MORE ABSORBERS IN HYPERSPACES

PAWEŁ KRUPSKI AND Alicja Samulewicz

Abstract. The family of all subcontinua that separate a compact connected n-manifold X (with or without boundary), n ≥ 3, is an $F_\sigma$-absorber in the hyperspace $C(X)$ of nonempty subcontinua of X. If $D_2(F_\sigma)$ is the small Borel class of spaces which are differences of two $\sigma$-compact sets, then the family of all $(n-1)$-dimensional continua that separate X is a $D_2(F_\sigma)$-absorber in $C(X)$. The families of nondegenerate colocally connected or aposyndetic continua in $I^n$ and of at least two-dimensional or decomposable Kelley continua are $F_\sigma\delta$-absorbers in the hyperspace $C(I^n)$ for $n \geq 3$. The hyperspaces of all weakly infinite-dimensional continua and of $C$-continua of dimensions at least 2 in a compact connected Hilbert cube manifold X are $\Pi_1^1$-absorbers in $C(X)$. The family of all hereditarily infinite-dimensional compacta in the Hilbert cube $I^\omega$ is $\Pi_1^1$-complete in $2^{I^\omega}$.

1. Introduction

The theory of absorbing sets was well developed in the eighties and nineties of the last century (see [1] and [24]). Since any two absorbers in a Hilbert cube $Q$ of a given Borel or projective class are homeomorphic via arbitrarily small ambient homeomorphisms of $Q$, it provides a powerful technique of characterizing some subspaces of the hyperspaces $2^X$ of all closed nonempty subsets of a nondegenerate Peano continuum X or $C(X)$ of all subcontinua of X (if X contains no free arcs). Nevertheless, the list of natural examples of such spaces is not too long and they usually are considered in X being a Euclidean or the Hilbert cube.

Let $I = [0,1]$ be the closed unit interval with the Euclidean metric. The following subspaces of a respective hyperspace are absorbers of Borel class $F_\sigma$ (otherwise known as cap-sets), so they are homeomorphic to the pseudo-boundary $B(I^\omega) = \{(x_i) \in I^\omega : \exists i (x_i \in \{0,1\})\}$, a standard $F_\sigma$-absorber in the Hilbert cube $I^\omega$:

Examples 1.1.

1. The subspace $D_n(X)$ of $2^X$ consisting of all compacta of covering dimensions $\geq n \geq 1$, where X is a locally connected continuum each of whose open non-empty subset has dimension $\geq n$ [5] (for $X = I^\omega$, see [11]),
2. The family of all decomposable continua in $I^n$, $n \geq 3$ [28],
3. The family of all compact subsets (subcontinua) with nonempty interiors in a locally connected nondegenerate continuum (containing no free arcs) [9],
4. The family of all compact subsets that block all subcontinua of a locally connected nondegenerate continuum which is not separated by any finite subset [15].

2010 Mathematics Subject Classification. Primary 57N20; Secondary 54H05, 54F45.

Key words and phrases. absorber, aposyndetic, Borel set, coanalytic set, colocally connected, continuum, C-space, Hilbert cube, Hilbert cube manifold, hyperspace, infinite-dimensional space, locally connected, manifold, weakly infinite-dimensional space.
Examples 1.2. Known $F_{\sigma \delta}$-absorbers include two standard ones ($B(I^n)$) $\omega$ \,
(in \,
(1) The subspace of $2^{I^n}$ of all infinite-dimensional compacta \,[11], \n(2) The subspace of $C(I^n)$ of all locally connected subcontinua of $I^n$, $n \geq 3$ \,[14], \n(3) The subspaces of $C(I^n)$ of all arcs \,[11] and \nExamples 1.3. If $D_2(F_\sigma)$ is the class of all subsets of the Hilbert cube that are \nare differences of two $F_\sigma$-sets, then a standard $D_2(F_\sigma)$-absorber is the subset $B(I^n) \times s$ \n$\sigma$-sets, then a standard $\Pi_1^1$-absorbers \,
Concerning $\Pi_1^1$-absorbers (coanalytic absorbers), the Hurewicz set $\mathcal{H}$ of all countable \nLet $\Pi_1^1$ and $\Sigma_1^1$ denote the classes of coanalytic and analytic sets, respectively. \nConcerning $\Pi_1^1$-absorbers (coanalytic absorbers), the Hurewicz set $\mathcal{H}$ of all countable \nExample 1.4. The subspace of $2^{\mathbb{R}^n}$, $n \geq 3$, of all compact ANR’s in $\mathbb{R}^n$ is a \nLet $\Pi_1^1$ and $\Sigma_1^1$ denote the classes of coanalytic and analytic sets, respectively. \nConcerning $\Pi_1^1$-absorbers (coanalytic absorbers), the Hurewicz set $\mathcal{H}$ of all countable \nExamples 1.5. \n(1) The subspace of $C(I^n)$, $3 \leq n \leq \omega$, consisting of all hereditarily decomposable \n(2) The subspaces $\mathcal{SCD}_k$ of $2^{I^n}$ of all strongly countable-dimensional compacta \n(3) The families of Wilder continua, of continuum-wise Wilder continua and of \nhereditarily arcwise connected nondegenerate continua in $I^n$, $n \geq 3$ \,[17]. \n
The main purpose of this paper is to add to the list some new examples of \nIn Section 4 we deal with three important in continuum theory classes $Col$ of \nIn Section 4 we deal with three important in continuum theory classes $Col$ of \nIn Section 5 we show that if $X$ is a locally connected continuum such that each \nIn Section 5 we show that if $X$ is a locally connected continuum such that each \n
nowhere dense continua which separate $X$ form $D_2(F_\sigma)$-absorbers in $2^X$ and $C(X)$, respectively.

One of the central notions in the theory of infinite dimension is that of a weakly infinite-dimensional space introduced by P. S. Alexandroff in 1948. In Section 6 we find two new coanalytic absorbers in $2^X$ $(C(X))$ for each locally connected continuum $X$ such that each open non-empty subset of $X$ contains a Hilbert cube: the family of weakly infinite-dimensional compacta (continua) in $X$ of dimensions $\geq n \geq 1$ ($\geq n \geq 2$) and the family of compacta (continua) which are $C$-spaces of dimensions $\geq n \geq 1$ ($\geq n \geq 2$).

We observe that the collection of hereditarily infinite-dimensional compacta in $I^\omega$ is a $\Pi^1_1$-complete subset of $2^{I^\omega}$ contained in a $\sigma Z$-set. We do not know however if it is a coanalytic absorber in $2^{I^\omega}$.

2. Preliminaries

All spaces in the paper are assumed to be metric separable. The hyperspace $2^X$ of nonempty compact subsets of $X$ is endowed with the Hausdorff metric $\text{dist}$ and the hyperspace $C(X)$ of continua in $X$ is considered as a subspace of $2^X$.

If $X$ is a locally connected nondegenerate continuum, then $2^X$ is homeomorphic to $I^\omega$ and if, additionally, $X$ contains no free arcs, then also $C(X)$ is a Hilbert cube [10].

A closed subset $C$ separates a space $X$ between two disjoint subsets $A$ and $B$ if there are open disjoint subsets $U$ and $V$ of $X$ such that $A \subseteq U$, $B \subseteq V$ and $X \setminus C = U \cup V$. Such set $C$ will be called a closed separator in $X$.

A subset $C$ cuts $X$ between two disjoint subsets $A$ and $B$ ($C$ is a cut between $A$ and $B$) if it is disjoint from $A \cup B$ and any continuum $D \subseteq X$ that meets both $A$ and $B$ intersects $C$.

If a compact space $X$ is locally connected, then a closed subset $C \subseteq X$ separates $X$ between $A$ and $B$ if and only is $C$ cuts $X$ between $A$ and $B$ [22, Theorem 1, p. 238].

Suppose $X$ is a closed subset of a space $Y$. It is known that for each closed separator $C$ in $X$ between disjoint closed subsets $A$ and $B$ of $X$, there is a closed separator $C'$ in $Y$ between $A$ and $B$ such that $C' \cap X = C$ [13, Lemma 1.2.9, Remark 1.2.10].

Combining these two classic facts, we get an easy observation.

