Special Case of Stability Condition on 4-Dimensional Upper-Half Plane Metric

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Abstract. In this paper, we consider special case of four-dimensional upper-half plane metric, which is a generalization of Joyce metric. The metric will only depend on one variable instead of two. We analyse the metric through Riemann-Hilbert action with general potential \( V(B) \) and find its stability condition through the Hessian matrix of Hamiltonian. In this special case, we find that the metric cannot be Einstein in critical point.

1 Introduction

Joyce metric, which have the self-duality properties, is a product of upper-half plane and torus manifold. The self-duality properties of Weyl tensor remain in the event of Conformal transformation. Previous works focus on the metric conditions imposed on self-dual metric which have torus symmetry. Some results are the differential equation for conformal factor \( W \) which has specific solution so that the self-dual metric can be Einstein.

While these results give some method on how to construct self-dual Einstein metric, they don't give insight on how these metric is constructed. The metric in discussion is already a combination of upper-half plane and torus symmetry. While these results may be useful, we can only use it in metric with Joyce form. We would like to know what happened when we generalize the problem without its symmetry, in other words, what happened when the torus symmetry is abandoned.

In this paper, we focus on upper-half plane metric, which is a generalization of Joyce metric. We abandon the torus symmetry and focus on stability conditions for the metric rather than metric conditions. To simplify the problem, we consider a special case where the scale function is a function of the other one. We start from Riemann-Hilbert action and work through the Lagrangian and Hamiltonian to find the Hessian matrix. We also discuss some of the properties of potential in critical points.

2 The Metric

With \( \rho, \eta, t \) and \( x \) as coordinates, we define upper-half plane metric as follows:

\[
g = A^2(\rho, \eta)(d\rho^2 + d\eta^2) + B^2(\rho, \eta)(\varepsilon dt^2 + dx^2),
\]  \hspace{1cm} (2.1)

where \( \varepsilon = \pm 1 \). Notice that the metric is static and only depends on \( \rho \) and \( \eta \). It can also become either Riemannian or pseudo-Riemannian depending on the value of \( \varepsilon \). In our special case, we define...
$A = A(B)$ where $B = B(\rho)$, making the metric only depends on one variable. The metric then becomes
\[ g = A^2(B)(d\rho^2 + d\eta^2) + B^2(\rho)(c dt^2 + dx^2), \] (2.2)
where $\varepsilon = \pm 1$. To calculate Riemann-Hilbert action, we need the Ricci scalar of the metric. Using
Levi-Civita connection with conventional method or Cartan calculus, one can calculate the Ricci scalar which has the following value:
\[ R = \frac{2}{A^4} \left[ A B B^2 - A \left( A_{\rho\rho} B_{\rho}^2 + A_{\eta\eta} B_{\eta}^2 \right) \right] - \frac{2}{A^2 B^2} \left( B_{\rho}^2 + 2 B_{\rho\rho} \right), \] (2.3)
where $A_{\rho} = \partial A / \partial \rho$. Here, we use subscript as the simplified version of derivatives. For example, in
above equation, $B_{\rho} = \partial B / \partial \rho$. We also use $\Delta$ as the notation for Laplacian. In other words, $\Delta = \nabla^2$.

3 Riemann-Hilbert Action

We define general potential $V(B)$ and use it in Riemann-Hilbert action:
\[ S = \int \sqrt{\varepsilon g} \left[ R - V(B) \right] d^4x, \] (3.1)
where $\sqrt{\varepsilon g}$ is the square root of the determinant of the metric, which, in our case, is given by $A^2 B^2$.
Thus, the action for the upper-half plane metric is given by:
\[ S = K \int \left[ \frac{2B^2}{A} \left[ A_{\rho}^2 B_{\rho}^2 - A \left( A_{\rho\rho} B_{\rho}^2 + A_{\eta\eta} B_{\eta}^2 \right) \right] - 2 \left( B_{\rho}^2 + 2 B_{\rho\rho} \right) - 2 \Delta B - V(B) \right] dS, \] (3.2)
where $V = \sqrt{\varepsilon g} V = A^2 B^2 V$ and $K$ is the area of $t$ and $x$ coordinates, since the Ricci scalar and
potential only depend on $t$ and $x$. The term inside the integral is called effective action because we only
integrate with respect to $dS$, which is the area element of $\rho$ and $\eta$ coordinate. Thus we have the
effective Lagrangian,
\[ L_{\text{eff}} = \frac{2B^2}{A} \left[ A_{\rho}^2 B_{\rho}^2 - A \left( A_{\rho\rho} B_{\rho}^2 + A_{\eta\eta} B_{\eta}^2 \right) \right] - 2 \left( B_{\rho}^2 + 2 B_{\rho\rho} \right) - 2 \Delta B - V(B). \] (3.3)

