SEVERAL APPLICATIONS OF BEZOUT MATRICES

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Abstract. The notion of Bezout matrix is an essential tool in studying broad variety of subjects: zeroes of polynomials, stability of differential equations, rational transformations of algebraic curves, systems of commuting nonselfadjoint operators, boundaries of quadrature domains etc. We present a survey of several properties of Bezout matrices and their applications in all mentioned topics. We use the framework of Vandermonde vectors because such approach allows us to give new proofs of both classical and modern results and in many cases to obtain new explicit formulas. These explicit formulas can significantly simplify various computational problems and, in particular, make the research of algebraic curves and their applications easier. In addition we wrote a Maple software package, which computes all the formulas. For instance, as Bezout matrices are used in order to compute the image of a rational transformation of an algebraic curve, we used these results to study some connections between small degree rational transformation of an algebraic curve and the braid monodromy of its image.

Introduction

Numerous works in operator theory shows that an algebraic curve given by a determinantal representation can be associated to a system of commuting nonselfadjoint operators. In the simplest case there are two commuting nonselfadjoint operators in a system and they are rational images of the same operator. This particular case gives a motivation and a framework for the studying the image of a complex line under a rational transformation. Using the notion of a determinantal representation of an algebraic curve there was found the explicit formula which describes the image of a complex line under a rational transformation. The image is given by a determinantal representation which uses the Bezout matrices of pairs of polynomials that define the rational transformation.

This property of Bezout matrices can be used for many purposes, i.e. to study the braid monodromy of the image of two intersecting lines under rational transformations of small degrees. Also using this property one can describe explicitly the boundary of a quadrature domain. This property and many others can be proved using the Vandermonde vectors, which are a natural framework to study Bezout matrices.

These properties were used by the authors of this paper to create a package of procedures for Maple software [21]. This package allows the computation of all

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classical and modern formulas that are mentioned in the paper, and to perform many different tasks.

1. Bezout matrices and Vandermonde vectors

First example of Bezout matrices for polynomials of small degree appeared in Euler’s work in 1748, [4]. Using this example Bezout gave a general definition of Bezout matrices for polynomials of any degree in 1764, [2]. The notation of Bezoutian matrix was introduced by Sylvester in 1853, [23]. The most common definition was given by Cayley in 1857, [3].

Definition 1.1. For two polynomials in one variable $p(x)$ and $q(x)$ of degree $n$ there exists uniquely determined $n \times n$ symmetric matrix $B(p, q) = (b_{ij})_{i,j=1}^n$ such that$$
\frac{p(x)q(y) - q(x)p(y)}{x - y} = \sum_{i,j=1}^n b_{ij} x^{i-1} y^{j-1}.
$$The matrix $B(p, q)$ is called the Bezout matrix of polynomials $p$ and $q$.

We will use a framework of Vandermonde vectors. The matrices composed of Vandermonde vectors appeared inexplicitly in Vandermonde’s work in 1772, [25]. The notation was attributed to Vandermonde by Weill in 1888, [27].

Definition 1.2.

$$
V_n(x) = \begin{pmatrix}
1 \\
x \\
\vdots \\
x^{n-1}
\end{pmatrix} = (x^i)_{i=0}^{n-1}
$$

$V_n(x)$ is called the Vandermonde vector of the length $n$.

Theorem 1.3. If $x_1, x_2, \ldots, x_n$ are pairwise distinct then vectors $V_n(x_1), V_n(x_2), \ldots, V_n(x_n)$ are linearly independent.

Proof. Let us consider $n \times n$ matrix

$$(V_n(x_1), V_n(x_2), \ldots, V_n(x_n)) = \begin{pmatrix}
1 & 1 & \cdots & 1 \\
x_1 & x_2 & \cdots & x_n \\
\vdots & \vdots & \ddots & \vdots \\
x_1^{n-1} & x_2^{n-1} & \cdots & x_n^{n-1}
\end{pmatrix}
$$

Let us suppose that $p = (p_0, p_1, \ldots, p_{n-1})$ is a row-vector from the left kernel of this matrix. Then the polynomial $p(x) = p_0 + p_1 x + \cdots + p_{n-1} x^{n-1}$ has $n$ zeroes. The degree of this polynomial is less than $n$. Hence, the kernel of this matrix is trivial and vectors $V_n(x_1), V_n(x_2), \ldots, V_n(x_n)$ are linearly independent.

