On the Resolution of Time Problem in Quantum Gravity Induced from Unconstrained Membranes

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ABSTRACT

The relativistic theory of unconstrained $p$-dimensional membranes ($p$-branes) is further developed and then applied to the embedding model of induced gravity. Space-time is considered as a 4-dimensional unconstrained membrane evolving in an $N$-dimensional embedding space. The parameter of evolution or the evolution time $\tau$ is a distinct concept from the coordinate time $t = x^0$. Quantization of the theory is also discussed. A covariant functional Schrödinger equations has a solution for the wave functional such that it is sharply localized in a certain subspace $P$ of space-time, and much less sharply localized (though still localized) outside $P$. With the passage of evolution the region $P$ moves forward in space-time. Such a solution we interpret as incorporating two seemingly contradictory observations: (i) experiments clearly indicate that space-time is a continuum in which events are existing; (ii) not the whole 4-dimensional space-time, but only a 3-dimensional section which moves forward in time is accessible to our immediate experience. The notorious problem of time is thus resolved in our approach to quantum gravity. Finally we include sources into our unconstrained embedding model. Possible sources are unconstrained worldlines which are free from the well known problem concerning the Maxwell fields generated by charged unconstrained point particles.

Short title: Resolution of Time Problem in Quantum Gravity
1. Introduction

Since the pioneering works of Sakharov [1] and Addler [2] there has been increasing interest in various models of the induced gravity [3]. A particularly interesting and promising seems to be the model in which spacetime is a 4-dimensional manifold (a "spacetime sheet") $V_4$ embedded in an $N$-dimensional space $V_N$ [4]-[7]. The dynamical variables are the embedding functions $\eta^a(x)$ which determine positions (coordinates) of points on $V_4$ with respect to $V_N$. The action is a straightforward generalization [6],[7] of the Dirac-Nambu-Goto action. The latter can be written in an equivalent form in which there appears the induced metric $g_{\mu\nu}(x)$ and $\eta^a(x)$ as variables which have to be varied independently. Quantization of such action enables one to express an effective action as a functional of $g_{\mu\nu}(x)$. The effective action is obtained in the Feynman path integral in which we functionally integrate over the embedding functions $\eta^a(x)$ of $V_4$, so that what remains is a functional dependence on $g_{\mu\nu}(x)$. Such an effective action contains the Ricci curvature scalar $R$ and its higher orders [3]. This theory was discussed more detailly in a previous work [7].

In the present paper we are going to generalize the above approach. The main problem with any reparametrization invariant theory is the presence of constraints relating the dynamical variables. Therefore there exist equivalence classes of functions $\eta^a(x)$ - related by reparametrizations of the coordinates $x^\mu$ - such that each member of an equivalence class represents the same spacetime sheet $V_4$. This must be taken into account in the quantized theory, e.g. when performing, for instance a functional integration over $\eta^a(x)$. Though elegant solution to such problems were found in string theories [8], the technical difficulties accumulate in the case of a $p$-dimensional membrane ($p$-brane) with $p$ greater than 2 [4].

We first discuss the possibility of removing constraints from a membrane ($p$-brane) theory. Such a generalized theory possesses additional degrees of freedom and contains the usual $p$-branes of the Dirac-Nambu-Goto type as a special case. It is an extension, from a point-particle to a $p$-dimensional membrane, of a theory which treats a relativistic particle without constraint, so that all coordinates $x^\mu$ and the conjugate momenta $p_\mu$ are independent dynamical variables which evolve along the invariant evolution parameter $\tau$ [10]-[13]. A membrane is then considered as a continuum of such point particles and has no constraints. It was shown [14],[15] that the extra degrees of freedom are related to variable stress and fluid velocity on the membrane, which is therefor, in general, a "wiggly membrane". Then we apply the concept of a relativistic membrane without constraints to the embedding model of induced gravity in which the whole spacetime is considered as a membrane in a flat embedding space.

In Sec. 2 we develop the theory of an unconstrained relativistic $p$-brane (also called simply membrane, with understanding that its dimension $p$ is arbitrary), denoted $V_p$ (in contrast to a constrained membrane $V_p$). To facilitate the introduction of our concepts we use the usual notation, where variables $X^\mu, \mu = 0, 1, 2, ..., D-1$, represent coordinates of a membrane living in a $D$-dimensional spacetime, and $\xi^a, a = 0, 1, 2, ..., d-1$, are parameters of a worldsheet $V_d$ swept by membrane (with $p = d - 1$).

In Sec. 3 we apply the theory of Sec.2 to the concept of an $(n-1)$- dimensional simultaneity surface $V_{n-1}$ (analogous to a $p$-brane of Sec.2) moving in an $N$-dimensional
embedding space \( V_N \) and thus sweeping a space-time sheet \( V_n \) (analogous to the worldsheet \( V_d \) of Sec.2). Notation is here changed, and in some sense reversed: \( \eta^a(x), a = 0, 1, ..., N-1 \) are positions of a spacetime surface (called also sheet) \( V_n \) in the embedding space \( V_N \), and \( x^\mu, \mu = 0, 1, ..., n-1 \) are parameters (coordinates) on \( V_n \).

In Sec. 4, we consider the theory in which the whole space-time is an \( n \)-dimensional unconstrained membrane \( V_n \) analogous to a \( p \)-brane \( V_p \) of Sec.2. The theory allows for motion of \( V_n \) in the embedding space \( V_N \). When considering the quantized theory it turns out that a particular wave packet functional exists such that:

(i) it approximately represents evolution of a simultaneity surface \( V_{n-1} \) (also denoted \( V_\Sigma \)), and

(ii) all possible space-time membranes \( V_p \) composing the wave packet are localized near an average space-time membrane \( V_n^{(c)} \) which corresponds to a classical space-time unconstrained membrane.

This approach gives both: the evolution of a state (to which classically there corresponds the progression of time slice) and a fixed spacetime as the expectation value. The notorious problem of time, as it occurs in a reparametrization invariant theory (for instance in general relativity), does not exist in our approach.

2. Relativistic membranes without constraints

2.1. A reformulation of the conventional p-brane action

Relativistic \( p \)-dimensional constrained membranes \([9]\), including strings \((p = 1)\) \([8]\) and point particles \((p = 0)\), are commonly described by an action which is invariant under reparametrizations of coordinates \( \xi^a, a = 0, 1, 2, ..., p, \) of the \( d = p + 1 \) dimensional worldsheet \( V_d \) swept by a \( p \)-dimensional membrane. Consequently, the dynamical variables \( X^\mu, \mu = 0, 1, 2, ..., D \) and the corresponding momenta are subjected to \( d \) primary constraints; not all \( X^\mu \) are independent, there are \( d \) relations among them.

