A new holographic limit of AdS$_5$⊗$S^5$

Machiko Hatsuda* and Warren Siegel*

*Theory Division, High Energy Accelerator Research Organization (KEK),
Tsukuba, Ibaraki, 305-0801, Japan
e-mail: mhatsuda@post.kek.jp
*C.N. Yang Institute for Theoretical Physics,
State University of New York, Stony Brook, NY 11794-3840, USA
e-mail: siegel@insti.physics.sunysb.edu

Abstract

We re-examine the projective lightcone limit of the gauge-invariant Green-Schwarz action on 5D anti-de Sitter ⊗ the five-sphere. It implies the usual holography for AdS$_5$, but also (a complex) one for $S^5$. The result is N=4 projective superspace, which unlike N=4 harmonic superspace can describe N=4 super Yang-Mills off shell.

Keywords: AdS/CFT correspondence, projective superspace, superparticle
1 Introduction

The anti-de Sitter/conformal field theory correspondence is an interesting attempt to relate string theory to quantum chromodynamics [1]. The states of the (Type IIB) string are identified as color-singlet bound states of Yang-Mills (N=4 supersymmetric, in the simplest version). Although the equivalence is still conjectural, its study has led to new insights on both theories. It is based on the assumption of holography, that one dimension of AdS$_5$ becomes irrelevant, so that the usual four dimensions of spacetime can be obtained. Holography has not been obtained dynamically, but is imposed as part of the definition of calculated quantities, and can be interpreted as an expansion about the boundary of AdS$_5$, whose convergence is yet to be determined.

The conventional correspondence identifies the Yang-Mills fields with those of a dual (Type I) open string whose ends are confined to the boundary (branes). Hence these fields are not seen explicitly in this approach. However, in string theory closed strings are not true bound states of open strings, but appear kinematically in the free theory, in a manner similar to bosonization/fermionization in free two-dimensional theories [2]. (They appear in one-loop graphs, not through ladder graphs representing multiple exchanges.) In a previous paper an alternative correspondence was proposed [3], which identifies the Yang-Mills fields with the matrix fields of the random-lattice approach to string theory, where planar Feynman diagrams of partons are identified with lattice discretizations of the string worldsheet path-integral through the 1/N expansion [4].

That paper also proposed a different approach to holography, based on treating anti-de Sitter space through an expansion about the projective lightcone (its zero-radius limit), the original higher-dimensional formulation of conformal symmetry [5]. In that limit the fifth dimension becomes nondynamical; effectively, rather than restricting the theory to the boundary, the boundary has been expanded into the whole space. When combined with the random lattice approach, the result is that in the path integral the fifth coordinate acts as Schwinger parameters (a fifth, “proper-time” coordinate), converting the Gaussian propagators of the partons in the usual random lattice approach into those of conventional field theory.

In that paper calculations were simplified by taking the projective lightcone limit only after choice of a special gauge that is not applicable to the general case [6]. Another consequence was that the gauge-invariant form of the limiting action was not examined. Here we take the limit before gauge fixing. While AdS$_5$ is contracted to the projective lightcone, S$^5$ is not contracted to a point, but also to a flat four-dimensional space, as expected from symmetry with AdS$_5$ under Wick rotation. Including the related contraction of the fermionic coordinates, the resulting superspace is that appropriate to the N=4 projective superspace description of off-shell N=4 super Yang-Mills, a direct generaliza-
tion of N=2 projective superspace \cite{7}. The modified reality conditions of superfields in this approach follow directly from this Wick rotation. (Such superspaces have also been applied to the harmonic superspace approach \cite{8}, but the result was always on shell \cite{9}, because the boundary conditions for the coordinates of the internal R symmetry differ in the two approaches \cite{10}.)

The resulting random lattice action for the superparton is identical to the usual Casalbuoni-Brink-Schwarz superparticle action (for D=4, N=4) after a simple redefinition, and thus describes 4D N=4 super Yang-Mills (as known, e.g., from lightcone quantization). (In \cite{3} this action was obtained after gauge fixing.) However, the form of the action before the redefinition directly implies projective superspace because of the appearance of the internal coordinates with derivatives, and half of the fermionic variables as Lagrange multipliers (i.e., without derivatives). The choice of off-shell superspace, and in particular the appearance of internal coordinates, thus depends on the treatment of the second-class constraints. (The on-shell superspace is always the same, up to trivial Fourier transformation in fermionic coordinates, because the physical spectrum is the same.) The appearance of internal coordinates also allows more general gauges, and the use of the infinite number of auxiliary fields required for off-shell N=4 supersymmetry.

The equations we give below as “constraints”, when applied in the harmonic formalism, imply the equations of motion (Klein-Gordon, Weyl, or Maxwell equations) due to the condition of regularity in the harmonic coordinates, as imposed by the harmonic expansion. On the other hand, in the projective formalism these constraints simply restrict dependence of the fields to the coordinates of projective superspace, which have a Laurent expansion in the projective coordinates.

This has already been demonstrated for the case of the N=2 scalar multiplet: In the harmonic case, only a subset of the constraints (“analyticity”) was imposed as constraints, the remainder were used as field equations. In the projective case, all were imposed, and solved off-shell, and the field equations following from the action were those equations called “field equations” below.

