INCONSISTENCY OF INACCESSIBILITY
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The work presents the brief exposition of the proof (in ZF) of inaccessible cardinals nonexistence. To this end in view there is used the apparatus of subinaccessible cardinals and its basic tools — reduced formula spectra and matrices and matrix functions and others. Much attention is devoted to the explicit and substantial development and cultivation of basic ideas, serving as grounds for all main constructions and reasonings.

In 1997 the author has proved

Main theorem (ZF): There are no weakly inaccessible cardinals.

The proof of this theorem was derived as a result of using the subinaccessible cardinal apparatus which the author has worked out since 1976, the preliminarily investigations were developed since 1973. The systematic exposition of this proof has been published in 2000, and the most complete and detailed form the proof of inaccessible cardinals nonexistence have received in works “Inaccessibility and Subinaccessibility”, Part I [1] and Part II [2] in 2008, 2010; these two works one can see also at arXiv sites [1], [2].

However, some criticism has been expressed that these works expose the material which is too complicated and too extensive and overloaded by the technical side of the matter, that should be avoided even when it uses in essence some new complicated apparatus. According to these views every result, even extremely strong, should be exposed on few pages, otherwise it causes doubts in its validity.

So, the present work constitutes the brief exposition of the whole investigation, called to overcome such criticism.

Preliminarily it is required to present the notions of various subinaccessibles and reduced matrices and matrix functions (Part I and half of Part II of specified works) - otherwise it is hardly possible even to sketch the idea of inaccessibles inconsistency proof.

In these works it is proved that the system \( ZF + \exists k \) (k is weakly inaccessible cardinal) is inconsistent. In what follows all the reasoning will be carried out in this system.

The idea of main theorem proof consists in the formation of matrix functions that are sequences of matrices, reduced to a fixed cardinal; such matrices are certain Boolean values in Lévy algebra, reduced to certain cardinal; on this foundations the simplest matrix function \( S_{\chi f} = (S_{\chi \tau})_{\tau} \) is introduced as the sequence of such matrices of special kind (see below).

This function has domain cofinal to inaccessible cardinal \( k \), it has range consisting of matrices reduced to some fixed cardinal \( \chi < k \) and defined as minimal in the sense of Gödel function \( Od \) on corresponding carriers; this property provides its monotonicity also in the same sense. The role of reducing cardinal \( \chi \) is played further mainly by the complete cardinal \( \chi^* < k \) (see below).

Now the idea of the main theorem proof comes out:

The required contradiction consists in creation of certain matrix function which possess inconsistent properties: it is monotone and at the same time it is deprived of its monotonicity.

Let’s turn to realization of this idea.

Weakly inaccessible cardinals become strongly inaccessible in Gödel constructive class \( L \) of values of Gödel constructive function \( F \) defined on the class of all ordinals. Every set
a ∈ L receives its ordinal number \( Od(a) = \min \{ \alpha : F(\alpha) = a \} \). The starting structure in the further reasoning is the countable initial segment \( \mathcal{M} = (L_\alpha, \in, =) \) of the class \( L \) serving as the standard model of the theory \( ZF + V = L + \exists k \ (k \text{ is inaccessible cardinal}) \).

Only the finite part of this theory will be used here because we shall consider only formulas of limited length (as it will be clear from what follows). So, the countability of this structure is required only for some technical convenience and it is possible to get along without it.

Further \( k \) is the minimal inaccessible cardinal in \( \mathcal{M} \). We shall investigate it “from inside”, considering the hierarchy of subinaccessible cardinals; the latter are “inaccessible” by means of formulas of certain elementary language. To receive this hierarchy rich enough it is natural to use some rich truth algebra \( B \). To receive this hierarchy rich enough it is natural to use some rich truth algebra \( B \). It is well-known that every Boolean algebra is embedded in an appropriate collapsing \( (\omega_0, \mu) \)-algebra and therefore it is natural to use as \( B \) the sum of the set of such algebras of power \( k \), that is Lévy \( (\omega_0, k) \)-algebra \( B \). This algebra can be introduced in the following way. Let’s apply the set \( P \in \mathcal{M} \) of forcing conditions that are finite functions \( p \subset k \times k \) such that for every limit \( \alpha < k \) and \( n \in \omega_0 \alpha + n \in \text{dom}(p) \rightarrow p(\alpha + n) < \alpha \); also let \( p(n) \leq n \) for \( \alpha = 0 \). The relation \( \leq \) of partial order is introduced on \( P \): \( p_1 \leq p_2 \leftrightarrow p_2 \subseteq p_1 \). After that \( P \) is densely embedded in the Boolean algebra \( B \in \mathcal{M} \) complete in \( \mathcal{M} \), consisting of regular sections \( \subseteq P \).

