Noncommutative Geometry and the Primordial Dipolar Imaginary Power Spectrum

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Abstract

We argue that an anisotropic dipolar imaginary primordial power spectrum is possible within the framework of noncommutative space-times. We show that such a spectrum provides a good description of the observed dipole modulation in CMBR data. We extract the corresponding power spectrum from data. The dipole modulation is related to the observed hemispherical anisotropy in CMBR data, which might represent the first signature of quantum gravity.

1 Introduction

The cosmic microwave background radiation (CMBR) shows a hemispherical power anisotropy [1–9], which can be parametrized as,

$$\Delta T(\hat{n}) = g(\hat{n})(1 + A \hat{\lambda} \cdot \hat{n})$$  \hspace{2cm} (1)

where $g(\hat{n})$ is an isotropic and Gaussian random field, $\hat{\lambda}$ the preferred direction and $A$ the amplitude of anisotropy. This model implies a dipole modulation [10–13] of the CMBR temperature field. The WMAP five year data leads to $A = 0.072 \pm 0.022$ with the dipole direction, $(\theta, \phi) = (224^\circ, 112^\circ) \pm 24^\circ$ for $l \leq 64$ in the galactic coordinates [1–4, 6, 8]. This anisotropy has been confirmed by PLANCK [7] with amplitude and direction similar to those found in WMAP.

The hemispherical anisotropy has also been probed at multipoles higher than 64 [4–5]. The signal is found to be absent at $l \sim 500$ [14–15] and also not seen in the large scale structures [16–17]. These observations may be accommodated in a model which proposes a scale dependent power spectrum [18], such that the effect is negligible at high-$l$.

Many theoretical models, such as, [19–40], have been proposed which aim to explain the observed hemispherical anisotropy as well as other signals of anisotropy seen in data [41–47]. An interesting possibility is that there might have been a phase of anisotropic expansion at very early time. The inflationary Big Bang cosmology is perfectly consistent with such an evolution. The
anisotropic modes, generated during this early phase may later re-enter the horizon \[48, 49\] and lead to the observed signals.

In this paper our objective is to determine a primordial power spectrum which may lead to dipole modulation and hence hemispherical anisotropy. It is possible to find a power spectrum based on an inhomogeneous model \[3, 16, 31, 50–52\] which is consistent \[53\] with the observed temperature anisotropy. However it is not clear how an anisotropic model might lead to a dipole modulation. The simplest model that one might construct leads to quadrupolar and not dipolar modulation \[53\]. The basic problem can be understood by considering the two point correlations in real space. Let \(\tilde{\delta}(\mathbf{x})\) be the density fluctuations in real space. Their two point correlation function, \(F(\mathbf{\Delta}, \mathbf{X})\), may be expressed as,

\[
F(\mathbf{\Delta}, \mathbf{X}) = \langle \tilde{\delta}(\mathbf{x})\tilde{\delta}(\mathbf{x}') \rangle
\]

where, \(\mathbf{\Delta} = \mathbf{x} - \mathbf{x}'\) and \(\mathbf{X} = (\mathbf{x} + \mathbf{x}')/2\). We are interested in a correlation which is anisotropic and hence depends on \(\mathbf{\Delta}\) besides the magnitude \(\Delta = |\mathbf{\Delta}|\). It is clear from the definition of the correlation function that, in a classical framework, it must satisfy,

\[
\langle \tilde{\delta}(\mathbf{x})\tilde{\delta}(\mathbf{x}') \rangle = \langle \tilde{\delta}(\mathbf{x}')\tilde{\delta}(\mathbf{x}) \rangle
\]

Hence it can only be an even function of \(\mathbf{\Delta}\). The simplest anisotropic function is, therefore,

\[
F(\mathbf{\Delta}, \mathbf{X}) = f_1(\Delta) + B_{ij}\Delta_i\Delta_j f_2(\Delta)
\]

where \(B_{ij}\), \(i, j = 1, 2, 3\) are parameters. It is clear that such a model cannot give rise to a dipole modulation, which requires a term linear in \(\Delta_i\).

