LAGRANGIAN FORMALISM OVER GRADED ALGEBRAS

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ABSTRACT. This paper provides a description of an algebraic setting for the Lagrangian formalism over graded algebras and is intended as the necessary first step towards the noncommutative C-spectral sequence (variational bicomplex). A noncommutative version of integration procedure, the notion of adjoint operator, Green's formula, the relation between integral and differential forms, conservation laws, Euler operator, Noether's theorem is considered.

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An outstanding progress which has in last years been made by noncommutative geometry stems out of shifting of the interests away from the original idea of geometrizing noncommutative rings using the language of noncommutative schemes. Now it has become clear that, bypassing the difficult problem of glueing noncommutative spectra, one can define directly differential geometric objects on a hypothetical “noncommutative space”. This is based on two essential points: the possibility of reformulating the classical notions of analysis and differential geometry in pure algebraic terms, so that differential calculus becomes an extension of the language of commutative algebra (for a very enlightening discussion see [1]), and the existence of several quite important cases in which one is able to go beyond the commutative case (see, e.g., [2]). These ideas have proven to be very successful and give an impetus to new researches whose aim is to transplant the classical tools of analysis and geometry into a noncommutative setting. An instance of this comes from the current work on noncommutative (and first of all supercommutative) theory of integration, conservation laws, and the Lagrangian formalism (see [3], [4], [5], [6], [7], [8], [9], [10], [11], [12], [13], [14], [15], [16], [17] and others). A key problem arises here is the description of integration procedure. The first question is the following: Given a commutative algebra, how to define the module of volume forms? An answer to this question must, in particular, give the possibility to define the Berezin volume forms on a supermanifold in usual fashion, i.e., by using the rule of signs. The well-known peculiarities of the Berezin integration show the character of the problem and result in the loss of a clear algebraic setting for integration procedure and related things.

The subject of this paper is an algebraic picture underlying the Lagrangian formalism and giving a solution of the problems discussed above. We work here over an arbitrary graded-commutative (with respect to a commutation factor) algebra to show that our constructions of volume forms, adjoint operators, the Euler operator, algebraic Green’s formula, Noether’s theorem, etc. can be extended
to such an algebra in a simple and straightforward manner. On the other hand, the
class of graded-commutative algebras is quite important for its own sake because it
includes supercommutative, colour-commutative (see [18], [19], [20], [21], [22], [23],
[24], [25] and references therein), and “quantum” algebras (see [26],[20], [27] and
others). We have chosen here not to accumulate formulas for specific algebras, but
to present a general scheme, using examples for illustrations only. The applications
will be described separately. Our ultimate goal, which is outside the present
paper, is to develop the super- and non-commutative generalizations of the $C$-
spectral sequence (variational bicomplex) (see [28], [29], [30], [31]), which is a
means for studying all aspects of the Lagrangian formalism: the inverse problem,
the description of conservation laws, characteristic classes and so on.

The paper is organized as follows. In Section 1 we outline the necessary defini-
tions and facts from calculus over graded algebras. In Section 2 the main objects
of this paper — adjoint operators and the Berezinian\footnote{We use this name for the module of volume forms.} — are defined. We want to
emphasize that these definitions, which are the \textit{deus ex machina} from that ev-
erything else follows, arose from an interplay between ideas and constructions of
work [32], where the Berezin forms have been explained in terms of $\mathcal{D}$-modules, and
works [33] and [29, part I], where structure of Lagrangian formalism for a smooth
commutative algebra has been clarified. In Section 3 we consider the Spencer
complexes, algebraic Green’s formula, and related staff. The main difference from
non-graded case (see [29] is the appearance of a new complex dual to the de Rham
complex: complex of integral forms (the name borrowed from the supergeometry).
The Lagrangian formalism, theory of conservation laws, and the Noether theorem
are developed in Section 4. In the Appendix we briefly describe, in a pure algebraic
manner, the formalism of right connections (see [34], [35]), which is closely related
with our subject.
1. A Sketch of Differential Calculus over Graded Algebras

1.1. We start with definitions of graded objects (see, e.g., [36]).

Let $G$ be an Abelian group written additively, which will serve as a grading group, and $K$ a commutative ring with unit.

Denote by $K^*$ the multiplicative group of invertible elements of $K$. Let us fix a commutation factor $\{\cdot, \cdot\} : G \times G \to K^*$, $g_1 \times g_2 \mapsto \{g_1, g_2\}$, i.e., a map satisfying two properties:

1. $\{g_1, g_2\}^{-1} = \{g_2, g_1\}$
2. $\{g_1 + g_2, g_3\} = \{g_1, g_3\}\{g_2, g_3\}$

We have also $\{g_1, g_2 + g_3\} = \{g_1, g_2\}\{g_1, g_3\}$ as a consequence of the definition.

**Example 1.** Let $G = \mathbb{Z}^n = \mathbb{Z} \oplus \ldots \oplus \mathbb{Z}$. Then it is easily shown that any commutation factor has the form:

$$\{g, h\} = \prod_{i=1}^{n} q_i^{g_i h_i} \cdot \prod_{i < j} (q_{ij}^{g_i h_j - h_i g_j})$$

where $g = (g_1, \ldots, g_n)$, $h = (h_1, \ldots, h_n) \in \mathbb{Z}^n$, $q_i = \pm 1$, $q_{ij} \in K^*$

**Example 2.** In particular, if $G = \mathbb{Z}$ there exist two commutation factors: the trivial one $\{g_1, g_2\} = 1$ and the standard superfactor $\{g_1, g_2\} = (-1)^{g_1 g_2}$.

**Example 3.** If $G = \mathbb{Z}^n$ and $K = \mathbb{C}$ then $\{g, h\} = \prod_{i=1}^{n} (\pm 1)^{g_i h_i} \cdot e^{\theta(g,h)}$, where $\theta : G \times G \to \mathbb{C}$ is an antisymmetric bilinear form.

Suppose $A = \bigoplus_{g \in G} A_g$ is a $G$-graded associative $K$-algebra with unit. $A$ is called **graded commutative** if

$$ab = \{a, b\}ba \quad \forall a, b \in A.$$

For this type of notation we always assume that $a$ and $b$ are homogeneous and that the symbol of graded object used as argument of the commutative factor denotes the grading of this object.
Example 4. A commutative algebra (graded or not) is graded commutative with respect to the trivial commutation factor \( \{g_1, g_2\} = 1 \).

Example 5. The algebra \( C^\infty(M) \) of smooth functions on a supermanifolds \( M \) is graded commutative with respect to the superfactor \( \{g_1, g_2\} = (-1)^{g_1 g_2}, G = \mathbb{Z}_2 \).

Example 6. Quantum superplane ([37], [2]). Let \( K \) be a field and \( K^n \) the \( n \)-dimensional coordinate vector space over \( K \). Picking up an arbitrary commutation factor over \( \mathbb{Z}^n \) (see Examples 1–3), the quadratic algebra \( A = T(K^n)/R \), where \( T(K^n) \) is the tensor algebra over \( K^n \) with the natural \( \mathbb{Z}^n \)-grading and \( R \) is the ideal in \( T(K^n) \) generated by the elements \( x_1 \otimes x_2 - \{x_1, x_2\} x_2 \otimes x_1, x_1, x_2 \in K^n \), is called the algebra of polynomial functions on the quantum superplane \( A_q \).

Example 7. Non-commutative torus ([38], [2]). Let \( K = \mathbb{C}, G = \mathbb{Z}^n \), and \( A \) be the space of all complex valued functions on \( \mathbb{Z}^n \) that decay faster than any polynomial. \( A \) has the natural \( \mathbb{Z}^n \)-grading. Denote by \( e(k) \) the function which is equal to 1 at \( k \in \mathbb{Z}^n \) and zero at all other points (a basic harmonic on a non-commutative torus). Then any element of \( A \) can be written as \( \sum_k a_k e(k) \). Given a skew bilinear form \( \theta: \mathbb{Z} \times \mathbb{Z} \to \mathbb{R} \), we define a multiplication on \( A \) by setting \( e(k)e(l) = e^{2\pi i \theta(k,l)} e(k+l) \). It is straightforward to check that the algebra \( A \), called the algebra of (smooth) functions on the non-commutative torus \( T_\theta \), is graded commutative with respect to the commutation factor from Example 3.

Remark. In [39], [40] Lychagin has shown that one can also define commutation factors over a non-commutative group \( G \) and introduce all notions of the \( G \)-graded differential calculus in these circumstances.

