Harnack Estimates for Nonlinear Backward Heat Equations in Geometric Flows

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Abstract

Let $M$ be a closed Riemannian manifold with a family of Riemannian metrics $g_{ij}(t)$ evolving by a geometric flow $\partial_t g_{ij} = -2S_{ij}$, where $S_{ij}(t)$ is a family of smooth symmetric two-tensors. We derive several differential Harnack estimates for positive solutions to the nonlinear backward heat-type equation

$$\frac{\partial f}{\partial t} = -\Delta f + \gamma f \log f + aSf$$

where $a$ and $\gamma$ are constants and $S = g^{ij}S_{ij}$ is the trace of $S_{ij}$. Our abstract formulation provides a unified framework for some known results proved by various authors, and moreover lead to new Harnack inequalities for a variety of geometric flows.

1 Introduction

The study of differential Harnack estimates for parabolic equations originated with the work of Li and Yau [15]. They proved a gradient estimate for the heat equation by using the maximal principle. By integrating the gradient estimate along a space-time path, a classical Harnack inequality was derived. Therefore, Li-Yau type gradient estimate is often called differential Harnack estimate. Similar techniques were used by Hamilton to prove Harnack estimates for the Ricci flow [10, 11] and the mean curvature flow [12].

Using similar techniques, many authors have proved a variety of Li-Yau-Hamilton’s Harnack estimates for various equations in different geometric flows, and we refer to the survey paper by Ni [21]. In the Ricci flow, Perelman proved a Harnack inequality for the fundamental solution of the conjugate heat equation [22] and later on different Harnack inequalities have been proved, to name but a few [1, 2, 3, 14, 17, 22, 24, 25]. Harnack inequalities for more general flows have been considered in [4, 5, 6, 13, 23].

The main purpose of the current article is, in the framework of a general geometric flow, to derive Li-Yau-Hamilton type differential Harnack estimates for positive solutions to a nonlinear backward heat-type equation generalizing Perelman’s conjugate heat equation.

Let $M$ be a closed Riemannian $n$-manifold with a one parameter family of Riemannian metrics $g(t)$ evolving by the geometric flow

$$\frac{\partial}{\partial t} g_{ij} = -2S_{ij}, \ t \in [0,T)$$

where $S_{ij}(t)$ is any smooth symmetric two-tensor on $(M, g(t))$. $f$ is a positive solution to

$$\frac{\partial f}{\partial t} = -\Delta f + \gamma f \log f + aSf.$$
where symbol $\Delta$ stands for the Laplacian of the evolving metric $g(t)$, $\gamma$ and $a$ are constants and $S = g^{ij}S_{ij}$ is the trace of $S_{ij}$. In the Ricci flow case, when $\gamma = 0$ and $a = 1$, (2) is the conjugate heat equation introduced by Perelman. The consideration of this nonlinear equation is motivated by gradient Ricci solitons. See [3, 24] for more details. In a forthcoming paper [7], we will consider the forward nonlinear heat-type equation.

To state the main results, we introduce evolving tensor quantities associated to the tensor $S_{ij}$.

**Definition 1** Let $g(t)$ evolve by (1) and $X = X^i \frac{\partial}{\partial x^i}$ be a vector field on $M$. For a constant $a \in \mathbb{R}$, we define

$$E_a(S_{ij}, X) = \left( a \frac{\partial S}{\partial \tau} + a \Delta S + 2|S_{ij}|^2 \right) - 2 \left( 2\nabla^i S_{i\ell} - \nabla_\ell S \right) X^\ell - 2(R^{ij} - S^{ij})X^iX^j,$$

where $\tau = T - t$, $R^{ij} = g^{ik}g^{j\ell}R_{k\ell}$, $S^{ij} = g^{ik}g^{j\ell}S_{k\ell}$, $S = g^{ij}S_{ij}$, $\nabla^i = g^{ij}\nabla_j$ and $X_k = g_{ik}X^i$.

**Remark 1** The quantity of $E_a(S_{ij}, X)$ is a generalization of $D(S_{ij}, X)$ defined by Reto Müller [19]. Indeed, $D(S_{ij}, X) = -E_1(S_{ij}, X)$. $E_a(S_{ij}, X)$ has also been implicitly discussed in recent papers of one of the authors and others [6, 8]. $E_a(S_{ij}, X)$ appears as the error term in our formulas under the flow (1), and when the flow is Hamilton’s Ricci flow $E_1(R_{ij}, X) = 0$.

In the following theorems A-E, we assume $(M, g(t)), t \in [0, T)$, is a solution to the geometric flow (1) on a closed oriented smooth $n$-manifold $M$. Denote $\tau := T - t$.

**Theorem A** 1. Suppose that

$$E_2(S_{ij}, X) + \frac{2S^2}{n} \leq 0$$

holds for all vector fields $X$ and all time $t \in [0, T)$ for which the flow exists. Let $f$ be a positive solution to the equation (2) with $\gamma = 1$ and $a = 2$,

$$\frac{\partial f}{\partial t} = -\Delta f + f \log f + 2Sf.$$

Then, for all time $t \in [0, T)$ it holds

$$2\Delta \log f + |\nabla \log f|^2 - 2S + 2\frac{n}{\tau} + \frac{n}{2} \geq 0.$$

2. Suppose that

$$E_1(S_{ij}, X) \leq 0, \quad S \geq 0$$

hold for all vector fields $X$ and all time $t \in [0, T)$ for which the flow exists. Let $f$ be a positive solution to the equation (2) with $\gamma = 1$ and $a = 1$, namely

$$\frac{\partial f}{\partial t} = -\Delta f + f \log f + Sf.$$

Then, for all time $t \in [0, T)$ it holds

$$2\Delta \log f + |\nabla \log f|^2 - S + 2\frac{n}{\tau} + \frac{n}{4} \geq 0.$$
**Theorem B** Let $f$ be a positive solution to

$$\frac{\partial f}{\partial t} = -\Delta f + f \log f + S f.$$  

Suppose that

$$E_1(S_{ij}, X) \leq 0,\ S \geq -\frac{n}{2t}$$  

(6)

hold for all vector fields $X$ and all time $t \in [T/2, T]$ for which the flow exists. Then for all time $t \in [T/2, T)$, it holds

$$2\Delta \log f + |\nabla \log f|^2 - S + 3\frac{n}{\tau} + \frac{n}{4} \geq 0.$$  

We are able to derive classical Harnack inequalities by integrating the above differential Harnack inequalities along space-time paths. For instance, the first case of Theorem A implies following. The proof is now standard (for example, see [1, 24]).

**Corollary 1** Suppose that (4) holds for all vector fields $X$ and all time $t \in [0, T)$ for which the flow exists. Let $f$ be a positive solution to the heat equation

$$\frac{\partial f}{\partial t} = -\Delta f + f \log f + 2Sf.$$  

Assume that $(x_1, t_1)$ and $(x_2, t_2)$ are two points in $M \times (0, T)$, where $0 < t_1 < t_2 < T$. Then the following holds:

$$e^{t_2} \log f(x_2, t_2) - e^{t_1} \log f(x_1, t_1) \leq \frac{1}{2} \int_{t_1}^{t_2} e^\tau (|\ell|^2 + 2S + \frac{n}{2} + \frac{n}{4}) d\tau,$$

where $\ell$ is any space-time path joining $(x_1, t_1)$ and $(x_2, t_2)$.

We notice that the second case of Theorem A and Theorem B also imply similar classical Harnack inequalities. We leave details to the interested readers.

