Analytic solution for one dimensional inverse heat conduction problem of semi-infinite bar

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Abstract

We present analytical formula along with its existence theorem for solution of inverse heat conduction problem of semi-infinite bar, equivalent to a Volterra integral equation of first kind, as an infinite series of fractional derivatives. The mathematical method is based on some properties of function space $M[0,T]$ (proved here) with respect to fractional integration and derivatives.

Keywords: Fractional derivatives; Function space; Volterra integral equation of first kind; Inverse heat conduction problem

1 Introduction

In this paper we consider the problem of finding analytic solution for one dimensional inverse heat conduction problem of semi-infinite bar (Cauchy problem (14)). This problem can turn into a problem of Volterra integral equation of first kind (VIE1),

$$\int_0^t K(x_0-x,t-\tau) u(x,\tau) d\tau = g(t), \quad 0 \leq x < x_0, \quad 0 < t \leq T,$$

where,

$$K(x,t) = \begin{cases} \frac{x}{2\sqrt{\pi}t^{3/2}} e^{-\frac{x^2}{4t}} & \text{if } 0 < t \leq T, \\ 0 & \text{if } t = 0. \end{cases} \quad \text{(1)}$$

The concern of this paper is only on finding the analytical solution along with an existence theorem for this problem. For regularization of this specific ill-posed problem the reader is referred to [8]. Although considerable amount of work has been done for analytical solution, existence condition and regularization of one dimensional inverse heat conduction problem of finite bar (Cauchy problem (14)), e.g. see [2], the analytical solution of Cauchy problem (14) or equivalently the above VIE1 problem, seems to be not treated in the literature.

The method we use to solve this problem is based on properties of function space $M[0,T]$; see definition 1 below. This function space is proved to be a useful tool in analyzing linear integro differential causal problems [5]. In this paper we prove and use different properties
of $M[0, T]$, regarding fractional derivatives and fractional integration. The importance of this function space to this problems is due to the fact that $K(x, t)$ (and all partial derivatives $\partial_x^j K(x, t)$) for every $T > 0$ as a function of $t$ belongs to $M[0, T]$ for all $x \neq 0$. From this it follows that $g(t)$ belongs to $M[0, T]$, when the solution $u(x, t)$ is assumed to be a continuous function of $t$. In section 2 of this paper we prove the fact that $M[0, T]$ is closed under fractional integrations and derivatives. In particular we show in part 1 of Theorem 1 that the Riemann-Liouville fractional derivatives and the Caputo fractional derivatives to all order exist and they coincide (for every fractional order) as elements of $M[0, T]$. The relation (10) in part 2 of Theorem 1 is used in section 3 to solve the above VIE1 which is a convolution type VIE1 with its kernel being an element of $M[0, T]$.

In the part 3.1, by using the results of section 2 we are able to deduce from the above VIE1 in Theorem 2 that,

$$\partial_x u(x, t)|_{x=x_0} = -\frac{1}{2} \partial_t^\frac{1}{2} g(t),$$

where $\partial_t^\frac{1}{2} g(t) = RL D_{0^+}^{\frac{1}{2}} g(t) = CD_{0^+}^{\frac{1}{2}} g(t)$. Using the result above, the problem of semi-infinite bar can turn into the problem of finite bar (equation (16)) and by which we find the answer as,

$$u(x, t) = \sum_{n=0}^{\infty} \partial_t^\frac{1}{2} (g(t)) \frac{(x_0 - x)^n}{(n)!}.$$ 

In the part 3.2 we discuss the convergence of this solution for $u(x, t)$ and derive an existence theorem for this problem using Holmgren classes. Finally in 3.3 a convergent example is provided for the solution $u(x, t)$, through a special case of initial value problem of one dimensional direct heat conduction equation of an infinite bar ($x \in (-\infty, \infty)$).

## 2 Function space $M[0, T]$ and fractional derivatives

For the rest of this paper we adopt the following notations and definitions.

$\mathbb{N} := \{1, 2, \ldots\}$ is the set of natural numbers. $\mathbb{Z}_+ := \{0, 1, 2, \ldots\}$ is the set of non-negative integers. $C^n[a, b]$ is the space of $n$ times continuously differentiable functions on $[a, b]$. $C^\infty[a, b]$ is the space of smooth functions on $[a, b]$.

Let us recall the definition of function space $M[0, T]$ from [5].

**Definition 1.** Vector space $M[0, T]$ is the space of all functions $\phi(x) \in C^\infty[0, T]$ for which,

$$\phi(0) = 0 \quad \text{and} \quad \frac{d^n}{dt^n} \phi(t)|_{t=0} = 0, \quad \forall n \in \mathbb{N}. \quad (2)$$

We also say $\phi(t) \in M[0, +\infty)$ if $\phi(t) \in M[0, T]$ for every $T > 0$. One can show for example that the function $K(x, t)$ given by (11) as a function of $t$ is in $M[0, +\infty]$ for $x \neq 0$. It can be also easily checked that function space $M[0, T]$ satisfies the following conditions.

$$\phi(t) \in M[0, T] \implies \partial_t \phi(t) \in M[0, T], \quad \int_0^t d\tau \phi(\tau) \in M[0, T]. \quad (3)$$
\( \phi(t) \in M[0, T], \ P(t) \in C[0, T] \implies \int_0^t d\tau \ \phi(t - \tau)P(\tau) \in M[0, T]. \) \hfill (4)

From (3) one finds, the function space \( M[0, T] \) is closed under actions of derivative, \( \partial_t : \phi(t) \mapsto \partial_t \phi(t) \), and integral, \( \int_0^t d\tau : \phi(t) \mapsto \int_0^t d\tau \phi(\tau) \) operators. In this paper, in many steps we use this property of \( M[0, T] \) in the integration by parts without mentioning. Also by (4) each \( \phi(t) \in M[0, T] \) can be viewed as a map from \( C[0, T] \) to \( M[0, T] \) via Volterra convolution operator.

