Unsteady rotational motion of a composite sphere in a viscous fluid using stress jump condition

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ABSTRACT
The unsteady rotational motion of a composite sphere, consisting of a solid core surrounded by a porous shell, in an incompressible viscous fluid is discussed using Brinkman’s formula. The composite sphere is assumed to rotate about the z-axis with time-dependent angular velocity. Laplace transform technique is utilized to obtain the solution of the problem. The inversion of Laplace transform is obtained analytically using contour integration. An analytical and numerical interpretation of the Torque exerted by the fluid on the porous surface is obtained. The effect of stress jump condition and permeability coefficient is discussed and some special cases are deduced.

1. Introduction
Rotational motion of a fluid is an essential part in fluid mechanics. The motion of a composite sphere consisting of a solid core and a porous shell is an interesting concept in many fields of engineering and sciences. According to the wide area of applications of viscous fluids with spherical boundaries, it has been discussed in the literature both analytically and numerically and has been reported in numerous studies. Keh and Chou [1] discussed an analytical study of creeping motions of a composite sphere in a concentric spherical cavity where the porous shell located at the centre of a spherical cavity filled with an incompressible Newtonian fluid. D. Srinivasacharya and M. Krishna Prasad [2] studied the steady rotation of a composite sphere in a concentric spherical cavity where both Brinkman’s model for the flow inside the composite sphere and the Stokes equation for the flow in the spherical container were used to study the motion. They also investigated the motion of a porous spherical shell in a bounded medium [3]. In the ref. [4] they discussed the slow steady rotation of a porous sphere located at the centre of a spherical container where the stress jump condition has been used as a boundary condition. Lately, D. Srinivasacharya and M. Krishna Prasad [5] introduced the rotation of a porous approximate sphere in an approximate spherical container. In the ref. [6] Ashmawy discussed the steady rotational motion of a porous spheroid in a viscous fluid. Also, he introduced the rotary oscillation of a composite sphere in a concentric spherical cavity using slip and stress jump conditions [7].

One of two models usually used to describe the fluid motion in a porous media is the Brinkman’s model. Many researchers are utilizing this model to discuss the behaviour of viscous fluid flows in porous regions e.g. [8–11]. Lately, the most important boundary condition that has widely used is the stress jump condition on a porous-fluid interface developed by Ochoa-Tapia and Whitaker [12,13]. It is mainly characterized by the stress jump coefficient that varies from −1 to 1. Stress jump condition has been discussed by many researchers [12–14]. The no-slip condition for viscous fluid assumes that at a solid boundary, the fluid will have zero velocity relative to the boundary. Recently, the classical no-slip condition has been widely used by many researchers to study the motion of viscous fluid [15].

In the present work, we consider the unsteady rotational motion of a composite sphere, consisting of a solid core and a porous shell in an incompressible viscous fluid. The clear fluid region is discussed by Stokes fluid flow where Brinkman equation is introduced in the flow in the porous region. Stress jump condition is applied in fluid-porous interface where the case of no-slip condition is assumed. An analytical formula of the Torque acting on the porous sphere has been discussed using Laplace transform contour integration. The obtained results are represented graphically.

2. Formulation of the problem
Let us consider the unsteady rotational motion of a composite sphere, consisting of a solid core of radius “a” surrounded by a concentric porous shell of radius “b” in
an incompressible viscous fluid. It is assumed that the composite sphere is rotating about z-axis with angular velocity $\Omega (t)$ (Figure 1).

To discuss such problem analytically, we use spherical polar coordinates denoted by $(r, \theta, \phi)$ with the origin at the centre of the solid sphere, region I ($r \geq b$) is occupied by Stokesian fluid flow and the porous region II ($b \geq r \geq a$) is governed by Brinkman’s equation.