In this paper we argue that in a noncommutative space-time, a term linear in \(\Delta_i\) is permissible. The power spectrum that we are interested in is applicable at very early time, perhaps even the time when quantum gravity effects were not negligible. At that time, we cannot assume that space-time is commutative \[54–58\]. Its noncommutativity may be expressed as,

\[
[\hat{x}_\mu, \hat{x}_\nu] = i\theta_{\mu\nu}
\]

where, \(\theta_{\mu\nu}\) are parameters and the coordinate functions, \(\hat{x}_\mu(x)\), depend on the choice of coordinate system. In a particular coordinate system, we may set,

\[
\hat{x}_\mu(x) = x_\mu
\]

In \[59\] the authors assume that this preferred system is the comoving coordinate system. In general, the noncommutativity can appear quite complicated in different systems. The effect of noncommutativity on cosmology has been considered earlier \[59, 70\]. However its relationship with dipole modulation has not been pointed out so far.

In \[55\], we have determined the power spectrum corresponding to an inhomogeneous model and shown that its spectral index is consistent with zero. In the present paper we determine the power spectrum of an anisotropic model based on noncommutative space-time.
2 Correlations induced by Dipole Modulation

In this section we review the correlations between different multipoles which are induced by the dipole modulation model, Eq. [1]. We may expand the CMBR temperature as,

$$\Delta T(\hat{n}) = \sum_{lm} a_{lm} Y_{lm}(\hat{n})$$

where, $a_{lm}$ are the spherical harmonic coefficients. Their two point correlation function may be expressed as, \[71\],

$$\langle a_{lm}a_{l'm'} \rangle = \langle a_{lm}a_{l'm'} \rangle_{iso} + \langle a_{lm}a_{l'm'} \rangle_{dm}$$

where,

$$\langle a_{lm}a_{l'm'} \rangle_{iso} = C_l \delta_{ll'} \delta_{mm'}$$

$$\langle a_{lm}a_{l'm'} \rangle_{dm} = A (C_{l'} + C_l) \xi_{lm,l'm'}^0$$

Here $C_l$ is the standard angular power spectrum, $\langle a_{lm}a_{l'm'} \rangle_{dm}$ is the contribution due to the anisotropic dipole modulation and

$$\xi_{lm,l'm'}^0 = \delta_{m'm} \left[ \frac{(l - m + 1)(l + m + 1)}{(2l + 1)(2l + 3)} \delta_{l',l+1} + \sqrt{\frac{(l - m)(l + m)}{(2l + 1)(2l - 1)}} \delta_{l',l-1} \right].$$

Hence the model leads to correlations between multipoles, $l$ and $l+1$. We define the statistic \[71\],

$$S_H(L) = \sum_{l=l_{min}}^L C_{l} (l+1) \sum_{m=-l}^l a_{lm} a_{l+1,m}$$

We maximize the statistic by varying over the direction parameters. The resulting statistic is labelled as $S_H^\text{data}$. This provides a measure of the signature of anisotropy seen in data. This can be compared with a theoretical power spectrum model in order to fix its parameters.

3 Anisotropic Power Spectrum

The relationship between the temperature fluctuations, $\Delta T(\hat{n})$, and the primordial density perturbations, $\delta(\hat{k})$, can be expressed as,

$$\Delta T(\hat{n}) = \int d^3k \sum_l \langle -i \rangle (2l + 1) \delta(\hat{k}) \Theta_l(\hat{k}) P_l(\hat{k} \cdot \hat{n})$$

where $P_l(\hat{k} \cdot \hat{n})$ are the Legendre polynomials,

$$P_l(\hat{n} \cdot \hat{n}') = \frac{4\pi}{2l + 1} \sum_m Y_{lm}(\hat{n}) Y_{lm}^*(\hat{n}')$$
and \( \Theta_l(k) \) the transfer function. Here we assume an approximate form of the transfer function, \( \Theta_l(k) = \frac{1}{10} j_l(k\eta_0) \) \( \cite{72} \), where \( j_l \) is the spherical Bessel function.

