AFFINE HERMITIAN GRASSMANN CODES

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Abstract
The Grassmannian is an important object in Algebraic Geometry. One of the many techniques used to study the Grassmannian is to build a vector space from its points in the projective embedding and study the properties of the resulting linear code.

We introduce a new class of linear codes, called Affine Hermitian Grassman Codes. These codes are the linear codes resulting from an affine part of the projection of the Polar Hermitian Grassmann codes. They combine Polar Hermitian Grassmann codes and Affine Grassmann codes. We will determine the parameters of these codes.

1 Introduction
Let \( q \) be a prime power and \( \mathbb{F}_q \) denote the finite field of \( q \) elements. The Grassmannian, \( \mathcal{G}_{\ell,m} \), is the collection of all subspaces of dimension \( \ell \) of a vector space \( V \) of length \( m \). We take \( V = \mathbb{F}_q^m \). The Grassmannian is a fascinating and well studied geometry with a rich algebraic structure.

It is well known that the Grassmannian may be embedded into a projective space through the Plücker embedding. The embedding is performed as follows. For each vector space \( W \in \mathcal{G}_{\ell,m} \), a matrix \( M_W \) is taken such that the rowspace of \( M_W \) is \( W \). We then fix some ordering of the \( \ell \)-minors of an \( \ell \times m \) matrix. Then \( M_W \) is mapped to the projective point \( p_W \), where the i-th position is the value of the i-th \( \ell \)-minor of \( M_W \). In order to study the properties of the Grassmannian we make use of the Grassmann code. The Grassmann code is defined as the linear code generated by taking all of the points \( p_W \) as the column vectors of a matrix.

In \[10\], Nogin studied the parameters of this linear code, denoted \( C(\ell,m) \). Later, in \[1\] Beelen, Ghorpade and Høholdt introduced the linear code associated to one of the affine maps of the Plücker embedding of the Grassmannian. These codes are known as affine Grassmann codes, \( C^A(\ell,m) \) for \( m \geq 2\ell \). Let \( \delta = \ell(m - \ell) \), they proved the parameters of \( C^A(\ell,m) \) are:

\[
[q^\delta, \binom{m}{\ell}, q^{\delta - \ell^2} \prod_{i=0}^{\ell-1} q^i - q^i]
\]

In that same paper, they studied the automorphisms of the code and counted the minimum distance codewords. Their duals, \( C^A(\ell,m)^\perp \), and related codes were studied in \[3\] and \[2\]. Moreover, in \[2\], the minimum weight codewords of the dual code were classified and counted.
In [4], Cardinali and Giuzzi introduced polar Grassmann codes, linear codes which are closely related to the Grassmann codes previously mentioned. The polar Grassmannian is a subvariety consisting of the subspaces of \( V \) isotropic under a given form. Cardinali and Giuzzi determined the parameters of polar Grassmann codes under a symplectic form (polar symplectic Grassmannian)[5], under a orthogonal form (polar orthogonal Grassmannian)[8][7] and under a Hermitian form (polar Hermitian Grassmannian)[6] for \( \ell = 2 \).

Much is known about both affine Grassmann codes and polar Grassmann codes. However, in this work we study the linear code associated to one of the affine maps of the polar Hermitian Grassmannian. We first determine the length, dimension and minimum distance of this code. We also determine some of its automorphisms and its dual code. Furthermore, we characterize the minimum weight codewords of the dual code.

## 2 Preliminaries

In this manuscript, we shall use the following notation. For positive integers \( m \) and \( n \), an \( m \times n \) matrix is a rectangular array of \( m \) rows and \( n \) columns of entries over \( \mathbb{F}_q \) or \( \mathbb{F}_{q^2} \). Matrices will be denoted by upper case letters such as \( A, B, C, M, N \). Generic matrices will be denoted by \( X, Y \) or \( Z \). In this work, we will focus mostly on square matrices over \( \mathbb{F}_q \) and \( \mathbb{F}_{q^2} \).

**Definition 1.** Let \( M \) be a square matrix. Suppose that \( I \) is a subset of the rows of \( M \) and \( J \) is a subset of the columns of \( M \). The minor \( \det_{I,J}(M) \) is the determinant of the submatrix of \( M \) obtained from the rows \( I \) and columns \( J \). In the case \( I = \{i\} \) and \( J = \{j\} \) we shall denote by \( \det_{i,j}(M) \) minor by the \((i, j)\)–th entry \( M_{i,j} \).

Example: Let \( M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, I = \{1, 2\} \) and \( J = \{2, 3\} \). Then \[
\det_{I,J}(M) = \begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix} = 0.
\]

As \( \mathbb{F}_{q^2} \) is an extension of degree 2 of the field \( \mathbb{F}_q \), for \( \alpha \in \mathbb{F}_{q^2} \) we denote the conjugate of \( \alpha, \alpha^q, \) by \( \bar{\alpha} = \alpha^q \). Likewise, we may conjugate matrices by conjugating each entry. Therefore we denote the conjugate of \( M \), the matrix \( [M^q_{i,j}] \) by \( M^{(q)} \). We recall the following definition:

**Definition 2.** For a matrix \( M \in M_{n \times n}(\mathbb{F}_{q^2}) \), we denote \( M^* \) as the conjugate transpose of \( M \), that is \( M^* = [m^q_{ji}] \).

Example: Let \( M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \) then \( M^* = \begin{bmatrix} a^q & c^q \\ b^q & d^q \end{bmatrix} \).

**Definition 3.** A matrix over \( \mathbb{F}_{q^2} \) is Hermitian if \( M^* = M \).

Example: Let \( q = 2 \), note \( M = \begin{bmatrix} 1 & \alpha & 1 \\ \alpha^2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \) is Hermitian. In general, a Hermitian matrix \( H \) over \( \mathbb{F}_{q^2} \) satisfies:

\[
H = \begin{cases} 
H_{i,j} \in \mathbb{F}_q & i = j \\
H_{i,j} \in \mathbb{F}_{q^2} & i > j \\
H_{i,j} = H_{j,i}^q & j < i 
\end{cases}
\]
2.1 affine Hermitian Grassman code

For \( \ell \geq 1 \) we define \( X = [X_{ij}] \) as the \( \ell \times \ell \) matrix of \( \ell^2 \) indeterminates over \( \mathbb{F}_{q^2} \). As we are interested in Hermitian matrices, we shall define \( X_{i,i} \) as an element of \( \mathbb{F}_q \) (satisfying \( X_{i,i}^q = X_{i,i} \)). For \( i < j \), \( X_{i,j} \) is an element of \( \mathbb{F}_{q^2} \) (satisfying \( X_{i,j}^q = X_{i,j} \)). Then \( X_{j,i} \) for \( i < j \) is the conjugate of its transposed position, that is, \( X_{j,i} = X_{i,j}^q \). In a similar manner, we define \( Z = [Z_{i,j}] \). \( Z \) is the generic \( \ell \times \ell \) matrix of \( \ell^2 \) indeterminates over \( \mathbb{F}_q \).

Definition 4. \( \Delta(\ell) \) is the set of all minors of the matrix \( X \). That is:

\[
\Delta(\ell) := \{ \det_{I,J}(X), I, J \subseteq [\ell], \#I = \#J \}
\]

We remark that \( \#\Delta(\ell) = \binom{2\ell}{\ell} \).

Example: For \( \ell = 2 \), let the matrix \( X \) be equal to

\[
X = \begin{bmatrix}
X_1 & X_2 \\
X_2^q & X_3
\end{bmatrix},
\]

Then \( \Delta(\ell) = \{ 1, X_1, X_2, X_3, X_2^q, X_1X_3 - X_2^{q+1} \} \).

Definition 5. We define \( \mathcal{F}(\ell) \) as the subspace of \( \mathbb{F}_{q^2} \)-linear combinations of elements of \( \Delta(\ell) \).

\[
\mathcal{F}(\ell) := \{ \sum_{I,J \subseteq [\ell], \#I = \#J} f_{I,J}\det_{I,J}(X)|f_{I,J} \in \mathbb{F}_{q^2} \}
\]

Example:
Let \( \ell = 2 \) then all elements of \( \mathcal{F}(\ell) \) are of the form:

\[
f_{\emptyset,\emptyset} + f_{\{1,\},\{1\}}X_1 + f_{\{1,\},\{2\}}X_2 + f_{\{2,\},\{1\}}X_3 + f_{\{2,\},\{2\}}X_2 + f_{\{1,2\},\{1,2\}}(X_1X_3 - X_2^{q+1})
\]

We remark that the minors \( \det_{I,J}(X) \) as polynomials on \( X_{i,j} \), it is simple to determine that \( \dim \Delta(\ell) = \binom{2\ell}{\ell} \). Also, in a similar manner to \( \mathcal{F}(\ell) \), \( \mathcal{F}(\ell)_Z \) may be defined as the subspace of \( \mathbb{F}_q \)-linear combinations of elements of \( \Delta(\ell) \) when taking \( Z \) as the generic matrix instead of \( X \).

Definition 6. \( \mathbb{H}^f(\mathbb{F}_{q^2}) \) denotes the space of all \( \ell \times \ell \) Hermitian matrices with entries in \( \mathbb{F}_{q^2} \). That is

\[
\mathbb{H}^f(\mathbb{F}_{q^2}) = \{ H \in \mathbb{M}^{\ell \times \ell}(\mathbb{F}_{q^2}) | M = M^* \}.
\]

Now we define, the evaluation of an element \( f \in \mathcal{F}(\ell) \) at any Hermitian matrix \( P \in \mathbb{H}^f(\mathbb{F}_{q^2}) \). For any \( f \in \mathbb{F}_{q^2}[X] \) and \( P \in \mathbb{H}^f(\mathbb{F}_{q^2}) \) the evaluation \( f(P) \) is obtained by replacing the variable \( X_{i,j} \) by the element \( P_{i,j} \). From now on, we shall denote \( n \) by \( n = q^{\ell^2} = \#\mathbb{H}^f(\mathbb{F}_{q^2}) \). For the evaluation map we fix an arbitrary enumeration \( P_1, P_2, ..., P_n \) of \( \mathbb{H}^f(\mathbb{F}_{q^2}) \).

Definition 7. The evaluation map of \( \mathbb{F}_{q^2}[X] \) is the map

\[
ev : \mathbb{F}_{q^2}[X] \rightarrow \mathbb{F}_{q^2}^n \text{ defined by } Ev(f) := (f(P_1), ..., f(P_n)).
\]

In order to prove the evaluation map is an injective function, we need the following lemma on the elements of \( \mathcal{F}(\ell) \).

Definition 8. Denote by \( I(\mathbb{H}) \) the ideal generated by \( X_{i,j}^q - X_{j,i} \) and, \( X_{i,i}^q - X_{i,i} \), where \( i < j \) and \( X_{i,i}^q = X_{i,i} \).

That is

\[
I(\mathbb{H}) := \langle X_{i,j}^q - X_{j,i}, X_{j,i}^q - X_{i,j}, X_{i,i}^q - X_{i,i} \rangle.
\]

The set of Hermitian matrices \( \mathbb{H}^f(\mathbb{F}_{q^2}) \) is precisely the set of solutions to all polynomial equations generating \( I(\mathbb{H}) \). We shall now prove the following lemma relating \( \mathcal{F}(\ell) \) and \( I(\mathbb{H}) \).
Lemma 1. There are no elements in common between \( F(\ell) \) and \( I(\mathbb{H}) \) except 0.

That is
\[
F(\ell) \cap I(\mathbb{H}) = \{0\}.
\]

Proof. Let \( f \in F(\ell) \). Consider \( f_H = f \mod I(\mathbb{H}) \), the reduction of \( f \) modulo the ideal \( I(\mathbb{H}) \). This may be attained simply by replacing the entry \( X_{j,i} \) for \( i < j \) with its conjugate transpose \( X_{i,j}^\dagger \). This is given by the polynomial equation \( X_{i,j}^\dagger - X_{j,i} = 0 \).

Any determinant will have at most degree \( q+1 \) on the variable \( X_{i,j} \) for \( i < j \) and degree at most 1 on the variable \( X_{i,i} \). This implies no single determinant is in \( I(\mathbb{H}) \) as that would imply the degree in \( X_{i,j} \) is at least \( q \) and the degree in \( X_{i,i} \) is at least \( q^2 \). Now we shall prove that even after setting \( X_{j,i} = X_{i,j}^\dagger \) all linear combinations in \( F(\ell) \) are still independent.

Let \( I = \{i_1 < i_2 < \cdots < i_m\} \) and \( J = \{j_1 < j_2 < \cdots < j_m\} \). The term \( \prod_{s=1}^m X_{i_s,j_s} \) is a term appearing only in \( \text{det}_{I,J} \). The reduction \( \prod_{s=1}^m X_{i_s,j_s} \mod I(\mathbb{H}) \) is obtained by replacing any \( X_{i_s,j_s} \) by \( X_{j_s,i_s}^\dagger \) whenever \( i_s > j_s \). Denote this monomial by \( M_{n_s} \).

