Stokes Structure and Direct Image of Irregular Singular $\mathcal{D}$-Modules

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Abstract: In this paper we will present a way of examining the Stokes structure of certain irregular singular $\mathcal{D}$-modules, namely the direct image of exponentially twisted regular singular meromorphic connections, in a topological point of view. This topological description enables us to compute Stokes data for an explicit example concretely.

1 Introduction – Preliminaries

Recently there was a lot of progress in proving an irregular Riemann-Hilbert correspondence in higher dimensions. Sabbah introduced the notion of good meromorphic connections and proved an equivalence of categories in this case \cite{Sab13}, which was extended to the general case by Mochizuki \cite{Moc09} and Kedlaya \cite{Ked10}. Furthermore d’Agnolo/Kashiwara proved an irregular Riemann-Hilbert correspondence for all dimensions using subanalytic sheaves \cite{DK}.

Nevertheless it is still difficult to describe the Stokes phenomenon for explicit situations and to calculate Stokes data concretely. In their recent article, Hien/Sabbah developed a topological way to determine Stokes data of the Laplace transform of an elementary meromorphic connection \cite{HS}. The techniques used by Hien/Sabbah can be adapted to other situations. Hence in this article we will present a topological view of the Stokes phenomenon for the direct image of an exponentially twisted meromorphic connection $\mathcal{M}$ in a 2-dimensional complex manifold. Namely we will consider the following situation:

Let $X = \Delta \times \mathbb{P}^1$ be a complex manifold, where $\Delta$ denotes an open disc in $0 \in \mathbb{C}$ with coordinate $t$. We denote the coordinate of $\mathbb{P}^1$ in 0 by $x$ and the coordinate in $\infty$ by $y = \frac{1}{x}$. Let $\mathcal{M}$ be a regular singular holonomic $\mathcal{D}_X$-module. We have the following projections:
Let $\mathcal{D}$ denote the singular locus of $\mathcal{M}$, which consists of $\{0\} \times \mathbb{P}^1 =: D$, $\Delta \times \{\infty\}$ and some additional components. We will distinguish between the components

- $S_{i \in I}$ ($I = \{1, \ldots, n\}$), which meet $D$ in the point $(0, \infty)$ and
- $\tilde{S}_{j \in J}$ ($J = \{1, \ldots, m\}$), which meet $\{0\} \times \mathbb{P}^1$ in some other point.

Furthermore we will require the following conditions on $\mathcal{D}$:

**Assumption 1.1:** Locally in $(0, \infty)$ the irreducible components $S_i$ of the divisor $\mathcal{D}$ achieve the following conditions:

- $S_i : \mu_i(t) y = t^{q_i}$, where $\mu_i$ is holomorphic and $\mu_i(0) \neq 0$.
- For $i \neq j$ either $q_i \neq q_j$ or $\mu_i(0) \neq \mu_j(0)$ holds.

**Assumption 1.2:** The irreducible components $\tilde{S}_j$ intersect $D$ in pairwise distinct points. Moreover we assume $\tilde{S}_j$ to be smooth, i.e. locally around the intersection point they can be described as

$$\tilde{S}_j : \mu_j(t) x = t^{q_j}.$$ 

We want to examine $p_+ (\mathcal{M} \otimes \mathcal{E}^q)$. This is a complex with $\mathcal{H}^k p_+ (\mathcal{M} \otimes \mathcal{E}^q) = 0$ for $k \neq -1, 0$. Furthermore one can show that even $\mathcal{H}^{-1} p_+ (\mathcal{M} \otimes \mathcal{E}^q) = 0$ on $\Delta^*$ (cf. [Sab08], p. 161), i.e. it is only supported in 0. Therefore, we will consider $\mathcal{H}^0 p_+ (\mathcal{M} \otimes \mathcal{E}^q)$. We will assume $\Delta$ small enough such that 0 is the only singularity of the $\mathcal{D}_{\Delta}$-module $\mathcal{H}^0 p_+ (\mathcal{M} \otimes \mathcal{E}^q)$ and we denote its germ at 0 by

$$\mathcal{N} := \left(\mathcal{H}^0 p_+ (\mathcal{M} \otimes \mathcal{E}^q)\right)_0.$$ 

In the following we will take a closer look at the Stokes-filtered local system $(\mathcal{L}, \mathcal{L}_{\leq \psi})$ (Chapter 2), which is associated to the $\mathcal{D}_{\Delta}$-module $\mathcal{N}$ by the irregular Riemann-Hilbert correspondence.
as mentioned above. We will use an isomorphism
\[ \Omega : \mathcal{L}_{\leq \psi} \overset{\cong}{\to} \mathcal{H}^1 R\tilde{\varphi}_* \text{DR}^\text{modD} \left( \mathcal{M} \otimes \mathcal{E}^+ \otimes \mathcal{E}^{-\psi} \right) \]
(proved by Mochizuki) to develop a topological description for $\mathcal{L}_{\leq \psi}$.

In Chapter 3 we will use this topological perspective to present a way of determining Stokes matrices for an explicit example, where the singular locus of a meromorphic connection $\mathcal{M}$ of rank $r$ only consists of two additional irreducible components, namely $(S_1 : y = t)$ and $(\tilde{S}_1 : x = 0)$. We will describe the Stokes-filtered local system $\mathcal{L}$ (which in this case will be of exponential type) in terms of linear data, namely a set of linear Stokes data of exponential type, defined as follows:

Let $\Phi = \{ \phi_i \mid i \in I \}$ denote a finite set of exponents $\phi_i$ of pole order $\leq 1$ and let $\theta_0 \in \mathbb{S}^1$ be a generic angle, i.e. it is no Stokes direction with respect to the $\phi_i$s. We get a unique ordering of the exponents $\phi_0 < \theta_0 < \phi_1 < \theta_0 < \phi_2 < \theta_0 < \cdots < \theta_0 < \phi_n$ and the reversed ordering for $\theta_1 := \theta_0 + \pi$.

**Definition 1.3** ([HS11], Def 2.6): The category of Stokes data of exponential type (for a set of exponents $\Phi$ ordered by $\theta_0$) has objects consisting of two families of $\mathbb{C}$-vector spaces $(G_{\phi_i}, H_{\phi_i})$ and two morphisms

\[ \bigoplus_{i=0}^{n} G_{\phi_i} \xrightarrow{S} \bigoplus_{i=0}^{n} H_{\phi_i}, \quad \bigoplus_{i=0}^{n} H_{\phi_{n-i}} \xrightarrow{S'} \bigoplus_{i=0}^{n} G_{\phi_{n-i}} \]

such that

1. $S$ is a block upper triangular matrix, i.e. $S_{ij} : G_{\phi_i} \to H_{\phi_j}$ is zero for $i > j$ and $S_{ii}$ is invertible (thus $S$ is invertible and $\dim G_{\phi_i} = \dim H_{\phi_i}$)

2. $S'$ is a block lower triangular matrix, i.e. $S'_{ij} : H_{\phi_{n-i}} \to G_{\phi_{n-j}}$ is zero for $i < j$ and $S'_{ii}$ is invertible (thus $S'$ is invertible)

A morphism consists of morphisms of $\mathbb{C}$-vector spaces $\chi^G_i : G_{\phi_i} \to G'_{\phi_i}$ and $\chi^H_i : H_{\phi_i} \to H'_{\phi_i}$, which are compatible with the corresponding diagrams.

The correspondence between Stokes-filtered local systems of exponential type and linear Stokes data is stated in the following theorem. For a proof we refer to [HS11], p. 12/13.

**Theorem 1.4:** There is an equivalence of categories between the Stokes-filtered local systems of exponential type and Stokes data of exponential type.

By associating the Stokes-filtered local system to a set of Stokes data via this equivalence of categories and using the isomorphism $\Omega$ we will finally get an explicit description of the Stokes
data in our concrete example. It is stated in the following

**Theorem 1.5:** Fix the following data:

- \( L_0 = \mathbb{V} \oplus \mathbb{V} \), \( L_1 = \mathbb{V} \oplus \mathbb{V} \)
- \( S_0^1 = N_n = \begin{pmatrix} -1 & 1 - ST^{-1} \\ 0 & -ST^{-1} \end{pmatrix} \), \( S_1^0 = (\mu_n^0 \circ \mu_n^0) \cdot N_0 = \begin{pmatrix} U & 0 \\ 0 & U \end{pmatrix} \cdot \begin{pmatrix} -TS^{-1} & 0 \\ 1 - TS^{-1} & -1 \end{pmatrix} \)

where \( \mathbb{V} \) is the generic stalk of the local system attached to \( \mathcal{M} \) and \( S, T \) denote the monodromies around the strict transforms of the irreducible components \( \tilde{S}_1, S_1 \) in the singular locus of \( \mathcal{M} \) and \( U \) denotes the monodromy around the component \( \{0\} \times \mathbb{P}^1 \). Then

\[
\left( L_0, L_1, S_0^1, S_1^0 \right)
\]
defines a set of Stokes data for \( \mathcal{H}^0 p_+ \left( \mathcal{M} \otimes \mathbb{E}^\mathbb{V} \right) \).

## 2 Stokes-filtered local system \((\mathcal{L}, \mathcal{L}_{\leq \psi})\)

In [Rou07] Roucairol examined the formal decomposition of direct images of \( \mathcal{D} \)-modules in the situation presented above. She determined the exponential factors as well as the rank of the corresponding regular parts appearing in the formal decomposition of \( \mathcal{N} \). In our case this leads to the following result:

**Theorem 2.1** ([Rou07], Thm 1): Let \( \mathcal{M} \) be a regular singular \( \mathcal{D}_X \)-module with singular locus \( \overline{D} \) which achieves the previous assumptions \( \text{[1.2]} \) and \( \text{[1.3]} \). Then \( \mathcal{N} := \left( \mathcal{H}^0 p_+ \left( \mathcal{M} \otimes \mathbb{E}^\mathbb{V} \right) \right)^\wedge \) decomposes as

\[
\mathcal{N} = R_0 \oplus \bigoplus_{i \in I} \left( R_i \otimes \mathbb{E}^{\psi_i(t)} \right),
\]

where \( R_0, R_i \ (i \in I) \) are regular singular \( \mathcal{D}_{\Delta} \)-modules and \( \psi_i(t) = \mu_i(t) t^{-\eta_i} \). Moreover we have:

- \( \text{rk} \ (R_i) = \dim \Phi_{P_i} \), where \( \Phi_{P_i} \) denotes the vanishing cycles of \( \text{DR}(e^+ \mathcal{M}) \) at the intersection point \( P_i \) of (a strict transform of) \( S_i \) with the exceptional divisor after a suitable blow up \( e \).
- \( \text{rk} \ (R_0) = \sum_{j \in J} \dim \Phi_{P_j} \), where \( \Phi_{P_j} \) denotes the vanishing cycles of \( \text{DR}(\mathcal{M}) \) at the intersection point \( P_j \) of \( \tilde{S}_j \) with \( D \).

