On self-similarity of $p$-adic analytic pro-$p$ groups of small dimension

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Abstract

Given a torsion-free $p$-adic analytic pro-$p$ group $G$ with $\dim(G) < p$, we show that the self-similar actions of $G$ on regular rooted trees can be studied through the virtual endomorphisms of the associated $\mathbb{Z}_p$-Lie lattice. We explicitly classify 3-dimensional unsolvable $\mathbb{Z}_p$-Lie lattices for $p$ odd, and study their virtual endomorphisms. Together with Lazard’s correspondence, this allows us to classify 3-dimensional unsolvable torsion-free $p$-adic analytic pro-$p$ groups for $p \geq 5$, and to determine which of them admit a faithful self-similar action on a $p$-ary tree. In particular, we show that no open subgroup of $SL_1^1(\Delta_p)$ admits such an action. On the other hand, we prove that all the open subgroups of $SL_2^2(\mathbb{Z}_p)$ admit faithful self-similar actions on regular rooted trees.

Introduction

The class of groups that admit a faithful self-similar action on some regular rooted $d$-ary tree $T_d$ contains many interesting and important examples such as the Grigorchuk 2-group [Gri80], the Gupta-Sidki $p$-groups [GS83], the affine groups $\mathbb{Z}^n \rtimes GL_n(\mathbb{Z})$ [BS98], groups obtained as iterated monodromy groups of self-coverings of the Riemann sphere by post-critically finite rational maps [Nek05], and so on. Recently there has been an intensive study on the self-similar actions of other important families of groups including abelian groups [BS10], finitely generated nilpotent groups [BS07], arithmetic groups [Kap12] and groups of type $FP_n$ [KS17]. Despite such an active research on the topic of self-similarity in recent years, very little is known about the self-similar actions of pro-$p$ groups. In this context one may ask the following question.

Question. Which pro-$p$ groups admit a faithful self-similar action on a regular rooted $p$-ary tree?

The pro-$2$ completion of the Grigorchuk group coincides with its topological closure as a subgroup of $\text{Aut}(T_2)$, so it is an example of a pro-$2$ group that admits a self-similar action on

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Mathematics Subject Classification (2010): Primary 20E18, 22E20, 20E08; Secondary 22E60.

Key words: self-similar group, $p$-adic analytic group, pro-$p$ group, virtual endomorphism, $p$-adic Lie lattice.
a binary tree. For similar reasons, the pro-$p$ completions of the Gupta-Sidki $p$-groups admit self-similar actions on a $p$-ary tree. Self-similar actions of some classes of finite $p$-groups where studied in [Sun11] and [BFFV16].

The main goal of the present paper is to study the self-similar actions of torsion-free $p$-adic analytic pro-$p$ groups, and we do so using Lie methods. It is not difficult to see that every non-trivial torsion-free $p$-adic analytic pro-$p$ group of dimension at most 2 admits a faithful self-similar action on a $p$-ary tree (see Proposition 1.5), but in dimension 3 or higher the problem becomes considerably more complex.

We say that a group $G$ is self-similar of index $d$ if $G$ admits a faithful self-similar action on $T_d$ that is transitive on the first level; moreover, we say that $G$ is self-similar if it is self-similar of some index $d$. A virtual endomorphism of $G$ is a group homomorphism $\varphi : D \to G$ where $D \leq G$ is a subgroup of finite index; the index of $\varphi$ is defined to be the index of $D$ in $G$ (cf. [Nek02]). A virtual endomorphism is said to be simple if there are no non-trivial normal subgroups of $G$ that are $\varphi$-invariant (cf. Definition 1.1). It is known that a group $G$ is self-similar of index $d$ if and only if it admits a simple virtual endomorphism of index $d$ (see Proposition 1.3). We will use the language of virtual endomorphisms in the context of Lie lattices. Given an $n$-dimensional Lie lattice $L$ over $\mathbb{Z}_p$, a virtual endomorphism of $L$ is a Lie algebra homomorphism $\varphi : M \to L$ where $M \subseteq L$ is a subalgebra of dimension $n$; the index of $\varphi$ is defined to be the index of $M$ in $L$. A virtual endomorphism $\varphi : M \to L$ is called simple if there are no non-trivial ideals of $L$ that are $\varphi$-invariant. Moreover, we say that $L$ is self-similar of index $d$ if $L$ admits a simple virtual endomorphism of index $d$. Using Lazard’s correspondence [Laz65], which is an isomorphism between the category of saturable pro-$p$ groups and the category of saturable Lie lattices over $\mathbb{Z}_p$, and results from [GK09], we prove the following proposition.

**Proposition A** Let $p$ be a prime and $k \in \mathbb{N}$. Let $G$ be a saturable $p$-adic analytic pro-$p$ group of dimension $\dim(G) \leq p$, and $L_G$ be the $\mathbb{Z}_p$-Lie lattice associated with $G$. Then $G$ is a self-similar group of index $p^k$ if and only if $L_G$ is a self-similar Lie lattice of index $p^k$.

In [GK09], González-Sánchez and Klopsch proved that any torsion-free $p$-adic analytic pro-$p$ group $G$ of dimension $\dim(G) < p$ is saturable, and that such groups correspond bijectively to residually nilpotent $\mathbb{Z}_p$-Lie lattices. Using this fact and the above proposition, it is possible to classify such groups and give their self-similarity properties, provided that one is able to do so for the corresponding Lie lattices, an easier task due to their linear nature. With this in mind, we prove a sequence of theorems on $\mathbb{Z}_p$-Lie lattices for $p$ odd. First of all, we give an explicit classification up to isomorphism of 3-dimensional unsolvable Lie lattices $L$ over $\mathbb{Z}_p$ (Theorem 2.32), complementing the classification of 3-dimensional solvable Lie lattices provided in [GK09]. The same theorem determines exactly which unsolvable Lie lattices are self-similar of index $p$. Next, we establish that, if $L \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \simeq sl_2(\mathbb{Q}_p)$ then $L$ is self-similar of some index $p^k$, and we give estimates for the self-similarity index of $L$ (Theorem 2.33); moreover, we prove the self-similarity of some notable subalgebras of $sl_2(\mathbb{Z}_p)$ (Theorem 2.34). On the other hand, let $D_p$ be a central simple $\mathbb{Q}_p$-division algebra of index 2, and let $sl_1(D_p)$ be the set of elements of reduced trace zero in $D_p$. We establish that, if $L \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \simeq sl_1(D_p)$ then $L$ is not self-similar of index $p$ (Theorem 2.35). We believe that such $L$’s are not self-similar of any index (Conjecture 2.36). Observe that $sl_2(\mathbb{Q}_p)$ and $sl_1(D_p)$ represent the two isomorphism classes of 3-dimensional unsolvable $\mathbb{Q}_p$-Lie algebras (for $p$ odd).

Combining Proposition A with our results on Lie lattices we get the main result of the paper.
Theorem B Let $p \geq 5$ be a prime, and fix $\rho \in \mathbb{Z}_p^*$ not a square modulo $p$. The following is a complete and irredundant list of 3-dimensional unsolvable torsion-free $p$-adic analytic pro-$p$ groups, up to isomorphism:

1. The pro-$p$ group $G_1(s_0, s_1, s_2, \varepsilon_1, \varepsilon_2)$, for $0 \leq s_0 < s_1 < s_2$ and $\varepsilon_1, \varepsilon_2 \in \{0,1\}$, associated with the $\mathbb{Z}_p$-Lie lattice presented by:

$$\langle x_0, x_1, x_2 \mid [x_1, x_2] = p^{s_0} x_0, [x_2, x_0] = \rho^{\varepsilon_1} p^{s_1} x_1, [x_0, x_1] = \rho^{\varepsilon_2} p^{s_2} x_2 \rangle.$$ 

2. The pro-$p$ group $G_2(s_0, s_2, \varepsilon_1)$, for $1 \leq s_0 < s_2$ and $\varepsilon_1 \in \{0,1\}$, associated with the $\mathbb{Z}_p$-Lie lattice presented by:

$$\langle x_0, x_1, x_2 \mid [x_1, x_2] = p^{s_0} x_0, [x_2, x_0] = -\rho^{\varepsilon_1} p^{s_1} x_1, [x_0, x_1] = p^{s_2} x_2 \rangle.$$ 

3. The pro-$p$ group $G_3(s_0, s_1, \varepsilon_2)$, for $0 \leq s_0 < s_1$ and $\varepsilon_2 \in \{0,1\}$, associated with the $\mathbb{Z}_p$-Lie lattice presented by:

$$\langle x_0, x_1, x_2 \mid [x_1, x_2] = p^{s_0} x_0, [x_2, x_0] = p^{s_1} x_1, [x_0, x_1] = -\rho^{\varepsilon_2} p^{s_2} x_2 \rangle.$$ 

4. The pro-$p$ group $G_4(s_0)$, for $1 \leq s_0$, associated with the $\mathbb{Z}_p$-Lie lattice presented by:

$$\langle x_0, x_1, x_2 \mid [x_1, x_2] = p^{s_0} x_0, [x_2, x_0] = p^{s_0} x_1, [x_0, x_1] = p^{s_2} x_2 \rangle.$$ 

Moreover, we have:

1. $G_1(s_0, s_1, s_2, \varepsilon_1, \varepsilon_2)$ is not self-similar of index $p$.

2. $G_2(s_0, s_2, \varepsilon_1)$ is self-similar of index $p$ if and only if $\varepsilon_1 = 0$.

3. $G_3(s_0, s_1, \varepsilon_2)$ is self-similar of index $p$ if and only if $\varepsilon_2 = 0$.

4. $G_4(s_0)$ is self-similar of index $p$.

Let $p$ be an odd prime. Let $\Delta_p$ denote the (unique) maximal $\mathbb{Z}_p$-order in $\mathbb{D}_p$ and write $\mathfrak{P}$ for the maximal ideal of $\Delta_p$. Let $SL_1(\mathbb{D}_p)$ be the set of elements of reduced norm 1 in $\mathbb{D}_p$, and, for $m \geq 1$, let $SL_1^m(\Delta_p) := SL_1(\mathbb{D}_p) \cap (1 + \mathfrak{P}^m)$. Any 3-dimensional unsolvable torsion-free $p$-adic analytic pro-$p$ group is isomorphic to an open subgroup of exactly one of the groups $SL_1^m(\Delta_p)$ and $SL_2^\omega(\mathbb{Z}_p)$, where $SL_2^\omega(\mathbb{Z}_p)$ is a Sylow pro-$p$ subgroup of $SL_2(\mathbb{Z}_p)$ (see [GK99, Section 7.3]).

The following two theorems present a dichotomy, with respect to self-similarity, among the open subgroups of $SL_2^\omega(\mathbb{Z}_p)$ and of $SL_1^m(\Delta_p)$; moreover, we believe that the ensuing conjecture is true. The self-similarity index $\sigma(G)$ of a self-similar pro-$p$ group $G$ is defined to be the least power of $p$, say $p^k$, such that $G$ is self-similar of index $p^k$.

Theorem C Let $p$ be a prime.

1. (a) For $p \geq 5$, any open subgroup of $SL_2^\omega(\mathbb{Z}_p)$ is self-similar.

     (b) For $p \geq 3$, any open subgroup of $SL_2^\omega(\mathbb{Z}_p)$ is self-similar.

The self-similarity index of each of these groups $G$ is estimated in Table 1 (page 21) through the $\mathbb{Z}_p$-Lie lattice $L_G$ of $G$. 

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whether the matrix is invertible over the given ring or not). We write diag(\(a\)) for a commutative ring (\(a\)), the diagonal matrix with diagonal entries in \(R\). A square matrix with coefficients in \(R\) is called non-degenerate if its determinant is not zero (regardless whether the matrix is invertible over the given ring or not). We write diag\((a_1, ..., a_n)\) for a diagonal matrix with diagonal entries \(a_1, ..., a_n\). For the commutator of two elements \(x, y\) of a group \(G\), we use the convention \([x, y] = x^{-1}y^{-1}xy\). For the lower central series we use the convention \(\gamma_0(G) = G\) and \(\gamma_{k+1}(G) = [G, \gamma_k(G)]\).

Acknowledgements. The first author would like to thank José Ramón-Marí for suggesting a useful reference. The second author thanks the Heinrich Heine University in Düsseldorf for its hospitality. The authors also thank the referee for useful comments.

1 Self-similar groups

Let \(d \geq 1\) be an integer, and let \(X := \{0, ..., d - 1\}\), a finite set with \(d\) elements. We denote by \(T_d\) the (regular) rooted tree associated with the alphabet \(X\). The vertices of \(T_d\) are the finite words in \(X\); with a slight abuse of notation, we write \(v \in T_d\) when \(v\) is such a word. There is an edge of \(T_d\) exactly between vertices of type \(v\) and \(vx\), for \(v \in T_d\) and \(x \in X\). The empty word is the root of \(T_d\). The automorphism group \(\text{Aut}(T_d)\) has a natural topology that makes it profinite; the \(n\)-th level stabilizers of \(T_d\) form a basis of open neighborhoods of the identity. With any \(g \in \text{Aut}(T_d)\) and any \(v \in T_d\) there is an associated restriction \(g_v \in \text{Aut}(T_d)\). For a left action \(G \times T_d \to T_d\) of a group \(G\) on \(T_d\) by rooted tree isomorphisms, we use the notation \(g \cdot v\) for the result of the action of \(g \in G\) on \(v \in T_d\). An action is said to be transitive on the first level (respectively, level transitive) if for all \(x, y \in X\) (respectively, for all \(v, w \in T_d\) of the
same length) there exists \( g \in G \) such that \( y = g \cdot x \) (respectively, \( w = g \cdot v \)). An action is said to be self-similar if for all \( g \in G \) and \( x \in X \) there exists \( h \in G \) such that for all \( v \in T_d \) we have \( g \cdot xv = (g \cdot x)(h \cdot v) \), where the right-hand side is a juxtaposition of words. For a comprehensive study of self-similar groups the reader may consult, for instance, [Nek05].

