An efficient finite strip procedure for initial post-buckling analysis of composite laminated members

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Abstract. An efficient procedure based on the semi-analytical finite strip method with newly introduced invariant matrices is developed to analyze the initial post-buckling of composite laminated members. The non-linear strain-displacement equations obtained from the Von-Karman assumption and three plate theories, which are classical thin plate theory, first-order shear deformation plate theory, and high-order shear deformation plate theory can be used to evaluate the initial post-buckling performance of the composite laminated members. According to the principle of the minimum potential energy, the formulations of the finite strip method can be deduced. In order to improve the computational efficiency, the pre-integrated invariant matrices are introduced, which can convert the complicated analytical integral calculation of the stiffness matrix into a relatively simple matrix multiplication calculation. Several benchmark problems are tested based on the proposed method and other conventional methods. The corresponding comparison results show that: (1) the proposed method is proved to be feasible and accurate for those three different theories; (2) compared with the other conventional finite strip methods, the proposed method is much more efficient since it requires the integration of the stiffness matrix only once no matter how many iterations are needed, (3) and the advantage of time-saving is increasingly remarkable as the number of iterations increases.

Key words: Finite strip method; Invariant matrix; Composite laminated members; Initial post-buckling

1. Introduction
Plate and thin-walled structure are main components of modern engineering structures, the primary failure mode of which can be attributed to the buckling instability [1]. Thus, it is important to accurately predict the buckling and post-buckling behavior of such structures [2], which has been analyzed by experimental, analytical and numerical methodologies [3]. What's more, the analysis and computation of the initial post-buckling behavior also is useful and important, to exploit the carrying capacity of plates or thin-walled structures [4, 5].

For the study of the initial post-buckling phenomenon of thin-walled structures, lots of finite strip methods have been developed and applied successfully [6]. The Semi-Analytical Finite Strip Method (SA-FSM) based on the harmonic functions satisfied the supported boundary condition and the interpolation functions from the Finite Element Method (FEM) was firstly developed and proved to be an efficient tool for the post-locally-buckled analysis of prismatic thin walled structures under end compression [7, 8]. For geometric nonlinear analysis of thin-walled structures the SA-FSM has been proposed by using the moderately large displacement assumption and non-linear strain-displacement...
relations, but linear curvature-displacement relations [9]. Then the SA-FSM is further developed with considering the effects of transverse shear deformation to analyze the large deflection [10], the post local buckling problems [11] of laminated composite plates with initial geometric imperfections [12] based on linear crosswise interpolation or cubic crosswise interpolation, and the results obtained by the SA-FSM and the spline FSM have been compared and discussed [13]. For the post-buckling analysis of laminated composite plates with initial geometric imperfection subjected to progressive end shortening, the higher-order SA-FSMS based on the higher-order shear deformation plate theory [14, 15, 16] and the SA-FSM for composite plates under combined compression and shear loading [17] are developed. The semi-energy SA-FSM for the post local buckling analysis of geometrically perfect thin-walled prismatic structures [18], the exact FSM for the buckling and initial post-buckling analyses of I-section struts based on the so-called full analytical method [19] and the semi-energy SA-FSM based on the concept of the first order shear deformation theory for the post-buckling solution for thin and relatively thick functionally graded plates [20] were developed and applied efficiently. Furthermore in order to decrease the computational complexity, the SA-FSM using parallel cloud computing for large displacement stability analysis of orthotropic prismatic shell structures has been discussed [21]. For the buckling problems of composite laminated cylinders subjected to deformation-dependent loads, which remains normal to the shell middle surface throughout the deformation process, the SA-FSM with polynomial functions in the meridional direction and truncated Fourier series in the circumferential direction has been presented recently [22]. To take a fully nonlinear compound strip with a transverse stiffener and non-uniform characteristics in the longitudinal direction into account, a new SA-FSM has been developed for geometric nonlinear static analysis of prismatic shells recently [23]. For the elastic-plastic large-deflection thin-walled structural stability problems of the folded-plate members, the SA-FSM [24] and the semi-energy SA-FSM [25] also have been developed and applied. Up to now, the SA-FSMs have become the powerful technology for the post buckling phenomenon of thin-walled structures.

Compare with the classical SA-FSM, the spline finite strip method (S-FSM) takes the place of the often used Fourier series by the spline function, in order to facilitate the description of local non-periodic buckles and oblique buckling modes [26, 27]. Because the S-FSM requires more unknown parameters than the SA-FSM, this method can be regarded as a compromise method between the SA-FSM and the FEM [28]. For the geometric nonlinear analysis of stiffened plates with arbitrary shape, the sub-parametric mapping technology and the S-FSM have been assembled [29]. In order to deal with the geometric nonlinear problems of the perforated flat and stiffened plates [30], the material inelastic subjects [31], and the inelastic buckling of the thin functionally graded material plates with cutout resting on an elastic foundation [32], the Isoparametric S-FSM has been developed. Two kinds of the relationships between the elastic force (or elastic deformation energy) and the nodal line displacements can be summarized from all of above FSMs, which are global forms [16, 17, 19] and incremental formulations respectively [21, 23, 26, 27, 31]. The incremental constitutive equation usually is used in the inelastic analysis [32]. For the global ones, the S-FSM often uses the numerical integration to obtain the global geometrical stiffness matrices. However, the analytical integration operator in the SA-FSM is applied even more frequently [33, 34]. Generally, the reliability of the S-FSM depends on its numerical integral accuracy [27], and the analytical integration in the SA-FSM usually yields high accuracy but needs huge amount of the symbolic or manual calculation [7, 9, 17] or the hybrid method of analytical integration of the trigonometric terms and Gauss quadrature integration in other terms considering the effects of numerical integral accuracy into account can be implemented [11, 14]. In order to achieve efficient post-buckling analysis of the thin-walled structures by the FSMs above, it is important to accurately and fast evaluate the elastic force (or elastic potential energy).

Fortunately we can find an efficacious procedure for evaluating the elastic forces and the elastic energy based on some invariant sparse matrices. The invariant sparse matrices are integrated in advance and have the property of transforming the evaluation of the elastic forces in a matrix multiplication process. With the assistance from the invariant sparse matrix, the full analytical
evaluated method of the stiffness matrices and the elastic energy of the SA-FSM will be developed for the post-buckling analysis of composite laminated members. The paper is organized as follows. In Section 2, the general theorem of the SA-FSM for buckling analysis of the composite laminated member is briefly described and the control equations of the structure will be given. The method based on the invariant matrices is then developed and discussed in Section 3. Numerical results calculated by the method proposed in this study and other state-of-the-art methods are presented in Section 4. Finally conclusions are given in Section 5.

2. Finite strip analysis
In this section, the fundamental equations of the large deflection plates are briefly outlined. The plates are assumed to be simply supported along all edges and the HSDPT is applied throughout this work. As a result of the HSDPT assumption, the normalcy condition is incorporated as the components \( u_0, \ v_0 \) and \( w_0 \) of displacement at a general point \((x, y, z)\) are

\[
\begin{align*}
    u_0(x, y, z) &= u(x, y) + \left(\theta_x - \chi \frac{4}{3} \left(\frac{z}{h}\right)^2 \left(\theta_y + \frac{\partial w(x, y)}{\partial y}\right)\right), \\
    v_0(x, y, z) &= v(x, y) + \left(\theta_y - \chi \frac{4}{3} \left(\frac{z}{h}\right)^2 \left(\theta_y + \frac{\partial w(x, y)}{\partial y}\right)\right), \\
    w_0(x, y, z) &= w(x, y),
\end{align*}
\]

where \( u, v, w, \ \theta_x, \) and \( \theta_y \) are a series of displacement components at the middle surfaces \((z=0)\), the strains of the plate associated with the displacement field given in equation (1) are

\[
\begin{align*}
    \varepsilon_x &= \frac{\partial u}{\partial x} + \frac{1}{2} \left(\frac{\partial w}{\partial x}\right)^2 + \frac{z}{2} \frac{\partial \theta_y}{\partial x} - \chi \frac{4}{3h^2} z^3 \left(\frac{\partial \theta_y}{\partial x} + \frac{\partial^2 w}{\partial x^2}\right), \\
    \varepsilon_y &= \frac{\partial v}{\partial y} + \frac{1}{2} \left(\frac{\partial w}{\partial y}\right)^2 + \frac{z}{2} \frac{\partial \theta_x}{\partial y} - \chi \frac{4}{3h^2} z^3 \left(\frac{\partial \theta_x}{\partial y} + \frac{\partial^2 w}{\partial y^2}\right), \\
    \gamma_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} + z \left(\frac{\partial \theta_x}{\partial y} + \frac{\partial \theta_y}{\partial x}\right) - \chi \frac{4}{3h^2} z^3 \left(\frac{\partial \theta_x}{\partial y} + \frac{\partial \theta_y}{\partial x}\right) + 2 \frac{\partial^3 w}{\partial x \partial y^2}, \\
    \gamma_{xz} &= \theta_x + \frac{\partial w}{\partial x} - \frac{4}{h} z^2 \theta_y, \\
    \gamma_{yz} &= \theta_y + \frac{\partial w}{\partial y} - \frac{4}{h} z^2 \theta_x,
\end{align*}
\]

where \( \chi \) is a tracer. If \( \chi = 1 \), equation (2) is for the case of the HSDPT which contains the same independent unknowns \((u, v, w, \ \theta_x, \) and \( \theta_y)\) as in the FSDPT. If \( \chi = 0 \), equation (2) is reduced to the case of the FSDPT. When \( \chi = 0 \), and the transverse shear strains can be ignored, an assumption \( \frac{\partial w}{\partial x} = -\theta_x = -\theta_y \) is introduced, equation (2) is reduced to the case of the CPT. It is generally considered that there are only three independent unknowns \((u, v, w)\) under the CPT.