**Proof:** The formal decomposition and the statement about the rank of the \( R_i \) is exactly Roucairol’s theorem applied to our given situation. With the same arguments as in Roucairol’s proof one can also show that \( \text{rk} \ (R_0) = \sum_{j \in J} \dim \Phi_{P_j} \). \( \square \)
Identify $S^1 = \{ \vartheta \mid \vartheta \in [0, 2\pi) \}$ and denote $\mathcal{P} := x^{-1}\mathbb{C}[x^{-1}]$. Let us recall the following definition of a Stokes-filtered local system.

**Definition 2.2** ([Sab13], Lemma 2.7): Let $\mathcal{L}$ be a local system of $\mathbb{C}$-vector spaces on $S^1$. We will call $(\mathcal{L}, \mathcal{L}_\leq)$ a Stokes-filtered local system, if it is equipped with a family of subsheaves $\mathcal{L}_{\leq \phi}$ (indexed by $\phi \in \mathcal{P} := x^{-1}\mathbb{C}[x^{-1}]$) satisfying the following conditions:

1. For all $\vartheta \in S^1$, the germs $\mathcal{L}_{\leq \phi, \vartheta}$ form an exhaustive increasing filtration of $\mathcal{L}_{\vartheta}$
2. $gr_\vartheta \mathcal{L} := \mathcal{L}_{\leq \phi}/\mathcal{L}_{< \phi}$ is a local system on $S^1$ (where $\mathcal{L}_{< \phi, \vartheta} := \sum_{\psi < \vartheta} \mathcal{L}_{\leq \psi, \vartheta}$)
3. $\dim \mathcal{L}_{\leq \phi, \vartheta} = \sum_{\psi \leq \vartheta} \dim gr_\psi \mathcal{L}_{\vartheta}$

Assume that $\mathcal{M}$ (and therefore the direct image $\mathcal{N}$ defined above) is a meromorphic connection. Let $\mathcal{L}'$ denote the local system on $\Delta^*$ corresponding to the meromorphic connection $\mathcal{N}$. Moreover let $\pi : \tilde{\Delta} \to \Delta, (r, e^{i\vartheta}) \mapsto r \cdot e^{i\vartheta}$ be the real oriented blow up of $\Delta$ in the singularity 0 and $j : \Delta^* \hookrightarrow \tilde{\Delta}$. Consider $j_*\mathcal{L}'$ and restrict it to the boundary $\partial \Delta$. We get a local system on $S^1 \cong \partial \tilde{\Delta}$ and define $\bar{\mathcal{L}} := j_*\mathcal{L}'|_{\partial \tilde{\Delta}}$. For $\phi \in \mathcal{P}$ define

$$\mathcal{L}_{\leq \phi} := \mathcal{H}^0 \text{DR}^{mod \geq 0}_{\partial \tilde{\Delta}}(\mathcal{N} \otimes \mathcal{E}^{-\phi}) \quad \text{and} \quad \mathcal{L}_{< \phi} := \mathcal{H}^0 \text{DR}^{< 0}_{\partial \tilde{\Delta}}(\mathcal{N} \otimes \mathcal{E}^{-\phi}),$$

where $\text{DR}^{mod \geq 0}$ resp. $\text{DR}^{< 0}$ denotes the moderate (resp. rapid decay) de Rham complex. According to the equivalence of categories between germs of a $\mathcal{O}_\Delta(\ast 0)$-connection and Stokes-filtered local systems stated by Deligne/Malgrange (cf. [Mal91]), $(\mathcal{L}, \mathcal{L}_\leq)$ forms the Stokes-filtered local system associated to $\mathcal{N}$.

Moreover by the Hukuhara-Turrittin-Theorem (cf. [Sab07], p. 109) the formal decomposition of $\mathcal{N}$ can be lifted locally on sectors to $\partial \tilde{\Delta} = S^1$. Thus to determine the filtration $\mathcal{L}_\leq$, it is enough to consider the set of exponential factors (respectively their polar part) appearing in the formal decomposition, since the moderate growth property of the solutions of an elementary connection $\mathcal{R} \otimes \mathcal{E}^\phi (\phi \in \mathcal{P})$ only depends on the asymptotical behavior of $e^\phi$. We denote the set of exponential factors of the formal decomposition of $\mathcal{N}$ by $\mathcal{P}_\mathcal{N} := \{ \psi_i \mid i \in I_0 \}$ whereby $I_0 := I \cup \{ 0 \}$ and $\psi_0 := 0$.

**Definition 2.3:** For $\vartheta \in S^1$ we define the following ordering on $\mathcal{P}$: $\phi \leq_\vartheta \psi :\Leftrightarrow e^{\phi \ast \psi} \in \mathcal{A}^{mod \geq 0}$.

**Remark 2.4:** 1. For $\psi_i, \psi_j \in \mathcal{P}_\mathcal{N}$ appearing in the formal decomposition of $\mathcal{N}$ we can determine $\vartheta \in S^1$, such that $\psi_i \leq_\vartheta \psi_j$ (cf. [Sab13], ex. 1.6):

$$\psi_i \leq_\vartheta \psi_j \Leftrightarrow \psi_i - \psi_j \leq_0 0 \Leftrightarrow \mu_i(t) t^{-\vartheta i} - \mu_j(t) t^{-\vartheta j} \leq_0 0.$$ 

There exists a finite set of $\vartheta \in S^1$, where $\psi_i$ and $\psi_j$ are not comparable, i.e. neither
\[ \psi_i \leq \varphi_j \text{ nor } \varphi_j \leq \psi_i \] holds. We call these angles the \textit{Stokes directions} of \((\psi_i, \psi_j)\).

2. For \(\vartheta_0 \in S^1\) not being a Stokes direction of any pair \((\psi_i, \psi_j)\) in \(\mathcal{P}_N\) we get a total ordering of the \(\psi_i\)s with respect to \(\vartheta_0\): \(\psi_0 < \vartheta_0 \ldots < \vartheta_0 \psi_N\).

**Corollary 2.5:** For \(\vartheta \in S^1\) we get the following statement:

1. \(\psi = 0\):
   \[ \dim (L_{\leq 0})_\vartheta = \sum_{j \in J} \Phi P_j + \sum_{\{i|\psi_i \leq \vartheta_0\}} \Phi P_i \]

2. \(\psi \neq 0\), \(\vartheta \in \left(\frac{\vartheta + \arg(-\mu(0))}{q}, \frac{\vartheta + \arg(-\mu(0))}{q}\right) \mod \frac{2\pi}{q}\):
   \[ \dim (L_{\leq \psi})_\vartheta = \sum_{j \in J} \Phi P_j + \sum_{\{i|\psi_i \leq \vartheta \psi\}} \Phi P_i \]

3. \(\psi \neq 0\), \(\vartheta \in \left(\frac{-\vartheta + \arg(-\mu(0))}{q}, \frac{\vartheta + \arg(-\mu(0))}{q}\right) \mod \frac{2\pi}{q}\):
   \[ \dim (L_{\leq \psi})_\vartheta = \sum_{\{i|\psi_i \leq \vartheta \psi\}} \Phi P_i \]

**Proof:** This is a direct consequence of the formal decomposition and the fact that for \(\psi \neq 0\) we have

\[ 0 \leq \vartheta \psi \Leftrightarrow -\mu(t) t^{-q} \leq \vartheta_0 \Leftrightarrow \arg(-\mu(0)) - q \vartheta \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right) \]

\[ \Leftrightarrow \vartheta \in \left(\frac{\vartheta + \arg(-\mu(0))}{q}, \frac{\vartheta + \arg(-\mu(0))}{q}\right) \mod \frac{2\pi}{q} \]

\[ \square \]

### 2.1 Topological description of the stalks

As before we assume that \(\mathcal{M}\) is a meromorphic connection with regular singularities along its divisor. Our aim is to determine the Stokes structure of \(\mathcal{N}\) by using a topological point of view, i.e. we will develop a topological description of the Stokes-filtered local system \((\mathcal{L}, \mathcal{L}_{\leq})\). Therefore we will use the following theorem:

**Theorem 2.6 (Moc, Cor. 4.7.5):** There is an isomorphism

\[ \Omega : \mathcal{L}_{\leq \psi} := \mathcal{H}^0 \text{DR}^{(\text{mod } 0)}_{\Delta} \left(\mathcal{N} \otimes \mathcal{E}^{-\psi}\right) \rightarrow \tilde{\mathcal{L}}_{\leq \psi} := \mathcal{H}^1 R\overline{p}_* \text{DR}^{(\text{mod } D)}_{\overline{X}(D)} \left(\mathcal{M} \otimes \mathcal{E}^{\psi} \otimes \mathcal{E}^{-\psi}\right) \]
Here $\tilde{X}(D)$ denotes the real-oriented blow up of $X$ along the divisor component $D$, $\text{DR}^{\text{mod}} D (\mathcal{M})$ denotes the moderate de Rham complex of a meromorphic connection $\mathcal{M}$ on $X$ and $\tilde{p} : \tilde{X}(D) \to \tilde{\Delta}$ corresponds to the projection $p$ in the real-oriented blow up space $\tilde{X}(D)$ along $D$.

Theorem 2.6 is a special case of [McO], Cor. 4.7.5. In the following we will develop a topological view of the right hand side of the above isomorphism, which enables us to describe the Stokes-filtered local system $(\mathcal{L}, \mathcal{L}_\leq)$ more explicitly.

First let us examine the behavior of $\tilde{\mathcal{L}}_{\leq \psi}$ with respect to birational maps.

