Abstract. We revisit the relegation algorithm by Deprit et al. (2001) in the light of the rigorous Nekhoroshev’s like theory. This relatively recent algorithm is nowadays widely used for implementing closed form analytic perturbation theories, as it generalises the classical Birkhoff normalisation algorithm. The algorithm, here briefly explained by means of Lie transformations, has been so far introduced and used in a formal way, i.e. without providing any rigorous convergence or asymptotic estimates. The overall aim of this paper is to find such quantitative estimates and to show how the results about stability over exponentially long times can be recovered in a simple and effective way, at least in the non-resonant case.

1. Introduction

The relegation algorithm, firstly introduced by Palacián [17] and fully stated in [5], is an algorithm suited for canonical simplifications of Hamiltonian problems and represents an extension of the classical Birkhoff normal form method. Given a system described by a perturbed Hamiltonian, this procedure aims to relegate the effect of a desired part of the perturbation to an arbitrarily small reminder, by constructing a suitable sequence of canonical changes of coordinates. The computation of such a transformed Hamiltonian would require an infinite sequence of canonical transformations which, in general, cannot be done since the resulting series turns out to be divergent. However, the asymptotic character of the transformation allows to progressively lower the influence of the so-called remainder, i.e. the not yet relegated part of the Hamiltonian, by considering a finite sequence of canonical transformations. This is very useful in practical applications, where one proceeds order-by-order until the remainder can be considered, in some sense, negligible.

Consider a Hamiltonian expanded in power series of a small parameter, namely $H = H_0 + \varepsilon H_1 + \ldots$, and in particular the case $H_0 = h_0 + f_0$ where $h_0$ is an integrable Hamiltonian and $f_0$ is for the moment a generic function. The relegation aims at constructing a first integral that at first order is essentially $h_0$. This is obtained by
constructing a canonical change of coordinates that puts the transformed Hamiltonian in a suitable normal form. In this respect the relegation procedure is different from that of Birkhoff, in which the normal form Hamiltonian is requested to commute with $H_0$. Indeed the effect of the function $f_0$ is reduced to a small contribution, i.e., relegated. The relegation algorithm has been successfully used in celestial mechanics\cite{18}\cite{2}\cite{4} and artificial satellite theory\cite{12}\cite{13}, with a particular focus on the dynamics close to asteroids (the so-called fast-rotating case) \cite{3}\cite{19}\cite{6}\cite{16}. Let us remark that a similar normal form approach has also been introduced in \cite{1}, where the authors studied the problem of the energy exchanges between a system of uncoupled harmonic oscillators and a generic other dynamical system, playing the role of the functions $h_0$ and $f_0$ of the relegation algorithm, respectively.

Following the usual tradition in celestial mechanics, the relegation algorithm has been introduced in a formal way. However, to our knowledge, the literature lacks of rigorous convergence or, at least, asymptotic estimates for the algorithm. We plan to do this in the present paper.

1.1 The relegation algorithm

Throughout the paper we will make a wide use of the formalism of Lie transforms which we briefly recall here. We refer to \cite{11} and \cite{7} for an exhaustive introduction.

The Lie transform $T_{\mathcal{X}}g$ of a generic function $g$ is defined as

$$T_{\mathcal{X}}g = \sum_{j=0}^{\infty} E_j g \quad \text{with} \quad E_0 g = g , \quad E_j g = \sum_{i=1}^{j} i \mathcal{L}_{\mathcal{X}_i} E_{j-i} g ,$$

where $\mathcal{X} = \{\mathcal{X}_1, \mathcal{X}_2, \ldots\}$ is a sequence of generating functions and $\mathcal{L}_{\mathcal{X}}g$ is the Lie derivative of $g$ with respect to $\mathcal{X}$, i.e., the Poisson bracket $\{\mathcal{X}, g\}$. There is also an explicit formula for the inverse, namely

$$T_{\mathcal{X}}^{-1} g = \sum_{j=0}^{\infty} D_j g \quad \text{with} \quad D_0 g = g , \quad D_j g = -\sum_{i=1}^{j} i \mathcal{L}_{\mathcal{X}_{j-i}} g .$$

The Lie transform is a generalisation of the Lie series, extremely useful in perturbation theory as it can represent every near the identity canonical transformation. Moreover Lie transforms, being defined in a recurrent explicit formula, are well suited to develop effective algorithms and perform computations using computer algebra (see, e.g., \cite{10}).

Consider a Hamiltonian

$$H = \sum_{s \geq 0} H_s ,$$

where $H_s$ is a term of order $s$ in some small parameter. As a general fact in perturbation theory one aims at transforming the Hamiltonian to a so-called normal form that we will denote as

$$Z = \sum_{s \geq 0} Z_s .$$
The condition of being in normal form is that $Z$ should commute with a certain function $h_0$, i.e. $\{Z, h_0\} = 0$. For instance, in Birkhoff normalisation $Z$ must commute with $h_0 = H_0$. The algorithm is developed by solving the equation

$$T_XZ = H,$$

for the sequence of generating functions, $X$, and the normalised Hamiltonian $Z$. In particular, at each order, we have to solve the so-called homological equation

$$L_{h_0}X_s + Z_s = \Psi_s,$$

where $\Psi_s$ is a known function collecting all the terms of order $s$, while $X_s$ and $Z_s$ are the unknowns to be determined.

In contrast with normalisation, in the relegation procedure $H_0$ is split in two functions, i.e., $H_0 = h_0 + f_0$, also asking $\{h_0, f_0\} = 0$ and require that the sole $h_0$ is integrable. Thus, in the relegation algorithm, the normal form terms $Z_s$, with $Z_0 = h_0 + f_0$, must commute with $h_0$ so that it becomes a formal first integral for the normal form. In this case, at each order, the homological equation takes the form

$$L_{h_0}X_s + Z_s = \Psi_s,$$

where, again, $\Psi_s$ is a known function collecting all the terms of order $s$, while $X_s$ and $Z_s$ are the unknowns to be determined.

As a matter of fact, in the relegation algorithm the homological equation (1) is solved via a recurrence formula that aims to counteract the effect of the terms generated by $f_0$. Thus $f_0$ plays a special role, as detailed in section 2. Again, due to lack of convergence we can only relegate the action of $f_0$ to a suitable order, thus making it as small as possible.

### 1.2 Statement of the results

We collect here our main results, i.e., the asymptotic properties of the relegation algorithm, together with estimates about the long time stability of the approximate first integrals. Precisely, we give rigorous bounds for a truncated sequence of generating functions $X$ and the corresponding transformed Hamiltonian $Z$. All details on the relegation algorithm and the quantitative estimates are included in section 2 and 4, respectively.