In the case of N=3 super Yang-Mills, only the harmonic case has been treated \cite{11}, and the same separation was made. However, quantization of this formalism proved difficult, and a Fermi-Feynman gauge was not found \cite{12}. Since the whole point of an off-shell formalism is quantization (otherwise one can simply look at solutions of the classical field equations), and the point of gauge invariance is to be able to choose arbitrary gauges (since different choices have different advantages, and gauge independence is part of unitarity), this theory still needs a formulation where Feynman rules for general gauges can be written. Furthermore, since this theory has an N=4 supersymmetry, ideally an N=4 superspace solution is desired. However, so far it has not been possible to give an off-shell
N=4 harmonic superspace formulation of this theory.

The AdS/CFT approach has shown promise in giving a superspace approach to this theory. It naturally leads to an N=4 formulation, and has been used previously in conjunction with on-shell N=4 harmonic superspace. However, as we will see below, manifesting as much superconformal invariance as possible leads naturally to projective superspace, as a direct supersymmetrization of the coset space approach used both for the standard representation of the conformal group on spacetime and for the AdS Green-Schwarz superstring action. In fact, the above limiting procedure leads directly to a classical mechanics action for N=4 super Yang-Mills on projective superspace. The projective approach solves the problem that prevented an N=4 formulation (in both harmonic and earlier approaches), by solving the constraints without going on shell. Although at this point we have quantized (and derived a field theory action) from this mechanics action only for the lightcone gauge (where it trivially reduces to the usual lightcone methods), these results are encouraging in their avoidance of earlier problems and relation to previous useful results, as well as their direct relation to the AdS/CFT correspondence.

2 Projective superspaces for D=4

2.1 Coset spaces

There is a general kind of coset-space construction that appears frequently in physics, but the relation of the various cases is never described. We will call them “half-coset” spaces because that describes both our construction and notation for them.

The general construction is: (1) Given a group G, consider its Cartan subalgebra, and pick from it one generator, perhaps a linear combination of the natural basis chosen. (2) Divide up the generators of G, in a basis of eigenvectors of the chosen U(1) generator, into those with positive eigenvalues, $G_+$ (raising operators), those with negative, $G_-$ (lowering operators), and those with vanishing eigenvalue, $G_0$ (which includes the Cartan subalgebra, and in particular the chosen U(1) generator, but in general will be nonabelian). (3) The subalgebra of interest is that generated by $G_+$. For notational and pedagogical purposes, it will usually be convenient to label this subalgebra by the half-coset notation as $G/G_0^+$, since G and $G_0$ will be more readily identifiable. It can also be interpreted as the coset with “isotropy” subgroup the semidirect product of $G_0$ and $G_-$. We first give some examples from string theory. (In the following subsections we describe cases more closely related to the present paper.) One familiar example is Gupta-Bleuler quantization of string theory. There G is the (super)Virasoro algebra, $G_0$ its zero-modes, and $G_+$ creates unphysical states from the vacuum, or from physical states.
gauge-equivalent ones. A similar method can be applied to general groups of constraints in Gupta-Bleuler quantization [13].

A case of recent interest is the “pure spinor”, which appears in Berkovits’ description of the superstring [14]. It has been described as the coset SO(10)/U(5), but more accurately it is SO(10)/U(5)+, where the relevant U(1) is unambiguous in U(5)=SU(5)U(1). (We omit the direct product symbol “⊗” in group products, just as the usual product symbol “×” is conventionally omitted in algebra.)

Another example, closely related algebraically to the ones relevant to this paper, where such a construction is implicit, is the manifestly T-dual formulation of the massless sector of oriented closed strings (graviton, axion, dilaton) [15]. There the half-coset is GL(2D)/GL(D)2+ in D dimensions, where the U(1) (actually GL(1)), from GL(D)=SL(D)GL(1), is not the diagonal one (which cancels the one from the GL(2D)). If we divide the defining-representation group element, a (2D)2 matrix, into four equal quadrants, the two diagonal quadrants give G0, while the lower-left quadrant gives G+, which is Abelian in this case. The G group element represents the fields of the theory (less the dilaton); in the gauge where G0 and G− have been used to gauge away three quadrants, the symmetric part of G+ corresponds to the metric and the antisymmetric part to the axion two-form. Previously this had been described in terms of two of the adjacent quadrants; that non-coset description follows from this one if G− and half of G0 are used to gauge away the other half of the matrix. The resulting D×2D matrix still represents GL(2D) by multiplication on the longer side, and one of the local GL(D)’s on the other. In that approach the ratio of the two surviving quadrant matrices (a “projective” space) is gauge invariant under the remaining gauge symmetry, which is equivalent to using that remaining symmetry to gauge away all but G+. (Here we refer to only the non-derivative, “tangent-space” gauge symmetry; for this particular case there remains the usual derivative gauge transformations of the metric and axion field.) Note that in this case, where G is a GL group, the advantage of working directly with group elements instead of algebra elements is that there is no need to exponentiate [16].

2.2 Superconformal cosets

Probably the most familiar use of such a construction is for the case of the usual coordinate representation of the conformal group (for spin zero). There the half-coset is SO(D,2)/SO(D−1,1)SO(1,1)+. The U(1) is the scale generator, the rest of G0 is the Lorentz generators, G− is conformal boosts, and G+ is translations, corresponding to the usual spacetime coordinates.

In D=4, if we replace these groups with their covering groups, the half-coset is SU(2,2)/SL(2,C)GL(1)+. For our purposes it will be more convenient to consider the
Wick rotation, the covering of SO(3,3)/SO(2,2)SO(1,1)+, which is GL(4)/GL(2)²+, where we have thrown in an extra GL(1) on top and bottom for convenience.

