The relativistic theories of light propagation are generalized by introducing two new parameters $\varsigma$ and $\eta$ in the second post-Newtonian (2PN) order, in addition to the parameterized post-Newtonian parameters $\gamma$ and $\beta$. This new 2PN parameterized (2PPN) formalism includes the non-stationary gravitational field and the influences of all kinds of relativistic effects. The multipolar components of gravitating bodies are taken into account as well at the first post-Newtonian order. The equations of motion and their solutions of this 2PPN light propagation problem are obtained. Started from the definition of a measurable quantity, a gauge-invariant angle between the directions of two incoming photons for a differential measurement in astrometric observation is discussed and its formula is derived. For a precision level of a few microarcsecond ($\mu$as) for space astrometry missions in the near future, we further consider a model of angular measurement, LATOR-like missions. In this case, all terms with aimed at the accuracy of $\sim 1\mu$as are estimated.

Keywords: astrometry; reference systems; relativity.

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1. Introduction

With the development of observational techniques and the improvement of measurement methods, it is time for astrometry to unfold a new era indubitably. Presently, the satellite laser ranging like laser geodynamics satellite (LAGEOS) has achieved accuracies of about one millimeter \cite{1} the precision of lunar laser ranging (LLR) has approached one millimeter \cite{2} and the very long baseline interferometry (VLBI) has attained to the precision of 0.1 mas or even better \cite{3}. Beyond the current stages, astrometric observation will be able to attain the accuracy of a few microarcsecond ($\mu$as) or higher for some astrometric missions in the future, such the Gaia\cite{4}, the Space Interferometry Mission (SIM)\cite{5}, the Square Kilometer Array (SKA)\cite{6}, the Laser Astrometric Test Of Relativity (LATOR)\cite{7} and the Beyond Einstein Advanced Coherent Optical Network (BEACON)\cite{8}. This tendency requires a practical framework that can satisfy the accuracy of $\mu$as.
In order to realize this purpose, there are three approaches: post-Newtonian (PN) method, post-Minkowskian (PM) method and Synge’s world function (SWF) method. Several authors have investigated the light propagation in the general relativity (GR) and alternative theories of gravity by PN method, which assumes gravitational fields are weak and motions are slow. Richter & Matzner studied 2PN light deflection for one body in parameterized post-Newtonian (PPN) formalism by introducing another parameter Λ at $c^{-4}$ of $g_{ij}$, the space-space component of metric. Hellings discussed the relativistic effects in astronomical timing measurements in the PPN formalism, which could be applied to VLBI, spacecraft ranging and pulsar timing in the gravitational field of a motionless body. Brumberg detailedly researched the 2PN light propagation of the Schwarzschild solution in three gauges (standard, harmonic and isotropic) through introducing two coordinate parameters. Brumberg also studies the angle between two incoming rays in a motionless $N$-body system in 1PN approximation of GR. Klioner & Kopeikin developed a practical relativistic model for the 2PN light propagation in the harmonic gauge of GR, in which the 1PN contributions from bodies in uniform motion and the 2PN contributions from the Sun were considered. Motivated by ESA’s Atomic Clock Ensemble in Space (ACES) on International Space Station (ISS), Blanchet et al. discussed time delay and frequency shift for one body in the isotropic gauge at 1PN of the framework of general relativity. Klioner presented a practical relativistic model for Gaia under 1PN formalism with PPN parameters $\beta$ and $\gamma$ for uniformly moving bodies. Xie & Huang deduced the 2PN approximation of Einstein-aether (Æ) theory in the form of both superpotentials and an $N$-point mass. Kopeikin derived an explicit Lorentz-invariant solution of the Einstein and null geodesic equations for data processing of the time delay and ranging experiments for a moving body in PPN formalism by introducing a parameter $\delta$ in spatial isotropic term of $c^{-4}$ for $g_{ij}$. Minazzoli & Chauvineau extended the IAU2000 resolutions to include all the $c^{-4}$ terms for the requirements from some space missions in GR. And then Kopeikin investigated scalar-tensor propagation of light in the inner Solar System including relevant $c^{-4}$ contributions for ranging and time transfer. Kliener & Zschocke formulated the light propagation in 2PN framework of a stationary gravitational field for the Schwarzschild metric (one body) in harmonic gauge by introducing one parameter $\epsilon_{xz}$ in spatial isotropic and anisotropic terms of $c^{-4}$ for $g_{ij}$. Xie & Deng investigated 2PN light propagation model at the $c^{-4}$ level in the framework of the scalar-tensor (ST) theory for a gravitational $N$-point mass system. And they found the parametrization with single parameter in spatial isotropic and anisotropic terms of $c^{-4}$ for $g_{ij}$ is not valid for the ST theory.

Within PM method, it only assumes the condition of weak gravitational fields, but not necessarily slow motion, and it expanded quantities in powers of the gravitational constant $G$. Kopeikin & Schäfer derived the light propagation at the first PM (1PM) linearized Einstein equations in the gravitational field of an arbitrary-moving $N$-body system. Kopeikin & Mashhoor researched the effects of bodies' spins on the light propagation in the gravitational field of an arbitrary-moving
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N-body system in the harmonic gauge to 1PM order under GR. A model of the celestial sphere and of the observables for next-generation astrometric missions was constructed by numerical simulations with the 1PM method in GR for N-body of the Solar System in two cases: static and dynamical. Kopeikin & Makarov took into account light deflection by a giant planet with multipoles in 1 μas astronomical measurements using the 1PM method for two-body (the Sun and Jupiter or Saturn).

In the method of SWF for the light propagation, the integration of geodesic equations can be avoided. Linet & Teyssandier derived time transfer and frequency shift to the order \( c^{-4} \) in the field of an axisymmetric rotating body with this method. Le Poncin-Lafitte, Linet & Teyssandier worked out a recursive procedure for expanding SWF method into a perturbative series of ascending powers of \( G \). Le Poncin-Lafitte & Teyssandier used SWF to investigate influence of mass multipolar moments on the deflection of a light ray by an isolated axisymmetric body. Teyssandier & Le Poncin-Lafitte derived the PM expansion of time transfer functions with SWF method. Recently, with the same method (SWF), Hees, Bertone & Le Poncin-Lafitte present a procedure to compute the relativistic coordinate time delay, Doppler and astrometric observables up to the second PM (2PM) order, and study light propagation in the field of a moving axisymmetric body for the Juno mission. Bertone & Minazzoli et al built the time of flight and tangent vectors in a closed form within the SWF formalism giving the case of a time-dependent metric and shown how to use this new approach to obtain a comparison of the time transfer with space astrometric modelings.

In this work, we use the PN method. Table 1 gives a comparison among these previous works and our present work. Firstly, it is worth emphasizing that we employ the same integral technique and iterative method in Refs. 13, 14, 17, 22. Aimed at the level of μas, our work has the following aspects:

- \( N \) gravitating bodies in the Solar System, including the effects of their quadrupole moments and spins;
- motions of the \( N \)-body. Kopeikin pointed out and discussed that these motions would contaminate the observed numerical value of PPN parameters. In our model, we separate them into two parts: the 1PN contributions from the rectilinear and uniform \( N \)-body (in a short time span) and the 2PN contributions from the motionless Sun, which is sufficient for near future missions;
- two new 2PN parameters \( \varsigma \) and \( \eta \) in addition to the two PPN parameters \( \beta \) and \( \gamma \). \( \varsigma \) and \( \eta \) respectively parametrize the spatial isotropic and anisotropic terms in \( g_{ij} \) at \( O(c^{-4}) \). In some cases, \( \varsigma \) and \( \eta \) are functions of \( \beta \) and \( \gamma \) but, in other cases, they can be independent of the PPN parameters.

In what follows, our conventions and notations generally follow those of Ref. 36. The plot of this paper is as follows. With the level of accuracy of \( \sim \) μas, we develop a 2PN parametrized (2PPN) framework for light propagation in section 2. Subsequently, in section 3, a gauge-invariant angle between incoming light-rays is derived.
and some special cases are shown. Then, we discuss a LATOR-like model of angle measurement: an observer measures the angle between two incoming rays emitted separately from two spacecrafts which are all at the Earth’s orbit circling the Sun. With the level of accuracy of $\sim 1\mu \text{as}$, we estimate the magnitudes of contributions from various physical sources in this case. Finally, conclusion and perspectives are outlined in section 4.

2. 2PPN framework for light propagation

2.1. 2PPN metric for light

In our investigation, we neglect the effects of our galaxy and external galaxies on the Solar System. This means the Solar System is an isolated system. Observations are modeled in the Solar System barycentric reference system (SSBRS), which can be mathematically described by a metric tensor.

For practical reasons, we only care about the effects more than $1\mu \text{as}$ in this work. It means the metric for light can be necessarily (but not over) simplified. The 2PPN metric for SSBRS reads as

$$g_{00} = -1 + \epsilon^2 \sum_A \left( \frac{G m_A}{r_A} + \frac{3}{2} \frac{G}{r_A^5} J^{<ik>}_A r_A^k \right) - \epsilon^4 \frac{G^2 m_{\odot}^2}{r_{\odot}^6} + \mathcal{O}(5), \quad (1)$$

$$g_{0i} = -\epsilon^3 (1 + \gamma) \sum_A \frac{G m_A}{r_A} v_i^A - \epsilon^3 (1 + \gamma) \sum_A \frac{G}{r_A^3} \epsilon_{ijk} S^j_A r_A^k + \mathcal{O}(5), \quad (2)$$

$$g_{ij} = \delta_{ij} + \epsilon^2 \delta_{ij} 2\gamma \sum_A \left( \frac{G m_A}{r_A} + \frac{3}{2} \frac{G}{r_A^5} J^{<kl>}_A r_A^l r_A^k \right)$$

$$+ \epsilon^4 \left\{ \delta_{ij} \varsigma \frac{G^2 m_{\odot}^2}{r_{\odot}^6} + \eta \frac{G^2 m_{\odot}^2}{r_{\odot}^5 \epsilon_{ijk} r_{\odot}^j} \right\} + \mathcal{O}(5), \quad (3)$$

where $\epsilon = 1/c$ and $\mathcal{O}(n)$ means of order $\epsilon^n$. $m_A$ and $S^i_A$ are the mass and intrinsic angular momentum (spin) of the body $A$; subscript “$\odot$” denotes the evaluation of the Sun; “$v^i_A$” is the coordinate velocity of the body $A$; $r^i_A = x^i - x^i_A(t)$ and the trajectory of the body $A$ is represented by $x_A(t)$ and $r_A = (r_A^i r_A^i)^{1/2}$. And $J^{<ik>} = J^{ik} - \frac{1}{2} \delta^{ik} J^s_A$ is the symmetric trace-free (STF) quadrupole moment of the body $A$. $\varsigma$ and $\eta$ are two new 2PPN parameters.

