The consistent form of the gauge anomaly is worked out at first order in $\theta$ for the noncommutative three-point function of the ordinary gauge field of certain noncommutative chiral gauge theories defined by means of the Seiberg-Witten map. We obtain that for any compact simple Lie group the anomaly cancellation condition of this three-point function reads
\[ \text{Tr} T^a T^b T^c = 0, \]
if one restricts the type of noncommutative counterterms that can be added to the classical action to restore the gauge symmetry to those which are renormalizable by power-counting. On the other hand, if the power-counting renormalizability paradigm is relinquished and one admits noncommutative counterterms (of the gauge fields, its derivatives and $\theta$) which are not power-counting renormalizable, then, the anomaly cancellation condition for the noncommutative three-point function of the ordinary gauge field becomes the ordinary one:
\[ \text{Tr} T^a \{ T^b, T^c \} = 0. \]
1 Introduction

The Seiberg-Witten map was introduced in ref. [1] to account, at least formally, for the physical equivalence of two formulations of the same theory. The authors of ref. [1] studied how noncommutative gauge fields and ordinary gauge fields arise in open string theory for U(n) groups. They showed that either type of gauge field can be obtained from the same world-sheet field theory by changing the regularization prescription. Since Physics cannot depend on the choice of regularization, and a change of the renormalization conditions on the string world-sheet corresponds to a field redefinition in space-time, Seiberg and Witten concluded that, generally, there must exist a map from the ordinary gauge field to its noncommutative counterpart intertwining with the gauges symmetries. However, this map does fail to exist in some instances [1, 2]. Then, the authors of refs. [3, 4, 5, 6] realized that one can take further advantage of the idea embodied in Seiberg-Witten map that a noncommutative gauge field can be defined in terms of its ordinary counterpart, and formulated gauge theories on noncommutative space-time for groups other than U(n); actually, for arbitrary gauge groups. Thus the standard model and GUTs were formulated at the tree level on noncommutative space-time [7, 8]. After a promising start [9] it turned out that the noncommutative gauge theories so defined might not be power-counting renormalizable in perturbation theory [10]. And yet, they may be phenomenologically useful if, as suggested in ref. [7], one embraces the effective field theory philosophy –see refs. [11, 12] for introductions to effective field theory. Or, it may well be that supersymmetry [13, 14] turned these models into power-counting renormalizable models in the perturbative expansion.

Several issues concerning the Seiberg-Witten map and the noncommutative field theories obtained by using it have been studied in the literature so far. The perturbative \( \theta \) – solution to the differential equation defining the Seiberg-Witten map has been obtained by employing several methods in refs. [15, 6, 16, 17, 18]. An exact expression for the inverse of Seiberg-Witten map was conjectured in ref. [19]. It was shown in refs. [20, 21, 22] that the conjecture is correct. In refs. [23, 24, 25] cohomological approaches to the Seiberg-Witten map were put forward. These approaches can be used to discuss the ambiguities affecting the Seiberg-Witten map which were pointed out in ref. [26]. It turns out that the Seiberg-Witten map is unique modulo gauge transformations and field redefinitions. This arbitrariness in the value of the Seiberg-Witten map is of the utmost importance in the the renormalization process [3]. As happens with the ordinary gauge anomaly, the Seiberg-Witten map also involves a (noncommutative) gauge group cocycle [27, 28]. How the Seiberg-Witten map acts on
topological nontrivial noncommutative gauge field configurations has been studied by several authors [29, 30, 31]. It so happens that noncommutative configurations constructed using projection operators map to “commutative” configurations that have delta-function singularities. Thus it can be exhibited that the physics of noncommutative gauge theories is rather different from that of their ordinary counterparts. On the phenomenological side the Seiberg-Witten map has been used to generate theories which lack, due the noncommutativity of space-time, particle Lorentz invariance [32]. Computations of the strength of the breaking of particle Lorentz invariance has led upon comparison with experimental data to bounds on the scale of the noncommutative parameter [33, 34]. However, no study of the gauge anomaly problem have we found in the literature –see [35] for the axial anomaly– in spite of its implications for model building as well as its bearing on the quantum consistency of chiral gauge theories in general. We shall try to remedy this situation in this paper.

The purpose of this article is to analyze the behaviour under gauge transformations of noncommutative gauge theories with chiral fermions carrying arbitrary finite dimensional unitary representations of compact simple Lie groups. Hence, the formalism put forward in refs. [3, 4, 5, 6] must be employed and use the Seiberg-Witten map to express the noncommutative fields in terms of their ordinary counterparts. We shall consider a noncommutative left-handed spinor whose ordinary counterpart carries an arbitrary finite dimensional unitary representation a compact simple Lie group, the generalization to more general instances being straightforward. We shall quantize the spinor field and keep the gauge field as a background field.

The lay out of this paper is as follows. In the first section we formulate our model and define a regularized action in terms of the ordinary fields. This action is obtained by applying the Seiberg-Witten map to an action written in terms of noncommutative fields. Thus the relation between noncommutative and ordinary fields established by the Seiberg-Witten map will not be spoiled by the regularization process. Section two is devoted to the diagrammatic computation of the anomaly carried by the noncommutative three-point function of the ordinary gauge field. This anomaly is the noncommutative sibling of the ordinary triangle gauge anomaly. In section three, we show by using a mixture of path integral and diagrammatic arguments what the consistent form of the gauge anomaly is at first order in $\theta$. We shall close the paper with a section in which comments and conclusions shall be given and an Appendix.
2 The model and its regularization

Let $G$ be a compact simple Lie group. Let $\psi_L$ denote a left-handed spinor on Minkowski space carrying a given finite dimensional unitary representation of $G$. Let $a_\mu$ denote the gauge field which couples to $\psi_L$. Then, the action that gives the interaction between $\psi$ and $a_\mu$ reads

$$S = \int d^4x \bar{\psi}_L i\mathcal{D}(a)\psi_L. \quad (1)$$

The symbol $i\mathcal{D}(a) = i\gamma^\mu D^\mu(a)$ denotes the Dirac operator, with $D^\mu(a)$ being the covariant derivative: $D^\mu(a)\psi_L = \partial^\mu\psi_L - ia_\mu\psi_L$. The gamma matrices, $\gamma^\mu$, $\mu = 0, 1, 2, 3$ are defined by

$$\{\gamma^\mu, \gamma^\nu\} = \eta^{\mu\nu}; \quad \eta_{00} = 1. \quad \text{As usual, } \bar{\psi}_L = \psi_L^\dagger \gamma^0.$$

The BRS operator, which is linear, commutes with $\partial^\mu$, satisfies the anti-Leibniz rule and is nilpotent ($s^2 = 0$). $D^\mu(a)$ is equal to $\partial^\mu - i[a_\mu, ]$ and $\lambda$ denotes the ghost field, which has ghost number 1. Both $a_\mu$ and $\psi$ have ghost number 0.

To construct the noncommutative counterpart of the ordinary theory defined by $S$, we shall employ the formalism developed in refs. [3, 4, 5, 6]. Let $A_\mu$, $\Psi_L$ and $\Lambda$ stand for the noncommutative gauge field, the noncommutative left-handed spinor field and the noncommutative ghost field, respectively. $A_\mu$, $\Psi_L$ and $\Lambda$ are defined in terms of $a_\mu$, $\psi_L$ and $\lambda$ by means of the Seiberg-Witten map. Modulo BRS transformations and field redefinitions, the Seiberg-Witten map at first order in $\theta$ reads

$$A_\mu(a, \theta) = a_\mu - \frac{1}{4} \theta^{\alpha\beta} \{a_\alpha, f_{\beta\mu} + \partial_\beta a_\mu\} + o(\theta^2),$$

$$\Psi_L(a, \psi_L, \theta) = \psi_L - \frac{1}{2} \theta^{\alpha\beta} a_\alpha \partial_\beta \psi_L + i \frac{1}{8} \theta^{\alpha\beta} [a_\alpha, a_\beta] \psi_L + o(\theta^2),$$

$$\Lambda(a, \lambda, \theta) = \lambda + \frac{1}{4} \theta^{\alpha\beta} \{\partial_\alpha \lambda, a_\beta\} + o(\theta^2). \quad (3)$$

Note that $A_\mu$ and $\Lambda$ are valued in the representation of the enveloping algebra of the Lie algebra of $G$ induced by the unitary representation of the latter algebra carried by $\psi_L$.

Let $\star$ denote the Moyal product of functions on Minkowski space:

$$(f \star g)(x) = \int \frac{d^4p}{(2\pi)^4} \int \frac{d^4q}{(2\pi)^4} e^{-i(p+q)x} e^{-\frac{i}{2} \theta^{\alpha\beta} p_{\alpha} q_{\beta}} \tilde{f}(p) \tilde{q}(q);$$

4
\( \hat{f}(p) \) and \( \hat{q}(q) \) being the Fourier transforms of \( f \) and \( g \), respectively. Then, the noncommutative, \( S_{nc} \), version of the action in eq. (1) reads

\[
S_{nc} = \int d^4x \bar{\Psi}_L \star i\mathcal{D}(A)\Psi_L,
\]

where \( \mathcal{D}(A) = \gamma^\mu D_\mu(A) \); \( D_\mu(A) \) being the noncommutative covariant derivative: \( D_\mu(A)\Psi_L = \partial_\mu\Psi_L - iA_\mu \star \Psi_L \). Again, \( \bar{\Psi}_L = \Psi_L^\dagger \gamma^0 \).

The noncommutative action \( S_{nc} \) is invariant under the action of the noncommutative BRS operator \( s \). \( s \) is a linear operator which commutes with \( \partial_\mu \), satisfies the anti-Leibniz rule and acts on the noncommutative fields as follows

\[
s A_\mu = D_\mu(A) \Lambda, \quad s \Psi_L = i\Lambda \star \Psi_L \quad \text{and} \quad s \Lambda = i\Lambda \star \Lambda.
\]

The symbol \( D_\mu(A) \) stands for \( \partial_\mu - i[A_\mu, ] \star \); \( [f, g]_s = f \star g - g \star f \). \( s \) is nilpotent.

