A lattice model related to the nonlinear Schrödinger equation

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This is a historical note. In 1981 we constructed a discrete version of quantum nonlinear Schrödinger equation. This led to our discovery of quantum determinant: it appeared in construction of anti-pod (11). Later these became important in quantum groups: it describes center of Yang-Baxter algebra. Our paper was published in Doklady Akademii Nauk vol 259, page 76 (July 1981) in Russian language.

1. Inverse scattering method is used (in classical [1, 2] and quantum cases [3]) to solve evolutionary equations of completely integrable dynamical systems. In quantum case we shall abbreviate the method to qism. It is based on Lax representation

\[ \partial_t L_n(\lambda) = M_{n+1}(\lambda)L_n(\lambda) - L_n(\lambda)M_n(\lambda) \]  

(1)

Entries of the matrices \( L_n(\lambda) \) and \( M_n(\lambda) \) are expressed in terms of dynamical variables of the lattice system (they also depend on spectral parameter \( \lambda \)). The monodromy matrix \( T^{n,m}(\lambda) = L_n(\lambda) \cdots L_m(\lambda) \ (n \geq m) \) and transfer matrix \( \tau(\lambda) = trT^{N,1}(\lambda) \) are important for qism. Recently an effective method of construction of action-angle variables was discovered, it is based on \( R \)-matrix. In classical case [4] it provides Poisson brackets of matrix elements of the monodromy matrix, while in quantum case [5, 6] it gives commutation relations:

\[ [T_c(\lambda) \otimes T_c(\mu)] = [T_c(\lambda) \otimes T_c(\mu), R_c(\lambda, \mu)] \]  

(2)

\[ R_q(\lambda, \mu) (T_q(\lambda) \otimes T_q(\mu)) = (I \otimes T_q(\mu)) (T_q(\lambda) \otimes I) R_q(\lambda, \mu) \]  

(3)

These equations lead to commutativity of transfer matrices (we use subindex c or q to distinguish classical from quantum). Meaning that \( \partial_t \ln \tau(\mu) \) is a generating functional of Hamiltonians for completely integrable systems.

In classical case E.K. Sklyanin proved [7] that corresponding equations of motion can be represented in the Lax form [11]. Here we prove that this is true in quantum case as well. Generating functional of the operators \( M_n(\lambda) \) is a matrix \( m_n(\lambda, \mu) \):

\[ m_n(\lambda, \mu) = i\tau^{-1}(\mu)\partial_\mu \tau(\mu) - iq_n^{-1}(\lambda, \mu)\partial_\mu q_n(\lambda, \mu) \]

\[ q_n(\lambda, \mu) = tr_2 (I \otimes T^{N,n}(\mu)) R_q^{-1}(\lambda, \mu) (I \otimes T^{n-1,1}(\mu)) \]

\[ i [\partial_\mu \ln \tau(\mu), L_n(\lambda)] = m_{n+1}(\lambda, \mu) - L_n(\lambda)m_n(\lambda, \mu) \]

2. Let consider nonlinear Schroedinger equation (nS). In continuous case it has a Hamiltonian

\[ H = \int dx \left( \partial_x \psi^\dagger \partial_x \psi + \kappa \psi^\dagger \psi^\dagger \psi \right) \]

\[ \{ \psi_n(x), \psi_k^\dagger(y) \} = i\delta(x - y), \quad [\psi_q(x), \psi_q^\dagger(y) = \delta(x - y)] \]  

(4)

It is integrable both in classical [8] and quantum [3, 9] cases. Corresponding \( R \)- matrix can be called quasi-classical

\[ R_q = I \otimes I - iR_c, \quad R_c = \frac{\kappa \Pi}{\lambda - \mu} \]  

(5)

Here \( I \) is identical matrix 2X2 and \( \Pi \) is permutation matrix.

Lattice generalization of nS has long attracted attention of the experts [12, 13]. We propose a new version of lattice nS both in classical and quantum cases. It is distinguishing feature is that \( R \)-matrix is the same as in
the continuous case and basic variables $\chi$ are canonical Bose fields. We start by suggesting the following $L_n$ operator:

$$
L_n(\lambda) = -\frac{i\lambda \Delta}{2} \sigma_3 / 2 + S_n^3 I + S_n^+ \sigma_+ + S_n^- \sigma_- ,
$$

$$
S^3 = 1 + \frac{\chi_n^\dagger}{2} \chi_n , \quad S_n^+ = -i \sqrt{\kappa} \chi_n^\dagger \sigma_+ , \quad S_n^- = i \sqrt{\kappa} \rho_n \chi_n
$$

$$
\rho_n^+ = \rho_n^\dagger (\chi_n^\dagger \chi_n) , \quad \rho_n^- = 1 + \frac{\chi_n^\dagger}{2} \chi_n
$$

$$
\{ \chi_{m}^c , \chi_{n}^c \} = i \Delta \delta_{m,n} , \quad [ \chi_{m}^c , \chi_{n}^c ] = \Delta \delta_{m,n}
$$

(6)

Here $\sigma$ are Pauli matrices. We consider repulsive case $\kappa > 0$ and put $\rho_n^+ = \rho_n^- = \rho_n$.