In region I, the fluid flow is governed by the continuity equation and Navier-Stokes equation

$$\nabla \vec{q}_c = 0, \quad \mu \nabla^2 \vec{q}_c - \nabla p_c = \rho \frac{\partial \vec{q}_c}{\partial t}, \quad (1)$$

In region II, the fluid flow is controlled by the continuity equation and Brinkman’s equation

$$\nabla \vec{q}_p = 0, \quad \mu_e \nabla^2 \vec{q}_p - \frac{\mu}{k} \vec{q}_p - \nabla p_p = \rho \frac{\partial \vec{q}_p}{\partial t}, \quad (2)$$

where $p_c, p_p, \vec{q}_c$ and $\vec{q}_p$ are the pressure and velocity of the flow in region I and II respectively. Also, $\mu, \mu_e, k$, and $\rho$ are denoting the viscosity in Stokes flow, the effective viscosity in porous region, the permeability and the density of the fluid, respectively.

According to the nature of the flow and the geometry of the boundaries, the velocity vectors of the two regions I and II are taking the forms

$$\vec{q}_c = (0, 0, w_c(r, \theta, t)), \quad \vec{q}_p = (0, 0, w_p(r, \theta, t)), \quad (3)$$

where $w_c$ and $w_p$ are the velocity components in the clear fluid region and porous region respectively.

3. Solution of the problem

After applying Laplace transform and making some straightforward mathematical calculations, Equations (1) and (2) reduce to

$$(E^2 - l_p^2)(r \sin \theta \tilde{w}_c(r, \theta, s)) = 0, \quad (4)$$

$$(E^2 - l_c^2)(r \sin \theta \tilde{w}_p(r, \theta, s)) = 0, \quad (5)$$

where

$$E^2 = \left[ \frac{\partial^2}{\partial r^2} - \frac{\cot \theta}{r^2} \frac{\partial}{\partial \theta} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \right], \quad l_c^2 = \frac{\rho \gamma}{k}, \quad l_p^2 = \frac{\mu_e}{k} \left( \frac{1}{l_p^2} + \frac{1}{l_c^2} \right). \quad (6)$$

The solution of Equations (4) and (5) are given by the following

$$\tilde{w}_c(r, \theta, s) = \frac{1}{\sqrt{r}} \Omega_{1/2}(l_c r) + \Omega_{3/2}(l_c r) \sin \theta, \quad (7)$$

$$\tilde{w}_p(r, \theta, s) = \frac{1}{\sqrt{r}} \Omega_{1/2}(l_p r) + \Omega_{3/2}(l_p r) \sin \theta, \quad (8)$$

where $\Omega_n(.)$ and $\Omega_p(.)$ are denoting the modified Bessel functions of the first and second kind, respectively, of order $n = 3/2$.

To obtain a complete solution of the problem, the following boundary conditions are proposed

On the impermeable core $(r = a)$

$$w_p(a, \theta) = a \Omega(t) \sin \theta. \quad (9)$$

On the surface of the porous shell $(r = b)$ the following stress jump condition is assumed

$$\tau_{r \theta}^p - \tau_{r \phi}^e = \frac{\mu \gamma}{k} \bar{w}_p(b, \theta), \quad (10)$$

where $\tau_{r \theta}^p, \tau_{r \phi}^e$ are the tangential stresses of the porous and clear fluid region, respectively, and $\gamma$ is the stress jump coefficient. The boundary condition (10) can be simplified to

$$\frac{\partial \bar{w}_p(b, \theta)}{\partial r} = \frac{\partial \tilde{w}_c(b, \theta)}{\partial r} = \frac{\gamma}{\sqrt{k}} \bar{w}_p(b, \theta). \quad (11)$$

Also, on the surface $(r = b)$, the continuity of velocity components is proposed by

$$w_p(b, \theta) = w_c(b, \theta). \quad (12)$$

Applying Laplace transform into boundary conditions, we get

$$\bar{w}_p(a, \theta) = a \bar{\Omega}(s) \sin \theta, \quad (13)$$

$$\frac{\partial \bar{w}_p(b, \theta)}{\partial r} - \frac{\partial \tilde{w}_c(b, \theta)}{\partial r} = \frac{\gamma}{\sqrt{k}} \bar{w}_p(b, \theta), \quad (14)$$

