Tunneling without tunneling: wavefunction reduction in a mesoscopic qubit

James A Nesteroff and Dmitri V Averin

Department of Physics and Astronomy, Stony Brook University, SUNY, Stony Brook, NY 11794-3800, USA
E-mail: dmitri.averin@stonybrook.edu

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Abstract
The transformation cycle and associated inequality are suggested for the study of basic properties of the wavefunction reduction in a mesoscopic qubit in measurements with quantum-limited detectors. Violation of the inequality would demonstrate directly that the qubit state changes in a way dictated by the probabilistic nature of the wavefunction and inconsistent with the dynamics of the Schrödinger equation: the qubit tunnels through an infinitely large barrier. Estimates show that the transformation cycle is within the reach of current experiments with superconducting qubits.

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1. Introduction: wavefunction reduction versus dynamic evolution in mesoscopic qubits

Can a quantum particle tunnel through a barrier which has vanishing transparency? The immediate answer to this question is ‘no’, as follows from the elementary properties of the Schrödinger equation. More careful consideration should of course remind us that evolution according to the Schrödinger equation is not the only way for a state of a quantum particle to change in time. The probabilistic nature of the wavefunction implies that it can also evolve due to the ‘wavefunction reduction’: a random process of realization of one specific outcome of a measurement. As suggested by its name, this process was originally envisaged as suppressing completely the quantum correlations described by the wavefunction. It was understood later, however (see e.g. [1–3]) that this is not true in the case of the ‘morally best’ [2, 3] or, in more modern and descriptive language, quantum-limited measurements, in which the wavefunction of the measured system changes coherently for any given outcome of the measurement. These changes can contradict the Schrödinger equation despite the fact that the dynamics of the measurement process as a whole is governed by this equation. They can be described formally as generic ‘quantum operations’ within the approach based on positive operator-valued measures (POVM) [4]. All ‘counterintuitive’ quantum-mechanical phenomena arise from such an evolution of the wavefunction in the measurement process. The best known example is given by the Einstein–Podolsky–Rosen (EPR) correlations [5], which violate the principle of ‘no action-at-a-distance’ as quantified by the Bell inequalities [6]. From the perspective of the wavefunction reduction, the EPR correlations appear for a specific random outcome of the local spin measurement. On average, the measurement does not create action-at-a-distance in a sense that the correlations do not violate the relativistic causality.

In different contexts, specific phenomena related to the processes of wavefunction reduction take form depending on the nature of the underlying physical system, e.g. ‘quantum jumps’ in optics [7]. In mesoscopic solid-state qubits, proposed and/or observed manifestations of the wavefunction reduction include violations of the ‘temporal’ Bell inequalities [8–11], which characterize quantum uncertainty in the qubit state in the process of coherent quantum oscillations between the two-qubit basis states; measurements of the ‘weak values’ of the operators, the outcomes of which can lie outside the limits of the operator eigenvalues [12–14]; stochastic reversal of the wavefunction reduction process [15]. The purpose of this work is to suggest a sequence of quantum transformations and the corresponding Bell-type inequality, which illustrates directly a contradiction between the wavefunction reduction
and dynamic evolution according to the Schrödinger equation by violating the very basic intuition based on this equation that the charge or magnetic flux cannot tunnel through an infinitely large barrier. For mesoscopic solid-state structures with their small geometric dimensions, violation of this intuition would provide, arguably, a more dramatic illustration of the wavefunction reduction than the non-locality of conventional Bell inequalities also discussed for mesoscopic structures—see e.g. [16–18]. A brief description of the suggested transformation cycle was given in [19].

2. Transformation cycle

The system we consider consists of a single qubit with the two basis states $|j\rangle$, $j=0,1$, distinguished by the average values of some quantity $x$, for instance, electric charge or magnetic flux [20–23] in the case of superconducting qubits. As the simplest example, $x$ can be viewed as a position of an individual elementary particle (electron in coupled quantum dots [24–26], Cooper pair on a superconducting island [20, 21], FQHE quasi-particle in a system of two quantum antidots [27, 28], or ultracold atom in a BEC junction [29, 30]), which can be localized on the opposite sides of a tunnel barrier separating the qubit basis states $|j\rangle$. In general, the barrier has finite transparency characterized by a tunnel amplitude $\Delta > 0$ between the states $|j\rangle$. The qubit is coupled through the coordinate $x$ to a detector (as illustrated in figure 1), which converts the information about $x$ into the classical output $q$. The main characteristics of the measurement process which depend, in particular, on the strength of the qubit–detector coupling and are relevant for the discussion below, are the probabilities $w_j(q)$ of obtaining the detector output $q$, when the qubit is in the state $|j\rangle$. For instance, in the example of a charge-based qubit, the conceptually simplest detector is the quantum point contact (QPC) (see e.g. [31] and references therein), in which the qubit controls the scattering characteristics of electrons in the contact, and therefore the current $I$ flowing through it (figure 1). In this case, the output $q$ is the total charge transferred through the contact during the time $\tau$ of the measurement.

