Differential Calculi on Quantum (Sub-) Groups and Their Classical Limit

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Abstract

For the two-parameter matrix quantum group $GL_{p,q}(2)$ all bicovariant differential calculi (with a four-dimensional space of 1-forms) are known. They form a one-parameter family. Here, we give an improved presentation of previous results by using a different parametrization. We also discuss different ways to obtain bicovariant calculi on the quantum subgroup $SL_q(2)$. For those calculi, we do not obtain the ordinary differential calculus on $SL(2)$ in the classical limit. The structure which emerges here can be generalized to a nonstandard differential calculus on an arbitrary differentiable manifold and exhibits relations with stochastic calculus and 'proper time' relativistic quantum theories.

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1 Introduction

Differential geometry of Lie groups plays an important role in the mathematical modelling of physical theories. In particular, this is the case for classical gauge theories formulated in terms of connections on principal fiber bundles. Since a Hopf algebra or quantum group can be regarded as a generalization of the notion of a group, it is tempting to also generalize the corresponding notions of differential geometry (see [1], in particular). Besides promising mathematical aspects of such a generalization, there is a hope to obtain interesting ‘deformations’ of physical models, like the gauge theory models of elementary particle physics.

More generally, a notion of differential forms has been introduced for an arbitrary associative algebra $A$. One can enlarge $A$ to a differential algebra. This is a $\mathbb{Z}$-graded associative algebra $\wedge(A) = \bigoplus_{r \geq 0} \wedge^r(A)$ where $\wedge^0 = A$ and the spaces $\wedge^r(A)$ of $r$-forms are generated as $A$-bimodules via the action of an exterior derivative $d : \wedge^r(A) \to \wedge^{r+1}(A)$. The latter is a linear operator acting in such a way that $d^2 = 0$ and $d(\omega \omega') = (d\omega) \omega' + (-1)^r \omega d\omega'$ where $\omega$ and $\omega'$ are $r$- and $r'$-forms, respectively. There are many differential algebras associated with an algebra $A$. But all of them can be obtained as a quotient of a maximal differential algebra, the universal differential envelope, by some ideal. In particular, if $A$ is the algebra of polynomials of $n$ independent elements, we might want the associated space of 1-forms to be $n$-dimensional as a left (or right) $A$-module. This does not restrict the possible differential algebras very much, however. In general, there seems not to be a kind of functorial way to associate such a differential algebra with a given algebra $A$. On the other hand, it turned out that different choices of differential algebras are actually of interest from a mathematical and physical point of view (cf the examples in [3, 4, 5, 6]).

For the case of matrix quantum groups, Woronowicz introduced the notions of left-, right- and bi-covariant differential calculus [7]. Bicovariance was soon accepted as the most natural condition for a differential algebra. Woronowicz gave two examples of bi-covariant differential algebras on $SU_q(2)$ (the so-called 4D-calculi) [7]. At that time, it was not known how many bicovariant differential calculi exist on $SU_q(2)$ (and other quantum groups). Later, it turned out that Woronowicz already found all bicovariant calculi on $SU_q(2)$ [8, 9]. In the meantime, a large number of papers appeared dealing with examples of bicovariant differential calculi on special (classes of) quantum groups (see [1] for an extensive list of references). However, according to our knowledge there are only few papers which go beyond examples and give a complete description of the possible bicovariant differential calculi on certain quantum groups [3, 10, 11]. In [10] all bicovariant differential calculi on the two-parameter quantum group $GL_{p,q}(2)$ were found (see also [8]). They form a family which depends on an additional parameter $s$ ($q, p$ and $s$ are complex numbers). In section 2, this family and its classical limit is described using a simplifying parametrization which greatly improves the presentation in [3, 10]. In section 3, we consider two ways in which the family of bicovariant calculi on $GL_{p,q}(2)$ induces corresponding calculi on the quantum subgroup $SL_q(2)$. Also the classical limit of bicovariant calculus on $SL_q(2)$ is discussed. Here we are led to a generalization [3, 4, 6] of the ordinary calculus of differential forms on a manifold on which we comment in section 4.
2 Bicovariant differential calculi on $GL_{p,q}(2)$