A suitable form of the action \([16]\) (equivalent to the Dirac-Nambu-Goto action \([17]\)) is

\[ I[X^\mu, \gamma^{ab}] = \frac{\kappa}{2} \int d^d \xi \sqrt{|\gamma|} \left( \gamma^{ab} \partial_a X^\mu \partial_b X^\nu g_{\mu\nu} + 2 - d \right) \]  

(1)

where \( X^\mu \) and \( \gamma^{ab} \) are to be varied independently. Variation of \( \gamma^{ab} \) gives the expression for the induced metric \( \gamma_{ab} = \partial_a X^\mu \partial_b X^\nu g_{\mu\nu} \) on \( V_d \). The Lagrange multipliers \( \gamma^{ab} \) are not all independent: there are \( d(d+1)/2 \) components of \( \gamma^{ab} \), while there are only \( d \) constraints.

In order to separate \( d \) independent Lagrange multipliers we perform an ADM-like \([18]\) decomposition of \( V_d \) such that \([19]\)

\[ \gamma^{ab} = \frac{n^a n^b}{n^2} + \bar{\gamma}^{ab} \]  

(2)

where \( n^a \) is the normal vector field to a chosen hypersurface \( \Sigma \) and \( \bar{\gamma}^{ab} \) the projection tensor which projects an arbitrary vector into \( \Sigma \). For instance, \( \bar{\gamma}^{ab} \) projects a derivative

\footnote{Usually \( n = 4 \), but if we wish to consider a Kaluza-Klein like theory, then \( n > 4 \).}
\[ \partial_a X^\mu \] into the tangent derivative:

\[ \bar{\partial}_a X^\mu = \bar{\gamma}^b_a \partial_b X^\mu = \partial_a X^\mu - n_a \partial X^\mu \] (3)

where \( \partial X^\mu \equiv n^b \partial_b X^\mu / n^2 \) is the normal derivative.

Let us take such a class of coordinates system in which covariant components of normal vectors are \( n_a = (1, 0, 0, ..., 0) \). Then we have

\[ \gamma^{00} = n^0 = n^a n_a, \quad \gamma^{0i} = n^i, \quad \gamma^{ij} = \bar{\gamma}_{ij} + n^i n^j / n^0 \] (4)

\[ \gamma_{00} = \frac{1}{n^0} + \bar{\gamma}_{ij} \frac{n^i n^j}{(n^0)^2}, \quad \gamma_{0i} = -\bar{\gamma}_{ij} n^j / n^0, \quad \gamma_{ij} = \bar{\gamma}_{ij}, \quad i, j = 1, 2, ..., p \] (5)

The decomposition (3) then becomes

\[ \partial_0 X^\mu = \partial X^\mu + \bar{\partial}_0 X^\mu \] (6)

\[ \partial_i X^\mu = \bar{\partial}_i X^\mu \] (7)

where

\[ \dot{X}^\mu \equiv \partial_0 X^\mu \equiv \frac{\partial X^\mu}{\partial \xi^0} \] (8)

\[ \partial X^\mu = \dot{X}^\mu + \frac{n^i \partial_i X^\mu}{n^0} \quad \partial_i X^\mu \equiv \frac{\partial X^\mu}{\partial \xi^i} \] (9)

As \( d \) independent Lagrange multipliers can be taken \( n^a = (n^0, n^i) \). We can now rewrite our action in terms of \( n^0 \) and \( n^i \). We insert (4) into (1) and take into account that

\[ |\gamma| = \frac{\bar{\gamma}}{n^0} \] (10)

where \( \gamma = \det \gamma_{ab} \) is the determinant of the worldsheet metric and \( \bar{\gamma} = \det \bar{\gamma}_{ij} \) the determinant of the metric \( \bar{\gamma}_{ij} = \gamma_{ij} \) on the hypersurface \( \Sigma \).

After using (4-10) our action (1) becomes a functional \( I[X^\mu, n^a, \bar{\gamma}^{ij}] \) of \( X^\mu \) and the Lagrange multipliers \( n^a, \bar{\gamma}_{ij} \). Variation of \( I[X^\mu, n^a, \bar{\gamma}^{ij}] \) with respect to \( \bar{\gamma}^{ij} \) gives the expression for the induced metric on the surface \( \Sigma \)

\[ \bar{\gamma}_{ij} = \partial_i X^\mu \partial_j X_\mu, \quad \bar{\gamma}^{ij} \bar{\gamma}_{ij} = d - 1 \] (11)

Using the latter expression (11) we can eliminate \( \bar{\gamma}^{ij} \) from \( I[X^\mu, n^a, \bar{\gamma}^{ij}] \) and we obtain (4-11) a functional of \( X^\mu \) and \( d \) independent Lagrange multipliers \( n^a = (n^0, n^i) \)

\[ I[X^\mu, n^a] = \frac{\kappa}{2} \int d\tau d^p \sigma \sqrt{\bar{f}} \left( \frac{\partial X^\mu \partial X_\mu}{\lambda} + \lambda \right), \quad \lambda \equiv \frac{1}{\sqrt{n^0}} \] (12)

where \( \partial X^\mu \) is given by (1) and \( \bar{f} \equiv \det \bar{f}_{ij}, \bar{f}_{ij} \equiv \partial_i X^\mu \partial_j X_\mu \). In eq.(12) the coordinates are split according to \( \xi^a = (\xi^0, \xi^i) \equiv (\tau, \sigma^i) \) and the volume element written as \( d^d \xi = d\tau d^p \sigma \). Instead of \( n^0 \) a new symbol \( \lambda \equiv 1/\sqrt{n^0} \) is introduced.
So we have arrived at an action which looks like the well known Howe-Tucker action for a point particle, apart from the integration over coordinates $\sigma^i$ of a space-like hypersurface $\Sigma$. Indeed, eq. (12) is an action of a continuous collection of point particles: it is a functional of a bundle $X^\mu(\tau, \sigma^i)$ of worldlines. Individual worldlines are distinguished by the values of parameters $\sigma^i$.

The equations of motion for the variables $X^\mu$ derived from (12) are exactly the equations of a minimal surface (as derived directly from (1)); the equations of ”motion” for $n^a$ are the worldsheet constraints [14], [19].

2.2 A generalized $p$-brane action

So far we had just a suitable reformulation of the well known Dirac-Nambu-Goto theory of minimal surfaces. Now we shall do a crucial step: let us fix $n^a = (n^0, n^i)$ in eq. (12) so that $n^a$ are no longer Lagrange multipliers, but given functions of $\tau$ and $\sigma$. In the conventional approaches such a fixing is interpreted as gauge fixing and the action (12) with fixed $n^a$ contains, besides the physical ones, also the unphysical degrees of freedom which must be compensated by an additional term, the so called gauge fixing term.

My aim is to go beyond the conventional theory. As an alternative to the conventional action I proposed [14], [15] the following new action:

\[ I[X^\mu] = \frac{\kappa}{2} \int d\tau d\sigma \sqrt{\bar{f}} \left( \frac{\dot{X}^\mu \ddot{X}^\mu}{\Lambda} + \Lambda \right) \]  

The old action (12) served only as a guidance for introducing the new action (13). While the action (12) is equivalent to the conventional action (1), the new action (13) is not equivalent to (1). The step from (13) to (12) by fixing $n^0 \neq 0$ and $n^i = 0$ brings a new physical content into the membrane’s theory. In this paper we consider a particular fixing of $n^i$, namely $n^i = 0$. A more general choice of $n^i$ is also possible [19].
of freedom of our unconstrained membrane $\mathcal{V}_p$ are related to the variable tension and fluid velocity of a wiggly membrane.

