The effect of strong quantising magnetic field on low density stellar matter is investigated using Thomas-Fermi-Dirac (TFD) model. The Wigner-Seitz cell structure is assumed for the low density matter. The significant changes in the properties of such low density matter in presence of strong magnetic fields are discussed. It is seen that the decay time scale for magnetic field decreases by at least two orders of magnitude in such model calculation.
1. INTRODUCTION

The study of low density stellar matter at the crustal region of a neutron star or in the case of white dwarf matter in presence of a strong magnetic field is extremely interesting both from the academic as well as from the Astrophysical point of view. The recent observational data of magnetars indicate the possibility of very strong surface magnetic field (up to $10^{15} G$) in some pulsars. The observed soft gamma repeaters (SGR) discovered in BATSE and KONUS experiments and X-ray source observed by ASCA and RXTE show strong surface magnetic field up to $10^{15} G$. These objects are called magnetars. They pose a great challenge to the existing models of magnetic field evolution since they require a very rapid field evolution in isolated neutron stars.

Now the neutron star crust plays most important role in the evolution of neutron star magnetic field. The neutron star crustal matter (at relatively low density) exhibits conventional lattice structure, which can be approximated by a regular arrangement of Wigner-Seitz cells, with a positively charged nucleus at the centre surrounded by spherical distribution of electron cloud. In the case of white dwarfs such approximation is also valid at low density region. In the non magnetic case, this kind of crystalline structure have already been studied in Solid state physics / Atomic Physics. Equation of state for low density stellar matter is also obtained using TFD model in the non magnetic case. Since the shell effect of orbital electrons significantly suppresses the statistical effect, it is not recommended to obtain equation of state for very low density solids in the laboratory using TFD model (e.g. metallic iron in the laboratory). On the other hand, in the case of stellar matter, since the density is high enough, such an issue does not arise. It was also argued that application of TFD model is valid for density $\leq 10^4 \text{ gm/cc}$. For higher densities, since electrons no longer remain bound within the cells, one uses Chandrasekhar’s ideal gas results for electrons with Coulomb lattice correction.

If magnetic field of strength $\sim 10^{14}-10^{15} G$ exists at the outer crust of a neutron star or inside a magnetised white dwarf(!) with Wigner-Seitz cell structure of the matter, the charge distribution of the electron cloud within such cells must be affected significantly. As a consequence, the size of Wigner-Seitz cells will change. This change in cell volume should affect the equation of state of low density stellar matter and reduces the width of outer crust of neutron stars. If such modified equation of state is considered for the neutron star crust, it will significantly affect the gross properties including the mass-radius relation of the star. To the best of our knowledge, such an important issue has not been discussed before.

In this article, we assume that the matter consists of fully ionised iron nuclei and are at rest at the centre of Wigner-Seitz cell. Since the matter is at relatively low density, the electrons surrounding the positive ions are assumed to be non-relativistic. It is further assumed that in the microscopic scale the magnetic field is constant and is along z-axis. As we have noticed that the qualitative nature of the results do not change if we consider carbon or oxygen instead of iron as the constituent of the matter.

The paper is organised in the following manner. In the next section we shall develop the basic formalism for Thomas-Fermi-Dirac model in presence of a quantising magnetic field. We have concluded our results and discussed the future perspective of this work at the last section. We have presented a brief outline of the derivation of electron-electron exchange interaction (Fock) term in presence of a strong magnetic field in the Appendix A.

2. TFD MODEL IN PRESENCE OF A STRONG MAGNETIC FIELD

In presence of a quantising magnetic field of strength $B$, the electron number density is given by

$$n_e = \frac{eB}{\pi^2 p_F} \tag{1}$$

where $p_F$ is the electron Fermi momentum, and $e$ is the magnitude of electronic charge. In the Thomas-Fermi model for statistical treatment of atomic structure, it is assumed that within the
Wigner-Seitz cell, the electrons move in a slowly varying spherically symmetric potential $V(r)$. Then the Fermi energy $\mu$ of an electron is given by,

$$\mu = -eV(r) + \frac{p_F^2}{2m}$$

where $m$ is the electron mass. The Fermi energy $\mu$ is independent of $r$, otherwise electron would migrate to a region of smaller $\mu$. In TFD model the electron Fermi energy is given by,

$$\mu = \frac{p_F^2}{2m} - e\phi - u_{ex}(p_F) = \text{constant}$$

where $u_{ex}$ is the exchange part of electron-electron interaction.

