CRITERIA FOR EXISTENCE OF RIESZ BASES
CONSISTING OF ROOT FUNCTIONS OF HILL AND 1D
DIRAC OPERATORS

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Abstract. We study the system of root functions (SRF) of Hill operator $Ly = -y'' + vy$ with a singular potential $v \in H^{-1}_{per}$ and SRF of 1D Dirac operator $Ly = i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \frac{dv}{dx} + vy$ with matrix $L^2$-potential $v = \begin{pmatrix} 0 & P \\ Q & 0 \end{pmatrix}$, subject to periodic or anti-periodic boundary conditions. Series of necessary and sufficient conditions (in terms of Fourier coefficients of the potentials and related spectral gaps and deviations) for SRF to contain a Riesz basis are proven. Equiconvergence theorems are used to explain basis property of SRF in $L^p$-spaces and other rearrangement invariant function spaces.

Keywords: Hill operators, singular potentials, Dirac operators, spectral decompositions, Riesz bases, equiconvergence

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1. Introduction

In the case of ordinary differential operators with strictly regular boundary conditions (bc) on a finite interval the system \{u_k\} of eigen- and

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associated functions could contain only finitely many linearly independent associated functions. The well-defined decompositions

$$\sum_k c_k(f)u_k = f \quad \forall f \in L^2([0,\pi]),$$

do converge; moreover, convergence is unconditional, i.e., \( \{u_k\}, \|u_k\| = 1, \) is a Riesz basis in \( L^2([0,\pi]) \). These facts and phenomena have been well understood in the early 1960’s after the works of N. Dunford \([15, 16]\), V. P. Mikhailov \([41]\) and G. M. Keselman \([27]\).

Maybe the simplest case of regular but not strictly regular \( \bc \) comes if we consider a Hill operator \( L_{bc} \).

$$Ly = -y'' + v(x)y, \quad 0 \leq x \leq \pi,$$

where \( v(x) = v(x+\pi) \) is a complex-valued smooth function, and \( \bc \) is periodic (\( \bc = \text{Per}^+ \)) or anti-periodic (\( \bc = \text{Per}^- \)), i.e.,

(a) periodic \( \text{Per}^+ \): \( y(0) = y(\pi), \ y'(0) = y'(\pi) \);
(b) anti-periodic \( \text{Per}^- \): \( y(0) = -y(\pi), \ y'(0) = -y'(\pi) \);

(Later we will consider non-smooth \( v \) as well, say \( v \in L^2 \) or \( L^1 \), and \( v \in H^{-1/2} \) or \( v \in H_{\text{per}}^{-1} \) – see in particular Section 4.1.)

Recently, i.e., in the 2000’s, many authors \([18, 26, 29, 34, 33, 35, 37, 38, 39, 40, 52]\) focused on the problem of convergence of eigenfunction (or more generally root function) decompositions in the case of regular but not strictly regular \( \bc \).

The free operators \( L_{bc}^0 = d^2/dx^2 \), with \( \bc = \text{Per}^\pm \) have infinitely many double eigenvalues \( \lambda_n^0 = n^2 \), (with \( n \) even for \( \bc = \text{Per}^+ \) and \( n \) odd for if \( \bc = \text{Per}^- \)), the corresponding two-dimensional eigenspaces \( E_n^0 \) are mutually orthogonal and we have the spectral decomposition of the space

$$L^2([0,\pi]) = \oplus E_n^0 \quad \text{or} \quad f = \sum_n P_n^0 f \quad \forall f \in L^2([0,\pi]),$$

where \( P_n^0 \) is the orthogonal projection on \( E_n^0 \). The operator \( L_{bc}(v) = L_{bc}^0 + v \) is a “perturbation” of the free operator; its spectrum is discrete and for large enough \( n \), say \( n > N \), close to \( \lambda_n^0 = n^2 \) there are exactly two eigenvalues \( \lambda_n^- = \lambda_n^+ \) (counted with multiplicity). Moreover, if \( E_n \) is the corresponding two-dimensional invariant subspace and \( P_n = \frac{1}{2\pi i} \int_{C_n} (z - L_{bc})^{-1} dz \) is the corresponding Cauchy projection, then we have the spectral decomposition

$$S_N f + \sum_{k>N} P_k f = f \quad \forall f \in L^2([0,\pi]),$$

where \( S_N \) is the (finite-dimensional) projection on the invariant subspace corresponding to ”small” eigenvalues of \( L_{bc}(v) \), and the series in \((1.3)\) converges unconditionally.

However, even if all eigenvalues \( \lambda_n^-, \lambda_n^+, \) \( n > N \) are simple, there is a question whether we could use the corresponding eigenfunctions to give an expansion like \((1.1)\). The same questions for \( \text{Per}^\pm \) in the case of 1D periodic Dirac operators could be asked. Interesting conditions on potentials \( v \) (or on
its Fourier coefficients), which guarantee basisness of \( \{ u_k \} \), – with or without additional assumptions about the structure or smoothness of a potential \( v \) – have been given by A. Makin [37, 38, 39, 40], A. Shkalikov [52], O. Veliev [59, 60, 61, 1], P. Djakov and B. Mityagin [4, 8, 12, 13, 14].

2. In our papers [24, 3, 4, 9, 14] we analyzed the relationship between smoothness of a potential \( v \) in (1.2) and the rate of “decay” of sequences of (1.4) spectral gaps \( \gamma_n = \lambda^+_n - \lambda^-_n \) and (1.5) deviations \( \delta_n = \mu_n - \frac{1}{2} (\lambda^+_n + \lambda^-_n) \).

This analysis is based on the Lyapunov-Schmidt projection method: by projecting on the \( n \)-th eigenvalue space \( E^0_n \) of the free operator \( L^0 \) the eigenvalue equation \( Ly = \lambda y \) is reduced locally, for \( \lambda = n^2 + z \) with \( |z| < n/2 \) to an eigenvalue equation for a \( 2 \times 2 \) matrix

\[
\begin{bmatrix}
\alpha_n(v, z) & \beta^-_n(v, z) \\
\beta^+_n(v, z) & \alpha_n(v, z)
\end{bmatrix}
\]

The entries of this matrix are functionals (depending analytically on \( v \) and \( z \)), which are given by explicit formulas in terms of the Fourier coefficients of the potential \( v \) (see (2.37) and (2.38) below). They played a crucial role in proving estimates for and inequalities between \( \gamma_n, \delta_n, \beta^+_n \) and (1.6)

\[
t_n(z) := |\beta^-_n(v, z)|/|\beta^+_n(v, z)|
\]

– see [4], Lemma 49 and Proposition 66.

Moreover, it turns out that there is an essential relation between the Riesz basis property of the system of root functions and the ratio functionals \( t_n(v, z) \) which made possible to give criteria for existence of (Riesz) bases consisting of root functions not only for Hill operators but for Dirac operators as well (see, for example, [13, Theorem 1] or [12, Theorem 2] for Hill, or [14, Theorem 12] for Dirac operators). These criteria are quite general and applicable to wide classes of potentials. For example, we proved that if

(1.7)\[ v(x) = 5e^{-4ix} + 2e^{2ix} - 3e^{2ix} + 4e^{4ix}, \]

then neither for \( bc = Per^+ \) nor for \( bc = Per^- \) the root function system of \( L_{bc} \) contains a basis in \( L^2([0, \pi]) \). To apply our criterion we had to overcome a few analytic difficulties. This was done on the basis of our results and techniques from [5].

In this paper we extend and slightly generalize these criteria. We claim, both for Hill operators with singular \( H^{-1}_{per} \)-potentials and Dirac operators with \( L^2 \)-potentials the following.

**Criterion.** The root system of functions of the operator \( L_{Per^\pm}(v) \) has the Riesz basis property (i.e., contains a Riesz basis) if and only

\[
(1.8) \exists C > 0 : \frac{1}{C} \leq t_n(z_n^*) \leq C \quad \text{if} \quad \lambda^-_n \neq \lambda^-_n, \quad n \in \Gamma_{bc}, \quad |n| > N_*.
\]

(See the definition of \( \Gamma_{bc} \) in Section 2, Formulas (2.10) and (2.26).)
3. Recently F. Gesztesy and V. Tkachenko [19, Theorem 1.2] gave – in the case of Hill operators with $L^2$ potentials - a criterion of basisness in the following form:

The system of root vectors for $bc = Per^+$ or $bc = Per^-$, contains a Riesz basis if and only if

$$R_{bc} = \sup \left\{ \frac{|\mu_n - \lambda_n^+|}{|\lambda_n^+ - \lambda_n^-|} : n \in \Gamma_{bc}, \lambda_n^+ \neq \lambda_n^- \right\} < \infty.$$ (1.9)

One can prove, by using the estimates of $|\lambda_n^+ - \lambda_n^-|$ and $|\mu_n - \lambda_n^+|$ in terms of $|\beta_n^-(v,z)|$ and $|\beta_n^+(v,z)|$ (see [4, Theorem 66, Lemma 49] and [9, Theorem 37, Lemma 21]) that the conditions (1.8) and (1.9) are equivalent.

However, we directly show (see Theorem 24 in Section 7), using the fundamental inequalities proven in [24, 3, 4, 9], that (1.9) gives necessary and sufficient conditions of Riesz basisness of root system with $bc = Per^+$ or $bc = Per^-$ both

(A) in the case of 1D periodic Dirac operators with $L^2$ potential, and

(B) in the case of Hill operators with potential in $H^{-1}_{per}$.

4. Criterion for $L^p$-spaces, $1 < p < \infty$, given in [19, Theorem 1.4] can be essentially improved and extended as well. We take any separable rearrangement invariant function space $E$ on $[0, \pi]$ (see [28, 32]) squeezed between $L^a$ and $L^b$, $1 < a \leq b < \infty$. If

$$\frac{1}{a} - \frac{1}{b} < 1/2$$ (1.10)

in the above cases (A) and (B) the root function system contains a basis in $E$ if and only if (1.9) holds. In the case of Hill operators with $v \in H^{-1/2}$ the hypothesis (1.10) could be weakened to

$$\frac{1}{a} - \frac{1}{b} < 1.$$ (1.11)

Of course for $L^p$, $1 < p < \infty$, we can put $a = b = p$, so (1.10) and (1.11) hold.

