New problems in universal algebraic geometry 
illustrated by boolean equations

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November 28, 2016

Abstract

We discuss new problems in universal algebraic geometry and explain them by boolean equations

MSC: 03G05 (boolean algebras), 03C98 (applications of model theory).

1 Introduction

The process of solving equations is the central part of mathematics. The most general and important problems in this area are the following.

1. Is a given equation consistent over an algebraic structure (algebra for shortness) $\mathcal{A}$?

2. Find all solution of a given equation over an algebra $\mathcal{A}$.

There are many surveys and papers devoted to equations in various classes of algebras. Let us just mention about the survey [1] for group equations.

However the recent achievements of universal algebraic geometry (see the papers [2, 3, 4] by E.Daniyarova, A.Miasnikov, V.Remeslennikov, and B.Plotkin) allow us to pose new problems about equations (all required definitions may be found in Section 2 of the current paper).

3. Systems of equations VS algebraic sets. Let $Y$ be an algebraic set over an algebra $\mathcal{A}$. Obviously, there exist more than one systems of equations (systems for shortness) with the solution set $Y$. Let us fix a family of systems $\mathcal{S}$, and let $S(Y) \subseteq \mathcal{S}$ be all systems with the solution set $Y$. It turns out that the numbers $|S(Y)|$ have a wide spread of values for almost all natural $\mathcal{S}$ (e.g. in [5] this fact was proved for semilattice equations). Thus, there arises a problem: is there an algebra $\mathcal{A}$ and a natural family $\mathcal{S}$ such that the variance of the set $\{|S(Y)|\}$ is minimal?

The sense of this problem is the following. Suppose we want to generate random algebraic sets over an algebra $\mathcal{A}$ by a random generation of systems from $\mathcal{S}$. If the variance of the set $\{|S(Y)|\}$ is small, the random distribution of algebraic sets becomes close to the uniform distribution.

*The author was supported by Russian Fund of Fundamental Research (project 14-01-00068, the results of Sections 6,7) and Russian Science Foundation (project 14-11-00085, the results of Section 4,5)
4. *Irreducible algebraic sets*. Let $Y$ be an algebraic set over an algebra $\mathcal{A}$. Is there an algorithm that decides whether $Y$ is irreducible or not? If $Y$ is reducible, can we find its irreducible components? Can you find the average number of irreducible components of all algebraic sets in $\mathcal{A}^n$?  

The importance of this problem is the following. According to [2, 3], the structure of irreducible algebraic sets over $\mathcal{A}$ determines the universal theory of $\mathcal{A}$. Moreover, if $\mathcal{A}$ is finite irreducible algebraic sets over $\mathcal{A}$ correspond to subalgebras of $\mathcal{A}$.

5. *Isomorphic algebraic sets*. In [2] it was defined isomorphisms between algebraic sets. Namely, isomorphic algebraic sets have the same properties with respect to universal algebraic geometry. For any algebra $\mathcal{A}$ one can pose the following problem: how many non-isomorphic algebraic sets are there in $\mathcal{A}^n$? The solution of this problem allows us to decide about the complexity of the class of all algebraic sets over $\mathcal{A}$.

6. *Equationally extremal algebras*. Let $\mathcal{A}_n$ be the class of $\mathcal{L}$-algebras of order $n$ (for example, $\mathcal{A}_n$ is the class of all semilattices of order $n$) and $\mathbf{S}$ a finite set of systems. The problem is the following: find an algebra $\mathcal{A} \in \mathcal{A}_n$ such that the number of consistent systems from $\mathbf{S}$ is maximal (minimal) for $\mathcal{A}$.  

Let us refer to the papers, where the problems above were solved for some algebras. In [6] we describe irreducible algebraic sets and compute the average number of irreducible components of algebraic sets over linearly ordered semilattices. In [5] for the class of semilattices of order $n$ it was described equationally extremal semilattices which have maximal (minimal) number of consistent equations. Above we mentioned about the paper [5], where we consider the 3rd problem for semilattices. The obtained results of all papers above show that the problems 3–6 are nontrivial even for simple algebras. However, there exists a class of algebras, the class of boolean algebras, where the problems above have nice solutions.

