How to avoid mass matrix for linear hyperbolic problems

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Abstract

We are interested in the numerical solution of linear hyperbolic problems using continuous finite elements of arbitrary order. It is well known that this kind of methods, once the weak formulation has been written, leads to a system of ordinary differential equations in \( \mathbb{R}^N \), where \( N \) is the number of degrees of freedom. The solution of the resulting ODE system involves the inversion of a sparse mass matrix that is not block diagonal. Here we show how to avoid this step, and what are the consequences of the choice of the finite element space. Numerical examples show the correctness of our approach.

1 Introduction

We are interested in the numerical approximation of the hyperbolic problem

\[
\frac{\partial u}{\partial t} + \text{div } f(x, u) = 0 \quad x \in \Omega \subset \mathbb{R}^d
\]

(1a)

by means of a finite element like technique. In this paper, we focus on the linear case where \( f(x, u) = a(x)u \).

The vector field \( a \) may depend on the spatial location \( x \). The problem \( (1a) \) is also supplemented with initial and boundary conditions:

\[
u(x, 0) = u_0(x) \quad (1b)\]

and

\[
u(x, 0) = g(x) \text{ if } x \in \partial \Omega. \quad (1c)\]

Obviously, \( (1c) \) has to be understood in the weak sense, i.e. that \( u = g \) on the inflow characteristics.

The physical space is covered by a conformal tessellation \( \mathcal{T} \). For ease of exposition, we assume that

\[\Omega = \cup_{K \in \mathcal{T}} K.\]

The solution of the problem is approximated by an element of the space \( V^h \) defined by:

\[V^h = \{u^h \in C^0(\Omega) \text{ such that for any } K, u^h_{|K} \text{ is a polynomial of degree } r\}.\]

We denote by \( \mathbb{P}^r \) the set of polynomials of degree \( r \). In this paper, we consider \( r = 1, 2 \) only.

It is well known that any finite element technique applied to \( (1a) \) will lead to a formulation of the type

\[M \frac{dU}{dt} + F = 0\]

where \( U \) denotes the vector of degrees of freedom, \( F \) is an approximation of the term \( \text{div } f \) and \( M \) is a mass matrix. In the case of continuous elements, this matrix is sparse but not block diagonal, contrarily to...
what happens for the Discontinuous Galerkin methods where the global continuity requirement is not made. Hence, in order to use any standard ODE solver, we need to invert $M$. This is considered cumbersome by many practitioners and this has been, in our opinion, one of the factors that has led to supremacy of DG methods in the current development of high order schemes.

Several researchers have proposed methods that avoid this step. More precisely, their methods are designed in such a way that the actual mass matrix is diagonal, so that the problem amounts to finding a “good” lumping integration formula. The first work we are aware of in that direction is [6], where the wave equation is considered, and the finite element space is made of functions belonging to a subspace of $P^{k+1}$ that contains $P^k$. This amounts to adding one degree of freedom to the “natural” quadratic elements. This work has been followed, in the same spirit, by [9] where higher accuracy could be obtained. However, the elements become more and more complex and, what is even more important, the stability condition on the time step becomes dramatically restrictive.

In these notes we describe some preliminary results about a new method for which no inversion of the mass matrix is needed, while a typical finite element approximation can be kept for the description of the divergence term. In this approach, there is no need to change the degrees of freedom. The method presented here can be seen as an extension of [10] where only $P^1$ elements and second order approximation in time have been considered.

The rest of the paper is organized as follows. In the first section, we describe the approximation of the divergence term of (1a). These are classical stabilized finite element methods. In the second section, we describe and somewhat justify our approach. The last section provides numerical examples that justify the correctness of our approach. A more involved analysis and and description will be made elsewhere. We conclude by giving some perspectives.

2 Description of the scheme

We start by describing the two spatial approximations we consider, then explain how to avoid the mass matrix inversion. We are given a triangulation of $\mathbb{R}^d$. Here we assume $d = 2$, but the discussion is general. The elements are denoted by $K$ and assumed to be simplices. In each element, we assume that the solution is approximated by a polynomial of degree $r$ and that the approximation is globally continuous. Let us denote the approximate solution by $u_h$. The function $u_h$ is fully defined by its control parameters $u_\sigma$ at all the degrees of freedom $\sigma$. We define by $S$ the set of degrees of freedom, so that

$$u_h = \sum_{\sigma \in S} u_\sigma \varphi_\sigma.$$  

We denote by $V_h = \text{span} \{ \varphi_\sigma, \sigma \in S \}$ For now, we can think of $u_\sigma$ as the value of $u_h$ at $\sigma$ and thus $\varphi_\sigma$ is the Lagrange basis, but we will need slightly less conventional approximation later.

