SUBCONVEXITY FOR $GL(3) \times GL(2)$ AND $GL(3)$ $L$-FUNCTIONS IN $GL(3)$ SPECTRAL ASPECT

PRAHLAD SHARMA

ABSTRACT. Let $f$ be a $SL(2, \mathbb{Z})$ holomorphic cusp form or the Eisenstein series $E(z, 1/2)$ and $\pi$ be a $SL(3, \mathbb{Z})$ Hecke-Maass cusp form with its Langlands parameter $\mu$ in generic position i.e. away from Weyl chamber walls and away from self dual for $ms$. We study an amplified second moment $\sum A(\pi_j) |L(1/2, \pi_j \times f)|^2$ and deduce the subconvexity bound

$$L(1/2, \pi \times f) \ll_{f, \epsilon} \|\mu\|^{3/2-1/1960+\epsilon}.$$  

As a corollary, when $f = E(z, 1/2)$, we also obtain the subconvexity bound

$$L(1/2, \pi) \ll_{\epsilon} \|\mu\|^{3/4-1/4044+\epsilon}.$$  

1. Introduction

Let $f$ be a $SL(2, \mathbb{Z})$ holomorphic cusp form with the $n$-th fourier coefficient $\lambda_f(n)$. For an $SL(3, \mathbb{Z})$ Hecke Maass cusp form $\pi$, we denote its Langlands parameter by $\mu = \mu_\pi = (\mu_1, \mu_2, \mu_3)$. This triplet satisfies $\mu_1 + \mu_2 + \mu_3 = 0$, normalised such that Ramanujan predicts $\mu \in (i\mathbb{R})^3$. Let $\pi_0$ be such that its Langlands parameter $\mu_0 = (\mu_{0,1}, \mu_{0,2}, \mu_{0,3})$ is in generic position, i.e. there exists constants $C > c > 0$ such that

$$c \leq \frac{|\mu_{0,j}|}{\|\mu_0\|} \leq C \quad (1 \leq j \leq 3), \quad \text{and} \quad c \leq \frac{|\mu_{0,j} - \mu_{0,i}|}{\|\mu_0\|} \leq C \quad (1 \leq j \leq 3).$$  

Note that for a suitable choice of $c$ and $C$, this set cover 99% of Maass forms. The Rankin-Selberg product of $\pi_0$ with $f$ is given by

$$L(s, \pi_0 \times f) = \sum_{r,n=1}^{\infty} A_{\pi_0}(r, n) \lambda_f(n) \frac{(nr^2)^s}{(n^2)^s},$$  

which converges absolutely for $\Re(s) > 1$. This series extends to a entire function and satisfies functional equation of the Reimman type and has conductor $\|\mu_0\|^6$. Consequently, the Phragmen-Lindelof principle yeilds the convexity bound

$$L(1/2, \pi_0 \times f) \ll_{\pi, f, \epsilon} \|\mu_0\|^{3/2+\epsilon}.$$  

The Lindelöf hypothesis asserts that the exponent $3/2+\epsilon$ can be replaced by any positive number. In this paper we prove the following subconvex bound.

**Theorem 1.1.** Let $\pi_0$ be an $SL(3, \mathbb{Z})$ Hecke-Maass cusp form with Langlands parameter $\mu_0$ in generic position. Let $f$ be a $SL(2, \mathbb{Z})$ holomorphic Hecke cusp form or the Eisenstein series $E(z, 1/2)$. Then

$$L(1/2, \pi_0 \times f) \ll_{f, \epsilon} \|\mu_0\|^{3/2-1/1960+\epsilon}.$$  

(1.3)
Note that when \( f(z) = E(z, 1/2) \),
\[
L(s, \pi_0 \times f) = \sum_{r,n=1}^{\infty} \frac{A_{\pi_0}(r,n)d(n)}{(nr^2)^s} = \left( \sum_{m=1}^{\infty} \frac{A_{\pi_0}(1,m)}{m^s} \right)^2 = (L(s, \pi_0))^2. \tag{1.4}
\]

Hence, as corollary we obtain following, which an improvement over the subconvex bound obtained in Blomer-Buttcane \[3\].

**Theorem 1.2.** Let \( \pi_0 \) be an \( SL(3, \mathbb{Z}) \) Hecke-Maass cusp form with Langlands parameter \( \mu_0 \) in generic position. Then
\[
L(1/2, \pi_0) \ll \|\mu_0\|^{3/4-1/3920+\epsilon}. \tag{1.5}
\]

Subconvexity for degree three \( L \)-functions was first obtained by X. Li \[12\] for a fixed self dual Maass form. In his series of papers \[17, 14, 15, 16, 18\], Munshi introduced a different approach to subconvexity through which he obtained subconvex bounds for more general degree three \( L \)-functions. In \[19\], Munshi adapted his new approach to obtain \( t \)-aspect subconvexity for \( GL(3) \times GL(2) \) \( L \)-functions, where the \( GL(3) \) form is any Hecke-Maass cusp form for \( SL(3, \mathbb{Z}) \). Using Munshi’s approach, Kumar-Mallesham-Singh \[10\] obtained subconvexity for \( GL(3) \times GL(2) \) in the \( GL(3) \) spectral aspect, where the Langlands parameter lie in a region complementary to ours. Similar results are available for twists by Dirichlet characters \[1, 21\]. In a recent breakthrough preprint, P. Nelson \[20\] established subconvexity for all ‘standard’ \( GL(n) \) \( L \)-functions in the spectral aspect having ‘uniform parameter growth’. His work is motivated by the fundamental work of Michel and Venkatesh \[13\].

Subconvexity using the spectral theory of \( GL(3) \) automorphic forms was first obtained by Blomer-Buttcane \[3\]. We apply this spectral theory to the Ranking–Selberg product of a \( GL(3) \) and a \( GL(2) \) form. The main tool is the \( GL(3) \) Kuznetsov formula as developed in \[2\] applied to the second moment of the Ranking–Selberg product \( L(1/2, \pi \times f) \). Note that Theorem \[14\] considerably improves upon the bound of Blomer-Buttcane and further generalizes to all \( GL(3) \times GL(2) \) \( L \)-functions in a single shot. The main technical difficulty in this paper (as was in \[3\]) is the analysis of the multi-dimensional oscillatory integral arising from the Fourier transform of the integral kernel \( (5.8) \). In \[3\], with the help of the Mellin-Barnes representation for the integral kernels, this analysis was reduced to obtaining non-trivial bounds for certain two-dimensional oscillatory integrals to which they apply, among other things, Morse theory in the form of a theorem of Milnor and Thom. Here we take a different route and proceed using the Bessel function representation (see \( 5.18 \)–\( 5.21 \)) for the integral kernels, which seems to fit well with the \( GL(2) \) coefficients. We are then led to studying Fourier transforms of Bessel functions with a certain non-linear twist which we provide in the appendix. We finally reduce our problem to obtaining a non-trivial bound for a one-dimensional oscillatory integral with essentially a degree 13 polynomial as the phase function. At this stage, just a simple application of a Van der Corput type of lemma serves as the endgame.

An important feature in the \( GL(3) \times GL(2) \) structure is that it allows us to achieve a crucial saving when the Kloosterman sums arising from the Kuznetsov formula get transformed into Ramanujan sum after the application of the \( GL(2) \) Voronoi formula. However, this saving becomes ineffective when the corresponding moduli are unbalanced.
Our analysis of the integral kernel takes care of this by gaining more in the “analytic part” in this case. The handling of the long Weyl Kloosterman sum becomes more subtle in our case (as compared to [3]) due to the coprimality conditions in the GL(2) Voronoi formula. We handle this by using a recent result of Kiral and Nakasuji [8], which gives a representation of the long Weyl Kloosterman sum as a finite sum of a product of two classical Kloosterman sums. Another feature of the GL(3) × GL(2) set-up is that off-diagonal after the Kuznetsov formula is reduced to only the long Weyl element, the others contributing negligibly small due to size considerations.

2. Sketch

We are interested in evaluating the second moment

\[ \sum_{\pi_j = \mu_0 + O(1)} |L(1/2, \pi_j \times f)|^2, \]  

which contains about \( T^3 \) terms. If we can show the off-diagonal term is \( \ll T^{3-\delta} \) for some \( \delta > 0 \), then an amplification to the above sum will establish subconvexity. For simplicity, we suppress the amplification part in the sketch. By the approximate functional equation, we roughly have

\[ |L(1/2, \pi_j \times f)|^2 \approx T^{-3} \sum_{m \gg T^3} \sum_{n \gg T^3} \lambda_f(m) \overline{\lambda_f(n)} A_{\pi_j}(1, m) A_{\pi_j}(1, n). \]  

Ignoring the positive contribution from the continuous spectrum, we then apply the Kuznetsov formula to the \( \mu_j \) sum in (2.1). The only non-negligible contribution in the off diagonal comes from the long Weyl element in the Kuznetsov formula which is of the form

\[ T^{-3} \sum_{m, n \gg T^3} \lambda_f(m) \overline{\lambda_f(n)} \sum_{D_1, D_2} S(m, 1, 1; n; D_1, D_2) \Phi_{w_6}(\frac{mD_2}{D_1}, nD_2), \]  

where \( S(m_1, m_2, n_1, n_2; D_1, D_2) \) is a certain \( SL_3 \) Kloosterman sum and \( \Phi_{w_6} \) is an integral transform of the form

\[ \Phi_{w_6}(y_1, y_2) = \int_{\mu = \mu_0 + O(1)} K(y_1, y_2; \mu) \text{spec}(\mu) d\mu, \]  

where \( \text{spec}(\mu) d\mu \approx \|\mu\|^3 d\mu \) is the spectral measure and \( K \) is the kernel function of the GL(3) Kuznetsov transform, an analogue of a Bessel \( K_{2it} \) or \( J_{2it} \) function. Lemma 5 of [3] gives a representation of this kernel as an integral over a product of two Bessel functions. This formula suggests that the typical size of \( K \) is \( T^{3/2} \); each Bessel function saves \( T^{1/2} \), and the \( u \)-integral saves \( T^{1/2} \) by stationary phase. Hence the size of \( \Phi_{w_6}(y_1, y_2) \) is roughly \( T^{3/2} \) and Lemma 6.2 suggests that it oscillates like \( e(y_1^{1/2}) e(y_2^{1/2}) \). Lemma 6.2 also restricts \( D_1, D_2 \ll T \). We then execute the \( m \) and \( n \) sum using the Voronoi summation formula. Note that for \( (D_1, D_2) = 1 \), we have

\[ S(m, 1, 1; n; D_1, D_2) = S(m, D_2; D_1) S(n, D_1; D_2), \]  

and for the general case, Theorem 5.1 allows us to treat it in a similar fashion as we do for the coprime case (2.3). Proceeding with (2.5), we execute the \( m \) sum using Voronoi summation which has conductor \( D_1^2 T^2 \). Hence the dual sum after Voronoi is roughly
bounded by $D_1 T$. Note that the Kloosterman sum $S(m, D_2; D_1)$ gets transformed into Ramanujan sum after the application of Voronoi summation and hence we save the whole modulus $D_1$ instead of just $D_1^{1/2}$ in the Kloosterman sum. This is a crucial point in the proof. The $n$ sum is handled similarly. Now taking the absolute values of the dual sums and executing the remaining sum trivially we see that (2.3) is bounded by $T^{-3} \times T^2 \times T^{3/2} = T^{(3-1/2)}$ and we win.

We should however admit that the above sketch barely touches the heart of the matter, and much more happens while furnishing the details. Firstly, for the case $f = E(z, 1/2)$, there is an additional main term of order $T^3$ in the off-diagonal of the Kuznetsov formula coming from the zero frequencies of the Voronoi summation for $d(n)$ and $d(m)$. We calculate the main term explicitly and save in the $L$-aspect of the amplifier. As observed in [3], this follows from the existence of a zero in the Mellin transform of the Kuznetsov kernel, which becomes apparent only after some non-trivial manipulations. Lastly and most importantly, the desired square root savings in $\Phi_w$ by stationary phase are very hard to show due to the complexity of algebraic equations defining the stationary points. In Theorem 6.3 we manage to obtain an upper bound of $T^{2-1/140}$ on an average instead of our idealistic estimate $T^{3/2}$. With this change of bound, our bound for the off-diagonal contribution becomes $T^{3-1/140}$.

3. Preliminaries

We recall some basic results which we require in the upcoming sections.

**Lemma 3.1 (GL(2) Voronoi summation formula).** Let $\lambda_f(n)$ be the Fourier coefficients of a holomorphic cusp form $f$ with weight $k_f$. Let $h$ be a compactly supported smooth function on the interval $(0, \infty)$. Let $q > 0$ an integer and $a \in \mathbb{Z}$ be such that $(a, q) = 1$. Then we have

$$\sum_{n=1}^{\infty} \lambda_f(n) e\left(\frac{an}{q}\right) h(n) = \frac{1}{q} \sum_{n=1}^{\infty} \lambda_f(n) e\left(-\frac{\bar{a}n}{q}\right) H_f\left(\frac{n}{q^2}\right),$$

(3.1)

where

$$H_f(y) = 2\pi i^{k_f} \int_{0}^{\infty} h(x) J_{k_f-1}(4\pi \sqrt{xy}) dx$$

and for $\lambda_f(n) = d(n)$ we have

$$\sum_{n=1}^{\infty} d(n) e\left(\frac{an}{q}\right) h(n) = \frac{2}{q} \int_{0}^{\infty} \left(\log \frac{\sqrt{x}}{q} + \gamma\right) h(x) dx$$

$$\frac{1}{q} \sum_{n=1}^{\infty} d(n) \int_{0}^{\infty} \left(-2\pi e\left(-\frac{\bar{a}n}{q}\right) Y_0\left(\frac{4\pi \sqrt{nx}}{q}\right) + 4e\left(\frac{\bar{a}n}{q}\right) K_0\left(\frac{4\pi \sqrt{nx}}{q}\right)\right) h(x) dx.$$

(3.2)

**Proof.** See appendix A.4 of [9].

We need the following asymptotics for the Bessel functions to extract the oscillation (see [22], p. 206).
Lemma 3.2. For $y > 0$, the Bessel functions $J_\nu(y)$ and $K_\nu(y)$, ($\nu \in \mathbb{R}$) satisfy the following oscillatory behaviour

$$J_\nu(y) = e^{iy}P_\nu(y) + e^{-iy}Q_\nu(y) \quad \text{and} \quad |y^kK_\nu^{(k)}(y)| \ll_k e^{-y}/\sqrt{y}, \quad (3.3)$$

where the function $P_\nu(y)$ (and similarly $Q_\nu(y)$) satisfies

$$y^jP_\nu^{(j)}(y) \ll_{j,\nu} y^{-j/2}. \quad (3.4)$$

We will frequently encounter integrals of the form

$$I = \int w(t)e(h(t))dt,$$

where $w$ is a smooth function supported on $[a, b]$. The next two lemma gives asymptotics of $I$ depending on the stationary point of $h(t)$ (see [4], Section 8).

Lemma 3.3. Let $Y \geq 1, X, Q, U, R > 0$, and suppose $w(t)$ and $h(t)$ satisfies

$$w^{(j)}(t) \ll_j XU^{-j} \quad (3.5)$$

and

$$|h'(t)| \geq R \quad \text{and} \quad h^{(j)}(t) \ll_j YQ^{-j}, \quad j = 2, 3, 4, \ldots \quad (3.6)$$

then we have

$$I \ll_A (b-a)X \left((QR/\sqrt{Y})^{-A} + (RU)^{-A}\right). \quad (3.7)$$

Lemma 3.4. Let $0 < \delta < 1/10, X, Y, V, Q > 0, Z := X + Y + b - a + 1$ and assume that

$$Y \geq Z^{3\delta}, \quad b - a \geq V \geq \frac{QZ^{2\delta}}{\sqrt{Y}}. \quad (3.8)$$

Assume that $w$ satisfies

$$w^{(j)}(t) \ll_j XV^{-j}$$

for $j \in \mathbb{N}_0$. Suppose that there exist unique $t_0 \in [a, b]$ such that $h'(t_0) = 0$, and furthermore

$$h''(t) \gg YQ^{-2}, \quad h^{(j)}(t) \ll_j YQ^{-j}, \quad \text{for } j = 1, 2, 3\ldots$$

Then the integral $I$ has an asymptotic expansion

$$I = \frac{e(h(t_0))}{\sqrt{h''(t_0)}} \sum_{n \leq 3\delta^{-1}A} p_n(t_0) + O_{\delta,A}(Z^{-A}), \quad p_n(t_0) = \frac{\sqrt{2\pi}e^{i\pi/4}}{n!} \left(\frac{i}{2h''(t_0)}\right)^n G^{(2n)}(t_0), \quad (3.9)$$

where

$$G(t) = w(t)e^{iH(t)}, \quad H(t) = h(t) - h(t_0) - \frac{1}{2}h''(t_0)(t - t_0)^2. \quad (3.9)$$

Furthermore, each $p_n$ is a rational function in $h'', h''', \ldots$, satisfying

$$\frac{d^j}{dt_0^j} p_n(t_0) \ll j, nX(V^{-j} + Q^{-j}) (\langle V^2Y/Q^2 \rangle^{-n} + Y^{-n/3}). \quad (3.9)$$

Using Lemma 3.2 we can deduce the following analogue of the Van der Corput lemma for oscillatory integrals.
Lemma 3.5. Let $T > 1$ and $Y \geq 1$, $X, Q, U, R > 0$ be parameters bounded by $T^{O(1)}$. Let $F(t)$ a compactly supported smooth function satisfying
\[ F^{(j)}(t) \ll_j XU^{-j}, \quad j \geq 1. \tag{3.10} \]
Suppose $\psi(t)$ is a smooth function such that
\[ \psi'(t) = A \cdot \frac{P(t)}{f(t)}, \tag{3.11} \]
where $P(t)$ is a degree $d(\geq 1)$ monic polynomial and $f(t)$ is smooth function with $f(t) \gg 1$ for $t$ in the support of $F$. Furthermore, suppose
\[ \psi^{(j)}(t) \ll_j YQ^{-j}, \quad j \geq 1. \]
Then
\[ \int F(t)e(T\psi(t)) \, dt \ll_T T^\epsilon \sup(F) \left( (AT)^{-1/(d+1)} + (ATU)^{-1/d} + (ATQ/Y^{1/2})^{-1/d} \right) + O_K(T^{-K}). \tag{3.16} \]
Proof. Fix a smooth function $w$ satisfying $w(t) = 1$ if $t \in [-1/2, 1/2]$ and $w(t) = 0$ if $|t| > 1$. Denote
\[ C := T^\epsilon \left( (AT)^{-1/(d+1)} + (ATU)^{-1/d} + (ATQ/Y^{1/2})^{-1/d} \right), \tag{3.12} \]
and define
\[ G(t) := \prod_{i=1}^{d} (1 - w(C^{-1}(t - \Re z_i))). \tag{3.13} \]
where $z_i, i = 1, 2, \ldots, d$, are the roots of $P(t)$. Note that $t \in \text{supp}(G)$ implies $t - \Re z_i \gg C$ for all $1 \leq i \leq d$. This in turn implies
\[ \psi'(t) = A \cdot \frac{P(t)}{f(t)} \gg AC^d, \quad t \in \text{supp}(G). \tag{3.14} \]
Now consider the integral
\[ \int G(t)F(t)e(T\psi(t)) \, dt. \]
Then from (3.13) and (3.10), we obtain
\[ \frac{\partial^j}{\partial t^j} G(t)F(t) \ll_j \max\{1, X\}(C + U)^{-j}. \tag{3.15} \]
Using (3.14), (3.15), (3.147) and invoking Lemma 3.3 we obtain
\[ \int G(t)F(t)e(T\psi(t)) \, dt \ll_K \max\{1, X\} \left( (TAC^d(C + U))^{-K} + (TAC^dQ/Y^{1/2})^{-K} \right). \tag{3.16} \]
From the definition of $C$ in (3.12), it follows
\[ TAC^{d+1} \gg TA \cdot T^\epsilon (AT)^{-1} = T^\epsilon, \quad \text{and} \quad TAC^dU \gg TA \cdot T^\epsilon (ATU)^{-1} \cdot U = T^\epsilon, \]
Hence
\[ TAC^d(C + U) \gg T^\epsilon. \]
Similarly, we obtain

\[ TACdQ/Y^{1/2} \gg T^*TAQ/Y^{1/2} \cdot (ATQ/Y^{1/2})^{-1} = T^*. \]

Substituting the last two obtained lower bounds into (3.16), we obtain

\[ \int G(t)F(t)e(T\psi(t)) \, dt \ll_K T^{-K}, \]

and consequently

\[ \int F(t)e(T\psi(t)) \, dt = \int G(t)F(t)e(T\psi(t)) \, dt + \int (1 - G(t))F(t)e(T\psi(t)) \, dt \]

\[ = \int (1 - G(t))F(t)e(T\psi(t)) \, dt + O_K(T^{-K}). \]

The lemma follows after executing the last integral trivially. \( \square \)

The next lemma generalises the chain rule to higher derivatives. This is will required to estimate derivatives of the weight functions obtained from stationary phase analysis.

**Lemma 3.6** (Faà di Bruno’s formula). Let \( f, g : \mathbb{R} \to \mathbb{R} \) be two smooth functions. Then

\[ \frac{d^n}{dx^n} f(g(x)) = \sum_{m_1!m_2! \cdots m_n!} \frac{n!}{m_1!m_2! \cdots m_n!} \cdot f^{(m_1+m_2+\cdots+m_n)}(g(x)) \cdot \prod_{j=1}^{n} (g^{(j)}(x))^{m_j}, \]

where the sum is over all \( n \)-tuples of non-negative integers \((m_1, \ldots, m_n)\) satisfying the constraint

\[ 1 \cdot m_1 + 2 \cdot m_2 + \cdots + n \cdot m_n = n. \]

### 4. The set up

For \( 0 \leq c \leq \infty \) let

\[ \Lambda_c := \{ \mu \in \mathbb{C}^3 \mid \mu_1 + \mu_2 + \mu_3 = 0, \quad |\Re \mu_j| \leq c \} \quad (4.1) \]

and

\[ \Lambda'_c := \{ \mu \in \Lambda_c \mid \{-\mu_1, -\mu_2, -\mu_3\} = \{\bar{\mu}_1, \bar{\mu}_2, \bar{\mu}_3\} \}. \quad (4.2) \]

In the Lie algebra \( \Lambda_{\infty} \) we will simultaneously use \( \mu \) and \( \nu = (\nu_1, \nu_2, \nu_3) \) coordinates, defined by

\[ \nu_1 = \frac{1}{3}(\mu_1 - \mu_2), \quad \nu_2 = \frac{1}{3}(\mu_2 - \mu_3), \quad \nu_3 = \frac{1}{3}(\mu_3 - \mu_1). \quad (4.3) \]

Let \( \mathcal{W} \) denote the Weyl group,

\[ \mathcal{W} := \left\{ I_3, w_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, w_3 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, w_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, w_5 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, w_6 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}, \]

which acts on \( \mu = (\mu_1, \mu_2, \mu_3) \) by permutations. Let \( \pi_0 \) be our preferred \( SL(3, \mathbb{Z}) \) Maass cusp from with Hecke eigenvalues \( A_{\pi_0}(1, n) \) and Langlands parameter \( (\mu_{0,1}\mu_{0,2}, \mu_{0,3}) \in \Lambda_0 \), and assume that

\[ |\mu_{0,j}| \asymp |\nu_{0,j}| \asymp T \quad (j = 1, 2, 3). \quad (4.4) \]
$L(1/2, \pi_0 \times f)$ has analytic conductor $T^3$. Hence, from the approximate functional equation (see Th. 5.3 and Prop. 5.4 in [7]) we essentially get

$$L \left( \frac{1}{2}, \pi_0 \times f \right) \ll \sum_{m=1}^{\infty} \frac{\lambda_{\pi_0 \times f}(m)}{m^{1/2}} V \left( \frac{m}{T^3} \right),$$

where

$$\lambda_{\pi_0 \times f}(m) = \sum_{n r^2 = m} A_{\pi_0}(r, n) \lambda_f(n),$$

and the smooth function $V$ satisfies

$$x^j V^{(j)}(x) = O_A(1 + |x|^{-A}).$$

**Lemma 4.1.**

$$L \left( \frac{1}{2}, \pi_0 \times f \right) \ll T^\epsilon \sup_{r \leq T^\theta} \sup_{\frac{T^{3-\theta}}{r^2} \leq N \leq \frac{T^{3+\epsilon}}{r^2}} \frac{|S_r(N)|}{N^{1/2}} + T^{(3-\theta)/2},$$

where

$$S_r(N) = \sum_{n=1}^{\infty} A_{\pi_0}(r, n) \lambda_f(n) V \left( \frac{n}{N} \right),$$

where $V$ is a smooth function supported in $[1,2]$ and satisfies $V^{(j)}(x) \ll_j 1$.

**Proof.** From the approximate functional equation (4.5), one has

$$L \left( \frac{1}{2}, \pi_0 \times f \right) \ll \left| \sum_{n, r = 1}^{\infty} A_{\pi_0}(r, n) \lambda_f(n) \frac{V \left( \frac{nr^2}{T^3} \right)}{(nr^2)^{1/2}} \right| \ll \left| \sum_{n r^2 \leq T^{3+\epsilon}} A_{\pi_0}(r, n) \lambda_f(n) \frac{V \left( \frac{nr^2}{T^3} \right)}{(nr^2)^{1/2}} \right| + T^{-2019} \tag{4.7}$$

$$= \left| \sum_{r \leq T^{(3+\epsilon)/2}} \frac{1}{r} \sum_{n \leq \frac{T^{3+\epsilon}}{r^2}} A_{\pi_0}(r, n) \lambda_f(n) \frac{V \left( \frac{nr^2}{T^3} \right)}{n^{1/2}} \right| + T^{-2019}.$$

We divide the range in the last summation as

$$\sum_{r \leq T^{(3+\epsilon)/2}} \sum_{n \leq \frac{T^{3+\epsilon}}{r^2}} = \sum_{r \leq T^\theta} \sum_{\frac{T^{3-\theta}}{r^2} \leq n \leq \frac{T^{3+\epsilon}}{r^2}} + \sum_{r \leq T^\theta} \sum_{n < \frac{T^{3-\theta}}{r^2}} + \sum_{r > T^\theta} \sum_{n \leq \frac{T^{3+\epsilon}}{r^2}}$$

where an optimal $\theta > 0$ will be chosen later. Using the Ramanujan bound on average

$$\sum_{n_1 n_2 \leq x} |A(n_1, n_2)|^2 \ll x^{1+\epsilon}$$

one sees that the last two ranges contributes at most $T^{(3-\theta)/2}$ to (4.7). Hence we have

$$L \left( \frac{1}{2}, \pi_0 \times f \right) \ll \left| \sum_{r \leq T^\theta} \frac{1}{r} \sum_{\frac{T^{3-\theta}}{r^2} \leq n \leq \frac{T^{3+\epsilon}}{r^2}} A_{\pi_0}(r, n) \lambda_f(n) \frac{V \left( \frac{nr^2}{T^3} \right)}{n^{1/2}} \right| + T^{(3-\theta)/2}. $$
Using a smooth dyadic partition of unity $W$, we see that inner sum above is at most
$$\sup_{T^{3-\theta} \leq N \leq T^{3+\epsilon}} \left| \sum_{n=1}^{\infty} \frac{A_{\pi_0}(r, n) \lambda_f(n)}{n^{1/2}} W\left( \frac{n}{N} \right) V\left( \frac{n r^2}{T^3} \right) \right|. $$

Using partial summation one sees that this is essentially bounded by
$$\sup_{T^{3-\theta} \leq N \leq T^{3+\epsilon}} \frac{|S_r(N)|}{N^{1/2}}$$

where
$$S_r(N) = \sum_{n=1}^{\infty} A_{\pi_0}(r, n) \lambda_f(n) V_{r,N} \left( \frac{n}{N} \right),$$

and $V_{r,N}(x) = W(x) V(N r^2 x / T^3)$. Note that $V_{r,N}(x)$ is supported on $[1, 2]$ and satisfies $V_{r,N}(x) \ll 1$ (bounds independent of $r, N$). Henceforth we ignore the dependence on $r, N$ and assume $V_{r,N}$ is the same function for all $r, N$ and call it $V(x)$ (abusing notation). The claim follows. 

As seen from the sketch above, the unamplified moment $\sum |L(1/2, \pi \times f)|^2$ fails to beat the convexity at the diagonal. To overcome this, we use the following amplifier introduced in [3].

