On the initial conditions of the 1-point PDF for incompressible 
Navier-Stokes fluids

M. Tessarotto\textsuperscript{1,2} and C. Asci\textsuperscript{1}

\textsuperscript{1}Department of Mathematics and Informatics, 
University of Trieste, Trieste, Italy

\textsuperscript{2}Consortium for Magnetofluid Dynamics, Trieste, Italy

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Abstract

An aspect of fluid dynamics lies in the search of possible statistical models for Navier-Stokes (NS) fluids described by classical solutions of the incompressible Navier-Stokes equations (INSE). This refers in particular to statistical models based on the so-called inverse kinetic theory (IKT). This approach allows the description of fluid systems by means a suitable 1-point velocity probability density function (PDF) which determines, in terms of suitable “moments”, the complete set of fluid fields which define the fluid state. A fundamental related issue lies in the problem of the unique construction of the initial PDF. The goal of this paper is to propose a solution holding for NS fluids. Our claim is that the initial PDF can be uniquely determined by imposing a suitable set of physical realizability constraints.

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I. INTRODUCTION: CSM-INSPIRED STATISTICAL MODELS

Fundamental aspects of fluid dynamics are related to construction of statistical models \( \{ f, \Gamma \} \) for fluid systems. These are sets \( \{ f, \Gamma \} \) formed by a suitable probability density function (PDF) and a phase-space \( \Gamma \) (subset of \( \mathbb{R}^n \)) on which \( f \) is defined. By definition, a statistical model \( \{ f, \Gamma \} \) of this type must permit the representation, via a suitable mapping

\[
\{ f, \Gamma \} \Rightarrow \{ Z \},
\]

(1)

of the fluid fields \( \{ Z \} \equiv \{ Z_i, i = 1, n \} \) which define the state of the same fluids. Depending whether the mapping provides the complete set or only a subset of \( \{ Z \} \) the statistical model is denoted respectively complete or incomplete. In the case of a Navier-Stokes (NS) fluid, i.e., an incompressible isothermal and isentropic Newtonian fluid described by the incompressible NS equations (INSE), the fluid fields are

\[
\{ Z \} \equiv \{ \rho_0, V, p_1, S_T \},
\]

(2)

where in particular \( \rho_0 \) (the mass density) and \( S_T \) (the thermodynamic entropy) are both assumed constant in \( \Omega \times I \) (with \( \Omega \) denoting the closure of the configuration domain \( \Omega \subseteq \mathbb{R}^3 \) and \( I \subseteq \mathbb{R} \) a time interval/axis). Moreover, \( V \) and \( p_1 \) denote the fluid velocity and the (strictly positive) kinetic pressure

\[
p_1(r, t) = p(r, t) + p_0(t) + \phi(r, t),
\]

(3)

where \( p(r, t), p_0(t) \) and \( \phi(r, t) \) represent respectively the fluid pressure, the (strictly-positive) pseudo-pressure and the (possible) potential associated to the conservative volume force density acting on the fluid. The construction of a statistical model involves the introduction of a suitable PDF of some sort \( f \), defined on an appropriate phase-space \( \Gamma \) in terms of which the fluid fields (or only a proper subset of them) can be represented via suitable velocity or phase-space moments of the PDF. A well-known example of incomplete statistical model, holding for incompressible NS fluids, is provided by the so-called statistical hydromechanics developed originally by Hopf [1] and later extended by Rosen [2] and Edwards [3] (HRE approach), based on the statistical model \( \{ f_H, \Gamma_1 \} \). This relies on the introduction of the 1–point (or local) velocity-space PDF, \( f_1 \), to be intended as the conditional PDF of the velocity \( v \) (kinetic velocity) with respect the remaining variables. In the HRE approach these
are identified with \((r,t)\), where \((r,t) \in \Omega \times I\), while \(f_1 \equiv f_1(r,u,t;Z)\), with \(u \equiv v - V(r,t)\) the relative kinetic velocity, is identified with

\[
f_H \equiv \delta(v - V(r,t))
\]  