2.1. Degree of freedom and shape function
In the finite strip method two right-handed coordinate systems, named the global and local systems, are used to separate a plate into many strips along longitudinal direction as shown in figure 1 (a). The global coordinate system is denoted as \(X-Y-Z\), with the \(Y\) axis parallel to the longitudinal axis of the
member. The local system is expressed as \(x-y-z\), which is always associated with a strip and \(z\) axis is perpendicular to the strip as shown in figure 1 (b). A numbering system of finite strip model is introduced as shown in figure 1 (a). One nodal line is arranged between each strip element, namely middle nodal line of a strip. Therefore the total number of the strips can be expressed as \(s\) and the total number of the nodal line is \(2s+1\) for the singular branched cross section member.

![Figure 1](image)

**Figure 1.** Coordinate systems and displacements vector.

Consider a rectangular plate made of composite materials which consists of arbitrary laminate layers. The length, width and total thickness of the plate are \(a\), \(B\) and \(h\). The analytical trigonometric functions of the longitudinal coordinate that satisfy the simply supported boundary condition of the loaded edges can be used to represent the strip’s deformation mode

\[
S_p = \sin \frac{p\pi y}{a}, \quad p = 1, 2, 3, \ldots, m, \tag{3}
\]

where \(p\) is the axial half-wave number, \(m\) is a finite positive integer which indicates the maximum half-wave number, \(y\) is the longitudinal coordinate in the local coordinate system.

In case of CPT, the shape function for the membrane DOFs \((u, v)\) uses a high-order polynomial interpolation function matrix along transverse direction

\[
h_i = \begin{bmatrix} h_1 & h_2 & h_3 \end{bmatrix}, \quad i = 1, 2, \ldots, m
\]

\[
h_1 = 1 - \frac{3x}{b} + \frac{2x^2}{b^2}, \quad h_2 = \frac{4x}{b} - \frac{4x^2}{b^2}, \quad h_3 = -\frac{x}{b} + \frac{2x^2}{b^2}, \tag{4}
\]

and six cubic polynomials selected as the shape functions to depict the bending displacement of the strip along transverse direction.
where $b$ is the width of the strip member as shown in figure 1 (b), $x$ is the horizontal coordinate in local coordinate.

Then by combining with the interpolation function and the shape function, the explicit expressions of two in-plane displacement vectors $u$, $v$, and one out-of-plane displacement $w$ can be given as follows in global system

$$[u,v,w] = \text{diag}(h_{L}S_{u}, h_{L}C_{v}, h_{w}S_{w})d,$$

where the shape function matrices $h_{L}$ and $h_{w}$ along $x$ direction, which correspond to the in-plane displacement vectors $d_{u}, d_{v}$, and the out-of-plane displacement vector $d_{w}$ in the nodal lines respectively, can be given by using equation (4) and equation (5) as

$$h_{L} = \begin{bmatrix} h_{1}, & h_{2}, & \cdots, & h_{m} \end{bmatrix},$$

$$h_{w} = \begin{bmatrix} \bar{h}_{1}, & \bar{h}_{2}, & \cdots, & \bar{h}_{m} \end{bmatrix}.$$

the shape function matrices $S_{u}$, $C_{v}$ and $S_{w}$ along $y$ direction, which correspond to the displacement vectors $d_{u}, d_{v}$, and $d_{w}$ in the nodal lines respectively, can be given by using equation (3) as

$$S_{u} = \text{diag}(S_{1}I_{3}, S_{2}I_{3}, \cdots, S_{m}I_{3}),$$

$$S_{w} = \text{diag}(S_{1}I_{6}, S_{2}I_{6}, \cdots, S_{m}I_{6}),$$

$$C_{v} = \text{diag}(C_{1}I_{3}, C_{2}I_{3}, \cdots, C_{m}I_{3}),$$

$$C_{p} = \cos\frac{p\pi y}{a}, \quad p = 1, 2, 3, \cdots, m$$

and $I_{j}$ is the $j \times j$ identity matrix, the displacement vector $d = [d_{u}^{T}, d_{v}^{T}, d_{w}^{T}]^{T}$ in the nodal lines for the strip can be given as

$$d_{u} = \begin{bmatrix} u_{1}^{1}, & u_{mid}^{1}, & u_{2}^{1}, & u_{mid}^{2}, & u_{mid}^{3}, & \cdots, & u_{m}^{1}, & u_{mid}^{m}, & u_{mid}^{m} \end{bmatrix}^{T},$$

$$d_{v} = \begin{bmatrix} v_{1}^{1}, & v_{mid}^{1}, & v_{2}^{1}, & v_{mid}^{2}, & v_{mid}^{3}, & \cdots, & v_{m}^{1}, & v_{mid}^{m}, & v_{mid}^{m} \end{bmatrix}^{T},$$

$$d_{w} = \begin{bmatrix} w_{1}, & \theta_{1}, & w_{mid}, & \theta_{mid}, & w_{1}, & \theta_{1}, & w_{2}, & \theta_{mid}, & \cdots, & w_{mid}, & \theta_{mid}, & w_{mid}, & \theta_{mid}, & \cdots, \theta_{mid}, \theta_{mid}, \theta_{mid} \end{bmatrix}^{T},$$

the subscripts $i$ and $j$ denote two nodal lines of one strip, $\text{mid}$ is middle nodal line of a strip, and $m$ is the maximum half-wave number employed in the analysis, which is a finite positive integer.

In case of FSDPT and HSDPT, the shape function for the all DOFs $(u, v, w, \theta_{x}, \theta_{y})$ uses a high-order polynomial interpolation function matrix as the equation(4) along transverse direction. Then the categorical expressions of two in-plane displacement vectors $u$, $v$, and three out-of-plane displacement $w$, $\theta_{x}$, and $\theta_{y}$ can be given as follows in global system
Similar to the equation (8), the shape function matrices along \( y \) direction, which correspond to the displacement vectors in the nodal lines, can be expressed as

\[
S_u = S_w = S_{\theta_x} = \text{diag}(S_{I_3}, S_{I_3}, \ldots, S_{I_3}),
\]

\[
C_v = C_{\theta_x} = \text{diag}(C_{I_3}, C_{I_3}, \ldots, C_{I_3}).
\]

The displacement vector \( \mathbf{d} = \left[ \mathbf{d}_u^T, \mathbf{d}_v^T, \mathbf{d}_w^T, \mathbf{d}_{\theta_x}^T, \mathbf{d}_{\theta_y}^T \right]^T \) in the nodal lines for the strip can be given as

\[
\mathbf{d}_S = \left[ \delta_1^i, \delta_{mid}^i, \delta_2^j, \delta_{mid}^j, \delta_2^j, \ldots, \delta_m^i, \delta_m^j \right]^T
\]

\[
(\delta = u, v, w, \theta_x, \theta_y),
\]

the subscripts \( i \) and \( j \) denote two nodal lines of one strip, \( mid \) is middle nodal line of a strip, and \( m \) is the maximum half-wave number employed in the analysis, which is a finite positive integer.

### 2.2. Elastic stiffness matrix

Under the three assumptions, combined with equation (6), the strain-displacement relationship of composite laminate plates in equation (2) can be rewritten uniformly as

\[
e_0 = \mathbf{D}(\mathbf{d}) \mathbf{d}.
\]

where \( \mathbf{D}(\mathbf{d}) \) represents the relationship between the strain and displacement vector. As for general elastic materials, the elastic deformation energy \( U = \frac{1}{2} \int \varepsilon_0^T \sigma \varepsilon \, dV \) can be expressed by using equation (13) as follows

\[
U = \frac{1}{2} \int \varepsilon_0^T Q \varepsilon_0 \, dV = \frac{1}{2} \mathbf{d}^T k_e(\mathbf{d}) \mathbf{d},
\]

where \( Q \) is elastic constant matrix of the composited laminates, which is determined by the number of layers of the laminated plate and the laying angle of each layer commonly. It is assumed that the laminate consists of orthotropic plates which are completely bonded together by the \( N \) layers. The stress-strain relationship of each layer can be expressed as

\[
\begin{bmatrix}
\sigma_x \\
\sigma_y \\
\tau_{xy}
\end{bmatrix} =
\begin{bmatrix}
\tilde{Q}_{11k} & \tilde{Q}_{12k} & \tilde{Q}_{16k} \\
\tilde{Q}_{21k} & \tilde{Q}_{22k} & \tilde{Q}_{26k} \\
\tilde{Q}_{61k} & \tilde{Q}_{62k} & \tilde{Q}_{66k}
\end{bmatrix}
\begin{bmatrix}
\varepsilon_x \\
\varepsilon_y \\
\gamma_{xy}
\end{bmatrix},
\]

\[
\begin{bmatrix}
\tau_{yz} \\
\tau_{sz}
\end{bmatrix} =
\begin{bmatrix}
\tilde{Q}_{44k} & \tilde{Q}_{45k} \\
\tilde{Q}_{54k} & \tilde{Q}_{55k}
\end{bmatrix}
\begin{bmatrix}
\gamma_{yz} \\
\gamma_{sz}
\end{bmatrix},
\]

where \( \tilde{Q}_k \) conversion elastic modulus matrix, that is,
\[
\bar{Q}_k = \begin{bmatrix}
\bar{Q}_{11k} & \bar{Q}_{12k} & \bar{Q}_{16k} & 0 & 0 \\
\bar{Q}_{21k} & \bar{Q}_{22k} & \bar{Q}_{26k} & 0 & 0 \\
\bar{Q}_{61k} & \bar{Q}_{62k} & \bar{Q}_{66k} & 0 & 0 \\
0 & 0 & 0 & \bar{Q}_{44k} & \bar{Q}_{45k} \\
0 & 0 & 0 & \bar{Q}_{54k} & \bar{Q}_{55k}
\end{bmatrix}
\]

