Dynkin Game of Stochastic Differential Equations with Random Coefficients, and Associated Backward Stochastic Partial Differential Variational Inequality *

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Abstract

A Dynkin game is considered for stochastic differential equations with random coefficients. We first apply Qiu and Tang’s maximum principle for backward stochastic partial differential equations to generalize Krylov estimate for the distribution of a Markov process to that of a non-Markov process, and establish a generalized Itô-Kunita-Wentzell’s formula allowing the test function to be a random field of Itô’s type which takes values in a suitable Sobolev space. We then prove the verification theorem that the Nash equilibrium point and the value of the Dynkin game are characterized by the strong solution of the associated Hamilton-Jacobi-Bellman-Isaacs equation, which is currently a backward stochastic partial differential variational inequality (BSPDVI, for short) with two obstacles. We obtain the existence and uniqueness result and a comparison theorem for strong solution of the BSPDVI. Moreover, we study the monotonicity on the strong solution of the BSPDVI by the comparison theorem for BSPDVI and define the free boundaries. Finally, we identify the counterparts for an optimal stopping time problem as a special Dynkin game.

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1 Introduction.

Throughout this paper, let \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})\) be a complete filtered probability space, on which we define two independent standard Brownian motions: \(d_1\)-dimension \(W \triangleq \{W_t\}_{t \geq 0}\) and \(d_2\)-dimension \(B \triangleq \{B_t\}_{t \geq 0}\). Denote by \(\mathbb{F}^W \triangleq \{\mathcal{F}^W_t\}_{t \geq 0}\) and \(\mathbb{F}^B \triangleq \{\mathcal{F}^B_t\}_{t \geq 0}\) the natural filtrations generated by \(W\) and \(B\), respectively. We assume that they contain all \(\mathbb{P}\)-null sets in \(\mathcal{F}\). Define \(\mathbb{F} \triangleq \mathbb{F}^W \vee \mathbb{F}^B\). Denote by \(\mathcal{P}\) and \(\mathcal{P}^B\) the \(\sigma\)-algebras of predictable sets in \(\Omega \times [0, T]\) associated with \(\mathbb{F}\) and \(\mathbb{F}^B\), respectively. Denote by \(\mathcal{B}(D)\) the Borel \(\sigma\)-algebra of the domain \(D\) in \(\mathbb{R}^d\).

Suppose that the state process \(X = (X_1, \cdots, X_d)\) is governed by the following stochastic differential equation (SDE, for short):

\[
X_{i,s} = x_i + \int_t^s \beta_i(u, X_u) \, du + \int_t^s \gamma_i(u, X_u) \, dW_u^i + \int_t^s \theta_i(u, X_u) \, dB_u^k, \quad (1.1)
\]

where \(i = 1, \cdots, d^*\) and \(x = (x_1, \cdots, x_{d^*}) \in \mathbb{R}^{d^*}\). The coefficients \(\beta, \gamma, \text{ and } \theta\) are \(\mathcal{P}^B \times \mathcal{B}(\mathbb{R}^{d^*})\)-measurable random fields taking values in proper spaces, and satisfying Assumptions D1 and D2 (see Section 2 below). Note that here and in the following we use repeated indices for summation. For example, the repeated subscript \(l\) implies summation over \(l = 1, \cdots, d_1\) and the repeated subscript \(k\) implies summation over \(k = 1, \cdots, d_2\). It is clear that SDE (1.1) has a unique strong solution \(X^{s,x}\).

Let \(\mathcal{U}_{t,T}\) be the class of all \(\mathbb{F}\)-stopping times which take values in \([t, T]\). For any \((\tau_1, \tau_2) \in \mathcal{U}_{t,T} \times \mathcal{U}_{t,T}\), we consider the payoff:

\[
R_t(x; \tau_1, \tau_2) = \int_t^{\tau_1 \wedge \tau_2} f_u(X_{u,x}^{t,x}) \, du + \mathbb{V}_{\tau_1} (X_{\tau_1}^{t,x}) \chi_{\{\tau_1 < \tau_2 \wedge T\}} + \mathbb{V}_{\tau_2} (X_{\tau_2}^{t,x}) \chi_{\{\tau_2 \leq \tau_1, \tau_2 < T\}} + \varphi(X_{T}^{t,x}) \chi_{\{\tau_1 \wedge \tau_2 = T\}}.
\]

The running cost \(f\) and terminal costs \(\mathbb{V}_\cdot\) are \(\mathcal{P}^B \times \mathcal{B}(\mathbb{R}^{d^*})\)-measurable random fields taking values in \(\mathbb{R}\), and \(\varphi\) is \(\mathcal{F}^B_T \times \mathcal{B}(\mathbb{R}^{d^*})\)-measurable random field taking values in \(\mathbb{R}\). They satisfy Assumptions V3 and V4 (see Section 4 below).