**Proposition 2.7:** Let $e : Z \to \Delta \times \mathbb{P}^1$ a birational map (i.e. a sequence of point blow-ups), $g(t, y) := \frac{1}{y} - \psi(t)$, $D_Z = e^{-1}(D)$. Then:

$$
\text{DR}^{\text{mod}} D (\tilde{X}(D)) (M \otimes \mathcal{E}^{g(l,y)}) \cong R\tilde{e}_* \text{DR}^{\text{mod}} D_Z (e^+ \mathcal{M} \otimes \mathcal{E}^{g(l,y)_{\text{loc}}})
$$

**Proof:** We know that $e$ is proper. Furthermore we assumed that $\mathcal{M}$ is a meromorphic connection with regular singularities along $D$, i.e. particularly that $M \otimes \mathcal{E}^{g(l,y)}$ is a holonomic $D_X$-module and localized along $D$. Thus, using the fact that $e_+ \left( e^+ \mathcal{M} \otimes \mathcal{E}^{g(l,y)} \right) \cong M \otimes \mathcal{E}^{g(l,y)}$, we can apply [Sab13], Prop. 8.9:

$$
\text{DR}^{\text{mod}} D (M \otimes \mathcal{E}^{g(l,y)}) \cong \text{DR}^{\text{mod}} (e^+ \left( e^+ \mathcal{M} \otimes \mathcal{E}^{g(l,y)_{\text{loc}}}) \right))
\cong R\tilde{e}_* \text{DR}^{\text{mod}} D_Z (e^+ \mathcal{M} \otimes \mathcal{E}^{g(l,y)}_{\text{loc}})
$$

\[\square\]

**Proposition 2.8:** Denote $D' := D \cup (\Delta \times \{\infty\})$. Let $D'_Z := e^{-1}(D')$ and $\overline{D}_Z := e^{-1}(\overline{D})$. If $\overline{D}_Z$ is a normal crossing divisor, we have isomorphisms

$$
R\tilde{e}_* \text{DR}^{\text{mod}} D_Z (e^+ \mathcal{M} \otimes \mathcal{E}^{g(l,y)_{\text{loc}}}) \cong R\tilde{e}_* \text{DR}^{\text{mod}} D'_Z (e^+ \mathcal{M} \otimes \mathcal{E}^{g(l,y)_{\text{loc}}})
$$

and

$$
R\tilde{e}_* \text{DR}^{\text{mod}} D_Z (e^+ \mathcal{M} \otimes \mathcal{E}^{g(l,y)_{\text{loc}}}) \cong R\tilde{e}_* \text{DR}^{\text{mod}} \overline{D}_Z (e^+ \mathcal{M} \otimes \mathcal{E}^{g(l,y)_{\text{loc}}})
$$

where $\tilde{e}$ denotes the induced map in the particular blow up spaces.

**Proof:** Assume $\overline{D}_Z$ a normal crossing divisor and consider the identity map $Z \overset{\text{Id}}{\to} Z$, which obviously induces an isomorphism on $Z \setminus \overline{D}_Z \to Z \setminus \overline{D}_Z$. We obtain ‘partial’ blow up maps

$$
\tilde{1}_d : \tilde{Z}(D'_Z) \to \tilde{Z}(D_Z) \text{ and } \tilde{1}_d : \tilde{Z}(\overline{D}_Z) \to \tilde{Z}(D_Z)
$$

7
which induce the requested isomorphisms. These are variants of Prop. 8.9 in [Sab13] (see also [Sab13], Prop. 8.7 and Rem. 8.8).

**Corollary 2.9:**

\[
\mathcal{H}^1 \left( R\tilde{p}_* \, \text{DR}^\text{mod}_D \left( \mathcal{M} \otimes \mathcal{E}^{g(t,y)} \right) \right) \cong \mathcal{H}^1 \left( R\left( p \circ e \right)_* \, \text{DR}^\text{mod}(\star) \left( e^+ \mathcal{M} \otimes \mathcal{E}^{g(t,y)\circ e} \right) \right)
\]

where \( \star = D_Z \) (resp. \( D'_{Z} \), resp. \( \overline{D}_Z \)).

As \((\mathcal{L}, \mathcal{L}_<)\) was defined on \( \partial \tilde{\Delta} \) we will restrict our investigation to the boundaries of the relevant blow up spaces, i.e. we consider \( \partial \tilde{\Delta}, \partial \tilde{X}(D), \partial \bar{Z}(D_Z), \partial \bar{Z}(D'_Z) \) and \( \partial \bar{Z}(\overline{D}_Z) \). We will take a closer look at the stalks:

**Lemma 2.10:** Let \( \vartheta \in \mathbb{S}^1 \). There is an isomorphism

\[
\left( \mathcal{H}^1 \left( R\left( p \circ e \right)_* \, \text{DR}^\text{mod}(D_Z) \left( e^+ \mathcal{M} \otimes \mathcal{E}^{g(t,y)\circ e} \right) \right) \right)_\vartheta \cong \mathcal{H}^1 \left( \left( p \circ e \right)^{-1}(\vartheta), \text{DR}^\text{mod}(D_Z) \left( e^+ \mathcal{M} \otimes \mathcal{E}^{g(t,y)\circ e} \right) \right)
\]

The same holds for \( D'_Z \) and \( \overline{D}_Z \) instead of \( D_Z \).

**Proof:** \( p \circ e \) is proper, thus the claim follows by applying the proper base change theorem (cf. [Dim04], Th. 2.3.26, p. 41).

In the following we will consider

\[
\mathcal{F}_\vartheta := \text{DR}^\text{mod}(D'_Z) \left( e^+ \mathcal{M} \otimes \mathcal{E}^{g(t,y)\circ e} \right)
\]
on the fiber \( (p \circ e)^{-1}(\vartheta) \) in \( \partial \bar{Z}(D'_Z) \), which in our case will be a 1-dimensional complex analytic space. Therefore we can assume \( \mathcal{F}_\vartheta \) to be a perverse sheaf, i.e. a 2-term complex \( \mathcal{F}_\vartheta : 0 \to \mathcal{F}_0 \to \mathcal{F}_1 \to 0 \) with additional conditions on the cohomology sheaves \( \mathcal{H}^0(\mathcal{F}_\vartheta) \) and \( \mathcal{H}^1(\mathcal{F}_\vartheta) \). Obviously \( \mathcal{H}^i(\mathcal{F}_\vartheta) = 0 \) for \( i \neq 0,1 \) (cf. [Dim04], Ex. 5.2.23, p. 139).

Furthermore we can restrict \( \mathcal{F}_\vartheta \) to the set \( B^\vartheta := \{ \zeta \in (p \circ e)^{-1}(\vartheta) \mid (\mathcal{H}^*(\mathcal{F}_\vartheta))_\zeta \neq 0 \} \), which is an open subset of \( (p \circ e)^{-1}(\vartheta) \). We denote the open embedding by \( \beta^\vartheta : B^\vartheta \hookrightarrow (p \circ e)^{-1}(\vartheta) \). Thus by interpreting \( \mathcal{F}_\vartheta \) as a complex of sheaves on \( B^\vartheta \), we have to compute

\[
\mathcal{H}^1 \left( (p \circ e)^{-1}(\vartheta), \beta^\vartheta_!, \mathcal{F}_\vartheta \right) \cong \mathcal{H}^1_c \left( B^\vartheta, \mathcal{F}_\vartheta \right).
\]
2.1.1 Construction of a Resolution of Singularities

In the following sections we will construct a suitable blow up map $e$, such that we can describe $(p \circ e)^{-1}(\vartheta)$ and $B^\vartheta_\psi$ more concretely.

**Lemma 2.11:** Let $g(t,y) = \frac{1}{y} - \psi(t)$. There exists a sequence of blow up maps $e$ such that

1. $g \circ e$ holomorphic or good, i.e. $(g \circ e)(u,v) = \frac{1}{u^{n_{\vartheta}}} \beta(u,v)$, whereby $\beta$ holomorphic and $\beta(0,v) \neq 0$.

2. For all $i$ the strict transform of $S_i$ intersects $D_Z$ in a unique point $P_i$.

**Proof:** The divisor components $S_i$ are given by $S_i : \mu_i(t)y = t^{q_i}$. Let $n := \max\{q_i\}$. We distinguish two cases:

1. $\psi = 0$, i.e. $g(t,y) = \frac{1}{y}$. After $n$ blow ups in $(0,0)$ we get an exceptional divisor $D_Z$ with local coordinates

$$t = u_kv_k, y = u_k^{k-1}v_k^k \text{ and } t = u^\ast_n, y = u_n^\ast v^\ast_n$$

and $g \circ e$ is good or holomorphic in every point.

![Diagram](image)

The intersection points $P_i$ of the strict transform of $S_i : \mu_i(t)y = t^{q_i}$ is given by

$$P_i = \left(0, \frac{1}{\mu_i(0)}\right)$$

in the proper coordinates (i.e. $(u_{q_i+1}, v_{q_i+1})$ for $q_i < n$ and $(\tilde{u}_n, \tilde{v}_n)$ for $q_i = n$)

2. $\psi \neq 0$: $\psi$ is given by $\psi(t) = \mu(t)t^{-q}$, so $g(t,y) = \frac{t^q - \mu(t)}{y^q}(g \circ e)$ is good in every point except:

- $q < n, k = q + 1 : P = \left(0, \frac{1}{\mu(0)}\right)$ with local coordinates $(u_{q+1}, v_{q+1})$
- $q = n, k = n : P = \left(0, \frac{1}{\mu(0)}\right)$ with local coordinates $(\tilde{u}_n, \tilde{v}_n)$

Let $q < n$. After changing coordinates $u' = u_k, v' = v_k - \frac{1}{\mu(u_kv_k)}$ and after $q$ blow-ups in $(0,0)$, $(g \circ e)$ is good for every point of $D_Z$ in local coordinates $(u'_q, v'_q)$ and holomorphic for every point of $D_Z$ in local coordinates $(\tilde{u}'_q, \tilde{v}'_q)$.
Let \( q = n \): After a change of coordinates \( u' = \tilde{u}_n, v' = \tilde{v}_n - \frac{1}{\mu(u_n)} \) and \( n \) blow-ups in \( (0, 0) \), \((g \circ e)\) is good for every point of \( D_Z \) in local coordinates \((u'_n, v'_n)\) and holomorphic for every point of \( D_Z \) in local coordinates \((\tilde{u}_n', \tilde{v}_n')\). As before the intersection points \( P_i \) of the strict transform of \( S_i : \mu_i(t)y = t^q \) with \( D_Z \) is given by

\[
P_i = \left(0, \frac{1}{\mu_i(0)}\right)
\]

in the suitable coordinates. Now let \( q_i = q \) and \( \mu_i(0) = \mu(0) \), i.e. we consider \( S : \mu(t)y = t^q \) (Notice, that \( S \) corresponds to our given \( \psi \! \))

- \( q < n \), i.e. \( k := q + 1 \neq n \): \( \overline{S} : u_{k}^{k-1}v_{k}^{k-1}(1 - \mu_i(u_kv_k)v_k) = 0 \) and by coordinate transform we have: \( \overline{S} : u_{k}^{k-1}v_{k}^{k-1}v' = 0 \). As \( v' = \tilde{v}_q\tilde{v}'_q \) we get the unique intersection point \( P = (0, 0) \).
- \( q = n \): In the same way we get the intersection point \( P = (0, 0) \).

