The Barwise-Schlipf Theorem

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In their seminal paper \cite{1}, Barwise and Schlipf initiated the study of recursively saturated models of PA with the following theorem.

\textbf{Theorem:} (Barwise-Schlipf \cite{1}) If $\mathcal{M} \models PA$ is nonstandard, then the following are equivalent:

(1) $\mathcal{M}$ is recursively saturated.

(2) There is $\mathcal{X}$ such that $(\mathcal{M}, \mathcal{X}) \models \Delta^1_1-CA_0$.

(3) $(\mathcal{M}, \text{Def}(\mathcal{M})) \models \Delta^1_1-CA_0 + \Sigma^1_1-AC_0$.

Their proof of (1) $\implies$ (3) (\cite{1} Theorem 2.2) uses Admissible Set Theory. In a reprise of this theorem by Smoryński \cite{5} Sect. 4, a more direct proof of this implication, attributed to Fefferman and Stavi (independently), is presented. This same proof is essentially repeated by Simpson \cite{4} Lemma IX.4.3. The implication (3) $\implies$ (2) is trivial. In the proof of the remaining implication (2) $\implies$ (1), it is claimed \cite{1} Theorem 3.1 that if $\mathcal{M}$ is nonstandard and not recursively saturated and $\text{Def}(\mathcal{M}) \subseteq \mathcal{X} \subseteq \mathcal{P}(\mathcal{M})$, then $(\mathcal{M}, \mathcal{X}) \not\models \Delta^1_1-CA_0$ because the standard cut $\omega$ is $\Delta^1_1$-definable \textsuperscript{[4]} in $(\mathcal{M}, \mathcal{X})$. To prove this, they consider a specific infinite set $Y \subseteq \omega$ and then show that it is $\Sigma^1_1$-definable in $(\mathcal{M}, \mathcal{X})$. However, their purported $\Pi^1_1$-definition of $Y$ does not work as it actually defines the set $Y \cup (\mathcal{M}\setminus\omega)$. Smoryński makes a similar error in his explicit claim \cite{5} Lemma 4.2 that if $\mathcal{M}$ is nonstandard and not recursively saturated and $(\mathcal{M}, \mathcal{X}) \models ACA_0$, then $\omega$ is $\Delta^1_1$-definable in $(\mathcal{M}, \mathcal{X})$. We will show in Theorem 2 that this approach is doomed since there are nonstandard models $\mathcal{M}$ that are not recursively saturated even though $\omega$ is

\textsuperscript{1}All usages of \textit{definable} in this paper should be understood as \textit{definable with parameters}.
not $\Delta^1_1$-definable in $(\mathcal{M}, \text{Def}(\mathcal{M}))$. Nevertheless, we are still able to give a proof (see Theorem 3) of $(2) \implies (1)$.

Suppose that $\mathcal{M} \models \text{PA}$ and $A \subseteq M$. Then, $A$ is **recursively $\sigma$-definable** if there is a recursive sequence $\langle \varphi_n(x) : n < \omega \rangle$ of formulas, each $\varphi_n(x)$ defining a subset $A_n \subseteq M$, such that $A = \bigcup_{n<\omega} A_n$. (For such a sequence to be recursive, it is necessary that there is a finite set $F \subseteq M$ such that any parameter occurring in any $\varphi_n(x)$ is in $F$.) For example, in every nonstandard $\mathcal{M} \models \text{PA}$, the standard cut $\omega$ is recursively $\sigma$-definable.

**Lemma 1:** Suppose that $\mathcal{M} \models \text{PA}$ and $A \subseteq M$.

(a) If $A$ is $\Sigma^1_1$-definable in $(\mathcal{M}, \text{Def}(\mathcal{M}))$, then $A$ is recursively $\sigma$-definable.

(b) If $\mathcal{M}$ is not recursively saturated, $\text{Def}(\mathcal{M}) \subseteq \mathfrak{X} \subseteq \mathcal{P}(\mathcal{M})$ and $A$ is recursively $\sigma$-definable, then $A$ is $\Sigma^1_1$-definable in $(\mathcal{M}, \mathfrak{X})$.

**Proof.** (a) Suppose that $A$ is $\Sigma^1_1$-definable in $(\mathcal{M}, \text{Def}(\mathcal{M}))$ by the formula $\exists X \theta(x, X)$. Let $\varphi_n(x)$ be the formula asserting: there is a $\Sigma_n$-definable subset $X$ such that $\theta(x, X)$. Then $\langle \varphi_n(x) : n < \omega \rangle$ is recursive and shows that $A$ is recursively $\sigma$-definable.