**Observation 2.1.** Let $X$ be a closed subset of a compact locally connected space $Y$. A closed set $C \subseteq X$ separates $X$ between closed subsets $A$ and $B$ of $X$ if and only if $C'$ cuts $Y$ between $A$ and $B$.

Recall that, given a class $\mathcal{M}$ of spaces, a subset $A$ of a complete space $Z$ is $\mathcal{M}$-complete if $A \in \mathcal{M}$ and $A$ is $\mathcal{M}$-hard, i.e., for any subset $C \in \mathcal{M}$ of a complete 0-dimensional space $Y$ there is a continuous mapping $\xi : Y \to Z$ (called a reduction of $C$ to $A$) such that $\xi^{-1}(A) = C$ [10].

A closed subset $B$ of a Hilbert cube ($Q, d$) is a $Z$-set in $Q$ if for any $\epsilon > 0$ there exists a continuous mapping $f : Q \to Q$ such that $f(Q) \cap B = \emptyset$ and $d(f, \text{id}_Q) = \sup \{d(f(x), x) : x \in Q\} < \epsilon$. A countable union of $Z$-sets in $Q$ is called a $\sigma Z$-set in $Q$.

Let $\mathcal{M}$ be a class of spaces which is topological (i.e., if $M \in \mathcal{M}$ then each homeomorphic image of $M$ belongs to $\mathcal{M}$) and closed hereditary (i.e., each closed
subset of $M \in \mathcal{M}$ is in $\mathcal{M}$). Following [11], we call a subset $A$ of a Hilbert cube $Q$ $\mathcal{M}$-universal if for each $M \subset I^\omega$ from the class $\mathcal{M}$ there is an embedding $f : I^\omega \to Q$ (a reduction of $M$ to $A$) such that $f^{-1}(A) = M$. $A$ is said to be strongly $\mathcal{M}$-universal if for each $M \subset I^\omega$ from the class $\mathcal{M}$ and each compact set $K \subset I^\omega$, any embedding $f : I^\omega \to Q$ such that $f(K)$ is a $Z$-set in $Q$ can be approximated arbitrarily closely by an embedding $g : I^\omega \to Q$ such that $g(I^\omega)$ is a $Z$-set in $Q$, $g|K = f|K$ and $g^{-1}(A) \setminus K = M \setminus K$.

Observe that a strongly $\mathcal{M}$-universal set is $\mathcal{M}$-universal and an $\mathcal{M}$-universal set is $\mathcal{M}$-hard. Often in practice, in order to show that a set $A \subset Q$ is $\mathcal{M}$-universal (or $\mathcal{M}$-hard), we choose an already known $\mathcal{M}$-universal ($\mathcal{M}$-hard) set $B \subset I^\omega$ and construct a continuous embedding $\xi : I^\omega \to Q$ such that $\xi^{-1}(A) = B$.

A subset $A$ of a Hilbert cube $Q$ is called an $\mathcal{M}$-absorber in $Q$ provided that:

1. $A \in \mathcal{M}$;
2. $A$ is contained in a $\sigma Z$-set in $Q$;
3. $A$ is strongly $\mathcal{M}$-universal.

3. Proving strong $\mathcal{M}$-universality in hyperspaces

We are now going to sketch two techniques for proving strong $\mathcal{M}$-universality. The first one, developed in [14], [27, Lemma 3.2], applies to subsets $A$ of $2^{I^\omega}$ or $C(I^n)$, $n \in \mathbb{N} \cup \{\omega\}$ (for simplicity we will consider the case $n \in \mathbb{N}$). The second, presented in [5], concerns subsets $A$ of the Hilbert cube $Q = 2^X$ or $Q = C(X)$ in a much more general case of a locally connected nondegenerate continuum $X$ (without free arcs) satisfying certain local properties. For some classes of continua located in such $X$ the second technique may fail while the first one still works if $X$ is a cube.

3.1. Approach I. Suppose that $K$ is a compact subset of $I^\omega$ and $f : I^\omega \to C(I^n)(2^{I^n})$ is an embedding such that $f(K)$ is a $Z$-set and $\epsilon > 0$. We have to find a $Z$-embedding $g : I^\omega \to C(I^n)(2^{I^n})$ which agrees with $f$ on $K$, is $\epsilon$-close to $f$ and satisfies

\begin{equation}
\label{eq:3.1}
g^{-1}(A) \setminus K = M \setminus K.
\end{equation}

For a construction of $g$ we need an auxiliary map $\theta : I^\omega \to C([-1,1]^n)$ sending $q = (q_i)$ to

\[ \theta(q) = \left( \left([-1,0] \times \{0\}\right) \cup S((-1/2,0); 1/2) \cup \bigcup_{i=1}^{\infty} S(a_i; r_i(q)) \right) \times \{(0,\ldots,0)\}, \]

where $S(x;r)$ denotes the circle in the plane centered at $x$ with radius $r$, $a_i = (-1 + 2^{-i}, 0) \in \mathbb{R}^2$ and $r_i(q) = 4^{-(i+1)}(1 + q_i)$ (see Figure [1]).

The set $\theta(q)$ is the union of disjoint circles $c_i = S(a_i; r_i(q))$ contained in $[-1,0] \times [-1,1] \times \{(0,\ldots,0)\}$ and of the diameter segment of the largest circle. The inner circles $c_i$ uniquely code the point $(q_i)$ and the map $\theta$ is a continuous embedding.

We also will exploit a continuous deformation $H_0 : 2^{I^n} \times I \to 2^{I^n}$ through finite sets such that, for any $(A,t) \in 2^{I^n} \times (0,1/2)$, $H_0(A,t)$ is finite,

\[ \text{dist}(A, H_0(A,t)) \leq 2t \quad \text{and} \quad H_0(A,t) \subset [t,1-t]^n. \]
Connecting points of $H_0(A,t)$, $t > 0$, one can define another deformation through finite graphs

$$H(A,t) = \begin{cases} \bigcup_{a,b \in H_0(A,t)} (\overline{ab} \cap (\overline{B(a;2t)} \cup \overline{B(b;2t)})) & \text{if } t > 0, \\ A & \text{if } t = 0, \end{cases}$$

where $\overline{B}(a;\alpha)$ is the closed $\alpha$-ball in $I^n$ around $a$ and $\overline{ab}$ is the line segment in $I^n$ from $a$ to $b$. For any $(A,t) \in C(I^n) \times (0,1/2)$, $H(A,t)$ is a connected graph in $[t,1-t]n$ and $\text{dist}(A,H(A,t)) \leq 4t$ (see [14] and [27] for details).

Assume now that, for each subset $M \subset I^\omega$ which belongs to class $M$, there exists a continuous map $\xi : I^\omega \to C(I^n)(2I^n)$ such that $\xi^{-1}(A) = M$ and $(0,0,\ldots) \in \xi(x)$ for every $x \in I^\omega$.

An embedding $g$ can be defined in the form

$$g(q) = H(f(q),\mu(q)) \cup \bigcup_{x \in H_0(f(q),\mu(q))} (x + \mu(q)\theta(q)) \cup \bigcup_{x \in H_0(f(q),\mu(q))} (x + \mu(q)\xi(q))$$

(we use linear operations of addition and scalar multiplication in $\mathbb{R}^n$), where

$$\mu(q) = \frac{1}{12} \min\{\epsilon, \min\{\text{dist}(f(q),f(z)) : z \in K\}\}.$$ 

If one is interested in the strong $M$-universality of $A$ in $2^{I^n}$ then the “graph” part $H(f(q),\mu(q))$ above is skipped. In Section 4.3 we modify $g$ by replacing $H(f(q),\mu(q))$ with the closed ball $\overline{B}(H(f(q),\mu(q));\frac{1}{2}\mu(q))$.

One can check that $g$ is always a $\mathbb{Z}$-embedding which agrees with $f$ on $K$ and $\epsilon$-approximates $f$ (see the proof of [27] Lemma 3.2; some details of it are sketched in the proof of Theorem 4.3). So, it only remains to verify property (3.1).

3.2. Approach II. Assume $X$ is a nondegenerate locally connected continuum without free arcs. As before, we are looking for a $\mathbb{Z}$-embedding $g$ that approximates $f$, agrees with $f$ on $K$ and satisfies (3.1), where the cube $I^n$ is replaced with $X$.