At the 1PN order, we keep all of the contributions from the mass monopoles of the Sun and the planets in the $\epsilon^2$ terms of $g_{00}$ and $g_{ij}$. Table 2 shows the their effects of the major bodies in the Solar System on the light ray when the light ray grazes the limb of one body. The biggest one comes from the Sun, which is about $10^{-6}$. We also include the quadrupole moments at this order to take account their leading contributions. Table 2 shows the effect of non-spherical part ($J_2$) of a body on light when a light ray grazes its limb. Although the fourth ($J_4$) and the sixth ($J_6$) zonal spherical harmonics for the giant planets, such as Jupiter and Saturn, can cause deflections larger than $1 \mu \text{as}$ for light deflection when the light ray arrived
their limbs, we leave them untouched in this work. In fact, their effects can be easily taken into account from the studies for Gaia mission.

In \( g_{0i} \) [see Eq. (2)], we include all of the contributions from the mass monopoles and their velocities of the Sun and the planets in the \( \epsilon^3 \) terms of \( g_{0i} \). Table 3 gives the estimation of \( g_{0i} \) related to a body’s orbital motion \( v_i^A \). In order to investigate the influence of the gravitomagnetic fields on light propagation, not only a body’s orbital motion but also its spin are taken into account in the metric of \( g_{0i} \) in our model.

For the 2PN terms in \( g_{00} \) [see Eq. (1)] and \( g_{ij} \) [see Eq. (3)], we only consider the contributions of the monopole of the Sun and neglect the 2PN contributions due to the planets as well as the nonlinear combinations between the planets and the Sun, and the terms \( \mathcal{O}(\epsilon^4 J_2^A) \) whose contributions are less than 1 \( \mu \)as. In the \( \epsilon^3 \) terms of \( g_{00} \) and \( g_{ij} \), we also omit the terms explicitly depending on \( v_\odot^2 \), such as \( G m_\odot c^{-4} r_\odot v_\odot^2 \), because their contributions to the deflection are at the level of femto-arcsecond.

It is very important to point out that the \( \epsilon^2 \) terms in \( g_{00} \) and \( g_{ij} \) and the \( \epsilon^3 \) terms in \( g_{0i} \) are all time-dependent, even including the terms associated with the Sun due to its barycentric motion \( x_\odot^i(t) \) and \( v_\odot^i(t) \). Kopeikin demonstrated that this motion of the Sun affects the measured values of the PPN parameters, and cannot be ignored. Moreover, the barycentric motion of the Sun was crucial in order to test general relativity in the Cassini experiment. Thus, when we integrate the trajectory of the light-ray (see Sec. 2.3), we will keep the velocities of all the bodies including the Sun.

At the 2PN order of \( g_{ij} \), we introduce two new parameters \( \varsigma \) and \( \eta \), which generalize the previous works. They have different values in different gravitational theories (see Table 5). When \( \varsigma = \eta = 1 \), the Eq. (3) reduces to GR. When \( \varsigma = 2\gamma^2 - \frac{1}{2} + 2\beta - \frac{5}{2} \) and \( \eta = \frac{1}{2}(1 + \gamma) \), the Eq. (3) returns to ST. When \( \varsigma = 1 + \frac{1}{2}c_{14} \) and \( \eta = 1 - \frac{1}{2}c_{14} \), the Eq. (3) coincides with \( \mathcal{E} \). When \( \varsigma = \eta \), the Eq. (3) goes back to the results of Ref. 22. When \( \varsigma = \frac{3}{2}\delta \) and \( \eta = 0 \), the Eq. (3) coincides with the result of Ref. 19.

**2.2. 2PPN equations of light**

Generally, for a photon propagating in a spacetime in which Einstein Equivalence Principle (EEP) is valid, the basic equations of light are

\[
g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} = 0, \tag{4}
\]

\[
\frac{d^2 x^\mu}{d\lambda^2} + \Gamma^\mu_{\nu\sigma} \frac{dx^\nu}{d\lambda} \frac{dx^\sigma}{d\lambda} = 0. \tag{5}
\]

We replace the affine parameter \( \lambda \) with coordinate time \( t \), and the equations become

\[
0 = g_{\mu\nu} \frac{dx^\mu}{dt} \frac{dx^\nu}{dt}. \tag{6}
\]
where the 1PN monopole component with the orbital motion is obtained the 2PPN equations of light propagation in SSBRS as follows (6). Then, by substituting \( \dot{\mathbf{x}} \) for analytical calculation, however, it is impractical. As an analytical ephemeris, our result Eq. (8) will be identical with Ref. 22. When we take GR into our model, in the calculation. Chebyshev polynomial could perfectly deal with the motion of the bodies by interpolated ephemerides. This has been guaranteed at DE/LE, iterative method used by Refs. 13, 14, 22.

When we consider a case of stationary gravitational field for one body and \( \gamma \equiv c s \mu = O(1) \) and \( \mu \cdot \mu = 1 \), we find the expression for \( s \) from Eq. (9). Then, by substituting \( \dot{\mathbf{x}} \cdot \dot{\mathbf{x}} \equiv c^2 s^2 \) and the metric Eqs. (11)-(13) into Eq. (7), we obtain the 2PPN equations of light propagation in SSBRS as follows

\[
\ddot{x}^i = F_{1PN}^i + F_{Q}^i + F_{S}^i + F_{2PN}^i,
\]

where the 1PN monopole component with the orbital motion is

\[
F_{1PN}^i = -(1 + \gamma) \sum_A \frac{G m_A}{r_A^2} \left\{ \left( 1 - 2 \frac{\dot{x} \cdot \mathbf{v}_A}{c^2} \right) r_A^i + 2 \frac{\dot{x} \cdot r_A}{c^2} v_A^i + \left[ \frac{r_A \cdot \mathbf{v}_A}{c} \right] \right\}.
\]

The influence of quadrupole moments of the bodies in the Solar System is

\[
F_{Q}^i = \frac{3(1 + \gamma)}{2} \sum_A \frac{G r_A^2}{r_A} \{ \left( \frac{\mathbf{S}_A \times \dot{\mathbf{x}}}{c^2} \right) + \frac{3}{2c^2 r_A^2} \left[ (\mathbf{S}_A \times \mathbf{r}_A) \cdot \dot{\mathbf{x}} \right] r_A^i - \frac{3}{2c^4 r_A^2} \left[ (\mathbf{S}_A \times \dot{\mathbf{x}} \cdot \mathbf{r}_A) \right] \dot{x}^i - \frac{3}{2c^2 r_A^2} (\mathbf{S}_A \times \mathbf{r}_A)^i \},
\]

and the 2PN monopole component of the Sun is

\[
F_{2PN}^i = 2(1 + \gamma) \sum_A \frac{G r_A^3}{c^2 r_A^2} \left\{ \frac{\mathbf{r}_A \cdot \mathbf{S}_A}{c^2} \right\} + \frac{3}{2c^2 r_A^2} \left[ (\mathbf{S}_A \times \dot{\mathbf{x}} \cdot \mathbf{r}_A) \right] \dot{x}^i - \frac{3}{2c^2 r_A^2} (\mathbf{S}_A \times \mathbf{r}_A)^i \},
\]

When we consider a case of stationary gravitational field for one body and \( \varsigma = \eta \), our result Eq. (8) will be identical with Ref. 22. When we take GR into our model \( (\gamma = 1) \), Eq. (8) will reduce to the result in Ref. 14.

### 2.3. Trajectory of the light-ray in the 2PPN Framework

The trajectory of the light-ray in the 2PPN framework can be obtained through integrating Eq. (8). However, it is difficult to derive it directly. So we adopt an iterative method used by Refs. 13, 14, 22.

But before that, we should find a way to describe the bodies’ motion analytically in the calculation. Chebyshev polynomial could perfectly deal with the motion of the bodies by interpolated ephemerides. This has been guaranteed at DE/LE, INPO and EPM ephemeris, due to its convenience for numerical calculation. For analytical calculation, however, it is impractical. As an analytical ephemeris,
VSOP\textsuperscript{42} used an iterative method by perturbations up to the eighth order of masses for the planets in order to deal with the motion of the bodies. But this approximate method is still lengthy for our investigation.

A Taylor expansion for the trajectory $x_A$ of the body $A$ at some moment $t_A$ is the simplest and practicable for our purpose. However, it leads to two sticky issues. One is the order of the Taylor expansion. The other is the determination of the moment $t_A$. For the first issue, we suppose that the motion of the bodies, including the Sun, is rectilinear and uniform, that is,

$$x_A(t) = x_A(t_A) + v_A(t_A)(t - t_A) + O[(t - t_A)^2].$$

(13)

Klioner \& Kopeikin\textsuperscript{14} have pointed out that the residual terms of Eq. (13) are negligible for the accuracy of $1 \mu$as through a reasonable choice of the moment $t_A$. This assumption, which means the path of a celestial body is approximated to a straight line, is enough for some measurements conducted in a short time span since the time of propagation of electromagnetic waves is very short with respect to the orbital period of a body in the Solar System. For instance, the time the light takes from the Sun to the Earth is about 8 minutes while the orbital period of the Earth is about 365 days.