Let \( p \) be a prime number. Observe that if \( G \) is a self-similar pro-\( p \) group of index \( d \) then \( d \) is a power of \( p \) (Proposition 1.3 and [DDMS03, Lemma 1.18]). If \( G \) is a nontrivial pro-\( p \) group, it is natural to ask whether \( G \) is self-similar of index \( p \) (or just whether it is self-similar), and to compute or estimate its self-similarity index. It is known that these questions can be tackled by looking at virtual endomorphisms (see Proposition 1.3). We state the next definition and lemma (whose proof is left to the reader) in the set theoretical setting, since these notions will be applied to \( \mathbb{Z}_p \)-Lie lattices as well.

**Definition 1.1** Let \( B \) be a subset of a set \( A \), and \( \varphi : B \to A \) be a function. A subset \( C \subseteq A \) is said to be \( \varphi \)-invariant if and only if \( C \subseteq B \) and \( \varphi(C) \subseteq C \). We define the domain of the powers of \( \varphi \) by \( D_0 := A \), and by \( D_{n+1} := \{ x \in B : \varphi(x) \in D_n \} \) for \( n \geq 0 \); we denote by \( D_\infty \) the intersection of all of these domains.

**Lemma 1.2** Let \( \varphi : B \to A \) be a function where \( B \subseteq A \), let \( C \subseteq A \), and let \( D_\infty \) be the intersection of the domains of the powers of \( \varphi \). If \( C \) is \( \varphi \)-invariant then \( C \subseteq D_\infty \).

**Proposition 1.3** Let \( G \) be a group and \( d \geq 1 \) be an integer. Then \( G \) is self-similar of index \( d \) if and only if \( G \) admits a simple virtual endomorphism of index \( d \).

**Proof:** We sketch the proof. First, assume that \( G \) admits a faithful self-similar action on \( T_d \) that is transitive on the first level. Define \( D \) to be the stabilizer of 0 in \( G \), and define \( \varphi : D \to G \) by \( \varphi(g) := g_0 \) (cf. [Nek05, Section 2.5.2]). Then \( G \) is a simple virtual endomorphism of index \( d \). On the other hand, let \( \varphi : D \to G \) be a simple virtual endomorphism of index \( d \), and let \( X = \{0, ..., d - 1\} \). Choose representatives \( f_0, ..., f_{d-1} \) of the left cosets of \( D \) in \( G \), and assume \( f_0 = 1 \). We define maps \( G \times X \to X \), \( (g, i) \mapsto g \cdot i \), and \( G \times X \to G \), \( (g, i) \mapsto g_i \) as follows. Given \( g \) and \( i \), there exists a unique \( j \) such that \( f_j^{-1}gf_i \in D \); define \( g \cdot i = j \) and \( g_i = \varphi(f_j^{-1}gf_i) \). These two maps define a reversible automaton, and with any initial state \( g \in G \), there is an associated automorphism \( \Psi_g \in \text{Aut}(T_d) \) (cf. [Nek05, Section 1.3]). One proves that the function \( \Psi : G \to \text{Aut}(T_d) \) so obtained is a faithful self-similar action of \( G \) on \( T_d \) that is transitive on the first level (cf. [Nek04, Theorem 3.1] and [Nek05, Proposition 2.7.4]). Hence, \( G \) is self-similar of index \( d \).

**Corollary 1.4** Let \( G \) be a group and \( d \geq 1 \) be an integer. Let \( H \) be a subgroup of \( G \) of finite index. If \( H \) is self-similar of index \( d \) then \( G \) is self-similar of index \( d'[G : H] \).

One may further specialize the problem of self-similarity by considering torsion-free \( p \)-adic analytic pro-\( p \) groups. For a comprehensive study of \( p \)-adic analytic pro-\( p \) groups, the reader may consult [DDMS03]. In low dimension matters are easy since such groups \( G \) may be classified.

**Proposition 1.5** Let \( p \) be a prime, \( k \geq 1 \) be an integer, and \( G \) be a non-trivial torsion-free \( p \)-adic analytic pro-\( p \) group with \( \dim(G) \leq 2 \). Then \( G \) is self-similar of index \( p^k \).

**Proof:** For each \( G \), we exhibit a simple virtual endomorphism \( \varphi : D \to G \) of index \( p^k \) and apply Proposition 1.3. If \( \dim(G) = 1 \) then \( G \) is isomorphic to \( \mathbb{Z}_p \). We define \( \varphi : p^k \mathbb{Z}_p \to \mathbb{Z}_p \) by \( \varphi(a) = p^{-ka} \); the simplicity of \( \varphi \) follows from Lemma 1.2 after observing that \( D_\infty = \{0\} \).
The case \( \dim(G) = 2 \) is richer. The groups under consideration are classified as follows (see [GK09, Propositions 7.1, 7.2]). Assume that \( s \in \mathbb{N} \cup \{ \infty \} \) and that \( s \geq 1 \) if \( p \geq 3 \), and \( s \geq 2 \) if \( p = 2 \). We define \( G_+(s) \) for all \( p \), and \( G_-(s) \) for \( p = 2 \) through the following pro-\( p \) presentations:

\[
G_+(s) = \langle x, y \mid [x, y] = x^p \rangle \quad \text{and} \quad G_-(s) = \langle x, y \mid [x, y] = x^{-2-2^s} \rangle,
\]

where, by convention, \( p^\infty = 0 \). We define the associated group homomorphism \( \Psi : \)

\[
\text{Corollary 1.6} \quad \text{if} \quad d \]

\[
\text{Proposition 1.7} \quad \text{admits a faithful self-similar action on} \quad \text{if} \quad \alpha, \beta \in \mathbb{Z}_p. \quad \text{Moreover, multiplication in} \quad \text{by} \quad \text{We divide the proof of self-similarity in three cases, according to whether} \quad \text{G} \quad \text{G} \quad \text{D}_{\infty} \text{is simple by Lemma 1.2.}
\]

\[
\text{(1) Case} \quad G = G_+(\infty) = \mathbb{Z}_p \times \mathbb{Z}_p. \quad \text{Let} \quad D = \langle x^p, y \rangle \text{and consider the homomorphism} \quad \varphi : \quad \text{given by} \quad \varphi(x^p) = y \quad \text{and} \quad \varphi(y) = x. \quad \text{Then} \quad D_{\infty} = \{1\}, \quad \text{so} \quad \varphi \text{ is simple by Lemma 1.2.}
\]

\[
\text{(2) Cases} \quad G = G_+(s) \quad \text{and} \quad G = G_-(s), \quad \text{with} \quad s \in \mathbb{N}. \quad \text{Let} \quad D = \langle x^p, y \rangle \quad \text{and consider the homomorphism} \quad \varphi : \quad \text{given by} \quad \varphi(x^p) = x \quad \text{and} \quad \varphi(y) = y. \quad \text{Then} \quad D_{\infty} = \langle y \rangle. \quad \text{Let} \quad N \quad \text{be a non-trivial normal subgroup of} \quad G \quad \text{and let} \quad 1 \neq y^\beta x^\alpha \in N. \quad \text{If} \quad \alpha \neq 0, \quad \text{then} \quad y^\beta x^\alpha \notin D_{\infty}. \quad \text{If} \quad \alpha = 0 \text{then} \quad \beta \neq 0, \quad \text{so} \quad z = xy^\beta x^{-1} = x^{u^2-1} \in N, \quad \text{where} \quad u = 1 + p^s \quad \text{or} \quad u = -1 - p^s \quad \text{(according to the case). Since} \quad u \neq \pm 1 \quad \text{and} \quad \beta \neq 0, \quad \text{we have} \quad u^2 - 1 \neq 0, \quad \text{so} \quad z \notin D_{\infty}. \quad \text{By Lemma 1.2,} \quad N \quad \text{is not} \quad \varphi\text{-invariant. Hence,} \quad \varphi \quad \text{is simple.}
\]

\[
\text{(3) Case} \quad G = G_-(\infty) \quad \text{for} \quad p = 2. \quad \text{Let} \quad D = \langle x, y^2 \rangle \quad \text{and consider the homomorphism} \quad \varphi : \quad \text{given by} \quad \varphi(x) = y \quad \text{and} \quad \varphi(y^2) = y. \quad \text{We observe that any element of} \quad \mathbb{Z}_p \quad \text{admits a faithful self-similar action on} \quad 1 \quad \text{and consider the homomorphism} \quad \varphi \quad \text{given by} \quad \varphi(x) = y \quad \text{and} \quad \varphi(y^2) = y. \quad \text{Then} \quad D_{\infty} = \langle y \rangle. \quad \text{Let} \quad N \quad \text{be a non-trivial normal subgroup of} \quad G \quad \text{and let} \quad 1 \neq y^\beta x^\alpha \in N. \quad \text{If} \quad \alpha = 0, \quad \text{then} \quad \beta \neq 0, \quad \text{so} \quad 1 \neq y^2 \beta \alpha \in N. \quad \text{On the other hand, if} \quad \alpha \neq 0 \text{then} \quad y^{-1}(y^2 \beta x^\alpha) y = y^2 \beta x^{-\alpha} \in N, \quad \text{and consequently}, \quad (y^2 \beta x^\alpha)(y^2 \beta x^{-\alpha})^{-1} = x^{2\gamma} \in N. \quad \text{Hence,} \quad \varphi(x^{2\gamma}) = y^{2\gamma} \in N. \quad \text{Let} \quad t \in \mathbb{N} \quad \text{be the minimum} \quad 2\text{-adic valuation of a} \quad \gamma \in \mathbb{Z}_2 \quad \text{such that} \quad y^t \in N \quad \text{(observe that} \quad t \geq k). \quad \text{Then} \quad \varphi(y^t) = y^{-2 \gamma} \in N \quad \text{and} \quad v_2(2^{-k} \gamma) = t - k \in \mathbb{N}, \quad \text{contradicting the minimality of} \quad t. \quad \square
\]

**Corollary 1.6** Let \( p \) be a prime and \( G \) be a compact \( p \)-adic analytic group of dimension \( \dim(G) \leq 2 \). Then \( G \) is self-similar.

Since \( p \)-adic analytic pro-\( p \) groups are topologically finitely generated, by a result of Serre their topology is determined by the group structure (see [DDMS03, Corollary 1.21]; see also [NS07]). However, this is not true in general for profinite groups that are not finitely generated. Modifying the definition to cover this case, we say that a **profinite group** \( G \) is **self-similar of index** \( d \) if \( G \) admits a faithful self-similar action on \( T_d \) that is transitive on the first level and moreover the associated group homomorphism \( \Psi : G \to \text{Aut}(T_d) \) is continuous. The next proposition is a topological analogue of Proposition 1.3; we leave its proof to the reader.

**Proposition 1.7** Let \( G \) be a profinite group and \( d \geq 1 \) be an integer. Then \( G \) is self-similar of index \( d \) (as a profinite group) if and only if there exists a continuous simple virtual endomorphism \( \varphi : D \to G \) of index \( d \) with \( D \) open in \( G \).
We close the section with a technical lemma which is used to show that the actions considered in Remark 3.3 are level-transitive and not just transitive on the first level.

**Lemma 1.8** Let \( p \) be a prime, and let \( G \) be a group such that the index of any finite-index subgroup of \( G \) is a power of \( p \). Let \( \varphi : D \to G \) be a virtual endomorphism of \( G \) of index \( p \), and let \((D_n)_{n \in \mathbb{N}}\) be the sequence of the domains of the powers of \( \varphi \). If \( \varphi(D_{n+1}) \not\subseteq D_{n+1} \) for all \( n \geq 0 \), then \( \varphi \) is regular, namely, \([D_n : D_{n+1}] = p\) for all \( n \geq 0 \).

**Proof:** Let \( d_n := [D_n : D_{n+1}] \) for all \( n \geq 0 \). Fix \( n \geq 0 \) for a while, and let \( \varphi_n : D_{n+1} \to D_n \) be the restriction of \( \varphi \). Then \( D_{n+2} = \varphi_n^{-1}(D_{n+1}) \) and \([D_{n+1} : D_{n+2}] = [\varphi(D_{n+1})D_{n+1} : D_{n+1}]\). Observe that the right-hand side of the last equality is the cardinality of the set of left cosets of \( D_{n+1} \) in \( D_n \) that are included in \( \varphi(D_{n+1})D_{n+1} \), so that \([\varphi(D_{n+1})D_{n+1} : D_{n+1}] \leq [D_n : D_{n+1}]\). It follows that \( d_{n+1} \leq d_n \). Since \( d_0 = p \), one may use induction on \( n \) to prove that \( d_n \leq p \) and that \([G : D_n] < \infty\) for all \( n \geq 0 \). Now, since \( \varphi(D_{n+1}) \not\subseteq D_{n+1} \), we have \( d_{n+1} > 1 \) and, consequently, \( d_n = p \) for all \( n \geq 0 \). \( \square \)

## 2 Self-similar Lie lattices

Let \( p \) be a prime number. We take \( \mathbb{Z}_p \) as the coefficient ring of our modules and algebras, and denote by \( v_p : \mathbb{Z}_p \to \mathbb{N} \cup \{\infty\} \) the \( p \)-adic valuation. Our main interest is on Lie lattices, namely, Lie algebras whose underlying module is a lattice (i.e., a finite-dimensional free module). At several points, we find more elegant and convenient to work with antisymmetric algebras. The “bracket” of two elements in such algebras is denoted by \([x, y]\). Let \( L \) be an \( n \)-dimensional antisymmetric \( \mathbb{Z}_p \)-algebra, \( n \in \mathbb{N} \). We say that \( L \) is self-similar if it is self-similar of index \( d \) for some positive integer \( d \). The **self-similarity index** \( \sigma(L) \) of a self-similar \( L \) is defined to be the least positive integer \( d \) such that \( L \) is self-similar of index \( d \). We say that \( L \) is just infinite if any non-zero ideal of \( L \) has dimension \( n \). Moreover, we say that \( L \) is hereditarily just infinite if any \( n \)-dimensional subalgebra of \( L \) is just infinite. For the lower central series of \( L \), we use the convention \( \gamma_0(L) = L \) and \( \gamma_{k+1}(L) = [L, \gamma_k(L)] \).

