpha_2) \\
(\alpha_1, \alpha_1 + 2\alpha_2) & \quad \longrightarrow \quad (\alpha_1 + 2\alpha_2, \alpha_2) \\
(\alpha_1, \alpha_1 + 3\alpha_2) & \quad \longrightarrow \quad (\alpha_1 + \alpha_2, \alpha_1 + 2\alpha_2) \\
(\alpha_1 + 2\alpha_1 + 3\alpha_2) & \quad \longrightarrow \quad (\alpha_1 + \alpha_2, \alpha_1 + 3\alpha_2) \\
(\alpha_1 + \alpha_2, 2\alpha_1 + 3\alpha_2) & \quad \longrightarrow \quad (\alpha_1 + 2\alpha_2, \alpha_1 + 3\alpha_2) \\
(\alpha_1 + 2\alpha_2, 2\alpha_1 + 3\alpha_2) & \quad \longrightarrow \quad (\alpha_1 + 3\alpha_2, 2\alpha_1 + 3\alpha_2) \\
(\alpha_1 + 3\alpha_2, 2\alpha_1 + 3\alpha_2) & \quad \longrightarrow \quad (\alpha_1 + 3\alpha_2, 2\alpha_1 + 3\alpha_2)
\end{align*}
\]
This yields an \( \eta_{a,b} \) component of \( \Omega^\Psi_1 \) equal to
\[
\eta_{a,1,2} [t_{a_1}, t_{a_2}] + \eta_{1+a_2,2} [t_{a_1+a_2} + t_{a_1}, t_{a_2}]
+ \eta_{1+a_2,2} [t_{a_1+a_2} + t_{a_1}, t_{a_2} + t_{a_1}]
+ \eta_{1+a_2,3} [t_{a_1}, t_{a_1+3}]
+ \eta_{1+a_2,2+2} [t_{a_1+a_2} + t_{a_1+2} + t_{a_2} + t_{a_1}]
\]
By (2.8), the first commutator is equal to \(-[t_{a_1}, t_{a_1+a_2}]\). By (Ψ) and (\( \Phi \) and (\( A_1 \times A_1 \subset G_2 \)), the third commutator is equal to \(-[t_{a_1+3}, t_{a_2}]\). The second commutator is therefore equal to
\[
-[t_{a_1+3} + t_{a_1+2}, t_{a_2} + t_{a_1+3}]
\]
where we used (2.5). By (\( A_2 \subset G_2 \)), the fourth commutator is equal to \(-[t_{a_1}, t_{2a_1+3}]\). Finally, the coefficient of \( \eta_{a_2,1+3} \) is equal to
\[
-[t_{a_1+a_2}, t_{a_1} + t_{2a_1+3}] + [t_{a_1}, t_{a_2}] = [t_{2a_1+3}, t_{a_1+a_2}]
\]
where we used (2.8). Using (2.17) shows that the right–hand side of (2.15) is equal to the \( \eta_{a_2,1} \) component of \( \Omega^\Psi_1 \). A similar use of relations (Φ), (\( A_2 \subset G_2 \)) and (\( A_1 \times A_1 \subset G_2 \)) shows that the \( \eta_a \wedge \eta_b \) component of \( \Omega^\Psi_1 \) is zero and therefore completes the proof.

2.18.

**Corollary.** Assume that the relations (11) and (17) hold. Then, for any rank 2 root subsystem \( \Psi \subset \Phi \), the curvature term \( \Omega^\Psi_1 \) is equal to
\[
\Omega^\Psi_1 = - \sum_{\alpha \in \Psi_+} \sum_{\beta \in \Psi_+ \cap w_\alpha \Psi_-} \eta_\alpha \wedge d\beta [t_\alpha, t_\beta]
\]
where \( w_\alpha \) is any element of the Weyl group of \( \Psi \) such that \( w_\alpha^{-1} \alpha \) is simple in \( \Psi \).

**Proof.** Lemma 2.17 and a simple case–by–case inspection show that \( \Omega^\Psi_1 \) does indeed have the above form for well–chosen elements \( w_\alpha \) (specifically, \( w_\alpha \) should be an element of shortest length such that \( w_\alpha^{-1} \alpha \) is simple in \( \Psi \)). But Lemma 2.16 applied to \( \Psi \) implies that the expression
\[
\sum_{\beta \in \Psi_+ \cap w_\alpha \Psi_-} \eta_\alpha \wedge d\beta [t_\alpha, t_\beta]
\]
is independent of the choice of \( w_\alpha \), hence the conclusion.

2.19. **Completion of the proof of (1) of Theorem 2.5.** Fix \( \alpha \in \Phi_+ \) and let \( \mathcal{R}_2(\alpha) \) be the set of complete, rank 2 subsystems \( \Psi \subset \Phi \) containing \( \alpha \) as a non–simple root. For \( \Psi \in \mathcal{R}_2(\alpha) \), denote the corresponding Weyl group by \( W(\Psi) \).

For any \( w \in W \), denote by \( N(w) \subset \Phi_+ \) the set
\[
N(w) = \{ \beta \in \Phi_+ | w_\beta \in \Phi_- \}
\]
Proposition. Let \( w \in W \) be such that \( w^{-1} \alpha \) is simple in \( \Phi \).

1. For any \( \Psi \in \mathcal{R}_2(\alpha) \), the intersection \( N(w^{-1}) \cap \Psi \) is non-empty.

2. The following holds

\[
N(w^{-1}) = \bigcup_{\Psi \in \mathcal{R}_2(\alpha)} N(w^{-1}) \cap \Psi
\]

3. For any \( \Psi \in \mathcal{R}_2(\alpha) \), there exists a unique \( w_\Psi \in W(\Psi) \) such that

\[
N(w^{-1}) \cap \Psi = N_\Psi(w_\Psi^{-1})
\]

Proof. For any pair of non-proportional roots \( \beta, \gamma \in \Phi \), let \( \langle \beta, \gamma \rangle \subset \Phi \) be the complete, rank 2 subsystem generated by \( \beta \) and \( \gamma \). We claim that the map \( \beta \to \langle \alpha, \beta \rangle \) induces a bijection

\[
\rho : N(w^{-1})/\sim \longrightarrow \mathcal{R}_2(\alpha)
\]

where \( \sim \) is the equivalence relation defined by \( \beta \sim \beta' \) if \( \langle \alpha, \beta \rangle = \langle \alpha, \beta' \rangle \).

This clearly proves (1) and (2) since \( \rho^{-1}(\Psi) = \Psi \cap N(w^{-1}) \).

To see this, we shall need some notation. For any complete subsystem \( \Psi \subset \Phi \), let \( \Psi^\perp = \bigcap_{\beta \in \Psi} \mathcal{H}_\beta \subset E \), where \( \mathcal{H}_\beta = \text{Ker}(\beta) \). Set \( E_\Psi = E/\Psi^\perp \) so that \( \Psi \) may be regarded as a root system in \( E_\Psi \), and let \( \pi_\Psi : E \to E_\Psi \) be the corresponding projection. If \( \beta \in N(w^{-1}) \), the wall \( \mathcal{H}_\beta \) separates the fundamental chamber \( C \subset E \) of \( \Phi \) and \( C' = w(C) \). Thus, if \( \Psi = \langle \alpha, \beta \rangle \), \( \pi_\Psi(\mathcal{H}_\beta) \) separates the fundamental chamber \( \pi_\Psi(C) \) of \( \Psi \) and \( \pi_\Psi(C') \). It follows that \( \pi_\Psi(C) \neq \pi_\Psi(C') \) so that \( \alpha \) is not simple in \( \Psi \) since \( \pi_\Psi(\mathcal{H}_\alpha) \) is a wall of \( \pi_\Psi(C') \), whence \( \langle \alpha, \beta \rangle \in \mathcal{R}_2(\alpha) \) and \( \rho \) is a well-defined embedding.

It is also surjective since if \( \Psi \in \mathcal{R}_2(\alpha) \), there exists a \( \beta \in \Psi^\perp \) such that \( \pi_\Psi(\mathcal{H}_\beta) \) separates \( \pi_\Psi(C) \) and \( \pi_\Psi(C') \) so that \( \mathcal{H}_\beta \) separates \( C \) and \( C' \) and therefore lies in \( N(w^{-1}) \).

Finally, for a given \( \Psi \in \mathcal{R}_2(\alpha) \), the set \( N(w^{-1}) \cap \Psi \) consists of those \( \beta \in \Psi^\perp \) which separate \( C \) and \( C' \) and therefore \( \pi_\Psi(C) \) and \( \pi_\Psi(C') \). It is therefore equal to \( \Psi^\perp \cap w_\Psi \Psi^{-} \) where \( w_\Psi \in W(\Psi) \) is the unique element such that \( \pi_\Psi(C') = w_\Psi \pi_\Psi(C) \).

\[\blacksquare\]

2.20. We now turn to part (2) of Theorem 2.15. We shall need the following

Lemma. The relations (11) imply that the following holds for any \( \alpha \in \Phi^+ \), \( w \in W \) such that \( w^{-1} \alpha \) is simple and \( v \in \mathfrak{h} \) such that \( \alpha(v) = 0 \)

\[
[t_\alpha, \sum_{\beta \in \Phi^+} \text{sign}(w^{-1} \beta) \beta(v) \xi_\beta] = 0 \quad (t^w t)
\]

where \( \text{sign}(\gamma) = \pm 1 \) depending on whether \( \gamma \in \pm \Phi^+ \).
Proof. Let $\mathcal{T}$ be the algebra generated by symbols $t_\alpha$, $\alpha \in \Phi$ subject to the relations $t_{-\alpha} = t_\alpha$ and (II). The Weyl group acts on $\mathcal{T}$ by $w t_\alpha = t_{w \alpha}$ and it is easy to check that the above relation holds for a triple $(\alpha, w, v)$ if, and only if, it holds for $(w^{-1} \alpha, 1, w^{-1} v)$. We may therefore assume that $\alpha$ is simple and that $w = 1$. Since the left–hand side of (II) may be written as

$$\sum_{\Psi} \sum_{\beta \in \Psi_+} [t_\alpha, \beta(v)t_\beta]$$

where $\Psi$ ranges over the complete, rank 2 subsystems of $\Phi$ containing $\alpha$, it is sufficient to prove (II) when $\Phi$ is of rank 2. In this case, it follows by a simple case–by–case verification. For example, if $\Phi_+ + \{\alpha_1, \alpha_2, \alpha_1 + 2\alpha_2, \alpha_1 + 3\alpha_2, 2\alpha_1 + 3\alpha_2\}$ is of type $G_2$, we have

$$[t_{\alpha_2}, \sum_{\beta \in \Phi_+} \beta(\lambda_\alpha) t_\beta] = [t_{\alpha_2}, \sum_{\beta \in \Phi_+} t_\beta] + [t_{\alpha_2}, t_{2 \alpha_1 + 3 \alpha_2}] = 0$$

by (III) and $(A_1 \times A_1 \subset G_2)$, while

$$[t_{\alpha_1}, \sum_{\beta \in \Phi_+} \beta(\lambda_\alpha) t_\beta] = [t_{\alpha_1}, \sum_{\beta \in \Phi_+} t_\beta] + [t_{\alpha_1}, t_{\alpha_1 + 2 \alpha_2}] + 2[t_{\alpha_1}, t_{2 \alpha_1 + 3 \alpha_2} + t_{\alpha_1 + 3 \alpha_2}]$$

which is equal to zero by (III), $(A_1 \times A_1 \subset G_2)$ and $(A_2 \subset G_2)$. 

2.21. Recall from §2.3 that $\delta : \mathfrak{h} \to A$ is defined by

$$\delta(v) = \tau(v) - \frac{1}{2} \sum_{\alpha \in \Phi_+} \alpha(v)t_\alpha$$

The following proves part (2) of Theorem 2.5

Proposition. Modulo the relations (II), the relations (II) are equivalent to

$$[t_\alpha, \delta(v)] = 0$$

(tδ)

for any $\alpha \in \Phi$ and $v \in \mathfrak{h}$ such that $\alpha(v) = 0$.

Proof. For any $w \in W$ (2.5), yields

$$\tau_w(v) = \tau(v) - \sum_{\alpha \in \Phi_+ \cap w \Phi_+} \alpha(v)t_\alpha = \delta(v) + \frac{1}{2} \sum_{\beta \in \Phi_+} \text{sign}(w^{-1} \beta)\beta(v)t_\beta(v)$$

The result now follows from Lemma 2.20.

2.22. Equivariance under $W$. Assume now that the algebra $A$ is acted upon by the Weyl group $W$ of $\Phi$.

Proposition.

(1) The connection $\nabla$ is $W$–equivariant if, and only if

$$s_i(t_\alpha) = t_{s_i \alpha}$$

$$s_i(\tau(x)) - \tau(s_i x) = (\alpha_i, x)t_{\alpha_i}$$

for any $\alpha \in \Phi$, simple reflection $s_i \in W$ and $x \in \mathfrak{h}$.
(2) Modulo \((2.18)\), the relation \((2.19)\) is equivalent to the \(W\)--equivariance of the linear map \(\delta : \mathfrak{h} \to A\) defined by \((2.6)\).

**Proof.** (1) Since \(s_i\) permutes the set \(\Phi_+ \setminus \{\alpha_i\}\) and, by \((2.4)\)
\[
\frac{1}{1 - e^{-\alpha_i}} = \frac{1}{e^{\alpha_i} - 1} + 1
\]
we get
\[
s_i^* \nabla = d - \sum_{\alpha \in \Phi_+} \frac{d\alpha}{e^\alpha - 1} s_i(t_{\alpha}) - d\alpha_i s_i(t_{\alpha}) - s_i(\tau(\alpha))d(s_i u_j)
\]
Requiring that \(s_i^* \nabla = \nabla\) and taking residues along each subtorus \(\{e^{\alpha_i} = 1\}\) yields \((2.18)\). To proceed, note that
\[
s_i(\tau(x))d(s_i u_j) = s_i(\tau(s_i u_j))d(u_j)
\]
since \(\tau(x)d(u_j)\) is independent of the choice of the dual bases \(\{x_i\}, \{u_j\}\). Thus, \(s_i^* \nabla = \nabla\) reduces to
\[
\tau(x)d(u_j) = s_i(\tau(s_i u_j))d(u_j) + t_{\alpha_i} d\alpha_i
\]
which, upon being contracted along the tangent vector \(u^k\) yields \((2.19)\) with \(x = s_i w^i\).