(16)

and \(\bar{Q}_{jk}(i,j=1, 2, 4, 5, 6)\) is conversion elastic constant, which can be defined as

\[
\begin{bmatrix}
\bar{Q}_{11k} \\
\bar{Q}_{22k} \\
\bar{Q}_{66k} \\
\bar{Q}_{44k} \\
\bar{Q}_{55k}
\end{bmatrix} = \begin{bmatrix}
c^4 & 2c^2s^2 & s^4 & 4c^2s^2 \\
c^2s^2 & c^4 + s^4 & c^2s^2 & -4c^2s^2 \\
c^4 & 2c^2s^2 & c^4 & 4c^2s^2 \\
c^3s & cs^3 - c^3s & -cs^3 & -2cs(c^2 - s^2) \\
c^3s & cs^3 - c^3s & -c^3s & 2cs(c^2 - s^2) \\
c^2s^2 & -2c^2s^2 & c^2s^2 & (c^2 - s^2)^2
\end{bmatrix} \begin{bmatrix}
Q_{11o} \\
Q_{12o} \\
Q_{22o} \\
Q_{44o} \\
Q_{55o}
\end{bmatrix},
\]

(17-1)

\[
\begin{bmatrix}
\bar{Q}_{44k} \\
\bar{Q}_{55k}
\end{bmatrix} = \begin{bmatrix}
c^2 & s^2 \\
-cs & cs \\
s^2 & c^2
\end{bmatrix} \begin{bmatrix}
Q_{44o} \\
Q_{55o}
\end{bmatrix},
\]

(17-2)

where \(Q_{11o} = \frac{E_x}{1 - \nu_{yx} \nu_{yx}}, \ Q_{12o} = \frac{\nu_{yx} E_y}{1 - \nu_{yx} \nu_{yx}}, \ Q_{22o} = \frac{E_y}{1 - \nu_{yx} \nu_{yx}}, \) and \(Q_{44o} = G_{xy}, \ Q_{55o} = k_c G_{xz}, \)

\(k_c = k_c G_{xz}; \ c = \cos \theta_c; \ s = \sin \theta_c; \ k = 1, 2, \ldots, N;\)

in which \(E_x, E_y, G_{xy}, G_{yz}, G_{xz}, \nu_{xy}, \nu_{yx}\) are Young's modulus, shear modulus, and the Poisson's ratios of the plate, and \(\theta_c\) is the lamination angle of the \(k\)th layer with respect to the plate \(X\)-axis, \(k_c\) is shear correction factor, which is equal to 1 under the assumption of HSDPT, meaning that it is no need to correct, while under the FSDPT, \(k_c\) is equal to 5/6 for the Reissner theory or \(\pi^2/12\) for the Mindlin theory. \(Q_{44o}\) is the elastic constant of the single layer plate at an angle of \(\theta_c\) with respect to the \(X\)-axis. However, in the case of the CPT, the shear effect will be ignored, meaning that the stress-strain relationship just needs to be written as equation (15-1).

Therefore, the elastic stiffness matrix \(k_c(d)\) of the strip is expressed as

\[
k_c(d) = \int D^c(d)QD(d) dV.
\]

(18)

In summary, these three theories have the similarities. According to the equation (18), the value of elastic stiffness matrix \(k_c(d)\) will be shifted once the displacement vector \(d\) changes. When the unknown displacement vector \(d\) changes, since it exists in integral operation, the elastic stiffness matrix \(k_c(d)\) is bound to induce the integral again, which will be a very cumbersome process undoubtedly. In order to avoid this complicated work, the invariant matrices will be exploited in this paper, and the specific analysis process will be illustrated in Section 3. In addition, the invariant matrices are only used in the elastic stiffness matrix.
2.3. Geometric stiffness matrix

As shown in figure 1 (b), if it can be assumed that the strip is loaded with linearly varying edge tractions, the membrane compressive loads can be expressed as

\[ T_e = T_i - (T_i - T_j) \frac{x}{b}, \]  

(19)

where \( T_i \) and \( T_j \) are the forces in two nodes of the strip, \( b \) is the width of the strip, \( x \) is the transverse coordinate in local coordinate system. Similar to the deduction of the elastic stiffness matrix, the potential energy induced by the membrane compressive loads in HSDPT and FSDPT can be expressed as

\[ W = \int \frac{1}{2} T_x \left[ \left( \frac{\partial (u + z\theta)}{\partial y} \right)^2 + \left( \frac{\partial (v + z\theta)}{\partial y} \right)^2 + \left( \frac{\partial w}{\partial y} \right)^2 \right] dV \]  

(20)

\[ = \frac{1}{2} \mathbf{d}^T \left( \int T_e \mathbf{G}^T \mathbf{G} dV \right) \mathbf{d} \]

where \( \mathbf{G} \) defines the relationship between the second order strain components and the displacement vector, which dimension is \((5m \times 15m)\).

\[ \mathbf{G} = \text{diag}(h_x \frac{\partial \mathbf{S}_u}{\partial y}, h_x \frac{\partial \mathbf{C}_v}{\partial y}, h_x \frac{\partial \mathbf{S}_w}{\partial y}, z h_x \frac{\partial \mathbf{C}_{\theta_z}}{\partial y}, z h_x \frac{\partial \mathbf{S}_{\theta_z}}{\partial y}), \]  

(21)

the geometric stiffness matrix of the strip element can be expressed as

\[ \mathbf{k}_g = \left[ T \mathbf{G}^T \mathbf{G} dV \right]. \]  

(22)

In the assumption of HSDPT and FSDPT, the dimension of the geometric stiffness matrix is \((15m \times 15m)\), which is consistent with the elastic stiffness matrix in HSDPT and FSDPT.

Additional, under the hypothesis of CPT, the potential energy induced by the membrane compressive loads is expressed as

\[ W = \int \frac{1}{2} T_x \left[ \left( \frac{\partial u}{\partial y} \right)^2 + \left( \frac{\partial v}{\partial y} \right)^2 + \left( \frac{\partial w}{\partial y} \right)^2 \right] dV = \frac{1}{2} \mathbf{d}^T \left( \int T_e \mathbf{G}^T \mathbf{G} dV \right) \mathbf{d} \]  

(23)

where \( \mathbf{G} \) defines the relationship between the second order strain components and the displacement vector, which dimension is \((3m \times 12m)\).

\[ \mathbf{G} = \text{diag}(h_x \frac{\partial \mathbf{S}_u}{\partial y}, h_x \frac{\partial \mathbf{C}_v}{\partial y}, h_x \frac{\partial \mathbf{S}_w}{\partial y}). \]  

(24)

Therefore, the expression of the geometric stiffness matrix of the strip element in CPT is same as equation (18) in HSDPT and FSDPT, but with different dimension, which is \((12m \times 12m)\). The dimension is consistent with the elastic stiffness matrix in CPT. According to the condition of the displacement continuum and the coordinate transformation, the relationship between the displacement vectors in the nodal line from the local coordinate to the global coordinate and a rotation angle \( \alpha \) as shown in figure 2, can be determined by the following equation,

\[ \mathbf{d}^p_{\text{global}} = \mathbf{T} \mathbf{d}^p_{\text{local}}. \]  

(25)
In the circumstances of HSDPT and FSDPT, the displacement vector \( \mathbf{d}^p \) can be expressed as
\[
\mathbf{d}^p = \begin{bmatrix} u_i, u_{mid}, u_j, v_i, v_{mid}, v_j, w_i, w_{mid}, w_j, \vartheta_{x,i}, \vartheta_{x,mid}, \vartheta_{x,j}, \vartheta_{y,i}, \vartheta_{y,mid}, \vartheta_{y,j} \end{bmatrix}^T,
\]
and under the assumption of CPT, the displacement vector \( \mathbf{d}^p \) in equation (25) is written as
\[
\mathbf{d}^p = \begin{bmatrix} u_i, u_{mid}, u_j, v_i, v_{mid}, v_j, w_i, w_{mid}, \vartheta_{x,i}, \vartheta_{x,mid}, \vartheta_{x,j}, \vartheta_{y,i}, \vartheta_{y,mid}, \vartheta_{y,j} \end{bmatrix}^T. \]
In addition, the coordinate transformation matrix \( \mathbf{T} \) under different theories will be given in appendix.

2.4. The Control Equation of Thin-walled Member

In the case of three theoretical assumptions, according to virtual work principle, the global control equation of the composite laminated member can be obtained by using the elastic deformation energy in equation (14) and the compression potential energy in equation (20) under the HSDPT and FSDPT, or equation (23) under the CPT, that is
\[
(K_e(D) - K_g)D = 0,
\]
where \( K_e(D) \) and \( K_g \) are global elastic stiffness matrix and global geometric stiffness matrix, which can be deduced by the stiffness matrix \( k_e(d) \) and \( k_g \) respectively.