A statement similar to Corollary 1.4 holds for Lie lattices.

**Lemma 2.1** Let \( L \) be an \( n \)-dimensional antisymmetric \( \mathbb{Z}_p \)-algebra and \( M \subseteq L \) be a subalgebra of dimension \( n \). Moreover, suppose that \( M \) is self-similar of index \( p^k \). Then \( L \) is self-similar of index \( p^k[L : M] \).

**Remark 2.2** Every non-trivial \( \mathbb{Z}_p \)-Lie lattice \( L \) of dimension \( n \leq 2 \) is self-similar of index \( p^k \) for all \( k \geq 1 \). We show this fact for \( n = 2 \); the case \( n = 1 \) has already been treated for \( \mathbb{Z}_p \) as a group. If \( L \) is a 2-dimensional \( \mathbb{Z}_p \)-Lie lattice, then there is some \( s \in \mathbb{N} \cup \{\infty\} \) such that \( L \) is isomorphic to \( L(s) = \langle x, y \mid [x, y] = p^s x \rangle \) (see, for instance, [GK09, Section 7.1]), where \( p^\infty := 0 \). Consider the subalgebra \( M = \langle p^s x, y \rangle \) of \( L(s) \). For \( s = \infty \) we define a homomorphism \( \varphi : M \to L(s) \) by \( \varphi(p^s x) = y \) and \( \varphi(y) = x \), while for \( s \in \mathbb{N} \) we define a homomorphism \( \varphi : M \to L(s) \) by \( \varphi(p^s x) = x \) and \( \varphi(y) = y \). It is not difficult to see that \( \varphi \) is a simple virtual endomorphism of \( L \) of index \( p^k \). Note that the virtual endomorphisms in this example are exactly those of Proposition 1.5 (except for the extra groups that appear for \( p = 2 \)).

Next we deal with 3-dimensional \( \mathbb{Z}_p \)-Lie lattices.

**Lemma 2.3** Let \( L \) be a 3-dimensional \( \mathbb{Z}_p \)-lattice, and let \( x = (x_0, x_1, x_2) \) be a basis of \( L \).
(1) The following formulas establish a bijection (which depends on $x$) between antisymmetric brackets on $L$ and $3 \times 3$ matrices $A$ with coefficients in $\mathbb{Z}_p$:

\[
\begin{align*}
[x_1, x_2] &= \sum_{i=0}^{2} A_{i0} x_i \\
[x_2, x_0] &= \sum_{i=0}^{2} A_{i1} x_i \\
[x_0, x_1] &= \sum_{i=0}^{2} A_{i2} x_i.
\end{align*}
\]

(2) Assume that an antisymmetric bracket over $L$ is given, and let $A$ be its matrix with respect to $x$. Let $M$ be 3-dimensional submodule of $L$, and $y = (y_0, y_1, y_2)$ be a basis of $M$ (the case $M = L$, change of basis, is included). Let $U$ be the matrix of $y$ with respect to $x$, namely, $y_j = \sum_i U_{ij} x_i$. Since $\det(U) \neq 0$, the formula

\[
B = \det(U)U^{-1}A(U^{-1})^T
\]

defines a matrix $B \in gl_3(\mathbb{Q}_p)$. The following properties hold.

(a) $M$ is a subalgebra of $L$ if and only if $B$ has coefficients in $\mathbb{Z}_p$.

(b) If $M$ is a subalgebra then $B$ is the matrix of $M$ with respect to $y$.

**Proof:** The proof is straightforward. The reader may want to consult [Jac62, page 13] for similar arguments.

**Definition 2.4** Let $L$ be a 3-dimensional antisymmetric $\mathbb{Z}_p$-algebra, and let $x$ be a basis of $L$. Denote by $A$ the matrix of $L$ with respect to $x$. We say that $x$ is **diagonalizing** if and only if $A$ is diagonal, say $A = \text{diag}(a_0, a_1, a_2)$. We say that $x$ is **well diagonalizing** if and only if it is diagonalizing and $v_\rho(a_0) \leq v_\rho(a_1) \leq v_\rho(a_2)$ (the case $v_\rho(a_i) = \infty$ is not excluded).

**Remark 2.5** A 3-dimensional antisymmetric $\mathbb{Z}_p$-algebra $L$ admits a diagonalizing basis if and only if it admits a well diagonalizing one. This follows from the following more general observation. Assume that $L$ admits a diagonalizing basis $x$ and that $A = \text{diag}(a_0, a_1, a_2)$ is the matrix of $L$ with respect to $x$. Through a “diagonal” change of basis, one can make an arbitrary permutation of the $a_i$’s, and multiply all the $a_i$’s by the same invertible element of $\mathbb{Z}_p$.

**Remark 2.6** Let $L$ be a 3-dimensional antisymmetric $\mathbb{Z}_p$-algebra. Then there exist $s_0, s_1, s_2 \in \mathbb{N} \cup \{\infty\}$ such that $L/[L, L] \cong \bigoplus_{i=0}^{2} (\mathbb{Z}/p^{s_i}\mathbb{Z})$ (where $p^\infty := 0$). The exponents $s_0, s_1, s_2$ of the invariant factors will be called the **s-invariants** of $L$ (they are all finite exactly when $L$ is unsolvable). Suppose that $L$ admits a diagonalizing basis and let $A = \text{diag}(a_0, a_1, a_2)$ be the associated matrix. Then the valuations $v_\rho(a_i)$ for $i \in \{0, 1, 2\}$ are the $s$-invariants of $L$.

**Proposition 2.7** Let $L$ be a 3-dimensional antisymmetric $\mathbb{Z}_p$-algebra. The following holds.

(1) $L$ is an unsolvable Lie lattice if and only if the matrix of $L$ with respect to some (equivalently, any) basis of $L$ is symmetric and non-degenerate.

(2) Assume that $p \geq 3$ and that $L$ is, moreover, an unsolvable Lie lattice. Then

(a) $L$ admits a well diagonalizing basis.

(b) $L$ is hereditarily just infinite.
Lemma 2.12 Let \( p \geq 3 \) be a prime, and let \( L \) be a 3-dimensional \( \mathbb{Z}_p \)-Lie lattice. Suppose that the matrix of \( L \) with respect to some basis \((x_0, x_1, x_2)\) is

\[
\begin{pmatrix}
  a & 0 & 0 \\
  0 & p^s & 0 \\
  0 & 0 & -p^s
\end{pmatrix}
\]

with \( a \in \mathbb{Z}_p \), \( a \neq 0 \), and \( s \in \mathbb{N} \). Then \( L \) is self-similar of index \( p \).
PROOF: Since \( p \) is odd, the formulas \( y_0 = 2x_0, y_1 = x_1 - x_2 \) and \( y_2 = x_1 + x_2 \) define a new basis of \( L \) whose matrix has the format of the one of Lemma 2.11 with \( b = 2p^s \). \( \square \)

We close the section by recording some results needed in the sequel. The proof of the lemma is a routine calculation which is left to the reader.

**Lemma 2.13** Let \( L \) be a 3-dimensional \( \mathbb{Z}_p \)-Lie lattice that admits a diagonalizing basis. Let \( s_0, s_1, s_2 \in \mathbb{N} \cup \{\infty\} \) be the \( s \)-invariants of \( L \), with \( s_0 \leq s_1 \leq s_2 \). The following holds.

1. Let \( (x_0, x_1, x_2) \) be a well diagonalizing basis of \( L \). Then the lower central series of \( L \) is given by
   \[
   \gamma_n(L) = \langle p^{s_i}x_i : i = 0, 1, 2 \rangle \quad (n \geq 0),
   \]
   where the exponents \( s_i^{(m)} \in \mathbb{N} \cup \{\infty\} \) are given by
   \[
   \begin{align*}
   &s_0^{(2m+1)} = (m+1)s_0 + ms_1, \\
   &s_1^{(2m+1)} = ms_0 + (m+1)s_1, \quad m \geq 0; \\
   &s_2^{(2m+1)} = ms_0 + ms_1 + s_2.
   \end{align*}
   \]

2. \( L \) is residually nilpotent if and only if \( s_1 \geq 1 \).

**Proposition 2.14** Let \( L \) be a 3-dimensional \( \mathbb{Z}_p \)-Lie lattice that admits a diagonalizing basis. Let \( s_0 = 0, s_1 = 1 \) and \( s_2 = 1 \) be the \( s \)-invariants of \( L \), and suppose that \( I \subseteq L \) is an ideal, \( k \in \mathbb{N} \) and \( I \not\subseteq \gamma_k(L) \). Then \( \gamma_k(L) \subseteq I \).

**Proof:** Let the variables \( u_0, u_1 \) and \( u_2 \) stand for elements of \( \mathbb{Z}_p^* \); their values may vary from one formula to another. From the assumptions and Remarks 2.5 and 2.6, there exists a basis \( (x_0, x_1, x_2) \) of \( L \) such that \( [x_1, x_2] = u_0x_0, [x_2, x_0] = pu_1x_1 \) and \( [x_0, x_1] = pu_2x_2 \). From \( I \not\subseteq \gamma_k(L) \), there exists \( w = a_0x_0 + a_1x_1 + a_2x_2 \in I \) such that \( w \not\in \gamma_k(L) \). We treat the case where \( k = 2m \) is even, \( v_p(a_0) \leq m - 1 \) and \( v_p(a_1), v_p(a_2) \geq m \); the other cases are left to the reader. From Lemma 2.13 it follows that \( \gamma_{2m}(L) = p^mL \), so that we only need to show that \( p^m x_i \in I \) for \( i \in \{0, 1, 2\} \). Since \( v_p(a_0) \leq m - 1 \), it is enough to show that \( a_0px_i \in I \) for \( i \in \{0, 1, 2\} \). From \([w, x_1], x_0] = a_0p^2u_1x_1 \in I \) it follows that \( p^{m+1}x_1 \in I \) and \( a_1px_1 \in I \). Finally, from \([w, x_2], x_1] = a_0pu_0x_0 + a_1pu_1x_1 \in I \) it follows that \( a_0px_0 \in I \), hence \( a_1x_0, a_2x_0 \in I \) and \( a_1x_0, a_2x_0, a_2x_1 \in I \), it follows that \( a_0px_1, a_0px_2 \in I \). \( \square \)

### 2.1 Classification of 3-dimensional unsolvable Lie lattices

We classify 3-dimensional unsolvable Lie lattices over \( \mathbb{Z}_p \) for \( p \) an odd prime. The classification is given in terms of 3 \times 3 matrices as explained below. Observe that the canonical forms that we will actually use in the paper are given in Remark 2.16.

With any matrix \( A \in \text{gl}_3(\mathbb{Z}_p) \), we associate a 3-dimensional antisymmetric \( \mathbb{Z}_p \)-algebra \( L_A \) as follows: the underlying lattice is \( \mathbb{Z}_p^3 \) endowed with the canonical basis; the bracket is induced by the matrix \( A \) as in Lemma 2.3. One can see that \( L_A \simeq L_B \) if and only if \( A \) is multiplicatively congruent to \( B \) (denoted \( A \sim_{MC} B \)), namely, there exist \( u \in \mathbb{Z}_p^* \) and \( V \in \text{GL}_3(\mathbb{Z}_p) \) such that \( B = uV^TAV \).
**Theorem 2.15** Let \( p \geq 3 \) be a prime, and fix \( \rho \in \mathbb{Z}_p^\ast \) not a square modulo \( p \). Then the antisymmetric \( \mathbb{Z}_p \)-algebras \( L_A \) associated with the diagonal matrices listed below in four families constitute a complete and irredundant list of 3-dimensional unsolvable Lie lattices over \( \mathbb{Z}_p \).

\[
(1) \quad A = \mathrm{diag}(p^{s_0}, \rho^{s_1}p^{s_1}, \rho^{s_2}p^{s_2}) \quad \text{with} \quad 0 \leq s_0 < s_1 < s_2 \quad \text{and} \quad \varepsilon_1, \varepsilon_2 \in \{0,1\}.
\]

\[
(2) \quad A = \mathrm{diag}(p^{s_0}, \rho^{s_1}p^{s_1}, p^{s_2}) \quad \text{with} \quad 0 \leq s_0 < s_2 \quad \text{and} \quad \varepsilon_1 \in \{0,1\}.
\]

\[
(3) \quad A = \mathrm{diag}(p^{s_0}, \rho^{s_1}, \rho^{s_2}p^{s_1}) \quad \text{with} \quad 0 \leq s_0 < s_1 \quad \text{and} \quad \varepsilon_2 \in \{0,1\}.
\]

\[
(4) \quad A = \mathrm{diag}(p^{s_0}, p^{s_0}, p^{s_0}) \quad \text{with} \quad 0 \leq s_0.
\]

**Proof:** We denote by \( \Omega \) the set of matrices listed in the statement. Observe that if \( A, B \in gl_3(\mathbb{Z}_p) \), \( A \sim_{MC} B \) and \( A \) is symmetric and non-degenerate, then \( B \) shares the same properties. By Proposition 2.7 and the observations preceding the statement of the theorem, it suffices to show that \( \Omega \) is a set of representatives for the symmetric non-degenerate matrices of \( gl_3(\mathbb{Z}_p) \) modulo the relation \( \sim_{MC} \) of multiplicative congruence. We write \( A \sim_C B \) for the congruence relation, that is, \( B = V^TAV \) for some \( V \in GL_3(\mathbb{Z}_p) \). Of course, \( A \sim_C B \) implies \( A \sim_{MC} B \).