The structure of this paper and the topics discussed in different sections are shown in Content, see p. 1.

2. LOCALIZATION OF SPECTRA AND RIEZ PROJECTIONS FOR HILL AND DIRAC OPERATORS

For basic facts of Spectral Theory of ordinary differential operators we refer to the books [30, 45, 50]. But let us introduce some notations and remind a few properties of Hill and Dirac operators on a finite interval.

1. We consider the Hill operator

$$Ly = -y'' + v(x)y, \quad x \in I = [0, \pi],$$ (2.1)
with a (complex-valued) potential \( v \in L^2(I) \), or more generally with a singular potential \( v \in H^{-1}_{\text{per}} \) of the form

\[
v = w', \quad w \in L^2_{\text{loc}}(\mathbb{R}), \quad w(x + \pi) = w(x).
\]

For \( v \in L^2 \), we consider the following boundary conditions:

(a) periodic \( \text{Per}^+ \): \( y(0) = y(\pi), \quad y'(0) = y'(\pi) \);

(b) anti-periodic \( \text{Per}^- \): \( y(0) = -y(\pi), \quad y'(0) = -y'(\pi) \);

(c) Dirichlet \( \text{Dir} \): \( y(0) = 0, \quad y(\pi) = 0 \).

For each \( bc = \text{Per}^\pm, \text{Dir} \) the operator \( L \) generates a closed operator \( L_{bc} \) with

\[
\text{Dom}(L_{bc}) = \{ f \in W^2_2(I) : f \text{ satisfies } bc \}.
\]

In the case of singular potentials (2.2) A. Savchuk and A. Shkalikov [47, 48] suggested to use the quasi-derivative \( y[1] = y' - wy \) in order to define properly the boundary conditions and corresponding operators. In particular, the periodic and anti-periodic boundary conditions \( \text{Per}^\pm \) have the form

(a*) \( \text{Per}^+ \): \( y(\pi) = y(0), \quad y[1](\pi) = y[1](0) \),

(b*) \( \text{Per}^- \): \( y(\pi) = -y(0), \quad y[1](\pi) = -y[1](0) \).

The Dirichlet boundary condition has the same form (c) as in the classical case. Of course, in the case where \( w \) is a continuous function, \( \text{Per}^+ \) and \( \text{Per}^- \) coincide, respectively, with the classical periodic boundary condition (a) and (b).

We refer the reader to our papers [6, 7, 9] for definitions of the operators \( L_{bc} \) and their domains in the case of \( H^{-1}_{\text{per}} \)-potentials. (We followed [47, 48] and further development of A. Savchuk – A. Shkalikov’s approach by R. Hryniv and Ya. Mykytyuk [21, 22, 23] to justify Fourier method in analysis of Hill-Schrödinger operators with singular potentials.

If \( v = 0 \) we denote by \( L^0_{bc} \) the corresponding free operator. Of course, it is easy to describe the spectra and eigenfunctions for \( L^0_{bc} \). Namely, we have

(i) \( \text{Sp}(L^0_{\text{Per}^+}) = \{ n^2, \ n = 0, 2, 4, \ldots \} \); its eigenspaces are \( E^0_n = \text{Span}\{ e^{\pm inx} \} \) for \( n > 0 \) and \( E^0_0 = \{ \text{const} \} \); \( \dim E^0_n = 2 \) for \( n > 0 \), and \( \dim E^0_0 = 1 \).

(ii) \( \text{Sp}(L^0_{\text{Per}^-}) = \{ n^2, \ n = 1, 3, 5, \ldots \} \); its eigenspaces are \( E^0_n = \text{Span}\{ e^{\pm inx} \} \), and \( \dim E^0_n = 2 \).

(iii) \( \text{Sp}(L^0_{\text{Dir}}) = \{ n^2, \ n \in \mathbb{N} \} \); each eigenvalue \( n^2 \) is simple; the corresponding normalized eigenfunction is

\[
s_n(x) = \sqrt{2} \sin nx,
\]

so the corresponding eigenspace is

\[
G^0_n = \text{Span}\{ s_n \}.
\]

2. Localization of spectra in the case of Hill operators.
Proposition 1. (localization of spectra) Consider \( L_{bc}(v) \) with \( bc = \text{Per}^\pm, \text{Dir} \) and with potential \( v \in L^2 \) or \( v \in (2.2) \). Then, for large enough \( N_* = N_*(v) \in 2\mathbb{N} \), we have

\[
\text{Sp}(L_{bc}) \subset \Pi_N \cup \bigcup_{n > N_*, n \in \Gamma_{bc}} D(n^2, r_n),
\]

where

\[
\Pi_N = \{ z = x + iy \in \mathbb{C} : |x|, |y| < N^2 + \frac{1}{2} N \},
\]

\[
D(a, r) = \{ z \in \mathbb{C} : |z - a| < r \},
\]

with

\[
r_n = N_*/2 \quad \text{if} \ v \in L^2, \quad r_n = n/4 \quad \text{if} \ v \in H_{\text{per}}^{-1},
\]

and

\[
\Gamma_{bc} = \begin{cases} 
\{0\} \cup 2\mathbb{N} & bc = \text{Per}^+, \\
2\mathbb{N} - 1 & bc = \text{Per}^-, \\
\mathbb{N} & bc = \text{Dir}.
\end{cases}
\]

With the resolvent \( R(z) = (z - L_{bc})^{-1} \) well defined in the complement of \( \text{Sp}(L_{bc}) \), we set

\[
S_{N_*} = \frac{1}{2\pi i} \int_{\partial \Pi_{N_*}} (z - L_{bc})^{-1} dz,
\]

\[
P_n = \frac{1}{2\pi i} \int_{|z-n^2|=r_n} (z - L_{bc})^{-1} dz, \quad n > N_*, \ n \in \Gamma_{bc},
\]

and

\[
S_N = S_{N_*} + \sum_{n = N_* + 1}^{N} P_n.
\]

Then

\[
\dim P_n = \begin{cases} 
2 & n \text{ even}, \ bc = \text{Per}^+, \\
2 & n \text{ odd}, \ bc = \text{Per}^-, \\
1 & n \in \mathbb{N}, \ bc = \text{Dir},
\end{cases}
\]

and

\[
\dim S_{N_*} = \begin{cases} 
N_* + 1 & bc = \text{Per}^+, \\
N_* & bc = \text{Per}^- \text{ or } \text{Dir}.
\end{cases}
\]

In each case the series

\[
S_{N_*} f + \sum_{n > N_*, n \in \Gamma_{bc}} P_n f = f \quad \forall f \in L^2(I)
\]

converges unconditionally, so the system of projections is a Riesz system.
The latter is true not only for potentials \( v \in L^2 \) but in the case \( v \in H^{-1}_{per} \) as well. It has been proven by A. Savchuk and A. Shkalikov [48, Theorem 2.8]. An alternative proof is given by the authors in [7], see Theorem 1 and Proposition 8.

3. Next we remind the basic fact about spectra decompositions and spectral decompositions for Dirac operators

\[
Ly = i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \frac{dy}{dx} + vy, \\
(2.17)
\]

\[
v(x) = \begin{pmatrix} 0 \\ Q(x) \end{pmatrix} P(x), \quad y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \\
(2.18)
\]

with \( L^2 \)-potential \( v \), i.e., \( P, Q \in L^2(I) \).

We consider three types of boundary conditions:

(a) periodic \( Per^+ \): \( y(0) = y(\pi) \), i.e., \( y_1(0) = y_1(\pi) \) and \( y_2(0) = y_2(\pi) \);

(b) anti-periodic \( Per^- \): \( y(0) = -y(\pi) \), i.e., \( y_1(0) = -y_1(\pi) \) and \( y_2(0) = -y_2(\pi) \);

(c) Dirichlet \( Dir \): \( y_1(0) = y_2(0), \ y_1(\pi) = y_2(\pi) \).

The corresponding closed operator with a domain

\[
\Delta_{bc} = \left\{ f \in (W^2(I))^2 : F = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \in (bc) \right\} \\
(2.19)
\]

will be denoted by \( L_{bc} \). If \( v = 0 \), i.e., \( P \equiv 0, Q \equiv 0 \), we write \( L^0_{bc} \). Of course, it is easy to describe the spectra and eigenfunctions for \( L^0_{bc} \):

(a) \( Sp(L^0_{Per^+}) = \{ n \text{ even} \} = 2\mathbb{Z} \); each number \( n \in 2\mathbb{Z} \) is a double eigenvalue, and the corresponding eigenspace is

\[
E^0_n = \text{Span}\{e^1_n, e^2_n\}, \\
(2.20)
\]

where

\[
e^1_n(x) = \begin{pmatrix} e^{-inx} \\ 0 \end{pmatrix}, \quad e^2_n(x) = \begin{pmatrix} 0 \\ e^{inx} \end{pmatrix}; \\
(2.21)
\]

(b) \( Sp(L^0_{Per^-}) = \{ n \text{ odd} \} = 2\mathbb{Z} + 1 \); the corresponding eigenspaces \( E^0_n \) are given by (2.20) and (2.21) but \( n \in 2\mathbb{Z} + 1 \);

(c) \( Sp(L^0_{Dir}) = \{ n \in \mathbb{Z} \} \); each eigenvalue \( n \) is simple. The corresponding normalized eigenfunction is

\[
g_n(x) = \frac{1}{\sqrt{2}} \left( e^1_n + e^2_n \right), \quad n \in \mathbb{Z}, \\
(2.22)
\]

so the corresponding (one-dimensional) eigenspace is

\[
G^0_n = \text{Span}\{g_n\}. \\
(2.23)
\]