4.1. The amplifier. Let the sequence $\{\pi_j\}$ of Maass cusp forms be an orthonormal basis for $L^2 \left( h^{3/2} / \Gamma \right)$. Consider the amplifier

$$A(\pi) = \sum_{j=1}^{3} \left| \sum_{l \sim L} \sum_{l \text{ prime}} A_{\pi}(1, l^j) x(l) \right|^2,$$

where
$$x(n) := \text{sgn}(A_{\pi_0}(1, n)) \in S^1 \cup \{0\}.$$  

Note that form the Hecke relation
$$A_{\pi}(1, l) A_{\pi}(1, l^2) = A_{\pi}(1, l^3) + A_{\pi}(1, l) A_{\pi}(l, 1) - 1$$

it follows $\max \{|A_{\pi}(1, l)|, |A_{\pi}(1, l^2)|, |A_{\pi}(1, l^3)|\} \gg 1$ and hence by the Cauchy-Schwarz inequality

$$A(\pi_0) \gg \left( \sum_{j=1}^{3} \sum_{l \sim L} \sum_{l \text{ prime}} |A_{\pi_0}(1, l^j)| \right)^2 \gg \left( \sum_{l \sim L} \sum_{l \text{ prime}} 1 \right)^2 \gg L^{2-\epsilon}. \quad (4.8)$$

We now insert the amplifiers $A(\pi_j)$ into Lemma 4.1 to get,

$$|L(1/2, \pi_0 \times f)|^2 \ll T^\epsilon \sup_{r \leq T^\theta} \sup_{T^{3-\theta} \leq N \leq T^{3+\epsilon}} \frac{\sum_{l_j = l_0 + O(T^\epsilon)} \lambda_f(n) \lambda_f(n) V_{r,j}(N)}{N L^2} \sup_{r \leq T^\theta} T^{3-\theta}. \quad (4.9)$$
We will make the spectral average \( \sum_{\mu_j = \mu_0 + O(T')} \) more precise in the upcoming sections. Our main object of study now becomes

\[
\sum_{\mu_j = \mu_0 + O(T')} A(\pi_j)|S_{\tau,j}(N)|^2, \quad (4.10)
\]

for which we will eventually establish

**Theorem 4.2.**

\[
\sum_{\mu_j = \mu_0 + O(T')} A(\pi_j)|S_{\tau,j}(N)|^2 \ll NT^3L + \frac{N^2\tau^2L^8}{T^{1/280}} + \sum_{k=1}^{3} \sum_{(k_0,k_1,k_2) \atop k_0+k_1+k_2=k} \sum_{L^{k_1} \ll r} \frac{NT^3L^2}{L^{k_2+k_0}}. \quad (4.11)
\]

**Proof of Theorem 4.2.** Substituting (4.11) in (4.9) we get

\[
|L(1/2, \pi_0 \times f)|^2 \ll \frac{T^3}{L} + T^{(3-1/280)}L^6 + T^{3-\theta} + \sup_{\frac{\theta}{2} \leq k \leq T^\theta} \sum_{(k_0,k_1,k_2) \atop k_0+k_1+k_2=k} \sum_{L^{k_1} \ll r} \frac{T^3}{L^{k_2+k_0}}. \quad (4.12)
\]

Equating the first two terms, we choose \( L = T^{1/1960} \). We now choose \( \theta = 1/1960 - \epsilon \). With these choices, note that \( k_1 \) must equal zero in the third term and consequently \( k_2 + k_0 \geq 1 \). Hence the third term is dominated by the first term and we get

\[
L(1/2, \pi_0 \times f) \ll \epsilon T^{(3/2-1/1960+\epsilon)}. \quad (4.13)
\]

\( \Box \)

To begin with the proof of Theorem 4.2 we open the absolute value square and apply the \( GL(3) \) Kuznetsov formula. To bring it in the proper set-up, we need to rearrange (4.10) further. From the Hecke relations, we get

\[
A_{\pi_j}(1, l^k)A_{\pi_j}(r, n) = \sum_{d_0,d_1,d_2=l^k \atop d_1|l} \sum_{d_2|n} \frac{rd_2}{d_1} \frac{nd_0}{d_2} = \sum_{(k_0,k_1,k_2) \atop k_0+k_1+k_2=k} \sum_{l^{k_1}|l, l^{k_2}|n} A_{\pi_j}(rl^{k_2-k_1}, nl^{k_0-k_2}). \quad (4.14)
\]

Hence by an application of the Cauchy-Schwarz inequality, (4.10) becomes

\[
\sum_{\mu_j = \mu_0 + O(T')} A(\pi_j)|S_{\tau,j}(N)|^2 \ll \sum_{k=1}^{3} \sum_{(k_0,k_1,k_2) \atop k_0+k_1+k_2=k} \sum_{\mu_j = \mu_0 + O(T')} |\tilde{S}_{k,r,j}(N)|^2, \quad (4.15)
\]

where

\[
\tilde{S}_{k,r,j}(N) = \sum_{l \geq L} \sum_{n \geq N \atop l^{k_2}|n} A_{\pi_j}(rl^{k_2-k_1}, nl^{k_0-k_2})\lambda(n)V\left(\frac{n}{N}\right).
\]

Opening the absolute value square we have
We are now in position to apply the Kuznetsov formula to the spectral sum above.

5. The \(GL(3)\) Kuznetsov

We first introduce some notations.

5.1. Kloosterman sums. For \(n_1, n_2, m_1, m_2, D_1, D_2 \in \mathbb{N}\), we have the following two types of Kloosterman sums

\[
\tilde{S}(n_1, n_2, m_1; D_1, D_2) := \sum_{\substack{C_1 (\text{mod } D_1), C_2 (\text{mod } D_2) \in \mathbb{Z}_{D_1/D_1} \times \mathbb{Z}_{D_2/D_1} \atop (C_1, D_1) = (C_2, D_2/D_1) = 1}} e \left( n_2 \frac{C_1 C_2}{D_1} + m_1 \frac{C_2}{D_2/D_1} + n_1 \frac{C_1}{D_1} \right)
\]

(5.1)

for \(D_1 | D_2\), and

\[
S(n_1, m_2, m_1, n_2; D_1, D_2)
= \sum_{\substack{B_1, C_1 (\text{mod } D_1) \in \mathbb{Z}_{D_1/D_1} \\
B_2, C_2 (\text{mod } D_2) \in \mathbb{Z}_{D_2/D_1} \\
D_1 C_1 + B_1 B_2 + D_2 C_2 \equiv 0 (\text{mod } D_1 D_2) \\
(B_1, C_1, D_1) = 1}} e \left( \frac{n_1 B_1 + m_1 (Y_1 D_2 - Z_1 B_2)}{D_1} + \frac{m_2 B_2 + n_2 (Y_2 D_1 - Z_2 B_1)}{D_2} \right),
\]

(5.2)

where \(Y_j B_j + Z_j C_j \equiv 1 (\text{mod } D_j)\) for \(j = 1, 2\).

We have the following result of E.M. Kiral and M. Nakasuji which gives a decomposition of \(S(n_1, m_2, m_1, n_2; D_1, D_2)\) as a sum of product of two classical \(SL_2\) Kloosterman sums.

**Theorem 5.1** (Kiral, Nakasuji). We have

\[
S(m_2, m_1, n_1, n_2; D_1, D_2)
= \sum_{f | (D_1, D_2)} f \sum_{y (\text{mod } f)} S \left( m_2, \frac{M_f(y)}{f^2} \frac{D_1}{f} \right) S \left( n_2, \frac{\bar{M}_f(y)}{f^2} \frac{D_2}{f} \right),
\]

(5.3)

where

\[
M_f(y) = \frac{n_1 D_2 + m_1 D_1 y}{f^2} \quad \text{and} \quad \bar{M}_f(y) = \frac{n_1 D_2 \bar{y} + m_1 D_1}{f^2}.
\]
5.2. Integral kernels. Following \[5\] Theorem 2 & 3, we define the following integral kernels in terms of Mellin-Barnes representations. For \( s \in \mathbb{C}, \mu \in \Lambda_\infty \), define the meromorphic function

\[
\tilde{G}^\pm(s, \mu) := \frac{\pi^{-3s}}{12288\pi^{7/2}} \left( \prod_{j=1}^{3} \frac{\Gamma\left(\frac{1}{2}(s - \mu_j)\right)}{\Gamma\left(\frac{1}{2}(1 - s + \mu_j)\right)} \pm i \prod_{j=1}^{3} \frac{\Gamma\left(\frac{1}{2}(1 + s - \mu_j)\right)}{\Gamma\left(\frac{1}{2}(2 - s + \mu_j)\right)} \right), \tag{5.4}
\]

and for \( s = (s_1, s_2) \in \mathbb{C}^2, \mu \in \Lambda_\infty \), define the meromorphic function

\[
G(s, \mu) := \frac{1}{\Gamma(s_1 + s_2)} \prod_{j=1}^{3} \Gamma(s_1 - \mu_j)\Gamma(s_2 + \mu_j). \tag{5.5}
\]

The latter is essentially the double Mellin transform of the \( GL(3) \) Whittaker function. We also define the following trigonometric functions:

\[
S^{++}(s, \mu) := \frac{1}{24\pi^2} \prod_{j=1}^{3} \cos\left(\frac{3}{2} \pi \nu_j\right),
\]

\[
S^{+-}(s, \mu) := -\frac{1}{32} \frac{\cos\left(\frac{3}{2} \pi \nu_2\right) \sin(\pi(s_1 - \mu_1)) \sin(\pi(s_2 + \mu_2)) \sin(\pi(s_2 + \mu_3))}{\sin\left(\frac{3}{2} \pi \nu_1\right) \sin\left(\frac{3}{2} \pi \nu_3\right) \sin(\pi(s_1 + s_2))},
\]

\[
S^{-+}(s, \mu) := -\frac{1}{32} \frac{\cos\left(\frac{3}{2} \pi \nu_1\right) \sin(\pi(s_1 - \mu_1)) \sin(\pi(s_1 - \mu_2)) \sin(\pi(s_2 + \mu_3))}{\sin\left(\frac{3}{2} \pi \nu_2\right) \sin\left(\frac{3}{2} \pi \nu_3\right) \sin(\pi(s_1 + s_2))},
\]

\[
S^{--}(s, \mu) := -\frac{1}{32} \frac{\cos\left(\frac{3}{2} \pi \nu_3\right) \sin(\pi(s_1 - \mu_2)) \sin(\pi(s_2 + \mu_2))}{\sin\left(\frac{3}{2} \pi \nu_2\right) \sin\left(\frac{3}{2} \pi \nu_1\right)}.
\]

For \( y \in \mathbb{R} \setminus 0 \) with \( \text{sgn}(y) = \epsilon \), define

\[
K_{w_4}(y; \mu) := \int_{-i\infty}^{i\infty} |y|^{-s} \tilde{G}^\epsilon(y)(s, \mu) \frac{ds}{2\pi i}, \tag{5.7}
\]

and for \( y = (y_1, y_2) \in (\mathbb{R} \setminus 0)^2 \) with \( \text{sgn}(y_1) = \epsilon_1, \text{sgn}(y_2) = \epsilon_2 \), define

\[
K^{\epsilon_1, \epsilon_2}_{w_6}(y; \mu) = \int_{-i\infty}^{i\infty} \int_{-i\infty}^{i\infty} 4\pi^2 y_1^{-s_1} 4\pi^2 y_2^{-s_2} G(s, \mu) S^{\epsilon_1, \epsilon_2}(s, \mu) \frac{ds_1 ds_2}{(2\pi i)^2}. \tag{5.8}
\]

5.3. Normalising factors. The following normalising appears in Kuznetsov formula:

\[
N(\pi) := \|\phi\|^2 \prod_{j=1}^{3} \cos\left(\frac{3}{2} \pi \nu_{\pi_j}\right), \tag{5.9}
\]

where \( \phi \) is the arithmetically normalized Maass form generating \( \mu \). That is, \( \phi(z) = \sum_{\gamma \in U \setminus SL_2(\mathbb{Z})} \sum_{m_1 = 1}^{\infty} \sum_{m_2 \neq 0} A_{\pi}(m_1, m_2) \frac{W_{\nu}^{\text{sgn}(m_2)}}{|m_1 m_2|} \mathcal{W}_{\nu}^{m_2} \left( \left( \frac{|m_1 m_2|}{m_1} \right)^{\gamma} z \right) \right), \tag{5.10}
\]

where \( U = \{ (1, *) \} \in SL_2(\mathbb{Z}) \} \) and \( \mathcal{W}_{\nu}^z(z) = e(x_1 \pm x_2) W_{\nu}^z(y_1, y_2) \), where \( W_{\nu}^z \) is the standard completed Whittaker function as in \([5\), Def. 5.9.2\] , and \( A_{\pi}(1, 1) = 1 \). By
Rankin-Selberg theory in combination with Stade's formula (see e.g. [2, Section 4]) and [11, Theorem 2]), it follows that

\[ N(\pi) \approx \text{res}_{s=1} L(s, \pi \times \tilde{\pi}) \ll \|\mu_\pi\|^c. \]  

(5.11)

The spectral measure is defined by

\[ \text{spec}(\mu) d\mu, \quad \text{spec}(\mu) d\mu := \prod_{i=1}^{3} \left( 3\nu_j \tan \left( \frac{3\pi}{2} \nu_j \right) \right), \]  

(5.12)

where \( d\mu = d\mu_1 d\mu_2 = d\mu_2 d\mu_3 = d\mu_1 d\mu_3. \)

With the above notations, we now state the Kuznetsov formula in the version of ([5, Theorem 2,3 & 4]).

5.4. The Kuznetsov formula. Let \( n_1, n_2, m_1, m_2 \in \mathbb{N} \) and let \( h \) be a function that is holomorphic on \( \Lambda_{1/2+\delta} \) for some \( \delta > 0 \), symmetric under the Weyl group, rapidly decreasing as \( |\Im \mu_j| \to \infty \) and satisfies

\[ h(3\nu_j \pm 1) = 0, \quad j = 1, 2, 3. \]  

(5.13)

Then we have

\[ C + E_{\min} + E_{\max} = \Delta + \sum_4 + \sum_5 + \sum_6, \]  

(5.14)

where

\[ C = \sum_j \frac{h(\pi_j)}{N_j} A_j(m_1, m_2) A_j(n_1, n_2), \]

\[ E_{\min} = \frac{1}{24(2\pi i)^2} \int \int_{\mathbb{R}(\mu) = 0} \frac{h(\mu)}{N_\mu} A_\mu(m_1, m_2) \overline{A_\mu(n_1, n_2)} d\mu_1 d\mu_2, \]  

(5.15)

\[ E_{\max} = \frac{1}{2\pi i} \sum_g \int_{\mathbb{R}(\mu) = 0} \frac{h(\mu + \mu_g, \mu - \mu_g, -2\mu)}{N_{\mu,g}} B_{\mu,g}(m_1, m_2) \overline{B_{\mu,g}(n_1, n_2)} d\mu, \]

and

\[ \Delta = \delta_{m_1 = n_1} \delta_{m_2 = n_2} \int_{\mathbb{R}(\mu) = 0} h(\mu) \text{spec}(\mu) d\mu, \]

\[ \sum_4 = \sum_{\epsilon = \pm 1} \sum_{D_1 | D_2} \sum_{n_2 D_1 = m_1 D_2^2} \tilde{S}(\epsilon m_2, n_2, n_1 : D_1, D_2) \Phi_{w_4} \left( \frac{em_1 n_2 m_2}{D_1 D_2} \right), \]

\[ \sum_5 = \sum_{\epsilon = \pm 1} \sum_{D_1 | D_2} \sum_{n_1 D_2 = m_2 D_1^2} \tilde{S}(\epsilon m_1, n_1, n_2 : D_1, D_2) \Phi_{w_5} \left( \frac{em_1 n_2}{D_1 D_2} \right), \]

\[ \sum_6 = \sum_{\epsilon_1, \epsilon_2 = \pm 1} \sum_{D_1, D_2} S(\epsilon_2 m_2, \epsilon_1 m_1, n_1, n_2; D_1, D_2) \Phi_{w_6} \left( -\frac{\epsilon_2 n_1 m_2 D_2}{D_1^2}, -\frac{\epsilon_1 n_2 m_1 D_1}{D_2^2} \right), \]
and

\[ \Phi_{w_4}(y) = \int_{\mathbb{R}_0} h(\mu) K_{w_4}(y; \mu) \text{spec}(\mu) d\mu, \]
\[ \Phi_{w_5}(y) = \int_{\mathbb{R}_0} h(\mu) K_{w_5}(-y; -\mu) \text{spec}(\mu) d\mu, \]
\[ \Phi_{w_6}(y_1, y_2) = \int_{\mathbb{R}_0} h(\mu) K_{w_6}^{\text{sgn}(y_1),\text{sgn}(y_2)}((y_1, y_2); \mu) \text{spec}(\mu) d\mu. \] (5.16)

Here \( A_\mu(m_1, m_2) \) denotes the Fourier coefficients of the minimal Eisenstein series \( E(z, \mu) \) and \( B_{\mu, g}(m_1, m_2) \) denotes the Fourier coefficient of \( E_{P_2}^g(z, \mu, g) \), the maximal Eisenstein series twisted by Maass form \( g \) (see Goldfeld’s text [6], eqn. (10.4.1), (10.11.1)). We ignore the positive contribution of \( \mathcal{E}_{\text{min}} \) and \( \mathcal{E}_{\text{max}} \).

5.5. Integral representations. We quote the following alternate expressions for the kernel functions given in definition (5.8) in terms of the Bessel functions. See section 5 of [3] for details. The stationary phase analysis seems to work better for these representations (at least in our set-up), and we will mainly work with these.

For \( y_1, y_2 \in \mathbb{R} \setminus \{0\} \) and \( \mu \in \Lambda_\mu \), define

\[ \mathcal{J}_1^\pm(y, \mu) = \left| \frac{y_1}{y_2} \right|^{\mu/2} \int_0^\infty J_{3\nu_3}^\pm \left( |y_1|^{1/2} \sqrt{1 + u^2} \right) J_{3\nu_3}^\pm \left( |y_2|^{1/2} \sqrt{1 + u^2} \right) u^{3\mu_2} \frac{du}{u}, \]
\[ \mathcal{J}_2(y, \mu) = \left| \frac{y_1}{y_2} \right|^{\mu/2} \int_1^\infty J_{3\nu_3}^- \left( |y_1|^{1/2} \sqrt{u^2 - 1} \right) J_{3\nu_3}^- \left( |y_2|^{1/2} \sqrt{1 - u^2} \right) u^{3\mu_2} \frac{du}{u}, \]
\[ \mathcal{J}_3(y, \mu) = \left| \frac{y_1}{y_2} \right|^{\mu/2} \int_0^\infty \tilde{K}_{3\nu_3} \left( |y_1|^{1/2} \sqrt{1 + u^2} \right) \tilde{K}_{3\nu_3} \left( |y_2|^{1/2} \sqrt{1 + u^2} \right) u^{3\mu_2} \frac{du}{u}, \]
\[ \mathcal{J}_4(y, \mu) = \left| \frac{y_1}{y_2} \right|^{\mu/2} \int_0^1 \tilde{K}_{3\nu_3} \left( |y_1|^{1/2} \sqrt{1 - u^2} \right) \tilde{K}_{3\nu_3} \left( |y_2|^{1/2} \sqrt{u^2 - 1} \right) u^{3\mu_2} \frac{du}{u}, \]
\[ \mathcal{J}_5(y, \mu) = \left| \frac{y_1}{y_2} \right|^{\mu/2} \int_0^\infty \tilde{K}_{3\nu_3} \left( |y_1|^{1/2} \sqrt{1 + u^2} \right) \tilde{K}_{3\nu_3} \left( |y_2|^{1/2} \sqrt{1 + u^2} \right) u^{3\mu_2} \frac{du}{u}, \] (5.17)

For \( y_1, y_2 > 0 \) we have

\[ K_{w_6}^{++}(y; \mu) = \frac{1}{12 \pi^2} \frac{\cos \left( \frac{3}{2} \pi \nu_1 \right) \cos \left( \frac{3}{2} \pi \nu_2 \right)}{\cos \left( \nu_3 \right)} \mathcal{J}_5(y, \mu); \] (5.18)

for \( y_1 > 0 > y_2 \) we have

\[ \sum_{w \in \{1, w_4, w_5\}} K_{w_6}^{+-}(y; w(\mu)) = \frac{1}{24 \pi^2} \sum_{w \in \{1, w_4, w_5\}} (\mathcal{J}_2(y; w(\mu)) + \mathcal{J}_3(y; w(\mu)) + \mathcal{J}_4(y; w(\mu))); \] (5.19)

for \( y_2 > 0 > y_1 \) we have

\[ K_{w_6}^{+-}((y_1, y_2); \mu) = K_{w_6}^{+-}((y_2, y_1); w_4(\mu)); \] (5.20)
and for \( y_1, y_2 < 0 \) we have
\[
\sum_{w \in \{I,w_4,w_5\}} K_{w_6}^- (y; w (\mu)) = \frac{1}{48\pi^2} \sum_{w \in \{I,w_4,w_5\}} \left( 4J_1^- (y; w (\mu)) + 4J_1^+ (y; w (\mu)) \right). \tag{5.21}
\]

6. Analytic properties of the integral transforms

The first two lemmas, which we directly quote from [3, Section 7], will be used to truncate various dual sums.

**Lemma 6.1.** Let \( 0 < y \leq T^{3-\epsilon} \). Then for any constant \( B \geq 0 \) one has
\[
\Phi_{w_4} (y) \ll_{\epsilon,B} T^{-B}. \tag{6.1}
\]
If \( T^{3-\epsilon} < |y| \), then
\[
|y|^2 \Phi_{w_4}^{(j)} (y) \ll_{j,\epsilon} T^{3+2\epsilon} (T + |y|^{1/3})^j, \tag{6.2}
\]
for any \( j \in \mathbb{N}_0 \).

**Lemma 6.2.** Let \( \Upsilon := \min \{|y_1|^{1/3} |y_2|^{1/2}, |y_1|^{1/6} |y_2|^{1/3}\} \). If \( \Upsilon \leq T^{1-\epsilon} \), then
\[
\Phi_{w_6} (y_1, y_2) \ll_{B,\epsilon} T^{-B}, \tag{6.3}
\]
for any fixed constant \( B \geq 0 \). If \( \Upsilon \geq T^{1-\epsilon} \), then
\[
|y_1|^{j_1} |y_2|^{j_2} \frac{\partial^{j_1}}{\partial y_1^{j_1}} \frac{\partial^{j_2}}{\partial y_2^{j_2}} \Phi_{w_6} (y_1, y_2) \ll_{j_1,j_2,\epsilon} T^{3+\epsilon} \left( T + |y_1|^{1/2} + |y_1|^{1/3} |y_2|^{1/6} \right)^{j_1} \left( T + |y_2|^{1/2} + |y_2|^{1/3} |y_1|^{1/6} \right)^{j_2}. \tag{6.4}
\]
for all \( j_1, j_2 \in \mathbb{N}_0 \).

The main effort of the paper goes into obtaining non-trivial estimates for integral transforms of the form
\[
\mathcal{K} (\theta_1, \theta_2, U, V) := \sum_{\epsilon_1, \epsilon_2 = \pm 1} \int_{\mathbb{R}} \int_{\mathbb{R}} U(x) V(y) \Phi_{w_6} (\epsilon_1 \theta_1 x^2, \epsilon_2 \theta_2 y^2)e(U x + V y)\, dx\, dy.
\]

By using the integral representations in term of the Bessel functions from (5.5), we reduce our problem to estimating each
\[
\mathcal{K}_i (\theta_1, \theta_2, U, V) = \int_{\mathbb{R}} \int_{\mathbb{R}} U(x) V(y) J_i (\epsilon_1 \theta_1 x^2, \epsilon_2 \theta_2 y^2; \mu)e(U x + V y)\, dx\, dy\, d\mu, \tag{6.5}
\]
where \( J_i \) as defined in (5.17). In this paper, we will restrict our attention to \( \mathcal{K}_4 (\theta_1, \theta_2, U, V) \) and provide a detailed analysis for the same. The arguments are robust and can be easily adapted to the remaining ones which provide us with the same (or smaller) contributions.

**Theorem 6.3.** Let \( \mu = (\mu_1, \mu_2, \mu_3) \) be in generic position with \( ||\mu|| \asymp T \). For \( U, V \in \mathbb{R}, \theta_1, \theta_2 > 0 \), denote
\[
c_1 := \frac{\pi |U|}{\theta_1^{1/2}}, \quad c_2 := \frac{\pi |V|}{\theta_2^{1/2}}, \quad A := |c_1 - 1|.
\]
Suppose
\[ \min\{\theta_1^{1/3}\theta_2^{1/6}, \theta_1^{1/6}\theta_2^{1/3}\} \gg T^{1-\epsilon}, \quad U \ll \theta_1^{1/2} + \theta_1^{1/3}\theta_2^{1/2}, \quad V \ll \theta_2^{1/2} + \theta_2^{1/3}\theta_1^{1/6} \]. \tag{6.6}
Then \( K_4(\theta_1, \theta_2, U, V) \) is negligibly small unless
\[ \theta_1 \gg \theta_2, \tag{6.7} \]
in which case,
\[ (\theta_1/\theta_2)^{1/6} K_4(\theta_1, \theta_2, U, V) \ll T^{1-1/140} + T\delta_{\min\{c_1, c_2\}} \ll T^{1-1/140} + T\delta_{\min\{c_1, c_2\}} \]. \tag{6.8}
Furthermore, if one of \( U \) or \( V \) is zero, we have
\[ (\theta_1/\theta_2)^{1/6} K_4(\theta_1, \theta_2, U, 0) \ll T^{1-1/140} + T\delta_{c_1} \ll T^{1-1/140} + T\delta_{c_1}, \tag{6.9} \]
and
\[ (\theta_1/\theta_2)^{1/6} K_4(\theta_1, \theta_2, 0, V) \ll T^{1-1/140} + T\delta_{c_2} \ll T^{1-1/140} + T\delta_{c_2}. \]

**Proof.** See section 13. \( \square \)

If both \( U = V = 0 \), we have the following estimate for our original integral transform \( K(\theta_1, \theta_2, U, V) \).

**Theorem 6.4.** We have
\[ \sum_{(f,F)=1}^{f \geq 1} \frac{1}{f} K\left( \frac{\theta_1}{f}, \frac{\theta_2}{f}, 0, 0 \right) \ll T^{3+4\epsilon}(\theta_1\theta_2)^{-1/2}. \tag{6.10} \]
for any positive integer \( F \).

•

**Proof.** See section 12. \( \square \)

7. **Applying the Kuznetsov Formula**

Before applying the Kuznetsov formula \( (5.14) \) to the spectral sum in \( (4.16) \), we need to specify the test function used to detect the spectral average.

The **test function** \( h(\mu) \). We use the same test function specified in \( [3] \). In particular, this test function satisfies the required properties for the Kuznetsov formula, is non-negative on \( \Lambda_{1/2}' \), satisfies \( h(\mu_0) \gg 1 \) and is negligibly small outside \( O(T^\epsilon) \)-balls about \( w(\mu_0) \) for \( w \in \mathcal{W} \), where \( \mathcal{W} \) is the Weyl group which acts on \( \mu \) by permutations. Firstly let us fix
\[ \psi(\mu) = \exp(\mu_1^2 + \mu_2^2 + \mu_3^2), \tag{7.1} \]
and let
\[ P(\mu) := \prod_{0 \leq n \leq A} \prod_{j=1}^{3} \frac{(\nu_j - \frac{1}{3}(1 + 2n))(\nu_j + \frac{1}{3}(1 + 2n))}{|\nu_0|^2}, \tag{7.2} \]
for some large fixed constant \( A \). This polynomial has zeroes at the poles of the spectral measure, which turns out to be convenient for contour shifts. Now we choose
\[ h(\mu) := P(\mu)^2 \left( \sum_{w \in \mathcal{W}} \psi \left( \frac{w(\mu) - \mu_0}{T^\epsilon} \right) \right)^2. \tag{7.3} \]
The $T^\epsilon$-radius provides us some extra space that will be convenient is later estimations. In particular, we have

$$D_j h(\mu) \ll T^{-j\epsilon},$$

(7.4)

for any differential operator $D_j$ of order $j$, which we will frequently use while integrating by parts. Moreover, we have

$$\int_{\mathbb{R}^\mu=0} h(\mu) \text{spec}(\mu) d\mu \ll T^{3+\epsilon}.$$  

(7.5)

Applying the $GL(3)$ Kuznetsov formula (5.14) with test function above, the $\mu_j$ sum in (4.16) becomes

$$\sum_j \frac{h(\pi_j)}{N_j} A_{\pi_j}(r_1^{k_2-k_1}, m_1^{k_0}) A_{\pi_j}(r_2^{k_2-k_1}, n_2^{k_0}) = \Delta + \sum_4 + \sum_5 + \sum_6 - \mathcal{E}_{\text{min}} - \mathcal{E}_{\text{max}}$$

(7.6)

where, by abuse of notations, the terms in the right hand side are as defined in (5.14) with the new variables variables $(m_1, m_2) = (r_1^{k_2-k_1}, m_1^{k_0})$, $(n_1, n_2) = (r_2^{k_2-k_1}, n_2^{k_0})$. Substituting (7.6) into (4.16), we obtain

$$\sum_j \frac{h(\pi_j)}{N_j} |\tilde{\mathcal{S}}_{k,r,j}(N)|^2 = S_0 + S_4 + S_5 + S_6 - S_{\text{max}} - S_{\text{min}},$$

(7.7)

where $S_0$ denotes the contribution corresponding to $\Delta$, $S_4$ corresponds to $\sum_4$, and so on. It remains to estimate each $S_j$.