Consider a system of differential equations with Hamiltonian

$$H(p, q, z, i\bar{z}) = h_0(p) + \mu f_0(p, q, z, i\bar{z}) + \varepsilon H_1(p, q, z, i\bar{z}),$$

with action-angle variables $p \in G \subseteq \mathbb{R}^{n_1}$, $q \in \mathbb{T}^{n_1}$, and conjugate canonical variables $(z, i\bar{z}) \in B \subseteq \mathbb{C}^{n_2}$, where both $G$ and $B$ are open sets containing the origin and $n_1$, $n_2$ are positive integers. The quantities $\mu, \varepsilon \in \mathbb{R}$ are two small parameters, with $\mu > \varepsilon$. The latter request is not essential for the correctness of the proof. However, in case $\mu < \varepsilon$, the term $\mu f_0$ can be moved to the perturbation and one can proceed by means of standard normalisation procedure (see section 4 in [5]).
We consider the domain $\mathcal{D} = \mathcal{G} \times \mathbb{T}^{n_1} \times \mathcal{B}$ and introduce the extended domains

$\mathcal{D}_{\varrho, \sigma, R} = \mathcal{G}_{\varrho} \times \mathbb{T}_{\sigma}^{n_1} \times \mathcal{B}_R$, where $\mathcal{G}_{\varrho} \subset \mathbb{C}^{n_1}$ and $\mathcal{B}_R \subset \mathbb{C}^{2n_2}$ are complex open balls centred at the origin with radii $\varrho$ and $R$, respectively, while the subscript $\sigma$, with $\sigma \in \mathbb{R}$ such that $\sigma > 0$, denotes the usual complex extension of the torus.

Let us consider a generic analytic function $g : \mathcal{D}_{\varrho, \sigma, R} \to \mathbb{C}$,

$$g(p, q, z, i\bar{z}) = \sum_{k \in \mathbb{Z}^{n_1}} g_k(p, z, i\bar{z}) e^{ik \cdot q},$$

where $g_k : \mathcal{G}_{\varrho} \times \mathcal{B}_R \to \mathbb{C}$. We define the supremum norm

$$|g|_{\varrho, \sigma, R} = \sup_{p \in \mathcal{G}_{\varrho}, q \in \mathbb{T}_{\sigma}^{n_1}} |g(p, q, z, i\bar{z})|.$$ 

and the weighted Fourier norm

$$\|g\|_{\varrho, \sigma, R} = \sum_{k \in \mathbb{Z}^{n_1}} |g_k|_{\sigma, R} e^{|k| \sigma},$$

where

$$|g_k|_{\sigma, R} = \sup_{p \in \mathcal{G}_{\varrho}} |g_k(p, z, i\bar{z})|.$$ 

Hereafter, we use the shorthand notations $| \cdot |_\alpha$ and $\| \cdot \|_\alpha$ for $| \cdot |_{\alpha(\varrho, R)}$ and $\| \cdot \|_{\alpha(\varrho, \sigma, R)}$.

The Hamiltonian (2) is characterised by 8 real parameters, namely $\varrho$, $\sigma$, $R$ (analytic parameters), $\varepsilon$, $\mu$ (perturbation parameters) and $\omega$, $\gamma$, $\tau$ (frequency parameters), where the latter are introduced below. We make the following hypotheses:

(i) $h_0(p) = \omega \cdot p$, where $\omega \in \mathbb{R}^{n_1}$ is a fixed frequency vector;

(ii) $h_0$, $f_0$ and $H_1$ are holomorphic bounded functions on the extended domain $\mathcal{D}_{\varrho, 2\sigma, R}$ with $\|f_0\|_{\varrho, 2\sigma, R} \leq G$ for some positive real $G$;

(iii) the functions $h_0$ and $f_0$ commute, i.e., $\{h_0, f_0\} = 0$.

Moreover, we introduce the so-called resonance module $\mathcal{M}_\omega$ associated to the fixed frequency vector $\omega \in \mathbb{R}^{n_1}$ as

$$\mathcal{M}_\omega = \{k \in \mathbb{Z}^{n_1} : k \cdot \omega = 0 \}.$$ 

We assume the following additional hypothesis:

(iv) for every positive integers $r$ and $K$, the frequency vector $\omega \in \mathbb{R}^{n_1}$ satisfies the Diophantine condition

$$\alpha_r = \min_{k \in \mathbb{Z}^{n_1} \setminus \mathcal{M}_\omega, |k| \leq rK} |k \cdot \omega| \geq \frac{\gamma}{|k| \tau},$$

with $\gamma > 0$ and $\tau > n_1$.

We remark that hypothesis (i) allows us to skip the so-called geometric part of Nekhoroshev’s theorem. The general case can be recovered using a suitable adaptation of

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1 Precisely, $\mathcal{G}_{\varrho} = \{z \in \mathbb{C}^{n_1} : \max_{1 \leq j \leq n_1} |z_j| < \varrho\}$, $\mathbb{T}_{\sigma}^{n_1} = \{q \in \mathbb{C}^{n_1} : \Re q_j \in \mathbb{T}, \max_{1 \leq j \leq n_1} |\Im q_j| < \sigma\}$, $\mathcal{B}_R = \{z \in \mathbb{C}^{2n_2} : \max_{1 \leq j \leq 2n_2} |z_j| < R\}$.
the geometric part of the Nekhoroshev’s theorem (see, e.g., [7]). Furthermore, assumption (iii) implies that the Fourier expansion of \( f_0 \) must have the special form, 
\[
\sum_{k \in \mathcal{M}} f_{0,k}(p, z, i\bar{z}) e^{ik \cdot q},
\]
i.e., it contains only resonant modes.
Concerning the small parameter \( \varepsilon \), let us remark that the Hamiltonian (2) is not already in the form of a function expanded in power series of a small parameter, e.g., in \( \varepsilon \). Actually, we perform an expansion in power series exploiting the exponential decay of the Fourier coefficients, as detailed in subsection 2.1, getting
\[
\varepsilon H_1 = \sum_{j \geq 1} h_j(p, q, z, i\bar{z}).
\]
We stress that in the expansion above the order in the small parameter is encoded in the index \( s \) of the function \( h_s \).

