Codes Endowed With the Rank Metric

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Abstract

We review the main results of the theory of rank-metric codes, with emphasis on their combinatorial properties. We study their duality theory and MacWilliams identities, comparing in particular rank-metric codes in vector and matrix representation. We then investigate the combinatorial structure of MRD codes and optimal anticodes in the rank metric, describing how they relate to each other.

Introduction

A $q$-ary rank-metric code is a set of matrices over $\mathbb{F}_q$ equipped with the rank distance, which measures the rank of the difference of a pair of matrices. Rank-metric codes were first studied in [3] by Delsarte for combinatorial interest.

More recently, codes endowed with the rank metric have been re-discovered for error correction in the context of linear network coding, and featured prominently in the coding theory literature.

In linear network coding, a source attempts to transmit information packets to multiple destinations via a network of intermediate nodes. The nodes compute and forward in the direction of the sinks linear functions of the received packets, rather than simply routing them. In [1], it was shown that linear network coding achieves the optimal multicast throughput over sufficiently large alphabets.

Rank-metric codes were proposed in [8, 18] for end-to-end error correction in noisy and adversarial networks. In this context, as shown in [17], the correction capability of a rank-metric code is measured by a fundamental parameter, called the minimum rank distance of the code.
In this work we survey the main results of the mathematical theory of rank-metric codes, with emphasis on their combinatorial structure.

In Section 1 we introduce the most important parameters of a rank-metric code, namely, the minimum distance, the weight distribution, and the distance distribution. We then define the trace-dual of a linear rank-metric code, and compare the duality theories of codes in matrix and vector representation. In particular, we show that the former generalizes the latter.

Section 2 is devoted to the duality theory of codes endowed with the rank metric. We study how combinatorial properties of a linear code relate to combinatorial properties of the dual code. In particular, we show that the weight distribution of a linear code and the weight distribution of its dual code determine each other via a MacWilliams-type transformation. We also show an application of the MacWilliams identities for the rank metric to an enumerative combinatorics problem.

In Section 3 we study codes that have the largest possible cardinality for their parameters. These are called Maximum Rank Distance codes (MRD in short) and have very remarkable properties. We first show the existence of linear MRD codes for all choices of the parameters and of the field size. Then we prove that the dual of a linear MRD code is MRD. Finally, we show that the distance distribution of a (possibly non-linear) rank-metric code is completely determined by its parameters.

Section 4 is devoted to rank-metric anticodes, i.e., sets of matrices where the distance between any two of them is bounded from above by a given integer. We study how codes and anticodes relate to each other, deriving in particular an upper bound for the cardinality of any anticode of given parameters. We conclude the section showing that the dual of an optimal linear anticode is an optimal anticode.

1 Rank-metric codes

In the sequel \( q \) denotes a fixed prime power, and \( \mathbb{F}_q \) the finite field with \( q \) elements. Moreover, \( k \) and \( m \) denote positive integers with \( k \leq m \) without loss of generality, and \( \mathbb{F}_{q}^{k \times m} \) is the space of \( k \times m \) matrices over \( \mathbb{F}_q \). Finally, for given integers \( a, b \in \mathbb{N} \) we denote by

\[
\binom{a}{b}_q
\]

the \( q \)-ary binomial coefficient of \( a \) and \( b \), which counts the number of \( b \)-dimensional subspaces of an \( a \)-dimensional space over \( \mathbb{F}_q \). See e.g. [19, Section 1.7] for details.

**Definition 1.** The **rank distance** is the function \( d : \mathbb{F}_q^{k \times m} \times \mathbb{F}_q^{k \times m} \rightarrow \mathbb{N} \) defined by \( d(M, N) = \text{rk}(M - N) \) for all \( M, N \in \mathbb{F}_q^{k \times m} \).

It is easy to check that \( d \) is indeed a distance function on \( \mathbb{F}_q^{k \times m} \).

**Definition 2.** A **(rank-metric) code** over \( \mathbb{F}_q \) is a non-empty subset \( C \subseteq \mathbb{F}_q^{k \times m} \). When \( |C| \geq 2 \), the **minimum distance** of \( C \) is the positive integer

\[
d(C) = \min \{d(M, N) \mid M, N \in C, \ M \neq N\}.
\]

A code \( C \) is **linear** if it is an \( \mathbb{F}_q \)-linear subspace of \( \mathbb{F}_q^{k \times m} \). In this case its **dual code** is defined as

\[
C^\perp = \{ N \in \mathbb{F}_q^{k \times m} \mid \text{Tr}(MN^t) = 0 \text{ for all } M \in C \} \subseteq \mathbb{F}_q^{k \times m},
\]

where \( \text{Tr}(\cdot) \) denotes the trace of a square \( k \times k \) matrix.

The map \( (M, N) \rightarrow \text{Tr}(MN^t) \in \mathbb{F}_q \) is a scalar product on \( \mathbb{F}_q^{k \times m} \), i.e., it is symmetric, bilinear and non-degenerate. In particular, the dual of a linear code is a linear code of dimension

\[
\dim(C^\perp) = km - \dim(C).
\]

Other fundamental parameters of a rank-metric code are the following.
Definition 3. The weight distribution and the distance distribution of a code $C$ are the collections $\{W_i(C) \mid i \in \mathbb{N}\}$ and $\{D_i(C) \mid i \in \mathbb{N}\}$ respectively, where

$$W_i(C) = |\{M \in C \mid \text{rk}(M) = i\}|, \quad D_i(C) = 1/|C| \cdot \{|(M, N) \in C^2 \mid d(M, N) = i\}|$$

for all $i \in \mathbb{N}$.

If $C$ is a linear code, then for all $P \in C$ there are precisely $|C|$ pairs $(M, N) \in C^2$ such that $M - N = P$. Therefore

$$D_i(C) = 1/|C| \cdot \sum_{P \in C, \text{rk}(P) = i} |\{(M, N) \in C^2 \mid M - N = P\}| = W_i(C)$$

for all $i \in \mathbb{N}$. Moreover, if $|C| \geq 2$ then $d(C) = \min\{|\text{rk}(M) \mid M \in C, \ M \neq 0\}$.

In [3], Gabidulin proposed independently a different notion of rank-metric code, in which the codewords are vectors with entries from an extension field $\mathbb{F}_{q^m}$ rather than matrices over $\mathbb{F}_q$.

Definition 4. The rank of a vector $v = (v_1, \ldots, v_k) \in \mathbb{F}_{q^m}^k$ is the dimension of the linear spaces generated over $\mathbb{F}_q$ by its entries, i.e., $\text{rk}_q(v) = \dim_{\mathbb{F}_q}(v_1, \ldots, v_k)$. The rank distance between vectors $v, w \in \mathbb{F}_{q^m}^k$ is $d_G(v, w) = \text{rk}_q(v - w)$.

One can check that $d_G$ is a distance function on $\mathbb{F}_{q^m}^k$.

Definition 5. A vector rank-metric code over $\mathbb{F}_{q^m}$ is a non-empty subset $C \subseteq \mathbb{F}_{q^m}^k$. When $|C| \geq 2$, the minimum distance of $C$ is the positive integer

$$d_G(C) = \min\{|d_G(v, w) \mid v, w \in C, \ v \neq w\}.$$

The code $C$ is linear if it is an $\mathbb{F}_{q^m}$-linear subspace of $\mathbb{F}_{q^m}^k$. In this case the dual of $C$ is defined as

$$C^\perp = \left\{w \in \mathbb{F}_{q^m}^k \mid \sum_{i=1}^k v_i w_i = 0 \text{ for all } v \in C\right\} \subseteq \mathbb{F}_{q^m}^k.$$

The map $(v, w) \mapsto \sum v_i w_i$ is an $\mathbb{F}_{q^m}$-scalar product on $\mathbb{F}_{q^m}^k$. Therefore for all linear vector rank-metric codes $C \subseteq \mathbb{F}_{q^m}^k$, we have

$$\dim_{\mathbb{F}_{q^m}}(C^\perp) = k - \dim_{\mathbb{F}_{q^m}}(C).$$

Definition 6. The weight distribution and the distance distribution of a vector rank-metric code $C$ are the integer vectors $(W_i(C) \mid i \in \mathbb{N})$ and $(D_i(C) \mid i \in \mathbb{N})$ respectively, where

$$W_i(C) = |\{v \in C \mid \text{rk}_q(v) = i\}|, \quad D_i(C) = 1/|C| \cdot |\{(v, w) \in C^2 \mid d_G(v, w) = i\}|$$

for all $i \in \mathbb{N}$.