For the second issue, Klioner \& Kopeikin\textsuperscript{14} indicated that $t_A$ can be used to minimize the error in the solution of the light propagation equations caused by the higher-order terms neglected in Eq. (13). Klioner \& Kopeikin\textsuperscript{14} also showed that the minimization procedure makes $t_A$ equal to the moment of the closest approach of the unperturbed light ray to the body deflecting the light ray. Kopeikin \& Schafer\textsuperscript{24} proved rigorously by solving Einstein and light ray propagation equations that at higher astrometric precision $t_A$ must be taken as the retarded instant of time corresponding to the retarded (Liénard-Wiechert) solution of linearized gravitational field equations. Klioner \& Peip\textsuperscript{43} used the numerical simulations and showed that it was sufficient to use the well-known solution for the light propagation in the field of a motionless mass monopole for the accuracy of $\sim 0.2 \mu$as and substituted in that solution the position of the body at the moment of closest approach. Klioner \& Peip\textsuperscript{43} found the post-Newtonian analytical solution for the body being at rest at its position at the moment of closest approach or at the retarded moment of time are virtually indistinguishable from each other for the Solar System applications and showed it attained an accuracy of $\sim 0.18 \mu$as if we took the form of uniformly moving bodies like Eq. (13). Therefore, we implement integration technique like Klioner \& Kopeikin\textsuperscript{14} to derive the trajectory of light-ray.

We assume the unperturbed light-ray as follows

$$x_N = x_0 + c(t - t_0)\hat{n},$$

(14)

where $\hat{n}$ is the initial direction of a light signal at the moment of emission $t_0$ and the position of the emission $x_0$. The photon’s coordinates can be written as sum of perturbations with respect to $x_N$:

$$x(t) = x_N + \delta x = x_N + \delta x_{1PN} + \delta x_Q + \delta x_S + \delta x_{2PN}. $$

(15)
For $\mathcal{F}_{1PN}$, we use the following assumption for motion of the bodies

$$\mathbf{r}_A(t) = \mathbf{x}(t) - \mathbf{x}_A(t) = \mathbf{x}(t) - \mathbf{x}_A(t_A) - \mathbf{v}_A(t_A)(t - t_A),$$

(16)

where $t_A$ is the moment of the closest approach between the body $A$ and the unperturbed light ray. Due to the smallness of $\mathcal{F}_Q$, $\mathcal{F}_S$ and $\mathcal{F}_{2PN}$, in comparison with $\mathcal{F}_{1PN}$, it is sufficient to suppose that

$$\mathbf{r}_A(t) = \mathbf{x}_N(t) - \mathbf{x}_A(t_A).$$

(17)

After these, we can obtain the following results

$$\frac{1}{c} \dot{\mathbf{x}}(t) = \hat{n} + \frac{1}{c} \delta \dot{x}_{1PN}(\mathbf{x}_N + \delta x_{1PN}) + \frac{1}{c} \delta \dot{x}_Q(\mathbf{x}_N) + \frac{1}{c} \delta \dot{x}_S(\mathbf{x}_N) + \frac{1}{c} \delta \dot{x}_{2PN}(\mathbf{x}_N),$$

(18)

$$\mathbf{x}(t) = \mathbf{x}_N(t) + \left[ \delta x_{1PN}(\mathbf{x}_N + \delta x_{1PN}) - \delta x_{1PN}(\mathbf{x}_0) \right] + \left[ \delta x_Q(\mathbf{x}_N) - \delta x_Q(\mathbf{x}_0) \right]$$

$$+ \left[ \delta x_S(\mathbf{x}_N) - \delta x_S(\mathbf{x}_0) \right] + \left[ \delta x_{2PN}(\mathbf{x}_N) - \delta x_{2PN}(\mathbf{x}_0) \right],$$

(19)

It is worthy of note that the 2PN terms in our solution actually have two sources, direct and indirect. The direct part comes from the 2PN order itself. The indirect part comes from the 1PN terms when the 1PN solution is iterated into itself in order to attain a 2PN accuracy, namely, we substitute $\mathbf{x}_N + \delta x_{1PN}$ into the trajectory of the light-ray in the 1PN approximation. During the procedure of iteration, we keep only the 2PN terms related to a motionless Sun. We can obtain that

$$\frac{1}{c} \delta \dot{x}_{1PN}(\mathbf{x}) = -(1 + \gamma) \sum_A \frac{G m_A}{c^2 r_A} \left\{ \frac{\hat{n} \times (\mathbf{r}_A \times \mathbf{k})}{kr_A - \mathbf{k} \cdot \mathbf{r}_A} + \mathbf{k} \right\},$$

(20)

$$\frac{1}{c} \delta \dot{x}_Q(\mathbf{x}) = (1 + \gamma) \sum_A \frac{G}{2c^2 r_A^3} \left\{ \frac{r_A(2r_A - \hat{n} \cdot \mathbf{r}_A)}{(r_A - \hat{n} \cdot \mathbf{r}_A)^2} \mathbf{q}_A + \left[ 1 - \frac{3(\hat{n} \cdot \mathbf{r}_A)^2}{r_A^2} \right] \mathbf{q}_A \right\},$$

(21)

$$\frac{1}{c} \delta \dot{x}_S(\mathbf{x}) = (1 + \gamma) \sum_A \frac{G}{c^3} \left\{ S_A \times \frac{d\mathbf{d}_A}{r_A^3} + \frac{d^2}{r_A^3} \left( r_A - \hat{n} \cdot \mathbf{r}_A \right)^2 \left[ (S_A \times \hat{n}) - \frac{d\mathbf{d}_A}{dA} \left( S_A \cdot (\hat{n} \times \mathbf{d}_A) \right) \right] \right\},$$

(22)

$$\frac{1}{c} \delta \dot{x}_{2PN}(\mathbf{x}) = -\frac{1}{2} \frac{G^2 m_\odot^2}{c^4 r_\odot} \left( \frac{\hat{n} \cdot \mathbf{r}_\odot}{r_\odot} \right) \mathbf{r}_\odot + \frac{G^2 m_\odot^2}{c^4} \frac{d\mathbf{d}_\odot}{r_\odot^2} \left\{ (1 + \gamma)^2 \cdot \frac{1}{r_\odot^2} \left( \frac{1}{r_\odot - \hat{n} \cdot \mathbf{r}_\odot} \right) + \frac{1}{r_\odot} \right\} \left[ \frac{\hat{n} \cdot \mathbf{r}_\odot}{r_\odot^2} + \frac{1}{r_\odot} \left( \frac{\pi}{2} + \arctan \frac{\hat{n} \cdot \mathbf{r}_\odot}{d_\odot} \right) \right] \right\}$$

$$+ \frac{G^2 m_\odot^2}{c^4 r_\odot} \left\{ (2\gamma(1 + \gamma) + \beta - \frac{1}{2}) \frac{1}{r_\odot} - \frac{1}{r_\odot^2} \right\}$$

(23)

and

$$\delta x_{1PN}(\mathbf{x}) = -(1 + \gamma) \sum_A \frac{G m_A}{c^2} \left\{ \frac{\hat{n} \times (\mathbf{r}_A \times \mathbf{k})}{kr_A - \mathbf{k} \cdot \mathbf{r}_A} + \mathbf{k} \ln \left( kr_A + \mathbf{k} \cdot \mathbf{r}_A \right) \right\},$$

(24)
Eqs. (18) and (19) will go back to the results of Ref. 17. When we consider GR.

3. Angular measurement

where \( d_A = \hat{n} \times (r_A \times \hat{n}) \) is an impact parameter for the body \( A \) which is the closest approach of the unperturbed light ray, \( d_A = |d_A|, r_A = x_N - x_A(t), r_A = |r_A|, k = \hat{n} - v_A(t)/c, k = |k| \) and

\[
\begin{align*}
    p_A^i &= 2J_A^{<ik>} \frac{d_A^i}{r_A} - 2J_A^{<kj>} \frac{d_A^i}{r_A} \hat{n}^k \hat{n}^i - J_A^{<jk>} \left( \hat{n}^i \hat{n}^k + \frac{4d_A^i d_A^k}{d_A^2} \right) \frac{d_A^j}{r_A}, \\
    q_A^i &= 2J_A^{<ik>} \frac{\hat{n}^j d_A^k}{d_A^2} d_A^i + J_A^{<jk>} \hat{n}^j \hat{n}^k - \frac{d_A^i d_A^k}{d_A^2} \frac{d_A^j}{r_A}, \\
    r_A^i &= 2J_A^{<ik>} \frac{d_A^k}{r_A} \hat{n}^i - J_A^{<jk>} \left( \hat{n}^i \hat{n}^k - \frac{d_A^i d_A^k}{d_A^2} \right) \frac{d_A^j}{r_A}, \\
    s_A^i &= 2J_A^{<ik>} \hat{n}^i - 4J_A^{<jk>} \frac{\hat{n}^j d_A^k}{d_A^2} d_A^i - J_A^{<jk>} \left( \hat{n}^i \hat{n}^k - \frac{2d_A^i d_A^k}{d_A^2} \right) \frac{d_A^j}{r_A}.
\end{align*}
\]

If we only consider a static case for one body and assume \( \eta = \gamma \), Eqs. 18 and 19 will return to the results of Ref. 17. When we neglect \( F_{2PN}^i \) and \( F_{S}^i \), Eqs. 13 and 17 will go back to the results of Ref. 10. When we consider GR (\( \gamma = \beta = \zeta = \eta = 1 \)), Eqs. 13 and 17 will reduce to the results of Ref. 10.

3. Angular measurement

3.1. Construction of a gauge-invariant angle in the 2PPN framework

In practical astrometric measurements, a differential measurement is a more powerful method. This concept is employed by LATOR mission through a skinny triangle formed by two spacecrafts and the ISS. In what follows, we mainly focus on discussing LAROR-like missions. Firstly, we construct a gauge-invariant angle \( \theta \) between the directions of the two incoming photons based on Ref. 13. It reads

\[
\cos \theta = \frac{h_{\alpha \beta} K_1^\alpha K_2^\beta}{\sqrt{h_{\alpha \beta} K_1^\alpha K_1^\beta / h_{\alpha \beta} K_2^\alpha K_2^\beta}}.
\]
where the spatial projection operator is
\[ h_{\alpha\beta} = g_{\alpha\beta} + u_\alpha u_\beta, \] (33)
which projects the two incoming photons onto the hypersurface orthogonal to the observer’s four-velocity \( u^\alpha \equiv dx^\alpha/cdt \) and \( K_1^\alpha \equiv dx_1^\alpha(t)/dt \), and \( K_2^\alpha \equiv dx_2^\beta(t)/dt \) are the tangent vectors of the paths \( x_1(t) \) and \( x_2(t) \) of the two incoming photons.