(2) It is easy to check that the map \(\tau(x) = 1/2 \sum_{\alpha \in \Phi_+} (x, \alpha)t_{\alpha}\) satisfies \((2.19)\). The result now follows since any two maps \(\tau_i : \mathfrak{h} \to A\) satisfying \((2.19)\) differ by a \(W\)--equivariant map. 

**2.23. Flatness and equivariance.** The following is a direct corollary of Theorem \(2.5\) and Proposition \(2.22\).

**Theorem.** The trigonometric connection
\[
\nabla = d - \sum_{\alpha \in \Phi_+} \frac{d\alpha}{e^\alpha - 1} t_{\alpha} - d(u) \tau(x)
\]
is flat and \(W\)--equivariant if, and only if the following relations hold

- For any rank 2 root subsystem \(\Psi \subset \Phi\) and \(\alpha \in \Psi\),
  \[
  [t_{\alpha}, \sum_{\beta \in \Psi_+} t_{\beta}] = 0
  \]
- For any \(u, v \in \mathfrak{h}\),
  \[
  [\tau(u), \tau(v)] = 0
  \]
- For any simple root \(\alpha_i\) and \(u \in \text{Ker}(\alpha_i)\),
  \[
  [t_{\alpha_i}, \tau(u)] = 0
  \]
- For any \(\alpha \in \Phi\) and simple reflection \(s_i \in W\),
  \[
  s_i(t_{\alpha}) = t_{s_i \alpha}
  \]
- For any \(\alpha \in \Phi\) and \(u \in \mathfrak{h}\),
  \[
  s_i(\tau(u)) - \tau(s_i u) = (\alpha_i, u)t_{\alpha_i}
  \]
Remark. Theorem 2.23 was first proved by Cherednik in the special case when $t_0$ is equal to the orthogonal reflection $s_0 \in W$ and shown to lead to the definition of the degenerate affine Hecke algebra of $W$ [Ch1 Ch2].

3. The trigonometric Casimir connection

3.1. The Yangian $Y(\mathfrak{g})$ [Dr1]. Let $\mathfrak{g}$ be a finite–dimensional, simple Lie algebra over $\mathbb{C}$ and $(\cdot, \cdot)$ a non–degenerate, invariant bilinear form on $\mathfrak{g}$. Let $h$ be a formal variable. The Yangian $Y(\mathfrak{g})$ is the associative algebra over $\mathbb{C}[h]$ generated by elements $x, J(x), x \in \mathfrak{g}$ subject to the relations

$$\lambda x + \mu y \quad (\text{in } Y(\mathfrak{g})) = \lambda x + \mu y \quad (\text{in } \mathfrak{g})$$
$$xy - yx = [x, y]$$
$$J(\lambda x + \mu y) = \lambda J(x) + \mu J(y)$$
$$[x, J(y)] = J([x, y])$$
$$[J(x), J([y, z])] + [J(z), J([x, y])] + [J(y), J([z, x])] = h^2([x, [x, y]], [y, [x, z]], [z, x]]) \{x^a, x^b, x^c\}$$
$$[[J(x), J(y)], [z, J(w)]] + [[J(z), J(w)], [x, J(y)]] = h^2([x, x_a], [y, x_b], [z, w], [w, x_c]) \{x^a, x^b, J(x^c)\}$$

for any $x, y, z, w \in \mathfrak{g}$ and $\lambda, \mu \in \mathbb{C}$, where $\{x_a\}, \{x^a\}$ are dual bases of $\mathfrak{g}$ with respect to $(\cdot, \cdot)$ and

$$\{z_1, z_2, z_3\} = \frac{1}{24} \sum_{\sigma \in S_3} z_{\sigma(1)} z_{\sigma(2)} z_{\sigma(3)}$$

$Y(\mathfrak{g})$ is an $\mathbb{N}$–graded $\mathbb{C}[h]$–algebra provided one sets $\deg(x) = 0$, $\deg(J(x)) = 1$ and $\deg(h) = 1$.

3.2. Drinfeld’s new realisation of $Y(\mathfrak{g})$ [Dr2]. Let $\mathfrak{h} \subset \mathfrak{g}$ be a Cartan subalgebra of $\mathfrak{g}$, $\Phi = \{\alpha\} \subset \mathfrak{h}^*$ the corresponding root system. Let $\{\alpha_i\}_{i \in I}$ be a basis of simple roots of $\Phi$ and $a_{ij} = 2(\alpha_i, \alpha_j)/(\alpha_i, \alpha_i)$ the entries of the Cartan matrix $A$ of $\mathfrak{g}$. Set $d_i = (\alpha_i, \alpha_i)/2$, so that $d_i a_{ij} = d_j a_{ji}$ for any $i, j \in I$.

Let $\nu : \mathfrak{h} \to \mathfrak{h}^*$ be the isomorphism determined by the inner product $(\cdot, \cdot)$ and set $t_i = \nu^{-1}(\alpha_i) = d_i \alpha_i^\vee$. For any $i \in I$, choose root vectors $x_i^\pm \in \mathfrak{g}_{\pm \alpha_i}$ such that $[x_i^+, x_i^-] = t_i$. Recall that $\mathfrak{g}$ has a (slightly non–standard) presentation in terms of the generators $t_i, x_i^\pm$ with relations

$$[t_i, t_j] = 0$$
$$[t_i, x_j^\pm] = \pm d_i a_{ij} x_j^\pm$$
$$[x_i^+, x_j^-] = \delta_{ij} t_i$$
$$\text{ad}(x_i^\pm)^{1-a_{ij}} x_j^\pm = 0$$
The Yangian $Y(A)$ is the associative algebra over $\mathbb{C}[\hbar]$ with generators $X^\pm_{i,r}, T_{i,r}, i \in I, r \in \mathbb{N}$ and relations

$$[T_{i,r}, T_{j,s}] = 0,$$

$$[T_{i,0}, X^\pm_{j,s}] = \pm d_i a_{ij} X^\pm_{j,s},$$

$$[T_{i,r+1}, X^\pm_{j,s}] - [T_{i,r}, X^\pm_{j,s+1}] = \pm \hbar (d_i a_{ij} (T_{i,r} X^\pm_{j,s} + X^\pm_{j,s} T_{i,r})),$$

$$[X^+_i, X^-_j] = \delta_{ij} T_{i,r+s},$$

$$[X^\pm_{i,r+1}, X^\pm_{j,s}] - [X^\pm_{i,r}, X^\pm_{j,s+1}] = \pm \frac{\hbar}{2} d_i a_{ij} (X^\pm_{i,r} X^\pm_{j,s} + X^\pm_{j,s} X^\pm_{i,r}),$$

$$\sum_{\pi} [X^\pm_{i,r_{\pi(1)}}, [X^\pm_{i,r_{\pi(2)}}, \cdots, X^\pm_{i,r_{\pi(m)}}, X^\pm_{j,s}], \cdots] = 0$$

where $i \neq j$ in the last relation, $m = 1 - a_{ij}, r_1, \ldots, r_m \in \mathbb{N}$ is any sequence of non-negative integers, and the sum is over all permutations $\pi$ of $\{1, \ldots, m\}$. $Y(A)$ is $\mathbb{N}$–graded by $\text{deg}(T_{i,r}) = r = \text{deg}(X^\pm_{i,r})$ and $\text{deg}(\hbar) = 1$.

3.3. Isomorphism between the two presentations [Dr2]. Choose root vectors $x_\alpha \in \mathfrak{g}_\alpha$ for any $\alpha \in \Phi$ such that $(x_\alpha, x_{-\alpha}) = 1$ and let

$$\kappa_\alpha = x_\alpha x_{-\alpha} + x_{-\alpha} x_\alpha$$

be the truncated Casimir operator of the $\mathfrak{sl}_2$–subalgebra of $\mathfrak{g}$ corresponding to $\alpha$. Then, the assignment

$$\varphi(t_i) = T_{i,0}, \quad \varphi(x^\pm_{i,0}) = X^\pm_{i,0},$$

$$\varphi(J(t_i)) = T_{i,1} + \hbar \varphi(v_i)$$

$$\varphi(J(x^\pm_{i})) = X^\pm_{i} + \hbar \varphi(w^\pm_{i})$$

extends to an isomorphism $\varphi : Y(\mathfrak{g}) \to Y(A)$, where

$$v_i = \frac{1}{4} \sum_{\beta \in \Phi_+} (\alpha_i, \beta) \kappa_\beta - t_i^2 / 2$$

$$w^\pm_i = \pm \frac{1}{4} \sum_{\beta \in \Phi_+} ([x^\pm_i, x_{\pm\beta}] x_{\mp\beta} + x_{\mp\beta} [x^\mp_i, x_{\pm\beta}]) - \frac{1}{4} (x^\pm_i t_i + t_i x^\pm_i)$$

3.4. The $W$–equivariant embedding $\delta_{a,b} : \mathfrak{h} \to Y(\mathfrak{g})$. The Yangian $Y(\mathfrak{g})$ is acted upon by the Lie algebra spanned by the elements $x \in \mathfrak{g}$ and is an integrable $\mathfrak{g}$–module under this action. In particular, the zero–weight subalgebra $Y(\mathfrak{g})^0$ is acted upon by the Weyl group $W$ of $\mathfrak{g}$. Moreover, for any $a, b \in \mathbb{C}$, the linear map

$$\delta_{a,b} : \mathfrak{h} \to Y(\mathfrak{g})^0, \quad \delta_{a,b}(t) = at + bJ(t)$$

is $W$–equivariant.
In terms of the new realisation of $Y'(g)$, the map $\delta_{a,b}$ becomes

$$\tilde{\delta}_{a,b} = \varphi \circ \delta_{a,b} : h \rightarrow Y(A)^h$$

$$\tilde{\delta}_{a,b}(t) = at + b \left( T(t) + \frac{h}{4} \sum_{\beta \in \Phi_+} \beta(t) \kappa_{\beta} - \frac{h}{2} \sum_i (t_i t'_i) t_i^2 \right) \quad (3.3)$$

where $\{t^i = \lambda^*_i\}$ is the basis of $h$ dual to $\{t_i\}$ given by the fundamental coweights, $T(-)_1 : h \rightarrow Y(A)$ is the embedding $t \rightarrow (t, t^i) T_{i,1}$ and, deviating slightly from the notation of §3.2, we identify $g \subset Y'(g)$ with the Lie subalgebra of $Y(A)$ spanned by $T_i, X_{\pm i,0}$.

3.5. The linear map $\tau_{a,b} : h \rightarrow Y(g)$. Let $\epsilon \in \mathbb{C}$. For any root $\alpha$, set

$$K_{\alpha} = \kappa_{\alpha} + \epsilon q_{\alpha} \quad \text{where} \quad q_{\alpha} = \frac{\nu^{-1}(\alpha)^2}{(\alpha, \alpha)} \quad (3.4)$$

and $\kappa_{\alpha}$ is the truncated Casimir given by (3.1). For any $a, b \in \mathbb{C}$, define a map $\tau_{a,b} : h \rightarrow Y(g)^h$ by

$$\tau_{a,b}(t) = \frac{h}{2} \sum_{\alpha \in \Phi_+} (t, \alpha) K_{\alpha} + \delta_{a,b}(t)$$

where $\delta_{a,b}$ is given by (3.2).

**Proposition.**

1. The elements $K_{\alpha}$ satisfy

$$K_{-\alpha} = K_{\alpha} \quad \text{and} \quad w(K_{\alpha}) = K_{w\alpha}$$

for any $w \in W$.

2. The following holds for any $t \in h$

$$s_i(\tau_{a,b}(t)) - \tau_{a,b}(s_i t) = h(\alpha_i, t) K_{\alpha_i}$$

3. If $b = -2$, then $\tilde{\tau}_{a,-2} = \varphi \circ \tau_{a,-2} : h \rightarrow Y(A)^h$ is given by

$$\tilde{\tau}_{a,-2}(t) = at - 2 \left( T(t) - \frac{h}{2} \sum_i (t_i t'_i) t_i^2 \right) + \epsilon \frac{h}{2} \sum_{\alpha \in \Phi_+} \alpha(t) q_{\alpha}$$

and satisfies in addition

$$[\tilde{\tau}_{a,-2}(t), \tilde{\tau}_{a,-2}(t')] = 0$$

for any $t, t' \in h$.

4. For any $\alpha \in \Phi_+$ and $t \in h$ such that $\alpha(t) = 0$

$$[K_{\alpha}, \delta_{a,b}(t)] = 0$$
Proof. (1) is obvious. (2) follows from the second part of Proposition 2.22 and the $W$-equivariance of $\delta_{a,b}$. For (3), we have by (3.3)

$$\tilde{\tau}_{a,b}(t) = \frac{\hbar}{2} \sum_{\alpha \in \Phi^+} \alpha(t)K_\alpha + at + b \left( T(t)_1 + \frac{\hbar}{4} \sum_{\beta \in \Phi^+} \beta(t)\kappa_\beta - \frac{\hbar}{2} \sum_i (t, t^i) t^i_1 \right)$$

$$= at + \frac{\hbar}{2} \sum_{\alpha \in \Phi^+} \alpha(t)(K_\alpha + \frac{b}{2}\kappa_\alpha) + b(T(t)_1 - \frac{\hbar}{2} \sum_i (t, t^i) t^i_1)$$

which, for $b = -2$ reduces to the claimed expression. The commutativity of $\tilde{\tau}_{a,-2}(t)$ and $\tilde{\tau}_{a,2}(t')$ then follows from that of the $T_i, 1$. (4) follows easily from the defining relations of $Y(g)$. □

3.6. For simplicity, we henceforth set $\epsilon = 0 = a$ and $b = -2$ in equations (3.2) and (3.4), although Theorem 3.8 below is true for any values of $a, \epsilon$. Thus, $K_\alpha = \kappa_\alpha$, $\delta = \delta_{0,-2} : \mathfrak{h} \to Y(g)$ is given by $\delta(t) = -2J(t)$ and the corresponding map $\tilde{\tau} = \tilde{\tau}_{0,-2} : \mathfrak{h} \to Y(A)^b$ by

$$\tilde{\tau}(t) = -2T(t)_1 + h(t, t^i) t^i_1$$

3.7. The trigonometric Casimir connection of $\mathfrak{g}$. Let $H_{reg} \subset H$ be given by (2.1) and $Y^b$ the trivial bundle over $H_{reg}$ with fibre $Y(g)^b \cong Y(A)^b$.