We will call $V_n(x)$ the Vandermonde vector of order zero. It is natural to define the Vandermonde vector of higher orders. For higher orders the definition is:$$
V_n^k(x) = \left(\frac{d^k}{dx^k}x^i\right)_{i=0}^{n-1}
$$We will call $V_n^k(x)$ the Vandermonde vector of order $k$.

Theorem 1.4. If $i_1 + i_2 + \cdots + i_m = n - 1$ and $x_1, x_2, \ldots, x_m$ are pairwise distinct then vectors $V_n(x_1), V_n^1(x_1), \ldots, V_n^i(x_1), V_n^1(x_2), V_n^1(x_2), \ldots, V_n^i(x_2), \ldots, V_n(x_m), V_n^1(x_m), \ldots, V_n^i(x_m)$ are linearly independent.

The prove is the same as of the above theorem.
Lemma 1.5. For every two polynomials $p(x)$ and $q(x)$

1. $$VT(x)B(p, q)V(y) = \frac{p(x)q(y) - q(x)p(y)}{x-y}$$

2. $$VT(x)B(p, q)V(y) = VT(x)(B(p, 1)q(y) - B(q, 1)p(y))V(y)$$

3. $$VT(x)B(p, q)V(x) = q(x)p'(x) - p(x)q'(x)$$

Proof. The first statement follows immediately from the definition of the Bezout matrix. The second statement follows from the next decomposition:

$$\frac{p(x)q(y) - q(x)p(y)}{x-y} = \frac{p(x)q(y) - p(y)q(y) + p(y)q(y) - q(x)p(y)}{x-y} =$$

$$q(y)VT(x)B(p, 1)V(y) - p(y)VT(x)B(q, 1)V(y)$$

To prove the third statement let us notice that

$$VT(x)B(p, 1)V(x) = \lim_{\epsilon \to 0} VT(x)B(p, 1)V(x + \epsilon) =$$

$$\lim_{\epsilon \to 0} \frac{p(x) - p(x + \epsilon)}{x - (x + \epsilon)} \epsilon = \lim_{\epsilon \to 0} \frac{p(x + \epsilon) - p(x)}{\epsilon} = p'(x)$$

This and the first statement of the lemma imply the third statement. \(\square\)

Corollary 1.6. For every polynomials $p(x)$ and $q(x)$, vector $w$ and point $y$

$$w^TB(p, q)V(y) = w^T(B(p, 1)q(y) - B(q, 1)p(y))V(y)$$

Proof. Vector $w$ can be represented as a linear combination of Vandermonde vectors. $w = \sum a_i V(z_i)$, where all points $z_i$ are different from $y$. If follows from the second statement of the previous lemma that

$$w^TB(p, q)V(y) = \left( \sum a_i V^T(z_i) \right) B(p, q)V(y) = \sum a_i (V^T(z_i)B(p, q)V(y)) =$$

$$\sum a_i \left( VT(z_i)(B(p, 1)q(y) - B(q, 1)p(y))V(y) \right) =$$

$$\left( \sum a_i V^T(z_i) \right) (B(p, 1)q(y) - B(q, 1)p(y))V(y) =$$

$$w^T(B(p, 1)q(y) - B(q, 1)p(y))V(y)$$

\(\square\)

One of the very well known classical applications of Bezout matrices is Jacobi–Darboux theorem. This theorem was proved independently by Jacobi in 1836, and by Darboux in 1876.

Theorem 1.7 (Jacobi–Darboux theorem). The number of common zeroes of two polynomials equals to the dimension of the kernel of the Bezout matrix of these polynomials.

