Worldline Path Integrals for Fermions with General Couplings †

Eric D’Hoker and Darius G. Gagné *

Department of Physics and Astronomy
University of California, Los Angeles
Los Angeles, CA 90024, USA

Abstract

We derive a worldline path integral representation for the effective action of a multiplet of Dirac fermions coupled to the most general set of matrix-valued scalar, pseudoscalar, vector, axial vector and antisymmetric tensor background fields. By representing internal degrees of freedom in terms of worldline fermions as well, we obtain a formulation which manifestly exhibits chiral gauge invariance.

Dedicated to the Memory of Shechao Feng

† Research supported in part by National Science Foundation grant PHY-92-18990
* E-mail addresses: dhoker@physics.ucla.edu and gagne@physics.ucla.edu
1. Introduction

The worldline path integral reformulation of quantum field theory produces an efficient alternative way of evaluating one loop Feynman diagrams and effective actions. The worldline formalism has a long history [1]; its close connection with string theory was studied in [2] and its use in the study of anomalies was demonstrated in [3]. The approach has known a revival of interest over the past few years, once it was realized that it gives rise to an improved method for computing large numbers of complicated Feynman diagrams [4,5].

Most of the investigations, though, have concentrated on QED-type diagrams with electrons and photons, and on QCD-type diagrams with gluons, ghosts and vector-like quarks [4,5]; only recently has a stronger interest developed in effective actions for fermions coupled to other fields as well. The case of couplings to a single scalar and pseudoscalar field was presented in [6], while the case of matrix-valued scalar, pseudoscalar and vector gauge fields was treated in [7]; the couplings to a single scalar, pseudoscalar, Abelian vector and Abelian axial vector have been recently discussed in [8], while the antisymmetric tensor coupling is discussed in [9].

More specifically, in our preceding work in Ref. [7], we derived, in a systematic way, the worldline path integral representation of the (one-loop) effective action for a multiplet of Dirac fermions coupled to general matrix-valued scalar, pseudoscalar, and vector gauge background fields. In a perturbative expansion in weak background fields, the real part of the Euclidean effective action generates Feynman graphs with an even number of $\gamma_5$-vertices while the imaginary part generates graphs with an odd number of $\gamma_5$-vertices. Real and imaginary parts of the (Euclidean space-time) effective action are reformulated in terms of worldline fermion integrals with anti-periodic and periodic boundary conditions respectively.

In the present paper, we use and extend the technology developed in [7] to derive systematically the worldline path integral representation for the effective action of a multiplet of Dirac fermions coupled to the most general set of matrix-valued scalar, pseudoscalar, vector, axial vector and antisymmetric tensor background fields. The presence of both vector and axial vector couplings allows us to deal also with chiral fermions, which is required in the study and computation of Feynman amplitudes in electro-weak theory and in grand-unified models. With the further presence of antisymmetric tensor background fields, we complete the list of all possible couplings of fermions to fields of spin less than or equal to one. Furthermore, when fermions are coupled to background geometry, i.e. background gravity, couplings to antisymmetric fields naturally appear through the appearance of the
spin connection.

The effective action for fermions in the presence of this general set of background fields is again obtained as the sum of real and imaginary parts, which are represented as path integrals over worldline fermions with anti-periodic and periodic boundary conditions respectively. This sum over different boundary conditions is analogous to the sum over spin structures familiar from superstring theory rules.

Throughout, we shall seek to maintain chiral (gauge) invariance as much as is possible. The real part of the effective action admits a completely chiral invariant formulation, while the imaginary part – which gives rise, amongst other contributions, to chiral anomalies – admits a formulation that involves an interpolation between chiral invariant Lagrangians, but is not itself manifestly chiral invariant. The imaginary part also requires an additional insertion operator (analogous to the worldsheet supercurrent insertion in superstring theory [10]), which converts the worldline fermion zero modes into a space-time Levi-Civita $\varepsilon$ tensor.

The remainder of this paper is arranged as follows. In Sect. 2, we express the fermion effective action in terms of a functional determinant of the Dirac operator $\mathcal{O}$, suitably continued to Euclidean space-time. In Sect. 3, its real part is reformulated in terms of a path integral involving worldline fermions over anti-periodic boundary conditions, and with manifest chiral (gauge) invariance. In Sect. 4, its imaginary part is re-expressed in terms of an integral over worldline fermions involving period boundary conditions, but manifest chiral (gauge) invariance is lost here. In Sect. 5, we show how the internal degrees of freedom can be represented by a new set of worldline anti-commuting degrees of freedom. In this way, the final worldline Lagrangians may be expressed as $c$-numbers.

In Appendix A, we present an alternative reformulation of the imaginary part of the fermion effective action, in terms of a worldline path integral which is manifestly invariant under vector-like gauge transformations, and is closer in spirit to our construction in [7].

2. Effective Action

We study the effective action of a multiplet of $N$ Dirac fermions coupled to the an arbitrary set of matrix-valued scalar $\Phi(x)$, pseudoscalar $\Pi(x)$, vector $A_\mu(x)$, axial vector $B_\mu(x)$ and antisymmetric tensor $K_{\mu\nu}(x)$ background fields. The most general, CPT
invariant, classical fermion action is given by

$$S[\bar{\Psi}, \Phi, \Pi, A, B, K, \Psi] = \int d^4x \bar{\Psi}^I[i\partial - \Phi + i\gamma^5\Pi + A + \gamma^5B + i\gamma^\mu\gamma^\nu K_{\mu\nu}]IJ^J. \quad (2.1)$$

The background fields are Hermitian; coupling constants have been absorbed into the definition of the background fields; the superscripts, $I$ and $J$, refer to the internal quantum numbers of the fermion multiplet and of the matrix-valued background fields. We might expect the presence of a further coupling term in (2.1) of the form $\gamma^5\gamma^\mu\gamma^\nu K'_{\mu\nu}$. However, this term need not be included, as it is simply related to the $K_{\mu\nu}$ term already present in (2.1). This fact can be seen from the identity

$$i\gamma^\mu\gamma^\nu K_{\mu\nu} = \gamma^5\gamma^\mu\gamma^\nu \tilde{K}_{\mu\nu}, \quad \text{with} \quad \tilde{K}_{\mu\nu} \equiv \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}K^{\rho\sigma}, \quad (2.2)$$

where $\tilde{K}$ is the dual of $K$. The largest internal unitary symmetry group of the action $S$ is $U(N)_L \times U(N)_R$, obtained e.g. for zero background fields; more generally, we shall assume that a subgroup $G_L \times G_R$ of $U(N)_L \times U(N)_R$ is gauged. It is convenient to introduce chiral fields, which transform simply under $G_L \times G_R$, as follows

$$\Psi^L = \frac{1}{2}(1 + \gamma^5)\Psi, \quad A^L = A \pm B, \quad H = \Phi - i\Pi, \quad K^s = K - i\tilde{K}. \quad (2.3)$$

The complex antisymmetric tensor field $K^s_{\mu\nu}$ is “self-dual” with $\tilde{K}^s_{\mu\nu} = iK^s_{\mu\nu}$ and provides a decomposition of an arbitrary real antisymmetric tensor into complex conjugate self-dual and anti-self-dual components: $2K = K^s + (K^s)^\dagger$. Under $G_L \times G_R$, $\Psi^L$ and $\Psi^R$ transform as $N$-dimensional representations $T_L \otimes 1$ and $1 \otimes T_R$ respectively, while $H$ and $K^s$ transform as the $N \times N$ dimensional representation $T_L \otimes T_R^*$.

