Deloping of relative exact categories

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Abstract

We introduce a delooping model of relative exact categories. It gives us a condition that the negative $K$-group of a relative exact category becomes trivial.

Keywords: Negative $K$-theory Derived category

Introduction

The negative $K$-theory $K(E)$ for an exact category $E$ is introduced in [Sch04], [Sch06] and [Sch11] by M. Schlichting, and also for a differential graded category and for a stable infinity category is innovated by C. Cisikini and G. Tabuada, A.J. Blumberg and D. Gepner and G. Tabuada in [CT11] and [BGT13] respectively. This generalizes the definition of Bass, Karoubi, Pedersen-Weibel, Thompson, Carter and Yao. The first motivation of our work is to investigate some vanishing conjectures of such negative $K$-groups:

(a) For any noetherian scheme $X$ of Krull dimension $d$, $K_{-n}(X)$ is trivial for $n > d$ ([Wei80]).
(b) The negative $K$-groups of a small abelian category is trivial ([Sch06]).
(c) For a finitely presented group $G$, $K_{-n}(ZG) = 0$ for $n > 1$ ([Hsi84]).

In [Sch06] Corollary 6, it was given a description of $K_{-1}(E)$ and a condition on vanishing of $K_{-1}(E)$ for an (essentially small) exact category $E$ in terms of its unbounder derived category $D(E)$: We have $K_{-1}(E) = K_0(D(E))$ and $K_{-1}(E)$ is trivial if and only if $D(E)$ is idempotent complete (= Karoubian in the sense of [TT90], A.6.1). To extend this observation, we shall introduce the notion of higher derived categories $D_n(E)$ and show the following theorem:

**Theorem 0.1** (A special case of Corollary 3.2). For an exact category $E$, we have $K_{-n}(E) = K_0(D_n(E))$. Moreover, $K_{-n}(E)$ is trivial if and only if $D_n(E)$ is idempotent complete.

Recall that the derived category $D(E)$ is the triangulated category obtained by formally inverting quasi-isomorphisms in the category of chain complexes $Ch(E)$. The pair $(Ch(E), qis)$ of the category of chain complexes on $E$ and the class of all quasi-isomorphisms in $Ch(E)$ forms a complicial Waldhausen exact category. More generally, for a pair $E = (E, w)$ of an exact category $E$ and a class of morphisms $w$ in $E$ which is closed under finite compositions and satisfies the strict axiom (cf. 1.1), we define a class of weak equivalences $qw$ in the category of chain complexes $Ch(E)$, which is called quasi-weak equivalences associated with $w$. If $w$ is the class of isomorphisms in $E$, then $qw$ is just the class of quasi-isomorphisms on $Ch(E)$. The derived category $D(E)$ of $E$ is obtained by formally inverting the quasi-weak equivalences in $Ch(E)$. Put $Ch(E) = (Ch(E), qw)$ and one can define the class of weak equivalences in $Ch_n(E) := Ch(CH_{n-1}(E))$ inductively. The $n$-th derived category $D_n(E)$ associated with $E$, is the derived category of $Ch_n(E)$. We also obtain the following theorems on the negative $K$-theory $K(E)$ for a very strict consistent relative exact category $E$ (for definition, see 1.1 and 1.7):

**Theorem 0.2** (A special case of Theorem 2.6). $K(Ch(E)) \cong \Sigma K(E)$, where $\Sigma$ is a suspension functor on the stable category of spectra.

The organization of this note is as follows: In Section 1 we define the derived categories of $E$ and introduce the notion of quasi-weak equivalences. We will prove Theorems 0.2 and 0.1 in Sections 2 and 3 respectively.
1 Derived categories

1.1 (Relative exact categories). (1) A relative exact category \( \mathcal{E} = (\mathcal{E}, w) \) is a pair of an exact category \( \mathcal{E} \) with a specific zero object \( 0 \) and a class of morphisms in \( \mathcal{E} \) which satisfies the following two axioms.

(Identity axiom). For any object \( x \in \mathcal{E} \), the identity morphism \( \text{id}_x \) is in \( w \).