\((4)\)

\(f_H\) denoting the three-dimensional Dirac delta defined in the velocity space \(U \subseteq \mathbb{R}^3\), with \(v\) belonging to \(U\) and \((r,t) \in \Omega \times I\) (with \(\Omega \subseteq \mathbb{R}^3\) and \(I\) the configuration domain and the time axis). Hence, \(f_H\) is defined by assumption on the set spanned by the state vector \(x = (r,v)\), i.e., the restricted phase space \(\Gamma_1 \equiv \Omega \times U\). It follows that, in this case, only one velocity moment of \(f_1\), corresponding to \(G = v\), is actually prescribed in terms of the fluid fields and reads

\[
\int_U d^3v G f_H(r,u,t;Z) = V(r,t).
\]  

\((5)\)

The statistical model \(\{f_H, \Gamma_1\}\), which was adopted also by to Monin [4] and Lundgren [5], belongs actually to a more general class of statistical models inspired by Classical Statistical Mechanics (CSM). In this context, a convenient alternative approach - which permits instead the representation in terms of \(f\) of the complete set of fluid fields (hence providing a complete statistical model) - is provided by with the so-called IKT-statistical model \(\{f_1, \Gamma_1\}\), developed in the framework of inverse kinetic theory (IKT; Tessarotto et al. [7–11]). This is based on the construction of a suitable 1-point velocity-space PDF, \(f_1\), defined in such a way to yield,, in terms of a prescribed set of velocity moments of the same PDF, the complete set of fluid fields characterizing a NS fluids.

A basic issue is therefore represented by the construction of the general solution \(f_1(t)\) for the IKT-statistical model, and in particular the initial PDF \(f_1(t_o)\), based on the available information (i.e., observables) on the fluid system.

The goal of this paper is to point out that \(f_1(t)\) can be uniquely determined at the initial time \(t = t_o\). The basic results are represented by THMs 1 and 2 (see Sections 4 and 5). We claim that, subject to the requirement of factorization at the initial time for all multi-point PDFs the initial PDF \(f_1(t_o)\) can be determined imposing the principle of entropy maximization (PEM [13]) on the Boltzmann-Shannon entropy [12]. In detail, we intend to show that the initial PDF \(f_1(t_o)\) can be determined in such a way to satisfy the following requirements:

1. (requirement #1) it satisfies the complete set of constraints placed at the initial time...
t = t_o by the observables defined for the fluid system (physical realizability conditions);

2. (requirement #2) it does not require the specification of higher-order PDFs, with s > 2;

3. (requirement #3) it provides the general solution for the initial PDF required by the IKT-statistical model \{f_1, \Gamma_1\}.

II. PHYSICAL OBSERVABLES OF A NS FLUIDS

An important preliminary task to accomplish is to determine the complete set of physical observables required to define the functional class of the PDF \{⟨f_1(t)⟩_Ω\}, to be prescribed at the initial time t_o. These include, in particular, the complete set of the fluid field which specify the state of the fluid. In the IKT approach [7–11] for an isothermal, isentropic and incompressible NS fluid the latter must be represented in terms of the set \(Ω\).

A further observable can, however, be defined in terms of the velocity field \(V(r,t)\). This is represented by the continuous velocity-frequency density (C-VFD), \(\hat{f}_1^{(freq)}(t, v)\), namely the velocity-frequency associated to the fluid velocity occurring in subsets of the fluid domain \(Ω \subseteq \mathbb{R}^3\), to be defined in particular so that

\[
\int_U d^3v \hat{f}_1^{(freq)}(t, v) = 1, \tag{6}
\]

\(U \equiv \mathbb{R}^3\) denoting the three-dimensional velocity space. For definiteness, let us assume in the remainder that:

- **Assumption #1**: the fluid domain \(Ω\) is a bounded subset of \(\mathbb{R}^3\) with finite measure (for example is a cube).

