A Summary of the Langlands-Shahidi Method of Constructing L-functions

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Abstract

These notes are from the Database of Automorphic L-functions at http://www.math.rutgers.edu/~sdmiller/l-functions. They were written up by Stephen Miller, and are based on discussions with Freydoon Shahidi, Purdue University. They are meant to serve as an introduction to the Langlands-Shahidi method of studying L-functions through the Fourier coefficients of Eisenstein series.

1 Introduction

The Langlands-Shahidi method uses information from spectral theory on non-compact spaces to obtain the functional equations and many analytic properties of L-functions, in particular several important examples of Langlands L-functions. The reason for the connection to spectral theory is that L-functions arise in the Fourier expansions of Eisenstein series. The simplest example of this phenomenon occurs already on $SL_2(\mathbb{Z}) \backslash \mathbb{H}$ for the non-holomorphic Eisenstein series

$$E(x + iy, s) = \sum_{\substack{m,n \in \mathbb{Z} \\backslash \{m,n\}=1}} \frac{y^s}{|m(x + iy) + n|^{2s}},$$

which has the Fourier expansion

$$E(x + iy, s) = \sum_{n \in \mathbb{Z}} a_n(y, s) e^{2\pi inx}, \quad (1.1)$$

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\[ a_0(y, s) = y^s + \frac{\xi(2s - 1)}{\xi(2s)} y^{1-s}, \]
\[ a_n(y, s) = \frac{2|n|^{s-1/2}\sigma_{1-2s}(|n|)\sqrt{y}K_{s-1/2}(2\pi|n|y)}{\xi(2s)}, \quad n \neq 0, \]

where \( \xi(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s) = \xi(1 - s) \) is the completed Riemann \( \zeta \) function, \( \sigma_s(n) = \sum_{d|n} d^s \), and \( K_s(y) \) is the K-Bessel function. One can deduce from the functional equation \( \xi(s) = \xi(1 - s) \) that

\[ E(z, s) = \frac{\xi(2s - 1)}{\xi(2s)} E(z, 1 - s). \quad (1.2) \]

The main point of using spectral theory here is Maass’ Lemma, which ultimately implies that because \( E(z, s) \) and \( E(z, 1 - s) \) both have Laplace eigenvalue \( s(1 - s) \), the two must be multiples of each other. The ratio can be found from the constant term \( a_0(y, s) \), and so the functional equation (1.2) can be proven without knowing the functional equation \( \xi(s) = \xi(1 - s) \).

Many results from functional analysis have been used to generalize the analytic continuation and functional equation of Eisenstein series, starting with Selberg and continued by Langlands. Their spectral methods in general do not assume any information about the L-functions (generalizing \( \zeta(s) \)) that are involved. In his Yale Monograph Euler Products, Langlands obtained the meromorphic continuation of a wide variety of L-functions using the theory of the constant term. His calculations led to the definition of the L-group, and ultimately to the formulation of his functoriality conjectures.

Furthermore, the functional equation (1.2) can be turned around to gain information about the symmetries of the terms in the Fourier expansion (1.1). In particular, an analysis of the first Fourier coefficient readily gives the functional equation \( \xi(s) = \xi(1 - s) \). This latter observation has been developed by Shahidi; in conjunction with Langlands’ constant term theory, he has applied an analysis to the non-constant terms of general Eisenstein series to obtain the functional equations and meromorphic continuation (with only a finite number of poles between 1/2 and 1) of the L-functions which occur in the Fourier expansions of Eisenstein series.

In general it has been a difficult challenge to prove the L-functions are entire. Kim has applied the following idea from representation theory to rule out the poles on the real axis between 1/2 and 1 in many cases. These potential poles are also singularities of Eisenstein series, and their residues are
$L^2$-automorphic forms, which always correspond to unitary representations that can be explicitly described by the Eisenstein series they came from. Kim remarked that known results about the unitary dual show that many of these representations do not occur, allowing one to conclude the holomorphy of the Eisenstein series and the L-functions at these points! When combined with [G-S], [Sh2], and [Sh3], this has recently led to new examples of entire L-functions with breakthrough applications to the Langlands functorality conjectures ([CKPSS],[K-S]).

2 An Outline of the Method

The following is a brief sketch of the main points of the method; a fuller introduction with more definitions and detailed examples can be found in [Sh1]. Detailed examples of constant term calculations can be found in many places, e.g. [L2],[L3],[G-S], and [M]. Though it is possible to describe the method without adeles (as was done in the introduction), their use is key in higher rank for factoring infinite product expansions into L-functions.

Let $F$ be a field, $\mathbb{A} = \mathbb{A}_F$ its ring of adeles, and $G$ a split group over $F$. Much carries over to quasi-split case as well, and we will highlight the technical changes needed for this at the end. Fix a Borel (= a maximal connected solvable) subgroup $B \subset G$, and a standard maximal parabolic $P \supset B$ defined over $F$.\[\] Decompose $B = TU$, where $T$ is a maximal torus. The parabolic can be also decomposed as $P = MN$, where the unipotent radical $N \subset U$, and $M$ is the unique Levi component containing $T$. Denote by $^L G$, $^L M$, $^L N$, etc. the Langlands dual L-groups (see [Sh1]).

One of the key aspects of this method is that it uses many possibilities of parabolics of different groups, especially exceptional groups. This is simultaneously a strength (in that there is a wide range of possibilities) and a limitation (in that there are only finitely many exceptional groups).

