A Probabilistic Approach to Driver Assistance for Delay Reduction at Congested Highway Lane Drops

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Abstract

This paper proposes an onboard advance warning system based on a probabilistic prediction model that advises vehicles on when to change lanes for an upcoming lane drop. The prediction model estimates the probability of reaching a goal state on the road using one or multiple lane changes. This estimate is based on several traffic-related parameters such as the distribution of inter-vehicle headway distances, as well as driver-related parameters like lane change duration. For an upcoming lane drop, the advance warning system uses the model and vehicle conditions at the moment to continuously estimate the probability of successfully changing lanes under those conditions before reaching the lane end, and advises the driver or autonomous vehicle to change lanes when that probability dips below a certain threshold. In a case study, the proposed system was used on a segment of the I-81 interstate highway with two lane drops - transitioning from four lanes to two lanes - to advise vehicles on avoiding the lane drops. The results show that the proposed system can reduce average delay up to 50% and maximum delay up to 33%, depending on traffic flow and the ratio of vehicles equipped with the advance warning system.

Keywords: Lane change, Probability estimation, Traffic simulation, Parameter analysis, Lane drop

1. Introduction

Lane changes are essential to highway driving, yet they can have a deteriorating effect on traffic flow and safety. These maneuvers are influenced by several factors, including driving behavior, state of nearby vehicles, and the urgency to change lanes (Brackstone et al., 1998). To successfully change lanes the driver (or autonomous vehicle) has to identify an acceptable gap in the target lane, adjust speed and maintain correct position relative to nearby vehicles, and navigate to the target lane while avoiding collision with other vehicles (Kesting et al., 2007). Because of this, any small mistake or unsafe driving behavior can result in an accident. In the United States, between four to ten percent of all reported motor vehicle crashes are due to unsafe lane changes. Apart from the fatalities, these crashes incur an economic loss by delaying traffic (Sen et al., 2003; Li-sheng et al., 2009; Van Dijck and van der Heijden, 2005). This can be partially mitigated by providing drivers with timely information of the road ahead and using assistant systems that help control the vehicle.

Lane changes can be classified as either discretionary or mandatory (Zhang et al., 1998). Discretionary lane changes are often performed to overtake slow traffic and move to a lane with a higher speed. In contrast, mandatory lane changes are required to follow a planned path or avoid an obstacle, for example a lane drop. Compared to discretionary lane changes, mandatory lane changes can have a disruptive impact on traffic. They can deteriorate traffic safety (Ahammed et al., 2008; Li and Sun, 2017) and cause traffic oscillation (Sarvi et al., 2007), traffic breakdown (Lv et al., 2013), and capacity drops (Cassidy and Rudjanakanoknad, 2005). Mandatory lane changes caused by lane drops have been shown to affect traffic flow in all lanes upstream by generating perturbation density waves (Munjal et al., 1971), forming queues (Bertini and Leal, 2005), and stop-and-go traffic patterns (Zhang and Shen, 2009; Yuan et al., 2017).

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To manage upstream traffic at a lane drop, past studies have focused on variable speed limit (VSL) strategies (Jin and Jin, 2015; Yu and Fan, 2018b,a), congestion assistants (Van Driel and Van Arem, 2010; Roncoli et al., 2017; Zhang et al., 2019), or a combination of both (Zhang and Ioannou, 2016). In approaches involving VSL, traffic flow is managed by dynamically modifying the speed limit upstream of the bottleneck using different control methods and optimization strategies - ranging from PI- and I- controllers (Jin and Jin, 2015) to tabu search algorithm (Yu and Fan, 2018b) and genetic algorithm (GA) (Yu and Fan, 2018a) - to mitigate a drop in capacity. While macroscopic simulations have shown that VSL strategies can achieve that goal and reduce total travel time (TTT) by up to 16% (Yu and Fan, 2018a), as Zhang and Ioannou (2016) notes because capacity drops and delays often happen due to lane changes close to the bottleneck, absent a lane assignment strategy the system often breaks down in microscopic simulations and fails to mitigate capacity drop. To that end, centralized lane assignment systems using connected vehicle technology, either standalone (Roncoli et al., 2017; Zhang et al., 2019) or combined with VSL strategies (Zhang and Ioannou, 2016), have been proposed and shown to reduce TTT by up to 40% (Zhang et al., 2019).

In this paper we take a different approach to delay reduction at lane drops and propose an advance warning system based on a probabilistic prediction model that advises vehicles on when to change lanes for an upcoming lane drop. Compared to past studies, our approach can be implemented as a simple, individualized, real-time onboard system using readily available information, removing the need for the hardware necessary for connected vehicle technology or a centralized lane assignment system. Furthermore, as our proposed system only advises vehicles on their lane change behavior, it can be coupled with VSL strategies for greater efficacy. Finally, we introduce a microscopic simulation setup to evaluate the performance of the proposed system and test its effectiveness for different traffic flow and penetration rate (ratio of vehicles with the onboard system) conditions.

The remainder of this paper is structured as follows. Section 2 presents the methodology, including a brief overview of the probability model underlying the system and the simulation setup used to evaluate its performance. Section 3 presents our findings and a discussion of the effects of the system on traffic flow and traffic efficiency. Finally, Section 4 concludes the findings of this paper.

2. Methodology

A road bottleneck is a location where downstream flow capacity is lower than upstream capacity (Roncoli et al., 2017). This can be the result of road features like lane drops and merges, or temporary blockages such as work zones and traffic accidents.