**Proposition 1:** Let the Hamiltonian (2) satisfy the hypotheses (i), (ii), (iii) and (iv) above. Take three positive integers \( K \geq 1, r \geq 1 \) and \( L \geq 1 \) and assume
\[
\frac{9r^2 L \Xi \mu G}{\alpha_r} \leq \frac{1}{2^r}, \quad \text{and} \quad \eta = \frac{\varepsilon r^4 A}{\alpha_r^2} + 4e^{-K\sigma/2} \leq \frac{1}{2},
\]
with \( \alpha_r \) as in (5),
\[
A = 2^{21} \Xi^2 \left( \frac{1 + e^{-\sigma/2}}{1 - e^{-\sigma/2}} \right)^n |H_1(p, q, z, i\bar{z})|_{\theta, 2\sigma, R}, \quad \Xi = \left( \frac{2}{\varepsilon \theta \sigma} + \frac{1}{R^2} \right).
\]
Then:
(i) there exist a truncated sequence of generating functions \( \mathcal{X}^{(r)} = \{\mathcal{X}_1, \ldots, \mathcal{X}_r, 0, \ldots\} \) that transforms the Hamiltonian into
\[
H^{(r)} = h_0 + \mu f_0 + Z_1 + \ldots + Z_r + R^{(r+1)},
\]
where the functions \( Z_1, \ldots, Z_r \) are in normal form, i.e., they commute with \( h_0 \). The term \( R^{(r+1)} \) is the reminder of the transformation, i.e., the collection of all the terms that are at least of order \( r + 1 \);
(ii) the generating sequence defines an analytic canonical transformation on the domain \( D^{(\frac{3}{4})}_{\frac{3}{4}(\theta, \sigma, R)} \) such that
\[
D^{(\frac{3}{4})}_{\frac{3}{4}(\theta, \sigma, R)} \subseteq T_{\mathcal{X}} D^{(\frac{3}{4})}_{\frac{3}{4}(\theta, \sigma, R)} \subseteq D^{(\frac{3}{4})}_{\frac{3}{4}(\theta, \sigma, R)}, \quad D^{(\frac{3}{4})}_{\frac{3}{4}(\theta, \sigma, R)} \subseteq T_{\mathcal{X}}^{-1} D^{(\frac{3}{4})}_{\frac{3}{4}(\theta, \sigma, R)} \subseteq D^{(\frac{3}{4})}_{\frac{3}{4}(\theta, \sigma, R)},
\]
and moreover in \( G^{(\frac{3}{4})}_{\frac{3}{4}} \) one has
\[
|T_{\mathcal{X}} p - p| \leq \frac{\theta}{16}, \quad |T_{\mathcal{X}}^{-1} p - p| \leq \frac{\theta}{16};
\]
(iii) the remainder is estimated by
\[
\|R^{(r+1)}\|_{\frac{3}{4}(\theta, \sigma, R)} \leq \varepsilon \left( \frac{A}{2^{18} \Xi^2} \right) \eta^r.
\]
Explicit bounds for \( \mu \) and \( \varepsilon \) are easily obtained from (6) using also (5).
Remark. The remainder $\mathcal{R}^{(r+1)}$ is composed of two kind of terms, the ones corresponding to the truncation at a finite order of the transformation itself, like in the usual normalisation process, and the ones obtained at each order as a consequence of solving the homological equation in a recursive manner. This is the key ingredient of the relegation algorithm and the main advantage with respect to the classical Birkhoff normal form. However, the estimate at point (iii) in proposition 1 somehow hides this distinction, being essentially the same as in the classical normalisation algorithm (see, e.g., [7]). We stress that the contribution due to the recursive solving of the homological equation is controlled by (6). Indeed the smallness condition on $\mu$ allows to perform $L$ steps in the recursive solving of the homological equations, getting a small remainder term of order $\mathcal{O}(\mu^{L+1})$.

We now should make a clear distinction between the resonant and non-resonant systems. In fact the formal scheme and the rigorous estimates in proposition 1 apply to both cases, but the result concerning the effective stability time exhibits a substantial difference.

In the resonant case, we have $n_1 - \dim \mathcal{M}_\omega$ approximate first integrals of the form $\Phi = T_{X^{(r)}_0} \Phi_0$, with $\Phi_0 = \lambda \cdot p$ where $\lambda \in \mathbb{R}^{n_1 - \dim \mathcal{M}_\omega}$ and satisfies $\lambda \perp \mathcal{M}_\omega$. We may assume that $|\lambda| < 1$ without loss of generality. We denote by $\Pi_{\mathcal{M}_\omega}(p)$ the plane through $p$ generated by the resonance module $\mathcal{M}_\omega$, namely

$$\Pi_{\mathcal{M}_\omega}(p) = \{p' \in \mathbb{R}^{n_1} : p' - p \in \text{span}(\mathcal{M}_\omega)\}.$$ Using the language introduced by Nekhoroshev[14][15], we call plane of fast drift the plane $\Pi_{\mathcal{M}_\omega}(p(0)) = p(0) + \text{span}(\mathcal{M}_\omega)$ and deformation the non-linear contributions to the functions $\Phi$, which cause the orbit to oscillate around the plane of fast drift. Finally, we say that the remainder generates a noise that may cause a slow motion of the orbit in a direction transversal to the plane of fast drift; we call this slow motion diffusion. In contrast with Birkhoff normal form, in the relegation procedure we do not have a strict control on the term $f_0$ and the dynamics described by $(z, i\dot{z})$. Therefore, in order to obtain a result concerning the long time stability of the approximate first integrals we have to assume that the evolution of $(z, i\dot{z})$ is confined.

Proposition 1 allows to obtain a bound for the diffusion time but, without further assumptions, we do not have any control on the fast drift along the resonant plane. Thus, in the resonant case, we can get an estimate of the stability time that must be combined with some a priori bound confining the orbits. The formal statement of the local stability is given by the following

**Lemma 1:** With the same hypotheses of proposition 1, the following statement holds true: if $(p(t), q(t), z(t), i\dot{z}(t))$ is an orbit lying in the domain $\mathcal{D}$ for $t \in [\tau^-, \tau^+] \subset \mathbb{R}$, with $\tau^- < 0 < \tau^+$, then one has

$$\text{dist}(p(t), \Pi_{\mathcal{M}_\omega}(p(0))) < \frac{\rho}{2}$$

for all $t \in [\tau^-, \tau^+] \cap [-t^*, t^*]$, with

$$t^* = \frac{2^{12} e^2 \rho^2 \bar{\Xi}^2}{A^2 \eta^r}.$$
where $\eta$, $A$ and $\Xi$ are defined as in (6) and (7), respectively.

Let us stress that an extension of the present result can be achieved constructing a suitable covering of boxed domains and studying the so-called geography of resonances. This constitutes the geometric part of the Nekhoroshev’s theorem and we refer to [7] for a detailed exposition on the subject. Such an extension requires a broad discussion, but does not contain any essential modification with respect to the Nekhoroshev’s theorem. Moreover, as the aim of this paper is to give a rigorous support for the relegation algorithm, we decided to state the results in the simplest framework.