By definition of Seiberg-Witten map, the following equations hold

\[
s A_\mu(a, \theta) = s_A_\mu(a) \Lambda, \quad s \Psi_L(a, \psi_L, \theta) = s \Psi_L \quad \text{and} \quad s \Lambda(a, \lambda, \theta) = s \Lambda.
\]

The action of \( s \) on \( A_\mu(a, \theta), \Psi_L(a, \psi_L, \theta) \) and \( \Lambda(a, \lambda, \theta) \) is computed by assuming first that these objects are formal power series of \( \theta \), the ordinary fields and their derivatives, and, then, using eq. (1). The right hand side of the identities in eq. (5) is given by eq. (4).

By expanding \( S_{nc} \) in terms of \( a \) and \( \psi_L \) to first order in \( \theta \), one obtains

\[
S_{nc} = \int d^4x \bar{\psi}_L \left\{ i\mathcal{D}(a) - \frac{1}{2} \theta^{\alpha\beta}\left[ \frac{1}{2} f_{\alpha\beta} i\mathcal{D}(a) + \gamma^\rho f_{\rho\alpha} iD_\beta(a) \right] \right\} \psi_L + o(\theta^2).
\]

From now on, we shall use the notation \( f_{\alpha\beta} = \partial_\alpha a_\beta - \partial_\beta a_\alpha - i[a_\alpha, a_\beta] \).

Upon quantizing \( \psi_L \), the previous action can be used to define a noncommutative quantum field theory on a background gauge field. For technical reasons—we will employ dimensional regularization as defined in ref. [36]—and thus we shall need the Dirac propagator for describing the free propagation of the fermionic degrees of freedom, we shall use an action which gives the very same interacting theory between gauge and fermion fields as the action of eq. (4), but whose kinetic term is that of a Dirac spinor. Let \( \psi \) denote an ordinary Dirac spinor such that \( \psi_L = \frac{1}{2}(1 - \gamma_5)\psi \), where \( \psi_L \) is the left-handed spinor introduced at the beginning of this section and \( \gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 \). We shall define the action on \( \psi \) of the BRS operator \( s \) as follows

\[
s \psi = i\lambda \mathcal{P}_- \psi,
\]

\[
s \psi = i\lambda \mathcal{P}_- \psi,
\]
where \( P_- = \frac{1}{2} (1 - \gamma_5) \).

Let \( \Psi \) be a noncommutative spinor which is a solution to the following Seiberg-Witten problem:

\[
\begin{align*}
    s\Psi(a, \psi, \theta) &= i \Lambda \Psi, \\
    \Psi(a, \psi, \theta = 0) &= \psi.
\end{align*}
\]

(9)

Modulo BRS transformations and field redefinitions, the solution to the previous equation reads

\[
\begin{align*}
    \Psi(a, \psi, \theta) &= \psi - \frac{1}{2} \theta^{\alpha \beta} a_{\alpha} P_\beta \psi + \frac{i}{8} \theta^{\alpha \beta} [a_{\alpha}, a_{\beta}] P_\beta \psi + o(\theta^2). \tag{10}
\end{align*}
\]

Let the noncommutative action describing the interaction between \( \Psi \) in eq. (10) and \( A_\mu \) in eq. (3) be given by

\[
S^{(-)}_{nc} = \int d^4x \bar{\Psi} i \tilde{D}(A) \Psi. \tag{11}
\]

The symbol \( \tilde{D}(A) \) denotes the following operator

\[
\tilde{D}(A) \Psi = \partial / \Psi - i A / \star P_- \Psi.
\]

The action in eq. (11), with \( A_\mu \) and \( \Psi \) given in eqs. (3) and (10), defines the same interacting theory as the action in eq. (1), with \( A_\mu \) and \( \Psi_L \) as in eq. (3), since \( \psi_R = \frac{1}{2} (1 + \gamma_5) \psi \) does not couple to the gauge field \( a_\mu \). Up to first order in \( \theta \), \( S^{(-)}_{nc} \) reads

\[
S^{(-)}_{nc} = \int d^4x \bar{\psi} \left\{ i \partial / \psi + \frac{1}{2} \theta^{\alpha \beta} \left[ \frac{1}{2} f_{\alpha \beta \gamma} \gamma^{\gamma} f_{\rho \delta} \partial (a_{\rho} P_- + \gamma^{\rho} f_{\rho \alpha} D_{\beta} (a_{\beta}) P_-) \right] \right\} \psi + o(\theta^2). \tag{12}
\]

Note that if we do perturbation theory with \( S^{(-)}_{nc} \), the free propagator for \( \psi \) is that of Dirac’s. Also note that, as in ordinary Quantum Field Theory [37], one can use \( S^{(-)}_{nc} \) to define the Wick rotated counterpart of the path integral

\[
\int d\bar{\psi} d\psi \ e^{iS^{(-)}_{nc}}
\]

as the determinant of the operator

\[
\mathcal{O} = i \partial + \star P_- - \frac{1}{2} \theta^{\alpha \beta} \left[ \frac{1}{2} f_{\alpha \beta \gamma} \gamma^{\gamma} f_{\rho \delta} \partial (a_{\rho} P_- + \gamma^{\rho} f_{\rho \alpha} D_{\beta} (a_{\beta}) P_-) \right].
\]

\( \mathcal{O} \) has a well-defined eigenvalue problem, at least at first order in \( \theta \), over Dirac spinors on Euclidean space. Let us remark that if we had used \( S_{nc} \) in eq. (7) instead of \( S^{(-)}_{nc} \), this definition of the path integral of the theory would have had no meaning: the operator in \( S_{nc} \) maps left-handed spinors to right-handed spinors so that its Euclidean version has no eigenvalue problem.
Now we come to one of chief issues in this paper: the choice of a regularization that does not spoil the noncommutative origin of the theory whose action is $S_{nc}^{(-)}$ in eq. (12). Since we lack a characterization of the noncommutative origin of the theory that only involved the ordinary fields $a_\mu$ and $\psi$ —e.g., some equation to be satisfied by the 1PI functional of the theory when expressed in terms of the ordinary fields—, the best we can do is to formulate an action in terms of the noncommutative fields which yields upon application of the Seiberg-Witten map a regularized action —i.e., an action which gives rise to regularized Feynman diagrams of $a_\mu$ and $\psi$. We shall do this by using Dimensional Regularization as systematized by the authors of ref. [36]. We shall thus use a non-anticommuting $\gamma_5$ object and employ the “hat-and-bar” notation of ref. [36]—see also refs. [38, 39].

We shall define the object $\theta^{\mu\nu}$ in dimensional regularization as an algebraic object which satisfies

$$\theta^{\mu\nu} = -\theta^{\nu\mu}, \quad \theta^{\mu\rho}\hat{g}_{\rho\nu} = 0.$$  

We introduce now the noncommutative regularized action

$$S_{nc,DR}^{(-)} = \int d^{2\omega}x \, \bar{\Psi} \ast \{ \partial /\Psi - iA_\mu \gamma^\mu \ast P_- \Psi \}.$$  \hspace{1cm} (13)$$

Let us generalize next to the $2\omega$-dimensional space of Dimensional Regularization the BRS transformations in four dimensions of $a_\mu$, $\lambda$, $\psi$, $A_\mu$, $\Lambda$ and $\Psi$ —see eqs. (2), (5), (8) and (9). We shall choose a straightforward generalization of the latter so that the BRS transformations look the same in “$2\omega$-dimensions” as in four. Hence, the Seiberg-Witten equations in the $2\omega$-dimensional space of dimensional regularization read

$$sA_\mu = D_\mu(A)\Lambda, \quad s\Psi = i\Lambda \ast P_- \Psi \quad \text{and} \quad s\Lambda = i\Lambda \ast \Lambda,$$  \hspace{1cm} (14)$$

where $A_\mu = A_\mu(a, \theta)$, $\Psi = \Psi(a, \psi, \theta)$ and $\Lambda = \Lambda(a, \lambda, \theta)$, and $s$ acts on the ordinary fields in “$2\omega$-dimensions” as it does on their counterparts in four dimensions:

$$sa_\mu = D_\mu(a)\lambda, \quad s\psi = i\lambda P_- \psi \quad \text{and} \quad s\lambda = i\lambda \lambda.$$  \hspace{1cm} (15)$$

The previous Seiberg-Witten equations —eq. (14)— solved for the appropriate boundary conditions —i.e., $A_\mu(a, \theta = 0) = a_\mu$, $\Psi(a, \psi, \theta = 0) = \psi$ and $\Lambda(a, \lambda, \theta = 0) = \lambda$— yield the Seiberg-Witten map in the $2\omega$-dimensional space of Dimensional Regularization. It is apparent that modulo field redefinitions and BRS transformations the Seiberg-Witten map obtained
from eqs. (14) will look the same as in four dimensions:

\[ A_\mu(a, \theta) = a_\mu - \frac{1}{4} \theta^{\alpha \beta} \{ a_\alpha, f_\beta \mu \} + \partial_\beta a_\mu + o(\theta^2), \]

\[ \Psi(a, \psi, \theta) = \psi - \frac{1}{2} \theta^{\alpha \beta} a_\alpha P_\alpha \partial_\beta \psi + \frac{1}{8} \theta^{\alpha \beta} [a_\alpha, a_\beta] P_\alpha \psi + o(\theta^2), \]

\[ \Lambda(a, \lambda, \theta) = \lambda + \frac{1}{4} \theta^{\alpha \beta} \{ \partial_\alpha \lambda, a_\beta \} + o(\theta^2). \]

Every object in the previous equations is an algebraic object in the 2\omega-dimensional space of dimensional regularization.