At a lane drop bottleneck, the nominal capacity is the maximum traffic flow that can be maintained downstream if the upstream traffic flow is no larger than that capacity. In other words, if upstream capacity for a road transitioning from $n$ lanes to $n - 1$ lanes, $n \geq 2$, is denoted by $C$, then the bottleneck capacity is $C_b = \frac{n - 1}{n} C$. However, if upstream traffic flow is larger than $C_b$ or if lane change maneuvers of vehicles in the blocked lane trying to get in other lanes cause vehicles to decelerate and disrupt traffic, the actual capacity, denoted by $C_a = \delta C$, is lower than the nominal capacity $C_b$. This reduction in capacity is called capacity drop and past studies have shown that actual capacity can be anywhere from 5% to 20% lower than the nominal capacity (Cassidy and Bertini, 1999; Chung et al., 2007). Capacity drop causes a disruption in traffic and results in higher delays for all vehicles.

To prevent or postpone capacity drop and reduce delay at lane drop bottlenecks, we propose an advance warning system based on a model that predicts the likelihood of reaching a near-term goal state using one or multiple lane changes (Mehr and Eskandarian, 2020). Using this model and driving conditions at each moment, vehicles equipped with the system in the blocked lane continuously calculate the probability of leaving that lane under those driving conditions before reaching the lane end. They change lanes when that probability dips below a certain threshold.

The proposed system is implemented in VISSIM® for a simulation of a section of the I-81 interstate highway transitioning from four lanes to two lanes to evaluate its performance in reducing delay under different traffic flows and penetration rates. In what follows, Section 2.1 gives a brief overview of the probability model introduced above, while Section 2.2 describes our implementation of the proposed advance warning system and the simulation setup used to evaluate its effectiveness.

2.1. Probability model

The model introduced in Mehr and Eskandarian (2020) estimates the probability of a vehicle reaching a near-term goal state using one or multiple lane changes. While a brief overview of the model is provided here for the sake of
completeness, detailed derivation and validation of the model can be found in Mehr and Eskandarian (2020).

Without loss of generality, consider a road with \( n \) lanes, numbered by 1 to \( n \) from left to right. Assume that the ego vehicle wants to reach a position on lane \( n \) a distance \( d \) ahead of its current position on lane 1. Denoting the success probability of doing so by \( P(S) \), the model estimates this probability by making a few assumptions. First, the model assumes that the velocity of all vehicles on lane \( i \), \( 1 \leq i \leq n \), is equal to \( v_i \), where \( v_i \) is the average velocity of all vehicles on that lane over a period of time. Second, the model assumes that inter-vehicle headway distances (front bumper to front bumper) on lane \( i \) are independent identically distributed (i.i.d) random variables from a common log-normal distribution defined by parameters \( \mu_i \) and \( \sigma_i \) (Mei and Bullen, 1993). Finally, the model assumes that vehicle lane changes can be approximated using the Gipps gap acceptance model (Gipps, 1986). That is, if the ego vehicle is on lane \( i \), it only changes lanes if the gap between its leading and trailing vehicles on the adjacent lane \( i \) is no smaller than a minimum acceptable (critical) gap \( g_i \). It takes \( t_i \) seconds to complete such a lane change. For better visualization, some of these assumptions are shown in Figure 1.

![Figure 1: Notations used in this paper for a road segment with three lanes. The red car is the ego vehicle and the red star shows the goal state.](image-url)

The model estimates the success probability \( P(S) \) based on the parameters defined above. In other words, for the case described above \( P(S) = f_n(d, v_{1,1}, \mu_{2,1}, \sigma_{2,1}, g_{2,1}, t_{2,1}) \), where \( w_{l,m} \) means \( w_l, w_{l+1}, \ldots, w_m \) for any parameter \( w \) and indices \( m \geq l \). \( P(S) \) is estimated recursively, with \( n = 2 \) as the base case. For the base case, \( P(S) \) is obtained from a look-up table of values calculated by Monte Carlo simulations of the problem normalized for unit distance, because a closed-form expression for the probability does not exist. For \( n > 2 \), \( P(S) \) is obtained recursively from

\[
f_n(d, v_{1,1}, \mu_{2,1}, \sigma_{2,1}, g_{2,1}, t_{2,1}) = \int_0^d f_2(d - x, v_{n-1,1}, \mu_n, \sigma_n, g_n, t_o) \frac{\partial}{\partial x} f_{n-1}(x, v_{1,n-1}, \mu_{2,n-1}, \sigma_{2,n-1}, g_{2,n-1}, t_{2,n-1}) dx
\]

which is based on the law of total probability (Leon-Garcia, 2017). Extensive traffic simulations in VISSIM® for a range of parameters showed that in most cases the model is accurate to within 4% of the actual probability (Mehr and Eskandarian, 2020).

### 2.2. Simulation setup

The advance warning system proposed in this paper uses the probability model to advise vehicles on when to change lanes to reach a particular goal state, here avoiding a lane drop. Specifically, when a vehicle approaches a lane drop and is in the blocked lane, the system uses traffic data and vehicle conditions (speed and distance to the lane end) to continuously calculate the probability of reaching the adjacent lane before the lane end under those conditions and instructs the vehicle to change lanes when that probability dips below a certain threshold. If an adequate portion of vehicles use this system, it can help them change lanes at the proper moment to reduce overall traffic delay.