2.1 The quantum general linear group in two dimensions. We recall that the quantum group $GL_{p,q}(2)$ is the Hopf algebra $A$ generated by $a, b, c, d$ and the unit $\mathbb{I}$, satisfying the commutation relations

\begin{align}
ac &= qca \\
bd &= qdb \\
cd &= pdc \\
bc &= (q/p)cb \\
b &= pba
\end{align}

where $p, q \in \mathbb{C} \setminus \{0\}$. The existence of an inverse $D^{-1}$ of the ‘quantum determinant’ $D := ad - pbc$ is required. The coproduct is the homomorphism determined by

$$\Delta \left( \begin{array}{cc}
a & b \\
c & d
\end{array} \right) = \left( \begin{array}{cc}
a & b \\
c & d
\end{array} \right) \otimes \left( \begin{array}{cc}
a & b \\
c & d
\end{array} \right) = \left( \begin{array}{cc}
a \otimes a + b \otimes c & a \otimes b + b \otimes d \\
c \otimes a + d \otimes c & c \otimes b + d \otimes d
\end{array} \right)$$

and the antipode is the anti-homomorphism $S : A \to A$ with

$$S \left( \begin{array}{cc}
a & b \\
c & d
\end{array} \right) = D^{-1} \left( \begin{array}{cc}
d & -b/q \\
-qc & a
\end{array} \right).$$

In addition, $\Delta(\mathbb{I}) = \mathbb{I} \otimes \mathbb{I}$ and $S(\mathbb{I}) = \mathbb{I}$.

2.2 Bicovariant differential calculus. The central object of first order differential calculus is the exterior derivative $d : A \to \Lambda^1(A)$ satisfying the Leibniz rule $d(fh) = (df)h + f dh$ ($\forall f, h \in A$). The space of 1-forms $\Lambda^1(A)$ is generated as an $A$-bimodule by the differentials of $a, b, c, d$. It is furthermore required that the differentials of $a, b, c, d$ form a basis of $\Lambda^1(A)$ as a left $A$-module. In order to achieve this, one has to find commutation relations between $a, b, c, d$ and their differentials which are consistent with the differential algebra structure.

A left-coaction $\Delta_L : \Lambda^1(A) \to A \otimes \Lambda^1(A)$ extends $\Delta$ as a bimodule homomorphism to 1-forms such that

$$\Delta_L \left( \begin{array}{cc}
da & db \\
dc & dd
\end{array} \right) = \left( \begin{array}{cc}
a & b \\
c & d
\end{array} \right) \otimes \left( \begin{array}{cc}
da & db \\
dc & dd
\end{array} \right).$$

In the same way a right-coaction $\Delta_R : \Lambda^1(A) \to \Lambda^1(A) \otimes A$ is a bimodule homomorphism with

$$\Delta_R \left( \begin{array}{cc}
da & db \\
dc & dd
\end{array} \right) = \left( \begin{array}{cc}
da & db \\
dc & dd
\end{array} \right) \otimes \left( \begin{array}{cc}
a & b \\
c & d
\end{array} \right).$$

If $\Delta_L$ and $\Delta_R$ exist, the (first order) differential calculus is called bicovariant [7].

Assuming the existence of $\Delta_L$, there is a basis of (left-coinvariant) Maurer-Cartan 1-forms $\theta^K$ in $\Lambda^1(A)$ given by

$$\left( \begin{array}{c}
\theta^1 \\
\theta^2 \\
\theta^3 \\
\theta^4
\end{array} \right) = S \left( \begin{array}{cc}
a & b \\
c & d
\end{array} \right) d \left( \begin{array}{cc}
a & b \\
c & d
\end{array} \right).$$

\footnote{Here and in the following we use a compact notation for $\Delta(a) = a \otimes a + b \otimes c$ etc.}
Commutation relations between the generators of $\mathcal{A}$ and their differentials can be expressed in terms of the Maurer-Cartan 1-forms,

$$\theta^K f = \Theta(f)^K_L \theta^L \quad (\forall f \in \mathcal{A}).$$

(2.7)

