A trajectorial approach to entropy dissipation for degenerate parabolic equations

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October 31, 2022

Abstract

We consider degenerate diffusion equations of the form \( \partial_t p_t = \Delta f(p_t) \) on a bounded domain and subject to no-flux boundary conditions, for a class of nonlinearities \( f \) that includes the porous medium equation. We derive for them a trajectorial analogue of the entropy dissipation identity, which describes the rate of entropy dissipation along every path of the diffusion. In line with the recent work [13], our approach is based on applying stochastic calculus to the underlying probabilistic representations, which in our context are stochastic differential equations with normal reflection on the boundary. This trajectorial approach also leads to a new derivation of the Wasserstein gradient flow property for nonlinear diffusions, as well as to a simple proof of the HWI inequality in the present context.

Keywords and Phrases: degenerate diffusion, porous medium equation, entropy dissipation, gradient flow, HWI inequality

AMS 2000 Subject Classifications: 60H30; 76S05

1 Introduction

In this paper, we are interested in a class of quasilinear degenerate parabolic equations with initial and no-flux boundary conditions of the following form:

\[
\begin{cases}
\partial_t p(t, x) = \Delta \left( f(p(t, x)) \right), & \text{for } (t, x) \in (0, T) \times U, \\
p(0, x) = p_0(x), & \text{for } x \in \overline{U}, \\
\frac{\partial p(t, x)}{\partial n(x)} = 0, & \text{for } (t, x) \in (0, T) \times \partial U,
\end{cases}
\]

(1.1)

for a fixed \( T \in (0, \infty) \), an open connected bounded domain \( U \subset \mathbb{R}^d \), and a given initial probability density function \( p_0 \) on \( \overline{U} \). Here, \( n(x) \) is the outward normal to the boundary \( \partial U \) at \( x \in \partial U \), and \( f : [0, \infty) \to \mathbb{R} \) is a function representing the nonlinearity. In particular, when \( f(u) = u^m \) for some \( m > 1 \), the partial differential equation of (1.1) becomes the porous medium equation.

Under suitable assumptions on \( f \), it is well known from [6] that the solution of (1.1) converges to a unique stationary distribution, i.e., a probability density function \( p_\infty \) satisfying \( \Delta \left( f(p_\infty(x)) \right) = 0 \), and

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that this convergence can be quantified by the rate of entropy dissipation. More precisely, let us define \( h : (0, \infty) \rightarrow \mathbb{R} \) and \( \Phi : [0, \infty) \rightarrow \mathbb{R} \) by

\[
h(u) := \int_1^u \frac{f'(s)}{s} \, ds, \quad \Phi(u) := \int_0^u h(s) \, ds.
\]

(1.2)

Define also the entropy functional

\[
\mathcal{F}(p) := \int_U \Phi(p(x)) \, dx,
\]

for any probability density function \( p \) on \( U \) such that the integral is finite. Then it can be shown that the stationary distribution is the minimizer of \( \mathcal{F} \). Also, by abbreviating \( p_t := p(t, \cdot) \), it is well known (see, e.g. [6, Equation (4)]) that

\[
\mathcal{F}(p_t) - \mathcal{F}(p_{t_0}) = -\int_{t_0}^t I(p_u) \, du
\]

(1.4)

holds for every \( 0 \leq t_0 \leq t \leq T \), where \( I \) is the entropy dissipation functional, defined by

\[
I(p) := \int_U \left| \Phi''(p(x)) \nabla p(x) \right|^2 p(x) \, dx,
\]

(1.5)

for any differentiable probability density function \( p \) such that the integral is finite. This identity measures the rate of entropy dissipation along the flow of the time-marginal distributions \( (p_t)_{t \in [0,T]} \), hence is known as the entropy dissipation identity. In particular, the entropy functional \( t \mapsto \mathcal{F}(p_t) \) is decreasing in time. See also [23, Lemma 18.14] and [7, Equation 3.4] for a specific form of this identity for the porous medium equation with drift.

The identity (1.4) describes the rate of entropy dissipation at the ensemble level of the diffusion modeled by (1.1), since it is formulated in terms of the probability distributions \( (p_t) \) of the diffusion. The main goal of this paper is to formulate a trajectorial analogue to this identity, which describes the rate of entropy dissipation at the level of the individual diffusive particle. To illustrate this, we begin with the following stochastic differential equation (SDE) with normal reflection at the boundary:

\[
X_t = X_0 + \int_0^t \sqrt{2f(p(s,X_s))} \frac{p(s,X_s)}{p(s,X_s)} \, dW_s - \int_0^t n(X_s) \, dL_s \in U, \quad X_0 \sim p_0.
\]

(1.6)

Here, \( W \) is a \( d \)-dimensional standard Brownian motion and \( L \) is a nondecreasing continuous process satisfying

\[
L_t = \int_0^t 1_{\{X_s \in \partial U\}} \, dL_s, \quad L_0 = 0.
\]

(1.7)

The stochastic process \( (X_t)_{0 \leq t \leq T} \) is probabilistic representation of (1.1), in the sense that its time-marginal probability density function is given by the solution \( p(t, \cdot) \) of (1.1). Intuitively, the diffusion \( X_t \) is reflected on the boundary \( \partial U \) in the direction \(-n(X_t)\). The reflecting term \( L \) is associated with a multidimensional analogue of the local time on \( \partial U \) [20]. With this probabilistic representation, the entropy at time \( t \) can then be expressed as an expectation

\[
\mathcal{F}(p_t) = \int_U v(t, x) \, p(t, x) \, dx = \mathbb{E}[v(t, X_t)], \quad \text{where} \quad v(t, x) := \frac{\Phi(p(t,x))}{p(t,x)}.
\]

(1.8)

\footnote{In the case of the porous medium equation, \( v \) is known as the pressure function.}
Using stochastic calculus, we shall derive the dynamics of the \textit{entropy process} \((v(t, X_t))_{t \in [0, T]}\), in terms of the \textit{semimartingale decomposition}

\[ v(t, X_t) - v(0, X_0) = M_t + F_t, \quad \text{for} \quad 0 \leq t \leq T, \tag{1.9} \]

where \(M\) is a martingale and \(F\) is a process of finite variation. This decomposition describes the evolution of the entropy process along every trajectory of the particle, thus it can be seen as a trajectorial analogue of (1.4). In fact, (1.4) can be recovered from (1.9) by averaging over these trajectories; in other words, by taking expectation.

Our work is much inspired from the recent work \([13]\), which provides a trajectorial approach to the relative entropy dissipation for Fokker-Planck equations. Subsequently, this approach has been extended to Markov chains \([12]\) and to McKean-Vlasov equations \([22]\). It is therefore natural to expect an adaptation of the approach for the porous media type equation (1.1). Compared with prior work, a key difficulty to Markov chains \([12]\) and to McKean-Vlasov equations \([22]\) is that our main trajectorial result (Theorem 2.5 below) is stated in the forward direction of time.

Along with our trajectorial approach come two applications. The first application is a new derivation of the Wasserstein gradient flow property of (1.1), which states that the curve of time-marginal probability density functions \(p_t\) descends in the steepest possible direction of the entropy functional \(\mathcal{F}\) in \(\mathcal{P}(\mathcal{U})\), the space of probability measures on \(U\). Here, \(\mathcal{P}(\mathcal{U})\) is equipped with the quadratic Wasserstein distance \(W_2\), defined by

\[ W_2(\mu, \nu) := \sqrt{\inf_{\pi} \int_{U \times U} |x - y|^2 \pi(dx, dy)}, \quad \text{for any} \quad \mu, \nu \in \mathcal{P}(\mathcal{U}), \tag{1.10} \]

where the infimum is taken over \(\pi \in \mathcal{P}(\mathcal{U} \times \mathcal{U})\) with marginals \(\mu\) and \(\nu\).

For the porous medium equation on \(\mathbb{R}^d\), this property was discovered by Otto in his seminal paper \([17]\), where he introduced a formal Riemannian structure on \(\mathcal{P}(\mathbb{R}^d)\). More recently, Ambrosio, Gigli and Savaré \([2]\) developed a rigorous theory of gradient flows on general metric spaces based on the notion of curves of maximal slopes. Similar results have been established for porous medium equations on discrete spaces \([9]\) and with fractional pressure \([16]\).