The Lagrangian is a scalar quantity. In other words, its value is unchanged regardless of the
coordinate system. The properties can be used when we perform coordinate transformation on the
metric. We can simplify the Lagrangian density further by remembering that we evaluate on the
surface. Further expansion, calculation, and simplification leads to
\[ L_{\text{eff}} = 4 \frac{B}{A} A_{\rho} B_{\rho}^2 - V(B). \] (3.4)

4 Equation of Motion

We now use Euler-Lagrange equation
\[ \frac{\partial L_{\text{eff}}}{\partial B} - \frac{d}{d\rho} \frac{\partial L_{\text{eff}}}{\partial B} = 0 \] (4.1)
to obtain the equation of motion. Calculating the first derivative of Lagrangian with respect to $B$ and
$B$, we have
\[ \frac{\partial L_{\text{eff}}}{\partial B} = \frac{4}{A} A_{\rho} B_{\rho}^2 - 4 \frac{B}{A^2} A_{\rho\rho} B_{\rho} + 4 \frac{B}{A} A_{\eta\eta} B^2 - V_{\rho}, \]
\[ \frac{\partial L_{\text{eff}}}{\partial B} = 8 \frac{B}{A} A_{\rho} B, \]
\[ \frac{\partial}{\partial \rho} \left( \frac{\partial L_{\text{eff}}}{\partial B} \right) = 8 \frac{A}{A} A_{\rho} B^2 - 8 \frac{B}{A} A_{\rho\rho} B_{\rho} + 8 \frac{B}{A} A_{\eta\eta} B^2 + 8 \frac{B}{A} A_{\rho} B. \] (4.2)
By combining them into Euler-Lagrange equation, we have the equation of motion:

$$
\frac{4 B}{A} A_y^2 B - 4 \frac{A_y}{A} B^2 + \frac{8 B}{A} A_y B + V_y = 0. \tag{4.3}
$$

5 Hamiltonian and Stability Conditions

Define $B \equiv B_y$. We can calculate the momentum:

$$
p = \frac{8 B}{A} A_y B, \tag{5.1}
$$

with its inverse equation:

$$
B = \frac{A}{8 B A_y} p. \tag{5.2}
$$

Then the Hamiltonian is

$$
H_{eff} = \frac{A}{16 B A_y} p^2 + V(B). \tag{5.3}
$$

The first derivatives of Hamiltonian are

$$
\frac{\partial H_{eff}}{\partial p} = \frac{A}{8 B A_y} p, \tag{5.4}
$$

$$
\frac{\partial H_{eff}}{\partial B} = p^2 \left[ 1 - \frac{A}{B A_y} - \frac{A}{B A_y} A_{BB} \right] + V_y,
$$

where $V_y \equiv \partial V / \partial B$ and $A_{BB} \equiv \partial^2 A / \partial B^2$. Then the Hessian matrix is given by

$$
H = \begin{pmatrix}
\frac{A}{8 B A_y} \\
p \left[ 1 - \frac{A}{B A_y} - \frac{A}{B A_y} A_{BB} \right] \\
p^2 \left[ -\frac{1}{8 B^2} + \frac{A}{8 B^3} A_y - \frac{A_{BB}}{16 B A_y} + \frac{A A_{BB}}{16 B A_y} + \frac{A A_{BB}^2}{8 B A_y} + \frac{A A_{BB}^3}{16 B A_y} + V_y \right]
\end{pmatrix} \tag{5.5}
$$

In critical point, we have

$$
\frac{\partial H_{eff}}{\partial p} = 0, \tag{5.6}
$$

$$
\frac{\partial H_{eff}}{\partial B} = 0. \tag{5.7}
$$

From equation (5.6), we have $\bar{p} = 0$, because $\bar{A} \neq 0$. From equation (5.7), we have $\bar{V}_y = 0$. Thus, we have the Hessian matrix in critical point:

$$
H = \begin{pmatrix}
\bar{A} \\
0 \\
0
\end{pmatrix}, \tag{5.8}
$$

where the bar above the variable signifies that the variable is evaluated in a critical point. The matrix above is already diagonal, thus the eigenvalues are

$$
\lambda_1 = \frac{\bar{A}}{8 B A_y}, \tag{5.9}
$$

$$
\lambda_2 = \bar{V}_{BB}.
$$

For a stable system, we must have $\lambda_1$ and $\lambda_2$ greater than zero. Thus, we have the stability condition:
6 Analysis

In this section, we analyze or result regarding the stability condition and critical points. The first section focuses on analysis near the critical point. We will expand the Hamiltonian near critical point and find the equation of motion. The second and third section will focus on imposing metric condition in its 2 and 4-dimensional metric.

