Brody curves omitting hyperplanes

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Abstract

A Brody curve, a.k.a. normal curve, is a holomorphic map $f$ from the complex line $\mathbb{C}$ to the complex projective space $\mathbb{P}^n$ such that the family of its translations $\{z \mapsto f(z + a) : a \in \mathbb{C}\}$ is normal. We prove that Brody curves omitting $n$ hyperplanes in general position have growth order at most one, normal type. This generalizes a result of Clunie and Hayman who proved it for $n = 1$.

Introduction

We consider holomorphic curves $f : \mathbb{C} \to \mathbb{P}^n$. The spherical derivative $\|f'\|$ measures the length distortion from the Euclidean metric in $\mathbb{C}$ to the Fubini–Study metric in $\mathbb{P}^n$. The explicit expression is

$$\|f'\|^2 = \|f\|^{-4} \sum_{i \neq j} |f_i'f_j - f_if_j'|^2,$$

where $(f_0, \ldots, f_n)$ is a homogeneous representation of $f$ (that is the $f_j$ are entire functions which never simultaneously vanish), and

$$\|f\|^2 = \sum_{j=0}^n |f_j|^2.$$

A holomorphic curve is called a Brody curve if its spherical derivative is bounded. This is equivalent to normality of the family of translations $\{z \mapsto f(z + a) : a \in \mathbb{C}\}$.

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Brody curves are important for at least two reasons. First one is the rescaling trick known as Zalcman’s lemma or Brody’s lemma: for every non-constant holomorphic curve $f: \mathbb{C} \to \mathbb{C}$ one can find a sequence of affine maps $a_k: \mathbb{C} \to \mathbb{C}$ such that the limit $f \circ a_k$ exists and is a non-constant Brody curve. Second reason is Gromov’s theory of mean dimension [4] in which a space of Brody curves is one of the main examples.

For the recent work on Brody curves we refer to [3, 10, 11, 12, 13]. A general reference for holomorphic curves is [6].

We recall that the Nevanlinna characteristic is defined by

$$T(r, f) = \int_0^r \frac{dt}{t} \left( \frac{1}{\pi} \int_{|z| \leq t} \|f'(z)\|^2 dm(z) \right),$$

where $dm$ is the area element in $\mathbb{C}$. So Brody curves have order at most two normal type, that is

$$T(r, f) = O(r^2). \tag{1}$$

Clunie and Hayman [2] found that Brody curves $\mathbb{C} \to \mathbb{P}^1$ omitting one point in $\mathbb{P}^1$ must have smaller order of growth:

$$T(r, f) = O(r). \tag{2}$$

A different proof of this fact is due to Pommerenke [8]. In this paper we prove that this phenomenon persists in all dimensions.

**Theorem.** Brody curves $f: \mathbb{C} \to \mathbb{P}^n$ omitting $n$ hyperplanes in general position satisfy (2).

Under the stronger assumption that a Brody curve omits $n+1$ hyperplanes in general position, the same conclusion was obtained by Berteloot and Duval [1] and Tsukamoto [11], with different proofs.

Combined with a result of Tsukamoto [10] our theorem implies

**Corollary.** Mean dimension in the sense of Gromov of the space of Brody curves in

$$\mathbb{P}^n \setminus \{n \text{ hyperplanes in general position}\}$$

is zero.

The condition that $n$ hyperplanes are omitted is exact: it is easy to show by direct computation that the curve $(f_0, f_1, 1, \ldots, 1)$, where $f_i$ are appropriately chosen entire functions such that $f_1/f_0$ is an elliptic function,
is a Brody curve, it omits $n - 1$ hyperplanes, and $T(r, f) \sim cr^2$, $r \to \infty$ where $c > 0$. This example will be discussed in the end of the paper.

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Preliminaries

Without loss of generality we assume that the omitted hyperplanes are given in the homogeneous coordinates by the equations \{\(w_j = 0\), \(1 \leq j \leq n\). We fix a homogeneous representation \((f_0, \ldots, f_n)\) of our curve, where \(f_j\) are entire functions without common zeros, and \(f_n = 1\). We assume without loss of generality that \(f_0(0) \neq 0\).

Then
\[
u = \log \sqrt{|f_0|^2 + \ldots + |f_n|^2}
\]
is a positive subharmonic function, and Jensen’s formula gives
\[
T(r, f) = \frac{1}{2\pi} \int_{-\pi}^{\pi} u(re^{i\theta})d\theta - u(0) = \int_0^r \frac{n(t)}{t}dt,
\]
where \(n(t) = \mu(\{z : |z| \leq t\})\), and \(\mu\) is the Riesz measure of \(u\), that is the measure with the density
\[
\frac{1}{2\pi} \Delta u = \frac{1}{\pi} \|f'|^2.
\]
Now positivity of \(u\) and (1) imply that all \(f_j\) are of order at most 2, normal type.