\( \square \)

2.1.2 Topology of \((p \circ e)^{-1}(\vartheta)\)

\( D'_Z \) is a normal crossing divisor, so locally at a crossing point \( D'_Z \) has the form \( \{uv = 0\} \) and at a smooth point \( D'_Z \) has the form \( \{u = 0\} \). Thus we can describe \( \partial \tilde{Z}(D'_Z) \) in local coordinates:

- real blow up with respect to \( \{u = 0\} \): \( \zeta = (0, \theta_u, |v| \cdot e^{i\theta_v}) \) (with \( \theta_v = |v| \cdot e^{i\theta_v} \))
- real blow up with respect to \( \{uv = 0\} \): \( \zeta = (0, \theta_u, 0, \theta_v) \)

Now we take the fiber of \( \vartheta \in \mathbb{S}^1 \), i.e. we consider \((p \circ e)^{-1}(\vartheta)\). For every fixed \( |v| \) we have a bijection \( \{(0, \theta_u, |v| \cdot e^{i\theta_v}) \mapsto \mathbb{S}^1 \} \), thus, following the nomenclature of C. Sabbah, we can interpret \((p \circ e)^{-1}(\vartheta)\) as a system of pipes, which furthermore is homeomorphic to a disc ( [Sab13], p. 203 )

Note, that for all \( i \) the strict transform of the divisor component \( S_i \) intersects the real blow up divisor \((p \circ e)^{-1}(\vartheta)\) in a unique point \( P_i \).
Remark 2.12: 1. Also the irreducible components $\tilde{S}_j$ intersect $(p \circ e)^{-1}(\vartheta)$ in distinct points $\tilde{P}_j$. This follows directly from Assumption 1.2.

2. According to Assumption 1.1 we have $(q_i \neq q_j$ or $\mu_i(0) \neq \mu_j(0)$ for $i \neq j$). This induces $P_i \neq P_j$.

3. Every intersection point $P_i, \tilde{P}_j$ may be interpreted as a 'leak' in the system of pipes $(p \circ e)^{-1}(\vartheta)$ (see [Sab13], p. 204). Thus topologically we can think of $(p \circ e)^{-1}(\vartheta)$ as a disc with singularities, which come from the intersection with $\tilde{S}_j$ and $S_i$.

2.1.3 Explicit Description of $B^\vartheta$ 

Remember the definition of $B^\vartheta_{\psi} := \{ \zeta \in (p \circ e)^{-1}(\vartheta) \mid (H^\vartheta F_{\psi})_\zeta \neq 0 \}$. Obviously we have:

$$\zeta \in B_{\psi}^\vartheta \iff (H^\vartheta F_{\psi})_\zeta \neq 0 \iff e^{(g \circ e)}(u,v) \in A_{\vartheta}^{mod D_Z} \text{ near } \zeta$$

(The second equivalence follows by considering $DR^{mod D_Z} e^{+} M \otimes E^{g(t,y) \circ e}$, where we know that it has cohomology in degree 0 at most (cf. Proposition 2.8 and Lemma 3.2).) Thus we need to take a closer look at the exponent $g \circ e$. Therefore we use the following Lemma (cf. [Sab13], 9.4):

Lemma 2.13: Let $u,v$ local coordinates of the divisor $D_Z$, such that $f(u,v)$ holomorphic or good, i.e.

$$f(u,v) = \frac{1}{u^m v^n} \beta(u,v), \text{ whereby } \beta \text{ holomorphic and } \beta(0,v) \neq 0.$$ 

Then $e^{f(u,v)} \in A_{\vartheta}^{mod D_Z}$ around a given point $\zeta \in (p \circ e)^{-1}(\vartheta)$ if and only if

$$f \text{ holomorphic in } \zeta \text{ or } \arg(\beta(0,v)) - mn u - m \theta_u - n \theta_v \in \left(\frac{\pi}{2}, \frac{3 \pi}{2}\right) \mod 2\pi.$$ 

Lemma 2.14: 1. Let $S_i : \mu_i(t) y = t^{\vartheta_i}, \psi_i(t) = \mu_i(t) t^{-\vartheta_i}$ with $\mu_i(0) \neq 0$ and $P_i$ the corresponding intersection point. Then we have: $P_i \in B^\vartheta_{\psi} \iff \psi_i \leq \psi$.

2. Let $\tilde{P}_j$ the intersection points of $\tilde{S}_j$ with $(p \circ e)^{-1}(\vartheta)$. Then we have: $\{ \tilde{P}_j \mid j = 1, \ldots, J \} \subset B^\vartheta_{\psi} \iff 0 \leq \psi$. 

11
Proof: This follows by examining goodness condition of $g \circ e$ near the intersection points and by Remark 2.24.

**Definition 2.15:** For abbreviation we define the following sets:

- $\mathcal{P}^\vartheta := \{ \tilde{P}_j \mid j = 1, \ldots, J \} \cup \{ P_i \mid i = 1, \ldots, I \} \subset (p \circ e)^{-1}(\vartheta)$
- $\mathcal{P}^\vartheta_\psi := \mathcal{P}^\vartheta \cap \mathcal{B}^\vartheta_\psi$

**Lemma 2.16:** Let $\psi = 0$. Then the fundamental group $\pi_1 \left( \mathcal{B}_0^\vartheta \setminus \mathcal{P}^\vartheta_0 \right)$ is a free group of rank $\# \left( \mathcal{P}^\vartheta_0 \right)$.

Proof: This follows from the fact, that $\mathcal{B}_0^\vartheta$ emerges from glueing the following sets of points:

- $M_1 = \{ \zeta = (0, \theta_{u_1}, |v_1|, \theta_{v_1}) \mid v_1 \neq 0 \}$
- $M_2 = \{ \zeta = (0, \theta_{u_1}, 0, \theta_{v_1}) \mid \theta_{v_1} \in \left( \frac{2\pi}{1}, \frac{3\pi}{1} \right) \}$
- $M_3 = \{ \zeta = (0, \theta_{u_k}, |v_k|, \theta_{v_k}) \mid \theta_{v_k} \in \left( \frac{2\pi}{1}, \frac{3\pi}{1} \right) \}$
- $M_4 = \{ \zeta = (0, \theta_{u_n}, |v_n|, \theta_{v_n}) \mid \theta_{v_n} \in \left( \frac{2\pi}{1} - n\vartheta, \frac{3\pi}{1} - n\vartheta \right) \}$

Since $M_1$ contains the $\tilde{P}_j$’s and $M_2, M_3, M_4$ are simply connected and contain the relevant $P_i$’s, this shows the claim.

**Lemma 2.17:** Let $\psi \neq 0$, i.e. $\psi$ is given by $\psi(t) = \mu(t)t^{-q}$. Then $\pi_1 \left( \mathcal{B}_\psi^\vartheta \setminus \mathcal{P}_\psi^\vartheta \right)$ is a free group of rank $\# \left( \mathcal{P}_\psi^\vartheta \right)$.

Proof: Explicitely we have to prove:

1. For $\vartheta \in \left( \frac{2\pi + \text{arg}(\mu(0))}{q}, \frac{3\pi + \text{arg}(\mu(0))}{q} \right) \mod \frac{2\pi}{q}$: $\pi_1 \left( \mathcal{B}_\psi^\vartheta \setminus \mathcal{P}_\psi^\vartheta \right)$ is a free group of rank $\# \left( \tilde{P}_j \right) + \# \{ P_i \mid P_i \in \mathcal{B}_\psi^\vartheta \}$

2. For $\vartheta \notin \left( \frac{2\pi + \text{arg}(\mu(0))}{q}, \frac{3\pi + \text{arg}(\mu(0))}{q} \right) \mod \frac{2\pi}{q}$: $\pi_1 \left( \mathcal{B}_\psi^\vartheta \setminus \mathcal{P}_\psi^\vartheta \right)$ is a free group of rank $\# \{ P_i \mid P_i \in \mathcal{B}_\psi^\vartheta \}$

For $k \leq q$ we have: $\zeta = (0, \theta_{u_k}, |v_k|, \theta_{v_k}) \in \mathcal{B}_\psi^\vartheta \Leftrightarrow \text{arg}(-\mu(0)) - q\vartheta \in \left( \frac{2\pi}{1}, \frac{3\pi}{1} \right)$. This explains the sub-division into the two cases above. As in the previous lemma $\mathcal{B}_\psi^\vartheta$ consists of glueing simply connected sets (additionally taking the sets of the branching part into account).

12
Descriptively, this means, that there are no other holes’ in $B^\vartheta_\psi$ than the singularities $P_i$, which arise from the intersection with $S_i$ and possibly – depending on the choice of $\vartheta$ if $\psi \neq 0$ – the singularities $\tilde{P}_j$, which arise from the intersection with $\tilde{S}_j$.

Remark, that an open interval of the boundary of $B^\vartheta_\psi$ lies in $B^\vartheta_\psi$. This holds because for $k = n$ we have

$$(0, \vartheta, 0, \theta_\vartheta) \in B^\vartheta_\psi \Leftrightarrow \theta_\vartheta \in \left(\frac{\pi}{2} - n\vartheta, \frac{3\pi}{2} - n\vartheta\right)$$

both if $q < n$ and if $q = n$.

