On polynomial invariants of several qubits

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Abstract. It is a recent observation that entanglement classification for qubits is closely related to local \( SL(2,\mathbb{C}) \)-invariants including the invariance under qubit permutations [1, 2, 3], which has been termed \( SL^* \) invariance. In order to single out the \( SL^* \) invariants, we analyze the \( SL(2,\mathbb{C}) \)-invariants of four resp. five qubits and decompose them into irreducible modules for the symmetric group \( S_4 \) resp. \( S_5 \) of qubit permutations. A classifying set of measures of genuine multipartite entanglement is given by the ideal of the algebra of \( SL^* \)-invariants vanishing on arbitrary product states. We find that low degree homogeneous components of this ideal can be constructed in full by using the approach introduced in Refs. [2, 4]. Our analysis highlights an intimate connection between this latter procedure and the standard methods to create invariants, such as the \( \Omega \)-process [5]. As the degrees of invariants increase, the alternative method proves to be particularly efficient.

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INTRODUCTION

The quantification and classification of multipartite entanglement takes an important part in quantum information theory and is subject to a lively discussion in the recent literature. Many different sets of measures of entanglement have been proposed for introducing some order and insight into the Hilbert space of multipartite systems. An important part of the discussion addresses the underlying invariance group the measures have to have. Local unitary invariance is clearly a minimal requirement but must be extended to invariance under local special linear transformations, when generalized local operations are admitted. For qubit systems this local invariance group is the \( SL(2,\mathbb{C}) \). The invariance group of \( q \) qubits is then given by \( SL_q := SL(2,\mathbb{C})^\otimes q \), where we will even omit the index wherever it doesn’t create confusion. Interestingly, the demand of invariance under \( SL \) operations on the measures of pure state entanglement readily implies that the induced measure on mixed states is an entanglement monotone when extended through its convex roof [6, 7]. The requirement of \( SL \)-invariance is restrictive enough to even single out a distinguished class of genuine multipartite entangled states: the nonzero SLOCC classes are made of all those states that do not vanish after infinitely many \( SL \) operations (the local filtering operations of e.g. Ref. [7]). Each such nonzero Stochas-

1 That is: where either it is clear what number of qubits we are talking about, or in generic statements applying to arbitrary number of qubits. We deviate from the standard definition \( SL_q := SL(q,\mathbb{C}) \); since we deal exclusively with qubits throughout the paper, this should not cause any confusion.
tic Local Operation and Classical Communication (SLOCC) class has a representative which can be considered as maximally entangled state within that class. Interestingly, for three qubits a maximally three-tangled state with no concurrence exists, and for four qubits there are three four-tangled states with neither three-tangle nor concurrence. It is an open question whether such representatives carrying only the genuine multipartite entanglement classes will exist in general.

It is worth emphasizing at this point that any function of the pure state coefficients that is invariant under $SL$ transformations will remain unchanged by such local filtering operations. Consequently, the complementary zero SLOCC class exists that contains all states for which all these invariant functions are zero. A prime example for a representative of the zero SLOCC class is the multiqubit W state $|W⟩ = \sum_ι α_i |i⟩$ with $|i⟩ = |0⋯1⋯0⟩$ being a state with all zeros but a single 1 placed at site number $i$ (or straightforward generalizations of it to higher local dimension). Notwithstanding its globally distributed entanglement of pairs we therefore would not call it genuinely multipartite entangled.

$SL$-invariance has been intensely studied for three qubit systems in Refs. [1, 8] and for four qubit systems in Refs. [3, 9], and geometric aspects of such invariants have been highlighted in Refs. [10, 11, 12]. Preliminary results for five qubits have been presented recently [5]. Independent of these approaches, a method based on local $SL(2, C)$-invariant operators has been suggested with emphasis on permutation invariance of the global entanglement measure [2, 4]. Permutation invariance has been highlighted as a demand on global entanglement measures already in Ref. [13] and later in Ref. [3], where the semidirect product of $SL_q$ and the symmetric group $S_q$ of qubit permutations has been termed $SL^*_q := SL(2, C)^\otimes q \times S_q$, which we will abbreviate as $SL^*$. In addition, Refs. [2, 4] focus on those invariants that vanish on all product states. These form an ideal in the ring of $SL^*$-invariants and are important for distinguishing genuine multipartite entangled states from tensor products of entangled states such as $|GHZ⟩ \otimes |GHZ⟩$. It lies within the nonzero SLOCC class of 6 qubit entangled states but is not genuinely 6 qubit entangled.

In this work we will use both local invariant operators as proposed in Refs. [2, 4] and the Cayley $Ω$-process to construct polynomial invariants. First we compare both approaches for the known complete set of invariants of four qubits [9] and those for five qubits up to degree 6 (see Ref. [5]). For these known cases we follow the notation of Refs. [9, 5] and express the invariants presented there in terms of combs and filters from Refs. [2, 4]. Then we go considerably ahead up to degree 12 with an outlook to degrees 14 and 16.

The manuscript is organized as follows: the next Section reviews the approach to $SL$-invariants using local invariant operators and fixes the notation. Section summarizes the main results for 4-qubit invariants from Ref. [9] and those for five qubits up to degree 6 (see Ref. [5]). For these known cases we follow the notation of Refs. [9, 5] and express the invariants presented there in terms of combs and filters from Refs. [2, 4]. Then we go considerably ahead up to degree 12 with an outlook to degrees 14 and 16.

After presenting an interesting connection between the Cayley $Ω$-process and local invariant operators (combs) in Section we draw our conclusions in Section. The appendix provides a detailed discussion of the concepts and notations of the comb based approach.
LOCAL INVARIANT OPERATORS AND NOTATION

Before we start with our analysis, we give a brief summary of the approach using local invariant operators. For a more detailed description of this approach see the appendix.

We will refer to multiple copies of a given quantum state simply as copies in what follows. The Hilbert space for \( m \) copies of a quantum state \( \psi \) of \( q \) qubits can be written as

\[
(H_{11} \otimes H_{12} \otimes \cdots \otimes H_{iq}) \otimes \cdots \otimes (H_{m1} \otimes H_{m2} \otimes \cdots \otimes H_{mq}),
\]

where \( H_{ij} \) is the Hilbert space for the \( j \)-th qubit in the \( i \)-th copy, and \( \otimes \) is used as the tensor product sign between the Hilbert spaces of different copies of \( \psi \).

We will often use the notion of an expectation value of an operator \( \hat{O} \), which for a pure state \( |\psi\rangle \) is defined as \( \langle \psi | \hat{O} | \psi \rangle \). A qubit comb has been defined in Refs. [2, 4] as an antilinear operator acting on a single or multiple copy of a pure single qubit state \( (q = 1) \) which has zero expectation value for all such states. We point out that combs are \( SL(2, \mathbb{C}) \)-invariant operators. Two independent combs \( \sigma_2 \mathcal{C} \) and \( \sigma_\mu \mathcal{C} \bullet \sigma^{\mu} \mathcal{C} \) have been identified in terms of the Pauli matrices

\[
\sigma_0 := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},
\]

where \( \mathcal{C} \) is the complex conjugation in the eigenbasis of \( \sigma_3 \), and the contraction is defined via the pseudo-metric \( G_{\mu\nu} := \delta_{\mu\nu} g_\mu \) as

\[
\sigma_\mu \bullet \sigma^{\mu} := \sum_{\mu=0}^{3} g_\mu \sigma_\mu \bullet \sigma_\mu
\]

\[
(g_0, g_1, g_2, g_3) := (-1, 1, 0, 1)
\]

Being \( SL \)-invariant, both combs admit the construction of antilinear operators acting on multiple copies of pure multiqubit states that are \( SL \)-invariant. Polynomial invariants are then constructed from multiqubit operators obtained from combs as their antilinear expectation values for a general multiqubit pure state. These invariants are homogeneous polynomials in the basis coefficients of the state \( |\psi\rangle \) (see the appendix for more details). We will use double brackets to denote the expectation value of an antilinear operator and we often omit the tensor product sign \( \otimes \):

\[
((\sigma_2 \sigma_2)) := \langle \psi^* | \sigma_2 \otimes \sigma_2 | \psi \rangle \quad ((\sigma_\mu \sigma_\nu \cdots \bullet \sigma^{\mu} \sigma^{\nu} \cdots)) := \langle \psi^* | \bullet \langle \psi^* | (\sigma_\mu \otimes \sigma_\nu \otimes \cdots) \bullet (\sigma^{\mu} \otimes \sigma^{\nu} \otimes \cdots) | \psi \rangle \bullet | \psi \rangle.
\]

Here, \( |\psi\rangle \) is a pure state of \( q \geq 2 \) qubits (expressible as a vector in \((\mathbb{C}^2)^\otimes q\)). The double bracket expressions in Eq. (5) can be evaluated by first calculating the (antilinear)
expectation values for the two copies of $|\psi\rangle$ and performing afterwards the contractions with the pseudometric. In (4) and (5) we have shown expressions for one and two copies of the state only. The extension to more than two copies (and hence higher degree of the invariant) is defined analogously. A measure of entanglement is then defined as the absolute value of such an invariant, e.g. $C = |\langle (\sigma_2 \sigma_2) \rangle|$ is the pure state concurrence [14].

By a product state we mean a state that can be written as a tensor product on some bipartition of the system. We slightly relax the use of the term “filter” as compared with Ref. [2], where also permutation invariance was included. We will call a filter an invariant which vanishes on all product states and reserve the term SL$^*$-filter for those which are also invariant under qubit permutations. The algebra of complex holomorphic polynomial SL-invariants resp. SL$^*$-invariants of $q$ qubits will be denoted by Inv$^\text{SL}$ resp. Inv$^\text{SL}^*$. The number of qubits, $q$, will be clear from the context. The filters form an ideal $I_0^\text{SL}$ of the algebra Inv$^\text{SL}$. The SL$^*$-filters form an ideal $I_0^\text{SL}^* := I_0^\text{SL} \cap \text{Inv}^\text{SL}^*$ of Inv$^\text{SL}^*$. The subspace of Inv$^\text{SL}$, Inv$^\text{SL}^*$, etc. consisting of homogeneous invariants of degree $d$ will be denoted by adding the degree as an index. E.g. Inv$_d^\text{SL}$ and Inv$_d^\text{SL}^*$.

Let us point out four important facts. First, Inv$_d^\text{SL} = 0$ whenever $d$ is odd. Second, Inv$_2^\text{SL} = 0$ if $q$ is odd and has dimension 1 if $q$ is even. These facts are special cases of more general results proved in [15, Prop. 11.1 and Cor. 11.2]. Third, the dimension of Inv$_4^\text{SL}$ is equal to $\left(2^q - 1 + (-1)^q\right)/3$. Fourth, the dimension of Inv$_4^\text{SL}^*$ is equal to $\left\lfloor \frac{q+5}{6} \right\rfloor$, where $\lfloor t \rfloor$ denotes the Gauss parenthesis, i.e. the largest integer $n \leq t$. These two results are proved in [15, Cor. 11.4 and Prop. 11.3].

We will also use the notion of a relative SL$^*$-invariant for an SL-invariant that is fixed up to a sign under all qubit permutations. Antisymmetric relative invariants will be termed odd relative invariants or SL$^*$-invariants. In this context we will sometimes need to either symmetrize or antisymmetrize a given invariant for obtaining the corresponding SL$^*$ and SL$^*$-invariants, respectively. For an operator $\hat{O}$, whose dependence on qubit permutations is indicated by the permutation operator as an index, we use the definitions

$$\langle \hat{O}_1 \rangle_s := \frac{1}{q!} \sum_{\pi \in S_q} \hat{O}_\pi,$$

$$\langle \hat{O}_1 \rangle_a := \frac{1}{q!} \sum_{\pi \in S_q} \text{sign} \pi \hat{O}_\pi.$$

An $S_q$ orbit of an invariant $\hat{I}$ will be denoted by $S_q \circ \hat{I}$, and $\pi_{ij}$ will denote the permutation operator that exchanges qubit numbers $i$ and $j$. Furthermore we will say that an invariant is generically of degree $d$, if it is not expressible as a polynomial in invariants of lower degrees. To simplify the notation, we set $V_d := \text{Inv}_d^\text{SL}$ and denote by $U_d$ the subspace of $V_d$ spanned by the products of homogeneous lower degree invariants, i.e. by $V_s V_{d-s}$ for $s = 1, \ldots, d - 1$. For the decomposition of the space $V_d$ (or $U_d$) into simple modules $X_i$ of the symmetric group $S_q$ we use the notation of Ref. [16].
**SL AND SL⁺-INVARINTS FOR FOUR QUBITS**

The Hilbert series for $SL_4$-invariants is [9]

$$h(t) = \frac{1}{(1-t^2)(1-t^4)^2(1-t^6)}$$

$$= 1 + t^2 + 3t^4 + 4t^6 + 7t^8 + 9t^{10} + 14t^{12} + 17t^{14} + 24t^{16} + 29t^{18} + \ldots$$

From theorem 4.2 of Ref. [3] we obtain immediately the Hilbert series for $SL_\ast^4$-invariants

$$h_{SL\ast}(t) = \frac{1}{(1-t^2)(1-t^6)(1-t^{12})}$$

$$= 1 + t^2 + t^4 + 2t^6 + 3t^8 + 3t^{10} + 5t^{12} + 6t^{14} + 7t^{16} + 9t^{18} + \ldots$$

We deduce that the algebra $Inv^{SL_4}$ is a polynomial algebra with generators of degree 2, 4, 4 and 6 and, similarly, that $Inv^{SL_\ast_4}$ is a polynomial algebra with generators of degree 2, 6, 8 and 12.