We used traffic simulations in VISSIM® to evaluate the performance of the proposed advance warning system in reducing delay at a highway segment with two consecutive lane drops. Simulations were performed for a variety of traffic conditions, obtained by changing peak traffic flow rate and the portion of vehicles equipped with the proposed system. For each case, we studied how different thresholds for the probability model (the value at which it advises the driver to start changing lanes) affect traffic behavior and average delay. Details of the simulation setup are presented in Section 2.2.1 to Section 2.2.4.
2.2.1. Simulation fundamentals

Traffic simulations were performed on a segment of the southbound I-81 interstate highway near Blacksburg, Virginia, shown in the left image of Figure 2. The segment is 12,210.91 ft (2.31 mi) long and transitions from four lanes to two lanes via two consecutive lane drops, as shown in the right image of Figure 2. It starts just after Exit 118 and ends just before the merge from 118A. It has one vehicle input and one vehicle output, located at either ends of the segment. The posted speed limit along the segment is 70 mph (roughly 112.7 km/h), though actual speeds vary based on traffic.

Figure 2: Left: Bird’s-eye view of the I-81 highway segment used for traffic simulations. This segment of the southbound interstate road starts on the top right corner of the image and ends on the bottom left corner. The segment is 12,210.91 ft (2.31 mi) long and transitions from four lanes to two lanes via two consecutive lane drops. Right: A section of the simulated highway segment. It shows the highway transitioning from four lanes to two lanes via two consecutive lane drops.

The simulated road segment was divided into five sections (links) for better assignment of driving behavior, with the endpoint of each link the same as the start of the next link. The first link started from the beginning of the segment and ended slightly before the first posted lane drop sign for the first lane drop, while the second link ended at the middle of the first lane drop taper. Similarly, the third link ended slightly before the first posted lane drop sign for the second lane drop, while the forth link ended at the middle of the second lane drop taper. The last link ended just before the merge from 118A. Link details are shown in Table 1.

| Link number | Length (ft) | Number of lanes |
|-------------|-------------|-----------------|
| 1           | 3275.312    | 4               |
| 2           | 2998.360    | 4               |
| 3           | 2490.507    | 3               |
| 4           | 1798.360    | 3               |
| 5           | 1632.506    | 2               |

Simulations were conducted according to Virginia Department of Transportation’s (VDOT) Traffic Operations and Safety Analysis Manual (TOSAM) and VISSIM® User Guide (Traffic Engineering Division, 2020a,b). The latter recommends running each simulation case 10 times with different - but consistent - random seeds and then averaging the results, but given our observation that for each case a few runs would crash before finishing1, each case was run 16 times to satisfy this recommendation. For each case the runs started from a random seed of 42 with an increment of 5 for each following run. Each run was set for 9000 simulation seconds. The first 1800 seconds were the seeding period and the following 7200 seconds were the analysis period, with the peak period defined as the time between 3600 and 7200 seconds. Simulation parameters were set for each 900 second interval (Traffic Engineering Division, 2020b).

For all cases, input vehicle flow \( q_i \) was set to 2400 vehicles per hour (veh/hr) for the entirety of the simulation except for the peak period where \( q_{i,p} \) was one of the parameters studied for its effects on overall system performance.

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1During some runs, when the advance warning system sent a signal for a vehicle to change lanes, VISSIM®’s internal model that controls driving behavior calculated a trajectory angle larger than 90 degrees, resulting in a crash.
was set to either 4400, 4600, or 4800 veh/hr, modeling temporary, rush-hour traffic (for example AM or PM peak flow or traffic after a football game at the nearby Lane Stadium). Traffic consisted of entirely North American vehicles (Traffic Engineering Division, 2020b) (slightly larger than European vehicles commonly used by default in VISSIM®, matching the composition of vehicles on the road in North America) divided into three different vehicle types: cars, smart cars, and heavy goods vehicles (HGVs). Smart cars were identical to cars, with the exception that their lane change initiation behavior was controlled by an external driver model (EDM), modeling vehicles that use the proposed advance warning system (see Section 2.2.3).

Vehicle composition varied from case to case and during the peak period. The ratio of HGVs was set to 15% during the peak period and to 20% at all other times (Virginia Commonwealth Transportation Board, 2018). The ratio of smart cars \( r \) (i.e. the penetration rate of the proposed system) was another parameter that was studied. For each value of \( q_{i,p} \) and \( r \) was set to either 10%, 40%, or 70%, resulting in 9 overall cases. The rest of the vehicles were simply cars. As an example, for a case with \( r = 40\% \), the ratio of cars, smart cars, and HGVs was set to 45%, 40%, and 15% during the peak period and to 40%, 40%, and 20% at all other times, respectively. All vehicle types had a desired speed distribution of 70 mph at the input, which meant assigning a desired speed between 67 mph and 80 mph to each vehicle at random with uniform probability (Traffic Engineering Division, 2020a).

To record traffic data during the simulation, sets of data collection points were defined at the midpoint of each link. They recorded time-stamped velocity of vehicles passing through them, modeling real-world loop detectors. Furthermore, a set of travel time measurements was defined to measure vehicle travel times and delays during the simulation. The measurement started at the beginning of the road segment and ended near its end, covering a total distance of 12,178.38 ft.