We assume that we have a good integrator of the steady version of (1), and that this scheme writes: for any degree of freedom $\sigma$, $u_h$ satisfies:

$$\sum_{K \ni \sigma} \Phi^K_x(u_h) = 0.$$  

Examples are given by:

1. The SUPG residual, [7,8]:

$$\Phi^K_\sigma(u_h) = \int_{\partial K} \varphi_\sigma f(u_h) \cdot n \, dl - \int_K \nabla \varphi_\sigma \cdot f(u_h) \, dx$$

$$+ h_K \int_K \left( \nabla u_h(u_h) \cdot \nabla \varphi_\sigma \right) \tau \left( \nabla u_h(u_h) \cdot \nabla u_h \right) \, dx$$

with $\tau > 0$. We take:

$$\left( h_K \tau \right)^{-1} = \sum_{\sigma \in K} |\nabla \varphi_\sigma|^2$$

2
We follow the main ideas of [5]. Between solution has continuous normal gradients. For steady solutions, both methods can be shown to converge as $h^{k+1/2}$, see [3, 8] for more details.

2. The Galerkin scheme with jump stabilisation [3]:

$$
\Phi_{\sigma}^K(u^h) = \int_{\partial K} \varphi \cdot f(u^h) \cdot n \, dl - \int_K \nabla \varphi \cdot f(u^h) \, dx + \sum_{edges} \frac{1}{\Gamma} \int_{e} \left| \nabla \varphi \right| \, dl
$$

with $\Gamma > 0$. Here, since the mesh is conformal, any edge (or face in 3D) is the intersection of the element $K$ and an other element denoted by $K^+$. We define $|\nabla u| = |\nabla u|_K - |\nabla u|_{K^+}$ and $|\nabla \varphi|^+ = (\varphi\varphi)|_{K^+}$. Here, we have taken $\Gamma = \max(\bar{\alpha} K, \bar{\alpha} K^+)$. See [3] for more details.

This streamline formulation implies formally that the exact solution cancels the residuals. In the case of the stabilisation by jumps, we can only write that

$$
\Phi_{\sigma}^K = \int_K \varphi \cdot \text{div} f(u) \, dx + R_{\sigma}(u^h)
$$

where $\sum_{\sigma \in K} R_{\sigma}(u^h) = 0$. The additional term $R_{\sigma}$ is non-zero, except for the exact solution unless this solution has continuous normal gradients. For steady solutions, both methods can be shown to converge as $h^{k+1/2}$, see [3, 8] for more details.

2.1 Formulation for unsteady problems

We use a deferred correction (DeC) approach. We start from the ODE:

$$
\frac{dy}{dt} = f(y, t), \quad y(0) = y_0.
$$

We follow the main ideas of [5]. Between $t_n$ and $t_{n+1}$, the solution of (4) satisfies

$$
y(t) = y(t_n) + \int_{t_n}^{t} f(y(s), s) \, ds.
$$

Given $0 = \xi_0 < \xi_1 < \ldots < \xi_l < \ldots < \xi_{M+1} = 1$, and we consider the times $t_{n,l} = t_n + \xi_l \Delta t$ with $\Delta t = t_{n+1} - t_n$. If we know $f_{n,l} \approx f(y(t_{n,l}), t_{n,l})$, we can consider the Lagrange interpolant $I_{M+1}$ of $f$ with data given by $(t_{n,l}, f_{n,l})$, therefore we get the approximation:

$$
y_{n,l} = y_{n,0} + \int_{t_n}^{t_{n,l}} I_{M+1}[f(y(s), s)](t_n + \xi \Delta t) \, d\xi.
$$

This is in general a non-linear implicit equation.

