On the fixed volume discrepancy of the Korobov point sets

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Abstract. This paper is devoted to the study of a discrepancy-type characteristic—the fixed volume discrepancy—of Korobov point sets in the unit cube. It has been observed recently that this new characteristic allows us to obtain an optimal rate of dispersion decay. This observation has motivated us to study this new version of discrepancy thoroughly; it also seems to have independent interest. This paper extends recent results due to Temlyakov and Ullrich on the fixed volume discrepancy of Fibonacci point sets.

Bibliography: 23 titles.

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§1. Introduction

This paper follows up the recent paper [1]. It is devoted to the study of a discrepancy-type characteristic—the fixed volume discrepancy—of a point set in the unit cube $\Omega_d := [0,1)^d$. We refer the reader to the books and survey papers [2]–[9] on discrepancy theory and numerical integration. Recently, an important new observation was made in [10]. This claims that a new version of discrepancy—$r$-smooth fixed volume discrepancy—allows us to obtain an optimal rate of dispersion decay from numerical integration results (see [11]–[19] for some recent results on dispersion). This observation motivated us to undertake a detailed study of this new version of discrepancy, which seems to be of independent interest.

The $r$-smooth fixed volume discrepancy takes into account two characteristics of the smooth hat function $h_B^r$, its smoothness $r$ and the volume of its support $v := \text{vol}(B)$ (see the definition of $h_B^r$ below). We now proceed to give a formal description of the problem setting and the formulation of results.

Let $\chi_{[a,b)}(x)$ denote the univariate characteristic function of the interval $[a,b) \subset \mathbb{R}$ and, for $r = 1, 2, 3, \ldots$, we define recursively

$$h_u^1(x) := \chi_{[-u/2,u/2)}(x)$$

and

$$h_u^r(x) := h_u^{r-1}(x) * h_u^1(x),$$

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where

\[ f(x) * g(x) := \int_{\mathbb{R}} f(x - y)g(y) \, dy. \]

Note that \( h_u^2 \) is the hat function, that is, \( h_u^2(x) = \max\{u - |x|, 0\} \).

Let \( \Delta_t f(x) := f(x) - f(x + t) \) be the first difference. We say that a univariate function \( f \) has smoothness 1 in \( L_1 \) if \( \|\Delta_t f\|_1 \leq C|t| \) for some absolute constant \( C < \infty \). In case \( \|\Delta_t^r f\|_1 \leq C|t|^r \), where \( \Delta_t^r := (\Delta_t)^r \) is the \( r \)th difference operator, \( r \in \mathbb{N} \), we say that \( f \) has smoothness \( r \) in \( L_1 \). Then \( h_u^r(x) \) has smoothness \( r \) in \( L_1 \) and has support \((-ru/2, ru/2)\).

Given a parallelepiped \( B \) of the form

\[ B := B(u, z) := \prod_{j=1}^{d} \left[ z_j - \frac{ru_j}{2}, z_j + \frac{ru_j}{2} \right) \]  

(1.1)

define

\[ h_u^r(x) := \prod_{j=1}^{d} h_{u_j}^r(x_j) \quad \text{and} \quad h_B^r(x) := h_u^r(x - z). \]  

(1.2)

We begin with nonperiodic \( r \)-smooth fixed volume discrepancy, which was introduced and studied in [10].

**Definition 1.1.** Let \( r \in \mathbb{N}, v \in (0, 1] \) and let \( \xi := \{\xi^\mu\}_{\mu=1}^{m} \subset [0, 1)^d \) be a point set. We define the \( r \)-smooth fixed volume discrepancy with equal weights by

\[ D^r(\xi, v) := \sup_{B \subset \Omega_d: \vol(B) = v} \left| \int_{\Omega_d} h_B^r(x) \, dx - \frac{1}{m} \sum_{\mu=1}^{m} h_B^r(\xi^\mu) \right|. \]  

(1.3)

It is well known that the Fibonacci cubature formulae are optimal in the sense of order for numerical integration of functions of two variables from different smoothness classes (see [7], [20] and [5], for example). We present a result from [10], which shows that a Fibonacci point set has good fixed volume discrepancy.