There are different proofs of this theorem. For the proof in terms of Sylvester matrices and Vandermonde vectors see [18].
2. Inverse of Bezout matrix and Hermite theorem

In 1974 Lander proved that the inverse of Bezout matrix is a matrix of Hankel type, [14]. This property, its applications and similar properties of structured matrices were studied by Gohberg and Olshevsky [8], Heinig and Hellinger [9], Tyrtyshnikov [24], Chen and Yang [5]. The framework of Vandermonde vectors allows not only to reprove the result of Landau but also to obtain explicit formulas for the coefficient of the inverse of Bezout matrix, [19].

**Theorem 2.1.** If two polynomials \( p(x) \) and \( q(x) \) of degree \( n \) have no common zeroes and polynomial \( p(x) \) has no multiple zeroes then the inverse of Bezout matrix \( B(p,q) \) is a Hankel type matrix \( H = (h_{ij}) \) and

\[
h_{ij} = \sum_{k=1}^{n} \frac{x_k^{i+j-2}}{q(x_k)p'(x_k)},
\]

where \( x_1, x_2 \ldots x_n \) are zeroes of \( p(x) \).

**Proof.** Let us construct a matrix \( V_p \) from Vandermonde vectors in zeroes of the polynomial \( p(x) \): \( V_p = (V(x_1), V(x_2), \ldots, V(x_n)) \). We will denote by \( D \) the matrix \( V_p^T B(p,q) V_p \). It is obvious that \( B(p,q) = (V_p^T)^{-1}D(V_p)^{-1} \) and therefore \( B^{-1}(p,q) = V_p D^{-1} V_p^T \).

Let us denote by \( d_{ij} \) coefficients of the matrix \( D \). If \( i \neq j \) then

\[
d_{ij} = V^T(x_i)B(p,q)V(x_j) = \frac{p(x_i)q(x_j) - q(x_i)p(x_j)}{x_i - x_j} = 0,
\]

since the numerator is zero and the denominator is not because all zeroes of \( p(x) \) are different. If \( i = j \) then by the third statement of the Lemma [15]

\[
d_{ii} = q(x_i)p'(x_i) - p(x_i)q'(x_i) = q(x_i)p'(x_i),
\]

therefore \( D \) is a diagonal matrix: \( D = diag(q(x_1)p'(x_1), \ldots, q(x_n)p'(x_n)) \) which means that

\[
D^{-1} = diag\left(\frac{1}{q(x_1)p'(x_1)}, \ldots, \frac{1}{q(x_n)p'(x_n)}\right)
\]

Therefore

\[
V_p D^{-1} = \left(\frac{x_k^{i-1}}{q((x_k)p'(x_k))_{ik}}\right)
\]

Clearly, \( V_p^T = (x_k^{j-1})_{kj} \). Hence, the inverse of Bezout matrix is

\[
B^{-1}(p,q) = V_p D^{-1} V_p^T = \left(\sum_{k=1}^{n} \frac{x_k^{i+j-2}}{q(x_k)p'(x_k)}\right)_{ij},
\]

which proves the theorem. \( \square \)

Hermite theorem was proved by Hermite in 1856, [10]. The theorem determines when all zeroes of a polynomial belong to the upper–half plane. This theorem can be proved in a framework of Vandermonde vectors, [19].

**Theorem 2.2** (Hermite theorem). All zeroes of the polynomial \( p(x) \) belongs to the upper–half plane if and only if the matrix \( \frac{1}{2i} B(p,p) \) is positive definite.
3. Modern applications

The simplest and very illustrative case of rational transformations of algebraic curves is a rational transformation of the complex line $\mathbb{C}$ into the complex plane $\mathbb{C}^2$. The image is a rational plane algebraic curve and the explicit formulas for this image in terms of the polynomials that define the rational transformation were obtained by Kravitsky in 1979, see [13].