In the non-resonant case proposition 1 is enough to guarantee an exponentially long-time stability for a suitable open set of initial data of the actions $p$. The results for non-resonant systems about the effective stability time is given by the following

**Theorem 1:** (non-resonant). Let the Hamiltonian (2) satisfy the same hypotheses of proposition 1 and assume that the frequency vector $\omega$ is non-resonant, i.e., $\mathcal{M}_\omega = \{0\}$. There exist positive real constants $\varepsilon^*$, $\mu^*$ and $T$ such that the following statement holds true: if $\varepsilon < \varepsilon^*$, for every orbit $(p(t), q(t), z(t), i\bar{z}(t))$ lying in the domain $\mathcal{D}$ at $t = 0$ with $(z(t), i\bar{z}(t)) \in B$ for $t \in [\tau^-, \tau^+] \subset \mathbb{R}$, with $\tau^- < 0 < \tau^+$ one has

$$\text{dist}(p(t), p(0)) < \frac{\theta}{2}$$

for all $t \in [\tau^-, \tau^+] \cap [-t^*, t^*]$, with

$$t^* \leq \frac{T}{\varepsilon} \exp\left(\frac{\gamma^2}{2\varepsilon A K^2\tau^4}\right),$$

where $A$ is defined as in (7).

Explicit estimates for the values of $\mu^*$, $\varepsilon^*$ and $T$ can be found in the proof.

Let us remark that if one is interested in actual applications to physical models, in general, the purely analytic estimates turn out to be too pessimistic and actually unpractical. Nevertheless, the use of Lie transforms provides a constructive normalisation algorithm that can be easily translated into a recursive scheme of estimates. Thus, using computer algebra in order to perform high-order perturbation expansions, one can produce estimates on the long-time stability for realistic models. For instance, see, e.g., [20] and [21] for the study of the effective resonant stability in the spin-orbit problem and [8], [22], [23] for the problem of the long-time stability of some of the giant planets of the Solar system.

The paper is organised as follows. In section 2 we reformulate the relegation algorithm via Lie transform in a suitable way in order to translate it into a scheme of recursive estimates. The analytic tools are reported in section 3 while the quantitative estimate are gathered in section 4 where we also report the proof of proposition 1, lemma 1 and theorem 1.

## 2. The relegation algorithm

The basis of our construction is the well known Birkhoff normal form for a Hamiltonian
system. We follow a quite standard approach, see, e.g., [7] for a detailed discussion or [9] for an exposition in a quite similar framework. The main problem we have to face is the presence of the so-called small divisors, thus we need to split the perturbation in such a way that, at every step of the normalisation process, we take into account only a finite number of Fourier harmonics.

In the following, a special role will be played by those functions which have a finite Fourier representation. Thus let us introduce some particular classes of functions.

**Definition 1:** Given $K_1, K_2 \in \mathbb{N}$, an analytic function $f(p, q, z, i\bar{z})$ is said to be of class $\mathcal{P}_{K_1, K_2}$ if

$$f = \sum_{k \leq K_2} c_k(p, z, i\bar{z}) e^{ik \cdot q},$$

and $c_k \in \mathbb{C}$ such that $c_k \neq 0$ if and only if $k = k' + k''$, with $k' \in \mathcal{M}_\omega$ and $|k''| \leq K_1$.

Let us stress that the classes of functions introduced above will be useful for both the formal scheme and the quantitative estimates. Indeed $K_1$ and $K_2$ control the accumulation of the small divisors and trigonometric degree, respectively.

### 2.1 Splitting of the Hamiltonian

We now split the perturbation in a suitable form. First let us consider the Fourier expansion of the term $\varepsilon H_1$ in (2)

$$\varepsilon H_1 = \sum_{k \in \mathbb{Z}^{n_1}} c_k(p, z, i\bar{z}) e^{ik \cdot q}.$$ 

Then, let us pick an arbitrary positive integer $K$ and write $H_1$ in the form

$$\varepsilon H_1 = \sum_{j \geq 1} h_j(p, q, z, i\bar{z}),$$

where

$$h_1(p, q, z, i\bar{z}) = \sum_{0 \leq |k| < K} c_k(p, z, i\bar{z}) e^{ik \cdot q},$$

$$h_s(p, q, z, i\bar{z}) = \sum_{(s-1)K \leq |k| < sK} c_k(p, z, i\bar{z}) e^{ik \cdot q}.$$

This procedure breaks down the classical scheme of series expansions in the perturbative parameter $\varepsilon$, introducing an arbitrary quantity $K$. However, at the end of the proof of theorem 1, we will see that there is a natural choice for $K$.

The splitting introduced above is based on the exponential decay of the Fourier coefficients of an analytic function, as stated by the following

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2 As reported in [7] (pag. 86, footnote 2), choosing the parameter $K$ by asking $e^{-K\sigma} \sim \varepsilon$ is not really convenient. As it will be evident from the optimisation of the parameters in the proof of theorem 1, the best choice is $K \sim 1/\sigma$. 
Lemma 2: Let $H_1(p, q, z, iz)$ be analytic in $D_{\varrho, 2\sigma, R}$ and
\[
|H_1(p, q, z, iz)|_{\varrho, 2\sigma, R} = \sup_{p \in \mathbb{G}_\varrho, \, q \in \mathbb{G}_{2\sigma}, \, (z, iz) \in \mathbb{B}_R} |H_1(p, q, z, iz)| < \infty.
\]
Then
\[
\|h_s\|_{\varrho, \sigma, R} \leq \zeta^{s-1} F, \quad s \geq 1,
\]
with
\[
\zeta = e^{-K\sigma/2}, \quad F = \varepsilon \left(\frac{1 + e^{-\sigma/2}}{1 - e^{-\sigma/2}}\right)^{n_1} |H_1(p, q, z, iz)|_{\varrho, 2\sigma, R}.
\]
The proof of lemma 2 is straightforward, see, e.g., lemma 5.2 in [7].

2.2 Formal scheme

Let us pick two positive integers $K, K'$ and write the Hamiltonian (2) as
\[
H^{(0)} = h_0 + \mu f_0 + h_1 + h_2 + \ldots + h_s + \ldots,
\]
with $h_0 = \omega \cdot p$, $f_0 \in \mathcal{P}_{0,K'}$ and $h_s \in \mathcal{P}_{sK,sK}$. We remark that with this splitting of the perturbation, the terms $h_s$ are of order $s$ in some small parameter, precisely they are of order $O(\varepsilon \zeta^{s-1})$.

We look for a sequence of generating functions $X^{(r)} = \{X_1, \ldots, X_r, 0, \ldots\}$, with $r$ arbitrary positive integer, and a function $Z^{(r)} = Z_0 + \ldots + Z_r$ such that
\[
T_{X^{(r)}} Z^{(r)} = h_0 + \mu f_0 + h_1 + \ldots + h_r + Q^{(r+1)}.
\]
The functions $Z_0, \ldots, Z_r$ must be determined so as to be in normal form, namely, they must commute with $h_0$. Instead, the term $Q^{(r+1)}$ is the unrelegated remainder, namely
\[
T_{X^{(r)}} Z^{(r)} - \sum_{s=0}^{r} E_s Z^{(r)} = Q^{(r+1)}.
\]
The remainder, being a term of order $r + 1$ in the small parameter, can be considered as a small term. Moreover, the asymptotic character of the transformation allows to progressively lower the influence of the remainder. However, it is important to recall that the resulting series are actually divergent, thus the sequence of canonical transformations must be finite.