2.2.2. Driving behavior

Driving behavior was defined according to Traffic Engineering Division (2020a,b) using data from a previous VDOT study (Virginia Department of Transportation, VDOT) with parameter values shown in Table 2. Per the recommendations of Traffic Engineering Division (2020b), the default Freeway (free lane selection) driving behavior - using the Wiedemann 99 driving model with default parameter values - was used for links 1, 3, and 5 where a large number of lane changes were not expected (Wiedemann, 1974). The Weave & Merge driving behavior was used on links 2 and 4 where we anticipated significant weaving and merging, given that their starting point marked the first posted sign for each lane drop.

| Parameter | Freeway (free lane selection) | Weave & Merge |
|-----------|-------------------------------|---------------|
| CC0 (Standstill Distance) (ft) | 4.92 | 4.92 |
| CC1 (Headway Time) (s) | 0.9 | 0.9 |
| CC2 (Following Variation) (ft) | 13.12 | 13.12 |
| Maximum Deceleration (Own Vehicle) (ft/s²) | -13.12 | -15.00 |
| Maximum Deceleration (Trailing Vehicle) (ft/s²) | -9.84 | -12.00 |
| Accepted Deceleration (Own Vehicle) (ft/s²) | -3.28 | -4.00 |
| Accepted Deceleration (Trailing Vehicle) (ft/s²) | -1.64 | -3.28 |
| Safety Distance Reduction Factor | 0.60 | 0.25 |
| Maximum Deceleration for Cooperative Braking (ft/s²) | -9.84 | -23.00 |
| Advanced Merging | On | On |
| Cooperative Lane Change | Off | On |

The value of Lane Change Distance (distance from a connector that vehicles anticipating a lane change start to act) for lane drop connectors (the connector between links 2 and 3, and the connector between links 4 and 5) was increased from the default value of 656.2 ft to the distance of that connector from the first posted lane drop sign. For the first connector, that distance was 2880 ft, while for the second connector it was 1740 ft. While the default value is usually enough for urban traffic simulations, it should be increased for highway modeling because a small value would result in artificial queues at the lane drop (Traffic Engineering Division, 2020b; Gomes et al., 2004). In the absence of experimental trajectory data to calibrate the model, the value used provides a good balance between preventing
artificial queues at each lane drop and forcing all merging vehicles out of the blocked lane much earlier than they are supposed too.

2.2.3. External driver model

VISSIM®'s External Driver Model (EDM) API grants control over various driving behavior aspects of all or a group of vehicles. For this study, the EDM was used to simulate an onboard advance warning system for an upcoming lane drop.

For each case, we first ran the simulation with all vehicles using VISSIM®'s internal model, serving as a baseline for later comparison. Using data from data collection points defined earlier, we calculated average values of parameters \( v_i, \mu_i, \) and \( \sigma_i, 2 \leq i \leq 3 \) as appropriate (depending on which lane the ego vehicle is on), for different road segments in 900-second intervals. For example, data from the first data collection point (defined at the midpoint of the first link) was used to calculate \( v_i, \mu_i, \) and \( \sigma_i \) for the first link for each 900-second interval, and so on. In the real world, this information can be collected once, either experimentally using loop detectors or through traffic simulations, and stored locally or in the cloud for future use by onboard warning systems. As for \( g_i \), it was set to \( \delta v_i + s_0 \) with \( \delta \) and \( s_0 \) set to 1.6 seconds and 1 meter, respectively. Though in reality the critical gap used by drivers is stochastic in nature and depends on a variety of factors - including relative speeds of leading and trailing vehicles in the adjacent lane and driver aggressiveness - our choice simplifies the model and its conservative nature (generally being larger than the actual critical gap) makes sure unsafe lane changes do not occur (Toledo et al., 2003). Finally, \( t_i \) was set to 3 seconds, as VISSIM®'s internal model completes a lane change in that time from when it is initiated (PTV et al., 2019). In the real world, both \( g_i \) and \( t_i \) can be tuned to match the lane change behavior of individual drivers.

In subsequent simulations, smart cars on the two leftmost lanes of links 1 and 2 or on the leftmost lane of links 3 and 4, used the EDM for advice on when to change lanes. Along with the values of \( v_i, \mu_i, \sigma_i, g_i, \) and \( t_i \), the EDM used each vehicle’s velocity as \( v_1 \) and its distance to the lane-end of that lane as \( d \). For example, if a vehicle was on the leftmost lane of link 1, \( d \) would be set to the distance of that vehicle to the first lane drop, whereas if the vehicle was on the second leftmost lane, \( d \) would be set to the distance of that vehicle to the second lane drop. The EDM was inactive when a vehicle was on the two rightmost lanes. The only exception to this entire process was when \( v_{i+1} \) was within the interval with endpoints \( v_i \pm v_3 \), with \( v_3 \) set to 4 m/s. In that case, \( v_{i+1} = v_i + v_3 \). This was done because our previous work in Mehr and Eskandarian (2020) showed that when \( v_i \) and \( v_{i+1} \), \( 1 \leq i \leq n - 1 \), are close to each other, due to a large reduction in the relative traveled distance the probability drops significantly which is unrealistic. Therefore, this modification was made to more accurately represent acceleration or deceleration behavior of drivers when looking for a gap in an adjacent lane to initiate a lane change.

