Classification of symmetry-protected topological phases in two-dimensional many-body-localized systems

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(Dated: August 13, 2019)

We use low-depth quantum circuits, a specific type of tensor networks, to classify two-dimensional symmetry-protected topological many-body localized (MBL) phases. The existence of strict MBL in strongly disordered two-dimensional systems is currently disputed, as very slow relaxation processes might eventually lead to thermalization. Our derivation shows that on the time scales within which two-dimensional MBL is observed, there are distinct symmetry-protected topological phases, which are robust to symmetry-preserving perturbations and cannot be connected without violating (short-time) MBL. For (anti-)unitary on-site symmetries we show that the topological phases can be labelled by the elements of the (generalized) third cohomology group of the symmetry group. Our approach might be adapted to supply arguments suggesting the same classification for two-dimensional symmetry-protected topological ground states with a rigorous proof.

I. INTRODUCTION

Many-body localization (MBL) occurs in isolated strongly disordered systems and is characterized by a lack of thermalization. This phenomenon was first conjectured by Anderson in 1958 as an interacting analogue of Anderson localization. Theoretical support was lacking until the last decade, when perturbation theory analyses various numerical studies and a rigorous proof put the phenomenon in one-dimensional lattice systems on a rigorous footing. In recent years, MBL was also observed in experiments of one-dimensional ultracold atomic gases and chains of trapped ions, superconducting qubits and NV-centers. Approaches to realizing MBL in solid state systems are currently being pursued.

In higher dimensions, theoretical arguments suggest the absence of MBL, i.e., arbitrarily strongly disordered systems should eventually thermalize. However, relaxation times might be astronomically long (e.g., exponential in the system size), which reconciles those arguments with recent very ultracold gas experiments, where two-dimensional MBL is observed. The notion of MBL-like behavior on short time scales has since been supported by numerical studies.

MBL systems are potentially technologically relevant for the storage and manipulation of quantum information. In one dimension, MBL systems with on-site symmetries are able to topologically protect qubits from decoherence caused by local noise at finite energy density. Two-dimensional MBL-like systems may display the same robustness on experimental time scales and furthermore be used to manipulate the stored quantum information.

One-dimensional MBL systems with (anti-)unitary on-site symmetry can be classified into different symmetry-protected topological (SPT) MBL phases. The different topological classes can be labeled by the elements of the (generalized) second cohomology group of the symmetry group. Note that the symmetry group must be abelian to be compatible with a stable MBL phase. In two dimensions, the expectation is thus that SPT MBL phases are classified by the elements of the third cohomology group, similarly to SPT ground states in two dimensions.

In this work, we use quantum circuits to carry out such a classification in two dimensions. Quantum circuits are a specific type of tensor networks and approximate the unitary diagonalizing the MBL Hamiltonian efficiently in one dimension, as indicated by numerical evidence and analytical considerations. Specifically, the error of the approximation decreases like an inverse polynomial of the computational time (and number of parameters of the approximation). The underlying reason is that all eigenstates of MBL systems fulfill the area law of entanglement and can thus be efficiently approximated by tensor network states (TNS). Under the above assumption on the error bound, it is possible to show rigorously that SPT MBL phases are robust to arbitrary symmetry-preserving perturbations and that topologically distinct phases cannot be connected without delocalizing the system. Furthermore, it follows that all eigenstates of SPT MBL systems have the same topological label as defined for ground states. Here, we use two-dimensional quantum circuits with four layers of unitaries acting on finite fractions of the overall system. While we cannot make similar assumptions on the error bound, as there might not be strict MBL in two dimensions, our results will hold on the time scales within which two-dimensional disordered systems behave MBL-like. Hence, we will show that within those time scales, two-dimensional SPT MBL phases are robust to symmetry-preserving perturbations and cannot be connected without destroying short-time MBL-like behavior. Again, it follows that all eigenstates must have the same topological label. We anticipate that our two-dimensional quantum circuit approach might be adapted to carry out a rigorous classification of two-dimensional SPT ground states, which is currently an outstanding problem. Note that our classification does not apply
to topologically ordered MBL systems as these cannot be represented by short-depth quantum circuits.

This article is structured as follows: In Section II we give a more formal introduction to the theoretical description of MBL systems in one and two dimensions, their SPT phases and tensor networks. Section III contains a non-technical summary of our results with the technical part provided in Sections IV (unitary on-site symmetries) and V (anti-unitary on-site symmetries). Section VI discusses the robustness of the obtained topological phases to symmetry-preserving perturbations and demonstrates that the only way of connecting topologically distinct MBL phases is by either breaking the symmetry or making the perturbation strong enough such as to destroy (short-time) MBL. In Section VII we summarize our results and present directions for future work. In the Appendix, we provide technical details on the interpretation of the elements of the second and third cohomology group in terms of projective and gerbal representations, respectively.

II. SYMMETRY-PROTECTED TOPOLOGICAL MANY-BODY LOCALIZED PHASES AND TENSOR NETWORKS

Here we briefly review the central ideas about many-body localization and symmetry-protected topological phases and introduce tensor network language. Readers already familiar with these topics may easily skip this Section. For a similar but slightly more complete review of SPT and MBL, see Section II of Ref. [38].

A. Many-body localization

MBL is the phenomenon of a quantum many-body system failing to thermalize and retaining some memory of its initial condition after arbitrary long times. While MBL is known to exist in one dimension, its existence in higher dimensions is under debate. We will briefly review MBL in one dimension before commenting on the two-dimensional case.

The canonical model of strongly disordered Hamiltonians that exhibits MBL is the random field Heisenberg model

\[ H = \sum_{i=1}^{N} (J S_i \cdot S_{i+1} + h_i S_i^z), \]

where \( h_i \) is sampled from a uniform distribution \([-W, W] \). \( \mathbf{1} \) displays a transition from the ergodic phase to the MBL phase as a function of the disorder strength controlled by \( W \). Close to the phase transition \((W \approx 3.5 \) for \( \mathbf{1} \)), \( \mathbf{1} \) exhibits a mobility edge: eigenstates in an energy window in the middle of the spectrum are volume law entangled, while eigenstates outside of this window are area law entangled. For SPT phases we are interested in the fully many-body localized (FMBL) phase \((W \geq 3.5 \) for \( \mathbf{1} \)), where all eigenstates are area law entangled. The FMBL phase is described by a complete set of local integrals of motion (LIOMs), \( \tau_i^\pm \). LIOMs form a set of effective spin degrees of freedom related to the physical spins by a local unitary transformation. Each LIOM is a quasi-local operator with a non-trivial support that concentrates on one site, and exponentially decays from that site. The corresponding decay length \( \xi_i \) is known as the localization length. All LIOMs commute with the Hamiltonian and with each other, and therefore form an emergent notion of integrability,

\[ [H, \tau_i^\pm] = [\tau_i^\pm, \tau_j^\pm] = 0 \]

for all \( i,j = 1,2, \ldots, N \). Hence, all eigenstates \(|\psi_{i_1i_2 \ldots i_N}\rangle \) of the Hamiltonian can be uniquely labeled by the expectation values (say \( i_1 = \pm 1 \), also known as l-bits) of the corresponding \( \tau_i^\pm \) operators. (Here we consider the case of spin-1/2 Hamiltonians, though the notion of LIOMs can be straightforwardly generalized to higher spin systems.)

Though it has not been rigorously shown that MBL phases exist in two dimensions, numerical and experimental evidence suggests that MBL-like behaviors are displayed at least on experimentally relevant time scales. For instance, numerical simulations on the disordered Bose-Hubbard model in two dimensions indicate MBL-like features on certain time scales. Within those no thermalization occurs and the description by LIOMs, Eq. (??), is thus appropriate. The results we derive in this paper on SPT phases will therefore hold on the time scales on which MBL-like behaviors are displayed.

B. Symmetry-protected topological phases

Quantum phases typically have to do with the ground states of gapped systems. A topological phase consists of the set of gapped local Hamiltonians that can be continuously deformed into each other without closing the energy gap, or equivalently, whose ground states can be evolved into each other with short-ranged quantum circuits with depth constant in the system size. A symmetry-protected topological phase is defined in the same way with the added constraint that all Hamiltonians along the connecting path must be invariant under the symmetry.

For MBL systems, we are interested in all eigenstates rather than only ground states, since the properties of the eigenstates constrain the dynamics of the system. We say that two FMBL Hamiltonians \( H_0 \) and \( H_1 \) are in the same MBL SPT phase if there exists a path \( H(\lambda) \) such that \( H_0 = H(0) \) and \( H_1 = H(1) \) and for all \( \lambda \in [0,1] \), \( H(\lambda) \) preserves the symmetry and is FMBL (or, in the \( D > 1 \) case, exhibits the MBL-like behavior on relevant time scales).

In the case of on-site symmetries, it was conjectured that the ground state SPT phases of \( D \)-dimensional spin
systems are labeled by the \((D + 1)\)th cohomology group of the symmetry group. In 2D it was shown that the SPT phases are indeed labeled by the elements of the second cohomology group in the ground state and MBL cases. In this paper, we will classify two-dimensional MBL SPT phases using the fact that the short-time dynamics of these two-dimensional systems can be efficiently described by low-depth quantum circuits.

C. Tensor networks

Tensor networks and the associated diagrammatic formulation are powerful tools for both analytical and numerical studies of quantum many-body physics. A tensor is an \(n\)-dimensional array of (complex) numbers, and is diagrammatically represented by a geometric shape with indices represented by outgoing legs. For example,

\[
A_{ijk} = \begin{array}{c}
A \\
ijk
\end{array}
\]

A contraction between different indices of (a single or multiple) tensor(s) is represented by connecting two corresponding legs, e.g.

\[
\sum_{ij} A_{ijk} B_{jkl} = \begin{array}{c}
A \\
ij
\end{array} \begin{array}{c}
B \\
k
\end{array} \begin{array}{c}
l
m
\end{array}
\]

Tensors can be blocked or grouped together to form a single tensor. The legs of a given tensor can be combined or split through reshaping. These operations are illustrated as follows,

\[
\begin{array}{c}
A \\
ijk
\end{array} = \begin{array}{c}
C \\
ij
\end{array} \begin{array}{c}
C
\end{array}
\]

The tensor product of two tensors is represented by placing two tensors together, e.g.

\[
\begin{array}{c}
A \\
ijk
\end{array} \begin{array}{c}
B \\
k
\end{array} = \begin{array}{c}
A \otimes B
\end{array}
\]

The trace operation is a contraction of two legs of the same tensor, e.g.

\[
\text{Tr}(A) = \begin{array}{c}
A
\end{array}
\]

A commonly cited problem in quantum many-body physics is the exponential increase of the dimension of the Hilbert space with the system size. However, many physically interesting states, such as the ground states of gapped systems, have area-law entanglement and lie in a small region of the Hilbert space, which only scales polynomially with the system size, and hence are expressible in terms of tensor networks.

