Thermodynamics and Spectroscopy of Charged Dilaton Black holes

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Abstract

The Bohr-Sommerfeld quantization rule is useful to study the area spectrum of black holes by employing adiabatic invariants. This method is extended to charged dilaton black holes in 2+1 dimensions. We put the background space-time into the Kruskal-like coordinate to find the period with respect to Euclidian time. Also assuming that the adiabatic invariant obeys Bohr-Sommerfeld quantization rule, detailed study of area and entropy spectrum has been done. It is dependent on the charge and is equally spaced as well. We also investigate the thermodynamics of the charged dilaton black hole.

1 Introduction

One of the most known solutions of Einstein equation is black hole solution. Black holes are the most surprising objects in general relativity. One of the important characteristics of a black hole is its thermodynamic behaviour. The discovery that the black hole laws are thermodynamic in nature leads to the notion that there should be an underlying statistical description of them in terms of microscopic states. Black hole thermodynamics is widely studied [1, 2]. The parameter ‘entropy’ which connects thermodynamics and statistical mechanics scales like the area of the event horizon, unlike its usual role in thermodynamic systems. This special behaviour of entropy makes the black hole thermodynamics curious. Apart from the area law of entropy, the surface area of event horizon shows a quantized behaviour which in turn quantizes the entropy. The quantization of the black hole horizon area is one of the most fascinating subjects of quantum gravity ever since Bekenstein first proposed in 1974 that the black hole area spectrum is equally spaced [1]. Bekenstein conjectured the possibility of a connection between quasinormal modes of black holes, which have been studied extensively, and the area spectrum. By considering minimum change of the horizon area in the process of assimilation of a test particle falling into a black hole, it was shown that the area spectrum should be linearly quantized, i.e. $A = \gamma m \hbar$, where $\gamma$ is an undetermined dimensionless constant [1] and $m$
is an integer. An important step in this direction was made by Hod\cite{2} in the semi-classical analysis of macroscopic oscillation modes of black holes.

Assuming that the horizon area of a black hole behaves as a classical adiabatic invariant, Bekenstein\cite{11} later showed that the quantized area spectrum has the following form \((\Delta A)_{\text{min}} = 8\pi l_p^2\), where \(l_p = (\frac{G\hbar}{c^3})^{1/2}\) is the Planck length. Subsequently many attempts have been made to derive the area spectrum and entropy spectrum directly utilizing the dynamical modes of this classical theory. Hod found that if one employs the correspondence principle of Bohr, the quantized area spectrum can be determined by the real part of quasinormal frequencies of the black hole. Hod suggested that the area spectrum is \((\Delta A)_{\text{min}} = 4\ln 3l_p^2\).

Based on the Bekenstein proposal for the adiabaticity of the black hole horizon area and the proposal suggested by Hod regarding the quasinormal frequencies, Kunstatter\cite{3} derived the area spectrum of d-dimensional spherically symmetric black holes. The specific result for the horizon area quantum is same as obtained by Hod and by Bekenstein and Mukhanov\cite{4}. The work done by Hod is an important step in this direction. The recent speculation of Maggiore\cite{6} that the periodicity of a black hole may be the origin of the area quantization law is confirmed. Zeng\cite{9} et al exclusively utilize the period of motion of an outgoing wave, which is shown to be related to the vibrational frequency of the perturbed black hole, to quantize the horizon areas of a Schwarzschild and a Kerr black holes. It is shown that the equally spaced area spectrum for both cases takes the same form and the spacing is the same as that obtained through the quasinormal mode frequencies.

In order to study the thermodynamics and spectroscopy of the system, we choose a suitable black hole in 2+1 dimensions\cite{8}. One particular class of interesting backgrounds is the charged dilaton black holes of the Einstein-Maxwell-dilaton theory in which the dilaton field\cite{14,15,16} is exponentially coupled with the gauge field, \(e^{2x\alpha\varphi}F^2\).

In the present work, the calculation employs the proposal of Zeng et al\cite{9} considering an outgoing wave which performs periodic motion outside the horizon. Since the gravity system is periodic with respect to Euclidean time with a period given by the inverse of the Hawking temperature, it is assumed that the frequency of the outgoing wave is given by this temperature. Thus the adiabatic invariant quantity can be found out using the surface gravity term, and using the method of Kunstatter, we can find the area and entropy spectrum. Recently the area spectrum of BTZ black holes is found by the periodicity method in ref.\cite{10} and the spectroscopy of rotating BTZ black holes is derived in ref.\cite{11} using the method introduced by Majhi and Vagenas\cite{12}. We also investigate the thermodynamics of the charged dilaton black hole. Recently Akbar et al\cite{13} studied the thermodynamics of charged rotating BTZ black holes.