The effective action $W[\Phi, \Pi, A, B, K]$ for fermions in the presence of the above background fields may be defined by a functional determinant in Minkowski space-time

$$iW[\Phi, \Pi, A, B, K] = \log \text{Det} i[i\partial - \Phi + i\gamma^5\Pi + A + \gamma^5B + i\gamma^\mu\gamma^\nu K_{\mu\nu}]. \quad (2.4)$$

To utilize heat-kernel methods we first analytically continue the theory to Euclidean space as in [7]. Although the $\gamma$-matrices are unaffected by the continuation, it is useful to change notation to Hermitian generators of an Euclidean Clifford algebra, $(\gamma_E)_j \equiv i\gamma_j$, $(\gamma_E)_4 \equiv \gamma_0$, and $(\gamma_E)_5 \equiv \gamma_5$, satisfying $\{(\gamma_E)_a, (\gamma_E)_b\} = 2\delta_{ab}$ with $a, b = \mu, 5$. From the Wick-rotation, $t \to -it$, it follows that

$$\phi \to i\phi_E, \quad A \to iA_E, \quad B \to iB_E, \quad \gamma^\mu\gamma^\nu K_{\mu\nu} \to -(\gamma_E)_\mu(\gamma_E)_\nu K_{\mu\nu}. \quad (2.5)$$

* Here, the Minkowski space-time metric $\eta$ has signature $(+ - - -)$ and we make use of the standard conventions $\{\gamma^\mu, \gamma^\nu\} = 2\eta^\mu\nu$, $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$, and $\varepsilon_{0123} = -\varepsilon^{0123} = 1$. 

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The expression (2.4) for the fermion effective action translates into the following Euclidean space-time functional determinant form:

\[-W[\Phi, \Pi, A, B, K] = \log \text{Det} [O], \tag{2.6}\]

where the operator $O$ is defined by

\[O \equiv p^\dagger - i\Phi(x) - \gamma_5 \Pi(x) - A(x) - \gamma_5 B(x) + \gamma_\mu \gamma_\nu K_{\mu\nu}(x). \tag{2.7}\]

As in [7], it is convenient to split the effective action into its real and imaginary parts: $W = W_R + iW_I$. Under parity transformations, both contributions are separately invariant. A perturbative expansion in weak fields reveals that graphs with an even number of $\gamma_5$-vertices are real and contribute to $W_R$ while those with an odd number of $\gamma_5$-vertices are imaginary and contribute to $W_I$.

### 3. Worldline Path Integral for the Real Part of the Effective Action

We propose to obtain a worldline path integral formulation for $W_R$ with manifest chiral gauge invariance, and a worldline Lagrangian that is even in worldline fermions. First, we double the fermion system in terms of a Hermitian operator $\Sigma$, as in [7]:

\[W_R = -\frac{1}{2} \ln \text{Det}[O^\dagger O] = -\frac{1}{2} \ln \text{Det} [\Sigma], \quad \Sigma \equiv \begin{pmatrix} 0 & O \\ O^\dagger & 0 \end{pmatrix}. \tag{3.1}\]

We introduce six-dimensional Hermitian Euclidean $\Gamma_A$ matrices, satisfying $\{\Gamma_A, \Gamma_B\} = 2\delta_{AB}I_8$, for $A, B = 1, \ldots, 6$, and choose a basis where *

\[\Gamma_\mu \equiv \begin{pmatrix} 0 & \gamma_\mu \\ \gamma_\mu & 0 \end{pmatrix}, \quad \Gamma_5 \equiv \begin{pmatrix} 0 & \gamma_5 \\ \gamma_5 & 0 \end{pmatrix}, \quad \Gamma_6 \equiv \begin{pmatrix} 0 & iI_4 \\ -iI_4 & 0 \end{pmatrix}. \tag{3.2}\]

For later use, we also introduce the Hermitian matrix $\Gamma_7$ with

\[\Gamma_7 \equiv -i \prod_{A=1}^{6} \Gamma_A = \begin{pmatrix} I_4 & 0 \\ 0 & -I_4 \end{pmatrix}, \quad \{\Gamma_A, \Gamma_7\} = 0 \text{ for } A = 1, \ldots, 6. \tag{3.3}\]

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† Henceforth, space-time is taken to be Euclidean and the subscript $E$ is dropped. Also, $p = -i\partial$ is the momentum conjugate to $x$, both of which are Hermitian operators.

* We take a basis where $\gamma_5 = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}$. $I_d$ denotes the $d \times d$ identity matrix.
The operator $\Sigma$ is simply expressed in terms of $\Gamma$-matrices, in analogy with [7]:

$$\Sigma = \Gamma_\mu (p_\mu - A_\mu) - \Gamma_6 \Phi - \Gamma_5 \Pi - i \Gamma_\mu \Gamma_5 \Gamma_6 B_\mu - i \Gamma_\mu \Gamma_\nu \Gamma_6 K_{\mu\nu} \quad . \quad (3.4)$$

The form of $\Sigma$ in (3.4) does not exhibit manifest chiral covariance, since the vector and axial gauge fields multiply independent $\Gamma$-matrices. To circumvent the problem, we notice that $i \Gamma_5 \Gamma_6$ has eigenvalues 1 and $-1$, and that within the corresponding eigenspaces, gauge fields couple in a manifestly chiral way through $A \pm B$. Thus, to render the chiral covariance manifest, we re-label $i \Gamma_5 \Gamma_6$ as an internal quantum number. This reduces the Clifford representation in (3.4) to a four-dimensional one, but doubles the dimension of the internal quantum numbers. This re-interpretation is equivalent to re-labeling all right-handed Weyl fermions as charge conjugates of left-handed Weyl fermions.

To render this re-labeling more explicit, it is convenient to work in a new $\Gamma$-matrix basis, with matrices denoted by $M^{-1} \Gamma M$ and arranged so that $M^{-1} i \Gamma_5 \Gamma_6 M$ is simple:

$$M^{-1} i \Gamma_5 \Gamma_6 M = \begin{pmatrix} I_4 & 0 \\ 0 & -I_4 \end{pmatrix}, \quad M = \begin{pmatrix} I_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_2 \\ 0 & 0 & I_2 & 0 \\ 0 & I_2 & 0 & 0 \end{pmatrix}. \quad (3.5)$$

In this basis, the operator $\Sigma$ takes on a manifestly chiral invariant form, as can easily be seen by expressing it in terms of the chiral fields of (2.3):

$$M^{-1} \Sigma M = \begin{pmatrix} \gamma_\mu (p_\mu - A_\mu)^L & \gamma_5 (-i H + \frac{1}{2} \gamma_\mu \gamma_\nu K_{\mu\nu}^s) \\ \gamma_5 (i H^\dagger - \frac{1}{2} \gamma_\mu \gamma_\nu (K_{\mu\nu}^s)^\dagger) & \gamma_\mu (p_\mu - A_\mu^R) \end{pmatrix}. \quad (3.6)$$

To represent the operator $\Sigma$ in terms of a worldline path integral, we must find a representation of $\gamma_\mu$ and $\gamma_5$ in terms of coherent states. But, we know from the analysis of [7] that this representation cannot be achieved in terms of $4 \times 4$ matrices. Instead, we shall double up the fermion system once more to achieve a good coherent state representation of the Clifford algebra.