Our action (13) is invariant with respect to reparametrizations $\sigma^i \to \sigma'^i(\sigma)$ of the parameters on a membrane $\mathcal{V}_p$. But, since this invariance does not involve the evolution parameter $\tau$, it does not imply constraints among the dynamical variables $X^\mu(\tau, \sigma)$ (and the corresponding momenta $p_\mu(\tau, \sigma)$).

We can introduce the auxiliary variables $\tilde{\gamma}^{ij}$ and write an action which is equivalent to (13):

$$I[X^\mu, \tilde{\gamma}^{ij}] = \frac{\kappa}{4} \int d\tau d^p\sigma \left( \sqrt{\tilde{\gamma}} |\tilde{\gamma}^{ij} \partial_i X^\mu \partial_j X_\mu + 2 - p) \right) \left( \frac{\dot{X}^2}{\Lambda} + \Lambda \right)$$

(15)

Variation of (15) with respect to $\tilde{\gamma}^{ij}$ gives the relation

$$\tilde{\gamma}_{ij} = \partial_i X^\mu \partial_j X_\mu$$

(16)

which tells that $\tilde{\gamma}^{ij}$ is the induced metric on the $p$-dimensional surface $\mathcal{V}_p$ in $D$-dimensional embedding space. Eq.(16) does not express constraints among $X^\mu$, it is merely the definition equation for the auxiliary variables $\tilde{\gamma}^{ij}$.

The initial conditions for our dynamical system described by the action (13) or (15) are given by

$$X^\mu(\tau = 0, \sigma)$$

(17)

$$\dot{X}^\mu(\tau = 0, \sigma)$$

(18)

Equation (17) determines a $p$-dimensional unconstrained membrane $\mathcal{V}_p$ at the initial value of the parameter $\tau$. Equation (18) determines a field of velocities at $\tau = 0$. A solution of the dynamical equations derived from (13) determines the membrane $\mathcal{V}_p$ at various values of parameter $\tau$; in other words, it determines membrane’s motion in space-time. When $\mathcal{V}_p$ moves it sweeps a $d$-dimensional surface $V_d$. An initial $\mathcal{V}_p$ is given arbitrarily, and its parametric equation is $x^\mu = X^\mu(\sigma^i), \mu = 0, 1, ..., D - 1; \ i, j = 0, 1, ..., d - 1$. Once $\mathcal{V}_p$ is given, we can calculate the tangential derivatives $\partial_i X^\mu$ and the induced metric (16). This illustrates that none of the tangential derivatives $\partial_i X^\mu$ (and consequently the induced metric $\tilde{\gamma}_{ij}$) is given independently as initial condition, and therefore eq.(16) indeed does not imply any constraint among the dynamical variables $X^\mu(\tau, \sigma)$. Though in our theory a local gauge group is present (invariance of the action (13) under reparametrizations of $\sigma^i$), yet solving the equations of motion derived from the action (13) with a given set of initial data (17),(18) constitutes a well-posed Cauchy problem. This is true, because the gauge group does not involve the evolution parameter $\tau$.

Let me try to further clarify this point. First, we assume that points of the membrane $\mathcal{V}_p$ are physically distinguishable. They can be marked and later, after the passage of evolution again identified. Next, we give ”house numbers” to the points on $\mathcal{V}_p$, that is, we choose a parametrization $\sigma^i$. The choice is arbitrary, and the theory is invariant under

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3 Analogous situation occurs in the description of non-relativistic motion of a usual 1-dimensional string or 2-dimensional membrane in 3-dimensional space, with the ordinary time $t$ as evolution parameter. The fact that one can arbitrarily parametrize string or membrane does not imply dynamical constraints in such a non-relativistic motion.
reparametrizations $\sigma^i \to \sigma'^i = \sigma'^i(\sigma^i)$. Suppose we have chosen an initial membrane and fixed a parametrization (i.e. choice of coordinates $\sigma^i$) on it, so that $X\mu(0, \sigma)$ is given. The dynamical equations of motion (straightforwardly derived from the action $\mathcal{I}$ and the initial data (13)) then determine $X\mu(\tau, \sigma)$ at arbitrary value of the evolution parameter. For different choices of initial velocities $\dot{X}\mu(0, \sigma)$, $\dot{X}'\mu(0, \sigma)$ we obtain different $X\mu(\tau, \sigma)$, $X'\mu(\tau, \sigma)$. In particular we can choose $\dot{X}'\mu(0, \sigma)$ such that $X'\mu(\tau, \sigma)$ describes from the mathematical point of view the same manifold $V$ as it is represented by $X\mu(\tau, \sigma)$. But physically, $X\mu(\tau, \sigma)$ and $X'\mu(\tau, \sigma)$ represent different objects: the former membrane is deformed in some way, and the latter membrane is deformed in some other way. This illustrates that our system is indeed a "wiggly" membrane.

The transformations at a fixed value of $\tau$

$$X\mu(\sigma) \to X'\mu(\sigma)$$

we interpret as active transformations of membrane’s positions (Fig.1) in spacetime. They transform one membrane’s configuration into another configuration which may be elastically deformed.

In general, the transformations at arbitrary values of $\tau$ and $\sigma^i$

$$X\mu(\tau, \sigma) \to X'\mu(\tau, \sigma)$$

we interpret as active transformations of membrane’s motion in spacetime. Kinematically all possible transformations of the type (20) (with certain restrictions concerning non-singularity and single-valuedness) are allowed, but dynamically (as relating solutions of the equations of motion) only a subclass is allowed.

When performing quantization by using path integral approach one calculates the transition amplitude given by the functional integral

$$<X_2(\sigma), \tau_2|X_1(\sigma), \tau_1> = \int e^{i\mathcal{I}[X\mu]} DX\mu(\tau, \sigma)$$

Different functions $X\mu(\tau, \sigma)$ over which the functional integration is performed are understood in the active sense (as described above). They represent various kinematically possible motions of the elastically deformed membrane. Since all $X\mu(\tau, \sigma)$ are physically distinguishable, we do not need to introduce ghosts.

On the contrary, in the local gauge theory of the usual $p$-branes, a class of functions $X\mu(\xi^a)$ which can be transformed one into the other by coordinate transformations of the worldsheet coordinates $\xi^a$ cannot be interpreted as representing physically different membrane’s motions. Therefore one needs to cancel the unphysical degrees of freedom, and a convenient way to do this is to take into account ghosts in order to treat functional integrals consistently.