Definition. The trigonometric Casimir connection of $\mathfrak{g}$ is the connection $\nabla$ on $Y^b$ given by either of the following forms

$$\nabla = d - \frac{\hbar}{2} \sum_{\alpha \in \Phi^+} e^\alpha + 1 e^{-\alpha} - 1 d\alpha \kappa_\alpha + 2du_i J(u^i)$$  \hspace{1cm} (3.5)

$$= d - h \sum_{\alpha \in \Phi^+} \frac{d\alpha}{e^\alpha - 1} \kappa_\alpha + 2du_i \left( T(u^i)_1 - \frac{\hbar}{2}(u^i, t^j) t^j_1 \right)$$  \hspace{1cm} (3.6)

whose equality follows from (2.3) and (3.6).

3.8.

Theorem. The trigonometric Casimir connection is flat and $W$-equivariant.

Proof. By Theorem 2.5, Proposition 2.22 and Proposition 3.5 we need only check that the elements $t_\alpha = \kappa_\alpha$ satisfy the relations (11). This follows as in [MTL, Thm. 2.3] and [TL1, Thm 2.2]. Specifically, if $\Psi \subset \Phi$ is a rank 2 root subsystem, the sum $\sum_{\alpha \in \Phi, \kappa_\alpha}$ is, up to Cartan terms, the Casimir operator for the rank 2 subalgebra $\mathfrak{g}_\Psi \subset \mathfrak{g}$ determined by $\Psi$ and therefore commutes with each summand $\kappa_\alpha, \alpha \in \Phi^+$. □
Remark. Since the relations of Theorem 2.5 and Proposition 2.22 are homogeneous, Theorem 3.8 proves in fact the flatness and $W$–equivariance of the one–parameter family of connections

$$\nabla = d - \lambda^{-1} \left( \frac{\hbar}{2} \sum_{\alpha \in \Phi_+} \frac{e^\alpha + 1}{e^\alpha - 1} d\alpha \kappa_\alpha - 2d u_i J(u^i) \right)$$

$$= d - \lambda^{-1} \left( \hbar \sum_{\alpha \in \Phi_+} \frac{d\alpha}{e^\alpha - 1} \kappa_\alpha - 2d u_i \left( T(u^i) - \frac{\hbar}{2} (u^i, t^j) \right) \right)$$

where $\lambda$ varies in $\mathbb{C}^\times$.

4. THE MONODROMY CONJECTURE

We show in this section that the monodromy of the trigonometric Casimir connection $\nabla$ gives rise to representations of the affine braid group $\hat{B}$ corresponding to $g$ and give a conjectural description of it in terms of the quantum Weyl group operators of the quantum loop algebra $U_\hbar(Lg)$.

4.1. Monodromy representation. Since $H$ is of simply–connected type, the Weyl group $W$ acts freely on $H$ and the fundamental group of the quotient $H/W$ is isomorphic to the affine braid group $\hat{B}$ [NVD, vdL].

Let $V$ be a finite–dimensional $Y(g)$–module and $\nabla$ the holomorphically trivial vector bundle over $H$ with fibre $V$. The connection $\nabla$ induces a flat connection on $V$. To push it down to the quotient by $W$ we use the ‘up and down’ trick of [MTL, p. 224] to circumvent the fact that $W$ does not in general act on $V$. To this end, we shall need a few basic results about Tits extensions of (affine) Weyl groups which are gathered in §8.

Specifically, since $V$ is an integrable $g$–module, the triple exponentials defined by a choice of simple root vectors $e_{\alpha_i}, f_{\alpha_i} \in g_{\alpha_i}$, $f_{\alpha_i} \in g_{-\alpha_i}$, are well–defined elements of $GL(V)$. They give rise to an action on $V$ of an extension of $W$ by the sign group $\mathbb{Z}_2^{\dim h}$ called the Tits extension $\hat{W}$ of $W$ (Definition 8.2 and Prop. 8.3). By Theorem 8.10, $\hat{W}$ is a quotient of the affine braid group $\hat{B}$ which may therefore be made to act on $V$. It is then easy to check that the pull–back of the flat vector bundle $(\nabla, V)$ to the universal cover of $H$ is equivariant under $\hat{B}$ acting by deck transformations on the base and through the $\hat{W}$–action on the fibres.

4.2. Let $Lg = g[t, t^{-1}]$ be the loop algebra of $g$ and $U_\hbar(Lg)$ the corresponding quantum loop algebra, viewed as a topological Hopf algebra over the ring of formal power series $\mathbb{C}[[h]]$. Thus, $U_\hbar(Lg)$ has Chevalley generators $E_i, F_i$, where $i$ ranges over the set $\hat{I} = \hat{I} \sqcup \{0\}$ of nodes of the affine Dynkin diagram of $g$ and a Cartan subalgebra isomorphic to $h$ and spanned by $H_i$, $H_0$. 

\[ \text{THE TRIGONOMETRIC CONNECTION OF A SIMPLE LIE ALGEBRA} \]
\( i \in I \) and \( H_0 = -H_\theta = -\sum_{i \in I} a_i H_i \), where \( \theta \in \mathfrak{h}^* \) is the highest root and the integers \( a_i \) are given by \( \theta^\vee = \sum a_i \alpha_i^\vee \).

### 4.3. Finite-dimensional representation

By a finite-dimensional representation of \( U_\hbar(Lg) \) we shall mean a module \( V \) which is topologically free and finitely-generated over \( \mathbb{C}[\hbar] \). Such a \( V \) is integrable and therefore endowed with a quantum Weyl group action of the affine braid group \( \hat{B} \). This action is given by letting the generator corresponding to \( i \in \hat{I} \) act by

\[
\mathcal{S}_i^\hbar v = \sum_{a,b,c \in \mathbb{Z} : a - b + c = -\lambda(\alpha_i^\vee)} (-1)^b q_i^{b-ac} E_i^{(a)} F_i^{(b)} E_i^{(c)} v
\]

where \( v \in V \) if of weight \( \lambda \in \mathfrak{h}^* \) and \( X_i^{(a)} \) is the divided power \( X^a/[a]! \) with

\[
q = e^\hbar, \quad q_i = q^{(\alpha_i,\alpha_i)/2}
\]

\[
[n]_i = \frac{q_i^n - q_i^{-n}}{q_i - q_i^{-1}} \quad \text{and} \quad [n]_i! = [n]_i[n - 1]_i \cdots [1]_i
\]

### 4.4. Monodromy conjecture

It is known that the Yangian \( Y(g) \) and the quantum loop algebra \( U_\hbar(Lg) \) have the same finite-dimensional representation theory (see [Va] and [GTL1]). By analogy with the quantum Weyl group description of the monodromy of the (rational) Casimir connection of \( g \) conjectured by De Concini (unpublished) and independently in [TL1, TL2], and proved in [TL1, TL3], we make the following

**Conjecture.** The monodromy of the trigonometric Casimir connection is equivalent to the quantum Weyl group action of the affine braid group \( \hat{B} \) on finite-dimensional \( U_\hbar(Lg) \)-modules.

We will return to this conjecture in forthcoming work in collaboration with S. Gautam [GTL2].

### 5. The trigonometric Casimir connection of \( \mathfrak{gl}_n \)

We consider in this section the Yangian \( Y(\mathfrak{gl}_n) \) of the Lie algebra \( \mathfrak{gl}_n \). The latter does not possess a presentation of the form given in [§3.1] but may be defined via a ternary, or \( RTT \) presentation. By exploiting the interplay between the latter and its loop presentation, we construct a flat, trigonometric connection with values in \( Y(\mathfrak{gl}_n) \). We then relate it to the corresponding connection for \( \mathfrak{sl}_n \) and show that, when it is taken with values in a tensor product of evaluation modules, it coincides with the trigonometric dynamical equations [TV].
5.1. **The RTT presentation of** \( Y(\mathfrak{gl}_n) \). The Yangian \( Y(\mathfrak{gl}_n) \) is the unital, associative algebra over \( \mathbb{C} \) generated by elements \( t_{ij}^{(r)} \), \( 1 \leq i, j \leq n \), \( r \geq 0 \), subject to the relations\(^3\)

\[\begin{split}
[&t_{ij}^{(r+1)}, t_{kl}^{(s)}] - [t_{ij}^{(r)}, t_{kl}^{(s+1)}] = t_{kj}^{(r)} t_{il}^{(s)} - t_{kj}^{(s)} t_{il}^{(r)} \\
&\quad \text{where } r, s \in \mathbb{N} \text{ and } t_{ii}^{(0)} = \delta_{ij}. 
\end{split}\] (5.1)

where \( E, t \) are again elementary matrices and the superscript \( V \) is used to stress the fact that they should be thought of as elements of the algebra \( \text{End}(V) \) rather than the underlying Lie algebra \( \mathfrak{gl}_n \).

Let \( V = \mathbb{C}^n \) with standard basis \( e_1, \ldots, e_n \) and let \( E_{ij} e_k = \delta_{jk} e_i \) be the corresponding basis of elementary matrices of \( \mathfrak{gl}_n \). The map \( i : E_{ij} \to t_{ij}^{(1)} \) defines an embedding of \( \mathfrak{gl}_n \) into \( Y(\mathfrak{gl}_n) \) and we will often identify \( \mathfrak{gl}_n \) with its image under \( i \). Moreover, for every \( s \geq 1 \), the subspace spanned by the elements \( t_{ij}^{(s)} \) transforms like the adjoint representation under the commutator action of \( \mathfrak{gl}_n \).

The above relations may be more compactly rewritten as follows. For any \( r \geq 0 \), let \( T^{(r)} \) be the \( n \times n \) matrix with values in \( Y(\mathfrak{gl}_n) \) given by

\[ T^{(r)} = \sum_{1 \leq i, j \leq n} E_{ij}^V \otimes t_{ij}^{(r)} \]

where \( E_{ij}^V \) are again elementary matrices and the superscript \( V \) is used to stress the fact that they should be thought of as elements of the algebra \( \text{End}(V) \) rather than the underlying Lie algebra \( \mathfrak{gl}_n \).

Let \( u \) be a formal variable and set

\[ T = \sum_{r \geq 0} T^{(r)} u^{-r} \in \text{End}(V) \otimes Y(\mathfrak{gl}_n)[[u^{-1}]] \]

Finally, let

\[ R(u) = 1 - Pu^{-1} \in \text{End}(V \otimes V)[[u^{-1}]] \]

be Yang’s \( R \)-matrix, where \( P \in \text{End}(V \otimes V) \) acts as the permutation of the two tensor factors. Then the relations (5.1) are equivalent to

\[ R(u - v) T_1(u) T_2(v) = T_2(v) T_1(u) R(u - v) \]

where \( T_1(u), T_2(u) \in \text{End}(V \otimes V) \otimes Y(\mathfrak{gl}_n)[[u^{-1}]] \) are given by

\[ T_1(u) = \sum_{i,j,r} E_{ij}^V \otimes 1 \otimes t_{ij}^{(r)} u^{-r} \quad \text{and} \quad T_2(v) = \sum_{i,j,r} 1 \otimes E_{ij}^V \otimes t_{ij}^{(r)} v^{-r} \]

5.2. **The loop presentation of** \( Y(\mathfrak{gl}_n) \). Let \( E(u), H(u), F(u) \) be the factors of the Gauss decomposition of \( T(u) \). Specifically,

\[ \begin{align*}
F(u) &= 1 + \sum_{i > j} E_{ij}^V \otimes f_{ij}(u) \\
E(u) &= 1 + \sum_{i < j} E_{ij}^V \otimes e_{ij}(u) \\
H(u) &= \sum_i E_{ii}^V \otimes h_i(u),
\end{align*} \]

\(^3\)we follow here the conventions of [Mo] and [NO].
are, respectively, the unique lower unipotent, upper unipotent and diagonal matrices with coefficients in $\text{Y}(\mathfrak{gl}_n)[[u^{-1}]]$ such that

$$T(u) = F(u)H(u)E(u) \quad (5.2)$$

Noting that $H(u), E(u), F(u) = 1 \mod u^{-1}$, write

$$h_i(u) = 1 + \sum_{r \geq 1} h_i^{(r)} u^{-r}, \quad f_{ij}(u) = \sum_{r \geq 1} f_{ij}^{(r)} u^{-r}, \quad e_{ij}(u) = \sum_{r \geq 1} e_{ij}^{(r)} u^{-r}$$

The coefficients of $e_{ii+1}(u), f_{ii+1}(u)$ and $h_i(u)$ give another system of generators of $\text{Y}(\mathfrak{gl}_n)$. Moreover, The elements $h_i(u)$ commute and their coefficients generate a maximal commutative subalgebra of $\text{Y}(\mathfrak{gl}_n)$ called the Gelfand–Zetlin subalgebra $H_n$.

5.3. The Gauss decomposition [5.2] yields in particular

$$t_{ij}^{(1)} = \begin{cases} e_{ij}^{(1)} & \text{if } i < j \\ h_i^{(1)} & \text{if } i = j \\ f_{ij}^{(1)} & \text{if } i > j \end{cases}$$

which we will use to identify the copies of $\mathfrak{gl}_n$ inside each presentation. Moreover,

$$t_{ii}^{(2)} = h_i^{(2)} + \sum_{j < i} E_{ij}E_{ji} = h_i^{(2)} + \frac{1}{2} \sum_{j < i} (\kappa_{\theta_j - \theta_i} - (E_{jj} - E_{ii})) \quad (5.3)$$

where $\theta_a$ is the linear form given by $\theta_a(E_{bb}) = \delta_{ab}$ and $\kappa_{\theta_a - \theta_b} = E_{ab}E_{ba} + E_{ba}E_{ab}$ is the truncated Casimir operator corresponding to the root $\theta_a - \theta_b$.