6.1 Analysis Near Critical Point

Near critical point, we can expand the Hamiltonian using Taylor series:

\[
H(p, B) = H_0 + \frac{\partial H}{\partial B} (B - \overline{B}) + \frac{1}{2} \frac{\partial^2 H}{\partial p^2} p^2 + \frac{1}{2} \frac{\partial^2 H}{\partial B^2} (B - \overline{B})^2 + \frac{1}{2} \frac{\partial^2 H}{\partial p \partial B} p (B - \overline{B}) + ...,
\]

where we have redefine \( H(0, \overline{B}) = H_0 \). The first derivative term are zero on critical point. The mixed second derivative is also zero on critical point. Thus, up until the second order, we have the Hamiltonian expansion:

\[
H(p, B) \approx H_0 + \frac{1}{2} \frac{\partial^2 H}{\partial p^2} p^2 + \frac{1}{2} \frac{\partial^2 H}{\partial B^2} (B - \overline{B})^2.
\]

Inserting the value from Hessian matrix at critical point, we have

\[
H(p, B) \approx H_0 + \frac{\overline{A}}{2 \cdot 8BA_g} p^2 + \frac{1}{2} \overline{F}_{ab} (B - \overline{B})^2.
\]

By inserting this result to Hamilton’s equations, we have

\[
p = -\frac{\partial H}{\partial B} = -\overline{F}_{ab} (B - \overline{B}), \quad (6.4)
\]

\[
B = \frac{\partial H}{\partial p} = \frac{\overline{A}}{8BA_g} p. \quad (6.5)
\]

By rearranging equation (6.5) and inserting it into equation (6.4), we have

\[
\frac{d}{d\rho} \left( \frac{8BA_g}{\overline{A}} B \right) = -\overline{F}_{ab} (B - \overline{B}), \quad (6.6)
\]

\[
B = -\frac{\overline{A} F_{ab}}{8BA_g} (B - \overline{B}).
\]

Define

\[
h^2 = \frac{\overline{A} F_{ab}}{8BA_g}, \quad (6.7)
\]

where we use \( h^2 \) to ensure the constant is positive, as stated in the stability condition. Then we have the differential equation for B:

\[
B + h^2 B = h B. \quad (6.8)
\]

The solution for this equation is given by:

\[
B(\rho) = C \cos(h\rho + \phi) + \overline{B}, \quad (6.9)
\]

where \( C \) and \( \phi \) are constants. Looking at equation (6.5), we can solve for \( p \):

\[
p(\rho) = -\sqrt{\frac{8BA_g F_{ab}}{\overline{A}}} \left[ C \sin(h\rho + \phi) \right]. \quad (6.10)
\]
For the sake of further analysis, we will replace all sinusoidal function with $B$, thus by rearranging, taking derivative, and using trigonometric identity, we have the following results:

$$\cos(h\rho + \phi) = \frac{B - \bar{B}}{C},$$

$$\sin(h\rho + \phi) = \frac{\sqrt{C^2 - (B - \bar{B})^2}}{C},$$

$$B_h = -h\sqrt{C^2 - (B - \bar{B})^2},$$

$$B_{\rho\rho} = -h^2(B - \bar{B}).$$

6.2 Analysis on 2-Dimensional Upper-Half Plane Curvature

The 2-dimensional upper-half plane metric given by

$$g = A^2(B)(d\rho^2 + d\eta^2)$$

has the following non-zero Ricci tensor components:

$$R_{11} = \frac{1}{A^2}(A^2 + B^2 - AA_{bb}B_{\rho}^2 - AA_{bb}B_{\eta\eta}),$$

$$R_{22} = \frac{1}{A^2}(A^2 + B^2 - AA_{bb}B_{\rho}^2 - AA_{bb}B_{\eta\eta}),$$  

with the following Ricci scalar:

$$R = \frac{2}{A^2}(A^2B_{\rho}^2 - AA_{bb}B_{\rho}^2 - AA_{bb}B_{\eta\eta}).$$

By inserting our solution of $B(\rho)$, and imposing Einstein condition for Ricci tensor and Ricci scalar constant condition for Ricci scalar, we have the Einstein condition:

$$\frac{\Lambda}{h^2}A^4 = \left[C^2 - (B - \bar{B})^2\right](A^2 - AA_{bb}) + h^2(B - \bar{B})AA_{bb},$$

and the Ricci scalar constant condition:

$$\frac{K}{2h^2}A^4 = \left[C^2 - (B - \bar{B})^2\right](A^2 - AA_{bb}) + h^2(B - \bar{B})AA_{bb},$$

where $\Lambda$ and $K$ are constants. These two equations are basically the same. We can also conclude that $\Lambda = K / 2$.

