Overlap lattice fermion in a gravitational field

Masashi HAYAKAWA,\textsuperscript{1,*} Hiroto SO\textsuperscript{2,**} and Hiroshi SUZUKI\textsuperscript{3,***}

\textsuperscript{1,3}Theoretical Physics Laboratory, RIKEN, Wako 2-1, Saitama 351-0198, Japan
\textsuperscript{2}Department of Physics, Niigata University, Ikarashi 2-8050, Niigata, 950-2181, Japan

Abstract

We construct a lattice Dirac operator of overlap type that describes the propagation of a Dirac fermion in an external gravitational field. The local Lorentz symmetry is manifestly realized as a lattice gauge symmetry, while it is believed that the general coordinate invariance is restored only in the continuum limit. Our doubler-free Dirac operator satisfies the conventional Ginsparg-Wilson relation and possesses $\gamma_5$ hermiticity with respect to the inner product, which is suggested by the general coordinate invariance. The lattice index theorem in the presence of a gravitational field holds, and the classical continuum limit of the index density reproduces the Dirac genus. Reduction to a single Majorana fermion is possible for $8k + 2$ and $8k + 4$ dimensions, but not for $8k$ dimensions, which is consistent with the existence of the global gravitational/gauge anomalies in $8k$ dimensions. Other Lorentz representations, such as the spinor-vector and the bi-spinor representations, can also be treated. Matter fields with a definite chirality (with respect to the lattice-modified chiral matrix) are briefly considered.

\textsuperscript{*}) E-mail: haya@riken.jp
\textsuperscript{**}) E-mail: so@muse.sc.niigata-u.ac.jp
\textsuperscript{***}) E-mail: hsuzuki@riken.jp
§1. Introduction

It is now well-known that a Dirac operator defined on a Euclidean lattice allows an exact chiral symmetry of a lattice-modified form if the lattice Dirac operator satisfies the Ginsparg-Wilson relation. Such Dirac operators in lattice gauge theory were discovered in the context of the perfect action approach and in the context of the overlap formulation. The lattice action of a massless Dirac fermion is exactly invariant under the modified lattice chiral transformation, and the functional integration measure is invariant under flavor non-singlet chiral transformations. Under a flavor-singlet $U(1)$ chiral rotation, the integration measure is not invariant and gives rise to a non-trivial Jacobian, just as in the Fujikawa method in the continuum theory; the continuum limit of the Jacobian reproduces the axial $U(1)$ anomaly (see also Ref. 12)). Moreover, the axial anomaly obtained from the Jacobian possesses a topological property and implies the lattice index theorem. (We reproduce the argument demonstrating this point by using our lattice Dirac operator for the gravitational interaction.) This topological property of the axial anomaly allows a cohomological analysis of the anomaly with finite lattice spacings, and this analysis is crucial in the formulation of lattice chiral gauge theories. See Refs. 15) and 16) for related works and a review of these developments.

The above describes the situation in lattice gauge theory. Now, it is natural to ask to what extent the above scenario holds in the presence of a gravitational field. In the continuum theory, it is well-known that the chiral symmetry suffers from the quantum anomaly also in the presence of a gravitational field, and this anomaly has a topological meaning expressed by the index theorem. The purpose of this paper is to determine how these points can be naturally understood in a framework of lattice field theory.

In this paper, we construct a lattice Dirac operator of overlap type that describes the propagation of a Dirac fermion in an external gravitational field. With suitable modifications for curved space, our lattice Dirac operator satisfies the Ginsparg-Wilson relation in its conventional form and possesses $\gamma_5$ hermiticity. These properties imply the validity of the lattice index theorem and a topological property of the axial $U(1)$ anomaly. Our formulation offers, in a well-defined lattice framework, a conceptually simple way to understand the axial anomaly in the presence of a gravitational field.

Here we do not intend to study the non-perturbative dynamics of gravity on a spacetime lattice. We also do not consider a possible non-trivial global topology of spacetime. These problems are beyond the scope of this paper. We believe, however, that studying the interaction of a lattice Dirac field to an external gravitational field will be useful in the investigation of the coupling of matter fields to dynamically discretized gravity. For previous studies of
the gravitational interaction of matter fields defined on a lattice, see Ref. 21) and references therein.

Our formulation is a natural generalization of the overlap-Dirac operator in lattice gauge theory to a system coupled to external gravitational fields. As easily imagined, the local Lorentz symmetry in curved space can be manifestly realized as an internal gauge symmetry on the lattice, while the general coordinate invariance is not manifest, because our base space takes the form of a conventional hypercubic lattice. With regard to the latter point, our goal is modest, seeking only the situation in which the general coordinate invariance is restored in the limit \( a \to 0 \), where \( a \) denotes the lattice spacing.

§2. Preliminaries

We consider a \( 2n \)-dimensional hypercubic lattice. We regard it as a discretized approximation of a family of coordinate curves on a Riemannian manifold with Euclidean signature. (We consider only torsion-free cases in this paper.) In particular, \( 4n \) links emanate from each of the lattice sites.\(^*)\) In this sense, our lattice is so primitive that it is not directly related to the Regge calculus or to dynamical triangulation.

The spacetime indices are denoted by Greek letters, \( \mu, \nu, \ldots \), and run from 0 to \( 2n-1 \), and the local Lorentz indices are denoted by Roman letters, \( a, b, \ldots \), and run also from 0 to \( 2n-1 \). Summation over repeated indices is implied for expressions in the continuum theory, while for expressions in the lattice theory, we always explicitly indicate a summation over indices. The unit vector along the \( \mu \)-direction, for example, is denoted by \( \hat{\mu} \). In most of expressions below, we set the lattice spacing to unity (i.e., \( a = 1 \)) for simplicity. The Euclidean gamma matrices satisfy the relations\(^{**})\)

\[
\{ \gamma^a, \gamma^b \} = 2\delta^{ab}, \quad (\gamma^a)^T = \gamma^a. \tag{2.1}
\]

The chiral matrix and the generator of \( \text{SO}(2n) \) in the spinor representation are defined by

\[
\gamma_5 \equiv i^n \gamma^0 \gamma^1 \cdots \gamma^{2n-1}, \quad (\gamma_5)^T = \gamma_5,
\sigma^{ab} \equiv \frac{1}{4}[\gamma^a, \gamma^b]. \tag{2.2}
\]

\(^*)\) In our present formulation, in which the vierbein \( e^a_\mu(x) \) exists on each lattice site \( x \), it is natural to regard the index \( \mu \) of \( e^a_\mu(x) \) as labeling the directions of the links starting from the site \( x \). Then, if the Lorentz index \( a \) runs from 0 to \( 2n-1 \), the index \( \mu \) should also run from 0 to \( 2n-1 \) for the matrix \( e^a_\mu(x) \) to be invertible. In this sense, the restriction to a hypercubic lattice is natural.