The idea of the DeC method is to consider the first order scheme, for $M \geq l \geq 1$:

$$
y_{n,l} = y_{n,l-1} + \alpha_l \Delta t f(y(t_{n,l-1}), t_{n,l-1}), \quad y_{n,0} \approx y(t_n)
$$

where $\alpha_l = \xi_l - \xi_{l-1}$. Then, we introduce the vector $v = (y_{n,1}, \ldots, y_{n,M+1})^T$. The first order scheme can be rewritten as $L^1(v) = 0$ where

$$
L^1(v) = \begin{pmatrix}
y_{n,1} - y_{n,0} - \Delta t \int_{0}^{\xi_t} I_0[f(y(s), s)](t_n + \xi \Delta t) \, d\xi \\
y_{n,2} - y_{n,1} - \Delta t \int_{0}^{\xi_2} I_0[f(y(s), s)](t_n + \xi_2 \Delta t) \, d\xi \\
\vdots \\
y_{n,M+1} - y_{n,0} - \Delta t \int_{0}^{\xi_t} I_0[f(y(s), s)](t_n + \xi_t \Delta t) \, d\xi
\end{pmatrix}
$$
where \( \mathcal{I}_0 \) is the first order interpolant of \( f \): for \( 1 \leq l \leq M + 1, \)
\[
\mathcal{I}_0[f(y(\cdot),\cdot)](s) = f(y_{n,t-l},t_{n,t-1}) \quad \text{for } s \in [t_{n,t-1},t_{n,t}].
\]
Note that \( L^1(v) = 0 \) can be solved explicitly.

Similarly, we define \( L^2 \) by:
\[
L^2(v) = \begin{pmatrix}
y_{n,1} - y_{n,0} - \Delta t \int_0^{\xi_1} \mathcal{I}_{M+1}[f(y(\cdot),\cdot)](t_{n} + \xi \Delta t) d\xi \\
y_{n,t} - y_{n,0} - \Delta t \int_0^{\xi_t} \mathcal{I}_{M+1}[f(y(\cdot),\cdot)](t_{n} + \xi \Delta t) d\xi \\
y_{n,M+1} - y_{n,0} - \Delta t \int_0^{\xi_{M+1}} \mathcal{I}_{M+1}[f(y(\cdot),\cdot)](t_{n} + \xi \Delta t) d\xi
\end{pmatrix}.
\]

The DeC formulation is defined as follows:
1. \( v^0 = (y_n, \ldots y_n)^T \) and \( y_n, 0 = y_n \),
2. For \( k = 1, \ldots M + 1, v^k \) is defined as
   \[
   L^1(v^k) = L^1(v^{k-1}) - L^2(v^{k-1})
   \]
Since \( L^1 \) is explicit, the method is completely explicit. One can show that \( L^2 - L^1 = O(\Delta t) \) so that the scheme is \( (M+1) \)-th order accurate.

Similar to what is done for ODEs, we could integrate \( \int \) in time and get:
\[
u(x, t_{n+1}) = u(x, t_n) + \int_{t_n}^{t_{n+1}} \text{div} f(u(x, s)) ds,
\]
This can be approximated by
\[
u(x, t_n + \xi \Delta t) \approx u(x, t_n) + \int_0^{\xi} \text{div} \mathcal{I}_{r+1}[f(u(x, \cdot)](t_n + \xi \Delta t) ds
\]
(5)
\[
= \Delta t \sum_{l=0}^{r} \omega_l \text{div} f(u(x, \xi_j) ds
\]
\( \mathcal{I}_{r+1}[f(u(x, \cdot))] \) is the Lagrange interpolant of \( f(u(x, \cdot)) \) at the points \( \{t_n, \ldots, \xi(t_{n+1} - t_n) \} \) and \( \omega_l \) are the weights.

This suggests the algorithm we describe now. For any \( V \in V_h^M, V^\sigma = (V_{1}^\sigma, \ldots V_{M+1}^\sigma)^T \) is a vector of control parameters at the degree of freedom \( \sigma \in S, V = \sum_{\sigma \in S} V^\sigma \varphi_\sigma \). Then, we can consider the following deferred correction approximation: we introduce \( t_{n,i} = t_n + \xi_i(t_{n+1} - t_n) \) so that \( t_{n,0} = t_n \) and \( t_{n,r+1} = t_{n+1}, \) and define
1. for any \( \sigma \in S, \) the operator \( L^1_\sigma \) as
   \[
   L^1_\sigma(V_1, \ldots, V_{r+1}) = \begin{pmatrix}
   |C_\sigma|(V_{r+1}^\sigma - V_0^\sigma) + \sum_{K \ni \sigma} \int_{t_{n,0}}^{t_{n,r+1}} \mathcal{I}_0[\Phi_\sigma^x](t_n + s \Delta t) \, ds
   \\
   |C_\sigma|(V_r^\sigma - V_0^\sigma) + \sum_{K \ni \sigma} \int_{t_{n,0}}^{t_{n,r}} \mathcal{I}_0[\Phi_\sigma^x](t_n + s \Delta t) \, ds
   \\
   \vdots
   \\
   |C_\sigma|(V_1^\sigma - V_0^\sigma) + \sum_{K \ni \sigma} \int_{t_{n,0}}^{t_{n,1}} \mathcal{I}_0[\Phi_\sigma^x](t_n + s \Delta t) \, ds
   \end{pmatrix}
   \]
   (6a)
   Here, \( V_0^\sigma = (u_0^\sigma, \ldots, u^n)^T \in \mathbb{R}^M \).
where the constant \( C \) is invertible. The overall cost is not larger than a standard Runge-Kutta method.