Furthermore, a complete set of invariants [9] and covariants [17] is known. With the focus of finding measures for genuine multipartite entanglement, three independent filter invariants have been constructed in Ref. [2]

$$\mathcal{F}_1^{(4)} = \left( (\sigma_\mu \sigma_\nu \sigma_2 \sigma_2 \bullet \sigma^\mu \sigma_2 \sigma_\lambda \sigma_2 \bullet \sigma_2 \sigma^\lambda \sigma_2 \right)$$

$$\mathcal{F}_2^{(4)} = \left( (\sigma_\mu \sigma_\nu \sigma_2 \sigma_2 \bullet \sigma^\mu \sigma_2 \sigma_\lambda \sigma_2 \bullet \sigma_2 \sigma^\nu \sigma_2 \sigma_\tau \bullet \sigma_2 \sigma_2 \sigma^\lambda \sigma^\tau ) \right)$$

$$\mathcal{F}_3^{(4)} = \frac{1}{2} \left( (\sigma_\mu \sigma_\nu \sigma_2 \sigma_2 \bullet \sigma^\mu \sigma^\nu \sigma_2 \sigma_2 \bullet \sigma_\rho \sigma_2 \sigma_\tau \sigma_2 \bullet \sigma^\rho \sigma_2 \sigma^\tau \sigma_2 \bullet \sigma_\kappa \sigma_2 \sigma_\lambda \sigma_2 \bullet \sigma^\kappa \sigma_2 \sigma^\lambda \sigma_2 ) \right).$$

We will use this Section to work out interrelations between the two approaches.

**Degree 2**

As mentioned in the end of Section , this smallest possible degree appears only for an even number of qubits $q$ and the corresponding space $V_2$ is one-dimensional with the $q$-tangle of Wong and Christensen [18] as generator. Here, for $q = 4$, this generator is the 4-tangle and has been termed $H$ in Ref. [9]

$$H(\psi) = \frac{1}{2} (\sigma_2 \sigma_2 \sigma_2 \sigma_2 ) =: \frac{1}{2} \mathcal{F}_2^{(4)}.$$

It does not vanish on tensor products of 2-qubit entangled states and so it is not a filter.
Summarizing, we have that

\[ \text{Inv}_2^{SL} = \text{Inv}_2^L = \text{span}\{H\} \]

\[ \mathcal{F}_{0:2}^{SL} = 0. \]

### Degree 4

Besides the one-dimensional space \( U_4 \) spanned by \( H^2 \), there exist three new invariants of degree 4, namely \( L, M, \) and \( N \) subject to the relation \( L + M + N = 0 \). Expressed in terms of the coefficients of the wave function

\[ |\psi\rangle = \sum_{i,j,k,l=0}^{2^l-1} \psi_{ijkl} |ijkl\rangle =: \sum_{n=0} a_n |n\rangle, \]

with the identification \( |ijkl\rangle \equiv |i + 2j + 4k + 8l\rangle \), they are given by the determinants

\[
L = \begin{vmatrix}
    a_0 & a_4 & a_8 & a_{12} \\
    a_1 & a_5 & a_9 & a_{13} \\
    a_2 & a_6 & a_{10} & a_{14} \\
    a_3 & a_7 & a_{11} & a_{15}
\end{vmatrix},
\]

\[
M = \begin{vmatrix}
    a_0 & a_8 & a_2 & a_{10} \\
    a_1 & a_9 & a_3 & a_{11} \\
    a_4 & a_{12} & a_6 & a_{14} \\
    a_5 & a_{13} & a_7 & a_{15}
\end{vmatrix},
\]

\[
N = \begin{vmatrix}
    a_0 & a_1 & a_8 & a_9 \\
    a_2 & a_3 & a_{10} & a_{11} \\
    a_4 & a_5 & a_{12} & a_{13} \\
    a_6 & a_7 & a_{14} & a_{15}
\end{vmatrix}.
\]

(6)

They can be expressed in terms of the following invariants obtained from local invariant operators

\[
\mathcal{C}^{(4)}_{4:(1,2)} := \left((\sigma_\mu \sigma_\nu \sigma_2 \sigma_2 \cdot \sigma_\mu \sigma_\nu \sigma_2 \sigma_2)\right),
\]

\[
\mathcal{C}^{(4)}_{4:(1,3)} := \left((\sigma_\mu \sigma_2 \sigma_\nu \sigma_2 \cdot \sigma_\mu \sigma_2 \sigma_\nu \sigma_2)\right),
\]

\[
\mathcal{C}^{(4)}_{4:(1,4)} := \left((\sigma_\mu \sigma_2 \sigma_2 \sigma_\nu \cdot \sigma_\mu \sigma_2 \sigma_2 \sigma_\nu)\right).
\]

Indeed we have

\[
L = \frac{1}{48} \left( \mathcal{C}^{(4)}_{4:(1,3)} - \mathcal{C}^{(4)}_{4:(1,4)} \right), \quad
M = \frac{1}{48} \left( \mathcal{C}^{(4)}_{4:(1,4)} - \mathcal{C}^{(4)}_{4:(1,2)} \right), \quad
N = \frac{1}{48} \left( \mathcal{C}^{(4)}_{4:(1,2)} - \mathcal{C}^{(4)}_{4:(1,3)} \right)
\]

and

\[
H^2 = \frac{1}{12} \left( \mathcal{C}^{(4)}_{4:(1,2)} + \mathcal{C}^{(4)}_{4:(1,3)} + \mathcal{C}^{(4)}_{4:(1,4)} \right).
\]

For analogously defined invariants \( \mathcal{C}^{(4)}_{4:(3,4)}, \mathcal{C}^{(4)}_{4:(2,4)} \) and \( \mathcal{C}^{(4)}_{4:(2,3)} \) we have the identities

\[
\mathcal{C}^{(4)}_{4:(1,2)} \equiv \mathcal{C}^{(4)}_{4:(3,4)}, \quad \mathcal{C}^{(4)}_{4:(1,3)} \equiv \mathcal{C}^{(4)}_{4:(2,4)}, \quad \mathcal{C}^{(4)}_{4:(1,4)} \equiv \mathcal{C}^{(4)}_{4:(2,3)}.
\]

It is interesting to mention at this point that further identities appear besides those stated above. Examples are

\[
((\sigma_\mu \sigma_\nu \sigma_\lambda \sigma_\tau \cdot \sigma_\mu \sigma_\nu \sigma_\lambda \sigma_\tau)) = 36H^2,
\]

(7)
and the identity for the three-tangle in [2]. We will report on such identities also for five qubit invariants. They suggest that double contractions \((\sigma_\mu \otimes \sigma_\nu) \cdot (\sigma^H \otimes \sigma^V)\) within a pair of copies could be somehow removed. However, the non-trivial example \((\sigma_2 \sigma_2 \sigma_2 \sigma_2) \neq (\sigma_\mu \sigma_\nu \sigma_2 \sigma_2) (\sigma^H \sigma^V \sigma_2 \sigma_2)\) demonstrates that double contractions cannot simply be removed. It would be worthwhile analyzing this curious observation in more detail and a rigorous reduction scheme would be highly desirable. The interrelation between the \(\Omega\)-process and the comb approach in Section singles out one origin for such identities. In particular it explains all the identities for degree 4 invariants mentioned here.

Summarizing, we have

\[
\begin{align*}
\text{Inv}_{4}^{SL} &= \text{span}\{\mathcal{C}_{4; (1,2)}^{(4)}, \mathcal{C}_{4; (1,3)}^{(4)}, \mathcal{C}_{4; (1,4)}^{(4)}\} \\
\text{Inv}_{4}^{SL^*} &= \text{span}\{H^2\} \subseteq U_4 \\
\mathcal{I}_{0; 4}^{SL} &= 0.
\end{align*}
\]

**Degree 6**

By invoking the Hilbert series, we deduce that \(\dim U_6 = 3\). We claim that \(V_6\) is spanned by \(U_6\) and the filter \(\mathcal{F}_{1}^{(4)}\) from Ref. [2]. As \(\dim V_6 = 4\), it suffices to observe that \(\mathcal{F}_{1}^{(4)} \notin U_6\).

Defining the \(SL^*\)-invariant \(W := D_{xy} + D_{xz} + D_{xt}\), the expressions for the \(D_{uv}\) from Ref. [9] give

\[
\begin{align*}
H(N - M) &= 3D_{xy} - W, \\
H(L - N) &= 3D_{xz} - W, \\
H(M - L) &= 3D_{xt} - W.
\end{align*}
\]

For comparison with Ref. [3] the correspondence for the invariants are \(D_{xt} \rightarrow D, D_{xy} \rightarrow E, D_{xz} \rightarrow F\) and \(W \rightarrow \Gamma\).

The subspace of \(SL\)-invariants of degree 6 is spanned by \(D_{xy}, D_{xz}, D_{xt}\), and \(H^3\) [9] and we find that \(\mathcal{F}_{1}^{(4)} = 8(4W - H^3)\).

From these relations all invariants in this subspace are readily expressed in terms of comb-based invariants. It is worth noticing that \(\mathcal{F}_{1}^{(4)}\) is a filter and that it spans the subspace \(\mathcal{I}_{0; 6}\).

Summarizing, we have

\[
\begin{align*}
\text{Inv}_{6}^{SL} &= \text{span}\{H^3, H\mathcal{C}_{4; (1,2)}^{(4)}, H\mathcal{C}_{4; (1,3)}^{(4)}, \mathcal{F}_{1}^{(4)}\} \\
\text{Inv}_{6}^{SL^*} &= \text{span}\{H^3, \mathcal{F}_{1}^{(4)}\} \\
\mathcal{I}_{0; 6}^{SL} &= \text{span}\{\mathcal{F}_{1}^{(4)}\}
\end{align*}
\]
All spaces $\text{Inv}_d^{SL}$ with $d > 6$ are built from generators of degree 2, 4, and 6. What we will focus on in the rest of this section is to construct a complete set of generators for the ideal $\mathcal{J}_0^{SL^*}$.

**Degree 8**

In this case we have $U_8 = V_8 = \text{Inv}_8^{SL}$ and $\dim V_8 = 7$. By proposition 0.1, which is proved below, the degree 8 component of $\mathcal{J}_0^{SL^*}$ is only two-dimensional. It is spanned by $H \mathcal{F}_1^{(4)}$ and the symmetrized filter $\langle \mathcal{F}_2^{(4)} \rangle_s$. Defining $\Sigma := L^2 + M^2 + N^2$ we find that

$$\mathcal{F}_2^{(4)} = 16 \left( H^4 + 4H^2(M - L) - 16HDxt - 16LM \right),$$

$$\langle \mathcal{F}_2^{(4)} \rangle_s = \frac{16}{3} \left( 8\Sigma - H^4 \right) - \frac{64}{3} H(4W - H^3).$$

The action of the symmetric group $S_4$ on the filter $\mathcal{F}_2^{(4)}$ produces three independent filter invariants. We have

$$\text{Inv}_8^{SL} = \text{span}\{H^4, H^2\mathcal{C}_4^{(4);(1,2)}, H^2\mathcal{C}_4^{(4);(1,3)}, H\mathcal{F}_1^{(4)}, S_4 \circ \mathcal{F}_2^{(4)} \}$$

$$\text{Inv}_8^{SL^*} = \text{span}\{H^4, H\mathcal{F}_1^{(4)}, \langle \mathcal{F}_2^{(4)} \rangle_s \}$$

$$\mathcal{J}_{0;8}^{SL^*} = \text{span}\{H \mathcal{F}_1^{(4)}, \langle \mathcal{F}_2^{(4)} \rangle_s \}.$$

We note that the orbit $S_4 \circ \mathcal{F}_2^{(4)}$ in the first formula above can be replaced by the three invariants: $(\mathcal{C}_4^{(4);(1,2)})^2$, $(\mathcal{C}_4^{(4);(1,3)})^2$ and $\mathcal{C}_4^{(4);(1,2)} \mathcal{C}_4^{(4);(1,3)}$.

