Mesoscopic scattering in the half-plane: squeezing conductance through a small hole

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We model the 2-probe conductance of a quantum point contact (QPC), in linear response. If the QPC is highly non-adiabatic or near to scatterers in the open reservoir regions, then the usual distinction between leads and reservoirs breaks down and a technique based on scattering theory in the full two-dimensional half-plane is more appropriate. Therefore we relate conductance to the transmission cross section for incident plane waves. This is equivalent to Landauer’s formula using a radial partial-wave basis. We derive the result that an arbitrarily small (tunneling) QPC can reach a p-wave channel conductance of $2e^2/h$ when coupled to a suitable reflector. If two or more resonances coincide the total conductance can even exceed this. This relates to recent mesoscopic experiments in open geometries. We also discuss reciprocity of conductance, and the possibility of its breakdown in a proposed QPC for atom waves.

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I. INTRODUCTION

The quantum point contact (QPC) has played a central role in the understanding of mesoscopic conductance. It is the simplest example of a 2DEG system where the quantum coherent nature of the electron controls the bulk transport properties. The Landauer-Büttiker (LB) formalism reduces the calculation of quantum conductance in the linear response regime to the evaluation of single-particle wavefunction transmission amplitudes. Traditionally, these amplitudes are measured between travelling wave basis states in the ‘leads’. Far from the scattering system the leads have constant profiles of finite-width, and support a finite number of transverse modes (channels). Eventually it is assumed that the leads are impedance-matched (that is, without reflection) into ‘reservoirs’ which act as thermalized sources of electrons at their respective potentials; these potentials are taken to reflect the measured bias voltage. Such theoretical constructs have been remarkably successful at describing transport phenomena, for instance conductance quantization because the scattering systems involved have generally had good lead-to-reservoir matching.

We consider ‘open’ 2-terminal mesoscopic systems, namely those where a QPC is non-adiabatic (possessing rapid longitudinal variation in transverse profiles) and has short or nonexistent leads (for instance if it suddenly abuts onto the ‘reservoir’ regions), or those where there can be scattering off nearby objects in the ‘reservoir’ region. We call such systems ‘open’ because the fully two-dimensional (2D) nature of the ‘reservoirs’ (i.e. the surrounding semi-infinite regions of free space) is important, and therefore they cannot be modelled using the quasi-1D approach described above. This includes a variety of recent mesoscopic experiments, for example the combination of QPCs with nearby resonator structures or with a nearby depletion region underneath an AFM tip. It also includes any QPC system where elastic backscattering from disorder in the reservoirs is significant, or generally where the lead-reservoir matching is bad. In such systems, the conventional quasi-1D picture does not apply: the scattering system is not coupled to leads in the usual sense, indeed the distinction between leads and reservoirs is no longer clear. The main aim of the present work is to introduce a 2D scattering theory approach which can handle such systems, and to apply it to the calculation of the maximum conductance of an open resonator structure of experimental relevance.

We imagine a geometry where a 2DEG exists in two semi-infinite half-plane regions, separated by an impenetrable potential barrier which we align with the y-axis (see Fig. Ia). Our general ‘QPC scattering system’ is any gap in this barrier which allows coupling of the wavefunction on the left and right sides. This gap can be defined by an arbitrary form of the elastic potential, and may include other nearby scattering objects or disorder (which would all be placed within the box shown in Fig. Ia). The only important limitation is that this coupling region (the ‘system’) be of finite y extent, so that electrons which leave the system do so via a well-defined terminal: either the left ($x < 0$) or the right ($x > 0$).

We also assume that the system size $L$ is much smaller than both the dephasing length $l_\phi$ and the momentum relaxation (elastic scattering) length $l_e$. The former requirement allows treatment using a coherent wavefunction across the system; the latter allows free-space elastic scattering concepts to be applied. We will stay within the non-interacting quasiparticle picture, consider zero applied magnetic field, and assume spin degeneracy of 2 throughout.

The conventional distinction between ‘reservoir’ and ‘lead’ is no longer applicable, however at short distances outside the system ($r > L$ but $r \ll l_\phi$ and $r \ll l_e$) the two semi-infinite free space regions behave like leads, since they support scattering-free ‘channels’ (see Section III). At large distances the same regions behave as reservoirs: for $r \gg l_\phi$ ergodicity ensures that the momentum distribution is uniform in angle, and for $r \gg l_e$ the energy is redistributed to ensure equilibrium at the
relevant (experimentally-measured) chemical potential of each terminal. In the intermediate region, there is a broad cross-over from lead to reservoir.

In this work we first derive a general relation between transmission cross section (a concept we define using scattering in the half-plane) and conductance for this open geometry, in Section II. In Section III we show that partial-wave type states, defined in the half-plane regions, can take the place of transverse lead modes in the Landauer formula. In Section IV we discuss the maximum conductance through an idealized, highly non-adiabatic QPC (a hole in a thin hard wall) which is reached when a resonator is placed on one side of the QPC. We find a universal result, namely a single conductance quantum, regardless how small the hole is. This illuminates the findings of a recent experiment in such an open geometry. In Section V we discuss attempts to exceed this universal quantum of conductance through a single channel. A reciprocity relation for cross section is derived in Section VI, and the possibility of breaking this reciprocity, due to a non-thermal reservoir occupation, is described. We discuss an application to matter-wave ‘conductance’ through a 3D QPC. We conclude in Section VII.