There exists a natural way to associate to a vector rank-metric code a code in matrix representation with the same cardinality and metric properties.

Definition 7. Let $\Gamma = \{\gamma_1, \ldots, \gamma_m\}$ be a basis of $\mathbb{F}_{q^m}$ over $\mathbb{F}_q$. The matrix associated to a vector $v \in \mathbb{F}_{q^m}^k$ with respect to $\Gamma$ is the $k \times m$ matrix $\Gamma(v)$ with entries in $\mathbb{F}_q$ defined by

$$v_i = \sum_{j=1}^m \Gamma(v)_{ij} \gamma_j \quad \text{for all } i = 1, \ldots, k.$$

The rank-metric code associated to a vector rank-metric code $C \subseteq \mathbb{F}_{q^m}^k$ with respect to $\Gamma$ is $\Gamma(C) = \{\Gamma(v) \mid v \in C\} \subseteq \mathbb{F}^{k \times m}_q$. 


Notice that in the previous definition the \( i \)-th row of \( \Gamma(v) \) is the expansion of the entry \( v_i \) over the basis \( \Gamma \).

The proof of the following result is standard and left to the reader.

**Proposition 8.** For every \( \mathbb{F}_q \)-basis \( \Gamma \) of \( \mathbb{F}_{q^m} \) the map \( v \mapsto \Gamma(v) \) is an \( \mathbb{F}_q \)-linear bijective isometry \( \mathbb{F}_{q^m} \to (\mathbb{F}_{q^m}^k, d_G) \to (\mathbb{F}_{q^m}^k, d) \).

In particular, if \( C \subseteq \mathbb{F}_{q^m} \) is a vector rank-metric code, then \( \Gamma(C) \) has the same cardinality, rank distribution and distance distribution as \( C \). Moreover, if \( |C| \geq 2 \) then \( d_G(C) = d(\Gamma(C)) \).

In the remainder of the section we compare the duality theories of matrix and vector rank-metric codes, showing that the former generalizes the latter. The following results appear in [12].

Given an \( \mathbb{F}_{q^m} \)-linear vector rank-metric code \( C \subseteq \mathbb{F}_{q^m} \) and a basis \( \Gamma \) of \( \mathbb{F}_{q^m} \) over \( \mathbb{F}_q \), it is natural to ask whether the codes \( \Gamma(C^\perp) \) and \( \Gamma(C)^\perp \) coincide or not. The answer is negative in general, as we show in the following example.

**Example 9.** Let \( q = 3 \), \( k = m = 2 \) and \( \mathbb{F}_{3^2} = \mathbb{F}_3[x] \), where \( \eta \) is a root of the irreducible primitive polynomial \( x^2 + 2x + 2 \in \mathbb{F}_3[x] \). Let \( \xi = \eta^2 + 1 = 0 \). Set \( \alpha = (\xi, 2) \), and let \( C \subseteq \mathbb{F}_{3^2} \) be the 1-dimensional vector rank-metric code generated by \( \alpha \) over \( \mathbb{F}_{3^2} \). Take \( \Gamma = \{1, \xi\} \) as basis of \( \mathbb{F}_{3^2} \) over \( \mathbb{F}_3 \). One can check that \( \Gamma(C) \) is generated over \( \mathbb{F}_3 \) by the two matrices

\[
\Gamma(\alpha) = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}, \quad \Gamma(\xi\alpha) = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}.
\]

Let \( \beta = (\xi, 1) \in \mathbb{F}_{3^2} \). We have \( \alpha_1\beta_1 + \alpha_2\beta_2 = 1 \neq 0 \), and so \( \beta \notin C^\perp \). It follows \( \Gamma(\beta) \notin \Gamma(C^\perp) \).

On the other hand,

\[
\Gamma(\beta) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},
\]

and it is easy to see that \( \Gamma(\beta) \) is trace-orthogonal to both \( \Gamma(\alpha) \) and \( \Gamma(\xi\alpha) \). Therefore \( \Gamma(\beta) \in \Gamma(C)^\perp \), hence \( \Gamma(C)^\perp \neq \Gamma(C^\perp) \).

Although the duality notions for matrix and vector rank-metric codes do not coincide, there is a simple relation between them via orthogonal bases of finite fields.

Let \( \text{Trace} : \mathbb{F}_{q^m} \to \mathbb{F}_q \) be the map defined by \( \text{Trace}(\alpha) = \alpha + \alpha^q + \cdots + \alpha^{q^{m-1}} \) for all \( \alpha \in \mathbb{F}_{q^m} \). Bases \( \Gamma = \{\gamma_1, \ldots, \gamma_m\} \) and \( \Gamma' = \{\gamma'_1, \ldots, \gamma'_m\} \) of \( \mathbb{F}_{q^m} \) over \( \mathbb{F}_q \) are called orthogonal if \( \text{Trace}(\gamma_i\gamma_j) = \delta_{ij} \) for all \( i, j \in \{1, \ldots, m\} \). It is well-known that every basis \( \Gamma \) of \( \mathbb{F}_{q^m} \) over \( \mathbb{F}_q \) has a unique orthogonal basis \( \Gamma' \) (see [10], page 54).

**Theorem 10.** Let \( C \subseteq \mathbb{F}_{q^m} \) be an \( \mathbb{F}_{q^m} \)-linear vector rank-metric code, and let \( \Gamma, \Gamma' \) be orthogonal bases of \( \mathbb{F}_{q^m} \) over \( \mathbb{F}_q \). We have

\[
\Gamma'(C^\perp) = \Gamma(C)^\perp.
\]

In particular, \( C \) has the same weight distribution as \( \Gamma(C) \), and \( C^\perp \) has the same weight distribution as \( \Gamma(C)^\perp \).

**Proof.** Write \( \Gamma = \{\gamma_1, \ldots, \gamma_m\} \) and \( \Gamma' = \{\gamma'_1, \ldots, \gamma'_m\} \). Let \( M \in \Gamma'(C^\perp) \) and \( N \in \Gamma(C) \). There exist \( \alpha \in C^\perp \) and \( \beta \in C \) such that \( M = \Gamma'(\alpha) \) and \( N = \Gamma(\beta) \). By Definition 7 we have

\[
0 = \sum_{i=1}^{k} \alpha_i \beta_i = \sum_{i=1}^{k} \sum_{j=1}^{m} M_{ij} \gamma'_j \sum_{t=1}^{m} N_{it} \gamma_t = \sum_{i=1}^{k} \sum_{j=1}^{m} \sum_{t=1}^{m} M_{ij} N_{it} \gamma'_j \gamma_t.
\]

Applying the function \( \text{Trace} : \mathbb{F}_{q^m} \to \mathbb{F}_q \) to both sides of equation (10) we obtain

\[
0 = \text{Trace} \left( \sum_{i=1}^{k} \sum_{j=1}^{m} \sum_{t=1}^{m} M_{ij} N_{it} \gamma'_j \gamma_t \right) = \sum_{i=1}^{k} \sum_{j=1}^{m} \sum_{t=1}^{m} M_{ij} N_{it} \text{Trace}(\gamma'_j \gamma_t) = \text{Tr}(MN^t).
\]
Therefore $\Gamma'(C^\perp) \subseteq \Gamma(C)^\perp$. Proposition \ref{prop:gamma-inequality} implies that $\Gamma'(C^\perp)$ and $\Gamma(C)^\perp$ have the same dimension over $\mathbb{F}_q$. Hence the two codes are equal. The second part of the statement follows from Proposition \ref{prop:gamma-inequality}.