Suppose that for each non-empty open subset $U$ of $X$ there is a continuous mapping $\varphi_U : I^\omega \to C(U)$ such that $\varphi_U^{-1}(A) = M$. If $X$ is a Peano continuum then each
nondegenerate aposyndetic continua and colocally connected continua in
topology are enough for our purposes to remark that the image \( \in \mathcal{U} \) of the diameter \( \text{diam} \mathcal{U} \geq 1 \).

It seems more convenient to deal with the complement of \( \text{Col} \) of \( X \), i.e. for any \( \epsilon > 0 \), each point \( x \in X \) has a neighborhood \( U \) of diameter \( \text{diam} U < \epsilon \) such that \( X \setminus U \) has finitely many components; if one requires that \( X \setminus U \) be connected then \( X \) is colocally connected.

Proposition 4.1. If \( Y \) is a compact space, then the families Apo\( (Y) \) and Col\( (Y) \) of nondegenerate aposyndetic continua and colocally connected continua in \( Y \), resp., are \( F_{\omega \delta} \)-subsets of \( C(Y) \). If \( Y \) contains a copy of \( I^2 \), then the families are \( F_{\omega \delta} \)-universal.

Proof. We first evaluate the Borel class of \( \text{Col}(Y) \). By compactness, \( X \in \text{Col}(Y) \) if and only if for each \( \epsilon > 0 \) there is a finite \( \epsilon \)-cover of \( X \) consisting of open subsets of \( X \) with connected complements. Passing to complements, this can be written in terms of closed subsets of \( Y \) as follows:

\[
X \in \text{Col}(Y) \quad \text{if and only if} \quad |X| > 1 \quad \land \quad (\forall n) \,( \exists m) \, (\exists K_1, \ldots, K_m \in C(Y)) \left( \bigcap_{i=1}^{m} K_i = \emptyset \quad \land \quad (\forall i \leq m) \,(K_i \subset X) \quad \land \quad \text{diam}(X \setminus K_i) < \frac{1}{n} \right).
\]

Since formula \( 4.1 \) yields merely analyticity of \( \text{Col}(Y) \), it needs further refinements. It seems more convenient to deal with the complement of \( \text{Col}(Y) \) in \( C(Y) \). Recall that the function

\[
f : C(Y) \times C(Y) \to 2^Y, \quad f(X, K) = X \setminus K
\]

is lower semi-continuous [221, p. 182]. Hence, the function

\[
(X, K) \mapsto \text{diam}(f(X, K))
\]
is of the first Borel class [22, Theorem 1, p. 70] which yields that the set
\[ \{ (X, K) \in C(Y) \times C(Y) : \text{diam}(X \setminus K) \geq \frac{1}{n} \} = \{ (X, K) \in C(Y) \times C(Y) : \text{diam}(X \setminus K) \geq \frac{1}{n} \} \]
is \( G_\delta \) in \( C(Y) \times C(Y) \) for each \( n \). Since \( X \in C(Y) \setminus Col(Y) \) if and only if
\[ |X| = 1 \quad \lor \quad (\exists n) \ (\forall m) \quad (\forall K_1, \ldots, K_m \in C(Y)) \]
\[ \bigcap_{i=1}^{m} K_i \neq \emptyset \quad \lor \quad (\exists i \leq m) \left( K_i \not\subset X \quad \lor \quad \text{diam}(X \setminus K_i) \geq \frac{1}{n} \right), \]
we get that \( C(Y) \setminus Col(Y) \) is \( G_{\delta \sigma} \) in \( C(Y) \).

The proof for the family \( Apo(Y) \) is similar.

Passing to the second part, we can assume that \( Y \) contains \( I^2 \). Let \( (J_{ij}) \), \( i, j = 1, 2, \ldots, \) be a double sequence of mutually disjoint nondegenerate closed intervals in \( I \) such that the length of \( J_{ij} \) is less than \( \frac{1}{4^i j} \) and, for each \( j \), \( \lim J_{ij} \to \{1\} \).

Consider rectangles
\[ R_{ij}(t) = J_{ij} \times \left[ 0, \frac{t}{j + 1} \right], \]
and define a continuous embedding \( \psi : I^\omega \to C(Y) \) by
\[ \psi((q_i)) = \partial(I^2) \cup \bigcup_{i,j} R_{ij}(q_i) \]
(Figure 2 instead of rectangles one can also use their boundaries).

It satisfies
\[ \psi^{-1}(Col(Y)) = \psi^{-1}(Apo(Y)) = \hat{c}_0 \]
and since \( \hat{c}_0 \) is an \( F_{\sigma\delta} \)-absorber in \( I^\omega \), the proof is complete.

\[ \square \]
Proposition 4.2. If \( D(Y) \) denotes the family of all decomposable continua in a space \( Y \), then \( \text{Col}(Y) \subset \text{Apo}(Y) \subset D(Y) \) and \( D(I^n) \) is a \( \sigma Z \)-set in \( C(I^n) \) for \( n \geq 3 \).

Proof. It is known that each nondegenerate aposyndetic continuum is decomposable. The last part of the proposition was proved in [27].

\[ \square \]

Theorem 4.3. \( \text{Apo}(I^n) \) and \( \text{Col}(I^n) \) are \( F_{\sigma \delta} \)-absorbers in \( C(I^n) \) for \( n \geq 3 \).

Proof. Concerning \( \text{Apo}(I^n) \), we can refer to the proof of the strong \( F_{\sigma \delta} \)-universality of the family of all Peano continua \( LC(I^n) \) due to Gladdines and van Mill [3]. It perfectly works for \( \text{Apo}(I^n) \). Below, we sketch an alternative construction that suits both families. Actually, we appropriately modify the construction of the embedding \( g \) from Subsection 3.1.

First, locate continua \( \psi((q_i)) \) from (4.3) in \( I^n \) as

\[ \psi^i((q_i)) = \psi((q_i)) \times \{(0, \ldots, 0)\}. \]

The set \( \bar{C}_0 \) being an \( F_{\sigma \delta} \)-absorber in \( I^\omega \), there is, for an \( F_{\sigma \delta} \)-set \( M \subset I^\omega \), a mapping \( \zeta : I^\omega \to I^\omega \) such that \( \zeta^{-1}(\bar{C}_0) = M \). Put

\[ (4.5) \quad \xi = \psi^i \zeta. \]

Second, since the “graph” ingredient \( H(f(q), \mu(q)) \) may spoil the colocal connectedness of \( g(q) \), we surround it by a closed ball

\[ \overline{B}(H(f(q), \mu(q)); \frac{1}{8}\mu(q)) \]

of a small enough radius (e. g., \( \frac{1}{8}\mu(q) \)) and redefine the embedding as follows:

\[ (4.6) \quad g'(q) = \overline{B}(H(f(q), \mu(q)); \frac{1}{8}\mu(q)) \cup \bigcup_{x \in H_0(f(q), \mu(q))} (x + \mu(q) \theta(q)) \cup \bigcup_{x \in H_0(f(q), \mu(q))} (x + \mu(q) \xi(q)). \]

