Hairy topological Black holes of dimensionally continued gravity coupled to double-Logarithmic electrodynamics

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ABSTRACT: A recently proposed model for double-Logarithmic electrodynamics has been minimally coupled to dimensionally continued gravity and magnetized topological black holes in the presence of conformal scalar field are studied. In this set up, the hairy topological black hole solution has been derived with magnetic monopole charge and the associated thermodynamic quantities such as Hawking temperature and heat capacity are calculated. The modified Smarr’s formula is constructed and it is shown that first law of thermodynamics can be verified for these black holes. In addition to this, thermodynamic stability and thermal phase transitions for these black holes are investigated. Finally, hairy magnetized black holes in general Lovelock-scalar gravity with double-Logarithmic electromagnetic source are also briefly studied.
1 Background

It’s known that Einstein’s general relativity (EGR) is non-renormalizable, however, it is believed that renormalizable theory [1] can be constructed by making higher derivative corrections in EGR, therefore, it can be highly motivated to consider high derivative theories of gravities. The study related to higher order gravities is important because inclusion of such theories could provide an alternative way to examine the universe’s accelerated expansion without giving any introduction to dark energy. One among the different high derivative modified theories is the well known Lovelock gravity theory (LTG) [2]. This theory contains dimensionally continued Euler characteristics and also possesses a unique property that EGR can be recovered from it in four dimensions. One beautiful thing in LTG is that the associated gravitational field equations donot contain any higher order metric derivatives more than 2nd order terms and it is for this reason appears to be ghost-free theory at linear level. It is also proven that a second order LTG known as Gauss-Bonnet (GB) gravity, comes out in the low energy limit of string theory [3, 4]. Since LTG contains many number of Lovelock coefficients, due to which, it becomes very difficult to find the explicit form of solution for equations of motion. However, Bañados, Teitelboim and Zanelli gives a way for the construction of an explicit solution and introduced a suitable choice for the Lovelock coefficients. Based upon this choice of coefficients, the new theory derived from LTG is known as dimensionally continued gravity (DCG) [5]. Several solutions of DCG field equations which describe uncharged and charged black holes have been constructed in [5–7]. Furthermore, thermodynamic properties and stability analysis associated to these DCG black holes are studied in [5–11].

Since, in Lovelock gravity, the nonlinear terms of curvature invariants should need to be taken in the gravity’s Lagrangian, therefore, it is quite natural to suppose the nonlinear terms also in the matter’s Lagrangian. Therefore, when an electromagnetic field is taken as a source for Lovelock gravity, then it is more convenient to considered the nonlinear electromagnetic coupling with gravity. In this work, we want to determine nonlinearly charged black hole solutions in DCG so for doing this we are assuming the minimal coupling of non-linear electrodynamics (NLED) with DCG. The idea of NLED is important because sometimes the linear Maxwellian theory does not seem to be workable in explaining electromagnetic phenomena. Heisenberg and Euler, in 1936, proposed a nonlinear NLED model for the description of the phenomena of quantum electrodynamics [12]. Also in 1930’s, an important model of NLED was established by Born and Infeld to cancel the divergences in the electron’s self-energy, this formulation is commonly named as Born-Infeld (BI) electrodynamics [13]. In [14, 15], it has been proved that action of BI theory could be reemerged in string theory. Recently, BI formalism has also been considered for the investigations related to dark energy, holographic superconductor and holographic entanglement entropy [16–18], etc. It is also worthwhile to note that in this theory, both electric field and electric potential come out to be finite at the center of charged particle. Therefore, it would be very handy to consider BI electromagnetic field as a matter source for gravitational field for the construction of regular metric function describing black holes. The first solution in EGR coupled to BI theory was derived by Hoffmann [19], it is also proven that this
solution is devoid of essential central singularity. In addition to this, different static black hole solutions with or without cosmological constant in the presence of BI electromagnetic field sources are found [20–26]. Thermodynamical properties possessed by the BI black hole solutions have also been examined in [27, 29–31]. There also exists several other models of NLED other than BI model, for example, one can study models introduced in [32–36]. Furthermore, several interesting black holes and their properties are discussed in different modified gravities coupled to BI or to other formulations such as arcsine, exponential, logarithmic, rational NLED models, for example, one can review the contents in [37–73]. Recently, the nonlinearly electric and magnetic black holes and their thermodynamics in DCG are investigated in [74, 75]. In addition to solution involving spherical symmetry, rotating black brane in GB theory and its thermodynamical properties is also studied in the context of NLED [76]. Furthermore, black holes in the presence of dilatonic scalar field non-minimally coupled to NLED are also investigated in [77, 78]. Moreover, instead of dark energy, theory of NLED could also be used to study the period of inflation in initial time of universe [79, 80]. It should also be noted that some models of NLED were also used to depict accelerated expansion of the universe [81–86]. In \( \Lambda \)-Cold Dark Matter model, it is concluded that the cosmological constant \( \Lambda \) generates the accelerated expansion of the universe, in which the non-zero trace of the matter tensor associated to NLED will play the role of cosmological constant [87, 88].