Then, we obtain
\[ \cos \theta = (\hat{n}_1 \cdot \hat{n}_2) + \left( f_{\text{obs}} + f_{\text{1PN}} + f_{\text{Q}} \right) + \left( f_{\text{obs}x1\text{PN}} + f_{\text{S}} \right) \]
\[ + \left( f_{\text{obs}x2\text{PN}} \right), \] (34)
where the first term is the angle between the unperturbed paths of the photons from two signals, \((n)\) denotes terms of order \( \epsilon^n \), the subscript “obs” denotes terms related to the observer’s velocity, subscript “1PN” denotes the contribution from the monopoles and orbital motions of gravitating bodies, subscript “Q” denotes the terms from their quadrupole moments, subscript “obs x 1PN” denotes the coupling terms of the bodies’ mass and observer’s velocity, subscript “S” denotes the terms of bodies’ spin and subscript “⊙” denotes the Sun monopole contribution. The expressions for these functions are presented in [Appendix A].

Obviously, the position of the photon at the moment \( t \) of observation coincides with the position of the spacecraft (observer) so that
\[ x_{\text{obs}} = x_{01} + c(t - t_{01})\hat{n}_1 + \delta x_1 = x_{02} + c(t - t_{02})\hat{n}_2 + \delta x_2, \] (35)
where \((t_{01}, x_{01})\) denotes the moment and position of the light signal 1 of emission and \((t_{02}, x_{02})\) for the light signal 2 respectively. Eq. (35) also give a constraint at the moment of measurement that
\[ r_{1A} = r_{2A} = r_{\text{obs}A} = r_{A}. \] (36)

Eq. (34) can be expanded with respect to the parameter \( \epsilon \) as
\[ \theta(t) = \vartheta_0 + \left( \vartheta_{\text{obs}} + \left( \vartheta_{\text{1PN}} + \vartheta_{\text{Q}} \right) \right) + \left( \vartheta_{\text{obs}x1\text{PN}} + \vartheta_{\text{S}} \right) \]
\[ + \left( \vartheta_{\text{obs}x2\text{PN}} \right), \] (37)
where \( \vartheta_0 \in (0, \pi) \) is the angle between the unperturbed light paths from two given sources
\[ \vartheta_{\text{obs}} = \arccos(\hat{n}_1 \cdot \hat{n}_2), \] (38)
and \((n)\) \( \vartheta_{\text{obs}} \) is the deflection angle due to the observer’s motion in terms of order \( \epsilon^n \), \( \vartheta_{\text{1PN}} \) is the 1PN deflection angle due to the spherically symmetric field of
gravitating bodies, $\vartheta_Q$ is due to quadrupole moment, $\vartheta_{OM}$ is due to the orbital motions of the bodies, $\vartheta_S$ is due to the spin of the bodies and $\epsilon^4 \vartheta_{2PN}$ is the 2PN deflection angle due to the spherically symmetric field of the Sun (see Appendix B for detailed expressions).

### 3.2. Special cases

As one kind of the verification, our results could reduce to some special cases and some known results, such as aberration of special relativity, 1PN light deflection, 2PN light deflection and Light deflection due to spins.

#### 3.2.1. Aberration

In Eq. (37), we pick up the terms due to the velocity of the observer

$$\theta = \vartheta_0 + \vartheta_{obs} + \vartheta_{obs},$$

and assume that one signal propagates along the direction of motion of the observer, namely, $v_{obs} = v\hat{n}_1$. Then, we obtain

$$\theta = \vartheta_0 + \epsilon v \sin \vartheta_0 + \epsilon^2 \frac{1}{2} v^2 \sin \vartheta_0 \cos \vartheta_0.$$  \hspace{1cm} (40)

This is just the aberration given by special relativity after ignoring $O(\epsilon^3)$.

#### 3.2.2. 1PN light deflection

The path of light is bent in a gravitational field, which represents by

$$\theta = \vartheta_{1PN}.$$  \hspace{1cm} (41)

We assume that only the Sun is considered and one of the two light rays is just along the line connecting the Sun and the observer and without any bending, namely source 2. Then, we obtain

$$\vartheta_{1PN} \approx \left( 1 + \gamma \right) \frac{4Gm_\odot}{c^2 d^{1}_\odot} \left( 1 + \cos \vartheta_0 \right).$$  \hspace{1cm} (42)

where we use $\hat{n}_2 \cdot \mathbf{d}_{1A}/d_{1A} = \sin \vartheta_0$, $\hat{n}_1 \cdot \mathbf{r}_A/r_A \simeq \cos \vartheta_0$ and $d_{1A}^2 = r_A^2 - (\hat{n}_1 \cdot \mathbf{r}_A)^2$.

For GR ($\gamma = 1$), When $\cos \vartheta_0 = 1$, our results reduces to the famous infinity-infinity light deflection formula.

Brumberg discussed the angle of the two incoming photons in 1PN general relativity for a $N$-body system. Now, we discuss our result about 1PN gravitational deflection, namely,

$$\cos \theta = (\hat{n}_1 \cdot \hat{n}_2) + \vartheta_{1PN}$$
When $\gamma = 1$, this is just the result of Ref. [13].

3.2.3. 2PN light deflection

By extending the previous calculation to 2PN order, we have $\arctan (\hat{n}_1 \cdot r_{\odot}/d_{1\odot}) \approx \pi/2$, $d_{1\odot}/r_{\odot} \approx 0$ and

$$(44) \quad \vartheta_{2\text{PN}}^{(4)} \approx -2(1 + \gamma)^2 + 2(1 + \gamma) - \beta + \frac{1}{4}(2\varsigma + \eta)]\pi \quad G^2 m_{\odot}^2 c^4 d_{1\odot}^2.$$

If the above result returns to GR ($\gamma = \beta = \varsigma = \eta = 1$), it coincides with the result of GR within harmonic gauge. If $(2\varsigma + \eta) \equiv 3\Lambda$, it will coincide with the result of Ref. [9]. In the case of $\varsigma = \eta$, it will reproduce the result of Ref. [22].

3.2.4. Light deflection due to spins

The spin of a body can also produce a light deflection. It belongs to one part of gravitomagnetic field. It reads as

$$\theta = \vartheta_S \quad (45)$$

Repeating previous calculations, we obtain

$$\theta \approx 2(1 + \gamma) \frac{|S_A|}{c^2 d_{1A}^2} + 4(1 + \gamma) \frac{|S_A \cdot (\hat{n}_1 \times d_{1A})|}{c^2 d_{1A}^2} \quad (46)$$

This is just the result of Ref. [25] with $\gamma = 1$ with turning their vector angle for deflection by reason of spin to a scalar quantity.

3.3. A MODEL FOR LATOR-LIKE MISSIONS

In this section, we apply our model to LATOR-like missions by qualitative estimate. We assume there are three spacecrafts in an orbit circling the Sun at the same distance as the Earth. Two of them carry a light signal emitter on board and the third one carries a light signal receiver. When the two spacecrafts and the third one are on the opposite side of the Sun, the experiment will be conducted. When a light signal 1 emitted by one of two spacecrafts passes by the limb of the Sun, another signal 2 is emitted by the other spacecraft which has quite a distance from the Sun. The receiver located on the third spacecraft which is on the opposite side of those two spacecrafts measures the angle between the two incoming photons. In the case of LATOR[27] the observer will be set on the ISS.
We assume that the velocity of the observer in SSBRP is near the orbital velocity of the Earth. In this case, we need to consider three bodies’ gravitational fields: the Sun, Mercury and Venus. We can neglect the effect of gravitational field of the Earth through making the position of the observer away from the Earth. However, for the case of ISS, the gravitational field of the Earth affects the motion of ISS and this effect can be represented by equations of motion for ISS. The influence of other gravitating bodies in the Solar System on the light propagation in this case can be neglected. For LATOR, signals are emitted inside the Solar System, but we could extended the trajectories to \( t = -\infty \) to make make \( \hat{n}_1 \) and \( \hat{n}_2 \) meaningful. With the angle between the initial emitting directions of two light signals 1 and 2

\[
\theta_0 = \hat{n}_1 \cdot \hat{n}_2 \approx 1^\circ ,
\]

it means that the distance between these two spacecrafts is about \( 5.22 \times 10^9 \) m. We can simplify Eq. (47) for this practical case as follows

\[
\theta = \frac{\theta_0}{2} + \epsilon (\hat{n}_2 \cdot \mathbf{v}_{obs}) \tan \left( \frac{\theta_0}{2} \right) - \epsilon^2 \mathbf{v}_{obs}^2 \tan \left( \frac{\theta_0}{2} \right) + \epsilon^2 \gamma (1 + \gamma) \frac{G m_{\odot}}{d_1} + \epsilon^2 \gamma (1 + \gamma) \frac{G m_{Mer}}{d_1} + \epsilon^2 \gamma (1 + \gamma) \frac{G m_{V}}{d_1} + \epsilon^3 \gamma (1 + \gamma) \frac{G m_{\odot}}{d_2} (\hat{n}_2 \cdot \mathbf{v}_{obs}) - \epsilon^4 (1 + \gamma)^2 \frac{G^2 m_{\odot}^2}{d_1^2} + \epsilon^4 [1 + (2\zeta + \eta - 2(1 + \gamma)^2 \frac{G^2 m_{\odot}^2}{d_1^2} + \arctan \left( \hat{n}_1 \cdot \mathbf{r}_{\odot} \right)] + \epsilon^4 [1 + (2\zeta + \eta - 2(1 + \gamma)^2 \frac{G^2 m_{\odot}^2}{d_2^2} + \arctan \left( \hat{n}_2 \cdot \mathbf{r}_{\odot} \right)] + O(\sim 1 \mu as), \tag{48}
\]

with the cut-off precision of \( \sim 1 \) \( \mu \)as. Here we consider the largest influence of Mercury (subscript “Mer”) and Venus (subscript “V”) on the measurement when the light signals might pass by the limbs of them. We estimate the order of these terms on Eq. (48) which are listed on Table [6].