To show the gradient flow property, we adopt the methodology in \([13]\) of \textit{perturbing} the SDE (1.6) from some time \(t_0 \in [0, T)\) by adding a gradient drift \(\nabla \beta\):

\[ X^\beta_t = X^\beta_{t_0} - \int_{t_0}^t \nabla \beta(X^\beta_s) \, ds + \int_{t_0}^t \sqrt{\frac{2f(p^\beta(s, X^\beta_s))}{p^\beta(s, X^\beta_s)}} \, dW^\beta_s - \int_{t_0}^t n(X^\beta_s) \, dL^\beta_s. \]

Here, \(p^\beta(t, \cdot)\) is the time-marginal probability density function for the solution \((X^\beta_t)_{t_0 \leq t \leq T}\) of this perturbed SDE. By deriving the dynamics of the associated perturbed entropy process, we obtain an analogous entropy dissipation identity for the perturbed diffusion. On the other hand, we can also explicitly compute the rates of changes of the Wasserstein distances along both the perturbed curve \((p^\beta_t)\) and the unperturbed curve \((p_t)\). Thus, the entropy dissipation rates can be measured not in terms of time elapsed,
but in terms of the Wasserstein distances traveled by the curve of time-marginal probability density functions, both in the perturbed and unperturbed settings. Comparing them allows us to establish the maximal rate of entropy dissipation for the unperturbed diffusion (1.1), by measuring the exact effect of each perturbation.

The second application of the trajectorial approach is a simple proof of the HWI inequality in the context of the nonlinear equation (1.1), which is a special case of [1, Theorem 4.2]. It is an interpolation inequality relating the entropy functional \( \mathcal{F} \), the Wasserstein distance \( W \) and the entropy dissipation functional \( I \). More precisely, it states that

\[
\mathcal{F}(\rho_0) - \mathcal{F}(\rho_1) \leq \sqrt{I(\rho_0)} W_2(\rho_0, \rho_1) \tag{1.11}
\]

holds for any \( \rho_0, \rho_1 \in \mathcal{P}(U) \). We will prove this inequality by applying a trajectorial approach similar to the one just described, but instead to the displacement interpolation between \( \rho_0 \) and \( \rho_1 \).

The rest of the paper is organized as follows. In Section 2.1, we introduce our setup and state some preliminary lemmas. Our main result is stated in Section 2.2. Section 2.3 formulates the gradient flow property, while Section 2.4 develops the HWI inequality. Proofs are provided in Section 3.

2 Setting and main results

2.1 Setup

We impose the following assumption on the initial distribution \( p_0 \) and the nonlinearity \( f \) of the degenerate parabolic equation (1.1).

**Assumption 2.1.**

(a) The domain \( U \) is an open connected bounded subset of \( \mathbb{R}^d \) for some \( d \geq 2 \), and the boundary \( \partial U \) is smooth.

(b) The initial datum \( p_0 \) is a smooth probability density function on \( U \). Moreover, it is non-degenerate, i.e., there exists \( \kappa > 1 \) for which \( \kappa^{-1} \leq p_0(x) \leq \kappa \) holds for all \( x \in U \). Also,

\[
\frac{\partial p_0(x)}{\partial n(x)} = 0, \quad \text{for all } x \in \partial U. \tag{2.1}
\]

where \( n(x) \) is the outward normal to the boundary \( \partial U \).

(c) The function \( f : [0, \infty) \to \mathbb{R} \) is smooth and strictly increasing. Moreover, \( f(0) = f'(0) = 0 \) and \( f'(u) > 0 \) for all \( u > 0 \). Also, \( f' \) is non-decreasing.

(d) The function \( h \), defined in (1.2), belongs to \( L^1_{\text{loc}}([0, \infty)) \).

**Remark 2.2.** Assumption (2.1) (c) – (d) covers the porous medium equations, in which case \( f(u) = u^m \) for some \( m > 1 \). The assumption \( f'(u) > 0 \) implies that (1.1) is a parabolic partial differential equation (PDE), while the assumption \( f'(0) = 0 \) implies that (1.1) is degenerate parabolic.

We first collect some basic properties of the solution to the PDE (1.1) in the following lemma. These properties are classical, and we refer to Chapter 3 of the monograph [23] and the references therein for a comprehensive overview.

\footnote{The letter “H” comes from the choice \( f(u) = u \) in (1.2), in which case the entropy functional defined in (1.3) satisfies \( \mathcal{F}(p) = H(p) - 1 \), where \( H(p) = \int_U p(x) \log p(x) \, dx \) is the (negative of the) differential entropy.}
Lemma 2.3. Under Assumption 2.1 there exists a smooth solution \( p \in C^\infty([0, T] \times \mathbb{U}) \) of (1.1). Moreover, \( \int_U p(t, x) \, dx = 1 \) for all \( t \in [0, T] \) and \( \kappa^{-1} \leq p(t, x) \leq \kappa \) for all \( (t, x) \in [0, T] \times \mathbb{U} \).

For the rest of the paper, we fix \( p \) as the solution given in Lemma 2.3. Fix also a filtered probability space \((\Omega, \mathbb{F}, \mathcal{F} = (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbb{P})\) supporting a \(\mathcal{F}\)-Brownian motion \( W \) and a \(\mathcal{F}_0\)-measurable random vector \( \xi : \Omega \to \mathbb{R}^d \) with \( \mathbb{P} \circ \xi^{-1} = \mathbb{P}_0 \). We shall denote by \( \mathbb{E} \) the expectation taken with respect to \( \mathbb{P} \). We will make use of the probabilistic representation of the porous medium equation (1.1), which is described in Lemma 2.4 below in terms of the solution of the following SDE with normal reflection on the boundary:

\[
\begin{aligned}
X_t &= X_0 + \int_0^t \sqrt{\frac{2f(p(s, X_s))}{p(s, X_s)}} \, dW_s - \int_0^t n(X_s) \, dL_s \in \mathbb{U}, \quad t \in [0, T], \\
X_0 &= \xi,
\end{aligned}
\]

(2.2)

For an introduction to SDEs with reflection and their connections with nonlinear parabolic PDEs, we refer to the lecture notes [18] and [21].

The following lemma shows that (2.2) is well-posed and provides the probabilistic representation of (1.1). Similar results are known for the porous medium equations [4] as well as for general nonlinear equations of the form (1.1) with discontinuous coefficients [3, 5], or with the half-line as the domain [8]. See also [10] for a martingale method for establishing gradient estimates for the porous medium and fast diffusion equations.

Lemma 2.4. Suppose Assumption 2.1 holds. Then the SDE with reflection (2.2) has a pathwise unique, strong solution \((X, L)\), for which the probability density functions of \((X_t)_{t \in [0, T]}\) are given by \( (p(t, \cdot))_{t \in [0, T]} \).

2.2 Trajectorial entropy dissipation of degenerate parabolic equation

Our first main result is the dynamics of entropy dissipation along every trajectory of the diffusion (2.2), formulated in terms of the semimartingale decomposition of the entropy process \((v(t, X_t))_{t \in [0, T]}\).

To begin, we define the entropy dissipation function

\[
D(t, x) := \left( \varphi'(p) \Delta f(p) + \frac{f(p)}{p} \Delta v \right)(t, x), \quad (t, x) \in [0, T] \times \mathbb{U}, \quad \text{where} \quad \varphi(u) := \frac{\Phi(u)}{u},
\]

(2.3)

and \( \Phi \) is defined in (1.2). This function will help provide the exact trajectorial rate of entropy dissipation, as will be explained in Remark 2.7 below.

Theorem 2.5. Suppose Assumption 2.1 holds. Then the entropy process \((v(t, X_t))_{t \in [0, T]}\) admits the semimartingale decomposition

\[
v(t, X_t) - v(0, X_0) = M_t + F_t, \quad \text{for} \quad t \in [0, T],
\]

(2.4)

where

\[
F_t := \int_0^t D(s, X_s) \, ds, \quad \text{and} \quad M_t := \int_0^t \left( \sqrt{\frac{2f(p(s, X_s))}{p(s, X_s)}} \nabla v(s, X_s), \, dW_s \right)
\]

(2.5)

is an \( L^2 \)-bounded martingale. Also, we have

\[
\mathbb{E}[F_t] = -\int_0^t I(p_t) \, dt > -\infty, \quad \text{for} \quad t \in [0, T].
\]

(2.6)
The proof of Theorem 2.5 will be given in Section 3.3. This result is the analogue of [13, Theorem 4.1] and [22, Theorem 3.1]. In contrast to them, Theorem 2.5 here is stated in the forward direction of time.