On the other hand, for the equation (2) in the case where $a = 0$, we shall prove the following result:

**Theorem C** Suppose $\gamma > 0$ and $0 < f < 1$ is a positive solution to

$$\frac{\partial f}{\partial t} = -\Delta f + \gamma f \log f.$$  

If

$$R_{ij}(t) + S_{ij}(t) \geq -\frac{1}{2} \gamma g_{ij}(t),$$  

(7)

then for all time $t \in [0, T)$ it holds:

$$|\nabla \log f|^2 + \frac{\log f}{\tau} \leq 0.$$  

For the equation (2) in the case where $a = 0$ and $\gamma = 0$, we prove
Theorem D Suppose that
\[ R_{ij}(t) + S_{ij}(t) \geq 0. \] (8)

Let \( 0 < f < 1 \) be a positive solution to the time-dependant heat equation
\[ \frac{\partial f}{\partial t} = -\Delta f. \]

Then for all time \( t \in [0, T) \), the following holds:
\[ |\nabla \log f|^2 + \frac{\log f}{\tau} \leq 0. \]

Moreover, we also prove

Theorem E Let \( f \) be a positive solution to the conjugate heat equation
\[ \frac{\partial f}{\partial t} = -\Delta f + Sf. \]

Suppose that
\[ \mathcal{E}_1(S_{ij}, X) \leq 0 \] (9)
holds for all vector fields \( X \) and all time \( t \in [0, T) \) for which the flow exists. Then
\[ \min_M \left( 2\Delta \log f + |\nabla \log f|^2 - S \right) \]
increases along the geometric flow (7).

The rest of this article is organized as follows. In Section 2 we apply the abstract results, Theorems A-E, to a variety of geometric flows. We shall justify that the technical assumptions (4, 5, 6, 7, 8, 9) are either automatically satisfied, or guaranteed by the geometric assumption at time \( t = 0 \). A variety of new Harnack inequalities are obtained. In Section 3 we shall derive general evolution equations. Sections 4 to 8 are devoted to proving Theorems A-E.

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2 Examples

(0) Static Riemannian manifold. In this case \( S_{ij} = 0 \) and thus for any \( a \)
\[ \mathcal{E}_a(0, X) = -2R^{ij}X_iX_j \]
As long as \((M, g)\) has nonnegative Ricci curvature it holds \( \mathcal{E}_a(0, X) \leq 0 \) and moreover the technical assumptions (4, 5, 6, 7, 8, 9) all hold. Applying our main theorems to this example, and reversing the time direction we have:

Corollary 2 Suppose \((M, g)\) is a compact static Riemannian manifold with nonnegative Ricci curvature. Let \( f \) be a positive solution to
\[ \frac{\partial f}{\partial t} = \Delta f - f \log f. \] (10)

Then for all \( t > 0 \)
\[ 2\Delta \log f + |\nabla \log f|^2 + \frac{2n}{t} + \frac{n}{4} \geq 0. \]
We note that gradient estimates of the corresponding elliptic version of (10) have been studied in [18].

By Theorem D and E and noting the time direction, we have

**Corollary 3** Suppose \((M, g)\) is a compact static Riemannian manifold with nonnegative Ricci curvature. Let \(0 < f < 1\) be a positive solution to

\[
\frac{\partial f}{\partial t} = \Delta f.
\]

Then, for all \(t\) it holds

\[
|\nabla \log f|^2 + \frac{\log f}{t} \leq 0.
\]

And

\[
\min_M \{2\Delta \log f + |\nabla \log f|^2\}
\]

decreases along time.

(1) **Hamilton’s Ricci flow.** Let \(g(t)\) be a solution to the Ricci flow:

\[
\frac{\partial}{\partial t} g_{ij} = -2R_{ij}.
\]

Namely, we have \(S_{ij} = R_{ij}\) and \(S = R\) the scalar curvature. Notice that it is known that the scalar curvature \(R\) evolves by \(\frac{\partial R}{\partial \tau} - \Delta R - 2|R_{ij}|^2 = 0\). Therefore

\[
\frac{\partial R}{\partial \tau} = -\Delta R - 2|R_{ij}|^2.
\]

Moreover, we have the twice contracted second Bianchi identity \(2\nabla^i R_{it} - \nabla_t R = 0\). Hence, we have

\[
\mathcal{E}_a(R_{ij}, \nabla v) = \left( a \frac{\partial R}{\partial \tau} + a\Delta R + 2|R_{ij}|^2 \right) = -2(a - 1)|R_{ij}|^2.
\]

This implies that \(\mathcal{E}_a(R_{ij}, \nabla v) \leq 0\) if \(a \geq 1\). In particular, (9) holds automatically. Moreover, (4) is equivalent to \(|R_{ij}|^2 \geq \frac{R^2}{n}\) which is automatically satisfied. Moreover, (6) is equivalent to

\[
R \geq -\frac{n}{2t}.
\]

(11)

Notice that the evolution of scalar curvature \(R\) under the Ricci flow satisfies

\[
\frac{\partial R}{\partial t} = \Delta R + 2|R_{ij}|^2 \geq \Delta R + \frac{2}{n}R^2.
\]

By the maximal principle, we get (11).

Summarizing, in Hamilton’s Ricci flow we have (4, 6, 9) hold automatically. (5) holds when \(R(0) \geq 0\), and (7, 8) can be guaranteed if the curvature operator at time \(t = 0\) is nonnegative. We remark that in the Ricci flow the corresponding results have been proved in [1, 3, 24].

(2) **List’s extended Ricci flow.** In [16], List introduced a geometric flow closely related to the Ricci flow:

\[
\frac{\partial}{\partial t} g_{ij} = -2R_{ij} + 4\nabla_i \psi \nabla_j \psi,
\]

\[
\frac{\partial \psi}{\partial t} = \Delta \psi,
\]

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where $\psi : M \to \mathbb{R}$ is a smooth function. In the extended Ricci flow $S_{ij} = R_{ij} - 2\nabla_i \psi \nabla_j \psi$ and $S = R - 2|\nabla \psi|^2$. List \[16\] pointed out that $S$ satisfies the following evolution equation:

$$\frac{\partial S}{\partial t} = \Delta S + 2|S_{ij}|^2 + 4|\Delta \psi|^2.$$ 

Therefore,

$$a \frac{\partial S}{\partial t} + a \Delta S + 2|S_{ij}|^2 = -2(a - 1)|S_{ij}|^2 - 4a|\Delta \psi|^2.$$ 

On the other hand, we have (see \[16\] and Section 2 in \[19\]): \(2\nabla^i S_{ij} - \nabla \ell S = -4\Delta \psi \nabla \ell \psi\). Therefore we obtain

$$\mathcal{E}_a(S_{ij}, X) = -2(a - 1)|S_{ij}|^2 - 4(a - 1)|\Delta \psi|^2 - 4|\Delta \psi - \nabla_X \psi|^2.$$ 

This tells that $\mathcal{E}_a(S_{ij}, X) \leq 0$ if $a \geq 1$.

On the other hand, if $a = 2$, then $\mathcal{E}_2(S_{ij}, X) = -2|S_{ij}|^2 - 4|\Delta \psi|^2 - 4|\Delta \psi - \nabla_X \psi|^2$ and we get the following:

$$\mathcal{E}_2(S_{ij}, X) + \frac{2}{n} S^2 = -2|S_{ij}|^2 + \frac{2}{n} S^2 - 4|\Delta \psi|^2 - 4|\Delta \psi - \nabla_X \psi|^2 
\leq \frac{-2}{n} S^2 + \frac{2}{n} S^2 - 4|\Delta \psi|^2 - 4|\Delta \psi - \nabla_X \psi|^2 
= -4|\Delta \psi|^2 - 4|\Delta \psi - \nabla_X \psi|^2 
\leq 0.$$ 

Namely, \((4)\) holds. Similar as in the Ricci flow, we get

$$\frac{\partial S}{\partial t} = \Delta S + 2|S_{ij}|^2 + 4|\Delta \psi|^2 \geq \Delta S + \frac{2}{n} S^2 + 4|\Delta \psi|^2 
\geq \Delta S + \frac{2}{n} S^2$$ 

and by applying the maximal principle, we get

$$S \geq -\frac{n}{2t}.$$ 

Summarizing, in List’s extended Ricci flow we have \((4)\) \((5)\) \((9)\) hold automatically. Notice that the positivity of $S = R - 2|\nabla \psi|^2$ is preserved by the flow, we get \((5)\) holds when $S(0) \geq 0$.