In the non magnetic case, $u_{ex}(p_F) = \frac{e^2}{\pi \hbar} p_F$ (4)

whereas in the case of a quantising magnetic field, if all the electrons are assumed to be at the lowest Landau level, the exchange energy is given by, (see Appendix A)

$$u_{ex}(p_F) = \alpha (1 - e^{-\beta p_F})$$

where the parameters $\alpha$ and $\beta$ are functions of magnetic field strength and are given in Table I. Rearranging eqn.(3) in the form (see also [11]),

$$\frac{p_F^2}{2m} + \alpha e^{-\beta p_F} = \mu^* + e\phi$$

where $\mu^* = \mu + \alpha$ is the modified form of Fermi energy of the electron. From eqn.(6) one can express Fermi momentum $p_F$ as a function of $\mu^* + e\phi$. The numerically fitted functional form is given by a simple power law,

$$p_F = C(\mu^* + e\phi)^\gamma$$

where, $C$ and $\gamma$ are constant parameters for a given magnetic field strength. In Table I, we have shown the variation of $C$ and $\gamma$ with the magnetic field strength $B$. The potential $\phi$ (which is the direct interaction term between electron-nucleus and electron-electron) is given by the Poisson’s equation

$$\nabla^2 \phi = 4\pi en_e + \text{nuclear contribution}$$

Since the nuclear contribution is a delta function about the origin, we can omit it for $r > 0$ and impose the boundary condition,

$$\lim_{r \to 0} r \phi(r) = Ze$$

Where $Z$ is the atomic number ($= 26$ for iron). The boundary condition at the cell wall of radius $r_s$ is that the electric field vanishes (neutral cell condition), which gives in spherical polar coordinate,

$$\frac{d\phi}{dr}|_{r=r_s} = 0$$

Now, using the empirically fitted form of $p_F$ given by eqn.(7), and writing the radial coordinate $r$ in the scaling from $r = ax$, we have the Poisson’s equation (from eqn. (7))

$$\frac{d^2 u}{dx^2} = x^{1-\gamma} u^\gamma$$

(11)
Where

$$\mu^* + e\phi = Z e^2 r u(r)$$

(12)

and

$$a^{3-\gamma} = \frac{\pi \hbar^2 c}{4CBe^{2\gamma+1}Z^{\gamma+1}}$$

(13)

The boundary condition at the cell boundary (eqn.(10)) gives,

$$\frac{du}{dx} = \frac{u}{x}$$

(14)

for \(x = x_s\). Since \(\gamma < 1\) for the whole range of magnetic field strength \((10^{14} G \leq B \leq 10^{17} G)\) of Astrophysical interest, unlike non-magnetic case, the Poisson’s equation (eqn.(11)) does not have singularity at the origin. Therefore, the numerical method prescribed by Feynman, Metropolis and Teller [10] is not necessary in the quantising magnetic field case. This qualitative change in the form of Poisson’s equation comes from the modified form of phase space integral of electron number density in presence of strong magnetic fields. The non-quantising magnetic field therefore can not make any qualitative change in the form of differential eqn.(11) or in other wards in the electron distribution within the cell. Standard fourth-order Runge-Kutta method has been used to obtain numerical solution of eqn.(11). In this case the initial value for the derivative

$$\frac{du}{dx} = v_0$$

(15)

are chosen by shooting method to match the boundary condition at the cell surface for three different magnetic field strengths, \(B = 10^{14}G, 10^{15}G, \) and \(10^{17}G\). The values for \(x_s\), the surface scaling parameter are given in table I for the above three magnetic field strengths. As we have noticed, the cell radius \(r_s = ax_s\) decreases with the increase of magnetic field strength and are about an order of magnitude smaller than the non magnetic value [8]. This squeezing of Wigner-Seitz cell in presence of strong quantising magnetic field is analogous to the well known magnetostriction phenomenon observed in classical magneto-statics. The variation of \(u(x)\) with \(x\) for a given magnetic field strength is given by the numerically fitted functional form

$$u(x) = \frac{u_0}{1 + \exp\{\xi (x - x_0)\}}$$

(16)

