Gauge invariance and mass gap in (2+1)-dimensional Yang-Mills theory

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Abstract

In terms of a gauge-invariant matrix parametrization of the fields, we give an analysis of how the mass gap could arise in non-Abelian gauge theories in two spatial dimensions.

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1. Introduction

In this talk, we shall be discussing pure non-Abelian gauge theories, i.e., with no matter fields, in two spatial dimensions. Specifically, we shall focus on a gauge-invariant analysis and the question of how a mass gap could arise in these theories. As is well-known, this issue was addressed long ago by Polyakov who considered an $SU(2)$-gauge theory spontaneously broken down to $U(1)$. If this were a $(3 + 1)$-dimensional theory, there will clearly be monopole solutions. For the $(2 + 1)$-dimensional theory, these “monopole” solutions can be considered as tunnelling configurations (or instantons) and a semiclassical analysis can be done by expanding the functional integral around these configurations. This leads to confinement and eventually a mass gap. It is believed that the results hold for the unbroken theory as well, although we go out of the regime of validity of the semiclassical expansion as the parameters are relaxed towards the unbroken phase. Here we would like to understand the mass gap directly in the unbroken theory. We will focus more on the geometry of the configuration space of the theory. After all, nonperturbative aspects of gauge theories are not well understood and exploring different points of view can be quite useful.

A question which might immediately arise is, why $(2 + 1)$ dimensions? Why not directly analyze the more physical case of $(3 + 1)$ dimensions? Apart from the fact that this is a conference devoted to low dimensional field theories, there is a good technical reason why we hope to make more progress in $(2 + 1)$ dimensions, at least as a start. In the latter case, in Hamiltonian analysis, many of the quantities of interest are defined in terms of two-dimensional fields and one can use known results from two dimensions, especially from conformal field theory. To set the stage and define a framework for the discussion, let us consider an $SU(N)$-gauge theory in the $A_0 = 0$ gauge. The gauge potential can be written as $A_i = -it^a A^a_i$, $i = 1, 2$, where $t^a$ are hermitian $N \times N$-matrices which form a basis of the Lie algebra of $SU(N)$ with $[t^a, t^b] = if^{abc} t^c$, $\text{Tr}(t^a t^b) = \frac{1}{2} \delta^{ab}$. The Hamiltonian in this case is given by

$$\mathcal{H} = T + V, \quad T = \frac{e^2}{2} \int d^2 x \, E_i^a E_i^a, \quad V = \frac{1}{2e^2} \int d^2 x \, B^a B^a$$

$$E_i^a = \partial_\alpha A_i^a = -i \frac{\delta}{\delta A_i^a}, \quad B^a = \frac{i}{2} \varepsilon_{ij} (\partial_i A_j^a - \partial_j A_i^a + f^{abc} A_b^i A_c^j)$$

The expectation value of the Hamiltonian for a physical state with wave function $\Psi[A]$ is given by

$$\langle \mathcal{H} \rangle = \int d\mu(C) \left[ \frac{e^2}{2} \frac{|\delta \Psi|^2}{|\delta A|} + \frac{1}{2e^2} B^2 |\Psi|^2 \right]$$

Wave functions for physical states are gauge-invariant and the integration in Eq.(2) is over all gauge-invariant field configurations. More precisely, let

$$\mathcal{A} = \{ \text{set of all gauge potentials } A_i^a(x) \text{ in } \mathbb{R}^2 \}$$

$$\mathcal{G}_* = \{ \text{set of all } g(x) : \mathbb{R}^2 \to SU(N), \quad g \to 1 \text{ as } |\vec{x}| \to \infty \}$$