It is well known that algebra \( B \) satisfies the \( k \)-chain condition, therefore it is possible to consider instead of values \( A \in B \) only sets \( P_A = \{ p \in A : \text{dom}(p) \subseteq \chi \} \) where \( \chi < k \), \( \chi = \min \{ \chi' : \forall p \in A \ p|_{\chi'} \leq A \} \), (here \( p|_{\chi'} \) is the restriction of \( p \) to \( \chi' \)). Since \( A = \sum P_A \), we shall always identify \( A \) and \( P_A \), that is we shall always consider \( P_A \) instead of \( A \) itself.

Just due to this convention and GCH in \( L_k \) all Boolean values \( A \in B \) are sets in \( L_k \), not classes, and this phenomenon will make possible all further reasoning as a whole.

We shall investigate the hierarchy of subinaccessible cardinals with the help of Boolean values in \( B \) of some propositions. The countability of the structure \( \mathcal{M} \) is needed here only to shorten the reasoning when using its generic extensions by means of \( \mathcal{M} \)-generic ultrafilters on \( B \). It is possible to get without it developing the corresponding reasoning in the Boolean-valued universe \( L^B \).

The main instrument of the further reasoning is the notion of a formula spectrum. Let’s use the usual elementary language \( L \) over the standard structure \( (L_k, [l], \in, =, l) \), where \( l \) is \( \mathcal{M} \)-generic ultrafilter on \( B \). Its alphabet consists of usual logic symbols, all names from Boolean-valued universe \( L^B_k \) serving as individual constants, symbols \( \in, = \) and \( l \), the canonical name of \( l \). So, all ordinal variables and constants will take values \( < k \); all formulas will be considered as formulas of the language \( L \) (if the other case is not meant by the context). Such formulas are arranged in the elementary Levy hierarchy \( \{ \Sigma_n(\overline{t}) ; \Pi_n(\overline{t}) \}_{n \in \omega_0} \) of formula classes, where \( \overline{t} \) is a train of individual constants. Further such classes and their formulas of some fixed level \( n > 3 \) are considered. This agreement is taken to have in hand further sufficiently large subinaccessible tools and also to use some auxiliary formulas, terms, relations and sets defined in \( L_k \) directly as additional relational constants in formulas denotations without raising their level. Obviously, in this way can be considered \( P, B, \) relations and operations on them mentioned above, and also the following:

\[ \prec \] – the relation of well-ordering on \( L_k : \ a \prec b \iff Od(a) < Od(b) \);
\( \prec \) – the corresponding relation on \( L_k \times k \) : \( a \prec \beta \iff Od(a) < \beta \).

Similarly one can use in \( L_k[l] \) G"odel constructive function \( F^l \) relatively to \( l \) and the function \( Od^l(a) = \min(\{ \alpha : F^l(\alpha) = a \} \) and also relations \( a \prec^l b \iff Od^l(a) < Od^l(b) \), \( a \prec^l \beta \iff Od^l(a) < \beta \).

We shall introduce the notion of spectrum only for propositions \( \varphi(\overrightarrow{a}, \overrightarrow{l}) \) with train of individual constants \( \overrightarrow{a} = (a_1, \ldots, a_m) \) consisting of ordinal constants (if the context does not point to another case). It is possible to manage without this convention replacing occurrences of each \( a_i \) by occurrences of the term \( F^l(\alpha_i) \) for the corresponding ordinal constant \( \alpha_i \).

Let’s also assume that every train \( \overrightarrow{a} = (\alpha_1, \ldots, \alpha_m) \) of ordinals \( < k \) is identified with the ordinal which is its image under the canonical order isomorphism of \( m^k \) onto \( k \).

For every formula \( \varphi \) and ordinal \( \alpha_1 \leq k \) by \( \varphi^{<\alpha_1} \) is denoted the formula obtained from \( \varphi \) by \( \prec \alpha \)-bounding of all its quantors by the ordinal \( \alpha_1 \), that is by replacing all occurrences of such quantors \( \exists x, \forall x \) by corresponding occurrences of \( \exists x (x \prec \alpha_1 \land \ldots), \forall x (x \prec \alpha_1 \rightarrow \ldots) \). In addition, if \( \alpha_1 < k \), then we say that \( \varphi \) is restricted to \( \alpha_1 \) or relativized to \( \alpha_1 \); if, in addition, the proposition \( \varphi^{<\alpha_1} \) holds, then we say that \( \varphi \) holds below \( \alpha_1 \) or that \( \varphi \) is preserved under this restriction or relativization to \( \alpha_1 \). In all such cases \( \alpha_1 \) is named respectively the \( \prec \alpha \)-bounding ordinal. It is obvious, that for \( \alpha_1 < k \) all formulas \( \varphi^{<\alpha_1} \) belong to the class \( \Delta_1 \).

If \( \alpha_1 = k \), then the upper index \( \prec \alpha_1 \) is omitted and such formulas, reasoning and constructions are named unrestricted or unrelativized.