II. CONDUCTANCE IN TERMS OF CROSS SECTION

We consider scattering of a single-quasiparticle wavefunction from the general 2-terminal system described in the Introduction (see Fig. 1a). The Hamiltonian is \( \mathcal{H} = -(\hbar^2/2m)\nabla^2 + V(r) \), for a quasiparticle mass \( m \). The elastic scattering potential \( V(r) \) completely defines the system. We imagine a monochromatic unit plane wave \( e^{i\mathbf{k}\cdot \mathbf{r}} \) incident from the free-space left-hand region. The wavevector is \( \mathbf{k} \equiv (k, \phi) \) in polar coordinates, \( \phi \) being the angle of incidence. The free-space wavevector magnitude is taken as \( k = k_F \) (corresponding to a total energy \( E = \hbar^2 k_F^2/2m \) equal to the Fermi energy), unless stated otherwise.

We are at liberty to choose our definition of the ‘un-scattered’ wave \( \psi_0 \). We take it to be the wavefunction which would result from reflection of the incident wave off a wall uniform in the \( y \) direction. We can imagine creating such a wall by replacing the ‘system box’ shown in Fig. 1a by the surrounding \( y \)-invariant wall profile. Note that \( \psi_0 \) exists only on the left side. In the left free-space region it is

\[
\psi_0 = e^{i(k_x x + k_y y)} - e^{-i(-k_x x + k_y y + \gamma_k)}
\]

where the first term is \( \psi_I \), and the angle-dependent reflection phase \( \gamma_k \) of the second term depends on both \( (k, \phi) \) and the wall profile. Upon introduction of our true system potential, the full wavefunction becomes

\[
\psi = \psi_0 + \psi_R + \psi_T,
\]

where the change in reflected wave \( \psi_R \) exists only on the left side, and the new transmitted wave \( \psi_T \) exists only on the right. These scattered waves have the asymptotic (\( r > L \) and \( kr \gg 1 \)) forms of 2D scattering theory,

\[
\psi_R = f_R(\theta) \frac{e^{ikr}}{\sqrt{r}}, \quad \psi_T = f_T(\theta') \frac{e^{ikr}}{\sqrt{r}}.
\]

See Fig. 1a for definitions of \( \theta \) and \( \theta' \).

The transmission cross section \( \sigma_T(k, \phi) \) is the ratio of \( \Gamma_T \), the transmitted particle flux (number per unit time), to \( j_I \), the incident particle flux per unit length normal to the incident beam:

\[
\sigma_T(k, \phi) \equiv \frac{\Gamma_T}{j_I}.
\]

Physically, \( \sigma_T(k, \phi) \) is the length required of an aperture oriented normal to the incident beam in order to transmit an equivalent flux of classical particles. (Note that \( \sigma_T(k, \phi) \) is proportional to the injection distribution which can be measured in mesoscopic systems.) It depends on the incident angle because \( V(r) \) has no radial symmetry. \( j_I \) is the magnitude of the incoming probability flux density vector \( j \equiv (\hbar/m) \text{Im}[\psi \nabla \psi] \), which for a unit wave gives \( j_I = v \), the particle speed. The transmitted flux is defined as
\[ \Gamma_T = \int dl \hat{n} \cdot j = \frac{\hbar}{m} \int dl \hat{n} \cdot \text{Im}[\psi_T^* \nabla \psi_T], \] (5)

where the line integral encloses the entire transmitted wave, and the (rightwards-pointing) surface normal is \( \hat{n} \). Applying this and (4) to the asymptotic form gives

\[ \sigma_T(k, \phi) = \int_{-\pi/2}^{\pi/2} d\theta |f_T(\theta')|^2, \] (6)

familiar from scattering theory apart from the restriction to the right half-plane. There is a corresponding form

\[ \sigma_R(k, \phi) = \int_{-\pi/2}^{\pi/2} d\theta |f_R(\theta)|^2, \] (7)

for the reflective cross section (removal from the unscattered wave without being transmitted).

We will calculate the conductance by assuming the chemical potential is slightly higher on the left side than the right, and as is usually considered only the left-to-right transport of the states in this narrow energy range. We take the left region to be a large \((\gg l_\phi)\) closed region of area \( A \) containing single-particle states, and find their decay rate through the QPC into the right side. Semiclassically each single-particle state occupies a phase-space volume \( \hbar^d \), where we have \( d = 2 \). Therefore the phase-space density in the 2DEG Fermi sea is \( 2/\hbar^2 \) where the factor of 2 comes from the spin degeneracy. We can project this density onto momentum space in order to find the effective number of plane-wave states impinging on the wall, this corresponds to a uniform density of states in \( k \)-space given by

\[ \rho(k, \phi) dk d\phi = \frac{A}{2\pi^2} kdk d\phi. \] (8)