### 8. The diagonal contribution $S_0$

We have

$$S_0 = \sum_{l \geq L} \sum_{m \geq N} |\lambda(m^{l_{k_2}})|^2 \int_{\mathbb{R}^\mu=0} h(\mu) \text{spec}(\mu) d\mu$$

Substituting the pointwise bound $\lambda(n) \ll n^\epsilon$ and using the bound $\text{spec}(\mu) \asymp T^3$ for the spectral measure, we obtain

$$S_0 \ll NT^3L.$$  

### 9. The contribution of $S_6$

We have

$$S_6 = \sum_{\epsilon_1, \epsilon_2 = \pm 1} \sum_{l_1, l_2 \leq L} \sum_{D_1, D_2} \frac{1}{D_1 D_2} \sum_{m, n \geq 1} \lambda(m^{l_{k_2}}) \overline{\lambda(n^{l_{k_2}})} S(\epsilon_2 m^{l_{k_0}} \epsilon_1 r^{l_{k_2-k_1}}_1, r^{l_{k_2-k_1}}_2, n^{l_{k_0}}; D_1, D_2) \Phi_{\omega_6} \left( -\frac{\epsilon_2 m r^{l_{k_0}}_1 l_{k_2-k_1} D_2}{D_1^2}, -\frac{\epsilon_1 n r l_{k_2-k_1} l_{k_2} D_1}{D_2^2} \right)$$

(9.1)

We apply Lemma 6.2 with $y_1 = m^{l_{k_0}} r^{l_{k_2-k_1}}_1 D_2 / D_1^2$ and $y_2 = n^{l_{k_0}} r l^{l_{k_2-k_1}}_2 D_1 / D_2^2$ to see that $\Phi_{\omega_6}(y_1, y_2)$ is negligibly small unless

$$\frac{1}{D_1^2} \gg T \frac{\lambda_{\epsilon_1, \epsilon_2}}{N^{\frac{1}{2}}} \text{ and } \frac{1}{D_2^2} \gg T \frac{\lambda_{\epsilon_1, \epsilon_2}}{N^{\frac{1}{2}}}.$$  

(9.2)
\[
D_1, D_2 \ll \frac{NL^{k_0+k_2-k_1}T}{f^2} =: D_0. \tag{9.3}
\]

We now substitute the decomposition (5.3) of the long Weyl Kloosterman sum. After substituting and writing \( D_1 = fd_1, D_2 = fd_2 \) and dividing the \( d_1, d_2 \) sum into dyadic blocks \( d_1 \asymp H_1 \ll D_0/f \) and \( d_2 \asymp H_2 \ll D_0/f \), we see that each dyadic box contributes at most

\[
\frac{1}{H_1H_2} \sum_{f \atop d_1 \asymp H_1} \sum_{d_2 \asymp H_2} \sum_{\epsilon_1, \epsilon_2 = \pm 1} \sum_{l_1, l_2 \asymp L} \sum_{y \pmod{f}} y
\]

\[
\times \sum_{m,n \asymp N/L^{k_2}} \lambda(ml_1^{k_2}) \lambda(nl_2^{k_2}) S(\epsilon_2ml_1^{k_0}, M_f(y); d_1) S(\epsilon_1nl_2^{k_0}, \tilde{M}_f(y); d_2)
\]

\[
\times V\left(\frac{ml_1^{k_2}}{N}\right) V\left(\frac{nl_2^{k_2}}{N}\right) \Phi_{w_6} \left( -\frac{\epsilon_2mr_1^{k_0}l_2^{k_2-k_1}d_2}{d_1^2 f}, -\frac{\epsilon_1nr_1^{k_0}l_1^{k_2-k_1}l_2^{k_0}d_1}{d_2^2 f} \right). \tag{9.4}
\]

To apply the \( GL(2) \) Voronoi summation to the \( m \) and \( n \) sums, we pull out the \( l_1, l_2 \) variables using the Hecke relation

\[
\lambda(ml_1^{k_2}) = \lambda(m)\lambda(l_1^{k_2}) - \sum_{d=1}^{k_2} \lambda\left( \frac{ml_1^{k_2}}{d^2} \right) \tag{9.5}
\]

where the \( r \)-th term in the bigger parenthesis occurs only if \( l_1 | m \). For simplicity, we only estimate the contribution of the first term. We remark that using (9.5) recursively the other terms can be handled similarly and provide us with smaller contributions. Hence the \( m, n \) sums in (9.4) is essentially

\[
\lambda(l_1^{k_2})\lambda(l_2^{k_2}) \sum_{m,n \asymp N/L^{k_2}} \lambda(m)\lambda(n) S(\epsilon_2ml_1^{k_0}, M_f(y); d_1) S(\epsilon_1nl_2^{k_0}, \tilde{M}_f(y); d_2)
\]

\[
\times V\left(\frac{ml_1^{k_2}}{N}\right) V\left(\frac{nl_2^{k_2}}{N}\right) \Phi_{w_6} \left( -\frac{\epsilon_2mr_1^{k_0}l_2^{k_2-k_1}d_2}{d_1^2 f}, -\frac{\epsilon_1nr_1^{k_0}l_1^{k_2-k_1}l_2^{k_0}d_1}{d_2^2 f} \right). \tag{9.6}
\]

We now apply \( GL(2) \) Voronoi to the \( m \) and \( n \) sum. We carry out the details for \( f \) cuspidal and \((d_1, l_1) = (d_2, l_2) = 1\) in this section. The case with \( l_1 | d_1 \) or \( l_2 | d_2 \) has smaller contribution due to lowering of the conductor and can be carried out in a similar fashion. The case \( \lambda(n) = d(n) \) contains an additional main term coming from the summation formula for \( d(n) \) and is addressed in the next section. Note that the Kloosterman sums get transformed into Ramanujan sums, saving the whole modulus \( d_1 \) and \( d_2 \). This is a crucial point in the proof. The dual sum is essentially of the form

\[
\frac{N^2}{L^{2k_2}d_2d_1} \sum_{c_2/d_2} c_1 c_2 \sum_{\hat{m}} \lambda(\hat{m})\lambda(\tilde{n}) I(\hat{m}, \tilde{n}), \tag{9.7}
\]
where

\[
I(\tilde{m}, \tilde{n}) = \sum_{\epsilon_1, \epsilon_2 = \pm 1} \int \int V(x)V(y)\Phi_{\nu_0}(-\epsilon_2 \theta_1 x, -\epsilon_1 \theta_2 y) J_{k_{f-1}}(U \sqrt{x}) J_{k_{f-1}}(V \sqrt{y}) \, dx \, dy,
\]

where

\[
\theta_1 = \frac{N_r l_1^{k_0-k_2} l_2^{k_2-k_1} d_2}{d_1^2 f}, \quad U = \frac{4\pi \sqrt{N \tilde{m}}}{l_1^{k_2/2} d_1},
\]

\[
\theta_2 = \frac{N_r l_1^{k_2-k_1} l_2^{k_0-k_2} d_1}{d_2^2 f}, \quad V = \frac{4\pi \sqrt{N \tilde{n}}}{l_2^{k_2/2} d_2}.
\]

Extracting the oscillations from the Bessel functions using (3.3) we see that \(I(\tilde{m}, \tilde{n})\) is essentially a sum of four terms of the form

\[
|UV|^{-1/2} \sum_{\epsilon_1, \epsilon_2 = \pm 1} \int \int W(x)W(y)\Phi_{\nu_0}(-\epsilon_2 \theta_1 x, -\epsilon_1 \theta_2 y)e(\pm U \sqrt{x} \pm V \sqrt{y}) \, dx \, dy,
\]

where \(W(x)\) is the updated smooth weight function which by (3.4) satisfies \(W^{(j)}(x) \ll j \) for \(j \leq 1\).

By (6.4) and repeated integration by parts in the \(x, y\) integral, it follows that \(I(\tilde{m}, \tilde{n})\) is negligibly small unless

\[
U \ll T + \theta_1^{1/2} + \theta_1^{1/3} \theta_2^{1/6}, \quad V \ll T + \theta_2^{1/2} + \theta_1^{1/6} \theta_2^{1/3},
\]

which by (6.3) implies

\[
U, V \ll \theta_1^{1/2} + \theta_2^{1/2}.
\]

Substituting the values (9.9), we obtain

\[
\tilde{m}, \tilde{n} \ll L_{k_0+k_2-k_1} r(d_1 + d_2) := M_0.
\]

Substituting the definition (5.16) into (9.10) we obtain

\[
I(\tilde{m}, \tilde{n}) \ll |UV|^{-1/2} |\mathcal{K}(\theta_1, \theta_2, \pm U, \pm V)|
\]

where

\[
\mathcal{K}(\theta_1, \theta_2, U, V) = \sum_{\epsilon_1, \epsilon_2 \in \pm 1} \int h(\mu) \text{spec}(\mu)
\]

\[
\int_{\mathbb{R}} \int_{\mathbb{R}} 2xW(x^2)2yW(y^2)K_{\nu_0}^{\epsilon_1, \epsilon_2}(\epsilon_1 \theta_1 x^2, \epsilon_2 \theta_2 y^2; \mu)e(Ux + Vy) \, dx \, dy \, d\mu.
\]

Now by the integral representations given in section 5.5 we get

\[
\mathcal{K}(\theta_1, \theta_2, U, V) \ll \sum_{i=1}^5 |\mathcal{K}_i(\theta_1, \theta_2, U, V)|
\]

where

\[
\mathcal{K}_i(\theta_1, \theta_2, U, V) = \int h(\mu) \text{spec}(\mu) \int_{\mathbb{R}} \int_{\mathbb{R}} U(x)U(y)J_i(\epsilon_1 \theta_1 x^2, \epsilon_2 \theta_2 y^2; \mu)e(Ux + Vy) \, dx \, dy \, d\mu,
\]

(9.14)
where $U(x) = 2xW(x^2)$ and $J_i$ as defined in (5.17). It remains to estimate the contribution of each $K_i$, $i = 1, \ldots, 5$. As remarked earlier, we only estimate the contribution of $K_4$, which is given by (5.8) of Theorem 6.3. The estimation of the rest is very similar and is just a matter of formality. In (5.8), it is enough to estimate the contribution of the last term on the right-hand side since the estimation for the rest are very similar and provide us with a smaller contribution. Substituting the last term of (5.8) and the expression for $U, V$ from (9.9) into (9.12), and trivially executing the $\mu$ integral we obtain

$$I(\tilde{m}, \tilde{n}) \leq \frac{TL^{k_2/2}}{N^{1/2}} \cdot \frac{d_1}{\tilde{m}^{1/4}\tilde{n}^{1/4}}$$

with the constraint

$$\left| \frac{2\pi U}{\theta_{1/2}^2} - 1 \right| = \left| \frac{\sqrt{\tilde{m}}}{b_2} - 1 \right| \ll T^{-1/40},$$

where $b := (8\pi^2)^{-1}(r_{l_1}^{k_0}l_2^{-k_1}/f)$. Hence taking absolute values, we see that (9.7) is dominated by

$$\frac{T N^{3/2} L^{k_2/2}}{L^{2k_2}} \sum_{l_1 \leq T} \frac{1}{f} \sum_{l_2 \geq T} \frac{1}{f_2} \sum_{c_2 \mid d_2} \sum_{c_1 \mid d_1} c_1 c_2 \sum_{\tilde{m} 
mid M_0} \sum_{\tilde{n} \nmid M_0} \frac{1}{\tilde{m}^{1/4}\tilde{n}^{1/4}}$$

where the prime over the $\tilde{m}$ sum denotes the restriction (9.16). Note that have replaced the absolute values of Fourier coefficients by their pointwise bound $T^\epsilon$. Substituting (9.17) in (9.4), and further dividing $\tilde{m} \asymp Z_1 \ll M_0$ and $\tilde{n} \asymp Z_2 \ll M_0$ into dyadic blocks, we arrive at

$$\frac{T N^{3/2} L^{k_2/2}}{L^{2k_2}} \sum_{l_1 \leq T} \frac{1}{f} \sum_{l_2 \geq T} \frac{1}{f_2} \sum_{c_2 \mid d_2} \sum_{c_1 \mid d_1} c_1 c_2 \sum_{\tilde{m} \asymp Z_1} \sum_{\tilde{n} \asymp Z_2} \sum_{y \mod f} y(f_{l_2^{k_2-k_1}d_2 + rl_1^{k_2-k_1}d_1})$$

We now count the number of points given by the last five set of summations. The last two condition on $\tilde{m}$ and $\tilde{n}$ implies

$$\tilde{m} f = c_1 k_1 + rl_2^{k_2-k_1}d_2 \quad \text{and} \quad \tilde{n} f = c_2 k_2 + rl_1^{k_2-k_1}d_1$$

We first fix $\tilde{m}, c_2, d_2$. Note that this determines $c_1$ (upto a divisor function). For this $c_1$, the number of $d_1$ is $\ll H_1/c_1$, and for each of these $d_1$, the no. of $k_2$ is $\ll 1 + fZ_2/c_2$. Also, the number of $y \mod f$ is $\ll (rl_2^{k_2-k_1}d_2, f) \leq (rl_2^{k_2-k_1}, f)(d_2, f)$. Hence (9.18) is dominated by

$$\frac{T N^{3/2} L^{k_2/2}}{L^{2k_2}} \sum_{l_1 \leq T} \sum_{l_2 \geq T} \sum_{f} \frac{(rl_l^{k_2-k_1}, f)}{f}$$

$$\times \frac{1}{H_2 Z_1^{1/4}Z_2^{1/4}} \sum_{d_2 \geq H_2} \sum_{c_2 \mid d_2} c_2 \left(1 + \frac{fZ_2}{c_2} \right) \sum_{\tilde{m} \asymp Z_1} \sum_{\tilde{n} \asymp Z_2} \sum_{y \mod f} y(f_{l_2^{k_2-k_1}d_2 + rl_1^{k_2-k_1}d_1})$$

(9.19)
Executing the $\tilde{m}$ sum with the constraint \((9.16)\), and the remaining sum trivially, we see that the second line in \((9.20)\) is dominated by
\[
T^{-1/140} \left( Z_1^{3/4} / Z_2^{1/4} \right) \left( H_2^2 + f Z_2 H_2 \right) \frac{T^{-1/140} Z_1^{1/4}}{Z_2^{1/4}} \left( Z_1^{1/2} + Z_2^{1/2} H_2 / Z_2 \right).
\]
\[\text{Eqn (9.21)}\]

The last bound is fine for us if \((Z_1/Z_2) \ll 1\). When \(Z_1/Z_2\) is large, we count the set of congruences \((9.19)\) by first fixing $\tilde{n}, c_1, d_1$. Following the rest of the arguments from earlier, we will arrive at the bound
\[
(Z_2/Z_1)^{1/4} \frac{N^{1/2} (L_{k_0+k_2-k_1} r)^2}{fT}.
\]
\[\text{Eqn (9.22)}\]

for the second line of \((9.20)\). Combining \((9.21)\) and \((9.22)\), we see that the second line of \((9.20)\) can be dominated by
\[
T^{-1/280} \frac{N^{1/2} (L_{k_0+k_2-k_1} r)^2}{fT}.
\]

Substituting the last bound and executing the remaining sum trivially, we see that \((9.20)\), and consequently
\[
S_0 \ll \frac{N^2 r^2 L^{2(1+k_0-k_1)+k_2/2}}{T^{1/280}} \ll \frac{N^2 r^2 L^8}{T^{1/280}}.
\]
\[\text{Eqn (9.23)}\]

10. Computations for $f(z) = E(z, 1/2)$

For the case $\lambda(m) = d(m)$, the $m$-sum (similarly the $n$-sum) in \((9.6)\) becomes
\[
\sum_{x(d_1)} e \left( \frac{x M_f(y)}{d_1} \right) \sum_m d(m) e \left( \frac{\epsilon \sqrt{x m l_{k_0}^2}}{d_1} \right) V \left( \frac{m l_{k_1}^2}{N} \right) \Phi_{w_6} \left( -\frac{\epsilon \sqrt{x m l_{k_0}^2}}{d_1 f}, -\frac{\epsilon \sqrt{x m l_{k_0}^2}}{d_2 f} \right)
\]
\[
= \frac{N (d_1, l_{k_0}^2)}{l_{k_1}^2 d_1} \sum_{x(d_1)} e \left( \frac{x M_f(y)}{d_1} \right) \left( 2 \int_{\mathbb{R}^+} \log((d_1, l_{k_0}^2) \sqrt{x}/d_1) + \gamma \right) h(x) dx
\]
\[\text{Eqn (9.1) for } f(z) = E(z, 1/2)\]

where $h(x) = V(x) \Phi_{w_6} \left( -\epsilon \sqrt{x} l_{k_0}^2, -\epsilon \sqrt{x} l_{k_1}^2 \right)$ and $U = (d_1, l_{k_0}^2)$, $U' = (d_1, l_{k_1}^2)$. The contribution of $Y_0(U' \sqrt{x})$ part will be the same as in \((9.7)\) since the asymptotics for the $Y_0$ Bessel function and the $J$ Bessel function are the same (for fixed orders). For the $K_0(U' \sqrt{x})$ part, note that from \((9.2)\) and \((9.3)\), $U' \gg U \gg f N^{1/2} / (L_{k_2}^2 D_0) \gg (T^2 / r T N^{1/2}) L^{-2(k_0+k_2+k_1)} \gg T^{1/2} L^{-6} \gg T^c$, by our choice of $L$. Hence from the exponential decay \((3.3)\), the contribution of the $K_0(U' \sqrt{x})$ part is negligibly small.
Thus for $f = E(z, 1/2)$, we just have to take care of the additional contribution $A_0 + A_1 + A_2$ towards (9.4), where

$$A_0 = \sum_{l_1, l_2 \geq L} \sum_{d_1, d_2 \geq 1} \frac{N^2}{l_1^2 l_2^2 d_1^2 d_2^2} \sum_{f \geq 1} \frac{1}{f} \sum_{y \equiv l_1^{k_2-k_1} d_1 y \pmod{f}} f(r_2^{k_2-k_1} d_2 + r_1^{k_2-k_1} d_1 y) \times \sum_{c_1 | (d_1, M_f(y))} c_1 \mu \left( \frac{d_1}{c_1} \right) c_2 \mu \left( \frac{d_2}{c_2} \right) I_0,$$

$$A_1 \ll \sum_{l_1, l_2 \geq L} \sum_{f \geq 1} \frac{1}{f} \sum_{d_1, d_2 \equiv D_0/f} \frac{N^2}{l_1^2 l_2^2 d_1^2 d_2^2} \sum_{y \equiv l_1^{k_2-k_1} d_2 + r_1^{k_2-k_1} d_1 y \pmod{f}} f(r_1^{k_2-k_1} d_2 + r_2^{k_2-k_1} d_1 y) \times \sum_{c_1 | (d_1, M_f(y))} c_1 c_2 \sum_{d_2 | (\tilde{n} + l_0 M_f(y))} d(\tilde{n}) I_1(\tilde{m}),$$

$$A_2 \ll \sum_{l_1, l_2 \geq L} \sum_{f \geq 1} \frac{1}{f} \sum_{d_1, d_2 \equiv D_0/f} \frac{N^2}{l_1^2 l_2^2 d_1^2 d_2^2} \sum_{y \equiv l_1^{k_2-k_1} d_2 + r_1^{k_2-k_1} d_1 y \pmod{f}} f(r_1^{k_2-k_1} d_2 + r_2^{k_2-k_1} d_1 y) \times \sum_{c_1 | d_1} c_1 c_2 \sum_{d_2 | (\tilde{\tilde{n}} + l_0 M_f(y))} d(\tilde{\tilde{n}}) I_2(\tilde{\tilde{m}}),$$

where

$$I_0 = \sum_{\epsilon_1, \epsilon_2 = \pm 1} \int \int V(x) V(y) \Phi_{w_0}(-\epsilon_2 \theta_1 x, -\epsilon_1 \theta_2 y) dx \ dy = K(\theta_1, \theta_2, 0, 0),$$

$$I_1(\tilde{m}) = \sum_{\epsilon_1, \epsilon_2 = \pm 1} \int \int V(x) V(y) \Phi_{w_0}(-\epsilon_2 \theta_1 x, -\epsilon_1 \theta_2 y) Y_0(U \sqrt{x}) \ dx \ dy,$$

and

$$I_2(\tilde{n}) = \sum_{\epsilon_1, \epsilon_2 = \pm 1} \int \int V(x) V(y) \Phi_{w_0}(-\epsilon_2 \theta_1 x, -\epsilon_1 \theta_2 y) Y_0(U \sqrt{y}) \ dx \ dy,$$

where $K$ as in (9.13) and $U, V, \theta_1, \theta_2$ as in (9.9). By abuse of notation, here $V(x)$ and $V(y)$ denotes the updated weight functions which also includes the log factors coming from (10.1), wherever occurring. Also, note that the expressions for $S_1$ and $S_2$ are under the assumptions $(d_1, l_1) = (d_2, l_2) = 1$ as earlier.

As we will see in a moment, $S_0$ contains a main term which we need to evaluate explicitly and save in the $L$ variable. For this reason all the sums are kept intact for this part.
10.1. Estimating $A_0$. Denote $a_1 = r l_1^{k_2-k_1}, a_2 = r l_2^{k_2-k_1}$. Recall $M_f(y) = (a_2 d_2 + a_1 d_1 y) / f$ so that

$$c_1|(d_1, M_f(y)) \Rightarrow c_1|(d_1, a_2 d_2 + a_1 d_1 y) \Rightarrow c_1|(d_1, a_2 d_2).$$

One similarly has $c_2|(a_1 d_1, d_2)$. Hence $S_0$ can be written as

$$S_0 = \sum_{l_1, l_2 < L} \sum_{d_1, d_2} \frac{N^2(d_1, l_1^{k_0})(d_2, l_2^{k_0})}{l_1^{k_2} l_2^{k_2} d_1^{2} d_2^{2}} \sum_{c_1|(d_1, a_2 d_2)} c_1 \mu \left( \frac{d_1}{c_1} \right) c_2 \mu \left( \frac{d_2}{c_2} \right) \frac{1}{f} \cdot I_0 \sum_{y \text{ (mod } f)} 1.$$  

We now count the number of solutions $y \text{ mod } f$ to the last set of congruence conditions. Write $(a_1 d_1, a_2 d_2) = \lambda_0, \lambda_1 = \lambda_0 / c_1, \lambda_2 = \lambda_0 / c_2$ and further write $(\lambda_1, f) = a, (\lambda_2, f) = b$. Now

$$f|(a_2 d_2 + a_1 d_1 y) \text{ and } c_1|M_f(y) \iff c_1|f|(a_2 d_2 + a_1 d_1 y) \iff (a_2 d_2 / \lambda_0) + (a_1 d_2 / \lambda_0) y = 0(\text{mod } f/a),$$

and similarly

$$f|(a_2 d_2 + a_1 d_1 y) \text{ and } c_2|\tilde{M}_f(y) \iff (a_2 d_2 / \lambda_0) + (a_1 d_2 / \lambda_0) y = 0(\text{mod } f/b).$$

Since $[f/a, f/b] = f/(a, b)$, we thus obtain

$$\sum_{y \text{ (mod } f)} 1 = \sum_{y \text{ (mod } f)} 1 = \sum_{y \text{ (mod } f)} \phi \left( \frac{y}{f/(a, b)} \right).$$

Since $(a_2 d_2 / \lambda_0, a_1 d_1 / \lambda_0) = 1$, the last congruence condition uniquely determines $y$ modulo $f/(a, b)$ which we call $y_{f/(a,b)}$. Hence we get $y = \lambda \cdot f/(a, b) + y_{f/(a,b)}, 1 \leq \lambda \leq (a, b)$. Note that $(y_{f/(a,b)}, f/(a, b)) = 1$ by definition. Hence our counting problem boils down to counting $1 \leq \lambda \leq (a, b)$ such that $(\lambda \cdot f/(a, b) + y_{f/(a,b)}, (a, b)) = 1$ which is easily seen to be

$$\sum_{y \text{ (mod } f)} 1 = \sum_{y \text{ (mod } f)} 1 = \phi \left( \frac{f}{(a, b)} \right) \phi \left( \frac{(a, b)}{(f/(a, b), (a, b))} \right).$$

Substituting we have

$$S_0 = \sum_{l_1, l_2 < L} \sum_{d_1, d_2} \frac{N^2(d_1, l_1^{k_0})(d_2, l_2^{k_0})}{l_1^{k_2} l_2^{k_2} d_1^{2} d_2^{2}} \sum_{c_1|(d_1, a_2 d_2)} c_1 \mu \left( \frac{d_1}{c_1} \right) c_2 \mu \left( \frac{d_2}{c_2} \right) \sum_{y \text{ (mod } f)} 1 \cdot \phi \left( \frac{f}{(a, b)} \right) \phi \left( \frac{(a, b)}{(f/(a, b), (a, b))} \right) \cdot I_0 \cdot (10.8)$$
The second line of (10.8) equals

$$\sum_{a|\lambda_1} \sum_{b|\lambda_2} \sum_{c|(a,b)} c \phi \left( \frac{(a,b)}{c} \right) \sum_{f \geq 1} \frac{1}{f} \cdot I_0.$$  \hfill (10.9)

For notational ease, we work the details only for $c = 1$. The general case is easily worked out in the same way provided extra space and notations. Now $(f, \lambda_1) = a$ implies $f = af', (f', \lambda_1/a) = 1$. Substituting this value of $f$ in $(f, \lambda_2) = b$ we obtain $(\lambda_2, a)|b, f' = (b/(\lambda_2, a))f''$ where $(f'', \lambda_2/b) = 1$. Combining these two we get $f = abf''/(\lambda_2, a)$ where $(f'', \lambda_1 \lambda_2/ab) = 1$ and $(b/(\lambda_2, a), \lambda_1/a) = 1$. Finally, the last condition $(f/(a,b), (a,b)) = 1$ translates to $(f'', (a,b)) = 1$ and $(ab/(\lambda_2, a)(a,b)), (a,b)) = 1$. Hence the contribution of $c = 1$ to (10.9) becomes

$$\sum_{a|\lambda_1} \sum_{(\lambda_2, a)|b|\lambda_2} \frac{(\lambda_2, a)\phi((a,b))}{ab} \sum_{f'' \geq 1} \frac{1}{f''} K(\theta_1, \theta_2, 0, 0)$$  \hfill (10.10)

where from (9.9)

$$\theta_1 = \frac{Nr_1^{k_0-k_2} N_2^{k_2-k_1} d_2}{d_1^2 f} = \left( \frac{Nr_1^{k_0-k_2} N_2^{k_2-k_1} d_2(\lambda_2, a)}{d_1^2 ab} \right) \frac{1}{f''},$$

and

$$\theta_2 = \frac{Nr_1^{k_2-k_1} N_2^{k_0-k_2} d_1}{d_2^2 f} = \left( \frac{Nr_1^{k_2-k_1} N_2^{k_0-k_2} d_1(\lambda_2, a)}{d_2^2 ab} \right) \frac{1}{f''}.$$  

