Scalar Field Quantization on the 2+1 Dimensional Black Hole Background

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Abstract

The quantization of a massless conformally coupled scalar field on the 2+1 dimensional Anti de Sitter black hole background is presented. The Green’s function is calculated, using the fact that the black hole is Anti de Sitter space with points identified, and taking into account the fact that the black hole spacetime is not globally hyperbolic. It is shown that the Green’s function calculated in this way is the Hartle-Hawking Green’s function. The Green’s function is used to compute $\langle T_{\mu\nu} \rangle$, which is regular on the black hole horizon, and diverges at the singularity. A particle detector response function outside the horizon is also calculated and shown to be a fermi type distribution. The back-reaction from $\langle T_{\mu\nu} \rangle$ is calculated exactly and is shown to give rise to a curvature singularity at $r = 0$ and to shift the horizon outwards. For $M = 0$ a horizon develops, shielding the singularity. Some speculations about the endpoint of evaporation are discussed.

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INTRODUCTION

The study of black hole physics is complicated by the many technical and conceptual problems associated with quantum field theory in curved spacetime. One serious difficulty is that exact calculations are almost impossible in 3+1 dimensions. In this paper we shall instead study some aspects of quantum field theory on a 2+1 dimensional black hole background. This enables us to obtain an exact expression for the Green’s function of a massless, conformally coupled scalar field in the Hartle-Hawking vacuum \[1\]. We use this Green’s function to study particle creation by the black hole, back-reaction and the endpoint of evaporation.

We shall work with the 2+1 dimensional black hole solution found by Bañados, Teitelboim and Zanelli (BTZ) \[2\]. It had long been thought that black holes cannot exist in 2+1 dimensions for the simple reason that there is no gravitational attraction, and therefore no mechanism for confining large densities of matter. This difficulty has been circumvented in the BTZ spacetime\[1\], but not surprisingly, their solution has some features that we do not normally associate with black holes in other dimensions, such as the absence of a curvature singularity. It is interesting to ask whether this spacetime behaves quantum mechanically in a way consistent with more familiar back holes.

The spinless BTZ spacetime has a metric \[2\]
\[
ds^2 = -N^2 dt^2 + N^{-2} dr^2 + r^2 d\phi^2
\]
where
\[
N^2 = \frac{r^2 - r_+^2}{\ell^2}, \quad r_+^2 = M\ell^2.
\]
Here \(M\) is the mass of the black hole. The metric is a solution to Einstein’s equations with a negative cosmological constant, \(\Lambda = -\ell^{-2}\), and the curvature of the black hole solution is constant everywhere. As a result there is no curvature singularity as \(r \to 0\). A Penrose diagram of the spacetime is given in Fig. \[\]

An important feature of the BTZ solutions is that the solution with \(M = 0\) (which we refer to as the vacuum solution), is not AdS\(_3\). Rather, it is a solution that is not globally Anti de Sitter invariant. It has no horizon, but does have an infinitely long throat for small \(r > 0\), which is reminiscent of the extreme Reissner-Nordström solution in 3+1 dimensions. It is worth noting that there are other similarities between the spinless BTZ black holes, \(M \geq 0\), and the Reissner-Nordström solutions for \(M \geq Q\). In particular, the temperature associated with the Euclidean continuation of the BTZ black holes has been computed in \[2\], and it was found to increase with the mass, and to decrease to zero as \(M \to 0\). Thus, if we carry over the usual notions from four dimensional black holes, the \(M = 0\) solution appears to be a stable endpoint of evaporation.

A feature of the BTZ solution that we shall make use of, is that the solution arises from identifying points in AdS\(_3\), using the orbits of a spacelike Killing vector field. It is

\[^1\]A charged black hole solution in 2+1 dimensions had previously been found in Ref. \[3\]. For further discussions on the BTZ black hole, see Refs. \[\]

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this property that is the starting point of our derivation of a Green’s function on the black
hole spacetime. We construct a Green’s function on AdS$_3$, and this translates to a Green’s
function on the black hole via the method of images.

A Green’s function constructed in this way is only interesting if we can identify the
vacuum with respect to which it is defined. We prove that our construction gives the Hartle-
Hawking Green’s function. It is worth noting that for the BTZ black hole, there is a limited
choice of vacua. Quantisation on AdS$_3$ is hampered by the fact that AdS$_3$ is not globally
hyperbolic, and this necessitates the use of boundary conditions at spatial infinity $\bar{V}$ (see
Fig. 2), as discussed in Appendix A. This problem carries over to the black hole solution,
and as a result, the value of the field at spatial infinity is governed by either Dirichlet or
Neumann type boundary conditions. Thus a Cauchy surface for the region $R$ of the BTZ
black hole is either the past or the future horizon only. With this knowledge, the natural
definition of the Hartle-Hawking vacuum is with respect to Kruskal modes on either horizon,
whereas there does not appear to be a natural definition of an Unruh vacuum (see [6,7] for
a discussion of the various eternal black hole vacua). The definition of an Unruh vacuum
might be possible given a description of the formation of a BTZ black hole from the vacuum
via some sort of infalling matter, but as far as we are aware, no such construction has been
found.

Having an explicit expression for the Hartle-Hawking Green’s function, we are able to
obtain a number of exact results. As a check, we show that it satisfies the KMS thermality
condition [8]. We then compute the expectation value of the energy-momentum tensor
and the response of a particle detector for both nonzero $M$ and for the vacuum solution.
For nonzero $M$ we address the issue of whether the response of the particle detector can
be interpreted as radiation emitted from the black hole, although a clear picture does not
emerge.

For the $M = 0$ solution, we find a non-zero energy-momentum tensor, although the
corresponding Green’s function is at zero temperature, and there is no particle detector
response. We interpret this as a sort of Casimir energy. Classically, the vacuum solution
appears to be similar to the extremal Reissner-Nordström solution, in the sense that we
expect that if any matter is thrown in, a horizon develops. Quantum mechanically, the
$M = 0$ solution appears to be unstable to the formation of a horizon, when the back-
reaction caused by the Casimir energy is taken into account. This suggests that the endpoint
of evaporation may not look like the classical $M = 0$ solution.

The paper is organized as follows. In section I we study the 1+1 dimensional solution
which arises from a dimensional reduction of the BTZ black hole [9], and show that the
vacuum defined by the Anti de Sitter (AdS) modes is the same as that defined by the
Kruskal modes; with this encouraging result we tackle the 2+1 case. Section II contains
a review of the essential features of the geometry of the BTZ black hole. In section III
we construct the Wightman Green’s functions on the black hole spacetime from the AdS$_3$
Wightman function, using the method of images. We then show that the Green’s function
coincides with the Hartle-Hawking Green’s function [1], by showing that it is analytic and
bounded in the lower half of the complex $\hat{V}$ plane on the past horizon ($\hat{U} = 0$), where
$\hat{V}, \hat{U}$ are the Kruskal null coordinates. We also compute the Wightman function for the
$M = 0$ solution, and compare this to the $M \to 0$ limit of the results for $M \neq 0$. Section IV
contains a calculation of $\langle T_{\mu\nu} \rangle$ for all $M \geq 0$. For the black hole solutions, it is regular on
the horizon, and for all $M$ it is singular as $r \to 0$. In Section V the response function of a stationary particle detector outside the horizon is calculated and shown to be of a fermi type distribution. A discussion is given of how this response might be interpreted. In Section VI we calculate the back reaction induced by $\langle T_{\mu\nu} \rangle$, and show that the spacetime develops a curvature singularity and a larger horizon for a given $M$. Throughout this paper we use metric signature ($-++$), and natural units in which $8G = \hbar = c = 1$.

I. 2-D BLACK HOLE

Let us begin by looking at quantum field theory on the region of Anti de Sitter spacetime in 1+1 dimensions described by the metric

$$ds^2 = -\left(\frac{r^2 - M\ell^2}{\ell^2}\right)dt^2 + \left(\frac{r^2 - M\ell^2}{\ell^2}\right)^{-1}dr^2 \quad 0 < r < \infty \quad -\infty < t < \infty,$$

where $M$ is the mass of the solution. This metric was discussed in [9] as the dimensional reduction of the spinless BTZ black hole, and can be thought of as being a region of AdS$_2$ in Rindler-type co-ordinates. Under the change of co-ordinates

$$r = \sqrt{M\ell^2} \sec \rho \cos \lambda, \quad \tanh \left(\frac{\sqrt{M}\ell t}{\ell}\right) = \frac{\sin \rho}{\sin \lambda},$$

where we shall call $(\lambda, \rho)$ AdS co-ordinates, the metric becomes

$$ds^2 = \ell^2 \sec^2 \rho (-d\lambda^2 + d\rho^2)$$

which for $-\frac{\pi}{2} \leq \rho \leq \frac{\pi}{2}$ and $-\infty < \lambda < \infty$ is just AdS$_2$ [10].

It is possible to define Kruskal-like co-ordinates for this black hole, which do not coincide with the usual AdS co-ordinates. For $r > M\ell^2$, they are:

$$U = \left(\frac{r - \sqrt{M}\ell}{r + \sqrt{M}\ell}\right)^{\frac{1}{2}} \cosh \left(\frac{\sqrt{M}\ell t}{\ell}\right)$$

$$V = \left(\frac{r - \sqrt{M}\ell}{r + \sqrt{M}\ell}\right)^{\frac{1}{2}} \sinh \left(\frac{\sqrt{M}\ell t}{\ell}\right).$$

The metric then takes the form

$$ds^2 = \frac{-2\ell^2}{1 + \tilde{U}\tilde{V}} \tilde{U}d\tilde{V}$$

where $\tilde{U} = V + U$, $\tilde{V} = V - U$, and the transformation between Kruskal and AdS co-ordinates is

$$\tilde{U} = \tan \left(\frac{\rho + \lambda}{2}\right) \quad \tilde{V} = \tan \left(\frac{\rho - \lambda}{2}\right)$$

which is valid over all the Kruskal manifold. The Kruskal co-ordinates cover only the part of AdS$_2$ with
We shall now show that the notion of positive frequency in \((\lambda, \rho)\) (AdS) modes coincides with that defined in \((U, V)\) (Kruskal) modes.

