Range Assignment of Base-Stations Maximizing Coverage Area without Interference

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September 15, 2017

Abstract

We study the problem of assigning non-overlapping geometric objects centered at a given set of points such that the sum of area covered by them is maximized. If the points are placed on a straight-line and the objects are disks, then the problem is solvable in polynomial time. However, we show that the problem is NP-hard even for simplest objects like disks or squares in $\mathbb{R}^2$. Eppstein [CCCG, pages 260–265, 2016] proposed a polynomial time algorithm for maximizing the sum of radii (or perimeter) of non-overlapping balls or disks when the points are arbitrarily placed on a plane. We show that Eppstein’s algorithm for maximizing sum of perimeter of the disks in $\mathbb{R}^2$ gives a 2-approximation solution for the sum of area maximization problem. We propose a PTAS for our problem. These approximation results are extendible to higher dimensions. All these approximation results hold for the area maximization problem by regular convex polygons with even number of edges centered at the given points.

Keywords: Quadratic programming, discrete packing, range assignment in wireless communication, NP-hardness, approximation algorithm, PTAS.

1 Introduction

Geometric packing problem is an important area of research in computational geometry, and it has wide applications in cartography, sensor network, wireless communication, to name a few. In the disk packing problem, the objective is to place maximum number of congruent disks (of a given radius) in a given region. Toth 1940 [3,12] first gave a complete proof that hexagonal lattice packing produces the densest of all possible disk packings of both regular and irregular regions. Several variations of this problem are possible depending

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* A preliminary version of the paper is to appear in the proceedings of 29th Canadian Conference on Computational Geometry, 2017.
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on various applications [11, 12]. In this paper, we will consider the following variation of
the packing problem:

**Maximum area discrete packing (MADP):** Given a set of points \( P = \{p_1, p_2, \ldots, p_n\} \)
in \( \mathbb{R}^2 \), compute the radii of a set of non-overlapping disks \( C = \{C_1, C_2, \ldots, C_n\} \), where
\( C_i \) is centered at \( p_i \in P \), such that \( \sum_{i=1}^n \text{area}(C_i) \) is maximum.

The problem can be formulated as a quadratic programming problem as follows. Let \( r_i \) be
the radius of the disk \( C_i \). Our objective is:

Maximize \( \sum_{i=1}^n r_i^2 \)

Subject to \( r_i + r_j \leq \text{dist}(p_i, p_j), \ \forall \ p_i, p_j \in P, \ i \neq j \).

Here, \( \text{dist}(p_i, p_j) \) denotes the Euclidean distance of \( p_i \) and \( p_j \). The motivation of the problem
stems from the range assignment problem in wireless networks. Here the inputs are the base-stations. Each base-station is assigned with a range, and it covers a circular area centered
at that base-station with radius equal to its assigned range. The objective is to maximize
the area coverage by these base-stations without any interference. In other words, the
area covered by two different base-stations should not overlap. Surprisingly, to the best of
our knowledge, there is no literature for the MADP problem. A related problem, namely
maximum perimeter discrete packing (MPDP) problem, is studied recently by Eppstein [4], where the objective is to compute the radii of the disks in \( C \) maximizing \( \sum_{i=1}^n r_i \) subject
to the same set of linear constraints. This is a linear programming problem for which
polynomial time algorithm exists [9]. In particular, here each constraint consists of only
two variables, and such a linear programming problem can be solved in \( O(mn^3 \log m) \) time
[8], where \( n \) and \( m \) are number of variables and number of constraints respectively. In
[4], a graph-theoretic formulation of the MPDP problem is suggested. Let \( G = (V, E) \) be a
complete graph whose vertices \( V \) correspond to the points in \( P \); the weight of edge \( (i, j) \in E \)
\( (i \neq j) \) is \( \text{dist}(p_i, p_j) \), which corresponds to the constraint \( r_i + r_j \leq \text{dist}(p_i, p_j) \). They
computed the minimum weight cycle cover of \( G \) in time \( O(mn + n^3 \log n) \) time. Since \( m = O(n^2) \) in our case, the time complexity of this algorithm is \( O(n^3) \). They further considered
the fact that a constraint \( r_i + r_j \leq \text{dist}(p_i, p_j) \) is useful if \( \delta(p_i) + \delta(p_j) \geq \text{dist}(p_i, p_j) \), where
\( \delta(p) \) is the distance of the point \( p \) and its nearest neighbor in \( P \); otherwise that constraint
is redundant. They also showed that the number of useful constraints is \( O(n) \), and thus the
overall time complexity becomes \( O(n^2 \log n) \). They used further graph structure to reduce
the time complexity. In \( \mathbb{R}^d \), the time complexity of this problem is shown to be \( O(n^{2\frac{d}{d-1}}) \).

It is well-known that if \( Q \) is a positive definite matrix, then the quadratic programming
problem which minimizes \( \hat{X}'Q\hat{X} \) subject to a set of linear constraints \( AX \leq \hat{b}, \hat{X} \geq 0 \) is
solvable in polynomial time [11]. However, if we present our maximization problem as a
minimization problem, the diagonal entries of the matrix \( Q \) are all \(-1\) and the off-diagonal
entries are all zero. Thus, all the eigen values of the matrix \( Q \) are \(-1\). It is already proved
that the quadratic programming problem is NP-hard when at least one of the eigen values

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of the matrix $Q$ is negative \[^{10}\]. This indicates that the MADP problem also seems to be computationally hard. For the minimization version of an NP-hard quadratic programming with $n$ variables and $m$ constraints, an $(1 - \frac{1}{(m(1+\epsilon))^2})$ factor approximation algorithm is proposed in \[^{6}\], which works for all $\epsilon \in (0, 1 - \frac{1}{\sqrt{2}})$. The time complexity of this algorithm is $O(n^3(m \log \frac{1}{\delta} + \log \log \frac{1}{\epsilon}))$, where $\delta$ is the radius of the largest ball inside the feasible region defined by the given set of constraints. For our MADP problem in $\mathbb{R}^2$, a 4-factor approximation algorithm is easy to obtain.

For each point $p_i \in P$, let $\mathcal{N}(p_i) \in P$ be its nearest neighbor, and $\ell_i = \text{dist}(p_i, \mathcal{N}(p_i))$. We assign $r_i = \frac{1}{2}\ell_i$ for each $i \in \{1, 2, \ldots, n\}$. Thus, all the constraints are satisfied. The approximation factor follows from the fact that in the optimum solution the radius $\rho_i$ of a disk centered at $p_i$ can take value at most $\ell_i$.

**Our contribution**

In Section \[^{3}\] we first show that if the points in $P$ are placed on a straight line, then the MADP problem can be optimally solved in $O(n^2)$ time. In Section \[^{4}\] we show that MADP problem in $\mathbb{R}^2$ is NP-hard. As a feasible solution of the MPDP problem is also a feasible solution of the MADP problem, it is very natural to ask whether an optimal solution of the MPDP problem is a good solution for the MADP problem, or not. In Section \[^{5}\] we answer this question in the affirmative. We show that the optimum solution for the MPDP problem proposed in \[^{4}\] is a 2-approximation result for the MADP problem. We also propose a PTAS for the MADP problem. In Section \[^{6}\] we show that the approximation results in Sections \[^{5}\] are extendible to higher dimensions. Finally, in Section \[^{7}\] we show that all these approximation results for the MADP problem in $\mathbb{R}^2$ hold for any regular convex polygon with even number of edges.

**2 Preliminaries**

A solution for the MADP problem consists of disks with center at each point in $P$. Their radii are all greater than or equal to zero\[^{1}\]. A solution of the MADP problem is said to be **maximal** if each disk touches some other disk (may be of radius 0) in the solution. From now onwards, by a **solution** of a MADP problem, we will mean it to be a maximal solution.

The nearest neighbor of a point $p_i \in P$ is denoted by $\mathcal{N}(p_i) \in P$. Here, a point $p_i \in P$ is said to be a **defining point** of the said solution if it appears on the boundary of some disk in the solution; otherwise it is said to be a **non-defining point**. A non-defining point $p_i \in P$ will be covered with a disk $C_i$ centered at point $p_i$, and its radius $r_i$ is either equal to or less than $\text{dist}(p_i, q_i)$, where $q_i = \mathcal{N}(p_i)$. In the former case, $C_i$ is said to have full-radius, and in the later case, $C_i$ is said to have part-radius since the boundary of $C_i$ does not have any

\[^{1}\]A disk with radius 0 implies that no disk is placed at that point.
point in \( P \). Let us consider a neighbor \( p_j \) of the point \( p_i \) which has a disk \( C_j \) of radius \( r_j \). We will use the term residue-distance to indicate a feasible radius for the disk \( C_i \) of length \( |\text{dist}(p_i, p_j) - r_j| \) for \( i \neq j \), if \( |\text{dist}(p_i, p_j) - r_j| \leq |\text{dist}(p_i, N(p_i))| \) (see Figure 1). Thus, the residue-distance of a disk \( C_i \) (centered at \( p_i \)) is zero if \( N(p_i) \) is a defining point. For each full-radius (resp. part-radius) of a disk \( C_i \) corresponding to \( p_i \), we define a full-radius interval (resp. part-radius interval) of length \( 2r_i \), where \( r_i \) is the radius of \( C_i \).

