Penalty Alternating Direction Methods for Mixed-Integer Optimization: A New View on Feasibility Pumps

Björn Geißler¹, Antonio Morsi¹, Lars Schewe¹, Martin Schmidt²

Abstract. Feasibility pumps are highly effective primal heuristics for mixed-integer linear and nonlinear optimization. However, despite their success in practice there are only few works considering their theoretical properties. We show that feasibility pumps can be seen as alternating direction methods applied to special reformulations of the original problem, inheriting the convergence theory of these methods. Moreover, we propose a novel penalty framework that encompasses this alternating direction method, which allows us to refrain from random perturbations that are applied in standard versions of feasibility pumps in case of failure. We present a convergence theory for the new penalty based alternating direction method and compare the new variant of the feasibility pump with existing versions in an extensive numerical study for mixed-integer linear and nonlinear problems.

Due to their practical relevance, mixed-integer nonlinear problems (MINLPs) form a very important class of optimization problems. One important part of successful algorithms for the solution of such problems is finding feasible solutions quickly. For this, typically heuristics are employed. These can be roughly divided into heuristics that improve known feasible solutions (e.g., local branching [25] or RINS [16]) and heuristics that construct feasible solutions from scratch. This article discusses a heuristic of the latter type: The algorithm of interest in this article is the so-called feasibility pump that has originally been proposed by Fischetti et al. in [24] for MIPs and that has been extended by many other researchers, e.g., in [1–3, 6, 7, 17, 19, 25, 26]. In addition, feasibility pumps have also been applied to MINLPs during the last years; see, e.g., [1, 8, 9, 13, 15, 39, 40]. A more detailed review of the literature about feasibility pumps is given in Section 1. For a comprehensive overview over primal heuristics for mixed-integer linear and nonlinear problems in general, we refer the interested reader to Berthold [4, 5] and the references therein.

In a nutshell, feasibility pumps work as follows: given an optimal solution of the continuous relaxation of the problem, the methods construct two sequences. The first one contains integer-feasible points, the second one contains points that are feasible w.r.t. the continuous relaxation. Thus, one has found an overall feasible point if these sequences converge to a common point. To escape from situations where the construction of the sequences gets stuck and thus do not converge to a common point, feasibility pumps usually incorporate randomized restarts.

The feasibility pumps described in the literature are difficult to analyze theoretically due to the use of random perturbations. These random perturbations are, however, crucial to the practical performance of the methods. The main object of the existing theoretical analysis is the idealized feasibility pump, i.e., the method without random perturbations. This is the method analyzed in the publications [17].

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To be more specific, De Santis et al. show in [17] that idealized feasibility pumps are a special case of the Frank–Wolfe algorithm applied to a suitable chosen concave and nonsmooth merit function and Boland et al. showed in [6] that idealized feasibility pumps can be seen as discrete versions of the proximal point algorithm. However, the analysis presented in both mentioned publications cannot be applied to feasibility pumps that use random perturbations. Recently, there has been progress in the understanding of the randomization step. In [19], the authors show that changing the randomization step can yield a method that, at least for certain instances, is guaranteed to produce feasible solutions with high probability.

Our approach is similar to these publications. We show that the idealized variant can be seen as an alternating direction method (ADM) applied to a special reformulation of the mixed-integer problem at hand. To this end, we extend the known theory on feasibility pumps by applying the convergence theory of general ADMs. We then go one step further: The necessity to use random perturbations comes from the need to escape from undesired points. We replace these random perturbations of the original feasibility pump by a penalty framework. This allows us to view feasibility pumps as penalty based alternating direction methods—a new class of optimization methods for which we also present convergence theory. In summary, we are able to give a convergence theory for a class of feasibility pumps that incorporates deterministic restart rules. Another advantage is that our method can be presented in a quite generic way that comprises both the case of mixed-integer linear and nonlinear problems.

We further give extensive computational results to show that our replacement of the random restarts also works well in practice. In particular, our method compares favorably with published variants of feasibility pumps for MIPs and MINLPs w.r.t. solution quality.

The paper is organized as follows: In Section 1 we review the main ingredients of feasibility pumps and give a more detailed literature survey. Afterward, we discuss general ADMs in Section 2 and show that idealized feasibility pumps can be seen as ADMs applied to certain equivalent reformulations of the original problem. In Section 3, we then present a penalty ADM, prove convergence results, and show how this new method can be used to obtain a novel feasibility pump algorithm that replaces random restarts with penalty parameter updates. In Section 4 we discuss important implementation issues and Section 5 finally presents an extensive computational study both for MIPs and MINLPs.

1. Feasibility Pumps

In this section we give an overview over feasibility pump algorithms for mixed-integer linear problems (MIPs) as well as for mixed-integer nonlinear problems (MINLPs). We start with the MIP case in Section 1.1 and afterward discuss generalizations for nonlinear problems in Section 1.2. General surveys on this topic can be found in Berthold [4] and Bonami et al. [10].

1.1. Feasibility Pumps for Mixed-Integer Linear Problems. The feasibility pump has been introduced by Fischetti et al. [24] for binary MIPs and has been extended to general MIPs by Bertacco et al. [3]. The goal of the feasibility pump is to find a feasible point of a mixed-integer linear problem of the general form

\[
\begin{align*}
\min \quad & c^\top x \\
\text{s.t.} \quad & Ax \geq b, \\
& x_i \in \mathbb{Z} \quad \text{for all } i \in I,
\end{align*}
\]
Algorithm 1 The basic feasibility pump for 0-1-MIPs

1: Compute $\tilde{x}^0 \in \text{argmin}\{c^\top x : x \in P\}$. 
2: if $\tilde{x}^0$ is integer feasible then 
3: \textbf{return} $\tilde{x}^0$ 
4: end if 
5: Set $\tilde{x}^0 = \lfloor \tilde{x}^0 \rfloor$ and $k \leftarrow 0$. 
6: while not termination condition do 
7: Compute $\tilde{x}^{k+1} \in \text{argmin}\{\|x_I - \tilde{x}_I^k\|_1 : x \in P\}$. 
8: if $\tilde{x}^{k+1}$ is integer feasible then 
9: \textbf{return} $\tilde{x}^{k+1}$ 
10: end if 
11: Set $\tilde{x}^{k+1} = \lceil \tilde{x}^{k+1} \rceil$. 
12: if algorithm stalls or cycles then 
13: perturb $\tilde{x}^{k+1}$ 
14: end if 
15: Set $k \leftarrow k + 1$. 
16: end while

where $c \in \mathbb{R}^n$, $b \in \mathbb{R}^m$, $A \in \mathbb{R}^{m \times n}$, and $\emptyset \neq I \subseteq \{1, \ldots, n\}$. Moreover, we assume that variable bounds $l_i \leq x_i \leq u_i$ with $-\infty < l_i \leq u_i < \infty$ for all $i \in I$ are part of $Ax \geq b$. We refer to the polyhedron of the LP relaxation of \textbf{[1]} by $P$, i.e., $P = \{x \in \mathbb{R}^n : Ax \geq b\}$. Throughout the paper we assume that $P \neq \emptyset$. The main idea of the feasibility pump is to create two sequences $(\tilde{x}^k)$ and $(\tilde{x}^k)$ such that $\tilde{x}^k \in P$ and $\tilde{x}^k$ is integer feasible, i.e., $\tilde{x}_i^k \in \mathbb{Z}$ for all $i \in I$. In addition, the construction of the sequences is tailored to minimize the distance of the pairs $\tilde{x}^k$ and $\tilde{x}^k$. The algorithm terminates after a given time, after an iteration limit has been reached, or if the distance is zero, i.e., $\tilde{x}^k = \tilde{x}^k$. In the latter case, the algorithm terminates with an MIP-feasible point, i.e., a point that is both in $P$ and integer feasible. We now describe the method in detail for the case of 0-1-MIPs, i.e., we replace $x_i \in \mathbb{Z}$ by $x_i \in \{0, 1\}$ in \textbf{[1]}. The initial point $\tilde{x}^0$ is computed to be an optimal solution of the LP relaxation of \textbf{[1]}, i.e., $\tilde{x}^0 \in \text{argmin}\{c^\top x : x \in P\}$, and the initial point $\tilde{x}^0$ of the other sequence is the rounding $\lfloor \tilde{x}^0 \rfloor$ of the integer components of $\tilde{x}^0$. Note that the rounding operator $\lfloor \cdot \rfloor$ only rounds integer components, i.e.,

\[ [x_i] := \begin{cases} [x_i + 0.5], & \text{if } i \in I, \\ x_i, & \text{otherwise.} \end{cases} \]

From then on, in each iteration $k$ the new iterate $\tilde{x}^{k+1}$ is the nearest point (w.r.t. the integer components) to $\tilde{x}^k$ in the $\ell_1$ norm, i.e.,

$\tilde{x}^{k+1} \in \text{argmin}\{\|x_I - \tilde{x}_I^k\|_1 : x \in P\}$

and $\tilde{x}^{k+1} := \lfloor \tilde{x}^{k+1} \rfloor$. Here and in what follows, $x_I$ denotes the sub-vector of $x$ only consisting of the components indicated by the index set $I$. After every rounding step a cycle and a stalling test decides whether a random perturbation of the integer part of $\tilde{x}^k$ is applied. The details can be found in Bertacco et al. \textbf{[3]} and Fischetti et al. \textbf{[24]}. A formal listing of the basic feasibility pump for binary MIPs is given in Algorithm \textbf{[4]}. In what follows, Line \textbf{[7]} of Algorithm \textbf{[1]} is referred to as the projection step and Line \textbf{[11]} is called the rounding step. Note that the projection step can be written as a linear program by reformulating the $\ell_1$ norm objective as

\[ \|x_I - \tilde{x}_I\|_1 = \sum_{i \in I, x_i = 0} x_i + \sum_{i \in I, x_i = 1} (1 - x_i). \]
This is also possible for general MIPs but then requires the introduction of auxiliary 
variables and constraints; see Bertacco et al. [3] for the details. The original feasibility 
pump is very successful in quickly finding feasible solutions. 
However, these solutions are often of minor quality. Thus, improvements of the 
original version with the goal of developing variants of the feasibility pump that are 
comparable in running time as well as success rate and provide solutions of better 
quality are studied in many publications. Fischetti and Salvagnin [26] improved the 
rounding step by replacing the simple rounding \( \tilde{x}^k = \lceil \bar{x}^k \rceil \) with a procedure based 
on constraint propagation. Other strategies of improving the rounding step are 
given in Baena and Castro [2], where simple rounding is replaced with rounding of 
different candidate points on a line segment between the solution of the projection 
step and the analytic center of the LP polyhedron, and in Boland et al. [7], where 
an integer line search is applied to find integer feasible points that are closer to \( P \) 
than the points achieved by simple rounding. In contrast to these approaches, 
Achterberg and Berthold [1] improved the projection step in order to achieve feasible 
solutions of better quality by replacing the \( \ell_1 \) norm objective 
\( \|x_I - \tilde{x}_I\|_1 \) with a convex combination of this distance measure and the original objective function: 
\[
(1 - \alpha)\|x_I - \bar{x}_I\|_1 + \alpha \beta c^T x.
\]
Here, \( \alpha \in [0, 1] \) is the convex combination parameter and \( \beta \in \mathbb{R}_{>0} \) is a problem 
data depending scaling parameter; see [1] for the details. 

None of the papers cited so far contains any theoretical results on the feasibility 
pump. This situation changed with a remark in Eckstein and Nediak [23] noticing that the idealized feasibility pump for binary 
MIPs can be interpreted as a Frank–Wolfe algorithm (see Frank and Wolfe [27]) applied to the minimization of 
a concave and nonsmooth objective function over a polyhedron. De Santis et al. [17] seized this idea and proved this correspondence, yielding the first theoretical 
result for the feasibility pump. To be more precise, they used the result shown by Mangasarian in [36] that the Frank–Wolfe algorithm applied to the above given 
situation terminates after a finite number of iterations and returns a so-called vertex 
stationary point of the problem. We discuss the relation of this result with our 
results in more detail in Section 2.1. In [18], De Santis et al. generalized their results 
to general MIPs. Another theoretical result is presented by Boland et al. in [6], 
where it is shown that the idealized feasibility pump can be interpreted as a discrete 
version of the proximal point algorithm. We remark that both cited theoretical 
investigations only consider the case of idealized feasibility pumps, i.e., the variants 
of feasibility pumps without random perturbations used to handle cycling or stalling 
isues.

1.2. Feasibility Pumps for Mixed-Integer Nonlinear Problems. We now 
turn to feasibility pumps for mixed-integer nonlinear problems of the form 
\[
\min_x \ f(x) \quad \text{(2a)} \\
\text{s.t.} \quad h(x) \geq 0, \quad \text{(2b)} \\
x_i \in \mathbb{Z} \cap [l_i, u_i] \quad \text{for all } i \in I, \quad \text{(2c)}
\]
where the objective function \( f : \mathbb{R}^n \to \mathbb{R} \) and the constraints function \( h : \mathbb{R}^n \to \mathbb{R}^p \) 
are continuous. The feasible set of the NLP relaxation is denoted by \( \Omega^r := \{ x \in \mathbb{R}^n : h(x) \geq 0 \} \). Again, we assume that \( \Omega^r \neq \emptyset \) holds and that \( \Omega^r \) is compact. Lastly, we 
assume that all bounds on the discrete variables are finite, i.e., \( -\infty < l_i \leq u_i < \infty \) 
for all \( i \in I \). The MINLP [3] is said to be convex if \( f \) and \(-h\) are convex. 

For convex problems, the direct generalization of the feasibility pump for MIPs 
to MINLPs is given in Bonami and Gonçalves [9]: the projection step LP is replaced 
by an NLP and the rounding step stays the same, i.e., \( x^{k+1} := \lceil x^k \rceil \). Cycling and
stalling issues are again handled by random perturbations of the integer components of \( x \). Variants of this method are then deduced by modifying the rounding step and by replacing the \( \ell_1 \) norm in the projection step. The feasibility pump for convex MINLP proposed in Bonami et al. [8] significantly differs from the version in [9] because it replaces the simple rounding step by a MIP relaxation of \( \ell_2 \) that is successively tightened by adding outer approximation cuts (see Duran and Grossmann [22]) based on the NLP feasible solutions obtained by solving the \( \ell_2 \) norm projection steps. Bonami et al. study two versions of their algorithm; a basic and an enhanced one. For their basic version it is shown that it cannot cycle if the LICQ holds for the NLP relaxation. For their enhanced version it is shown that it cannot cycle and that it is an exact method for solving convex MINLPs if all integer variables are bounded.

For nonconvex MINLPs the convergence results of Bonami et al. [8] do not hold because the outer approximation cuts cannot be applied to the nonconvex NLP relaxation and because the projection step is now a nonconvex problem, which is too hard to be solved to global optimality in general. The article D’Ambrosio et al. [15] is the first presentation of a feasibility pumps for nonconvex MINLPs. The above mentioned issues are “resolved” by solving the nonconvex projection step NLP via a multistart heuristic using local NLP solvers and the rounding step is realized by a MIP using outer approximation cuts if they are globally valid for the nonconvex problem. In the subsequent paper [14], D’Ambrosio et al. interpreted feasibility pumps for general nonconvex MINLPs as variants of successive projection methods (SPMs). Despite their strong similarity, the authors observe that typical feasibility pumps do not fall exactly into the class of SPMs, which is why their convergence theory is not applicable.

Finally, a generalization of the objective feasibility pump of Achterberg and Berthold [1] is given in [39, 40] for convex MINLP and Berthold [4] discusses some new algorithmic ideas for nonconvex MINLPs.

2. Alternating Direction Methods

In this section, we first briefly review classical alternating direction methods (ADMs) and afterward prove that idealized feasibility pumps, i.e., the basic feasibility pump (Algorithm 1) without random perturbations, can be seen as a special case of alternating direction methods. This gives new theoretical insights since the complete theory of ADMs can be applied to idealized feasibility pumps.

To this end, we consider the general problem

\[
\min_{x,y} \; f(x, y) \quad (3a)
\]

s.t.

\[
g(x, y) = 0, \quad h(x, y) \geq 0, \quad (3b)
\]

\[
x \in X, \quad y \in Y, \quad (3c)
\]

for which we make the following assumption:

**Assumption 1.** The objective function \( f : \mathbb{R}^{n_x + n_y} \to \mathbb{R} \) and the constraint functions \( g : \mathbb{R}^{n_x + n_y} \to \mathbb{R}^m, h : \mathbb{R}^{n_x + n_y} \to \mathbb{R}^p \) are continuous and the sets \( X \) and \( Y \) are non-empty and compact.

The feasible set is denoted by \( \Omega \), i.e.,

\[
\Omega = \{(x, y) \in X \times Y : g(x, y) = 0, \; h(x, y) \geq 0\} \subseteq X \times Y,
\]

and the corresponding projections onto \( X \) and \( Y \) are denoted by \( \Omega_X \) and \( \Omega_Y \), respectively. Classical alternating direction methods are extensions of Lagrangian methods and have been originally proposed in Gabay and Mercier [28] and Glowinski and Marroco [30]. More recently, ADM-type methods have seen a resurgence; see,
Algorithm 2 A Standard Alternating Direction Method

1: Choose initial values $(x^0, y^0) \in X \times Y$.

2: for $k = 0, 1, \ldots$ do

3: Compute

   \[ x^{k+1} \in \arg\min_x \{ f(x, y^k) : g(x, y^k) = 0, h(x, y^k) \geq 0, \ x \in X \}. \]

4: Compute

   \[ y^{k+1} \in \arg\min_y \{ f(x^{k+1}, y) : g(x^{k+1}, y) = 0, h(x^{k+1}, y) \geq 0, \ y \in Y \}. \]

5: Set $k \leftarrow k + 1$.

6: end for

e.g., [11] for a general overview and [29] for an application of ADMs to nonconvex MINLPs from gas transport including heat power constraints. The latter application also provided the motivation for this article. ADMs solve Problem (3) by solving two simpler problems: Given an iterate $(x^k, y^k)$, they solve Problem (3) for $y$ fixed to $y^k$ into the direction of $x$, yielding a new $x$-iterate $x^{k+1}$. Afterward, $x$ is fixed to $x^{k+1}$ and Problem (3) is solved into the direction of $y$, yielding a new $y$-iterate $y^{k+1}$.

A formal listing is given in Algorithm 2. Note that we do not state a practical termination criterion in Algorithm 2 in order to facilitate a more streamlined analysis. For the implementation details we refer to Section 4.

If the optimization problem in Line 3 or Line 4 of Algorithm 2 has a unique solution for all $k$, it is known that ADMs converge to so-called partial minima of Problem (3), i.e., to points $(x^*, y^*) \in \Omega$ for which

\[ f(x^*, y^*) \leq f(x, y^*) \] for all $(x, y^*) \in \Omega,

\[ f(x^*, y^*) \leq f(x^*, y) \] for all $(x^*, y) \in \Omega.

holds; see Gorski et al. [31] for the following result:

**Theorem 2.1.** Let $\{(x^i, y^i)\}_{i=0}^\infty$ be a sequence with $(x^{i+1}, y^{i+1}) \in \Theta(x^i, y^i)$, where $\Theta(x^i, y^i) := \{ (x^*, y^*) : \forall x \in X. f(x^*, y^i) \leq f(x, y^i); \forall y \in Y. f(x^i, y^*) \leq f(x^i, y) \}$. Suppose that Assumption 7 holds and that the solution of the first optimization problem is always unique. Then every convergent subsequence of $\{(x^i, y^i)\}_{i=0}^\infty$ converges to a partial minimum. For two limit points $z, z'$ of such subsequences it holds that $f(z) = f(z')$.

