AN EXPLICIT EXPRESSION OF THE LÜROTH INARIANT

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Abstract. In this short note, we give an algorithm to get an explicit expression of the Lüroth invariant in terms of the Dixmier-Ohno invariants. We also get the explicit factorized expression on the locus of Ciani quartics in terms of the coefficients. Finally, we answer two open questions on sub-loci of singular Lüroth quartics.

1. Introduction

A Lüroth quartic is a plane quartic \( C \) containing the ten vertices of a complete pentalateral, which is to say that there exist 5 lines \( \ell_1, \ldots, \ell_5 \) such that \( \bigcup_{i \neq j} (\ell_i = 0) \cap (\ell_j = 0) \) consists of 10 points contained in \( C \). Over the complex field, the closure of the locus of Lüroth quartics in \( \mathbb{P}^4 \) is known to be an irreducible hypersurface given by an \( \text{SL}_3(\mathbb{C}) \)-invariant equation, called the Lüroth invariant and denoted \( L \) in the sequel. The classical study of the Lüroth invariant culminated in 1919 with the work of Morley [7], which showed that the degree of \( L \) is 54. More recently, after the seminal work of [1], several authors have revived the subject in [2, 11, 10, 9] (see also [12] on the undulation invariant). However, an explicit expression of \( L \) was still missing. The present note explains how we were able to compute such an expression. Our luck was mainly being acquainted with [8], an unfortunately unpublished article in which Ohno gives a complete set of generators for the invariants of ternary quartics under the action of \( \text{SL}_3(\mathbb{C}) \), completing the set of primary invariants found in [3]. An indirect source is however [4] and the implementation of these invariants in Magma software by Kohel.

In subsequent sections, we also confirm a decomposition of Lüroth invariant on Ciani quartics btained in [5, Sec.5] and answer two questions raised in [11] on the existence of invariants for sub-loci of singular Lüroth quartics.

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2. An algorithm to find an expression of the Lüroth invariant

Let $I_3, I_6, I_9, I_{12}, I_{15}, I_{18}, I_{21}, I_{27}$ be the primary invariants for ternary quartics under the action of $\text{SL}_3(\mathbb{C})$ found by Dixmier in [3]. Let $J_9, J_{12}, J_{15}, J_{18}, J_{21}$ be the secondary invariants found by Ohno [8]. In both cases, the index specifies the degree of the invariant. Together these invariants form a complete set of generators for the ring of invariants of ternary quartics under the action of $\text{SL}_3(\mathbb{C})$.

The method used to work out the expression of $L$ is straightforward:

1. We take the 1380 monomials $I = \{I_3^3, I_6^3, \ldots, I_{27}^3\}$ of degree 54 that generate the $\mathbb{C}$-vector space of invariants of degree 54.
2. We generate a sufficiently large finite set $Q$ of cardinality $q$ of random plane quartics with rational coefficients.
3. We generate a sufficiently large finite set $L$ of cardinality $l$ of random Lüroth quartics of the form
   \[ \ell_1 \ell_2 \ell_3 \ell_4 + c_1 \cdot \ell_2 \ell_3 \ell_4 \ell_5 + c_2 \cdot \ell_1 \ell_3 \ell_4 \ell_5 + c_3 \cdot \ell_1 \ell_2 \ell_4 \ell_5 + c_4 \cdot \ell_1 \ell_2 \ell_3 \ell_5 \]
   where $\ell_1 = x, \ell_2 = y, \ell_3 = z, \ell_4 = x + y + z, \ell_5$ is a line with random rational coefficients and $c_i$ are rational coefficients.
4. We compute the matrix $M_1 = (\langle I(q) \rangle_{I \in I, q \in Q})$, evaluating the monomials in $I$ at the generic quartics in $Q$.
5. We compute the matrix $M_2 = (\langle I(q) \rangle_{I \in I, q \in L})$, evaluating the monomials in $I$ at the generic Lüroth quartics in $L$.
6. We compute the 215-dimensional kernel $N_1$ of $M_1$. This gives a basis of the homogeneous relations of degree 54 that are satisfied by the invariants of all ternary quartics. Note that $1380 - 215 = 1165$ is indeed the dimension of the vector space of invariants of degree 54, as can be computed from the Poincaré series given in [13, p.1045].
7. We compute the 216-dimensional kernel $N_2$ of $M_2$. This gives a basis of the homogeneous relations of degree 54 that are satisfied by all Lüroth quartics.
8. A non-zero element in the complement of $N_2$ in $N_1$ is an expression for $L$ in terms of the Dixmier-Ohno invariants.

