VIRASORO CENTRAL CHARGES FOR NICHOLS ALGEBRAS

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ABSTRACT. Virasoro central charge associated with Nichols algebras is invariant under the Weyl groupoid action and takes very suggestive values for some items in Heckenberger’s list of rank-2 Nichols algebras. In particular, this might be taken as an indication of the existence of reasonable logarithmic extensions of $W_3 \equiv WA_2, WB_2,$ and $WG_2$ models of conformal field theory. In the $W_3$ case, the construction of an octuplet extended algebra is outlined.

0. INTRODUCTION

In [1], we described a paradigm treating screening operators in conformal field theory as a braided Hopf algebra, a Nichols algebra $[2, 3, 4, 5, 6, 7, 8, 9]$. This immediately suggests that the inverse relation may also exist. Is any finite-dimensional Nichols algebra (with diagonal braiding at least) an algebra of screenings in some conformal model? This is a fascinating problem, especially considering the recent remarkable development in the theory of Nichols algebras—originally a “technicality” in Andruskiewitsch and Schneider’s program of classification of pointed Hopf algebras, which has grown into a beautiful theory in and of itself (in addition to the papers cited above and the references therein, also see [10, 11, 12, 8, 13, 14, 15, 16, 17]). I assume diagonal braiding from now on.

As many “inverse” problems, that of identifying conformal field theories “underlying” a given Nichols algebra is not necessarily well defined. I restrict myself to Nichols algebras of rank two (already a fairly large number in terms of the possible conformal models). All of these were listed by Heckenberger [18] (the general classification was achieved in [7], and was reproduced in a different and independent way in [15, 16, 17]). These notes are in fact a compilation of the original Heckenberger’s list with explicit results on the presentation of some Nichols algebras (obtained in [15] for the standard type and in [19] in several nonstandard cases), and with several conformal field theory constructions added. As regards these last, it is of course well known that screenings can be used to define “consistent” models of conformal field theory. My emphasis here is on placing the Nichols-algebra and CFT pieces of knowledge into a common context. Passing from Nichols algebras to CFT is ideologically simple but involves various ambiguities, as I now briefly describe.
For any Nichols algebra $\mathfrak{B}(X)$ in Heckenberger’s list, the braiding matrix of basis elements $F_i$ of the two-dimensional braided linear space $(X, \Psi)$ is some

\[
\begin{pmatrix}
  q_{11} & q_{12} \\
  q_{21} & q_{22}
\end{pmatrix}, \quad \Psi(F_i \otimes F_j) = q_{i,j} F_j \otimes F_i.
\]

Relation to conformal field theory is based on representing the generators as screening operators

\[
F_1 \equiv F_\alpha = \oint e^{\alpha \cdot \varphi} \quad \text{and} \quad F_2 \equiv F_\beta = \oint e^{\beta \cdot \varphi}
\]

acting in a space of two bosonic fields (or simply “bosons”). Here, $\varphi(z) = (\varphi^1(z), \varphi^2(z))$ is a two-component boson field with OPEs (16) (Appendix V), the dot denotes Euclidean scalar product, and $\alpha$ and $\beta$ are two-dimensional vectors such that the screenings have the self-braidings and the monodromy coincident with those in (1):

\[
e^{i\pi \alpha \cdot \alpha} = q_{1,1}, \quad e^{2i\pi \alpha \cdot \beta} = q_{1,2} q_{2,1}, \quad e^{i\pi \beta \cdot \beta} = q_{2,2}.
\]

The ambiguities inherent in passing from a braiding matrix to screenings realized in terms of free bosons are numerous. Already the “two-boson space” on which $F_\alpha$ and $F_\beta$ act can be chosen differently, e.g., by allowing or not allowing certain types of exponentials of the bosons, yielding different results. Furthermore, solving relations (3) for $\alpha, \beta \in \mathbb{C}^2$ involves taking logarithms, which introduces up to three arbitrary integer parameters.

And yet the idea to look for conformal models corresponding to given Nichols algebras is not altogether meaningless because the Virasoro central charge is invariant under the Weyl groupoid action. I go into some detail here because the statement implicitly refers to a procedure to deal with the ambiguities such that the invariance be nevertheless ensured.

For noncollinear $\alpha$ and $\beta$, two screenings (2) uniquely define a Virasoro algebra in their centralizer in the space of differential polynomials in the $\partial \varphi^j(z)$ ($\partial = \partial/\partial z$). This Virasoro algebra is characterized by its central charge

\[
c = 2 - 3 \frac{(4 + (\alpha \cdot \alpha)(\beta \cdot \beta))(\alpha - \beta) \cdot (\alpha - \beta) + 4(\alpha - \beta) \cdot ((\alpha \cdot \alpha)\beta - (\beta \cdot \alpha)\alpha)}{(\alpha \cdot \alpha)(\beta \cdot \beta) - (\alpha \cdot \beta)^2}
\]

On the Nichols algebra side, the Weyl groupoid action is defined for a finite-dimensional rank-$\theta$ Nichols algebra with diagonal braiding as follows [10, 20, 21, 22]. There exists a generalized Cartan matrix $(a_{i,j})_{1 \leq i,j \leq \theta}$ such that $a_{i,i} = 2$ and

\[
q_{i,j}^{a_{i,j}} = q_{i,j} q_{j,i} \quad \text{or} \quad q_{i,i}^{1-a_{i,j}} = 1
\]

\[\text{Notably, the conditions on the } q_{i,j} \text{ selecting the different rank-2 Nichols algebras involve only the self-braidings } q_{1,1} \text{ and } q_{2,2} \text{ and the monodromy } q_{1,2} q_{2,1}. \]
holds for each pair \( i \neq j \) (the \( q_{i,j} \) are, of course, the entries of a braiding matrix, a \( \theta \times \theta \) counterpart of \( (1) \)). A Weyl reflection on the set of braiding matrices is defined for any \( k \), \( 1 \leq k \leq \theta \). The reflected braiding matrix has the entries

\[
\mathcal{R}^{(k)}(q_{i,j}) = q_{i,j} - \frac{a_{k,j} - a_{k,i}a_{k,j}}{q_{k,j}q_{k,k}}
\]

(it may or may not have the same generalized Cartan matrix).\(^2\) The use of this tool has remarkably resulted in the classification of Nichols algebras with diagonal braiding \(^7\).

Continuing with the rank-\( \theta \) case, I define the screening momenta \( \alpha_i \in \mathbb{C}^\theta \), \( 1 \leq i \leq \theta \), by imposing the relations

\[
e^{i\pi \alpha_i \alpha_i} = q_{i,i},
\]

\[
e^{2i\pi \alpha_i \alpha_j} = q_{i,j}q_{j,i}, \quad i \neq j.
\]

Conditions \( (5) \) are then “lifted” to the scalar products as the condition

\[
2\alpha_i \alpha_j = a_{i,j} \alpha_i \alpha_i \quad \text{or} \quad (1 - a_{i,j}) \alpha_i \alpha_i = 2
\]

to be satisfied for each pair \( i \neq j \). Several particular choices have been made in writing this, for example, the 2 in the second relation could in principle be replaced with other even integers (nonzero, to try to keep things interesting). The Weyl reflections are now lifted to the scalar products similarly, by “naively taking the logarithm” of \( (6) \):

\[
\mathcal{R}^{(k)}(\alpha_i, \alpha_j) = \alpha_i - \alpha_{k,j} \alpha_i \alpha_k - a_{k,i} \alpha_k \alpha_j + a_{k,i} a_{k,j} \alpha_k \alpha_k.
\]

Returning to the rank-2 case, with \( \alpha = \alpha_1 \) and \( \beta = \alpha_2 \), Weyl-reflecting central charge \( (4) \) amounts to replacing each \( \alpha_i \alpha_j \) with \( \mathcal{R}^{(k)}(\alpha_i, \alpha_j) \). It then follows that

\[
\mathcal{R}^{(1)}(c) - c = \frac{3}{(\alpha_1, \alpha_1)(\alpha_2, \alpha_2) - (\alpha_1, \alpha_2)^2} \left(2\alpha_1 \alpha_2 - a_{1,2} \alpha_1 \alpha_1\right) \left((a_{1,2} - 1) \alpha_1 \alpha_1 + 2\right)
\]

\[
= \left(a_{1,2}^2 \alpha_1 \alpha_1 - a_{1,2} \alpha_1 \alpha_1 - 2a_{1,2} \alpha_1 \alpha_1 + 2\alpha_2 \alpha_2 + 2a_{1,2} - 4\right).
\]

The product \( (2\alpha_1 \alpha_2 - a_{1,2} \alpha_1 \alpha_1) \left((a_{1,2} - 1) \alpha_1 \alpha_1 + 2\right) \) vanishes whenever \( (7) \) holds for \( i = 1 \) and \( j = 2 \), and hence \( c \) is indeed invariant under \( \mathcal{R}^{(1)} \). The invariance under \( \mathcal{R}^{(2)} \) is shown similarly.