5.4. Define the elements $D_i \in H_n$ by

$$D_i = 2h_i^{(2)} - \sum_{j < i} (E_{jj} - E_{ii}) - E_{ii}^{2} = 2t_{ii}^{(2)} - \sum_{j < i} \kappa_{\theta_j - \theta_i} - E_{ii}^{2} \quad (5.4)$$

The symmetric group $S_n$ acts by algebra automorphisms on $\text{Y}(\mathfrak{gl}_n)$ by

$$\sigma(t_{ij}^{(r)}) = t_{\sigma(i)\sigma(j)}^{(r)} \quad (5.6)$$

**Lemma.** The following holds

1. $[D_i, D_j] = 0$.
2. $(i+1)D_j = D_j$ if $j \notin \{i, i+1\}$.
3. $(i+1)D_i - D_{i+1} = \kappa_{\theta_i - \theta_{i+1}}$.

**Proof.** (1) follows from (5.4) and the fact that the $h_i^{(r)}$ commute. (2) and (3) follows from (5.5).
5.5. The following is a direct consequence of (5.5) and (5.3)

**Lemma.** The element $D = D_1 + \cdots + D_n$ is given by

$$D = 2 \sum_i t^{(2)}_{ii} - C_{\mathfrak{gl}_n}$$

$$= 2 \sum_i h^{(2)}_{ii} - 2\rho^\vee - \sum_i E^2_{ii}$$

where

$$C_{\mathfrak{gl}_n} = \sum_{i<j} \kappa_{\theta_i - \theta_j} + \sum_i E^2_{ii}$$

and

$$2\rho^\vee = \sum_{i<j} (E_{ii} - E_{jj})$$

are the Casimir operator and sum of the positive coroots of $\mathfrak{gl}_n$.

5.6. **The trigonometric Casimir connection of $\mathfrak{gl}_n$.** Let $D_i$ be given by (5.5) and define elements $\Delta_i \in H_n$ by (cf. §2.3)

$$\Delta_i = D_i - \frac{1}{2} \sum_{a<b} (\theta_a - \theta_b)(E_{ii}) \kappa_{\theta_a - \theta_b}$$

$$= 2t^{(2)}_{ii} - \frac{1}{2} \sum_{j \neq i} \kappa_{\theta_i - \theta_j} - E^2_{ii}$$

Let $H \subset \text{GL}_n$ be the maximal torus consisting of diagonal matrices, $H_{\text{reg}}$ its set of regular elements and $\mathcal{Y}(\mathfrak{gl}_n)$ the trivial $\mathcal{Y}(\mathfrak{gl}_n)$–bundle over $H_{\text{reg}}$.

**Definition.** The trigonometric Casimir connection of $\mathfrak{gl}_n$ is the connection on $\mathcal{Y}(\mathfrak{gl}_n)$ given by either of the following forms

$$\nabla = d - \sum_{i<j} \frac{d(\theta_i - \theta_j)}{e^{\theta_i - \theta_j} - 1} \kappa_{\theta_i - \theta_j} - \sum_{i=1}^n d\theta_i D_i$$

$$= d - \frac{1}{2} \sum_{i<j} \frac{e^{\theta_i - \theta_j} + 1}{e^{\theta_i - \theta_j} - 1} d(\theta_i - \theta_j) \kappa_{\theta_i - \theta_j} - \sum_{i=1}^n d\theta_i \Delta_i$$

5.7. Let the symmetric group $\mathfrak{S}_n$ act on the vector bundle $\mathcal{Y}(\mathfrak{gl}_n)$ by permutations of the base and automorphisms (5.6) of the fibre.

**Theorem.** The trigonometric Casimir connection of $\mathfrak{gl}_n$ is flat and equivariant under $\mathfrak{S}_n$.

**Proof.** Let $\overline{U}, U \subset H$ be the subtori consisting respectively of diagonal matrices of determinant 1 and multiples of the identity, and let $\overline{\mathfrak{h}}, \mathfrak{u} = \mathbb{C}\mathbf{1}_n$ and $\mathfrak{h}$ be their Lie algebras, where $\mathbf{1}_n = \sum_{i=1}^n E_{ii}$. Thus

$$\mathfrak{h} = \overline{\mathfrak{h}} \oplus \mathbb{C}\mathbf{1}_n$$

and

$$\mathfrak{h}^* \cong \overline{\mathfrak{h}}^* \oplus \mathbb{C} \text{ tr}$$

where $\text{tr} : \mathfrak{h} \to \mathbb{C}$ is the trace. Clearly, $H \cong \overline{U} \times U$ and the connection $\nabla$ decomposes as the product of the following $\mathcal{Y}(\mathfrak{gl}_n)$–valued connections on
Rational form.

5.8. follows easily from (5.5).

Connections on $GL_n$ type for the connection 5.6. ($\tau$)

\[ (\text{the latter commutes with the coefficients of } \tau) \]

3.8. The relations (\ref{eq:relations}) and the equivariance relations (2.19) follow from Lemma 5.5 on $\nabla_U$. Since the latter acts trivially on $U$ and, by Lemma 5.5 on $\nabla_D$, the flatness and equivariance of $\nabla_U$ reduces to that of $\nabla$, which, in turn is determined by Theorem 2.5 and Proposition 2.22. The relations (\ref{eq:relations}) have already been checked in the proof of the flatness of the trigonometric Casimir connection for $\mathfrak{sl}_n$ in Theorem 3.8. The relations (\ref{eq:relations}) and the equivariance relations (2.19) follow from Lemma 5.7. There remains to check that, for any $i = 1, \ldots, n - 1$ and $u \in \text{Ker}(\theta_i - \theta_{i+1})$, $[\kappa_{\theta_i - \theta_{i+1}}, D(u)] = 0$. This reduces to checking that $[\kappa_{\theta_i - \theta_{i+1}}, D_j] = 0$ for $j \notin \{i, i+1\}$ and that $[\kappa_{\theta_i - \theta_{i+1}}, D_i + D_{i+1}] = 0$ which follows easily from (5.5).

5.8. **Rational form.** It is well known that trigonometric connections of type $GL_n$ may be put into a rational form, thus expressing them as KZ type connections on $n + 1$ points, with one frozen to 0. We carry this step below for the connection 5.6.

Let $z_i = e^{\theta_i}$, $i = 1, \ldots, n$ be the standard coordinates on the torus $H \cong (\mathbb{C}^*)^n$. Since $d\theta_i = dz_i/z_i$ and

\[ d(\theta_i - \theta_j) = \frac{d(z_i - z_j)}{z_i - z_j} - \frac{dz_i}{z_i} \]

the connection 5.6 is equal to

\[ \nabla = d - \sum_{i<j} \frac{d(z_i - z_j)}{z_i - z_j} \kappa_{\theta_i - \theta_j} - \sum_i \frac{dz_i}{z_i} \bar{D}_i \]

where

\[ \bar{D}_i = D_i - \sum_{j>i} \kappa_{\theta_i - \theta_j} = 2t_{ii}^{(2)} - \sum_{j \neq i} \kappa_{\theta_i - \theta_j} - E_{ii}^2 \]

5.9. **Relation to $Y(\mathfrak{sl}_n)$.** Following Olshanski and Drinfeld, we realise the Yangian $Y(\mathfrak{sl}_n)$ as a Hopf subalgebra of $Y(\mathfrak{gl}_n)$ as follows (see [Mo §1.8]). Let $\Lambda = 1 + u^{-1}\mathbb{C}[u^{-1}]$ be the abelian group of formal power series in $u^{-1}$ with constant term 1. $\Lambda \ni f$ acts on $Y(\mathfrak{gl}_n)$ by Hopf algebra automorphisms given by

\[ T(u) \rightarrow f(u)T(u) \]

The Hopf subalgebra $Y(\mathfrak{gl}_n)^\Lambda \subset Y(\mathfrak{gl}_n)$ of elements fixed by $\Lambda$ is isomorphic to $Y(\mathfrak{sl}_n)$.

\[ \text{note that these differ from the coordinates } z_i = e^{\alpha_i} = e^{\theta_i - \theta_{i+1}} \text{ used in (2.12)}. \]
5.10. The generators $T_{ir}$ of the presentation of $Y(\mathfrak{sl}_n)$ described in §3.2 may be obtained within the RTT realisation of $Y(\mathfrak{gl}_n)$ as follows [BK, Rk. 5.12]. Consider their generating function $T_i(u) = 1 + \sum_{r \geq 0} T_{ir} u^{-r-1}$. Then,

$$T_i(u) = h_i(u - \frac{i-1}{2})^{-1} \cdot h_{i+1}(u - \frac{i-1}{2})$$

To spell this out, consider a formal power series $a(u) = 1 + a_1 u^{-1} + a_2 u^{-2} + \cdots$ Then, for any $\lambda \in \mathbb{C}$ one has

$$a(u - \lambda) = 1 + a_1 u^{-1}(1 - \frac{\lambda}{u})^{-1} + a_2 u^{-2}(1 - \frac{\lambda}{u})^{-2} + \cdots$$

$$= 1 + a_1 u^{-1} + (a_2 + \lambda a_1)u^{-2} + \cdots$$

and therefore

$$(a(u - \lambda))^{-1} = 1 - a_1 u^{-1} - (a_2 + \lambda a_1 - a_1^2)u^{-2} + \cdots$$

It follows that $T_{i,0} = -(E_{ii} - E_{i+1,i+1})$ and

$$T_{i,1} = -(h^{(2)}_i - h^{(2)}_{i+1}) - \frac{i-1}{2} (E_{ii} - E_{i+1,i+1}) + E_{ii}^2 - E_{ii} E_{i+1,i+1}$$

(5.9)

5.11. Let $\mathcal{H} \subset H$ be the torus of $SL_n$ consisting of diagonal matrices with determinant 1.

**Proposition.** The restriction of the trigonometric Casimir connection of $\mathfrak{gl}_n$ to $\mathcal{H}_{\text{reg}}$ takes values in $Y(\mathfrak{sl}_n)$ and is equal to the sum of the trigonometric Casimir connection of $\mathfrak{sl}_n$ with the $\mathcal{H} \subset Y(\mathfrak{sl}_n)$–valued, closed one–form

$$-\sum_{i=1}^n d\lambda_i (E_{ii} - E_{i+1,i+1})$$

where $\{\lambda_i\}$ are the fundamental weights of $\mathfrak{sl}_n$.

**Proof.** The restriction of the trigonometric Casimir connection of $\mathfrak{gl}_n$ to $\mathcal{H}_{\text{reg}}$ is given by (5.7), namely

$$\nabla = \partial - \sum_{i<j} \frac{d(\theta_i - \theta_j)}{\theta_i - \theta_j - 1} \kappa_{\theta_i - \theta_j} - du^i D(u^i)$$

where $D : \mathfrak{h} \to Y(\mathfrak{gl}_n)$ is given by $D(E_{ii}) = D_i$ and $\{u_i\}, \{u^i\}$ are dual bases of $\mathfrak{h}$ and $\mathfrak{h}$ respectively. Choosing $u_i = \lambda_i$ so that $u^i = E_{ii} - E_{i+1,i+1}$, $i = 1, \ldots, n - 1$ and comparing with the form (3.6) we need to show that

$$D_i - D_{i+1} = -2T_{i,1} + (E_{ii} - E_{i+1,i+1})^2 + (E_{ii} - E_{i+1,i+1})$$

By (3.4), the left–hand side is equal to

$$2(h^{(2)}_i - h^{(2)}_{i+1}) + i (E_{ii} - E_{i+1,i+1}) - E_{ii}^2 + E_{i+1,i+1}^2$$

and the result follows from (5.9).
5.12. **Evaluation homomorphism.** The Yangian $Y(gl_n)$ possesses an evaluation homomorphism $ev : Y(gl_n) \to Ugl_n$ defined by
\[
ev(t_{ij}(u)) = \delta_{ij} + E_{ij}u^{-1}
\]
where $t_{ij}(u) = \sum_{r \geq 0} t_{ij}^{(r)}u^{-r}$. When composed with the translation automorphisms $\tau_a$, $a \in \mathbb{C}$ given by $\tau_a T(u) = T(u-a)$, that is
\[
\tau_a T^{(r)} = \delta_{r0} + \sum_{s=1}^{r} T^{(s)} \left( \frac{r-1}{r-s} \right) a^{r-s}
\]
this yields a one–parameter family of evaluation homomorphisms $ev_a = ev \circ \tau_a$ given by
\[
ev_a(t_{ij}^{(r)}) = \delta_{r0}\delta_{ij} + E_{ij}a^{r-1}
\]
(5.10)

5.13. **Hopf algebra structure.** $Y(gl_n)$ is a Hopf algebra with coproduct
\[
\Delta(t_{ij}(u)) = \sum_{k=1}^{n} t_{ik}(u) \otimes t_{kj}(u)
\]
For any $m \geq 2$, let $\Delta^{(m)} : Y(gl_n) \to Y(gl_n)^{\otimes m}$ denote the corresponding iterated coproduct. Then,
\[
\Delta^{(m)}(t_{ij}^{(2)}) = \sum_{p=1}^{m} (t_{ij}^{(2)})_p + \sum_{1 \leq k \leq n \atop 1 \leq p < q \leq m} (E_{ik})_p (E_{kj})_q
\]
(5.11)
where $X_p = 1^{\otimes (p-1)} \otimes X \otimes 1^{\otimes (m-p)}$.