The goal of the transformation cycle developed in this work is to demonstrate that the evolution of the wavefunction in the measurement process is 'real' to the same extent as the dynamics of the wavefunction governed by the Schrödinger equation, i.e. it describes the evolution of the physical quantities and not only information about them, even if the specific form of this evolution is qualitatively different. This is accomplished by combining the two types of evolution in one cycle, which can be arranged so that they completely compensate each other, ideally leaving the initial qubit state unchanged. The first part of the cycle is the partial wavefunction reduction in weak quantum-limited measurement, which coherently changes the amplitudes $c_j$, ‘mimicking’ the tunneling between the qubit states $|j\rangle$ in the situation when the corresponding tunneling amplitude is vanishing, $\Delta = 0$. The second part is the regular tunneling between the states $|j\rangle$ with non-vanishing $\Delta \neq 0$. The fact that no charge or flux is transferred through the tunneling barrier in the whole cycle means that the wavefunction reduction induces tunneling even without the corresponding tunneling amplitude.

The starting point of the cycle is a qubit with vanishing average bias $\epsilon = 0$ between the two basis states $|j\rangle$. The Hamiltonian of such a qubit is

$$H = -\frac{1}{2}[v\sigma_z + \Delta\sigma_x],$$

where $\sigma_x, \sigma_z$ are the Pauli matrices, and the bias $v$ represents the low-frequency noise characteristic for mesoscopic solid-state qubits [32–36]. Due to assumed weak, but unavoidable, relaxation processes and low temperature $T \ll \Delta$, the qubit is in the instantaneous ground state of (1) before the cycle. The initial qubit density matrix $\rho_i$ in the $\sigma_z$ basis is then:

$$\rho_i = \left(\begin{array}{cc} c_0^2 & c_0 c_1^* \\ c_0^* c_1 & c_1^2 \end{array}\right),$$

where the probability amplitudes $c_j$ are $c_{0,1} = [(1 \pm v/\Omega^2)/2]^{1/2}$ and $\Omega = (v^2 + \Delta^2)^{1/2}$. One needs to consider the density matrix dependent on the specific noise realizations $v$ and average over them only at the end, since some of the transformations of the discussed cycle (see e.g. equations (3) and (4) below) are nonlinear in the matrix elements of the density matrix. For $\Delta \gg v_0$, where $v_0$ is the rms magnitude of noise $v$, the amplitudes $c_j$ are $c_{0,1} = 1/\sqrt{2}$, and the state (2) reduces to the ideal noiseless version of initial state for our transformation cycle, the eigenstate $|\sigma_z = 1\rangle$ of the $\sigma_z$ operator. In principle, the effect of noise on the initial qubit state can also be eliminated if the state is prepared by strong projective measurement of $\sigma_z$ and selection of the $\sigma_z = 1$ result. However, as will be shown in section 3, optimization of the cycle as a whole (and not just the initial state) for better stability against the noise can require keeping the ratio $\Delta/v_0$ finite, which reduces the dephasing effect of the noise.

The cycle begins by rapidly raising the tunneling barrier so that the tunnel amplitude vanishes, $\Delta \rightarrow 0$, while the qubit wavefunction remains distributed between the two basis states $|j\rangle$. In the presence of non-vanishing noise, the rise time $\tau$ of the signal suppressing $\Delta$ needs to be short on the characteristic scale of the noise amplitude, $\tau \ll h/v_0$. In this case, the qubit remains in its initial state (2) even at the end of the transition process, when $\Delta = 0$.

The second step of the cycle is a weak measurement of the $\sigma_z$ operator performed on the qubit state (2). The cycle in its ideal form requires the detector performing the measurement to be quantum-limited. Qualitatively, this means that the fluctuations underlying the probability distributions $w_j(q)$ of the detector output are themselves quantum, so that the evolution of the detector+qubit system leading to any given value $q$ of the output is quantum coherent (see e.g. [37]). The simplest example of the detector that under some natural conditions can be quantum-limited is the QPC electrometer [31, 38] (figure 1(b)). For the QPC in this regime, the distribution of the transferred charge $q$ is created by the shot noise that is generated by the quantum-coherent scattering of electrons at the contact. Quantum evolution of the detector implies that while observation of a particular outcome $q$ yields some information about the state of the qubit, the qubit retains its coherence in this process. The amount of information gained by the measurement depends
on the ratio of the probabilities \( w_j(q) \) for the observed output \( q \) (as illustrated in figure 1(a)), and the condition of quantum coherence mandates then the following evolution of the initial qubit amplitudes \( c_j \) into the output-dependent values \( c_j(q) \) [38, 39]:

\[
c_j(q) = \frac{c_j\sqrt{w_j(q)}}{\sqrt{w_0(q)|c_0|^2 + w_1(q)|c_1|^2}}.
\]

This evolution of the qubit amplitudes is caused by the information acquisition in the measurement process as reflected in the probability-amplitude nature of the wavefunction. Probabilistic transformation (3) is clearly different from the dynamic evolution governed by the Schrödinger equation. In particular, in the setup considered in this work, it implies the redistribution of the wavefunction amplitudes \( c_j \) through the barrier with \( \Delta = 0 \).