Executing the $f''$-sum using Theorem 6.4 we get

$$\sum_{f'' \geq 1} \frac{1}{f''} K(\theta_1, \theta_2, 0, 0) \ll T^3 \left( \frac{d_1^{1/2} d_2^{1/2} ab}{N(rL^{-k_1}) L^{k_0}(\lambda_2, a)} \right).$$

Substituting in (10.10) and executing the remaining sum we see that (10.10) is bounded by

$$\frac{T^3 d_1^{1/2} d_2^{1/2}}{N(rL^{-k_1}) L^{k_0}} \sum_{a|\lambda_1} \sum_{b|\lambda_2} \phi((a,b)) \ll \frac{T^3 d_1^{1/2} d_2^{1/2}}{N(rL^{-k_1}) L^{k_0}(\lambda_1, \lambda_2)} \frac{T^3 d_1^{1/2} d_2^{1/2}}{N(rL^{-k_1}) L^{k_0}} \frac{\lambda_0(c_1, c_2)}{c_1, c_2}.$$
Substituting the above bound with the value \( \lambda_0 = (rl_1^{k_2-k_1}d_1, rl_2^{k_2-k_1}d_2) \) for the second line of (10.8) we obtain
\[
A_0 \ll \frac{NT^3}{(rL-L_1^{k_1})L^{2k_2+k_0}} \sum_{l_1, l_2 \leq L, d_1, d_2} (d_1, l_1^0)(d_2, l_2^0)(rl_1^{k_2-k_1}d_1, rl_2^{k_2-k_1}d_2) \sum_{c_1, c_2} (c_1, c_2) \]
\[
\ll \frac{NT^3}{(rL-L_1^{k_1})L^{2k_2+k_0}} \sum_{l_1, l_2 \leq L, d_1, d_2} (d_1, l_1^0)(d_2, l_2^0)(rl_1^{k_2-k_1}d_1, rl_2^{k_2-k_1}d_2) d_3^{3/2} d_2^{3/2}
\]
\[
\ll \frac{NT^3}{L^{2k_2+k_0}} \sum_{l_1, l_2 \leq L} (1 + L^{k_2+k_0} \delta_{l_1=l_2}) \ll NT^3 L^2 \left( \frac{1}{L^{2k_2+k_0}} + \frac{1}{L} \right).
\]

(10.11)

10.2. Estimating \( A_1 \) and \( A_2 \). By symmetry, it is enough to consider \( S_1 \). By extracting the oscillations from the Bessel functions using (3.3), and substituting the definition of \( \Phi_{w_0} \), we obtain
\[
I_1(\tilde{m}) \ll |U|^{-1/2} K(\theta_1, \theta_2, \pm U, 0),
\]
where \( K(\theta_1, \theta_2, U, V) \) as defined in (9.13). From the representations given in section 5.5 we have
\[
K(\theta_1, \theta_2, U, 0) \ll \sum_{i=1}^5 |K_i(\theta_1, \theta_2, U, 0)|,
\]
where \( K_i(\theta_1, \theta_2, U, V) \) as defined in (9.14). As earlier, we consider the contribution of only \( K_i(\theta_1, \theta_2, U, 0) \), towards which we have the estimate (6.9). It is enough to provide the calculation for the second line in the right hand side of (6.9). Substituting the bound into (10.12), we obtain
\[
I_1(\tilde{m}) \ll \frac{\left| \frac{2\pi U}{\theta_1^{1/2}} - 1 \right|}{\sqrt{\tilde{m}}} \ll T^{-1/40},
\]
with the constraint
\[
\left| \frac{2\pi U}{\theta_1^{1/2}} - 1 \right| = \left| \frac{\sqrt{\tilde{m}}}{bd_2} - 1 \right| \ll T^{-1/40},
\]
where \( b := (8\pi^2)^{-1}(rl_1^0, l_2^0, k_2-k_1) / f \). Dividing into dyadic blocks \( d_1 \approx H_1 \ll D_0 / f, d_2 \approx H_2 \ll D_0 / f \) and \( \tilde{m} \approx Z \ll M_0 \) we see that the contribution of this dyadic block towards \( A_1 \) at most
\[
\frac{N^{7/4} L^{k_2/4}}{L^{2k_2}} \sum_f \frac{1}{d_1, d_2 \geq L} \lambda(l_1^0) \lambda(l_2^0)
\]
\[
\times \frac{1}{H_1 H_2^2 Z^{1/4}} \sum_{d_1 \ll H_1} \sum_{d_2 \ll H_2} y \mod f \sum_{c_1, c_2} c_1 c_2 \sum_{c_1 | d_1} c_2 | d_2, M_f(y)} \sum_{\tilde{m} \approx Z} \sum_{c_1 | \tilde{m}} m_1 c_1 = 1,
\]

(10.15)
where the prime over the \( \tilde{m} \) sum denotes the restriction \((10.14)\). Note that \( c_2|(d_2, M_f(y)) \) implies \( c_2|(d_2, r l_1^{k_2-k_1} d_1) \) and \( c_1|\tilde{m} + l_1^{k_0} M_f(y) \) implies \( c_1|\tilde{m} f + r l_1^{k_0} l_2^{k_2-k_1} d_2 \). Executing the \( c_2 \) sum and rearranging, we get that the second line of \((10.15)\) is bounded by

\[
\frac{1}{H_1 H_2^{5/2} Z^{1/4}} \sum_{\tilde{m} \geq Z} (f, r l_1^{k_2-k_1} d_2) \sum_{c_1|\tilde{m} f + r l_1^{k_0} l_2^{k_2-k_1} d_2} c_1 \sum \left( d_2, r l_1^{k_2-k_1} d_1 \right). \tag{10.16}
\]

The last sum in \((10.16)\) is bounded by \((H_1/c_1)(d_2, r l_1^{k_2-k_1} c_1)\), which by the congruence condition is at most \((H_1/c_1)(d_2, r l_1^{k_2-k_1} \tilde{m} f)\). Summing over \( \tilde{m} \) with the restriction \((10.14)\), we get that \((10.16)\) is at most

\[
\frac{T^{-1/140} Z^{3/4}}{H_2^{5/2}} \sum_{d_2 \geq H_2} (f, r l_1^{k_2-k_1} d_2) (d_2, r l_1^{k_2-k_1} f) \ll \frac{T^{-1/140} Z^{3/4}}{H_2^{5/2}} \sum_{d_2 \geq H_2} (f, d_2)^2 \ll \frac{T^{-1/140} Z^{3/4}}{H_2^{1/2}}. \tag{10.17}
\]

Substituting \( Z \) from \((9.11)\), we see that \((10.15)\) is bounded by

\[
(r L^{k_0+k_2-k_1})^{3/4} \frac{T^{-1/140} N^{7/4} L^{k_2/4} L^2}{L^{2k_2}} \sum_{f \leq D_0} \frac{1}{f^{7/4}} \sup_{H_1, H_2 \ll D_0/f} H_2^{1/4} \ll (r L^{k_0+k_2-k_1})^{3/4} \frac{T^{-1/140} N^{7/4} L^{k_2/4} L^2}{L^{2k_2}} D_0^{1/4} \ll (r L^{k_0+k_2-k_1})^{5/4} \frac{N^2 L^{k_2/4} L^2}{T^{1/2+1/140} L^{2k_2}}, \tag{10.18}
\]

which is clearly smaller than \((9.23)\).

11. Remaining contributions

First let us consider \( S_4 \) which involves the transform \( \Phi_{w_4}(n_1 n_2 m_2/(D_1 D_2)) \) with the constraints \( D_2 D_1, \ n_2 D_1 = m_1 D_2^2 \), where \( (m_1, m_2) = (r l_1^{k_2-k_1}, m_1^{k_0}) \) and \( (n_1, n_2) = (r l_2^{k_2-k_1}, n_2^{k_0}) \) in our case. From \((6.1)\), this transform negligibly small unless

\[
\frac{n_1 n_2 m_2}{D_1 D_2} \gg T^3. \tag{11.1}
\]

Writing \( D_1 = D_2 q \), we get \( D_2 = n_2 q / m_1 \) and hence we must have

\[
\frac{n_1 m_2^2 m_2}{n_2 q^2} \gg T^3. \tag{11.2}
\]

Substituting the values \( (m_1, m_2) = (r l_1^{k_2-k_1}, m_1^{k_0}) \) and \( (n_1, n_2) = (r l_2^{k_2-k_1}, n_2^{k_0}) \), the above gives

\[
\frac{(r L^{k_2-k_1})^3 m}{n q^2} \gg T^3. \tag{11.3}
\]

Since \( m \approx n \approx N \) and \( r, L \) are small powers of \( T \), it is clear that \((11.3)\) can never happen. Hence the contribution of \( S_4 \) is negligible. One can similarly show that the contribution of \( S_5 \) is also negligible.
Substituting $\mathcal{E}_{\min}, \mathcal{E}_{\max}$ from (3.15) into (7.7), we have

$$S_{\min} = \frac{1}{24(2\pi i)^2} \int \int_{\Re(\mu) = 0} \frac{h(\mu)}{N_\mu} \left| \sum_{l \geq L} \sum_{n \geq N} \sum_{l' \leq l} \sum_{n' \leq n} \lambda_\mu(l^{k_2-k_1}, nl^{k_0-k_2}) \lambda(n) V \left( \frac{n}{N} \right) \right|^2 d\mu_1 d\mu_2 \geq 0,$$

$$S_{\max} = \frac{1}{2\pi i} \sum_g \int \int_{\Re(\mu) = 0} \frac{h(\mu + \mu_g, \mu - \mu_g, -2\mu)}{N_{\mu_g}} \times \left| \sum_{l \geq L} \sum_{n \geq N} \sum_{l' \leq l} \sum_{n' \leq n} \lambda_{\mu,g}(l^{k_2-k_1}, nl^{k_0-k_2}) \lambda(n) V \left( \frac{n}{N} \right) \right|^2 d\mu \geq 0.$$

Hence we ignore $S_{\min}$ and $S_{\max}$ due to their negative contribution towards (7.7).

**Proof of Theorem 4.2** Follows after combining (4.15) with the estimates (10.18), (10.11) and (9.23).

It remains to prove Theorem 6.4 and Theorem 6.3 which we do in the last two sections below.

### 12. Proof of Theorem 6.4

Shifting the contours in (5.13) to $\Re s_1 = \Re s_2 = -1/2$, we have (upto a negligible error term)

$$\frac{1}{f} K \left( \frac{\theta_1}{f}, \frac{\theta_2}{f}, 0, 0 \right) = \frac{1}{6} \int_{\Re(\mu) = 0} h(\mu) \text{spec}(\mu) \int_{\Re} \int_{\Re} U(x)V(y) \sum_{\epsilon_1, \epsilon_2 = \pm 1} \sum_{w \in W} \int_{(-1/2)} \int_{(-1/2)} f^{s_1+s_2-1} |4\pi^2 \theta_1 x^2|^{-s_1} |4\pi^2 \theta_2 y^2|^{-s_2} G(s, \mu) S^{\epsilon_1, \epsilon_2}(s, w(\mu)) \frac{ds}{(2\pi i)^2} dx dy d\mu \quad (12.1)$$

Summing over $f$ with $(f, F) = 1$, the inner $s = (s_1, s_2)$ integral becomes

$$\int_{(-1/2)} \int_{(-1/2)} \zeta(1-s_1-s_2) W_F(1-s_1-s_2)|4\pi^2 \theta_1 x^2|^{-s_1} |4\pi^2 \theta_2 y^2|^{-s_2} G(s, \mu) S^{\epsilon_1, \epsilon_2}(s, w(\mu)) \frac{ds}{(2\pi i)^2} \quad (12.2)$$

where $W_F(s) = \prod_{p | F} (1-p^{-s})$. We now shift the contours in (12.2) to $\Re s_1 = \Re s_2 = 1/2 - \epsilon$, right past the possible pole at $s_1 + s_2 = 0$. We claim that the residues corresponding to $\epsilon_1, \epsilon_2 \in \{\pm 1\}^2, w \in W$ all add up to zero (upto a negligible error). Indeed, we first note that

$$G((s,-s), \mu) S^{++}((s,-s), \mu) = G((s,-s), \mu) S^{--}((s,-s), \mu) = 0 \quad (12.3)$$
by the definition in (5.5) and (5.6). So, we are left with
\[ \sum_{w \in W} (S^+((s,-s), w(\mu)) + S^-((s,-s), w(\mu))) G((s,-s), \mu). \] (12.4)

We have
\[ S^+((s,-s), w(\mu)) G((s,-s), \mu) = \frac{\cos \left( \frac{\pi}{2}(\mu_2 - \mu_3) \right)}{32 \sin \left( \frac{\pi}{2}(\mu_1 - \mu_2) \right) \sin \left( \frac{\pi}{2}(\mu_1 - \mu_3) \right) (\mu_1 - s)(s - \mu_2)(s - \mu_3)}. \] (12.5)

This is negligibly small unless
\[ \Im(\mu_2) < \Im(\mu_1) < \Im(\mu_3) \text{ or } \Im(\mu_3) < \Im(\mu_1) < \Im(\mu_2), \] (12.6)
in which case it equals, up to a negligible error term,
\[ \frac{1}{16(\mu_1 - s)(s - \mu_2)(s - \mu_3)}. \] (12.7)

One can similarly show that
\[ S^-((s,-s), w(\mu)) G((s,-s), \mu) = -\frac{1}{16(\mu_1 - s)(s - \mu_2)(s - \mu_3)} + O_A(T^{-A}), \] (12.8)
if
\[ \Im(\mu_1) < \Im(\mu_3) < \Im(\mu_2) \text{ or } \Im(\mu_2) < \Im(\mu_3) < \Im(\mu_1), \] (12.9)
and is negligibly small otherwise. Adding the four relevant Weyl chambers with non-negligible contribution, it is now clear that (12.4) is essentially zero. Hence the theorem follows after shifting the contours to \( \Re s_1 = \Re s_2 = 1/2 - \epsilon \), and then estimating trivially using the Stirling’s formula.

13. Proof of Theorem 6.3

Let us start by recalling
\[ \mathcal{J}_4(y, \mu) = \frac{|y_1|^{\mu_2/2}}{|y_2|^{\mu_2/2}} \int_0^1 \tilde{K}_{3\mu_3} \left( |y_1|^{1/2} \sqrt{1-u^2} \right) \tilde{K}_{3\mu_3} \left( |y_2|^{1/2} \sqrt{u^2-1} \right) u^{3\mu_3} \frac{du}{u}, \] (13.1)
and the object we want to bound in this section is
\[ \mathcal{K}_4(\theta_1, \theta_2, U, V) = \int_{\Re \mu = 0} h(\mu) \text{spec}(\mu) \int \mathcal{U}(x) \mathcal{V}(y) \mathcal{J}_4(\theta_1 x^2, \theta_2 y^2; \mu) e(Ux + V y) dx dy d\mu. \] (13.2)

After substituting (13.1), we can write the above as
\[ \mathcal{K}_4(\theta_1, \theta_2, U, V) = \int_{\Re \mu = 0} h(\mu) \left( \frac{\theta_1}{\theta_2} \right)^{\mu_2/2} \text{spec}(\mu) \int_0^1 I(U, \alpha(u)) I(V, \beta(u)) u^{3\mu_3} \frac{du}{u}, \] (13.3)
where
\[ I(U, \alpha(u)) = \int \mathcal{U}(x)x^{\nu_2} \tilde{K}_{3\mu_3} (\alpha(u)x) e(Ux) dx, \alpha(u) = \theta_1^{1/2} \sqrt{1-u^2}, \]
\[ \tilde{I}(V, \beta(u)) = \int_{\mathbb{R}} V(y) y^{-\mu} K_{3\nu_3} (\beta(u)y) e(V y) \, dx, \quad \beta(u) = \theta_2^{1/2} \sqrt{u^2 - 1}. \]

Now, the \( \mu_2 \) integral in (13.3) is given by

\[ \int_{\mathbb{R}_{\mu_2} = 0} h(\mu) \text{spec}(\mu) \left| \frac{u^3 \theta_1^{1/2}}{\theta_2^{1/2} y} \right|^{\mu_2} \, d\mu_2, \quad (13.4) \]

and by integration by parts and 7.4, this integral is negligible unless

\[ u = \frac{\theta_2^{1/2} y}{\theta_1^{1/2} x} \left( 1 + O(T^{-\epsilon}) \right). \]

(13.5)

Since \( u \leq 1 \), the first claim (6.7) follows.

The rest of the section is devoted to the estimation of the \( u \)-integral in (13.3). For convenience, we denote \( T_1 = \Im(3\nu_3) = \Im(\mu_3 - \mu_1) \) and \( T_2 = \Im\mu_2 \). Note that by our hypothesis

\[ T_1 \asymp T_2 \asymp T. \]

(13.6)

Denote

\[ c_1 := \left( \frac{U\pi}{\theta_1^{1/2}} \right) \left( \frac{\epsilon_2 T_1}{\epsilon_1 (\epsilon_2 T_1 - T_2)} \right), \quad c_2 := \left( \frac{V\pi}{\theta_2^{1/2}} \right) \left( \frac{\epsilon'_2 T_1}{\epsilon'_1 (\epsilon'_2 T_1 - T_2)} \right) \]

and

\[ c_0 := \frac{\epsilon_2 T_1 + T_2}{\epsilon_2 T_2 - T_2}, \quad \tilde{c}_0 := \frac{\epsilon'_2 T_1 + T_2}{\epsilon'_2 T_2 - T_2}. \]

We introduce several dyadic partition of unity and insert localising factors

\[ G(u) := F_1(T^\delta_1 (1 - u^2)) F_2(T^\delta_2 u) F_3 \left( T^\delta_3 \left( \frac{c_1}{1 - u^2} - 4c_0 \right) \right) F_4 \left( T^\delta_4 \left( \frac{c_2}{u^2 - 1} - 4\tilde{c}_0 \right) \right) \]

(13.8)

to the \( u \)-integral in (13.3), so that

\[ \mathcal{K}_4(\theta_1, \theta_2, U, V) \ll T^3 \sup_{\mu = \mu_0 + O(T^\epsilon)} \left| \int_0^1 \frac{G(u)}{u} I(U, \alpha(u)) \tilde{I}(V, \beta(u)) u^{3\mu_2} \, du \right|. \]

(13.9)

To estimate the last \( u \)-integral, we substitute of asymptotics (13.96) for \( I(U, \alpha(u)) \) and \( \tilde{I}(V, \beta(u)) \) from Lemma (13.8) to get

\[ \int_0^1 \frac{G(u)}{u} I(U, \alpha(u)) \tilde{I}(V, \beta(u)) u^{3\mu_2} \, du \sim \sum \frac{1}{T} \int_0^1 \frac{G(u)}{u} I^{\epsilon_1, \epsilon_2}_{R_1}(U, \alpha(u)) I^{\epsilon'_1, \epsilon'_2}_{R_2}(V, \beta(u)) u^{3\epsilon_2 T_2} \, du, \quad (13.10) \]

where \( I^{\epsilon_1, \epsilon_2}_{R_1} \) and \( I^{\epsilon'_1, \epsilon'_2}_{R_2} \) as in (13.94). Our object of study is now

\[ \frac{1}{T} \int_0^1 \frac{G(u)}{u} I^{\epsilon_1, \epsilon_2}_{R_1}(U, \alpha(u)) I^{\epsilon'_1, \epsilon'_2}_{R_2}(V, \beta(u)) u^{3\epsilon_2 T_2} \, du. \]

(13.11)
Note that from (13.3), we can assume
\[ u^{-1} \asymp T^{\delta_2} \asymp (\theta_1/\theta_2)^{1/6}. \] (13.12)

**Lemma 13.1.** We have
\[ G^j(u) \ll ((1 + T^{\delta_1} + T^{\delta_2})T^{\delta_1+\delta_2}j, \ j \geq 0. \] (13.13)

**Proof.** We first note that since \(1 - u^2 \asymp T^{-\delta_1}\), from Faà di Bruno’s formula (3.6) we get
\[
\frac{\partial^j}{\partial u^j}(T^{\delta_1}(c_1^2(1 - u^2)^{-1} - 4c_0)) \ll_j T^{\delta_1}c_1^2 \sum_{j_1+j_2=j} |1 - u^2|^{-1-j_1-j_2}|u|^{j_1}
\]
\[
\ll T^{\delta_1}c_1^2 \sum_{j_2 \leq j/2} (T^{-\delta_1}-1-j_2(T^{-\delta_2})^{-2j_2} \ll \sum_{j_2 \leq j/2} T^{\delta_1}c_1^2 T^{\delta_1} T^{(\delta_1-\delta_2)}(1 + T^{-\delta_1+2\delta_2})^{j/2}
\]
\[
\ll T^{\delta_1}c_1^2 T^{\delta_1}(T^{\delta_1-\delta_2} + T^{\delta_1/2})^j \ll (1 + T^{\delta_1})T^{\delta_1}. \] (13.14)

Hence
\[
\frac{\partial^j}{\partial u^j} F_3(T^{\delta_1}(c_1^2(1 - u^2)^{-1} - 4c_0)) \ll_j T^{\delta_1}(1 + T^{\delta_1})^j. \] (13.15)

Similarly, using the Faà di Bruno’s formula we get
\[
\frac{\partial^j}{\partial u^j}(T^{\delta_1}(c_2^2(u^2-1)^{-1} - 4\tilde{c}_0)) \ll_j T^{\delta_1}c_2^2 \sum_{j_1+j_2+j_3+j_4=j} |u^{-2} - 1|^{-1} - \sum j_k |u|^{-2} \sum j_k - j
\]
\[
\ll T^{\delta_1}c_2^2 \sum_{j_1+j_2+j_3+j_4=j} (T^{-\delta_1+2\delta_2} - 1 - \sum j_k (T^{-\delta_2})^{-2} \sum j_k - j
\]
\[
\ll T^{\delta_1}c_2^2 T^{-\delta_1+2\delta_2} T^{\delta_1} \sum_{j_1+j_2+j_3+j_4=j} T^{\delta_1} \sum j_k \ll T^{\delta_1}c_2^2 T^{-\delta_1+2\delta_2} T^{\delta_1+\delta_2} j
\]
\[
\ll (1 + T^{\delta_1})T^{(\delta_1+\delta_2)}j,
\]
and consequently
\[
\frac{\partial^j}{\partial u^j} F_4(T^{\delta_1}(c_2^2(u^2-1)^{-1} - 4\tilde{c}_0)) \ll_j T^{(\delta_1+\delta_2)}j(1 + T^{\delta_1})^j. \] (13.16)

Using (13.15), (13.16) and the simple inequalities \(F_1^{(j)}(T^{\delta_1}(1-u^2)) \ll_j T^{\delta_1} \) and \(F_2^{(j)}(T^{\delta_2}u) \ll_j T^{\delta_2}\) we arrive at
\[ G^j(u) \ll ((1 + T^{\delta_1} + T^{\delta_2})T^{\delta_1+\delta_2})^j, \ j \geq 0. \]
\[ \Box \]

**Case 1 : The generic case.** \(R_1^{-1} \ll T^\beta, \ R_2^{-1} \ll T^\beta, \)
\[ T^{-\delta_3/2} \gg T^{-1/3+2\varepsilon} \left( \frac{T}{\alpha(u)} + 1 \right) \] (13.17)
and
\[ T^{-\delta_4/2} \gg T^{-1/3+2\varepsilon} \left( \frac{T}{\beta(u)} + 1 \right). \] (13.18)
Note that we have
\[ T^{\delta_2} \asymp \theta_1^{1/6} / \theta_2^{1/6} = (\theta_1^{1/2} / T)(T/\theta_1^{1/3}\theta_2^{1/6}) \ll T^{\epsilon} T^{\delta_1/2}(\alpha(u)/T), \]  
where we have used the first condition of (6.6). Since \( T/\alpha(u) \asymp R_1 + 1 > 1 \), we conclude
\[ T^{\delta_2} \ll T^{\epsilon} \Rightarrow T^{\delta_2} \ll T^{\epsilon}. \]  

In this case we claim that the condition (13.104) holds for the corresponding roots of\( I^\epsilon_{R_1,R_2}(U,\alpha(u)) \) and \( \bar{I}^\epsilon_{R_1,R_2}(U,\beta(u)) \). It is enough to prove the claim for \( I^\epsilon_{R_1,R_2}(U,\alpha(u)) \) since the argument for the other is identical. Recall that \( z_1(\epsilon_1 \alpha(u)) \) and \( z_2(\epsilon_1 \alpha(u)) \) are the two roots of the equation
\[ \frac{\partial}{\partial v} \left( -T_2 \log(\phi_R(\epsilon_1 \alpha(u), v)) + \epsilon_2 T_1 \log v \right) = 0, \]  
where
\[ \phi_R(\alpha, v) = \frac{-2\pi}{T_2} \left( \frac{\alpha}{2\pi} \left( Rv - \frac{1}{Rv} \right) + U \right). \]  

Expanding we get
\[ \frac{\partial}{\partial v} \left( -T_2 \log(\phi_R(\epsilon_1 \alpha(u), v)) + \epsilon_2 T_1 \log v \right) \]
\[ = -\frac{T_2 \epsilon_1 \alpha(u) (v^2 + R^{-2})}{v(\epsilon_1 \alpha(u) (v^2 - R^{-2}) + R^{-1}2\pi Uv)} + \frac{\epsilon_2 T_1}{v} \]
\[ = \frac{\epsilon_1 \alpha(u)(\epsilon_2 T_1 - T_2)v^2 + R^{-1}(2\pi U)(\epsilon_2 T_1)v - \epsilon_1 \alpha(u)R^{-2}(\epsilon_2 T_1 + T_2)}{v(\epsilon_1 \alpha(u) (v^2 - R^{-2}) + R^{-1}2\pi Uv)} \]

Hence \( z_1(\epsilon_1 \alpha(u)) \) and \( z_2(\epsilon_1 \alpha(u)) \) are the roots of the quadratic equation
\[ v^2 + R^{-1} \left( \frac{2\pi U}{\alpha(u)} \right) \left( \frac{\epsilon_2 T_1}{\epsilon_2 T_1 - T_2} \right) v - R^{-2} \left( \frac{\epsilon_2 T_1 + T_2}{\epsilon_2 T_1 - T_2} \right). \]

Solving we get
\[ z_1(\epsilon_1 \alpha(u)) = R^{-1} \frac{c_1}{\sqrt{u^2 - 1}} + R^{-1} \sqrt{\frac{c_1^2}{1 - u^2} - c_0} \]  
and
\[ z_2(\epsilon_1 \alpha(u)) = R^{-1} \frac{c_1}{\sqrt{u^2 - 1}} - R^{-1} \sqrt{\frac{c_1^2}{1 - u^2} - c_0}, \]

where \( c_1 \) and \( c_0 \) as in (13.7). Hence by the assumption (13.17)
\[ z_1(\epsilon_1 \alpha(u)) - z_2(\epsilon_1 \alpha(u)) \asymp R^{-1} \sqrt{\frac{c_1^2}{1 - u^2} - c_0} \asymp R^{-1}T^{-\delta_1/2} \gg T^{2\epsilon} R^{-1}(T/\alpha(u) + 1)T^{-1/3}. \]

Since we are in the range \( R + 1 \asymp T/\alpha(u) \), we get
\[ z_1(\epsilon_1 \alpha(u)) - z_2(\epsilon_1 \alpha(u)) \gg T^{2\epsilon} R^{-1}(T/\alpha(u) + 1)T^{-1/3} \gg T^{2\epsilon} (R^3 T/(R + 1)^3)^{-1/3} \]
\[ \asymp T^{2\epsilon} (R^3 \alpha/(R + 1)^2)^{-1/3}. \]  

(13.23)
Hence we can substitute asymptotic expansion (13.105) in place of \( I_{R_1}^{\epsilon_1, \epsilon_2}(U, \alpha(u)) \) in (13.11). By similar arguments, the asymptotic expansion holds for \( \tilde{I}_{R_2}^{\epsilon_1', \epsilon_2'}(V, \beta(u)) \) as well. Let \( x_l(\epsilon'_1 \beta(u)) \) be the corresponding roots of the phase function of \( \tilde{I}_{R_2} \). Substituting these approximations we get

\[
\frac{1}{T} \int_0^1 \frac{G(u)}{u} I_{R_1}^{\epsilon_1, \epsilon_2}(U, \alpha(u)) \tilde{I}_{R_2}^{\epsilon_1', \epsilon_2'}(V, \beta(u)) u^{3T_2} \, du \\
\approx \frac{1}{T} \cdot \frac{T^{\delta_1/4 + \delta_2/4}}{\theta_1^{1/4}} \cdot \frac{T^{\delta_1/4 + \delta_2/4 - \delta_2/2}}{\theta_2^{1/4}} \sum_{k,l} \int_0^1 H_{k,l}(u) e \left( \frac{T_1 \cdot \phi_{k,l}(u)}{2\pi} \right) \, du,
\]