3 MADP problem on a line

In this section, we will consider a constrained version of the MADP problem, where the point set \( P = \{p_1, p_2, \ldots, p_n\} \) lie on a given line \( L \), which is assumed to be the \( x \)-axis. We also assume \( \{p_1, p_2, \ldots, p_n\} \) is sorted in left to right order. We use \( d_i \) to denote the distance of the pair of points \( (p_i, p_{i+1}) \), \( i = 1, 2, \ldots, n - 1 \). Our objective is to place non-overlapping disks centered at each point \( p_i \in P \) such that the sum of the area formed by those disks is maximized. We will use \( r_i \) to denote the radius of the disk centered at the point \( p_i \), where \( r_i \geq 0 \) for \( i = 1, 2, \ldots, n \).

Lemma 1. In the optimum solution of the MADP problem on a line, at least one of the leftmost or rightmost point in \( P \) must be either a defining point or its corresponding disk has full radius.

Proof. For the contradiction, let the leftmost point \( p_1 \) in \( P \) has radius \( r_1 \) satisfying \( 0 < r_1 < \text{dist}(p_1, N(p_1)) \) (see Figure 2). If \( r_2 = d_2 < d_1 - r_1 \), then we can increase \( r_1 \), indicating the non-optimality of the solution. If \( r_2 = d_1 - r_1 \), then \( r_3 = \min(d_3, (d_2 - (d_1 - r_1))) \). Assuming \( r_3 = d_2 - (d_1 - r_1) \) and proceeding similarly, we may reach one of the following two situations:

1. \( r_k = d_{k-1} - (d_{k-2} - (\ldots(d_1 - r_1))) \ldots \), and the values of \( r_{k+1}, \ldots, r_n \) are independent of \( r_1 \).
2. \( r_{n-1} = d_{n-2} - (d_{n-3} - (\ldots (d_1 - r_1)\ldots )) \) and \( r_n = d_{n-1} - r_{n-1} \).

Below, we show that in Case 1, \( S_k = r_1^2 + r_2^2 + \ldots + r_k^2 \) can be increased while keeping the values of \( r_{k+1}, \ldots, r_n \) unchanged.

\[
S_k = \pi \cdot (r_1^2 + (d_1 - r_1)^2 + (d_2 - (d_1 - r_1))^2 + \ldots + (d_k - (d_{k-1} - (\ldots (d_1 - r_1)\ldots )))^2)
\]
\[
= \pi \cdot (k \cdot r_1^2 - 2r_1 \cdot c_2 + c_1),
\]
where \( c_1 = d_1^2 + (d_2 - d_1)^2 + \ldots + (d_k - (d_{k-1} - (\ldots + (-1)^k \cdot d_1))^2,
\]
and \( c_2 = (d_1 - (d_2 - d_1) + \ldots + (-1)^{k-1}(d_k - (d_{k-1} - (\ldots + (-1)^k \cdot d_1)))) \).

Thus, \( S_k \) is a parabolic function whose minimum is attained at \( r_1 = \frac{d_2}{k} \), and it attains maximum at the boundary values of the feasible region of \( r_1 \), i.e either at \( r_1 = 0 \) or \( d_1 \).

In Case 2, if \( r_n > r_{n-1} \), we can increase the sum \( S_n \) by setting \( r_n = d_{n-1} \), \( r_{n-1} = 0 \) and keeping \( r_1, r_2, \ldots, r_{n-2} \) unchanged. Now, \( r_1^2 + r_2^2 + \ldots + r_{n-2}^2 \) can further be increased as in Case 1. Similarly, if \( r_1 > r_2 \) then also \( S_n \) can be increased by setting \( r_1 = d_1 \) and \( r_2 = 0 \), and then maximizing \( r_1^2 + r_2^2 + \ldots + r_n^2 \) as in Case 1. If \( r_n \leq r_{n-1} \) and \( r_1 \leq r_2 \), then also \( S_n \) is a parabolic function of \( r_1 \), and it is maximized at either \( r_1 = 0 \) or \( r_1 = \min(d_1, \alpha) \) where \( \alpha = \) value of \( r_1 \) for which \( r_{n-1} = d_{n-1} \).

![Figure 2: An instance, considering k = 3](image)

Lemma\(^2\) says that in an optimum solution all the disks have either full-radius or zero radius or has radius equal to the residue distance with respect to the radius of its neighboring points.

Full-radius disks (intervals) are easy to get. For each point \( p_i \), find its nearest neighbor \( \mathcal{N}(p_i) = p_{i-1} \) or \( p_{i+1} \), and define an interval of length equal to \( 2 \cdot \text{dist}(p_i, \mathcal{N}(p_i)) \), centered at \( p_i \). We now describe the generation of all possible part-radius intervals for each point \( p_i \in P \) considering them in left to right order.

- For both the points \( p_1 \) and \( p_2 \), there is no part-radius interval.
- If \( \mathcal{N}(p_2) = p_1 \), then for point \( p_3 \), there is a part-radius interval of length \( 2(d_2 - d_1) \), centered at \( p_3 \); otherwise there is no part-radius interval for the point \( p_3 \).

\(^2\)Here right-end of the feasible region of \( r_1 \) is obtained by placing a disk of radius \( d_{n-1} \) at \( p_n \), and placing disks at points \( p_{n-1}, \ldots, p_2 \) touching those of \( p_n, \ldots, p_3 \), and then placing the disk of radius \( \alpha \) at \( p_1 \) that touches the disk at \( p_2 \). Here surely \( \alpha \leq d_1 \).
In general, for an arbitrary point \( p_k \) if there are \( m \) number of part-radius intervals \( I_1, I_2, \ldots, I_m \) of lengths \( 2\delta_1, 2\delta_2, \ldots, 2\delta_m \) respectively, then each of these intervals \( I_j \) gives birth to a part-radius interval for the point \( p_{k+1} \) with center at \( p_{k+1} \) and of length \( 2(d_k - \delta_j) \).

In addition, if \( N(p_k) = p_k - 1 \), then for point \( p_{k+1} \), there is another part-radius interval centered at \( p_{k+1} \) and of length \( 2(d_k - d_{k-1}) \).

Finally, we have \( \mathcal{I} = \bigcup_{i=1}^{n} I_i \). A similar process is performed to generate part-radius intervals \( \mathcal{J} \) by considering the points in \( P \) in right to left order.

**Lemma 2.** For a set \( P \) of \( n \) points lying on a line \( L \), the maximum number of intervals generated by the above procedure is \( \Theta(n^2) \).

**Proof.** Let us first consider the forward pass as explained above. Here, for each point \( p_i \) (in order) a full-radius interval is generated, and the full-radius interval for point \( p_i \) may generate a part-radius interval for each point \( p_j, j = i + 1, \ldots, n \). Thus, for all the points in \( P \), we may get \( O(n^2) \) intervals. To justify the number of intervals is \( \Omega(n^2) \), see the demonstration in Figure 3. Here the points \( p_i = (x_i, 0), i = 1, 2, \ldots, n \) are placed on the \( x \)-axis, where \( x_1 = 0, x_2 = 1 \) and \( x_i = x_{i-1} + (x_{i-1} - x_{i-2}) + 0.5, i = 3, 4, \ldots, n \). Here for each generated interval at \( p_i \), a part-radius interval for the points \( p_j, j = i + 1, \ldots, n \) will be generated. The same argument follows for the reverse pass also.  

![Figure 3: An \( \Omega(n^2) \) instance of full and part radius intervals](image)

For each of these intervals we assign weight equal to the square of their half-length. We sort the right end points of these intervals. For this sorted set of weighted intervals, we find the maximum weight independent set. This leads us to the following theorem.

**Theorem 1.** Given a set \( P \) of \( n \) points on a line \( L \), one can place non-overlapping disks maximizing sum of their area in \( O(n^2) \) time.

**Proof.** We can generate the intervals in \( O(n^2) \) time as follows. Given a set of intervals \( \mathcal{I}_i \) (of full- and part-radius) generated for a point \( p_i \) which are sorted by their right end-points, we can generate the set of part-radius intervals \( \mathcal{I}_{i+1} \) for the point \( p_{i+1} \) in \( O(i) \) time. Thus, total time for interval generation is \( O(n^2) \) in the worst case. Since intervals for each point \( p_i \) are generated in sorted manner, ordering them with respect to their end-points also takes
Finally, computing the maximum weight independent set of the sorted set of intervals $\cup_{i=1}^n I_i$ using dynamic programming needs $O(n^2)$ time [7].