2.1.4 Dimension of $H^1_c \left( B^\vartheta_\psi, F^\vartheta_\psi \right)$

Proposition 2.18: On $B^\vartheta_\psi \setminus P^\vartheta_\psi$ the perverse sheaf $F^\vartheta_\psi$ has cohomology in degree 0 at most.

Proof: We know that $e^+ M$ has regular singularities along the divisor and that $\mathcal{E}^{\vartheta(t, y)}$ is good or even holomorphic. Moreover the divisor is normal crossing except possibly at the intersection points $P_i, \tilde{P}_j$. Thus we can apply [Sab13], Corollary 11.22.

Lemma 2.19 ([Sab], Prop 1.1.6): Let $\Delta$ be an open disc and $\Delta'$ an open subset of the closure $\overline{\Delta}$, consisting of $\Delta$ and a connected open subset of $\partial \overline{\Delta}$. Let $C$ be a finite set of points in $\Delta$. Let $\mathcal{M}$ be a regular singular $\mathcal{D}_{\Delta'}$-module and consider a perverse sheaf $\mathcal{F}$ with singularities in $c \in C$ only. Then $H^k_c(\Delta', \mathcal{F}) = 0$ for $k \neq 1$ and $\dim H^1_c(\Delta', \mathcal{F})$ is equal to the sum of the dimensions of the vanishing cycle spaces at $c \in C$.

Proof: For a proof we refer to [Sab].

As a direct consequence we see:

Theorem 2.20: For $\vartheta \in S^1_1$ we have $\dim (\mathcal{L}_{\leq \vartheta})_{\vartheta} = \dim \left( \tilde{\mathcal{L}}_{\leq \vartheta} \right)_{\vartheta}$.

Proof: This follows from Corollary 2.19 and Lemma 2.19.

Remark 2.21: In an analogous way we can determine $\mathcal{L}$ on $S^1_1$. First let us recall that $\mathcal{L} = j_* \mathcal{L}'$, whereby $\mathcal{L}'$ denotes the local system associated to $\mathcal{N} := \mathcal{H}^0 p_+ \left( \mathcal{M} \otimes \mathcal{E}^{\vartheta} \right)$. Consider a small circle $S^1_\epsilon$ around $0 \in \Delta$. Then we have

$\mathcal{L}|_{S^1_\epsilon} \cong \mathcal{L}'|_{S^1_\epsilon}$

On $S^1_\epsilon$ we can identify $\mathcal{L}' = \mathcal{H}^0 DR^\Delta_{\mathcal{E}^{\vartheta}}(\mathcal{N})$ (since the growing condition mod 0 is irrelevant
outside of $\partial \Delta$). Furthermore in this situation, the isomorphism
\[
\Omega : H^0_{DR}^{mod 0} (\mathcal{N})|_{S^1_\epsilon} \cong H^1 R\tilde{p}_* DR^{mod D} \left( \mathcal{M} \otimes \mathcal{E}^\frac{1}{\tau} \right)|_{S^1_\epsilon}
\]
holds (analogously to Theorem 2.6). Now as before we can describe the right hand side
in topological terms. Since $\mathcal{E}^\frac{1}{\tau}$ is a good (or even holomorphic) connection, we know that
$DR^{mod D} \left( \mathcal{M} \otimes \mathcal{E}^\frac{1}{\tau} \right)$ has cohomology in degree zero at most and thus corresponds to a sheaf. By taking the stalk at a point $\rho \in S^1_\epsilon$ we get
\[
\left( H^0_{DR}^{mod 0} (\mathcal{N}) \right)_\rho \cong H^1_c \left( \tilde{p}^{-1}(\rho), DR^{mod D} \left( \mathcal{M} \otimes \mathcal{E}^\frac{1}{\tau} \right) \right).
\]
$\tilde{p}^{-1}(\rho)$ corresponds to the projective line with a real blow up in the point $\infty$ (denoted by $S^1_\infty$).

We can interpret this space as a disc with boundary $S^1_\infty$. Furthermore by choosing $\epsilon$ small enough, we know that every irreducible component $S_i$ resp. $\tilde{S}_j$ of $SS(\mathcal{M})$ meets the (inner of the) disc in exactly one point ($P_i$ resp. $\tilde{P}_j$).

The sheaf is supported everywhere away from the boundary, on the boundary its supported on an open hemisphere of $S^1_\infty$ (i.e. the points where it fulfills the moderate growth condition). Thus the dimension of $H^1_c \left( \tilde{p}^{-1}(\rho), DR^{mod D} \left( \mathcal{M} \otimes \mathcal{E}^\frac{1}{\tau} \right) \right)$ depends on the dimension of the vanishing cycle spaces in the points $P_i$ and $\tilde{P}_j$:
\[
\dim H^1_c \left( \tilde{p}^{-1}(\rho), DR^{mod D} \left( \mathcal{M} \otimes \mathcal{E}^\frac{1}{\tau} \right) \right) = \dim \sum_{i \in I} \Phi_{P_i} + \sum_{j \in J} \Phi_{\tilde{P}_j}.
\]

We will use this topological description in the following chapter, supply it to the explicit example and determine a set of linear Stokes data.

3 Proof of Theorem 1.5: Explicit example for the determination of Stokes data

As already mentioned in the introduction, we will consider the following explicit example:

Let $X = \Delta \times \mathbb{P}^1$ and $\mathcal{M}$ a meromorphic connection on $X$ of rank $r$ with regular singularities along its divisor. Let the singular locus $SS(\mathcal{M})$ be of the following form: $SS(\mathcal{M}) = \{ t \cdot y \cdot (t - y) \cdot x = 0 \}$. Denote the irreducible components ($S_1 : y = t$) and ($\tilde{S}_1 : x = 0$).
The following chapter is devoted to prove Theorem 1.5 and thereby determine a set of Stokes data to $\mathcal{H}^0 p_+ \left( \mathcal{M} \otimes \mathcal{E}^\perp \right)$.

### 3.1 Stokes-filtered local system

According to Theorem 2.1, $\hat{N} = \mathcal{H}^0 p_+ \left( \mathcal{M} \otimes \mathcal{E}^\perp \right)$ can be decomposed to:

$$\hat{N} \cong R_0 \oplus \left( R_1 \otimes \mathcal{E}^\perp \right)$$

whereby $rk \left( R_0 \right) = rk \left( R_1 \right) = r$. Now we will describe the Stokes structure via the isomorphism $\Omega$ (Theorem 2.6). $\Omega$ identifies $(\mathcal{L}_{\leq \psi}) \vartheta \cong H^1 \left( (p \circ e)^{-1} (\vartheta), \beta^{\vartheta}_{\psi, \xi}, \mathcal{F}_\psi \right)$.

**Lemma 3.1:** There exists a sequence of point blow-ups $e: Z \to X$, such that

1. the singular support of $\mathcal{M}$ becomes a normal crossing divisor
2. for both $\psi = 0$ and $\psi = \frac{1}{t}$ the exponent $g \circ e$ is holomorphic or good in every point.

**Proof:** The following blow up of the divisor satisfies the requested conditions:

(i.e. a point-blow-up in $(t, y) = 0$ and a second point-blow-up in $(\tilde{u}_1, \tilde{v}_1) = (0, 1)$)
We will denote the resulting normal crossing divisor by $\mathcal{D}_Z$. The fiber over $\vartheta$ with respect to the blow up along $\mathcal{D}_Z$ is homeomorphic to a closed disc with two 'holes'. We will denote it by $\mathcal{A} \times \{\vartheta\}$.

**Lemma 3.2:** $\text{DR}^{\text{mod}}_{\mathcal{D}_Z} (e^+ \mathcal{M} \otimes \mathcal{E}^{\text{geo}})$ has cohomology in degree 0 at most.

**Proof:** Since $\mathcal{D}_Z$ is normal crossing and $g \circ e$ is good or holomorphic (Lemma 3.1) the claim follows by [Sab13], Prop. 8.15 and Cor. 11.22. □

Consider the map $\kappa : \tilde{Z} \left( \mathcal{D}_Z \right) \to \tilde{Z} (\mathcal{D}_Z)$. Restricting it to a fiber $\kappa_{\vartheta} : \mathcal{A} \times \{\vartheta\} \to (\rho \circ e)^{-1} (\vartheta)$, it is just the identity except at the points $\tilde{P}_1$ and $P_1$, where it describes the 'collapse' of the real blow ups of $\tilde{P}_1$ and $P_1$ back to these points. $\text{DR}^{\text{mod}}_{\mathcal{D}_Z} (e^+ \mathcal{M} \otimes \mathcal{E}^{\text{geo}})$ has cohomology in degree 0 at most and therefore corresponds to a local system $\mathcal{G}_\psi$. Then $\text{DR}^{\text{mod}}_{\mathcal{D}_Z} (e^+ \mathcal{M} \otimes \mathcal{E}^{\text{geo}})$ corresponds to the perverse sheaf $\mathcal{F}_\psi$ given by $0 \to \kappa_{\vartheta} \mathcal{G}_\psi \to 0 \to 0$. Let $\mathcal{G}_{\psi}^{\vartheta}$ the restriction to the support $\kappa^{-1} (B_{\psi}^{\vartheta})$ in the fiber $\mathcal{A} \times \{\vartheta\}$ and $\mathcal{B}_{\psi}^{\vartheta} : \kappa^{-1} (B_{\psi}^{\vartheta}) \to \mathcal{A} \times \{\vartheta\}$ the open inclusion. Because of Proposition 2.8 we have:

$$H^1 \left( \mathcal{A} \times \{\vartheta\}, \mathcal{B}_{\psi}^{\vartheta} \mathcal{G}_{\psi}^{\vartheta} \right) \cong H^1 \left( (p \circ e)^{-1} (\vartheta), \mathcal{B}_{\psi}^{\vartheta} \mathcal{F}_{\psi}^{\vartheta} \right)$$