In the more realistic case of the detector that is not strictly quantum-limited, time evolution of the qubit+detector system is not completely coherent. This means that such a non-ideal detector induces partial dephasing of the qubit state during the measurement process even under the condition of a given specific detector output \( q \). Physically, the origin of this dephasing is the loss of information in the measurement process. In the example of the QPC detector, the non-ideality can be caused by finite detector temperature \( T \), which creates the scattering events for electrons incident at the contact from both opposite directions. Since only the total transferred charge \( q \) is reflected in the detector output, the information about the source of the scattered electron is lost in the output \( q \) (i.e. more than one pattern of the scattering events correspond to the same value of the transferred charge \( q \)). The detector non-ideality of this type can be described in general by the suppression factor \( e^{-v} \) of the non-diagonal elements of the qubit density matrix \( \rho \) in the measurement basis. Taking this into account, one can write the density matrix after weak \( \sigma_z \) measurement as

\[
\rho_m = \left( \begin{array}{cc}
c_0^2(q) & c_0(q)c_1(q)e^{-\eta-\text{i}p_1(t)} \\
c_0(q)c_1(q)e^{-\eta+\text{i}p_1(t)} & c_1^2(q)
\end{array} \right).
\]

The factor \( e^{\text{i}p_1(t)} \) in the off-diagonal elements of this matrix represents the phase accumulation by the qubit states because of the bias noise \( v \) acting on the qubit during the measurement time \( \tau \):

\[
\psi(\tau) = \frac{1}{\hbar} \int_0^\tau v(t) \, dt.
\]

Besides the suppression factor \( e^{-v} \), a non-ideal detector can also induce a similar, in general \( q \)-dependent, phase difference between the qubit states. In contrast to the noise-induced phase, however, such a phase is set by the detector characteristics. Therefore, in measurements distinguishing individual outcomes \( q \), as discussed in this work, it can be compensated for and does not produce then an extra dephasing.

The aim of the next step of the cycle is to reverse the changes in the qubit amplitudes \( c_j \) due to ‘tunneling’ through barrier with vanishing transparency in the wavefunction reduction process (3) by regular tunneling according to the Schrödinger equation. This is done by creating a non-vanishing tunneling amplitude for some appropriate period of time, i.e. realizing a fraction of the regular coherent oscillations in which the charge or flux goes back and forth between the qubit basis states. In the situation with no disturbances (vanishing noise and quantum-limited detector), this can be done precisely, returning the qubit from the state

\[
|\psi_m\rangle = \sqrt{w_0(q)}|0\rangle + \sqrt{w_1(q)}|1\rangle
\]

obtained as a result of the measurement, to the initial state \( |\psi_i\rangle = |0\rangle + |1\rangle/\sqrt{2} \) before the measurement. If the amplitudes of \( |\psi_m\rangle \) are written in the form usual for the spin-1/2 as \( \cos(\theta/2) \) and \( \sin(\theta/2) \), both states can be represented in the spin-1/2 diagrams shown in figure 2. One can see from figure 2(a) that the transformation reversing directly the wavefunction reduction (3) is the rotation about the \( y \)-axis. For mesoscopic qubits, such a rotation corresponds to tunneling with complex amplitude \( \Delta' \) which has a \( \pi/2 \)-phase: \( \arg \Delta' = \pi/2 \). The rotation angle depends on the detector output \( q \), which determines the degree of localization of the qubit state in one of the basis state according to equation (3). Quantitatively, figure 2(a) shows that the rotation angle required to reverse the wavefunction reduction is

\[
\int |\Delta'(t)| \, dt/h = \pi/2 - \theta, \quad \theta = 2 \tan^{-1}\left[ \sqrt{w_1(q)/w_0(q)} \right].
\]

Typically, the qubit structure allows only for the real tunnel amplitude \( \Delta \)—see the Hamiltonian (1), which realizes the \( x \)-axis rotations \( R_x = \exp[\text{i} \sigma_x \Delta(t) \, dt/2\hbar] \).
of the qubit. In this case, the y-axis rotation $R_y = \exp[-i\sigma_y \Delta(t)] dt/2\hbar]$ defined by equation (6) can be simulated directly by the x-rotation of the same magnitude (6), if it is preceded and followed by the rotations around the z-axis: $R_z = R_z^{-1} R_x R_z$. The z-rotations $R_z^{\pm \epsilon} = \exp[\pm i\sigma_z \epsilon/4]$ are created by the pulses of the qubit bias: $\int \epsilon(t) dt/h = \pm \epsilon/2$. Such a three-step sequence can be simplified into two steps (figure 2(b)) by interchanging the order and magnitude of rotations: first, the x-rotation by $\pi/2$, then one z-rotation:

$$\int (\Delta(t) dt/h = \pi/2, \int \epsilon(t) dt/h = \pi/2 - \theta. \quad (7)$$

Here $\theta$ is given by the same expression as in equation (6).