(13.24)

where

\[
\phi_{k,l}(u) := -(T_2/T_1) \log \left( \frac{\phi_{R_1}(\epsilon_1 \alpha(u), z_k(\epsilon_1 \alpha(u)))}{\psi_{R_2}(\epsilon'_1 \beta(u), x_l(\epsilon'_1 \beta(u)))} \right) + \epsilon_2 \log(z_k(\epsilon_1 \alpha(u))) + \epsilon'_2 \log(x_l(\epsilon'_1 \beta(u))) + 3(T_2/T_1) \log u,
\]

(13.25)

where

\[
\phi_{R_1}(\alpha, v) = \frac{-2\pi}{T_2} \left( \frac{\alpha}{2\pi} \left( R_1 v - \frac{1}{R_1 v} \right) + U \right),
\]

\[
\psi_{R_2}(\alpha, v) = \frac{-2\pi}{T_2} \left( \frac{\alpha}{2\pi} \left( R_2 v - \frac{1}{R_2 v} \right) + V \right),
\]

and

\[
H_{k,l}(u) := \frac{G(u)}{u} H_k(u) \tilde{H}_l(u),
\]

(13.26)

where \( H_k(u) \) is the weight function coming from \( I_{R_1}^{\epsilon_1, \epsilon_2} \) and \( \tilde{H}_l(u) \) is the weight function coming from \( \tilde{I}_{R_2}^{\epsilon_1', \epsilon_2'} \), that is,

\[
H_k(u) = (T^{\delta_1}(1 - u^2))^{-1/4} \frac{\omega_{R_1}(\epsilon_1 \alpha(u), z_k(\epsilon_1 \alpha(u)))}{(RT^{\delta_1/2}|z_1(\epsilon_1 \alpha(u)) - z_2(\epsilon_1 \alpha(u))|^{1/2})}
\]

(13.27)

and

\[
\tilde{H}_l(u) = (T^{\delta_1 - 2\delta_2}(u^{-2} - 1))^{-1/4} \frac{\omega_{R_2}(\epsilon'_1 \beta(u), x_l(\epsilon'_1 \beta(u)))}{(RT^{\delta_1/2}|x_1(\epsilon'_1 \beta(u)) - x_2(\epsilon'_1 \beta(u))|^{1/2})},
\]

(13.28)

where \( \omega_{R_1}, \omega_{R_2} \) as in (13.99). From the derivative bounds (13.106), (13.13) we deduce

**Lemma 13.2.** We have

\[
\frac{d^j H_{k,l}(u)}{du^j} \ll_j T^{\delta_2} \cdot ((B_1 + B_2)T^{\delta_1 + \delta_2})^j, \quad j \geq 1,
\]

(13.29)

where

\[
B_1 := (1 + R_1^{-1} T^{-\delta_1/2})(1 + T^{\delta_1})
\]

\[
B_2 := (1 + R_2^{-1} T^{-\delta_2/2})(1 + T^{\delta_2}).
\]
Proof. We first note that since $1 - u^2 \asymp T^{-\delta_1}$, $u \asymp T^{-\delta_2}$, from Faá di Bruno’s formula we get
\[
\frac{\partial^j}{\partial u^j} \alpha(u) = \frac{\partial^j}{\partial u^j}(\theta_1^{1/2}(1 - u^2)^{1/2}) \ll_j \theta_1^{1/2} \sum_{j_1, j_2} |1 - u^2|^{1/2-j_1-j_2} |u|^{j_1} \\
\ll \theta_1^{1/2} \sum_{j_2 \leq j/2} (T^{-\delta_1})^{1/2-j_2} (T^{-\delta_2})^{j_2-2j_2} \\
\ll \theta_1^{1/2} T^{-\delta_1/2} T^{j(\delta_1-\delta_2)} (1 + T^{-\delta_1+2\delta_2})^{j/2} \\
\ll \theta_1^{1/2} T^{-\delta_1/2} (T^{\delta_1-\delta_2} + T^{\delta_1/2})^j. 
\]

Using (13.30), (13.106) and invoking Faá di Bruno’s formula once again we obtain
\[
\begin{align*}
\frac{\partial^j}{\partial u^j} & \left( \frac{\omega_{R_1}(\epsilon_1 \alpha(u), z_k(\epsilon_1 \alpha(u)))}{RT^{\delta_1/2}|z_1(\epsilon_1 \alpha(u)) - z_2(\epsilon_1 \alpha(u))|^{1/2}} \right) \\
& \ll_j \sum_{j_1, j_2} \frac{1}{(|\alpha(u)|^{-1} B_1)} \delta_k \delta_j \ldots \delta_j \ldots \delta_j \ldots \delta_j \sum_k T^{-\delta_1/2} (T^{\delta_1-\delta_2} + T^{\delta_1/2})^j \\
& \ll (T^{\delta_1-\delta_2} + T^{\delta_1/2})^j \sum_{j_1, j_2} B_1 \delta_k \delta_j \ldots \delta_j \ldots \delta_j \ldots \delta_j \ldots \delta_j \ll (B_1 T^{\delta_1-\delta_2})^j \ll (B_1 T^{\delta_1+\delta_2})^j. 
\end{align*}
\]

Using the Faá di Bruno’s formula one similarly obtains
\[
\begin{align*}
\frac{\partial^j}{\partial u^j} (T^{\delta_1}(1 - u^2))^{-1/4} & \ll_j \sum_{j_1, j_2} (T^{\delta_1})^{j_1} (T^{\delta_1})^{j_2} \ll \sum_{j_2 \leq j/2} (T^{\delta_1-\delta_2})^{j_2} (T^{\delta_1})^{j_2} \\
& \ll (T^{\delta_1-\delta_2} + T^{\delta_1/2})^j \\
& \ll T^{j(\delta_1+\delta_2)}. 
\end{align*}
\]

Combining (13.31) and (13.32), we obtain
\[
\frac{\partial^j}{\partial u^j} H_k(u) \ll_j (B_1 T^{\delta_1+\delta_2})^j. 
\]

For $\beta(u) = \theta_2^{1/2}(u^2 - 1)^{1/2} = \theta_2^{1/2} u^{-1}(1 - u^2)^{1/2}$, using (13.30) and the fact that $(\partial^j/\partial u^j) u^{-1} \ll_j T^{\delta_2} T^{j^2}$, we obtain that for $j \geq 1$,
\[
\frac{\partial^j}{\partial u^j} \beta(u) \ll_j \theta_2^{1/2} T^{-\delta_1/2 + \delta_2} (T^{\delta_1-\delta_2} + T^{\delta_1/2} + T^{\delta_2})^j \ll \theta_2^{1/2} T^{-\delta_1/2 + \delta_2} T^{j(\delta_1+\delta_2)}. 
\]

Doing a similar calculation as in (13.31) and (13.32), we arrive at
\[
\frac{\partial^j}{\partial u^j} \left( \frac{\omega_{R_2}(\epsilon_1' \beta(u), x_1(\epsilon_1' \beta(u)))}{R_2 T^{\delta_1/2}|x_1(\epsilon_1' \beta(u)) - x_2(\epsilon_1' \beta(u))|^{1/2}} \right) \ll_j (B_2 T^{\delta_1+\delta_2})^j, 
\]
and
\[
\frac{\partial^j}{\partial u^j} (T^{\delta_1-2\delta_2}(u^2 - 1))^{-1/4} \ll_j T^{j(\delta_1+\delta_2)}. 
\]
from which it follows
\[
\frac{\partial^j}{\partial u^j} H_l(u) \ll_j (B_2 T^{\delta_1 + \delta_2})^j. \tag{13.34}
\]
Using Lemma 13.1, it is easy to see that
\[
\frac{\partial^j}{\partial u^j} G(u) \ll_j T^{\delta_2}((1 + T^{\delta_3} + T^{\delta_4})T^{\delta_1 + \delta_2})^j, \quad j \geq 1. \tag{13.35}
\]
From (13.33), (13.34) and (13.35), it follows
\[
\frac{\partial^j}{\partial u^j} H_{k,l}(u) = \frac{\partial^j}{\partial u^j} G(u) H_k(u)H_l(u) \ll_j T^{\delta_2}((B_1 + B_2)T^{\delta_1 + \delta_2})^j.
\]

The lemma follows.

\[\□\]

Lemma 13.3. We have
\[
\frac{d^j \phi_{k,l}(u)}{du^j} \ll_j ((B_1 + B_2)T^{\delta_1 + \delta_2})^j, \quad j \geq 1, \tag{13.36}
\]
where \(B_1, B_2\) as in Lemma 13.2.

Proof. Same as the proof of the previous lemma after using the derivative bounds (13.142) and (13.144) for each individual term in (13.25). Note the term (13.35) is not present in this case and hence the absence of the factor \(T^{\delta_2}\) in the statement of the lemma.

\[\□\]

Lemma 13.4. Let \(\delta_1, \delta_2, \delta_3\) and \(\delta_4\) as in (13.3). Then \(T^{-\delta_3}, T^{-\delta_4} \ll T^{\delta_1}\).

Proof. By definition, we have
\[
T^{-\delta_3} \asymp \frac{c_1^2}{1 - u^2} - 4c_0, \quad T^{-\delta_4} \asymp \frac{c_2^2}{u^2 - 1} - 4\bar{c}_0 \tag{13.37}
\]
Now
\[
\frac{c_1^2}{1 - u^2} \asymp T^{\delta_3} u^2 / \theta_1 \ll T^{\delta_1} (1 + (\theta_2 / \theta_1)^{1/3}) \ll T^{\delta_1} (1 + T^{-2\delta_2}),
\]
and
\[
\frac{c_2^2}{u^2 - 1} \asymp T^{\delta_1 - 2\delta_2} V^2 / \theta_2 \asymp T^{\delta_1} V^2 / (\theta_1^{1/3} \theta_2^{2/3}) \ll T^{\delta_1} (1 + (\theta_2 / \theta_1)^{1/3}) \ll T^{\delta_1} (1 + T^{-2\delta_2}),
\]
where we have used (6.6) and (13.12) in the last two inequalities. The claim follows after substituting the above bounds into (13.37) and observing that \(\delta_2 \geq 0\).

\[\□\]

Lemma 13.5. Define the quantities
\[
A := \left| 4c_1^2/(c_0 - 1)^2 - 1 \right| = \left| (\pi U/\theta_1^{1/2})^2 - 1 \right|, \tag{13.38}
\]
\[
B_3 := (B_1 + B_2)T^{\delta_1 - \delta_2}((1 + T^{-\delta_1/2 - \delta_4/2}).
\]
We can write
\[
\int_0^1 H_{k,l}(u)e \left( \frac{T_1 \phi_{k,l}(u)}{2\pi} \right) du = \int_{\mathbb{R}} F(t)e(T_1 \psi(t)) dt, \tag{13.39}
\]
where \(F(t)\) is a compactly supported smooth function and the phase function \(\psi(t)\) is such that
\[
\psi'(t) = K_0 \frac{Q(t)}{h(t)}, \quad K_0 := AT^{\delta_1 - 12\delta_2},
\]
Lemma 13.6. We begin with the second term of (13.40).

Proof. Using the fact that $x_t := x_t(\epsilon_1' \beta(u))$ satisfies (13.21), we get

$$\frac{1}{\psi_R^2(\epsilon_1' \beta(u), x_t(\epsilon_1' \beta(u)))} = -\frac{\epsilon_2' T_1}{\beta(u)(R_2 x_t + (R_2 x_t)^{-1})},$$

and hence

$$T_2 \epsilon_1' \beta'(u) \left( \frac{1}{\psi_R^2 (\alpha, v)} \right) \bigg|_{(\alpha, v) = (\epsilon_1' \beta(u), x_t(\epsilon_1' \beta(u)))} = -\frac{\epsilon_2' T_1 \left( x_t^2 - R_2^{-2} \right)}{u(1 - u^2)x_t^2 R_2^{-2}}.$$

where $Q(t)$ is a degree 13 monic polynomial in $t$ and $h(t)$ is a smooth bounded function for $t \in \text{supp}(F)$. Furthermore, for $t \in \text{supp}(F)$ we have the following derivative bounds.

$$F^{(j)}(t) \ll j T^{-\delta_2}(1 + T^{-\delta_1/2 - \delta_1/2})B_3^j, \quad j \geq 1,$$

and

$$\psi^{(j)}(t) \ll j B_3^j, \quad j \geq 1.$$

Proof. It follows from (13.21) and (13.25) that

$$\phi_{k,l}(u) = - (T_2/T_1) \epsilon_1' \alpha'(u) \left( \frac{1}{\psi_{R_1}^2} \frac{\partial \phi_{R_1}(\alpha, v)}{\partial \alpha} \right) \bigg|_{(\alpha, v) = (\epsilon_1 \alpha(u), z_k(\epsilon_1 \alpha(u)))} + (T_2/T_1) \epsilon_1' \beta'(u) \left( \frac{1}{\psi_{R_2}^2} \frac{\partial \psi_{R_2}(\alpha, v)}{\partial \alpha} \right) \bigg|_{(\alpha, v) = (\epsilon_1' \beta(u), x_t(\epsilon_1' \beta(u)))} + \frac{3(T_2/T_1)}{u}.$$
Substituting the roots (see (13.22) in case of $z_k$), we get

\[
\frac{x_t^2 - R_2^{-2}}{x_t^2 + R_2^{-2}} = \frac{(\tilde{c}_0 + 1)(u^{-2} - 1) - 2c_2^2 \pm 2c_2\sqrt{c_2^2 - \tilde{c}_0(u^{-2} - 1)}}{(\tilde{c}_0 - 1)(u^{-2} - 1) - 2c_2^2 \pm 2c_2\sqrt{c_2^2 - \tilde{c}_0(u^{-2} - 1)}}. \tag{13.46}
\]

In terms of $t = \sqrt{c_2^2 - \tilde{c}_0(u^{-2} - 1)}$, the numerator becomes

\[
(\tilde{c}_0 + 1)(u^{-2} - 1) - 2c_2^2 \pm 2c_2\sqrt{c_2^2 - \tilde{c}_0(u^{-2} - 1)} = -(\tilde{c}_0 + 1)t^2/\tilde{c}_0 \pm 2c_2t - (\tilde{c}_0 - 1)c_2^2/\tilde{c}_0 = Q_1(t),
\]

and the denominator

\[
(\tilde{c}_0 - 1)(u^{-2} - 1) - 2c_2^2 \pm 2c_2\sqrt{c_2^2 - \tilde{c}_0(u^{-2} - 1)} = -(\tilde{c}_0 - 1)t^2/\tilde{c}_0 \pm 2c_2t - (\tilde{c}_0 + 1)c_2^2/\tilde{c}_0 = Q_2(t).
\]

We also get

\[
u^{-1} = \sqrt{-t^2/\tilde{c}_0 + (1 + c_2^2/\tilde{c}_0)} = \sqrt{P_1(t)},
\]

\[
u^{-2} - 1 = (c_2^2 - t^2)/\tilde{c}_0 = P_2(t).
\]

The first part of the claim follows after substituting the above transformations into (13.45). For the sizes of the polynomials, note from the definitions it follows $P_1(t) = u^{-2} \asymp T^{2\delta_2}$ and $P_2(t) = u^{-2} - 1 \asymp T^{-\delta_1 + 2\delta_2}$. For $Q_1, Q_2$, we have from (13.46),

\[Q_1(t) = R_2^2(u^{-2} - 1)(x_t^2 - R_2^{-2}), Q_2(t) = R_2^2(u^{-2} - 1)(x_t^2 + R_2^{-2}).\]

Now from the support of $\omega_{R_2}(\epsilon_1'\beta(u), x_t(\epsilon_1'\beta(u)))$ function in (13.28), it follows $x_t \ll 1 + R_2^{-1}$. Hence

\[Q_1(t), Q_2(t) \ll T^{-\delta_1 + 2\delta_2}(1 + R_2)^2 \asymp T^{-\delta_1 + 2\delta_2}(T/\beta(u))^2 \ll T^{2\delta_2 + \epsilon}, \tag{13.47}
\]

since $\beta(u) \asymp T^{-\delta_1/2}T^{\delta_2}\theta_2^{1/2} \asymp T^{-\delta_1/2}\theta_1^{1/6}\theta_2^{1/3} \gg T^{-\delta_1/2}T^{1-\epsilon}$. □

**Lemma 13.7.** We have

\[-(T_2/T_1)\epsilon_1\alpha'(u) \left( \frac{1}{\phi_{R_1}} \frac{\partial \phi_{R_1}(\alpha, v)}{\partial \alpha} \right)_{(\alpha, v) = (\epsilon_1, \alpha(u), z_k(\epsilon_1, \alpha(u)))}\]

\[
= \frac{\epsilon_2\sqrt{P_1(t)}(Q_3(t) \pm c_1\sqrt{P_1(t)P_3(t)})}{P_2(t)(Q_4(t) \pm c_1\sqrt{P_1(t)P_3(t)})},
\]

where

\[P_3(t) = (c_1^2 - c_0)P_1(t) - c_0, \quad Q_3(t) = (c_0 + 1)P_2(t) - 2c_1^2P_1(t),
\]

\[Q_4(t) = (c_0 - 1)P_2(t) - 2c_1^2P_1(t),
\]

where $P_1(t)$ and $P_2(t)$ as in (13.43). The sizes are given by

\[(Q_3(t) \pm c_2\sqrt{P_1(t)P_3(t)}), (Q_4(t) \pm c_2\sqrt{P_1(t)P_3(t)}) \ll T^{-\delta_1} \ll T^\epsilon. \tag{13.49}
\]
Proof. Using the fact that \( z_k := z_k(\epsilon_1 \alpha(u)) \) satisfies (13.21) we get
\[
\frac{1}{\phi_{R_1}(\epsilon_1 \alpha(u), z_k(\epsilon_1 \alpha(u)))} = -\frac{\epsilon_2 T_1}{\alpha(u)(R_1 z_k + (R_1 z_k)^{-1})}
\]
and hence
\[
-T_2 \epsilon_1 \alpha'(u) \left( \frac{1}{\phi_{R_1}} \frac{\partial \phi_{R_1}(\alpha, v)}{\partial \alpha} \right)_{(\alpha, v) = (\epsilon_1 \alpha(u), z_k(\epsilon_1 \alpha(u)))} = \frac{\epsilon_2 T_1 u(z_k^2 - R_1^{-2})}{(1 - u^2)(z_k^2 + R_1^{-2})}
\]
Substituting the roots from (13.22) we get
\[
\frac{(z_k^2 - R_1^{-2})}{(z_k^2 + R_1^{-2})} = \frac{(c_0 + 1)(1 - u^2) - 2c_1^2 \pm 2c_1\sqrt{c_1^2 - c_0(1 - u^2)}}{(c_0 - 1)(1 - u^2) - 2c_1^2 \pm 2c_1\sqrt{c_1^2 - c_0(1 - u^2)}} = \frac{(c_0 + 1)(u^2 - 2) - 2c_1^2 u^{-2} \pm 2c_1 u^{-1} \sqrt{c_1^2 u^{-2} - c_0(u^{-2} - 1)}}{(c_0 - 1)(u^2 - 1) - 2c_1^2 u^{-2} \pm 2c_1 u^{-1} \sqrt{c_1^2 u^{-2} - c_0(u^{-2} - 1)}}
\]
In terms of \( t = \sqrt{c_1^2 - c_0(u^{-2} - 1)} \), the numerator becomes
\[
(c_0 + 1)(u^{-2} - 1) - 2c_1^2 u^{-2} \pm 2c_1 u^{-1} \sqrt{c_1^2 u^{-2} - c_0(u^{-2} - 1)} = (c_0 + 1)P_2(t) - 2c_1^2 P_3(t) \pm 2c_1 \sqrt{P_1(t)}(c_1^2 - c_0)P_1(t) + c_0 = Q_3(t) \pm 2c_1 \sqrt{P_1(t)P_3(t)}
\]
and the denominator
\[
(c_0 - 1)(u^{-2} - 1) - 2c_1^2 u^{-2} \pm 2c_1 u^{-1} \sqrt{c_1^2 u^{-2} - c_0(u^{-2} - 1)} = (c_0 - 1)P_2(t) - 2c_1^2 P_3(t) \pm 2c_1 \sqrt{P_1(t)}(c_1^2 - c_0)P_1(t) + c_0 = Q_4(t) \pm 2c_1 \sqrt{P_1(t)P_3(t)}.
\]
The first part of claim follows after substituting these expressions in (13.50). For the last part, following the calculation in (13.47), we obtain
\[
(Q_3(t) \pm 2c_1 \sqrt{P_1(t)P_3(t)}), (Q_4(t) \pm 2c_1 \sqrt{P_1(t)P_3(t)}) \ll T^{-\delta_1} (1 + R_1)^2 \ll T^\epsilon.
\]
Finally the third term in (13.40) in terms of \( t = \sqrt{c_1^2 - c_0(u^{-2} - 1)} \) becomes
\[
\frac{3T_2}{u} = 3T_2 \sqrt{P_1(t)}
\]
where \( P_1(t) \) as in (13.43). Combining (13.40), (13.42), (13.48) and (13.51) we obtain
\[
\phi_{k,\epsilon}^r(u) = -\frac{\epsilon_2 T_1 (P_1(t))^{3/2} Q_1(t)}{P_2(t) Q_2(t)} + \frac{\epsilon_2 T_1 \sqrt{P_1(t)} (Q_3(t) \pm 2c_1 \sqrt{P_1(t)P_3(t)})}{P_2(t) (Q_4(t) \pm 2c_1 \sqrt{P_1(t)P_3(t)})} + 3T_2 \sqrt{P_1(t)}.
\]
We now rationalise the expression in (13.52) so that numerator becomes a polynomial in $t$. Let $P_i, Q_i$ denote the shorthand for $P_i(t), Q_i(t)$. Simplifying we get

$$
\phi'_{k,i}(u) = \frac{T_1 P_1^{1/2} \left( 2c_1 P_1^{1/2} P_3^{1/2} F_\pm + G \right)}{P_2 Q_2 (Q_4 \pm 2c_1 P_1^{1/2} P_3^{1/2})},
$$

(13.53)

where $F_\pm := \mp c_1 P_1 Q_1 \mp c_2 Q_2 \mp c P_2 Q_2$ and $G := -c_2 P_1 Q_1 Q_4 + c_2 Q_2 Q_3 + (3T_2/T_1)P_2 Q_2 Q_4$. Multiplying and dividing $2c_1 P_1^{1/2} P_3^{1/2} F_\pm - G$, we get

$$
\phi'_{k,i}(u) = \frac{T_1 P_1^{1/2} \left( 4c_1^2 P_1 P_3 F_\pm^2 - G^2 \right)}{P_2 Q_2 (Q_4 \pm 2c_1 P_1^{1/2} P_3^{1/2})(2c_1 P_1^{1/2} P_3^{1/2} F_\pm - G)}.
$$

(13.54)

Note that $4c_1^2 P_1 P_3 F_\pm^2$ is a degree 12 polynomial in $t$ with leading coefficient

$$
\frac{4c_1^2(c_1^2 - c_0)(c_0^2 + 1) - (3T_2/T_1)(c_0 - 1))^2}{c_0^6} = \left( \frac{4T_2}{c_0^3(T_2 + c_0 T_1)} \right)^2 4c_1^2(c_1^2 - c_0),
$$

and $G^2$ is also degree 12 polynomial with leading coefficient

$$
\frac{(2c_1^2 - (c_0 - 1))^2(c_0^2 + 1) - (3T_2/T_1)(c_0 - 1))^2}{c_0^6} = \left( \frac{4T_2}{c_0^3(T_2 + c_0 T_1)} \right)^2 (2c_1^2 - (c_0 - 1))^2.
$$

Hence, the numerator $c_1^2 P_1 P_3 F_\pm^2 - G^2$ in (13.54) is a degree 12 polynomial in $t$ with leading coefficient

$$
\left( \frac{4T_2}{c_0^3(T_2 + c_0 T_1)} \right)^2 (4c_1^2(c_1^2 - c_0) - (2c_1^2 - (c_0 - 1))^2) \asymp (4c_1^2/(c_0 - 1)^2 - 1) := A
$$

(13.55)

Also, from the bounds in (13.44), (13.49), we get

$$
2c_1 P_1^{1/2} P_3^{1/2} F_\pm \pm G \ll T^{6\delta_2 + \epsilon}.
$$

(13.56)

(13.56) combined with the bounds from (13.44) and (13.49) shows that the denominator in (13.54) is bounded by

$$
T^{10\delta_2 - 6\delta_1 + \epsilon}.
$$

(13.57)

Summarising the above, we have

$$
\phi'_{k,i}(u) = T_1 \left( A T \delta_1 - 9\delta_2 - \epsilon \right) \frac{P(t)}{f(t)},
$$

where $P(t)$ is a degree 12 monic polynomial in $t$ and $f(t) \ll 1$ is a smooth bounded function. Hence, after the change of variable $\sqrt{c_1^2 - c_0(u - 2)} \mapsto t$, the above discussion boils down to the equality

$$
\int_0^1 H_{k,i}(u) e \left( \frac{T_1 \phi_{k,i}(u)}{2\pi} \right) du = \int_{\mathbb{R}} F(t) e(T_1 \psi(t)) dt,
$$

where

$$
F(t) = -\frac{P_1(t) H_{k,i}(P_1(t))^{-1/2}}{2(P_1(t))^{3/2}}, \quad \psi(t) = \frac{\phi_{k,i}(P_1(t))^{-1/2}}{2\pi}.
$$
such that
\[ \psi'(t) \lesssim K_0 \cdot \frac{t P(t)}{T^{-3\delta_2}(P_1(t))^{3/2} f(t)}, \]  
where
\[ K_0 := AT^{\delta_1-12\delta_2} \]  
Here \( P_1(t) \) is as in \( \text{(13.44)} \) with size \( T^{2\delta_2} \), so that the denominator in \( \text{(13.58)} \) is a smooth bounded function, and the numerator, \( t P(t) \), is a degree 13 monic polynomial. This completes the proof of first part of the lemma. For the second part, note that for \( \alpha \in \mathbb{R} \) and \( P_1(t) = -t^2/\bar{c}_0 + (1 + c_2/\bar{c}_0) \approx T^{2\delta_2}, t \approx T^{-\delta_1/2-\delta_1/2+\delta_2}, \) using the Faà di Bruno’s formula \( \text{3.6} \) we obtain
\[ \frac{d^j}{dt^j} P_1(t)^\alpha \ll_j \sum_{j_1+2j_2=j} (T^{2\delta_2})^\alpha \cdot T^{j_1(-\delta_4/2-\delta_1/2+\delta_2)} \]
\[ = \sum_{j_1+2j_2=j} T^{2\delta_2\alpha} (T^{2\delta_2})^{j_1} \cdot T^{j_2(-\delta_4/2-\delta_1/2+\delta_2)} \ll T^{2\delta_2\alpha} T^{-\delta_2 j} \sum_{j_2 \leq j/2} T^{j_2(\delta_4+\delta_1)} (13.59) \]
\[ \ll T^{2\delta_2\alpha} (T^{-\delta_2} (1 + T^{\delta_1/2+\delta_1/2}))^j, \]
for \( j \geq 0 \). Using the above with \( \alpha = -1/2 \), and using Lemma \( \text{13.2} \) and the Faà di Bruno’s formula \( \text{3.6} \) we obtain
\[ \frac{d^j}{dt^j} H_{k,l}((P_1(t))^{1/2}) \ll_j T^{\delta_2} (T^{-\delta_2} (1 + T^{\delta_1/2+\delta_4/2}))^j \sum_{j_1+2j_2=j} \sum_{j_1+2j_2=j} ((B_1 + B_2) T^{\delta_1+\delta_2}) \]
\[ \ll T^{\delta_2} (B_1 + B_2) T^{\delta_1-\delta_2} (1 + T^{\delta_1/2+\delta_4/2})^j. \]
Using the last inequality and \( \text{(13.59)} \) with \( \alpha = -3/2 \), we deduce
\[ \frac{d^j}{dt^j} H_{k,l}((P_1(t))^{1/2}) \ll_j T^{-\delta_2} ((B_1 + B_2) T^{\delta_1-\delta_2} (1 + T^{\delta_1/2+\delta_4/2}))^j. \]
Since \( P_1(t) = -2t/\bar{c}_0 \approx T^{-\delta_4/2-\delta_1/2+\delta_2} \), we finally obtain
\[ F^{(j)}(t) \ll_j |t|^{d^j} \frac{H_{k,l}((P_1(t))^{1/2})}{P_1(t)^{3/2}} + \left| \frac{d^{j-1}}{dt^{j-1}} H_{k,l}((P_1(t))^{1/2}) \right| \]
\[ \ll_j T^{-\delta_2} (1 + T^{-\delta_1/2-\delta_4/2}) ((B_1 + B_2) T^{\delta_1-\delta_2} (T^{-\delta_1/2-\delta_4/2} + 1))^j \]
\[ = T^{-\delta_2} (1 + T^{-\delta_1/2-\delta_4/2}) B_3^j. \]
Similarly, using Lemma \( \text{13.3} \) one can deduce
\[ \psi^{(j)}(t) \ll_j ((B_1 + B_2) T^{\delta_1} (T^{-\delta_2-\delta_1/2-\delta_4/2} + T^{-\delta_2}))^j, \]
\[ = B_3^j. \]
For the last part, note that for \( u \in \text{supp}(G(u)) \), where \( G \) as in \( \text{(13.3)} \),
\[ t = \sqrt{c_2^2 - \bar{c}_0 (u^2 - 1)} \approx T^{-\delta_1/2-\delta_4/2+\delta_2}. \]
Similarly for \( H_{k,l} \) as in (13.26) we have \( H_{k,l}(u) \ll T^{\delta_2} \). Lastly from (13.34) we have \( P_1(t) \asymp T^{2\delta_2} \). Hence we get

\[
\begin{align*}
F(t) &= -\frac{P_1'(t) H_{k,l}((P_1(t))^{-1/2})}{2(P_1(t))^{3/2}} \ll T^{-\delta_2-\delta_1/2-\delta_4/2}.
\end{align*}
\]

This completes the proof the the lemma. \( \square \)

Applying Lemma 3.5 to the right hand side of (13.39) we obtain

\[
\int_0^1 H_{k,l}(u) e \left( \frac{\phi_{k,l}(u)}{2\pi} \right) du \ll T^{-\delta_2-\delta_1/2-\delta_4/2} \left( (TK_0)^{1/14} + \left( \frac{TK_0}{B_3} \right)^{-1/13} + \left( \frac{T^{1/2}K_0}{B_3} \right)^{-1/13} \right)
\]

\[\ll T^{-\delta_2-\delta_1/2-\delta_4/2} T^{1/26} B_3^{1/13} (K_0^{-1/13} + K_0^{-1/14}). \tag{13.60}\]

Note that we also have the trivial bound

\[
\int_0^1 H_{k,l}(u) e \left( \frac{\phi_{k,l}(u)}{2\pi} \right) du \ll \min\{T^{-\delta_1}, T^{-\delta_2+\delta_3}, T^{-\delta_4}\} \tag{13.61}\]

coming from the size of support of \( H_{k,l}(u) \). We now substitute the above obtained bounds in (13.24) according to the following subcases.