*Thus, the aim of this paper is the solution of problems 3–5 in the class of finite boolean algebras* (the sixth problem is unreasonable for boolean algebras, since $|\mathcal{A}_n| \leq 1$ for any $n \in \mathbb{N}$). So the reader may consider this paper as a vast example for problems above.

Let us explain the plan of our paper. In Section 2 we give basics notions of universal algebraic geometry. Section 3 contains the rules of transformations of equations over boolean algebras. Actually, any boolean system $S(X)$ in $n$ variables $X$ can be equivalently reduced to an orthogonal system $S'(Z)$ in $2^n$ variables $Z$. Solving the 3rd problem, we prove that any algebraic set defined by a system in $n$ variables is isomorphic to the solution set of a unique orthogonal system in $2^n$ variables.

In Section 4 we describe irreducible algebraic sets over finite boolean algebras and decompose any algebraic set into a finite union of irreducible ones. In Section 5 we count the average number of irreducible components of algebraic sets over finite boolean algebras. In Section 6 we give the definition of a rank of irreducibility $IR(S)$ of a system $S$ and count the average rank of irreducibility of all orthogonal systems in $2^n$ variables. Thus, Sections 4–6 solve the 4th problem for finite boolean algebras.

In Section 7 we study the 5th problem and directly compute the number of pairs of isomorphic algebraic sets defined by orthogonal systems in $2^n$ variables.
2 Basic notions

Let $\mathcal{L} = \{\vee^{(2)}, \cdot^{(2)}, \neg^{(1)}, 0, 1\}$ be a language of binary functional symbols $\vee, \cdot$ (join and meet), unary symbol $\neg$ (complement) and constant symbols $0, 1$. Clearly, boolean algebras are algebraic structures of the language $\mathcal{L}$ with natural interpretation of functional and constant symbols (see [7] for more details).

Recall that for any finite boolean algebra $\mathcal{B}$ there exists a number $r \geq 1$ such that $\mathcal{B}$ is isomorphic to the power set algebra on $r$ elements ($|\mathcal{B}| = 2^r$). The number $r$ is called the rank of a boolean algebra $\mathcal{B}$. We assume below that any boolean algebra $\mathcal{B}$ is nontrivial, i.e. $|\mathcal{B}| > 1$.

An element $a$ is an atom (co-atom) if $\{ab | b \in \mathcal{B}\} = \{0, a\}$ (respectively, $\{a \lor b | b \in \mathcal{B}\} = \{a, 1\}$). Remark that the rank of a finite boolean algebra is equal to the number of atoms (co-atoms).

Following [2], let us give the basic notions of algebraic geometry over boolean algebras.

Let $X = \{x_1, x_2, \ldots, x_n\}$ be a finite set of variables. A term $t(X)$ of the language $\mathcal{L}$ is called an $\mathcal{L}$-term. The set of all $\mathcal{L}$-terms in variables $X$ is denoted by $\mathcal{T}_{\mathcal{L}}(X)$. A boolean equation is an atomic formula $\tau(X) = \sigma(X)$ of the language $\mathcal{L}$ ($\tau, \sigma$ are $\mathcal{L}$-terms). The examples of boolean equations are the following expressions: $x_i = x_j$, $x_1 x_2 = x_3 \lor x_4$, $\bar{x}_1 \lor x_2 = \bar{x}_3 x_4$.

A system of equations (system for shortness) is an arbitrary set of boolean equations. The set of all solutions (solution set) of a system $\mathcal{S}$ over a boolean algebra $\mathcal{B}$ is denoted by $\mathcal{V}_\mathcal{B}(\mathcal{S})$.

A set $Y \subseteq \mathcal{B}^n$ is algebraic over a boolean algebra $\mathcal{B}$ if there exists a system $\mathcal{S}$ such that $Y = \mathcal{V}_\mathcal{B}(\mathcal{S})$. A nonempty algebraic set $Y$ is irreducible if it is not a finite proper union of other algebraic sets. According to [2], it follows that each algebraic set $Y \subseteq \mathcal{B}^n$ is decomposable into a finite union of irreducible algebraic sets

$$Y = Y_1 \cup Y_2 \cup \ldots \cup Y_m \ (Y_i \nsubseteq Y_j \text{ for } i \neq j), \quad (1)$$

and the decomposition (1) is unique up to the permutation of the sets $Y_i$. The sets $Y_i$ in (1) are called the irreducible components of a set $Y$.