A classic example is the matrix product state (MPS) in one dimension. The state of an \(N\)-site spin chain,

\[
|\psi\rangle = \sum_{i_1 \cdots i_N} \psi_{i_1 i_2 i_3 \cdots i_N} |i_1 i_2 i_3 \cdots i_N\rangle
\]

can be written in the form of an MPS,

\[
|\psi\rangle = \sum_{i_1 \cdots i_N} \text{Tr}(A_{i_1}^{(1)} A_{i_2}^{(2)} A_{i_3}^{(3)} \cdots A_{i_N}^{(N)}) |i_1 i_2 i_3 \cdots i_N\rangle
\]

if we decompose \(\psi_{i_1 \cdots i_N}\) as

\[
\begin{array}{c}
\psi
\end{array} = \begin{array}{c}
A^{(1)}
\end{array} \begin{array}{c}
A^{(2)}
\end{array} \begin{array}{c}
A^{(3)}
\end{array} \cdots \begin{array}{c}
A^{(N)}
\end{array}
\]

Such a decomposition can always be found using, say, a singular value decomposition (SVD). This procedure is not always useful since the maximum dimension of the legs of \(A^{(n)}\), or the “bond dimension”, can be exponentially large. However, for area-law entangled states, there exist accurate MPS representations with small bond dimensions. Furthermore, in a few cases such as the AKLT model, exact MPS representations can be found with fixed bond dimensions. Another example of tensor network states is projected entangled pair states (PEPS). PEPS are \(D\)-dimensional versions of MPS, with each site represented by a tensor with one “physical” leg and 2\(D\) bond legs on a square lattice. In this paper we will work mostly with unitary quantum circuits (or simply “quantum circuits”), which is a sequence of unitary quantum gates and can be diagrammatically represented in the tensor network notation.

III. NON-TECHNICAL SUMMARY OF RESULTS

A. Main results

Here we give an overview of the main ideas and results. We assume that the Hamiltonian describes a two-dimensional strongly disordered spin system, such that
it behaves FMBL on relevant time scales. We work with a spin system on an $N \times N$ square lattice with periodic boundary conditions. Furthermore, we assume that the system is invariant under an on-site symmetry $v_g$ with abelian symmetry group $G \ni g$, that is
\begin{equation}
H = v_g^\otimes N^2 H (v_g^\dagger)^\otimes N^2.
\end{equation}
$v_g$ forms a representation of the group, i.e., $v_g v_h = v_{gh}$. The symmetry has to be abelian, as non-abelian symmetries are incompatible with MBL\cite{38} (even on short time scales). Abelian symmetries do not protect degeneracies. We can thus assume that all exact degeneracies have been lifted by a small perturbation, which implies that the unitary $U$ fulfills (for details, see Sec. IV B)
\begin{equation}
v_g^\otimes N^2 U = U \Theta_g,
\end{equation}
where $\Theta_g$ is a diagonal matrix where each diagonal element is a complex number of magnitude 1.

We assume that $U$ may be efficiently approximated by a four-layer quantum circuit of the form of Fig. 1, where each unitary acts on plaquettes of $\ell \times \ell$ sites. We take $\ell \propto N$.

As strongly disordered two-dimensional systems might not be fully localized, it is not known whether $U$ can be efficiently approximated by a short-depth quantum circuit, which has strictly local correlations. However, we assume that any long-range entanglement present in the eigenstates contained in $U$ is so small that it affects the dynamics of the system only on extremely long time scales\cite{39}. Hence, we represent $U$ as a short-depth quantum circuit, whose topological classification will yield a set of two-dimensional SPT MBL phases which exist on experimental time scales\cite{37}. The quantum circuit to be used consists of four layers of unitaries acting on $\ell \times \ell$ sites as shown in Fig. 1. It is the natural generalization of the two-layer quantum circuit with long gates used in one dimension to represent MBL systems\cite{37,38,39}. It consists of parallel one-dimensional two-layer quantum circuits, which are themselves coupled with each other in a two-layer quantum circuit structure,
\begin{equation}
U = \begin{array}{ccccc}
U_1 & V_1 & V_2 & V_3 & \cdots \\
U_4 & V_4 & V_5 & V_6 & \cdots \\
& U_7 & U_8 & U_9 & \cdots \\
& & U_{10} & U_{11} & \cdots \\
\end{array}
\end{equation}

Here, we blocked together sites as in Fig. 1 i.e., each tensor leg corresponds to $\ell \times \ell$ sites. The unitaries of $U_k$ are located in the first two layers of Fig. 1 (i.e., Fig. 1a,b), the unitaries of $V_k$ in the second two layers (Fig. 1c,d).

Specifically, we will take $\ell = cN$ with $c < 1/4$ a fixed fraction. This allows to represent MBL-like systems with no LIOM having a localization length $\xi$ of order $O(N)$ in the thermodynamic limit\cite{38,39}. While for two-dimensional strongly disordered systems this may never be strictly the case, by our assumption, for the description of relevant time scales we can neglect the contribution of tails of the integrals of motion which extend over the whole system (making them actually non-local). Hence, $\ell = cN$ is sufficient to represent the relevant short-time MBL behavior. Note that on the other hand, the corresponding quantum circuit is still sufficiently local as to now allow for entanglement over the whole system (which would enable us to connect all unitaries continuously leading to only one topological phase), since sites, which are further than $2\ell$ sites apart have no entanglement. For the derivation below we also assume that one-dimensional unitaries with strict short-range entanglement can be efficiently approximated by one-dimensional two-layer quantum circuits with long gates, which corresponds to the assumption that one-dimensional MBL systems can be efficiently approximated by such unitaries.

It follows from Eqs. (12) and (13) that $\Theta_g$ can likewise be written as a four-layer quantum circuit (see Sec. IV B for details), thus making Eq. (12) an equality of two short-depth quantum circuits.

Next, we perform manipulations with the quantum circuits. We collapse the quantum circuits of Eq. (12) along the $y$-direction, so that (12) becomes an equality of two one-dimensional quantum circuits, which are stretched out along the $x$-direction. The obtained equation (Eq. (10) below) is identical to an equation that arises in the study of SPT phases in one-dimensional MBL systems\cite{38}. This allows us to employ results from Ref. 38. Let $g$ be an element of the symmetry group $G$, and suppose $j$ labels the position along $x$-direction. Then, there exist operators $W^g_j$ acting on the boundary of a patch of the quantum circuit (13), similarly to the situation for tensor network states as shown in Fig. 2.

The $W^g_j$ are projective representations of $G$, i.e.,
\begin{equation}
W^g_j = \beta(g,h) W^{gh}_j
\end{equation}
with $|\beta(g,h)| = 1$. In our two-dimensional case, each $W^g_j$ is a tensor that extends along the $y$-direction and can be
FIG. 1. An illustration of the 4-layer quantum circuit representing the unitary $U$. The $xy$-plane is parallel to the plane where the sites of the system are located. The unitaries are stacked from bottom to top in the order (a), (b), (c), (d) (parallel to the $z$-axis). (a) represents the top view of the first layer, (b) of the second layer, and so on. A dot represents a group of $\ell^2 \times \ell^2$ sites. A red box represents a unitary gate. The quantum circuit periodically extends beyond the region defined by the dashed lines.

The $j$ subscript and indices corresponding to the position along the $y$-direction have been suppressed on the right hand side.

In section IV D we prove the following lemma: two-layer quantum circuit projective representations of a group $G$ have a topological index given by an element of the third cohomology group $H^3(G,U(1))$ of $G$. Together with the existence of the $W_j(g)$ acting on the boundary, the lemma implies that two-dimensional SPT MBL phases are labeled by the elements of the third cohomology group.

The $v(g)$ operator in Fig. 2 cannot be a group representation in the usual sense, since given two symmetry operations $v(g)$ and $v(h)$, their composition would correspond to an MPO with a larger bond dimension, whose tensors thus are different from those of $v(gh)$. Rather, we need a “combining” operator $X_L,R(g,h)$

$\chi$ is an arbitrary complex number. However, we have to exclude $\chi = 0$ (and $\chi = \infty$) such that $X_L(g,h)$ and $X_R(g,h)$ remain well-defined. This is topologically equivalent to constraining to $|\chi| = 1$, i.e., no rescaling of $X_L(g,h)$ and $X_R(g,h)$ is allowed. For the quantum circuit case we focus on in this paper, $\chi(g,h)$ will appear as a result of a gauge degree of freedom in the quantum circuit unitaries.

B. Intuitive overview of the proof of the lemma

Here we give an intuitive overview of the ideas behind the proof of the lemma in Section IV D. Following Refs. 62 and 63, we review the “pentagon equation” (20), which applies to the tensor network symmetry operator that acts on an edge of a two-dimensional symmetric tensor network state as depicted in Fig. 2, such as matrix product operators (MPOs) acting on a PEPS.

In the technical part, we demonstrate that $W_j(g)$ satisfies the pentagon equation and consequently two-dimensional MBL SPT phases can be labeled by the elements of the third cohomology group.

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For three group elements we then have

\[
\begin{align*}
  v(ghk) &= \begin{array}{ccc}
  X_L(g, h) & v(g) & X_R(g, h) \\
  v(h) & & X_R(gh, k) \\
  v(k) & & X_R(h, k)
  \end{array} \\
  &= \begin{array}{ccc}
  X_L(gh, k) & v(g) & X_R(gh, k) \\
  v(h) & & X_R(h, k)
  \end{array} \\
  &= \begin{array}{ccc}
  X_L(gk, h) & v(g) & X_R(gk, h) \\
  v(k) & & X_R(h, k)
  \end{array} .
\end{align*}
\]

(17)

If one considers operating on the left edge and right edge separately, one may deduce (if \(v(g)\) is injective) that

\[
\begin{align*}
  X_L(g, h) &= \alpha(g, h, k) \\
  X_R(gh, k) &= X_R(h, k)
\end{align*}
\]

(18)

as well as a similar equation for the \(X_L(g, h)\) with a factor \(1/\alpha(g, h, k)\). Since no rescaling of \(X_L(g, h)\) and \(X_R(g, h)\) is allowed, \(|\alpha(g, h, k)| = 1\).