5.14. **Evaluation modules.** For any $\underline{a} = (a_1, \ldots, a_m) \in \mathbb{C}^m$ define $ev_{\underline{a}} : Y(gl_n) \to Ugl_n^{\otimes m}$ by
\[
ev_{\underline{a}} = ev_{a_1} \otimes \cdots \otimes ev_{a_m} \circ \Delta^{(m)}
\]

**Proposition.** The image of the trigonometric Casimir connection of $gl_n$ under the homomorphism $ev_{\underline{a}}$ is the $Ugl_n^{\otimes m}$–valued connection given by
\[
\nabla_{\underline{a}} = d - \sum_{i<j} \frac{d(\theta_i - \theta_j)}{e^{\theta_i - \theta_j} - 1} \Delta^{(m)}(\kappa_{\theta_i - \theta_j}) - \sum_{i=1}^{n} d\theta_i D_{i,\underline{a}}
\]
where
\[
D_{i,\underline{a}} = 2 \sum_{p=1}^{m} a_p (E_{ii})_p + 2 \sum_{1 \leq j \leq n \atop 1 \leq p < q \leq m} (E_{ij})_p (E_{ji})_q - \sum_{j<i} \Delta^{(m)}(\kappa_{\theta_j - \theta_i}) - \Delta^{(m)}(E_{ii}^2)
\]

**Proof.** By construction $D_{i,\underline{a}} = ev_{\underline{a}}(D_i)$ and is given by the above expression by [5.5], [5.11] and [5.10].
5.15. The trigonometric dynamical differential equations for $\mathfrak{gl}_n$. In [TV], Tarasov and Varchenko considered differential operators $D_1, \ldots, D_n$ in the variables $z_1, \ldots, z_n \in \mathbb{C}^n$ with coefficients in $U \mathfrak{g}_n \otimes^m$ given by

$$D_i = z_i \partial_{z_i} + \lambda L_i(a, z)$$

where $\lambda \in \mathbb{C}$, $a = (a_1, \ldots, a_m) \in \mathbb{C}^m$, $z = (z_1, \ldots, z_n)$ and

$$L_i(a, z) = \frac{\Delta^{(m)}(E_{ii})^2}{2} - \sum_{p=1}^{m} a_p(E_{ii})_p - \sum_{1 \leq j \leq n, \ 1 \leq p < q \leq m} \frac{z_j}{z_i - z_j} \Delta^{(m)}(E_{ij}E_{ji} - E_{ii})$$

Set $z_i = e^{\theta_i}$ so that $z_i \partial_{z_i} = \theta_i$ and

$$\frac{z_j}{z_i - z_j} = \frac{1}{e^{\theta_i} - \theta_j - 1} = -\left(\frac{1}{e^{\theta_j} - \theta_i - 1} + 1\right)$$

Since

$$E_{ij}E_{ji} - E_{ii} = \frac{1}{2}(\kappa_{\theta_i - \theta_j} - (E_{ii} + E_{jj}))$$

the operators $D_i$ are the covariant derivatives for the connection

$$\nabla'_a = d - \frac{\lambda}{2} \left( \sum_{i<j} \frac{d(\theta_i - \theta_j)}{e^{\theta_i} - \theta_j - 1} \Delta^{(m)}(\kappa_{\theta_i - \theta_j} - (E_{ii} + E_{jj})) + \sum_{i=1}^{n} d\theta_i D'_{i,a} \right)$$

where

$$D'_{i,a} = -\Delta^{(m)}(E_{ii})^2 + 2 \sum_{p=1}^{m} a_p(E_{ii})_p + 2 \sum_{1 \leq j \leq n, \ 1 \leq p < q \leq m} (E_{ij})_p(E_{ji})_q - \sum_{j<i} \Delta^{(m)}(\kappa_{\theta_j - \theta_i} - (E_{ii} + E_{jj}))$$

By Proposition 5.14, $\nabla'_a$ is the image of the trigonometric Casimir connection for $\mathfrak{gl}_n$ under the homomorphism $\text{ev}_a : Y(\mathfrak{gl}_n) \rightarrow (U \mathfrak{g}_n) \otimes^m$ plus the $\mathfrak{h}$-valued, closed one-form

$$\frac{\lambda}{2} \Delta^{(m)} \left( \sum_{i<j} \frac{d(\theta_i - \theta_j)}{e^{\theta_i} - \theta_j - 1} (E_{ii} + E_{jj}) - \sum_{i} d\theta_i \sum_{j<i} (E_{ii} + E_{jj}) \right)$$

<sup>5</sup> when the latter is scaled by a factor of $\lambda/2$, as in Remark [8].
6. Bispectrality

We show in this section that the trigonometric Casimir connection with values in a tensor product of \( Y(\mathfrak{g}) \)-modules commutes with the \( q \)-KZ difference equations of Frenkel–Reshetikhin determined by the rational \( R \)-matrix of \( Y(\mathfrak{g}) \). This was checked by Tarasov–Varchenko for \( \mathfrak{g} = \mathfrak{gl}_n \) in the case where all representations are evaluation modules \([TV]\).

6.1. Hopf algebra structure \([Dr1]\). If \( \mathfrak{g} \) is simple, \( Y(\mathfrak{g}) \) is a Hopf algebra with coproduct \( \Delta : Y(\mathfrak{g}) \to Y(\mathfrak{g}) \otimes Y(\mathfrak{g}) \) given on generators by

\[
\Delta(x) = x \otimes 1 + 1 \otimes x
\]

\[
\Delta(J(x)) = J(x) \otimes 1 + 1 \otimes J(x) + \frac{\hbar}{2}[x \otimes 1, t]
\]

where \( t = \sum_a x_a \otimes x^a \in (\mathfrak{g} \otimes \mathfrak{g})^\mathfrak{g} \), with \( \{x_a\}, \{x^a\} \) dual bases of \( \mathfrak{g} \) with respect to the given inner product. Thus, if \( \Delta^{(n)} : Y(\mathfrak{g}) \to Y(\mathfrak{g})^{\otimes n} \) is the iterated coproduct, then

\[
\Delta^{(n)}(x) = \sum_{i=1}^{n} x^{(i)}
\]

\[
\Delta^{(n)}(J(x)) = \sum_{i=1}^{n} J(x)^{(i)} + \frac{\hbar}{2} \sum_{1 \leq i < j \leq n} [x^{(i)}, t^{ij}]
\]

where \( x^{(i)} = 1 \otimes (i-1) \otimes x \otimes 1 \otimes (n-i) \) and \( t^{ij} = \sum_{a} x_{a}^{(i)} (x^a)^{(j)} \).

6.2. Translation automorphisms \([Dr1]\). \( Y(\mathfrak{g}) \) possesses a one–parameter group of Hopf algebra automorphisms \( T_v, v \in \mathbb{C} \) given by

\[
T_v x = x \quad \text{and} \quad T_v J(x) = J(x) + vx
\]

If \( v_1, \ldots, v_n \in \mathbb{C} \), we set \( T_{v_1} \cdots T_{v_n} = T_{v_1} \otimes \cdots \otimes T_{v_n} \in \text{Aut}(Y(\mathfrak{g})^{\otimes n}) \) and

\[
\Delta_{v_1, \ldots, v_n} = T_{v_1} \cdots T_{v_n} \circ \Delta^{(n)} : Y(\mathfrak{g}) \to Y(\mathfrak{g})^{\otimes n}
\]

6.3. The universal \( R \)-matrix of \( Y(\mathfrak{g}) \) \([Dr1]\). Let \( R(u) \in Y(\mathfrak{g}) \otimes Y(\mathfrak{g})[[u^{-1}]] \) be the universal \( R \)-matrix of \( Y(\mathfrak{g}) \). \( R(u) \) satisfies

\[
\Delta \otimes \text{id}(R(u)) = R^{13}(u)R^{23}(u)
\]

\[
\text{id} \otimes \Delta(R(u)) = R^{13}(u)R^{12}(u)
\]

\[
\Delta^{21}(x) = R(u)\Delta(x)R(u)^{-1}
\]

\[
T_{v,w} R(u) = R(u + v - w)
\]

The above relations imply that \( R \) satisfies the quantum Yang–Baxter equations (QYBE) with spectral parameter

\[
R^{12}(u)R^{13}(u + v)R^{23}(v) = R^{23}(v)R^{13}(u + v)R^{12}(u)
\]

and the more general form of (6.6)

\[
R(u)(T_{v,w}\Delta(x))R(u)^{-1} = T_{v,w} \Delta^{21}(x)
\]
6.4. The rational qKZ equations [FR]. Let $V_1, \ldots, V_n$ be $Y(g)$–modules and $d_j \in GL(V_j)$ be such that $d_i d_j R^{ij}(u) = R^{ij}(u) d_i d_j$ for any $1 \leq i < j \leq n$. Fix a step $\kappa \in \mathbb{C}^\times$, let $a_1, \ldots, a_n \in \mathbb{C}$ be distinct and define operators

$$A_i = A_i(a_1, \ldots, a_n) \in \text{End}(V_1 \otimes \cdots \otimes V_n)$$

for $i = 1, \ldots, n$ by

$$A_i = R^{i-1}(a_{i-1} - a_i - \kappa)^{-1} R^{i-2}(a_{i-2} - a_i - \kappa)^{-1} \cdots R^1(a_1 - a_i - \kappa)^{-1} \cdot d_i \cdot R^{n}(a_i - a_n) R^{n-1}(a_i - a_{n-1}) \cdots R^{i+1}(a_i - a_{i+1})$$

The (rational) qKZ equations of Frenkel–Reshetikhin are the system of difference equations $T_i f = A_i f$ where $f$ takes values in $V_1 \otimes \cdots \otimes V_n$ and

$$T_i f(a_1, \ldots, a_n) = f(a_1, \ldots, a_{i-1}, a_i + \kappa, a_{i+1}, \ldots, a_n)$$

They are a consistent system in that $[T_i, T_j] = 0$, where $T_i = A_i^{-1} T_i$ are the covariant difference operators.

6.5.

Lemma. The following holds for any $i = 1, \ldots, n$

$$\mathcal{T}_1 \mathcal{T}_2 \cdots \mathcal{T}_i = (\tilde{A}_i)^{-1} T_1 \cdots T_i$$

where $\tilde{A}_i = \tilde{A}_i(a_1, \ldots, a_n)$ is given by

$$\tilde{A}_i = d_1 \cdots d_i \left( R^n(a_1 - a_n) \cdots R^n(a_i - a_n) \right)$$

$$\left( R^{n-1}(a_1 - a_{n-1}) \cdots R^{n-1}(a_i - a_{n-1}) \right) \cdots$$

$$\left( R^{i+1}(a_1 - a_{i+1}) \cdots R^{i+1}(a_i - a_{i+1}) \right)$$

(6.8)

Thus, $\tilde{A}_n = d_1 \cdots d_n$ and for any $i \leq n - 1$

$$\tilde{A}_i = d_1 \cdots d_i \Delta^{(i)}_{a_{i-1}, \ldots, a_{i-1}, a_i, a_i} \Delta^{(n-i)}_{a_{i+1}, a_{i+1}, \ldots, a_{i+1}, a_n, a_n} (R(a_i - a_n))$$

(6.9)

Proof. Clearly, $\mathcal{T}_1 \mathcal{T}_2 \cdots \mathcal{T}_i = (T_i \cdots T_{i-1} A_i) \cdots T_1 (A_2) A_1)^{-1} T_1 \cdots T_i$

and, for any $j \leq i$,

$$T_1 \cdots T_{j-1} (A_j) = (R^{i-1})^{-1}(R^{i-2})^{-1} \cdots (R^1)^{-1} \cdot d_i \cdot R^{i+n} R^{i+n-1} \cdots R^{i+1}$$

where $R^{kl}$ is shorthand for $R^{kl}(a_k - a_l)$. The first claimed identity now follows by induction on $i$ using the QYBE. The second follows from the relations (6.3)–(6.5) and (6.7). ■
6.6. Bispectrality. To couple the \( qKZ \) and trigonometric Casimir connection equations with values in the tensor product \( V_1 \otimes \cdots \otimes V_n \), assume that each \( V_i \) is an integrable \( g \)-module and that \( d_i \) is the \( GL(V_i) \)-valued function on the torus \( H \) given by

\[
d_i(e^u) = (e^{-u})^{(i)}
\]

(6.10)

Let also \( \nabla_{\mathfrak{g}} \) be the trigonometric Casimir connection with values in the \( \mathcal{Y}(g) \)-module \( T^{* a_1, \ldots, a_n} V_1 \otimes \cdots \otimes V_n \) and scaled by a factor of \( 2\kappa \) as in Remark 3.8. Thus, \( \nabla_{\mathfrak{g}} \) is the \( \text{End}(V_1 \otimes \cdots \otimes V_n) \)-valued connection given by

\[
\nabla_{\mathfrak{g}} = d - \frac{1}{2\kappa} \Delta_{a_1, \ldots, a_n}^{(n)}(B)
\]

where

\[
B = \frac{\hbar}{2} \sum_{\alpha \in \Phi_+} \frac{e^\alpha + 1}{e^\alpha - 1} d\alpha - 2du_i J(u^i)
\]

Theorem. The \( qKZ \) operators \( T_i \) commute with the trigonometric Casimir connection \( \nabla_{\mathfrak{g}} \).

Proof. It suffices to prove that \( [\nabla_{\mathfrak{g}}, T_1 \cdots T_i] = 0 \) for any \( i = 1, \ldots, n \). Since

\[
[d - (2\kappa)^{-1} \Delta_{a_1, \ldots, a_n}^{(n)}(B), (\tilde{A}_i)^{-1} T_1 \cdots T_i] (T_1 \cdots T_i)^{-1}
\]

\[
= d\tilde{A}_i^{-1} - (2\kappa)^{-1} \tilde{A}_i^{-1}(\text{id} - \text{Ad}(T_1 \cdots T_i))\Delta_{a_1, \ldots, a_n}^{(n)}(B)
\]

\[
= (2\kappa)^{-1}[\text{Ad}(T_1 \cdots T_i) - \text{id}]\Delta_{a_1, \ldots, a_n}^{(n)}(B)
\]

the claim follows from the two lemmas below.

6.7.