Under the experimentally realistic assumption that the measurement time $\tau$ is much longer than the time duration of the control pulses, the noise-induced distortions of the qubit states will be dominated by the phase accumulation during the measurement that was taken into account in equation (4). In this case, the qubit density matrix $\rho = U \rho_0 U^\dagger$ after the unitary transformation $U = R_z(\pi/2 - \theta) R_x(\pi/2)$ effected by the pulses (7) has the following matrix elements:

$$\rho_{11} = \frac{1}{2} + c_0(q) c_1(q) e^{-\gamma} \sin \varphi(t),$$

$$\rho_{12} = \frac{c_1(q) - c_0(q)}{2} i c_0(q) c_1(q) e^{-\gamma} \cos \varphi(t) e^{-i\theta}. \quad (8)$$

The rest of the matrix elements are defined by the usual conditions on the density matrix: $\rho_{22} = 1 - \rho_{11}$ and $\rho_{12} = \rho_{21}^\dagger$.

As the last step in the transformation cycle, one needs to check that the qubit is brought back to the ideal form of the initial state $\sigma_z = 1$. This is done directly by performing strong projective measurement of $\sigma_z$, and obtaining, upon sufficiently large number of cycles, the error probability $p$ of the qubit reaching the wrong state $\sigma_z = -1$. This probability can be calculated as

$$p = Tr[PP] = \frac{1}{2} \begin{pmatrix} 1 & 1 \\
0 & 1 \end{pmatrix}. \quad (9)$$

where $P$ is the projection operator on the $\sigma_z = -1$ state. If $p = 0$, i.e. steps (II) and (III) of the cycle return the qubit with certainty to the state $\sigma_z = 1$, the cycle is closed. This means that the shift of the qubit amplitudes due to the state reduction process with $\Delta = 0$ is precisely compensated for by a fraction of a period of coherent qubit oscillations with $\Delta \neq 0$ that reverses this shift. Since coherent qubit oscillations actually transfer the charge or flux between the two-qubit basis states, the fact that the cycle as a whole is closed shows that the changes in the qubit state caused by the wavefunction reduction cannot be interpreted only as the changes in our knowledge of the probabilities of the state of the qubit, but rather involve actual transfer of charge or flux in the absence of the tunneling amplitude.

3. Analysis of imperfections: Bell-type inequality

In the presence of finite noise and detector non-idealities, the error probability $p$ is non-vanishing, $p \neq 0$. Still, the probability $1 - p$ with which the cycle is closed can be incompatible in this case with the classical description of the measurement process, demonstrating that the actual qubit evolution in measurement is governed by the counterintuitive wavefunction reduction (3) which contradicts the dynamics of the Schrödinger equation. The alternative classical description would be based on the assumption that before the measurement (e.g. during the switching off of the tunneling amplitude $\Delta$) some unspecified process localizes the qubit in one of the basis states on one or the other side of the tunnel barrier. This means that at the beginning of the measurement, the qubit is in the classical state that is an incoherent mixture of the basis states with some undetermined probability $r$ for the qubit to be in the state $|0\rangle$ i.e. has the density matrix, $\rho^{(cl)}_r$:

$$\rho_r^{(cl)} = r|0\rangle\langle 0| + (1 - r)|1\rangle\langle 1|. \quad (10)$$

In this classical alternative to the wavefunction reduction (3), the qubit state is ‘objectively’ well defined and coincides with one of the basis states. It is, however, unknown, and the measurement provides information about this unknown state, i.e. changes the probabilities $r$ and $1 - r$. Quantitatively, the changes in the classical probabilities due to information acquisition depend on the observed outcome $q$ through the probabilities $w_j(q)$:

$$r \rightarrow r(q) = rw_0(q)/(|rw_0(q) + (1 - r)w_1(q)|).$$

Applying the same transformations as to the state (2) to the classical incoherent mixture of the basis states with the probabilities $r(q)$ and $1 - r(q)$, one obtains the probability $p^{(cl)}(q)$ for the cycle to end in the wrong state $|\sigma_z = -1\rangle$ given the measurement outcome $q$:

$$p^{(cl)}(q) = \frac{w_0(q)w_1(q)}{|w_0(q) + w_1(q)| (rw_0(q) + (1 - r)w_1(q))}. \quad (11)$$

Since the final error probability $p^{(cl)}$ is found by repeating the cycle many times, one needs to average equation (11) over different detector outcomes $q$ and over the bias noise $v$ which, in principle, can affect the initial probability $r$. For classical initial state (10), the probability $\sigma(q)$ of obtaining the detector outcome $q$ in many cycles is

$$\sigma(q) = rw_0(q) + (1 - r)w_2(q).$$

The average of $p^{(cl)}(q)$ over all possible outcomes is then

$$p^{(cl)} = \int \sigma(q) p^{(cl)}(q) dq = \int dq \frac{w_0(q)w_1(q)}{w_0(q) + w_1(q)}. \quad (12)$$

Equation (12) shows that the probability $p^{(cl)}$ for the cycle with classical measurement to end in the wrong state is independent of the assumed initial probability $r$. This means, in particular, that the noise cannot change this result. The probability (12) is also independent of the degree of detector non-ideality as the off-diagonal elements of the initial state (10) vanish in the classical case.