Now, substituting eq. (16) in eq. (13) and expanding at first order in \( \theta \), one obtains a regularized version of \( S_{nc}^{(-)} \) in eq. (12):

\[ S_{nc, DR}^{(-)} = \bar{S}_{nc} + \hat{S}_{nc}, \]

\[ \bar{S}_{nc} = \int d^{2\omega} x \bar{\psi} \left( i \partial /\psi + \bar{\phi} P_\alpha - \frac{1}{2} \theta^{\alpha \beta} \left[ \frac{1}{2} f_{\alpha \beta \mu} \bar{\gamma}_\mu \right] P_\alpha + \bar{\gamma}^\rho f_{\rho \alpha \beta} D_\beta(a) P_\alpha \right) \psi \]

\[ \hat{S}_{nc} = -\frac{i}{2} \theta^{\alpha \beta} \int d^{2\omega} x \bar{\psi} \left[ \partial_\alpha a_\beta + a_\beta \partial_\alpha - i a_\alpha a_\beta \right] \bar{\theta} P_+ \psi \]

\[ + \frac{i}{2} \theta^{\alpha \beta} \int d^{2\omega} x \bar{\psi} \left[ \bar{\partial} a_\beta \partial_\alpha \right] \psi \left[ \bar{\partial} a_\alpha a_\beta + a_\alpha \bar{\partial} a_\beta \right] + (a_\beta \partial_\alpha + i a_\alpha a_\beta) \bar{\theta} \right] P_\alpha \psi. \]

We have used the following notation: \( \bar{\phi} = a_\mu \bar{\gamma}^\mu \), \( \bar{D}(a) = \bar{\gamma}^{\alpha} D_\alpha(a) \) and \( \bar{\theta} = \bar{\gamma}^{\mu} \partial_\mu \).

Furnished with the action in eq. (17) and employing standard Feynman diagram techniques, we can set up a dimensionally regularized perturbative quantum field theory. Explicit computations will be carried out below.

Before we close this section we would like to make two comments. First, let us stress that by using a regularized action for the ordinary fields –the action in eq. (17)– which comes via the Seiberg-Witten map from a noncommutative object –the action in eq. (13)–, we make sure that the regularization method does not erase (partially or totally) the noncommutative origin of our theory. If the regularized action in terms of the ordinary fields could not be obtained via the Seiberg-Witten map from a noncommutative object, then there would be no guarantee that the renormalized theory based on that regularized action would have a noncommutative interpretation. However, it might well happen –as it the case in the algebraic renormalization of gauge theories– that by adding appropriate counterterms to the action of this renormalized theory, a theory having a noncommutative content can be worked out. Probably, the counterterms needed to restore, and compensate for the lack of it, the noncommutative origin of the theory will not have a noncommutative content. Of course, anomalies in the Seiberg-Witten
map might arise, if no regularized action with a noncommutative interpretation is found. To settle all these issues in a regularization independent way, we need an equation (or equations) involving only the ordinary fields which tells us when a given 1PI functional defines a theory having a noncommutative origin. We lack such an equation (or set of equations); so, for the time being the only way to proceed is as it is done in this paper. Second, the dimensionally regularized theory defined by the action in eq. (17) is not BRS invariant since \( s \) in eq. (15) fails to annihilate \( S_{nc,DR}^{(-)} \). This befits the occurrence of gauge anomalies. We shall begin to explore the consequences that this BRS-breaking brings about in the next section.

3 The anomaly in the three-point function of the gauge field

Let \( \Gamma[a, \theta] \) be the renormalized noncommutative effective action for the ordinary gauge field which is defined as follows

\[
e^{i\Gamma[a, \theta]} = \mathcal{R}\left\{ \int d\bar{\psi} d\psi e^{iS_{nc,DR}^{(-)}} \right\}.
\]

\( \mathcal{R}\{\cdots\} \) stands for renormalization of the sum of dimensionally regularized diagrams represented by the formal path integral between the curly brackets.

Since \( S_{nc}^{(-)} \) in eq. (12) is BRS invariant, one aims at constructing –by choosing appropriate counterterms– a \( \Gamma[a, \theta] \) which be BRS invariant as well, i.e.,

\[
s\Gamma[a, \theta] = 0.
\]

(18)

See eqs. (2) and (8) for the definition of \( s \). We shall show below that eq. (18) cannot hold for any finite dimensional unitary representation of any compact simple Lie group, if only noncommutative power-counting renormalizable counterterms are allowed. But we shall also show that if power-counting renormalizability is given up and the ordinary anomaly cancellation condition [40] is satisfied, then, at least the noncommutative three-point function of \( a_\mu \) is anomaly free.

For the Fourier transform of the two-point and three-point functions, eq. (18) boils down to

\[
a) \quad ip^\mu \Gamma_{\mu\nu}^{ab}(p) = 0,
\]

\[
b) \quad ip_3^\mu \Gamma_{\mu_1\mu_2\mu_3}^{a_1a_2a_3}(p_1, p_2, p_3) = f^{a_2a_3c}_{\mu_1\mu_2} \Gamma_{\mu_1\mu_2}^{a_1c}(p_1) - f^{a_3a_1c}_{\mu_2\mu_1} \Gamma_{\mu_2\mu_1}^{a_2c}(p_2),
\]

(19)
with $p_1 + p_2 + p_3 = 0$. Since any contribution involving an even number of $\gamma_5$’s to a Feynman diagram can be dimensionally regularized as in a vector-like theory, it turns out that only the parity violating contributions – contributions with an odd number of $\gamma_5$’s – to a Feynman diagram can yield truly anomalous contributions. Hence, if $a$) and $b$) in eq. (19) are violated, it is the contributions to $\Gamma_{\mu}^{ab}(p)$ and $\Gamma_{\mu_1\mu_2\mu_3}^{\alpha\beta\gamma}(p_1, p_2, p_3)$ involving $\varepsilon_{\mu_1\mu_2\mu_3\mu_4}$ which give rise to such a breaking of BRS invariance.

Let

$$\Gamma_{DR}^{(\text{odd}) \ ab \ \mu}(p) \quad \text{and} \quad \Gamma_{DR}^{(\text{odd}) \ a_1 a_2 a_3 \ \mu_1 \mu_2 \mu_3}(p_1, p_2, p_3)$$

denote, respectively, the dimensionally regularized contributions to the noncommutative two-point and three-point functions of $a_\mu$ which depend on $\varepsilon_{\mu_1\mu_2\mu_3\mu_4}$. These regularized contributions are calculated with the action in eq. (17). We have found that up to first order in $\theta$ the two-point function reads

$$\Gamma_{DR}^{(\text{odd}) \ ab \ \mu}(p) = 0 + o(\omega - 2). \quad (20)$$

As for the three-point function we have obtained the following results

$$i p_3^{\mu_3} \Gamma_{DR}^{(\text{odd}) \ a_1 a_2 a_3 \ \mu_1 \mu_2 \mu_3}(p_1, p_2, p_3) = i p_3^{\mu_3} \Gamma_{DR}^{(\text{odd}) \ a_1 a_2 a_3 \ \mu_1 \mu_2 \mu_3}(p_1, p_2, p_3)_{\text{triangle}} + i p_3^{\mu_3} \Gamma_{DR}^{(\text{odd}) \ a_1 a_2 a_3 \ \mu_1 \mu_2 \mu_3}(p_1, p_2, p_3)_{\text{swordfish}} + i p_3^{\mu_3} \Gamma_{DR}^{(\text{odd}) \ a_1 a_2 a_3 \ \mu_1 \mu_2 \mu_3}(p_1, p_2, p_3)_{\text{jellyfish}}, \quad (21)$$

where

$$i p_3^{\mu_3} \Gamma_{DR}^{(\text{odd}) \ a_1 a_2 a_3 \ \mu_1 \mu_2 \mu_3}(p_1, p_2, p_3)_{\text{triangle}} = \text{Tr} \left\{ \left( T^{a_1}, T^{a_2} \right) T^{a_3} \right\} \frac{i}{24\pi^3} \varepsilon_{\mu_1 \mu_2 \rho \sigma} p_1^\rho p_2^\sigma,$$

$$-i \text{Tr} \left[ \left\{ T^{a_1}, T^{a_2} \right\} T^{a_3} \right] \left\{ I(p_2^2) \left( \varepsilon_{\mu_1 \mu_2 \rho \sigma} p_1^\rho p_2^\sigma + \varepsilon_{\mu_1 \rho \sigma \tau} p_1^\rho p_2^\sigma \theta_{\mu_2}^\tau \right) + I(p_2^2) \left( \varepsilon_{\mu_1 \mu_2 \rho \sigma} p_2^\rho p_3^\sigma + \varepsilon_{\mu_2 \rho \sigma \tau} p_2^\rho p_3^\sigma \theta_{\mu_2}^\tau \right) + \frac{i}{16\pi^3} \varepsilon_{\mu_1 \mu_2 \rho \sigma} p_1^\rho p_2^\sigma \theta_{\alpha \beta} p_1^\alpha p_2^\beta \right\} + o(\omega - 2),$$

$$i p_3^{\mu_3} \Gamma_{DR}^{(\text{odd}) \ a_1 a_2 a_3 \ \mu_1 \mu_2 \mu_3}(p_1, p_2, p_3)_{\text{swordfish}} =$$

$$+ i \text{Tr} \left[ \left\{ T^{a_1}, T^{a_2} \right\} T^{a_3} \right] \left\{ I(p_2^2) \left( \varepsilon_{\mu_1 \mu_2 \rho \sigma} p_1^\rho p_3^\sigma + \varepsilon_{\mu_1 \rho \sigma \tau} p_1^\rho p_3^\sigma \theta_{\mu_2}^\tau \right) + I(p_2^2) \left( \varepsilon_{\mu_1 \mu_2 \rho \sigma} p_2^\rho p_3^\sigma + \varepsilon_{\mu_2 \rho \sigma \tau} p_2^\rho p_3^\sigma \theta_{\mu_2}^\tau \right) \right\} + o(\omega - 2),$$

$$i p_3^{\mu_3} \Gamma_{DR}^{(\text{odd}) \ a_1 a_2 a_3 \ \mu_1 \mu_2 \mu_3}(p_1, p_2, p_3)_{\text{jellyfish}} = 0.$$
Here, every \( o(\theta^2) \)-contribution has been dropped and the following shorthand has been adopted

\[
\tilde{p}^\mu = \theta^\mu\nu \, p^\nu, \quad \Gamma(p^2) = \frac{i}{192\pi^2} \left( \frac{1}{\omega - 2} + \ln \left( -\frac{p^2}{4\pi \mu^2} \right) + \gamma - \frac{8}{3} \right) p^2.
\] (23)