By aggregating this trajectorial result, i.e., taking expectation, we recover the entropy dissipation identity (1.4) and its differential version (2.7). Furthermore, by taking conditional expectation, we obtain below the conditional trajectorial rate of entropy dissipation (2.8).

**Corollary 2.6.** Suppose Assumption 2.1 holds. Then for every $0 \leq t_0 \leq t \leq T$, the entropy dissipation identity (1.4) holds. The corresponding differential version

$$\frac{d}{dt} \bigg|_{t=t_0} \mathcal{F}(p_t) = -I(p_{t_0})$$

also holds for all $t_0 \in [0, T]$. Moreover, for all $t_0 \in [0, T]$, the conditional trajectorial rate of entropy dissipation is given by

$$\lim_{t \downarrow t_0} \mathbb{E}[v(t, X_t) \big| \mathcal{F}_{t_0}] - v(t_0, X_{t_0}) = D(t_0, X_{t_0}),$$

where the limit exists in $L^1(\mathbb{P})$.

**Remark 2.7.** From (2.6), we see that $\mathbb{E}[D(t_0, X_{t_0})] = -I(p_{t_0})$ holds. This explains why (2.8) constitutes a conditional trajectorial version of the entropy dissipation identity (2.7).

### 2.3 Gradient flow property of the degenerate parabolic equation, via perturbation analysis

In this subsection, we discuss how our trajectorial approach leads to a new interpretation of the Wasserstein gradient flow property of the degenerate parabolic equation (1.1). Following the method of [13, 22], we shall *perturb* the degenerate parabolic equation.

To this effect, let $\beta$ be a perturbation potential satisfying the following assumption.

**Assumption 2.8.** The perturbation potential $\beta : \Omega \to \mathbb{R}$ is smooth. Moreover, $\nabla \beta(x) = 0$ for $x \in \partial \Omega$.

For the rest of the paper, we fix a $t_0 \in [0, T)$ and a perturbation $\beta$ satisfying Assumption 2.8. Consider the following Neumann problem, which can be viewed as a perturbed version of (1.1):

\[
\begin{equation}
\begin{aligned}
\partial_t p^\beta(t, x) &= \text{div} \left( \nabla f(p^\beta(t, x)) + p^\beta(t, x) \nabla \beta(x) \right), & \text{for } (t, x) \in (t_0, T] \times \Omega, \\
p^\beta(t_0, x) &= p(t_0, x), & \text{for } x \in \Omega, \\
\frac{\partial p^\beta(t, x)}{\partial n(x)} &= 0, & \text{for } x \in \partial \Omega.
\end{aligned}
\end{equation}
\]

(2.9)

The following result, which is the “perturbed analogue” of Lemma 2.3, shows the existence of a strictly positive smooth solution to (2.9) in a short time interval.

**Lemma 2.9.** Under Assumptions 2.7 and 2.8 there exists $T_\beta \in (t_0, T]$ such that (2.9) has a smooth solution in $p^\beta \in C^\infty([t_0, T_\beta] \times \Omega)$. Moreover, $\int_\Omega p^\beta(t, x) \, dx = 1$ for all $t \in [t_0, T_\beta]$ and $\frac{1}{\kappa t} \leq p^\beta(t, x) \leq \frac{1}{2\kappa}$ for all $(t, x) \in [t_0, T_\beta] \times \Omega$. 

For the rest of the paper, we fix $p^\beta$ as given by Lemma 2.9. The corresponding probabilistic representation of the perturbed PDE (2.9) is the following SDE with reflection:

$$
\begin{align*}
X_t^\beta &= X_{t_0}^\beta - \int_{t_0}^t \nabla \beta (X_s^\beta) \, ds + \int_{t_0}^t \sqrt{\frac{2f(p^\beta(s, X^\beta_s))}{p^\beta(s, X^\beta_s)}} \, dW_s - \int_{t_0}^t n(X_s^\beta) \, dL_s^\beta \in \overline{U}, \quad t \in [t_0, T_{\beta}], \\
X_{t_0}^\beta &= X_0, \\
L_{t_0}^\beta &= 0.
\end{align*}
$$

By analogy with Lemma 2.4, the following result ensures that (2.10) is well-posed and is the stochastic counterpart of (2.9).

**Lemma 2.10.** Under Assumptions 2.1 and 2.8, the SDE with reflection (2.10) has a pathwise unique, strong solution $(X_t^\beta, L_t^\beta)_{t \in [t_0, T_{\beta}]}$, for which the probability density functions of $(X_t^\beta)_{t \in [t_0, T_{\beta}]}$ are given by $(p^\beta(t, \cdot))_{t \in [t_0, T_{\beta}]}$.

As before, let us abbreviate $p_t^\beta := p^\beta(t, \cdot)$. In parallel to (1.8), we can express the entropy for the perturbed diffusion at time $t$ as

$$
\mathcal{F}(p_t^\beta) = \int_U v^\beta(t, x) p^\beta(t, x) \, dx = \mathbb{E} \left[ v^\beta(t, X_t^\beta) \right], \quad \text{where} \quad v^\beta(t, x) := \frac{\Phi(p^\beta(t, x))}{p^\beta(t, x)}.
$$

Therefore, we similarly call $(v^\beta(t, X_t^\beta))_{t \in [t_0, T_{\beta}]}$ the **perturbed entropy process**. The following result, which is the perturbed counterpart of Theorem 2.5, derives the dynamics of this process. By analogy with (2.3), we introduce the perturbed entropy dissipation function

$$
D^\beta(t, x) := \left( \varphi' (p^\beta) \right) \text{div} \left( \nabla f(p^\beta) + p^\beta \nabla \beta(x) \right) + \frac{f(p^\beta)}{p^\beta} \Delta v^\beta - \langle \nabla v^\beta, \nabla \beta \rangle (t, x),
$$

for $(t, x) \in [t_0, T_{\beta}] \times \overline{U}$.

**Theorem 2.11.** Suppose Assumptions 2.1 and 2.8 hold. Then the perturbed entropy process $(v^\beta(t, X_t^\beta))_{t \in [t_0, T_{\beta}]}$ admits the semimartingale decomposition

$$
v^\beta(t, X_t^\beta) - v^\beta(t_0, X_{t_0}^\beta) = M_t^\beta + F_t^\beta, \quad \text{for} \quad t \in [t_0, T_{\beta}],
$$

where

$$
F_t^\beta := \int_{t_0}^t D^\beta(s, X_s^\beta) \, ds, \quad \text{and} \quad M_t^\beta := \int_{t_0}^t \left( \sqrt{\frac{2f(p^\beta(s, X_s^\beta))}{p^\beta(s, X_s^\beta)}} \nabla v^\beta(s, X_s^\beta), \, dW_s \right)
$$

is an $L^2$-bounded martingale. Also, we have for $t \in [t_0, T_{\beta}]$,

$$
\mathbb{E} \left[ F_t^\beta \right] = -\int_{t_0}^t I(p_s^\beta) \, ds - \int_{t_0}^t \mathbb{E} \left[ \langle \nabla h(p^\beta(s, X_s^\beta)), \nabla \beta(X_s^\beta) \rangle \right] \, ds > -\infty.
$$

Once again, by averaging this trajectorial result, we obtain the following perturbed entropy dissipation identity and its conditional trajectorial version, to which a comment similar to Remark 2.7 applies.
Corollary 2.12. Suppose Assumptions 2.7 and 2.8 hold. For every $t \in [t_0, T_\beta]$, the following perturbed entropy dissipation identity holds:

$$\mathcal{F}(p_t^\beta) - \mathcal{F}(p_{t_0}^\beta) = - \int_{t_0}^{t} I(p_s^\beta) \, ds - \int_{t_0}^{t} \mathbb{E} \left[ \langle \nabla h(p_{t_0}^\beta(s), X_s^\beta) \rangle, \nabla \beta(X_s^\beta) \rangle \right] \, ds.$$  \hfill (2.16)

The corresponding differential version also holds:

$$\frac{d}{dt} \bigg|_{t=t_0^+} \mathcal{F}(p_t^\beta) = -I(p_{t_0}) - \mathbb{E} \left[ \langle \nabla h(p(t_0, X_{t_0})), \nabla \beta(X_{t_0}) \rangle \right].$$  \hfill (2.17)

Moreover, the conditional trajectorial rate of entropy dissipation for the perturbed diffusion is given by

$$\lim_{t \downarrow t_0} \frac{\mathbb{E} \left[ v^\beta(t, X_t^\beta) \, | \, F_{t_0} \right] - v^\beta(t_0, X_{t_0})}{t - t_0} = D^\beta(t_0, X_{t_0}),$$  \hfill (2.18)

where the limit exists in $L^1$.