\textbf{(3) Müller’s Ricci flow coupled with harmonic map flow.} Let $(Y, h)$ be a fixed Riemannian manifold. Let $(g(t), \phi(t))$ be the couple consisting of a family of metric $g(t)$ on $M$ and a family of maps $\phi(t)$ from $M$ to $Y$. We call $(g(t), \phi(t))$ a solution of Müller’s flow \[20\] (also known as Ricci flow coupled with harmonic map heat flow) with coupling function $\alpha(t) \geq 0$ if

$$\frac{\partial}{\partial t} g_{ij} = -2R_{ij} + 2 \alpha(t) \nabla_i \phi \nabla_j \phi,$$

$$\frac{\partial \phi}{\partial t} = \tau_\phi \phi,$$

where $\tau_\phi \phi$ is the tension field of the map $\phi$ with respect to the metric $g(t)$. List’s flow is a special case of this flow. In this example $S_{ij} = R_{ij} - \alpha(t) \nabla_i \phi \nabla_j \phi$, and $S = R - \alpha(t) |\nabla \phi|^2$. Müller \[20\] proved that $S$ satisfies

$$\frac{\partial S}{\partial t} = \Delta S + 2|S_{ij}|^2 + 2 \alpha(t) |\tau_\phi \phi|^2 - (\frac{\partial \alpha(t)}{\partial t}) |\nabla \phi|^2.$$
Therefore, we get
\[
\alpha \frac{\partial S}{\partial t} + a \Delta S + 2|S_{ij}|^2 = -2(a - 1)|S_{ij}|^2 - 2a\alpha(t)|\tau_g \phi|^2 + a\left(\frac{\partial \alpha(t)}{\partial t}\right)|\nabla \phi|^2.
\]

On the other hand, we have (see [20] and Section 2 in [19]):
\[
2 \left( 2\nabla^i S_{ij} - \nabla_i S \right) X^\ell = -4\alpha(t)\tau_g \psi \nabla \ell \phi X^\ell
\]
and moreover
\[
\mathcal{E}_a(S_{ij}, X) = -2(a - 1)|S_{ij}|^2 - 2a\alpha(t)|\tau_g \phi|^2 + a\left(\frac{\partial \alpha(t)}{\partial t}\right)|\nabla \phi|^2 + 4\alpha(t)\tau_g \psi \nabla \ell \phi X^\ell
- 2\alpha(t)\nabla_i \phi \nabla_j \phi X^i X^j
= -2(a - 1)|S_{ij}|^2 - 2a\alpha(t)(a - 1)|\tau_g \phi|^2 + a\left(\frac{\partial \alpha(t)}{\partial t}\right)|\nabla \phi|^2
- 2\alpha(t) \left( |\tau_g \phi|^2 - 2\tau_g \phi \nabla \ell \phi X^\ell + \nabla_i \phi \nabla_j \psi \nabla \phi X^i X^j \right)
= -2(a - 1)|S_{ij}|^2 - 2a\alpha(t)(a - 1)|\tau_g \phi|^2 - 2\alpha(t)|\tau_g \phi - \nabla X \phi|^2 + a\left(\frac{\partial \alpha(t)}{\partial t}\right)|\nabla \phi|^2.
\]
Suppose that \(a \geq 1\) and \(\frac{\partial \alpha(t)}{\partial t} \leq 0\), we get
\[
\mathcal{E}_a(S_{ij}, X) \leq 0.
\]
In particular, (9) holds automatically. Moreover, (8) is equivalent to \(S \geq 0\).

On the other hand, suppose that \(a = 2\) and \(\frac{\partial \alpha(t)}{\partial t} \leq 0\). Then the above computation tells us that
\[
\mathcal{E}_2(S_{ij}, X) = -2|S_{ij}|^2 - 2a\alpha(t)|\tau_g \phi|^2 - 2\alpha(t)|\tau_g \phi - \nabla X \phi|^2 + 2\left(\frac{\partial \alpha(t)}{\partial t}\right)|\nabla \phi|^2.
\]
Hence
\[
\mathcal{E}_2(S_{ij}, X) + \frac{2}{n} S^2 = -2|S_{ij}|^2 + \frac{2}{n} S^2 - 2\alpha(t)|\tau_g \phi|^2 - 2\alpha(t)|\tau_g \phi - \nabla X \phi|^2 + 2\left(\frac{\partial \alpha(t)}{\partial t}\right)|\nabla \phi|^2
\leq \frac{2}{n} S^2 + \frac{2}{n} S^2 - 2\alpha(t)|\tau_g \phi|^2 - 2\alpha(t)|\tau_g \phi - \nabla X \phi|^2 + 2\left(\frac{\partial \alpha(t)}{\partial t}\right)|\nabla \phi|^2
= -2\alpha(t) \left( |\tau_g \phi|^2 + |\tau_g \phi - \nabla X \phi|^2 \right) + 2\left(\frac{\partial \alpha(t)}{\partial t}\right)|\nabla \phi|^2
\leq 0,
\]
where notice that \(\alpha(t) \geq 0\) and \(\frac{\partial \alpha(t)}{\partial t} \leq 0\). Namely, (11) holds.

On the other hand, we get the following under \(\alpha(t) \geq 0\) and \(\frac{\partial \alpha(t)}{\partial t} \leq 0\):
\[
\frac{\partial S}{\partial t} = \Delta S + 2|S_{ij}|^2 + 2\alpha(t)|\tau_g \phi|^2 - \left(\frac{\partial \alpha(t)}{\partial t}\right)|\nabla \phi|^2
\geq \Delta S + \frac{2}{n} S|^2 + 2\alpha(t)|\tau_g \phi|^2 - \left(\frac{\partial \alpha(t)}{\partial t}\right)|\nabla \phi|^2
\geq \Delta S + \frac{2}{n} |S_{ij}|^2.
\]
By applying the maximal principle, we get
\[
S \geq \frac{n}{2t}.
\]
Therefore (6) holds, where notice that we have \(\mathcal{E}_1(S_{ij}, X) \leq 0\).

Summarizing, in Müller’s flow we have (11) (6) (9) hold automatically. Notice that the positivity of \(S\) is again preserved by the flow, we get (5) holds when \(S(0) \geq 0\).
3 General evolution equations

In this section, we shall prove general evolution equations of general Harnack quantities under the geometric flow, which are useful to prove the main results in the backward case. See Theorems 1 and 2 stated below. In the Ricci flow case, such general evolution equations are firstly proved by Cao [1]. Theorems 1 and 2 can be seen as generalizations of Lemma 2.1 and Lemma 3.4 in [1] respectively.

3.1 Case of $u = -\log f$

Let $M$ be a closed Riemannian manifold with a Riemannian metric $g_{ij}(t)$ evolving by a geometric flow $\partial_t g_{ij} = -2S_{ij}$. Let $f$ be a positive solution of the following nonlinear backward heat equation with potential term $-cS$:

$$\frac{\partial f}{\partial t} = -\Delta f + \gamma f \log f - cSf,$$

where $c$ and $\gamma$ are constants. In what follows, let $u = -\log f$. By a direct computation, we see that $u$ satisfies

$$\frac{\partial u}{\partial t} = -\Delta u + |\nabla u|^2 + cS + \gamma u.$$  \hfill (12)

Let $\tau = T - t$. Then $f$ satisfies

$$\frac{\partial f}{\partial \tau} = \Delta f + cSf - \gamma f \log f$$

and $u$ satisfies

$$\frac{\partial u}{\partial \tau} = \Delta u - |\nabla u|^2 - cS - \gamma u.$$  \hfill (13)