**Definition 1**

1) Let \( \varphi(\overrightarrow{a}, \overrightarrow{l}) \) be a proposition \( \exists x \varphi_1(x, \overrightarrow{a}, \overrightarrow{l}) \) and \( \alpha_1 \leq k \). For every \( \alpha < \alpha_1 \) let us introduce the following Boolean values:

\[
A^{<\alpha_1}_\varphi(\alpha, \overrightarrow{a}) = \| \exists x \prec \alpha \varphi^{<\alpha_1}_1(x, \overrightarrow{a}, \overrightarrow{l}) \|; \quad \Delta^{<\alpha_1}_\varphi(\alpha, \overrightarrow{a}) = A^{<\alpha_1}_\varphi(\alpha, \overrightarrow{a}) - \sum_{\alpha' < \alpha} A^{<\alpha_1}_\varphi(\alpha', \overrightarrow{a}).
\]

2) We name the following function \( S^{<\alpha_1}_\varphi(\overrightarrow{a}) \) the spectrum of \( \varphi \) on the point \( \overrightarrow{a} \) below \( \alpha_1 \):

\[
S^{<\alpha_1}_\varphi(\overrightarrow{a}) = \{(\alpha, \Delta^{<\alpha_1}_\varphi(\alpha, \overrightarrow{a})) : \alpha < \alpha_1 \wedge \Delta^{<\alpha_1}_\varphi(\alpha, \overrightarrow{a}) > 0\}.
\]

**Projections** \( \text{dom} (S^{<\alpha_1}_\varphi(\overrightarrow{a})) \), \( \text{rng} (S^{<\alpha_1}_\varphi(\overrightarrow{a})) \) are named respectively the ordinal and the Boolean spectra of \( \varphi \) on the point \( \overrightarrow{a} \) below \( \alpha_1 \).

3) If \( (\alpha, \Delta) \in S^{<\alpha_1}_\varphi(\overrightarrow{a}) \), then \( \alpha \) is named the jump ordinal of this formula and spectra, while \( \Delta \) is named its Boolean value on the point \( \overrightarrow{a} \) below \( \alpha_1 \).

4) The ordinal \( \alpha_1 \) itself is named the carrier of these spectra.

If a train \( \overrightarrow{a} \) is empty, then we omit it in notations and omit other mentionings about it.

The investigation of propositions is natural by means of their spectra, so one can develop more fine analysis using their two-dimensional, three-dimensional spectra and so on.

All spectra introduced possess the following simple properties:

**Lemma 2**

Let \( \varphi \) be a proposition \( \exists x \varphi_1(x, \overrightarrow{a}, \overrightarrow{l}) \), \( \varphi_1 \in \Pi_{n-1} \), \( \alpha_1 \leq k \), then \( \sup \text{dom} (S^{<\alpha_1}_\varphi(\overrightarrow{a})) < k \).

Here this lemma comes directly from \( k \)-chain property of \( B \). This and all other spectra basic properties will remain at all their further transformations.

The so called universal spectrum is singled out among all other spectra. It is well known that the class \( \sum_n(\overrightarrow{a}) \) for \( n > 0 \) contains the formula which is universal for this class; let’s denote it by \( U_n^\Sigma(n, \overrightarrow{a}, \overrightarrow{l}) \). Its universality means that for any \( \sum_n(\overrightarrow{a}) \)-formula \( \varphi(\overrightarrow{a}, \overrightarrow{l}) \) there is a natural \( n \) (the Gödel number of \( \varphi \)) such that \( \varphi(\overrightarrow{a}, \overrightarrow{l}) \iff U_n^\Sigma(n, \overrightarrow{a}, \overrightarrow{l}) \).
The dual formula universal for $\Pi_n(\bar{d})$ is denoted by $U_n^\Pi(n, \bar{d}, \bar{\ell})$. For some convenience we shall use $U_n^\Sigma$ in the form $\exists x U_{n-1}^\Pi(n, x, \bar{d}, \bar{\ell})$. In this notation the upper indices $\Sigma, \Pi$ will be omitted in the case when they can be restored from the context.

We name as the spectral universal for the class $\Sigma_n$ formula of level $n$ the formula $u_n^\Sigma(\bar{d}, \bar{\ell})$ obtained from the universal formula $U_n^\Sigma(n, \bar{d}, \bar{\ell})$ by replacing all occurrences of the variable $n$ by occurrences of the term $\ell(\omega_0)$. The spectral universal for the class $\Pi_n$ formula $u_n^\Pi(\bar{d}, \bar{\ell})$ is introduced in the dual way. Thus we shall use $u_n^\Pi(\bar{d}, \bar{\ell}) = \exists x u_{n-1}^\Pi(x, \bar{d}, \bar{\ell})$, where $u_{n-1}^\Pi(x, \bar{d}, \bar{\ell})$ is the spectral universal for the class $\Pi_{n-1}$ formula.