In particular,
\[
f_j = e^{P_j}, \quad 1 \leq j \leq n,
\]
where \(P_j\) are polynomials of degree at most two.

First we state a lemma which is the core of our arguments. It is a refined version of Lemma 1 in [2]. We denote by \(B(a, r)\) the open disc of radius \(r\) centered at the point \(a\).

**Lemma 1.** Let \(u\) be a non-negative harmonic function in the closure of the disc \(B(a, R)\), and assume that \(u(z_1) = 0\) for some point \(z_1 \in \partial B(a, R)\). Then
\[
|\nabla u(z_1)| \geq \frac{u(a)}{2R}.
\]
Proof. The function 
\[ b(r) = \min_{|z-a|=r} u(z) \]
is decreasing and \( b(R) = 0 \). Harnack’s inequality gives
\[ b(t) \geq \frac{R-t}{R+t} u(a), \quad 0 \leq t \leq R. \]
As
\[ b(t) = |b(R) - b(t)| \leq (R-t) \max_{[t,R]} |b'|, \]
we conclude that for every \( t \in (0, R) \) there exists \( r \in [t, R] \) such that
\[ |b'(r)| \geq \frac{1}{R-t} \frac{R-t}{R+t} u(a) = \frac{u(a)}{R+t}. \]
According to Hadamard’s three circle theorem, \( rb'(r) \) is a negative decreasing function, so
\[ |Rb'(R)| \geq |rb'(r)| \geq r \frac{u(a)}{R+t} \geq t \frac{u(a)}{R+t}, \]
and the last expression tends to \( u(a)/2 \) as \( t \to R \). So we have \( |b'(R)| \geq u(a)/(2R) \). On the other hand, \( |\nabla u(z_1)| \geq \left| \frac{du}{dn}(z_1) \right| \geq |b'(R)| \), where \( d/dn \) is the normal derivative. This completes the proof.

Proof of the theorem

We may assume without loss of generality that \( f_0 \) has at least one zero. Indeed, we can compose \( f \) with an automorphism of \( \mathbb{P}^n \), for example replace \( f_0 \) by \( f_0 + cf_1, \ c \in \mathbb{C} \) and leave all other \( f_j \) unchanged. This transformation changes neither the \( n \) omitted hyperplanes nor the rate of growth of \( T(r, f) \) and multiplies the spherical derivative by a bounded factor.

Put \( u_j = \log |f_j| \), and
\[ u^* = \max_{1 \leq j \leq n} u_j. \]
Here and in what follows max denotes the pointwise maximum of subharmonic functions. We are going to prove first that
\[ u_0(z) \leq u^*(z) + 4(n+1)|z| \sup_C \|f'\|. \quad (5) \]
for $|z|$ sufficiently large.

Let $a$ be a point such that $u_0(a) > u^*(a)$. Consider the maximal disc $B(a, R)$ centered at $a$ where the inequality $u_0(z) > u^*(z)$ still holds. If $z_0$ is a zero of $f_0$ then $u_0(z_0) = -\infty$ and we have

$$R \leq |a| + |z_0| \leq 2|a|,$$

for $|a| > |z_0|$. There is a point $z_1 \in \partial B(a, R)$ and an integer $k \in \{1, \ldots, n\}$ such that

$$u_0(z_1) = u^*(z_1) = u_k(z_1) \geq u_j(z_1),$$

for all $j \in \{1, \ldots, n\}$. Applying Lemma 1 to the positive harmonic function $u_0 - u_k$ in $B(a, R)$ we obtain

$$|\nabla (u_0 - u_k)(z_1)| \geq \frac{u_0(a) - u_k(a)}{2R},$$

or

$$u_0(a) \leq u_k(a) + 2R |\nabla u_0(z_1) - \nabla u_k(z_1)| .$$

On the other hand, $|f_0(z_1)| = |f_k(z_1)| \geq |f_j(z_1)|$ for all $j \in \{1, \ldots, n\}$, so

$$\|f'(z_1)\| \geq \frac{|f_0'(z_1)f_k(z_1) - f_0(z_1)f_k'(z_1)|}{|f_0(z_1)|^2 + \ldots + |f_n(z_1)|^2} \geq (n + 1)^{-1} \left| \frac{f_0'(z_1)}{f_0(z_1)} - \frac{f_k'(z_1)}{f_k(z_1)} \right| .$$

Combining (8), (9) and (6), and taking into account that $|\nabla \log |f|| = |f'/f|$, we obtain (5).