The correctness of the algorithm follows from the fact that, if there is an interval $\theta$ corresponding to point $p_i$ in the optimum solution that does not belong to $I \cup J$, then it is not generated by any interval in $I_{i-1}$ and $J_{i+1}$. As a result it does not touch any interval of $I_{i+1}$ and also $J_{i-1}$. Thus, interval $\theta$ can be elongated to increase the total covering area. $\square$

4 MADP problem in $\mathbb{R}^2$ is NP-hard

Here, we show that the MADP problem in $\mathbb{R}^2$ is NP-hard by a polynomial time reduction of planar rectilinear monotone 3-SAT (PRM-3SAT) problem to this problem.

**Definition 1.** A planar rectilinear monotone 3-SAT (PRM-3SAT) is a 3-SAT formula $\theta$ such that in every clause of $\theta$, either all the literals are positive, or all the literals are negative. Furthermore, $\theta$ has an embedding $\xi$ in $\mathbb{R}^2$ with the following properties:

(i) The variables and clauses of $\theta$ are represented in $\xi$ by axis parallel squares and rectangles respectively.

(ii) All the squares representing the variables have the same size and they lie on the x-axis.

(iii) All the rectangles representing the clauses have the same height. But their lengths may vary.

(iv) The rectangles for positive clauses are above the x-axis while the rectangles for the negative clauses are below the x-axis.

(v) The paths joining variables to their respective clauses are just vertical lines, called clause-variable connecting path (CVC-path, in short).

(vi) The corners of the squares and rectangles representing the variables and clauses respectively, and the end-points of all the CVC-paths in the embedding $\xi$ are lattice points.

Given a PRM-3SAT formula $\theta$, its embedding $\xi$, as stated above, can be obtained in polynomial time. In [2], it is shown that PRM-3SAT problem is NP-complete.

We call a set of non-intersecting disks centered at the given points, a disk configuration. A disk configuration that gives the maximum area is called a maximum disk configuration. A disk is said to be on a point if it is centered at that point.

4.1 The reduction

We start with an embedding of a planar monotone rectilinear 3-SAT formula $\theta$, as in Definition 1 with variables $\{x_1, x_2, \ldots, x_n\}$ and clauses $\{C_1, C_2, \ldots, C_m\}$ (see Figure 4 for an
example). Observe that, a clause with two literals can be made a three literal clause by duplicating its any one of the literals. Thus, we can assume that all clauses in $\Theta$ have three literals. We replace each clause with a clause-gadget and each variable with a variable-gadget using point sets. Also we put points along the CVC-paths connecting each clause with the variables in it. For convenience we use points of three colors, namely red, green and blue, in our reduction. Our configuration of points will contain the following sub-configurations.

**Clause gadget**

A clause gadget corresponding to any clause $C_\alpha$ of $\Theta$ has eight green and four blue points. Let the coordinate of one green point is $p_1 = (\mu, \nu)$. The coordinates of the other seven green points are $p_2 = (\mu, \nu + a)$, $p_3 = (\mu + a, \nu + a)$, $p_4 = (\mu + a, \nu)$, $p_5 = (\mu + \frac{5a}{2}, \nu)$, $p_6 = (\mu + \frac{5a}{2}, \nu - a)$, $p_7 = (\mu + \frac{7a}{2}, \nu)$ and $p_8 = (\mu + \frac{7a}{2}, \nu - a)$. The coordinates of the blue points are $p_9 = (\mu + a, \nu + \frac{3a}{2})$, $p_{10} = (\mu + a, \nu + \frac{3a}{2} + b)$, $p_{11} = (\mu + \frac{7a}{2} + \frac{a}{10}, \nu)$ and $p_{12} = (\mu + \frac{7a}{2} + \frac{a}{10} + b, \nu)$ (see Figure 5(a)). Other than these points, there are three blue points $v_i$, $v_j$, and $v_k$, which are at a distance of $a$ units to the left of $p_1$, to the left of $p_2$, and to the right of $p_7$ respectively. These are the points on the CVC-path from the variable-gadgets $x_i$, $x_j$ and $x_k$ appearing in this clause. We choose $b$ and $a$ later depending on the number of variables $n$ and number of clauses $m$ of $\Theta$. We have the following observation on clause-gadgets, which we will prove in Lemma 6.

**Observation 1.** The total area of the disks centered at the points of a clause-gadget for a clause with three literals is maximized only if there is a disk of radius $a$ at some green point of that clause-gadget touching the last blue point of at least one CVC-path reaching to that clause-gadget (see Figures 5(b–f)).

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Figure 4: A planar monotone rectilinear 3-SAT formula.
Figure 5: (a) A clause-gadget. (b,c) Two maximum disk configurations touching only \(w\). (d) A maximum disk configuration touching both \(v\) and \(w\). (e) A maximum disk configuration touching only \(v\). (f) A maximum disk configuration touching only \(u\). (g) A maximum disk configuration touching only \(u\) and \(w\).

**Variable-gadget**

A variable-gadget corresponds both to the positive and negative literals associated with the variable \(x_i\). For each variable \(x_i\), since each of the literals \(x_i\) and \(\overline{x_i}\) may appear in each of the \(m\) clauses of \(\theta\) at most twice, we may need a total of \(2m\) points for both \(x_i\) and \(\overline{x_i}\) in the variable-gadget. We create the variable gadget as follows:

- It is a rectangle \(r_i\) of size \((4m + 3)a \times 5a\),

- Assuming the coordinate of the bottom left corner of this rectangle as \((0, 0)\), \(8m + 4\) points placed along the boundary of another rectangle \(r'_i\) of size \((4m + 1)a \times 3a\) inside \(r_i\) at coordinate points \(((i + 1)a, 4a), i = 1, 2, \ldots, 4m\}, (4m + 2)a, 2a), (4m + 2)a, 3a\}, (((i + 1)a, a), i = 1, 2, \ldots, 4m\}, (a, 2a), (a, 3a) (see Figure 5(a)).

- Points are labeled with \(x_i\) and \(\overline{x_i}\) alternately around the boundary of the rectangle \(r'_i\).
Figure 6: (a) A variable-gadget with $4m$ points, (b-c) Two configurations of disks achieving maximum area, (d) A non-optimum configuration of disks for those points

in clockwise order starting from the point at the location having coordinate $(2a, 4a)$.

We have the following observation on variable-gadgets, which we will prove in Lemma 8.

**Observation 2.**

(a) The total area of non-overlapping disks on the points of the variable-gadget for $x_i$ is maximized if and only if either all the points representing $x_i$ have disks of radius $a$ on them, or all the points representing $\overline{x_i}$ have disks of radius $a$ on them (see Figures 6(b) and 6(c)). In this case, the total area covered is $(4m + 2)a^2$

(b) If disks are placed at both $x_i$ and $\overline{x_i}$ in non-overlapping manner, then the total area covered is strictly less than $(4m + 2)a^2$ (see Figure 6(d)).

We construct the configuration of points $P$ for the PRM-3SAT formula $\theta$ using the following steps:

(a) Consider an embedding $\xi$ of $\theta$. The variables are represented by squares of size $(4m + 1)a \times 4a$ centered on the $x$-axis, and the clauses are represented by rectangles of size $4a \times 3a$. The horizontal distance between two consecutive squares on the $x$-axis (representing variables) is $2a$. The vertical distance between two rectangles defining two different clauses in the embedding (if any) is also $2a$.

(b) Replace each clause-rectangle by a clause-gadget, as follows.
Figure 7: (a) A CVC-path. (b) A maximum disk configuration of the connecting path. (c) A unique maximum disk configuration of the connecting path when a red disk touches its bottommost point. (d) A unique disk configuration of the connecting path when a green disk touches its leftmost point.

In the original embedding $\xi$ of $\theta$, all paths from a clause to its variables are vertical lines. Each clause $C = (u+v+w)$ in the embedding has three literals, namely left-literal, middle-literal and right-literal respectively. Consider the middle-literal $v$ of a positive clause $C$ embedded above the $x$-axis in the embedding of $\theta$. Among all the positive clauses having literal $v$, let $C$ be the $k$-th one from the left in our embedding $\xi$. Then place the clause-gadget $C$ so that the $x$-coordinate of its left-most point ($p_2$ in Figure 5(a)) is greater than the $x$-coordinate of the $(4k-1)^{th}$ red point in the top boundary of the variable-gadget for the variable $v$ by a multiple of $a$.

If the path is from a variable to a negative clause, then follow an analogous procedure of placing the corresponding clause-gadget such that the $x$-coordinate of its left-most point ($p_2$ in Figure 5(a)) is greater than the $x$-coordinate of the $4k^{th}$ red point in the bottom boundary of the variable-gadget of the variable $v$ (i.e., $k$-th $\overline{v}$ from the left).