It is familiar according to no-hair theorems that there does not exist asymptotically flat black holes in EGR conformally coupled to scalar field [89, 90]. However, for choosing cosmological constant to be zero, black holes having conformal scalar hair in \((3+1)\)-dimensional spacetime are found but then the scalar field configuration gives divergences at the horizon [91]. In case of higher dimensional spacetimes more than four, this type of gravitating objects simply do not exist [92]. When non-zero value of cosmological constant is considered, hairy black holes with regular conformal scalar field on/outside the horizon in \((2 + 1)\) and \((3 + 1)\) dimensional geometries have been determined in [93, 94]. However, until recently black hole solution in this regard were not constructed for \(d > 4\) where no-go results were reported [95]. But recently, gravity is coupled conformally to the real scalar field [96, 97], this theory is described by the following action

\[
I_S = \int d^d x \sqrt{-g} \sum_{p=0}^{n-1} \left( b_p \phi^{d-4p} \delta^{\mu_1 \ldots \mu_{2p}}_{\nu_1 \ldots \nu_{2p}} S^{\nu_1 \nu_2}_{\mu_1 \mu_2 \ldots} S^{
u_{2p-1} \nu_{2p}}_{\mu_{2p-1} \mu_{2p}} \right),
\]

(1.1)

where \( \delta^{\mu_1 \ldots \mu_{2p}}_{\nu_1 \ldots \nu_{2p}} \) is the generalized Kronecker delta and

\[
S^{\gamma \alpha}_{\mu \nu} = \phi^2 R^{\gamma \alpha}_{\mu \nu} - 2 \delta^{[\gamma}_{[\mu} \delta^{\alpha]}_{\nu]} \phi \nabla^\rho \phi - 4 \phi \delta^{[\gamma}_{[\mu} \nabla^\rho \phi \nabla^\sigma \phi - 8 \phi \delta^{[\gamma}_{[\mu} \nabla^\rho \phi \nabla^\sigma \phi.
\]

(1.2)

It appears to be most general one of gravity theories coupled with conformal scalar field. This theory is also ghost free because second order equations of motion for metric as well as for scalar field are produced in this context. For \((3 + 1)\)-dimensions or when \( p = 1 \), the action given by (1.1) gives a gravity coupled to conformal scalar field having potential
\[ V(\phi) = \frac{\lambda}{4!} \phi^4 \] where non-minimal coupling term is \((-1/12)R \phi^2\) [97]. Solutions representing Hairy black holes in this subject have been discussed in [97–101], where the scalar field configuration comes out to be analytic or regular at the horizon as well as in exterior spacetime. These black holes are considered as the first ones for \(d > 4\) whose solutions are obtained in this context. It is expected that the hairy black holes come out to be more thermodynamically stable systems as compared with the black holes having no hairs [98, 99].

The nonlinearly electric charged black hole with scalar hair and its thermal phase transitions in DCG with BI NLED source are investigated in [102]. In the similar manners, hairy DCG solutions with magnetic monopole charges are also derived in the presence of exponential NLED and power-Yang-Mills theory [75]. Therefore, on the basis of above mentioned concepts of NLED, we are highly motivated to study magnetized black holes of DCG with conformal scalar field where the source of gravity is double-Logarithmic electrodynamics, a new NLED model which is recently proposed in [36]. Black holes are considered as one of the most interesting objects which are predicted in EGR and in modified theories, for instance, LTG as well. These great gravitating objects exhibit many interestingly physical consequences. The pioneering work of Bekenstein and Hawking showed that the black hole behave like thermodynamic system whose entropy is described as the area of event horizon while its temperature is given by the the quantity known as surface gravity evaluated at the event horizon [103–107]. In 1980’s, Hawking and Page showed that there can be possible phase transitions between AdS Schwarzschild black hole and thermal AdS space, these transitions are commonly known as the Hawking–Page transitions in the literature [108]. The AdS/CFT correspondence says that Hawking–Page transition appeared as the gravitational dual for the confinement/deconfinement phase transition [109–112]. Thus, due to importance of thermodynamical aspects in black holes, we have also calculated important thermodynamic quantities for obtained black hole solutions in this paper.

The paper is arranged as follows. In coming section, we provide a brief introduction to dimensionally continued gravity minimally coupled to double-Logarithmic electrodynamics while conformally coupled to real scalar field. In this setup, the DCG gravitational field equations are solved and a new hairy black holes are investigated. In Section 3 we studied black hole thermodynamics by calculating different thermodynamic quantities in terms of event horizon and magnetic monopole charge. The validity of first law is examined and the modified Smarr’s relation is also derived. In Section 4, magnetized hairy black holes in general Lovelock gravity are briefly analyzed and thermodynamic quantities are find out. Atlast, our results are summarized with some concluding remarks in Section 5.
2 DCG and Magnetized black holes

The action corresponding to Lovelock gravity conformally coupled to scalar field in the framework of double-Logarithmic NLED is given by

\[
I = \frac{1}{16\pi G} \int d^d x \sqrt{-g} \left( \sum_{p=0}^{n-1} \frac{1}{2p} \delta^{\mu_1 \ldots \mu_{2p}}_{\nu_1 \ldots \nu_{2p}} \left( a_p F_{\mu_1 \mu_2} \cdots R_{\mu_{2p-1} \mu_{2p}}^{\nu_{2p-1} \nu_{2p}} + 16\pi G b_p \phi^{d-4p} \right) \times S_{\mu_1 \mu_2}^{\nu_{2p-1} \nu_{2p}} \right) + 4\pi G L_M(P),
\]