In this article we work in the category \( \text{Mod}_A \) of all \( G \)-graded modules over a graded commutative algebra \( A \). Clearly any left module \( P \) can be transformed canonically into a right module, \( pa = \{p, a\} ap \) for \( a \in A, p \in P \), so that we may consider \( P \) as a bimodule.

Remark. The category \( \text{Mod}_A \) is a closed tensor category (see, e.g., [2]) with respect
to the commutativity constraint

\[ S_{P,Q} : P \otimes Q \to Q \otimes P, \quad p \otimes q \mapsto \{p, q\} q \otimes p. \]

A generalization of our constructions to an arbitrary Abelian closed tensor category appears to be very interesting. Significant results along this line, dealing with basic concepts of the differential calculus, may be found in [41], [42].

1.2. We now introduce basic objects of the differential calculus (for details see [43], [44], [45], [46]).

Let \( \Delta \in \text{Hom}_K (P, Q) \) be a \( K \)-homomorphism, \( P \) and \( Q \) being \( A \)-modules.

For every element \( a \in A \) define a \( K \)-homomorphism \( \delta_a (\Delta) : P \to Q \) by setting

\[ \delta_a (\Delta) (p) = \{a, \Delta\} \Delta (ap) - a \Delta (p), \quad p \in P. \]

Obviously, \( \delta_a \circ \delta_b = \delta_b \circ \delta_a \quad \forall a, b \in A. \)

Put \( \delta_{a_0, \ldots, a_k} = \delta_{a_0} \circ \cdots \circ \delta_{a_k}. \)

**Definition.** A \( K \)-homomorphism \( \Delta \in \text{Hom}_K (P, Q) \) is called a *differential operator* (d.o.) of order \( \leq k \), if for all \( a_0, \ldots, a_k \in A \) we have \( \delta_{a_0, \ldots, a_k} (\Delta) = 0. \)

The set of all d.o.’s of order \( \leq k \), from \( P \) to \( Q \), may be endowed with two \( A \)-module structures by putting \( a\Delta = a \circ \Delta \) or \( a\Delta = \{a, \Delta\} \Delta \circ a \), where \( a \in A \) is understood as the operator of multiplication by \( a \). The modules that arise in this way are denoted by \( \text{Diff}_k (P, Q) \) and \( \text{Diff}^+ (P, Q) \) respectively. Clearly, \( \text{Diff}_k^+ (P, Q) \subset \text{Diff}^+ (P, Q) \) for \( k \leq l \), so that we may consider the union \( \text{Diff}^+ (P, Q) = \bigcup_{k \geq 0} \text{Diff}^+_k (P, Q) \).

**Proposition.**

1. If \( \Delta_1 \in \text{Diff}_k (P, Q), \ \Delta_2 \in \text{Diff}_l (Q, R), \ P, Q, R \) are \( A \)-modules, then

\( \Delta_2 \circ \Delta_1 \in \text{Diff}_{k+l} (P, R) \).

2. The maps \( \text{Diff}_k (P, Q) \to \text{Diff}^+_k (P, Q) \) and \( \text{Diff}^+_k (P, Q) \to \text{Diff}_k (P, Q) \)

generated by the identity map are d.o.’s of order \( \leq k \).

3. There exists a canonical isomorphism

\[ \text{Diff}^+_k (P, Q) \to \text{Hom}_A (P, \text{Diff}^+_k (Q)), \]
where $\text{Diff}_k^+ (Q) = \text{Diff}_k^+ (A, Q)$. To every d.o. $\Delta: P \to Q$ corresponds the homomorphism $\varphi_\Delta \in \text{Hom}_A (P, \text{Diff}_k^+ (Q))$, $\varphi_\Delta (p)(a) = \Delta (pa)$ under this isomorphism. The inverse mapping takes a homomorphism $\varphi: P \to \text{Diff}_k^+ (Q)$ to an operator $\mathcal{D}_k \circ \varphi$, where $\mathcal{D}_k: \text{Diff}_k^+ (Q) \to Q$ is a d.o. of order $\leq k$ defined by the formula $\mathcal{D}_k (\nabla) = \nabla (1)$, $\nabla \in \text{Diff}_k^+ (Q)$.

**Proof** consists of a series of automatic verifications. 

This proposition has the following:

**Corollary.** The commutative diagram

$$
\begin{array}{ccc}
\text{Diff}_k^+ (\text{Diff}_l^+ (P)) & \xrightarrow{\mathcal{D}_k} & \text{Diff}_l^+ (P) \\
\downarrow c_{k,l} & & \downarrow \mathcal{D}_l \\
\text{Diff}_{k+l}^+ (P) & \xrightarrow{\mathcal{D}_{k+l}} & P
\end{array}
$$

uniquely defines the operator $c_{k,l}$, which is said to be glueing operator.

**Definition.** A $k$-multilinear over $K$ mapping $\nabla: A \times \ldots \times A \to P$, $P$ being an $A$-module, is said to be a **skew multiderivation** if the following conditions hold:

1. $\nabla (a_1, \ldots, a_i, a_{i+1}, \ldots, a_k) = -\{a_i, a_{i+1}\} \nabla (a_1, \ldots, a_{i+1}, a_i, \ldots, a_k)$
2. $\nabla (a_1, \ldots, a_{i-1}, ab, a_{i+1}, \ldots, a_k) =$
   
   $\{\nabla, a\} \{a_1 \ldots a_{i-1}, a\} a \nabla (a_1, \ldots, b, \ldots, a_k) +$
   
   $\{\nabla, b\} \{a_1 \ldots a_{i-1}a, b\} b \nabla (a_1, \ldots, a, \ldots, a_k)$

The set of all skew multiderivations is an $A$-module denoted by $\text{D}_k (P)$. Obviously $\text{D}_1 (P)$ is a submodule of $\text{Diff}_1 (A, P)$.

The *Spencer* Diff-operator $S: \text{D}_k (\text{Diff}_l^+ (P)) \to \text{D}_{k-1} (\text{Diff}_{l+1}^+ (P))$ is defined by the formula

$$
S (\nabla) (a_1, \ldots, a_{k-1}) (a) = \nabla (a_1, \ldots, a_{k-1}, a) (1),
$$

$\nabla \in \text{D}_k (\text{Diff}_l^+ (P))$. 
Proposition.

(1) $S$ is a d.o. of order $\leq 1$;
(2) $S^2 = 0$.

Proof is straightforward. □

The complex $0 \leftarrow P \xleftarrow{D} \text{Diff}^+ (P) \xleftarrow{S} D_1 (\text{Diff}^+ (P)) \xleftarrow{S} D_2 (\text{Diff}^+ (P)) \leftarrow \ldots$ is said to be the Spencer Diff-complex.

1.3. The operations $Q \mapsto D_k (Q)$, $Q \mapsto \text{Diff} (P, Q)$, and $Q \mapsto \text{Diff}^+ (Q, P)$ are functors from the category of all $A$-modules $\text{Mod}_A$ to itself. We have seen that the latter functor is representable. The following proposition shows that two others functors are also representable.

Proposition (see [43], [45], [46]). There exists a module $\Lambda^k$ (resp. $\mathcal{J}^k (P)$), which is called the module of $k$-form (resp. $k$-jets) over $A$, such that the functor $Q \mapsto D_k (Q)$ (resp. $Q \mapsto \text{Diff}_k (P, Q)$) is isomorphic to the functor $Q \mapsto \text{Hom}_A (\Lambda^k, Q)$ (resp. $Q \mapsto \text{Hom}_A \left( \mathcal{J}^k (P), Q \right)$).

Let us denote by $j_k = j_k (P) : P \to \mathcal{J}^k (P)$ the d.o. of order $\leq k$ that corresponds under the isomorphism $\text{Diff}_k \left( P, \mathcal{J}^k (P) \right) = \text{Hom}_A \left( \mathcal{J}^k (P), \mathcal{J}^k (P) \right)$ to the identical map $id_{\mathcal{J}^k (P)}$. Proposition 1.3 implies that the operator $j_k$ is universal, i.e., for every d.o. $\Delta \in \text{Diff}_k (P, Q)$ there exists a unique homomorphism $\psi_\Delta : \mathcal{J}^k (P) \to Q$ such that $\Delta = \psi_\Delta \circ j_k$.

Now let $\mathcal{M}$ be a subcategory of category $\text{Mod}_A$ of all $A$-modules that is closed under the action of the above-discussed functors. Then it is quite natural to ask if these functors are representable in $\mathcal{M}$.