Theorem \ref{thm:dual-proposition} shows that the duality theory of $\mathbb{F}_q$-linear rank-metric codes in matrix representation can be regarded as a generalization of the duality theory of $\mathbb{F}_m$-linear vector rank-metric codes. For this reason, in the sequel we only treat rank-metric codes in matrix representation.

## 2 MacWilliams identities for the rank metric

This section is devoted to the duality theory of codes endowed with the rank metric. We concentrate on linear rank-metric codes, and show that the weight distributions of a code $C$ and its dual code $C^\perp$ determine each other via a MacWilliams-type transformation. This result was established by Delsarte in \cite[Theorem 3.3]{Delsarte} using the machinery of association schemes, and may be regarded as the rank-analogue of a celebrated theorem by MacWilliams on the weight distribution of linear codes endowed with the Hamming metric (see \cite{MacWilliams}). In this section we present a lattice-theoretic proof inspired by \cite[Theorem 27]{Delsarte}.

**Notation 11.** We denote by $\text{colsp}(M) \subseteq \mathbb{F}_q^{k \times m}$ the $\mathbb{F}_q$-space generated by the columns of a matrix $M \in \mathbb{F}_q^{k \times m}$. Given a code $C \subseteq \mathbb{F}_q^{k \times m}$ and an $\mathbb{F}_q$-subspace $U \subseteq \mathbb{F}_q^k$, we let

$$C(U) = \{ M \in C \mid \text{colsp}(M) \subseteq U \} \subseteq \mathbb{F}_q^{k \times m}$$

be the set of matrices in $C$ whose columnspace is contained in $U$.

Note that for all $M, N \in \mathbb{F}_q^{k \times m}$ we have $\text{colsp}(M + N) \subseteq \text{colsp}(M) + \text{colsp}(N)$. As a consequence, if $U \subseteq \mathbb{F}_q^k$ is an $\mathbb{F}_q$-linear subspace and $C \subseteq \mathbb{F}_q^{k \times m}$ is a linear code, then $C(U)$ is a linear code as well.

We start with a series of preliminary results. In the sequel we denote by $U^\perp$ the orthogonal (or dual) of an $\mathbb{F}_q$-vector space $U \subseteq \mathbb{F}_q^k$ with respect to the standard inner product of $\mathbb{F}_q$. It will be clear from context if by "$\perp$" we denote the trace-dual in $\mathbb{F}_q^{k \times m}$ or the standard dual in $\mathbb{F}_q^k$.

**Lemma 12.** Let $U \subseteq \mathbb{F}_q^k$ be a subspace. The following hold.

1. $\dim(\mathbb{F}_q^{k \times m}(U)) = m \cdot \dim(U)$.
2. $\mathbb{F}_q^{k \times m}(U^\perp) = \mathbb{F}_q^{k \times m}(U^\perp)$.

**Proof.**

1. Let $s = \dim(U)$ and $V = \{(x_1, \ldots, x_k) \in \mathbb{F}_q^k \mid x_i = 0$ for $i > s\} \subseteq \mathbb{F}_q^k$. There exists an $\mathbb{F}_q$-isomorphism $g : \mathbb{F}_q^k \to \mathbb{F}_q^k$ that maps $U$ to $V$. Let $G \in \mathbb{F}_q^{k \times k}$ be the invertible matrix associated to $g$ with respect to the canonical basis $\{e_1, \ldots, e_k\}$ of $\mathbb{F}_q^k$, i.e.,

$$g(e_j) = \sum_{i=1}^k G_{ij} e_i \quad \text{for all } j = 1, \ldots, k.$$

The map $M \mapsto GM$ is an $\mathbb{F}_q$-isomorphism $\mathbb{F}_q^{k \times m}(U) \to \mathbb{F}_q^{k \times m}(V)$. Property \ref{property} of the lemma now directly follows from the definition of $\mathbb{F}_q^{k \times m}(V)$.

2. Let $N \in \mathbb{F}_q^{k \times m}(U^\perp)$ and $M \in \mathbb{F}_q^{k \times m}(U)$. Using the definition of trace-product one sees that $\text{Tr}(MN^t) = \sum_{i=1}^m \langle M_i, N_i \rangle$, where $\langle \cdot, \cdot \rangle$ is the standard inner product of $\mathbb{F}_q^k$ and $M_i, N_i$ denote the $i$-th column of $M$ and $N$ (respectively). Each column of $N$ belongs to $U^\perp$, and each column of $M$ belongs to $U$. Therefore $\text{Tr}(MN^t) = 0$, hence $\mathbb{F}_q^{k \times m}(U^\perp) \subseteq \mathbb{F}_q^{k \times m}(U^\perp)$. By property \ref{property} the two spaces $\mathbb{F}_q^{k \times m}(U^\perp)$ and $\mathbb{F}_q^{k \times m}(U^\perp)$ have the same dimension over $\mathbb{F}_q$. Therefore they are equal.

\qed
Now assume that for all $U \subseteq \mathbb{F}_q^k$, where the last equality immediately follows from the definition of $u$. Therefore

$$|C(U)| = \frac{|C|}{q^{m(k-u)}}|C^\perp(U^\perp)|.$$  

**Proof.** We have $C(U^\perp) = (C \cap \mathbb{F}_q^{k\times m}(U))^\perp = C^\perp + \mathbb{F}_q^{k\times m}(U^\perp) = C^\perp + \mathbb{F}_q^{k\times m}(U^\perp)$, where the last equality follows from part 2 of Lemma 12. Therefore

$$|C(U)| \cdot |C^\perp + \mathbb{F}_q^{k\times m}(U^\perp)| = q^{km}. \quad (2)$$

On the other hand, part 1 of Lemma 12 gives

$$\dim(C^\perp + \mathbb{F}_q^{k\times m}(U^\perp)) = \dim(C^\perp) + m \cdot \dim(U^\perp) - \dim(C^\perp(U^\perp)).$$

As a consequence,

$$|C^\perp + \mathbb{F}_q^{k\times m}(U^\perp)| = \frac{q^{km} \cdot q^{m(k-u)}}{|C| \cdot |C^\perp(U^\perp)|}. \quad (3)$$

Combining equations (2) and (3) one obtains the proposition. \qed

We will also need the following preliminary lemma, which is an explicit version of the Möbius inversion formula for the lattice of subspaces of $\mathbb{F}_q^k$. We include a short proof for completeness. See [19, Sections 3.7 – 3.10] for details.

**Lemma 14.** Let $\mathcal{P}(\mathbb{F}_q^k)$ be the set of all $\mathbb{F}_q$-subspaces of $\mathbb{F}_q^k$, and let $f : \mathcal{P}(\mathbb{F}_q^k) \rightarrow \mathbb{Z}$ be any function. Define $g : \mathcal{P}(\mathbb{F}_q^k) \rightarrow \mathbb{Z}$ by $g(V) = \sum_{U \subseteq V} f(U)$ for all $V \subseteq \mathbb{F}_q^k$. Then for all $i \in \{0, \ldots, k\}$ and for any subspace $V \in \mathcal{P}(\mathbb{F}_q^k)$ with $\dim(V) = i$ we have

$$f(V) = \sum_{u=0}^{i} (-1)^{i-u} q^{(i-u)} \sum_{U \subseteq V} g(U).$$

**Proof.** Fix an integer $i \in \{0, \ldots, k\}$ and a vector space $V \in \mathcal{P}(\mathbb{F}_q^k)$ with $\dim(V) = i$. We inductively define a function $\mu : \{U \in \mathcal{P}(\mathbb{F}_q^k) \mid U \subseteq V\} \rightarrow \mathbb{Z}$ by $\mu(U) = 1$ if $U = V$, and $\mu(U) = -\sum_{U \subseteq S \subseteq V} \mu(S)$ if $U \subsetneq V$. By definition of $g$ we have

$$\sum_{U \subseteq V} \mu(U) g(U) = \sum_{U \subseteq V} \mu(U) \sum_{S \subseteq U} f(S) = \sum_{S \subseteq V} f(S) \sum_{S \subseteq U \subseteq V} \mu(U) = f(V),$$

where the last equality immediately follows from the definition of $\mu$. Therefore it suffices to show that for all $U \subseteq V$ we have

$$\mu(U) = (-1)^{i-u} q^{(i-u)}, \quad (4)$$

where $u = \dim(U)$. We proceed by induction on $i-u$. If $i = u$ then equation (4) is trivial. Now assume $i > u$. By definition of $\mu$ and the induction hypothesis we have

$$\mu(U) = -\sum_{U \subseteq S \subseteq V} \mu(S) = -\sum_{s=u+1}^{i} (-1)^{i-s} q^{(i-s)} \binom{i-j}{s-u} q^{(s-u)}$$

$$= -\sum_{s=u+1}^{i} (-1)^{i-s} q^{(i-s)} \binom{i-u}{i-s} q^{(i-u)}$$

$$= -\sum_{s=0}^{i-u} (-1)^{s} q^{(i-s)} \binom{i-u}{s} + (-1)^{i-u} q^{(i-u)}$$

$$= (-1)^{i-u} q^{(i-u)},$$

where the last equality follows from the $q$-Binomial Theorem (see [19, page 74]). \qed
We can now prove the main result of this section, first established by Delsarte in [3, Theorem 3.3]. A proof for the special case of $F_q^m$-linear vector rank-metric codes using different techniques can be found in [6].