Considering fractional integral operator defined as,

\[
J_\alpha^t(P(t)) = \frac{1}{\Gamma(\alpha)} \int_0^t d\tau (t - \tau)^{\alpha-1} P(\tau), \quad 0 < \alpha < 1, \ P(t) \in C[c, T],
\]

where \( 0 \leq c < t \), in the following lemma we prove \( M[0, T] \) is closed under action of fractional integral operator \( J_\alpha^t \) and also, the operators \( J_\alpha^t \) and \( \partial_t \) commute on elements of \( M[0, T] \).

**Lemma 1.** If \( \phi(t) \in M[0, T] \) and \( P(t) \in C[0, T] \) then for any \( 0 < \alpha < 1 \) we have,

\[
J_\alpha^t(\phi(t)) \in M[0, T], \quad \partial_t(J_\alpha^t(\phi(t))) = J_\alpha^t(\partial_t \phi(t)),
\]

\[
J_\alpha^t(\int_0^t \phi(t - \tau)P(\tau) d\tau) = \int_0^t J_\alpha^{t-\tau}(\phi(t - \tau))P(\tau)d\tau.
\] \hfill (5)

**Proof.** For the first part we note,

\[
J_\alpha^t(\phi(t)) = \frac{1}{\Gamma(\alpha)} \int_0^t d\tau (t - \tau)^{\alpha-1} \phi(\tau) = \frac{1}{\Gamma(\alpha)\alpha} \int_0^t d\tau (t - \tau)^{\alpha} \partial_\tau \phi(\tau).
\]

Thus \( J_\alpha^t(\phi(t))|_{t=0} = 0 \) and \( J_\alpha^t(\phi(t)) \) is continuous on \([0, T]\). Furthermore by last line we have,

\[
\partial_t(J_\alpha^t(\phi(t))) = \frac{1}{\Gamma(\alpha)} \int_0^t d\tau (t - \tau)^{\alpha-1} \partial_\tau \phi(\tau) = \frac{1}{\Gamma(\alpha)\alpha} \int_0^t d\tau (t - \tau)^{\alpha} \partial_\tau^2 \phi(\tau).
\]

So, \( \partial_t(J_\alpha^t(\phi(t)))|_{t=0} = 0 \) and \( \partial_t(J_\alpha^t(\phi(t))) \) is continuous on \([0, T]\). Continuing in the same way one can show,

\[
\partial_\tau^n(J_\alpha^t(\phi(t))) = \frac{1}{\Gamma(\alpha)\alpha} \int_0^t d\tau (t - \tau)^{\alpha} \partial_\tau^{n+1} \phi(\tau).
\] \hfill (6)

Therefore,

\[
\forall n \in \mathbb{Z}_+, \quad \partial_t^n(J_\alpha^t(\phi(t)))|_{t=0} = 0 \quad \text{and} \quad J_\alpha^t(\phi(t)) \in C^n[0, T].
\]

Thus, \( J_\alpha^t(\phi(t)) \in M[0, T] \).

For the second part using (5) we have,

\[
\partial_t(J_\alpha^t(\phi(t))) = \frac{1}{\Gamma(\alpha)\alpha} \int_0^t d\tau (t - \tau)^{\alpha} \partial_\tau^2 \phi(\tau) = \frac{1}{\Gamma(\alpha)} \int_0^t d\tau (t - \tau)^{\alpha-1} \partial_\tau \phi(\tau)
\]

\[
= J_\alpha^t((\partial_t \phi(t))).
\]
For the third part by taking \( F(t) = (\int_0^t \phi(t - \tau)P(\tau)\,d\tau) \), we have
\[
\partial_t F(t) = (\int_0^t \partial_t \phi(t - \tau)P(\tau)\,d\tau)
\]
and also from (1), \( F(t) \in M[0, T] \). Now by starting from the left hand side of (5) we have,
\[
J_0^\alpha (F(t)) = \frac{1}{\Gamma(\alpha)} \int_0^t d\tau (t - \tau)^\alpha \partial_\tau F(\tau)
\]
\[
= \frac{1}{\Gamma(\alpha)} \int_0^t d\tau (t - \tau)^\alpha \int_0^\tau \partial_\tau \phi(\tau - z)P(z)\,dz
\]
\[
= \frac{1}{\Gamma(\alpha)} \int_0^t dz G(t, z) P(z)\,dz.
\]

In the last line we use Volterra repeated integral formula (that is if \( B(t, \tau) \) and \( L(t, \tau) \) are continuous on \( \Delta = \{(t, \tau) \mid 0 \leq \tau \leq t \leq T \} \) and \( Q(t) \) is a continuous function on \( [0, T] \) then
\[
\int_0^t d\tau B(t, \tau) \int_0^\tau dz L(\tau, z)Q(z) = \int_0^t dz H(t, z)Q(z), \text{ where } H(t, z) = \int_0^t d\tau B(t, \tau)L(\tau, z),
\]
so \( G(t, z) \) is given by,
\[
G(t, z) = \int_z^t d\tau (t - \tau)^\alpha \partial_\tau \phi(\tau - z) = \alpha \int_z^t d\tau (t - \tau)^{\alpha - 1} \phi(\tau - z)
\]
\[
= \alpha \int_0^t d\tau' (t - \tau' - z)^{\alpha - 1} \phi(\tau'). = \alpha \Gamma(\alpha) J_0^\alpha (\phi(t - z)).
\]

Thus by inserting the expression for \( G(x, z) \) into (1) we have,
\[
J_0^\alpha (F(t)) = \int_0^t dz J_0^\alpha (\phi(t - z)) P(z)\,dz.
\]

For any \( \nu > 0 \), with \([\nu] = m\) (that is, \( m \leq \nu < m + 1 \) and \( m \in \mathbb{Z}_+ \)), the Riemann-Liouville fractional derivative and the Caputo fractional derivative are defined for \( P(t) \in C^\infty[c, T] \) as (e.g. see [10]),
\[
RLD_{c,t}^\nu P(t) = \partial_t^{m+1} (J_{c,t}^{m-\nu+1} (P(t))), \quad (7)
\]
\[
CD_{c,t}^\nu P(t) = (J_{c,t}^{m-\nu+1} (\partial_t^{m+1} P(t))). \quad (8)
\]

Now we state the following theorem regrading fractional derivatives \( RLD_{0,t}^\nu \) and \( CD_{0,t}^\nu \) of elements of \( M[0, T] \).