The present paper is organized as follows. In Sec. 2, we study the thermodynamics of the charged dilaton black hole in 2+1 dimensions. We calculate the area and entropy spectrum of the system in Sec. 3. And the results and conclusion of the present study are summarized in Sec. 4.
2 Thermodynamics of charged dilaton black hole

In this work we consider a class of black hole solution obtained by Chan and Mann\cite{5}. The solution represents static charged black hole with a dilaton field. The Einstein-Maxwell-dilaton action considered by Chan and Mann\cite{5} is given below:

\[ S = \int d^3x \sqrt{-g} \left[ R - \frac{B}{2} (\nabla \phi)^2 + e^{-4a\phi} F_{\mu\nu} F^{\mu\nu} + 2e^{b\phi} \Lambda \right], \]  

(1)

where \( \Lambda \) is treated as the cosmological constant. The constants \( a, b \) and \( B \) are arbitrary coupling constants. \( \phi \) is the dilaton field, \( R \) is the scalar curvature and \( F_{\mu\nu} \) is the Maxwell field strength. This action is conformally related to the low-energy string action in 2 + 1 dimensions for \( B = 8, b = 4 \) and \( a = 1 \). These black holes have very interesting properties.

Exact solutions to this field equations were found in \cite{5} and the most general such metric has two degrees of freedom, and can be written in the form

\[ ds^2 = -U(r)dt^2 + \frac{d\rho^2}{U(r)} + H^2(r)d\theta^2. \]  

(2)

We now consider the ansatz

\[ H^2 = \gamma^2 r^{N}, \]  

(3)

as a more generic case and further assume that

\[ \phi = k \ln \left( \frac{r}{\beta} \right), \]  

(4)

where \( k \) is a real number. The solutions of Eq. (2) with the above ansatz depend on the dimensionless couplings \( a \) and \( b \) (or alternatively \( N \)). By performing the coordinate transformation \( \gamma^2 r^{-N} \rightarrow r^2 \), Eq. (2) becomes

\[ ds^2 = -\left( Ar^{\frac{2}{3N-1}} + \frac{8\Lambda r^2}{(3N-2)N} + \frac{8Q^2}{N(2-N)} \right) dt^2 + \frac{4r^{\frac{2}{3N-1}}}{N^2 \gamma^2} \left( Ar^{\frac{2}{3N-1}} + \frac{8\Lambda r^2}{(3N-2)N} + \frac{8Q^2}{N(2-N)} \right) dr^2 + r^2 d\theta^2. \]  

(5)
Figure 2: Variation of mass of the black hole with respect to the entropy for different values of charge(Q)

From now on \( r \) denotes the usual radial coordinate. A family of static solutions with rotational symmetry for the action which is mentioned above were thus derived. With \( N = 1 \), \( A = -\frac{2M}{N} = -2M \), Eq. (5) will be

\[
ds^2 = -(-2Mr + 8\Lambda r^2 + 8Q^2)dt^2 + \frac{4r^2 dr^2}{(-2Mr + 8\Lambda r^2 + 8Q^2)} + r^2 d\theta^2. \tag{6}
\]

with \( \phi = \frac{1}{4} ln(\frac{r}{\beta}) \) and \( F_{rt} = \frac{Q}{r} \). For \( M \geq 8Q\sqrt{\Lambda} \), the space time represents a black hole and its horizons are given by

\[
r_+ = \frac{M + \sqrt{M^2 - 64Q^2\Lambda}}{8\Lambda}, \tag{7}
\]

\[
r_- = \frac{M - \sqrt{M^2 - 64Q^2\Lambda}}{8\Lambda}. \tag{8}
\]

We can establish the relation between mass of black hole and horizon radius straightforwardly from the above equation and is given by

\[
M = \frac{4Q^2}{r_+} + 4r_+\Lambda. \tag{9}
\]

Redefining this equation in terms of entropy, with the help of the notion that

\[
S = 4\pi r_+, \tag{10}
\]

we can find,

\[
M = \frac{SA}{\pi} + \frac{16Q^2\pi}{S}. \tag{11}
\]