Let us now have a look at the invertibility of \( L \) as \( \sigma \) varies. It is easy to see that \( L \) is invertible, we need \( \sigma L \) to be invertible, we need \( \sigma L \) to be invertible.

Last, we define the operators \( L^1 \) and \( L^2 \) on the finite element set \( V_h \) as

\[
L^1 = (L^1_\sigma)_{\sigma \in \Sigma}, \quad L^2 = (L^2_\sigma)_{\sigma \in \Sigma}.
\]

The step of the method between \( t_n \) and \( t_{n+1} \) is defined as follows.

1. Knowing \( u^n_\sigma \), we set \( V^0_\sigma = (u^n_\sigma, \ldots, u^n_\sigma) \).
2. For \( k = 1, \ldots, M \), we construct \( V^k \) as the solution of

\[
L^1(V^{k+1}) = L^1(V^k) - L^2(V^k).
\]

3. Then we define \( u^{n+1}_\sigma \) as

\[
u^{n+1}_\sigma = (V^{n+1}_\sigma)^M.
\]

This methods provides a decent approximation of the solution because one can show that, for the \( L^2 \) norm,

\[
||L^1 - L^2|| \leq C \Delta t,
\]

where the constant \( C \) depends only on the mesh. Then, using standard results for deferred correction methods, one can show that we have an \( (r+1) \)-th order accurate scheme if \( M = r+1 \), provided \( L^1 \) is invertible. The overall cost is not larger than a standard Runge-Kutta method.

Let us now have a look at the invertibility of \( L^1 \). Not every finite element approximation can work. The reason is that we have not yet specified what should be the parameters \( C_\sigma \) in relation (6a). It is easy to see that we must have

\[
C_\sigma = \int_{\Omega} \phi_\sigma d\mathbf{x},
\]

and in order that \( L^1 \) be invertible, we need \( C_\sigma > 0 \). For \( P^1 \) elements, there is no problem because the basis functions are positive, but it is well known that this condition is not met for higher order finite elements. For example, in the case of two-dimensional quadratic Lagrange interpolation, we have six basis functions. Three of them are associated to the vertices, and it is well known that their integral vanishes, so that in the end \( C_\sigma = 0 \) for the vertices. For other finite elements, we can have \( C_\sigma < 0 \).

In order to circumvent this restriction, and since we are interested in the approximation order and not on the practical representation, i.e. the physical meaning of the degrees of freedom, a simple way is to replace classical Lagrange elements of degree \( r \) by their Bezier counterparts. If

\[
\left( \sum_{j=1}^{d+1} x_j \right)^r = \sum_{\sum_{j=1}^{d+1} j_k = r} \theta_{j_1, \ldots, j_{d+1}}^r x_{j_1}^{j_1} \ldots x_{j_{d+1}}^{j_{d+1}}
\]

is the binomial expansion, then the Bezier polynomials are simply

\[
B_{j_1, \ldots, j_{d+1}}^r = \theta_{j_1, \ldots, j_{d+1}}^r \Lambda_{j_1}^{j_1} \ldots \Lambda_{j_{d+1}}^{j_{d+1}}
\]
where the $\Lambda_j$ are the standard barycentric coordinates. Since
\[
\int_K B_{j_1 \ldots j_{d+1}}(x)dx > 0,
\]
and since this family is a basis of $\mathbb{P}^r$, there are no more problems. In the simulations done in this paper, we have chosen $\mathbb{P}^1$ elements (i.e. Bézier of degree 1), and quadratic Bézier elements. Note that this kind of approximation has already been used for steady problems [2], and has some links with isogeometrical analysis [4], but for completely different reasons.