For LATOR mission, it will attain the precision of 0.01 \( \mu \)as. It needs to consider a complete form at \( \epsilon^{-4} \) of \( g_{ij} \) and \( g_{00} \). Various terms at \( \epsilon^{-4} \) of \( g_{ij} \) and \( g_{00} \) respectively are \( \sum_A \frac{G m_A^2}{c^2 v_A^2} \), \( \sum_A \frac{G^2 m_A^2}{c^4 v_A^2} \), \( \sum_B \sum_{A \neq B} \frac{G m_A m_B}{c^2 v_A v_B} \), \( \sum_A \sum_B \frac{G m_A m_B}{c^2 v_A v_B} \),
\[
\sum_{A} \frac{G m_A}{c^2 r_A} v_A^i u_A^j, \quad \sum_{A} \frac{G m_A}{c^2 r_A} r_A^i r_A^j, \quad \sum_{A} \sum_{B \neq A} \frac{G m_A m_B}{c^2 (r_A + r_B + r_{AB})^2} r_{AB}^i r_{AB}^j,
\]
and
\[
\sum_{A} \sum_{B \neq A} \frac{G m_A m_B}{c^2 (r_A + r_B + r_{AB})^2} (r_{AB}^i r_{AB}^j - \frac{r_A^i r_A^j}{r_{AB}} - \frac{r_B^i r_B^j}{r_{AB}} + \frac{r_{AB}^i r_{AB}^j}{r_{AB}^2}).
\]
These terms have been shown by Ref. [23]. These terms can be considered as different combinations between terms in Table 2 and in Table 4. If the model will be done in the 2PPN framework with the accuracy of 0.01\(\mu\)as, it comes down to deal with the problems for integral of these two-body terms in the 2PN order analytically and 2PN parametrized issue in these coupling terms. Besides, LATOR also measures laser ranging for three sides of the triangle, it needs to derive the light time solution. And the spacecrafts in LATOR mission are constantly moving. The triangle formed by the spacecrafts are also changing. \(\vartheta_0 \approx 1^\circ\) is just a special case which presents the maximal effect of this measurement. We will study these issues numerically in our next move.

4. CONCLUSIONS AND PROSPECTS

In this paper, we present light propagation and a gauge-invariant angular measurement in the 2PPN framework by introducing two new parameters \(\varsigma\) and \(\eta\) besides the two PPN parameters \(\gamma\) and \(\beta\). In the framework, we consider all kinds of relativistic effects on light propagation in SSBRS, which are monopole and quadrupole moments of the bodies in the Solar System, their motions and their gravitomagnetic fields. With the derivation of a gauge-invariant angle between the directions of two incoming photons, we further discuss a practical astronomic observation, namely, an observer on a spacecraft measures the angle between the two incoming photons emitted separately two spacecrafts which are all at an orbit circling the Sun at the same distance as the Earth. Given attaining a level of \(\sim 1\ \mu\)as for space astrometry missions in the near future, like LATOR mission, the terms at this level are listed.

In this work, we take planetary motions linearly for simplicity in our analytic calculations. It makes our model does not work for a long time which depends on specific cases. Our next move is to study this problem numerically by integrating the equations of motion of \(N\)-body and light simultaneously. How to parameterize the 2PN coupling terms in these experiments is another issue that will be investigated.

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Appendix A. Functions in \(\cos \theta\)

The expressions of the functions in Eq. (34) are

\[
(\text{A.1})
\]

\[
\hat{f}_{\text{obs}}^{(1)} = \epsilon \sum_{k=1}^{2} (\hat{n}_k \cdot \mathbf{v}_{\text{obs}}) \left[ (\hat{n}_1 \cdot \hat{n}_2) - 1 \right],
\]
The second post-Newtonian Light Propagation and Its Astrometric Measurement in the Solar System

\[ f_{obs} = \epsilon^2 \left[ \sum_{k=1}^{2} (\hat{n}_k \cdot v_{obs})^2 + \frac{1}{2} \sum_{k,j=1 \atop k \neq j}^{2} (\hat{n}_k \cdot v_{obs})(\hat{n}_j \cdot v_{obs}) - v_{obs}^2 \right] (\hat{n}_1 \cdot \hat{n}_2) \] (A.2)

\[ f_{1PN} = \sum_{h,j=1 \atop h \neq j}^{2} \hat{n}_j \cdot \frac{1}{c} \delta x_{1PN}(x_N + \delta x_{1PN})|_h \]

\[- \sum_{j=1}^{2} (\hat{n}_1 \cdot \hat{n}_2) \hat{n}_j \cdot \frac{1}{c} \delta x_{1PN}(x_N + \delta x_{1PN})|_j, \] (A.3)

\[ f_{Q} = \sum_{h,j=1 \atop h \neq j}^{2} \hat{n}_j \cdot \frac{1}{c} \delta x_Q(x_N)|_h - \sum_{j=1}^{2} (\hat{n}_1 \cdot \hat{n}_2) \hat{n}_j \cdot \frac{1}{c} \delta x_Q(x_N)|_j, \] (A.4)

\[ f_{obs} = \epsilon^3 \left[ \sum_{k=1}^{2} v_{obs} \cdot \frac{1}{c} \delta x_{1PN}(x_N)|_k \left[ (\hat{n}_1 \cdot \hat{n}_2) - I \right] + \epsilon \sum_{j,h=1 \atop h \neq j}^{2} (\hat{n}_j \cdot v_{obs}) \hat{n}_j \cdot \frac{1}{c} \delta x_{1PN}(x_N)|_h \]

\[- 3\epsilon \sum_{j=1}^{2} (\hat{n}_1 \cdot \hat{n}_2) (\hat{n}_j \cdot v_{obs})(\hat{n}_j \cdot \frac{1}{c} \delta x_{1PN}(x_N)|_j \]

\[- \epsilon \sum_{j,h=1 \atop h \neq j}^{2} (\hat{n}_1 \cdot \hat{n}_2)(\hat{n}_j \cdot v_{obs}) \hat{n}_h \cdot \frac{1}{c} \delta x_{1PN}(x_N)|_h, \] (A.6)

\[ f_S = \sum_{h,j=1 \atop j \neq h}^{2} \hat{n}_j \cdot \frac{1}{c} \delta x_S(x_N)|_h - \sum_{j=1}^{2} (\hat{n}_1 \cdot \hat{n}_2) \hat{n}_j \cdot \frac{1}{c} \delta x_S(x_N)|_j, \] (A.7)

\[ f_{obs} = \epsilon^4 \left[ \sum_{k=1}^{2} (\hat{n}_k \cdot v_{obs})^4 - \sum_{k=1}^{2} (\hat{n}_k \cdot v_{obs})^2 v_{obs}^2 - \frac{1}{2} \sum_{k,j=1 \atop k \neq j}^{2} (\hat{n}_k \cdot v_{obs})(\hat{n}_j \cdot v_{obs}) v_{obs}^2 + \right. \]

\[ + \sum_{k,j=1 \atop k \neq j}^{2} (\hat{n}_k \cdot v_{obs})^3 \hat{n}_j \cdot v_{obs} + \frac{1}{2} \sum_{k,j=1 \atop k \neq j}^{2} (\hat{n}_k \cdot v_{obs})^2 (\hat{n}_j \cdot v_{obs})^2 \] \left[ (\hat{n}_1 \cdot \hat{n}_2) \right] (A.8)

\[ f_{obs} = -(1 + \gamma) \epsilon^4 \sum_{k=1}^{2} \frac{Gm_{\odot}}{r_{obs}^{(\odot)}} (\hat{n}_k \cdot v_{obs})^2 (\hat{n}_1 \cdot \hat{n}_2) + (1 + \gamma) \epsilon^4 \sum_{k,j=1 \atop k \neq j}^{2} \frac{Gm_{\odot}}{r_{obs}^{(\odot)}} (\hat{n}_k \cdot v_{obs})(\hat{n}_j \cdot v_{obs}) \]

\[- \epsilon^2 \sum_{j=1}^{2} v_{obs}^2 \hat{n}_j \cdot \frac{1}{c} \delta x_{1PN}(x_N)|_h + 3\epsilon^2 \sum_{j=1}^{2} (\hat{n}_1 \cdot v_{obs})(\hat{n}_2 \cdot v_{obs}) \hat{n}_j \cdot \frac{1}{c} \delta x_{1PN}(x_N)|_j \]
\[\begin{align*}
+3c^2 \sum_{j=1}^2 (\hat{n}_j \cdot \text{v}_{\text{obs}})^2 \hat{n}_j \cdot \frac{1}{c} \delta \hat{x}_{1PN}(x_N)|_j &+ c^2 \sum_{j,h=1 \atop j \neq h}^2 (\hat{n}_j \cdot \text{v}_{\text{obs}})^2 \hat{n}_j \cdot \frac{1}{c} \delta \hat{x}_{1PN}(x_N)|_h \\
+ c^2 \sum_{j,h=1 \atop j \neq h}^2 (\hat{n}_1 \cdot \text{v}_{\text{obs}})(\hat{n}_2 \cdot \text{v}_{\text{obs}})\hat{n}_j \cdot \frac{1}{c} \delta \hat{x}_{1PN}(x_N)|_h \\
- 2c^2 \sum_{j=1}^2 (\hat{n}_j \cdot \text{v}_{\text{obs}})\text{v}_{\text{obs}} \cdot \frac{1}{c} \delta \hat{x}_{1PN}(x_N)|_j - c^2 \sum_{j,h=1 \atop j \neq h}^2 (\hat{n}_j \cdot \text{v}_{\text{obs}})\text{v}_{\text{obs}} \cdot \frac{1}{c} \delta \hat{x}_{1PN}(x_N)|_h \\
+ 2c^2 \sum_{j=1}^2 \text{v}_{\text{obs}}^2(\hat{n}_1 \cdot \hat{n}_2)\hat{n}_j \cdot \frac{1}{c} \delta \hat{x}_{1PN}(x_N)|_j + 2c^2 \sum_{j=1}^2 (\hat{n}_1 \cdot \hat{n}_2)(\hat{n}_j \cdot \text{v}_{\text{obs}})\text{v}_{\text{obs}} \cdot \frac{1}{c} \delta \hat{x}_{1PN}(x_N)|_j \\
+ c^2 \sum_{j,h=1 \atop j \neq h}^2 (\hat{n}_1 \cdot \hat{n}_2)(\hat{n}_j \cdot \text{v}_{\text{obs}})\text{v}_{\text{obs}} \cdot \frac{1}{c} \delta \hat{x}_{1PN}(x_N)|_h \\
- c^2 \sum_{j,h=1 \atop j \neq h}^2 (\hat{n}_1 \cdot \hat{n}_2)(\hat{n}_j \cdot \text{v}_{\text{obs}})^2 \hat{n}_h \cdot \frac{1}{c} \delta \hat{x}_{1PN}(x_N)|_h \\
- 3c^2 \sum_{j=1}^2 (\hat{n}_1 \cdot \text{v}_{\text{obs}})(\hat{n}_2 \cdot \text{v}_{\text{obs}})(\hat{n}_1 \cdot \hat{n}_2)\hat{n}_j \cdot \frac{1}{c} \delta \hat{x}_{1PN}(x_N)|_j \\
- 6c^2 \sum_{j=1}^2 (\hat{n}_1 \cdot \hat{n}_2)(\hat{n}_j \cdot \text{v}_{\text{obs}})^2 \hat{n}_j \cdot \frac{1}{c} \delta \hat{x}_{1PN}(x_N)|_j,
\end{align*}\]