We can deduce the thermodynamic quantities from the above expression of mass in terms of entropy.

\[
T = \left( \frac{\partial M}{\partial S} \right)_Q, \tag{12}
\]
\[ V = \left( \frac{\partial M}{\partial Q} \right)_S, \]  
\[ C_Q = T \left( \frac{\partial S}{\partial T} \right)_Q. \]  

From Eqs. (11) and (12) we will get the black hole temperature as,

\[ T = \left( \frac{\partial M}{\partial S} \right)_Q = \frac{\Lambda}{\pi} - \frac{16Q^2\pi}{S^2}. \]  

This is the temperature-entropy relation we obtained and we have plotted the variations in Fig.1. We could see the rapid change of entropy in a certain small interval of temperature. And we take the variation for different charges and the charge of a back hole is taken in terms of mass.

Fig.2 represents the variation of the mass of the charged dilaton black hole with entropy as in Eq. (11). We take the variation for different values of black hole charge, an approximate linear relation exists between \( M \) and \( S \) only asymptotically for \( S \gg \frac{Q}{\sqrt{\Lambda}} \).

Now, we look at the potential difference between the horizon and infinity. That will follow Eq. (13),

\[ V = \left( \frac{\partial M}{\partial Q} \right)_S = \frac{32\pi Q}{S}. \]  

Replacing the entropy in terms of \( r_+ \), we obtain

\[ V_+ = \frac{8Q}{r_+}. \]  

Now the heat capacity is calculated as in Eq. (14),

\[ C_Q = T \left( \frac{\partial S}{\partial T} \right)_Q = \frac{2Q\pi^2T}{(\Lambda - \pi T)^2}. \]
The variation of the heat capacity is examined and found that heat capacity diverges as temperature increases. Heat capacity of a system is positive. According to black hole thermodynamics\cite{17}, the more mass and energy a black hole absorbs, the colder it becomes, which implies that Schwarzschild black hole gets hotter as they radiate out energy since it possesses a negative heat capacity. Fig(3) represents the variation of heat capacity with temperature for the charged dilaton black holes. The heat capacity becomes singular at $T = \frac{\Lambda}{\pi}$. We have plotted the graph for different values of $Q$.

3 Spectroscopy of Charged Dilaton Black hole

In this part of the work we discuss the properties of entropy quantization and the area spectrum. We find the area spectrum by evaluating the adiabatic invariant integral which vary very slowly compared to variations of the external perturbations of the system. Given a system with energy $E$, the first law of thermodynamics for charged black hole is\cite{20}:

$$dM = \frac{TH}{4}dA + VdQ.$$ \hfill (19)

We exclusively utilize the period of motion of outgoing wave, which is shown to be related to the vibrational frequency of the perturbed black hole, to quantize the area of the charged dilaton black hole. It is well known that the gravity system in Kruskal coordinate is periodic with respect to the Euclidean time. Particle’s motion in this periodic gravity system also owns a period, which has been shown to be the inverse Hawking temperature\cite{19}. To find the area spectrum via the periodicity method, we use Eq.(6) in KG equation,

$$g^\mu\nu \partial_\mu \partial_\nu \Phi - \frac{m^2}{\hbar^2} \Phi = 0.$$ \hfill (20)

By adopting the wave equation ansatz for the scalar field we can get the solution of wave equation. On the other hand we can also obtain the solution from the Hamilton-Jacobi equation.

$$g^\mu\nu \partial_\mu S \partial_\nu S + m^2 S = 0,$$ \hfill (21)

where $S$ is the action and $S$ and $\Phi$ are related by,

$$\Phi = \exp \left[ \frac{i}{\hbar} S(t, r) \right].$$ \hfill (22)

The action can be decomposed as\cite{9, 18}

$$S(t, r) = -Et + W(r),$$ \hfill (23)

where $E$ is the energy of the emitted particle measured by an observer at infinity and \cite{6} gives the function $W$ near the horizon as

$$W(r) = \frac{i\pi E}{f'(r_H)}.$$ \hfill (24)
where we only consider the outgoing wave near the horizon. In this case, it is obvious that the wave function \( \Phi \) outside the horizon can be expressed as the form,

\[
\Phi = \exp \left[ -\frac{i}{\hbar} E t \right] \psi(r_H),
\]

where

\[
\psi(r_H) = \exp \left[ -\frac{\pi E}{\hbar f'(r_H)} \right],
\]

and from the above equation it is clear that \( \Phi \) is a periodic function with period

\[
\tau = \frac{2\pi \hbar}{E}.
\]