In the quantum circuit case, a similar equation as Eq. (18) holds, because the quantum circuits are only short-range correlated and hence the left and right parts can be separated.

We note that the \(X_R(g, h)\) (or equivalently the \(X_L(g, h)\)) in Eq. (15) in some sense form a “representation” of the group \(G \ni g, h\), but with not one but two group elements associated to each operator. This kind of representation is sometimes called a gerbal representation and has been studied in the mathematics literature.

We can use the gauge degree of freedom of \(X_R(g, h)\) to show that \(\alpha(g, h, k)\) is only defined up to a 3-coboundary

\[
\alpha(g, h, k) \rightarrow \alpha'(g, h, k) = \alpha(g, h, k) \chi(g, hk)\chi(h, k) / \chi(g, h)\chi(h, k) .
\]

(19)

Using Eq. (18), we can perform the following sequence of manipulations on the combination of \(v(g), v(h), v(k)\) and \(v(l)\) leading to the same result in two different ways (cf. pentagon equation in topological quantum field the-

This implies that the incurred phases \(\alpha(g, h, k)\) must fulfill the following consistency relation

\[
\frac{\alpha(g, h, k)\alpha(g, hk, l)\alpha(h, k, l)}{\alpha(gh, k, l)\alpha(g, h, kl)} = 1,
\]

(21)

which is known as a 3-cocycle. Recall that the cohomology group \(H^n(G, U(1))\) consists of the equivalence classes of \(n\)-cocycles that differ by only an \(n\)-coboundary (Eq. (19) in our case). So, we have essentially shown that a projective representation in the form of a one-dimensional tensor network state acting on the edge of a two-dimensional tensor network corresponds to an element of the third cohomology group of the symmetry group.

For the case where \(v(g)\) is an injective MPO, the above calculation is the complete argument. In the context of two-dimensional SPT MBL, \(v(g)\) is a quantum circuit, and it is not obvious how to define a combining operation in terms of \(X_L, R(g, h)\) tensors such as those in Eq. (16). In Sec. IV D we construct a suitable combining operation and show that it satisfies the corresponding pentagon equation and hence the 3-cocycle condition. Thus, SPT MBL phases in two dimensions are also labeled by an element of the third cohomology group. We explicitly demonstrate below that all eigenstates of the MBL system must correspond to the same element...
of the third cohomology group, just as in one dimension. Intuitively, this can be seen from the fact that symmetry-preserving, local $l$-bit flips transform between eigenstates. However, symmetry-preserving, local operations cannot change topological labels. Hence, all eigenstates must have the same topological label.

IV. CLASSIFICATION OF TWO-DIMENSIONAL SPT MBL PHASES WITH QUANTUM CIRCUITS

We will show that two-dimensional MBL SPT phases are labeled by the group elements of the cohomology classes of the symmetry group $G$. Due to a mathematical result (proven in Sec. IV D), this reduces to the problem of finding a projective representation of $G$ in terms of quantum circuits. This follows from projecting the two-dimensional problem into one dimension and then applying the results of the calculations for the classification of SPT phases in one-dimensional MBL systems, as done in Ref. 38.

Consider a two-dimensional spin system on a square lattice. We shall work with a strongly disordered Hamiltonian invariant under an on-site abelian symmetry. As elaborated on above, one can capture its short-time dynamics by approximating the unitary which diagonalizes the Hamiltonian with a four-layer quantum circuit with gates acting on plaquettes of $\ell \times \ell$ sites, cf. Fig. 1. As detailed above, we can choose $\ell \propto N$ to carry out our classification.

A. Quantum circuits

Many of the calculations of this paper will involve manipulation of unitary quantum circuit tensor networks. In this section we will derive a few basic results to be used later. First let us recall a simple but important result that follows when we have two equivalent two-layer quantum circuits, as in

\begin{align}
V_n & \quad V_1 \quad V_2 \quad \cdots \quad V_n \\
U_1 & \quad U_2 \quad \cdots \quad U_n \\
= & \\
V_n' & \quad V_1' \quad V_2' \quad \cdots \quad V_n'
U_1' & \quad U_2' \quad \cdots \quad U_n'.
\end{align}

Then there exist unitaries $W_1, \ldots, W_{2n}$ such that

\begin{align}
U_k & = W_{2k-1} \quad W_{2k} \\
U_k' & = W_{2k} \quad W_{2k+1}.
\end{align}

(23)

(24)

Combining the above two equations, we can derive the following useful relation between the primed and unprimed $u$ and $v$ tensors

\begin{align}
V_k' & = V_{2k-1} \quad V_k \\
U_k & = U_{2k-1} \quad U_k \\
& \quad W_{2k+1}.
\end{align}

(25)

The tensors $W_1, \ldots, W_{2n}$ are not fixed but have an overall “gauge” degree of freedom. We may make the change of variables $W_{2k+1} \rightarrow W_{2k+1} e^{i\chi}$ and $W_{2k} \rightarrow W_{2k} e^{-i\chi}$. The parameter $\chi$ is the global gauge degree of freedom.

In this paper, we will suppress the indices of the tensors (e.g., of $U_k$ and $V_k$), when there are no ambiguities, but we emphasize that the quantum circuits are typically not translational invariant. As an example, the left side of Eq. (22) would be written with all the upper layer tensors labeled $V$ and all the lower layer tensors labeled $U$.

B. 2D MBL systems with an on-site symmetry

We assume the strongly disordered “MBL-like” Hamiltonian $H$ to be invariant under a local unitary symmetry operator $v_g$, for $g \in G$. That is, $H$ commutes with the symmetry operator

\begin{align}
H & = v_g^{\otimes N^2} H (v_g^\dagger)^{\otimes N^2}.
\end{align}

(26)

Let $U$ be the unitary matrix that diagonalizes the Hamiltonian, and $E$ the diagonal matrix of energies, i.e. $H = U E U^\dagger$. By the same line of reasoning as in Ref. 37 one can derive the action of the symmetry on $U$. Eq. (26) implies that

\begin{align}
E & = U^\dagger v_g^{\otimes N^2} U E U^\dagger (v_g^\dagger)^{\otimes N^2} U.
\end{align}

(27)
As the symmetry group is abelian, \( E \) cannot have any symmetry-enforced degeneracies. Assuming \( E \) to be non-degenerate, Eq. (27) implies

\[
\Theta_g = U^\dagger v_g^{\otimes N^2} U,
\]

with \( \Theta_g \) being a diagonal matrix whose diagonal elements have magnitude 1. Accidental degeneracies can be removed and are treated explicitly in Section VI. For the assumed abelian symmetry, there are no symmetry-enforced degeneracies.

Note that the eigenstates \(|\psi_{l_1\ldots l_{N^2}}\rangle\) can be obtained by fixing the lower indices of the unitary \( U \) to the corresponding l-bit labels \( l_1, l_2, \ldots, l_{N^2} = \pm 1 \),

\[
|\psi_{l_1,l_2\ldots l_{N^2}}\rangle = U_{l_1,l_2\ldots l_{N^2}}. \tag{29}
\]

1. Quantum circuit representation of the \( \Theta_g \) matrix

Next we will show that the tensor \( \Theta_g \) can be written as a four-layer quantum circuit as in Fig. 1 (recall that \textit{a priori} only \( U \) is assumed to have that property). The derivation is the two-dimensional version of the one-dimensional case in Ref. [37].

Let us set up a coordinate system where \( k \in \mathbb{Z}^2 \) labels a block of \( \ell \times \ell \) sites, or equivalently, a \( \ell \)-tensor in the lowest layer of \( U \) (red squares in Fig. 1a). Let \( l_k \) denote the \( l \)-bit indices associated with the legs at \( k \). Making the definition \( Z_{g,k} = V_k^\dagger (v_g^{\otimes \ell^2}) V_k \), we write the diagonal elements of \( \Theta_g \) as (note that we use the convention that multiplication order left to right in algebraic notation corresponds to top to bottom in diagrammatic notation)

\[
\theta_g^* (l_k, \{ l_r \mid \forall r \neq k \}) \theta_g (l'_k, \{ l_r \mid \forall r \neq k \}) = \theta_g (l_k, \{ l_r \mid \forall r \neq k \}) \theta_g^* (l'_k, \{ l_r \mid \forall r \neq k \}). \tag{31}
\]

Note that (30) is the projected view onto the \( xz \)-plane of a two-dimensional seven-layer quantum circuit where the locations of the unitaries in the individual layers are as illustrated in panels (a,b,c,d c,b,a) of Fig. 1 respectively. (The uppermost layer Fig. 1d) can be combined with \( v_g^{\otimes N^2} \) and its adjoint.

Consider for some \( k \), the product of numbers \( \theta_g^* (l_k, \{ l_r \mid \forall r \neq k \}) \theta_g (l'_k, \{ l_r \mid \forall r \neq k \}) \), which can be written diagrammatically (with the same convention as in Eq. (30) and with implicit subscripts) as

\[
\theta_g^* (l_k, \{ l_r \mid \forall r \neq k \}) \theta_g (l'_k, \{ l_r \mid \forall r \neq k \}) =
\]

\[
\begin{array}{c}
\theta_g (l_k, \{ l_r \mid \forall r \neq k \}) \\
\theta_g^* (l'_k, \{ l_r \mid \forall r \neq k \})
\end{array}
\]