On the other hand, it is known that $\Delta(|\nabla u|^2) = 2\nabla^i u \Delta(\nabla_i u) + 2|\nabla \nabla u|^2$. Since we also have $\Delta(\nabla_i u) = \nabla_i(\Delta u) + R_{ij} \nabla^j u$, we obtain $\Delta(|\nabla u|^2) = 2\nabla^i u(\nabla_i(\Delta u) + R_{ij} \nabla^j u) + 2|\nabla \nabla u|^2$. Equivalently, we get

$$2\nabla^i(\Delta u) \nabla_i u = \Delta(|\nabla u|^2) - 2R_{ij} \nabla^i u \nabla^j u - 2|\nabla \nabla u|^2.$$  \hfill (14)

Then we have

**Lemma 1** Under the above situation, the following holds:

$$\frac{\partial}{\partial \tau}(\Delta u) = \Delta(\Delta u) - \Delta(|\nabla u|^2) - c\Delta S - 2S^{ij} \nabla_i \nabla_j u - \left(2\nabla^i S_{i\ell} - \nabla_\ell S\right) \nabla_\ell u - \gamma \Delta u,$nabla^i \nabla_j u - 2\nabla^i(|\nabla u|^2) \nabla_i u - 2c\nabla^i S \nabla_i u - 2(R^{ij} + S^{ij}) \nabla_i u \nabla_j u - 2\gamma |\nabla u|^2.$$

**Proof.** These formulas follow from direct computations as follows. First of all, recall that

$$\Gamma^{k}_{ij} = \frac{1}{2} g^{k\ell} \left( \frac{\partial}{\partial x^i} g_{j\ell} + \frac{\partial}{\partial x^j} g_{i\ell} - \frac{\partial}{\partial x^\ell} g_{ij} \right).$$

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By working with a normal coordinate, we obtain

\[
\frac{\partial}{\partial t} \Gamma^k_{ij} = \frac{1}{2} g^{kl} \left( \nabla_i (-2S_{j\ell} + \nabla_j (-2S_{\ell i}) - \nabla_\ell (-2S_{ij}) \right)
\]

\[
= -g^{kl} \left( \nabla_i S_{j\ell} + \nabla_j S_{\ell i} - \nabla_\ell S_{ij} \right).
\]

Therefore, the following holds:

\[
g^{ij} \left( \frac{\partial}{\partial t} \Gamma^k_{ij} \right) = -g^{ij} g^{kl} \left( \nabla_i S_{j\ell} + \nabla_j S_{\ell i} - \nabla_\ell S_{ij} \right) = -g^{kl} \left( \nabla^i S_{j\ell} + \nabla^\ell S_{ij} - \nabla^\ell S \right)
\]

\[
= -g^{kl} \left( 2 \nabla^i S_{\ell i} - \nabla^\ell S \right).
\]

By using this and (12), we have

\[
\frac{\partial}{\partial \tau} (\Delta u) = -2S^{ij} \nabla_i \nabla_j u - \Delta \left( \frac{\partial u}{\partial t} \right) + g^{ij} \left( \frac{\partial}{\partial t} \Gamma^k_{ij} \right) \nabla_k u
\]

\[
= -2S^{ij} \nabla_i \nabla_j u - \Delta (-\Delta u + |\nabla u|^2 + cS + \gamma u)
\]

\[
- g^{kl} (2 \nabla^i S_{\ell i} - \nabla^\ell S) \nabla_k u
\]

\[
= \Delta (\Delta u) - \Delta (|\nabla u|^2) - c \Delta S - \gamma \Delta u - 2S^{ij} \nabla_i \nabla_j u
\]

\[
- 2 \nabla^i S \nabla_i u - 2 \gamma |\nabla u|^2,
\]

where we used (12). Since we have (14), we obtain the desired formula.

By using Lemma 1, we prove the following which is used to prove Theorems C and D

**Proposition 1** Let \( g(t) \) be a solution to the geometric flow (1) and \( u \) satisfies (13). Let

\[
H_S = \alpha \Delta u - \beta |\nabla u|^2 + aS + b \frac{u}{\tau} + d \frac{n}{\tau},
\]

where \( \alpha, \beta, a, b \) and \( d \) are constants. Then \( H_S \) satisfies the following evolution equations:

\[
\frac{\partial H_S}{\partial \tau} = \Delta H_S - 2 \nabla^i H_S \nabla_i u + 2(a + \beta c) \nabla^i S \nabla_i u - 2(\alpha - \beta)|\nabla \nabla u|^2
\]

\[
- 2 \alpha S^{ij} \nabla_i \nabla_j u + \frac{b}{\tau} |\nabla u|^2 - \frac{b}{\tau} cS - \frac{b}{\tau^2} u - \frac{d}{\tau^2} u
\]

\[
+ \frac{\partial S}{\partial \tau} - (a + \alpha c) \Delta S - \alpha \left( 2 \nabla^i S_{\ell i} - \nabla^\ell S \right) \nabla_\ell u
\]

\[
- 2 \left( aR^{ij} - \beta (R^{ij} + S^{ij}) \right) \nabla_i u \nabla_j u - \alpha \gamma \Delta u + 2 \beta \gamma |\nabla u|^2 + \gamma \frac{u}{\tau}.
\]

**Proof.** Notice that we have the following by the definition of \( H_S \):

\[
\Delta H_S = \alpha \Delta (\Delta u) - \beta \Delta (|\nabla u|^2) + a \Delta S + \frac{b}{\tau} \Delta u.
\]
By (13) and Lemma 1 we obtain
\[
\frac{\partial H_S}{\partial \tau} = \frac{\partial}{\partial \tau}(\Delta u) - \beta \frac{\partial}{\partial \tau}(|\nabla u|^2) + a \frac{\partial S}{\partial \tau} + b \frac{u}{\tau} - \frac{b}{\tau^2} - \frac{d}{\tau^2}
\]
\[
= \alpha \left( \Delta(\Delta u) - 2\Delta(|\nabla u|^2) - c\Delta S - 2S^{ij}\nabla_i \nabla_j u - (2\nabla^i S_{\ell \ell} - \nabla S)\nabla^\ell u - \gamma \Delta u \right)
\]
\[
- \beta \left( \Delta(|\nabla u|^2) - 2|\nabla \nabla u|^2 - 2\nabla^i(|\nabla u|^2)\nabla_i u - 2c\nabla^i S \nabla_i u \right)
\]
\[
- 2(R^{ij} + S^{ij})\nabla_i u \nabla_j u - 2\gamma |\nabla u|^2 \right) + a \frac{\partial S}{\partial \tau} + b \frac{u}{\tau} \left( \Delta u - |\nabla u|^2 - cS - \gamma u \right)
\]
\[
- \frac{b}{\tau^2} - \frac{d}{\tau^2}
\]
\[
= \Delta H_S - \alpha \Delta(|\nabla u|^2) - \alpha c \Delta S - 2\alpha S^{ij}\nabla_i \nabla_j u + 2\beta |\nabla u|^2
\]
\[
+ 2\beta \nabla^i(|\nabla u|^2)\nabla_i u + 2b \frac{u}{\tau} - \frac{d}{\tau^2} + \alpha \left( 2\nabla^i S_{\ell \ell} - \nabla S \right)\nabla^\ell u
\]
\[
- a \gamma \Delta u + 2\beta \gamma |\nabla u|^2 - b \gamma \frac{u}{\tau}
\]

On the other hand, we also have the following by the definition of $H_S$:
\[
\nabla^i H_S = \alpha \nabla^i(\Delta u) - \beta \nabla^i(|\nabla u|^2) + a \nabla^i S + \frac{b}{\tau} \nabla^i u
\]

This and (14) imply
\[
-2\nabla^i H_S \nabla_i u = -2\alpha \nabla^i(\Delta u)\nabla_i u + 2\beta \nabla^i(|\nabla u|^2)\nabla_i u - 2a \nabla^i S \nabla_i u - \frac{2b}{\tau} |\nabla u|^2
\]
\[
= -\alpha \Delta(|\nabla u|^2) + 2\alpha R^{ij} \nabla^i u \nabla^j u + 2\alpha |\nabla \nabla u|^2 + 2\beta \nabla^i(|\nabla u|^2)\nabla_i u
\]
\[
- 2a \nabla^i S \nabla_i u - \frac{2b}{\tau} |\nabla u|^2.
\]

Equivalently, we have
\[
-\alpha \Delta(|\nabla u|^2) = -2\nabla^i H_S \nabla_i u - 2\alpha R^{ij} \nabla^i u \nabla^j u - 2\alpha |\nabla \nabla u|^2 - 2\beta \nabla^i(|\nabla u|^2)\nabla_i u
\]
\[
+ 2a \nabla^i S \nabla_i u + \frac{2b}{\tau} |\nabla u|^2.
\]

The desired result now follows from the above equation on $\frac{\partial H_S}{\partial \tau}$ and this equation. 