\(^{**})\) Here we intentionally use a combination of the complex conjugation symbol \( * \) and the transpose operation symbol \( T \) with respect to the spinor indices instead of the dagger \( \dagger \), because we want to use the symbol \( \dagger \) to represent hermitian conjugation with respect to the inner product \( \langle \cdot, \cdot \rangle \). This avoids possible confusion concerning the chirality constraint in curved space (see §5).
In addition to the lattice Dirac fermion field $\psi(x)$, the vielbein $e^a_\mu(x)$ is also defined on the lattice sites. The determinant of the vielbein is denoted by

$$e(x) \equiv \det \{ e^a_\mu(x) \},$$

(2.3)

and it is assumed to satisfy

$$e(x) > 0, \quad \text{for all } x$$

(2.4)
as in the continuum theory. The inverse matrix of the vielbein is denoted by $e^a_\mu(x)$.

As a lattice counterpart of the spin connection, we introduce gauge variables on the lattice links:

$$U(x, \mu) \in \text{spin}(2n).$$

(2.5)

In this paper, the vielbein $e^a_\mu(x)$ and the link variables $U(x, \mu)$ are treated as external non-dynamical fields. They are, however, not independent of each other and are subject to a certain constraint, which is analogous to the metric condition. This constraint is explored in the next subsection. We often use the abbreviated notation

$$\gamma^\mu(x) \equiv \sum_a e^\mu_a(x) \gamma^a.$$  

(2.6)

Note that $\gamma^a$ and $\gamma^\mu(x)$ do not commute with the link variables $U(x, \mu)$, i.e., $[\gamma^\mu(x), U(y, \nu)] \neq 0$.

The local Lorentz transformation has a natural realization on the lattice as follows:

$$\psi(x) \to g(x)\psi(x), \quad \bar{\psi}(x) \to \bar{\psi}(x)g(x)^{-1},$$

$$U(x, \mu) \to g(x)U(x, \mu)g(x + \hat{\mu})^{-1},$$

$$\gamma^\mu(x) \to g(x)\gamma^\mu(x)g(x)^{-1}, \quad e^a_\mu(x) \to \frac{1}{2^n} \sum_b \text{tr}\{\gamma_a g(x)\gamma^b g(x)^{-1}\} e^a_\mu(x),$$

(2.7)

where the gauge transformation $g(x) \in \text{spin}(2n)$ is defined on each site $x$. Note that $e(x)$ is invariant under local Lorentz transformations. Covariance under this transformation is manifest in our construction, exactly as in the case of an internal gauge symmetry in lattice gauge theory. Contrastingly, invariance or covariance under general coordinate transformations is very difficult to implement in a lattice spacetime. We conjecture that this is restored only in the continuum limit. Below, we examine whether the correct index in the continuum limit is reproduced as the classical continuum limit, $a \to 0$, of the lattice chiral anomaly. Our result indicates that the violation of general coordinate invariance is $O(a)$ in our quantum theory of fermion fields coupled to non-dynamical gravitational fields.
§3. Overlap-Dirac operator for the Lorentz spinor field

3.1. Inner product, hermiticity and the metric condition

We start our analysis with a lattice Dirac operator which is defined in terms of the nearest-neighbor forward covariant difference:

\[
\nabla \psi(x) \equiv \sum_\mu \gamma^\mu(x) \{ U(x, \mu) \psi(x + \hat{\mu}) - \psi(x) \}. \tag{3.1}
\]

This clearly behaves covariantly, i.e., \( \nabla \psi(x) \to g(x) \nabla \psi(x) \), under the local Lorentz transformation (2.7). Then, setting

\[
U(x, \mu) = \mathcal{P} \exp \left\{ \int_0^1 dt \frac{1}{2} \sum_{ab} \omega_{\mu ab}(x + (1 - t)\hat{\mu}) \sigma^{ab} \right\}, \tag{3.2}
\]

with a smooth spin connection field \( \omega_{\mu ab}(x) \), the naive continuum limit of \( \nabla \) coincides with the Dirac operator in the continuum, denoted by \( \mathcal{D} \). More precisely, we have \( \lim_{a \to 0} \nabla \psi(x) = \mathcal{D} \psi(x) \) for any fields \( \psi(x), e^a_\mu(x) \) and \( \omega_{\mu ab}(x) \) which vary slowly over the lengthscale \( a \).

Now to ensure that the lattice index theorem holds in lattice gauge theory, in addition to the Ginsparg-Wilson relation, the \( \gamma_5 \) hermiticity of a lattice Dirac operator is very important.\(^{16}\) We are therefore naturally led to attempt to clarify the meaning of hermiticity in the presence of a gravitational field, because the concept of hermiticity involves the definition of the inner product. In the continuum theory, the inner product is modified in curved space for general coordinate invariance.

We introduce the inner product of two functions on the lattice with spinor indices as

\[
(f, g) \equiv \sum_x e(x) f(x)^T g(x), \tag{3.3}
\]

by using the determinant of the vielbein, \( e(x) \). The inner product \( (f, g) \) is a natural lattice counterpart of the general coordinate invariant inner product in curved space, and the norm of any non-zero function with respect to this inner product is positive definite. We believe that inclusion of the vielbein \( e(x) \) is important for the restoration of general coordinate invariance in the continuum limit.

