EULER–KRONECKER CONSTANTS FOR CYCLOMATIC FIELDS

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Abstract

The Euler–Mascheroni constant \( \gamma = 0.5772 \ldots \) is the \( K = \mathbb{Q} \) example of an Euler–Kronecker constant \( \gamma_K \) of a number field \( K \). In this note, we consider the size of the \( \gamma_q = \gamma_{K_q} \) for cyclotomic fields \( K_q := \mathbb{Q}(\zeta_q) \). Assuming the Elliott–Halberstam Conjecture (EH), we prove uniformly in \( Q \) that

\[
\frac{1}{Q} \sum_{Q < q \leq 2Q} |\gamma_q - \log q| = o(\log Q).
\]

In other words, under EH, the \( \gamma_q / \log q \) in these ranges converge to the one point distribution at 1. This theorem refines and extends a previous result of Ford, Luca and Moree for prime \( q \). The proof of this result is a straightforward modification of earlier work of Fouvry under the assumption of EH.

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1. Introduction

For a number field \( K \), the Euler–Kronecker constant \( \gamma_K \) is given by

\[
\gamma_K := \lim_{s \to 1} \left( \frac{\zeta_K'(s)}{\zeta_K(s)} + \frac{1}{s - 1} \right),
\]

where \( \zeta_K(s) \) is the Dedekind zeta-function for \( K \). The Euler–Mascheroni constant \( \gamma = 0.5772 \ldots \) is the \( K = \mathbb{Q} \) case, where \( \zeta_{\mathbb{Q}}(s) = \zeta(s) \) is the Riemann zeta-function. We consider the constants \( \gamma_q = \gamma_{K_q} \) for cyclotomic fields \( K_q := \mathbb{Q}(\zeta_q) \), where \( q \in \mathbb{Z}^+ \) and \( \zeta_q \) is a primitive \( q \)th root of unity.

The recent interest in the distribution of the \( \gamma_q \) is inspired by work of Ihara [4, 5]. He proposed, for every \( \varepsilon > 0 \), that there is a \( Q(\varepsilon) \) for which

\[
(c_1 - \varepsilon) \log q \leq \gamma_q \leq (c_2 + \varepsilon) \log q
\]

for every integer \( q \geq Q(\varepsilon) \), where \( 0 < c_1 \leq c_2 < 2 \) are absolute constants. This conjecture was disproved by Ford et al. in [2] assuming a strong form of the

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Hardy–Littlewood $k$-tuple conjecture. However, assuming the Elliott–Halberstam conjecture (see [1]), these same authors also proved that the conjecture holds for almost all primes $q$, with $c_1 = c_2 = 1$. We recall the Elliott–Halberstam Conjecture as formulated in terms of the Von Mangoldt function $\Lambda(n)$, the Chebyshev function $\psi(x)$ and Euler’s totient function $\varphi(n)$.

**Elliott–Halberstam Conjecture (EH).** If we let

$$E(x; m, a) := \sum_{\substack{p \equiv a \pmod{m} \text{ prime} \atop p \leq x}} \Lambda(p) - \frac{\psi(x)}{\varphi(m)},$$

then for every $\varepsilon > 0$ and $A > 0$, we have

$$\sum_{m \leq x^{1-\varepsilon}} \max_{a(m)=1} |E(x; m, a)| \ll_{A, \varepsilon} \frac{x}{(\log x)^A}.$$

Assuming EH, Ford et al. proved (see [2, Theorem 6(i)]), for every $\varepsilon > 0$, that

$$1 - \varepsilon < \frac{\gamma_q}{\log q} < 1 + \varepsilon$$

for almost all primes $q$ (that is, the number of exceptional $q \leq x$ is $o(\pi(x))$ as $x \to \infty$). Here we extend and refine this result to all integers $q$.

**Theorem 1.1.** Under EH, for $Q \to +\infty$, we have

$$\frac{1}{Q} \sum_{Q < q \leq 2Q} |\gamma_q - \log q| = o(\log Q),$$

where the sum is over integers $q$.