Let \( \{b_n\}_{n=0}^\infty, \ b_0 = b_1 = 1 \) and \( b_n = b_{n-1} + b_{n-2}, n \geq 2, \) be the Fibonacci numbers. Denote the \( n \)th Fibonacci point set by

\[ \mathcal{F}_n := \left\{ \left( \frac{\mu}{b_n}, \frac{\mu b_{n-1}}{b_n} \right) : \mu = 1, \ldots, b_n \right\}. \]

In this definition \( \{a\} \) is the fractional part of the number \( a \). The cardinality of the set \( \mathcal{F}_n \) is \( b_n \). In [10] we proved the following upper bound.

**Theorem 1.1.** Let \( r \geq 2. \) Then there exist positive constants \( c \) and \( C \) such that for any \( v \geq c/b_n \)

\[ D^r(\mathcal{F}_n, v) \leq C \frac{\log(b_n v)}{b_n^r}. \]  

(1.4)

The main object of interest in [1] was the periodic \( r \)-smooth \( L^p \)-discrepancy of Fibonacci point sets. To look at this, we define the periodization \( \tilde{f} \) (with period 1 in each variable) of a function \( f \in L_1(\mathbb{R}^d) \) with compact support by

\[ \tilde{f}(x) := \sum_{m \in \mathbb{Z}^d} f(m + x), \]

and we let \( \tilde{h}_u^r \) be the periodization of \( h_u^r \) from (1.2).
We now define the periodic $r$-smooth $L_p$-discrepancy.

**Definition 1.2.** For $r \in \mathbb{N}$, $1 \leq p \leq \infty$ and $v \in (0, 1]$, the periodic $r$-smooth fixed volume $L_p$-discrepancy of a point set $\xi$ is

$$
\tilde{D}_p^r(\xi, v) := \sup_{u: \text{vol}(B(u, 0)) = v} \left\| \int_{\Omega_d} \tilde{h}_u^r(x - z) \, dx - \frac{1}{m} \sum_{\mu=1}^{m} \tilde{h}_u^r(\xi^\mu - z) \right\|_p,
$$

(1.5)

where the $L_p$-norm is taken with respect to $z$ over the unit cube $\Omega_d = [0, 1)^d$.

In the case when $p = \infty$ this concept was introduced and studied in [21].

The following upper bound for $1 \leq p < \infty$ was proved in [1].

**Theorem 1.2.** Let $r \in \mathbb{N}$ and $1 \leq p < \infty$. Then there exist positive constants $c$ and $C$ such that for any $v \geq c/b_n$ we have

$$
\tilde{D}_p^r(\mathcal{F}_n, v) \leq C \frac{\sqrt{\log (b_nv)}}{b_n^r}.
$$

In the case $p = \infty$ a weaker upper bound was proved in [1].

**Theorem 1.3.** Let $r \geq 2$. Then there exist positive constants $c$ and $C$ such that for any $v \geq c/b_n$

$$
\tilde{D}_\infty^r(\mathcal{F}_n, v) \leq C \frac{\log (b_nv)}{b_n^r}.
$$

Some comments in [1] established that Theorems 1.2 and 1.3 are sharp in a certain sense.

Our main interest in this paper is the Korobov cubature formulae, rather than the Fibonacci cubature formulae, from the point of view of the fixed volume discrepancy. We prove a conditional result under the assumption that the Korobov cubature formulae are exact on a certain subspace of trigonometric polynomials with frequencies in a hyperbolic cross. There are results that guarantee the existence of such cubature formulae (see §3 for a discussion).

Let $m \in \mathbb{N}$ and let $a := (a_1, \ldots, a_d)$, where $a_1, \ldots, a_d \in \mathbb{Z}$. We consider the cubature formulae

$$
P_m(f, a) := m^{-1} \sum_{\mu=1}^{m} f \left( \left\{ \frac{\mu a_1}{m} \right\}, \ldots, \left\{ \frac{\mu a_d}{m} \right\} \right),
$$

which are called the **Korobov cubature formulae**. In the case when $d = 2$, $m = b_n$ and $a = (1, b_{n-1})$ we have

$$
P_m(f, a) = \Phi_n(f) := \frac{1}{b_n} \sum_{y \in \mathcal{F}_n} f(y).
$$

Set

$$
y^\mu := \left( \left\{ \frac{\mu a_1}{m} \right\}, \ldots, \left\{ \frac{\mu a_d}{m} \right\} \right), \quad \mu = 1, \ldots, m, \quad \mathcal{K}_m(a) := \{ y^\mu \}_{\mu=1}^m.
$$
The set $K_m(a)$ is called a Korobov point set. Further, denote

$$S(k, a) := P_m(e^{i2\pi(k,x)}, a) = m^{-1} \sum_{\mu=1}^{m} e^{i2\pi(k,y^{\mu})}, \quad k \in \mathbb{Z}^d.$$  