The last ingredients we need are the rates of change of the Wasserstein distances along the curve of the marginal distributions $(p_t)$ and $(p_t^\beta)$. The following result is a consequence of the general theory of Wasserstein metric derivatives for absolutely continuous curves [2, Chapter 8], but our setting allows for a more direct proof, which we provide in Section 3.5. We recall the definitions of $h$ in (1.2) and the perturbation potential $\beta$ described at the beginning of Section 2.3.

Lemma 2.13. Suppose Assumptions 2.7 and 2.8 hold. Then the Wasserstein metric slope along the unperturbed curve $(p_t)$ is given by

$$\lim_{t \downarrow t_0} \frac{W_2(p_t, p_{t_0})}{t - t_0} = \left\| \nabla h(p(t_0, X_{t_0})) \right\|_{L^2}.$$  \hfill (2.19)

Similarly, the Wasserstein metric slope along the perturbed curve $(p_t^\beta)$ is given by

$$\lim_{t \downarrow t_0} \frac{W_2(p_t^\beta, p_{t_0}^\beta)}{t - t_0} = \left\| \nabla h(p(t_0, X_{t_0})) + \nabla \beta(X_{t_0}) \right\|_{L^2}.$$  \hfill (2.20)

Combining the entropy dissipation identities (2.7) and (2.17) with the Wasserstein derivatives (2.19)–(2.20) allows us to derive the gradient flow property. We define the Wasserstein metric slopes of the entropy functional $\mathcal{F}$ along the unperturbed curve $(p_t)$ and along the perturbed curve $(p_t^\beta)$, respectively, by

$$|\partial \mathcal{F}|_{W_2(p_{t_0})} := \lim_{t \downarrow t_0} \frac{\mathcal{F}(p_t) - \mathcal{F}(p_{t_0})}{W_2(p_t, p_{t_0})} \quad \text{and} \quad |\partial \mathcal{F}|_{W_2(p_{t_0}^\beta)} := \lim_{t \downarrow t_0} \frac{\mathcal{F}(p_t^\beta) - \mathcal{F}(p_{t_0}^\beta)}{W_2(p_t^\beta, p_{t_0}^\beta)}.$$  \hfill (2.21)

The following theorem computes both of these slopes explicitly, which shows in particular that the unperturbed slope $|\partial \mathcal{F}|_{W_2(p_{t_0})}$ is always steeper than the perturbed slope $|\partial \mathcal{F}|_{W_2(p_{t_0}^\beta)}$.

Theorem 2.14. Suppose Assumptions 2.7 and 2.8 hold. Then the Wasserstein metric slope of the entropy functional along the unperturbed curve $(p_t)$ is given by

$$|\partial \mathcal{F}|_{W_2(p_{t_0})} = -\left\| \nabla h(p(t_0, X_{t_0})) \right\|_{L^2} = -\sqrt{I(p_{t_0})}.$$  \hfill (2.22)
Similarly, if \( \|\nabla h(p(t_0, X_{t_0}))) + \nabla \beta(X_{t_0})\|_{L^2} > 0 \), then the Wasserstein metric slope along the perturbed curve \((p_t^\beta)\) is given by
\[
|\partial \mathcal{F}|_{W_2(p_0^\beta)} = -\left\langle \nabla h(p(t_0, X_{t_0})), \frac{\nabla h(p(t_0, X_{t_0}))) + \nabla \beta(X_{t_0})}{\|\nabla h(p(t_0, X_{t_0}))) + \nabla \beta(X_{t_0})\|_{L^2}} \right\rangle_{L^2}.
\] (2.23)

In particular,
\[
|\partial \mathcal{F}|_{W_2(p_0)} \leq |\partial \mathcal{F}|_{W_2(p_0^\beta)},
\] (2.24)
and equality holds if and only if \( \nabla h(p(t_0, X_{t_0}))) + \nabla \beta(X_{t_0}) \) is a.s. a scalar multiple of \( \nabla h(p(t_0, X_{t_0})) \).

### 2.4 HWI Inequality

In this subsection, we apply a similar trajectorial approach to derive the HWI inequality (1.11). Let us fix two probability density functions \( \rho_0, \rho_1 \in \mathcal{P}(\Omega) \) and impose the following assumption.

**Assumption 2.15.**

(a) Both \( \rho_0, \rho_1 \) are strictly positive and smooth. Also, \( \rho_0(x) = \rho_1(x) \) for all \( x \in \partial U \).

(b) The function \( f : [0, \infty) \to \mathbb{R} \) is smooth. Moreover, the function \( h, \) defined in (1.2), belongs to \( L_{loc}^1([0, \infty)) \).

(c) The function \( r \mapsto r^d \Phi(r^{-d}) \) is convex nonincreasing on \((0, \infty)\), where \( \Phi \) is defined in (1.2).

**Remark 2.16.** Assumption (b), (c) is satisfied by the porous medium equation, see [24] Examples 5.19.

By Brenier’s theorem [24, Theorem 2.12(ii)], there exists a convex function \( \psi : \Omega \to \mathbb{R} \) such that \( \nabla \psi \) is the optimal transport map from \( \rho_0 \) to \( \rho_1 \), i.e.,
\[
W_2^2(\rho_0, \rho_1) = \int_U |x - \nabla \psi(x)|^2 \rho_0(\text{d}x).
\] (2.25)

Let \((\rho_t)_{t \in (0,1)}\) denote the displacement interpolation between \( \rho_0 \) and \( \rho_1 \), i.e.,
\[
\rho_t = \rho_0 \circ ((1-t) \text{Id} + t \nabla \psi)^{-1}, \quad \text{for } t \in (0,1).
\]

It is known that each \( \rho_t \) has a probability density function [24, Remarks 5.13(ii)]. For the following result, we recall the entropy functional \( \mathcal{F} \) defined in (1.3), the nonlinearity \( f \) satisfying Assumption 2.15(b), and the convex function \( \psi \) described just above.

**Proposition 2.17.** Suppose Assumption 2.1(a) and Assumption 2.15(a–b) hold. Then the rate of change of \( t \mapsto \mathcal{F}(\rho_t) \) at \( t = 0 \) is given by
\[
\frac{d}{dt} \big|_{t=0} \mathcal{F}(\rho_t) = \int_U \langle \nabla f(\rho_0(z)), \nabla \psi(z) - z \rangle \text{d}z.
\] (2.26)

Using this proposition and the displacement convexity of the entropy functional \( \mathcal{F} \), we obtain the HWI inequality (1.11).