Now let \( I' = \{i'_1 < i'_2 < \cdots < i'_m\} \) and \( J' = \{j'_1 < j'_2 < \cdots < j'_m\} \). We shall prove that if \( I \neq I' \) or \( J \neq J' \) then the monomial \( M_{n_s} \neq M_{n'_s} \).

Clearly if the sizes of \( I \) and \( J \) are different from the size of \( I' \) and \( J' \) then both \( M_{n_s} \) and \( M_{n'_s} \) will have a different number of monomials. We may assume \( \#I \neq \#J \neq \#I' \neq \#J' \). Suppose now that \( I \neq I' \) or \( J \neq J' \). We may assume there exists a pair \( (i_s,j_s) \neq (i'_s,j'_s) \). If \( (i_s,j_s) \neq (i'_s,j'_s) \) then \( M_{n_s} \) contains a different variable than \( M_{n'_s} \). If \( (i_s,j_s) = (i'_s,j'_s) \) implies \( i'_s = j_s \) and \( j_s = i'_s \). The reduction \( \mod I(\mathbb{H}) \) will map \( X_{j_s,i_s} \) to \( X_{i'_s,j'_s}^\dagger \), which implies \( M_{n_s} \neq M_{n'_s} \).

As all reductions of all determinants modulo \( I(\mathbb{H}) \) are independent, the statement follows.

Lemma 2. The evaluation map ev is injective.

Proof. Note that the ideal \( I(\mathbb{H}) \) is precisely the ideal of polynomial functions which vanish on \( \mathbb{H}^\ell(\mathbb{F}_q^2) \). That is \( ev(f) = 0 \) if and only if \( f \in I(\mathbb{H}) \). As Lemma 1 implies there is no nonzero element in both \( F(\ell) \) and \( I(\mathbb{H}) \), we then have that \( \text{Ker}(ev) = F(\ell) \cap I(\mathbb{H}) = \{0\} \). Thus the evaluation map is injective.

We are now ready to define affine Hermitian Grassmann codes.

Definition 9. The affine Hermitian Grassmann code \( C^{\mathbb{H}}(\ell) \) is the image of \( F(\ell) \) under the evaluation map \( ev \). That is
\[
C^{\mathbb{H}}(\ell) := \{(f(P_1), f(P_2), \ldots, f(P_n)) | f \in F(\ell)\}
\]

We remark that if \( A \) is skew-Hermitian, then \( A = \alpha H \) where \( H \) is Hermitian and \( \alpha^q = -\alpha \). This implies \( \text{det}_{\ell,J}(A) = \alpha^{#J} \text{det}_{\ell,J}(H) \). Which, in turn, implies that all evaluations under \( ev \) for skew-Hermitian matrices are scalar multiples of evaluations for \( \mathbb{H}^\ell(\mathbb{F}_q^2) \). Therefore they generate the same code, that is the affine Hermitian Grassmann code and the affine skew-Hermitian Grassmann code are equivalent.

As a direct consequence of the Rank–Nullity theorem and Lemma 2 we obtain the following corollary.

Corollary 3.
\[
\dim C^{\mathbb{H}}(\ell) = \binom{2\ell}{\ell}.
\]

Now we compare our code \( C^{\mathbb{H}}(\ell) \) to the affine Grassmann code \( C^{\mathbb{A}}(\ell, 2\ell) \) introduced by Beelen, Ghorpade and Høholdt. It turns out that both codes are very similar to each other. However some key differences between \( C^{\mathbb{H}}(\ell) \) and \( C^{\mathbb{A}}(\ell, 2\ell) \) remain. Now we state the definition of the code \( C^{\mathbb{A}}(\ell, 2\ell) \).
Definition 10. \([7]\)

The affine Grassmann code \(C^k(\ell, 2\ell)\) is obtained by evaluating \(f \in \mathcal{F}(\ell)\) onto all \(\ell \times \ell\) matrices over \(\mathbb{F}_q\).

\[
C^k(\ell, 2\ell) := \{(f(P_1), f(P_2), \ldots, f(P_n)) | f \in \mathcal{F}(\ell), P_i \in \mathbb{F}_q^\ell \}
\]

where \(P_1, P_2, \ldots, P_n\) are the elements of \(\mathbb{M}^{\ell \times \ell}(\mathbb{F}_q)\) in some order.

We remark that \(\#\mathbb{M}^{\ell \times \ell}(\mathbb{F}_q) = \#\mathbb{H}^{\ell}(\mathbb{F}_q^2) = q^{\ell^2}\). Thus both \(C^k(\ell, 2\ell)\) and \(C^{\Omega}(\ell)\) have the same length. They also have the same dimension, \(\binom{\ell q}{\ell} \). Although \(C^k(\ell, 2\ell)\) is defined over \(\mathbb{F}_q\) and \(C^{\Omega}(\ell)\) is defined over \(\mathbb{F}_q^2\), we shall prove that \(C^{\Omega}(\ell)\) has a basis over the subfield \(\mathbb{F}_q\) and in fact, may be considered as a \(\mathbb{F}_q\) code. We begin by defining the following map.

Definition 11. Let \(\mathbb{F}_{q^m}\) be a finite field containing \(\mathbb{F}_q\). If \(c \in \mathbb{F}_{q^m}\), we define

\[
c^q := (c_1^q, c_2^q, \ldots, c_n^q).
\]

This definition is extended to codes as follows.

Definition 12. Let \(\mathbb{F}_{q^m}\) be a finite field containing \(\mathbb{F}_q\). If \(C \subseteq \mathbb{F}_{q^m}\), we define

\[
C^q := \{c^q | c \in C\}.
\]

We state without proof that \(C^q\) is also a \(\mathbb{F}_{q^m}\) linear code whenever \(C\) is.

Proposition 1. \([7]\) Let \(\mathbb{F}_{q^m}\) be a finite field containing \(\mathbb{F}_q\). Let \(C\) be a linear code over \(\mathbb{F}_{q^m}\). Then \(C\) has a basis over \(\mathbb{F}_q\) if and only if \(C = C^q\).

Proposition 2. \([7]\) Let \(\mathbb{F}_{q^m}\) be a finite field containing \(\mathbb{F}_q\). Let \(C\) be a linear code over \(\mathbb{F}_{q^m}\). If \(C = C^q\), then all of its minimum distance codewords are multiples of a codeword in the subfield \(\mathbb{F}_q\).

Definition 13. Let \(f \in \mathcal{F}(\ell)\). We denote the conjugate of \(f\) by

\[
f_{\text{conj}} = \sum_{I, J \subseteq [\ell], \# I = \# J} f_{1, J}^q \det_{I, J}(X).
\]

Lemma 4. The code \(C^{\Omega}(\ell)\) satisfies

\[
C^{\Omega}(\ell) = C^{\Omega}(\ell)^q.
\]

Proof. We only need to prove that if \(c \in C^{\Omega}(\ell)\) then so is \(c^q \in C^{\Omega}(\ell)\). Recall that for any Hermitian matrix \(H \in \mathbb{H}^{\ell}(\mathbb{F}_q^2)\) \(\det_{I, J}(H) = \det_{I, J}(H)^q\). Hence if \(f \in \mathcal{F}(\ell)\) satisfies

\[
f = \sum_{I, J \subseteq [\ell], \# I = \# J} f_{I, J} \det_{I, J}(X).
\]

Then for any \(P_i \in \mathbb{H}^{\ell}(\mathbb{F}_q^2)\), \(f^q\) satisfies

\[
f(P_i)^q = \sum_{I, J \subseteq [\ell], \# I = \# J} f_{I, J}^q \det_{I, J}(P_i)^q.
\]

However, as \(P_i\) is Hermitian we have that

\[
f(P_i)^q = \sum_{I, J \subseteq [\ell], \# I = \# J} f_{I, J}^q \det_{I, J}(P_i).
\]

Therefore the linear combination

\[
f_{\text{conj}} = \sum_{I, J \subseteq [\ell], \# I = \# J} f_{I, J}^q \det_{I, J}(X)
\]

satisfies

\[
ev(f_{\text{conj}}) = ev(f)^q.
\]
Now we calculate the $\mathbb{F}_q$-basis for the code $C^\mathbb{H}(\ell)$.

**Lemma 5.** Let $\{\alpha, \alpha^q\}$ be a basis of $\mathbb{F}_{q^2}$ over $\mathbb{F}_q$. Then $C^\mathbb{H}(\ell)$ is generated as a code over $\mathbb{F}_q$ by the functions

\[ ev(\alpha \det_{I,J}(X) + \alpha^q \det_{I,I}(X)) \]

and

\[ ev(\alpha^q \det_{I,J}(X) + \alpha \det_{I,I}(X)) \]

**Proof.** The definition of the code $C^\mathbb{H}(\ell)$ implies that the code $C^\mathbb{H}(\ell)$ is generated by $ev_{\mathbb{H}(\mathbb{F}_q^2)}(\alpha f_{I,J} \det_{I,J}(X))$ where the coefficient $f_{I,J} \in \mathbb{F}_{q^2}$.

Note that in the case $I = J$ then $ev_{\mathbb{H}(\mathbb{F}_q^2)}(\det_{I,J}(X))$ is clearly an $\mathbb{F}_q$-valued function because $X^{I,I}$ is a principal determinant of a Hermitian matrix satisfying $(X^{I,I})^T = (X^{I,I})^q$.

If $I \neq J$ then note that the vector space

\[ \langle \det_{I,J}(X), \det_{I,I}(X) \rangle \]

is spanned by

\[ \langle \det_{I,J}(X) + \alpha^q \det_{I,I}(X), \alpha^q \det_{I,J}(X) + \alpha \det_{I,I}(X) \rangle. \]

When evaluating $\det_{I,J}(X)$ on a nonprincipal minor of a Hermitian matrix we obtain that $\det_{I,J}(X)^q = \det_{I,I}(X)$. Therefore $ev_{\mathbb{H}(\mathbb{F}_q^2)}(\alpha \det_{I,J}(X) + \alpha^q \det_{I,I}(X))$ and $ev_{\mathbb{H}(\mathbb{F}_q^2)}(\alpha^q \det_{I,J}(X) + \alpha \det_{I,I}(X))$ are $\mathbb{F}_q$-valued functions. Thus we’ve found a basis for $C^\mathbb{H}(\ell)$ of the required form. □

We have the following corollary on the codewords of $C^\mathbb{H}(\ell)$ which take values exclusively on the subfield $\mathbb{F}_q$.

**Corollary 6.** Let $c = ev_{\mathbb{H}(\mathbb{F}_q^2)}(f)$ be a codeword of $C^\mathbb{H}(\ell)$. Denote by $c_H$ the position of $c$ indexed by $H \in \mathbb{H}(\mathbb{F}_q^2)$. Then $c_H^q = c_H \ \forall H \in \mathbb{H}(\mathbb{F}_q^2)$ if and only if $f_{I,J} = f_{I,I}$.

**Proof.** Note that for $c = ev_{\mathbb{H}(\mathbb{F}_q^2)}(f)$ the position $c_H$ is equal to $f(H)$. Lemma 5 implies $f(H) \in \mathbb{F}_q$ if and only if $f_{I,J} = f_{I,I}^q$. □

By Proposition 3 we may assume our minimum weight codewords are over $\mathbb{F}_q$. Thus, by Corollary 6 we may assume our coefficients meet $f_{I,J} = f_{I,I}^q$.

**Definition 14.** Let $C$ be a code of length $n$. We say that a permutation $\sigma \in S_n$ is an automorphism of $C$ if and only if

\[ (c_1, c_2, \ldots, c_n) \in C \text{ if and only if } (c_{\sigma(1)}, c_{\sigma(2)}, \ldots, c_{\sigma(n)}) \in C. \]

The group of such automorphisms is determined by $\text{Aut}(C)$.

We state the following automorphisms of $C^\mathbb{H}(\ell, 2\ell)$ from [3].

**Proposition 3.** [3 Lemma 7]

The automorphism group $\text{Aut}(C^\mathbb{H}(\ell, 2\ell))$ contains the following permutations:

- For $A \in GL_\ell(\mathbb{F}_q)$, $X \mapsto AX$.
- For $B \in GL_\ell(\mathbb{F}_q)$, $X \mapsto XB$.
- For $M \in M^{\ell \times \ell}(\mathbb{F}_q)$, $X \mapsto X + M$.
- $X \mapsto X^T$.

Some of the automorphisms of $C^\mathbb{H}(\ell, 2\ell)$ are also automorphisms of $C^\mathbb{H}(\ell)$ as seen in the following lemma.
Lemma 7. The automorphism group $Aut(C_{q}(\ell))$ contains the group generated by the following permutations

- For $A \in GL_{\ell}(F_{q^{2}})$, $X \mapsto A(q^{T})X A$.
- For $M \in H_{\ell}(F_{q^{2}})$, $X \mapsto X + M$.
- $X \mapsto X^{T}$.