**Remark 1.2.** Theorem 1.1 shows that EH implies that the distribution of $\gamma_q/\log q$ in $[Q, 2Q]$ converges to the one point distribution supported on 1.

To prove Theorem 1.1, we use the work of Fouvry [3] that allowed him to unconditionally prove that

$$\frac{1}{Q} \sum_{Q < q \leq 2Q} \gamma_q = \log Q + O(\log \log Q).$$

Our conditional result is a point-wise refinement of Fouvry’s asymptotic formula under EH.

**2. Proof of Theorem 1.1**

For brevity, we shall assume that the reader is familiar with Fouvry’s paper [3]. The key formula is (see (3) of [3]) the following expression for $\gamma_q$ in terms of logarithmic derivatives of Dirichlet $L$-functions:
Here the inner sum runs over the primitive Dirichlet characters $\chi^*$ modulo $q^*$. We follow the strategy and notation in [3], which makes use of the modified Chebyshev function

$$\psi(x; q, a) := \sum_{n \leq x \atop n \equiv a \pmod{q}} \Lambda(n),$$

and the integral

$$\Phi_{\chi^*}(x) := \frac{1}{x-1} \int_1^x \left( \sum_{n \leq t} \frac{\Lambda(n)}{n} \chi^*(n) \right) dt.$$

However, we replace the sums $\Gamma_i(Q)$ and $\Gamma_{1,j}(Q)$ defined in [3] with the pointwise terms $\gamma_i(q)$ and $\gamma_{1,j}(q)$. Following the approach in [3], which is based on (2.1), we have

$$\gamma_q = \gamma + A(q) + B(q) - \gamma_2(q) - \gamma_3(q) - (\gamma_{1,1}(q) + \gamma_{1,2}(q) + \gamma_{1,3}(q)),$$

where

$$A(q) = \sum_{q' | q \atop q' \equiv q \pmod{q^*}} \frac{L'}{L}(1, \chi^*) + \Phi_{\chi^*}(x),$$

$$B(q) = \sum_{\chi \pmod{q} \atop \chi \equiv \chi_0 \pmod{q^*}} \Phi_{\chi}(x) - \sum_{q' | q \atop q' \equiv q \pmod{q^*}} \sum_{\chi^*} \Phi_{\chi^*}(x),$$

$$\gamma_2(q) = \frac{1}{x-1} \int_1^x \frac{\varphi(q) \psi(t; q, 1) - \psi(t)}{t} dt,$$

$$\gamma_3(q) = \frac{1}{x-1} \int_1^x \sum_{n \leq t \atop (n,q) \neq 1} \frac{\Lambda(n)}{n} dt,$$

$$\gamma_{1,1}(q) = \frac{1}{x-1} \int_1^x \int_1^{\text{min}(q,t)} \left( \frac{\varphi(q) \psi(u; q, 1) - \psi(u)}{u^2} \right) du dt,$$

$$\gamma_{1,2}(q) = \frac{1}{x-1} \int_1^x \int_1^{\text{min}(x_1,t)} \left( \frac{\varphi(q) \psi(u; q, 1) - \psi(u)}{u^2} \right) du dt,$$

$$\gamma_{1,3}(q) = \frac{1}{x-1} \int_1^x \int_1^{\text{min}(x_1,t)} \left( \frac{\varphi(q) \psi(u; q, 1) - \psi(u)}{u^2} \right) du dt.$$

To complete the proof, for $\varepsilon > 0$, we let $x := q^{100}$ and $x_1 := q^{1+\varepsilon}$. Apart from $\gamma_{1,1}(q)$, which gives the $-\log q$ terms in Theorem 1.1, we shall show that these summands are all small.
Estimation of $A(q)$: By Proposition 1 and Remark (i) of [3],
\[
\sum_{q=Q}^{2Q} |A(q)| = O(Q).
\]