The AdS modes for a conformally coupled scalar field are normalized solutions of \(\Box \psi = 0\), subject to the boundary conditions

\[
\phi \left( \rho = \frac{\pi}{2} \right) = \phi \left( \rho = \frac{-\pi}{2} \right) = 0.
\]

The positive frequency modes are then

\[
\phi_m = \frac{1}{\sqrt{m\pi}} e^{-im\lambda} \sin m\rho \quad m \text{ even } \geq 0
\]

and these define a vacuum state \(|0\rangle_A\) in the usual way.

The Kruskal modes are solutions of \(\Box \psi = 0\) with the boundary condition \(\psi(\bar{U}, \bar{V} = -1) = 0\). Positive frequency solutions are given by

\[
\psi_\omega = N_\omega \left( e^{-i\omega \bar{U}} - e^{i\omega/\bar{V}} \right) \quad \omega > 0
\]

where \(N_\omega = (8\pi\omega)^{-1/2}\), and these define \(|0\rangle_K\). These modes are analytic and bounded in the lower half of the complex \(\bar{U}, \bar{V}\) plane. In order to show equivalence of the two vacua \(|0\rangle_A\) and \(|0\rangle_K\), it is enough to show that the positive frequency AdS modes can be written as a sum of only positive frequency Kruskal modes. Because of the analyticity properties of the Kruskal modes, it is enough to show that the AdS modes are analytic and bounded in the lower half of the complex \(\bar{U}, \bar{V}\) plane. Changing co-ordinates, we have

\[
\phi_m = \frac{1}{\sqrt{\pi m^2 i}} \left( e^{-2im \arctan \bar{V}} - e^{-2im \arctan \bar{U}} \right) \quad m \text{ even } \geq 0
\]

\[
\phi_m = \frac{1}{\sqrt{\pi m^2 i}} \left( e^{-2im \arctan \bar{V}} + e^{-2im \arctan \bar{U}} \right) \quad m \text{ odd } \geq 0
\]

Using the definition \(\arctan z = \frac{1}{2i} \ln \left[ \frac{1+i z}{1-i z} \right]\), (1.3) and (1.4) become

\[
\phi_m = \frac{1}{2\sqrt{\pi m i}} \left[ \left( \frac{1-i \bar{V}}{1+i \bar{V}} \right)^m \mp \left( \frac{1-i \bar{U}}{1+i \bar{U}} \right)^m \right]
\]

where \(\pm\) is for \(m\) odd or even. These modes can easily be seen to be bounded and analytic in the lower half of the complex \(\bar{U}, \bar{V}\) plane for all \(m\). This establishes that the vacuum defined by the AdS modes is the same as that defined by the Kruskal modes. Thus a Green’s function defined on this spacetime using AdS co-ordinates \((\lambda, \rho)\) corresponds to a Hartle-Hawking Green’s function, in the sense discussed in the Introduction.
II. THE GEOMETRY OF THE 2+1 DIMENSIONAL BLACK HOLE

In this paper, we shall be working only with the spinless black hole solution in 2+1 dimensions

\[ ds^2 = -N^2 dt^2 + N^{-2} dr^2 + r^2 d\phi^2 \]  

(2.1)

where

\[ N^2 = \frac{r^2 - r_+^2}{\ell^2}, \quad r_+^2 = M\ell^2. \]

As was shown in [2], this metric has constant curvature, and is a portion of three dimensional Anti de Sitter space with points identified. The identification is made using a particular killing vector \( \xi \), by identifying all points \( x_n = e^{2\pi n\xi}x \). In order to see this most clearly, it is useful to introduce different sets of co-ordinates on AdS_3.

AdS_3 can be defined as the surface \(-v^2 - u^2 + x^2 + y^2 = -\ell^2 \) embedded in \( R^4 \) with metric \( ds^2 = -du^2 - dv^2 + dx^2 + dy^2 \). A co-ordinate system \((\lambda, \rho, \theta)\) which covers this space, and which we shall refer to as AdS co-ordinates, is defined by [3]

\[
\begin{align*}
  u &= \ell \cos \lambda \sec \rho \\
  x &= \ell \tan \rho \cos \theta \\
  y &= \ell \tan \rho \sin \theta
\end{align*}
\]

where \( 0 \leq \rho \leq \frac{\pi}{2}, \; 0 < \theta \leq 2\pi, \; \text{and} \; 0 < \lambda < 2\pi. \) In these co-ordinates, the AdS_3 metric becomes

\[
ds^2 = \ell^2 \sec^2 \rho (-d\lambda^2 + d\rho^2 + \sin^2 \rho d\theta^2).\]

AdS_3 has topology \( S^1 \) (time) \( \times R^2 \) (space) and hence contains closed timelike curves. The angle \( \lambda \) can be unwrapped to form the covering space of AdS_3, with \(-\infty < \lambda < \infty\), which does not contain any closed timelike curves. Throughout this paper we work with this covering space, and this is what we henceforth refer to as AdS_3. As mentioned in the Introduction, even this covering space presents difficulties since it is not globally hyperbolic (see the discussion in Appendix A).

The identification taking AdS_3 into the black hole (2.1) is most easily expressed in terms of co-ordinates \((t, r, \phi)\), related in an obvious way to those used above, and defined on AdS_3 by

\[
\begin{align*}
  u &= \sqrt{A(r)} \cosh \left( \frac{r_+}{\ell} \phi \right) \\
  x &= \sqrt{A(r)} \sinh \left( \frac{r_+}{\ell} \phi \right) \\
  y &= \sqrt{B(r)} \cosh \left( \frac{r_+}{\ell^2} t \right) \\
  v &= \sqrt{B(r)} \cosh \left( \frac{r_+}{\ell^2} t \right)
\end{align*} \quad r > r_+ \]

\[
\begin{align*}
  u &= \sqrt{A(r)} \cosh \left( \frac{r_+}{\ell} \phi \right) \\
  x &= \sqrt{A(r)} \sinh \left( \frac{r_+}{\ell} \phi \right) \\
  y &= -\sqrt{-B(r)} \cosh \left( \frac{r_+}{\ell^2} t \right) \\
  v &= \sqrt{-B(r)} \cosh \left( \frac{r_+}{\ell^2} t \right)
\end{align*} \quad 0 > r > r_+.
\]
Note that $-\infty < \phi < \infty$. Under the identification $\phi \to \phi + 2\pi n$, where $n$ is an integer, these regions of AdS$_3$ become regions R and F of the black hole. Regions P and L are defined in an analogous way [4] (see Fig. 1 for a definition of regions F (future), P (past), R (right) and L (left)). The $r = 0$ line is a line of fixed points under this identification, and hence there is a singularity there in the black hole spacetime of the Taub-NUT type [2, 10].

Finally, it is possible to define Kruskal co-ordinates on the black hole. The relation between the Kruskal co-ordinates $V$ and $U$ and the black hole co-ordinates $t$ and $r$ is precisely as in (1.1) and (1.2). $U$, $V$ and an unbounded $\phi$ cover the region of AdS$_3$ which becomes the black hole after the identification.

III. GREEN’S FUNCTIONS ON THE 2+1 DIMENSIONAL BLACK HOLE

In this section we derive a Green’s function on the black hole spacetime, by using the method of images on a Green’s function on AdS$_3$. We then show that the resulting Green’s function is thermal, in that it obeys a KMS condition [8]. Using the analyticity properties discussed in the Introduction, the Green’s function is also shown to be defined with respect to a vacuum state corresponding to Kruskal co-ordinates on both the past and future horizons of the black hole. We therefore interpret it as a Hartle-Hawking Green’s function. Finally we derive the Green’s function for the $M = 0$ solution directly from a mode sum, and compare it with the $M \to 0$ limit of the black hole Green’s function.

A. Deriving the Green’s Functions

Since the black hole spacetime is given by identifying points on AdS$_3$ using a spacelike Killing vector field, we can use the method of images to derive the two point function on the black hole spacetime. Given the two point function $G_{+}^{+}(x, x'; \delta)$ on AdS$_3$,

$$G_{BH}^{+}(x, x'; \delta) = \sum_{n} e^{-i\delta n} G_{+}^{+}(x, x'_n)$$

Here $x'_n$ are the images of $x'$ and $0 \leq \delta < \pi$ can be chosen arbitrarily. For a general $\delta$ the modes of the scalar field on the black hole background will satisfy $\phi_m(e^{2\pi n\delta}x) = e^{-i\delta n} \phi_m(x)$. $\delta = 0$ for normal scalar fields and $\delta = \pi$ for twisted fields. From now on we will restrict ourselves to $\delta = 0$.

This definition of the Green’s function on the black hole spacetime means that when summing over paths to compute the Feynman Green’s function $G_F(x, x')$, we sum over all paths in AdS$_3$. Hence paths that cross and recross the singularities must be taken into account (compare this with the results of Hartle and Hawking [1]).

As explained in Appendix A, boundary conditions at infinity must be imposed on any Green’s function on AdS$_3$ in order to deal with the fact that AdS$_3$ is not globally hyperbolic. From Appendix A, we have

$$G_{+}^{+} = G_{+}^{+1} \pm G_{+}^{+2}$$

(3.1)

where $+(-)$ corresponds to Neumann (Dirichlet) boundary conditions (from now on, it should be assumed that the upper (lower) sign is always for Neumann (Dirichlet) boundary conditions unless otherwise stated). The individual terms in (3.1) are given by
two point function on the black hole background becomes

\[ G_{A1}(x, x') = \frac{1}{4\sqrt{2\pi\ell}} (\cos(\Delta\lambda - i\epsilon) \sec \rho' \sec \rho - 1 - \tan \rho \tan \rho' \cos \Delta\theta)^{-\frac{1}{2}} \]

\[ G_{A2}(x, x') = \frac{1}{4\sqrt{2\pi\ell}} (\cos(\Delta\lambda - i\epsilon) \sec \rho' \sec \rho + 1 - \tan \rho \tan \rho' \cos \Delta\theta)^{-\frac{1}{2}}. \]

\( \Delta\lambda \) is defined as \( \lambda - \lambda' \), and similarly for all other co-ordinates.

