Almost complete revivals in quantum many-body systems

Igor Ermakov\textsuperscript{1,2,3,4} and Boris V. Fine\textsuperscript{3,4}

\textsuperscript{1} Skolkovo Institute of Science and Technology, Skolkovo Innovation Center 3, Moscow 143026, Russia.
\textsuperscript{2} Department of Mathematical Methods for Quantum Technologies, Steklov Mathematical Institute of Russian Academy of Sciences, 8 Gubkina St., Moscow 119991, Russia.
\textsuperscript{3} Laboratory for the Physics of Complex Quantum Systems, Moscow Institute of Physics and Technology, Institutsky per. 9, Dolgoprudny, Moscow region, 141700, Russia. and
\textsuperscript{4} Institute for Theoretical Physics, University of Leipzig, Brüderstrasse 16, 04103 Leipzig, Germany.

(Dated: April 22, 2021)

Revivals of initial non-equilibrium states is an ever-present concern for the theory of dynamic thermalization in many-body quantum systems. Here we consider a nonintegrable lattice of interacting spins 1/2 and show how to construct a quantum state such that a given spin 1/2 is maximally polarized initially and then exhibits an almost complete recovery of the initial polarization at a predetermined moment of time. An experimental observation of such revivals may be utilized to benchmark quantum simulators with a measurement of only one local observable. We further propose to utilize these revivals for a delayed disclosure of a secret.

An important task of the dynamic thermalization research is to determine the applicability limits of the standard assumption that a nonintegrable many-body quantum system reaches thermal equilibrium. A practical way to search for these limits is to propose examples exhibiting unconventional thermalization behavior or no thermalization at all. Such examples are often based on special quantum Hamiltonians that may exhibit proximity to integrability \cite{1,2}, many-body localization \cite{3}, glassy behavior \cite{4,5}, long-range interactions \cite{6,7}, quantum scars \cite{8–15}, and driving \cite{16,17}. An alternative way to induce unconventional thermalization behavior is to use special initial conditions without putting significant restrictions on a system itself \cite{18,19}. One example of the latter kind was recently mentioned by Dymarsky \cite{19}: namely, for an isolated many-body system, where a local observable $\hat{O}$ has equilibrium expectation value $\langle \hat{O}\rangle_{\text{eq}} = 0$ but initially deviates from it, one can initiate a revival of a non-equilibrium value of $\langle \hat{O}\rangle$ at any given moment of time $\tau$ by using the initial state:

$$|\Psi(0)\rangle = \frac{|\Psi(0)\rangle + |\Psi(-\tau)\rangle}{\sqrt{2}},$$

where $|\Psi(0)\rangle$ is a many-body wave function representing a “conventional” nonequilibrium state such that $\langle \hat{O}\rangle_{\text{eq}} = 0$, $|\Psi(-\tau)\rangle$ is the state that the system should have at time $\tau = -\tau$ in order to arrive to $|\Psi(0)\rangle$ at $\tau = 0$. For the initial state $|\Psi(0)\rangle$, both the initial value $\langle \hat{O}(0)\rangle$ and the revived one $\langle \hat{O}(\tau)\rangle$ are close to $\langle \hat{O}\rangle_{\text{max}}/2$.

In the present paper, we make a step beyond the ansatz \cite{1} and show that, if $\hat{O}$ is a spin-1/2 operator, then one can construct a nonequilibrium state such that $\langle \hat{O}(0)\rangle = \langle \hat{O}\rangle_{\text{max}}$, while $\langle \hat{O}(\tau)\rangle$ is exponentially close to $\langle \hat{O}\rangle_{\text{max}}$. This behavior is to be referred to below as “almost complete revival” (ACR). We argue that ACR may serve as an efficient tool for benchmarking the performance of quantum computers or quantum simulators. We further show that ACR can be used as a method for a \textit{delayed disclosure of a secret}.

Let us now construct the ACR state for a lattice of $L$ interacting spins 1/2 described by spin operators $\{S^z_i\}$, where $i$ is the lattice index and $\alpha = x, y, z$ is the spin projection index. As a local observable we choose the $z$-projection of the spin on an arbitrary site labelled by index 1, i.e. $\hat{O} = S^z_1$. We refer to the other $L - 1$ spins as the “reservoir”.