**Theorem 2.18.** Suppose Assumption 2.1(a) and Assumption 2.15 hold. Then
\[
\mathcal{F}(\rho_0) - \mathcal{F}(\rho_1) \leq -\int_U \langle \nabla f(\rho_0(z)), \nabla \psi(z) - z \rangle \text{d}z \leq \sqrt{I(\rho_0)} W_2(\rho_0, \rho_1),
\] (2.27)
where \( I \) is the entropy dissipation functional defined in (1.5).
3 Proofs

3.1 Proofs of Lemmas 2.3 and 2.9

Proof of Lemma 2.3 We adopt the same method as in the proof of [23, Theorem 3.1], which exploits the nondegeneracy of the initial condition of Assumption 2.1(b) in a crucial manner. Let \( \tilde{f} : [0, \infty) \to \mathbb{R} \) be a smooth function satisfying \( \tilde{f}(u) = f(u) \) for \( \kappa^{-1} \leq u \leq \kappa \), \( \tilde{f}'(u) > \epsilon^{-1} \) for \( 0 \leq u \leq \kappa^{-1} \) and \( \tilde{f}'(u) < \epsilon \) for \( u \geq \kappa \), where \( \epsilon > 1 \) is some fixed constant. Consider the PDE

\[
\partial_t p(t, x) = \Delta \left( \tilde{f}(p(t, x)) \right), \quad \text{for} \ (t, x) \in (0, T] \times U \tag{3.1}
\]

subject to the same initial and Neumann boundary conditions as in (1.1). By Assumption 2.1(c), \( \tilde{f}' \) is increasing, so \( \tilde{f}'(u) > \epsilon^{-1} \) for all \( u \geq 0 \). This implies that (3.1) is uniformly parabolic. We can therefore apply standard quasilinear theory [23, Chapter 3.1] to obtain a smooth solution \( p \in C^\infty([0, T] \times U) \) to (3.1). By the comparison principle, \( \kappa^{-1} \leq p \leq \kappa \), so \( p \) also satisfies (1.1).

Finally, the mass conservation law [23, Chapter 3.3.3] implies that the total mass \( \int_U p(t, x) \, dx = 1 \) is conserved over time \( t \in [0, T] \).

Proof of Lemma 2.9 The proof is similar to that of Lemma 2.3. Let \( \bar{f} : [0, \infty) \to \mathbb{R} \) be a smooth function satisfying \( \bar{f}(u) = f(u) \) for \( \frac{1}{2\kappa} \leq u \leq \kappa + \frac{1}{2\kappa} \), \( \bar{f}'(u) > \epsilon^{-1} \) for \( 0 \leq u \leq \frac{1}{2\kappa} \) and \( \bar{f}'(u) < \epsilon \) for \( u \geq \kappa + \frac{1}{2\kappa} \), where \( \epsilon > 1 \) is some fixed constant. Consider the PDE

\[
\partial_t p^\beta(t, x) = \text{div} \left( \nabla \bar{f}(p^\beta(t, x)) + p^\beta(t, x) \nabla \beta(x) \right), \quad \text{for} \ (t, x) \in (t_0, T] \times U \tag{3.2}
\]

subject to the same initial and Neumann boundary conditions as in (2.9). Again since \( \tilde{f}'(u) > \epsilon^{-1} \) for all \( u \geq 0 \), this PDE is uniformly parabolic. Therefore, standard quasilinear theory implies that there exists a smooth solution \( p^\beta \in C^\infty([t_0, T] \times \overline{U}) \) to (3.2).

For a fixed \( \delta > 0 \), we define

\[
\lambda_\delta := \max_{t \in [t_0, t_0 + \delta], \ x \in \overline{U}} |\partial_t p^\beta(t, x)| < \infty \quad \text{and} \quad \tau := \min \left( \delta, \frac{1}{2\kappa \lambda_\delta} \right) > 0.
\]

Let \( T_\beta := T \wedge (t_0 + \tau) \). Since \( p^\beta_{t_0} = p_{t_0} \) by construction in (2.9),

\[
|p^\beta(t, x) - p(t_0, x)| = |p^\beta(t, x) - p^\beta(t_0, x)| \leq \int_{t_0}^{T_\beta} |\partial_t p^\beta(s, x)| \, ds \leq \tau \lambda_\delta \leq \frac{1}{2\kappa},
\]

for every \((t, x) \in [t_0, T_\beta] \times \overline{U}\). From Lemma 2.3, we have \( \kappa^{-1} \leq p_{t_0} \leq \kappa \), thus \( (2\kappa)^{-1} \leq p^\beta_t \leq \kappa + (2\kappa)^{-1} \) for all \( t \in [t_0, T_\beta] \). This implies that \( \bar{f}(p^\beta(t, x)) = f(p^\beta(t, x)) \) holds for every \((t, x) \in [t_0, T_\beta] \times \overline{U}\) and therefore \( p^\beta \) also solves (2.9).

Finally, for mass conservation, note that integration by parts gives us

\[
\frac{d}{dt} \int_U p^\beta(t, x) \, dx = \int_U \text{div} \left( \nabla \bar{f}(p^\beta(t, x)) + p^\beta(t, x) \nabla \beta(x) \right) \, dx
\]

\[
= \int_{\partial U} \left( \nabla \bar{f}(p^\beta(t, x)) + p^\beta(t, x) \nabla \beta(x) \right) \cdot \nu(x) \, dx = 0,
\]

where the last step follows from the no-flux boundary condition in (2.9) as well as Assumption 2.8.
3.2 Proofs of Lemmas 2.4 and 2.10

We will only prove Lemma 2.10 as the proof of Lemma 2.4 is completely analogous.

For every \((\tau, y) \in [t_0, T_{\beta}] \times \overline{U}\), consider the SDE with reflection (2.10) conditional on the initial position \(y\) at time \(\tau\):

\[
X_t^{\beta, \tau, y} = y - \int_{t_0}^t \nabla \beta(X_s^{\beta, \tau, y}) \, ds + \int_{t_0}^t \sqrt{2f\left(p^\beta(s, X_s^{\beta, \tau, y})\right)} \, dW_s - \int_{\tau}^t n(X_s^{\beta, \tau, y}) \, dL_s^{\beta, \tau, y} \in U, \quad t \in [\tau, T_{\beta}],
\]

\[
L_t^{\beta, \tau, y} = \int_{\tau}^t 1_{\{X_s^{\beta, \tau, y} \in \partial U\}} \, dL_s^{\beta, \tau, y},
\]

\[
L^{\beta, \tau, y}_\tau = 0 \text{ and } t \mapsto L^{\beta, \tau, y}_t \text{ is nondecreasing and continuous.}
\]

(3.3)

From Lemma 2.9 and Assumption 2.1(c), it is straightforward to check that diffusion coefficient is uniformly Lipschitz in the spatial variable. Thus by [15, Theorem 3.1 and Remark 3.3], the SDE with reflection (3.3) has a pathwise unique, strong solution. Let \(\xi\) be an independent \(\overline{U}\)-valued random variable. Consider the process \(X^{\beta}\) given by \(X^{\beta}_{t_0} = \xi\) and \(X^{\beta}_t = X^{\beta, t_0, \xi}_t\) for \(t \in (t_0, T_{\beta}]\).

Similarly, let \(L^{\beta}\) be specified by \(L^{\beta}_{t_0} = 0\) and \(L^{\beta}_t = L^{\beta, t_0, \xi}_t\) for \(t \in (t_0, T_{\beta})\). Then \((X^{\beta}, L^{\beta})\) is the unique strong solution to (2.10). This completes the proof of the first part of the lemma.

Turning to the proof of the second part, we borrow ideas from [18, Remark 3.1.2] and [14, Chapter 5.7.B]. Recall that \(p^\beta\) is fixed as the solution given in Lemma 2.9. Consider the following backward Kolmogorov equation:

\[
\begin{align*}
\partial_t q^\beta(\tau, y) + &\frac{f(p^\beta(\tau, y))}{p^\beta(\tau, y)} \Delta_y q^\beta(\tau, y) - \left\langle \nabla_y \beta(y), \nabla_y q^\beta(\tau, y) \right\rangle = 0, & \text{for } (\tau, y) \in (t_0, T_{\beta}) \times U, \\
\frac{\partial q^\beta(\tau, y)}{\partial n(y)} &= 0, & \text{for } y \in \partial U.
\end{align*}
\]

(3.4)

It follows from [11] that (3.4) has a fundamental solution \(G^{\beta}(\tau, y; t, x)\) defined for \(t_0 \leq \tau < t \leq T_{\beta}\) and \(x, y \in \overline{U}\). In particular, \(G^{\beta}\) is nonnegative and for every \(\phi \in C(\overline{U})\) and \(t \in (\tau, T_{\beta})\), the function

\[
q^\beta(\tau, y) := \int_U G^{\beta}(\tau, y; t, x) \phi(x) \, dx,
\]

(3.5)

satisfies (3.4) and the terminal condition

\[
\lim_{\gamma \uparrow t} q^\beta(\tau, y) = \phi(y), \quad \text{for all } y \in U.
\]

(3.6)

If furthermore \(\phi\) satisfies the no-flux boundary condition \(\partial\phi/\partial n = 0\), then the above convergence holds uniformly in \(U\).