(2.1)

where coefficients \( a_p \) and \( b_p \) in (2.1) are arbitrary constants. Also \( G \) is the Newtonian constant, \( R_{\mu\nu}^{\alpha\beta} \) are the curvature tensor components and \( S_{\mu\nu}^{\alpha\beta} \) are the components of the 4th rank tensor given by (1.2) which transforms homogeneously under the conformal transformation such that \( g_{\mu\nu} \rightarrow \Omega^2 g_{\mu\nu}, \phi \rightarrow \Omega^{-1} \phi \) when \( S_{\mu\nu}^{\alpha\beta} \rightarrow \Omega^{-4} S_{\mu\nu}^{\alpha\beta} \). Further, \( L_M(P) \) denotes the Lagrangian density which describes the double-Logarithmic electrodynamics and is given as

\[
L_M(P) = \frac{1}{2\beta} \left( 1 - \sqrt{-2\beta P} \right) \log \left( 1 - \sqrt{-2\beta P} \right) + \left( 1 + \sqrt{-2\beta P} \right) \times \log \left( 1 + \sqrt{-2\beta P} \right),
\]

(2.2)

where \( P = F_{\mu\nu} F^{\mu\nu} = 2 \left( B^2 - E^2 \right) \), \( E \) represents the electric field, \( B \) is the magnetic field and \( F_{\mu\nu} \) denotes the Maxwell tensor which in terms of gauge potential \( A_\mu \) can be defined as \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \). The action function (2.1) describes dimensionally continued gravity when the coefficients \( a_p \) are chosen as \([5, 8]\)

\[
a_p = \binom{n-1}{p} \frac{(d-2p-1)!}{(d-2)! l^{2(n-p-1)}},
\]

(2.3)

The equations of motion corresponding to LTG can be obtained in a similar manners, i.e., under the variation of action (2.1) with respect to the metric tensor, \( g_{\mu\nu} \), we have

\[
- \sum_{p=0}^{n-1} \frac{g_p}{2p+1} \delta^{\mu \nu \lambda_1 \ldots \lambda_{2p}} \frac{\rho_1 \rho_2 \cdots \rho_{2p}}{R_{\lambda_1 \lambda_2}^{\rho_1 \rho_2} \cdots R_{\lambda_{2p-1} \lambda_{2p}}^{\rho_{2p-1} \rho_{2p}}} = 16\pi G T_{\mu\nu}^{(M)} + 16\pi G T_{\mu\nu}^{(S)},
\]

(2.4)

where \( T_{\mu\nu}^{(M)} \) are the components of matter tensor associated to double-Logarithmic electromagnetic field given by

\[
T_{\mu\nu}^{(M)} = \frac{1}{2\beta} \left( 1 - \sqrt{-2\beta P} \right) \log \left( 1 - \sqrt{-2\beta P} \right) + \left( 1 + \sqrt{-2\beta P} \right) \times \log \left( 1 + \sqrt{-2\beta P} \right) g_{\mu\nu} - \frac{2F_{\mu\lambda} F_{\nu}^{\lambda} \log \left( 1 - \sqrt{-2\beta P} \right)}{\sqrt{-2\beta P}},
\]

(2.5)

and \( T_{\mu\nu}^{(S)} \) is the matter tensor associated with scalar field defined by

\[
T_{\mu\nu}^{(S)} = \sum_{p=0}^{n-1} \frac{b_p}{2p+1} \phi^{d-4p} \delta^{\nu \lambda_1 \ldots \lambda_{2p}} \frac{\rho_1 \rho_2 \cdots \rho_{2p}}{S_{\lambda_1 \lambda_2}^{\rho_1 \rho_2} \cdots S_{\lambda_{2p-1} \lambda_{2p}}^{\rho_{2p-1} \rho_{2p}}}.
\]

(2.6)
The double-Logarithmic electromagnetic field equations are arised due to varying Eq. (2.1) with respect to gauge potential $A_{\mu}$ as

$$\partial_{\mu} \left[ \sqrt{-g} \sqrt{-2\beta P} \log \frac{1 - \sqrt{-2\beta P}}{1 + \sqrt{-2\beta P}} F_{\mu\nu} \right] = 0. \tag{2.7}$$

It can be noted that, by imposing limit $\beta \to 0$, both Lagrangian density (2.2) and the above equations reduce to the ones corresponding to Maxwell’s theory. Similarly, variation of (2.1) with respect to scalar field yields the equations of motion

$$n - 1 \sum_{p=0}^{n-1} \frac{(d - 2p)b_p}{2p} \phi^{d - 4p - 1} \delta_{\rho_1...\rho_{2p}}^{\lambda_1...\lambda_{2p}} \phi^{\rho_1...\rho_p} \phi^{\rho_{2p-1}...\rho_{2p}} = 0. \tag{2.8}$$

It can be noted from the above equation (2.8) that it makes the trace of $T^{(S)}_{\mu\nu}$ to be zero, which confirms the conformal coupling of scalar field with gravity. Since, we want to derive static spherically symmetric solution, so it is convenient to consider metric ansatz as