Each state has an amplitude \( A^{-1/2} \) due to the requirement of unity area normalization in the left region, so has incoming flux density \( j_1 = v/A \). Substituting this into (4) gives the decay rate of a state \( i \) as

\[ \Gamma_T^{(i)} = \frac{v}{A} \sigma_T(k_i, \phi_i). \] (9)

We can now sum the decay rates of all the left-hand states in a given wavevector range \( k_F \) to \( k_F + \delta k \), to get the current

\[ \delta I = e \sum_i \Gamma_T^{(i)} = \frac{ev}{A} \int_{-\pi/2}^{\pi/2} d\phi \int_{k_F}^{k_F + \delta k} kdk \rho(k, \phi) \sigma_T(k, \phi) \]

\[ = \frac{ev k_F\delta k}{2\pi^2} \int_{-\pi/2}^{\pi/2} d\phi \sigma_T(k_F, \phi), \] (10)

where the last step incorporates the linear-response assumption that \( \sigma_T \) is constant over the range \( \delta k \).

When a potential difference \( \delta V \) is applied across the QPC, the energy range carrying current is \( \delta E = e \delta V \), which we can equate with \( \hbar v \delta k \) using the dispersion relation. This can be used with (10) to write the conductance

\[ G \equiv \frac{\delta I}{\delta V} = \frac{2e^2}{h} \cdot \frac{1}{\lambda_F} \int_{-\pi/2}^{\pi/2} d\phi \sigma_T(k_F, \phi) \]

\[ = \frac{2e^2}{h} \cdot \frac{k_F \delta k}{2} \langle \sigma_T \rangle_\phi, \] (11a)

where the particle wavelength is \( \lambda_F \equiv 2\pi/k_F \). The latter form is written in terms of the angle-averaged cross section at the Fermi energy. The weighting of this average is uniform because of the ergodic assumption that incoming states are uniformly distributed in angle.

Eq. (11) is a key result of this paper (an independent derivation is given by Barnett[13]). Like the Landauer formula, it directly connects conductance and scattering. In a scattering measurement from the left side, \( \sigma_T \) appears to be the QPC’s inelastic cross section (since the transmitted waves never return to this side). In a current measurement the corresponding conductance is given by (3). Our derivation was for temperature \( T = 0 \), but it applies at a finite \( T \) as long as \( \sigma_T \) does not change significantly over the energy range \( k_BT \). This can be seen by generalizing the above to include integration over the Fermi distribution.

In the limit where a QPC is adiabatic, its conductance is known to be quantized, \( G = (2e^2/h)N \) where \( N \) is the integer number of open channels at the Fermi energy. Looking at (11a), this corresponds to quantization of the angular integral of the cross section in units of \( \lambda_F \).

### III. PARTIAL-WAVE CHANNEL MODES FOR A 2-TERMINAL SYSTEM

In free-space scattering theory, partial waves form a basis in which to decompose the asymptotic \((r \to \infty)\) form of the full wavefunction \( \psi \) into incoming and outgoing states of definite angular momentum \( l \). In 2D the basis functions are the cylindrical solutions to the free-space wave equation; the \( S \)-matrix which takes incoming to outgoing waves can then be written in this basis. Because there is only a single set of incoming channels and a single set of outgoing channels, this is equivalent to a scattering system (a ‘stub’) connected to a single ‘lead’, with an infinite number of open channel modes. This contrasts the open two-terminal geometry we study, where we need to account for two new related facts: 1) in the \( r \to \infty \) limit the potential \( V \) no longer preserves angular-momentum, and 2) there are now distinct ways the particle can enter and exit the system, via different leads.

We define a ‘half-plane partial-wave basis’ as the subset of the cylindrical free-space solutions which go to zero on the entire \( y \)-axis. This gives independent basis functions existing on either the left or right side of the \( y \)-axis. The
basis is expressed in terms of Hankel functions on either side

\[ \begin{align*}
\phi_{l}^{+L}(kr) & \equiv H_{l}^{(2)}(kr) \sin[l(\frac{\pi}{2} - \theta)] \\
\phi_{l}^{-L}(kr) & \equiv H_{l}^{(1)}(kr) \sin[l(\frac{\pi}{2} - \theta)] \\
\phi_{l}^{+R}(kr) & \equiv H_{l}^{(2)}(kr) \sin[l(\frac{\pi}{2} - \theta')] \\
\phi_{l}^{-R}(kr) & \equiv H_{l}^{(1)}(kr) \sin[l(\frac{\pi}{2} - \theta')]
\end{align*} \tag{12} \]

where on the left (L) side \( \theta \) is the angle from the negative \( x \)-axis and on the right (R) side \( \theta' \) is the angle from the positive \( x \)-axis (see Fig. [4]). The channel index is \( l = 1, 2, \ldots, \infty \), and \(+(-)\) refers to outgoing (incoming) travelling waves. We note that the s-wave \( l = 0 \) is excluded because of the \( y \)-axis barrier, leaving the first channel as the \( p \)-wave \( H_{1}(kr) \cos(\theta) \). Assuming the width of the barrier is finite and constant as \( |y| \to \infty \) (see Fig. [4]), then any wavefunction in the \( r \to \infty \) limit can be written as a sum of the above basis functions. The separability of this basis in \((r, \theta)\) is directly analogous to the separability of conventional (constant-width) lead basis states into a product of transverse modes and longitudinal travelling waves.

