AVERAGE REPRESENTATION NUMBERS FOR SPINOR GENERA

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Abstract. In this paper we establish a formula for the average of representation numbers of ternary quadratic forms in a spinor genus over a totally real number field. The significant fact about the formula is the fact that it is given in terms of local quantities. Such a formula has already been established by Kneser [7], Hsia [5] and Schulze-Pillot [10] by a different method.

1. Introduction

The problem which motivates this work is that of calculating the representation numbers for ternary forms. For a ternary form $Q$ with integral coefficients over a totally real field $F$ and an integer $m$ the representation number $r_Q(m)$ is the number of integral solutions to $Q(x, y, x) = m$.

Recall the well known Seigel formula which gives a weighted sum of representation numbers $r_{Q_i}$ for forms $Q_1 \ldots Q_t$ in a genus of quadratic forms. Roughly speaking, a genus of quadratic forms is a finite collection of forms which are equivalent over all completions $F_v$. The weights in the sum are given by the size of stabilizers $O(Q_i)$ of the forms in a certain group which permutes the forms in a genus. The Seigel formula is

\begin{equation}
\sum_{i=1}^t \frac{r_{Q_i}(m)}{|O(Q_i)|} = \left( \sum_{i=1}^t \frac{1}{|O(Q_i)|} \right) \prod_v \beta_v(m)
\end{equation}

where the factors $\beta_v$ are local densities which are calculated over the completions $F_v$.

Let $\chi = \chi_\kappa$ for $\kappa \in F^\times$ be a quadratic character of $\mathbb{A}^\times / F$ given by the product of Hilbert symbols $\chi(a) = \prod_v (-\kappa, a_v)_v$. Our formula differs from the Seigel formula by the presence of a twist by a certain such $\chi$.

\begin{equation}
\sum_{i=1}^t \frac{\chi(\nu(h_i))}{|O(Q_i)|} r_{Q_i}(m) = \begin{cases} C|x|_\infty \prod_v f_{0,v}(x) & \text{if } m = \kappa x^2 \\ 0 & \text{if } m \notin \kappa(F^\times)^2 \end{cases}
\end{equation}
Where as in the case of the Seigel formula, the right hand side is given in terms of locally calculated factors $f_{0,v}$. Furthermore, only a finite collection of the factors $f_{0,v}$ is not identically 1 so the product is in fact finite. The $h_i$ are elements of the group $H := \text{GSpin}(V)$ which permutes the forms $Q_i$ and $\nu : H(\mathbb{A}) \to \mathbb{A}^\times / \mathbb{A}^{\times,2}$ is the reduced norm which in this case is also the spinor norm.

The composition of $\chi$ with the spinor norm $\nu$, lifts $\chi$ to a character of $H(\mathbb{A})$. Those $\chi$ for which (1.2) is valid must be constant on the stabilizers $O(Q_i)$ of the forms $Q_i$. The subsets of $h_1 \ldots h_t$ for which $\nu(h_i) = \nu(h_j)$ form spinor genera and evidently $\chi \circ \nu$ is really a character of the group which permutes the spinor genera. Thus the characters $\chi$ for which (1.2) is valid form a finite group which indexes the spinor genera in a genus.

By summing over the formulas for all such characters we can use finite group Fourier inversion to obtain the partial sums over the spinor genera. That is, we can obtain expressions of the form

$$R(\text{Spn}(Q), m) := \sum_i \frac{r_Q(m)}{|O(Q_i)|} = \prod_v (\text{local data})$$

where the sum is over the spinor genus of $Q$.

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2. Notation and Basic Definitions

We will generally follow the notation in $\text{Spin}$. Let $F$ be a number field and $O$ its ring of integers. Let $\mathbb{A} = \mathbb{A}_F$ be the ring of adeles of $F$ and denote a completion of $F$ by $F_v$. Let $\sigma_i : F \to \mathbb{R}$ $i = 1 \ldots d$ be the real embeddings of $F$. Also, fix a totally positive quadratic form $Q \in F[x_1, x_2, x_3]$ with integral coefficients.