where, \(u_0, \xi, x_0\) are constant parameters for a given magnetic field strength. The variation of these parameters with magnetic field strength are shown in Table I. In presence of strong quantising magnetic field, the variation of \(u\) with \(x\) is entirely different from the non magnetic case. The variation is more or less like the radial distribution of matter in neutron stars. Now the equation of state of such cold degenerate low density matter is that due to nucleons and electrons present in the system. The pressure contribution mainly comes from the electrons. The nuclei are at rest at the centre of each Wigner-Seitz cell, therefore we can ignore their contribution to kinetic pressure of the system. On the other hand, the energy density of the system is mainly dominated by rest mass of the ions. The energy contribution from the electronic sector is about five-six orders of magnitude less than the ionic part, therefore one can discard the energy contribution in the equation of state from electronic part. The mass density \(\rho\) is simply given by the rest mass of nucleons inside the cell, then we have

$$\rho = \frac{3Am_B}{4\pi a^3x_s^3}$$

(17)

where \(m_B = 1.66057 \times 10^{-24}g\), the effective nucleon mass. In Table I we have shown the variation of matter density with the magnetic field strength \(B\). Since the radius of Wigner-Seitz cells decrease
with the increase of $B$, there is an increase in mass density with the increase in magnetic field strength. The expression for kinetic pressure from electron sector is given by

$$P = \frac{eB}{\pi^2} \left[ \frac{p_F^3}{3m} + \alpha \exp(\beta p_F) \left( p_F + \frac{1}{\beta} \right) - \frac{\alpha}{\beta} \right]$$

(18)

where the Fermi momentum $p_F$ has already been expressed as a function of dimensionless surface parameter $x_s$ (see eqn.(7)). Eqns.(17) and (18) give the equation of state $P = P(\rho)$ in terms of the surface parameter $x_s$.

In fig.1 we have shown the equation of state of such low density matter in presence of strong magnetic fields for three different cases: the upper curve is for $B = 10^{14}G$, middle one is for $B = 10^{15}G$, and the lower one is for $B = 10^{17}G$. As we can see from the figure that the softness of the matter increases with the increase of magnetic field strength. Which also means that the matter becomes energetically more stable with respect to non-magnetic case. We have further noticed from the upper and the middle curves that at high density since a large number of Landau levels are populated for electrons, which is equivalent to the non-quantising picture of external magnetic field, these two curves coincide at high density. The effect of magnetic field is completely washed out completely at very high density.

3. CONCLUSION

The outer crust of a neutron star, particularly in the case of a strong magnetic field (magnetars?) plays a crucial role in the evolution of pulsar magnetic field. It is really a great challenge to explain field evolution in these strongly magnetised objects using existing models of field evolution. These objects require a very rapid field evolution. Now the TFD model for low density matter in presence of strong magnetic fields shows an over all contraction of the outer crust. Since the Ohmic decay of magnetic field in a conducting material depends on the thickness of the region, a decrease in width of the outer crust by an order of magnitude will cause a rapid decay of magnetic field (at least two orders of magnitude decrease in decay time scale). The equation of state curves (fig.(1)) indicates that electrons within the Wigner-Seitz cells are more strongly bound to the positively charged nuclei in presence of strong quantising magnetic fields than the non-magnetic (or non-quantising) case. Such strong binding of electrons within the cells may decrease the electrical conductivity of the matter. Which will further reduce the time scale for Ohmic decay of magnetic field in the outer crust of these strongly magnetised stellar objects.
APPENDIX A:

Exchange energy is given by (with \( \hbar = c = 1 \))

\[
U_{ex} = \frac{e^2}{2} \sum_{j=1}^{Z} \int d^3r' d^3r'' \frac{1}{|r'' - r'|} \psi^*(r') \psi(r') \psi(r''). \tag{A1}
\]

For the zeroth Landau level, the wave function \( \psi(r) \) for electron is given by

\[
\psi(r) = \frac{1}{\sqrt{L_y L_z}} \left( \frac{eB}{\pi} \right)^{1/4} \exp \left[ -\frac{eB}{2} (x - \frac{p_y}{eB})^2 \right] \exp \left[ i(p_y y + p_z z) \right] \tag{A2}
\]

In this particular case the sum over \( j \) can be replaced by \( L_y L_z dp'_y dp'_z \), where \( L_y \) and \( L_z \) are the linear dimensions of the box along \( y \) and \( z \) directions respectively. The volume element \( d^3r' = dx' dy' dz' \).

Then following Lee [12], we have

\[
\int dy' dz' \frac{1}{|r'' - r'|} \exp[-i(p_y - p'_y)(y - y') - i(p_z - p'_z)(z - z')] = \frac{4\pi}{2K} \exp(-K \mid x - x' \mid) \tag{A3}
\]

where \( K = \sqrt{(p_y - p'_y)^2 + (p_z - p'_z)^2} \).