We next examine

\[
(f, \nabla g) = \sum_x e(x) f(x)^T \sum_\mu \gamma^\mu(x) \{ U(x, \mu) g(x + \hat{\mu}) - g(x) \}
\]

\[
= \sum_{x, \mu} \left\{ e(x - \hat{\mu}) f(x - \hat{\mu})^T \gamma^\mu(x - \hat{\mu}) U(x - \hat{\mu}, \mu) - e(x) f(x)^T \gamma^\mu(x) \right\} g(x)
\]
\[-\sum_{x,\mu} e(x-\hat{\mu}) \left[ U(x-\hat{\mu},\mu)^{-1} \gamma^\mu(x-\hat{\mu}) U(x-\hat{\mu},\mu) \right. \\
\times \left\{ f(x) - U(x-\hat{\mu},\mu)^{-1} f(x-\hat{\mu}) \right\} \right]^{T^*} g(x) \\
- \sum_{x,\mu} \left\{ e(x)\gamma^\mu(x) f(x) - e(x-\hat{\mu}) U(x-\hat{\mu},\mu)^{-1} \gamma^\mu(x-\hat{\mu}) U(x-\hat{\mu},\mu) f(x) \right\}^{T^*} g(x) \\
eq - (\nabla^* f, g) - \sum_x \left\{ \sum_\mu \nabla^*_\mu \{ e(x)\gamma^\mu(x) \} f(x) \right\}^{T^*} g(x), \tag{3.4} \]

where, in the second equality, we have shifted the coordinate $x$ to $x-\hat{\mu}$. This manipulation is justified for a lattice with infinite extent or for a finite size lattice with periodic boundary conditions. In the last line, we have introduced a lattice Dirac operator defined in terms of the backward covariant difference,

\[ \nabla^* \psi(x) \equiv \sum_\mu e(x)^{-1} e(x-\hat{\mu}) U(x-\hat{\mu},\mu)^{-1} \gamma^\mu(x-\hat{\mu}) U(x-\hat{\mu},\mu) \]
\[ \times \left\{ \psi(x) - U(x-\hat{\mu},\mu)^{-1} \psi(x-\hat{\mu}) \right\}, \tag{3.5} \]

and the covariant divergence of $e(x) \gamma^\mu(x)$,

\[ \sum_\mu \nabla^*_\mu \{ e(x)\gamma^\mu(x) \} \]
\[ \equiv \sum_\mu \left\{ e(x)\gamma^\mu(x) - e(x-\hat{\mu}) U(x-\hat{\mu},\mu)^{-1} \gamma^\mu(x-\hat{\mu}) U(x-\hat{\mu},\mu) \right\}. \tag{3.6} \]

Although its structure is somewhat complicated, $\nabla^*$ behaves covariantly under the local Lorentz transformation (2.7) and coincides with the continuum Dirac operator $\partial_\mu \{ e(x)\gamma^\mu(x) \}$ in the naive continuum limit.

We note that under the parametrization (3.2), the continuum limit of Eq. (3.6) becomes

\[ \lim_{a \to 0} \sum_\mu \nabla^*_\mu \{ e(x)\gamma^\mu(x) \} = \partial_\mu \{ e(x)\gamma^\mu(x) \} + e(x) \frac{1}{2} \omega_{\mu ab}(x) [\sigma^a, \gamma^\mu(x)], \tag{3.7} \]

which is the covariant divergence of $e(x) \gamma^\mu(x)$ in the continuum, $\nabla_\mu \{ e(x)\gamma^\mu(x) \}$. This combination vanishes identically if the spin connection and the vielbein are related through the metric condition. Thus, as a lattice counterpart of this property, we postulate that the external vielbein and the link variables satisfy the following constraint:

\[ \sum_\mu \nabla^*_\mu \{ e(x)\gamma^\mu(x) \} = 0, \quad \text{for all } x. \tag{3.8} \]
The equality (3.4) then shows that the combination $\nabla^*$ in eq. (3.5) is precisely the opposite of the hermitian conjugate of $\nabla$ in Eq. (3.1) with respect to the inner product (3.3). Using the symbol $\dagger$ to express this conjugation, we have

$$ \nabla^\dagger = -\nabla^*, \quad (\nabla^*)^\dagger = -\nabla. \quad (3.9) $$

The structure of $\nabla^*$, defined in Eq. (3.5), is more complicated than that of $\nabla$, defined in Eq. (3.1), and, partially due to this fact, any explicit analytical calculations involving our Dirac operator are expected to become quite involved. Although our choice of $\nabla$ and $\nabla^*$ given above is by no means unique, it turns out that one or the other of $\nabla$ and $\nabla^*$ constituting such a conjugate pair must be somewhat complicated, as for the choice we consider. Noting this point, we adopt the above choice in this paper.

Having clarified the meaning of the hermitian conjugation, it is straightforward to construct a lattice Dirac operator of overlap type with the desired properties. We first define the Wilson-Dirac operator by

$$ D_w = \frac{1}{2} \left\{ \nabla + \nabla^* - \frac{1}{2}(\nabla^* \nabla + \nabla \nabla^*) \right\}, \quad (3.10) $$

where the second term is the Wilson term in curved space, which is hermitian with respect to the inner product (3.3). This Wilson term ensures the absence of species doubling in our overlap-type Dirac operator constructed below. The naive continuum limit of $D_w$ coincides with the continuum Dirac operator $\not{D}$. Due to the conjugation property (3.9), we have the $\gamma_5$ hermiticity of the Wilson-Dirac operator

$$ D^\dagger_w = \gamma_5 D_w \gamma_5, \quad (3.11) $$

which is crucial in the following analysis.

### 3.2. Overlap-Dirac operator, Ginsparg-Wilson relation and the lattice index theorem

In analogy to the case of lattice gauge theory, we define the overlap-Dirac operator from the Wilson-Dirac operator (3.10) as

$$ D = 1 - A(A^\dagger A)^{-1/2}, \quad A \equiv 1 - D_w. \quad (3.12) $$

The naive continuum limit of the operator $D$ is $\not{D}$. In the flat space limit, in which we have $e_{\mu}^a(x) = \delta_{\mu}^a$ and $U(x, \mu) = 1$, the free Dirac operator in the momentum space reads

$$ \tilde{D}(k) = \frac{1}{a} - \frac{1}{a} - \sum_{\mu} (i \gamma^\mu \tilde{k}_\mu + \frac{1}{2} a \tilde{k}_\mu^2) \left[ 1 + \frac{1}{2} a \sum_{\mu<\nu} \tilde{k}_\mu^2 \tilde{k}_\nu^2 \right]^{1/2}, \quad (3.13) $$

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*Our result can also be obtained by using the “weighted variables” $\tilde{\psi}(x) \equiv e(x)^{1/2} \psi(x)$, $\tilde{\psi}(x) \equiv e(x)^{1/2} \psi(x)$ and the “naive” inner product $(f, g)_{\text{naive}} \equiv \sum_x f(x)T^* g(x)$. In that case, the hermitian conjugation with respect to the naive inner product is just $T^*$. Then, re-expressing the Dirac operators acting on the weighted variables as ones acting on the original variables, we obtain the expressions appearing above.*
where the lattice spacing $a$ has been restored and we have

$$
\hat{k}_\mu \equiv \frac{1}{a} \sin(ak_\mu), \quad \hat{k}_\mu \equiv \frac{2}{a} \sin \left( \frac{ak_\mu}{2} \right).
$$

(3.14)

We thus see that $\tilde{D}(k)\dagger \tilde{D}(k) = 0$ implies $k = 0$, and the free Dirac operator does not suffer from species doubling.