(c) In the original embedding of $\theta$, all paths from clauses to variables are vertical lines. Consider such a vertical line $\ell$ in the embedding of $\theta$. Suppose the path connects a positive clause $C$ and a variable $v$. Also assume that among all such vertical paths from positive clauses to the variable $v$, this path is the $k^{th}$ one from the left. Translate $C$ horizontally, so that it is vertically above the $(2k-1)^{th}$ red point representing the variable $v$ on the top boundary of the variable gadget (rectangle) for $v$. If the path corresponds to the left, middle or right literal in the clause, then add a vertical line
segment of length $3a$, $2a$ or $a$ respectively above it, and after that a horizontal line segment of adequate length such that the path is horizontally $a$ distance away from its corresponding green point of clause $C$.

The case when $C$ is a negative clause and the vertical line $\ell$ connecting $C$ and its middle-literal $v$ is the $k$-th one among the vertical lines incident on the bottom boundary of the variable gadget corresponding to $v$ in the embedding $\xi$, then we translate $C$ horizontally to align $\ell$ with the $2k$-th red point representing $v$ in the bottom boundary of the variable gadget of $v$. Next, we follow the same procedure (increasing the length of $\ell$ vertically downwards and adding a horizontal line segment of required length) to connect $v$ with the corresponding green point of clause $C$.

(d) Now we express $a$ and $b$ in terms of $m$ and $n$. The height (span in vertical direction) of the embedding is upper bounded by that of $m$ clause rectangles (assuming that they are in different layers in the embedding $\xi$), vertical gap between layers, and a variable rectangle. These make a total of $3ma + 2ma + 5a = 5(m + 1)a = B$ (say). The upper bound on the length ($L$) of the embedding is $(4m + 5)na$. So, the length of a CVC-path connecting a clause with a literal is upper bounded by $K = L + B$.

There are at most $3m$ CVC-paths in our point set. We want to set $a$ and $b$ such that the sum of the areas of all blue disks is a small fraction of the area of a single green or red disk. We want the area of a single green or blue disk to be 100 times that of the sum of areas of all blue disks. So, we set $a$ such that $100(3m)K\pi b^2 \leq \pi a^2$. Or, in other words, $b \leq \frac{a}{10\sqrt{K}} = \frac{a}{10\sqrt{3m(5m + 5 + 4mn + 5n)}} = \frac{\sqrt{a}}{10\sqrt{3m(5m + 5 + 4mn + 5n)}}$. Choosing $b = 1$ gives $a \geq 300m(5m + 5 + 4mn + 5n)$. Since $5m + 5 + 4mn + 5n \geq 20mn$, we set $a = 300m(20mn) = 6000m^2n$.

(e) Note that here $\frac{a}{10}$ is an integer. Thus, the point set consisting of all the variable-gadgets and all the clause-gadgets can be placed at points with integer coordinates.

(f) Replace each CVC-path from a clause to a vertex with blue points at unit distance apart along that path, except at the turning point (see Figure 7). The vertical and horizontal lengths of the paths are multiples of $a$, which is an even number due to our choice of $a$. Hence, the number of lattice points on each path is odd. Since we do not put blue point on the turning point, the number of blue points on each path is also even. As mentioned earlier, the end-point of a CVC-path closer to a clause is referred to as the last blue point of the said path (points $u$, $v$ and $w$ shown in Figure 5(a)).

(g) We use a total number of $K' (= m(3 \times 100K + 4))$ blue points, where $K'$ depends only on $\theta$ and not an embedding of $\theta$. Let the total number of blue points used so far on paths and clause-gadgets be $K_b$. Since each path and clause-gadget has even number of blue points, $K_b$ is even. Put $K' - K_b$ blue points on a separate vertical line, with consecutive points at unit distance apart. These will be referred to as the excess points from the clause-gadgets. Note that $K' - K_b$ is also even.
Figure 8: A planar monotone rectilinear 3-SAT embedding transformed to a point set.

See Figure 8 for the point set embedding $P$ of the PRM-3SAT formula shown in Figure 4. Here the coordinates of each point are integer.

4.2 Properties of the point configuration

Denote the point set constructed in the previous section by $P$. Denote by $\Delta(P)$ the maximum sum of areas of non-intersecting disks centered at the points of $P$.

Lemma 3. If $Q_1$ and $Q_2$ are disjoint subsets of a point set $Q$, then $\Delta(Q) \leq \Delta(Q_1) + \Delta(Q_2)$.

Proof. Assume on the contrary that $\Delta(Q) > \Delta(Q_1) + \Delta(Q_2)$ for some choice $S$ of disks, and $S = S_1 \cup S_2$, where $S_1$ are centered at points in $Q_1$ and $S_2$ are centered at points in $Q_2$. Observe that the total area of $S_1$ (resp. $S_2$) is smaller than $\Delta(Q_1)$ (resp. $\Delta(Q_2)$), leading to a contradiction. 

Lemma 4. If $Q$ is a set of $k > 1$ collinear points on the plane, placed uniformly unit distances apart, then $\Delta(Q) = \pi \lceil \frac{k}{2} \rceil$, and it can be realized only by a configurations of disks of unit and zero radii at alternate points.

Proof. Follows from Lemma 1 where the points on a line are equidistant.

Lemma 5. If $Q$ is a set of four points on four vertices of a unit square, then $\Delta(Q) = \pi(4 - 2\sqrt{2})$, and can be realized only by either two diagonally opposite disks of radius $\frac{\sqrt{2}}{2}$ and two other diagonally opposite disks of radius $1 - \frac{\sqrt{2}}{2}$, or two diagonally opposite disks of radii $1$ and $\sqrt{2} - 1$, respectively.

Proof. We prove this result by exhaustive case analysis. Let the top left point of $Q$ be $q_1$, and the other points are named as $q_2$, $q_3$ and $q_4$ in a counterclockwise order (see Figure 9(a)). Let the disks on these points be named as $d_1$, $d_2$, $d_3$ and $d_4$, and their radii be $r_1$, $r_2$, $r_3$ and $r_4$ respectively, where $r_i \geq 0$ for $i = 1, 2, 3, 4$. 

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disks touch each other. Since for any two points, the larger disk can be expanded and the

Now consider the remaining case, where at least one disk touches all the other three disks. Then the total area of each disk is $\pi a^2 + \pi(\sqrt{2} - 1)^2 a^2 + \pi b^2$. (c) Another choice for maximum area of disks. (d) A non-optimum area covered by disks on the points.

In a configuration achieving the maximum area, each disk $d_i$ must touch some other disk in $d_j, j \neq i, i, j = 1, \ldots, 4$. Suppose that in a configuration, there are three or less disks having strictly positive radii. Let the point $q_1$ has no disk (i.e., $r_1 = 0$). It must be touched by some other disk, say $d_2$ at point $q_2$, having $r_2 > 0$. Implying, $r_2 = 1$, and it touches another point $q_3$ as well. The disk $d_4$ can have a radius up to $\sqrt{2} - 1$ to avoid intersection with $d_2$. Adding the area of $d_2$ and $d_4$, we have $\Delta(Q) \geq \pi(4 - 2\sqrt{2})$ (see Figure 9(b)).

Now suppose that there are fours disks on the four points with maximum possible total area. Suppose that one of these disks, say $d_1$, touches all other three disks. Then the total area $A(r_1) = \pi(r_1^2 + 2(1 - r_1)^2 + (\sqrt{2} - r_1)^2) = 4\pi r_1^2 - (4 + 2\sqrt{2})\pi r_1 + 4\pi$ is a convex function of $r_1$. It attains minimum at $r_1^* = \frac{1}{2} + \frac{\sqrt{2}}{4}$, and increases in both the sides of $r_1^*$. We also have $r_1 + r_2 = 1, r_1 + r_3 = 1$, and $r_1 + r_3 = \sqrt{2}$, and $r_2 + r_3 \leq 1$. Thus, $r_1 + r_2 + r_3 \leq \frac{1 + \sqrt{2}}{2}$. Implying $r_1 \geq \frac{\sqrt{2}}{2}$. But $A(\frac{\sqrt{2}}{2}) = \pi(4 - 2\sqrt{2}) = A(1)$ (see Figure 9(c)).

Now we consider the case where no disk touches all the three other three disks. So, each disk must touch either one or two other points or disks. Here two cases need to be considered.

A pair of diagonally opposite disks, say $d_1$ and $d_3$ touch each other.

If $r_1 > r_3$, then $r_2$ and $r_4$ can be set appropriately such that $d_2$ and $d_4$ touch $d_1$, leading to a contradiction. The same argument holds for $r_1 < r_3$.

If $r_1 = r_3$, then as before, each of $d_2$ and $d_4$ must touch both $d_1$ and $d_3$, a contradiction.

No pair of diagonally opposite disks are touching. Let $d_1$ touch $d_2$, $d_2$ touch $d_3$, and $d_3$ touch $d_4$. Implying $r_1 + r_2 = 1, r_2 + r_3 = 1$, and $r_3 + r_4 = 1$. Thus $r_1 + r_4 = 1$.