4. Localization of spectra in the case of Dirac operators.
Proposition 2. (localization of spectra) For Dirac operators $L_{bc}(v)$ with $bc = \text{Per}^\pm, \text{Dir}$, there is $N_* = N_*(v)$, such that

\begin{equation}
\text{Sp}(L_{bc}) \subset \Pi_{N_*} \cup \bigcup_{n > N_*, n \in \Gamma_{bc}} D(n^2, 1/4),
\end{equation}

where

\begin{equation}
\Pi_{N} = \{z = x + iy \in \mathbb{C} : |x|, |y| < N^2 + \frac{1}{4},
\end{equation}

and

\begin{equation}
\Gamma_{bc} = \begin{cases} 
2\mathbb{Z} & \text{bc} = \text{Per}^+ , \\
1 + 2\mathbb{Z} & \text{bc} = \text{Per}^- , \\
\mathbb{Z} & \text{bc} = \text{Dir} .
\end{cases}
\end{equation}

With the resolvent $R(z) = (z - L_{bc})^{-1}$ well defined in the complement of $\text{Sp}(L_{bc})$, we set

\begin{equation}
S_{N_*} = \frac{1}{2\pi i} \int_{\partial \Pi_{N_*}} (z - L_{bc})^{-1} dz ,
\end{equation}

\begin{equation}
P_n = \frac{1}{2\pi i} \int_{|z - n| = 1/4} (z - L_{bc})^{-1} dz , \quad |n| > N_*, n \in \Gamma_{bc} ,
\end{equation}

and

\begin{equation}
S_N = S_{N_*} + \sum_{N_* + 1 \leq |n| \leq N} P_n .
\end{equation}

Then

\begin{equation}
dim P_n = \begin{cases} 
2 & n \text{ even}, \text{bc} = \text{Per}^+ , \\
2 & n \text{ odd}, \text{bc} = \text{Per}^- , \\
1 & n \in \mathbb{Z} , \text{bc} = \text{Dir} ,
\end{cases}
\end{equation}

and

\begin{equation}
dim S_{N_*} = \begin{cases} 
2N_* + 2 & \text{bc} = \text{Per}^+ , \\
2N_* & \text{bc} = \text{Per}^- , \\
2N_* + 1 & \text{bc} = \text{Dir} .
\end{cases}
\end{equation}

In each case the series

\begin{equation}
S_N f + \sum_{|n| > N_*, n \in \Gamma_{bc}} P_n f = f \quad \forall f \in L^2(I)
\end{equation}

converges unconditionally, so

\begin{equation}
\{S_{N_*}, P_n, |n| > N_*, n \in \Gamma_{bc}\}
\end{equation}

is a Riesz system of projections.
The latter is proven in [8, Theorem 5.1]. (Under more restrictive assumption on the potential \( v \in H^\alpha, \alpha > 1/2 \), the fact that (2.33) is a Riesz system of projections has been proven in [13, Theorem 8.8].)

Propositions 1 and 2 guarantee the existence of the level \( N_s = N_s(v) \) when all formulas for \( P_n, S_N \), etc. become valid if \( n > N_s, n \in \mathbb{N} \) (or \( |n| > N_s, n \in \mathbb{Z} \) in the Dirac case). In the next sections, there are other formulas which are valid for large enough \( n \) and require different levels \( N_s = N_s(v) \).

But throughout the paper we use one and the same letter \( N \) with (2.34) the matrix potentials [9, Formulas (3.21)–(3.33)].

Lemma 3. (a) Let \( L \) be a Dirac operator with a potential \( v \in L^2 \) or \( v \in H^1_{\text{per}} \). Then, for large enough \( n \in \mathbb{N} \), there are functions \( \alpha_n(v, z) \) and \( \beta_n^\pm(v; z) \), \( |z| < n \) such that a number \( \lambda = n^2 + z, \ |z| < n/4 \), is a periodic (for even \( n \)) or anti-periodic (for odd \( n \)) eigenvalue of \( L \) if and only if \( z \) is an eigenvalue of the matrix

\[
\begin{bmatrix}
\alpha_n(v, z) & \beta_n^-(v; z) \\
\beta_n^+(v; z) & \alpha_n(v, z)
\end{bmatrix}
\]

(2.34)

(b) Let \( L \) be a Dirac operator with a potential \( v \in L^2 \). Then, for large enough \( |n|, n \in \mathbb{Z} \), there are functions \( \alpha_n(v, z) \) and \( \beta_n^\pm(v; z) \), \( |z| < 1 \) such that a number \( \lambda = n + z, \ |z| < 1/4 \), is a periodic (for even \( n \)) or anti-periodic (for odd \( n \)) eigenvalue of \( L \) if and only if \( z \) is an eigenvalue of the matrix (2.34).

(c) A number \( \lambda = n^2 + z^*, \ |z| < n/4 \), (respectively, \( \lambda = n + z, \ |z| < 1/4 \) in the Dirac case) is a periodic (for even \( n \)) or anti-periodic (for odd \( n \)) eigenvalue of \( L \) of geometric multiplicity 2 if and only if \( z^* \) is an eigenvalue of the matrix (2.34) of geometric multiplicity 2.

The functionals \( \alpha_n(z; v) \) and \( \beta_n^\pm(z; v) \) are well defined for large enough \( |n| \) by explicit expressions in terms of the Fourier coefficients of the potential (see for Hill operators with \( L^2 \)-potentials [4, Formulas (2.16)–(2.33)], for Dirac operators [4, Formulas (2.59)–(2.80)], and for Hill operators with \( H^1_{\text{per}} \)-potentials [9, Formulas (3.21)–(3.33)]).

Here we provide formulas only for \( \beta_n^\pm(v; z) \) in the case of Hill operators with \( H^1_{\text{per}} \)-potentials. Let \( v \) be a singular potential as in (2.2), and

\[
v = w', \quad w = \sum_{m \in \mathbb{Z}} W(m) e^{imx}.
\]

(2.35)
Then the Fourier coefficients of \( v \) are given by
\[
V(m) = \text{im} W(m), \quad m \in 2\mathbb{Z},
\]
and by \([9] \text{ Formulas (3.21)–(3.33)}\) we have
\[
\beta_n^\pm(v; z) = V(\pm 2n) + \sum_{k=1}^{\infty} S_k^\pm(n; z),
\]
with
\[
S_k^\pm(n, z) = \sum_{j_1, \ldots, j_k \neq \pm n} \frac{V(\pm n - j_1)V(j_1 - j_2) \cdots V(j_{k-1} - j_k)V(j_k \pm n)}{(n^2 - j_1^2 + z) \cdots (n^2 - j_k^2 + z)}.
\]

Next we summarize some basic properties of \( \alpha_n(z; v) \) and \( \beta_n^\pm(z; v) \).

**Proposition 4.** Let \( v \) be a \( H_{\text{per}}^{-1} \)-potential of the form \((2,2)\), and let \( L_{\text{per}^\pm} \) be the corresponding Hill operator.

(a) The functionals \( \alpha_n(z; v) \) and \( \beta_n^\pm(z; v) \) depend analytically on \( z \) for \( |z| < n \). There exists a sequence of positive numbers \( \varepsilon_n \to 0 \) such that for large enough \( n \)
\[
|\alpha_n(v; z)| + |\beta_n^\pm(v; z)| \leq n \cdot \varepsilon_n, \quad |z| \leq n/2,
\]
and
\[
\left| \frac{\partial \alpha_n}{\partial z}(v; z) \right| + \left| \frac{\partial \beta_n^\pm}{\partial z}(v; z) \right| \leq \varepsilon_n, \quad |z| \leq n/4.
\]

(b) For large enough \( n \) (even, if \( bc = \text{Per}^+ \) or odd, if \( bc = \text{Per}^- \)), a number \( \lambda = n^2 + z \), \( |z| < n/4 \), is an eigenvalue of \( L_{\text{per}^\pm} \) if and only if \( z \) satisfies the basic equation
\[
(z - \alpha_n(z; v))^2 = \beta_n^+(z; v)\beta_n^-(z; v).
\]

(c) For large enough \( n \), the equation \((2.41)\) has exactly two roots in the disc \( |z| < n/4 \) counted with multiplicity.

**Proof.** Part (a) is proved in \([9] \text{ Proposition 15} \). Lemma \(\text{E} \) implies Part (b). By \((2.39)\), \( \sup\{\frac{1}{z} \alpha_n(z), \ |z| = n/4\} \to 0 \) and \( \sup\{\frac{1}{z} \beta_n^\pm(z), \ |z| = n/4\} \to 0 \). Therefore, Part (c) follows from the Rouché theorem. \(\Box\)

**Proposition 5.** Let \( L_{\text{per}^\pm} \) be a Dirac operator with \( L^2 \)-potential.

(a) The functionals \( \alpha_n(z; v) \) and \( \beta_n^\pm(z; v) \) depend analytically on \( z \) for \( |z| < 1 \). There exists a sequence of positive numbers \( \varepsilon_n \to 0 \) such that for large enough \( |n| \)
\[
|\alpha_n(v; z)| + |\beta_n^\pm(v; z)| \leq \varepsilon_n, \quad |z| \leq 1/2,
\]
and
\[
\left| \frac{\partial \alpha_n}{\partial z}(v; z) \right| + \left| \frac{\partial \beta_n^\pm}{\partial z}(v; z) \right| \leq \varepsilon_n, \quad |z| \leq 1/4.
\]
(b) For large enough $|n|$, (n even, if $bc = Per^+$ or odd, if $bc = Per^-$), the number $\lambda = n + z$, $z \in D = \{\zeta : |\zeta| \leq 1/4\}$, is an eigenvalue of $L_{Per^\pm}$ if and only if $z \in D$ satisfies the basic equation

\[(z - \alpha_n(z; v))^2 = \beta_n^\pm(z; v)\beta_n^-(z, v),\]

(c) For large enough $|n|$, the equation (2.44) has exactly two (counted with multiplicity) roots in $D$.