The procedure is then the same as for T-dual axionic gravity as described above (but now we deal with coordinates rather than fields). The resulting representation of the conformal group is the same as with orthogonal groups, but in spinor notation: It is the “quaterionic projective” version of ADHM twistors [17] (again if we identify by using half the matrix).

Such representations of conformal groups have been generalized to superconformal groups [18], and in particular for chiral [19] and other [9] “subsuperspaces”. For our purposes, it will be sufficient to consider the cases GL(4|N)/GL(2|n)GL(2|N-n)+, which are direct generalizations of the bosonic (N=n=0) case. Now only the two diagonal blocks need be square; e.g., n=0 describes chiral superspace. However, the cases N=2n are actually the most relevant ones, since they describe CPT self-conjugate multiplets (the N=2 scalar multiplet, N=4 Yang-Mills, and N=8 supergravity). In general these half-cosets are Abelian subgroups: These subspaces of the full superspace have no torsion.

### 2.3 Superspaces

Now we consider these spaces GL(4|N)/GL(2|n)GL(2|N-n)+ in more detail, and the description of supersymmetric theories on them. We divide up the indices as

$$\mathcal{A} = A, A'$$

where $\mathcal{A}$ is the (4|N) index, $A$ is (2|n), and $A'$ is (2|N-n). We then separate out the bosonic (2-component Weyl spinor) and fermionic (internal) indices as

$$A = \alpha, a; \quad A' = \dot{\alpha}, a'$$

Writing the G superspace coordinates (GL(4|N) group element in the defining representation) as $z_{\mathcal{A} M}$ and its inverse as $z_{M \mathcal{A}}$, where the superconformal group generators $G_{\mathcal{M} N}$ act on the index $\mathcal{M}$, and thus the covariant derivatives $D_{\mathcal{A} B}$ act on the other side, on the index $\mathcal{A}$, we have

$$G_{\mathcal{M} N} = z_{\mathcal{A} N} \partial_{\mathcal{M} A}, \quad D_{\mathcal{A} B} = z_{\mathcal{M} A} \partial_{\mathcal{M} B}$$

with the usual implicit sign factors for ordering of fermionic indices, where $\partial_{\mathcal{M} A} = \partial / \partial z_{\mathcal{A} M}$. The two are then linearly related as

$$D_{\mathcal{A} B} = z_{\mathcal{A} M} z_{N B} G_{\mathcal{M} N}.$$ 

5
We now consider the steps necessary to restrict the coordinates $z^M_A$ that appear as arguments of the field to the half-coset. We begin by imposing the constraints (which also generate the gauge transformations of the isotropy group)

$$D_A^B = D_A'B' = D_A'B = 0 \quad (2.5)$$
on the field. (This will define the superspace. Below we will consider the more general case of superfields with indices, where $D_A^B$ and $D_A'B'$ will be set to equal matrix representations, which will lead to the same coordinates, but add spin operators to the group generators.) This leaves covariant derivatives

$$D_A'B' = p_\alpha \dot{\theta}_\beta , d_{\alpha}^{\dot{a}}, \bar{\theta}_\dot{a} t_{\beta}^{\alpha} \quad (2.6)$$
corresponding to the surviving coordinates.

We now solve the constraints explicitly, starting with

$$0 = D_A^B = z^M_A \partial_M^B \quad \Rightarrow \partial_M^A = 0 \quad (2.7)$$
explicitly eliminating dependence on $z^M_A$. (At this point we have the “half-matrix” mentioned above that appeared in earlier approaches to some theories.) The remaining constraint $D_A'B' = 0$ then says that the field is invariant under general linear transformations on the $A'$ index, so that the remaining coordinates $z^{A'}$ can appear in the field only in the combination

$$w_{A'}^{M} \equiv (z^{A'}_{M})^{-1}z^{A'}_{N} \quad (2.8)$$
where “$(z^{A'}_{M})^{-1}$” means to take only the inverse of the matrix $z^{A'}_{M}$ and not the corresponding part of the inverse of $z^M_A$. These can be separated into

$$w_{A'}^{M} = x_{\mu}^{\nu}, \theta_{m'}^{\nu}, \bar{\theta}_{\mu}^{\dot{n}}, y_{m}^{n} \quad (2.9)$$
where $x$ are the usual spacetime coordinates, $\theta$ and $\bar{\theta}$ are half of the anticommuting coordinates of the full superspace, and $y$ are the internal coordinates. When acting on fields depending on only $w$, the superconformal generators take the form, using the notation $\partial_{M}^{N'} \equiv \partial/\partial w_{N'M}$,

$$G_{M}^{N'} = \partial_{M}^{N'}, \quad G_{M}^{N} = w_{P'}^{N} \partial_{P'}^{M'}, \quad G_{M'}^{N'} = -w_{M'}^{P} \partial_{P'}^{N'}, \quad (2.10)$$
$$G_{M'}^{N} = -w_{M'}^{P} w_{Q'}^{N} \partial_{Q'}^{P'} \quad (2.11)$$