Example. The most important example of the closed (in the above sense) category is that of geometrical modules $\text{Mod}_A^g$. A module $P$ is called geometrical if $\widetilde{P} = \bigcap_{\psi, k} \psi^k P = 0$, where intersection is taken over all powers of prime ideals of $A$. There is the geometrization functor $P \mapsto P/\widetilde{P}$ from $\text{Mod}_A$ to $\text{Mod}_A^g$. It can easily be checked that the functors $D_k$ and $\text{Diff}_k (P, \cdot)$ are representable in $\text{Mod}_A^g$. 
the geometrization functor transforming the representing objects in \( \text{Mod}_A \) into the corresponding representing objects in \( \text{Mod}_A^{\mathbb{Z}} \).

1.4. A natural transformation of the functors \( D_k, \text{Diff}_k \), and their compositions by duality gives rise to operators between the corresponding representing objects.

**Example 1.** The natural inclusion \( D_{k+l}(P) \rightarrow D_k(D_l(P)) \) defines the wedge product of forms over \( A \): \( \Lambda^k \otimes \Lambda^l \rightarrow \Lambda^{k+l} \).

**Example 2.** The Spencer operator \( S: D_k(P) \rightarrow \hat{D}_{k-1}(\text{Diff}^+_1(P)) \), where the superscribed dot means that the \( K \)-module \( D_{k-1}(\text{Diff}^+_1(P)) \) is supplied with \( A \)-module structure by putting \( a\theta = D_{k-1}(\text{Diff}^+_1(a))\theta, \theta \in D_{k-1}(\text{Diff}^+_1(P)) \), induces the homomorphism \( \mathcal{J}^1(\Lambda^{k-1}) \rightarrow \Lambda^k \). The composition of \( \Lambda^{k-1} \xrightarrow{\mathcal{J}^1} \Lambda^k \) is called the **exterior differentiation** operator and is denoted by \( d: \Lambda^{k-1} \rightarrow \Lambda^k \). Using the fact that \( S^2 = 0 \), one can easily prove that \( d^2 = 0 \).

The complex \( 0 \rightarrow A \xrightarrow{d} \Lambda^1 \xrightarrow{d} \Lambda^2 \xrightarrow{d} \ldots \) is said to be the **de Rham complex** of the algebra \( A \).

1.5. The above described algebraic formalism can be realized geometrically, the algebra \( A \) being the algebra \( C^\infty(M) \) of smooth real functions on a supermanifold \( M \) (for the supergeometry see, e.g., [47], [48], [35], [49], [50]). In this situation it may be shown in the same way as in non-graded case that the standard notions of differential operator, forms, jets, etc. coincide with the ones introduced above. Having this in mind, in the next section we shall illustrate constructions under consideration by giving their local coordinate description.

### 2. Adjoint Operators and Berezinian

2.1. Given an \( A \)-module \( P \), consider the complex of homomorphisms

\[
0 \rightarrow \text{Diff}^+(P, A) \xrightarrow{w} \text{Diff}^+(P, \Lambda^1) \xrightarrow{w} \text{Diff}^+(P, \Lambda^2) \xrightarrow{w} \ldots ,
\]

where \( w(\nabla) = d \circ \nabla \in \text{Diff}^+(P, \Lambda^k), \nabla \in \text{Diff}^+(P, \Lambda^{k-1}) \). Let us denote the cohomology module in a term \( \text{Diff}^+(P, \Lambda^n) \) by \( \hat{P}_n, n \geq 0 \). Every d.o. \( \Delta: P \rightarrow Q \)
generates the natural map of the complexes:

\[
\begin{array}{cccc}
\ldots & \longrightarrow & \text{Diff}^+ (Q, \Lambda^{k-1}) & \longrightarrow \text{Diff}^+ (Q, \Lambda^k) & \longrightarrow & \ldots \\
\Delta \downarrow & & \Delta \downarrow & & \Delta \downarrow & \\
\ldots & \longrightarrow & \text{Diff}^+ (P, \Lambda^{k-1}) & \longrightarrow \text{Diff}^+ (P, \Lambda^k) & \longrightarrow & \ldots,
\end{array}
\]

where \( \tilde{\Delta} (\nabla) = \{ \Delta, \nabla \} \nabla \circ \Delta \in \text{Diff}^+ (P, \Lambda^k) \), \( \nabla \in \text{Diff}^+ (Q, \Lambda^k) \).

**Definition.** The operator \( \Delta^*_n : \hat{Q}_n \rightarrow \hat{P}_n \) induced by \( \tilde{\Delta} \) is called the \((n\text{-th})\) adjoint operator.

Below we assume that an integer \( n \) is fixed and omit the corresponding index to simplify notation.

**Proposition.**

1. \( \Delta^* \) has the same grading as \( \Delta \).
2. If \( \Delta \in \text{Diff}_k (P, Q) \) then \( \Delta^* \in \text{Diff}_k (\hat{Q}, \hat{P}) \).
3. For all \( \Delta_1 \in \text{Diff} (P, Q) \) and \( \Delta_2 \in \text{Diff} (Q, R) \) we have

\[
(\Delta_2 \circ \Delta_1)^* = \{ \Delta_2, \Delta_1 \} \Delta_1^* \circ \Delta_2^*.
\]

**Proof.**

1. Obvious.
2. Denote by \( [\nabla] \) the cohomologous class of an operator \( \nabla \in \text{Diff}^+ (P, \Lambda^n) \),
\[
w (\nabla) = 0. \text{ Then } \Delta^* ([\nabla]) = \{ \Delta, \nabla \} [\nabla \circ \Delta] \text{ and we have }
\delta_a (\Delta^*) ([\nabla]) = \{ a, \Delta^* \} \Delta^* (a [\nabla]) - a \Delta^* ([\nabla]) =
\{ a, \Delta^* \} [a, \nabla] [\Delta, \nabla \circ a] [\nabla \circ a \circ \Delta] - \{ \Delta, \nabla \} [a, \nabla \circ \Delta] [\nabla \circ \Delta \circ a] =
(a \circ \Delta)^* ([\nabla]) - \{ a, \Delta \} (\Delta \circ a)^* ([\nabla]) = -\delta_a (\Delta)^* ([\nabla]),
\]
i.e., \( \delta_a (\Delta^*) = -\delta_a (\Delta)^* \). Thus \( \delta_{a_0, \ldots, a_k} (\Delta^*) = (-1)^{k+1} \delta_{a_0, \ldots, a_k} (\Delta)^* = 0 \).
3. \( (\Delta_2 \circ \Delta_1)^* ([\nabla]) = \{ \Delta_2 \circ \Delta_1, \nabla \} [\nabla \circ \Delta_2 \circ \Delta_1] =
\{ \Delta_2, \nabla \} [\Delta_2, \Delta_1] \Delta_1^* (\nabla \circ \Delta_2) = \{ \Delta_2, \Delta_1 \} \Delta_1^* (\Delta_2^* (\nabla)).\]

\[\square\]

Let us consider some examples of adjoint operators.
Example 1. Let $a: P \rightarrow P$ be the operator of multiplication by $a \in A$. Then $a^* ([\nabla]) = \{a, \nabla \} [\nabla \circ a] = a [\nabla]$, i.e., $a^* = a$.

Example 2. Let $p: A \rightarrow P$ be the operator $p(a) = pa$, $a \in A$, $p \in P$. Then $p^* ([\nabla]) = \{p, \nabla \} [\nabla \circ p]$, $p^* \in \text{Hom}_A \left( \hat{P}, \hat{A} \right)$. Thus we have a natural pairing $\langle \cdot , \cdot \rangle : P \otimes \hat{P} \rightarrow \hat{A}$, $\langle p, \hat{p} \rangle = p^* (\hat{p})$, $\hat{p} \in \hat{P}$.

Example 3. Berezinian and Integral forms. Let \ldots $\rightarrow P_{k-1} \xrightarrow{\Delta_k} P_k \rightarrow \ldots$ be a complex of d.o.’s. Since $\Delta^*_k \circ \Delta^*_{k+1} = \{\Delta_k, \Delta_{k+1} \} (\Delta_{k+1} \circ \Delta_k)^* = 0$, we get a complex \ldots $\leftarrow \hat{P}_{k-1} \xleftarrow{\hat{\Delta}^*_k} \hat{P}_k \leftarrow \ldots$, which is called dual to given one. The complex dual to the de Rham complex is said to be the complex of integral forms and is denoted by

$$0 \leftarrow \Sigma_0 \leftarrow \Sigma_1 \leftarrow \ldots,$$

where $\Sigma_i = \hat{\Lambda}^i$, $\delta = d^*$. The module $\Sigma_0 = \hat{A}$ is called the Berezinian (or the module of volume forms) and will be denoted by $B$.