FIG. 3. Layers of the lower half of the causal cone ordered from top to bottom as denoted by arrows. The unitaries of the respective upper layer are indicated by red dashed lines. Each dot corresponds to \( \frac{1}{4} \times \frac{1}{4} \) sites.
where we have used the fact that Eq. (30) is diagonal, and where the operator $|k\rangle⟨k'|$ acts non-trivially only on the block of sites labeled by $k$. All the unitaries outside the causal cone (blue dashed line) cancel. The causal cone also has a finite extension along $y$-direction and its lower half is shown in detail in Fig. 3. Consequently, the product becomes a phase that depends only on the degrees of freedom that lie within the causal cone in Eq. (31),

$$
\begin{align*}
\theta_g^* & \begin{pmatrix}
\cdots & l_k - x + 2\gamma & l_k + 2\gamma & l_k + x + 2\gamma \\
l_k - x + \gamma & l_k & l_k + x + \gamma \\
l_k - x - \gamma & l_k - x - 2\gamma & \cdots
\end{pmatrix} \\
\theta_g & \begin{pmatrix}
\cdots & l_k - x + 2\gamma & l_k + 2\gamma & l_k + x + 2\gamma \\
l_k - x + \gamma & l_k & l_k + x + \gamma \\
l_k - x - \gamma & l_k - x - 2\gamma & \cdots
\end{pmatrix} \\
= & \exp \left[ -ip^g_k \begin{pmatrix}
l_k - x + 2\gamma & l_k + 2\gamma & l_k + x + 2\gamma \\
l_k - x + \gamma & l_k & l_k + x + \gamma \\
l_k - x - \gamma & l_k - x - 2\gamma & \cdots
\end{pmatrix} \right], \quad (32)
\end{align*}
$$

for some functions $p^g_k \in \mathbb{R}$. Note that the arguments of $\theta_g$ and $p^g_k$ were written out in a two-dimensional array, such that the dependence on the l-bit indices within the causal cone of $|k\rangle⟨k'|$ in (32) is apparent. Let us introduce $f_g(|k\rangle\langle k|)$ defined by $\theta_g(|k\rangle\langle k|) = \exp(f_g(|k\rangle\langle k|))$, so that we have

$$
\begin{align*}
\begin{pmatrix}
\cdots & l_k - x + 2\gamma & l_k + 2\gamma & l_k + x + 2\gamma \\
l_k - x + \gamma & l_k & l_k + x + \gamma \\
l_k - x - \gamma & l_k - x - 2\gamma & \cdots
\end{pmatrix} \\
\begin{pmatrix}
\cdots & l_k - x + 2\gamma & l_k + 2\gamma & l_k + x + 2\gamma \\
l_k - x + \gamma & l_k & l_k + x + \gamma \\
l_k - x - \gamma & l_k - x - 2\gamma & \cdots
\end{pmatrix} \\
\begin{pmatrix}
\cdots & l_k - x + 2\gamma & l_k + 2\gamma & l_k + x + 2\gamma \\
l_k - x + \gamma & l_k & l_k + x + \gamma \\
l_k - x - \gamma & l_k - x - 2\gamma & \cdots
\end{pmatrix} \\
\end{align*}
$$

and

$$
\begin{align*}
\begin{pmatrix}
\cdots & l_k - x + 2\gamma & l_k + 2\gamma & l_k + x + 2\gamma \\
l_k - x + \gamma & l_k & l_k + x + \gamma \\
l_k - x - \gamma & l_k - x - 2\gamma & \cdots
\end{pmatrix} \\
\begin{pmatrix}
\cdots & l_k - x + 2\gamma & l_k + 2\gamma & l_k + x + 2\gamma \\
l_k - x + \gamma & l_k & l_k + x + \gamma \\
l_k - x - \gamma & l_k - x - 2\gamma & \cdots
\end{pmatrix} \\
\begin{pmatrix}
\cdots & l_k - x + 2\gamma & l_k + 2\gamma & l_k + x + 2\gamma \\
l_k - x + \gamma & l_k & l_k + x + \gamma \\
l_k - x - \gamma & l_k - x - 2\gamma & \cdots
\end{pmatrix} \\
\end{align*}
$$

where in the second equation we act with $|k - y\rangle⟨k - y'|$ on the block of sites at $k - y$ instead of $k$. We sweep column-by-column through the lattice and write down analogous equations corresponding to cases where that operator acts on other blocks. As an example, at an intermediate step, we have, at some point $r$
Adding up all of these equations leads to

\[ f_g(\{ l_k \}) - f_g(\{ l'_k \}) = \sum_r p^0_r \left( \begin{array}{cccc}
I_{0+2\theta} & I_{0+2\phi} & I_{0+2\theta+2\phi} \\
I_{0+2\phi} & I_{0+\theta} & I_{0+\theta+2\phi} \\
I_{0+\theta} & I_{0+\phi} & I_{0+\theta+\phi} \\
I_{0+\phi} & I_{0} & I_{0+\phi+\theta} \\
I_{0} & I_{-\theta} & I_{0+\phi+\theta+\phi} \\
I_{-\theta} & I_{-\phi} & I_{0+\phi+\theta+\phi} \\
I_{-\phi} & I_{-\theta} & I_{0+\phi+\theta+\phi} \\
I_{-\theta} & I_{-\phi} & I_{0+\phi+\theta+\phi} \end{array} \right) + \text{boundary terms mod } 2\pi. \]  

(35)

Now we set all the primed indices to zero, i.e. let \( l'_k = (0, 0, 0, 0) \) for all \( k \). This implies that there exist functions of five \( l_k \) indices \( q^0_r \) such that we can write

\[ f_g(\{ l_k \}) = \sum_r q^0_r \left( \begin{array}{cccc}
I_{0+2\theta} & I_{0+2\phi} & I_{0+2\theta+2\phi} \\
I_{0+2\phi} & I_{0+\theta} & I_{0+\theta+2\phi} \\
I_{0+\theta} & I_{0+\phi} & I_{0+\theta+\phi} \\
I_{0+\phi} & I_{0} & I_{0+\phi+\theta} \\
I_{0} & I_{-\theta} & I_{0+\phi+\theta+\phi} \\
I_{-\theta} & I_{-\phi} & I_{0+\phi+\theta+\phi} \\
I_{-\phi} & I_{-\theta} & I_{0+\phi+\theta+\phi} \\
I_{-\theta} & I_{-\phi} & I_{0+\phi+\theta+\phi} \end{array} \right). \]  

(36)

But we could have just as well applied the above argument sweeping row-by-row, which leads to

\[ f_g(\{ l_k \}) = \sum_r q^0_r \left( \begin{array}{cccc}
I_{0+2\theta} & I_{0+2\phi} & I_{0+2\theta+2\phi} \\
I_{0+2\phi} & I_{0+\theta} & I_{0+\theta+2\phi} \\
I_{0+\theta} & I_{0+\phi} & I_{0+\theta+\phi} \\
I_{0+\phi} & I_{0} & I_{0+\phi+\theta} \\
I_{0} & I_{-\theta} & I_{0+\phi+\theta+\phi} \\
I_{-\theta} & I_{-\phi} & I_{0+\phi+\theta+\phi} \\
I_{-\phi} & I_{-\theta} & I_{0+\phi+\theta+\phi} \\
I_{-\theta} & I_{-\phi} & I_{0+\phi+\theta+\phi} \end{array} \right). \]  

(37)

Comparing the last two equations shows that there must exist functions \( s^0_r \) of six \( l_k \) indices such that we can write

\[ f_g(\{ l_k \}) = \sum_r s^0_r \left( \begin{array}{cccc}
I_{0+2\theta} & I_{0+2\phi} & I_{0+2\theta+2\phi} \\
I_{0+2\phi} & I_{0+\theta} & I_{0+\theta+2\phi} \\
I_{0+\theta} & I_{0+\phi} & I_{0+\theta+\phi} \\
I_{0+\phi} & I_{0} & I_{0+\phi+\theta} \\
I_{0} & I_{-\theta} & I_{0+\phi+\theta+\phi} \\
I_{-\theta} & I_{-\phi} & I_{0+\phi+\theta+\phi} \\
I_{-\phi} & I_{-\theta} & I_{0+\phi+\theta+\phi} \\
I_{-\theta} & I_{-\phi} & I_{0+\phi+\theta+\phi} \end{array} \right). \]  

(38)

(39)

Therefore \( \Theta_\theta \) can be expressed as a four-layer quantum circuit whose unitary matrices \( \theta_{g,k} \) are all diagonal and can be arranged as shown in Fig. 4. Those unitaries act on plaquettes of \( 2\ell \times 3\ell \) sites.

C. Reduction to one dimensional problem

We have \( g^\otimes N^2 U = U\Theta_\theta \), where the left hand side (LHS) is a four-layer quantum circuit like Fig. 1 and the right hand side (RHS) is an eight-layer quantum circuit. We then reduce the two-dimensional quantum circuit to a one-dimensional one by blocking unitaries along the
y-direction. We then obtain, along the x-direction

\[
\Theta^g_k = \begin{pmatrix}
\theta_g & \theta_g & \theta_g & \cdots & \theta_g \\
\theta_g & \theta_g & \cdots & \theta_g \\
\theta_g & \theta_g & \cdots & \theta_g \\
\end{pmatrix}.
\]

(41)

Note that in Eq. (40) we have used indices on the U, V, and Θ-tensors to emphasize the non-translation invariance, while we have employed the index-free notation on the RHS of Eq. (41).

Eq. (40) also appears in the exact same form in the one-dimensional classification of SPT MBL phases.\(^{38}\) Using the blocking indicated by dashed lines in Eq. (40) reveals that it is an equation relating two one-dimensional two-layer quantum circuits. Hence, we can use the results of Sec. (VA) and deduce the existence of gauge tensors \(W_k\), which transform unitaries of both sides of the equation into each other. These unitaries depend on the group element \(g\), and we refer to them as \(W^g_k\). The result (see Ref. [8] for details) is that the \(W^g_k\) have to fulfill

\[
W^g_{2k-1} = W^g_{2k} = W^g_{2k+1} = W^g_{2k+2},
\]

(42)

\[
U_{4k-2} \quad U_{4k-1} \quad \Theta^g_{4k-3} \quad \Theta^g_{4k-1} \\
\Theta^g_{4k-2} \quad \Theta^g_{4k-1} \\
U_{4k-2}^\dagger \quad U_{4k-1}^\dagger
\]

where the differing numbers of legs of \(W^g_k\) for \(k\) even and odd are due to the blocking scheme used in that calculation, and also

\[
W^g_{2k} = W^g_{2k+1},
\]

(43)

\[
U_{4k} \quad U_{4k+1}
\]

Since all unitaries on the RHS of Eqs. (42) and (43) are quantum circuits along the y-direction, the \(W^g_k\) must also be strictly short-range correlated along that direction. Thus, they can at least be efficiently approximated by two-layer one-dimensional quantum circuits along the y-direction (cf. assumptions made in the beginning of Section III).

It can also be shown (again see Ref. [8] for details) that the \(W^g_k\) form a projective representation of \(G\), i.e. for all
$k$, it is the case that $W_k^g W_k^h = \rho_k(g, h) W_k^{gh}$ for some $\beta_k(g, h) \in U(1)$. For the two dimensional classification of SPT MBL phases, we will use this result combined with the lemma below.