Clearly, \( g' \) is an \( \varepsilon \)-approximation of \( f \) and \( g'|K = f|K \). So, it is 1-to-1 on \( K \). For distinct \( q, q' \in I^\omega \setminus K \), coefficients \( \mu(q) \) and \( \mu(q') \) are positive. Let \( x, x' \) be the minimal points in \( H_0(f(q), \mu(q)) \) and \( H_0(f(q'), \mu(q')) \), respectively, in the lexicographic order on \( I^n \). Then \( g'(q) \neq g'(q') \) because the inner circles in copies \( x + \mu(q) \theta(q) \) and \( x' + \mu(q') \theta(q') \) remain disjoint from the rest of \( g'(q) \) and \( g'(q') \), respectively. If \( q \in K \), \( q' \notin K \), then \( g'(q) \neq g'(q') \) by a similar simple estimation of \( \text{dist}(g'(q), g'(q')) \) as for the original approximation \( g \). Moreover, \( g'(K) = f(K) \) is a \( Z \)-set in \( C(I^n) \) by assumption and \( g'(I^\omega) \setminus g'(K) \), as an open subset of \( g'(I^\omega) \), is \( F_{\sigma} \) in \( C(I^n) \). Notice that \( g'(q) \), for each \( q \in I^\omega \setminus K \), contains an open in \( g'(q) \), one-dimensional subset of \( x + \mu(q) \theta(q) \) (e. g., the inner circles in there). Therefore, the deformation

\[ h : C(I^n) \times I \to C(I^n), \quad h(A, t) = \overline{B}(A; t) \]

maps \( g'(I^\omega \setminus K) \) off the set \( I^\omega \setminus K \) arbitrarily closely to the identity map on \( C(I^n) \). It means that \( g'(I^\omega) \setminus g'(K) \) is a \( \sigma Z \)-set and, consequently, \( g'(I^\omega) \) is a \( Z \)-set in \( C(I^n) \).
Finally, we need to check the property
\[(g')^{-1}(\text{Col}(I^n)) \setminus K = (g')^{-1}(\text{Apo}(I^n)) \setminus K = M \setminus K.\]

So, suppose \(q \notin K\). Then \(\mu(q) > 0\). If \(q \in M\), then one can easily see that \(g'(q) \in \text{Col}(I^n)\). If \(q \notin M\), it is convenient to consider the maximal point \(y = (y_1, y_2, \ldots, y_n) \in H_0(f(q), \mu(q))\) in the lexicographic order \(\preceq\) on \(I^n\). The copy \(y + \mu(q)\xi(q)\) of \(\psi(\xi(q))\) is not aposyndetic by \((4.4)\), and it can be intersected by at most finitely many other isometric copies \(x + \mu(q)\xi(q), x \in H_0(f(q), \mu(q))\), where \(x = (x_1, x_2, y_3, \ldots, y_n) \prec y\). Neither these copies nor adding the part
\[
\overline{B}(H(f(q), \mu(q)); \frac{1}{8}\mu(q)) \cup \bigcup_{x \in H_0(f(q), \mu(q))} (x + \mu(q)\theta(q))
\]

affect the non-semi-local connectedness caused by \(y + \mu(q)\xi(q)\). So, \(g'(q) \notin \text{Apo}(I^n)\) and \((4.4)\) is satisfied.

Recall that a continuum \(Y\) is called a *Kelley continuum* if for each point \(z \in Y\), each sequence of points \(z_n \in Y\) converging to \(z\) and each subcontinuum \(Z \subseteq Y\) such that \(z \in Z\), there is a sequence of subcontinua \(Z_n \subseteq Y, z_n \in Z_n\), that converge to \(Z\) (in the sense of the Hausdorff distance).

One can easily observe that the proof of the strong \(F_{\sigma\delta}\)-universality of \(\text{Col}(I^n)\) presented above applies directly to family \(K(I^n)\), i.e., \((4.7)\) is satisfied if \(\text{Col}(I^n)\) is substituted with \(K(I^n)\). Moreover, if \(X\) is a locally connected continuum each of whose non-empty open subset contains a copy of \(I^n, n \geq 2\), then \(K(X)\) is strongly \(F_{\sigma\delta}\)-universal in \(C(X)\) because Approach II \((5.2)\) applies to \(K(X)\) with mappings \(\varphi_U\) being compositions of \(\xi\) \((4.5)\) with embeddings \(I^n \hookrightarrow U\). Exactly the same observation concerns families \(\text{Apo}(X)\) and \(\text{LC}(X)\) of locally connected subcontinua of \(X\). We do not know if \(K(I^n)\) is contained in a \(\sigma Z\)-set in \(C(I^n)\) but if we restrict the family to decomposable continua, then the condition is satisfied for \(n \geq 3\), since \(D(I^n)\) is a \(\sigma Z\)-set in \(C(I^n)\) \((4.4)\). Concerning the more general case of a Peano continuum \(X\) as above, it is not known if decomposable subcontinua of \(X\) form a \(\sigma Z\)-set in \(C(X)\), so, instead, we can restrict \(K(X)\) (and \(\text{Apo}(X)\) and \(\text{LC}(X)\)) to \(D_2(X)\) which is a \(\sigma Z\)-set in \(C(X)\) by Example \((4.4)\) (1)). Summarizing, we get the following theorem.

**Theorem 4.4.** Let \(X\) be a locally connected continuum each of whose non-empty open subset contains a copy of \(I^n, n \geq 2\). Then \(K(X), \text{Apo}(X)\) and \(\text{LC}(X)\) are strongly \(F_{\sigma\delta}\)-universal in \(C(X)\). The families \(K(X) \cap D_2(X), \text{Apo}(X) \cap D_2(X), \text{LC}(X) \cap D_2(X)\) are \(F_{\sigma\delta}\)-absorbers in \(C(X)\) and \(K(I^n) \cap D(I^n)\) is an \(F_{\sigma\delta}\)-absorber in \(C(I^n)\) for \(n \geq 3\).

**Remark 4.5.** The more general Approach II \((5.2)\) cannot be directly applied for \(\text{Col}(X)\) with mappings \(\varphi_U\) taken as copies of the reduction \(\xi\) \((4.5)\), since the 1-dimensional part \(A(q)\) of \(g(q)\) can spoil the colocal connectedness of \(g(q)\) and the remedy of surrounding it by a small closed ball (similarly as in the proof of Theorem \((4.4)\) may kill the “one-to one” property of \(g(q)\).

5. Closed separators of \(I^n\)

**Proposition 5.1.** If \(X\) is a locally connected continuum, then the family \(S(X)\) of all compact separators of \(X\) is an \(F_{\sigma}\)-subset of \(2^X\).
Proof. Since $X$ is locally connected, a closed subset $S$ separates $X$ if and only if
$S$ cuts $X$ between two points. Let $E$ be a countable dense subset of the family
{\{(C, x, y) \in C(X) \times X^2 : x \neq y, \, x, y \in C\}}. Denote by $proj_1$ and $proj_2$ the
projections of $C(X) \times X^2$ onto $C(X)$ and $X^2$, respectively. We can now express
the definition of a closed separator (= cut) using $E$:
\[ S \text{ is a closed separator of } X \text{ if and only if } \]
(5.1) $\exists(x, y) \in proj_2(E)$ \{ $(x, y) \cap S = \emptyset$ \} and
\[ \forall C \in proj_1(E \cap (proj_2)^{-1}(x, y)) \quad (C \cap S \neq \emptyset). \]
Since the two quantifiers in (5.1) are taken over countable sets, the conclusion
follows. 

\[ \square \]

Proposition 5.2. If a space $X$ contains an open subset homeomorphic to the
combinatorial interior $\text{int}(I^n) = (0, 1)^n$ of $I^n$, $2 \leq n < \infty$, then $S(X) \cap C(X)$ is
$F_\sigma$-universal.

