Lorentz-violating Euler-Heisenberg effective action

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(Dated: January 3, 2014)

Abstract

In this work, we study the radiative generation of the Lorentz-violating Euler-Heisenberg action, in the weak field approximation. For this, we first consider a nonperturbative calculation in the coefficient $c_{\mu\nu}$, however, by assuming rotational invariance. Within this approach, we also obtain the results of the amplitude for the photon triple splitting. In the following, we take into account the perturbative approach, where $c_{\mu\nu}$ is treated as an insertion in the propagator and a new vertex. The partial results are in fact an expansion up to first order in $c_{\mu\nu}$ of the nonperturbative ones, with $c_{00} = \kappa$ and $c_{0i} = c_{ij} = 0$. This suggests that the complete results obtained in the nonperturbative approach can be used in both treatments.

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I. INTRODUCTION

The Euler-Heisenberg effective action \([1, 2]\) has been widely investigated in the last decades, in various contexts (see \([3]\) and references therein). Nevertheless, studies related to the Lorentz-symmetry violation \([4–6]\) have been considered so far in the context of photon splitting \([7–12]\). The effective theory used in these analyses was mainly the Lorentz-violating QED \([13]\), a subset of the minimal standard-model extension \([4, 5]\). For an overview and references on this extended QED, see the review \([14]\) and the incomplete list of works \([15–22]\).

In this paper, we are interested in studying the radiative generation of the Lorentz-violating Euler-Heisenberg action, from the CPT-even derivative term \(i\bar{\psi}c_{\mu\nu}\gamma^\mu(\partial^\nu + ieA^\nu)\psi\) of the Lorentz-violating QED extension. The CPT-odd term \(\bar{\psi}b_{\mu}\gamma^\mu\gamma_5\psi\) also generates nonlinear corrections to the Maxwell theory, however, these calculations will be presented in a next work. With the exact expressions for these results we will be able to calculate specific scattering amplitudes, e.g., the photon scattering in the electromagnetic field of a nucleus (Delbrück scattering) \([23, 24]\) and the photon splitting in a strong magnetic field \([25–28]\), in order to numerically estimate the coefficients for the Lorentz violation.

Recently it has been argued that the photon-photon scattering (also a process calculated from the results of the Euler-Heisenberg action) can be observed at the Large Hadron Collider \([29]\) as an opportunity to discuss the noncommutative interactions \([30]\), among others, and therefore also the Lorentz violation effects.

The structure of the paper is as follows. In Sec. II we consider a nonperturbative calculation in the coefficient \(c_{\mu\nu}\), by assuming rotational invariance in which \(c_{\mu\nu}\) is reduced to the product of timelike vectors. We also recover the results of the amplitude for vacuum photon splitting, previously obtained in \([9]\). In Sec. III we take into account the perturbative approach, where the coefficient \(c_{\mu\nu}\) is treated as an insertion in the propagator as well as a new vertex. Sec. IV contains a summary of our results.

II. NONPERTURBATIVE APPROACH

The starting fermion Lagrangian that we are interested is given by

\[
\mathcal{L}_f = \bar{\psi}(i\tilde{\partial}_\mu \gamma^\mu - m - e\tilde{A}_\mu \gamma^\mu)\psi,
\]

where we have written \(\tilde{\partial}_\mu = (g_{\mu\nu} + c_{\mu\nu})\partial^\nu\) and \(\tilde{A}_\mu = (g_{\mu\nu} + c_{\mu\nu})A^\nu\). As we have mentioned above, in our nonperturbative approach the coefficient \(c_{\mu\nu}\) is reduced to the product of two unit timelike
vectors, i.e., $c_{\mu\nu} = \kappa u_\mu u_\nu$, where $u_\mu = (1, 0, 0, 0)$ and $\kappa$ is the coefficient that determine the scale of Lorentz violation. Hence, the corresponding Feynman rules are the fermion propagator,

$$\langle p | = \frac{i}{\not{p} - m},$$

and the fermion-photon vertex,

$$\langle \gamma^\mu | = -ie\gamma^\mu,$$

with $\not{p} = \not{p}_u \gamma^\mu$ and $\not{p}_u = ((1 + \kappa)p_0, p_i)$. Thus, the resulting effective action takes the form

$$S^{(4)}_{\text{eff}} = \frac{1}{4} \int d^4x \int d^4k_1 d^4k_2 d^4k_3 d^4k_4 e^{i (k_1 + k_2 + k_3 + k_4) \cdot x} \frac{1}{6} G^{\mu_1 \mu_2 \mu_3 \mu_4}(k_1, k_2, k_3, k_4) \tilde{A}_{\mu_1}(k_1) \tilde{A}_{\mu_2}(k_2) \tilde{A}_{\mu_3}(k_3) \tilde{A}_{\mu_4}(k_4),$$

where

$$G^{\mu_1 \mu_2 \mu_3 \mu_4}(k_1, k_2, k_3, k_4) = 2 T_1^{\mu_1 \mu_2 \mu_3 \mu_4}(k_1, k_2, k_3, k_4) + 2 T_2^{\mu_1 \mu_2 \mu_3 \mu_4}(k_1, k_2, k_3, k_4) + 2 T_3^{\mu_1 \mu_2 \mu_3 \mu_4}(k_1, k_2, k_3, k_4),$$

with

$$T_1^{\mu_1 \mu_2 \mu_3 \mu_4}(k_1, k_2, k_3, k_4) = i T_1^{\mu_1 \mu_2 \mu_3 \mu_4}(k_1, k_2, k_3, k_4)$$

$$= -(-ie)^4 \int \frac{d^4p}{(2\pi)^4} \text{tr} \frac{i}{\not{p} - m} \frac{i}{\not{p}_1 - m} \frac{i}{\not{p}_2 - m} \frac{i}{\not{p}_3 - m} \gamma^{\mu_4},$$

$$T_2^{\mu_1 \mu_2 \mu_3 \mu_4}(k_1, k_2, k_3, k_4)$$

$$= i T_2^{\mu_1 \mu_2 \mu_3 \mu_4}(k_1, k_2, k_3, k_4)$$

$$= -(-ie)^4 \int \frac{d^4p}{(2\pi)^4} \text{tr} \frac{i}{\not{p} - m} \frac{i}{\not{p}_2 - m} \frac{i}{\not{p}_3 - m} \frac{i}{\not{p}_123 - m} \gamma^{\mu_4},$$

$$T_3^{\mu_1 \mu_2 \mu_3 \mu_4}(k_1, k_2, k_3, k_4)$$

$$= i T_3^{\mu_1 \mu_2 \mu_3 \mu_4}(k_1, k_2, k_3, k_4)$$

$$= -(-ie)^4 \int \frac{d^4p}{(2\pi)^4} \text{tr} \frac{i}{\not{p} - m} \frac{i}{\not{p}_34 - m} \frac{i}{\not{p}_24 - m} \frac{i}{\not{p}_234 - m} \gamma^{\mu_1}. $$
In the above expressions, $\tilde{A}_\mu(k) = ((1 + \kappa)A_0(k), A_i(k))$, the global factor 2 in Eq. (5) means the two orientations of the fermion loop, and $p'_\nu = p' + k'_\nu$, $p_1'' = p' + k_1'' + k_2''$, and so on.