**Theorem 1.** 1-If \( \phi(t) \in M[0, T] \) then for any \( \nu > 0 \), with \([\nu] = m\), the Riemann-Liouville fractional derivative and the Caputo fractional derivative exist and coincide in \( M[0, T] \), in other word,
\[
\partial_{0,t}^\nu \phi(t) =: RL D_{0,t}^\nu \phi(t) = CD_{0,t}^\nu \phi(t)
\]
\[
= \frac{\partial_t^{m+1} \int_0^t ds (t - s)^{m-\nu} \phi(s)}{\Gamma(m - \nu + 1)}
\]
\[
= \frac{\int_0^t ds (t - s)^{m-\nu} \partial_s^{m+1} \phi(s)}{\Gamma(m - \nu + 1)} \in M[0, T]. \quad (9)
\]
2-For any \( P(x) \in C[0, T] \) and \( \phi(x) \in M[0, T] \),
\[
\partial_{0,t}^\nu \int_0^t \phi(t-\tau)P(\tau)d\tau = \int_0^t (\partial_{0,t-\tau}^\nu \phi(t-\tau))P(\tau)d\tau. \tag{10}
\]

**Proof.** Part 1 is the consequence of the fact that \( M[0, T] \) is closed under actions of both derivative \( \partial_t \) and fractional integration \( J_0^\alpha \) and also the fact that derivative operator \( \partial_t \) and fractional integration operator \( J_0^\alpha \) commute.

For part 2 we have,
\[
\partial_{0,t}^\nu \int_0^t \phi(t-\tau)P(\tau)d\tau = \partial_{0,t}^{m+1}(J_0^m J_0^\nu)(\int_0^t \phi(t-\tau)P(\tau)d\tau) = \int_0^t (\partial_{0,t-\tau}^\nu \phi(t-\tau))P(\tau)d\tau. \tag{11}
\]

By Lemma 1 we have \( J_0^m J_0^\nu \phi(t) \in M[0, T] \), it follows that,
\[
\forall n \in Z_+, \quad \partial_t^n (J_0^m J_0^\nu \phi(t-\tau))|_{t=\tau} = 0,
\]
therefore in \( \tag{11} \), \( \partial_{0,t}^{m+1} \) can go inside the integral,
\[
\partial_{0,t}^\nu \int_0^t \phi(t-\tau)P(\tau)d\tau = \int_0^t \partial_{0,t-\tau}^{m+1}(J_0^m J_0^\nu \phi(t-\tau))P(\tau)d\tau = \int_0^t (\partial_{0,t-\tau}^\nu \phi(t-\tau))P(\tau)d\tau.
\]

We state the following Lemma regarding some required properties of \( K(x,t) \) as our last needed mathematical tool for the following section.

**Lemma 2.** For \( K(x,t) \) given by (1) we have,
\[
\partial_x K(x,t) = \partial_{0,t}^2 K(x,t), \quad x > 0, \quad t \geq 0, \tag{11}
\]
\[
\lim_{x \to 0^+} \int_0^t d\tau K(x,t-\tau)P(\tau) = P(t), \quad P(t) \in C[0, T]. \tag{12}
\]

**Proof.** Equations \( \tag{11} \) can be easily verified directly. By applying \( \partial_{0,t}^2 \) (or \( \partial_x \)) again on \( \tag{11} \) one can reach to more familiar relation, \( \partial_t K(x,t) = \partial_{0,t}^2 K(x,t) \). For equation \( \tag{12} \) for example see \( \tag{4.2.3} \) lemma (4.2.3).

### 3 One dimensional inverse heat conduction problem of semi-infinite bar

In this section for convenience we denote,
\[
\partial_{0,t}^\nu := \partial_t^\nu, \quad \text{for } \nu > 0, \tag{13}
\]
for elements of \( M[0, T] \). Here we consider finding analytic solution for Cauchy problem of determining a function \( u(x, t) \) that satisfies,

\[
\begin{align*}
\partial_x^2 u(x, t) &= \partial_t u(x, t), & 0 \leq x < \infty, \quad 0 \leq t \leq T, \\
u(x_0, t) &= g(t), & x_0 \in (0, \infty), \quad 0 \leq t \leq T, \\
u(x, 0) &= 0, & 0 \leq x < \infty.
\end{align*}
\]

(14)

The equation (14) can be considered as the formulation of one dimensional inverse heat problem for retrieving \( u(x, t) \), the temperature at time \( t \) \((0 \leq t \leq T)\) and at point \( x \) \((x \geq 0)\) along semi-infinite conducting bar \((x \in [0, \infty))\) on non-negative \( x \)-axis which is initially at constant temperature zero \((u(x, 0) = 0\) for \( x \in [0, \infty))\). The bar is insulated all the way except at \( x = 0 \) (where it is subjected to unknown heat source) and our given data, \( g(t) = u(x_0, t) \), is the temperature function at specific point \( x_0 \in (0, \infty) \) for the time period, \( 0 \leq t \leq T \).