Proof. Note that $C_{H}(\ell)$ is obtained from $C_{A}(\ell, 2\ell)$ by removing all matrices which are not Hermitian. That is the code $C_{H}(\ell)$ is a puncturing of the code $C_{A}(\ell, 2\ell)$ at the matrices in $M_{\ell \times \ell}(F_{q^{2}}) \setminus H_{\ell}(F_{q^{2}})$ to obtain $C_{H}(\ell)$. Therefore the permutations in $Aut(C_{A}(\ell, 2\ell))$ fixing $H_{\ell}(F_{q^{2}})$ setwise are permutations of $Aut(C_{H}(\ell))$. If $H$ is a Hermitian matrix, then so are the matrices $A(q^{T})HA$, $H + H'$ where $H' \in H_{\ell}(F_{q^{2}})$ and $HT$. Therefore these permutations are automorphisms of $C_{H}(\ell)$.

One of the most important parameters of a linear code is its minimum distance. We state its definition as follows:

Definition 15. The Hamming distance of the vectors $x = (x_{1}, x_{2}, \ldots, x_{n})$ and $y = (y_{1}, y_{2}, \ldots, y_{n})$ is the number of positions in which $x$ and $y$ differ. That is:

$$d(x, y) := \# \{i \mid x_{i} \neq y_{i} \}$$

For example: let $x = (1010)$ and $y = (1100)$

Then $d(x, y) = 2$ because they differ in the second and third position.

Definition 16. The weight of $x = (x_{1}, x_{2}, \ldots, x_{n})$ is the number of positions in which $x_{i} \neq 0$. That is:

$$w(x) := \# \{i \mid x_{i} \neq 0 \}$$

Note that the weight of a vector is the same as its distance to the zero vector. That is $w(x) = d(x, 0)$.

We now state the definition of the minimum distance of a code.

Definition 17. Let $C$ be a code. Then the minimum distance of $C$ is the minimum number of positions in which any two distinct elements of $C$ differ.

$$d(C) = \min_{x, y \in C} d(x, y).$$

For linear codes, as is our case, the distance can be calculated in terms of the weights of the vectors. One of our remarkable findings is that not only that $C_{H}(\ell)$ may be considered as a $F_{q}$ code, but also that $C_{H}(\ell)$ has a much better minimum distance than $C_{A}(\ell, 2\ell)$. The distance of affine Grassmann codes (see $[1]$) is

$$d(C_{A}(\ell, 2\ell)) = \#GL_{\ell}(F_{q}) = \prod_{i=0}^{\ell-1} q^{i} - q^{i}.$$ 

Now we compare with our main result.

Claim 1. Suppose that $\ell \geq 2$. Then

$$d(C_{H}(\ell)) = q^{\ell^{2}} - q^{\ell^{2} - 1} - q^{2\ell - 3}.$$ 

In the next two sections of the paper we work out a proof of this claim by induction. We use polynomial evaluation and bounds from the fundamental theorem of Algebra to determine $d(C_{H}(\ell))$. 

7
3 Determining $d(C^H_\ell)$ for $2 \leq \ell \leq 3$

Our calculation of $d(C^H_\ell)$ depends on mathematical induction. The base cases $\ell = 2$ and $\ell = 3$ are the two cases which best illustrate our proof. We include several examples. Once we establish certain bounds on $wt(ev(f))$ over $H(F_{q^2})$, we then derive respective bounds on $wt(ev(f))$ for the Hermitian matrices of size $\ell + 1 \times \ell + 1$. We begin with the following lemma counting the number of zeroes of a certain quadratic equation.

Lemma 8. Let $a, b, \lambda \in F_q$, where $\lambda \neq 0$. Then

- The equation $(X_1 + a)(X_2 + b) = 0$ has $2q - 1$ solutions over $F_q$.
- The equation $(X_1 + a)(X_2 + b) = \lambda$ has $q - 1$ solutions over $F_q$.

Proof. We begin with the case $(X_1 + a)(X_2 + b) = 0$. In this case then either $(X_1 + a) = 0$ or $(X_2 + b) = 0$. If $X_1 = -a$, any of the $q$ values for $X_2$ is a solution to the equation. Similarly, for $X_2 = -b$, any of the $q$ values for $X_1$ is a solution to the equation. The solution $X_1 = -a, X_2 = -b$ is counted twice this implies we have $q + q - 1 = 2q - 1$ total solutions to the equation.

Now we consider $(X_1 + a)(X_2 + b) = \lambda \neq 0$. If $X_1 = -a$ then the equation becomes $(a - a)(X_2 + b) = \lambda$ which has no solution. If $X_1 = \alpha$ where $\alpha$ is any element of $F_q$ except $\alpha = -a$, then for there is exactly one value of $X_2$ (namely $X_2 = \frac{\lambda}{a + \alpha} - b$) such that the equation is satisfied.

As for any of $q - 1$ values $\alpha \neq -a$ for $X_1$, we find exactly one value of $X_2$ such that $(X_1 + a)(X_2 + b) = \lambda$ is satisfied, it is established that $(X_1 + a)(X_2 + b) = \lambda$ has $q - 1$ solutions. \qed

We shall also need the following lemma on the number of solutions to a particular system of polynomial equations over $F_{q^2}$.

Lemma 9. Let $a_1, a_2, \ldots, a_n \in F_q$ and for $1 \leq i, j \leq n$ let $b_{i,j} \in F_{q^2}$. Then the system of polynomial equations given by

$$
X_i^{q+1} = a_i, 1 \leq i \leq n
$$

$$
X_i X_j^q = b_{i,j}
$$

has at most $q+1$ solutions.

Proof. If the $a_i$’s satisfy $a_i = 0$, then the only possible solution is $X_i = 0$. If there is some $a_s \neq 0$, then there are at most $q+1$ values for $X_s$ which satisfy $X_s^{q+1} = a_s$. In this case, for each of the $q+1$ solutions of $X_s^{q+1} = a_s$, there is at most one value (namely $X_s = c_s$, $X_i = \frac{b_{i,s}}{c_s}$) which satisfies the equation $X_i X_s^q = b_{i,s}$. Therefore there are at most $q+1$ solutions to the system of equations. \qed

Now we define the support of a combination of minors $f \in F(\ell)$ and the concept of a maximal “term”. This concept will be akin to the degree in order to make the induction proof for the minimum distance.

Definition 18. Let $f \in F(\ell)$, where

$$
f = \sum_{1 \leq i,j \leq \ell, \# I = \# J} f_{i,j} det_{i,j}(X).
$$

The support of $f$ is defined as

$$
supp(f) = \{det_{i,j}(X) \mid f_{i,j} \neq 0\}.
$$

Example: Let $X = \begin{bmatrix} X_1 & X_2 \\ X_2^q & X_3 \end{bmatrix}$

If $f = 1 + X_1 X_3 - X_2^{q+1}$, then $supp(f) = \{det_{0,0}(X), det_{\{1,2\},\{1,2\}}(X)\}$
We say a minor \( \det_{I,J} \in \text{supp}(f) \) is maximal if and only if for any other minor \( \det_{I',J'} \in \text{supp}(f) \) we have that \( I \nsubseteq I' \) or \( J \nsubseteq J' \). That is the columns and rows of \( \det_{I,J} \) are not contained in the rows and columns of any other determinant in \( \text{supp}(f) \).

Example: Let \( X = \begin{bmatrix} X_1 & X_2 & X_3 \\ X_2 & X_4 & X_5 \\ X_3 & X_5 & X_6 \end{bmatrix} \) and

\[
\det = \det_{\emptyset,\emptyset}(X) + \det_{\{1\},\{2\}}(X)\det_{\{1,2\},\{1,2\}}(X) + \det_{\{1,2\},\{2,3\}}(X),
\]

and

\[
\text{supp}(f) = \{ \det_{\emptyset,\emptyset}(X), \det_{\{1\},\{2\}}(X), \det_{\{1,2\},\{1,2\}}(X), \det_{\{1,2\},\{2,3\}}(X) \}.
\]

The minors \( \det_{\{1,2\},\{1,2\}}(X) \) and \( \det_{\{1,2\},\{2,3\}}(X) \) are maximal. However, the minors \( \det_{\emptyset,\emptyset}(X) \) and \( \det_{\{1\},\{2\}}(X) \) are not maximal, because their row and column sets are contained in the row and column sets of other minors. Moreover, we remark that \( f \neq f_{\text{conj}} \), because \( \det_{\{2,3\},\{1,2\}}(X) \notin \text{supp}(f) \).

The following lemma relating translations and self–conjugate combinations gives insight on the structure of \( f \in \mathcal{F}(\ell) \). We shall use the automorphisms of the form \( f(X) \mapsto f(X + H), H \in \mathbb{H}(\mathbb{F}_q^2) \) to simplify the calculation of \( \text{wt}(\text{ev}(f)) \). Now we prove that that under a certain translation, we may consider \( f \) has no “terms” of second highest degree.

Lemma 10. Let \( f \in \mathcal{F}(\ell) \) where \( f = f_{\text{conj}} \). Suppose \( \det_{I,J} \) is a maximal minor of size \( k \) in \( f \). There exists a matrix \( H \in \mathbb{H}(\mathbb{F}_q^2) \) such that \( f(X + H) \) has no \#I − 1 × \#I − 1 minors in its support whose rows and columns are contained in \( I \).

Proof. Let \( H \in \mathbb{H}(\mathbb{F}_q^2) \). If \( \det_{I,J}(X + H) \) expanded is via minor expansions along a row \( i \in I \) it may be seen that \( \det_{I,J}(X + H) \) is a combination of \( \det_{I,J}(X), \det_{I \setminus \{i\}, J \setminus \{j\}}(X) \) and other minors of smaller size. However, the minor given by rows \( I \setminus \{i\} \) and columns \( I \setminus \{j\} \) appears with coefficient \((-1)^{i+j+1}H_{i,j}\).

Suppose \( f = f_{\text{conj}} \) and that \( \det_{I,J} \) is a maximal minor. Note that as \( f \) is self–conjugate, for any \( i, j \in I \), \( f_{I \setminus \{i\}, J \setminus \{j\}} = f_{I \setminus \{j\}, J \setminus \{i\}} \). Without loss of generality, we assume \( f_{I,J} = 1 \). Suppose \( H_f \in \mathbb{H}(\mathbb{F}_q^2) \) where

\[
h_{i,j} = (-1)^{i+j+1}f_{I \setminus \{i\}, J \setminus \{j\}}.
\]

Because \( f \) is self–conjugate, the matrix is Hermitian. Clearly the expansion of \( \det_{I,J}(X + H) + \det_{J,I}(X + H) \) has \( \det_{I \setminus \{i\}, J \setminus \{j\}}(X) \) in its support, appearing with coefficient \(-f_{I \setminus \{i\}, J \setminus \{j\}}\).

After the expansion of \( f(X + H) \) we see that the \#I − 1 × \#I − 1 minors from \( \det_{I,J}(X + H) \) cancel the \#I − 1 × \#I − 1 minors in \( f \), which implies \( f(X + H) \) has no \#I − 1 × \#I − 1 minors whose rows and columns are contained in \( I \). \qed

3.1 Calculating \( d(C_{\mathbb{F}_2}(2)) \)

For the case \( \ell = 2 \), we shall consider a generic matrix \( X \in \mathbb{H}(\mathbb{F}_q^2) \) as a matrix of the form

\[
X = \begin{bmatrix} X_1 & X_2 \\ X_2 & X_3 \end{bmatrix},
\]

where \( X_1, X_3 \in \mathbb{F}_q, X_2 \in \mathbb{F}_q^2 \).

In this case a function \( f \in \mathcal{F}(\ell) \) is of the form

\[
f = f_0 + f_{1,1}X_1 + f_{1,2}X_2 + f_{2,1}X_2^2 + f_{2,2}X_3 + f_{\{1,2\},\{1,2\}}(X_1X_3 - X_2^{q+1}), f_{i,j} \in \mathbb{F}_q^2.
\]
More specifically, as we assume $f = f_{\text{conj}}$, Corollary \ref{cor:conj} implies $f$ is of the form

$$f = f_0 + f_{1,1}X_1 + f_{1,2}X_2 + f_{2,1}X_2^q + f_{2,2}X_3 + f_{(1,2), (1,2)}(X_1X_3 - X_2^{q+1})$$

Where $f_0, f_{1,1}, f_{2,2}, f_{(1,2), (1,2)} \in \mathbb{F}_q$ and $f_{1,2} \in \mathbb{F}_{q^2}$.

We split our proof in two cases: $f_{(1,2), (1,2)} = 0$ or $f_{(1,2), (1,2)} \neq 0$.

**Lemma 11.** Let $l = 2$. Suppose $f \in \mathcal{F}(\ell)$ where $f$ is of the form

$$f = f_0 + f_{1,1}X_1 + f_{1,2}X_2 + f_{2,1}X_2^q + f_{2,2}X_3.$$ 

Then $\text{wt}(\text{ev}(f)) \geq q^4 - q^3$.