**Theorem 3.1.** Three polynomials in two variables $p_0(x)$, $p_1(x)$ and $p_2(x)$ map complex line $\mathbb{C}$ into complex plane $\mathbb{C}^2$:

$$x \rightarrow \begin{pmatrix} p_1(x) \\ p_2(x) \\ p_0(x) \end{pmatrix} \frac{p_0(x)}{p_0(x)}$$

The image is a rational curve defined by a polynomial

$$\Delta(x_1, x_2) = \det(B(p_1, p_2) + x_1 B(p_2, p_0) + x_2 B(p_0, p_1))$$

**Proof.** Let $x$ and $y$ be two points on the curve, $V(x)$ and $V(y)$ be two Vandermonde vectors.

$$V^T(x)(p_0(x)B(p_1, p_2) + p_1(x)B(p_0, p_2) + p_2(x)B(p_0, p_1))V(y) = p_0(x)V^T(x)B(p_1, p_2)V(y) + p_1(x)V^T(x)B(p_2, p_0)V(y) + p_2(x)V^T(x)B(p_0, p_1)V(y) = (x - y)(p_0(x)(p_1(x)p_2(y) - p_2(x)p_1(y)) + (p_2(x)p_0(y) - p_0(x)p_2(y)) + (p_0(x)p_1(y) - p_1(x)p_0(y))) = 0$$

This identity holds for arbitrary $y$. Hence, $(p_0(x)B(p_1, p_2) + p_1(x)B(p_2, p_0) + p_2(x)B(p_0, p_1))V(x) = 0$ and therefore

$$\det \left( B(p_1, p_2) + \frac{p_1(x)}{p_0(x)} B(p_2, p_0) + \frac{p_2(x)}{p_0(x)} B(p_0, p_1) \right) = 0,$$

which implies the theorem. \hfill \square

3.1. Nonselfadjoint operators. The work of M. S. Livšic and his collaborators in operator theory associates to a system of commuting nonselfadjoint operators an algebraic curve (called the discriminant curve) given by a determinantal representation, see [15]. This discovery leads to a very fruitful interplay between operator theory and algebraic geometry: problems of operator theory lead to problems of algebraic geometry and vice versa.

A natural problem in operator theory is to define properly the notion of a rational transformation of a system of commuting nonselfadjoint operators. This arises whenever one wants to study the algebra generated by a given system of commuting nonselfadjoint operators. It may also allow representing the given system of commuting nonselfadjoint operators in terms of another system which is simpler in some sense (e.g., it contains fewer operators, or the operators have a smaller nonhermitian rank). A related problem in algebraic geometry is to find an image of an algebraic curve given by a determinantal representation under a rational transformation.

To formulate the main problem more precisely, we have to introduce some notation. We shall use the framework of commutative vessels [15] which turns out to be very convenient in the study of commuting nonselfadjoint operators; it generalizes to the multi–operator case the framework of colligations (nodes) which has been extensively used in the study of a single nonselfadjoint (or non-unitary) operator, see, e.g., [4].

Let $H$ be a Hilbert space (finite– or infinite–dimensional) and let $E$ be a finite–dimensional Hilbert space.
Definition 3.2. An operator node is a collection
\[ C = (A, H, \Phi, E, \sigma) \]
where \( A : H \to H \) is a bounded linear operator, \( \Phi \) is a bounded linear mapping from \( H \) to \( E \) with the adjoint mapping \( \Phi^* : E \to H \), \( \sigma \) is a bounded selfadjoint operator in \( E \), such that \( \Phi^* \sigma \Phi = \frac{1}{i}(A - A^*) \).

Definition 3.3. A commutative vessel is a collection
\[ V = (A_1, A_2, H, \Phi, E, \sigma_1, \sigma_2, \gamma^{in}, \gamma^{out}) \]
where \( A_1, A_2 : H \to H \) are bounded linear commuting operators, \( \Phi \) is a bounded linear mapping from \( H \) to \( E \) with the adjoint mapping \( \Phi^* : E \to H \), \( \sigma_1, \sigma_2, \gamma^{in}, \gamma^{out} \) are bounded selfadjoint operators in \( E \), such that \( \gamma^{in} = -\gamma^{in}, \gamma^{out} = -\gamma^{out} \), and
\[
\Phi^* \sigma_1 \Phi = \frac{1}{i}(A_i - A_i^*) \\
\gamma^{in} \Phi = \sigma_1 \Phi A_i^* - \sigma_2 \Phi A_i^* \\
\gamma^{out} \Phi = \sigma_1 \Phi A_2 - \sigma_2 \Phi A_1 \\
\gamma^{out} = \gamma^{in} + i(\sigma_1 \Phi \Phi^* \sigma_2 - \sigma_2 \Phi \Phi^* \sigma_1)
\]