(Composition closed axiom). For any composable morphisms \( \bullet \xrightarrow{a} \bullet \xrightarrow{b} \bullet \) in \( \mathcal{E} \), if \( a \) and \( b \) are in \( w \), then \( ba \) is also in \( w \).

(2) A relative exact functor between relative exact categories \( f : \mathcal{E} = (\mathcal{E}, w) \rightarrow (\mathcal{F}, v) \) is an exact functor \( f : \mathcal{E} \rightarrow \mathcal{F} \) such that \( f(w) \subset v \) and \( f(0) = 0 \). We denote the category of relative exact categories and relative exact functors by \( \text{RelEx} \).

(3) We write \( \mathcal{E}^w \) for the full subcategory of \( \mathcal{E} \) consisting of those object \( x \) such that the canonical morphism \( 0 \xrightarrow{} x \) is in \( w \). We consider the following axioms.

(Strict axiom). \( \mathcal{E}^w \) is an exact category such that the inclusion functor \( \mathcal{E}^w \rightarrow \mathcal{E} \) is exact and reflects exactness.

(Very strict axiom). \( \mathcal{E} \) satisfies the strict axiom and the inclusion functor \( \mathcal{E}^w \rightarrow \mathcal{E} \) induces a fully faithful functor \( D^b(\mathcal{E}^w) \rightarrow D(\mathcal{E}) \) on the bounded derived categories.

We denote the category of strict (resp. very strict) relative exact categories by \( \text{RelEx}_{\text{strict}} \) (resp. \( \text{RelEx}_{\text{vs}} \)).

(4) A relative natural equivalence \( \theta : f \rightarrow f' \) between relative exact functors \( f, f' : \mathcal{E} = (\mathcal{E}, w) \rightarrow \mathcal{E}' = (\mathcal{E}', w') \) is a natural transformation \( \theta : f \rightarrow f' \) such that \( \theta(x) \) is in \( w' \) for any object \( x \) in \( \mathcal{E} \).

Relative exact functors \( f, f' : \mathcal{E} \rightarrow \mathcal{E}' \) are weakly homotopic if there is a zig-zag sequence of relative natural equivalences connecting \( f \) to \( f' \). A relative functor \( f : \mathcal{E} \rightarrow \mathcal{E}' \) is a homotopy equivalence if there is a relative exact functor \( g : \mathcal{E}' \rightarrow \mathcal{E} \) such that \( gf \) and \( fg \) are weakly homotopic to identity functors respectively.

(5) A functor \( F \) from a full subcategory \( \mathcal{R} \) of \( \text{RelEx} \) to a category \( \mathcal{C} \) is categorical homotopy invariant if for any relative exact functors \( f, f' : \mathcal{E} \rightarrow \mathcal{E}' \) such that \( f \) and \( f' \) are weakly homotopic, we have the equality \( F(f) = F(f') \).

Proposition 1.2. Let \( f : \mathcal{F} \rightarrow \mathcal{G} \) be an exact functor between exact categories. If the induced functor \( D^b_f : D^b(\mathcal{F}) \rightarrow D^b(\mathcal{G}) \) is fully faithful for some \# \in \{b, \pm, \text{nothing}\}, then \( D^b(\mathcal{F}) \) is fully faithful for any \# \in \{b, \pm, \text{nothing}\}.