Compatibility with $\Delta_L$ leads to

$$\Theta \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \left( \begin{array}{cc} A & B \\ C & D \end{array} \right)$$

(2.8)

where $A, B, C, D$ are 4 × 4 matrices (with complex entries). Associativity of $\Lambda^1(\mathcal{A})$ and (2.7) require $\Theta(fh) = \Theta(f) \Theta(h)$ which means that $A, B, C, D$ have to form a representation of $a, b, c, d$. (2.7) and (2.8) imply

$$\theta^K a = (a A^K_L + b C^K_L) \theta^L, \quad \theta^K b = (a B^K_L + b D^K_L) \theta^L$$

(2.9)

and the corresponding relations with $a$ replaced by $c$ and $b$ replaced by $d$. The consistency conditions for first order bicovariant differential calculus [9] were completely solved for $GL_{p,q}(2)$ in [10] using computer algebra (see also [12]). We found that there is a one-parameter set of such calculi.

**Theorem 10**

Let $r := pq \neq 0, \pm 1$ and $t \in \mathcal{C}$, $t \neq 0$ and $t \neq r(1 + r)/(1 + r^2)$. ² All bicovariant first order differential calculi on $GL_{p,q}(2)$ – for which the differentials of $a, b, c, d$ form a basis of $\Lambda^1(\mathcal{A})$ as a left $\mathcal{A}$-module – are given by

$$A = \begin{pmatrix} A_1 & 0 & 0 & A_4 \\ 0 & t/p & 0 & 0 \\ 0 & 0 & t/q & 0 \\ A_4 & 0 & 0 & 1 - r A_4 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & t - 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ B_1 & 0 & 0 & B_4 \\ 0 & t/r - 1 & 0 & 0 \end{pmatrix}$$

$$C = \begin{pmatrix} 0 & 0 & (q/p)B_1 & 0 \\ 0 & 0 & 0 & (q/p)B_4 \\ (q/p)B_1 & 0 & 0 & 0 \\ 0 & 0 & t/r - 1 & 0 \end{pmatrix}, \quad D = \begin{pmatrix} 1 - A_4 & 0 & 0 & D_4 \\ 0 & t/p & 0 & 0 \\ 0 & 0 & t/q & 0 \\ r A_4 & 0 & 0 & D_4 \end{pmatrix}$$

(2.10)

where

$$A_1 = \frac{r^2(r t - 1)(t - 1) + r t (t - 2) + t^2}{(r^3 N)}, \quad A_4 = \frac{(r - t)(t - 1)}{(r^2 N)}$$

²The last condition ensures that $N \neq 0$. In the case excluded by this condition, there are no calculi when $r \neq \pm 1$ [9]. $t = 0$ has to be excluded because in that case one finds $\theta^K D = 0$ which conflicts with the existence of $D^{-1}$. We may admit $r = 1$ in the theorem, but in that case additional calculi exist [10].

³Here we use a different parametrization as in [9, 10]. The reason is that writing $A_4 = t (1 + r)/r + r A_4 - 1$ with a new parameter $t$, the quadratic relation between $A_1^2$ and $A_4$ which was obtained in [10] (equation (6.25) therein) simply becomes the expression for $A_4$ in (2.11). $A_4$ is the complex parameter $s$ in [9, 10]. $t$ is the parameter $s$ in [13].
\[A^1_4 = (t - r)(r^2 - t^2 - r^2 + t)/(r^3 N)\]
\[B^1_3 = t(r - t)(r - 1)/(q r^2 N)\]
\[B^2_3 = t p(r - 1)(t - 1)/(r^2 N)\]
\[D^1_4 = (t - 1)(r^2 t - r^2 - r t + t)/(r^2 N)\]
\[D^4_1 = [r^3(t - 1)^2 + r^3 t(t - 1) - r t + t^2]/(r^3 N)\]  \hspace{1cm} (2.11)

with \(N := [t(1 + r^2) - r(1 + r)]/r^2\).