The values $A_0^{\alpha_1}(\alpha, \bar{d}), \Delta_0^{\alpha_1}(\alpha, \bar{d})$ and the spectrum $S_0^{\alpha_1}(\bar{d})$ of the formula $\varphi = u_n^\Sigma(\bar{d}, \bar{\ell})$ and its projections will be named the universal Boolean values and spectra of the level $n$ on the point $\bar{d}$ below $\alpha_1$ and they are denoted by $A_n^{\alpha_1}(\alpha, \bar{d}), \Delta_n^{\alpha_1}(\alpha, \bar{d}), S_n^{\alpha_1}(\bar{d})$.

Here the term “universal spectra” is justified by the fact: for every $\varphi = \exists x \varphi_1(x, \bar{d}, \bar{\ell})$, $\varphi_1 \in \Pi_{n-1}$ there holds $\text{dom} (S_1^{\alpha_1}(\bar{d})) \subseteq \text{dom} (S_n^{\alpha_1}(\bar{d}))$.

All further reasoning is conducted in $L_k$ (or in $\mathfrak{M}$). Let’s introduce the central notion of subinaccessibility – the inaccessibility by means of our language. The “meaning” of propositions is contained in their spectra and therefore it is natural to define this inaccessibility by means of the spectra of all propositions of a given level:

**Definition** 3 Let $\alpha_1 \leq k$. We name an ordinal $\alpha < \alpha_1$ subinaccessible of a level $n$ below $\alpha_1$ iff it fulfills the formula $\forall \bar{d} < \alpha \text{ dom}(S_n^{\alpha_1}(\bar{d})) \subseteq \alpha$ denoted by $\text{SIN}_n^{\alpha_1}(\alpha)$. The set $\{\alpha < \alpha_1 : \text{SIN}_n^{\alpha_1}(\alpha)\}$ of all these ordinals is denoted by $\text{SIN}_n^{\alpha_1}$ and they are named $\text{SIN}_n^{\alpha_1}$-ordinals.

As usual, for $\alpha_1 < k$ we say that subinaccessibility of $\alpha$ is restricted by $\alpha_1$ or relativized to $\alpha_1$; for $\alpha_1 = k$ the upper indices $\alpha_1, \alpha_1$ are dropped. This definition obviously follows to the following:

let $\alpha < \alpha_1 \leq k$, $\alpha \in \text{SIN}_n^{\alpha_1}$ and a proposition $\exists x \varphi(x, \bar{d}, \bar{\ell})$ has $\bar{d} < \alpha$, $\varphi \in \Pi_{n-1}$, then for any $\mathfrak{M}$-generic $l$ $L_k[l] \models (\exists x < \alpha \alpha \varphi^{\alpha_1}(x, \bar{d}, \bar{\ell}) \rightarrow \exists x < \alpha \alpha \varphi^{\alpha_1}(x, \bar{d}, \bar{\ell}))$.

In this case we shall say that below $\alpha_1$ the ordinal $\alpha$ restricts or relativizes the proposition $\exists x \varphi$.

Considering the same in the inverted form for $\varphi \in \Sigma_{n-1}$: $L_k[l] \models (\forall x < \alpha \alpha \varphi^{\alpha_1}(x, \bar{d}, \bar{\ell}) \rightarrow \forall x < \alpha \alpha \varphi^{\alpha_1}(x, \bar{d}, \bar{\ell}))$, we shall say that below $\alpha_1$ the ordinal $\alpha$ extends or prolongs the proposition $\forall x \varphi$ up to $\alpha_1$.

Obviously, the cardinal $k$ is subinaccessible itself of any level, if we define this notion for $\alpha = \alpha_1 = k$. So, the comparison of inaccessibility and subinaccessibility notions naturally arises in a following way:

The cardinal $k$ is weakly inaccessible, since it is uncountable and cannot be reached by means of smaller powers in the sense that: 1) it is regular and 2) it is closed under operation of passing to next power: $\forall \alpha < k \alpha^+ < k$. Turning to subinaccessibility of the ordinal $\alpha < k$ of the level $n$, one can see that the property of regularity is dropped now, but $\alpha$ still cannot be reached, but by more mighty means: condition 2) is strengthened and $\alpha$ is closed under more mighty operations of passing to jump ordinals of universal spectrum: $\forall \bar{d} < \alpha \forall \alpha' < \text{dom} S_n(\bar{d}) \alpha' < \alpha$, that is by means of ordinal spectra of all propositions of level $n$. Hence, this ordinal $\alpha$ is closed under all $\Pi_{n-1}$-functions in all extensions $L_k[l], \in = l$, not only under operation of power successor in $L_k$. In particular, for every $n \geq 2 \alpha = \omega_\alpha$ (in $L_k$). Besides that the set $\text{SIN}_{\alpha_1}$ is closed in $\alpha_1$, that is for any $\alpha < \alpha_1$ $\sup(\alpha \cap \text{SIN}_{\alpha_1}) \subseteq \text{SIN}_{\alpha_1}$, and the set $\text{SIN}_n$ is unbounded in $k$, 4
sup $SIN_n = k$.