In the circumstances of HSDPT and FSDPT, \( D \) is \((15m \times 1)\) global displacement vector of the composite laminated plate, which can be expressed as
\[
D = \begin{bmatrix} \mathbf{D}^1_u, \mathbf{D}^2_u, \cdots, \mathbf{D}^m_u, \mathbf{D}^1_v, \mathbf{D}^2_v, \cdots, \mathbf{D}^m_v, \mathbf{D}^1_w, \mathbf{D}^2_w, \cdots, \mathbf{D}^m_w, \mathbf{D}^1_{\vartheta_x}, \mathbf{D}^2_{\vartheta_x}, \cdots, \mathbf{D}^m_{\vartheta_x}, \mathbf{D}^1_{\vartheta_y}, \mathbf{D}^2_{\vartheta_y}, \cdots, \mathbf{D}^m_{\vartheta_y} \end{bmatrix}^T,
\]
while under the assumption of CPT, the global displacement vector \( D \) in equation (27) is written as
\[
D = \begin{bmatrix} \mathbf{D}^1_u, \mathbf{D}^2_u, \cdots, \mathbf{D}^m_u, \mathbf{D}^1_v, \mathbf{D}^2_v, \cdots, \mathbf{D}^m_v, \mathbf{D}^1_w, \mathbf{D}^2_w, \cdots, \mathbf{D}^m_w \end{bmatrix}^T,
\]
which dimension is \((12m \times 1)\). In equation (28) and (27), \( \mathbf{D}^p_u \) and \( \mathbf{D}^p_v \) are the in-plane displacement vectors of the \( p \)th axial half-wave with respect to the global coordinate \( X \) and \( Y \) correspondingly, which is same in three theories. \( \mathbf{D}^p_{\vartheta_x}, \mathbf{D}^p_{\vartheta_y} \) and \( \mathbf{D}^p_w \) are the out-of-plane displacement vector of the \( p \)th axial half-wave with respect to the global coordinate \( Z \) under HSDPT and FSDPT, which dimension is \((1 \times 3)\), cause of \( \vartheta_x \) and \( \vartheta_y \) are independent of each other, that is
\[
\mathbf{D}^p_{\vartheta_x} = \begin{bmatrix} \delta_{x,i}, \delta_{x,mid}, \delta_{x,j} \end{bmatrix}^T (\delta=w, \vartheta_x, \vartheta_y), \]
while in the hypothesis of CPT, the dimension of out-of-plane displacement vector \( \mathbf{D}^p_w \) is \((1 \times 6)\), it can be expressed as
\[
\mathbf{D}^p_w = \begin{bmatrix} w_i, \vartheta_{x,i}, w_{mid}, \vartheta_{x,mid}, w_j, \vartheta_{x,j} \end{bmatrix}^T,
\]
and \( p = 1, 2, 3, \ldots, m \). Assuming \( T_0(x) \) is the initial axial force, the real axial force in the geometric stiffness matrix can be expressed as

\[
T(x) = \lambda T_0(x),
\]

(31)

where \( \lambda \) is the load factor. Therefore, the geometric stiffness matrix \( K_g \) can be rewritten as the proportionate function of the initial geometric stiffness matrix \( K_g \biggr|_{T_0} \) caused by initial axial force \( T_0(x) \) and the load factor \( \lambda \), that is

\[
K_g = \lambda K_g \biggr|_{T_0}.
\]

(32)

By substituting equation (32) into equation (26), the control equation of buckling analysis of the composite laminated plate can be rewritten as

\[
\left( K_e(D) - \lambda K_g \biggr|_{T_0} \right) D = 0.
\]

(33)

Equation (33) gives the nonlinear relationship of the global displacement vector \( D \) and the generalized axial force described by the load factor \( \lambda \) and the initial geometric stiffness matrix \( K_g \biggr|_{T_0} \).

For the initial post-buckling analysis of the composite laminated members, only the geometrical nonlinear strain-displacement relationship has been taken into account here.

3. Invariant Matrices for Stiffness Computation

In this section, the invariant matrices in the elastic stiffness matrix in these three theories will be extracted and utilized. According to the strain-displacement relationship given in equation (13) and equation (2), \( D(d) \) can be rewritten as

\[
D(d) = D_1 + D_2(d) + zD_3 + z^2D_4 + z^3D_5.
\]

(34)

Therefore, the elastic stiffness matrix \( k_e(d) \) of the strip under the HSDPT in equation (18) can be expressed as

\[
k_e(d) = \int \left( D_1 + D_2(d) + zD_3 + z^2D_4 + z^3D_5 \right)^T Q \left( D_1 + D_2(d) + zD_3 + z^2D_4 + z^3D_5 \right) dV,
\]

(35)

According to equation (35) and equation (15), it can be seen that \( Q \) is elastic constant matrix of the composited laminates. Thus, a series of coefficients which is superimposed along the thickness direction is defined as

\[
(A, B, C, D, E, F, H) = \int z Q(1, z, z^2, z^3, z^4, z^5, z^6)dz
\]

(36)

\[
= \sum_{k=1}^N \int_{z_k}^{z_{k+1}} \bar{Q}_k(1, z, z^2, z^3, z^4, z^5, z^6)dz.
\]

Substituted equation (36) into equation (35), the elastic stiffness matrix \( k_e(d) \) can be rewritten as

\[
k_e(d) = k_{e1} + k_{e2}(d) + k_{e3}(d)
\]

(37)

where

\[
k_{e1} = \int_z \left( D_1^T A D_1 + D_1^T B D_3 + D_1^T C D_4 + D_1^T D D_5 \right.
\]

\[
+ D_2^T B D_1 + D_2^T C D_3 + D_2^T D D_4 + D_2^T E D_5
\]

\[
+ D_3^T C D_1 + D_3^T D D_3 + D_3^T E D_4 + D_3^T F D_5
\]

\[
+ D_4^T D D_1 + D_4^T E D_3 + D_4^T F D_4 + D_4^T H D_5 \right) dS
\]

(38-1)
According to equation (37) it can be seen that the elastic stiffness matrix \( k_e(d) \) can be divided into three parts, i.e., \( k_{e1}, \ k_{e2}(d), \) and \( k_{e3}(d). \) While under the assumption of FSDPT and CPT, the variable parameter \( \chi \) in equation (2) is equal to 0, \( \mathbf{D}_4 \) and \( \mathbf{D}_5 \) will become zero matrices. Thus \( k_{e1}, \ k_{e2}(d), \) \( k_{e3}(d) \) will be simplified as

\[
k_{e1} = \int_S D_1^T \mathbf{A} D_1 + D_2^T \mathbf{B} D_3 + D_3^T \mathbf{C} D_4 + D_4^T \mathbf{D} D_5 \ dS,
\]
\[
k_{e2}(d) = \int_S D_1^T \mathbf{A} D_2 \ dS,
\]
\[
k_{e3}(d) = \int_S D_2^T \mathbf{A} D_2 \ dS.
\]  

Overall, the similarity exists in these three theories. According to the elastic stiffness matrix \( k_e(d) \) given in equation (38-1) to (38-3) in HSDPT, equation (39-1) to (39-3) in FSDPT and CPT, the first item \( k_{e1} \) in the right integral of the equal sign denotes the constant elastic stiffness coefficient, the second item \( k_{e2}(d) \) in the right side of the equal sign can take the first order nonlinear stiffness with respect to the displacement vector \( d \) into account, the third item \( k_{e3}(d) \) will express the effect of the second order nonlinear stiffness with respect to the displacement vector \( d. \)

Therefore, the unknown displacement vector \( d \) can be found clearly, which induces the integral will be changed and must be evaluated again in the initial post-buckling analysis of the member for the difference configurations. The invariant matrices will be used in this section, which will directly avoid the time consumption caused by repeated integration of stiffness matrix.

### 3.1. Invariant matrices for HSDPT

In order to narrate conveniently, it will be analyzed under the circumstances of HSDPT, firstly. According to the relationship of the strain-displacement of the strip element, it can be clearly shown that only \( \mathbf{D}_2(d) \) contains \( d, \) thus \( \mathbf{D}_2(d) \) can be written as

\[
\mathbf{D}_2(d) = \begin{bmatrix}
\mathbf{O}_{3\times 6m} & \mathbf{D}_2^w & \mathbf{O}_{3\times 6m} \\
\mathbf{O}_{2\times 6m} & \mathbf{O}_{2\times 3m} & \mathbf{O}_{2\times 6m}
\end{bmatrix},
\]

and

\[
\mathbf{D}_2^w = \begin{bmatrix}
\frac{1}{2} \left( \frac{\partial h_l}{\partial x} d_w \right) S_w \\
\frac{1}{2} \left( h_l \frac{\partial S_w}{\partial x} d_w \right) S_w \\
\frac{1}{2} \left( h_l \frac{\partial S_w}{\partial y} d_w \right) S_w
\end{bmatrix}.
\]
To analyze equation (40) and equation (41), it should be seen that $\mathbf{D}_2^w$ is variant matrix, it will be different because it needs the repeated integral calculation once the displacement vector changes. Therefore, in order to avoid this repeated integral calculation, it would take the unknown displacement vector $\mathbf{d}$ out the integral calculation. Firstly the three $1 \times 3m$ row matrices are defined,

$$
\xi = [\xi_1, \xi_2, \cdots, \xi_{3m}] = \frac{1}{2} S_w^T \frac{\partial h_i^T}{\partial x} \frac{\partial h_i^T}{\partial x} S_w,
$$

$$
\psi = [\psi_1, \psi_2, \cdots, \psi_{3m}] = \frac{1}{2} \frac{\partial S_w^T}{\partial y} h_i h_i \frac{\partial S_w}{\partial y},
$$

$$
\zeta = [\zeta_1, \zeta_2, \cdots, \zeta_{3m}] = S_w^T \frac{\partial h_i^T}{\partial x} h_i \frac{\partial S_w}{\partial y},
$$

where $\xi_i, \psi_i, \zeta_i (i = 1, 2, \cdots, 3m)$ are the column matrices, then equation (41) can be rewritten as

$$
\mathbf{D}_2^w = \begin{bmatrix}
(d_i^T \xi) & (d_i^T \psi) & (d_i^T \zeta)
\end{bmatrix}^T.
$$