Our basis is chosen such that unit amplitude coefficients carry equal fluxes in all incoming and outgoing channels, so flux conservation implies the unitarity of the \( S \)-matrix when written in this basis. As with a conventional transverse lead mode basis, the familiar Landauer formula

\[ G = \frac{2e^{2}}{h} \text{Tr}(t^\dagger t), \tag{13} \]

holds. The transmission matrix \( t \) is defined by \( q_{l}^\dagger = \sum_{m} t_{lm} p_{m} \), where the outgoing (incoming) amplitude coefficients are \( p_{l}^\dagger (p_{l}) \) on the left and \( q_{l}^\dagger (q_{l}) \) on the right. Note that is possible to ‘mix and match’ different basis set types (for instance define a transmission matrix between transverse lead modes on the left side and partial-wave modes on the right), as long as equal-flux normalisation, and transverse orthogonality, are preserved.

**IV. POINT CONTACT COUPLED TO A RESONATOR**

Fig. 2 illustrates a QPC-plus-reflector system whose conductance has been experimentally measured. The circular arc reflector and the vertical wall together form a cavity which can support long-lived resonances; the energy of these resonances can be swept by sweeping the reflector gate voltage. The classical condition for stability of the cavity modes is that the arc center must lie at, or to the left of, the wall \((x = 0)\). The cavity modes are coupled to the left terminal via the QPC, and to the right terminal via leakage of the modes out through the cavity top and bottom. The system is interesting because it is ‘open’ in the sense that it has no Coulomb blockade, but ‘closed’ in the sense that the dwell time is much greater than the ballistic time (the resonances are long-lived). It has also been studied recently in our laboratory using microwave measurements.

The actual potential in a mesoscopic experiment differs from the illustration: it has soft walls (on the scale \( 1/k_F \)), may have deviations from the circle due to lithographic error, and it has modulations of the background potential due to elastic disorder. However, we will not be interested in details of the resonator on the right-hand side. Rather, we will adopt the view of a 2D scattering-theorist ‘looking’ from the left-hand side. In this section we discuss the maximum conductance of this system, when the ‘bare’ QPC (i.e. without the reflector) is in the tunneling regime (conductance \( \ll 2e^2/h \)).

We use an idealized slit QPC model (see Fig. 1) in which the potential \( V \) is zero everywhere except along a hard, thin wall where it is taken as infinite. The QPC is a gap in the wall of size \( 2a \). This model is highly non-adiabatic (see Ref. 17 for a review of its transmission properties). The hard wall simplifies the treatment of the left-hand side scattering problem, and we do not believe it alters our basic conclusion. We consider the ‘unsattered’ wave to be the incident plus reflected wave Eq. (11) when the QPC is closed \( (a = 0) \). This we expand in Bessel functions,

\[ \psi_{0}(r) = e^{i(k_x x + k_y y)} - e^{-i(k_x x + k_y y)} 
= -4iJ_1(kr) \cos(\theta) \cos(\delta) + \text{higher order terms}. \tag{14} \]

The first term in the expansion is the incoming plus outgoing \( p \)-wave, which in the tunneling limit will dominate in our consideration of the absorption.

Now we open the slit, and replace \( 2J_1(kr) \) in the above by \( H_1^{(2)}(kr) + e^{2i\delta} H_1^{(1)}(kr) \), where \( \delta \) follows the usual definition of partial-wave phase shift. The closed slit corresponds to \( \delta = 0 \). An open slit leading into a closed resonator (imagine extending the arc in Fig. 2 to seal off the entire right side), in the case of infinite dephasing length, corresponds to \( \delta = \text{real} \), and would appear from the left side as an elastic dipole scatterer. An open slit with an open resonator corresponds to complex \( \delta \) with positive imaginary part, and would appear as a general...
inelastic dipole scatterer. Therefore transmission though the QPC appears, to an observer on the left side, to be absorption of incident waves. \( \sigma_T \) is interpreted as an ‘inelastic’ cross section (since exiting the right-hand terminal is equivalent to leaving in a new channel), and \( \sigma_R \) as an ‘elastic’ one. \( \sigma_T(k, \phi) \) can be found from integrating the net incoming flux as in Eq. (14) of the total wavefunction on the left side. Substitution into (14) then gives \( \sigma_T(k, \phi) = \frac{1}{2} (1 - |t_{11}|^2) \cos^2(\phi) \). For \( \delta \to i \infty \) the maximal cross section is reached,

\[
\sigma_{T,\text{max}}(k, \phi) = \frac{4}{k} \cos^2(\phi). \tag{15}
\]

This corresponds to an effective classical ‘area’ (size) \( a_{\text{eff}} = \lambda_F/2 \). This is analogous to the fact that in 3D the effective area of an arbitrarily-small electromagnetic dipole aerial can be of order \( \lambda^2 \). To an observer on the left side who was able to ‘see’ the electron waves living in the energy range \( eV \) responsible for conductance, the QPC would stand out as a ‘black dot’ of size \( \sim \lambda_F \) against the surrounding uniform ‘grey’ thermal luminosity reflected in the vertical wall mirror.

