Criticality and finite size effects in a simple realistic model of stock market

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We discuss a simple model based on the Minority Game which reproduces the main \textit{stylized facts} of anomalous fluctuations in finance. We present the analytic solution of the model in the thermodynamic limit and show that stylized facts arise only close to a line of critical points with non-trivial properties. By a simple argument, we show that, in Minority Games, the emergence of critical fluctuations close to the phase transition is governed by the interplay between the signal to noise ratio and the system size. These results provide a clear and consistent picture of financial markets as critical systems.

Understanding the origin of the anomalous collective fluctuations arising in stock markets poses novel and fascinating challenges in statistical physics. Stock market prices are characterized by anomalous collective fluctuations – known as \textit{stylized facts} \cite{1} – which are strongly reminiscent of critical phenomena. Prices do not follow a simple random walk process, but rather price increments are fat tailed and their absolute value exhibits long range auto-correlations, called volatility clustering.

The connection with critical phenomena is natural, because financial markets are indeed complex systems of many interacting degrees of freedom — the traders. By means of agent based modeling, it has been realized \cite{2–6} that stylized facts are due to the way in which the trading activity of agents interacting in a market “dresses” the fluctuations arising from economic activity – the so-called \textit{fundamentals}. Ref. \cite{6} has shown that very simple models based on the Minority Game \cite{7} can reproduce a quite realistic and rich behavior. Their simplicity makes an analytical approach to these models possible, using tools of statistical physics. Whether Minority Games describe interacting traders is a matter of debate \cite{8,10}. At any rate, the emergence of anomalous fluctuations in these models, besides providing a scenario for the behavior of real markets, poses questions in statistical physics which deserve interest of their own.

In this Letter, we first introduce the simplest possible Grand Canonical Minority Game (GCMG) which reproduces the main stylized facts, i.e. fat tails and volatility clustering. Then we present the analytic solution of this model in the relevant thermodynamic limit. It shows that the behavior of GCMG, in this limit, exhibits Gaussian fluctuation for all parameter values but on a line of critical points which marks a discontinuous phase transition. For finite size systems, numerical simulations reveal that stylized facts emerge close to the transition line, but they abruptly disappear as the system size increases. Remarkably, the vanishing of stylized facts when the system’s size increases also occurs in a variety of models of financial markets \cite{11}. We present a theory of finite size effects which is fully confirmed by numerical simulations. This allows us to conclude that anomalous fluctuations are properties of the critical point in GCMG. The phase transition is quite unique as it mixes features which are typical of first order phase transitions – as discontinuities and phase coexistence – and of second order phase transitions – such as the divergence of correlation volumes and finite size effects.

In the market described by the Minority Game \cite{7}, agents $i = 1, \ldots, N$ submit a bid $b_i(t)$ to the market in every period $t = 1, 2, \ldots$. Agents whose bid has the opposite sign of the total bid $A(t) = \sum_i b_i(t)$, win whereas the others lose. The bids of agents depend on the value $\mu(t)$ of a public information variable, which is drawn uniformly from the integers $1, \ldots, P$. In other words, agents have \textit{trading strategies} which prescribes to agent $i$ a bid $a_i^\mu(t) \pm 1$ for each information $\mu$. Each agent is assigned one such strategy, randomly chosen from the set of $2^P$ possible strategies of this type. Agents are adaptive and may decide to refrain from playing if their strategy is not good enough \cite{3,4}. More precisely, the bids of agents take the form $b_i(t) = \phi_i(t)a_i^\mu(t)$ where $\phi_i(t) = 1$ or $0$ according to whether agent $i$ trades or not. In order to assess the performance of their strategy, agents assign scores $U_i(t)$ which they update by

$$U_i(t+1) = U_i(t) - a_i^\mu(t)A(t) - \epsilon_i.$$  \hspace{1cm} (1)

where

$$A(t) = \sum_{i=1}^{N} b_i(t) = \sum_{i=1}^{N} \phi_i(t)a_i^\mu(t).$$  \hspace{1cm} (2)

So if $-\epsilon_i a_i^\mu(t)A(t)$ is large enough, i.e., larger than $\epsilon_i$, the score $U_i$ increases. The larger $U_i$, the more likely it is that the agent trades ($\phi_i = 1$). Here we suppose that \cite{12}

$$\text{Prob}\{\phi_i(t) = 1\} = \frac{1}{1 + e^{\Gamma U_i(t)}}$$  \hspace{1cm} (3)

where $\Gamma > 0$ is a constant. A good strategy prescribes bids $a_i^\mu$ which tend to coincide with those $b_i(t) = -\text{sign} A(t)$ of the minority of agents. The connection with markets goes along the lines of Refs. \cite{4–6,10}, which show that $A(t)$ is proportional to the difference of price logarithms; here, we take $\log p(t+1) = \log p(t) + A(t)$.