**Subcase 1.**

\[ A > T^{-\gamma}. \tag{13.62} \]

Let us recall that

\[
B_3 = (B_1 + B_2) T^{\delta_1-\delta_2} (1 + T^{-\delta_1/2-\delta_4/2}),
\]

where

\[
B_1 = (1 + T^{\delta_1}) (1 + R_1^{-1}T^{-\delta_3/2}), \quad B_2 = (1 + T^{\delta_4}) (1 + R_2^{-1}T^{-\delta_4/2})
\]

and

\[
K_0 = AT^{\delta_1-12\delta_2}.
\]

Let \( \delta_{\max} = \max\{\delta_1, \delta_3, \delta_4\} \) and \( \delta_{\min} = \min\{\delta_1, \delta_3, \delta_4\} \). Suppose first that \( \delta_{\max} \leq a \), for some \( a > 0 \) to be chosen in a moment. Note that from Lemma 13.4 we have \( T^{-\delta_1} \ll T^{\delta_1} \) and \( T^{-\delta_4} \ll T^{\delta_1} \). Consequently,

\[
B_3 \ll T^{-\delta_2+3\delta_{\max}+\beta}, \quad K_0 = AT^{\delta_1-12\delta_2}. \tag{13.63}\]

Hence from (13.60) we obtain

\[
\int_0^1 H_{k,l}(u) e \left( \frac{\phi_{k,l}(u)}{2\pi} \right) du \ll T^{-\delta_2-\delta_1/2-\delta_4/2} T^{1/26-\delta_1/14+12\delta_2/13} T^{-\delta_2/13+3\delta_{\max}/13+\beta/13} (A^{-1/13} + A^{-1/14}).
\]
Substituting, we see that the contribution of this case towards (13.24) is bounded by

\[
\frac{T^{-\delta_2/2}}{T(\theta_1,\theta_2)^{1/4}} \cdot T^{\delta_3/4+\delta_4/4+\delta_1/2} \cdot T^{-\delta_3/2-\delta_1/2}T^{-\lceil \frac{1-2(\beta+\gamma)}{26} \rceil}T^{3\delta_{\max}/13} \leq \frac{T^{-\delta_2/2}}{T(\theta_1,\theta_2)^{1/4}} \cdot T^{\delta_{\max}/4}T^{-\lceil \frac{1-2(\beta+\gamma)}{26+3\delta_{\max}} \rceil} \leq \frac{T^{-\delta_2/2}}{T(\theta_1,\theta_2)^{1/4}} \cdot T^{-\lceil \frac{1-2(\beta+\gamma)}{26+a/2} \rceil} \leq (\theta_2/\theta_1)^{1/6}T^{-1}(\theta_1^{1/6}\theta_2^{1/3})^{-1}T^{-\lceil \frac{1-2(\beta+\gamma)}{26+a/2} \rceil} \leq (\theta_2/\theta_1)^{1/6}T^{-a/8+\epsilon},
\]

where we have used the fact that \( T^{\delta_2} \approx (\theta_2/\theta_1)^{1/6} \) and the assumption (6.6). Now suppose \( \delta_{\max} > a > 0 \) and \( \delta_{\max} - \delta_{\min} \geq \delta_{\max}/2 \). Then, using the trivial bound (13.61), we see that the contribution of this case towards (13.24) is bounded by

\[
\frac{T^{-\delta_2/2}}{T(\theta_1,\theta_2)^{1/4}} \cdot T^{\delta_3/4+\delta_4/4+\delta_1/2} \cdot T^{-\delta_3/2-\delta_1/2}T^{-\lceil \frac{1-2(\beta+\gamma)}{26} \rceil}T^{3\delta_{\max}/13} \leq \frac{T^{-\delta_2/2}}{T(\theta_1,\theta_2)^{1/4}} \cdot T^{\delta_{\max}/4}T^{-\lceil \frac{1-2(\beta+\gamma)}{26+3\delta_{\max}} \rceil} \leq \frac{T^{-\delta_2/2}}{T(\theta_1,\theta_2)^{1/4}} \cdot T^{-\lceil \frac{1-2(\beta+\gamma)}{26+a/2} \rceil} \leq (\theta_2/\theta_1)^{1/6}T^{-2-a/8+\epsilon}.
\]

Next suppose \( \delta_{\max} > a \) and \( \delta_{\max} - \delta_{\min} < \delta_{\max}/2 \). This implies \( \delta_{\max}/2 < \delta_i \leq \delta_{\max} \) for each \( i = 1, 3, 4 \). By the definitions of \( T^{\delta_1}, T^{\delta_4} \), this forces \( c_1^2T^{\delta_1} \leq 1, c_2^2T^{\delta_1} \leq 1 \) and consequently,

\[
c_1^2 \ll T^{-\delta_1} \ll T^{-a/2} \quad \text{and} \quad c_2^2 \ll T^{-\delta_1} \ll T^{-a/2}.
\]

In this case we use the trivial bound (13.61) to obtain that the contribution of this case towards (13.24) is bounded by

\[
\frac{T^{-\delta_2/2}}{T(\theta_1,\theta_2)^{1/4}} \cdot T^{\delta_3/4+\delta_4/4+\delta_1/2} \cdot T^{-\delta_3/2-\delta_1/2}T^{-\lceil \frac{1-2(\beta+\gamma)}{26} \rceil}T^{3\delta_{\max}/13} \leq \frac{T^{-\delta_2/2}}{T(\theta_1,\theta_2)^{1/4}} \cdot T^{\delta_{\max}/4}T^{-\lceil \frac{1-2(\beta+\gamma)}{26+3\delta_{\max}} \rceil} \leq \frac{T^{-\delta_2/2}}{T(\theta_1,\theta_2)^{1/4}} \cdot T^{-\lceil \frac{1-2(\beta+\gamma)}{26+a/2} \rceil} \leq (\theta_2/\theta_1)^{1/6}T^{-2+\epsilon}.
\]

Summarising the above, the contribution of this sub-case towards (13.24) is

\[
\ll (\theta_2/\theta_1)^{1/6}T^{-2+\epsilon}
\]

if \( c_1 \ll T^{-a/4}, c_2 \ll T^{-a/4} \) and is

\[
\ll (\theta_2/\theta_1)^{1/6}T^{-2+\epsilon} \left( T^{-\lceil \frac{1-2(\beta+\gamma)}{26+a/2} \rceil} + T^{-a/8} \right)
\]

otherwise. Equating the powers of \( T \) in the last inequality, we choose \( a = 4(1 - 2(\beta + \gamma))/65 \). Substituting, we finally obtain the contribution of this sub-case towards (13.24) is

\[
\ll (\theta_2/\theta_1)^{1/6}T^{-2+\epsilon}
\]
if \( \min\{c_1, c_2\} \ll T^{-(1-2(\beta+\gamma))/6} \), and is
\[
\ll (\theta_2/\theta_1)^{1/6} T^{-2-(1-2(\beta+\gamma))/130+\epsilon}
\]  
otherwise.

**Subcase 2.**

\[ A \leq T^{-\gamma}. \]  
(13.67)

In this case we use the trivial estimate (13.61) to see that its contribution towards (13.24) is at most
\[
\ll T^{-1}(\theta_1/\theta_2)^{-1/4} T^{\delta_1/2+\delta_3/4+\delta_4/4-\delta_2/2} \langle \min\{T^{\delta_1}, T^{-\delta_1+\delta_2}, T^{-\delta_3}\} \rangle
\]
\[
\ll T^{-1+\delta_2}(\theta_1/\theta_2)^{-1/4} T^{-\delta_2/2} \ll T^{1+\epsilon}(\theta_1/\theta_2)^{-1/4} (\theta_2/\theta_1)^{1/12} = T^{-1+\epsilon}(\theta_2/\theta_1)^{1/6} (\theta_1^{1/6} \theta_2^{1/3})^{-1}
\]
\[
\ll (\theta_2/\theta_1)^{1/6} T^{-2+\epsilon}.
\]  
(13.68)

**Case 2 :** \( T^{-\beta} \min\{\alpha^{-1}, T^{-1}\} \ll R_1, T^{-\beta} \min\{\beta^{-1}, T^{-1}\} \ll R_2 \) and either
\[
T^{-\delta_3/2} \ll \left( \frac{T}{\alpha(u)} + 1 \right) T^{-1/3+2\epsilon}
\]  
(13.69)
or
\[
T^{-\delta_4/2} \ll \left( \frac{T}{\beta(u)} + 1 \right) T^{-1/3+2\epsilon}.
\]  
(13.70)

Note that from the bounds in (13.105) and (13.108), we have
\[
\tilde{I}_{R_1}^{\ell_1, \ell_2}(U, \alpha(u)) \ll \frac{T^{\delta_1/4}}{\alpha(u)^{1/2}} + T^{-1/3} \lesssim \frac{T^{\delta_1/4+\delta_3/4}}{\theta_1^{1/4}} + T^{-1/3} \ll T^{-1/2+\delta_1/4+\delta_3/4} + T^{-1/3}.
\]  
(13.71)

Similarly, we have
\[
\tilde{I}_{R_2}^{\ell_1, \ell_2}(V, \beta(u)) \ll T^{-1/2+\delta_1/4+\delta_3/4} + T^{-1/3}.
\]  
(13.72)

Let \( 0 < a < 1/3 \) be a quantity to be chosen in a moment.

**Subcase 1.** \( \delta_1 < 1/3 - a \).

WLOG, suppose (13.69) holds. Then, using \( \delta_1 < 1/3 - a \), we obtain
\[
T^{-\delta_3} \ll T^{-2/3+4\epsilon} \left( \frac{T}{\alpha(u)} + 1 \right)^2 \ll T^{-2/3+4\epsilon+\delta_1} \left( \frac{T}{\theta_1^{1/2}} + 1 \right)^2 \ll T^{-1/3-a+4\epsilon}.
\]  
(13.73)

Similar bounds holds in case (13.70) holds. Hence in any case, if \( \delta_{\max} = \max\{\delta_1, \delta_3, \delta_4\} \), then we have
\[
T^{-\delta_{\max}} \ll T^{-1/3-a+4\epsilon}.
\]  
(13.74)
Using the above, the bounds (13.71), (13.72) for the integrals and then trivially execute the \( u \)-integral in (13.11) we get
\[
\frac{1}{T} \int_{0}^{1} \frac{G(u)}{u} I_{R_1}^{\epsilon_1,\epsilon_2}(U, \alpha(u)) I_{R_2}^{\epsilon_1',\epsilon_2'}(V, \beta(u)) u^{3T_1} \, du \\
\ll T^{-1} (T^{-1/2+\delta_1/4+\delta_{\text{max}}/4} + T^{-1/3}) \min \{ T^{-\delta_1}, T^{-\delta_1+\delta_2}, T^{-\delta_1} \} \\
\ll T^{-1+\delta_2} (T^{-1+\delta_1/2+\delta_{\text{max}}/2} + T^{-2/3}) T^{-\delta_{\text{max}}}
\]  

(13.75)

Using \( \delta_1 < 1/3 - a \), \( \delta_2 \leq \epsilon \) and \( T^{-\delta_{\text{max}}} \ll T^{-1/3-a+4\epsilon} \) we obtain
\[
\frac{1}{T} \int_{0}^{1} \frac{G(u)}{u} I_{R_1}^{\epsilon_1,\epsilon_2}(U, \alpha(u)) I_{R_2}^{\epsilon_1',\epsilon_2'}(V, \beta(u)) u^{3T_1} \, du \ll T^{-2-a+5\epsilon} \ll (\theta_2/\theta_1)^{1/6} T^{-2-a+6\epsilon}.
\]

Subcase 2. \( 1/3 - a \leq \delta_1 \leq 1/3 \) and \( \delta_2 = 0 \).

WLOG, suppose (13.69) holds. If \( c_1 T^{\delta_1} \gg 1 \), then from the definition (13.37) it follows \( T^{-\delta_1} \gg c_1 T^{\delta_1} \). Substituting this in (13.69) we obtain
\[
c_1 T^{\delta_1/2} \ll \left( \frac{T}{\alpha(u)} + 1 \right) T^{-1/3+2\epsilon} \ll T^{\delta_1/2-1/3+2\epsilon},
\]

that is
\[
c_1 \ll T^{-1/3+2\epsilon}.
\]

(13.76)

On the other hand if \( c_1 T^{\delta_1} \ll 1 \), then by the assumption \( \delta \geq 1/3 - a \) we get
\[
c_1 \ll T^{-1/6+a/2}.
\]

(13.77)

So, in any case we have
\[
c_1 \ll T^{-1/6+a/2}.
\]

(13.78)

in this case. For the integral bound in this case, note that the arguments leading up to (13.74) gives
\[
T^{-\delta_{\text{max}}} \ll T^{-1/3+4\epsilon}
\]
in this case. Consequently, from the bound (13.75) we get
\[
\frac{1}{T} \int_{0}^{1} \frac{G(u)}{u} I_{R_1}^{\epsilon_1,\epsilon_2}(U, \alpha(u)) I_{R_2}^{\epsilon_1',\epsilon_2'}(V, \beta(u)) u^{3T_1} \, du \ll T^{-2+4\epsilon} \ll (\theta_2/\theta_1)^{1/6} T^{-2+4\epsilon}
\]
since \( (\theta_2/\theta_1)^{1/6} \ll T^{-\delta_2} = 1 \) in this sub-case. For the case where (13.70) holds, it is clear that the only change is that the bound (13.78) holds for \( c_2 \) instead of \( c_1 \). Summarising, we obtain that for \( 1/3 - a \leq \delta_1 \), either
\[
c_1 \ll T^{-1/6+a/2},
\]

(13.79)
or
\[
c_2 \ll T^{-1/6+a/2}
\]

(13.80)

holds, and furthermore
\[
\frac{1}{T} \int_{0}^{1} \frac{G(u)}{u} I_{R_1}^{\epsilon_1,\epsilon_2}(U, \alpha(u)) I_{R_2}^{\epsilon_1',\epsilon_2'}(V, \beta(u)) u^{3T_1} \, du \ll (\theta_2/\theta_1)^{1/6} T^{-2+4\epsilon}
\]

(13.81)
We summarise the last two sub-cases above by choosing \(a\) such that \(-1/6 + a/2 = -a\), i.e., \(a = 1/9\) to get

\[
\frac{1}{T} \int_{0}^{1} \frac{G(u)}{u} I_{R_1}^{\epsilon_1, \epsilon_2}(U, \alpha(u)) \tilde{I}_{R_2}^{\epsilon_1', \epsilon_2'}(V, \beta(u)) u^{3\delta_2} du \ll (\theta_2/\theta_1)^{1/6} T^{-2+4\epsilon},
\]

(13.82)

if \(\min\{c_1, c_2\} \ll T^{-1/9}\), and

\[
\frac{1}{T} \int_{0}^{1} \frac{G(u)}{u} I_{R_1}^{\epsilon_1, \epsilon_2}(U, \alpha(u)) \tilde{I}_{R_2}^{\epsilon_1', \epsilon_2'}(V, \beta(u)) u^{3\delta_2} du \ll (\theta_2/\theta_1)^{1/6} T^{-2-1/9+6\epsilon}
\]

(13.83)

otherwise.

**Subcase 3.** \(\delta_1 > 1/3\) and \(\delta_2 = 0\).

Following the above calculations, it is clear that the conclusion [13.79] and [13.80] holds for this case with \(a = 0\), i.e., either

\[
c_1 \ll T^{-1/6} \quad \text{or} \quad c_2 \ll T^{-1/6}
\]

(13.84)

holds. For the integral bound, since \(\delta_1 > 1/3\), we have \(T^{-\delta_{\max}} \ll T^{-1/3}\) and consequently the bound (13.75) gives

\[
\frac{1}{T} \int_{0}^{1} \frac{G(u)}{u} I_{R_1}^{\epsilon_1, \epsilon_2}(U, \alpha(u)) \tilde{I}_{R_2}^{\epsilon_1', \epsilon_2'}(V, \beta(u)) u^{3\delta_2} du \ll T^{-2+\epsilon} \asymp (\theta_2/\theta_1)^{1/6} T^{-2+\epsilon}
\]

since \((\theta_2/\theta_1)^{1/6} \asymp T^{-\delta_2} = 1\) in this sub-case. Hence this case gets absorbed into the previous one.

**Remark 1.** Here we outline the necessary modification required when one of \(U\) or \(V\) is zero, i.e., one of \(c_1\) or \(c_2\) is zero. It enough to consider when \(c_1 = 0\). Note that by definition (13.8), \(c_1 = 0\) implies \(\delta_3 = 0\). Suppose first \(\delta_1 > 2/3 - 100\epsilon\). Now, using the bounds (13.71), (13.72) with \(\delta_3 = 0\) and then trivially execute the \(u\)-integral we obtain

\[
\frac{1}{T} \int_{0}^{1} \frac{G(u)}{u} I_{R_1}^{\epsilon_1, \epsilon_2}(U, \alpha(u)) \tilde{I}_{R_2}^{\epsilon_1', \epsilon_2'}(V, \beta(u)) u^{3\delta_2} du
\]

\[
\ll T^{-1} \left( T^{-1+\delta_1/2+\delta_2/4} + T^{-2/3} \right) \min\{T^{-\delta_1}, T^{-\delta_1}\}
\]

\[
\ll T^{-1} \left( T^{-1-\delta_1/4} + T^{-2/3-\delta_1} \right) \ll T^{-2-1/6+100\epsilon} \asymp (\theta_2/\theta_1)^{1/6} T^{-2-1/6+100\epsilon},
\]

which gets absorbed into previous upper bounds (13.83 for example). If \(\delta_1 \leq 2/3 - 100\epsilon\), then from (13.73) it is clear that (13.69) cannot hold since \(\delta_3 = 0\). Hence only (13.70) can hold in this case and we will be left with only the condition (13.80) and the second condition in (13.84) in the sub-cases above.

**Case 3 :** \(T^{\beta} \min\{\alpha^{-1}, T^{-1}\} \ll R_1\), \(T^{\beta} \min\{\beta^{-1}, T^{-1}\} \ll R_2\),

\[
T^{-\delta_1/2} \gg T^{-1/3+2\epsilon} \left( \frac{T}{\alpha(u)} + 1 \right),
\]

(13.85)

\[
T^{-\delta_2/2} \gg T^{-1/3+2\epsilon} \left( \frac{T}{\beta(u)} + 1 \right),
\]

(13.86)
and \( \min\{R_1, R_2\} \ll T^{-\beta} \).

Suppose that \( T^\beta \min\{\alpha^{-1}, T^{-1}\} \ll R_1 \ll T^{-\beta} \). Then from part 3 of the lemma we have the restriction
\[
\left| \left( \frac{\pi U}{\alpha(u)} \right) \left( \frac{T_1}{T_2} \right) - 1 \right| \ll T^\epsilon (R_1 + (R_1 \alpha(u))^{-1}) \ll T^{-\beta+\epsilon} 
\] (13.87)
and the bound
\[
I^{\epsilon_1, \epsilon_2}_{R_1} (U, \alpha(u)) \ll \alpha(u)^{-1/2} \ll \frac{T^{\delta_1/4}}{\theta_1^{1/4}}. 
\] (13.88)

Also, since \( T^{-\delta_4} \) satisfies (13.86), arguing as in (13.23), we have the bound
\[
I^{\epsilon_1, \epsilon_2}_{R_2} (V, \beta(u)) \ll \frac{T^{\delta_3/4}}{\beta(u)^{1/2}} \ll \frac{T^{-\delta_2/2+\delta_4/4}}{\theta_2^{1/4}}. 
\] (13.89)

Note that we still have the condition \( T^\delta \ll T^\epsilon \) from (13.20). Using the bounds (13.88), (13.89) and trivially executing the \( u \)-integral with the restriction (13.87) we obtain
\[
\frac{1}{T} \int_0^1 \frac{G(u)}{u} I^{\epsilon_1, \epsilon_2}_{R_1} (U, \alpha(u)) I^{\epsilon_1, \epsilon_2}_{R_2} (V, \beta(u)) u^{3T_2} du \ll \frac{R_1 R_2 T^{-\delta_1}}{T^{\delta_1/2} T^{\delta_1/2} T^{\delta_2/4} T^{\delta_4/4} \min\{T^{-\delta_1}, T^{-\delta_2+\delta_4}, T^{-\delta_4}, T^{-\beta+\epsilon+\delta_4}\}} \ll \frac{R_1 R_2 T^{-\delta_1}}{(\theta_2/\theta_1)^{1/6} T^{-2^{\beta/4+3\epsilon}}.} 
\] (13.90)

The same conclusion holds if we instead assume \( T^\beta \min\{\beta^{-1}, T^{-1}\} \ll R_1 \ll T^{-\beta} \).

**Case 4**: \( R_1 \ll T^\beta \min\{\alpha^{-1}, T^{-1}\} \) or \( R_2 \ll T^\beta \min\{\beta^{-1}, T^{-1}\} \).

Suppose first that both \( R_1 \ll T^\beta \min\{\alpha^{-1}, T^{-1}\} \) and \( R_2 \ll T^\beta \min\{\beta^{-1}, T^{-1}\} \) holds. Then a trivial estimation gives
\[
\frac{1}{T} \int_0^1 \frac{G(u)}{u} I^{\epsilon_1, \epsilon_2}_{R_1} (U, \alpha(u)) I^{\epsilon_1, \epsilon_2}_{R_2} (V, \beta(u)) u^{3T_2} du \ll \frac{R_1 R_2 T^{-\delta_1}}{T^{\delta_1-2^{\beta-1}} T^{-\delta_1} T^{-\delta_2+\delta_1-\delta_2} T^{-3+3\beta}} \ll \frac{R_1 R_2 T^{-\delta_1}}{(\theta_2/\theta_1)^{1/6} T^{-3+2\beta}}. 
\] (13.91)

Now suppose \( R_1 \ll T^\beta \min\{\alpha^{-1}, T^{-1}\} \) and \( R_2 \gg T^\beta \min\{\beta^{-1}, T^{-1}\} \). Then from the bounds in (13.105) and (13.108), we have
\[
I^{\epsilon_1, \epsilon_2}_{R_2} (V, \beta(u)) \ll \frac{T^{\delta_4/4}}{\beta(u)^{1/2}} + T^{-1/3} \ll T^{-1/2+\delta_1/4+\delta_4/4} + T^{-1/3}. 
\]
Using this and the trivial bound \( R_1 \) for \( I_{R_1}^{I_1,\epsilon_2}(U, \alpha(u)) \) we obtain

\[
\frac{1}{T} \int_0^1 \frac{G(u)}{u} I_{R_1}^{I_1,\epsilon_2}(U, \alpha(u)) \tilde{I}_{R_2}^{I_1,\epsilon_2}(V, \beta(u)) u^{3T_2} du \\
\ll \frac{R_1}{T}(T^{-1/2+\delta_1/4+\delta_4/4} + T^{-1/3}) \min\{T^{-\delta_1}, T^{-\delta_4}\}
\]

\[
\ll \frac{T^{-1+\beta}}{\alpha(u)} (T^{-1/2+\delta_1/4+\delta_4/4} + T^{-1/3}) \min\{T^{-\delta_1}, T^{-\delta_4}\}
\]

\[
\ll T^{-\delta_2+\epsilon} T^{-2+\beta} (T^{-1/2+3\delta_1/4+\delta_4/4} + T^{-1/3+\delta_1/2}) \min\{T^{-\delta_1}, T^{-\delta_4}\}
\]

\[
\ll T^{-\delta_2+\epsilon} T^{-2-1/3+\beta} \ll (\theta_2/\theta_1)^{1/6} T^{-2-1/3+\beta+\epsilon},
\]

where we have used the inequality \( T/\alpha(u) \ll T^{\delta_1/2-\delta_2+\epsilon} \) from (13.19). Finally suppose \( R_1 \gg T^\beta \min\{\alpha^{-1}, T^{-1}\} \) and \( R_2 \ll T^\beta \min\{\beta^{-1}, T^{-1}\} \), then proceeding similarly as above we get

\[
\frac{1}{T} \int_0^1 \frac{G(u)}{u} I_{R_1}^{I_1,\epsilon_2}(U, \alpha(u)) \tilde{I}_{R_2}^{I_1,\epsilon_2}(V, \beta(u)) u^{3T_2} du \\
\ll \frac{T^{-1+\beta}}{\beta(u)} (T^{-1/2+\delta_1/4+\delta_4/4} + T^{-1/3}) \min\{T^{-\delta_1}, T^{-\delta_3+\delta_2}\}
\]

\[
\ll T^{-2+\beta+\epsilon} T^{\delta_1/2} (T^{-1/2+\delta_1/4+\delta_4/4} + T^{-1/3}) \min\{T^{-\delta_1}, T^{-\delta_3+\delta_2}\}
\]

\[
\ll T^{-2+\beta+\epsilon} (T^{-1/2+3\delta_1/4+\delta_4/4} + T^{-1/3+\delta_1/2}) \min\{T^{-\delta_1}, T^{-\delta_3+\epsilon}\}
\]

\[
\ll T^{-2-1/3+\beta+\epsilon} \ll (\theta_2/\theta_1)^{1/6} T^{-2-1/3+\beta+3\epsilon},
\]

where we have used the inequalities \( \beta(u) \ll T^{-\delta_1/2} (T^{\delta_2} T^{1/2}) \gg T^{-\delta_1/2} T^{1-\epsilon} \) and \( \delta_2 \ll T^\epsilon \)
following from (13.20).