3. Here we shall discuss lattice model (6) in classical case. Simple calculations lead to

$$
T(\lambda) \sigma_2 T^t(\lambda) \sigma_2 = d_{-m+1}^n (\lambda) I
$$

$$
d_c(\lambda) = \det L_n(\lambda) = 1 + \lambda^2 \Delta^2 / 4
$$

(7)

This shows that at $\lambda = \nu = -2i / \Delta$ the $L_n(\lambda)$ operator turns into one dimensional projector. This makes it possible to calculate explicitly logarithmic derivatives of $\tau(\lambda)$ at this point, which can be represented as a sum of local densities:

$$
\partial^n \ln \tau(\lambda)|_{\lambda=\nu} = \sum_{k=1}^N h_{k,n},
$$

$$
h_{k,n} = D^n \ln \text{tr} L_{k+n} (\nu) L_{k+n-1} (\nu) \ldots L_k (\nu) L_{k-1} (\nu)|_{\lambda=\nu}
$$

Here $D^n$ is a differential operator. For small $n$ it is:

$$
D^1 = \partial_k = \frac{d}{d \nu} , \quad D^2 = 2 \partial_{k+1} \partial_k + \partial_k^2
$$

$$
D^3 = 6 \partial_{k+2} \partial_{k+1} \partial_k + 6 \partial_{k+2}^2 \partial_{k+1} + 6 \partial_{k+1} \partial_k^2 - 6 \partial_{k+2} \partial_k^2 + \partial_k^3
$$

(9)

We use this notations to define lattice classical Hamiltonian of nS

$$
H_c = D_c(\lambda) \ln \left[ (1 + \lambda / \nu)^{-N} \tau(\lambda) \right] + \text{complex conjugate}
$$

$$
D_c(\lambda) = \left( \frac{d}{d \lambda} \right)^3
$$

(10)

The explicit expression shows that this Hamiltonian describes interaction of five nearest neighbors on the lattice. In the continuous limit $[\chi_n = \psi_n \Delta; \psi_{n+1} - \psi_n = O(\Delta); \Delta \rightarrow 0, N \rightarrow \infty$ but $N \Delta = \text{const}$] it goes to the correct Hamiltonian of the continuous model and $L_n$ operator turns into correct continuous $L_n$ operator, see [8].

4. Here we construct quantum lattice nS model. Quantum analog of (7) is given by

$$
T(\lambda) \sigma_2 T^t(\lambda + i \kappa) \sigma_2 = d_{-m+1}^n (\lambda) I
$$

$$
d_q(\lambda) = \Delta^2 (\lambda - \nu)(\lambda - \nu + i \kappa) / 4
$$

(11)

This defines quantum determinant:

$$
\text{det}_q T(\lambda) = T_{11}(\lambda) T_{22}(\lambda + i \kappa) - T_{12}(\lambda) T_{21}(\lambda + i \kappa) = d_{-m+1}^n (\lambda)
$$

To define Hamiltonian of the model let us add quantum correction like in [3]:

$$
H_q = \left( D_c(\lambda) + \frac{i \kappa}{6} \frac{d}{d \lambda} \right) \ln \left[ (1 + \lambda / \nu)^{-N} \tau(\lambda) \right] + \text{hermitian conjugate}
$$

(12)

The model can be solved by qism [3]. The pseudo-vacuum $\Omega$ is annihilated by lattice Bose fields $\chi_n \Omega = 0$. The eigenvectors are given by algebraic Bethe ansatz:

$$
\Psi(\lambda_1 \ldots \lambda_n) = B(\lambda_1) \ldots B(\lambda_n) \Omega , \quad B(\lambda) = T_{12}(\lambda)
$$

These $\lambda_j$ satisfy a system of Bethe equations:

$$
\left( \frac{1 - i \lambda_j \Delta / 2}{1 + i \lambda_j \Delta / 2} \right)^N = \prod_{k \neq j} \frac{\lambda_j - \lambda_k - i \kappa}{\lambda_j - \lambda_k + i \kappa}
$$

(13)
Corresponding eigenvalue of $\tau(\lambda)$ is

$$
(1 - \frac{i\lambda\Delta}{2})^N \prod_{k=1}^{n} \frac{\lambda - \lambda_k + i\kappa}{\lambda - \lambda_k} + (1 + \frac{i\lambda\Delta}{2})^N \prod_{k=1}^{n} \frac{\lambda_k - \lambda + i\kappa}{\lambda_k - \lambda} \tag{14}
$$

From here we obtain energy levels [eigenvalues of the Hamiltonian]

$$
H_q \Psi = (\sum_{k=1}^{n} E(\lambda_k)) \Psi, \quad E(\mu) = f(\mu) + f(\bar{\mu})
$$

$$
f(\mu) = (D_c + \frac{i\kappa}{\mu - \lambda}) \ln \left(\frac{\mu - \lambda + i\kappa}{\mu - \lambda}\right) |_{\lambda = \nu}
$$

The Hamiltonian has correct continuous limit \cite{4} and $E(\mu) \rightarrow \mu^2$.

5. Quantum nS model constructed above can be considered as a generalization of XXX model with negative spin $-2/\kappa\Delta$. We can rewrite the $L_n$ operator \cite{6} in the way similar to XXX:

$$
L_n^X = -\sigma_3 L_n = i\lambda + t_n^k \otimes \sigma_k
$$

Here $t_n^k$ are simple linear combinations of $S_n^k$ from \cite{3}. They form an infinite dimensional representation of $SU(2)$ algebra, see \cite{14}.

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