For the simplest case, when a system of operators consists of a single operator and the discriminant curve is a line, these problems were solved by Kravitsky in 1979. Let us consider an operator node \( C = (A, H, \Phi, E, \sigma) \) and three polynomials \( p_0(x), p_1(x) \) and \( p_2(x) \) of degree \( n \), such that \( p_0(A) \) is an invertible operator. We define \( A_1 = p_1(A) p_0^{-1}(A) \) and \( A_2 = p_2(A) p_0^{-1}(A) \).

Theorem 3.4. A collection \( V = (A_1, A_2, H, \Phi', E', \sigma_1, \sigma_2, \gamma^{in}, \gamma^{out}) \), where
\[
E' = E^\otimes n \\
\Phi' = P_E(\Phi A_1^{i_1} A_2^{i_2} p_0^{-1}(A_1^i A_2^j))_{i_1,i_2} \\
\sigma_1 = B(p_0, p_1) \otimes \sigma \\
\sigma_2 = B(p_0, p_2) \otimes \sigma \\
\gamma^{in} = B(p_1, p_2) \otimes \sigma \\
\gamma^{out} = \gamma^{in} + i(\sigma_1 \Phi \Phi^* \sigma_2 - \sigma_2 \Phi \Phi^* \sigma_1)
\]
is a vessel and the discriminant curve of this vessel is defined by the equation
\[ \det^n(B(p_1, p_2) + x_1 B(p_2, p_0) + x_2 B(p_0, p_1)) = 0 \]

Explicit formulas for an image of a plane algebraic curve given by a determinantal representation under a rational transformation were obtained using generalization of Bezout matrices in [13]. These results allowed to define properly the rational image of a system of two operators where the discriminant curve is a plane algebraic curve, [20].

3.2. Quadrature domains. The last two decades have witnessed a renewed interest and constant progress in the theory of quadrature domains. Questions such as the constructions and parametrization of quadrature domains with prescribed distribution \( u \), the algebraic structure of the boundary or various functional analytic characterizations of quadrature domains have been successfully investigated, see [22], [16]. One of possible ways to find a polynomial that defines the boundary
of a quadrature domain is to consider such rational transformations of a complex 
plane that the image is a Riemann surface equipped with an involution.

The domain \( \Omega \subset \mathbb{C} \) is called a quadrature domain if there exists a distribution 
\( u \) with finite support in \( \Omega \) such that

\[
\int_\Omega f dA = u(f),
\]

for every integrable analytic function \( f \) in \( \Omega \). To be more specific, there are points 
\( \lambda_j \in \Omega \) and constants \( \gamma_{jk}, 0 \leq k \leq m(j) - 1, 1 \leq j \leq m \) such that

\[
\int_\Omega f dA = \sum_{j=1}^{m} \sum_{k=0}^{m(j)-1} \gamma_{jk} f^{(k)}(\lambda_j),
\]

where \( m(j) \) is the multiplicity of the point \( \lambda_j \). The quadrature domain is an image 
of the unit disk \( D \) under polynomial transformation defined by a polynomial \( q(z) \) and 
the boundary of the quadrature domain is given by the equation \( \Delta(z, \overline{z}) = 0 \).
To find the polynomial \( \Delta \) we consider a rational transformation of the complex plane: 
\( z \to \left(q(z), q(\overline{z})\right) \).