(b) Let $\text{Sat}(x, X)$ be a formula asserting that $X$ is a satisfaction class for all formulas of length at most $x$. Let $A$ be recursively $\sigma$-definable by the recursive sequence $\langle \varphi_n(x) : n < \omega \rangle$. We can assume that $\ell(\varphi_n(x)) < \ell(\varphi_{n+1}(x))$ for all $n < \omega$, where $\ell(\varphi(x))$ is the length of $\varphi(x)$ (by replacing $\varphi_n(x)$ with $\bigvee_{i \leq n} \varphi_i(x)$). The sequence $\langle \varphi_n(x) : n < \omega \rangle$ is coded in $\mathcal{M}$, so let $d \in M$ be nonstandard such that $\langle \varphi_n(x) : n < d \rangle$ extends $\langle \varphi_n(x) : n < \omega \rangle$ and $\ell(\varphi_n(x))$ is standard iff $n$ is. Then $A$ is $\Sigma^1_1$-definable in $(\mathcal{M}, \text{Def}(\mathcal{M}))$ by the formula $\exists X \theta(x, X)$, where

\[
\theta(x, X) = \exists z [\text{Sat}(z, X) \land \exists n < d (\ell(\varphi_n) \leq z \land \langle \varphi_n, x \rangle \in X)].
\]

Thus, $A$ is $\Sigma^1_1$-definable in $(\mathcal{M}, \text{Def}(\mathcal{M}))$. The same definition works in $(\mathcal{M}, \mathfrak{X})$. \(\square\)

**Theorem 2:** Every completion $T$ of PA has a nonstandard, finitely generated (so not recursively saturated) model $\mathcal{M}$ such that $\omega$ is not $\Delta^1_1$-definable in $(\mathcal{M}, \text{Def}(\mathcal{M}))$.

**Proof.** Let $T$ be a completion. According to [3, Corollary 2.8], there is a finitely generated $\mathcal{M} \models T$ such that, in the terminology of [3], $\omega$ is not recursively definable. Clearly, $\mathcal{M} \setminus \omega$ is not recursively $\sigma$-definable. By Lemma 1(a), $\omega$ is not $\Pi^1_1$-definable in $(\mathcal{M}, \text{Def}(\mathcal{M}))$. \(\square\)

**Theorem 3:** If $\mathcal{M}$ is nonstandard and $(\mathcal{M}, \mathfrak{X}) \models \Delta^1_1\text{CA}_0$, then $\mathcal{M}$ is recursively saturated.

**Proof.** We will show that if $\mathcal{M}$ is nonstandard and not recursively saturated and $\mathfrak{X} \subseteq \mathcal{P}(\mathcal{M})$, then $(\mathcal{M}, \mathfrak{X}) \not\models \Delta^1_1\text{CA}_0$. We can assume that $(\mathcal{M}, \mathfrak{X}) \models \text{ACA}_0$. There are two cases depending on whether $\mathcal{M}$ is short or tall.
**M** is short: Let $a \in M$ be such that the elementary submodel of $M$ generated by $a$ is cofinal in $M$. Let $\langle \varphi_n(x) : n < \omega \rangle$ be a recursive sequence of formulas (with $a$ as the only parameter) such that $\varphi_n(x)$ defines $d_n \in M$, where $d_n$ is the least element not definable from $a$ by a $\Sigma_n$ formula. Thus, $\langle d_n : n < \omega \rangle$ is a strictly increasing, unbounded sequence. Let $D = \{d_n : n < \omega\}$. Since $(M, \mathcal{X}) \models ACA_0$, then $D \not\in \mathcal{X}$ as otherwise $\omega \in \mathcal{X}$. Clearly, $D$ is recursively $\sigma$-definable; its complement also is (using the recursive sequence $\langle \psi_n(x) : n < \omega \rangle$, where $\psi_0(x)$ is $x < d_0$ and $\psi_{n+1}(x)$ is $d_n < x < d_{n+1}$). By Lemma 1(b), $D$ is $\Delta^1_1$-definable in $(M, \mathcal{X})$.

**M** is tall: Since $M$ is tall and not recursively saturated, there is a recursive sequence $\langle \varphi_n(x) : n < \omega \rangle$ of formulas, among which is a formula $x < b$, that is finitely realizable in $M$ but not realizable in $M$. According to [2, Lemma 2.4], we can assume that each $\varphi_n(x)$ defines an interval $[a_n, b_n]$, where $a_n < a_{n+1} < b_{n+1} < b_n$. Then, the cut $I = \sup\{a_n : n < \omega\} = \inf\{b_n : n < \omega\}$, so both $I$ and its complement are recursively $\sigma$-definable. Lemma 1 implies $I$ is $\Delta^1_1$-definable in $(M, \mathcal{X})$. Since $I \not\in \mathcal{X}$, then $(M, \mathcal{X}) \not\models \Delta^1_1$-$CA_0$. □

We conclude with several remarks concerning the Theorem.

It is well known that $\Sigma^1_k$-$AC_0$ implies $\Delta^1_k$-$CA_0$ for all $k < \omega$. An easy proof can be found in [1, Lemma VII.6.6(1)]. Apparently, when Barwise and Schlipher were writing [1], they were unaware of this, but by the time Smoryński wrote [5], this became well known, as he describes as “evident” that $\Sigma^1_1$-$AC_0$ implies $\Delta^1_1$-$CA_0$.

Barwise and Schlipher go out of their way to point out [1, Remark, p. 52] that their (erroneous) proof of $(2) \implies (1)$ shows the slightly stronger implication in which $\Delta^1_1$-$CA_0$ is replaced by its counterpart $\Delta^1_1$-$CA_0^{-}$ in which there are no set parameters. The same is true of our proof of $(2) \implies (1)$.

It is rather ironic that the impression one gets from reading [1] is that $(1) \implies (3)$ is the deep direction of the Theorem, whereas $(2) \implies (1)$ is the straightforward one. In retrospect, the exact opposite is the case: the hard direction is $(2) \implies (1)$ while $(1) \implies (3)$ is fairly routine.

**References**

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