For any rank \( \theta \), the central charge defined by \( \theta \) screenings in a \( \theta \)-boson space is also invariant under \( (8) \) if conditions \( (7) \) are imposed. Showing this requires some more work and is relegated to Appendix [V].

\(^2\)If the diagonal braiding is of Cartan type, then Weyl reflections preserve the Cartan matrix. If a generalized Cartan matrix (not of Cartan type) is the same for the entire class of Weyl-reflecting braided matrices, then such a generalized Cartan matrix and the braiding matrix are said to belong to the standard type. Nonstandard braidings do exist \([23, 15]\).
0.5. From Virasoro to extended algebras. The central charge value alone does not specify a conformal field theory uniquely. In “good” cases, however—when the central charge found from (4) is a function of a (discrete) parameter—the form of this dependence does suggest what type of operators extend the Virasoro algebra and therefore what the resulting conformal model is; and the centralizer of the screenings then turns out to be sufficiently ample for an interesting conformal field theory to live in it. An exemplary case is the $W_3$ algebra, which centralizes two screenings associated with a braiding matrix such that $q_{1,1} = q_{2,2} = q$ and $q_{1,2}q_{2,1} = q^{-1}$, with a primitive root of unity $q$. This fact is of course well known in the nonlogarithmic context. From the logarithmic perspective, this $W_3$ algebra is a nonextended algebra, playing the same role in relation to the extended algebra as the Virasoro algebra plays in relation to the triplet algebras of $(p, 1)$ [24, 25] and $(p, p')$ [26] logarithmic models. Specifically in the $W_3$ case, that biggest algebra is the octuplet algebra in Appendix [W]. Similar constructions are expected in other good cases; I am optimistic about the fact that the same generalized Dynkin diagram gives rise to a finite-dimensional Nichols algebra and to an interesting conformal field theory. The intricate machinery underlying the finite dimensionality of the corresponding Nichols algebra may manifest itself in constructing new logarithmic models.\footnote{Recall that rational conformal field theories are generally defined as the cohomology of a complex associated with the screenings, whereas logarithmic models are defined by the kernel (cf. [27, 28, 26, 29]). In particular, this allows interesting logarithmic conformal models to exist in the cases where the rational model is nonexistent (the $(p, 1)$ series) or trivial (the $(2, 3)$ model).}

In what follows, I therefore reproduce Heckenberger’s list of rank-2 finite-dimensional Nichols algebras [18], with the only difference that I enumerate, not itemize the subitems. For several items, I also add the presentations known from [15] and explicitly borrowed from [19], starting with the indication of the case number in that paper. From the Nichols-algebra data, I move toward conformal field theory by analyzing the conditions on the screening momenta. When it is clear what current algebra extends the Virasoro algebra with the central charge obtained from (4), I recall the explicit construction, presenting it in the form that manifestly refers to the corresponding pair of screenings (once again, all extended algebras before Appendix [W] are not logarithmic extensions, but rather starting points for such extensions).

0.9. Points to note.

0.91. In conformal field theory, fermionic screenings are often interesting. Their Nichols-algebra counterparts are the diagonal entries $-1$ in braiding matrices. But given a $q_{i,i} = -1$ and trying to reconstruct a screening in general leads to $\alpha_i \alpha_i = 1 + 2m$ for the screening momentum, with $m \in \mathbb{Z}$. For the corresponding screening current $f(z) = e^{\alpha_i \phi(z)}$, it then follows that $f(z)f(w)$ develops a $(1 + 2m)$th-order zero as $z \to w$. The cases where this zero is actually a pole are somewhat pathological
from the CFT standpoint; as regards the cases of an actual higher-order zero, I am unaware of any such examples of screenings. Only $m = 0$ is a “good” value. Remarkably, solving conditions (7) with $q_{i,j} = -1$ has the tendency to select the value $m = 0$, thus ensuring a true fermionic screening.

0.92. Other integers appearing in “taking the logarithms” are not disposed of that easily. There are solutions of (7) where these integers vanish (and the central charge depends on another integer parameter, the order of a root of unity); such solutions are referred to as “regular” in what follows. But there also exist “peculiar” solutions of (7) where some of these parasitic integers persist, and which have somewhat reduced chances to correspond to interesting CFT models. In fact, some peculiar solutions are eliminated already by the conditions in Heckenberger’s list: in some items, the order of the corresponding root of unity must not be too small, and the peculiar solutions do require just one of those excluded values. This might suggest that peculiar solutions should somehow be eliminated altogether, but if so, then I have overlooked the argument.

0.93. Things get worse with the many items in the list that do not involve a free discrete parameter such as the order of a root of unity. Isolated central charge values are by no means illuminating, and remain entirely unsuggestive when expanded into families by the “parasitic integers.”

The unwieldiness of the “peculiar” central charges also thwarted my original intention to provide each item in the list with a central charge. This can be done, but the results are not indicative of anything. The corresponding items in the list are therefore left in their original form given in [18].

0.94. In the “regular” cases, I choose a primitive $p$th root of unity as $e^{\frac{2\pi i}{p}}$ (or $e^{-\frac{2\pi i}{p}}$). This might unnecessarily restrict the generality, but the cases that follow with this choice are already interesting. In “peculiar” cases, by contrast, I try to work out the cases with $e^{\frac{2\pi r i}{p}}$, where $r$ is coprime with $p$. The $r$ parameter sometimes survives till the central charge, but that’s where the story ends, because I do not construct any current algebra generators beyond Virasoro in peculiar cases.

Mostly, I take the logarithm of relations such as $e^{i\pi x} = e^{\frac{2\pi r}{p}}$ (where $x$ is typically a linear combination of scalar products) “honestly,” as $x = \frac{2r}{c} + 2\ell$, $\ell \in \mathbb{Z}$. In some cases, however, the ensuing dependence on $\ell$ turns out to be “inessential” (something like a shift of the level of the affine Lie algebra with which the corresponding conformal field theory is associated—which interestingly corresponds to a twist equivalence of the braiding matrix), and I sometimes omit it.

\footnote{It might be that those cases are “accidental degenerations” of some higher-rank cases; in the higher rank, where more conditions must be satisfied, the unwanted integers might be fixed uniquely, possibly simultaneously with some good parameter dependence introduced instead.}
0.95. Strictly speaking, identifying a CFT model from its central charge that depends
on a parameter is an ill-defined procedure in the sense that given a central charge
c = f(k) and redefining the parameterization by an arbitrary function, k' = g(k),
changes the “functional form” of c arbitrarily. It is tacitly understood that some
“natural” parameterizations are considered and very limited reparameterizations
are allowed (typically those that are known to occur in some CFT constructions).

0.99. Notation. The notation $R_l$ for the set of primitive $\ell$th roots of unity is copied from
[18] as part of the defining conditions in the list items. A braiding matrix (1) is encoded
in a generalized Dynkin diagram $\begin{array}{cccc} q_{1,1} & m_{1,2} & q_{2,2} \end{array}$, where $m_{1,2} = q_{1,2}q_{2,1}$. The $A_2$, $B_2$, and $G_2$
Cartan matrices are $\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$, $\begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix}$, and $\begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}$.

1. The list, item 1

The defining conditions are

$$q_{12}q_{21} = 1 \quad \text{and} \quad q_{11}, q_{22} \in \bigcup_{a = 2}^{\infty} R_a.$$  

This is the “trivial” $A_1 \times A_1$ case. The corresponding CFT model is the product $(p', 1) \times
(1, p)$) of two “(p, 1)” models [28, 30], or, in the degenerate case where $\alpha$ and $\beta$ are
collinear (and hence only one boson is needed), the $(p', p)$ model [26, 31].