For \( u(x, t) \), the solution of this problem, it can be proved (e.g. see [1]),

\[
\int_0^t K(x' - x, t - \tau) u(x', \tau) d\tau = u(x', t), \quad x < x',
\]

where \( K(x, t) \) is given by (1). This leads to the following equivalent problem of Volterra integral equation of first kind to equation (14), for \( 0 < x \leq x_0 \),

\[
\int_0^t K(x_0 - x, t - \tau) u(x, \tau) d\tau = g(t), \quad 0 \leq x < x_0 \quad 0 \leq t \leq T,
\]

(15)

and the following answer for \( x_0 < x \),

\[
u(x, t) = \int_0^t K(x - x_0, t - \tau) g(\tau) d\tau, \quad x_0 < x, \quad 0 \leq t \leq T.
\]

The function \( K(x, t) \) \((x \neq 0)\) as a function of \( t \) is smooth on \([0, T]\) and satisfies,

\[
\forall n \in \mathbb{Z}_+, \quad \partial_t^n K(x, t)|_{t=0} = 0, \quad x \in \mathbb{R} \setminus \{0\}.
\]

where \( \mathbb{Z}_+ = \{0, 1, 2, \ldots\} \). In equation (15) if we assume that \( u(x, t) \), as a function of \( t \), is a continuous function on \([0, T]\), then it follows (by differentiating (15)) that \( g(t) \) is smooth on \([0, T]\) and it also satisfies,

\[
\forall n \in \mathbb{Z}_+, \quad \partial_t^n g(\tau)|_{t=0} = 0,
\]

thus \( g(t) \in M[0, T] \).

### 3.1 Analytic Solution

Let us first begin by stating the problem and solution of analogous case of inverse heat problem for finite bar \((x \in [0, l]\)),

\[
\begin{align*}
\partial_x^2 w(x, t) &= \partial_t w(x, t), & 0 \leq x \leq l, \quad 0 \leq t \leq T, \\
w(l, t) &= h(t), & 0 \leq t \leq T, \\
\partial_x w(x, t)|_{x=l} &= f(t), & 0 \leq t \leq T.
\end{align*}
\]

(16)
Equation (16) is the formulation of inverse heat conduction problem of retrieving \(w(x,t)\), the temperature at point \(x\) \((0 \leq x \leq l)\) and time \(t\) \((0 \leq t \leq T)\) of a finite bar (located between points \(x = 0\) and \(x = l\) of \(x\)-axis, insulated for \(0 < x < l\) and is subjected to unknown heat source at \(x = 0\)) from measuring the temperature, \(h(t) = w(l,t)\), and heat flux \(f(t) = \partial_x w(x,t)|_{x=l}\) at point \(x = l\) for \(0 \leq t \leq T\). The solution to this problem, equation (16), can be derived easily just by assuming the solution in the from \(\sum_{j=0}^{\infty} a_j(t)(l-x)^j\) (e.g. see [1] chapter 2) as,

\[
w(x,t) = \sum_{m=0}^{\infty} \partial_t^m h(t) \frac{(l-x)^{2m}}{(2m)!} - \sum_{m=0}^{\infty} \partial_t^m f(t) \frac{(l-x)^{2m+1}}{(2m + 1)!}.
\] (17)

Now we apply the results of Theorem (1) and Lemma (2) to equation (15) to prove the following theorem.

**Theorem 2.** For \(g(t) \in M[0,T]\) from equation (15) we have,

\[
\partial_x u(x,t)|_{x=x_0} = -\partial_t^{\frac{1}{2}} g(t).
\] (18)

**Proof.** By applying \(\partial_t^{\frac{1}{2}}\) to both sides of equation (15) we have,

\[
\partial_t^{\frac{1}{2}} \int_0^t K(x_0 - x, t - \tau) u(x, \tau) \, d\tau = \partial_t^{\frac{1}{2}} g(t).
\] (*)

Assuming \(u(x,t)\) as a function of \(t\) is smooth on \([0,T]\), then for the left hand side of above equation using (10) we have,

\[
\partial_t^{\frac{1}{2}} \int_0^t K(x_0 - x, t - \tau) u(x, \tau) \, d\tau = \int_0^t \left(\partial_{t-\tau}^{\frac{1}{2}} K(x_0 - x, t - \tau)\right) u(x, \tau) \, d\tau
\]
\[
= \int_0^t \partial_x K(x_0 - x, t - \tau) u(x, \tau) \, d\tau
\]
\[
= -\int_0^t K(x_0 - x, t - \tau) \left(\partial_x u(x, \tau)\right) \, d\tau,
\] (**)

where in the second line we used (11) and in the third line we used,

\[
\int_0^t \partial_x K(x_0 - x, t - \tau) u(x, \tau) \, d\tau + \int_0^t K(x_0 - x, t - \tau) \left(\partial_x u(x, \tau)\right) \, d\tau = 0,
\]

which comes from differentiating (15) with respect to variable \(x\). Thus from (*) and (**) we get,

\[
\int_0^t K(x_0 - x, t - \tau) \left(\partial_x u(x, \tau)\right) \, d\tau = -\partial_t^{\frac{1}{2}} g(t).
\]

The above equations is valid for \(x \in [0, x_0]\) so one can take the limit \(x \rightarrow x_0^-\) and use equation (12) to get,

\[
\lim_{x \rightarrow x_0^-} \int_0^t K(x_0 - x, t - \tau) \left(\partial_x u(x, \tau)\right) \, d\tau = -\partial_t^{\frac{1}{2}} g(t)
\]

\[
\partial_x u(x,t)|_{x=x_0} = -\partial_t^{\frac{1}{2}} g(t).
\]
The mathematical consequence of Theorem 2 is that it converts the problem of semi-infinite bar (14) into,
\[
\begin{align*}
\partial^2_x u(x, t) &= \partial_t u(x, t), & 0 \leq x < x_0, & 0 \leq t \leq T, \\
u(x_0, t) &= g(t), & 0 \leq t \leq T, \\
\partial_x u(x, t)|_{x=x_0} &= -\partial^\frac{1}{2}_t g(t), & 0 \leq t \leq T, \\
u(x, 0) &= 0, & 0 \leq x < x_0.
\end{align*}
\] (19)