$\mathcal{G}_*$ acts on $\mathcal{A}$ in the usual manner of gauge transformations, viz.,

$$A_i^a(x) = g^{-1} A_i g + g^{-1} \partial_i g$$

The gauge-invariant configuration space $\mathcal{C}$ is then given by $\mathcal{A}/\mathcal{G}_*$, viz., all gauge potentials modulo gauge transformations.
Many years ago Feynman tried to develop qualitative arguments regarding the wave functions and mass gap for this theory, similar in spirit to what he did so successfully for liquid Helium. It is easy to see that the ground state wave function can be taken to be a real, positive function on \( C \) since the Hamiltonian operator is real (and not just hermitian). The argument is similar to the standard argument of ‘no-nodes’ for the ground state wave function of a quantum mechanical system without velocity-dependent potentials. It then follows that an excited state wave function \( \Psi_{exc} \), being orthogonal to the ground state, must be positive in some regions of \( C \) and negative in some other regions. The kinetic energy term in Eq.(2), which is a gradient energy on \( C \), will roughly go like \( 1/L^2 \) where \( L \) is some measure of the distance between the region where \( \Psi_{exc} \) is positive and where it is negative. Feynman suggested that, for the gauge theory, \( L \) cannot be made arbitrarily large and the gradient energy would have a definite nonzero lower bound; this would be the mass gap. The reasoning is very similar to what leads to a discrete spectrum for the Laplacian on a finite dimensional compact manifold. Indeed the kinetic energy term in Eq.(1) is the Laplacian on \( C \), but Feynman’s argument cannot immediately be proved since \( C \) is infinite dimensional and developing notions of compactness requires great care. Singer and others have given a careful formulation and discussion of some of the geometrical issues involved.

2. A matrix parametrization of fields

Fifteen years have passed since Feynman’s paper and we have learnt many things about two-dimensional field theories and what we would like to do here is to reanalyze this line of reasoning. For this we need to understand the geometry of \( C = A/G \) better; we shall need gauge-invariant variables for describing \( C \), we shall need to calculate the metric, volume element and Laplacian on \( C \). A good parametrization of the fields which allows the explicit calculation of \( d\mu(C) \) is a first step. We shall combine the spatial coordinates \( x_1, x_2 \) into complex combinations \( z = x_1 - ix_2, \bar{z} = x_1 + ix_2 \); correspondingly we have \( A \equiv A_z = \frac{1}{2}(A_1 + iA_2), \bar{A} \equiv A_{\bar{z}} = \frac{1}{2}(A_1 - iA_2) = -(A_z)\dagger \). The parametrization we use is given by

\[
A_z = -\partial_z M M^{-1}, \quad A_{\bar{z}} = M^\dagger \partial_{\bar{z}} M^\dagger \quad (4)
\]

Here \( M, M^\dagger \) are complex \( SL(N, \mathbb{C}) \)-matrices (for gauge group \( SU(N) \)). Such a parametrization is possible and is standard in many discussions of two-dimensional gauge fields. Indeed for any \( A, \bar{A} \), it is easily checked that a choice of \( M, M^\dagger \) is given by

\[
M(x) = 1 - \int G(x, z_1)A(z_1) + \int G(x, z_1)A(z_1)G(z_1, z_2)A(z_2) - \ldots
\]

\[
= 1 - \int_y D^{-1}(x, y)A(y) \quad (5)
\]

\[
M^\dagger(x) = 1 - \int_y \bar{A}(y)\bar{D}^{-1}(y, x)
\]

(Here \( D = \partial + A, \bar{D} = \bar{\partial} + \bar{A}, G(x, x') = \frac{1}{\pi(z-x)}, G(x, x') = \frac{1}{\pi(z-x)}, \partial_z G(x, y) = \partial_y G(x, y) = \delta^{(2)}(x - y) \). There may be many choices for \( M, M^\dagger \); we shall discuss this question later.) From the definition (4), it is clear that a gauge transformation (3) is expressed in terms of \( M, M^\dagger \) as

\[
M \rightarrow M^{(g)} = gM, \quad M^{\dagger(g)} = M^\dagger g^{-1} \quad (6)
\]

for \( g(x) \in SU(N) \). In particular, if we split \( M \) into a unitary part \( U \) and a hermitian part \( \rho \) as \( M = U\rho \), then \( U \) is the ‘gauge part’, so to speak; it can be removed by a gauge transformation and \( \rho \) represents the
gauge-invariant degrees of freedom. Alternatively, we can use $H = M^1 M = \rho^2$ as the gauge-invariant field parametrizing $C$. Since $M \in SL(N, C)$, $\rho$, and hence $H$, belong to $SL(N, C)/SU(N)$. 