2.1 Cuspidal Eisenstein Series

Recall that an automorphic form in $L^2(\Gamma \backslash G)$ is associated to a (unitary) automorphic representation of $G$ mapping $\phi(h) \mapsto \phi(hg)$. Let $\pi = \otimes_v \pi_v$ be a cuspidal automorphic representation of $M(\mathbb{A})$; we may assume that almost\[\]

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all components $\pi_v$ are spherical unitary representations (meaning that they have a vector fixed by $G(O_v)$, where $O_v$ is the ring of integers of the local field $F_v$). For these places $v$ the equivalence class of the unitary representation $\pi_v$ is determined by a semisimple conjugacy class $t_v \in L^G$, the L-group. This conjugacy class is used to define the L-functions below in (2.3).

A maximal parabolic has a modulus character $\delta_P$, which is the ratio of the Haar measures on $M \cdot N$ and $N \cdot M$. It is related to the simple root of $G$ which does not identically vanish on $P$. For any automorphic form $\phi$ in the representation space of $\pi$, we can define the Eisenstein series

$$E(\pi, s, g) = \sum_{\gamma \in P(F)\backslash G(F)} \phi(\gamma g)\delta_P(\gamma g)^s,$$  \hspace{1cm} (2.1)

and their constant terms

$$c(\pi, s, g) = \int_{N'(F)' \backslash N'(A)} E(\phi, s, ng)dn,$$  \hspace{1cm} (2.2)

where $N'$ is the opposite parabolic to $N$ (it is related by the longest element in the Weyl group).

**2.2 Langlands L-functions**

If $\rho$ is a finite-dimensional complex representation of $L^G$ and $S$ is a finite set including the archimedean and ramified places, then the partial Langlands L-function is

$$L_S(s, \pi, \rho) = \prod_{v \notin S} \det(I - \rho(t_v)q_v^{-s})^{-1}.$$  \hspace{1cm} (2.3)

Here $q_v$ is the cardinality of the residue field of $F_v$, a prime power. The full, completed, L-function involves extra factors for places in $S$, whose definition is technical and in general difficult.

**2.3 The Constant Term Formula**

The constant term formula involves the sum of two terms. The first, which only occurs when the parabolic is its own opposite, is essentially just $\phi(g)$.
Langlands showed that the map to the second term is given by an operator

\[
M(s, \pi) = \left( \prod_{j=1}^{m} \frac{L(a_j s, \tilde{\pi}, r_j)}{L(1 + a_j s, \tilde{\pi}, r_j)} \right) \otimes_{v \in S} A(s, \pi_v),
\]

where \(A(s, \pi_v)\) are a finite collection of operators, \(r\) the adjoint action of \(L^M\) on the lie algebra of \(L^N\), \(r_1, \ldots, r_m\) the irreducible representations it decomposes into, and \(a_j\) integers which are multiples of each other (coming from roots related to the \(r_j\)). See [Sh1] for a fuller discussion along with an example for the Lie group \(G_2\) and the symmetric cube \(L\)-function. Tables listing Lie groups and the representations \(r_j\) occurring for them can be found in [L2] and [Sh3].

### 2.4 The Non-Constant Term: Local Coefficients

We must now make a further restriction on the choice of \(\pi\) involved, namely that it be generic, i.e. have a Whittaker model. This means that if \(\psi\) is a generic unitary character of \(U(F)\backslash U(A)\), we need to require

\[
W_v(g, \psi) = \int_{U_M(F)\backslash U_M(A)} \phi(n g) \overline{\psi(n) dn} \neq 0, \quad U_M = U \cap M
\]

for some \(\phi\) and \(g\).

Shahidi’s formula uses the Casselman-Shalika formula for Whittaker functions to express the non-constant term as

\[
\int_{N'(F)\backslash N'(A)} E(n g) \overline{\psi(n) dn} = \prod_{v \in S} W_v(1) \prod_{j=1}^{m} \frac{1}{L(1 + a_j s, \tilde{\pi}, r_j)}. \tag{2.5}
\]

Applying the functional equation of the Eisenstein series (which has the constant-term ratio involved), one gets the “crude” functional equation for the product of \(m\) \(L\)-functions

\[
\prod_{j=1}^{m} L_S(a_j s, \tilde{\pi}, r_j) = \prod_{j=1}^{m} L_S(1 - a_j s, \pi, r_j) \prod_{v \in S} (\text{local factors}). \tag{2.6}
\]
Shahidi’s papers [Sh2] and [Sh3] match all the local factors above to the desired L-functions (as in the remark after (2.3)). This gives the full functional equation for these $m$ L-functions, but only when multiplied together. His 1990 paper [Sh3] uses an induction to isolate each of the $m$ factors above separately.

2.5 Analytic Properties and the Quasi-Split Case

It remains to prove that the L-functions are entire, except perhaps at $s = 0$ and 1 where the order of the poles is understood. The theory of Eisenstein series provides this full analyticity except for when $\pi$ satisfies a self-duality condition; even in this case, it can be shown that the L-functions have only a finite number of poles, all lying on the real axis between $\frac{1}{2}$ and 1. Kim’s observation of using the unitary dual has worked in many cases. It is also always possible to remove the potential poles by twisting by a highly-ramified $GL(1)$ character of $A_F$; this has been crucial for applications to functorality by the converse theorem [CPS].

The main difference in the quasi-split case is that the action of the Galois group $G_F$ is no longer trivial. The L-groups are potentially disconnected as a semi-direct product of a connected component and $G_F$. Also, the representation $\rho$ used to define the Langlands L-functions may also depend on the place $v$.

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