Stronger results can be obtained if additional assumptions are made on $f$ and $\Omega$: If $f$ is continuously differentiable, Algorithm 2 converges to a stationary point (in the classical sense of nonlinear optimization). If, in addition, $f$ and $\Omega$ are convex it is easy to show that partial minimizers are also global minimizers of Problem (3). For more details on the convergence theory of classical ADMs, see Gorski et al. [31] as well as Wendell and Hurter [41].

### 2.1. Feasibility Pumps as ADMs

Recall that the basic feasibility pump algorithm [1] tries to find a feasible solution for the binary variant of MIP [1]. We now consider the idealized feasibility pump, i.e., we omit the perturbation step in Line 13 of Algorithm [1] and show that the idealized feasibility pump is a special case of the ADM (Algorithm 2) applied to a certain reformulation of MIP [1]. To this end, we...
duplicate the variables $x_1$ using the new variable vector $y \in \{0,1\}^I$, yielding
\[
\min_{x,y} c^\top x \\
\text{s.t. } x \in X := \{x \in \mathbb{R}^n : Ax \geq b, x_1 \in [0,1]^I\}, \\
y \in Y := \{0,1\}^I, \quad g(x,y) = x_1 - y = 0,
\]
which is obviously equivalent to the original MIP. Note that, compared to the general problem (3), we do not explicitly require the inequality constraints vector $h$.

The feasibility pump is only interested in feasibility and thus ignores the objective function. By deleting the objective from the reformulated model and instead moving an $\ell_1$ penalty term of the coupling condition $y = x_1$ into the objective, we obtain
\[
\min_{x,y} \|x_1 - y\|_1 \\
\text{s.t. } x \in X := \{x \in \mathbb{R}^n : Ax \geq b, x_1 \in [0,1]^I\}, \quad y \in Y := \{0,1\}^I. \tag{4a}
\]
If we define initial values $(x^0,y^0)$ by $x^0 := \arg\min \{c^\top x : x \in X\}$ and $y^0 := \lfloor x^0 \rfloor$, it can be easily seen that solving Problem (4) with the ADM algorithm exactly corresponds to the idealized feasibility pump algorithm. To be more precise, finding a new iterate within the ADM coincides with the projection step and finding the new $y$-iterate corresponds to the rounding step.

In the context of Algorithm 2, we say that the sequence of iterates $z^k$ cycles if there exists an iteration $k$ and an $l \geq 2$ with $z^k = z^{k+l}$. Next, we prove that the ADM cannot cycle and thus terminates after a finite number of iterations. To this end, we make the following observations: First, $X$ and $Y$ in (4) are non-empty and compact sets. Second, we can assume uniqueness of the rounding step by resolving tie-breaks choosing lexicographically minimal solutions. Thus, by using that norms are continuous, we have the following result.

**Lemma 2.2.** Algorithm 2 does not cycle.

**Proof.** Assume the contrary, i.e., there exists an iteration $k$ and an $l \geq 2$ such that $z^k, z^{k+1}, \ldots, z^{k+l} = z^k$. Since $f(x^{k+1},y^{k+1}) \leq f(x^{k+1},y^k) \leq f(x^k,y^k)$ holds for all iterations $k$ we directly see that $f(z^k) = f(z^{k+1}) = \cdots = f(z^{k+l-1})$ holds. This, however, implies that $z^k$ is already a partial minimum at which the algorithm stops. □

We note that this lemma is equivalent to Proposition 1 of De Santis et al. [17]. There, the authors show that the idealized feasibility pump for binary MIP’s is equivalent to the Frank–Wolfe algorithm (using an unitary stepsize) applied to the problem
\[
\min_{x \in \mathcal{P}} \sum_{i \in I} \min \{x_i, 1 - x_i\}, \tag{5}
\]
where the objective function is a concave and nonsmooth merit function for measuring integrality. Applying the convergence theorem from [36] then also yields finite termination at so-called vertex stationary points. Since we have now proven that the ADM (Algorithm 2) applied to (4) is equivalent to the above mentioned special case of the Frank–Wolfe method, we have also shown that partial minima of Problem (4) are exactly the vertex stationary points of Problem (5).

From the theory reported above and the last lemma we can directly deduce the following convergence theorem for the idealized feasibility pump.

**Theorem 2.3.** The idealized feasibility pump terminates at a partial minimum $(x^*, y^*)$ of Problem (4) after a finite number of iterations. If the partial minimum $(x^*, y^*)$ has objective value $\|x^*_1 - y^*\|_1 = 0$, the point $(x^*, y^*)$ is feasible for the MIP (1).
This theorem also gives us a new view of the random perturbation steps of feasibility pump algorithms: They can be interpreted as an attempt to escape from non-integral partial optima of Problem (4).

So far, we have only discussed the case of binary MIPs. However, our theory is still applicable as long as the optimization problems in Line 3 and 4 of Algorithm 2 are solved to global optimality. This is a realistic assumption for convex MINLPs of type (2). The suitable generalization of Problem (4) for this problem class reads

$$\min_{x,y} \|x_I - y\|_1$$ (6a)

s.t. $x \in X := \{x \in \mathbb{R}^n : h(x) \geq 0, x_I \in [0,1]^I\}, \quad y \in Y := \{0,1\}^I$. (6b)

We note that the resulting ADM is exactly the method presented in Bonami and Gonçalves [9]. Using the same techniques as above, we get the following convergence theorem for the idealized feasibility pump for convex MINLP.

**Theorem 2.4.** The idealized feasibility pump for convex MINLP (2) is equivalent to the ADM algorithm 2 applied to Problem (6). Thus, it terminates at a partial minimum $(x^*, y^*)$ of Problem (6) after a finite number of iterations. If this partial minimum has objective value $\|x_I^* - y^*\|_1 = 0$, the point $(x^*, y^*)$ is feasible for the convex MINLP (2).

We close this section with two remarks. First, we note that we presented the results in this section for binary MI(NL)Ps only for improving readability. The extension to general mixed-integer problems is straightforward; see Section 4 for the details. Second, we again want to highlight that the theoretical results presented in this section only hold if the optimization problems in Line 3 and 4 of Algorithm 2 are solved to global optimality. Since this is typically not possible for general nonconvex MINLPs, the results are only practically valid for convex mixed-integer problems.

3. THE PENALTY ALTERNATING DIRECTION METHOD

In this section we first present a new penalty alternating direction method in Section 3.1 and afterward prove the convergence results in Section 3.2. Finally, in Section 3.3, we show how the new method can be used to obtain a novel feasibility pump algorithm for general mixed-integer optimization. This new penalty alternating direction method based feasibility pump replaces random perturbations with a theoretically analyzeable penalty framework for escaping from undesirable intermediate points. Thus, the complete theory presented for the new method also applies to the new feasibility pump variant.

3.1. The Algorithm. We now present the novel weighted $\ell_1$ penalty method based on the classical ADM framework given in Section 2. To this end, we define the $\ell_1$ penalty function

$$\phi_1(x, y; \mu, \rho) := f(x, y) + \sum_{i=1}^{m} \mu_i |g_i(x, y)| + \sum_{i=1}^{p} \rho_i [h_i(x, y)]^-, $$

where $[\alpha]^- := \max\{0, -\alpha\}$ holds and $\mu = (\mu_i)_{i=1}^m, \rho = (\rho_i)_{i=1}^p \geq 0$ are the penalty parameters for the equality and inequality constraints. Note that we allow for different penalty parameters for the constraints instead of a single penalty parameter as it is often the case for penalty methods.

The penalty ADM now proceeds as follows. Given a starting point and initial values for all penalty parameters, the alternating direction method of Algorithm 2 is used to compute a partial minimum of the penalty problem

$$\min_{x,y} \phi_1(x, y; \mu, \rho) \quad \text{s.t.} \quad x \in X, \ y \in Y. \quad (7)$$
Algorithm 3 The ℓ₁ Penalty Alternating Direction Method

1: Choose initial values \((x^{0,0}, y^{0,0}) ∈ X × Y\) and penalty parameters \(\mu^0, \rho^0 ≥ 0\).
2: for \(k = 0, 1, \ldots \) do
3: \hspace{1em} Set \(l = 0\).
4: \hspace{1em} while \((x^{k,l}, y^{k,l})\) is not a partial minimum of \((7)\) with \(\mu = \mu^k\) and \(\rho = \rho^k\) do
5: \hspace{2em} Compute \(x^{k,l+1} ∈ \arg\min_x \{φ_1(x, y^{k,l}; \mu^k, \rho^k) : x ∈ X\}\).
6: \hspace{2em} Compute \(y^{k,l+1} ∈ \arg\min_y \{φ_1(x^{k,l+1}, y; \mu^k, \rho^k) : y ∈ Y\}\).
7: \hspace{1em} Set \(l ← l + 1\).
8: \hspace{1em} end while
9: \hspace{1em} Choose new penalty parameters \(\mu^{k+1} ≥ \mu^k\) and \(\rho^{k+1} ≥ \rho^k\).
10: end for

Afterward, the penalty parameters are updated and the next penalty problem is solved to partial minimality. Thus, the algorithm produces a sequence of partial minima of a sequence of penalty problems of type \((7)\). More formally, the method is specified in Algorithm 3.

3.2. Convergence Theory. We now present the convergence results for the penalty ADM algorithm. We start by proving that partial minima of the penalty problems are partial minima of the original problem if they are feasible.

Lemma 3.1. Assume that \((x^*, y^*)\) is a partial minimum of \(φ_1(x, y; \mu, \rho)\) for arbitrary but fixed \(\mu, \rho ≥ 0\) and let \((x^*, y^*)\) be feasible for Problem \((3)\). Then \((x^*, y^*)\) is a partial minimum of Problem \((3)\).

Proof. Let \(x ∈ X\) such that \((x, y^*)\) is feasible for Problem \((3)\). Then it holds

\[
f(x, y^*) = f(x, y^*) + \sum_{i=1}^{m} \mu_i |g_i(x, y^*)| + \sum_{i=1}^{p} \rho_i |h_i(x, y^*)|^{−}
= φ_1(x, y^*; \mu, \rho) ≥ φ_1(x^*, y^*; \mu, \rho)
= f(x^*, y^*) + \sum_{i=1}^{m} \mu_i |g_i(x^*, y^*)| + \sum_{i=1}^{p} \rho_i |h_i(x^*, y^*)|^{−}
= f(x^*, y^*).
\]

The analogous inequality holds for all \(y ∈ Y\) such that \((x^*, y)\) is feasible. Thus, \((x^*, y^*)\) is a partial minimum of Problem \((3)\). □

For the next theorem we need some more notation. Let \(χ\) be the ℓ₁ feasibility measure of Problem \((3)\), which we define as

\[
χ(x, y) := \sum_{i=1}^{m} |g_i(x, y)| + \sum_{i=1}^{p} |h_i(x, y)|^{−}.
\]

Obviously, \(χ(x, y) ≥ 0\) holds and \(χ(x, y) = 0\) if and only if \((x, y)\) is feasible w.r.t. \(g\) and \(h\). Moreover, we define the weighted ℓ₁ feasibility measure as

\[
χ_{\mu, \rho}(x, y) := \sum_{i=1}^{m} \mu_i |g_i(x, y)| + \sum_{i=1}^{p} \rho_i |h_i(x, y)|^{−},
\]

i.e., our ℓ₁ penalty function can be stated as

\[
φ_1(x, y; \mu, \rho) = f(x, y) + χ_{\mu, \rho}(x, y).
\]

The next theorem states that the sequence of partial minima of the iteratively solved penalty problems converges to a partial minimum of \(χ_{\mu, \rho}\).
Lemma 3.2. Suppose that Assumption [4] holds and that $\mu^k \not \to \infty$ for all $i = 1, \ldots, m$ and $\rho^k \not \to \infty$ for all $i = 1, \ldots, p$. Moreover, let $(x^k, y^k)$ be a sequence of partial minima of (7) (for $\mu = \mu^k$ and $\rho = \rho^k$) generated by Algorithm 3 with $(x^k, y^k) \to (x^*, y^*)$. Then there exist weights $\bar{\mu}, \bar{\rho} \geq 0$ such that $(x^*, y^*)$ is a partial minimizer of the feasibility measure $\chi_{\bar{\mu}, \bar{\rho}}$.

Proof. Let $(x^k, y^k)$ be a partial minimizer of $\phi_1(x, y; \mu^k, \rho^k)$, i.e.,

$\phi_1(x, y^k; \mu^k, \rho^k) \geq \phi_1(x^k, y^k; \mu^k, \rho^k)$ for all $x \in X$,
$\phi_1(x^k, y; \mu^k, \rho^k) \geq \phi_1(x^k, y^k; \mu^k, \rho^k)$ for all $y \in Y$,

which is equivalent to

$$f(x, y^k) + \sum_{i=1}^m \mu^k_i |g_i(x, y^k)| + \sum_{i=1}^p \rho^k_i [h_i(x, y^k)]^{-} \geq f(x^k, y^k) + \sum_{i=1}^m \mu^k_i |g_i(x^k, y^k)| + \sum_{i=1}^p \rho^k_i [h_i(x^k, y^k)]^{-},$$

(8)

for all $x \in X$ and

$$f(x^k, y) + \sum_{i=1}^m \mu^k_i |g_i(x^k, y)| + \sum_{i=1}^p \rho^k_i [h_i(x^k, y)]^{-} \geq f(x^k, y^k) + \sum_{i=1}^m \mu^k_i |g_i(x^k, y^k)| + \sum_{i=1}^p \rho^k_i [h_i(x^k, y^k)]^{-},$$

(9)

for all $y \in Y$. The sequence $(\mu^k, \rho^k) \subseteq \mathbb{R}^{m+p}$ is unbounded but the normalized sequence

$$\frac{(\mu^k, \rho^k)}{\|(\mu^k, \rho^k)\|} \subseteq \mathbb{R}^{m+p},$$

is bounded. Thus, there exists a subsequence (indexed by $l$) of the normalized sequence such that

$$\frac{(\mu^l, \rho^l)}{\|(\mu^l, \rho^l)\|} \to (\bar{\mu}, \bar{\rho}) \text{ for } l \to \infty.$$

Division of (8) and (9) by $\|(\mu^l, \rho^l)\|$ yields

$$\frac{1}{\|(\mu^l, \rho^l)\|} f(x, y^k) + \sum_{i=1}^m \frac{\mu^l_i}{\|(\mu^l, \rho^l)\|} |g_i(x, y)| + \sum_{i=1}^p \frac{\rho^l_i}{\|(\mu^l, \rho^l)\|} [h_i(x, y)]^{-} \geq \frac{1}{\|(\mu^l, \rho^l)\|} f(x^k, y^k) + \sum_{i=1}^m \frac{\mu^l_i}{\|(\mu^l, \rho^l)\|} |g_i(x^k, y^k)| + \sum_{i=1}^p \frac{\rho^l_i}{\|(\mu^l, \rho^l)\|} [h_i(x^k, y^k)]^{-},$$

for all $x \in X$ and

$$\frac{1}{\|(\mu^l, \rho^l)\|} f(x^k, y) + \sum_{i=1}^m \frac{\mu^l_i}{\|(\mu^l, \rho^l)\|} |g_i(x^k, y)| + \sum_{i=1}^p \frac{\rho^l_i}{\|(\mu^l, \rho^l)\|} [h_i(x^k, y)]^{-} \geq \frac{1}{\|(\mu^l, \rho^l)\|} f(x^k, y^k) + \sum_{i=1}^m \frac{\mu^l_i}{\|(\mu^l, \rho^l)\|} |g_i(x^k, y^k)| + \sum_{i=1}^p \frac{\rho^l_i}{\|(\mu^l, \rho^l)\|} [h_i(x^k, y^k)]^{-},$$

for all $y \in Y$. Finally, by using that the limit preserves non-strict inequalities, linearity of the limit, and continuity of $f, g,$ and $h$, for $l \to \infty$ we obtain

$$\sum_{i=1}^m \bar{\mu}_i |g_i(x, y^*)| + \sum_{i=1}^p \bar{\rho}_i [h_i(x, y^*)]^{-} \geq \sum_{i=1}^m \bar{\mu}_i |g_i(x^*, y^*)| + \sum_{i=1}^p \bar{\rho}_i [h_i(x^*, y^*)]^{-}.$$
for all \( x \in X \) and
\[
\sum_{i=1}^{m} \bar{\mu}_i |g_i(x^*, y)| + \sum_{i=1}^{p} \bar{\rho}_i |h_i(x^*, y)| \leq \sum_{i=1}^{m} \bar{\mu}_i |g_i(x^*, y^*)| + \sum_{i=1}^{p} \bar{\rho}_i |h_i(x^*, y^*)|
\]
for all \( y \in Y \). This is equivalent to
\[
\chi_{\bar{\mu}, \bar{\rho}}(x^*, y^*) \geq \chi_{\bar{\mu}, \bar{\rho}}(x^*, y^*) \quad \text{for all } x \in X,
\]
\[
\chi_{\bar{\mu}, \bar{\rho}}(x^*, y) \geq \chi_{\bar{\mu}, \bar{\rho}}(x^*, y^*) \quad \text{for all } y \in Y
\]
and thus completes the proof. \( \square \)

The two preceding lemmas now enable us to characterize the overall convergence behavior of the penalty ADM algorithm.

**Theorem 3.3.** Suppose that Assumption holds and that \( \mu_k \not\rightarrow \infty \) for all \( i = 1, \ldots, m \) and \( \rho^k \not\rightarrow \infty \) for all \( i = 1, \ldots, p \). Moreover, let \((x^k, y^k)\) be a sequence of partial minima of (7) (for \( \mu = \mu^k \) and \( \rho = \rho^k \)) generated by Algorithm 3 with \((x^k, y^k) \rightarrow (x^*, y^*)\). Then there exist weights \( \bar{\mu}, \bar{\rho} \geq 0 \) such that \((x^*, y^*)\) is a partial minimizer of the feasibility measure \( \chi_{\bar{\mu}, \bar{\rho}} \).

If, in addition, \((x^*, y^*)\) is feasible for the original problem, the following holds:

a) If \( f \) is continuous, then \((x^*, y^*)\) is a partial minimum of (3).

b) If \( f \) is continuously differentiable, then \((x^*, y^*)\) is a stationary point of (3).

c) If \( f \) is continuously differentiable and \( f \) and \( \Omega \) are convex, then \((x^*, y^*)\) is a global optimum of (3).

**Proof.** The first part always holds by Lemma 3.2. If, in addition, the obtained partial minimum of \( \chi_{\bar{\mu}, \bar{\rho}} \) satisfies \( \chi_{\bar{\mu}, \bar{\rho}}(x^*, y^*) = 0 \), we can apply Lemma 3.1 and obtain the statements a)–c). \( \square \)

In the next theorem we generalize the classical result on the exactness of the \( \ell_1 \) penalty function (see, e.g., [33, 37]) to the setting of partial minima. For the case of presentation, we state and prove this result only for the case without inequality constraints. However, the result can also be applied to problems including inequality constraints by using standard reformulation techniques to translate inequality constrained to equality constrained problems. Beforehand, we need two assumptions:

**Assumption 2.** The objective function \( f : X \times Y \rightarrow \mathbb{R} \) of Problem (3) is locally Lipschitz continuous in the direction of \( x \) and of \( y \), i.e., for every \((x^*, y^*) \in \Omega \) there exists an open set \( N(x^*, y^*) \) containing \((x^*, y^*)\) and a constant \( L \geq 0 \) such that
\[
|f(x, y^*) - f(x^*, y^*)| \leq L\|x - x^*\| \quad \text{for all } x \in N(x^*, y^*),
\]
\[
|f(x^*, y) - f(x^*, y^*)| \leq L\|y - y^*\| \quad \text{for all } y \in N(x^*, y^*).
\]

Note that if one set, say \( Y \), is discrete, the corresponding condition is trivially satisfied. In this case any set of the form \( \{U, \{y^*\}\} \), where \( U \subseteq X \) is an open neighborhood around \( x^* \), is an open neighborhood around \( (x^*, y^*) \).