- **Assumption #2**: the fluid velocity \(V(r,t)\) is bounded. Hence there exists a positive constant \(V_B \in \mathbb{R}\), with \(V_B(t) = 2 \sup \{V_k(r,t); k = 1, 2, 3; r \in Ω\}\) and we can define the bounded subset of \(\mathbb{R}^3\)

\[
U = \left\{ \mathbf{v} : |v_i| \leq \frac{1}{2} V_B; \mathbf{v} \in \mathbb{R}^3 \right\},
\]

to be identified with the velocity space of the statistical model.
Let us require for simplicity that, thanks to Assumption #1, $\Omega$ is partitioned in $N$ like cubic cells. Then let us introduce the notion of continuous velocity-frequency density (C-VFD) $\hat{f}_{1}^{(freq)}(t, \mathbf{v})$ to be defined as the limit function

$$\hat{f}_{1}^{(freq)}(t, \mathbf{v}) = \lim_{N \to \infty} \hat{f}_{1,N}^{(freq)}(t, \mathbf{v}),$$

(7)

$\hat{f}_{1,N}^{(freq)}(t, \mathbf{v})$ indicating the corresponding (discrete) velocity-frequency density function (V-FDF) defined on a $N$-cell partition of $\Omega$

$$\hat{f}_{1,N}^{(freq)}(t, \mathbf{v}) \equiv \frac{1}{N} \sum_{i=1,N} N_1(\mathbf{r}_i, \mathbf{v}, t).$$

(8)

To define $N_1(\mathbf{r}_i, \mathbf{v}, t)$ let us notice that - thanks to Assumption #2 - for each component of the fluid velocity $V_k(\mathbf{r}, t)$ (with $k = 1, 2, 3$) the inequality

$$|V_k(\mathbf{r}, t)| \leq \frac{1}{2} V_B$$

holds. Hence, the velocity space $U$ can be partitioned in $M$ like cubic cells, with $M \in \mathbb{N}_0$ denoting an arbitrary integer. Therefore, if $\mathbf{r}_i$ denotes the position of the center of mass for the $i$-th configuration-space cell, $N_1(\mathbf{r}_i, \mathbf{v}, t)$ can be defined as the frequency of occurrence of the velocity fluid field $\mathbf{v}$, assumed to belong to the velocity-space cell defined (for $k = 1, 2, 3$) by the inequalities $|V_k(\mathbf{r}_i, t) - v_k| \leq \frac{V_B}{2M}$. Thus, $N_1(\mathbf{r}_i, \mathbf{v}, t)$ can be defined as

$$N_1(\mathbf{r}_i, \mathbf{v}, t) = \frac{1}{c} \prod_{k=1,2,3} \Theta_{ik}(\mathbf{v}),$$

(10)

$$\Theta_{ik}(\mathbf{v}) \equiv \Theta(V_k(\mathbf{r}_i, t) - v_k - \frac{V_B}{2M}),$$

$$\Theta(v_k - V_k(\mathbf{r}_i, t) + \frac{V_B}{2M}),$$

(11)

with $\Theta(x)$ denoting the Heaviside theta function, $c \in \mathbb{R}$ the normalization constant

$$c = \int_{U} d^3 v \frac{1}{N} \sum_{i=1,N} \prod_{k=1,2,3} \Theta_{ik}(\mathbf{v})$$

(12)

and $M, N \in \mathbb{N}_0$ arbitrary integers. We stress that they can always be defined so that $M = M(N)$ is a strictly monotonic function of $N$ (in particular, $M$ and $N$ can be, for example, so that $N = M^3$). Then, thanks to positions (10)-(12), by construction $\hat{f}_{1,N}^{(freq)}(\mathbf{v}, t)$ satisfies the normalization condition

$$\int_{U} d^3 v \hat{f}_{1,N}^{(freq)}(\mathbf{v}, t) = 1.$$
This manifestly implies for the limit function $\hat{f}_1^{(\text{freq})}(v,t)$ the analogous condition of normalization (6).