**Degree 10**

The degree 10 homogeneous component $\mathcal{J}_{0;10}^{SL^*}$ is two-dimensional and is spanned by $H^2 \mathcal{F}_1^{(4)}$ and $H \langle \mathcal{F}_2^{(4)} \rangle_s$. The last missing ideal generator is obtained from degree 12.

**Degree 12 and beyond**

The $SL^*$-invariants of degree 12 are to be built from $H$, $W$, $\Sigma$ and $\Pi := (L - M)(M - N)(N - L)$. The filter $\mathcal{F}_3^{(4)} = \frac{1}{2} \mathcal{C}_4^{(4);(1,2)} \mathcal{C}_4^{(4);(1,3)} \mathcal{C}_4^{(4);(1,4)}$ is invariant under qubit permutations, i.e. it is an $SL^*$-filter. We find that

$$\mathcal{F}_3^{(4)} = -96H^2(8\Sigma - H^4) - 64(32\Pi + H^6).$$
Proposition 0.1 The ideal \( \mathcal{J}_0^{SL^*} \) is generated by the invariants \( 4W - H^3, 8\Sigma - H^4, \) and \( 32\Pi + H^6 \).

Proof: First, it is easy to check that these three invariants belong to \( \mathcal{J}_0^{SL^*} \). Next, let \( f \in \mathcal{J}_0^{SL^*} \) be arbitrary. Note that \( f \) is a polynomial in the generators \( H, W, \Sigma \) and \( \Pi \) of the algebra \( \text{Inv}^{SL^*} \). Without any loss of generality we can assume that \( f \) is homogeneous of degree \( 2d \). The above three invariants can be used to eliminate \( W, \Sigma \) and \( \Pi \) from \( f \). Then the corresponding reduced element \( f_0 \in \mathcal{J}_0^{SL^*} \) is a homogeneous polynomial in \( H \) only.

Consequently, \( f_0 = cH^d \) for some constant \( c \). In particular, \( f_0 \) must vanish on arbitrary product states. Since \( H \) however does not vanish on arbitrary product states, this implies \( c = 0 \) and completes the proof.

Equivalently, the same ideal is generated by the \( SL^* \)-filters \( \mathcal{F}_1^{(4)}, \left\langle \mathcal{F}_2^{(4)} \right\rangle_s \) and \( \mathcal{F}_3^{(4)} \) which are functionally independent [2]. This follows immediately from

\[
\begin{align*}
4W - H^3 &= \frac{1}{8} \mathcal{F}_1^{(4)}, \\
8\Sigma - H^4 &= \frac{3}{16} \left( \left\langle \mathcal{F}_2^{(4)} \right\rangle_s + \frac{8}{3} H \mathcal{F}_1^{(4)} \right), \\
32\Pi + H^6 &= -\frac{1}{64} \left( \mathcal{F}_3^{(4)} + 18H^2 \left\langle \mathcal{F}_2^{(4)} \right\rangle_s + 48H^3 \mathcal{F}_1^{(4)} \right).
\end{align*}
\]

As an important and often cited invariant, we briefly consider the hyperdeterminant, \( \text{Det} \), of four qubits. It has degree 24 and is given by

\[
2^{8 \cdot 3^3} \text{Det} = (H^3 - 4W)A + (8\Sigma - H^4)B - 4(32\Pi + H^6)^2,
\]

where

\[
\begin{align*}
A &= 5H^9 + 20WH^6 - 144\Sigma H^5 + 16(5W^2 - 24\Pi)H^3 \\
&\quad - 960\Sigma WH^2 + 1536\Sigma^2 H + 192W(3W^2 + 8\Pi), \\
B &= H^8 - 136\Sigma H^4 + 384\Pi H^2 + 256\Sigma^2.
\end{align*}
\]

This can be translated into an expression in terms of \( H \) and the filters \( \mathcal{F}_1^{(4)}, \left\langle \mathcal{F}_2^{(4)} \right\rangle_s \) and \( \mathcal{F}_3^{(4)} \) in a straightforward manner.

The decomposition of \( \text{Inv}^{SL}_d \) for even \( d, 2 \leq d \leq 12 \), into irreducible \( S_4 \)-modules is given in table 1. Note that the \( SL^* \) Hilbert series confirms the multiplicities of the trivial module \( X_1 \).

| degree | degree | degree |
|--------|--------|--------|
| 2 \( X_1 \) | 4 \( X_1 + X_3 \) | 6 \( 2X_1 + X_3 \) |
| 8 \( 3X_1 + 2X_3 \) | 10 \( 3X_1 + 3X_3 \) | 12 \( 5X_1 + 4X_3 + X_3 \) |
It is interesting to briefly focus on specific multipartite entangled four qubit states. One prominent class of states is formed by the so called graph states \([19, 20]\). They are created from a fully polarized state in e.g. \(x\)-direction by successive action of the two-qubit entangling operator \(U_{ij} := \frac{1}{2}(I + \sigma_{3;i} + \sigma_{3;j} - \sigma_{3;i}\sigma_{3;j})\) which is also known as the control-\(\sigma_3\) gate. A complete characterization of graph states for up to seven qubits can be found in Ref. [20]. In the case of four qubits, only two graph state classes exist. Representatives are the GHZ state and the 4-qubit cluster state \([19]\). A genuinely entangled four qubit state that falls out of this classification, namely \(|X\rangle := \frac{1}{\sqrt{6}}(|1111\rangle + |1000\rangle + |0100\rangle + |0010\rangle + |0001\rangle\), has been presented in Ref. \([2, 4]\) together with an evaluation of the filters \(F_1^{(4)}\), \(F_2^{(4)}\), and \(F_3^{(4)}\) on these three states. It is worth noticing that the three filters admit for 7 \(= 2^3 - 1\) classes of genuine four-party entanglement (in the nonzero SLOCC class). Representative states for these seven classes of entanglement can be obtained as coherent superpositions of GHZ, cluster, and \(X\) state from the table of filter values in Ref. \([4]\). In this sense these three maximally entangled states form a basis for the whole nonzero SLOCC class. Consequently, a classification of genuine multipartite entanglement in terms of graph states alone is not complete. The account of the complementary set of non-graph states as a resource for quantum information processing is largely unexplored.

**SL AND SL\(^*\)-INVARIANTS FOR FIVE QUBITS**

The \(SL\) Hilbert series for five qubits has been determined in Ref. \([5]\) as

\[
h(t) = \frac{1 + 16t^8 + 9t^{10} + 82t^{12} + \cdots + 82t^{22} + 9t^{24} + 16t^{96} + t^{104}}{(1 - t^4)^5(1 - t^6)(1 - t^8)(1 - t^{10})(1 - t^{12})^5} \tag{8}
\]

\[
= 1 + 5t^4 + t^6 + 36t^8 + 15t^{10} + 228t^{12} + 231t^{14} + 1313t^{16} + 1939t^{18} + \cdots \tag{9}
\]

We have verified the values of the coefficients \(c_{2d}\) of \(t^{2d}\) in Eq. (9) by using the formula \(c_{2d} = (1/(2d)!)\sum_{\pi \in S_{2d}} \chi(\pi)^5\) where \(\chi(\pi)\) is the character of the irreducible representation of \(S_{2d}\) corresponding to the partition \([d, d]\) of the integer \(2d\). This is a special case of the formula from Eq. (29) where we replace \(d\) with \(2d\) and insert the local Hilbert space dimension \(n = 2\) and the number of qubits \(k = 5\). Both numerator and denominator in Eq. (8) are even palindromic polynomials of degrees 104 and 136 respectively. The expanded Hilbert series tells us that there are 5 invariants of degree 4, a single invariant of degree 6, 36 invariants of degree 8, 15 invariants of degree 10, 228 invariants of degree 12, etc. In Ref. \([5]\) the invariants up to degree 6 have been determined together with 5 invariants of degree 8.

The first terms of the \(SL^*\) Hilbert series are (for details see Section )

\[
h_{SL^*}(t) = 1 + t^4 + 4t^8 + 12t^{12} + 2t^{14} + 39t^{16} + 21t^{18} + 130t^{20} + 115t^{22} + \cdots .
\]

In this section we will give a complete characterization of invariants up to degree 12 and establish a connection with the invariants from Ref. \([4]\). Since the Hilbert series shows that no invariant of degree 2 exists, we start our analysis with degree 4.
Degree 4

A straightforward calculation shows that the 5 linearly independent invariants \( D_v \) of degree 4 \((v = x, y, z, t, u)\) from Ref. [5] can be written as

\[
D_1 := D_x = (\sigma_\mu \sigma_2 \sigma_2 \sigma_2 \sigma_2 \cdot \sigma^\mu \sigma_2 \sigma_2 \sigma_2) \tag{10}
\]
\[
D_2 := D_y = (\sigma_2 \sigma_\mu \sigma_2 \sigma_2 \sigma_2 \cdot \sigma_2 \sigma_\mu \sigma_2 \sigma_2) \tag{11}
\]
\[
D_3 := D_z = (\sigma_2 \sigma_2 \sigma_2 \sigma_\mu \sigma_2 \cdot \sigma_2 \sigma_2 \sigma_\mu \sigma_2) \tag{12}
\]
\[
D_4 := D_t = (\sigma_2 \sigma_2 \sigma_2 \sigma_2 \sigma_\mu \sigma_2 \cdot \sigma_2 \sigma_2 \sigma_2 \sigma_\mu \sigma_2 \sigma_2) \tag{13}
\]
\[
D_5 := D_u = (\sigma_2 \sigma_2 \sigma_2 \sigma_2 \sigma_\mu \sigma_2 \cdot \sigma_2 \sigma_2 \sigma_2 \sigma_2 \sigma_\mu) \tag{14}
\]

Interestingly, these invariants and their generalizations to higher odd number of qubits already appeared in Ref. [18]. They form an \( S_5 \)-orbit, which is nicely seen from their explicit forms (10)–(14). The unique \( SL^* \)-invariant of degree 4 is then

\[
P := \sum_{i=1}^{5} D_i .
\]

It does not vanish on all product states. Therefore, \( \mathcal{I}_{SL^*}^{0;4} = 0 \) for five qubits. It is not obvious whether this observation can be extended to larger number of qubits.

The investigation of the full \( S_5 \)-orbits of a given set of invariants will be a major tool for the construction of the complete space of invariants and the determination of the \( SL^* \)-invariants. In the present case, we only needed e.g. \( D_1 \) in order to create all degree 4 invariants from its orbit. The decomposition into irreducible \( S_5 \)-modules is \( V_4 = X_1 + X_2 \).

It is an interesting consequence of the completeness of (10)–(14) as generators of invariants of degree 4 that additional contractions lead to identities. Two examples are

\[
((\sigma_\mu \sigma_\nu \sigma_2 \sigma_2 \sigma_2 \cdot \sigma^\mu \sigma^\nu \sigma_2 \sigma_2 \sigma_2)) = 3(D_4 + D_5) - P , \tag{15}
\]
\[
((\sigma_\mu \sigma_\nu \sigma_\lambda \sigma_\tau \sigma_\rho \cdot \sigma^\mu \sigma^\nu \sigma^\lambda \sigma^\tau \sigma^\rho)) = -3P , \tag{16}
\]

but also note the above mentioned identities for four qubits. Up to the prefactor, equation (16) readily follows from the obvious permutation symmetry.

Degree 6 and singly even \( q \)

The unique invariant \( F \) of degree 6 has been created in Ref. [5] by invoking the \( \Omega \)-process. It is an odd function under qubit permutations, corresponding to the irreducible \( S_5 \)-module \( V_6 = X_7 \). It cannot be created from the combs. However, see section 6. Remembering degree 2, where we know from the Hilbert series that \( \text{Inv}_{SL^*}^{0;2} = 0 \) for an odd number of qubits, it seems that singly even \( q \) is peculiar. This is particularly true as far as the comb-based method is concerned. The only invariant of degree 2 that can be created from local invariant operators is the \( q \)-tangle [18] \( (((\sigma^\otimes q))) \), which however is identically zero when \( q \) is odd. This phenomenon draws wider circles, as expressed in

**Theorem 0.1** For an odd number of qubits, all nonzero \( SL \)-invariants that can be constructed from combs have doubly even degree.
Proof: An expectation value \( \langle (\sigma_{a_1} \sigma_{a_2} \ldots) \rangle \) vanishes if it contains an odd number of \( \sigma_2 \)'s [4]. Since the contraction with the pseudo-metric \( G_{\mu \nu} \) does not contain \( \sigma_2 \) this implies that for an odd number of qubits there must be an odd number of contractions in each copy, leading to an odd number of contractions where all contractions in each of the copies are counted separately. Since the copies are always contracted in pairs, this number of contractions must be even. Therefore, the number of copies for an odd number of qubits has to be even, leading to a degree divisible by 4, hence doubly even. This completes the proof.