Let us denote the bases of one-spin Hilbert spaces as $|1\rangle$ and $|0\rangle$, such that $\langle 1|S^z_1|1\rangle = 1/2$ and $\langle 0|S^z_1|0\rangle = -1/2$. We define the basis $B$ for the entire lattice as $B = B^+ \cup B^-$, where

$$B^+ = \{|1_1 1_2 \ldots 1_L\rangle, \ldots, |1_1 0_2 0_3 \ldots 0_L\rangle\},$$

$$B^- = \{|0_1 1_2 \ldots 1_L\rangle, \ldots, |0_1 0_2 0_3 \ldots 0_L\rangle\},$$

represent the subspaces with the first spin being “up” or “down” respectively. Each of the two subspaces thus has dimension $N = 2^{L-1}$. For the entire basis $B$ we also use the notation $\{|\varphi_n\rangle\}$, where $n = 1, \ldots, N$ represents the basis $B^+$ and $n = N + 1, \ldots, 2N$ the basis $B^-$. We search for the ACR state such that it initially has the form of a tensor product $|\Phi_{\text{ACR}}(0)\rangle = |1_1\rangle \otimes |\Psi_{\text{res}}\rangle$, where $|\Psi_{\text{res}}\rangle$ is the state of the reservoir. Such a state can be parameterized as

$$|\Phi_{\text{ACR}}(0)\rangle = \sum_{n=1}^{N} A_n |\varphi_n\rangle,$$

where $A_n$ are the complex amplitudes to be determined later. As follows from our indexing convention, amplitudes $A_n$ have non-zero values only for the basis states belonging to $B^+$. This choice guarantees that $\langle S^z_1(0)\rangle$ is equal to its maximum possible value $\langle S^z_1\rangle_{\text{max}} = 1/2$. 
To obtain ACR at time $\tau$, we now find such $A_n$ that $|\Phi_{\text{ACR}}(\tau)\rangle$ has the form:
$$
|\Phi_{\text{ACR}}(\tau)\rangle \equiv e^{-i\mathcal{H}\tau}|\Phi_{\text{ACR}}(0)\rangle = \sum_{n=1}^{N} C_n|\varphi_n\rangle + \delta |\varphi_{N+1}\rangle,
$$
where $\mathcal{H}$ is the Hamiltonian of the system, while $\{C_n\}$ and $\delta$ are some complex amplitudes. The principal feature of the ansatz (5) is that only state $|\varphi_{N+1}\rangle$ participates in the expansion with amplitude $\delta$, while the basis $B^-$ is fully represented by the set of amplitudes $\{C_n\}$. As we show below this leads to the ACR.

Ansatz (5) implies unambiguous prescription for finding $\{A_n\}$, $\{C_n\}$, and $\delta$. In order to do this, one needs first to define the matrix $u_{mn}$ of the time evolution operator $u \equiv e^{-i\mathcal{H}\tau}$ in the the basis $B$. Then, to make sure that only state $|\varphi_{N+1}\rangle$ from the subspace $B^-$ contributes to $|\Phi_{\text{ACR}}(\tau)\rangle$ one needs to satisfy to the following system of $N$ equations:
$$
\begin{align*}
u_{N+1,1}A_1 + \cdots + u_{N+1,N}A_N &= \delta, \\
u_{N+2,1}A_1 + \cdots + u_{N+2,N}A_N &= 0, \\
\vdots \\
u_{N,N+1}A_1 + \cdots + u_{N,N}A_N &= 0.
\end{align*}
$$
From this system one can find $N$ variables $\{A_n\}$ as a function of yet unknown parameter $\delta$, and then find $\delta$ by normalizing $\{A_n\}$. Finally, one can substitute the result into the system of equations
$$
\begin{align*}
u_{1,1}A_1 + \cdots + u_{1,N}A_N &= C_1, \\
u_{2,1}A_1 + \cdots + u_{2,N}A_N &= C_2, \\
\vdots \\
u_{N,1}A_1 + \cdots + u_{N,N}A_N &= C_N,
\end{align*}
$$
thereby obtaining the set of amplitudes $\{C_n\}$.

The central result of the present work is that the above prescription implies ACR, because, generically, the values $|\delta|$ and all $|C_n|$ are of the order of $1/\sqrt{N}$, and, as a result,
$$
\langle S_1^z(\tau) \rangle = \frac{1}{2} \left( \sum_{i=1}^{N} |C_i|^2 - |\delta|^2 \right) = 1/2 - O(1/N),
$$
which, in turn, means that the revived $\langle S_1^z(\tau) \rangle$ is exponentially close to $\langle S_1^z \rangle_{\text{max}}$.

While the estimate $|C_n| \sim 1/\sqrt{N}$ in the above construction is by no means surprising, the generic validity of $\delta \sim 1/\sqrt{N}$ requires a justification. Our justification is based mainly on the direct numerical solution of system (6) but also it is supported by the following analytical argument.