However, while the symmetry generators $G_{M}^{N'}$ depend only on the surviving coordinates and their derivatives (i.e., those coordinates are a realization of the full group),
the remaining covariant derivatives, while containing derivatives with respect to only the (half-)coset coordinates, can have coordinate dependence on some of the coordinates of the isotropy group. For example, in chiral superspace, in the chiral representation, the supersymmetry generators contain no $\bar{\theta}$’s nor $\partial/\partial \bar{\theta}$’s, but the surviving spinor covariant derivative does have a $\bar{\theta}$ term. To simplify the resulting expressions it is useful to extend the change of variables from $z$ to $w$ as represented by the matrix decomposition

\[ \begin{pmatrix} z_A^M \\ z_{A'}^{M'} \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} u \\ u' \end{pmatrix} \begin{pmatrix} I & 0 \\ w & I \end{pmatrix} \]  
(2.12)

or

\[ z_{A'}^{M'} = u_{A'}^{M'}, \quad z_{A'}^{M} = u_{A'}^{N'} w_{N'}^{M}, \quad z_{A}^{M'} = v_{A}^{B'} u_{B'}^{M'}. \]  
(2.13)

\[ z_{A}^{M} = u_{A}^{M} + v_{A}^{B'} u_{B'}^{N'} w_{N'}^{M}. \]  
(2.14)

Later we will need also the inverse

\[ \begin{pmatrix} z_M^A \\ z_{M'}^{A'} \end{pmatrix} = \begin{pmatrix} I & 0 \\ -w & I \end{pmatrix} \begin{pmatrix} u^{-1} & 0 \\ 0 & u'^{-1} \end{pmatrix} \begin{pmatrix} I & -v \\ 0 & I \end{pmatrix} \]  
(2.15)

\[ z_M^A = u_M^A, \quad z_{M'}^{A} = -w_{M'}^{N} u_{N}^{A}, \quad z_{M'}^{A'} = -u_{M}^{B'} v_{B}^{A'}. \]  
(2.16)

\[ z_{M'}^{A'} = u_{M'}^{A'} + w_{M'}^{N} u_{N}^{B} v_{B}^{A'}. \]  
(2.17)

where $u_M^A$ is the matrix inverse to $u_A^M$, etc. ($u$ and $u'$ now act as the vielbeins for GL($2|n$) and GL($2|N-n$).) In terms of these the surviving covariant derivatives act on the fields as

\[ D_{A}^{B'} = u_{A}^{M} u_{N'}^{B'} \partial_{M}^{N'}. \]  
(2.18)

### 2.4 Free field equations

The free superconformal equations of motion (field equations for field strengths) on general superspaces can be written as either [20]

\[ G_{[M}^{\mathcal{N}} G_{P]}^{\mathcal{Q}} - \text{traces} = 0 \quad \text{or} \quad D_{[A}^{[B} D_{C]}^{D] - \text{traces} = 0} \]  
(2.19)

using graded antisymmetrization $[\ldots]$, where $z_A^{M}$ can be used to convert between the two forms. (A familiar example is that a field equation common to all massless supersymmetric theories, the generator of $\kappa$ symmetry, can be written as either $\dot{p} d = 0$ or $\dot{p} q = 0$ in linear combination with $p^2 = 0$.) These equations are those related to the masslessness condition $p^2 = 0$ by superconformal transformations.
On projective superspaces the field equations remaining after imposing the (isotropy) constraints are

\[ D_{[A}^B D_{C]}^{D'} = 0 \quad . \tag{2.20} \]

These can be expanded as (using antisymmetrization in spinor indices to contract them), in order of decreasing (scale) dimension,

\[ p^2 = 0 \quad . \tag{2.21} \]

\[ p\bar{d} = \bar{p}d = 0 \quad . \tag{2.22} \]

\[ (d^2)^{(a'b')} = (d^2)_{(ab)} = \bar{d}_a^{\beta} \bar{d}_a^{\beta} - \bar{p}_a^{\beta} t_a^{b'} = 0 \quad . \tag{2.23} \]

\[ t_a^{(b'} d_a^{c')} = t_{(a'}^{b'} \bar{d}_c^{\beta} )^{\beta} = 0 \quad . \tag{2.24} \]

\[ t_{(a'}^{b'} t_c^{d')} = 0 \quad . \tag{2.25} \]

Field equations of higher dimension follow from those of lower dimension by commutation with the constraints; this corresponds to expansion in the isotropy coordinates. In fact, such an expansion of the lowest-dimension field equation yields the corresponding equations in terms of the generators of the half-coset subgroup:

\[ G_{[M}^{[N'} G_{P]}^{Q']} = 0 \quad \Rightarrow \quad \partial_{[M}^{[N'} \partial_{P]}^{Q')} = 0 \quad . \tag{2.26} \]

For the harmonic superspace approach, these constraints alone are enough to put the cases N=2n=2 [8] or 4 [9] on shell. (That approach for N=2 uses one of these constraints as a field equation instead. There is also an off-shell harmonic N=3 formalism of N=4 [11], but it lacks the explicit N=4 superspace used in treatments of AdS/CFT. In reference [21] the AdS/CFT representation of the superconformal group was gauge fixed and reduced to N=2, but since no dynamics were considered, as either string or field theory action or field equations, and the result was gauge dependent, it is impossible for them to distinguish between N=2 harmonic and projective superspaces.) But in the projective superspace approach for N=2n=2, these constraints only define the space (off shell), and the field equations for the scalar (“arctic”) multiplet, whose field strength (the field itself) is projective, are those given above. These equations also describe the free part of N=4 super Yang-Mills for N=2n=4, although a field theory action has not yet been written. This can be seen easily by Taylor expanding the superfield in y (the definition of the projective boundary conditions), and in \( \theta \). A simpler (covariant) way is to use
supertwistors \[22\]: Since the equations in terms of the generators involve only partial derivatives, Fourier transformation gives the algebraic solution
\[ G_{M',N'} = \zeta_M \zeta_{N'} \Rightarrow \Phi(w) = \int d\zeta \exp \left( w_{M'N'} \zeta_N \zeta^{M'} \right) \chi(\zeta) + h.c. \tag{2.27} \]
where the components of \( \zeta \) with spinor indices are the usual Penrose twistors and the rest are fermions, corresponding to the \( \theta \)'s of lightcone superfields (giving the usual helicity expansion).