Similarly \( d^3r = dx dy dz \). The integral \( \int dy dz = L_y L_z \). Then we have

\[
U_{ex} = \frac{1}{2} \left( \frac{eB}{\pi} \right) 4\pi e^2 \int dp'_y dp'_z dx dx' \frac{1}{2K} \exp(-K \mid x - x' \mid) \exp \left[ -\frac{eB}{2} \left( \left( x - \frac{p_y}{eB} \right)^2 + \left( x' - \frac{p'_y}{eB} \right)^2 \right) + \left( p_z - \frac{p'_z}{eB} \right)^2 \right] \tag{A4}
\]

To evaluate the integrals over \( x \) and \( x' \), we change the integration variables to \( X \) and \( Y \), where \( X = x - x' \) and \( Y = (x + x')/2 \).

Now

\[
\int_{-\infty}^{\infty} dX \exp(-K \mid x \mid) \exp\left( -\frac{eB}{2} X^2 \right) = \sqrt{\frac{2\pi}{eB}} \exp\left( \frac{K^2}{2eB} \right) \text{erfc} \left( \frac{K}{\sqrt{2eB}} \right) \tag{A5}
\]

where \( \text{erfc}(x) \) is the complimentary error function.

Then we have

\[
U_{ex} = \frac{e^2 B}{2} \int \frac{1}{K} dp'_y dp'_z dY \sqrt{\frac{2\pi}{eB}} \exp\left( \frac{K^2}{2eB} \right) \text{erfc} \left( \frac{K}{\sqrt{2eB}} \right) \exp \left[ -\frac{eB}{2} \left( 4Y^2 + \frac{2p_y^2}{e^2B^2} + \frac{2p'_y^2}{e^2B^2} - \frac{4p_y Y}{eB} - \frac{4p'_y Y}{eB} \right) \right] \tag{A6}
\]

The \( Y \) integral is given by

\[
\int_{-\infty}^{\infty} dY \exp\left[ -\frac{eB}{2} \left( 2Y - \frac{p_y + p'_y}{eB} \right)^2 \right] = \sqrt{\frac{\pi}{2eB}} \tag{A7}
\]

Then we have after changing the integration variables from \( p'_y \) and \( p'_z \) to \( P_y = p_y - p'_y \) and \( P_z = p_z - p'_z \).
\[ U_{ex} = e^{2\pi} \int dP_y dP_z \frac{1}{\sqrt{P_y^2 + P_z^2}} \exp \left( \frac{P_z^2}{2eB} \right) \text{erfc} \left( \sqrt{\frac{P_y^2 + P_z^2}{2eB}} \right) \]  

(A8)

where the limit of \( P_y \) is from \(-\infty\) to \( \infty \) and \( P_z \) is from 0 to 2\( p_F \) for \( p_z = p_F \).

Again putting \( P_y = P_z \tan \theta \), we have

\[ U_{ex} = e^{2\pi} \int_0^{2p_F} dP_z \int_0^{\pi/2} \sec \theta \ d\theta \ \text{erfc} \left( \frac{P_z}{\sqrt{2eB}} \sec \theta \right) \exp \left( \frac{P_z^2}{2eB} \right) \]  

(A9)

These double integrals have been evaluated numerically as a function of Fermi momentum \( p_F \). The fitted functional form of \( U_{ex} \) is given by

\[ U_{ex} = \alpha [1 - \exp(-\beta p_F)] \]  

(A10)

where the parameters \( \alpha \) and \( \beta \) vary with magnetic field strength \( B \) and are shown in Table I.
Table I

|       | $10^{14}$ | $10^{15}$ | $10^{17}$ |
|-------|-----------|-----------|-----------|
| $B$ (Gauss) |           |           |           |
| $\alpha$ (MeV) | 0.568 | 1.796 | 17.909 |
| $\beta$ MeV$^{-1}$ | 3.412 | 1.067 | 0.109 |
| $\gamma$ | 0.506 | 0.527 | 0.658 |
| $C$ | 0.973 | 0.870 | 0.386 |
| $x_s$ | 3.096 | 3.170 | 4.404 |
| $r_s$ (Å) | 0.402 | 0.203 | 0.123 |
| $v_0$ | $-0.938556$ | $-0.937365$ | $-0.936123$ |
| $u_0$ | 1.633 | 1.651 | 1.944 |
| $\xi$ | 2.097 | 2.071 | 1.755 |
| $x_0$ | 0.213 | 0.204 | 0.031 |
| $\rho$ (gm/cc) | 72.79 | 572.29 | 962.14 |

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