Furthermore, since the transpose of the scalar quantity equals to itself, the items in the right side of the equal sign of equation (43) can be given as

$$
d_w^T \mathbf{U} = [u_1^T d_w, u_2^T d_w, \cdots, u_{3m}^T d_w] = [u_1^T, u_2^T, \cdots, u_{3m}^T] d_w,
$$

$$
(U = \xi, \psi, \zeta; u_i = \xi_i, \psi_i, \zeta_i; i = 1, 2, \cdots, 3m).
$$

Define the $9m^2 \times 3m$ nodal line coordinate matrix

$$
\mathbf{X}_w = \text{diag}(d_w, d_w, \ldots, d_w),
$$

the nodal line coordinate $d_w$ will be shift from the front to the back of the row matrices $\xi, \psi, \zeta$ in equation (46),

$$
\mathbf{D}_2^w = \mathbf{J} \mathbf{X}_w,
$$

where the $3 \times 9m^2$ spare matrix $\mathbf{J}$ can be defined as

$$
\mathbf{J} = \begin{bmatrix}
\mathbf{\xi}^T & \mathbf{\psi}^T & \mathbf{\zeta}^T
\end{bmatrix}^T,
$$

and

$$
\mathbf{U}_w = [u_1^T, u_2^T, \cdots, u_{3m}^T],
$$

$$
(U = \xi, \psi, \zeta; u_i = \xi_i, \psi_i, \zeta_i; i = 1, 2, \cdots, 3m).
$$

Thirdly, substituted the equation (46) into the first order nonlinear stiffness item $\mathbf{k}e_2(d)$, owing to elastic constant matrix $\mathbf{Q}$ is symmetrical, which leads to the a series of coefficients $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}, \mathbf{E}, \mathbf{F}, \mathbf{H})$ are also symmetric matrices, the intermediate items

$$
\left(\int s D_2^T \mathbf{A} D_2(d) dS\right)^T = \int s D_2^T (d) \mathbf{A} D_2(d) dS, \quad \left(\int s D_2^T (d) \mathbf{B} D_2(d) dS\right)^T = \int s D_2^T \mathbf{B} D_2(d) dS, \quad \text{and}
$$

$$
\left(\int s D_2^T (d) \mathbf{D} D_2(d) dS\right)^T = \int s D_2^T \mathbf{D} D_2(d) dS.
$$

In addition, according to specific expression form of $\mathbf{D}_1$ and $\mathbf{D}_2(d)$, it should be given that the intermediate items $\int s D_1^T \mathbf{C} D_2(d) dS$ and $\int s D_1^T \mathbf{D} D_2(d) dS$ in equation (38) are zero matrices $\mathbf{O}_{15m \times 15m}$. Therefore, the stiffness item $\mathbf{k}e_2(d)$ of equation (38) can be rewritten as
\[ \mathbf{ke}_2(d) = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \end{bmatrix}_{15m^2 \times 6m} \begin{bmatrix} \mathbf{I}_{n1} + \mathbf{I}_{n2} + \mathbf{I}_{n3} \\ \mathbf{9m^2 \times 3m} \end{bmatrix} \mathbf{X}_w \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \end{bmatrix}_{15m^2 \times 6m} + \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \end{bmatrix}_{15m^2 \times 6m} \begin{bmatrix} \mathbf{I}_{n1} + \mathbf{I}_{n2} + \mathbf{I}_{n3} \\ \mathbf{9m^2 \times 3m} \end{bmatrix} \mathbf{X}_w \mathbf{e}^T \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \end{bmatrix}_{15m^2 \times 6m}, \]

where

\[ \mathbf{I}_{n1} = \int_S \begin{bmatrix} \mathbf{D}_1^T \mathbf{A} \end{bmatrix} \mathbf{J} dS, \quad \mathbf{I}_{n2} = \int_S \begin{bmatrix} \mathbf{D}_2^T \mathbf{B} \end{bmatrix} \mathbf{J} dS, \quad \mathbf{I}_{n3} = \int_S \begin{bmatrix} \mathbf{D}_3^T \mathbf{D} \end{bmatrix} \mathbf{J} dS, \]

and

\[ \mathbf{A}'_{5 \times 3} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} & \mathbf{A}_{16} \\ \mathbf{A}_{21} & \mathbf{A}_{22} & \mathbf{A}_{26} \\ \mathbf{A}_{31} & \mathbf{A}_{32} & \mathbf{A}_{36} \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \end{bmatrix}, \quad \mathbf{B}'_{5 \times 3} = \begin{bmatrix} \mathbf{B}_{11} & \mathbf{B}_{12} & \mathbf{B}_{16} \\ \mathbf{B}_{21} & \mathbf{B}_{22} & \mathbf{B}_{26} \\ \mathbf{B}_{31} & \mathbf{B}_{32} & \mathbf{B}_{36} \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \end{bmatrix}, \quad \mathbf{D}'_{5 \times 3} = \begin{bmatrix} \mathbf{D}_{11} & \mathbf{D}_{12} & \mathbf{D}_{16} \\ \mathbf{D}_{21} & \mathbf{D}_{22} & \mathbf{D}_{26} \\ \mathbf{D}_{31} & \mathbf{D}_{32} & \mathbf{D}_{36} \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \end{bmatrix}. \]

For the post-buckling analysis of the composite laminates, since these \(15m^2 \times 9m^2\) invariant matrices \(\mathbf{I}_{n1}, \mathbf{I}_{n2}, \mathbf{I}_{n3}\) defined in equation (50) are independent on the displacement vector \(\mathbf{d}\), they need to be integrated analytically only once in advance. Analyze the stiffness matrix defined by equation (37) and equation (38-2). It should be noticed that the integral calculus of the matrix \(\mathbf{ke}_2(d)\) given in equation (38-2) is depend on the unknown displacement vector \(\mathbf{d}\), however which can be evaluated by the multiplication of these three invariant matrices \(\mathbf{I}_{n1}, \mathbf{I}_{n2}, \mathbf{I}_{n3}\), and the nodal line coordinate matrix \(\mathbf{X}_w\).

Similarly, substitute equation (43) into the second order nonlinear stiffness matrix item \(\mathbf{ke}_3(d)\) in equation (38-3), the expression can be given as

\[ \mathbf{ke}_3(d) = \int_S \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \end{bmatrix}_{6m \times 6m} \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \end{bmatrix}_{6m \times 6m} \begin{bmatrix} \mathbf{D}_2^T \mathbf{A} \end{bmatrix} \mathbf{J} \mathbf{D}_2 \mathbf{X}_w \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \end{bmatrix}_{6m \times 6m} dS. \]

where

\[ \mathbf{A}'_{3 \times 3} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} & \mathbf{A}_{16} \\ \mathbf{A}_{21} & \mathbf{A}_{22} & \mathbf{A}_{26} \\ \mathbf{A}_{31} & \mathbf{A}_{32} & \mathbf{A}_{36} \end{bmatrix}. \]

then substitute equation (46) into equation (52), it will be induced

\[ \mathbf{ke}_3(d) = \int_S \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \end{bmatrix}_{6m \times 6m} \mathbf{X}_w^T \begin{bmatrix} \mathbf{J}^T \mathbf{A}' \mathbf{J} \end{bmatrix} \mathbf{X}_w \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \end{bmatrix}_{6m \times 6m} dS. \]

Because zeros matrix and the nodal line coordinate matrix \(\mathbf{X}_w\) are independent on the integral calculation, equation (54) can be rewritten as
where the other invariant matrix $I_{n4}$ of the stiffness matrix can be defined as

$$I_{n4} = \int_S J^T A^* J dS.$$  \hspace{1cm} (56)

The invariant matrix $I_{n4}$ is a $9m^2 \times 9m^2$ constant matrix and can be calculated analytically beforehand once only. Then the integral of the matrix $k_{e3}(d)$ defined in equation (55) can be transformed as the three matrices $X^T_w$, $I_{n4}$, and $X_w$ multiples in turn.

### 3.2. Invariant matrices for FSDPT

This method of extracting the invariant from variant stiffness matrix also can be utilized under the circumstances of FSDPT. According to the assumption of FSDPT, it should be shown that $\chi$ in equation (2) is equal to 0. Reasonably, $D_4$ and $D_4'$ in equation (34) is converted to zero matrices, and the shear correction factor $k$ will be shifted. The expression of three items $k_{e1}$, $k_{e2}(d)$, and $k_{e3}(d)$ in the elastic stiffness matrix has given in equation (38-1) to equation (38-3).