Using the fact that the gravity system in Kruskal coordinate is periodic with respect to the Euclidian time, particle’s motion in this periodic gravity system also owns a period, which has been shown to be the inverse Hawking temperature. Thus the relation between \( \tau \) & \( T \) can be written as

\[
\tau = \frac{2\pi}{\kappa_r} = \frac{\hbar}{T_H},
\]

hence

\[
T_H = \frac{\hbar \kappa_r}{2\pi},
\]

where \( \kappa_r \) is the surface gravity of the black hole. The expression of surface area (here it will be the circumference since is is a 2+1 black hole) of the event horizon is

\[
A = 2\pi r_+.
\]

from the above equation the change in the horizon area of a charged dilaton black hole can be written as

\[
\Delta A = 2\pi dr_+.
\]

Now, Eq. (9) gives the differential

\[
dM = \left[ 4\Lambda - \frac{4Q^2}{r_+^2} \right] dr_+ + \frac{8Q}{r_+} dQ.
\]

Using the lapse function we can calculate the surface gravity as follows

\[
\kappa_{r_+} = \frac{1}{2} \frac{df(r)}{dr} |_{r=r_+} = \left[ 2\Lambda - \frac{2Q^2}{r_+^2} \right],
\]

where \( f(r) = -M + \frac{4Q^2}{r_+^2} + 4Ar_+ \). Using Eq. (33) in Eq. (32)

\[
dM = 2\kappa_{r_+} dr_+ + \frac{8Q}{r_+} dQ.
\]

and substituting Eq. (17)

\[
dr_+ = \frac{dM - V_c dQ}{2\kappa_{r_+}}.
\]
Now we can use Eq. (31)

\[ \Delta A = \pi \left[ \frac{dM - V_+ dQ}{\kappa_{r_+}} \right]. \]  

(36)

Now, the adiabatic invariant integral for the charged 2+1 black hole can be written as

\[ I = \pi \int \frac{dM - V_+ dQ}{\kappa_{r_+}}. \]  

(37)

Now we can write the surface gravity, \( \kappa_{r_+} \), in terms of \( M, Q \) and \( r_+ \) to carry out the integration, and is given by

\[ \kappa_{r_+} = \frac{\sqrt{M^2 - 64Q^2 \Lambda}}{2r_+}. \]  

(38)

The adiabatic invariant integral will read as

\[ I = \int \frac{dM - V_+ dQ}{\sqrt{M^2 - 64Q^2 \Lambda}}. \]  

(39)

The integral can be solved and using the value of the potential given in Eq. (16)

\[ I = \frac{1}{4\Lambda} \int \frac{M}{\sqrt{M^2 - 64Q^2 \Lambda}} dM + \frac{1}{4\Lambda} \int dM - 16 \int \frac{Q}{\sqrt{M^2 - 64Q^2 \Lambda}} dQ. \]  

(40)

Integrating, we get

\[ I = \frac{M + 2\sqrt{M^2 - 64Q^2}}{4\Lambda}. \]  

(41)

Replacing \( M \) in terms of \( r_+ \) using Eq. (9) and applying Bohr-Sommerfeld quantization condition we find

\[ I = 3r_+ - \frac{Q^2}{\Lambda r_+} \simeq n\hbar. \]  

(42)

Finding the circumference term, \( 2\pi r_+ \), (since this is a 2+1 black hole), in the above expression we can write the circumference spectrum as

\[ A_n \simeq \frac{2\pi n\hbar}{3} - \frac{2\pi Q^2}{3\Lambda r_+}. \]  

(43)

Thus we see that the circumference of the charged dilaton black hole is discrete and the spacing is equidistant. In this system, the circumference spectrum depends on the black hole parameters, but the ‘circumference spacing’ is equal and is independent of the black hole parameters. Recalling that the black hole entropy is proportional to the black hole horizon area (here it is the circumference), it is clear that the entropy is also quantized.
4 SUMMARY AND CONCLUSION

We have investigated the thermodynamic and spectroscopic aspects of charged dilaton black hole in 2+1 dimensions. We have obtained the variation of entropy with temperature, mass with entropy and heat capacity with temperature. The heat capacity is found to be positive. We have found the circumference spectrum. It is found that the though the circumference spectrum is black hole parameter dependent, the circumference spacing is independent of the black hole parameters.

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