From its construction and the $\gamma_5$ hermiticity of the Wilson-Dirac operator, we find that the operator satisfying the Ginsparg-Wilson relation in the conventional form,

$$
\gamma_5 D + D\gamma_5 = D\gamma_5 D.
$$

(3.15)

Or, in terms of the lattice-modified chiral matrix,

$$
\hat{\gamma}_5 \equiv \gamma_5 (1 - D), \quad (\hat{\gamma}_5)\dagger = \hat{\gamma}_5, \quad (\hat{\gamma}_5)^2 = 1,
$$

(3.16)

the relation is expressed as

$$
D\hat{\gamma}_5 = -\gamma_5 D.
$$

(3.17)

The Dirac operator $D$ is also $\gamma_5$ hermitian, i.e.,

$$
D\dagger = \gamma_5 D\gamma_5,
$$

(3.18)

with respect to the inner product.

Thus we have obtained a lattice Dirac operator of overlap type that describes the propagation of a single Dirac fermion in external gravitational fields. Since the forms of the Ginsparg-Wilson relation and the $\gamma_5$ hermiticity are identical to those in lattice gauge theory, we can repeat the same argument for the index theorem in the latter theory with slight modifications for curved space.

A natural lattice action for the massless Dirac fermion in curved space is

$$
S_F = \sum_x e(x)\overline{\psi}(x)D\psi(x).
$$

(3.19)

The invariance of this action under the local Lorentz transformation is obvious. This action is also invariant under the modified chiral transformation

$$
\psi(x) \to (1 + i\theta \hat{\gamma}_5) \psi(x), \quad \overline{\psi}(x) \to \overline{\psi}(x) (1 + i\theta \gamma_5),
$$

(3.20)

*) From this $\gamma_5$ hermiticity, it follows that the Dirac determinant $\int \prod_x d\psi(x)d\overline{\psi}(x) e^{-S_F}$ is real.
where $\theta$ is an infinitesimal constant parameter, due to the Ginsparg-Wilson relation (3.17). The functional integration measure is, however, not invariant under this transformation and gives rise to a non-trivial Jacobian $J$:

$$\prod_x d\psi(x)d\overline{\psi}(x) \rightarrow J \prod_x d\psi(x)d\overline{\psi}(x),$$

(3.21)

where

$$\ln J = -i\theta \sum_x \text{tr} \{ \hat{\gamma}_5(x,x) + \gamma_5 \delta_{xx} \}$$

$$\equiv -2i\theta \sum_x \text{tr} \Gamma_5(x,x)$$

(3.22)

and

$$\Gamma_5(x,y) \equiv \gamma_5 \left( \delta_{xy} - \frac{1}{2} D(x,y) \right).$$

(3.23)

The operator $\Gamma_5$ anti-commutes with the hermitian operator $H$, again due to the Ginsparg-Wilson relation:

$$\{H, \Gamma_5\} = 0, \quad H \equiv \gamma_5 D, \quad H^\dagger = H.$$  (3.24)

We introduce the eigenfunctions of $H$ as

$$H \varphi_r(x) = \lambda_r \varphi_r(x), \quad \lambda_r \in \mathbb{R}$$  (3.25)

and assume that the eigenfunctions are normalized with respect to the inner product (3.3):

$$(\varphi_r, \varphi_r) = 1, \quad \text{for all } r.$$  (3.26)

Then, by the hermiticity of $H$ with respect to the inner product, we have

$$(\varphi_r, \varphi_s) = 0, \quad \text{for } \lambda_r \neq \lambda_s.$$  (3.27)

The completeness of eigenmodes is expressed as

$$\sum_r \varphi_r(x) \varphi_r(y)^* = \delta_{xy} e(y)^{-1}.$$  (3.28)

Now we can prove the lattice index theorem. The quantity

$$\sum_x \text{tr} \Gamma_5(x,x) = \sum_x \text{tr} \sum_y \Gamma_5(x,y) \delta_{yx}$$

$$= \sum_x e(x) \sum_r \varphi_r(x)^* \sum_y \Gamma_5(x,y) \varphi_r(y)$$

$$= \sum_r (\varphi_r, \Gamma_5 \varphi_r)$$

(3.29)
in the Jacobian (3.22) is an \textit{integer} even for finite lattice spacings (i.e., before taking the continuum limit). This can easily be seen by noting that the eigenvalue of $\Gamma_5\varphi_r$ is $-\lambda_r$ and thus $\Gamma_5\varphi_r$ is orthogonal to $\varphi_r$ when $\lambda_r \neq 0$. Hence only zero eigenmodes contribute to the sum over $r$ in Eq. (3.29). Noting finally that $\Gamma_5 \to \gamma_5$ for zero modes of $H$ and the normalization (3.26), we have

$$\sum_x \text{tr} \Gamma_5(x,x) = n_+ - n_-,$$

where $n_\pm$ denote the numbers of zero eigenmodes with positive and negative chiralities, respectively. This is the lattice index theorem in the presence of a gravitational field.

### 3.3. Classical continuum limit of the lattice index density

We next consider the classical continuum limit of the lattice index (3.29). We consider the classical continuum limit to be the $a \to 0$ limit in which derivatives of the external gravitational fields are assumed to be $O(a^0)$.\footnote{Note that we cannot take the naive continuum limit of $\Gamma_5$ in Eq. (3.29) \textit{a priori}, because the function on which $\Gamma_5$ acts (namely $\delta_{yx}$) varies rapidly in lengthscale $a$.} It is desirable to evaluate the continuum limit directly in the same way as in lattice gauge theory,\textsuperscript{7,9–11} but the actual calculation is quite involved. Instead, here we resort to a powerful argument due to Fujikawa,\textsuperscript{8} which utilizes the topological nature of the lattice index (3.29). We slightly modify the original argument to make the reasoning more transparent.

