Semisimple Lie groups satisfy property RD, a short proof

Les groupes de Lie semi-simples ont la propriété RD, une preuve courte

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1. Introduction

A length function \( L : G \to \mathbb{R}^*_+ \) on a locally compact group \( G \) is a measurable function satisfying:

(i) \( L(e) = 0 \) where \( e \) is the neutral element of \( G \),
(ii) \( L(g^{-1}) = L(g) \),
(iii) \( L(gh) \leq L(g) + L(h) \).

A unitary representation \( \pi : G \to U(H) \) on a complex Hilbert space has property RD with respect to \( L \) if there exists \( C > 0 \) and \( d \geq 1 \) such that for each pair of unit vectors \( \xi \) and \( \eta \) in \( H \), we have:

\[
\int_G \frac{|\langle \pi(g)\xi, \eta \rangle|^2}{(1 + L(g))^d} \, dg \leq C
\]

where \( dg \) is a (left) Haar measure on \( G \), see [10]. We say that \( G \) has property RD if its regular representation has property RD with respect to \( L \). First established for free groups by U. Haagerup in [6], property RD has been introduced and studied as such by P. Jolissaint in [8], who notably established it for groups of polynomial growth, and for classical hyperbolic groups. See [12, Chap. 8, p. 69] for more details.

If \( \pi \) denotes a unitary representation on a Hilbert space \( H \), then \( \overline{\pi} \) denotes its conjugate representation on the conjugate Hilbert space \( \overline{H} \). The process of linearisation consists in working with \( \sigma : G \to U(\overline{H} \otimes H) \) the unitary representation \( \sigma = \pi \otimes \overline{\pi} \), see [3, Section 2.2].
A connected semisimple real Lie group with finite center can be written $G = KP$, where $K$ is a compact connected subgroup, and $P$ a closed amenable subgroup. We denote by $\Delta_P$ the right-modal function of $P$. Extend to $G$ the map $\Delta_P$ of $P$ as $\Delta : G \to \mathbb{R}^+_\times$ with $\Delta(g) = \Delta(kp) := \Delta_P(p)$. It is well defined because $K \cap P$ is compact (observe that $\Delta_{P(K\cap P)} = 1$).

The quotient $G/P$ carries a unique quasi-invariant measure $\mu$, such that the Radon–Nikodym derivative at $(g,x) \in G \times G/P$, denoted by $c(g,x) = \frac{d\mu_{G/P}}{d\mu}(x)$, with $g,\mu(A) = \mu(g^{-1}A)$, satisfies $\frac{d\mu_{G/P}}{d\mu}(x) = \frac{\Delta(x)}{\Delta(y)}$ for all $g \in G$ and $x, y \in G/P$ (notice that for all $g \in G$, the function $x \in G/P \mapsto \Delta(x) \in \mathbb{R}^+_\times$ is well defined). We refer to [2, Appendix B, Lemma B.1.3, pp. 344–345] for more details. Consider the quasi-regular representation $\lambda_{G/P} : G \to U(L^2(G/P))$ associated with $P$, defined by $((\lambda_{G/P}(g)\xi))(x) = c(g^{-1},x)\frac{\xi(g^{-1}x)}{\xi(x)}$. Denote by $dk$ the Haar measure on $K$, and under the identification $G/P = K/(K \cap P)$, denote by $d[k]$ the measure $\mu$ on $G/P$.

The well-known Harish-Chandra function is defined by $\Xi(g) := (\lambda_{G/P}(g)\xi)\nu_{G/P}$ where $\nu_{G/P}$ denotes the characteristic function of the space $G/P$.

In the rest of the paper, we set $\sigma = \frac{\lambda_{G/P}}{\lambda_{G/P}} \otimes \lambda_{G/P}$. Observe that $L^2(G/P) \otimes L^2(G/P) \cong L^2(G/P \times G/P)$, via: $\xi \otimes \eta \mapsto ((x,y) \mapsto \xi(x)\eta(y))$. Notice that $\sigma$ preserves the cone of positive functions on $L^2(G/P \times G/P)$.