The simplest way to achieve this is by replacing the $4 \times 4$ Clifford matrices $\gamma$ everywhere in (3.6) by the $8 \times 8$ Clifford matrices $\Gamma$, as defined in (3.2). The new operator, thus obtained, will be denoted by $\tilde{\Sigma}$, and we have

$$W_R = -\frac{1}{4} \ln \text{Det} [\tilde{\Sigma}] \quad , \quad (3.7)$$

$$\tilde{\Sigma} = \Gamma_\mu (p_\mu - A_\mu) - \Gamma_5 \mathcal{H} - i \Gamma_\mu \Gamma_\nu \Gamma_5 K_{\mu\nu} \quad . \quad (3.8)$$
The $2N \times 2N$ Hermitian background fields are defined in terms of the chiral fields of (2.3)

\[
A_\mu \equiv \begin{pmatrix} A^L_\mu & 0 \\ 0 & A^R_\mu \end{pmatrix}, \quad H \equiv \begin{pmatrix} 0 & iH \\ -iH^\dagger & 0 \end{pmatrix}, \quad K_{\mu\nu} \equiv \begin{pmatrix} 0 & iK^s_{\mu\nu} \\ -i(K^s_{\mu\nu})^\dagger & 0 \end{pmatrix}. \quad (3.9)
\]

Finally, the operator $\tilde{\Sigma}$ is manifestly chiral gauge covariant, which will guarantee that the real part of the Euclidean effective action is manifestly invariant.

To obtain a well-defined heat-kernel regularization of the determinant defining $W_\Re$, we replace the self-adjoint operator $\tilde{\Sigma}$ by its square, which is automatically positive, and find \footnote{We denote by Tr functional traces of operators and reserve tr for traces of finite dimensional internal matrices.}

\[
W_\Re = -\frac{1}{8} \ln \det [\tilde{\Sigma}^2] = \frac{1}{8} \int_0^\infty \frac{dT}{T} \text{Tr} e^{-\frac{T}{4} \tilde{\Sigma}^2}, \quad (3.10)
\]

where the operator $\tilde{\Sigma}^2$ works out to be

\[
\tilde{\Sigma}^2 = (p - A)^2 + H^2 + \frac{1}{2} K_{\mu\nu} K_{\mu\nu} + \frac{i}{2} \Gamma_\mu \Gamma_\nu (F_{\mu\nu} + \{H, K_{\mu\nu}\}) + i[K_{\mu\rho}, K_{\nu\sigma}] + i\Gamma_\mu \Gamma_5 (D_\mu H + \{p_\nu - A_\nu, K_{\mu\nu}\}) - \frac{1}{2} \Gamma_{\mu\rho\sigma} \Gamma_5 D_\mu K_{\rho\sigma} - \frac{1}{4} \Gamma_{\mu\nu\rho\sigma} K_{\mu\nu} K_{\rho\sigma}. \quad (3.11)
\]

Here, $\Gamma_{A_1 \cdots A_k} \equiv \Gamma_{[A_1 \cdots A_k]}$ denotes the anti-symmetrized product of $k$ $\Gamma$ matrices. Covariant derivatives and fields strengths have been defined relative to the field $A_\mu$ :

\[
F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i [A_\mu, A_\nu], \quad D_\mu H = \partial_\mu H - i[A_\mu, H]. \quad (3.12)
\]

and similarly for $D_\mu K_{\rho\sigma}$. We have now achieved a second order formulation of the fermion determinant problem that manifestly exhibits the chiral gauge symmetries of the original Lagrangian while only involving terms that are even in the number of $\Gamma_a$ matrices $a = 1, \cdots, 5$, and it remains to derive a worldline path integral formulation for it.

In a preceding paper [7], we constructed a worldline fermion reformulation of the Clifford algebra with the help of the coherent state formalism. We showed that under this correspondence, $\Gamma_A \Gamma_B \rightarrow 2 \psi_A \psi_B$, when A and B are different; analogously, it is equally easy to show $\Gamma_A \Gamma_B \Gamma_C \Gamma_D \rightarrow 4 \psi_A \psi_B \psi_C \psi_D$, when A, B, C and D are all different. By inspection of (3.11), it is clear that these are the only Clifford multi-linears needed to express $W_\Re$ as a path integral involving worldline fermions. Since each term in $\tilde{\Sigma}^2$ is even in powers of $\Gamma$-matrices, we are guaranteed that our final worldline Lagrangian will have even Grassmann grading as shown in [7].
Denoting the dependence of the operator $\tilde{\Sigma}^2$ on momentum, background fields and the matrices $\Gamma_a$, $a = 1, \ldots, 5$, by $\tilde{\Sigma}^2(p, A, H, K, \Gamma_a)$, the trace of the exponential in (3.10) may easily be written with the help of the formalism of [7]:

$$\text{Tr} e^{-\frac{\xi}{4} T \tilde{\Sigma}^2} = \int \mathcal{D} p \mathcal{D} x \int \mathcal{D} \psi \mathcal{P} e^{\int_0^T d\tau \left[ i \dot{x}_\mu p_\mu - \frac{1}{2} \psi_a \dot{\psi}_a - \frac{\xi}{4} \tilde{\Sigma}^2(p, A, H, K, \sqrt{2} \psi_a) \right]}.$$ (3.13)

Here, $\mathcal{P}$ and tr denotes path ordering of and tracing over internal representation matrices. In Sect. 5, we shall show that the trace may be recast in terms of worldline Grassmann integrations as well. The boundary conditions on the worldline loop are periodic for $x$, while anti-periodic (AP) for $\psi_a$. Notice that $\psi_6$ does not enter in $\tilde{\Sigma}^2$ and may be integrated out of (3.13); thus, the measure reduces to $\mathcal{D} \psi = \mathcal{D} \psi_\mu \mathcal{D} \psi_5$.

The terms in the path integral which involve the momentum function $p$ may be rearranged by completing the square in $p$. Using the results of Appendix A in [7], the momentum may then be decoupled from all fields by a suitable shift $^{\dagger}$ and yields a simple field-independent normalization factor $\mathcal{N}$.

$$\mathcal{N} = \int \mathcal{D} p e^{-\frac{\xi}{4} \int_0^T d\tau \, \dot{p}^2(\tau)}$$ (3.14)

As a result, we find the following path integral representation for the real part $W_\Re$ of the effective action

$$W_\Re = \frac{1}{8} \int_0^\infty \frac{dT}{T} \mathcal{N} \int \mathcal{D} x \int \mathcal{D} \psi \mathcal{P} e^{-\int_0^T d\tau \mathcal{L}(\tau)}.$$ (3.15)

The worldline Lagrangian, $\mathcal{L}(\tau)$, is given by

$$\mathcal{L}(\tau) = \frac{\dot{x}^2}{2\xi} + \frac{1}{2} \psi_a \dot{\psi}_a - i \dot{x}_\mu A_\mu + \frac{\xi}{2} H^2 - \frac{\xi}{4} K_{\mu\nu} K_{\mu\nu} + i \psi_\mu \psi_5 (\xi D_\mu H + 2i \dot{x}_\nu K_{\mu\nu})$$

$$+ \frac{i}{2} \xi \psi_\mu \psi_\nu (F_{\mu\nu} + \{H, K_{\mu\nu}\}) - \xi \psi_\mu \psi_\nu \psi_\rho (\psi_5 D_\mu K_{\nu\rho} + \frac{1}{2} \psi_\sigma K_{\mu\nu} K_{\rho\sigma})$$ (3.16)

The path integral of (3.15-16) is manifestly chiral gauge invariant. When $H = K_{\mu\nu} = 0$, the fermion multiplet is massless, and the worldline Lagrangian in (3.16) correctly reduces to the standard form [5,7] for a $2N$-component multiplet of massless fermions coupled to a background non-Abelian vector gauge field, $A$.