The d.o.’s $D: \text{Diff}^+ (\Lambda^k) \rightarrow \Lambda^k$ induce the cohomology map: $\int: B \rightarrow H^n (\Lambda^*)$, so that to any element $\omega \in B$ (volume form) corresponds the $n$-th de Rham cohomology class $\int \omega$. This is an algebraic version of the integration operation. Clearly, $\int [\nabla] = \nabla (1) \mod d\Lambda^{n-1}$, $\nabla \in \text{Diff} (A, \Lambda^n)$.

Remark. The construction and the whole line of this subsection, as it was mentioned in the Introduction, are principally motivated by Penkov’s explanation of the Berezin forms on a supermanifold [32]. Close approaches to the construction of the Berezin forms on a supermanifold were suggested in [51] and [52].

2.2. Proposition. $\int \delta \omega = 0$, where $\omega \in \Sigma_1$.

Proof. Suppose $\omega = [\nabla]$, then $\delta \omega = [\nabla \circ d]$. Therefore $\int \delta \omega = |\nabla \circ d (1)| = 0$. □

2.3. Proposition (“Integration by parts”). For any d.o. $\Delta: P \rightarrow Q$ and $p \in P$, $\tilde{q} \in \tilde{Q}$, we have

$$\int \langle \Delta (p), \tilde{q} \rangle = \{\Delta, p\} \int \langle p, \Delta^* (\tilde{q}) \rangle$$
Proof. Suppose \( \hat{q} = [\nabla], \nabla : Q \to \Lambda^n \). Then
\[
\int \langle \Delta (p), \hat{q} \rangle = \int \{ \Delta (p), \nabla \} [\nabla \circ \Delta (p)] = \int \{ \Delta, \nabla \} \{ \Delta, p \} [\nabla \circ \Delta] = \{ \Delta, p \} \int \langle p, \Delta^* (\hat{q}) \rangle.
\]
\( \square \)

2.4 Coordinates. Let \( M \) be a smooth supermanifold of dimension \( s|t \), \( G = \mathbb{Z}_2 \), 
\( K = \mathbb{R} \) or \( \mathbb{C} \), \( A = C^\infty_K (M) \), \( P = \Gamma (\alpha) \) and \( Q = \Gamma (\beta) \) the modules of smooth 
sections of vector bundles over \( M \). Suppose 
\[ x = (y_i, \xi_j), \quad i = 1, \ldots, s, \quad j = 1, \ldots, t, \]
\[ x_1 = y_1, \ldots, x_s = y_s, \quad x_{s+1} = \xi_1, \ldots, x_{s+t} = \xi_t \]
is a coordinate system on a domain \( U \subset M \).

First of all let us compute the Berezinian, i.e., the cohomology of complex (1) 
for \( P = A \).

**Theorem ([32]).**

1. \( \hat{A}_n = 0 \) for \( n \neq s \).

2. \( \hat{A}_s \) is the module of sections for the bundle of volume forms \( \text{Ber} (M)^2 \).

Proof. The assertion is local, so we can consider the domain \( U \) and split the 
complex \( \text{Diff}^+ (\Lambda^*) \) in a tensor product of complexes \( \text{Diff}^+ (\Lambda^*)_{\text{even}} \otimes \text{Diff}^+ (\Lambda^*)_{\text{odd}} \), 
where \( \text{Diff}^+ (\Lambda^*)_{\text{even}} \) is complex (1) on the underlying even domain of \( U \) and 
\( \text{Diff}^+ (\Lambda^*)_{\text{odd}} \) is complex (1) for the Grassmann algebra in variables \( \xi_1, \ldots, \xi_t \).

It is known that \( H^i (\text{Diff}^+ (\Lambda^*)_{\text{even}}) = 0 \) for \( i \neq s \) and \( H^i (\text{Diff}^+ (\Lambda^*)_{\text{even}}) = \Lambda^s_u \), 
where \( \Lambda^s_u \) is the module of \( s \)-form on the underlying even domain of \( U \) (see [29, 
section 2]). To compute the cohomology of \( \text{Diff}^+ (\Lambda^*)_{\text{odd}} \) consider the quotient 
complexes
\[
0 \to \text{Smbl}_k (A)_{\text{odd}} \to \text{Smbl}_{k+1} (\Lambda^1)_{\text{odd}} \to \ldots,
\]
where \( \text{Smbl}_k (P)_{\text{odd}} = \text{Diff}^+_k (P)_{\text{odd}} / \text{Diff}^+_{k-1} (P)_{\text{odd}} \). An easy calculation shows 
that these complexes are the Koszul complexes, hence \( H^i (\text{Diff}^+ (\Lambda^*))_{\text{odd}} = 0 \) for 
\( i > 0 \) and \( H^0 (\text{Diff}^+ (\Lambda^*)) \) is a module of rank 1. Therefore \( \hat{A}_s = H^i (\text{Diff}^+ (\Lambda^*)) = \ldots \)

Recall that locally sections of \( \text{Ber} (M) \) are written in the form \( f (x) D (x) \), where \( f \in C^\infty (U) \) 
and \( D \) is a basis local section that is multiplied by the Berezin determinant of the Jacobi matrix 
under the change of coordinates. The Berezin determinant of an even matrix
\[
\begin{pmatrix}
A & B \\
C & D
\end{pmatrix}
\]
is equal to \( \det (A - BD^{-1}C) (\det D)^{-1} \).
0 for \( i \neq s \) and the only operators that represent non-trivial cocycles have the form \( f(y, \xi) \, dy_1 \wedge \ldots \wedge dy_s \frac{\partial t}{\partial \xi_1 \ldots \partial \xi_t} \).

To complete the proof it remains to check that \( \hat{A}_s \) is precisely \( \text{Ber}(M) \), i.e., that changing coordinates we obtain:

\[
f dy_1 \wedge \ldots \wedge dy_s \frac{\partial t}{\partial \xi_1 \ldots \xi_t} = f \text{Ber} J \left( \frac{x}{z} \right) dv_1 \wedge \ldots \wedge dv_s \frac{\partial t}{\partial \eta_1 \ldots \eta_t} + T,
\]

where \( z = (v_i, \eta_j) \) is a new coordinate system on \( U \), \( \text{Ber} \) denotes the Berezin determinant, \( J \left( \frac{x}{z} \right) \) is the Jacobi matrix, \( T \) is cohomologous to zero. This is an immediate consequence of the following:

Claim. If \( X = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \) and \( X^{-1} = \begin{pmatrix} \tilde{A} & \tilde{B} \\ \tilde{C} & \tilde{D} \end{pmatrix} \) are mutually inverse matrices written in the standard format, then \( \text{Ber} X = \det A \cdot \det \tilde{D} \).

Proof. Obviously, \( A\tilde{B} + B\tilde{D} = 0 \) and \( C\tilde{B} + D\tilde{D} = 1 \), whence \( D = \tilde{D}^{-1} + CA^{-1}B \). Therefore

\[
\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A & B \\ C & \tilde{D}^{-1} + CA^{-1}B \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ CA^{-1} & 1 \end{pmatrix} \begin{pmatrix} A & B \\ 0 & \tilde{D}^{-1} \end{pmatrix}
\]

and we get

\[
\text{Ber} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \text{Ber} \begin{pmatrix} 1 & 0 \\ CA^{-1} & 1 \end{pmatrix} \text{Ber} \begin{pmatrix} A & B \\ 0 & \tilde{D}^{-1} \end{pmatrix} = \det A \cdot \det \tilde{D}.
\]

\[\Box\]

Now let us consider the coordinate expression for adjoint operator. Suppose \( \Delta \in \text{Diff}_k(A, A) \) is a scalar operator: \( \Delta = \sum_\sigma a_\sigma \frac{\partial^{\mid \sigma \mid}}{\partial x_\sigma} \), where \( \sigma = (i_1, \ldots, i_r) \) is an ordered set of integers \( 1 \leq i_j \leq s + t \), \( \mid \sigma \mid = r \), \( \frac{\partial^{\mid \sigma \mid}}{\partial x_\sigma} = \frac{\partial^{\mid \sigma \mid}}{\partial x_{i_1} \ldots \partial x_{i_r}} \). Then we have

\[
\Delta^* = \sum_\sigma (-1)^\Theta \frac{\partial^{\mid \sigma \mid}}{\partial x_\sigma} \circ a_\sigma,
\]

where \( \Theta = \mid \sigma \mid + \tilde{a}_\sigma \sum_{j=1}^{\mid \sigma \mid} \tilde{x}_{i_j} + \sum_{1 \leq j < k \leq \mid \sigma \mid} \tilde{x}_{i_j} \tilde{x}_{i_k} \) and tilde over an object denotes the parity of the object. For a matrix d.o. \( \Delta = \| \Delta_{ij} \| \) one has \( (\Delta^*)_{ij} = (-1)^{ij} (\Delta_{ji})^* \).
3. Spencer Complexes and Green’s Formula

3.1. From now onwards we assume that the module $\Lambda^1$ is of finite type and projective. This implies that the same is true for the modules $\Lambda^k$ and $\mathcal{J}^k (A)$ and, also, that for any projective module the Spencer Diff-complex is exact (see [45]). In this case it is expedient to consider another variant of construction of adjoint operator.