To see the incompatibility of the classical description of the transformation cycle with the actual qubit evolution, one needs to compare equation (12) with the error probability $p$ for the quantum version of the cycle based on the wavefunction reduction (3). Substituting equations (8) into
the definition of the error probability (9), we obtain (before the average over the detector outcome and noise):

\[ p = \frac{1}{2} - \frac{c_j^2(q) - c_j(q)}{2} \cos \theta - c_0(q) c_1(q) e^{-i \sin \theta \cos \varphi} \tau. \]  

(13)

Using equation (5) for the phase \( \varphi(\tau) \), equation (3) for the amplitudes \( c_j(q) \) with the initial values \( c_j \), and the definition (6) of the angle \( \theta \) which implies that

\[ \cos \theta = \frac{w_0(q) - w_1(q)}{w_0(q) + w_1(q)}, \quad \sin \theta = 2 \sqrt{w_0(q) w_1(q)} \frac{w_0(q) - w_1(q)}{w_0(q) + w_1(q)}. \]

we reduce the average over the detector outcome and noise of the error probability (13) to a simple form

\[ p = p^{(c)} \left[ 1 - e^{-\eta} \left( \Delta \left( \frac{\cos \int_0^\tau v(t) \, dt}{h} \right) \right) \right]. \]  

(14)

Here \( \langle \cdots \rangle \) denotes the average over the noise \( v(t) \).

One can see from equation (14) that the quantum error probability \( p \) of the cycle is not closed is always smaller than the classical probability \( p^{(c)} \), but the relative difference between the two, \( 1 - p/p^{(c)} \) is suppressed by increasing detector non-ideality and noise amplitude. From the definition (12) of the probability \( p^{(c)} \), one can also see that this probability characterizes the measurement strength, with \( p^{(c)} \to 0 \) in the limit of strong projective measurement. Indeed, since the projective measurement provides definite information about the qubit state \( |j\rangle \), there should be no overlap of the two probability distributions \( w_j(q) \) of the detector output \( q \) in this case, i.e. \( w_0(q) w_1(q) = 0 \). This means that \( p^{(c)} = 0 \) for projective measurement, and weak measurements are essential for observation of the non-trivial features of the wavefunction reduction (3). Otherwise the absolute difference between \( p^{(c)} \) and \( p \) vanishes together with these probabilities themselves, and the state obtained as a result of the transformation cycle can be accounted for by the classical description of the measurement. In the limit of very weak measurements, the two probability distributions \( w_{0,1}(q) \) nearly coincide giving \( p^{(c)} = 1/2 \).

Equation (14) means that if the noise and the detector non-ideality are not very large, it should be possible to observe the error probability \( p \) that is smaller than the value explainable by the classical model of the measurement process:

\[ p < p^{(c)}. \]  

(15)

For any finite measurement strength, i.e. when the distributions \( w_0(q) \) and \( w_1(q) \) are not identical and there is a measurement actually performed, observation of this inequality would mean that a larger number of the transformation cycles are closed than can be explained classically, without invoking the redistribution of the probability amplitudes \( c_j \) across the infinite barrier in the process of wavefunction reduction (3). Combined with the non-vanishing transfer of charge or flux during the ‘oscillation’ step (6) or (7) of the cycle, this implies that the wavefunction reduction induces similar transfer across the tunnel barrier separating the qubit basis states even if the corresponding tunnel amplitude is zero.

As the last part of this work, we make the limitations on the noise and detector characteristics more quantitative. The simplest general model of the detector output is given by the Gaussian probability distributions \( w_{0,1}(q) \) of the same width \( \sigma \) shifted by some amount, \( 2a\delta q \), along the \( q \)-axis: \( w_{0,1}(q) = (2\pi \sigma)^{-1/2} e^{-[(q - \delta q)/2\sigma]^2} \). This model describes, for instance, the QPC detector (figure 1(b)) weakly coupled to the qubit [31]. In this case, the average currents \( I_j \) in the QPC for the qubit state \( |j\rangle \) are slightly different, \( \delta I = (I_1 - I_0)/2 \ll I_j = (I_1 + I_0)/2 \), so that \( \delta q = \delta I \tau \). Because of the weak coupling, the noticeable charge difference \( \delta q \) is created only for sufficiently large measurement times \( \tau \), when the binomial statistics of the charge transfer through the QPC reduces effectively to the Gaussian distribution. The width of this distribution is determined by the intensity of the short noise \( S_I \) of the QPC, \( \sigma = S_I \tau \). In the general Gaussian case, the probability \( p^{(c)} \) is

\[ p^{(c)} = \frac{e^{-\eta / 4 \pi^2}}{\sqrt{\pi \tau}} \int e^{-\xi^2 / 4\tau^2} \cos \frac{\Delta}{\xi} \left( \int_0^\tau v(t) \, dt \right) \, d\xi. \]  

(16)

where \( \eta = (\delta q)^2 / 2\sigma \) is the information about the qubit state obtained as a result of the measurement, i.e. the degree of certainty with which these states can be distinguished based on the detector output. Equation (16) shows how the classical error probability decreases with the information \( s \) which describes the transition from weak (\( s \ll 1 \)) to strong (\( s \gg 1 \)) measurements. Similar transition exists for other detector models as well—for example [40]. Another quantitative characteristics of the detector is its non-ideality. It is defined by the ratio \( \eta / s \) of the degree \( \eta \) of suppression of the qubit coherence by the detector during the measurement process to the information \( s \) gained in it.