The threshold $\epsilon_i$ in Eq. (1) models the incentives of agents for trading in the market. Some investors may have incentives to trade because they need the market for
exchanging goods or assets. This corresponds to $\epsilon_i < 0$. On the contrary, speculators who only trade for profiting of price fluctuations typically have $\epsilon_i > 0$. Of course there may be prudent investors with $\epsilon_i > 0$ or risk-lover speculators with $\epsilon_i < 0$ and a whole range of other type of traders. Here we focus, for simplicity, on the case

$$
\epsilon_i = \epsilon \quad \text{for } i \leq N_s \quad \epsilon_i = -\infty \quad \text{for } N_s < i \leq N
$$

The $N_p = N - N_s$ agents who have $\epsilon_i = -\infty$ — we call them producers after Refs. [13,14] — trade no matter what, whereas the remaining $N_s$ — the speculators — trade only if the cumulated performance of their active strategy increases more rapidly than $et$.

If the conditional time average $\langle A|\mu\rangle$ of $A(t)$ given $\mu(t) = \mu$ is non-zero, then the knowledge of $\mu(t)$ allows a statistical prediction of the sign of $A(t)$. A measure of predictability is hence given by

$$
H_0 = \langle A \rangle^2 - 1 \frac{P}{\mu=1} \langle A|\mu \rangle^2
$$

where we introduced the notation $\langle \ldots \rangle$ for averages over $\mu$ ($\langle \ldots \rangle$ denotes averages on the stationary state). When $H_0 = 0$ the market is unpredictable or informationally efficient. Volatility is instead defined as $\sigma^2 = \langle A^2 \rangle$ and it measures market’s fluctuations. A further quantity of interest is the number $N_{act}(t) = \sum_i \langle \phi_i(t) \rangle$ of active speculators in the market.

Exact results can be obtained in the thermodynamic limit, which is defined as the limit $N_s,N_p,P \rightarrow \infty$, keeping constant the reduced number of speculators and producers $n_s = N_s/P, n_p = N_p/P$. In this limit, both $\sigma^2$ and $H_0$ diverge with the system size, since $A(t) \sim \sqrt{N}$. Hence we shall consider the rescaled quantities $H_0/P$ or $\sigma^2/P$. A detailed account of the calculation will be given elsewhere [15]. Here we just discuss the main step and the results. Following Ref. [16], we derive an Ito stochastic differential equations for the strategy scores $y_i(\tau) = U_i(t)$ in the rescaled continuous time $\tau = t/N$

$$
\frac{dy_i}{d\tau} = -a_i \langle A \rangle - \epsilon + \eta_i.
$$

Here $\eta_i$ is a zero average Gaussian noise term with

$$
\langle \eta_i(\tau)\eta_j(\tau') \rangle = \frac{1}{N} a_i a_j \langle A^2 \rangle \delta(\tau - \tau').
$$

In Eqs. (4,5) averages $\langle \ldots \rangle_y$ are taken on the distribution of $\phi_i(t)$ in Eq. (3), which depends on $y_i(\tau)$ in a non-linear way: $\text{Prob}\{\phi_i(t) = 1\} = 1/[1 + e^{\epsilon y_i(\tau)}]$. Hence Eq. (4) is a quite complex system of non-linear equations with a noise strength proportional to the time dependent volatility $\langle A^2 \rangle_y$. This feedback will be responsible for the emergence of volatility build-ups.

Following Refs. [17,16] we find that the fraction $\langle \phi_i \rangle$ of times that agent $i$ plays his active strategy in the stationary state is the solution of the minimization of the function

$$
H_\epsilon = \frac{1}{\rho} \sum_{\mu=1}^P \left[ \sum_{i=1}^N \langle \phi_i \rangle a_i^\mu + \sum_{i=N_s+1}^N a_i^\mu \right] + 2\epsilon \sum_i \langle \phi_i \rangle
$$

with respect to $\langle \phi_i \rangle$. Note that for $\epsilon = 0$ this function reduces to the predictability $H_0$. For $\epsilon \neq 0$, the solution to this problem, and hence the stationary state, is unique. An exact statistical mechanics description of the solution $\{\langle \phi_i \rangle\}$ can be carried out with the replica method, because the replica symmetric ansatz is exact. Furthermore the solution to the Fokker-Planck equation corresponding to Eq. (4) can be well approximated by a factorized ansatz for $\epsilon \neq 0$. This means that the off-diagonal correlations vanish $\langle \langle \phi_i - \langle \phi_i \rangle \rangle \langle \phi_j - \langle \phi_j \rangle \rangle \rangle = 0$ for $i \neq j$ and, as a consequence, the volatility turns out to be given by $\sigma^2 = \langle A^2 \rangle = H_0 + \sum_{i=1}^N \langle \phi_i \rangle (1 - \langle \phi_i \rangle)$. The solution $\{\langle \phi_i \rangle\}$ of the minimization of $H_\epsilon$ provides a complete description of the model in the limit $N \rightarrow \infty$ for $\epsilon > 0$. In particular the behavior is independent of $\Gamma$.