Let $Y = \mathcal{V}_\mathcal{B}(\mathcal{S})$ be an algebraic set over a boolean algebra $\mathcal{B}$, and $\mathcal{S}$ depends on variables $X = \{x_1, x_2, \ldots, x_n\}$. One can define an equivalence relation $\sim_Y$ on $\mathcal{T}_{\mathcal{L}}(X)$ as follows:

$$t(X) \sim_Y s(X) \iff t(P) = s(P) \text{ for each point } P \in Y.$$

The set of $\sim_Y$-equivalence classes is called the coordinate algebra of $Y$ and denoted by $\Gamma_\mathcal{B}(Y)$. By the results of [2], it follows that $\Gamma_\mathcal{B}(Y)$ is a boolean algebra and generated by the elements $x_1, x_2, \ldots, x_n$. In other words, all coordinate algebras are finitely generated, and, therefore, all coordinate algebras of algebraic sets over boolean algebras are finite. The following statement describes the properties of coordinate algebras of irreducible algebraic sets.

**Theorem 2.1.** An algebraic set $Y$ is irreducible over a boolean algebra $\mathcal{B}$ iff $\Gamma_\mathcal{B}(Y)$ is embedded into $\mathcal{B}$

**Proof.** Actually, in [2] (Theorem A) it was proved that $\Gamma_\mathcal{B}(Y)$ is discriminated by $\mathcal{B}$ iff the algebraic set $Y$ is irreducible. Since $\Gamma_\mathcal{B}(Y)$ is finite, the discrimination is equivalent to the embedding of $\Gamma_\mathcal{B}(Y)$ into $\mathcal{B}$. \qed
Therefore, \( \Gamma \) gives that the term \( \overline{x_1x_2} \) equals 0 in \( \Gamma_B(Y) \). The direct computations give that \( \Gamma_B(Y) \) consists of 8 elements (\(~\gamma\)-equivalence classes)

\[
0, 1, x_1, x_2, x_1\overline{x}_2, \overline{x}_1, \overline{x}_2, x_2 \lor \overline{x}_1.
\]

Therefore, \( \Gamma_B(Y) \) is isomorphic to a boolean algebra of rank 3, and the elements \( x_2, x_1\overline{x}_2, \overline{x}_1 \) (\( x_1, \overline{x}_1 \lor x_2, \overline{x}_2 \) are atoms (respectively, co-atoms) of \( \Gamma_B(Y) \)).

### 3 Transformations of boolean equations

Let \( X = \{x_1, x_2, \ldots, x_n\} \) be a finite set of variables. Let us define new variables \( Z = \{z_\alpha | \alpha \in \{0,1\}^n\} \) indexed by all \( n \)-tuples \( \alpha \in \{0,1\}^n \) (\(|Z| = 2^n\)). Following [8], the variables \( Z \) are called orthogonal. By \( \pi_i(\alpha) \) \((1 \leq i \leq n)\) we denote the projection of a tuple \( \alpha \in \{0,1\}^n \) onto the \( i \)-th coordinate. The substitution of the variables \( X = \{x_1, x_2, \ldots, x_n\} \) is the following

\[
x_i = \bigvee_{\pi_i(\alpha) = 1} z_\alpha. \tag{2}
\]

For example, the set \( X = \{x_1, x_2\} \) gives \( Z = \{z_{(0,0)}, z_{(0,1)}, z_{(1,0)}, z_{(1,1)}\} \) and

\[
x_1 = z_{(1,0)} \lor z_{(1,1)}, \quad x_2 = z_{(0,1)} \lor z_{(1,1)}.
\]

According to the axioms of boolean algebras, it follows that the variables \( Z \) are obtained from \( X \) by the following rules:

\[
z_\alpha = \overline{x}_1^{a_1} \overline{x}_2^{a_2} \cdots \overline{x}_n^{a_n}, \tag{3}
\]

where \( \alpha = (a_1, a_2, \ldots, a_n) \), \( a_i \in \{0,1\} \) and

\[
x_i^{a_i} = \begin{cases} x_i & \text{if } a_i = 1, \\ \overline{x}_i & \text{if } a_i = 0. \end{cases} \tag{4}
\]