**Theorem 15** (MacWilliams identities for the rank metric). Let $C \subseteq F_k^m$ be an linear rank-metric code. For all $i \in \{0, \ldots, k\}$ we have

$$W_i(C^\perp) = \frac{1}{|C|} \sum_{j=0}^{k} W_j(C) \sum_{u=0}^{k} (-1)^{i-u} q^{mu+(\frac{i-u}{2})} \left[ \begin{array}{c} k-u \\ k-i \\ \end{array} \right]_q \left[ \begin{array}{c} k-j \\ u \end{array} \right]_q.$$

**Proof.** For all subspaces $V \subseteq F_k^m$ define

$$f(V) = |\{M \in C^\perp \mid \colsp(M) = V\}|, \quad g(V) = \sum_{U \subseteq V} f(U) = |C^\perp(V)|.$$

By Lemma [14] for any $i \in \{0, \ldots, k\}$ and for any vector space $V \subseteq F_k^m$ of dimension $i$ we have

$$f(V) = \sum_{u=0}^{i} (-1)^{i-u} q^{\frac{i-u}{2}} \sum_{\dim(U)=u} |C^\perp(U)|$$

$$= \sum_{u=0}^{i} (-1)^{i-u} q^{\frac{i-u}{2}} \sum_{T \supseteq V^\perp \subseteq F_k^m \atop \dim(T)=k-u} |C^\perp(T^\perp)|$$

$$= \frac{1}{|C|} \sum_{u=0}^{i} (-1)^{i-u} q^{mu+(\frac{i-u}{2})} \sum_{T \supseteq V^\perp \subseteq F_k^m \atop \dim(T)=k-u} |C(T)|,$$

where the last equality follows from Proposition [13]. Now observe that

$$W_i(C^\perp) = \sum_{V \subseteq F_k^m \atop \dim(V)=i} f(V)$$

$$= \frac{1}{|C|} \sum_{u=0}^{i} (-1)^{i-u} q^{mu+(\frac{i-u}{2})} \sum_{V \subseteq F_k^m \atop \dim(V)=i} \sum_{T \supseteq F_k^m \atop \dim(T)=k-u} |C(T)|$$

$$= \frac{1}{|C|} \sum_{u=0}^{i} (-1)^{i-u} q^{mu+(\frac{i-u}{2})} \sum_{V \subseteq F_k^m \atop \dim(V)=i} \sum_{T \supseteq F_k^m \atop \dim(V)=k-u} |C(T)|$$

$$= \frac{1}{|C|} \sum_{u=0}^{i} (-1)^{i-u} q^{mu+(\frac{i-u}{2})} \left[ \begin{array}{c} k-u \\ i-u \end{array} \right]_q \sum_{T \subseteq F_k^m \atop \dim(T)=k-u} |C(T)|. (5)$$

On the other hand,

$$\sum_{T \subseteq F_k^m \atop \dim(T)=k-u} |C(T)| = \sum_{T \subseteq F_k^m \atop \dim(T)=k-u} \sum_{j=0}^{k-u} \sum_{S \subseteq T \atop \dim(S)=j} |\{M \in C \mid \colsp(M) = S\}|$$
\[
\sum_{j=0}^{k-u} \sum_{S \subseteq \mathbb{F}_q^k, \dim(S) = j, T \supseteq S} \sum_{T \subseteq \mathbb{F}_q^k, \dim(T) = k-u} |\{M \in C \mid \text{colsp}(M) = S\}| = \sum_{j=0}^{k-u} \sum_{M \in \mathbb{F}_q^k} \binom{k-j}{u}_q W_j(C).
\]

Combining equations (5) and (6) one obtains the desired result. \hfill \Box

**Example 16.** Let \( q = 2, \ k = 2, \ m = 3 \). Let \( C \subseteq \mathbb{F}_q^{k \times m} \) be the 2-dimensional linear code generated over \( \mathbb{F}_5 \cong \mathbb{Z}/5\mathbb{Z} \) by the matrices
\[
\begin{bmatrix}
1 & 0 & 2 \\
0 & 2 & 4
\end{bmatrix}, \quad
\begin{bmatrix}
2 & 3 & 0 \\
1 & 4 & 0
\end{bmatrix}.
\]

We have \( W_0(C) = 1, \ W_1(C) = 8 \) and \( W_2(C) = 16 \). Applying Theorem 15 one can easily compute \( W_0(C^\perp) = 1, \ W_1(C^\perp) = 65 \) and \( W_2(C^\perp) = 560 \). Observe that \( C^\perp \) has dimension \( 6 - 2 = 4 \), and that \( 1 + 64 + 560 = 625 = 5^4 \), as expected.

We now present a different formulation of the MacWilliams identities for the rank metric. The following result is [12, Theorem 31].

**Theorem 17.** Let \( C \subseteq \mathbb{F}_q^{k \times m} \) be a linear code. For all \( 0 \leq \nu \leq k \) we have
\[
\sum_{i=0}^{k-\nu} W_i(C) \begin{bmatrix}
k - i \\
\nu
\end{bmatrix}_q = \frac{|C|}{q^{m \nu}} \sum_{j=0}^{\nu} W_j(C^\perp) \begin{bmatrix}
k - j \\
\nu - j
\end{bmatrix}_q.
\]

**Proof.** Proposition 13 gives
\[
\sum_{U \subseteq \mathbb{F}_q^k, \dim(U) = k-\nu} |C(U)| = \frac{|C|}{q^{m \nu}} \sum_{U \subseteq \mathbb{F}_q^k, \dim(U) = \nu} |C^\perp(U)|.
\]

Observe that
\[
\sum_{U \subseteq \mathbb{F}_q^k, \dim(U) = k-\nu} |C(U)| = |\{(U, M) \mid U \subseteq \mathbb{F}_q^k, \dim(U) = k-\nu, \ M \in C, \ \text{colsp}(M) \subseteq U\}|
\]
\[
= \sum_{M \in C} |\{U \subseteq \mathbb{F}_q^k, \dim(U) = k-\nu, \ \text{colsp}(M) \subseteq U\}|
\]
\[
= \sum_{i=0}^{k} \sum_{M \in C, \ \text{rk}(M) = i} |\{U \subseteq \mathbb{F}_q^k, \dim(U) = k-\nu, \ \text{colsp}(M) \subseteq U\}|
\]
\[
= \sum_{i=0}^{k} \sum_{M \in C, \ \text{rk}(M) = i} \begin{bmatrix}
k - i \\
\nu - i
\end{bmatrix}_q = \sum_{i=0}^{k-\nu} W_i(C) \begin{bmatrix}
k - i \\
\nu
\end{bmatrix}_q.
\]

Using the same argument with \( C^\perp \) and \( k - \nu \) one shows that
\[
\sum_{U \subseteq \mathbb{F}_q^k, \dim(U) = \nu} |C^\perp(U)| = \sum_{j=0}^{k-\nu} W_j(C^\perp) \begin{bmatrix}
k - j \\
\nu - j
\end{bmatrix}_q.
\]

The result now follows combining equations (7), (8) and (9). \hfill \Box
The two formulations of the MacWilliams identities for the rank metric given in Theorems 10 and 17 are equivalent. See [5, Corollary 1 and Proposition 3] and [12, Theorem 61] for details.