$$\bar{w}_p(b, \theta) = \bar{w}_c(b, \theta). \quad (15)$$

Substituting by the proposed boundary conditions into Equations (7) and (8), we deduce the following
formulas

\[ D_{k/2}(lp) + E_{k/2}(lp) = a\sqrt{\alpha}\Theta(s), \quad (16) \]

\[ A[lkpK_{k/2}(lp) + 2K_{k/2}(lp)] - D [lpK_{k/2}(lp)] \]
\[ + \left( 2 + \frac{\gamma}{\sqrt{k}} \right) K_{k/2}(lp) + E [lpK_{k/2}(lp)] \]
\[ - \left( 2 + \frac{\gamma}{\sqrt{k}} \right) I_{k/2}(lp) = 0, \quad (17) \]

\[ AK_{k/2}(lp) - DK_{k/2}(lp) - E_{k/2}(lp) = 0. \quad (18) \]

Solving the algebraic system of Equations (16)–(18) we get \( A, D, \) and \( E \)

\[ A = \frac{a\sqrt{\alpha}\Theta(s)}{lpK_{k/2}(lp)F_{2} - K_{k/2}(lp)F_{1}}, \quad (19) \]

\[ D = \frac{a\sqrt{\alpha}\Theta(s) (lpK_{k/2}(lp)I_{k/2}(lp) + lpK_{k/2}(lp)K_{k/2}(lp))}{lpK_{k/2}(lp)F_{2} - K_{k/2}(lp)F_{1}}, \quad (20) \]

\[ E = \frac{a\sqrt{\alpha}\Theta(s) (-lpK_{k/2}(lp)K_{k/2}(lp))}{lpK_{k/2}(lp)F_{2} - K_{k/2}(lp)F_{1}}. \quad (21) \]

where

\[ F_{1} = \frac{b\gamma}{\sqrt{k}} F_{2} - lpF_{3}, \quad (22) \]

\[ F_{2} = K_{k/2}(lp)I_{k/2}(lp) - K_{k/2}(lp)I_{k/2}(lp), \quad (23) \]

\[ F_{3} = K_{k/2}(lp)I_{k/2}(lp) + K_{k/2}(lp)I_{k/2}(lp). \quad (24) \]

The torque exerted by the fluid on the porous spherical shell can be evaluated using the following formula

\[ T = 2\pi^{3} \int_{0}^{\infty} \tau_{\gamma}^{(c)}(b, \theta, \phi) \sin^{2}\theta d\theta, \quad (25) \]

where the shear stress of the clear Stokesian fluid flow region is given by

\[ \tau_{\gamma}^{(c)}(b, \theta, \phi) = \mu \left( \frac{\partial \tilde{w}_{c}(r, \theta)}{\partial r} - \frac{\tilde{w}_{c}(r, \theta)}{r} \right). \quad (26) \]

Substituting Equation (26) into (25), we get

\[ \bar{T} = -\frac{8\pi}{3} \mu a^{2} b^{2} \sqrt{k} \Theta(s) \left[ \frac{\tilde{f}_{1}(s)}{\tilde{f}_{2}(s) + \tilde{f}_{3}(s)} \right], \quad (27) \]

where,

\[ \tilde{f}_{1}(s) = lp \sqrt{lp} (l_{c}^{2} b^{2} + 3(1 + l_{c})b), \quad (28) \]

\[ \tilde{f}_{2}(s) = lp \cosh(lp(b - a)) \left[ l_{c}^{2} (ab\sqrt{k} + ab^{2}l_{c}\sqrt{k}) \right. \]
\[ + (b - a) \left( l_{c}^{2} b\sqrt{k} - \gamma (1 + l_{c} b) \right) \left], \quad (29) \]

\[ \tilde{f}_{3}(s) = \sinh(lp(b - a)) \left[ l_{c}^{2} (ab^{2}\sqrt{k} + l_{c} b^{2}h + bh) \right. \]
\[ + l_{c} b \left( \gamma - l_{c}\sqrt{k} \right) + \gamma \right]. \quad (30) \]