For example, the sets \( X = \{x_1, x_2\}, Z = \{z_{(0,0)}, z_{(0,1)}, z_{(1,0)}, z_{(1,1)}\} \) give \( z_{(0,0)} = \overline{x}_1x_2, z_{(0,1)} = \overline{x}_1\overline{x}_2, z_{(1,0)} = x_1\overline{x}_2, z_{(1,1)} = x_1x_2. \)

By (2), any system \( S' \) in variables \( X = \{x_1, x_2, \ldots, x_n\} \) can be written as

\[
S = \{z_\alpha = 0 | \alpha \in A\} \cup \bigcup_{\alpha \neq \beta} \{z_\alpha z_\beta = 0\} \cup \{\bigvee_\alpha z_\alpha = 1\}. \tag{5}
\]
where \( A \subseteq \{0, 1\}^n \) and \( \bigvee_\alpha z_\alpha \) is the join of all variables \( z_\alpha \in Z \) (see [9] for more details).

Moreover, in [9] it was proved that the algebraic sets \( V_B(S') \), \( V_B(S) \) are isomorphic. A system of the form \( (5) \) is called orthogonal.

**Example 3.1.** The set \( Y = V_B(x_1x_2 = x_2) \) (\( B \) is an arbitrary boolean algebra) is isomorphic to the solution set of a system

\[
\begin{align*}
z(0,1) &= 0, \\
z(0,0)z(1,0) &= z(0,0)z(1,1) = z(0,1)z(1,1) = z(1,0)z(1,1) = 0, \\
z(0,0) \lor z(0,1) \lor z(1,0) \lor z(1,1) &= 1
\end{align*}
\]

since

\[
x_1x_2 = x_2 \iff \bar{x}_1x_2 = 0 \iff z(0,1) = 0.
\]

**Statement 3.2.** The coordinate algebra of the solution set of an orthogonal system \( S \) (5) is isomorphic to the boolean algebra of rank \( m - a \), where \( m = |Z| = 2^n \) and \( a = |A| \).

**Proof.** Since all points \( P_\alpha = (p_\beta : \beta \in \{0, 1\}^n) \) (\( \alpha \notin A \)),

\[
p_\beta = \begin{cases} 1 & \text{if } \beta = \alpha, \\ 0 & \text{otherwise} \end{cases}
\]

belong to \( Y = V_B(S) \), the definition of the \( \sim_Y \)-equivalence gives that the elements \( z_\alpha \) (\( \alpha \notin A \)) are nonzero in \( \Gamma_B(Y) \) and \( z_\alpha \neq z_{\alpha'} \) for distinct \( \alpha, \alpha' \notin A \). The equations \( z_\alpha z_\beta = 0 \in S \) imply that the elements \( z_\alpha \) (\( \alpha \notin A \)) are exactly the atoms of the boolean algebra \( \Gamma_B(Y) \). Since the rank of a boolean algebra is equal to the number of atoms, \( \Gamma_B(Y) \) is isomorphic to the boolean algebra of rank \( m - a \).

**Example 3.3.** According to Statement 3.2, the coordinate algebra of the solution set of an orthogonal system \( S \) (6) is isomorphic to the boolean algebra of rank 3 (in Example 2.2 we directly obtained the same result). Using Theorem 2.1, we obtain that the set \( Y \) is irreducible over any boolean algebra of rank \( r \geq 3 \).

If \( B \) is the boolean algebra of rank 2 the solution set of (6) is decomposable into the union of solution sets of the following systems

\[
S_1 = S \cup \{z(0,0) = 0\}, \ S_2 = S \cup \{z(1,0) = 0\}, \ S_3 = S \cup \{z(1,1) = 0\}.
\]

For the boolean algebra \( B \) of rank 2 there are not nonzero elements \( z_1, z_2, z_3 \in B \) with \( z_iz_j = 0 \) (\( i \neq j \)). Therefore, for any solution of \( S \) one of the following equalities holds \( z(0,0) = 0, z(1,0) = 0, z(1,1) = 0 \). Thus, \( V_B(S) \) can be decomposed into a union of solution sets of \( S_1, S_2, S_3 \).