**Assumption 3.** For every constraint \( g_i, i = 1, \ldots, m \), there exists a constant \( l_i > 0 \) such that
\[
l_i\|x - x^*\| \leq |g_i(x, y^*) - g_i(x^*, y^*)| \quad \text{for all } x \in N(x^*, y^*),
\]
\[
l_i\|y - y^*\| \leq |g_i(x^*, y) - g_i(x^*, y^*)| \quad \text{for all } y \in N(x^*, y^*).
\]

Note that in the case of existing directional derivatives of \( g_i \), the latter assumption states that the directional derivatives of the \( g_i \) both in the direction of \( x \) and of \( y \) are bounded away from zero. Before we state and proof the exactness theorem we briefly
discuss the latter assumption. In the context of ADMs, the constraints \( g(x, y) = 0 \) are mostly so-called copy constraints of the type 
\[
g(x, y) = A(x - y) = 0
\]
that are used to decompose the genuine problem formulation such that it fits into the framework of Problem (10); see, e.g., Nowak [38]. If the matrix \( A \) is square and has full rank—as it is typically the case for copy constraints—the constraints \( g \) are bi-Lipschitz and thus fulfill Assumption 3. Now, we are ready to state and prove the theorem on exactness of the \( \ell_1 \) penalty function w.r.t. partial minima.

**Theorem 3.4.** Let \((x^*, y^*)\) be a partial minimizer of 
\[
\min_{x, y} f(x, y) \quad \text{s.t.} \quad g(x, y) = 0, \ x \in X, \ y \in Y,
\]
and suppose that Assumptions 2 and 3 hold. Then there exists a constant \( \bar{\mu} > 0 \) such that \((x^*, y^*)\) is a partial minimizer of 
\[
\min_{x, y} \phi_1(x, y; \mu) \quad \text{s.t.} \quad x \in X, \ y \in Y
\]
for all \( \mu \geq \bar{\mu} \) and 
\[
\phi_1(x, y; \mu) := f(x, y) + \sum_{i=1}^{m} \mu_i |g_i(x, y)|.
\]

**Proof.** Since \((x^*, y^*)\) is a partial minimizer of Problem (10), it holds that 
\[
\begin{align}
    f(x, y^*) &\geq f(x^*, y^*) \quad \text{for all } (x, y^*) \in \Omega, \\
    f(x^*, y) &\geq f(x^*, y^*) \quad \text{for all } (x^*, y) \in \Omega,
\end{align}
\]
where \( \Omega \) is the feasible region of Problem (10). First, assume that \((x, y^*)\) is feasible for Problem (10). Using (11) we obtain 
\[
\phi_1(x^*, y^*; \mu) = f(x^*, y^*) + \sum_{i=1}^{m} \mu_i |g_i(x^*, y^*)| = f(x^*, y^*) \leq f(x, y^*)
\]
for all \( \mu \). The inequality 
\[
\phi_1(x^*, y^*; \mu) \leq \phi_1(x^*, y; \mu)
\]
can be shown analogously assuming that \((x^*, y)\) is feasible.

We now consider the case that \((x^*, y^*)\) is not feasible for Problem (10). We set \( \bar{\mu} := L/(m\ell) e > 0 \), where \( \ell := \min_{i=1, \ldots, m} \{ l_i \} \), \( e = (1, \ldots, 1)^\top \in \mathbb{R}^m \), and show that for all \( \mu \geq \bar{\mu} \) the inequality 
\[
f(x, y^*) + \sum_{i=1}^{m} \mu_i |g_i(x, y^*)| \geq f(x^*, y^*) + \sum_{i=1}^{m} \mu_i |g_i(x^*, y^*)|
\]
holds for all \( x \in X \). Since \((x^*, y^*)\) is feasible for Problem (10), Inequality (12) is equivalent to 
\[
\sum_{i=1}^{m} \mu_i |g_i(x, y^*)| \geq f(x^*, y^*) - f(x, y^*).
\]
In the following, we use that for each \((x, y) \in X \times Y\), Assumption 2 also implies the global Lipschitz continuity of \(f\) on the compact sets \(X \times \{y\}\) and \(\{x\} \times Y\). From the definition of \(\bar{\mu}\) and the Assumptions 2 and 3 we obtain

\[
\sum_{i=1}^{m} \mu_i \left| g_i(x, y^*) \right| \geq \bar{\mu} \sum_{i=1}^{m} \left| g_i(x, y^*) \right| = \bar{\mu} \sum_{i=1}^{m} \left| g_i(x^*, y^*) - g_i(x^*, y^*) \right|
\]

\[
\geq \bar{\mu} \sum_{i=1}^{m} l_i \|x - x^*\| \geq \bar{\mu} m \bar{\mu} l \|x - x^*\| = L \|x - x^*\|
\]

\[
\geq |f(x^*, y^*) - f(x, y^*)| \geq f(x^*, y^*) - f(x, y^*).
\]

Thus, (12) holds for all \(\mu \geq \bar{\mu}\). Analogously, it can be shown that the inequality

\[
f(x^*, y) + \sum_{i=1}^{m} \mu_i \left| g_i(x^*, y) \right| \geq f(x^*, y^*) + \sum_{i=1}^{m} \mu_i \left| g_i(x^*, y^*) \right|
\]

holds for all \(y \in Y\). \(\square\)

### 3.3. The Penalty ADM as a Feasibility Pump

In this section we discuss the application of the proposed penalty alternating direction method as a new variant of a feasibility pump algorithm for convex MINLPs. The motivation is the following: We have seen in the last section that idealized, i.e., perturbation-free, feasibility pumps for convex MINLPs terminate at partial minima after a finite number of iterations. However, it is possible that the obtained partial minimum is not integer feasible. Feasibility pumps typically try to resolve this problem by applying a random perturbation of the integer components. This procedure has the significant drawback that it renders a convergence theory of the overall method (almost) impossible. In contrast to these random perturbations, the method we propose uses a theoretically analyzable penalty framework to escape integer infeasible partial minima.

We start by rewriting Problem (2) by again duplicating the integer components \(x_I\) of \(x\) and obtain

\[
\min_{x, y} \ f(x) \ \text{s.t.} \ h(x) \geq 0, \ x_I = y, \ y \in \mathbb{Z}^I \cap [l_I, u_I]. \tag{13}
\]

With the compact sets

\[
X := \{x : h(x) \geq 0\}, \quad Y := \mathbb{Z}^I \cap [l_I, u_I]
\]

and the additional equality constraints

\[
g(x, y) = x_I - y = 0
\]

we can apply Algorithm 3 to Problem (13). Note that the problem interfaced to the penalty ADM does not contain any inequality constraints explicitly since we moved them to the set \(X\). This also simplifies the \(\ell_1\) penalty function \(\phi_1(x, y; \mu, \rho)\) to \(\phi_1(x, y; \mu)\). In the \(lth\) ADM iteration of the \(kth\) penalty iteration of Algorithm 3 the two subproblems being solved are

\[
\min_{x \in X} \phi_1(x, y^{k,l}; \mu^k),
\]

which can be written as

\[
\min_{x} \ f(x) + \sum_{i \in I} \mu_i \left| x_i - y_i^{k,l} \right| \ \text{s.t.} \ h(x) \geq 0, \tag{14}
\]

and

\[
\min_{y \in Y} \phi_1(x^{k,l+1}, y; \mu^k),
\]

which can be written as

\[
\min_{y} \ f(x^{k,l+1}) + \sum_{i \in I} \mu_i \left| x_i^{k,l+1} - y_i \right| \ \text{s.t.} \ y \in \mathbb{Z}^I \cap [l_I, u_I]. \tag{15}
\]
Note that Problem (14) is the NLP relaxation of (2), where the original objective function is augmented by a weighted \( \ell_1 \) penalty term. Note further that solving Problem (15) simply means to apply a weighted rounding of the variables \( y_i, i \in I \).

4. Implementation Issues

In this section we comment on important implementation issues. First, we rewrite Problem (13) by replacing the coupling equality \( x_I = y \) by inequality constraints in order to be able to penalize a coupling error \( x_i > y_i, i \in I \), different than the error \( y_i > x_i \). In addition, we also explicitly state all variable bounds from now on. Thus, we obtain

\[
\begin{align*}
\min_{x,y} & \quad f(x) \\
\text{s.t.} & \quad h(x) \geq 0, \quad x \in [l, u], \\
& \quad x_i \geq y_i, \quad y \geq x_i, \quad y \in \mathbb{Z}^l \cap [l, u],
\end{align*}
\]

In other words, we slightly modified the compact and non-empty constraint sets to

\[
X := \{ x \in [l, u] : h(x) \geq 0 \}, \quad Y := \mathbb{Z}^l \cap [l, u]
\]

and replaced the coupling equalities by coupling inequalities. The two subproblems (14) and (15) that are solved within the \( k \)th ADM iteration of the \( k \)th penalty iteration are now given by

\[
\begin{align*}
\min_x & \quad f(x) + \sum_{i \in I} \left( \mu^k_i |x_i - y_i|^+ + \bar{\mu}_i^k |y_i - x_i|^+ \right) \\
\text{s.t.} & \quad h(x) \geq 0, \quad x \in [l, u],
\end{align*}
\]

and

\[
\begin{align*}
\min_y & \quad f(x^{k,l+1}) + \sum_{i \in I} \left( \mu^k_i |x_i^{k,l+1} - y_i|^+ + \bar{\mu}_i^k |y_i - x_i^{k,l+1}|^+ \right) \\
\text{s.t.} & \quad y \in \mathbb{Z}^l \cap [l, u],
\end{align*}
\]

i.e., we also replaced the single penalty parameters \( \mu_i \) for the equality coupling constraints in Problem (13) by two new penalty parameters \( \mu_i \) and \( \bar{\mu}_i \) for the lower and upper violation of the coupling.

In order to actually implement the penalty ADM based feasibility pump for MINLPs, we follow Achterberg and Berthold [1] and scale the objective function of Problem (16) such that the impact of the \( \ell_1 \) penalty terms and the original objective function \( f \) can be balanced. This balancing between feasibility and optimality is done using the parameter \( \alpha^k \in [0, 1] \). Additionally, we again rewrite the \( \ell_1 \) penalty terms in the objective function. To this end, we denote the set of indices of binary variables by \( B \subseteq I \), introduce the variables \( d_i^+ \), \( d_i^- \geq 0 \) for all \( i \in I \setminus B \), and rewrite Problem (16) as

\[
\begin{align*}
\min_{x,d} & \quad \alpha^k \frac{\sqrt{|I|}}{\|\nabla f(x, 0, 0)\|} \sum_{i \in I} \left( \mu^k_i x_i - y_i \right) + \sum_{i \in I} \bar{\mu}_i^k (1 - x_i) + \sum_{i \in I \setminus B} \left( \mu^k_i d_i^+ + \bar{\mu}_i^k d_i^- \right) \\
\text{s.t.} & \quad h(x) \geq 0, \quad x \in [l, u], \\
& \quad d_i^+ \geq x_i - y_i \quad \text{for all } i \in I \setminus B, \\
& \quad d_i^- \geq y_i - x_i \quad \text{for all } i \in I \setminus B, \\
& \quad d_i^-, d_i^+ \geq 0
\end{align*}
\]

with

\[
\tilde{\chi}(x_B, d_{I \setminus B}; \rho^B_I) := \sum_{i \in B_0} \rho^B_i x_i + \sum_{i \in B_1} \bar{\rho}_i^B (1 - x_i) + \sum_{i \in I \setminus B} \left( \rho^B_i d_i^+ + \bar{\rho}_i^B d_i^- \right),
\]
where \( B_0 := \{ i \in B : y_{i}^{k,l} = 0 \} \) and \( B_1 := \{ i \in B : y_{i}^{k,l} = 1 \} \). For binary MINLPs, i.e., \( I = B \), only the objective function of Problem \( \text{(18)} \) may change from one iteration to the next. This is of special importance for mixed-integer linear programs since then the optimal simplex basis obtained in iteration \( k \) yields a primal feasible starting basis for iteration \( k + 1 \). However, when \( I \neq B \), the optimal basis obtained from iteration \( k \) is generally (primal and dual) infeasible for the LP that has to be solved in iteration \( k + 1 \).

Since \( f(x^{k,l+1}) \) is constant and \( \alpha^k \in [0,1] \), solving Problem \( \text{(17)} \) is equivalent to solving the \(|I|\) independent problems

\[
y_{i}^{k,l+1} := \arg\min_{y_i} \left\{ \rho^{k}_i[x_{i}^{k,l+1} - y_i]^{-} + \rho^{k}_i[y_i - x_{i}^{k,l+1}]^{-} \mid y_i \in \mathbb{Z} \cap [l_i, u_i] \right\}, \quad i \in I.
\]

The solutions to these problems can be stated explicitly:

\[
y_{i}^{k,l+1} = \begin{cases} [x_{i}^{k,l+1}], & \text{if } \rho^{k}_i([x_{i}^{k,l+1}] - x_{i}^{k,l+1}) \leq \rho^{k}_i(x_{i}^{k,l+1} - [x_{i}^{k,l+1}]), \\ [x_{i}^{k,l+1}], & \text{otherwise.} \end{cases}
\]

Finally, we have a look on the update of the penalty parameters. An update takes place, whenever the inner ADM loop terminates. In our implementation, we terminate the \( k \)th penalty iteration if \( \| (x^{k,l}, y^{k,l}) - (x^{k,l-1}, y^{k,l-1}) \|_{\infty} \leq \epsilon \) holds, where \( \epsilon = 10^{-5} \). For the actual update of the penalty parameters, we set

\[
\rho^{k+1}_i = \begin{cases} \text{inc}(\rho^{k}_i), & \text{if } y_{i}^{k,l+1} = [x_{i}^{k,l+1}], \\ \rho^{k}_i, & \text{otherwise,} \end{cases}
\]

\[
\lambda^{k+1} = \begin{cases} \text{inc}(\lambda^{k}), & \text{if } y_{i}^{k,l+1} = [x_{i}^{k,l+1}], \\ \lambda^{k}, & \text{otherwise,} \end{cases}
\]

where the penalty parameter update operator \( \text{inc}(a) \) may be any function with \( \text{inc}(a) > a \), e.g., \( \text{inc}(a) = a + 1 \) or \( \text{inc}(a) = 10a \) are used in our computational study. This way, unsuccessful rounding down of the same variable is eventually followed by rounding up of this variable due to increasingly penalizing rounding down and vice versa. Similarly, the conventional feasibility pump algorithm tries to escape from repeated rounding in the “wrong” direction by randomly switching the rounding direction from time to time; see Fischetti et al. [24]. Finally, we set \( \alpha^{k+1} = \lambda \alpha^k \) with \( \lambda \in (0,1) \) whenever the penalty parameters are updated.

We note that choosing \( \alpha_0 = 0 \) in our penalty alternating direction method based feasibility pump is similar to the feasibility pump presented in Bertacco et al. [3] and Fischetti et al. [24], while choosing \( \alpha_0 = 1 \) yields an algorithm the behaves similar to the objective feasibility pump algorithm presented by Achterberg and Berthold [1]. Lastly, we note that for \( \alpha_0 = 0 \), we also have \( \alpha_k = 0 \) for all \( k > 0 \). In this case the first term of the objective function of Problem \( \text{(18)} \) vanishes for all \( k,l \). In any other case, we can divide the entire objective function by \( \alpha_k \) to be conformal to the theoretical setting presented in previous sections.

5. Computational Results

In this section we present extensive numerical results for the penalty ADM based feasibility pump introduced in Section 3 and 4. Since our method is completely generic in terms of the problem type to which it is applied, we present computational results both for MIPs in Section 5.1 and for (convex as well as nonconvex) MINLPs in Section 5.2.

Throughout this section we use log-scaled performance profiles as proposed by Dolan and Moré [20] to compare running times and solution quality. As it is always the case for log-scaled performance profiles the axes have the following meaning: If \((x,y)\) lies on a profile curve, this means that the respective solver is
Figure 1. Non-dominated parameterizations $\lambda \in \{0.9, 0.95, 0.99\}$, inc $\in \{\text{inc}_m, \text{inc}_a\}$ (all with activated Gurobi presolve) for the penalty ADM based feasibility pump for MIPs. Top: primal-dual gap. Bottom: running times.

not more than $2^t$-times worse than the best solver on 100y% of the problems of the test set. Running times are always given in seconds and, following [35], solution quality is measured by the primal-dual gap defined by

$$\text{gap} = \frac{b_p - b_d}{\inf\{|z| : z \in [b_d, b_p]|}$$

(19)

where $b_p$ is the primal and $b_d$ is the dual bound, respectively. Additionally, we set gap $= \infty$ whenever $b_d < 0 \leq b_p$ and gap $= 0$ if $b_d = b_p = 0$.

All computational experiments have been executed on a 12 core Xeon 5650 “Westmere” chip running at 2.66 GHz with 12 MB shared cache per chip and 24 GB of DDR3-1333 RAM. The time limit is set to $t^+ = 1$ h without any limit on the number of iterations for the outer penalty and the inner ADM loop. Additional information about the computational setup and the implementation details are given in the respective sections.

5.1. Mixed-Integer Linear Problems. We start with discussing the results of our algorithm applied to mixed-integer linear problems. For MIPs, our algorithm is implemented in C++ and uses Gurobi 6.5.0 [32] for solving the LP subproblems. We use Gurobi’s option deterministic concurrent for the first LP and solve all succeeding LPs using the primal simplex method; see Section 4. The C++ code has been compiled with gcc 4.8.4 using the optimization flag -O3. First, we present a parameter study. Our penalty ADM based feasibility pump can be instantiated using different choices for certain algorithmic parameters: The initial convex combination parameter $\alpha^0$ for weighting the objective function and the distance function $\tilde{\chi}$ (see Problem (18)) is always set to $\alpha^0 = 1$, emulating the
objective feasibility pump of Achterberg and Berthold [1]. The parameter $\lambda$ for updating the convex combination parameter is varied in the set \{0.9, 0.95, 0.99\} in our parameter study and the penalty parameter update operator can be chosen to be the additive variant $\text{inc}_a(x) = x + 1$ or the multiplicative variant $\text{inc}_m(x) = 10x$. In addition, we also tested the impact of (de)activating the MIP presolve of Gurobi before applying our method. Thus, combining the different choices for $\lambda$, inc, and the (de)activation of Gurobi’s presolve leads to 12 parameter combinations. We applied every of these 12 variants of our method to solve the MIPLIB 2010 benchmark test set excluding the infeasible instances ash608gplia-3col, enlight14, and ns1766074. This yields a test set of 84 instances; see Koch et al. [35]. In order to determine the best parameterization, we compare all 12 variants using performance profiles, where the performance measure is chosen as defined in (19). We then exclude a parameterization $p$ if another parameterization $p'$ exists that dominates $p$. Here, domination is defined by a performance profile completely left-above the other one. This yields the exclusion of deactivating Gurobi’s presolve and, thus, 6 remaining parameterizations; $\lambda \in \{0.9, 0.95, 0.99\}$ and $\text{inc} \in \{\text{inc}_m, \text{inc}_a\}$. The corresponding performance profiles are given in Figure 1. It can be seen that lower values for $\lambda$ yield more robust instantiations of the algorithm, i.e., the number of instances for which a feasible solution can be found is larger. Additionally, all tested variants solve 5 out of 84 instances to global optimality, except for the variant with $\lambda = 0.95$ and $\text{inc}_a$ penalty parameter update rule, which solves 6 instances to global optimality. Altogether, the six parameter choices are quite comparable. Turning to running times, it can be clearly seen that smaller values of $\lambda$ also lead to shorter running times. Thus, our parameter study suggests to activate the MIP presolve of Gurobi, to choose $\lambda = 0.9$, and to leave the choice of the penalty parameter update rule inc $\in \{\text{inc}_m, \text{inc}_a\}$ as an option for the user. Figure 2 shows the performance profiles for solution quality (left) and running times (right) for these “winning” parameterizations. We again see that some instances are solved to global optimality[1] and that both parameterizations of our algorithm find a feasible solution for approximately 90% of the MIPLIB 2010 benchmark instances (75 out of 84 instances for the $\text{inc}_a$ update rule and 76 for the multiplicative rule $\text{inc}_m$). Moreover, the multiplicative update operator $\text{inc}_m$ yields a slightly more robust algorithm, i.e., it finds a feasible solution for a few more instances than the additive

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1The instances triptim1, pigeon-10, enlight13, ex9, and ns1758913 are solved to global optimality using the $\text{inc}_m$ penalty update rule and ns1208400, triptim1, acc-tight5, enlight13, ex9, and ns1758913 are solved to global optimality using $\text{inc}_a$. 
version \( \text{inc}_a \). The right part of Figure 2 compares the two winner instantiations w.r.t. running times. It can be seen that the additive \( \text{inc}_a \) update operator tends to result in a faster algorithm for significantly more instances than the multiplicative version (59.5% vs. 38.1%).