Further details can be found in the Appendix. Eqs. (21) and (22) lead finally to

\[

i p^\mu_3 \, \Gamma^{(\text{odd})}_{\mu_1\mu_2\mu_3} (p_1, p_2, p_3) =
\frac{i}{24\pi^2} \left( \text{Tr} \left( \{T^a_1, T^a_2\} T^a_3 \right) - \frac{\pi}{3} \theta_{\alpha\beta} p^\alpha_1 p^\beta_2 \text{Tr} \left( [T^a_1, T^a_2] T^a_3 \right) \right) \epsilon_{\mu_1\mu_2\rho\sigma} p^\rho_1 p^\sigma_2 + O(\omega - 2).
\] (24)

We should stress that the simplicity of the this eq. belies the complexity of the intermediate computations involved in its calculation. It is the fact that \( S^{(-)}_{nc,DR} \) in eq. (17) is obtained from \( S^{(-)}_{nc,DR} \) in eq. (13) upon Seiberg-Witten mapping –the Seiberg-Witten connection between noncommutative and commutative fields being thus manifestly preserved– which should be held responsible for the simplicity of the final outcome –eq. (24)– of our computations. Indeed, it can be seen in the Appendix that if we define the regularized theory by employing just \( \tilde{S} \) in eq. (17) instead of using the full \( S^{(-)}_{nc,DR} \) disaster sets in. The \( o(\theta) \)-contribution to the left hand side of eq. (24) becomes –see Appendix– the following ugly expression:

\[
\text{Tr} \left( \{T^a_1, T^a_2\} T^a_3 \right) \frac{i}{24\pi^2} \epsilon_{\mu_1\mu_2\rho\sigma} p^\rho_1 p^\sigma_2 \\
+ \text{Tr} \left( [T^a_1, T^a_2] T^a_3 \right) \frac{1}{16\pi^2 (\omega - 2)} \left( \frac{1}{24} \right) \left\{ \epsilon_{\mu_1\rho\sigma\tau} p^\rho_1 p^\sigma_2 \theta_{\mu_2\tau} \left( \tilde{p}^2_3 + \tilde{p}^3_2 \right) - \epsilon_{\mu_2\rho\sigma\tau} p^\rho_1 p^\sigma_2 \theta_{\mu_1\tau} \left( \tilde{p}^2_3 + \tilde{p}^3_2 \right) - \epsilon_{\mu_1\rho\sigma\tau} p^\rho_1 p^\sigma_2 \theta_{\mu_2\tau} \left( \tilde{p}^2_3 + \tilde{p}^3_2 \right) - \epsilon_{\mu_2\rho\sigma\tau} p^\rho_1 p^\sigma_2 \theta_{\mu_1\tau} \left( \tilde{p}^2_3 + \tilde{p}^3_2 \right) \right\} \\
- \text{Tr} \left( [T^a_1, T^a_2] T^a_3 \right) \frac{1}{16\pi^2} \left( \frac{1}{24} \right) \left( \epsilon_{\mu_1\rho\sigma\tau} p^\rho_1 p^\sigma_2 \theta_{\mu_2\tau} + \epsilon_{\mu_2\rho\sigma\tau} p^\rho_1 p^\sigma_2 \theta_{\mu_1\tau} \right) \\
+ \frac{1}{16\pi^2} \left( \frac{1}{24} \right) \epsilon_{\mu_1\mu_2\rho\sigma} p^\rho_1 p^\sigma_2 \theta_{\alpha\beta} p^\alpha_1 p^\beta_2 + O(\omega - 2),
\] (25)

where \( p_1 + p_2 + p_3 = 0 \), \( \tilde{p}^\mu_i = \bar{g}^\mu\nu p_i \nu \), \( i = 1, 2, 3 \), \( \hat{p}^\mu = \bar{g}^\mu\nu p_i \nu \) and \( \bar{p}^\mu_i = \theta^\mu\nu p_3 \nu \). Notice that the difference between eq. (25) and the right hand side of eq. (24) is nonetheless a local expression; as corresponds to the fact that they come from different regularizations of the same theory. General theorems in renormalization theory tell us that one can retrieve eq. (24) from
by introducing appropriate local counterterms of the field \( a_\mu \) and its derivatives. We shall not be concerned with the actual value of these counterterms, but we shall point out that the coefficient of the term

\[
\text{Tr} \left( [T^{a_1}, T^{a_2}] T^{a_3} \right) \varepsilon_{\mu_1 \mu_2 \rho \sigma} p_1^\rho p_2^\sigma \theta_{\alpha \beta} p_1^\alpha p_2^\beta
\]  

is not same in both equations. Hence, there must exist a local polynomial of \( a_\mu \) (and its derivatives) whose BRS variation yields a contribution proportional to the expression in eq. (26). This casts doubts on the \( \theta \)-dependent term of eq. (24) as being a truly anomalous contribution. We shall analyse this issue below.

Eqs. (19), (20) and (24) leads to the following candidate for anomalous BRS equation

\[
s\Gamma[A[a, \theta], \theta] = -\frac{i}{24\pi^2} \int d^4x \varepsilon^{\mu_1 \mu_2 \mu_3 \mu_4} \text{Tr} \left( \partial_{\mu_1} \Lambda A_{\mu_2} \partial_{\mu_3} A_{\mu_4} \right) + \frac{1}{48\pi^2} \int d^4x \varepsilon^{\mu_1 \mu_2 \mu_3 \mu_4} \theta^\alpha \beta \text{Tr} \left( \partial_{\mu_1} \Lambda \partial_{\alpha} a_{\mu_2} \partial_{\beta} a_{\mu_4} \right) + o(a^3) + o(\theta^2)
\]

where \( \Lambda = \Lambda[\lambda, a, \theta] \) and \( A_\mu = A_\mu[a, \theta] \) are defined in eq. (3). And yet, for the right hand side of the previous equation to be a true anomaly, we must show that there is no integrated \( * \)-polynomial of the noncommutative field \( A(a, \theta) \) and its derivatives –let us call it \( \Gamma_{ct}[A, \theta] \) – such that its BRS variation, \( s\Gamma_{ct}[A, \theta] \), is, upon applying the Seiberg-Witten map, equal to the right hand side of eq. (27) up to first order in \( \theta \) and up to two fields \( a_\mu \). If only a renormalizable by power-counting at the noncommutative level \( \Gamma_{ct}[A, \theta] \) is allowed, then there is only one \( \Gamma_{ct}[A, \theta] \) that might do the job, namely

\[
\Gamma_{ct}[A[a, \theta], \theta] = c \int d^4x \varepsilon^{\mu_1 \mu_2 \mu_3 \mu_4} \text{Tr} \left( \partial_{\mu_1} A_{\mu_2} A_{\mu_3} A_{\mu_4} \right).
\]

\( c \) is an appropriate number and, again, \( \Lambda = \Lambda[\lambda, a, \theta] \) and \( A_\mu = A_\mu[a, \theta] \) are as in eq. (3). Unfortunately, for this \( \Gamma_{ct}[A[a, \theta], \theta] \), we have

\[
s\Gamma_{ct}[A[a, \theta], \theta] = s_* \Gamma_{ct}[A[a, \theta], \theta] = 0 + o(A^3),
\]

where \( s \) and \( s_* \) are defined by eqs. (3) and (3), respectively. Hence, if we want to save renormalizability by power-counting at the noncommutative level, the only way that the right hand side of eq (27) would vanish is that

\[
\text{Tr} [T^a, T^b] T^c = 0 \quad \text{and} \quad \text{Tr} \{ T^a, T^b \} T^c = 0.
\]
And thus, unlike in ordinary space-time, the theories we are considering present a breach of gauge invariance even if the ordinary condition

$$\text{Tr}\{T^a, T^b\} T^c = 0$$  \hfill (29)

is satisfied by the representation of the simple gauge group carried by the matter content of the theory. This result leads immediately to the conclusion that the $SU(2)$ part of the noncommutative standard model of ref. [7] and the noncommutative $SU(5)$ and $SO(10)$ models of ref. [8] cannot be rendered anomaly free, if power-counting renormalizability is not given up at the noncommutative level. However, to demand that noncommutative field theories – at least if they are defined by means of the Seiberg-Witten – be renormalizable by power-counting seems to be too strong a requirement and not in keeping with current ideas on the renormalizability of gauge theories. Indeed, on the one hand, even noncommutative QED fails to be renormalizable by power-counting, as shown in ref. [10]; and, on the other hand, if we adopt the effective field theory viewpoint, there is nothing wrong with losing power-counting renormalizability provided BRS invariance is preserved [41]. If we give up the power-counting-renormalizability paradigm, an interesting phenomenon takes place: the term in eq. (27) linear in $\theta$ and involving two gauge fields can be canceled by adding to the classical noncommutative action of our theory an appropriate counterterm, $\Gamma_{ct}[A[a, \theta], \theta]$. It is not difficult to show that

$$\Gamma_{ct}[A[a, \theta], \theta] = -\frac{1}{48\pi^2} \int d^4x \; \varepsilon^{\mu_1 \mu_2 \mu_3 \mu_4} \theta^{\alpha \beta} \text{Tr} \left( \partial_{\alpha} \partial_{\mu_1} A_{\mu_2} \star \partial_{\mu_3} A_{\mu_4} \star A_{\beta} \right)$$  \hfill (30)

satisfies

$$s\Gamma_{ct}[A[a, \theta], \theta] = -\frac{1}{48\pi^2} \int d^4x \; \varepsilon^{\mu_1 \mu_2 \mu_3 \mu_4} \theta^{\alpha \beta} \text{Tr} \left( \partial_{\alpha} \lambda \partial_{\mu_1} a_{\mu_2} \partial_{\mu_3} \partial_{\beta} a_{\mu_4} \right) + o(a^3) + o(\theta^2).$$  \hfill (31)