Proof. Part (a) is proved in [4, Proposition 35]. Lemma 3 implies Part (b). By (2.42), $\sup_D |\alpha_n(z)| \to 0$ and $\sup_D |\beta_n^\pm(z)| \to 0$ as $n \to \infty$. Therefore, Part (c) follows from the Rouché theorem.

\[\square\]

3. Elementary geometry of bases in a Banach space

In this section we give a few well-known facts about geometry and bases in Banach and Hilbert spaces – see [25, 31, 32, 2, 28].

1. Let $\{u_k \in X, \psi_k \in X'\}_{k \in \mathbb{N}}$ be a biorthogonal system in a Banach space $X$, i.e.,

\[\psi_k(u_j) = \begin{cases} 1, & k = j, \\ 0, & k \neq j \end{cases}, \quad j, k \in \mathbb{N}.\]

The system $\{u_k\}$ is called a basis, or a Schauder basis in $Y$, its closed linear span if

\[\lim_{N \to \infty} \sum_{k=1}^{N} \psi_k(y)u_k = y, \quad \forall y \in Y.\]

Put

\[Q_m = q_{2m-1} + q_{2m}, \quad \text{where} \quad q_j(x) = \psi_j(x)u_j, \quad j \in \mathbb{N}\]

are one-dimensional projections so

\[\|q_j\| = \|u_j\| \cdot \|\psi_j\|\]

Let us assume that

\[\lim_{M \to \infty} \sum_{m=1}^{M} Q_my = y \quad \forall y \in Y.\]

In this case, certainly

\[\sup_m \|Q_m\| = C < \infty.\]

Notice that partial sums in (3.5) are equal to partial sums in (3.2) with even indices. But

\[\sum_{k=1}^{2t+1} \psi_k(y)u_k = \left(\sum_{m=1}^{t} Q_my\right) + \psi_{2t+1}(y)u_{2t+1}.\]

These elementary identities together with (3.1) explain the following.
Lemma 6. If \( \{u_k\}_1^\infty \) is a basis in \( Y \), i.e., (3.2) holds then
\[
T \equiv \sup_j \|q_j\| < \infty.
\]
Under the assumption (3.5) if (3.8) holds then \( \{u_k\}_1^\infty \) is a basis in \( Y \).

2. What does happen inside of 2D subspaces \( E_m = \text{Ran} \ Q_m, \ m \in \mathbb{N} \)?

Let \( \{u_1, u_2\}, \|u_j\| = 1 \), be a basis in \( E_m \) and let \( \psi_1, \psi_2 \) be the corresponding biorthogonal functionals, so
\[
h = \psi_1(h)u_1 + \psi_2(h)u_2 \quad \forall h \in E_m.
\]
To avoid any confusion let us notice that for \( j = 2m - 1, 2m \)
\[
\psi_j(y) = \psi_j(Q_m y) \quad \forall y \in Y,
\]
and if (3.5) holds then with (3.6)
\[
\|Q_m y\| \leq C \|y\|.
\]
Therefore,
\[
\|\psi_j\| \geq \sup \{|\psi_j(w)| : \|w\| = 1, \ w \in E_m\}
\]
\[
\geq \sup \{\langle \frac{1}{C}Q_m y, \psi_j \rangle : \|y\| = 1, \ y \in Y\} = \frac{1}{C}\|\psi_j\|,
\]
so
\[
\|\psi_j\| \leq C \kappa_j, \quad \kappa_j \leq \|\psi_j\|,
\]
i.e.,
\[
\kappa_j \equiv \|\psi_j|_{E_m}\| \leq \|\psi_j|_{Y}\| \leq C \kappa_j.
\]
In a Hilbert space case, elementary straightforward estimates show that for \( j = 1, 2 \)
\[
\kappa_j = \sup \{|\psi_j(w)| : \|w\| = 1, \ w \in \mathbb{C}^2\} = \left(1 - |\langle u_1, u_2 \rangle| \right)^{-1/2}.
\]
We use this fact when analyzing subspaces \( E_m \) and their bases \( \{u_{2m-1}, u_{2m}\}, \ m \in \mathbb{N} \).

3. Now we consider separable Hilbert spaces \( H \). We say that the system
\( \{Q_m\} \in (3.3) \) is a Riesz system, or an unconditional 2D-block basis in \( Y \) if
for some \( C > 0 \)
\[
\| \sum_{m \in F} Q_m \| \leq C \quad \text{for any finite subset } F \subset \mathbb{N}.
\]

Lemma 7. Assume the system of 2D projections \( Q_m \in (3.3) \) in a Hilbert space \( H \) is a Riesz system, i.e., (3.16) holds. If \( \{u_k\}_1^\infty \) is a basis in \( Y \subset H \) then it is an unconditional basis in \( Y \).

Proof. Proof is based on the Orlicz [46] lemma:
Lemma 8. \((3.16)\) holds for the system \(Q_m \in (3.3)\) in a Hilbert space if and only if for some constant \(C_1 > 0\)

\[
\frac{1}{C_1^2} \|y\|^2 \leq \sum_m \|Q_m y\|^2 \leq C_1^2 \|y\|^2 \quad \forall y \in Y.
\]

By Lemma 6 and \((3.8), (3.4)\) the norms of 1D projections \(q_j\) are uniformly bounded. By \((3.1)\)

\[
(3.18) \quad q_j Q_m = \begin{cases} q_j & \text{if } j = 2m - 1, 2m \\ 0 & \text{otherwise} \end{cases}
\]

so for \(j = 2m - 1, 2m\)

\[
(3.19) \quad \|q_j y\| = \|q_j Q_m y\| \leq M \|Q_m y\| \leq (\|q_{2m-1} y\| + \|q_{2m} y\|) \leq 2M \|Q_m y\|.
\]

Therefore

\[
(3.20) \quad \frac{1}{4M^2} (\|q_{2m-1} y\|^2 + \|q_{2m} y\|^2) \leq \|Q_m y\|^2 \leq 2M^2 (\|q_{2m-1} y\|^2 + \|q_{2m} y\|^2)
\]

and with \(C_1 = 2M\) the condition \((3.16)\) holds for the system of 1D projections \(\{q_j\}\). It guarantees that \(\{q_j\}\) is a Riesz system and \(\{u_k\}\) is an unconditional basis in \(Y\). \(\square\)

4. Now we are ready to claim the following.

Criterion 9. With notations \((3.1), (3.3)\) let us assume that the system of 2D projections \(\{Q_m\}\) is a Riesz system in a Hilbert space. If a normalized system

\[
(3.21) \quad \{u_k\}, \quad \|u_k\| = 1,
\]

is a basis in \(Y\) then

\[
(3.22) \quad \kappa := \sup \{1 - |\langle u_{2m-1}, u_{2m} \rangle|^2 : m \in \mathbb{N}\} < \infty.
\]

If the condition \((3.22)\) holds then \(\{u_k\}\) is a normalized unconditional basis, that is a Riesz basis in \(Y\).

Corollary 10. If \((3.16)\) holds in a Hilbert space \(H\) the system \(\{u_k\}_{1}^{\infty} \in (3.21), (3.1)\) is a Riesz basis if and only if it is a basis.

4. Moving from geometric criterion to Hill and Dirac operators

1. The basic assumption in the geometric Criterion 9 is the property of a system of projections \(\{Q_m\}\) in a Hilbert space to be a Riesz system.

When we analyze systems of projections \(\{P_n, |n| \geq N_n\}\) coming from Hill or Dirac operators, then it is a fundamental fact that they are Riesz systems.
If \( v \in L^2 \) this has been understood since 1980’s ([49], [50], [51]). To make technically formal reference let us mention [3, Proposition 5], where it is shown that

\[
\|P_n - P_0^n\|_{2 \to \infty} \leq C \frac{\|v\|_2}{n},
\]

so certainly

\[
\sum_{|n| \geq N} \|P_n - P_0^n\|_{2 \to 2} < \infty
\]

and with

\[
\dim S_N = \dim S_0^N
\]

the Bari-Markus theorem [20, Ch. 6, Sect. 5.3, Theorem 5.2] implies that the series converge unconditionally.

A. Savchuk and Shkalikov [48, Theorem 2.4] showed that (4.2) - (4.3) hold if \( v \in H^{-1}_per \) and \( bc = Per^\pm \). An alternative proof has been given by the authors – see Theorem 1 and Proposition 8 in [10].

Finally, in the case of one dimensional Dirac operators we proved (4.2) - (4.3) if \( v \in L^2 \) and \( bc = Per^\pm \) or \( Dir \) (see [8], Theorems 3.1 and 5.1). Later we proved (4.2) - (4.3) for arbitrary regular boundary condition – see Theorems 15 and 20 in [11]; however, we do not use these results from [11] in the present paper. Certainly in all these cases

\[
\|P_n - P_0^n\|_{2 \to 2} \to 0 \quad \text{and} \quad \|P_n\|_2 \leq \frac{3}{2} \quad \text{for} \quad |n| > N^*.
\]

These bibliography references justify applicability of Criterion 9 when we are trying to give different analytic criteria for Riesz basis property of the root function system of specific differential operators.

Of course, Corollary 10 indicates that in a Hilbert space there is no separate question about Schauder basis property. If \( \{Q_m\} \), or \( \{S_N; P_n, |n| \geq N\} \) is a Riesz system such that \( \dim Q_m = 2, \dim P_n = 2 \), then the properties of the system \( \{u_{2m-1}, u_{2m}\} \) to be a Riesz basis or to be a Schauder basis are identical. Therefore, to talk about two properties is semantically artificial.

2. Let us define the root function system \( \{u_j\} \) which will play a special role in our analysis in Sections 5 and 6 and in Main Theorem (Theorem 24). Section 3 and Criterion 9 use an indexation by natural numbers, i.e., \( m \in \mathbb{N} \). But in the case of Riesz bases (or unconditional convergence of series) it means that we can ignore the ordering in \( \mathbb{N} \), consider any countable set of indices and use all related statements from Section 3. Of course, in the case of bases which are not Riesz bases we should be accurate when we use statements from Section 3 – this is important in Section 6.