To simulate the proposed advance warning system, the EDM was programmed to continuously calculate the probability of successfully avoiding the lane drop under momentary conditions for each smart car not on the two rightmost lanes. If the probability dipped below a certain threshold, the EDM instructed that vehicle to change lanes. Along with the values of \( v_i, \mu_i, \sigma_i, g_i, \) and \( t_i \), the EDM was used to calculate \( v_{i+1} \) for the first link for each 900-second interval, and so on. In the real world, this information can be collected once, either experimentally using loop detectors or through traffic simulations, and stored locally or in the cloud for future use by onboard warning systems. As for \( g_i \), it was set to \( \delta v_i + s_0 \) with \( \delta \) and \( s_0 \) set to 1.6 seconds and 1 meter, respectively. Though in reality the critical gap used by drivers is stochastic in nature and depends on a variety of factors - including relative speeds of leading and trailing vehicles in the adjacent lane and driver aggressiveness - our choice simplifies the model and its conservative nature (generally being larger than the actual critical gap) makes sure unsafe lane changes do not occur (Toledo et al., 2003). Finally, \( t_i \) was set to 3 seconds, as VISSIM®'s internal model completes a lane change in that time from when it is initiated (PTV et al., 2019). In the real world, both \( g_i \) and \( t_i \) can be tuned to match the lane change behavior of individual drivers.

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One problem that we faced was that when the EDM instructed any vehicle to change lanes, VISSIM® would immediately start to do so without first checking to see if it was safe, leading to bizarre situations where vehicles passed through each other. To solve this problem, the EDM first checked to see if conducting a lane change was safe before instructing a vehicle to do so. It used the velocity of the ego vehicle relative to its leading and trailing vehicles in the adjacent lane to calculate the leading and trailing critical gaps given in Equation 2 and Equation 3 and compared them to the actual relative distance between the ego vehicle and its leading and trailing vehicles in the adjacent lane (Toledo et al., 2003). If both distances were larger than the critical gap, it would proceed with the lane change.

\[
g_i^{lead} = \exp(1.353 - 2.700 \max[0, \Delta v_i^{lead}] - 0.231 \min[0, \Delta v_i^{lead}] + \epsilon^{lead}), \quad (2)
\]

\[
g_i^{lag} = \exp(1.429 + 0.471 \max[0, \Delta v_i^{lag}] + \epsilon^{lag}), \quad (3)
\]

where \( \epsilon^{lead} \sim N(0, 1.112^2) \) and \( \epsilon^{lag} \sim N(0, 0.742^2) \). In the equations above, \( g^{lead} \) refers to the gap between the front bumper of the ego vehicle and the rear bumper of the leading vehicle in the adjacent lane and \( g^{lag} \) refers to the gap between the rear bumper of the ego vehicle and the front bumper of the trailing vehicle in the adjacent lane. Similarly, \( \Delta v^{lead} \) refers to the velocity of the leading vehicle in the adjacent lane relative to the velocity of the ego vehicle and
results and discussion

for a case the number of runs that completed successfully is smaller than 10, that case is marked and the number of run crashed at 4896 seconds, data up to the 4500 second mark was retained. For the results reported in Section 3, if up to the nearest 900-second interval before the run crashed was retained and used for averaging. For example, if a run crashed at 4896 seconds, data up to the 4500 second mark was retained. For the results reported in Section 3, if for a case the number of runs that completed successfully is smaller than 10, that case is marked and the number of successful runs is indicated.

3. Results and Discussion

Statistical characteristics of traffic delay results for all vehicles during the analysis period are tabulated in Table 3. The results are divided into blocks according to $q_{i,p}$ in the first column and $r$ in the first row. For each block the second column shows different values used for $p_i$, with baseline being the case where no advance warning system was present. In each block the three columns present the average, standard deviation of, and maximum delay. For each row other than baseline, the numbers in parenthesis show percentage change relative to the respective value for the baseline case. For example, the third, fourth, and fifth columns of the fifth row show the average, standard deviation of, and maximum delay for the simulation case with $q_{i,p} = 4400$ veh/hr, $r = 10\%$, and $p_i = 0.99$, with the numbers in parenthesis showing percentage change relative to the baseline case in the third row. For this example, average and maximum delay were improved by 18.8% and 13.2% relative to the baseline case, respectively. Finally, for the few cases where the number of simulations that crashed exceeded 6 (as mentioned before), the number of simulations that did not crash and were used for averaging are indicated as a superscript for the average, standard deviation, and maximum delay values for that case.

An overall look at the results shows that in all cases (combinations of $q_{i,p}$ and $r$), for at least one value of $p_i$ the system was successful at reducing average traffic delay, but its behavior varied for different cases. Therefore, after discussing some broad trends in the results, we look at two cases in more detail: the case with $q_{i,p} = 4600$ veh/hr and $r = 40\%$ where all values of $p_i$ result in a sizable reduction in delay; and the case with $q_{i,p} = 4800$ veh/hr and $r = 70\%$ where only a few $p_i$ values result in minor reductions in delay. From here on, the former is called Case A and the latter Case B.

Table 3 shows that as $q_{i,p}$ increases, so does average delay for the baseline case. This is because as vehicle density per lane increases, for vehicles traveling in the blocked lanes finding an acceptable gap in an adjacent lane becomes harder and changing lanes results in additional traffic disruptions and delays. In the same light, the results show that as $q_{i,p}$ increases and the road becomes congested beyond the capacity of a two-lane highway, the impact of the system is reduced. For baseline cases, as the advance warning system is not active its penetration rate $r$ does not have an impact on the results. When it is active, for $r = 10\%$ the results match our expectations that regardless of $q_{i,p}$, because of a low penetration rate the system would only have a modest impact on delay. For $q_{i,p} = 4400$ veh/hr, reduction in delay is largest when $r = 70\%$, with improvements reaching as high as 50%. This is reversed, however, for the other two values of $q_{i,p}$, because for higher penetration rates the system tries to force more traffic onto non-blocked lanes earlier, causing additional delay. Finally, broadly speaking to the effects of $p_i$, when it is higher (for example 0.999) the system warns drivers much earlier than it would when $p_i$ is lower, which depending on $q_{i,p}$ and $r$ can have a positive or negative impact on traffic, as will be discussed next. For all parameters, similar trends can be observed for maximum and standard deviation values of delay.