Proof. Since we have the fully faithful embeddings \( D^b(\mathcal{F}) \rightarrow D^b(\mathcal{F}) \) and \( D^b(\mathcal{F}) \rightarrow D(\mathcal{F}) \) and the equality \( D^+(\mathcal{F}) = (D^+ \mathcal{F})^{\text{op}} \), we only need to check that if \( D^b(f) \) (resp. \( D^b(f) \)) is fully faithful, then \( D^+(f) \) (resp. \( D(f) \)) is also. For any objects \( x \) and \( y \) in \( \text{Ch}_-(\mathcal{F}) \) (resp. \( \text{Ch}_+(\mathcal{F}) \)), there are sequences \( a = \{a_0 \rightarrow a_1 \rightarrow a_2 \rightarrow \cdots \} \) and \( b = \{b_0 \rightarrow b_1 \rightarrow b_2 \rightarrow \cdots \} \) of inflations in \( \text{Ch}_-(\mathcal{F}) \) (resp. \( \text{Ch}_+(\mathcal{F}) \)) such that \( x \xrightarrow{a} y \xrightarrow{b} \) in \( T(\mathfrak{S} \text{Ch}_-(\mathcal{F}), \text{qis}) \) (resp. \( T(\mathfrak{S} \text{Ch}_+(\mathcal{F}), \text{qis}) \)). Since we have the fully faithful embeddings \( D^b(\mathcal{F}) \rightarrow T(\mathfrak{S} \text{Ch}_-(\mathcal{F}), \text{qis}) \) and \( T(\mathfrak{S} \text{Ch}_+(\mathcal{F}), \text{qis}) \) we have the natural isomorphism \( \text{Hom}_{D^b(\mathcal{F})}(x, y) \xrightarrow{\sim} \text{Hom}_{T(\mathfrak{S} \text{Ch}_ -(\mathcal{F}), \text{qis})}(x, y) \). On the other hand, Proposition 1 and Theorem 3, we regard both \( x \) and \( y \) as objects in \( T(\mathfrak{S} \text{Ch}_+(\mathcal{F}), \text{qis}) \) we have the natural isomorphism \( \text{Hom}_{D^b(\mathcal{F})}(x, y) \xrightarrow{\sim} \text{Hom}_{T(\mathfrak{S} \text{Ch}_+(\mathcal{F}), \text{qis})}(x, y) \).
other hand, the induced functor \( T(\mathfrak{g} \text{Ch}^\#(f), \text{qis}) : T(\mathfrak{g} \text{Ch}^\#(\mathcal{F}), \text{qis}) \to T(\mathfrak{g} \text{Ch}^\#(\mathcal{G}), \text{qis}) \) where \( \# = b \) (resp. \( \# = + \)) is fully faithful by \cite{Sch06} Corollary 2 and Proposition 1. Hence we obtain the result.

1.3 (Derived category). We define the derived categories of a strict relative exact category \( E = (\mathcal{E}, w) \) by the following formula

\[
D^\#(E) := \text{Coker}(D^\#(\mathcal{E}^w) \to D^\#(\mathcal{E}))
\]

where \( \# = b, \pm \) or nothing. Namely \( D^\#(E) \) is a Verdier quotient of \( D^\#(\mathcal{E}) \) by the thick subcategory of \( D^\#(\mathcal{E}) \) spanned by the complexes in \( \text{Ch}^\#(\mathcal{E}^w) \).

Definition 1.4 (Exact sequence). A sequence \( E \to F \to G \) of strict relative exact categories is exact if the induced sequence of triangulated categories \( D^b(E) \to D^b(F) \to D^b(G) \) is exact.

We sometimes denote the sequence above by \((u, v)\). For a full subcategory \( \mathcal{R} \) of \( \text{RelEx}_{\text{strict}} \), we let \( E(\mathcal{R}) \) denote the category of exact sequences in \( \mathcal{R} \). We define three functors \( s^R, m^R \) and \( q^R \) from \( E(\mathcal{R}) \) to \( \mathcal{R} \) which sends an exact sequence \( E \to F \to G \) to \( E, F \) and \( G \) respectively.

1.5 (Quasi-weak equivalences). Let \( F^\# : \text{Ch}^\#(\mathcal{E}) \to D^\#(E) \) be the canonical quotient functor. We denote the pull-back of the class of all isomorphisms in \( D^\#(E) \) by \( \text{qW}^\# \) or simply \( \text{qW} \). We call a morphism in \( \text{qW} \) a quasi-weak equivalence. We write \( \text{Ch}^\#(E) \) for a pair \((\text{Ch}^\#(\mathcal{E}), \text{qW})\). We can easily prove that \( \text{Ch}^\#(E) \) is a complicial biWaldhausen category in the sense of \cite{T90}. In particular, it is a relative exact category. The functor \( F^\# \) induces an equivalence of triangulated categories \( T(\text{Ch}^\#(\mathcal{E}), \text{qW}) \to D^\#(E) \) (See \cite{Sch11} 3.2.17). If \( w \) is the class of all isomorphisms in \( \mathcal{E} \), then \( \text{qW} \) is just the class of all quasi-isomorphisms in \( \text{Ch}^\#(\mathcal{E}) \) and we denote it by \( \text{qis} \).