The sign of the \( i\epsilon \) is proportional to \( \text{sign} (\sin \Delta\lambda) \). It is only important for timelike separated points, for which the argument of the square root is negative. In the three dimensional Kruskal co-ordinates on AdS3, the identification is only in the angular direction. For timelike separated points, \( \text{sign} \Delta\lambda = \text{sign} \Delta V \), where \( V \) is the Kruskal time. It follows that for all identified points the sign of \( i\epsilon \) in \( G(x, x'_n) \) is the same.

We now work in the black hole co-ordinates \((t, r, \phi)\), so that the identification taking AdS3 into the black hole spacetime is given by \( \phi \rightarrow \phi + 2\pi n \). Under this identification, the two point function on the black hole background becomes

\[ G^+(x, x') = \frac{1}{4\sqrt{2\pi\ell}} \left[ G_1^+(x, x') \pm G_2^+(x, x') \right] \]

where for \( x, x' \in \text{region R} \)

\[ G_1^+(x, x') = \sum_{n=-\infty}^{\infty} \left[ \frac{rr'}{r_+^2} \cosh \frac{r_+ (\Delta\phi + 2\pi n)}{\ell} - 1 - \frac{(r_+^2 - r_-^2)^{\frac{1}{2}} (r_+^2 - r_-^2)^{\frac{1}{2}}}{r_+^2} \cosh \frac{r_+ (\Delta t - i\epsilon)}{\ell^2} \right]^{-\frac{1}{2}} \] (3.2)

\[ G_2^+(x, x') = \sum_{n=-\infty}^{\infty} \left[ \frac{rr'}{r_+^2} \cosh \frac{r_+ (\Delta\phi + 2\pi n)}{\ell} + 1 - \frac{(r_+^2 - r_-^2)^{\frac{1}{2}} (r_+^2 - r_-^2)^{\frac{1}{2}}}{r_+^2} \cosh \frac{r_+ (\Delta t - i\epsilon)}{\ell^2} \right]^{-\frac{1}{2}}. \] (3.3)

For \( x, x' \in \text{region F} \),

\[ G_1^+(x, x') = \sum_{n=-\infty}^{\infty} \left[ \frac{rr'}{r_+^2} \cosh \frac{r_+ (\Delta\phi + 2\pi n)}{\ell} - 1 + \frac{(r_+^2 - r_-^2)^{\frac{1}{2}} (r_+^2 - r_-^2)^{\frac{1}{2}}}{r_+^2} \cosh \frac{r_+ \Delta t}{\ell^2} + i\epsilon \text{ sign } \Delta V \right]^{-\frac{1}{2}} \]

\[ G_2^+(x, x') = \sum_{n=-\infty}^{\infty} \left[ \frac{rr'}{r_+^2} \cosh \frac{r_+ (\Delta\phi + 2\pi n)}{\ell} + 1 + \frac{(r_+^2 - r_-^2)^{\frac{1}{2}} (r_+^2 - r_-^2)^{\frac{1}{2}}}{r_+^2} \cosh \frac{r_+ \Delta t}{\ell^2} + i\epsilon \text{ sign } \Delta V \right]^{-\frac{1}{2}}. \]

Of course in this region \( \Delta V \neq \text{sign } \Delta t \). For \( x \in \text{region R} \) and \( x' \in \text{region F} \), we have

\[ G_1^+(x, x') = \sum_{n=-\infty}^{\infty} \left[ \frac{rr'}{r_+^2} \cosh \frac{r_+ (\Delta\phi + 2\pi n)}{\ell} - 1 - \frac{(r_+^2 - r_-^2)^{\frac{1}{2}} (r_+^2 - r_-^2)^{\frac{1}{2}}}{r_+^2} \sinh \frac{r_+ \Delta t}{\ell^2} + i\epsilon \text{ sign } \Delta V \right]^{-\frac{1}{2}} \]

\[ G_2^+(x, x') = \sum_{n=-\infty}^{\infty} \left[ \frac{rr'}{r_+^2} \cosh \frac{r_+ (\Delta\phi + 2\pi n)}{\ell} + 1 - \frac{(r_+^2 - r_-^2)^{\frac{1}{2}} (r_+^2 - r_-^2)^{\frac{1}{2}}}{r_+^2} \sinh \frac{r_+ \Delta t}{\ell^2} + i\epsilon \text{ sign } \Delta V \right]^{-\frac{1}{2}}. \]

In all of these expressions, \( G^-(x, x') = \langle 0 | \phi(x') \phi(x) | 0 \rangle \) is obtained by reversing the sign of \( i\epsilon \). All of these expressions are uniformly convergent for \( x, x' \) real, and \( r, r' > \delta, \delta > 0 \). Notice that as \( M \rightarrow 0 (r_+ \rightarrow 0) \), \( G(x, x') \) will diverge like \( \sum \frac{1}{n} \) unless we take the Dirichlet boundary conditions.
From these expressions the Feynman Green’s function can easily be constructed and in fact has exactly the same form, but with the sign of $ie$ being strictly positive. It should be noted that none of these Green’s functions are invariant under Anti de Sitter transformations, as the Killing vector field defining the identification does not commute with all the generators of the AdS group.

B. KMS condition

A thermal noise satisfies a skew periodicity in imaginary time called Kubo-Martin-Schwinger (KMS) condition [8]

$$g_\beta(\Delta \tau - \frac{i}{T}) = g_\beta(-\Delta \tau)$$

where $g_\beta(\Delta \tau) = G_\beta^+(x(\tau), x(\tau'))$ and $G_\beta^+ = \langle 0_\beta | \phi(x) \phi(x') | 0_\beta \rangle |_{x^0 = -\epsilon}$ with the world line $x(\tau)$ taken to be the one at rest with respect to the heat bath (for a more extensive discussion of the KMS condition, see [12]). We will show that $g(\Delta \tau) = G_\beta^+(x(\tau), x(\tau'))$ with $x(\tau) = (\frac{\tau}{T}, r, \phi)$ and $b = (r^2 - r_+^2)^{1/2}/\ell$, satisfies this condition outside the horizon, with a local temperature $T = \frac{\ell}{2\pi (\tau^2 - \tau_+^2)^{1/2}}$, which agrees with the Tolman relation [13] $T = (g_\infty)^{-1/2} T_0$, with $T_0 = r_+/2\pi \ell^2$ the temperature of the black hole.

$$g(\Delta \tau) = \frac{1}{4\sqrt{2\pi \ell}} (g_1(\Delta \tau) \pm g_2(\Delta \tau))$$

where

$$g_1(\Delta \tau) = \sum_{n=-\infty}^{\infty} \left[ \frac{r_+^2}{r_+^2} \cosh \frac{2\pi n r_+}{\ell} - 1 - \frac{(r^2 - r_+^2)}{r_+^2} \cosh \left( \frac{r_+^2}{\ell^2} \left( \frac{\Delta \tau}{b} - i\epsilon \right) \right) \right]^{-\frac{1}{2}}$$

$$g_2(\Delta \tau) = \sum_{n=-\infty}^{\infty} \left[ \frac{r_+^2}{r_+^2} \cosh \frac{2\pi n r_+}{\ell} + 1 - \frac{(r^2 - r_+^2)}{r_+^2} \cosh \left( \frac{r_+^2}{\ell^2} \left( \frac{\Delta \tau}{b} - i\epsilon \right) \right) \right]^{-\frac{1}{2}}.$$

We will demonstrate the KMS property for each term in these sums.

Take a typical term in the sum. It has singularities at

$$\Delta \tau_n = \pm \Delta \tau_n^0 + \frac{i}{T} p + i\epsilon$$

where $p$ is an integer. These singularities are square root branch points and the branch cuts go from $(\Delta \tau_n^0 + 2\pi \frac{i}{T} p + i\epsilon \rightarrow \infty + \frac{i}{T} p + i\epsilon)$ and $(\Delta \tau_n^0 + 2\pi \frac{i}{T} p + i\epsilon \rightarrow -\infty - 2\pi \frac{i}{T} p + i\epsilon)$. In any region without the branch cuts, $g_1$ and $g_2$ are analytic. Going around a branch point gives an additional minus sign. Now for a given $n$, if $\Delta \tau$ is such that the expression inside the square root is positive, then $|\text{Real } \Delta \tau| < \Delta \tau_n$. In this region, $g_1^n$ and $g_2^n$ are analytic and periodic in $\frac{\tau}{T}$. What’s more $g^n(-\Delta \tau) = g^n(\Delta \tau)$ as $\epsilon \rightarrow 0$. If on the other hand the expression inside the square root is negative, then because of the branch cuts, $g^n(\Delta \tau - \frac{i}{T}) = -g^n(\Delta \tau)$. As $g^n(\Delta \tau) = (-A + i\epsilon \text{ sign } \Delta \tau)^{-\frac{1}{2}}$ and because our definition of the square root is with a branch cut along the negative real axis, we see that $g^n(\Delta \tau) = -g^n(-\Delta \tau)$ ($A$ is only a function of $|\Delta \tau|$). This shows that the KMS condition is satisfied, and hence that $G^+$ is a thermal Green’s function.
C. Identifying the Vacuum State

In the region $R$ where $r > r_+$, the Kruskal co-ordinates are defined as

\[ U = \left(\frac{r - r_+}{r + r_+}\right)^{\frac{1}{2}} \cosh \frac{r_+ t}{\ell^2} \]

\[ V = \left(\frac{r - r_+}{r + r_+}\right)^{\frac{1}{2}} \sinh \frac{r_+ t}{\ell^2}. \]

Defining $\bar{V} = V - U$ and $\bar{U} = V + U$, $r$ is given by

\[ \frac{r}{r_+} = \frac{1 - \bar{U}\bar{V}}{1 + \bar{U}\bar{V}}. \]

In these co-ordinates the two point function becomes

\[ G_j^+(\bar{U}, \bar{V}, \phi; \bar{U}'\bar{V}', \phi') = \frac{1}{\sqrt{24\pi\ell}} \sum_n \left\{ \frac{1}{(1 + \bar{U}\bar{V})(1 + \bar{U}'\bar{V}')} \times \right. \]

\[ \left[ (1 - \bar{U}\bar{V})(1 - \bar{U}'\bar{V}') \cosh \left( \frac{\bar{r}_+}{\ell}(\Delta \phi + 2\pi n) \right) \mp (1 + \bar{U}\bar{V})(1 + \bar{U}'\bar{V}') + 2(\bar{U}\bar{V}' + \bar{U}'\bar{V}) + 2i\epsilon \text{ sign} \Delta V \right]\}

\[ \left. \right\}^{-\frac{1}{2}}. \]

where $\mp$ is for $J = 1, 2$. Here the sign of $i\epsilon$ is the same as sign $\Delta V$ which is the same as

\[ \text{sign} \Delta \lambda \] for timelike separated points. For $x, x' \in R$, this is just sign$\Delta t = \text{sign}(\bar{V}\bar{U}' - \bar{U}\bar{V}')$.