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Sigma^2_{(d-2)}, \tag{2.9}$$

where $d\Sigma^2_{(d-2)}$ defines the line element of $(d - 2)$-dimensional hyper-surface of constant curvature equal to $(d^2 - 5d + 6)\alpha$ such that $\alpha$ is a constant and takes the values $\alpha = 0, +1, -1$, associated to flat, spherical and hyperbolic horizon topologies, respectively. The volume of this submanifold is represented by $\Sigma_{(d-2)}^{(\alpha)}$, which for the case of spherical horizon topology yields the value $\Sigma_{(d-2)}^{(+1)} = \frac{2\pi^{(d-1)/2}}{\Gamma[(d-1)/2]}$. Now, for the construction of a solution which describes pure magnetized hairy black hole, we will take only the magnetic field contributions in maxwell’s invariant, i.e., $B \neq 0$ and $E = 0$, so that the Maxwell’s invariant can be expressed as $P = \frac{2Q^2}{r^{d-1}}$ and $Q$ represents the magnetic monopole charge. Furthermore, if the scalar configuration is taken in the form as $\phi(r) = \frac{N}{r}$, (2.10)

then this form of function $\phi(r)$ will satisfy the scalar field equations (2.8), if $N$ satisfies the equations

$$\sum_{p=1}^{n-1} pb_p \frac{(d - 1)!}{(d - 2p - 1)!} N^{2 - 2p} = 0, \tag{2.11}$$

and

$$\sum_{p=1}^{n-1} b_p \frac{(d - 1)!(d^2 - d + 4p^2)}{(d - 2p - 1)!} N^{-2p} = 0. \tag{2.12}$$

It can be clearly seen that as there is only one unknown $N$ while the number of equations are two, so it implies that one of the equation is playing the role of constraint on the conformal coupling constants $b_p$. Thus, by using the definition of coefficients $a_p$ given by
(2.3), the assumption of pure magnetic field in matter tensor components of NLED and Eqs. (2.9)-(2.12) for scalar field configuration, we can find the following solution for the gravitational field equations:

\[
f(r) = \alpha + r^2 \left[ \frac{16\pi GM}{\Sigma_{d-2}^\alpha r^{d-1}} + \frac{\delta_{d,2n-1}}{r^{d-1}} + \frac{16\pi(d-2)GH}{r^{d}} \right] - \frac{16\pi G}{\beta(d-2)(d-1)^2} \\
\times \log \left( \frac{1 + \frac{4\beta Q^2}{r_h^{d-4}}}{\sqrt{\beta(d-1)(d-2)(d-3)}} \right) \arctan \left( \frac{2Q\sqrt{\beta}}{r_h^{d-2}} \right) \\
- \frac{128\pi dQ^2G}{(d-3)(d-1)^2r_h^{2d-4}} F_1 \left( \frac{d}{2d-4}, \frac{d}{2d-4}, \frac{4\beta Q^2}{r_h^{2d-4}} \right) \right]^{\frac{1}{n-1}},
\]

(2.13)

where

\[
H = \sum_{p=0}^{n-1} b_p \frac{(d-2)!N^{d-2p}}{(d-2p-2)!},
\]

(2.14)

The constant of integration \(M\) in Eq.(2.13) is associated to mass of black hole and the additive constant \(\delta_{d,2n-1}\) is chosen such that the limit \(M \to 0\) implies the shrinking of a black hole into a single point. Fig. (1) shows the behavior of metric function (2.13), the value of radial coordinate \(r\) at which the curve intersects horizontal axis indicates the position of black hole’s horizon. It is easy to seen that for choosing \(b_p = 0\) and in the limit \(H \to 0\), the new class of topological black holes in DCG without scalar hair in the framework of double-Logarithmic electrodynamics can be obtained. The black hole solutions of DCG with Maxwell source can be regained in the weak field limit i.e. when \(\beta \to 0\). Moreover, the neutral black holes in DCG can be obtained when \(Q = 0\) and \(H \to 0\). It is worthwhile to mentioned here that this type of hairy black hole solutions given by (2.13) are possible.

![Figure 1](image.png)

**Figure 1.** Plots of metric function \(f(r)\) (Eq. (2.13)) vs \(r\) for fixed values of \(Q = 10\), \(M = 100\), \(\delta_{d,2n-1} = 5\), \(\beta = 0.1\), \(\Sigma_{d-2} = 100\), \(l = 1\) and \(H = 1\).
only for $d \geq 5$, since for the case $d = 4$, the equations Eqs. (2.11)-(2.12) are satisfied only when all the coupling constants $b_p = 0$. Hence, it can be concluded that in this theory, hairy black hole solutions are not possible to derive for $d = 4$. In general, the Ricci and Kretschmann scalars for the line element (2.9) are defined by the following expressions

$$R(r) = \left[(d-2)(d-3)\left(\frac{\alpha - f(r)}{r^2}\right) - \frac{d^2 f}{dr^2} - \frac{2(d-2)}{r} \frac{df}{dr}\right],$$

(2.15)

and

$$K(r) = \left[2(d-2)(d-3)\left(\frac{\alpha - f(r)}{r^2}\right)^2 - \left(\frac{d^2 f}{dr^2}\right)^2 + \frac{2(d-2)}{r^2} \left(\frac{df}{dr}\right)^2\right].$$

(2.16)

So, by using the metric function (2.13), one can verified that both Ricci and Kretschmann scalars diverge at the center $r = 0$, which confirms the existence of a true curvature singularity at this point and hence the resulting gravitating object is a black hole.