Estimation of $B(q)$: For $B(q)$, by (26) and Lemma 3 of [3], we simplify
\[
B(q) = -\frac{1}{x-1} \int_1^x \sum_{q'|q} \sum_{x^* \mod q'} \sum_{\substack{n \leq t \n (n,q)>1}} \frac{\Lambda(n)x^*(n)}{n} dt
\]
\[
= -\frac{1}{x-1} \int_1^x \sum_{q'|q} \sum_{x^* \mod q'} \sum_{\substack{p^* \leq t \n p^* \not| q}} \frac{\log p \cdot x^*(p^*)}{p^*} dt
\]
\[
= -\frac{1}{x-1} \int_1^x \sum_{q'|q} \sum_{p^* \leq t \n p^* \not| q} \frac{\log p}{p^*} \cdot \varphi(d) \mu\left(\frac{q^*}{d}\right) dt
\]
\[
= -\frac{1}{x-1} \int_1^x \sum_{p^* \leq t \n p^* \not| q} \frac{\log p}{p^*} \cdot \varphi(d) \sum_{\substack{q'|q \n d|q^* \n d|p^* \not| q}} \mu\left(\frac{q^*}{d}\right) dt.
\]

We note that the innermost sum
\[
\sum_{\substack{q'|q \n d|q^* \n d|p^* \not| q}} \mu\left(\frac{q^*}{d}\right)
\]
is always 0 or 1, so we conclude that $B(q) \leq 0$ for any $q$. Proposition 2 of [3] gives
\[
\sum_{q=Q}^{2Q} B(q) = O(Q),
\]
and so we have
\[
\sum_{q=Q}^{2Q} |B(q)| = O(Q).
\]

Estimation of $\gamma_2(q)$: By Lemma 8 of [3], uniformly in $Q$ with $u \geq 1$, we have
\[
\sum_{q=Q}^{2Q} \psi(u; q, 1) \ll u.
\]
Therefore,
\[
\sum_{q=Q}^{2Q} |\varphi(q)\psi(t; q, 1) - \psi(t)| = O(Qt).
\]
and so we conclude that
\[
\sum_{q=Q}^{2Q} |\gamma_2(q)| = O(Q).
\]

**Estimation of \(\gamma_3(q)\):** By definition, \(\gamma_3\) is positive, so by (36) of [3],
\[
\sum_{q=Q}^{2Q} |\gamma_3(q)| = O(Q).
\]

**Estimation of \(\gamma_{1,1}(q)\):** Since \(\psi(u; q, 1) = 0\) for \(u < q\), we have
\[
\gamma_{1,1}(q) = - \frac{1}{x-1} \int_{1}^{x} \left( \int_{1}^{\min(q, t)} \frac{\psi(u)}{u^2} \, du \right) \, dt.
\]
Dividing both sides of (41) of [3] by \(Q\),
\[
\gamma_{1,1}(q) = - \log q + O(1).
\]

**Estimation of \(\gamma_{1,2}(q)\):** By the same proof as (42) of [3], we have
\[
\sum_{q=Q}^{2Q} |\gamma_{1,2}(q)| \ll \epsilon Q \log Q.
\]

Summing the above estimates, we conclude unconditionally that
\[
\frac{1}{Q} \sum_{q=Q}^{2Q} |\gamma_q - \log q| = \frac{1}{Q} \sum_{q=Q}^{2Q} |\gamma_{1,3}(q)| + O(\epsilon \log Q).
\]

**Estimation of \(\gamma_{1,3}(q)\):** If we assume Conjecture EH holds, then we have (as in Lemma 7 of [3]) that
\[
\sum_{q \leq 2Q \atop (q, a) = 1} \varphi(q) \left| \psi(x; q, a) - \frac{\psi(x)}{\varphi(q)} \right| = O_A(Qx(\log x)^{-A+2}).
\]
Therefore,
\[
\frac{1}{Q} \sum_{q=Q}^{2Q} |\gamma_{1,3}(q)| = O_{\epsilon, A}(\log^{-A} Q).
\]

By combining these estimates, we obtain the main result
\[
\frac{1}{Q} \sum_{q=Q}^{2Q} |\gamma_q - \log q| = o(\log Q).
\]
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