**Proof.** To determine $w(f)$ we count the solutions to

$$f = f_0 + f_{1,1}X_1 + f_{1,2}X_2 + f_{2,1}X_2^q + f_{2,2}X_3 = 0.$$ 

Suppose that $f_{1,1} \neq 0$. Then, for any of the $q$ values of $X_3$ and any of the $q^2$ values of $X_2$, there are at least $q - 1$ values of $X_1$ which make $f \neq 0$. The case $f_{2,2} \neq 0$ is similar.

Now we consider $f_{1,1} = f_{2,2} = 0$. As $X_1, X_3$ represent elements of $\mathbb{F}_q$, we may replace them with arbitrary values from $\mathbb{F}_q$. In that case, the equation $f = 0$ becomes $f_{1,2}^*X_2^q + f_{1,2}X_2 = -f_0$. As this is a polynomial of degree at most $q$ over $\mathbb{F}_q$, it has at most $q$ zeros for each of the $q^2$ total possible values of $X_1$ and $X_3$. Consequently there are at most $q^3$ matrices in $H^2(\mathbb{F}_q^2)$ such that $f(H) = 0$ and $w(f) \geq q^4 - q^3$. \hfill $\square$

Now consider $f \in \mathcal{F}(\ell)$ where the $2 \times 2$ minor of $X$ appears in $\text{supp}(f)$. Without loss of generality we assume $f_{(1,2), (1,2)} = 1$

**Lemma 12.** Let $l = 2$. Suppose $f \in \mathcal{F}(\ell)$ where $f$ is of the form

$$f = f_0 + f_{1,1}X_1 + f_{1,2}X_2 + f_{2,1}X_2^q + f_{2,2}X_3 + X_1X_3 - X_2^{q+1}.$$ 

Then $\text{wt}(\text{ev}(f)) \geq q^4 - q^3 - q$.

**Proof.** As in the previous case we count the number of solutions to

$$f = f_0 + f_{1,1}X_1 + f_{1,2}X_2 + f_{2,1}X_2^q + f_{2,2}X_3 + X_1X_3 - X_2^{q+1} = 0.$$ 

We move the terms with $X_1, X_3$ to one side and obtain:

$$f_{1,1}X_1 + f_{2,2}X_3 + X_1X_3 = X_2^{q+1} + f_{1,2}^*X_2^q - f_{1,2}X_2 - f_0.$$ 

Now we add $f_{1,1}f_{2,2}$ to both sides:

$$f_{1,1}f_{2,2} + f_{1,1}X_1 + f_{2,2}X_3 + X_1X_3 = X_2^{q+1} - f_{1,2}^*X_2^q - f_{1,2}X_2 - f_0 + f_{1,1}f_{2,2}.$$ 

The left hand side of the equation factors as:

$$(X_1 + f_{2,2})(X_3 + f_{1,1}) = X_2^{q+1} - f_{1,2}^*X_2^q - f_{1,2}X_2 - f_0 + f_{1,1}f_{2,2}.$$ 

The right hand side of the equation is an univariate polynomial in $X_2$ of degree $q + 1$. Denote by

$$P(X_2) = X_2^{q+1} - f_{1,2}^*X_2^q - f_{1,2}X_2 - f_0 + f_{1,1}f_{2,2}.$$ 

Let $S = \{ \lambda \in \mathbb{F}_{q^2} | P(\lambda) = 0 \}$ denote the set of zeroes of $P(X_2)$. Note that $\# S \leq q + 1$. Let $\alpha \in S$. In this case $P(\alpha) = 0$. Lemma \ref{lem:univ} implies that there are $2q - 1$ values of $X_1$ and $X_3$ such that the equation

$$(X_1 + f_{2,2})(X_3 + f_{1,1}) = P(\alpha)$$
is satisfied. This implies both sides are 0 for exactly \#S(2q - 1) values.

Now assume \( \alpha \in \mathbb{F}_{q^2} \setminus S \). In this case Lemma 8 implies that there are \( q - 1 \) values of \( X_1 \) and \( X_3 \) such that the equation

\[
(X_1 + f_{2,2})(X_3 + f_{1,1}) = P(\alpha)
\]

is satisfied. This implies there are \((q^2 - \#S)(q - 1)\) solutions to the equation where \( P(\alpha) \neq 0 \).

Therefore there are

\[\#S(2q - 1) + (q^2 - \#S)(q - 1)\]

elements of \( \mathbb{H}^\ell(\mathbb{F}_{q^2}) \) such that \( f = 0 \). As \( \#S \leq q + 1 \), we have that \( f \) has at most

\[(q + 1)(2q - 1) + (q^2 - q - 1)(q - 1) = q^3 + q\]

Consequently, \( w(f) \geq q^4 - q^3 - q \).

It is very useful to classify the codewords in \( C^k(\ell, 2\ell) \) for \( \ell = 2 \) where the \( 2 \times 2 \) determinant is in \( \text{supp}(f) \).

**Lemma 13.** Let \( \ell = 2 \). Suppose \( f \in \mathcal{F}(\ell) \) where \( f \) is of the form

\[f = f_0 + f_{1,1}X_1 + f_{1,2}X_2 + f_{1,3}^qX_2^q + f_{2,2}X_3 + X_1X_3 - X_2^{q+1}.\]

The following statements are true:

- If \( f_0 + f_{1,2}^{q+1} - f_{1,1}f_{2,2} = 0 \) then \( \text{wt}(f) = q^4 - q^3 + q^2 - q \).
- If \( f_0 + f_{1,2}^{q+1} - f_{1,1}f_{2,2} \neq 0 \) then \( \text{wt}(f) = q^4 - q^3 - q \).

**Proof.** Suppose \( f \) is of the form

\[f = f_0 + f_{1,1}X_1 + f_{1,2}X_2 + f_{1,3}^qX_2^q + f_{2,2}X_3 + X_1X_3 - X_2^{q+1}.\]

As in the proof of Lemma 12 we write \( f \) as

\[f = (X_1 + f_{2,2})(X_3 + f_{1,1}) + P(X_2)\]

where

\[P(X_2) = X_2^{q+1} + f_{1,2}^qX_2^q + f_{1,2}X_2 + f_0 - f_{1,1}f_{2,2}\]

is a polynomial of degree \( q + 1 \).

Now we shall change the variable \( X_2 \) to the variable \( T_2 \) where \( X_2 = T_2 - f_{1,2}^q \). In this case

\[P(T_2 - f_{1,2}^q) = (T_2 - f_{1,2}^q)^{q+1} + f_{1,2}^q(T_2 - f_{1,2}^q)^q + f_{1,2}(T_2 - f_{1,2}^q) + f_0 - f_{1,1}f_{2,2}.\]

Expanding and eliminating like terms we obtain

\[P(T_2 - f_{1,2}^q) = T_2^{q+1} - f_{1,2}^{q+1} + f_0 - f_{1,1}f_{2,2}.\]

Note that if \( f_{1,2}^{q+1} - f_0 + f_{1,1}f_{2,2} = 0 \), then \( P(T_2 - f_{1,2}^q) \) has exactly one zero, namely \( T_2 = f_{1,2}^q \).

Since \( f = f_{\text{conjugate}} \) we assume \( f_0, f_{1,1}, f_{2,2} \) are in \( \mathbb{F}_q \). As \( f_{1,2}^{q+1} - f_0 + f_{1,1}f_{2,2} \neq 0 \) it assumes values over \( \mathbb{F}_q \). Hence \( P(T_2 - f_{1,2}^q) \) has \( q + 1 \) zeroes over \( \mathbb{F}_q \). Note that the number of zeroes of \( P(T_2 - f_{1,2}^q) \) is precisely the same number of solutions to \( P(X_2) = 0 \)

When \( P(X_2) = 0 \), there are \( 2q - 1 \) values of \( X_1 \) and \( X_3 \) which make \( f = 0 \). When \( P(X_2) \neq 0 \), there are \( q - 1 \) values of \( X_1 \) and \( X_3 \) which make \( f = 0 \).

Therefore, \( f = 0 \) for \( (2q - 1)(q - 1) = q^3 - q^2 - q + 1 + 2q - 1 = q^3 - q^2 + q \) and \( \text{wt}(f) = q^4 - q^3 + q^2 - q \).

If \( f_{1,2}^{q+1} - f_0 + f_{1,1}f_{2,2} \neq 0 \), then there are \( q + 1 \) values of \( X_2 \) which make \( P(X_2) = 0 \). There are \( 2q - 1 \) values of \( X_1 \) and \( X_3 \) which make \( f = 0 \). When \( P(X_2) \neq 0 \), there are \( q - 1 \) values of \( X_1 \) and \( X_3 \) which make the equation true. Therefore \( f = 0 \) for \( (2q - 1)(q + 1) + (q^2 - q - 1)(q - 1) = q^3 + q \) and \( \text{wt}(f) = q^4 - q^3 - q \). 

\[\square\]
3.2 Calculating \( d(C_{\mathbb{F}}(3)) \)

The technique used in [1] to find the minimum distance of \( C^A(\ell, 2\ell) \) was to specialize from the \( \ell \times \ell \) case down to the \( (\ell - 1) \times (\ell - 1) \) case. As the matrices in the affine Grassmann code are generic, one can perform a partial evaluation on any row and any column while still preserving the structure of the code \( C^A(\ell, 2\ell) \). In the case of Hermitian matrices, the Hermitian property of the matrices must be preserved. This means that a partial evaluation on a column also fixes the corresponding row. In order to refine the concept of the size of a minor and to simplify our induction proof we introduce the following definition.

**Definition 20.** Let \( I, J \subseteq [\ell] \). We define the spread of the minor \( X^I,J \) as the set \( I \cup J \).

Consider the matrix \( X = \begin{bmatrix} X_1 & X_2 & Y_1 \\ X_2^T & Y_2 & Y_3 \\ Y_1^T & Y_2^T & Y_3 \end{bmatrix} \) and the minor given by rows \( \{1, 2\} \) and columns \( \{2, 3\} \). That is the minor \( f = \det_{\{1, 2\},\{2,3\}}(X) = \begin{bmatrix} X_2 & Y_1 \\ X_3 & Y_2 \end{bmatrix} \). The spread of the minor is \( I \cup J = \{1, 2\} \cup \{2, 3\} = \{1, 2, 3\} \).

The following lemma will prove that in several cases we can view the \( \ell = 3 \) case as several \( \ell = 2 \) cases.

**Lemma 14.** Let \( f \in \mathcal{F}(\ell) \). Suppose that \( \ell = 3 \) and \( f \) has a maximal minor whose spread has size \( \leq 2 \). Then \( \text{wt}(f) \geq q^3 - q^2 - q \).

**Proof.** Let \( f \) be as in the statement of the lemma. There is a row and a column which does not appear in the spread of the maximal minor. Then for any of the \( q^5 \) values one can put on this column, \( f \) specializes to a combination of \( 2 \times 2 \) determinants with the same maximal minors (though others may be changed due to the specialization). As each specialization has weight at least \( q^3 - q^2 - q \) and there are \( q^2 \) specializations, the statement follows.

To calculate the distance of our code, we need to count the invertible Hermitian matrices. We shall use group actions with the following sets:

- The set of \( \ell \times \ell \) Hermitian matrices
  \[ HL_\ell(\mathbb{F}_{q^2}) = \{ H \in \mathbb{H}^{\ell \times \ell}(\mathbb{F}_{q^2}) | \det(H) \neq 0 \} \]
- The General Linear group over \( \mathbb{F}_{q^2} \),
  \[ GL_\ell(\mathbb{F}_{q^2}) = \{ M \in \mathbb{M}^{\ell \times \ell}(\mathbb{F}_{q^2}) | \det(M) \neq 0 \} \]
- The group of Unirary matrices over \( \mathbb{F}_{q^2} \),
  \[ U_\ell(\mathbb{F}_{q^2}) = \{ U \in GL_\ell(\mathbb{F}_{q^2}) | U \times U^* = I \} \]

The cardinality of the classic finite groups is known:

- \[ \#GL_\ell(\mathbb{F}_{q^2}) = q^{2\binom{\ell}{2}} \prod_{i=1}^{\ell} (q^{2i} - 1) \]
- \[ \#U_\ell(\mathbb{F}_{q^2}) = q^{\binom{\ell}{2}} \prod_{i=1}^{\ell} (q^{i} - (-1)^i) \]

**Proposition 4.** The cardinality of \( HL_\ell(\mathbb{F}_{q^2}) = \{ H \in \mathbb{H}^{\ell \times \ell}(\mathbb{F}_{q}) | \det(H) \neq 0 \} \) is given by: \( q^{\binom{\ell}{2}} \prod_{i=1}^{\ell} (q^{i} + (-1)^i) \).
Proof. Recall that $GL_\ell(F_{q^2})$ acts upon $HL_\ell(F_{q^2})$ under the group action defined as $G \cdot A = G^* A G$. By [9], this action has exactly one orbit. Moreover, the orbit stabilizer, that is the subgroup that fixes the identity, is $U_\ell(F_{q^2})$. By the Orbit-Stabilizer Theorem:

$$#HL_\ell(F_{q^2}) = #GL_\ell(F_{q^2}) / #U_\ell(F_{q^2})$$

(1)

$$= q^{2\ell} \prod_{i=1}^{\ell}(q^{2i} - 1)$$

(2)

$$= q^{2\ell} \prod_{i=1}^{\ell}(q^i - 1)$$

(3)

$$= q^{2\ell} \prod_{i=1}^{\ell}(q^i + 1)(q^i - 1)$$

(4)

Now we shall assume $X$ is of the form

$$X = \begin{bmatrix} X_1 & X_2 & Y_1 \\ X_2 & X_3 & Y_2 \\ Y_1 & Y_2 & Y_3 \end{bmatrix},$$

where $X_1, X_3, Y_2$ are variables of elements in $F_q$ and $X_2, Y_1, Y_3$ are variables of elements in $F_{q^2}$. Now we shall study the $q^5$ possible specializations of $Y_1, Y_2, Y_3$ and their effect on $f \in F(\ell)$ on the remaining unspecialized $2 \times 2$ submatrix. We remind the reader of the following minor expansion:

Proposition 5.

$$det(X) = det_{\{1,2,1\}}(X)Y_3 - det_{\{1,3,1\}}(X)Y_2 + det_{\{2,3,1\}}(X)Y_1$$

Let us denote by $f_{a,b,c}(X)$ the minor combination obtained by the partial evaluation of $f(X)$ at $Y_1 = a$, $Y_2 = b$ and $Y_3 = c$.