Substituted equation (46) into equation (40), $D_2(d)$ is transformed as

$$D_2(d) = \begin{bmatrix} O_{6 \times 6} & JX^T_w & O_{3 \times 6} \\ O_{3 \times 6} & I_{n1} + I_{n2} & X_w \\ O_{6 \times 3} & O_{3 \times 6} \\ \end{bmatrix}. \hspace{1cm} (57)$$

Take equation (38-1) to (38-3) and equation (57) into account, the variable stiffness items $k_{e2}(d)$, and $k_{e3}(d)$ can be expressed by exploiting the invariant matrices. Owing to $D_4 = D_5 = O_{5 \times 15m}$ in the FSDPT, which load to the invariant matrix $I_{n3} = O_{15m \times 9m^2}$. Therefore stiffness item can be rewritten as

$$k_{e2}(d) = \begin{bmatrix} O_{15m \times 6m} & (I_{n1} + I_{n2}) & X_w \\ O_{15m \times 9m^2} & 9m^2 \times 3m & O_{15m \times 6m} \\ \end{bmatrix} \hspace{1cm} (58)$$

where

$$I_{n1} = \int S D^T_1 A' J dS, \hspace{0.5cm} I_{n2} = \int S D^T_1 B' J dS,$$ \hspace{1cm} (59)

and the expression of $A'$ and $B'$ are same as equation (51).

Distinctly, since $D_1$ and $D_2(d)$ are same in the HSDPT and FSDPT, and there is no influence of the shear correction factor $k$ in $A^*$, the form of second order nonlinear stiffness matrix item $k_{e3}(d)$ expressed by invariant matrix is the same as equation (55) and equation (56).
3.3. Invariant matrices for CPT

Finally, in the case of CPT, the expression of three items $k_{e1}$, $k_{e2}(d)$, and $k_{e3}(d)$ can be given in equation (39-1) to equation (39-3), this method of extracting the invariant from variant stiffness matrix also can be utilized. In CPT, the dimension of the relationship of the strain and displacement is different from the dimension of HSDPT and FSDPT. The strain-displacement relationship $D_2(d)$ which contains of unknown displacement vector $d$ can be written as

$$D_2(d) = \begin{bmatrix} O & D_w^{0} \\ 3 \times 6m & 3 \times 6m \end{bmatrix}, \tag{60}$$

and

$$D_w^{0} = \begin{bmatrix} \frac{1}{2} \left( \frac{\partial h_w}{\partial \xi} S_w d_w \right) & \frac{\partial h_w}{\partial \psi} S_w \\ \frac{1}{2} \left( \frac{\partial h_w}{\partial \xi} S_w d_w \right) & \frac{\partial h_w}{\partial \psi} S_w \end{bmatrix}. \tag{61}$$

Redefine three $1 \times 6m$ row matrices, these row matrices is written as

$$\xi = [\xi_1, \xi_2, \ldots, \xi_{6m}] = \frac{1}{2} S_w^{T} \frac{\partial h_w^{T}}{\partial \xi} \frac{\partial h_w}{\partial \psi} S_w,$$

$$\psi = [\psi_1, \psi_2, \ldots, \psi_{6m}] = \frac{1}{2} \frac{\partial h_w^{T}}{\partial \psi} S_w,$$

$$\zeta = [\zeta_1, \zeta_2, \ldots, \zeta_{6m}] = S_w^{T} \frac{\partial h_w}{\partial \xi} \frac{\partial h_w}{\partial \psi}, \tag{62}$$

where $\xi_i, \psi_i, \zeta_i (i = 1, 2, \ldots, 6m)$ are the column matrices. Substituted equation (62) into equation (61), $D_w^{0}$ have the same expression as equation (43), however with the different dimension.

Since the transpose of the scalar quantity equals to itself, the items in the right side of the equal sign of equation (44) can be rewritten as

$$d^T w \dot{U} = [d^T w_1, d^T w_2, \ldots, d^T w_{6m}] = [u^T w_1, u^T w_2, \ldots, u^T w_{6m} d_w], \tag{63}$$

$$\dot{U} = \dot{\xi}, \dot{\psi}, \dot{\zeta}; u_i = \xi_i, \psi_i, \zeta_i; i = 1, 2, \ldots, 6m).$$

Thus the dimension of $D_w^{0} = J X_w$ will be changed, where nodal line coordinate matrix $X_w$ will be shifted as

$$X_w = diag(d_w, d_w, \ldots, d_w), \tag{64}$$

while the $3 \times 36m^2$ spare matrix $J$ can be redefined as

$$J = \begin{bmatrix} \overline{\xi} & \overline{\psi} & \overline{\zeta} \end{bmatrix}, \tag{65}$$

and

$$\overline{\dot{U}} = [u^T \dot{w}_1, u^T \dot{w}_2, \ldots, u^T \dot{w}_{6m}], \tag{66}$$

$$\dot{U} = \dot{\xi}, \dot{\psi}, \dot{\zeta}; \dot{u}_i = \dot{\xi}_i, \dot{\psi}_i, \dot{\zeta}_i; i = 1, 2, \ldots, 6m).$$
According to in equation (64) to equation (66), and a series of coefficients in equation (36), the first order nonlinear stiffness matrix item $k_{e1}(d)$ and the second order nonlinear stiffness matrix item $k_{e3}(d)$ can be rewritten as

$$k_{e2}(d) = \begin{bmatrix} O_{12m \times 6m} & (I_n + I_{n/2})_W X_w \end{bmatrix} + \begin{bmatrix} O_{12m \times 6m} & (I_n + I_{n/2})_W X_w \end{bmatrix}^T,$$

$$k_{e3}(d) = \begin{bmatrix} O_{6m \times 6m} & X_w \end{bmatrix} \begin{bmatrix} O_{6m \times 6m} & I_{n/3} X_w \end{bmatrix},$$

where $I_n = \int D_1^T AJdS$, $I_{n/2} = \int D_3^T BJDdS$, $I_{n/3} = \int F^T AJdS$, which dimension of these invariant matrices in CPT are different in the assumption of HSDPT and FSDPT. Thus the elastic stiffness matrix $k_e(d) = k_{e1} + k_{e2}(d) + k_{e3}(d)$ has the different dimension with the case of HSDPT and FSDPT, which is $(12m \times 12m)$.

In summary, it can be concluded that the method for extracting the invariant matrices can be utilized diffusely both in these three theories. Furthermore, it is clearly found that in the first and second nonlinear stiffness matrices, there are sparse matrices which contain numbers of zero elements and only non-zero elements need to be evaluated. In other words, only part of elements in $k_{e2}(d)$ and $k_{e3}(d)$ should be calculated. The principle of this procedure is to transform the integral operation into the matrix multiplication calculation, by calculated these invariant matrices beforehand. Compared with the integral calculus of the stiffness matrices of $k_{e2}(d)$ and $k_{e3}(d)$, the matrix multiple will be more efficient clearly which can be verified by the example in section 4.3. In addition, the detailed calculation process analysis will be given in the next section.

### 3.4. Computational process analysis

In above section, under the three assumptions, the invariant matrices and the nodal line coordinate matrix $X_w$ are introduced to improve the computational efficiency of the stiffness matrix in FSM for the initial post-buckling analysis of the compositend laminates. In this section, corresponding flowcharts is demonstrated to clearly signify the solution process with invariant matrices, which can be found in figure 3. The flow chart can be divided into the following three stages:

In stage 1, all computational parameters of the structural system will be set firstly, including the length $a$ and the width $b$ of the strip, the number $n$ of the strips, the thickness $h$ of the strip element, initial axial force of each strip $T_0$, the maximum load factor $\lambda_{max}$, and load factor increment $\Delta \lambda$, and a series of coefficients $(A, B, C, D, E, F, H)$ by given the number $N$ of the layers, the angle $\alpha$ of each layer, Young’s modulus, Shear modulus, Poisson’s ratios of the composite laminates. Then the linear term of elastic stiffness matrix and geometric stiffness matrix will be calculated in advance. The initial test solution $D_0$ and critical buckling load factor will be evaluated by the control equation of linear derivation system which neglects the stiffness items $K_2(d)$ and $K_3(d)$ assembled by $k_{e2}(d)$ and $k_{e3}(d)$. 

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In stage 2, the invariant matrices will be evaluated firstly. In different theory assumptions, the invariant matrices will also be given in different form. Then set the initial load factor \( \lambda \) and substitute it into the nonlinear control equation, calculate the elastic stiffness matrices \( k_{c2}(d) \) and \( k_{c3}(d) \) by the matrix multiplication operations of invariant matrices and nodal line coordinate matrix \( X_w \), and assemble the elastic stiffness matrix items \( K_{c2}(d), K_{c3}(d) \). On the other hand for the traditional method, the elastic stiffness matrix items \( k_{c2}(d) \), and \( k_{c3}(d) \) will be integrated and these integral operation must be implemented in each step of the iterative calculation for the nonlinear algebra equation. Using these invariant matrices, however, transforms the integral operations into the matrix multiplication operations of the invariant matrices and the displacement vector matrices as shown in above section.
Stage 3 describes specifically describes the iterative process. Firstly the global stiffness matrix can be evaluated, the test solution can be substituted into the nonlinear control equation and the solution $D_{new}$ of the control equation is obtained by the iterative update based on Newton's method. Then check the convergence of the iterative process. If the convergence condition induced from equation

$$\lg \left[ \left( K_e(D_0) - \lambda K_e \right) D_{new} \right] \leq -12 \quad (69)$$

is satisfied, the solution $D_{new}$ can be regarded as the requested one and saved. Otherwise, the solution $D_{new}$ will be set as initial test solution $D_0$ into the nonlinear control equation for further iterative calculations until it satisfies the convergence condition. The obtained solution here is the displacement vector corresponding to the load factor given in stage 2. Hereafter, substitute the new load factor $\lambda + \Delta \lambda$ into stage 2 to start the next calculation loop with $\Delta \lambda$ being as the load factor increment until the new load factor reaches the maximum load factor $\lambda_{max}$.