Next, we compare our penalty ADM based feasibility pump with the objective feasibility pump of Achterberg and Berthold [1]. In order to achieve a fair comparison, we extend our method with a local branching strategy with an additional \( k \)-opt neighborhood constraint with \( k = \min \{20, \lceil |I|/10 \rceil \} \) and a time limit \( t^+ - t^- \); see Fischetti and Lodi [25]. Here \( t^* \) denotes the time spent in the penalty ADM based feasibility pump itself. Thus, the local branching stage serves as an improvement heuristic as it is also the case in [1]. As test instances we use all MIPLIB 2010 and MIPLIB 2003 instances that have been used in [1] as well. Figure 3 shows the primal-dual gap performance profiles of the two winner parameterizations (\( \lambda = 0.9 \) and \( \text{inc} \in \{\text{inc}_m, \text{inc}_a\} \)) with and without local branching applied as an additional improvement heuristic as well as the corresponding performance profile curve based on the results reported by Achterberg and Berthold [1]. First of all, it can be seen that all five methods find a feasible solution for at least 90.2% of the tested instances, which underpins the strength of feasibility pumps in general. Comparing only the different parameterization of our method we see that the \( \text{inc}_m \) update rule with local branching outperforms the version without local branching and both variants using the additive penalty update rule. The latter also performs similar independent of whether local branching is used or not, whereas the local branching stage significantly improves the solution quality when the \( \text{inc}_m \) rule is used (in which case we find a global optimal solution for 11.5%; compared to 8.2% for the objective feasibility pump of Achterberg and Berthold [1]). One sees that the multiplicative update rule together with local branching slightly outperforms the objective feasibility pump of Achterberg and Berthold [1] w.r.t. solution quality.

We now turn to a comparison with the MIP solver SCIP 3.2.1. We used SCIP instead of, e.g., Cplex or Gurobi, for our comparison with a state-of-the-art MIP solver because SCIP is the only solver that allows to completely de-activate all other components of the solution process such that we can compare solution quality and running times. To this end, we de-activated SCIP’s presolve, all cuts, and all heuristics except for the feasibility pump\(^2\). We also use a time limit of 1 h, stop the algorithm after finding the first feasible solution, and use Cplex 12.6 as the internal

\(^2\)The feasibility pump specific SCIP parameters are `heuristics/feaspump/maxloops=-1`, `heuristics/feaspump/maxlpiterofs=2147483647`, `heuristics/feaspump/maxlpiterquot=1e10`, and `heuristics/feaspump/maxlpiterquot=1e10`. 

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**Figure 3.** The two winner parameterizations (see Figure 2) with and without local branching compared to the objective feasibility pump (OFP) by Achterberg and Berthold [1].
The two winner parameterizations (see Figure 2) with and without local branching compared to the feasibility pump implementation for MIPs of SCIP: primal dual gap (left) and running times (right).

Figure 5. Left: Cumulative distribution function of absolute running times (x-axis; in s) for two winner parameterizations inc ∈ {inc_m, inc_a} on all MIPLIB 2010 instances. Right: Cumulative distribution function of absolute running times (x-axis; in s) on all MINLPLib2 instances.

To be comparable with our feasibility pump implementation that uses Gurobi’s preprocessing, we presolved all 84 MIPLIB 2010 instances with Gurobi and solved these presolved instances with SCIP’s feasibility pump implementation. The results are given in Figure 4. The left figure shows the performance profile of the primal-dual gap. We see that we are again comparable in terms of solution quality and that our solutions tend to have a slightly better objective value. Moreover, we find a feasible solution for up to approximately 90% of all instances, whereas SCIP finds a feasible point for slightly more than 75%. However, this comes at the price of significantly larger running times; see Figure 4 (right). SCIP is faster for almost all instances and solves most of them within approximately 10 s. Although we did not try to tune our code extensively so far, the latter comparison shows that there is still a strong potential and a lot of work to do if our code should be competitive with a state-of-the-art heuristic w.r.t. running times.

In Figure 5 (left) cumulative distribution functions for absolute running times are given for both winner parameterizations of our algorithm for MIPs. It can be seen that for approximately 25% of all instances a feasible solution is found in at most

and heuristics/feaspump/maxstallloops=-1. They are used to avoid a too early stopping of SCIP’s feasibility pump.
1 s and 50% to 60% of all instances have been solved to feasibility within 10 s. The geometric mean of the running times taken over all instances for which a feasible solution has been found within the time limit is 4.56 s for inc = inc_{in} and 4.94 s for inc = inc_{a}. The running times for all instances can be found in Appendix A.

We close this section with an exemplary discussion of the course of integer (in)feasibility during the iterations of our method. Figure 6 shows the approximately 50 last iterations of our method applied to the MIPLIB 2010 instance rococoC10-001000. Dots correspond to numbers of fractional integer components and the solid line represents the course of the total fractionality measure $\sum_{i \in I} |x_i - \lfloor x_i + 0.5 \rfloor|$ of solutions $x$ of the continuous subproblems over the subsequent ADM iterations. The small black lines on top of the ADM iteration axis denote iterations at which the penalty parameters are updated.

First of all, we see that penalty parameter updates are applied whenever the ADM of the inner loop stalls, i.e., whenever the ADM of the inner loop entered an undesired integer infeasible partial minimum. The method stops after 295 ADM iterations with an integer feasible partial minimum. As expected, we typically see a sawtooth phenomenon: The total fractionality decreases between two consecutive penalty parameter updates and increases after a penalty parameter update. The number of fractional integer components follows this behavior qualitatively. The number of ADM iterations between two consecutive penalty parameter updates varies between 3 to 6 iterations. Thus, convergence to partial minima does not seem to be challenging for this specific instance.

5.2. Mixed-Integer Nonlinear Problems. We now turn to mixed-integer nonlinear programs. The penalty based ADM for this class of models has been implemented in C++ using the so-called GAMS Expert-level API with GAMS 24.5.4 [13]. The continuous relaxation models are solved with CONOPT 3.17A [21]. According to the results from Section 5.1 we choose the parameters $\alpha^0 = 1$ and $\lambda = 0.9$. The penalty parameter update rule is chosen to be inc_{a}(x) = x + 1 since this variant turned out to be favorable for MINLPs. We set the time limit to 1 h as for the MIP experiments and we do not incorporate any iteration limits for the inner ADM and the outer penalty loop. Throughout this section we declare an MINLP instance as solved to
Figure 7. Left: Performance profiles for the primal-dual gap of the penalty ADM based feasibility pump (red) and all six feasibility pump variants (blue) for MINLPs presented in [14] on subset of 61 MINLPLib instances. Right: Performance profiles for the primal-dual gap of the penalty ADM based feasibility pump (red) and different feasibility pump variants (blue) proposed by Berthold [4]

feasibility if CONOPT finds a feasible solution (w.r.t. its default tolerances) with all integer components fixed to integral values.

We compare the results of our method with other recently published numerical results concerning feasibility pump algorithms for convex and nonconvex MINLPs. Most of the results from the literature that we use for these comparisons are based on the first and second version of the MINLPLib; see Bussieck et al. [12]. Since a reasonable comparison of running times is not possible due to differences in the used hardware, we focus on the comparison of success rates and solution quality.

We also compared the obtained results separately for convex and nonconvex instances. As expected, the results of our method are slightly better for the convex case. However, the results are qualitatively rather similar and we thus present the following analysis of our computational results without distinguishing between convex and nonconvex instances.

First, we start with a comparison of different feasibility pump versions presented by D’Ambrosio et al. in [14] and our method on selected MINLPLib instances. D’Ambrosio et al. test their method on 65 MINLPLib instances. However, it turned out in the meantime that the used test set contained 4 duplicate instances. Thus, the following comparison is carried out on 61 MINLPLib instances. Figure 7 (left) displays the performance profiles using the primal-dual gap as the performance measure; see [19]. It can be clearly seen that the penalty ADM based feasibility pump outperforms all feasibility pump variants presented in [14], although we used a time limit of 1 h in contrast to 2 h used in [14]. Although the number of instances solved to optimality is comparably low for all algorithms, the overall solution quality of our penalty ADM based algorithm is significantly higher than the quality of solutions obtained in D’Ambrosio et al. [14]. Additionally, the number of instances for which we found a feasible solution is 60.7% whereas the percentage of instances solved to feasibility in D’Ambrosio et al. [14] ranges from 54.1% to only 16.4%.

Complementing this comparison on the MINLPLib test set we also tested our method on 889 out of 1385 instances of the more recent MINLPLib2 test set. Here, we neglected all instances containing only continuous variables. Again, our method behaves quite satisfactory. It computes a feasible solution for 642 instances (72.2%), leaving 247 instances unsolved. We compare our solutions with the dual bound given in the MINLPLib2. The penalty ADM based feasibility pump computes solutions
with a vanishing primal-dual gap for 83 instances; i.e., for 9.3% of all instances of the test set. The running times on all MINLPLib2 instances are given in the cumulative distribution function in Figure 5 (right). Remarkably, the results are quite comparable with those for the MIP instances: For approximately 50% of all MINLPLib2 instances a feasible solution has been found within 10s and slightly more than 25% of the instances are solved to feasibility within 1s. The geometric mean of the running times taken over all instances for which a feasible solution has been found within the time limit is 5.36s. Nevertheless, the running times are fast enough so that the method could, in principle, be embedded in a global solver as a primal heuristic. As for the MIP results, a table with all running times and additional information is given in the appendix.

Finally, we discuss the most recent (at least to the best of our knowledge) results on a new variant of feasibility pumps for nonconvex mixed-integer nonlinear problems that are published by Berthold in his PhD thesis [4]. The test set used by Berthold is neither a sub- nor a superset of the current MINLPLib. Thus, we compare our method with the algorithms proposed by Berthold on all instances of his test set that are also part of the current MINLPLib version, yielding a test set of 154 instances. Again, we compare the methods by performance profiles of the primal-dual gap, see Figure 7 (right). The penalty ADM based feasibility pump significantly outperforms all variants of the feasibility pump for MINLPs proposed by Berthold. First, the methods of Berthold do not solve any instance to optimality, whereas we close the primal-dual gap for 15.6% of the instances. Second, our method finds a feasible solution for 79.9% of the tested instances, whereas the methods of Berthold solve approximately 43.5% to 64.9% to feasibility.

6. Summary

In this paper we have shown that idealized feasibility pumps, i.e., feasibility pumps without random perturbations, can be seen as alternating direction methods applied to a special reformulation of the original mixed-integer problem. This yields that idealized feasibility pumps converge to a partial minimum of the reformulated problem. If this partial minimum is not an integer feasible point, feasibility pumps apply a random perturbation to escape this undesired point. We replace this random restart with a penalty framework that encompasses the alternating direction method in the inner loop and that replaces random perturbations by tailored penalty parameter updates. This way it is possible for the first time to perform a theoretical study for a variant of the feasibility pump including restarts. The resulting penalty based alternating direction method can be applied to both MIPs and MINLPs. Our numerical results indicate that this new version of the feasibility pump is comparable (w.r.t. most recent publications) for the case of MIPs and clearly outperforms other feasibility pump algorithms on MINLPs in terms of solution quality.

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References

[1] T. Achterberg and T. Berthold. “Improving the feasibility pump.” In: *Discrete Optimization* 4.1 (2007). Mixed Integer Programming IMA Special Workshop on Mixed-Integer Programming, pp. 77–86. doi: [10.1016/j.disopt.2006.10.004](10.1016/j.disopt.2006.10.004)
[2] D. Baena and J. Castro. “Using the analytic center in the feasibility pump.” In: Operations Research Letters 39.5 (2011), pp. 310–317. DOI: 10.1016/j.orl.2011.07.005

[3] L. Bertacco, M. Fischetti, and A. Lodi. “A feasibility pump heuristic for general mixed-integer problems.” In: Discrete Optimization 4.1 (2007). Mixed Integer Programming IMA Special Workshop on Mixed-Integer Programming, pp. 63–76. DOI: 10.1016/j.dsi.2006.10.001

[4] T. Berthold. “Heuristic algorithms in global MINLP solvers.” PhD thesis. Technische Universität Berlin, 2014.

[5] T. Berthold. “Primal Heuristics for Mixed Integer Programs.” Diploma Thesis. TU Berlin, Sept. 2006.

[6] N. L. Boland, A. C. Eberhard, F. Engineer, and A. Tsoukalas. “A New Approach to the Feasibility Pump in Mixed Integer Programming.” In: SIAM Journal on Optimization 22.3 (2012), pp. 831–861. DOI: 10.1137/110823596

[7] N. L. Boland, A. C. Eberhard, F. G. Engineer, M. Fischetti, M. W. P. Savelsbergh, and A. Tsoukalas. “Boosting the feasibility pump.” In: Mathematical Programming Computation 6.3 (2014), pp. 255–279. DOI: 10.1007/s12532-014-0068-9

[8] P. Bonami, G. Cornuéjols, A. Lodi, and F. Margot. “A Feasibility Pump for mixed integer nonlinear programs.” In: Mathematical Programming 119.2 (2009), pp. 331–352. DOI: 10.1007/s10107-008-0212-2

[9] P. Bonami and J. P. M. Gonçalves. “Heuristics for convex mixed integer nonlinear programs.” In: Computational Optimization and Applications 51.2 (2012), pp. 729–747. DOI: 10.1007/s10589-010-9350-6

[10] P. Bonami, M. Kilinç, and J. Linderoth. “Algorithms and Software for Convex Mixed Integer Nonlinear Programs.” In: Mixed Integer Nonlinear Programming. Ed. by J. Lee and S. Leyffer. Vol. 154. The IMA Volumes in Mathematics and its Applications. Springer New York, 2012, pp. 1–39. DOI: 10.1007/978-1-4614-1927-3_1

[11] S. Boyd, N. Parikh, E. Chu, B. Peleato, and J. Eckstein. “Distributed Optimization and Statistical Learning via the Alternating Direction Method of Multipliers.” In: Found. Trends Mach. Learn. 3.1 (Jan. 2011), pp. 1–122. DOI: 10.1561/22000000016

[12] M. R. Bussieck, A. S. Drud, and A. Meeraus. “MINLPLib - A Collection of Test Models for Mixed-Integer Nonlinear Programming.” In: INFORMS Journal on Computing 15.1 (2003), pp. 114–119. DOI: 10.1287/ijoc.15.1.114.15159

[13] G. D. Corporation. General Algebraic Modeling System (GAMS) Release 24.5.4. Washington, DC, USA. 2015. URL: http://www.gams.com

[14] C. D’Ambrosio, A. Frangioni, L. Liberti, and A. Lodi. “A storm of feasibility pumps for nonconvex MINLP.” In: Mathematical Programming 136.2 (2012), pp. 375–402. DOI: 10.1007/s10107-012-0608-x

[15] C. D’Ambrosio, A. Frangioni, L. Liberti, and A. Lodi. “Experiments with a Feasibility Pump Approach for Nonconvex MINLPs.” In: Experimental Algorithms. Ed. by P. Festa. Vol. 6049. Lecture Notes in Computer Science. Springer Berlin Heidelberg, 2010, pp. 350–360. DOI: 10.1007/978-3-642-13193-6_30

[16] E. Danna, E. Rothberg, and C. L. Pape. “Exploring relaxation induced neighborhoods to improve MIP solutions.” In: Mathematical Programming 102.1 (2005), pp. 71–90. DOI: 10.1007/s10107-004-0518-7

[17] M. De Santis, S. Lucidi, and F. Rinaldi. “A New Class of Functions for Measuring Solution Integrality in the Feasibility Pump Approach.” In: SIAM Journal on Optimization 23.3 (2013), pp. 1575–1606. DOI: 10.1137/110855351
[18] M. De Santis, S. Lucidi, and F. Rinaldi. “Feasibility Pump-like heuristics for mixed integer problems.” In: *Discrete Applied Mathematics* 165 (2014). 10th Cologne/Twente Workshop on Graphs and Combinatorial Optimization (CTW 2011), pp. 152–167. DOI: 10.1016/j.dam.2013.06.018.

[19] S. S. Dey, A. Iroume, M. Molinaro, and D. Salvagnin. “Improving the Randomization Step in Feasibility Pump.” In: *arXiv preprint arXiv:1609.08121* (2016).

[20] E. D. Dolan and J. J. Morè. “Benchmarking Optimization Software with Performance Profiles.” In: *Mathematical Programming* 91 (2 2002), pp. 201–213. DOI: 10.1007/s101070100263.

[21] A. Drud. “CONOPT—A Large-Scale GRG Code.” In: *ORSA Journal on Computing* 6.2 (1994), pp. 207–216.

[22] M. A. Duran and I. E. Grossmann. “An outer-approximation algorithm for a class of mixed-integer nonlinear programs.” In: *Mathematical Programming* 36.3 (1986), pp. 307–339. DOI: 10.1007/BF02592064.

[23] J. Eckstein and M. Nediak. “Pivot, Cut, and Dive: a heuristic for 0-1 mixed integer programming.” In: *Journal of Heuristics* 13.5 (2007), pp. 471–503. DOI: 10.1007/s10732-007-9021-7.

[24] M. Fischetti, F. Glover, and A. Lodi. “The feasibility pump.” In: *Mathematical Programming* 104.1 (2005), pp. 91–104. DOI: 10.1007/s10107-004-0570-3.

[25] M. Fischetti and A. Lodi. “Local branching.” In: *Mathematical Programming* 98.1-3 (2003), pp. 23–47. DOI: 10.1007/s10107-003-0395-5.

[26] M. Fischetti and D. Salvagnin. “Feasibility pump 2.0.” In: *Mathematical Programming Computation* 1.2-3 (2009), pp. 201–222. DOI: 10.1007/s12532-009-0007-3.

[27] M. Frank and P. Wolfe. “An algorithm for quadratic programming.” In: *Naval Research Logistics Quarterly* 3.1-2 (1956), pp. 95–110. DOI: 10.1002/nav.3800030109.

[28] D. Gabay and B. Mercier. “A dual algorithm for the solution of nonlinear variational problems via finite element approximation.” In: *Computers & Mathematics with Applications* 2.1 (1976), pp. 17–40. DOI: 10.1016/0898-1221(76)90003-1.

[29] B. Geißler, A. Morsi, L. Schewe, and M. Schmidt. “Solving power-constrained gas transportation problems using an MIP-based alternating direction method.” In: *Computers & Chemical Engineering* 82 (2015), pp. 303–317. DOI: 10.1016/j.compchemeng.2015.07.005.

[30] R. Glowinski and A. Marroco. “Sur l’approximation, par éléments finis d’ordre un, et la résolution, par pénalisation-dualité d’une classe de problèmes de Dirichlet non linéaires.” In: *ESAIM: Mathematical Modelling and Numerical Analysis - Modélisation Mathématique et Analyse Numérique* 9.R2 (1975), pp. 41–76. URL: http://eudml.org/doc/193269.