One can prove that for any algebraic set \( Y \subseteq B^n \) there exists a unique orthogonal system \( S \) in \( m = 2^n \) variables with the solution set isomorphic to \( Y \). Therefore, there arises a one-to-one correspondence between algebraic sets in \( B^n \) and orthogonal systems in \( m = 2^n \) variables. It allows us to below to identify the class of algebraic sets in \( B^n \) and the class of all orthogonal systems in \( m = 2^n \) variables.
4 Irreducible components of algebraic sets

Let $Y$ be the solution set of $\mathcal{S}$ (5) over the boolean algebra $\mathcal{B}$ of rank $r$. Let $m = |Z| = 2^n$, $a = |A|$.

Lemma 4.1. If $m - a \leq r$, then $Y$ is irreducible.

Proof. It directly follows from Statement 3.2 and Theorem 2.1.

Lemma 4.2. Let $m - a > r$ then $Y$ is a union of solution sets of the following orthogonal systems

$$\mathcal{S}_B = \{ z_\alpha = 0 \mid \alpha \in B \} \cup \bigcup_{\alpha \neq \beta} \{ z_\alpha = 0 \} \cup \bigvee \alpha \{ z_\alpha = 1 \}$$

(7)

where $B \subseteq \{0, 1\}^n$, $B \supseteq A$, $|B| = m - r$. Moreover, the sets $Y_B = V_B(\mathcal{S}_B)$ are irreducible components of $Y$.

Proof. Actually, the statement of this lemma was demonstrated in Example 3.3, where the solution set of $\mathcal{S}$ over the boolean algebra of rank 2 is a union of the solution sets of the systems $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3$. For the systems $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3$ the set $B$ respectively equals $\{(0, 0), (0, 1)\}$, $\{(1, 0), (0, 1)\}$, $\{(1, 1), (0, 1)\}$.

The proof of the lemma follows from the statements below.

1. Let us prove $Y = \bigcup_B Y_B$. Since the systems $\mathcal{S}_B$ contain new equalities $z_\alpha = 0$, then obviously $V_B(\mathcal{S}_B) \subseteq V_B(\mathcal{S})$ and $\bigcup_B Y_B \subseteq Y$.

Let us prove the inverse inclusion $Y \subseteq \bigcup Y_B$. Let $P = (p_\alpha \mid \alpha \in \{0, 1\}^n) \in Y$. Since $p_\alpha p_\beta = 0$ for all $\alpha \neq \beta$, then $P$ contains at most $r$ nonzero coordinates (and at least $m - r$ zero coordinates). Therefore, there exists a set $B \subseteq \{0, 1\}^n$, $|B| = m - r$, $B \supseteq A$ such that $p_\beta = 0$ for all indexes $\beta \in B$, and therefore $P \in Y_B$.

2. Statement 3.2 implies that the coordinate algebras of algebraic sets $Y_B$ are isomorphic to $\mathcal{B}$. By Theorem 2.1, all sets $Y_B$ are irreducible.

3. Let us prove that $Y_B \not\subseteq Y_{B'}$ for distinct sets $B, B'$. Let $\beta \in B \setminus B'$. Then the point $P = (p_\alpha \mid \alpha \in \{0, 1\}^n)$ with coordinates

$$p_\alpha = \begin{cases} 1 & \text{if } \alpha = \beta \\ 0 & \text{otherwise} \end{cases}$$

belongs to $Y_{B'}$, but $P \notin Y_B$.

5 Average number of irreducible components

In this section we obtain a formula for the average number of irreducible components of algebraic sets defined by orthogonal systems (5) over the boolean algebra $\mathcal{B}$ of rank $r$. Let $m$ be the number of variables in the orthogonal system $\mathcal{S}$ (5) and $a = |A|$. According to Lemmas 4.1, 4.2, the number of irreducible components $\text{Irr}(\mathcal{S})$ of the solution set of $\mathcal{S}$ equals

$$\text{Irr}(\mathcal{S}) = \begin{cases} 1 & \text{if } m - a \leq r \\ \binom{m-a}{r} & \text{otherwise} \end{cases}$$
The number of orthogonal systems for fixed $m, a$ equals $\binom{m}{a}$. The number of all orthogonal systems is $2^m$, therefore the average number of irreducible components of algebraic sets defined by orthogonal systems in $m$ variables equals

$$\overline{\text{Irr}} = \frac{1}{2^m} \left( \sum_{a=m-r}^{m} \binom{m}{a} + \sum_{a=0}^{m-r-1} \binom{m-r}{a} \binom{m-a}{r} \right).$$