D. Quantum circuit representations and the third cohomology group

We now prove our main statement: The quantum circuit projective representations of a given group $G$ are labeled by the elements of the third cohomology group $H^3(G,U(1))$. That is, quantum circuits corresponding to different third cohomology classes cannot be continuously connected with each other while preserving the fact that they projectively represent the symmetry group. We note that this statement is related to the result from Ref. 62 that injective matrix product operator (MPO) representations of $G$ likewise correspond to the elements of $H^3(G,U(1))$. Let us also mention that (for a group $G$ and $G$-module $F$), the third cohomology group $H^3(G,F)$ has a representation theoretic interpretation: similar to how elements of $H^2(G,F)$ correspond to projective representations of $G$, elements of $H^3(G,F)$ correspond to gerbal representations of $G$; see Appendix A for details.

Each $g \in G$ is associated a quantum circuit, which we denote as

$$
\begin{array}{ccccccc}
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\
v_1^g & v_2^g & v_3^g & \cdots & v_n^g & u_1^g & u_2^g & \cdots & u_n^g \\
\end{array}
$$

(44)

To reduce clutter let us adopt a shorthand notation where we label all the $u_k^g$ and $v_k^g$ tensors as simply $g$. That is, we would write Eq. (44) as

$$
\begin{array}{ccccccc}
g & g & g & \cdots & g \\
g & g & g & \cdots & g \\
\end{array}
$$

(45)

Let the quantum circuits associated with $g, h \in G$ be a projective representation of $G$, i.e.

$$
\begin{array}{ccccccc}
g & g & g & \cdots & g \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
h & h & h & \cdots & h \\
\end{array} =\beta(g,h)\begin{array}{ccccccc}
g & g & g & \cdots & g \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
h & h & h & \cdots & h \\
\end{array}.
$$

(46)

for some $\beta(g, h) \in U(1)$. In the following steps of the calculation, the factors of $\beta(g, h)$ will only lead to factors like $\beta(g, h)\beta(gh, k)\beta(h, k)$, which are equal to 1, due to the 2-cocycle condition for projective representations. So we omit all the factors of $\beta(g, h)$ hereafter.

Consider blocking unitaries in (45) as follows,

$$
\begin{array}{ccccccc}
g & g & g & \cdots & g \\
k & k & k & \cdots & k \\
\end{array}
$$

(47)

$$
\begin{array}{ccccccc}
g & g & g & \cdots & g \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
h & h & h & \cdots & h \\
\end{array} = \begin{array}{ccccccc}
gh & gh & gh & \cdots & gh \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
gh & gh & gh & \cdots & gh \\
\end{array}.
$$

(48)

In the language of Sec. 4.2.1 (as in Eq. 22), let the blocked tensors on the LHS be $U'$, $V'$ and the ones on the RHS be $U$, $V$. We deduce the existence of the $W_k$ tensors (which are functions of two group elements here) and plug in Eq. (25) to obtain
where we have temporarily reverted from the abbreviated notation for clarity. Let us now return to the abbreviated notation, and denote $W_{4k-3}(g, h)$ as $W_L(g, h)$ and $W_{4k+1}(g, h)$ as $W_R(g, h)$. Eq. (49) can be rearranged into
whence it becomes clear that we can define new tensors $W(g, h)$ and $W'(g, h)$ such that we have

\[ W(g, h) \quad \begin{array}{cccccc}
g & g & g & g & g & g \\
g & g & g & g & g & g \\
h & h & h & h & h & h \\
h & h & h & h & h & h \\
h & h & h & h & h & h \\
W'(g, h) \end{array} = \begin{array}{cccccc}
g & g & g & g & g & g \\
g & g & g & g & g & g \\
k & k & k & k & k & k \\
k & k & k & k & k & k \\
k & k & k & k & k & k \\
W'(g, h) \end{array} \quad (51) \]

$W(g, h)$ and $W'(g, h)$ act as gerbal representation operators: Informally, $W(g, h)$ and $W'(g, h)$ “convert” a combination of a section of the $g$ quantum circuit and the $h$ quantum circuit into a section of the $gh$ quantum circuit, playing a role analogous to that of $X_L(g, h)$ and $X_R(g, h)$ in Eq. (16), respectively.

To show that the quantum circuit projective representations of $G$ satisfy the pentagon equation (20), we must find an associated function of three group elements $\alpha(g, h, k) \in U(1)$ which is a 3-cocycle invariant up to multiplication by a 3-coboundary. Consider three group elements $g, h, k \in G$: 

\[ W(g, k) \quad \begin{array}{cccccc}
g & g & g & g & g & g \\
g & g & g & g & g & g \\
k & k & k & k & k & k \\
k & k & k & k & k & k \\
k & k & k & k & k & k \\
W'(g, k) \end{array} = \begin{array}{cccccc}
g & g & g & g & g & g \\
g & g & g & g & g & g \\
k & k & k & k & k & k \\
k & k & k & k & k & k \\
k & k & k & k & k & k \\
W'(g, k) \end{array} \quad (52) \]

Canceling out the middle sections as indicated by the red
lines, the second equality implies

\[
\begin{array}{c}
W(g, h, k) \\
W(g, h) \\
g \\
g \\
W(h, k)
\end{array}
\otimes
\begin{array}{c}
W'(g, h) \\
k \\
k \\
W'(h, k) \\
W'(g, h, k)
\end{array}
= \begin{array}{c}
k \\
k \\
g \\
g \\
w'(g, h, k)
\end{array}
\]

which means there must be some phase factor \(\alpha(g, h, k)\) such that

\[
\begin{array}{c}
W(g, h, k) \\
W(g, h) \\
g \\
g \\
W(h, k)
\end{array}
= \begin{array}{c}
\alpha(g, h, k) \\
g \\
g \\
W(h, k)
\end{array}
\]

\[
\alpha(g, h, k) = \alpha(g, h, k) \chi(g, h, k)
\]

and, with a slight abuse of notation, we write, for example, Eq. (54) algebraically as

\[
W(g, h)W(g, h)X(g) = \alpha(g, h, k)W(g, h)X(g)W(h, k).
\]

(57)

\(W(g, h)\) inherits the gauge degree of freedom of the old \(W_L(g, h)\), so it is invariant up to a transformation \(W(g, h) \to \chi(g, h)W(g, h)\) for \(\chi(g, h) \in U(1)\). After the transformation, we have Eq. (57) but with \(\alpha(g, h, k)\) replaced by

\[
\alpha'(g, h, k) = \alpha(g, h, k) \chi(g, h, k) \chi(h, k) \chi(gh, k).
\]

(58)

Thus \(\alpha(g, h, k)\) is defined up to a 3-coboundary.

Now we show that \(\alpha(g, h, k)\) satisfies an analogue of Eq. (20), and therefore is a 3-cocycle. Consider the following expression involving four group elements,

\[
W(gh, k)W(g, h)X(g) = \alpha(g, h, k)W(g, h)X(g)W(h, k).
\]

(57)

\[
W(gh, k)W(g, h)X(g)X(h)
= \alpha(g, h, k)W(gh, k)W(g, h)X(g)W(h, k)X(h)
= \alpha(g, h, k)\alpha(g, h, k)\alpha(g, h, l)\alpha(h, k, l)\chi(h, k, l) \times W(g, hkl)X(g)X(h)W(h, kl)X(k, l)
\]

(59)

where we have used Eq. (57) repeatedly. Let us introduce a new shorthand notation where, for example, Eq. (51) is written as

\[
\begin{array}{c}
gh \\
g \\
g \\
g
\end{array}
= \begin{array}{c}
g \\
g
\end{array}
\]

(60)

\(X(g)\) is the function of three group variables that we are looking for. Before proceeding further let first simplify the notation. Define

\[
X(g) = \begin{array}{c}
g \\
g \\
g \\
g
\end{array}
\]

(56)

Using this notation, we have the following expression in-
To show that $\alpha(g,h,k)$ is a 3-cocycle, we only need to consider an expression consisting of the top parts of the RHS of the above equation. We then repeatedly apply Eq. (57) to the left edge of that expression. There are two ways to do this. First, we can apply Eq. (59) (converted back into diagrammatic form) and immediately obtain

$$W'_{ghk} = \alpha(g,h,k) \times \alpha(g,hk,l) \times \alpha(h,k,l) \times \alpha(gh,kl) = 1.$$  

(62)

Alternatively, we can calculate via a different route

$$W'_{ghk} = \alpha(gh,k,l) \times \alpha(g,h,k) \times \alpha(g,hk,l) \times \alpha(h,k,l) \times \alpha(gh,kl) = 1.$$  

(63)

Comparing the above two final expressions we find that indeed

$$\frac{\alpha(g,h,k)\alpha(g,hk,l)\alpha(h,k,l)}{\alpha(gh,k,l)\alpha(g,h,kl)} = 1.$$  

(64)

Note that we have only considered $W(g,h)$ but the same argument applies to the right edge and $W'(g,h)$, which from the $\alpha^{-1}$ in Eq. (55) is associated with the inverse element of $H^3(G, U(1))$.

To complete the argument, we need to show that the cohomology class does not depend on the position of the block; recalling the non-translational-invariance of the original quantum circuit, we need to show that the $W(g,h)$ and $W'(g,h)$ of the adjacent block are associated with the same element of $H^3(G, U(1))$. We also need to show that there is no dependence on the blocking scheme.
FIG. 5. Three ways of blocking: original 4-blocking (red), 4-blocking shifted by one block (green), and 5-blocking (blue)

For instance, we can use block sizes of larger than four (though it is easy to see that the above arguments would not work for block sizes of three or smaller). Fig 5 depicts different ways of blocking.

The argument is as follows. Suppose we are looking at a 4-blocking starting from a certain index, such as $4k - 3$ as in Eq. (49). Then let us consider a larger blocking also starting from the same index. (For example, we could consider the red 4-blocking and the blue 5-blocking in Fig 5.) Applying Eq. (23), we have

$$= 0.$$ (65)

From the larger blocking, we have, separately (using the ellipsis notation to we emphasize that this works for an arbitrarily large blocking):

$$= W_L(g, h) W_M(g, h).$$ (66)

The two above equations taken together imply that $W_L(g, h)$ and $W'_L(g, h)$ are the same up to a phase. So the $W_L(g, h)$ and hence $W(g, h)$ are the same up to a phase in either blocking. Hence two blocks of different sizes that start at the same point along the quantum circuit have the same cohomology class. The same argument applies to $W_R(g, h)$ and $W'(g, h)$ of different blocks that share the same right edge. This then implies that the entire quantum circuit is associated with a single cohomology class $a \in H^3(G, U(1))$, because we can then use the above results to argue that any any two blocks in the quantum circuit correspond to the same cohomology class: We may deduce from the schematic picture Fig. 6 that $a = \tilde{a} = \tilde{a}$, i.e. they are the same element of $H^3(G, U(1))$ while the corresponding functions $\alpha(g, h, k)$, $\tilde{\alpha}(g, h, k)$, and $\tilde{\alpha}(g, h, k)$ would be equal up to a 3-coboundary. It is easy to see that this generalizes to show that any block (from any blocking scheme) produces $W(g, h)$ and $W'(g, h)$ labeled by the same $a$ and $a^{-1} \in H^3(G, U(1))$, respectively. The entire quantum circuit representation of $G$ is associated with one particular element of $H^3(G, U(1))$, completing the proof of the lemma.