This expression is valid all over the Kruskal manifold.

As discussed in the Introduction, the Hartle-Hawking Green’s function is defined to be analytic and bounded in the lower half complex plane of $\bar{V}$ on the past horizon ($\bar{U} = 0$), when $\bar{U}', \bar{V}', \phi, \phi'$ are real, or in the lower half plane of $\bar{U}$ on the future horizon ($\bar{V} = 0$).

On the past horizon $\bar{U} = 0$ we have

\[ G_j^+ = \frac{1}{\sqrt{24\pi\ell}} \sum_n \left\{ \frac{1}{1 + \bar{U}'\bar{V}'} \left[ (1 - \bar{U}'\bar{V}') \cosh \left( \frac{\bar{r}_+}{\ell}(\Delta \phi + 2\pi n) \right) \mp (1 + \bar{U}'\bar{V}') + 2\bar{V}\bar{U}' + 2i\epsilon \text{ sign} \Delta V \right]\}^{-\frac{1}{2}}. \]

In order to prove analyticity and boundedness we will show that the singularities occur in the upper half plane of $\bar{V}$. Hence every term in the sum is a holomorphic function in the lower half plane. We will then use Wierstrass’s Theorem on sums of holomorphic functions in order to prove that the $G_j^+$ are analytic in the lower half of the complex $\bar{V}$ plane.

$G_j^+$ has singularities when

\[ \bar{V} = \pm(1 + \bar{U}'\bar{V}') - (1 - \bar{U}'\bar{V}') \cosh \left( \frac{\bar{r}_+}{\ell}(\Delta \phi + 2\pi n) \right), \]

Now suppose that $x' \in R$, then $-1 \leq \bar{U}'\bar{V}' < 0$ and $\bar{U}' > 0$. Defining $\bar{U}\bar{V}' = -a$, $(1 > a > 0)$, the singularity occurs at

\[ \bar{V} = \pm(1 - a) - (1 + a) \cosh \left( \frac{\bar{r}_+}{\ell}(\Delta \phi + 2\pi n) \right), \]
We see that when $\epsilon \to 0$, $\bar{V}$ is real and negative. Hence $\bar{V} = \frac{-A}{1 + i\epsilon}$ with $A > 0$, so that the singularities are in the upper half plane. Similarly, for the future horizon $\bar{V} = 0$, there are singularities when

$$\bar{U} = \pm (1 + \bar{U}'\bar{V}') - (1 - \bar{U}'\bar{V}') \cosh(\Delta \phi + 2\pi n)$$

with $A > 0$, so that the singularities are in the upper half plane. Similarly, for the future horizon $\bar{V} = 0$, there are singularities when $\bar{U} = \bar{U}' = \pm(1 + \bar{U}'\bar{V}') - (1 - \bar{U}'\bar{V}') \cosh(\Delta \phi + 2\pi n)$.

For $x' \in \mathbb{R}$, then $-1 < \bar{U}'\bar{V}' < 0$, and $\bar{V}' < 0$, so that $\bar{U} = \frac{-A}{1 - i\epsilon} = A + i\epsilon$, with $A > 0$, so the singularities are in the upper half plane of $\bar{U}$. At this point it should be noted that for $G_J$ we get singularities in the lower half plane of $\bar{U}$ on the surface $\bar{V} = 0$, and singularities in the lower half plane of $\bar{V}$ on the surface $\bar{U} = 0$.

For $x' \in \mathbb{F}$, if $\bar{U} = 0$ and $x$ and $x'$ connected by a null geodesic, then $\Delta V < 0$. This is the case because for timelike and null separations, $\text{sign} \Delta V = -\text{sign} \Delta r$ ($r$ is a timelike co-ordinate in $\mathbb{F}$) and $\Delta r$ is always positive if $x$ is on the horizon. Then it can be checked that the singularities are again in the upper half plane of either $\bar{U}$ or $\bar{V}$.

Now that we have established that each term in the infinite sum is holomorphic in the lower half plane of $\bar{V}$ on the past horizon (and in $\bar{U}$ on the future horizon) we will use Weierstrass’s Theorem. This states that if a series with analytic terms $f(z) = f_1(z) + f_2(z) + \cdots$ converges uniformly on every compact subset of a region $\Omega$, then the sum $f(z)$ is analytic in $\Omega$, and the series can be differentiated term by term. It is easily seen that unless $\bar{U}'\bar{V}' = 1$, i.e. $x'$ is at the singularity, the sum converges uniformly on every compact subset of the lower half plane. For $\bar{U}'\bar{V}' = 1$ the sum diverges and the Green’s function becomes singular at $r = 0$. This is because $r = 0$ is a fixed point of the identification.

To conclude we have shown that our Green’s function is analytic on the past horizon in the lower half $\bar{V}$ complex plane, and similarly on the future horizon in the lower half $\bar{U}$ complex plane. Its singularities occur when $x, x'$ can be connected by a null geodesics either directly or after reflection at infinity (see Appendix A and Ref. [7]). We conclude that the Green’s function we have constructed is the Hartle-Hawking Green’s function as defined in the Introduction, for both Neumann and Dirichlet boundary conditions.

**D. The $M = 0$ Green’s function**

The black hole solution as $M \to 0$ is the spacetime with metric

$$ds^2 = -\left(\frac{T}{r}\right)^2 dt^2 + \left(\frac{\ell}{r}\right)^2 dr^2 + r^2 d\phi^2$$

with $r > 0$, and $t$ and $\phi$ as in (2.1). Defining $z = \frac{\ell}{T}$ and $\gamma = \frac{T}{\ell}$ the metric becomes

$$ds^2 = \frac{\ell^2}{z^2}(-dt^2 + dz^2 + d\phi^2).$$

The modes for a massless conformally coupled scalar field are solutions of the equation
\[ \Box \phi - \frac{1}{8} R \phi = 0 \]

where again \( R = -6\ell^{-2} \), and are given by

\[ \phi_{km} = N_\omega \sqrt{\frac{\ell}{l}} e^{-i\omega t} e^{im\phi} e^{ikz} \]

where \( \omega^2 = k^2 + m^2 \), \( m \) is an integer, and \( N_\omega = (8\pi^2 \omega)^{\frac{1}{2}} \) is a normalization constant such that \( \langle \psi_{mk}, \psi_{m'k'} \rangle = \delta_{mm'} \delta(k - k') \).

As in quantization on AdS_3, care must be taken at the boundary \( z = 0 \), which is at spatial infinity. The metric (3.4) is conformal to Minkowski spacetime with one spatial coordinate periodic and the other restricted to be greater than zero. As in the case of AdS_3, we impose the boundary conditions

\[ \frac{1}{\sqrt{z}} \psi = 0 \quad \text{or} \quad \frac{\partial}{\partial z} \left( \frac{1}{\sqrt{z}} \psi(z) \right) = 0 \]

at \( z = 0 \), corresponding to Dirichlet or Neumann boundary conditions in the conformal Minkowski metric. Our approach will be to first calculate the Green’s function without boundary conditions and then use the method of images to impose them.

Summing modes, we obtain the two point function

\[ \tilde{G}(x, x') = \frac{1}{8\pi^2 \omega} \frac{\sqrt{zz'}}{\ell} \sum_m \int_k e^{-i\omega \Delta t} e^{im\Delta \phi} e^{ik\Delta z} dk \]

\[ = \frac{1}{2\pi} \frac{\sqrt{zz'}}{\ell} \sum_m e^{im\Delta \phi} G_2(y, y', m) \]

where \( G_2(y, y', m) \) is the massive 1+1 dimensional Green’s function and \( y = (\gamma, z) \).