Again, we emphasise that the relegation algorithm represents a variazione of the classical Birkhoff normal form. The difference is the special role played by the term $f_0$. Thus, we introduce the additional parameter $L$ so as to take into account the peculiar character of $f_0$. The role of $L$ will be clear from the definition of the generating functions.

For $r = 1$, we have to solve the following equations
\[
Z_{1,0} - \mathcal{L}_{h_0} X_{1,0} = h_1, \\
Z_{1,j} - \mathcal{L}_{h_0} X_{1,j} = \mathcal{L}_{\mu f_0} X_{1,j-1}, \quad j = 1, \ldots, L.
\]
We define the generating function $X_1$ as
\[
X_1 = X_{1,0} + X_{1,1} + \ldots + X_{1,L}.
\]
Explicit expressions for $X_1$ are easy to obtain. We report here the sole expression of $X_{1,0}$, that follows from (9)

$$X_{1,0}(p,q,z,i\bar{z}) = \sum_{k \in \mathbb{Z} \setminus \{0\} \mid k \leq |k| < K} \frac{c_k(p,z,i\bar{z})}{ik \cdot \omega} e^{ik \cdot q},$$

similar expressions can be obtained for $X_{1,j}$ with $j = 1, \ldots, L$. Let us remark that $X_{1,j} \in P_{K,K+LK'}$, $X_1 \in P_{K,K+L'K'}$.

The relegated term $Z_1$ is given by

$$Z_1 = Z_{1,0} + Z_{1,1} + \ldots + Z_{1,L}.$$ 

Thus for $r = 1$ we get

$$Z_1 - L_{Z_0}X_1 = h_1 + L_{X_{1,L}} \mu f_0.$$ 

Proceeding by induction, for $r > 1$, for all orders $s \leq r$ we need to solve

$$Z_{s,0} - L_{h_0}X_{s,0} = \Psi_r,$$

$$Z_{s,j} - L_{h_0}X_{s,j} = L_{\mu f_0} X_{s,j-1}, \quad j = 1, \ldots, L,$$

where

$$\Psi_1 = h_1,$$

$$\Psi_2 = h_2 - \frac{1}{2} (L_X h_1 + E_1 Z_1) - \frac{1}{2} L_X L_{X_{1,L}} \mu f_0 - L_{X_{1,L}} \mu f_0,$$

$$\Psi_s = h_s = \sum_{j=1}^{s-1} \frac{j}{s} (L_{X_j} h_{s-j} + E_{s-j} Z_j)$$

$$- \sum_{j=1}^{s-1} \frac{j}{s} L_{X_j} (L_{X_{s-j,L}} \mu f_0 - L_{X_{s-j-1,L}} \mu f_0) - L_{X_{s-1,L}} \mu f_0, \quad s > 2,$$

with $\Psi_s \in P_{sK,s(K+L'K')}$. Let us remark that we cannot let either $L$ or $r$ go to infinity, as the radius of convergence shrinks to zero; this is in agreement with the classical Birkhoff normal form.

### 3. Analytic tools

In order to make the paper self-contained, we report in this section all the technical tools needed to prove proposition 1, lemma 1 and theorem 1. Again we recall that we use the shorthand notations $| \cdot |_\alpha$ and $\| \cdot \|_\alpha$ for $| \cdot |_{(\theta,R)}$ and $\| \cdot \|_{(\theta,\sigma,R)}$. 
3.1 Estimates for multiple Poisson brackets

Some Cauchy estimates on the derivatives in the restricted domains will be useful.

**Lemma 3:** Let $d \in \mathbb{R}$ such that $0 < d < 1$ and $g$ be an analytic function with bounded norm $\|g\|_1$. Then one has

\[
\left\| \frac{\partial g}{\partial p_j} \right\|_{1-d} \leq \frac{\|g\|_1}{d}, \quad \left\| \frac{\partial g}{\partial q_j} \right\|_{1-d} \leq \frac{\|g\|_1}{ed}, \quad \left\| \frac{\partial g}{\partial z_j} \right\|_{1-d} \leq \frac{\|g\|_1}{dR}.
\]

Of course, the latter inequality holds true also by replacing $z_j$ with $i\bar{z}_j$.

The proof of lemma 3 is straightforward and it is left to the reader.

**Lemma 4:** Let $d, d' \in \mathbb{R}$ such that $d > 0$, $d' \geq 0$ and $d + d' < 1$, and $g, g'$ be two analytic functions with bounded norms $\|g\|_{1-d-d'}$ and $\|g'\|_{1-d'}$, respectively. Then, for all $\delta \in \mathbb{R}$ such that $\delta > 0$ and $d + d' + \delta < 1$ one has

\[
\|\{g, g'\}\|_{1-d-d'-\delta} \leq \frac{\Xi}{(d + \delta)\delta} \|g\|_{1-d-d'} \|g'\|_{1-d'}, \quad \Xi = \left( \frac{2}{eg\sigma} + \frac{1}{R^2} \right).
\]

**Proof.** We separately consider the parts of the Poisson bracket involving the $(p, q)$ variables and the $(z, i\bar{z})$ ones. For the first part we have

\[
\left\| \sum_{j=1}^{n_1} \left( \frac{\partial g}{\partial q_j} \frac{\partial g'}{\partial p_j} - \frac{\partial g}{\partial p_j} \frac{\partial g'}{\partial q_j} \right) \right\|_{1-d-d'-\delta} \leq \sum_{k \in \mathbb{Z}^{n_1}} \sum_{k' \in \mathbb{Z}^{n_1}} \left[ \left( \frac{|k||g_k|_{1-d-d'}|g'_{k'}|_{1-d'}}{(d + \delta)\delta} \right) e^{(|k|+|k'|)(1-d-d'-\delta)\sigma} \right.
\]

\[
\left. \quad + \frac{|g_k|_{1-d-d'}|k'| |g'_{k'}|_{1-d'}\delta}{\delta \sigma} \right]
\leq \frac{2}{e\sigma} \frac{1}{(d + \delta)\delta} \|g\|_{1-d-d'} \|g'\|_{1-d'},
\]

being $g'_{k'} = g'_{k'}(p, z, i\bar{z})$ as in the expansion (3). Here the Cauchy estimate (12) and the elementary inequality $ae^{-ab} \leq 1/(eb)$, for positive $a$ and $b$ have been used.