Hence, we may define a new renormalized action

$$\Gamma_{new}[A[a, \theta], \theta] = \Gamma[A[a, \theta], \theta] + \Gamma_{ct}[A[a, \theta], \theta]$$  \hfill (32)

satisfying

$$s\Gamma_{new}[A[a, \theta], \theta] = -\frac{i}{24\pi^2} \int d^4x \; \varepsilon^{\mu_1 \mu_2 \mu_3 \mu_4} \text{Tr} \left( \partial_{\mu_1} \lambda a_{\mu_2} \partial_{\mu_3} a_{\mu_4} \right) + o(a^3) + o(\theta^2).$$  \hfill (33)

The latter equation implies that the anomaly cancellation condition for the noncommutative three-point function of the gauge field $a_{\mu}$ is the ordinary one given in eq. (29). Now the
SU(2) part of the noncommutative standard model of ref. [7] and the noncommutative SU(5) and SO(10) models of ref. [8] carry no anomaly in the noncommutative three-point function of the ordinary gauge fields, since their fermion representations satisfy eq. (29). Whether the cancellation mechanism discussed above can be extended to the remaining Green functions and at any order in $\theta$ shall not be discussed here. Indeed, any feasible way of proving or disproving it shall require the extension of the theorems on local BRS cohomology in ref. [42] to the case at hand. For further comments the reader is referred to the last two paragraphs of the Appendix.

Let us close this section by rewriting eq. (33) in terms of the noncommutative fields:

$$s \Gamma_{\text{new}}[A, \theta] = -\frac{i}{24\pi^2} \int d^4x \varepsilon^{\mu_1 \mu_2 \mu_3 \mu_4} \text{Tr} \left( \partial_{\mu_1} \Lambda \times A_{\mu_2} \partial_{\mu_3} \times A_{\mu_4} \right)$$

$$- \frac{1}{48\pi^2} \text{Tr} \int d^4x \varepsilon^{\mu_1 \mu_2 \mu_3 \mu_4} \theta^{\alpha \beta} \text{Tr} \left( \partial_{\mu_1} \Lambda \times \partial_{\alpha} \times A_{\mu_2} \partial_{\beta} \times A_{\mu_4} \right) + o(a^3) + o(\theta^2),$$

where $\Lambda = \Lambda[\lambda, a, \theta]$ and $A_{\mu} = A_{\mu}[a, \theta]$ as in eq. (3).

4 The gauge anomaly and the ambiguity of the Seiberg-Witten map

The issue we shall address in this section is the change, if any, of the results presented in the previous section induced by the freedom in choosing a concrete realization of the Seiberg-Witten map. Using the techniques of ref. [23], it is not difficult to show that the most general solution [26, 43, 8, 9] to eqs. (6) and (9) –the Seiberg-Witten equations– are the following:

$$\Lambda^{(\text{gen})}(a, \lambda, \theta) = \Lambda(a, \lambda, \theta) + (2 \kappa_1 - i \kappa_1) \theta^{\alpha \beta}[a_\alpha, \partial_\beta \lambda] + o(\theta^2),$$

$$A^{(\text{gen})}_\mu(a, \theta) = A_\mu(a, \theta) + \kappa_4 \theta^{\alpha \beta} D_\mu(a) f_{\alpha \beta} + \kappa_4 \theta^{\alpha \beta} D^\rho(a) f_{\rho \beta} + (\kappa_2 - \frac{i}{2} \kappa_1) \theta^{\alpha \beta} D_\mu(a)[a_\alpha, a_\beta] + o(\theta^2),$$

$$\Psi^{(\text{gen})}(a, \psi, \theta) = \Psi(a, \psi, \theta) + t^{\mu \nu}(\eta, \gamma^\sigma, \theta) D_\mu(a) D_\nu(a) P_- \psi + t^\mu_+(\eta, \gamma^\sigma, \theta) \partial_\mu \partial_\nu P_+ \psi + i \kappa_1 \theta^{\alpha \beta} D_\alpha(a) a_\beta P_- \psi + i \kappa_2 \theta^{\alpha \beta}[a_\alpha, a_\beta] P_- \psi + o(\theta^2).$$

(35)

$\Lambda(a, \lambda, \theta)$, $A_\mu(a, \theta)$ and $\Psi(a, \psi, \theta)$ are as in eqs. (3) and (10), respectively. In the equation above, $t^{\mu \nu}(\eta, \gamma^\sigma, \theta)$ and $t^\mu_+(\eta, \gamma^\sigma, \theta)$ are arbitrary Lorentz tensors constructed out of the
Minkowski metric, $\eta_{\mu\nu}$, the Moyal matrix $\theta^{\mu\nu}$ and the Dirac matrices $\gamma_\mu$. These tensors take values on the Clifford algebra generated by $\gamma_\mu$, but their actual values will be of no relevance to our discussion. Note that we are taking for granted that $\Lambda^{(\text{gen})}(a,\lambda,\theta)$, $A_\mu^{(\text{gen})}(a,\theta)$, $\Psi_L^{(\text{gen})} = P_- \Psi^{(\text{gen})}(a,\psi,\theta)$ and $\Psi_R^{(\text{gen})} = P_+ \Psi^{(\text{gen})}(a,\psi,\theta)$ transforms under parity as their ordinary counterparts do [8], this is why the Levi-Civita pseudotensor does not occur in eq. (35). $\kappa_1$, $\kappa_2$, $\kappa_3$ and $\kappa_4$ are numbers. The requirement that $\Lambda^{(\text{gen})}(a,\lambda,\theta)$ and $A_\mu^{(\text{gen})}(a,\theta)$ be hermitian imposes obvious constraints on these numbers [8].

The regularized effective action, $\Gamma[A^{(\text{gen})}[a,\theta], \theta]_{\text{DR}}$, of the noncommutative theory in dimensional regularization is given by the diagrammatic expansion of the following path integral

$$e^{i\Gamma[A^{(\text{gen})}[a,\theta], \theta]_{\text{DR}}} = \int d\bar{\psi} d\psi e^{iS_{\text{nc,DR}}(A^{(\text{gen})}[a,\theta], \Psi^{(\text{gen})}[a,\psi,\theta], \bar{\Psi}^{(\text{gen})}[a,\bar{\psi},\theta])}.$$  

(36)

Now $S_{\text{nc,DR}}(a,\psi,\bar{\psi})$ is obtained by substituting first eq. (35) in eq. (13) and then expanding the result in powers of $\theta$. As in section 2, the Seiberg-Witten map in the $2\omega$-dimensional space of Dimensional Regularization is obtained by replacing each object in eq (35) with its counterpart in the Dimensional Regularization scheme systematized in ref. [36].

In this section we will not compute explicitly the Feynman diagrams with $a_\mu$ in the external legs that may give anomalous contributions to the noncommutative three-point function of the latter gauge field. Rather, we shall take advantage of the fact that our Dimensional Regularization scheme satisfies the Quantum Action Principle –see [36]– and use the path integral as much as possible. Indeed, any formal manipulation of the path integral in eq. (36) is mathematically sound when spelt out in terms of Feynman diagrams. Now, the following change of fermionic variables

$$\Upsilon = (\mathbb{I} + M(a,\theta,\partial)) \psi,$$

(37)

where the operator $M(a,\theta,\partial)$ is given by

$$M(a,\theta,\partial) = \left[ -\frac{1}{2} \theta^{\alpha\beta} a_\alpha \partial_\beta + i(\frac{1}{8} + \kappa_2) \theta^{\alpha\beta}[a_\alpha, a_\beta] + i\kappa_1 \theta^{\alpha\beta} D_\alpha(a) a_\beta + t^{\mu\nu}(\theta) D_\mu(a) D_\nu(a) \right] P_- + t^+_{\mu\nu}(\theta) \partial_\mu \partial_\nu P_+,$$

(38)

leaves invariant the path integral in eq. (36). Hence, the following equation holds up to first
order in $\theta$:

$$\int d\bar{\psi} d\psi \ e^{iS^{(-)}_{nc, DR}(A^{(gen)}[a, \theta], \Psi^{(gen)}[a, \psi, \theta], \bar{\Psi}^{(gen)}[a, \bar{\psi}, \theta])} =$$

$$\int d\bar{\Upsilon} d\Upsilon \ \det \left[ \mathbb{I} + M(a, \theta, \partial) \right] \ \det \left[ \mathbb{I} + \bar{M}(a, \theta, \partial) \right] \ e^{iS^{(-)}_{nc, DR}(A^{(gen)}[a, \theta], \Upsilon, \bar{\Upsilon})}. \quad (39)$$

The determinants in the previous equation are defined as the sum of the appropriate dimensionally regularized Feynman diagrams; $M(a, \theta, \partial)$ and $\bar{M}(a, \theta, \partial)$ being understood as perturbations of $\mathbb{I}$. The operator $\mathbb{I} + \bar{M}(a, \theta, \partial)$ yields the change of $\bar{\psi}$, $\bar{\Upsilon} = (\mathbb{I} + \bar{M}(a, \theta, \partial)) \bar{\psi}$, induced by the change of $\psi$ in eq. (37).