3. Metric and volume element

Let us now consider the metrics or distance functions on the relevant spaces. From comparing the action $S = \int \frac{1}{4} F^2 = \frac{1}{2} \int \partial_\alpha A^\alpha \partial_\beta A^\beta$ to a standard form like $\frac{1}{2} g_{\alpha\beta} \dot{Q}^\alpha \dot{Q}^\beta$, we see that the relevant metric on $A$ is given by

$$ ds^2_A = \int d^2 x \, \delta A^a \delta A^\alpha = -8 \int \text{Tr}(\delta A_\alpha \delta A_\alpha) $$

$$ = 8 \int \text{Tr}[D(\delta M M^{-1}) \bar{D}(M^1 \delta M^1)] $$

where in the last line we have used the parametrization (4) and $D, \bar{D}$ are in the adjoint representation. This is a simple Euclidean metric for $A$ and the volume element $d\mu(A)$ for this space is the standard Euclidean one, $d\mu(A) = (dA(x)) \prod_x dA(x)$.

We now turn to the matrices $M, M^1$; these are elements of $SL(N, C)$ and we have the Cartan-Killing metric for $SL(N, C)$, viz., $ds^2 = 8 \text{Tr}(\delta M M^{-1} \, M^1 \delta M^1)$. For $SL(N, C)$-valued fields, we thus have

$$ ds^2_{SL(N, C)} = 8 \int \text{Tr}[(\delta M M^{-1}) (M^1 \delta M^1)] $$

(8)

We denote the corresponding volume element, the Haar measure, by $d\mu(M, M^1)$; we do not need an explicit expression for this at this stage. However, from Eq.(7) we can see that $d\mu(A) = \text{det}(D \bar{D})d\mu(M, M^1)$.

Finally we need to consider $SL(N, C)/SU(N)$. Since this is a coset space we can start from the $SL(N, C)$-metric and obtain a metric for the quotient in a standard way. A simple way to do this is to write an $SU(N)$-invariant version of Eq.(8) by introducing an auxiliary ‘gauge field’ $\alpha$, viz.,

$$ ds^2 = 8 \int \text{Tr}[(\delta M + \alpha M) M^{-1} M^1 (\delta M^1 + M^1 \alpha)] $$

(9)

Eliminating $\alpha$ by its equation of motion we get

$$ ds^2_{\bar{H}} = 2 \int \text{Tr}(H^{-1} \delta H)^2 = \int r_{ak} r_{bk} \delta \varphi^a \delta \varphi^b $$

(10)

where we parametrize $H$ in terms of the real field $\varphi^a(x)$, $H^{-1} \delta H = \delta \varphi^a r_{ak}(\varphi) t_k$. The corresponding volume element or Haar measure is given by

$$ d\mu(H) = (\text{det } r)[\delta \varphi] $$

(11)

We are now ready to consider the volume element $d\mu(C)$ on the configuration space. This is obtained from $d\mu(A)$ by factoring out the volume of gauge transformations. We thus have

$$ d\mu(C) = \frac{d\mu(A)}{\text{vol}(G_s)} = \frac{dA_\alpha dA_\beta}{\text{vol}(G_s)} $$

$$ = (\text{det } D_\alpha D_\beta) \frac{d\mu(M, M^1)}{\text{vol}(G_s)} = (\text{det } D \bar{D}) d\mu(H) $$

(12)

The problem is thus reduced to calculating the determinant of the two-dimensional operator $D \bar{D}$ \cite{6,7}. We have

$$ (\text{det } D \bar{D}) = \text{constant } \times \exp[2c_A S(H)] $$

(13)
where \( c_A \delta^{ab} = f^{amn}f^{bmn} \) and \( S(H) \) is the Wess-Zumino-Witten (WZW) action for the hermitian matrix field \( H \) given by

\[
S(H) = \frac{1}{2\pi} \int \text{Tr}(\partial H \bar{\partial} H^{-1}) + \frac{i}{12\pi} \int \epsilon^{\mu\nu\sigma} \text{Tr}(H^{-1} \partial_\mu HH^{-1} \partial_\nu HH^{-1} \partial_\sigma H) \tag{14}
\]