From the Feynman-Kac representation [18, Theorem 3.1.1], for any \(\phi \in C(\overline{U})\),

\[
q^\beta(\tau, y) = \mathbb{E}\left[\phi(X^{\beta, \tau, y}_t)\right].
\]

(3.7)

Comparing (3.5) with (3.7), we deduce that the transition probability density of \(X^{\beta, \tau, y}\) is given by \(G^{\beta}\), i.e.,

\[
P(X^{\beta, \tau, y}_t \in A) = \int_A G^{\beta}(\tau, y; t, x) \, dx, \quad \text{for every Borel } A \subseteq \overline{U}.
\]

(3.8)
For any fixed \( y \in \overline{U} \), the function \( \Psi^\beta(t, x) := G^\beta(t_0, y; t, x) \) satisfies the forward Kolmogorov equation
\[
\partial_t \Psi^\beta(t, x) = \text{div}_x \left( \nabla_x \left( \frac{f(p^\beta(t, x))}{p^\beta(t, x)} \Psi^\beta(t, x) \right) + \Psi^\beta(t, x) \nabla_x \beta(t, x) \right),
\]
which is the adjoint of (3.4). The probability density function of \( X^\beta_t \) with initial distribution \( p_{t_0} \) is then
\[
\tilde{p}^\beta(t, x) := \int_U G^\beta(t_0, y; t, x) p_{t_0}(y) \, dy.
\]
Together with (2.1) of Assumption 2.1(c), we see that \( p^\beta \) satisfies the linear uniformly parabolic PDE
\[
\begin{cases}
\partial_t p^\beta(t, x) = \text{div} \left( \nabla f \left( \frac{f(p^\beta(t, x))}{p^\beta(t, x)} \right) \tilde{p}^\beta(t, x) + p^\beta(t, x) \nabla \beta(t, x) \right), & \text{for } (t, x) \in (t_0, T_\beta] \times U, \\
p^\beta(t_0, x) = p(t_0, x), & \text{for } x \in U, \\
\frac{\partial p^\beta(t, x)}{\partial n(x)} = 0, & \text{for } x \in \partial U.
\end{cases}
\]
Since the solution to this PDE is unique, we deduce from Lemma 2.9 that \( \tilde{p}^\beta = p^\beta \).

3.3 Proof of Theorems 2.5 and 2.11

We will only prove Theorem 2.11 as similar arguments can be used to show Theorem 2.5. We first prove (2.13). Recall the function \( \psi \) defined in (2.3) and note for later use the simple identities
\[
\varphi(u) = h(u) - \frac{f(u)}{u}, \quad h(u) = \varphi'(u)u + \varphi(u), \quad \Phi''(u) = \varphi''(u)u + 2\varphi'(u) \quad \text{for all } u > 0.
\]
By writing \( v^\beta(t, x) = \varphi(p^\beta(t, x)) \), we deduce from (2.9) that
\[
\partial_t v^\beta(t, x) = \varphi'(p^\beta(t, x)) \partial_t p^\beta(t, x) = \varphi'(p^\beta(t, x)) \text{div} \left( \nabla f \left( \frac{f(p^\beta(t, x))}{p^\beta(t, x)} \right) + p^\beta(t, x) \nabla \beta(t, x) \right).
\]
Using Itô's lemma along with (2.10) and (3.12), we see that the dynamics of the perturbed entropy process satisfies
\[
\begin{align*}
dv^\beta(t, X^\beta_t) &= \left( \varphi'(p^\beta) \text{div} \left( \nabla f \left( \frac{f(p^\beta)}{p^\beta} \right) + p^\beta \nabla \beta \right) + \frac{f(p^\beta)}{p^\beta} \Delta v^\beta - \langle \nabla v^\beta, \nabla \beta \rangle \right)(t, X^\beta_t) \, dt \\
&\quad + \left( \int \sqrt{\frac{2f(p^\beta)}{p^\beta} \nabla v^\beta}(t, X^\beta_t), \, \mathcal{W}_t \right) - \langle \nabla v^\beta(t, X^\beta_t), n(X^\beta_t) \rangle \, dL^\beta_t \\
&= D^\beta(t, X^\beta_t) \, dt + M^\beta_t - \varphi'(p^\beta(t, X_t)) \langle \nabla p^\beta(t, X^\beta_t), n(X^\beta_t) \rangle \, dL^\beta_t.
\end{align*}
\]
Note that the last line in (2.2) ensures that the reflecting term \( L^\beta \) only increases when \( X^\beta \) is on the boundary. In conjunction with the no-flux boundary condition in (2.9), we see that the last term above is zero. This completes the proof of (2.13).

Next, to see that the local martingale \( M^\beta \) in (2.14) is in fact a true \( L^2 \)-martingale, note that the quadratic variation of \( M^\beta \) is given by
\[
\mathbb{E} \left[ \langle M^\beta, M^\beta \rangle_{T_\beta} \right] = \mathbb{E} \int_{t_0}^{T_\beta} \left( \frac{2f(p^\beta)}{p^\beta} \left| \nabla v^\beta \right|^2 \right)(t, X^\beta_t) \, dt = \mathbb{E} \int_{t_0}^{T_\beta} \left( \frac{2f(p^\beta)^2}{(p^\beta)^5} \left| \nabla p^\beta \right|^2 \right)(t, X^\beta_t) \, dt.
\]
We claim that the above quantity is finite. Indeed, due to the properties of the solution \( p^\beta \) in Lemma 2.9, the above expectation is bounded by

\[
2(T_\beta - t_0)(2\kappa)^5 f \left( \frac{3}{2\kappa} \right)^3 \max_{[t_0, T_\beta] \times \mathbb{R}} |\nabla p^\beta(t, x)|^2 < \infty.
\]

Therefore, it follows from [19, Corollary IV.1.25] that \( (M^\beta_t) \) is an \( L^2 \)-martingale.

Finally, in order to show (2.15), we take expectation in the first equation in (2.14) and use Fubini’s theorem to get

\[
\mathbb{E}[F^\beta_t] = \int_{t_0}^t \mathbb{E}[\mathcal{D}^\beta(s, X^\beta_s)] \, ds = \sum_{i=1}^3 \int_{t_0}^t \mathbb{E}[\mathcal{D}^\beta_i(s, X^\beta_s)] \, ds
\]

where

\[
\mathcal{D}^\beta_i := \varphi'(p^\beta) \nabla f(p^\beta) + p^\beta \nabla \beta, \quad \mathcal{D}^\beta_2 := \frac{f(p^\beta)}{p^\beta} \Delta v^\beta, \quad \text{and} \quad \mathcal{D}^\beta_3 := -(\nabla v^\beta, \nabla \beta).
\]

We now evaluate each of the expectations in (3.13). Integrating by parts, we have

\[
\mathbb{E}[\mathcal{D}^\beta_i(t, X^\beta_t)] = \int_U \varphi'(p^\beta)p^\beta \nabla f(p^\beta + p^\beta \nabla \beta)(t, x) \, dx
\]

\[
= -\int_U \langle \nabla (\varphi'(p^\beta)p^\beta), \nabla f(p^\beta) + p^\beta \nabla \beta \rangle(t, x) \, dx + C,
\]

where \( C \) is the boundary term given by

\[
C := \int_{\partial U} \varphi'(p^\beta)f'(p^\beta)p^\beta \langle \nabla p^\beta + p^\beta \nabla \beta, n \rangle(t, x) \, dx.
\]

From the no-flux boundary condition in (2.9) and Assumption 2.8, we see that \( C = 0 \). Similarly, we see that \( \mathbb{E}[\mathcal{D}^\beta_2(t, X^\beta_t)] \) and \( \mathbb{E}[\mathcal{D}^\beta_3(t, X^\beta_t)] \) are respectively equal to

\[
-\int_U \langle \nabla f(p^\beta), \nabla v^\beta \rangle(t, x) \, dx, \quad \text{and} \quad -\int_U \langle \nabla v^\beta, \nabla \beta \rangle(t, x) \, dx.
\]