Proof. We can assume, without loss of generality, that $I^n \subset X$ and $(0, 1)^n$ is open
in $X$. Since the pseudo-boundary $B(I^\omega)$ is an $F_\sigma$-absorber, it is enough to construct
an embedding $\Psi : I^\omega \to C(X)$ such that
\[ (5.2) \quad \Psi((q_i)) \in S(X) \text{ if and only if } (q_i) \in B(I^\omega). \]
Denote $J_i = \left[ \frac{1}{2i+1}, \frac{1}{2i} \right]$ and let $\partial(\Delta_i)$ be the combinatorial boundary of the cube
\[ \Delta_i = J_i \times \left( \frac{1}{2} + \frac{1}{2} + \frac{1}{4i(i+1)} \right)^{n-1}, \quad i = 1, 2, \ldots. \]
For each $i$, there is a deformation $h_i(A, t)$ of $2\Delta_i$ through finite sets in
\[ \left[ \frac{1}{2i+1} + t, \frac{1}{2i} - t \right] \times \left[ \frac{1}{2} + t, \frac{1}{2} + \frac{1}{4i(i+1)} - t \right]^{n-1}, \quad 0 < t < \frac{1}{8i(i+1)} \]
(look at $h_i$ as $H_0$ considered in Section 3 with $I^n$ replaced by $\Delta_i$). Given $t \in I$, choose points
\[ x_i(t) = \left( \frac{1}{2i+1} + t, \frac{1}{2i+1} + \frac{1}{2} \right) \] \[ \text{and} \]
\[ y_i(t) = \left( \frac{1}{2i+1} + t, \frac{1}{2i+1} + \frac{1}{2} \right). \]
For $0 < t < \frac{1}{8i(i+1)}$ and each $i$, connect points of the set $h_i(A, t) \cup \{ x_i \}$ following
formula (3.2):
\[ (5.3) \quad H_i(A, t) = \bigcup_{a,b \in h_i(A, t) \cup \{ x_i \}} (ab \cap (B(a; 2t) \cup B(b; 2t))) \]
and put $H_i(A, 0) = A$. Thus $H_i$ is a deformation $2\Delta_i$ through finite graphs in $\Delta_i$
that meet the edge $J_i \times \{ (\frac{1}{2}, \ldots, \frac{1}{2}) \}$ at the single point $x_i$. The function $\Psi : I^\omega \to C(I^n)$ defined by
\[ (5.4) \quad \Psi((q_i)) = \]
\[ I \times \left\{ (\frac{1}{2}, \ldots, \frac{1}{2}) \right\} \cup \bigcup_i \left( H_i(\partial(\Delta_i), \frac{q_i(1-q_i)}{8i(i+1)}) \cup \bigcup_i x_i(q_i) y_i(q_i) \right) \]
Figure 3. $\Psi((0,1,0,\ldots))$

(see Figure 3) is continuous and it is 1-1 since the correspondence $(q_i) \mapsto (x_i(q_i),y_i(q_i))$ is 1-1. It satisfies \(q_{n-1}(\partial I^2)\) for \(n \geq 3\) because if \((q_i) \in B(I^\omega)\), then $\Psi((q_i))$ contains $\partial(\Delta_i)$ for some $i$ which separates $\text{int}(I^n)$ — consequently, by construction, $\Psi((q_i))$ separates $X$; otherwise, $\Psi((q_i))$ is one-dimensional, so it does not separate $I^n$, $n \geq 3$.

If $n = 2$, the above construction does not work, since $\Psi((q_i))$ might separate $I^2$ for $(q_i)$ in the pseudo-interior $s$. In this case we can, however, define an appropriate embedding $\Psi$ differently much easier. Given $(q_i) \in I^2$, denote $p_{2i} = x_i(q_i)$, $p_{2i-1} = x_i(1 - q_i)$, $d_i = (\frac{1}{2i+1}, \frac{1}{2})$, $A_i = \overline{d_i}p_i \setminus \{d_i, p_i\}$. Put

$$\Psi((q_i)) = I \times \left\{ \frac{1}{2} \right\} \cup \bigcup_i (\partial(\Delta_i) \setminus A_i).$$

\[\square\]

**Theorem 5.3.** Let $X$ be a locally connected continuum such that each open non-empty subset of $X$ contains a copy of $(0,1)^n$, $3 \leq n < \infty$, as an open subset and no subset of dimension \(\leq 1\) separates $X$. Then the families $S(X)$ and $S(X) \cap C(X)$ are $F_{\sigma}$-absorbers in $2^X$ and $C(X)$, respectively.

**Proof.** Since being an $F_{\sigma}$-absorber in a Hilbert cube is equivalent, for an $F_{\sigma}$-set, to being strongly $F_{\sigma}$-universal [4 Theorem 5.3], it remains to prove the strong $F_{\sigma}$-universality. To this end, we are going to use Approach II [3.2]. The pseudoboundary $B(I^\omega)$ being strongly $F_{\sigma}$-universal, there exists for each $M \subset I^\omega$, $M \in F_{\sigma}$, a mapping $\chi : I^\omega \to I^\omega$ such that $\chi^{-1}(B(I^\omega)) = M$. Hence, the composition $\phi = \Psi\chi : I^\omega \to C(I^n)$ ($\Psi$ as in (5.4)) satisfies $\phi^{-1}(S(I^n)) = M$ for every $q \in I^\omega$. For each open non-empty subset $U$ of $X$ and an open copy of $(0,1)^n$ in $U$, let $\phi_U : I^\omega \to C(U)$ be a composition of $\phi$ with an embedding of $C(I^n)$ into the hyperspace of that copy. Then $\phi_U(q)$ separates $X$ iff $\phi(q)$ separates $I^n$ iff $q \in B(I^\omega)$.

The properties of $q$ in Approach II [3.2] yield that $g$ satisfies (3.1) for $A = S(X)$ as well as for $A = S(X) \cap C(X)$. In fact, for $q \notin K$, $g(q)$ separates $X$ if and only if $\phi_U(q)$ does, because neither the zero- nor one-dimensional part $A(q)$ of $g(q)$ do affect separation of $X$ by copies $\phi_U(q)$ for $n \geq 3$. This completes the proof. \[\square\]
Corollary 5.4. The families \( \mathcal{S}(X) \) and \( \mathcal{S}(X) \cap C(X) \) are \( F_\sigma \)-absorbers in \( 2^X \) and \( C(X) \), respectively, if \( X \) is a continuum which is an \( n \)-manifold (with or without boundary), \( 3 \leq n < \infty \).

Denote by \( \mathcal{N}(X) \) the family of all nowhere dense closed subsets of \( X \). The following proposition is well known and has a straightforward proof.

Proposition 5.5. For any compact space \( X \), the subspace of \( 2^X \) consisting of all closed subsets of \( X \) with non-empty interiors is an \( F_\sigma \)-set.

The next fact follows from Propositions 5.1 and 5.5.

Proposition 5.6. If \( X \) is a locally connected continuum, then

\[
\mathcal{S}(X) \cap \mathcal{N}(X) \in D_2(F_\sigma) \quad \text{and} \quad \mathcal{S}(X) \cap \mathcal{N}(X) \cap C(X) \in D_2(F_\sigma).
\]

Proposition 5.7. If a space \( X \) contains an open subset homeomorphic to the combinatorial interior \( \operatorname{int}(I^n) = (0,1)^n \) of \( I^n \), \( 2 \leq n < \infty \), then \( \mathcal{S}(X) \cap \mathcal{N}(X) \cap C(X) \) is \( D_2(F_\sigma) \)-universal.

Proof. Assume again that \( I^n \subset X \) and \( (0,1)^n \) is an open subset of \( X \). First, replace in the definition \( \text{Proposition 5.4} \) of embedding \( \Psi \) the boundaries \( \partial(D_i) \) with the cubes \( \Delta_i \) themselves for all \( i \)'s and denote thus obtained embedding by \( \Psi_0 \). We have

\[
\Psi_0: I^\omega \to C(I^n) \quad \text{and} \quad \Psi_0^{-1}(\mathcal{N}(X)) = s,
\]

where \( s \) is the pseudo-interior of \( I^\omega \).

Let \( \alpha: I^n \to [-1,0] \times I^{n-1} \) be the reflection

\[
\alpha(x_1,x_2,\ldots,x_n) = (-x_1,x_2,\ldots,x_n).
\]

Fix an embedding \( f: [-1,1] \times I^{n-1} \to \operatorname{int}(I^n) \subset X \). Define a continuous embedding

\[
\Phi: I^\omega \times I^\omega \to C(X), \quad \Phi((q_i),(t_i)) = f \left( \Psi \left( \Psi_0((q_i)) \cup \alpha((t_i)) \right) \right).
\]

For \( n \geq 3 \), it follows from \text{Proposition 5.2} and \text{Proposition 5.3} that

\[
\Phi((q_i),(t_i)) \in \mathcal{S}(X) \cap \mathcal{N}(X) \cap C(X) \quad \text{if and only if} \quad ((q_i),(t_i)) \in B(I^\omega) \times s.
\]

In case \( n = 2 \) we define map \( \Phi \) by formula \text{Proposition 5.8} in which \( \Psi \) is the map from \text{Proposition 5.5} and \( \phi \) is defined in the following way.