Let $V$ be the quadratic space $F^3$ with a basis $e_1, e_2, e_3$ and let $Q$ also denote the quadratic form on $V$ defined by $Q(a_1e_1 + e_2v_2 + e_3v_3) = Q(a_1, a_2, a_3)$. A full rank Lattice $L \subset V$ with basis $\{l_1, l_2, l_3\}$ defines a quadratic form $Q_L \in F[x_1, x_2, x_3]$ by $Q_L(a_1, a_2, a_3) = Q(a_1l_1 + a_2l_2 + a_3l_3)$. In fact, a lattice $L$ corresponds to a set of integrally equivalent forms which correspond to different choices of a basis for $L$. Evidently, the lattice $L_0 := O\{e_1, e_2, e_3\}$ gives rise to integral equivalence class of the form $Q$. For an integer $m \in O$, the number of points $x \in O^3$ such that $Q(x) = m$ is called the representation number of $m$ with respect to $Q$ and is denoted by $r_Q(m)$. Notice that the representation numbers
of integrally equivalent lattices are equal and therefore lattices are the correct objects to consider when looking for representation numbers.

**Definition 2.1.** Let \( L \subset V \) be a lattice of full rank. The genus of \( L \) is the set of lattices \( K \subset V \) such that the local lattices \( K_v := K \otimes \mathcal{O}_v \) and \( L_v := L \otimes \mathcal{O}_v \) are isometric with respect to \( Q \) for all local places \( v \).

Note the local equivalence does not mean that \( L \) is isometric to \( K \) over \( F \).

**Definition 2.2.** Let \( Q_1 \) and \( Q_2 \) be two quadratic forms corresponding to two lattices \( L_1 \) and \( L_2 \). \( Q_1 \) and \( Q_2 \) belong to the same genus of forms (or simply a genus) if the lattices \( L_1 \) and \( L_2 \) belong to the same genus.

A convenient way to describe the genus of \( L \) is as the orbit of the Adelic orthogonal group of \((V, Q)\), \( O(A) \) under the action \( L \rightarrow hL \) for \( h = (h_v) \in O(A) \), where \( hL \otimes \mathcal{O}_v = h_vL_v \). The stabilizer of \( L_0 \) under this action is denoted by \( O_h(L_0) \) and \( O(A) \) can be decomposed as the coset space

\[
O(A) = \bigcup_{i=1}^t O(F)h_iO_h(L_0).
\]

The \( h_i \) in this decomposition are the representatives of the lattices in the genus of \( L_0 \) which we denote by \( L_i := h_i \cdot L_0 \) for \( i = 1 \ldots t \). The corresponding forms are denoted by \( Q_i := h_i \cdot Q \).

For every ternary quadratic form \( Q \), there is a quaternion algebra \( \text{Quat}(a, b) := B(F) \) such that, with the right basis, \( Q \) is a multiple of the reduced norm of the algebra on the space of trace zero quaternions. We let \( B \) be the algebra that corresponds to the form \( Q \) which we fixed and identify \( V \) with the space of trace zero quaternions. The group \( H(A) := B^\times(A) \) acts on \( V(A) \subset B(A) \) by conjugation \( h \cdot x = hxh^{-1} \). This action identifies \( A^\times \backslash H(A) \) with \( \text{SO}(3)(A) \).

There are two possible isomorphism classes of the local algebras \( B_v = B(F_v) \). For almost all \( v \) we have \( B_v \simeq M_2(F_v) \), the algebra of \( 2 \times 2 \) matrices in which case \( B_v \) is called unramified. At a finite (even) number of places \( v \), the algebra \( B_v \) is isomorphic to a division algebra and we say that \( B_v \) is ramified.

**Definition 2.3.** The spinor norm \( \theta : O(F_v) \rightarrow F_v^\times/(F_v^\times)^2 \) or \( \theta : O(F) \rightarrow F^\times/(F^\times)^2 \) will be used to denote the spinor norm regardless of the underlying field. With the identification we made between \( A \backslash H(A) \) and \( \text{SO}(3)(A) \) the spinor norm for \( h_v \in \text{SO}(3)(F_v) \) is \( Q(h_v) \).
3. The Calculation

3.1. Initial Formula. The basic setup for our calculation is the same as in $\text{SP1}$ and $\text{DSP}$. We use a lift of a one dimensional automorphic representation (automorphic character) from the group $\text{O}(A)$ to the group $\tilde{\text{SL}}_2(A)$. We will then use the correspondence between automorphic forms and classical modular forms (see $\text{B}$) compare Fourier coefficients and extract the arithmetic information.

