MEAN CURVATURE OF THE INDICATRIX OF FINSLER MANIFOLD

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Abstract. Fundamental function in Finsler manifold defines a metrics that depend on a point and a direction. At any point tangent space is a Riemannian and an indicatrix is a convex hypersurface. In this paper a mean curvature of the indicatrix is expressed in terms of fundamental function.

1. Introduction

Finsler manifold is generalization of the Riemannian one, in the same way as Riemann manifold is for the Euclidean. A metric depends on the point and the direction.

Let $M$ be an $n$-dimensional differential manifold and $F$ smooth, real nonnegative and 1-homogeneous function that acts on the tangent bundle of $M$, such that a Hessian of $F^2$ is nonnegative [9][6][1]. $F$ is called a fundamental function of Finsler manifold $(M,F)$.

For a fixed point $x \in M$, the tangent vector space $T_xM$ can be considered as a Riemannian space with metric generated by $F$,

$$g_{ij} = g_{ij}(y) = \frac{1}{2} \frac{\partial^2 F^2(x, y)}{\partial y^i \partial y^j};$$

but also, $T_xM$ is Minkowski space with a norm $F(x, y)$. Collection of vectors

$$I_x = \{y \in T_xM | F(x, y) = 1\}$$

is called an indicatrix, and might be considered twofold: as a hypersurface and as an unit sphere. The first viewpoint appears in [10][11][7] and the second one in [4][5].

As a Riemannian hypersurface, the indicatrix is convex and orientable [9][6], and it has the tangent bundle containing Euclid spaces as fibres. From the theory of hypersurfaces [12], it stems that the indicatrix $I_x$ is totally umbilical with constant unit mean curvature [5][6].

In [10][11] the indicatrix is represented in certain local frame. Covariant differential of that representation is given, and its integrability conditions. Necessary and sufficient conditions, for any hypersurfaces to be homothetic with the indicatrix, contain a mean curvature as a characteristic of the shape.

A mean curvature of the indicatrix $I_x$ can be expressed as a trace of second fundamental tensor, arithmetic mean of principal curvatures, but in this paper, it will be connected directly with Hessian of fundamental function. The idea comes from a

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paper of Nishimura and Hashiguchi [7], where the Gauss curvature of hypersurface is expressed in terms of its defining function.

2. MEAN CURVATURE OF A HYPERSURFACE

Let I be an oriented hypersurface in a Riemannian space $V^n$, given implicitly by a $C^\infty$-differentiable function $f$,

$$I = \{x \in V^n | f(x) = 0\}.$$  

As usual, notation for coordinate map in Riemannian space is $x = (x^1, x^2, ..., x^n)$ and an abbreviation for partial derivatives is used, $f_i = \partial_i f = \frac{\partial f}{\partial x^i}$.

Differentiability of at least second order provides nonvanishing gradient

$$\nabla f : I \to \mathbb{R}^n, \quad \nabla f(x) \neq 0, \quad \nabla f = [\nabla f]_{n \times 1} = (f_1, f_2, ..., f_n).$$

Tangent space on the hypersurface at each point $p \in I$ is a real vector Euclidean space $T_pI$ and has dimension $n - 1$ [8][3]. So, it is a hyperplane and with the hypersurface has common unit normal vector at the point $p$. Unit normal vector field on the hypersurface is continuous and globally defined

$$N = \frac{\varepsilon}{|\nabla f|} \nabla f; \quad (\varepsilon = \pm 1)$$

Element of the tangent hyperplane $T_pI$ is a vector tangent on $I$, but also on $V^n$, $v \in T_pI \subset T_pV^n$, so it has $n$ components $v = (v^1, v^2, ..., v^n)$; a basis in $T_pI$ has $n - 1$ elements $\{X_1, X_2, \ldots, X_{n-1}\}$ and there is a decomposition $v = v^\alpha X_\alpha$. Precisely, Latin indices run $\{1, 2, ..., n\}$ and mean components in $T_pV^n$ and Greek ones run $\{1, 2, ..., n-1\}$ and consider decomposition over the basis of $T_pI$.