We consider the following Nash equilibrium of our non-Markovian zero-sum Dynkin game (denoted by \(\mathcal{D}_{tx}\) hereafter): Find a pair \((\tau_1^*, \tau_2^*) \in \mathcal{U}_{t,T} \times \mathcal{U}_{t,T}\) such that for any \(\tau_1, \tau_2 \in \mathcal{U}_{t,T}\), the following inequalities hold:

\[
\mathbb{E} \left[ R_t(x; \tau_1^*, \tau_2^*) \bigg| \mathcal{F}_t \right] \geq \mathbb{E} \left[ R_t(x; \tau_1^*, \tau_2) \bigg| \mathcal{F}_t \right] \geq \mathbb{E} \left[ R_t(x; \tau_1, \tau_2^*) \bigg| \mathcal{F}_t \right].
\]

Such a pair \((\tau_1^*, \tau_2^*)\), if it exists, is called a Nash equilibrium point or saddle point of Problem \(\mathcal{D}_{tx}\), and the random variable \(V_t(x) \triangleq \mathbb{E} \left[ R_t(x; \tau_1^*, \tau_2^*) \big| \mathcal{F}_t \right]\) is called the value of Problem \(\mathcal{D}_{tx}\). We have

\[
V_t(x) = \operatorname{ess.inf}_{\tau_2 \in \mathcal{U}_{t,T}} \operatorname{ess.sup}_{\tau_1 \in \mathcal{U}_{t,T}} \mathbb{E} \left[ R_t(x; \tau_1, \tau_2) \big| \mathcal{F}_t \right] = \operatorname{ess.sup}_{\tau_1 \in \mathcal{U}_{t,T}} \operatorname{ess.inf}_{\tau_2 \in \mathcal{U}_{t,T}} \mathbb{E} \left[ R_t(x; \tau_1, \tau_2) \big| \mathcal{F}_t \right].
\]

The value \(V_t(x)\) is unique if it exists. In general, a Nash equilibrium point may not be unique, and we shall always mean it by the smallest one.
Dynkin games were initially introduced by Dynkin and Yushkevich [10], and have received many studies, see among others [1, 6, 7, 24, 32]. Many interesting problems arising from the theory of probability, mathematics statistics are reformulated as Dynkin games (see [25]). Recently, many new financial problems are formulated as Dynkin games, and are turned into partial differential variational inequalities (PDVI, for short) or free boundary problems, which are then studied via the PDE theory (see [13] for example).

The existence of saddle points of Dynkin games has been discussed, either via a pure probabilistic approach, such as Snell’s envelope and martingale method, or by means of a PDE method. If the coefficients $\beta, \gamma$, and $\theta$ of the state equation (1.1) are all deterministic functions, then the state $X$ is Markovian, and the value $V_t(x)$ is a deterministic function of $(t, x)$ if the cost functions $f, V$, $V_0$, $\varphi$ are deterministic function, too. Moreover, it can be proved that $V_t(x)$ coincides with the strong solution of the associated PDVI by the dynamic programming principle under proper assumptions. See Friedman [13] for more details. Nowadays very general results on a Dynkin game have been established for a right-continuous strong Markov process (see Ekström and Peskir [11] and Peskir [24]). In contrast, there are fewer studies on a Dynkin game for a non-Markov process (see Lepeltiet and Maingueneau [19] and Cvitanic and Karatzas [7]).