2. The list, items 2.

The defining conditions are

$$q_{12}q_{21}q_{22} = 1 \quad \text{and} \quad q_{12}q_{21} \neq 1,$$

plus any of conditions 2.1–2.7. In terms of the momenta $\alpha, \beta \in \mathbb{C}^2$ of the screenings, the
common condition for all these cases takes the form

$$2\alpha.\beta + \beta.\beta = 2m \ (m \in \mathbb{Z}).$$

2.1 (5.7(1)[19]). $q_{11}q_{12}q_{21} = 1$, $q_{12}q_{21} \in \bigcup_{a = 2}^{\infty} R_a$, Cartan type $A_2$, $\begin{array}{ccc} q & q^{-1} & q \end{array}$.

In terms of scalar products, the conditions are

$$\alpha.\alpha + 2\alpha.\beta = 2n \ (n \in \mathbb{Z}), \quad 2\alpha.\beta = -\frac{2}{p} + 2j, \quad |p| \geqslant 2 \ (j \in \mathbb{Z})$$

The braiding matrix (which is stable under Weyl reflections) is then parameterized as

$$\begin{pmatrix} e^{2\pi i p} & (1)^i e^{-\frac{2\pi i}{p}} \\ (1)^i e^{-\frac{2\pi i}{p}} & e^{2\pi i p} \end{pmatrix}.$$  

None of the two screenings is fermionic unless $|p| = 2$.

Conditions (7) are not satisfied for all $m$, $n$, $j$, and $p$. They have several “peculiar”
solutions and a “regular” solution. The peculiar solutions are $(m = 0, n = k - \frac{3}{2}, p = 2,$
where $k = -k - \frac{3}{2}$, $(m = 0, n = -k - \frac{3}{2}, p = -2, j = -k - \frac{5}{2})$, $(m = -k - \frac{3}{2}, n = 0, p = 2, j = -k - \frac{5}{2})$, $(m = -k - \frac{3}{2}, n = 0, p = 2, j = -k - \frac{5}{2})$, $(m = k + \frac{3}{2}, n = k + \frac{3}{2}, p = 2, j = k + \frac{3}{2})$, and $(n = m = k + \frac{3}{2}, k + \frac{3}{2}, p = -2, j = k + \frac{1}{2})$ with half-integer $k$ in all cases; with this parameterization, the resulting central charge is in each case equal to $\frac{3k}{k+2} - 1$, which is the central charge of the $\hat{sl}(2)_k/\hbar$ coset (more on it is to be said below, when it occurs as a “regular” solution).

The regular solution is $m = n = 0$, yielding the central charge

$$c = 50 - \frac{24}{k+3} - 24(k+3),$$

where $k + 3 = \frac{1}{p} - j$ (or, in view of the structure of the formula, $\frac{1}{k+3} = \frac{1}{p} - j$). This is the central charge of the $W_3$ algebra parameterized in terms of the level $k$ of the $\hat{sl}(3)$ affine Lie algebra from which $W_3$ can be obtained by Hamiltonian reduction.

The centralizer of the screenings does indeed contain a dimension-3 primary field $W(z)$ (unique up to an overall factor) in the space of differential polynomials in the fields

$$\partial \varphi_\alpha(z) = \alpha \partial \varphi(z) \quad \text{and} \quad \partial \varphi_\beta(z) = \beta \partial \varphi(z).$$

Explicitly, setting $j = 0$ for simplicity and omitting the $(z)$ arguments in the right-hand side for brevity,

$$W(z) = \partial \varphi_\alpha \partial \varphi_\alpha \partial \varphi_\alpha + \frac{3}{2} \partial \varphi_\alpha \partial \varphi_\alpha \partial \varphi_\beta - \frac{3}{2} \partial \varphi_\alpha \partial \varphi_\beta \partial \varphi_\beta - \partial \varphi_\beta \partial \varphi_\beta \partial \varphi_\beta - \frac{9(p-1)}{2p} \partial^2 \varphi_\alpha \partial \varphi_\alpha - \frac{9(p-1)}{4p} \partial^2 \varphi_\alpha \partial \varphi_\beta + \frac{9(p-1)}{4p} \partial^2 \varphi_\beta \partial \varphi_\alpha + \frac{9(p-1)}{2p} \partial^3 \varphi_\alpha \partial \varphi_\beta + \frac{9(p-1)^2}{4p^2} \partial^3 \varphi_\alpha - \frac{9(p-1)^2}{4p^2} \partial^3 \varphi_\beta.$$

The Nichols algebra $\mathcal{B}(X)$ (of the two-dimensional braided vector space $X$ with basis $F_1$ and $F_2$ with braiding matrix (1)) is in this case the quotient $[19]

$$\mathcal{B}(X) = T(X)/\left([F_1, F_1, F_2], [F_1, F_2, F_2], F_1^p, [F_1, F_2]^p, F_2^p\right)$$

if $p \geq 3$. Here and hereafter, square brackets denote an iterated $q$-commutator. If $p = 2$ (the screenings are fermionic!), the triple-bracket generators of the ideal are absent. The dimension is $\dim \mathcal{B}(X) = p^3$. The elements $F_1^p$ and $F_2^p$ in the ideal indicate the “positions” of the long screenings

$$\mathcal{E}_\alpha = \int e^{-\alpha \cdot \varphi} = \int e^{-\alpha \cdot \varphi} \quad \text{and} \quad \mathcal{E}_\beta = \int e^{-\beta \cdot \varphi} = \int e^{-\beta \cdot \varphi}, \quad (\alpha \cdot \beta = \frac{2\alpha}{\alpha \cdot \alpha}),$$

i.e., $F_1^p$ and $F_2^p$ “tend to be” the operators “opposite” to the respective long screening.

Generally, these long screenings are to produce $m$-plet structures in logarithmic models, similarly to how the triplet structure of the $(p, 1)$ logarithmic models [24, 25] is generated by the corresponding long screening [27]. For the current $W_3$-case, some details are given in Appendix [W].
2.2 (5.7(3)\cite{19}). \( q_{11} = -1, \ q_{12}q_{21} \in \bigcup_{\alpha=3}^{\infty} R_{\alpha} \). Cartan type \( A_2, -\text{co}_{1}q^{-1}q^{1} \).

The conditions are restated in terms of the scalar products as

\[ \alpha.\alpha = 1 + 2n \ (n \in \mathbb{Z}), \quad 2\alpha.\beta = -\frac{2}{p} + 2j, \ |p| \geq 3 \ (j \in \mathbb{Z}). \]

The braiding matrix is then parameterized as

\[\begin{pmatrix}
-1 \\
(-1)i\frac{e^{i\pi}}{p} \\
(-1)i\frac{e^{i\pi}}{p} \\
-1
\end{pmatrix}, \]

and both of its Weyl reflections are

\[\begin{pmatrix}
-1 \\
(-1)i\frac{e^{i\pi}}{p} \\
-1
\end{pmatrix}.

The first screening “wants to be fermionic.” Remarkably, conditions (7) hold only if \( m = n = 0 \) (would-be solutions with nonzero \( m \) or \( n \) require \( |p| = 2 \)). In particular, the \(-1\) in the braiding matrix does indeed correspond to a fermionic screening in the standard sense—an operator of the form \( \frac{1}{2}F(z) \), where \( F(z)F(w) \) has a first-order, not a higher-order, zero as \( z \to w \).

The solution for the scalar products with \( m = n = 0 \) yields the central charge

\[ c = \frac{3k}{k+2} - 1, \]

where \( k+2 = \frac{1}{p} - j \). This is the central charge of the \( \hat{\mathfrak{sl}}(2)_{k}/\mathfrak{h} \) coset (where \( \mathfrak{h} \) is the Heisenberg subalgebra). The two currents \( j^+(z) \) and \( j^-(z) \) that are in the centralizer of the screenings and generate the coset algebra can be expressed in terms of the two bosons “in the direction” of each screening as

\[ j^+(z) = e^{-\frac{1}{2}(2\varphi_u(z)+\varphi_\beta(z))}, \]

\[ j^-(z) = -(\partial\varphi_\alpha(z)\partial\varphi_\beta(z) + \partial\varphi_\alpha(z)\partial\varphi_\alpha(z) + (k+1)\partial^2\varphi_\alpha(z))e^{\frac{1}{2}(2\varphi_u(z)+\varphi_\beta(z))}. \]

The exponentials are assumed to be normal ordered, and the second line involves the (standard) abuse of notation: nested normal ordering from right to left is in fact understood after the expression is expanded. Adding a boson \( \chi(z) \) associated with the \( \mathfrak{h} \) algebra immediately yields the three \( \hat{\mathfrak{sl}}(2)_k \) currents

\[ j^\pm(z) = j^\pm(z)\sqrt{\frac{k}{2}}\chi(z), \quad j^0(z) = \sqrt{\frac{k}{2}}\partial\chi(z). \]

For \( p \geq 3 \), the Nichols algebra is the quotient \[19\]

\[ \mathfrak{B}(X) = T(X)/([F_1,F_2,F_2], F_1^2, F_2^p), \]

with \( \dim \mathfrak{B}(X) = 4p. \)

A long screening here is

\[ \mathcal{E}_\beta = \oint e^{-p\beta}\varphi. \]
2.3 (5.11(3))\footnote{[19]}. \( q_{11} \in R_3 \), \( q_{12} q_{21} \in \bigcup_{a=2}^{\infty} R_a \), \( q_{11} q_{12} q_{21} \neq 1 \), Cartan type \( B_2 \), \( \frac{\zeta}{\delta} \frac{q^{-1} q}{\delta} \), \( R_3 / \frac{\zeta}{\delta} \neq q \).