We next propose the following form of the anisotropic power spectrum in real space,

\[
F(\Delta) = f_1(\Delta) + \hat{\lambda} \cdot \Delta f_2(\Delta) \tag{14}
\]

where \( \hat{\lambda} \) represents the preferred direction and \( f_1 \) and \( f_2 \) depend only on the magnitude \( \Delta \). Such a form is generally not permissible since the correlation function must satisfy Eq. 3. However this does not follow in noncommutative space-time \( \cite{59} \). In this case the relevant quantity is the deformed quantum field. Let \( \phi_0(\vec{x}, t) \) be a self-adjoint scalar field. The deformed quantum field is defined as \( \cite{59} \),

\[
\varphi_\theta = \varphi_0 e^{\frac{i}{\hbar} \hat{\theta} \wedge P} \tag{15}
\]

where

\[
\hat{\theta} \wedge P \equiv \hat{\theta}_{\mu} \theta^{\mu\nu} P_{\nu}. \tag{16}
\]

For a deformed field,

\[
\Phi_\theta(x, t)\Phi_\theta(x', t') \neq \Phi_\theta(x', t')\Phi_\theta(x, t) \tag{17}
\]

for space like separations \( \cite{59} \). Hence Eq. 3 does not follow and a correlation function, Eq. 14, which depends linearly on \( \vec{\Delta} \), is permissible.

The correlation function of the Fourier transform, \( \delta(\vec{k}) \), of \( \tilde{\delta}(\vec{x}) \) may be expressed as,

\[
\langle \delta(\vec{k})\delta^*(\vec{k}') \rangle = \int \frac{d^3X}{(2\pi)^3} \frac{d^3\Delta}{(2\pi)^3} e^{i(\vec{k}+\vec{k}') \cdot \vec{X}} e^{i(\vec{k}-\vec{k}') \cdot \vec{X}} F(\vec{\Delta}, \vec{X}) \tag{18}
\]

Using the model given in Eq. 14 we obtain,

\[
\langle \delta(\vec{k})\delta^*(\vec{k}') \rangle = \delta^3(\vec{k} - \vec{k}') \int d^3\Delta e^{i(\vec{k}+\vec{k}') \cdot \vec{\Delta}/2} F(\vec{\Delta}) \tag{19}
\]

This leads to,

\[
\langle \delta(\vec{k})\delta^*(\vec{k}') \rangle = \delta^3(\vec{k} - \vec{k}') P(k) [1 + i(\vec{k} \cdot \hat{\lambda}) g(k)] \tag{20}
\]

where the delta function arises due to spatial translational invariance and \( P(k) \) is the standard power spectrum,

\[
P(k) = k^{n-4} A_\phi/(4\pi) \tag{21}
\]

Here we set the parameters, \( n = 1 \) and \( A_\phi = 1.16 \times 10^{-9} \) \( \cite{72} \). In Eq. 20 \( g(k) \) is a real function which depends only on the magnitude \( k = |\vec{k}| \) and represents the violation of statistical isotropy. A more detailed fit is postponed to future work.

We may compare this for the power spectrum obtained in \( \cite{68} \), for the commutator,

\[
\frac{1}{2} [\phi_\theta(\vec{x}, \eta), \phi_\theta(\vec{y}, \eta)]_\pm = \frac{1}{2} (\phi_\theta(\vec{x}, \eta)\phi_\theta(\vec{y}, \eta) - \phi_\theta(\vec{y}, \eta)\phi_\theta(\vec{x}, \eta)) \tag{22}
\]
where $\eta$ is the conformal time. In Fourier space the correlator is given by Eq. 17 of [68], reproduced here for convenience,

$$
\frac{1}{2} < 0| [\phi_\theta (\vec{k}, \eta), \phi_\theta (\vec{k'}, \eta)] - | 0 >_{\text{horizon crossing}} = (2\pi)^3 P(k)i \sinh(\bar{H} \delta^0 \cdot \vec{k}) \delta(\vec{k} + \vec{k'})
$$

where we correct a crucial typographical error in [68] regarding the presence of the imaginary $i$. In this equation $P(k)$ is the standard power spectrum, given in Eq. 21 and $\bar{\theta}^0 = (\theta^{01}, \theta^{02}, \theta^{03})$ are three parameters. The argument of the Dirac delta function is $(\vec{k} + \vec{k'})$ instead of $(\vec{k} - \vec{k'})$ in Eq. 19 since here we take the correlation between $\phi_\theta(\vec{k}, \eta)$ and $\phi_\theta(\vec{k'}, \eta)$ instead of $\phi_\theta(\vec{k}, \eta)$ and $\phi_\theta(\vec{k'}, \eta)$. In the limit of small anisotropy parameters, $\bar{\theta}^0$, we can expand the sinh function and keep only the leading order term. Comparing with Eq. 20 we identify,