Remark 11. In the case of Hill operators, \( \Gamma_{bc} \in (2.10) \) as a subset of \( \mathbb{N} \) has a natural ordering and we have no confusion in defining the sum in (2.10).
- this is
\[ \lim_{N \to \infty} \sum_{N_* < n \leq N} n \in \Gamma_{bc} \]

if this limit does exists. However for Dirac operators \( \Gamma_{bc} \in (2.26) \) are subsets in \( \mathbb{Z} \); we have to accept convention to define the sum in (2.32) as
\[ \lim_{N \to \infty} \sum_{N_* < |n| \leq N} n \in \Gamma_{bc} \] and
\[ \lim_{N \to \infty} \sum_{-N < n \leq N + 1} n \in \Gamma_{bc}, |n| > N_* \]

if both these limits exist and are equal. Such understanding is in accordance with the choice of contours in (2.25) and (2.27).

But in all four cases – Per\(^+\) and Per\(^-\) for both Hill and Dirac operators – the systems of projections
\[ (4.5) \quad \{ S_{N_*}, P_n, |n| > N_*, n \in \Gamma_{bc} \} \]
given in (2.11) - (2.15) or (2.27) - (2.31) are Riesz systems of projections as (2.16) and (2.32) tell us.

Now we define three sets of indices:
\[ (4.6) \quad M = \{ m \in \Gamma_{bc} : |m| > N_*, \lambda^+_m - \lambda^-_m \neq 0 \}, \]
\[ (4.7) \quad M_1 = \{ m \in \Gamma_{bc} : |m| > N_*, \lambda^+_m - \lambda^-_m = 0, \ P_m L_{bc} P_m = \lambda \cdot 1_{E_m} \}, \]
i. e., \( \lambda^+_m \) is a double eigenvalue of algebraic and geometric multiplicities 2;
\[ (4.8) \quad M_2 = \{ m \in \Gamma_{bc} : |m| > N_*, \lambda^+_m - \lambda^-_m = 0, \ P_m L_{bc} P_m \text{ is a Jordan matrix} \}, \]
i. e., \( \lambda^+_m \) is a double eigenvalue of algebraic multiplicity 2 and geometric multiplicity 1.

If \( m \in M \), we choose \( (u_{2m-1}, u_{2m}) \) in such a way that
\[ (4.9) \quad Lu_{2m} = \lambda^+_m u_{2m}, \quad Lu_{2m-1} = \lambda^-_m u_{2m-1}, \]
\[ (4.10) \quad \|u_j\| = 1, \quad j \in \mathbb{N}. \]

If \( m \in M_1 \) choose any pair of orthogonal normalized vectors in \( E_m \)
\[ (4.11) \quad (u_{2m-1}, u_{2m}) = 0. \]

3. For \( m \in M_2 \) we consider two different options to choose root functions for a basis.

**Option 1.** If \( m \in M_2 \), then there is only one (up to constant factor) normalized eigenvector \( f \in E_m \),
\[ (4.12) \quad Lf = \lambda^+_m f, \quad \|f\| = 1, \]
so we choose
\[ (4.13) \quad u_{2m} = f, \quad u_{2m-1} \perp u_{2m}, \quad \|u_{2m-1}\| = 1. \]
Such a pair \((u_{2m-1}, u_{2m}), m \in \mathcal{M}_2\) – as for \(m \in \mathcal{M}_1\) – is a nice basis in \(E_m\), so it will not be an obstacle for Riesz basisness of the larger system (see Lemmas 7 and 8) which contains \\{u_{2m-1}, u_{2m}\}.

**Option 2.** We choose \(u_{2m}\) as in Option 1, and we choose \(u_{2m-1} \in (4.11)\) to be an associated function, i.e.,
\[
L_{bc} u_{2m} = \lambda_m^+ u_{2m}, \quad L_{bc} u_{2m-1} = \lambda_m^+ u_{2m-1} + u_{2m}.
\]
Since we choose \(u_{2m-1}\) to satisfy (4.14) and (4.11), it is uniquely defined but its norm \(\|u_{2m-1}\|\) is out of our control.

For Hill operators with potentials in \(L^1\) A. Shkalikov and O. Veliev \[52, Theorem 1, Step 1\] observed that if \(\mathcal{M}_2\) is infinite then
\[
\|u_{2m-1}\| \to \infty \quad \text{as} \quad m \to \infty, \quad m \in \mathcal{M}_2.
\]
For potentials \(v \in L^2\) this has been proven in \[24, Ine. (3.29)\]. Formula (4.15) implies that \(\{u_{2m-1}, u_{2m}, m \in \mathcal{M}_2\}\) could not be a subset of a Riesz basis.

However, if a potential \(v\) is singular it may happen that \(\mathcal{M}_2\) is infinite but with the choices determined by Option 2 we have
\[
\exists C > 0 \quad 0 < \frac{1}{C} \leq \|u_{2m-1}\| \leq C < \infty, \quad \forall m \in \mathcal{M}_2.
\]

**Example 12.** Take Gasymov type \[17\] singular potential
\[
v(x) = \sum_{k=1}^{\infty} c(k) e^{2ikx},
\]
with
\[
\exists A > 0 : \quad 1/A \leq |c(k)| \leq A \quad \forall k \in \mathbb{N}.
\]
Then we have:
(i) \(\mathcal{M}_2 = \Gamma_{bc} \cap \{n : n > N_*\}\) for \(bc = \text{Per}^+\) and \(\text{Per}^-\), i.e., all \(E_m\) with \(m > N_*\) are Jordan;
(ii) with choices by Option 2 the condition (4.16) holds, and the system of eigen- and associated functions \(\{u_{2m-1}, u_{2m}, m \in \mathcal{M}_2\}\) is a Riesz basis in \(L^2\).

This example is in a quite curious contrast with the case \(v \in L^2\) or \(v \in L^1\) – see (4.15) above. We prove the claims (i) and (ii) in Section 6, where other examples of \(H^{-1}_{\text{per}}\) potentials are considered as well.

4. Now we declare our canonical choice of vectors in Jordan blocks:
\[
\text{from now on our special system } \{u_j\} \text{ is chosen by Option 1}.
\]

**Remark 13.** The choice (4.19) guarantees that the total system \(\{u_j\}\) of root functions has the Riesz basis property if and only if its subsystem
\[
U_M = \{u_{2m-1}, u_{2m}, m \in \mathcal{M}\}
\]
is a Riesz basis in its closed linear span.
But still we need to define $u_j$ for small $j$, $|j| \leq N_*$. This system will be a basis in $E_* = \text{Ran}S_N$. Of course $\dim E_* < \infty$, so this choice has no bearing on whether the entire system will or will not be a Riesz basis (or a basis) in $L^2$ or another function space. We want it to be a system of root functions, so we choose the system of eigen- and associated functions of a finite-dimensional operator $S_*L_{bc}S_*$, $S_* = S_{N_*}$ (We omit elementary linear algebra details.)

5. $L^p$-spaces and other rearrangement invariant function spaces

1. In Sections 3 and 4 we discussed (criteria of) convergence of decompositions

$$S_{N_*}f + \sum_{n>N_*,n \in \Gamma_{bc}} P_nf = f \quad \forall f \in L^2$$

in $L^2$. Convergence of such series or of eigenfunction decompositions in $L^p$, $p \neq 2$, or other rearrangement invariant function spaces (see [28, 41]) is not an independent from convergence in $L^2$ question because of the following two reasons of very general nature:

(A) In the case of free operator $L^0$ its decompositions (5.1) are standard (or slight variations of) Fourier series. These decompositions

$$S_{N_*}^0f + \sum_{n>N_*,n \in \Gamma_{bc}} P_n^0f = f \quad \forall f \in E$$

converge in $E$ if $E$ is a separable rearrangement invariant function space where Hilbert transform is bounded. This is certainly the case if

$$L^a \supset E \supset L^b$$

for some $a, b$ with $1 < a \leq b < \infty$.

See [28, Theorem 2.7.2], [41, 62], and more about Boyd indices in [32, Theorem 2.c.16 and Proposition 2.b.3 there.

(B) Put

$$S_N = S_{N_*} + \sum_{n>N_*,n \in \Gamma_{bc}} P_n.$$  

There are different versions of equiconvergence – see the survey paper of A. Minkin [42]. For example, J. Tamarkin [54, 55] and M. Stone [53] proved the following.

**Lemma 14.** If $v \in L^1$ then for any $f \in L^1$

$$\| (S_N - S_N^0)f \|_\infty \to 0.$$  

This lemma helps to cover the case of Hill operator with $v \in L^1$. For $v \in H^{-1}_{pe}$, see Proposition 16 below.

Equiconvergence in the case of Dirac operator with potentials $v \in L^c$, $c > 4/3$, is proven in [43, Theorem 6.2 (a)]. As a corollary it is noticed there [43]...
Theorem 6.4, (6.105)] that the series \([5.6]\) converges in \(L^p(I, C^2)\) \(1 < p < \infty\).

2. Now we can combine (A) and (B) to conclude the following.

**Proposition 15.** If \(v \in L^2\) and (5.3) holds then

\[
S_N^* f + \sum_{n > N, n \in \Gamma_{bc}} P_n f = f \quad \forall f \in E
\]

**Proof.** Indeed

\[
S_N f = S_N^0 f + (S_N - S_N^0) f
\]

but with (5.3) \(\|g\|_E \leq \|g\|_\infty\) so for \(f \in L^1\)

\[
\|(S_N - S_N^0) f\|_E \leq \|(S_N - S_N^0) f\|_\infty \to 0.
\]

Now (5.2) and (5.8) together imply (5.6). \(\square\)

3. Of course in the case of Hill operators we want to cover potentials \(v \in H_{per}^{-1}\) as well. This is possible because the following *equiconvergence* statement is true.