Corollary 1.6. Let \( E = (\mathcal{E}, w) \) be a very strict relative exact category and \( \# \in \{b, \pm, \text{nothing}\} \). Then

1. The inclusion functor \( \mathcal{E}^w \to \mathcal{E} \) induces a fully faithful embedding \( D^\#(\mathcal{E}^w) \to D^\#(\mathcal{E}) \).
2. The inclusion functor \( \text{Ch}^\#(\mathcal{E}^w) \to \text{Ch}^\#(\mathcal{E}) \) and the identity functor on \( \text{Ch}^\#(\mathcal{E}) \) induce an exact sequence of relative exact categories \( (\text{Ch}^\#(\mathcal{E}^w), \text{qW}) \to (\text{Ch}^\#(\mathcal{E}), \text{qW}) \to (\text{Ch}^\#(\mathcal{E}), \text{qis}) \to (\text{Ch}^\#(\mathcal{E}), \text{qis}) \to (\text{Ch}^\#(\mathcal{E}), \text{qis}) \).

1.7 (Consistent axiom). Let \( E = (\mathcal{E}, w) \) be a strict relative exact category. There exists the canonical functor \( i_E^\# : \mathcal{E} \to \text{Ch}^\#(\mathcal{E}) \) where \( i_E^\#(x)_k = x \) if \( k = 0 \) and \( 0 \) if \( k \neq 0 \). We say that \( w \) (or \( E \)) satisfies the consistent axiom if \( i_E^\#(w) \subset \text{qW} \). We denote the full subcategory of consistent relative exact categories in \( \text{RelEx} \) by \( \text{RelEx}_{\text{consist}} \).

Examples 1.8. (cf. \cite{Moc13}). (1) A pair \((\mathcal{E}, i_\mathcal{E})\) of an exact category \( \mathcal{E} \) with the class of all isomorphisms \( i_\mathcal{E} \) is a very strict consistent relative exact category.

2. In particular we denote the trivial exact category by \( 0 \) and we also write \((0, i_0)\) for \( 0 \) is the zero objects in the category of consistent relative exact categories.

3. A complicial exact category with weak equivalences in the sense of \cite{Sch11} 3.2.9] is a consistent relative exact category. In particular for any relative exact category \( E \), \( \text{Ch}^\#(E) \) is a very strict consistent relative exact category.

Theorem 1.9 (Derived Gillet-Waldhausen theorem). (cf. \cite{Moc13} 4.15). Let \( E \) be a consistent relative exact category. Then

1. The canonical functor \( i_{\text{Ch}^\#(\mathcal{E})} : \text{Ch}^\#(\mathcal{E}) \to \text{Ch}^b(\text{Ch}^\#(\mathcal{E})) \) induces an equivalence of triangulated categories \( D^\#(E) \to T(\text{Ch}^\#(\mathcal{E}), \text{qW}) \to D^b(\text{Ch}^\#(\mathcal{E})) \).
2. In particular, the canonical functor \( i_E^\# : \mathcal{E} \to \text{Ch}^b(\mathcal{E}) \) induces an equivalence of triangulated categories \( D^b(E) \to D^b(\text{Ch}^b(\mathcal{E})) \).