Note that

$$P_m(f, a) = \sum_{k} \hat{f}(k) S(k, a), \quad \hat{f}(k) := \int_{[0,1)^d} f(x) e^{-i2\pi(k,x)} \, dx, \quad (1.6)$$

where for simplicity we assume that $f$ is a trigonometric polynomial. It is clear that (1.6) holds for $f$ with absolutely convergent Fourier series.

It is easy to see that the following relation holds:

$$S(k, a) = \begin{cases} 1, & k \in \mathcal{L}(m, a), \\ 0, & k \notin \mathcal{L}(m, a), \end{cases} \quad (1.7)$$

where

$$\mathcal{L}(m, a) := \{k: (a, k) \equiv 0 \pmod{m}\}.$$  

For the rest of the paper, if $M \subset \mathbb{Z}^d$ we set $M' := M \setminus \{0\}$.

For $N \in \mathbb{N}$ define the hyperbolic cross by

$$\Gamma(N) = \Gamma(N, d) := \left\{k = (k_1, \ldots, k_d) \in \mathbb{Z}^d: \prod_{j=1}^{d} \max(|k_j|, 1) \leq N\right\}.$$  

Set

$$T(N, d) := \left\{f: f(x) = \sum_{k \in \Gamma(N, d)} c_k e^{i2\pi(k,x)} \right\}.$$  

It is easy to see that the condition

$$P_m(f, a) = \hat{f}(0), \quad f \in T(N, d), \quad (1.8)$$

is equivalent to

$$\Gamma(N, d) \cap \mathcal{L}(m, a)' = \emptyset. \quad (1.9)$$

**Definition 1.3.** We say that the Korobov cubature formula $P_m(\cdot, a)$ is exact on $T(N, d)$ if condition (1.8) (equivalently, condition (1.9)) is satisfied.

**Theorem 1.4.** Suppose that $P_m(\cdot, a)$ is exact on $T(L, d)$ for some $L \in \mathbb{N}$, $L \geq 2$. Let $r \in \mathbb{N}$ and $1 \leq p < \infty$. Then there exist positive constants $c(r, d)$ and $C(r, d, p)$ such that for any $v \geq c(r, d)/L$

$$\tilde{D}^{r}_p(K_m(a), v) \leq C(r, d, p) L^{-r} (\log(Lv))^{(d-1)/2}.$$  

For $p = \infty$ we prove a weaker upper bound.

**Theorem 1.5.** Let $r \geq 2$. Suppose that $P_m(\cdot, a)$ is exact on $T(L, d)$ for some $L \in \mathbb{N}$, $L \geq 2$. Then there exist positive constants $c(r, d)$ and $C(r, d)$ such that for any $v \geq c(r, d)/L$

$$\tilde{D}^{r}_\infty(K_m(a), v) \leq C(r, d) L^{-r} (\log(Lv))^{d-1}.$$
§2. Proofs of Theorems 1.4 and 1.5

The proofs of both theorems go along the same lines. We give a detailed proof of Theorem 1.4 and point out the modifications of this proof that give Theorem 1.5. For continuous functions of \( d \) variables that are \( 1 \)-periodic in each variable consider the Korobov cubature formula \( P_m(\cdot, a) \).

For univariate test functions \( h^r_u \), from the properties of convolution we obtain

\[
\hat{h}^r_u(y) = \hat{h}^r_u(y)\hat{h}^1_u(y), \quad y \in \mathbb{R},
\]

which implies that

\[
\hat{h}^r_u(y) = \left( \frac{\sin(\pi yu)}{\pi y} \right)^r
\]

for \( y \neq 0 \). Therefore,

\[
\left| \hat{h}^r_u(k) \right| \leq \min \left( |u|^r, \frac{1}{|k|^r} \right) = \left( \frac{|u|}{k'} \right)^{r/2} \min \left( |k'u|^{r/2}, \frac{1}{|k'u|^{r/2}} \right),
\]

where \( k' := \max\{1, |k|\} \). (Here, we are using \( \hat{h} \) for the Fourier transform of \( h \) on \( \mathbb{R} \). This should not lead to any confusion.)