The next theorem is [24, Theorem 27], and shows that the weight distribution of a linear code is determined by its parameters, together with the number of codewords of small weight. We state it without proof. An application of this result will be given in Section 3 (see Corollary 52).

**Theorem 19.** Let \( C \subseteq \mathbb{F}_q^{k \times m} \) be a linear code with \( 1 \leq \dim(C) \leq km - 1 \), minimum distance \( d = d(C) \), and dual minimum distance \( d^\perp = d(C^\perp) \). Let \( \varepsilon = 1 \) if \( C \) is MRD, and \( \varepsilon = 0 \) otherwise. For all \( 1 \leq i \leq d^\perp \) we have

\[
W_{k-d^\perp+i}(C) = (-1)^i q^{\lfloor \frac{i}{t} \rfloor} \sum_{u=d^\perp}^{k-d} \binom{u}{d^\perp-i} q \left[ 1 - u \right]_{q^i} q \left[ 1 - u \right]_{q^i} W_{k-u}(C).
\]

In particular, \( k \), \( m \), \( t \), \( d \), \( d^\perp \) and \( W_d(C), \ldots, W_{k-d^\perp}(C) \) completely determine the weight distribution of \( C \).

We conclude this section showing how MacWilliams identities for the rank metric can be employed to solve certain enumerative problems of matrices over finite fields. The following result is [13, Corollary 52].

**Corollary 20.** Let \( I \subseteq \{ (i, j) \in \{ 1, \ldots, k \} \times \{ 1, \ldots, m \} \mid i = j \} \) be a set of diagonal entries. For all \( 0 \leq r \leq k \) the number of \( k \times m \) matrices \( M \) over \( \mathbb{F}_q \) having rank \( r \) and \( M_{ij} = 0 \) for all \( (i, j) \in I \) is

\[
q^{-|I|} \sum_{t=0}^{k} \binom{|I|}{t} (q-1)^t \sum_{u=0}^{k} (-1)^r u q^{mu + \left( \frac{r-u}{q} \right)} \binom{k-u}{k-r} \binom{k-t}{u}.
\]

**Proof.** Define the linear code \( C = \{ M \in \mathbb{F}_q^{k \times m} \mid M_{ij} = 0 \text{ for all } (i, j) \notin I \} \subseteq \mathbb{F}_q^{k \times m} \). Then \( \dim(C) = |I| \), \( W_t(C) = 0 \) for \( |I| < t \leq k \), and

\[
W_t(C) = \binom{|I|}{t} (q-1)^t
\]

for \( 0 \leq t \leq |I| \). Moreover, \( C^\perp = \{ M \in \mathbb{F}_q^{k \times m} \mid M_{ij} = 0 \text{ for all } (i, j) \in I \} \). Therefore the number of matrices \( M \in \mathbb{F}_q^{k \times m} \) having rank \( r \) and \( M_{ij} = 0 \) for all \( (i, j) \in I \) is precisely \( W_r(C^\perp) \). The corollary now follows from Theorem 15.

## 3 MRD codes

In this section we study rank-metric codes that have the largest possible cardinality for their parameters. We start with a Singleton-type bound for the cardinality of a rank-metric code of given minimum distance. A code is called MRD if it attains the bound. We then show that for any admissible choice of the parameters there exists a linear MRD code with those parameters.

In the second part of the section we study general structural properties of MRD codes. We first prove in Theorem 29 that the dual of a linear MRD code is MRD. Then we show in Theorem 28 that the weight distribution of a possibly non-linear MRD code \( C \subseteq \mathbb{F}_q^{k \times m} \) with \( 0 \notin C \) is determined by \( k \), \( m \) and \( d(C) \). As a corollary, we prove that these three parameters completely determine the distance distribution of any MRD code. Our proofs are inspired by the lattice-theory approach to the weight functions of coding theory proposed in [14] and [13].
Theorem 21 (Singleton-like bound). Let $C \subseteq \mathbb{F}_q^{k \times m}$ be a rank-metric code with $|C| \geq 2$ and minimum distance $d$. Then $|C| \leq q^{m(k-d+1)}$.

Proof. Let $\pi : C \rightarrow \mathbb{F}_q^{(k-d+1) \times m}$ denote the projection on the last $k-d+1$ rows. Since $C$ has minimum distance $d$, the map $\pi$ is injective. Therefore

$$|C| = |\pi(C)| \leq q^{m(k-d+1)}.$$

A code is MRD if its parameters attain the Singleton-like bound.

Definition 22. We say that $C \subseteq \mathbb{F}_q^{k \times m}$ is an MRD code if $|C| = 1$, or $|C| \geq 2$ and $|C| = q^{m(k-d+1)}$, where $d = d(C)$.

We now prove that for any choice of $q$, $k$, $m$ and $d$ there exists a linear rank-metric code $C \subseteq \mathbb{F}_q^{k \times m}$ that attains the bound of Theorem 21. This result was first shown by Delsarte in [11], and rediscovered independently by Gabidulin in [12] and by Kötter and Kschischang in [13] in the context of linear network coding.

Theorem 23. For all $1 \leq d \leq k$ there exists an $\mathbb{F}_q^m$-linear vector rank-metric code $C \subseteq \mathbb{F}_q^{k \times m}$ with $d_C(C) = d$ and $\dim_{\mathbb{F}_q^m}(C) = k - d + 1$. In particular, there exists a linear MRD code $C \subseteq \mathbb{F}_q^{k \times m}$ with $d(C) = d$.

We include an elegant proof for Theorem 23 from [8]. Recall that a linearized polynomial $p$ over $\mathbb{F}_q^m$ is a polynomial of the form

$$p(x) = \alpha_0 x + \alpha_1 x^q + \alpha_2 x^{q^2} + \cdots + \alpha_s x^{q^s}, \quad \alpha_i \in \mathbb{F}_q^m, \quad i = 0, \ldots, s.$$

The degree of $p$, denoted by $\deg(p)$, is the largest integer $i \geq 0$ such that $\alpha_i \neq 0$. The $\mathbb{F}_q^m$-vector space of linearized polynomials over $\mathbb{F}_q^m$ of degree at most $s$ is denoted by $\text{Lin}_q(m, s)$. It is easy to see that $\dim_{\mathbb{F}_q^m}(\text{Lin}_q(m, s)) = s+1$.

Remark 24. The roots of a linearized polynomial $p$ over $\mathbb{F}_q^m$ form an $\mathbb{F}_q$-vector subspace of $\mathbb{F}_q^m$ (see [10], Theorem 3.50), which we denote by $V(p) \subseteq \mathbb{F}_q^m$ in the sequel. Clearly, for any non-zero linearized polynomial $p$ we have $\dim_{\mathbb{F}_q^m} V(p) \leq \deg(p)$ by the Fundamental Theorem of Algebra.

Proof of Theorem 23. Let $E = \{\beta_1, \ldots, \beta_k\} \subseteq \mathbb{F}_q^m$ be a set of $\mathbb{F}_q$-independent elements. These elements exist as $k \leq m$ by assumption. Define the $\mathbb{F}_q^m$-linear map

$$\text{ev}_E : \text{Lin}_q(m, k-d) \rightarrow \mathbb{F}_q^k, \quad \text{ev}_E(p) = (p(\beta_1), \ldots, p(\beta_k)) \text{ for } p \in \text{Lin}_q(m, k-d).$$

We claim that $C = \text{ev}_E(\text{Lin}_q(m, k-d)) \subseteq \mathbb{F}_q^k$ is a vector rank-metric code with the desired properties.