4 Again we have the analogy with a usual non-relativistic elastic string or membrane. It can be elastically deformed in such a way that the mathematical manifold $V_p$ ($p = 1$ or 2) remains the same, but nevertheless a deformed object $V'_p$, described by $\mathbf{x}'(\sigma)$, is physically different from the "original" object $V_p$ described by $\mathbf{x}(\sigma)$. Both $\mathbf{x}(\sigma)$ and $\mathbf{x}'(\sigma)$ describe the same mathematical manifold $V_p$, but $\mathbf{x}'(\sigma)$ now represents positions of an elastically deformed string or membrane.
Some more details about quantized unconstrained $p$-branes we discuss in Sec. 4. Here let us just mention that a $p$-brane’s state can be represented by a wave functional $\psi[\tau, X^\mu(\sigma)]$ which evolves along the evolution parameter $\tau$. A wave functional is in general a wave packet localized around a "centroid" $p$-brane (Fig. 2). As in the case of unconstrained point particle [11]-[13], a wave packet is localized in space-time and the region of localization proceeds forward along a time-like direction while the evolution parameter $\tau$ increases. During such a motion in space-time the centroid $p$-brane describes a $p+1$ dimensional worldsheet.

The wave functional is normalized in space-time so that at any $\tau$ we have

$$\int \psi^*[\tau, X^\mu(\sigma)] \psi[\tau, X^\mu(\sigma)] \mathcal{D}X(\sigma) = 1 \quad (22)$$

and consequently the evolution operator $U$ which brings $\psi(\tau) \to \psi(\tau') = U \psi(\tau)$ is unitary. In the particular case of a 0-brane (i.e. a point particle) the above expression (22) reads $\int \psi^*(\tau, x)\psi(\tau, x)d^4x = 1$, but in a generic case ($p \geq 1$) the measure $d^4x$ is replaced by

$$\mathcal{D}X(\sigma) = \prod_{\sigma,\mu} dX^\mu(\sigma) \bar{\gamma}^{1/4} \quad (23)$$

where $\bar{\gamma}$ is the determinant of the induced metric on $V_p$. (For details about the reparametrization invariant measure in curved space and the origin of $\gamma^{1/4}$ see Ref. [21].)

3. **Application to the embedding model of induced gravity: a spacetime sheet generated by a 3-brane motion**

The ideas that we have developed so far may be used to describe elementary particles as extended objects - unconstrained $p$-branes $V_p$ - living in spacetime. In the following we are going to follow yet another application of $p$-branes: to represent spacetime itself! Spacetime is considered as a surface -called also spacetime sheet $V_n$ embedded in a higher dimensional space $V_N$. For details about this model see Refs. [4]-[7]. In this section we consider a particular model in which an $n-1$ dimensional surface, called simultaneity surface $V_\Sigma$ moves in the embedding space according to the unconstrained theory of Sec. 2 and sweeps an $n$-dimensional spacetime sheet $V_n$.

Since we are now talking about spacetime which is conventionally parametrized by coordinates $x^\mu$, the notation of Sec. 2 is not appropriate. For this particular application of the $p$-brane theory we use different notation. Coordinates denoting position of a spacetime sheet $V_n$ (alias worldsheet) in the embedding space $V_N$ are

$$\eta^a, \quad a = 0, 1, 2, ..., N - 1 \quad (24)$$

whilst parameters denoting positions of points on $V_n$ are

$$x^\mu, \quad \mu = 0, 1, 2, ..., n - 1 \quad (25)$$
The parametric equation of a spacetime sheet is
\[ \eta^a = \eta^a(x) \] (26)

Parameters on a simultaneity surface \( V_\Sigma \) are
\[ \sigma^i, \quad i = 1, 2, \ldots, n - 1 \] (27)
and its parametric equation is \( \eta^a = \eta^a(\sigma) \). In particular, one may choose such a parametrization of \( V_n \) that \( x^i = \sigma^i \). A moving \( V_\Sigma \) is described by the variables \( \eta^a(\tau, \sigma) \).

The formal theory goes along the similar lines as in Sec. 2. The action is given by
\[ I[\eta^a(\tau, \sigma)] = \frac{1}{2} \int \omega \, d\tau \, d^{n-1}\sigma \sqrt{\bar{f}} \left( \frac{\dot{\eta}^\mu \dot{\eta}_\mu}{\Lambda} + \Lambda \right) \] (28)
where \( \bar{f} \equiv \text{det} \bar{f}_{ij}, \bar{f}_{ij} \equiv \frac{\partial \eta^a}{\partial \sigma^i} \frac{\partial \eta^a}{\partial \sigma^j} \) is the determinant of the induced metric on \( V_\Sigma \), and \( \Lambda = \Lambda(\tau, \sigma) \) a fixed function. The tension \( \kappa \) is now replaced by the symbol \( \omega \). The latter may be a constant. However, in the proposed embedding model of spacetime \[6\],[7] we admit \( \omega \) to be a function of position in \( V_N \):
\[ \omega = \omega(\eta) \] (29)

In the case when \( \omega \) is a constant we have a spacetime without "matter" sources. When \( \omega \) is a function of \( \eta^a \), we have in general a spacetime with sources (see Ref. \[7\] and Sec. 4.3)

A solution to the equations of motion derived from (28) represents a motion of a simultaneity surface \( V_\Sigma \). This is analogous to motion of a \( p \)-brane discussed in Sec. 2. Here again we see a big advantage of such an unconstrained theory: it predicts actual motion of \( V_\Sigma \) and evolution of a corresponding quantum state with \( \tau \) being the evolution parameter or historical time \[11\]. The latter is a distinct concept from the coordinate time \( t \equiv x^0 \). The existence (and progression) of a time slice is automatically incorporated in our unconstrained theory. It not need be separately postulated, as it is in the usual, constrained relativistic theory \[14\]. And, since an observer cannot perceive the whole spacetime at once, one cannot simply dispense with the existence of a time slice. What an observer directly experiences or is aware of, are events on \( V_\Sigma \). He has only (fading) memories of the past events and expectation of future events, but he doesn’t experience neither past nor future events. Later we shall see that even the concept of "time slice" is provisory and can be replaced by a suitably generalized concept.

The theory based on the action (28) is satisfactory in several respects. However, it still cannot be considered as a complete theory, because it is not manifestly invariant with respect to general coordinate transformations of spacetime coordinates (which include

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5 To simplify notation we use the same symbol \( \eta^a \) to denote coordinates of an arbitrary point in \( V_N \) and also to denote the embedding variables (which are functions of \( x^\mu \)).

6 More or less explicit assumption of the existence of a time slice (associated with the perception of "now") is manifest in conventional relativistic theories from the very fact that the talk is about "point-particles" or "strings" which are objects in three dimensions.
Lorentz transformations). In the next section we shall ”improve” the theory and explore some of its consequences. We shall see that the theory of motion of a time slice $\mathcal{V}_\Sigma$, based on the action (28), comes out as a particular case (solution) of the generalized theory which is fully relativistic, i.e. invariant with respect to reparametrizations of $x^\mu$. Yet it incorporates the concept of state evolution.

4. Spacetime as a moving 4-dimensional membrane in $V_N$

4.1. General considerations

Experimental basis\footnote{The crucial is the fact that different observers (in relative motion) determine different sets of events as being simultaneous and thus those events must exist in a 4-dimensional spacetime in which time is just one of the coordinates.} on which rests the special relativity and its generalization to curved spacetime clearly indicates that spacetime is a continuum in which events are existing. On the contrary, our subjective experience clearly tells us that not the whole 4-dimensional spacetime, but only a 3-dimensional section of it is accessible to our immediate experience. How to reconcile those seemingly contradictory observations?