It is convenient for us to use the following abbreviated notation for a product:

\[
pr(u) := \prod_{j=1}^d u_j.
\]

We have

\[
\hat{h}^r_B(k) = e^{-i2\pi(k,u)} \hat{h}^r_u(k).
\]

Therefore, we obtain the upper bound

\[
\left| \hat{h}^r_B(k) \right| \leq \prod_{j=1}^d \left( \frac{|u_j|}{k'_j} \right)^{r/2} \min \left( |k'_j u_j|^{r/2}, \frac{1}{|k'_j u_j|^{r/2}} \right).
\]

For \( s \in \mathbb{N}_0^d \) we set

\[
\rho(s) := \{ k \in \mathbb{Z}^d : [2^{s_j-1}] \leq |k_j| < 2^{s_j}, \ j = 1, \ldots, d \},
\]

where \([a]\) denotes the integer part of \( a \), and, for \( k \in \rho(s) \), we obtain

\[
\left| \hat{h}^r_B(k) \right| \leq H^r_u(s) := \left( \frac{2^d pr(u)}{2\|s\|_1} \right)^{r/2} \prod_{j=1}^d \min \left( (2^{s_j} u_j)^{r/2}, \frac{1}{(2^{s_j} u_j)^{r/2}} \right).
\]

Below we will need certain sums of these quantities. First, consider

\[
\sigma^r_u(t) := \sum_{\|s\|_1=t} \prod_{j=1}^d \min \left( (2^{s_j} u_j)^{r/2}, \frac{1}{(2^{s_j} u_j)^{r/2}} \right), \quad t \in \mathbb{N}_0.
\]

The following technical lemma is part (I) of Lemma 6.1 in [10].
**Lemma 2.1.** Let \( r, t \in \mathbb{N} \) and \( u \in \mathbb{R}^d_+ \) be such that \( \text{pr}(u) \geq 2^{-t} \). Then

\[
\sigma^r_u(t) \leq C(d) \left( \frac{(\log (2^{t+1} \text{pr}(u)))^{d-1}}{(2^t \text{pr}(u))^{r/2}} \right).
\]

This lemma, together with (2.2), implies that

\[
\sum_{\|s\|_1 = t} H^r_u(s)^2 \leq \left( \frac{2^d \text{pr}(u)}{2^t} \right)^r \sigma^{2r}_u(t) \leq C_1(r, d) 2^{-2rt} \left( \log (2^t v) \right)^{d-1}
\]

(2.3)

for all \( r \geq 1 \) and \( t \in \mathbb{N}_0 \) such that \( v \geq r^d 2^{-t+1} \) and some constant \( C_1(r, d) < \infty \), where \( v := \text{vol}(B) = r^d \text{pr}(u) \).

In addition, we need a result from harmonic analysis, which is a corollary of the Littlewood-Paley theorem. Set

\[
\delta_s(f, x) := \sum_{k \in \rho(s)} \hat{f}(k) e^{i2\pi(k,x)}.
\]

Then it is known that for \( p \in [2, \infty) \)

\[
\|f\|_p \leq C(d, p) \left( \sum_{s \in \mathbb{N}^d_0} \|\delta_s(f)\|_p^2 \right)^{1/2}.
\]

(2.4)

Note that in the proof of Theorem 1.5 we use the simple triangle inequality

\[
\|f\|_\infty \leq \sum_s \|\delta_s(f)\|_\infty
\]

(2.5)

instead of (2.4).

Let

\[
E^r_u(z) := \frac{1}{m} \sum_{\mu=1}^m \tilde{h}^r_u(y^\mu - z) - \int_{[0,1]^d} \tilde{h}^r_u(x) \, dx
\]

(2.6)

so that

\[
\tilde{D}^r_u(K_m(a), v) = \sup_{u: \text{vol}(B(u, 0)) = v} \|E^r_u\|_p.
\]

From formulae (1.6), (1.7) and (2.1) we obtain

\[
E^r_u(z) = \sum_{k \neq 0} \tilde{h}^r_u(k) S(k, a)e^{-i2\pi(k,z)} = \sum_{k \in \mathcal{L}(m,a)'} \tilde{h}^r_u(k) e^{-i2\pi(k,z)}.
\]

It is apparent from (2.4) that it remains to bound \( \|\delta_s(E^r_u)\|_p \).