**Proposition 16.** Let \(v \in H_{per}^{-1}\), \(W\) be coming from (2.35) and (2.36), and

\[
1 < a \leq b < \infty \quad \text{with} \quad \delta = 1/2 - (1/a - 1/b) > 0.
\]

Then for any \(N > N_*(v)\)

\[
\|S_N - S_N^0 : L^a \to L^b\| \leq C(\delta) \left[ N^{-\tau} + E_N(W) \right],
\]

where

\[
\tau = \begin{cases} 
\delta & \text{if } 1 < a \leq 2 \leq b < \infty; \\
1 - 1/a & \text{if } 1 < a \leq b \leq 2; \\
1/b & \text{if } 2 \leq a \leq b < \infty.
\end{cases}
\]

and

\[
E_N(W) = \left( \sum_{|m| \geq N} |W(m)|^2 \right)^{1/2}.
\]

Proof with all details is to be given in another paper we will submit shortly.

**Proposition 17.** If \(v \in H_{per}^{-1}\) and \(E\) is a s.r.i.f.s. such that (5.3) and (5.9) hold then (5.6) hold.
Proof. Now with \( \|g\|_a \leq \|g\|_E \leq \|g\|_b \) \( (5.10) \) and \( (5.7) \) imply (5.13)
\[
\|(S_N - S_N^0)f\|_E \leq \|(S_N - S_N^0)f\|_{L^b} \leq \|S_N - S_N^0\| : L^a \rightarrow L^b \|f\|_{L^b} \leq \varepsilon(N) \|f\|_E,
\]
where
\[
\varepsilon(N) = C(\delta) \left[ N^{-\delta} + E_N(w) \right] \rightarrow 0,
\]
so \( (5.6) \) holds. \( \square \)

4. Terms \( P_m f \) in \( (5.6) \) are vectors in two-dimensional subspaces
\[ E_m = \text{LinSpan}\{u_{2m-1}, u_{2m}\}, \]
with \( \{u_j\} \) defined in Section 4.2, \( (4.19) \).

**Fact (C).** In these 2D subspaces \( L^1 \) norms and \( L^\infty \) norms are uniformly equivalent, i.e., with \( B = B(v) < \infty \)
\[ \|F\|_\infty \leq B \|F\|_1 \quad \text{if} \quad F \in E_m, \ m \geq N(v) \]
This is proven in \[43\] Theorem 8.4, p.185 for Dirac operators with \( V \in L^p, \ 1 < p, \) and in \[9\] Theorem 51, p.159 for Hill operators with \( v \in H^{-1}_{\text{per}}. \)

Section 4.2 explains that with conditions \( (3.5) \) and \( (3.6) \)
\[ \|\psi_j|E_m\| \leq \|\psi_j|E\| \leq C \|\psi_j|E_m\|. \]
– see \( (3.11) - (3.14) \). By Lemma 6, the system \( \{u_j\} \) is a basis in \( Y \subset E \) if and only if
\[ \sup_j \|u_j\|_E \cdot \|\psi_j\|_E < \infty. \]
But Fact (C) shows that \( (5.16) \) holds – or does not hold – for all s.r.i.f.s. \( E \) such that
\[ L^1([0,\pi]) \supset E \supset L^\infty([0,\pi]) \]
simultaneously. Any condition which is good to guarantee basisness in one \( E \) is automatically good for all \( E'\)s. Therefore, we can immediately to claim the following.

**Theorem 18.** Let \( E \) be a separable r.i.f.s. and
\[ L^a([0,\pi]) \supset E \supset L^b([0,\pi]), \ 1 < a \leq b < \infty. \]
The system \( \{u_j\} \) defined in \( (4.19) \) is a basis in \( E \) (or \( E^2 \)) if and only if \( \{u_j\} \) is a basis in \( L^a([0,\pi]) \) (or \( (L^a([0,\pi]))^2 \).
6. Criteria in terms of Fourier coefficients of potentials

1. Let \( L = L_{Per}^\pm(v) \) be a Hill operator with \( H_{Per}^{-1} \)-potential, or Dirac operator with \( L^2 \)-potential, subject to periodic \( Per^+ \) or anti-periodic \( Per^+ \) boundary conditions.

Recall that the eigenvalues \( \lambda_n^\pm, \mu_n \) and the related functions \( \beta_n^\pm(v, z) \) are well defined for large enough \( |n| \). Let

\[
(6.1) \quad t_n(z) = \begin{cases} 
|\beta_n^+(z)/\beta_n^-(z)| & \text{if } \beta_n^+(z) \neq 0, \\
\infty & \text{if } \beta_n^+(z) = 0, \beta_n^-(z) \neq 0, \quad |n| > N_*, \\
1 & \text{if } \beta_n^+(z) = 0, \beta_n^-(z) = 0.
\end{cases}
\]

Then the following criterion for existence of a Riesz basis consisting of root functions of \( L \) holds.

**Theorem 19.** Let \( \mathcal{M} = \{ n : |n| \geq N_*, \lambda_n^- \neq \lambda_n^+ \} \), and let \( \{u_{2n-1}, u_{2n}\} \) be a pair of normalized eigenfunctions corresponding to the eigenvalues \( \lambda_n^-, \lambda_n^+ \).

(a) The system \( \{u_{2n-1}, u_{2n}, n \in \mathcal{M}\} \) is a Riesz basis in its closed linear span if and only if

\[
(6.2) \quad 0 < \liminf_{n \in \mathcal{M}} t_n(z_n^*), \quad \limsup_{n \in \mathcal{M}} t_n(z_n^*) < \infty,
\]

where \( z_n^* = \frac{1}{2}(\lambda_n^- + \lambda_n^+) - \lambda_n^0 \) with \( \lambda_n^0 = n^2 \) for Hill operators and \( \lambda_n^0 = n \) for Dirac operators.

(b) The system of root functions of \( L \) contains a Riesz basis if and only if (6.2) holds.

This theorem implies that Condition (7) in Theorem 24 is equivalent to Conditions (1) - (6) there.

**Proof.** In view of Remark 13 we need to prove only (a).

For Dirac operators, [14, Theorem 12] proves, in the case \( \mathbb{N} \setminus \mathcal{M} \) is finite, that Condition (6.2) implies the existence of a Riesz basis in \( L^2((0, \pi), C^2) \) which consists of eigenfunctions and at most finitely many associated functions of the operator \( L_{Per}^\pm(v) \). The same proof explains that (6.2) implies (a) for arbitrary infinite set of indices \( \mathcal{M} \) not only for Dirac operators but also for Hill operators with \( H_{Per}^{-1} \)-potentials.

If (6.2) fails, then one may follow, with a slight modification, the proof of [4, Theorem 71] in order to show that (a) fails. We provide all details of such a modification below.

Suppose (6.2) fails. Then there is a subsequence of indices \( (n_k) \) in \( \mathcal{M} \) such that either

\[
(6.3) \quad t_{n_k}(z_{n_k}^*) \to 0 \quad \text{as } k \to 0,
\]
or \( t_{n_k}(z_{n_k}^*) \to \infty \). Next we consider only the case (6.3) because the other one is symmetric – if \( 1/t_{n_k}(z_{n_k}^*) \to 0 \), then one may exchange the roles of \( \beta_n^+ \) and \( \beta_n^- \) and use the same argument.
Lemma 20. In the above notations, if (6.5) holds then there is a sequence \((\eta_k)\) of positive numbers such that

\[ t_{n_k}(z) \leq \eta_k \to 0 \quad \forall z \in [z_{n_k}^-, z_{n_k}^+], \]

where \([z_{n_k}^-, z_{n_k}^+]\) denotes the segment with end points \(z_{n_k}^-\) and \(z_{n_k}^+\).

Proof. By [9, Lemma 20] (in the case of Hill operators with \(H_{\text{per}}^{-1}\) potentials) or by [4, Lemma 40] (in the case of Dirac operators), for large enough \(|n|\) we have

\[ |\gamma_n| \leq 2(|\beta_n^- (z_n^+)| + |\beta_n^+ (z_n^-)|). \]

Therefore, (6.3) implies that for large enough \(k\)

\[ |\gamma_{n_k}| \leq 2(|\beta_{n_k}^- (z_{n_k}^+)| + |\beta_{n_k}^+ (z_{n_k}^-)|) \leq 4|\beta_{n_k}^+ (z_{n_k}^-)|. \]

In view of (2.40) in Proposition 4 or (2.43) in Proposition 5, for each \(z \in [z_{n_k}^-, z_{n_k}^+]\) and all \(n \in \mathcal{M}\) with large enough \(|n|\) we have

\[ |\beta_{n_k}^+ (z) - \beta_{n_k}^+ (z_n^+)| \leq \sup_{[z_{n_k}^-, z_{n_k}^+]} \left| \frac{\partial \beta_{n_k}^+}{\partial z} (z) \right| \cdot |z - z_{n_k}^+| \leq \varepsilon_n |\gamma_{n_k}|, \]

with \(\varepsilon_n \to 0\) as \(|n| \to \infty\). Therefore, by (6.5) and (6.6) it follows that

\[ |\beta_{n_k}^+ (z)| \geq |\beta_{n_k}^+ (z_{n_k}^+)| - 4\varepsilon_n |\beta_{n_k}^+ (z_{n_k}^-)| = (1 - 4\varepsilon_n) |\beta_{n_k}^+ (z_{n_k}^-)|. \]

On the other hand, (6.5) and (6.6) imply that

\[ |\beta_{n_k}^- (z)| \leq |\beta_{n_k}^+ (z) - \beta_{n_k}^+(z_{n_k}^+)| + |\beta_{n_k}^- (z_{n_k}^+)| \leq 4\varepsilon_n |\beta_{n_k}^+ (z_{n_k}^-)| + |\beta_{n_k}^- (z_{n_k}^+)|. \]

Thus, since \(\varepsilon_{n_k} \to 0\), we obtain

\[ \frac{|\beta_{n_k}^- (z)|}{|\beta_{n_k}^+ (z)|} \leq \frac{4\varepsilon_{n_k} |\beta_{n_k}^+ (z_{n_k}^+)| + |\beta_{n_k}^- (z_{n_k}^+)|}{(1 - 4\varepsilon_{n_k}) |\beta_{n_k}^+ (z_{n_k}^-)|} = \frac{4\varepsilon_{n_k} + t_{n_k} (z_{n_k}^+)}{1 - 4\varepsilon_{n_k}} \to 0, \]

i. e., (6.4) holds with \(\eta_k = \frac{4\varepsilon_{n_k} + t_{n_k} (z_{n_k}^+)}{1 - 4\varepsilon_{n_k}}\).