FIG. 6. A schematic diagram depicting two adjacent blocks and a large block encompassing both of them, and their associated phase factors. Note that the adjacent blocks do not have to be of the same length.

E. Invariance of the topological index across the 2D system

The above lemma applies separately to each $W^g_j$ that appears on the LHS of Eq. (42) and Eq. (43) and also to the overall quantum circuits of those equations. To complete the argument for our two-dimensional MBL phase classification, we must show that the different $W^g_j$ along the $x$-direction have the same 3rd cohomology class. This can be done by showing that $W^g_j \otimes W^g_{j+1}$ is topologically trivial, that is, corresponds to the identity element of $H^3(G, U(1))$.

Because $W^g_j$ takes a different form for odd and even $j$, we have two points to show, that $W^g_{2k-1} \otimes W^g_{2k}$ is topologically trivial, and that $W^g_{2k} \otimes W^g_{2k+1}$ is topologically trivial as well.
1. \( W^{g}_{2k-1} \otimes W^{g}_{2k} \) is topologically trivial

From Eq. (42), we have

\[
\tilde{W}^{g}_{2k-1} \otimes \tilde{W}^{g}_{2k} = \tilde{\Theta}^{g}_{2k-1} \otimes \tilde{\Theta}^{g}_{2k}
\]

\[
= \tilde{U}_{2k-2} \otimes \tilde{U}_{2k-1} \\otimes \tilde{\Theta}^{g}_{4k-2} \otimes \tilde{\Theta}^{g}_{4k-1}
\]

\[
= \tilde{U} \tilde{\Theta}(g) \tilde{U}^\dagger = \tilde{W}(g)
\]

\( \Theta^{g}_{j} \) can be chosen in such a way that\(^{35} \Theta^{g}_{j} \Theta^{h}_{j} = \Theta^{gh}_{j} \), i.e., it is a linear representation of the group \( G \). Since \( \tilde{W}(g) \) is unitarily equivalent to a product of \( \Theta^{g} \)-quantum circuits, \( \tilde{W}(g) \) must be a linear representation, too,

\[
\tilde{W}(g) \tilde{W}(h) = \tilde{W}(gh).
\]

The third cohomology class is a topological label of quantum circuits which are a projective representation of the group \( G \). Hence, two quantum circuits corresponding to different third cohomology classes cannot be continuously connected while preserving the fact that they projectively represent the group \( G \). Keeping that in mind, we note that \( \tilde{W}(g) \) can be continuously connected to \( \tilde{\Theta}(g) \) by defining \( \tilde{W}_{\lambda}(g), \lambda \in [0,1] \) via

\[
u_{j,\lambda} = e^{iL_{j}(1-\lambda)},
\]

\[
u_{j,\lambda} = e^{iM_{j}(1-\lambda)}
\]

with \( L_{j} = L_{j}^{1}, M_{j} = M_{j}^{1} \) and the original unitaries \( u_{j} = e^{iM_{j}} \). Hence, \( \tilde{W}_{0}(g) = \tilde{W}(g) \) and \( \tilde{W}_{1}(g) = \tilde{\Theta}(g) \) and since for all \( \lambda \) \( \tilde{W}_{\lambda}(g) \tilde{W}_{\lambda}(h) = \tilde{W}_{\lambda}(gh) \), \( \tilde{W}(g) \) and \( \tilde{\Theta}(g) \) must correspond to the same element of the third cohomology group. Finally, we show that \( \tilde{\Theta}(g) \) corresponds to the identity of the third cohomology group. This can be most easily seen by combining \( \theta_{g} \)'s and \( \bar{\theta}_{g} \)'s by commuting them through each other and combining four and two adjacent legs to respectively one. We call the newly obtained unitaries \( \theta_{u}^{g} \) and \( \theta_{v}^{g} \). The \( \theta_{u}^{g} \) and \( \theta_{v}^{g} \) commute with each other, that is

\[
\theta_{u}^{g} \theta_{u}^{h} = \theta_{u}^{gh},
\]

\[
\theta_{v}^{g} \theta_{v}^{h} = \theta_{v}^{gh}.
\]

Furthermore, \( \Theta^{g}_{j} \Theta^{h}_{j} = \Theta^{gh}_{j} \) can be used to show in the same way as in Ref.\(^{38} \) that also the \( \theta_{g} \)'s (and \( \bar{\theta}_{g} \)'s) can be gauged such that \( \theta_{g} \theta_{h} = \theta_{gh} \), which implies \( \theta_{u}^{g} \theta_{u}^{h} = \theta_{u}^{gh} \) and \( \theta_{v}^{g} \theta_{v}^{h} = \theta_{v}^{gh} \). Using this and Eqs. (71) and (72), reveals via Eq. (51) that \( W(g,h) = W'(g,h) = 1 \), i.e., after the deformation \( \alpha(g,h,k) = 1 \). Thus, \( W^{g}_{2k-1} \otimes W^{g}_{2k} \) is topologically trivial, as claimed.

2. \( W^{g}_{2k} \otimes W^{g}_{2k+1} \) is topologically trivial

From Eq. (43), we have

\[
\tilde{W}^{g}_{2k} \otimes \tilde{W}^{g}_{2k+1} = \tilde{U} \tilde{\Theta}(g) \tilde{U}^\dagger = \tilde{W}(g)
\]

\( \Theta^{g}_{j} \) can be chosen in such a way that\(^{35} \Theta^{g}_{j} \Theta^{g}_{j} = \Theta^{gh}_{j} \), i.e., it is a linear representation of the group \( G \). Since \( \tilde{W}(g) \) is unitarily equivalent to a product of \( \Theta^{g} \)-quantum circuits, \( \tilde{W}(g) \) must be a linear representation, too,

\[
\tilde{W}(g) \tilde{W}(h) = \tilde{W}(gh).
\]

The third cohomology class is a topological label of quantum circuits which are a projective representation of the group \( G \). Hence, two quantum circuits corresponding to different third cohomology classes cannot be continuously connected while preserving the fact that they projectively represent the group \( G \). Keeping that in mind, we note that \( \tilde{W}(g) \) can be continuously connected to \( \tilde{\Theta}(g) \) by defining \( \tilde{W}_{\lambda}(g), \lambda \in [0,1] \) via

\[
u_{j,\lambda} = e^{iL_{j}(1-\lambda)},
\]

\[
u_{j,\lambda} = e^{iM_{j}(1-\lambda)}
\]

with \( L_{j} = L_{j}^{1}, M_{j} = M_{j}^{1} \) and the original unitaries \( u_{j} = e^{iM_{j}} \). Hence, \( \tilde{W}_{0}(g) = \tilde{W}(g) \) and \( \tilde{W}_{1}(g) = \tilde{\Theta}(g) \) and since for all \( \lambda \) \( \tilde{W}_{\lambda}(g) \tilde{W}_{\lambda}(h) = \tilde{W}_{\lambda}(gh) \), \( \tilde{W}(g) \) and \( \tilde{\Theta}(g) \) must correspond to the same element of the third cohomology group. Finally, we show that \( \tilde{\Theta}(g) \) corresponds to the identity of the third cohomology group. This can be most easily seen by combining \( \theta_{g} \)'s and \( \bar{\theta}_{g} \)'s by commuting them through each other and combining four and two adjacent legs to respectively one. We call the newly obtained unitaries \( \theta_{u}^{g} \) and \( \theta_{v}^{g} \). The \( \theta_{u}^{g} \) and \( \theta_{v}^{g} \) commute with each other, that is

\[
\theta_{u}^{g} \theta_{u}^{h} = \theta_{u}^{gh},
\]

\[
\theta_{v}^{g} \theta_{v}^{h} = \theta_{v}^{gh}.
\]

Furthermore, \( \Theta^{g}_{j} \Theta^{h}_{j} = \Theta^{gh}_{j} \) can be used to show in the same way as in Ref.\(^{38} \) that also the \( \theta_{g} \)'s (and \( \bar{\theta}_{g} \)'s) can be gauged such that \( \theta_{g} \theta_{h} = \theta_{gh} \), which implies \( \theta_{u}^{g} \theta_{u}^{h} = \theta_{u}^{gh} \) and \( \theta_{v}^{g} \theta_{v}^{h} = \theta_{v}^{gh} \). Using this and Eqs. (71) and (72), reveals via Eq. (51) that \( W(g,h) = W'(g,h) = 1 \), i.e., after the deformation \( \alpha(g,h,k) = 1 \). Thus, \( W^{g}_{2k-1} \otimes W^{g}_{2k} \) is topologically trivial, as claimed.

\[
\tilde{W}^{g}_{2k-1} \otimes \tilde{W}^{g}_{2k} = \tilde{U} \tilde{\Theta}(g) \tilde{U}^\dagger = \tilde{W}(g)
\]

\( \Theta^{g}_{j} \) can be chosen in such a way that\(^{35} \Theta^{g}_{j} \Theta^{h}_{j} = \Theta^{gh}_{j} \), i.e., it is a linear representation of the group \( G \). Since \( \tilde{W}(g) \) is unitarily equivalent to a product of \( \Theta^{g} \)-quantum circuits, \( \tilde{W}(g) \) must be a linear representation, too,

\[
\tilde{W}(g) \tilde{W}(h) = \tilde{W}(gh).
\]