All these computations were done with Magma software. Over finite fields $\mathbb{F}_p$ with a prime $p = 2017, 10007, 100003$ or even $1000003$, computations can be done in less than a minute. However, getting the result over the rationals is more challenging. The main concern is to deal with matrices $M_1$ and $M_2$ whose coefficients are as small as possible. So, at Step (2) of the algorithm, we generate plane quartics with random integer coefficients only equal to $-1, 0, 1$. Similarly, we restrict Step (3) to Lüroth forms defined by integer coefficients $c_i$ bounded in absolute value by 4. We can estimate the feasibility of the final computation through the Hadamard bounds for our matrices $M_1$ and $M_2$: the quartics under consideration yield bounds slightly smaller than $2^{2000000}$ for $M_1$ and $2^{3500000}$ for $M_2$. As a sanity check before running the code over the rationals, we verify that these restrictions still allow us to obtain a valid result modulo small primes. Most of the time is spent at Step (6) and Step (7) of the algorithm, precisely 5 and 9 hours on our laptop (based on an Intel Core i7 M620 2.67GHz processor).

The program to get the result is available at [http://iml.univ-mrs.fr/~ritzenth/programme/luroth/luroth.m](http://iml.univ-mrs.fr/~ritzenth/programme/luroth/luroth.m)
It uses the implementation of the Dixmier-Ohno invariants in Magma by Kohel available at
\url{http://echidna.maths.usyd.edu.au/kohel/alg/index.html}

The 1.4Mb result is available at
\url{http://iml.univ-mrs.fr/~ritzenth/programme/luroth/LurothInvF.m}

It is given by 1164 monomials with rational coefficients, the largest of which is a quotient of a 680-digit integer by a coprime 671-digit integer. Modulo 1000003, it starts as

\[
I_3^{18} + 469313I_5^2I_6^5 + 710780I_6^9 + 969230I_3^3I_6^4I_9 + 374233I_5I_6^7I_9 + 276144I_2I_6^9I_9^2 + 602674I_6^2I_9^2 + 527614I_3^3I_6^3I_9^3 + 538637I_3I_6^4I_9^4 + 392526I_3I_6I_9^4 + 645841I_2^3I_6^2I_9^4 + 91422I_3^4I_9^4 + 207808I_3^3I_9^5 + 31577I_3I_6I_9^5 + 635768I_9^6 + 668878I_3^5I_9^6 + 507293I_3^3I_6^7I_9 + 318476I_3I_6^7I_9^2 + 59775I_3^2I_6^8I_9^3 + 581086I_6^8I_9^4I_9 + 830307I_3^3I_6^3I_9^2J_9 + 804817I_3I_6^2I_9^4I_9 + 6418I_3^3I_6^3I_9^3I_9 + 578316I_3I_6I_9^3I_9^3I_9 + 538637I_6^2I_9^4I_9 + 452974I_3I_6^3I_9^3I_9 + 36214I_3^2I_6^3I_9^3I_9 + 522408I_3I_6^2I_9^4I_9 + 253043I_5I_9^2I_9^4 + 469299I_3^2I_6^2I_9^2 + \ldots
\]

3. Lüroth invariants for Ciani quartics

We call a Ciani quartic a plane quartic of the form

\[ax^4 + bx^2y^2 + cxy^2z + dy^4 + ey^2z^2 + fz^4.\]