3 Thermodynamics of hairy black holes

Next we want to calculate thermodynamic quantities for the DCG hairy black holes which are introduced in previous section. These quantities can be computed in terms of the event horizon $r_+$ which satisfies the equation $f(r_+) = 0$. Thus, in terms of horizon’s radius $r_+$, mass of black hole can be written as

$$M = \frac{\Sigma_d}{16\pi G} \left[r_+^{d-1} \left(\frac{1}{r_+^2} + \frac{1}{r^2}\right)^{n-1} - \delta_{d,2n-1} \frac{16\pi GH(d-2)}{r_+}\right]
+ \frac{128\pi GQ^2d}{(d-3)(d-1)^2r_+^{d-3}} F_1\left[1, \frac{d-3}{2d-4}, \frac{3d-7}{2d-4}, \frac{4\beta Q^2}{r_+^{2d-4}}\right]
+ \frac{16\pi G r_+^{d-1}}{\beta (d-2)(d-1)^2}
\times \log \left(1 + \frac{4\beta Q^2}{r_+^{2d-4}}\right) - \frac{64\pi G Q^2 r_+}{\sqrt{\beta} (d-1)(d-2)(d-3)} \arctan \left(\frac{2Q\sqrt{3}}{r_+^{d-2}}\right).$$

(3.1)

The behavior of mass as a function of horizon radius is shown in Fig. (2). It is shown that for these type of hairy black hole solutions given by (2.13), there exists one or more horizons. The values of $r_+$ for which the above mass give positive values in its range corresponds to horizon for the chosen values of parameters involved in metric function. However, those values of $r_+$ which assign negative values of $M$ do not corresponds to horizon radius.

The Hawking temperature [107] of black holes described by (2.13), can be defined in terms of surface gravity $\kappa_s$ as $T_H = \kappa_s/2\pi$. Thus, using the definition of surface gravity,
The graph of Hawking temperature is plotted in Fig. (3.2) for different values of dimensionality parameter $d$. Those values of $r_+$ for which Hawking temperature is negative implies that hairy black holes with horizon radii in this region are unstable. The point at which the temperature changes sign corresponds to phase transition of type I and the region in which there exists positivity of Hawking temperature implies thermodynamic stability of black holes.

The entropy corresponding to the hairy black hole can be obtained by using Wald’s method [113, 114], this formulation implies that

$$ S = -2\pi \int d^{d-2}x \sqrt{\gamma} \frac{\partial L}{\partial R_{abcd}} \epsilon_{ab} \epsilon_{cd}, \quad (3.3) $$

in which $L$ is the Lagrangian density corresponding to gravitational field, $\gamma$ represents the determinant of the induced metric defined on the horizon and $\epsilon_{ab}$ defines the binormal to
Figure 3. Plots of temperature $T_H$ (Eq. (3.2)) vs $r_+$ for fixed values of $Q = 10$, $\beta = 0.1$, $\Sigma_{d-2} = 100$, $l = 1$ and $H = 1$.

the horizon. Thus, for the solution (2.13), entropy takes the form as

$$S = \frac{\Sigma_{d-2}}{4G} \sum_{p=1}^{n-1} \frac{p b_p (d-2)! N^{d-2p}}{(d-2p)!} + \frac{(n-1) \Sigma_{d-2} r_+^d}{4G(d-2n+2)} \left( \frac{1}{r_+^2} + \frac{1}{l^2} \right)^{n-1}$$

$$\times F_1 \left[ 1, \frac{d}{2}, \frac{d-2n+4}{2}, \frac{-r_+^2}{l^2} \right].$$

The extended first law in this scenario takes the form

$$dM = T_H dS + \Phi_Q dQ + W_\beta d\beta + \sum_{p=0}^{n-1} B^{(p)} db_p,$$  \hspace{1cm} (3.5)

where the conjugate quantity

$$\Phi_Q = \frac{8\pi r_+ \Sigma_{d-2}}{(d-3)(d-2)(d-1)^2} \left[ \frac{(d^2 - 2d - 3 + Qd - Q)r_+^{d-2}}{(r_+^{2d-4} + 4\beta Q^2)} \right] - \frac{(d-1)}{\sqrt{\beta}} \arctan \left( \frac{2Q\sqrt{\beta}}{r_+^{d-2}} \right)$$

$$+ \frac{d(d-1)}{r_+^{d-2}} F_1 \left[ 1, \frac{d-3}{2d-4}, \frac{3d-7}{2d-4}, -\frac{4\beta Q^2}{r_+^{2d-4}} \right],$$  \hspace{1cm} (3.6)

is the magnetic potential associated to magnetic charge $Q$. Similarly, the conjugate quantity
relative to NLED parameter $\beta$ is given by

$$W_\beta = \frac{\Sigma_{d-2}}{(d-2)(d-1)} r_+^2 \left[ \frac{2Q^2 \sqrt{3} r_+^2}{r_+^{d-2}} \right] (d-1) \arctan \left( \frac{2Q \sqrt{3}}{r_+^{d-2}} \right)$$

$$+ \frac{2 \sqrt{3} r_+^{d-2}}{r_+^{2d-4} + 4 \beta Q^2} \left( 7 - 3d + 2d^2 - 4 \right) - r_+^d \log \left( 1 + \frac{4 \beta Q^2}{r_+^{2d-4}} \right) - 4dQ^2 \beta F_1 \left( 1, \frac{3 - d}{4 - 2d}, \frac{7 - 3d}{4 - 2d} - \frac{4 \beta Q^2}{r_+^{2d-4}} \right).$$