Lemma 15. Let $f \in F(\ell)$. Suppose that for all $q^5$ partial evaluations we have that $f_{a,b,c}(X) \neq 0$. Then $wt(f) \geq q^9 - q^8 - q^6$

Proof. Suppose that $f$ is as in the hypothesis of the lemma. Then for any given partial evaluation, $f_{a,b,c}(X)$ is a nonzero combination of minors in the $2 \times 2$ case. As $wt(f_{a,b,c}) \geq q^4 - q^3 - q$ and all $q^5$ combinations are nonzero, the result follows.

We are now ready to find $wt(f)$ for $\ell = 3$. We begin with the full $3 \times 3$ determinant first.

Lemma 16. Let $f \in F(\ell)$ where $f = f_{1,2,3}$. Suppose that the maximal minor of $f$ is the full determinant $det_{\{1,2,3\},\{1,2,3\}}(X)$. Then we may assume $f$ is of the form $det_{\{1,2,3\},\{1,2,3\}}(X) + a_1X_{1,1} + a_2X_{2,2} + a_3X_{3,3} + a_4$.

Proof. Recall $f_{\{1,2,3\},\{1,2,3\}} \neq 0$ implies we may assume without loss of generality that $f_{\{1,2,3\},\{1,2,3\}} = 1$. Then, Lemma [14] implies that we may assume without loss of generality that $f$ has no $2 \times 2$ minors. As $f = f_{conj}$ we have that $f_{i,j} = f'_{i,j}$. Moreover, we may express $f_{i,j}X_a + f'_{i,j}X_a^\ell$ as $Tr(f_{i,j}X_a)$. This implies that $f$ is of the form:

$$\begin{bmatrix} X_1 & X_2 & Y_1 \\ X_2 & X_3 & Y_2 \\ Y_1 & Y_2 & Y_3 \end{bmatrix} + f_{1,1,1}X_1 + Tr(f_{1,2}X_2) + f_{2,2}X_3 + Tr(f_{1,3}Y_1) + Tr(f_{2,3}Y_2) + f_{3,3}Y_3 + f_0$$
We change the matrix of variables where Lemma 16 implies

\[ f(X) = \det_{\{1,2,3\},\{1,2,3\}}(X) + \text{Tr}(F^T X) + f_0. \]

If \( F \) is a Hermitian matrix of rank \( r \), there exists a nonsingular matrix \( A \) such that \( A^{(q)^r} X A = F \). We rewrite

\[ f(X) = \det_{\{1,2,3\},\{1,2,3\}}(X) + \text{Tr}(A^{(q)^r} X A) + f_0. \]

Recall that any nonsingular matrix \( A^{-1} \) induces a permutation of \( \mathbb{F}_q^r \) via the map \( H \mapsto A^{-1} H (A^{-1})^{(q)^t} \). We change the matrix of variables \( X \) to \( Y \) where \( X = (A^{-1}) Y (A^{-1})^{(q)^t} \).

In this case \( f \) becomes

\[ f(Y) = \frac{1}{a^{q+1}} \det_{\{1,2,3\},\{1,2,3\}}(Y) + \text{Tr}(A^{(q)^t} Y A (A^{-1})(A^{-1})^{(q)^t}) + f_0. \]

Which in turn it equals

\[ f(Y) = \frac{1}{a^{q+1}} \det_{\{1,2,3\},\{1,2,3\}}(Y) + \text{Tr}(A^{(q)^t} Y (A^{-1})(A^{-1})^{(q)^t}) + f_0. \]

As the matrix \( A^{(q)^t} Y (A^{-1})(A^{-1})^{(q)^t} \) is similar to \( I_r Y \), we have that

\[ f(Y) = \frac{1}{a^{q+1}} \det_{\{1,2,3\},\{1,2,3\}}(Y) + \text{Tr}(I_r Y) + f_0 \]

and the result follows. \( \Box \)

**Lemma 17.** Let \( f \in F(\ell) \). Suppose \( f = f_{\text{conj}} \). Suppose that the maximal minor of \( f \) is the full determinant \( \det_{\{1,2,3\},\{1,2,3\}}(X) \). Then

\[ \text{wt}(f) \geq q^9 - q^8 + q^6 - q^4 + q^3 \]

**Proof.** Lemma 16 implies \( f \) is of the form

\[ \det_{\{1,2,3\},\{1,2,3\}}(X) + f_{1,1} X_1 + f_{2,2} X_3 + f_{3,3} Y_3 + f_0. \]

We shall consider what happens when we evaluate \( f \) along the third row and column.

Suppose now that \( f_{a,b,c} \) is the partial evaluation of \( f \) with \( Y_1 = a \), \( Y_2 = b \) and \( Y_3 = c \neq 0 \). Then \( f_{a,b,c} \) looks as such:

\[ c(X_1 X_3 - X_2^{q+1}) + \text{Tr}(a^{q} b X_2) - b^{q+1} X_1 - a^{q+1} X_3 + f_{1,1} X_1 + f_{2,2} X_3 + f_{3,3} c + f_0. \]

Now we shall apply Lemma 13 to determine the weight of each partial evaluation. Lemma 13 implies that if the coefficients of \( f_{a,b,c} \) satisfy

\[ \binom{a^{q} b}{c}^{q+1} - \frac{f_{1,1} - b^{q+1}}{c} f_{2,2} - \frac{a^{q+1}}{c} f_{3,3} + \frac{f_0}{c} = 0, \]

where

- Coefficient of \( X_1 \): \( \frac{f_{1,1} - b^{q+1}}{c} \)
- Coefficient of \( X_2 \): \( \binom{a^{q} b}{c} \)
- Coefficient of \( X_3 \): \( \binom{a^{q} b}{c}^{q} \)
The number of such matrices where

\( f(a, X) \)

then the partial evaluation has weight \( q(q - 1)(q^2 + 1) \), and otherwise it has weight \( q^4 - q^3 - q \).

After evaluating the parenthesis, and multiplying by \( c^2 \) we obtain

\[
a^{q+1}b^{q+1} - (f_{1,1} - b^{q+1})(f_{2,2} - a^{q+1}) + c^2f_{3,3} + cf_0 = 0
\]

or

\[
-f_{1,1}f_{2,2} + f_{1,1}a^{q+1} + f_{2,2}b^{q+1} + c^2f_{3,3} + cf_0 = 0.
\]

Now we shall consider the different cases, depending on if \( f_{1,1} = 0 \) or \( f_{1,1} \neq 0 \).

If \( f_{1,1} = 0 \), then Lemma 10 implies we may assume \( f_{2,2} = f_{3,3} = 0 \) too. In this case \( f = det_{\{1,2,3\},\{1,2,3\}}(X) + f_0 = 0 \).

If \( f_0 = 0 \), by Proposition 14 we have

\[
wt(f) = q^3(q - 1)(q^2 + 1)(q^3 - 1) = q^9 - q^8 + q^7 - 2q^6 - q^4 + q^3.
\]

If \( f_0 \neq 0 \), we count the \( 3 \times 3 \) Hermitian matrices such that \( det_{\{1,2,3\},\{1,2,3\}}(H) = -f_0 \neq 0 \) for \( f_0 \in \mathbb{F}_q \). Recall that, by hypothesis, \( f \) is self–conjugate which implies \( f_0 = f_0^q \). Recall that the number of \( 3 \times 3 \) Hermitian matrices with \( det_{\{1,2,3\},\{1,2,3\}}(H) \neq 0 \) is

\[
(q - 1)(q^2 + 1)(q^3 - 1)q^3.
\]

The number of such matrices where \( det_{\{1,2,3\},\{1,2,3\}}(H) = -f_0 \) is \( (q^2 + 1)(q^3 - 1)q^3 \) which implies then

\[
wt(f) = q^9 - q^3(q^2 + 1)(q^3 - 1) = q^9 - q^8 - q^4 + q^3.
\]

There is always one value of \( a \) and \( b \) which make

If \( f_{1,1} \neq 0 \), but \( f_{2,2} = f_{3,3} = 0 \), if \( f_0 = 0 \) there is one value of \( a \) such that

\[
f_{1,1}f_{2,2} + f_{1,1}a^{q+1} + f_{2,2}b^{q+1} + c^2f_{3,3} + cf_0 = 0.
\]

In this case there are \( (q - 1)q^2(q^2 - 1) \) partial evaluations of weight \( q^4 - q^3 - q \) and \( (q - 1)q^2 \) partial evaluations of weight \( q(q - 1)(q^2 + 1) \).

If \( f_0 \neq 0 \) there are \( q + 1 \) values of \( a \) such that

\[
f_{1,1}f_{2,2} + f_{1,1}a^{q+1} + f_{2,2}b^{q+1} + c^2f_{3,3} + cf_0 = 0.
\]

In this case there are \( (q - 1)q^2(q^2 - 1) \) partial evaluations of weight \( q^4 - q^3 - q \) and \( (q - 1)q^2(q + 1) \) partial evaluations of weight \( q(q - 1)(q^2 + 1) \).

Now we shall count the number of zeroes of \( f \) when specializing \( X_{3,3} = 0 \). The specialization \( X_{1,3} = a, X_{2,3} = b, X_{3,3} = 0 \) is of the form

\[
f_{a,b,0}(X) = \begin{bmatrix} X_1 & X_2 & a \\ X_2^q & X_3 & b \\ a^{q+1} & b^q & 0 \end{bmatrix} + f_{1,1}X_1 + f_{2,2}X_3 + f_{3,3}(0) + f_0.
\]

Expanding the \( 3 \times 3 \) determinant we obtain:

\[
f_{a,b,0}(X) = a^q bX_2 + ab^q X_3^q - a^{q+1}X_3 - b^{q+1}X_1 + f_{1,1}X_1 + f_{2,2}X_3 + f_0.
\]

Note that the coefficient of the \( (1,1) \)–minor of the partial specialization \( f_{a,b,0} \) is \( f_{1,1} - b^{q+1} \), the coefficient of the \( (2,2) \)–minor is \( f_{2,2} - a^{q+1} \), and the coefficients of the \( (1,2) \)–minor and the \( (2,1) \)–minor are \( a^{q}b \) and \( ab^q \) respectively. Lemma 14 implies there are at most \( q + 1 \) partial specializations such that all three coefficients are 0. Therefore, for \( q^4 - q - 1 \) specializations we get a nonzero polynomial with at most \( q^2 \) zeroes.

In the previous cases we obtain
\[ \text{wt}(f) \geq (q-1)q^2(q^2-1)(q^4-q^3-q) + (q-1)q^2(q(q-1)(q^2+1)) + (q^4-q-1)(q^4-q^3) \]
\[ \text{wt}(f) \geq q^9 - q^8 - q^6 + q^5 - q^4 + q^3 \]

\[ \square \]

**Lemma 18.** Let \( f \in \mathcal{F}(\ell) \). Suppose that \( f \) has no maximal \( 3 \times 3 \) minors in its support. Nor that it has a \( 2 \times 2 \) principal minor in its support. It does have \( 2 \times 2 \) nonprincipal minors. Moreover, \( f = f_{\text{conj}} \). Then \( \text{wt}(f) \geq q^9 - q^8 - q^6 + q^5 \).