In this paper, we are concerned with the Dynkin game for a non-Markovian process. We suppose that the state $X$ is driven by two independent standard Brownian motions $W$ and $B$ and the drift coefficient $\beta$ and the diffusion coefficients $\gamma, \theta$ only depend on the path of $B$ in a predictable way, that is, they are all $\mathcal{P}^B$-predictable and independent of $W$. The structural assumption is used to guarantee the super-parabolic condition (see Assumption V2 in Section 2 below) of the associated backward stochastic partial differential variational inequality (BSPDVI, in short). In this context, the value $V_t(x)$ further depends on, in addition to $(t, x)$, the path of Brownian motion $B$ up to time $t$. Hence, it is a random field. We show that it is characterized by the unique strong solution of the associated Hamilton-Jacobi-Bellman-Isaacs (HJBI) equation, which is the following type of BSPDVI:

$$
\begin{cases}
    dV_t = -(L V_t + \mathcal{M}^k Z_t^k + f_t) dt + Z_k^k dB_t^k, & \text{if } V_t < V_t < \nabla_t; \\
    dV_t \leq -(L V_t + \mathcal{M}^k Z_t^k + f_t) dt + Z_k^k dB_t^k, & \text{if } V_t = \nabla_t; \\
    dV_t \geq -(L V_t + \mathcal{M}^k Z_t^k + f_t) dt + Z_k^k dB_t^k, & \text{if } V_t = V_t; \\
    V_T(x) = \varphi(x),
\end{cases}
$$

(1.2)

where the repeated superscript $k$ is summed from 1 to $d_2$, and

$$L \triangleq a_{ij} D_{ij} + b^i D_i + c, \quad \mathcal{M}^k \triangleq \sigma^{ik} D_i + \mu^k, \quad i, j = 1, 2, \cdots, d^*, \quad k = 1, 2, \cdots, d_2. \quad (1.3)$$

The coefficients $a, b, c, \sigma, \mu$, the upper obstacle $\nabla$, and the lower obstacle $V$ are $\mathcal{P}^B \times \mathcal{B}(\mathbb{R}^{d^*})$-measurable random fields taking values in proper spaces. The terminal value $\varphi$ is $\mathbb{F}^B_T \times (\mathbb{R}^{d^*})$-measurable random field.

When the above coefficients are all deterministic, BSPDVI (1.2) (with the second unknown process $Z$ vanishing) is reduced to a deterministic PDVI. There is a huge literature concerning deterministic PDVIs, and see Lions and Stampacchia [20] and Brezis [5] among the pioneers and Bensoussan and Lions [2] and Friedman [14] among the monographs. On
the contrary, there are very few studies on BSPDVI (1.2) with random coefficients. We note that BSPDVIs with one obstacle has been discussed in Chang, Pang and Yong [6] in connection with an optimal stopping problem for an SDE with random coefficients, as the associated Hamilton-Jacobi-Bellman (HJB, in short) equation, and in Øksendal, Sulem and Zhang [26] in connection with a singular control of SPDEs problem, as a system of backward stochastic partial differential equations (BSPDEs, for short) with only one reflection, which is a more precise formulation of BSPDVI with only one obstacle. However, they only concern weak solution of BSPDVIs. In this paper, we concern strong solution of BSPDVI (1.2), which enables us to interpret the derivatives $DV, D^2V, DZ$ almost everywhere in $\Omega \times [0, T] \times \mathbb{R}^d$, and therefore (1.2) can be understood point-wisely in $x \in \mathbb{R}^d$. The connection to the associated BSPDVI of the value field $V$ is extended to a wider context of the strong solution. The existence of such a strong solution requires the super-parabolic condition.

BSPDVI is in fact a singular or constrained BSPDE. It can also be regarded as a reflected backward stochastic differential equation (RBSDE, in short) in an infinite-dimensional space. Its analysis depends heavily on the state of arts of BSPDEs, which is referred to e.g. [3, 8, 9, 12, 21, 22, 23, 29, 30, 34]. In particular, we make use of an estimate by Du and Tang [9] on the square-integrable strong solution theory of BSPDEs in a $C^2$ domain. Solution of BSPDVI (1.2) is obtained by a conventional penalty method. We consider the penalized approximating BSPDEs (5.11), and show that BSPDVI (1.2) is their limit in the strong sense. The key is to prove the convergence of the nonlinear penalty term. In [1, 13], a deterministic PDVI is concerned and the convergence is obtained by the compact imbedding theorem for Sobolev spaces, which fails to hold in our stochastic Sobolev space. In [6], a BSPDVI with one obstacle is concerned and the convergence is obtained via the monotonicity of the approximating BSPDEs’ weak solution $V_n$ in $n$. Our difficulty has two folds. One comes from the feature of two obstacles in our BSPDVI (1.2), which destroys the monotonicity of the one-parameterized approximating BSPDEs. The other comes from the strong solution of BSPDVI (1.2), which requires an extra higher order (second-order) estimate than that requested for the weak solution. We show the convergence by observing that $V_n$ is a Cauchy sequence in a proper space, which is an extension of the method for RBSDE in [7].

