Maxwell-Chern-Simons Hydrodynamics for the Chiral Magnetic Effect

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The rate of vacuum changing topological solutions of the gluon field, sphalerons, is estimated to be large at the typical temperatures of heavy-ion collisions, particularly at the Relativistic Heavy Ion Collider. Such windings in the gluon field are expected to produce parity-odd bubbles, which cause separation of positively and negatively charged quarks along the axis of the external magnetic field. This chiral magnetic effect can be mimicked by Chern-Simons modified electromagnetism. Here we present a model of relativistic hydrodynamics including the effects of axial anomalies via the Chern-Simons term.

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Introduction

The ground state of strong interactions is believed not to break CP invariance owing to a theorem by Vafa and Witten [1]. However, it has been suggested that the Vafa-Witten theorem does not prejudice formation of metastable P- and CP-odd domains [2–5]. These P-odd bubbles are expected to occur randomly anywhere in space-time in hot quark-gluon plasma [6], and they have the property that \( \langle \vec{E}_a \cdot \vec{B}_a \rangle \neq 0 \), where the fields are color electric and magnetic fields [6].

P-odd bubbles may yield observable effects on quarks in the presence of a strong external (electromagnetic) magnetic field because of the winding of the gluon field. The necessary setup for observation might be achieved in heavy-ion collisions when two nuclei collide with a nonzero impact parameter, which results in a strong magnetic field normal to the collision plane.

In the deconfinement phase of quark-gluon plasma, the chiral limit \( m_{\text{quark}} = 0 \) is a reasonable approximation. In this limit quarks have definite helicity eigenstates as the left-handed and right-handed helicity states do not mix in the absence of a mass term in the Lagrangian. Accordingly, the momentum and spin will be parallel for the right-handed massless quarks and antiparallel for the left-handed ones. In the meantime, if the gauge field has a nonvanishing winding number, the chirality will not be conserved owing to the axial anomaly, and vacuum-changing topological solutions will create net chirality. If there are more left-handed or right-handed quarks than the other, then when they align their spins parallel or antiparallel to the external magnetic field according to their charge, charge separation will occur in the direction of the magnetic field. As a result, a net electric current perpendicular to the collision plane will be produced. This is called the chiral magnetic effect (CME) [8–12]. Charge separation creates an electric dipole moment and it causes CP violation.

As chirality nonconservation is purely a nonperturbative effect, any evidence for this effect in the data would be a valuable observation of the nontrivial topology of Yang-Mills theories. The CME has been verified in lattice calculations [13–15]. According to the STAR Collaboration, data from RHIC indicate that charge separation has indeed been observed [16–19].

Vacuum of Yang-Mills Theories

The vacuum of non-Abelian gauge theories are infinitely degenerate; there are infinitely many pure gauge configurations of the gauge field. Different vacua are distinguished by an integer winding number, defined as

\[
Q_w = \frac{g^2}{32\pi^2} \int d^4x G^{\mu\nu} \tilde{G}_{\mu\nu}^a,
\]

where \( \tilde{G}_{\mu\nu}^a = \frac{i}{2} \epsilon^{\mu\nu\alpha\beta} G_{\alpha\beta}^a \) is the dual field strength.

No small perturbation around the minimum of a vacuum can change the winding number; perturbation theory is blind to the many-vacua structure of the non-Abelian theory. Instantons are finite solutions of the Euclidian action and they are localized in space-time with the feature of tunneling between two different vacua. Their effect, however, is suppressed exponentially at any temperature. Another possibility of the vacuum changing solution is the sphaleron. It interpolates from one vacuum to another by hopping over the energy barrier, whose height is on the order of \( \Lambda_{QCD}^4 \). This is possible at high temperatures because there will be enough energy to surmount the barrier without tunneling. Although sphalerons originally appeared in the electroweak theory, their importance also arises in quark-gluon plasma [20]. The approximate sphaleron rate is (see Ref. [21], and references therein)

\[
\Gamma = \frac{dN}{d^4x dt} \approx 386\alpha_s^2 T^4,
\]

where \( \alpha_s = g^2/4\pi \) and \( T \) is the temperature. As sphalerons do not tunnel, the rate is not suppressed at high temperatures and an appreciable number of transitions is expected.
QED Coupled to QCD with Topological Charge