If \( t \neq 0 \) is such that \( 2^t \leq L \) then for \( s \) with \( \|s\|_1 = t \) we have \( \rho(s) \subset \Gamma(L) \). Then our assumption (1.9) implies that \( S(k, a) = 0 \) for \( k \in \rho(s) \) and therefore \( \delta_s(E^r_u) = 0 \).

Let \( t_0 \in \mathbb{N} \) be the smallest number satisfying \( 2^{t_0} > L \), that is, \( t_0 > \log L \). Then from (2.4), for \( p \in [2, \infty) \) we have

\[
\|E^r_u\|_p \leq C(d, p) \left( \sum_{t = t_0}^\infty \sum_{\|s\|_1 = t} \|\delta_s(E^r_u)\|_p^2 \right)^{1/2}.
\]

(2.7)
Moreover, (1.9) implies that for $t \geq t_0$ we have
\[
\#(\rho(s) \cap \mathcal{L}(m, a)) \leq C_2(d)2^{t-t_0}, \quad \|s\|_1 = t. \tag{2.8}
\]
The bound in (2.8) is a corollary of the following simple fact.

**Proposition 2.1.** Let $P$ be a parallelepiped such that $P^- := \{k - n : k, n \in P\}$ does not contain points from $\mathcal{L}(m, a)'$. Then for any shifted parallelepiped $\tilde{P} := \{k : k' = k + n, k \in P\}$ the inequality $|\mathcal{L}(m, a)' \cap \tilde{P}| \leq 1$ holds.

**Proof.** In fact, if there are two distinct points $k'_1 = k_i + n$, $i = 1, 2$, in $\mathcal{L}(m, a)' \cap \tilde{P}$, then $k_1 - k_2$ belongs to $\mathcal{L}(m, a)' \cap P^-$. This contradicts our assumption on $P^-$. The proposition is proved.

Now, (2.8) follows since the set $\rho(s)$ is the union of $O(d)$ dyadic parallelepipeds such as $\{k \in \mathbb{Z}^d : 2^{s_j - 1} \leq k_j < 2^{s_j}, j = 1, \ldots, d\}$ and each of these is the union of $2^{t-t_0}$ shifts of similar parallelepipeds lying in $\Gamma(L)$.

By Parseval’s identity we obtain
\[
\|\delta_s(E^n_u)\|_2 = \sqrt{\sum_{k \in \rho(s) \cap \mathcal{L}(m, a)} |\widehat{h^n_u}(k)|^2} \leq \sqrt{\#(\rho(s) \cap \mathcal{L}(m, a)) \cdot H^n_u(s)},
\]
and, by the triangle inequality,
\[
\|\delta_s(E^n_u)\|_\infty \leq \#(\rho(s) \cap \mathcal{L}(m, a)) H^n_u(s).
\]
Hence, using the inequality
\[
\|f\|_p \leq \|f\|_2^{2/p}\|f\|_\infty^{1-2/p}
\]
for $2 \leq p \leq \infty$ we have
\[
\|\delta_s(E^n_u)\|_p \leq (\#(\rho(s) \cap \mathcal{L}(m, a)))^{1-1/p} H^n_u(s).
\]
Combining this with (2.3), (2.7) and (2.8), for all $v = \text{vol}(B) \geq 2r^d 2^{-t_0}$ and $p \in [2, \infty)$ we finally obtain
\[
\|E^n_u\|_p \leq C \left( \sum_{t=t_0}^{\infty} 2^{2(t-t_0)(1-1/p)} \sum_{\|s\|_1 = t} H^n_u(s)^2 \right)^{1/2} \leq C' \left( \sum_{t=t_0}^{\infty} 2^{2(t-t_0)(1-1/p)} 2^{-2rt} (\log(2^t v))^{d-1} \right)^{1/2} = C'' 2^{-r t_0} \left( \sum_{t=0}^{\infty} 2^{2t(1-1/p-r)} (\log(2^{t_0} v))^{d-1} \right)^{1/2} \leq C''' 2^{-r t_0} (\log(2^{t_0} v))^{(d-1)/2} \left( \sum_{t=0}^{\infty} t^2 2^{2t(1-1/p-r)} \right)^{1/2}.
\]
Using the inequalities \( t_0 \geq \log L \) and \( \| E^r_u \|_p \leq \| E^r_u \|_2 \) for \( p < 2 \), this implies Theorem 1.4. (Here we have used the fact that, clearly, \( r > 1 - 1/p \) for \( p < \infty \).)