The congruence classes of symmetric non-degenerate matrices (i.e., non-degenerate quadratic forms) are classified (see [Cas78, Theorem 3.1, page 115]). We can write the congruence classes in the form:

\[
(1) \quad A = \mathrm{diag}(\rho^{s_0}p^{s_0}, \rho^{s_1}p^{s_1}, \rho^{s_2}p^{s_2}) \quad \text{with} \quad 0 \leq s_0 < s_1 < s_2 \quad \text{and} \quad \varepsilon_0, \varepsilon_1, \varepsilon_2 \in \{0,1\}.
\]

\[
(2) \quad A = \mathrm{diag}(p^{s_0}, \rho^{s_1}p^{s_1}, \rho^{s_2}p^{s_2}) \quad \text{with} \quad 0 \leq s_0 < s_2 \quad \text{and} \quad \varepsilon_1, \varepsilon_2 \in \{0,1\}.
\]

\[
(3) \quad A = \mathrm{diag}(\rho^{s_0}p^{s_0}, p^{s_1}, \rho^{s_2}p^{s_1}) \quad \text{with} \quad 0 \leq s_0 < s_1 \quad \text{and} \quad \varepsilon_0, \varepsilon_2 \in \{0,1\}.
\]

\[
(4) \quad A = \mathrm{diag}(p^{s_0}, p^{s_0}, \rho^{s_2}p^{s_0}) \quad \text{with} \quad 0 \leq s_0 \quad \text{and} \quad \varepsilon_2 \in \{0,1\}.
\]

In some sense, there are twice as many congruence classes as elements of \( \Omega \): in the list of the congruence classes compared to the list of elements of \( \Omega \), there is an extra factor \( \rho^{s_1} \). We proceed as follows.

First of all, we show that any symmetric non-degenerate \( B \) is multiplicatively congruent to some \( A \in \Omega \). Indeed, some congruence puts \( B \) in one of the forms given above, and we complete the argument for the second family, namely, we assume that \( B \) is congruent to \( C = \mathrm{diag}(p^{s_0}, \rho^{s_1}p^{s_1}, \rho^{s_2}p^{s_2}) \); the argument for the other families is similar. If \( \varepsilon_2 = 0 \) we are done. If \( \varepsilon_2 = 1 \) then we multiply \( C \) by \( \rho \), getting \( B \sim_{MC} \mathrm{diag}(\rho p^{s_0}, \rho p^{s_1}p^{s_0}, \rho^{s_2}p^{s_2}) \). Applying [Cas78, Lemma 3.4, page 115], we “discharge” the factor \( \rho \) in the first diagonal entry to the second, which is done through a congruence, so that \( B \sim_{MC} \mathrm{diag}(p^{s_0}, \rho^2 \rho^{s_1}p^{s_0}, \rho^{s_2}p^{s_2}) \). Another (obvious) congruence eliminates the factors \( \rho^2 \) on the second and third diagonal entries, getting one of the forms in the statement, namely, \( B \sim_{MC} \mathrm{diag}(p^{s_0}, \rho^{s_1}p^{s_0}, \rho^{s_2}p^{s_2}) \).

The last thing to be proven is that no two distinct matrices in \( \Omega \) are multiplicatively congruent to each other. Formally, take \( A, B \in \Omega \) and assume that they are multiplicatively congruent. We have to show that \( A = B \). First of all, observe that no two distinct elements of \( \Omega \) are congruent. In other words, if we show that \( A \sim_{C} B \) then we are done. By assumption, there exist \( u \in \mathbb{Z}_p^\ast \) and \( V \in GL_3(\mathbb{Z}_p) \) such that \( B = uV^TAV \). Hence, \( B \sim_{C} uA \). If \( u \) is a square then \( uA \sim_{C} A \) and we are done. We now show that the other possibility, namely, that \( u \) is not a square, leads to a contradiction. Since \( u = \rho v^2 \) for some \( v \in \mathbb{Z}_p^\ast \), we see that \( uA \sim_{C} \rho A \). Summarizing, \( B \sim_{C} \rho A \), where \( A, B \in \Omega \). Again, we should analyze four cases, depending on which family \( A \) belongs to. We will do the case of the second family, the others being similar.
The matrix $\rho A$ is $\text{diag}(\rho p s_0, \rho \rho^2, \rho p s_2)$, whose congruence class is shown to be represented by $\text{diag}(p s_0, \rho^2 p s_0, \rho p s_2)$ after discharging the first $\rho$ as above. Hence, $B = \text{diag}(p s_0, \rho^2 p s_0, \rho p s_2)$, but this is a contradiction, since $\text{diag}(p s_0, \rho^2 p s_0, \rho p s_2)$ is not an element of $\Omega$. □

**Remark 2.16** The classification given in Theorem 2.15 is a natural one if one takes the classification of quadratic forms as a starting point. On the other hand, in order to formulate the main theorem of this section (Theorem 2.32) in a more “uniform” way, we change some of the representative matrices as follows. We recall that $\rho \in \mathbb{Z}_p^*$ represents matrices as follows. We recall that $\rho \in \mathbb{Z}_p^*$ is a fixed non-square modulo $p$, and that $p$ is assumed to be odd. The “new” matrices are:

1. $A = \text{diag}(p s_0, \rho \rho^2, \rho p s_2)$ with $0 \leq s_0 < s_1 < s_2$ and $\varepsilon_1, \varepsilon_2 \in \{0, 1\}$.
2. $A = \text{diag}(p s_0, -\rho \rho^2, p s_2)$ with $0 \leq s_0 < s_2$ and $\varepsilon_1 \in \{0, 1\}$.
3. $A = \text{diag}(p s_0, p s_1, -\rho \rho^2)$ with $0 \leq s_0 < s_1$ and $\varepsilon_2 \in \{0, 1\}$.
4. $A = \text{diag}(p s_0, p s_0, p s_0)$ with $0 \leq s_0$.

Observe that we have just inserted two ‘minus’ signs, one in family (2) and the other in family (3). We explain which kind of replacement has been performed by treating family (2); the argument for family (3) is completely analogous. Recall that $-1$ is a square in $\mathbb{Z}_p^*$ if and only if $p \equiv 1 \pmod{4}$. Below, on the left we write the representative with respect to the original classification, while on the right we write the corresponding representative in the new classification, depending on the residue class of $p$ modulo 4. We have:

$$\text{diag}(p s_0, p s_0, p s_2) \sim_{MC} \begin{cases} 
\text{diag}(p s_0, -p s_0, p s_2) & \text{if } p \equiv 1 \pmod{4} \\
\text{diag}(p s_0, -p p s_0, p s_2) & \text{if } p \equiv 3 \pmod{4}
\end{cases}$$

and

$$\text{diag}(p s_0, p p s_0, p s_2) \sim_{MC} \begin{cases} 
\text{diag}(p s_0, -p p s_0, p s_2) & \text{if } p \equiv 1 \pmod{4} \\
\text{diag}(p s_0, -p s_0, p s_2) & \text{if } p \equiv 3 \pmod{4}
\end{cases}$$

As a matter of terminology, each of the matrices in the enumeration above will be called the **canonical matrix** of the corresponding isomorphism class of $\mathbb{Z}_p$-Lie lattices. A basis of a lattice $L$ whose associated matrix is canonical will be called a **canonical basis** of $L$, and the corresponding presentation (i.e., the commutation relations of the basis elements) will be called the **canonical presentation** of $L$. We will also use the terminology **canonical form** in order to refer generically to the canonical matrix or to the canonical presentation.

**Example 2.17** We give the canonical matrix of some relevant $\mathbb{Z}_p$-Lie lattices, where $p$ is assumed to be odd. Before doing that, we recall some definitions from [Klo03, Section 2]. Choose $\rho \in \{1, 2, \ldots, p-1\}$ not a square modulo $p$ and let

$$\mathbb{D}_p = \mathbb{Q}_p + \mathbb{Q}_p u + \mathbb{Q}_p v + \mathbb{Q}_p uv$$

be the quaternion algebra defined by the multiplication rules

$$u^2 = \rho, \quad v^2 = p, \quad uv = -vu;$$
then $\mathbb{D}_p$ is a central division algebra of index 2 over $\mathbb{Q}_p$. For an element $z = \alpha + \beta u + \gamma v + \delta uv \in \mathbb{D}_p$, the reduced norm and reduced trace are given by

$$N(z) = \alpha^2 - \rho\beta^2 - (\gamma^2 - \rho\delta^2)p \quad \text{and} \quad T(z) = 2\alpha.$$ 

The ring of integers $\Delta_p = \mathbb{Z}_p + \mathbb{Z}_p u + \mathbb{Z}_p v + \mathbb{Z}_p uv$ of $\mathbb{D}_p$ forms the unique maximal $\mathbb{Z}_p$-order in $\mathbb{D}_p$, and $\mathfrak{P} = v\Delta_p$ is the maximal ideal of $\Delta_p$. Let $SL_1(\mathbb{D}_p)$ be the set of elements of reduced norm 1 in $\mathbb{D}_p$; this is a 3-dimensional compact $p$-adic analytic group. Note that $SL_1(\mathbb{D}_p)$ coincides with the group $SL_1(\Delta_p) = SL_1(\mathbb{D}_p) \cap \Delta_p$. For any integer $m \geq 1$, let $SL_1^{m}(\Delta_p) = SL_1(\Delta_p) \cap (1 + \mathfrak{P}^m)$ be the $m$-th congruence subgroup of $SL_1(\Delta_p)$. Similarly, let $sl_1(\mathbb{D}_p)$ denote the set of elements of reduced trace zero in $\mathbb{D}_p$. Then $sl_1(\Delta_p) = sl_1(\mathbb{D}_p) \cap \Delta_p$, and $sl_1^{m}(\Delta_p) = sl_1(\Delta_p) \cap \mathfrak{P}^m$ is the $m$-th congruence subalgebra of $sl_1(\Delta_p)$.

On the right column of the list below, where $k \in \mathbb{N}$, $sl^k_2(\mathbb{Z}_p) = p^k sl_2(\mathbb{Z}_p)$ and $sl^0_1(\Delta_p) = sl_1(\Delta_p)$, we display the canonical matrix of the Lie lattices appearing on the left.

| Lattice | Matrix |
|---------|--------|
| $sl_2(\mathbb{Z}_p)$ | $\text{diag}(1,1,1)$ |
| $sl^2_2(\mathbb{Z}_p)$ | $\text{diag}(p^k,p^k,p^k)$ |
| $sl^2_2(\mathbb{Z}_p)$ | $\text{diag}(1,p,-p)$ |
| $\gamma_{2k}(sl^2_2(\mathbb{Z}_p))$ | $\text{diag}(p^k,p^{k+1},-p^{k+1})$ |
| $\gamma_{2k+1}(sl^2_2(\mathbb{Z}_p))$ | $\text{diag}(p^{k+1},-p^{k+1},p^{k+2})$ |
| $sl_1(\Delta_p)$ | $\text{diag}(1,-\rho,p)$ |
| $sl^2_1(\Delta_p)$ | $\text{diag}(p^k,-\rho p^k,p^{k+1})$ |
| $sl^{2k+1}_1(\Delta_p)$ | $\text{diag}(p^k,p^{k+1},-\rho p^{k+1})$. |

We prove the claims given in the list. Let $(x_0, x_1, x_2)$ be the basis of $sl_2(\mathbb{Z}_p)$ given by $x_0 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, $x_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $x_2 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$. The matrix of $sl_2(\mathbb{Z}_p)$ with respect to the new basis $(2^{-1}x_0, 2^{-1}(x_1 + x_2), 2^{-1}(-x_1 + x_2))$ is diag$(1,1,-1)$, from which it follows that the $s$-invariants are $s_0 = s_1 = s_2 = 0$, so that the matrix of $sl_2(\mathbb{Z}_p)$ with respect to some basis $(y_0, y_1, y_2)$ is the identity (see Remark 2.16). Since $sl^k_2(\mathbb{Z}_p) = p^k sl_2(\mathbb{Z}_p)$, the matrix of $sl^k_2(\mathbb{Z}_p)$ with respect to the basis $(p^k y_0, p^k y_1, p^k y_2)$ is diag$(p^k,p^k,p^k)$.