Now [11],

\[ G_2(y, y', m) = \frac{1}{2\pi} K_0(|m| d) \quad m \neq 0 \]

\[ = -\frac{1}{2\pi} \log d + \lim_{n \to 2} \frac{\Gamma \left( \frac{n}{2} - 1 \right)}{n \pi^{\frac{n}{2}}} \quad m = 0 \]

where \( d = \epsilon + i\Delta \text{ sign } \Delta t \), with \( \Delta = (\Delta \gamma)^2 - (\Delta z)^2 \frac{1}{2} \) for timelike separation, and \( d = ((\Delta z)^2 - (\Delta \gamma - i\epsilon)^2)^{\frac{1}{2}} \) for spacelike separation. Here \( K_0 \) is a modified Bessel function. It follows that

\[ \tilde{G}(x, x') = \frac{1}{4\pi^2} \frac{\sqrt{zz'}}{\ell} \left[ 2 \sum_{m>0} \cos(m\Delta \beta) K_0(md) - \log d \right] \]

where the infinite constant in the \( m = 0 \) expression was dropped to regularize the infrared divergences of the 1+1 dimensional Green’s function. Using [13]

\[ \sum_{m=1}^{\infty} K_0(mx) \cos(mxt) = \frac{1}{2} \left( c + \ln \frac{x}{4\pi} \right) + \frac{\pi}{2\sqrt{x^2 + (xt)^2}} + \frac{\pi}{2} \sum_{l \neq 0} \left( x^2 + (2\ell \pi - tx)^2 \right)^{-\frac{1}{2}} - \frac{1}{2\ell \pi}. \]
Here $c$ is the Euler constant. This expression is valid for $x > 0$ and real $t$, and gives the following expression for spacelike separated $x$ and $x'$:

$$
\tilde{G}(x, x') = \frac{1}{4\pi} \sqrt{z'z} \left[ \sum_n \left[ (\Delta z)^2 + (\Delta \phi + 2\pi n)^2 - (\Delta \gamma - i\epsilon)^2 \right]^{-\frac{1}{2}} - \sum_{n \neq 0} \frac{1}{2\pi n} + c_1 \right] = \frac{1}{4\pi} \sqrt{z'z'} F(x, x')
$$

Here $c_1 = c - \ln 4\pi$. Although the above formula was only true for $x > 0$ and real $t$, the result is analytic for every real $\Delta z$, $\Delta \gamma$ and $\Delta \phi$. Hence it is also correct for timelike separated points.

This is just what is expected from the conformality to the Minkowski space, other than the $\sum \frac{1}{2\pi n} - c_1$ which regularizes $\tilde{G}$. Now the boundary condition can be easily put in by writing

$$
G^+(x, x') = \frac{1}{4\pi} \sqrt{z'z'} \left( F(x, x') \pm F(x, \bar{x}') \right)
$$

where $\bar{x}' = (\gamma', -z', \phi)$. Notice that for Dirichlet boundary conditions, this agrees with the $M \to 0$ limit of (3.2) and (3.3).

Going to the $(t, r, \phi)$ co-ordinates we have

$$
G^+_{M=0} = \frac{1}{4\pi} (rr')^{-\frac{1}{2}} (G^+_1 \pm G^+_2)
$$

with

$$
G^+_1 = \sum_n \left[ \ell^2 \left( \frac{r'}{rr'} \right)^2 + (\Delta \phi + 2\pi n)^2 - (\Delta t - i\epsilon \ell) \right]^{-\frac{1}{2}} - \sum_{n \neq 0} \frac{1}{2\pi n} + c_1
$$

$$
G^+_2 = \sum_n \left[ \ell^2 \left( \frac{r'}{rr'} \right)^2 + (\Delta \phi + 2\pi n)^2 - (\Delta t - i\epsilon \ell) \right]^{-\frac{1}{2}} - \sum_{n \neq 0} \frac{1}{2\pi n} + c_1
$$

E. Computation of $\langle \phi^2 \rangle$

$\langle \phi^2 \rangle$ is defined as $\langle \phi^2 \rangle = \lim_{x \to x'} \frac{1}{2} G_{\text{Reg}}(x, x')$ where $G = G^+ + G^-$ is the symmetric Green’s function. In order to compute $\langle \phi^2 \rangle$, we need to regularize $G$. Now only the $n = 0$ term in $G^+_1$ is infinite and is just a Green’s function on AdS$_3$. Hence, we can use the Hadamard development in AdS$_3$ to regularize $G$ [11]:

$$
G_{\text{Had}} = \frac{-i \Delta \frac{1}{2}}{2\sqrt{2\pi} \sigma^2}
$$

where
\[ \sigma = \frac{\ell^2}{2} \left[ ar \cos Z \right]^2, \quad \Delta^{-\frac{1}{2}} = \frac{\sin \left( \frac{2\rho}{\ell} \right)^{\frac{1}{2}}}{\left( \frac{2\rho}{\ell} \right)^{\frac{1}{2}}} \quad \text{and} \quad Z = \frac{\cos \Delta \lambda - \sin \rho \sin \rho' \cos \Delta \theta}{\cos \rho \cos \rho'} \]

(here \( \Delta \) is the Van Vleck determinant). Defining

\[ G_{\text{Reg}}(x, x') = G_{\text{BH}}(x, x') - G_{\text{Had}}(x, x') \]

we get

\[ \langle \phi^2 \rangle = \frac{1}{4\sqrt{2\pi^2 / \ell}} r_+ \left[ \sum_{n \neq 0} \left( \cosh \left( \frac{r_+}{\ell} 2\pi n \right) - 1 \right)^{-\frac{1}{2}} \pm \sum_{n} \left( \cosh \left( \frac{r_+}{\ell} 2\pi n \right) - 1 + 2 \left( \frac{r_+}{r} \right)^2 \right)^{-\frac{1}{2}} \right] \]

which, for Dirichlet boundary conditions, can be seen to be regular as \( M \to 0 \) (that is \( r_+ \to 0 \)), and to coincide in this limit with the \( M = 0 \) result for Dirichlet boundary conditions.

### IV. THE ENERGY-MOMENTUM TENSOR

The energy-momentum tensor for a massless conformally coupled scalar field in \( \text{AdS}_3 \) is given by the expression

\[ T_{\mu \nu}(x) = \frac{3}{4} \partial_\mu \phi(x) \partial_\nu \phi(x) - \frac{1}{4} g_{\mu \nu} g^{\rho \sigma} \partial_\rho \phi(x) \partial_\sigma \phi(x) - \frac{1}{4} \nabla_\mu \partial_\nu \phi(x) \phi(x) + \frac{1}{96} g_{\mu \nu} R \phi^2(x) \]

where \( R = -6\ell^{-2} \). In order to compute \( \langle T_{\mu \nu} \rangle \) one differentiates the symmetric two-point function \( G = \langle 0 | \phi(x) \phi(x') + \phi(x') \phi(x) | 0 \rangle \) \[^{11}\], and then takes the coincident point limit. This makes \( \langle T_{\mu \nu} \rangle \) divergent and regularization is needed. A look at our Green’s function reveals that only the \( n = 0 \) term in \( G_1 \) diverges as \( x \to x' \), so only the \( \langle T_{\mu \nu} \rangle \) derived from it should be regularized.

The \( n = 0 \) term is just the Green’s function in \( \text{AdS}_3 \) in accelerating co-ordinates. The vacuum in which this Green’s function is derived is symmetric under the Anti de Sitter group and \( \text{AdS}_3 \) is a maximally symmetric space. Hence \[^{10}\] \( \langle T_{\mu \nu} \rangle = \frac{1}{3} g_{\mu \nu} \langle T \rangle \) where \( \langle T \rangle = g^{\mu \nu} \langle T_{\mu \nu} \rangle \). For a conformally coupled massless scalar field \( \langle T \rangle = 0 \) (there is no conformal anomaly in 2+1 dimensions) so \( \langle T_{\mu \nu}^{\text{AdS}} \rangle = 0 \).

Having shown that we may drop the \( n = 0 \) term in \( G_1 \), after a somewhat lengthy calculation we arrive at the result for \( M \neq 0 \),

\[ \langle T_{\mu}^\nu(x) \rangle = \frac{1}{16\pi^3 r^3} \sum_{n > 0} \left\{ \left[ \frac{r^2}{2} f_n^{-1} \left[ 1 \pm \left( 1 + (fnr)^{-2} \right)^{-\frac{1}{2}} \right] + f_n^{-3} \right] \text{ diag}(1, 1, -2) \right\} \]

\[ \pm \frac{3}{2} \left( 1 - \frac{r_+^2}{r^2} \right) f_n^{-3} \left( 1 + (fnr)^{-2} \right)^{-\frac{5}{2}} \text{ diag}(1, 0, -1) \]  

(4.1)

where \( f_n = \sinh(\frac{r_+}{\ell} 2\pi n)/r_+ \). \( \text{diag}(a, b, c) \) is in \( (t, r, \phi) \) co-ordinates. As expected the \( n = 0 \) term from \( G_2 \) did not contribute.

For \( M = 0 \) we get from Sec. \[^{III D}\]
\[ \langle T^\mu_\nu(x) \rangle = \frac{1}{16\pi r^3} \sum_{n>0} \left\{ \frac{1}{(n\pi)^3} \text{diag}(1,1,-2) \pm \frac{3}{2(n\pi)^3} \left(1 + (f_n r)^{-2}\right)^{-\frac{3}{2}} \text{diag}(1,0,-1) \right\} \] (4.2)

where now \( f_n = \pi n/\ell \). Note that the \( M = 0 \) result agrees with the \( M \rightarrow 0 \) limit of (4.1).

Some properties of \( \langle T^\mu_\nu \rangle \) are:

- As we can see from (4.1), far away from the black hole, \( \langle T^\mu_\nu \rangle \) obeys the strong energy condition [10] only for the Dirichlet boundary conditions, while for the Neumann boundary conditions, the energy density is negative in this limit.

- For Dirichlet boundary conditions, as \( M \) decreases, although the temperature decreases, the energy density increases; just the opposite occurs for Neumann boundary conditions.

- In the limit \( M \rightarrow \infty \), \( \langle T^\mu_\nu \rangle \rightarrow 0 \) for both sets of boundary conditions, which suggests the presence of a Casimir effect.

- On the horizon, \( \langle T^\mu_\nu \rangle \) is regular, and hence in the semiclassical approximation, the horizon is stable to quantum fluctuations; on the other hand, at \( r = 0 \), \( \langle T^\mu_\nu \rangle \) diverges.

- Our Green’s function was thermal in \((t,r,\phi)\) co-ordinates, but although \( \langle T^\mu_\nu \rangle \sim T^3_{\text{loc}} \) for large \( r \), it is not of a thermal type [13].