Let $G$ be a (non-compact) connected semisimple real Lie group. Let $g$ be its Lie algebra. Let $\theta$ be a Cartan involution. Define the bilinear form denoted by $(X, Y)$ such that for all $X, Y \in g$, $(X, Y) = -B(X, \theta(Y))$, where $B$ is the Killing form. Set $|\lambda| = \sqrt{(X, X)}$. Write $g = l \oplus p$ the eigenvector space decomposition associated with $\theta$ ($l$ for the eigenvalue 1). Let $K$ be the compact subgroup defined as the connected subgroup whose Lie algebra $l$ is the set of fixed points of $\theta$. Fix $a \subset p$ a maximal Abelian subalgebra of $p$. Consider the roots system $\Sigma$ associated with $a$ and let $\Sigma^+$ be the set of positive roots, and define the corresponding positive Weyl chamber as:

$$a^+ := \{H \in a, \alpha(H) > 0, \forall \alpha \in \Sigma^+\}.$$

Let $A^+ = Cl(\exp(a^+))$, where $Cl$ denotes the closure of $\exp(a^+)$. Consider the corresponding polar decomposition $KA^+K$. Then define the length function:

$$L(g) = L(k_1e^Hk_2) := |H|$$

where $g = k_1e^Hk_2$ with $e^H \in A^+$. Notice that $L$ is $K$ bi-invariant. The desintegration of the Haar measure on $G$ according to the polar decomposition is:

$$dg = dk \ J(H) \ dH \ dk$$

where $dk$ is the Haar measure on $K$, $dH$ the Lebesgue measure on $a^+$, and

$$J(H) = \prod_{\alpha \in \Sigma^+} \left( e^{\alpha(H)} - e^{-\alpha(H)} \right)^{n_{\alpha}}$$

where $n_{\alpha}$ denotes the dimension of the root space associated with $\alpha$. See [9, Chap. V, Section 5, Proposition 5.28, pp. 141–142], [5, Chap. 2, §2.2, p. 65] and [5, Chap. 2, Proposition 2.4.6, p. 73] for more details.

The aim of this note is to give a short proof of the following known result [4,7].

**Theorem 1 (C. Herz).** Let $G$ be a connected real semisimple Lie group with finite center. Then $G$ has property RD with respect to $L$.

See [4, Proposition 5.5 and Lemma 6.3] for the case $G$ has infinite center.

**2. Proof**

**Proof.** We shall prove that the quasi-regular representation has property RD with respect to $L$ defined above. This implies that the regular representation has property RD with respect to $L$ by Lemma 2.3 in [11]. Write $G = KP$, where $K$ is a compact subgroup and $P$ is a closed amenable subgroup of $G$. It is sufficient to prove that there exists $d_0 \geq 1$ and $C_0 \geq 0$ such that $\int_{\mathbb{R}^+} \frac{\nu_{G/P}(\xi)}{\parallel \xi \parallel^{1+1/2d_0}} \ d\xi < C_0$, for positive functions $\xi$, with $\parallel \xi \parallel = 1$.

Take $\xi \in L^2(G/P)$ such that $\xi \geq 0$, and $\parallel \xi \parallel = 1$. Define the function:

$$F : G/P \times G/P \to \mathbb{R}_+, \ (x, y) \mapsto \int_K \sigma(k)(\xi \otimes \xi)(x, y) \ dk.$$
For all \( (x, y) \in G/P \times G/P \), we have by the Cauchy–Schwarz inequality:

\[
\int_K \sigma(k)(\xi \otimes \xi)(x, y) \, dk = \int_K \xi(k^{-1}x)\xi(k^{-1}y) \, dk \\
\leq \left( \int_K \xi^2(k^{-1}x) \, dk \right)^{\frac{1}{2}} \left( \int_K \xi^2(k^{-1}y) \, dk \right)^{\frac{1}{2}}.
\]

Observe that the function \( f : x \in G/P \mapsto \int_K \xi^2(k^{-1}x) \, dk \in \mathbb{R}_+ \) is constant. Indeed, fix \( x \in G/P \) and let \( y \in G/P \). Write \( y = hx \) for some \( h \in K \) (as \( K \) acts transitively on \( G/P \)). By invariance of the Haar measure, we have \( f(y) = \int_K \xi^2(k^{-1}y) \, dk = \int_K \xi^2(h^{-1}hx) \, dk = \int_K \xi^2(k^{-1}x) \, dk = f(x) \). If \( e \) is the neutral element in \( G \), we write \([e] \in G/P \). We have, for all \( x \in G/P \), \( f(x) = f([e]) \).