In terms of the screening momenta, the conditions become
\[
\alpha \cdot \alpha = \frac{2s}{3}, \quad 2\alpha \cdot \beta = -\frac{2}{p} + 2j, \quad |p| \geq 2 \quad (j \in \mathbb{Z}),
\]
where \( s \) is coprime with 3. The braiding matrix is parameterized as
\[
\begin{pmatrix}
e^{\frac{2\pi i}{p}} & (-1)^{j}e^{-\frac{2\pi i}{p}} \\
(-1)^{j}e^{-\frac{2\pi i}{p}} & e^{\frac{2\pi i}{p}}
\end{pmatrix}
\]
and its Weyl reflections are
\[
\begin{pmatrix}
e^{\frac{2\pi i}{p}} & (-1)^{j}e^{-\frac{\pi i}{p}} \\
(-1)^{j}e^{-\frac{\pi i}{p}} & e^{\frac{2\pi i}{p}}
\end{pmatrix}
\text{and}
\begin{pmatrix}
e^{\frac{2\pi i}{p}} & (-1)^{j}e^{-\frac{2\pi i}{p}} \\
(-1)^{j}e^{-\frac{2\pi i}{p}} & e^{\frac{2\pi i}{p}}
\end{pmatrix}.
\]
Conditions (7) can be satisfied only if \( (m = 0, p = 3, s = 3\ell - 1, j = 1 - 2\ell) \) or \( (m = 0, p = -3, s = 3\ell + 1, j = -1 - 2\ell) \) (two peculiar solutions), or \( (m = 0, s = 1) \) (the regular solution).

In both peculiar cases, the central charge is \( 86 - 60(k + 3) - \frac{30}{k + 3} \), where \( k + 3 = -\frac{1}{3} + \ell \) (or \( \frac{k}{1 + \ell} = -\frac{2}{3} + 2\ell \)) in the first case and \( k + 3 = \frac{1}{3} + \ell \) (or \( \frac{k}{1 + \ell} = \frac{2}{3} + 2\ell \)) in the second case. The central charge is that of the \( WB_2 \) algebra, discussed in more detail below when it appears as a “regular” solution.

In the regular case \( m = 0 \), the central charge is
\[
c = -\frac{21}{2} - 6z - \frac{27}{2(4z - 3)},
\]
where \( \frac{1}{z} = \frac{1}{p} - j \) or \( \frac{1}{z} = -\frac{1}{p} + \frac{4}{3} + j \).

The condition \( q_{11} q_{12} q_{21} \neq 1 \) excludes the value \( p = 3 \).

If \( p \geq 4 \), and \( p' = \text{ord}(q_{11} q_{21}^{-1}) = \text{ord}(e^{\frac{2\pi i}{p} - \frac{2\pi i}{p'}}) \), then \([19]\)
\[
\mathcal{B}(X) = T(X) / ([F_1, F_2, F_2], F_1^3, [F_1, F_1, F_2]p', F_2^p),
\]
with \( \dim \mathcal{B}(X) = 9pp' \).

2.4. \( q_{11} \in \bigcup_{a=4}^{\infty} R_a \), with two subcases listed below. To identify the central charges below, we use the formula [32]
\[
c(k) = \ell - 12 \left| (k + h^\vee)\rho^\vee - \rho \right|^2
\]
for the central charge of a \( W \)-algebra obtained by Hamiltonian reduction of a level-\( k \) affine Lie algebra; \( h^\vee \) is the dual Coxeter number, \( \rho \) is half the sum of positive roots, \( \rho^\vee \) half the sum of their duals, and \( \ell \) is the rank of the corresponding finite-dimensional Lie algebra.

2.4.1 (5.11(1))\footnote{[19]}: \( q_{12} q_{21} = q_{11}^{-2} \), Cartan type \( B_2 \), \( \frac{q^{-2} q^2}{\delta} \).

In terms of scalar products, we then have
\[
\alpha \cdot \alpha = \frac{2}{p} + 2j, \quad |p| \geq 4 \quad (j \in \mathbb{Z}), \quad 2\alpha \cdot \beta + 2\alpha \cdot \alpha = 2n \quad (n \in \mathbb{Z}).
\]
The braiding matrix (stable under Weyl reflections) is
\[ \begin{pmatrix} e^{\frac{2\pi i}{p}} & (1)^n e^{-\frac{2\pi i}{p}} \\ (1)^n e^{-\frac{2\pi i}{p}} & e^{\frac{4\pi i}{p}} \end{pmatrix}. \]

Conditions (7) hold in two peculiar cases and one regular case. The peculiar cases are \((m = -2j, n = 0, p = 4)\) with \(c = -1 - \frac{24}{4j+1} + \frac{24}{4j-1}\) and \((m = 1 - 2j, n = 0, p = -4)\) with \(c = -1 - \frac{24}{4j+1} + \frac{24}{4j-3}\).

The regular case is \((m = n = 0, p = 30)\),

\[ c = 86 - 60(k + 3) - \frac{30}{k+5}, \]

where \(k + 3 = \frac{1}{p} + j\) (or \(\frac{1}{k+3} = \frac{2}{p} + 2j\)). This is the central charge of the \(WB_2\) algebra [33, 34, 35] (also see [36]) obtained by Hamiltonian reduction of the level-
\((k)\) \(B_2^{(1)}\) (by formula (14), with \(|\rho|^2 = \frac{5}{2},|\rho^\gamma|^2 = 5\), and \(\langle \rho, \rho^\gamma \rangle = \frac{1}{2}\) for \(B_2\)).

The \(WB_2\) algebra contains a unique primary field of dimension 4. Explicitly, it is a rather long (20 terms) differential polynomial in \(\partial \phi_\alpha(z)\) and \(\partial \phi_\beta(z)\).

\[
\begin{align*}
(p - 3)p(27p - 32)\partial \phi_\alpha \partial \phi_\alpha \partial \phi_\alpha & + 2(p - 3)p(27p - 32)\partial \phi_\alpha \partial \phi_\alpha \partial \phi_\beta \\
- 21p (p^2 - 2) \partial \phi_\alpha \partial \phi_\alpha \partial \phi_\beta \partial \phi_\beta & - p(3p - 2)(16p - 27) \partial \phi_\alpha \partial \phi_\alpha \partial \phi_\beta \partial \phi_\beta \\
- \frac{p}{4} (3p - 2)(16p - 27) \partial \phi_\beta \partial \phi_\beta \partial \phi_\beta & + \frac{p}{4} (3p - 2)(16p - 27) \partial \phi_\beta \partial \phi_\beta \partial \phi_\beta \\
+ \ldots &
\end{align*}
\]

\((-p-3)(3p-4)(30p^3 - 115p^2 + 144p - 60) \partial^4 \phi_\alpha + \frac{2(p-3)(3p-2)(15p^3 - 72p^2 + 115p - 60)}{3p^2} \partial^4 \phi_\beta \]

(all coefficients are polynomials in \(p\) with integer coefficients after the overall renormalization by \(12p^2\)).

If \(p \geq 5\) is odd, then [19]

\[ \mathfrak{B}(X) = T(X) / ([F_1, F_1, F_1, F_2], [F_1, F_2, F_2], F_1^p, [F_1, F_1, F_2]^p, [F_1, F_2]^p, F_2^p), \]

with \(\dim \mathfrak{B}(X) = p^4\). If \(p \geq 6\) is even, then

\[ \mathfrak{B}(X) = T(X) / ([F_1, F_1, F_1, F_2], [F_1, F_2, F_2], F_1^p, [F_1, F_1, F_2]^p, [F_1, F_2]^p, F_2^p) \]

and \(\dim \mathfrak{B}(X) = \frac{p^4}{4}\) (the second generator of the ideal is absent for \(p = 4\)).