Next, we quantify the low-frequency noise \( v(t) \) acting on the qubit. The noise can be modeled with good accuracy as a classical Gaussian random variable with the high-frequency cutoff \( \omega_0 \) that is low on the frequency scale of the control pulses. The longer measurement time \( \tau \) can still be comparable to \( \omega_0^{-1} \). Although the noise characteristics do not have a qualitative effect on our conclusions, for completeness, we consider two opposite limits of the noise dynamic properties.

First, if \( \tau \omega_0 \gg 1 \), and the noise spectral density \( S(\omega) \) is such that \( \omega_0 \) represents characteristic noise frequencies, one can treat the noise values at the beginning of the cycle and during the measurement as uncorrelated, splitting the noise average in equation (14) into two.

\[ \left\langle \frac{\Delta}{(v^2 + \Delta^2)^{1/2}} \cos \int_0^\tau v(t) \, dt \right\rangle = \frac{\Delta}{(v^2 + \Delta^2)^{1/2}} \left\langle \cos \int_0^\tau v(t) \, dt \right\rangle. \]

The first part gives:

\[ \left\langle \frac{\Delta}{(v^2 + \Delta^2)^{1/2}} \right\rangle = \frac{1}{\sqrt{2\pi \omega_0}} \int d\omega e^{-\omega^2/2\omega_0^2} \frac{\Delta}{(v^2 + \Delta^2)^{1/2}} \]

\[ = \frac{2\Delta}{\pi} e^{\omega_0^2 / 2 \omega_0^2} K_0(\omega_0) \]

\[ = \left\{ \begin{array}{ll} 1, & \Delta \gg \omega_0, \\ (2/\pi)^{1/2}(\Delta / \omega_0)[\ln(\omega_0 / \Delta) + \ln 2 - \gamma / 2], & \Delta \ll \omega_0. \end{array} \right. \]  

(17)
where $K_0$ is the modified Bessel function, $a \equiv \Delta^2/(4v_0^2)$ and $\gamma \simeq 0.577$ is Euler’s constant. Using the standard approach to averaging over the Gaussian noise (see e.g. [33, 36]), one obtains the second, dephasing part of the noise average as:

$$\left\{ \cos \int_0^\tau v(t) \, dt / h \right\} = \exp \left\{- \frac{1}{\pi \hbar^2} \int d\omega S(\omega) \frac{\sin^2(\omega \tau/2)}{\omega^2} \right\}.$$  \hspace{1cm} (18)

For large time $\tau$, this result reduces to $e^{-\gamma \tau}$ with $\gamma = S(0)/2\hbar^2$. If the spectral density of noise diverges at zero frequency, $S(0)$ should be roughly taken at the frequency $1/\tau$.

For $\tau_0 \ll 1$, the noise is static during the measurement time $\tau$, and the noise average in equation (14), with the phase $\varphi(\tau) = v\tau/h$ for the static noise, can be written as

$$F \equiv \frac{1}{\sqrt{2\pi \tau v_0}} \int dv e^{-v^2/2v_0^2} \frac{\Delta \cos(v\tau/h)}{(v^2 + \Delta^2)^{1/2}}.$$  \hspace{1cm} (19)

In terms of $F$, the error probability $p$ is

$$p = p^{(\text{cl})}[1 - e^{-\gamma F}],$$  \hspace{1cm} (20)

i.e. the factor $F$ shows how strongly the difference between the classical description of the measurement process and the quantum wavefunction reduction is suppressed by noise. For $F \to 1$ and ideal detector ($\eta \to 0$), the errors of the quantum cycle are suppressed, $p \to 1$, regardless of the measurement strength characterized by $p^{(\text{cl})}$.

Equation (19) can be evaluated analytically in several limits. If the noise-induced phase $\varphi \equiv v\tau/h$ is eliminated by a spin echo technique (as, in principle, can be done for the static noise), or if $v_0\tau/h \ll 1$ because of the short measurement time, the factor $F$ is given by equation (17). In the opposite limit, for large initial values of the qubit tunnel amplitude, $\Delta \gg v_0$, the noise effect on the initial state disappears, and equation (19) reduces for arbitrary $v_0\tau/h$ to

$$F = e^{-\Delta^2 t^2/2\hbar^2},$$  \hspace{1cm} (21)

as follows from (18) for $\tau_0 \ll 1$. In the case of $\Delta \ll v_0$ and non-vanishing $\tau$, one can neglect the exponential factor in the integral (19) to obtain

$$F = \left( \frac{2}{\pi} \right)^{1/2} \frac{\Delta}{v_0} K_0(t)$$

$$= \Delta \left\{ \frac{(2/\pi)^{1/2} \ln(2t) - \gamma}{(1/\pi)^{1/2} e^{-t}}, \right\} \quad t \ll 1,$$

$$= \frac{\Delta}{v_0} \{ \left(1/t\right)^{1/2} e^{-t}, \right\} \quad t \gg 1,$$

where $t = \Delta \tau/h$. The $t \ll 1$ part of this equation cannot be extended directly to the point $\tau = 0$. It coincides, however, with the $\tau = 0$ result (17) for $\Delta \ll v_0$, when $\tau \simeq h/v_0$.