**Theorem 3.5.** If a quadrature domain \( \Omega \) is an image of the unit disk \( D \) under polynomial 
transformation defined by a polynomial \( q(z) \) then there exist three polynomials \( p_0(z), p_1(z) \) and \( p_2(z) \) such that \( q(z) = \frac{p_1(z)}{p_0(z)} \) and \( \overline{q(\frac{1}{z})} = \frac{p_2(z)}{p_0(z)} \). The boundary 
of \( \Omega \) is defined by the equation \( \det(B(p_1, p_2) + zB(p_2, p_0) + \overline{z}B(p_0, p_1)) = 0 \).

For the proof see [17].

### 3.3. Braid monodromy

In 1937 Zariski laid down the foundations for the braid monodromy of curves in \( \mathbb{CP}^2 \). The braid monodromy is a homomorphism between the fundamental group of a punctured disk and the braid group. We give here a short description of the braid monodromy.

Let us recall that the braid group \( B_n \) is the \( \text{MC} \Gamma \) (mapping class group) of the \( n \)-puncture disk. We distinguish some important elements in the braid group \( B_n \) which are called half-twists. Let \( D \) be a closed disk and let \( K = \{ k_1, \cdots, k_n \} \subset D \setminus \partial D \). Choose \( u \in \partial D \). Let \( a, b \) be two points of \( K \). We denote \( K_{a,b} = K \setminus \{a, b\} \). Let \( \sigma \) be a simple path in \( D \setminus (\partial D) \cup K_{a,b} \) connecting \( a \) with \( b \). Choose a small regular neighborhood \( U \) of \( \sigma \) and an orientation preserving diffeomorphism \( f: \mathbb{R}^2 \to \mathbb{C} \) such that \( f(\sigma) = [-1, 1], f(U) = \{ z \in \mathbb{C} \mid |z| < 2 \} \).

Let \( \alpha(x), 0 \leq x \) be a real smooth monotone function such that:

\[
\alpha(x) = \begin{cases} 
1, & 0 \leq x \leq \frac{1}{2} \\
0, & 2 \leq x
\end{cases}
\]

Define a diffeomorphism \( h: \mathbb{C} \to \mathbb{C} \) as follows: for \( z = re^{i\varphi} \in \mathbb{C} \) let \( h(z) = re^{i(\varphi + \alpha(\varphi))} \).

For the set \( \{ z \in \mathbb{C} \mid 2 \leq |z| \} \), \( h(z) = \text{Id} \), and for the set \( \{ z \in \mathbb{C} \mid |z| \leq \frac{3}{2} \} \), \( h(z) \) is a rotation by \( 180^\circ \) in the positive direction.

Considering \( (f \circ h \circ f^{-1})|_D \) (we will compose from left to right) we get a diffeomorphism of \( D \) which switches \( a \) and \( b \) and is the identity on \( D \setminus U \). Thus it defines an element of \( B_n[D, K] \).

The diffeomorphism \( (f \circ h \circ f^{-1})|_D \) defined above induces an automorphism on \( \pi_1(D \setminus K, u) \), that switches the position of two generators of \( \pi_1(D \setminus K, u) \).
Definition 3.6. Let $H(\sigma)$ be the braid defined by $(f \circ h \circ f^{-1})|_D$. We call $H(\sigma)$ the positive half-twist defined by $\sigma$.

Let $C$ be a real curve in $\mathbb{C}^2$ of degree $n$. Denote by $pr_1 : C \rightarrow \mathbb{C}$ and by $pr_2 : C \rightarrow \mathbb{C}$ the projections to the first and second coordinate, defined in the obvious way. For $x \in \mathbb{C}$ we denote $K(x)$ the projection of the points in $C$ which lie with $x$ as their first coordinate to the second coordinate (i.e., $K(x) = pr_2(pr_1^{-1}(x))$).

Let $N \subset \mathbb{C}$ be the set $N = N(C) = \{ x \in \mathbb{C} \mid |K(x)| < n \} = \{ x_1, \cdots, x_p \}$. We restrict ourselves only to the cases where $N$ is finite. Take $E$ to be a closed disc in $\mathbb{C}$ for which $N \subset E \setminus \partial E$. In addition take $D$ to be a closed disc in $\mathbb{C}$ for which $D$ contains all the points $\{ K(x) \mid x \in E \}$. That means that when restricted to $E$, we have $C \subset E \times D$.