Now one may follow p. 754 in [4] (in Russian original p. 170) in order to complete the proof.

2. Theorem 19 provides a general criterion for Riesz basis property of the system of root functions of Hill operator or Dirac operator subject to periodic or anti-periodic boundary conditions. It extends and slightly generalizes [13, Theorem 1] (or [12, Theorem 2]) in the case of Hill operators, and [14, Theorem 12] in the case of Dirac operators.

Theorem 19 is an effective criterion for analyzing the existence or non-existence of Riesz bases consisting of root functions of Hill or Dirac operators. We refer to our papers [12, 13, 14] for concrete applications (see also [4, Theorem 71]).

Now we give examples of classes of Hill operators with singular potentials which system of root functions has (or has not) the Riesz basis property.
Example 21. Let $\mathcal{A} \subset (0, \pi)$ be countable, and let

$$v(x) = \sum_{k \in \mathbb{Z}} \sum_{\alpha \in \mathcal{A}} g(\alpha) \delta(x - \alpha - k\pi) - \frac{1}{\pi} \sum_{\alpha \in \mathcal{A}} g(\alpha)$$

with

$$\exists \alpha^* : |g(\alpha^*)| > \sum_{\alpha \in \mathcal{A}\setminus\{\alpha^*\}} |g(\alpha)|.$$ 

Then the system of root functions of $L_{\text{Per}}^{\pm}(v)$ has the Riesz basis property.

(The function $v$ in (6.8) lies in $H_{\text{per}}^{-1}$ as it follows from [21, Theorem 3.1 and Remark 2.3] or [7, Proposition 1].)

Proof. Indeed, (6.8) implies that the Fourier coefficients of $v$

$$V(k) = \frac{1}{\pi} \sum_{\alpha \in \mathcal{A}} g(\alpha)e^{ik\alpha}, \quad k \in 2\mathbb{Z},$$

satisfy

$$\exists A > 0 : \frac{1}{A} \leq |V(k)| \leq A, \quad \forall k \in 2\mathbb{Z}.$$ 

Recall that by (2.37) $\beta_n^\pm(v, z) = V(\pm 2n) + \sum_{k=1}^\infty S_k^\pm$, with $S_k$ defined by (2.38). In view of (2.38) and (6.11),

$$|S_k^\pm| \leq \sum_{j_1, \ldots, j_k \neq \pm n} \frac{A^{k+1}}{|n^2 - j_1^2 + z| \cdots |n^2 - j_k^2 + z|}.$$ 

For $|z| < n/2$, we have

$$|n^2 - j^2 + z| \geq |n^2 - j^2| - n/2 \geq \frac{1}{2}|n^2 - j^2| \quad \text{for } j \neq \pm n, \ j - n \in 2\mathbb{Z}.$$ 

Therefore,

$$|S_k^\pm| \leq \sum_{j_1, \ldots, j_k \neq \pm n} \frac{(2A)^{k+1}}{|n^2 - j_1^2| \cdots |n^2 - j_k^2|} \leq (2A)^{k+1} \left( \sum_{j \neq \pm n} \frac{1}{|n^2 - j^2|} \right)^k.$$ 

Now, by the elementary inequality

$$\sum_{j \neq \pm n} \frac{1}{|n^2 - j^2|} \leq \frac{2 \log n}{n}, \quad n \geq 3,$$

it follows that

$$|S_k^\pm| \leq (4A)^{k+1} \left( \frac{\log n}{n} \right)^k.$$ 

Thus, $\sum_{k=1}^\infty |S_k^\pm| = O((\log n)/n)$, so we obtain

$$\beta_n^\pm(v, z) = V(\pm 2n) + O((\log n)/n).$$ 

In view of (6.11) the latter formula implies (6.2), thus the system of root functions of $L_{\text{Per}}^{\pm}(v)$ has the Riesz basis property. \qed
3. Next we use (6.12) to explain the claims in Example 12.

Proof of Claims (i) and (ii) in Example 12

Proof of (i). In view of (4.17), the Fourier coefficients $V(m)$, $m \in 2\mathbb{Z}$, of the potential $v$ in Example 12 are given by

$$V(m) = \begin{cases} 0 & m \leq 0, \\ c(m/2) & m > 0. \end{cases}$$

Since $V(m) = 0$ for $m \leq 0$, one can easily see from Formulas (2.37) and (2.38) that

$$\beta_n^-(v; z) \equiv 0 \quad \forall n > N_*, \ |z| \leq n.$$

On the other hand, by (4.18),

$$\exists A > 0 : \frac{1}{A} \leq |V(m)| \leq A \quad \forall m \in 2\mathbb{N},$$

so the same argument as above proves that (6.12) holds. Since, by (4.18), we have $|V(2n)| > 1/A$, it follows that

$$\beta_n^+(v; z) = V(\pm 2n) + O((\log n)/n) \neq 0 \quad \text{if} \quad n > N_*.$$

Fix an $n > N_*$. By Proposition 4, the equation (2.41), that is

$$(z - \alpha_n(z; v))^2 = \beta_n^+(v; z)\beta_n^-(v; z),$$

has exactly two (counted with multiplicity) roots in the disc $|z| < n/4$. Since $\beta_n^-(v; z) \equiv 0$, now this equation has one double root, say $z_n^*$, and the matrix

$$\begin{bmatrix} \alpha_n(v, z^*_n) - z^*_n & \beta_n^-(v, z^*_n) \\ \beta_n^+(v, z^*_n) & \alpha_n(v, z^*_n) \alpha_n(v, z^*_n) - z^*_n \end{bmatrix}$$

is Jordan. In view of Lemma 3(c), this implies that all $E_m$ with $m > N_*$ are Jordan, i.e., (i) in Example 12 holds.

Proof of (ii). By the proof of (i) we have, for large enough $n$,

$$\gamma_n = 0, \quad \beta_n^-(v; z^*_n) = 0, \quad \frac{1}{2A} \leq |\beta_n^+(v; z^*_n)| \leq 2A.$$

Therefore, by [9 Theorem 37, (7.30)] it follows for $n > N_*$ that

$$\frac{1}{144A} \leq \frac{1}{72} |\beta_n^+(v; z^*_n)| \leq |\mu_n - \lambda_n^+| \leq 58|\beta_n^+(v; z^*_n)| \leq 116A.$$

We set

$$f_n = u_{2n}, \quad \xi_n = \|u_{2n-1}\|^{-1}, \quad \varphi_n = \xi_n \cdot u_{2n-1}.$$  

Then (4.14) takes the form

$$L f_n = \lambda_n^+ f_n, \quad L \varphi_n = \lambda_n^+ \varphi_n + \xi_n \cdot f_n,$$

so now we are using the notations of [9 Lemma 30] (or [4 Lemma 59]) and can apply the related Fundamental Inequalities.

By the inequalities

$$|\mu_n - \lambda_n^+| \leq 4\xi_n + 4|\gamma_n| \quad \xi_n \leq 4|\gamma_n| + 2(|\beta_n^-(v; z^*_n)| + |\beta_n^+(v; z^*_n)|),$$

we have

$$L \varphi_n = \lambda_n^+ \varphi_n + \xi_n \cdot f_n, \quad L f_n = \lambda_n^+ f_n, \quad L \varphi_n = \lambda_n^+ \varphi_n + \xi_n \cdot f_n.$$
(see [4], p. 741; p. 156 in Russian original) it follows, in view of (6.14) and (6.15), that
\[ \xi_n \sim |\mu_n - \lambda_n^+| \sim |\beta_n^+(v; z_n^*|)|. \]
Therefore,
\[ 0 < \inf\{\xi_n\}, \quad \sup\{\xi_n\} < \infty, \]
so the system \(\{u_{2n}, u_{2n-1}, n > N_*\}\) is a Riesz basis in its closed linear span.
This completes the proof of Claim (ii) in Example 12.

7. Fundamental inequalities and criteria for Riesz basis property

1. Now we have to analyze carefully 2D-blocks, \(P_m, E_m = \text{Ran} P_m\) and pairs of root-functions \(\{u_{2m-1}, u_{2m}\}\).

As a matter of fact it has been done – just in the form which perfectly fits to our needs coming from Criterion [3] – in our papers [24, 3, 4, 9]. T. Kappeler and B. Mityagin [24, Theorem 4.5], in the case of Hill operator with \(L^2\)-potential proved the inequality
\[ |\mu - \lambda^+| \leq 2K_{10}(|\xi| + 2|\gamma|) \]
(see notations in (7.4) - (7.9) below). P.Djakov and B. Mityagin [3, Lemma 10, Inc. (4.32)] succeeded to go to the opposite direction and proved the inequality
\[ |\xi| \leq 6|\gamma| + 8|\mu - \lambda^+| \]
(Notice that the constants may change because in [24] and [3] the interval \(I = [0, 1]\), not \([0, \pi]\) as in the present paper.)