Definition 1.10 (Quotient of consistent relative exact categories). For any fully faithful relative functor \( f : E = (\mathcal{E}, w) \to F = (\mathcal{F}, v) \) between consistent relative exact categories such that induced functor \( D^b(f) \) is also fully faithful, we define a quotient \( F / E := (\text{Ch}^b(F), v/w) \) of \( F \) by \( E \) (along \( f \)) as follows. There exists a canonical quotient morphism \( P_{F/E} : \text{Ch}^b(F) \to D^b(F) / D^b(E) \). We write \( v/w \) for the pull back of all isomorphisms in \( D^b(F) / D^b(E) \) by \( P_{F/E} \). We put \( F / E := (\text{Ch}^b(F), v/w) \). \( F / E \) is again a consistent relative exact category by \cite{Moc13} 4. Let us have the canonical relative functor \( i_F^E : F \to F / E \).
2 Localizing theory

In this section, we will prove Theorem 2.2

2.1. Let $E = (\mathcal{E}, w)$ be a relative exact category. We denote the exact category of admissible short exact sequences in $\mathcal{E}$ by $E(\mathcal{E})$. There exist three exact functors $s$, $t$ and $q$ from $E(\mathcal{E}) \to \mathcal{E}$ which send an admissible exact sequence $x \twoheadrightarrow y \twoheadrightarrow z \to x$, $y$ and $z$ respectively. We write $w_E[1]$, for the class of morphisms $s^{-1}(w) \cap t^{-1}(w) \cap q^{-1}(w)$ and put $E(\mathcal{E}) := (E(\mathcal{E}), w_E[1])$. We can easily prove that $E(\mathcal{E})$ is a relative exact category and the functors $s$, $t$ and $q$ are relative exact functors from $E(\mathcal{E})$ to $E$. Moreover we can easily prove that if $E$ is consistent, then $E(\mathcal{E})$ is also consistent.

Now we give a definition of additive theories which is slightly different from [Moc13, 6.9, 7.8].

**Definition 2.2 (Additive theory).** (1) A full subcategory $\mathcal{R}$ of RelEx is closed under extensions if $\mathcal{R}$ contains the trivial relative exact category $0$ and if for any $E$ in $\mathcal{R}$, $E(\mathcal{E})$ is also in $\mathcal{R}$.

(2) Let $F$ be a functor from a full subcategory $\mathcal{R}$ of RelEx closed under extensions to an additive category $\mathcal{B}$. We say that $F$ is an additive theory if for any relative exact category $E$ in $\mathcal{R}$, the following projection is an isomorphism

$$
\begin{pmatrix}
F(s) \\
F(q)
\end{pmatrix} : F(E(\mathcal{E})) \to F(E) \oplus F(E).
$$

**Lemma 2.3 (Eilenberg Swindle).** Let $\mathcal{R}$ be a full subcategory of RelEx strict closed under extensions, $F$ a categorical homotopy invariant additive theory from $\mathcal{R}$ to an additive category $\mathcal{B}$ and $E$ a strict relative exact category in $\mathcal{R}$. We assume that $\text{Ch}^+ E$ (resp. $\text{Ch}^- E$) is also in $\mathcal{R}$. Then $F(\text{Ch}^+ E)$ (resp. $F(\text{Ch}^- E)$) is trivial.

**Proof.** We only give a proof for $\text{Ch}^+ E$. We denote $f : \text{Ch}^+ E \to \text{Ch}^+ E$ to be a relative exact functor by sending an object $x$ to $\bigoplus_{n \geq 0} x[2n]$. Then we have the equality $F(f[2]) + F(\text{id}_{\text{Ch}^+ E}) = F(f)$ and $F(f[2]) = F(f)$ by [Moc13] Proposition 7.9. Hence we obtain the result. \(\Box\)

**Definition 2.4 (Localization theory).** A localizing theory $(F, \partial)$ from a full subcategory $\mathcal{R}$ of RelEx strict to a triangulated category $(\mathcal{T}, \Sigma)$ is a pair of functor $F : \mathcal{R} \to \mathcal{T}$ and a natural transformation $\partial : Fq \to \Sigma Fs$ between functors $E(\mathcal{R}) \rightarrow \mathcal{R} \rightarrow \mathcal{T}$ which sends a exact sequence $E \xrightarrow{f} F \xrightarrow{g} G$ in $\mathcal{R}$ to a distinguished triangle $F(E) \xrightarrow{F(i)} F(F) \xrightarrow{F(g)} F(G) \xrightarrow{\partial(f,g)} \Sigma F(E)$ in $\mathcal{T}$.