First we note that in the expressions of the index (3.29), $\Gamma_5$ can be replaced by $\Gamma_5 e^{-H^2/M^2}$ with an arbitrary mass $M$, because only zero eigenmodes of $H$ contribute to the index. Thus we have

$$\sum_x \text{tr} \Gamma_5(x,x) = \sum_x \text{tr} \sum_y \Gamma_5 e^{-H^2/M^2}(x,y) \delta_{yx} \equiv a^2 \sum_x \mathcal{A}_5(x),$$

where the lattice spacing $a$ has been restored, and the index density has been defined by

$$\mathcal{A}_5(x) \equiv \frac{1}{a^{2n}} \text{tr} \sum_y \Gamma_5 e^{-H^2/M^2}(x,y) \delta_{yx}.$$  \hspace{1cm} (3.32)

From this point, we consider the index density on a lattice with infinite extent, as usual for the classical continuum limit. Then we have

$$\mathcal{A}_5(x) = \int_B \frac{d^{2n}k}{(2\pi)^{2n}} e^{-ikx} \text{tr} \sum_y \Gamma_5 e^{-H^2/M^2}(x,y) e^{iky},$$

$$= \int_B \frac{d^{2n}k}{(2\pi)^{2n}} e^{-ikx} \text{tr} \left( \gamma_5 - \frac{a}{2} H \right) e^{-H^2/M^2} e^{ikx},$$

\hspace{1cm} (3.33)
where the Brillouin zone is denoted by $\mathcal{B}$:

$$
\mathcal{B} \equiv \left\{ k \in \mathbb{R}^{2n} \mid -\frac{\pi}{a} < k_\mu \leq \frac{\pi}{a} \right\}.
$$

(3.34)

Now suppose that we expand the integrand of Eq. (3.33) with respect to the external fields $h_\mu^a(x) \equiv e_\mu^a(x) - \delta_\mu^a$, $\omega_{\mu ab}(x)$ and their derivatives. Formally, this yields

$$
\begin{align*}
& e^{-ikx} \text{tr} \left( \gamma_5 - \frac{a}{2} H \right) e^{-H^2/M^2} e^{ikx} \\
& = \sum \mathbf{I} c_I(k; a, M) \partial^{\alpha_1} h_{\mu_1}^{a_1}(x) \cdots \partial^{\alpha_p} h_{\mu_p}^{a_p}(x) \partial^{\beta_1} \omega_{\nu_1 b_1 c_1}(x) \cdots \partial^{\beta_q} \omega_{\nu_q b_q c_q}(x),
\end{align*}
$$

(3.35)

where the superscripts of the partial derivatives $\alpha_i$ and $\beta_j$ are used to label multiple partial derivatives [for example, $\partial^x = (\partial_0)^{\alpha_0} \cdots (\partial_{2n-1})^{\alpha_{2n-1}}$], and $I$ denotes the collection of indices: $I = \{ \alpha_1, \ldots, \alpha_p; a_1, \ldots, a_p; \beta_1, \ldots, \beta_q; b_1, \ldots, b_q; c_1, \ldots, c_q \}$. One can then confirm, through some examination, that the coefficient $c_I(k; a, M)$ has the structure

$$
c_I(k; a, M) = p_I(\hat{k}; k, a, M) \exp \left\{ -\frac{2}{a^2 M^2} \left( 1 - \frac{1 - \frac{1}{2} a^2 \sum \kappa_\mu^2}{[1 + \frac{1}{2} a^4 \sum_{\mu<\nu} \kappa_\mu^2 \kappa_\nu^2]^{1/2}} \right) \right\},
$$

(3.36)

where $p_I(\hat{k}; k, a, M)$ is a polynomial in $\hat{k}_\mu$ whose coefficients are bounded functions of $k \in \mathcal{B}$ and of $O(a^0)$. Noting this structure and the inequality

$$
\frac{2}{\pi} |\kappa_\mu| \leq |\hat{k}_\mu| \leq |\kappa_\mu|, \quad \text{for all } k \in \mathcal{B},
$$

(3.37)

we infer that there exists a polynomial of $|\kappa_\mu|$ with positive coefficients, $b_I(k; M)$, and some positive number $\varepsilon$ such that, for any $a \leq \varepsilon$, we have

$$
|c_I(k; a, M)| \leq b_I(k; M) \exp \left\{ -\frac{4}{\pi^2 M^2} \sum \kappa_\mu^2 \right\}, \quad \text{for all } k \in \mathcal{B}.
$$

(3.38)

Next, we write

$$
\int_{\mathcal{B}} d^{2n} k c_I(k; a, M) = \int_{\Delta} d^{2n} k c_I(k; a, M) + \int_{\mathcal{B} - \Delta} d^{2n} k c_I(k; a, M),
$$

(3.39)

where $\Delta$ is a box of size $\Lambda$ in the Brillouin zone (from this point, the lattice spacing $a$ is assumed to be smaller than $\varepsilon$)

$$
\Delta \equiv \left\{ k \in \mathbb{R}^{2n} \mid |\kappa_\mu| \leq \Lambda < \pi/\varepsilon \right\}.
$$

(3.40)
From the bound (3.38), we have

\[
\left| \int_{\mathcal{B} - \Delta} d^{2n}k c_I(k; a, M) \right| \leq \int_{\mathcal{B} - \Delta} d^{2n}k b_I(k; M) \exp \left\{ -\frac{4}{\pi^2M^2} \sum_{\mu} k_{\mu}^2 \right\}, \tag{3.41}
\]

and thus

\[
\lim_{A \to \infty} \lim_{a \to 0} \int_{\mathcal{B} - \Delta} d^{2n}k c_I(k; a, M) = 0, \tag{3.42}
\]

because \(b_I(k; M) \exp \left\{ -4 \sum_{\mu} k_{\mu}^2 / (\pi^2M^2) \right\}\) is integrable in \(\mathbb{R}^{2n}:

\[
\int_{\mathbb{R}^{2n}} d^{2n}k b_I(k; M) \exp \left\{ -\frac{4}{\pi^2M^2} \sum_{\mu} k_{\mu}^2 \right\} < \infty. \tag{3.43}
\]

We thus conclude from Eq. (3.39) that

\[
\lim_{a \to 0} \int_{\mathcal{B}} d^{2n}k c_I(k; a, M) = \lim_{A \to \infty} \lim_{a \to 0} \int_{\mathcal{B} - \Delta} d^{2n}k c_I(k; a, M)
= \int_{\mathbb{R}^{2n}} d^{2n}k \lim_{a \to 0} c_I(k; a, M). \tag{3.44}
\]

That is, the \(a \to 0\) limit of the integral of \(c_I(k; a, M)\) over \(\mathcal{B}\) is given by an \(\mathbb{R}^{2n}\) integration of the \(a \to 0\) limit of \(c_I(k; a, M)\). The latter should be regarded as the *naive* continuum limit, because the momentum \(k\) carried by the plane wave \(e^{ikx}\) is kept fixed in this \(a \to 0\) limit.