Regarding $sl^2_2(\mathbb{Z}_p)$, its matrix with respect to the basis $(z_0, z_1, z_2) = (2^{-1}px_0, 2^{-1}(x_1 + px_2), 2^{-1}(-x_1 + px_2))$ is diag$(1,p,-p)$, as desired. From Lemma 2.13, we have:

$$\begin{cases} 
\gamma_{2k}(L) = (p^k z_0, p^k z_1, p^k z_2) \\
\gamma_{2k+1}(L) = (p^k z_0, p^{k+1} z_1, p^{k+1} z_2).
\end{cases}$$

One computes that the matrix of $\gamma_{2k}(L)$ with respect to the displayed basis is the desired canonical form, while the matrix of $\gamma_{2k+1}(L)$ is diag$(p^{k+2},p^{k+1},-p^{k+1})$, which can be easily put into the desired canonical form (see Remark 2.5).

Finally, we consider $sl_1(\Delta_p)$ and its congruence subalgebras. The ordered set $(x_0, x_1, x_2) = (2^{-1}uv, 2^{-1}v, 2^{-1}u)$ is a basis of $sl_1(\Delta_p)$ over $\mathbb{Z}_p$, and the commutation relations are given by

$$[x_1,x_2] = -x_0, \quad [x_2,x_0] = \rho x_1, \quad [x_0,x_1] = px_2.$$ 

More generally, for all $k \geq 0$, $(p^k x_0, p^k x_1, p^k x_2)$ is a basis of $sl^{2k}_1(\Delta_p)$, while $(p^k x_0, p^k x_1, p^{k+1} x_2)$ is a basis of $sl^{2k+1}_1(\Delta_p)$ (see [Klo03]). One may take these bases and apply [Cas78, Lemma 3.4, page 115] to compute the desired canonical matrix of the respective lattices.
2.2 Non-self-similarity theorem

Definition 2.18 Let $L$ be a 3-dimensional antisymmetric $\mathbb{Z}_p$-algebra, and let $x$ be a basis of $L$. Denote by $A$ the matrix of $L$ with respect to $x$. We say that $x$ satisfies the non-self-similarity condition if and only if $x$ is well diagonalizing and, denoting $A = \text{diag}(a_0, a_1, a_2)$, $Z_0 = \{0, ..., p - 1\}$ and $Z_1 = \{1, ..., p - 1\}$, the following conditions are satisfied:

1. $v_p(e^2a_1 + a_1) = v_p(a_0)$, for all $e \in Z_1$.
2. $v_p(e^2a_0 + f^2a_1 + a_2) = v_p(a_0)$, for all $e \in Z_1$ and $f \in Z_0$.
3. $v_p(f^2a_1 + a_2) = v_p(a_1)$, for all $f \in Z_1$.

Theorem 2.19 Let $p \geq 3$ be a prime, and let $L$ be a 3-dimensional unsolvable $\mathbb{Z}_p$-Lie lattice. Suppose that there exists a basis of $L$ that satisfies the non-self-similarity condition, and that the $s$-invariants $s_0 \leq s_1 \leq s_2$ of $L$ are not all equal (i.e., $s_0 < s_2$). Then $L$ is not self-similar of index $p$.

Before proving the theorem above, we need some preparation.

Lemma 2.20 Let $L$ be a 3-dimensional unsolvable $\mathbb{Z}_p$-Lie lattice that admits a diagonalizing basis, and let $x$ be a well diagonalizing basis of $L$. Let $A = \text{diag}(a_0, a_1, a_2)$ be the matrix of $L$ with respect to $x$, and let $u_i = u_ip^{s_i}$ with $s_i \in \mathbb{N}$, $u_i \in \mathbb{Z}_p^*$, and $s_0 \leq s_1 \leq s_2$. Then the basis $x$ satisfies the non-self-similarity condition whenever one of the following conditions is satisfied:

1. $s_0 < s_1 < s_2$; or
2. $s_0 = s_1 < s_2$ and $-u_0u_1$ is not a square modulo $p$; or
3. $s_0 < s_1 = s_2$ and $-u_1u_2$ is not a square modulo $p$.

Proof: Clearly, the elements of $\{1, ..., p - 1\}$ are invertible in $\mathbb{Z}_p$. Note that some of the desired conditions of Definition 2.18 follow directly from one of the basic properties of valuations: if $v_p(a) < v_p(b)$ then $v_p(a + b) = v_p(a)$. Assume that (2) holds, and let $e \in Z_1$ and $f \in Z_0$. Then $e^2u_0 + f^2u_1 + u_2p^{s_2-s_0}$ is invertible in $\mathbb{Z}_p$; indeed, if not, then reducing modulo $p$ we have $-u_0u_1 \equiv (u_1 f e^{-1})^2$, which yields a contradiction, since by assumption $-u_0u_1$ is not a square modulo $p$. Now, $e^2a_0 + f^2a_1 + a_2 = (e^2u_0 + f^2u_1 + u_2p^{s_2-s_0})p^{s_0}$ implies that $v_p(e^2a_0 + f^2a_1 + a_2) = s_0 = v_p(a_0)$. This proves condition (2) of Definition 2.18. The proof of the other cases is similar and is left to the reader.

Remark 2.21 For $p \geq 3$, we consider the classification of 3-dimensional unsolvable $\mathbb{Z}_p$-Lie lattices given in Remark 2.16. Observe that, by assumption, the basis corresponding to each of the canonical forms is well diagonalizing. One may apply Lemma 2.20 to see that a canonical basis of a Lie lattice presented by any of the following canonical forms satisfies the non-self-similarity condition:

1. $A = \text{diag}(p^{s_0}, p^{s_1}p^{s_2}, p^{s_1}p^{s_2})$ with $s_0 < s_1 < s_2$.
2. $A = \text{diag}(p^{s_0}, -p^{s_0}, p^{s_2})$ with $s_0 < s_1 < s_2$.
3. $A = \text{diag}(p^{s_0}, p^{s_1}, -p^{s_1})$ with $s_0 < s_1$. 

It will turn out that, among the 3-dimensional unsolvable ones, these Lie lattices are exactly the ones that are not self-similar of index $p$.

**Definition 2.22**

1. Let $\Xi$ be the set of symbols $\xi$ of the form $(\cdot, (e), (e, f))$, where $e, f \in Z_0 := \{0, 1, \ldots, p - 1\}$. We make a partition of the set $\Xi$ as follows, where $Z_1 := \{1, \ldots, p - 1\}$:

   (a) $\Xi_0 = \{(\cdot)\} \cup \{(e) : e \in Z_1\} \cup \{(e, f) : e \in Z_1, f \in Z_0\}$.

   (b) $\Xi_1 = \{(0)\} \cup \{(0, f) : f \in Z_1\}$.

   (c) $\Xi_2 = \{(0, 0)\}$.

2. With any $\xi \in \Xi$ we associate a matrix $U_\xi$:

   $\begin{align*}
   U(\cdot) &= \begin{bmatrix} p & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},
   U(e) &= \begin{bmatrix} 1 & 0 & 0 \\ e & p & 0 \\ 0 & 0 & 1 \end{bmatrix},
   U(e, f) &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ e & f & p \end{bmatrix}.
   \end{align*}$

3. Let $L$ be a 3-dimensional $\mathbb{Z}_p$-lattice endowed with a basis $x = (x_0, x_1, x_2)$. We define a submodule $L^\xi = \langle y_0, y_1, y_2 \rangle$ of $L$ by $y_i = \sum_{j=0}^{2}(U_\xi)_{ji}x_j$, for $i = 0, 1, 2$.

**Lemma 2.23** Let $L$ be a 3-dimensional $\mathbb{Z}_p$-lattice endowed with a basis $(x_0, x_1, x_2)$. The map that associates with any $\xi \in \Xi$ the submodule $L^\xi$ of $L$ is a bijection from $\Xi$ to the set of index-$p$ submodules of $L$.

**Proof:** Given $U \in gl_3(\mathbb{Z}_p)$, we define a triple $y = (y_0, y_1, y_2)$ as in Definition 2.22 (and vice versa, any triple $y$ corresponds to a unique matrix $U$). The submodule $M$ generated by $y$ has dimension 3 if and only if $\det(U) \neq 0$; in this case, $y$ is a basis of $M$. Moreover, $M$ has index $p$ in $L$ if and only if $v_p(\det(U)) = 1$. Two matrices $U$ and $U'$ determine the same submodule if and only if there exists $V \in GL_3(\mathbb{Z}_p)$ such that $U' = UV$. Finally, note that Gaussian reduction along the columns of a matrix $U$ with $v_p(\det(U)) = 1$ reduces $U$ to some $U_\xi$ for exactly one value of $\xi$ (see [AW92, Theorem 2.9 on page 302, Theorem 2.13 on page 304] for details). \(\Box\)

**Lemma 2.24** Let $L$ be a 3-dimensional $\mathbb{Z}_p$-Lie lattice that admits a diagonalizing basis $x = (x_0, x_1, x_2)$, and denote by $A = \text{diag}(a_0, a_1, a_2)$ the corresponding matrix. For $\xi \in \Xi$, let $L^\xi$ be the index-$p$ submodule of $L$ of Definition 2.22, which comes endowed with a basis $y = (y_0, y_1, y_2)$. We denote by $B_\xi$ the associated matrix $B$ of Equation (2.1) of Lemma 2.3. Then the matrix $B_\xi$ is given as follows, where $e, f \in \{0, \ldots, p - 1\}$ and the vertical bars enhance readability:

$$
B(\cdot) = \begin{bmatrix}
  p^{-1}a_0 & 0 & 0 \\
  0 & pa_1 & 0 \\
  0 & 0 & pa_2
\end{bmatrix},

B(e) = \begin{bmatrix}
  p a_0 & -e a_0 & 0 \\
  -e a_0 & -p^{-1}(e^2 a_0 + a_1) & 0 \\
  0 & 0 & pa_2
\end{bmatrix},

B(e, f) = \begin{bmatrix}
  p a_0 & 0 & -e a_0 \\
  0 & -f a_1 & -e a_0 \\
  -f a_1 & p^{-1}(e^2 a_0 + f^2 a_1 + a_2) & 0
\end{bmatrix}.
$$

**Proof:** The proof is a straightforward computation using Equation (2.1). \(\Box\)

**Lemma 2.25** Let $L$ be a 3-dimensional $\mathbb{Z}_p$-Lie lattice that admits a basis satisfying the non-self-similarity condition, and choose one such basis. Let $s_0 \leq s_1 \leq s_2$ be the $s$-invariants of $L$. Take $i \in \{0, 1, 2\}$ and $\xi \in \Xi_i$. The following holds.
(1) \(L^s\) is a subalgebra of \(L\) if and only if \(s_i \geq 1\).

(2) If \(L^s\) is a subalgebra and, moreover, \(L\) is unsolvable, then the \(s\)-invariants of \(L^s\) are:

(a) \(s_0 - 1, s_1 + 1, s_2 + 1\), when \(i = 0\).
(b) \(s_0 + 1, s_1 - 1, s_2 + 1\), when \(i = 1\).
(c) \(s_0 + 1, s_1 + 1, s_2 - 1\), when \(i = 2\).

**Proof:** Let \(\{x_0, x_1, x_2\}\) be a basis of \(L\) satisfying the non-self-similarity condition. Part (1) follows from Lemmas 2.3 and 2.24, and the conditions of Definition 2.18. For part (2), let \(y = (y_0, y_1, y_2)\) be the basis of \(L^s\) given in Definition 2.22, and let \(B\) be the matrix of \(L^s\) with respect to \(y\). Defining \(z_0 = [y_1, y_2], z_1 = [y_2, y_0]\) and \(z_2 = [y_0, y_1]\) we have \([L^s, L^s] = \{z_0, z_1, z_2\}\) and \(z_i = \sum_{j=0}^2 B_{ij} y_j, i \in \{0, 1, 2\}\). Hence, the \(s\)-invariants of \(L^s\) are obtained by computing the Smith normal form of \(B\), which can be achieved through Gaussian reduction along the rows and the columns (cf. [AW92, Theorem 3.1 on page 307, Remark 3.4 on page 308]). We consider the case of \(\xi = (e)\) with \(e \in \{1, \ldots, p - 1\}\), so that \(i = 0\); the others are similar and left to the reader. Recall that \(A = \text{diag}(a_0, a_1, a_2)\) is the notation for the matrix of \(L\) with respect to \(x\). Observe that, since \(L^s\) is a subalgebra and part (1) holds, we have \(s_0 \geq 1\). The matrix \(B = B(e)\) is block diagonal (Lemma 2.24), and we see at once that \(s_2 + 1\) is one of the \(s\)-invariants. We have to compute the Smith normal form of the \(2 \times 2\) block

\[
\begin{bmatrix}
pa_0 & -ea_0 \\
-ea_0 & p^{-1}(e^2a_0 + a_1)
\end{bmatrix} = \begin{bmatrix}
u_0p^{s_0+1} & -cu_0p^{s_0} \\
-cu_0p^{s_0} & up^{s_0-1}
\end{bmatrix},
\]

where \(a_j = u_j p^{s_j}\) with \(u_j \in \mathbb{Z}_p^*\) and \(u = e^2 u_0 + u_1 p^{s_1-s_0}\). It follows from the non-self-similarity condition that \(u\) is invertible in \(\mathbb{Z}_p^*\). Multiplying the matrix displayed above by \(\begin{bmatrix}1 & 0 \\ cu_0^{-1} & 1\end{bmatrix}\) on the right, and by \(\begin{bmatrix}1 & cu_0^{-1}u^{-1} p \\ 0 & 1\end{bmatrix}\) on the left, we get \(\begin{bmatrix}u_0u_1p^{s_1}u^{-1} & 0 \\ 0 & up^{s_0-1}\end{bmatrix}\). It follows that the other two \(s\)-invariant are \(s_0 - 1\) and \(s_1 + 1\), as desired. \(\square\)

The following is the key lemma for the proof of Theorem 2.19.