Hence, for all \( x \in G/P \) we have:

\[
\int_K \xi^2(k^{-1}x) \, dk = \int_K \xi^2([k^{-1}]) \, dk \\
= \int_{K/K \cap P} \xi^2([k^{-1}]) \, dk \\
= \|\xi\|^2 = 1.
\]

Therefore \( \|F\|_\infty := \sup\{F(x, y) \mid (x, y) \in G/P \times G/P\} \leq 1 \). Hence \( 0 \leq F \leq 1_{G/P \times G/P} \), where \( 1_{G/P \times G/P} \) denotes the characteristic function of \( G/P \times G/P \).

Let \( r \) be the number of indivisible positive roots in \( a \). We know that there exists \( C > 0 \) such that, for all \( H \in a \) where \( e^H \in A^+ \), we have:

\[
\Xi(e^H) \leq Ce^{-\rho(H)(1 + L(e^H))}.
\]

with \( \rho = \frac{1}{2} \sum_{a \in \Sigma} n_a \alpha \in a^+ \), see [5, Chap. 4, Theorem 4.6.4, p. 161]. Hence for \( d_0 > \dim(a) + 2r \), we have:

\[
\int_{a^+} \frac{\Xi^2(e^H)}{(1 + L(e^H))^{d_0}} J(H) \, dH < \infty.
\]

We obtain for all \( d \geq 0 \) and for all positive functions \( \xi \), with \( \|\xi\| = 1 \):

\[
\int_G \frac{\langle \lambda_{G/P}(g)\xi, \xi \rangle^2}{(1 + L(g))^d} \, dg = \int_G \frac{\langle \lambda_{G/P}(g)\xi, \lambda_{G/P}(g)\xi \rangle}{(1 + L(g))^d} \, dg \\
= \int_G \frac{\langle \sigma(g)\xi \otimes \xi, \xi \otimes \xi \rangle}{(1 + L(g))^d} \, dg \\
= \int_K \int_{a^+} \int_{a^+} \int_{a^+} \frac{\langle \sigma(k_1 e^H k_2)\xi \otimes \xi, \xi \otimes \xi \rangle}{(1 + L(k_1 e^H k_2))^d} J(H) \, dk_1 \, dk_2 \, dH \\
= \int_K \int_{a^+} \int_{a^+} \int_{a^+} \frac{\langle \sigma(e^H)\sigma(k_2)(\xi \otimes \xi), \sigma(k_1^{-1})(\xi \otimes \xi) \rangle}{(1 + L(e^H))^d} J(H) \, dk_1 \, dk_2 \, dH \\
= \int_{a^+} \int_{a^+} \int_{a^+} \frac{\langle \sigma(e^H)\left(\int_{a^+} \sigma(k_2)(\xi \otimes \xi) \, dk_2 \right) \sigma(k_1^{-1})(\xi \otimes \xi) \rangle}{(1 + L(e^H))^d} J(H) \, dk_1 \, dH \\
= \int_{a^+} \int_{a^+} \frac{\langle \sigma(e^H)F, F \rangle}{(1 + L(e^H))^d} J(H) \, dH \\
\leq \int_{a^+} \frac{\langle \sigma(e^H)1_{G/P \times G/P}, 1_{G/P \times G/P} \rangle}{(1 + L(e^H))^d} J(H) \, dH.
\]
\[
\int_{\mathfrak{a}^+} \frac{(\lambda G/P(e^H)1_{G/P}1_{G/P})^2}{(1 + L(e^H))^d} J(H) \, dH \\
= \int_{\mathfrak{a}^+} \frac{\Xi^2(e^H)}{(1 + L(e^H))^d} J(H) \, dH.
\]

Take \(d_0 > \dim(\mathfrak{a}) + 2r\) and \(C_0 = \int_{\mathfrak{a}^+} \frac{\Xi^2(e^H)}{(1 + L(e^H))^d} J(H) \, dH\). We have found \(d_0 \geq 1\) and \(C_0 > 0\) such that for all positive functions \(\xi\) in \(L^2(G/P)\) with \(\|\xi\| = 1\), we have:

\[
\int_{G} \frac{(\lambda G/P(g)\xi, \xi)^2}{(1 + L(g))^{d_0}} \, dg \leq C_0.
\]

Remark 1. The same approach applies to algebraic semisimple Lie groups over local fields. See [1, Section 1, (1.3)] and [13, Lemma II.1.5].

Remark 2. It’s not hard to see that this approach shows that the representations of the principal series of \(G\) (of class one, see [5, (3.1.12), p. 103]) satisfy also property RD.

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