Here, we observe that the contributions (7) and (8) can be obtained from (6), when we perform the interchanges:

\begin{align}
T_{2}^{\mu_1\mu_2\mu_3\mu_4}(k_1, k_2, k_3, k_4) &= T_{1}^{\mu_2\mu_1\mu_3\mu_4}(k_2, k_1, k_3, k_4), \quad (9a) \\
T_{3}^{\mu_1\mu_2\mu_3\mu_4}(k_1, k_2, k_3, k_4) &= T_{1}^{\mu_2\mu_1\mu_3\mu_4}(k_1, k_2, k_3, k_4). \quad (9b)
\end{align}

Therefore, we just need to calculate the expression (9), in which, by using first the Feynman parameterization, we arrive at the expression

\begin{equation}
T_{1}^{\mu_1\mu_2\mu_3\mu_4}(k_1, k_2, k_3, k_4) = \int \frac{d^4 p}{(2\pi)^4} \frac{6ie^4}{(p^2 - M^2)^4} \times \text{tr}[g_{(12)}m^4(g_{12} + m)\gamma_\mu(g_{12} + m)\gamma_\nu(g_{12} + m)\gamma_\mu], \quad (10)
\end{equation}

where

\begin{equation}
M^2 = m^2 - x_1(1 - x_1)\tilde{k}_1^2 - x_{12}(1 - x_{12})\tilde{k}_2^2 - x_{123}(1 - x_{123})\tilde{k}_3^2
\end{equation}

\begin{equation}
-2x_1(1 - x_{12})\tilde{k}_1 \cdot \tilde{k}_2 - 2x_1(1 - x_{123})\tilde{k}_1 \cdot \tilde{k}_3 - 2x_{12}(1 - x_{123})\tilde{k}_2 \cdot \tilde{k}_3, \quad (11)
\end{equation}

and

\begin{align}
q_\mu &= p_\mu - (1 - x_1)k_1^\mu - (1 - x_{12})k_2^\mu - (1 - x_{123})k_3^\mu, \\
q_1' &= p_1' - (1 - x_1)k_1'^\mu - (1 - x_{12})k_2'^\mu - (1 - x_{123})k_3'^\mu, \quad (12) \\
q_1'' &= p_1'' - (1 - x_1)k_1''^\mu - (1 - x_{12})k_2''^\mu - (1 - x_{123})k_3''^\mu, \quad (13)
\end{align}

with $x_{12} = x_1 + x_2$, $x_{123} = x_1 + x_2 + x_3$, and $I = 1, 12, 123$.

In order to calculate the momentum integrals in (10), we will use the fact that

\begin{equation}
\int \frac{d^4 p}{(2\pi)^4} \frac{p_\nu_1 p_\nu_2 \cdots p_\nu_\alpha}{(p^2 - M^2)^\alpha} = (1 + \kappa)^{-1} \int \frac{d^4 \tilde{p}}{(2\pi)^4} \frac{\tilde{p}_\nu_1 \tilde{p}_\nu_2 \cdots \tilde{p}_\nu_\alpha}{(\tilde{p}^2 - M^2)^\alpha} \\
= (1 + \kappa)^{-1} \int \frac{d^4 \tilde{p}}{(2\pi)^4} \frac{\tilde{p}_\nu_1 \tilde{p}_\nu_2 \cdots \tilde{p}_\nu_\alpha}{(\tilde{p}^2 - M^2)^\alpha}, \quad (14)
\end{equation}

where $d^4 \tilde{p} = dp_0 dp_1 dp_2 dp_3$ and $dp_0 = (1 + \kappa)dp_0$. Consequently, the effect of Lorentz violation will be concentrated in the overall factor $(1 + \kappa)^{-1}$ and into the mass parameter $M$. Thus, by calculating the trace over Dirac matrices and the integrals, we obtain, up to order $1/m^4$, the following results:

\begin{equation}
T_{1}^{\mu_1\mu_2\mu_3\mu_4}(k_1, k_2, k_3, k_4) = T_{1g}^{\mu_1\mu_2\mu_3\mu_4}(k_1, k_2, k_3, k_4) + \sum_{i=1}^{4} T_{1g_i}^{\mu_1\mu_2\mu_3\mu_4}(k_1, k_2, k_3, k_4)
\end{equation}

\begin{equation}
+ \sum_{i=1}^{12} T_{1g_{ik}}^{\mu_1\mu_2\mu_3\mu_4}(k_1, k_2, k_3, k_4) + \sum_{i=1}^{9} T_{1kk_i}^{\mu_1\mu_2\mu_3\mu_4}(k_1, k_2, k_3, k_4), \quad (15)
\end{equation}
where

\[
T_{1ge}^{\mu_1\mu_2\mu_3\mu_4} = - \left[ \frac{e^4}{6\pi^2} - \frac{e^4}{12\pi^2} \ln \left( \frac{m^2}{\mu^2} \right) \right] (1 + \kappa)^{-1} 
\times \left( g^{\mu_1\mu_2} g^{\mu_3\mu_4} - 2 g^{\mu_1\mu_3} g^{\mu_2\mu_4} + g^{\mu_1\mu_4} g^{\mu_2\mu_3} \right),
\]

(16)

with \( \epsilon = 4 - D \) and \( \mu^2 = 4\pi\mu^2 e^{-\gamma - i\pi} \), and

\[
T_{1gg_1}^{\mu_1\mu_2\mu_3\mu_4} = - \frac{e^4}{240m^2\pi^2} (1 + \kappa)^{-1} 
\times \left( g^{\mu_1\mu_2} g^{\mu_3\mu_4}(13\tilde{k}_1^2 + 9\tilde{k}_1 \cdot \tilde{k}_2 + 17\tilde{k}_1 \cdot \tilde{k}_3 + 4\tilde{k}_2^2 - \tilde{k}_2 \cdot \tilde{k}_3 + 8\tilde{k}_3^2) 
+ g^{\mu_1\mu_3} g^{\mu_2\mu_4}(-17\tilde{k}_1^2 - 16\tilde{k}_1 \cdot \tilde{k}_2 - 18\tilde{k}_1 \cdot \tilde{k}_3 - 16\tilde{k}_2^2 - 16\tilde{k}_2 \cdot \tilde{k}_3 - 17\tilde{k}_3^2) 
+ g^{\mu_1\mu_4} g^{\mu_2\mu_3}(8\tilde{k}_1^2 - \tilde{k}_1 \cdot \tilde{k}_2 + 17\tilde{k}_1 \cdot \tilde{k}_3 + 4\tilde{k}_2^2 + 9\tilde{k}_2 \cdot \tilde{k}_3 + 13\tilde{k}_3^2) \right),
\]