As we have assumed that diagonally opposite disks are non-touching, all the four disks must have radius less than $\frac{\sqrt{2}}{2}$. Let $r_1 = r$, and the total area becomes $A(r) = 2\pi(r^2 + (1 - r)^2) = 2\pi(2r^2 - 2r + 1)$, which is an unimodal function. It attains minimum at $r = \frac{1}{2}$, and monotonically increases in both the sides (see Figure 9(d)). Due to our constraints, $r \in [1 - \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}]$. Also note that, $A(1 - \frac{\sqrt{2}}{2}) = A(\frac{\sqrt{2}}{2}) = \pi(4 - 2\sqrt{2})$.

Now consider the remaining case, where $d_1$ touches $d_2$ and $d_3$ touches $d_4$, and no other two disks touch each other. Since for any two points, the larger disk can be expanded and the
other one shrinks to increase their total area, the larger among \( d_1 \) and \( d_2 \) can be expanded to touch \( d_3 \) or \( d_4 \), giving a greater total area, a contradiction.

We have considered all the possibilities of maximum area of disks, and these give only two configurations: either two diagonally opposite disks are of radius \( \frac{\sqrt{2}}{2} \) and the other two diagonally opposite disks are of radius \( 1 - \frac{\sqrt{2}}{2} \), or two diagonally opposite disks of radii \( 1 \) and \( \sqrt{2} - 1 \). In both the cases, we have \( \Delta(Q) = \pi(4 - 2\sqrt{2}) \).

**Lemma 6.** The maximum area covered by a disk configuration of the clause-gadget is \( 2\pi(4 - 2\sqrt{2})a^2 + 2\pi b^2 \).

**Proof.** We divide the points \( P_C \) in the clause-gadget \( C \) into the subsets: \( P_1 = \{p_1, p_2, p_3, p_4\} \), \( P_2 = \{p_5, p_6, p_7, p_8\} \) (of green points), and \( P_3 = \{p_9, p_{10}\} \), \( P_4 = \{p_{11}, p_{12}\} \) (of blue points). By Lemma 5, \( \Delta(P_4) = \Delta(P_2) = \pi(4 - 2\sqrt{2})a^2 \). By Lemma 4, \( \Delta(P_3) = \Delta(P_1) = 2\pi b^2 \). Using Lemma 3, we have \( \Delta(P_C) \leq 2\pi(4 - 2\sqrt{2})a^2 + 2\pi b^2 \). The disks, if any, on \( p_2 \) and \( p_7 \) cannot intersect the disks on \( p_{10} \) and \( p_{12} \) respectively, as \( a^2 + \left(\frac{\sqrt{2}}{2} - 1\right)^2 < (a + b)^2 \). Thus, \( \Delta(P_C) = 2\pi(4 - 2\sqrt{2})a^2 + 2\pi b^2 \), which can be achieved by all five configurations in Figure 5.

**Lemma 7.** Each of the maximum disk configurations in a clause-gadget (see Figure 5) must touch the blue point of at least one of the three CVC-path (namely \( u, v, w \)). Moreover, given any one of the three such points, there is a maximum disk configuration of the clause-gadget that touches only that point.

**Proof.** From Lemma 5, we know that in the optimal disk configuration of the clause-gadget of a clause \( C \), either two diagonally opposite disks of radius \( \frac{\sqrt{2}}{2}a \) and the other two diagonally opposite disks of radius \( 1 - \frac{\sqrt{2}}{2}a \) or two diagonally opposite disks of radii \( a \) and \( \sqrt{2} - 1 \) can be placed on each of the green point sets \( P_1 \) and \( P_2 \) of \( P_C \) to get the maximum area for \( P_1 \cup P_2 \). Hence in an optimum covering for \( P_C \), these remain the only choices for the green points. Since the possible four radii for the green points for attaining optimality are greater than \( \frac{\sqrt{2}}{10} \), no disk can be drawn on \( p_8 \). Again, due to the presence of the blue point \( p_9 \) at a distance \( \frac{\sqrt{2}}{2} \) from \( p_3 \), \( p_3 \) can have only a disk of radius \( 1 - \frac{\sqrt{2}}{2}a \). Thus, the possible configurations of disks at the green points of \( Q \) are:

(a) \( p_5 \) and \( p_7 \) have disks of radii \( a \) and \( \sqrt{2} - 1 \) respectively, and \( p_2 \) and \( p_4 \) have disks of radii \( a \) and \( \sqrt{2} - 1 \) respectively: This is a valid configuration, and the disk of radius \( a \) on \( p_2 \), touches \( u \) (see Figure 5(f)).

(b) \( p_5 \) and \( p_7 \) have disks of radii \( \sqrt{2} - 1 \) and \( a \) respectively, and \( p_2 \) and \( p_4 \) have disks of radii \( a \) and \( \sqrt{2} - 1 \) respectively: This is a valid configuration, and the disk of radius \( a \) on \( p_7 \), touches \( w \) (see Figure 5(g)).

(c) \( p_5 \) and \( p_7 \) have disks of radii \( a \) and \( \sqrt{2} - 1 \) respectively, and \( p_2 \) and \( p_4 \) have disks of radii \( \sqrt{2} - 1 \) and \( a \) respectively: This is an invalid configuration since the disks of radius \( a \) on \( p_4 \) and \( p_5 \) intersect.
Figure 10: Disks drawn according to a satisfying assignment of a PRM-3SAT formula θ; here $x_1, x_2, x_5, x_6 = 1$, and $x_3, x_4 = 0$

(d) $p_5$ and $p_7$ have disks of radii $(\sqrt{2} - 1)a$ and a respectively, and $p_2$ and $p_4$ have disks of radii $(\sqrt{2} - 1)a$ and $a$ respectively: This is a valid configuration and the disk of radius $a$ on $p_7$, touches $w$ (see Figure 5(b)).

(e) $p_5$ and $p_7$ have disks of radii $a$ and $(\sqrt{2} - 1)a$ respectively, and $p_1$ and $p_3$ have disks of radii $a$ and $(\sqrt{2} - 1)a$ respectively: This is a valid configuration and the disk of radius $a$ on $p_1$, touches $v$ (see Figure 5(e)).

(f) $p_5$ and $p_7$ have disks of radii $(\sqrt{2} - 1)a$ and $a$ respectively, and $p_1$ and $p_3$ have disks of radii $a$ and $(\sqrt{2} - 1)a$ respectively: This is a valid configuration and both the disks of radius $a$ on $p_1$ and $p_7$ touch $v$ and $w$ respectively (see Figure 5(d)).

(g) $p_5$ and $p_7$ have disks of radii $(\sqrt{2} - 1)a$ and $a$ respectively, and $p_1, p_2, p_3, p_4$ have disks of radii $(1 - \sqrt{2}a), \sqrt{2}a, (1 - \sqrt{2}a), \sqrt{2}a$: This is a valid configuration where the disk of radius $a$ on $p_7$ touches $w$ (see Figure 5(c)).

(h) $p_5$ and $p_7$ have disks of radii $a$ and $(\sqrt{2} - 1)a$ respectively, and $p_1, p_2, p_3, p_4$ have disks of radii $(1 - \sqrt{2}a), \sqrt{2}a, (1 - \sqrt{2}a), \sqrt{2}a$: This is an invalid configuration since $d_4$ and $d_5$ overlap.

For the second part of the lemma, observe that the configurations of Figures 5(b), 5(e) and 5(f) touch only $w$, $v$ and $u$ respectively.

**Lemma 8.** The total area of disks on the points of the variable-gadget for a variable $x_i$ is maximized if and only if either all the points representing $x_i$ have disks of radius $a$ on them, or all the points representing $\overline{x_i}$ have disks of radius $a$ on them, and is equal to $2\pi ma^2$.

**Proof.** Follows from Lemma 7.

It is already mentioned in the earlier subsection that every CVC-path is of even length. The following lemma gives the optimum area of disks centered at the points on each CVC-path.
Lemma 9. Every CVC-path of length \( k \) has exactly three distinct maximum disk configurations, each having an area of \( \pi \frac{k^2}{2} b^2 \). If we are not allowed to draw a disk on any one of the end points of a connecting path, then it has exactly one possible maximum disk configuration, also having an area of \( \pi \frac{k^2}{2} b^2 \).

Proof. Consider a CVC-path \( \chi \) of \( k \) blue points; the set of blue points on its vertical and horizontal parts are denoted as \( \chi_V \) and \( \chi_H \) respectively. By Lemma 3, \( \Delta \chi \leq \Delta \chi_V + \Delta \chi_H = \pi \frac{k^1}{2} b^2 + \pi \frac{k^1}{2} b^2 = \pi \frac{k}{2} b^2 \). Equality is attained by each of the disk configurations in Figure 7.