To connect Problem $\mathcal{D}_{tx}$ with BSPDVI (1.2), we have to use Itô-Kunita-Wentzell’s formula. The existing one in the literature (see Lemma 2.1) requires that the random test function should be twice continuously differentiable. The strong solution of BSPDVI (1.2) only guarantees that $D^2V$ is integrable, and not necessarily continuous in general in $x$. Therefore, Lemma 2.1 fails to be directly applied to our computation of $V_t(X_t)$, and has to be extended to more general random test functions. We first use Qiu and Tang [28]’s maximum principle for quasilinear BSPDEs to generalize Krylov estimate for the distribution of a Markov process to that of a non-Markov process. Then using a smoothing method and the generalized Krylov estimate, we prove a generalized Itô-Kunita-Wentzell’s formula.

The rest of the paper is organized as follows. We introduce some notations and results about Itô-Kunita-Wentzell’s formula and BSPDEs in Section 2. In Section 3, we state our hypotheses and generalize Itô-Kunita-Wentzell’s formula. In Section 4, we prove the verification theorem that the Nash equilibrium point and the value of the Dynkin game
are characterized by the strong solution of the associated Hamilton-Jacobi-Bellman-Isaacs equation, which is currently a BSPDVI with two obstacles. In section 5, we establish the existence and uniqueness result and a comparison theorem for strong solution of the BSPDVI, via the strong solution theory of BSPDE. In Section 6, we use the comparison theorem for BSPDVI to derive properties of the strong solution of BSPDVI (1.2), and define its stochastic free boundaries under proper assumptions. In the last section, we show that the optimal stopping time problem is a special case of a Dynkin game, and therefore similar results hold true here.

2 Preliminaries.

In this section, we introduce notations and collect results about Itô-Kunita-Wentzell’s formula and BSPDEs.

Denote by \( \mathbb{N} \) and \( \mathbb{N}_+ \) the set of all nonnegative and positive integers, respectively. Denote by \( E \) a Euclidean space like \( \mathbb{R} \) or \( \mathbb{R}^{d_1} \times \mathbb{R}^{d_2} \). Moreover, for any \( x \in \mathbb{R}^{d*} \), \( \gamma \in \mathbb{R}^{d*} \times \mathbb{R}^{d_1} \), \( \theta \in \mathbb{R}^{d*} \times \mathbb{R}^{d_2} \) and \( a \in \mathbb{R}^{d*} \times \mathbb{R}^{d*} \), define

\[
|x| \triangleq \left( \sum_{i=1}^{d*} x_i^2 \right)^{\frac{1}{2}}, \quad |\gamma| \triangleq \left( \sum_{i=1}^{d*} \sum_{l=1}^{d_1} \gamma_{il}^2 \right)^{\frac{1}{2}},
\]

\[
|\theta| \triangleq \left( \sum_{i=1}^{d*} \sum_{k=1}^{d_2} \theta_{ik}^2 \right)^{\frac{1}{2}}, \quad \text{and} \quad |a| \triangleq \left( \sum_{i,j=1}^{d*} a_{ij}^2 \right)^{\frac{1}{2}}.
\]

Define

\[
D_i \triangleq \partial_{x_i}; \quad D_{ij} \triangleq \partial_{x_i x_j}, \quad i, j = 1, 2, \ldots, d*; \quad D^\alpha \triangleq \partial_{x_1}^{\alpha_1} \cdots \partial_{x_{d*}}^{\alpha_{d*}}; \quad \text{and} \quad |\alpha| \triangleq \sum_{i=1}^{d*} \alpha_i
\]

for any multi-index \( \alpha = (\alpha_1, \ldots, \alpha_{d*}) \) with \( \alpha_i \in \mathbb{N} \). Denote by \( D\eta \) and \( D^2\eta \) respectively the gradient and the Hessian matrix for a function \( \eta : E \to \mathbb{R} \).