(17a)

\[
T_{1gg_2}^{\mu_1\mu_2\mu_3\mu_4} = - \frac{e^4}{5040m^4\pi^2} g^{\mu_1\mu_2} g^{\mu_3\mu_4}(1 + \kappa)^{-1} \left[ 48\tilde{k}_1^4 + (79\tilde{k}_1 \cdot \tilde{k}_2 + 113\tilde{k}_1 \cdot \tilde{k}_3 + 48\tilde{k}_2^2 + 13\tilde{k}_2 \cdot \tilde{k}_3 + 65\tilde{k}_2^2)\tilde{k}_1^2 + 40(\tilde{k}_1 \cdot \tilde{k}_2)^2 + 74(\tilde{k}_1 \cdot \tilde{k}_3)^2 + 9\tilde{k}_3^4 - 2(\tilde{k}_2 \cdot \tilde{k}_3)^2 
+ 27\tilde{k}_3^4 + 156\tilde{k}_1 \cdot \tilde{k}_3\tilde{k}_2^2 + 131\tilde{k}_1 \cdot \tilde{k}_3\tilde{k}_2 \cdot \tilde{k}_3 - 3\tilde{k}_2^2\tilde{k}_2 \cdot \tilde{k}_3 + (92\tilde{k}_1 \cdot \tilde{k}_3 + 27\tilde{k}_2^2 + 16\tilde{k}_2 \cdot \tilde{k}_3)\tilde{k}_3^2 + \tilde{k}_1 \cdot \tilde{k}_2(159\tilde{k}_1 \cdot \tilde{k}_3 + 39\tilde{k}_2^2 - 24\tilde{k}_2 \cdot \tilde{k}_3 + 20\tilde{k}_3^2) \right],
\]

(17b)

\[
T_{1gg_3}^{\mu_1\mu_2\mu_3\mu_4} = \frac{e^4}{5040m^4\pi^2} g^{\mu_1\mu_3} g^{\mu_2\mu_4}(1 + \kappa)^{-1} \left[ 57\tilde{k}_1^4 + (110\tilde{k}_1 \cdot \tilde{k}_2 + 118\tilde{k}_1 \cdot \tilde{k}_3 + 92\tilde{k}_2^2 + 71\tilde{k}_2 \cdot \tilde{k}_3 + 96\tilde{k}_2^2)\tilde{k}_1^2 + 72(\tilde{k}_1 \cdot \tilde{k}_2)^2 + 80(\tilde{k}_1 \cdot \tilde{k}_3)^2 + 54\tilde{k}_2^4 + 72(\tilde{k}_2 \cdot \tilde{k}_3)^2 
+ 57\tilde{k}_3^4 + 180\tilde{k}_1 \cdot \tilde{k}_3\tilde{k}_2^2 + 177\tilde{k}_1 \cdot \tilde{k}_3\tilde{k}_2 \cdot \tilde{k}_3 + 108\tilde{k}_2^2\tilde{k}_2 \cdot \tilde{k}_3 + 2(59\tilde{k}_1 \cdot \tilde{k}_3 + 46\tilde{k}_2^2 + 55\tilde{k}_2 \cdot \tilde{k}_3)\tilde{k}_3^2 + \tilde{k}_1 \cdot \tilde{k}_2(177\tilde{k}_1 \cdot \tilde{k}_3 + 108\tilde{k}_2^2 + 66\tilde{k}_2 \cdot \tilde{k}_3 + 71\tilde{k}_3^2), \right]
\]

(17c)

\[
T_{1gg_4}^{\mu_1\mu_2\mu_3\mu_4} = \frac{e^4}{5040m^4\pi^2} g^{\mu_1\mu_4} g^{\mu_2\mu_3}(1 + \kappa)^{-1} \left[ 27\tilde{k}_1^4 + (16\tilde{k}_1 \cdot \tilde{k}_2 + 92\tilde{k}_1 \cdot \tilde{k}_3 + 27\tilde{k}_2^2 
+ 20\tilde{k}_2 \cdot \tilde{k}_3 + 65\tilde{k}_2^2)\tilde{k}_1^2 - 2(\tilde{k}_1 \cdot \tilde{k}_2)^2 + 74(\tilde{k}_1 \cdot \tilde{k}_3)^2 + 9\tilde{k}_3^4 + 40(\tilde{k}_2 \cdot \tilde{k}_3)^2 
+ 48\tilde{k}_3^4 + 156\tilde{k}_1 \cdot \tilde{k}_3\tilde{k}_2^2 + 159\tilde{k}_1 \cdot \tilde{k}_3\tilde{k}_2 \cdot \tilde{k}_3 + 39\tilde{k}_2^2\tilde{k}_2 \cdot \tilde{k}_3 + (113\tilde{k}_1 \cdot \tilde{k}_3 + 48\tilde{k}_2^2 + 79\tilde{k}_2 \cdot \tilde{k}_3)\tilde{k}_3^2 + \tilde{k}_1 \cdot \tilde{k}_2(131\tilde{k}_1 \cdot \tilde{k}_3 - 3(\tilde{k}_2^2 + 8\tilde{k}_2 \cdot \tilde{k}_3) + 13\tilde{k}_3^2) \right].
\]

(17d)

The other contributions of Eq. (15), \( T_{1gk} \) and \( T_{1kk} \), have been omitted for the sake of brevity. Note that we have adopted dimensional regularization by extending the spacetime from 4 to \( D \) dimensions, so that \( d^4p/(2\pi)^4 \) goes to \( \mu^{4-D}d^Dp/(2\pi)^D \), where \( \mu \) is an arbitrary parameter that identifies the mass scale. As we will see below, from these results (15) we will be able to obtain the Lorentz-violating Euler-Heisenberg action in both nonperturbative and perturbative approaches.
Let us now recover the (perturbative in $c_{\mu\nu}$) amplitude obtained in $\text{9}$ for the photon triple splitting, from the above nonperturbative results $\text{(15)}$, in the collinear limit. For this, if we consider that the incident on-shell photon has energy $E_1$ and momentum $\vec{k}_1$, from the momentum conservation $\vec{k}_1 = \vec{k}_2 + \vec{k}_3 + \vec{k}_4$, we have the inequation

$$|\vec{k}_1| = |\vec{k}_2 + \vec{k}_3 + \vec{k}_4| \leq |\vec{k}_2| + |\vec{k}_3| + |\vec{k}_4|.$$  \hfill (18)

Then, as $k_i^2 = E_i^2 - \vec{k}_i^2 = 0$ or $|\vec{k}_i| = E_i$ (with $i = 1, 2, 3, 4$), in order to also satisfy the energy conservation $E_1 = E_2 + E_3 + E_4$, all the momenta $\vec{k}_i$ must be aligned. Therefore, the incident photon and the decay photons must be collinear, such that the four-momenta of all photons are mutually orthogonal, $k_i^\mu k_j\mu = 0$. By considering that these four-momenta are proportional to some four-momentum $k_0^\mu$, satisfying $k_0^2 = 0$, we can write $k_i^\mu = k_i k_0^\mu$, so that $k_i^\mu k_j\mu = k_i k_j k_0^\mu = 0$, in which $k_i$ are now scalar coefficients, instead four-momenta. With regard to the transversality condition of the polarization four-vectors, as usual, $\epsilon_{ij\mu} k_i^\mu = k_j \epsilon_{ij\mu} k_0^\mu = 0$ (or $A_{ij\mu} (k_i) k_0^\mu = 0$). Thus, due to the requirement of collinearity, we also have that $\epsilon_{ij\mu} k_j^\mu = k_j \epsilon_{ij\mu} k_0^\mu = 0$ (or $A_{ij\mu} (k_i) k_0^\mu = 0$).