**Lemma 2.26** Let \(L\) be a 3-dimensional unsolvable \(\mathbb{Z}_p\)-Lie lattice that admits a basis \((x_0, x_1, x_2)\) satisfying the non-self-similarity condition. Let \(s_0 \leq s_1 \leq s_2\) be the \(s\)-invariants of \(L\), \(i \in \{0, 1, 2\}\), \(\xi \in \Xi_i\), and suppose that \(M := L^s\) is a subalgebra of \(L\). Then

\[
[M, M] + p^{s_0}M = p[L, L] + p^{s_0}L.
\]

**Proof:** Let \(x = (x_0, x_1, x_2)\), and let \(y = (y_0, y_1, y_2)\) be the basis of \(M\) given in Definition 2.22. In coordinates with respect to \(x\), the submodule \(p[L, L] + p^{s_0}L\) is generated by the columns of the \(3 \times 6\) matrix given in block form by \(A = [pA | p^{s_0}I]\), where \(A = \text{diag}(a_0, a_1, a_2)\) is the matrix of \(L\) with respect to \(x\), and \(I\) is the \(3 \times 3\) identity matrix. Likewise, in coordinates with respect to \(y\), the submodule \([M, M] + p^{s_0}M\) is generated by the columns of the \(3 \times 6\) matrix given in block form by \(B = [B | p^{s_0}I]\), where \(B\) is the matrix of \(M\) with respect to \(y\). In order to compare \(p[L, L] + p^{s_0}L\) with \([M, M] + p^{s_0}M\), we have to write generators of the latter in coordinates with respect to \(x\). This is achieved by multiplying \(B'\) by \(U = U_e\) (Definition 2.22) on the left. It follows that \(p[L, L] + p^{s_0}L = [M, M] + p^{s_0}M\) if and only if \(A'\) and \(UB'\) have the same Hermite normal form (reducing along the columns); see, for instance, [AW92, Theorem 2.9 on page 302, Theorem 2.13 on page 304]. The Hermite normal form can be computed through
Gaussian reduction [AW92, Remark 3.4 on page 308]. We treat the case \( i = 0 \) and \( \xi = (e) \) with \( e \in \{1, \ldots, p - 1\} \); the other cases are left to the reader. The matrix \( A' \) is easily reduced:

\[
A' = \begin{bmatrix}
   p a_0 & 0 & 0 & p a_0 & 0 & 0 \\
   0 & p a_1 & 0 & p a_0 & 0 & 0 \\
   0 & 0 & p a_2 & 0 & 0 & p a_0 \\
\end{bmatrix} \sim \begin{bmatrix}
   p a_0 & 0 & 0 & 0 & 0 & 0 \\
   0 & p a_0 & 0 & 0 & 0 & 0 \\
   0 & 0 & p a_0 & 0 & 0 & 0 \\
\end{bmatrix},
\]

where \( \sim \) denotes multiplication on the right by a \( 3 \times 3 \) matrix that is invertible over \( \mathbb{Z}_p \). Observe that this is a fancy way to say that \( p[L, L] + p^{s_0} L = p^{s_0} L \) (for \( i = 1, 2 \) a similar phenomenon does not happen, a fact that does not change the difficulty of the proof). On the other hand

\[
UB' = \begin{bmatrix}
p a_0 & -e a_0 & 0 & p a_0 & 0 & 0 \\
   0 & a_1 & 0 & e p a_0 & p a_0 + 1 & 0 \\
   0 & 0 & p a_2 & 0 & 0 & p a_0 \\
\end{bmatrix}.
\]

The entry \( p a_0 \) on the sixth column is used to eliminate the entry \( p a_2 \) on the third one. Moreover, the entry \( p a_0 \) on the fourth column is used to eliminate the entries \( p a_0 \) and \(-e a_0 \) on the first line. We get

\[
UB' \sim \begin{bmatrix}
   0 & e^2 a_0 + a_1 & 0 & 0 & p a_0 & 0 \\
   0 & e^2 a_0 + a_1 & 0 & 0 & e p a_0 & p a_0 + 1 \\
   0 & 0 & 0 & 0 & 0 & p a_0 \\
\end{bmatrix}.
\]

From the non-self-similarity condition, \( v_p(e^2 a_0 + a_1) = v_p(a_0) = s_0 \), so that the entry \( e^2 a_0 + a_1 \) may be used to eliminate all the other entries on the second row. After having done that, one may permute the columns and get the same normal form as for \( A' \).

**Proof of Theorem 2.19.** Let \( M \) be a subalgebra of \( L \) of index \( p \), and let \( \varphi : M \to L \) be a virtual endomorphism of \( L \). We will show that \( \varphi \) is not simple.

Suppose that \( \varphi \) is not injective. Then by item (2b) of Proposition 2.7, \( \ker(\varphi) \) is 3-dimensional, so there is some \( k \) such that \( I := p^k L \subseteq \ker(\varphi) \) is a non-trivial \( \varphi \)-invariant ideal of \( L \). Hence, \( \varphi \) is not simple. Now assume that \( \varphi \) is injective. Then \( \varphi \) establishes an isomorphism of \( M \) with its image \( M' := \varphi(M) \). By Proposition 2.8, \( M' \) has index \( p \) in \( L \) as well. Take a basis \( x \) of \( L \) satisfying the non-self-similarity condition. Recall the notation of Definition 2.22. There are (unique) \( \xi, \eta \in \Xi \) such that \( M = L^\xi \) and \( M' = L^\eta \) (Lemma 2.23). Hence, there are unique \( i, j \in \{0, 1, 2\} \) such that \( \xi \in \Xi_i \) and \( \eta \in \Xi_j \). The rest of the proof, divided in four cases, relies on Lemma 2.26.

**Case 1:** \( i = j \). We take \( I := p[L, L] + p^{s_0} L \), a non-trivial ideal of \( L \). We show that \( I \) is \( \varphi \)-invariant. From Lemma 2.26, we have \([M, M] + p^{s_0} M = I = [M', M'] + p^{s_0} M'\). Observe that \( I \subseteq M \). Since \( \varphi : M \to M' \) is an isomorphism of Lie algebras, we have \( \varphi([M, M] + p^{s_0} M) = [M', M'] + p^{s_0} M' \), so that \( \varphi(I) = I \).

**Case 2:** \( \{i, j\} = \{0, 1\} \). We take \( I := p[L, L] + p^{s_0} L \), a non-trivial ideal of \( L \). We show that \( I \) is \( \varphi \)-invariant. First of all, we observe that \( s_i = s_j \); indeed, we can assume \( i = 0 \) and \( j = 1 \), so that the minimum \( s \)-invariant of \( M \) is \( s_0 - 1 \) (see Lemma 2.25); if it was \( s_0 < s_1 \) then all the \( s \)-invariants of \( M' \) would be greater then \( s_0 - 1 \), and \( M \) and \( M' \) could not be isomorphic. From Lemma 2.26 we have \([M, M] + p^{s_0} M = p[L, L] + p^{s_0} L \) and \([M', M'] + p^{s_0} M' = p[L, L] + p^{s_0} L \). Since \( s_i = s_j = s_0 \), the same arguments as in case 1 permit to conclude that \( I \subseteq M \) and \( \varphi(I) = I \).

**Case 3:** \( \{i, j\} = \{1, 2\} \). Taking \( I := p[L, L] + p^{s_1} L \), the argument is very similar to the one of case 2. The only difference is that, in proving that \( s_i = s_j \), one has to consider the maximum value of the \( s \)-invariants instead of the minimum.

**Case 4:** \( \{i, j\} = \{0, 2\} \). We show that in this case we have a contradiction. By assumption, \( s_0 < s_2 \). We can assume \( i = 0 \) and \( j = 2 \). Observe that the minimum \( s \)-invariant of \( M \), namely
$s_0 - 1$, is less then all the $s$-invariants of $M'$ (Lemma 2.25), so that $M \not\cong M'$, contrary to the assumptions. \hfill \Box

2.3 Extension of scalars

Throughout this section $p$ denotes an odd prime. Given a 3-dimensional unsolvable Lie lattice $L$ over $\mathbb{Z}_p$, we compute the isomorphism class of $L \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$, a 3-dimensional unsolvable Lie algebra over $\mathbb{Q}_p$ (Proposition 2.31). To achieve our goal, we define a $(\mathbb{Z}/2\mathbb{Z})$-valued function of a symmetric non-degenerate matrix $A$. This function is defined through the classical discriminant and $\varepsilon$-invariant of $A$, where $A$ is thought of as a non-degenerate quadratic form (see Section IV.2 of [Ser73]). Actually, we will use the additive $\varepsilon$-invariant, which is defined through the additive version of the Hilbert symbol, the latter being denoted by $[a, b]$ in [Ser73, Remark on page 23].

**Definition 2.27** Let $A \in \text{gl}_3(\mathbb{Q}_p)$ be symmetric and non-degenerate. Let $d(A) \in \mathbb{Q}_p^*/(\mathbb{Q}_p^*)^2$ and $e(A) \in \mathbb{Z}/2\mathbb{Z}$ be the discriminant and the additive $\varepsilon$-invariant of $A$. The $\eta$-invariant of $A$ is defined to be

$$
\eta(A) := \delta_p v_p(d(A)) + e(A) \quad \quad \eta(A) \in \mathbb{Z}/2\mathbb{Z},
$$

where $\delta_p$ is the residue of $(p-1)/2$ modulo 2; we observe that the $p$-adic valuation of an element of $\mathbb{Q}_p^*/(\mathbb{Q}_p^*)^2$ is well defined in $\mathbb{Z}/2\mathbb{Z}$.

**Lemma 2.28** Let $A = \text{diag}(u_0p^{s_0}, u_1p^{s_1}, u_2p^{s_2})$ with $u_i \in \mathbb{Z}_p^*$ and $s_i \in \mathbb{Z}$ for $i \in \{0, 1, 2\}$. The following holds.

1. $\eta(A) \equiv 1 \pmod{2}$ if and only if two of the $s$-invariants have the same parity while the third $s$-invariant has a different one, say $s_i \equiv s_j \not\equiv s_k \pmod{2}$, and $-u_i u_j$ is not a square modulo $p$.

2. For the following specific forms of $A$ (which include the canonical forms of Remark 2.16), the $\eta$-invariant is given by the following formulas, where $\rho \in \mathbb{Z}_p^*$ is not a square modulo $p$, $\varepsilon_1, \varepsilon_2 \in \{0, 1\}$, $\delta_p \equiv (p-1)/2 \pmod{2}$, and the congruences are modulo 2:

a) If $A = \text{diag}(p^{s_0}, \rho e_1 p^{s_1}, \rho e_2 p^{s_2})$ then

$$
\eta(A) \equiv \delta_p(s_0 + s_1 + s_2 + s_0 s_1 + s_0 s_2 + s_1 s_2) + (\varepsilon_1 + \varepsilon_2)s_0 + \varepsilon_2 s_1 + \varepsilon_1 s_2.
$$

b) If $A = \text{diag}(p^{s_0}, -\rho e_1 p^{s_0}, p^{s_2})$ then $\eta(A) \equiv \varepsilon_1(s_0 + s_2)$.

c) If $A = \text{diag}(p^{s_0}, p^{s_1}, -\rho e_2 p^{s_1})$ then $\eta(A) \equiv \varepsilon_2(s_0 + s_1)$.

d) If $A = \text{diag}(p^{s_0}, p^{s_0}, p^{s_0})$ then $\eta(A) \equiv 0$.

**Proof:** Given $A$, define $\varepsilon_i = 0$ if $u_i$ is a square modulo $p$, and $\varepsilon_i = 1$ if $u_i$ is not a square modulo $p$. A straightforward computation shows that

$$
\eta(A) \equiv \delta_p \left( \sum_i s_i + \sum_{i<j} s_i s_j \right) + \sum_{i<j} (\varepsilon_i s_j + \varepsilon_j s_i) \pmod{2},
$$

where the indices $i, j$ take values in $\{0, 1, 2\}$. (For the computation of the Hilbert symbols involved see, for instance, [Ser73, Remark on page 23].) Both items of the lemma are a direct consequence of this formula. For item (1) one has to observe that $-u_i u_j$ is a square modulo $p$ if and only if $\delta_p + \varepsilon_i + \varepsilon_j \equiv 0 \pmod{2}$. \hfill \Box

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Proposition 2.29 Let \( A, B \in \text{gl}_3(\mathbb{Q}_p) \) be symmetric and non-degenerate. Then \( A \) is multiplicatively congruent to \( B \) (over \( \mathbb{Q}_p \)) if and only if \( \eta(A) = \eta(B) \).

Proof: By definition, \( A \) is multiplicatively congruent to \( B \) if and only if there exists \( u \in \mathbb{Q}_p^* \) such that \( uA \) is congruent to \( B \) (cf. Remark 2.30). Moreover, ‘\( uA \) congruent to \( B \)’ is equivalent to ‘\( d(uA) = d(B) \) and \( e(uA) = e(B) \)’. Since \( d(uA) = ud(A) \) and \( e(uA) = \delta_p v_p(u) + e(A) \), one can see that ‘there exists \( u \in \mathbb{Q}_p^* \) such that \( uA \) is congruent to \( B \)’ is equivalent to ‘\( \eta(A) = \eta(B) \)’.