\[\begin{align*}
(f_{2PN}^{(4)}) &= \frac{1}{2} \eta^4 \sum_{k,j=1 \atop k \neq j}^2 \frac{G^2 m_0^2}{r_{\text{obs}}^4}(\hat{n}_k \cdot \text{r}_{\text{obs}})(\hat{n}_j \cdot \text{r}_{\text{obs}}) - \frac{1}{2} \eta^4 \sum_{k=1}^2 \frac{G^2 m_0^2}{r_{\text{obs}}^4}(\hat{n}_k \cdot \text{r}_{\text{obs}})^2(\hat{n}_1 \cdot \hat{n}_2) \\
- \frac{1}{2} \sum_{j,l,n=1 \atop j \neq n}^2 \sum_{l,l,n=1 \atop j \neq l}^2 \delta \hat{x}_{1PN}(x_N)|_j \hat{n}_l \cdot \frac{1}{c} \delta \hat{x}_{1PN}(x_N)|_n \\
- \frac{1}{2} \sum_{j=1}^2 (\hat{n}_1 \cdot \hat{n}_2)\hat{n}_j \cdot \frac{1}{c} \delta \hat{x}_{1PN}(x_N)|_j \hat{n}_l \cdot \frac{1}{c} \delta \hat{x}_{1PN}(x_N)|_j \\
+ \frac{3}{2} \sum_{j=1}^2 (\hat{n}_1 \cdot \hat{n}_2)\hat{n}_j \cdot \frac{1}{c} \delta \hat{x}_{1PN}(x_N)|_j \hat{n}_j \cdot \frac{1}{c} \delta \hat{x}_{1PN}(x_N)|_j + \frac{1}{c} \delta \hat{x}_{1PN}(x_N)|_1 \cdot \frac{1}{c} \delta \hat{x}_{1PN}(x_N)|_2 \\
+ (\hat{n}_1 \cdot \hat{n}_2)\hat{n}_j \cdot \frac{1}{c} \delta \hat{x}_{1PN}(x_N)|_2 \hat{n}_1 \cdot \frac{1}{c} \delta \hat{x}_{1PN}(x_N)|_1 + \sum_{j,h=1 \atop j \neq h}^2 \hat{n}_j \cdot \frac{1}{c} \delta \hat{x}_{2PN}(x_N)|_h \\
- \frac{3}{2} \sum_{j=1}^2 (\hat{n}_1 \cdot \hat{n}_2)\hat{n}_j \cdot \frac{1}{c} \delta \hat{x}_{2PN}(x_N)|_j,
\end{align*}\]
Appendix B. Angle expressions

We list the expressions of the deflection angles in Eq. (37).

\[
\vartheta_\text{obs}^{(1)} = \varepsilon \left[ \hat{n}_1 \cdot \mathbf{v}_\text{obs} + (\hat{n}_2 \cdot \mathbf{v}_\text{obs}) \right] \frac{1 - \cos \vartheta_0}{\sin \vartheta_0}, \quad (B.1)
\]

\[
\vartheta_\text{obs}^{(2)} = \varepsilon^2 \frac{\sin \vartheta_0}{(1 + \cos \vartheta_0)^2} \left[ \left( (\hat{n}_1 \cdot \mathbf{v}_\text{obs})^2 + (\hat{n}_2 \cdot \mathbf{v}_\text{obs})^2 \right) \left( 1 + \frac{1}{2} \cos \vartheta_0 \right) + (\hat{n}_1 \cdot \mathbf{v}_\text{obs})(\hat{n}_2 \cdot \mathbf{v}_\text{obs}) - v_\text{obs}^2 (1 + \cos \vartheta_0) \right], \quad (B.2)
\]

\[
\vartheta_{1PN}^{(2)} = (1 + \gamma)^2 \sum A \frac{G m_A}{r_A \sin \vartheta_0} \left\{ \frac{\hat{n}_1 \cdot [\hat{n}_2 \times (r_A \times \hat{n}_1)]}{r_A - \hat{n}_1 \cdot r_A} + \frac{\hat{n}_2 \cdot [\hat{n}_1 \times (r_A \times \hat{n}_1)]}{r_A - \hat{n}_1 \cdot r_A} \right\}, \quad (B.3)
\]

\[
\vartheta_Q^{(2)} = -(1 + \gamma)^2 \sum A \frac{G}{r_A \sin \vartheta_0} \left\{ \frac{J_A^{i<k>}[\hat{n}_1\cdot\mathbf{d}_2]+(\hat{n}_1\cdot\mathbf{d}_2^k)}{2r_A^2} \left[ \begin{array}{c} 2r_A - \hat{n}_2 \cdot r_A \\ (r_A - \hat{n}_2 \cdot r_A)^2 \end{array} \right] d_{2A}^k - \hat{n}_2 \cdot r_A \right\}
\]

\[
\vartheta_{\text{obs}}^{(3)} = \frac{G m_A}{6 \sin^3 \vartheta_0} \left[ (\hat{n}_1 \cdot \mathbf{v}_\text{obs}) + (\hat{n}_2 \cdot \mathbf{v}_\text{obs}) \right]^3
\]

\[
+ \frac{G m_A}{6 \sin^3 \vartheta_0} \left[ (\hat{n}_1 \cdot \mathbf{v}_\text{obs}) + (\hat{n}_2 \cdot \mathbf{v}_\text{obs}) \right] \left[ v_\text{obs}^2 - (\hat{n}_1 \cdot \mathbf{v}_\text{obs})^2 - (\hat{n}_2 \cdot \mathbf{v}_\text{obs})^2 \right]
\]

\[
- (\hat{n}_1 \cdot \mathbf{v}_\text{obs})(\hat{n}_2 \cdot \mathbf{v}_\text{obs}) + \left( \frac{1 - \cos \vartheta_0}{\sin \vartheta_0} \right) \left[ (\hat{n}_1 \cdot \mathbf{v}_\text{obs})^3 + (\hat{n}_2 \cdot \mathbf{v}_\text{obs})^3 \right]
\]

\[
(1 + \gamma)^3 \frac{G m_A}{6 \sin^3 \vartheta_0} \left\{ \left( \hat{n}_1 \cdot \mathbf{v}_\text{obs} \right) + \left( \hat{n}_2 \cdot \mathbf{v}_\text{obs} \right) \right\} \left[ \begin{array}{c} v_\text{obs} \cdot \mathbf{d}_{1A} \\ r_A - \hat{n}_1 \cdot r_A \\ r_A - \hat{n}_2 \cdot r_A \end{array} \right] + (1 + \gamma)^3 \frac{G m_A}{6 \sin \vartheta_0} \left\{ \left( \hat{n}_1 \cdot \mathbf{v}_\text{obs} \right) + \left( \hat{n}_2 \cdot \mathbf{v}_\text{obs} \right) \right\} \left[ \begin{array}{c} v_\text{obs} \cdot \mathbf{d}_{2A} \\ r_A - \hat{n}_2 \cdot r_A \\ r_A - \hat{n}_1 \cdot r_A \end{array} \right] \quad (B.4)
\]

\[
\vartheta_{OM}^{(3)} = (1 + \gamma)^3 \sum A \frac{G m_A}{\sin \vartheta_0} \left[ \left( \frac{\hat{n}_1 \cdot \mathbf{d}_{2A}}{r_A (r_A - \hat{n}_2 \cdot r_A)} - \frac{\hat{n}_2 \cdot \mathbf{d}_{1A}}{r_A (r_A - \hat{n}_1 \cdot r_A)} + \frac{\hat{n}_1 \cdot \mathbf{d}_{2A}}{r_A (r_A - \hat{n}_2 \cdot r_A)} \right) \right], \quad (B.5)
\]
\begin{equation}
\frac{\hat{\mathbf{n}}_2 \cdot \mathbf{d}_1}{r_A^2} + \frac{\hat{\mathbf{n}}_1 \cdot \mathbf{v}_A}{r_A} = \frac{\hat{\mathbf{n}}_1 \cdot \mathbf{d}_2}{r_A^2} + \frac{\hat{\mathbf{n}}_2 \cdot \mathbf{v}_A}{r_A} = \frac{\hat{\mathbf{n}}_1 \cdot \mathbf{d}_1}{r_A^2} \frac{\hat{\mathbf{n}}_2 \cdot \mathbf{v}_A}{r_A} + \frac{\hat{\mathbf{n}}_1 \cdot \mathbf{v}_A}{r_A} + \frac{\hat{\mathbf{n}}_2 \cdot \mathbf{v}_A}{r_A} + \frac{\hat{\mathbf{n}}_1 \cdot \mathbf{d}_2}{r_A^2} \frac{\hat{\mathbf{n}}_2 \cdot \mathbf{v}_A}{r_A} + \frac{\hat{\mathbf{n}}_1 \cdot \mathbf{v}_A}{r_A} + \frac{\hat{\mathbf{n}}_2 \cdot \mathbf{v}_A}{r_A} \right], \quad (B.6)
\end{equation}