Let $\chi$ be a quadratic character of $A^\times$ given by the product of Hilbert symbols $\chi(a) = \chi_v(a) = \Pi_v(a_v, -\kappa_v)$ for some $\kappa \in F^\times$ determined up to a square class. The automorphic character of $\text{O}(A)$ is an extension of the composition $\chi \circ Q : A^\times \backslash H(A) \simeq \text{SO}(3)(A) \to \{\pm 1\}$.

Recall that the representation corresponding to the character $\chi \circ Q$ is a space of automorphic forms on $\tilde{\text{SL}}_2(A)$, each constructed from an integral of a Schwartz function $f \in S(V(A))$ multiplied by a theta kernel. The function $f$ which gives rise to the modular form from which we extract information is given by $f := \prod_v f_v$. For every finite $v$ the factor $f_v$ is the characteristic function of $L_{0,v}$ and at the real places we define $f_\infty(x) = \exp(-2\pi \text{trace}(Q(x)))$.

Definition 3.1. Let $\omega$ be the Weil representation of $\tilde{\text{SL}}_2(A)$ on the space $S(V(A))$. The theta kernel defined for a Schwartz function $f \in S(V(A))$ a group element $g \in \tilde{\text{SL}}_2(A)$ and an $h \in \text{O}(A)$ is

$$\theta_V(g, h, f) := \sum_{x \in V(F)} \omega(g) f(h^{-1}x)$$

With that we can define the theta lift of the character $\chi \circ Q$.

Definition 3.2. In the lift of the automorphic character $\chi \circ Q$ the Schwartz function $f$ gives rise to the automorphic form of $g \in \tilde{\text{SL}}_2(A)$

$$\Theta_f(\chi \circ Q)(g) := \int_{\text{O}(F) \backslash \text{O}(A)} \chi \circ Q(h) \theta_V(g, h, f) dh$$

In order to benefit from the decomposition of $\text{O}(A)$ the character $\chi$ must be compatible with $Q$ in the following way:

Definition 3.3. The form $Q$ and the character $\chi = \chi_\kappa$ are said to be compatible if

1. $Q$ represents an element in the square class of $\kappa$ over $F$ and
2. for every $h \in \text{O}(L_0)$ we have $\chi(Q(h)) = 1$, i.e. $\theta(\text{O}_h(L_0) \subset \ker(\chi_\kappa)$.
Remark 3.4. $\chi$ is compatible with $Q$ if and only if it is compatible with all the forms in the genus of $Q$. Therefore we would sometimes say that $Q$ is compatible with the genus rather than with $Q$.

The modular form from whose Fourier coefficients contains the information we seek is the following.

**Definition 3.5.** For $Z = (z_1, \ldots, z_d) \in \mathcal{H}^d$ where $\mathcal{H}$ is the upper half plane and $d = |F : Q|$. For $m \in F$ the trace $\text{trace}(mz)$ is given by

$$\text{trace}(mz) = \sum_{i=1}^{d} \sigma_i(m) z_i$$

where $\sigma_i : F \to \mathbb{R}$ are the different embeddings of $F$ in $\mathbb{R}$. Using that, we define the function

$$\theta_Q(Z, L_0) = \sum_{x \in L_0} \exp(2\pi i \text{trace}(Q(x)Z)).$$

This function is a modular form of weight $(3/2, \ldots, 3/2)$ and it is not too hard to see that its Fourier expansion is

$$\theta_Q(Z, L_0) = \sum_{m \in \mathcal{O}} r_Q(m) \exp(2\pi i \text{trace}(mZ)).$$