We compile the all the above cases to reach at our conclusion (5.8). Let

\[
\alpha = 1 - 2(\beta + \gamma), \quad A = \left| \frac{\pi |U|}{\theta_1^{1/2}} - 1 \right|.
\]

Combining the estimates from (13.65), (13.66), (13.68), (13.82), (13.83), (13.84), (13.90), (13.91), (13.92), (13.93) and substituting in (13.9) we obtain

\[
T^{-100k} (\theta_1/\theta_2)^{1/6} K_4(\theta_1, \theta_2, U, V) \ll T^3 \left( T^{-2-\alpha/130} + T^{-2\delta_{\min\{c_1,c_2\}}/T^{-\alpha/65}} + T^{-2\delta_{\lambda-T^{-\gamma}}} + T^{-2\delta_{\min\{c_1,c_2\}}/T^{-1/9}} + T^{-2-\beta/4} + T^{-2-1/3+\beta} + T^{-3+2\beta} \right)
\]

Comparing the exponents of the first, third and the fifth term, we choose \( \beta, \gamma \) such that

\[
\frac{\alpha}{130} = \gamma = \frac{\beta}{4}.
\]

Solving we obtain

\[
\gamma = \frac{\beta}{4} = \frac{1}{140}.
\]
Note that with this choice of \( \beta \) and \( \gamma \), the first, third and the fifth term of \((13.94)\) dominates the others. This completes the proof of Theorem 6.3.

**Appendix: An average of Bessel functions with non-linear twist**

For this section we fix a smooth partition of unity

\[
\sum_R F \left( \frac{r}{R} \right) = 1 \quad \text{for } r \in (0, \infty)
\]

consisting of a sequence of numbers \( R \in \mathbb{R}_{>0} \) and a smooth function \( F \) supported on \([1, 2]\) and satisfying \( F^{(j)}(x) \ll j \).

**Lemma 13.8.** Let \( \alpha > 0, U \in \mathbb{R} \) and \( \epsilon > 0 \) be small enough. Let \( U \) be a smooth weight function compactly supported on \( \mathbb{R}_{>0} \) and satisfying \( U^{(j)}(x) \ll j \). Define

\[
I(U, \alpha) := \int_{\mathbb{R}} U(x) e^{iT_2 T_1 (\alpha x) e(U(x))} dx,
\]

where \( K_{iT_1} \) is one of the Bessel function \( \tilde{K}_{iT_1} \) or \( J_{iT_1} \). Then there exists smooth weight functions \( W \) (depending on \( U \)) compactly supported on \( \mathbb{R}_{>0} \), such that

\[
I(U, \alpha) \sim e^{\left( -T_2 \frac{2\pi}{T_1} U \right)} \sum_{\epsilon_1, \epsilon_2 = \pm 1} \sum_R I_R^{(\epsilon_1, \epsilon_2)}(U, \alpha),
\]

where

\[
I_R^{(\epsilon_1, \epsilon_2)}(U, \alpha) = \int_{\mathbb{R}} v^{-1} \omega_R(\epsilon_1 \alpha, v) e \left( \frac{-T_2 \log \phi_R(\epsilon_1 \alpha, v) + \epsilon_2 T_1 \log v}{2\pi} \right) dv
\]

and

\[
\phi_R(\alpha, v) = -\frac{2\pi}{T_2} \left( \frac{\alpha}{2\pi} \left( Rv \pm \frac{1}{Rv} \right) + U \right).
\]

The ‘\( \pm \)’ in \((13.98)\) is + for \( K_{iT_1} = \tilde{K}_{iT_1} \) and is − for \( K_{iT_1} = J_{iT_1} \).

For \( R \gg T^\epsilon \min \{ \alpha^{-1}, T^{-1} \} \), \( I_R^{(\epsilon_1, \epsilon_2)} \) is negligibly small unless

\[
(R + 1)\alpha \asymp T, \quad \text{if} \quad K_{iT_1} = \tilde{K}_{iT_1},
\]

\[
R \alpha \asymp T, \quad \text{if} \quad K_{iT_1} = J_{iT_1}.
\]

Let \( z_1(\alpha), z_2(\alpha) \) be the the two roots of the corresponding phase function derivative

\[
\frac{\partial}{\partial v} \left( -T_2 \log(\phi_R(\alpha, v)) + \epsilon_2 T_1 \log v \right) = 0.
\]

Suppose

\[
z_1(\epsilon_1 \alpha) - z_2(\epsilon_1 \alpha) \asymp R^{-1} T^{-\delta/2},
\]

for some \( \delta \in \mathbb{R} \). We have the following asymptotics for \( I_R \):
(1) If
\[ R \gg T^{3\epsilon} \min\{\alpha^{-1}, T^{-1}\} \]
and
\[ z_1(\epsilon_1 \alpha) - z_2(\epsilon_1 \alpha) \asymp R^{-1} T^{-\delta/2} \gg T^{2\epsilon} \left( \frac{R^3 \alpha}{(R + 1)^2} \right)^{-1/3} \]
then \( I_R^{(\epsilon_1, \epsilon_2)}(U, \alpha) \) is negligibly small unless the roots \( z_1(\epsilon_1 \alpha), z_2(\epsilon_1 \alpha) \) are real, in which case,
\[
I_R^{(\epsilon_1, \epsilon_2)}(U, \alpha) \\ \approx \frac{T^{\delta/4}}{|\alpha|^{1/2}} \sum_{k=1,2} \left( \frac{\omega_R(\epsilon_1 \alpha, z_k(\epsilon_1 \alpha))}{(RT^{\delta/2} |z_1(\epsilon_1 \alpha) - z_2(\epsilon_1 \alpha)|)^{1/2}} \right)^{1/2} (\phi_R(\epsilon_1 \alpha, z_k(\epsilon_1 \alpha)))^{-iT_2} (z_k(\epsilon_1 \alpha))^{i\zeta T_1}.
\]
The weight function satisfy
\[ |\alpha|^j \frac{d^j}{d\alpha^j} \frac{\omega_R(\epsilon_1 \alpha, z_k(\epsilon_1 \alpha))}{(RT^{\delta/2} |z_1(\epsilon_1 \alpha) - z_2(\epsilon_1 \alpha)|)^{1/2}} \ll_{j, R, \epsilon} A^j, \quad j \geq 0, \]
where
\[ A := (1 + R^{-1} T^{\delta/2})(1 + T^{\delta}). \]

(2) If \( R \gg T^{3\epsilon} \min\{\alpha^{-1}, T^{-1}\} \) and
\[ z_1(\epsilon_1 \alpha) - z_2(\epsilon_1 \alpha) \asymp R^{-1} T^{-\delta/2} \ll T^{2\epsilon} \left( \frac{R^3 \alpha}{(R + 1)^2} \right)^{-1/3}, \]
then
\[ I_R^{(\epsilon_1, \epsilon_2)}(U, \alpha) \ll ((R + 1)\alpha)^{-1/3}. \]

(3) If \( T^\epsilon \min\{\alpha^{-1}, T^{-1}\} \ll R \ll T^{-\epsilon} \), then \( I_R^{(\epsilon_1, \epsilon_2)}(U, \alpha) \) is negligibly small unless
\[ \left| \left( \frac{\pi U}{\alpha} \right) \left( \frac{T_1}{T_2} \right) - 1 \right| \ll T^\epsilon (R + (R\alpha))^{-1}, \quad \text{if } K_{iT_1} = \tilde{K}_{iT_1}, \]
\[ \left| \left( \frac{\pi U}{\alpha} \right) - 1 \right| \ll T^\epsilon (R + (R\alpha))^{-1}, \quad \text{if } K_{iT_1} = \tilde{J}_{iT_1}. \]
in which case
\[ I_R^{(\epsilon_1, \epsilon_2)}(U, \alpha) \ll \alpha^{-1/2}. \]

Remark 2. The “\( \sim \)” symbol in (13.105) means equal up to lower order terms. The lower order terms can be calculated explicitly by following the proof and can be handled similarly as the main term. These have smaller contributions and their details are suppressed for the sake of simplicity.

Proof. Let us begin with the proof of (13.96) for \( K_{iT_1} = \tilde{K}_{iT_1} \). The modifications required for \( J_{iT_1}^- \) will be pointed out at appropriate places in the course of the proof below.
Consider the integral representation
\[
K_{iT_1}(\alpha x) = \frac{1}{2} \sum_{\pm} \int_{-\infty}^{\infty} e^{\left( \pm 2\alpha x \sinh u + T_1 u \right) / 2\pi} du
\]
\[
= \frac{1}{2} \sum_{\epsilon_1, \epsilon_2 \in \{\pm 1\}^2} \int_{0}^{\infty} e^{\left( 2\epsilon_1 \alpha x \sinh u + \epsilon_2 T_1 u \right) / 2\pi} du,
\]
for \(\alpha, x > 0\). Changing variable \(e^u \mapsto v\), we obtain
\[
\tilde{K}_{iT_1}(\alpha x) = \frac{1}{2} \sum_{\epsilon_1, \epsilon_2 \in \{\pm 1\}^2} \int_{1}^{\infty} e^{\left( \epsilon_1 \alpha x (v - v^{-1}) + \epsilon_2 T_1 \log v \right) / 2\pi} dv.
\]
Inserting the dyadic partition \(U((v - 1)/R), R > 0\) of the interval \((1, \infty)\) into the last integral we obtain
\[
K_{iT_1}(\alpha x) = \frac{1}{2} \sum_{\epsilon_1, \epsilon_2 \in \{\pm 1\}^2} \sum_{R > 0} \int_{R}^{\infty} U((v - 1)/R) e^{\left( \epsilon_1 \alpha x (v - v^{-1}) + \epsilon_2 T_1 \log v \right) / 2\pi} dv.
\]
Changing variable \(v \mapsto Rv\) we obtain
\[
\tilde{K}_{iT_1}(\alpha x)
\]
\[
= \frac{1}{2} \sum_{\epsilon_1, \epsilon_2 \in \{\pm 1\}^2} \sum_{R > 0} e(\epsilon_2 T_1 \log(R)) \int_{R}^{\infty} U(v - R^{-1}) e^{\left( \epsilon_1 \alpha x (Rv - R^{-1}v^{-1}) + \epsilon_2 T_1 \log v \right) / 2\pi} dv.
\]
Note that \(U(v - 1/R)\) has support inside \([1 + R^{-1}, 2 + R^{-1}]\) and satisfies \(\frac{\partial}{\partial v} U(v - R^{-1}) \ll_1 1\). Note that the phase function in (13.111) has the first derivative
\[
(2\pi)^{-1} (R\epsilon_1 \alpha x (1 + (Rv)^{-2}) + \epsilon_2 T_1 v^{-1})
\]
(13.112)
For \(v \in \text{supp} U(v - R^{-1})\) we have
\[
R\epsilon_1 \alpha x (1 + (Rv)^{-2}) \asymp R\alpha \quad \text{and} \quad \epsilon_2 T_1 v^{-1} \asymp T(1 + R^{-1})^{-1}.
\]
So if we assume \(RT \gg T^\epsilon\), it follows that the integral (13.111) is negligibly small unless
\[
(R + 1)\alpha \asymp T.
\]
(13.113)
This completes the proof of the claim (13.100).

**Remark 3.** In the case \(K_{iT_1} = J_{iT_1}\), (13.112) becomes \((R\epsilon_1 \alpha x (1 - (Rv)^{-2}) + \epsilon_2 T_1 v^{-1})\).
So we must have \(R^2 \alpha / (R + 1) \asymp R\epsilon_1 \alpha x (1 - (Rv)^{-2}) \asymp \epsilon_2 T_1 v^{-1} \asymp RT/(R + 1)\), that is, \(R\alpha \asymp T\), for non-negligible contribution.

Substituting (13.111) in (13.95), we obtain
\[
I(U, \alpha) = \frac{1}{2} \sum_{\epsilon_1, \epsilon_2 \in \{\pm 1\}^2} \sum_{Rw} e(\epsilon_2 T_1 \log(R)) \int_{R}^{\infty} U(v - R^{-1}) e^{\left( \epsilon_2 T_1 \log v \right) / 2\pi} dv.
\]
\[
\int_{R}^{\infty} U(x) x^{T_2} e^{\left( \epsilon_1 \alpha (Rv - R^{-1}v^{-1})x + Ux \right) / 2\pi} dx dv.
\]
(13.114)
Let
\[
\phi_R(\alpha, v) := -2\pi T_2^{-1} \left( \frac{\alpha(Rv - R^{-1}v^{-1})}{2\pi} + U \right),
\]
then by a simple stationary phase analysis, the last \(x\)-integral is essentially
\[
\int_{\mathbb{R}} U(x) x^2 e \left( -\frac{T_2}{2\pi} \cdot \phi_R(\epsilon_1 \alpha, v)x \right) \, dx
\]
\[
\sim \left(\frac{2\pi}{T_2}\right)^{1/2} \cdot \left(\phi_R(\epsilon_1 \alpha, v)\right)^{-1} U((\phi_R(\epsilon_1 \alpha, v))^{-1}) \cdot e \left( -\frac{T_2 \log(\phi_R(\epsilon_1 \alpha, v))}{2\pi} - T_2 \right).
\]
Substituting the last approximation of the \(x\) integral into (13.114), we obtain
\[
I(U, \alpha) \sim \frac{e \left(\frac{-T_2}{T^{1/2}}\right)}{T^{1/2}} \sum_{\epsilon_1, \epsilon_2 = \pm 1} e(\epsilon_2 T_1 \log(R)) I_R^{(\epsilon_1, \epsilon_2)}(U, \alpha),
\]
where
\[
I_R^{(\epsilon_1, \epsilon_2)}(U, \alpha) = \int_{\mathbb{R}} v^{-1} \omega_R(\epsilon_1 \alpha, v) e \left( -\frac{T_2 \log \phi_R(\epsilon_1 \alpha, v) + \epsilon_2 T_1 \log v}{2\pi} \right) \, dv,
\]
where
\[
\omega_R(\alpha, v) = U(v - R^{-1}) \mathcal{W}(\phi_R(\epsilon_1 \alpha, v)), \quad \mathcal{W}(y) = \frac{2\pi T^{1/2}}{|T_2|^{1/2}} \cdot y^{-1} U(y^{-1}).
\]
To complete the proof of part (13.96), it remains to prove \(U \ll T\). Note that from the definition (13.115)
\[
U/T \ll |\phi_r(\alpha, v)| + R\alpha/T \ll |\phi_r(\alpha, v)| + 1 \ll 1,
\]
for \(\phi_r(\alpha, v) \in \text{supp} \mathcal{W}\). This completes the proof upto (13.96).

Let us proceed for the proof of (13.105). We have
\[
\frac{\partial}{\partial v} \left( -T_2 \log(\phi_R(\epsilon_1 \alpha, v)) + \epsilon_2 T_1 \log v \right)
\]
\[
= -\frac{T_2 \epsilon_1 \alpha (v^2 + R^{-2})}{v(\epsilon_1 \alpha (v^2 - R^{-2}) + R^{-1}2\pi Uv)} + \frac{\epsilon_2 T_1}{v}
\]
\[
= \frac{\epsilon_1 \alpha (\epsilon_2 T_1 - T_2) v^2 + R^{-1}2\pi U(\epsilon_2 T_1) v - \epsilon_1 \alpha (u) R^{-2}(\epsilon_2 T_1 + T_1)}{v(\epsilon_1 \alpha (v^2 - R^{-2}) + R^{-1}2\pi Uv)}
\]
Hence \(z_1(\epsilon_1 \alpha)\) and \(z_2(\epsilon_1 \alpha)\) are the roots of the quadratic equation
\[
v^2 + R^{-1} \left( \frac{\pi U}{\epsilon_1 \alpha} \right) \left( \frac{2\epsilon_2 T_1}{(\epsilon_2 T_1 - T_2)} \right) v - R^{-2} \left( \frac{\epsilon_2 T_1 + T_2}{\epsilon_2 T_1 - T_2} \right),
\]
and we have
\[
\frac{\partial}{\partial v} \left( -T_2 \log(\phi_R(\epsilon_1 \alpha, v)) + \epsilon_2 T_1 \log v \right) = \frac{R\alpha(\epsilon_2 T_1 - T_2)(v - z_1(\epsilon_1 \alpha))(v - z_2(\epsilon_1 \alpha))}{T_2 v^2 \phi_R(\epsilon_1 \alpha, v)}
\]
(13.118)

Denote \(A_0 := R^3 \alpha/(R + 1)^2 \gg T^\epsilon\).
Consider the integral with $G$

Furthermore, using the simple inequalities (13.120) is negligibly small. For the second integral in (13.120), we further introduce

It is easy to show that for $A := |z_1(\epsilon_1 \alpha) - z_2(\epsilon_1 \alpha)| \gg T^{2k} A_0^{-1/3}$. We can write

$$I_R^{(\epsilon_1, \epsilon_2)}(U, \alpha) = \int_{\mathbb{R}} G_1(v) e(\psi(v)/2\pi) \, dv + \int_{\mathbb{R}} G_2(v) e(\psi(v)/2\pi) \, dv$$

where

$$G_1(v) := w\left(A^{-1}(v - \Re(z_1))\right) v^{-1} \omega_R^{(\epsilon_1, \epsilon_2)}(\alpha, v), \quad G_2(v) = (1-w(A^{-1}(v-\Re(z_1)))) v^{-1} \omega_R^{(\epsilon_1, \epsilon_2)}(\alpha, v).$$

Consider the integral with $G_1$. For $v$ in support of $G_1$, from (13.118) we obtain

$$\psi'(v) \gg A_0 \left(|v - \Re(z_1)|^2 + |z_1 - z_2|^2\right) \gg A_0 A^2.$$ 

Furthermore, using the simple inequalities

$$\frac{\partial^j}{\partial v^j} \phi_R(\alpha, v) \ll_j \frac{\alpha}{T} \left(\frac{R^j}{(R+1)^{j-1}}\right) \approx (R/(R+1))^j, \quad j \geq 1, \quad \frac{\partial^j}{\partial v^j} (1/v^2) \ll (R/(R+1))^{j+2}, \quad j \geq 0$$

it is easy to show that for $v$ in support of $\omega_R^{(\epsilon_1, \epsilon_2)}(\alpha, v)$,

$$\frac{\partial^j}{\partial v^j} (1/v^2 \phi_R(v, \alpha)) \ll_j (R/(R+1))^{j+2}, \quad j \geq 0,$$

so that using the above and (13.118), we obtain

$$\psi''(v) \ll A_0 (A + A^2), \quad \psi^{(j)}(v) \ll_j A_0 (1 + A + A^2), \quad j \geq 3.$$ 

On the other hand, using (13.121) and (13.116), we obtain that

$$G_1^{(j)}(v) \ll_j (1 + A^{-1})^j.$$ 

Applying Lemma 3.3 with the parameters $(X, U^{-1}) = (1, 1 + A^{-1})$, $W = A_0 A^2, Y = \min\{1, A\}^2 A_0 (A + A^2), Q = \min\{1, A\}$, we obtain

$$I_n^{(\epsilon_1, \epsilon_2)}(U, \alpha, \delta) \ll A \left(\frac{A_0 A^2}{(A_0 (A + A^2))^{1/2}}\right)^{-A} + \left(\frac{A_0 A^2}{1 + A^{-1}}\right)^{-A}. \quad (13.123)$$

Using the fact that $A \gg T^{2k} A_0^{-1/3}$ and $A_0 \gg T^c$, it can be easily verified that each term inside the parenthesis above is $\gg T^c$. Hence we conclude that the first integral in (13.120) is negligibly small. For the second integral in (13.120), we further introduce dyadic partition and insert localising factors $F(T^\delta(v - \Re(z)))$, where $T^{-\delta} \gg A$. Each dyadic part is of the form

$$\int_{\mathbb{R}} F(T^\delta(v - \Re(z))) G_2(v) e(\psi(v)/2\pi) \, dv. \quad (13.124)$$

Then for $v$ in the support of $F(T^\delta(v - \Re(z))) G_2(v)$, we have

$$\psi'(v) \gg A_0 \left(|v - \Re(z_1)|^2 + |z_1 - z_2|^2\right) \gg A_0 T^{-2\delta},$$
and arguing as earlier we have
\[ \psi''(v) \ll A_0(T^{-\delta} + T^{-2\delta}), \quad \psi^{(j)}(v) \ll_j A_0(1 + T^{-\delta} + T^{-2\delta}), \quad j \geq 3. \]

Now arguing similarly as in (13.123) with \( T^{-\delta} \) in place of \( A \), and using the fact that \( T^{-\delta} \gg A \), we can conclude (13.124) is negligibly small.

Hence, for rest of the calculation we assume that the roots \( z_1, z_2 \) are real. Fix a smooth function \( w \) such that \( w(t) = 1, t \in [-1/2, 1/2] \) and \( \text{supp}(w) \subseteq [-1, 1] \). We divide the range of integration of \( I_n^{(\epsilon_1, \epsilon_2)}(U, \alpha) \) into two pieces (localising around the two roots)

\[ I_R^{(\epsilon_1, \epsilon_2)}(U, \alpha) = \int_{v \in \mathbb{R}} v^{-1} \omega_R^{(\epsilon_1, \epsilon_2)}(\alpha, v)e(\psi(v)/2\pi)\,dv = I_1 + I_2, \tag{13.125} \]

where

\[ I_1 := \int_{\mathbb{R}} (1 - w(T^{-\epsilon}A_0^{1/3}(v - z_1)))v^{-1} \omega_R^{(\epsilon_1, \epsilon_2)}(\alpha, v)e(\psi(v)/2\pi)\,dv, \]

and

\[ I_2 := \int_{\mathbb{R}} w(T^{-\epsilon}A_0^{1/3}(v - z_1))v^{-1} \omega_R^{(\epsilon_1, \epsilon_2)}(\alpha, v)e(\psi(v)/2\pi)\,dv. \]

Consider the \( I_1 \) part first. Note that \( v \in \text{supp}(1 - w(T^{-\epsilon}A_0^{1/3}(v - z_1))) \) implies \( v - z_1 \gg T^\epsilon A_0^{-1/3} \). We fix the size of \( v - z_1 \) by introducing dyadic partition of unity and inserting localising factors \( F(T^\delta(v - z_1)) \), where

\[ T^\epsilon A_0^{-1/3} \ll T^{-\delta} \tag{13.126} \]

Note that since \( \omega_R^{(\epsilon_1, \epsilon_2)}(\alpha, v) \subseteq [1 + R^{-1}, 1 + 2R^{-1}] \) and \( |z_1| \ll 1 + R^{-1} \) which follows from (13.117), we can also assume

\[ T^\epsilon A_0^{-1/3} \ll T^{-\delta} \ll 1 + R^{-1}. \tag{13.127} \]

Each dyadic part is of the form

\[ I_1(\delta) := \int_{v \in \mathbb{R}} F(T^\delta(v - z_1))(1 - w(T^{-\epsilon}A_0^{1/3}(v - z_1)))v^{-1} \omega_R^{(\epsilon_1, \epsilon_2)}(\alpha, v) \]

\[ e\left(-T_2 \log \phi_R(\epsilon_1 \alpha, v) + \epsilon_2 T_1 \log v\right)2\pi^{-1}dv. \]

Changing variable \( T^\delta(v - z_1) \mapsto v_1 \), we obtain

\[ I_1(\delta) = T^{-\delta} \int_{v_1} G(v_1)e(\phi(v_1)/2\pi)\,dv_1, \tag{13.128} \]

where

\[ G(v_1) = F(v_1)(1 - w(T^{-\epsilon}A_0^{1/3}T^{-\delta}v_1))(T^{-\delta}v_1 + z_1)^{-1}\omega_R^{(\epsilon_1, \epsilon_2)}(\alpha, T^{-\delta}v_1 + z_1) \]

and \( \phi(v_1) := \psi(T^{-\delta}v_1 + z_1) \) which using (13.118) satisfy

\[ \phi'(v_1) = \frac{R\alpha(\epsilon_2 T_1 - T_2)T^{-3\delta}v_1(v_1 + T^\delta(z_1 - z_2))}{T_2(T^{-\delta}v_1 + z_1)^2\phi_n(\epsilon_1 \alpha, T^{-\delta}v_1 + z_1)} \tag{13.129} \]
Lemma 13.9. $I_1(\delta)$ is negligibly small unless $T^\delta(z_1 - z_2) \ll 1$ for some large enough implied constant.

Proof. Suppose $T^\delta(z_1 - z_2) \gg 1$ for some large enough implied constant. Then from (13.129), for $v_1$ in the support of $G(v_1)$,

$$\phi'(v_1) \gg A_0 T^{-3\delta} (T^\delta(z_1 - z_2)),$$

$$\phi^{(j)}(v_1) \ll_j A_0 T^{-3\delta} (T^\delta(z_1 - z_2)), \ j \geq 2.$$

where we have used (13.122) and (13.127). Furthermore,

$$G^{(j)}(v_1) \ll_j (1 + T^{-\delta} + A_0^{1/3} T^{-\delta})^j.$$

Applying Lemma 5.3 with $(X, U^{-1}) = (1, 1 + T^{-\delta} + T^{-\delta} A_0^{1/3}), (Y, Q^{-1}) = (A_0 T^{-3\delta} (T^\delta(z_1 - z_2)), 1)$ and $W = A_0 T^{-3\delta} (T^\delta(z_1 - z_2))$, we obtain

$$I_1(\delta) \ll_A \left( \frac{A_0 T^{-3\delta} (T^\delta(z_1 - z_2))}{(A_0 T^{-3\delta} (T^\delta(z_1 - z_2)))^{1/2}} \right)^{-A} + \left( \frac{A_0 T^{-3\delta} (T^\delta(z_1 - z_2))}{1 + T^{-\delta} + T^{-\delta} A_0^{1/3}} \right)^{-A}$$

$$\ll (A_0 T^{-3\delta})^{-A/2} + \left( \frac{A_0 T^{-3\delta}}{1 + T^{-\delta} + T^{-\delta} A_0^{1/3}} \right)^{-A}$$

(13.130)

Using $T^{-\delta} \gg T^\epsilon A_0^{-1/3}$ and $A_0 \gg T^\epsilon$, it can be easily shown that the term inside each parenthesis above is $\gg T^\epsilon$. The lemma follows. \hfill \Box

The next lemma localises the integral $I_1(\delta)$ around the root $-T^\delta(z_1 - z_2)$ of its phase function.