For an integer \( k \in \mathbb{N} \), \( p \in [1, +\infty) \), \( q \in [1, +\infty) \), a smooth domain \( D \) in \( \mathbb{R}^{d*} \), and a positive number \( T \), we introduce the following spaces:

- \( C^k(D) \) : the set of all functions \( \eta : D \to E \) such that \( \eta \) and \( D^\alpha \eta \) are continuous for all \( 1 \leq |\alpha| \leq k \);
- \( C^k_0(D) \) : the set of all functions in \( C^k(D) \) with compact support in \( D \);
- \( H^{k,p}(D) \) : the completion of \( C^k(D) \) under the norm

\[
|\eta|_{k,p} \triangleq \left( \int_D |\eta|^p \, dx + \sum_{|\alpha|=1}^{k} \int_D |D^\alpha \eta|^p \, dx \right)^{\frac{1}{p}};
\]

- \( H^{k,p}_0(D) \) : the completion of \( C^k_0(D) \) under the norm \( |\eta|_{k,p} \);
- \( L^{k,p}(D) \) : the set of all \( H^{k,p}(D) \)-valued and \( \mathbb{F}_T \)-measurable random variables such that
\[ \mathbb{E}(\varphi_{k,p}^p) < \infty; \]

- \( L_{0}^{k,p}(D) \): the set of all \( H_{0}^{k,p}(D) \)-valued and \( \mathbb{F}_{t}^{B} \)-measurable random variables such that \( \mathbb{E}(\varphi_{k,p}^p) < \infty; \)

- \( \mathcal{L}^{p} \): the set of all \( \mathcal{P} \)-predictable stochastic processes taking values in \( E \) with the norm

\[ \|X\|_p \triangleq \left[ \mathbb{E}\left( \int_{0}^{T} |X_t|^p dt \right) \right]^{\frac{1}{p}}; \]

- \( \mathcal{S}^{p} \): the set of all \( H_{0}^{k,p}(D) \)-valued \( \mathcal{P} \)-predictable stochastic processes with values in \( F \)

\[ \|\|X\|\|_p \triangleq \left[ \mathbb{E}\left( \sup_{t \in [0,T]} |X_t|^p \right) \right]^{\frac{1}{p}}; \]

- \( \mathcal{L}_{\mathcal{P}^{B}}^{p,q}(E) \): the set of all \( \mathcal{P}^{B} \)-predictable stochastic processes with values in Banach space \( E \) with the norm

\[ \|V\|_{\mathcal{L}_{\mathcal{P}^{B}}^{p,q}(E)} \triangleq \left[ \mathbb{E}\left( \int_{0}^{T} \|V_t\|_{E}^q dt \right) \right]^{\frac{1}{q}}; \]

- \( \mathbb{H}^{k,p}(D) \triangleq \mathcal{L}_{\mathcal{P}^{0}}^{p}(H^{k,p}(D)) \) with the norm \( \|V\|_{k,p} \triangleq \|V\|_{\mathcal{L}_{\mathcal{P}^{0}}^{p}}(H^{k,p}(D)); \)

- \( \mathbb{H}_{0}^{k,p}(D) \triangleq \mathcal{L}_{\mathcal{P}^{0}}^{p}(H_{0}^{k,p}(D)) \) with the norm \( \|V\|_{k,p}; \)

- \( \mathbb{S}^{k,p}(D) \): the set of all path continuous \( \mathcal{P}^{B} \)-predictable stochastic processes in \( \mathbb{H}^{k,p}(D) \) equipped with the norm

\[ \|\|V\|\|_{k,p} \triangleq \left[ \mathbb{E}\left( \sup_{t \in [0,T]} |V_t|^p \right) \right]^{\frac{1}{p}}; \]

- \( \mathbb{M}^{p}(D) \): the subspace of \( \mathbb{H}^{0,p}(D) \) equipped with the norm

\[ \|V\|_{\mathbb{M}^{p}} \triangleq \text{ess.sup}_{(w,t) \in \Omega \times [0,T]} \left[ \mathbb{E}\left( \int_{t}^{T} |V_u|^p_{0,p} du \right| \mathcal{F}_{t}^{B} \right) \right]^{\frac{1}{p}}. \]