It turns out that this is naturally achieved by joining the formal theory of membrane motion (Sec. 2) with the concept of spacetime embedded in a higher dimensional space $V_N$ (Sec. 3). Let us assume that the spacetime is an unconstrained 4-dimensional membrane $V_4$ which evolves (or moves) in the embedding space $V_N$. What was a membrane (or p-brane) in Sec. 2 is now a spacetime sheet $V_4$. In other words, $V_4$ (or $V_n$ in general) is now analogous to $V_p$ of Sec. 2. Positions of points on $V_4$ at a given instant of the evolution time $\tau$ are described by embedding variables $\eta^a(\tau, x^\mu)$. The latter now depend not only on the spacetime sheet parameters (coordinates) $x^\mu$, but also on $\tau$. Let us at the moment just accept such a possibility that $V_4$ evolves, and we shall later see how the quantized theory brings a physical sense to such an evolution.

The action which is analogous to one of eq.(28) (which in turn is analogous to eq.(13) is

$$I[\eta^a(\tau, x)] = \frac{1}{2} \int \omega d\tau d^4x \sqrt{|f|} \left( \frac{\dot{\eta}^a \dot{\eta}_a}{\Lambda} + \Lambda \right)$$  \hspace{1cm} (30)

$$f \equiv \det f_{\mu\nu} \ , \ f_{\mu\nu} \equiv \partial_{\mu} \eta_{a} \partial_{\nu} \eta_{a}$$  \hspace{1cm} (31)

where $\Lambda = \Lambda(\tau, x)$ is a fixed function of $\tau$ and $x^\mu$ (like a ”background field”) and $\omega = \omega(\eta)$.

The action (30) is invariant with respect to arbitrary transformations of spacetime coordinates $x^\mu$. But it is not invariant under reparametrizations of the evolution parameter $\tau$. Again we use analogous reasoning as in Sec. 2. Namely, the freedom of choice of parametrization on a given initial $V_4$ is trivial and it does not impose any constraints among the dynamical variables $\eta^a$ which depend also on $\tau$. In other words, we consider spacetime $V_4$ as a physical continuum, the points of which can be identified and their $\tau$-evolution in the embedding space $V_N$ followed. For a chosen parametrization $x^\mu$ of the points on $V_4$ different functions $\eta^a(x), \eta'^a(x)$ (at arbitrary $\tau$) represent different physically
deformed spacetime continua $\mathcal{V}_4$, $\mathcal{V}_4'$. Different functions $\eta^a(x)$, $\eta'^a(x)$, even if denoting positions on the same mathematical surface $V_4$ will be interpreted as describing physically distinct spacetime continua, $\mathcal{V}_4$, $\mathcal{V}_4'$, locally deformed in different ways. An evolving physical spacetime continuum $\mathcal{V}_4$ is not identical concept to a mathematical surface $V_4$. Let us now start developing some basic formalism. The canonically conjugate variables belonging to the action (30) are

$$\eta^a(x), \quad p_a(x) = \frac{\partial L}{\partial \dot{\eta}^a} = \omega \sqrt{|f|} \frac{\dot{\eta}^a}{\Lambda}$$  \hfill (32)

The Hamiltonian is

$$H = \frac{1}{2} \int d^4x \sqrt{|f|} \frac{\Lambda}{\omega} \left( \frac{p^a p_a}{\sqrt{|f|}} - \omega^2 \right)$$  \hfill (33)

The theory can be straightforwardly quantized by considering $\eta^a(x)$, $p_a(x)$ as operators satisfying the equal $\tau$ commutation relations

$$[\eta^a(x), p_b(x')] = \delta^a_b \delta(x - x')$$  \hfill (34)

In the representation in which $\eta^a(x)$ are diagonal the momentum operator is given by the functional derivative

$$p_a = -i \frac{\delta}{\delta \eta^a(x)}$$  \hfill (35)

A quantum state is represented by a wave functional $\psi[\tau, \eta^a(x)]$ which depends on the evolution parameter $\tau$ and the coordinates $\eta^a(x)$ of a physical spacetime sheet $\mathcal{V}_4$, and satisfies the functional Schrödinger equation

$$i \frac{\partial \psi}{\partial \tau} = H \psi$$  \hfill (36)

The Hamiltonian operator is given by eq.(33) in which $p_a$ are now operators (35). A possible solution to eq.(36) is a linear superposition of states with definite momentum $p_a(x)$ which are taken as constant functionals of $\eta^a(x)$, so that $\delta p_a / \delta \eta^a = 0$ :  

$$\psi[\tau, \eta(x)] = \int \mathcal{D}p \, c(p) e^{-iH\tau} e^{i \int p_a(x) \eta^a(x) \, d^4x}$$  \hfill (37)

where $H$ is given by (33) and $p_a(x)$ are now eigenvalues of the corresponding operators. The expectation value of an operator $A$ is

$$< A >= \int \psi^*[\tau, \eta(x)] A \psi[\tau, \eta(x)] \, \mathcal{D}\eta$$  \hfill (38)

\footnote{A strict notation would then require a new symbol, for instance $\tilde{\eta}^a(x)$ for the variables of the physical continuum $\mathcal{V}_4$, to be distinguished from the embedding functions $\eta'^a(x)$ of a mathematical surface $V_4$. We shall not use this distinction in notation, since the meaning will be clear from the context.}
The measures $D\eta$ and $Dp$ should be invariant under reparametrizations of $x^\mu$. This is achieved if we define (following Ref. \[21\])

\[ D\eta \equiv \prod_{a,x} |f|^{1/4} d\eta^a(x) \]  

\[ Dp \equiv \prod_{a,x} |f|^{-1/4} dp_a(x) \]

The above expressions result if we take into account the following invariant scalar products

\[ \int d\eta^a(x) d\eta_a(x) \sqrt{|f|} d^4x \]  

\[ \int \frac{dp^a(x) dp_a(x)}{|f|} \sqrt{|f|} d^4x \]

so that the metrics are $|f|^{1/2} \eta_{ab} \delta(x - x')$ and $|f|^{-1/2} \eta_{ab} \delta(x - x')$, respectively. Into the definition of the invariant volume elements $D\eta$ and $Dp$ then enter the square roots of the determinants of the corresponding metric.

### 4.2 A physically interesting solution

Let us now pay attention to eq.(37). It defines a wave functional packet spread over a continuum of functions $\eta^a(x)$. The expectation value of $\eta^a(x)$ is

\[ \langle \eta^a(x) \rangle = \eta^a_{\text{c}}(\tau, x) \]  

where $\eta^a_{\text{c}}(\tau, x)$ represents motion of the centroid spacetime sheet $V_4^{(\text{c})}$ which is the ”centre” of the wave functional packet. This is illustrated in Fig. 3.