The associated maximum conductance is found easily using (13) and (14) to be

\[
G_{\text{max}} = \frac{2e^2}{h}, \tag{16}
\]

the universal quantum of conductance (for 2 spin channels), independent of the size of the QPC hole, even for an arbitrarily small hole \( (ka \to 0) \). This universal resonant-tunnelling maximum conductance was first found numerically \(^{23} \) and \(^{24} \), however our system differs from those of Xue \textit{et al.} \(^{23} \) and Kalmeyer \textit{et al.} \(^{24} \) because the resonance does not involve transmission through an isolated (zero-dimensional) quantum dot. The dramatic increase over the conductance of the bare QPC (which vanishes as \( (ka)^4 \), see Ref. \(^{24} \)) runs counter to the naive classical expectation, namely that the reflector would decrease the left-to-right flow of electrons because it sends back into the QPC particles which would otherwise exit to the right.

How do we know that it is possible to build a resonant geometry which corresponds to \( \delta \to i \infty \)? The reflector can be described by \( r \), the amplitude with which it returns an outgoing p-wave back to the QPC as an incoming p-wave. If \( |r|^2 = 1 - |t_{11}|^2 \), where the p-wave transmission of the QPC is \( t_{11} \) as defined in Section 11, then the p-wave channel becomes a 1D Fabry-Perot resonator with mirrors of matched reflectivity. Sweeping the round-trip phase then produces peaks of complete transmission (corresponding to complete p-wave absorption on the left side). The ratio of peak separation to peak width is the quality factor \( Q \sim 1/|t_{11}|^2 \). Such peaks, with heights much greater than the bare tunneling QPC conductance, were observed in the experiments of Katine \textit{et al.} \(^{24} \). However, Eq. (16) has not yet been tested quantitatively because of the difficulty of matching the Fabry-Perot reflectivities in a real 2DEG experiment. Note that the maximum conductance \( G_{\text{max}} \) also follows immediately from the Landauer formula when we realize that there can be complete transmission of the incoming \( l=1 \) channel state (from Section 11).

An interesting possibility arises when we realize \(^{24} \) that higher \( l \) channels are still slightly transmitted by the bare QPC, when \( ka \ll 1 \), even though they are increasingly evanescent. If the resonator has a high enough reactivity for these modes, then additional Fabry-Perot conductance peaks will be produced \(^{24} \). The peaks may be extremely narrow, but can carry a full quantum of conductance because they can transmit another incoming \( l \) channel. By careful arrangement of the cavity, one or more of these peaks could be brought into conjunction with an already-existing \( l=1 \) peak at the Fermi energy. (For instance, the \( l=1 \) and \( l=2 \) resonances are in different symmetry classes in Fig. 2 so there can be an exact level crossing). Therefore, we have the surprising result that, in theory, a conductance of \( (2e^2/h)/n \) can pass through an arbitrarily small QPC hole if \( n \) resonances (from \( n \) different channels) coincide at the Fermi energy. However, due to their extremely small width, such large conductance peaks are unlikely to be observable in a real mesoscopic tunneling QPC due to finite dephasing length and finite-temperature smearing.

Finally, we should not overlook the fact that our expressions for partial cross sections are a factor of 4 greater than those conventionally arising in 2D scattering theory from a radial potential \(^{24} \) because we are measuring cross section on the reflective boundary of a semi-infinite half plane. For instance, the maximum inelastic partial cross section for a single channel in free space \(^{24} \) is \( \sigma_r = 1/k \), compared to our maximum ‘inelastic’ cross section per channel Eq. (15). Similarly, the maximum elastic result in free space is \( \sigma_e = 4/k \), compared to our maximum (normal-incidence) ‘elastic’ cross section per channel \( \sigma_{R,\text{max}} = 16/k \). This latter case occurs when \( \delta = (\text{integer} + \frac{1}{2})\pi \).

**V. WHAT IS THE MAXIMUM CONDUCTANCE OF A SINGLE QUANTUM CHANNEL?**

The surprising theoretical results of the previous section might lead one to question the conductance limit \( 2e^2/h \) for a single quantum channel (by which we mean a single transverse mode for which the longitudinal degree of freedom is a 1D Fermi gas; this includes both conventional and partial-wave basis sets). For this gedanken-experiment we will consider conventional electron waveguides which are single-mode and long enough that evanescent waves are negligible, but which are also \( \ll l_o \). We try to encourage more current to pass down a single-mode channel (E) by connecting it to a reservoir via multiple routes (A,B,C,D), as shown in Fig 3b, where two routes are used on each side. It is possible to match the junctions so that a wave entering down A,B,C, or D has no
of flux conservation. A consequence is that the entrances to A and B can at most appear ‘half black’ to the observer, due to waves which enter A then exit B and vice versa.