Even though the result (13) is well-known, we shall review it briefly here since we shall need one of the steps in its evaluation for later purposes. Defining \( \Gamma = \log \det D \bar{D} \), we have

\[
\delta \Gamma \bigg|_{A^a} = -i \text{Tr} \left[ (x, y) T^a \right]_{y \to x} \tag{15}
\]

Here \( (T^a)_{mn} = -if^{amn} \) are the generators of the Lie algebra in the adjoint representation. The coincident-point limit of \( \bar{D}^{-1}(x, y) \) is, of course, singular and needs regularization. Since \( d\mu(C) \) must be gauge-invariant, a gauge-invariant regularization is appropriate here. With a gauge-invariant regulator such as covariant point-splitting we have

\[
\text{Tr} \left[ (x, y) \left( \bar{D}^{-1}_{\text{reg}} \right) \right]_{y \to x} = \frac{2c_A}{\pi} \text{Tr} \left[ (A - M^{-1} \partial M^{-1}) T^a \right] \tag{16}
\]

Using this result in Eq. (15), and with a similar result for the variation of \( \Gamma \) with respect to \( A^a \), and integrating we get \( \Gamma = 2c_A S(H) \).

Using the expression (13) in Eq. (12), we get the volume element on the configuration space \( C \) as \(^7\)\(^8\)

\[
d\mu(C) = d\mu(H) e^{2c_A S(H)} = [\delta \varphi][(\det r)] e^{2c_A S(H)} \tag{17}
\]

The inner product for physical states is given by

\[
\langle 1 | 2 \rangle = \int d\mu(H) e^{2c_A S(H)} \Psi_1^*(H) \Psi_2(H) \tag{18}
\]

Here we begin to see how conformal field theory can be useful; this formula shows that all matrix elements in \((2 + 1)\)-dimensional \( SU(N) \)-gauge theory can be evaluated as correlators of the \( SL(N, C)/SU(N) \)-WZW model for the hermitian matrix \( H \). (For a general gauge group \( G \), we will have a \( G^C/G \)-WZW model, \( G^C \) being the complexification of \( G \).)

4. Spectrum of \( T \), a first look

Some interesting observations follow from Eq. (17). The integral of \( d\mu(C) \) is the partition function for the WZW-model. We can evaluate this as

\[
\int d\mu(C) = \int d\mu(H) e^{2c_A S} = \left[ \frac{\text{det}' \partial \bar{\partial}}{d^2x} \right]^{-\text{dim} G} \tag{19}
\]

where \( \text{dim} G = N^2 - 1 \) for \( G = SU(N) \). Regularizing with a cutoff on the number of modes we see that the result is finite; i.e., the total volume of \( C \) with an appropriate regulator is finite. This is to be contrasted with the Abelian case which has \( c_A = 0 \) and where the integral diverges for each mode. The “finiteness” of the volume is clearly a step in the right direction as regards the mass gap although we are far from any statement of compactness for \( C \).

We can also see, in an intuitive way, how the exponential factor \( \exp(2c_A S) \) can influence the spectrum. Writing \( \Delta E, \Delta B \) for the root mean square fluctuations of the electric field \( E \) and the magnetic field \( B \), we
have, from the canonical commutation rules \([E^a_i, A^b_j] = -i\delta_{ij}\delta^{ab}\), \(\Delta E \Delta B \sim k\), where \(k\) is the momentum variable. This gives an estimate for the energy

\[
E = \frac{1}{2} \left( \frac{e^2 k^2}{\Delta B^2} + \frac{\Delta B^2}{e^2} \right)
\]  

(20)