We have

$$\sum_{a=0}^{m-r-1} \binom{m-r}{a} \binom{m-a}{r} = \sum_{a=0}^{m-r-1} \binom{m}{a} \binom{m-a}{r} = (\binom{m}{r})^{2^{m-r}-1},$$

and, therefore, the average number of irreducible components is

$$\overline{\text{Irr}} = \frac{1}{2^m} \left( \sum_{a=m-r}^{m} \binom{m}{a} + 2^{m-r} \binom{m}{r} \right) = \frac{1}{2^m} \left( \sum_{i=0}^{r-1} \binom{m}{i} + 2^{m-r} \binom{m}{r} \right).$$

For a fixed $r$ and $m \to \infty$ we have $\frac{1}{2^m} \sum_{i=0}^{r-1} \binom{m}{i} \to 0$ and

$$\overline{\text{Irr}} \sim 2^{-r} \binom{m}{r} \text{ for } m \to \infty.$$

6 Ranks of irreducibility

According to Lemmas 4.1, 4.2, the solution set of a system $S (5)$ may be reducible over the boolean algebra of rank $r$, but the solution set of $S$ becomes irreducible over the boolean algebras of higher ranks. We say that a system $S (5)$ has the rank of irreducibility $\text{IR}(S)$ if the solution set of $S$ is irreducible over the boolean algebra of rank $\text{IR}(S)$, but solution set of $S$ is reducible over each boolean algebra of rank $r < \text{IR}(S)$ (if $S$ is inconsistent over any boolean algebra we put $\text{IR}(S) = 0$). Below we compute the average rank of irreducibility of orthogonal systems in $m$ variables.

By Lemmas 4.1, 4.2, we have that $\text{IR}(S)$ of a system $S (5)$ equals $m-a$, $a = |A|$. The number of orthogonal systems in $m$ variables with the rank of irreducibility $m-a$ is equal to $\binom{m}{a}$. Therefore, the average rank of irreducibility of orthogonal systems in $m$ variables is

$$2^{-m} \sum_{a=0}^{m} (m-a) \binom{m}{a} = 2^{-m} \left( m \sum_{a=0}^{m} \binom{m}{a} - \sum_{a=0}^{m} a \binom{m}{a} \right) = 2^{-m} \left( m2^m - m2^{m-1} \right) = m/2.$$

7 Pairs of isomorphic algebraic sets

In this section we compute the number of pairs $(Y_1, Y_2)$ such that the algebraic sets $Y_i$ are isomorphic to each other and $Y_i$ are defined by orthogonal systems in $m$ variables.

Suppose algebraic sets $Y_i$ are defined by the following orthogonal systems

$$S_i = \{z_\alpha = 0 \mid \alpha \in A_i\} \cup \bigcup_{\alpha \neq \beta} \{z_\alpha z_\beta = 0\} \cup \bigcup_{\alpha} \{z_\alpha = 1\}, \quad (8)$$
where \( A_i \subseteq \{0, 1\}^n \)

The following lemma is a simple corollary of Statement 3.2.

**Lemma 7.1.** Algebraic sets \( Y_1, Y_2 \) defined by orthogonal systems \( S_1, S_2 \) (8) are isomorphic to each other iff \(|A_1| = |A_2|\).

**Proof.** In [2] (Corollary 5.7) it was proved that that \( Y_1, Y_2 \) are isomorphic iff their coordinate algebras are the isomorphic. The application of Statement 3.2 concludes the proof. \( \square \)

The number of pairs \((S_1, S_2)\) with \(|A_1| = |A_2|\) is equal to

\[
\sum_{i=0}^{m} \binom{m}{i} \binom{m}{i} = \binom{2m}{m}.
\]

Since there are exactly \(2^m 2^m = 4^m\) pairs of algebraic sets defined by orthogonal systems in \(m\) variables, two random algebraic sets are isomorphic with the following probability

\[
\frac{\binom{2m}{m}}{2^m 2^m}.
\]

Applying Stirling formula to the expression \(\binom{2m}{m}\), we obtain that the required probability asymptotically equals \(\frac{1}{\sqrt{\pi m}}\).

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