![Fig. 1. Theory and numerical simulations: $n_{act}$ (top) and $\sigma^2/P$ and $H/P$ (bottom) as a function of $n_s$ for $\epsilon = 0.1$ (solid line) and $\epsilon = -0.01$ (dashed line). Numerical results for $\epsilon = 0.1$ (open symbols) and $\epsilon = -0.01$ (full symbols) are averages over 200 runs, with $N_sP = 10000$ fixed and $\Gamma = \infty$.](image-url)
a constant. This means that the market becomes highly selective: Only a negligible fraction of speculators trade \((\phi_i(t) = 1)\) whereas the majority is inactive \((\phi_i(t) = 0)\). The volatility \(\sigma^2\) also remains constant in this limit.

For \(\epsilon < 0\) we see a markedly different behavior: The number of active speculators continues growing with \(n_a\) even if the market is unpredictable \(H_0 \approx 0\). The volatility \(\sigma^2\) has a minimum and then it increases with \(n_a\) in a way which depends on \(\Gamma\). In other words, \(\epsilon = 0\) for \(n_a \geq n_a^c(n_p) (= 4.15 \ldots \text{ for } n_p = 1)\) is the locus of a first order phase transition across which \(N_{\text{act}}\) and \(\sigma^2\) exhibit a discontinuity. This same picture applies to a wider range of GCMG models such as that of Ref. [6].

\[ \text{FIG. 2. Probability distribution of } A(t) > 0 \text{ for } n_a = 10 \text{ (continuous line), } 20, 50, 100, 200 \text{ (dash-dotted line) } (PN_a = 16000, n_p = 1, \epsilon = 0.01, \Gamma = \infty). \text{ Inset: time series of returns } A(t) \text{ showing volatility clustering for } n_a = 20 \text{ (lower curve), but not for } n_a = 200 \text{ (upper curve).} \]

Numerical simulations reproduce anomalous fluctuations similar to those of real financial markets close to the phase transition line. As shown in Fig. 2, the distribution of \(A(t)\) is Gaussian for small enough \(n_a\), and has fatter and fatter tails as \(n_a\) increases; the same behavior is seen for decreasing \(\epsilon\). In particular the distribution of \(A(t)\) shows a power law behavior \(P(|A| > x) \sim x^{-\beta}\) with an exponent which we estimated as \(\beta \simeq 2.8, 1.4\) for \(n_a = 20, 200\) respectively and \(\epsilon = 0.01\). Note that a realistic value \(\beta \simeq 3\) [19] is obtained for \(n_a = 20\).

This is inconsistent, at first sight, with the theoretical results discussed previously for \(N \to \infty\). Indeed, if the distribution of \(\phi_i\) factorizes, \(A(t)\) is the sum of \(N_a\) independent contributions and it satisfies the Central Limit Theorem. This implies that for \(\epsilon \neq 0\) the variable \(A(t)/\sqrt{N}\) converges in distribution to a Gaussian variable with zero average and variance \(\sigma^2/N\) in the limit \(N \to \infty\). There are no anomalous fluctuations and no stylized facts. Fig. 3 indeed shows that the anomalous fluctuations of Fig. 2 are finite size effects which disappear abruptly as the system size increases (or if \(\Gamma\) is small).

\[ \text{FIG. 3. Kurtosis of } A(t) \text{ in simulations with } \epsilon = 0.01, n_a = 70, n_p = 1 \text{ and several different system sizes } P \text{ for } \Gamma = 1, 10 \text{ and } \infty. \]

In order to understand these finite size effects, we note that volatility clustering arises because the noise strength in Eqs. (4,5) is proportional to the time dependent volatility \((\dot{A})^2\). The noise term is a source of correlated fluctuations because \(a_i a_j \langle A^2 \rangle_y /N \sim 1/\sqrt{N}\) is small but non zero, for \(i \neq j\). It is reasonable to assume that the dynamics will sustain collective correlated fluctuations in the \(y_i\) only if the correlated noise is larger than the signal \(-a_i \langle A \rangle_y - \epsilon\) which agents receive form the deterministic part of Eq. (4). Time dependent volatility fluctuations would be dissipated by the deterministic dynamics otherwise. A quantitative translation of this insight goes as follows: The noise correlation term is of order \(a_i a_j \langle A^2 \rangle_y /N \sim \sigma^2/P^{3/2}\) for \(i \neq j\). This should be compared to the square of the deterministic term of Eq. (4) \([a_i \langle A \rangle_y + \epsilon]^2 \sim \left[\sqrt{H_0/P} + \epsilon\right]^2\). Rearranging terms, we find that volatility clustering sets in when

\[ \frac{H_0}{\sigma^2} + 2\epsilon \sqrt{\frac{H_0 P}{P \sigma^2} + \epsilon^2} \frac{P}{\sigma^2} \simeq \frac{K}{\sqrt{P}} \]  