As we pointed out above, in the proof of Theorem 1.5 we use inequality (2.5) instead of (2.4). Moreover, instead of (2.3) we use

\[
\sum_{|s|_1 = t} H^*_u(s) \leq C_1(r, d)2^{-rt} (\log(2^t v))^{d-1},
\]

for all \( r \geq 1 \). However, note that we need \( r > 1 \) for the last series in the above computation to be finite. This implies

\[
\| E^r_u \|_\infty \leq C(r, d)L^{-r} (\log(Lv))^{d-1}.
\]

§3. Dispersion of Korobov point sets

We recall the definition of dispersion. Let \( d \geq 2 \) and let \([0, 1)^d\) be the \( d\)-dimensional unit cube. For \( x, y \in [0, 1)^d \) with \( x = (x_1, \ldots, x_d) \) and \( y = (y_1, \ldots, y_d) \) we write \( x < y \) if this inequality holds coordinatewise. For \( x < y \) we write \([x, y)\) for the axis-parallel box \([x_1, y_1) \times \cdots \times [x_d, y_d)\) and set

\[
B := \{ [x, y) : x, y \in [0, 1)^d, x < y \}.
\]

For \( n \geq 1 \) let \( T \) be a set of points in \([0, 1)^d\) with cardinality \(|T| = n\). The volume of the largest empty (that is, containing no points in \( T \)) axis-parallel box that can be inscribed in \([0, 1)^d\) is called the dispersion of \( T \):

\[
\text{disp}(T) := \sup_{B \in B : B \cap T = \emptyset} \text{vol}(B).
\]

An interesting extremal problem is to find (estimate) the minimum dispersion of point sets with fixed cardinality:

\[
\text{disp}^*(n, d) := \inf_{T \subseteq [0, 1)^d, |T| = n} \text{disp}(T).
\]

It is known that

\[
\text{disp}^*(n, d) \leq \frac{C^*(d)}{n}. \tag{3.1}
\]

Inequality (3.1) with \( C^*(d) = 2^{d-1} \prod_{i=1}^{d-1} p_i \), where \( p_i \) denotes the \( i \)-th prime number, was proved in [13] (see also [14]). The authors of [13] used the Halton-Hammersley set of \( n \) points (see [3]). Inequality (3.1) with \( C^*(d) = 2^{d+1} \) was proved in [11].

The authors of [11], following Larcher, used \((t, r, d)\)-nets (see [22] and [3] for results on \((t, r, d)\)-nets and see Definition 3.1 below).

**Definition 3.1.** A \((t, r, d)\)-net (in base 2) is a set \( T \) of \( 2^r \) points in \([0, 1)^d\) such that each dyadic box \([a_1-1/2^{s_1}, a_12^{-s_1}) \times \cdots \times [a_d-1/2^{s_d}, a_d2^{-s_d})\), \( 1 \leq a_j \leq 2^{s_j}, j = 1, \ldots, d\), of volume \( 2^{-r} \) contains exactly \( 2^t \) points of \( T \).

It was demonstrated in [10] that good upper bounds for fixed volume discrepancy can be used to prove good upper bounds for dispersion. This fact is one of the motivations for studying the fixed volume discrepancy. Theorem 3.1 below was derived from Theorem 1.1 (see [10]). The upper bound in Theorem 3.1, in combination with the trivial lower bound, shows that the Fibonacci point sets provide an optimal rate of decay for the dispersion.
Theorem 3.1. There exists an absolute constant \( C \) such that for all \( n \)
\[
\text{disp}(\mathcal{F}_n) \leq \frac{C}{b_n}.
\] (3.2)

The reader can find further recent results on dispersion in [15]–[17] and [12].

We now turn to some new results on the dispersion of Korobov point sets. We need the following simple statement.