In [5, Sec.5], using different techniques, Hauenstein and Sottile obtained the factorization on Ciani quartics of the Lüroth invariant as

\[G^4HJ^2\]

with \(G, H, J \in \mathbb{C}[a, b, c, d, e, f]\) homogeneous of respective degree 6, 9 and 12. Using our expression, it is easy to confirm their decomposition

\[G = a \cdot d \cdot f \cdot (adf - 1/4ae^2 - 1/4b^2f - 1/4bcf - 1/4c^2d),\]

\[H = (adf - 1/4ae^2 \cdot (adf - 1/4ae^2 - 3/4b^2f + 1/4bcf - 1/4c^2d) \cdot \left((adf + 3/4ae^2 - 1/4b^2f + 1/4bcf - 1/4c^2d) \cdot (adf + 3/4ae^2 + 3/4b^2f + 1/4bcf - 1/4c^2d)\right),\]

\[J = a^4d^4f^4 - 1/49a^4d^2e^2f^3 + 51/19208a^4d^2e^2f^2 - 1/38416a^3d^6f + 1/614656a^4e^8 - 1/49a^3d^2e^3f^4 - 205/960a^3d^2e^2f^3 - 3/38416a^3d^2e^4f^2 + 1/153664a^3d^2e^6f + \ldots\]

The product \(G^4H^2J\) has 1695 monomials. Note that the total amount of weighted monomials in \(a, b, c, d, e\) and \(f\) in a degree 54 invariant is 3439.

4. Singular Lüroth quartics

Let \(V\) be a three-dimensional vector space over \(\mathbb{C}\), and let \(P = \mathbb{P}(S^4V^*)\) be the space of quartic forms on \(V\). The vanishing locus of the Lüroth invariant \(L\) defined above determines a hypersurface \(L\) in the 14-dimensional projective space \(P\) that is independent of the choice of basis of \(V\). We are now interested in studying the locus \(L\) of singular Lüroth quartics, or in other words the intersection \(L \cap D\), where \(D\) is the hypersurface in \(P\) defined by the vanishing of the discriminant invariant \(I_{27}\). Work by Le Potier and Tikhomirov ([6]) shows that

\[L \cap D = L_1 \cup L_2,\]
where $\mathcal{L}_1$ and $\mathcal{L}_2$ are irreducible subschemes of $P$ of codimension 2 whose respective degrees as subschemes of $\mathcal{L}$ equal 24 and 30. Moreover, while $\mathcal{L}_1$ is reduced, the reduced subscheme $(\mathcal{L}_2)_{\text{red}}$ of $\mathcal{L}_2$ is of degree 15.

In [11], Ottaviani and Sernesi showed that no new degree 15 invariant vanishes on $(\mathcal{L}_2)_{\text{red}}$, which implies that this scheme is not a principal hypersurface in $\mathcal{L}$. By generating singular quartics in $\mathcal{L}_1$ and $\mathcal{L}_2$ and proceeding as above, we were able to verify that neither is there any new (apart from $I_3 \cdot I_{27}$) degree 30 invariant vanishing on $\mathcal{L}_2$, which therefore is not a principal hypersurface in $\mathcal{L}$ either. A similar conclusion holds of $\mathcal{L}_1$ as there is no degree 24 invariant vanishing on $\mathcal{L}_1$. The Magma program to check this is available at

http://iml.univ-mrs.fr/~ritzenth/programme/luroth/SingularLurothInv.m

A word is needed on how to generate singular quartics corresponding to points of $\mathcal{L}_1$ and $\mathcal{L}_2$. For $\mathcal{L}_2$, we can proceed by using Remark 3.3 in [11]: we now choose the lines in Step (3) above such that three of them have a common point of intersection.

For $\mathcal{L}_1$, things are slightly more involved, but basically it is a compilation of other constructions and results in [11]. The program is available at

http://iml.univ-mrs.fr/~ritzenth/programme/luroth/GenerateL1.m

We proceed as follows.