6.3 Analysis on Curvature

By using Levi-Civita connection, we can calculate the Ricci tensor and Ricci scalar of the static upper half plane metric

$$g = A^2(B)(d\rho^2 + d\eta^2) + B^2(\rho)(d\rho^2 + d\eta^2).$$

The non-zero Ricci tensor components are given by

$$R_{11} = \frac{1}{A^2}(A^2B_{\rho}^2 - AA_{bb}B_{\rho}^2 - AA_{bb}B_{\eta\eta}) + \frac{2}{AB}(A^2B_{\rho}^2 - AB_{\eta\eta}),$$

$$R_{22} = \frac{1}{A^2}(A^2B_{\rho}^2 - AA_{bb}B_{\rho}^2 - AA_{bb}B_{\eta\eta}) + \frac{2}{AB}(A^2B_{\rho}^2 - AB_{\eta\eta}),$$

$$R_{33} = -\frac{e}{A^2}(B_{\rho}^2 + BB_{\eta\eta}),$$
The Ricci scalar is given by
\[ R_{44} = -\frac{1}{A^2}(B_\rho^2 + BB_{\rho\rho}). \] (6.21)

The Ricci scalar is given by
\[ R = \frac{2}{A^4}(A^2 b^\rho B_\rho^2 - A b b B_\rho^2 - A A b B_{\rho\rho}) - \frac{2}{A^2 B^2}(B_\rho^2 + 2BB_{\rho\rho}). \] (6.22)

By inserting the solution of \( B \), and imposing Einstein condition, we have
\[ \Lambda^2 B = \left[h^2 C^2 - h^2 (B - B)^2 \right] \Lambda^2 B \left( B A^2 - A B A b + 2 B A A b \right) + h^2 \left( B - B \right) \left( B A A b + 2 A^2 \right), \] (6.23)
\[ \Lambda^2 B = \left[h^2 C^2 - h^2 (B - B)^2 \right] \Lambda^2 B \left( B A^2 - A B A b - 2 A A b \right) + h^2 \left( B - B \right) \left( B A A b \right), \] (6.24)
\[ \frac{\Lambda^2 B^2}{h^2} = 2 B + B B + B^2 - C^2. \] (6.25)

These equations have no solution for it is an overdetermined system of equations, with three equation and two variable (\( A \) and \( B \)). We will prove it in the following paragraph.

We will now do a little manipulation to simplify the equations. Combining equation (6.23) and (6.24), we have
\[ 2 \left[ C^2 - (B - B)^2 \right] A B + (B - B) A = 0. \] (6.26)

Rearranging equation (6.25), we have the explicit expression of \( A(B) \):
\[ A(B) = \pm \frac{h}{B} \left[ 2 B - B B + B^2 - C^2 \right]^{1/2}, \] (6.27)
which is inconsistent with equation (6.26). Thus, we conclude that the metric cannot be Einstein in critical point. This result concludes that there is no stable Einstein metric in upper-half plane where the scale factor of the two coordinates is a function of the scale factor of the other two coordinates.

We now turn our attention on the constant Ricci scalar condition. This condition cannot be overdetermined because there’s only one equation. The constant Ricci scalar condition gives:
\[ A^4 K = 2 A^2 B_\rho^2 - 2 A A b B_\rho^2 - 2 A A b B_{\rho\rho} - 2 \frac{A^2 B_\rho^2}{B^2} - 4 \frac{A B}_{\rho\rho}. \] (6.28)

We can rearrange the equation according to the derivative of \( B \):
\[ A^4 K = 2 B_\rho^2 \left[ A^2 b - A A b - \frac{A^2}{B^2} \right] - 2 A B_{\rho\rho} \left[ A b + \frac{A}{B} \right]. \] (6.29)

In critical point, the constant Ricci Scalar equation becomes:
\[ A^4 B^2 K = \left[ h^2 C^2 - h^2 (B - B)^2 \right] \left[ B^2 A^2 - A^2 B^2 A b - A^2 \right] + h^2 (B - B) \left( A B^2 A b + 2 A^2 B \right). \] (6.30)

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