$^{\dagger}$ Since the shift here involves worldline fermions it must actually be performed at the level of the $\Gamma$-matrices.
The perturbative expansion of the path integral (3.15) proceeds by decomposing $x(\tau) = x'(\tau) + x^o$, where the zero mode $x^o$ is constant and $x'$ is orthogonal to constants. The $x'$ and $\psi_A$ propagators are:

$$
\langle x'(\tau_1)x'(\tau_2) \rangle = \mathcal{E} \left( \frac{(\tau_1 - \tau_2)^2}{2T} - \frac{\mathcal{E}}{2} |\tau_1 - \tau_2| \right) + \text{constant}.
$$

$$
\langle \psi_{A_1}(\tau_1)\psi_{A_2}(\tau_2) \rangle = \frac{1}{2} \delta_{A_1A_2} \text{sign}(\tau_1 - \tau_2).
$$

The normalization of the $\mathcal{D}x$ integral produces an extra overall factor of $1/(2\pi\mathcal{E}T)^2$, while the fermion integral produces an extra overall normalization factor of 8, accounting for the dimensionality of the Clifford algebra.

4. Worldline Path Integral for the Imaginary Part of the Effective Action

In this Section, we propose to obtain a worldline path integral formulation for the imaginary part of the effective action $W_3$. While it is not possible to preserve manifest chiral invariance for $W_3$, (which generates, amongst other things, the chiral anomaly), we shall seek a worldline Lagrangian that is as close as possible to the one for $W_\mathcal{R}$, with as much manifest chiral symmetry as possible. First, we double the fermion system in terms of a non-Hermitian operator $\Omega$, as in [7]:

$$
W_3 = -\text{argDet}[\mathcal{O}] = -\frac{1}{2} \text{argDet} [\Omega], \quad \Omega \equiv \begin{pmatrix} 0 & \mathcal{O} \\ \mathcal{O} & 0 \end{pmatrix}.
$$

The matrix $\Omega$ is simply represented in terms of the $\Gamma$-matrices of (3.2-3) and we have

$$
\Omega = \Gamma_\mu (p_\mu - A_\mu) - \Gamma_7 \Gamma_6 \Phi - \Gamma_5 \Pi - i\Gamma_7 \Gamma_\mu \Gamma_5 \Gamma_6 B_\mu - i\Gamma_7 \Gamma_\mu \Gamma_\nu \Gamma_6 K_{\mu\nu}
\quad = \frac{1}{2} (\Sigma - \Gamma_6 \Sigma \Gamma_6) + \frac{1}{2} \Gamma_7 (\Sigma + \Gamma_6 \Sigma \Gamma_6).
$$

Next, as in Sect. 3, we need to re-interpret $i\Gamma_5 \Gamma_6$ as an internal quantum number, which will reduce the dimension of the Clifford matrices back to $4 \times 4$, and which will double the dimension of the internal degrees of freedom. Then, as in Sect. 3, we shall have to double the system once more to obtain a good coherent state representation for the Clifford algebra. In the present case of the operator $\Omega$, this procedure is complicated by the presence of $\Gamma_7$, and some additional care is needed here.

To exhibit these operations most clearly, it turns out to be most convenient to double the system first, and then transform to a basis where the $\pm 1$ eigenvalues of the doubled
\[ i \Gamma_5 \Gamma_6 \text{ are arranged in } 8 \times 8 \text{ blocks. This may be achieved in the following way} \]
\[ \tilde{\Omega} = \tilde{M}^{-1} \begin{pmatrix} M^{-1} \Omega M & 0 \\ 0 & M^{-1} \Omega M \end{pmatrix} \tilde{M} , \quad \tilde{M} = \begin{pmatrix} I_4 & 0 & 0 & 0 \\ 0 & 0 & I_4 & 0 \\ 0 & I_4 & 0 & 0 \\ 0 & 0 & 0 & I_4 \end{pmatrix} . \] (4.3)

Once this form is obtained, we may re-label the eigenvalues of the doubled \( i \Gamma_5 \Gamma_6 \) as internal quantum numbers, and re-express the operator \( \tilde{\Omega} \) in terms of the \( 8 \times 8 \) \( \Gamma \)-matrices, defined in (3.2) and the chiral fields defined in (3.9). The result is conveniently re-expressed in terms of the operator \( \tilde{\Sigma} \) of (3.8)
\[ \tilde{\Omega} = \frac{1}{2} (\Sigma - \Sigma^c) i \Gamma_6 \Gamma_7 + \frac{i}{2} \Gamma_5 \Gamma_6 \Gamma_7 \chi (\tilde{\Sigma} + \tilde{\Sigma}^c) i \Gamma_6 \Gamma_7 . \] (4.4)

Here, the operator \( \tilde{\Sigma}^c \) stands for \( \tilde{\Sigma} \) of (3.8), in which the fields \( A, \mathcal{H} \) and \( K \) have been replaced by their chiral conjugates. This transformation amounts to letting \( A^{L,R}_\mu \mapsto A^{R,L}_\mu \), \( H \mapsto -H^\dagger \) and \( K^s_{\mu \nu} \mapsto -(K^s_{\mu \nu})^\dagger \) and yields the following fields
\[ A^c_\mu \equiv \begin{pmatrix} A^R_\mu & 0 \\ 0 & A^L_\mu \end{pmatrix} , \quad \mathcal{H}^c \equiv \begin{pmatrix} 0 & -iH^\dagger \\ iH & 0 \end{pmatrix} , \quad K^{c c}_{\mu \nu} \equiv \begin{pmatrix} 0 & -i(K^s_{\mu \nu})^\dagger \\ iK^s_{\mu \nu} & 0 \end{pmatrix} . \] (4.5)

The matrix \( \chi \) acts only on internal degrees of freedom, and in this basis is given by
\[ \chi = \begin{pmatrix} I_N & 0 \\ 0 & -I_N \end{pmatrix} . \] (4.6)

We are now in a position to evaluate the functional determinant of (4.1). As in [7], we shall make use of the heat-kernel regularization for the arg-function,
\[ W_3 = -\frac{1}{8} \text{arg Det}[\tilde{\Omega}^2] = \frac{i \mathcal{E}}{64} \int^1_{-1} d\alpha \int_0^\infty dT \text{ tr} \{(\tilde{\Omega}^2 - \tilde{\Omega}^\dagger)^2 e^{-\frac{T}{2} \tilde{\Sigma}^2_{(\alpha)}} \} . \] (4.7)