(3.7)

The quantities $B^{(p)}$ conjugate to constants $b_p$ are given by

$$B^{(p)} = -\frac{4 \Sigma_{d-2}}{r_+} \sum_{p=0}^{n-1} \frac{(d-2)! N^{d-2p}}{(d-2p-2)!}.$$ 

(3.8)

Similarly, by using the above quantities, the generalized Smarr’s relation can be followed as

$$(d-3)M = (d-2)T_H S + (d-3)\Phi_Q Q + 2 \beta W_\beta d \beta + (d-2) \sum_{p=0}^{n-1} B^{(p)} b_p.$$ 

(3.9)

Since all $b_p^*$s are not independent for the present solution (2.13) due to the constraint equantions (2.11) and (2.12), thus it should be emphasized that the variations of coupling constant $b_p$ in the first law (3.5) are not all independent.

The heat capacity is defined by the relation

$$C_Q = T_H(r_+) \frac{dS}{dT_H} |_Q.$$ 

(3.10)

Differentiation of Eq. (2.15) with respect to $r_+$ gives

$$\frac{\partial T_H}{\partial r_+} = \frac{1}{4\pi} \left[ \Xi_1(r_+) + \Xi_2(r_+) + \Xi_3(r_+) + \Xi_4(r_+) + \frac{2}{\beta} \right],$$ 

(3.11)

$$\Xi_1(r_+) = \frac{2(d-1)(2-n) r_+^2}{(n-1)} - 2(d-1) l^2 + \frac{(2n-3)(d-1)}{(n-1)(r_+^2 + l^2)}$$

$$+ 16\pi G l^{2n-4} (r_+^2 + l^2)^{3-n},$$

(3.12)

$$\Xi_2(r_+) = - \frac{16\pi G l^{2n-4} (r_+^2 + l^2)^{1-n}}{(d-2)(d-1)(n-1)(r_+^{2d-4} + 4 \beta Q^2)} \left[ (8d - 16) \beta Q^2 (r_+^2 + l^2) 

- (r_+^2 + (2n-3)l^2) (r_+^{2d-4} + 4 \beta Q^2) \log \left( 1 + \frac{4 \beta Q^2}{r_+^{2d-4}} \right) \right],$$

(3.13)

$$\Xi_3(r_+) = \frac{64\pi G Q l^{2n-4} (r_+^2 + l^2)^{1-n} r_+^{2n-d-6}}{(d-3)(d-1)(d-2)(n-1) \sqrt{3}(r_+^{2d-4} + 4 \beta Q^2)} \left[ 2Q(d-2) \sqrt{3} 

+ r_+^2 ((d-3)r_+^2 + (d-2n+1)l^2) (r_+^{2d-4} + 4 \beta Q^2) \arctan \left( \frac{2Q \sqrt{3}}{r_+^{2d-4}} \right) \right],$$

(3.14)
and
\[
\Xi_4(r_+) = \frac{128\pi GQ^2 l^{2n-4} r_+^{2n-4} (d^2 - 2d - 3 - Qd + Q)}{(d-3)(d-1)^2(n-1)(l^2 + r_+^2)^{n-1}(r_+^{2d-4} + 4\beta Q^2)^2} \left[ ((2d-5)r_+^2 + l^2(2d - 2n - 1))r_+^{2d-4} - 4\beta Q^2 (r_+^2 + (2n - 3)l^2) \right] \tag{3.15}
\]

Hence, by using Eq. (3.11)-(3.15) along with the expressions of entropy and temperature in (3.10), we can get heat capacity as
\[
C_Q = \frac{(n-1)\Sigma_{d-2}(r_+^2 + l^2) r_+^{d-2n+1} \chi(r_+)}{4G l^{2n}(d-2n+2)(d-2n+4)\sqrt{\beta}(n-1)(d-1)(d-2)} \times \frac{\sqrt{\beta}(n-1)(d-1)(d-2)(r_+^2 + l^2)^{n-2}(\Delta_1 + \Delta_3) + 16\pi Gr_+^{2n-3} l^{2n-4} \Delta_2}{2r_+^2 + \Xi_1(r_+) + \Xi_2(r_+) + \Xi_3(r_+) + \Xi_4(r_+)} \tag{3.16}
\]
where
\[
\Delta_1(r_+) = \frac{(d-1)r_+^{2n-3} l^{2n-4}}{(n-1)(r_+^2 + l^2)^{n-2}} \left[ \left( \frac{1}{r_+^2} + \frac{1}{l^2} \right)^{n-1} + \frac{16\pi Gd}{(d-1)r_+^2} \right] - \frac{2}{r_+}, \tag{3.17}
\]
\[
\Delta_2(r_+) = \frac{1}{\sqrt{\beta}} \log \left( 1 + \frac{4\beta Q^2}{r_+^{2d-4}} \right) - \frac{4Q^2}{(d-3)r_+^{d-2}} \arctan \left( \frac{2Q\sqrt{\beta}}{r_+^{d-2}} \right), \tag{3.18}
\]
\[
\Delta_3(r_+) = \frac{128\pi G l^{2n-4} r_+^{2n-3} (Q(d-1) - (d-3)(d+1))}{(d-3)(d-1)^2(n-1)(r_+^2 + l^2)^{n-2}(r_+^{2d-4} + 4\beta Q^2)}, \tag{3.19}
\]
and
\[
\chi_1(r_+) = \frac{l^2(d-2n+4)(r_+^2 d + l^2(d-2n+2))}{F_1 \left[ \frac{d}{2}, \frac{d-2n+4}{2}, -\frac{r_+^2}{l^2} \right]} \left[ 2, 1 + \frac{d}{2}, 3 - n + \frac{d}{2} \right] \tag{3.20}
\]