\begin{equation}
\delta_S = -(1 + \gamma) \epsilon^3 \sum_A \frac{G}{\sin \theta_0} \left\{ - \frac{S_A \cdot (\hat{\mathbf{n}}_1 \times \hat{\mathbf{n}}_2)}{d_A^2} \right\} \left[ 1 + \frac{\hat{\mathbf{n}}_2}{r_A} \frac{\hat{\mathbf{n}}_1}{r_A} - \frac{\hat{\mathbf{n}}_1}{r_A} \frac{\hat{\mathbf{n}}_2}{r_A} \right] + \frac{\hat{\mathbf{n}}_1 \cdot \mathbf{d}_1 A}{d_A^2} \frac{\mathbf{d}_2 A}{d_A^2} + \frac{\hat{\mathbf{n}}_2 \cdot \mathbf{d}_1 A}{d_A^2} \frac{\mathbf{d}_2 A}{d_A^2} \right\} \left[ 1 + \frac{\hat{\mathbf{n}}_1 + \mathbf{d}_1 A}{r_A} \frac{\hat{\mathbf{n}}_2 + \mathbf{d}_2 A}{r_A} \right] + \frac{\hat{\mathbf{n}}_1 \cdot \mathbf{d}_1 A}{r_A^2} \frac{\mathbf{d}_2 A}{r_A^2} + \frac{\hat{\mathbf{n}}_2 \cdot \mathbf{d}_1 A}{r_A^2} \frac{\mathbf{d}_2 A}{r_A^2} \right\} \left[ 1 + \frac{\hat{\mathbf{n}}_1}{r_A} \frac{\hat{\mathbf{n}}_2}{r_A} \right] + \frac{\hat{\mathbf{n}}_1}{r_A} \frac{\hat{\mathbf{n}}_2}{r_A} \right]\right\}, \quad (B.7)
\end{equation}

\begin{equation}
\delta_{\text{obs}} = -\epsilon^4 \cos \theta_0 (3 + 2 \cos^2 \theta_0) (1 - \cos \theta_0)^4 \left\{ (\hat{\mathbf{n}}_1 \cdot \mathbf{v}_{\text{obs}}) + (\hat{\mathbf{n}}_2 \cdot \mathbf{v}_{\text{obs}}) \right\}^4 + \epsilon^4 \left\{ (1 + 2 \cos^2 \theta_0) (1 - \cos \theta_0)^3 \right\} \left\{ (\hat{\mathbf{n}}_1 \cdot \mathbf{v}_{\text{obs}}) + (\hat{\mathbf{n}}_2 \cdot \mathbf{v}_{\text{obs}}) \right\}^2 (\hat{\mathbf{n}}_1 \cdot \mathbf{v}_{\text{obs}})^2 + (\hat{\mathbf{n}}_2 \cdot \mathbf{v}_{\text{obs}})^2 + (\hat{\mathbf{n}}_1 \cdot \mathbf{v}_{\text{obs}})^2 + (\hat{\mathbf{n}}_2 \cdot \mathbf{v}_{\text{obs}})^2\right\}
\end{equation}
\[ +(1 + \gamma)^2 \epsilon^4 \cot \vartheta_0 \frac{Gm_\odot}{r_\odot} \left[ 2 \hat{v}_{\text{obs}}^2 - 2(\hat{n}_1 \cdot \hat{v}_{\text{obs}})^2 - 2(\hat{n}_2 \cdot \hat{v}_{\text{obs}})^2 - 2(\hat{n}_1 \cdot \hat{v}_{\text{obs}}) (\hat{n}_2 \cdot \hat{v}_{\text{obs}}) \right] \]
\[ + \frac{(\hat{n}_2 \cdot \hat{v}_{\text{obs}})(\hat{v}_{\text{obs}} \cdot \hat{d}_{1\odot})}{r_\odot - \hat{n}_1 \cdot r_\odot} + \frac{(\hat{n}_1 \cdot \hat{v}_{\text{obs}})(\hat{v}_{\text{obs}} \cdot \hat{d}_{2\odot})}{r_\odot - \hat{n}_2 \cdot r_\odot} + 2 \frac{(\hat{n}_2 \cdot \hat{v}_{\text{obs}})(\hat{v}_{\text{obs}} \cdot \hat{d}_{2\odot})}{r_\odot - \hat{n}_2 \cdot r_\odot} \]
\[ + 2 \frac{(\hat{n}_1 \cdot \hat{v}_{\text{obs}})(\hat{v}_{\text{obs}} \cdot \hat{d}_{1\odot})}{r_\odot - \hat{n}_1 \cdot r_\odot} \]
\[ - \frac{2(1 + \gamma)^4 \epsilon^4}{\sin \vartheta_0} \frac{Gm_\odot}{r_\odot} \left[ 2 \hat{n}_1 \cdot \hat{d}_{1\odot} - \hat{n}_1 \cdot r_\odot \right] + \frac{2(\hat{n}_2 \cdot \hat{v}_{\text{obs}})(\hat{v}_{\text{obs}} \cdot \hat{d}_{2\odot})}{r_\odot - \hat{n}_2 \cdot r_\odot} \]
\[ + \frac{2(\hat{n}_1 \cdot \hat{v}_{\text{obs}})(\hat{v}_{\text{obs}} \cdot \hat{d}_{1\odot})}{r_\odot - \hat{n}_1 \cdot r_\odot}, \] (B.8)

\[ \dot{\vartheta}_{2PN} = -(1 + \gamma)^2 \epsilon^4 \frac{G^2 m_\odot^2}{\sin \vartheta_0} \left[ \frac{\hat{n}_2 \cdot \hat{d}_{1\odot}}{d_{1\odot}^3} \left( 1 + \frac{\hat{n}_1 \cdot \hat{r}_\odot}{r_\odot} \right) + \frac{\hat{n}_1 \cdot \hat{d}_{2\odot}}{d_{2\odot}^3} \left( 1 + \frac{\hat{n}_2 \cdot \hat{r}_\odot}{r_\odot} \right) \right] \]
\[ - \beta \frac{1}{4} (2\kappa + \eta) + \gamma^2 - 1 \epsilon^4 \frac{G^2 m_\odot^2}{d_{1\odot}^2} \left[ \hat{n}_1 \cdot \hat{r}_\odot + \frac{\hat{n}_2 \cdot \hat{r}_\odot}{r_\odot} \right] \]
\[ - \beta \frac{1}{4} (2\kappa + \eta) - 2(1 + \gamma) \epsilon^4 \frac{G^2 m_\odot^2}{d_{1\odot}^3} \left[ \frac{\pi}{2} + \arctan \left( \frac{\hat{n}_1 \cdot \hat{r}_\odot}{d_{1\odot}} \right) \right] \]
\[ + \frac{\hat{n}_1 \cdot \hat{d}_{1\odot}}{d_{1\odot}^3} \left[ \frac{\pi}{2} + \arctan \left( \frac{\hat{n}_1 \cdot \hat{r}_\odot}{d_{1\odot}} \right) \right]. \] (B.9)

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The second post-Newtonian Light Propagation and Its Astrometric Measurement in the Solar System

Table 1. Differences among the metric between some previous works and our work for light propagation. These include the order of expansion, whether or not to be parametrized, whether or not to consider spin, quadrupole moment, $N$-body and bodies' motion.

| Ref. | Parametrized | Parameter in $c^{-2}$ | Parameter in $c^{-4}$ | $N$-body | Bodies' motions | Quadrupole moment | Spin |
|------|--------------|-----------------------|-----------------------|----------|----------------|-------------------|------|
| 1    | Yes          | $\gamma$              | $\beta, \Lambda$     | No       | No             | Yes               | Yes  |
| 2    | Yes          | $\gamma$              | $\beta$              | No       | No             | No                | No   |
| 3    | No           | --                    | --                   | No       | No             | No                | No   |
| 4    | No           | --                    | --                   | Yes      | No             | No                | No   |
| 5    | No           | --                    | --                   | Yes      | Yes            | Yes               | No   |
| 6    | No           | --                    | --                   | No       | No             | No                | No   |
| 7    | Yes          | $\gamma$              | $\beta$              | Yes      | Yes            | Yes               | No   |
| 8    | No           | --                    | $\gamma_{14}$        | Yes      | No             | No                | No   |
| 9    | Yes          | $\gamma$              | $\beta, \delta$      | No       | Yes            | No                | No   |
| 10   | No           | --                    | --                   | Yes      | Yes            | No                | No   |
| 11   | No           | --                    | --                   | Yes      | Yes            | Yes               | No   |
| 12   | Yes          | $\gamma$              | $\beta, \epsilon_{k \alpha}$ | No | No | No | No |
| 13   | No           | $\gamma, \beta$       | Yes                  | No       | No             | No                | No   |
| 14   | No           | --                    | --                   | Yes      | Yes            | Yes               | No   |
| 15   | No           | --                    | --                   | No       | No             | Yes               | No   |
| 16   | Yes          | $\gamma$              | $\beta$              | No       | No             | No                | No   |
| 17   | Yes          | $\gamma$              | $\beta, \epsilon_{k \alpha}$ | No | No | No | No |
| 18   | No           | $\gamma, \beta$       | Yes                  | No       | No             | No                | No   |
| 19   | No           | --                    | --                   | Yes      | Yes            | Yes               | No   |
| 20   | No           | --                    | --                   | Yes      | Yes            | Yes               | No   |
| 21   | Yes          | $\gamma$              | --                   | No       | No             | Yes               | Yes  |
| 22   | Yes          | $\gamma$              | $\beta, \delta$      | No       | No             | No                | No   |
| 23   | Yes          | $\gamma$              | --                   | Yes      | Yes            | No                | No   |
| 24   | Yes          | $\gamma$              | $\beta, \epsilon_{k \alpha}$ | No | No | No | No |
| 25   | No           | --                    | --                   | No       | Yes            | Yes               | No   |
| 26   | Yes          | $\gamma$              | --                   | Yes      | Yes            | Yes               | No   |
| 27   | Yes          | $\gamma$              | $\beta, \delta$      | No       | No             | No                | No   |
| 28   | Yes          | $\gamma$              | --                   | Yes      | Yes            | Yes               | No   |
| 29   | Yes          | $\gamma$              | $\beta, \epsilon_{k \alpha}$ | No | No | No | No |
| 30   | Yes          | $\gamma$              | $\beta$              | No       | No             | No                | No   |
| 31   | Yes          | $\gamma$              | --                   | Yes      | Yes            | Yes               | No   |
| 32   | Yes          | $\gamma$              | $\beta, \delta$      | No       | No             | No                | No   |
| 33   | Yes          | $\gamma$              | $\beta, \epsilon_{k \alpha}$ | No | No | No | No |
| 34   | No           | --                    | --                   | No       | Yes            | Yes               | No   |
| 35   | Yes          | $\gamma$              | --                   | Yes      | Yes            | Yes               | No   |
| 36   | Yes          | $\gamma$              | $\beta, \delta$      | No       | No             | No                | No   |