It seems that this fact is intimately related to the observed permutation antisymmetry of the invariants of singly even degree (we anticipate here that the one-dimensional \( S_5 \)-modules of the generic invariants of degree 6 and 10 are spanned by odd functions under qubit permutations). The symmetry of the combs under permutations of the copies might hinder asymmetry under qubit exchange, even though there is a profound difference between copy and qubit exchange. Indeed, as we will see later, nonzero antisymmetrizations of comb-based invariants do exist. Nevertheless, it is natural to ask for local invariants that are antisymmetric under the permutation of copies; it turns out that no such construction exists that connects two or three copies, i.e. there are no antisymmetric combs of order two or three. Also notice that no independent symmetric combs exist up to degree four.

Degree 8

We next proceed with a complete discussion of degree 8 invariants. Looking at the Hilbert series, the dimension of this space is 36. The 15 products \( D_i D_j \), \( 1 \leq i \leq j \leq 5 \), form a basis of the subspace \( U_8 \). This implies the existence of 21 independent invariants that are generically of degree 8. Five of these have been constructed in Ref. [5] by using the \( \Omega \)-process. We will at first give the decomposition of \( U_8 \) and \( V_8 \) into irreducible \( S_5 \)-modules and then establish the connection with these five invariants \( H_v \). We find that

\[
U_8 = 2X_1 + 2X_2 + X_3, \quad (17)
\]

\[
V_8 = 4X_1 + 3X_2 + 3X_3 + X_5. \quad (18)
\]

The dimension of \( \text{Inv}_{8}^{SL^*} \) for 5 qubits is consequently 4, which agrees with the \( SL^* \) Hilbert series.

The filter

\[
\mathcal{F}_1^{(5)} = \langle (\sigma_{\mu_1} \sigma_{\mu_2} \sigma_{\mu_3} \sigma_{a_2} \sigma_2 \cdot \sigma_{\mu_1} \sigma_{\mu_2} \sigma_2 \sigma_{\mu_4} \sigma_2 \cdot \sigma_{\mu_5} \sigma_2 \sigma_{\mu_3} \sigma_2 \cdot \sigma_{\mu_5} \sigma_2 \sigma_2 \sigma_2 \sigma_2) \rangle \quad (19)
\]

\[\text{2} \] A detailed presentation of this result will appear elsewhere.
has been introduced in Ref. [4]. We add two new \( SL \)-invariants

\[
\mathcal{R}^{(5)}_5 = \left( (\sigma_\mu \sigma_2 \sigma_3 \sigma_4 \bullet \sigma_\mu \sigma_5 \sigma_6 \sigma_7 \bullet \\
\sigma_2 \sigma^\rho \sigma_3 \sigma_4 \sigma_5 \bullet \sigma_2 \sigma_6 \sigma^\tau \sigma_7 \sigma^\kappa) \right),
\]

(20)

\[
\mathcal{R}^{(5)}_6 = 3\left( (\sigma_\mu \sigma_2 \sigma_3 \sigma_4 \bullet \sigma_\mu \sigma^\nu \sigma_5 \sigma_6 \bullet \sigma_2 \sigma_7 \bullet \\
\sigma^\tau \sigma_2 \sigma_3 \sigma_4 \sigma_5 \bullet \sigma_6 \sigma^\rho \sigma_7 \sigma^\kappa) \right) + \left( (\sigma_\mu \sigma_2 \sigma_3 \sigma_4 \bullet \sigma_\mu \sigma^\nu \sigma_5 \sigma_6 \bullet \\
\sigma_2 \sigma_7 \sigma_3 \sigma_4 \sigma_5 \bullet \sigma^\tau \sigma_2 \sigma_6 \sigma^\rho \sigma_7 \sigma^\kappa) \right).
\]

(21)

Notice that the second summand in (21) is an element of \( U_8 \). More precisely,

\((\sigma_\mu \sigma_\nu \sigma_\lambda \sigma_2 \sigma_3 \bullet \sigma_\mu \sigma^\nu \sigma_\lambda \sigma_2 \sigma_3) = 3(D_4 + D_5) - P.\)

We claim that these two invariants are filters. It is straightforward to check this claim for the invariant (20). To see that also (21) is a filter it suffices to show that it vanishes on product states. The only partitions that lead to a nonzero value for both terms in the above sum are those factoring out either qubits \((2,3)\) or qubits \((4,5)\). The nonzero value is a product of powers of concurrence and three-tangle \((\text{e.g. } C^{(2,3)}_{2}; \tau^{2}_{3;1,4,5})\), and the prefactor is found to be independent of which of the two distinct partitions we take. It is then straightforward to check that the above combination vanishes also for these factorizations, proving the filter property.

We will show next that, together with \( P^2 \), the permutation averages of the above three filters span the space \( \text{Inv}^{SL^*}_8 \). The \( S_3 \)-submodule generated by the filter \( \mathcal{R}^{(5)}_1 \) has dimension 24 and meets \( U_5 \) in an \( X_2 \), a 4-dimensional subspace. Thus by selecting 20 suitable qubit permutations of this filter, we obtain altogether \( 15 \times 20 = 35 \) linearly independent invariants in \( V_8 \). To obtain a basis of \( V_8 \), we have to add also the filter \( \mathcal{R}^{(5)}_6 \).

It is interesting that the filter (19) resp. (20) creates a 24-dimensional space of invariants \( V_{8;1} \) resp. \( V_{8;2} \). These two spaces have a 23-dimensional overlap \( K \). Thus \( V_{8;i} = T_i + K; \ i = 1,2, \) where \( T_i \) are one-dimensional subspaces of \( \text{Inv}^{SL^*} \). Furthermore, also the space created from (21), which we will call \( V_{8;3} \) can be expressed as \( V_{8;3} = T_5 + \kappa \), where \( \kappa \subseteq K \) and the subspace \( T_3 \subseteq \text{Inv}^{SL^*} \) is one-dimensional. Since these spaces have been created from filters, the \( T_i (i = 1,2,3) \) are already the elements in \( \mathcal{R}^{SL^*}_0 \) we have been looking for. These particular invariants are given by

\[
T_{1;0} := \left\langle \mathcal{R}^{(5)}_1 \right\rangle_s, \ T_{2;0} := \left\langle \mathcal{R}^{(5)}_5 \right\rangle_s, \ T_{3;0} := \left\langle \mathcal{R}^{(5)}_6 \right\rangle_s.
\]

A detailed analysis of the characters of the resulting irreducible \( S_5 \)-modules leads to the decomposition (18).

We now give the expression of the three invariants in \( \mathcal{R}^{SL^*}_0 \) in terms of the invariants obtained in Ref. [5]. To this end, we define a second \( SL^* \)-invariant in \( U_8 \), namely

\[
Q := \sum_{i=1}^{5} D_i^2
\]
and use the sum of all the $H_i$

$$H_0 := \sum_{i=1}^{5} H_i.$$  

We find that

$$T_{2;0} = P^2 - 3Q \in U_8, \quad T_{3;0} = H_0 + P^2 - 6Q.$$  

Summarizing, $\text{Inv}^{SL^*}_8$ is spanned by $P^2, T_{1;0}, T_{2;0}, T_{3;0}$, or equivalently by $P^2, Q, T_{1;0}, T_{3;0}$. The subspace $\mathcal{J}^{SL^*}_{0;8}$ is spanned by $T_{1;0}, T_{2;0}, T_{3;0}$.

**Degree 10**

From the Hilbert series we extract that there are 15 independent invariants of degree 10, where 5 independent elements of $U_{10}$ are obtained by multiplying the 5 invariants $D_i, i = 1, \ldots, 5$ with the invariant $F$ of degree 6. Hence, $U_{10} = X_6 + X_7$ as an $S_5$-module.

The missing ten invariants are in the $S_5$-orbit of $\mathcal{G}^{(5)}_{10}$ which we construct by invoking the following $\Omega$-process (for the notation used here see Section 3).

\begin{align*}
B_{00222} &:= (f, f)^{11000} \\
B_{20022} &:= (f, f)^{01100} \\
B_{20202} &:= (f, f)^{01001} \\
C_{20222} &:= (B_{20022}, B_{00222})^{00011} \\
D_{11131} &:= (C_{20222}, f)^{10101} \\
E_{20222} &:= (D_{11131}, f)^{01010} \\
F_{11311} &:= (E_{20222}, f)^{10011} \\
H_{11111} &:= (F_{11311}, B_{20202})^{10201} \\
\mathcal{G}^{(5)}_{10} &:= (H_{11111}, f)^{11111} \tag{22}
\end{align*}

$\mathcal{G}^{(5)}_{10}$ spans a 14-dimensional space which has a 4-dimensional intersection with $U_{10}$. In terms of irreducible $S_5$-modules the space of degree 10 invariants decomposes as

$$V_{10} = X_5 + 2X_6 + 2X_7$$

with dimension counting $15 = 5 + 2 \times 4 + 2 \times 1$. This shows that there are no $SL^*$-invariants; however, there are two odd symmetric invariants: $P \cdot F \in U_{10}$ and the antisymmetrization of $\mathcal{G}^{(5)}_{10}$. Both are in the ideal $\mathcal{J}^{SL^*}_{0;10}$.

Summarizing, $\text{Inv}^{SL^*}_{10} = \mathcal{J}^{SL^*}_{0;10} = \text{span}\{ P \cdot F, \langle \mathcal{G}^{(5)}_{10} \rangle \}$.

---

3 We follow the notation of Ref. [5].
Degree 12

From the Hilbert series we see that the space $V_{12}$ of degree 12 invariants has dimension 228, where a 141-dimensional subspace $U_{12}$ emerges from lower degrees. The latter space decomposes as

$$U_{12} = 7X_1 + 10X_2 + 8X_3 + 5X_4 + 4X_5 + X_6.$$  

Hence there are 87 invariants that are generically of degree 12. For the complete reconstruction and decomposition of this space into irreducible $S_5$-modules we use the filters rather than employing the $\Omega$-process, since this reduces significantly the computational complexity. The origin of this reduction in computational complexity can be understood from the analysis in Section.

We claim that the $S_5$-orbits of the five invariants

$$\mathcal{F}_{12;1}^{(5)} = \left( (\sigma_{\mu_1} \sigma_{\mu_2} \sigma_{\mu_3} \sigma_2 \sigma_2 \bullet \sigma_{H_1} \sigma_2 \sigma_2 \sigma_2 \sigma_2 \sigma_2 \bullet 
\sigma_2 \sigma_2 \sigma_2 \sigma_{H_2} \sigma_{H_3} \sigma_{H_4} \sigma_{H_5} \sigma_{H_6} \sigma_{H_7} ) \right)$$  

(23)

$$\mathcal{F}_{12;2}^{(5)} = \left( (\sigma_{\mu_1} \sigma_{\mu_2} \sigma_{\mu_3} \sigma_2 \sigma_2 \bullet \sigma_{H_1} \sigma_{H_2} \sigma_2 \sigma_2 \sigma_2 \bullet 
\sigma_2 \sigma_2 \sigma_2 \sigma_{H_3} \sigma_2 \sigma_2 \sigma_2 \sigma_2 \sigma_{H_7} ) \right)$$  

(24)

$$\mathcal{F}_{12;4}^{(5)} = \left( (\sigma_{\mu_1} \sigma_{\mu_2} \sigma_{\mu_3} \sigma_2 \sigma_2 \bullet \sigma_{H_1} \sigma_2 \sigma_2 \sigma_2 \sigma_2 \sigma_{H_6} \bullet 
\sigma_{H_7} \sigma_2 \sigma_2 \sigma_{H_3} \sigma_2 \sigma_2 \sigma_{H_8} ) \right)$$  

(25)

$$\mathcal{F}_{12;2}^{(5)} = \left( (\sigma_{\mu_1} \sigma_{\mu_2} \sigma_{\mu_3} \sigma_2 \sigma_2 \bullet \sigma_{H_1} \sigma_{H_2} \sigma_2 \sigma_2 \sigma_2 \bullet 
\sigma_{H_5} \sigma_2 \sigma_{H_3} \sigma_2 \sigma_2 \sigma_2 \sigma_{H_8} ) \right)$$  

(26)

$$\mathcal{F}_{12;6}^{(5)} = \left( (\sigma_{\mu_1} \sigma_{\mu_2} \sigma_{\mu_3} \sigma_{\mu_4} \sigma_2 \sigma_2 \bullet \sigma_{H_1} \sigma_{H_2} \sigma_2 \sigma_2 \sigma_2 \bullet 
\sigma_{H_7} \sigma_{H_3} \sigma_2 \sigma_{H_6} \sigma_{H_8} \sigma_{H_5} ) \right)$$  

(27)

generate the full set of 87 new invariants. The symbol $\mathcal{F}$ indicates that the invariant has the filter property. The filters (23) and (24) are taken from [4], whereas the invariants (25), (26) and (27) are new.