2.4.2: \(q_{12}q_{21} = q_{11}^3\), Cartan type \(G_2\), \(q_1q^{-3}q_3^3\).

In terms of scalar products, we now have

\[ \alpha.\alpha = \frac{2}{p} + 2j, \quad |p| \geq 4 \quad (j \in \mathbb{Z}), \quad 2\alpha.\beta + 3\alpha.\alpha = 2n \quad (n \in \mathbb{Z}). \]

The braiding matrix is parameterized as
\[ \begin{pmatrix} e^{\frac{2\pi i}{p}} & (1)^n e^{-\frac{2\pi i}{p}} \\ (1)^n e^{-\frac{2\pi i}{p}} & e^{\frac{4\pi i}{p}} \end{pmatrix} \]

and is stable under Weyl reflections. Conditions (7) hold in the peculiar cases \((j = 0, m = 0, p = 4)\) with \(c = -10 - \frac{54}{4n+1} + \frac{24}{4n-3}\), \((m = -3j, n = 0, p = 6)\) with \(c = -2 + \)
2.5. $q_{12} q_{21} \in R_8$, $q_{11} = (q_{12} q_{21})^2$, Cartan type $G_2$, $\frac{\zeta^2}{\alpha} - \frac{\zeta^{-1}}{\alpha} \in R_8$.

The conditions reformulate in terms of scalar products as

$$2\alpha \beta = \frac{r}{4} + 2j \quad (j \in \mathbb{Z}), \quad \alpha \alpha - 4\alpha \beta = 2n,$$

where $r$ is odd. The braiding matrix is

$$\begin{pmatrix}
i^r & (-1)^j e^{\frac{i \pi r}{4}} \\
(-1)^j e^{-\frac{i \pi r}{4}} & e^{-\frac{i \pi r}{4}}
\end{pmatrix}$$

and both of its Weyl reflections are given by

$$\begin{pmatrix}
i^r & (-1)^j e^{-\frac{i \pi r}{4}} \\
(-1)^j e^{\frac{i \pi r}{4}} & (-1)^l
\end{pmatrix}.$$ Conditions (7) can be satisfied only if $(m = 0, r = 1 - 8j - 4n), \text{yielding the central charge } c = -10 - \frac{48}{4m-1} + \frac{108}{4m-2}. \text{ For } n = 0, \text{this gives the celebrated central charge value}$

$$c = 26.$$ 

2.6. $q_{12} q_{21} \in R_{24}$, $q_{11} = (q_{12} q_{21})^6$, $\frac{\zeta^6}{\alpha} - \frac{\zeta^{-1}}{\alpha} \in R_{24}$.

The conditions for the scalar products are

$$2\alpha \beta = \frac{r}{12} + 2j \quad (j \in \mathbb{Z}), \quad \alpha \alpha - 12\alpha \beta = 2n,$$

where $r$ is coprime with 2 and 3. The braiding matrix, parameterized as

$$\begin{pmatrix}
i^r & (-1)^j e^{\frac{i \pi r}{6}} \\
(-1)^j e^{-\frac{i \pi r}{6}} & e^{-\frac{i \pi r}{6}}
\end{pmatrix},$$

has the $G_3$ Cartan matrix, but both Weyl reflections are

$$\begin{pmatrix}
i^r & (-1)^j e^{-\frac{i \pi r}{6}} \\
(-1)^j e^{\frac{i \pi r}{6}} & e^{\frac{i \pi r}{6}}
\end{pmatrix},$$

with the associated generalized Cartan matrix $(\frac{-3}{2})$; various other (generalized) Cartan matrices are produced under further Weyl reflections.

Conditions (7) are satisfied only if $(m = 0, r = 1 - 24j - 4n), \text{with } c = -10 - \frac{144}{4m-1} + \frac{324}{4m-25}$ (that $r$ be coprime with 2 and 3 selects the values $n = 2 + 3\ell$ or $n = 3 + 3\ell, \ell \in \mathbb{Z}$).
2.7. \( q_{12} q_{21} \in R_{30}, \ q_{11} = (q_{12} q_{21})^{12}, \ \xi^{12} = \frac{\xi}{\zeta} \cdot \xi^{-1}, \ \zeta \in R_{30}. \)

In terms of scalar products,
\[
2\alpha,\beta = \frac{r}{15} + 2 j \ (j \in \mathbb{Z}), \ \alpha,\alpha - 24\alpha,\beta = 2n,
\]
where \( r \) is coprime with 30. The braiding matrix is
\[
\begin{pmatrix}
\ e^{i\frac{2\pi}{3}} & (-1)^i e^{i\frac{2\pi}{3}} \\
-1 & e^{i\frac{2\pi}{3}}
\end{pmatrix}
\]
with the generalized Cartan matrix
\[
\begin{pmatrix}
\frac{2}{3} & \frac{-4}{3} \\
\frac{-4}{3} & \frac{2}{3}
\end{pmatrix}
\]. Both Weyl reflections are
\[
\begin{pmatrix}
\ e^{i\frac{2\pi}{3}} & (-1)^i e^{i\frac{2\pi}{3}} \\
-1 & e^{i\frac{2\pi}{3}}
\end{pmatrix}
\]
(with the same generalized Cartan matrix, but other (generalized) Cartan matrices are produced under further Weyl reflections). Conditions (7) are solved by \((m = 0, n = 1 + 2\ell, r = -2 - 5\ell - 30j), \) where \( \ell = 1 + 6u \) or \( \ell = 3 + 6u, u \in \mathbb{Z} \), respectively yielding the incomprehensible \( c = \frac{-62}{5} + \frac{2916}{5(30u - 17)} - \frac{180}{30u + 17} \)
and \( c = \frac{-62}{5} + \frac{2916}{5(30u - 17)} - \frac{180}{30u + 17} \).

3. The list, items 3.

The defining conditions are\[ q_{12} q_{21} \neq 1, \ q_{11} q_{12} q_{21} \neq 1, \ q_{12} q_{21} q_{22} \neq 1, \ q_{22} = -1, \ q_{11} \in R_{2} \cup R_{3}, \]
plus any of conditions 3.1–3.7. In terms of the screening momenta, the common conditions are
\[
\beta,\beta = 1 + 2m \ (m \in \mathbb{Z}), \ \alpha,\alpha = 1 \text{ or } \frac{2s}{3},
\]
where \( s \) is coprime with 3.

3.1 (5.7(4)[19]). \( q_{11} = -1, \ q_{12} q_{21} \in \bigcup_{n=3}^{\infty} R_{n}, \) Cartan type \( A_{2}, \)
\[
\begin{pmatrix}
-1 & q \cdot q^{-1} \\
-1 & q \cdot q^{-1}
\end{pmatrix}
\].

In terms of scalar products of the screening momenta, these conditions are
\[
\alpha,\alpha = 1 + 2n \ (n \in \mathbb{Z}), \ 2\alpha,\beta = \frac{2}{p} + 2j, \ |p| \geq 3 \ (j \in \mathbb{Z}).
\]
Both screenings “want to be fermionic.” The braiding matrix is
\[
\begin{pmatrix}
-1 & (-1)^i e^{i\frac{20\pi}{3}} \\
-1 & e^{i\frac{20\pi}{3}}
\end{pmatrix}
\]
and both of its Weyl reflections are
\[
\begin{pmatrix}
-1 & (-1)^i e^{i\frac{20\pi}{3}} \\
-1 & e^{i\frac{20\pi}{3}}
\end{pmatrix}
\]. Remarkably, conditions (7) are satisfied (with \(|p| \geq 3\)) only for \( m = n = 0 \) (no “peculiar” solutions!), yielding the \( \hat{\ell}(2)_{k/6} \) central charge
\[
c = \frac{3k}{k+2} - 1,
\]
where \( k + 1 = \frac{1}{p} + j. \) For \( j = 0, \) in particular, this relation between \( k \) and \( p \) takes the form
\[
\frac{1}{p + 1} + \frac{1}{k + 2} = 1.
\]
This “duality” between two levels, $k$ and $p - 1$, was extensively used in [39, 40] (see also the references therein); in particular,

$$\hat{s}_k(2)/\hat{h} = \hat{s}_0(2|1)_{p-1}/\hat{g}_l(2)_{p-1},$$

offering another view on what the CFT counterpart of the Nichols algebra is.