**Remark 2.5.** (1) The non-connective $K$-theory on RelEx consist studied in [Moc13] is a categorical homotopy invariant localization theory.

(2) (cf. [Moc13, 7.9]). Let $F$ be a localization theory on a full subcategory $\mathcal{R}$. Then

(i) $F$ is a derived invariant functor. Namely if a morphism $\mathcal{E} \to \mathcal{F}$ in $\mathcal{R}$ induces an equivalence of triangulated categories $D^b \mathcal{E} \to D^b \mathcal{F}$, then the induced morphism $F(\mathcal{E}) \to F(\mathcal{F})$ is an isomorphism. In particular if $\iota^b : \mathcal{E} \to \text{Ch}^b \mathcal{E}$ is in $\mathcal{R}$, then $F(\iota^b)$ is an isomorphism.

(ii) If further we assume that $\mathcal{R}$ is closed under extensions and if $F$ is categorical homotopy invariant, then we can easily prove that $F$ is an additive theory.

**Theorem 2.6.** Let $(F, \partial)$ be a categorical homotopy invariant localization theory from a full subcategory $\mathcal{R}$ closed under extensions to a triangulated category $(\mathcal{T}, \Sigma)$, $E$ a very strict relative exact category in $\mathcal{R}$. Assume that $\text{Ch}^# F$ is also in $\mathcal{R}$ for any $\# \in \{b, \pm, \text{nothing}\}$ and for any $\mathcal{F} \in \{\mathcal{E}, (\mathcal{E}, \iota_\mathcal{E}), (\mathcal{E}^w, \iota_{\mathcal{E}^w})\}$. Then there is an isomorphism $F(\mathcal{F} \mathcal{E}) \to \Sigma F(\mathcal{F} \mathcal{E})$. In particular if further we assume that $E$ is consistent, then we have an isomorphism $F(\mathcal{F} \mathcal{E}) \to \Sigma F(\mathcal{F} \mathcal{E})$.

**Proof.** First assume that $w$ is the class of all isomorphisms in $\mathcal{E}$. Then the fully faithful embeddings $D^b(\mathcal{E}) \to D^b(\mathcal{E})$ and $D^b(\mathcal{E}) \to D(\mathcal{E})$ yield the commutative diagram of distinguished triangles

$$
\begin{align*}
F(\text{Ch}^b \mathcal{E}, \text{qis}) & \to F(\text{Ch}^+ \mathcal{E}, \text{qis}) \to F((\text{Ch}^+ \mathcal{E}, \text{qis})/(\text{Ch}^b \mathcal{E}, \text{qis})) \to \Sigma F(\text{Ch}^b \mathcal{E}, \text{qis}) \\
\quad \downarrow \quad & \quad \downarrow \quad & \quad \downarrow \quad \\
F(\text{Ch}^- \mathcal{E}, \text{qis}) & \to F(\text{Ch}^- \mathcal{E}, \text{qis}) \to F((\text{Ch}^- \mathcal{E}, \text{qis})/(\text{Ch}^- \mathcal{E}, \text{qis})) \to \Sigma F(\text{Ch}^- \mathcal{E}, \text{qis}).
\end{align*}
$$
Here the morphisms \( I \) and \( II \) are isomorphisms by triviality of \( F(\text{Ch}^b\mathcal{E}, \text{qis}) \) and the morphism \( III \) is also an isomorphism by [Moc13] Proposition 7.10 (2). We denote the compositions of the morphisms \( I \) and the inverse of \( III \) and \( II \) by \( \Delta_{\mathcal{E}} : F(\text{Ch}\mathcal{E}, \text{qis}) \to \Sigma F(\text{Ch}^b\mathcal{E}, \text{qis}) \). Then \( \Delta_{\mathcal{E}} \) is functorial on \( \mathcal{E} \).