The on-shell form of the superconformal generators (acting on \( \chi \)) is then
\[ G_M^{N'} = \zeta_M \zeta^{N'}, \quad G_M^N = \zeta_M \partial^N, \quad G_{M',N'} = \zeta^{N'} \partial_{M'}, \quad G_{M',N} = \partial_{M'} \partial^N \tag{2.28} \]
where
\[ [\partial_{M'}, \zeta^{N'}] = \delta^{N'}_{M'}, \quad [\zeta_M, \partial^N] = \delta^N_M \]

### 2.5 Superspin

So far we have considered just coordinates, sufficient to define scalar superfields. For more general representations we may want to consider superfields with indices, and thus “superspin” (in addition to the spin that comes from expanding the superfields in the anticommuting coordinates). In terms of the supertwistor transform, these correspond to
\[ \Phi_{M...N'...}(w) = \int d\zeta \exp \left( w_{M'N'} \zeta_N \zeta^{M'} \right) \zeta_M ... \zeta^{N'} ... \chi(\zeta) + h.c. \tag{2.29} \]
The supertwistor form of the superconformal generators is then the same as for the scalar superfield given above.

These representations follow from the constraint approach by the modification
\[ D_A^B = s_A^B, \quad D_{A'}^{B'} = s_{A'}^{B'}, \quad D_{A'}^B = 0 \tag{2.30} \]
where the \( s \)'s are matrix representations of \( G_0 \). The extra field equations involving the now nonvanishing \( D_A^B \) and \( D_{A'}^{B'} \) then fix the only representations as \( \Phi_{A...B'...} \), where
\[ \Phi_{A...B'...} = u_A^M ... u_{N'}^{B'} ... \Phi_{M...N'...}(w) \tag{2.31} \]

Off shell, we can consider more general representations. For example, for the defining representations of the GL groups we are considering, for the superfields \( \Phi_{A'} \) and \( \Phi^{A'} \) we still have
\[ 0 = D_A^B = z_A^M \partial_M^B \Rightarrow \partial_M^A = 0 \tag{2.32} \]
but are free to use $z_{A M'} = u_{A' M'}$ to define
\[
\Phi_{A'} = u_{A' M'} \Phi_{M'}, \quad \Phi_{A'} = \Phi_{M'} u_{M' A'}. \tag{2.33}
\]
The remaining constraint $D_{A B'} = s_{A B'}$ then determines that $\Phi_{M'}$ and $\Phi_{M'}$ depend only on $w$. We can apply a similar construction to $\Phi_{A}$ and $\Phi_{A}$ if we first note that we can also write
\[
D_{A B} = -z_{M}^{B} \frac{\partial}{\partial z_{M}^{A}} \tag{2.34}
\]
to show
\[
D_{A B} = 0 \quad \Rightarrow \quad \frac{\partial}{\partial z_{M}^{A}} = 0 \tag{2.35}
\]
and thus use $z_{M}^{A} = u_{M}^{A}$ to define
\[
\Phi_{A} = \Phi_{M} u_{M}^{A}, \quad \Phi_{A} = u_{A M} \Phi_{M} \tag{2.36}
\]
where $\Phi_{A}$ and $\Phi_{A}$ also depend only on $w$. (Note also that $\Phi_{A'} \Phi_{A'} = \Phi_{M'}^{2}$ and $\Phi_{A'} \Phi_{A'} = \Phi_{M}^{2} \Phi_{M}$.) These relations agree with the on-shell ones for the allowed cases.

Another way to derive the modification to the $D$’s is from the modified $G$’s: Starting with the supertwistor expressions above, we find in terms of $\Phi_{M N' ... N'}(w)$:
\[
G_{M N'} = \partial_{M}^{N'}, \quad G_{M}^{N} = w_{P}^{N} \partial_{M}^{P} + s_{M}^{N}, \quad G_{M'}^{N'} = -w_{M'}^{P} \partial_{P}^{N'} + s_{M'}^{N'}, \tag{2.37}
\]
\[
G_{M'}^{N} = -w_{M'}^{P} w_{Q'}^{N} \partial_{P}^{Q'} - w_{M'}^{P} s_{P}^{N} + s_{M'}^{P} w_{P}^{N}. \tag{2.38}
\]
where these $s$’s act on the $\mathcal{M}$ indices of the fields that arise from the supertwistor transform. (Since these symmetries are global, they take the same form on shell and off.) The original field equations now also imply (from the new terms in $G$)
\[
s_{[M}^{N} \partial_{P}^{Q']} = s_{M'}^{N'} \partial_{P}^{Q'} = 0 \tag{2.39}
\]
which restrict the indices to those following from supertwistors. Multiplying the $G$’s by $z$’s to “flatten” the indices, we then find for the $D$’s, when acting on constrained superfields (but either on or off shell),
\[
D_{A'}^{B} = 0 \quad \tag{2.40}
\]
\[
D_{A}^{B} = u_{A}^{M} s_{M}^{N} u_{N}^{B} \equiv s_{A}^{B}, \quad D_{A}^{B'} = u_{A'}^{M'} s_{M'}^{N'} u_{N'}^{B'} \equiv s_{A'}^{B'}, \tag{2.41}
\]
\[
D_{A}^{B'} = u_{A}^{M} u_{N'}^{B'} \partial_{M}^{N'} - s_{A}^{C} u_{C}^{B'} + v_{A}^{C'} s_{C'}^{B'}. \tag{2.42}
\]
3 Projective lightcone limit