In general, the theory admits an arbitrary motion $\eta^a_{\text{c}}(\tau, x)$ which is a solution of the classical equations of motion derived from the action (30). But in particular, a wave packet (37) which is a solution of the Schrödinger equation (36) can be such that its centroid spacetime sheet is either

(i) at rest in the embedding space $V_N$, i.e. $\dot{\eta}^a_{\text{c}} = 0$

or, more generally:

(ii) it moves ”within itself” so that its shape does not change with $\tau$. More precisely, at every $\tau$ and $x^\mu$ there exist a displacement $\Delta x^\mu$ such that $\eta^a_{\text{c}}(\tau + \Delta \tau, x^\mu) = \eta^a_{\text{c}}(\tau, x^\mu + \Delta x^\mu)$, which implies $\dot{\eta}^a_{\text{c}} = \partial_\mu \eta^a_{\text{c}} \dot{x}^\mu$. Therefore $\dot{\eta}^a_{\text{c}}$ is always tangent to a fixed mathematical surface $V_4$ which does not depend on $\tau$.

Now let us consider a special form of the wave packet as illustrated in Fig. 4. Within the effective boundary $B$ a given function $\eta^a(x)$ is admissible with high probability, outside
\[ B \] with low probability. Such is, for instance, a Gaussian wave packet which, at the initial \[ \tau = 0 \], is given by

\[
\psi[0, \eta^a(x)] = N e^{- \int d^4x \sqrt{|f| \frac{\eta^a(x) - \eta^a_c(x)}{2\sigma(x)}}^2}
\]  

(44)

where the function \( \sigma(x) \) vary with \( x^\mu \) so that the wave packet corresponds to Fig. 4.

Of special interest in Fig. 4. is the region \( P \) around a spacelike hypersurface \( \Sigma \) on \( \mathcal{V}_4^{(c)} \). In that region the wave functional is much more sharply localized than in other regions (that is, at other values of \( x^\mu \)). This means that in the neighborhood of \( \Sigma \) a spacetime sheet \( \mathcal{V}_4 \) is relatively well defined. On the contrary, in the regions that we call past or future, space-time is not so well defined, because the wave packet is spread over a relatively large range of functions \( \eta^a(x) \) (each representing a possible spacetime sheet \( \mathcal{V}_4 \)).

The above situation holds at a certain, let us say initial value of the evolution parameter \( \tau \). Our wave packet satisfies the Schrödinger equation and is therefore subjected to evolution. The region of sharp localization depends on \( \tau \), and so it moves as \( \tau \) increases. In particular, it can move within the mathematical spacetime surface \( V_4 \) which corresponds to such a ”centroid“ (physical) spacetime sheet \( \mathcal{V}_4^{(c)} = < \mathcal{V}_4 > \) which ”moves within itself“ (case (ii) above). Such a solution of the Schrödinger equation provides, on the one hand the existence of a fixed spacetime \( V_4 \), defined within the resolution of the wave packet (see Fig. 4), and on the other hand the existence of a moving region \( P \) in which the wave packet is more sharply localized. The region \( P \) represents the ”present“ of an observer.

We assume that an observer in principle measures the embedding positions \( \eta^a(x) \) of the entire spacetime sheet. Every \( \eta^a(x) \) is in principle possible. However, in the practical situations available to us, a possible measurement procedure is expected to be such that only the embedding positions \( \eta^a(x^\mu_{\Sigma}) \) of the simultaneity surface \( \Sigma \) are measured with high precision\(^{10}\), whereas the embedding positions \( \eta^a(x) \) of all other regions of spacetime sheet are measured with low precision. As a consequence of such a measurement a wave packet like one of Fig. 4 and Eq.(44) is formed and it is then subjected to the unitary \( \tau \)-evolution given by the covariant functional Schrödinger equation (36).

Using our theory, which is fully covariant with respect to reparametrizations of spacetime coordinates \( x^\mu \), we have thus arrived in a natural way at the existence of time slice \( \Sigma \) which corresponds to the ”present“ experience and which progresses forward in spacetime. The theory of Sec. 3 is just a particular case of this more general theory. This can be seen by taking the limit \( 1/\sigma(x) \propto \delta^4(x^\mu - x^\mu_{\Sigma}) \) in the wave packet (44). Then the integration over the \( \delta \)-function gives in the exponent the expression

\[
\int d^3x \sqrt{|f|} (\eta^a(i) - \eta^a_c(i))^2/2\sigma(x^i) = 1, 2, 3 \text{ so that eq.(44) becomes a wave functional of 3-dimensional membranes } \eta^a(x^i)
\]

So far we have taken that the region of sharp localization \( P \) of a wave functional packet is situated around a space-like surface \( \Sigma \), and so we obtained a time slice. But there is a difficulty with the concept of ”time slice“ related to the fact that an observer in practice never have the access to the experimental data on an entire spacelike hypersurface. Since

\(^{10}\)Another possibility is to measure the induced metric \( g_{\mu\nu} \) on \( \mathcal{V}_4 \), and measure \( \eta^a(x) \) merely with a precision at a cosmological scale or not measure at all.
the signals travel with the final velocity of light, there is a delay in getting information. Therefore, the greater is a portion of a space-like hypersurface, the longer is the delay. This imposes limits to the extent of a space-like region within which the wave functional packet (44) can be sharply localized. The situation in Fig. 4 is just an idealization. A more realistic wave packet is illustrated in Fig. 5. It can still be represented by the expression (44) with a suitably width function $\sigma(x)$.

A possible interpretation is that such a wave packet of Fig. 5 represents a private wave function(al) of an observer. The region of sharp localization is mainly within his brain. An observer has a relatively good knowledge of his brain state at a given moment of the evolution time $\tau$, whilst the outside spacetime is less well definite. It is important that the outside spacetime is not completely indefinite; its definiteness is given by the wave packet. So an "outside" or "objective" space-time is given within the resolution of the wave packet. If we assume that the wave packet moves according to the case (ii) (at the beginning of Sec. 4.2), then the average physical spacetime sheet $<V_4> \leftrightarrow \eta^a_\xi(\tau,x)$ moves in such a way that all its points are within a mathematical 4-surface $V_4$ which remain constant in $\tau$. Positions on $V_4$ can be represented by $\tau$-independent embedding functions $\eta^a(x)$.

This model thus predicts:

(i) an objective existing outside spacetime $V_4$ without evolution in $\tau$ (such is a space-time of the conventional special and general relativity);

(ii) a region $P$ of spacetime which changes its position on $V_4$ while the evolution time $\tau$ increases (this is a subjective region situated mainly within the brain of an observer).

The division into an "objective" and "subjective" part of spacetime is, of course, merely explanatory. It serves to explain the fact that the single basic object, the wave functional packet which moves in $\tau$, has two qualitatively different features, as described above.

4.3. Inclusion of sources

In a previous publication [7] we included the point-particle sources into the embedding model of gravity (which was based on the usual constrained membrane theory). This was achieved by including in the action for a spacetime sheet a function $\omega(\eta)$ which consists of a constant part and a $\delta$-function part. In the analogous way we can introduce sources into our unconstrained embedding model which has explicit $\tau$-evolution.