The charge separation may be examined from a more formal viewpoint. The Lagrangian for QED coupled to QCD with topological charge is given by [22]

\[
\mathcal{L}_{\text{QCD+QED}} = \sum_{f} \bar{\psi}_{f} \left[ i \gamma_{\mu} (\partial_{\mu} - ig A_{\mu}^{a} t^{a} - iq f A_{\mu}) - m_{f} \right] \psi_{f} - \frac{1}{4} G_{\mu\nu}^{a} G_{\mu\nu}^{a} - \frac{\theta}{32\pi^{2}} g^{2} G_{\mu\nu}^{a} \tilde{G}_{\mu\nu}^{a} - \frac{1}{4} F_{\mu\nu}^{a} F_{\mu\nu}^{a}, \quad (3)
\]

where \( A_{\mu}^{a} \) is the gluon field, \( A_{\mu} \) is the electromagnetic field, and \( G_{\mu\nu}^{a} \) and \( F_{\mu\nu}^{a} \) are the gluon and electromagnetic field strength tensors, respectively. Although the term \( F_{\mu\nu}^{a} \tilde{F}_{\mu\nu}^{a} \) does not exist in the Lagrangian, it will be induced by the topological charge of the gluon field via quark loops [23]. To focus on the electromagnetic sector of this theory we can start with the effective Maxwell-Chern-Simons Lagrangian [24–26]:

\[
\mathcal{L}_{\text{MCS}} = -\frac{1}{4} F_{\mu\nu}^{a} F_{\mu\nu}^{a} - A_{\mu} J^{\mu} - \frac{c}{4} \theta F_{\mu\nu}^{a} \tilde{F}_{\mu\nu}^{a}, \quad (4)
\]

where \( \theta = \theta(\vec{x}, t) \) will mimic \( P \)-odd bubbles, \( c = \sum f q_{f}^{2} e^{2}/(2\pi^{2}) \) and \( J^{\mu} \) is the electric current of the quarks. The last term in Eq. (4) is not a total derivative and does not vanish because \( \theta \) is a field and is determined by fluctuations of the topological charge. The equation of motion following from (4) is

\[
\partial_{\mu} F_{\mu\nu} = J^{\nu} - c \tilde{F}_{\sigma\nu} \partial_{\sigma} \theta. \quad (5)
\]

Together with \( \partial_{\mu} \tilde{F}_{\mu\nu} = 0 \), the set of Maxwell-Chern-Simons equations can be written as

\[
\nabla \times \vec{B} - \frac{\partial \tilde{E}}{\partial t} = \vec{J} + c \left( \theta \vec{B} + \nabla \theta \times \vec{E} \right), \quad (6)
\]

\[
\nabla \cdot \vec{E} = \rho - c \nabla \theta \cdot \vec{B}, \quad (7)
\]

\[
\nabla \times \vec{E} + \frac{\partial \tilde{B}}{\partial t} = 0, \quad (8)
\]

\[
\nabla \cdot \tilde{B} = 0. \quad (9)
\]

These reduce to Maxwell’s equations when the field \( \theta \) is 0. The modification of electromagnetism can be attributed to the vacuum changing solutions in the full QED × QCD theory.

Maxwell-Chern-Simons Hydrodynamics

A very strong magnetic field is created in heavy ion collisions when two highly charged nuclei collide with a nonzero impact parameter. Considering typical RHIC parameters, the magnetic field generated normal to the collision plane is of the order \( eB \sim (10 - 100 \text{ MeV})^{2} \) during the first moments (\( \tau \sim 1 \text{ fm}/c \)) of the collision. This magnetic field is even stronger than the ones in magnetized neutron stars and magnetars, where typical magnetic fields are about \( eB \approx (2 \text{ MeV})^{2} \), or \( 10^{10} \text{ T} \).

Relativistic hydrodynamics is the prime tool being used to describe the time evolution of quark-gluon plasma. Let us start with the conservation of energy and momentum to derive the hydrodynamic model of quarks coupled to Chern-Simons modified electromagnetism.

\[
\partial_{\mu} (T^{\mu\nu} + \Theta^{\mu\nu}) = 0. \quad (10)
\]

Here the energy-momentum tensor of the ideal fluid is given by

\[
T^{\mu\nu} = (\epsilon + P) u^{\mu} u^{\nu} - g^{\mu\nu} P, \quad (11)
\]

where \( \epsilon \) and \( P \) are the local energy density and pressure. The fluid four-velocity is \( u^{\mu} = (\gamma, \gamma \vec{v}) \), and in the rest frame of the fluid it reduces to \( u^{\mu} = (1, 0, 0, 0) \). We use the natural units \( \hbar = c = k_{B} = 1 \) and the metric \( g^{\mu\nu} = \text{diag}(+1, -1, -1, -1) \). The energy-momentum tensor of the free electromagnetic field is given by

\[
\Theta^{\mu\nu} = F_{\lambda}^{\mu} F_{\lambda}^{\nu} + \frac{1}{4} g^{\mu\nu} F_{\alpha\beta} F^{\alpha\beta}. \quad (12)
\]