In terms of the differentials, the commutation relations \((2.7)\) for the bicovariant differential calculi are not quadratic relations if \(t \neq 1, r\). For example,
\[d a a = (A^1_4 + A^1_4 p r^2 D^{-1} b c) a da - A^1_4 D^{-1} a^2 (q c db + p b dc - a dd)\]  \hspace{1cm} (2.12)

The differential of an element \(f \in \mathcal{A}\) can be expressed as \([7, 10]\)
\[d f = \frac{N}{N} [\vartheta, f]\]  \hspace{1cm} (2.13)

where \(N\) is defined in the theorem and
\[\vartheta := \theta^1 + \frac{1}{r} \theta^4\]  \hspace{1cm} (2.14)

is a bi-coinvariant 1-form. Bicovariant first order differential calculi always admit an extension to higher orders \([4]\). Differential forms of higher order are obtained by applying \(d\) to 1-forms (and then also higher forms) using \(d^2 = 0\) and the graded Leibniz rule. Bicovariance guarantees that there are commutation relations between the 1-forms which are compatible with these structures. \((2.13)\) then holds more generally with \(f\) replaced by any form if the commutator is replaced by a graded commutator \([7]\). We refer to \([11]\) for the corresponding results in the case of \(GL_{pq}(2)\).

2.3 An R-matrix formulation. In terms of the new basis of left-coinvariant 1-forms
\[\omega^1_1 = (p/r^2 N t) [(r - t) \theta^1 + r (t - 1) \theta^4]\]
\[\omega^2_2 = (p/r^2 N t) [(t^2 r - r + 1) - r) \theta^1 + (r - t) \theta^4]\]
\[\omega^1_2 = -(p/q t) \theta^2\]
\[\omega^2_1 = -(1/t) \theta^3\]  \hspace{1cm} (2.15)

the commutation relations with elements of \(\mathcal{A}\) are given by
\[\omega^1_1 a = (t/r) a \omega^1_1\]
\[\omega^1_2 a = t [p^{-1} a \omega^1_2 + r^{-1}(1 - r) b \omega^1_1]\]
\[\omega^2_1 a = (t/q) a \omega^2_1\]
\[\omega^2_2 a = t [a \omega^2_2 + q^{-1}(1 - r) b \omega^2_1]\]
\[\omega^1_1 b = t b \omega^1_1\]
\[\omega^1_2 b = (t/p) b \omega^1_2\]
\[\omega^2_1 b = t [q^{-1} b \omega^2_1 + r^{-1}(1 - r) a \omega^1_1]\]
\[\omega^2_2 b = (t/r) [b \omega^2_2 + (r - 1)^2 b \omega^1_1 + q (1 - r) a \omega^1_2].\]  \hspace{1cm} (2.16)

For \(p = q\), these relations can be found in \([13]\). In terms of the 1-forms \(\omega^i_j, \ i, j = 1, 2\), the commutation relations look much simpler than the corresponding relations with \(\theta^K\). In particular, the parameter \(t\) only appears as a common factor on the right hand sides of \((2.16)\). However, one has to keep in mind that the 1-forms \(\omega^i_j\) – when expressed in terms
of the differentials of $a, b, c, d$ or the Maurer-Cartan 1-forms – depend on $t$ (and $p, q$) in a rather complicated way. The relations (2.16) can be expressed in terms of the $R$-matrix of $GL_{p,q}(2)$ as follows$^4$

$$\omega^i_j T^k_\ell = t (q/p) T^k_m (R^{-1})^{mn} u_{ij} (R^{-1})^{iu}_{\nu \ell} \omega^\nu_n$$

(2.17)

where $T$ is the matrix with entries $a, b, c, d$ and

$$R^{-1} = \begin{pmatrix}
q^{-1} & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & q^{-1} - p & p/q & 0 \\
0 & 0 & 0 & q^{-1}
\end{pmatrix}.$$  

(2.18)