We start our construction of a basis of $V_{12}$ by choosing a basis of the subspace $U_{12}$ (141 elements). Next we make use of the filter $\mathcal{F}_{12;4}^{(5)}$. The $S_5$-module that it generates has dimension 112 and intersects $U_{12}$ in a 44-dimensional submodule. Thus we can construct the next 68 basis elements by applying suitable qubit permutations to this filter. The next 15 elements of the basis are obtained similarly from $\mathcal{F}_{12;2}^{(5)}$, and 2 more from $\mathcal{F}_{12;2}^{(5)}$. This gives in total $141 + 68 + 15 + 2 = 226$ basis elements. A full basis of $V_{12}$ is obtained by adjoining the invariants $\mathcal{F}_{12;1}^{(5)}$ and $\mathcal{F}_{12;6}^{(5)}$. This proves the claim made above.
It is straightforward to construct filters from \( \mathcal{G}_{12:2}^{(5)} \) and \( \mathcal{G}_{12:6}^{(5)} \) by subtracting suitable elements of \( U_{12} \). In both cases there is a single partition for which the invariant does not vanish on corresponding product states: for the partition \((1,2)(3,4,5)\) we have \( \mathcal{G}_{12:2}^{(5)} = 9C_{1,2}^6 \cdot \tau_{3,3,4,5}^3 \) whereas for \((2,4)(1,3,5)\) we obtain that \( \mathcal{G}_{12:6}^{(5)} = 3C_{2,3}^6 \cdot \tau_{3,1,4,5}^3 \).

Filters are constructed by subtracting the \( U_{12} \)-elements

\[
\Delta \mathcal{G}_{12:2}^{(5)} = \left( (\sigma_{\mu_1} \sigma_{\mu_2} \sigma_{\mu_3} \sigma_2 \sigma_2 \bullet \sigma_{\mu_1} \sigma_{\mu_2} \sigma_{\mu_3} \sigma_2 \sigma_2) \right) \cdot
\left( (\sigma_{\mu_5} \sigma_{\mu_6} \sigma_{\mu_7} \sigma_2 \sigma_2 \bullet \sigma_{\mu_5} \sigma_{\mu_6} \sigma_{\mu_7} \sigma_2 \sigma_2) \right)
\]

\[
\Delta \mathcal{G}_{12:6}^{(5)} = -\frac{1}{9} \left( (\sigma_{\mu_1} \sigma_{\mu_2} \sigma_{\mu_3} \sigma_{\mu_5} \bullet \sigma_{\mu_1} \sigma_{\mu_2} \sigma_{\mu_3} \sigma_{\mu_5}) \right) \cdot
\left( (\sigma_{\mu_6} \sigma_{\mu_7} \sigma_2 \sigma_2 \bullet \sigma_{\mu_6} \sigma_{\mu_7} \sigma_2 \sigma_2) \right)
\]

from \( \mathcal{G}_{12:2}^{(5)} \) and \( \mathcal{G}_{12:6}^{(5)} \), respectively.

It is interesting to mention here that an \( SL^* \)-filter can readily be constructed from combs as follows

\[
\mathcal{F}_0^{(5)} = \left( (\sigma_{\mu_1} \sigma_{\mu_2} \sigma_{\mu_3} \sigma_{\mu_5} \bullet \sigma_{\mu_1} \sigma_{\mu_2} \sigma_{\mu_3} \sigma_{\mu_5} \sigma_2 \sigma_2 \sigma_2) \right)
\]

It turns out that \( \mathcal{F}_0^{(5)} \) is equivalent modulo \( U_{12} \) to the symmetrization of \( \mathcal{F}_{12:1}^{(5)} \).

We find that the 7-dimensional space \( U_{12} \cap \text{Inv}_{12}^{SL^*} \) is spanned by \( P^3, PT_{j,0} \) \((j=1,2,3)\), \( F^2, \sum_i D_i^3 \), and \( \left< D_1 \mathcal{F}_{1}^{(5)} \right>_s \). Besides \( PT_{j,0} \) \((j=1,2,3)\), \( \left< D_1 \mathcal{F}_{1}^{(5)} \right>_s \) and \( \mathcal{F}_0^{(5)} \), also \( P^3 - 9 \sum_i D_i^3 \) is in \( \mathcal{F}_0^{SL^*} \).

The complementary 5-dimensional space in \( \text{Inv}_{12}^{SL^*} \) is spanned by

\[
\left< \mathcal{F}_{12:1}^{(5)} \right>_s , \left< \mathcal{F}_{12:2}^{(5)} \right>_s , \left< \mathcal{F}_{12:4}^{(5)} \right>_s , \left< \mathcal{G}_{12:6}^{(5)} \right>_s , \left< \mathcal{G}_{12:2}^{(5)} \right>_s .
\]

The two antisymmetrized filters

\[
\left< \mathcal{F}_{12:2}^{(5)} \right>_a , \left< \mathcal{F}_{12:4}^{(5)} \right>_a
\]

span the space of \( SL^* \)-invariants of degree 12, whereas \( 3\mathcal{G}_{12:6}^{(5)} - \mathcal{G}_{12:2}^{(5)} \) is in \( \mathcal{F}_0^{SL^*} \). It is worthwhile noticing that the comb-based invariants also create the \( SL^* \)-invariants in \( U_{12} \); those of degree 6 and 10, which are not accessible by the comb approach, are not needed.
Summarizing we have a 10-dimensional space $\mathcal{I}^{SL^*}_{0;12}$ inside a 12-dimensional space $\text{Inv}^{SL^*}_{12}$. In addition there are two $SL^*_-$-invariants, both belong to the filter ideal. Thus,

$$\mathcal{I}^{SL^*}_{0;12} = \text{span} \left\{ P_j T_j | j = 1, \ldots, 3, P^2, \left\langle \mathcal{P}_1^{(5)} \right\rangle_s, \left\langle \mathcal{P}_0^{(5)} \right\rangle_s, P^3 - 9 \sum D_i \left\langle 3 \mathcal{P}_1^{(5)} - \mathcal{P}_2^{(5)} \right\rangle_s \right\}$$

$$\mathcal{I}^{SL^*}_{0;12} = \text{span} \left\{ \left\langle \mathcal{P}_2^{(5)} \right\rangle_a, \left\langle \mathcal{P}_3^{(5)} \right\rangle_a \right\}.$$ 

The complete decomposition into irreducible $S_5$-modules is given in table 2.

**TABLE 2.** The space of polynomial invariants of degree 2 up to 12 into irreducible $S_5$-modules.

| degree | degree |
|--------|--------|
| 2      | 0      | 4 | $X_1 + X_2$ |
| 6      | $X_7$  | 8 | $4X_1 + 3X_2 + 3X_3 + X_5$ |
| 10     | $X_5 + 2X_6 + 2X_7$ | 12 | $12X_1 + 15X_2 + 14X_3 + 6X_4 + 8X_5 + 2X_6 + 2X_7$ |

**Beyond degree 12**

We add here a couple of remarks about the next two degrees, 14 and 16. To this end, let $Y$ denote a minimal set of (homogeneous) generators of the algebra $\text{Inv}^{SL}$. We know that $Y$ is a finite set, but its cardinality is not known. It is a disjoint union of the subsets $Y_d := Y \cap \text{Inv}^{SL}_d$. From the Hilbert series we know that $|Y_d| = 0$ for odd $d$ and for $d = 2$. Our computations show that for $d = 4, 6, 8, 10, 12, 14, 16$ we have $|Y_d| = 5, 1, 21, 10, 87, 145, 247$.

Let us assume that the conjecture made in Ref. [5] regarding the Cohen–Macaulay ring structure of $\text{Inv}^{SL}$ is correct, i.e., that the primary invariants consist of five polynomials of degree 4, one of degree 6, five of degree 8, one of degree 10, and five of degree 12. Then $\text{Inv}^{SL}$ would be a free module of rank 3 014 400 over the algebra generated by the primary invariants (a polynomial algebra in 17 variables). Moreover, the coefficients of the numerator of the Hilbert series give, for each degree, the number of basis elements of this free module. The first six nonzero coefficients are 1, 16, 9, 82, 145, 383 and the degrees of the corresponding basis elements are 0, 8, 10, 12, 14, 16, respectively (see Table 1 in Ref. [5]). For instance, for $d = 16$ we have 383 basis elements. We may assume that this basis contains $Y_{d}$. Consequently there must be $383 - 247 = 136$ basis elements of degree 16 that come from the products of basis elements of degree 8. As there are sixteen basis elements of degree 8, the number of different products of two of them (including the squares) is indeed 136. This may be interpreted as additional evidence for the validity of the above mentioned conjecture.

It is interesting to examine again the class of graph states. For five qubits there exist four inequivalent graph states: numbers 5 through 8 in Ref. [20] (see figure 1). They are distinguished already by the degree 4 invariants $D_i$, $i = 1, \ldots, 5$. Among $SL^*_-$-invariants up to degree 8, the state d) of figure 1 is only detected by $T_{0;3}$ (and hence also by $H_0$). It
is worth mentioning that the \( SL^* \)-invariant \( F \) does not detect any of these states. Besides the two maximally entangled states from Ref. [4] \(|1\rangle + |2\rangle + |4\rangle + |24\rangle + \sqrt{2} |31\rangle \) and \( W_4 + \sqrt{3} |31\rangle \), also superpositions of these two states and possibly including the four graph states fall out of the graph state classification. This basis of 6 states would thereby admit for already \( 2^6 - 1 = 63 \) distinct SLOCC classes of 5-qubit states. Only four of these are covered by graph states.

\section*{Character Computations and the Hilbert Series for \( SL^*_5 \)}

It is interesting to mention that the coefficients of the Hilbert series for the symmetry group \( SL \) and \( SL^* \) can be obtained directly using the results of Ref. [15]. Here, we recall some results from that work and use them to compute the dimension of the space of \( SL^* \)-invariants of degree \( 2d \). We also do the same for the \( SL^*_n \) invariants.

As in the cited reference, we shall be more general. First, instead of qubits we may work with qudits, i.e., we consider the vector representation of \( SU(n) \) or \( SL(n, \mathbb{C}) \) on \( V = \mathbb{C}^n \). By taking \( k \) copies of \( SL(n, \mathbb{C}) \) and \( k \) copies of \( V \) and tensoring, we obtain the standard representation of \( G = SL(n, \mathbb{C}) \otimes k \) on \( V \otimes k \). Let us denote by \( \mathcal{R}_{n,k} \) the algebra of holomorphic polynomial functions on \( V \otimes k \), and by \( \mathcal{R}_{n,k,d} \) its subspace consisting of the homogeneous polynomials of degree \( d \). Next, denote by \( \mathcal{R}_{n,k,d}^G \) its subspace consisting of \( G \)-invariant polynomials. If \( d \) is not divisible by \( n \), then \( \mathcal{R}_{n,k,d}^G = 0 \) by [15, Proposition 11.1].

Assume now that \( d = nr \) and let \( \pi = [r^d] \) be the partition of \( d \) into \( n \) equal parts. Denote by \( E_{\pi} \) the irreducible module of the symmetric group \( S_d \) which corresponds to \( \pi \), and let \( \chi \) be its character. Then by [15, Corollary 11.1] we have an isomorphism

\[
\mathcal{R}_{n,k,d}^G \cong \left( E_{\pi} \otimes k \right)^{S_d}
\]  

(28)
of $S_k$-modules. The superscript $S_d$ means that one has to form the space of invariants of $S_d$, i.e., the largest trivial $S_d$ submodule of $E^\otimes_k$. The other symmetric group, $S_k$, acts on both sides by permuting the tensor factors.

This formula is very useful. For instance, one obtains immediately the following formula for the dimension of the space of $G$ invariants of degree $d$:

$$\dim \mathcal{R}_n^G = \frac{1}{d!} \sum_{g \in S_d} \chi(g)^k.$$  \hspace{1cm} (29)

By symmetrization, i.e., by taking the $S_k$-invariants on both sides of Eq. (28), and taking into account that the actions of $S_k$ and $S_d$ commute, we obtain that

$$\left( \mathcal{R}_n^G \right)^{S_k} \cong \left( S^k(E_\pi) \right)^{S_d}$$

as complex vector spaces. (By $S^k(E_\pi)$ we denote the $k$-th degree piece of the symmetric algebra $S(E_\pi)$ of the module $E_\pi$.)

By performing anti-symmetrization instead of the symmetrization, one obtains a similar formula for the dimension of the space of odd invariants of $S_k$ in $\mathcal{R}_n^G$. Then on the right hand side one should replace the symmetric power $S^k(E_\pi)$ by the exterior power $\wedge^k(E_\pi)$.