$$
g(k) = H k | \bar{\theta}^0 |
$$

and the direction, $\hat{\lambda} = \bar{\theta}^0$. The precise form of the correlation predicted within the framework of noncommutative geometry is model dependent. In particular, the basic equation, Eq. 5, depends on the choice of coordinates which obey this simple relationship. Here we don’t confine ourselves to a particular model and instead extract the anisotropic power directly from data. For this purpose, we assume the following parametrization of $g(k)$,

$$
g(k) = g_0 (k \eta_0)^{-\alpha}.
$$

We next compute the two point temperature correlations,

$$
\langle \Delta T(\hat{n}) \Delta T(\hat{n'}) \rangle = T_0^2 \int d^3 k \sum_{l,l'=0}^{\infty} (-i)^{l-l'} \times (2l+1)(2l'+1) \Theta_l(k) \Theta_{l'}(k) \times \delta_{l,l'} \delta_{m,m'} \langle \Delta T(\hat{n}) \Delta T(\hat{n'}) \rangle
$$

Setting $z$-axis as the preferred direction, we obtain $\hat{k} \cdot \hat{\lambda} = \cos \theta$. The correlations of the spherical harmonic coefficients can be expressed as,

$$
\langle a_{lm} a_{l'm'}^\ast \rangle = \int d\Omega_\hat{n} d\Omega_\hat{n'} Y_{lm}^\ast (\hat{n}) Y_{l'm'} (\hat{n'}) \langle \Delta T(\hat{n}) \Delta T(\hat{n'}) \rangle
$$

We finally obtain,

$$
\langle a_{lm} a_{l'm'}^\ast \rangle = \langle a_{lm} a_{l'm'}^\ast \rangle_{\text{iso}} + \langle a_{lm} a_{l'm'}^\ast \rangle_{\text{aniso}},
$$

where,

$$
\langle a_{lm} a_{l'm'}^\ast \rangle_{\text{iso}} = (4\pi)^2 \frac{9 T_0^2}{100} \delta_{l,l'} \delta_{m,m'} \int_0^\infty k^2 dk j_l^2 (k \eta_0) P_{iso}(k),
$$

$$
\langle a_{lm} a_{l'm'}^\ast \rangle_{\text{aniso}} = (-i)^{l-l'+1} (4\pi)^2 \frac{9 T_0^2}{100} G_{l,l'} \xi_{l,m,l'm'},
$$
\( \xi^0_{lm;lm'} \) is defined in Eq. 10 and

\[
G_{ll'} = \int_0^\infty k^2 dk P(k) j_l(k \eta_0) j_{l'}(k \eta_0) g(k).
\]  

(31)

Using Eq. 25, we obtain,

\[
G_{ll'} = \frac{g_0 A_\phi}{4 \pi} \int_0^\infty \frac{ds}{s^{1+\alpha}} j_l(s) j_{l'}(s).
\]  

(32)

Hence the anisotropic power spectrum, Eq. 14, leads to a correlation between \( l \) and \( l \pm 1 \). This allows us to obtain the theoretical prediction of the statistic, \( S_H(L) \), which can be compared to to \( S_{data}^H \) in order to determine the best fit value of power spectrum parameters, \( g_0 \) and \( \alpha \).

### 4 Data Analysis

We use the cleaned CMB map, ILC, based on WMAP 9 year data [73] (hereafter WILC9) and SMICA, provided by the PLANCK team [74]. We use the KQ85 and CMB-union mask in order to eliminate foreground contaminated regions for the WMAP and PLANCK data respectively. We generate a full sky map from the masked data by filling the masked portion with simulated data. The simulated CMB maps contain contribution due to the dipole modulation. We first generate a full sky CMB map by using isotropic and Gaussian random field. This map is generated at high resolution with \( N_{side} = 2048 \). The resulting map is multiplied with the dipole modulation term, \((1 + A \hat{\lambda} \cdot \hat{n})\) in order to generate a full sky map which has same properties as the real data. The data from this map is used to fill the gaps in the real map. This data map is downgraded to a lower resolution with \( N_{side} = 32 \) after applying appropriate Gaussian beam to smooth the mask boundary [71]. Hence any breaks that might be introduced at the boundary of the masked region get eliminated. We also use the SMICA in-painted map, in which the in-painting procedure [75, 76] has been used to reconstruct the masked regions, provided by the PLANCK team.