Clearly, $C$ is $\mathbb{F}_q^m$-linear. Now let $p \in \text{Lin}_q(m, k-d)$ be a non-zero linearized polynomial, and let $W \subseteq \mathbb{F}_q^m$ denote the space generated over $\mathbb{F}_q$ by the evaluations $p(\beta_1), \ldots, p(\beta_k)$. The polynomial $p$ induces an $\mathbb{F}_q$-linear evaluation map $p : \langle \beta_1, \ldots, \beta_k \rangle_{\mathbb{F}_q} \rightarrow \mathbb{F}_q^m$. The image of $p$ is $W$, and therefore by the rank-nullity theorem we have $\dim_{\mathbb{F}_q} W = k - \dim_{\mathbb{F}_q} V(p)$. By Remark 23 we conclude $\dim_{\mathbb{F}_q} W \geq k - (k-d) = d$. This shows that $d_C(C) \geq d$. In particular, as $d \geq 1$, the map $\text{ev}_E$ is injective, and the dimension of $C$ is $\dim_{\mathbb{F}_q^m}(C) = k-d+1$. Combining Proposition 8 and Theorem 21 we obtain $d(C) = d$.

The second part of the theorem immediately follows from Proposition 8.

The MRD code construction in the proof of Theorem 23 was later generalized by Sheekey in [14], introducing a new class of MRD codes.

The reminder of the section is devoted to the structural properties of MRD codes. We start with a preliminary result from [14] Chapter 7].
Lemma 25. Let \( \mathcal{C} \subseteq \mathbb{F}_{q}^{k \times m} \) be an MRD code with \( |\mathcal{C}| \geq 2 \) and minimum distance \( d \). For all subspaces \( U \subseteq \mathbb{F}_{q}^{k} \) with \( u = \dim(U) \geq d - 1 \) we have

\[
|\mathcal{C}(U)| = q^{m(u-d+1)}.
\]

Proof. As in Lemma 14 define the space \( V = \{(x_1, ..., x_k) \in \mathbb{F}_{q}^{k} \mid x_i = 0 \text{ for } i > u \} \subseteq \mathbb{F}_{q}^{k} \). Let \( g : \mathbb{F}_{q}^{k} \to \mathbb{F}_{q}^{k} \) be an \( \mathbb{F}_{q} \)-isomorphism with \( f(U) = V \). Denote by \( G \in \mathbb{F}_{q}^{k \times k} \) the matrix associated to \( g \) with respect to the canonical basis of \( \mathbb{F}_{q}^{k} \). Define the rank-metric code \( \mathcal{D} = GC = \{GM \mid M \in \mathcal{C} \} \). Clearly, \( \mathcal{D} \) has the same dimension and minimum distance as \( \mathcal{C} \). In particular, it is MRD. Observe moreover that \( \mathcal{C}(U) = \mathcal{D}(V) \).

Now consider the maps

\[
\mathcal{D} \xrightarrow{\pi_1} \mathbb{F}_{q}^{(k-d+1) \times m} \xrightarrow{\pi_2} \mathbb{F}_{q}^{(k-u) \times m},
\]

where \( \pi_1 \) is the projection on the last \( k - d + 1 \) coordinates, and \( \pi_2 \) is the projection on the last \( k - u \) coordinates. Since \( d(\mathcal{D}) = d \), \( \pi_1 \) is injective. Since \( \mathcal{D} \) is MRD, we have \( \log_{q}(|\mathcal{D}|) = m(k - d + 1) \). Therefore \( \pi_1 \) is bijective. The map \( \pi_2 \) is \( \mathbb{F}_{q} \)-linear and surjective. Therefore

\[
|\pi_2^{-1}(0)| = |\pi_2^{-1}(M)| = q^{m(u-d+1)} \quad \text{for all } M \in \mathbb{F}_{q}^{(k-u) \times m}.
\]

Since \( \pi_1 \) is bijective and \( \pi_2 \) is surjective, the map \( \pi = \pi_2 \circ \pi_1 \) is surjective. Moreover,

\[
|\pi^{-1}(0)| = |\pi^{-1}(M)| = q^{m(u-d+1)} \quad \text{for all } M \in \mathbb{F}_{q}^{(k-u) \times m}.
\]

The lemma now follows from the identity \( \mathcal{C}(U) = \mathcal{D}(V) = \pi^{-1}(0) \).

We can now show that the dual of a linear MRD code is MRD. The next fundamental result is [13, Theorem 5.5].

Theorem 26. Let \( \mathcal{C} \subseteq \mathbb{F}_{q}^{k \times m} \) be a linear MRD code. Then \( \mathcal{C}^\perp \) is MRD.

Proof. The result is immediate if \( \dim(\mathcal{C}) \in \{0, km\} \). Assume \( 1 \leq \dim(\mathcal{C}) \leq km - 1 \), and let \( d = d(\mathcal{C}), d^\perp = d(\mathcal{C}^\perp) \). Applying Theorem 24 to \( \mathcal{C} \) and \( \mathcal{C}^\perp \) we obtain

\[
\dim(\mathcal{C}) \leq m(k - d + 1), \quad \dim(\mathcal{C}^\perp) \leq m(k - d^\perp + 1).
\]

Therefore \( km = \dim(\mathcal{C}) + \dim(\mathcal{C}^\perp) \leq 2mk - m(d + d^\perp) + 2m \), i.e.,

\[
d + d^\perp \leq k + 2. \tag{10}
\]

Let \( U \subseteq \mathbb{F}_{q}^{k} \) be any \( \mathbb{F}_{q} \)-subspace with \( \dim(U) = k - d + 1 \). By Proposition 13 we have

\[
|\mathcal{C}^\perp(U)| = \frac{|\mathcal{C}^\perp|}{q^{|\mathcal{C}^\perp|(d-1)}} |\mathcal{C}(U^\perp)|. \tag{11}
\]

Since \( \dim(U^\perp) = d - 1 \), by Lemma 25 we have \( |\mathcal{C}(U^\perp)| = |\mathcal{C}|/q^{m(k-d+1)} = 1 \), where the last equality follows from the fact that \( \mathcal{C} \) is MRD. Therefore (11) becomes

\[
|\mathcal{C}^\perp(U)| = \frac{|\mathcal{C}^\perp|}{q^{|\mathcal{C}^\perp|(d-1)}} = \frac{q^{km}/q^{m(d-1)}}{q^m} = 1.
\]

Since \( U \) is arbitrary with \( \dim(U) = k - d + 1 \), this shows \( d^\perp \geq k - d + 2 \). Using (10) we conclude \( d^\perp = k - d + 2 \). The theorem now follows from

\[
\dim(\mathcal{C}^\perp) = km - \dim(\mathcal{C}) = km - m(k - d + 1) = m(k - d^\perp + 1).
\]

\[\square\]
The proof of Theorem 26 also shows the following useful characterization of linear MRD codes in terms of their minimum distance and dual minimum distance.

**Proposition 27.** Let $C \subseteq \mathbb{F}_q^{k \times m}$ be a linear code with $1 \leq \dim(C) \leq km - 1$. The following are equivalent.

1. $C$ is MRD,
2. $C^\perp$ is MRD,
3. $d(C) + d(C^\perp) = k + 2$.

In the remainder of the section we concentrate on the weight and distance distributions of (possibly non-linear) MRD codes. We start with a result on the weight distribution of MRD codes containing the zero vector (see [14, Theorem 7.46]).

**Theorem 28.** Let $C$ be an MRD code with $|C| \geq 2$ and $0 \in C$. Let $d = d(C)$. Then $W_0(C) = 1$, $W_i(C) = 0$ for $1 \leq i \leq d - 1$, and

$$W_i(C) = \sum_{u=0}^{d-1} (-1)^{i-u} q^{\binom{i-u}{2}} \sum_{U \subseteq V \atop \dim(U) = u} f(U).$$

for $d \leq i \leq k$.