If all polynomials $P_j$ are linear then inequality (5) completes the proof. Suppose now that some $P_j$ is of degree 2.

Consider again the subharmonic functions $u_j = \log |f_j|$, $0 \leq j \leq n$. For each $j \in \{0, \ldots, n\}$, the family

$$\{r^{-2}u_j(rz) : r > 0\}$$

in uniformly bounded from above on compact subsets of the plane, and bounded from below at 0. By [5, Theorem 4.1.9] these families are normal (from every sequence one can choose a subsequence that converges in $L^1_{\text{loc}}$). Take a sequence $r_k$ such that

$$\lim_{k \to \infty} \frac{1}{r_k^2} \int_{-\pi}^{\pi} u(r_ke^{i\theta})d\theta > 0,$$
where $u$ is defined in (3). Such sequence exists because we assume that at least one of the $P_j$ is of degree two.

Then we choose a subsequence (still denoted by $r_k$) such that

$$r_k^{-2} u_j(r_k z) \to v_j, \quad 0 \leq j \leq n,$$

and $r_k^{-2} u(r_k z) \to v$, where $v_j, v$ are some subharmonic functions in $\mathbb{C}$. Then

$$v = \max\{v_0, \ldots, v_n\} \neq 0$$

is a non-negative subharmonic function. Let $\nu$ be the Riesz measure of $v$. Notice that $\nu \neq 0$ because $v$ is non-negative and $v \neq 0$. We have weak convergence

$$\nu = \lim_{k \to \infty} \mu_{r_k},$$

where

$$\mu_{r_k}(E) = r_k^{-2} \mu(r_k E)$$

for every Borel set $E$. Now (4) and the condition that $\|f'\|$ is bounded imply

**Lemma 2.** $\nu$ is absolutely continuous with respect to Lebesgue’s measure in the plane, with bounded density.

**Proof.** For every disc $B(a, \delta)$ we have

$$\nu(B(a, \delta)) \leq \liminf_{k \to \infty} r_k^{-2} \mu(r_k a, r_k \delta) \leq \delta^2 \sup_C \|f'\|^2.$$

Now we invoke our inequality (5). It implies that

$$v_0 \leq v^* = \max(v_1, \ldots, v_n),$$

so $v = v^*$. Thus the measure $\nu$ is supported by finitely many rays. This contradiction with Lemma 2 shows that all polynomials $P_j$ are in fact linear. This completes the proof.

**Example**

Let $\Gamma_0 = \{n + im : n, m \in \mathbb{Z}\}$ be the integer lattice in the plane, and $\Gamma_1 = \Gamma + (1 + i)/2$. For $j \in \{0, 1\}$, let $f_j$ be the Weierstrass canonical products of genus 2 with simple zeros on $\Gamma_j$. Then the $f_j$ are entire functions
of completely regular growth in the sense of Levin–Pfluger and their zeros satisfy the \( R \)-condition in [7, Theorem 5, Ch. 2]. This theorem of Levin implies that

\[
\log |f_j(re^{i\theta})| = (c + o(1))r^2,
\]

as \( r \to \infty \), \( re^{i\theta} \notin C_0 \) where \( C_0 \) is a union of discs of radius \( 1/4 \) centered at the zeros of \( f_j \). It follows that

\[
|f_0(z)|^2 + |f_1(z)|^2 \to \infty, \quad z \to \infty.
\]

Cauchy’s estimate for the derivative and (11) give

\[
\log |f_j'(z)| \leq (c + o(1))|z|^2, \quad z \to \infty.
\]

So for the curve \( f = (f_0, f_1, 1, \ldots, 1) \) we obtain

\[
\|f'\|^2 = \sum_{i \neq j} |f'_i f_j - f_i f'_j|^2 = \frac{(|f_0 f_1 - f_0 f'_1|^2 + n(|f_0'|^2 + |f_1'|^2))}{(|f_0|^2 + |f_1|^2)^2} \\
= \frac{|g'|^2}{(1 + |g|^2)^2} + o(1).
\]

The spherical derivative of \( g \) is bounded because \( g \) is an elliptic function. Thus \( f \) is a Brody curve that omits \( n - 1 \) hyperplanes in general position. Evidently \( T(r, f) \sim c_1 r^2 \).

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