Taking into account these considerations, we find

$$G_{\text{coll}}^{\mu_1\mu_2\mu_3\mu_4} = - \frac{e^4}{120m^2\pi^2} \vec{k}_0^2 (1 + \kappa)^{-1}$$

$$\times [g^{\mu_1\mu_2} g^{\mu_3\mu_4} (9k_1^2 - 8k_2^2 - k_3^2 + 2k_1 k_2 + 16k_1 k_3 - 18k_2 k_3)$$

$$+ g^{\mu_1\mu_3} g^{\mu_2\mu_4} (4k_1^2 - 8k_2^2 + 4k_3^2 - 8k_1 k_2 + 16k_1 k_3 - 8k_2 k_3)$$

$$+ g^{\mu_1\mu_4} g^{\mu_2\mu_3} (-k_1^2 - 8k_2^2 + 9k_3^2 - 18k_1 k_2 + 16k_1 k_3 + 2k_2 k_3)]$$

$$+ \mathcal{O}(\vec{k}_0^4) + \mathcal{O}(\vec{k}_0^\mu),$$  \hfill (19)

where we have interchanged $\mu_1$ and $\mu_2$ and thereafter $\mu_1$ and $\mu_4$, however without changing the momentum indices, in order to obtain $T_2$ and $T_3$ (Eqs. $\text{(7)}$ and $\text{(8)}$) from $T_1$ (Eq. $\text{(15)}$), following the prescription given in Ref. $\text{9}$. Note that, as expected, the divergent term $\mathcal{O}(\vec{k}_0^4)$ vanishes, and, for simplicity, we have not included the terms of $\mathcal{O}(\vec{k}_0^\mu)$ and $\mathcal{O}(\vec{k}_0^\mu)$. Now, when we expand up to first order in $\kappa$, the Lorentz-violating contributions become

$$\vec{k}_0^2 (1 + \kappa)^{-1} = [(k_0^2)^2 (1 + \kappa^2) - (k_0^\mu)^2]^2 (1 + \kappa)^{-1}$$

$$= 2\kappa (k_0^\mu)^2 + \mathcal{O}(\kappa^2).$$  \hfill (20)

Therefore, the perturbative result is readily recovered (see Eq. (10) of Ref. $\text{9}$), which is given by

$$G_{\text{coll,}\kappa}^{\mu_1\mu_2\mu_3\mu_4} = - \frac{e^4 c_{\mu\nu} k_0^{\mu_1} k_0^{\mu_2} k_0^{\mu_3} k_0^{\mu_4}}{60\pi^2 m^2}$$

$$\times [g^{\mu_1\mu_2} g^{\mu_3\mu_4} (9k_1^2 - 8k_2^2 - k_3^2 + 2k_1 k_2 + 16k_1 k_3 - 18k_2 k_3)$$

$$+ g^{\mu_1\mu_3} g^{\mu_2\mu_4} (4k_1^2 - 8k_2^2 + 4k_3^2 - 8k_1 k_2 + 16k_1 k_3 - 8k_2 k_3)$$

$$+ g^{\mu_1\mu_4} g^{\mu_2\mu_3} (-k_1^2 - 8k_2^2 + 9k_3^2 - 18k_1 k_2 + 16k_1 k_3 + 2k_2 k_3)],$$  \hfill (21)
with \( c_{\mu \nu} k_0^\mu k_0^\nu = \kappa (k_0^0)^2 \), however, for \( c_{00} \neq 0 \) and \( c_{0i} = c_{ij} = 0 \).

We will now discuss the generation of the Lorentz-violating Euler-Heisenberg action, in the first order correction, \( \alpha^2 = e^4 / 16\pi^2 \). For this, we must calculate \( G \) (Eq. (5)) without regard to the collinear limit. Thus, as pointed out at the beginning, \( T_2 \) and \( T_3 \) are obtained from \( T_1 \), when we interchange the uncontracted indices as well as the momentum indices, see Eq. (9). Proceeding in this way, we obtain

\[
G^{\mu_1 \mu_2 \mu_3 \mu_4}(k_1, k_2, k_3, k_4) = G_{gg}^{\mu_1 \mu_2 \mu_3 \mu_4}(k_1, k_2, k_3, k_4) + \sum_{i=1}^{6} G_{g_k g_i}^{\mu_1 \mu_2 \mu_3 \mu_4}(k_1, k_2, k_3, k_4) + G_{k_k}^{\mu_1 \mu_2 \mu_3 \mu_4}(k_1, k_2, k_3, k_4),
\]

(22)

where

\[
G_{gg}^{\mu_1 \mu_2 \mu_3 \mu_4} = \frac{e^4}{180\pi^2 m^4} (1 + \kappa)^{-1}
\times \left[g^{\mu_1 \mu_2} g^{\mu_3 \mu_4}(7 \bar{k}_4 \cdot \bar{k}_4 k_2 \cdot \bar{k}_3 + 7 \bar{k}_4 \cdot \bar{k}_3 k_2 \cdot \bar{k}_4 - 10 \bar{k}_1 \cdot \bar{k}_2 \bar{k}_3 \cdot \bar{k}_4)
+ g^{\mu_1 \mu_3} g^{\mu_2 \mu_4}(7 \bar{k}_4 \cdot \bar{k}_4 k_2 \cdot \bar{k}_3 - 10 \bar{k}_1 \cdot \bar{k}_3 k_2 \cdot \bar{k}_4 + 7 \bar{k}_1 \cdot \bar{k}_2 \bar{k}_3 \cdot \bar{k}_4)
+ g^{\mu_1 \mu_4} g^{\mu_2 \mu_3}(-10 \bar{k}_1 \cdot \bar{k}_4 k_2 \cdot \bar{k}_3 + 7 \bar{k}_1 \cdot \bar{k}_3 k_2 \cdot \bar{k}_4 + 7 \bar{k}_1 \cdot \bar{k}_2 \bar{k}_3 \cdot \bar{k}_4)]
\]

(23)

(24a)

(24b)

(24c)

(24d)

(24e)

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with \( \tilde{k}_4 = -\tilde{k}_1 - \tilde{k}_2 - \tilde{k}_3 \). We observe that all the contributions of order \( 1/m^2 \) completely vanish, as expected. In fact, the Euler-Heisenberg actions are all proportional to order \( 1/m^4 \), in the first order correction.