The new operator \( \tilde{\Sigma}^2_{(\alpha)} \) depends on the parameter \( \alpha \), and is defined by
\[ \tilde{\Sigma}^2_{(\alpha)} = \frac{1}{4}(\tilde{\Omega} + \tilde{\Omega}^\dagger)^2 + \frac{\alpha}{2} [\tilde{\Omega}, \tilde{\Omega}^\dagger] - \frac{\alpha^2}{4}(\tilde{\Omega} - \tilde{\Omega}^\dagger)^2 \] (4.8)

To compute \( \tilde{\Sigma}^2_{(\alpha)} \), it is helpful to observe that
\[ \tilde{\Omega}^\dagger = \frac{1}{2}(\tilde{\Sigma} - \tilde{\Sigma}^c) i \Gamma_6 \Gamma_7 - \frac{1}{2} \Gamma_5 \Gamma_6 \Gamma_7 \chi (\tilde{\Sigma} + \tilde{\Sigma}^c) i \Gamma_6 \Gamma_7 . \] (4.9)
Furthermore, as explained in [7], due to the symmetry of the $\alpha$-integration, the factor of $i\Gamma_5\Gamma_6\Gamma_7\chi$, which appears in the commutator $[\tilde{\Omega}, \tilde{\Omega}^\dagger]$, is immaterial and may be omitted in the calculation of $\tilde{\Sigma}^2_{(\alpha)}$. Finally, the overall factor of $i\Gamma_6\Gamma_7$ cancels out of the square in $\tilde{\Sigma}^2_{(\alpha)}$. Putting all together, we discover that $\tilde{\Sigma}^2_{(\alpha)}$ in the integration in (4.7), can be re-expressed in terms of $\tilde{\Sigma}$ of (3.8), but where the fields $\Phi$, $B$ and $K$ have been rescaled by a factor of $\alpha$:

$$
\tilde{\Sigma}^2_{(\alpha)} = \tilde{\Sigma}^2|_{\Phi \to \alpha\Phi, \ B \to \alpha B, \ K \to \alpha K}.
$$

(4.10)

Notice that for $\alpha = 1$, we recover the original fields, and the original operator $\tilde{\Sigma}^2$ of (3.11), while for $\alpha = -1$, we recover all fields with chirality reversed, and the operator $(\tilde{\Sigma}^c)^2$, defined through (4.5).

Next, we derive an expression for the insertion operator $(\tilde{\Omega}^2 - \tilde{\Omega}^\dagger)^2$ in (4.7). From the structure of $\tilde{\Omega}$ in (4.4), it appears natural to break up the operator as

$$
\tilde{\Omega}^2 - \tilde{\Omega}^\dagger^2 = i \Gamma_7 \chi(p, A, H, K, \Gamma_a)\Gamma_6,
$$

$$
\omega = \Gamma_5[\tilde{\Sigma}, \tilde{\Sigma}^c].
$$

(4.11)

The reduced insertion operator $\omega$ may easily be worked out in terms of the chiral fields of (3.9), and we find

$$
\omega = \Gamma_\mu (i\{D^c_\mu, H\} - i\{D_\mu, H^c\} + \{D^c_\nu, K_{\mu\nu}\} - \{D_\nu, K^{c\mu}_{\nu}\}) - \frac{1}{4} \Gamma_{\mu\nu\rho\sigma} \Gamma_5[K_{\mu\nu}, K^{c\rho\sigma}]
$$

$$
+ \Gamma_5(-[D_\mu, D^c_\mu] + [H, H^c] + \frac{1}{2}[K_{\mu\nu}, K^{c\mu}_{\nu}]) + \frac{1}{2} \Gamma_{\mu\nu\rho} (D_\mu K^{c\rho\nu} - D^c_\mu K_{\nu\rho})
$$

$$
+ \Gamma_{\mu\nu} \Gamma_5(-\{D_\mu, D^c_\nu\} + \frac{i}{2}[H, K^{c\mu}_{\nu}] - \frac{i}{2}[H^c, K_{\mu\nu}] - \{K_{\mu\rho}, K^{c\rho\nu}\})
$$

(4.12)

The chiral fields $A$, $H$ and $K$ were defined in (3.9), while their charge conjugates are given in (4.5). Covariant derivatives $D_\mu$ are taken with respect to $A$, while $D^c_\mu$ is taken with respect to $A^c$.

The heat-kernel representation of (4.7), with the above results giving $\tilde{\Sigma}^2_{(\alpha)}$ and the insertion $\tilde{\Omega}^2 - \tilde{\Omega}^\dagger^2$ in terms of $\Gamma$-matrices, allows for an immediate coherent state reformulation. As we have shown in [7], $\Gamma_A$, $A = 1, \cdots, 6$ is represented by $\sqrt{2}\psi_A$, while the overall factor of $\Gamma_7$ acts as a fermion number operator and flips the worldsheet fermion boundary conditions to periodic ones. Putting all together, we obtain

$$
W_{\tilde{\Sigma}} = -\frac{i\mathcal{E}}{64} \int_{-1}^{1} d\alpha \int_{0}^{\infty} dT \int DpDx \int_{p} D\psi \text{tr} \chi(\omega(p, A, H, K, \sqrt{2}\psi_a)\sqrt{2}\psi_a)((\tau = 0)
$$

$$
\times \mathcal{P} \exp \left\{ \int_{0}^{T} d\tau \left[ i\dot{x}_\mu p_{\mu} - \frac{1}{2} \psi_A \dot{\psi}_A - \frac{\mathcal{E}}{2} \tilde{\Sigma}^2_{(\alpha)}(p, A, H, K, \sqrt{2}\psi_a) \right] \right\}.
$$

(4.13)
The boundary conditions on the worldline loop are periodic (P) for both $x$ and $\psi_A$. The $\psi_6$ insertion may be integrated since it is absent from $\hat{\Sigma}_{(\alpha)}^2$ and $\omega$; this allows us to change $\psi_A \tilde{\psi}_A \rightarrow \tilde{\psi}_a \tilde{\psi}_a$ and view the measure as $\mathcal{D} \psi = \mathcal{D} \psi_\mu \mathcal{D} \psi_5$. Now by shifting the momentum integration as in the Sect. 3, we obtain

$$W_3 = \frac{-i\mathcal{E}}{64} \int_{-1}^{1} d\alpha \int_{0}^{\infty} dT \mathcal{N} \int_{P} \mathcal{D}x \mathcal{D} \psi \text{tr} \chi \tilde{\omega}(0) \mathcal{P} e^{-\int_{0}^{T} d\tau \mathcal{L}_{(\alpha)}^2(\tau)}.$$  

(4.14)

The worldline Lagrangian, $\mathcal{L}_{(\alpha)}(\tau)$, is given in terms of the worldline Lagrangian of the previous Section, $\mathcal{L}(\tau)$, by

$$\mathcal{L}_{(\alpha)}(\tau) = \mathcal{L}(\tau) \bigg|_{\Phi \rightarrow \alpha \Phi , \ H \rightarrow \alpha H , \ K \rightarrow \alpha K}.$$  