Now, by applying Eq. (3.44) to all terms of the expansion (3.35), we obtain

\[
\lim_{a \to 0} \mathcal{A}_5(x) = \int_{\mathbb{R}^{2n}} \frac{d^{2n}k}{(2\pi)^{2n}} \left. \lim_{a \to 0} e^{-ikx} \text{tr} \left( \gamma_5 - \frac{a}{2}H \right) e^{-H^2/M^2} e^{ikx} \right|_{x = 0} = \int_{\mathbb{R}^{2n}} \frac{d^{2n}k}{(2\pi)^{2n}} e^{-ikx} \text{tr} \gamma_5 e^{D^2/M^2} e^{ikx}, \tag{3.45}
\]

because \(\lim_{a \to 0} H = \gamma_5 \mathcal{D}\) in the naive continuum limit. It is interesting that the term proportional to \(aH/2\) does not contribute in this calculational scheme. In this way, we obtain the index density in the continuum theory. The underlying important point in our argument is that the lattice free Dirac operator does not possess doubler’s zero, and \(e^{-H^2/M^2}\) acts as a suppression factor at the boundary of the Brillouin zone for \(a \to 0\).

The calculation of the index density in the continuum (which can be evaluated in the \(M \to \infty\) limit) is well-known. (For a calculation in the plane wave basis, see Ref. 23.) The result is given by the so-called Dirac genus,

\[
\lim_{a \to 0} a^{2n} \sum_x \text{tr} \Gamma_5(x, x) = \int_{M_{2n}} \lim_{a \to 0} \mathcal{A}_5(x) = \int_{M_{2n}} \det \left\{ \frac{i\hat{R}/4\pi}{\sinh(i\hat{R}/4\pi)} \right\}^{1/2}. \tag{3.46}
\]
where the curvature 2-form is defined by

$$\left( \hat{R} \right)_a{}^b = \frac{1}{2} R_{\mu \nu a}{}^b \, dx^\mu \wedge dx^\nu,$$

and the determinant, det, is taken with respect to the Lorentz indices of $\hat{R}$. Thus as expected from the absence of the species doubling (which implies the correct number of degrees of freedom) and the topological properties of the lattice index, we find that our formulation reproduces the correct expression of the axial U(1) anomaly in a gravitational field. This demonstration can be regarded as a test of the restoration of the general coordinate invariance in the classical continuum limit.

As noted in the introduction, the gauge invariance and the topological nature of the axial anomaly in lattice gauge theory allow a cohomological analysis of the anomaly, as in the continuum theory. It might be possible to carry out this kind of analysis in the present lattice formulation for the gravitational interaction, because the index density (3.32) is local, Lorentz invariant and possesses a topological property. The absence of a manifest general coordinate invariance in our lattice formulation, however, could be an obstacle for such an analysis.

3.4. Reduction to the Majorana fermion

We now briefly comment on a reduction of the Dirac fermion to the Majorana fermion with our lattice Dirac operator. We can apply the prescription in Euclidean field theory, which is reviewed in Ref. 25) also to the present case of a gravitational interaction. Naively, one expects that the Majorana fermion can be defined in $8k$, $8k+2$ and $8k+4$ dimensions (and in the odd number of dimensions $8k+1$ and $8k+3$), because the spinor representation of the Lorentz group is real for these number of dimensions.

The prescription starts by setting

$$\psi = \frac{1}{\sqrt{2}} (\chi + i \eta), \quad \bar{\psi} = \frac{1}{\sqrt{2}} (\chi^T B - i \eta^T B)$$

in Eq. (3.19), where $B$ denotes the charge conjugation matrix, either $B_1$ or $B_2$. (We shall use the notation of Ref. 25)). From the definition (3.12) and properties of $B$, it is straightforward to confirm the skew-symmetricity expressed by

$$e(y)(BD)^T(y, x) = -e(x)BD(x, y).$$

* For a lattice with finite extent, $a^{2n} \sum_x \delta A_5(x) = 0$ clearly holds, from the index theorem, where $\delta$ denotes an arbitrary variation of gravitational fields. Even for a lattice with infinite extent, this equation is meaningful, provided that the variation $\delta$ has finite support and the Dirac operator is local. (We did not prove the latter for gravitational fields of finite strength.) Then, this topological property can be shown directly from the Ginsparg-Wilson relation for any $M$. 13
where the transpose \( T \) acts on the spinor indices, with \( B = B_1 \) for \( 8k + 2 \) dimensions and \( B = B_2 \) for \( 8k + 4 \) dimensions. (To show this, we have to use the constraint (3.49).) Due to this skew-symmetric property, the action for the Dirac fermion (3.19) decomposes into two pieces, describing mutually independent systems,

\[
S_F = \frac{1}{2} \sum_x e(x)\chi^T(x)BD\chi(x) + \frac{1}{2} \sum_x e(x)\eta^T(x)BD\eta(x),
\]

for \( 8k + 2 \) and \( 8k + 4 \) dimensions. Thus, taking either of these two pieces as the action, say,

\[
S_M = \frac{1}{2} \sum_x e(x)\chi^T(x)BD\chi(x),
\]

we can define the Majorana fermion in a gravitational field as half of the Dirac fermion. Moreover, by repeating the argument of Ref. 25), we can show that the partition function of the Majorana fermion (which is manifestly local and Lorentz invariant),

\[
Pf\{eBD\} = \int \prod_x d\chi(x) e^{-S_M},
\]

is semi-positive definite.

However, this prescription does not work for \( 8k \) dimensions, because our lattice Dirac operator does not possess the property (3.49) for use of either \( B_1 \) or \( B_2 \). (This is basically due to the presence of the Wilson term.) This is expected, as argued in Ref. 25), because a single Majorana fermion in \( 8k \) dimensions may suffer from global gravitational/gauge anomalies. Specifically, if there exists a simple local Lorentz invariant lattice formulation of the Majorana fermion in \( 8k \) dimensions, it would not reproduce the global gravitational/gauge anomalies.

\[\text{§4. Other Lorentz representations}\]

We now consider a generalization of our construction to systems with matter fields in other Lorentz (reducible) representations. This is easily achieved. The link variable in the vector representation is given by

\[
U^b_a(x, \mu) \equiv \frac{1}{2^n} \text{tr}\{\gamma_a U(x, \mu)\gamma^b U(x, \mu)^{-1}\} \in SO(2n),
\]

which reads \( U^b_a(x, \mu) = \delta^b_a + \omega_{\mu a}^b(x) + \cdots \) with the parametrization (3.2). For example, the lattice Dirac operators for the spinor-vector field \( \psi_a(x) \) can be defined by

\[
\nabla^a \psi_a(x) \equiv \sum_{\mu} \gamma^\mu(x)\{U(x, \mu)\sum_b U^b_a(x, \mu)\psi_b(x + \hat{\mu}) - \psi_a(x)\},
\]