The argument is based on the assumption that, in a generic nonintegrable system, the time evolution operator $u$ for sufficiently large times $\tau$ is similar to a random rotation in the $2N$-dimensional Hilbert space. The matrix $u_{mn}$ can then be viewed as being composed of a set of $2N$ normalized vectors $\{u_{1n}\}$, $\{u_{2n}\}$, etc., where the typical matrix element has absolute value $|u_{mn}| \sim 1/\sqrt{2N}$ and largely random phase. The system of equations (6) involves only half of the components of each vector $\{u_{N+1,n}\}$, $\{u_{N+2,n}\}$, etc. It implies that the $N$-dimensional vector $\{A_n^\star\}$ must be orthogonal to $N-1$ “half-vectors” $\{u_{N+2,n}\}$, ..., $\{u_{2,n}\}$, while the value of $\delta$ is the projection of the half-vector $\{u_{N+1,n}\}$ onto the direction of $\{A_n^\star\}$. If the half-vector $\{u_{N+1,n}\}$ were uncorrelated with “half-vectors” $\{u_{N+2,n}\}$, ..., $\{u_{2,n}\}$, then it should also be uncorrelated with $\{A_n^\star\}$, which means that the relative orientation of $\{u_{N+1,n}\}$ and $\{A_n^\star\}$ is random and thus the left-hand-side of the first equation in system (6) can be estimated as $A_0u_0/\sqrt{N}$, where $A_0 \sim 1/\sqrt{N}$ and $u_0 \sim 1/\sqrt{2N}$ are the rms values of $A_n$ and $u_{mn}$ respectively. Such an estimate indeed gives $|\delta| \sim 1/\sqrt{N}$. The same kind of estimate can also be used for each line of the system (7), which would give $|C_n| \sim 1/\sqrt{N}$.

If the above assumptions were fully correct, they would imply that, once the rms values of $|C_n|$ and $|\delta|$ averaged over different not-too-small $\tau$ were exactly equal to each other. However, our numerical tests show that, even though both $C_n$ and $\delta$ are indeed of the order $1/\sqrt{N}$, there is a systematic difference between them, which is, presumably, related to subtle correlations between $u_{mn}$, for which we have no explanation.

**Example of ACR**

Let us now construct ACR for a translationally invariant periodic chain of $L$ spins $1/2$ described by the Hamiltonian:
mean value

Equilibrium fluctuations. Another remarkable observation is that the Hamiltonian (9) is far from integrability as evidenced by the energy level-spacing statistics [20][21].

An example of the ACR behavior for the observable \( S_z^i \) in a 12-spin chain is presented in Fig. 1. The initial state in this case was obtained by solving the system of equations (6) for the revival time \( \tau = 10 \), and, indeed, the expected revival at \( t = \tau \) was observed.

In the same figure, we compare the ACR behavior with the one of a fully polarized spin in a “random reservoir” associated with the initial state \( |\Phi_{in}\rangle = |1\rangle \otimes |\Psi_{int}\rangle \), where \( |\Psi_{int}\rangle \) is sampled from the infinite temperature ensemble for the remaining \( L-1 \) spins [22][23]. As seen in Fig. 1 the value of \( \langle S_z^i(t) \rangle \) in the case of random reservoir quickly relaxes to zero as expected for the infinite temperature equilibrium. We note that the \( \langle S_z^i(t) \rangle \) for the ACR state and for the random reservoir state nearly coincide over an extended initial time interval, yet the former evolves to exhibit a revival at time \( t = \tau \), while the latter shows just featureless equilibrium fluctuations. Another remarkable observation is that the almost complete revival around \( t = \tau \) has the character of a nearly complete time reversal, even though the time reversal as such was not explicitly targeted by the procedure based on system (6). We note in this regard that, as shown in the inset of Fig. 1, the fidelity of the many-body wave function \( |\langle \Phi_{ACR}(0) |\Phi_{ACR}(t) \rangle| \) does not exhibit a revival at \( t = \tau \). Yet, the observed time-reversed behavior of \( \langle S_z^i(t) \rangle \) during ACR is consistent with the statistical argument of Ref.[24] that the most likely behavior of strong fluctuations is that of a time-reversed relaxation.