In general, $F$ can be calculated numerically from equation (19). The results are shown in figure 3. One can see that both the larger noise $v_0$, and longer measurement time $\tau$ in the presence of the finite noise, suppress the difference between quantum and classical error probabilities. Figure 3 shows also that, in agreement with equations (17) and (21), the noise-induced dephasing (21), which is controlled by the time $\tau$, results in a stronger suppression of $F$ than the noise effect on the initial qubit state (2) for $\tau = 0$ described by equation (17). This means, in particular, that in order to obtain larger values of $F$, i.e. to maximize the difference between the quantum and classical error probabilities, one should chose smaller values of the $\Delta/v_0$ ratio. Although the reduction of $\Delta$ makes the noise effect on the initial state more pronounced, at the same, it effectively suppresses decoherence: the instances with the larger values of the noise which would result in a stronger decoherence become irrelevant because the qubit state for such strong noise is already localized in one of the $\sigma_z$ eigenstates and does not contribute to the wavefunction reduction process.

Both the strength of the qubit coupling to the detector, which determines the measurement time $\tau$, and coupling to the noise which determines the magnitude of the qubit random bias $v_0$ depend on the same physical qubit parameters, e.g. for flux qubits, the difference between the average flux in the two $\sigma_z$ eigenstates. This means that optimization of the qubit parameters for maximum $F$ should be done for fixed values of the $v_0\tau/h$ product which represents a more fundamental quality of the qubit–detector system. Figure 4 shows $F$ as a function of the $\Delta/v_0$ ratio for several values of $v_0\tau/h$. One can see that for sufficiently strong noise, $v_0\tau/h > 2$, the $\Delta$-dependence of $F$ is non-monotonic, with the maximum at small $\Delta/v_0$ ratios. This maximum represents the suppression of dephasing by small values of $\Delta$ discussed above. In the large-$\Delta$ limit, the curves in figure 4 saturate at $F$ given by equation (21), which is exponentially small for strong noise. As discussed above, large realizations of the noise are effectively removed by small $\Delta$, lifting this exponential suppression of coherence. On the other hand, since the probability of obtaining useful noise realizations scales as $\Delta/v_0$, the factor $F$ is also suppressed for $\Delta \to 0$. This gives the peak of $F$ at small $\Delta/v_0$ seen in figure 4 and described analytically by equation (22) if $v_0\tau/h \gg 1$. This maximum of $F$ may be useful in experimental realization of the wavefunction reduction cycle discussed in this work.
4. Discussion and estimates

In summary, we have proposed a transformation cycle as a very basic demonstration of the two main qualitative features of the wavefunction reduction process in quantum measurements using mesoscopic solid-state qubits. One is the fact that the wavefunction evolves in the reduction process in a way that explicitly contradicts the dynamics of the Schrödinger equation: a particle can be transferred through an infinitely large barrier. The other feature is that, to the same extent as the Schrödinger equation, the wavefunction reduction affects not only the probability distributions of a dynamic variable (e.g. electric charge or magnetic flux), but the physical variable itself. Of course, different forms of the wavefunction evolution in the reduction process imply that the particle transfer through the infinite barrier cannot be interpreted directly as regular Schrödinger-equation tunneling. For instance, the operator of tunneling current through the barrier in the Schrödinger equation is proportional to the tunneling amplitude $\Delta$ and vanishes for $\Delta = 0$. Still, as demonstrated by the cycle described in this work, the charge or flux can be transferred through such a barrier.

Experimental realization of this cycle and observation of the inequality (15) is not very far from the possibilities of the current experiments with superconducting, in particular flux, qubits. Indeed, figure 4 shows that by adjusting the initial value of the tunneling amplitude $\Delta$, one can obtain a noticeable difference between the classical and quantum error probabilities even for the cycle times $\tau$ on the order of $\sim3$ dephasing times (21) $\hbar/v_0$. The dephasing times for the qubit states that differ by the average flux or charge values are about 5 ns—see e.g. [23, 41]. This value would be improved by using the spin echo technique that in principle can be directly incorporated into the transformation cycle. Without this, one has roughly 15 ns time interval for performing three operations of the cycle during which the qubit is subject to dephasing: measurement and two compensating pulses (7). In superconducting qubits of all types, simple control pulses are regularly performed on the few-nanosecond timescale, and can be fit into this time interval. The new requirement presented by the transformation cycle described in this work is the need to perform a variant of the feed-back control, when the applied pulses depend on the result of the previous qubit measurement. Rapid non-destructive qubit read-out necessary for this task apparently still needs to be developed. A possible way of addressing this problem would be to use the circuits of the classical superconductor electronics adopted to qubit control as described, e.g. in [39, 42].