All these results are presented in [4] and the proofs are written in the way which covers the case of 1D Dirac operator as well – see Section 4.2 and 4.3 there. Moreover, these proofs could be extended to the case of Hill operators with \(H^{-1}_{\text{per}}\) potentials as soon as we prove (4.4) for the deviations \(P_n - P_0^n\). This is done in [9, Section 9.2, Proposition 44 and Theorem 45] even in a stronger form
\[ \|P_n - P_0^n\|_{L^1 \to L^\infty} \to 0 \quad \text{as} \quad n \to \infty \]
– see [9] (9.7), (9.8), (9.84)]. Analogues of the inequalities (7.1) and (7.2) are inside of the proof of Lemma 30 there.

2. We fix \(m\) to consider \(E = E_m = \text{Ran} P_m, \dim E = 2\), with \(m\) large enough. For a while we suppress an index \(m\) and write
\[ f = u_{2m}, \quad h = u_{2m-1}, \quad \gamma = \lambda_m^+ - \lambda_m^- \neq 0 \]
with
\[ L_{bc} f = \lambda^+ f, \quad L_{bc} h = \lambda^- h, \quad \|f\| = \|h\| = 1 \]
and such a normalization that
\[ h = af + b\varphi, \quad \langle \varphi, f \rangle = 0, \quad a \geq 0, \quad b > 0, \quad a^2 + b^2 = 1. \]
Notice that
\begin{equation}
\langle u_{2m}, u_{2m-1} \rangle = \langle f, h \rangle = a, \quad \kappa := (1 - a^2)^{-1/2} = 1/b.
\end{equation}
Moreover,
\begin{equation}
L_{bc}\varphi = (\lambda^+ - \gamma)\varphi + \xi f, \quad \xi = -\frac{a}{b}\gamma.
\end{equation}
For \( \mu = \mu_m \) put
\begin{equation}
L_{Dir}g = \mu g, \quad \|g\| = 1.
\end{equation}
Then – see [4, formula (4.32)] and the lines which follow – for some \( \tau, \frac{1}{2} \leq |\tau| \), by [4, (4.28)]
\begin{equation}
\tau(\mu - \lambda^+)g = b(\xi P_{Dir}f - \gamma P_{Dir}\varphi).
\end{equation}
Put
\begin{equation}
r = \frac{|\mu - \lambda^+|}{|\lambda^+ - \lambda^-|}, \quad \text{i.e.,} \quad |\gamma| = \frac{1}{r} |\mu - \lambda^+|;
\end{equation}
then
\begin{equation}
\mu - \lambda^+ = \frac{1}{r} b(\xi(\xi P_{Dir}f, g) - \gamma(\xi P_{Dir}\varphi, g))
\end{equation}
and with \( \|P_{Dir}\| \leq 3/2 \) by (4.4) we have
\begin{equation}
|\mu - \lambda^+| \leq 2 \left( \frac{3}{2} |\xi| + \frac{3}{2} \cdot \frac{1}{r} |\mu - \lambda^+| \right).
\end{equation}
If \( r \geq 6 \) it follows that
\begin{equation}
|\mu - \lambda^+| \leq 6 |\xi| = \frac{6a}{b} |\gamma| \leq \frac{6}{b} |\gamma|,
\end{equation}
and
\begin{equation}
r \leq 6\kappa, \quad \kappa \in (7.7).
\end{equation}
If \( r \leq 6 \) of course (7.15) holds because \( \kappa \geq 1 \).
These relations (7.14)–(7.15) hold for any \( m \in M \),
\begin{equation}
M = \{ n : \gamma_n = \lambda^+_n - \lambda^-_n \neq 0, \quad n \geq N_\ast \}.
\end{equation}
For \( \Delta \subset M \) set
\begin{equation}
U_\Delta = \{ u_{2m-1}, u_{2m} : m \in \Delta \}
\end{equation}
and
\begin{equation}
H_\Delta = \text{the closure of Lin Span } U_\Delta.
\end{equation}

**Proposition 22.** If the system \( U_\Delta \) is a basis in \( H_\Delta \) then
\begin{equation}
\kappa(\Delta) = \sup \{ (1 - |\langle u_{2m-1}, u_{2m} \rangle|^2)^{-1/2} : m \in \Delta \} < \infty
\end{equation}
is finite, and
\begin{equation}
R_\Delta = \sup_{m \in \Delta} \frac{|\mu - \lambda^+|}{|\lambda^+ - \lambda^-|} \leq 6\kappa(\Delta) < \infty.
\end{equation}
Proof. With proper adjustments of indexation (see the remark in the first paragraph of Section 4.2) Criterion 9, Formula (3.22), imply that if \( U_\Delta \) is a basis then (7.19) holds. By (7.14)–(7.15) for each individual \( m \in \Delta \)

\[
(7.21) \quad r_m = \frac{|\mu_m - \lambda_m^+|}{|\lambda_m^+ - \lambda_m^-|} \leq 6 \kappa_m.
\]

Taking supremum over \( m \in \Delta \) we get (7.20). \( \square \)

3. Now we want to complement the inequality (7.14)–(7.15) with estimates of \( \kappa = 1/b \) from above in terms of \( r \in (7.11) \) (\( m \) is suppressed). It immediately follows from the inequality

\[
(7.22) \quad |\xi| \leq 8|\gamma| + 36|\mu - \lambda^+| - \text{ see lines after formula (4.59) on p. 745 in [4] (p. 161 in Russian original).}
\]

Indeed with \( \gamma \neq 0 \) (7.22) together with (7.8) and (7.11) imply

\[
(7.23) \quad |\xi| \leq \frac{1}{b} |\gamma| \leq (8 + 36r)|\gamma|
\]

so

\[
b \geq \frac{\sqrt{3}}{2}, \quad \frac{1}{b} \leq \frac{2}{\sqrt{3}} < 2, \quad \text{or} \quad b \leq \frac{\sqrt{3}}{2},
\]

and

\[
(7.24) \quad \frac{1}{2b} \leq \frac{\sqrt{1 - b^2}}{b} \leq 4(2 + 9r);
\]

Therefore, in either case

\[
(7.25) \quad \kappa = \frac{1}{b} \leq 16 + 72r.
\]

With these inequalities Criterion 9, its second part, implies with notations (7.19), (7.20) the following.

**Proposition 23.** If \( R_\Delta < \infty \) then

\[
(7.26) \quad \kappa(\Delta) \leq 16 + 72R_\Delta
\]

and the system \( U_\Delta \) is a Riesz basis in \( H_\Delta \).

Proof. Again, individual inequalities

\[
(7.27) \quad \kappa_m \leq 16 + 72r_m, \quad m \in \Delta
\]

hold by (7.25). With \( R_\Delta \) being finite if we take supremum over \( m \in \Delta \) in (7.25) we get (7.26). Then Criterion 9 claims that \( U_\Delta \) is a Riesz basis in \( H_\Delta \). \( \square \)

4. **Fundamental inequalities** (7.14) and (7.22) for individual \( m \) and Propositions 22 and 23 where a subset \( \Delta \) could be chosen as we wish emphasize that neither Dirichlet eigenvalues \( \mu_m, m \notin \Delta \), nor \( \text{Per}^+ \) or \( \text{Per}^- \) eigenvalues \( \lambda^\pm \) for \( m \notin \Delta \) could have any effect on \( R_\Delta \) or \( \kappa(\Delta) \). In particular, Dirichlet eigenvalues with even (or odd) indices have no effect whatsoever.
when convergence of spectral decompositions related to $\text{Per}^-$ (or $\text{Per}^+$ correspondingly) is considered.

We can combine Propositions 22 and 23 and claim (for all four cases listed in Section 4.2 in the line prior to (4.5)) the following.

**Theorem 24.** Let $L_{\text{Per}^\pm}(v)$ be either the Hill operator with $L^2$ or $H_{\text{per}}^{-1}$-potential $v$ or the Dirac operator with $L^2$-potential $v$, subject to periodic $\text{Per}^+$ or anti-periodic $\text{Per}^-$ boundary conditions. Then the following conditions are equivalent:

1. The system of root functions of $L_{\text{Per}^\pm}(v)$ contains a Riesz basis in $L^2([0, \pi])$ (respectively in $L^2([0, \pi])^2$).

2. The system $\{u_j\}$ defined in (4.19) is a Riesz basis in $L^2([0, \pi])$ (respectively in $(L^2([0, \pi]))^2$).

3. The system $\{u_j\}$ is a basis in $L^2([0, \pi])$ (respectively in $(L^2([0, \pi]))^2$).

4. $\kappa(M) := \sup \{(1 - |\langle u_{2m-1}, u_{2m}\rangle|^2)^{-1/2} : m \in M\} < \infty$.

5. $R(M) := \sup \{|\frac{\mu_m - \lambda^+_m}{\lambda_m - \lambda_m^+}| : m \in M\} < \infty$.

6. The system $\{u_j\}$ is a basis in a separable r.i.f.s. $E$ such that for some $1 < a \leq b < \infty$

   \[ L^a \supset E \supset L^b, \quad \|g\|_{L^a} \leq \|g\|_E \leq \|g\|_{L^b} \quad \forall g \in L^\infty. \]

7. With $\beta^\pm_n(v, z)$ defined in (2.34), and $t_n(z) = |\beta^-(v, z)/\beta^+(v, z)|$

   \[ 0 < \liminf_{n \in M} t_n(z_n^*), \quad \limsup_{n \in M} t_n(z_n^*) < \infty, \]

where $z_n^* = \frac{1}{2}(\lambda^+_n + \lambda^-_n) - n^2$ in the Hill case and $z_n^* = \frac{1}{2}(\lambda^+_n + \lambda^-_n) - n$ in the case of Dirac operators.

(Recall that $\beta^\pm_n(v; z)$ are introduced in Section 2.5, Lemma 3; see their basic properties in Propositions 4 and 5).

*Proof.* The equivalence of Conditions (1) – (5) follows from Propositions 22 and 23 and Corollary 10. Conditions (6) and (7), and their equivalence to (1) – (5) are explained in Sections 5, Theorem 18 and Section 6, Theorem 19. □
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