When formulas are equivalently transformed their spectra can change. It is possible to use this phenomenon for the analysis of subinaccessible cardinals. To this end we shall introduce the universal formulas with ordinal spectra containing only subinaccessible cardinals of smaller level. For more clearness formulas without individual constants will be considered. Let's start with the spectral universal formula for the class $\Sigma_n$. The upper indices $\Sigma, \Pi$ will be omitted as usual (if it will not cause misunderstanding).

In what follows bounding ordinals $\alpha$ are always assumed to be $SIN_{n-1}$-ordinals or $\alpha = k$.

**Definition 4** 1) We name as the monotone spectral universal for the class $\Sigma_n$ formula of the level $n$ the $\Sigma_n$-formula $\tilde{u}_n(\underline{l}) = \exists x \tilde{u}_{n-1}(x, \underline{l})$ where $\tilde{u}_{n-1}(\underline{l}) \in \Pi_{n-1}$ and $\tilde{u}_{n-1}(x, \underline{l}) \leftrightarrow \exists x' \leq x \ u^n_{n-1}(x', \underline{l})$.

2) We name as the subinaccessibly universal for the class $\Sigma_n$ formula of the level $n$ the $\Sigma_n$-formula $\tilde{u}_n^{\sin}(\underline{l}) = \exists x \tilde{u}_{n-1}^{\sin}(x, \underline{l})$ where $\tilde{u}_{n-1}^{\sin}(\underline{l}) \in \Pi_{n-1}$ and $\tilde{u}_{n-1}^{\sin}(x, \underline{l}) \leftrightarrow SIN_{n-1}(x) \wedge \tilde{u}_{n-1}(x, \underline{l})$.

The subinaccessibly universal for the class $\Pi_n$ formula is introduced in the dual way.

3) The Boolean values $A^{\varphi \alpha_1}_n(\alpha), \Delta^{\varphi \alpha_1}_n(\alpha)$, and the spectrum $S^{\varphi \alpha_1}_n$ of the formula $\varphi = \tilde{u}_n^{\sin}$ and its projections (see definition 1 where $\leq$ should be replaced with $\leq$) are named subinaccessibly universal of the level $n$ below $\alpha_1$ and are denoted respectively by $\tilde{A}^{\varphi \alpha_1}_n(\alpha), \tilde{\Delta}^{\varphi \alpha_1}_n(\alpha), \tilde{S}^{\varphi \alpha_1}_n$.

Obviously $u^\Sigma_n(\underline{l}) \leftrightarrow \tilde{u}^{\sin}_n(\underline{l})$ and $\text{dom}(\tilde{S}^{\varphi \alpha_1}_n) \subseteq SIN^{\leq}_n \cap \text{dom}(\tilde{S}^{\varphi \alpha_1}_n)$.

Our aim is to “compare” universal spectra with each other on different carriers $\alpha$ disposed cofinally to $k$ in order to introduce monotone matrix functions. To this end it is natural to do it by means of using values of function $Od$ for such spectra. Also it is natural to find some estimates of “informational complexity” of these spectra by means of estimates of their order types. But the required comparison of such spectra can be hardly carried out in a proper natural way because they are “too much differ” from each other for arbitrary great carriers $\alpha_1$.

So, there is nothing for it but to consider further spectra reduced to some fixed cardinal and, next, reduced matrices. So here we start to form the main material for building matrix functions – reduced matrices. With this end in view first we shall consider the necessary preliminary constructions – reduced spectra.

For an ordinal $\chi \leq k$ let $P_\chi$ denote the set $\{ p \in P : \text{dom}(p) \subseteq \chi \}$ and $B_\chi$ denote the subalgebra of $B$ generated by $P_\chi$ in $L_k$. For every $A \in B$ let’s introduce the set $A[\chi] = \{ p \in P_\chi : \exists q \ (p = q|\chi \wedge q \leq A) \}$ which is named the value of $A$ reduced to $\chi$. It is known that $B_\chi = \{ \sum X : X \subseteq P_\chi \}$ and therefore every $A \in B_\chi$ coincides with $\sum A[\chi]$. Therefore let’s identify every $A \in B_\chi$ with its reduced value $A[\chi]$; so, here one should point out again, that due to the chain property of $B$ and $GCH$ in $L_k$ every value $A \in B_\chi$ is the set in $L_k$, not class, and $B_\chi$ is considered as the set of such values, $B_\chi \subseteq L_k$ for $\chi < k$.