In Dimensional Regularization, we have

$$\det \left[ \mathbb{I} + M(a, \theta, \partial) \right] = 1 = \det \left[ \mathbb{I} + \bar{M}(a, \theta, \partial) \right],$$

for the diagrammatic expansions of these determinants always lead to integrals of the type

$$\int \frac{d^{2\omega}p}{(2\pi)^{2\omega}} \ p_{\mu_1} \cdots p_{\mu_n}.$$ 

These integrals vanish in Dimensional regularization. We thus come to the conclusion that in perturbation theory and at first order in $\theta$ $\Gamma[A^{(gen)}[a, \theta]]_{DR}$ in eq. (36) is also given by the diagrammatic expansion of

$$\frac{1}{i} \ln \left\{ \int d\bar{\Upsilon} d\Upsilon \ e^{iS^{(-)}_{1, DR}(A^{(gen)}[a, \theta], \Upsilon, \bar{\Upsilon}, \theta)} \right\}, \quad (40)$$

with $S^{(-)}_{1, DR}(A^{(gen)}[a, \theta], \Upsilon, \bar{\Upsilon}, \theta)$ being given by the expansion of

$$S^{(-)}_{nc, DR}(A^{(gen)}[a, \theta], \Upsilon, \bar{\Upsilon}) = \int d^{2\omega}x \ \bar{\Upsilon} \star \left\{ \partial \Psi - (iA^{(gen)}_{\mu}[a, \theta]) \bar{\gamma}^{\mu} \star P \Upsilon \right\} \quad (41)$$

up to first order in $\theta$. The latter expansion reads

$$S^{(-)}_{1, DR}(A^{(gen)}[a, \theta], \Upsilon, \bar{\Upsilon}, \theta) = \int d^{2\omega}x \ \bar{\Upsilon} \{ iA^{(gen)}_{\mu}[a, \theta] - \frac{1}{2} \theta^{\alpha\beta} \partial_{\alpha}a_{\mu\beta} \} \bar{\gamma}^{\mu} P \Upsilon. \quad (42)$$

It is understood here that $A^{(gen)}_{\mu}[a, \theta]$ is given by the right hand side of the second equation in eqs. (33), provided we forget about the $o(\theta^2)$ contributions.

In view of eqs. (39), (41) and (42) one concludes that $\Gamma[A^{(gen)}[a, \theta]]_{DR}$ can be obtained from the diagrams contributing to the noncommutative $U(n)$ with a left-handed fermion –see refs. [44, 45] – as follows:
i) Take a diagram contributing to the effective action of the noncommutative $U(n)$ theory in question. Such a diagram, which is always planar, has the generic form
\[ \text{Tr}_{U(n)} \int d^{2n}x_1 \cdots d^{2n}x_n A_{\mu \alpha}(x_{\pi(1)}) \cdots A_{\mu \alpha}(x_{\pi(n)}) \Gamma^{\mu_1 \cdots \mu_n}(x_1, \cdots, x_n; \theta), \]
with $\pi(1) \cdots \pi(n)$ being a appropriate permutation of $1 \cdots n$ and with
\[ \Gamma^{\mu_1 \cdots \mu_n}(x_1, \cdots, x_n; \theta) = (-1)^{n+1} \int \prod_{i=1}^{n} \frac{d^{2n}p_i}{(2\pi)^{2n}} e^{-i \sum_{i=1}^{n} p_i \cdot x_i} e^{-\frac{i}{2} \sum_{1 \leq i < j < n} \theta^{\alpha \beta} p_i a_{\alpha} p_j a_{\beta}} 
(2\pi)^{2n} \delta(p_1 + \cdots + p_n) \int \frac{d^{2n}q}{(2\pi)^{2n}} \frac{\text{tr}(f^{\alpha \beta} P_- (q - p_1) \cdots (q - \sum_{i=1}^{n-1} p_i) \cdots (q - p_n) \gamma^{\alpha \beta} P_-)}{q^2 (q - p_1)^2 \cdots (q - \sum_{i=1}^{n-1} p_i)^2}. \]

Then, expand at first order in $\theta$ the global Moyal phase of the diagram. Call the result \textit{Diagram}.

ii) Replace in \textit{Diagram} the noncommutative $U(n)$ field in the fundamental representation, $A_\mu$, which only occurs as a background field, with $A^{(gen)}_\mu$ defined in eq. \textcolor{red}{(33)}. And also replace $\text{Tr}_{U(n)}$, the trace in the fundamental representation of $U(n)$, with the trace in the representation of our simple gauge group. Call the result \textit{Diagram}, again.

iii) Sum over all \textit{Diagrams} obtained these way. Replace $A^{(gen)}_\mu$ with its value –given in eq. \textcolor{red}{(33)}— in terms of $a_\mu$ and get rid of any contribution of order $\theta^2$.

Note that the dimensionally regularized action of the chiral noncommutative $U(n)$ gauge theory of refs. \textcolor{red}{[14, 15]} is the action in eq. \textcolor{red}{(13)}, provided $A_\mu$ is a $U(n)$ field in the fundamental representation. Hence, the process just spelt out converts the action in eq. \textcolor{red}{(13)} for $U(n)$ into the action in eq. \textcolor{red}{(12)} for our compact simple group, $G$. So it is no wonder that $i)$–$iii)$ yields the result we sought.

After all these preparations it does not come as a surprise that, up to first order in $\theta$, the candidate for anomalous contribution to BRS variation of the renormalized action $\Gamma[A^{(gen)}[a, \theta], \theta]$, obtained from our regularized action $\Gamma[A^{(gen)}[a, \theta], \theta]_{DR}$—see eqs. \textcolor{red}{(36)} and \textcolor{red}{(40)}—reads
\[ s\Gamma[A^{(gen)}[a, \theta], \theta] = -\frac{i}{24\pi^2} \int d^4x \, \varepsilon^{\mu_1 \mu_2 \mu_3 \mu_4} \text{Tr} \left( \partial_{\mu_1} \lambda \left( a_{\mu_2} \partial_{\mu_3} a_{\mu_4} - \frac{i}{2} a_{\mu_2} a_{\mu_3} a_{\mu_4} \right) \right) 
- \frac{i}{24\pi^2} \int d^4x \, \varepsilon^{\mu_1 \mu_2 \mu_3 \mu_4} \text{Tr} \left( \partial_{\mu_1} \partial \lambda \left( a_{\mu_2} \partial_{\mu_3} a_{\mu_4} - \frac{i}{2} a_{\mu_2} a_{\mu_3} a_{\mu_4} \right) + \partial_{\mu_2} \lambda \{ \delta A_{\mu_2}, \partial_{\mu_3} a_{\mu_4} \} \right) 
- \frac{1}{48\pi^2} \int d^4x \, \varepsilon^{\mu_1 \mu_2 \mu_3 \mu_4} \text{Tr} \left( \partial_{\mu_1} \lambda \left( \{ \delta A_{\mu_2}, a_{\mu_3} \} a_{\mu_4} + a_{\mu_2} a_{\mu_3} \delta A_{\mu_4} \right) \right) 
+ \frac{1}{48\pi^2} \int d^4x \, \varepsilon^{\mu_1 \mu_2 \mu_3 \mu_4} \theta^{\alpha \beta} \text{Tr} \left( \partial_{\mu_1} \lambda \left( \partial_\alpha a_{\mu_2} \partial_\beta a_{\mu_3} \partial_\gamma a_{\mu_4} - \frac{i}{2} \{ \partial_\alpha a_{\mu_2} \partial_\beta a_{\mu_3}, a_{\mu_4} \} \right) \right). \]
(43)
The symbols \( \delta \Lambda \) and \( \delta A_\mu \) stand for

\[
\delta \Lambda = \frac{i}{4} \theta^{\alpha \beta} \{ \partial_\alpha \lambda, a_\beta \} + (2 \kappa_2 - i \kappa_3) \theta^{\alpha \beta} [a_\alpha, \partial_\beta \lambda] \quad \text{and}
\]

\[
\delta A_\mu = - \frac{1}{4} \theta^{\alpha \beta} \{ a_\alpha, f_\beta + \partial_\beta a_\mu \} + \kappa_3 \theta^{\alpha \beta} D_{\mu} f_{\alpha \beta} + \kappa_4 \theta_\mu^{\beta} D^\rho f_{\rho \beta} + (\kappa_2 - \frac{i}{2} \kappa_1) \theta^{\alpha \beta} D_\mu [a_\alpha, a_\beta],
\]

respectively. Obviously, the result in eq. (43) can be retrieved from the consistent form of the U(n) gauge anomaly obtained in refs. [44, 46], by applying the process i)–iii) and replacing each ghost field in each diagram of the U(n) theory with \( \Lambda^{(gen)}(a, \lambda, \theta) \) given in eq. (35).