Table 2. The order of $\frac{Gm_{A}}{c^{2}r_{AB}}$ between bodies in the Solar System. This table shows that the bodies at columns affect the bodies at rows when a light ray arrives their limbs.

| $\frac{Gm_{A}}{c^{2}r_{AB}}$ | Sun   | Mercury | Venus | Earth | Mars   | Jupiter | Saturn | Uranus | Neptune |
|-----------------------------|-------|---------|-------|-------|--------|---------|--------|--------|---------|
| Sun                         | $2.1 \times 10^{-6}$ | $2.6 \times 10^{-8}$ | $1.4 \times 10^{-8}$ | $9.9 \times 10^{-9}$ | $6.5 \times 10^{-9}$ | $1.9 \times 10^{-9}$ | $1.0 \times 10^{-9}$ | $5.1 \times 10^{-10}$ | $3.3 \times 10^{-10}$ |
| Mercury                    | $4.2 \times 10^{-15}$ | $1.0 \times 10^{-10}$ | $4.9 \times 10^{-15}$ | $2.7 \times 10^{-15}$ | $1.4 \times 10^{-15}$ | $3.4 \times 10^{-16}$ | $1.8 \times 10^{-16}$ | $8.7 \times 10^{-17}$ | $5.5 \times 10^{-17}$ |
| Venus                      | $3.3 \times 10^{-14}$ | $7.2 \times 10^{-14}$ | $6.0 \times 10^{-10}$ | $8.7 \times 10^{-14}$ | $3.0 \times 10^{-14}$ | $5.4 \times 10^{-15}$ | $2.7 \times 10^{-15}$ | $1.3 \times 10^{-15}$ | $8.2 \times 10^{-16}$ |
| Earth                      | $3.0 \times 10^{-14}$ | $4.8 \times 10^{-14}$ | $1.1 \times 10^{-13}$ | $7.0 \times 10^{-10}$ | $5.7 \times 10^{-14}$ | $7.1 \times 10^{-15}$ | $3.5 \times 10^{-15}$ | $1.6 \times 10^{-15}$ | $1.0 \times 10^{-15}$ |
| Mars                       | $2.1 \times 10^{-15}$ | $2.8 \times 10^{-15}$ | $4.0 \times 10^{-15}$ | $6.1 \times 10^{-15}$ | $1.4 \times 10^{-10}$ | $8.7 \times 10^{-15}$ | $4.0 \times 10^{-16}$ | $1.8 \times 10^{-16}$ | $1.1 \times 10^{-16}$ |
| Jupiter                    | $1.8 \times 10^{-12}$ | $2.0 \times 10^{-12}$ | $2.1 \times 10^{-12}$ | $2.2 \times 10^{-12}$ | $2.6 \times 10^{-12}$ | $2.0 \times 10^{-8}$ | $2.2 \times 10^{-12}$ | $6.7 \times 10^{-13}$ | $3.8 \times 10^{-13}$ |
| Saturn                     | $3.0 \times 10^{-13}$ | $3.1 \times 10^{-13}$ | $3.2 \times 10^{-13}$ | $3.3 \times 10^{-13}$ | $3.5 \times 10^{-13}$ | $6.5 \times 10^{-13}$ | $7.0 \times 9$ | $2.9 \times 10^{-13}$ | $1.4 \times 10^{-13}$ |
| Uranus                     | $2.3 \times 10^{-14}$ | $2.3 \times 10^{-14}$ | $2.3 \times 10^{-14}$ | $2.4 \times 10^{-14}$ | $2.4 \times 10^{-14}$ | $3.1 \times 10^{-14}$ | $4.5 \times 10^{-14}$ | $2.5 \times 10^{-9}$ | $4.0 \times 10^{-14}$ |
| Neptune                    | $1.7 \times 10^{-14}$ | $1.7 \times 10^{-14}$ | $1.7 \times 10^{-14}$ | $1.8 \times 10^{-14}$ | $1.8 \times 10^{-14}$ | $2.0 \times 10^{-14}$ | $2.5 \times 10^{-14}$ | $4.7 \times 10^{-14}$ | $1.4 \times 10^{-9}$ |

Table 3. The relativistic effect of the oblateness of a body on light at its surface, $\frac{Gm_{A}}{c^{2}a_{A}}J_{2A}$. Where $m_{A}$, $a_{A}$ and $J_{2A}$ are respectively the mass, the equatorial radius and the dynamical form factor for body $A$.

|        | Sun     | Mercury | Venus   | Earth   | Mars    | Jupiter | Saturn  | Uranus  | Neptune |
|--------|---------|---------|---------|---------|---------|---------|---------|---------|---------|
| 2.1 $\times 10^{-13}$ | $6.0 \times 10^{-15}$ | $2.4 \times 10^{-15}$ | $7.5 \times 10^{-13}$ | $2.8 \times 10^{-13}$ | $2.9 \times 10^{-10}$ | $1.4 \times 10^{-11}$ | $8.4 \times 10^{-12}$ | $5.2 \times 10^{-12}$ |
Table 4. The order of $\frac{Gm}{c^2 r_{AB}} v_A$ between bodies in the Solar System. This table shows that the bodies at columns affect the bodies at rows when a light ray arrives their limbs.

| $\frac{Gm}{c^2 r_{AB}} v_A$ | Sun       | Mercury   | Venus     | Earth     | Mars      | Jupiter   | Saturn    | Uranus    | Neptune   |
|-----------------------------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|
| Sun                         | $1.1 \times 10^{-13}$ | $1.3 \times 10^{-15}$ | $6.8 \times 10^{-16}$ | $4.9 \times 10^{-16}$ | $3.2 \times 10^{-16}$ | $9.5 \times 10^{-17}$ | $5.2 \times 10^{-17}$ | $2.6 \times 10^{-17}$ | $1.6 \times 10^{-17}$ |
| Mercury                    | $6.8 \times 10^{-19}$ | $1.6 \times 10^{-14}$ | $7.8 \times 10^{-19}$ | $4.3 \times 10^{-19}$ | $2.3 \times 10^{-19}$ | $5.4 \times 10^{-20}$ | $2.9 \times 10^{-20}$ | $1.4 \times 10^{-20}$ | $8.8 \times 10^{-21}$ |
| Venus                      | $3.9 \times 10^{-18}$ | $8.4 \times 10^{-18}$ | $7.0 \times 10^{-14}$ | $1.0 \times 10^{-17}$ | $3.5 \times 10^{-18}$ | $6.3 \times 10^{-19}$ | $3.2 \times 10^{-19}$ | $1.5 \times 10^{-19}$ | $9.6 \times 10^{-20}$ |
| Earth                      | $2.9 \times 10^{-18}$ | $4.8 \times 10^{-18}$ | $1.1 \times 10^{-17}$ | $6.9 \times 10^{-14}$ | $5.6 \times 10^{-18}$ | $7.0 \times 10^{-19}$ | $3.5 \times 10^{-19}$ | $1.6 \times 10^{-19}$ | $1.0 \times 10^{-19}$ |
| Mars                       | $1.7 \times 10^{-19}$ | $2.3 \times 10^{-19}$ | $3.2 \times 10^{-19}$ | $4.9 \times 10^{-19}$ | $1.1 \times 10^{-14}$ | $7.0 \times 10^{-20}$ | $3.2 \times 10^{-20}$ | $1.5 \times 10^{-20}$ | $9.0 \times 10^{-21}$ |
| Jupiter                    | $7.9 \times 10^{-17}$ | $8.5 \times 10^{-17}$ | $9.2 \times 10^{-17}$ | $9.8 \times 10^{-17}$ | $1.1 \times 10^{-16}$ | $8.6 \times 10^{-13}$ | $9.5 \times 10^{-17}$ | $2.9 \times 10^{-17}$ | $1.7 \times 10^{-17}$ |
| Saturn                     | $9.5 \times 10^{-18}$ | $9.9 \times 10^{-18}$ | $1.0 \times 10^{-17}$ | $1.1 \times 10^{-17}$ | $1.1 \times 10^{-17}$ | $2.1 \times 10^{-17}$ | $2.3 \times 10^{-13}$ | $9.4 \times 10^{-18}$ | $4.4 \times 10^{-18}$ |
| Uranus                     | $5.1 \times 10^{-19}$ | $5.2 \times 10^{-19}$ | $5.3 \times 10^{-19}$ | $5.4 \times 10^{-19}$ | $5.5 \times 10^{-19}$ | $7.0 \times 10^{-19}$ | $1.0 \times 10^{-18}$ | $5.7 \times 10^{-14}$ | $9.0 \times 10^{-19}$ |
| Neptune                    | $3.1 \times 10^{-19}$ | $3.1 \times 10^{-19}$ | $3.1 \times 10^{-19}$ | $3.2 \times 10^{-19}$ | $3.2 \times 10^{-19}$ | $3.7 \times 10^{-19}$ | $4.5 \times 10^{-19}$ | $8.5 \times 10^{-19}$ | $2.8 \times 10^{-14}$ |

Table 5. $\zeta$ and $\eta$ at the spatial isotropic and anisotropic parts of $c^{-4}$ for $\theta_{ij}$ in harmonic gauge in different theories. $\beta$ and $\gamma$ are PPN parameters and $c_4 = c_1 + c_4$, where $c_1$ and $c_4$ are constant parameters in $\mathcal{E}$ theory.

| Parameter | GR | ST | $\mathcal{E}$ |
|-----------|----|----|----------------|
| $\zeta$   | $1$ | $2\gamma^2 - \frac{1}{2}\gamma + 2\beta - \frac{\beta}{\gamma}$ | $1 + \frac{1}{2}c_{14}$ |
| $\eta$    | $1$ | $\frac{1}{2}(1 + \gamma)$ | $1 - \frac{1}{2}c_{14}$ |

Table 6. The estimation of the terms in Eq. 48.

| Term1 | Term2 | Term3 | Term4 | Term5 | Term6 | Term7 | Term8 | Term9 | Term10 | Term11 | Term12 |
|-------|-------|-------|-------|-------|-------|-------|-------|-------|--------|--------|--------|
| 3.14mas | 18.02µs | 1'' .75 | 0'' .47 | 83.06µs | 492.76µs | 4.13µs | 1.53µs | 3.05µs | 7.44µs | 11.32µs | 1.20µs |