As in the calculation in [12], when the character $\chi$ is compatible with $Q$ we get that for $g = (g_\infty, 1 \ldots)$

$$\Theta_f(\chi \circ Q)(g) = \sum_{i=1}^{t} \frac{\chi(Q(h_i))}{|O(h_iL_0)|} \sum_{x \in h_iL_0} \omega(g_\infty) \exp(-2\pi i \text{trace}(Q(x)))$$

where $O(h_iL_0)$ is the finite group of integral automorphisms of the lattice $h_iL_0$. Next we have as in the bottom of page 3 in [12] that the corresponding modular form is

$$\Theta_f(\chi \circ Q)(Z) := \sum_{i=1}^{t} \chi(Q(h_i)) \frac{\theta_{Q_i}(Z, L_0)}{|O(Q_i)|}$$

$$= \sum_{i=1}^{t} \frac{\chi(Q(h_i))}{|O(Q_i)|} \sum_{m \in \mathcal{O}} r_{Q_i}(m) \exp(2\pi i \text{trace}(mZ)).$$

In the notation of [6], equation (3.2) is the modular form corresponding to the automorphic form $I_V(\chi, f)$ where $f = f_{L_0}$ defined above. The main result of [6] is that

$$I_V(\chi, f) = I_U(\mu, f_0)$$

Where $I_U(\mu, f_0)$ is an automorphic form coming from the lift of the character $\mu$ from the one dimensional space $U$. Therefore we would
like to describe $I_U(\mu, f_0)$ and the modular form corresponding to it. From [5] we have that

$$I_U(\mu, f_0)(g) = \frac{1}{2} \sum_{x \in F} \omega_0(g) f_0(x)$$

where $\omega_0$ is the Weil representation of $\tilde{SL}_2(\mathbb{A})$ on the Schwartz space $S(U(\mathbb{A}))$. Another result of [5] is that the function $f_0 = \prod_v f_{0,v}$ is defined by the factors $f_{0,v} \in S(U(F_v))$ given by the local equation

$$\varphi_0 f_{0,v}(r) = |r|_v \chi_v(r) \int_{T_{x_0}(F_v) \setminus H(F_v)} f(rh^{-1} \cdot x_0) \chi_v(Q_v(h)) dh$$

Where the element $x_0$ is any vector in $V(F)$ such that $Q(x_0)$ is in the square class of $\kappa$, the quantity that defines $\chi$ and $T_{x_0}$ is the stabilizer of $x_0$ in $H(F_v)$. If no such $x_0$ exists then $f_{0,v} = 0$ for all $v$.

To simplify (3.3) we need to calculate $f_{0,v}$ for all $v$. For $x_0 \in V$, at almost all places $x_{0,v} \in L_{0,v}$ but for $r \notin \mathcal{O}_v$ we have $rx_0 \notin L_{0,v}$. In addition, at almost all places $\chi_v$ and $B_v$ are unramified and by lemmas 39 in [5] (the fundamental lemma) we get that $f_{0,v}$ is the characteristic function of $\mathcal{O}_v$ at all such places. In addition, by lemma 36 in [5], at places $v$ where $B_v$ is ramified, if $|x|_v$ is small enough then $f_{0,v}(x) = 0$. Furthermore, by [5] we have that for $f_{\infty}(x) = \exp(-2\pi Q(x))$ when $Q$ is positive definite we get $f_{0,\infty}(r) = r \exp(-2\pi \kappa r^2)$. Putting that together with (3.3) and evaluating at $g = (g_{\infty}, 1, 1, \ldots)$ we get that

$$I_U(\mu, f_0)(g) = \frac{C}{2} \sum_{x \in \mathcal{O}} \prod_{v \in S} f_{0,v}(x) |x|_{\infty} \omega_0(g_{\infty}) \exp(-2\pi \text{trace}(\kappa x^2))$$

Where $S$ is the finite set of non archimedean places which include the dyadic places and where $f_0$ is not the characteristic function of $\mathcal{O}_v$. Note also that the expression is up to scalar $C$ which depends on the normalization of the measure on $H(\mathbb{A})$.