Next we consider the general case. By virtue of Corollary 1.6 (2), there is a commutative diagram of distinguished triangles

\[
\begin{array}{ccc}
F(\text{Ch}\mathcal{E}^w, \text{qis}) & \longrightarrow & F(\text{Ch}\mathcal{E}, \text{qis}) & \longrightarrow & F(\text{Ch}\mathcal{E}) & \longrightarrow & \Sigma F(\text{Ch}\mathcal{E}^w, \text{qis}) \\
\Delta_{\mathcal{E}^w} & \downarrow & \Delta_{\mathcal{E}} & \downarrow & \Sigma \Delta_{\mathcal{E}^w} \\
\Sigma F(\text{Ch}^b\mathcal{E}^w, \text{qis}) & \longrightarrow & \Sigma F(\text{Ch}^b\mathcal{E}, \text{qis}) & \longrightarrow & \Sigma F(\text{Ch}^b\mathcal{E}) & \longrightarrow & \Sigma^2 F(\text{Ch}^b\mathcal{E}^w, \text{qis}).
\end{array}
\]

Then there exists the dotted morphism \( IV \) in the diagram above which makes the diagram commutative and it is an isomorphism by 5-lemma.

**Remark 2.7.** The full subcategory \( \text{Re} I \text{Ex}_{\text{consist}} \) satisfies the assumption in Theorem 2.6 by [Moc13]. In particular we obtain Theorem 0.2.

### 3 Higher derived categories

In this section, we assume that \( E \) is a very strict consistent relative exact category.

**3.1 (Higher derived categories).** Let us denote \( n \)-th times iteration of \( \text{Ch} \) for \( E \) by \( \Sigma^n E \) and \( D_n(E) := D^b(\Sigma^n E) \) the \( n \)-th **higher derived category** of \( E \). Then

1. \( D_0(E) \) is just the usual bounded derived category \( D^b(E) \) of \( E \).
2. For any positive integer \( n \), \( D_n(E) \) is the unbounded derived category \( D(\Sigma^{n-1} E) \) of \( \Sigma^{n-1} E \) by Theorem 1.9 (1). In particular, \( D_1(E) \) is just the unbounded derived category \( D(E) \) of \( E \).

As the following corollary, we can consider the negative \( K \)-groups as an obstruction group of idempotent completeness of the higher derived categories:

**Corollary 3.2.** For any positive integer \( n \), we have

1. \( K_{-n}(E) \simeq K_0(D_n(E)) \).
2. \( K_{-n}(E) \) is trivial if and only if \( D_n(E) \) is idempotent complete.

**Proof.** By Theorem 0.2 we have \( K_{-n}(E) \simeq K_0(\text{Ch}_n(E)) \simeq K_0(D_n(E)) \). Then Proposition 3.3 below leads the desired assertion.

**Proposition 3.3.** (1) For an essentially small triangulated category \( T \), if \( K_0(T) = K_0(T^\sim) \) is trivial, then \( T \) is idempotent complete.

(2) The derived category \( D(E) \) is idempotent complete if and only if the Grothendieck group \( K_0(D(E)) \simeq K_0(D(E)^\sim) \) is trivial.

**Proof.** (1) Since the map \( K_0(T) \to K_0(T^\sim) \) is injective by [Tho97] Corollary 2.3, now \( K_0(T) \) is also trivial. Applying the Thomason classification theorem of (strictly) dense triangulated subcategories in essentially small triangulated categories [Tho97] Theorem 2.1 for \( T^\sim \), the inclusion functor \( T \to T^\sim \) must be an equivalence.

(2) We have the equalities \( K_0(\text{Ch}(E^w), \text{qis}) = K_0(\text{Ch}(E), \text{qis}) = 0 \) as in the proof of Collorary 6 in [Sch06]. Therefore \( K_0(D(E)) = K_0(\text{Ch}(E)) = 0 \) by the canonical fibration sequence associated to the exact sequence in Corollary 1.6 (2) for \( \text{Ch} \ E \). If \( D(E) \) is idempotent complete, that is, \( D(E) \to D(E)^\sim \), then we have \( K_0(D(E)) = K_0(D(E)^\sim) = K_0(D(E)) = 0 \). The converse is followed from (1).
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