Furthermore we denote by $\mathcal{K}$ the local system on $\tilde{Z} \left( \mathcal{D}_Z \right)$ corresponding to the pullback connection $e^+ \mathcal{M}$ of rank $\text{rk} (\mathcal{M})$ (see Cor. 8.3 in [Sab13]) and by $\mathcal{K}^{\vartheta}$ its restriction to $\mathcal{A} \times \{\vartheta\}$. Then obviously $\mathcal{B}_{\psi}^{\vartheta} \mathcal{G}_{\psi}^{\vartheta}$ equals $\mathcal{B}_{\psi}^{\vartheta} \mathcal{K}^{\vartheta}$ for all $\psi$. For brevity we will write $\beta_{\psi}^{\vartheta} \mathcal{K}^{\vartheta}$ instead of $\mathcal{B}_{\psi}^{\vartheta} \mathcal{K}^{\vartheta}$ in the following. Thus it is enough to examine $(\mathcal{L}_{\leq \psi})_{\vartheta} \cong H^1 \left( \mathcal{A} \times \{\vartheta\}, \mathcal{B}_{\psi}^{\vartheta} \mathcal{K}^{\vartheta} \right)$. With the blow up $e$ constructed in the proof of Lemma 3.1 we can determine the open subset $B_{\vartheta}^{\psi} \subset (p \circ e)^{-1} (\vartheta)$ and we receive the following pictures, which show $(p \circ e)^{-1} (\vartheta)$ (resp. $\mathcal{A} \times \{\vartheta\}$) and the subsets $B_{\vartheta}^{\psi}$, $B_{\vartheta}^{0}$. One can see very clearly that by passing the Stokes directions $\pm \frac{\pi}{2}$, the relation of the subsets $B_{\vartheta}^{0}$ and $B_{\vartheta}^{0}$ changes from $B_{\vartheta}^{0} \subset B_{\vartheta}^{0}$ to $B_{\vartheta}^{0} \subset B_{\vartheta}^{0}$ and vice versa.

\[
\begin{array}{cccc}
\vartheta = 0 & \vartheta = \frac{\pi}{2} - \epsilon & \vartheta = \frac{\pi}{2} + \epsilon & \vartheta = \pi \\
\vartheta = \pi & \vartheta = \frac{3\pi}{2} - \epsilon & \vartheta = \frac{3\pi}{2} + \epsilon & \vartheta = 2\pi
\end{array}
\]
In this situation we can compute the first cohomology groups $H^1(\overline{A} \times \{\vartheta\}, \overline{\beta}_{\psi, i} K^\vartheta)$ in another way, namely by using Čech cohomology.

Therefore consider the following curves $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ in $\overline{A}$:

Then for $\vartheta_0 \in [0, \frac{\pi}{2})$ the closed covering $\mathfrak{A} = A_1 \cup A_2 \cup A_3$ of $\overline{A}$ defines a Leray covering of $\beta_{\psi, i}^\vartheta K^\vartheta_0$ and for $\beta_{\psi, i}^\vartheta K^\vartheta_1$, whereby $\vartheta_1 := \vartheta_0 + \pi$. This can be proved easily by using the exact sequence of Chapter 3.3.4 and doing the same calculations for the restriction to the intersections $\alpha_k$.

For $\vartheta_0 \in [\frac{\pi}{2}, \pi)$ the following curves define a Leray covering for $\beta_{\psi, i}^\vartheta K^\vartheta_0$ and $\beta_{\psi, i}^\vartheta K^\vartheta_1$:

Thus we can conclude:

**Lemma 3.3:** For every pair of angles $(\vartheta_0, \vartheta_1 := \vartheta_0 + \pi)$ of $\mathbb{S}^1$ the construction above defines a closed covering $\mathfrak{A}$ of $\overline{A}$, such that $\mathfrak{A}$ is a common Leray covering of $\beta_{\psi, i}^\vartheta K^\vartheta$ (i.e., $i = 1, 2$ and $\psi = 0, \frac{\pi}{2}$). Consequently we get an isomorphism

$$H^1(\overline{A} \times \{\vartheta\}, \overline{\beta}_{\psi, i}^\vartheta K^\vartheta) \to \check{H}^1(\mathfrak{A}, \overline{\beta}_{\psi, i}^\vartheta K^\vartheta)$$

In the following lemma we will compute the cohomology groups concretely for $\vartheta = 0$ and $\vartheta = \pi$.

**Lemma 3.4:**

- $H^1(\overline{A} \times \{0\}, \overline{\beta}_{0, i}^0 K^0) \cong K^0_{x_1}$
- $H^1(\overline{A} \times \{0\}, \overline{\beta}_{i, 0}^0 K^0) = H^1(\overline{A} \times \{0\}, \overline{\beta}_{0}^0 K^0) \cong K^0_{x_1} \oplus K^0_{x_3}$
- $H^1(\overline{A} \times \{\pi\}, \overline{\beta}_{\pi, i}^\pi K^\pi) \cong K^\pi_{x_4}$
- $H^1(\overline{A} \times \{\pi\}, \overline{\beta}_{\pi}^\pi K^\pi) = H^1(\overline{A} \times \{\pi\}, \overline{\beta}_{\pi}^\pi K^\pi) \cong K^\pi_{x_2} \oplus K^\pi_{x_4}$

**Proof:** Just notice that $\check{C}^0 = 0$ and thus $\check{H}^1 = \check{C}^1$. □
3.2 Stokes data associated to $\mathcal{L}$

Using the functor constructed in [HS11] we can associate a set of Stokes data to the Stokes-filtered local system $\mathcal{L}$ described above.

**Construction 3.5:** Fix two intervals

$$I_0 = (0 - \epsilon, \pi + \epsilon), \quad I_1 = (-\pi - \epsilon, 0 + \epsilon)$$

of length $\pi+2\epsilon$ on $S^1$, such that the intersection $I_0 \cap I_1$ consists of $(0 - \epsilon, 0 + \epsilon)$ and $(\pi - \epsilon, \pi + \epsilon)$. Observe, that the intersections do not contain the Stokes directions $-\frac{\pi}{2}, \frac{\pi}{2}$. To our given local system $\mathcal{L}$ we associate:

- a vector space associated to the angle 0, i.e. the stalk $\mathcal{L}_0$. It comes equipped with the Stokes filtration.
- a vector space associated to the angle $\pi$, i.e. the stalk $\mathcal{L}_\pi$, coming equipped with the Stokes filtration.
- vector spaces associated to the intervals $I_0$ and $I_1$, i.e. the global sections $\Gamma(I_0, \mathcal{L})$, $\Gamma(I_1, \mathcal{L})$
- a diagram of isomorphisms (given by the natural restriction to the stalks):

$$
\begin{array}{ccc}
\mathcal{L}_0 & \xrightarrow{a_0} & \mathcal{L}_\pi \\
\downarrow{a_0} & & \downarrow{a_1'} \\
\Gamma(I_0, \mathcal{L}) & & \Gamma(I_1, \mathcal{L})
\end{array}
$$

The filtrations on the stalks are opposite with respect to the maps $a_0^\dagger a_0^{-1}$ and $a_1 a_1'^{-1}$, i.e.

$$
\mathcal{L}_0 = \bigoplus_{\phi \in \{0, \frac{\pi}{2}\}} L_{\leq \phi, 0} \cap a_0^\dagger a_0^{-1}(L_{\leq \phi, \pi}), \quad \mathcal{L}_\pi = \bigoplus_{\phi \in \{0, \frac{\pi}{2}\}} L_{\leq \phi, \pi} \cap a_1 a_1'^{-1}(L_{\leq \phi, 0})
$$

Furthermore we know, by using the isomorphisms $\Omega$ and $\Gamma$, that $\mathcal{L}_0 \cong \mathcal{K}_{x_1} \oplus \mathcal{K}_{x_3}$ whereby $\mathcal{L}_{\leq 0, 0} \cong \mathcal{K}_{x_1}$. Thus we have a splitting $\mathcal{L}_0 = G_0 \oplus G_\dagger$ with $G_0 = \mathcal{L}_{\leq 0, 0}$. The same holds for $\mathcal{L}_\pi$: $\mathcal{L}_\pi = H_0 \oplus H_\dagger$ with $H_\dagger = \mathcal{L}_{\leq \frac{\pi}{2}, \pi}$.

Thus the set of data $(G_0, G_\dagger, H_0, H_\dagger, S_0^\pi, S_\pi^0)$ with:

- $\mathcal{L}_0 = G_0 \oplus G_\dagger$ and $\mathcal{L}_\pi = H_\dagger \oplus H_0$
- $S_0^\pi : G_0 \oplus G_\dagger \xrightarrow{a_0^{-1}} \Gamma(I_0, \mathcal{L}) \xrightarrow{a_0^\dagger} H_0 \oplus H_\dagger$
• $S^0_\pi : H^+ \oplus H_0 \xrightarrow{\alpha^{-1}} \Gamma (I_1, \mathcal{L}) \xrightarrow{\alpha^1} G^+ \oplus G_0$

describes a set of Stokes data associated to the local system $\mathcal{L}$.

By exhaustivity of the filtrations on $\mathcal{L}_\vartheta$ ($\vartheta = 0, \pi$) the isomorphism $\Omega$ induces

\[
\mathcal{L}_\vartheta \cong H^1 (\mathfrak{A} \times \vartheta, \overline{\beta}_\vartheta \mathcal{K}^\vartheta)
\]

where $\overline{\beta}^\vartheta : B^\vartheta \hookrightarrow \mathfrak{A} \times \vartheta$ corresponds to the $\overline{\beta}_\vartheta^\vartheta$ with $\vartheta \geq \vartheta$ for $\psi, \phi \in \Phi = \{0, \frac{1}{\pi}\}$ and $B^\vartheta := B^\vartheta_\psi \supset B^\vartheta_\phi$.