The character $\chi^{(k)}$ of the $S_d$-module $S^k(E_\pi)$ is given by the classical formula [21, 22]

$$\chi^{(k)}(g) = \sum_{(i)} \prod_{\alpha=1}^{k} \frac{\chi^{(\alpha)}(g^\alpha)^{i_\alpha}}{i_\alpha! \alpha^{i_\alpha}},$$  \hspace{1cm} (30)

where the summation is over all sequences $(i) = (i_1, i_2, \ldots, i_k)$ of nonnegative integers such that

$$\sum_{\alpha} \alpha i_\alpha = k.$$  

This is valid for all permutations $g \in S_d$.

Similarly, the $S_d$-character $\chi^{[k]}$ of the $k$-th exterior power $\wedge^k(E_\pi)$ is given by the formula

$$\chi^{[k]}(g) = \sum_{(i)} \prod_{\alpha=1}^{k} \frac{(-1)^{i_\alpha - 1} \chi^{(\alpha)}(g^\alpha)^{i_\alpha}}{i_\alpha! \alpha^{i_\alpha}}.$$  \hspace{1cm} (31)

The values of the irreducible characters of $S_d$ are easily available, say in James and Kerber book [16] or in software systems such as Maple or GAP. Hence, we obtain the following formula for the space of joint $G$ and $S_k$-invariants of degree $d = nr$:

$$\dim \left( \mathcal{R}_n^G \right)^{S_k} = \frac{1}{d!} \sum_{g \in S_d} \chi^{(k)}(g).$$

In our case we have $n = 2$, since we work with qubits, and $k = 5$, i.e., the number of qubits is 5. In that case there are exactly seven sequences $(i)$ having the required
property. Explicitly, they are: \((5, 0, 0, 0, 0), (3, 1, 0, 0, 0), (1, 2, 0, 0, 0), (2, 0, 1, 0, 0), (0, 1, 1, 0, 0), (1, 0, 0, 1, 0)\) and \((0, 0, 0, 0, 1)\). Formula (30) then reads as

\[
120 \chi^{(5)}(g) = \chi(g)^5 + 10\chi(g)^3\chi(g^2) + 15\chi(g)\chi(g^2)^2 + 20\chi(g)^2\chi(g^3) + 20\chi(g^2)\chi(g^3) + 30\chi(g)\chi(g^4) + 24\chi(g^5).
\]

For instance, if \(d = 8 = 2 \cdot 4\) we have \(r = 4\), \(\pi = [4, 4]\), \(E_{\pi}\) is the module \(X_8\), and the values of \(\chi\) on the representatives of the 22 conjugacy classes of \(S_8\) are

\[
\begin{align*}
14, & 4, 2, 0, 6, -1, 1, -1, 2, -2, -2, 0, 2, 1, 2, -1, -1, -1, 0, 0, 0, 0 \\
\end{align*}
\]

(see James and Kerber, p. 351). By using the above formula, we find that the values of the character \(\chi^{(5)}\) on the same representatives are

\[
8568, 216, 72, 0, 536, 0, 0, 0, 18, 0, -12, 0, 12, 0, 24, 3, 1, 0, 0, 2, 0, 0.
\]

Then the multiplicity of the principal character (i.e., the character of the trivial \(S_8\)-module \(X_1\)) in \(\chi^{(5)}\) is equal to the dimension of the space of \(SL^*\)-invariants of degree 8. Hence we have

\[
\dim \text{Inv}_{S_8}^{SL^*} = \frac{1}{8!} \sum_{g \in S_8} \chi^{(5)}(g).
\]

The evaluation of this sum confirms our finding that this dimension is 4.

In conclusion, we summarize the results of our computations. The number of linearly independent \(SL^*\)-invariants in degrees 0, 2, 4,..., 22 is 1, 0, 1, 0, 4, 0, 12, 2, 39, 21, 130, 115 respectively. The number of linearly independent relative \(SL^*\)-invariants in degrees 0, 2, 4,..., 22 is 1, 0, 1, 4, 2, 14, 11, 49, 58, 185, 269 respectively.

**CONNECTON BETWEEN THE \(\Omega\)-PROCESS AND THE COMB APPROACH**

In this section we present a rephrasing of elements of Cayley’s \(\Omega\)-process in terms of local invariant antilinear operators. The central operations in the \(\Omega\)-process are determinants of derivatives

\[
\Omega_x = \det \left| \begin{array}{cc} \partial_{x'} \psi_{i_1} & \partial_{x''} \psi_{i_1} \\ \partial_{x'} \psi_{i_2} & \partial_{x''} \psi_{i_2} \end{array} \right| \quad (32)
\]

with subsequent “trace” \(\text{tr} : x', x'' \to x\) applied to functions of the wave function coefficients \(\psi_{i_1, \ldots, i_q}\) dressed with auxiliary variables \(z_{i_j}^{(j)}\) such that a wave function \(|\Psi\rangle := \sum_{i_1, \ldots, i_q} |i_1, \ldots, i_q\rangle\) is mapped to the function \(f := \sum_{i_1, \ldots, i_q} z_{i_1}^{(1)} \cdots z_{i_q}^{(q)}\). A typical step in the \(\Omega\)-process is then prescribed as [5]

\[
(P, Q)^{\varepsilon_1, \ldots, \varepsilon_q} := \text{tr} \Omega_{z^{(1)}} \cdots \Omega_{z^{(q)}} P(z') Q(z'').
\]

The key observation is that the action of \(\Omega_x\) (32) amounts to a contraction of two of the wave function coefficients with the antisymmetric tensor \(\varepsilon_{kl}, k, l \in \{0, 1\}\) with \(\varepsilon_{01} := 1\)
It is seen that the pairs of wave function copies to be contracted with the antisymmetric tensor where we used Einstein sum convention and contraction via (34) reproduces precisely this unique invariant all those, whose functions contractions of the idexes whose absolute value is the three-tangle [13]. For obtaining the second equality in (32) we illustrate this procedure in the most simple example

\[ B_{22200} = (f,f)^{0,0,0,1,1} \]

where we used Einstein sum convention and contraction via \( \varepsilon \). The above so-called transvectant \( B_{22200} \), which is bilinear in the \( z^{(j)} \) \( (j = 1, 2, 3) \), coincides with the subsequently shown antilinear expectation value after setting all \( z^{(j)}_{ij} = 1 \); here \( J = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \).

It is seen that the pairs of wave function copies to be contracted with \( \varepsilon \) by \( \Omega \) (32) are all those, whose functions \( f \) contain the variable \( z^{(j)} \) and \( z^{(j)} \), respectively. Since both these variables might occur in more than one function \( f \), the action of \( \Omega \) will in general lead to a sum over such \( \varepsilon \) contractions involving different pairs of copies of the wave function.

Though it is clear now that each invariant constructed by the \( \Omega \)-process can be directly transcribed into a sum of complete contractions of the wave function coefficients via the antisymmetric tensor \( \varepsilon = i\sigma_2 \), it cannot be directly written in terms of antilinear expectation values of \( \sigma_2 \). A simple three qubit counterexample is the invariant

\[ \tau_3 = -2\psi_{a_1,a_2,a_3}\psi_{b_1,b_2}^\dagger \psi_{b_1,b_2,a_4}^\dagger \psi_{b_1,b_2,a_4} = (\langle \sigma_2\sigma_2\sigma_\mu \bullet \sigma_2\sigma_2\sigma_\mu \rangle) \] (33)

whose absolute value is the three-tangle [13]. For obtaining the second equality in Eq. (33) we grouped in pairs the first and the last two wave function coefficients. The contractions of the indexes \( a_1, a_2, b_1 \) and \( b_2 \) then appear inside these pairs and we will call them inner contractions. On the other hand the contractions of the indexes \( a_3 \) and \( a_4 \) involve two different pairs of coefficients, and we will call them cross-contractions.

As stated above, for more complicated invariants produced by the \( \Omega \)-process, this correspondence is not given by a single full contraction; however, each of those complete contractions contained in the \( \Omega \)-process is an invariant. E.g. the invariant \( F \) of degree 6 (see Ref. [5] for its construction from an \( \Omega \)-process) is equivalently expressed as

\[ F = 96\psi_{i_1,i_2,i_3,i_4,i_5}\psi_{i_6,i_7,i_8,i_9,i_{10}}\psi_{i_1}^i\psi_{i_2}^j\psi_{i_3}^k\psi_{i_4}^l\psi_{i_5}^m\psi_{i_6}^n\psi_{i_7}^o\psi_{i_8}^p\psi_{i_9}^q\psi_{i_{10}}^r \] (34)

Since the space of degree 6 invariants for five qubits is one-dimensional, the expression (34) reproduces precisely this unique invariant \( F \) (up to a prefactor). For the invariant of degree 10 as constructed in Eq. (22), a possible transcription is

\[ G^{(10)} = \psi_{i_1,i_2,i_3,i_4,i_5,i_6,i_7,i_8,i_9,i_{10}}\psi_{i_1}^i\psi_{i_2}^j\psi_{i_3}^k\psi_{i_4}^l\psi_{i_5}^m\psi_{i_6}^n\psi_{i_7}^o\psi_{i_8}^p\psi_{i_9}^q\psi_{i_{10}}^r \] (35)

\[ \psi_{i_1}^1, \psi_{i_2}^1, \psi_{i_3}^1, \psi_{i_4}^1, \psi_{i_5}^1, \psi_{i_6}^1, \psi_{i_7}^1, \psi_{i_8}^1, \psi_{i_9}^1, \psi_{i_{10}}^1 \]
\[ \psi_{i_1}^2, \psi_{i_2}^2, \psi_{i_3}^2, \psi_{i_4}^2, \psi_{i_5}^2, \psi_{i_6}^2, \psi_{i_7}^2, \psi_{i_8}^2, \psi_{i_9}^2, \psi_{i_{10}}^2 \]
\[ \psi_{i_1}^3, \psi_{i_2}^3, \psi_{i_3}^3, \psi_{i_4}^3, \psi_{i_5}^3, \psi_{i_6}^3, \psi_{i_7}^3, \psi_{i_8}^3, \psi_{i_9}^3, \psi_{i_{10}}^3 \]
\[ \psi_{i_1}^4, \psi_{i_2}^4, \psi_{i_3}^4, \psi_{i_4}^4, \psi_{i_5}^4, \psi_{i_6}^4, \psi_{i_7}^4, \psi_{i_8}^4, \psi_{i_9}^4, \psi_{i_{10}}^4 \]
\[ \psi_{i_1}^5, \psi_{i_2}^5, \psi_{i_3}^5, \psi_{i_4}^5, \psi_{i_5}^5, \psi_{i_6}^5, \psi_{i_7}^5, \psi_{i_8}^5, \psi_{i_9}^5, \psi_{i_{10}}^5 \]
\[ \psi_{i_1}^6, \psi_{i_2}^6, \psi_{i_3}^6, \psi_{i_4}^6, \psi_{i_5}^6, \psi_{i_6}^6, \psi_{i_7}^6, \psi_{i_8}^6, \psi_{i_9}^6, \psi_{i_{10}}^6 \]
\[ \psi_{i_1}^7, \psi_{i_2}^7, \psi_{i_3}^7, \psi_{i_4}^7, \psi_{i_5}^7, \psi_{i_6}^7, \psi_{i_7}^7, \psi_{i_8}^7, \psi_{i_9}^7, \psi_{i_{10}}^7 \]
\[ \psi_{i_1}^8, \psi_{i_2}^8, \psi_{i_3}^8, \psi_{i_4}^8, \psi_{i_5}^8, \psi_{i_6}^8, \psi_{i_7}^8, \psi_{i_8}^8, \psi_{i_9}^8, \psi_{i_{10}}^8 \]
\[ \psi_{i_1}^9, \psi_{i_2}^9, \psi_{i_3}^9, \psi_{i_4}^9, \psi_{i_5}^9, \psi_{i_6}^9, \psi_{i_7}^9, \psi_{i_8}^9, \psi_{i_9}^9, \psi_{i_{10}}^9 \]
\[ \psi_{i_1}^{10}, \psi_{i_2}^{10}, \psi_{i_3}^{10}, \psi_{i_4}^{10}, \psi_{i_5}^{10}, \psi_{i_6}^{10}, \psi_{i_7}^{10}, \psi_{i_8}^{10}, \psi_{i_9}^{10}, \psi_{i_{10}}^{10} \]

It is worth emphasizing that \( \varepsilon = i\sigma_2 \) has the physical interpretation of a spinor-metric [23].
The space of degree 10 invariants has dimension 15, and the tilde indicates that the expression (35) cannot be expected to coincide with $\mathcal{G}^{(5)}_{10}$ as created from the $\Omega$-process. The latter is rather a sum over all possible $\epsilon$-contractions emerging from the given $\Omega$-process, and $\tilde{\mathcal{G}}^{(5)}_{10}$ is only one element of this sum. This transcription therefore already reduces the computational complexity of such invariants. Interestingly, the symmetric group $S_5$ generates from $\tilde{\mathcal{G}}^{(5)}_{10}$ a 14-dimensional subspace where only $PF$ is missing to give the whole 15-dimensional space $V^{(5)}_{10}$.