Let \( f_i: 2^I \times I \to 2^I \) be a deformation through finite sets. Denote

\[
W_i(t) = f_i(J_i,t) \times \left[ \frac{1}{2} \times \frac{1}{2^2 + \frac{1}{4i(i+1)}} \right],
\]

\[
V_{2i-1} = W_i((q_i)), \quad V_{2i} = W_i((1-q_i))
\]

for \( (q_i) \in I^\omega \). Observe that \( W_i(0) = \Delta_i \). Now, the map

\[
\phi((q_i)) = I \times \left\{ \frac{1}{2} \right\} \cup \bigcup_i V_i
\]

is an embedding which satisfies \text{Proposition 5.9} (Figure 4).

Since \( B(I^\omega) \times s \) is strongly \( D_2(F_\sigma) \)-universal (see Example 1.3), the proof is complete. \( \square \)
Theorem 5.8. Assume $X$ satisfies hypotheses of Theorem 5.3. Then $S(X) \cap N(X)$ is an $D_2(F_\sigma)$-absorber in $2^X$ and $S(X) \cap N(X) \cap C(X)$ is an $D_2(F_\sigma)$-absorber in $C(X)$ for $n \geq 3$.

Proof. Since

$$S(X) \cap N(X) \subset S(X), \quad S(X) \cap N(X) \cap C(X) \subset S(X) \cap C(X)$$

and $S(X)$ and $S(X) \cap C(X)$ are $\sigma Z$-sets in $2^X$ and $C(X)$, respectively (by Theorem 5.3), it remains to show the strong $D_2(F_\sigma)$-universality of both families. There exists, for each subset $M \subset I^\omega$ from the class $D_2(F_\sigma)$, a mapping

$$\zeta : I^\omega \to I^\omega \times I^\omega$$

such that $\zeta^{-1}(B(I^\omega) \times s) = M$.

The composition $\varphi = \Phi \zeta$, where $\Phi$ is the mapping (5.8), provides a map $I^\omega \to C(I^n) \subset C(X)$ satisfying

$$\varphi^{-1}((S(X) \cap N(X) \cap C(X))) = M$$

and mappings $\varphi_U : I^\omega \to C(U)$, for every open non-empty subset $U$ of $X$ and an open copy of $(0,1)^n$ in $U$, which are compositions of $\varphi$ with an embedding of $C(I^n)$ into the hyperspace of that copy. Now, the construction of the embedding $g$ in Approach 5.2 works for both families, i.e., if $q \notin K$, then

$$g(q) \in S(X) \cap N(X) \cap C(X)$$

if and only if

$$\varphi(q) \in S(I^n) \cap N(I^n) \cap C(I^n)$$

and similarly for $S(X) \cap N(X)$, because neither the zero- nor one-dimensional part $A(q)$ of $g(q)$ destroy the separation properties of copies $\varphi_U(q)$ in $g(q)$ or change their status of being nowhere dense in $X$ for $n \geq 3$.

Corollary 5.9. If $X$ is a continuum which is an $n$-manifold (with or without boundary), $3 \leq n < \infty$, then the family $S(X)_{n-1}$ of all $(n-1)$-dimensional closed separators of $X$ is a $D_2(F_\sigma)$-absorber in $2^X$ and $S(X)_{n-1} \cap C(X)$ is a $D_2(F_\sigma)$-absorber in $C(X)$.

Proof. It is well known that each such $X$ is a Cantor $n$-manifold ($\subseteq$ no subset of dimension $\leq n-2$ separates $X$) and $Y \in N(X)$ iff $\dim Y \leq n-1$. This means that

$$S(X) \cap N(X) = S(X)_{n-1}.$$
6. INFINITE-DIMENSIONAL COMPACTA

Recall that a space $X$ is strongly infinite-dimensional if there exists a sequence $(A_n, B_n)_n$ of closed disjoint subsets of $X$ such that for each sequence $(C_n)_n$ of closed separators of $X$ between $A_n$ and $B_n$ we have $\bigcap_n C_n \neq \emptyset$.

A space is weakly infinite-dimensional if it is not strongly infinite-dimensional. The collection of all weakly infinite-dimensional compacta in a space $Y$ will be denoted by $\mathcal{W}(Y)$.

It was proved in [26, Section 4, p. 173] that strongly infinite-dimensional compacta form an analytic subset of $2^{I^Y}$. Below, we provide a different and elementary proof of this fact for such compacta in an arbitrary locally connected compact space.

**Proposition 6.1.** The family of strongly infinite-dimensional compacta in a compact locally connected space $Y$ is an analytic subset of $2^Y$.

**Proof.** In view of Observation 2.1 we have the following claim.

**Claim 6.1.1.** A compact space $X \subset Y$ is strongly infinite-dimensional if and only if

\begin{equation}
\exists (A_n, B_n)_n \in (2^Y \times 2^Y)^\omega \\
\forall n \ (A_n \subset X, B_n \subset X, A_n \cap B_n = \emptyset) \quad \text{and} \quad \forall (C_n) \in (2^Y)^\omega \\
\quad \quad \quad \text{(if} \quad \forall n \ (C_n \text{ cuts } Y \text{ between } A_n \text{ and } B_n), \text{ then } X \cap \bigcap_n C_n \neq \emptyset) \text{).}
\end{equation}

A rough evaluation of the projective complexity of formula (6.1) gives merely class $\Pi^2_2$. Therefore we need to refine the cutting condition in (6.1).

Let $\mathcal{E}$ be a countable dense subset of the family

$$\{(C, A, B) \in (2^Y)^3 : A \cap B = \emptyset, \ C \text{ cuts } Y \text{ between } A \text{ and } B\}$$

and let $\mathcal{E}_1 = \text{proj}_1(\mathcal{E})$, where \text{proj} is the projection of $(2^Y)^3$ onto the first factor.

**Claim 6.1.2.** A compact space $X \subset Y$ is strongly infinite-dimensional if and only if

\begin{equation}
\exists (A_n, B_n)_n \in (2^Y \times 2^Y)^\omega \\
\forall n \ (A_n \subset X, B_n \subset X, A_n \cap B_n = \emptyset) \quad \text{and} \quad \forall k \forall (C_1, \ldots, C_k) \in (\mathcal{E}_1)^k \\
\quad \quad \quad \text{(if} \quad \forall i \leq k \ (C_i \text{ cuts } Y \text{ between } A_i \text{ and } B_i), \text{ then } X \cap \bigcap_{i \leq k} C_i \neq \emptyset) \text{).}
\end{equation}

In order to show the less obvious implication $\Leftarrow$, assume that $C_n$ cuts $Y$ between $A_n$ and $B_n$ for each $n \in \mathbb{N}$. For each $k$, approximate $(C_i, A_i, B_i)$ by $(C_i', A_i', B_i') \in \mathcal{E}$, $i \leq k$. Then, by the local connectedness of $Y$, $C_i'$ cuts $Y$ between $A_i$ and $B_i$ if the approximation is sufficiently close. Then $X \cap \bigcap_{i \leq k} C_i' \neq \emptyset$. So, there is a point $x_k \in X \cap \bigcap_{i \leq k} C_i'$ and the distances $d(x_k, C_1), \ldots, d(x_k, C_k)$ can be made arbitrarily small. Let $x$ be an accumulation point of sequence $(x_k)$. Then $x \in X \cap \bigcap_{n \in \mathbb{N}} C_n \neq \emptyset$ which completes the proof of Claim 6.1.2.

We are now ready to evaluate the complexity of formula (6.2). First, write down the sentence

\begin{equation}
\end{equation}
MORE ABSORBERS IN HYPERSPACES

$C_i$ cuts $Y$ between $A_i$ and $B_i$

as

$$C_i \cap A_i = \emptyset, \quad C_i \cap B_i = \emptyset \quad \text{and} \quad \forall D \in C(Y) \quad (D \cap A_i \neq \emptyset, \quad D \cap B_i \neq \emptyset) \Rightarrow D \cap C_i \neq \emptyset$$

and observe that its Borel complexity is $G_\delta$. Next, notice that all quantifiers in formula (6.2), preceding the sentence, except for the first existential one, are taken over at most countable sets of variables, hence the formula following the first existential quantifier describes a Borel set. Now, the whole formula (6.2) gives an analytic set, as a continuous projection of a Borel set.