4. Inversion of Laplace transform

In order to get an exact formula to the torque experienced by the fluid flow on the surface of the porous shell, we apply the complex inversion formula together with contour integration to the formula (27) and make the use of the convolution theorem. Thus, we obtain

\[ \bar{T} = -\frac{8\pi}{3} \mu a^{2} b^{2} \sqrt{k} \int_{0}^{t} \Omega(t - \tau) G(\tau) d\tau, \quad (31) \]

we have

\[ G(\tau) = L^{-1} \tilde{f}(s), \tilde{f}(s) = \frac{\tilde{f}_{1}(s)}{\tilde{f}_{2}(s) + \tilde{f}_{3}(s)}, s = x + iy. \quad (32) \]

We consider the case of flow due to sudden motion of the composite sphere.

In this case, we assume that the composite sphere is set in a sudden motion with a velocity \( \Omega_{0}H(t) \), where \( \Omega_{0} \) is a constant and \( H(t) \) is the Heaviside unit step function, defined by

\[ H(t) = \begin{cases} 1 & \text{if } t > 0 \\ 0 & \text{otherwise} \end{cases}. \]

In order to get \( G(\tau) \) we have to use the complex inversion formula where we noticed that there are two branch points at \( s_{0} = 0 \) and \( s_{1} = -\mu / pk = -\theta^{2} \). The contour \( \Gamma \) consists of two large circular arcs (BCD,LMA) of radii \( R \), a small circular arc (GHI) of radius \( \omega \) around the branch point \( s_{0} \) and two small arcs (EF, JK) of small radius around the branch point \( s_{1} \). In addition to four lines segments (DE, FG, IJ and KL) and a straight line AB as shown in Figure 2.

\[ \int_{\Gamma} e^{i\theta} \tilde{f}(s) ds = 0, \quad (33) \]

\[ \lim_{R \to \infty} \int_{AB} e^{i\theta} \tilde{f}(s) ds = 2\pi i \epsilon(t), \quad (34) \]

\[ \lim_{R \to \infty} \left( \int_{BC} e^{i\theta} \tilde{f}(s) ds + \int_{LA} e^{i\theta} \tilde{f}(s) ds \right) = 0, \quad (35) \]

For the arcs (EF and JK) assume that \( s = \epsilon e^{i\theta}, ds = \epsilon ie^{i\theta} d\theta \)

\[ \lim_{\epsilon \to 0} \left( \int_{EF} e^{i\theta} \tilde{f}(s) ds + \int_{JK} e^{i\theta} \tilde{f}(s) ds \right) = 0, \quad (36) \]

For the small circular arc (GHI) assume that \( \omega e^{i\theta}, ds = \omega e^{i\theta} d\theta \)

\[ \lim_{\omega \to 0} \int_{GHI} e^{i\theta} \tilde{f}(s) ds = 0, \quad (37) \]

For the line segments DE and FG, suppose that \( s = re^{i\theta} \)

\[ \lim_{r \to \infty} \int_{DE} e^{i\theta} \tilde{f}(s) ds \]
\[ = -i \int_{0}^{2\pi} e^{i\theta} (m_{1} + im_{2}) l_{c}^{4} \frac{dx}{\cosh(\lambda_{1}(b - a))(m_{3} + im_{4}) + \sinh(\lambda_{1}(b - a))(m_{5} + im_{6})}, \quad (38) \]
For the line segments IJ and KL, suppose that \( s = \infty \omega \pi \rightarrow 0 \), \( \rho \rightarrow 0 \), \( \beta \), \( \int_0^1 \int_0^1 e^{-\pi}f(s)ds \) for the effective viscosity on the torque experienced by the composite sphere. Hence we can get the exact formula of \( G(t) \) by substituting equations (34-48) into (33)

\[
G(t) = -\frac{1}{\pi} \int_0^2 h(x)dx - \frac{1}{\pi} \int_0^\infty g(x)dx,
\]