Therefore, by considering these results (22), the effective action (1) takes the following form

\[
S_{EH} = -\frac{\alpha^2}{180 m^4 T} (1 + \kappa)^{-1} \int d^4 x \int d^4 k_1 d^4 k_2 d^4 k_3 d^4 k_4 \, e^{i(k_1 + k_2 + k_3 + k_4) \cdot x} G(k_1, k_2, k_3, k_4),
\]

where

\[
G(k_1, k_2, k_3, k_4) = G^{\mu_1 \mu_2 \mu_3 \mu_4}(k_1, k_2, k_3, k_4) \tilde{A}_{\mu_1}(k_1) \tilde{A}_{\mu_2}(k_2) \tilde{A}_{\mu_3}(k_3) \tilde{A}_{\mu_4}(k_4),
\]

so that we can write

\[
G(k_1, k_2, k_3, k_4) = \frac{5}{3} \tilde{F}_{\mu \nu}(k_1) \tilde{F}^{\mu \nu}(k_2) \tilde{F}_{\lambda \rho}(k_3) \tilde{F}^{\lambda \rho}(k_4) + \frac{5}{3} \tilde{F}_{\mu \nu}(k_1) \tilde{F}^\mu_{\nu}(k_3) \tilde{F}_{\lambda \rho}(k_2) \tilde{F}^{\lambda \rho}(k_4)
+ \frac{5}{3} \tilde{F}_{\mu \nu}(k_1) \tilde{F}^{\mu \nu}(k_4) \tilde{F}_{\lambda \rho}(k_2) \tilde{F}^{\lambda \rho}(k_3) - \frac{14}{6} F_{\mu \nu}(k_1) \tilde{F}_{\nu \lambda}(k_2) \tilde{F}_{\lambda \rho}(k_3) \tilde{F}^{\mu \rho}(k_4)
- \frac{14}{6} F_{\mu \nu}(k_1) \tilde{F}^{\nu \lambda}(k_2) \tilde{F}_{\lambda \rho}(k_3) \tilde{F}^{\mu \rho}(k_3) - \frac{14}{6} F_{\mu \nu}(k_1) \tilde{F}^{\nu \lambda}(k_3) \tilde{F}_{\lambda \rho}(k_2) \tilde{F}^{\mu \rho}(k_4)
- \frac{14}{6} F_{\mu \nu}(k_1) \tilde{F}^{\nu \lambda}(k_3) \tilde{F}_{\lambda \rho}(k_2) \tilde{F}^{\mu \rho}(k_3) - \frac{14}{6} F_{\mu \nu}(k_1) \tilde{F}^{\nu \lambda}(k_4) \tilde{F}_{\lambda \rho}(k_2) \tilde{F}^{\mu \rho}(k_3),
\]

with \( \tilde{F}^{\mu \nu}(k_1) = \tilde{k}_1^\mu \tilde{A}^\nu(k_1) - \tilde{k}_1^\nu \tilde{A}^\mu(k_1) \), and so on. Then, inverting the Fourier transform in the expression (26), the Lorentz-violating Euler-Heisenberg action becomes

\[
S_{EH} = -\frac{\alpha^2}{180 m^4 T} (1 + \kappa)^{-1} \int d^4 x (5 \tilde{F}_{\mu \nu} \tilde{F}^{\mu \nu} \tilde{F}_{\lambda \rho} \tilde{F}^{\lambda \rho} - 14 \tilde{F}_{\mu \nu} \tilde{F}^{\mu \lambda} \tilde{F}^{\nu \rho})
\]

where now \( \tilde{F}^{\mu \nu} = \tilde{\partial}^{\mu} \tilde{A}^\nu(x) - \tilde{\partial}^{\nu} \tilde{A}^\mu(x) \). The above result is somewhat unusual, although it resembles the conventional Euler-Heisenberg action (31). From this expression (29) (or from the equations (15)) we are now able to calculate some scattering amplitudes of interest in the presence of Lorentz violation (see Sec. 11 for some proposals).
It is worth mentioning that the Lagrangian of Eq. (29) is the leading term of the expansion of the Euler-Heisenberg effective Lagrangian, in the weak-field limit, which, written in terms of the dual field strength \( \tilde{F}_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\lambda\rho} \tilde{F}^{\lambda\rho} \) and the critical field strength \( E_c = m^2 e^3/\hbar e = 1.3 \times 10^{16} \text{ V/cm} \), can be reexpressed as

\[
\mathcal{L}^{(2)}_{\text{EH}} = \kappa \left( 1 + \kappa \right)^{-1} \left( 4 \tilde{F}_{\mu\nu} \tilde{F}^{\mu\nu} \tilde{F}^{\lambda\rho} + 7 \tilde{F}_{\mu\nu} * \tilde{F}^{\mu\nu} \tilde{F}^{\lambda\rho} \right),
\]

where

\[
\kappa = \frac{2\alpha^2 \hbar^3}{45 m^4 c^5} = \frac{\alpha}{90\pi E_c^2},
\]

with \( \alpha = e^2/4\pi\hbar c \) (note that we have reinstated \( \hbar \) and \( c \)). In this limit, the electric and magnetic fields \( E \) and \( B \) are weak in comparison with the critical field \( E_c \). In fact, the limit \( E \ll E_c \) is essential for the validity of the effective Lagrangian as well as imposes severe restrictions on the real (electron-positron) pair production. We refer the reader to Ref. [32], for a more complete discussion.

While the above expression (30) is sufficient to analyze the processes of (four-photon) box diagrams, for more external legs, we can infer from the Euler-Heisenberg effective Lagrangian the other expressions, e.g., for (six-photon) hexagon diagrams, we can write

\[
\mathcal{L}^{(3)}_{\text{EH}} = -\xi \left( 1 + \kappa \right)^{-1} \tilde{F}_{\alpha\beta} \tilde{F}^{\alpha\beta} \left( 8 \tilde{F}_{\mu\nu} \tilde{F}^{\mu\nu} \tilde{F}^{\lambda\rho} + 13 \tilde{F}_{\mu\nu} * \tilde{F}^{\mu\nu} \tilde{F}^{\lambda\rho} \right),
\]

where

\[
\xi = \frac{32\pi \alpha^3 \hbar^6}{315 m^8 c^{10}} = \frac{2\alpha}{315\pi E_c^2}.
\]

In particular, the Lagrangiana (32) is that responsible for the first non-zero contribution to the photon splitting in an (constant and spatially uniform) external magnetic field \( \vec{B} \) [25, 26], in which \( \vec{B} \sim B_c = m^2 c^3/\hbar e = 4.42 \times 10^{13} \text{ G} \) (i.e., a strong field). The limit of very strong magnetic fields \( (B \gg B_c) \) is also interesting, however, it must be taken into account in the complete Euler-Heisenberg effective Lagrangian. We believe that the Lorentz-violating version of this effective Lagrangian resembles its original shape, except for the overall factor \( (1 + \kappa)^{-1} \) and for the field strength \( \tilde{F}^{\mu\nu} = ((1 + \kappa)F^{0i}, F^{ij}) \) and its dual. Nevertheless, this will be confirmed in a forthcoming study.