The currents generating the $\hat{s}_k(2)/\hat{h}$ coset algebra are given by

$$j^+ (z) = \hat{\partial} \phi_\beta (z) e^{\frac{i}{\hat{h}} (\phi_\alpha (z) - \phi_\beta (z))},$$

$$j^- (z) = \hat{\partial} \phi_\alpha (z) e^{-\frac{i}{\hat{h}} (\phi_\alpha (z) - \phi_\beta (z))}$$

(as before, $\phi_\alpha (z) = \alpha \cdot \varphi (z)$ and $\phi_\beta (z) = \beta \cdot \varphi (z)$ are the boson fields “in the direction” of the corresponding screening). With an extra boson $\chi (z)$ added to account for the missing $\hat{h}$, the $\hat{s}_k(2)$ algebra currents are reconstructed as

$$J^\pm (z) = j^\pm (z) e^{\pm \sqrt{2} \chi (z)}, \quad J^0 (z) = \sqrt{\frac{k}{2}} \hat{\partial} \chi (z).$$

The $\hat{s}_k(2)$ algebra is well known, since the “old” studies of the Wakimoto bosonization, to be described as a centralizer of two fermionic screenings “at an angle” to each other. In this item in the list, we see again that imposing relations (7) implies that both $-1$ in the braiding matrix translate exactly into true fermionic screenings.

For $p \geq 3$, the Nichols algebra is given by [19]

$$\mathfrak{B} (X) = T (X) / (F_1^2, [F_1, F_2]^p, F_2^2)$$

with $\dim \mathfrak{B} (X) = 4p$.

### 3.2. There are two subcases.

**3.2.1:** $q_{11} \in \mathbb{R}, \quad q_{12} q_{21} = q_{11}$, Cartan type $B_2$, \begin{align*}
\xi & : \frac{\xi}{\tilde{\xi}} = -1, \quad \xi \in \mathbb{R}.
\end{align*}

This reformulates in terms of scalar products of the screening momenta as

$$\alpha . \alpha = \frac{2s}{\pi} + 2\ell \quad (\ell \in \mathbb{Z}), \quad 2\alpha . \beta = \alpha . \alpha + 2n \quad (n \in \mathbb{Z}),$$

where $s$ is coprime with 3. The braiding matrix, parameterized as

$$\begin{pmatrix}
\frac{e^{\frac{2\pi i}{s}}}{\sqrt{3}} & e^{\frac{2\pi i}{s}} \\
(-1)^{\ell + \ell} e^{\frac{2\pi i}{s}} & -1
\end{pmatrix},$$

has the Cartan type $B_2$ and is stable under Weyl reflections.

Once again, conditions (7) ensure that the tentative fermionic screening is such indeed, i.e., $m = 0$: the conditions can be solved only if $(m = 0, s = 1 - 3\ell)$ or $(m = 0, s = 1 - 3\ell)$.

---

5 This coset equivalence is related to a vast subject discussed in [41].

6 The Wakimoto bosonization [42] yields two essentially different three-boson realizations of $\hat{s}_k(2)$—the “symmetric” and the “nonsymmetric” ones, respectively centralizing two fermionic screenings and one bosonic plus one fermionic screening. The names refer to the “$j^+ \leftrightarrow j^-$ symmetric” structure of [15] and the “asymmetric” structure of [13]. Somewhat broader, the “variously symmetric” realizations are discussed in [40].
\(-n - 3\ell\). These cases respectively yield the unilluminating central charges \(2 - \frac{6(12\alpha - 7)}{9n + 6n - 3}\) and \(-1 - \frac{36}{2n+3} + \frac{18}{n}\).

### 3.2.2 (5.11(4)\(^{[19]}\))

\(q_{11} \in R_3, \quad q_{12}q_{21} = -q_{11}\), Cartan type \(B_2\), \(\zeta - \zeta - 1\), \(\zeta \in R_3\).

Then
\[
\alpha, \beta = \frac{2s}{3}, \quad 2\alpha, \beta - \alpha, \alpha = 1 + 2n,
\]
where \(s\) is coprime with \(3\). The braiding matrix
\[
\begin{pmatrix}
e^{\frac{2is}{3}} & (-1)^{n+\alpha}e^{\frac{i\alpha}{2}} \\
(-1)^{n+\beta}e^{\frac{i\beta}{2}} & -1
\end{pmatrix}
\]
is of Cartan type \(B_2\) (its off-diagonal elements change sign under both Weyl reflections). Conditions \((7)\) are solved only if \((m = 0, s = 1 - 3\ell)\), which leaves us with another incomprehensible \(c = 2 - \frac{24(12n-1)}{36n^2+60n+1}\) (which is \(c = 26\) at \(n = 0\), however).

The Nichols algebra is given by the quotient \([19]\)
\[
\mathcal{B}(X) = T(X)/([F_1, F_1, F_2, F_1, F_2], F_1^3, F_2^3)
\]
with \(\dim \mathcal{B}(X) = 36\).

The remaining subcases are equally unsuggestive, and the details are omitted.

### 3.3. \(q_0 := q_{11}q_{12}q_{21} \in R_{12}\), \(q_{11} = q_{12}^4\).

This translates into
\[
\alpha, \alpha + 2\alpha, \beta = \frac{2r}{12} + 2j, \quad \alpha, \alpha = 4\alpha, \alpha + 8\alpha, \beta + 2n.
\]

### 3.4. \(q_{12}q_{21} \in R_{12}\), \(q_{11} = -(q_{12}q_{21})^2\), or
\[
2\alpha, \beta = \frac{2r}{12} + 2j, \quad \alpha, \alpha = 4\alpha, \beta + 1 + 2n.
\]

### 3.5. \(q_{12}q_{21} \in R_9\), \(q_{11} = (q_{12}q_{21})^{-3}\), or
\[
2\alpha, \beta = \frac{2r}{9} + 2j, \quad \alpha, \alpha = -6\alpha, \beta + 2n.
\]

### 3.6. \(q_{12}q_{21} \in R_{24}\), \(q_{11} = -(q_{12}q_{21})^4\), or
\[
2\alpha, \beta = \frac{2r}{24} + 2j, \quad \alpha, \alpha = 8\alpha, \beta + 2n.
\]

### 3.7. \(q_{12}q_{21} \in R_{30}\), \(q_{11} = -(q_{12}q_{21})^5\), or
\[
2\alpha, \beta = \frac{2r}{30} + 2j, \quad \alpha, \alpha = 10\alpha, \beta + 1 + 2n.
\]

### 4. THE LIST, ITEMS 4.

The conditions are
\[
q_{12}q_{21} \neq 1, \quad q_{11}q_{12}q_{21} \neq 1, \quad q_{12}q_{21}q_{22} \neq 1, \quad q_{22} = -1, \quad q_{11} \notin R_2 \cup R_3,
\]
plus any of the conditions in cases 4.1–4.8.
In terms of the screening momenta, the common condition is
\[ \beta \cdot \beta = 1 + 2m, \]
showing that \( F_\beta \) is a candidate for a fermionic screening.

4.1 \((5.11(2))^{[19]}\). \( q_{11} \in \bigcup_{a=5}^{\infty} R_a, \) \( q_{12}q_{21} = q_{11}^{-2} \), Cartan type \( B_2 \), \( q = q^{-2} \).\(^1\)

In terms of screenings, \( \alpha \cdot \alpha = \frac{2}{p} + 2j, \, |p| \geq 5 \, (j \in \mathbb{Z}), \, 2\alpha \cdot \alpha + 2\alpha \cdot \beta = 2n. \)

Then the braiding matrix (which invariant under Weyl reflections) is parameterized as
\[
\begin{pmatrix}
e^{\frac{2\pi i}{p}} & -1 \\
-1 & e^{-\frac{2\pi i}{p}}
\end{pmatrix}^{(-1)^n} \begin{pmatrix}
(-1)^n e^{-\frac{2\pi i}{p}} \\
-1
\end{pmatrix}^{(-1)^n}. \]
Remarkably, once again, conditions \((7)\) are satisfied only for \( m = n = 0 \) (tentative “peculiar” solutions are with \( |p| \leq 4 \)). Hence, \( F_\beta \) is indeed a standard fermionic screening. The central charge is then given by
\[ c = -25 + \frac{24}{k+3} + 6(k+3), \]
where \( \frac{1}{p} + j = -\frac{1}{k+1} \) (or \( \frac{1}{p} + j = \frac{1}{2} + \frac{1}{k+1} \)). Somewhat mysteriously, this is minus the central charge
\[ c_{W_3^{(2)}} = 25 - \frac{24}{k+3} - 6(k+3) \]
of the \( W_3^{(2)} \) algebra, which can be obtained by a “partial” Hamiltonian reduction of \( \tilde{s}\ell(3)_k \) and which has a three-boson realization \([43][44][40]\).