**Definition 5** 1) For every $\alpha < \alpha_1$ let’s introduce the Boolean values and the spectrum:

$$
\tilde{A}^{\sin \alpha_1}_n(\alpha)[\chi]; \quad \tilde{\Delta}^{\sin \alpha_1}_n(\alpha)[\chi] = \tilde{A}^{\sin \alpha_1}_n(\alpha)[\chi] \wedge \sum_{\alpha' < \alpha} \tilde{A}^{\sin \alpha_1}_n(\alpha')[\chi];
$$

$$
\tilde{S}^{\sin \alpha_1}_n[\chi] = \{ (\alpha; \tilde{A}^{\sin \alpha_1}_n(\alpha)[\chi]) : \alpha < \alpha_1 \wedge \tilde{\Delta}^{\sin \alpha_1}_n(\alpha)[\chi] > 0 \}.
$$
2) These values, spectrum and its first and second projections are named subinaccessibly universal reduced to $\chi$ of the level $n$ below $\alpha_1$.

In a similar way multi-dimensional reduced spectra can be introduced.

From this definition comes that reduced spectra possess previous properties, for instance, for $\alpha < \alpha_1$, $\chi \leq k$ there holds $\sup \text{dom} \left( S_n^{\infty, \alpha} \upharpoonright \chi \right) < k$, $\text{dom} \left( \tilde{S}_n^{\infty, \alpha} \upharpoonright \chi \right) \subseteq \text{SIN}_{n-1} \cap \cap \text{dom} \left( S_n^{\alpha_1} \right)$ and so on.

The main role further is played by matrices and spectra reduced to complete cardinals; their existence comes out from $k$-chain property of algebra B:

**Definition 6** We name as a complete ordinal of level $n$ every ordinal $\chi$ such that $\exists x \tilde{u}^{\infty}_{n-1} (x, \bar{l}) \leftrightarrow \exists x < \chi \tilde{u}^{\infty}_{n-1} (x, \bar{l})$. The least of these ordinals is denoted by $\chi^*$. From this definition it comes $\chi^* = \sup \text{dom} \left( \tilde{S}_n^{\infty} \right) = \sup \text{dom} \left( S_n \right) < k$ and $\text{SIN}_{n-1} (\chi^*)$ and so on.

Let's turn to order spectrum types. If $X$ is a well ordered set, then its order type is denoted by $OT(X)$; if $X$ is a function with well ordered domain, then we assume $OT(X) = OT(\text{dom}(X))$. The obvious rough upper estimate of spectrum types comes from $|P| = |\chi|$ and GCH in $L_k : \text{OT} (\tilde{S}_n^{\infty, \alpha}) \upharpoonright \chi < \chi^+$.

Now let's discuss estimates of such types from below. Here comes out the lemma essential for the proof of main theorem: it shows, that as soon as an ordinal $\delta < \chi^+$ is defined through some jump ordinals of the subinaccessibly universal spectrum reduced to $\chi^*$, the order type of this spectrum exceeds $\delta$ under certain natural conditions:

**Lemma 7** (About spectrum type) Let ordinals $\delta$, $\alpha_0$, $\alpha_1$ be such that:

(i) $\delta < \chi^+ < \alpha_0 < \alpha_1 \leq k$;

(ii) $\text{SIN}_{n-2} (\alpha_1) \land \text{OT} (\text{SIN}_{n-1}^{\leq \alpha_1}) = \alpha_1$;

(iii) $\alpha_0 \in \text{dom} \left( \tilde{S}_n^{\infty, \alpha_1} \upharpoonright \chi^* \right)$;

(iv) $\delta$ is defined in $L_k$ through $\alpha_0$, $\chi^*$ by a formula $\in \Sigma_n \cup \Pi_{n-2}$.

Then

$$\delta < \text{OT} (\tilde{S}_n^{\infty, \alpha_1} \upharpoonright \chi^*)$$

But still there is the following essential inconvenience: such spectra, taken on their different carriers, can be hardly compared with each other in view to their basic properties, because their domains (ordinal spectrums) can contain an arbitrary great cardinals, when these carriers are increasing up to $k$. In order to avoid this obstacle we shall transform them to reduced matrices. These matrices comes from reduced spectra by easy isomorphic enumeration of their domains:

**Definition 8** 1) We name as a matrix reduced to an ordinal $\chi$ every function $M$ defined on some ordinal and with $\text{rng}(M) \subseteq B_\chi$.

2) Let $M$ be a matrix and $M_1 \subseteq k \times B$. We name as a superimposition of $M$ onto $M_1$ an order isomorphism $f$ of $\text{dom}(M)$ onto $\text{dom}(M_1)$ such that $\forall \alpha, \alpha' \forall \Delta, \Delta' (f(\alpha) = \alpha' \land (\alpha, \Delta) \in M \land (\alpha', \Delta') \in M_1 \rightarrow \Delta = \Delta')$; in this case we say that $M$ is superimposed onto $M_1$ and write $M \Rightarrow M_1$.