The third cohomology class is a topological label of quantum circuits which are a projective representation of the group \( G \). Hence, two quantum circuits corresponding to different third cohomology classes cannot be continuously connected while preserving the fact that they projectively represent the group \( G \). Keeping that in mind, we
where the boxes labeled by $u_g$, $v_y$ indicate blocks of unitaries $u_j$, $v_j$ and $v_g$, and we have combined legs. In the last part of the equation, we used that due to the diagonality of the corresponding indices, the $\theta_g$'s commute with the quantum circuit comprised of the unitaries $u_g$

$$= \begin{array}{cccccccc}
\theta_g & \theta_g & \theta_g & \theta_g & \theta_g & \theta_g \\
\theta_g & \theta_g & \theta_g & \theta_g & \theta_g & \theta_g \\
v_g & v_g & v_g & v_g & v_g & v_g \\
u_g & u_g & u_g & u_g & u_g & u_g \\
\theta_g & \theta_g & \theta_g & \theta_g & \theta_g & \theta_g \\
u_g & u_g & u_g & u_g & u_g & u_g \\
v_g & v_g & v_g & v_g & v_g & v_g \\
\end{array}
$$

(73)

and $v_g$. Due to its local structure, this implies also

$$= \begin{array}{cccccccc}
v_g & v_g & v_g & v_g & v_g & v_g \\
u_g & \theta_h & \theta_h & \theta_h & \theta_h & \theta_h \\
v_g & v_g & v_g & v_g & v_g & v_g \\
u_g & u_g & u_g & u_g & u_g & u_g \\
v_g & v_g & v_g & v_g & v_g & v_g \\
u_g & u_g & u_g & u_g & u_g & u_g \\
\end{array}
= \begin{array}{cccc}
v_g & v_g \\
u_g & u_g \\
u_g & u_g \\
\theta_h & \theta_h \\
u_g & u_g \\
u_g & u_g \\
\end{array}
$$

(74)

We now show that this implies that $u_g$ and $v_g$ can also be gauged in such a way that they individually commute with $\theta_h$. From the previous equation it follows that

$$= \begin{array}{cccccccc}
u_g & \nu_g & \nu_g & \nu_g \\
u_g & \theta & \theta & \theta & \theta & \theta \\
u_g & v_g & v_g & v_g & v_g & v_g \\
u_g & v_g & v_g & v_g & v_g & v_g \\
u_g & v_g & v_g & v_g & v_g & v_g \\
u_g & v_g & v_g & v_g & v_g & v_g \\
\end{array}
= \begin{array}{cccc}
u_g & \nu_g \\
u_g & \nu_g \\
u_g & \nu_g \\
\theta & \theta & \theta & \theta \\
u_g & \nu_g & \nu_g \\
u_g & \nu_g & \nu_g \\
\end{array}
$$

(75)

where we have replaced $\theta_h$ by a diagonal matrix $\theta$, which has the diagonal structure common to all $\theta_h$'s, but whose non-trivial phase factors can be chosen arbitrarily. These correspond to the indices of the forth to seventh leg from the left in the second part of Eq. (78). We now choose $\Theta = \theta \otimes 1$, such that Eq. (75) simplifies to

$$= \begin{array}{cccccccc}
u_g & \nu_g & \nu_g & \nu_g \\
u_g & \nu_g & \nu_g & \nu_g \\
u_g & \nu_g & \nu_g & \nu_g \\
u_g & \nu_g & \nu_g & \nu_g \\
u_g & \nu_g & \nu_g & \nu_g \\
u_g & \nu_g & \nu_g & \nu_g \\
\end{array}
= \begin{array}{cccc}
u_g & \nu_g \\
u_g & \nu_g \\
u_g & \nu_g \\
u_g & \nu_g \\
u_g & \nu_g \\
u_g & \nu_g \\
\end{array}
$$

(76)

This implies that $[X_0^y \otimes 1, X_0^y \otimes 1] = 0$, i.e., $[X_0^y, X_0^y] = 0$. Since $X_0^y$ and $X_0^y$, are unitaries, they can be diagonalized by the same matrix $u^y$. The result of the diagonalization would be $\theta$, i.e., $X_0^y = w_g \theta w_g^\dagger$. Hence, if we use a gauge transformation as in Eqs. (23) and (21) to replace $u^y$ by $(w_g \otimes 1) u^y$, the RHS of Eq. (76) is $1 \otimes \theta \otimes 1 \otimes 1$. Moreover, in Eq. (75), we could instead have set $\theta = 1 \otimes \theta$ leading to

$$= \begin{array}{cccccccc}
u_g & \nu_g & \nu_g & \nu_g \\
u_g & \nu_g & \nu_g & \nu_g \\
u_g & \nu_g & \nu_g & \nu_g \\
u_g & \nu_g & \nu_g & \nu_g \\
u_g & \nu_g & \nu_g & \nu_g \\
u_g & \nu_g & \nu_g & \nu_g \\
\end{array}
= \begin{array}{cccc}
u_g & \nu_g \\
u_g & \nu_g \\
u_g & \nu_g \\
u_g & \nu_g \\
u_g & \nu_g \\
u_g & \nu_g \\
\end{array}
$$

(77)
Similarly, it follows that $Y_\vartheta^g$ can be diagonalized by a unitary matrix $\tilde{w}_g$ which does not depend on $\vartheta$. Hence, the gauge transformation $u_g \to (w_g \otimes \tilde{w}_g)u_g$ (and the corresponding one for $v_g$) ensures that the new $u_g$ commutes with $\vartheta \otimes \vartheta'$ for all $\vartheta, \vartheta'$. Hence, it must also commute with $\theta$ (which could be written as $\sum_i \vartheta_i \otimes \vartheta'_i$ if we relax the condition that $\vartheta$ and $\vartheta'$ have diagonal elements of magnitude 1, which is not needed for the above derivation). In the new gauge, $[u_g, \theta_h] = 0$ and the second part of Eq. (74) implies that in that gauge $[v_g, \theta_h] = 0$ as well. In other words, we can choose $u_g$ and $v_g$ such that they all commute with $\theta_h$, i.e., the $\theta_h$’s can be moved through them in all the diagrams. We now take advantage of the fact that the last expression of Eq. (78) can be written as a two-layer quantum circuit after blocking unitaries, such that Eq. (51) implies

$$W'(g, h) = W(g, h)$$

We can gauge $\theta_g$ such that $\theta_g \theta_h = \theta_{gh}$ (see above), i.e., in the new gauge of $u_g$ and $v_g$, all $\theta$’s can be canceled out, leading to

$$W'(g, h) = W(g, h)$$

(79)
Hence, the $W(g,h)$ and $W'(g,h)$ are the same (up to a phase) as the ones corresponding to the quantum circuit (78) without the $\theta_{g}$'s. That is, the third cohomology group of $W_{2k}^{g} \otimes W_{2k+1}^{g}$ is the same as the one of $\alpha_{L}$. (80)

For this quantum circuit, we can use the same approach as in the previous subsection and continuously deforming the $u$'s and $v$'s to 1 while preserving the property that it forms a linear representation of the group $G$ due to $v_{g}v_{h} = v_{gh}$. Eventually, one is left with $v_{g} \otimes N^{2}_{2g}$, which is topologically trivial. Thus, $\alpha_{L} = 1$ after the deformation, and $W_{2k}^{g} \otimes W_{2k+1}^{g}$ is topologically trivial, too.

**F. Equivalence of the topological label across eigenstates**

One important point is that the three-leg-wide $W_{2k-1}^{g}$ as in Eq. (42) or the first expression in Eq. (67) is actually diagonal in its first (left) two indices, and the five-leg-wide $W_{2k}^{g}$ is likewise diagonal in its last (right) two indices. This follows immediately from Eq. (43).

Say in the second expression of Eq. (67), we fix the first two and last two indices to $L_{1}, L_{2}, L_{3}$, and $L_{4}$. These indices correspond to the l-bit configuration of the eigenstates which are being approximated, since those indices are lower indices in Eq. (40), which according to Eq. (29) are eigenstate labels. Hence, a priori $W_{2k-1}^{g,L_{1}L_{2}}$ has cohomology class $a_{L_{1}L_{2}}$ depending on the indices $L_{1}$, $L_{2}$ (and thus on the eigenstates). Similarly, $W_{2k}^{g,L_{3}L_{4}}$ has cohomology class $a_{L_{3}L_{4}}$ again depending on the l-bits. However, since together they are topologically trivial, we must have $a_{L_{1}L_{2}}, a_{L_{3}L_{4}} = 1$. By fixing $L_{1}$, $L_{2}$ we conclude that the cohomology class cannot depend on $L_{3}$, $L_{4}$, and by fixing $L_{3}$, $L_{4}$ we conclude that the cohomology class cannot depend on $L_{1}$, $L_{2}$. Hence the topological label must be the same for all eigenstates.

**V. ANTI-UNITARY SYMMETRIES**

The above treatment can be generalized by allowing as well for anti-unitary symmetries. That is, for some group elements $g \in G$ we have

$$H = v_{g}^{\otimes N^{2}}H^{*}(v_{g})^{\otimes N^{2}},$$

which analogously leads to

$$\Theta_{g} = U^{*}v_{g}^{\otimes N^{2}}U.$$
Due to $v_g[v_h]^{\gamma(g)} = v_{gh}$, we thus have $W_j(g)[W_j(h)]^{\gamma(g)} = \beta_k(g,h)W_j(gh)$. Therefore, when approximating them by quantum circuits, we have (cf. Eq. (47))

$$\cdots \cdots = \beta(g,h) (\cdots g h g \cdots),$$

(87)

$\beta(g,h) \in U(1)$. Using the same line of reasoning as in Sec. [IV D] we obtain for a patch of the quantum circuit

$$= \cdots$$

(88)
This finally results in

\[ W(gh,k)W(g,h)X(g) = \alpha(g,h,k)W(h,k)X(h) \gamma(g). \] (90)

Thus, the gauge transformation \( W(g,h) \rightarrow \chi(g,h)W(g,h), \chi(g,h) \in U(1) \) corresponds to

\[ \alpha'(g,h,k) = \alpha(g,h,k) \frac{\chi(g,hk)\chi(h,k)\gamma(g)}{\chi(g,h)\chi(h,k)}. \] (91)

which is a redefinition of \( \alpha(g,h,k) \) by a generalized 3-coboundary. Eq. (90) implies

\[ W(ghl)W(g,h)X(g)W(h,k)X(h) \gamma(g) = \alpha(g,h,k)W(h,kl)X(h)\gamma(g)[W(h,k)]\gamma(g)X(h)\gamma(g). \] (92)

Again, we can reach a similar relation using a different sequence of manipulations,
Comparing the above two final expressions leads to
\[
\frac{\alpha(g, h, k)\alpha(g, h, l)\alpha(h, k, l)}{\alpha(g, k, l)\alpha(g, h, l)} = 1. \tag{95}
\]
Together with Eq. (91), this defines elements \(\alpha(g, h, k)\) of the generalised third cohomology group.