The projective lightcone limit of AdS$_5 \otimes$S$^5$ [3] can be defined as
\[
\begin{align*}
  z_A^M &\to \sqrt{R} z_A^M, & z_{A'}^M &\to \frac{1}{\sqrt{R}} z_{A'}^M, & R &\to 0
\end{align*}
\]
which preserves the superconformal group, but not the isotropy group. (Effectively, the $M$ indices are left invariant, while the $A$ and $A'$ indices are scaled oppositely.) Actions and metrics can be written in terms of the superconformally invariant (but not isotropic) differential forms
\[
J_A^B = (dz_A^M)z_M^B
\]
whose limit is
\[
J_A^B' \to RJ_A^B', & J_A^B \to J_A^B, & J_{A'}^B \to J_{A'}^B, & J_{A'}^B \to \frac{1}{R} J_{A'}^B .
\]
In this limit, the leading contribution to actions and metrics is given by just $J_A'{}^B$. In terms of the above change of variables (for which the factored matrix expression is convenient),
\[
J_A'{}^B = u_{A'}{}^M(du_{M'}{}^N)u_N^B .
\]
To relate to previous results, we first examine the projective lightcone limit for just AdS$_5$. Corresponding to the superstring, we use the covering groups GL(4)/Sp(4)GL(1), rather than SO(3,3)/SO(3,2). The limit is defined [3] by scaling the coordinate “$x_0$” (which vanishes on the boundary) with respect to the remaining coordinates. In spinor (covering group) notation, this is the special case $A = \alpha$, $A' = \dot{\alpha}$ of the above. The limit leaves GL(4) intact, while contracting Sp(4) to the Poincaré group. The metric of AdS$_5$ in terms of the GL(4)-invariant forms is
\[
ds^2 = R^2 J_A{}^B J_C^D \Omega^{AC} \Omega_{BD}
\]
using the Sp(4) metric $\Omega_{AB}$ and its inverse. The limiting form is then
\[
ds^2 \to J_{\dot{\alpha}}{}^{\beta} J_{\dot{\gamma}}{}^{\delta} C^{\dot{\alpha}\dot{\gamma}} C_{\beta\delta}
\]
where
\[
J_{\dot{\alpha}}{}^{\beta} = u_{\dot{\alpha}}{}^{\mu}(dx_{\mu})u_{\nu}{}^{\beta}
\]
\[
\Rightarrow \quad ds^2 = \frac{dx_0^2}{(x_0)^2} ; \quad dx_0^2 = dx_{\dot{\alpha}}{}^{\beta} dx_{\dot{\gamma}}{}^{\delta} C^{\dot{\alpha}\dot{\gamma}} C_{\beta\delta}, \quad x_0 = \left| \frac{1}{2} u_{\alpha}{}^{\mu} u_{\beta}{}^{\nu} C^{\alpha\beta} C_{\mu\nu} \right|
\]
and we have used the fact that after Wick rotation complex conjugation relates $u'$ to $u^{-1}$ (not $u$).

The Green-Schwarz action for $\text{AdS}_5 \otimes S^5$ [23] can be expressed as quadratic [24] in the above currents $J_{A'B'}$, with an overall factor of $R^2$. Upon taking the limit, where only $J_{A'B'}$ survives, the only term remaining is again

$$L \rightarrow J_{\dot{\alpha}}^{\beta} J_{\dot{\gamma}}^{\delta} C^{\dot{\alpha}\dot{\beta}} C_{\dot{\gamma}\delta}$$  \hspace{1cm} (3.9)

but now

$$J_{\dot{\alpha}}^{\beta} = u_{\dot{\alpha}}^{M'} (dw_{M'}^N) u_N^{\beta}$$  \hspace{1cm} (3.10)

Plugging into $L$, the $\text{SL}(2)$ parts of $u$ and $u'$ are again canceled, and the $\text{GL}(1)$ parts again combine into $x_0$, leaving only the fermions. Explicitly, we can write

$$u_M^\alpha = (\delta_\mu^\nu, \vartheta_m^\nu) u_\nu^\alpha, \quad u_{\dot{\alpha}}^{M'} = u_{\dot{\alpha}}^{\dot{\nu}} (\delta_{\dot{\nu}}^{\dot{\mu}}, \vartheta_{\dot{m}}^{\dot{\mu}})$$  \hspace{1cm} (3.11)

We then have

$$L = \frac{(J_{\mu}^{\nu})^2}{(x_0)^2}, \quad J_{\mu}^{\nu} = dx_{\mu}^{\nu} + \bar{\vartheta}_{m^\mu}^{\nu} d\theta_{m^\nu}^{\nu} + (d\bar{\vartheta}_{\mu}^{m} \vartheta_{m^\nu}^{\nu} + \bar{\vartheta}_{\mu}^{m'} (d y_{m'}^{n}) \vartheta_{n^\nu}^{\nu}).$$  \hspace{1cm} (3.12)

Note that our definition of the projective lightcone limit for the supersymmetric case treats $S^5$ in a similar way to $\text{AdS}_5$, in that a five-dimensional space of constant curvature has been contracted to a flat four-dimensional space. Previously we contracted the sphere to a point, as the only obvious contraction that preserved the internal symmetry. This new contraction actually produces a complex internal space, but such spaces are standard in harmonic and projective approaches to extended supersymmetry, and reality properties of superfields are preserved by the usual generalization of complex conjugation for fields whose coordinates have been Wick rotated.