This is similar to problem of finite bar (equation (16) for \(w(x,t)\)). So by doing the substitutions
\[
\begin{align*}
l & \rightarrow x_0, \\
h(t) & \rightarrow g(t) \\
f(t) & \rightarrow -\partial^\frac{1}{2}_t g(t)
\end{align*}
\]
in expression for \(w(x,t)\) (equation (17)), one can get the following answer for \(u(x,t)\),
\[
\begin{align*}
u(x, t) &= \sum_{n=0}^{\infty} \partial^n_t (g(t)) \frac{(x_0 - x)^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \partial^n_t (\partial^\frac{1}{2}_t g(t)) \frac{(x_0 - x)^{2n+1}}{(2n+1)!}, \\
&= \sum_{n=0}^{\infty} \partial^n_T (g(t)) \frac{(x_0 - x)^n}{(n)!}.
\end{align*}
\] (20)

Because of the appearance of extra half derivative in (20) the existence Theorem in comparison with solution of finite bar problem (17) is a bit different. We discuss this matter in the next part where we present a simple existence Theorem for equation (14).

**Remark 1.** Before moving to the next section here we just notice that a similar method can be used to find analytical solution of more general Cauchy problem,
\[
\begin{align*}
\partial^2_t v(x, t) &= \partial_t v(x, t), & 0 \leq x < \infty, & 0 \leq t \leq T, \\
v(x_0, t) &= g(t), & x_0 \in (0, \infty), & 0 \leq t \leq T, \\
v(x, 0) &= \psi(x), & 0 \leq x < \infty.
\end{align*}
\] (21)

That is the inverse heat conduction of semi-infinite bar with initial temperature distribution \(v(x,0) = \psi(x)\). For \(v(x,t)\), the solution of this problem, it can be proved (e.g. see [1] equation 4.1.1),
\[
\int_0^t K(x'-x, t-\tau) v(x, \tau) d\tau + \int_x^{\infty} dy N(x', y, t) \psi(y) = v(x', t), \quad x < x',
\]
for \(0 < t \leq T\), where \(N(x,y,t) = \Phi_1(x - y, t) - \Phi_1(x + y, t)\) and
\[
\Phi_1(x, t) = \begin{cases} 
\frac{1}{(4\pi t)^2} e^{-\frac{x^2}{4t}} & \text{if } x \in \mathbb{R}, \ t > 0, \\
0 & \text{if } x \in \mathbb{R}, \ t = 0.
\end{cases}
\]
This leads to the following equivalent problem of Volterra integral equation of first kind for \(0 < x \leq x_0\) and \(0 < t \leq T\),

\[
\int_0^t K(x_0 - x, t - \tau) v(x, \tau) \, d\tau = g(t) - \int_x^\infty dy \, N(x_0, y, t) \psi(y), \quad 0 \leq x < x_0, (22)
\]

and the following answer for \(x_0 < x\) and \(0 < t \leq T\),

\[
v(x, t) = \int_0^t K(x - x_0, t - \tau) g(\tau) \, d\tau + \int_{x_0}^\infty dy \, N(x, y, t) \psi(y), \quad x_0 < x.
\]

For Equation (22) we may apply half derivative and do the same procedure as in Theorem (2) to get,

\[- \int_0^t K(x_0 - x, t - \tau) (\partial_x v(x, \tau)) \, d\tau = \partial_t^\frac{1}{2} \left( g(t) - \int_x^\infty dy \, N(x_0, y, t) \psi(y) \right), \]

for \(0 \leq x < x_0\) and \(0 < t \leq T\). By taking the limit \(x \to x_0\) one find the heat flux at \(x = x_0\) as,

\[\partial_x(v(x, t))|_{x=x_0} = \partial_t^\frac{1}{2} \left( g(t) - \int_x^\infty dy \, N(x_0, y, t) \psi(y) \right)|_{x=x_0}, \quad 0 < t \leq T.\]

By substituting \(v(x_0, t)\) and \(\partial_x v(x_0, t)\) with \(h(t)\) and \(f(t)\) in (17) respectively one finds the analytical solution for Cauchy problem (21).

### 3.2 Existence Theorem

A non-zero smooth function \(P(t)\) on \([0, T]\) which satisfies (2) can not be analytic every where in \([0, T]\). Due to Cauchy estimate Theorem, the useful condition, \(\exists M, R > 0, \forall n \in \mathbb{Z}_+, \sup_{t \in [0, T]} |Q^{(n)}(t)| < M^{(n)} R^n\), is valid and only valid, if \(Q(t)\) is analytic in \([0, T]\) (where \(Q^{(n)}(t) = \partial_t^n Q(t)\)). Thus one needs to extend the definition of analyticity for smooth functions such that it includes functions which satisfy (2) and also provide some useful condition like above. This is classically done by the use of Gevrey classes ([4], [6], [7]). A special class of interest for heat equation is Gevrey class two, \(G^2([0, t])\), called Holmgren functions, which can be defined by condition of,

\[\exists M, R > 0, \forall n \in \mathbb{Z}_+, \sup_{t \in [0, T]} |P^{(n)}(t)| < M^{(2n)} R^{2n}.\]

Further more Holmgren functions can be classified according to constants in the above condition. For the matter of consistency with literature we bring definition of Holmgren classes from [1] (definition 2.2.1)

**Definition 2. (Holmgren)** For the positive constants \(\gamma_1, \gamma_2\) and \(C\), the Holmgren class \(H(\gamma_1, \gamma_2, C, t_0)\) is set of smooth functions \(\phi\) defined on \(|t - t_0| < \gamma_2\) that satisfy

\[\forall n \in \mathbb{Z}_+, \sup_{|t-t_0|<\gamma_2} |\phi^{(n)}(t)| < C \frac{(2n)!}{\gamma_1^{2n}}.\]
In this paper we have chosen our interval such that $t$ start from zero ($t \in [0, T]$), this is equivalent of choosing $t_0 = \gamma_2$ in above definition. So for convenience we take the notation,

$$H_t(\gamma_1, \gamma_2, C) := H(\gamma_1, \gamma_2, C, \gamma_2).$$

(23)

Thus by definition $\phi(t) \in H_t(\gamma_1, \gamma_2, C)$ is defined for $t \in [0, 2\gamma_2]$.