If we consider the modular form we obtain, from the above automorphic form we get that

$$\Theta_f(\chi \circ Q)(Z) = \frac{C}{2} \sum_{x \in \mathcal{O}} |x|_{\infty} \prod_{v \in S} f_{0,v}(x) \exp(2\pi i \text{trace}(\kappa x^2) Z)$$

In [5] it is proved that $f_0$ defined above is even, using that and a comparison of (onedim 3.5) with (onedim 3.2) we get our result:

**Theorem 3.6.** Let $Q$ be an integral quadratic form with lattice $L_0$ such that $Q$ is compatible with $\chi = \chi_\kappa$ in the sense of definition 3.3. Then
AVERAGE REPRESENTATION NUMBERS FOR SPINOR GENERA

for \( m \in \mathcal{O} \)

\[
\sum_{i=1}^{t} \frac{\chi(Q(h_i))}{|O(Q_i)|} r_{Q_i}(m) = \begin{cases} 
C |x|_{\infty} \prod_{v \in S} f_{0,v}(x) & m = \kappa x^2 \\
0 & m \notin \kappa(F^\times)^2
\end{cases}
\]

Where \( S \) is the finite set of non-archimedean places which includes the
diadic places and places where the space \( V_v \) is not split or the character
\( \chi_{\kappa} \) is ramified.

3.2. Averages Over Spinor Genera. It is possible to use formula
(\ref{res-sn}) and others like it to obtain information about the average number
of representations over a spinor genus. Recall that two forms \( Q_i \) and
\( Q_j \) are in the same spinor genus if and only if the rotation relating
them is in the kernel of the spinor norms, i.e. \( Q(h_{i,v} h_{j,v}) \in (F_v^\times)^2 \)
for all \( v \). This means that \( \chi \) is constant on each spinor genera and the sum
in (\ref{res-spn}) breaks up into sums over the different spinor genera.

\[
\sum_{j=1}^{l} \chi(Q(h'_j)) R(Spn(Q'_j), m) = \begin{cases} 
C |x|_{\infty} \prod_{v \in S} f_{0,v}(x) & m = \kappa x^2 \\
0 & m \notin \kappa(F^\times)^2
\end{cases}
\]

where we have relabeled \( h'_j \) as the representatives of the spinor genera
\( Q'_j := h'_j \cdot Q \quad j = 1 \ldots l \).

The number of spinor genera in a genus corresponds to the number
of elements of the Abelian group \( G = R/ST \) defined in \( \ref{comp} \) p. 209. It
is immediate from the definition of \( G \) that its characters correspond to
characters which satisfy the second condition of compatibility with \( Q \)
in definition \( \ref{comp} \). Furthermore, The characters of \( G \) are quadratic and
therefore they all correspond to \( \chi_{\kappa_j} \) for some \( \kappa_j \in F^\times \).

From the definition of \( f_{0,v} \) we can see that a character \( \chi \) of \( G \) is
compatible with the form \( Q \) if and only if the right hand side of equation
(\ref{res-spn}) is not identically 0. Whether or not \( \chi \) is compatible with the
genus, there will be \( l \) equations of the form of (\ref{res-spn}) with \( \chi \) being one of
\( \chi_{\kappa_j} \). For those \( \chi_{\kappa_j} \) which are not compatible, the right hand side
of (\ref{res-sn}) will be zero but in any case the orthogonality relations of the
characters of \( G \) allow us to solve for each one of the \( R(Spn(Q'_j), m) \) for
all \( m \in \mathcal{O} \) and \( j = 1 \ldots l \). For the sake of clarity is should be said
that the question of compatibility is only relevant because it tells us in
which cases the right hand side of (\ref{res-sn}) is already known.

Lemma 3.7. When \( F = \mathbb{Q} \) the character \( \chi = \chi_{\kappa_j} \) as above is always
compatible with the form \( Q \). When \( F \neq \mathbb{Q} \), checking the condition of
compatibility reduces to checking for compatibility at the diadic places.
Proof. It is a fact that $Q$ represents an element from the square class of $\kappa_j$ if and only if $\kappa_j$ is locally represented at all places. The question of local representability of $\kappa_j$ almost always follows from the second condition of compatibility of $Q$ and $\chi_{\kappa_j}$. The spinor norms of $\SO(3)(F_v)$ are given in [4] and [14] and they allow us to verify that at least at the odd primes, when $\theta(\SO_v(L_{0,v})) \subset \ker(\chi_{\kappa_j})$ then $\kappa$ is represented by $L_{0,v}$.