The expression of heat capacity is significant in the sense because this quantity can play a crucial role in thermodynamic stability of black holes. Fig. (4) shows the plot of heat capacity $C_Q$ with respect to $r_+$. Note that, the positivity (negativity) of this quantity in the region indicates stability (instability) of the black hole. The values of $r_+$ for which it changes sign represents the phase transition of type I while those values at which it is infinite gives phase transition of type II. Note that, phase transitions of type II can also be analyzed from Fig. (5) because the condition $dT_H/dr_+ = 0$, implies the infiniteness and divergences of heat capacity. Hence, those values of $r_+$ at which the curve corresponding to $dT_H/dr_+$ crossed the horizontal axis determines type II transition points.
Figure 4. Plots of temperature $C_Q$ (Eq. (3.16)) vs $r_+$ for fixed values of $Q = 10$, $\beta = 0.1$, $\Sigma_{d-2} = 100$, $l = 1$ and $H = 1$.

Figure 5. Plots of $\frac{dT_H}{dr_+}$ (Eq. (3.11)) vs $r_+$ for fixed values of $Q = 10$, $\beta = 0.1$, $\Sigma_{d-2} = 100$, $l = 1$ and $H = 1$. 


4 Hairy Lovelock black holes

The polynomial equation corresponding to general hairy magnetized black holes in general LTG can be obtained as

\[ \sum_{p=0}^{n-1} \frac{(d-1)!a_p}{(d-2p-1)!} \left( \frac{\alpha - f(r)}{r^2} \right)^p = \frac{16\pi GM(d-1)}{(d-2)\Sigma_{d-2}^{(n)}} \frac{H}{r^d} + \frac{16\pi G(d-1)(d-2)H}{r^d} + \frac{64Q\pi G r^{2-d}}{\sqrt{\beta}(d-2)(d-3)} \arctan \left( \frac{2Q\sqrt{\beta}}{r^{d-2}} \right) - \frac{16\pi G \beta}{\beta(d-2)(d-1)} \log \left( 1 + \frac{4\beta Q^2}{r^{2d-4}} \right) \] (4.1)

\[ - \frac{128Q^2d\pi G}{(d-3)(d-1)r^{2d-4}} \times F_1 \left[ 1, \frac{d-3}{2d-4}, \frac{3d-7}{2d-4}, 4\beta Q^2, - \left( \frac{4\beta Q^2}{r^{2d-4}} \right)^2 \right]. \]

Note that, in the derivation of above polynomial equation we have used the scalar configuration in the form (2.10) along with the assumption of pure magnetic field. The finite mass of black hole associated to the above polynomial equation is given by

\[ M = \frac{\Sigma_{d-2}^{(n)}}{16\pi G(d-1)} \left[ \sum_{p=0}^{n-1} \frac{(d-1)!a_p\alpha^p}{(d-2p-1)!r_+^{-(d-2p-1)}} - \frac{16\pi G(d-1)(d-2)H}{r_+} \right. 
\]

\[ + \frac{16\pi G r^{d-1}}{\beta(d-2)(d-1)} \times \log \left( 1 + \frac{4\beta Q^2}{r^{2d-4}} \right) - \frac{64\pi G Q^2 r_+ \arctan \left( \frac{2Q\sqrt{\beta}}{r_+^{d-2}} \right)}{\sqrt{\beta}(d-2)(d-3)} \]

\[ + \frac{128\pi G Q^2 d}{(d-3)(d-1)r_+^{d-3}} F_1 \left[ 1, \frac{d-3}{2d-4}, \frac{3d-7}{2d-4}, 4\beta Q^2, - \left( \frac{4\beta Q^2}{r_+^{2d-4}} \right)^2 \right]. \] (4.2)

The associated Hawking temperature of Lovelock black holes governed by the polynomial equation (2.11) can be computed as

\[ T_H(r_+) = \frac{1}{4\pi Z(r_+)} \left[ \sum_{p=0}^{n-1} \frac{(d-1)!a_p\alpha^p}{(d-2p-2)!r_+^{2p+1}} + \frac{16\pi GH(d-1)(d-2)}{r_+^{d+1}} \right. 
\]

\[ + \frac{16\pi G}{(d-2)r_+^\beta} \log \left( 1 + \frac{4\beta Q^2}{r_+^{2d-4}} \right) - \frac{64\pi G Q^2 \arctan \left( \frac{2Q\sqrt{\beta}}{r_+^{d-2}} \right)}{\sqrt{\beta}(d-2)(d-3)r_+^{d-1}} \]

\[ + \frac{128\pi G Q^2 (Q(d-1) - (d + 1)(d-3))}{(d-1)(d-3)r_+(r_+^{2d-4} + 4\beta Q^2)} \right]. \] (4.3)

where \( Z(r_h) \) is defined by

\[ Z(r_h) = \sum_{p=0}^{n-1} \frac{p a_p \alpha^{p-1}(d-1)!}{(d-2p-1)! r_+^{2p}}. \] (4.4)