In this procedure, the fundamental idea is to transform the integral operation of $k_{e2}(d)$ and $k_{e3}(d)$ into the matrix multiplication calculation. In addition, it can be found that only one fifth of non-zero elements in $k_{e2}(d)$ and a quarter of non-zero elements in $k_{e3}(d)$ need to be calculated. Therefore, these sparse matrices have great benefits for reducing storage space in post-buckling analysis.

4. Examples and analysis

In this section, several benchmark examples are studied and the corresponding results are compared with those available in the literature or by the FEM, to demonstrate the feasibility, accuracy, and efficiency of the method with MATLAB program.

4.1. Illustrations of laminated thin plate

Firstly, we will introduce a laminated thin plate. As far as author known, only the influence of the first-order shear is considered in the modeling of the commercial finite element software shell element. Therefore, in this section, we will use FEM software ABAQUS to simulate thin plate and moderately thick plate. The thin laminated plate is assumed to be constructed from three plies of equal thickness, and the lay-up configuration in laminates is assumed to be symmetric angle-ply $[0 \ 90]_3$, with the material properties as follows:

$$E_s = 40\text{GPa}; \quad E_s / E_y = 40; \quad G / E_y = 0.5; \quad \mu = 0.3.$$ 

And the other basic parameters are given: Width: $a = 300\text{mm}$; Plate thickness: $h = 3\text{mm}$; Layer thickness: $t = 1\text{mm}$; Critical force: $T_0 = \pi E_y h^3 / 12a^2(1 - \mu^2)$; Dimensionless deflection: $W = w / h$; Dimensionless load factor: $\lambda = T / T_0$. The finite element model can be built up by software ABAQUS. The shell element in ABAQUS has been used. Here this benchmark example is studied by the proposed FSM and the square thin plate member is divided into 3 strips evenly along the loaded edge.

Figure 4 shows the relationships between the dimensionless deflection $W$ at the point $O_1$ of the laminated thin plate and the dimensionless load factor $\lambda$. The point $O_1$ is located at the maximum deflection. It’s known that the absolute value of the maximum deflection on the half wave is equal. These two kinds of results are calculated by FSM and FEM, respectively. Moreover, the FSM in this example is under the assumption of CPT. The comparison shows that the proposed FSM has good precision for the initial post-buckling analysis of laminated thin plate members under the boundary condition of simply supported each edge.
Figure 4. Comparison of initial post-buckling deflection-load curve of thin plate.

It is seen in figure 4 that for both cases of the studies, the two sets of results compare very closely and hence that the FSM under the CPT for the post buckling behaviors of laminated thin plates under uniform pressure are verified.

4.2. Illustrations of laminated moderately thick plate

Secondly, a moderately thick plate is taken into account. The moderately thick plate has the same material parameters with above. The shear correct factor is equal to 5/6. It is assumed that the plate is constructed from three plies of equal thickness, and the lay-up configuration in laminates is assumed to be symmetric angle-ply [0 90]. The length to thickness ratios of the plate is varied as $a/h=20$, and the basic parameters are as follow: Width: $a = 300mm$; Plate thickness: $h = 15mm$; Layer thickness: $t = 5mm$; Critical force: $T_0 = \pi E_y h^3 / 12a^2(1-\mu^2)$; Dimensionless deflection: $W = w / h$; Dimensionless load factor: $\lambda = T / T_0$. The laminated moderately thick plate member is divided into 3 strips evenly along the loaded edge.

According to Shen Huishen, when the length to thickness ratio of plate is relatively large, the CPT has sufficient accuracy. Due to the high ratio of the plate, the effects of through-the-thickness shear stresses on the post-buckling behavior of the plate have appeared to be insignificant. As the length-to-thickness ratio of the plate decreases, the effect of the thickness shear stress on the post-buckling behavior of the plate is enhanced gradually.

Figure 5 shows the relationships between the dimensionless deflection $W$ at the center point $P_1$ of the laminated moderately thick plate and the dimensionless load factor $\lambda$. The point $P_1$ is located at the maximum deflection. The absolute value of the maximum deflection on the half wave is equal, which mode is same as above example. These three kinds of result are calculated by FEM and two species of FSM, respectively. These two FSMs are under the assumption of CPT and FSDPT. The comparison shows that the FSM under the FSDPT has good precision with the FEM result, while the
results under CPT have deviated from the FEM. This is because that CPT based on Kirchhoff ignores the influence of shear deformation, which leads to the obtained deflection under the same load is smaller than that under the theory considering the shear effect. Thus, the present FSDPT for the post-buckling analysis of thicker laminates is more suitable than the CPT. In other words, for the more accurate post-buckling analysis of relatively thick plates, the effects of through-the-thickness shear stresses should be taken into account.

Figure 5. Comparison of initial post-buckling deflection-load curve of moderately thick plate.

From figure 5, it could be concluded that for both cases of the studies, the two sets of results compare very closely. Thus, it should be concluded that the FSM under the FSDPT is feasible for the initial post-buckling analysis of laminated moderately thick plate members under the boundary condition of simply supported each edge. In addition, we may find that the proposed FSM can achieve the similar accuracy with fewer strip units than FEMs with many elements, which could prove that FSM is an effective and efficient technology.

4.3. Illustrations of laminated thick plate

Thirdly, in order to further study the FSM under the three theories, a thick plate is introduced. The length to thickness ratio is $a/h=10$. As for the plate with small ratio, the influence of the first-order shear is not sufficient to achieve the required accuracy. Thus, the HSDPT is introduced in this example. In the modeling process of the FEM software shell element, however, only the influence of the first-order shear is considered and the high-order shear is ignored. Thus, we can no longer model with shell elements of FEM software to compare with FSM in this paper.

Under the hypothesis of FSDPT, the determination of the shear correction factor has quite complex external factors. Furthermore, Librescu and Stein (1991) reported that the post-buckling load-deflection curves are sensitive to the selection of the shear correction factor. Hence, HSDPT no longer needs to introduce shear factor, and the calculation accuracy is also improved. Shen huishen
studied the post-buckling load-deflection curves for \([0\ 90]_4\) laminated square plate with \(a/h = 10\) subjected to uniaxial compression. Material parameters are as follows:
\[
E_x / E_y = 40; \quad G_{xy} / E_y = G_{xz} / E_y = 0.6; \quad G_{yz} / E_y = 0.5; \quad \mu_{xy} = 0.25.
\]
Thus, we use FSM to build a laminated thick plate model with the same material parameters. The FSM is based on the assumption of CPT, FSDPT, and HSDPT, respectively. Then the basic parameters are as follow: Width: \(a = 300\ mm\); Plate thickness: \(h = 30\ mm\); Layer thickness: \(t = 10\ mm\); Critical force: \(T^* = E_y h^2 / a^2\). The dimensionless deflection \(W = w / h\) and the dimensionless load factor \(\lambda = T / T^*\). According to Shen’s result, only one half-wave appears in the laminated square plate, and the maximum deflection is at the center of the plate, under this circumstance. Similarly, the laminated thick plate member is divided into 3 strips evenly along the loaded edge.

Figure 6 clearly shows the relationships between the dimensionless deflection \(W\) at the center point \(Q_1\) of the laminated moderately thick plate and the dimensionless load factor \(\lambda\). These four kinds of result are calculated by Shen’s theoretical calculation and three species of FSM, respectively. These three FSMs are under the assumptions of CPT, FSDPT, and HSDPT. The comparison shows that the FSM under the HSDPT has good precision with the Shen’s result, and the results under FSDPT have slightly deviated from Shen’s, while the results under CPT have diverge from Shen’s completely. From this figure, it can be concluded that the influence of shear stress for laminated thick plates haven’t be ignored. Taking Shen’s theoretical calculations as a standard, it can be found that under the same load factor, the FSM’s results of HSDPT are the closest, followed by FSDPT’s.

![Figure 6. Comparison of initial post-buckling deflection-load curve of thick plate.](image)

Thus, it should be concluded that the FSM under the HSDPT is feasible for the initial post-buckling analysis of laminated thick plate members under the boundary condition of simply supported each edge. In summary, from these three examples, it can be clearly seen that the difference between the results of FSM under these three theories has been significantly noticeable. Thus, in the case of
composite laminates with different length to thickness ratios, it is strongly recommended to consider the FSM under different theories.

4.4. Illustration of composite laminated columns
As for the post-buckling behavior of composite laminated columns, the method proposed in this paper is also applicable. Considered columns are made as 8 layers GFRP laminate with different layer arrangement configurations. Assume the member is subjected to uniaxial uniform pressure, with the load edges are simply supported and the unload edge are free, namely recorded as “SSff”. The common dimensions of columns under investigation are presented in figure 7.