(4.15)

The insertion operator, $\bar{\omega}$, is given by

$$\bar{\omega} = \sqrt{2} \omega(p_\mu \rightarrow \left[ \frac{i\hat{\mu}}{\mathcal{E}} \right] (\alpha + A_\mu + \alpha_- A_\mu^c) + i\Gamma_5 \Gamma_5(\alpha_+ \kappa_{\mu\nu} + \alpha_- \kappa_{\mu\nu}^c), A, H, K, \Gamma_\alpha \bigg)$$

(4.16a)

followed by $\Gamma_\alpha \rightarrow \sqrt{2} \tilde{\psi}_a$. This works out to be $^\dagger$

$$\begin{align*}
\bar{\omega} &= 2\psi_\mu \left( -\frac{2i\hat{\mu}}{\mathcal{E}} (H - H^c) - \frac{2i\nu}{\mathcal{E}} (K_{\mu\nu} - K_{\mu\nu}^c) - \{(A_\mu - A_{\mu}^c), (\alpha_+ H + \alpha_- H^c)\} \
&+ 2\psi_5 \left( -i\partial_\mu (A_\mu - A_{\mu}^c) + [A_\mu, A_{\mu}^c] + [H, H^c] + \frac{3}{2} [K_{\mu\nu}, K_{\mu\nu}^c] \right) \
&+ 2\psi_\mu \psi_\nu \psi_\rho \left( \mathcal{D}_\mu K_{\nu\rho}^c - \mathcal{D}_\mu K_{\nu\rho} + 2i \{(A_\mu - A_{\mu}^c), (\alpha_+ K_{\nu\rho} + \alpha_- K_{\nu\rho}^c)\} \right) \
&+ 2\psi_\mu \psi_\nu \psi_5 \left( \frac{4i\hat{\mu}}{\mathcal{E}} (A_\nu - A_{\nu}^c) + i[H, K_{\mu\nu}^c] - i[H^c, K_{\mu\nu}] - 4\{K_{\mu\rho}, K_{\rho\nu}^c\} \
&- 2i \{(H - H^c), (\alpha_+ K_{\mu\nu} + \alpha_- K_{\mu\nu}^c)\} - \psi_\rho \psi_\sigma \{K_{\mu\nu}, K_{\rho\sigma}^c\} \right)
\end{align*}$$

(4.16b)

with

$$\alpha_\pm \equiv \frac{1 \pm \alpha}{2}.$$  

(4.17)

It is clear that the introduction of the parameter $\alpha$ breaks the manifest chiral invariance, but it seems that this is the price to pay for obtaining a well-defined heat-kernel representation of $W_3$.

In [7], we discussed the perturbation expansion of (4.14), and we quote some of the result here. Formulas for the bosonic sector are given exactly as in (3.17). Writing $\psi_\Lambda(\tau) = \psi_\Lambda^0(\tau) + \psi_\Lambda^1(\tau)$ where $\psi_\Lambda^0 = 0$ and $\int_{0}^{T} d\tau \psi_\Lambda^1(\tau) = 0$, the worldline fermion propagator is

$$\langle \psi_{\Lambda_1}^0(\tau_1) \psi_{\Lambda_2}^0(\tau_2) \rangle = \frac{1}{2} \delta_{\Lambda_1 \Lambda_2} \left( \text{sign}(\tau_1 - \tau_2) - \frac{(\tau_1 - \tau_2)}{T} \right).$$  

(4.18)

$^\dagger$ In the insertion, the shift of the momentum by non-commuting background fields must always be performed under the anti-commutator.
The integration over $D\psi'$ produces an extra overall normalization factor of $-1$. The integration over the worldline fermion zero modes is given by

$$\int d^6\psi \psi_\mu^o \psi_\nu^o \psi_\sigma^o \psi_5^o \psi_6^o = \varepsilon_{\mu\nu\rho\sigma},$$

(4.19)

where our Euclidean space convention for the Levi-Civita tensor is $\varepsilon_{1234} = 1$.

5. Worldline Treatment for the Internal Degrees of Freedom

To complete our analysis, we derive the worldline formulation that also promotes the internal degrees of freedom into Grassmann-valued integrals. This representation has already been discussed extensively in the literature [1], but we shall give an independent derivation of it here. We begin by showing that the trace of the path ordered exponential over an $n \times n$ Hermitian traceless matrix $M(\tau)$,

$$z[M] \equiv \text{tr} \mathcal{P} e^{i \int_0^T d\tau M(\tau)},$$

(5.1)

may be re-expressed as a worldline path integral

$$z[M] = \left(\frac{\pi}{T}\right)^n \sum_{\varphi} \int AP \mathcal{D}\lambda^\dagger \mathcal{D}\lambda \ v^{i\varphi(\lambda^\dagger \lambda + \frac{\pi}{2} - 1)} e^{-\int_0^T d\tau \left[\lambda^\dagger \dot{\lambda} - i\lambda^\dagger M\lambda\right]}.$$

(5.2)

The summation symbol $\sum_{\varphi}$ may either be taken to be the integral over the angle $\varphi$ from 0 to $2\pi$, or, in the case of $n \times n$ matrices, the sum over the values $2\pi k/n$ with $k = 1, \ldots, n$, in either case, suitably normalized to unity. The $\varphi$-integration (or sum), must be included to properly project the intermediate states in the path integral on coherent states of occupation number 1. (When the discrete sum is used, this projection is analogous to the GSO projection in string theory.) The operator $\lambda^\dagger \lambda$ in the exponent of (5.2) may be inserted at any arbitrary value of $\tau$ which we shall take to be $\tau = 0$.

To show (5.1), we may perform the integration over $\lambda$ and $\lambda^\dagger$ explicitly and obtain:

$$z[M] = \left(\frac{\pi}{T}\right)^n \sum_{\varphi} \text{Det} \left[\frac{d}{d\tau} - M - \frac{i\varphi}{T}\right] v^{i\varphi(\frac{\pi}{2} - 1)}.$$

(5.3)

We evaluate the functional determinant in (5.3) as the infinite product of the eigenvalues of the operator $\frac{d}{d\tau} - M - \frac{i\varphi}{T}$ with anti-periodic boundary conditions on the eigenfunctions. This leads to

$$z[M] = \left(\frac{\pi}{T}\right)^n \frac{T}{2\pi} \sum_{\varphi} \prod_{k=1}^{n} \prod_{l=1}^{\infty} \left[1 - \frac{(m_k + \varphi)^2}{\pi^2(2l - 1)^2}\right] v^{i\varphi(\frac{\pi}{2} - 1)}.$$

(5.4)
where the \( m_k \) are defined by
\[
\mathcal{P} e^{i \int_0^T d\tau M(\tau)} = e^{i \text{Diag}(m_1, \ldots, m_n)}, \quad \sum_{k=1}^n m_k = 0. \tag{5.5}
\]

Finally, using the product formula for the cosine function, we easily obtain the desired result,
\[
z[M] = \sum_{\varphi} \prod_{k=1}^n [1 + e^{i \varphi} e^{i m_k}] e^{-i \varphi} = \text{tr} \mathcal{P} e^{i \int_0^T d\tau M(\tau)}. \tag{5.6}
\]

Now, we could have perfectly well used periodic boundary conditions for the integration over the worldline fermions \( \lambda \) and \( \lambda^\dagger \) in (5.2). In this case, the path integral (5.2) would reduce to the path ordered exponential (5.1) provided we replace the normalization of \((\pi/T)^n\) in (5.2) by \(- (2\pi^2)^n\).