Let us now focus on the second part of the Poisson bracket. For every point $(p, z, i\bar{z}) \in \mathcal{G}_{(1-d-d'-\delta)q} \times \mathcal{B}_{(1-d-d'-\delta)R}$ and for all pairs of vectors $k, k' \in \mathbb{Z}^{n_1}$, we introduce an auxiliary function

\[ W_{(p,z,i\bar{z});k,k'}(t) = g_k\left( p, z - t\frac{\partial g_{k'}}{\partial (i\bar{z})}, i\bar{z} + t\frac{\partial g_{k'}}{\partial z} \right). \]

Since $g_k$ is analytic on $\mathcal{G}_{(1-d-d')q} \times \mathcal{B}_{(1-d-d')R}$, then $W_{(p,z,i\bar{z});k,k'}$ is analytic for $|t| \leq \tilde{t}$, with

\[
\tilde{t} = \frac{\delta R}{\max_{1 \leq j \leq n_2} \left\{ \left| \frac{\partial g_{k'}}{\partial z_j} \right|_{1-d-d'-\delta}, \left| \frac{\partial g_{k'}}{\partial (i\bar{z})} \right|_{1-d-d'-\delta} \right\}}.
\]
Thus, by the Cauchy’s estimate we get
\[ |\{g_k, g'_k\}|_{1-d-d'-\delta} \leq \left| \frac{d}{dt} W_{p, z, i\bar{z}}; k, k'(t) \right|_{t=0} \leq \frac{|g_k|_{1-d-d'} t}{1-d-d'}. \]

By the definition (4) of the norm, we get
\[ (15) \quad \left\| \sum_{j=1}^{n^2} \left( \frac{\partial g}{\partial (i\bar{z}_j)} \frac{\partial g'}{\partial z_j} - \frac{\partial g}{\partial z_j} \frac{\partial g'}{\partial (i\bar{z}_j)} \right) \right\|_{1-d-d'-\delta} \leq \frac{1}{R^2} \frac{\|g\|_{1-d-d'} \|g'\|_{1-d'}}{(d+\delta)\delta}. \]

The wanted inequality (13) follows by adding up (14) and (15).

Q.E.D.

**Lemma 5:** Let \( d, d' \in \mathbb{R} \) such that \( d > 0, d' \geq 0 \) and \( d + d' < 1 \) and \( \mathcal{X}, g \) be two analytic functions with bounded norms \( \|\mathcal{X}\|_{1-d-d'} \) and \( \|g\|_{1-d-d'} \), respectively. Then, for \( j \geq 1 \), we have
\[ \left\| \mathcal{L}^j \mathcal{X} g \right\|_{1-d-d'} \leq \frac{j!}{e^2} \left( \frac{e^2}{d^2} \Xi \right)^j \|\mathcal{X}\|_{1-d-d'}^j \|g\|_{1-d-d'}, \quad \Xi = \left( \frac{2}{e^2 \sigma} + \frac{1}{R^2} \right). \]

**Proof.** For \( j \geq 1 \) let \( \delta = d/j \). By repeated application of lemma 4, we get the recursive chain of inequalities
\[ \left\| \mathcal{L}^j \mathcal{X} g \right\|_{1-d-d'} \leq \frac{\Xi}{j^d} \|\mathcal{X}\|_{1-d-d'} \left\| \mathcal{L}^{j-1} \mathcal{X} g \right\|_{1-d-d'-(j-1)d} \leq \ldots \leq \frac{j!}{e^2} \left( \frac{e^2}{d^2} \Xi \right)^j \|\mathcal{X}\|_{1-d-d'}^j \|g\|_{1-d-d'}. \]

In the last row, we used the trivial inequality \( j^j \leq j! e^{j-1} \), holding true for \( j \geq 1 \).

Q.E.D.

### 3.2 Analyticity of Lie transform

We report here two main results concerning the analyticity of the Lie transform.

**Proposition 2:** Let the generating sequence \( \mathcal{X} = \{\mathcal{X}_s\}_{s \geq 1} \) be analytic on the domain \( D_{\varrho, \nu, R} \), and assume
\[ (16) \quad \|\mathcal{X}_s\|_{\varrho, \nu, R} \leq \frac{b^{s-1}}{s} G, \]
with \( b, G \in \mathbb{R} \) such that \( b \geq 0 \) and \( G > 0 \). Then, for every positive \( d < 1/2 \) the following statement holds true: if the condition
\[ (17) \quad e^2 \frac{\Xi}{d^2} G + b \leq \frac{1}{2}, \quad \Xi = \left( \frac{2}{e^2 \sigma} + \frac{1}{R^2} \right) \]
is satisfied, then the operator $T_X$ and its inverse $T_X^{-1}$ define an analytic canonical transformation on the domain $\mathcal{D}_{(1-d)(\varrho,\sigma,R)}$ with the properties

$$
\mathcal{D}_{(1-2d)(\varrho,\sigma,R)} \subset T_X \mathcal{D}_{(1-d)(\varrho,\sigma,R)} \subset \mathcal{D}_{\varrho,\sigma,R} ;
\mathcal{D}_{(1-2d)(\varrho,\sigma,R)} \subset T_X^{-1} \mathcal{D}_{(1-d)(\varrho,\sigma,R)} \subset \mathcal{D}_{\varrho,\sigma,R}.
$$

The proof of the proposition is based on the following

**Lemma 6:** Let a function $f$ and the generating sequence $\mathcal{X} = \{X_s\}_{s \geq 1}$ be analytic on the domain $\mathcal{D}_{\varrho,\sigma,R}$, and assume that $\|f\|_{\varrho,\sigma,R}$ is finite. Let the generating sequence satisfy (16) and assume that (17) holds true. Then the series $T_X f$, $T_X^{-1} f$, $T_X p$, $T_X^{-1} p$, $T_X q$ and $T_X^{-1} q$ are absolutely convergent on $\mathcal{D}_{(1-d)(\varrho,\sigma,R)}$, and for any integer $r > 0$ one has

(i) the operators $E_s$ and $D_s$ are estimated by

$$
\|E_s f\|_{(1-d)(\varrho,\sigma,R)} \leq \left( e^2 \frac{\Xi}{d^2} G + b \right)^{s-1} \frac{\Xi}{d^2} G \|f\|_{\varrho,\sigma,R} ;
\|D_s f\|_{(1-d)(\varrho,\sigma,R)} \leq \left( e^2 \frac{\Xi}{d^2} G + b \right)^{s-1} \frac{\Xi}{d^2} G \|f\|_{\varrho,\sigma,R} ;
$$

(ii) the Lie transform and its inverse are estimated by

$$
\|T_X f\|_{(1-d)(\varrho,\sigma,R)} \leq 2 \|f\|_{\varrho,\sigma,R} ;
\|T_X^{-1} f\|_{(1-d)(\varrho,\sigma,R)} \leq 2 \|f\|_{\varrho,\sigma,R} ;
$$