Lemma 13.10. Suppose $w$ is a smooth function such that $w(t) = 1$, $t \in [-1/2, 1/2]$ and $\text{supp}(w) \subseteq [-1, 1]$. Let $A := A_0^{1/2} T^{-3\delta/2} - \epsilon (\gg T^{\epsilon/2})$ and

$$H(v_1) := (1 - w(A(v_1 + T^\delta(z_1 - z_2)))) G(v_1).$$

Then

$$I := \int_{v_1} H(v_1) e(\phi(v_1)/2\pi) \, dv_1 \ll_A T^{-A}.$$

Proof. Using (13.122), (13.127) and (13.129), it is easy to see that for $v_1$ in the support of $1 - w(A(v + T^\delta(z_1 - z_2)))$, we have

$$\phi'(v_1) \gg T^\epsilon A_0^{1/2} T^{-3\delta/2}, \ \phi^{(j)}(v_1) \ll A_0 T^{-3\delta}, \ j \geq 2,$$

and

$$H^{(j)}(v_1) \ll_j (A + T^{-\delta} + T^{-\delta} A_0^{1/3})^j, \ j \geq 1.$$

Applying Lemma 5.3 with the parameters $(X, U^{-1}) = (1, A + T^{-\delta} + T^{-\delta} A_0^{1/3}), W = T^\epsilon A_0 T^{-3\delta/2}, (Y, Q^{-1}) = (A_0 T^{-3\delta}, 1)$, we obtain

$$I \ll_A T^{-A\epsilon} \left( \frac{A_0^{1/2} T^{-3\delta/2}}{A + T^{-\delta} + T^{-\delta} A_0^{1/3}} \right)^{-A} + T^{-A\epsilon/2}.$$  

(13.131)

Using $A = A_0^{1/2} T^{-3\delta/2 - \epsilon}, T^{-\delta} \gg T^\epsilon A_0^{-1/3}$ and $A_0 \gg T^\epsilon$, the term inside the parenthesis is easily seen to be $\gg 1$ and the claim follows. \hfill \Box
It follows from (13.128) and Lemma 3.10 that

\[ I_1(\delta) \approx T^{-\delta} \int_{v_1} w(A(v_1 + T^\delta(z_1 - z_2)))G(v_1)e(\phi(v_1)/2\pi) \, dv_1. \]  

(13.132)

Note that using (13.129), (13.122) and (13.127), for \( v_1 \) in the support of \( w(A(v_1 + T^\delta(z_1 - z_2)))G(v_1) \), we get

\[ \phi''(v_1) > A_0 T^{-3\delta} (1 + O(A^{-1})) \approx A_0 T^{-3\delta} \]

It is also clear that for \( v_1 \) in the support of \( w(A(v_1 + T^\delta(z_1 - z_2)))G(v_1) \),

\[ \phi'(v_1) \ll T^\delta A_0^{1/2} T^{-3\delta/2}, \quad \phi^{(j)}(v_1) \ll_j A_0 T^{-3\delta}, \quad j \geq 3. \]  

(13.133)

Let \( Y_0 = A_0 T^{-3\delta}, Q_0 = Y_0^{1/2} T^{-\epsilon} \). Since \( Q_0 \gg T^{3\epsilon T^{-\epsilon} = T^{2\epsilon} \), the above inequalities implies that

\[ \phi^{(j)}(v_1) \ll_j Y_0 Q_0^{-j} \]

holds for all \( j \geq 1 \). We also have

\[ \frac{\partial^j}{\partial v_1^j} w(A(v_1 + T^\delta(z_1 - z_2)))G(v_1) \ll_j (A + T^{-\delta} + T^{-\delta} A_0^{1/3})^j, \]

As earlier, plugging in \( A = A_0^{1/2} T^{-3\delta/2} \), \( T^{-\delta} \gg T^\epsilon A_0^{-1/3} \) and \( A_0 \gg T^{\epsilon} \), we can easily verify

\[ A + T^{-\delta} + T^{-\delta} A_0^{1/3} \ll T^{-\epsilon/2} Y_0^{1/2} \]

We are now ready to apply 3.4 to the integral \( I_1(\delta) \) in (13.132) with the parameters,

\( (X, V^{-1}) = (1, A + T^{-\delta} + T^{-\delta} A_0^{1/3}), (Y, Q^{-1}) = (Y_0 Q_0^{-2}, Q_0) \), with the unique root \( t_0 = -T^\delta(z_1 - z_2) \), to get the main term

\[ I_1(\delta) \sim T^{-\delta} e(\phi(t_0)/2\pi) G(t_0). \]

Evaluating from (13.129), we have

\[ \phi''(t_0) = \phi''(-T^\delta(z_1 - z_2)) = \left( -\frac{(\epsilon_2 T_1 - T_2)}{T_2 \phi_R(\epsilon_1 \alpha, z_2)} \right) \frac{R^\alpha T^{-2\delta(z_1 - z_2)}}{z_2^2}. \]

From their definitions, we also have

\[ \phi(t_0) = \phi(-T^\delta(z_1 - z_2)) = \psi(z_2) = -T_2 \log \phi_R(\epsilon_1 \alpha, z_2) + \epsilon_2 T_1 \log z_2, \]

and

\[ G(t_0) = G(-T^\delta(z_1 - z_2)) = F(T^\delta(z_2 - z_1)) z_2^{-1} \omega^{(\epsilon_1, \epsilon_2)}(\alpha, z_2). \]

Substituting these expressions we obtain

\[ I_1(\delta) \sim \frac{1}{\alpha^{1/2}} \left( -\frac{(\epsilon_2 T_1 - T_2)}{T_2 \phi_R(\epsilon_1 \alpha, z_2)} \right)^{-1/2} \frac{F(T^\delta(z_2 - z_1)) \omega^{(\epsilon_1, \epsilon_2)}(\alpha, z_2)}{(\phi(t_0)/2\pi)} e(\psi(z_2)/2\pi). \]

Finally, summing over all the dyadic parts and substituting in (13.125), we obtain

\[ I_1 \sim \frac{1}{\alpha^{1/2}} \left( -\frac{(\epsilon_2 T_1 - T_2)}{T_2 \phi_R(\epsilon_1 \alpha, z_2)} \right)^{-1/2} \frac{\omega^{(\epsilon_1, \epsilon_2)}(\alpha, z_2)}{(\phi(t_0)/2\pi)} e(\psi(z_2)/2\pi). \]

This complete the analysis for the \( I_1 \) part.
Next consider the $I_2$ integral
\[
I_2 = \int_{\mathbb{R}} w(T^{-\epsilon} A_0^{1/3} (v - z_1)) v^{-1} \omega_R^{(e_1, e_2)}(\alpha, v)\epsilon(\psi(v)/2\pi) \, dv. \tag{13.135}
\]
Using (13.118) and (13.122), for $v$ in the support of $w(T^{-\epsilon} A_0^{1/3} (v - z_1))\omega_R(\epsilon_1 \alpha, v)$, we have
\[
\psi''(v) \asymp A_0 \left( |z_1 - z_2| + O(T^\epsilon A_0^{1/3}) + O((R/(R + 1))|z_1 - z_2| T^\epsilon A_0^{-1/3}) \right)
\]
Since $|z_1 - z_2| \gg T^{2\epsilon} A_0$ from (13.119), and $R/(R + 1) A_0^{-1/3} T^\epsilon = T^\epsilon ((R + 1) \alpha)^{-1/3} \asymp T^{-1/3 + \epsilon}$, it follows that
\[
\psi''(v) \asymp A_0 |z_1 - z_2|.
\]
Furthermore, we have
\[
\psi'(v) \ll A_0 |z_1 - z_2| T^{\epsilon} A_0^{-1/3} \ll A_0 |z_1 - z_2|
\]
since $A_0 \gg T^{3\epsilon}$, and using (13.122) we have
\[
\psi^{(j)}(v) \ll_j A_0 |z_1 - z_2|, \quad j \geq 3. \tag{13.136}
\]
For the weight function, we have
\[
\frac{\partial^j}{\partial v^j} w(T^{-\epsilon} A_0^{1/3} (v - z_1)) v^{-1} \omega_R(\epsilon_1 \alpha, v) \ll_j (1 + T^{-\epsilon} A_0^{-1/3})^j, \quad j \geq 1.
\]
Since $|z_1 - z_2| \gg T^{2\epsilon} A_0$ and $A_0 \gg T^{3\epsilon}$, it follows that
\[
1 + T^{-\epsilon} A_0^{-1/3} \ll T^{-\epsilon} (A_0 |z_1 - z_2|)^{1/2}.
\]
Hence, we can apply Lemma 3.3 to the $I_2$ integral in (13.135) with the parameters $(X, V^{-1}) = (1, 1 + T^{-\epsilon} A_0^{1/3})$, $(Y, Q^{-1}) = (A_0 |z_1 - z_2|, 1)$ and with the unique root $z_1$, to get
\[
I_2 \sim \frac{e(\psi(z_1)/2\pi)}{\sqrt{\psi''(z_1)}} z_1^{-1} \omega_R(\alpha, z_1).
\]
From (13.118), we get
\[
\psi''(z_1) = \frac{(\epsilon_2 T_1 - T_2)}{(T_2 \phi_R(\epsilon_1 \alpha, z_1))} R \alpha (z_1 - z_2) z_1^2
\]
Substituting the above and $\psi(z_1) = -T_2 \log \phi_R(\epsilon_1 \alpha, z_1) + \epsilon_2 T_1 \log z_1$, we obtain
\[
I_2 \sim \frac{1}{\alpha^{1/2}} \left( \frac{(\epsilon_2 T_1 - T_2)}{T_2 \phi_R(\epsilon_1 \alpha, z_1)} \right)^{-1/2} \omega_R(\alpha, z_1) \left( \frac{\omega_R(\epsilon_1 \alpha, z_1)}{R(z_1 - z_2)} \right)^{1/2} \left( \phi_R(\epsilon_1 \alpha, z_1(\epsilon_1 \alpha)) \right)^{-iT_2} (z_1(\epsilon_1 \alpha))^{iT_2 T_1}. \tag{13.137}
\]
Proof upto (13.105) of Lemma 13.8 is now complete after combining (13.134) and (13.137). It remains to prove (13.106) to complete the proof of part II.

Note that from (13.117), we have
\[
z_1(\epsilon_1 \alpha) = R^{-1} c_1 \left( \frac{\pi U}{\alpha} \right) + \sqrt{f(\alpha)}, \quad z_2(\epsilon_1 \alpha) = R^{-1} c_1 \left( \frac{\pi U}{\alpha} \right) - \sqrt{f(\alpha)}. \tag{13.138}
\]
where
\[ f(\alpha) = R^{-2} \left( \frac{\pi U}{\alpha} \right)^2 + R^{-2} c_0 \]
and \( c_1 = \frac{\epsilon_1 T_1}{\epsilon_1 (\epsilon_2 T_1 - T_2)} \approx 1, \ c_0 = \frac{\epsilon_2 T_1 + T_2}{\epsilon_2 T_1 - T_1} \approx 1. \) It is easy to see that
\[ \frac{\partial^j}{\partial \alpha^j} f(\alpha) \ll_j |\alpha|^{-j} R^{-2} \left( \frac{U}{|\alpha|} \right)^2, \ j \geq 1. \quad (13.139) \]

Recall the hypothesis \( f(\alpha) \approx R^{-2} T^{-\delta}. \) Using these informations, one deduces from the Faà di Bruno’s formula 3.6, that
\[ \frac{\partial^j}{\partial \alpha^j} \sqrt{f(\alpha)} \ll_j \sum_{j_1, j_2, \ldots, j_n} (R^{-2} T^{-\delta})^{1/2 - \sum j_i} \prod_{i=1}^n \left( |\alpha|^{-i} R^{-2} \left( \frac{U}{|\alpha|} \right)^2 \right)^{j_i}, \]
where the sum is over all \( n \)-tuples of non-negative integers \((j_1, j_2, \ldots, j_n)\) satisfying
\[ 1 \cdot j_1 + 2 \cdot j_2 + 3 \cdot j_3 + \cdots + n \cdot j_n = j. \]

Simplifying we obtain
\[ |\alpha|^j \frac{\partial^j}{\partial \alpha^j} \sqrt{f(\alpha)} \ll_j (R^{-2} T^{-\delta})^{1/2} \sum (T^\delta \left( \frac{U}{|\alpha|} \right)^2)^{\sum j_i}. \]

Since \( \sum j_i \leq j, \) we conclude
\[ |\alpha|^j \frac{\partial^j}{\partial \alpha^j} \sqrt{f(\alpha)} \ll_j (R^{-2} T^{-\delta})^{1/2} \left( 1 + T^\delta \left( \frac{U}{|\alpha|} \right)^2 \right)^j. \quad (13.140) \]

On the other hand we have
\[ |\alpha|^j \frac{\partial^j}{\partial \alpha^j} \left( R^{-1} \frac{\pi U}{\alpha} \right) \ll_j R^{-1} \frac{U}{|\alpha|}. \quad (13.141) \]

By the simple AM-GM inequality, the right hand side of (13.141) gets absorbed into (13.140). Substituting in (13.138) we obtain
\[ |\alpha|^j \frac{\partial^j}{\partial \alpha^j} z_i(\epsilon_1 \alpha) \ll_j R^{-1} T^{-\delta/2} \left( 1 + T^\delta \left( \frac{U}{|\alpha|} \right)^2 \right)^j, \ j \geq 1. \quad (13.142) \]

One can similarly show that
\[ |\alpha|^j \frac{\partial^j}{\partial \alpha^j} z_i^{-1}(\epsilon_1 \alpha) \ll_j R^{-1} T^{-\delta/2} \left( 1 + T^\delta \left( \frac{U}{|\alpha|} \right)^2 \right)^j, \ j \geq 1. \quad (13.143) \]

Now
\[ \phi_R(\alpha, z_i(\epsilon_1 \alpha)) = -\frac{R \alpha}{T_2} \left( z_i(\epsilon_1 \alpha) - \frac{1}{R^2 z_i(\epsilon_1 \alpha)} \right) - \frac{2\pi U}{T_2}, \]
so that
\[
\frac{\partial^j}{\partial \alpha^j} \phi_R(\alpha, z_i(\epsilon_1 \alpha)) = -\frac{R \alpha}{T_2} \frac{\partial^j}{\partial \alpha^j} \left( z_i(\epsilon_1 \alpha) - \frac{1}{R^2 z_i(\epsilon_1 \alpha)} \right) - \frac{R}{T_2} \frac{\partial^{j-1}}{\partial \alpha^{j-1}} \left( z_i(\epsilon_1 \alpha) - \frac{1}{R^2 z_i(\epsilon_1 \alpha)} \right).
\]

Substituting the bounds (13.142) and (13.143) into above, we obtain
\[
\frac{\partial^j}{\partial \alpha^j} \phi_R(\alpha, z_i(\epsilon_1 \alpha)) \ll_j T^{-\delta/2}(\alpha/T)|\alpha|^{-j}(1 + T^\delta(U/\alpha)^2)^j + T^{-\delta/2} |\alpha|^{-j+1}(1 + T^\delta(U/\alpha)^2)^{j-1}.
\]

Since \(1 + T^\delta(U/\alpha)^2 > 1\), we conclude
\[
|\alpha|^j \frac{\partial^j}{\partial \alpha^j} \phi_R(\alpha, z_i(\epsilon_1 \alpha)) \ll_j T^{-\delta/2}(\alpha/T)(1 + T^\delta(U/\alpha)^2)^j, \quad j \geq 1. \tag{13.144}
\]

Using the last inequality, (13.142) and the Faà di Bruno’s formula\(^3\), we conclude that for
\[
\omega_R(\epsilon_1 \alpha, z_i(\epsilon_1 \alpha)) = F(z_i(\epsilon_1 \alpha))U(z_i(\epsilon_1 \alpha) - R^{-1})W(\phi_R(\epsilon_1 \alpha, z_i(\epsilon_1 \alpha))),
\]
\[
|\alpha|^j \frac{\partial^j}{\partial \alpha^j} \omega_R(\epsilon_1 \alpha, z_i(\epsilon_1 \alpha)) \ll_j \left((1 + T^{-\delta/2}(R^{-1} + \alpha/T))(1 + T^\delta(U/\alpha)^2)\right)^j, \quad j \geq 1. \tag{13.145}
\]

Finally, since \((z_1(\epsilon_1 \alpha) - z_2(\epsilon_1 \alpha)) = \sqrt{f(\alpha)} \approx R^{-1}T^{-\delta/2}\), using the derivative bound (13.139), and the Faà di Bruno’s formula\(^3\), we deduce
\[
\frac{\partial^j}{\partial \alpha^j}(RT^{\delta/2}(z_1(\epsilon_1 \alpha) - z_2(\epsilon_1 \alpha)))^{-1/2} \ll_j \sum_{j_1, j_2, \ldots, j_n} (RT^{\delta/2}|\alpha|^{-k}R^{-2}(U/|\alpha|)^2)^{j_k} = |\alpha|^{-j} \sum_{j_1, j_2, \ldots, j_n} (R^{-1}T^{\delta/2}(U/|\alpha|)^2)^{j_k} \ll |\alpha|^{-j} \left(1 + R^{-1}T^{\delta/2}(U/|\alpha|)^2\right)^j \tag{13.146}
\]

Combining (13.145) and (13.146) we obtain
\[
|\alpha|^j \frac{\partial^j}{\partial \alpha^j} \left( \frac{\omega_R(\epsilon_1 \alpha, z_i(\epsilon_1 \alpha))}{RT^{\delta/2}(z_1(\epsilon_1 \alpha) - z_2(\epsilon_1 \alpha)))^{1/2}} \right) \ll_j \left((1 + T^{-\delta/2}(R^{-1} + \alpha/T))(1 + T^\delta(U/\alpha)^2)\right)^j.
\]

The proof of (13.106) is complete after observing that \(\alpha/T \asymp (R + 1)^{-1}\) and
\[
T^\delta(U/\alpha)^2 \ll 1 + T^\delta
\]
which is trivially true if \(U/\alpha \ll 1\), and follows the definition \(T^{-\delta} \asymp (\pi U/\alpha)^2 - c_0\) if \(U/\alpha \gg 1\).

Let us proceed to prove part 2 (13.108), which is the third derivative bounds for the integral \(I_R^{(\epsilon_1, \epsilon_2)}(U, \alpha)\). Recall that
\[
I_R^{(\epsilon_1, \epsilon_2)}(U, \alpha) = \int_{v \in \mathbb{R}} v^{-1} \omega_R(\epsilon_1 \alpha, v)e(\psi(v)/2\pi) dv
\]
and from (13.118)

\[ \psi'(v) = \frac{R\alpha(e_2T_1 - T_2)(v - z_1(e_1\alpha))(v - z_2(e_1\alpha))}{T_2v^2\phi_R(e_1\alpha, v)}. \] (13.147)

The standing assumptions for the case under consideration are

\[ R \gg T^{3\varepsilon} \min\{\alpha^{-1}, T^{-1}\} \quad \text{and} \quad z_1(e_1\alpha) - z_2(e_1\alpha) \ll T^{2\varepsilon} A_0^{-1/3}, \]

where \( A_0 = R^3\alpha/(R + 1)^2 \). It is enough to work around one of root \( z_1 \) (say) and furthermore, by introducing dyadic partition of unity, it is enough to prove the claim for each dyadic piece

\[ I(\delta) := \int_{v \in \mathbb{R}} F(T^\delta(v - \Re(z_1)))v^{-1}\omega_R(e_1\alpha, v)e(\psi(v)/2\pi) \, dv, \]

where we can assume \( T^{-\delta} \ll 1 + R^{-1} \) following from the earlier remark (13.127).

**Lemma 13.11.** \( I(\delta) \) is negligibly small unless

\[ T^{-\delta} \ll T^{2\varepsilon} A_0^{-1/3}, \]

for some large enough implied constant.

**Proof.** Suppose that for some large enough implied constant,

\[ T^{-\delta} \gg T^{2\varepsilon} A_0^{-1/3}. \]

Then from (13.147), for \( v \) in the support of \( F(T^\delta(v - \Re(z_1)))\omega_R^{(e_1, e_2)}(\alpha, v), \)

\[ \psi'(v) \gg A_0 T^{-2\delta}. \]

Furthermore, using (13.122) and \( T^{-\delta} \ll 1 + R^{-1} \), we get

\[ \psi''(v) \ll A_0 \left( T^{-\delta} + T^{-2\delta}(R/(R + 1)) \right) \ll A_0 T^{-\delta} \]

\[ \psi^{(j)}(v) \ll_j A_0 \left( T^{-2\delta}(R/(R + 1))^{j-1} + (R/(R + 1))^{j-2}T^{-\delta} + (R/(R + 1))^{j-3} \right) \ll A_0, \quad j \geq 3. \]

We also have

\[ \frac{\partial^j}{\partial v^j}F(T^\delta(v - \Re(z_1)))v^{-1}\omega_R(e_2\alpha, v) \ll_j (1 + T^\delta)^j, \quad j \geq 1. \]

Applying Lemma 3.3 with \((X, U) = (1, 1 + T^\delta), W = A_0 T^{-2\delta} \) and \((Y, Q) = (A_0 T^{-\varepsilon} \min\{1, T^{-2\delta}\}, \max\{1, T^\delta\})\), we obtain

\[ I(\delta) \ll_A \left( \frac{A_0 T^{-2\delta}}{1 + T^\delta} \right)^{-A} + \left( \frac{A_0 T^{-2\delta}}{A_0^{1/2} T^{-\delta/2}} \right)^{-A} \]

As earlier, using the \( T^{-\delta} \gg T^{2\varepsilon} A_0^{-1/3} \) and the basic inequality \( A_0 \gg T^\varepsilon \), one can verify that both the term inside the parenthesis above is \( \gg T^\varepsilon \). The lemma follows. \( \square \)
Assuming $T^{-\delta} \ll T^{2\varepsilon} A_0^{1/3}$, trivially executing the $v$ integral one obtains

$$I(\delta) \ll T^{-\delta} \sup v^{-1} \omega_R(\alpha, v) \ll T^{2\varepsilon} A_0^{1/3} (R/(R + 1)) = T^{2\varepsilon}((R + 1)\alpha)^{-1/3}$$

This completes the proof of part 2.

It remains to prove part 3. Recall

$$I_R^{(\varepsilon_1, \varepsilon_2)}(U, \alpha) = \int_{\mathbb{R}} v^{-1} \omega_R(\varepsilon_1 \alpha, v) e\left(\frac{\psi(v)}{2\pi}\right) dv,$$

where

$$\psi(v) = -T_2 \log \phi_R(\varepsilon_1 \alpha, v) + \varepsilon_2 T_1 \log v, \quad \phi_R(\alpha, v) := -2\pi T_2^{-1} \left(\frac{\alpha Rv - R^{-1}v^{-1}}{2\pi} + U\right),$$

and $\omega_R(\varepsilon_1 \alpha, v) = U(v - R^{-1}) W(\phi(\varepsilon_1 \alpha, v))$ for some compactly supported smooth functions $U, W$ with bounded derivatives. Now,

$$\psi'(v) = \frac{\varepsilon_1 \alpha (R + (Rv)^{-1})}{\phi_R(\alpha, v)} + \frac{\varepsilon_2 T_1}{v}$$

and

$$= \frac{\varepsilon_1 \alpha (Rv + (Rv)^{-1}) - (\varepsilon_2 T_1/T_2)(2\pi U) - (\varepsilon_2 T_1/T_2) \alpha (Rv - (Rv)^{-1})}{v \phi_R(\alpha, v)}.$$  \hspace{1cm} (13.148)

Since $R \ll T^{-\varepsilon}, Rv = 1 + O(R)$, it follows

$$\psi'(v) = \frac{2\varepsilon_1 \alpha - (\varepsilon_2 T_1/T_2)(2\pi U) + O(R\alpha)}{v \phi_R(\alpha, v)}.$$  \hspace{1cm} (13.149)

Also, using the fact that $\frac{\phi'}{\phi^j}(1/v \phi_R(\alpha, v)) \ll (R/(R + 1))^j$, which can be proved in a similar fashion as \hspace{1cm} (13.122), it is easy to see that

$$\psi^{(j)}(v) \ll_j R^2 \alpha, \quad j \geq 2.$$ 

So if $|\varepsilon_1 \alpha - (\varepsilon_2 T_1/T_2)(2\pi U)| \gg T^\varepsilon (R\alpha + R^{-1})$, then

$$\psi'(v) \gg T^\varepsilon (1 + R^2 \alpha).$$

Appealing to Lemma 3.3 with the parameters $(X, U^{-1}) = (1, 1), W = T^\varepsilon (1 + R^2 \alpha)$ and $(Y, Q^{-1}) = (R^2 \alpha, 1)$, it follows that $I_R^{(\varepsilon_1, \varepsilon_2)}(U, \alpha)$ is negligibly small.

Now in the case $|\varepsilon_1 \alpha - (\varepsilon_2 T_1/T_2)(2\pi U)| \ll T^\varepsilon (R\alpha + R^{-1})$, from (13.149) we get

$$\psi'(v) = \frac{\varepsilon_1 \alpha (R + (Rv)^{-1})}{\phi_R(\alpha, v)} + \frac{\varepsilon_2 T_1}{v} \ll T^\varepsilon (1 + R^2 \alpha),$$

and from (13.148) we have

$$\psi''(v) = \frac{(\alpha^2/T_2)(R + (Rv)^{-1})^2}{\phi_R^2(\varepsilon_1 \alpha, v)} - \frac{(2\varepsilon_1 \alpha)/(Rv^3)}{\phi_R(\varepsilon_1 \alpha, v)} - \frac{\varepsilon_2 T_1}{v^2}$$

and

$$= \frac{(\alpha^2/T_2)(R + (Rv)^{-1})^2}{\phi_R^2(\varepsilon_1 \alpha, v)} - v^{-1} \left( \frac{\varepsilon_1 \alpha (R + (Rv)^{-1})}{\phi_R(\alpha, v)} + \frac{\varepsilon_2 T_1}{v} + O(R^2 \alpha) \right)$$

$$\gg \left| \frac{R^2 \alpha^2}{T} - T^\varepsilon (R + R^3 \alpha) \right| \gg R^2 \alpha^2 / T \asymp R^2 \alpha,$$
since we are in the range \( \min\{T^{-1}, \alpha^{-1}\} \ll R \ll T^{-\epsilon} \) and \( T/\alpha \approx 1 + R \approx 1 \). Finally, observing that \( v^{-1}\omega_R(\epsilon_1, v) \ll R, \frac{\partial}{\partial v}(v^{-1}\omega_R(\epsilon_1, v)) \ll R \), and using the second derivative bound we obtain

\[
I_R^{(\epsilon_1, \epsilon_2)}(U, \alpha) \ll R(R^2\alpha)^{-1/2} = \alpha^{-1/2}.
\]

\[\Box\]

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References

[1] V. Blomer. Subconvexity for twisted \( L \)-functions on \( GL(3) \). Amer. J. Math., 134:1385–1421, 2012.
[2] V. Blomer. Applications of the Kuznetsov formula on \( GL(3) \). Invent. math., 194:673–729, 2013.
[3] V. Blomer and J. Buttcane. On the subconvexity problem for \( L \)-functions on \( GL(3) \). Ann. Sci. ENS, 53, 2020.
[4] V. Blomer, R. Khan, and M. Young. Distribution of mass of holomorphic cusp forms. Duke Math. J., 162(14):2609–2644, 2013.
[5] J. Buttcane. The Spectral Kuznetsov Formula on \( SL(3) \). Trans. Amer. Math. Soc., 368:6683–6714, 2016.
[6] D. Goldfeld. Automorphic forms and \( L \)-functions for the group \( GL(n, R) \). Cambridge studies in advanced mathematics, 99, 2006.
[7] H. Iwaniec and E. Kowalski. Analytic number theory. American Mathematical Society Colloquium Publications. American Mathematical Society, Providence, RI, 53, 2004.
[8] E. M. Kiral and M. Nakasuji. Parametrization of Kloosterman sets and \( SL_3 \)-Kloosterman sums. \texttt{arXiv:2001.01936}, 2020.
[9] E. Kowalski, P. Michel, and J. VanderKam. Rankin-–Selberg \( L \)-functions in the level aspect. Duke Math. J., 141(1):123—191, 2002.
[10] S. Kumar, K. Mallesham, and S. Singh. Subconvexity bound for \( GL(3) \times GL(2) \)-functions: \( GL(3) \)-spectral aspect. \texttt{arXiv:2006.07819}, 2020.
[11] X. Li. Upper bounds on \( L \)-functions at the edge of the critical strip. International Mathematics Research Notices, 2010:727–755, 2009.
[12] X. Li. Bounds for \( GL(3) \times GL(2) \)-functions and \( GL(3) \)-functions. Ann. of Math. (2), 173:301–336, 2011.
[13] P. Michel and A. Venkatesh. The subconvexity problem for \( GL(2) \). Publ. Math. Inst. Hautes Études Sci., 111:171–271, 2010.
[14] R. Munshi. The circle method and bounds for \( L \)-functions - I. Math. Annalen, 358:389–401, 2014.
[15] R. Munshi. The circle method and bounds for \( L \)-functions - III. \( t \)-aspect subconvexity for \( GL(3) \)-functions. J. Amer. Math. Soc., 28:913–938, 2015.
[16] R. Munshi. The circle method and bounds for \( L \)-functions - IV. Subconvexity for twists of \( GL(3) \)-functions. Ann. of Math. (2), 182(2):617–672, 2015.
[17] R. Munshi. The circle method and bounds for \( L \)-functions-II: Subconvexity bounds for twists of \( GL(3) \)-functions. American J. Math., 137:791–812, 2015.
[18] R. Munshi. Twists of \( GL(3) \)-functions. \texttt{arXiv:1601.08000}, 2016.
[19] R. Munshi. Subconvexity for \( GL(3) \times GL(2) \)-functions in \( t \)-aspect. J. Eur. Math. Soc., 24(5):1543–1566, 2022.
[20] P. Nelson. Bounds for standard \( L \)-functions. \texttt{arXiv:2109.15230}.
[21] P. Sharma. Subconvexity for \( GL(3) \times GL(2) \) twists (with an appendix by W. Sawin). to appear in Advances in Mathematics.
[22] G. Watson. A Treatise on the Theory of Bessel Functions. Reprint of the second (1944) edition. Cambridge University Press, 1995.
SUBCONVEXITY FOR $GL(3) \times GL(2)$ AND $GL(3)$ $L$-FUNCTIONS IN $GL(3)$ SPECTRAL ASPECT

School of Mathematics, Tata Institute of Fundamental Research, Mumbai

Email address: prahlad@math.tifr.res.in