[31] J. Gorski, F. Pfeuffer, and K. Klamroth. “Biconvex sets and optimization with biconvex functions: a survey and extensions.” In: *Math. Methods Oper. Res.* 66.3 (2007), pp. 373–407. DOI: 10.1007/s00186-007-0161-1.

[32] Gurobi Optimization, Inc. *Gurobi Optimizer Reference Manual, Version 6.5*. 2015.

[33] S.-P. Han and O. L. Mangasarian. “Exact penalty functions in nonlinear programming.” In: *Mathematical Programming* 17.1 (1979), pp. 251–269. DOI: 10.1007/BF01588250.

[34] S. Hanafi, J. Lazić, and N. Mladenović. “Variable Neighbourhood Pump Heuristic for 0-1 Mixed Integer Programming Feasibility.” In: *Electronic Notes in Discrete Mathematics* 36 (2010). ISCO 2010 - International Symposium on
REFERENCES

Combinatorial Optimization, pp. 759–766. doi: 10.1016/j.endm.2010.05.096

[35] T. Koch, T. Achterberg, E. Andersen, O. Bastert, T. Berthold, R. E. Bixby, E. Danna, G. Gamrath, A. M. Gleixner, S. Heinz, A. Lodi, H. Mittelmann, T. Ralphs, D. Salvagnin, D. E. Steffy, and K. Wolter. “MIPLIB 2010.” In: Mathematical Programming Computation 3.2 (2011), pp. 103–163. doi: 10.1007/s12532-011-0025-9

[36] O. L. Mangasarian. “Solution of general linear complementarity problems via nondifferentiable concave minimization.” In: Acta Mathematica Vietnamica 22.1 (1997), pp. 199–205.

[37] J. Nocedal and S. J. Wright. Numerical Optimization, 2nd ed. Springer Series in Operations Research and Financial Engineering. New York: Springer Verlag, 2006. doi: 10.1007/978-0-387-40065-5

[38] I. Nowak. Relaxation and decomposition methods for mixed integer nonlinear programming. 2005. URL: urn:nbn:de:kobv:11-10038479

[39] S. Sharma. “Mixed-integer nonlinear programming heuristics applied to a shale gas production optimization problem.” Master’s thesis. Norwegian University of Science and Technology, 2013.

[40] S. Sharma, B. R. Knudsen, and B. Grimstad. “Towards an objective feasibility pump for convex MINLPs.” In: Computational Optimization and Applications 63.3 (2015), pp. 737–753. doi: 10.1007/s10589-015-9792-y

[41] R. E. Wendell and A. P. Hurter Jr. “Minimization of a non-separable objective function subject to disjoint constraints.” In: Operations Research 24.4 (1976), pp. 643–657. doi: 10.1287/opre.24.4.643
### Appendix A. Detailed Results for the MIPLIB 2010

Table 1: Detailed numerical results for the penalty ADM based feasibility pump on all 84 feasible MIPLIB 2010 instances; see Section 5.1 Mixed-Integer Linear Problems. Objective function value, running times (s), outer penalty iterations (#Pen.), and inner ADM iterations (#ADM) for the method with $\lambda = 0.9$, multiplicative penalty parameter update rule $\text{inc}_m$, activated Gurobi presolve, and deactivated local branching.

| Instance       | Objective | Time  | #Pen. | #ADM |
|----------------|-----------|-------|-------|------|
| 30n20b8        | 906.00    | 40.42 | 892   | 3828 |
| acc-tight5     |           |       |       |      |
| allow40b       | 3565.00   | 2.48  | 146   | 437  |
| air04          | 71018.00  | 60.42 | 41    | 190  |
| app1-2         | -30.00    | 3189.86 | 4133 | 15567 |
| bab5           | -78587.20 | 34.25 | 178   | 540  |
| beasleyC3      | 863.00    | 0.19  | 24    | 67   |
| biella1        | 3743460.00| 65.50 | 26    | 238  |
| bienst2        | 73.25     | 0.21  | 20    | 76   |
| binkar10-1     | 7263.57   | 0.23  | 32    | 112  |
| bley_xll       | 285.00    | 4.62  | 37    | 149  |
| bnatt350       |           |       |       |      |
| core2536-691   | 692.00    | 31.37 | 19    | 106  |
| cov1075        | 120.00    | 0.17  | 12    | 24   |
| cshed010       |           |       |       |      |
| danoint        | 78.00     | 1.70  | 24    | 101  |
| dfn-gwn-UUM    | 125512.00 | 0.28  | 66    | 166  |
| eil33-2        | 1373.60   | 0.74  | 19    | 64   |
| eilB101        | 1513.00   | 0.94  | 17    | 66   |
| enlight13      | 0.00      | 0.00  | 1     | 1    |
| ex9            | 0.00      | 0.00  | 1     | 1    |
| glass4         | 3390030000.00 | 0.28 | 103 | 353 |
| gnu-35-40      | -2159090.00 | 1.80 | 298 | 1073 |
| iis-100-0-cov  | 100.00    | 0.27  | 12    | 24   |
| iis-bupa-cov   | 100.00    | 2.16  | 12    | 38   |
| iis-pima-cov   | 74.00     | 3.23  | 12    | 44   |
| lecstched-4-obj| 9.00      | 3.46  | 66    | 287  |
| m100n500k4r1   | -20.00    | 0.76  | 30    | 203  |
| macrophage     | 522.00    | 0.32  | 14    | 30   |
| map18          | -280.00   | 250.17 | 120  | 317  |
| map20          | -371.00   | 159.76 | 127  | 306  |
| mcsched        | 228737.00 | 1.37  | 13    | 56   |
| mik-250-1-100-1| 284980.00 | 0.48  | 71    | 202  |
| mine-166-5     | -22751900.00 | 1.16 | 22   | 54   |
| mine-90-10     | -558839000.00 | 1.69 | 58   | 173  |
| msc98-ju       | 25797700.00 | 146.92 | 48   | 307  |
| msspp16        | 407.00    | 1399.15 | 91   | 249  |
| mzzv11         | -16438.00 | 94.84 | 102  | 440  |
| n3div36        | 170600.00 | 13.36 | 103  | 321  |
| n3seq24        | 53600.00  | 271.77 | 71   | 315  |
| n4-3           | 13980.00  | 0.28  | 13    | 35   |
Table 1: Detailed numerical results for the penalty ADM based feasibility pump on all 84 feasible MIPLIB 2010 instances; see Section 5.1 Mixed-Integer Linear Problems. Objective function value, running times (s), outer penalty iterations (#Pen.), and inner ADM iterations (#ADM) for the method with $\lambda = 0.9$, multiplicative penalty parameter update rule $\text{inc}_m$, activated Gurobi presolve, and deactivated local branching.

| Instance      | Objective | Time  | #Pen. | #ADM |
|---------------|-----------|-------|-------|------|
| neos-1109824  | 687.00    | 3.93  | 93    | 295  |
| neos13        | -28.04    | 5.54  | 23    | 91   |
| neos-1337307  | -201818.00| 2.99  | 29    | 100  |
| neos-1396125  |           |       |       |      |
| neos-1601936  | 23409.00  | 266.56| 66    | 629  |
| neos18        | 17.00     | 0.87  | 21    | 86   |
| neos-476283   | 407.01    | 66.98 | 78    | 226  |
| neos-686190   | 16410.00  | 1.87  | 47    | 143  |
| neos-849702   |           |       |       |      |
| neos-916792   | 44.56     | 5.64  | 100   | 335  |
| neos-934278   | 264.00    | 1091.80| 57   | 225  |
| net12         | 337.00    | 29.06 | 53    | 217  |
| netdiversion  |           |       |       |      |
| newdano       | 89.75     | 0.44  | 19    | 77   |
| noswot        | -38.00    | 0.17  | 61    | 240  |
| ns1208400     |           |       |       |      |
| ns1688347     | 35.00     | 6.14  | 49    | 227  |
| ns1758913     | -1454.67  | 18.37 | 5     | 15   |
| ns1830653     |           |       |       |      |
| opm2-27-s2    | -1519.00  | 35.30 | 13    | 37   |
| pg5_34        | -12628.50 | 0.76  | 65    | 195  |
| pigeon-10     | -9000.00  | 2.51  | 1091  | 2732 |
| pw-myciel4    | 13.00     | 3.99  | 20    | 130  |
| qiu           | 1235.01   | 0.44  | 19    | 51   |
| rail507       | 183.00    | 30.66 | 35    | 158  |
| ran16x16      | 4734.00   | 0.12  | 52    | 144  |
| reblock67     | -18739300.00| 1.14 | 51    | 169  |
| rmatri100-p10 | 494.00    | 2.49  | 12    | 41   |
| rmatri100-p5  | 1327.00   | 5.18  | 11    | 39   |
| rmine6        | -239.11   | 3.21  | 32    | 93   |
| rocII-4-11    | -0.52     | 18.74 | 292   | 1162 |
| rococcoC10-001000 | 34598.00 | 1.43 | 68    | 295  |
| roll3000      | 18404.00  | 2.83  | 48    | 207  |
| satellites1-25| 33.00     | 44.26 | 24    | 98   |
| sp98ic        | 558066000.00| 12.55| 84    | 313  |
| sp98ir        | 244287000.00| 4.73 | 49    | 168  |
| tanglegram1   | 6478.00   | 114.05| 16    | 35   |
| tanglegram2   | 1445.00   | 2.51  | 16    | 34   |
| tinttab1      | 1415540.00| 0.27  | 41    | 153  |
| triptim1      | 22.87     | 317.56| 7     | 16   |
| unitcal_7     | 204266000.00| 40.69| 112   | 400  |
| vphhard       | 44.00     | 85.31 | 38    | 242  |
| zib54-UUE     | 131644000.00| 0.44 | 27    | 75   |
Table 2: Detailed numerical results for the penalty ADM based feasibility pump on all 889 MINLPLib2 instances with integer variables. Objective function value, running times (s), outer penalty iterations (#Pen.), and inner ADM iterations (#ADM) for the method with $\lambda = 0.9$, additive penalty parameter update rule $\text{inc}_a$, and deactivated local branching.

| Instance          | Objective | Time | #Pen. | #ADM |
|-------------------|-----------|------|-------|------|
| 4stufen           | 118114.00 | 9    | 74    | 169  |
| alan              | 3.00      | 0    | 5     | 7    |
| autocorr_bern20-03| -64.00    | 0    | 1     | 1    |
| autocorr_bern20-05| -396.00   | 2    | 1     | 1    |
| autocorr_bern20-10| -2912.00  | 1    | 1     | 1    |
| autocorr_bern20-15| -5936.00  | 1    | 1     | 1    |
| autocorr_bern25-03| -80.00    | 0    | 1     | 1    |
| autocorr_bern25-06| -936.00   | 0    | 3     | 4    |
| autocorr_bern25-13| -7984.00  | 0    | 1     | 1    |
| autocorr_bern25-19| -14472.00 | 0    | 1     | 1    |
| autocorr_bern25-25| -10352.00 | 0    | 1     | 1    |
| autocorr_bern30-04| -288.00   | 0    | 1     | 1    |
| autocorr_bern30-08| -2912.00  | 0    | 1     | 1    |
| autocorr_bern30-15| -15384.00 | 0    | 1     | 1    |
| autocorr_bern30-23| -30240.00 | 1    | 1     | 1    |
| autocorr_bern30-30| -22640.00 | 1    | 1     | 1    |
| autocorr_bern35-04| -344.00   | 0    | 1     | 1    |
| autocorr_bern35-09| -4976.00  | 0    | 1     | 1    |
| autocorr_bern35-18| -30712.00 | 0    | 1     | 1    |
| autocorr_bern35-26| -54960.00 | 0    | 1     | 1    |
| autocorr_bern35-35| -40272.00 | 1    | 1     | 1    |
| autocorr_bern40-05| -908.00   | 0    | 1     | 1    |
| autocorr_bern40-10| -8192.00  | 0    | 1     | 1    |
| autocorr_bern40-20| -50228.00 | 1    | 5     | 7    |
| autocorr_bern40-30| -94040.00 | 1    | 1     | 1    |
| autocorr_bern40-40| -66832.00 | 1    | 1     | 1    |
| autocorr_bern45-05| -1004.00  | 0    | 3     | 4    |
| autocorr_bern45-11| -12532.00 | 0    | 1     | 1    |
| autocorr_bern45-23| -8484.00  | 0    | 1     | 1    |
| autocorr_bern45-34| -151768.00| 0    | 1     | 1    |
| autocorr_bern45-45| -108528.00| 1    | 1     | 1    |
| autocorr_bern50-06| -2072.00  | 1    | 3     | 4    |
| autocorr_bern50-13| -23176.00 | 0    | 1     | 1    |
| autocorr_bern50-25| -123764.00| 1    | 3     | 4    |
| autocorr_bern50-38| -232808.00| 0    | 1     | 1    |
| autocorr_bern50-50| -166168.00| 1    | 1     | 1    |
| autocorr_bern55-06| -2288.00  | 0    | 3     | 4    |
| autocorr_bern55-14| -32280.00 | 1    | 3     | 4    |
| autocorr_bern55-28| -189404.00| 0    | 1     | 1    |
| autocorr_bern55-41| -335980.00| 0    | 1     | 1    |
| autocorr_bern55-55| -238296.00| 1    | 1     | 1    |
| autocorr_bern60-08| -6712.00  | 0    | 1     | 1    |
Table 2: Detailed numerical results for the penalty ADM based feasibility pump on all 889 MINLPLib 2 instances with integer variables. Objective function value, running times (s), outer penalty iterations (#Pen.), and inner ADM iterations (#ADM) for the method with $\lambda = 0.9$, additive penalty parameter update rule $\text{inc}_a$, and deactivated local branching.

| Instance          | Objective | Time  | #Pen. | #ADM |
|-------------------|-----------|-------|-------|------|
| autocorr_bern60-15| -44368.00 | 0     | 1     | 1    |
| autocorr_bern60-30| -258304.00| 0     | 1     | 1    |
| autocorr_bern60-45| -476456.00| 1     | 1     | 1    |
| autocorr_bern60-60| -347372.00| 1     | 1     | 1    |
| batch             | 309205.00 | 2     | 22    | 43   |
| batch0812         | 2838520.00| 5     | 64    | 107  |
| batch0812_nc      | 3534900.00| 4     | 32    | 70   |
| batchdes          | 185769.00 | 0     | 7     | 13   |
| batch_nc          | 363583.00 | 5     | 60    | 105  |
| batch101006m      | 776397.00 | 5     | 24    | 65   |
| batch121208m      | 1336450.00| 7     | 22    | 73   |
| batch151208m      | 1588930.00| 12    | 37    | 120  |
| batch201210m      | 2408440.00| 17    | 72    | 157  |
| bchoco05          | 0.95      | 1     | 3     | 4    |
| bchoco06          | 0.96      | 0     | 3     | 4    |
| bchoco07          | 0.96      | 2     | 3     | 4    |
| bchoco08          | —         | —     | —     | —    |
| beuster           | 128512.00 | 15    | 99    | 254  |
| blend029          | —         | —     | 68919 | 68961|
| blend146          | 29.93     | 8     | 76    | 110  |
| blend480          | -8.24     | 56    | 655   | 753  |
| blend531          | —         | —     | 53222 | 53555|
| blend718          | 1.23      | 6     | 30    | 77   |
| blend721          | -0.87     | 4     | 27    | 56   |
| blend852          | 46.13     | 9     | 87    | 126  |
| blendgap          | -0.00     | 0     | 1     | 1    |
| cardqp_inlp       | 3843.61   | 0     | 1     | 1    |
| cardqp_iqp        | 3843.61   | 0     | 1     | 1    |
| carton7           | 303.88    | 368   | 4645  | 5010 |
| carton9           | 340.75    | 15    | 77    | 155  |
| casctanks         | 9.16      | 3     | 40    | 46   |
| case_1scv2        | 7791.24   | 16    | 33    | 86   |
| cecil_13          | -115564.00| 9     | 65    | 92   |
| chp_partload      | —         | —     | 25492 | 25701|
| clay0203h         | 41709.80  | 81    | 1498  | 1519 |
| clay0203m         | 41737.50  | 50    | 1002  | 1022 |
| clay0204h         | 7830.00   | 5     | 50    | 82   |
| clay0204m         | 10340.00  | 3     | 35    | 48   |
| clay0205h         | 23484.20  | 43    | 609   | 715  |
| clay0205m         | 9715.00   | 5     | 57    | 86   |
| clay0303h         | 36613.00  | 56    | 954   | 996  |
| clay0303m         | 41737.50  | 8     | 101   | 170  |
| clay0304h         | 61315.90  | 52    | 764   | 859  |
| clay0304m         | 61831.50  | 38    | 694   | 748  |
Table 2: Detailed numerical results for the penalty ADM based feasibility pump on all 889 MINLPLib 2 instances with integer variables. Objective function value, running times (s), outer penalty iterations (#Pen.), and inner ADM iterations (#ADM) for the method with $\lambda = 0.9$, additive penalty parameter update rule $inc_a$, and deactivated local branching.

| Instance         | Objective | Time | #Pen. | #ADM |
|------------------|-----------|------|-------|------|
| clay0305h        | 43405.70  | 10   | 132   | 161  |
| clay0305m        | 24593.10  | 7    | 92    | 121  |
| contvar          | 829318.00 | 7    | 8     | 35   |
| crossdock_15x7   | 14467.00  | 6    | 62    | 99   |
| crossdock_15x8   | 16765.00  | 1856 | 26313 | 27298|
| crudeoil_lee1_05 | 79.35     | 75   | 808   | 851  |
| crudeoil_lee1_06 | 78.75     | 50   | 555   | 655  |
| crudeoil_lee1_07 | 78.75     | 2868 | 24641 | 25051|
| crudeoil_lee1_08 | 79.35     | 18   | 72    | 106  |
| crudeoil_lee1_09 | 78.75     | 918  | 5556  | 5841 |
| crudeoil_lee1_10 | 78.75     | 32   | 103   | 153  |
| crudeoil_lee2_05 | 90.00     | 31   | 117   | 158  |
| crudeoil_lee2_06 | 97.59     | 114  | 265   | 401  |
| crudeoil_lee2_07 | 90.00     | 329  | 949   | 1114 |
| crudeoil_lee2_08 | —         | —    | —     | —    |
| crudeoil_lee2_09 | —         | —    | —     | —    |
| crudeoil_lee2_10 | —         | —    | 7412  | 8007 |
| crudeoil_lee3_05 | 82.00     | 173  | 996   | 1033 |
| crudeoil_lee3_06 | 84.49     | 125  | 574   | 606  |
| crudeoil_lee3_07 | 82.90     | 29   | 35    | 65   |
| crudeoil_lee3_08 | —         | —    | 13112 | 13645|
| crudeoil_lee3_09 | 82.75     | 66   | 85    | 108  |
| crudeoil_lee3_10 | 77.50     | 144  | 145   | 198  |
| crudeoil_lee4_05 | 132.48    | 13   | 15    | 21   |
| crudeoil_lee4_06 | 132.49    | 16   | 15    | 22   |
| crudeoil_lee4_07 | 132.55    | 170  | 272   | 333  |
| crudeoil_lee4_08 | 131.54    | 175  | 192   | 299  |
| crudeoil_lee4_09 | —         | —    | 5326  | 5417 |
| crudeoil_lee4_10 | —         | —    | 4329  | 4802 |
| crudeoil_li01    | 4852.37   | 1247 | 19259 | 19368|
| crudeoil_li02    | —         | —    | —     | —    |
| crudeoil_li03    | —         | —    | 31637 | 32220|
| crudeoil_li05    | 3030.59   | 34   | 225   | 312  |
| crudeoil_li06    | 3303.84   | 111  | 869   | 947  |
| crudeoil_li11    | —         | —    | 26706 | 27063|
| crudeoil_li21    | —         | —    | 19307 | 19619|
| csched1           | -30174.60 | 0    | 4     | 5    |
| csched1a          | -29903.30 | 1    | 7     | 13   |
| csched2           | -160668.00| 2    | 9     | 20   |
| csched2a          | -162047.00| 7    | 44    | 120  |
| deb10             | 209.43    | 1    | 9     | 13   |
| deb6              | 251.66    | 1    | 3     | 4    |
| deb7              | 176.08    | 2    | 3     | 4    |
| deb8              | 176.08    | 1    | 3     | 4    |
Table 2: Detailed numerical results for the penalty ADM based feasibility pump on all 889 MINLPLib 2 instances with integer variables. Objective function value, running times (s), outer penalty iterations (#Pen.), and inner ADM iterations (#ADM) for the method with $\lambda = 0.9$, additive penalty parameter update rule inc$_a$, and deactivated local branching.