Let us introduce

**Definition 2** Let $g(t)$ evolve by (1) and let $S$ be the trace of $S_{ij}$. Let $X = X^i \frac{\partial}{\partial x^i}$ be a vector field on $M$. We define
\[
\mathcal{E}_{(a,c,\alpha,\beta)}(S_{ij}, X) = a \frac{\partial S}{\partial \tau} - (a + c) \Delta S + \frac{\alpha^2}{2(\alpha - \beta)} |S_{ij}|^2 - \alpha \left( 2\nabla^i S_{\ell \ell} - \nabla S \right) X^\ell
\]
\[
- 2 \left( \alpha R^{ij} - \beta (R^{ij} + S^{ij}) \right) X_i X_j,
\]

where $a, c, \alpha, \beta$ are constants and $\alpha \neq \beta$.

Then we get
Theorem 1 Suppose that $\alpha \neq 0$ and $\alpha \neq \beta$. Then, the evolution equation in Proposition 1 can be rewritten as follows:

$$\frac{\partial H_S}{\partial \tau} = \Delta H_S - 2\nabla^i H_S \nabla^i u - 2(\alpha - \beta)\nabla_i \nabla_j u + \frac{\alpha}{2(\alpha - \beta)} S_{ij} - \frac{\lambda}{2\tau} g_{ij} \bigg|^2
$$

$$+ 2(a + \beta c) \nabla^i u \nabla_i S - \frac{2(\alpha - \beta)}{\alpha} \frac{\lambda}{\tau} H_S + (\alpha - \beta) \frac{n \lambda^2}{2\tau^2} + \left( b - \frac{2(\alpha - \beta) \lambda \beta}{\alpha} \right) \frac{\nabla u^2}{\tau}
$$

$$+ \left( \frac{2(\alpha - \beta)}{\alpha} a \lambda - \alpha \lambda - bc \right) S - \left( \frac{2(\alpha - \beta)}{\alpha} \lambda \beta - (\alpha - \beta) \lambda - 1 \right) \frac{b}{\tau^2} u + \left( \frac{2(\alpha - \beta) \lambda \beta}{\alpha} - 1 \right) \frac{d}{\tau^2} n
$$

where $\lambda$ is a constant.

Proof. Notice that a direct computation implies

$$-2(\alpha - \beta) \bigg| \nabla_i \nabla_j u + \frac{\alpha}{2(\alpha - \beta)} S_{ij} - \frac{\lambda}{2\tau} g_{ij} \bigg|^2 = -2(\alpha - \beta) \nabla u^2 - 2\alpha S_{ij} \nabla_i \nabla_j u
$$

$$+ 2(\alpha - \beta) \frac{\lambda}{\tau} \Delta u - \frac{\alpha^2}{2(\alpha - \beta)} |S_{ij}|^2
$$

$$+ \frac{\lambda}{\tau} \alpha S - (\alpha - \beta) \frac{\lambda^2}{2\tau^2} n.
$$

By this and Proposition 1 we obtain

$$\frac{\partial H_S}{\partial \tau} = \Delta H_S - 2\nabla^i H_S \nabla^i u - 2(\alpha - \beta)\nabla_i \nabla_j u + \frac{\alpha}{2(\alpha - \beta)} S_{ij} - \frac{\lambda}{2\tau} g_{ij} \bigg|^2
$$

$$- 2 \left( \alpha R^{ij} - \beta (R^{ij} + S^{ij}) \right) \nabla_i u \nabla_j u + 2(a + \beta c) \nabla^i S \nabla_i u + (\alpha - \beta) \frac{\lambda^2}{2\tau^2} n
$$

$$- 2(\alpha - \beta) \frac{\lambda}{\tau} \Delta u - \frac{\lambda}{\tau} \alpha S + \frac{b}{\tau} \nabla u^2 + \frac{a}{\tau} \frac{\partial S}{\partial \tau} - \left( a + \alpha c \right) \Delta S - \alpha \left( 2\nabla^i S_{ij} - \nabla \nabla^i S \right) \nabla^j u
$$

$$+ \frac{\alpha^2}{2(\alpha - \beta)} |S_{ij}|^2 - \frac{b}{\tau} c S - \frac{b}{\tau^2} u - d \frac{n}{\tau^2} - \alpha \gamma \Delta u + 2\beta \gamma |\nabla u|^2 - b \gamma \frac{u}{\tau}.
$$

The desired result now follows from the above equation, Definition 2 and the following which follows from the definition of $H_S$:

$$-\frac{2(\alpha - \beta) \frac{\lambda}{\alpha}}{\tau} H_S = -2(\alpha - \beta) \frac{\lambda}{\tau} \Delta u + \frac{2(\alpha - \beta) \frac{\lambda \beta}{\tau}}{\alpha} \nabla u^2 - \frac{2(\alpha - \beta) \frac{\lambda}{\alpha}}{\tau} S \nabla u
$$

$$- \frac{2(\alpha - \beta) \frac{\lambda \beta}{\alpha}}{\tau} \frac{u}{\tau^2} - \frac{2(\alpha - \beta) \frac{\lambda}{\alpha}}{\tau} \frac{d n}{\tau^2}.
$$

3.2 Case of $v = -\log f - w(\tau)$

As in Subsection 3.1 let $f$ be a positive solution of the following nonlinear backward heat equation with potential term $-cS$:

$$\frac{\partial f}{\partial t} = -\Delta f + \gamma f \log f - cS f.
$$

In what follows, let

$$v := -\log f - w(\tau),
$$

where $w(\tau)$ is any $C^\infty$ function on $\tau = T - t$. Then we have the following:
Theorem 2 Let \( g(t) \) be a solution to the geometric flow \( \Box \). Let

\[
P_S = \alpha \Delta v - \beta |\nabla v|^2 + aS + \frac{b}{\tau} + \frac{d}{\tau},
\]

where \( \alpha, \beta, a, b \) and \( d \) are constants, \( \alpha \neq 0 \) and \( \alpha \neq \beta \). Then \( P_S \) satisfies

\[
\frac{\partial P_S}{\partial \tau} = \Delta P_S - 2\nabla^i P_S \nabla_i v - 2(\alpha - \beta) \left| \nabla_i \nabla_j v + \frac{\alpha}{2(\alpha - \beta)} S_{ij} - \frac{\lambda}{2\tau} g_{ij} \right|^2 + 2(a + \beta c) \nabla^i v \nabla_i S - \frac{2(\alpha - \beta)\lambda}{\alpha} \frac{\tau}{\tau} P_S + (\alpha - \beta) \frac{n\lambda^2}{2\tau^2} + \left( b - \frac{2(\alpha - \beta)\lambda}{\alpha} \right) \frac{\tau}{\tau} |\nabla v|^2 + \left( \frac{2\alpha(\alpha - \beta)}{\alpha} a\lambda - \alpha\lambda - bc \right) \frac{\tau}{\tau} 2(\alpha - \beta) \frac{\tau}{\tau} n - \alpha\gamma \Delta v + 2\beta \gamma |\nabla v|^2 - \frac{b}{\tau} - \frac{c}{\tau} - \frac{d}{\tau} \frac{\partial w(\tau)}{\partial \tau}.
\]

where \( \lambda \) is a constant.