This suggests another way to try and defeat the conductance limit: direct the incoming plane waves in a narrow enough angular distribution so that waves always come down A and B in phase, and this will double the conductance. (This is similar to experiments where the series resistance of two QPCs was found to be less than the sum of the individual QPC resistances, because collimation at the exit of the first QPC illuminated the second with a narrow beam, increasing its conductance). However, this beam is no longer a thermal occupation of incoming states. This illustrates the inextricable link between thermal Fermi occupation of reservoir states and the universal quantum of conductance. At $T=0$, thermal occupation at a given chemical potential difference implies that all quantum states lying in the appropriate energy range are filled in the left reservoir and empty in the right. Semiclassically, this corresponds to a uniform distribution in phase space, or when projected into momentum states, uniform in angle, as exemplified by Eq. (11). The semiclassical viewpoint allows one to see that since transformations in phase space cannot change the phase space density (Liouville’s theorem), neither can the universal conductance per quantum channel be changed. This reminds us that unitarity in quantum mechanics is analogous to Liouville’s theorem in classical mechanics.

VI. RECIPROCITY AND ‘CONDUCTANCE’ OF ATOM WAVES

We can ask if the conductance (11) computed using transmission of left-side reservoir plane wave states through the QPC is equal to that using right-side reservoir states. Since the two directions correspond to opposite signs of $\delta V$, then in order to have linear response (well-defined constant $G$ around $\delta V = 0$) we would hope that they are equal. That the angular average of transmission cross section is equal from the left and right sides is not immediately apparent in a general asymmetric system. For instance, consider Fig. 3b which has a small acceptance angle from the left but a large from the right, and very wide on the right. Such a mesoscopic 2DEG system would exhibit symmetric conductance, however, in an atom beam context the conductance can become unsymmetric.

reflection back along the same lead. In this case we might guess that the hypothetical left-side observer (from the previous section) would see the single-mode entrances to guides A and B as two ‘black dots’, giving twice the effective absorption cross section, and therefore infer a conductance of twice $2e^2/h$. We might also justify this by saying that waves travelling down A and B will meet and continue down E, and since they have no particular phase relation, their currents will add to give a doubled current through E, as would be necessary.

However there is a fundamental flaw in the above reasoning. The ABE junction can be designed so that if waves come down A and B in phase, they will be adiabatically transformed into the lowest transverse mode of E, so will propagate through to the right side without reflection, carrying a current of twice that of a usual single-mode guide. However, if A and B are $\pi$ out of phase, the same adiabatic transformation must result in the second transverse mode, which is evanescent. So this latter wave will reflect perfectly back out of the left side, and carry no current. Plane waves are impinging from the left reservoir uniformly over all angles, and because of the $> \lambda$ separation of the entrances, an average over angles gives an average over relative phase in A and B. Thus we are left with no increase above the single channel conductance. This property of the ABE junction is not merely practical; rather, it is easy to show that its $3 \times 3$ S-matrix cannot be unitary if a junction is to couple both A→E and B→E with unity transmissions. Such an appealing junction is therefore ruled out on the grounds

![FIG. 3.](image)

a) An attempt to increase conductance through a single channel by multiple connections feeding from the reservoirs. All channels are single-mode and sufficiently long that the evanescent tunneling of higher modes is negligible. b) An illustrative hard-walled exponential horn system which has differing acceptance angles on each side: very narrow on the left, and very wide on the right. Such a mesoscopic 2DEG system would exhibit symmetric conductance, however, in an atom beam context the conductance can become unsymmetric.
the same phase space area is transmitted right-to-left. When it is realised that the angle-averaged cross section is proportional to the transmitted phase space area on a PS, then the symmetry of the angle-averaged classical cross sections follows.

The same symmetry is not obvious for quantum cross sections, but it also holds true. Comparing (11a) with (3) gives

$$\int_{-\pi/2}^{\pi/2} d\phi \sigma_T^{-R}(k, \phi) = \lambda F \text{Tr}(t^1 t),$$  (17)

where $t$ is measured from left to right states. It is instructive to derive this directly\cite{33}. This relation ties together the cross section and Landauer views of conductance. Time-reversal invariance and flux conservation together imply\cite{33} that $\text{Tr}(t^1 t)$ is unchanged by swapping the labelling of the leads,\cite{33} thus we immediately have from (13) the reciprocity of angle-integrated quantum cross section

$$\int_{-\pi/2}^{\pi/2} d\phi \sigma_T^{-R}(k, \phi) = \int_{-\pi/2}^{\pi/2} d\phi \sigma_T^{-L}(k, \phi).$$  (18)

So in Fig. 3b it is now clear that the ratio of acceptance angles must be balanced by the ratio of effective areas.