The Wald entropy corresponding to these black holes is given by

\[ S = \frac{(d-2)\Sigma_{d-2}^{(n)}}{4G} \sum_{p=1}^{n-1} p c p^{p-1} \left( \frac{(d-2p-1)!a_p r_+^{d-2p}}{(d-2)!} + \frac{b_p(d-3)!N^{d-2p}}{(d-2p)!} \right). \] (4.5)
And so, by using the above entropy and temperature, heat capacity at constant charge $Q$ is given by the relation

$$C_Q = \frac{\sum^{(n)}_{d-2} \sum^{n-1}_{p=1} p\alpha^{p-1}(d-2p)!a_\alpha p^d 2p^{-1}}{4G(d-1)! \left( A(r_+) + \frac{16\pi G(d-1)(d-2)}{r_+^{d+1}} + \sum^{n-1}_{p=0} \frac{(d-1)!a_\alpha p^d}{(d-2p-2)!r_+^{2p+1}} \right)^{-1}} \times \left[ \frac{dA}{dr_+} - \sum^{n-1}_{p=0} (d-1)!a_\alpha p^d (2p+1) \frac{16\pi G(d-1)(d-2)}{r_+^{d+2}} - \frac{16\pi G(d^2 - 1)(d - 2)}{r_+^{d+2}} \right]^{-1},$$

(4.6)

$$A(r_+) = \frac{16\pi G}{(d-2)r_+^d} \log \left( 1 + \frac{4\beta Q^2}{r_+^{2d-4}} \right) + \frac{128\pi GQ^2(Q(d-1) - (d+1)(d-3))}{(d-1)(d-3)r_+^{2d-4} + 4\beta Q^2}$$

$$- \frac{64\pi GQ^2}{\sqrt{\beta}(d-2)(d-3)r_+^{2d-1}} \arctan \left( \frac{2Q\sqrt{\beta}}{r_+^{d-2}} \right),$$

(4.7)

and

$$\frac{dA}{dr_+} = \frac{256\pi GQ^2}{(d-3)(d-1)r_+^{2d-4} + 4\beta Q^2} \left( (d^2 - (Q + 4)d^2 + (3Q + 2)d \right.$$}

$$+ (3 - 2Q)r_+^{2d-4} + 4(d - 3)\beta Q^2) \left. + \frac{64\pi G(d-1)Q^2}{\sqrt{\beta}(d-2)(d-3)r_+^{d}} \arctan \left( \frac{2Q\sqrt{\beta}}{r_+^{d-2}} \right) \right)$$

$$- \frac{16\pi G}{\beta r_+^{2d} (d-2)} \log \left( 1 + \frac{4\beta Q^2}{r_+^{2d-4}} \right).$$

(4.8)

5 Summary and conclusion

In this manuscript we work out for hairy black holes in DCG coupled to double-Logarithmic electrodynamics. After making set up for Lovelock gravity conformally coupled to scalar field and minimally with NLED contents, the gravitational field equations are solved in the presence of special choice of Lovelock coefficients. During this process, the metric function (2.13) which describe magnetized hairy black holes in DCG is computed. We mainly focused on pure magnetically charged black holes, since the case of electric or dyonic objects can never yields the solution in terms of elementary functions in this framework. In addition to this, thermodynamics of these hairy black holes are examined and the quantities like mass, Hawking temperature and heat capacity corresponding to the obtained hairy black holes are calculated. It is shown that these quantities are satisfying first law and the corresponding Smarr’s relation are also well defined in this case. These quantities are plotted in different dimensions and their behaviors are analyzed. The region where the heat capacity and Hawking temperature are positive (negative) is identified which implies the black hole’s thermal stability (instability). It can be clearly seen from the plots of heat capacity and Hawking temperature that both types of thermal phase transitions are
possible for the resulting black hole solutions. The phase transition of type I is associated
with that value of $r_+$ at which the heat capacity changes sign, while the type II phase
transitions of black holes corresponds to the roots of $dT_H/dr_+ = 0$, or to those values at
which the heat capacity fails to be convergent. Finally, the hairy black holes in Lovelock
gravity are also briefly investigated within the chosen model of double-Logarithmic NLED
and the associated thermodynamic quantities are calculated.

It should be noted that when $\beta \to 0$ in each case, the resulting metric function gives the
hairy black hole solution with Maxwell’s electromagnetic source. The black holes without
scalar hair can also be recovered by simply putting either $H = 0$ or $b'_s = 0$. However, for
choosing $Q$ equal to zero, the calculated metric function (2.13) implies neutral black holes
in DCG.

It would be interesting to study Hawking radiations, thermal fluctuations and grey body
factors for the hairy black holes obtained in this paper. Further, the investigation of the
causal structure and causality conditions will also give useful insights into the black hole so-
lutions obtained in this paper. In addition to this, one can also used the double-Logarithmic
electrodynamics model used in this paper for the analysis of accelerated expansion of the
universe.

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