| Instance       | Objective | Time | #Pen. | #ADM |
|----------------|-----------|------|-------|------|
| deb9           | 176.08    | 1    | 3     | 4    |
| densitymod     | —         | —    | 4     | 8    |
| dosemind2l     | 173.98    | 11   | 5     | 11   |
| dosemind3d     | 1.32      | 29   | 7     | 13   |
| du-opt         | 5.34      | 0    | 3     | 4    |
| du-opt5        | 112.02    | 0    | 5     | 8    |
| edgecross10-010| 4.00      | 0    | 1     | 1    |
| edgecross10-020| 19.00     | 0    | 1     | 1    |
| edgecross10-030| 53.00     | 0    | 3     | 4    |
| edgecross10-040| 142.00    | 0    | 3     | 4    |
| edgecross10-050| 301.00    | 1    | 1     | 1    |
| edgecross10-060| 470.00    | 0    | 1     | 1    |
| edgecross10-070| 735.00    | 0    | 1     | 1    |
| edgecross10-080| 1048.00   | 0    | 1     | 1    |
| edgecross10-090| 1387.00   | 0    | 1     | 1    |
| edgecross14-019| 5.00      | 0    | 1     | 1    |
| edgecross14-039| 123.00    | 1    | 1     | 1    |
| edgecross14-058| 391.00    | 0    | 1     | 1    |
| edgecross14-078| 725.00    | 0    | 1     | 1    |
| edgecross14-098| 1392.00   | 0    | 1     | 1    |
| edgecross14-117| 2168.00   | 1    | 3     | 4    |
| edgecross14-137| 2880.00   | 0    | 1     | 1    |
| edgecross14-156| 4342.00   | 0    | 3     | 4    |
| edgecross14-176| 5956.00   | 1    | 1     | 1    |
| edgecross20-040| 73.00     | 1    | 1     | 1    |
| edgecross20-080| 530.00    | 1    | 1     | 1    |
| edgecross22-048| 97.00     | 2    | 1     | 1    |
| edgecross22-096| 980.00    | 4    | 3     | 4    |
| edgecross24-057| 213.00    | 3    | 1     | 1    |
| edgecross24-115| 1429.00   | 3    | 1     | 1    |
| eg_all_s       | 8.67      | 11   | 20    | 44   |
| eg_disc2_s     | 5.68      | 3    | 3     | 4    |
| eg_disc_s      | 6.03      | 6    | 7     | 15   |
| eg_int_s       | 8.32      | 7    | 6     | 13   |
| elf            | 1.68      | 1    | 1     | 1    |
| eniplac        | -120713.00| 43   | 636   | 764  |
| enpro48pb      | 188887.00 | 1    | 10    | 23   |
| enpro56pb      | 266762.00 | 3    | 24    | 56   |
| ethanolh       | -157.59   | 0    | 4     | 5    |
| ethanolm       | -31.27    | 9    | 116   | 168  |
| ex1221         | 7.67      | 0    | 1     | 1    |
| ex1222         | 1.08      | 0    | 6     | 10   |
| ex1223         | 5.81      | 1    | 6     | 10   |
| ex1223a        | 5.81      | 1    | 5     | 8    |
Table 2: Detailed numerical results for the penalty ADM based feasibility pump on all 889 MINLPLib 2 instances with integer variables. Objective function value, running times (s), outer penalty iterations (#Pen.), and inner ADM iterations (#ADM) for the method with $\lambda = 0.9$, additive penalty parameter update rule $\text{inc}_a$, and deactivated local branching.

| Instance | Objective | Time  | #Pen. | #ADM |
|----------|-----------|-------|-------|------|
| ex1223b  | 5.81      | 1     | 6     | 10   |
| ex1224   | -0.88     | 1     | 12    | 19   |
| ex1225   | 31.00     | 0     | 6     | 9    |
| ex1226   | -17.00    | 0     | 1     | 1    |
| ex1233   | 201540.00 | 3     | 21    | 49   |
| ex1243   | 135552.00 | 2     | 18    | 47   |
| ex1244   | 95046.40  | 1     | 13    | 17   |
| ex1252   | —         | —     | —     | —    |
| ex1252a  | 143555.00 | 3     | 23    | 54   |
| ex1263   | 69.60     | 86    | 1665  | 1773 |
| ex1263a  | 38.60     | 55    | 1101  | 1142 |
| ex1264   | 31.60     | 11    | 185   | 222  |
| ex1264a  | 10.00     | 5     | 90    | 108  |
| ex1265   | 22.30     | 16    | 231   | 343  |
| ex1265a  | 15.10     | 4     | 60    | 87   |
| ex1266   | —         | —     | 62770 | 64607|
| ex1266a  | 16.30     | 1     | 7     | 13   |
| ex3pb    | 103.58    | 2     | 19    | 37   |
| ex4      | 659.57    | 4     | 46    | 77   |
| fac1     | 172954000.00 | 9 | 85    | 185  |
| fac2     | 407585000.00 | 5 | 35    | 97   |
| fac3     | 347895000.00 | 3 | 23    | 53   |
| faclay20h| 16941.00  | 0     | 1     | 1    |
| faclay25 | 5107.00   | 1     | 1     | 1    |
| faclay30 | 8970.00   | 3     | 1     | 1    |
| faclay30h| 50775.00  | 3     | 1     | 1    |
| faclay33 | 67677.00  | 6     | 1     | 1    |
| faclay35 | 77040.00  | 8     | 1     | 1    |
| faclay60 | 1564130.00 | 15  | 1     | 1    |
| faclay70 | 1656440.00 | 2316| 1     | 1    |
| faclay75 | —         | —     | 1     | 0    |
| faclay80 | —         | —     | 1     | 0    |
| feedtray | -13.41    | 1     | 1     | 1    |
| feedtray2| —         | —     | —     | —    |
| fin2bb   | 0.00      | 2     | 19    | 26   |
| flay02h  | 37.95     | 2     | 20    | 40   |
| flay02m  | 37.95     | 2     | 24    | 45   |
| flay03h  | 48.99     | 4     | 26    | 60   |
| flay03m  | 48.99     | 3     | 29    | 61   |
| flay04h  | 54.99     | 6     | 50    | 100  |
| flay04m  | 54.99     | 4     | 42    | 85   |
| flay05h  | 80.99     | 7     | 36    | 97   |
| flay05m  | 64.50     | 5     | 41    | 96   |
| flay06h  | 106.88    | 11    | 38    | 121  |
Table 2: Detailed numerical results for the penalty ADM based feasibility pump on all 889 MINLPLib 2 instances with integer variables. Objective function value, running times (s), outer penalty iterations (#Pen.), and inner ADM iterations (#ADM) for the method with $\lambda = 0.9$, additive penalty parameter update rule inc$_a$, and deactivated local branching.

| Instance      | Objective | Time  | #Pen. | #ADM  |
|---------------|-----------|-------|-------|-------|
| flay06m       | 66.93     | 6     | 43    | 102   |
| fo7           | 25.60     | 87    | 1643  | 1691  |
| fo7_2         | 40.43     | 15    | 171   | 236   |
| fo7_ar2_1     | 55.38     | 317   | 5449  | 5627  |
| fo7_ar25_1    | —         | —     | 65860 | 66079 |
| fo7_ar3_1     | 34.49     | 13    | 97    | 212   |
| fo7_ar4_1     | 41.66     | 1369  | 25281 | 25479 |
| fo7_ar5_1     | 38.09     | 7     | 53    | 110   |
| fo8           | 47.40     | 7     | 66    | 99    |
| fo8_ar2_1     | —         | —     | 62624 | 62999 |
| fo8_ar25_1    | —         | —     | 61119 | 61451 |
| fo8_ar3_1     | 45.34     | 38    | 469   | 590   |
| fo8_ar4_1     | 38.66     | 36    | 384   | 529   |
| fo8_ar5_1     | 51.84     | 370   | 5769  | 5963  |
| fo9           | 52.78     | 7     | 60    | 109   |
| fo9_ar2_1     | —         | —     | 52908 | 53324 |
| fo9_ar25_1    | —         | —     | 57149 | 57561 |
| fo9_ar3_1     | 49.40     | 46    | 446   | 592   |
| fo9_ar4_1     | —         | —     | 54254 | 54687 |
| fo9_ar5_1     | 51.17     | 11    | 72    | 131   |
| fuel          | 8566.12   | 2     | 16    | 42    |
| fuzzy         | —         | —     | 1     | 10135 |
| gams01        | 26878.30  | 36    | 100   | 261   |
| gams02        | 99281400.00 | 3427 | 7309  | 8005  |
| gams03        | —         | —     | 10477 | 10747 |
| gasnet        | 6999380.00 | 10   | 97    | 194   |
| gasprod_sarawak01 | -31599.40 | 1   | 5   | 7           |
| gasprod_sarawak16 | -31399.40 | 5   | 11  | 16         |
| gasprod_sarawak81 | -31399.40 | 60  | 11  | 16         |
| gastrans      | —         | —     | 71678 | 71681 |
| gastrans040   | 0.00      | 1     | 3    | 4     |
| gastrans135   | 0.00      | 63    | 517   | 524   |
| gastrans582_cold13 | —         | —     | 24865 | 24973 |
| gastrans582_cold13_95 | —         | —     | 26008 | 26017 |
| gastrans582_cold17 | 0.00      | 12    | 34    | 40    |
| gastrans582_cold17_95 | 0.00      | 13    | 38    | 45    |
| gastrans582_cool12 | 0.00      | 10    | 22    | 27    |
| gastrans582_cool12_95 | —         | —     | 25893 | 25904 |
| gastrans582_cool14 | —         | —     | 6717  | 20655 |
| gastrans582_cool14_95 | —         | —     | 24012 | 24195 |
| gastrans582_freezing27 | —         | —     | 21961 | 22878 |
| gastrans582_freezing27_95 | —         | —     | 649   | 19415 |
| gastrans582_freezing30 | —         | —     | 25209 | 25221 |
| gastrans582_freezing30_95 | 0.00      | 108   | 728   | 736   |
Table 2: Detailed numerical results for the penalty ADM based feasibility pump on all 889 MINLPLib 2 instances with integer variables. Objective function value, running times (s), outer penalty iterations (#Pen.), and inner ADM iterations (#ADM) for the method with $\lambda = 0.9$, additive penalty parameter update rule inc$_a$, and deactivated local branching.

| Instance             | Objective | Time | #Pen. | #ADM |
|----------------------|-----------|------|-------|------|
| gastrans582_mild10   | —         | —    | 23990 | 23997|
| gastrans582_mild10_95| —         | —    | 25798 | 25810|
| gastrans582_mild11   | 0.00      | 49   | 295   | 304  |
| gastrans582_mild11_95| —         | —    | 25515 | 25527|
| gastrans582_warm15   | —         | —    | 24775 | 25032|
| gastrans582_warm15_95| —         | —    | 24674 | 24934|
| gastrans582_warm31   | 0.00      | 44   | 271   | 279  |
| gastrans582_warm31_95| —         | —    | 26392 | 26403|
| gbd                  | 2.20      | 0    | 1     | 1    |
| gear                 | 0.00      | 1    | 3     | 4    |
| gear2                | 0.01      | 0    | 4     | 6    |
| gear3                | 0.00      | 1    | 3     | 4    |
| gear4                | 120.67    | 5    | 70    | 87   |
| genpooling_lec1      | —         | —    | —     | —    |
| genpooling_lec2      | —         | —    | —     | —    |
| genpooling_meyer04   | —         | —    | —     | —    |
| genpooling_meyer10   | —         | —    | —     | —    |
| genpooling_meyer15   | —         | —    | —     | —    |
| ghg_1veh             | 7.78      | 0    | 1     | 1    |
| ghg_2veh             | 7.78      | 0    | 1     | 1    |
| ghg_3veh             | 7.77      | 0    | 4     | 5    |
| gkocis               | -1.41     | 1    | 7     | 9    |
| graphpart_2g-0044-1601| -789955.00| 1    | 1     | 1    |
| graphpart_2g-0055-0062| —         | —    | 68626 | 68626|
| graphpart_2g-0066-0066| -2082170.00| 11   | 110   | 215  |
| graphpart_2g-0077-0077| —         | —    | 66436 | 66436|
| graphpart_2g-0088-0088| -5701630.00| 0    | 1     | 1    |
| graphpart_2g-0099-9211| -4495840.00| 0    | 4     | 5    |
| graphpart_2g-1010-0824| -6583360.00| 0    | 1     | 1    |
| graphpart_2pm-0044-0044| -11.00    | —    | —     | —    |
| graphpart_2pm-0055-0055| —         | —    | 70312 | 70312|
| graphpart_2pm-0066-0066| -27.00    | 1    | 6     | 9    |
| graphpart_2pm-0077-0777| —         | —    | 64537 | 64537|
| graphpart_2pm-0088-0888| -46.00    | 0    | 4     | 5    |
| graphpart_2pm-0099-0999| -56.00    | 1    | 18    | 22   |
| graphpart_3g-0234-0234| —         | —    | 69513 | 69513|
| graphpart_3g-0244-0244| -2702200.00| 0    | 1     | 1    |
| graphpart_3g-0333-0333| —         | —    | 66677 | 66677|
| graphpart_3g-0334-0334| -3279970.00| 0    | 1     | 1    |
| graphpart_3g-0344-0344| —         | —    | 66770 | 66770|
| graphpart_3g-0444-0444| -6621100.00| 0    | 1     | 1    |
| graphpart_3pm-0234-0234| —         | —    | 74786 | 74786|
| graphpart_3pm-0244-0244| -27.00    | 0    | 1     | 1    |
| graphpart_3pm-0333-0333| —         | —    | 71653 | 71653|
Table 2: Detailed numerical results for the penalty ADM based feasibility pump on all 889 MINLPlib 2 instances with integer variables. Objective function value, running times (s), outer penalty iterations (#Pen.), and inner ADM iterations (#ADM) for the method with $\lambda = 0.9$, additive penalty parameter update rule $\text{inc}_\lambda$, and deactivated local branching.

| Instance              | Objective | Time | #Pen. | #ADM |
|-----------------------|-----------|------|-------|------|
| graphpart_3pm-0334-0334 | -33.00    | 0    | 1     | 1    |
| graphpart_3pm-0344-0344 | —         | —    | 68708 | 68708|
| graphpart_3pm-0444-0444 | -57.00    | 1    | 4     | 5    |
| graphpart_clique-20    | 147.00    | 0    | 1     | 1    |
| graphpart_clique-30    | 495.00    | 0    | 1     | 1    |
| graphpart_clique-40    | 1183.00   | 0    | 1     | 1    |
| graphpart_clique-50    | —         | —    | 60889 | 60889|
| graphpart_clique-60    | 4010.00   | 2    | 34    | 36   |
| graphpart_clique-70    | —         | —    | 50784 | 50784|
| lda                   | -4322.55  | 6    | 18    | 39   |
| heatexch_gen1         | 410804.00 | 2    | 19    | 42   |
| heatexch_gen2         | 739019.00 | 1    | 20    | 26   |
| heatexch_gen3         | 109097.00 | 4    | 28    | 46   |
| heatexch_spec1        | 219858.00 | 2    | 18    | 42   |
| heatexch_spec2        | 849922.00 | 0    | 3     | 4    |
| heatexch_spec3        | 319465.00 | 1    | 8     | 13   |
| heatexch_trigen       | 977262.00 | 144  | 2367  | 2602 |
| hmittelman            | 16.00     | 2    | 22    | 39   |
| hybriddynamic_fixed   | 1.47      | 0    | 7     | 12   |
| hybriddynamic_var     | 1.54      | 0    | 5     | 9    |
| hydroenergy1          | 207178.00 | 2    | 29    | 39   |
| hydroenergy2          | 369251.00 | 4    | 31    | 42   |
| hydroenergy3          | 742404.00 | 7    | 31    | 44   |
| ibr2                  | 4.88      | 1032 | 55    | 125  |
| jit1                  | 173983.00 | 0    | 3     | 4    |
| johnall               | -222.37   | 5    | 54    | 57   |
| kport20               | 33.50     | 2    | 14    | 37   |
| kport40               | 42.02     | 70   | 1184  | 1289 |
| lip                   | 5428650.00| 0    | 3     | 6    |
| lop97ic               | 4553.18   | 7    | 10    | 31   |
| lop97icx              | 4590.48   | 3    | 13    | 31   |
| m3                    | 55.80     | 2    | 21    | 32   |
| m6                    | 123.98    | 13   | 105   | 206  |
| m7                    | —         | —    | 66377 | 66664|
| m7_ar2_1              | —         | —    | 65520 | 65796|
| m7_ar25_1             | —         | —    | 69125 | 69424|
| m7_ar3_1              | —         | —    | 66925 | 67206|
| m7_ar4_1              | 450.97    | 127  | 1993  | 2166 |
| m7_ar5_1              | 511.21    | 61   | 857   | 1025 |
| mbtd                  | 5.58      | 66   | 13    | 41   |
| meanvarx              | 14.37     | 1    | 23    | 26   |
| meanvarxsc            | —         | —    | —     | —    |
| milinfract            | 2.63      | 8    | 5     | 7    |
| minlphix              | 345.51    | 3    | 53    | 63   |
Table 2: Detailed numerical results for the penalty ADM based feasibility pump on all 889 MINLPLib 2 instances with integer variables. Objective function value, running times (s), outer penalty iterations (#Pen.), and inner ADM iterations (#ADM) for the method with $\lambda = 0.9$, additive penalty parameter update rule inc$_a$, and deactivated local branching.