**Proof.** We may assume that \( f_{0,0,0} = 0 \) and that all determinants in \( \text{supp}(f) \) contain either row 3 or column 3. For any partial specialization \( f_{a,b,c} \) we may assume without loss of generality that \( f_{a,b,c}(X) \) is a polynomial with terms \( X_1, X_2, X_3 \) and \( X_3 \) (and a constant over \( \mathbb{F}_q \)). Without loss of generality, we assume \( f_{\{1,2\},\{2,3\}} \neq 0 \)

- Coefficient of \( X_1 \): \( f_{\{1,2\},\{2,3\}}b + f_{\{1,2\},\{2,3\}}b^q \)
- Coefficient of \( X_2 \): \( f_{\{1,2\},\{2,3\}}a + f_{\{1,2\},\{2,3\}}b + f_{\{1,3\},\{2,3\}}c \)
- Coefficient of \( X_3 \): \( f_{\{1,2\},\{2,3\}}a - f_{\{1,2\},\{2,3\}}a^q \)

We shall compute \( \text{wt}(f) \) by splitting the specializations into two exclusive cases: \( \text{Tr}(f_{\{1,2\},\{2,3\}}b) \neq 0 \) and \( \text{Tr}(f_{\{1,2\},\{2,3\}}b) = 0 \)

**Case 1:** \( \text{Tr}(f_{\{1,2\},\{2,3\}}b) \neq 0 \):

If \( \text{Tr}(f_{\{1,2\},\{2,3\}}b) \neq 0 \), then for any of the \( q^6 \) specializations possible for \( X_2, X_3, a, c \), we can find \( q - 1 \) values of \( X_1 \) such that \( f \neq 0 \). Note we have \( q^2 - q \) values of \( b \) such that \( \text{Tr}(f_{\{1,2\},\{2,3\}}b) \neq 0 \) and we have \( q^6 \) remaining values for \( X_2, X_3, a, c \) and \( q - 1 \) values for \( X_1 \). This implies there are at least \( q^6(q^2 - q)(q - 1) \) nonzero values in case 1.

**Case 2:** \( \text{Tr}(f_{\{1,2\},\{2,3\}}b) = 0 \):

If \( \text{Tr}(f_{\{1,2\},\{2,3\}}b) = 0 \), there are \( q^2-1 \) values for \( a \), such that \( a \neq 0 \). Thus the polynomial \( \text{Tr}(f_{\{1,2\},\{2,3\}}bX_2) \neq 0 \) for \( q^2 - q \) values of \( X_2 \). This holds for any of the \( q^3 \) possible specialization of \( X_3, X_1, c \). For the \( q^2 - 1 \) nonzero values of \( a \), the partial specialization \( f_{a,b,c} \) is a polynomial of at most degree \( q \) in \( X_2 \). Thus, \( f_{a,b,c} \neq 0 \) for at least \( q^2 - q \) values of \( X_2 \). Therefore together with the \( q \) values of \( b \) where \( \text{Tr}(f_{\{1,2\},\{2,3\}}b) = 0 \), thus there are at least \( q^4(q^2 - 1)(q^2 - q) \) nonzero values in case 2.

Adding the two exclusive cases together where either \( \text{Tr}(f_{\{1,2\},\{2,3\}}b) = 0 \) or \( \text{Tr}(f_{\{1,2\},\{2,3\}}b) \neq 0 \) we get obtain:

\[ w(f) \geq q^9(q^2 - q)(q - 1) + q^4(q^2 - 1)(q^2 - q) \]
\[ w(f) \geq q^9 - q^8 - q^6 + q^5 \]

\[ \square \]

### 4 Finding \( d(C^{\ell_1}(\ell)) \) for \( \ell \geq 4 \)

Now we generalize some of the previous arguments for any \( \ell \). These generalizations will establish certain relations between \( C^{\ell_1}(\ell) \) and \( C^{\ell_1}(\ell + 1) \). These relations determine the minimum distance in the general case. First, we recall the following notation:

**Definition 21.** We denote the elementary matrix for row addition operations \( L_{i,j}(m) \) as the corresponding matrix obtained by adding \( m \) times row \( j \) to row \( i \).
Example:

Let $\ell = 3$, $L_{1,2}(\alpha) = \begin{bmatrix} 1 & \alpha & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

From the cases with $\ell = 2$ and $\ell = 3$, we see that the weight of a function $wt(ev(f))$ for $f \in \mathcal{F}(\ell)$ depends on three aspects: the ambient matrix space $H^\ell(F_\ell)$, the size of the maximal minor on $supp(f)$ and the spread of the maximal minor on $supp(f)$. We now introduce the following notation:

**Definition 22.** Denote by $w_{n,k,s}$ the minimum weight of a function $ev(f)$ such that $f$ is obtained by evaluating $n \times n$ matrices with maximal minor size $k \times k$ and the smallest spread size among all $k \times k$ minors in $supp(f)$ is $s$.

Instead of inducting on the degree of the determinant functions, as in the case of affine Grassmann codes $C^h(\ell, 2\ell)$, we will find the minimum distance of the code $C^H(\ell)$ by inducting on $k$ and then $s$. Now we generalize some previous lemmas using the new notation.

**Lemma 19.** Let $f \in \mathcal{F}(\ell)$. Suppose that $f$ has a maximal minor of size $k$ whose spread has size $s$. Then $wt(f) \geq q^{\ell^2-s^2}w_{s,k,s}$.

**Proof.** Let $f$ be as in the statement of the lemma. This implies there are $\ell - s$ rows and columns which do not appear in the spread of the maximal minor. Then for any of the $q^{\ell^2-s^2}$ values one can put on these columns, $f$ specializes to a combination of $s \times s$ determinants with the same maximal minors (though others may be changed due to the specialization). As each specialization has weight $w_{s,k,s}$ and there are $q^{\ell^2-s^2}$ specializations, the statement follows. \qed

Note the following behavior in the $3 \times 3$ case:

Let $X = \begin{bmatrix} X_1 & X_2 & Y_1 \\ X_3 & X_4 & Y_2 \\ Y_1 & Y_2 & Y_3 \end{bmatrix}$ and $f = det_{(1,2),(2,3)}(X) = \begin{vmatrix} X_2 & Y_1 \\ X_3 & Y_2 \end{vmatrix}$

Note that $f \in \mathcal{F}(\ell)$ is a minor of size $2$ and spread $3$. Also recall that the following map $X \mapsto A^{(q)T}X$ is an automorphism which preserves the weight of of a codeword, that is $wt(f(X)) = wt(f(A^{(q)T}X))$.

Let $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, the map $X \mapsto A^{(q)T}X$ is as follows:

\[
\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} X_1 & X_2 & Y_1 \\ X_3 & X_4 & Y_2 \\ Y_1 & Y_2 & Y_3 \end{bmatrix} = \begin{bmatrix} X_1 & X_2 & \dots \\ X_3 & X_4 & \dots \\ \dots & \dots & \dots \end{bmatrix}
\]

Then

\[
f = \begin{vmatrix} X_2 & X_1 + Y_1 \\ X_3 & X_2 + Y_2 \\ X_4 & X_3 + Y_3 \end{vmatrix} = \begin{vmatrix} X_2 & Y_1 \\ X_3 & Y_2 \end{vmatrix} = det_{(1,2),(2,3)}(X) - det_{(1,2),(1,2)}(X)
\]

Note that applying this map, $f(A^{(q)T}X)$ is combination of minors in $\mathcal{F}(\ell)$ containing a maximal minor of size $2$ and spread $2$. We may transform a function of a given size and spread to another function of the same weight with the same size but with smaller spread. The matrix $A$ we used in the transformation may be obtained from elementary matrices. In this example we used $L_{1,3}(1)$. In a more general sense, we state the following:

**Lemma 20.** Let $f \in \mathcal{F}(\ell)$. Suppose that $f$ has a maximal minor of minimal spread size, $M$ of size $k$ and spread $= s > k$. Then there exists $\tilde{f} \in \mathcal{F}(\ell)$ of the same weight with a maximal minor with size $k$ and spread size $\leq s - 1$.

**Proof.** Let $\mathcal{M} = det_{I,J}(X)$ be a maximal minor of size $k$ and spread size $s > k$ in the support of $f$. Applying a suitable permutation of rows and columns, Without loss of generality, we may assume

$I = \{k\}$ and $J = \{s-k+1, s-k+2, \ldots, s\}$.
Let $\lambda \in \mathbb{F}_q^*$ such that

$$-\lambda f_{I,J} - \lambda^q f_{I \cup \{s\} - \{1\}, J \cup \{1\} - \{s\}} \neq 0.$$ 

We take $L_{1,s}(\lambda) \in GL_\ell(\mathbb{F}_q)$. Consider the generic matrix $Y = L_{1,s}(\lambda^q)X L_{s,1}(\lambda)$. We shall prove that the codeword given by $\bar{f} = f(Y) = f(L_{1,s}(\lambda^q)X L_{s,1}(\lambda))$ has a maximal determinant with a smaller support. The multilinearity of the determinant implies that

$$\det_{I,J}(Y) = \det_{I,J}(X) - \det_{I,J \cup \{1\} - \{s\}}(X).$$

Note that the only minors of the form $\det_{A,B}(Y)$ such that $\det_{I,J \cup \{1\} - \{s\}}(X)$ may appear in are the following:

$$Q = \det_{I \cup \{s\} - \{1\}, J \cup \{1\} - \{s\}}(Y), \quad P = \det_{I \cup \{s\} - \{1\}, J}(Y).$$

Note the spread of $P$ has size $s - 1$, which contradicts that $M$ is of minimal spread size. This implies $f_{I \cup \{s\} - \{1\}, J} = 0$. This implies we only need to worry about $Q$ and its corresponding $f_{I \cup \{s\} - \{1\}, J \cup \{1\} - \{s\}}$. Thus the coefficient of $\det_{I,J \cup \{1\} - \{s\}}(X)$ in $\bar{f}$ is $-\lambda f_{I,J} - \lambda^q f_{I \cup \{s\} - \{1\}, J \cup \{1\} - \{s\}} \neq 0$. Therefore $N = \det_{I,J \cup \{1\} - \{s\}}(X) \in \text{supp}(f)$ where $N$ is size $k$ and has spread $s - 1$.

As a direct consequence of the previous lemma, we obtain the following corollaries:

**Corollary 21.** Let $f \in \mathcal{F}(\ell)$. Suppose that $f$ has a maximal minor of minimal spread, $\mathcal{M}$ of size $k$ and spread $= s > k$. Then we may find $\bar{f}$ of the same weight with a maximal minor of minimal spread, $\mathcal{N}$ with size $k$ and spread $k$.

**Proof.** Simply apply Lemma 20 repeatedly.

**Corollary 22.** Let $f \in \mathcal{F}(\ell)$. Suppose that $f$ has a maximal minor of size $k$ whose spread has size $s$. Then $wt(f) \geq q^{k^2-k^2}(w_{k,k,k})$.

**Proof.** Let $f$ have a maximal minor of size $k$ whose spread has size $s$. By Corollary 21 we may apply an automorphism such that we transform $f$ to a function of the same weight with a maximal minor of minimal spread, with size $k$ and spread $k$. By Lemma 19 we may then state that $wt(f) \geq q^{k^2-k^2}(w_{k,k,k})$.

### 4.1 Finding $w_{n,k}$ via induction

Having determined $w_{2,2}$, $w_{3,2}$ and $w_{3,3}$ (the base cases for $2 \leq \ell \leq 3$), we shall now calculate $w_{k,k}$ for general $\ell$. Note that by Corollary 21 we may assume that $f$ has a maximal minor of size $k$ whose spread is $k$. This implies we may assume we have a principal $k \times k$ minor in its support.

Therefore we shall define $w_{n,k}$.

**Definition 23.** $w_{n,k}$ denotes the minimum weight of a function $f \in \mathcal{F}(\ell)$ evaluated on $n \times n$ matrices, where $\text{supp}(f)$ has a maximal minor of size $k \times k$.

**Lemma 23.** $w_{k,k} \geq q^{k^2} - q^{k^2-1} - q^{k^2-3} + q^{k^2-2k} - 1$

**Proof.** In Lemma 13 and Lemma 17 the base cases for $k = 2$ and $k = 3$ are established. We assume the statement of the lemma is true for $2 \leq k \leq K$ and prove that the statement holds for $K + 1$. Without loss of generality we may assume $f$ has a maximal $K + 1 \times K + 1$ principal minor in its support.

As we did in the argument for the base cases, we specialize $\det_{[K+1], [K+1]}(X)$ along the $(K+1)-th$ column. Note there are exactly $q^{2(K+1)-1} - q^{2(K+1)-2}$ values for the $(K+1) - th$ column such that $x_{K+1,K+1} \neq -f_{[K], [K]}$. This leaves us with a non-trivial combination in the $K$ case and with a $K \times K$ minor in its support. Therefore, there are $q^{2(K+1)-1} - q^{2(K+1)-2}$ values for the specialization of the $(K+1) - th$ row and column where we specialize into the case $w_{K,K}$. These specializations contribute $(q^{2(K+1)-1} - q^{2(K+1)-2})w_{K,K}$ to $w_{K+1,K+1}$. 18
We shall now consider the specialization where we do not obtain such a $K \times K$ maximal minor. This is the exclusive case where $x_{K+1,K+1} = -f_{[K],[K]}$. Hence we may assume there is no $K \times K$ minor in its support. Now we consider the possibilities for the $K-1 \times K-1$ minors.