1. We start with 6 rational points $p_1, \ldots, p_6$ in the plane that are sufficiently general in the sense that the complete linear system $C$ of cubics passing through them has dimension 4. Choosing a basis $c_1, \ldots, c_4$ of $C$, we construct the Clebsch rational map $c = (c_1, c_2, c_3, c_4) : \mathbb{P}^2 \to \mathbb{P}^3$ and determine its image $S$, which is the vanishing locus of a quaternary cubic form $F$. Note that $S$ will have 27 rational lines. The rational map $c$ restricts to a birational map between $\mathbb{P}^2$ and $S$, and we can use it to construct two skew lines $l, m$ on $S$, for example by taking $l$ to be the image of the line between $p_1$ and $p_2$ and taking $m$ to be the image of the line between $p_1$ and $p_3$.

2. Next we determine one of the involutory points $\mathcal{Q}$ associated with the pair $l, m$ of lines on $S$ by the construction on page 1759 of [11].

3. By Proposition 3.1(i) of [11], we obtain a quartic in $\mathcal{L}_1$ by constructing the ramification locus of the projection $S \to \mathbb{P}(T_\mathcal{Q}S)$ from the point $Q \in S$.

All of these steps can be implemented using very little beyond linear algebra:

1. This part is straightforward. The final equation $F$ can be determined by constructing the degree 3 homogeneous expressions in the $c_i$ and applying some linear algebra.

2. We choose coordinates on $l$ by taking two points $l_1, l_2 \in l$ and sending $(x : y) \in \mathbb{P}^2$ to $xl_1 + yl_2$, and similarly on $m$ by choosing $m_1, m_2 \in m$. To determine the morphism $f : A \to B$ explicitly in these coordinates, we choose two equations $M_1 = M_2 = 0$ defining $m$. Given $(x : y) \in \mathbb{P}^2$, the point $f(p) = T_pS \cap B$ corresponds to the vector space that is the kernel of the matrix whose rows are given by $M_1, M_2$ and the partial derivatives of $F$. A generating vector for this space will be combination of $m_1$ and $m_2$ with homogeneous quadratic coefficients $f_1(x, y), f_2(x, y)$ in $(x, y)$. The morphism $f$ now corresponds to the map $\mathbb{P}^1 \to \mathbb{P}^1$ given by $(f_1, f_2)$. Similarly, one determines $g$.

To construct the $f'$ in [11] over the ground field, first compute the branch divisor $D$ of $g$ by deriving and pushing forward. If $D$ consists of two rational points $(x_1 : y_1)$...
and $(x_2 : y_2)$, then one can take
\[ f'(x : y) = ((y_1 x - x_1 y)^2 : (y_2 x - x_2 y)^2). \]
The case where the points of $D$ are defined over a proper quadratic extension is most easily illustrated by using the affine coordinate $t = x/y$. Suppose that using this coordinate, $D = [d] + [\overline{d}]$ consists of two conjugate points, and suppose moreover that these points do not sum to 0. The $f' = ((t - d)/(t - \overline{d}))^2$ of above now has the property that $\overline{f'} = 1/f'$. But then one verifies that \((df' + \overline{d})/(\overline{df'} + d)\) is a fractional linear transformation of $f'$ that is stable under conjugation and hence defines a morphism over the ground field. If $d$ and $\overline{d}$ sum to 0, then one can use $((1 + d)f' + (1 + \overline{d}))/((1 + d)f' + (1 + \overline{d}))$ instead. More invariantly, if $D$ is defined by an equation of the form $rx^2 + sxy + ty^2 = 0$, then
- if $s = 0$, then one can take
  \[ f'(x : y) = (rx^2 - 2txy - ty^2 : rx^2 + 2txy - ty^2); \]
- if $s \neq 0$, then one can take
  \[ f'(x : y) = (r^2sx^2 + 2r(s^2 - 2rt)xy + (s^3 - 3rst) : r(\lambda y^2 + 4rtxy + sty^2)). \]