For low lying states, we minimize \(E\) with respect to \(\Delta B^2\), \(\Delta B^2_{\text{min}} \sim e^2 k\), giving \(E \sim k\). This is, of course, the standard photon or perturbative gluon. For the non-Abelian theory, this is inadequate since \(\langle H \rangle\) involves the factor \(e^{2c_A S}\). In fact,

\[
\langle H \rangle = \int d\mu(H) e^{2c_A S} \frac{1}{2}(e^2 E^2 + B^2/e^2)
\]

(21)

Since \(S(H) \approx -(c_A/\pi)^2 \int B(1/k^2)B + ...\), we see that \(B\) follows a Gaussian distribution of width \(\Delta B^2 \approx \pi k^2/c_A\), for small values of \(k\). This Gaussian dominates near small \(k\) giving \(\Delta B^2 \sim k^2(\pi/c_A)\). In other words, even though \(E\) is minimized around \(\Delta B^2 \sim k\), probability is concentrated around \(\Delta B^2 \sim k^2(\pi/c_A)\).

For the expectation value of the energy, we then find \(E \sim (e^2 c_A/2\pi) + O(k^2)\). Thus the kinetic term in combination with the measure factor \(e^{2c_A S}\) could lead to a mass gap of order \(e^2 c_A\). The argument is not rigorous; many terms, such as the non-Abelian contributions to the commutators and \(S(H)\), have been neglected. Nevertheless, we expect this to capture the essence of how a mass gap could arise.

Before taking up the construction of the Laplacian on \(E\) we need some more properties of the hermitian WZW-model. These can be obtained by comparison with the \(SU(N)\) model defined by \(\exp(kS(U))\), \(U(x) \in SU(N)\). The quantity which corresponds to \(e^{kS(U)}\) for the hermitian model is \(\exp[(k + 2c_A)S(H)]\). The hermitian analogue of the renormalized level \(\kappa = k + 2c_A\) of the \(SU(N)\)-model is \(-k + 2c_A\). Correlators can be calculated from the Knizhnik-Zamolodchikov equation. Since the latter involves only the renormalized level \(\kappa\), we see that the correlators of the hermitian model (of level \((k + 2c_A)\)) can be obtained from the correlators of the \((level \ k)\) \(SU(N)\)-model by the analytic continuation \(\kappa \rightarrow -\kappa\). For the \(SU(N)_k\)-model there are the so-called integrable representations whose highest weights are limited by \(k\) (spin \(\leq k/2\) for \(SU(2)\), for example). Correlators involving the nonintegrable representations vanish. For the hermitian model the corresponding statement is that the correlators involving nonintegrable representations are infinite. (This is not a regularization problem, the correlators have singularities at certain values of \(k\), the coupling constant.) In our case, \(k = 0\), and we have only one integrable representation corresponding to the identity operator (and its current algebra descendents). All matrix elements of the (2 + 1)-dimensional gauge theory being correlators of the hermitian WZW-model, we must conclude that all wave functions (with finite inner product and norm) can be taken to be functions of the current

\[
J^a(x) = \frac{c_A}{\pi} \left( \partial H H^{-1} \right)^a(x) = \frac{c_A}{\pi} [iM^{ab}(x)A^b(x) + (\partial M^a \ M^{-1})^a(x)]
\]

(22)

where \(M^{ab} = 2\text{Tr}(t^a M^b \ M^{-1})\) is the adjoint representation of \(M^a\).

Although we have gone through the line of reasoning which follows from conformal field theory, this conclusion is, in the end, not surprising. The Wilson loop operator can be written in terms of the current as

\[
W(C) = \text{Tr} P e^{\oint_C A dz + \bar{A} d\bar{z}} = \text{Tr} P e^{-(\pi/c_A) \oint_C J}
\]

(23)

In principle, all gauge-invariant functions of \((A, \bar{A})\) can be constructed from \(W(C)\) and hence it suffices to consider wave functions as functions of \(J^a\).
5. The Laplacian and $T$

We can now start looking at the spectrum of $T = (e^2/2) \int E^2$. Since $T$ is positive and $E = -i\delta/\delta A$, the ground state is given by $\Psi_0 = 1$. This seems too trivial an observation but the key point is that $\Psi_0$ is normalizable with the inner product (18).