Note that all such minors are of the form $\det_{[K+1]-\{i,K+1\}}[K+1]-\{j,K+1\}(X)$. Note that if the minor given by $[K+1]-\{i,K+1\},[K+1]-\{j,K+1\}$ does not appear in the partial evaluation, then the coefficients of the partial evaluation must satisfy the equation

$$f_{[K+1]-\{i,K+1\},[K+1]-\{j,K+1\}} = x_{i,K+1}x_{j,K+1}q.$$

Lemma \ref{lem:bound} implies that the system of polynomial equations as stated above has at most $q+1$ solutions. Thus there are at least $q^{2(K+1)-2} - q - 1$ values for the partial evaluation on the $(K+1)-th$ column such that we have a non-trivial combination with a $K-1 \times K-1$ minor in its support. These specializations contribute $q^{2K-1}(w_{K-1,K-1})$ to $w_{K+1,K+1}$. We put together all the inequalities and we obtain:

$$w_{K+1,K+1} \geq (q^{2(K+1)-1} - q^{2(K+1)-2})w_{K,K} + (q^{2(K+1)-2} - q - 1)q^{2(K+1)-3}w_{K-1,K-1}$$

$$= (q^{2K+1} - q^{2K})w_{K,K} + (q^{2K} - q - 1)(q^{2K-1})w_{K-1,K-1}$$

$$\geq (q^{2K+1} - q^{2K})(q^{K^2} - q^{K^2-1} - q^{K^2-3} + q^{K^2-2K-1}) + (q^{2K} - q - 1)(q^{2K-1} - q^{2K-2K-2} + q^{2K-2} - q^{2K-4}) + 1$$

$$\geq q^{K^2+2K+1} - q^{K^2+2K} - q^{K^2+2K-2} + q^{K^2+1} - q^{2K-1}$$

$$+ q^{K^2-2} + q^{K^2-3} - q^{K^2-2K+2} - q^{2K-1} + 2q^{2K} - q^{2K-1}$$

$$\geq q^{K^2+2K+1} - q^{K^2+2K} - q^{K^2+2K-2} + q^{K^2-1} - 1$$

$$\geq q^{(K+1)^2} - q^{(K+1)^2-1} - q^{(K+1)^2-3} + q^{(K+1)^2-2(K+1)} - 1$$

Therefore, by the principle of strong mathematical induction, the bound is met.

\[\square\]

**Proposition 6.** $w_{k,k} \geq q^k - q^{k^2-1} - q^{k^2-3}$

**Proof.** By Lemma \ref{lem:bound} we have $w_{k,k} \geq q^k - q^{k^2-1} - q^{k^2-3} + q^{k^2-2k} - 1$. Note that for $k \geq 2$ we have $q^k - q^{k^2-1} - q^{k^2-3} + q^{k^2-2k} - 1 \geq q^k - q^{k^2-1} - q^{k^2-3}$. In fact, the equality is met for $k = 2$. Therefore:

$$w_{k,k} \geq q^k - q^{k^2-1} - q^{k^2-3}$$

\[\square\]

Now we are finally ready to prove the main result of this paper

**Theorem 24.** Suppose that $\ell \geq 2$. Then

$$d(C_{\ell}^G) = q^\ell - q^{\ell^2-1} - q^{\ell^2-3}.$$ 

**Proof.** Assume $\ell \geq 2$. Let $f \in F(\ell)$ with a maximal minor of size $k$ and spread $s$. By Corollary \ref{cor:bound} $wt(f) \geq q^{s-k^2}(w_{k,k,k})$. By Proposition \ref{prop:bound}

$$w_{k,k,k} \geq q^k - q^{k^2-1} - q^{k^2-3}.$$ 

Taken together

$$wt(f) \geq q^{s-k^2}(q^k - q^{k^2-1} - q^{k^2-3}) = q^\ell - q^{\ell^2-1} - q^{\ell^2-3}.$$ 

This implies $d(C_{\ell}^G) \geq q^\ell - q^{\ell^2-1} - q^{\ell^2-3}$. Equality is obtained by

$$f = \det_{\{1,2\},\{1,2\}}(X) + 1.$$ 

\[\square\]
Corollary 25. Let $\ell \geq 2$, $d(C_{\ell}^S(\ell)) \geq d(C_{\ell}(\ell, 2\ell))$. That is

$$q^{2\ell} - q^{2\ell-1} - q^{2\ell-3} \geq \prod_{i=0}^{\ell-1} (q^i - q)$$

Proof. Note that $q^{2\ell} - q^{2\ell-1} - q^{2\ell-3} \geq (q^2 - 1)(q^2 - q)q^{2\ell-4} \geq \prod_{i=0}^{\ell-1} (q^i - q)$.

The most remarkable fact about the affine Hermitian Grassman code is that it has better minimum distance than the affine Grassmann code while having the same length and dimension. In the tables below, we compare the parameters of these codes for $\ell = 2$ and $\ell = 3$.

$$
\begin{array}{|c|c|c|c|c|c|}
\hline
q & n & k & d(C_{\ell}(2,4)) & d(C_{\ell}(2)) \\
\hline
2 & 16 & 6 & 6 & 6 \\
3 & 81 & 6 & 48 & 51 \\
4 & 256 & 6 & 180 & 188 \\
5 & 625 & 6 & 480 & 495 \\
7 & 2,401 & 6 & 2,016 & 2,051 \\
8 & 4,096 & 6 & 3,528 & 3,576 \\
9 & 6,561 & 6 & 5,760 & 5,823 \\
\hline
\end{array}
\begin{array}{|c|c|c|c|c|}
\hline
q & n & k & d(C_{\ell}(3,6)) & d(C_{\ell}(3)) \\
\hline
2 & 512 & 20 & 168 & 192 \\
3 & 19,683 & 20 & 11,232 & 12,393 \\
4 & 262,144 & 20 & 181,440 & 192,512 \\
5 & 1,953,125 & 20 & 1,488,000 & 1,546,875 \\
7 & 40,353,607 & 20 & 33,784,128 & 34,471,157 \\
8 & 134,217,728 & 20 & 115,379,712 & 117,178,368 \\
9 & 387,420,489 & 20 & 339,655,680 & 343,842,327 \\
\hline
\end{array}
$$

## 5 Dual Code

In this section we study some properties of the dual code $C_{\ell}^S(\ell)^\perp$ and its relation to the dual affine Grassmann code $C^A(\ell, 2\ell)^\perp$. In particular, we prove that the minimum distance codewords of the dual codes are similar for both the dual of affine Grassmann codes and dual affine Hermitian Grassmann codes. We begin by defining a dual code. Please recall the following:

**Definition 24.** Let $C$ be an $[n, k]$ linear code, then its dual code $C^\perp$ is its orthogonal complement as a vector space. That is, $C^\perp$ is an $[n, n-k]$ code such that:

$$C^\perp := \{ h \in \mathbb{F}_q^n \mid \sum_{i=1}^{n} c_i h_i = 0 \ \forall c \in C \}$$

Let $C^H(\ell)^\perp$ denote the Dual of the affine Hermitian Grassmann code. Recall that

$$C^H(\ell)^\perp := \{ h \in \mathbb{F}_{q^2}^{H(\mathbb{F}_q^2)} \mid \sum_{H \in H(\mathbb{F}_q^2)} h_H f(H) = 0 \ \forall f \in \mathcal{F}(\ell) \}$$

Then we know the following

$C^H(\ell)^\perp$ is an $[q^{2\ell}, q^{2\ell} - (\ell^2)]$ code and $C^H(\ell)^\perp = C^H(\ell)^\perp(q)$. The latter fact is due to \[11\].

In fact, we know more about the minimum distance of $C^H(\ell, 2\ell)^\perp$.

**Proposition 7.** [Theorem 17]

Let $\ell \geq 2$. The minimum distance $d(C^H(\ell, 2\ell)^\perp)$ of the code $C^H(\ell, 2\ell)^\perp$ satisfies:

$$d(C^H(\ell, 2\ell)^\perp) = \begin{cases} 3 & q > 2 \\ 4 & q = 2 \end{cases}$$

In subsequent work, one of the named authors along with P. Beelen characterized the minimum distance codewords of $C^H(\ell, 2\ell)^\perp$.

**Definition 25.** Let $f \in C^H(\ell)^\perp$ supp($f$) = $\{ H \in H \mid c_H \neq 0 \}$

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Conversely, given are 0. That is, nonzero coordinates satisfy support $C$.

Theorem 26. Let $k \leq \ell$, we denote $I_k$ as the matrix with an $k \times k$ identity block and the remaining entries are 0. That is, $a_{i,j} = 1$ if $1 \leq i \leq k$.

Definition 27. We denote $E_{i,j}$ to be the $\ell \times \ell$ matrix which all entries equal 0 except the $(i,j)$ - th entry which equals 1.

Proposition 8. [2 Theorem 8] Let $\ell \geq 2$, let $q > 2$ and let $c \in C^k(\ell,2\ell)^\perp$ be a weight 3 codeword with support $\text{supp}(c) = \{N_1,N_2,N_3\}$. Then there exists an automorphism such that we may map $c \rightarrow c'$ where $\text{supp}(c') = \{0,I_1,\alpha I_1\}$ and $\alpha = (\frac{c_{N_1} + c_{N_2}}{c_{N_1}})$.

Conversely, given $\alpha \in \mathbb{F}_q \setminus \{0,1\}$, there exists a codeword $c \in C^k(\ell,2\ell)^\perp$ with $\text{supp}(c) = \{0,I_1,\alpha I_1\}$. Its nonzero coordinates satisfy

$c_{I_1} = \frac{-\alpha}{\alpha - 1} c_0$ and $c_{\alpha I_1} = \frac{1}{\alpha - 1} c_0$

Proposition 9. [2 Theorem 15] Let $\ell \geq 2$. Let $q = 2$ and let $c$ be a codeword of $C^k(\ell,2\ell)^\perp$ of weight 4. Suppose that $\text{supp}(c) = \{M_1,M_2,M_3,M_4\}$. Then there exists an automorphism such that we may map $c \rightarrow c'$ where $\text{supp}(c')$ is one of the following:

i) $\{0,E_{1,1},E_{1,2},I_1 + E_{1,2}\}$

ii) $\{0,E_{1,1},E_{2,1},I_1 + E_{2,1}\}$

iii) $\{0,E_{1,1},E_{1,2} + E_{2,1},E_{1,1} + E_{1,2} + E_{2,1}\}$

Definition 28. Let $C$ be a linear code and $T$ be a set of coordinates in $C$. We define the puncturing of $C$ on $T$ as the resulting linear code $C^T$ from deleting all coordinates in $T$ in each codeword of $C$.

Definition 29. Let $C$ be a linear code, $T$ be a set of coordinates in $C$ and $C(T)$ the set of codewords which are 0 on $T$. We define the shortening of $C$ as the puncturing of $C(T)$ on $T$.

We remark that these code operations are duals of each other. That is, the dual code of puncturing $C$ is shortening $C^\perp$.

With the fact that $C^H(\ell)$ is a puncturing of $C^k(\ell,2\ell)$ on the positions outside of $H^\perp(F_q^2)$, and that $C^H(\ell)^\perp$ is the linear code obtained by shortening $C^k(\ell,2\ell)^\perp$ at the positions outside of $H^\perp(F_q^2)$, we shall determine the minimum distance of $C^H(\ell)^\perp$.

Theorem 26. Let $\ell \geq 2$. The minimum distance $d(C^H(\ell)^\perp)$ of the code $C^H(\ell)^\perp$ satisfies:

$$d(C^H(\ell)^\perp) = \begin{cases} 3 & q > 2 \\ 4 & q = 2 \end{cases}$$

Proof. Recall we are puncturing the code $C^k(\ell,2\ell)$ at the matrices in $M^{\ell \times \ell}(F_{q^2}) \setminus H^\perp(F_{q^2})$ to obtain $C^H(\ell)$.

This implies we are shortening the code $C^k(\ell,2\ell)^\perp$ to obtain $C^H(\ell)^\perp$. By [2], this implies we have a lower bound

$$d(C^H(\ell)^\perp) \geq \begin{cases} 3 & q > 2 \\ 4 & q = 2 \end{cases}$$

If $q > 2$, there exists $c \in C^k(\ell,2\ell)^\perp$ such that $\text{supp}(c) = \{0,E_{1,1},\alpha E_{1,1}\}$ and $\alpha \in \mathbb{F}_q, \alpha \neq 0, 1$. Because all 3 matrices are Hermitian, this implies that when shortening we have a codeword $c' \in C^H(\ell)^\perp$ such that $\text{supp}(c) = \text{supp}(c')$. Therefore, for $q > 2$, $d(C^H(\ell)^\perp) = 3$.