(iii) the remainder of a $r$-th order truncated transformation is estimated by

$$
\|T_X f - \sum_{s=0}^{r} E_s f\|_{(1-d)(\varrho,\sigma,R)} \leq \frac{2}{e^2} \left( e^2 \frac{\Xi}{d^2} G + b \right)^{r+1} \|f\|_{\varrho,\sigma,R} ;
\|T_X^{-1} f - \sum_{s=0}^{r} D_s f\|_{(1-d)(\varrho,\sigma,R)} \leq \frac{2}{e^2} \left( e^2 \frac{\Xi}{d^2} G + b \right)^{r+1} \|f\|_{\varrho,\sigma,R} ;
$$

(iv) the change of coordinates is estimated by

$$
\|T_X p - p\|_{(1-d)(\varrho,\sigma,R)} \leq \frac{1}{2\varepsilon^2} d\varrho ;
\|T_X^{-1} p - p\|_{(1-d)(\varrho,\sigma,R)} \leq \frac{1}{2\varepsilon^2} d\varrho ;
\|T_X q - q\|_{(1-d)(\varrho,\sigma,R)} \leq \frac{1}{2\varepsilon^2} d\sigma ;
\|T_X^{-1} q - q\|_{(1-d)(\varrho,\sigma,R)} \leq \frac{1}{2\varepsilon^2} d\sigma ;
\|T_X z - z\|_{(1-d)(\varrho,\sigma,R)} \leq \frac{1}{2\varepsilon^2} dR ;
\|T_X^{-1} z - z\|_{(1-d)(\varrho,\sigma,R)} \leq \frac{1}{2\varepsilon^2} dR .
$$

The proof of the previous lemma and proposition 2 are just a straightforward adaptation of proposition 4.3 and lemma 4.4 in [7].

4. **Quantitative estimates**

We now translate the formal scheme introduced in section 2 into recursive estimates.

The quantitative estimates for the truncated sequence of generating functions are collected in the following
Lemma 7: Consider the Hamiltonian (10) and let $\|h_s\|_1 \leq \zeta^{s-1}F$, for some real $F > 0$ and $\zeta \geq 0$, and assume $\|f_0\|_{\rho,\sigma,R} \leq G$. Furthermore, for any positive $d < 1$, let $\mu$ satisfy the smallness condition

$$\frac{9r^2L\Xi\mu G}{\alpha_r d^2} \leq \frac{1}{2}, \quad \Xi = \left(\frac{2}{e\rho \sigma} + \frac{1}{R^2}\right),$$

with $\alpha_r$ as in (5). Then the truncated sequence of generating functions $\mathcal{X}^{(r)}$ that gives the Hamiltonian the normal form (11) satisfies

$$\|\Psi_s\|_{(1-d)(\rho,\sigma,R)} \leq \frac{b^{s-1}}{s} F, \quad \|\mathcal{X}_s\|_{(1-d)(\rho,\sigma,R)} \leq \frac{b^{s-1} 2F}{s \alpha_r},$$

with

$$b = 4 \left(\frac{27r^4F \Xi^2}{\alpha_r^2 d^4} + \zeta\right).$$

Proof. In order to produce recursive estimates of the norm of Poisson brackets, we need to define a suitable sequence of restrictions of the domain. Fix the final restriction, $d$, and define the following sequence

$$(18) \quad d_i = \frac{i}{3r}d, \quad i = 1, \ldots, 3r,$$

where $r$ is the maximum relegation order. We now look for the estimates of $\|\Psi_s\|_{1-d_{3s-2}}$ and $\|\mathcal{X}_s\|_{1-d_{3s-1}}$. Again, here we omit the $(\rho,\sigma,R)$ in the subscript of the norms.

For $s = 1$ we have

$$\|\Psi_1\|_{1-d_1} \leq F,$$

from which we immediately get

$$\|\mathcal{X}_{1,0}\|_{1-d_1} \leq \frac{F}{\alpha_r}.$$

Iterating the previous estimates, for $l \geq 1$ we get

$$\|\mathcal{X}_{1,l}\|_{1-d_1 - l \frac{d_2 - d_1}{L}} \leq \frac{1}{\alpha_r} \left( d_1 + l \frac{d_2 - d_1}{L} \right) \frac{d_2 - d_1}{L} \|\mathcal{X}_{1,l-1}\|_{1-d_1 - (l-1) \frac{d_2 - d_1}{L}} \|\mu f_0\|_1$$

$$\leq \frac{9r^2L\Xi}{\alpha_r d^2} \|\mathcal{X}_{1,l-1}\|_{1-d_1 - (l-1) \frac{d_2 - d_1}{L}} \|\mu f_0\|_1,$$

where we used the elementary inequality

$$\left( d_1 + l \frac{d_2 - d_1}{L} \right) \frac{d_2 - d_1}{L} \geq d_1 \frac{d_2 - d_1}{L} \geq \frac{d^2}{9r^2L}.$$

Plugging in the condition

$$\frac{9r^2L\Xi\mu G}{\alpha_r d^2} \leq \frac{1}{2},$$

we get

$$\|\mathcal{X}_1\|_{1-d_2} \leq \frac{2F}{\alpha_r}.$$
Summing up, for $s = 1$, we have

$$
\|\Psi_1\|_{1-d_1} \leq F, \quad \|Z_1\|_{1-d_1} \leq F, \quad \|X_1\|_{1-d_2} \leq \frac{F}{\alpha r}.
$$

We now look for two real sequences $\{\eta_s\}_{1 \leq s \leq r}$ and $\{\theta_{s,j}\}_{0 \leq s \leq r, 1 \leq j \leq r}$ such that

$$
\|\Psi_s\|_{1-d_{3s-2}} \leq \eta_s F, \quad \|E_sZ_j\|_{1-d_{3(s+j)-2}} \leq \theta_{s,j} F,
$$

and we remark that we may choose $\eta_1 = 1$ and set $\theta_{0,j} = \eta_j$.