$\Delta v_{lag}$ refers to the velocity of the trailing vehicle in the adjacent lane relative to the velocity of the ego vehicle. Finally, $\epsilon$ is a random term associated with lane utility (Toledo et al., 2003).

2.2.4. Data processing and evaluation

Average delay, defined as the difference between actual travel time and travel time under free flow speed, was selected as our measure of effectiveness (MoE) (Traffic Engineering Division, 2020b). Using the travel time measurement defined in Section 2.2.1, VISSIM® automatically calculated the delay for each individual vehicle. Using that data, we calculated the average $m_i$, standard deviation $s_i$, and maximum delay $a_i$ for each run $i$, $1 \leq i \leq 16$, reporting the average of those values over all runs for each combination of $q_{i,p}$, $r$, and $p_i$ both for the entire analysis period and for each 900-second interval. In other words, for each combination of $q_{i,p}$, $r$, and $p_i$ we reported $m = \frac{1}{16} \sum_{i=1}^{16} m_i$, $s = \frac{1}{16} \sum_{i=1}^{16} s_i$, and $a = \frac{1}{16} \sum_{i=1}^{16} a_i$ for both the analysis period and each 900-second interval.

As noted before, in some cases a few of the 16 runs crashed before finishing. They were excluded from the calculation of $m$, $s$, and $a$ for the analysis period. When calculating $m$, $s$, and $a$ for each 900-second interval, data up to the nearest 900-second interval before the run crashed was retained and used for averaging. For example, if a run crashed at 4896 seconds, data up to the 4500 second mark was retained. For the results reported in Section 3, if for a case the number of runs that completed successfully is smaller than 10, that case is marked and the number of successful runs is indicated.
### Table 3: Statistical characteristics of traffic delay results.

| q_p (veh/hr) | p_l | r = 10% | r = 40% | r = 70% |
|--------------|-----|---------|---------|---------|
|              |     | avg.    | std.    | max.    | avg.    | std.    | max.    | avg.    | std.    | max.    |
| **baseline** |     | 29.2    | 33.8    | 153.4   | 29.2    | 33.8    | 153.4   | 29.2    | 33.8    | 153.4   |
| 0.999        |     | 29.4    | 33.7    | 159.9   | 21.6    | 26.6    | 131.0   | 14.0    | 18.1    | 91.4    |
| 0.9          |     | 23.8    | 29.8    | 133.2   | 13.0    | 21.4    | 110.6   | 1.4     | 18.6    | 84.8    |
| 0.97         |     | 30.1    | 34.9    | 155.4   | 13.0    | 18.5    | 138.2   | 9.9     | 23.0    | 117.9   |
| 0.95         |     | 23.8    | 31.0    | 143.4   | 21.2    | 25.0    | 129.2   | 15.8    | 30.3    | 138.4   |
| 0.9          |     | 33.3    | 37.4    | 165.4   | 20.9    | 24.2    | 117.6   | 23.4    | 31.2    | 114.7   |
| 0.85         |     | 29.2    | 35.4    | 162.9   | 20.1    | 24.1    | 117.6   | 23.3    | 25.8    | 132.5   |
| 0.8          |     | 29.3    | 35.7    | 164.0   | 22.7    | 28.2    | 129.1   | 15.9    | 21.9    | 109.9   |
| 0.75         |     | 25.2    | 30.5    | 141.9   | 27.0    | 33.1    | 142.8   | 6.9     | 21.6    | 115.2   |

### Table 4: Average delay for q_p = 4600 veh/hr and r = 40% (Case A).

| Time       | interval (s) | baseline | q | 0.999 | 0.9 | 0.85 | 0.8 | 0.75 |
|------------|--------------|----------|----|-------|----|------|----|-----|
|            |              | 89.2     | 87.0 | 355.1 | 89.2 | 87.0 | 355.1 | 89.2 | 87.0 |
| 4400       | 0.999        | 78.0     | 80.7 | 305.3 | 85.4 | 86.0 | 330.8 | 95.2 | 107.1 |
|            | 0.9          | 90.5     | 90.5 | 343.7 | 83.7 | 85.8 | 327.6 | 85.4 | 86.5 |
|            | 0.97         | 78.1     | 81.5 | 324.3 | 81.3 | 81.8 | 342.8 | 71.8 | 74.6 |
|            | 0.95         | 81.4     | 82.0 | 316.5 | 76.2 | 75.8 | 296.9 | 69.0 | 72.2 |
|            | 0.9          | 75.0     | 78.9 | 307.9 | 77.7 | 76.7 | 319.1 | 81.5 | 83.9 |
|            | 0.85         | 80.0     | 82.5 | 312.1 | 71.3 | 74.8 | 286.8 | 84.8 | 86.3 |
|            | 0.8          | 82.0     | 86.1 | 333.4 | 73.3 | 75.2 | 295.6 | 72.3 | 77.0 |
|            | 0.75         | 80.8     | 84.6 | 330.6 | 64.2 | 68.9 | 288.9 | 80.4 | 86.3 |
| 4600       | 0.999        | 161.8    | 147.2 | 621.9 | 161.8 | 147.2 | 621.9 | 161.8 | 147.2 |
|            | 0.9          | 167.8    | 157.1 | 649.4 | 170.6 | 154.6 | 653.7 | 659.6 | 656.1 |
|            | 0.97         | 165.1    | 150.3 | 640.9 | 162.2 | 146.0 | 632.6 | 172.9 | 155.8 |
|            | 0.95         | 155.2    | 142.8 | 624.7 | 166.9 | 150.3 | 627.6 | 174.3 | 157.1 |
|            | 0.9          | 166.9    | 149.4 | 644.5 | 163.8 | 148.9 | 656.6 | 150.6 | 137.1 |
|            | 0.85         | 161.3    | 146.6 | 621.8 | 136.9 | 120.7 | 553.7 | 167.0 | 152.1 |
|            | 0.8          | 158.4    | 144.6 | 623.9 | 162.8 | 148.0 | 630.4 | 157.7 | 146.7 |
|            | 0.75         | 150.6    | 137.7 | 590.8 | 165.3 | 149.4 | 638.7 | 158.7 | 146.1 |