For $\omega$ we can choose the following function of the embedding space coordinates $\eta^a$ :

$$\omega(\eta) = \omega_0 + \sum_i \int m_i \delta^N(\eta - \hat{\eta}_i) \sqrt{\hat{f}} \, d^m \hat{x}$$

where $\eta^a = \hat{\eta}_{a}^i(\hat{x})$ is the parametric equation of an $m$-dimensional surface $\hat{V}_m^{(i)}$, called matter sheet, also embedded in $V_N$; $\hat{x}^{\hat{a}}$ are parameters (coordinates) on $\hat{V}_m^{(i)}$ and $\hat{f}$ is the determinant of the induced metric tensor on $\hat{V}_m^{(i)}$. If we take $m = N - 4 + 1$, then

\[11\] This implies also a knowledge or sharp localization of those outside regions of the spacetime sheet which are coupled to our observer’s brain by his sensory organs.
the intersection of $V_4$ and $\hat{V}_m^{(i)}$ can be a (one-dimensional) line, i.e. a worldline $C_i$ on $V_4$. If $V_4$ moves in $V_N$, then also the intersection $C_i$ moves. A moving spacetime sheet was denoted by $V_4$ and described by $\tau$-dependent coordinate functions $\eta^\mu(\tau, x^\nu)$. Let a moving worldline be denoted $C_i$. It can be described either by the coordinate functions $\eta^\mu(\tau, u)$ in the embedding space $V_N$ or by the coordinate functions $X^\mu(\tau, u)$ in the moving spacetime sheet $V_4$. Besides the evolution parameter $\tau$ we have also a one dimensional worldline parameter $u$ which has the analogous role as the spacetime sheet parameters $x^\nu$ in $\eta^\mu(\tau, x^\nu)$. At a fixed $\tau$, $X^\mu(\tau, u)$ gives a one dimensional worldline $X^\mu(u)$. If $\tau$ increases monotonically, then the worldlines continuously change or move. In the expression \( m - 1 \) coordinates $\hat{x}^{\mu}$ can be integrated out and we obtain

$$\omega = \omega_0 + \sum_i \int m_i \frac{\delta^4(x - X_i)}{\sqrt{|f|}} \left( \frac{dX^\mu_i}{du} \frac{dX^\nu_i}{du} f_{\mu\nu} \right)^{1/2} du$$ \hspace{1cm} (46)$$

where $x^\mu = X^\mu(\tau, u)$ is the parametric equation of a ($\tau$-dependent) worldline $C_i$, $u$ an arbitrary parameter on $C_i$, $f_{\mu\nu} \equiv \partial_\mu \eta^\nu \partial_\nu \eta_\mu$ the induced metric on $V_4$ and $f \equiv \det f_{\mu\nu}$. By inserting Eq. (46) into the membrane’s action (30) we obtain the following action

$$I[X^\mu(\tau, u)] = I_0 + I_m = \frac{\omega_0}{2} \int d\tau d^4x \sqrt{|f|} \left( \frac{\dot{\eta}^{\mu} \dot{\eta}_\mu}{\Lambda} + \Lambda \right) + \int d\tau d^4x \sqrt{|f|} \sum_i \left( \frac{\dot{X}_i^\mu \dot{X}_i^\nu f_{\mu\nu}}{\Lambda} + \Lambda \right) \frac{\delta^4(x - X_i)}{\sqrt{|f|}} \left( \frac{dX^\mu_i}{du} \frac{dX^\nu_i}{du} f_{\mu\nu} \right)^{1/2} d\lambda$$ \hspace{1cm} (47)$$

In a special case when the membrane $V_4$ is static with respect to the evolution in $\tau$, i.e. all $\tau$ derivatives are zero, then we obtain the usual Dirac-Nambu-Goto 4-dimensional membrane coupled to point particle sources

$$I[X^\mu(u)] = \omega_0 \int d^4x \Lambda \sqrt{|f|} + \int du \sum_i m_i \left( \frac{dX^\mu_i}{du} \frac{dX^\nu_i}{du} f_{\mu\nu} \right)^{1/2}$$ \hspace{1cm} (48)$$

However, the action (47) is more general than (48) and it allows for solutions which evolve in $\tau$. The first part $I_0$ describes a 4-dimensional membrane which evolves in $\tau$, whilst the second part $I_m$ describes a system of (1-dimensional) worldlines which evolve in $\tau$. After performing the integration over $x^\mu$, the "matter" term $I_m$ becomes - in the case of one particle - analogous to the membrane’s term $I_0$:

$$I_m = \int d\tau du \left( \frac{dX^\mu}{du} \frac{dX^\nu}{du} f_{\mu\nu} \right)^{1/2} \left( \frac{\dot{X}^\mu \dot{X}^\nu f_{\mu\nu}}{\Lambda} + \Lambda \right)$$ \hspace{1cm} (49)$$

Instead of 4 parameters (coordinates) $x^\mu$ we have in (48) a single parameter $u$, instead of the variables $\eta^\mu(\tau, x^\nu)$ we have $X^\mu(\tau, u)$, and instead of the determinant of the 4-dimensional induced metric $f_{\mu\nu} \equiv \partial_\mu \eta^\nu \partial_\nu \eta_\mu$ we have $(dX^\mu/du)(dX^\mu/du)$. All what we said about the theory of an unconstrained $n$-dimensional membrane evolving in $\tau$ can be straightforwardly applied to a worldline (which is a special membrane with $n = 1$).
After inserting the matter function $\omega(\eta)$ of eq.(13) into the Hamiltonian (33) we obtain

$$H = \frac{1}{2} \int d^4x \sqrt{|f|} \frac{\Lambda}{\omega_0} \left( \frac{P_0^2(x)P_{0a}(x)}{|f|} - \omega_0^2 \right) + \frac{1}{2} \sum_i \int du \left( \frac{dX^\mu}{du} \frac{dX^\nu}{du} \right)^{1/2} \frac{\Lambda}{m_i} \left( P^{(i)}_\mu P^{(i)\mu} - m_i^2 \right)$$

(50)

where $p_{0a} = \omega_0\Lambda^{-1} \sqrt{|f|} \eta_0$ is the membrane’s momentum everywhere except on the intersections $V_4 \cap \hat{V}_m^{(i)}$, and $P^{(i)}_\mu = m_i \dot{X}_\mu / \Lambda$ is the membrane momentum on the intersections $V_4 \cap \hat{V}_m^{(i)}$. In other words, $P^{(i)}_\mu$ is the momentum of a worldline $C^i$. The contribution of the worldlines is thus explicitly separated out in the Hamiltonian (50).