The usual quantization of this string action is expected to be difficult, as it was before the limit, particularly since the Wess-Zumino term has dropped out (but will contribute to corrections higher-order in $R$). Instead, we want to consider random-lattice quantization of this action [3], which effectively means that it is treated as the action for a superparton of the superstring (so dropping the Wess-Zumino term is actually an advantage). In the following section we consider this and related superparticle actions.

### 4 Proposed actions for $N=4$ YM

#### 4.1 Holographic action

We now apply the above action to the superparticle. In addition to the coordinates expected for projective superspace (in this case, $\text{GL}(4|4)/\text{GL}(2|2)^2$), we have found
“Lagrange multipliers” $x_0$ and $\vartheta$. The former appear with derivatives, the latter without: By the usual canonical quantization, this means the fields depend only on the projective superspace coordinates. ($\partial/\partial(\text{Lagrange multiplier})=0.$) However, there are still the usual second-class constraints: In fact, without the $y$ terms, this is just the usual Casalbuoni-Brink-Schwarz action for the superparticle, up to a redefinition of $x$. (We identify $(x_0)^2$ as the worldline metric; the conformal construction has guaranteed its positivity.) So, this action is a generalization of that to projective superspace.

In particular, the case of chiral superspace ($n=0$) has no $y$’s, and this treatment of the usual superparticle action merely suggests the separation of first- and second-class constraints: The usual $\bar{\theta}$’s are now exactly the $\bar{\vartheta}$’s, so the restriction to chiral superspace follows from interpretation of the $\bar{\vartheta}$’s as Lagrange multipliers. The redefinition of $x$ that led to their appearing without derivatives was just the usual (complex) coordinate transformation to the chiral representation. (Of course, antichiral superspace, $n=N$, is also equivalent to the usual superparticle, after the opposite transformation.)

In the general case, if we make the redefinitions (in matrix notation)

$$
\theta \rightarrow \theta - y\vartheta, \quad \bar{\theta} \rightarrow \bar{\theta} - \bar{\vartheta}y, \quad x \rightarrow x + \bar{\vartheta}y\vartheta
$$

(4.1)

we find

$$
J \rightarrow dx + \bar{\vartheta}d\theta + (d\bar{\theta})\vartheta
$$

(4.2)

i.e., the only effect is to remove $y$. Furthermore, the redefinition

$$
x \rightarrow x - \frac{1}{2}(\bar{\vartheta}\theta + \overline{\theta}\vartheta)
$$

(4.3)

gives the usual superparticle action (with the usual reality properties), since

$$
J \rightarrow dx + \frac{1}{2}(\bar{\vartheta}\theta + \overline{\theta}\vartheta)
$$

(4.4)

The addition of pure gauge degrees of freedom (in this case, $y$) to the classical mechanics formulation of a theory, and the resulting nonminimal terms to the first-quantized Becchi-Rouet-Stora-Tyutin operator, is often required for covariant second-quantization, and thus for the gauge- and Lorentz-covariant formulation of the classical field theory [25]. In the harmonic and projective approaches to superspace, the addition of the internal coordinates for CPT self-conjugate multiplets is necessary to introduce the infinite number of auxiliary fields required for a manifestly supersymmetric formulation. On the other hand, in terms of on-shell lightcone superfields, all formulations are the same, up to trivial Fourier transformation in some of the lightcone $\theta$’s (which trades a $\theta$ for a $\bar{\theta}$), since the on-shell field content is unique. This also can be seen covariantly from the supertwistor transform given above: In terms of the same supertwistor field $\chi(\zeta)$, reassigning which
of the anticommuting components belong to $\zeta^M$ and which to $\zeta^{M'}$ results in a different choice of coordinates $w$ in $\Phi(w)$.

The constraints following from a canonical analysis of the system (3.12) are

$$\left(D_{\dot{\mu}} \dot{\nu}\right)^2 = \left(\frac{\partial}{\partial x_{\dot{\nu}}^\mu}\right)^2 = 0$$  \hspace{1cm} (4.5)

which comes from variation of $x^0$, as well as the usual mixed first- and second-class constraints that follow from the definition of the canonical fermionic momenta,

$$D_{\dot{\mu}}^{n'} = \frac{\partial}{\partial \theta_{n'}^{\mu}} + \bar{\theta}_{n'}^{\nu} \frac{\partial}{\partial x_{\dot{\nu}}^\mu} = 0, \quad D_{\dot{\mu}} = \frac{\partial}{\partial \theta_{m}^{\mu}} - \frac{\partial}{\partial x_{\dot{\nu}}^\mu} \bar{\theta}_{m}^{\mu} = 0, \quad (4.6)$$

$$D_{\mu}^{n} = \frac{\partial}{\partial \vartheta_{n}^{\mu}} = 0, \quad D_{m'}^{\dot{\nu}} = \frac{\partial}{\partial \bar{\vartheta}_{m'}^{\nu}} = 0 \quad (4.7)$$

and the one following from the definition of the canonical conjugate of $y$,

$$\frac{\partial}{\partial y_{m'}^{m}} + \bar{\theta}_{m'}^{\nu} \frac{\partial}{\partial x_{\dot{\nu}}^{m'}} \bar{\theta}_{m}^{\nu} = 0 . \quad (4.8)$$