Assembling them gives

\[
\mathbb{E}[\mathcal{D}^\beta(t, X^\beta_t)] = -\int_U \langle \nabla (\varphi'(p^\beta)p^\beta + v^\beta), \nabla f(p^\beta) \rangle(t, x) \, dx
\]

\[
-\int_U \langle \nabla (\varphi'(p^\beta)p^\beta) + \nabla v^\beta, p^\beta \nabla \beta \rangle(t, x) \, dx.
\]

Using the third identity in (3.11), we see that the first integrand above is equal to

\[
(\varphi''(p^\beta)p^\beta + 2\varphi'(p^\beta)) f'(p^\beta)|\nabla p^\beta|^2 = \Phi''(p^\beta)f'(p^\beta)|\nabla p^\beta|^2,
\]

so the first integral in (3.16) is

\[
-\int_U \langle |\Phi''(p^\beta)|\nabla p^\beta|^2 p^\beta \rangle(t, x) \, dx = -I(p^\beta_t) > -\infty,
\]

where the last step follows from the boundedness of \( p^\beta \) and \( \nabla p^\beta \) implied by Lemma 2.9. Similarly, using the second identity in (3.11), we see that the second integral in (3.16) is equal to

\[
-\mathbb{E} \left[ \langle \nabla (h(p^\beta)), \nabla \beta \rangle(t, X^\beta_t) \right] > -\infty.
\]

Putting them together completes the proof of (2.15).
3.4 Proofs of Corollaries 2.6 and 2.12

We will only prove Corollary 2.12, as the proof of Corollary 2.6 proceeds in the same way.

Taking expectation in (2.13) and using the martingale property of $M^\beta$ in (2.14) as well as (2.15), we have

$$\mathcal{F}(p^\beta) - \mathcal{F}(p^\beta_{t_0}) = \mathbb{E} \left[ v^\beta(t, X_t) - v^\beta(t_0, X_{t_0}) \right]$$

$$= -\int_{t_0}^t I(p^\beta_s) \, ds - \int_{t_0}^t \mathbb{E} \left[ \langle \nabla h(p^\beta(s, X_s^\beta)), \nabla \beta(X_s^\beta) \rangle \right] \, ds,$$  

(3.17)

which proves (2.16).

Turning to the proof of (2.17), note that from (1.2) we have

$$\mathbb{E} \left[ \langle \nabla h(p^\beta(s, X_s^\beta)), \nabla \beta(X_s^\beta) \rangle \right] = \int_U f'(p^\beta(s, x)) \langle \nabla p^\beta(s, x), \nabla \beta(x) \rangle \, dx.$$  

(3.18)

From the continuity of $(s, x) \mapsto f'(p^\beta(s, x)) \langle \nabla p^\beta(s, x), \nabla \beta(x) \rangle$, we see that the expression of (3.18) is continuous as a function of $s$, thus

$$\frac{d}{dt} \bigg|_{t=t_0^+} \int_{t_0}^t \mathbb{E} \left[ \langle \nabla h(p^\beta(s, X_s^\beta)), \nabla \beta(X_s^\beta) \rangle \right] \, ds = \mathbb{E} \left[ \langle \nabla h(p^\beta(t_0, X_{t_0})), \nabla \beta(X_{t_0}) \rangle \right],$$

(3.19)

where the last equality is due to the fact that $X_{t_0}^\beta = X_{t_0}$ by construction in (2.9). Similarly, from the continuity of $t \mapsto I(p^\beta_t)$, we have

$$\frac{d}{dt} \bigg|_{t=t_0^+} \int_{t_0}^t I(p^\beta_u) \, du = I(p^\beta_t) = I(p^\beta_{t_0})$$

(3.20)

and the identity (2.17) follows.

Finally, to show (2.18), the martingale property of $M^\beta$ implies that the numerator on the left-hand side of (2.18) is equal to

$$\mathbb{E} \left[ F^\beta_t - F^\beta_{t_0} \big| \mathcal{F}_{t_0} \right] = \mathbb{E} \left[ \int_{t_0}^t D^\beta(u, X_u^\beta) \, du \big| \mathcal{F}_{t_0} \right].$$

From the continuity of $u \mapsto D^\beta(u, X_u)$, we have

$$\lim_{t \uparrow t_0} \frac{1}{t - t_0} \int_{t_0}^t D^\beta(u, X_u^\beta) \, du = D^\beta(t_0, X_{t_0}), \quad \text{a.s.}$$

Moreover, the properties of $p^\beta$ from Lemma 2.9 implies that $D^\beta$ is uniformly bounded on $[t_0, T_\beta] \times \overline{U}$. Therefore, by the bounded convergence theorem,

$$\lim_{t \uparrow t_0} \mathbb{E} \left[ \frac{1}{t - t_0} \int_{t_0}^t D^\beta(u, X_u^\beta) \, du \right] = \mathbb{E} \left[ D^\beta(t_0, X_{t_0}) \right].$$

We now apply Schefé’s lemma to obtain

$$\lim_{t \uparrow t_0} \left\| \frac{1}{t - t_0} \int_{t_0}^t D^\beta(u, X_u^\beta) \, du - D^\beta(t_0, X_{t_0}) \right\|_{L_1} = 0.$$  

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Therefore, by setting
\[
\left\| \mathbb{E} \left[ \frac{1}{t-t_0} \int_{t_0}^t D^\beta(u, X^\beta_u) \, du \bigg| \mathcal{F}_t \right] - D^\beta(t_0, X_{t_0}) \right\|_{L^1} \leq \left\| \frac{1}{t-t_0} \int_{t_0}^t D^\beta(u, X^\beta_u) \, du - D^\beta(t_0, X_{t_0}) \right\|_{L^1},
\]
and the $L^1$-convergence in (2.18) follows.

\[\square\]

### 3.5 Proof of Lemma 2.13

Since the proofs of (2.19) and (2.20) are very similar, we will only prove (2.20). We first rewrite the PDE in (2.9) as a continuity equation
\[
\partial_t p^\beta(t, x) + \text{div} \left( p^\beta(t, x) \tilde{v}^\beta(t, x) \right) = 0, \quad \text{for } (t, x) \in (t_0, T_\beta) \times U, \tag{3.21}
\]
where $\tilde{v}^\beta : [t_0, T_\beta] \times \overline{U} \to \mathbb{R}^d$ is the velocity field, defined by
\[
\tilde{v}^\beta(t, x) := -\nabla \left[ \beta + h(p^\beta) \right](t, x). \tag{3.22}
\]

We see from Assumption 2.8 and Lemma 2.9 that $\tilde{v}^\beta(t_0, \cdot)$ is the gradient of a smooth function. For each $x \in \overline{U}$, consider the curved flow $\Lambda^\beta_t$ associated with $\tilde{v}^\beta$, specified by
\[
\frac{d}{dt} \Lambda^\beta_t(x) = \tilde{v}^\beta \left( t, \Lambda^\beta_t(x) \right), \quad t \in [t_0, T_\beta], \quad \Lambda^\beta_{t_0}(x) = x, \tag{3.23}
\]
By the Cauchy-Lipschitz theorem, there exists a unique solution $t \mapsto \Lambda^\beta_t(x) \in \overline{U}$ to (3.23). Moreover, it follows from [24, Theorem 5.34] that $\Lambda^\beta_t$ pushes forward $p^\beta_{t_0}$ to $p^\beta_t$, in the sense that $p^\beta_{t_0} \circ (\Lambda^\beta_t)^{-1} = p^\beta_t$.