Rows and columns of the matrix are numbered by $(1,1)$, $(1,2)$, $(2,1)$, $(2,2)$. To this expression for the bicovariant differential calculi one is led by applying the recipe of [15] (based on the techniques of [16]) with the slight generalization given in [17, 13]. It is interesting that this procedure already exhausts the possible bicovariant calculi. This also holds (with a further refinement) for the quantum group $GL_q(3)$ for which all bicovariant differential calculi have recently been obtained [18] using the methods of [10]. Also in this case we have a one-parameter family and half of it was already found in [14]. This suggests that, more generally, on $GL_q(n)$ ($n \geq 2$) the bicovariant calculi form a one-parameter set. There are indeed partial results [19] substantiating this conjecture.

2.4 The classical limit. In terms of $x^1 := a$, $x^2 := b$, $x^3 := c$, $x^4 := d$, the commutation relations between $x^\mu$ and $dx^\nu$ (cf (2.12) and [9]) take the following form in the classical limit $p, q \to 1$,

$$[x^\mu, dx^\nu] = \tau g^{\mu\nu}$$

(2.19)

with

$$\tau := -s \vartheta = s [dx^4 x^4 - dx^2 x^3 - dx^3 x^2 + dx^4 x^1] =: dx^\mu \tau_\mu$$

(2.20)

$$g^{\mu\nu} := (x^1 x^4 - x^2 x^3)^{-1} x^\mu x^\nu + 4 [\delta_2(\mu, \nu) - \delta_1(\mu, \delta_4(\nu))]$$

(2.21)

where indices in brackets are symmetrized. Here we have

$$s = (1 - t)/2$$

(2.22)

for $t \neq 1$ (cf the assumptions in the theorem) if we regard $t$ as a parameter which does not depend on $p$ or $q$ (otherwise the limit will depend on the choice of $t$ as a function of $p$ and $q$, cf section 3.3). The matrix $g$ is degenerate since $g^{\mu\nu} \tau_\nu = 0$ which reminds us of a ‘Galilei structure’ (see also [8], appendix B). The 1-form $\tau$ commutes with $x^\mu$, $\mu = 1, \ldots, 4$, anticommutes with all 1-forms and satisfies $d\tau = 0$.

$^4$See also [14] for the case of $GL_q(3)$. 

6
3 From differential calculus on $GL_q(2)$ to differential calculus on $SL_q(2)$

In this section we restrict the deformation parameters of $GL_{p,q}(2)$ by $p = q$. The quantum group is then called $GL_q(2)$. In this case the quantum determinant $\mathcal{D}$ (see section 2.1) becomes central, i.e. it commutes with all elements of $\mathcal{A}$. The condition

$$\mathcal{D} = a d - q b c = 1$$

then defines the quantum subgroup $SL_q(2)$. In the following it is shown that there are two different ways to obtain bicovariant differential calculi on $SL_q(2)$ from the family of bicovariant calculi on $GL_q(2)$ (see also [9]).

3.1 The direct way. Imposing the condition (3.1) on the family of bicovariant differential algebras on $GL_q(2)$ requires that all 1-forms commute with $\mathcal{D}$. This means that

$$1 = A D - q B C = (t/q)^2 1$$

(where $1$ is the $4 \times 4$ unit matrix) and restricts the parameter $t$ to the values

$$t_{\pm} = \pm q$$

(3.3)

For the general bicovariant calculus on $GL_q(2)$ one finds

$$d \mathcal{D} = -\frac{(t-q)(t+q)}{q^2 N} \mathcal{D} \vartheta$$

(3.4)

with $N$ and $\vartheta$ defined in section 2. Differentiation of (3.1) leads to the constraint $d \mathcal{D} = 0$ which is identically satisfied when $t = t_{\pm}$. Hence, there are two bicovariant differential calculi on $GL_q(2)$ which are consistent with the constraint (3.1):

(1) For $t = t_+$ the matrices $A, B, C, D$ take the following form.