| Instance            | Objective | Time | #Pen. | #ADM |
|---------------------|-----------|------|-------|------|
| multiplants_mtg1a   | 363.57    | 2    | 17    | 35   |
| multiplants_mtg1b   | 212.21    | 33   | 602   | 617  |
| multiplants_mtg1c   | 406.98    | 5    | 31    | 52   |
| multiplants_mtg2    | 7051.41   | 4    | 65    | 75   |
| multiplants_mtg5    | 4706.88   | 75   | 1228  | 1258 |
| multiplants_stgl1   | 250.44    | 1000 | 16363 | 16375|
| multiplants_stgl1a  | 38.85     | 19   | 267   | 284  |
| multiplants_stgl1b  | 136.38    | 35   | 498   | 506  |
| multiplants_stgl1c  | —         | —    | 61799 | 61808|
| multiplants_stgl5   | —         | —    | 52743 | 52952|
| multiplants_stgl6   | —         | —    | 45323 | 46702|
| ndcc12              | 108.11    | 8    | 59    | 106  |
| ndcc12persp         | —         | —    | —     | —    |
| ndcc13              | 94.17     | 6    | 17    | 56   |
| ndcc13persp         | —         | —    | —     | —    |
| ndcc14              | 130.16    | 19   | 70    | 247  |
| ndcc14persp         | —         | —    | —     | —    |
| ndcc15              | 95.28     | 4    | 22    | 50   |
| ndcc15persp         | —         | —    | —     | —    |
| ndcc16              | 131.63    | 7    | 28    | 67   |
| ndcc16persp         | —         | —    | —     | —    |
| netmod_doll1        | -0.01     | 18   | 22    | 40   |
| netmod_doll2        | -0.49     | 37   | 16    | 44   |
| netmod_karl1        | 0.00      | 3    | 17    | 31   |
| netmod_karl2        | 0.00      | 3    | 17    | 31   |
| no7_ar2_1           | —         | —    | 66159 | 66423|
| no7_ar25_1          | 175.87    | 156  | 2635  | 2871 |
| no7_ar3_1           | 149.22    | 13   | 142   | 201  |
| no7_ar4_1           | 188.94    | 141  | 2040  | 2246 |
| no7_ar5_1           | 153.84    | 148  | 2440  | 2576 |
| nous1               | 1.57      | 0    | 3     | 4    |
| nous2               | 0.63      | 0    | 3     | 4    |
| nuclear104          | —         | —    | —     | —    |
| nuclear10a          | —         | —    | 1737  | 2056 |
| nuclear10b          | -1.15     | 1604 | 31    | 124  |
| nuclear14           | -1.13     | 3    | 4     | 5    |
| nuclear14a          | -1.13     | 1    | 1     | 1    |
| nuclear14b          | -1.09     | 13   | 51    | 73   |
| nuclear25           | -1.12     | 7    | 22    | 29   |
| nuclear25a          | -1.12     | 5    | 22    | 29   |
| nuclear25b          | -1.09     | 29   | 63    | 153  |
| nuclear49           | -1.15     | 30   | 13    | 17   |
| nuclear49a          | -1.15     | 26   | 25    | 33   |
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| Instance     | Objective | Time  | #Pen. | #ADM |
|--------------|-----------|-------|-------|------|
| nuclear49b   | -1.13     | 133   | 25    | 74   |
| nuclearva    | -1.01     | 3     | 33    | 45   |
| nuclearvb    | -1.02     | 1     | 1     | 1    |
| nuclearvc    | -0.98     | 5     | 49    | 59   |
| nuclearvd    | -1.04     | 10    | 63    | 75   |
| nuclearve    | -1.02     | 1     | 7     | 9    |
| nuclearvf    | —         | —     | —     | —    |
| nvs01        | 13.10     | 1     | 5     | 7    |
| nvs02        | 5.96      | 2     | 35    | 38   |
| nvs03        | 16.00     | 0     | 11    | 14   |
| nvs04        | —         | —     | —     | —    |
| nvs05        | 5.47      | 0     | 3     | 4    |
| nvs06        | 1.86      | 1     | 3     | 4    |
| nvs07        | 4.00      | 1     | 5     | 6    |
| nvs08        | 23.83     | 1     | 10    | 14   |
| nvs09        | -43.13    | 0     | 1     | 1    |
| nvs10        | -308.40   | 0     | 4     | 7    |
| nvs11        | -416.40   | 3     | 69    | 74   |
| nvs12        | -477.00   | 0     | 3     | 5    |
| nvs13        | -585.20   | 1     | 3     | 5    |
| nvs14        | -40358.20 | 2     | 35    | 38   |
| nvs15        | 1.00      | 0     | 3     | 4    |
| nvs16        | 14.20     | 7     | 55    | 108  |
| nvs17        | -1078.20  | 3     | 52    | 57   |
| nvs18        | -778.40   | 0     | 3     | 5    |
| nvs19        | -1070.00  | 1     | 10    | 17   |
| nvs20        | 230.92    | 1     | 7     | 13   |
| nvs21        | -4.27     | 8     | 172   | 174  |
| nvs22        | 6.06      | 0     | 4     | 6    |
| nvs23        | -1078.40  | 1     | 3     | 5    |
| nvs24        | -1001.00  | 1     | 7     | 14   |
| o7           | 190.62    | 25    | 374   | 418  |
| o7_2         | 161.38    | 17    | 147   | 281  |
| o7_ar2_1     | —         | —     | 61789 | 62056|
| o7_ar25_1    | 160.47    | 173   | 2936  | 3100 |
| o7_ar3_1     | 162.83    | 34    | 446   | 562  |
| o7_ar4_1     | 182.29    | 69    | 971   | 1138 |
| o7_ar5_1     | 166.99    | 85    | 1274  | 1443 |
| o8_ar4_1     | 345.47    | 22    | 130   | 256  |
| o9_ar4_1     | 341.57    | 55    | 497   | 691  |
| oacr         | -1.92     | 1     | 4     | 5    |
| oil          | -0.87     | 2     | 3     | 4    |
| oil2         | -0.73     | 1     | 3     | 4    |
| ortez        | -9532.04  | 2     | 13    | 20   |
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| Instance       | Objective | Time  | #Pen. | #ADM |
|----------------|-----------|-------|-------|------|
| parallel       | —         | —     | 65520 | 65520|
| pb302035       | 4052260.00| 28    | 25    | 34   |
| pb302055       | 4087020.00| 152   | 155   | 200  |
| pb302075       | 4624370.00| 62    | 49    | 77   |
| pb302095       | —         | —     | 3207  | 4701 |
| pb351535       | 5474850.00| 180   | 240   | 306  |
| pb351555       | 5256790.00| 26    | 21    | 41   |
| pb351575       | 6457260.00| 16    | 15    | 21   |
| pb351595       | 11847500.00| 915  | 919   | 1524 |
| pooling_epa1   | -280.81   | 8     | 110   | 134  |
| pooling_epa2   | -4268.72  | 3     | 36    | 44   |
| pooling_epa3   | —         | —     | —     | —    |
| portfol_buyin  | 0.04      | 1     | 3     | 4    |
| portfol_card   | 0.03      | 1     | 5     | 8    |
| portfol_classical050_1 | -0.09    | 1     | 10    | 21   |
| portfol_classical200_2 | -0.06    | 8     | 18    | 50   |
| portfol_robust050_34 | -0.00   | 1     | 11    | 16   |
| portfol_robust100_09  | -0.08 | 2     | 11    | 23   |
| portfol_robust100_09  | -0.08 | 12    | 16    | 44   |
| portfol_robust200_03  | 0.15 | 3412  | 5488  | 69538|
| portfol_roundlot | -1.06    | 1     | 10    | 15   |
| portfol_shortfall050_68 | -1.07 | 3     | 13    | 25   |
| portfol_shortfall100_04 | -1.06 | 7     | 14    | 27   |
| portfol_shortfall200_05 | 112235.00 | 1     | 10    | 13   |
| portfol_shortfall050_68 | -1.06 | 1     | 10    | 15   |
| portfol_robust050_34 | -0.00 | 1     | 11    | 16   |
| portfol_robust100_09  | -0.08 | 2     | 11    | 23   |
| portfol_robust200_03  | -0.08 | 12    | 16    | 44   |
| portfol_roundlot | 0.15     | 3412  | 5488  | 69538|
| prod02         | 112235.00 | 1     | 10    | 13   |
| prod03         | 11.00     | 2     | 18    | 31   |
| prod10         | 3.45      | 0     | 3     | 4    |
| procsel        | -1.41     | 0     | 5     | 7    |
| product        | -2075.33  | 16    | 56    | 103  |
| product2       | —         | —     | —     | —    |
| qap            | 411560.00 | 6     | 13    | 28   |
| qapw           | 395664.00 | 48    | 106   | 345  |
| ravempb        | 269590.00 | 1     | 10    | 23   |
| risk2pb        | -55.48    | 1     | 17    | 21   |
| routingdelay_bigm | —       | —     | 27714 | 27777|
| routingdelay_proj | 149.70    | 25    | 145   | 170  |
| rsyn0805h      | 1280.01   | 1     | 17    | 26   |
| rsyn0805m      | 928.36    | 3     | 24    | 50   |
| rsyn0805m02h   | 2121.61   | 5     | 9     | 24   |
| rsyn0805m02m   | 923.55    | 6     | 34    | 80   |
| rsyn0805m03h   | 2996.90   | 7     | 8     | 19   |
| rsyn0805m03m   | 2256.38   | 8     | 33    | 77   |
| rsyn0805m04h   | 7145.11   | 7     | 5     | 11   |
| rsyn0805m04m   | 4641.20   | 10    | 32    | 78   |
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| Instance           | Objective | Time | #Pen. | #ADM |
|--------------------|-----------|------|-------|------|
| rsyn0810h          | 1675.12   | 2    | 18    | 26   |
| rsyn0810m          | 801.54    | 3    | 21    | 49   |
| rsyn0810m02h       | 1631.39   | 5    | 9     | 24   |
| rsyn0810m02m       | 987.64    | 6    | 32    | 76   |
| rsyn0810m03h       | 2671.49   | 8    | 7     | 18   |
| rsyn0810m03m       | 2355.46   | 20   | 162   | 226  |
| rsyn0810m04h       | 6472.92   | 23   | 12    | 26   |
| rsyn0810m04m       | 4525.62   | 11   | 25    | 71   |
| rsyn0815h          | 1262.05   | 1    | 11    | 21   |
| rsyn0815m          | 923.47    | 3    | 26    | 52   |
| rsyn0815m02h       | 1579.85   | 6    | 11    | 30   |
| rsyn0815m02m       | 598.63    | 6    | 32    | 71   |
| rsyn0815m03h       | 2702.87   | 10   | 8     | 20   |
| rsyn0815m03m       | 2697.07   | 10   | 35    | 81   |
| rsyn0815m04h       | 3220.26   | 21   | 9     | 24   |
| rsyn0815m04m       | 1329.02   | 13   | 30    | 79   |
| rsyn0820h          | 1138.41   | 2    | 9     | 17   |
| rsyn0820m          | 723.84    | 4    | 24    | 56   |
| rsyn0820m02h       | 1005.44   | 6    | 8     | 18   |
| rsyn0820m02m       | 258.97    | 7    | 32    | 84   |
| rsyn0820m03h       | 1984.43   | 14   | 8     | 16   |
| rsyn0820m03m       | 1800.99   | 17   | 52    | 113  |
| rsyn0820m04h       | 2363.62   | 25   | 8     | 23   |
| rsyn0820m04m       | 1642.15   | 18   | 30    | 89   |
| rsyn0830h          | 506.70    | 2    | 10    | 19   |
| rsyn0830m          | 203.31    | 4    | 30    | 64   |
| rsyn0830m02h       | 661.04    | 4    | 8     | 18   |
| rsyn0830m02m       | -273.98   | 6    | 33    | 68   |
| rsyn0830m03h       | 1431.22   | 10   | 8     | 18   |
| rsyn0830m03m       | 137.73    | 12   | 34    | 85   |
| rsyn0830m04h       | 2284.01   | 21   | 9     | 19   |
| rsyn0830m04m       | 128.87    | 19   | 36    | 90   |
| rsyn0840h          | 297.64    | 2    | 6     | 14   |
| rsyn0840m          | 54.73     | 5    | 30    | 69   |
| rsyn0840m02h       | 612.11    | 5    | 10    | 18   |
| rsyn0840m02m       | -362.91   | 7    | 29    | 70   |
| rsyn0840m03h       | 2664.97   | 15   | 11    | 20   |
| rsyn0840m03m       | 2121.79   | 12   | 30    | 72   |
| rsyn0840m04h       | 2327.32   | 21   | 8     | 16   |
| rsyn0840m04m       | -18.45    | 21   | 32    | 84   |
| saa_2              | 12.75     | 35   | 50    | 78   |
| sepI               | -510.08   | 0    | 5     | 7    |
| sepasequ_complex   | 538.16    | 12   | 55    | 97   |
| sepasequ_convent   | 514.98    | 11   | 24    | 34   |
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| Instance          | Objective | Time | #Pen. | #ADM |
|-------------------|-----------|------|-------|------|
| sfacloc1_2_80     | 13.55     | 1    | 3     | 4    |
| sfacloc1_2_90     | 29.57     | 0    | 3     | 4    |
| sfacloc1_2_95     | 18.85     | 0    | 1     | 1    |
| sfacloc1_3_80     | 9.10      | 0    | 3     | 4    |
| sfacloc1_3_90     | 11.88     | 0    | 3     | 4    |
| sfacloc1_3_95     | 12.46     | 0    | 1     | 1    |
| sfacloc1_4_80     | 8.46      | 1    | 3     | 4    |
| sfacloc1_4_90     | 12.00     | 0    | 3     | 4    |
| sfacloc1_4_95     | 11.24     | 1    | 3     | 4    |
| sfacloc2_2_80     | —         | —    | 38561 | 38572|
| sfacloc2_2_90     | 29.96     | 7    | 77    | 109  |
| sfacloc2_2_95     | 30.30     | 92   | 1745  | 1788 |
| sfacloc2_3_80     | 25.20     | 15   | 82    | 127  |
| sfacloc2_3_90     | 29.73     | 9    | 103   | 143  |
| sfacloc2_3_95     | 25.17     | 3    | 27    | 54   |
| sfacloc2_4_80     | 21.03     | 156  | 1403  | 1510 |
| sfacloc2_4_90     | 28.72     | 87   | 1272  | 1390 |
| sfacloc2_4_95     | 26.74     | 682  | 11503 | 11784|
| slay04h           | 14763.90  | 3    | 20    | 44   |
| slay04m           | 12200.30  | 1    | 20    | 37   |
| slay05h           | 24164.50  | 3    | 16    | 38   |
| slay05m           | 23290.00  | 1    | 14    | 28   |
| slay06h           | 38850.80  | 3    | 19    | 46   |
| slay06m           | 33168.50  | 2    | 16    | 35   |
| slay07h           | 69708.50  | 5    | 26    | 58   |
| slay07m           | 70870.80  | 3    | 27    | 60   |
| slay08h           | 105595.00 | 7    | 28    | 71   |
| slay08m           | 85921.30  | 3    | 25    | 52   |
| slay09h           | 156117.00 | 10   | 30    | 83   |
| slay09m           | 160234.00 | 5    | 31    | 86   |
| slay10h           | 197463.00 | 18   | 37    | 106  |
| slay10m           | 176821.00 | 4    | 29    | 73   |
| smallinvDAXr1b010-011 | —       | —    | —     | —    |
| smallinvDAXr1b020-022 | —       | —    | —     | —    |
| smallinvDAXr1b050-055 | —       | —    | —     | —    |
| smallinvDAXr1b100-110 | —       | —    | —     | —    |
| smallinvDAXr1b150-165 | —       | —    | —     | —    |
| smallinvDAXr1b200-220 | —       | —    | —     | —    |
| smallinvDAXr2b010-011 | —       | —    | —     | —    |
| smallinvDAXr2b020-022 | —       | —    | —     | —    |
| smallinvDAXr2b050-055 | —       | —    | —     | —    |
| smallinvDAXr2b100-110 | —       | —    | —     | —    |
| smallinvDAXr2b150-165 | —       | —    | —     | —    |
| smallinvDAXr2b200-220 | —       | —    | —     | —    |
Table 2: Detailed numerical results for the penalty ADM based feasibility pump on all 889 MINLPLib 2 instances with integer variables. Objective function value, running times (s), outer penalty iterations (#Pen.), and inner ADM iterations (#ADM) for the method with $\lambda = 0.9$, additive penalty parameter update rule $\text{inc}_\text{a}$, and deactivated local branching.

| Instance        | Objective | Time | #Pen. | #ADM |
|-----------------|-----------|------|-------|------|
| smallinvDAXr3b010-011 | —         | —    | —     | —    |
| smallinvDAXr3b020-022 | —         | —    | —     | —    |
| smallinvDAXr3b050-055 | —         | —    | —     | —    |
| smallinvDAXr3b100-110 | —         | —    | —     | —    |
| smallinvDAXr3b150-165 | —         | —    | —     | —    |
| smallinvDAXr3b200-220 | —         | —    | —     | —    |
| smallinvDAXr4b010-011 | —         | —    | —     | —    |
| smallinvDAXr4b020-022 | —         | —    | —     | —    |
| smallinvDAXr4b050-055 | —         | —    | —     | —    |
| smallinvDAXr4b100-110 | —         | —    | —     | —    |
| smallinvDAXr4b150-165 | —         | —    | —     | —    |
| smallinvDAXr4b200-220 | —         | —    | —     | —    |
| smallinvDAXr5b010-011 | —         | —    | —     | —    |
| smallinvDAXr5b020-022 | —         | —    | —     | —    |
| smallinvDAXr5b050-055 | —         | —    | —     | —    |
| smallinvDAXr5b100-110 | —         | —    | —     | —    |
| smallinvDAXr5b150-165 | —         | —    | —     | —    |
| smallinvDAXr5b200-220 | —         | —    | —     | —    |
| smallinvSNPr1b010-011 | —         | —    | —     | —    |
| smallinvSNPr1b020-022 | —         | —    | —     | —    |
| smallinvSNPr1b050-055 | —         | —    | —     | —    |
| smallinvSNPr1b100-110 | —         | —    | —     | —    |
| smallinvSNPr1b150-165 | —         | —    | —     | —    |
| smallinvSNPr1b200-220 | —         | —    | —     | —    |
| smallinvSNPr2b010-011 | —         | —    | —     | —    |
| smallinvSNPr2b020-022 | —         | —    | —     | —    |
| smallinvSNPr2b050-055 | —         | —    | —     | —    |
| smallinvSNPr2b100-110 | —         | —    | —     | —    |
| smallinvSNPr2b150-165 | —         | —    | —     | —    |
| smallinvSNPr2b200-220 | —         | —    | —     | —    |
| smallinvSNPr3b010-011 | —         | —    | —     | —    |
| smallinvSNPr3b020-022 | —         | —    | —     | —    |
| smallinvSNPr3b050-055 | —         | —    | —     | —    |
| smallinvSNPr3b100-110 | —         | —    | —     | —    |
| smallinvSNPr3b150-165 | —         | —    | —     | —    |
| smallinvSNPr3b200-220 | —         | —    | —     | —    |
| smallinvSNPr4b010-011 | —         | —    | —     | —    |
| smallinvSNPr4b020-022 | —         | —    | —     | —    |
| smallinvSNPr4b050-055 | —         | —    | —     | —    |
| smallinvSNPr4b100-110 | —         | —    | —     | —    |
| smallinvSNPr4b150-165 | —         | —    | —     | —    |
| smallinvSNPr4b200-220 | —         | —    | —     | —    |
| smallinvSNPr5b010-011 | —         | —    | —     | —    |
| smallinvSNPr5b020-022 | —         | —    | —     | —    |
Table 2: Detailed numerical results for the penalty ADM based feasibility pump on all 889 MINLPLib 2 instances with integer variables. Objective function value, running times (s), outer penalty iterations (#Pen.), and inner ADM iterations (#ADM) for the method with \( \lambda = 0.9 \), additive penalty parameter update rule inc_{\lambda}, and deactivated local branching.