With these definitions in hand we may define the braid monodromy of a projective curve:

Definition 3.7. Let $C$ be a projective curve of degree $n$ in $\mathbb{CP}^2$, $L$ be a generic line at infinity such that $|L \cap C| = n$, and $(x,y)$ is an affine coordinate system for $\mathbb{C}^2 = \mathbb{C}^2 \setminus L$ such that the projection of $C$ to the first coordinate is generic. For $E, D, N$ defined as above, let $M \in \partial E \cap \mathbb{R}$ be the base point of $\pi_1(E \setminus N)$, and let $\sigma$ be an element of $\pi_1(E \setminus N)$. To $\sigma$ there are $n$ lifts in $C$, each one of them begins and ends in the points of $M \times K(M)$. Projecting these lifts using $pr_2 : C \rightarrow \mathbb{C}$ we get $n$ paths in $D$ which begin and end in the points of $K(M)$. These induce a diffeomorphism of $\pi_1(D \setminus K(M))$ which is the braid group $B_n$ as defined earlier. We call the homomorphism $\varphi : \pi_1(E \setminus N) \rightarrow B_n$ the braid monodromy of $C$ with respect to $L, E \times D, pr_1$, and $M$.

It is natural to ask questions about the connection between the rational transformation and the braid monodromy induced by its image. For example one can formulate the following questions:

Question 3.8. Study the singular points of the image of a rational transformation, and define conditions on $p_0(x,y), p_1(x,y), p_2(x,y)$ which induce specific braid monodromy results.

Question 3.9. Let $C$ be a curve. let $r$ be a rational transformation. Classify all braid monodromy results which may result.

Question 3.10. Formulate necessary and sufficient conditions for a braid monodromy to be of rational curve.

Question 3.11. Given a braid monodromy, which satisfies the sufficient condition above. Formulate a family of rational curves which will induce such braid monodromy.

Question 3.12. Given two isomorphic rational curves. What can be said on their braid monodromies.

Of course, these questions are more than wide, and at this point may not be completely answered. In [12] we established two results concerning degree 2 rational transformations, as follows:

Let us consider the rational transformation

$$(x, y) \mapsto (p_0(x, y), p_1(x, y), p_2(x, y)).$$
In order to compute the local braid monodromy at the point \((p_0(x_0, y_0), p_1(x_0, y_0), p_2(x_0, y_0))\), we assume that \(p_0(x_0, y_0) \neq 0\). We define: \(r_1(x, y) = \frac{p_1(x, y)}{p_0(x, y)}\), \(r_2(x, y) = \frac{p_2(x, y)}{p_0(x, y)}\), and recursively

\[
D_1(x) = r'_2(x, 0) \cdot \frac{1}{r'_1(x, 0)} \\
D_n(x) = D'_{n-1}(x, 0) \cdot \frac{1}{r'_1(x, 0)} \\
E_1(y) = r'_2(0, y) \cdot \frac{1}{r'_1(0, y)} \\
E_n(y) = D'_{n-1}(0, y) \cdot \frac{1}{r'_1(0, y)}
\]

**Corollary 3.13.** Let \((p_0(x_0, y_0), p_1(x_0, y_0), p_2(x_0, y_0))\) be one of the intersection points of the two conics at the image \(r(C)\). Let \(i\) be the minimal index for which \(D_i(x_0) \neq E_i(y_0)\). Then, the multiplicity of the intersection point is \(i + 1\), and thus the local braid monodromy at this intersection point is \((i + 1)\) full twists of two strings.

**Theorem 3.14.** Let \(C\) be a curve which consists of two intersecting lines, and let \(r\) be a real rational transformation of degree 2. Then, the braid monodromy of \(r(C)\) is completely defined by the number and multiplicity of its real self intersection points.

Theorem 3.14 gives a full classification of the braid monodromy of the image of two intersecting lines under degree 2 rational transformations.
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SEVERAL APPLICATIONS OF BEZOUT MATRICES

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