**Proof.** Since $0 \in C$, we have $W_0(C) = 1$ and $W_i(C) = 0$ for $1 \leq i \leq d - 1$. For all subspaces $V \subseteq \mathbb{F}_q^k$ define

$$f(V) = |\{ M \in C \mid \text{colsp}(M) = V\}|, \quad g(V) = \sum_{U \subseteq V} f(U) = |\mathcal{C}(V)|.$$

Fix $0 \leq i \leq k$ and a vector space $V \subseteq \mathbb{F}_q^k$ of dimension $i$. By Lemma 14 we have

$$f(V) = \sum_{u=0}^{i} (-1)^{i-u} q^{\binom{i-u}{2}} \sum_{U \subseteq V \atop \dim(U) = u} g(U).$$

Using Lemma 25 and the fact that $C$ is MRD with $0 \in C$ we obtain

$$g(U) = \begin{cases} 1 & \text{if } 0 \leq \dim(U) \leq d - 1, \\ q^{m(u-d+1)} & \text{if } d \leq \dim(U) \leq k. \end{cases}$$

Therefore

$$f(V) = \sum_{u=0}^{d-1} (-1)^{i-u} q^{\binom{i-u}{2}} \sum_{U \subseteq V \atop \dim(U) = u} g(U).$$

The result now follows from the identity

$$W_i(C) = \sum_{V \subseteq \mathbb{F}_q^k \atop \dim(V) = i} f(V).$$

Different formulas for the weight distribution of linear MRD codes were obtained in [4] using elementary methods.

Theorem 28 implies the following [3, Theorem 5.6], which states that the distance distribution of any MRD code is determined by its parameters.
Corollary 29. Let $C \subseteq \mathbb{F}_{q}^{k \times m}$ be an MRD code with $|C| \geq 2$ and minimum distance $d$. We have $D_{0}(C) = 1$, $D_{i}(C) = 0$ for $1 \leq i \leq d - 1$, and

$$D_{i}(C) = \sum_{u=0}^{d-1} (-1)^{i-u} q^{\left\lfloor \frac{i}{2} \right\rfloor} \left[ \begin{array}{c} k \\ i \\ u \end{array} \right] q^{\left\lfloor \frac{i}{2} \right\rfloor} + \sum_{u=d}^{\infty} (-1)^{i-u} q^{\left\lfloor \frac{i}{2} \right\rfloor + m(u-d+1)} \left[ \begin{array}{c} k \\ i \\ u \end{array} \right] q^{\left\lfloor \frac{i}{2} \right\rfloor}$$

for $d \leq i \leq k$.

Proof. Fix an $i$ with $d \leq i \leq k$. For $N \in \mathcal{C}$ define $C - N = \{M - N \mid M \in C\}$. By definition of distance distribution we have

$$|C| \cdot D_{i}(C) = \left| \{ (M, N) \in C^{2} \mid \text{rk}(M - N) = i \} \right| = \sum_{N \in C} W_{i}(C - N).$$

For all $N \in \mathcal{C}$ the code $C - N$ is MRD. Moreover, $0 \in C - N$. The result now easily follows from Theorem [28].

Corollary 29 shows in particular that the weight distribution of a linear MRD code is determined by $k$, $m$ and $d(C)$. Recall from Proposition [27] that an MRD code $C \subseteq \mathbb{F}_{q}^{k \times m}$ is characterized by the property $d(C) + d(C^\perp) = k + 2$. We now prove that the weight distribution of a linear code $C$ with $d(C) + d(C^\perp) = k + 1$ is determined by $k$, $m$ and $\dim(C)$. The following result is [29 Corollary 28].

Corollary 30. Let $C \subseteq \mathbb{F}_{q}^{k \times m}$ be a linear rank-metric code with $1 \leq \dim(C) \leq km - 1$ and $d(C) + d(C^\perp) = k + 1$. Then

$$\dim(C) \neq 0 \mod m \quad \text{and} \quad d(C) = k - \left\lceil \dim(C)/m \right\rceil + 1.$$ 

Moreover, for all $d \leq i \leq k$ we have

$$W_{i}(C) = \left[ \begin{array}{c} k \\ i \\ q \end{array} \right] \sum_{u=0}^{i-d(C)} (-1)^{i-u} q^{\left\lfloor \frac{i}{2} \right\rfloor} \left[ \begin{array}{c} i \\ u \end{array} \right] q^{\dim(C)-m(k+u-i) - 1}.$$

Proof. Assume by contradiction that $\dim(C) = \alpha m$ for some $\alpha$. Applying Theorem [21] to $C$ and $C^\perp$ we obtain

$$d(C) \leq k - \alpha + 1, \quad d(C^\perp) \leq \alpha + 1. \quad (12)$$

By Proposition [27] the two inequalities in (12) are either both equalities, or both strict inequalities. Since $d(C) + d(C^\perp) = k + 1$ by assumption, they must be both strict inequalities. Therefore

$$d(C) \leq k - \alpha, \quad d(C^\perp) \leq \alpha,$$

hence $d(C) + d(C^\perp) \leq k$, a contradiction. This shows that $\dim(C) \not\equiv 0 \mod m$.

Now write $\dim(C) = \alpha m + \beta$ with $1 \leq \beta \leq m - 1$. Applying again Theorem [21] to $C$ and $C^\perp$ one finds

$$d(C) \leq k - \left\lceil \frac{\alpha m + \beta}{m} \right\rceil + 1 = k - \alpha, \quad d(C^\perp) \leq k - \left\lceil \frac{km - \alpha m - \beta}{m} \right\rceil = \alpha + 1.$$ 

Since $d(C) + d(C^\perp) = k + 1$, we must have

$$d(C) = k - \left\lceil \frac{\alpha m + \beta}{m} \right\rceil + 1 = k - \left\lceil \frac{\dim(C)}{m} \right\rceil + 1,$$

as claimed. The last part of the statement follows from Theorem [19].
4 Rank-metric anticodes

This section is devoted to rank-metric anticodes, i.e., rank-metric codes in which the distance between any two matrices is bounded from above by a given integer $\delta$.

In Theorem 33 we give a bound for the cardinality of a (possibly non-linear) anticode, using a code-anticode-type bound. We also characterize optimal anticodes in terms of MRD codes. Then we show that the dual of an optimal linear anticode is an optimal linear anticode. The main results of this section appear in [12] and [14].

**Definition 31.** Let $0 \leq \delta \leq k$ be an integer. A (rank-metric) $\delta$-anticode is a non-empty subset $A \subseteq \mathbb{F}_q^{k \times m}$ such that $d(M, N) \leq \delta$ for all $M, N \in A$. We say that $A$ is linear if it is an $\mathbb{F}_q$-linear subspace of $\mathbb{F}_q^{k \times m}$.

**Example 32.** Any $A \subseteq \mathbb{F}_q^{k \times m}$ with $|A| = 1$ is a 0-anticode. The ambient space $\mathbb{F}_q^{k \times m}$ is a $k$-anticode. The vector space of $k \times m$ matrices over $\mathbb{F}_q$ whose last $k - \delta$ rows are zero is a linear $\delta$-anticode of dimension $m\delta$.

In the sequel we work with a fixed integer $0 \leq \delta \leq k$. Moreover, for $A, C \subseteq \mathbb{F}_q^{k \times m}$ we set $A + C = \{M + N \mid M \in A, N \in C\}$.

**Theorem 33.** Let $A \subseteq \mathbb{F}_q^{k \times m}$ be a $\delta$-anticode. Then $|A| \leq q^{m\delta}$. Moreover, if $\delta \leq k - 1$ then the following are equivalent.

1. $|A| = q^{m\delta}$.
2. $A + C = \mathbb{F}_q^{k \times m}$ for some MRD code $C$ with $d(C) = \delta + 1$.
3. $A + C = \mathbb{F}_q^{k \times m}$ for all MRD codes $C$ with $d(C) = \delta + 1$.

**Proof.** Let $C \subseteq \mathbb{F}_q^{k \times m}$ be any MRD code with $d(C) = \delta + 1$. Such a code exists by Theorem 28.