### Table 5: Average delay for q_p = 4800 veh/hr and r = 70% (Case B).

| Time       | interval (s) | baseline | q | 0.999 | 0.9 | 0.85 | 0.8 | 0.75 |
|------------|--------------|----------|----|-------|----|------|----|-----|
| 4400       | 0.999        | 7200    | 8100 | 162.6 | 126.3 | 674.5 | 172.9 | 157.1 |
|            | 0.9          | 6300    | 7200 | 190.0 | 173.4 | 286.8 | 184.8 | 171.7 |
|            | 4500         | 5400    | 6300 | 229.1 | 227.9 | 288.9 | 225.4 | 225.1 |
|            | 3600         | 4500    | 5400 | 30.9  | 26.0  | 29.3  | 25.8  | 25.5  |
|            | 2700         | 3600    | 4500 | 140.5 | 138.5 | 217.1 | 145.0 | 140.3 |
|            | 1800         | 2700    | 3600 | 316.9 | 317.5 | 333.9 | 321.4 | 308.2 |
|            | 1200         | 1800    | 2700 | 347.5 | 362.6 | 378.1 | 368.7 | 356.8 |
|            | 800          | 1200    | 1800 | 366   | 53.8  | 65.3  | 63.4  | 56.8  |

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Figure 3: Time-space plots of density and speed of all vehicles and lane departure of smart cars for the fourth (leftmost) lane. The top half plots belong to Case A and the bottom half to Case B.
Figure 4: Time-space plots of density and speed of all vehicles and lane departure of smart cars for the third lane (from right). The top half plots belong to Case A and the bottom half to Case B.
Before moving forward, we need to define a parameter called lane departure density denoted by $d_l$. For a blocked lane, if $N$ vehicles depart that lane for the final time in a specific road span $D$ and time span $T$ during the simulation, $d_l$ for that $T-D$ time-space block is defined as

$$d_l = \frac{N}{DT},$$

with (lane departure)/(ft.s) as its unit. In other words, $d_l$ quantifies the time-space rate of vehicles leaving a blocked lane for the last time during the simulation. For simplicity, in this paper we assume that $D = 100$ ft and $T = 100$ s.

Detailed average delay results for Case A and Case B are shown in Table 4 and Table 5, respectively. In both tables, the results for the baseline case are shown in the second column. Starting from the third column, each one presents the results for a $p_l$ value, starting from $p_l = 0.999$ in descending order. Each row represents a 900-second simulation time interval during the analysis period. As before, numbers in parenthesis show change in average delay relative to the baseline case of the respective time interval.

To analyze the performance of the proposed system, time-space plots of density (in veh/mi) and speed (in mph) for all vehicles and log$(K_{\text{veh}}^d + 1)$ for smart cars are shown in Figure 3 for lane 4 and in Figure 4 for lane 3. Time-space plots of density and speed for lane 2 are shown in Figure 5. In each figure, the top half plots belong to Case A and
the bottom half to Case B. For each case, plot rows represent lane departure, density, and speed, while plot columns represent the baseline case and cases with $p_l = 0.99$, 0.9, and 0.75, respectively. For better referencing, plot rows of each figure are assigned a letter from A to F from top to bottom (A to D in the case of Figure 5) and plot columns are assigned a number from 1 to 4 from left to right. So for example, Figure 4C3 refers to the plot on the third row and third column of Figure 4, showing time-space variation of speed for a simulation of Case A where $p_l = 0.9$.

The time axis of each plot spans the entire analysis period (1800 to 9000 seconds) and the distance axis spans the entire distance of the respective lane, i.e. 6277 ft for the fourth lane, 10575 ft for the third lane, and 12208 ft for the second lane. Furthermore, $\log(K + 1)$ was plotted instead of $d_l$ to a) make the parameter independent of $q_i$ (in units of veh/s) and $r$, b) simplify presentation by using a constant scaling factor of $K = 10000$, and c) better display differences between cases by using logarithmic values. Finally, note that while speed and density plots are made using data from all vehicles, lane departure plots are made using data from only smart cars. This is because turning the proposed advance warning system off (as in the baseline case) or on (as in other cases) only affects the behavior of smart cars and the behavior of other cars should not be statistically different from that shown for the baseline case when the system is turned off.

Lane departure plots of both Cases A and B in Figure 3 show a narrow horizontal band at around 3600 ft where lane departure from the fourth lane peaks. That area corresponds to the road segment just after the lane-end sign for the first lane drop and shows that a large portion of smart cars (and other cars) leave that lane after seeing the sign, conditioned on traffic density and speed of the adjacent lane. Similar bands can be seen in lane departure plots of Figure 4, this time at around 3600 ft and 9000 ft. Same as before, the latter corresponds to the road segment just after the lane-end sign for the second lane drop. As for the former, it indicates cars that move from the third lane to the second lane to make room for vehicles that are coming from the fourth lane.