**Lemma 3.** 1- If $P(t) \in H_t(\gamma_1, \gamma_2, C)$ then, $\partial_t P(t) \in H_t(\sqrt{\frac{2}{5}} \gamma_1, \gamma_2, C_1)$ where $C_1 = \frac{24}{5} \gamma_1^{-2} C$

2- If $R(t) \in M[0, T]$ and $R(t) \in H_t(\gamma_1, \gamma_2, C)$ then, $\partial_t^2 R(t) \in H_t(\sqrt{\frac{2}{5}} \gamma_1, \gamma_2, C_2)$ where $C_2 = \frac{48\sqrt{\frac{2}{5}}}{5\sqrt{\pi}} \gamma_1^{-2} C$

**Proof.** Let us first show the following inequality for $1 \leq \beta \leq \frac{12}{5}$,

$$(2j + 1)(2j + 2) \leq 2\beta \left(\frac{6}{\beta}\right)^j, \quad j = 0, 1, 2, \ldots$$

Assuming $(2j + 1)(2j + 2) \leq c(d)^j$ then the parametrization, $c = 2\beta$, $d = 6/\beta$ for $1 \leq \beta \leq \frac{12}{5}$ can be derived matching the inequality for $j = 0, 1, 2$. The rest can be proved by induction.

Now for part 1 assuming $P(t) \in H_t(\gamma_1, \gamma_2, C)$ then for $Q(t) = \partial_t P(t)$ one gets,

$$\sup_{t \in [0, T]} |Q^{(n)}(t)| = \sup_{t \in [0, T]} |P^{(n+1)}(t)| \leq C \frac{(2n + 2)!}{\gamma_1^{2n+2}} = (C \gamma_1^{-2})(2n)! \frac{(2n)!}{\gamma_1^{2n}(2n + 1)(2n + 2)}.$$

Now by using the above inequality, for $1 \leq \beta \leq \frac{12}{5}$ we have,

$$\sup_{t \in [0, T]} |Q^{(n)}(t)| \leq (C \gamma_1^{-2})\frac{(2n)!}{\gamma_1^{2n}}(2\beta)(6/\beta)^n = 2\beta \gamma_1^{-2} C \frac{(2n)!}{\gamma_1^{2n}},$$

where $\gamma = \gamma_1 \sqrt{\frac{2}{5}}$. Therefore $\partial_t P(t) \in H_t((\gamma_1 \sqrt{\frac{2}{5}}, \gamma_2, (2\beta \gamma_1^{-2} C))$ for $1 \leq \beta \leq \frac{12}{5}$ and thus particularly for $\beta = \frac{12}{5}$ we have, $\partial_t P(t) \in H_t((\sqrt{\frac{2}{5}} \gamma_1, \gamma_2, \frac{24}{5} \gamma_1^{-2} C))$.

For part 2, since $R(t) \in M[0, T]$ by using Theorem [1] part one we can write,

$$|\partial_t^1(\partial_t^{\frac{3}{2}} R(t))| = |\partial_t^{\frac{3}{2}}(\partial_t^1 R(t))| = \frac{1}{\sqrt{\pi}} |\int_0^t d\tau \left(\frac{\partial_t^{n+1} R(\tau)}{\sqrt{t-\tau}}\right)| = \frac{2\sqrt{\gamma_2}}{\sqrt{\pi}} |\partial_t^{n+2} R(\tau)|.$$

But since $\sqrt{t-\tau} \leq 2\sqrt{\gamma_2}$ for $t \in [0, 2\gamma_2]$ therefore,

$$|\partial_t^n(\partial_t^{\frac{3}{2}} R(t))| \leq \left|\int_0^t d\tau \frac{2\sqrt{\gamma_2}}{\sqrt{\pi}} (\partial_t^{n+2} R(\tau))\right| = \frac{2\sqrt{\gamma_2}}{\sqrt{\pi}} |\partial_t^{n+1} R(t)|.$$

Since $R(t) \in H_t(\gamma_1, \gamma_2, C)$ by part 1 we have, $\partial_t R(t) \in H_t((\gamma_1 \sqrt{\frac{2}{5}}, \gamma_2, \frac{24}{5} \gamma_1^{-2} C)$ thus, $|\partial_t^{n+1} R(t)| < \frac{24}{5} C \gamma_1^{-2} (\frac{2n!}{\gamma_1^{2n}})$ where $\gamma_1 = \gamma_1 \sqrt{\frac{2}{5}}$. By inserting this in to the last line we get,

$$|\partial_t^n(\partial_t^{\frac{3}{2}} g(t))| < \left(\frac{48\sqrt{\gamma_2}}{5\sqrt{\pi}} \gamma_1^{-2} C\right) \frac{(2n)!}{\gamma_1^{2n}}.$$

□

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Theorem 3. If $g(t) \in H_l(\gamma_1, \gamma_2, C)$ and $g(t) \in M[0, 2\gamma_2]$ then the power series,

$$u(x, t) = \sum_{n=0}^{\infty} \partial_t^n(g(t)) \frac{(x_0 - x)^{2n}}{(2n)!} + \sum_{n=0}^{\infty} (\partial_t^\frac{n+1}{2} g(t)) \frac{(x_0 - x)^{2n+1}}{(2n + 1)!},$$

converges uniformly and absolutely for $|x_0 - x| \leq r < \sqrt{\frac{2}{5}} \gamma_1$ and $u(x, t)$ is the solution of equation (14) for $t \in [0, 2\gamma_2]$ and $|x_0 - x| \leq r < \sqrt{\frac{2}{5}} \gamma_1$.