Set $P^\circ = \text{Hom} (P, B)$. Obviously, $\Sigma_i = \Lambda_i = (\Lambda^i)\circ = D_i (B)$, $\mathcal{J}^k (A) = \mathcal{J}^k (A)^\circ = \text{Diff}_k (A, B)$.

**Definition.** For an operator $\Delta \in \text{Diff}_k (A, B)$ we define the operator $\Delta^\circ: A \to B$, $\Delta^\circ (a) = \{\Delta, a\} j_k^* (a\Delta)$, $a \in A$, which, for economy of language, will be called adjoint.

**Proposition.**
1. The grading of $\Delta^\circ$ is equal to the one of $\Delta$.
2. If $\Delta \in \text{Diff}_k (A, B)$ then $\Delta^\circ \in \text{Diff}_k (A, B)$.
3. $\omega^\circ = \omega$, $\omega \in B = \text{Diff}_0 (A, B)$.
4. If $X \in D_1 (B)$ then $X + X^\circ \in \text{Diff}_0 (A, B) = B$ and $X + X^\circ = \delta (X)$.
5. $(a\Delta)^\circ = \{a, \Delta\} \Delta^\circ \circ a$, $\Delta \in \text{Diff} (A, B)$, $a \in A$.
6. If $\Delta (a) = [\nabla_a]$, $\Delta \in \text{Diff} (A, B)$, $a \in A$, $\nabla_a \in \text{Diff}^+ (\Lambda^n)$, then $\Delta^\circ (a) = [\Box_a]$, where $\Box_a (a') = \{a, a'\} \nabla_{a'} (a)$.
7. $(\Delta^\circ)^\circ = \Delta$.

**Proof.**
1. Obvious.
2. $\delta\circ (\Delta^\circ) (a') = \{a, \Delta^\circ\} \Delta^\circ (a a') - a \Delta^\circ (a') =$
   $\{\Delta^\circ, a'\} j_k^* (a a' \Delta) - \{\Delta^\circ, a'\} a j_k^* (a a' \Delta) =$
   $\{\Delta^\circ, a'\} \delta a (j_k^* (a a' \Delta))$, so $\delta a, \ldots, a_k (\Delta^\circ) (a') =$
   $\{\Delta^\circ, a'\} \delta a, \ldots, a_k (j_k^* (a a' \Delta)) = 0$.
3. Obvious.
4. Clearly, $\delta\circ (j_1) = j_1 (a) - a j_1 (1) \in \mathcal{J}^1 (A)$, hence
   $$(\delta\circ (j_1))^\circ (\Delta) = \{a, \Delta\} \Delta (a) - a \Delta (1) = \delta a (\Delta) (1), \quad \Delta \in \text{Diff}_1 (A, A).$$
Thus $\delta_a (X + X^\circ) (1) = \delta_a (X) (1) + \delta_a (j_1^*) (X) = \delta_a (X) (1) - \delta_a (j_1)^* (X) = 0$. Further, $\delta (X) = j_1^* (\gamma^* (X))$, where $\gamma: J^1 (A) \to \Lambda^1$ is the natural projection. Obviously, $\gamma^*: D_1 (B) \to \text{Diff}_1 (A, B)$ is the natural inclusion, so that $\delta (X) = X^\circ (1) = X + X^\circ$.

(5) $a \Delta^\circ (a') = \{a \Delta, a'\}j_k^* (a'a \Delta) = \{a \Delta, a'\}\{a'a, \Delta\} \Delta^\circ (a'a) = \{a, \Delta\} (\Delta^\circ \circ a)(a')$.

(6) Obvious.

(7) Let $\Delta (a) = [\nabla_a]$ and $\Delta^\circ (a) = [\Box_a]$. Then $(\Delta^\circ)^\circ (a) = [\Box_a]$, where $\Box_a (a') = \{a, a'\}\Box_a (a') = \nabla_a (a')$, i.e., $(\Delta^\circ)^\circ (a) = [\nabla_a] = \Delta (a)$.

Now let us define the adjoint operator $\Delta^\circ$ in the general case when $\Delta \in \text{Diff} (P, Q^\circ)$, the $P$ and $Q$ being $A$-modules. As in non-graded case consider the family of operators $\Delta (p, q) \in \text{Diff} (A, B)$, $p \in P$, $q \in Q$, $\Delta (p, q) (a) = \{p, a\}\{q, a\} \langle \Delta (ap), q \rangle$, $a \in A$; the brackets $\langle \cdot, \cdot \rangle$ denote the natural pairing $Q \otimes Q^\circ \to B$. Then set $\langle \Delta^\circ (q), p \rangle = \{q, p\} \Delta (p, q)^\circ (1)$, $\Delta^\circ \in \text{Diff} (Q, P^\circ)$.

**Proposition.** (1) For any $\Delta \in \text{Diff}_k (P, Q^\circ)$ the operator $\Delta^\circ$ is well-defined and of order $\leq k$.

(2) For every $\Delta \in \text{Diff} (P, Q^\circ)$ we have $(\Delta^\circ)^\circ = \Delta$.

(3) If the modules $P$, $Q$, $Q^\circ$ are projective, then for $\Delta \in \text{Diff} (P, Q^\circ)$ the adjoint operator $\Delta^\circ: Q \to P^\circ$ coincides with the composition of $Q \to Q^{\circ\circ} \xrightarrow{\Delta^*} P^\circ$, where $Q \to Q^{\circ\circ}$ is the natural homomorphism.

**Proof.** Statements (1) and (2) are straightforward.

(3) Take $\Delta: P \to Q^\circ$, $p \in P$, $q \in Q$. We have $\Delta = \psi_\Delta j_k$, so $\langle \Delta^* (q), p \rangle = \langle j_k^* \psi_\Delta^* (q), p \rangle = j_k^* (\psi_\Delta (q) \circ p) = (\psi_\Delta^* (q) \circ p)^\circ (1) = \{q, p\} \Delta (p, q)^\circ (1)$.

Now let us consider one more example of adjoint operators.
3.2 Example: The Spencer Operators. The complex

\[ 0 \rightarrow P \xrightarrow{j} J^\infty (P) \xrightarrow{s} J^\infty (P) \otimes \Lambda^1 \xrightarrow{s} J^\infty (P) \otimes \Lambda^2 \xrightarrow{s} \ldots, \]

where \( s (j (p) \otimes \omega) = j (p) \otimes d\omega \), is called the Spencer \( J \)-complex (for details see [45]). Note that \( s \) is a d.o. of order \( \leq 1 \). Let us compute the dual complex. Clearly, for the projective module \( P \) we have \( \hat{J}^\infty (P) \otimes \Lambda^i = (J^\infty (P) \otimes \Lambda^i)^0 = D_i (\text{Hom} (J^\infty (P), B)) = D_i (\text{Diff} (P, B)) = D_i (\text{Diff}^+ (P^\circ)) \). To describe \( s^* \) we need the following remark.