The primary constraint (4.8) can be recast into the first-class constraint

$$D_{m'}^{n'} = \frac{\partial}{\partial y_{m'}^{m}} + \bar{\theta}_{m'}^{n'} \frac{\partial}{\partial x_{\dot{\nu}}^{m'}} \bar{\theta}_{m}^{\nu} - \vartheta_{m}^{n'} D_{\mu}^{n'} - D_{m'}^{\dot{\nu}} \bar{\theta}_{m}^{\nu} = 0 , \quad (4.9)$$

since $y$ is pure gauge. These constraints satisfy the usual relations $\{D_{\dot{\mu}}^{n'}, D_{m'}^{\dot{\nu}}\} = \delta_{m'}^{n'} \delta_{\dot{\nu}}^{\mu}$ and $\{D_{m}^{\dot{\nu}}, D_{\mu}^{n}\} = -\delta_{m}^{n} \delta_{\dot{\nu}}^{\mu}$, expressing their second-class nature. They can be reduced to the first-class constraints $D_{m'}^{[m'} D_{N']^{n]}}, D_{\mu}^{n},$ and $D_{m'}^{\dot{\nu}}$. The linear constraints, $D_{\mu}^{n}$ and $D_{m'}^{\dot{\nu}}$, are imposed as isotropy constraints, implying that the superfields are independent of $\vartheta$ and $\bar{\vartheta}$. The quadratic first-class constraints are the projective superspace field equations (2.20). They include the generator of $\kappa$ symmetry transformations just as in the Casalbuoni-Brink-Schwarz action. The constraints (4.5)-(4.9) are consistent with the superconformal field equations (2.26).

### 4.2 Dual Hamiltonian action

This action, and its relation to projective superspace, suggests alternative actions that do not suffer from second-class constraints. First, noting the similarity ("duality" in the differential geometry sense) between the expressions for the forms $J_{A'}^{B'}$ and the covariant derivatives $D_{A'}^{B'}$, we are led to propose the action

$$L' = \dot{w}_{M'}^{N} p_{N'}^{M'} - \left(D_{\dot{\alpha}} \dot{\beta}\right)^2, \quad D_{\dot{\alpha}} \dot{\beta} = u_{\dot{\alpha}}^{M} D_{M'}^{N'} u_{N'}, \quad (4.10)$$
Separating the $u$’s in a way dual to the previous,

$$u_\alpha^M = u_\alpha^{\prime \nu} (\delta^\mu_\nu, \phi_\nu^m), \quad u_{M'}^{\dot{\alpha}} = (\delta_\mu^\dot{\nu}, \bar{\phi}_m^{\dot{\nu} \nu}) u_\nu^{\dot{\alpha}}$$  \hfill (4.11)

we have

$$L' = \dot{w}_{M'N} p_{N'M} - (x_0)^2 (D_\mu \phi_\nu^m)^2, \quad D_\mu \phi_\nu^m = p_\mu \phi_\nu^m + \phi_\mu^m \bar{\phi}_m^{\dot{\nu} \nu} + \bar{\phi}_m^{\dot{\nu} \nu} \pi_\mu^m + \bar{\phi}_m^{\dot{\nu} \nu} \pi_\mu^m + \phi_\mu^m T_{m'}^{n'} \pi_\nu^{n'}$$  \hfill (4.12)

where $p, \pi, T$ are conjugate to $x, \theta, y$.

An expansion in $\phi$, as follows from the canonical $\partial/\partial \phi = 0$, leads exactly to the superconformal equations of motion (2.26).

### 4.3 Quadratic multiplier action

Another suggested alternative follows from noticing that the Lagrange multipliers $u$ appear quartically in the previous actions, although there can be some simplification because they are multiplied directly as $u^2$ and $u'^2$. Then a manifestly $\text{GL}(2|n)\text{GL}(2|N-n)$-covariant Hamiltonian action can be written as

$$L'' = \dot{w}_{M'N} p_{N'M} - e^{[MN]} e_{[M'N']} p_{M'} p_{N'}.$$  \hfill (4.13)

Again the positivity of $x_0 = |\frac{1}{2} e^{i\omega C_{\mu\nu}}|$ is guaranteed. Many of the terms are similar to those in $L'$, but now there are no terms higher than quartic.

### 5 Conclusions

We have shown that $N=4$ super Yang-Mills is the underlying theory of the random lattice approach to quantization of the superstring about the $\text{AdS}_5 \otimes \text{S}^5$ background. The addition of internal coordinates to the Casalbuoni-Brink-Schwarz action implies the use of projective superspace for off-shell superfields. While the physical content of the action is unchanged, the introduction of the auxiliary internal variables may allow a simpler treatment of covariant first-quantization, just as it does for second-quantization. Effectively, the new coordinates are gauge degrees of freedom at the first-quantized level, but introduce auxiliary fields at the second-quantized level, useful for covariant treatments. Alternatively, one of the proposed related actions might prove useful in avoiding second-class constraints.

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