Without loss of generality, let us assume that the CVC-path \( \chi \) travels vertically upward and then turns left to meet the clause-gadget. Since the topmost blue point of \( \chi_V \) and the rightmost blue point of \( \chi_H \) are only \( \sqrt{2} \) distance apart, both of them can not have disks on them in a maximum disk configuration. As the number of blue points on both \( \chi_V \) and \( \chi_H \) are even, if there is no disk on the bottommost blue point of \( \chi_V \) in a maximum disk configuration, then there must be a disk on the topmost blue point of \( \chi_V \). This implies, there is no disk on the rightmost blue point of \( \chi_H \) and hence a disk is present on the leftmost blue point of \( \chi_H \), giving an optimal configuration (see Figure 7(c)). Similarly, if there is no disk on the leftmost blue point in \( \chi_H \), then there is a disk on the bottommost blue point of \( \chi_H \). Thus, the other two optimal configurations are formed with disk on the lowest blue point in \( \chi_V \) and the leftmost blue point in \( \chi_H \) (see Figure 7(b)), and disk on the lowest blue point in \( \chi_V \) and the rightmost blue point of \( \chi_H \) (see Figure 7(d)). \( \square \)

Now, we consider the set of points \( P \) in all the clause-gadgets, variable-gadgets, and CVC-paths created for a PRM-3SAT formula \( \theta \). The following two lemmas give estimates of the total area in the optimum solution of the MADP problem for \( P \) for the case where \( \theta \) is satisfied, and \( \theta \) is unsatisfied.

Lemma 10. If \( \theta \) has a satisfying assignment, then there is a choice of non-intersecting disks on the points of \( P \) such that their total area is exactly equal to \( \pi((2n + (8 - 4\sqrt{2}))ma^2 + (K_i^2 + 2m)) \).

Proof. Consider a satisfying assignment of \( \theta \). For each variable-gadget \( u \), if \( u = 1 \) then draw the disk centered at \( u \), otherwise draw the disk centered at \( u \). We also draw the disks of radius \( b = 1 \) for the half of the extra \( K' - K_b \) points. Thus, half of the excess blue points for each clause contains disks. For each clause-gadget, say \( C_\alpha = x_i + x_j + x_k \), draw an optimum disk configuration satisfying Lemma 7 such that one disk of radius \( a \) on a green point must touch the last blue point of exactly one satisfying variable (literal), say \( x_i \). Thus, if \( x_i = 1 \) satisfies the clause \( C_\alpha \) then the disk at the last point \( u \) of the CVC-path from \( x_i \) to the clause-gadget of \( C_\alpha \) can not be drawn. Now, we can put \( \frac{\pi}{2} \) blue disks (of radius \( b = 1 \)) on the CVC-path connecting \( C_\alpha \) to the variable gadget of \( x_i \) since \( x_i = 1 \) and so the disks at variable \( x_i \) are put on the points marked as \( \pi_7 \). For the other two variables, namely \( x_j \) and \( x_k \) also, we can put exactly \( \frac{\pi k}{2} \) disks on their corresponding CVC-path irrespective of whether the disks are put at \( x_j \) or \( \pi_7 \) (resp. \( x_k \) or \( \pi_k \)) for the variable \( x_j \) (resp. \( x_k \)) since the
last point $v_j$ (resp. $v_k$) near to the clause $\theta_\alpha$ may contain a disk. Thus, the total area for all clause-gadgets is $\pi(2mb^2 + 2m(4 - 2\sqrt{2})a^2)$ (see Lemma 8), total area for $n$ variable-gadgets is $2\pi nma^2$ (see Lemma 8), and total area for all CVC-paths is $\frac{K'}{2}\pi b^2$. See Figure 10 for the demonstration. Thus, the total area is $\pi((8 - 4\sqrt{2})ma^2 + 2mb^2 + 3nma^2 + \frac{K'}{2}b^2)$. Putting $b = 1$ and simplifying, the result follows.

**Lemma 11.** If a PRM-3SAT formula $\theta$ is not satisfiable then the total area of the corresponding MADP problem is less than $\pi((2n + (8 - 4\sqrt{2}))ma^2 + (\frac{K'}{2} + 2m))$.

**Proof.** Let $\theta$ is not satisfiable. There is no difficulty to have a total area of $3\pi nma^2$ from the variable-gadgets corresponding to $n$ variables since the used disks at the red points of the variable gadgets are much larger than the disks used for the blue points of the CVC-paths near them. Similarly, the disks used for the green points are much larger than the disks used for the blue points in it and also the last point on its adjacent three CVC-paths. Moreover, among the four blue points $p_9, p_{10}, p_{11}$ and $p_{12}$, $p_{10}$ and $p_{12}$ can always be used for placing disks of radius $b = 1$. Also, exactly two such disks can be placed irrespective of any arbitrary assignment of disks among the other 8 green points in that clause-gadget. Thus, for each clause-gadget exactly $\pi(2m + 2m(4 - 2\sqrt{2})a^2)$ area is achieved in the optimum solution in the MADP problem with the point set $P$ corresponding to $\theta$. Now, let us consider the CVC-paths. As $\theta$ is not satisfiable, for each truth assignment $X$ of the variables there is at least one clause, say $\theta_\alpha$, that is not satisfiable. In the optimum disk assignment of the green points of $\theta_\alpha$, at least one of the green disks must touch the last (blue) point of the corresponding CVC-path. For the CVC-path, that is not touched by any green disk, one can put a disk (of radius $b = 1$) at its last point, and a total area of $\pi\frac{K'}{2}$ is achievable on the blue points along that path even if disk can not be placed at the other end of that CVC-path. But, if the last vertex of a CVC-path is touched by a green disk, no blue disk can be placed at its either end. Thus only a total area of $\pi(\frac{K'}{2} - 1)$ is achievable. Thus, for the truth assignment $X$, the total area obtained in the optimum solution of the MADP problem is at most $\pi((11 - 4\sqrt{2})ma^2 + (\frac{K'}{2} + 2m) - \beta)$, where $\beta$ is the number of non-satisfied clause(s). The result follows from the fact that if $\theta$ is not satisfiable, $\beta \geq 1$ for every truth assignment of the variables.

Thus, we can check the satisfiability of a PRM-3SAT formula with $m$ clauses and $n$ variables by generating the points $P$ as described, and then observing whether the total area of the disks in the optimum solution of the MADP problem on the point set $P$ is equal to or less than $\pi((2n + (8 - 4\sqrt{2}))ma^2 + (\frac{K'}{2} + 2m))$. As PRM-3SAT problem is NP-complete \cite{2}, we have the following result.

**Theorem 2.** The problem of finding disks of a maximum total area centered on a given set of points, is NP-hard.
4.3 MADP for axis-parallel squares

Now, we demonstrate that the MADP problem remains NP-hard when the objects are axis-parallel squares instead of disks.

Our reduction, as before, follows from PRM-3SAT. In fact, we just modify the point configuration for disks to get the reduction for squares. Our clause patterns are now simplified, with only six points, shown in Figure 11(a). Only three maximum configurations are possible, shown in Figures 11(b), 11(c) and 11(d). The variable patterns remain identical with only two possible maximum configurations, as shown in Figures 12(a) and 12(b). The CVC-paths also remain identical with only two possible maximum configurations having no square around one of the end points, as shown in Figures 13(a), 13(b), 13(c) and 13(d). The reduction proceeds as before, with the squares giving a maximum area of $4(4ma^2 + 2mna^2 + 50K + m)$ if and only if the corresponding PRM-3SAT formula is satisfiable.

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Figure 11: Clause gadget: MADP problem for squares

Figure 12: Variable gadget: MADP problem for squares
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5 Approximation algorithm

In this section, we show that the optimum solution for the MPDP problem proposed in [4] gives a 2-factor approximation result for the MADP problem. We also propose a PTAS for the problem.
5.1 2-factor approximation algorithm

Given a set of points $P$ in $\mathbb{R}^2$, let $R = \{r_i, i = 1, 2, \ldots, n\}$ be the set of radii of the points in $P$ obtained by the optimum solution for MPDP problem [4]. It is clear that any feasible solution of the MPDP is a feasible solution of the MADP problem. We show that an optimal radii returned by the MPDP problem produce at most $2 \times OPT$ area for the corresponding MADP problem, where $OPT$ is the optimum solution of that MADP problem.

**Lemma 12.** [4] The maximum sum of radii of non-overlapping disks, centered at points $p_i \in P$, equals half of the minimum total edge length of a collection of vertex-disjoint cycles (allowing 2-cycles) spanning the complete geometric graph on the points $p_i \in P$ with each edge having length equal to the distance between the end-points of that edge.

**Lemma 13.** [4] In the minimum total edge length of a collection of vertex-disjoint cycles mentioned in Lemma 12, each cycle is either of odd length or a 2-cycles (i.e., a single edge).