Local bending (shape) of the hypersurface is expressed by infinitesimal changes of the hypersurfaces normal. A shape operator applied to the tangent space $T_pI$ is a linear transformation and for a tangent vector $v \in T_pI$ determines negative derivative of the normal in the direction of $v$ [2],

$$S : T_pI \to T_pI, \quad S(v) = -D_v N.$$  

Calculation of a directional derivative $D_v N$ means partial derivatives of the normal. They are tangent on the hypersurface and Wiengartens equations $\partial_\beta N = -h_\beta^\alpha X_\alpha$ allowed matrix notation

$$S(v) = [h_\beta^\alpha]_{(n-1) \times (n-1)} [v^\alpha]_{(n-1) \times 1}.$$  

A mean curvature of the hypersurface is arithmetical mean for diagonal elements in matrix $[h_\beta^\alpha]$ of normal’s derivatives decompositions over the tangent space, i.e. is proportional to the trace of the shape operator

$$H = \frac{1}{n-1} \text{tr} [h_\beta^\alpha] = \frac{1}{n-1} \text{tr} S.$$  

$T_pI$ is $(n-1)$-dimensional Euclidean space and scalar product is defined as an intrinsic form. But, if one of the arguments is obtained by the shape operator, scalar product is quadratic form in ambient space $T_pV^n$, [7].

Proposition 2.1. The shape operator of an oriented hypersurface $(I, N)$, given by (2.1), satisfies the following condition

$$u \cdot S(v) = -\frac{\varepsilon}{|\nabla f|} f_{ij} u^i u^j.$$
for any vectors $u = (u^i)$ and $v = (v^i)$ tangent on the hypersurface $I$. An abbreviation is used, $f_{ij} = \partial_i \partial_j f$.

Proof. $\nabla f$ is vector field on an open neighbourhood of $I$, so directional derivation of (2.2) gives

$$\partial_j (\nabla f) v^j = \varepsilon D_v(||\nabla f||) N - \varepsilon ||\nabla f|| S(v).$$

$\partial_j (\nabla f)$ is a quadratic matrix of order $n$ with second partial derivatives. Observing $N$, $S(v)$ and $u$ as elements of $T_p V^n$, scalar product is

$$u \cdot f_{ij} v^j = -\varepsilon ||\nabla f|| u \cdot S(v).$$

□

According to the previous proposition, algebraic invariants of the linear transformation matrix in the hyperspace can be expressed by corresponding matrix of higher order. This idea is used in [7] to calculate Gauss curvature of the hypersurface. Some constructions in the proof of next lemma are very similar.

**Lemma 2.2.** Let $V$ be a $n$–dimensional real vector space, $T$ a linear transformation of hyperspace $W$ and $N$ unit vector orthogonal to $W$. If for any two vectors $u, v \in W$ scalar product $u \cdot T(v)$ is expressed by matrix of order $n$, $A = [a_{ij}]$ in the following way

$$u \cdot T(v) = u Av^T = a_{ij} u^i v^j,$$

then a trace of $T$ is given by

$$tr T = tr A - NAN^T.$$  (2.4)

Proof. Let $\{X_1, X_2, \ldots, X_{n-1}\}$ be a basis of hyperspace $W$ and $Y_\alpha = T(X_\alpha)$. Decompositions of the images $Y_\alpha$ over the chosen basis

$$Y_\alpha = b_{\beta \alpha} X_\beta; \quad B = [b_{\alpha \beta}]$$

give a matrix $B$ of the linear transformation $T$ in $W$, such that

$$tr T = tr B.$$  (2.2)

Vectors $X_\alpha, Y_\alpha$ actually are row matrices of dimension $1 \times n$, and $B$ is of order $n-1$. Now, one needs matrices of order $n$:

$$\tilde{B} = \begin{bmatrix} B & 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad X = \begin{bmatrix} X_1 \\ \vdots \\ X_{n-1} \\ N \end{bmatrix}, \quad Y = \begin{bmatrix} Y_1 \\ \vdots \\ Y_{n-1} \\ N \end{bmatrix};$$

and matrices of order $n + 1$:

$$\tilde{A} = \begin{bmatrix} A & N^T \\ N & 0 \end{bmatrix}, \quad \tilde{X} = \begin{bmatrix} X & 0 \\ 0 & 1 \end{bmatrix}.$$  (2.3)