Proof. A similar computation with Theorem 1 enables us to prove this result. In fact, notice that we have \( v = u - w(\tau) \). Therefore we get \( \nabla u = \nabla v \) and \( \Delta u = \Delta v \). We also have

\[
P_S = H_S - \frac{b}{\tau} w(\tau).
\]

Then Theorem 1 and direct computations imply

\[
\frac{\partial P_S}{\partial \tau} = \frac{\partial H_S}{\partial \tau} + \frac{b}{\tau^2} w(\tau) - \frac{b}{\tau} \frac{\partial w(\tau)}{\partial \tau} = \Delta H_S - 2\nabla^i H_S \nabla_i u - 2(\alpha - \beta) \left| \nabla_i \nabla_j u + \frac{\alpha}{2(\alpha - \beta)} S_{ij} - \frac{\lambda}{2\tau} g_{ij} \right|^2 + 2(a + \beta c) \nabla^i u \nabla_i S - \frac{2(\alpha - \beta)\lambda}{\alpha} \frac{\tau}{\tau} H_S + (\alpha - \beta) \frac{n\lambda^2}{2\tau^2} + \left( b - \frac{2(\alpha - \beta)\lambda}{\alpha} \right) \frac{\tau}{\tau} |\nabla u|^2 + \left( \frac{2\alpha(\alpha - \beta)}{\alpha} a\lambda - \alpha\lambda - bc \right) \frac{\tau}{\tau} 2(\alpha - \beta) \frac{\tau}{\tau} n - \alpha\gamma \Delta u + 2\beta \gamma |\nabla u|^2 - \frac{b}{\tau} - \frac{c}{\tau} - \frac{d}{\tau} \frac{\partial w(\tau)}{\partial \tau} - \frac{b}{\tau} \frac{\partial w(\tau)}{\partial \tau}.
\]

Since we have \( \nabla u = \nabla v \) and \( \Delta u = \Delta v \), this implies

\[
\frac{\partial P_S}{\partial \tau} = \Delta P_S - 2\nabla^i P_S \nabla_i v - 2(\alpha - \beta) \left| \nabla_i \nabla_j v + \frac{\alpha}{2(\alpha - \beta)} S_{ij} - \frac{\lambda}{2\tau} g_{ij} \right|^2 + 2(a + \beta c) \nabla^i v \nabla_i S - \frac{2(\alpha - \beta)\lambda}{\alpha} \frac{\tau}{\tau} \left( P_S + \frac{b}{\tau} w(\tau) \right) + (\alpha - \beta) \frac{n\lambda^2}{2\tau^2} + \left( b - \frac{2(\alpha - \beta)\lambda}{\alpha} \right) \frac{\tau}{\tau} |\nabla v|^2 + \left( \frac{2\alpha(\alpha - \beta)}{\alpha} a\lambda - \alpha\lambda - bc \right) \frac{\tau}{\tau} 2(\alpha - \beta) \frac{\tau}{\tau} n + \alpha\gamma \Delta v + 2\beta \gamma |\nabla v|^2 - \frac{b}{\tau} - \frac{c}{\tau} - \frac{d}{\tau} \frac{\partial w(\tau)}{\partial \tau}.
\]

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By Theorem 2 in the case where

Proof. In particular, if

Then, the following holds:

Hence we obtained the desired result.

As a corollary of Theorem 2, we get

Corollary 4 Let $g(t)$ be a solution to the geometric flow (7) and $f$ be a positive solution of the following:

Let $v = -\log f - w(\tau)$, $\tau = T - t$ and

Then, the following holds:

In particular, if $0 \leq \gamma \leq 1$, then

Proof. By Theorem 2 in the case where $\alpha = 2$, $\beta = 1$, $\lambda = 2$, $a = -c$, $\lambda = 2$, $b = 0$, we obtain

By the definition of $P_S$, we have

On the other hand, we also have

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Therefore, we obtain
\[
\frac{\partial P_s}{\partial \tau} \leq \Delta P_s - 2\nabla^i P_s \nabla_i v - \left(\frac{2}{\tau} + 2\gamma\right) P_s - \frac{2}{\tau} |\nabla v|^2 + E_{(a,-a,2,1)}(S_{ij}, \nabla v) + 2(a - 2)S - \frac{2}{n} \left(\Delta v + S - \frac{n}{\tau}\right)^2 + 2\gamma \left(\Delta v + S - \frac{n}{\tau}\right) + 2\gamma \left(a - 1\right) S + \frac{n}{\tau} \left(\frac{1}{\tau} (d + 2) + 2\gamma (d + 1)\right)
\]

On the other hand, we get the following by a direct computation.
\[
2\gamma \left(\Delta v + S - \frac{n}{\tau}\right) = -\frac{2}{\tau} \gamma \left(\Delta v + S - \frac{n}{\tau} - \frac{n}{2}\right)^2 + \frac{2}{\tau} \gamma \left(\Delta v + S - \frac{n}{\tau}\right)^2 + \frac{2}{\tau} \gamma.
\]

By using this, we get the desired result.

\section{Proof of Theorem A}

In this section, we shall prove Theorem A.

\subsection{The first case}

Suppose that \(d \leq -2\) and \(0 \leq \gamma \leq 1\). We have the following by Corollary 4
\[
\frac{\partial P_s}{\partial \tau} \leq \Delta P_s - 2\nabla^i P_s \nabla_i v - \left(\frac{2}{\tau} + 2\gamma\right) P_s + \frac{n}{2} \gamma + 2S \left(\frac{a - 2}{\tau} + \gamma(a - 1)\right) + E_a(S_{ij}, \nabla v),
\]
where notice that \(E_{(a,-a,2,1)}(S_{ij}, \nabla v) = E_a(S_{ij}, \nabla v)\) holds. Since we assumed \(a = 2\), this implies
\[
\frac{\partial P_s}{\partial \tau} \leq \Delta P_s - 2\nabla^i P_s \nabla_i v - \left(\frac{2}{\tau} + 2\gamma\right) P_s + \frac{n}{2} \gamma + 2S \gamma + E_2(S_{ij}, \nabla v).
\]

On the other hand, a direct computation tells us that
\[
2S \gamma = -\frac{2}{\tau} \gamma \left(S - \frac{n}{2}\right)^2 + \frac{2}{\tau} \gamma S^2 + \frac{n}{2} \gamma.
\]

Hence, we get
\[
\frac{\partial P_s}{\partial \tau} \leq \Delta P_s - 2\nabla^i P_s \nabla_i v - \left(\frac{2}{\tau} + 2\gamma\right) P_s + n\gamma - \frac{2}{\tau} \gamma \left(S - \frac{n}{2}\right)^2 + \frac{2}{\tau} \gamma S^2 + E_2(S, \nabla v)
\]
\[
\leq \Delta P_s - 2\nabla^i P_s \nabla_i v - \left(\frac{2}{\tau} + 2\gamma\right) P_s + n\gamma + E_2(S_{ij}, \nabla v) + \frac{2}{\tau} \gamma S^2.
\]

On the other hand, we also have
\[
-\left(\frac{2}{\tau} + 2\gamma\right) P_s = -\left(\frac{2}{\tau} + 2\gamma\right) \left(P_s - \frac{n}{2} \gamma\right) - \frac{n\gamma}{\tau} - n\gamma^2.
\]

Adding \(-\frac{2}{\tau} \gamma\) to \(P_s\), we get
\[
\frac{\partial}{\partial \tau} \left(P_s - \frac{n}{2} \gamma\right) \leq \Delta \left(P_s - \frac{n}{2} \gamma\right) - 2\nabla^i \left(P_s - \frac{n}{2} \gamma\right) \nabla_i v - \left(\frac{2}{\tau} + 2\gamma\right) \left(P_s - \frac{n}{2} \gamma\right) - \frac{n\gamma}{\tau}
\]
\[
- n(\gamma^2 - \gamma) + E_2(S_{ij}, \nabla v) + \frac{2}{n} \gamma S^2
\]
\[
\leq \Delta \left(P_s - \frac{n}{2} \gamma\right) - 2\nabla^i \left(P_s - \frac{n}{2} \gamma\right) \nabla_i v - \left(\frac{2}{\tau} + 2\gamma\right) \left(P_s - \frac{n}{2} \gamma\right)
\]
\[
- n\gamma(\gamma - 1) + E_2(S_{ij}, \nabla v) + \frac{2}{n} \gamma S^2.
\]
Finally, by taking $\gamma = 1$ and using (4), we obtain

$$\frac{\partial}{\partial \tau} \left( P_S - \frac{n}{2} \right) \leq \Delta \left( P_S - \frac{n}{2} \right) - 2 \nabla^i \left( P_S - \frac{n}{2} \right) \nabla_i v - \left( \frac{2}{\tau} + 2 \right) \left( P_S - \frac{n}{2} \right).$$

Since

$$P_S - \frac{n}{2} < 0$$

holds for $\tau$ small enough which depends on $d$, the maximal principle implies the desired result.