| Instance          | Objective | Time | #Pen. | #ADM |
|-------------------|-----------|------|-------|------|
| smallinvSNPr5b050-055 | —         | —    | —     | —    |
| smallinvSNPr5b100-110 | —         | —    | —     | —    |
| smallinvSNPr5b150-165 | —         | —    | —     | —    |
| smallinvSNPr5b200-220 | —         | —    | —     | —    |
| space25           | 919.99    | 279  | 3235  | 4234 |
| space25a          | 657.73    | 5    | 80    | 106  |
| space900          | 7985000.00| 106  | 39    | 111  |
| spectra2          | 14.04     | 3    | 25    | 53   |
| sporttournament06 | 10.00     | 0    | 3     | 4    |
| sporttournament08 | 22.00     | 0    | 1     | 1    |
| sporttournament10 | 38.00     | 0    | 1     | 1    |
| sporttournament12 | 58.00     | 0    | 1     | 1    |
| sporttournament14 | 80.00     | 0    | 1     | 1    |
| sporttournament16 | 100.00    | 0    | 3     | 4    |
| sporttournament18 | 134.00    | 0    | 1     | 1    |
| sporttournament20 | 162.00    | 1    | 3     | 4    |
| sporttournament22 | 198.00    | 0    | 3     | 4    |
| sporttournament24 | 230.00    | 0    | 1     | 1    |
| sporttournament26 | 282.00    | 0    | 1     | 1    |
| sporttournament28 | 298.00    | 0    | 3     | 4    |
| sporttournament30 | 318.00    | 0    | 1     | 1    |
| sporttournament32 | 384.00    | 0    | 1     | 1    |
| sporttournament34 | 438.00    | 1    | 1     | 1    |
| sporttournament36 | 472.00    | 0    | 1     | 1    |
| sporttournament38 | 526.00    | 0    | 1     | 1    |
| sporttournament40 | 590.00    | 0    | 1     | 1    |
| sporttournament42 | 666.00    | 0    | 1     | 1    |
| sporttournament44 | 706.00    | 0    | 1     | 1    |
| sporttournament46 | 806.00    | 0    | 1     | 1    |
| sporttournament48 | 904.00    | 0    | 1     | 1    |
| sporttournament50 | 948.00    | 1    | 1     | 1    |
| spring            | 1.35      | 1    | 19    | 27   |
| squfl010-025      | 233.69    | 3    | 20    | 41   |
| squfl010-025persp | 214.11    | 247  | 3     | 776  |
| squfl010-040      | 283.98    | 5    | 21    | 43   |
| squfl010-040persp | 240.60    | 72   | 1     | 100  |
| squfl010-080      | 523.50    | 8    | 22    | 45   |
| squfl010-080persp | 509.70    | 1418 | 4     | 1684 |
| squfl015-060      | 422.51    | 5    | 24    | 49   |
| squfl015-060persp | 366.62    | 522  | 8     | 367  |
| squfl015-080      | 596.93    | 10   | 29    | 56   |
| squfl015-080persp | 407.13    | 994  | 3     | 608  |
| squfl020-040      | 286.55    | 4    | 23    | 45   |
| squfl020-040persp | 221.02    | 339  | 11    | 249  |
Table 2: Detailed numerical results for the penalty ADM based feasibility pump on all 889 MINLPLib 2 instances with integer variables. Objective function value, running times (s), outer penalty iterations (#Pen.), and inner ADM iterations (#ADM) for the method with $\lambda = 0.9$, additive penalty parameter update rule $\text{inc}_a$, and deactivated local branching.

| Instance            | Objective | Time | #Pen. | #ADM |
|---------------------|-----------|------|-------|------|
| squfl020-050        | 323.25    | 7    | 25    | 50   |
| squfl020-050persp   | 244.05    | 331  | 11    | 193  |
| squfl020-150        | 1131.79   | 113  | 36    | 73   |
| squfl020-150persp   | 562.07    | 2348 | 15    | 384  |
| squfl025-025        | 244.44    | 4    | 22    | 43   |
| squfl025-025persp   | 178.63    | 719  | 10    | 645  |
| squfl025-030        | 215.34    | 3    | 19    | 37   |
| squfl025-030persp   | 215.34    | 1367 | 10    | 1057 |
| squfl025-040        | 299.01    | 7    | 24    | 47   |
| squfl025-040persp   | 225.39    | 650  | 29    | 385  |
| squfl030-100        | 766.47    | 36   | 33    | 66   |
| squfl030-100persp   | —         | —    | 3     | 535  |
| squfl030-150        | 628.11    | 374  | 31    | 61   |
| squfl030-150persp   | —         | —    | 10    | 307  |
| squfl040-080        | 294.95    | 50   | 22    | 43   |
| squfl040-080persp   | 578.72    | 2651 | 23    | 378  |
| sssd08-04           | 203796.00 | 2    | 12    | 36   |
| sssd08-04persp      | 413575.00 | 2    | 18    | 38   |
| sssd12-05           | 421690.00 | 3    | 18    | 51   |
| sssd12-05persp      | 458644.00 | 2    | 21    | 52   |
| sssd15-04           | 266257.00 | 1    | 14    | 37   |
| sssd15-04persp      | 281947.00 | 3    | 21    | 47   |
| sssd15-06           | 782066.00 | 4    | 21    | 61   |
| sssd15-06persp      | 553100.00 | 2    | 12    | 28   |
| sssd15-08           | 882079.00 | 5    | 26    | 77   |
| sssd15-08persp      | 599059.00 | 3    | 17    | 49   |
| sssd16-07           | 666186.00 | 4    | 25    | 78   |
| sssd16-07persp      | 421588.00 | 2    | 14    | 29   |
| sssd18-06           | 438633.00 | 3    | 14    | 41   |
| sssd18-06persp      | 484779.00 | 2    | 18    | 47   |
| sssd18-08           | 949385.00 | 5    | 46    | 100  |
| sssd18-08persp      | 1054060.00| 4    | 21    | 67   |
| sssd20-04           | 402023.00 | 7    | 16    | 42   |
| sssd20-04persp      | 403765.00 | 3    | 20    | 58   |
| sssd20-08           | 522836.00 | 3    | 14    | 59   |
| sssd20-08persp      | 568422.00 | 3    | 21    | 55   |
| sssd22-08           | 600312.00 | 5    | 22    | 59   |
| sssd22-08persp      | 567598.00 | 3    | 22    | 59   |
| sssd25-04           | 302373.00 | 2    | 19    | 46   |
| sssd25-04persp      | 300946.00 | 2    | 30    | 55   |
| sssd25-08           | 585480.00 | 3    | 19    | 55   |
| sssd25-08persp      | 601061.00 | 3    | 20    | 51   |
| st_e13              | —         | —    | —     | —    |
| st_e14              | 5.81      | 0    | 6     | 10   |
Table 2: Detailed numerical results for the penalty ADM based feasibility pump on all 889 MINLPLib 2 instances with integer variables. Objective function value, running times (s), outer penalty iterations (#Pen.), and inner ADM iterations (#ADM) for the method with $\lambda = 0.9$, additive penalty parameter update rule inc\textsubscript{a}, and deactivated local branching.

| Instance                  | Objective | Time | #Pen. | #ADM |
|---------------------------|-----------|------|-------|------|
| st_e15                    | —         | —    | —     | —    |
| st_e27                    | 2.00      | 0    | 1     | 1    |
| st_e29                    | -0.88     | 1    | 12    | 19   |
| st_e31                    | —         | —    | —     | —    |
| st_e32                    | -0.72     | 14   | 243   | 289  |
| st_e35                    | 71468.10  | 1    | 3     | 4    |
| st_e36                    | —         | —    | 8225  | 51928|
| st_e38                    | 7197.73   | 0    | 1     | 1    |
| st_e40                    | —         | —    | —     | —    |
| st_miqp1                  | 380.50    | 1    | 15    | 28   |
| st_miqp2                  | 2.00      | 1    | 12    | 16   |
| st_miqp3                  | —         | —    | —     | —    |
| st_miqp4                  | —         | —    | —     | —    |
| st_miqp5                  | -333.89   | 0    | 1     | 1    |
| stockcycle                | 260786.00 | 12   | 147   | 218  |
| st_test1                  | 0.00      | 1    | 6     | 11   |
| st_test2                  | —         | —    | —     | —    |
| st_test3                  | —         | —    | —     | —    |
| st_test4                  | —         | —    | —     | —    |
| st_test5                  | -110.00   | 2    | 28    | 46   |
| st_test6                  | 664.00    | 3    | 46    | 75   |
| st_test8                  | -29605.00 | 0    | 4     | 5    |
| st_testgr1                | -12.76    | 1    | 11    | 15   |
| st_testgr3                | -20.50    | 2    | 19    | 29   |
| st_testph4                | -80.50    | 0    | 3     | 5    |
| super1                    | —         | —    | —     | —    |
| super2                    | —         | —    | —     | —    |
| super3                    | —         | —    | —     | —    |
| super3t                   | —         | —    | —     | —    |
| supplychain               | —         | —    | —     | —    |
| supplychainp1_020306      | 593975.00 | 0    | 3     | 5    |
| supplychainp1_022020      | 3378850.00| 5    | 5     | 9    |
| supplychainp1_030510      | 1215690.00| 0    | 3     | 5    |
| supplychainp1_053050      | 8543940.00| 111  | 5     | 11   |
| supplychainr1_020306      | 628623.00 | 0    | 5     | 9    |
| supplychainr1_022020      | 4050140.00| 8    | 14    | 43   |
| supplychainr1_030510      | 1568450.00| 1    | 10    | 16   |
| supplychainr1_053050      | 8588080.00| 180  | 32    | 122  |
| syn05h                    | 837.73    | 0    | 3     | 4    |
| syn05m                    | 831.45    | 1    | 10    | 18   |
| syn05m02h                 | 3032.74   | 0    | 3     | 4    |
| syn05m02m                 | 3026.74   | 1    | 13    | 25   |
| syn05m03h                 | 4027.37   | 0    | 3     | 4    |
| syn05m03m                 | 3987.03   | 1    | 10    | 22   |
Table 2: Detailed numerical results for the penalty ADM based feasibility pump on all 889 MINLPLib 2 instances with integer variables. Objective function value, running times (s), outer penalty iterations (#Pen.), and inner ADM iterations (#ADM) for the method with $\lambda = 0.9$, additive penalty parameter update rule inc$_a$, and deactivated local branching.

| Instance  | Objective | Time | #Pen. | #ADM |
|-----------|-----------|------|-------|------|
| syn05m04h| 5510.39   | 0    | 3     | 4    |
| syn05m04m| 5499.39   | 1    | 13    | 28   |
| syn10h   | 1267.35   | 0    | 3     | 4    |
| syn10m   | 560.25    | 0    | 10    | 14   |
| syn10m02h| 2310.30   | 1    | 3     | 4    |
| syn10m02m| 2275.85   | 2    | 13    | 25   |
| syn10m03h| 3354.68   | 1    | 3     | 4    |
| syn10m03m| 2773.64   | 2    | 25    | 38   |
| syn10m04h| 4557.06   | 0    | 3     | 4    |
| syn10m04m| 3814.77   | 3    | 26    | 41   |
| syn15h   | 853.28    | 0    | 3     | 4    |
| syn15m   | 503.63    | 1    | 10    | 25   |
| syn15m02h| 2832.75   | 0    | 3     | 4    |
| syn15m02m| 2777.75   | 1    | 14    | 25   |
| syn15m03h| 3850.18   | 0    | 3     | 4    |
| syn15m03m| 3795.18   | 3    | 17    | 34   |
| syn15m04h| 4937.48   | 1    | 3     | 4    |
| syn15m04m| 4882.48   | 2    | 18    | 31   |
| syn20h   | 924.26    | 1    | 3     | 4    |
| syn20m   | 722.64    | 2    | 8     | 19   |
| syn20m02h| 1752.13   | 1    | 3     | 4    |
| syn20m02m| 1693.13   | 2    | 23    | 38   |
| syn20m03h| 2646.95   | 1    | 3     | 4    |
| syn20m03m| 2574.01   | 3    | 18    | 41   |
| syn20m04h| 3532.74   | 1    | 3     | 4    |
| syn20m04m| 3475.74   | 3    | 18    | 35   |
| syn30h   | 134.03    | 0    | 3     | 5    |
| syn30m   | -38.75    | 2    | 15    | 28   |
| syn30m02h| 393.25    | 1    | 3     | 5    |
| syn30m02m| -13.34    | 2    | 16    | 29   |
| syn30m03h| 646.05    | 2    | 3     | 5    |
| syn30m03m| 31.70     | 3    | 15    | 34   |
| syn30m04h| 859.05    | 3    | 3     | 5    |
| syn30m04m| 111.41    | 5    | 13    | 33   |
| syn40h   | 58.66     | 1    | 3     | 5    |
| syn40m   | -25.08    | 2    | 27    | 46   |
| syn40m02h| 379.76    | 1    | 3     | 4    |
| syn40m02m| 149.17    | 3    | 11    | 30   |
| syn40m03h| 390.15    | 4    | 4     | 6    |
| syn40m03m| -12.17    | 6    | 28    | 51   |
| syn40m04h| 896.96    | 4    | 3     | 4    |
| syn40m04m| 609.95    | 6    | 12    | 32   |
| synheat  | 219858.00 | 2    | 18    | 43   |
| synthes1 | 7.09      | 1    | 8     | 14   |
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| Instance          | Objective | Time | #Pen. | #ADM |
|-------------------|-----------|------|-------|------|
| synthes2         | 80.29     | 2    | 17    | 34   |
| synthes3         | 82.37     | 1    | 14    | 28   |
| tanksize         | 1.27      | 0    | 6     | 10   |
| telecomsp_metro  | —         | —    | 1     | 1    |
| telecomsp_njata  | —         | —    | 1     | 2    |
| telecomsp_nor_sun| —         | —    | 1     | 0    |
| telecomsp_pacbell| 336210.00 | 3146 | 35    | 176  |
| th12              | 315.60    | 207  | 2860  | 3350 |
| th2               | 23.30     | 15   | 284   | 313  |
| th4               | 9.30      | 5    | 92    | 103  |
| th5               | 16.50     | 3    | 47    | 65   |
| th6               | 20.50     | 13   | 177   | 209  |
| th7               | 39.00     | 157  | 3173  | 3239 |
| tloss             | 24.30     | 2    | 41    | 62   |
| tls12             | —         | —    | 37596 | 45964|
| tls2              | 11.30     | 193  | 3385  | 3543 |
| tls4              | 22.00     | 492  | 9538  | 9781 |
| tls5              | 15.50     | 81   | 1319  | 1590 |
| tls6              | 36.10     | 152  | 2418  | 2859 |
| tls7              | 48.80     | 1693 | 27458 | 28995|
| tlr               | 54.60     | 2    | 23    | 32   |
| transswitch0009p  | —         | —    |       |      |
| transswitch0009r  | —         | —    |       |      |
| transswitch0010p  | —         | —    |       |      |
| transswitch0014p  | —         | —    |       |      |
| transswitch0014r  | —         | —    |       |      |
| transswitch0030p  | —         | —    |       |      |
| transswitch0030r  | —         | —    |       |      |
| transswitch0039p  | —         | —    |       |      |
| transswitch0039r  | —         | —    |       |      |
| transswitch0057p  | —         | —    |       |      |
| transswitch0057r  | —         | —    |       |      |
| transswitch0118p  | —         | —    |       |      |
| transswitch0118r  | —         | —    |       |      |
| transswitch0300p  | —         | —    |       |      |
| transswitch0300r  | —         | —    |       |      |
| transswitch2383wpp| —         | —    | 1     | 0    |
| transswitch2383wpr| —         | —    |       |      |
| transswitch2736spp| —         | —    | 1     | 0    |
| transswitch2736spr| —         | —    |       |      |
| tspu05            | 191.25    | 0    | 1     | 1    |
| tspu08            | 290.57    | 0    | 1     | 1    |
| tspu10            | 225.13    | 0    | 1     | 1    |
| tspu12            | 270.90    | 0    | 1     | 1    |
| tspu15            | 334.73    | 2    | 3     | 6    |
Table 2: Detailed numerical results for the penalty ADM based feasibility pump on all 889 MINLPLib 2 instances with integer variables. Objective function value, running times (s), outer penalty iterations (#Pen.), and inner ADM iterations (#ADM) for the method with $\lambda = 0.9$, additive penalty parameter update rule $inc_a$, and deactivated local branching.

| Instance         | Objective | Time | #Pen. | #ADM |
|------------------|-----------|------|-------|------|
| unitcommit1      | 585256.00 | 14   | 35    | 58   |
| unitcommit2      | 632283.00 | 44   | 23    | 90   |
| useinear         |           |      |       |      |
| util             | 999.84    | 2    | 34    | 39   |
| var_con10        | 452.69    | 1    | 5     | 7    |
| var_con5         | 285.87    | 1    | 3     | 4    |
| waste            | 1169.08   | 18   | 82    | 139  |
| wastepaper3      | 0.03      | 3    | 16    | 32   |
| wastepaper4      | 0.03      | 2    | 22    | 32   |
| wastepaper5      | 0.04      | 2    | 24    | 42   |
| wastepaper6      | 0.01      | 2    | 19    | 37   |
| water3           |           |      |       |      |
| water4           | 1323.97   | 70   | 1294  | 1381 |
| watercontamination0202 | 129.15 | 467  | 43    | 95   |
| watercontamination0202r | 2690.24 | 25   | 97    | 196  |
| watercontamination0303 | 580.69 | 811  | 60    | 137  |
| watercontamination0303r | 8654.50 | 91   | 131   | 277  |
| waterful2        |           |      |       |      |
| waternd1         |           |      |       |      |
| waternd2         |           |      |       |      |
| waternd_blacksburg | 518613.00 | 21   | 271   | 315  |
| waternd_fossiron | 346062.00 | 1035 | 14379 | 14507|
| waternd_fosspoly0 | 93204600.00 | 10   | 61    | 126  |
| waternd_fosspoly1 |          | 10   |       | 33692|
| waternd_hanoi    | 7438010.00 | 36   | 525   | 640  |
| waternd_modena   | 346062.00 | 1080 | 14379 | 14507|
| waternd_pescara  | 4575970.00 | 1060 | 11611 | 11743|
| waternd_shamir   | 434000.00 | 1    | 10    | 15   |
| waterno1_01      | 306.82    | 10   | 181   | 187  |
| waterno1_02      | 331.56    | 5    | 50    | 58   |
| waterno1_03      | 1257.91   | 9    | 106   | 117  |
| waterno1_04      | 1202.02   | 63   | 817   | 840  |
| waterno1_06      | 1093.12   | 7    | 33    | 54   |
| waterno1_09      | 1516.33   | 17   | 50    | 76   |
| waterno1_12      | 1826.62   | 35   | 140   | 166  |
| waterno1_18      | 2398.41   | 40   | 46    | 74   |
| waterno1_24      | 3130.12   | 1745 | 6004  | 6042 |
| waterno2_01      | 29.08     | 4    | 63    | 69   |
| waterno2_02      | 156.66    | 7    | 86    | 94   |
| waterno2_03      | 134.20    | 8    | 81    | 105  |
| waterno2_04      | 257.23    | 11   | 81    | 118  |
| waterno2_06      | 442.50    | 16   | 88    | 137  |
| waterno2_09      | 1422.16   | 31   | 87    | 159  |
| waterno2_12      | 3119.55   | 48   | 99    | 211  |
Table 2: Detailed numerical results for the penalty ADM based feasibility pump on all 889 MINLPLib 2 instances with integer variables. Objective function value, running times (s), outer penalty iterations (#Pen.), and inner ADM iterations (#ADM) for the method with $\lambda = 0.9$, additive penalty parameter update rule $\text{inc}_{\lambda}$, and deactivated local branching.

| Instance          | Objective | Time | #Pen. | #ADM |
|-------------------|-----------|------|-------|------|
| waterno2_18       | 7359.11   | 105  | 107   | 253  |
| waterno2_24       | 9711.81   | 301  | 477   | 653  |
| waters            | —         | —    | —     | —    |
| watersbp          | —         | —    | —     | —    |
| watersym1         | —         | —    | —     | —    |
| watersym2         | —         | —    | —     | —    |
| watertreatnd_conc | 349525.00 | 19   | 294   | 297  |
| watertreatnd_flow | 315610.00 | 1575 | 13551 | 26704|
| waterx            | 934.86    | 0    | 7     | 12   |
| waterz            | 0.25      | 2    | 26    | 28   |

1Björn Geißler, Antonio Morsi, Lars Schewe, Friedrich-Alexander-Universität Erlangen-Nürnberg (FAU), Discrete Optimization, Cauerstr. 11, 91058 Erlangen, Germany, 2Martin Schmidt, (a) Friedrich-Alexander-Universität Erlangen-Nürnberg (FAU), Discrete Optimization, Cauerstr. 11, 91058 Erlangen, Germany; (b) Energie Campus Nürnberg, Fürther Str. 250, 90429 Nürnberg, Germany

E-mail address: 1{bjoern.geissler,antonio.morsi,lars.schewe}@math.uni-erlangen.de
E-mail address: 2mar.schmidt@fau.de