When we restrict to $F = \mathbb{Q}$ then we can also verify the above condition at $\mathbb{Q}_2$ when $a$ or $b$ which define the quaternion algebra $\text{Quat}(a, b)$ are units. Therefore, at least over $\mathbb{Q}$, when $2$ does not divide $a$ or $b$, then all the characters of $G$ are compatible with the genus. □

It is the author’s intention to carry the calculation further in a follow up paper. In particular, it is possible to transform the orbital integrals defining $f_{0,v}$ into line integrals and simplify them significantly. It is also possible to use the formulas in [15] to simplify the local densities $\beta_v$.

In some situations (when the spinor genera consist of a single class) it is possible to use this result even to obtain formulas for representation numbers of individual forms.

4. Example

We start with the form

\[Q_1(x, y, z) = 4x^2 + 16y^2 + 64z^2.\]

Using the Kneser neighbor method [8] implemented by a computer program of Schulze-Pillot [11] we get that the genus of this form has one additional form

\[Q_2(x, y, z) = 20x^2 + 16xy + 16y^2 + 16z^2.\]

and $|O(Q_1)| = |O(Q_2)| = 8$. In order to find a character compatible with the form we need to consider the set $\theta(\SO_{\mathbb{Q}_v}(L_{0,v}))$ for every $v$. These are described in [4] and [2]. In particular, lemma 3.3 p. 208 in [2] says that for the form $Q$, at all odd primes $v$, $\theta(\SO_{\mathbb{Q}_v}(L_{0,v})) = \mathcal{O}_v^\times(\mathbb{Q}_v^\times)^2$. This means that in order for a character $\chi_{\kappa}$ to be compatible with the form $Q$ it must be trivial on $\mathcal{O}_v^\times(\mathbb{Q}_v^\times)^2$ for all odd $v$. That is, $(-k, a)_v = 1$ iff ord($a$) is even, i.e. ord($-\kappa$) is also even. The spinor norms of the stabilizers of the lattice over $\mathbb{Q}_2$ is given by theorem 2.7 in [4] and in our case it is equal to the kernel of $\chi_1$. Therefore the character $\chi = \chi_4$ is trivial on the stabilizer of the lattice and since $Q_1$ represents 4 the character is compatible with $Q_1$. To further evaluate equation (3.6) we use the ‘fundamental lemma’ of [6] which says that
whenever $L_{0,v}$ is a split lattice and the character $\chi_\kappa$ is unramified then $f_0, v$ is the characteristic function of $O_v$. The lattice of the form $Q_1$ is split at all odd primes and the character $\chi_4$ is unramified at all odd primes. Therefore for the genus of two forms $Q_1$ and $Q_2$, equation (4.4) becomes.

\[
\text{ressimp (4.3)} \quad r_{Q_1}(m) - r_{Q_2}(m) = \begin{cases} 
8C \sqrt{m} f_{0,2}(\sqrt{m}) & m \in (\mathbb{Z}^\times)^2 \\
0 & m \notin (\mathbb{Z}^\times)^2 
\end{cases}
\]

where

\[
\text{f02 (4.4)} \quad f_{0,2}(x) = |x|_2(-1, x)_2 \int_{T_x_0(Q_2) \setminus H(Q_1)} \text{Char}_{L_{0,v}}(xh^{-1} \cdot x_0)\chi_2(Q_2(h))dh
\]

We can simplify the evaluation of equation (4.3) by noticing the dependence of (4.4) on $x$. It is not hard to see that if $n$ is an odd number then

\[
\text{trans (4.5)} \quad f_{0,2}(nx) = (-1, n)_2 f_{0,2}(x)
\]

Next, it is clear that neither of the forms $Q_1$ or $Q_2$ can represent odd numbers. These observations give us the right hand side of (4.3) for all integers $m = 2^j n$ for odd all odd powers $j$ as well as $j = 0, 2$. For the remaining even powers of $j$ we use lemma 5.11 in (6) which says that at a prime $v$ where $\chi$ is non trivial and the lattice $L_{0,v}$ is non split (such is the case at $v = 2$ in this example) then $f_{0,2}(2^j n)$ vanishes for large enough $j$. The value $j$ such that $f_{0,2}(2^j n) = 0$ for all $l \geq j$ can be determined following the proof of lemma 5.11 in (6). The condition for vanishing is that if $v \in V(Q_2)$ and $Q_1(v) = 2^j n$ then $v \in L_{0,2}$. In our case this amounts to verifying that the solutions to $4a^2 + 16\beta^2 + 64\gamma^2 = 2^j n$ must be 2-adic integers for all $j \geq 6$. The remaining cases of $m = 4, 16$ are calculated by directly calculating $r_{Q_1}(4) = 0, r_{Q_2}(16) = 4, r_{Q_1}(4) = 2, r_{Q_2}(16) = 4$. Using equation (4.3) this gives that $8C f_{0,2}(2) = 1$ and $8C f_{0,2}(4) = 0$ and we finally get that