If $q = 2$, suppose by way of contradiction that $\#\text{supp}(c) = 3$. Let $A,B,C \in \text{supp}(c)$. This implies $c_A = c_B = c_C = 1$. This implies $c \cdot \mathbb{I} = c_A + c_B + c_C = 1 \neq 0$ Note $\mathbb{I} \notin C^H(\ell)$, which contradicts that $c \in C^H(\ell)^\perp$. Therefore $\#\text{supp}(c) \neq 3$.

Note there exists $c \in C^k(\ell,2\ell)^\perp$ such that the support set consists of the matrices $\text{supp}(c) = \{0,E_{1,1},E_{1,2} + E_{2,1},E_{1,2} + E_{2,1} + E_{1,1}\}$. Because all 4 matrices are Hermitian, this implies that when shortening we have a codeword $c' \in C^H(\ell)^\perp$ such that $\text{supp}(c) = \text{supp}(c')$. Therefore, for $q = 2$, $d(C^H(\ell)^\perp) = 4$. 

\[\square\]
As in the case for affine Grassmann codes, we characterize all minimum distance codewords of \( C^H(\ell) \). The case for \( q > 2 \) can be done simply by considering the shortening operation. As there are minimum distance codewords of \( C^H(\ell, 2\ell) \) over \( \mathbb{F}_2^2 \) whose support is entirely of Hermitian matrices. It is these codewords which are the minimum distance codewords of \( C^{\ell_2}(\ell) \). However, when \( q = 2 \), the corresponding code \( C^H(\ell, 2\ell) \) is actually a quaternionic code of distance 3. In this case, shortening from the affine Grassmannian into the affine Hermitian Grassmannian eliminates all codewords of weight 3 from \( C^{\ell_2}(\ell) \). Interestiongly, despite the code \( C^{\ell_2}(\ell) \) being quaternionic, it behaves similar to the binary case for the affine Grassmann codes. Remarkably, we characterize all the minimum distance codewords of \( C^{\ell_2}(\ell) \) with the same technique used in \([9]\).

**Lemma 27.** Let \( q = 2, \ell \geq 2 \) and \( c \in C^{\ell_2}(\ell) \) and \( \text{supp}(c) = \{ A, B, C, D \} \). Then there exists an automorphism \( \sigma \in \text{Aut}(C^{\ell_2}(\ell)) \) such that \( \text{supp}(\sigma(c)) = \{ M, N, P, 0 \} \).

*Proof.* Let \( A, B, C, D \in \text{supp}(c) \). Now we translate the support by \( D \) applying the automorphism \( f(X) \mapsto f(X + D) \). Then, let \( M = A + D, N = B + D \) and \( P = C + D, \text{supp}(\sigma(c)) = \{ M, N, P, 0 \} \).

**Lemma 28.** Let \( q = 2, \ell \geq 2 \) and \( c \in C^{\ell_2}(\ell) \) and \( \text{supp}(c) = \{ A, B, C, 0 \} \). Then there exists an automorphism \( \sigma \in \text{Aut}(C^{\ell_2}(\ell)) \) such that \( \text{supp}(\sigma(c)) = \{ 0, I_k, E_{1,2} + E_{2,1}, I_k + E_{1,2} + E_{2,1} \} \) where \( k = \min\{\text{rank}(A), \text{rank}(B), \text{rank}(C)\} \).

*Proof.* Let \( A, B, C, 0 \in \text{supp}(c) \). As \( C^{\ell_2}(\ell) \) is a \( 2 \)-invariant code, its minimum distance codewords are multiples of binary vectors. Thus, without loss of generality, we may assume \( c_A = c_B = c_C = c_0 = 1 \). Without loss of generality, we may assume \( A \) is of minimum nonzero rank.

Note that for \( f \in F(\ell), c \cdot \text{ev}(f) = 0 \) implies \( A^{i,j} + B^{i,j} + C^{i,j} + D^{i,j} = 0 \) for all minors \( \text{det}_{1,j}(X) \). As this also holds for all \( 1 \times 1 \) minors we obtain \( C = A + B \).

Without loss of generality, we may assume \( C_{1,1} = 1 \) with \( A_{1,1} = 1 \) and \( B_{1,1} = 0 \). Because \( A \) is Hermitian, by \([9]\), there exists \( P \in GL_\ell(\mathbb{F}_{q^2}) \) such that \( P(q)^T AP = I_k \). By applying this automorphism \( \sigma_1, \text{supp}(\sigma_1(c)) = \{ A', B', C', 0 \} \) where \( C'_{1,1} = 1 \) with \( A'_{1,1} = 1 \) and \( B'_{1,1} = 0 \). Note \( \text{rank}(A) \geq 1 \) implies \( A_{1,i} = A_{i,1} = 0 \) for \( i > 1 \).

Now that we have established that \( A \) has a special form, we will use the condition \( c \cdot \text{ev}(f) = 0 \) to determine the entries of \( A \) and \( B \).

First let \( f = \text{det}(\{1,i\}, \{1,j\}) \) with \( i \geq 2 \) and \( j \geq 2 \). Note:

\[
\begin{align*}
\text{f}(A') &= A_{1,1}' A_{i,j}' - A_{i,1}' A_{1,j}' = A_{i,j}', \\
\text{f}(B') &= B_{1,1}' B_{i,j}' - B_{1,j}' B_{1,1}' = B_{i,j}' B_{1,1}', \\
\text{f}(C') &= C_{1,1}' C_{i,j}' - C_{1,j}' C_{1,1}' = (A_{i,j}' + B_{i,j}') + (B_{i,j}' B_{1,1}')
\end{align*}
\]

Then \( c \cdot \text{ev}(f) = 0 \) implies

\[
A_{i,j}' + B_{i,j}' B_{1,1}' + (A_{i,j}' + B_{i,j}') + (B_{i,j}' B_{1,1}') = 0.
\]

This implies \( B_{i,j}' = 0 \) for all \( i, j \) such that \( i \geq 2 \) or \( j \geq 2 \).

Because \( B' \neq 0 \), there must be a nonzero entry in row or column 1. Without loss of generality after permuting rows or columns we may assume this is \( B_{1,2}' \). Recall that because \( B' \) is Hermitian, \( B_{1,2}'^2 \neq 0 \). Without loss of generality we may assume \( B_{1,2}' = B_{2,1}' = 1 \). This is because \( x^3 = 1 \) for all nonzero values in \( \mathbb{F}_4 \). Thus, automorphisms defined by multiplying a nonzero scalar to a row and its conjugate to a column do not affect \( I_k \). Then, through proper elementary row operations, we may use the entry in the second column of \( B' \), \( B_{1,2}' = 1 \) to eliminate all other entries in row 1 and column 1. As \( A_{1,2}' = A_{2,1}' = 0 \) these operations do not change \( A' \). Thus we may assume \( B' = E_{1,2} + E_{2,1} \). Therefore \( \text{supp}(\sigma(c)) = \{ 0, I_k, E_{1,2} + E_{2,1}, I_k + E_{1,2} + E_{2,1} \} \).

We remark that since all automorphism were from multiplying by invertible matrices, then \( k = \min\{\text{rank}(I_k), \text{rank}(E_{1,2} + E_{2,1}), \text{rank}(I_k + E_{1,2} + E_{2,1})\} \). Moreover, this implies \( k \leq 2 = \text{rank}(E_{1,2} + E_{2,1}) \).
Lemma 29. Let \( q = 2, \ell \geq 2 \) and \( c \in C^H(\ell)^1 \) and

\[
\text{supp}(c) = \{0, I_{k}, E_{1,2} + E_{2,1}, I_{k} + E_{1,2} + E_{2,1}\}
\]

where

\[
k = \min\{\text{rnk}(I_{k}), \text{rnk}(E_{1,2} + E_{2,1}), \text{rnk}(I_{k} + E_{1,2} + E_{2,1})\}.
\]

Then \( k = 1 \).

Proof. Let \( c \) be as stated above. Recall \( k \leq 2 \), thus we need only show \( k \neq 2 \). Let’s assume that \( k = 2 \). This implies that \( I_{k} + E_{1,2} + E_{2,1} = I_{2} + E_{1,2} + E_{2,1} \). This implies \( \text{rnk}(I_{k} + E_{1,2} + E_{2,1}) = 1 \), which contradicts the previous remark that \( k = \min\{\text{rnk}(I_{k}), \text{rnk}(E_{1,2} + E_{2,1}), \text{rnk}(I_{k} + E_{1,2} + E_{2,1})\} \). Therefore, \( k = 1 \).

Lemma 30. Let \( q = 2, \ell \geq 2 \) and \( c \in C^H(\ell)^1 \) and \( \text{supp}(c) = \{A, B, C, D\} \). Then there exists an automorphism \( \sigma \in \text{Aut}(C^H(\ell)) \) such that \( \text{supp}(\sigma(c)) = \{0, I_{1}, E_{1,2} + E_{2,1}, I_{1} + E_{1,2} + E_{2,1}\} \).

Proof. Let \( c \) be as stated above, by Lemma 27 and Lemma 28, there exists an automorphism \( \sigma \in \text{Aut}(C^H(\ell)) \) such that \( \text{supp}(\sigma(c)) = \{0, I_{1}, E_{1,2} + E_{2,1}, I_{1} + E_{1,2} + E_{2,1}\} \). By Lemma 29, \( k = 1 \). Therefore, \( \text{supp}(\sigma(c)) = \{0, I_{1}, E_{1,2} + E_{2,1}, I_{1} + E_{1,2} + E_{2,1}\} \).

Lemma 31. Let \( q = 2 \). The support of a minimum weight codeword of \( C^H(\ell)^1 \) is contained in the class:

\[
\{H, H + a_1^* a_1, H + a_2^* a_1 + a_1^* a_2, H + a_1^* a_1 + a_2^* a_1 + a_1^* a_2\}
\]

Where \( H \in \mathbb{H}^2(\mathbb{F}_{q^2}) \), meanwhile \( a_1, a_2 \) are linearly independent vectors in \( \mathbb{F}_{q^\ell} \).

Proof. By Lemma 30, we may apply an automorphism of \( C^H(\ell)^1 \) to map any of the possible support sets \( \{A, B, C, D\} = \text{supp}(c) \) to a codeword with support \( \{0, I_{1}, E_{1,2} + E_{2,1}, I_{1} + E_{1,2} + E_{2,1}\} \). By acting on \( \{0, I_{1}, E_{1,2} + E_{2,1}, I_{1} + E_{1,2} + E_{2,1}\} \) with an invertible matrix \( A \) satisfying \( e_1 A = a_1, e_2 A = a_2 \) via the action \( M \mapsto A^* MA \), we map the set \( \{0, I_{1}, E_{1,2} + E_{2,1}, I_{1} + E_{1,2} + E_{2,1}\} \) to the set \( \{A^* 0A, A^* I_{1} A, A^* (E_{1,2} + E_{2,1}) A, A^* (I_{1} + E_{1,2} + E_{2,1}) A\} \) which is equal to the set \( \{0, a_1^* a_1, a_2^* a_1 + a_1^* a_2, a_1^* a_1 + a_2^* a_1 + a_1^* a_2\} \) where \( a_1 \) is the first row of \( A \) and \( a_2 \) is the second row of \( A \). Similarly, \( a_1^* \) is the first column of \( A^* \) and \( a_2^* \) is the second column of \( A^* \). Finally, we act by adding \( H \in \mathbb{H}^2(\mathbb{F}_{q^2}) \), thus obtaining

\[
\{H, H + a_1^* a_1, H + a_2^* a_1 + a_1^* a_2, H + a_1^* a_1 + a_2^* a_1 + a_1^* a_2\}
\]

6 Conclusion

In this paper we have introduced the affine Hermitian Grassmann codes. These are linear codes associated to the affine part of the polar Hermitian Grassmannian, defined in the same way affine Grassmann codes are defined from the Grassmannian. As might be expected, the affine Hermitian Grassmann code is very similar to the affine Grassmann code. The affine Hermitian Grassmann code for \( q \) is defined over the field \( \mathbb{F}_{q^2} \) and obtained from puncturing \( C^H(\ell, 2\ell) \) as a code over \( \mathbb{F}_{q^2} \). However, \( C^H(\ell) \) is a \( q \)-invariant code. This implies \( C_H(\ell) \) has a basis over \( \mathbb{F}_q \). Furthermore it has the same dimension and better minimum distance than the corresponding affine Grassmann code over \( \mathbb{F}_q \). The similarities are deeper, as the dual code \( C^H(\ell, 2\ell)^1 \) is similar to \( C^H(\ell, 2\ell)^1 \). Even the support of the minimum weight codewords are nearly identical in a case by case basis. We hope this remarkable similarity holds for other codes related to the Grassmannian.

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