□

Proposition 6.2. Let $Y$ be a compact, locally connected space containing a Hilbert cube. Then, for each integer $n \geq 0$, the subsets $W_n(Y)$ of $2^Y$ and $W_n(Y) \cap C(Y)$ of $C(Y)$ consisting of weakly infinite-dimensional compacta and continua, respectively, of dimensions $\geq n$ are $\Pi^1_1$-complete.

Proof. It is well known that the family $D_n(Y)$ of all closed subsets of $Y$ that have dimension at least $n$ is $F_\sigma$ in $2^Y$ (for any compact $Y$). Since $W_n(Y) = W(Y) \cap D_n(Y)$, the set $W_n(Y)$ is coanalytic by Proposition 6.1. Hence $W_n(Y) \cap C(Y)$ is also coanalytic. The $\Pi^1_1$-hardness of $W(Y) \cap C(Y)$ for $Y = I^\omega$ was established in [13, Corollary 3.3] and an analogous argument gives the hardness for each set $W_n(Y) \cap C(Y)$ and arbitrary $Y$ as in the hypothesis. It follows immediately that each $W_n(Y)$ is $\Pi^1_1$-hard as well.

□

Recall that a space $X$ is a $C$-space if for each sequence $U_1, U_2, \ldots$ of open covers of $X$ there exists a sequence $\mathcal{V}_1, \mathcal{V}_2, \ldots$, of families of pairwise disjoint open subsets of $X$ such that each $\mathcal{V}_i$ refines $U_i$ and $\bigcup_{i=1}^{\infty} \mathcal{V}_i$ is a cover of $X$.

If $(X, d)$ is compact, then covers $U_1, U_2, \ldots$ can be replaced by a sequence of positive reals $\epsilon_i \to 0$, if $i \to \infty$, and the cover $\bigcup_{i=1}^{\infty} \mathcal{V}_i$ can be replaced by a finite subcover. Each family $\mathcal{V}_i$ is then finite and its elements have diameters $< \epsilon_i$. The definition can be rewritten as follows:

$$\forall(n_1, n_2, \ldots) \in \mathbb{N}^\omega \exists k \in \mathbb{N} \exists(\mathcal{V}_1, \mathcal{V}_2, \ldots, \mathcal{V}_k) \forall i \leq k$$

$[\mathcal{V}_i$ is finite, consists of open subsets of $X$ and $[V, V' \in \mathcal{V}_i, V \neq V' \Rightarrow V \cap V' = \emptyset]$

and $[V \in \mathcal{V}_i, x, y \in V \Rightarrow d(x, y) < \frac{1}{n_i}]$ and $\bigcup_{i \leq k} \mathcal{V}_i = X].$

Proposition 6.3. For each $n \geq 0$, the families $C_n(Y)$ of $C$-compacta of dimensions $\geq n$ in a compact space $Y$ and $C_n(Y) \cap C(Y)$ are coanalytic subsets of $2^Y$. 

(6.4) $\forall(n_1, n_2, \ldots) \in \mathbb{N}^\omega \exists k \in \mathbb{N} \exists(\mathcal{V}_1, \mathcal{V}_2, \ldots, \mathcal{V}_k) \forall i \leq k$

$[\mathcal{V}_i$ is finite, consists of open subsets of $X$ and $[V, V' \in \mathcal{V}_i, V \neq V' \Rightarrow V \cap V' = \emptyset]$

and $[V \in \mathcal{V}_i, x, y \in V \Rightarrow d(x, y) < \frac{1}{n_i}]$ and $\bigcup_{i \leq k} \mathcal{V}_i = X].$
Proof. In view of (6.4), the definition of a $C$-compactum $(X,d)$ in terms of closed subsets of $Y$ runs as follows:

\[(6.5) \forall (n_1, n_2, \ldots) \in \mathbb{N}^\omega \exists k \in \mathbb{N} \exists (\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_k) \in (2^{(2^X)})^k \]

\[\forall i \leq k [\mathcal{F}_i \text{ is finite and } \forall F \in \mathcal{F}_i (F \subset X) \]

\[\text{and } \forall F, F' \in \mathcal{F}_i (F \neq F' \Rightarrow F \cup F' = X) \]

\[\text{and } \forall F \in \mathcal{F}_i (x,y \in X \setminus F \Rightarrow d(x,y) < \frac{1}{n_i}) \text{ and } \bigcap_{i \leq k} \mathcal{F}_i = \emptyset.\]

Formula (6.5) easily yields that the set $C(Y)$ of all $C$-compacta in $Y$ is coanalytic. Hence, each of $C_n(Y) = C(Y) \cap D_n(Y)$ and $C_n(Y) \cap C(Y)$ is also coanalytic.

\[\square\]

Proposition 6.4. Let $Y$ be a compact space containing a Hilbert cube. Then, for each integer $n \geq 0$, the families $C_n(Y)$ and $C_n(Y) \cap C(Y)$ are $\Pi^1_1$-complete.

Proof. The argument for the $\Pi^1_1$-hardness of each of the above families is the same as in the proof of Proposition (6.2). So, the conclusion follows from Proposition (6.3).

\[\square\]

Proposition 6.5. Let $Y$ be a locally connected continuum.

1. The families $W_n(Y)$, $W_n(Y) \cap C(Y)$, $C_n(Y)$ and $C_n(Y) \cap C(Y)$, $n \geq 1$, are contained in $D_1(Y)$, a $\sigma Z$-set in $2^Y$.

2. If each non-empty open subset of $Y$ contains an $n$-cell, $n \geq 2$, then the families $W_n(Y) \cap C(Y)$ and $C_n(Y) \cap C(Y)$ are contained in $D_2(Y) \cap C(Y)$, a $\sigma Z$-set in $C(Y)$.

Proof. The set $D_1(Y)$ is a $\sigma Z$-set in $2^Y$ since there is a deformation $2^Y \times I \rightarrow 2^Y$ through finite sets (8). The set $D_2(Y) \cap C(Y)$ is a $\sigma Z$-set in $C(Y)$ as it is an $F_\sigma$-absorber in $C(Y)$ (see Examples (1.1) (1)).

\[\square\]

Denote by $SCD_n(X)$ the family of all strongly countable-dimensional compacts of dimension $\geq n$ in a space $X$.

Theorem 6.6. Let $X$ be a locally connected continuum such that each non-empty open subset of $X$ contains a copy of the Hilbert cube (in particular, $X$ can be a Hilbert cube manifold).

Families $SCD_n(X)$, $W_n(X)$, and $C_n(X)$ are coanalytic absorbers in $2^X$ for $n \geq 1$.

Families $SCD_n(X) \cap C(X)$, $W_n(X) \cap C(X)$ and $C_n(X) \cap C(X)$ are coanalytic absorbers in $C(X)$ for $n \geq 2$.

Proof. We have $SCD_n(X) \subset C_n(X) \subset W_n(X)$ (see (13)). In view of Propositions (6.2), (6.3) and (6.4), it suffices to check the strong $\Pi^1_1$-universality of the families. In the case when $X = I^\omega$, an Approach I-construction used in the proof of (20) Theorem 3.1 for the strong $\Pi^1_1$-universality of $SCD_n(I^\omega)$ applies to other families without any change. Its main ingredient is a continuous mapping $\xi: I^\omega \rightarrow C(I^\omega)$ such that

- $\xi(q)$ is a strongly countable-dimensional continuum of dimension $\geq n$ for $q \in M$ (actually, it is a countable union of Euclidean cubes of dimensions $\geq n$) and
\[ \xi(q) \text{ contains a Hilbert cube for } q \notin M, \]
where \( M \) is a coanalytic subset of \( I^\omega \). In other words, \( \xi \) is a continuous reduction of \( M \) to \( SCD_n \cap C(I^\omega) \).

Observe that \( \xi \) can also be viewed as a reduction of \( M \) to any of the families in our theorem, since their members do not contain Hilbert cubes.