We now discuss a case in which non-thermal occupation of incoming states is possible: the rapidly developing field of coherent matter-wave optics, in which potentials are defined by microfabricated structures\cite{33,34}. There is a recent proposal\cite{35} for observation of quantization of atomic flux through a micron-sized 3D QPC defined by the Zeeman effect potential of a magnetic field. The device is illuminated by a beam of atoms passing through a vacuum, whose angular distribution is an experimental parameter (for instance, a collimated oven source or a dropped cloud of cold atoms\cite{33}). The atomic flux transmitted (per unit $k$, at wavevector $k$) will be $F(k) = G_{\text{atom}}(k) J_0(k)$ where $J_0(k)$ is the flux incident per unit wall area, and we define the atomic ‘conductance’ by

$$G_{\text{atom}}(k) \equiv \int d\Omega w(k, \Omega) \sigma_T(k, \Omega).$$  (19)

As before, the quantum transmission cross section is $\sigma_T(k, \Omega)$, but now there is a weighting function $w(k, \Omega)$ which defines the angular distribution of the incident beam\cite{33}. The weight has the normalization $\int d\Omega w(k, \Omega) \cos(\theta) = 1$. [All integrals over solid angle $\Omega \equiv (\theta, \phi)$ are over a range of $2\pi$ appropriate for the half-sphere]. Following the analogy of Thywissen\cite{33}, $F(k)$ plays the role of current, $J_0(k)$ that of bias voltage. However, the name ‘conductance’ does not imply any definite chemical potential difference as in the 2DEG case. For classical particles, the ‘conductance’ of an aperture of area $A_{\text{eff}}$ in a thin wall is simply $G_{\text{atom}}(k) = A_{\text{eff}}$, regardless of the incident angular distribution. Thus $G_{\text{atom}}(k)$ gives the effective area $A_{\text{eff}}$ of a QPC, in an analogous fashion to $a_{\text{eff}}$ in 2D.

For an integer number of quantum channels, the 2D quantization of $a_{\text{eff}}$ in units of $\lambda/2$ becomes in 3D the quantization of $A_{\text{eff}}$ in units of $\lambda^2/\pi$, a result well known from work on 3D metallic point contacts\cite{33}. As stated by Thywissen\cite{33}, this accurate flux quantization requires the incident beam width to be much larger than the QPC acceptance angle.

Eq. (19) is the matter-wave equivalent of Eq. (11a), with the important difference that it has a general weight function. Possible non-uniformity of this weight function leads to a key result: that asymmetry of the conductance is possible given identical illumination on either side, even though the (center of mass) motion is time-reversal invariant. For example, if the incident flux used to illuminate the horn QPC of Fig. 3b is narrow in angular spread, then the left-to-right conductance will be much larger than the right-to-left conductance. This contrasts with the 2DEG case where the conductance is always symmetric.

Finally, it is interesting to note that for the non-thermal incident (reservoir) distributions discussed above, the Landauer formula takes the modified form

$$G \propto \text{Tr}(t^1 \rho)$$  (20)

where $\rho$ is the density matrix of the incident beam.

VII. CONCLUSIONS

Quantum scattering theory in the 2D half-plane can provide an alternative description of the mesoscopic conductance of non-interacting particles. It is especially useful in ‘open’ systems (e.g. those with nearby scatterers in the reservoir regions) where the usual transverse-channel approach is inappropriate. We have considered elastic potentials in zero magnetic field, in linear response in the low temperature limit. Conductance is proportional to the transmission cross section integrated over all incident angles, Eq. (17). We also define a half-plane partial-wave basis applicable with the usual Landauer formula, and relate this to our transmission cross section result. A difference between this and previous work is the ability to treat a direct ‘leadless’ connection to the reservoir.

Using the example of a slit QPC combined with an open cavity structure, we show that an arbitrarily small QPC can carry up to a single quantum of conductance via resonant tunnelling (equal to the limit in the closed-dot resonant tunnelling case). This requires a resonance at the Fermi energy. If $n$ coincident resonances occur for different incoming channels, then $n$ conductance quanta can in theory be achieved through this same tunneling QPC, a result which we believe has not been noted until now.

We emphasize that conductance is proportional to phase-space density of the reservoir states. Therefore
the universal quantum of conductance $e^2/h$ per spin in Fermi gas systems is a direct result of the uniform phase-space density (angular distribution) in a thermal occupation of the Fermi sea. This insight is supported by discussion of attempts to exceed this universal value. When the reservoir occupation differs from thermal, the conductance formula requires generalization: an angle-dependent weight is included in the cross section integral $\sigma_0$; equivalently for 2DEG systems the Landauer formula requires inclusion of the incoming ensemble (20). This result, and our approach in general, is relevant to the emerging field of matter-wave conductance by micro-fabricated structures (for instance, a quantum point contact in 3D), under general illumination by atom waves. We hope this work provides new tools for the study of coherent electron and matter-wave systems.