Additional observables, to be defined in analogy to $\hat{f}_1^{(\text{freq})}(v,t)$, are however represented by:

1. the 2-point velocity difference-frequency density function (VD-FDF) $\hat{f}_2^{\text{freq}}(r,v,t_o)$ to be identified with the frequency of the velocity difference $v = V_1 - V_2$ occurring between two positions $r_1$ and $r_2$ with displacement $r = r_1 - r_2$ and subject to the normalization

\[
\int_U d^3v \hat{f}_2^{\text{freq}}(r,v,t_o) = 1; \quad (14)
\]

2. as well as the analogous $s$-point velocity-difference FDFs which can be defined in principle for arbitrary $s \geq 2$.

It is obvious that in principle all the constraints provided by the infinite set of velocity-difference PDFs should be satisfied by the initial 1-point $f_1(t_o)$ [which defines the IKT-statistical model \{\(f_1, \Gamma_1\)\}]!

III. PHYSICAL REALIZABILITY CONDITIONS FOR $f_1(t_o)$

In this reference, a natural question arises, i.e., whether the arbitrariness [in the definition of \{\(f_1, \Gamma_1\)\}] can be used, by proper prescription on its functional class \{(\(f_1(t_o)\))_\Omega\}, to determine it uniquely consistent not only with INSE but also with the relevant physical observables. An important preliminary task to accomplish is to establish the relationship of $f_1$ with the fluid fields. More precisely, here we state that, besides the complete set of fluid fields evaluated at the initial time \{\(\mathbf{V}(r,t_o), p_{1}(r,t_o), S_T(t_o)\)\}, the PDF must also be suitably related to the initial 1-point velocity-frequency density function $\hat{f}_1^{(\text{freq})}(v,t_o)$.

In the following we shall require that $f_1$ satisfies the following constraints (to be intended as physical realizability conditions for $f_1$):

Realizability condition #1: it admits for all $(r,t) \in \Omega \times I$ (including the initial time $t_o$) and $G = 1, v, u^2/2, uu, uu^2/2, \ln f_1(r, v, t)$ the velocity and phase-space moments $\int d\mathbf{v} G f_1$
and \( \int d\mathbf{v} f_1 \ln f_1 \) and satisfies the constraint equations (denoted as correspondence principle):

\[
\int d\mathbf{v} G f_1(\mathbf{r}, \mathbf{v}, t) = 1, V(\mathbf{r}, t), p_1(\mathbf{r}, t),
\]

(15)

\[
S(f_1(t)) = S_T,
\]

(16)

with \( S(f_1(t)) = -\int d\mathbf{x} f_1(\mathbf{r}, \mathbf{v}, t) \ln f_1(\mathbf{r}, \mathbf{v}, t) \) denoting the Boltzmann-Shannon statistical entropy associate to \( f_1(t) \) and \( \Gamma_1 \) the phase space \( \Gamma_1 = \Omega \times U \);

**Realizability condition #2**: at the initial time \( t = t_o \) it satisfies the constraint:

\[
\langle f_1(t_o) \rangle_\Omega = \tilde{f}_{1, freq}(\mathbf{v}, t_o).
\]

(17)

where \( f_1(t) \equiv f_1(\mathbf{r}, \mathbf{v}, t; Z) \) and \( \langle f_1(t) \rangle_{r, \Omega} \) denotes the \( \Omega \)–average at time \( t \)

\[
\langle a(\mathbf{r}, \mathbf{v}, t) \rangle_{r, \Omega} \equiv \frac{1}{\mu(\Omega)} \int d^3 \mathbf{r} a(\mathbf{r}, \mathbf{v}, t).
\]

(18)

**Realizability condition #3**: at the initial time \( t = t_o \), we require that the 2-point PDF \( f_2(1, 2, t_o) \) satisfies the constraint:

\[
\tilde{f}_2(\mathbf{r}, \mathbf{v}, t_o) = \tilde{f}_{2, freq}(\mathbf{r}, \mathbf{v}, t_o),
\]