**Remark 2.1.** The space \( \mathbb{M}^{p} \) is a Banach space. See [4, 28].

The space notations \( C_{0}^{k}(\mathbb{R}^{d}) \), \( H_{k,p}^{*}(\mathbb{R}^{d}) \) and \( L_{k,p}^{*}(\mathbb{R}^{d}) \), \( \mathbb{H}^{k,p}(\mathbb{R}^{d}) \), \( \mathbb{S}^{k,p}(\mathbb{R}^{d}) \), \( \mathbb{M}^{p}(\mathbb{R}^{d}) \) will be abbreviated as \( C_{0}^{k}, H_{k,p}, L_{k,p}, \mathbb{H}^{k,p}, \mathbb{S}^{k,p}, \mathbb{M}^{p} \) if there is no any confusion.

**Remark 2.2.** We have

\[ H_{k,p}^{*}(\mathbb{R}^{d}) = H_{0}^{k,p}(\mathbb{R}^{d}), \quad L_{k,p}^{*}(\mathbb{R}^{d}) = L_{0}^{k,p}(\mathbb{R}^{d}), \quad \mathbb{H}^{k,p}(\mathbb{R}^{d}) = \mathbb{H}_{0}^{k,p}(\mathbb{R}^{d}). \]

The following special case of Itô-Kunita-Wentzell’s formula (see [15, 23]) is the key to connect Problem \( \mathcal{Q}_{ix} \) and BSPDVI (1.2).
Lemma 2.1. Suppose that the random field $V : \Omega \times [0, T] \times \mathbb{R}^{d^*} \to \mathbb{R}$ satisfies the following:

(i) $V(w, \cdot)$ is continuous with respect to $(t, x)$ a.s. in $\Omega$.

(ii) $V(w, t, \cdot)$ is twice continuously differentiable with respect to $x$ for any $t \in [0, T]$ a.s. in $\Omega$.

(iii) For each $x \in \mathbb{R}^{d^*}$, $V(\cdot, x)$ is a continuous semi-martingale of form:

$$V_t(x) = V_0(x) + \int_0^t U_s(x) \, ds + \int_0^t Z^k_s(x) \, dB^k_s \text{ for any } t \in [0, T] \text{ a.s. in } \Omega,$$

where $U(\cdot, x)$ and $Z(\cdot, x)$ are $\mathbb{F}$-adapted with values in $\mathbb{R}$, $\mathbb{R}^{d^*}$ for any $x \in \mathbb{R}^{d^*}$, and $Z(w, t, \cdot)$ is continuously differentiable with respect to $x$ for any $t \in [0, T]$ a.s. in $\Omega$.

Let $X$ be a continuous semi-martingale of form (2.1). Then we have

$$V_t(X_t) = V_0(X_0) + \int_0^t (LV_s + M^k Z^k_s + U_s)(X_s) \, ds + \int_0^t (Z^k_s + M^k V_s)(X_s) \, dB^k_s \text{ for any } t \in [0, T] \text{ a.s. in } \Omega,$$

where the repeated superscript $l$ is summed from $1$ to $d_1$ and the repeated superscript $k$ is summed from $1$ to $d_2$, and

$$L \triangleq \frac{1}{2} (\gamma_{ij} \gamma_{ji} + \theta_{ik} \theta_{jk}) D_{ij} + \beta_i D_i, \quad M^k \triangleq \theta_{ik} D_i, \quad N^l \triangleq \gamma_{il} D_i, \quad i, j = 1, 2, \ldots, d^*. \quad (2.2)$$

In this paper, we make the following assumptions on the coefficients $\beta$, $\gamma$, and $\theta$ of SDE (2.1).