The Nichols algebra is the quotient \([19]\)
\[ \mathfrak{B}(X) = T(X) / (\{F_1, F_1, F_2, F_1^p, [F_1, F_2]^p, F_2^2\}), \]
where \( p' = \text{ord}(-e^{\frac{2\pi i}{p}}) \), with \( \text{dim} \mathfrak{B}(X) = 4pp' \).

None of the remaining cases currently seems illuminating in any respect.

4.2. \( q_{11} \in R_5 \cup R_8 \cup R_{12} \cup R_{14} \cup R_{20}, q_{12}q_{21} = q_{11}^{-3}. \)

4.3. \( q_{11} \in R_{10} \cup R_{18}, q_{12}q_{21} = q_{11}^{-4}. \)

4.4. \( q_{11} \in R_{14} \cup R_{24}, q_{12}q_{21} = q_{11}^{-5}. \)

4.5. \( q_{12}q_{21} \in R_8, q_{11} = (q_{12}q_{21})^{-2}. \)

4.6. \( q_{12}q_{21} \in R_{12}, q_{11} = (q_{12}q_{21})^{-3}. \)

4.7. \( q_{12}q_{21} \in R_{20}, q_{11} = (q_{12}q_{21})^{-4}. \)

4.8. \( q_{12}q_{21} \in R_{30}, q_{11} = (q_{12}q_{21})^{-6}. \)
5. THE LIST, ITEMS 5.

The basic conditions delimiting this case are

\[ q_{12}q_{21} \neq 1, \quad q_{11}q_{12}q_{21} \neq 1, \quad q_{12}q_{21}q_{22} \neq 1, \quad q_{11} \neq -1, \quad q_{22} \in \mathbb{R}_3, \]

to which further conditions in 5.1.–5.5. are to be added one by one. I merely reproduce them.

5.1. \( q_0 := q_{11}q_{12}q_{21} \in \mathbb{R}_{12}, \quad q_{11} = q_0^4, \quad q_{22} = -q_0^2. \)

5.2. \( q_{12}q_{21} \in \mathbb{R}_{12}, \quad q_{11} = q_{22} = -(q_{12}q_{21})^2. \)

5.3. \( q_{12}q_{21} \in \mathbb{R}_{24}, \quad q_{11} = (q_{12}q_{21})^{-6}, \quad q_{22} = (q_{12}q_{21})^{-8}. \)

5.4. \( q_{11} \in \mathbb{R}_{18}, \quad q_{12}q_{21} = q_{11}^{-2}, \quad q_{22} = -q_{11}^3. \)

5.5. \( q_{11} \in \mathbb{R}_{30}, \quad q_{12}q_{21} = q_{11}^{-3}, \quad q_{22} = -q_{11}^5. \)

\( \mathbb{R}_0. \) CONCLUSIONS

Some additions to the above list (including the currently uninteresting items?!) might hopefully follow in the future. A variety of isolated central charge values for lower-rank \( W \)-algebras can be found in [45] (also see [46]), with interesting possibilities of an overlap with the isolated values which I deemed uninteresting. I know nothing about a CFT counterpart of one “regular” case, 2.3 (a \( G(3) \) reduction?). More presentations of nonstandard type appeared recently in [47].

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APPENDIX V. VIRASORO ALGEBRA

In CFT, the Virasoro algebra

\[ [L_m, L_n] = (m - n)L_{m+n} + \frac{c}{12}(m - 1)m(m + 1), \quad m, n \in \mathbb{Z} \]

standardly appears as the energy–momentum tensor \( T(z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2} \)—a (chiral) field on the complex plane that satisfies the OPEs

\[ T(z) T(w) = \frac{c/2}{(z-w)^2} + \frac{2T(z)}{(z-w)^2} + \frac{\partial T(z)}{z-w}. \]
The $c$ parameter (understood to be multiplied by the unit operator whenever necessary) is called the central charge.

For $\theta$ bosonic fields $\varphi(z) = (\varphi^1(z), \ldots, \varphi^\theta(z))$ with the OPEs
\begin{equation}
\varphi^i(z) \varphi^j(w) = \delta^{ij} \log(z - w),
\end{equation}
which are also frequently used in calculations in the form
\[ \partial \varphi^i(z) \partial \varphi^j(w) = \frac{\delta^{ij}}{(z-w)^2}, \]
the energy–momentum tensors are parameterized by $\xi \in \mathbb{C}^\theta$.
\begin{equation}
T_\xi(z) = \frac{1}{2} \partial \varphi(z), \partial \varphi(z) + \xi \cdot \partial^2 \varphi(z).
\end{equation}
The corresponding central charge is
\begin{equation}
c_\xi = \theta - 12 \varphi \cdot \xi.
\end{equation}
The OPE of $T_\xi(z)$ with a vertex operator $e^{\mu, \varphi(z)}$ is
\[ T_\xi(z) e^{\mu, \varphi(w)} = \frac{\Delta e^{\mu, \varphi(w)}}{(z-w)^2} + \frac{\partial e^{\mu, \varphi(w)}}{z-w}, \quad \Delta = \frac{1}{2} \mu \cdot \mu - \xi \cdot \mu \]
A screening operator is, by definition, any expression $\hat{S}V(\cdot)$, where $V(z)$ is a field of dimension $\Delta = 1$ (and the contour integration is essentially equivalent to taking a residue “after the action of $V(z)$ is evaluated”). For $\theta = 2$, any two exponentials $e^{\alpha, \varphi(z)}$ and $e^{\beta, \varphi(z)}$ with noncollinear $\alpha, \beta \in \mathbb{C}^2$ define screening operators with respect to the energy–momentum tensor
\[ T(z) = \frac{1}{2} \partial \varphi(z), \partial \varphi(z) \]
\[ - \frac{(2 + \alpha \cdot \beta - \alpha \cdot \alpha) \beta \cdot \beta - 2 \alpha \cdot \beta}{2 \delta^2} \partial^2 \varphi_{\alpha}(z) - \frac{(2 + \alpha \cdot \beta - \beta \cdot \beta) \alpha \cdot \alpha - 2 \alpha \cdot \beta}{2 \delta^2} \partial^2 \varphi_{\beta}(z), \]
where $\partial \varphi_{\alpha}(z) = \alpha \cdot \partial \varphi(z)$ and $\partial \varphi_{\beta}(z) = \beta \cdot \partial \varphi(z)$, and $\delta^2 = (\alpha \cdot \alpha)(\beta \cdot \beta) - (\alpha \cdot \beta)^2$. This gives formula (4) for the central charge.

Next, I show that the central charge of the $\theta$-boson energy–momentum tensor that centralizers $\theta$ screenings $\hat{S}e^{\alpha_i, \varphi(z)}$, $1 \leq i \leq \theta$, with linearly independent momenta is invariant under Weyl reflections (8) if Eqs. (7) hold.

Given the $\alpha_i$, $1 \leq i \leq \theta$, the condition that all the exponentials $e^{\alpha_i, \varphi(z)}$ have dimension 1 is expressed by the system of equations for $\xi$
\[ \frac{1}{2} \alpha_i \cdot \alpha_i - \xi \cdot \alpha_i = 1, \quad 1 \leq i \leq \theta. \]
With $\xi$ written as $\xi = \sum_{j=1}^{\theta} x_j \alpha_j$, this becomes a system for the $x_j$,
\begin{equation}
\frac{1}{2} \alpha_i \cdot \alpha_i - \sum_{j=1}^{\theta} x_j \alpha_j \cdot \alpha_i = 1, \quad 1 \leq i \leq \theta,
\end{equation}
the original solution is solved by the ansatz (20).