In the quantized theory a membrane’s state is represented by a wave functional which satisfies the Schrödinger equation (36). A wave packet (e.g. one of eq.(44)) contains, in the case of $\omega(\eta)$ given by Eq.(13), a separate contribution of the membrane’s portion outside and on the intersection $V_4 \cap \hat{V}_m^{(i)}$:

$$\psi[0, \eta(x)] = \psi_0[0, \eta(x)]\psi_m[0, X(u)]$$

$$= Ne^{-\omega_0 \int d^4x \sqrt{|f|} |\psi(x) - \psi_0(x)|^2 \frac{1}{2\sigma(x)} e^{-\sum_i \int du \frac{m_i}{\Lambda} \left( \frac{dX_\mu}{du} \frac{dX_\nu}{du} f_{\mu\nu} \right)^{1/2} \left( X_\nu(u) - X_\nu^0(u) \right)^2 \frac{1}{2\sigma(u)}}}$$

(51)

In the second factor of Eq.(51) the wave packets of worldlines are expressed explicitly. For a particular $\sigma(x)$, such that a wave packet has the form as sketched in Fig. 4 or Fig. 5, there exists a region $P$ of parameters $x^\mu$ at which membrane $V_4$ is much more sharply localized then outside $P$. The same is true for the intersections (which are worldlines): any such a worldline $C_i$ is much more sharply localized in a certain interval of the worldline parameter $u$. With the passage of the evolution time $\tau$ the region of sharp localization on a worldline moves in space-time $V_4$. In a suitable limit\(^\text{12}\) this becomes equivalent to motion of a wave packet of a point particle (event) localized in space-time. The latter particle is just an unconstrained point particle, a particular case (for $p = 0$) of a generic unconstrained $p$-dimensional membrane described in Sec. 2.

The unconstrained theory of point particles has a long history. It was considered by Fock, Stueckelberg, Schwinger, Feynman, Horwitz, Fanchi, Enatsu, and many others.\(^\text{10} - \text{13}\) Quantization of the theory appeared under various names, for instance the Schwinger proper time method or the parametrized relativistic quantum theory. The name unconstrained theory is used in Ref. [12] both for the classical and the quantized theory.

Such an unconstrained point particle theory implies that a wave packet is in general localized in space-time and the region of localization moves in space-time. What exists is a point particle like event in space-time, not a worldline. Then there is a problem of how to obtain the physically observed Maxwell fields which are such that they can only be generated by charged worldline sources, and not by charged events in space-time. A single charged event generates the electromagnetic potential field proportional to $e \delta[(x - X)^2] \dot{X}^\mu$, whereas a charged worldline generates the field given by the latter

\(^\text{12}\)An analogous limit is given in Sec. 4.2 where a moving (unconstrained) 3-dimensional membrane was obtained as a limiting case of a moving wave packet functional of a 4-dimensional membrane with a region $P$ of sharp localization.
expression integrated over the parameter \( \tau \) (which is usually assumed to be identical with the worldline parameter).

While it is not quite clear whether the unconstrained point particle theory can consistently deal with the observed electromagnetic fields, there is no such a problem in the theory of an unconstrained space-time membrane \( \mathcal{V}_4 \) which, for the action (47), contains worldlines \( X^\mu(\tau, u) \). The latter, in addition to being existing objects in space-time (parameter \( u \)), also move in space-time (parameter \( \tau \)). A wave packet (51), with \( \sigma(u) \) corresponding to Fig. 4, is expected to give the Maxwell field containing (i) a \( \tau \)-independent term (as is usually observed) and (ii) a \( \tau \)-dependent term due to the moving region \( P \) of sharp localization. A complete treatment of the electromagnetic fields as solutions of the dynamical equations is beyond the scope of this paper and will be given elsewhere.

5. Conclusion

We have formulated a reparametrization invariant and Lorentz invariant theory of \( p \)-dimensional membranes without constraints among the dynamical variables. This is possible if we assume a generalized form of the Dirac-Nambu-Goto action, such that the dependence of the dynamical variables on an extra parameter, the evolution time \( \tau \), is admitted.

Such a membrane's theory manifests its full power in the embedding model of gravity, in which space-time is treated as a 4-dimensional unconstrained membrane, evolving in an \( N \)-dimensional embedding space. The embedding model was previously discussed within the conventional theory of constrained membranes. Release of the constraints and introduction of the \( \tau \) evolution brings new insight into the quantization of the model. A particularly interesting is a state, represented by a functional of 4-dimensional membranes \( \mathcal{V}_4 \), localized around an average space-time membrane \( \mathcal{V}_4^{(c)} \), and even more sharply localized around a space-like surface \( \Sigma \) on \( \mathcal{V}_4^{(c)} \). Such a state incorporates the existence of a classical space-time continuum and the evolution. The notorious problem of time is thus resolved in our approach to quantum gravity. The space-time coordinate \( x^0 = t \) is not time at all! Time must be separately introduced, and this was achieved in our theory in which the action depends on the evolution time \( \tau \).

The importance of the evolution time was considered, in the case of a point particle, by many authors [10]-[13]. But a charged event in space-time generates an electromagnetic field which does not agree with the experimentally observed field. The latter requires a worldline as a source. Worldlines occur in our embedding model as intersections of space-time membranes \( \mathcal{V}_4 \) with \( (N - 4 + 1) \)-dimensional "matter" sheets. In the quantized

\footnote{For a discussion of the problem see Ref. [1].}

\footnote{In this sentence "time" stands for the parameter of evolution. Such is the meaning of the word "time" adopted by the authors who discuss the problem of time in general relativity. What they want to say is essentially just that there is a big problem, since the coordinate \( x^0 \) cannot have the role of evolution parameter (or "time" in short). In our work, following Horwitz [11], we make explicite distinction between the coordinate \( x^0 \) and the parameter of evolution \( \tau \). These two distinct concepts are usually mixed and given the same name "time". In order to distinguish them, we use the names "coordinate time" and "evolution time".}
theory the state of a worldline can be represented by a wave functional $\psi_m[\tau, X^\mu(u)]$, which may be localized around an average worldline (in the quantum mechanical sense of the expectation value). Moreover, at a certain value $u = u_P$ of the worldline parameter the wave functional may be much more sharply localized than at other values of $u$, thus approximately imitating the wave function of a point particle (or event) localized in space-time. And since $\psi_m[\tau, X^\mu(u)]$ evolves with $\tau$, also the point $u_P$ changes with $\tau$.

The embedding model, based on the theory of unconstrained membranes satisfying the action (30), appears to be a promising candidate for the theoretical formulation of quantum gravity including the bosonic sources. Incorporation of fermions is expected to be achieved by taking into account the Grassmann coordinates.

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Fig. 1. To different sets of initial velocities $\dot{X}^\mu(0,\sigma)$, $\dot{X}'^\mu(0,\sigma)$ belong different configurations $X^\mu(\tau,\sigma)$, $X'^\mu(\tau,\sigma)$ of membrane’s motion. They may lie on the same mathematical manifold $V_d$.

Fig. 2. A $p$-brane’s wave functional is in general a wave packet localized around a ”centroid” $p$-brane. Its position in space-time depends on the Lorentz-invariant evolution parameter $\tau$.

Fig. 3. Quantum mechanically a state of our space-time membrane $\mathcal{V}_4$ is given by a wave packet which is a functional of $\eta^a(x)$. Its ”centre” $\eta^a_c(\tau,x)$ is the expectation value $<\eta^a(x)>$ and moves according to the classical equations of motion (as derived from the action (30)).

Fig. 4. The wave packet representing a quantum state of a space-time membrane is localized within an effective boundary $B$. The form of the latter may be such that the localization is significantly sharper around a space-like surface $\Sigma$.

Fig. 5. Illustration of a wave packet with a region of sharp localization $P$. 
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