Proposition 3.1. Suppose that \( \text{disp}(\xi) \geq v \). Then \( D^r(\xi, v) \geq (vr^{-d})^r \).

Proposition 3.1 is an extension of the trivial inequality
\[
D^1(\xi) := \sup_{v \leq 1} D^1(\xi, v) \geq \text{disp}(\xi).
\] (3.3)

Proof of Proposition 3.1. Let \( B \in \mathcal{B} \) be an empty box, that is, \( B \cap \xi = \emptyset \). Set \( v = \text{vol}(B) \). Note that for \( B \in \mathcal{B} \) we have \( \tilde{h}_B^r(x) = h_B^r(x) \). Then, from Definition 1.1 of \( D^r(\xi, v) \) it follows that
\[
\left| \int_{\Omega_d} h_B^r(x) \, dx - \frac{1}{m} \sum_{\mu=1}^m h_B^r(y^\mu) \right| \leq D^r(\xi, v).
\] (3.4)

Our assumption that \( B \) is an empty box implies that
\[
\frac{1}{m} \sum_{\mu=1}^m h_B^r(\xi^\mu) = 0,
\]
and therefore, from (3.4) we obtain
\[
\left| \int_{\Omega_d} h_B^r(x) \, dx \right| \leq D^r(\xi, v).
\] (3.5)

Since by the definition of \( h_B^r(x) \) we have
\[
\left| \int_{\Omega_d} h_B^r(x) \, dx \right| = (vr^{-d})^r,
\] (3.6)
the proof is complete.

Theorem 3.2. Suppose that \( P_m(\cdot, a) \) is exact on \( T(L, d) \) for some \( L \in \mathbb{N}, L \geq 2 \). Then there exists a positive constant \( C_1(d) \) such that
\[
\text{disp}(\mathcal{K}_m(a)) \leq C_1(d) L^{-1}.
\]

Proof. The proof is based on Theorem 1.5. We specify some \( r \in \mathbb{N}, r \geq 2 \), say, \( r = 2 \). If \( v \leq c(2, d)L^{-1} \), where \( c(2, d) \) is from Theorem 1.5, then Theorem 3.2 with \( C_1(d) \geq c(2, d) \) follows. Assume that \( v \geq c(2, d)L^{-1} \). Then by Theorem 1.5 we have
\[
\tilde{D}_\infty^2(\mathcal{K}_m(a), v) \leq C(2, d)L^{-2}(\log(Lv))^{d-1}.
\]
We apply Proposition 3.1 with \( r = 2 \) and the inequality \( D^r(\xi, v) \leq \tilde{D}_\infty^r(\xi, v) \) and obtain the estimate \( (v2^{-d})^2 \leq C(2, d)L^{-2}(\log(Lv))^{d-1} \). Therefore, \( Lv \leq C'(2, d) \). Setting \( C_1(d) := \max(c(2, d), C'(2, d)) \) we complete the proof of Theorem 3.2.
Now we make some comments.

**Fibonacci point sets.** As we mentioned above, in the case when \( d = 2, m = b_n \) and \( \mathbf{a} = (1, b_{n-1}) \) we have \( P_m(f, \mathbf{a}) = \Phi_n(f) \). In this case set \( L(n) := L(m, \mathbf{a}) \).

In other words
\[
L(n) := \{ k = (k_1, k_2) \in \mathbb{Z}^2 : k_1 + b_{n-1}k_2 \equiv 0 \pmod{b_n} \}.
\]

The following lemma is well known (for instance, see [5], p.274).

**Lemma 3.1.** There exists an absolute constant \( \gamma > 0 \) such that, for any \( n > 2 \), the two-dimensional hyperbolic cross satisfies
\[
\Gamma(\gamma b_n, 2) \cap L(n)' = \emptyset.
\]

Combining Theorem 3.2 with Lemma 3.1 gives Theorem 3.1.

**Special Korobov point sets.** Fix \( L \in \mathbb{N} \). We are interested in the smallest possible \( m \) such that a Korobov cubature formula exact on \( T(L, d) \) exists. In the case when \( d = 2 \) the Fibonacci cubature formula is an ideal choice in a certain sense. No Korobov cubature formulae are known for \( d \geq 3 \) that are as good as the Fibonacci cubature formula for \( d = 2 \). We formulate some known results in this direction. Consider the special case \( \mathbf{a} = (1, a, a^2, \ldots, a^{d-1}) \), \( a \in \mathbb{N} \). In this case, in the notation for \( K_m(a) \) and \( P_m(\cdot, \mathbf{a}) \) we put a scalar \( a \) instead of a vector \( \mathbf{a} \), namely, \( K_m(a, d) \) and \( P_m(\cdot, a, d) \). The following Lemma 3.2 is a simple well-known result (for instance, see [5], p.285).