(49)

\[
e^{-\pi}l_3 \frac{\cosh(\lambda_3(b-a))(m_1m_8 - m_2m_7)}{\sinh(\lambda_3(b-a))(m_7m_8 - m_2m_9) + \sinh(\lambda_3(b-a))(m_2m_9 - m_1m_8)}
\]

(50)

\[
e^{-\pi}l_3 \frac{\cos(\lambda_1(b-a))(m_1m_3 + m_2m_4)}{\sin(\lambda_1(b-a))(m_3m_4 - m_1m_5) + \sin(\lambda_1(b-a))(m_1m_5 - m_3m_4)}
\]

(51)

5. Numerical results and discussion

In this section, we represent the normalized torque \( T^* = T/T_0, T_0 = -8\pi \mu \alpha^2 \Omega_0 \), graphically for the case of flow due to sudden motion of the composite sphere.

Figure 3 shows the distribution of the torque against time (in seconds) for different values of the permeability parameter at fixed values of \( \mu e/\mu = 1, \gamma = 0.5 \) and \( b/a = 1.1 \). It is observed that, the torque increases suddenly to reach its peak for a small time, then decreases with the increase of time to reach steady state for large time. It can be seen also, that the increase in the permeability value results in an apparent decrease in the value of the torque. Figure 4 shows the influence of the effective viscosity on the torque experienced by

\[
m_6 = \lambda_2 b \left( \gamma - \lambda_3^2(b - \alpha + \sqrt{k}) \right),
\]

(45)

\[
m_7 = \lambda_3 \left( \lambda_3^2ab \sqrt{k} - (b - a) \left( \lambda_3^2b \sqrt{k} + \gamma \right) \right),
\]

(46)

\[
m_8 = \lambda_3 \lambda_2 b \left( \lambda_3^2b \sqrt{k} - \gamma(b - a) \right),
\]

(47)

\[
m_9 = \lambda_3 b \left( -\gamma a + \sqrt{k} - \lambda_3^2ab \sqrt{k} + \lambda^2b \sqrt{k} + \gamma \right),
\]

(48)

\[
m_{10} = \lambda_2 b \left( \lambda_3^2b \left( -\gamma a + \sqrt{k} + \gamma \right) \right).
\]

(49)

\[
\int_0^\infty \int_0^1 e^{-\pi}f(s)ds
\]

(42)

\[
m_1 = -\lambda_2^2b^2 + 3, m_2 = 3\lambda_2 b,
\]

(43)

\[
m_3 = \lambda_1 \lambda_2 b \left( \lambda_3^2ab \sqrt{k} + \gamma(b - a) \right),
\]

(44)

\[
m_4 = -\lambda_1 \left( \lambda_3^2ab \sqrt{k} + (b - a) \left( \lambda_3^2b \sqrt{k} + \gamma \right) \right),
\]

(45)
Figure 4. Normalized Torque variation versus time at $k = 1$, $\gamma = 0.5$ and $b/a = 1.1$ for different values of $\mu_e/\mu$.

Figure 5. Normalized Torque variation versus time at $k = 1$, $\mu_e/\mu = 1$ and $b/a = 1.1$ for different values of stress jump coefficient.

Figure 6. Normalized Torque variation versus time at $k = 1$, $\mu_e/\mu = 1$ and $\gamma = 0.5$ for different values of $b/a$.

Figure 7. Normalized Torque variation versus time at $\gamma = 0.5$, $\mu_e/\mu = 1$ and $b/a = 1.1$ for different values of time (in seconds).

Figure 8. Normalized Torque variation versus $k$ at $b/a = 1.1$, $t = 3$ sec and $\gamma = 0.5$ for different values of $\mu_e/\mu$.