The dependence of the revived value of \( \langle S_z^i \rangle \) on the revival time \( \tau \) and on the size of the lattice is illustrated in Fig. 2(a): different points of the plot \( \langle S_z^i(\tau) \rangle \) are obtained from different initial states \( |\Phi_{ACR}(0)\rangle \) computed for the fixed time \( \tau \) with the help of Eqs.(6). There one can observe that, for smaller systems, the revived values of \( \langle S_z^i \rangle \) exhibit stronger fluctuations as a function of \( \tau \). However, the amplitude of these fluctuations rapidly decreases with the system size \( L \). To quantify this decrease, we further observe that the fluctuation amplitudes for all system sizes \( L \) are already stationary for \( \tau > 5 \), which allows us to characterize the typical fluctuations of \( \langle S_z^i(\tau) \rangle \) by a \( \tau \)-averaged quantity \( 1/2 - \langle \langle S_z^i \rangle \rangle_{\tau} \), where

\[
\langle \langle S_z^i \rangle \rangle_{\tau} = \frac{1}{\tau_1 - \tau_0} \int_{\tau_0}^{\tau_1} \langle S_z^i(\tau) \rangle d\tau
\]

with \( \tau_0 = 5 \) and \( \tau_1 = 30 \). The dependence of \( \langle \langle S_z^i \rangle \rangle_{\tau} \) on the system size is plotted in Fig. 2(b). Finally, in Fig. 2(c), we present the semi-logarithmic plot of the \( \tau \)-averaged value \( \langle |\delta|^2 \rangle_{\tau} = 1/2 - \langle \langle S_z^i \rangle \rangle_{\tau} \) as a function of \( L \). This is an important plot because it shows that, for sufficiently large revival times \( \tau \), the typical value of \( \delta \) decreases exponentially with the system size.

Benchmarking quantum simulators.

One possible application of ACR is to benchmark the performance of engineered many-qubit systems, such as quantum computers or quantum simulators. The observation of ACR amounts to a comprehensive test of quantum coherence and quantum control of the system. The larger the revival time \( \tau \), the more stringent is the test and the greater is the coverage of the many-qubit Hilbert space probed by the wave function in the course of the dynamical evolution. In particular, for nonintegrable systems, one can hope that the time evolution of the many-qubit wavefunction before ACR would amount to a reasonably fair sampling of the system’s Hilbert space. We further note that the observation of ACR of only one qubit for sufficiently large \( \tau \) indicates that the overlap between the desired initial many-qubit state and the experimentally prepared one is close to 1.

Delayed disclosure of a secret

Imagine that one needs to share a piece of valuable information in the form of a string of \( K \) classical bits. However, the information must not be disclosed to anyone before a certain moment of time in the future. Below we propose a scheme, which allows one to implement such a delayed disclosure of a secret with the help of the ACR states.

Let us first consider only one classical bit. The state of this bit is to be encoded as \( S_z^i = \pm 1/2 \) for a given spin 1/2.
(a qubit), interacting with a finite “reservoir” of 10-50 other spins 1/2. The state of the reservoir $|\Psi_{\text{res}}\rangle$ is to be prepared using the solutions of the system of equations (6) such that $\langle S^z_1(t) \rangle$ exhibits a revival at $t = \tau$ (see Fig. 3). After the quantum evolution is launched, there are two possible scenarios: Either one measures $S^z_1$ at $t = \tau$ and thereby obtains the encoded bit value with probability close to one, or the measurement is performed at a wrong time (or someone has interfered with the evolution of the system) and, therefore, the measured value of $S^z_1$ is, most likely, uncorrelated with the encoded one.

If one were to be transmitting only one bit of information, then one realization of the above procedure would not be sufficient: the occurrence of ACR on a single spin 1/2 would need to be verified either by repeating the procedure several times, or by running it simultaneously for several identical groups of spins 1/2. In this regard, the need to transmit a larger number of bits makes the verification of ACR more efficient: namely, one only needs to transmit two copies of the string of $K$ bits. If $K$ is sufficiently large and the two recovered strings are identical, then this indicates that the information was transmitted as intended.

In conclusion, we have shown how to generate an almost complete recovery of a fully polarized state of a given spin 1/2 belonging to a larger lattice of interacting spins 1/2. We have discussed possible applications of ACR to the benchmarking of quantum simulators and also proposed to utilize ACR for a delayed disclosure of a secret.