A view back onto Eq. (33) suggests a connection between the $\Omega$-process and the invariant construction from combs; namely that the cross-contraction on the third qubit might be substituted by the comb of second order $\sigma_\mu \bullet \sigma^\mu$, possibly including a term proportional to $\sigma_2 \bullet \sigma_2$. In order to make this connection a rigorous statement we translate the index contraction into an antilinear expectation value. The symmetry of antilinear expectation values

$$\langle \psi^* | \hat{A} | \varphi \rangle = \langle \varphi^* | \hat{A}^\dagger | \psi \rangle = \langle \varphi^* | \hat{A} | \psi \rangle$$

(36)

for Hermitean operators is crucial for this to work. The procedure is best explained graphically in figure 2. The three-qubit wavefunction coefficients are sketched as a staple of three circles. Each contraction with the antisymmetric tensor $\epsilon = i \sigma_2$ is visualized by a line connecting two circles. Arranged in pairs, an expectation value with $\epsilon$ corresponds to an inner contraction. Each contraction with the antisymmetric tensor $\epsilon = i \sigma_2$ is visualized by a line connecting two circles. Arranged in pairs, an expectation value with $\epsilon$ corresponds to an inner contraction. The three-qubit wavefunction coefficients are sketched as a staple of three circles. Each contraction with the antisymmetric tensor $\epsilon = i \sigma_2$ is visualized by a line connecting two circles. Arranged in pairs, an expectation value with $\epsilon$ corresponds to an inner contraction.

of three circles there. They are arranged in pairs, and an expectation value corresponds to an inner contraction of that pair – i.e. a contraction of wavefunction indexes inside such a pair. Each contraction with the antisymmetric tensor $\epsilon = i \sigma_2$ is visualized by a line connecting two circles. The cross-contractions are not yet expressed in terms of expectation values. Fortunately, suitable permutations of copies, which are however local in the qubits, do exist as to transform also the cross-contractions into expectation values without disturbing the inner contractions. In the following we describe this iterative procedure. The first equality in Fig. 2 is due to the symmetry (36) for antilinear expectation values of Hermitean operators. The second equality is formally expressed as

$$\langle \psi^* | \bullet \langle \psi^* | 3 \mathcal{P} \sigma_2 \sigma_2 \bullet \sigma_2 \sigma_2 | \psi' \rangle \bullet | \psi \rangle = \langle \psi^* | \bullet \langle \psi^* | \sigma_2 \sigma_2 \bullet \sigma_2 \sigma_2 \mathcal{P} | \psi' \rangle \bullet | \psi \rangle = -\psi_{a_1 b_1} a_2 b_2 a_3 b_3 \psi_{a_1 a_2 a_3 a_4} \psi_{b_1 b_2 a_3 b_4}$$

where $\mathcal{P}$ is the symbol for a copy permutation operator and the number three on top of $\mathcal{P}$ indicates that this permutation operator acts non-trivially only on the third qubit. Using $\mathcal{P} = \frac{1}{2} \sum_{\mu=0}^3 \sigma_\mu \bullet \sigma_\mu$, a straightforward calculation produces

$$\langle \sigma_2 \bullet \sigma_2 | \mathcal{P} = M_{\mu \nu} \sigma_\mu \bullet \sigma_\nu$$

$$= \frac{1}{2} (\sigma_\mu \bullet \sigma^\mu - \sigma_2 \bullet \sigma_2)$$

(37)
where \( M_{\mu \nu} = \delta_{\mu \nu} m_{\mu} \), \((m_0, m_1, m_2, m_3) = (1, -1, 1, -1)/2\). The resulting antilinear expectation value of \( M_{\mu \nu} \sigma_{\mu} \cdot \sigma_{\nu} \) is then indicated graphically by a double line connecting the copies. For completeness we mention that

\[
(\sigma_{\mu} \cdot \sigma^{\mu}) P = -\frac{1}{2} (\sigma_{\mu} \cdot \sigma^{\mu} + 3\sigma_{2} \cdot \sigma_{2})
\]

which readily follows from the identity \( (\sigma_{2} \cdot \sigma_{2}) PP = (\sigma_{2} \cdot \sigma_{2}) \) and Eq. (37).

In order to see in how far every invariant constructed with an \( \Omega \)-process can be expressed in terms of expectation values of local invariant operators, the identity described in Fig. 3 is helpful.

![FIGURE 3. Using copy permutation operators that act separately on demand on specific qubits, the above graphical expression for an identity is obtained. It shows that the rephrasing in terms of antilinear expectation values can indeed be obtained in this way.](image)

It means that the translation of the \( \Omega \)-process into antilinear expectation values can be performed iteratively qubit per qubit: an apparent incompatibility of a contraction pattern with a fixed ordering of wave function coefficients (vertical rows of circles) is resolved iteratively making use of the symmetry (36). It is worth mentioning at this point that grouping the wavefunction coefficients in pairs is arbitrary and therefore we obtain the same invariant when changing this order by permuting the order of the wave function coefficients. This however changes the contraction scheme for the invariant and leads to identities for invariants as observed above. For degree 4 invariants this freedom means that interchanging

\[
\sigma_{2} \cdot \sigma_{2} \leftrightarrow \frac{1}{2} (\sigma_{2} \cdot \sigma_{2} - \sigma_{\mu} \cdot \sigma^{\mu})
\]

on each qubit leads to the same invariant. This immediately leads to the identities (7), those mentioned just above it, and the identities (15) and (16).

For higher degree invariants, more such pair permutations may occur. Since the symmetric group is generated from nearest neighbor exchanges \( \pi_{j, j+1} \) and by virtue of the relations \( \pi_{12} \pi_{23} \pi_{12} = \pi_{23} \pi_{12} \pi_{23} \), \( \pi_{i j}^{2} = I \), it is sufficient to consider the results for up to three permutation operators. We find that

\[
(\sigma_{2} \cdot \sigma_{2} \cdot \sigma_{2}) P_{12} P_{23} = \frac{1}{4} \left[ \begin{array}{c}
\sigma_{2} \cdot \sigma_{2} \cdot \sigma_{2} - \\
(\sigma_{\mu} \cdot \sigma^{\mu} \cdot \sigma_{2} + \sigma_{\mu} \cdot \sigma_{2} \cdot \sigma^{\mu} + \sigma_{2} \cdot \sigma_{\mu} \cdot \sigma^{\mu}) - \\
i\epsilon_{k l m} \tau_{k} \cdot \tau_{l} \cdot \tau_{m}
\end{array} \right] \tag{38}
\]

and

\[
(\sigma_{2} \cdot \sigma_{2} \cdot \sigma_{2}) P_{23} P_{12} = \frac{1}{4} \left[ \begin{array}{c}
\sigma_{2} \cdot \sigma_{2} \cdot \sigma_{2} - \\
(\sigma_{\mu} \cdot \sigma^{\mu} \cdot \sigma_{2} + \sigma_{\mu} \cdot \sigma_{2} \cdot \sigma^{\mu} + \sigma_{2} \cdot \sigma_{\mu} \cdot \sigma^{\mu}) + \\
i\epsilon_{k l m} \tau_{k} \cdot \tau_{l} \cdot \tau_{m}
\end{array} \right], \tag{39}
\]
where $\tau_1 := \sigma_0$, $\tau_2 := \sigma_1$, $\tau_3 := \sigma_3$. A further permutation acting on the antisymmetric part $i\varepsilon_{klm} \tau_k \bullet \tau_l \bullet \tau_m$ leads to terms of the type $(\sigma_{\mu} \bullet \sigma_{\mu} \bullet \sigma_2 + \sigma_2 \bullet \sigma_{\mu} \bullet \sigma_{\mu} \bullet \sigma_2)$ and vice versa.

Besides an antisymmetric part in the exchange of copies, only $\sigma_2$ and $\sigma_{\mu} \bullet \sigma_{\mu}$ appear and the set of locally invariant operators is closed. The antisymmetric part is not captured by the two combs but in our analysis it appeared to be irrelevant for the search of $SL^*$-invariants. It leads, however, to invariants that are antisymmetric under qubit permutations (see e.g. the invariants of degree 6 and 10 for five qubits). Since an entanglement measure is defined as the modulus of an invariant, the requirement for a class-specific global entanglement measure must be relaxed to include also $SL^*$-invariants.

**CONCLUSIONS**

We have presented a thorough analysis of the polynomial $SL$-invariants of four and five qubits with particular emphasis on the filter ideal, i.e. on those invariants that vanish on all product states. Within the complete ring of invariants the filter ideal plays an outstanding role because only these invariants can clearly discriminate the genuine multipartite entanglement from that within parts of the system. It therefore hosts candidates for a class specific quantification of genuine multipartite entanglement and its knowledge plays a key role for a systematic analysis and a deeper understanding of the structure of entanglement in multipartite systems. A reasonable measure for global entanglement should also be invariant under qubit permutations. In order to determine the dimension of the subspace of permutation invariant elements in the filter ideal, we analyzed the decomposition of the space of polynomial invariants into irreducible modules of the symmetric group.

In the case of four qubits, the standard approach from invariant theory, employing the well established $\Omega$-process by Cayley, has already lead to the construction of a complete set of $SL$-invariants [5]. We have compared this approach to an alternative proposal based on local invariant operators, termed combs [2, 4]. We could demonstrate that also the latter approach generates a complete set of invariants, and we provide a full dictionary for expressions from both approaches. We have furthermore established the equivalence of the $\Omega$-process and the contraction with the spinor metric and provide the missing link between the $\Omega$-process and the construction from local $SL$-invariant operators. This implies that major part of the ring of invariants can be generated using the comb based method. Indeed we find that all $SL^*$-invariants and even many $SL^*$-invariants in this work are accessible to the comb based approach. In addition, the interrelation between $\Omega$-process and the comb approach implies the existence of interesting identities among sets of invariants and readily explains those identities observed among invariants of degree 4.

We single out two major advantages of the approach from local invariant operators.

1 **Control:** The comb based approach admits a high degree of control over specific important properties of the invariants that are to be constructed. Of particular relevance is the ab initio knowledge about the set of product states for which the
invariant will vanish. This is a key quality that admits a systematic construction of filter invariants; this provides a targeted construction of the filter ideal. In contrast, from the $\Omega$-process, and equivalently from the contraction with the spinor metric, there is no a priori knowledge about the invariant’s value on product states.

II Computational complexity: The interrelation between $\Omega$-process, $\varepsilon$ contractions and the comb based approach explains the notable difference in computational complexity we observed. It is clear from these interrelations that the computational complexity of the comb based approach is significantly lower than that using contractions with the spinor metric $\varepsilon$, which itself already constitutes a speed-up as compared to the $\Omega$-process. This discrepancy grows more important with increasing degree of the invariants.

These advantages permit us to go considerably further in a thorough analysis of invariants, and we demonstrate this for five qubits: we give a complete analysis of $SL^*$-invariants up to degree 12, and provide an outlook on the situation for degrees 14 and 16. Although the five qubit case is still far from being completed, we have presented a straightforward technique for how to proceed; we are confident that a minimal set of generators can be obtained in the way described in this manuscript. All results have been cross-checked with predictions from the Hilbert series. To this end we also derived the first terms of the Hilbert series for relative $SL^*$-invariants.

We hope that the high degree of control paired with the significantly lower computational complexity in the generation of invariants have future impact in both quantum information and invariant theory. Further analysis would be necessary in order to find an expression of all $SL^*$-invariants in terms of (antilinear) expectation values. It would be also interesting to extend an analysis along the lines proposed e.g. in Refs. [12, 11] in order to see whether the filter ideal has a distinguished geometrical interpretation.

**COMB-BASED INVARIANTS**

In this appendix we give a detailed elucidation how comb-based invariants are calculated.

Let the pure $q$-qubit quantum state $|\psi\rangle$ be expressed in terms of a basis $\mathcal{B}$ made of tensor products of eigenstates $|-\rangle$ and $|1\rangle$ of the Pauli spin operator $\sigma_3$, such that $\sigma_3 |s\rangle = s |s\rangle$ for $s = \pm 1$. That is, we have

$$\mathcal{B} = \{|s_1\rangle \otimes \cdots \otimes |s_q\rangle : s_j = \pm 1\}$$

In this basis the Pauli spin operators (consider $q = 1$ for the sake of simplicity) assume the matrix representations $\sigma_{ij}^{s_i} := \langle s'_i | \sigma_i | s \rangle$ as given in Eq. (1). Matrix elements of $q$-qubit operators are then defined in the standard way for arbitrary $q$-qubit pure states $|\varphi\rangle$, $|\psi\rangle$ as

$$\langle \varphi | \sigma_{i_1} \otimes \cdots \otimes \sigma_{i_q} | \psi \rangle := \langle \varphi_{s_1, \ldots, s_q} | \sigma_{i_1}^{s_1} \cdots \sigma_{i_q}^{s_q} | \psi_{s_1, \ldots, s_q} \rangle$$

$$\equiv [\mathcal{C} \varphi]_{s_1, \ldots, s_q} \sigma_{i_1}^{s_1} \cdots \sigma_{i_q}^{s_q} \psi_{s_1, \ldots, s_q}$$
within Einstein summation convention, and \([C\varphi]_{s_1,\ldots,s_q} := \langle s_1,\ldots,s_q | C | \varphi \rangle\).