The fluid on the porous spherical surface at fixed values of $k = 1$, $\gamma = 0.5$ and $b/a = 1.1$. It can be noticed that the torque increases with the increase permeability coefficient. In Figure 5 the distribution of the torque is represented versus time (in seconds) for different values of the stress jump coefficient $\gamma$, at fixed values of $k = 1$, $\mu_e/\mu = 1$ and $b/a = 1.1$. It can be observed that the increase in stress jump coefficient results to a slight increase of torque. However, the variation of the torque with respect to time (in seconds) for different values of $b/a$, at fixed values of $\mu_e/\mu = 1$, $\gamma = 0.5$ and $k = 1$ is represented in Figure 6, it is shown that as the values of $b/a$ increases there is a mild decrease in the torque. Moreover, it is noticed in Figure 7 the strictly decrease of the values of the torque with the increase in permeability to approach the steady state. In Figure 8, the graph of torque with respect to permeability coefficient for different values of effective viscosity, at fixed values of $b/a = 1.1$, $t = 3$ sec. and $\gamma = 0.5$ shows that the torque slightly increases at small values of $k$ then strictly decreases as the permeability coefficient increase, also it is seen that the increase in $\mu_e/\mu$ leads to an apparent increase in torque.

6. Conclusion

The problem of rotational motion of a composite sphere, consisting of a solid core surrounded by a porous shell, in an incompressible viscous fluid is discussed and solved analytically using Navier-Stokes model and Brinkman model. The case of no-slip condition is assumed and the stress jump condition is applied on the fluid-porous interface. The effect of the stress jump coefficient and the permeability coefficient is discussed numerically. It can be observed that the permeability coefficient decreases the values of torque for different values of time, effective viscosity and stress jump coefficient.

Disclosure statement

No potential conflict of interest was reported by the authors.

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References

[1] Keh C. Creeping motions of a composite sphere in a concentric spherical cavity. Chem Eng Sci. 2004;59(2): 407--415.
[2] Srinivasacharya D, Krishna Prasad M. Steady rotation of a composite sphere in a concentric spherical cavity. Acta Mech Sin. 2012;28(3):653--658.
[3] Srinivasacharya D, Krishna Prasad M. Motion of a porous spherical shell in a bounded medium. Adv Theor Appl Mech. 2012;15(6):247--256.
[4] Srinivasacharya D, Krishna Prasad M. Slow steady rotation of a porous sphere located at the center of a spherical container. J Porous Med. 2015;15(12):1105–1110.
[5] Srinivasacharya D, Krishna Prasad M. Rotation of a porous approximate sphere in an approximate spherical container. Lat Am Appl Res. 2015;45(2):107–112.
[6] Ashmawy EA. Steady rotational motion of a porous spheroid in a viscous fluid. Eur J Mech B Fluids. 2015;5:1–12.
[7] Ashmawy EA. Rotary oscillation of a composite sphere in a concentric spherical cavity using slip and stress jump conditions. Eur Phys J Plus. 2015;130(8):1–12.
[8] Wook RH, Advani SG. Numerical simulations of Stokes–Brinkman equations for permeability prediction of dual scale fibrous porous media. Phys Fluids. 2010;22(11):1–10.
[9] Ng CO, Wang CY. Darcy–Brinkman flow through a corrugated channel. Trans in Porous Med. 2010;85(8):605–618.
[10] Guta L, Sundar S. Navier-Stokes-Brinkman system for interaction of viscous waves with a submerged porous structure. Tamkang Journal of Mathematics. 2010;41(3):217–243.
[11] Ashmawy EA. Steady rotation of an axially symmetric porous particle about its axis of revolution in a viscous fluid using Brinkman model. Eur J Mech B Fluids. 2015;50:147–155.
[12] Ochoa Tapia JA, Whitaker S. Momentum transfer at the boundary between a porous-medium and a homogeneous fluid 1. Int J Heat Mass Transfer. 1995;38(14):2635–2646.
[13] Ochoa Tapia JA, Whitaker S. Momentum-transfer at the boundary between a porous-medium and a homogeneous fluid 2. Int J Heat Mass Transfer. 1995;38(14):2647–2655.
[14] Kuznetsov AV. Analytical investigation of Couette flow in a composite channel partially filled with a porous medium and partially with a clear fluid. Int J Heat Mass Transfer. 1998;41(16):2556–2560.
[15] Day MA. The no-slip condition of fluid dynamics. Kluwer Acad Pub. 2004;33(3):285–296.