Lemma.

\[
(d\tilde{A}_i)\tilde{A}_i^{-1} = (2\kappa)^{-1}(\text{Ad}(T_1 \cdots T_i) - \text{id})\Delta_{a_1, \ldots, a_n}^{(n)}(B)
\]

(6.11)

Proof. By (6.8), the left–hand side of (6.11) is equal to

\[
d(d_1 \cdots d_i) (d_1 \cdots d_i)^{-1} = -\sum_{j=1}^i du_a(u^a)^{(j)}
\]

where we used (6.10). Write \( B = B_1 + B_2 \) where

\[
B_1 = \frac{\hbar}{2} \sum_{\alpha \in \Phi_+} \frac{e^\alpha + 1}{e^\alpha - 1} d\alpha - 2du_i J(u^i)
\]

\[
B_2 = -2du_i J(u^i)
\]

Since \( B_1 \) takes values in \( Ug \), \( \Delta_{a_1, \ldots, a_n}(B_1)^{(n)} \) is independent of \( a_1, \ldots, a_n \) and the right–hand side of (6.11) is equal to \( (2\kappa)^{-1}(\text{Ad}(T_1 \cdots T_i) - \text{id})\Delta_{a_1, \ldots, a_n}(B_2) \).

By (6.2), for any \( x \in \mathfrak{g} \),

\[
(\text{Ad}(T_1 \cdots T_i) - \text{id})\Delta_{a_1, \ldots, a_n}(J(x)) = \sum_{j=1}^i \kappa x^{(i)}
\]
so that the right–hand side of (6.11) is equal to \(-\sum_{j=1}^{i} du_{a}(u^{a})^{(j)}\). 

6.8.

Lemma.

\[ [\tilde{A}_{i}, \Delta_{a_{1}, \ldots, a_{n}}^{(n)}(B)] = 0 \]

Proof. For any \( x \in Y(\mathfrak{g}) \) and \( 1 \leq i \leq n \),

\[ \Delta_{a_{1}, \ldots, a_{n}}^{(n)}(x) = \Delta_{a_{1}-a_{i}, \ldots, a_{i-1}-a_{i}, 0}^{(i)} \otimes \Delta_{a_{i+1}-a_{n}, \ldots, a_{n-1}-a_{n}, 0}^{(n-i)} \]  

so that, by (6.9) and the fact that \( d_{1} \cdots d_{i} = \Delta^{(i)}(d_{1}) \), it suffices to prove the claimed identity for \( n = 2 \) and \( i = 1 \). We have

\[
d_{1}^{-1} [d_{1}R(a_{1} - a_{2}), \Delta_{a_{1}, a_{2}}(B)] R(a_{1} - a_{2})^{-1} = (\text{id} - \text{Ad}(d_{1}^{-1})) \Delta_{a_{1}, a_{2}}(B) + (\text{Ad}(R(a_{1} - a_{2})) - \text{id}) \Delta_{a_{1}, a_{2}}(B) \quad (6.12)
\]

Let \( u \in \mathfrak{h} \). By (6.2),

\[
(\text{id} - \text{Ad}(d_{1}^{-1})) \Delta_{a_{1}, a_{2}}(J(u)) = \frac{h}{2} (\text{id} - \text{Ad}(d_{1}^{-1}))[u^{(1)}]_{e} t
\]

and, by (6.3),

\[
(\text{Ad}(R(a_{1} - a_{2})) - \text{id}) \Delta_{a_{1}, a_{2}}(J(u)) = \frac{h}{2} [u^{(2)} - u^{(1)}]_{e} t = -h [u^{(1)}]_{e} t
\]

Thus, the right–hand side of (6.12) with \( B \) replaced by \( B_{2} = -2du_{a}J(u^{a}) \)

is equal to

\[
h du_{a}(\text{id} + \text{Ad}(d_{1}^{-1}))[u^{a(1)}]_{e} t
\]

Since \( R(a_{1} - a_{2}) \) commutes with \( \Delta_{a_{1}, a_{2}}(B_{1}) \), we have left to compute

\[
(\text{id} - \text{Ad}(d_{1}^{-1})) \Delta_{a_{1}, a_{2}}(B_{1}) = h \sum_{\alpha \in \Phi_{+}} da \frac{e^{\alpha} + 1}{e^{\alpha} - 1} (\text{id} - \text{Ad}(d_{1}^{-1})) \bar{t}_{\alpha}
\]

where \( \bar{t}_{\alpha} = x_{\alpha} \otimes x_{-\alpha} + x_{-\alpha} \otimes x_{\alpha} \), with \( x_{\pm \alpha} \in \mathfrak{g}_{\pm \alpha} \) such that \( (x_{\alpha}, x_{-\alpha}) = 1 \), so that \( \Delta(\kappa_{\alpha}) = \kappa_{\alpha} \otimes 1 + \frac{1}{2} \kappa_{\alpha}. \) By (6.10),

\[
(\text{id} - \text{Ad}(d_{1}^{-1})) \bar{t}_{\alpha} = (1 - e^{\alpha})(x_{\alpha} \otimes x_{-\alpha} - e^{-\alpha} x_{-\alpha} \otimes x_{\alpha})
\]

so that, for any \( \alpha \in \Phi_{+} \) and \( u \in \mathfrak{h} \),

\[
\alpha(u) \frac{e^{\alpha} + 1}{e^{\alpha} - 1} (\text{id} - \text{Ad}(d_{1}^{-1})) \bar{t}_{\alpha}
\]

\[
= -\alpha(u) \left((e^{\alpha} + 1) x_{\alpha} \otimes x_{-\alpha} - (e^{-\alpha} + 1) x_{-\alpha} \otimes x_{\alpha}\right)
\]

\[
= -[u^{(1)}]_{e}, (\text{id} + \text{Ad}(d_{1}^{-1}))) \bar{t}_{\alpha}
\]

whence the claimed result.

Remark. The proof of Theorem 6.6 works almost verbatim for \( \mathfrak{g} = \mathfrak{gl}_{n} \) and gives the commutation of the rational \( q \)-KZ connection and trigonometric Casimir connections for \( Y(\mathfrak{g}l_{n}) \).
7. THE AFFINE KZ CONNECTION

We show in this section that the degenerate affine Hecke algebra $\mathcal{H}'$ of $W$ is, very roughly speaking, the ‘Weyl group’ of the Yangian $\mathcal{Y}(g)$. More precisely, we show that if $V$ is a $\mathcal{Y}(g)$–module whose restriction to $g$ is small, the canonical action of $W$ on the zero weight space $V[0]$ extends to one of $\mathcal{H}'$. Moreover, the trigonometric Casimir connection with values in $V[0]$ coincides with Cherednik’s affine KZ connection with values in this $\mathcal{H}'$–module.

7.1. The degenerate affine Hecke algebra. Let $K$ be the vector space of $W$–invariant functions $\Phi \to \mathbb{C}$ and denote the natural linear coordinates on $K$ by $k_\alpha, \alpha \in \Phi/W$. Recall [Lu] that the degenerate affine Hecke algebra $\mathcal{H}'$ associated to $W$ is the algebra over $\mathbb{C}[K]$ generated by the group algebra $\mathbb{C}W$ and the symmetric algebra $S\mathfrak{h}$ subject to the relations

$$s_i x_u - x_{s_i(u)} s_i = k_\alpha \alpha_i(u)$$

(7.1)

for any simple reflection $s_i \in W$ and linear generator $x_u, u \in \mathfrak{h}$, of $S\mathfrak{h}$.

7.2. The affine KZ connection. The AKZ connection is the trigonometric, $\mathcal{H}'$–valued connection given by

$$\nabla = d - \sum_{\alpha \in \Phi_+} \frac{d\alpha}{e^{\alpha} - 1} k_\alpha s_\alpha - du_i x_u$$

(7.2)

where $\{u_i\}, \{u^i\}$ are dual bases of $\mathfrak{h}^*, \mathfrak{h}$ respectively. This connection was defined by Cherednik in [Ch1, Ch2] and proved to be flat and $W$–equivariant. This may also be obtained as a consequence of Theorem 2.23. Indeed, the relations (7.1), with $t_\alpha = k_\alpha s_\alpha$ are easily verified and, as pointed out by Cherednik, the remaining relations are precisely those defining $\mathcal{H}'$.

Remark. The $\delta$–form (2.7) of the AKZ connection corresponds to Drinfeld’s presentation of $\mathcal{H}'$ in terms of $\mathbb{C}W$ and non–commuting elements $y_u$ which transform like the reflection representation of $W$ (see [Dr3] and [RS]). Indeed, it is given by

$$\nabla = d - \frac{1}{2} \sum_{\alpha \in \Phi_+} \frac{e^{\alpha} - 1}{e^{\alpha} - 1} d\alpha k_\alpha s_\alpha - du_i y_u$$

where the elements $y_u$ are defined by (2.6) as

$$y_u = x_u - \frac{1}{2} \sum_{\alpha \in \Phi_+} \alpha(u) k_\alpha s_\alpha$$

(7.3)

and therefore satisfy $s_i y_u s_i = y_{s_i(u)}$ by Proposition 2.22.

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6I owe this observation to Pavel Etingof.
7.3. **W-action on zero weight spaces of g-modules.** Let $G$ be the complex, simply-connected Lie group with Lie algebra $\mathfrak{g}$, $H$ the maximal torus with Lie algebra $\mathfrak{h}$ and $N(H) \subset G$ its normaliser. If $V$ is an integrable $\mathfrak{g}$-module, the action of $N(H)$ on $V$ permutes the weight spaces compatibly with the action of $N(H)$ on $H$. In particular, it acts on the zero weight space $V[0]$ and this action factors through $W = N(H)/H$.

7.4. **Small g-modules.** Recall that a $\mathfrak{g}$-module $V$ is small if $2\alpha$ is not a weight of $V$ for any root $\alpha$. If $V$ is a small $\mathfrak{g}$-module with a non-trivial zero weight space $V[0]$, the restriction to $V[0]$ of the square $e_\alpha^2$ of a raising operator maps to the weight space $V[2\alpha]$ and is therefore zero. This implies the following result [11.2, Prop. 9.1]

**Lemma.** If $V$ is an integrable, small $\mathfrak{g}$-module, the following holds on the zero weight space $V[0]$

$$\kappa_\alpha = (\alpha, \alpha)(1 - s_\alpha)$$

where the right-hand side refers to the action of the reflection $s_\alpha \in W$ on $V[0]$.

7.5. Let $\mathcal{H}'_h$ be the degenerate affine Hecke algebra of $W$ with parameters

$$k_\alpha = -h(\alpha, \alpha)$$

(7.4)

**Theorem.** Let $V$ be a finite-dimensional $Y(\mathfrak{g})$-module whose restriction to $\mathfrak{g}$ is small.

1. The canonical $W$-action on the zero weight space $V[0]$ together with either of the equivalent assignments

$$x_u \to -2T(u)_1 + \frac{1}{2} \sum_{\alpha \in \Phi_+} k_\alpha \alpha(u)$$

$$y_u \to -2J(u)$$

(7.5)

yield an action of $\mathcal{H}'_h$ on $V[0]$.

2. The trigonometric Casimir connection of $\mathfrak{g}$ with values in $\text{End}(V[0])$ is equal to the sum of the AKZ connection with values in the $\mathcal{H}'_h$-module $V[0]$ and the scalar valued one-form

$$A = \frac{1}{2} \sum_{\alpha \in \Phi} k_\alpha \frac{d\alpha}{e^\alpha - 1}$$

(7.7)

**Proof.** The trigonometric Casimir connection with values in $\text{End}(V[0])$ reads, by Lemma 7.4

$$\nabla = d - h \sum_{\alpha \in \Phi_+} \frac{d\alpha}{e^\alpha - 1} (\alpha, \alpha)(1 - s_\alpha) + 2du_i T(u^i)_1$$

$$= d - \sum_{\alpha \in \Phi_+} \frac{d\alpha}{e^\alpha - 1} k_\alpha s_\alpha + 2du_i T(u^i)_1 + \sum_{\alpha \in \Phi_+} k_\alpha \frac{d\alpha}{e^\alpha - 1}$$
where the weights $k_\alpha$ are given by (7.4). By (2.4)
\[
\sum_{\alpha \in \Phi} (\alpha, \alpha) \frac{d\alpha}{e^{\alpha} - 1} = 2 \sum_{\alpha \in \Phi_+} (\alpha, \alpha) \frac{d\alpha}{e^{\alpha} - 1} + \sum_{\alpha \in \Phi_+} (\alpha, \alpha) d\alpha
\]
so that if $A$ is given by (7.7), then
\[
\nabla - A = d - \sum_{\alpha \in \Phi_+} k_\alpha \frac{d\alpha}{e^{\alpha} - 1} s_\alpha + du_i \left( 2T(u^i)_1 - \frac{1}{2} \sum_{\alpha \in \Phi_+} k_\alpha \alpha(u^i) \right)
\]
Applying Proposition 2.22 to $\nabla - A$ which is $W$–equivariant since $\nabla$ and $A$ are, shows that the map (7.5) gives an action of $H'$ on $V[0]$. This proves (1) and (2). The equivalence of (7.5) and (7.6) follows easily from §3.3 and (7.3). ■

Remark. Theorem 7.5 extends to the trigonometric setting the relation between the rational Casimir and KZ connections proved in [TL2, Prop. 9.1].

7.6. The adjoint representation. Drinfeld proved that, for any simple $\mathfrak{g}$, the direct sum $\overline{\mathfrak{g}} = \mathfrak{g} \oplus \mathbb{C}$ of the adjoint and trivial representations of $\mathfrak{g}$ admits an extension to an action of $Y(\mathfrak{g})$ on $\overline{\mathfrak{g}}$ [Dr2, Thm. 8]. It is easy to check that the corresponding action of $H'_h$ on $\overline{\mathfrak{g}}[0] = \mathfrak{h} \oplus \mathbb{C}$ given by Theorem 7.5 coincides with its action on affine linear functions on $\mathfrak{h}^*$ given by rational Dunkl operators (see, e.g. [Kr]).

7.7. The case of $\mathfrak{sl}_n$. Let $\mathfrak{g} = \mathfrak{sl}_n$ and $V = \mathbb{C}^n$ its vector representation. A simple inspection shows that $V^{\otimes n}$ is a small [Re2]. The zero weight space $V^{\otimes n}[0]$ possesses two actions of the symmetric group: one arising from the Weyl group action of $\mathfrak{S}_n$, the other from the permutation of the tensor factors, under which it identifies with the group algebra $\mathbb{C}\mathfrak{S}_n$.