In the same way, for an open interval $I \subset \mathbb{S}^1$ let $\mathcal{K}^I$ be the restriction of $\mathcal{K}$ to $\mathfrak{A} \times I$ and define $\overline{\beta}^I : B^I \hookrightarrow \mathfrak{A} \times I$ the inclusion of the subspace $B^I$, which is the support of $\text{DR}^{\text{mod}D} (\mathcal{M} \otimes \mathcal{E}^\frac{1}{\pi})$ according to Remark 2.21. Notice that $\overline{\beta}^I|_{\mathfrak{A} \times \vartheta} = \overline{\beta}^\vartheta$. Then, by the isomorphism $\Omega$, we identify

\[
\Gamma (I, \mathcal{L}) \cong H^1 (\mathfrak{A} \times I, \overline{\beta}_I^I \mathcal{K}^I)
\]

With these isomorphisms we have restriction morphisms (according to the restrictions to the stalks)

\[
\rho_\vartheta : H^1 (\mathfrak{A} \times I_0, \overline{\beta}^\vartheta_0 \mathcal{K}^I_0) \cong H^1 (\mathfrak{A} \times \vartheta, \overline{\beta}^\vartheta_0 \mathcal{K}^\vartheta) \text{ for } \vartheta \in I_0
\]

\[
\rho'_\vartheta : H^1 (\mathfrak{A} \times I_1, \overline{\beta}^\vartheta_1 \mathcal{K}^I_1) \cong H^1 (\mathfrak{A} \times \vartheta, \overline{\beta}^\vartheta_1 \mathcal{K}^\vartheta) \text{ for } \vartheta \in I_1
\]

This yields to a new way of describing Stokes data associated to the local system $\mathcal{L}$:

**Theorem 3.6:** Fix the following data:

• vector spaces $L_0 := H^1 (\mathfrak{A} \times \{0\}, \overline{\beta}_I^0 \mathcal{K}^0)$ and $L_1 := H^1 (\mathfrak{A} \times \{\pi\}, \overline{\beta}_I^\pi \mathcal{K}^\pi)$

• morphisms $\sigma^0_\pi := \rho_\pi \circ \rho^{-1}_0$ and $\sigma^0_\pi := \rho'_0 \circ \rho'^{-1}_\pi$, with $\rho_0$ and $\rho'_0$ defined as above.

Then

\[
(L_0, L_1, \sigma^0_\pi, \sigma^0_\pi)
\]

defines a set of Stokes data for $\mathcal{H}^0 p_+ (\mathcal{M} \otimes \mathcal{E}^\frac{1}{\pi})$.

**Proof:** The isomorphism $\Omega : \mathcal{L}_\vartheta \to H^1 (\mathfrak{A} \times \{\vartheta\}, \overline{\beta}_I^\vartheta \mathcal{K}^\vartheta)$ passes the filtrations on $\mathcal{L}_0$ and $\mathcal{L}_\pi$ to $L_0$ and $L_1$ and therefore yields to a suitable graduation of $L_0 = \tilde{G}_0 \oplus \tilde{G}_\pi \cong \mathcal{K}^0_{x_1} \oplus \mathcal{K}^0_{x_3}$ and $L_1 = \tilde{H}_1 \oplus \tilde{H}_0 \cong \mathcal{K}^\pi_{x_1} \oplus \mathcal{K}^\pi_{x_3}$.

On the level of graduated spaces $\Omega : G_0 \oplus G_\pi \to \tilde{G}_0 \oplus \tilde{G}_\pi$ (resp. $H_+ \oplus H_0 \to \tilde{H}_+ \oplus \tilde{H}_0$) is obviously described by a block diagonal matrix. Furthermore, by definition of $\rho$ and $\rho'$, the maps $\sigma^0_{\pi}$ and $\sigma^0_{\pi}$ can be read as $\sigma^0_\pi = \Omega \circ S^0_\pi \circ \Omega^{-1}$ and $\sigma^0_\pi = \Omega \circ S^0_\pi \circ \Omega^{-1}$. Since, according to the construction in 3.5, $S^0_\pi$ is upper block triangular (resp. $S^0_\pi$ is lower block triangular) the same holds for $\sigma^0_\pi$ (resp. $\sigma^0_\pi$). 

\[\square\]
3.3 Explicit computation of the Stokes matrices

The determination of the Stokes data thus corresponds to the following picture:

\[
H^1 \left( \mathcal{A} \times \{0\}, \pi^* \mathcal{K}^0 \right) \cong H^1 \left( \mathcal{A} \times I_1, \pi^* \mathcal{K}^1 \right) \]

\[
H^1 \left( \mathcal{A} \times \{0\}, \pi^* \mathcal{K}^0 \right) \cong H^1 \left( \mathcal{A} \times \{\pi\}, \pi^* \mathcal{K}^{\pi} \right) \]

In Lemma 3.3 we have already computed the cohomology groups for \( \vartheta = 0 \) and \( \vartheta = \pi \) using the Leray covering \( \mathfrak{A} \). Now for \( l = 0, 1 \) we fix a diffeomorphism \( \mathcal{A} \times I_l \cong \mathcal{A} \times I_l \) by lifting the vector field \( \partial_\vartheta \) to \( \mathcal{A} \times S^1 \) such that the lift is equal to \( \partial_\vartheta \) away from a small neighborhood of \( \partial \mathcal{A} \) and such that the diffeomorphism induces \( B^l \cong B^{l+1} \times I_l \) where \( \vartheta_1 = \pi \) and \( \vartheta_2 = 0 \).

It induces a diffeomorphism \( \mathcal{A} \times \{\vartheta_l\} \cong \mathcal{A} \times \{\vartheta_{l+1}\} \) and an isomorphism between the push forward of \( \mathcal{K}^{\vartheta_l} \) and \( \mathcal{K}^{\vartheta_{l+1}} \). Moreover it sends the boundary \( \partial B^{\vartheta_l} \) to \( \partial B^{\vartheta_{l+1}} \) (i.e. the boundaries are rotated in counter clockwise direction by the angle \( \pi \)). Via this diffeomorphism the curves \( \alpha_i \subset \mathcal{A} \times \vartheta_l \) are sent to curves \( \tilde{\alpha}_i \subset \mathcal{A} \times \vartheta_{l+1} \) and therefore induce another Leray covering \( \tilde{\mathfrak{A}} \) of \( H^1 \left( \mathcal{A} \times \{\vartheta_{l+1}\}, \pi^* \mathcal{K}^{\vartheta_{l+1}} \right) \).

Explicitly, for \( l = 0 \) we get a Leray covering \( \tilde{\mathfrak{A}} \) of \( H^1 \left( \mathcal{A} \times \{\pi\}, \pi^* \mathcal{K}^{\pi} \right) \) and, as in the previous chapter, we get an isomorphism \( \tilde{\Gamma}_\pi : H^1 \left( \mathcal{A} \times \{\pi\}, \pi^* \mathcal{K}^{\pi} \right) \rightarrow \tilde{H}^1 \left( \tilde{\mathfrak{A}}, \pi^* \mathcal{K}^{\pi} \right) \), which gives us:

\[
H^1 \left( \mathcal{A} \times \{\pi\}, \pi^* \mathcal{K}^{\pi} \right) \cong \mathcal{K}^{\pi} \left( \tilde{\alpha}_1 \right) \oplus \mathcal{K}^{\pi} \left( \tilde{\alpha}_3 \right) \cong \mathcal{K}_2^{\pi} \oplus \mathcal{K}_3^{\pi}
\]
Thus the above diffeomorphism leads to an isomorphism

\[ \mu_0^\pi : K^0_{x_1} \oplus K^0_{x_3} \xrightarrow{\cong} K^\pi_{x_1} \oplus K^\pi_{x_3} \]

Furthermore let us fix the following vector space \( V := K^\pi_c \) to be the stalk of the local system \( K \) at the point \( c := (0, \pi, \frac{1}{2}, 0) \) (which is obviously a point in the fiber \( \overline{A} \times \{\pi\} \)). By analytic continuation we can identify every non-zero stalk \( K^\vartheta \) with \( V \) for all \( \vartheta \).

Now consider the following diagram of isomorphisms:

It remains to determine the map \( \nu_\pi \) (respectively \( N_\pi \)). Therefore we will combine the coverings \( \mathfrak{A} \) and \( \hat{\mathfrak{A}} \) of \( \overline{A} \times \{\pi\} \) to a refined covering \( \mathfrak{B} \). We get refinement maps \( \text{ref}_{\mathfrak{A} \rightarrow \mathfrak{B}} \) and \( \text{ref}_{\hat{\mathfrak{A}} \rightarrow \mathfrak{B}} \) and receive the following picture:
Since $\mathfrak{B}$ is again a Leray covering the refinement maps induce isomorphisms on the cohomology groups. Thus we will extend the above diagram of isomorphisms in the following way:

\[
\begin{align*}
\check{H}^1(\mathfrak{A}, \overline{\beta}_i^\pi K^\pi) & \xrightarrow{\nu^i} \check{H}^1(\mathfrak{A}, \overline{\beta}_i^\pi K^\pi) \\
H^1(\overline{A} \times \{\pi\}, \overline{\beta}_i^\pi K^0) & \xrightarrow{\cong} \check{H}^1(\mathfrak{A}, \overline{\beta}_i^\pi K^\pi) \\
\check{H}^1(\mathfrak{A}, \overline{\beta}_i^\pi K^\pi) & \xrightarrow{\cong} H^1(\overline{A} \times \{\pi\}, \overline{\beta}_i^\pi K^\pi)
\end{align*}
\]

To determine the refinement maps on the first cohomology groups we have to consider the Čech complex for $\mathfrak{B}$.

We set the following index sets

- $I := \{1, 2, 3\}$ (indices corresponding to the covering $\mathfrak{A} = \bigcup_{i \in I} A_i$)
- $\tilde{I} := \{1, 2, 3\}$ (corresponding to $\mathfrak{A} = \bigcup_{i \in \tilde{I}} \tilde{A}_i$)
- $J := \{1, 2, 3, \ldots, 11\}$ (corresponding to $\mathfrak{B} = \bigcup_{j \in J} B_j$)
- $K := I \times J$
- $J' := \{2, 3, 9, 10, 11\} \subset J$
- $K' = \{(1, 2), (1, 9), (1, 11), (2, 3), (2, 8), (2, 9), (2, 10), (2, 11), (3, 9), (3, 10), (3, 11), (4, 9), (4, 10), (4, 11), (5, 9), (5, 10), (6, 9), (8, 9), (9, 10), (10, 11)\} \subset K$

The Čech complex for $\mathfrak{B}$ is given by:

\[
\begin{align*}
\bigoplus_{j \in J'} \check{H}^0(B_j, \overline{\beta}_i^\pi K^\pi) & \xrightarrow{d_0} \check{C}^1 \\
\bigoplus_{(i,j) \in K'} \check{H}^0(B_i \cup B_j, \overline{\beta}_i^\pi K^\pi) & \xrightarrow{d_1} \check{C}^2 \\
& \vdots
\end{align*}
\]

To determine the map $d_0$ we will use the following identification of $\check{C}^1$.