3) Let matrix $M$ be superimposed onto the spectrum $\tilde{S}_n^{\infty, \alpha}$ then $M$ is named the matrix of this spectrum or the subinaccessibly universal matrix of the level $n$ reduced to $\chi$ on $\alpha$.

4) If $(\alpha', \Delta) \in \tilde{S}_n^{\infty, \alpha}$, then $\alpha'$ is named the jump cardinal of the matrix $M$, while $\Delta$ is named its Boolean value on $\alpha$. 5) In this case the cardinal $\alpha$ is named the carrier of the matrix $M$. 

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It is obvious that for $M \Rightarrow \overline{S}_{n}^{\sin<\alpha}$ there holds $OT(M) = dom(M) \leq Od(M) < \chi^{+}$. This property shows that reduced matrices can be compared (in the sense of function $Od$) within $L_{\kappa^{+}}$ only and it will help to define matrix functions with required properties. Further the very special role is played by the so called singular matrices:

**Definition 9** We denote by $\sigma(\chi, \alpha)$ the following formula:

$SIN_{n-2}(\alpha) \land (\chi$ is a limit cardinal $< \alpha) \land OT(SIN_{n-1}^{<\alpha}) = \alpha \land \sup \{ \overline{S}_{n}^{\sin<\alpha} | \chi \} = \alpha$.

And let $\sigma(\chi, \alpha, M)$ denote the formula $\sigma(\chi, \alpha) \land (M \Rightarrow \overline{S}_{n}^{\sin<\alpha} | \chi)$ . The matrix $M$ and the spectrum $\overline{S}_{n}^{\sin<\alpha} | \chi$ reduced to $\chi$ are named singular on a carrier $\alpha$ iff $\sigma(\chi, \alpha, M)$ is fulfilled. The symbol $S$ will be used for the common denotation of singular matrices.

Further all matrices will be singular on their carriers; all reasoning will be conducted in $L_{\kappa}$ (or in $\mathfrak{M}$).

By this definition jump cardinals of singular matrix on its carrier $\alpha$ are disposed cofinally to $\alpha$. Due to the last fact it is possible to introduce the following important cardinals:

**Definition 10** Let $\sigma(\chi, \alpha, S)$ fulfills, then we name as jump cardinal and prejump cardinal after $\chi$ of the matrix $S$ on the carrier $\alpha$, or, briefly, of the cardinal $\alpha$, the following cardinals respectively:

$\alpha^{+}_{\chi} = \min \{ \alpha' \mid \chi, \alpha : \overline{A}_{n}^{\sin<\alpha} (\chi) \cap \chi < \overline{A}_{n}^{\sin<\alpha} (\alpha') \}$;  
$\alpha^{\#}_{\chi} = \sup \{ \alpha' < \alpha^{+}_{\chi} : \overline{A}_{n}^{\sin<\alpha} (\chi) \cap \chi < \overline{A}_{n}^{\sin<\alpha} (\alpha') \}$.

It is not hard to see, that in this situation these $\alpha^{+}_{\chi}$, $\alpha^{\#}_{\chi}$ really do exist and

$\alpha^{+}_{\chi} = \min \{ \alpha' > \chi : \alpha' \in dom (\overline{S}_{n}^{\sin<\alpha} | \chi) \}$;  
$\alpha^{\#}_{\chi} < \alpha^{+}_{\chi} < \alpha; \quad [\alpha^{\#}_{\chi}, \alpha^{+}_{\chi}] \cap SIN_{n-1}^{<\alpha} = \emptyset$.

The basic instruments of main theorem proof are matrix functions that are sequences of reduced singular matrices. The following lemma makes it possible to build such functions:

**Lemma 11** $\forall \chi \forall \alpha_{0}((\chi$ is a limit cardinal $> \omega_{0}) \rightarrow \exists \alpha_{1} > \alpha_{0} \sigma(\chi, \alpha_{1}))$ .

The building of matrix functions relies on the following enumeration (in $L_{\kappa}$) of subinaccessible $SIN_{n-1}$-cardinals below $\alpha_{1}$:

**Definition 12** The function $\gamma^{<\alpha_{1}} = (\gamma^{<\alpha_{1}})_{\tau}$ is defined by recursion on $\tau < \alpha_{1} \leq k$:

$\gamma^{<\alpha_{1}}_{0} = 0$;  
for $\tau > 0$ $\gamma^{<\alpha_{1}}_{\tau} = \min \{ \gamma^{<\alpha_{1}} \mid SIN_{n-1}^{<\alpha_{1}} (\gamma) \land \forall \tau' < \tau : \gamma^{<\alpha_{1}}_{\tau'} < \gamma \}$.