The $\mathfrak{sl}_n$–module $V^{\otimes n}$ may be endowed with an action of $Y(\mathfrak{g})$ depending on $a_1, \ldots, a_n \in \mathbb{C}$ obtained by composing the coproduct $\Delta^{(n)} : Y(\mathfrak{g}) \to Y(\mathfrak{g})^{\otimes n}$ with the evaluation homomorphisms $ev_{a_i} : Y(\mathfrak{g}) \to U\mathfrak{g}$. It is easy to check that the action of $H'_h$ on $V^{\otimes n}[0]$ given by Theorem 7.5 coincides with that on the induced representation $\text{ind}_{\mathfrak{g}_n}\mathbb{C}_{a_1,\ldots,a_n}$.

8. Appendix: Tits extensions of affine Weyl groups

In this appendix, we review the definition of the Tits extension $\tilde{W}$ of a Weyl group $W$. We then define the reduced Tits extension $\tilde{W}_\text{red}$ of $W$ and show that, when $W$ is an affine Weyl group, $\tilde{W}_\text{red}$ is isomorphic to the semi–direct product of the Tits extension of the finite Weyl group underlying $W$ by the corresponding coroot lattice (Theorem 8.10).
8.1. Weyl groups and braid groups. Let \( A = (a_{ij})_{i,j \in I} \) be a generalised Cartan matrix and \((\mathfrak{h}, \Delta, \Delta^\vee)\) its unique realisation. Thus, \( \mathfrak{h} \) is a complex vector space of dimension \( 2|I| - \text{rank}(A) \), \( \Delta = \{ \alpha_i \}_{i \in I} \subset \mathfrak{h}^* \) and \( \Delta^\vee = \{ \alpha_i^\vee \}_{i \in I} \subset \mathfrak{h} \) are linearly independent sets and
\[
\langle \alpha_i^\vee, \alpha_j \rangle = a_{ij}
\]
Recall that the Weyl group \( W = W(A) \) attached to \( A \) is the subgroup of \( GL(\mathfrak{h}^*) \) generated by the reflections \([3.7]\)
\[
s_i(\lambda) = \lambda - \langle \lambda, \alpha_i^\vee \rangle \alpha_i
\]
or, equivalently, the subgroup of \( GL(\mathfrak{h}) \) generated by the dual reflections
\[
s_i^\vee(t) = t - \langle t, \alpha_i \rangle \alpha_i^\vee
\]
By \([3.13]\), the defining relations of \( W \) are
\[
s_i^2 = 1
\]
\[
(s_is_j)^{m_{ij}} = 1
\]
where for any \( i \neq j \), \( m_{ij} \) is equal to 2, 3, 4, 6 or \( \infty \) according to whether \( a_{ij}a_{ji} \) is equal to 0, 1, 2, 3 or \( \geq 4 \).

The braid group \( B = B(A) \) attached to \( A \) is the group with generators \( S_i, i \in I \) and relations
\[
\underbrace{S_iS_j\cdots}_{m_{ij}} = \underbrace{S_jS_i\cdots}_{m_{ij}}
\]
for any \( i \neq j \).

8.2. Tits extensions of Weyl groups.

Definition \((\text{Ti})\). The Tits extension of \( W \) is the group \( \tilde{W} \) with generators \( \tilde{s}_i, i \in I \) and relations
\[
\underbrace{\tilde{s}_i\tilde{s}_j\cdots}_{m_{ij}} = \underbrace{\tilde{s}_j\tilde{s}_i\cdots}_{m_{ij}}
\]
(8.1)
\[
\tilde{s}_i^4 = 1
\]
(8.2)
\[
\tilde{s}_i^2\tilde{s}_j^2 = \tilde{s}_j^2\tilde{s}_i^2
\]
(8.3)
\[
\tilde{s}_i^2\tilde{s}_j^2\tilde{s}_i^{-1} = \tilde{s}_j^2(\tilde{s}_i^2)^{-a_{ji}}
\]
(8.4)

8.3. Let \( \mathfrak{g} = \mathfrak{g}(A) \) be the Kac–Moody algebra corresponding to the Cartan matrix \( A \) with generators \( t \in \mathfrak{h} \) and \( e_i, f_i, i \in I \). Recall that a representation of \( \mathfrak{g} \) is integrable if \( \mathfrak{h} \subset \mathfrak{g} \) acts semi–simply with finite–dimensional eigenspaces and \( e_i, f_i \) act locally nilpotently. The next two results explain the relevance and structure of the Tits extension \( \tilde{W} \).

Proposition. Let \( V \) be an integrable representation of \( \mathfrak{g} \). Then, the triple exponentials
\[
r_i = \exp(e_i) \exp(-f_i) \exp(e_i)
\]
are well–defined elements of $GL(V)$ and the assignment $\tilde{s}_i \to r_i$ yields a representation of $\tilde{W}$ on $V$ mapping $\tilde{s}_i^2$ to $\exp(\pi\sqrt{-1}\alpha_i^\vee)$.

**Proof.** The $r_i$ are clearly well defined and satisfy

$$r_i \cdot t \cdot r_i^{-1} = s_i^\vee(t)$$

for any $t \in h$ and $r_i^2 = \exp(\pi\sqrt{-1}\alpha_i^\vee)$ \cite[3.8]{K}, from which (8.2)–(8.4) readily follow. Let now $i \neq j$ be such that $m_{ij} < \infty$. Then, the Lie subalgebra $g_{ij}$ of $g$ generated by $e_i, f_i, \alpha_i^\vee$ and $e_j, f_j, \alpha_j^\vee$ is finite–dimensional and semi–simple and $V$ integrates to a representation of the complex, connected and simply–connected Lie group $G_{ij}$ with Lie algebra $g_{ij}$. By \cite{T}, $r_i$ and $r_j$ satisfy the braid relations (8.1) when regarded as elements of $G_{ij}$, and these therefore hold in $GL(V)$.

8.4. Let $Q^\vee \subset \mathfrak{h}$ be the lattice spanned by the coroots $\alpha_i^\vee, i \in I$.

**Proposition (\cite{T}).** $\widetilde{W}$ is an extension of $W$ by the abelian group $Z$ generated by the elements $\tilde{s}_i^2$. $Z$ is isomorphic, as $W$–module to $Q^\vee/2Q^\vee \cong \mathbb{Z}_2^{|I|}$.

**Proof.** Let $K \supset Z$ the kernel of the canonical projection $\widetilde{W} \to W$. By (8.4), $Z$ is a normal subgroup of $\widetilde{W}$ and $\widetilde{W}/Z$ is generated by the images $\tilde{s}_i$ of $\tilde{s}_i$ which, in addition to the braid relations (8.1), satisfy $\tilde{s}_i^2 = 1$. Thus, $\widetilde{W}/Z$ is a quotient of $W$, $K = Z$ and $\widetilde{W}/Z \cong W$. Note next that, by (8.2)–(8.4), the assignment $\alpha_i^\vee \to \tilde{s}_i^2$, $\tilde{s}_i^2$ extends to a $W$–equivariant surjection $Q^\vee/2Q^\vee \to Z$. To prove that this is an isomorphism it suffices to exhibit, for any $i \in I$ a $\mathbb{Z}_2$–valued character $\chi_i$ of $Z$ such that $\chi_i(\tilde{s}_i^2) = (-1)^{\delta_{ij}}$. Let $\lambda_i$ be the $i$th fundamental weight of $g$, so that $\langle \lambda_i, \alpha_j^\vee \rangle = \delta_{ij}$, $V_i$ the irreducible $g$–module with highest weight $\lambda_i$ and $v_i \in V_i$ a nonzero highest weight vector. $V_i$ is integrable and since $r_i^2 = \exp(\sqrt{-1}\pi\alpha_i^\vee)$, we have $r_i^2 v_i = (-1)^{\delta_{ij}} v_i$.

8.5. **Reduced Tits extensions.** For any $v = \sum_i m_i \alpha_i^\vee \in Q^\vee$, set

$$\tilde{s}_v^2 = \prod_i (\tilde{s}_i^2)^{m_i} \in Z$$

so that for any $w \in W$ and lift $\tilde{w} \in \widetilde{W}$, $\tilde{w}s_v^2\tilde{w}^{-1} = \tilde{s}_v^2$. By \cite[Prop. 1.6]{K}, the center $c$ of $g$ is equal to

$$c = \{ t \in h | \langle t, \alpha_i \rangle = 0 \text{ for any } i \in I \}$$

(8.5)

The Weyl group operates trivially on $c \subset \mathfrak{h}$ and it follows from (8.4) that the subgroup $Z_c \subset Z$ generated by the elements $\tilde{s}_v^2$, with $v \in c \cap Q^\vee$ lies in the centre of $\widetilde{W}$.

**Definition.** The reduced Tits extension $\widetilde{W}^{\text{red}}$ of $W$ is the quotient $\widetilde{W}^{\text{red}} = \widetilde{W}/Z_c$.

\footnote{Tits’ argument is reproduced in the proof of (i) of Proposition 8.9}
By Proposition 8.4, \( \widetilde{W}^{\text{red}} \) is an extension of \( W \) by \( Q^\vee/(2Q^\vee + \mathfrak{c} \cap Q^\vee) \cong \mathbb{Z}_2^{\operatorname{rank}(A)} \).

### 8.6. Reduced Tits extensions of affine Weyl groups.

Assume henceforth that \( A = (a_{ij})_{0 \leq i,j \leq n} \) is an affine Cartan matrix of untwisted type. Altering our notations, we denote by \( \mathfrak{g} \) the underlying complex, semi–simple Lie algebra and by \( \mathfrak{h} \), \( \{\alpha_i\}_{i=1}^n \), \( \{\alpha_i^\vee\}_{i=1}^n \), \( W \) and \( Q^\vee \) its Cartan subalgebra, simple roots, simple coroots, Weyl group and coroot lattice respectively. Thus, for any \( 1 \leq i,j \leq n \),

\[
a_{ij} = \langle \alpha_i^\vee, \alpha_j \rangle, \quad a_{0j} = -\langle \theta^\vee, \alpha_j \rangle \quad \text{and} \quad a_{j0} = -\langle \alpha_j^\vee, \theta \rangle
\]

where \( \theta \in \mathfrak{h}^* \) is the highest root of \( \mathfrak{g} \). It is well known that the (affine) Weyl group \( W_a \) attached to \( A \) is isomorphic to the semi–direct product \( W \ltimes Q^\vee \) [Kac prop. 6.5]. The isomorphism is given by mapping \( s_i \) to \((s_i, 0)\) for \( i \geq 1 \) and \( s_0 \) to \((s_0, -\theta^\vee)\).

The subspace \( \mathfrak{c} \) defined by (8.5) is spanned by the element

\[
K = \alpha_0^\vee + \sum_{i=1}^n m_i \alpha_i^\vee
\]

where the \( m_i \) are the positive integers such that \( \theta^\vee = \sum_{i=1}^n m_i \alpha_i^\vee \) [Kac Prop. 6.2]. It follows that the reduced Tits extension \( \widetilde{W}_a \) of \( W_a \) is the quotient of \( \widetilde{W}_a \) by the relation

\[
\frac{-s_0}{s_0^2} \cdot \prod_{i=1}^n \frac{s_i}{s_i^2} = 1
\]  

### 8.7. Loop groups.

The structure of the reduced Tits extension of \( W_a \) will be determined in [8.7]–[8.10] by embedding \( \widetilde{W}_a \) into the loop group corresponding to \( \mathfrak{g} \).

Let \( L_\mathfrak{g} = \mathfrak{g}[z, z^{-1}] \) be the loop algebra of \( \mathfrak{g} \) and \( d \) the derivation of \( L_\mathfrak{g} \) defined by \( dx(m) = mx(m) \), where \( x(m) = x \otimes z^m \). Then, \( L_\mathfrak{g} \ltimes \mathbb{C}d \) is the quotient of the Kac–Moody algebra corresponding to \( A \) by the central element \( K \) defined above. Let \( G \) be the complex, connected and simply connected Lie group with Lie algebra \( \mathfrak{g} \) and \( LG = G(\mathbb{C}[z, z^{-1}]) \) the group of polynomial loops into \( G \). Let \( H \subset G \) be the maximal torus with Lie algebra \( \mathfrak{h} \). The group \( \mathbb{C}^* \) acts on \( LG \) by reparametrisation fixing \( G \supset H \) and \( H \times \mathbb{C}^* \) is a maximal abelian subgroup of the semi–direct product \( LG \ltimes \mathbb{C}^* \). By [PS], Prop. 5.2, the normaliser of \( H \times \mathbb{C}^* \) in \( LG \ltimes \mathbb{C}^* \) is equal to \( (N(H) \ltimes H^\vee) \times \mathbb{C}^* \) where \( N(H) \) is the normaliser of \( H \) in \( G \) and \( H^\vee = \operatorname{Hom}_\mathbb{Z}(\mathbb{C}^*, H) \subset LG \) is isomorphic to the coroot lattice \( Q^\vee \) by

\[
\lambda \in Q^\vee \rightarrow \left( z \rightarrow z^\lambda = \exp_H(-\ln(z)\lambda) \right)
\]

The quotient \( N(H \times \mathbb{C}^*)/H \times \mathbb{C}^* \) is therefore isomorphic to the affine Weyl group \( W_a = W \ltimes H^\vee \).
8.8. For each real root $\tilde{\alpha} = (\alpha,n)$ of $LG$, the subalgebra $\mathfrak{sl}_2^\alpha$ of $Lg$ spanned by

$$e_\tilde{\alpha} = e_\alpha(n), \quad f_\tilde{\alpha} = f_\alpha(-n) \quad \text{and} \quad h_\alpha$$

is the Lie algebra of a closed subgroup of $LG$ isomorphic to $SL_2(\mathbb{C})$. This is obvious if $n = 0$ and follows in the general case from the fact that $\mathfrak{sl}_2^{(\alpha,n)}$ is conjugate to $\mathfrak{sl}_2^{(\alpha,0)}$. Indeed, any element $\gamma_\lambda$ of the coweight lattice $\text{Hom}(\mathbb{C}^*, H/Z) \subset L(G/Z)$ induces by conjugation an automorphism of $LG$ such that

$$\text{Ad}(\gamma_\lambda)e_\alpha(n) = e_\alpha(n - \langle \lambda, \alpha \rangle) \quad \text{and} \quad \text{Ad}(\gamma_\lambda)f_\alpha(n) = f_\alpha(n + \langle \lambda, \alpha \rangle)$$

8.9. Let now $\alpha_i = (\alpha_i,0)$, $i = 1 \ldots n$ and $\alpha_0 = (-\theta,1)$ be the simple roots of $LG$. For each $i = 0 \ldots n$, let $SL_2(\mathbb{C}) \cong G_i \subset LG$ be the corresponding subgroup, $H_i \subset G_i$ its torus and $N_i$ the normaliser of $H_i$ in $G_i$. Note that any element of $N_i \setminus H_i$ is of the form

$$\exp(e_i)\exp(-f_i)\exp(e_i) = \exp(-f_i)\exp(e_i)\exp(-f_i)$$

for some choice of root vectors $e_i \in (Lg)_{\alpha_i}, f_i \in (Lg)_{-\alpha_i}$ such that $[e_i,f_i] = \alpha_i^\vee$ if $i \geq 1$ and $-\theta^\vee$ if $i = 0$.