For all $M \in A$ let $|M| = M + C = \{M + N \mid N \in C\}$. Then $|M| \cap |M'| = \emptyset$ for all $M, M' \in A$ with $M \neq M'$. Moreover, by definition of MRD code we have $||M|| = |C| = q^{m(k-\delta)}$ for all $M \in A$, hence

$$|\mathbb{F}_q^{k \times m}| \geq \left| \bigcup_{M \in A} |M| \right| = \sum_{M \in A} ||M|| = |A| \cdot |C| = |A| \cdot q^{m(k-\delta)}.$$ 

Therefore $|A| \leq q^{m\delta}$, and equality holds if and only if

$$\mathbb{F}_q^{k \times m} = \bigcup_{M \in A} |M| = A + C.$$

A similar argument shows that properties 1, 2, and 3 are equivalent. \[\square\]

**Definition 34.** We say that a $\delta$-anticode $A$ is (cardinality)-optimal if it attains the bound of Theorem 33.

**Remark 35.** Example 32 shows the existence of optimal linear $\delta$-antcodes for all choices of the parameter $\delta$.

In the remainder of the section we prove that the dual of an optimal linear $\delta$-anticode is an optimal $(k - \delta)$-anticode. The result may be regarded as the analogue of Theorem 26 in the context of rank-metric anticodes. We start with a preliminary result on the weight distribution of MRD codes.

**Lemma 36.** Let $C \subseteq \mathbb{F}_q^{k \times m}$ be an MRD code with $0 \in C$, $|C| \geq 2$ and $d(C) = d$. Then $W_{d+\ell}(C) > 0$ for all $0 \leq \ell \leq k - d$. 


Proof. By Theorem 28, we shall prove the lemma for a given MRD code $C \subseteq \mathbb{F}_{q}^{k \times m}$ of our choice with $|C| \geq 2$, minimum distance $d$, and $0 \in C$. We will first produce a convenient MRD code with the prescribed properties.

Let $C \subseteq \mathbb{F}_{q}^{k \times m}$ be the vector rank-metric code constructed in the proof of Theorem 28 with evaluation set $E = \{\beta_1, \ldots, \beta_k\}$ and evaluation map $ev_E$. Let $\Gamma$ be a basis of $\mathbb{F}_{q}^{m}$ over $\mathbb{F}_{q}$. By Proposition 8, the set $C = \Gamma(C) \subseteq \mathbb{F}_{q}^{k \times m}$ is a linear code with $\dim(C) = m(k - d + 1)$ and the same weight distribution as $C$. In particular, $C$ is a non-zero linear MRD code of minimum distance $d$.

Now we prove the lemma for the MRD code $C$ constructed above. Fix $\ell$ with $0 \leq \ell \leq k - d$. Define $t = k - d - \ell$, and let $U \subseteq \mathbb{F}_{q}^{m}$ be the $\mathbb{F}_{q}$-subspace generated by $\{\beta_1, \ldots, \beta_t\}$. If $t = 0$ we set $U$ to be the zero space. By 10, Theorem 3.32,

$$pu = \prod_{\gamma \in U} (x - \gamma)$$

is a linearized polynomial over $\mathbb{F}_{q^{m}}$ of degree $t = k - d - \ell \leq k - d$, i.e., $pu \in \text{Lin}_{\mathbb{F}_{q}}(n, k - d)$. Therefore by Proposition 8 it suffices to show that $ev_E(pu) = (pu(\beta_1), \ldots, pu(\beta_k))$ has rank $d + \ell = k - t$. Clearly, $V(pu) = U$. In particular we have $ev_E(pu) = (0, \ldots, 0, pu(\beta_{t+1}), \ldots, pu(\beta_k))$. We will show that $pu(\beta_{t+1}), \ldots, pu(\beta_k)$ are linearly independent over $\mathbb{F}_{q}$. Assume that there exist $a_{t+1}, \ldots, a_k \in \mathbb{F}_{q}$ with $\sum_{i=t+1}^{k} a_i pu(\beta_i) = 0$. Then we have $pu \left( \sum_{i=t+1}^{k} a_i \beta_i \right) = 0$, i.e.,

$$\sum_{i=t+1}^{k} a_i \beta_i \in V(pu) = U. \text{ It follows that there exist } a_1, \ldots, a_t \in \mathbb{F}_{q} \text{ such that } \sum_{i=1}^{t} a_i \beta_i = \sum_{i=t+1}^{k} a_i \beta_i, \text{ i.e., } \sum_{i=1}^{t} a_i \beta_i - \sum_{i=t+1}^{k} a_i \beta_i = 0. \text{ Since } \beta_1, \ldots, \beta_k \text{ are independent over } \mathbb{F}_{q}, \text{ we have } a_i = 0 \text{ for all } i = 1, \ldots, k. \text{ In particular } a_i = 0 \text{ for } i = t + 1, \ldots, k. \text{ Hence } pu(\beta_{t+1}), \ldots, pu(\beta_k) \text{ are linearly independent over } \mathbb{F}_{q^m}, \text{ as claimed.}$

The following proposition characterizes optimal linear anticodes in terms of their intersection with linear MRD codes.

**Proposition 37.** Assume $0 \leq \delta \leq k - 1$, and let $A \subseteq \mathbb{F}_{q}^{k \times m}$ be a linear code with $\dim(C) = m\delta$. The following are equivalent.

1. $A$ is an optimal $\delta$-anticode.
2. $A \cap C = \{0\}$ for all non-zero MRD linear codes $C \subseteq \mathbb{F}_{q}^{k \times m}$ with $d(C) = \delta + 1$.

**Proof.** By Theorem 28, it suffices to show that if $A \cap C = \{0\}$ for all non-zero MRD linear codes $C \subseteq \mathbb{F}_{q}^{k \times m}$ with $d(C) = \delta + 1$, then $A$ is a $\delta$-anticode.

By contradiction, assume that $A$ is not a $\delta$-anticode. Since $A$ is linear, by definition of $\delta$-anticode there exists $N \in C$ with $\text{rk}(N) \geq \delta + 1$. Let $D$ be a non-zero linear MRD code with $d(D) = \delta + 1$ (see Theorem 28 for the existence of such a code). By Lemma 36 there exists $M \in D$ with $\text{rk}(M) = \text{rk}(N)$. There exist invertible matrices $A$ and $B$ of size $k \times k$ and $m \times m$, resp., such that $N = AMB$. Define $C = ADB = \{APB \mid P \in D\}$. Then $C \subseteq \mathbb{F}_{q}^{k \times m}$ is a non-zero linear MRD code with $d(C) = \delta + 1$ and such that $N \in A \cap C$. Since $\text{rk}(N) \geq \delta + 1 \geq 1$, $N$ is not the zero matrix. Therefore $A \cap C \neq \{0\}$, a contradiction.

We conclude the section showing that the dual of an optimal linear anticode is an optimal linear anticode.

**Theorem 38.** Let $A \subseteq \mathbb{F}_{q}^{k \times m}$ be an optimal linear $\delta$-anticode. Then $A^\perp$ is an optimal linear $(k - \delta)$-anticode.

**Proof.** Let $A \subseteq \mathbb{F}_{q}^{k \times m}$ be an optimal linear $\delta$-anticode. If $\delta = k$ then the result is trivial. From now on we assume $0 \leq \delta \leq k - 1$. By Definition 34 we have $\dim(A) = m\delta$, hence $\dim(A^\perp) = m(k - \delta)$. Therefore by Proposition 8 it suffices to show that $A^\perp \cap C = \{0\}$ for all non-zero linear MRD codes $C \subseteq \mathbb{F}_{q}^{k \times m}$ with $d(C) = k - \delta + 1$. Let $C$ be such a code. Then

$$\dim(C) = m(k - (k - \delta + 1) + 1) = m\delta < mk.$$
Combining Theorem 26 and Proposition 27 one shows that $C^\perp$ is a linear MRD code with $d(C^\perp) = k - (k - \delta + 1) + 2 = \delta + 1$. By Proposition 37 we have $\mathcal{A} \cap C^\perp = \{0\}$. Since $\dim(\mathcal{A}) + \dim(C^\perp) = m\delta + m(k - (\delta + 1) + 1) = mk$, we have $\mathcal{A} \oplus C^\perp = \mathbb{F}_q^{k \times m}$. Therefore $\{0\} = (\mathbb{F}_q^{k \times m})^\perp = (\mathcal{A} \oplus C^\perp)^\perp = \mathcal{A}^\perp \cap C$. This shows the theorem.

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