V. THE RESPONSE OF A PARTICLE DETECTOR

In this section we calculate the response of a particle detector which is stationary in the black hole co-ordinates \((t,r,\phi)\), and outside the black hole. The simplest particle detector can be described by an idealized point monopole coupled to the quantum field through an interaction described by \( L_{\text{int}} = cm(\tau) \phi[x(\tau)] \) where \( \tau \) is the detector’s proper time, and \( c \ll 1 \). The probability per unit time for the detector to undergo a transition from energy \( E_1 \) to \( E_2 \) [11] is \( R(E_1/E_2) = c^2 |\langle E_2|m(0)|E_1 \rangle|^2 F(E_2 - E_1) \) to lowest order in perturbation theory, where

\[
F(\omega) = \lim_{\tau_0 \to \infty} \lim_{s \to 0} \frac{1}{2\tau_0} \int_{-\tau_0}^{\tau_0} d\tau \int_{-\tau_0}^{\tau_0} d\tau' e^{-i\omega(\tau-\tau')-S|\tau|+S|\tau'|} g(\tau,\tau').
\]

\( g(\tau,\tau') = G^+(x(\tau),x(\tau')) \) and \( x(\tau) \) is the detector trajectory.

\( F(\omega) \) is called the response function. It represents the bath of particles that the detector sees during its motion [17]. We take \( x(\tau) = (\frac{\tau}{b},r,\phi) \) where \( b = \left(\frac{r^2-r^2}{\ell^2}\right)^{\frac{1}{2}} \). Because \( g(\tau,\tau') = g(\Delta \tau) \), then

\[
F(\omega) = \int_{-\infty}^{\infty} e^{-i\omega \Delta \tau} g(\Delta \tau) d(\Delta \tau)
\] (5.1)

where \( g(\Delta \tau) = g_1(\Delta \tau) \pm g_2(\Delta \tau) \). Here
\[ g_1(\Delta \tau) = \frac{r^+_+}{\sqrt{24\pi \ell}} (r^+ - r^2)^{-\frac{1}{2}} \sum_n \left[ \frac{r^2}{r^2 - r^+} \left( \frac{r^2}{r^+} \cosh \left( \frac{r^+_+ 2\pi n}{\ell} \right) \mp 1 \right) - \cosh \frac{r^+_+}{\ell^2} (b - i\epsilon) \right]^{-\frac{1}{2}}. \]  

(5.2)

In this expression \(-(+)\) is for \(g_1(g_2)\).

Defining

\[ \cosh \alpha_n = \frac{r^2}{r^2 - r^2} \left( \frac{r^2}{r^+} \cosh \left( \frac{r^+_+ 2\pi n}{\ell} \right) - 1 \right) \]

and

\[ \cosh \beta_n = \frac{r^2}{r^2 - r^2} \left( \frac{r^2}{r^+} \cosh \left( \frac{r^+_+ 2\pi n}{\ell} \right) + 1 \right) \]

we have from Appendix B that

\[ F(\omega) = \frac{1}{2} \frac{1}{e^{\omega/T} + 1} \sum_n \left( P_{\frac{\omega}{2\pi T} - \frac{1}{2}} (\cosh \alpha_n) \pm P_{\frac{\omega}{2\pi T} + \frac{1}{2}} (\cosh \beta_n) \right) \]

where \( T = \frac{r^+_+}{2\pi (r^2 - r^2)^2} \) is the local temperature. This looks like a fermion distribution with zero chemical potential and a density of states

\[ D(\omega) = \frac{\omega}{2\pi} \sum_n \left( P_{\frac{\omega}{2\pi T} - \frac{1}{2}} (\cosh \alpha_n) \pm P_{\frac{\omega}{2\pi T} + \frac{1}{2}} (\cosh \beta_n) \right). \]

Notice that \( F(\omega) \) is finite on the horizon in contrast with black holes in two and four dimensions (see [18]). This seems to be a consequence of the Fermi type distribution. Statistical inversion in odd-dimensional flat spacetime was first noted in Ref. [12].

If the mass of the black hole satisfies \( e^{2\pi \sqrt{M}} \gg 1 \) and \( 2\ell^{-1} > \omega \gg T \), then far from the horizon, \( r \gg r^+ \), we can sum the series, and for Dirichlet boundary conditions we obtain

\[ F(\omega) \approx 2\pi^2 \ell^2 T^2 \frac{1}{e^{\omega/T} + 1} \left[ \left( \frac{\omega}{2\pi T} \right)^2 + \frac{1}{4} + \frac{8e^{-3\pi \sqrt{M}}}{\pi} \left( \frac{\omega}{T} \right)^\frac{3}{2} \left( \sin \frac{w\sqrt{M}}{T} - \cos \frac{w\sqrt{M}}{T} \right) \right]. \]

A similar result holds for Neumann boundary conditions at large \( r \),

\[ F(\omega) \approx \frac{1}{e^{\omega/T} + 1} \left[ 1 + 4 \left( \frac{\omega}{T} \right)^{-\frac{1}{2}} e^{-\pi \sqrt{M}} \left( \sin \frac{w\sqrt{M}}{T} + \cos \frac{w\sqrt{M}}{T} \right) \right] \]

where the approximation improves for large \( M \) as before.

It seems clear that the particle detector response will consist of a Rindler-type effect [11], and, if present, a response due to Hawking radiation (real particles). The former is due to the fact that a stationary particle detector is actually accelerating, even when \( r \to \infty \) (there is no asymptotically flat region). This is reflected for instance in the fact that for some range of \( \omega \), the behaviour of \( F(\omega) \) for \( r \gg r^+ \) and \( M \gg 1 \) is governed by the \( n = 0 \) term in \( G^+ \), which is AdS invariant. Hence all observers connected by an AdS transformation (a subgroup of the asymptotic symmetry group) register the same response, even though
they might be in relative motion; this means that $F(\omega)$ as a whole cannot be interpreted as real particles (see [13] [20] for a discussion of this point). Unfortunately, one cannot filter out these effects in a simple way, and further work is needed in order to find the spectrum of the Hawking radiation.

Finally, for $M = 0$, we may again define

$$F_i(\omega) = \int_{-\infty}^{\infty} e^{-i\omega \Delta \tau} g_i(\Delta \tau) d\Delta \tau.$$  

Now, however, $g_1$ and $g_2$ are analytic in the lower half complex plane of $\Delta \tau$. Hence for $\omega > 0$ we can close the integral in an infinite semicircle in the lower half plane and by Cauchy’s theorem $F_i(\omega) = 0$ for $\omega > 0$ so that no particles are detected by a stationary particle detector.

VI. BACK-REACTION

In this section we shall discuss the back-reaction on the BTZ solutions due to quantum fluctuations, using the energy momentum tensor $\langle T_{\mu\nu} \rangle$ derived in Sec. [V]. We shall show that for all $M$, including $M = 0$, divergences in the energy momentum tensor cause the curvature scalar $R_{\mu\nu}R^{\mu\nu}$ to blow up at $r = 0$ (note that since the energy momentum tensor is traceless, $R$ does not blow up). It is also interesting to consider the effects of back-reaction on the location of the horizon, even though this is only an order $\hbar$ effect. It is possible to show for all $M \neq 0$ that the horizon shifts outwards under the effect of quantum fluctuations. For $M = 0$, the effect is that a horizon develops at a radius of order $\hbar$, but where we may still be able to trust the semi-classical approximation.

We compute the back-reaction in the usual way by inserting the expectation value of the energy-momentum tensor (4.1) or (4.2), into Einstein’s equations,

$$G_{\mu\nu} = \ell^{-2}g_{\mu\nu} + \pi \langle T_{\mu\nu} \rangle.$$  

The first thing to note is that although the external solution is of constant curvature everywhere, the perturbed solution is not, and the curvature scalar $R_{\mu\nu}R^{\mu\nu}$ diverges at the origin, $r = 0$. Einstein’s equations give

$$R_{\mu\nu}R^{\mu\nu} = (\pi \langle T_{\mu\nu} \rangle - 2\ell^{-2}\delta_{\mu}^{\nu})(\pi \langle T_{\mu}^{\nu} \rangle - 2\ell^{-2}\delta_{\mu}^{\nu})$$

$$= \pi^2 \langle T_{\mu\nu} \rangle \langle T_{\mu\nu} \rangle + 12\ell^{-4}$$

$$> \pi^2 \langle T_{r}^{r} \rangle^2 + 12\ell^{-4} = 12\ell^{-4} + \frac{1}{64\ell^6 r^6} \left( \sum_{n>0} f_n^{-3} \right)^2$$

The sum in the last expression is a constant depending only on $M$. In the limit as $M \to 0$, the curvature still diverges as $1/r^6$. Although the divergence in the curvature scalar occurs precisely where the semi-classical approximation is unreliable, the result does say that we must go beyond semi-classical physics in order to describe the region near $r = 0$. This seems to be the natural notion of a singularity at the semi-classical level.

Having shown that the back-reacted metric becomes singular, it remains to look at horizons. We begin with a general static, spherically symmetric metric, which we take to be
\[ ds^2 = -N^2 \, dt^2 + \frac{dr^2}{N^2} + e^{2A} \, d\phi^2, \]

where \( N \) and \( A \) are functions of \( r \) only. A linear combination of Einstein’s equations implies that

\[ (N^2)'' = 2\ell^{-2} + 2\pi \langle T^\phi_\phi \rangle. \]  

(6.1)

Integrating Eq. (6.1) once, and inserting (4.1), we obtain the result

\[ (N^2)' = 2r \ell^{-2} + \frac{1}{8\ell^3 r} \sum_{n>0} \left\{ \frac{r_n^2 f_n^{-1}}{2} \left[ 1 \pm \left( 1 + (f_n r)^{-2} \right)^{-\frac{3}{2}} \right] + f_n^{-3} \left[ 1 \pm \frac{f_n^2 r_n^2}{2} \left( 1 - \left( 1 + (f_n r)^{-2} \right)^{-\frac{3}{2}} \right) \right] \right\} \]

(6.2)

where an integration constant has been included to make the result finite. A second integration gives

\[ N^2 = \frac{r^2}{\ell^2} - M - \frac{1}{8\ell^3 r} \sum_{n>0} \left\{ \frac{r_n^2 f_n^{-1}}{2} \left[ 1 \pm \left( 1 + (f_n r)^{-2} \right)^{-\frac{3}{2}} \right] + f_n^{-3} \left[ 1 \pm \frac{f_n^2 r_n^2}{2} \left( 1 + (f_n r)^{-2} \right)^{-\frac{3}{2}} \right] \right\} \]

(6.3)

where the second integration constant has been set to \( M \), and is the ADM mass of the solution \([2]\). The two integration constants ensure that \( N^2 \to r^2/\ell^2 - M + o(1/r) \) as \( r \to \infty \).