Block matrices multiplications give:

$$XX^T \tilde{B} = \begin{bmatrix} X_\alpha Y_\beta & 0 \\ 0 & 1 \end{bmatrix},$$  (2.5)

$$\tilde{X} \bar{A} \tilde{X}^T = \begin{bmatrix} X_\alpha AX_\beta^T & X_\alpha AN^T & 0 \\ NAX_\beta^T & NAN^T & 1 \\ 0 & 1 & 0 \end{bmatrix}. $$  (2.6)
Without loss of generality, an assumption of orthogonality of the basis \( \{X_1, X_2, \ldots, X_{n-1}\} \) may be used. Then, \( X \) and \( \tilde{X} \) are orthogonal matrices and according to properties of the trace, two previous equations (2.5) and (2.6) result

\[
\text{tr} \left[ X_{\alpha} Y_{\beta} \right] = \text{tr} \tilde{B} - 1 = \text{tr} T
\]
\[
\text{tr} A = \text{tr} \left[ X_{\alpha} A X_{\beta}^T \right] + N A N^T.
\]

These equations with (2.3) give (2.4).

This result is used to express a mean curvature of the hypersurface (2.1) by its defining function.

**Theorem 2.3.** A mean curvature of the oriented hypersurface \((I, N) (2.1), (2.2)\) in Riemannian space is

\[
H = \frac{1}{n-1} \left( \text{tr} \left[ f_{ij} \right] - N [f_{ij}] N^T \right).
\]

Used notation is \( f_i = \frac{\partial f}{\partial x^i} \), \( f_{ij} = \frac{\partial^2 f}{\partial x^i \partial x^j} \).

**Proof.** Proposition 2.1 provides conditions of the lemma 2.2. By putting \( a_{ij} = f_{ij} \), the proof is obvious. \( \square \)

### 3. Indicatrix of a Finsler manifold

According to properties of fundamental function, indicatrix of the Finsler manifold \((M, F)\),

\[
I_x = \{ y \in T_x M | F(x, y) = 1 \}
\]

is the convex oriented hypersurface [9][1][6] of Riemann space \( T_x M \), so results of the previous section can be applied. Defining function of the hypersurface (2.1) can be presented in few forms:

\[
f(y) = F(x, y) - 1
\]
(3.1)

and unit normal is just radius vector [3][9]

\[
N(y) = y.
\]

Second order derivative of defining function (3.1) can be related with fundamental function or metric tensor,

\[
\frac{\partial^2 f}{\partial y^i \partial y^j} = \left. F_{y^i y^j} \right|_{y^i} = g_{ij},
\]

where an abbreviation is used, \( F_{y^i y^j} = \left. \frac{\partial^2 F}{\partial y^i \partial y^j} \right|_{y^i} \).

Considering these facts, the mean curvature theorem 2.3 can be adapted for Finsler manifold.

**Theorem 3.1.** A mean curvature of the indicatrix of Finsler manifold is

\[
H = \frac{1}{n-1} \left( \text{tr} \left[ F_{y^i y^j} \right] - 1 \right).
\]
Any Riemannian space is locally Euclidean, so in $T_x M$ a coordinates can be chosen such that $g_{ij} = \delta_{ij}$, which means $tr[g_{ij}] = n$. A trace of the matrix is an algebraic invariant, therefore the last relation is coordinate free. In that way, a necessary condition for the fundamental function is obtained.

**Proposition 3.2.** If $(M, F)$ is Finsler manifold, then a trace of second derivatives (Hessian) of fundamental function is just a dimension of basic manifold,

$$tr[F_{y'y'}] = \text{dim} M.$$  

Combination of the previous property with theorem 3.1 gives final result:

**Theorem 3.3.** A mean curvature of the indicatrix of Finsler manifold is constant and identically equal 1,

$$H = 1$$

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