(7)

where \(K\) is a constant. This prediction is remarkably well confirmed by Fig. 4: In the lower panel we plot the two sides of Eq. (7) as a function of \(n_a\), for different system sizes. The upper panel shows that the volatility \(\sigma^2/N\) starts deviating from the analytic result exactly at the crossing point \(n_a^c(P)\) where Eq. (7) holds true. Furthermore the inset shows that the region \(n_a > n_a^c(P)\) is described by a novel type of scaling limit. Indeed the curves of Fig. 4 collapse one on top of the other when plotted against \(n_a/n_a^c(P)\).

The non-linearity of the response of agents is crucial for the onset of volatility time dependence. If \(\Gamma\) is small the response becomes smooth and anomalous fluctuations disappear (see Fig. 3). This picture is not affected by the introduction of a finite memory in the learning process of agents, as e.g. in Ref. [18]. In particular the exponents of Fig. 2 do not depend on the memory.
same generic argument: When the signal to noise ratio of the standard MG \[7,15\] are indeed explained by the Games. Finite size effects close to the phase transition are usually typical of first order phase transitions. Dow shrinks as \(dν\) the GCMG with system shows the normal fluctuations of a paramagnet. and the discontinuous nature of the transition at \(\alpha ≃ 1.1132\) in this plot. The intersection defines \(n^c_\nu(P)\) (parallel dashed lines) as a function of \(n^c_\nu(P)\). Inset: Collapse plot of \(σ^2/N\) as a function of \(n^c_\nu(P)\).

The fact that, in finite systems, stylized facts arise only close to the phase transition is reminiscent of finite size scaling in the theory of critical phenomena: In \(d\)-dimensional Ising model, for example, at temperature \(T = T_c + \epsilon\) critical fluctuations (e.g. in the magnetization) occur as long as the system size \(N\) is smaller than the correlation volume \(\sim \epsilon^{-dν}\). But for \(N \gg \epsilon^{-dν}\) the system shows the normal fluctuations of a paramagnet.

Eq. (7) and \(H_0/P \sim \epsilon^2\) imply that the same occurs in the GCMG with \(dν = 4\). In other words, the critical window shrinks as \(N^{-1/4}\) when \(N \to \infty\). However, because of the long range nature of the interaction, anomalous fluctuations either concern the whole system or do not affect it at all, as clearly shown in Fig. 3. In the critical region the Gaussian phase coexists probabilistically with a phase characterized by anomalous fluctuations. This and the discontinuous nature of the transition at \(\epsilon = 0\), are usually typical of first order phase transitions.

The picture of a phase transition controlled by the signal to noise ratio appears to be universal for Minority Games. Finite size effects close to the phase transition of the standard MG \[7,15\] are indeed explained by the same generic argument. When the signal to noise ratio \(H_0/σ^2\) is of order \(1/√P\) self-sustained collective fluctuations arise. Volatility clustering in real markets is known to be due to wild fluctuations in the volume of trades \[19\]. Volume is the number of active traders \(N_{act} + N_p\) in the GCMG. Hence wild volume fluctuations require correlated collective fluctuations in the behavior of agents which only arise close to criticality. This suggests that real markets operate close to a phase transition. Numerical simulations suggest that exponents vary continuously on the line of critical points. This raises the question of why real markets self-organize close to the critical surface with \(α ≃ 3\).

We conclude that the GCMG exhibits a quite peculiar type of phase transition which mixes properties of continuous and discontinuous transitions. Finite size effects clearly relate the occurrence of stylized facts to the analytic nature of the phase transition. The extension of renormalization group approaches to this system promises to be a quite interesting challenge.

FIG. 4. Onset of the anomalous dynamics for different system sizes. Top: \(σ^2/N\) for different series of simulations with \(L = PN_s\) constant: \(PN_s = 1000\) (circles), 2000 (squares), 4000 (diamonds), 8000 (up triangles) and 16000 (left triangles). In all simulations \(n_s = 1\), \(ε = 0.1\) and \(Γ = \infty\). Bottom: L.H.S. of Eq. (7) (full line) from the exact solution and \(K/√P = K(n_s/L)^{1/4}\) (parallel dashed lines) as a function of \(n_s\) (\(K ≃ 1.1132\) in this plot). The intersection defines \(n_s^c(P)\).