Having obtained an expression for \( N \), it is also necessary to look at the \( g_{\phi\phi} \) component given by \( A \). \( A \) is given in terms of \( N \) by the equation

\[ A' = \frac{16\ell r^3 + \sum_{n>0} \left[ \frac{r_n^2 f_n^{-1}}{2} \left[ 1 \pm \left( 1 + (f_n r)^{-2} \right)^{-\frac{3}{2}} \right] + f_n^{-3} \right]}{8\ell^3 r^3 (N^2)'}, \]

which we shall not attempt to integrate, although it is easy to see that as \( r \to \infty \), \( A \to \ln r \). The important thing to notice is that \( A' \) diverges only at \( r = 0 \) or where \( (N^2)' = 0 \). If the singularity at \( r = 0 \) is to be taken seriously, it is important that \( (N^2)' \) should not vanish for any finite, non-zero \( r \). To see that this is indeed the case, note that since the quantity inside the curly brackets of Eq. (6.2), is positive for all \( r > 0 \), then so is \( (N^2)' \).

Having checked that the backreaction does not cause a qualitative change in \( g_{\phi\phi} \), and having found the exact change in \( N \), we may examine the horizon structure of the new solutions. Note that each term in the sum in \( (6.3) \) is strictly positive, and behaves as \( 1/r \) at infinity and near the origin, for any \( M \). Hence, the horizon of the \( M \neq 0 \) solutions is pushed out by quantum fluctuations, as compared with the classical solution of the same ADM mass.

The \( M = 0 \) solution, which acquires a curvature singularity due to backreaction, also develops a horizon. We regard this result as being indicative of the fact that the \( M = 0 \) solution is unstable, in the sense that the qualitative features of the solution are changed by quantum fluctuations. Recall that \( \langle T^\nu_\mu \rangle \) in this case appears to be just the Casimir energy of the spacetime as it is associated with a zero temperature Green’s function. The appearance
of a horizon may be contrasted in an obvious way with 4-dimensional Minkowski spacetime, regarded as the \( M = 0 \) limit of the Schwarzschild solution. Minkowski spacetime has no Casimir energy associated with it, and is stable in the above sense.

Note that as \( M \to 0 \), the horizon is located in a region sufficiently close to \( r = 0 \) that the semi-classical approximation may break down, i.e. fluctuations in \( \langle T_{\mu}^{\nu} \rangle \) will be of the order of \( \langle T_{\mu}^{\nu} \rangle \). However, if there are \( n \) independent scalar fields present, then the ratio of the fluctuations to \( \langle T_{\mu}^{\nu} \rangle \) becomes negligible in the vicinity of the horizon, as \( n \) becomes large.

The size of the perturbation on the metric near where the horizon develops may also be estimated. It is approximately an order of magnitude smaller than the curvature of the original solution, a result which is independent of \( n \).

We end with a speculation about the endpoint of evaporation. Notice that although classically there is a clear but puzzling distinction between the black hole solutions of BTZ, with \( M \geq 0 \), the solutions with conical singularities of Deser and Jackiw \cite{21} corresponding to \(-1 < M < 0\), and AdS\(_3\) \((M = -1)\), semiclassically the difference between the small \( M \) and negative \( M \) solutions is not so marked. Our results for \( M = 0 \) are qualitatively similar to those of Refs. \cite{22}, where it is shown that quantum fluctuations on a conical spacetime generate a singularity at the apex of the cone, shielded by an order \( \bar{h} \) horizon. One might speculate from this similarity that evaporation could continue beyond the \( M = 0 \) solution, perhaps ending at AdS\(_3\).

**VII. CONCLUSIONS**

In this paper we presented some aspects of quantization on the 2+1 dimensional black hole geometry. We obtained an exact expression for the Green’s function in the Hartle-Hawking vacuum and for the expectation value of the energy-momentum tensor, but we found some difficulty in interpreting the particle detector response as Hawking radiation. We feel that further investigation on this question is required. If the black hole evaporates, the results of section \[\text{VI}\] suggest the possibility that due to quantum fluctuations, the endpoint of evaporation may not look like the classical \( M = 0 \) solution.

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**NOTE ADDED**

After this work was completed, we received three related preprints: K. Shiraishi and T. Maki, Akita Junior College preprint AJC-HEP-18, and A. R. Steif, Cambridge preprint DAMTP93/R20, hep-th/9308032, in which a Green’s function on the BTZ black hole spacetime is computed without the use of boundary conditions at infinity; and K. Shiraishi and T. Maki, Akita Junior College preprint AJC-HEP-19.
APPENDIX A: SCALAR FIELD QUANTIZATION ON ADS$_3$

The derivation of a scalar field propagator on AdS$_3$ is reviewed. This computation is complicated by the fact that AdS$_3$ is not globally hyperbolic. In the AdS co-ordinate system defined in Sec. II, spatial infinity is the $\rho = \frac{\pi}{2}$ surface which is seen to be timelike (see Fig. 2). Information can escape or leak in through this surface in a finite co-ordinate time, spoiling the composition law property of the propagator. In order to resolve this problem and define a good quantization scheme on AdS$_3$, we follow [3] and use the fact that AdS$_3$ is conformal to half of the Einstein Static Universe (ESU) $R \times S^2$.

The metric of ESU is

$$ds^2 = -d\lambda^2 + d\rho^2 + \sin^2 \rho \, d\theta^2$$

where $-\infty < \lambda < \infty$, $0 < \rho \leq \pi$, and $0 < \theta \leq 2\pi$. Positive frequency modes on ESU are solutions of

$$\Box \psi^E - \frac{1}{8} R \psi^E = 0$$

where $R = 2$, and are given by

$$\psi_{\ell m}^E = N_{\ell m} e^{-i\omega \lambda} Y_{\ell m}(\rho, \theta) \quad \omega > 0$$

(A1)

where $Y_{\ell m}$ are the spherical harmonics, $\omega = \ell + \frac{1}{2}$, $m$ and $\ell$ are integers with $\ell \geq 0$, $|m| \leq \ell$, and $N_{\ell m} = \frac{1}{\sqrt{2\ell + 1}}$. These modes are orthonormal in the inner product $[\Box]$

$$(\psi_1, \psi_2) = -i \int_\Sigma \psi_1^\dagger \mathcal{J}_\mu \psi_2^* [-g_\Sigma(x)]^{1/2} d\Sigma^\mu$$

where $\Sigma$ is a spacelike Cauchy surface. i.e. $(\psi_{\ell m}, \psi_{\ell m'}) = \delta_{\ell \ell'} \delta_{mm'}$, $(\psi_{\ell m}, \psi_{\ell m'}^*) = 0$, and $(\psi_{\ell m}^*, \psi_{\ell m'}^*) = -\delta_{\ell \ell'} \delta_{mm'}$. As usual the field operator is expanded in these modes $\phi = \sum_{\ell, m} \psi_{\ell m} a_{\ell m} + \psi_{\ell m}^+ a_{\ell m}^+$ so that $a, a^+$ destroy and create particles, and define the vacuum state $|0\rangle_E$.

The two point function is defined as

$$G_E^+(x, x') = \langle 0 | \phi(x) \phi(x') | 0 \rangle_E = \sum \psi_{\ell m}^E(x) \psi_{\ell m}^* (x') .$$

Inserting (A1),

$$G_E^+(x, x') = \sum_{\ell} \frac{1}{2\ell + 1} e^{-i(\ell + \frac{1}{2})(\lambda - \lambda')} \sum_m Y_{\ell m}(\rho, \theta) Y_{\ell m}^*(\rho', \theta').$$

Using $(Y_{\ell m}^*)^* = (-1)^m Y_{\ell m}$ and $\sum_{m=0}^{\ell} (-1)^m Y_{\ell m}(x) Y_{\ell m}(x') = \frac{2\ell + 1}{4\pi} P_\ell (\cos \alpha)$ where $\alpha$ is the angle between $(\rho, \theta)$ and $(\rho', \theta')$, we get

$$G_E^+(x, x') = \frac{1}{4\pi} e^{-i(\lambda - \lambda')} \sum_{\ell=0} e^{-i\ell(\lambda - \lambda')} P_\ell (\cos \alpha).$$

Further, using $\sum_{n=0}^{\infty} P_n (x) z^n = (1 - 2xz + z^2)^{-1/2}$ for $-1 < x < 1$ and $|z| < 1$ and as usual giving $\Delta \lambda$ a small negative imaginary part for convergence, we get
\[ G_E^+ = \frac{1}{4\sqrt{2\pi}} (\cos(\Delta \lambda - i\epsilon) - \cos \rho \cos \rho' - \sin \rho \sin \rho' \cos \Delta \theta)^{-\frac{1}{2}} \]

where the square root is defined with a branch cut along the negative real axis and the argument function is between \((-\pi, \pi]\). From now we shall call this two point function \(G_{1,E}^+\) and define \(G_{2,E}^+(x,x') = G_{1,E}^+(\bar{x},x')\) where \(\bar{x} = (\lambda, \pi - \rho, \theta)\). Then,

\[ G_{2,E}^+ = \frac{1}{4\sqrt{2\pi}} (\cos(\Delta \lambda - i\epsilon) + \cos \rho \cos \rho' - \sin \rho \sin \rho' \cos \Delta \theta)^{-\frac{1}{2}} \]

and \(G_{2,E}^+\) satisfies also the homogeneous equation \((\Box - \frac{1}{8}R)G = 0\). Conformally mapping these solutions to \(\text{AdS}_3\), where \(G_A^+ = \sqrt{\cos \rho \cos \rho'} G_E^+\) we get

\[ G_{1,A}^+(x,x') = \frac{1}{4\sqrt{2\pi \ell}} (\cos(\Delta \lambda - i\epsilon) \sec \rho \sec \rho' - 1 - \tan \rho \tan \rho' \cos \Delta \theta)^{-\frac{1}{2}} \]

and

\[ G_{2,A}^+(x,x') = \frac{1}{4\sqrt{2\pi \ell}} (\cos(\Delta \lambda - i\epsilon) \sec \rho \sec \rho' + 1 - \tan \rho \tan \rho' \cos \Delta \theta)^{-\frac{1}{2}}. \]

It can be seen that \(G_{1,A}^+\) and \(G_{2,A}^+\) are functions of \(\sigma(x,x') = \frac{1}{2}[(u-u')^2 + (v-v')^2 + (x-x')^2 + (y-y')^2]\), which is the distance between the spacetime points \(x, x'\) in the 4-dimensional embedding space.