Remark 2.30 Analogous statements to the ones given at the beginning of Section 2.1 hold over \( \mathbb{Q}_p \). In particular, the map \( A \mapsto L_A \) induces a bijection between the set of multiplicative congruence classes of matrices of \( \text{gl}_3(\mathbb{Q}_p) \) and the set of isomorphism classes of 3-dimensional antisymmetric algebras over \( \mathbb{Q}_p \). Also, the analogue of item (1) of Proposition 2.7 holds, so that, under \( A \mapsto L_A \), symmetric non-degenerate matrices correspond exactly to unsolvable Lie algebras. Recall from Example 2.17 that there exist bases of \( \text{sl}_2(\mathbb{Z}_p) \) and \( \text{sl}_1(\Delta_p) \) such that the associated matrices are \( A_0 = \text{diag}(1, 1, 1) \) and \( A_1 = \text{diag}(1, -\rho, p) \), respectively. After tensoring by \( \mathbb{Q}_p \), the same is true for \( \text{sl}_2(\mathbb{Q}_p) \simeq \text{sl}_2(\mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \) and \( \text{sl}_1(\Delta_p) \simeq \text{sl}_1(\Delta_p) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \). One computes \( \eta(A_0) = 0 \) and \( \eta(A_1) = 1 \) (Lemma 2.28), from which we see that both values of \( \eta \), namely 0 and 1, are represented.

The above observations, together with Proposition 2.29, (re)prove that there are exactly two isomorphism classes of 3-dimensional unsolvable Lie algebras over \( \mathbb{Q}_p \), and that \( \text{sl}_2(\mathbb{Q}_p) \) and \( \text{sl}_1(\Delta_p) \) represent these two classes.

Proposition 2.31 Let \( p \geq 3 \) be a prime, and \( L \) be a 3-dimensional unsolvable \( \mathbb{Z}_p \)-Lie lattice. Let \( A \in \text{gl}_3(\mathbb{Z}_p) \subseteq \text{gl}_3(\mathbb{Q}_p) \) be the (symmetric non-degenerate) matrix of \( L \) associated with some basis. The following holds.

(1) \( L \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \simeq \text{sl}_2(\mathbb{Q}_p) \) if and only if \( \eta(A) = 0 \).

(2) \( L \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \simeq \text{sl}_1(\Delta_p) \) if and only if \( \eta(A) = 1 \).

Proof: Observe that if the given basis of \( L \) is \( (x_0, x_1, x_2) \) then \( (x_0 \otimes 1, x_1 \otimes 1, x_2 \otimes 1) \) is a basis of \( L \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \) whose corresponding matrix is \( A \) as well. The proposition follows from Proposition 2.29 and Remark 2.30. \( \square \)

2.4 Main results on Lie algebras

We collect here the main results on Lie algebras.

Theorem 2.32 Let \( p \geq 3 \) be a prime, and \( L \) be a 3-dimensional unsolvable \( \mathbb{Z}_p \)-Lie lattice. Fix \( \rho \in \mathbb{Z}_p^* \) not a square modulo \( p \). Then \( L \) is isomorphic to exactly one of the following \( \mathbb{Z}_p \)-Lie lattices:

(1) \( \text{L}_1(s_0, s_1, s_2, \varepsilon_1, \varepsilon_2) \) for \( 0 \leq s_0 < s_1 < s_2 \) and \( \varepsilon_1, \varepsilon_2 \in \{0, 1\} \), presented by:

\[
\{ x_0, x_1, x_2 \mid [x_1, x_2] = p^{s_0} x_0, \ [x_2, x_0] = \rho^{\varepsilon_1} p^{s_1} x_1, \ [x_0, x_1] = \rho^{\varepsilon_2} p^{s_2} x_2 \}.
\]

(2) \( \text{L}_2(s_0, s_2, \varepsilon_1) \) for \( 0 \leq s_0 < s_2 \) and \( \varepsilon_1 \in \{0, 1\} \), presented by:

\[
\{ x_0, x_1, x_2 \mid [x_1, x_2] = p^{s_0} x_0, \ [x_2, x_0] = -\rho^{\varepsilon_1} p^{s_0} x_1, \ [x_0, x_1] = p^{s_2} x_2 \}.
\]
(3) $L_3(s_0, s_1, \varepsilon_2)$ for $0 \leq s_0 < s_1$ and $\varepsilon_2 \in \{0, 1\}$, presented by:

$$\langle x_0, x_1, x_2 \mid [x_1, x_2] = p^{s_0} x_0, [x_2, x_0] = p^{s_1} x_1, [x_0, x_1] = -p^2 p^{s_1} x_2 \rangle.$$  

(4) $L_4(s_0)$ for $0 \leq s_0$, presented by:

$$\langle x_0, x_1, x_2 \mid [x_1, x_2] = p^{s_0} x_0, [x_2, x_0] = p^{s_0} x_1, [x_0, x_1] = p^{s_0} x_2 \rangle.$$  

Moreover, we have:

(1) $L_1(s_0, s_1, s_2, \varepsilon_1, \varepsilon_2)$ is not self-similar of index $p$.

(2) $L_2(s_0, s_2, \varepsilon_1)$ is self-similar of index $p$ if and only if $\varepsilon_1 = 0$.

(3) $L_3(s_0, s_1, \varepsilon_2)$ is self-similar of index $p$ if and only if $\varepsilon_2 = 0$.

(4) $L_4(s_0)$ is self-similar of index $p$.

**Proof:** The presentations in the statement correspond exactly to the canonical forms given in Remark 2.16, which indeed represent all the 3-dimensional unsolvable $\mathbb{Z}_p$-Lie lattices in a non-redundant way. We proceed in two steps.

First, we show that the lattices that are stated to be self-similar of index $p$ admit the required simple virtual endomorphism of index $p$. For this, we apply Lemma 2.12. For family (3) and $\varepsilon_2 = 0$, the lemma can be applied directly. For family (2) and $\varepsilon_1 = 0$, one can apply the lemma after making a change of basis that permutes the diagonal entries; see Remark 2.5. For family (4), one applies [Cas78, Lemma 3.4, page 115] in order to get the form $\text{diag}(-p^{s_0}, p^{s_0}, -p^{s_0})$, so that the lemma can be applied.

Second, from Remark 2.21 and Theorem 2.19, we see that the lattices that are stated not to be self-similar of index $p$ are indeed of such a type. This completes the proof.  

**Theorem 2.33** Let $p \geq 3$ be a prime, and $L$ be a 3-dimensional unsolvable $\mathbb{Z}_p$-Lie lattice. If $L \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \simeq \mathfrak{sl}_2(\mathbb{Q}_p)$ then $L$ is self-similar. Moreover, Table 1 gives the value or an estimate for the self-similarity index $\sigma$ of $L$. All the 3-dimensional unsolvable $\mathbb{Z}_p$-Lie lattices with $L \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \simeq \mathfrak{sl}_2(\mathbb{Q}_p)$ are represented exactly once in the table.

**Proof:** Observe that one may use Lemma 2.28 in order to compute the $\eta$-invariant of the matrices corresponding to the canonical presentations (appearing, for instance, in Theorem 2.32) of all the 3-dimensional unsolvable $\mathbb{Z}_p$-Lie lattices. One can check that the canonical presentations appearing in the table are exactly the ones for which $\eta = 0$. It follows (Proposition 2.31) that all the 3-dimensional unsolvable Lie lattices with $L \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \simeq \mathfrak{sl}_2(\mathbb{Q}_p)$ are represented exactly once.

Observe that lines 1, 2 and 4 of the table correspond exactly to the index-$p$ self-similar lattices respectively of cases 4, 3 and 2 of Theorem 2.32. These are exactly the cases where $L$ is self-similar of index $p$, so that the given value of $\sigma$ is correct, and the estimate $p^2 \leq \sigma$ of the other lines of the table is correct as well.

It remains to prove the validity of the given upper bounds for $\sigma$. Here the idea is to exhibit a 3-dimensional subalgebra $M$ of $L$ that is self-similar of index $p$. Applying Lemma 2.1, we get that $L$ is self-similar of index $p(L : M)$. We proceed as follows. Let $(x_0, x_1, x_2)$ be a basis of $L$ that puts $L$ in its canonical form. We denote by $A = \text{diag}(a_0, a_1, a_2)$ the corresponding matrix. In each case, we define $M = \langle p^{k_0} x_0, p^{k_1} x_1, p^{k_2} x_2 \rangle$, where the exponents $k_0, k_1, k_2 \in \mathbb{N}$ are given
| Presentation | Conditions |
|--------------|------------|
| (\(p^{s_0}, p^{s_0}, p^{s_0}\)) | \(\sigma = p\) |
| (\(p^{s_0}, p^{s_1}, -p^{s_1}\)) | \(\sigma = p\) |
| (\(p^{s_0}, p^{s_1}, -p^{s_1}\)) | \(p^2 \leq \sigma \leq p^{s_1-s_0+1}\) \(s_0 \equiv_2 s_1\) |
| (\(p^{s_0}, -p^{s_0}, p^{s_2}\)) | \(\sigma = p\) |
| (\(p^{s_0}, -p^{s_0}, p^{s_2}\)) | \(p^2 \leq \sigma \leq p^{\frac{s_2-s_0+1}{2}}\) \(s_0 \equiv_2 s_2\) |
| (\(p^{s_0}, \rho^{s_1}p^{s_1}, \rho^{s_2}p^{s_2}\)) | \(p^2 \leq \sigma \leq p^{\frac{s_1-s_0+1}{2}}\) \(s_0 \equiv_2 s_1 \equiv_2 s_2\) |
| (\(p^{s_0}, \rho^{s_1}p^{s_1}, \rho^{s_2}p^{s_2}\)) | \(p^2 \leq \sigma \leq p^{\frac{s_1-s_0+1}{2}}\) \(s_0 \equiv_2 s_1 \not\equiv_2 s_2\) \(\varepsilon_1 + \delta_p \equiv_2 0\) |
| (\(p^{s_0}, \rho^{s_1}p^{s_1}, \rho^{s_2}p^{s_2}\)) | \(p^2 \leq \sigma \leq p^{\frac{s_1-s_0+1}{2}}\) \(s_1 \equiv_2 s_2 \not\equiv_2 s_0\) \(\varepsilon_1 + \varepsilon_2 + \delta_p \equiv_2 0\) |
| (\(p^{s_0}, \rho^{s_1}p^{s_1}, \rho^{s_2}p^{s_2}\)) | \(p^2 \leq \sigma \leq p^{\frac{s_1-s_0+1}{2}}\) \(s_0 \equiv_2 s_2 \not\equiv_2 s_1\) \(\varepsilon_2 + \delta_p \equiv_2 0\) |

Table 1: Estimates for the self-similarity index as in Theorem 2.33. In the table, \(\rho \in \mathbb{Z}_p^*\) is a fixed non-square modulo \(p\), and \(\delta_p \equiv_2 (p-1)/2\), where the symbol \(\equiv_2\) denotes congruence modulo 2. The variables appearing in each line of the table (i.e., the parameters of the classification) take values: \(s_0, s_1, s_2, s_3 \in \mathbb{N}\) with \(s_0 < s_1 < s_2\) and \(\varepsilon_1, \varepsilon_2 \in \{0, 1\}\). The first column of the table gives the diagonal entries of the (diagonal) matrix associated with the canonical presentation of \(L\) as in Theorem 2.32. The second column gives the value of \(\sigma\) or an estimate for it. The third column gives conditions on the parameters in order the given estimate to be true.

Below case by case. Observe that, for any choice of these exponents, the matrix \(B\) of formula (2.1) of Lemma 2.3 (where \(U = \text{diag}(p^{k_0}, p^{k_1}, p^{k_2})\)) is

\[
B = \text{diag}(a_0p^{-k_0+k_1+k_2}, a_1p^{k_0-k_1+k_2}, a_2p^{k_0+k_1-k_2}).
\]

For line 9 of the table we make the full proof while, for the other relevant lines, we just give the exponents, leaving the details to the reader.

Line 3. Take \((k_0, k_1, k_2) = (0, \frac{s_1-s_0}{2}, \frac{s_1-s_0}{2})\).

Line 5. Take \((k_0, k_1, k_2) = (0, 0, \frac{s_2-s_0}{2})\).

Line 6. Take \((k_0, k_1, k_2) = (0, \frac{s_1-s_0}{2}, \frac{s_2-s_0}{2})\).

Line 7. Take \((k_0, k_1, k_2) = (0, \frac{s_1-s_0}{2}, 0)\).
Line 8. Take \((k_0, k_1, k_2) = \left(0, 0, \frac{s_2 - s_0}{2}\right)\).