$$A = \begin{pmatrix} q^4 + q^3 + q^2 + 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -q^4 - q^3 - q^2 - q - 1 & 0 & 0 & q^2 + q + 1 \end{pmatrix}$$

$$B = \begin{pmatrix} 0 & q - 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{q + 1}{q^2 + q + 1} & 0 & 0 & \frac{q(q+1)}{q^2 + q + 1} \\ 0 & \frac{1}{q} & 1 & 0 & 0 \end{pmatrix}$$

(3.5)

$$C = \begin{pmatrix} 0 & 0 & q - 1 & 0 \\ \frac{q+1}{q^2 + q + 1} & 0 & 0 & \frac{q(q+1)}{q^2 + q + 1} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$D = \begin{pmatrix} \frac{q(q+1)}{q^2 + q + 1} & 0 & 0 & \frac{q^2 - q - 1}{q^2 + q + 1} \\ \frac{q+1}{q^2 + q + 1} & 0 & 0 & \frac{q(q+1)}{q^2 + q + 1} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{q^2}{q^2 + q + 1} & 0 & 0 & \frac{q^2 + q^2 + q + 1}{q(q^2 + q + 1)} \end{pmatrix}$$

The factor $\mathcal{D}$ is missing on the rhs of (4.7) in [9].
Theorem 3.1
Let \( q \neq 0, \pm 1, \pm i \). The \( t_\pm \) calculi (with \( q^2 \pm q + 1 \neq 0 \)) are the only bicovariant differential calculi on \( \text{SL}_q(2) \).

The two calculi on \( \text{SL}_q(2) \) induce the \( 4D_\pm \) calculi \( \[7\] \) on \( \text{SU}_q(2) \). The uniqueness of the latter has been shown in \( \[8\] \).

3.2 An indirect way. There is another simple way to obtain a differential calculus on \( \text{SL}_q(2) \) from a calculus on \( \text{GL}_q(2) \). For the special differential calculus with \( t = 1 \) it has been considered in \( \[20\] \). Let \( T \) denote the matrix with entries \( a, b, c, d \) satisfying the \( \text{GL}_q(2) \) commutation relations. Furthermore, let us assume that \( D^{-1/2} \) exists and commutes with all elements of \( \text{GL}_q(2) \) (note that \( D \) is central). Then

\[
\theta^K D^{-1/2} = \pm(q/t) D^{-1/2} \theta^K
\]

(for \( t \neq 0 \)). The entries of

\[
\hat{T} := D^{-1/2} T =: \begin{pmatrix} \hat{a} & \hat{b} \\ \hat{c} & \hat{d} \end{pmatrix}
\]

satisfy the \( \text{GL}_q(2) \) commutation relations and furthermore \( \hat{D} = \det_q \hat{T} = 1 \). They generate \( \text{SL}_q(2) \) as a subalgebra of \( \text{GL}_q(2) \) and the differential calculus can be restricted to it. We can introduce corresponding Maurer-Cartan 1-forms

\[
\begin{pmatrix} \hat{\theta}^1 & \hat{\theta}^2 & \hat{\theta}^3 & \hat{\theta}^4 \end{pmatrix} := S(\hat{T}) d\hat{T} = \pm(q/t) \begin{pmatrix} \theta^1 & \theta^2 & \theta^3 & \theta^4 \end{pmatrix} + \frac{1}{N} (\pm q/t - 1) \begin{pmatrix} \varphi & 0 \\ 0 & \vartheta \end{pmatrix}
\]

To derive the last expression, we made use of \( \[2.13\] \) and \( \[3.4\] \). It allows us to calculate commutation relations between the 1-forms \( \hat{\theta}^K \) and the entries of \( T \) from the corresponding commutation relations of a bicovariant differential calculus on \( \text{GL}_q(2) \). In this way, each bicovariant differential calculus on \( \text{GL}_q(2) \) induces two corresponding bicovariant differential calculi on the subalgebra \( \text{SL}_q(2) \). In accordance with the theorem in section 3.1,
the latter do not dependent on the value of \( t \). More generally, one obtains the following result about the structure of the bicovariant calculi on \( GL_q(2) \).