If there is an insertion in the trace, our method easily generalizes to give
\[
\text{tr} \omega(0) \mathcal{P} e^{i \int_0^T d\tau M(\tau)} = N_{\text{BC}} \sum_{\varphi} \int_{\text{BC}} D\lambda^\dagger D\lambda \{ \lambda^\dagger \omega \lambda(0) \} e^{i \varphi (\lambda^\dagger \lambda + \frac{3}{2} - 1)} e^{- \int_0^T d\tau \left[ \lambda^\dagger \lambda - i \lambda^\dagger M \lambda \right]}, \tag{5.7}
\]
where \( N_{\text{BC}} = (\pi/T)^n \) or \(- (2\pi^2)^n\) depending on whether the boundary conditions for \( \lambda \) and \( \lambda^\dagger \) are anti-periodic or periodic, respectively.

We may now recast our previous results for the path integral reformulation of the one-loop effective action where the trace over internal degrees of freedom shall be represented by a worldline path integral over a multiplet of \( 2N \) worldline fermions, \( \lambda \). Combining the real and imaginary parts of the effective action obtained in the previous two Sections with a sum over the boundary conditions (BC) of the worldline fermions, namely periodic (P) or anti-periodic (AP) boundary conditions on the worldloop gives
\[
W[\Phi, \Pi, A, B, K] = \frac{1}{8} \int_0^\infty \frac{dT}{T} N \int D\lambda^\dagger D\lambda \sum_{\text{BC}} \int D\psi D\lambda^\dagger D\lambda \\
\times \sum_{\varphi} e^{i \varphi (\lambda^\dagger \lambda + N - 1)} I_{\text{BC}}(0) e^{- \int_0^T d\tau \left[ \mathcal{L}_K(s, \tau) + \lambda^\dagger \mathcal{L}_I(\alpha, \tau) \lambda \right]}, \tag{5.8}
\]
with the worldline insertions given by
\[
I_{\text{AP}}(0) = \left( \frac{\pi}{T} \right)^{2N} \delta_{\alpha, 1}, \tag{5.9a}
\]
and
\[
I_P(0) = -(2\pi^2)^{2N} \frac{ET}{8} \int_{-1}^1 d\alpha \lambda^\dagger \chi \omega \lambda(0). \tag{5.9b}
\]
Recall that $\bar{\omega}(\tau = 0)$ is the previous worldline insertion defined in (4.16) and $\chi$ is the fermion number defined in (4.6). The worldline Lagrangians are

$$L_{K}(s, \tau) = \frac{\dot{x}^2}{2E} + \frac{1}{2} \psi_{\mu} \dot{\psi}_{\mu} + \frac{1}{2} \psi_{5} \dot{\psi}_{5} + \lambda^\dagger \lambda, \quad (5.10a)$$

and

$$L_{I}(\alpha, \tau) = \left[ -i \dot{x}_{\mu} A_{\mu} + \frac{\mathcal{E}}{2} \mathcal{H}^2 - \frac{\mathcal{E}}{4} K_{\mu\nu} K_{\mu\nu} + i \psi_{\mu} \psi_{5} (\mathcal{E} \mathcal{D}_{\mu} \mathcal{H} + 2i \dot{x}_{\nu} K_{\mu\nu}) \right. \right.$$

$$\left. + \frac{i \mathcal{E}}{2} \psi_{\mu} \psi_{\nu} \left( \mathcal{F}_{\mu\nu} + \{ \mathcal{H}, K_{\mu\nu} \} \right) \right. \left. \right.$$

$$\left. - \mathcal{E} \psi_{\mu} \psi_{\nu} \psi_{\rho} \psi_{5} D_{\mu} K_{\nu\rho} + \frac{1}{2} \psi_{\sigma} \psi_{\mu\nu} K_{\rho\sigma} \right] \Phi \rightarrow \alpha \Phi, \ B \rightarrow \alpha B, \ K \rightarrow \alpha K. \quad (5.10b)$$

Letting $\lambda$ transform under the representation $(T_L \otimes 1) \oplus (1 \otimes T_R)$ of the chiral gauge group $G_L \times G_R$, it is clear that for $\alpha = 1$ these Lagrangians are manifestly $G_L \times G_R$ invariant.

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**Appendix A : Alternative Worldline Path Integral Reformulation**

The path integral reformulation of the effective action can be carried out in a way that does not exhibit chiral symmetry manifestly but makes the coupling to the background fields, $\Phi$, $\Pi$, $A$, $B$ and $K$, more explicit. We are again interested in the field theory defined classically by (2.1). Then from (3.1) and (4.1), we have for the Euclidean one-loop effective action

$$W[\Phi, \Pi, A, B, K] = -\frac{1}{2} \ln \text{Det}[\Sigma] - \frac{i}{2} \text{arg Det}[\Omega], \quad (A.1)$$

where explicitly, $\Sigma$ and $\Omega$ are given by (3.4) and (4.2) respectively. Regularizing the real and imaginary parts of (A.1) directly as in [7], gives

$$W = \frac{1}{4} \int_0^\infty \frac{dT}{T} \text{Tr} \left\{ e^{-\frac{\mathcal{E} T}{2} \Sigma^2} - \frac{\mathcal{E} T}{8} \int_{-1}^1 d\alpha (\Omega^2 - \Omega^\dagger 2) e^{-\frac{\mathcal{E} T}{4} \left( \frac{1}{2} (\Omega + \Omega^\dagger)^2 + \frac{3}{4} [\Omega, \Omega^\dagger] - \frac{3}{4} (\Omega - \Omega^\dagger)^2 \right) \right\}. \quad (A.2)$$

By introducing a multiplet of $N$ worldline fermions, the path integral reformulation [7] of (A.2) may be expressed with a sum over the boundary conditions (BC) of the worldline fermions, namely periodic (P) or anti-periodic boundary conditions (AP) on the worldloop:

$$W = \frac{1}{4} \int_{-1}^1 d\alpha \int_0^\infty \frac{dT}{T} N \int D\chi \sum_{\text{BC}} D\psi D\lambda^\dagger D\lambda \sum_\varphi e^{i\varphi \lambda \lambda^\dagger \lambda - 1} J_{\text{BC}}(0) e^{-\int_0^T d\tau L_{\alpha}(\tau)}, \quad (A.3)$$
with the worldline insertions given by

$$J_{\mathrm{AP}}(0) = \left( \frac{\pi}{T} \right)^N \delta(\alpha - 1) ,$$  \hspace{1cm} (A.4a)$$