In general, calculations involving Lorentz violation are performed in the first order of the coefficient that controls the violation. Thus, by expanding the action (29) up to first order in \( \kappa \), we
obtain

\[ S_{\text{EH}} = -\frac{\alpha^2}{180m^4} \int d^4x \left( 5F_{\mu\nu}F^{\mu\nu}F_{\lambda\rho}F^{\lambda\rho} - 14F_{\mu\nu}F^{\mu\lambda}F_{\lambda\rho}F^{\rho\mu} \right) \]

\[ -\frac{\alpha^2}{90m^4} \int d^4x \left( k_F)_{\mu\nu\alpha\beta} (5F^{\alpha\beta}F^{\mu\nu}F_{\lambda\rho}F^{\lambda\rho} - 14F^{\alpha\beta}F^{\mu\lambda}F_{\lambda\rho}F^{\rho\mu}) \right) \]

\[ + \alpha^2 \frac{m^4}{180} \int d^4x \left( k_F)_{\mu\nu\alpha\beta} (5F^{\alpha\beta}F^{\mu\nu}F_{\lambda\rho}F^{\lambda\rho} - 14F^{\alpha\beta}F^{\mu\lambda}F_{\lambda\rho}F^{\rho\mu} + O(c^2) \right) \]

where we have introduced the coefficient \((k_F)_{\mu\nu\alpha\beta}\), given by

\[ (k_F)_{\mu\nu\alpha\beta} = g_{\mu\alpha}c_{\nu\beta} + g_{\nu\beta}c_{\mu\alpha} - g_{\mu\beta}c_{\nu\alpha} - g_{\nu\alpha}c_{\mu\beta}. \]

In the next section we will see that the second and third contributions of Eq. (34) are in fact found from the perturbative approach, however, for a generic coefficient \(c_{\mu\nu}\). An interesting point about the decomposition of \((k_F)_{\mu\nu\alpha\beta}\) (Eq. (35)), in terms of a symmetry tensor \(c_{\mu\nu}\), is that it is also found in the Born-Infeld electrodynamics, with an (Lorentz-violating) external field [33]. In particular, this decomposition restricts the Lorentz-violating electrodynamics to a nonbirefringence sector [34].

III. PERTURBATIVE APPROACH

In order to take into account the perturbative approach, let us first rewrite the fermion Lagrangian (1) as follows

\[ L_f = \bar{\psi} \left( i\partial_\mu \gamma^\mu + ic_{\mu\nu}\partial_\mu \gamma^\nu - m - eA_\mu \gamma^\mu - eA_\mu c^{\mu\nu} \gamma^\nu \right) \psi, \]

in which, from now on, \(c_{\mu\nu}\) is a generic coefficient for Lorentz violation. Now, by expanding the propagator up to first order in \(c_{\mu\nu}\),

\[ \frac{i}{\not{p} + c_{\mu\nu}p^\mu \gamma^\nu - m} = \frac{i}{\not{p} - m} + \frac{i}{\not{p} - m} ic_{\mu\nu}p^\mu \gamma^\nu \frac{i}{\not{p} - m} + \cdots, \]

so that we can consider \(ic_{\mu\nu}p^\mu \gamma^\nu\) as an insertion in the propagator \(iS(p) = i(\not{p} - m)^{-1}\), and treating the last term in (36) as a new vertex, the effective action (4) then becomes

\[ S_{\text{eff}}^{(4)} = \frac{1}{4} \int d^4x \int d^4k_1 d^4k_2 d^4k_3 d^4k_4 e^{i(k_1 + k_2 + k_3 + k_4) \cdot x} \]

\[ \times \frac{1}{6} G_{c_1}^{\mu_1\mu_2\mu_3\mu_4}(k_1, k_2, k_3, k_4) A_{\mu_1}(k_1) A_{\mu_2}(k_2) A_{\mu_3}(k_3) A_{\mu_4}(k_4) + O(c^2) \]

with

\[ G_{c_1}^{\mu_1\mu_2\mu_3\mu_4}(k_1, k_2, k_3, k_4) = 2 T_{c_1}^{\mu_1\mu_2\mu_3\mu_4}(k_1, k_2, k_3, k_4) + 2 T_{c_2}^{\mu_1\mu_2\mu_3\mu_4}(k_1, k_2, k_3, k_4) \]

\[ + 2 T_{c_3}^{\mu_1\mu_2\mu_3\mu_4}(k_1, k_2, k_3, k_4), \]

\[ (39) \]
where now each graph above is subdivided into eight graphs,

\[ T_{c1}^{\mu_1 \mu_2 \mu_3 \mu_4} (k_1, k_2, k_3, k_4) = T_{c11}^{\mu_1 \mu_2 \mu_3 \mu_4} (k_1, k_2, k_3, k_4) + T_{c12}^{\mu_1 \mu_2 \mu_3 \mu_4} (k_1, k_2, k_3, k_4) + T_{c13}^{\mu_1 \mu_2 \mu_3 \mu_4} (k_1, k_2, k_3, k_4) + T_{c14}^{\mu_1 \mu_2 \mu_3 \mu_4} (k_1, k_2, k_3, k_4), \]  

(40)

given by

\[
T_{c11}^{\mu_1 \mu_2 \mu_3 \mu_4} = ie^4 \int \frac{d^4p}{(2\pi)^4} \text{tr} S(p) c^{\mu_1 \mu_2} \gamma_\mu S(p_1) \gamma^{\mu_3} S(p_12) \gamma^{\mu_4} S(p_123) - ie^4 \int \frac{d^4p}{(2\pi)^4} \text{tr} S(p) c^{\mu_1 \mu_2} \gamma_\mu S(p_1) \gamma^{\mu_3} S(p_12) \gamma^{\mu_4} S(p_123),
\]

(41)

\[
T_{c12}^{\mu_1 \mu_2 \mu_3 \mu_4} = ie^4 \int \frac{d^4p}{(2\pi)^4} \text{tr} S(p) \gamma^{\mu_1} S(p_1) c^{\mu_2 \mu_3} \gamma_\mu S(p_12) \gamma^{\mu_4} S(p_123) - ie^4 \int \frac{d^4p}{(2\pi)^4} \text{tr} S(p) \gamma^{\mu_1} S(p_1) c^{\mu_2 \mu_3} \gamma_\mu S(p_12) \gamma^{\mu_4} S(p_123),
\]

(42)

\[
T_{c13}^{\mu_1 \mu_2 \mu_3 \mu_4} = ie^4 \int \frac{d^4p}{(2\pi)^4} \text{tr} S(p) \gamma^{\mu_1} S(p_1) \gamma^{\mu_2} S(p_12) c^{\mu_3 \mu_4} \gamma_\mu S(p_123) - ie^4 \int \frac{d^4p}{(2\pi)^4} \text{tr} S(p) \gamma^{\mu_1} S(p_1) \gamma^{\mu_2} S(p_12) c^{\mu_3 \mu_4} \gamma_\mu S(p_123),
\]