Let \( \Delta (P) : F (P) \rightarrow G (P) \) be a natural d.o., \( F \) and \( G \) be functors on a category of projective modules over \( A \). For any functors \( F \) denote by \( \hat{F} (P) \) the abelian group \( F (P) \) supplied with \( A \)-module structure induced by the \( A \)-module structure in \( P \).

\[ \text{Then } \Delta \text{ generates the natural homomorphism } \hat{\Delta} : \hat{F} \rightarrow \hat{G}. \]

**Lemma.** \( (\Delta^*)^\circ = (\hat{\Delta})^* \)

**Proof** is trivial. \( \square \)

The lemma immediately implies that the complex dual to the Spencer \( J \)-complex for module \( P \) is the Spencer Diff-complex for module \( P^\circ \). On the other hand we have \( J^\infty (P) \otimes \Lambda^k = \text{Hom} (J^\infty (P), D_k (B)) = \text{Diff} (P, \Sigma_k) \) and the operator \( s^* : \text{Diff} (P, \Sigma_k) \rightarrow \text{Diff} (P, \Sigma_{k-1}) \) has the form \( s^* (\nabla) = \delta \circ \nabla, \nabla \in \text{Diff} (P, \Sigma_k); j^* (\nabla) = \nabla^\circ (1) \). Bringing all this together we can state the following:

**Theorem.** For a projective module \( P \) there is an isomorphism of complexes:

\[ 0 \leftarrow P^\circ \xleftarrow{D} \text{Diff}^+ (P^\circ) \xleftarrow{S} D_1 (\text{Diff}^+ (P^\circ)) \xleftarrow{S} D_2 (\text{Diff}^+ (P^\circ)) \leftarrow \ldots \]

\[ \begin{array}{ccc}
| & \psi & | \\
\downarrow & & \downarrow \psi \\
0 & \leftarrow P^\circ & \leftarrow \text{Diff} (P, B) & \leftarrow \text{Diff} (P, \Sigma_1) & \leftarrow \text{Diff} (P, \Sigma_2) & \leftarrow \ldots, \\
& & \omega & \omega & \omega & ,
\end{array} \]

where \( \omega (\nabla) = \delta \circ \nabla, \nabla \in \text{Diff} (P, \Sigma_k), \mu (\nabla) = \nabla^\circ (1), \) the maps \( \psi \) are the composition of isomorphisms:

\[ D_k (\text{Diff}^+ (P^\circ)) \xrightarrow{D_k (\circ)} D_k (\text{Diff} (P, B)) \rightarrow \text{Diff} (P, D_k (B)) \rightarrow \text{Diff} (P, \Sigma_k). \]

This theorem has the following:
Corollary (Green’s Formula). If the Spencer Diff-cohomology of the Berezinian $B$ in the term $\text{Diff}^+(B)$ is trivial, then for any $\Delta \in \text{Diff}(P,Q^\circ), p \in P, q \in Q$, we have

$$\langle \Delta (p) , q \rangle - \{ \Delta , p \} \langle p , \Delta^\circ (q) \rangle = \delta G,$$

where $G \in \Sigma_1$ is an integral 1-form.

Proof. Suppose $\nabla \in \text{Diff}^+(B)$; then $(\nabla - \nabla (1)) \in \ker D$, hence there exists $\square \in D_1(\text{Diff}^+(B))$ such that $S(\square) = \nabla - \nabla (1)$. It follows from the theorem that $\nabla^* (1) - \nabla (1) = \psi \circ S(\square) = \omega (\psi (\square))$. Therefore $\nabla^* (1) - \nabla (1) = \delta G$, where $G = \psi (\square) (1)$. Letting $\nabla = \Delta (p,q)$, we get the Green formula. □

Remark. If $B$ is projective then there exists an $A$-homomorphism $\varkappa: \ker D \to D_1(\text{Diff}^+(B))$, such that $\psi \circ \varkappa = id$, and we can take $\square = \varkappa (\Delta - \Delta (1))$ and, therefore, $G = G_{\varkappa} = \psi (\varkappa (\Delta (p,q) - \langle \Delta (p) , q \rangle)) (1)$. Since the Spencer complex of $B$ is exact, for any two homomorphisms $\varkappa$ and $\varkappa'$ one can find a homomorphism $f: \ker D \to D_2(\text{Diff}^+(B))$ such that $\varkappa - \varkappa' = S \circ f$. Hence $G_{\varkappa} - G_{\varkappa'} = \delta F$, where $F = \psi (f (\Delta (p,q) - \langle \Delta (p) , q \rangle)) (1)$.

3.3. In this subsection we describe a spectral sequence, which establishes the relationship between the de Rham cohomology and the homology of the complex of integral forms.

Proposition (Poincaré duality). There is a spectral sequence, $(E^r_{*,*}, d^r)$, with

$$E^2_{p,q} = H_p \left( (\Sigma_*)_{-q} \right),$$

the homology of complexes of integral forms, and converging to the de Rham cohomology $H (\Lambda^*)$.

Proof. Let $K_{p,q} = D_p(\text{Diff}^+(\Lambda^q))$, $d'$ be the Spencer operator $d': K_{p,q} \to K_{p-1,q}$, and $d'' = (-1)^p D_p(\text{Diff}^+(d'))$, $d''': K_{p,q} \to K_{p,q-1}$. Then $\{K_{*,*}, d', d''\}$ is a double complex. Obviously, $H^l (K) = H(K_{*,*}, d') = \Lambda^{-q}$ for $p = 0$ and 0 for $p \neq 0$. Therefore in the second spectral sequence $H^2_{p,q} = H^r_{p,q} (H (K)) = \ldots$
\( H_{p,q} \left( t H_{*,*} (K), d'' \right) = H^{-q} (\Lambda^*) \) for \( p = 0 \) and \( H_{p,q} \left( t H \right) = 0 \) for \( p \neq 0 \). Thus the second spectral sequence converge to the de Rham cohomology. From the first spectral sequence we get \( I E_{2}^{p,q} = H_{p,q} \left( (II H (K)) \right) = H_{p,q} \left( D_{*} (B), d' \right) = H_{p} \left( (\Sigma_{s})_{-q} \right) \).

**Corollary.** If \( \hat{A}_{i} = 0 \) for all \( i \neq n \), then \( H_{i} (\Sigma_{s}) = H^{n-i} (\Lambda^*) \).

### 4. Quadratic Lagrangians, the Euler Operator, and the Noether theorem

In this section \( P \) and \( Q \) are projective modules.

#### 4.1.
Consider the complex

\[
0 \leftarrow \text{Diff}^{\text{sym}} (P, P^{o}) \xleftarrow{\mu} \text{Diff}^{\text{sym}}_{(2)} (P, B) \xleftarrow{\omega} \text{Diff}^{\text{sym}}_{(2)} (P, \Sigma _{k}) \xleftarrow{\omega} \ldots ,
\]

where \( \text{Diff}^{\text{sym}}_{(2)} (P, Q) \) denotes the submodule of \( \text{Diff} (P, \text{Diff} (P, Q)) \) consisting of all symmetric bidifferential \( Q \)-valued operators \( \nabla (p_{1}) (p_{2}) = \{ p_{1}, p_{2} \} \nabla (p_{2}) (p_{1}) \), \( \nabla \in \text{Diff} (P, \text{Diff} (P, Q)) \). \( \text{Diff}^{\text{sym}} (P, P^{o}) \) is the module of all self-adjoint operators from \( \text{Diff} (P, P^{o}) \), \( \omega (\nabla) = \delta \circ \nabla \), \( \nabla \in \text{Diff}^{\text{sym}}_{(2)} (P, \Sigma _{k}) \), \( k > 0 \), and \( \mu (\nabla) (p) = (\nabla (p))^{\circ} (1), p \in P \).