For $s \geq 2$ we get

$$
\|\Psi_s\|_{1-d_{3s-2}} \leq \zeta^{s-1}F + \sum_{j=1}^{s-1} \frac{j}{s} \left( \frac{\Xi}{d_{3s-2}(d_{3s-2} - d_{3j-1})} \right) \frac{2\eta_j F}{\alpha} \zeta^{s-j-1} + \theta_{s-j,j} F
$$

$$
+ \sum_{j=1}^{s-1} \frac{j}{s} \left( \frac{\Xi}{d_{3j}(d_{3j} - d_{3(s-j)-1})} \right) \frac{2\eta_j \eta_{s-j} F^2}{\alpha^2} + \frac{\Xi}{d_{3s-2}(d_{3s-2} - d_{3s-4})} \frac{2\eta_{s-1} F^2}{\alpha r},
$$

and, by the definition of $E_s$, for $s \geq 1$ we have

$$
\|E_sZ_j\|_{1-d_{3(s+j)-3}} \leq \sum_{l=1}^{s} \frac{l}{s} \frac{2\eta_l \theta_{s-l,j} \Xi}{(d_{3(s+j)-2} - d_{3l-1})(d_{3(s+j)-2} - d_{3(s+j-l)-2})} \frac{F^2}{\alpha r}.
$$

In view of (18), for $1 \leq j \leq s - 1$ one has

$$
\frac{1}{(d_{3s-2})(d_{3s-2} - d_{3j-1})} \leq \frac{3r^2}{2d^2},
$$

and

$$
\frac{1}{(d_{3s-2} - d_{3j-1})(d_{3s-2} - d_{3(s-j)})(d_{3(s-j)} - d_{3(s-j)-1})} \leq \frac{27r^4}{2d^4},
$$

while for $s \geq 1$, $j \leq s$ and $1 \leq l \leq s$ one gets

$$
\frac{1}{(d_{3(s+j)-2} - d_{3l-1})(d_{3(s+j)-2} - d_{3(s+j-l)-2})} \leq \frac{3r^2}{2d^2}.
$$

Thus the sequences $\{\eta_s\}$ and $\{\theta_{s,j}\}$ may be defined as

$$
\eta_s = \zeta^{s-1} + \frac{C_r}{s} \sum_{j=1}^{s-1} j \eta_j \zeta^{s-j-1} + \frac{C_r}{s} \sum_{j=1}^{s-1} j \eta_j \eta_{s-j} + \frac{1}{s} \sum_{j=1}^{s-1} j \theta_{s-j,j},
$$

$$
\theta_{s,j} = \frac{C_r}{s} \sum_{l=1}^{s} l \eta_l \theta_{s-l,l},
$$
with
\[ C_r = \frac{2^7 r^4 F \Xi^2}{\alpha^2 d^4}. \]

We are led to study the behaviour of the two sequences with initial values \( \eta_1 = 1 \) and \( \theta_{0,j} = \eta_j \). Let us remark that the second sequence can be rewritten as \( \theta_{s,j} = \eta_j \theta_{s,1} \), hence we set \( \theta_{s,1} = \theta_s \) and consider the double sequence

\[
\begin{align*}
\eta_s &= \zeta^{s-1} + \frac{C_r}{s} \sum_{j=1}^{s-1} j \eta_j \zeta^{s-j-1} + \frac{C_r}{s} \sum_{j=1}^{s-1} j \eta_j \eta_{s-j} + \frac{1}{s} \sum_{j=1}^{s-1} j \eta_j \theta_{s-j}, \\
\theta_s &= \frac{C_r}{s} \sum_{j=1}^{s} j \eta_j \theta_{s-j},
\end{align*}
\]

(19)

with initial values \( \eta_1 = \theta_0 = 1 \). Subtracting from the second equation the first one multiplied by \( C_r \) we get

\[ \theta_s = C_r \eta_s - C_r \zeta^{s-1} - \frac{C_r^2}{s} \sum_{j=1}^{s-1} j \eta_j \zeta^{s-j-1}. \]

Let us remark that \( \theta_s \) just depends by \( \eta_1, \ldots, \eta_{s-1} \). Replacing the previous expression in the first of (19) we get

\[ \eta_s \leq \zeta^{s-1} + C_r \sum_{j=1}^{s-1} \eta_j \eta_{s-j}, \]

and we have

\[ \eta_s \leq (C_r + \zeta)^{s-1} \nu_s, \]

(20)

where the real sequence \( \{\nu_s\}_{s \geq 1} \) is the so-called Catalan sequence

\[ \nu_1 = 1, \quad \nu_s = \sum_{j=1}^{s-1} \nu_j \nu_{s-j} \leq \frac{4^{s-1}}{s}. \]

The claim follows replacing \( \nu_s \leq 4^{s-1}/s \) in (20) and collecting the estimates previously obtained.

Q.E.D.

**Proof of proposition 1.** The proof is a straightforward application of the results previously obtained, having set \( d = 1/8 \), and it is left to the reader.

Q.E.D.

**Proof of lemma 1.** Using the elementary estimate

\[ |\Phi_0(t) - \Phi_0(0)| \leq |\Phi_0(t) - \Phi(t)| + |\Phi(t) - \Phi(0)| + |\Phi(0) - \Phi_0(0)|. \]

By means of (8) we easily bound the sum of the first and third terms, being smaller than \( \varrho/8 \). Coming to the second term we have

\[ |\hat{\Phi}| \leq \| T \chi \{ \Phi_0, \mathcal{R}^{(r+1)} \} \|_\frac{1}{2} \leq \frac{16}{\varepsilon \sigma} \left( \frac{\varepsilon A}{2^{18} \Xi^2} \right) \eta_r \leq \frac{\varepsilon A}{2^{14} \varepsilon \sigma \Xi^2} \eta_r. \]
Thus
\[ |\Phi(t) - \Phi(0)| \leq |t| \frac{A}{2^{14}e\sigma^2} \varepsilon \eta^r, \]
which is smaller than \( \varrho/4 \) if \( |t| < t^* \) as claimed. \( \text{Q.E.D.} \)

The proof of theorem 1 follows directly from lemma 1, it is just a matter of making a clever choice of the parameters.

**Proof of Theorem 1.** It remains to choose the parameters \( K \geq 1 \) and \( r \geq 1 \) as functions of the parameters \( \varrho, \sigma, R \) and \( \varepsilon \) that characterise the Hamiltonian. The aim is to make a good choice, so that the stability time is as large as possible.

Assuming the smallness condition \( \eta < 1/e \), namely
\[
\varepsilon r^4 A \frac{\alpha^2}{\alpha^2_r} + 4e^{-K\sigma/2} \leq \frac{1}{e},
\]
that is satisfied if
\[
\varepsilon r^4 A \frac{\alpha^2}{\alpha^2_r} \leq \frac{1}{2e}, \quad e^{-K\sigma/2} \leq \frac{1}{8e},
\]
where \( A \) is defined as in (7). Using the Diophantine condition
\[
\alpha_r \leq \frac{\gamma}{(rK)^r},
\]
it is natural to choose
\[
r = \left( \frac{\gamma^2}{2\varepsilon eAK^{2r}} \right)^{\frac{1}{1+2r}}, \quad K = \left[ \frac{2(1 + 3 \log 2)}{\sigma} \right].
\]
We have \( K \geq 1 \) by definition, while the condition \( r \geq 1 \) is satisfied provided
\[
\varepsilon \leq \frac{\gamma^2}{2eAK^{2r}}.
\]
The claim follows by setting \( T = 2^{12}e\varrho\sigma\Xi^2/A \).

\( \text{Q.E.D.} \)

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