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\nabla^a \psi_a(x) \equiv \sum_{\mu} \gamma^\mu(x)\{U(x, \mu)\sum_b U^b_a(x, \mu)\psi_b(x + \hat{\mu}) - \psi_a(x)\},
\]
\[ \mathcal{D} \psi_a(x) \equiv \sum_{\mu} e(x)^{-1} e(x - \hat{\mu}) U(x - \hat{\mu}, \mu)^{-1} \gamma^\mu(x - \hat{\mu}) U(x - \hat{\mu}, \mu) \]
\[ \times \{ \psi_a(x) - U(x - \hat{\mu}, \mu)^{-1} \sum_b U^{-1} \right|_a b (x - \hat{\mu}, \mu) \psi_b(x - \hat{\mu}) \}, \quad (4.2) \]

and these operators are conjugate to each other with respect to the following natural inner product for the spinor-vector fields:

\[ (f, g)_{sv} \equiv \sum_x e(x) \sum_a f_a(x)^T g_a(x). \quad (4.3) \]

Thus the overlap-Dirac operator for the spinor-vector fields can be defined by combinations analogous to Eqs. (3.12) and (3.10). The lattice action is given by

\[ S_{sv} = \sum_x e(x) \sum_a \bar{\psi}_a(x) D \psi_a(x). \quad (4.4) \]

Then, the argument for the axial anomaly and the lattice index proceeds in the same manner as in §3. The continuum limit is easily obtained because, as is clear from Eq. (4.2), the link variables in the vector representation \( U_{ab} \) can be regarded as gauge fields associated with an internal \( \text{SO}(2n) \) gauge symmetry. Thus its contribution to the index is given by the Chern character (for a direct calculation with the overlap-Dirac operator, see Ref. 11)), and we have

\[ \lim_{a \to 0} \sum_x \text{tr} \Gamma_5(x, x) = \int_{M_{2n}} \det \left\{ \frac{i \hat{R}/4\pi}{\sinh(i \hat{R}/4\pi)} \right\}^{1/2} \text{tr} \left\{ e^{-i \hat{R}/(2\pi)} \right\}. \quad (4.5) \]

The last trace \( \text{tr} \) is defined with respect to the Lorentz indices of \( \hat{R} \). This is the well-known expression of the index for the spinor-vector representation.\(^{18-20}\) Here, we do not claim that we have constructed a lattice action of the Rarita-Schwinger spin \( 3/2 \) field. The action for the spinor-vector field \(^{12}\), however, might be regarded as a gauge-fixed form of the action of the Rarita-Schwinger field, when supplemented with appropriate ghost fields.

Similarly, for the bi-spinor field \( \psi(x) \), which has two spinor indices, we may take

\[ \mathcal{D} \psi(x) \equiv \sum_{\mu} \gamma^\mu(x) \{ U(x, \mu) \psi(x + \hat{\mu}) U(x, \mu)^{-1} - \psi(x) \}, \]
\[ \mathcal{D}^* \psi(x) \equiv \sum_{\mu} e(x)^{-1} e(x - \hat{\mu}) U(x - \hat{\mu}, \mu)^{-1} \gamma^\mu(x - \hat{\mu}) U(x - \hat{\mu}, \mu) \]
\[ \times \{ \psi(x) - U(x - \hat{\mu}, \mu)^{-1} \psi(x - \hat{\mu}) U(x - \hat{\mu}, \mu) \}. \quad (4.6) \]

It can be verified that these two operators are conjugate to each other with respect to a natural inner product for the bi-spinor field,

\[ (f, g)_{bs} \equiv \sum_x e(x) \text{tr} \{ f(x)^T g(x) \}, \quad (4.7) \]
if the constraint for the external fields, Eq. (3.8) is satisfied. The construction of the overlap-operator can be repeated, and the action for the bi-spinor field is given by

\[ S_{bs} = \sum_x e(x) \text{tr} \{ \overline{\psi}(x) D \psi(x) \}. \]  

(4.8)

As is clear from Eq. (4.6), the link variables on the right-hand side of \( \psi \) (which act on one of the spinor indices of the bi-spinor) can be regarded as gauge fields associated with an internal spin(2n) gauge symmetry. Thus, the continuum limit of the index* is obtained from Eq. (4.5) by considering the Chern character in the spinor representation, and we have

\[ \lim_{a \to 0} \sum_x \text{tr} \Gamma_5(x,x) = \int_{M_2n} \det \left\{ \frac{2i\hat{R}/4\pi}{\tanh(i\hat{R}/4\pi)} \right\}^{1/2}. \]  

(4.9)

This is also the well-known expression of the index for the bi-spinor representation.18–20)

On the basis of the correspondence of the bi-spinor field with a collection of anti-symmetric fields (differential forms), this formulation could be useful for defining anti-symmetric tensor fields coupled to gravity on a lattice.

§5. Matter fields with a definite chirality

Finally, we briefly consider the chirality projection with our overlap-type lattice Dirac operator. For simplicity, we treat only the Weyl fermion. Generalization to other cases, such as a spinor-vector or a bi-spinor with a definite chirality, is straightforward. We follow the formulation given in Ref. 14) for lattice chiral gauge theories.

As in lattice gauge theory, we introduce chirality projection operators in the forms

\[ \hat{P}_\pm = \frac{1}{2} (1 \pm \hat{\gamma}_5), \quad P_\pm = \frac{1}{2} (1 \pm \gamma_5), \]  

(5.1)

where \( \hat{\gamma}_5 \) is given in Eq. (3.16). The chirality for the fermion is asymmetrically imposed as \( \hat{P}_- \psi = \psi \) and \( \overline{\psi} P_+ = \overline{\psi} \). The partition function of the Weyl fermion in an external gravitational field is then given by

\[ \int \text{D}[\psi] \text{D}[\overline{\psi}] e^{-S_F}. \]  

(5.2)

(The lattice action \( S_F \) takes a form identical to that in Eq. (3.19).) We next introduce basis vectors \( v_j(x) \), \( j = 1, 2, 3, \ldots \), which satisfy

\[ \hat{P}_- v_j = v_j, \quad \sum_x e(x)v_k(x)^* v_j(x) = \delta_{kj}, \]  