Proof. Since $g(t)$ belongs to $H_l(\gamma_1, \gamma_2, C)$, it follows from Lemma (3) that $\partial_t^\frac{1}{2} g(t) \in H_l(\sqrt{\frac{2}{5}} \gamma_1, \gamma_2, C_2)$. Therefore both $g(t)$ and $\partial_t^\frac{1}{2} g(t)$ are in $H_l(\sqrt{\frac{2}{5}} \gamma_1, \gamma_2, C_3)$ where $C_3 = \max\{C, C_2\}$. Considering equation (9), the series for $u(x, t)$ can be rewritten as,

$$u(x, t) = \sum_{n=0}^{\infty} \left( \partial_t^n(g(t)) \frac{(x_0 - x)^{2n}}{(2n)!} + \partial_t^n(\partial_t^\frac{1}{2} g(t)) \frac{(x_0 - x)^{2n+1}}{(2n + 1)!} \right).$$

The rest of proof is similar to Holmgren result for problem (16) (e.g. see [1] Theorem (2.3.1)).

It should be noted that a solution $u(x, t)$ with the conditions in Theorem 3 covers the whole $x \in [0, x_0]$ when,

$$x_0 < \sqrt{\frac{2}{5}} \gamma_1,$$

and thus one can retrieve, $u(0, t)$, the unknown heat source at $x = 0$. Otherwise,

$$x_0 \geq \sqrt{\frac{2}{5}} \gamma_1 \implies u(x, t) \text{ is convergent for } x_0 - \sqrt{\frac{2}{5}} \gamma_1 < x \leq x_0.$$

There are also functions like,

$$\psi(t) = \begin{cases} \exp(\frac{t}{T}) & \text{if } 0 < t \leq T, \\ 0 & \text{if } t = 0, \end{cases}$$

which satisfy extra property that for each $\gamma_1 > 0$ there exists a $C_1 = C_1(\gamma_1) > 0$ such that $\psi \in H_l(\gamma_1, \gamma_2, C_1(\gamma_1))$ for fixed $\gamma_2$ (e.g. see [11] or [1] section 2.4). Therefore for such functions by Theorem 3, the series, $\sum_{n=0}^{\infty} \partial_t^n(\psi(t)) \frac{(x_0 - x)^{2n}}{(2n)!}$, always converges uniformly and absolutely for any $x \in [0, x_0]$ and $t \in [0, 2\gamma_2]$.

### 3.3 An example via 1d direct heat equation

The relation between heat flux, $u_x(x, t)$ and time semi-derivative of temperature $u(x, t)$ has been noted practically in electrochemistry (in the context of diffusion equation where $u(x, t)$ represents density) and is justified using direct heat equation with special boundary
conditions (e.g. see [9], [10]). In the following we bring a case of direct one dimensional heat conduction equation of infinite bar \((x \in (-\infty, +\infty))\) with an initial distribution, \(g(x)\), being confined to \(x < 0\). In other word the initial distribution is zero for \(x \geq 0\), thus it provides an example for testing formulas (18) and (20), since the left part of infinite bar \((x < 0)\) can be thought as external source at \(x = 0\) for the right part \((x \geq 0)\). Our method here again is based on rigorous properties of heat kernel with respect to fractional derivatives. For the record, in the following we keep thermal diffusivity constant, \(\kappa\), in the heat conduction equation.

Lemma 4. Considering the following initial value problem,

\[
\begin{aligned}
\partial_t u(x, t) - \kappa \partial_x^2 u(x, t) &= 0, & (x \in \mathbb{R}, & \ t > 0), \\
u(x, 0) &= g(x), & (x \in \mathbb{R}),
\end{aligned}
\tag{25}
\]

where \(g(x)\) is bounded continuous function on \(\mathbb{R} \ (g(x) \in C^b(\mathbb{R}))\) and

\[
g(x) = 0, \quad x \geq 0. \tag{26}
\]

then \(u(x, t)\) for \(x > 0\), as a function of \(t\), is in \(M[0, +\infty]\) and we have,

\[
\partial_x u(x, t) = \frac{1}{\sqrt{\kappa}} \partial_t \frac{1}{t} u(x, t), \quad t > 0, \quad x > 0,
\tag{27}
\]

\[
\begin{aligned}
u(x, t) &= \sum_{n=0}^{\infty} (a - x)^n \frac{\partial_x^n (u(a, t))}{(n)!} \frac{\kappa}{t^{\frac{n}{2}}} & t > 0, \quad a > 0, \quad x \in (0, a], \\
\int_0^t K(x' - x, t - \tau) u(x, \tau) d\tau &= u(x', t), \quad x' > x > 0,
\tag{28}
\end{aligned}
\]

where \(K(x, t) = \frac{e^{-\frac{x^2}{4\kappa t}}}{2\sqrt{\pi \kappa} t^{\frac{3}{2}}}\).

Proof. With condition \(g(x) \in C^b(\mathbb{R})\) the solution for equation (25) exists in \(C^\infty(\mathbb{R} \times (0, \infty))\), given by,

\[
u(x, t) = \int_{\mathbb{R}} \Phi_1(x - y, t) g(y) dy,
\]

where,

\[
\Phi_1(x, t) = \begin{cases} \\
\frac{1}{(4\pi \kappa t)^{\frac{3}{2}}} e^{-\frac{|x|^2}{4\kappa t}} & \text{if } x \in \mathbb{R}, \ t > 0, \\
0 & \text{if } x \in \mathbb{R}, \ t = 0,
\end{cases}
\]

(e.g. see [3], section 2.3, Theorem 1). It is easy to show that the function \(\Phi_1(x, t)\) for \(x \neq 0\) as a function of \(t\) is in \(M[0, +\infty]\) and we have,

\[
\partial_x \Phi_1(x, t) = \frac{1}{\sqrt{\kappa}} \partial_t \frac{1}{t} \Phi_1(x, t), \quad x > 0, \ t \geq 0, \quad (\star)
\]