(19)

where \( \tilde{f}_2(\mathbf{r}, \mathbf{v}, t_o) \) denotes the velocity-difference 2-point PDF defined as

\[
\tilde{f}_2(\mathbf{r}, \mathbf{v}, t_o) = \frac{1}{\mu(\Omega)} \int d\mathbf{R} \int d\mathbf{V} f_2(1, 2, t_o).
\]

(20)

Then requiring that \( f_2(1, 2, t_o) \) is factorized

\[
f_2(1, 2, t_o) = f_1(1, t_o) f_1(2, t_o)
\]

(21)

this implies the constraint

\[
\frac{1}{\mu(\Omega)} \int d\mathbf{R} \int d\mathbf{V} f_1(1, t_o) f_1(2, t_o) = \tilde{f}_{2, freq}(\mathbf{r}, \mathbf{v}, t_o).
\]

(22)

**IV. CONSEQUENCES OF PEM**

Let us now show how \( f_1(1, t_o) \equiv f_1(\mathbf{x}_1, t_o; Z) \) can be determined for the IKT-statistical model \( \{ f_1, \Gamma_1 \} \) by imposing \[8, 9\] that at \( t = t_o \) it satisfies the constrained maximal variational principle (also known as principle of entropy maximization or PEM; Jaynes, 1957
\[ \delta S(f_1(1, t)) = 0, \] 

(23)
together with the \textit{physical realizability conditions} defined above. For definiteness, let us assume that \( f_1(1, t_o) \) is an ordinary, strictly positive function of the general form

\[ f_1(1, t_o) = \tilde{f}_1(1, t_o) h(1, t_o) \langle f_1(1, t) \rangle_{\Omega} h(1, t) \langle h(1, t) \rangle_{\Omega}, \] 

(24) 

with

\[ \tilde{f}_1(1, t_o) = \langle f_1(1, t_o) \rangle_{\Omega} \langle h(1, t_o) \rangle_{\Omega}. \] 

(25) 

\( \langle f_1(1, t_o) \rangle_{\Omega} \) determined by the constraint Eq. (17) and \( h(t_o) \) to be assumed as a strictly positive and regular real function.

Let us determine the initial PDF \( f_1(1, t_o) \) subject to the whole set of constraints placed by the aforementioned physical realizability conditions. Invoking the factorization condition (21) and the position (24) the extremal 1-point PDF \( f_1(1, t_o) \) can be determined in an equivalent way either from the variational principle (23) of for the 2-point entropy principle

\[ \delta S(f_2(t_o)) = 0. \] 

(26)

In fact thanks to (21) it follows that \( S(f_2) = -2\mu(\Omega) \int dx_1 f_1(1, t_o) \ln f_1(1, t_o) \equiv 2\mu(\Omega) S(f_1). \) Hence one obtains

\[ \delta S(f_2) = -2\mu(\Omega) \int dx_1 \tilde{f}_1(1, t_o) \delta h(1, t_o) \left[ 1 + \ln h(1, t_o) + \lambda_o(1, t_o) + \lambda_2(1, t_o) u(t_1, t_o)^2 + \int d x_1 \lambda_3(r, v, t_o) f_1(r, v) \right] = 0, \] 

(27)

where \( \lambda_3(r, v, t_o) \) is a suitable Lagrange multiplier to be determined in such a way to fulfill the constraint (21). Then, considering variations of \( h(1, t_o) \) of the form \( \delta f_1(1, t_o) = \tilde{f}_1(1, t_o) \delta h(1, t_o) \), i.e., defined so that there results identically \( \delta \tilde{f}_1(1, t_o) \equiv 0 \), the previous variational principle requires that \( h(1, t_o) \) must fulfill the variational equation

\[ \int_{\Gamma_1} d x_1 \delta h(1, t_o) \langle f_1(1, t_o) \rangle_{\Omega} \left[ 1 + \ln h(1, t_o) + \lambda_o(1, t_o) + \lambda_2(1, t_o) u(t_1, t_o)^2 + \int d x_1 \lambda_3(r, v, t_o) f_1(r, v) \right] = 0, \] 