and

$$J_{\mathrm{P}}(0) = \frac{i T}{2} (2\pi^2)^N \lambda \left[ \psi_\mu \psi_\nu (2E D_\nu K_{\mu \nu} - 2i \dot{x}_\mu \Phi - iE[\Pi, B_\mu]) + \varepsilon \psi_5 \psi_6 \left\{ \{ \Phi, \Pi \} - D_\mu B_\mu \right\} 
+ \psi_\mu \psi_\nu \varepsilon(2i\alpha[\Phi, K_{\mu \nu}] + 4\alpha\{K_{\mu \rho}, K_{\rho \nu}\}) + \psi_\mu \psi_5 \psi_6 \left\{ i\alpha[\Phi, B_\mu] + 2\alpha\{B_\nu, K_{\mu \nu}\} \right\} 
+ \psi_\mu \psi_\nu \psi_\rho \left( 4\dot{x}_\mu \psi_6 K_{\nu \rho} + 4\alpha \varepsilon \psi_\alpha [K_{\mu \nu}, K_{\rho \sigma}] - 2\alpha \varepsilon \psi_5 [B_\mu, K_{\nu \rho}] \right) 
+ \psi_\mu \psi_\nu \psi_5 \psi_6 \left( 4\dot{x}_\mu B_\nu + 2iE \{\Pi, K_{\mu \nu}\} \right) \right] \lambda(0) ,$$  \hspace{1cm} (A.4b)$$

and the worldline Lagrangian given by

$$L_\alpha(\tau) = \frac{x^2}{2\varepsilon} + \frac{1}{2} \psi_\mu \dot{\psi}_\mu + \frac{1}{2} \psi_5 \dot{\psi}_5 + \frac{1}{2} \psi_6 \dot{\psi}_6 + \lambda^\dagger \lambda 
+ \lambda^\dagger \left[ -i\dot{x}_\mu A_\mu + \frac{\varepsilon}{2} \Pi^2 + \frac{\varepsilon}{2} \alpha^2 \Phi^2 - \varepsilon \alpha^2 K_{\mu \nu} K_{\mu \nu} + \psi_\mu \psi_\nu \left( iD_\mu \Pi + i\alpha^2 \{\Phi, B_\mu\} \right) 
+ \alpha \psi_\mu \psi_6 \left( iE D_\mu \Phi - 4\dot{x}_\mu K_{\nu \rho} - iE \{\Pi, B_\mu\} \right) + \alpha \psi_5 \psi_6 \left( 2\dot{x}_\mu B_\nu - \varepsilon[\Phi, \Pi] \right) 
+ \frac{\varepsilon}{2} \psi_\mu \psi_\nu \left( iF_{\mu \nu} + \alpha^2 \{B_\mu, B_\nu\} + 2i\alpha^2 \{\Phi, K_{\mu \nu}\} \right) 
- 2\varepsilon \alpha \psi_\mu \psi_\nu \psi_\rho \psi_5 \left( D_\mu B_\nu - i[\Pi, K_{\mu \nu}] \right) - 2\varepsilon \alpha \psi_\mu \psi_\nu \psi_\rho \psi_6 D_\mu K_{\nu \rho} 
- 2\varepsilon \alpha^2 \psi_\mu \psi_\nu \psi_\rho \psi_5 \left( B_\mu, K_{\nu \rho} \right) - \varepsilon \alpha^2 \psi_\mu \psi_\nu \psi_\rho \psi_6 \left( K_{\mu \nu}, K_{\rho \sigma} \right) \right] \lambda ,$$  \hspace{1cm} (A.5)$$

where the covariant derivatives are now taken only with respect to the vector gauge field $A_\mu$, and $F_{\mu \nu}$ is the vector field strength

$$D_\mu = \partial_\mu - iA_\mu , \quad F_{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu] .$$  \hspace{1cm} (A.6)$$

We have checked that the imaginary part of (A.3) correctly reproduces the chiral anomaly in four dimensions. We have also checked that our Lagrangian (A.4b) matches term by term with the result proposed in [8] in the abelian limit (with $K = 0$). We unfortunately were unable to determine whether or not our result for the antisymmetric tensor couplings in (A.4b) agrees with the result presented in [9].

**References**

[1] R.P. Feynman, Phys. Rev. 80 (1950) 440; R.P. Feynman, Phys. Rev. 84 (1951) 108; J. Schwinger, Phys. Rev. 82 (1951) 664; E. S. Abers and B. W. Lee, Phys. Rep. 9C (1973) 1; R. Casalbuoni, J. Gomis and G. Longhi, Nuovo Cimento 24A (1974) 249; F. A. Berezin and M. S. Marinov, JETP Lett. 21 (1975) 320; R. Casalbuoni, Nuovo
Cimento 33A (1976) 389; L. Brink, P. DiVecchia and P. Howe, Nucl. Phys. B118 (1977) 76; A.P. Balachandran, P. Salomson, B. Skagerstam and J. Winnberg, Phys. Rev. D15 (1977) 2308; A. Barducci, R. Casalbuoni and L. Lusanna, Nucl. Phys. B124 (1977) 93; M.B. Halpern, P. Senjanovic and A. Jevicki, Phys. Rev. D16 (1977) 2476; M.B. Halpern and W. Siegel, Phys. Rev. D16 (1977) 2486; A. Barducci, F. Bordi and R. Casalbuoni, Nuovo Cimento 64B (1981) 287; D. Cangemi, E. D’Hoker and G. Dunne, Phys. Rev D51 (1995) R2513; A. G. Morgan, Phys. Lett. B351 (1995) 249.

[2] A. M. Polyakov, Gauge Fields and Strings, Harwood Academic Pub., 1987.

[3] S. Cecotti and L. Girardello, Phys. Lett. 110B (1982) 39; L. Alvarez-Gaumé, Commun. Math. Phys. 90 (1983) 161; L. Alvarez-Gaumé and E. Witten, Nucl. Phys. B234 (1983) 269; D. Friedan and P. Windey, Nucl. Phys. B235 (1984) 395.

[4] Z. Bern and D.A. Kosower, Phys. Rev. D38 (1988) 1888; Z. Bern and D.A. Kosower, Phys. Rev. Lett. 66 (1991) 1669; Z. Bern and D.A. Kosower, Nucl. Phys. B379 (1992) 451.

[5] M.J. Strassler, Nucl. Phys. B385 (1992) 145; D. G. C. McKeon, Ann. Phys. 224 (1993) 139; D. G. C. McKeon and A. Rebhan, Phys. Rev. D48 (1993) 2891.

[6] M. G. Schmidt and C. Schubert, Phys. Lett. 318 (1993) 438; M. G. Schmidt and C. Schubert, Phys. Lett. 331 (1994) 69. M. Mondragón, L. Nellen, M.G. Schmidt and C. Schubert, Phys. Lett. 351B (1995) 200.

[7] E. D’Hoker and D. G. Gagné, Preprint UCLA/95/TEP/22 (hep-th/9508131), UCLA, 1995.

[8] M. Mondragón, L. Nellen, M.G. Schmidt and C. Schubert, Preprint IASSNA-HEP-95/74, HD-THHEP-95-43, HUB-EP-95/17 (hep-th/9510036), Heidelberg, 1995.

[9] J.W. van Holten, Preprint NIKHEF-95-055 (hep-th/9510021), NIKHEF, 1995.

[10] M.B. Green, J.H. Schwarz and E. Witten, Superstring Theory, Cambridge University Press, 1987; E. D’Hoker and D.H. Phong, Rev. Mod. Phys. 60 (1988) 917.