In this sense the antilinear expectation values as defined in Eq. (5) are specific matrix elements of an antilinear operator \(A = L_A C\). Here, \(C\) is the complex conjugation as defined in Eq. (41), and \(L_A\) is the linear operator associated to \(A\). In all this work, the operators \(A\) and \(L_A\) are antilinear Hermitean and Hermitean, respectively. In the case of a single copy of the state we then have

\[
((L_A)) := \langle \psi | A^\dagger | \psi \rangle^* = \langle \psi | A | \psi \rangle^* = \langle \psi^* | L_A | \psi \rangle
\]

which is a matrix element as defined in Eq. (40) where \(|\varphi\rangle \rightarrow |\psi^*\rangle\) (see Eq. (41)). For any indexes \(i_1, i_2, \ldots, i_q \in \{0, 1, 2, 3\}\) we therefore define a bilinear form

\[
\langle \sigma_1, \sigma_2, \ldots, \sigma_q \rangle : \mathcal{H}_q \times \mathcal{H}_q \rightarrow \mathbb{C},
\]

whose value at \((\varphi, \psi)\) is the multiple sum (using the Einstein convention)

\[
\langle \sigma_1, \sigma_2, \ldots, \sigma_q \rangle (\varphi, \psi) = \langle \varphi^* | \sigma_1 \otimes \sigma_2 \otimes \cdots \otimes \sigma_q | \psi \rangle.
\]

This can be also expressed as

\[
\langle \sigma_1, \sigma_2, \ldots, \sigma_q \rangle (\varphi, \psi) = \langle \varphi^* | \sigma_i \otimes \sigma_2 \otimes \cdots \otimes \sigma_q | \psi \rangle.
\]

As the first example, we set \(q = 1\) and \(i_1 = 2\) and we obtain the \(SL(2,C)\)-invariant bilinear form

\[
\langle \sigma_2 \rangle (\varphi, \psi) = -i\begin{vmatrix} \varphi_1 & \varphi_2 \\ \psi_1 & \psi_2 \end{vmatrix}.
\]

(42)

However, in this case we have \((\langle \sigma_2 \rangle) := \langle \sigma_2 \rangle (\psi, \psi) = 0\) for all \(\psi\), which is the comb property of the operator \(\sigma_2\).

As another example we take \(q = 2\) and \(i_1 = i_2 = 2\). Since \(\sigma_2 = -i\epsilon\), we have

\[
\langle \sigma_2 \sigma_2 \rangle (\varphi, \psi) = -\epsilon^{a_1,b_1} \epsilon^{a_2,b_2} \psi_{a_1,a_2} \psi_{b_1,b_2}
\]

\[
= \begin{vmatrix} \varphi_{2,1} & \varphi_{2,2} \\ \psi_{1,1} & \psi_{1,2} \end{vmatrix} - \begin{vmatrix} \varphi_{1,1} & \varphi_{1,2} \\ \psi_{2,1} & \psi_{2,2} \end{vmatrix},
\]

an \(SL\)-invariant bilinear form. In the case when \(\varphi = \psi\), we obtain the nonzero \(SL\)-invariant quadratic form

\[
((\langle \sigma_2 \sigma_2 \rangle) := \langle \sigma_2 \sigma_2 \rangle (\psi, \psi) = -2\begin{vmatrix} \psi_{1,1} & \psi_{1,2} \\ \psi_{2,1} & \psi_{2,2} \end{vmatrix}.
\]

For operators acting on \(m\) copies of the state just replace \(|\psi\rangle\) by \(|\psi\rangle \cdots \cdots |\psi\rangle \equiv |\psi\rangle^m\) and the corresponding expression for \(\langle \psi^* \rangle\). To outline this in more detail, let \(\mathcal{H}\) denote the Hilbert space of a single qubit, and \(\mathcal{H}_q = \mathcal{H}^\otimes q\) the one for the system of \(q\) qubits. We shall also use the Hilbert space for \(m\) copies of this multipartite system. In that case we use \(\bullet\) to denote tensor products of Hilbert spaces of different copies. Let us now take a collection of \(m\) bilinear forms of the above type,

\[
\langle \sigma_{i_1}^{(k)} \sigma_{i_2}^{(k)} \cdots \sigma_{i_q}^{(k)} \rangle, \quad k = 1, 2, \ldots, m,
\]
and let us form their product

\[ \langle \sigma_{i_1}^{(1)} \sigma_{i_2}^{(1)} \cdots \sigma_{i_q}^{(1)} \rangle \langle \sigma_{i_1}^{(2)} \sigma_{i_2}^{(2)} \cdots \sigma_{i_q}^{(2)} \rangle \cdots \langle \sigma_{i_1}^{(m)} \sigma_{i_2}^{(m)} \cdots \sigma_{i_q}^{(m)} \rangle, \]  

(43)

which is a bilinear form

\[ \mathcal{H}_q^m \times \mathcal{H}_q^m \to \mathbb{C}. \]  

(44)

The value of this bilinear form on the special elements

\[ (\varphi^{(1)} \cdot \varphi^{(2)} \cdots \varphi^{(m)}, \psi^{(1)} \cdot \psi^{(2)} \cdots \psi^{(m)}) \]

is equal to

\[ \prod_{k=1}^{m} \langle \sigma_{i_1}^{(k)} | \sigma_{i_2}^{(k)} \cdots \sigma_{i_q}^{(k)} \rangle \left( \varphi^{(k)}, \psi^{(k)} \right). \]

If we further specialize \( \varphi^{(k)} = \psi^{(k)} = \psi \) for all \( k \), we obtain the \( 2^q \)-ary form of degree \( 2m \) in the complex components of \( \psi \):

\[ \left( \prod_{k=1}^{m} \sigma_{i_1}^{(k)} \cdots \sigma_{i_q}^{(k)} \right) := \prod_{k=1}^{m} \langle \sigma_{i_1}^{(k)} | \cdots \sigma_{i_q}^{(k)} \rangle \left( \psi, \psi \right). \]

We refer to this form of degree \( 2m \) as the associated form of the bilinear form (43). This definition extends immediately to any bilinear form (44).

In general, the forms of degree \( 2m \) constructed above are not \( SL \)-invariant, but we can use their suitable linear combinations to obtain \( SL \)-invariant forms.

In order to do that we proceed as follows. First we select a site, say \( s \), \( 1 \leq s \leq q \), of our multipartite system and choose two different copies of the state, say copies \( p \) and \( q \), \( 1 \leq p < q \leq m \). Next we replace in (43) the Pauli matrices \( \sigma_{i_s}^{(p)} \) and \( \sigma_{i_s}^{(q)} \) with symbols \( \sigma_\mu \) and \( \sigma^\mu \), respectively. This is to indicate that the two indexes \( \mu \) are to be contracted by using the pseudo-metric \( G_{\mu \nu} \). We now interrupt our description to give an example.

When \( q = 3 \) and \( m = 2 \) the expression (43) has the form

\[ \langle \sigma_{i_1}^{(1)} \sigma_{i_2}^{(1)} \sigma_{i_3}^{(1)} \rangle \langle \sigma_{i_1}^{(2)} \sigma_{i_2}^{(2)} \sigma_{i_3}^{(2)} \rangle \]

We now choose \( p = 1 \), \( q = 2 \) and \( s = 1 \). By applying the above instruction, we obtain the expression

\[ \langle \sigma_\mu \sigma_{i_2}^{(1)} \sigma_{i_3}^{(1)} \rangle \langle \sigma^\mu \sigma_{i_2}^{(2)} \sigma_{i_3}^{(2)} \rangle . \]

By fixing \( \sigma_{i_2}^{(1)} = \sigma_{i_3}^{(1)} = \sigma_{i_2}^{(2)} = \sigma_{i_3}^{(2)} = \sigma_2 \) and performing the \( \mu \)-contraction this gives the linear combination

\[ -\langle \sigma_0 \sigma_2 \sigma_2 \rangle \langle \sigma_0 \sigma_2 \sigma_2 \rangle + \langle \sigma_1 \sigma_2 \sigma_2 \rangle \langle \sigma_1 \sigma_2 \sigma_2 \rangle + \langle \sigma_3 \sigma_2 \sigma_2 \rangle \langle \sigma_3 \sigma_2 \sigma_2 \rangle. \]

The associated quartic form is then obtained as

\[ \langle \sigma_\mu \sigma_2 \sigma_2 \cdot \sigma^\mu \sigma_2 \sigma_2 \rangle := \sum_{\mu=0}^{3} g_\mu \langle \psi^* | \sigma_\mu \sigma_2 \sigma_2 | \psi \rangle^2 \]

\[ = \sum_{\mu=0}^{3} g_\mu \left| \psi_{s_1, s_2, s_3} \sigma_\mu \sigma_2 \sigma_2 \psi_{s_1, s_2, s_3} \right|^2. \]
It generates the $SL^*$-invariants for three qubits; its modulus is the three-tangle [13].

To continue our description, we choose a collection of triples $(s_i, p_i, q_i)$, $i = 1, 2, \ldots, t$ such that $1 \leq p_i < q_i \leq m$ and whenever $s_i = s_j$, with $i \neq j$, we require that the four integers $p_i, q_i, p_j, q_j$ be all distinct. For each index $i$, we replace the Pauli matrices on the site $s_i$ and copies $p_i$ and $q_i$ with the symbols $\sigma_i$ and $\sigma^\mu_i$, respectively. Next we replace all other Pauli matrices in (43) with the matrix $\sigma_2$. Finally, by using the pseudo-metric $G_{\mu\nu}$, we perform the $\mu_i$ contractions for each $i$, $1 \leq i \leq t$. We obtain a linear combination of bilinear forms of the type given by (43). We refer to these linear combinations as comb-based bilinear forms.

These comb-based forms are homogeneous multilinear expressions in the (complex) state coefficients, which are $SL_q$-invariant. This invariance harkens back to the $SL(2, \mathbb{C})$ invariance of the antilinear single qubit combs $\sigma_2 \mathcal{C}$ and $\sigma_\mu \mathcal{C} \cdot \sigma^\mu \mathcal{C}$. We formulate this statement in

**Theorem 2.** Any comb-based bilinear form (and, consequently, also its associated form) is an $SL$-invariant.

It has been stated in [2, 4] that the combs are $SL$-invariant, but there is only implicit reference to the fact that this derives from the central comb property to have zero expectation value on all the local Hilbert spaces. Here we sketch a proof for this connection.

**Proof:** The comb property for the operator $\sigma_2$, namely that $\langle \sigma_2 \rangle (\psi, \psi) = 0$ for all single qubit states $\psi$, can be read off directly from Eq. (42), and it can be checked by direct calculation that it is the unique operator with this property up to rescaling. Also by direct calculation we find that $\langle \sigma_i \cdot \sigma^\mu \rangle (\psi \cdot \psi, \psi \cdot \psi) = 0$ for all single qubit states $\psi$. Furthermore, this is the unique form (up to rescaling) on $\mathcal{H}^m_q$ satisfying this condition which is symmetric under copy-permutation and orthogonal to $\langle \sigma_2 \cdot \sigma_2 \rangle$ in the sense of the vanishing scalar product $\text{tr} (\sigma_2 \cdot \sigma_2) \cdot (\sigma_\mu \cdot \sigma^\mu) = 0$. For arbitrary $S \in SL(2, \mathbb{C})$ we then find that

$$0 = \langle \sigma_2 \rangle (S\psi, S\psi) = \langle S' \sigma_2 S \rangle (\psi, \psi)$$

for all $\psi$ and, due to the uniqueness property for the operator $\sigma_2$, this implies $S' \sigma_2 S = \sigma_2$. Analogously we have

$$0 = \langle \sigma_\mu \cdot \sigma^\mu \rangle (S\psi \cdot S\psi, S\psi \cdot S\psi)$$

$$= \langle (S \cdot S)' \sigma_\mu \cdot \sigma^\mu (S \cdot S) \rangle (\psi \cdot \psi, \psi \cdot \psi)$$

This proves that the two comb operators are $SL(2, \mathbb{C})$-invariant. Consequently, a $q$-qubit form constructed from those is seen to be $SL_q$-invariant by wrapping a transformation $S^{(q)} = S_1 \otimes \cdots \otimes S_q$ with $S_j \in SL(2, \mathbb{C})$ back onto the states. This completes the proof.

We refer to the $SL$-invariants constructed in this manner as the comb-based invariants. In some cases this invariant may be zero.

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