**Theorem**

Let \( q \neq 0, \pm 1, \pm i, \pm 2, \pm 3 \) and \( t \neq 0 \) and \( t \neq q^2(1 + q^2)/(1 + q^4) \). In terms of the \( (SL_q(2) \) Maurer-Cartan) 1-forms \( \theta \) and the algebra elements \( \hat{a}, \hat{b}, \hat{c}, \hat{d}, \hat{D} \) all bicovariant differential calculi on \( GL_q(2) \) are determined by

\[
\begin{align*}
\hat{\theta}^K \hat{a} &= (\hat{a} A^K_L + \hat{b} C^K_L) \hat{\theta}^L, \\
\hat{\theta}^K \hat{b} &= (\hat{a} B^K_L + \hat{b} D^K_L) \hat{\theta}^L, \\
\hat{\theta}^K \hat{D}^{1/2} &= \pm (t/q) \hat{D}^{1/2} \hat{\theta}^K
\end{align*}
\]

and the first two relations with \( \hat{a}, \hat{b} \) replaced by \( \hat{c}, \hat{d} \), respectively. For the plus sign in the last equation the matrices \( A, B, C, D \) are now given by \( (3.3) \). In case of the minus sign they are given by \( (3.6) \).

3.3 The classical limit. For the \( t_+ \) calculus we obtain in particular \( \theta^K a = -a \theta^K \) for \( K = 2, 3 \) when \( q = 1 \) which is far away from the ordinary differential calculus on \( SL(2) \). Let us therefore turn to the \( t_+ \) calculus. For \( q = 1 \) we obtain

\[
\begin{pmatrix}
\theta^1 \\
\theta^2 \\
\theta^3 \\
\theta^4
\end{pmatrix}
= \begin{pmatrix}
\theta^1 + \frac{1}{3} \vartheta \\
\theta^2 \\
\theta^3 \\
\theta^4 - \frac{1}{3} \vartheta
\end{pmatrix} + \begin{pmatrix}
0 \\
0 \\
0 \\
0
\end{pmatrix}
\]

(3.10)

and

\[
\begin{pmatrix}
\theta^1 \\
\theta^2 \\
\theta^3 \\
\theta^4
\end{pmatrix}
= \begin{pmatrix}
\theta^1 - \frac{1}{3} \vartheta \\
\theta^2 \\
\theta^3 \\
\theta^4 + \frac{1}{3} \vartheta
\end{pmatrix} + \begin{pmatrix}
0 \\
0 \\
0 \\
\frac{2}{3} \vartheta
\end{pmatrix}
\]

(3.11)

where now \( \vartheta = \theta^1 + \theta^4 \). In terms of \( x^\mu \) (see section 2.4), these relations can be expressed in the form \( (2.19) \) with \( (2.20) \) and \( (2.21) \) where now \( s = 1/3 \) and the differential calculus on \( SL(2) \) is not the ordinary one.\footnote{The parameter \( t \) only enters the last relation. It appears, however, implicitly in the relation between the \( GL_q(2) \) and the \( SL_q(2) \) Maurer-Cartan forms.}

One of the 'coordinates' \( x^\mu \) is redundant because of the constraint \( \mathcal{D} = 1 \). Let us consider the subalgebra generated by only three of them, say \( x^i \) where \( i = 1, 2, 3 \). Then

\[
[x^i, dx^j] = \tau g^{ij}, \quad g^{ij} = x^i x^j + 4 \delta^i_2 \delta^j_3.
\]

(3.12)

Since \( \det(g^{ij}) = -4 (x^1)^2 \), \( g \) is a non-degenerate symmetric matrix if \( x^1 \neq 0 \). The latter is just the condition allowing us to solve the determinant constraint for \( x^4 \). An attempt to express \( \tau \) in the form \( \tau = \sum_{i=1}^3 dx^i f_i \) with \( f_i \in \mathcal{A} \) using \( x^4 = (1 + x^2 x^3)/x^1 \) fails.\footnote{One might have expected that we simply had to insert the value of \( t_+ \) at \( q = 1 \) in \( (2.22) \) which would indeed lead to the ordinary differential calculus on \( SL(2) \). This would not be correct, however, since before taking the limit \( q \to 1 \) we have to identify \( t = q \) (rather than treating \( t \) as independent of \( q \) as we did at the end of section 2).}
Therefore, $dx^i$ and $\tau$ are linearly independent in $\Lambda^1(\mathcal{A})$, regarded as a right $\mathcal{A}$-module (see also [21]).