Line 9. Take \((k_0, k_1, k_2) = \left(0, 0, \frac{s_2 - s_0}{2}\right)\). Define \(l := \frac{s_2 - s_0}{2}\) and observe that \(l \geq 1\) is an integer. For the corresponding \(M\) (a priori just a submodule of \(L\)) we have \([L : M] = p^1\). We have to show that \(M\) is a subalgebra of \(L\) and that \(M\) is self-similar of index \(p\). One computes \(B = \text{diag}(\rho^{s_0+l}, \rho^{s_1+s_1+l}, \rho^{s_2+p^{s_0+l}})\). The entries of \(B\) are in \(\mathbb{Z}_p\), so that \(M\) is a subalgebra of \(L\). We will show that the canonical matrix of \(L\) is \(\text{diag}(p^{s_0+l}, p^{s_1+l}, p^{s_1+l})\), which implies that \(M\) is self-similar of index \(p\) (Theorem 2.32). Indeed, we can permute the diagonal entries of \(B\), multiply them by \(\rho^s\) (see Remark 2.5) and “eliminate” a square (through a congruence), getting \(\text{diag}(\rho^{s_0+l}, p^{s_1+l}, p^{s_1+l}, p^{s_1+l})\) as the matrix of \(M\) with respect to some basis. With other changes of basis we can first “move” the factor \(\rho^{s_1}\) of the first diagonal entry to the second (cf. [Cas78, Lemma 3.4, page 115]), and then eliminate the resulting square \(\rho^{2s_1}\). By writing \(-1 = \rho^b u^2\) with \(u \in \mathbb{Z}_p^*\) and eliminating \(u^2\), we get \(\text{diag}(p^{s_0+l}, p^{s_1+l}, p^{s_1+l}, p^{s_1+l})\). Since \(\epsilon_2 + \delta_p \equiv \epsilon_2 0\), possibly eliminating another square, we arrive at the desired presentation, and the proof of this case is complete. \(\square\)

**Theorem 2.34** Let \(p \geq 3\) be a prime. The following holds.

1. The \(\mathbb{Z}_p\)-Lie lattice \(sl_2(\mathbb{Z}_p)\) and its congruence subalgebras \(sl_k^p(\mathbb{Z}_p) = p^k sl_2(\mathbb{Z}_p)\), \(k \geq 1\), are self-similar of index \(p\).

2. (a) The \(\mathbb{Z}_p\)-Lie lattice \(sl_2^p(\mathbb{Z}_p)\) and the terms \(\gamma_k(sl_2^p(\mathbb{Z}_p))\), \(k \geq 1\), of its lower central series are self-similar of index \(p\).

   (b) If \(I \subseteq L\) is a non-zero ideal of \(sl_2^p(\mathbb{Z}_p)\) then \(I\) has dimension \(3\) and it is self-similar of index \(p\) or \(p^2\).

**Proof:** First, the canonical forms of \(sl_2(\mathbb{Z}_p)\), \(sl_2^p(\mathbb{Z}_p)\), \(sl_2^p(\mathbb{Z}_p)\) and \(\gamma_k(sl_2^p(\mathbb{Z}_p))\) have been computed in Example 2.17. According to Theorem 2.32, they all correspond to lattices that are self-similar of index \(p\).

Second, since \(L\) is a 3-dimensional unsolvable Lie lattice then any non-zero ideal \(I\) of \(L\) is 3-dimensional (item (2b) of Proposition 2.7). If \(I = L\) then \(I\) is self-similar of index \(p\). Assume \(I \neq L\). Since \(\gamma_2(I) = p^m I\) for all \(m \geq 0\) (Lemma 2.13), there exists \(k \in \mathbb{N}\) such that \(\gamma_k(I) \subseteq I\) (where \(I \neq L\) implies that \(k \geq 1\)). Taking the least such \(k\), we have \(\gamma_{k-1}(I) \subseteq I\), so that by Proposition 2.14 we have \(I \subseteq \gamma_{k-1}(I)\). By minimality of \(k\), we end up with \(\gamma_k(I) \subseteq I \subseteq \gamma_{k-1}(I)\), where \(\subset\) denotes strict inclusion. Observe that \(\gamma_{k-1}(I) : \gamma_k(I))\) is \(p\) if \(k\) is even and it is \(p^2\) if \(k\) is odd. In any case \([I : \gamma_k(I)]\) is \(1\) or \(p\). Since \(\gamma_k(I)\) is self-similar of index \(p\) then \(I\) is self-similar of index \(p\) or \(p^2\) (Lemma 2.1). \(\square\)

**Theorem 2.35** Let \(p \geq 3\) be a prime, and \(L\) be a 3-dimensional unsolvable \(\mathbb{Z}_p\)-Lie lattice. Suppose that \(L \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \simeq sl_1(\mathbb{D}_p)\). Then \(L\) is not self-similar of index \(p\).

**Proof:** Let \(A\) be the matrix of \(L\) associated with the canonical presentation given in Theorem 2.32. From Proposition 2.31, we see that \(\eta(A) = 1\). From Lemma 2.28, the value of \(\eta(A)\) can be explicitly computed. We have to show that the canonical presentation of \(L\) belongs to one of the families that is proven to be non self-similar of index \(p\) in Theorem 2.32. If \(s_0 = s_1 = s_2\) then \(\eta(A) = 0\), a contradiction. If \(s_0 = s_1 < s_2\) then \(\eta(A) = 1\) implies \(\epsilon_1(s_0 + s_2) \equiv 1\) (mod 2), so that \(\epsilon_1 = 1\), as desired. If \(s_0 < s_1 = s_2\) then \(\epsilon_2(s_0 + s_1) \equiv 1\) (mod 2), so that \(\epsilon_2 = 1\), as desired. Finally, if \(s_0 < s_1 < s_2\) then there is nothing to prove, since no such \(L\) is self-similar of index \(p\). \(\square\)
We expect a stronger version of Theorem 2.35 to be true.

Conjecture 2.36 Let \( p \geq 3 \) be a prime, and \( L \) be a 3-dimensional unsolvable \( \mathbb{Z}_p \)-Lie lattice. If \( L \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \simeq \mathfrak{sl}_1(\mathbb{D}_p) \) then \( L \) is not self-similar.

3 Proofs of the main theorems

We quickly recall Lazard’s correspondence, which is the main tool in this section. With a saturable pro-\( p \) group \( G \) one may associate a saturable \( \mathbb{Z}_p \)-Lie lattice \( L_G \) in the following way: \( G \) and \( L_G \) are identified as sets, and the Lie operations are defined by

\[
g + h = \lim_{n \to \infty} (g^{p^n}h^{p^n})^{p^{-n}}, \quad [g, h]_{\text{Lie}} = \lim_{n \to \infty} [g^{p^n}, h^{p^n}]^{p^{-2n}} = \lim_{n \to \infty} (g^{-p^n} h^{-p^n} g^{p^n} h^{p^n})^{p^{-2n}}.
\]

On the other hand, if \( L \) is a saturable \( \mathbb{Z}_p \)-Lie lattice, then the Campbell-Hausdorff formula induces a group structure on \( L \); the resulting group is a saturable pro-\( p \) group. If this construction is applied to the \( \mathbb{Z}_p \)-Lie Lattice \( L_G \) associated with a saturable group \( G \), one recovers the original group. Indeed, the assignment \( G \to L_G \) gives an isomorphism between the category of saturable pro-\( p \) groups and the category of saturable \( \mathbb{Z}_p \)-Lie lattices. See [Laz65, IV (3.2.6)], [Klo05, Section 2] and [GK09] for more details.

Theorem 3.1 Let \( p \) be a prime and \( G \) be a saturable \( p \)-adic analytic pro-\( p \) group of dimension \( \dim(G) \leq p \), and \( L_G \) be the \( \mathbb{Z}_p \)-Lie lattice associated with \( G \). Let \( D, N \subseteq G \) be subsets, and denote them by \( L_D, L_N \) when regarded as subsets of \( L_G \). Then the following holds.

(1) If \( D \) is a closed subgroup of \( G \) then \( D \) is saturable.

(2) \( D \) is an open subgroup of \( G \) if and only if \( L_D \) is a finite-index subalgebra of \( L_G \). If this is the case, then \( [G : D] = [L_G : L_D] \).

(3) The set \( N \) is a closed normal subgroup of \( G \) if and only if \( L_N \) is an ideal of \( L_G \).

(4) If \( D \) is an open subgroup of \( G \) and \( \varphi : D \to G \) is a function, then:

(a) \( \varphi \) is a group homomorphism if and only if \( \varphi \) is a Lie algebra homomorphism (denoted by \( L_\varphi \)).

(b) If \( \varphi \) is a group homomorphism, then \( \varphi \) is simple if and only if \( L_{L_\varphi} \) is simple.

Proof: Item (1) follows from [GK09, Theorem A] and [GK09, Proposition D]. Let \( D \) be an open subgroup of \( G \). By (1), \( D \) is saturable and \( L_D \) is a saturable subalgebra of \( L_G \). Since \( G \) is a saturable pro-\( p \) group, it admits a valuation \( \omega : G \to \mathbb{R}_{>0} \cup \{\infty\} \) such that \((G, \omega)\) is a saturable pro-\( p \) group (cf. [Klo05, Section 2]). In particular, the sets \( G_\mu = \{g \in G \mid \omega(g) \geq \mu\} \), \( \mu \in \mathbb{R}_{>0} \), are open normal subgroups of \( G \) and form a basis of neighborhoods of the identity. It follows that there exists \( \nu \in \mathbb{R}_{>0} \) such that \( G_\nu \) is contained in \( D \), so that we have a chain \( G_\nu \subseteq D \subseteq G \) of saturable groups. From Lazard’s correspondence there is an associated chain \( L_{G_\nu} \subseteq L_D \subseteq L_G \) of Lie algebras. Now, from [Klo05, Proposition A.2], for every \( y \in G \) (in particular, for every \( y \in D \)) the multiplicative coset \( yG_\nu \) is equal to the additive coset \( y + L_{G_\nu} \). Hence, \( [G : G_\nu] = [L_G : L_{G_\nu}] \) and \([D : G_\nu] = [L_D : L_{G_\nu}]\), which yields \( [G : D] = [L_G : L_D] \).

Moreover, from this fact and [GK09, Theorem E] it follows that \( D \) is an open subgroup of \( G \) whenever \( L_D \) is a finite index subalgebra of \( L_G \). This establishes (2). Note that (3) follows from
[GK09, Theorem E]. Part (a) of (4) follows from (1) and Lazard’s correspondence. Finally, to establish part (b) of (4) it suffices to observe that the $\varphi$-invariance of a subset $N \subseteq G$ is a purely set theoretical notion, namely, it does not depend on whether we look at $N$ in the group context or in the Lie-algebra context.

**Proof of Proposition A.** The proposition is a direct consequence of Proposition 1.3 and Theorem 3.1.

**Proof of Theorem B.** The Lie lattices presented in the statement form a subset of the ones given in Theorem 2.32 (where all the isomorphism classes of 3-dimensional unsolvable $\mathbb{Z}_p$-Lie lattices are represented in a non-redundant way). More precisely, it is the subset of the residually nilpotent lattices (cf. Lemma 2.13). Since $p$ is greater than the dimension of the groups and algebras involved, it follows from [GK09, Theorem B] that the groups given in the statement constitute the claimed complete and irredundant list. The conclusions about self-similarity of index $p$ follow directly from Proposition A and Theorem 2.32.

**Proof of Theorem C.** For $p \geq 5$, respectively, $p \geq 3$, the groups $SL_2^2(\mathbb{Z}_p)$, respectively, $SL_2^2(\mathbb{Z}_p)$ are 3-dimensional unsolvable saturable $p$-adic analytic pro-$p$ groups. The theorem follows from Proposition A and Theorems 2.33 and 2.34 (see also Theorem 3.1 and, for the $\mathbb{Z}_p$-Lie algebras associated with the groups involved, [GK09, Section 7.3], [Il99, page 158] and [Gon07, Theorem B, item (3)]).

**Proof of Theorem D.** For $p \geq 5$, respectively, $p \geq 3$, the groups $SL_1^2(\mathbb{Z}_p)$, respectively, $SL_1^2(\Delta_p)$ are 3-dimensional unsolvable saturable $p$-adic analytic pro-$p$ groups. The theorem follows from Proposition A and Theorem 2.35 (for the $\mathbb{Z}_p$-Lie algebras associated with the groups involved, see [GK09, Section 7.3] and [Klo03, Section 2]).

**Corollary 3.2** Let $p \geq 3$ be a prime and let $G$ be a compact $p$-adic analytic group whose associated $\mathbb{Q}_p$-Lie algebra $\mathcal{L}_G$ is isomorphic to $\mathfrak{sl}_2(\mathbb{Q}_p)$. Then $G$ is self-similar.

**Proof:** Since $G$ is a compact $p$-adic analytic group, there exists an open subgroup $H$ of $G$ that is a saturable pro-$p$ group; moreover, $L_H \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \cong \mathcal{L}_G \cong \mathfrak{sl}_2(\mathbb{Q}_p)$ (cf. [DDMS03, Section 9.5]). Now the corollary follows from Proposition 2.33, Proposition A and Corollary 1.4.

**Remark 3.3** Let $p \geq 3$ be a prime and let $G$ be a 3-dimensional unsolvable torsion-free $p$-adic analytic pro-$p$ group that is self-similar of index $p$. Take a faithful self-similar action of $G$ on $T_p$ that is transitive on the first level. We claim that this action is level transitive. Indeed, there exists a basis $(x_0, x_1, x_2)$ of $L_G$ such that $L_D = \langle x_0, px_1, x_2 \rangle$ is a subalgebra of $L_G$ of index $p$ and the homomorphism $\varphi : L_D \rightarrow L_G$ given by $\varphi(x_0) = x_0$, $\varphi(px_1) = x_1$ and $\varphi(x_2) = px_2$ is a simple virtual endomorphism of index $p$ (cf. Lemma 2.11). By Theorem 3.1, $\varphi : D \rightarrow G$ is a simple virtual endomorphism of $G$ of index $p$. It is easy to see that $\varphi(D_{n+1}) \subseteq D_{n+1}$ for all $n \geq 0$, where $D_n$ is the domain of the $n$-th power of $\varphi$. By Lemma 1.8, $\varphi$ is regular, so the action of $G$ on $T_p$ is level transitive (cf. [Nek02, Proposition 4.20]).

**References**

[AW92] Adkins W. A., Weintraub S. H., *Algebra - An Approach Via Module Theory*, Graduate Texts in Mathematics, vol. 136, Springer-Verlag, 1992.