Obviously, if $\alpha_{1} = k$ then range and domain of this function are both cofinal to $k$.

The proof of main theorem consists in creation in $L_{\kappa}$ of the special matrix functions possessing inconsistent properties; here are such functions of the main kind:

**Definition 13** We name as a matrix function of the level $n$ below $\alpha_{1}$ reduced to $\chi$ the function $S_{x}^{<\alpha_{1}} = (S_{x}^{<\alpha_{1}})_{\tau}$ taking values for $\tau < k$:

$S_{x}^{<\alpha_{1}} = \min \{ S : \exists \alpha < \alpha_{1} (\gamma^{<\alpha_{1}} \cap S_{x}^{<\alpha_{1}} (\chi, \alpha, S)) \}$.

So, these values are matrices $S_{x}^{<\alpha_{1}}$ reduced to $\chi$ singular on these carriers $\alpha$ that are $\leq$-minimal for $\gamma^{<\alpha_{1}} < \chi$. In the same sense all these values are bounded by $\chi^{+} < k$ due to $GCH$ in $L_{\kappa}$: $Od(S_{x}^{<\alpha_{1}}) < \chi^{+}$.

Everywhere further $\chi = \chi^{\ast}$; the lower index $\chi^{\ast}$ can be omitted in notations, for instance $\alpha^{+}_{\chi^{\ast}}$, $\alpha^{\#}_{\chi^{\ast}}$, $S_{x}^{f}$, $S_{x}^{\tau}$ will be denoted through $\alpha^{+}$, $\alpha^{\#}$, $S_{f}$, $S_{\tau}$ (if the context will not mean another case) and so on.

The following two lemmas represent the mainstream of all further reasoning: they establish that such matrix functions have properties of definiteness and of $\leq$-monotonicity which comes from $\leq$-minimality of their values. From lemma 11 there follows directly:
Lemma 14 (About matrix function definiteness) The unrelativized function $S_{\chi^*f}$ is defined on the final segment of $k$: $\text{dom}(S_{\chi^*f}) = \{ \tau : \chi^* \leq \gamma_\tau < k \}$.

Lemma 15 This function $S_{\chi^*f}$ is $\leq$-monotone: $\chi^* < \tau_1 < \tau_2 < k \rightarrow S_{\chi^*\tau_1} \leq S_{\chi^*\tau_2}$.

Hence this function stabilizes, that is there is an ordinal $\tau^*$ such that for every $\tau \geq \tau^*$ there exist constant value $S_{\chi^*\tau} = S_{\chi^*\tau^*}$. Thus for all $\gamma_\tau \geq \chi^*$ values $S_{\chi^*\tau}$ are bounded by the fixed ordinal $Od(S_{\chi^*\tau^*}) < \chi^*+$.  

Now everything is ready to present the main theorem proof mode. The monotonicity of the simplest matrix function $S_{\chi^*f}$ is established already. So, the required contradiction should be find out in its nonmonotonicity on the following way:

The lower index $\chi^*$ will be dropped for some brevity. Let’s consider the matrix function $S_f$ in its state of stabilizing, that is consider an arbitrary sufficiently great $\tau_0 > \tau^*$ and the matrix $S_{\tau_0}$ on some carrier $\alpha_0 \in ]\gamma_{\tau_0}, k[$, the prejump cardinal $\alpha^0 = \alpha_0^\downarrow$ and the relativized function $S_f^{<\alpha_0}$ below $\alpha^0$. Remind, all values of $S_f$ are bounded by the fixed ordinal $Od(S_{\tau^*}) < \chi^*+$.

Standing on the jump cardinal $\alpha_0^\downarrow$ after $\chi^*$ of this matrix on this carrier, one should observe the behavior of this very function, but in its relativized to $\alpha^0 = \alpha_0^\downarrow$ form $S_f^{<\alpha_0}$ below $\alpha^0$. The function $S_f$ is monotone and this relativized function $S_f^{<\alpha_0}$ is monotone too by the same reasons.

But it is excluded. To see it one should apply lemma 7 about spectrum type to this situation, considering it for $\delta = \sup_\tau Od(S_f^{<\alpha_0})$, $\alpha_0$ - the jump cardinal $\alpha_0^\downarrow$, $\alpha_1$ – the carrier $\alpha_0$.

Let’s consider $\delta < \chi^*+$, then one can define $\delta$ standing on $\alpha_0^\downarrow$ and can see that all conditions of this lemma are fulfilled and therefore it implies the contradiction:

$Od(S_{\tau^*}) \leq \delta < OT(S_{\tau_0}) \leq Od(S_{\tau^*})$.

Hence, $S_f$ is nonmonotone and it constitutes the required contradiction.

The case $\delta = \chi^*+$ can be eliminated by certain transformation of matrix functions $S_f^{<\alpha_1}$.

References

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