It should be noted that function \(\Phi_1(x, t)\) does not belong to \(M[0, +\infty]\) for \(x = 0\) and also for \(x < 0\) the relation \((\star)\) changes sign. Considering the condition (26) for \(g(x)\) we have,

\[
u(x, t) = \int_{y<0} \Phi_1(x - y, t) g(y) dy. \quad (\star\star)
\]
Now if \( x > 0 \) then, \( \Phi_1(x - y, t) \) appearing in the integral (⋆⋆), belongs to \( M[0, +\infty] \) since \( (x - y) > 0 \). It follows that for any integer \( n \geq 0 \),

\[
\partial_n^x u(x, t)|_{t=0} = \int_{y<0} \partial_n^x \Phi_1(x - y, t)|_{t=0} g(y) \, dy = 0, \quad x > 0.
\]

Thus for \( x > 0 \), \( u(x, t) \) as a function of \( t \) is in \( M[0, +\infty] \).

For \( x > 0 \) we have,

\[
\partial_x u(x, t) = \int_{-\infty}^{0} \partial_x \Phi_1(x - y, t) \, g(y) \, dy,
\]

\[
= -\frac{1}{\sqrt{\kappa}} \int_{-\infty}^{0} \partial_1^x \Phi_1(x - y, t) \, g(y) \, dy,
\]

\[
= -\frac{1}{\sqrt{\kappa}} \int_{-\infty}^{0} dy \, \frac{2}{\Gamma(\frac{1}{2})} \left( \int_{0}^{t} d\tau (t - \tau)^{\frac{1}{2}} \partial_2^x \Phi_1(x - y, \tau) \right) g(y),
\]

\[
= -\frac{1}{\sqrt{\kappa}} \frac{2}{\Gamma(\frac{1}{2})} \int_{0}^{t} d\tau (t - \tau)^{\frac{1}{2}} \partial_2^x (\int_{-\infty}^{0} g \Phi_1(x - y, \tau) \, g(y) \, dy,
\]

\[
= -\frac{1}{\sqrt{\kappa}} \frac{1}{\Gamma(\frac{1}{2})} \int_{0}^{t} d\tau (t - \tau)^{-\frac{1}{2}} \partial \left( \int_{-\infty}^{0} g \Phi_1(x - y, \tau) \, g(y) \, dy,
\]

\[
= -\frac{1}{\sqrt{\kappa}} (\partial_1^x u(x, t)),
\]

where in the first line we applied Leibniz integral rule, in the third line we used equation (10) for semi-fractional derivative of \( \Phi_1(x - y, t) \) and in the line four the integrations order is changed, using Fubini theorem.

For equation (28), for \( t > 0, \ a > 0, \ x \in (0, a] \) we have,

\[
\sum_{n=0}^{\infty} \frac{(a - x)^n}{(n)!} \partial_1^n \Phi_1(x - y, t) \right) |_{x=a} \, g(y),
\]

\[
= \int_{-\infty}^{0} dy \left\{ \sum_{n=0}^{\infty} \frac{(x-a)^n}{(n)!} \right\} \partial_1^n \Phi_1(x - y, t) \right) |_{x=a} \, g(y),
\]

\[
= \int_{-\infty}^{0} dy \Phi_1(x - y, t) \right) |_{x=a} \, g(y) = u(x, t),
\]

where in the second line above we used the relation (⋆). The series appearing in the second line, is just the Taylor series of function \( \Phi_1(x - y, t) \) around \( x = a \) which is convergent (for \( x \in (0, a] \)) to itself since \( \Phi_1(x - y, t) \) is an analytic function with respect to \( x \).

The last part, equation (29), is just brought to make sure that solution of 1d direct heat equation (⋆⋆⋆), for right part of the line \( (x > 0) \), satisfies the same integral relation of inverse heat equation of semi-infinite bar (14). Considering solution (⋆⋆⋆), it is enough to show,

\[
\int_{0}^{t} K(x' - x - \tau) \Phi_1(x - y, \tau) \, d\tau = \Phi_1(x' - y, t), \quad x' > x > y. \quad (⋆⋆⋆)
\]
Using \( K(x, t) = -2\kappa \partial_x \Phi_1(x, t) \) and
\[
\int_{t'}^t d\tau K(x' - x, t - \tau) K(x - z, \tau - t') = K(x' - x, t - t'),
\]
by starting from left hand side we have, for \( x' > x > y \),
\[
\int_0^t K(x' - x, t - \tau) \Phi_1(x - y, \tau) d\tau
\]
\[
= \frac{1}{2\kappa} \int_0^t d\tau K(x' - x, t - \tau) \int_y^y dz K(x - z, \tau)
\]
\[
= \frac{1}{2\kappa} \int_{-\infty}^y dz \int_0^t d\tau K(x' - x, t - \tau) K(x - z, \tau)
\]
\[
= \frac{1}{2\kappa} \int_{-\infty}^y dz K(x' - z, t) = \Phi_1(x' - y, t).
\]

\[\square\]

4 Conclusion

In this paper we considered solving Volterra integral equation of first kind \([15]\). We examined some properties of function space \( M[0, T] \) with regards to fractional integrals and derivatives relevant to this problem. In Theorem \([1]\) we found relation \([10]\) which is an extension of Leibnitz integral rule to derivatives of fractional order in a special case. In Theorem \([2]\) we used results of previous section to deduce a new boundary condition \([18]\) for equivalent formulation of problem in term of Cauchy problem of parabolic partial differential equation \([14]\), by which the solution is found by the series \([20]\). In Theorem \([3]\) an existence theorem is found for solution \([20]\) using Holmgren classes. In section \([3.3]\) we provided an example for this problem through a special case of initial value problem given in Lemma \([4]\).

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