Under a Weyl groupoid operation $\mathcal{R}^{(k)}$ in (8), the scalar products change and the solution $(x_j)$ also changes. The “old” and “new” central charges are

$$c = \theta - 12 \sum_{\ell,j=1}^{\theta} x_\ell x_j \alpha_\ell \cdot \alpha_j \quad \text{and} \quad \mathcal{R}^{(k)}(c) = \theta - 12 \sum_{\ell,j=1}^{\theta} \tilde{x}_\ell \tilde{x}_j \mathcal{R}^{(k)}(\alpha_\ell \cdot \alpha_j),$$

where the $\tilde{x}_j$ solve the system “$\mathcal{R}^{(k)}(19)$.” With $\tilde{x}_j = x_j + y_j$, this system becomes

$$\frac{1}{2} \alpha_i \cdot \alpha_i - a_{k,i} \alpha_i \cdot \alpha_k + \frac{1}{2} a_{k,i}^2 \alpha_k \cdot \alpha_k$$

$$- \sum_{j=1}^{\theta} (x_j + y_j) (\alpha_j \cdot \alpha_i - a_{k,j} \alpha_j \cdot \alpha_k - a_{k,j} \alpha_k \cdot \alpha_i + a_{k,j} a_{k,i} \alpha_k \cdot \alpha_k) = 1, \quad 1 \leq i \leq \theta$$

(for a chosen $k$). The claim is that this system of equations for the “deformation” of the original solution is solved by the ansatz $y_j = \delta_{j,k} y$. Indeed, substituting such $y_j$ and using (19) in the resulting equations gives the equations

$$a_{k,j} \left( \frac{1}{2} \alpha_k \cdot \alpha_k (a_{k,j} + 1) - \alpha_i \cdot \alpha_k - 1 \right) + \left( y + \sum_{j=1}^{\theta} x_j a_{k,j} \right) (\alpha_k \cdot \alpha_i - a_{k,j} \alpha_k \cdot \alpha_k) = 0$$

that must be satisfied for all $1 \leq i \leq \theta$. Remarkably, if (7) holds for the $\alpha_j$, then all the equations are solved by

$$y = 1 - \frac{2}{\alpha_k \cdot \alpha_k} \sum_{j=1}^{\theta} x_j a_{k,j}. \quad (20)$$

It remains to find the new central charge. With $\tilde{x}_j = x_j + \delta_{j,k} y$,

$$\sum_{\ell,j=1}^{\theta} \tilde{x}_\ell \tilde{x}_j \mathcal{R}^{(k)}(\alpha_\ell \cdot \alpha_j) = \sum_{\ell,j=1}^{\theta} x_\ell x_j (\alpha_\ell \cdot \alpha_j - 2 a_{k,\ell} \alpha_j \cdot \alpha_k + a_{k,i} a_{k,j} \alpha_k \cdot \alpha_k)$$

$$+ 2 \sum_{j=1}^{\theta} y x_j (a_{k,j} \alpha_k \cdot \alpha_k - \alpha_k \cdot \alpha_j) + y^2 \alpha_k \cdot \alpha_k$$

and yet another use of (19) shows that this is

$$= \sum_{\ell,j=1}^{\theta} x_\ell x_j \alpha_\ell \cdot \alpha_j + \left( y + \sum_{j=1}^{\theta} x_j a_{k,j} \right) \left( y \alpha_k \cdot \alpha_k + 2 - \alpha_k \cdot \alpha_k + \sum_{\ell=1}^{\theta} x_\ell a_{k,\ell} \alpha_k \cdot \alpha_k \right),$$

where the last factor vanishes by virtue of (20). The central charge is invariant.
APPENDIX W. $W_3$ LOGARITHMIC OCTUPLET ALGEBRAS

With the two screenings as in case 2.1 (the “regular” solution there, with central charge 9 and the $W(z)$ field 10), I propose a $W_3$ counterpart of the $(1, p)$ triplet algebra 24,25 by closely following the constructions in 27.

An octuplet of primary fields is generated from the field $e^{\gamma \phi(z)}$ with $\mu \in \mathbb{C}^2$ such that $\gamma.\alpha^\vee = p$ and $\gamma.\beta^\vee = p$, i.e., from the field

$$W(z) = e^{\gamma \phi(z)}, \quad \gamma = \alpha^\vee + \beta^\vee$$

(which is in the kernel of the two screenings 2). This is a Virasoro primary field of dimension $\Delta = 3p - 2$, that is,

$$L_n W(z) = 0, \quad n \geq 1,$$
$$L_0 W(z) = \Delta W(z), \quad \Delta = 3p - 2,$$

and, moreover, a $W_3$ primary: as is easy to verify, the modes of the dimension-3 field $W(z) = \sum_{n \in \mathbb{Z}} W_n z^{-n-3}$ in 10 act on $W(z)$ such that

$$W_n W(z) = 0, \quad n \geq 0.$$

Then the long screenings 12 generate the octuplet

Here, $W_{\alpha}(z) = \mathcal{E}_{\alpha} W(z)$, $W_{\beta\alpha}(z) = \mathcal{E}_{\beta} W_{\alpha}(z)$, and so on, and $W_{\alpha\beta\alpha}(z) = \mathcal{E}_{\beta} W_{\alpha\beta}(z) = \mathcal{E}_{\alpha} W_{\beta\alpha\beta}(z)$; the dashed arrows represent maps to the target field up to a nonzero overall factor $(\frac{(-1)^p}{2})$. All the fields in the diagram are $W_3$-algebra primaries, with the same Virasoro dimension. All fields below the top are of the form $W_*(z) = P_*(\partial \phi(z)) \phi^{\mu_*.\phi(z)}$, where the momenta $\mu_*$ are immediately read off from the diagram as $\mu_{\alpha} = \gamma - \alpha^\vee$,
\[ \mu_{\alpha\beta} = \mu_{\beta\alpha} = \gamma - \alpha^\gamma - \beta^\gamma = 0, \] and so on, and the \( P_\alpha(\partial \phi(z)) \) are differential polynomials in \( \partial \phi_\alpha(z) \) and \( \partial \phi_\beta(z) \), of the orders \( \text{ord}(P_\alpha) = \text{ord}(P_\beta) = p - 1, \text{ord}(P_{\alpha\beta}) = 3p - 2, \text{ord}(P_{\alpha\beta\alpha}) = 3p - 3, \) and \( \text{ord}(P_{\alpha\alpha\beta\beta}) = 4p - 4. \)

Calculations in particular examples show the OPE

\[ W(z)W_{\alpha\alpha\beta\beta}(w) = \frac{c_1}{(z-w)^{3p-1}} + \frac{c_2 T(w)}{(z-w)^{3p-5}} + \frac{c_2/2 \partial T(w)}{(z-w)^{3p-7}} + \ldots \]

with nonzero coefficients (and no dimension-3 \( W(w) \) field), and the OPEs \( W_\alpha(z)W_{\alpha\beta\beta}(w) \) and \( W_\beta(z)W_{\alpha\alpha\beta}(w) \) that start very similarly. The adjoint-\( \mathfrak{sl}(3) \) nature of the octuplet manifests itself in the OPEs such as

\[
W_{\alpha}(z)W_{\beta}(w) = \frac{c_2 W(w)}{(z-w)^{3p-2}} + \ldots,
\]

\[
W_{\alpha}(z)W_{\alpha\beta\alpha}(w) = \mathcal{O}(z-w),
\]

\[
W_{\beta}(z)W_{\beta\beta\alpha}(w) = \mathcal{O}(z-w),
\]

\[
W_{\alpha\beta\alpha}(z)W_{\beta\alpha\beta}(w) = \frac{c_2 W_{\alpha\alpha\beta\beta}(w)}{(z-w)^{3p-2}} + \ldots.
\]

It may be worthwhile to comment in this case, apparently the simplest in the list, on the third element to the \( p \)th power in (11). Is it also associated with a long screening, as \( F_1^p \) and \( F_2^p \) are? The answer is interesting. The candidate expression \( e^{(-\alpha^\gamma - \beta^\gamma) \cdot \phi(z)} \) has the dimension \( 2 - p \) and cannot be used as a screening current. However, there is a degree-(\( p - 1 \)) differential polynomial \( P(\partial \phi(z)) \) in the two bosons such that \( \mathcal{E} = \int P(\partial \phi(z))e^{(-\alpha^\gamma - \beta^\gamma) \cdot \phi(z)} \) is a screening for the \( W_3 \) algebra (and, importantly, the integrand is not a total derivative!). But it is not a screening for the octuplet algebra.

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