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1 For a review see C.W.J. Beenakker and H. van Houten, Solid State Physics 44, 1 (1991).
2 T. Dittrich, P. Hänggi, G.-L. Ingold, B. Kramer, G. Schön, W. Zweger, Quantum Transport and Dissipation, (Wiley-VCH, Weinheim, 1998).
3 R. Landauer, IBM J. Res. Dev. 1, 233 (1957); ibid. Z. Phys. B 68, 217 (1987); M. Büttiker, Phys. Rev. Lett. 57, 1761 (1986).
4 S. Datta, Electronic Transport in Mesoscopic Systems, (Cambridge University Press, NY, 1995).
5 B.J. van Wees et al., Phys. Rev. Lett. 60, 848 (1988).
6 By non-adiabatic, we mean that even at a QPC's narrowest region the transverse profile is changing rapidly. Clearly every QPC becomes 'non-adiabatic' at the coupling to infinite-width reservoirs: this type of non-adiabaticity we do not include because it does not cause significant impedance mismatch, as explained by Yacoby and Imry (7).
7 A. Yacoby and Y. Imry, Phys. Rev. B 41, 5341 (1990).
8 J. A. Katine et al., Phys. Rev. Lett. 79, 4806 (1997).
9 M. A. Topinka, et al. Science 289, 2323 (2000).
10 A.K. Geim et al., Phys. Rev. B 49, 2265 (1994).
11 Although the quasi-1D approach can be retained by modelling very wide leads attached to such systems, following A. Szafer and A. D. Stone, Phys. Rev. Lett. 62, 300 (1989), this has both numerical and conceptual limitations.
12 Of course, throughout this paper we could imagine the incident wave on the right-hand side, and the same conductance would result (since we are in linear response); see Section 7.
13 We could equally well imagine that the QPC can be 'closed off' (no transmission) by varying a parameter (this is often true experimentally), and define $\psi_0$ as the full wavefunction in this closed-off state. Thus $\psi_0$ would be the sum of an incident plane wave and a more complicated outgoing wave. This alternative definition may be better in systems where the wall has disorder, or where there is more complicated structure on the left-hand side than shown in Fig. 1.
14 This argument can also be verified in the more specific case of the left region being a rectangular Dirichlet box, in which case the exact eigenfunctions are known and can be written explicitly in terms of a sum of $\psi_0$ for incidences $\phi$ and $-\phi$. However the phase-space presentation is more general, and applies to the real situation where the left region is chaotic (diffusive elastic scattering).
15 K. L. Shepard, M. L. Roukes, B. P. Van der Gaag, Phys. Rev. Lett. 68, 2660 (1992).
16 This argument can also be verified in the more specific case of the left region being a rectangular Dirichlet box, in which case the exact eigenfunctions are known and can be written explicitly in terms of a sum of $\psi_0$ for incidences $\phi$ and $-\phi$. However the phase-space presentation is more general, and applies to the real situation where the left region is chaotic (diffusive elastic scattering).
17 A. H. Barnett, Ph.D. thesis, Harvard University, 2000.
18 G. Arfken, Mathematical Methods for Physicists, 2nd Ed., (Academic Press, 1985).
19 D. S. Fisher and P. A. Lee, Phys. Rev. B 23, 6851 (1981).
20 A. D. Stone and A. Szafer, IBM J. Res. Develop. 32, 317 (1988).
21 J. S. Hersch, M. R. Haggerty, and E. J. Heller, Phys. Rev. Lett. 83, 5342 (1999); Phys. Rev. E. 62, 4873 (2000).
22 J. D. Jackson, Classical Electrodynamics, (Wiley, N.Y., 1975).
23 W. Xue and P. A. Lee, Phys. Rev. B 38, 3913 (1988).
24 V. Kalmeyer and R. B. Laughlin, Phys. Rev. B 35, 9805 (1987).
25 G. W. Bryant, Phys. Rev. B 39, 3145 (1989).
26 D. A Wharam, M. Pepper, H. Ahmed, J. E. F. Frost, D. G. Hasko, D. C. Peacock, D. A. Richie, and G. C. Jones, J. Phys. C 21, L887 (1998); also see the review and references within.
27 M. C. Gutzwiller, Chaos in Classical and Quantum Mechanics, (Springer-Verlag, N.Y., 1990), Ch. 7.
28 With $B\neq 0$, the conductance is still symmetric under swapping the leads. This results from the 2-terminal special case of unitarity sum rules, namely that the rows and columns of the matrix of absolute-value-squared S-matrix elements must all sum to 1. Thus the reciprocity derived here is preserved for $B\neq 0$. How the classical argument from the previous paragraph generalizes for $B\neq 0$ is not known by the authors.
Because we wish to consider general illumination and general \( st(k, \Omega) \), our definition of ‘conductance’ coincides with that of Thywissen\(^\text{30}\) only in the case of isotropic illumination \( w(k, \Omega) = 1/\pi \). The beam brightness per unit \( k \) range, that is, its phase-space density, is assumed uniform in position space, and is proportional to \( J_0(k) w(k, \Omega) \). This is also proportional to \( a(k) \) defined by Thywissen.

Yu. V. Sharvin, J. Exptl. Theoret. Phys. (U.S.S.R.) 48, 984-985, (1965) [trans. in Sov. Phys. JETP 21, 655 (1965)].

J.M. Krans, J. M. van Ruitenbeek, V. V. Flsun, I. K. Yan-
son, and L. J. de Jongh, Nature 375, 767 (1995); P. García-
Mochales, P. A. Serena, N. García, and J. L. Costa-Krämer;
Phys. Rev. B 53, 10268 (1996).