The authors are grateful to O. Lychkovskiy for useful discussions. The work of I. E. was funded by the Ministry of Science and Higher Education of the Russian Federation (grant number 075-15-2020-788)

---

[1] Toshiya Kinoshita, Trevor Wenger, and David S Weiss, “A quantum newton’s cradle,” Nature 440, 900–903 (2006).
[2] Mari Carmen Bañuls, J Ignacio Cirac, and Matthew B Hastings, “Strong and weak thermalization of infinite non-integrable quantum systems,” Physical review letters 106, 050405 (2011).
[3] Rahul Nandkishore and David A Huse, “Many-body localization and thermalization in quantum statistical mechanics,” Annu. Rev. Condens. Matter Phys. 6, 15–38 (2015).
[4] Louk Rademaker and Dmitry A Abanin, “Slow nonthermalizing dynamics in a quantum spin glass,” Physical Review Letters 125, 260404 (2020).
[5] Daniel L Stein and Charles M Newman, Spin glasses and complexity, Vol. 4 (Princeton University Press, 2013).
[6] Zhe-Xuan Gong and Lu-Ming Duan, “Prethermalization and dynamic phase transition in an isolated trapped ion spin chain,” New Journal of Physics 15, 113051 (2013).
[7] Brian Neyenhuis, Jiehang Zhang, Paul W Hess, Jacob Smith, Aaron C Lee, Phil Richerme, Zhe-Xuan Gong, Alexey V Gorshkov, and Christopher Monroe, “Observation of prethermalization in long-range interacting spin chains,” Science advances 3, e1706672 (2017).
[8] Christopher J Turner, Alexios A Michailidis, Dmitry A Abanin, Maksym Serbyn, and Zlatko Papić, “Weak ergodicity breaking from quantum many-body scars,” Nature Physics 14, 745–749 (2018).
[9] Hannes Bernien, Sylvain Schwartz, Alexander Keesling, Harry Levine, Ahmed Omran, Hannes Pichler, Soonwon Choi, Alexander S Zibrov, Manuel Endres, Markus Greiner, et al., “Probing many-body dynamics on a 51-atom quantum simulator,” Nature 551, 579–584 (2017).
[10] Cheng-Ju Lin and Olexei I. Motrunich, “Exact quantum many-body scar states in the rydberg-blockaded atom chain,” Phys. Rev. Lett. 122, 173401 (2019).
[11] Thomas Fädecola and Michael Schecter, “Quantum many-body scar states with emergent kinetic constraints and finite-entanglement revivals,” Physical Review B 101, 024306 (2020).
[12] Sambuddha Chattopadhyay, Hannes Pichler, Mikhail D Lukin, and Wen Wei Ho, “Quantum many-body scars from virtual entangled pairs,” Physical Review B 101, 174308 (2020).
[13] Hongzheng Zhao, Joseph Vovrosh, Florian Mintert, and Johannes Knolle, “Quantum many-body scars in optical lattices,” Physical Review Letters 124, 160604 (2020).
[14] Kyungmin Lee, Ronald Melendrez, Arijit Pal, and Hitesh J Changlani, “Exact three-colored quantum scars from geometric frustration,” Physical Review B 101, 241111 (2020).
[15] AA Michailidis, CJ Turner, Z Papić, DA Abanin, and Maksym Serbyn, “Stabilizing two-dimensional quantum scars by deformation and synchronization,” Physical Review Research 2, 022065 (2020).
[16] Kai Ji and Boris VFine, “Nonthermal statistics in isolated quantum spin clusters after a series of perturbations,” Physical review letters 107, 050401 (2011).
[17] Kai Ji and Boris VFine, “Suppression of heating in quantum spin clusters under periodic driving as a dynamic localization effect,” Physical review letters 121, 050602 (2018).
[18] Sheldon Goldstein, Takashi Hara, and Hal Tasaki, “Time scales in the approach to equilibrium of macroscopic quantum systems,” Physical review letters 111, 140401 (2013).
[19] Anatoly Dymarsky, “Mechanism of macroscopic equilibration of isolated quantum systems,” Physical Review B 99, 224302 (2019).
[20] YY Atas, E Bogomolny, O Giraud, and G Roux, “Distribution of the ratio of consecutive level spacings in random matrix ensembles,” Physical review letters 110, 084101 (2013).
[21] The $r$-value defined in Ref. [20] is equal to $r_{L=12} = 0.5474$, while, for the Gaussian Orthogonal Ensemble, it is $r_{\text{GOE}} = 0.5359$, and for the Poisson Ensemble, it is $r_p = 0.3862$.
[22] Jochen Gemmer, Mathias Michel, and Günter Mahler, Quantum thermodynamics: Emergence of thermodynamic behavior within composite quantum systems, Vol. 784 (Springer, 2009).
[23] Boris V Fine, “Typical state of an isolated quantum system with fixed energy and unrestricted participation of eigenstates,” Physical Review E 80, 051130 (2009).
[24] MI Dykman and IB Schwartz, “Large rare fluctuations in systems with delayed dissipation,” Physical Review E 86, 031145 (2012).