**Theorem.** This complex is acyclic.

**Proof.** From theorem 3.2 it follows easily that the complex

\[
0 \leftarrow \text{Diff} (P, P^{o}) \xleftarrow{\mu} \text{Diff} (P, \text{Diff} (P, B)) \xleftarrow{\omega} \text{Diff} (P, \text{Diff} (P, \Sigma _{1})) \xleftarrow{\omega} \ldots ,
\]

where \( \omega (\nabla) = \delta \circ \nabla \), \( \mu (\nabla) (p) = (\nabla (p))^{\circ} (1) \), is acyclic. To prove the theorem it is sufficient to show that this complex split into the symmetric and the skew-symmetric parts. To do this let us check that the involution \( \rho \) of complex (3), \( \rho (\nabla) (p_{1}) (p_{2}) = \{ p_{1}, p_{2} \} \nabla (p_{2}) (p_{1}) \), \( \nabla \in \text{Diff} (P, \text{Diff} (P, \Sigma _{k})) \) and \( \rho (\nabla) = \nabla^{\circ}, \nabla \in \text{Diff} (P, P^{o}) \), is an authormorphism of this complex. The fact that \( \omega \circ \rho = \rho \circ \omega \) is obvious. Let us verify that \( \mu \circ \rho = \rho \circ \mu \). Take \( \Delta \in \text{Diff} (P, \text{Diff} (P, B)) \) and let \( \delta (p_{1}) (p_{2}) = [\nabla_{p_{1},p_{2}}] \), \( \nabla_{p_{1},p_{2}} \in \text{Diff}^{+} (\Lambda^{n}) \). It follows from proposition 3.1
that \( \langle \tilde{\mu} (\Delta) (p_1), p_2 \rangle = [\square] \), where \( \square \in \text{Diff}^+ (\Lambda^n) \), \( \square (a) = \{ p_2, a \} \nabla_{p_1, ap_2} (1) \), \( p_1, p_2 \in P \). Therefore \( \langle \rho \tilde{\mu} (\Delta) (p_1), p_2 \rangle = \langle \tilde{\mu} (\Delta) \circ (p_1), p_2 \rangle = \{ p_1, p_2 \} [\square'] \), where \( \square' (a) = \{ p_1, a \}{p_2, a} \nabla_{p_1} \), \( \rho (\Delta) (p_1), p_2 \rangle = [\nabla'_{p_1, p_2}] \). On the other hand \( \langle \tilde{\mu} \rho (\Delta) (p_1), p_2 \rangle = [\square''] \), where \( \square'' (a) = \{ p_2, a \} \nabla'_{p_1, ap_2} (1) \rangle \). Clearly, \( \nabla'_{p_1, p_2} = \{ p_1, p_2 \} \nabla_{p_1, p_2} \), hence \( \square'' (a) = \{ p_2, a \} \{ p_1, ap_2 \} \nabla_{ap_2, p_1} (1) \).

\[ 4.2 \] Lagrangian Formalism.

**Definition.** The space \( \text{Lag} (P) \) of quadratic Lagrangians on \( P \) is defined as the cokernel of the operator \( \omega: \text{Diff}^{\text{sym}} (2) (P, \Sigma_1) \rightarrow \text{Diff}^{\text{sym}} (2) (P, B) \). An operator \( L \in \text{Diff}^{\text{sym}} (2) (P, B) \) is called the density of quadratic Lagrangian \( L \) if \( L = L \mod \text{im} \omega \).

From theorem 4.1 we see that the operator \( \mu \) gives rise to an isomorphism of \( \text{Lag} (P) \) to the submodule of the module \( \text{Diff} (P, P^\circ) \) consisting of self-adjoint operators. This isomorphism is said to be the Euler operator and denoted by \( \mathcal{E} \).

**4.3 Conservation laws.** Let \( \Delta \in \text{Diff}^k (P, Q) \) and \( E = \{ p \in P | \Delta (p) = 0 \} \) is the corresponding equation. The operator \( \Delta \) generates the chain map \( \Omega_{\Delta} \) of the complexes (2):

\[
\begin{array}{ccccccc}
0 & \leftarrow & \text{Diff} (Q, B) & \leftarrow & \text{Diff} (Q, \Sigma_1) & \leftarrow & \text{Diff} (Q, \Sigma_2) & \leftarrow & \cdots \\
& & \downarrow \Omega_{\Delta} & & \downarrow \Omega_{\Delta} & & \downarrow \Omega_{\Delta} & \\
0 & \leftarrow & \text{Diff} (P, B) & \leftarrow & \text{Diff} (P, \Sigma_1) & \leftarrow & \text{Diff} (P, \Sigma_2) & \leftarrow & \cdots \\
\end{array}
\]

(Linear) conservation laws for the equation \( E \) are defined by analogy with non-graded case (see [29]) as classes of 1-dimensional homology of the complex \( \text{coker} \Omega_{\Delta} \). Let us denote the group of linear conservation laws for the equation \( \Delta = 0 \) by \( \text{Cl} (\Delta) = H_1 (\text{coker} \Omega_{\Delta}) \). The following theorem and the corollary describe the group \( \text{Cl} (\Delta) \).

**Theorem.** There exists an exact sequence

\[
0 \rightarrow H_1 (\text{im} \Omega_{\Delta}) \rightarrow H_0 (\text{ker} \Omega_{\Delta}) \rightarrow \ker \Delta^* \rightarrow \text{Cl} (\Delta) \rightarrow 0.
\]
Proof. It follows from theorem 4.1 that exact homology sequences corresponding to the short exact sequences of complexes

\[ 0 \to \ker \Omega_{\Delta} \to \text{Diff} (Q, \Sigma_{*}) \to \im \Omega_{\Delta} \to 0 \]

\[ 0 \to \im \Omega_{\Delta} \to \text{Diff} (p, \Sigma_{*}) \to \coker \Omega_{\Delta} \to 0 \]

have the form

\[ 0 \to H_1 (\im \Omega_{\Delta}) \to H_0 (\ker \Omega_{\Delta}) \xrightarrow{i_1} Q^o \xrightarrow{i} H_0 (\im \Omega_{\Delta}) \to 0 \]

\[ 0 \to H_1 (\coker \Omega_{\Delta}) \to H_0 (\im \Omega_{\Delta}) \xrightarrow{j_2} P^o \to H_0 (\coker \Omega_{\Delta}) \to 0. \]

It is straightforward to check that the composition \( j \circ i : Q^o \to P^o \) coincides with the adjoint operator \( \Delta^* : Q^o \to P^o \). Hence \( \ker \Delta^* / \im i_1 \) is isomorphic to \( \ker j = \text{Cl} (\Delta) \) and we get the desired exact sequence. \( \square \)

**Corollary.** If \( \ker \Omega_{\Delta} = 0 \) then the group of linear conservation laws \( \text{Cl} (\Delta) \) is isomorphic to \( \ker \Delta^* \).

Let us give an explicit expression for the map \( \ker \Delta^* \to \text{Cl} (\Delta) \). Suppose \( q^o \in \ker \Delta^* \subset Q^o \). Then, choosing a homomorphism \( \varkappa \) (see Remark 3.2), we have a d.o. from \( P \) to \( \Sigma_1 \), \( p \mapsto G_{\varkappa} (\Delta (p, q^o)) \). The Green formula yields \( \langle \Delta (p), q^o \rangle = \delta G_{\varkappa} (\Delta (p, q^o)) \). Hence the operator \( p \mapsto G_{\varkappa} (\Delta (p, q^o)) \) is an 1-cocycle of the complex \( \text{coker} \Omega_{\Delta} \) and we obtain a map \( \chi : \ker \Delta^* \to \text{Cl} (\Delta) \), where \( \chi (q^o) \) is the conservation law corresponding to the operator \( p \mapsto G_{\varkappa} (\Delta (p, q^o)) \). Let us show that this is the map under consideration. If \( \nabla \) is an 1-cocycle of \( \text{coker} \Omega_{\Delta} \), the element \( q^o \) from \( \ker \Delta^* \subset Q^o \) corresponding to it according to the proof of theorem 4.3 can be found as \( q^o = \mu (\square) \), where \( \square \in \text{Diff} (Q, B) \) satisfy the relation \( \square \circ \Delta = \delta \circ \nabla \). If \( \nabla \) is the operator \( p \mapsto G_{\varkappa} (\Delta (p, q^o)) \) then \( \square = q^o \in \text{Diff} (P, B) \) and \( \mu (\square) = \mu (q^o) = q^o \).

**4.4 The Noether Theorem.** We start with a description of transformations of the objects that Noether’s theorem includes.
Let \( \text{Der} (P) = \{(X_P, X) | X \in D_1 (A), X_P \in \text{Diff}_1 (P, P) \}, \) and \( \forall a \in A, \forall p \in P \)
\( X_P (ap) = \{X_P, a\} a X_P (p) + X (a) p \). If \( X \in D_1 (A) \) then \( (X_B, X) \in \text{Der} (B) \),
where \( X_B = -X^* \) (more generally, if \( (X_P, X) \in \text{Der} (P) \) then one can define \( (X_P^*, X) \in \text{Der} (P^*), \) \( X_P^* = -(X_P)^* \)). Given \( (X_P, X) \in \text{Der} (P) \), define
\( (X_{\text{Diff}}, X) \in \text{Der} (\text{Diff} (P, B)) \) by the formula
\( X_{\text{Diff}} (\Delta) = [X, \Delta] = X_B \circ \Delta - \{X, \Delta\} \Delta \circ X_P \). For \( L \in \text{Diff} (P, \text{Diff} (P, B)) \) we put
\( X_P (L) = X_{\text{Diff}} \circ L - \{X, L\} L \circ X_P \). If \( L \in \text{Diff}_{sym}^* (P, B) \) then \( X_P (L) \in \text{Diff}_{sym}^* (P, B) \) and is called the variation
of \( L \) under the infinitesimal transformation \( X_P \). Clearly, \( X_P \) generates the map
of Lagrangians on \( P \): \( X_P : \text{Lag} (P) \to \text{Lag} (P) \).