Note also that the Jacobian of $\Lambda^\beta_t$ is given by
\[
\nabla \Lambda^\beta_t(x) = I - \int_{t_0}^t \nabla^2 \left( (\beta + h(p^\beta))(s, \Lambda^\beta_s(x)) \right) \, ds.
\]
Therefore, by setting
\[
K := \max \left\{ \left| \partial_{ij} \left( \beta + h(p^\beta) \right)(t, x) \right| : t \in [t_0, T_\beta], x \in \overline{U}, i, j = 1, \ldots, n \right\},
\]
we see that for any $t \in [t_0, t_0 + K^{-1}]$, the Jacobian $\nabla \Lambda^\beta_t(x)$ is positive-semidefinite for all $x \in \overline{U}$, so $\Lambda^\beta_t$ is the gradient of a convex function. Hence, by Brenier’s theorem [24, Theorem 2.12(ii)], $\Lambda^\beta_t$ is the optimal transport map from $p^\beta_{t_0}$ to $p^\beta_t$, i.e.,
\[
\mathcal{W}^2_2(p^\beta_{t_0}, p^\beta_t) = \mathbb{E} \left[ \left| \Lambda^\beta_t(X^\beta_{t_0}) - X_{t_0} \right|^2 \right] = \mathbb{E} \left[ \left| \int_{t_0}^t \tilde{v}^\beta(s, \Lambda^\beta_s(X^\beta_{t_0})) \, ds \right|^2 \right].
\]
Now, the continuity of $t \mapsto \tilde{v}^\beta(t, \Lambda^\beta_t(x))$ implies
\[
\left| \frac{1}{t-t_0} \int_{t_0}^t \tilde{v}^\beta(s, \Lambda^\beta_s(X^\beta_{t_0})) \, ds \right|^2 \xrightarrow{t \uparrow t_0} \left| \tilde{v}^\beta(t_0, X_{t_0}) \right|^2, \quad \text{a.s.} \tag{3.24}
\]

Moreover, by Jensen’s inequality, the random variable on the left-hand side above is bounded by
\[
\frac{1}{t - t_0} \int_{t_0}^t |\bar{v}^\beta(s, \Lambda_s^\beta(X_{t_0}^\beta))|^2 ds \leq \max \{ |\bar{v}^\beta(t, x)|^2 : t \in [t_0, T_\beta], x \in \bar{U} \} < \infty,
\]
where the last step follows from the aforementioned fact that \( \bar{v}^\beta(t, \cdot) \) is the gradient of a smooth function.

Consequently, by the bounded convergence theorem,
\[
\lim_{t \uparrow t_0} W_2(p_t^\beta, p_{t_0}^\beta) = \lim_{t \uparrow t_0} \frac{1}{t - t_0} \int_{t_0}^t \bar{v}^\beta(s, \Lambda_s^\beta(X_{t_0}^\beta)) ds = \|\bar{v}^\beta(t_0, X_{t_0})\|_{L^2},
\]
where the last step follows from the fact that \( X_{t_0}^\beta = X_{t_0}^\beta \) by construction in (2.9).

3.6 Proof of Theorem 2.14

The identity (2.22) follows from (2.7) of Corollary 2.6 and (2.19) of Lemma 2.13. Similarly, for (2.23), we deduce from (2.17) of Corollary 2.12 and (2.20) of Lemma 2.13 that
\[
|\partial \mathcal{F}|_{W^2}(p_t^\beta) = -\frac{I(p_{t_0}) + \mathbb{E}\left[ \langle \nabla h(p(t_0, X_{t_0})), \nabla \beta(X_{t_0}) \rangle \right]}{\|\nabla h(p(t_0, X_{t_0})) + \nabla \beta(X_{t_0})\|_{L^2}}.
\]

Moreover, we see from (1.5) and (1.2) that the entropy dissipation functional can be expressed as
\[
I(p_{t_0}) = \mathbb{E}\left[ |\nabla h(p(t_0, X_{t_0}))|^2 \right].
\]

Putting this back into (3.26) yields (2.23). Finally, the inequality (2.24) follows from the Cauchy-Schwarz inequality.

3.7 Proof of Proposition 2.17

For \( t \in [0, 1) \), let us denote by
\[
T_t := (1 - t) \text{Id} + t \nabla \psi
\]
the optimal transport map from \( \rho_0 \) to \( \rho_t \). Note that \( T_t \) is injective by [24 Theorem 5.49], so its inverse exists. By [24 Theorem 5.51(ii)], the probability density functions \( \rho_t \) satisfy the continuity equation
\[
\partial_t \rho_t(x) + \text{div}(\rho_t(x) \nabla \psi_t(x)) = 0,
\]
where \( \nabla : [0, 1) \times \bar{U} \to \mathbb{R}^d \) is the velocity field defined by
\[
\nabla_t(x) := (\nabla \psi - \text{Id}) \circ (T_t)^{-1}(x).
\]

In conjunction with (3.27), we see that \( T_t \) satisfies the integral equation
\[
T_t(x) = x + \int_0^t \nabla s(T_s(x)) ds.
\]
We now switch to probabilistic notations. On a sufficiently rich probability space, let $Z_0$ be a random variable with distribution $\rho_0$ and let $Z_t := T_t(Z_0)$ for $t \in (0, 1)$. On account of (3.30), we have
\[
Z_t = Z_0 + \int_0^t \hat{v}_s(Z_s) \, ds.
\] (3.31)
Together with (3.28), we deduce
\[
d\rho_t(Z_t) = \partial_t \rho_t(Z_t) \, dt + \langle \nabla \rho_t(Z_t), dZ_t \rangle = -\rho_t(Z_t) \text{div}(\hat{v}_t(Z_t)) \, dt.
\]
Recalling the function $\varphi$ in (2.3), we have
\[
d\varphi(\rho_t(Z_t)) = -\varphi'(\rho_t(Z_t)) \rho_t(Z_t) \text{div}(\hat{v}_t(Z_t)) \, dt = -\frac{f(\rho_t(Z_t))}{\rho_t(Z_t)} \text{div}(\hat{v}_t(Z_t)) \, dt,
\]
where the last step follows from the identity $f(u) = uh(u) - \Phi(u)$, valid for all $u \geq 0$. Integrating from 0 to $t$ and taking expectation yield
\[
\mathcal{F}(\rho_t) - \mathcal{F}(\rho_0) = -\int_0^t \int_U f(\rho_s(z)) \text{div}(\hat{v}_s(z)) \, dz \, ds
\]
\[
= \int_0^t \int_U \langle \nabla f(\rho_s(z)), \hat{v}_s(z) \rangle \, dz \, ds - \int_0^t \int_{\partial U} f(\rho_s(z)) \langle \hat{v}_s(z), n(z) \rangle \, dz \, ds. \quad (3.32)
\]
It follows from Assumption 2.15(b) that $\nabla \psi(x) = x$ for all $x \in \partial U$. Therefore, we see from (3.29) that $\hat{v}_t(z) = 0$ for all $x \in \partial U$. Hence, the last integral in (3.32) vanishes. Letting $t \downarrow 0$ yields
\[
\frac{d}{dt} \bigg|_{t=0} \mathcal{F}(\rho_t) = \int_U \langle \nabla f(\rho_0(z)), \hat{v}_0(z) \rangle \, dz
\]
Recalling the definition of $\hat{v}_0$ in (3.29) completes the proof. \hfill \square

### 3.8 Proof of Theorem 2.18

From [24, Theorem 5.15(i)], Assumption 2.15(c) implies that the entropy functional $\mathcal{F}$ is displacement convex. In other words,
\[
\frac{d^2}{dt^2} \mathcal{F}(\rho_t) \geq 0, \quad \text{for} \quad t \in [0, 1].
\]
Therefore, Taylor’s theorem and Proposition 2.17 give us
\[
\mathcal{F}(\rho_1) = \mathcal{F}(\rho_0) + \frac{d}{dt} \bigg|_{t=0} \mathcal{F}(\rho_t) + \int_0^1 (1 - t) \frac{d^2}{dt^2} \mathcal{F}(\rho_t) \, dt
\]
\[
\geq \mathcal{F}(\rho_0) + \int_U \langle \nabla f(\rho_0(z)), \nabla \psi(z) - z \rangle \, dz,
\]
which proves the first inequality in (2.27). The second inequality is a simple consequence of the Cauchy-Schwarz inequality; the expression in the middle of (2.27) is bounded from above by
\[
\sqrt{\int_U \frac{|\nabla f(\rho_0(z))|^2}{\rho_0(z)} \, dz} \sqrt{\int_U |\nabla \psi(z) - z|^2 \rho_0(z) \, dz} = \sqrt{I(\rho_0)} \mathcal{W}_2(\rho_0, \rho_1).
\] \hfill \square
Acknowledgements

We are indebted to Ioannis Karatzas for suggesting this problem and for many helpful discussions. L.C. Yeung is partially supported by the National Science Foundation (NSF) under grant NSF-DMS-20-04997.

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