\[
\text{resfin (4.6)} \quad r_{Q_1}(m) - r_{Q_2}(m) = \begin{cases} 
2n(-1, 2n)_2 & m = 4n^2 \text{ for odd } n \\
0 & \text{otherwise} 
\end{cases}
\]

We can combine (4.5) and the Seigel formula (1.3) to find expressions for $r_{Q_1}(m)$ and $r_{Q_2}(m)$ which depend only on local calculations. The Seigel formula for the genus in this example is

\[
\text{sgex (4.7)} \quad r_{Q_1}(m) + r_{Q_2}(m) = 2 \prod_v \beta_v(m)
\]
Where the product is taken over all primes. The $\beta_v(m)$ are the local densities defined by Seigel and they were calculated explicitly in \[15\]. Thus we have obtained the following

$$r_1(m) = \begin{cases} n(-1, 2n)_2 + 2 \prod_v \beta_v(m) & m = 4n^2 \text{ for odd } n \\ 2 \prod_v \beta_v(m) & \text{otherwise} \end{cases} \quad (4.8)$$

and

$$r_2(m) = \begin{cases} 2 \prod_v \beta_v(m) - n(-1, 2n)_2 & m = 4n^2 \text{ for odd } n \\ 2 \prod_v \beta_v(m) & \text{otherwise} \end{cases} \quad (4.9)$$

These formulas give the representation numbers in terms of local factors but we can do a little better with a recursion formula and reduce the calculation to a finite number of factors. First of all, the product of the archimedean factors is given as $C\sqrt{m}$ where $C$ is some constant which does not depend on $m$. Next, in the notation in \[15\] $\beta_v(m) = \alpha_v(m, S)$ where $S$ is the gram matrix of the form $Q_1$. It follows from the definition of the local densities that if $u \in \mathbb{Z}^* \times v$ then $\beta_v(u^2m) = \beta_v(m)$. Therefore we obtain that if $d = p_1^{2k_1} p_2^{2k_2} \cdots p_r^{2k_r}$ for distinct primes $p_1 \ldots p_r$ then

$$\prod_v \beta_v(dm) = \sqrt{d} \prod_{i=1}^r \frac{\beta_{p_i}(nm)}{\beta_{p_i}(m)} \prod_v \beta_v(m) \quad (4.10)$$

$$= \sqrt{d} \prod_{i=1}^r \frac{\beta_{p_i}(dm)}{\beta_{p_i}(m)} \left[ r_{Q_1}(m) + r_{Q_2}(m) \right] / 2 \quad (4.11)$$

and our final formulas are

$$r_{Q_1}(dm) = \begin{cases} n(-1, 2n)_2 + \sqrt{d} \prod_{i=1}^r \frac{\beta_{p_i}(dm)}{\beta_{p_i}(m)} R(m) & dm = 4n^2 \text{ for odd } n \\ \sqrt{d} \prod_{i=1}^r \frac{\beta_{p_i}(dm)}{\beta_{p_i}(m)} R(m) & \text{otherwise} \end{cases} \quad (4.12)$$

and

$$r_{Q_2}(dm) = \begin{cases} \sqrt{d} \prod_{i=1}^r \frac{\beta_{p_i}(dm)}{\beta_{p_i}(m)} R(m) - n(-1, 2n)_2 & m = 4n^2 \text{ for odd } n \\ \sqrt{d} \prod_{i=1}^r \frac{\beta_{p_i}(dm)}{\beta_{p_i}(m)} R(m) & \text{otherwise} \end{cases} \quad (4.13)$$

where $R(m) = r_{Q_1}(m) + r_{Q_2}(m)$

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11

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