Finally, we have to show how, given the morphisms $f, f' : \mathbb{P}^1 \to \mathbb{P}^1$, we can compute a $q \in \mathbb{P}^1$ such that the fiber of $f$ over $q$ is also a fiber of $f'$ over a point $q'$. For this, write
\[ f(x : y) = (a_1x^2 + b_1xy + c_1y^2 : a_2x^2 + b_2xy + c_2y^2) \]
and
\[ f'(x : y) = (a'_1x^2 + b'_1xy + c'_1y^2 : a'_2x^2 + b'_2xy + c'_2y^2). \]
If we let $q = (\lambda_1, \lambda_2)$ and $q' = (\lambda'_1, \lambda'_2)$, then this question reduces to finding solutions of the equation
\[
\lambda_2(a_1x^2 + b_1xy + c_1y^2) - \lambda_1(a_2x^2 + b_2xy + c_2y^2) = \\
\lambda'_2(a'_1x^2 + b'_1xy + c'_1y^2) - \lambda'_1(a'_2x^2 + b'_2xy + c'_2y^2),
\]
which corresponds to the determination of the kernel of the matrix
\[
\begin{pmatrix}
  a_1 & -a_2 & -a'_1 & a'_2 \\
  b_1 & -b_2 & -b'_1 & b'_2 \\
  c_1 & -c_2 & -c'_1 & c'_2
\end{pmatrix}.
\]

(3) We suppose $Q = (1 : 0 : 0 : 0)$ for simplicity. We get an induced coordinatization of $T_QS$ by sending $(x : y : z)$ to the tangent direction given by the line through $(1 : 0 : 0 : 0)$ and $(1 : x : y : z)$. Then a point $(x : y : z)$ is a ramification point of the projection $S \to \mathbb{P}(T_QS)$ if and only if the equation $F(1, xt, yt, zt) = 0$ has a double root outside 0, or in other words if the discriminant of the quadratic polynomial $F(1, xt, yt, zt)/t$ vanishes. This discriminant is a homogeneous quartic form in the variables $x, y, z$, which defines the quartic in $L_1$ that we were looking for.

Having generated a sufficiently large database\(^1\) of curves in $L_1$ by choosing random 6-tuples \(\{p_1, \ldots, p_6\}\), we can again proceed as in Section 2, which quickly yields our results.

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\(^1\)10,000 curves over $\mathbb{Q}$ are available at [http://iml.univ-mrs.fr/~ritzenth/programme/luroth/L1Database.m](http://iml.univ-mrs.fr/~ritzenth/programme/luroth/L1Database.m)
Remark 4.1. We also tried to generate quartics in $L_1$ by using Remark 3.4 of [11]. We take cubics $S$ of the form

$$t^2x + t(ax^2 + 2bxy + 2cxz + dy^2 + 2eyz + fz^2) + g(x, y, z)$$

with $g$ a random degree 3 homogeneous polynomial, such that $S$ is non singular and $e^2 = df$. The last condition ensures that $p = (0 : 0 : 0 : 1)$ belongs to the Hessian $H$ of $S$ and, after checking that $p$ is non singular, we take quartics which are tangent plane sections of $H$ at $p$. Unfortunately, it seems that these quartics are special in $L_1$ since there are degree 24 relations between their invariants (there is a 27 dimensional space of relations in degree 24 between randomly generated quartics of this form).

5. Open questions

The expression $L$ of Lüroth invariant that we found depends on several arbitrary choices that may explain its cumbersomeness. First, there is the choice of the basis of invariants. Though some of the Dixmier invariants have geometrical interpretations that are ‘natural’, the same is far from evident for the new Ohno invariants. Secondly, our choice can be modified by any element of the kernel $N_1$. Beyond a cancellation of the coefficients of 215 of these monomials that we have already accomplished by simple linear algebra, further minimization of the number of monomials in the expression for $L$ could in theory be achieved by techniques based on coding theory. Still, the parameters seem too large to make this feasible in practice.

The negative answers on the existence of degree 24 and 30 invariants in Sec.4 leave us clueless on a possible decomposition of $L$ as the one hoped in [11, p.1764]. An expression in terms of the 15 coefficients of the generic quartic would of course be canonical. However, it is not even practically achievable to formally express all Dixmier-Ohno invariants, since these contain too many monomials, as is for instance the case for the discriminant. A count of weighted monomials in 15 variables for degree 54 invariants leads to a total of 62,422,531,333. Of course only a fraction of these monomials may occur in the final expression of $L$, but we could not figure out their number, let alone the Newton polytope of $L$.

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