Consider now an excited state with wave function $J^a(x)$. We have

$$T J^a(x) = -\frac{e^2}{2} \int \delta^2 y \frac{\delta^2 J^a(x)}{\delta A^b(y) \delta A^b(y)} = \frac{e^2 c_A}{2\pi} M^{amn} \Tr [T^m D^{-1}(y, x)]_{y \to x}$$

In Eq.(24a) we encounter the same coincident-point limit as in the calculation of $(\det D D)$. We have used the same regulator and the result (16) to obtain Eq.(24b), with $m = (e^2 c_A/2\pi)$.

Eq.(24) shows that a state with wave function proportional to $J^a$, say $\Psi_1 = \int d^2 x f(x) J^a(x)$ has a mass gap $m$, a rather nice result. However, $J^a$ is not an acceptable state. The reason is that there are many choices for $M$ for a given potential $A$. In particular $M$ and $M V(x)$, where $V(x)$ is an antiholomorphic function, lead to the same $A = -\partial M M^{-1}$. Of course, there are no globally defined antiholomorphic functions except the constant and if we impose $M \to 1$ at spatial infinity, this ambiguity can be eliminated. However $M$’s corresponding to some $A$’s will have singularities. One can eliminate singularities in $M$ by defining it separately on coordinate patches and using the antiholomorphic transformation as transition functions, i.e., $M_1 = M_2 V_{12}$, etc., or $H_1 = V_{12} H_2 V_{12}$ in terms of $H = M^1 M$. Since this is an ambiguity of choice of field variables, the wave functions must be invariant under this. (The ambiguity in the choice of $M$ or $H$ and the need for (anti)holomorphic transition functions are related to the geometry of $A$ as a $\mathcal{G}$-bundle over $C$ and the Gribov problem. For a discussion of these issues, see reference 1.) $J^a$ (or $\Psi_1$) by itself does not satisfy the required condition of “holomorphic invariance”; we need at least two $J$’s. We should thus consider the action of the kinetic energy operator $T$ on two $J$’s or more generally we need $T$ as an operator on any function of $J$’s.

5. The Laplacian and $T$

In constructing the operator $T$, first let us consider the change of variables from $A, \bar{A}$ to $M, M^1$. The metric on $A$ can be written as

$$ds^2_A = -8 \int \Tr(\delta A \delta \bar{A}) = \int g_{a\bar{b}}(x, y) \delta \theta^a(x) \delta \bar{\theta}^b(y) + h.c. \tag{25}$$

where $\theta, \bar{\theta}$ are parameters defining $M$ and $M^1$ respectively and

$$g_{a\bar{b}}(x, y) = 2 \int_{u, v} \partial_u[\delta(u - x) R_{ar}(u)] M_{kr}(u) M_{ab}^1(v) \partial_v[\delta(v - y) R_{ar}^*(v)] \tag{26}$$

$$M^{-1} M = \delta \theta^a R_{ab}(\theta) t_b, \quad \delta M^1 M^{-1} = \delta \bar{\theta}^a R_{ab}^*(\bar{\theta}) t_b \tag{27}$$

The metric on $A$ is a Kähler metric since $ds^2 = \delta_A \delta_{\bar{A}}[-8 \int \Tr(A \bar{A})]$. The Laplacian $\Delta$ has the general form

$$\Delta = g^{-1} \left( \partial_a g^{a\bar{a}} g \partial_{\bar{a}} + \partial_{\bar{a}} g^{a\bar{a}} g \partial_a \right) \tag{28}$$

where $g = \det(g_{a\bar{b}})$. Eq.(26) then leads to

$$T = -\frac{e^2}{2} \Delta = \frac{e^2}{4} \int_x e^{-2c_A S(H)} \left[ \tilde{G} p_a(x) K_{ab}(x) e^{2c_A S(H)} G p_b(x) + G p_a(x) K_{ba}(x) e^{2c_A S(H)} \tilde{G} p_b(x) \right] \tag{29}$$
where $p_a$ generates right-translations on $M$ and $\bar{p}_a$ generates left-translations on $M^\dagger$, i.e.,

$$[p_a(x), M(y)] = M(y)(-it_a) \delta(y - x)$$

$$[\bar{p}_a(x), M^\dagger(y)] = (-it_a) M^\dagger(y) \delta(y - x)$$

(30)