As above, by analytic continuation we can identify each component $\check{H}^0(B_i \cup B_j, \overline{\beta}_i^\pi K^\pi)$ with
\( \psi^{k(i,j)} := \bigoplus_{i=1}^{k(i,j)} \psi \) where \( k(i,j) \) denotes the number of connected components of \((B_i \cap B_j)\).

\( k(i,j) = 1 \) for all \((i,j) \in J\) except for \((3,9)\) and \((6,9)\) where it is equal to 2. So we have an isomorphism

\[
\tilde{C}^1 \cong \bigoplus_{(i,j) \in K'} \psi^{k(i,j)}
\]

Now if we take a section \( \beta_0 \) of \( B_9 \) (which is the intersection with one of the bordering \( B_i \)'s) and identify it with \( \mathbb{V} \), we have to take care about the monodromies \( S, T \) around the two leaks in \( \hat{A} \times \{ \pi \} \). The following picture shows, by restriction to which boundary component we receive monodromy.

![Diagram](image)

**Remark 3.7:** If we follow \( S \) and \( T \) in the coordinates of \( \hat{A} \times \{ \vartheta \} \), we can also describe them in terms of monodromy around the divisor components: \( S \) can be described by a path \( \gamma_S : [0,1] \to \hat{A} \times \{ \vartheta \}, \tau \mapsto (0, \vartheta, 0, \tau \cdot 2\pi) \) in the coordinates \((|t|, \vartheta, |x|, \theta_x)\). If we look at this path in the (complex) blow up of the singular locus of \( M \), \( \gamma_S \) corresponds to the monodromy around \( \hat{S}_1 \).

In the same way \( T \) is given by \( \gamma_T : [0,1] \to \hat{A} \times \{ \vartheta \}, \tau \mapsto (0, \vartheta, 0, \tau \cdot 2\pi) \) in the coordinates \((|\tilde{u}'_1|, \vartheta, |\tilde{v}'_1|, \theta_{\tilde{v}'_1})\) and it corresponds to the monodromy around the strict transform of \( S_1 \). Note that \( S \) and \( T \) do not depend on \( \vartheta \).

Now one can write down easily the map \( d_0 \) of the Čech complex and the refinement maps \( \text{ref}_{\hat{A}\to \hat{B}} \) with respect to the above analytic continuation.

Furthermore let \((a_2, a_4)\) be a base of \( \tilde{H}^1 \left( \hat{A}, \beta_1^p K^p \right) \cong K_{x_2}^p \oplus K_{x_3}^p \cong \mathbb{V} \oplus \mathbb{V} \) and \((\hat{a}_1, \hat{a}_3)\) a base of \( \tilde{H}^1 \left( \hat{A}, \beta_3^p K^p \right) \cong K_{x_1}^p \oplus K_{x_3}^p \cong \mathbb{V} \oplus \mathbb{V} \). We receive two bases of \( \tilde{H}^1 \left( \beta, \beta_1^p K^p \right) \), namely \( \text{ref}_{\hat{A}\to \hat{B}} (\hat{a}_1, \hat{a}_3) \) and \( \text{ref}_{\hat{A}\to \hat{B}} (a_2, a_4) \). Thus \( \nu' \) is determined by representing the base \( \text{ref}_{\hat{A}\to \hat{B}} (\hat{a}_1, \hat{a}_3) \) in terms of \( \text{ref}_{\hat{A}\to \hat{B}} (a_2, a_4) \).

As before we identify \( \tilde{C}^1 \left( \beta, \beta_1^p K^p \right) \cong \bigoplus \psi^{k(i,j)} \) and we end up in solving the following equation for each \((i,j) \in K'\):

\[
\left( \text{ref}_{\hat{A}\to \hat{B}} (\hat{a}_1, \hat{a}_3) \right)_{(i,j)} \equiv \left( \text{ref}_{\hat{A}\to \hat{B}} (a_2, a_4) \right)_{(i,j)} \mod \text{im} \left( d_0 \right)
\]

We get the following result:

\[
\hat{a}_1 = -a_2 + (1 - ST^{-1})a_4, \quad \hat{a}_3 = -ST^{-1}a_4
\]
and consequently the map \( N_\pi \) is given by the matrix
\[
\begin{pmatrix}
-1 & 1 - ST^{-1} \\
0 & -ST^{-1}
\end{pmatrix}
\]

For calculating \( S^0_\pi : \mathcal{L}_\pi \to \mathcal{L}_0 \), we will use exactly the same procedure, except that we have to take care about the continuation to the vector space \( \mathcal{V} \).

First we fix another vector space \( \mathcal{W} := \mathcal{K}_0^0 \) where \( c = (0, 0, \frac{1}{2}, 0) \in \overline{A} \times \{0\} \) and consider the following diagram:

\[
\begin{array}{c}
\mathcal{L}_\pi \cong H^1(\overline{A} \times \{\pi\}, \overline{\partial}^\pi_{\overline{A}} \mathcal{K}^\pi) \rightarrow H^1(\overline{A} \times \{0\}, \overline{\partial}^0_{\overline{A}} \mathcal{K}^0) \\
\cong \Gamma_0 \\
K_{x_2}^\pi + K_{x_4}^\pi \rightarrow K_{x_2}^0 + K_{x_4}^0 \rightarrow K_{x_1}^0 + K_{x_3}^0 \\
\mathcal{W} \oplus \mathcal{W} \rightarrow N_0 \rightarrow \mathcal{W} \oplus \mathcal{W}
\end{array}
\]

As before, for the determination of the map \( \nu_0 \) (respectively \( N_0 \)) we combine the coverings \( \mathfrak{A} \) and \( \mathfrak{A} \) of \( \overline{A} \times \{0\} \) to the refined covering \( \mathfrak{B} \) and get the refinement maps \( \text{ref}_{\mathfrak{A} \to \mathfrak{B}} \) and \( \text{ref}_{\mathfrak{A} \to \mathfrak{B}} \) which induce isomorphisms on the cohomology groups.

We get:
\[
N_0 = \begin{pmatrix}
-TS^{-1} & 0 \\
1 - TS^{-1} & -1
\end{pmatrix}
\]

Now we extend \( \nu_0 \) respectively \( N_0 \) to the vector space \( \mathcal{V} \oplus \mathcal{V} \) by \( \mu_\pi^0 \) (which does not affect \( N_0 \)):

\[
\begin{array}{c}
K_{x_2}^\pi + K_{x_4}^\pi \rightarrow K_{x_2}^0 + K_{x_4}^0 \rightarrow K_{x_1}^0 + K_{x_3}^0 \rightarrow \mathcal{V} \oplus \mathcal{V} \\
\mathcal{V} \oplus \mathcal{V} \rightarrow \mathcal{W} \oplus \mathcal{W} \rightarrow N_0 \rightarrow \mathcal{W} \oplus \mathcal{W} \rightarrow \mathcal{V} \oplus \mathcal{V}
\end{array}
\]

Let us fix two isomorphisms \( \Sigma_0 \) and \( \Sigma_{\pi} \), which we will call the standard identification of \( H^1(\overline{A} \times \{0\}, \overline{\partial}^0_{\overline{A}} \mathcal{K}^0) \), respectively \( H^1(\overline{A} \times \{\pi\}, \overline{\partial}^\pi_{\overline{A}} \mathcal{K}^\pi) \) with the vector space \( \mathcal{V} \oplus \mathcal{V} \).

\[
\Sigma_0 : H^1(\overline{A} \times \{0\}, \overline{\partial}^0_{\overline{A}} \mathcal{K}^0) \rightarrow K_{x_1}^0 + K_{x_3}^0 \rightarrow \mathcal{V} \oplus \mathcal{V} \\
\Sigma_{\pi} : H^1(\overline{A} \times \{\pi\}, \overline{\partial}^\pi_{\overline{A}} \mathcal{K}^\pi) \rightarrow K_{x_2}^\pi + K_{x_4}^\pi \rightarrow \mathcal{V} \oplus \mathcal{V}
\]

24
Summarizing the previous calculations, we can now prove Theorem 1.5.

3.4 Conclusion: Proof of Theorem 1.5

At first remark that $\mu^0_\pi \circ \mu^0_\pi : \mathcal{K}^\pi_x \oplus \mathcal{K}^\pi_y \rightarrow \mathcal{K}^0_x \oplus \mathcal{K}^0_y \rightarrow \mathcal{K}^\pi_x \oplus \mathcal{K}^\pi_y$ is the isomorphism arising from varying the angel $\vartheta$ via the path $\gamma_U : [0,1] \rightarrow S^1, \tau \mapsto \pi + \tau \cdot 2\pi$. This corresponds to the monodromy $U$ around the divisor component $\{0\} \times \mathbb{P}^1$.

Furthermore from Theorem 3.6 we know that

$$(H^1(\mathbb{A} \times \{0\}, \beta^0 \mathcal{K}^0), H^1(\mathbb{A} \times \{\pi\}, \beta^\pi \mathcal{K}^\pi), \sigma_0^\pi, \sigma^0_\pi)$$

defines a set of Stokes data. With the standard identifications of our vector spaces we get the following diagram:

Since $\Sigma_0$ and $\Sigma_\pi$ respect the given filtrations of the vector spaces $H^1(\mathbb{A} \times \{0\}, \beta^0 \mathcal{K}^0)$ and $H^1(\mathbb{A} \times \{\pi\}, \beta^\pi \mathcal{K}^\pi)$, it follows that the induced filtrations on $V \oplus V$ are mutually opposite with respect to $S_0^1 = \Sigma_\pi \circ \sigma_0^\pi \circ \Sigma_0^{-1}$ and $S_\pi^0 = \Sigma_0 \circ \sigma^0_\pi \circ \Sigma_\pi^{-1}$. Thus we conclude that $(L_0, L_1, S_0^1)$ defines a set of Stokes data for $\mathcal{H}^0(p_+ (\mathcal{M} \otimes \mathcal{E}^\perp))$.

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