From the definition of variation it follows that
\[
X_P (L) (p_1) (p_2) = X_B (L (p_1) (p_2)) - \{X, L\} \{X, p_1\} L (p_1) (X_P (p_2)) - \{X, L\} L (X_P (p_1)) (p_2) .
\]
Further, using proposition 3.1(4) we get
\[
X_B (L (p_1) (p_2)) = -\{X, L\} \{X, p_1\} \{X, p_2\} \delta (L (p_1) (p_2) \circ X) .
\]
It follows from Green’s formula that \( \forall p_1, p_2 \in P \)
\[
L (p_1, p_2) = \langle \mathcal{E}_L (p_1), p_2 \rangle + \delta G_\gamma (L (p_1) (p_2, 1)) .
\]
Combining these formulae, we obtain the following formula for the first variation:
\[
X_P (L) (p_1) (p_2) = -\{L, p_2\} \{p_1, p_2\} \langle X_P (p_2), \mathcal{E}_L (p_1) \rangle - \{L, p_1\} \langle X_P (p_1), \mathcal{E} (p_2) \rangle - \delta n_\gamma (p_1, p_2) ,
\]
where \( n_\gamma (p_1, p_2) = \{X, L\} \{X, p_1\} \{X, p_2\} L (p_1) (p_2) \circ X + \{X, L\} \{X, p_1\} G_\gamma (p_1, X_P (p_2)) + \{X, L\} \{X, p_2\} \{p_1, p_2\} G_\gamma (p_2, X_P (p_1)) .
\]

**Definition.** \( X_P \in \text{Der} (P) \) is said to be a symmetry of a Lagrangian \( \mathcal{L} \) if
\( X_P (\mathcal{L}) = 0 \).

From theorem 4.1 it follows that a symmetry of \( \mathcal{L} \) is a symmetry of the operator
\( \mathcal{E}_\mathcal{L} = \mathcal{E} (\mathcal{L}) \), i.e.,
\[
X_P (\mathcal{E}_\mathcal{L}) = X_P^* \circ \mathcal{E}_\mathcal{L} - \{X, \mathcal{E}_\mathcal{L}\} \mathcal{E}_\mathcal{L} \circ X_P = 0 ,
\]
and conversely.
If $X_P$ is a symmetry of $\mathcal{L}$, then $X_P(L) = \omega(L')$, where $L$ is a density of $\mathcal{L}$, $L' \in \text{Diff}^{*\text{sym}}_{(2)}(P, \Sigma_1)$. Consider the integral 1-form $n_{\kappa}(p_1, p_2) + L'(p_1)(p_2)$. The first variation formula implies that this integral form is closed whenever $p_1, p_2 \in \ker \mathcal{E}_L$. Clearly, its homological class does not depend on the choice of $\kappa$ and $L'$. Thus we have proved the following:

**Theorem (Noether).** If $X_P$ is a symmetry of the Lagrangian $\mathcal{L} = L \mod \text{im} \omega$ and the module $P$ is projective, then the map $p \mapsto n_{\kappa}(p, p) + L'(p)(p), p \in P$, gives rise to a conservation law of the equation $\mathcal{E}_L = 0$.

**APPENDIX: RIGHT CONNECTIONS**

This appendix provides a lightening sketch of the algebraic formalism of right connections.

*Remark.* This issue is also treated in [53], the approach, although different, being close to the approach suggested here.

1. We begin with a collection of a few facts about (usual) linear connections.

One can consider a connection on an $A$-module $P$ as an $A$-homomorphism of grading zero $\gamma: P \mapsto \mathcal{J}^1(P)$ satisfying $\nu \circ \gamma = id_P$, where $\nu: \mathcal{J}^1(P) \rightarrow P$ is the natural projection.

The composition $\Lambda^i \otimes P \xrightarrow{id \otimes \gamma} \Lambda^i \otimes \mathcal{J}^1(P) \xrightarrow{s} \Lambda^{i+1} \otimes P$ of $\gamma$ and the Spencer operator $s$ is called the de Rham operator associated with $\gamma$ and denoted by $d: \Lambda^i \otimes P \rightarrow \Lambda^{i+1} \otimes P$. The first de Rham operator $d: P \rightarrow \Lambda^1 \otimes P$ gives rise in a natural way to a covariant derivative, i.e., a homomorphism $\nabla: D_1(A) \rightarrow \text{Diff}_1(P, P)$, $X \mapsto \nabla_X$, such that $\nabla_X(ap) = \{X, a\}a \nabla_X(p) + X(a)p, X \in D_1(A)$. The de Rham sequence with values in $P$

$$0 \rightarrow P \xrightarrow{d} \Lambda^1 \otimes P \xrightarrow{d} \Lambda^2 \otimes P \xrightarrow{d} \ldots$$

is not a complex in general. The operator $d^2: \Lambda^i \otimes P \rightarrow \Lambda^{i+2} \otimes P$ is said to be curvature of the connection $\gamma$. It is straightforward to check that $d^2$ is a $\Lambda^*$-linear
operator and, therefore, for $P$ projective the curvature $d^2$ is multiplication by a $R_\gamma \in \Lambda^2 \otimes \text{Hom}_A (P,P)$.

2. We define a right connection in a dual way.

**Definition.** A right connection on $P$ is defined to be an $A$-homomorphism of grading zero $\lambda: \text{Diff}_1^+ (P) \to P$ satisfying $\lambda \circ \iota = id_P$, where $\iota: P \to \text{Diff}_1^+ (P)$ is the natural inclusion.

Given a module $P$ with a right connection $\lambda$, one can carry out the construction of the sequence of integral form with values in $P$ by letting the operator $\delta: D_{i+1} (P) \to D_i (P)$ be the composition $D_{i+1} (P) \xrightarrow{S} D_i \left( \text{Diff}_1^+ (P) \right) \xrightarrow{D_i (\lambda)} D_i (P)$ of the Spencer operator $S$ and $D_i (\lambda)$:

$$0 \leftarrow P \xleftarrow{\delta} D_1 (P) \xleftarrow{\delta} D_2 (P) \xleftarrow{\delta} \ldots$$

The first operator $\delta: D_1 (P) \to P$ provides a right covariant derivative, i.e., a homomorphism $\nabla: D_1 (A) \to \text{Diff}_1^+ (P,P)$, $X \mapsto \nabla_X$, such that

$$\nabla_X (ap) = \{X, a\} a \nabla_X (p) - X (a) p, \quad X \in D_1 (A). \quad (1)$$

**Remark.** This “right Leibniz rule” may be reformulated by the following way. The operator $\nabla$ can be extended to an $A$-homomorphism $\nabla: \text{Diff}_1 (A,A) \to \text{Diff}_1^+ (P,P)$ by putting $\nabla_{id_A} = id_P$. Then (1) means that the map $\nabla: \text{Diff}_1^+ (A,A) \to \text{Diff}_1 (P,P)$ is also an $A$-homomorphism.

The operator $\delta^2: D_{i+2} (P) \to D_i (P)$ is called the curvature of the right connection $\lambda$. A direct calculation shows that this is a $D_\ast (A)$-linear operator and, so, for $P$ projective the curvature $\delta^2$ can be interpreted as inner product with a $R_\lambda \in \Lambda^2 \otimes \text{Hom}_A (P,P)$.

**Example.** If on a projective module $P$ there is a connection $\gamma: P \to \mathcal{J}^1 (P)$, then on $P^\circ$ there is a right connection $\lambda = \gamma^*: \left( \mathcal{J}^1 (P) \right)^\circ = \text{Diff}_1^+ (P^\circ) \to P^\circ$. In particular, the obvious flat connection on $A \gamma: A \to \mathcal{J}^1 (A)$, $\gamma (a) = aj_1 (1)$, provides the canonical flat right connection on the Berezinian $B$. The complex of integral forms with values in $P^\circ$ is dual to the de Rham complex with coefficients in $P$. 


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