Its divergence can be found by using the equation of motion (5),

\[
\partial_{\mu} \Theta^{\mu\nu} = - F^{\nu\lambda} J_{\lambda} - c F^{\nu\lambda} \tilde{F}_{\lambda\sigma} \partial^{\sigma} \theta. \quad (13)
\]

Substituting this into Eq. (10) gives

\[
\partial_{\mu} T^{\mu\nu} = F^{\nu\lambda} J_{\lambda} + c F^{\nu\lambda} \tilde{F}_{\lambda\sigma} \partial^{\sigma} \theta. \quad (14)
\]

The electric current of the quarks in the fluid is defined as \( J^{\mu} = n u^{\mu} \), where \( n \) is the net electric charge density. It is conserved, as can be seen by taking the divergence of both sides of (10)

\[
\partial_{\mu} J^{\mu} = 0. \quad (15)
\]

By using the definitions \( E^{\mu} \equiv F^{\mu\nu} u_{\nu} \) and \( B^{\mu} \equiv \tilde{F}^{\mu\nu} u_{\nu} \), Eq. (13) can be written as
\[ \partial_\nu T^{\mu \nu} = nE^\nu - cE^\lambda B_\lambda \partial^\nu \theta, \quad (16) \]

where \( E^\lambda B_\lambda = -\vec{E} \cdot \vec{B} \). We refer to the right-hand side of Eq. (16) as the Lorentz-Chern-Simons force density.

Now we calculate the entropy production in the system by contracting Eq. (16) with \( u_\nu \). Using the fact that \( u_\mu E^\mu = 0 \), current conservation, and the normalization condition of the four-velocity \( u^\mu u_\mu = 1 \), we find

\[ nu^\mu \left( \partial_\mu \left( \frac{\epsilon + P}{n} \right) - \frac{1}{n} \partial_\mu P \right) = -cE^\lambda B_\lambda u_\nu \partial^\nu \theta. \quad (17) \]

By using the Gibbs relation,

\[ d \left( \frac{\epsilon + P}{n} \right) = Td \left( \frac{\sigma}{n} \right) + \frac{1}{n} dP, \quad (18) \]

Eq. (17) can be transformed into

\[ \partial_\mu s^\mu = -\frac{1}{T}cE^\lambda B_\lambda u_\nu \partial^\nu \theta, \quad (19) \]

where \( \sigma \) is the entropy density, and the entropy current is \( s^\mu = \sigma u^\mu \). Here we see that the Chern-Simons term can increase or decrease the entropy as it can be positive or negative. Although it may appear to be a violation of the second law of thermodynamics, we point out that entropy can decrease locally. However, the change in the entropy of the whole system cannot be negative. This situation is similar to that of Maxwell’s demon, where the demon can collect fast-moving and slow-moving gas molecules on two separate sides of the container by controlling the door in the middle of the container and, consequently, decrease the entropy of the gas by driving the system into more ordered phases. However, this does not violate the second law of thermodynamics. The entropy produced by the demon cannot be less than the entropy decrease of the gas, which means that the total entropy change of the whole system, the gas and the demon, cannot be less than 0. In Eq. (19) we isolated the external field \( \theta(x,t) \) and calculated the entropy for the rest of the system.

Discussion of the entropy also relates to the vacuum structure of the \( SU(N) \) gauge theories. It is well known that the vacuum of non-Abelian gauge theories is infinitely degenerate. Topological solutions, such as instantons and sphalerons, interpolate between different vacua, and what enumerates that is the winding number. As positive and negative change in the winding number are equally possible, its behavior is completely determined by a one-dimensional random walk. Therefore, a net change in the winding number can happen in a particular event by means of fluctuations. The parallel electric and magnetic fields produced by the term \( G^{\mu \nu} \bar{G}_{\mu \nu} \sim \vec{E} \cdot \vec{B} \) correspond to increase or decrease in the entropy locally; however, when infinitely many degenerate vacua are considered, the total entropy does not decrease. We note that in Ref. [27] an extra term was added to the current to cure the negative entropy change problem. Here we take a different approach.