In order to determine the power spectrum parameters, we first set \( \alpha = 0 \) and determine the best fit value of \( g_0 \) over the entire multipole range \( 2 \leq l \leq 64 \). The maximum value of the statistic, \( S_H(L) \), in this multipole range is determined by maximizing over the preferred direction parameters. The resulting value of the statistic depends on the random realization used to fill the masked regions. Hence the maximum value of \( S_H(L) \) and \((\theta, \phi)\), are obtained by taking an average over 100 full sky data maps. Here \((\theta, \phi)\) are the direction parameters in polar coordinates. The resulting statistic is compared with theoretical prediction in order to determine the best fit value of \( g_0 \) with the constraint, \( \alpha = 0 \).

We next determine the best fit values of both \( g_0 \) and \( \alpha \) by splitting data into three multipole bins, \( l = 2 - 22, 23 - 43, 44 - 64 \). In this case we find it convenient to fix the direction parameters to be same as those obtained over the entire multipole range. As shown in [53], these show some dependence on the multipole bin, but the dependence is relatively mild and we ignore it for present analysis.
5 Results

For the entire multipole range, $2 \leq l \leq 64$ the best fit value of $g_0$ is found to be, 
$g_0 = 0.32 \pm 0.08$ and $g_0 = 0.30 \pm 0.08$ for WILC9 and SMICA respectively. Here 
we have assumed that the spectral index $\alpha = 0$. Hence the function, $g(k)$ is 
equal to a constant, $g_0$. We have verified that the results obtained for the case 
of the SMICA in-painted map are in good agreement with those for SMICA and 
WILC9.

We next extract the function, $g(k)$, using data in the three multipole bins, 
$l = 2 - 22, 23 - 43$ and $l = 44 - 64$. We parametrize it in terms of $g_0$ and 
the spectral index $\alpha$. Setting $\alpha = 0$, the best fit value of $g_0$ is found to be, 
$g_0 = 0.32 \pm 0.06$ with $\chi^2 = 0.45$ for WILC9 and $g_0 = 0.30 \pm 0.05$ with $\chi^2 = 0.41$ 
for SMICA. Hence we find that a zero spectral index for the anisotropic part of 
the power spectrum provides a good fit to data. The resulting fit is shown in 
Fig. 1 as the dotted line. Allowing a non-zero value of $\alpha$ we find that the $1\sigma$ 
limit on this parameter is, $-0.13 < \alpha < 0.15$ and $-0.16 < \alpha < 0.19$ for WILC9 
and SMICA respectively.

![Figure 1: The statistic, S_{H}^{data}, as a function of the multipole l for WILC9. Here the statistic in the three bins is extracted by fixing the direction parameters to be equal to the mean direction over the entire multipole range. The dotted line corresponds to the theoretical fit corresponding to $\alpha = 0, g_0 = 0.32 \pm 0.06$.](image)

6 Conclusion

We show that an anisotropic power spectrum model, derived on the basis of 
noncommutative geometry provides a description of the observed hemispherical 
anisotropy. This anisotropy can be parametrized in terms of a dipole modulation 
model, which leads to correlations among the multipoles corresponding to $l$ and 
$l+1$. The noncommutative anisotropic power spectrum model also leads to such 
a correlation. The anisotropic power spectrum is parameterized by the function, 
g(k). We first fit the data by assuming that $g(k)$ is a constant equal to $g_0$. We 
determine the value of $g_0$ by making first making a fit over the entire multipole 
range, $2 - 64$. The best fit value is found to be $g_0 = 0.32 \pm 0.08$. We next 
assume a power law form of $g(k) = g_0(k)\eta_0^{\alpha}$ and extract the corresponding 
amplitude, $g_0$ and spectral index $\alpha$ by making a fit over the three multipole
bins, $l = 22, 23 - 43$ and $l = 44 - 64$. Setting $\alpha = 0$, the best fit leads to $\alpha = 0.32 \pm 0.06$ for WILC9. This leads to a good fit to data with $\chi^2 = 0.45$. Hence the data suggests that the anisotropic power, $g(k)$, is independent of $k$. Furthermore we find the one sigma limit on $\alpha$ to be, $-0.13 < \alpha < 0.15$ for WILC9.

We conclude that the observed hemispherical anisotropy might represent the first observational signature of noncommutative geometry and hence of quantum gravity.

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