Eq. (43) readily leads to

\[
s \Gamma[A^{(gen)}[a, \theta], \theta] = - \frac{i}{24 \pi^2} \int d^4x \, \varepsilon^{\mu_1 \mu_2 \mu_3 \mu_4} \text{Tr} \left( \partial_{\mu_1} \lambda a_{\mu_2} \partial_{\mu_3} a_{\mu_4} \right)
\]

\[
- \frac{i \kappa_4}{24 \pi^2} \int d^4x \, \varepsilon^{\mu_1 \mu_2 \mu_3 \mu_4} \text{Tr} \left( \partial_{\mu_1} \lambda \{ \theta^{\mu_2}_{\beta} (\partial^2 a_{\beta} - \partial_{\beta} \partial^\rho a_\rho), \partial_{\mu_3} a_{\mu_4} \} \right)
\]

\[
+ \frac{1}{48 \pi^2} \int d^4x \, \varepsilon^{\mu_1 \mu_2 \mu_3 \mu_4} \alpha^\beta \text{Tr} \left( \partial_{\mu_1} \lambda \partial_\alpha a_{\mu_2} \partial_{\mu_3} \partial_\beta a_{\mu_4} \right) + o(a^3 \lambda).
\]

Let us introduce next the nonrenormalizable noncommutative counterterm

\[
\Gamma^{(3)}_{ct}[A^{(gen)}, \theta] = \frac{i \kappa_4}{24 \pi^2} \int d^4x \, \varepsilon^{\mu_1 \mu_2 \mu_3 \mu_4} \text{Tr} \left( A^{(gen)}_{\mu_1} \ast \{ \theta^{\mu_2}_{\beta} (\partial^2 A^{(gen)}_\beta - \partial_{\beta} \partial^\rho A^{(gen)}_\rho), \partial_{\mu_3} A^{(gen)}_{\mu_4} \} \ast \right)
\]

\[
\frac{1}{48 \pi^2} \int d^4x \, \varepsilon^{\mu_1 \mu_2 \mu_3 \mu_4} \text{Tr} \left( a_{\mu_1} \{ \theta^{\mu_2}_{\beta} (\partial^2 a_{\beta} - \partial_{\beta} \partial^\rho a_\rho), \partial_{\mu_3} a_{\mu_4} \} \right) + o(a^4) + o(\theta^2),
\]

with the notation \( \{ f, g \}_s = f \ast g + g \ast f \). The BRS variation of this counterterm reads

\[
s \Gamma^{(3)}_{ct}[A^{(gen)}[a, \theta], \theta] = \frac{i \kappa_4}{24 \pi^2} \int d^4x \, \varepsilon^{\mu_1 \mu_2 \mu_3 \mu_4} \text{Tr} \left( \partial_{\mu_1} \lambda \{ \theta^{\mu_2}_{\beta} (\partial^2 a_{\beta} - \partial_{\beta} \partial^\rho a_\rho), \partial_{\mu_3} a_{\mu_4} \} \right) + o(a^3 \lambda).
\]

As we did in the previous section, we may define now a new renormalized effective action, \( \Gamma_{new}[A^{(gen)}[a, \theta], \theta] \), which satisfies eq. (33):

\[
\Gamma_{new}[A^{(gen)}[a, \theta] = \Gamma[A^{(gen)}[a, \theta], \theta] + \Gamma_{ct}[A^{(gen)}[a, \theta], \theta] + \Gamma^{(3)}_{ct}[A^{(gen)}[a, \theta], \theta].
\]

\( \Gamma_{ct}[A^{(gen)}[a, \theta], \theta] \) is obtained by replacing \( A \) with \( A^{(gen)} \) in eq. (30). For this new effective action \( \Gamma_{new}[A^{(gen)}[a, \theta], \theta] \), the anomaly cancellation condition up to order three in the number of \( a_\mu \)-fields is the ordinary cancellation condition:

\[
\text{Tr} \left[ T^a, T^b \right] T^c = 0.
\]

Hence, we conclude that at first order in \( \theta \) and at least for the three-point functions of the ordinary fields, the models formulated in refs. [7] and [8] are anomaly free. It remains to be
shown that the procedure introduced above can be successfully implemented at first order in $\theta$ for the remaining Green functions of $a_\mu$ and, then, show that it also holds at any order in $\theta$.

Note that strictly speaking the counterterm $\Gamma^{(3)}_{ct}[A^{(gen)}[a, \theta], \theta]$ above is not needed, for the would-be anomalous term it cancels vanishes upon imposing the ordinary anomaly cancellation condition. In general, it will suit our purposes to show that the sum of terms in the right hand side of eq. (43) which are not set to zero by imposing eq. (47) is BRS exact.

5 Summary and Conclusions

In this paper we have computed, using diagrammatic techniques and at first order in $\theta$, the consistent form of the gauge anomaly carried by the noncommutative three-point function of the ordinary gauge field –call it $a_\mu$– of certain noncommutative chiral gauge theories defined by means of the Seiberg-Witten map. We have considered only noncommutative theories whose ordinary matter content is a left-handed spinor carrying an arbitrary finite dimensional unitary representation of a given compact simple gauge group; the gauge group being arbitrary as well. Our computations have led to the following conclusions:

1.- If only noncommutative counterterms which are renormalizable by power-counting are admitted –in an attempt to not to spoil power-counting renormalizability–, then, there is an anomalous noncommutative correction linear in $\theta$, besides the ordinary anomalous contribution, to the ordinary gauge field three-point function. To cancel both these anomalous contributions, the two conditions in eq. (28) should be satisfied by the representation of the gauge group carried by the left-handed spinor of our theory. This is impossible. Hence, the “safe” representations and and “safe” groups [47] of ordinary gauge theories are totally “unsafe” for noncommutative space-time. Actually, “safe” representations in the sense of ref. [47] always carry, for noncommutative space-time, a gauge anomaly if the fermions of the theory all have the same type of handedness. However, if there are both left-handed and right-handed fermions in the noncommutative theory, then, anomaly freedom can be achieved provided the $\theta$ dependent piece of the right-handed and left-handed anomaly cancel each other, i.e.,

$$\text{Tr}[T^a_L, T^b_L] T^c_L = \text{Tr}[T^a_R, T^b_R] T^c_R.$$
Here we have used an obvious notation. Note that the anomalous contributions furnished by a right-handed spinor is obtained by multiplying by $-1$ the right hand side of eq. (27) --the same for eq. (43)-- so that the anomalous cancellation conditions in the case at hand read

$$\text{Tr}\{T^a_L, T^b_L\} T^c_L - \text{Tr}\{T^a_R, T^b_R\} T^c_R = 0,$$

and

$$\text{Tr}\{T^a_L, T^b_L\} T^c_L - \text{Tr}\{T^a_R, T^b_R\} T^c_R = 0.$$

Let us recall that the action of a noncommutative right-handed spinor cannot be expressed as the action of its charge conjugate left-handed spinor unless $\theta$ is replaced with $-\theta$ --see ref. [8] for further details.

2.- If renormalizability by power-counting is given up and renormalizability in the broader sense of ref. [11] is called forth, then, nonrenormalizable noncommutative counterterms can be added to the classical action to cancel any, linear in $\theta$, would-be anomalous contribution to the noncommutative three-point function of $a_\mu$ --see eqs. (30)–(34) and eqs. (44)–(46). The anomaly cancellation condition for the ordinary gauge field noncommutative three-point function is then the ordinary one given in eq (47). Hence, within the framework of effective field theory, the noncommutative models of refs. [7, 8] stand a fat chance of being anomaly free models and thus becoming phenomenologically useful. It remains to show that the procedure described above can be still carried out successfully for the other Green functions of the theory, at least upon imposing eq. (47). In other words, it remains to see whether the whole contribution linear in $\theta$ to the right hand side of eq. (13) is (perhaps for the constraint in eq. (17) ) BRS exact. The antifield formalism [12] may prove an invaluable tool in such a task, which is not altogether hopeless as shows the results presented here and the fact that the noncommutative Chern-Simons action for a noncommutative field configuration is equal to the ordinary Chern-Simons of the ordinary field configuration obtained from the former by a non-singular Seiberg-Witten map [18].

Finally, the results presented in this paper along with the analysis in ref. [10] back up the suggestion made in ref. [7] that noncommutative field theories defined by means of the Seiberg-Witten map must be formulated within the framework of effective field theories.
Figure 1: Types of diagrams involved in computation of the noncommutative three-point function of $a_\mu$: a) triangle, b) swordfish and c) jellyfish diagrams.

6 Appendix

Let $i p_3^{(\mu_3} \Gamma_{DR}^{(odd)} a_1 a_2 a_3 (p_1, p_2, p_3)_\text{triangle}$ denote the contribution to the right hand side of eq. (21) coming from the triangle-type diagrams in fig. 1. Both $\tilde{S}_{nc}$ and $\hat{S}_{nc}$ in eq. (17) contribute to this distribution. The bit of it which involves only vertices from $\tilde{S}_{nc}$ is given by

$$
\begin{align*}
&\text{Tr} \left( \{ T^{a_1}, T^{a_2} \} T^{a_3} \right) \frac{i}{24 \pi^2} \varepsilon_{\mu_1 \mu_2 \rho \sigma} p_1^\rho p_2^\sigma \\
- &i \text{Tr} \left( [T^{a_1}, T^{a_2}] T^{a_3} \right) \left\{ 1(p_1^2) \left( \varepsilon_{\mu_1 \mu_2 \rho \sigma} p_1^\rho p_3^\sigma + \varepsilon_{\mu_1 \rho \sigma \tau} p_1^\rho p_2^\sigma \theta_{\mu_2}^\tau \right) \\
&\quad + 1(p_2^2) \left( \varepsilon_{\mu_1 \mu_2 \rho \sigma} p_2^\rho p_3^\sigma + \varepsilon_{\mu_2 \rho \sigma \tau} p_1^\rho p_2^\sigma \theta_{\mu_2}^\tau \right) \\
&\quad - \frac{i}{16 \pi^2 (\omega - 2)} \left( \frac{1}{24} \right) \left( \hat{p}_2^2 + \hat{p}_3^2 - \hat{p}_2 \cdot \hat{p}_3 \right) \left( \varepsilon_{\mu_1 \mu_2 \rho \sigma} p_1^\rho p_3^\sigma + \varepsilon_{\mu_1 \rho \sigma \tau} p_1^\rho p_2^\sigma \theta_{\mu_2}^\tau \right) \\
&\quad - \frac{i}{16 \pi^2 (\omega - 2)} \left( \frac{1}{24} \right) \left( \hat{p}_1^2 + \hat{p}_3^2 - \hat{p}_1 \cdot \hat{p}_3 \right) \left( \varepsilon_{\mu_1 \mu_2 \rho \sigma} p_2^\rho p_3^\sigma + \varepsilon_{\mu_2 \rho \sigma \tau} p_1^\rho p_2^\sigma \theta_{\mu_2}^\tau \right) \\
&\quad + \frac{i}{16 \pi^2} \left( \frac{1}{24} \right) \left( p_1^2 + p_2^2 + p_1 \cdot p_2 \right) \left( \varepsilon_{\mu_1 \rho \sigma \tau} p_1^\rho p_2^\sigma \theta_{\mu_2}^\tau + \varepsilon_{\mu_2 \rho \sigma \tau} p_1^\rho p_2^\sigma \theta_{\mu_1}^\tau \\
&\quad \quad - \varepsilon_{\mu_1 \mu_2 \rho \sigma} p_1^\rho p_3^\sigma \right) \\
&\quad + \frac{i}{16 \pi^2} \left( \frac{1}{24} \right) \varepsilon_{\mu_1 \mu_2 \rho \sigma} p_2^\rho p_3^\sigma \theta_{\alpha \beta} p_1^\alpha p_1^\beta \right\} + o(\omega - 2),
\end{align*}
$$