(28)
with summation understood on the index $i$ (for $i = 1, 2, 3$). The result, which provides the sought solution for $f_1(t_o)$, has the flavor of:

**THM.1 - General solution for $f_1(t_o)$**

The general solution of the PEM variational principle [Eq.(23)] in the functional class \{\langle f_1(t_o) \rangle_\Omega \} determined by realizability conditions #1-#3 is of the form (24) with $h(1, t_o)$ taking the form of a non-Gaussian distribution

$$h(1, t_o) = \exp \left\{ -1 - \lambda_o(\mathbf{r}_1, t_o) - \lambda_2(\mathbf{r}_1, t_o)u(1, t_o)^2 - \hat{\lambda}_3(1, t_o) \right\},$$

where $\lambda_o(\mathbf{r}, t_o)$ and $\lambda_2(\mathbf{r}, t_o)$ suitable Lagrange multipliers to be determined imposing at $t = t_o$ the moment equations (15), while $\hat{\lambda}_3(1, t_o)$ denotes

$$\hat{\lambda}_3(1, t_o) \equiv \hat{\lambda}_3(\mathbf{r}_1, \mathbf{v}_1, t_o) = \int_{\Gamma_1} d\mathbf{x}_2 \lambda_3(\mathbf{r}, \mathbf{v}, t_o)f_1(\mathbf{r}_2, \mathbf{v}_2, t_o)$$

and $\lambda_3(\mathbf{r}, \mathbf{v}, t_o)$ is determined in such a way to fulfill the constraint (22).

It follows that:

1. $f_1(1, t_o)$ is strictly positive and determined as velocity and phase-space moments the fluid velocity $\mathbf{V}(\mathbf{r}, t_o)$, the kinetic pressure $p_1(\mathbf{r}, t_o)$ as well as the thermodynamic entropy $S_T$ [8–10];

2. due to the arbitrariness of $\langle f_1(1, t_o) \rangle_\Omega$, $f_1(1, t_o)$ is generally non-Gaussian and non-isotropic in velocity space;

3. the PDF $f_1(t_o)$ does not require the specification of constraints on any higher-order multi-point PDF of order $s > 2$.

**PROOF**

In fact there it results identically

$$\delta \int_{\Gamma_1} d\mathbf{x} h(t_o) \frac{\langle f_1(1, t_o) \rangle_\Omega}{\langle h(1, t_o) \rangle_\Omega} \ln \frac{\langle f_1(1, t_o) \rangle_\Omega}{\langle h(1, t_o) \rangle_\Omega} = 0. \quad (31)$$

From the Euler-Lagrange equation it is immediate to reach Eq.(29) for $h(1, t_o)$ [see the Euler-Lagrange (28) above]. To prove 1) and 2) we notice that, by construction $f_1(1, t_o)$ is strictly positive its moments satisfy the correspondence principle constraints, while from
and \((29)\) it follows that \(f_1(1,t_o)\) is generally non-Maxwellian. Finally to prove 3) it is sufficient to notice that constraints of the type \((22)\) to be placed \(s\)-point PDFs \((s > 3)\) leave unchanged the Euler-Lagrange equation \((28)\). Q.E.D.

As a consequence of THM.1, Eqs. \((24)\) and \((29)\) provide the general solution for the initial PDF \(f_1(1,t_o)\), fulfilling all the physical realizability conditions defined above [see Sec.3]. Remarkably, the solution determined in this way does not require the specification of possible infinite set of additional constraints, analogous to Eq.\((22)\), to be placed on the initial multi-multipoint PDFs of order \(s > 2\).