Further, $K_{ab} = 2\text{Tr}(t_a H t_b H^{-1})$ is the adjoint representation of $H$ and $G_{p_a}(x) = \int_u G(x, u)p_a(u)$ (and similarly for $\bar{G}\bar{p}_a$).

Rather than constructing the Laplacian as above, we can also write $T\Psi = (-e^2/2)(\delta^2\Psi/\delta A\delta\bar{A})$ and make the change of variables to $M, M^\dagger$. This gives the expression

$$T = \frac{e^2}{2} \int_x K_{ab}(x)\bar{G}\bar{p}_a(x)Gp_b(x)$$

(31)

For a finite dimensional Kähler manifold, we have the identity $\partial_b(g^{a\alpha}g) = 0$ and this suffices to prove that the two forms of $T$, viz., Eqs.(29,31), are in fact identical. In our case, even though the metric is Kähler, the equality of the two expressions does not immediately follow since the space is infinite dimensional; we need to regularize these expressions. A regularized expression preserving gauge and holomorphic invariance is given by

$$T = \frac{e^2}{2} \int_{u,v} \Delta_{ab}(u,v)\bar{p}_a(u)p_b(v)$$

(32)

$$\Delta_{ab}(u,v) = \int_x \bar{G}_{ma}(x, u)K_{ma}(x)G_{nb}(x, v)$$

$$\bar{G}_{ma}(x, u) = \bar{G}(x, u)[\delta_{ma} - e^{-|x-u|^2/\epsilon} (K(x, \bar{u})K^{-1}(u, \bar{u}))_{ma}]$$

(33)

$$G_{nb}(x, v) = G(x, v)[\delta_{nb} - e^{-|x-v|^2/\epsilon} (K^{-1}(v, \bar{x})K(v, \bar{v}))_{nb}]$$

The regularized Green’s functions $\bar{G}, \bar{G} \to G, \bar{G}$ (times the Kronecker delta ) as the regularizing parameter $\epsilon \to 0$ and $G(x, x) = \bar{G}(x, x) = 0$ for finite $\epsilon$.

One can check that, for $T$ defined by Eqs.(32,33), we have

$$T = \frac{e^2}{4} \int_x e^{-2c_A S(\epsilon, H)}[(\bar{G}\bar{p})_a(x)K_{ab}(x)e^{2c_A S(\epsilon, H)}(\bar{G}p)_b(x) +$$

(34)

$$+ (\bar{G}p)_a(x)K_{ba}(x)e^{2c_A S(\epsilon, H)}(\bar{G}\bar{p})_b(x)]$$

where $S(\epsilon, H) = S(H) + O(\epsilon)$.

The above considerations apply to the space $\mathcal{A}$. The restriction to $\mathcal{C}$ is, however, straightforward. Using $M = U\rho$, the operator $p_a$ can be written as

$$p_a = -iR^{-1}_{ab}\delta/\delta\bar{g}^b = (1 + M)^{-1}_{ab}(\alpha_b + I_b)$$

(35)

where $\alpha_b$ generates right-translations on $\rho$ and $I_b$ generates translations on $U$. (There is a similar expression for $\bar{p}_a$.) On gauge-invariant functions which are independent of $U$, $I_b = 0$. Setting $I_b = 0$ in Eqs.(32,34), we get the required expressions on $\mathcal{C}$. When $I_b = 0$, $p_a = -i\tau^{-1}_{ab}\delta/\delta\bar{g}^b$ and $\bar{p}_a = K_{ab}p_b$. With this understanding and using Eq.(32) for $T$ we find

$$T \Psi(J) = m \left[ \int_w \omega^a(w) \frac{\delta}{\delta J^a(w)} + \int_{w,z} \Omega^{ab}(w, z) \frac{\delta}{\delta J^a(w)} \frac{\delta}{\delta J^b(z)} \right] \Psi(J)$$

(36)
We see that as $\epsilon \to 0$, the first term in $T$ gives the number of $J$’s in $\Psi(J)$ while the second replaces pairs of $J$’s by the lowest terms of the operator product expansion for currents in the WZW-model.