**Lemma 3.2.** Let \( m \) and \( L \) be a prime and a natural number, respectively, such that
\[
|\Gamma(L, d)| < \frac{m-1}{d}.
\]

Then there is a natural number \( a \in [1, m) \) such that for all \( k \in \Gamma(L, d) \), \( k \neq 0 \)
\[
k_1 + ak_2 + \cdots + a^{d-1}k_d \not\equiv 0 \pmod{m}.
\]

Combining Theorem 3.2 with Lemma 3.2 yields the following Proposition 3.2.

**Proposition 3.2.** There exists a positive constant \( C_2(d) \), which depends only on \( d \), with the following property. For any \( L \in \mathbb{N}, L \geq 2 \), there exist a prime number \( m \leq C_2(d)|\Gamma(L, d)| \) and a natural number \( a \in [1, m) \) such that
\[
\text{disp}(K_m(a, d)) \leq C_1(d)L^{-1}.
\]

Since \( |\Gamma(L, d)| \asymp L(\log L)^{d-1} \), Proposition 3.2 implies that for any prime \( m \) there exists a natural number \( a \in [1, m) \) such that
\[
\text{disp}(K_m(a, d)) \leq C(d)m^{-1}(\log m)^{d-1}.
\]

Combining inequality (3.3) with a very interesting result from [23], which states that for any \( m \) there exists an \( \mathbf{a} \) such that
\[
D^1(K_m(\mathbf{a})) \leq C(d)m^{-1}(\log m)^{d-1} \log \log m,
\]

\[
\text{disp}(K_m(\cdot, \mathbf{a})) \leq C(d)m^{-1}(\log m)^{d-1} \log \log m.
\]
for this \( a \) we obtain

\[
\text{disp}(\mathcal{K}_m(a)) \leq C(d)m^{-1}(\log m)^{d-1} \log \log m.
\]

We now present a corollary of Proposition 3.2.

**Corollary 3.1.** Let \( C_1(d) \) be the number from Theorem 3.2 and let \( p \) be a prime number. Then there exist a natural number \( a \in [1, p) \) such that for any segments of the natural number series \( I_1, \ldots, I_d \subset [1, p] \) satisfying

\[
\prod_{j=1}^{d} |I_j| \geq C_1(d)p^{d-1}(\log p)^{d-1},
\]

there exists a natural number \( \mu \in [1, p] \) such that

\[
\begin{cases}
\mu \in I_1 \pmod{p}, \\
\mu a \in I_2 \pmod{p}, \\
\vdots \\
\mu a^{d-1} \in I_d \pmod{p}.
\end{cases}
\]

(3.9)

**Proof.** Take

\[
L = \frac{C(d)p}{(\log p)^{d-1}},
\]

where \( C(d) \) is small enough to satisfy (3.7). By Lemma 3.2 there exists a natural number \( a \in [1, p) \) such that \( P_p(\cdot, a) \) is exact on \( T(L, d) \).

Set \( I_j := [x_j, y_j], j = 1, \ldots, d \). Then for \( \tilde{I}_j := [x_j/p, y_j/p] \) we have

\[
\prod_{j=1}^{d} |\tilde{I}_j| \geq \frac{C_1(d)(\log p)^{d-1}}{p}.
\]

By Theorem 3.2 the set \( \mathcal{K}_p(a) \) intersects the box \( \tilde{I}_1 \times \cdots \times \tilde{I}_d \) at at least one point. Then there exists a natural number \( \mu \in [1, p] \) such that

\[
\begin{cases}
\left\{ \frac{\mu}{p} \right\} \in \tilde{I}_1, \\
\left\{ \frac{\mu a}{p} \right\} \in \tilde{I}_2, \\
\vdots \\
\left\{ \frac{\mu a^{d-1}}{p} \right\} \in \tilde{I}_d,
\end{cases}
\]

which implies (3.9).
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