3 Numerical illustrations

3.1 Parameters

In the numerical experiments we present, we have chosen a temporal scheme that is third order in time. It is based on the Lagrange interpolation in $[0, 1]$, where the data are given at the points $t = 0, \frac{1}{2}$ and 1. This results in the following formula that defines the operator $L^2$:
\[
\int_0^{1/2} I_2(f)ds = \frac{5}{24} f(0) + \frac{1}{3} f(\frac{1}{2}) - \frac{1}{24} f(1)
\]
\[
\int_0^1 I_2(f)ds = \frac{1}{6} f(0) + \frac{4}{6} f(\frac{1}{2}) + \frac{1}{6} f(1)
\]

We have used the same temporal scheme for $\mathbb{P}^1$ and $\mathbb{B}^2$ elements.

3.2 Simulations

The velocity field at $(x, y)$ is given by $a = 2\pi (-y, x)$. The initial condition is given by:
\[
u_0(x, y) = e^{-40(x^2 + y^2)}.
\]

The domain is a circle with center $(0, 0)$ and radius $R = 1$. The mesh representing all the degrees of freedom is displayed in Figure 1: The quadratic elements have 6 degrees of freedom (the vertices and the mid-points of the edges). These degrees of freedom are also used for the linear element just by mesh refinement. There are 7047 degrees of freedom here, so $h \approx \sqrt{\frac{7047}{\pi}} \approx 0.021$ which is relatively coarse. On the same figure, we represent the exact solution. The time step is evaluated as the minimum of the $\Delta t_K$ defined by:
\[
\Delta t_K = CFL \frac{h_K}{||\bar{a}_K||}
\]

where $h_K$ is the length of the smallest edge of $K$ and $\bar{a}_K$ is the speed at the centroid. Since the elements for the $\mathbb{P}^1$ simulations are obtained from those of the $\mathbb{B}^2$ simulation by splitting, the parameter $h_K$, for the $\mathbb{P}^1$ simulations, is half of the one for the $\mathbb{B}^2$ simulations. For that reason, the CFL number for the quadratic approximation is half of the one chosen for the linear simulations, namely 0.6 instead of 0.3: we run with the same time step. By the way, we have not yet conducted a rigorous study of the CFL condition, but all experiments indicate that the quadratic simulations can be safely run with $CFL = 0.5$.

Figure 2 displays the results for the $\mathbb{P}^1$ approximation, while Figure 3 shows those obtained for the quadratic approximation. The baseline schemes are the SUPG and the Galerkin scheme with jumps.

In Figure 2, the same isolines are represented for the three results. We can see that after 10 rotations, the results of the Galerkin+jump scheme look pretty good despite the coarse resolution. The minimum and maximum are $-0.012$ and $0.762$. For the SUPG results, after 1 rotation, the minimum/maximum are $-0.004$ and $1.02$. After 2 rotations we have $-0.047$ and $1.02$. This is better that what is obtained for Figure 2(c),

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but the dispersive effects are much more important for the SUPG scheme as it can be seen on Figure 2(b): this is why we have not shown further results for the SUPG/P1 case.

In Figure 3 we show similar results obtained with the quadratic approximation. Again, the Galerkin+jump method is way less dispersive that the SUPG (stopped after only one rotation this time). We have found that if we perform 4, 6 or 8 iterations of the defect correction, the quality of the SUPG improves a lot, but the cost becomes prohibitive with respect to the Galerkin+jump method for which, after 10 rotations, the min/max are $-0.0044$ and $0.95$. We also see that the solution improves a lot with respect to linear elements, for example in terms of min/max values. There is however some dispersion, if we compare with the exact solution.

4 Conclusions, perspectives

The paper deals with the numerical approximation of linear scalar hyperbolic problems. We have shown, by carefully choosing the spatial approximation, and by using a non standard time step discretization, that it is possible to avoid the use of mass matrix in this problem, contrarily to what is usually thought about. The cost, on paper, is similar to a standard Runge-Kutta scheme, at least if we consider second and third order in time. In a preliminary work, we have had similar results for the 1D advection problem, which are not shown here. We had also obtained the expected convergence slope.

A lot remains to be done. First, we have found experimental CFL conditions but this has to be rationalized by a numerical analysis. This method needs to be extended to non-linear problems. Preliminary results seems promising, but the results need to be checked on a wider range of problems, this is why we have not reported them here. Last, this method needs to be extended to systems, for example the Euler equations of fluid mechanics.

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Figure 2: Results for the $\mathbb{P}^1$ approximation: (a) with SUPG, after 1 rotation, (b) with SUPG after 2 rotations, (c) with Galerkin+Jump after 10 rotations. The same isolines are represented.
Figure 3: Results for the $B^2$ approximation: (a) with Galerkin+Jump after 10 rotations, (b) with SUPG after 1 rotations, (c). The same isolines are represented.