### 4.2 The second case

Suppose that $d \leq -2$ and $0 \leq \gamma \leq 1$. As in the first case, the following holds:

$$\frac{\partial P_S}{\partial \tau} \leq \Delta P_S - 2 \nabla^i P_S \nabla_i v - \left( \frac{2}{\tau} + 2 \gamma \right) P_S + \frac{n}{2} \gamma + 2 S \left( \frac{a - 2}{\tau} + \gamma(a - 1) \right) + \mathcal{E}_a(S_{ij}, \nabla v).$$

Since we assumed $a = 1$, we have

$$\frac{\partial P_S}{\partial \tau} \leq \Delta P_S - 2 \nabla^i P_S \nabla_i v - \left( \frac{2}{\tau} + 2 \gamma \right) P_S + \frac{n}{2} \gamma.$$

Assume (5) holds. Then this implies

$$\frac{\partial P_S}{\partial \tau} \leq \Delta P_S - 2 \nabla^i P_S \nabla_i v - \left( \frac{2}{\tau} + 2 \gamma \right) P_S + \frac{n}{2} \gamma.$$

By adding $-\frac{n}{4} \gamma$ to $P_S$, we get

$$\frac{\partial}{\partial \tau} \left( P_S - \frac{n}{4} \gamma \right) \leq \Delta \left( P_S - \frac{n}{4} \gamma \right) - 2 \nabla^i \left( P_S - \frac{n}{4} \gamma \right) \nabla_i v - \left( \frac{2}{\tau} + 2 \gamma \right) \left( P_S - \frac{n}{4} \gamma \right) - \frac{n}{2} \left( \gamma^2 - \gamma \right)$$

$$\leq \Delta \left( P_S - \frac{n}{4} \gamma \right) - 2 \nabla^i \left( P_S - \frac{n}{4} \gamma \right) \nabla_i v - \left( \frac{2}{\tau} + 2 \gamma \right) \left( P_S - \frac{n}{4} \gamma \right)$$

Finally, by taking $\gamma = 1$,

$$\frac{\partial}{\partial \tau} \left( P_S - \frac{n}{4} \right) \leq \Delta \left( P_S - \frac{n}{4} \right) - 2 \nabla^i \left( P_S - \frac{n}{4} \right) \nabla_i v - \left( \frac{2}{\tau} + 2 \right) \left( P_S - \frac{n}{4} \right).$$

Notice that

$$P_S - \frac{n}{4} < 0$$

holds for $\tau$ small enough which depends on $d$. By using the maximal principle, we get the desired result.

### 5 Proof of Theorem B

Suppose that $0 \leq \gamma \leq 1$. We get the following by Corollary 4:

$$\frac{\partial P_S}{\partial \tau} \leq \Delta P_S - 2 \nabla^i P_S \nabla_i v - \left( \frac{2}{\tau} + 2 \gamma \right) P_S + \frac{n}{2} \gamma + 2 S \left( \frac{a - 2}{\tau} + (a - 1) \gamma \right)$$

$$+ \frac{n}{\tau} \left( \gamma(d + 2) + 2 \gamma(d + 1) \right) + \mathcal{E}_a(S_{ij}, \nabla v).$$
By taking $a = 1$ and $d = -3$, we get
\[
\frac{\partial P_S}{\partial \tau} \leq \Delta P_S - 2\nabla^2 P_S \nabla_i v - \left(\frac{2}{\tau} + 2\gamma\right) P_S - \frac{2}{\tau}\left(S + \frac{n}{2\tau}\right) - \frac{4n}{2\tau} + \frac{n}{2} + \mathcal{E}_1(S, \nabla v)
\]
\[
\leq \Delta P_S - 2\nabla^2 P_S \nabla_i v - \left(\frac{2}{\tau} + 2\gamma\right) P_S - \frac{2}{\tau}\left(S + \frac{n}{2\tau}\right) + n\gamma + \mathcal{E}_1(S_{ij}, \nabla v)
\]
Assume that $\mathcal{E}_1(S_{ij}, \nabla v) \leq 0$ holds. Then we obtain
\[
\frac{\partial P_S}{\partial \tau} \leq \Delta P_S - 2\nabla^2 P_S \nabla_i v - \left(\frac{2}{\tau} + 2\gamma\right) P_S - \frac{2}{\tau}\left(S + \frac{n}{2\tau}\right) + \frac{n}{2}.
\]
On the other hand, assume that $S \geq -\frac{\partial}{\partial \tau}$ holds for all time $t \in [\frac{T}{2}, T)$. Since we have $\frac{1}{\tau} \geq \frac{1}{4}$, we get
\[
S \geq -\frac{n}{2\tau} \geq -\frac{n}{2\gamma}.
\]
Therefore, we obtain
\[
\frac{\partial P_S}{\partial \tau} \leq \Delta P_S - 2\nabla^2 P_S \nabla_i v - \left(\frac{2}{\tau} + 2\gamma\right) P_S + \frac{n}{2}\gamma.
\]
By adding $-\frac{n}{4}\gamma$ to $P_S$, we have
\[
\frac{\partial}{\partial \tau}\left(P_S - \frac{n}{4}\gamma\right) \leq \Delta\left(P_S - \frac{n}{4}\gamma\right) - 2\nabla^2\left(P_S - \frac{n}{4}\gamma\right) \nabla_i v - \left(\frac{2}{\tau} + 2\gamma\right)\left(P_S - \frac{n}{4}\gamma\right) - \frac{n}{2}\gamma - \frac{n}{2}(\gamma^2 - \gamma)
\]
\[
\leq \Delta\left(P_S - \frac{n}{4}\right) - 2\nabla^2\left(P_S - \frac{n}{4}\right) \nabla_i v - \left(\frac{2}{\tau} + 2\right)\left(P_S - \frac{n}{4}\right) - \frac{n}{2}\gamma(\gamma - 1).
\]
Finally, by taking $\gamma = 1$, we obtain
\[
\frac{\partial}{\partial \tau}\left(P_S - \frac{n}{4}\right) \leq \Delta\left(P_S - \frac{n}{4}\right) - 2\nabla^2\left(P_S - \frac{n}{4}\right) \nabla_i v - \left(\frac{2}{\tau} + 2\right)\left(P_S - \frac{n}{4}\right).
\]
Notice that
\[
P_S - \frac{n}{4} < 0
\]
holds for $\tau$ small enough. By using the maximal principle, we get Theorem $\mathbb{B}$.