Let $B_a$ be the (affine) braid group corresponding to $A$ and $S_0, S_1, \ldots, S_n$ its generators.

**Proposition.**

1. For any choice of $\sigma_i \in N_i \setminus H_i$, $i = 0 \ldots n$, the assignment $S_i \mapsto \sigma_i$ extends uniquely to a homomorphism $\sigma : B_a \rightarrow N(H) \times H^\vee$.

2. $\sigma$ factors through an isomorphism of the reduced Tits extension $\tilde{W}_a^\text{red}$ onto its image in $N(H) \times H^\vee$.

3. If $\sigma, \sigma' : \tilde{W}_a^\text{red} \rightarrow N(H) \times H^\vee$ are the homomorphisms corresponding to the choices $\{\sigma_i\}$ and $\{\sigma'_i\}$ respectively, there exists $t \in H \times \mathbb{C}^*$ such that, for any $\tilde{s} \in \tilde{W}_a^\text{red}$, $\sigma(\tilde{s}) = t\sigma'(\tilde{s})t^{-1}$.

**Proof.** (1) the following argument is due to Tits [11]. Let $i \neq j$ be such that $m_{ij}$ is finite and set $s_{ij} = s_is_j \cdots \in W_a$ and $\sigma_{ij} = \sigma_i\sigma_j \cdots \in N(H) \times H^\vee$ where each product has $m_{ij} - 1$ factors. The braid relations in $W_a$ may be written as $s_{ij}s_{j'} = s_{j}s_{ij}$ where $j' = j$ or $i$ according to whether $m_{ij}$ is even or odd. Thus, $s_{ij}^{-1}s_{ij} = s_{j}$ and therefore,

$$\Delta_{ij} = \sigma_j^{-1}\sigma_{ij}^{-1}\sigma_j\sigma_{ij} \in H \cap \left(\sigma_{j'}^{-1}\sigma_{ij}^{-1}N_j\sigma_{ij}\right) = H \cap \sigma_j^{-1}N_{j'} = H_{j'}$$

Repeating the argument with $i$ and $j$ permuted, we find that $\Delta_{ji} \in H_{i'}$ with $i' = i$ or $j$ according to whether $m_{ij}$ is even or odd. Thus, $\Delta_{ij} = \Delta_{ji}^{-1} \in H_{i'} \cap H_{j'} = \{1\}$ where the latter assertion follows by follows from the simple connectedness of $G$.

(2) The $\sigma_i$ satisfy (8.2)–(8.4) and (8.6) since, for any $x_j \in N_j \setminus H_j$, $x_j^2 = \exp(i\pi\alpha_j^\vee)$ for $j \geq 1$ and $x_0^2 = \exp(-i\pi\theta^\vee)$. Thus $\sigma$ descends to $W_a^{\text{red}}$. 


Since the diagram
\[ \tilde{W}_a^{\text{red}} \to N(H) \rtimes H^\vee \to W_a \]
is commutative, the kernel of \( \sigma \) is contained in \( Z/Z_c \cong \mathbb{Z}_2^n \) and is therefore trivial since, due to the simple-connectedness of \( G \), the subgroup of \( G \) generated by \( \sigma_j^2 = \exp(\pi i \alpha_j^\vee) \), \( j = 1 \ldots n \) is isomorphic to \( \mathbb{Z}_2^n \).

(3) For \( i = 1, \ldots, n \), let \( t_i \in H_i \) be such that \( \sigma_i = t_i \sigma_i' \) and choose \( c_i \in \mathbb{C} \) such that \( \exp(c_i h_{\alpha_i}) \). Since \( s_i \lambda_j^\vee = \lambda_j^\vee - \delta_{ij} \alpha_i^\vee \), where the \( \lambda_j^\vee \in \mathfrak{h} \) are the fundamental coweights of \( \mathfrak{g} \), we find, with \( \tilde{t} = \exp(\sum_{j=1}^n c_j \lambda_j^\vee) \in H \),
\[ \tilde{t} \cdot \sigma_i' \cdot \tilde{t}^{-1} \exp(c_i h_{\alpha_i}) \cdot \sigma_i' = \sigma_i \]
Let now \( t_0 = \exp(c_0 h_{\theta}) \in H_0 \) be such that \( \sigma_0 = t_0 \sigma_0' \). Since for any \( x \in \mathbb{C} \),
\[ \exp(x d) \sigma_0' \exp(-x d) = \exp(-x h_{\theta}) \sigma_0' \]
we find, with \( y = \sum_j c_j \langle \lambda_j^\vee, \theta \rangle - c_0 \), that
\[ \tilde{t} \exp(yd) \sigma_0' \exp(-yd) \tilde{t}^{-1} = \exp((-y + \sum_j c_j \langle \lambda_j^\vee, \theta \rangle) h_{\theta}) \sigma_0' = \sigma_0 \]
so that \( t = \tilde{t} \exp(yd) \) is the required element.  

8.10. The following is the main result of this appendix.

**Theorem.** The inclusion \( \tilde{W} \hookrightarrow \tilde{W}_a^{\text{red}} \) extends to an isomorphism \( \tilde{W} \times Q^\vee \to \tilde{W}_a^{\text{red}} \) making the following a commutative diagram
\[ \begin{array}{ccc}
\tilde{W} \times Q^\vee & \longrightarrow & \tilde{W}_a^{\text{red}} \\
\downarrow & & \downarrow \\
W \times Q^\vee & \longrightarrow & W_a
\end{array} \]

**Proof.** We wish to construct a \( \tilde{W} \)-equivariant section \( s \) to the restriction to \( Q^\vee \) of the extension
\[ 1 \to Z/Z_c \to \tilde{W}_a^{\text{red}} \to W_a \to 1 \]
Identify for this purpose \( \tilde{W}_a^{\text{red}} \) with its image inside \( N(H \times \mathbb{C}^*) \cap LG \) by using Proposition [8.9]. We claim that there exists \( x \in \mathbb{C} \) such that, for any \( \lambda \in Q^\vee \), \( \exp_H(x \lambda) \cdot z^\lambda \) lies in \( \tilde{W}_a^{\text{red}} \). It is then clear that \( s(\lambda) = \exp_H(x \lambda) \cdot z^\lambda \) yields the required section.

Let \( \alpha_i^\vee \) be a short simple coroot and \( w \in W \) an element such that \( \theta^\vee = w \alpha_i^\vee \). Let \( \tilde{w} \in \tilde{W} \) be a lift of \( w \) and lift \( \theta^\vee \in Q^\vee \) to
\[ \tau^{\theta^\vee} = \tilde{s}_0 \tilde{w} \tilde{s}_i \tilde{w}^{-1} \in \tilde{W}_a^{\text{red}} \]
Let \( e_\theta, f_\theta \in \mathfrak{g}_\theta, \mathfrak{g}_- \) be root vectors such that \([e_\theta, f_\theta] = \theta^\vee\) and denote by \( \rho_{\theta^\vee} : SL_2 \rightarrow G\) the embedding whose differential maps \( e, f, h \in \mathfrak{sl}_2\) to \( e_\theta, f_\theta, \theta^\vee\). We may assume \( e_\theta, f_\theta \) chosen so that \( \tilde{s}_0 \) is of the form
\[
\exp(f_\theta \otimes z) \exp(-e_\theta \otimes z^{-1}) \exp(f_\theta \otimes z) = \rho_{\theta^\vee}(\begin{pmatrix} 0 & -z^{-1} \\ z & 0 \end{pmatrix})
\]
Since \( \tilde{w} \tilde{s}_1 \tilde{w}^{-1} \in N_\theta \setminus H_\theta \) is necessarily of the form
\[
\exp(te_\theta) \exp(-t^{-1}f_\theta) \exp(te_\theta) = \rho_{\theta^\vee}(\begin{pmatrix} 0 & t \\ -t^{-1} & 0 \end{pmatrix})
\]
for some \( t \in \mathbb{C}^\ast \), we find that
\[
\tau^{\theta^\vee} = \rho_{\theta^\vee}(\begin{pmatrix} (tz)^{-1} & 0 \\ 0 & tz \end{pmatrix}) = \exp(x\theta^\vee) \cdot z^{\theta^\vee}
\]
with \( x = -\ln(t) \) which proves our claim for \( \lambda = \theta^\vee \). Let now \( w \in \tilde{W} \) with lift \( \tilde{w} \in \tilde{W} \), then
\[
\tilde{w} \tau^{\theta^\vee} \tilde{w}^{-1} = \exp(xw(\theta^\vee)) \cdot z^{w\theta^\vee}
\]
so that that \( \exp(x\alpha_i^\vee) \cdot z^{\alpha_i^\vee} \in \tilde{W}_a^{\text{red}} \) for any short coroot \( \alpha_i^\vee \). Since the short coroots span \( Q^\vee \), the claim holds for any \( \lambda \in Q^\vee \).

Since \( s(Q^\vee) \) is free abelian and \( \tilde{W} \) is finite, their intersection is trivial and the map \( \tilde{W} \times Q^\vee \rightarrow \tilde{W}_a^{\text{red}}, (\tilde{w}, \lambda) \rightarrow \tilde{w}s(\lambda) \) is injective. It is moreover surjective since \( Z/Z_c \) is generated by \( \tilde{s}_i^2, i = 1 \ldots n \) and therefore lies in \( \tilde{W} \).

8.11.

**Remark.** Unlike \( \tilde{W}_a^{\text{red}} \), the (non–reduced) Tits extension \( \tilde{W}_a \) of \( W_a \) is not a semi–direct product in general. For example, for \( \mathfrak{g} = \mathfrak{sl}_2 \), with affine Cartan matrix
\[
A = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}
\]
\( \tilde{W}_a \) is generated by \( \tilde{s}_0, \tilde{s}_1 \) with relations \( \tilde{s}_1^4 = 1 \),
\[
\tilde{s}_0 \tilde{s}_1 \tilde{s}_0^{-1} = \tilde{s}_1^2(\tilde{s}_0^2)^2 = \tilde{s}_1^2 \quad \text{and} \quad \tilde{s}_1 \tilde{s}_0 \tilde{s}_1^{-1} = \tilde{s}_0^2
\]
In particular, the group \( Z \cong \mathbb{Z}_2^2 \) generated by \( \tilde{s}_0, \tilde{s}_1^2 \) lies in the centre of \( \tilde{W}_a \). Any lift in \( \tilde{W}_a \) of the generator of \( Q^\vee \cong \mathbb{Z} \) is of the form \( \tau = z \tilde{s}_0 \tilde{s}_1 \), for some \( z \in Z \) and gives rise a \( \tilde{W} \)–equivariant section \( Q^\vee \rightarrow \tilde{W}_a \) if, and only if, \( \tilde{s}_1 \tilde{s}_0 \tilde{s}_1^{-1} = \tau^{-1} \). Since \( z = z^{-1} \) is central, such a section exists iff \( \tilde{s}_1(\tilde{s}_0 \tilde{s}_1) \tilde{s}_1^{-1} = \tilde{s}_1^{-1} \tilde{s}_0^{-1} \) and therefore iff \( \tilde{s}_1^2 \tilde{s}_0^2 = 1 \) which holds in \( \tilde{W}_a^{\text{red}} \) but not in \( \tilde{W}_a \).

**Remark.** The section \( Q^\vee \rightarrow \tilde{W}_a^{\text{red}} \) constructed in Theorem 8.10 does not in general coincide with that obtained from the canonical section \( Q^\vee \rightarrow B_a \) [Mc, §3.2–3.3]. For example, for \( \mathfrak{g} = \mathfrak{sl}_3 \), the canonical lift of \( \theta^\vee \in Q^\vee \) in \( B_a \) is \( T^{\theta^\vee} = S_0S_1S_2S_1 \). When regarded as an element \( \tau^{\theta^\vee} \) of \( \tilde{W}_a^{\text{red}} \), this does
not give rise to a \( \tilde{W} \)-equivariant section since \( \text{Ad}(\tilde{s}_\theta)\tau^{\theta'} \neq (\tau^{\theta'})^{-1} \) where \( \tilde{s}_\theta = \tilde{s}_1 \tilde{s}_2 \tilde{s}_1 \) is a lift in \( \tilde{W} \) of the reflection \( s_\theta \). Indeed, \( \text{Ad}(\tilde{s}_2)\tilde{s}_1^2 = \tilde{s}_1^2 \tilde{s}_2 \) in \( \tilde{W} \), so that

\[
\tilde{s}_\theta^2 = \tilde{s}_1 \text{Ad}(\tilde{s}_2)(\tilde{s}_1^2 \tilde{s}_2 \tilde{s}_1) = \tilde{s}_1^3 \tilde{s}_2 \tilde{s}_1 = 1
\]

Thus, since \( \tau^{\theta'} = \tilde{s}_0 \tilde{s}_\theta \),

\[
\text{Ad}(\tilde{s}_\theta)\tau^{\theta'} = \tilde{s}_0 \tilde{s}_\theta \quad \text{while} \quad (\tau^{\theta'})^{-1} = \tilde{s}_0 \tilde{s}_\theta^{-1}
\]

which are different elements of \( \tilde{W}_a^{\text{red}} \) by (8.6).

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