![Figure 7. Overall shape and dimensions of considered columns.](image)

Obtained values of all determined material properties for singular layer of GFRP are presented in table 1.

| Table 1. Material properties for GFRP (S-glass reinforced epoxy resin) lamina. |
|---------------------------------|---------------------------------|
| Young modulus |
| In fibre direction | $E_1 = 38.5Gpa$ |
| In transverse to fibre direction | $E_2 = 8.1Gpa$ |
| Poisson’s ratio | $\mu_{12} = 0.27$ |
| Kirchhoff modulus | $G_{12} = 1.9Gpa$ |

The GFRP are composed of eight layers with total laminate thickness equals to 2 mm, with the thickness of each GFRP layer is 0.25 mm. In case of GFRP laminates two different layer arrangements have been considered:
This problem was previously studied by experimental test and numerical calculation. Tomasz Kubiak and Radoslaw J. Mania dealt with post-buckling behavior of short columns of channel cross section subjected to uniform axial compression [37]. Employed elements are based on First Shear Deformation Theory, while the shear correct factor $k_c$ is not given herein. The composite columns were subjected to axial uniform compression. Modelled in FEM boundary conditions of loaded edges of considered column correspond to simply supported designed in experiments. This model can be established by the FSM proposed in this paper. In FSM, the model is divided into 8 strip elements along the loaded edge evenly, with the width of each strip at 20mm. Moreover, the FSM in this example is under the assumption of FSDPT and HSDPT. Under the assumption of FSDPT, the shear correct factor is taken as 5/6, while there is no need to introduce shear correction factor in HSDPT, meaning that the factor is equal to 1. According to coordinate transformation in equation (25), the rotation angles $\alpha$ of the strips are set as $[90^\circ, 90^\circ, 0^\circ, 0^\circ, 0^\circ, 0^\circ, 0^\circ, 90^\circ]$ in turn. The comparison of the relationship between maximum deflection of the web and the load factor is presented in figure 11. The abscissa in this figure is dimensionless deflection: $W = w / h$, and the ordinate is Dimensionless load factor: $\lambda = P / P_{cr}$, where $P_{cr}$ is linear buckling critical load. The comparison results show that the proposed FSM has good precision for the initial post-buckling analysis of plate members. The very good agreement between the results in this paper and the numerical value [37] proves the accuracy of the semi-analytical finite strip calculation method.
In summary, from these several examples, it could be confirmed that the method proposed in this paper is an applicable, effective, and precise method for solving the initial post-buckling of composite laminated plate and composite laminated structures.

4.5. Comparative analysis
According to the comparative results of above cases, it should be noticed that the FSM with invariant matrices possesses sufficient precision for the initial post-buckling analysis. In order to elaborated the efficiency of the FSM with invariant matrices, a series of composite laminated structure mentioned in section 4.2, with “SSff” boundary condition and the uniaxial uniform pressure is taken into account. The comparative results of CPU time by the FSM under three different theoretical assumptions with and without invariant matrices are presented.

From this table, it can be noticed that each model is divided into different strip members to carry on post-buckling analysis under the different theoretical assumption, respectively. In addition, the load factor \( \lambda \) changes from 0 to 1 (0.5, 1.5 and 2.5) with an interval of 0.1. If the CPU times of the FSM with and without invariant matrices is donated as \( T_{\text{with}} \) and \( T_{\text{without}} \), then the ratio of the CPU time \( \alpha \) between the FSMs without and with invariant matrices is defined as

\[
\alpha = \frac{T_{\text{without}}}{T_{\text{with}}},
\]

The program is implemented by MATLAB platform, and the program code is implemented in the Lenovo computer with Intel(R) Core(TM) i7-4790CPU @ 3.60GHz and Windows 7 system.

**Table 2.** Comparison of the CPU time of the plates under several parameter types with and without invariant matrix.
In order to efficiently evaluate the stiffness matrix and the elastic force in the initial post-buckling analysis of composite laminated members, the invariant matrices have been deduced and applied to transform the analytical integral of the stiffness matrix into the matrix multiple calculation. Several benchmark examples are studied and the results are compared with those available in the literature or by the FEM, to demonstrate the feasibility, accuracy and efficiency of the proposed method. The results determined by different methods can verify the proposed FSM with invariant matrices is an effective and efficient technology to analyze the initial post-buckling of composite laminated members.

In addition, the essential advantage of the proposed FSM with invariant matrices are summarized as: 1) the proposed procedure is much more efficient since it requires the integration of the stiffness matrix only once no matter how many iterations are needed; 2) The analytical integral of the stiffness

| Modal | Theory | Strip number | Load factor $\lambda$ | Invariant matrix | CPU time(s) | Multiple of consumed time $\alpha$ |
|-------|--------|--------------|-----------------------|------------------|------------|----------------------------------|
| CPT   | FSDPT  | 3            | $[0, 0.1, 0.2, ..., 1]$ | with | $2.49 \times 10^3$ | 10.08 |
| GFRP-1|        | 4            | $[0, 0.1, 0.2, ..., 1.5]$ | with | $2.51 \times 10^4$ | 15.07 |
| HSDPT |        | 8            | $[0, 0.1, 0.2, ..., 2.5]$ | without | $1.30 \times 10^3$ | 25.4 |
| CPT   | FSDPT  | 4            | $[0, 0.1, 0.2, ..., 1]$ | without | $1.96 \times 10^4$ | 25.5 |
| GFRP-2|        | 6            | $[0, 0.1, 0.2, ..., 1.5]$ | without | $1.73 \times 10^3$ | 15.0 |
| HSDPT |        | 8            | $[0, 0.1, 0.2, ..., 2.5]$ | without | $4.41 \times 10^4$ | 25.5 |

From the first example in table 2, it can be seen that the CPU time of the FSM under the different theoretical assumption without invariant matrices is $2.51 \times 10^4$ ($1.96 \times 10^4$ and $4.41 \times 10^4$) seconds, while under the corresponding assumption, the CPU time without the invariant matrices is $2.49 \times 10^3$ ($1.30 \times 10^3$ and $1.73 \times 10^3$) seconds, respectively. According to equation (70), it can be obtained the corresponding ratio of the CPU time $\alpha$ with and without invariant matrices under the assumption of CPT, FSDPT, HSDPT, which value is 10.08 (15.07 and 25.4), separately. Similarly, it also can be seen that in the second example, the ratio of the CPU time $\alpha$ under the different theoretical assumption is 10.08 (15.0 and 25.5).

It can be clearly seen that for these two examples, the ratio of the consumption time with and without the invariant matrix is close to the number of load factor calculation in table 2, which is 11 (16 and 26). It is because that the CPU time is mainly consumed in the integral calculation of the stiffness matrix, while the traditional FSM requires huge CPU time to calculate the stiffness matrix once the load factor changes. In comparison, the FSM with invariant matrices just need to be pre-integrated only once, no matter how many times the load factor changes. This measure economizes the CPU time significantly, and makes the calculation obviously optimized. Thus, we can conclude that the FSM by using the invariant matrices can improve the computational efficiency obviously and as the number of calculated load factor increase this advantage of the FSM is more apparent. As far as the authors known, there is no notice in currently available bibliography of the use of these invariant matrices that considerably reduce computational time in the post buckling analysis of composite laminated members. The tremendously boosted efficiency of FSM, which under different theoretical assumption, can be extensively used in engineering field.

5. Conclusion

In order to efficiently evaluate the stiffness matrix and the elastic force in the initial post-buckling analysis of composite laminated members, the invariant matrices have been deduced and applied to transform the analytical integral of the stiffness matrix into the matrix multiple calculation. Several benchmark examples are studied and the results are compared with those available in the literature or by the FEM, to demonstrate the feasibility, accuracy and efficiency of the proposed method. The results determined by different methods can verify the proposed FSM with invariant matrices is an effective and efficient technology to analyze the initial post-buckling of composite laminated members.

In addition, the essential advantage of the proposed FSM with invariant matrices are summarized as: 1) the proposed procedure is much more efficient since it requires the integration of the stiffness matrix only once no matter how many iterations are needed; 2) The analytical integral of the stiffness
matrix can be transformed into the matrix multiple calculation; 3) The advantage of time-saving is increasingly remarkable as the number of iterations increases so the high efficient analysis of the initial post-buckling of composite laminated members can be implemented.

Acknowledgements
The research is financed by Science Challenge Project in China (JCKY2016212A 506-0104), Natural Science Foundation of China(11472135), Natural Science Foundation of Jiangsu Province, China(BK20130911).

Appendix
The expression of the coordinate transformation matrix $T$ under CPT in equation (25):

$$
T = \begin{bmatrix}
\cos \alpha & 0 & 0 & 0 & 0 & -\sin \alpha & 0 & 0 & 0 & 0 \\
0 & \cos \alpha & 0 & 0 & 0 & 0 & -\sin \alpha & 0 & 0 & 0 \\
0 & 0 & \cos \alpha & 0 & 0 & 0 & 0 & 0 & 0 & -\sin \alpha \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
\sin \alpha & 0 & 0 & 0 & 0 & \cos \alpha & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & \sin \alpha & 0 & 0 & 0 & 0 & 0 & \cos \alpha & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & \sin \alpha & 0 & 0 & 0 & 0 & \cos \alpha \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

The expression of the coordinate transformation matrix $T$ under HSDPF and FSDPT in equation (25):

$$
T = \begin{bmatrix}
\cos \alpha & 0 & 0 & 0 & 0 & -\sin \alpha & 0 & 0 & 0 & 0 \\
0 & \cos \alpha & 0 & 0 & 0 & 0 & -\sin \alpha & 0 & 0 & 0 \\
0 & 0 & \cos \alpha & 0 & 0 & 0 & 0 & 0 & 0 & -\sin \alpha \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
\sin \alpha & 0 & 0 & 0 & 0 & \cos \alpha & 0 & 0 & 0 & 0 \\
0 & 0 & \sin \alpha & 0 & 0 & 0 & 0 & 0 & \cos \alpha & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \sin \alpha & 0 & 0 & 0 & 0 & \cos \alpha \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$
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