To calculate the entropy of the whole system, we use the modified Gibbs relation (see also Ref. [28]):

\[ d \left( \frac{\epsilon + P}{n} \right) = Td \left( \frac{\sigma}{n} \right) + \frac{1}{n} dP + \frac{1}{n} R_\theta d\theta, \quad (20) \]

where the generalized force \( R_\theta \) is defined as

\[ R_\theta \equiv \left( \frac{\partial \Omega}{\partial \theta} \right)_{T,\mu}, \]

and \( \Omega \) is the thermodynamic potential density. In terms of the partition function of the model given in Eq. (4),

\[ Z = \int [D\bar{q}][Dq][DA_\mu]\exp \left( \int_0^\beta d\tau \int d^3x \mathcal{L} \right) \quad (21) \]

where \( \beta = 1/T \), the thermodynamic potential can be written as

\[ \Omega = -P = \frac{T}{V} \ln Z. \quad (22) \]

The term \(-\frac{1}{2} \theta F^{\mu \nu} \bar{F}_{\mu \nu} = c\theta \vec{E} \cdot \vec{B} \) in the Lagrangian, Eq. (4), involves fields such as \( \vec{B} \), which is generated by the colliding nuclei, and \( \vec{E} \), which is induced by the field \( \theta(x,t) \). Here we assume that the fields vary so slowly in space and time that they can be taken outside the integration to give

\[ Z = \exp \left( -\frac{V}{T} c\theta E^\lambda B_\lambda \right) \times \int [D\bar{q}][Dq][DA_\mu]\exp \left( \int_0^\beta d\tau \int d^3x \mathcal{L} \right) \quad (23) \]

This assumption is no better or worse than the application of hydrodynamics to heavy-ion collisions, where it is necessary to assume that the matter is locally thermalized and that the temperature and flow velocity vary slowly in space and time. The functional integration in \( Z \) involves fluctuations \( A_\mu \) around the externally imposed \( \vec{B} \) field and the induced \( \vec{E} \) field. Hence \( R_\theta = -cE^\lambda B_\lambda \).

Now we rewrite Eq. (17) as

\[ nu^\mu \left( \partial_\mu \left( \frac{\epsilon + P}{n} \right) - \frac{1}{n} \partial_\mu P + \frac{1}{n} cE^\lambda B_\lambda \partial_\mu \theta \right) = 0. \quad (24) \]

Using Eq. (20) we deduce that entropy is conserved:
\[ \partial_{\mu}s^\mu = 0. \]  

In this calculation we used ideal hydrodynamics and ignored the dissipative terms. It is well-known that the inclusion of dissipative terms in \( T^{\mu\nu} \) and \( J^\mu \) does not result in a decrease in entropy.

In Eq. (17) we calculated the component of the energymomentum conservation equation in the direction of \( u_{\nu} \). Now we find the projection of Eq. (16) perpendicular to \( u_{\nu} \). Multiplying both sides of Eq. (16) by the projection tensor \( \Delta^\alpha_\nu = \delta^\alpha_\nu - u^\alpha u_{\nu} \), we get

\[
(\epsilon + P)u^\mu\partial_\mu u^\alpha - (\delta^\alpha_\nu - u^\alpha u_{\nu})\partial^\mu P = nE^\alpha - cE^\lambda B_{\lambda}D^\alpha \theta. \tag{26}
\]

Defining the comoving derivative as \( D = u_{\mu}\partial^\mu \), the normal derivative as \( D^\alpha = \Delta^\alpha_\nu\partial^\nu \), and using the definition \( w = \epsilon + P \), we can write Eqs. (17) and (25) as

\[
D\epsilon + w\partial_\mu u^\mu = -cE^\lambda B_{\lambda}D\theta, \tag{27}
\]

\[
wDu^\alpha - D^\alpha P = nE^\alpha - cE^\lambda B_{\lambda}D^\alpha \theta. \tag{28}
\]

Equation (28) is reminiscent of the nonrelativistic Euler’s equation.

**Conclusion**

We have considered Chern-Simons modified electromagnetism in the context of the Chiral Magnetic Effect. We formulated the hydrodynamic model for the purpose of incorporating axial anomalies into relativistic hydrodynamics. The equations we found should be solved numerically by using realistic values for the magnitude of the external magnetic produced in heavy ion collisions and the size and rate of the sphaleron transitions. This work is now in progress.

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