(48)
whereas the part of it which involves only one vertex from \( S_{nc} \) reads

\[
-i \text{Tr} \left( [T^{a_1}, T^{a_2}] T^{a_3} \right) \left\{ \frac{i}{16\pi^2 (\omega - 2)} \left( \frac{1}{24} \right) (\hat{p}_1^2 + \hat{p}_2^2 - \hat{p}_1 \cdot \hat{p}_2) \left( \varepsilon_{\mu_1\mu_2\rho\sigma} p_1^\rho p_2^\sigma + \varepsilon_{\mu_1\rho\sigma\tau} p_1^\rho p_2^\sigma \theta_{\mu_2}^\tau \right) \right. \\
+ \frac{i}{16\pi^2 (\omega - 2)} \left( \frac{1}{24} \right) (\hat{p}_1^2 + \hat{p}_3^2 - \hat{p}_1 \cdot \hat{p}_3) \left( \varepsilon_{\mu_1\mu_3\rho\sigma} p_2^\rho p_3^\sigma + \varepsilon_{\mu_3\rho\sigma\tau} p_1^\rho p_2^\sigma \theta_{\mu_1}^\tau \right) \\
- \frac{i}{16\pi^2} \left( \frac{1}{24} \right) (p_1^2 + p_2^2 + p_1 \cdot p_2) \left( \varepsilon_{\mu_1\rho\sigma\tau} p_1^\rho p_2^\sigma \theta_{\mu_2}^\tau + \varepsilon_{\mu_2\rho\sigma\tau} p_1^\rho p_2^\sigma \theta_{\mu_1}^\tau \\
- \varepsilon_{\mu_1\mu_2\rho\sigma} p_1^\rho p_2^\sigma \theta_{\mu_3} \right) \\
+ \frac{i}{16\pi^2} \left( \frac{1}{12} \right) \varepsilon_{\mu_1\mu_2\rho\sigma} p_1^\rho p_2^\sigma \theta_{\alpha\beta} p_1^\alpha p_2^\beta \right) + o(\omega - 2).
\]

(49)

We are using the notation \( \hat{p}_i^\mu = \hat{g}^{\mu\nu} p_{i\nu} \), \( \tilde{p}_i^\mu = \hat{g}^{\mu\nu} p_{i\nu} \), \( i = 1, 2, 3 \), \( \tilde{p}_3^3 = 0 \), and \( p_3 = -p_1 - p_2 \). \( I(p^2) \) is given in eq. (23).

\[ i p_3^{\alpha_3} \Gamma^{(odd)}_{DR\ \mu_1\mu_2\mu_3} (p_1, p_2, p_3)_{\text{swordfish}} \] in eq. (24) is obtained by summing over all diagrams of swordfish type –see b) in fig. 1. These diagrams only carry nonvanishing contributions if the vertices come all from \( S_{nc} \) in eq. (17). The sum of these swordfish-type diagrams is

\[
+i \text{Tr} \left( [T^{a_1}, T^{a_2}] T^{a_3} \right) \left\{ I(p_1^2) \left( \varepsilon_{\mu_1\mu_2\rho\sigma} p_1^\rho \tilde{p}_3^\sigma + \varepsilon_{\mu_1\rho\sigma\tau} p_1^\rho p_2^\sigma \theta_{\mu_2}^\tau \right) \\
+ I(p_2^2) \left( \varepsilon_{\mu_1\mu_2\rho\sigma} p_2^\rho \tilde{p}_3^\sigma + \varepsilon_{\mu_2\rho\sigma\tau} p_1^\rho p_2^\sigma \theta_{\mu_1}^\tau \right) \right\} .
\]

(50)

The jellyfish-type diagrams in c) of fig. 1, which vanish in dimensional regularization, give rise to \( i p_3^{\mu_3} \Gamma^{(odd)}_{DR\ \mu_1\mu_2\mu_3} (p_1, p_2, p_3)_{\text{swordfish}} = 0 \) in eq. (25).

The sum of the expressions in eqs. (18), (19) and (50) yield the result in eq. (24). This final outcome is the contribution the noncommutative three-point function of the ordinary field computed with the regularized action \( S_{nc,DR}^{(-)} \) in eq. (17). This action is obtained from the action in eq. (13) by using the Seiberg-Witten map. It is for the latter action and for the noncommutative \( U(n) \) groups that the gauge anomaly has been computed without using the Seiberg-Witten map formalism in a number of papers –see refs. [44, 46, 45]. Note that for the \( U(n) \) case eq. (24) can be obtained by applying the Seiberg-Witten map to the results presented in refs. [44, 46, 45]. This is consistent with the fact that the Seiberg-Witten map is explicitly preserved by our regularization procedure and that in the aforementioned papers the computation of the anomaly is carried out over the space of \( \ast \)-polynomials of the noncommutative gauge field and its derivatives where \( \theta^{\mu\nu} \) only occurs in the Moyal product, i.e.,
polynomials like

$$\theta^\alpha{}^\beta \partial_\alpha \partial_\mu A_{\mu_2} \star \partial_\mu_3 A_{\mu_4} \star A_\beta,$$

(51)

$A_\mu$ denoting the noncommutative gauge field, were not allowed: renormalizability by power-counting was a constraint.

Now, the sum of the results in eq. (48) and eq. (50) is the ugly expression in eq. (25), which for the $U(n)$ case cannot be obtained from the results in [14, 46, 15] by applying the Seiberg-Witten map technique. Note that eq. (25) corresponds to the regularization of the theory achieved by just using $\bar{S}_{nc}$ in eq. (17) as the regularized action. The noncommutative ancestor of this action seems to involve $\star$-polynomials with $\theta$ coefficients. That is to say,

$$\int d^{2\omega} x \bar{\Psi} \star \{ \hat{\partial} \Psi - i A_\mu \varepsilon^{\mu \mu_1 \mu_2} \star P_- \Psi \} + \frac{i}{2} \theta^{\alpha\beta} \int d^{2\omega} x \bar{\Psi} \star [ \partial_\alpha A_\beta + A_\beta \partial_\alpha - \frac{i}{2} A_\alpha \star A_\beta ] \hat{\partial} \star P_+ \Psi$$

$$- \frac{i}{2} \theta^{\alpha\beta} \int d^{2\omega} x \bar{\Psi} \star [ \hat{\partial} A_\beta \partial_\alpha + \frac{i}{2} ( \hat{\partial} A_\alpha \star A_\beta + A_\alpha \star \hat{\partial} A_\beta ) + ( A_\beta \partial_\alpha + \frac{i}{2} A_\alpha \star A_\beta ) \hat{\partial} ] \star P_- \Psi$$

(52)

yields $\bar{S}_{nc}$ in eq. (17) upon using the Seiberg-Witten map. One may now compute the breaking of gauge invariance in the triangle diagrams of the theory defined by minimal subtraction of the triangle diagrams of the action in eq. (52). This breaking is equal to

$$\frac{i}{24\pi^2} \left( \text{Tr} T^{a_1} T^{a_2} T^{a_3} e^{-\frac{i}{2} \theta^{\alpha\beta} p_1 A_{\alpha} p_2 A_{\beta}} + \text{Tr} T^{a_2} T^{a_1} T^{a_3} e^{\frac{i}{2} \theta^{\alpha\beta} p_1 A_{\alpha} p_2 A_{\beta}} \right) \varepsilon_{\mu_1 \mu_2 \rho \sigma} p_1^\rho p_2^\sigma$$

$$- \frac{1}{16\pi^2} \left( \frac{1}{24} \right) \left( \text{Tr} T^{a_1} T^{a_2} T^{a_3} e^{-\frac{i}{2} \theta^{\alpha\beta} p_1 A_{\alpha} p_2 A_{\beta}} - \text{Tr} T^{a_2} T^{a_1} T^{a_3} e^{\frac{i}{2} \theta^{\alpha\beta} p_1 A_{\alpha} p_2 A_{\beta}} \right) \left[ 2 \varepsilon_{\mu_1 \mu_2 \rho \sigma} p_1^\rho p_2^\sigma \theta_{\alpha \beta} p_1^\alpha p_2^\beta \right.$$  

$$+ (\varepsilon_{\mu_1 \nu \tau} p_1^\nu p_2^\tau \theta_{\mu_2}^\tau + \varepsilon_{\mu_2 \nu \tau} p_1^\nu p_2^\tau \theta_{\mu_1}^\tau - \varepsilon_{\mu_1 \mu_2 \rho \sigma} p_1^\rho p_3^\sigma (p_1^2 + p_2^2 + p_1 \cdot p_2)) \right],$$

(53)

and it agrees, as it must be, up to first order in $\theta$ with the expression in eq. (25) once the latter has been minimally subtracted. To sort out which terms in eq. (53) are truly anomalous, if any, one should solve the Wess-Zumino consistency condition on the space of $\star$-polynomials with $\theta^{\mu\nu}$ dependent coefficients; a problem not studied as yet. An instance of the $\star$-polynomials with $\theta^{\mu\nu}$ dependent coefficients relevant to our problem is given in eq. (51).

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