The implication of Lemma 12 and 13 is that in the optimum solution of the MPDP problem, each disk touches its neighboring disk(s) in the cycle in which it appears.

In [4], an $O(n^{1.5})$ time algorithm is proposed to compute the minimum length cycle cover $C$ of the complete geometric graph $G$ with a set $P$ of $n$ points on the plane. From the geometric property of the Euclidean distances, they show that if a subgraph $G'$ of $G$ is formed by removing all the edges $(p_i, p_j)$ satisfying $\text{dist}(p_i, N(p_i)) + \text{dist}(p_j, N(p_j)) < \text{dist}(p_i, p_j)$, then the minimum weight cycle cover of $G'$ remains same as that in $G$.

**Lemma 14.** For a given set of points $P$ arbitrarily placed in $\mathbb{R}^2$, the radii $\{r_i, i = 1, 2, \ldots, n\}$ in the optimum solution of the MPDP problem is a 2-approximation result for the MADP problem for the point set $P$. 

Figure 13: CVC-path gadget: MADP problem for squares
respectively, and $O$ the solution and in our algorithm for the MWIS problem of 

$$14).$$

Each disk is assigned weight equal to its area. Now we consider all the disks 

$$\cup$$

and use the algorithm of [5] to compute the maximum weight independent set (MWIS) 

$$\alpha$$

For each point $p_i$ we will use this algorithm in designing our PTAS.

Combining Lemma 14 with the time complexity result in [4], we have the following result.

**Theorem 3.** For a given set of points $P$ arbitrarily placed in $\mathbb{R}^2$, one can compute a 2-approximation result of the MADP problem in $O(n^{3})$ time.

### 5.2 PTAS

In this section, we propose a PTAS for the MADP problem. In [5], Erlebach et al. proposed a $(1 + \frac{1}{k})$-factor approximation algorithm for the maximum weight independent set for the intersection graph of a set of weighted disks of arbitrary size. It runs in $n^{O(k^2)}$ time. We will use this algorithm in designing our PTAS.

For each point $p_i \in P$, let the maximum possible radius be $\ell_i = \text{dist}(p_i, N(p_i))$. Thus, the maximum possible area be $\alpha_i = \pi \ell_i^2$. Given an integer $k$, we compute $h_i = \frac{\alpha_i}{k}$, and define $k + 1$ circles $C_i = \{C_i^0, C_i^1, \ldots, C_i^k\}$ centered at $p_i$ with area $\{0, h_i, 2h_i, \ldots, kh_i\}$ (see Figure 14). Each disk is assigned weight equal to its area. Now we consider all the disks $\cup_{i=1}^{n} C_i$, and use the algorithm of [5] to compute the maximum weight independent set (MWIS) $A$. Note that the number of disks centered at any point $p_i$ present in both the optimum solution and in our algorithm for the MWIS problem of $\cup_{i=1}^{n} C_i$ is exactly one.

Let $o_i$ and $a_i$ be the disks centered at $p_i$ in the optimum solution and in our solution ($A$) respectively, and $O_i, A_i$ be their respective area. Let $\Theta = \sum_{i=1}^{n} A_i$ be the solution obtained
by our algorithm, and $OPT = \sum_{i=1}^{n} O_i$ be the value of the optimum solution. We need to analyze the bound on $\frac{OPT}{OPT}$. Let $\tilde{OPT}$ be the optimum solution of the MWIS problem among the set of disks $\bigcup_{i=1}^{n} C_i$. Thus, $\frac{OPT}{OPT} \times \tilde{OPT}$ is a feasible solution of the MWIS problem among the disks $\bigcup_{i=1}^{n} C_i$. Following [5], $\tilde{OPT}$ is the optimum solution of the MWIS problem among the set of disks $\bigcup_{i=1}^{n} C_i$. Thus, $OPT \leq \tilde{OPT}$.

In other words, $\frac{OPT}{OPT} \leq 1 + \frac{k-4}{4}$, where $k' = \frac{k-4}{4}$. Thus, $\frac{OPT}{OPT} \leq (1 + \frac{1}{k})(1 + \frac{1}{k'}) \leq (1 + \frac{1}{k''})$, where $k'' = \frac{k-4}{5}$. Thus, we have the following result.

**Theorem 4.** Given a set of points $P$ in $\mathbb{R}^2$ and a positive integer $k$, we can get a $(1 + \frac{1}{k})$-approximation algorithm with time complexity $(nk)^O(k^2)$.

### 6 MADP in higher dimension

In this section, we first propose a constant factor approximation algorithm for the MADP problem where the points $P = \{p_1, p_2, \ldots, p_n\}$ are distributed in $\mathbb{R}^d$, and then we propose a PTAS for the same.
6.1 Approximation algorithm

Let $R = \{r_i, i = 1, 2, \ldots, n\}$ and $R = \{r_i, i = 1, 2, \ldots, n\}$ be the set of radii of the points in $P$ for the solution given by MPDP [4] and the optimum of MADP problem, respectively. As noted in Section 3, $R$ is a feasible solution for MADP problem, and the disks in $R$ can be partitioned into cycles where each disk is touching with two adjacent disks in the cycle.

**Lemma 15.** For a given set of points $P$ in $\mathbb{R}^d$, the radii $\{r_i, i = 1, 2, \ldots, n\}$ is a $2^{d-1}$-factor approximation result for the MADP problem.

**Proof.** Consider the cycles $C = \{C_1, C_2, \ldots, C_k\}$ generated by MPDP algorithm as in the proof of Lemma 14. We prove that, for each cycle $C_i$, we have $\sum_{r \in C_i} r_{\alpha}^d \geq \frac{1}{m^{d-1}} \sum_{r \in C_i} \rho_{\alpha}^d$. Aggregating these relations for all the cycles $C_i, i = 1, 2, \ldots, k$, we will have the desired result. As in Lemma 14, consider the following two cases.

**$C_i$ is a 2-cycle $(p_{\alpha}, p_{\beta})$:** Let $r = \text{dist}(p_{\alpha}, p_{\beta})$; $r_{\alpha} = \frac{r}{2} - \delta$ and $r_{\beta} = \frac{r}{2} + \delta$ be the radii for the two disks $C_{\alpha}', C_{\beta}'$ that maximizes the sum of square of the radii of these two disks, where $-\frac{r}{2} \leq \delta \leq \frac{r}{2}$. Thus, $r_{\alpha}^d + r_{\beta}^d = (\frac{r}{2} - \delta)^2 + (\frac{r}{2} + \delta)^2 \geq 2((\frac{r}{2})^2 + (\frac{r}{2})^2 - 2\delta^2 + \ldots) \geq \frac{1}{m^{d-1}} r^d$. The upper bound for any two non-overlapping disks having their centers $r$ distance apart is $\rho_{\alpha}^d + \rho_{\beta}^d \leq (\rho_{\alpha} + \rho_{\beta})^d \leq r^d$. Thus, for the two-cycle $C_i = (p_{\alpha}, p_{\beta})$, we have $r_{\alpha}^d + r_{\beta}^d \geq \frac{1}{m^{d-1}} r^d \geq \frac{1}{m^{d-1}} (\rho_{\alpha}^d + \rho_{\beta}^d)$.

**$C_i$ is an odd cycle:** Let the length of the cycle be $m$. Without loss of generality, assume that the vertices be $p_1, p_2, \ldots, p_m$. For each edge $(p_{\alpha}, p_{\alpha+1})$ of this cycle (where the indices are numbered modulo $m$), we have $r_{\alpha}^d + r_{\alpha+1}^d \geq \frac{1}{m^{d-1}} (\rho_{\alpha}^d + \rho_{\alpha+1}^d)$ (as explained in the earlier case). Adding these inequalities for $\alpha = 1, 2, \ldots, m$, we have $2 \sum_{\alpha=1}^{m} r_{\alpha}^d \geq \frac{1}{m^{d-1}} [2 \sum_{\alpha=1}^{m} \rho_{\alpha}^d]$. Ignoring 2 in both sides, we have the result.

As the time complexity of solving MPDP problem in $\mathbb{R}^d$ is $O(n^{2 - \frac{1}{d}})$, we have the following result.

**Theorem 5.** For a given set of points $P$ arbitrarily placed in $\mathbb{R}^d$, one can compute a $2^{d-1}$-approximation result of the MADP problem in $O(n^{2 - \frac{1}{d}})$ time.

6.2 PTAS

The same scheme of designing PTAS as in Section 5.2 also works in higher dimension due to the following reasons:

- The algorithm $(1 + \frac{1}{d})$-factor for the maximum weighted independent set in a disk graph with geometric layout of the disks and for a given $k$ also works in higher dimension in $n^{O(k^{2d-2})}$ time [3].
\( \theta \)

\[ \delta(p_1, p_2) \]

\( p_1 \)

\( p_2 \)

\( p_3 \)

\[ \theta \]

\[ \delta(p_1, p_2) \]

Figure 15: Property of the distance function \( \delta \)

- \( \text{OPT} \geq \frac{1}{d} \sum_{i=1}^{n} \ell_i^d \) in \( \mathbb{R}^d \), where \( \ell_i = \text{dist}(p_i, N(p_i)) \) using the same argument as in Section 5.2 since the volume of a ball with radius \( \frac{\ell_i}{2} \) in \( \mathbb{R}^d \) is proportionate to \( \frac{\ell_i^d}{2^d} \).