(43)

\[
T_{c14}^{\mu_1 \mu_2 \mu_3 \mu_4} = ie^4 \int \frac{d^4p}{(2\pi)^4} \text{tr} S(p) \gamma^{\mu_1} S(p_1) \gamma^{\mu_2} S(p_12) c^{\mu_3 \mu_4} \gamma_\mu S(p_123) - ie^4 \int \frac{d^4p}{(2\pi)^4} \text{tr} S(p) \gamma^{\mu_1} S(p_1) \gamma^{\mu_2} S(p_12) c^{\mu_3 \mu_4} \gamma_\mu S(p_123),
\]

(44)

Once more, it is easy to see that \( T_{c2} \) and \( T_{c3} \) are obtained from \( T_{c1} \), as well as \( T_{c12}, T_{c13}, \) and \( T_{c14} \) from \( T_{c11} \), when we also perform the ciclic interchanges:

\[
T_{c12}^{\mu_1 \mu_2 \mu_3 \mu_4} (k_1, k_2, k_3, k_4) = T_{c11}^{\mu_1 \mu_2 \mu_3 \mu_4} (k_4, k_1, k_2, k_3), \]

(45a)

\[
T_{c13}^{\mu_1 \mu_2 \mu_3 \mu_4} (k_1, k_2, k_3, k_4) = T_{c11}^{\mu_1 \mu_2 \mu_3 \mu_4} (k_3, k_4, k_1, k_2), \]

(45b)

\[
T_{c14}^{\mu_1 \mu_2 \mu_3 \mu_4} (k_1, k_2, k_3, k_4) = T_{c11}^{\mu_1 \mu_2 \mu_3 \mu_4} (k_2, k_3, k_4, k_1). \]

(45c)

Therefore, we must focus our attention on the two graphs of \( T_{c11} \) (Eq. 41), in which, by considering first the Feynman parameterization, we obtain

\[
T_{c11}^{\mu_1 \mu_2 \mu_3 \mu_4} = \int_0^1 dx_1 \int_0^{1-x_1} dx_2 \int_0^{1-x_12} dx_3 \int \frac{d^4p}{(2\pi)^4} \frac{6ie^4}{(p^2 - M^2)^4} \times \text{tr}[(g + m) c^{\mu_1 \mu_2} \gamma_\mu (g_1 + m) \gamma^{\mu_2} (g_12 + m) \gamma^{\mu_3} (g_123 + m) \gamma^{\mu_4}] - \int_0^1 dx_1 \int_0^{1-x_1} dx_2 \int_0^{1-x_12} dx_3 \int \frac{d^4p}{(2\pi)^4} \frac{24ixe^4}{(p^2 - M^2)^5} \times \text{tr}[(g + m) c^{\mu_1 \mu_2} \gamma_\mu (g + m) \gamma^{\mu_1} (g_1 + m) \gamma^{\mu_2} (g_12 + m) \gamma^{\mu_3} (g_123 + m) \gamma^{\mu_4}].
\]

(46)
Following the standard procedure, after we calculate the trace over Dirac matrices and the corresponding integrals, up to order $1/m^4$, we arrive at

\[
T^{\mu_1\mu_2\mu_3\mu_4}_{c_{11}(k_1, k_2, k_3, k_4)} = T^{\mu_1\mu_2\mu_3\mu_4}_{c_{11}g_e}(k_1, k_2, k_3, k_4) + \sum_{i=1}^{8} T^{\mu_1\mu_2\mu_3\mu_4}_{c_{11}g_{g_i}}(k_1, k_2, k_3, k_4) + \sum_{i=1}^{24} T^{\mu_1\mu_2\mu_3\mu_4}_{c_{11}g_{k_i}}(k_1, k_2, k_3, k_4) + \sum_{i=1}^{18} T^{\mu_1\mu_2\mu_3\mu_4}_{c_{11}k_{k_i}}(k_1, k_2, k_3, k_4),
\]

where

\[
T^{\mu_1\mu_2\mu_3\mu_4}_{c_{11}g_e} = -\left[ \frac{e^4}{12\pi^2} - \frac{e^4}{24\pi^2} \ln \left( \frac{m^2}{\mu^2} \right) \right] \left[ -g^{\mu_1\mu_2}g^{\mu_3\mu_4} + e^{\mu_1\mu_2}g^{\mu_3\mu_4} - 4e^{\mu_1\mu_3}g^{\mu_2\mu_4} + 2e^{\mu_1\mu_4}g^{\mu_2\mu_3} + 2g^{\mu_1\mu_4}e^{\mu_2\mu_3} + c_{\mu\nu}g^{\mu\nu}g^{\mu_1\mu_3}g^{\mu_2\mu_4} - c_{\mu\nu}g^{\mu\nu}g^{\mu_1\mu_4}g^{\mu_2\mu_3} \right],
\]

\[
T^{\mu_1\mu_2\mu_3\mu_4}_{c_{11}g_{g_i}} = \frac{e^4 c_{\mu\nu}g^{\mu\nu}}{1440m^4\pi^2}g^{\mu_1\mu_2}g^{\mu_3\mu_4}(27k_1^2 + 17k_1 \cdot k_2 + 37k_1 \cdot k_3 + 9k_2 \cdot k_3 + 10k_3^2) + g^{\mu_1\mu_3}g^{\mu_2\mu_4}(-39k_1^2 - 37k_1 \cdot k_2 - 41k_1 \cdot k_3 - 24k_2^2 - 39k_2 \cdot k_3 - 26k_3^2) + g^{\mu_1\mu_4}g^{\mu_2\mu_3}(24k_1^2 + 11k_1 \cdot k_2 + 37k_1 \cdot k_3 + 12k_2^2 + 33k_2 \cdot k_3 + 25k_3^2),
\]

\[
T^{\mu_1\mu_2\mu_3\mu_4}_{c_{11}g_{k_i}} = \frac{e^4 c_{\mu\nu}g^{\mu\nu}}{10080m^4\pi^2}g^{\mu_1\mu_2}g^{\mu_3\mu_4}(39k_1^4 + (63k_1 \cdot k_2 + 93k_1 \cdot k_3 + 30k_2^2 + 22k_2 \cdot k_3 + 45k_3^2)k_1^2 + 31(k_1 \cdot k_2)^2 + 61(k_1 \cdot k_3)^2 + 15(k_2 \cdot k_3)^2 + 14k_4 + 93k_1 \cdot k_3 k_2^2 + 96k_1 \cdot k_3 k_2 \cdot k_3 + 12k_2^2 k_3 \cdot k_3 + 2(33k_1 \cdot k_3 + 7k_2^2 + 15k_2 \cdot k_3)k_3^2 + k_1 \cdot k_2(127k_1 \cdot k_3 + 23k_2^2 + 11k_2 \cdot k_3 + 31k_3^2)],
\]