We now return to the consideration of the states. The quantity $\partial J^a(x)\partial J^a(x)$ has both gauge and holomorphic invariance and we can construct a physical state

$$\Psi_2(J) = \int_x f(x) \left[ \partial J_a(x)\partial J_a(x) + \frac{c_A \dim G}{\pi^2} \partial_x \partial_y \delta(x - y) \right]$$

(38)

The second (c-number) term in this expression is necessary to orthogonalize this with respect to the ground state. It is also what is needed for normally ordering the term $\bar{\partial}J^a(x)\partial J^a(x)$. From the above formula for the action of $T$ we find

$$T \Psi_2 = 2m \Psi_2$$

(39)

This is the lowest excited eigenstate of $T$. One can form more general combinations, for example,

$$\Psi = \int f(x_1, x_2) : \text{Tr}[\partial J(x_1)U(x_1, x_2)\partial J(x_2)]: + \int f(x_1, x_2, x_3) : \text{Tr}[\partial J(x_1)U(x_1, x_2)\partial J(x_2)U(x_2, x_3)\partial J(x_3)U(x_3, x_1)]: + \cdots$$

(40)

where $U(x, y) = K(x, y)K^{-1}(y, y)$. By requiring that this be an eigenstate of $T$, we obtain a hierarchy of coupled equations for the functions $f(x_1, x_2)$, $f(x_1, x_2, x_3)$, etc.. The solutions will give series of eigenstates and eigenvalues, a series for $f(x_1, x_2)$, a series for $f(x_1, x_2, x_3)$, etc.; we expect these to be analogous to Regge trajectories.

So far we have talked about the spectrum of the kinetic energy operator $T$. How do we include the potential term? We expect that this can be done perturbatively, $1/\epsilon^2$ being the expansion parameter. Since $m = (\epsilon^2 c_A/2\pi)$, this will be an expansion in $1/m$, say in powers of $p/m$ where $p$ is a typical momentum. For example, up to the first order in perturbation theory we find

$$\mathcal{H} J^a = (T + V)J^a \approx \left( m + \frac{p^2}{2m} \right) J^a + \mathcal{O}\left( \frac{1}{m^2} \right)$$

(41)

We get the first correction to the energy beyond the mass $m$ in a small momentum expansion. Since the theory is Lorentz invariant higher order corrections are expected to sum up to the relativistic expression $(p^2 + m^2)^\dagger$. The situation is similar to what occurs for solitons; there again one finds a similar expansion which sums up to the relativistic formula $^9$. Notice that the state $\Psi_2$, for example, is degenerate, having the same eigenvalue for $T$ for all $f(x)$ (and similarly for the more general states of Eq.(40)). This degeneracy is lifted by the potential term. The inclusion of the potential term is thus necessary to obtain a proper hierarchy of equations for the functions $f(x_1, x_2)$, $f(x_1, x_2, x_3)$, etc..

6. Concluding remarks
From what we have said so far it should be clear that this is a fruitful line of investigation and there are many remaining questions of interest. The construction of a complete set of eigenstates for $T$, the inclusion of the potential term and the derivation of the hierarchy of equations for $f(x_1, x_2)$, $f(x_1, x_2, x_3)$, etc. are of prime importance. We are currently investigating these questions. Another very interesting possibility is the following. Once we have the Hamiltonian $H$ as an operator on functions on $C$, we can obtain a functional integral, starting from $e^{-iHt}$, which is defined directly on $C$ without the need for gauge fixing. There are also interesting generalizations of the pure gauge theory, such as the inclusion of quarks or a Chern-Simons mass term, which can be investigated using our approach.

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