### 6 Proof of Theorem $\mathbb{C}$

By Proposition $\mathbb{C}$ in the case where $\alpha = 0$, $\beta = -1$, $a = 0$, $c = 0$, $b = -1$ and $d = 0$, we get
\[
H_S = |\nabla u|^2 - \frac{u}{\tau}
\]
and
\[
\frac{\partial H_S}{\partial \tau} = \Delta H_S - 2\nabla^2 H_S \nabla_i u - \frac{1}{\tau}H_S - 2|\nabla \nabla u|^2 - 2\left(R^{ij} + S^{ij}\right) \nabla_i u \nabla_j u
\]
\[
\leq \Delta H_S - 2\nabla^2 H_S \nabla_i u - \left(\frac{1}{\tau} + \gamma\right)H_S - 2|\nabla \nabla u|^2 - 2\left(R^{ij} + S^{ij}\right) \nabla_i u \nabla_j u
\]
\[
\leq \Delta H_S - 2\nabla^2 H_S \nabla_i u - \left(\frac{1}{\tau} + \gamma\right)H_S - 2\left(R^{ij} + S^{ij}\right) \nabla_i u \nabla_j u - \gamma|\nabla u|^2.
\]
Assume that $R_{ij}(t) + S_{ij}(t) \geq -Ag_{ij}$, where $0 \leq A \leq \frac{1}{2}\gamma$. Then we have

$$-2\left(R^{ij} + S^{ij}\right)\nabla_i u \nabla_j u - \gamma|\nabla u|^2 \leq (2A - \gamma)|\nabla u|^2 \leq 0.$$ 

Therefore we obtain

$$\frac{\partial H_S}{\partial \tau} \leq \Delta H_S - 2\nabla^i H_S \nabla_i u - \left(\frac{1}{\tau} + \gamma\right)H_S.$$ 

If $\tau$ is small, then $H_S < 0$. By applying the maximal principle to the above inequality, Theorem C follows.

On the other hand, it is also natural to consider the following equation:

$$\frac{\partial f}{\partial \tau} = \Delta f - \gamma(\tau)f \log f - aSf,$$

where $\gamma(\tau)$ is any $C^\infty$ function of $\tau - T - t$. Let $u = -\log f$. Then $u$ satisfies

$$\frac{\partial u}{\partial \tau} = \Delta u - |\nabla u|^2 + aS - \gamma(\tau)u. \quad (15)$$

Then, computations which are similar to Proposition 1 tell us that the following holds:

**Proposition 2** Let $g(t)$ be a solution to the geometric flow (1) and $u$ satisfies (15). Let

$$H_S = \alpha \Delta u - \beta |\nabla u|^2 + aS + b\frac{u}{\tau} + d\frac{n}{\tau},$$

where $\alpha, \beta, a, b$ and $d$ are constants. Then the following holds:

$$\frac{\partial H_S}{\partial \tau} = \Delta H_S - 2\nabla^i H_S \nabla_i u + 2(a + \beta c)\nabla^i S \nabla_i u - 2(\alpha - \beta)|\nabla \nabla u|^2$$

$$- 2\alpha S^{ij} \nabla_i \nabla_j u + \frac{b}{\tau}|\nabla u|^2 - \frac{b}{\tau}cS - \frac{b}{\tau^2}u - \frac{d}{\tau^2}$$

$$+ a \frac{\partial S}{\partial \tau} - (a - \alpha a)\Delta S - \alpha \left(2\nabla^i S_{\ell \ell} - \nabla \nabla S\right) \nabla^i u$$

$$- 2\left(\alpha R^{ij} - \beta(R^{ij} + S^{ij})\right) \nabla_i u \nabla_j u - \alpha \gamma(\tau)\Delta u + 2\beta \gamma(\tau)|\nabla u|^2 - b\gamma(\tau)\frac{u}{\tau}$$

holds.

By using this, we are able to prove a generalization of Theorem C as follows:

**Theorem 3** Suppose that $g(t)$, $t \in [0, T)$, evolves by the geometric flow (1) on a closed oriented smooth $n$-manifold $M$ and let $0 < f < 1$ be a positive solution to

$$\frac{\partial f}{\partial \tau} = \Delta f - \gamma(\tau)f \log f,$$

where $\gamma(\tau) > 0$ is any $C^\infty$ function on $\tau = T - t$. Suppose that there is a constant $A$ such that

$$R_{ij}(t) + S_{ij}(t) \geq -Ag_{ij}(t),$$

where $0 \leq A \leq \frac{1}{2}\gamma(\tau)$ and Let $u = -\log f$. Then for all time $t \in [0, T)$,

$$|\nabla u|^2 - \frac{u}{\tau} \leq 0$$

holds.

The proof is similar to that of Theorem C. Instead of Proposition 1 use Proposition 2.
7 Proof of Theorem D

By taking $\alpha = 0$, $\beta = -1$, $\gamma = a = c = 0$, $b = -1$ and $d = 0$ in Proposition we have

$$H_S = |\nabla u|^2 - \frac{u}{\tau}$$

and

$$\frac{\partial H_S}{\partial \tau} = \Delta H_S - 2\nabla^i H_S \nabla_i u - \frac{1}{\tau} H_S - 2|\nabla \nabla u|^2 - 2(R^{ij} + S^{ij}) \nabla_i u \nabla_j u$$

$$\leq \Delta H_S - 2\nabla^i H_S \nabla_i u - \frac{1}{\tau} H_S - 2(R^{ij} + S^{ij}) \nabla_i u \nabla_j u.$$ 

Assume that $R_{ij}(t) + S_{ij}(t) \geq 0$. Then we have

$$-2(R^{ij} + S^{ij}) \nabla_i u \nabla_j u \leq 0.$$ 

Therefore, the following holds:

$$\frac{\partial H_S}{\partial \tau} \leq \Delta H_S - 2\nabla^i H_S \nabla_i u - \frac{1}{\tau} H_S.$$ 

If $\tau$ is small, then $H_S < 0$. Then the maximal principle implies the desired result.

8 Proof of Theorem E

Let $P_S := 2\Delta v - |\nabla v|^2 + S - 2\frac{n}{\tau}$. By Corollary in the case where $\gamma = 0$, $a = 1$ and $d = -2$, we obtain

$$\frac{\partial P_S}{\partial \tau} \leq \Delta P_S - 2\nabla^i P_S \nabla_i v - \frac{2}{\tau} P_S - \frac{2}{n} \left(\Delta v + S - \frac{n}{\tau}\right)^2$$

$$\geq \frac{2}{\tau} |\nabla v|^2 - \frac{2S}{\tau} + E_1(S_{ij}, \nabla v)$$

$$= \Delta P_S - 2\nabla^i P_S \nabla_i v - \frac{2}{\tau} \left(P_S + |\nabla v|^2 + S\right)$$

$$- \frac{2}{n} \left(\Delta v + S - \frac{n}{\tau}\right)^2 + E_1(S_{ij}, \nabla v).$$

where notice that $E_{(1, -1, 2, 1)}(S, \nabla v) = E_1(S, \nabla v)$. Since we also have

$$P_S + |\nabla v|^2 + S = 2 \left(\Delta v + S - \frac{n}{\tau}\right),$$

we obtain

$$\frac{\partial P_S}{\partial \tau} \leq \Delta P_S - 2\nabla^i P_S \nabla_i v - \frac{2}{\tau} \left(P_S + |\nabla v|^2 + S\right)$$

$$- \frac{1}{2n} \left(P_S + |\nabla v|^2 + S\right)^2 + E_1(S_{ij}, \nabla v)$$

$$= \Delta P_S - 2\nabla^i P_S \nabla_i v - \frac{1}{2n} \left(P_S + |\nabla v|^2 + S + \frac{2n}{\tau}\right)^2 + \frac{2n}{\tau^2}$$

$$+ E_1(S_{ij}, \nabla v)$$

$$\leq \Delta P_S - 2\nabla^i P_S \nabla_i v + \frac{2n}{\tau^2},$$

where we used $E_1(S_{ij}, \nabla u) \leq 0$. By adding $\frac{2n}{\tau}$ to $P_S$, we get

$$\frac{\partial}{\partial \tau} \left(P_S + \frac{2n}{\tau}\right) \leq \Delta \left(P_S + \frac{2n}{\tau}\right) - 2\nabla^i \left(P_S + \frac{2n}{\tau}\right) \nabla_i v.$$ 

This implies max $\frac{\partial}{\partial \tau} \left(P_S + \frac{2n}{\tau}\right) = \max \left(2\Delta v - |\nabla v|^2 + S\right)$ decreases as $\tau$ increases, which means that this quantity increases as $t$ increases.
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