V. PARTICULAR SOLUTIONS

It is immediate to show that particular solutions are respectively provided by:

a) the generally non-Gaussian PDF:

\[
f_1(1,t_o) = \langle f_1(1,t_o) \rangle_{\Omega} \frac{h(1,t_o)}{\langle h(1,t_o) \rangle_{\Omega}},
\]

with \(h(1,t_o)\) the Gaussian PDF

\[
h(1,t_o) = \exp \left\{ -1 - \lambda_3(r_{1o},t_o) - \lambda_2(r_{1o},t_o)u(1,t_o)^2 \right\};
\]

b) the Gaussian PDF

\[
f_1(1,t_o) = f_M(u(r_1,t_o); p_1(r_1,t_o)),
\]

with

\[
f_M(u(r,t_o); p_1(r,t_o)) = \frac{1}{\pi^{3/2}v_{thp}(r_1,t_o)^3} \exp \left\{ -\frac{u(r_1,t_o)^2}{v_{thp}(r_1,t_o)^2} \right\},
\]

and \(v_{thp}(r_1,t_o) = \sqrt{2p_1(r_1,t_o,\alpha)/\rho_0}\) denoting the thermal velocity associated to the kinetic pressure.

THM.2 - Particular solutions for \(f_1(t_o)\)

\(f_1(t_o)\) coincides either with \((32)\) or \((34)\) respectively if:

A) there results identically:

\[
\lambda_3(r_1,v_1,t_o) \equiv 0;
\]

B) besides \((36)\), also the equation

\[
\frac{\langle f_1(t_o) \rangle_{\Omega}}{\langle h(t_o) \rangle_{\Omega}} = 1
\]
holds identically.

PROOF

In validity of the constraint (36) Eqs. (32) and (33) follow from Eqs. (29) and (31). In validity of Eq. (37) too, it is immediate to prove that PEM implies that for arbitrary variations \( \delta h(r, u, t_o) \) it must result identically:

\[
\int_{\Gamma_1} d\mathbf{x} \delta h(r, v, t_o) \{ 1 + \ln h(r, v, t_o) + \lambda_o + \lambda_2 u^2 \} = 0.
\]

It follows that the Lagrange multipliers \( \lambda_o(r, t_o) \) and \( \lambda_2(r, t_o) \), determined by imposing the correspondence principle, necessarily require

\[
f_1(t_o) = f_M(v - V(r, t_o); p_1(r, t_o)).
\]

Q.E.D.

VI. CONCLUSIONS

An axiomatic approach, based on the IKT-statistical model \( \{ f_1, \Gamma_1 \} \), has been developed to determine the initial condition for the 1-point PDF \( f_1 \) which characterizes an isothermal and isentropic incompressible NS fluid.

In particular, we have proven that - extending the statistical approach earlier developed \[7–11\] - the initial 1-point PDF can be uniquely determined by invoking the principle of entropy maximization. The present theory is based on the assumption that the initial PDF satisfies suitable physical realizability conditions. In detail, this requires that the initial PDF \( f_1(t_o) \) be constructed in such a way to satisfy the physical requirements placed by:

- the correspondence principle [in particular Eqs. (15)], prescribing that the PDF determines in terms of suitable moments the complete set of fluid fields;
- the velocity-frequency density function (V-FDF) \( \hat{f}_1^{\text{freq}}(v, t_o) \) (17);
- the 2-point velocity difference-frequency density function (VD-FDF) \( \hat{f}_2^{\text{freq}}(r, v, t_o) \), in term of the constraint (22).

The theory has several important consequences:
1. THM.1 provides the general form of the initial 1-point PDF satisfying the previous physical requirements;

2. the initial 1-point PDF is generally non-Gaussian and non-isotropic in velocity-space (see THM. 1).

3. particular solutions include, including the Gaussian 1-point PDF, are discussed in THM.2.

In addition, the present theory provides an answer to the requirements placed in Sec.1, namely that the initial PDF $f_1(t_o)$ obtained in this way is unique. In fact we have proven (THM.1) that the determination of $f_1(t_o)$ does not require the specification of higher-order PDFs of order $s > 2$ and thus provides the general solution for the initial $f_1(t_o)$ of the IKT-statistical model $\{f_1, \Gamma_1\}$.

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