Let * be an antilinear involution on $\mathcal{A}$ (which on complex numbers acts as complex conjugation). The reality conditions $(x^\mu)^* = x^\mu$ are compatible with the $SL_q(2)$ commutation relations only when $|q| = 1$. These conditions define the quantum group $SL_q(2, \mathbb{R})$. Assuming the rule $(f \, dh)^* = d(h^*) f^*$ $(\forall f, h \in \mathcal{A})$, the $t_+$ calculus on $SL_q(2)$ is compatible with the reality conditions [3]. In the classical limit $(q = 1)$, we then have (3.12) with real functions $x^i$ and a real metric $g$ which turns out to be the maximally symmetric Lorentzian metric on $SL(2, \mathbb{R})$ with negative constant curvature [9].

4 Comments

We have obtained a considerable simplification of some of the results in [9, 10]. Particular emphasis has been given to the fact that the classical limit of a bicovariant differential calculus on a quantum group does not coincide, in general, with the ordinary differential calculus. In particular in view of possible applications of bicovariant differential calculus on quantum groups in physics, it is interesting that the resulting ‘deformed’ calculus exhibits relations to various branches of mathematical physics. This will be discussed briefly in the following. A generalization of (3.12) is given by

$$[x^i, dx^j] = \tau g^{ij}$$

where $x^i$ are coordinates on a (smooth) manifold $\mathcal{M}$, $g$ a contravariant symmetric tensor field (e.g., a metric), and $\tau$ a 1-form on $\mathcal{M}$ which commutes with $x^i$, anticommutes with 1-forms and satisfies $d\tau = 0$. One can show that these commutation relations are well defined on $\mathcal{M}$, i.e. independent of the choice of coordinates. Such a calculus has been considered before [4, 3] with $\tau = dt$, the ordinary differential of a real parameter $t$. As a consequence of (4.1) we then have

$$df = dt \left( \frac{\partial}{\partial t} + \frac{1}{2} g^{ij} \frac{\partial}{\partial x^i} \frac{\partial}{\partial x^j} \right) f + dx^i \frac{\partial}{\partial x^i} f$$

for a function $f(t, x^i)$. Note that the differential of $f$ involves a second order differential operator. This hints towards applications of this calculus in the context of stochastics (diffusion equation), quantum mechanics (Schrödinger equation) and ‘proper time’ relativistic quantum theories (see [22] for a review). For a real (positive definite) tensor field $g$, the first order calculus was indeed shown to be equivalent to the (Itô) calculus of stochastic differentials [3] (see also [21]). Here $t$ is the stochastic time. The other aspects mentioned above were discussed in [4, 4].

In (3.12) the 1-form $\tau$ is not of the form $dt$ with a parameter $t$ independent of the $x^i$. Therefore, there is no (extra) ‘time’ parameter in this case and we have

$$df = \tau \frac{1}{2} g^{ij} \frac{\partial}{\partial x^i} \frac{\partial}{\partial x^j} f + dx^i \frac{\partial}{\partial x^i} f$$

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instead of (1.2).

In this work we have only considered bicovariant differential calculus on the quantum groups $GL_{p,q}(2), GL_q(2), SL_q(2)$ and $SU_q(2)$. For the corresponding higher-dimensional quantum groups, one does not have a complete knowledge of the bicovariant calculi yet (with the exception of $GL_q(3)$ [8] for which the calculi induced on quantum subgroups are now being studied). But the existing examples also exhibit a nonstandard classical limit, in general. This will be discussed in more detail elsewhere.

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