\[
T^{\mu_1\mu_2\mu_3\mu_4}_{c_{11}g_{g_3}} = -\frac{e^4 c_{\mu\nu}g^{\mu\nu}}{10080m^4\pi^2}g^{\mu_1\mu_3}g^{\mu_2\mu_4}(48k_1^4 + (93k_1 \cdot k_2 + 299k_1 \cdot k_3 + 65(k_2^2 + k_2 \cdot k_3) + 68k_3^2)k_1^2 + 61(k_1 \cdot k_2)^2 + 67(k_1 \cdot k_3)^2 + 27k_2^4 + 57(k_2 \cdot k_3)^2 + 29k_4^4 + 113k_1 \cdot k_3 k_2^2 + 132k_1 \cdot k_3 k_2 \cdot k_3 + 75k_2^2 k_3 \cdot k_3 + 2(43k_1 \cdot k_3 + 25k_2^2 + 40k_2 \cdot k_3)k_3^2 + k_1 \cdot k_2(145k_1 \cdot k_3 + 79(k_2^2 + k_2 \cdot k_3) + 69k_3^2)],
\]

\[
T^{\mu_1\mu_2\mu_3\mu_4}_{c_{11}g_{g_4}} = \frac{e^4 c_{\mu\nu}g^{\mu\nu}}{10080m^4\pi^2}g^{\mu_1\mu_4}g^{\mu_2\mu_3}(27k_1^4 + (29k_1 \cdot k_2 + 79k_1 \cdot k_3 + 27k_2^2 + 35k_2 \cdot k_3 + 52k_3^2)k_1^2 + 11(k_1 \cdot k_2)^2 + 61(k_1 \cdot k_3)^2 + 9k_2^4 + 45(k_2 \cdot k_3)^2 + 28k_4^4 + 95k_1 \cdot k_3 k_2^2 + 116k_1 \cdot k_3 k_2 \cdot k_3 + 48k_2^2 k_3 \cdot k_3 + (80k_1 \cdot k_3 + 36k_2^2 + 71k_2 \cdot k_3)k_3^2 + k_1 \cdot k_2(111k_1 \cdot k_3 + 17k_2^2 + 29k_2 \cdot k_3 + 41k_3^2)].
\]

In the above results we have omitted some contributions of $T_{c_{11}g}$ and all of $T_{c_{11}gk}$ and $T_{c_{11}kk}$, because they are very lengthy. In fact, the expressions (49) are also extensive, however, it is worth considering them here, as we will see below.
Finally, by taking into account the interchanges (45), the first graph $T_{c1}$ takes the form:

$$T_{c1}^{\mu_1 \mu_2 \mu_3 \mu_4}(k_1, k_2, k_3, k_4) = T_{c1g_1}^{\mu_1 \mu_2 \mu_3 \mu_4}(k_1, k_2, k_3, k_4) + \sum_{i=1}^{8} T_{c1g_{1i}}^{\mu_1 \mu_2 \mu_3 \mu_4}(k_1, k_2, k_3, k_4)$$

$$+ \sum_{i=1}^{24} T_{c1g_{2i}}^{\mu_1 \mu_2 \mu_3 \mu_4}(k_1, k_2, k_3, k_4) + \sum_{i=1}^{18} T_{c1g_{3i}}^{\mu_1 \mu_2 \mu_3 \mu_4}(k_1, k_2, k_3, k_4),$$

where

$$T_{c1g_1}^{\mu_1 \mu_2 \mu_3 \mu_4} = - \left[ \frac{e^4}{6\pi^2\epsilon} - \frac{e^4}{12\pi^2} \ln \left( \frac{m^2}{\mu^2} \right) \right] \times (2g^{\mu_1 \mu_4} c^{\mu_2 \mu_3} + 2g^{\mu_1 \mu_2} c^{\mu_3 \mu_4} - 4g^{\mu_1 \mu_3} c^{\mu_2 \mu_4}$$

$$+ 2c_{\mu_4} g^{\mu_2 \mu_3} - 4g^{\mu_2 \mu_3} c^{\mu_2 \mu_4} + 2g^{\mu_2 \mu_3} c^{\mu_1 \mu_4}$$

$$- c_{\mu_4} g^{\mu_2 \mu_3} - 2g^{\mu_1 \mu_3} g^{\mu_2 \mu_4} + g^{\mu_1 \mu_2} g^{\mu_3 \mu_4})$$

and

$$\sum_{i=1}^{4} T_{c1g_{1i}}^{\mu_1 \mu_2 \mu_3 \mu_4} = \sum_{i=1}^{4} T_{c1g_{1i}}^{\mu_1 \mu_2 \mu_3 \mu_4} \bigg|_{(\kappa \rightarrow c_{\mu_4} g^{\mu_4 \nu}, \kappa^2 \rightarrow 0, \hat{k}_i \rightarrow k_i)}.$$  (52)

Thus, comparing the above perturbative expressions (52) with the nonperturbative ones (17) we observe that they are similar, being the results of Eq. (50) an expansion up to first order in $c_{\mu\nu}$ of the equation (15), with $c_{00} = \kappa$ and $c_{0i} = c_{ij} = 0$. Note that, as expected, the expression (24) for the vacuum photon splitting is obtained from (52), when we consider the collinear limit.

Therefore, we can easily deduce the action coming from $T_{c1}$ (50) whose expression is in fact the perturbative Lorentz-violating Euler-Heisenberg action, previously obtained in (34), nevertheless here, for a generic coefficient $c_{\mu\nu}$, given by

$$S_{cEH} = -\frac{\alpha^2}{90m^4} \int d^4x \left( k_F \right)_{\mu\nu\alpha\beta} (5F^{\alpha\beta} F^{\mu\nu} F^{\lambda\rho} - 14F^{\alpha\beta} F^{\nu\lambda} F^{\mu\rho})$$

$$+ \frac{\alpha^2}{180m^4} \int d^4x \left( c_{\alpha\beta} g^{\alpha\beta} (5F_{\mu\nu} F^{\mu\nu} F^{\lambda\rho} - 14F_{\mu\nu} F^{\nu\lambda} F^{\rho\mu}) \right),$$  (53)

where $(k_F)_{\mu\nu\alpha\beta}$ is defined in Eq. (35).

IV. SUMMARY

In this work, we have studied the radiative generation of the Lorentz-violating Euler-Heisenberg action, in the weak field approximation. Firstly, we have considered a nonperturbative calculation in the coefficient $c_{\mu\nu}$, however, by assuming rotational invariance, such that only $c_{00} \neq 0$. The preliminary results are presented in Eq. (15), resulting then in the action (29). From these expressions,
we are now able to calculate some scattering amplitudes relating to four-photon diagrams, in order to numerically estimate the coefficient for Lorentz violation. Within this approach, we have also recovered the results of the amplitude for the photon triple splitting (21), previously obtained in the literature. Finally, we have taken into account the perturbative approach, where $c_{\mu\nu}$ is treated as an insertion in the propagator and a new vertex. The partial results are shown in Eq. (50), which are in fact an expansion up to first order in $c_{\mu\nu}$ of Eq. (15), with $c_{00} = \kappa$ and $c_{0i} = c_{ij} = 0$. This suggest that the complete results (15) obtained in the nonperturbative approach can be used in both treatments. An extension of this work would be to consider higher-derivative terms, e.g., see the Lorentz-violating QED with operator of mass dimension six of Ref. [35].

Acknowledgements. This work was supported by Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq).

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