In a general case, we use Approach II \( \text{II.2} \). So, for each nonempty open subset \( U \) of \( X \), choose an embedding \( \eta_U : I^\omega \to U \) and let \( \varphi_U(q) = \eta_U(\xi(q)) \). We have mappings \( \varphi_U : I^\omega \to C(U) \) satisfying \( \varphi_U^{-1}(A) = M \), where \( A \) is any of the considered families. Since taking finite unions of copies \( \varphi_U(q) \) and adding zero- or one-dimensional part \( A(q) \) of \( g(q) \) do not change dimension properties of members of each family, we have that \( g(q) \in A \text{ iff } q \in M \text{ for } q \in I^\omega \setminus K. \)

\[ \square \]

Other intriguing spaces studied in dimension theory are hereditarily infinite-dimensional compacta in the sense of Henderson, i.e., infinite-dimensional compact spaces whose nonempty closed subspaces are either infinite-dimensional or zero-dimensional.

**Proposition 6.7.** The collection \( \mathcal{HID}(Y) \) of all hereditarily infinite-dimensional compacta in a compact space \( Y \) is a coanalytic subset of \( 2^Y \).

**Proof.** We have \( X \in \mathcal{HID}(Y) \) if and only if
\[
\forall F \in 2^Y \quad (\text{if } F \subseteq X, \text{ then } \dim F = 0 \text{ or } \forall n \ \dim F \geq n).
\]
A direct evaluation of the projective complexity of (6.6) shows that \( X \) is coanalytic.

\[ \square \]

**Theorem 6.8.** The families \( \mathcal{HID}(I^\omega) \) and \( \mathcal{HID}(I^\omega) \cap C(I^\omega) \) are \( \Pi^1_1 \)-complete subsets of \( 2^{I^\omega} \) and of \( C(I^\omega) \), respectively.

**Proof.** The proof of \( \Pi^1_1 \)-hardness will use an idea of R. Pol presented in [25, Lemma 6.3].

Take a continuum \( L \in \mathcal{HID}(I^\omega) \) and the set \( \mathcal{L} \) of all topological copies of \( L \) in \( I^\omega \). Let \( \mathcal{P} \) be the collection of all pseudoarcs in \( I^\omega \). Families \( \mathcal{P} \) and \( \mathcal{L} \) are dense in \( C(I^\omega) \) and \( \mathcal{P} \) is a \( G_\delta \)-subset of \( C(I^\omega) \). Let \( Q \) be a countable dense subset of a Cantor set \( C \subseteq I \).

Let \( \varphi : C \to \{\mathcal{P}, \mathcal{L}\} \) be a function defined by
\[
\varphi(x) = \begin{cases} 
\mathcal{P}, & \text{for } x \in C \setminus Q; \\
\mathcal{L}, & \text{for } x \in Q.
\end{cases}
\]

By [23, Corollary 1.61], \( \varphi(x) \) has a continuous selection
\[
\sigma : C \to C(I^\omega), \quad \sigma(x) \in \varphi(x).
\]
Define the map \( \pi : 2^C \to 2^{I^\omega \times \top} \) by
\[
\pi(A) = \bigcup_{x \in A} \{x\} \times \sigma(x).
\]
Observe that \( \pi^{-1}(\mathcal{HID}(I^\omega)) = 2^Q \) is the Hurewicz set of all compacta in \( Q \), a standard coanalytic complete set. In other words, \( \pi \) is a continuous reduction of \( 2^Q \) to \( \mathcal{HID}(I^\omega) \).
If we identify to a point the Cantor set level of each \( \pi(A) \) and respectively modify \( \pi \), then we get a continuous reduction of \( 2^Q \) to \( \mathcal{HID}(I^\omega) \cap C(I^\omega) \).

Families \( \mathcal{HID}(I^\omega) \) and \( \mathcal{HID}(I^\omega) \cap C(I^\omega) \) are contained, respectively, in \( D_2(I^\omega) \) and \( D_2(I^\omega) \cap C(I^\omega) \) which are \( \sigma Z \)-sets in \( 2^I \) and in \( C(I^\omega) \).

We do not know if \( \mathcal{HID}(I^\omega) \) is strongly \( \Pi^1_1 \)-universal in \( 2^I \).

REFERENCES

1. T. Banakh, T. Radul and M. Zarichnyi, Absorbing sets in Infinite-Dimensional Manifolds, VNTL Publishers, Lviv, 1996.
2. J. Baars, H. Gladdines and J. van Mill, Absorbing systems in infinite-dimensional manifolds, Topology Appl. 50 (1993), 147-182.
3. R. Cauty, Caractérisation topologique de l’espace des fonctions dérivables, Fund. Math. 138 (1991), 35-58.
4. R. Cauty, L’espace des arcs d’une surface, Trans. Amer. Math. Soc. 332 (1992), 193-209.
5. R. Cauty, Suites \( F_\sigma \)-absorbantes en théorie de la dimension, Fund. Math. 159 (1999), 115–126.
6. R. Cauty, T. Dobrowolski, H. Gladdines and J. van Mill, Les hyperespaces des rétractions absolus et des rétractions absolus de voisinage du plan, Fund. Math. 148 (1995), 257-282.
7. T. A. Chapman, Lectures on Hilbert cube manifolds, AMS, Providence, Rhode Island, 1976.
8. D. W. Curtis, Hyperspaces of finite subsets as boundary sets, Topology Appl. 22 (1986), 97-107.
9. D. Curtis and M. Michael, Boundary sets for growth hyperspaces, Topology Appl. 25 (1987), 269-283.
10. D. Curtis and M. Schori, Hyperspaces of Peano continua are Hilbert cubes, Fund. Math. 101 (1978), 19–38.
11. J. J. Dijkstra, J. van Mill and J. Mogilski, The space of infinite-dimensional compacta and other topological copies of \( (\ell^2)^\omega \), Pacific J. Math. 152 (1992), 255–273.
12. T. Dobrowolski and L. R. Rubin, The space of ANRs in \( \mathbb{R}^n \), Fund. Math. 146 (1994), 31-58.
13. R. Engelking, Theory of dimensions: Finite and Infinite, Heldermann Verlag, 1995.
14. H. Gladdines and J. van Mill, Hyperspaces of Peano continua of Euclidean spaces, Fund. Math. 142 (1993), 173-188.
15. A. Illanes and P. Krupski, Blockers in hyperspaces, Topology Appl. 158 (2011), 653-659.
16. A. Kechris, Classical descriptive set theory, Springer, 1995.
17. K. Królicki and P. Krupski, Wilder continua and their subfamilies as coanalytic absorbers, Topology Appl. (to appear), [arXiv:1512.05802]
18. P. Krupski, More non-analytic classes of continua, Topology Appl. 127 (2003), 299–312.
19. P. Krupski, Families of continua with the property of Kelley, arc continua and curves of pseudo-arcs, Houston J. Math. 30 (2004), 459–482.
20. P. Krupski and A. Samulewicz, Strongly countable dimensional compacta form the Hurewicz set, Topology Appl. 154 (2007), 996-1001.
21. K. Kuratowski, Topology, Vol. I, Academic Press-PWN, 1966.
22. K. Kuratowski, Topology, Vol. II, Academic Press-PWN, 1968.
23. E. Michael, Some refinements of a selection theorem with 0-dimensional domain, Fund. Math. 140 (1992), 279–287.
24. J. van Mill, The Infinite-Dimensional Topology of Function Spaces, North-Holland, 2002.
25. E. Pol, On infinite-dimensional Cantor manifolds, Topology Appl. 71 (1996), 265–276.
26. R. Pol, On classification of weakly infinite-dimensional compacta, Fund. Math. 116 (1983), 169–188.
27. A. Samulewicz, The hyperspace of hereditarily decomposable subcontinua of a cube is the Hurewicz set, Topology Appl. 154 (2007), 985-995.
28. A. Samulewicz, The hyperspace of indecomposable subcontinua of a cube, Bol. Soc. Mat. Mexicana (3) 17 (2011), no. 1, 89-91.
29. B. E. Wilder, Between aposyndetic and indecomposable continua, Topology Proc. 17 (1992), 325–331.
E-mail address: Pawel.Krupski@math.uni.wroc.pl

Mathematical Institute, University of Wroclaw, pl. Grunwaldzki 2/4, 50–384 Wroclaw, Poland

E-mail address: Alicja.Samulewicz@polsl.pl

Institute of Mathematics, Faculty of Applied Mathematics, Silesian University of Technology, ul. Kaszubska 23, 44-101 Gliwice, Poland