Thus, \( \frac{\text{OPT}}{\text{OPT}_d} \leq (1 + \frac{1}{k})(1 + \frac{2^d}{k-2^d}) = (1 + \frac{1}{k''}) \), where \( k'' = \frac{k-2^d}{2^d+1} \).

**Theorem 6.** Given a set of points \( P \) in \( \mathbb{R}^d \) and a positive integer \( k \), we can get a \((1 + \frac{1}{k})\)-approximation algorithm in time \( (nk)^{O(k^{d-2})} \).

7 MADP problem for \( 2m \)-regular convex polygons

In this section, we will show that both 2-approximation and PTAS results explained in Section 5.1 and 5.2, respectively, can be generalized for the MADP problem when the objective is to place non-overlapping \( 2m \)-regular convex polygons (\( m \geq 2 \)) of fixed orientation centered at the given set of points \( P \) such that the sum of area covered by them is maximized.

For a \( 2m \) regular convex polygon \( S \), the *width* of \( S \), denoted by \( w(S) \), is defined as the radius of the circle inscribed in \( S \) which touches all the edges of boundary of \( S \). Note that the area of \( S \) is \( 2m(w(S))^2 \tan \frac{\pi}{2m} \).

Now, we define the distance \( \delta(p_1, p_2) \) between two points \( p_1, p_2 \in \mathbb{R}^2 \) as the width of the minimum width \( 2m \)-regular convex polygon centered at the point \( p_1 \) containing the point \( p_2 \) (see Figure 15).

**Lemma 16.** The distance function is symmetric\(^5\), i.e., \( \delta(p_1, p_2) = \delta(p_2, p_1) \), where \( p_1 \) and \( p_2 \) are two points in \( \mathbb{R}^2 \).

**Proof.** Let \( S_1 \) be the \( 2m \)-regular polygon centered at \( p_1 \) and containing the point \( p_2 \) on its boundary. Note that \( 2m \)-regular polygon is symmetric. Thus, if we translate the polygon \( S_1 \) such that the center moves to \( p_2 \), then it will also contain the point \( p_1 \) on its boundary. Let this translated copy be \( S_2 \). According to the definition, \( \delta(p_1, p_2) = w(S_1) \) and \( \delta(p_2, p_1) = w(S_2) \). As \( w(S_1) = w(S_2) \), hence the property follows. \( \square \)

\(^5\)This property does not hold for odd regular convex polygon.
Lemma 17. Let $p_1$ and $p_3$ be any two points in $\mathbb{R}^2$ and let $p_2$ be any point on the line segment $\overline{p_1, p_3}$, then $\delta(p_1, p_3) = \delta(p_1, p_2) + \delta(p_2, p_3)$.

Proof. Let $S_1, S_2$ and $S_3$ be three $2m$-regular polygons centered at $p_1$, $p_2$ and $p_1$, respectively. Their widths are $\delta(p_1, p_2), \delta(p_2, p_3)$ and $\delta(p_1, p_3)$, respectively (see Figure 15). Without loss of generality, assume that the line segment $\overline{p_1, p_3}$ intersects the $i$-th side of these polygons. Let $d_j$ be the perpendicular from the center to the $i$-th side of the polygon $S_j$, for $j \in \{1, 2, 3\}$, and $\overline{p_1, p_3}$ makes an angle $\theta$ with the perpendicular $d_j$ ($j \in \{1, 2, 3\}$). Note that $w(S_1) = \text{dist}(p_1, p_2) \cos \theta$, $w(S_2) = \text{dist}(p_2, p_3) \cos \theta$ and $w(S_3) = \text{dist}(p_1, p_3) \cos \theta$, where $\text{dist}(p_i, p_j)$ is the Euclidean distance between two points $p_i$ and $p_j$. As $p_1, p_2$ and $p_3$ are co-linear, so $\text{dist}(p_1, p_3) = \text{dist}(p_1, p_2) + \text{dist}(p_2, p_3)$. Further, $w(S_1) = w(S_2) = w(S_3)$ and $\delta(p_1, p_2) = \delta(p_1, p_3) - \delta(p_2, p_3)$.

Hence the lemma follows.

Lemma 18. The distance function follows the triangular inequality, i.e., $\delta(p_1, p_3) \leq \delta(p_1, p_2) + \delta(p_2, p_3)$, where $p_1$, $p_2$ and $p_3$ are any three points in $\mathbb{R}^2$.

Proof. Let $S_2$ and $S_3$ be two $2m$-regular convex polygons centered at $p_1$ and containing the points $p_2$ and $p_3$, respectively, in their boundary. Now, if the width of $S_3$ is less than equal to the width of $S_2$, then the lemma holds true. So, without loss of generality, assume that $\delta(p_1, p_3) > \delta(p_1, p_2)$. Let $p$ be the intersection point of $S_2$ with the line segment $\overline{p_1, p_3}$, and $S$ be the smallest $2m$-regular polygon centered at $p$ containing the point $p_3$. The width of $S$ is $\delta(p_1, p_3) - \delta(p_1, p_2)$ (follows from Lemma 17). If $p_2$ does not coincide with $p$, then the translated copy of $S$ does not cover the point $p_3$ (see Figure 15). As a result, in this case, we need a $2m$-regular polygon $S'$ centered at $p_2$ of width at least $\delta(p_1, p_3) - \delta(p_1, p_2)$ to have the point $p_3$ on its boundary. Thus, the claim follows.

Lemma 19. The distance function $\delta$ satisfies the metric properties.

Proof. From the definition of the distance function, it is obvious that $\delta(p_1, p_2) = 0$ if and only if $p_1 = p_2$. Thus, the proof follows from Lemmata 16 and 18.

Combining Lemma 16 and 17, we have the following lemma.

Lemma 20. Let $p_1$ and $p_3$ be any two points in $\mathbb{R}^2$ and let $p_2$ be any point on the line segment $\overline{p_1, p_3}$, then $\delta(p_1, p_3) = \delta(p_3, p_1) = \delta(p_1, p_2) + \delta(p_3, p_2)$.
Above lemma along with the fact that the area of a $2m$-regular convex polygon with width $w$ is $2mw^2 \tan \frac{\pi}{2m}$ implies that the MADP problem for $2m$-regular polygon can be formulated as a quadratic programming problem as follows.

Maximize $\sum_{i=1}^{n} w_i^2$
Subject to $w_i + w_j \leq \delta(p_i, p_j)$, $\forall p_i, p_j \in P, i \neq j$.

Similarly, we can formulate the linear programming problem of MPDP problem where the objective is to maximize $\sum_{i=1}^{n} w_i$ with the same set of constraints. Note that Eppstein’s result [4] for MPDP problem holds when distance function is metric. The only difference is in time complexity which takes $O(n^3)$. Throughout our proof in Section 5.1 and 5.2 we have not used any special property of disk other than the metric property of the Euclidean distance function. Thus, using the distance function $\delta$ instead of Euclidean distance and assuming that $\delta(p_i, p_j)$ can be computed in constant time (which depends only on $m$ which is constant), we have the following results.

**Theorem 7.** For a given set of points $P$ arbitrarily placed in the plane, one can compute a 2-approximation result of the MADP problem for $2m$-regular convex polygons in $O(n^3)$ time.

**Theorem 8.** Given a set of points $P$ in $\mathbb{R}^2$ and a fixed integer $m$, where we have to place non-overlapping $2m$-regular convex polygons to maximize the area covered by them, we can get a $(1 + \frac{1}{k})$-approximation algorithm in time $(nk)^{O(k^2)}$.

Furthermore, note that the approach given in Section 6 also works for $2m$-regular convex polygons in fixed dimension $d$ with same approximation guarantee.

### 8 Conclusion

Following Eppstein’s work [4] on placing non-overlapping disks for a set of given points on the plane to maximize perimeter, we study the area maximization problem under the same setup. If the points are placed on a straight line, then the area maximization problem is solvable in polynomial time. Though the perimeter maximization problem in $\mathbb{R}^2$ is polynomially solvable, the area maximization problem is shown to be NP-hard. We also observe that the solution of the perimeter maximization problem gives a $2^{d-1}$-factor approximation result of the area maximization problem in $\mathbb{R}^d$. A PTAS for the MADP problem in $\mathbb{R}^d$ is also proposed. Finally, we show that these results for MADP problem can be generalized for different types of objects: squares, and regular convex polygons with even number of edges.

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