In order to deal with the problem of global hyperbolicity, it was shown in [3] that imposing boundary conditions on the ESU modes gives a good quantization scheme on the half of ESU with \(\rho \leq \frac{\pi}{2}\), thus inducing a good quantization scheme on \(\text{AdS}_4\). It may be checked that this method also works in 2+1 dimensions. The boundary conditions on the ESU modes are either Dirichlet

\[ \psi_{\ell,m}^E(\rho = \frac{\pi}{2}) = 0 \quad \text{obeyed by } \psi_{\ell,m} \text{ with } \ell + m = \text{ odd} \]

or Neumann

\[ \frac{\partial}{\partial \rho} \psi_{\ell,m}^E(\rho = \frac{\pi}{2}) = 0 \quad \text{obeyed by } \psi_{\ell,m} \text{ with } \ell + m = \text{ even}. \]

It is easily verified that the combination \(G_E^+ = G_{1,E}^+ \pm G_{2,E}^+\) has the right boundary condition where the \(+(-)\) signs are for Neumann (Dirichlet) boundary conditions.

Some remarks are in order: if \(x, x'\) are restricted such that \(-\pi < \lambda(x) - \lambda(x') < \pi\) then

1. \(G_{1,E}^+\) is real for spacelike points, imaginary for timelike points and singular for \(x, x'\) which can be connected by a null geodesic.

2. \(G_{2,E}^+\) has the same property when \(x \to \bar{x}\), and if \(0 \leq \rho(x'), \rho(x) < \frac{\pi}{2}\) then \(G_2\) has singularities when \(x, x'\) can be connected by a null geodesic bouncing off \(\rho = \frac{\pi}{2}\) boundary.
From this we see that if we take the modes in AdS$_3$ as
\[
\psi^{A}_{\ell,m} = (\cos \rho)^{1/2} e^{-i(\ell + \frac{1}{2})\lambda} Y^m_\ell(\rho, \theta)
\]
then these modes give rise to a well-behaved propagator \[5\]. The two point function is then
\[
G_A^+ = \sqrt{\cos \rho \cos \rho'} (G_{1,A}^+ \pm G_{2,A}^+)
\]
where $+(-)$ are for Neumann (Dirichlet). The two point function has singularities whenever $x, x'$ can be connected by a null geodesic directly or by a null geodesic bouncing off infinity (null geodesics remain null geodesics by a conformal transformation). All other properties listed before also stay the same.

Note that it is possible to define a quantization scheme on AdS$_3$ without using boundary conditions (i.e. just using $G_{1,A}^+$), which is referred to as transparent boundary conditions in Ref. \[5\]. However this requires the use of a two-time Cauchy surface, and its physical interpretation is unclear.

**APPENDIX B: CALCULATING THE RESPONSE FUNCTION**

We are interested in an integral of the type
\[
J(\omega) = \frac{\ell^2 b}{r_+} \int_{-\infty}^{\infty} e^{-\frac{\omega t}{2\pi T}} (\cosh \alpha_n - \cosh(t - i\epsilon))^{-\frac{1}{2}} dt
\]
where $T = \frac{r_+}{2\pi \ell (r^2 - r_+^2)^{\frac{1}{2}}}$ is the local temperature. $J(\omega) = I_1(\omega) + I_2(\omega) + I_3(\omega)$ where $I_1$ is the integral from $-\infty$ to $-\alpha_n$, $I_2$ is from $-\alpha_n$ to $\alpha_n$, and $I_3$ is from $\alpha_n$ to $\infty$. Recall that the square root is defined with the cut along the negative real axis. Then
\[
I_1 = \frac{\ell^2 b}{ir_+} \int_{-\infty}^{-\alpha_n} e^{-\frac{\omega t}{2\pi T}} (\cosh t - \cosh \alpha_n)^{-\frac{1}{2}}
\]
\[
I_3 = \frac{\ell^2 b}{i r_+} \int_{\alpha_n}^{\infty} e^{-\frac{\omega t}{2\pi T}} (\cosh t - \cosh \alpha_n)^{-\frac{1}{2}}
\]
\[
I_2 = \frac{2\ell^2 b}{r_+} \int_{0}^{\alpha_n} \cos \frac{\omega t}{2\pi T} (\cosh \alpha_n - \cosh t)^{-\frac{1}{2}} dt.
\]

Using \[15\]
\[
\int_{-\alpha}^{\infty} e^{-\left(\nu + \frac{1}{2}\right)t} (\cosh t - \cosh \alpha)^{-\frac{1}{2}} = \sqrt{2} Q_\nu(\cosh \alpha) \quad \text{Re} \nu > -1 \quad \alpha > 0
\]
\[
\int_{0}^{\alpha} \cosh(\nu + \frac{1}{2})t (\cosh \alpha - \cosh t)^{-\frac{1}{2}} = \frac{\pi}{\sqrt{2}} P_\nu(\cosh \alpha) \quad \alpha > 0
\]
where $P_\nu$ and $Q_\nu$ are associated Legendre functions of the first and second kind respectively, we get

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\[ I_3 = -\frac{i\sqrt{2}l^2b}{r_+}Q_{\frac{\omega}{2\pi T} - \frac{1}{2}}(\cosh \alpha_n) \]
\[ I_2 = \frac{\sqrt{2}\pi l^2b}{r_+}P_{\frac{\omega}{2\pi T} - \frac{1}{2}}(\cosh \alpha_n) \]
\[ I_1 = \frac{i\sqrt{2}l^2b}{r_+}Q_{\frac{\omega}{2\pi T} - \frac{1}{2}}(\cosh \alpha_n). \]

Now using \( Q_\nu(z) - Q_{-\nu-1}(z) = \pi \cot(\nu\pi) P_\nu(z) \) \[13\]
\[ J(\omega) = \frac{2\sqrt{2}\pi l^2b}{r_+} P_{\frac{\omega}{2\pi T} - \frac{1}{2}}(\cosh \alpha_n) \frac{1}{e^{\omega/T} + 1}, \]
and \( F_{1,2}(\omega) \), defined by (5.1) and (5.2) in an obvious way, are given by
\[ F_1(\omega) = \frac{1}{2}\frac{1}{e^{\omega/T} + 1} \sum_{n \neq 0} P_{\frac{\omega}{2\pi T} - \frac{1}{2}}(\cosh \alpha_n) \]
\[ F_2(\omega) = \frac{1}{2}\frac{1}{e^{\omega/T} + 1} \sum_{n} P_{\frac{\omega}{2\pi T} - \frac{1}{2}}(\cosh \beta_n). \]

Notice that although the formulae that we used were not correct when \( \alpha = 0 \), nevertheless the \( \alpha_0 \) term came out correctly, since
\[ \int_{-\infty}^{\infty} e^{-i\omega t} (1 - \cosh(2\pi Tt - ie))^{-\frac{1}{2}} \]
\[ = i\int_{-\infty}^{0} e^{-i\omega t} \left| \sqrt{2} \sinh(\pi Tt - ie) \right|^{-1} - i \int_{0}^{\infty} e^{-i\omega t} \left| \sqrt{2} \sinh(\pi Tt - ie) \right|^{-1} \]
\[ = \int_{-\infty}^{\infty} e^{-i\omega t} \left( \sqrt{2} i \sinh(\pi Tt) + e \right)^{-1}. \]
This gives \[12\]
\[ F_{1,0}(\omega) = \frac{T}{2} \int_{-\infty}^{\infty} e^{-i\omega t} (2i \sinh(\pi Tt) + e)^{-1} = \frac{1}{2} \frac{1}{e^{\omega/T} + 1} \]
which is exactly what we got before as \( P_1(1) = 1. \) Combining the results for \( F_1 \) and \( F_2 \), we have
\[ F(\omega) = \frac{1}{2}\frac{1}{e^{\omega/T} + 1} \sum_{n} \left( P_{\frac{\omega}{2\pi T} - \frac{1}{2}}(\cosh \alpha_n) \pm P_{\frac{\omega}{2\pi T} - \frac{1}{2}}(\cosh \beta_n) \right). \]
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FIGURES

FIG. 1. A Penrose diagram of (a) the $M \neq 0$ black hole, and (b) the $M = 0$ solution. Information can leak through spatial infinity, unless we impose boundary conditions at $r = \infty$.

FIG. 2. A Penrose diagram of AdS$_3$. Information can leak in or out through spatial infinity, and thus $\Sigma$ is not a Cauchy surface unless we impose boundary conditions at $r = \infty$. 
This figure "fig1-1.png" is available in "png" format from:

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