(5.3)

* Here, the \( \gamma_5 \) chirality is understood to be defined with respect to the left spinor index of the bi-spinor.
and $v_k(x)$, $k = 1, 2, 3, \ldots$, which satisfy
\begin{equation}
\mathbf{\tau}_k P_+ = \mathbf{v}_k, \quad \sum_x e(x)\mathbf{\tau}_k(x)\mathbf{v}_j(x)^T = \delta_{kj}.
\end{equation}

Then we expand the Weyl fermion in terms of these bases as $\psi(x) = \sum_j v_j(x)c_j$ and $\bar{\psi}(x) = \sum_k \bar{c}_k \bar{v}_k(x)$. The functional integration measure is then defined by $D[\psi]D[\bar{\psi}] \equiv \prod_j dc_j \prod_k d\bar{c}_k$. The partition function is thus given by $\det M$, where the matrix $M$ is
\begin{equation}
M_{kj} \equiv \sum_x e(x)\mathbf{v}_k(x)Dv_j(x).
\end{equation}

Because covariance under the local Lorentz transformation is manifest in our construction, it is easy to study how the partition function changes under the local Lorentz transformation; this provides a lattice counterpart of the local Lorentz anomaly,\(^{28}\) which is one facet\(^{29}\) of the gravitational anomalies.\(^{26,30}\) Under an infinitesimal local Lorentz transformation,
\begin{equation}
\delta_\eta U(x, \mu) = \Theta(x)U(x, \mu) - U(x, \mu)\Theta(x + \hat{\mu}),
\end{equation}
where $\Theta(x) = \frac{1}{2} \sum_{ab} \Theta_{ab}(x)\sigma^{ab}$, the Dirac operator behaves covariantly, as $\delta_\eta D = [\Theta, D]$. Then, noting the completeness relations\(^*)
\begin{align}
\sum_j v_j(x)v_j(y)^T &= \hat{P}_-(x,y)e(y)^{-1}, \quad \sum_k \mathbf{\tau}_k(x)^T\mathbf{\tau}_k(y) = P_+\delta_{xy}e(y)^{-1},
\end{align}
we have
\begin{align}
\delta_\eta \ln \det M &= \sum_x \text{tr}\{\Theta(x)\Gamma_5(x, x)\} + \sum_j \sum_x e(x)v_j(x)^T\delta_\eta v_j(x) \\
&\equiv i \sum_x \frac{1}{2} \sum_{ab} \Theta_{ab}(x)A^{ab}(x) - i\mathfrak{L}_\eta,
\end{align}
where
\begin{equation}
A^{ab}(x) \equiv -i \text{tr}\{\sigma^{ab}\Gamma_5(x, x)\}, \quad \mathfrak{L}_\eta \equiv i \sum_j (v_j, \delta_\eta v_j).
\end{equation}

The first part, $A^{ab}(x)$, which is covariant under the local Lorentz transformation, corresponds to the covariant form\(^{29,31}\) of the local Lorentz anomaly. The second part, $\mathfrak{L}_\eta$, is the so-called measure term,\(^{14}\) which parametrizes the manner in which the basis vectors change
\(^*)\text{In the first expression, the argument of } e^{-1} \text{ must be } y. \text{ This is consistent with the facts that the “conjugate” of a basis vector } v_j \text{ satisfies the constraint } \sum_y v_j(y)^T\bar{e}(y)\hat{P}_-(y,z)e(z)^{-1} = v_j(z)^T \text{ and that } (\hat{P}_-)^2 = \hat{P}_-.
under a variation of a gravitational field (which in the present context is a variation of an infinitesimal local Lorentz transformation). The measure term cannot be completely arbitrary, because it is subject to the integrability condition,\textsuperscript{14} and in Eq. (1.8), it provides the lattice counterpart of (the divergence of) the Bardeen-Zumino current,\textsuperscript{29} which gives rise to a difference between the covariant form and the consistent form\textsuperscript{29,32} of anomalies. For the local Lorentz invariance to hold, basis vectors for which $\delta \eta^a \det M = 0$ must be used. Whether this is possible is the central question.\textsuperscript{13,14} In any case, the cancellation of the local Lorentz anomaly in the continuum is necessary for this anomaly cancellation in the lattice theory.

The covariant anomaly $A^{ab}$ in our lattice formulation is given by the expression of the axial $U(1)$ anomaly \textsuperscript{6,22} with an additional generator of the gauge transformation $\sigma^{ab}$ inserted. This correspondence of the two anomalies also exists in the continuum theory. The evaluation of the classical continuum limit of $A^{ab}(x)$, however, can be quite involved, because a lattice sum of $A^{ab}(x)$ is not a topologically invariant quantity, and we cannot apply the argument of Ref. 8). We do not study this problem in the present paper.

The local Lorentz anomaly is only one facet of the gravitational anomaly,\textsuperscript{29} and it is closely related to the anomaly in the general coordinate invariance.\textsuperscript{26} However, a treatment of this anomaly in our lattice formulation, unfortunately, would not be so simple, because the covariance (or invariance) under general coordinate transformations is not manifest in our formulation. This is also a subject of future study.

A more ambitious program employing the present formulation is to consider the global gravitational anomalies,\textsuperscript{26,33} as done in Ref. 34) for the global gauge anomaly\textsuperscript{35} in lattice gauge theory.

\section{6. Conclusion}

In this paper, we constructed a lattice Dirac operator of overlap type that describes the propagation of a single Dirac fermion in an external gravitational field. The operator satisfies the conventional Ginsparg-Wilson relation \textsuperscript{5,15} and possesses the $\gamma_5$ hermiticity \textsuperscript{5,18} with respect to the inner product \textsuperscript{5,3}. The lattice index theorem holds, as in lattice gauge theory, and the classical continuum limit of the index density reproduces the Dirac genus. Thus, the situation is, as far as the $\gamma_5$ symmetry and associated anomalies are concerned, analogous to the case of lattice gauge theory.

However, there remain many points to be clarified. The first is to determine a sufficient condition for the Dirac operator with gravitational fields of finite strength to be local; this is known as the admissibility in the case of lattice gauge theory.\textsuperscript{36} To test the restoration of the
general coordinate invariance is another important problem. We observed such restoration in a computation of the classical continuum limit of the lattice index. This would not be a good example, however, because the restoration can be guessed from the topological properties and the local Lorentz invariance. One interesting test is to examine the trace anomaly, because its correct form is fixed only when the general coordinate invariance is imposed.

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