Keeping in mind that \(W^g_j W^{h,i} = \beta(g, h) W^{h,i}_j\), one can use a similar line of reasoning as in Sec. 1VI E to show that \(W^{g, 2k-1} W^{g, 2k}\) and \(W^{g, 2k} W^{g, 2k+1}\) are also topologically trivial in the current setup. This shows that for anti-unitary symmetries, all eigenstates of SPT MBL-like phases in two dimensions share the same topological label, which corresponds to an element of the generalised third cohomology group.

VI. ROBUSTNESS TO PERTURBATIONS

Here we show that the cohomology class is invariant under a symmetry preserving perturbation. The discussion here is very similar to the argument for the 1D case.\textsuperscript{57,58} Let us again consider the picture of Hamiltonians along a continuous path. Say we have a family of strongly disordered MBL-like Hamiltonians \(H(\lambda)\) invariant under the symmetry, for all \(\lambda \in [0, 1]\), and let \(U_{\text{ex}}(\lambda)\) be a matrix that diagonalises \(H(\lambda)\). It is always possible to find a \(U_{\text{ex}}(\lambda)\) that is a continuous function of \(\lambda\), which can easily be verified by writing \(H(\lambda') = U_{\text{ex}}(\lambda') E(\lambda') U_{\text{ex}}^\dagger(\lambda')\) and taking the limit \(\lambda' \rightarrow \lambda\) from above and from below.

At this point we note that for the rest of this section, all equations are correct up to corrections which play a role only on an extremely large time scale. There exists at each point \(\lambda\) a quantum circuit \(U(\lambda)\) which approximately diagonalizes the Hamiltonian with an error which only has an effect on the very long-time dynamics. In the absence of degeneracies, \(U(\lambda)\) can furthermore differ from the defined \(U_{\text{ex}}(\lambda)\) by up to a permutation matrix \(P(\lambda)\) whose nonzero elements may be phases, that is
\[
U(\lambda) = U_{\text{ex}}(\lambda) P(\lambda). \tag{96}
\]
For any finite \(N\) and abelian symmetry groups \(G\) there are only accidental degeneracies, which can be removed by small symmetry-preserving perturbations (which do not violate the quasi-MBL condition). Along the evolution via \(\lambda\) any degeneracies occur at isolated points \(\lambda_i\), which can be excluded in the following argument. From Eq. (96) it follows that \((\epsilon > 0)\)
\[
U(\lambda - \epsilon) P(\lambda - \epsilon) = U_{\text{ex}}(\lambda - \epsilon) U_{\text{ex}}^\dagger(\lambda + \epsilon) U(\lambda + \epsilon) P(\lambda + \epsilon). \tag{97}
\]
Taking the \(\epsilon \rightarrow 0\) limit, we have that \(U_{\text{ex}}(\lambda - \epsilon) U_{\text{ex}}^\dagger(\lambda + \epsilon) \rightarrow 1\), implying that
\[
U(\lambda - \epsilon) P(\lambda - \epsilon) P(\lambda + \epsilon) = U(\lambda + \epsilon), \tag{98}
\]
i.e. \(U(\lambda + \epsilon)\) and \(U(\lambda - \epsilon)\) are equivalent up to a permutation, or in other words, the eigenstates encoded by them are the same up to phase factors and a relabelling. Since all eigenstates have the same topological index, this shows that the overall topological index of the quantum circuit and thus also of the actual physical system is unchanged along the evolution.

VII. CONCLUSION

We have shown that given a two-dimensional strongly disordered MBL-like spin system invariant under an on-site symmetry, the symmetry-protected topological phases are classified by the elements of the third cohomology group of the symmetry group.

Though we have only considered the bosonic case, the ideas and results from the one-dimensional version of this problem\textsuperscript{35} imply that the classification is likely to be the same as for ground states also for fermionic systems.

One potential direction for further research is to investigate whether the method presented here can be adapted to rigorously show the correctness of the classification of ground state SPT phases in two-dimensional gapped systems, which is currently an open problem\textsuperscript{25}.

Another potential direction for further investigation is the extension of our classification to three and higher dimensions, though an obvious difficulty would be the challenge of working with higher dimensional tensor network diagrams. This case would also be particularly interesting as there is currently some doubt as to whether the cohomology conjecture is true in three and higher dimensions.\textsuperscript{67}

Finally, we note that the calculations here do not preclude the existence of additional topological indices in the 2D MBL case that do not exist for ground states. Specifically, although we have shown that quantum circuits belonging to different elements of the third cohomology group cannot be continuously connected, we have not shown the converse of this statement. In other words, we have not demonstrated the completeness of our classification, i.e., there may exist additional SPT MBL phases.

ACKNOWLEDGEMENTS

TBW acknowledges support by the European Commission under the Marie Curie Programme. AC received support by the EPSRC Grant No. EP/N01930X/1. The contents of this article reflect only the authors’ views and not the views of the European Commission.

Appendix A: Projective and gerbal representations

An important idea in the study of one dimensional SPT phases is the relation between second cohomology groups and projective representations. Here we briefly introduce
gerbal representations, the third cohomology analogue, which are relevant in the context of two-dimensional SPT phases. For an introduction to group cohomology as applied to the physics of SPT phases and definitions of cocycles and coboundaries, see Ref. [40].

A projective representation satisfies

$$u(g)u(h) = \omega(g,h)u(gh), \quad (A1)$$

where $$\omega(g,h) \in U(1)$$ is the factor system. Two projective representations are equivalent if their factor systems are related by

$$\omega'(g,h) = \frac{\chi(g)\chi(h)}{\chi(gh)} \omega(g,h), \quad (A2)$$

$$\chi \in U(1),$$ i.e. if they differ by a 2-coboundary. We also observe that an expression like $$u(g)u(h)u(k)$$ can written in two different ways, namely

$$u(g)u(h)u(k) = \omega(g,h)\omega(gh,k)u(ghk) = \omega(g,hk)\omega(h,k)u(ghk). \quad (A3)$$

So we obtain the result that the factor system of a projective representation must satisfy the following rule:

$$\frac{\omega(g,h)\omega(gh,k)}{\omega(g,hk)\omega(h,k)} = 1, \quad (A4)$$

i.e. it must be a 2-cocycle. While elements of the second cohomology group $$H^2(G, U(1))$$ correspond to projective representations of $$G$$, elements of the third cohomology group $$H^3(G, U(1))$$ correspond to gerbal representations of $$G$$. A gerbal representation associates an operator $$w(g,h)$$ to each pair of group elements $$g, h$$ rather than to a single group element. $$w(g,h)$$ does not act on a vector space, but on a space of functors of an abelian category. (A category consists of objects linked by arrows, also known as morphisms. There exists an identity arrow for each object, and a binary operation of to compose arrows associatively. An abelian category is one in which the objects and morphisms can be added. A functor is a homomorphism between categories.)

First, we need to consider another, “auxiliary” representation of $$G$$ that is a representation over an abelian category. In that representation, $$g \in G$$ is associated with a functor $$f_g$$. The functor $$f_g$$ essentially behaves as function $$f_g()$$ with the peculiar feature that the composition $$f_g \circ f_h() = f_g(f_h())$$ does not live in the same space as $$f_g$$. $$f_e$$ is the identity map (i.e. being the identity element of $$G$$). Since we cannot demand $$f_g \circ f_h$$ be equal to $$f_{gh}$$, we instead demand that they be related by an isomorphism. The gerbal representation operator $$w(g,h)$$ is then defined to be the isomorphism map, i.e. $$w(g,h) : f_g \circ f_h \leftrightarrow f_{gh}$$. When acting on compositions of many functors the action is defined to be $$w(g,h,k) : f_g \circ f_k \circ f_{hk} \leftrightarrow f_{gk}$$, and $$w(h,k) : f_h \circ f_k \leftrightarrow f_{hk}$$, and so on. From this we see that $$w(a,b)$$ commutes with $$w(c,d)$$ if $$a, b, c, d$$ are different group elements, since

$$w(a,b)w(c,d)(f_a \circ f_b \circ f_c \circ f_d) = w(c,d)w(a,b)(f_a \circ f_b \circ f_c \circ f_d) = f_{ab} \circ f_{cd} \quad (A5)$$

Now let us see how $$w(g,h)$$ represents the group $$G$$. For that, consider $$f_g \circ f_h \circ f_k$$ which is isomorphic to $$f_{ghk}$$. We can get $$f_{ghk}$$ by acting on $$f_g \circ f_h \circ f_k$$ with either $$w(g,h,k)w(g,h)$$ or $$w(g,hk)w(h,k)$$. We can demand they be equal, or we can relax that slightly and instead demand

$$w(g,h,k)w(g,h) = \alpha(g,h,k)w(g,hk)w(h,k), \quad (A6)$$

for some $$\alpha(g,h,k) \in U(1)$$ acting as the factor system.

We can derive a relation similar to Eq. (A4) that the $$\alpha$$ must satisfy, by considering that $$f_{abcd}$$ can be obtained by acting on $$f_a \circ f_b \circ f_c \circ f_d$$ with either $$w(a,b)w(ab,c)w(abcd)$$ or $$w(c,d)w(b,cd)w(a,bcd)$$. We can go $$w(a,b)w(ab,c)w(abcd) \leftrightarrow w(c,d)w(b,cd)w(a,bcd)$$ by repeatedly applying Eq. (A6) via two different routes (i.e. start from the left, or start from the right). In each case we obtain a prefactor, and we require both of them to be equal, leading to

$$\frac{\alpha(a,b,c)\alpha(a,bc,d)\alpha(b,c,d)}{\alpha(ab,c,d)\alpha(a,b,cd)} = 1, \quad (A7)$$

i.e. $$\alpha$$ is a 3-cocycle. As with projective representations, two gerbal representations are equivalent if they differ by only a phase, i.e. $$w$$ and $$v$$ are equivalent if $$v(g,h) = \chi(g,h)w(g,h)$$ for some $$\chi \in U(1)$$. The analogue of Eq. (A2), then, is that two gerbal representations are equivalent if their factor systems are related by a 3-coboundary,

$$\alpha'(g,h,k) = \frac{\chi(g,h)\chi(h,k)}{\chi(g,hk)\chi(h,k)} \alpha(g,h,k). \quad (A8)$$

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