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References

[1] Lüders G 1951 Ann. Phys., Lpz. 8 322
[2] Goldberger M L and Watson K M 1964 Phys. Rev. 134 B919
[3] Bell J S and Nauenberg M 1966 Preludes in Theoretical Physics (North Holland: Amsterdam) p 279
[4] Peres A 1993 Quantum Theory (Dordrecht: Kluwer)
[5] Einstein A, Podolsky B and Rosen N 1935 Phys. Rev. 47 777
[6] Bell J S 1965 Physics 1 195
[7] Pleinio M B and Knight P L 1998 Rev. Mod. Phys. 70 101
[8] Leggett A J and Garg A 1985 Phys. Rev. Lett. 54 857
[9] Korotkov A N and Averin D V 2001 Phys. Rev. B 64 165310
[10] Ruskov R, Korotkov A N and Mizel A 2006 Phys. Rev. Lett. 96 200404
[11] Jordan A N, Korotkov A N and Buttiker M 2006 Phys. Rev. Lett. 97 026805
[12] Aharonov Y, Albert D Z and Vaidman L 1988 Phys. Rev. Lett. 60 1351
[13] Williams N S and Jordan A N 2008 Phys. Rev. Lett. 100 026804
[14] Romito A, Gefen Y and Blanter Y M 2008 Phys. Rev. Lett. 100 056801
[15] Katz N et al 2008 Phys. Rev. Lett. 101 200401
[16] Chetshkatchev N M, Blatter G, Lesovik G B and Martin Th 2002 Phys. Rev. B 66 161320
[17] Beenakker C W J, Emary C, Kindermann M and van Velsen J L 2003 Phys. Rev. Lett. 91 147901
[18] Samue1sson P, Sukhurov E V and Büttiker M 2004 Phys. Rev. Lett. 92 026805
[19] Nesteroff J A and Averin D V 2009 arXiv:0909.4343
[20] Nakamura Y, Pashkin Yu A and Tsai J S 1999 Nature 398 786
[21] Pashkin Yu A, Yamamoto T, Astafiev O, Nakamura Y, Averin D V and Tsai J S 2003 Nature 421 823
[22] Friedman J R, Patel V, Chen W, Tolpygo S K and Lukens J E 2000 Nature 406 43
[23] Plantenberg J H, de Groot P C, Harmans C J P M and Mooij J E 2007 Nature 447 836
[24] Elzerman J M, Hanson K, Greidanus J S, van Beveren L H W, De Franceschi S, Vandersypen L M K and Kouwenhoven L P 2003 Phys. Rev. B 67 161308
[25] Hayashi T, Fujisawa T, Cheong H D, Jeong Y H and Hirayama Y 2003 Phys. Rev. Lett. 91 226804
[26] Buttiker M 2005 Phys. Rev. B 72 081310
[27] Maasila I J and Goldman V J 2000 Phys. Rev. Lett. 84 1776
[28] Averin D V and Goldman V J 2002 Solid State Commun. 121 25
[29] Fölling S, Trotzky S, Cheinet P, Feld M, Saers R, Widera A, Miller T and Bloch I 2007 *Nature* **448** 1029
[30] Averin D V, Bergeman T, Hosur P R and Bruder C 2008 *Phys. Rev.* A **78** R031601
[31] Averin D V and Sukhorukov E V 2005 *Phys. Rev. Lett.* **95** 126803
[32] Falci G, D’Arrigo A, Mastellone A and Paladino E 2005 *Phys. Rev. Lett.* **94** 167002
[33] Ithier G *et al* 2005 *Phys. Rev.* B **72** 134519
[34] Koch R H, DiVincenzo D P and Clarke J 2007 *Phys. Rev. Lett.* **98** 267003
[35] Harris R *et al* 2008 *Phys. Rev. Lett.* **101** 117003
[36] Bennett D A, Longobardi L, Patel V, Chen W, Averin D V and Lukens J E 2009 *Quantum Inf. Process.* **8** 217
[37] Averin D V 2003 *Quantum Noise in Mesoscopic Physics* ed Yu V Nazarov (Dordrecht: Kluwer) p 229
[38] Averin D V 2003 arXiv:cond-mat/0301524
[39] Korotkov A N 1999 *Phys. Rev.* B **60** 5737
[40] Averin D V, Rabenstein K and Semenov V K 2006 *Phys. Rev.* B **73** 094504
[41] Ashhab S, You J Q and Nori F 2009 *Phys. Rev.* A **79** 032317
[42] Duty T, Gunnarsson D, Bladh K and Delsing P 2004 *Phys. Rev.* B **69** 140503
[43] Kaplunenko V K and Ustinov A V 2004 *Europhys. J.* B **38** 3