To understand the effect of $p_l$ on the change in average delay, we first take a look at Case A. As previously mentioned, when $p_l$ is higher the system warns drivers much earlier than it would when $p_l$ is lower. This can be seen when comparing Figure 4A2 to Figure 4A4. When $p_l$ is 0.99, lane departures happen much earlier than when it is 0.9 or 0.75, as indicated by a smaller red area near the 9000 ft band. This initially helps reduce average delay during the peak period by pushing some lane departures away from the concentration at the 9000 ft band, as evidenced from Table 4 where average delay for the 3600 - 4500-second time interval is reduced by 30% compared to the baseline when $p_l$ is 0.99. However, because peak traffic flow is near the capacity of a two-lane highway, the congestion at the second lane drop eventually grows and this pushes the point where the system warns drivers further and further back, until it coincides with the 3600 ft band where vehicles are already moving from the third lane to the second lane because of those moving from the fourth lane to the third lane. This increased volume of vehicles moving to the second lane causes congestion and reduces speed in that lane, as evidenced by Figure 5A2 and Figure 5B2, increasing the delay. In comparison, $p_l = 0.9$ and 0.75 do a better job of distributing lane departures over the segment of the third lane between the 3600 ft and 9000 ft bands, reducing average delay by more in the end even though they may have lagged initially. In comparison to the baseline case, all three cases are successful in reducing average delay because they significantly delay or slow the growth of congestion at the second lane drop.

A similar story plays out for Case B. Same as before, when $p_l$ is 0.99, lane departures happen much earlier than when it is 0.9 or 0.75, as evidenced by the smaller red area near the 9000 ft band of Figure 4D2. By pushing some lane departures away from the concentration at the 9000 ft band, the system helps reduce average delay during the 3600 - 4500-second time interval by around 16%, much larger than the 4% reduction for $p_l = 0.9$ or the 7% reduction for $p_l = 0.75$. As in Case A, the congestion eventually grows and pushes the point of warning further and further back until it reaches the 3600 ft band. Denser traffic and higher ratio of smart cars compared to Case A, combined with the volume of vehicles already departing the third lane to make room for those departing the fourth lane, causes a big surge in the density of the second lane and slows traffic down. This can be seen in Figure 5C2 and Figure 5C3, where the boundary of the red area is much steeper in the former than it is in the latter. Between $p_l$ values of 0.75 and 0.9, for the former the warning to change lanes comes too late and too close to the 9000 ft band, while the latter does the best overall job of distributing lane departures to balance the increase in density in the second lane. Compared to the baseline case, $p_l = 0.99$ increases average delay by 5% while the other two decrease it, though not by much when $p_l$ is 0.75.

So how should we select $p_l$ for a different case? As discussed above, the answer depends on traffic flow and system penetration rate, but given that larger values of $p_l$ tend to push some lane departures away from concentration zones after lane-end signs and distribute them more evenly, a rule of thumb for general cases (involving one lane drop)
would be to use larger values in the range of 0.9 to 0.99. For other, more complex cases like the one discussed here, the answer may require additional traffic simulation. Another possible solution may be to dynamically assign $p_i$ based on various traffic flow characteristics, though this strategy needs further research.

Compared to other methods, the proposed system has two main advantages. The first is that it can be implemented in a simple, cost-effective way. A real-world implementation would only need traffic information, distance to the lane drop, and vehicle velocity to calculate the probability. The first one can be obtained from real-time or existing traffic data, possibly stored as a database where the system can search based on vehicle location and time of day. The second one can be calculated based on the position of the vehicle, and the last one can be obtained directly from the vehicle. This means the proposed system can be directly integrated with the in-vehicle navigation system. The second advantage is that it can be used together with other delay reduction methods such as VSLs, as it only affects the lane change behavior of vehicles and not their longitudinal behavior, though more research is needed in this area.

4. Conclusions and Outlook

In this work we proposed an onboard advance warning system based on a probabilistic prediction model that advised vehicles on when to change lanes for an upcoming lane drop. The prediction model estimated the probability of reaching a goal state on the road using one or multiple lane changes. This estimate was based on several traffic-related parameters such as the distribution of inter-vehicle headway distances, as well as driver-related parameters like lane change duration. For an upcoming lane drop, the proposed system would use the prediction model to continuously calculate the probability of departing the blocked lane before reaching the lane-end and advise the driver to change lanes when that probability dipped below a certain threshold. We used the proposed system in a case study on a segment of the I-81 interstate highway with two lane drops - transitioning from four lanes to two lanes - to advise a portion of the vehicles on when to change lanes. The results showed that the proposed system was successful at reducing average delay, but the reduction depended on the probability threshold, traffic flow, and the ratio of vehicles using the proposed system. We concluded that larger probability thresholds are favored for a general case with one lane drop, while traffic simulations are needed to determine the proper probability threshold for more complex cases. We also noted that the proposed system could be simply implemented through the in-vehicle navigation systems and could be combined with other methods (such as VSL strategies) for further efficacy.

Building upon the results of this study, future work will focus on studying the impact of this system on driving behavior using full-cabin driving simulators. Future research will also examine dynamic assignment of the probability threshold, as well as possible integration of the proposed system with other delay reduction methods for increased performance.

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