Assumption D1. (Boundedness) $\beta$, $\gamma$, $\theta$ are $\mathcal{P}^B \times \mathcal{B}(\mathbb{R}^{d^*})$-measurable with values in $\mathbb{R}^{d^*}$, $\mathbb{R}^{d^* \times d_1}$, $\mathbb{R}^{d^* \times d_2}$, respectively. Moreover, they are bounded by a positive constant $K$, i.e.,

$$|\beta(\cdot, x)| + |\gamma(\cdot, x)| + |\theta(\cdot, x)| \leq K \text{ for any } x \in \mathbb{R}^{d^*}, \text{ a.e. in } \Omega \times (0, T).$$

Assumption D2. (Lipschitz continuity and non-degeneracy) $\beta$, $\gamma$, and $\theta$ satisfy Lipschitz condition in $x$ with constant $K$, i.e.,

$$|\beta(\cdot, x_1) - \beta(\cdot, x_2)| + |\gamma(\cdot, x_1) - \gamma(\cdot, x_2)| + |\theta(\cdot, x_1) - \theta(\cdot, x_2)| \leq K |x_1 - x_2|$$

for any $x_1, x_2 \in \mathbb{R}^{d^*}$, almost everywhere in $\Omega \times (0, T)$. Moreover, there exists a positive constant $\kappa$ such that

$$\gamma_{ij} \xi_i \xi_j \geq \kappa |\xi|^2 \text{ for any } \xi \in \mathbb{R}^{d^*}, \text{ a.e. in } \Omega \times (0, T) \times \mathbb{R}^{d^*}.$$

It is clear that SDE (2.1) has a unique strong solution $X \in \mathcal{S}^p$ for any $p \geq 1$.

Consider the following semilinear BSPDEs in the domain $D$:

$$\begin{aligned}
&dV_t(x) = -(\mathcal{L}V_t(x) + \mathcal{M}^k Z^k_t(x) + F(t, x, V_t(x))) \, dt + Z^k_t(x) \, dB^k_t, \\
&V_t(x) = 0, \quad (t, x) \in [0, T] \times \partial D; \quad V_T(x) = \varphi(x), \quad x \in \partial D, \quad (t, x) \in [0, T] \times \partial D; \\
&V_t(x) \in \mathcal{S}^p \text{ for any } p \geq 1.
\end{aligned} \quad (2.3)$$

where $\partial D$ is the boundary of $D$, and the operators $\mathcal{L}$ and $\mathcal{M}$ are defined in (2.3).
**Definition 2.1.** (strong solution) The pair of random fields \((V, Z) \in (\mathbb{H}^{2,2}(D) \cap \mathbb{H}^{1,2}_0(D)) \times \mathbb{H}^{1,2}(D)\) is called a strong solution of BSPDE (2.3), if almost surely for all \(t \in [0, T]\), the following holds

\[
V_t(x) = \varphi(x) + \int_t^T \left[ \mathcal{L}V_s(x) + \mathcal{M}^k Z_s^k(x) + F(s, x, V_s(x)) \right] ds - \int_t^T Z_s^k(x) dB_s^k
\]

for a.e. \(x \in D\).

Consider the following assumptions on the coefficients in (1.3).

**Assumption V1.** (Boundedness) The given functions \(a, b, c, \sigma, \mu\) are \(\mathcal{P}^B \times \mathcal{B}(\mathbb{R}^{d'})\)-measurable, and are bounded by a constant \(K\), taking values in the set of real symmetric \(d' \times d'\) matrices, in the spaces \(\mathbb{R}^{d'}, \mathbb{R}, \mathbb{R}^{d' \times d_2}, \mathbb{R}^{d_2}\), respectively. \(Da\) and \(D\sigma\) exist almost everywhere and are bounded by \(K\).

**Assumption V2.** (Super-parabolicity) There exist two positive constants \(\kappa\) and \(K\) such that:

\[
\kappa |\xi|^2 + |\sigma^* \xi|^2 \leq 2 \xi^*(a^j) \tilde{\xi} \leq K |\xi|^2 \quad \text{for any } \xi \in \mathbb{R}^{d'}, \text{ a.e. in } \Omega \times (0, T) \times \mathbb{R}^{d'}.
\]

In view of [9] Theorem 5.3 and Corollary 3.4], we have

**Lemma 2.2.** Let Assumptions V1 and V2 be satisfied. Assume that \(\varphi \in \mathbb{L}^{1,2}_{0}(D)\) and the random field \(F : \Omega \times [0, T] \times D \times \mathbb{R} \to \mathbb{R}\) satisfies the following:

(i) \(F(\cdot, u)\) is \(\mathcal{P}^B \times \mathcal{B}(D)\)-measurable for any \(u \in \mathbb{R}\);

(ii) \(F(\cdot, 0) \in \mathbb{H}^{0,2}(D)\) and \(F\) is Lipschitz continuity with respect to \(u\), i.e.,

\[
|F(\cdot, v) - F(\cdot, u)| \leq K |v - u| \quad \text{for any } v, u \in \mathbb{R}, \text{ a.e. in } \Omega \times (0, T) \times D.
\]

Then BSPDE (2.3) has a unique strong solution \((V, Z)\) such that \(V \in \mathcal{S}^{1,2}(D)\). Moreover, we have the following estimate:

\[
\|V\|_{2,2}^2 + \|V\|_{1,2}^2 + \|Z\|_{1,2}^2 \leq C(\kappa, K, T) \left( \mathbb{E} \left[ |\varphi|^2_{1,2} \right] + \|F(\cdot, V(\cdot))\|_{0,2}^2 \right).
\]

According to [30], Theorem 5.1], we give the existence and regularity result for strong solution of the linear BSPDE.

**Lemma 2.3.** Let Assumptions V1 and V2 be satisfied. Assume that \(F(\cdot, u) \equiv F(\cdot)\) is independent of \(u\) and \(F \in \mathbb{H}^{k,2}(D)\) with \(k > d' + 2\). Moreover, we suppose that there exists a constant \(M\) such that the coefficients and terminal value satisfy the following

\[
\sum_{1 \leq |\alpha| \leq k} |D^\alpha a| + |D^\alpha b| + |D^\alpha c| + |D^\alpha \sigma| + |D^\alpha \mu| \leq M, \quad \varphi \in \mathbb{L}^{1,2}_0(D) \cap \mathbb{L}^{k+1,2}(D).
\]

Then BSPDE (2.3) has a unique strong solution \((V, Z)\) such that \(V \in \mathcal{S}^{1,2}(D) \cap \mathcal{L}^{2,2}_{\mathcal{F}B}(C^2(D'))\) and \(Z \in \mathcal{L}^{2,2}_{\mathcal{F}B}(C^1(D'))\) for any domain \(D' \subset \subset D\).

Now, we recall a special case of the maximum theorem for BSPDE of Qiu and Tang [28, Theorem 5.8].
Lemma 2.4. Let Assumptions V1 and V2 be satisfied. Assume that $F(\cdot, u) \equiv F(\cdot)$ is independent of $u$, $F \in \mathcal{M}^p(D) \cap \mathcal{M}^2(D)$ with $p > d^* + 2$, and the terminal value satisfies:

$$\text{ess.sup}_{(w, x) \in \Omega \times D} |\varphi(w, x)| < \infty.$$ 

let $(V, Z)$ be the strong solution of BSPDE \[2.3\]. Then we have the following estimate:

$$\text{ess.sup}_{(w, t, x) \in \Omega \times [0, T] \times D} |V_t(w, x)| \leq C(p, \kappa, K, T) \left( \|F\|_{\mathcal{M}^p} + \|F\|_{\mathcal{M}^2} + \text{ess.sup}_{(w, x) \in \Omega \times D} |\varphi(w, x)| \right).$$

Finally, we recall the following backward version Itô’s formula for BS PDEs, which is the special case of Theorem 3.2 in [18]:

Lemma 2.5. Let Assumption V1 be satisfied. Assume that $(v, Z) \in (H^{2,2}(D) \cap H_0^{1,2}(D)) \times H^{1,2}(D)$, $f \in H^{0,2}(D)$, $\varphi \in L_0^{1,2}(D)$ and the following equality holds for every $\eta \in C^2_0(D)$,

$$\int_D \eta v_t \, dx = \int_D \eta \varphi \, dx + \int_t^T \int_D \eta (Lv_s + M^k Z_s^k + f_s) \, dx \, ds - \int_t^T \int_D \eta Z_s^k \, dx \, dB_s^k$$

almost everywhere in $\Omega \times [0, T]$. Then there exists a new version $V$ of $v$ such that $V \in S^{1,2}(D)$ and $(V, Z)$ is a strong solution of the following BSPDE:

$$V_t(x) = \varphi(x) + \int_t^T \left[ LV_s(x) + M^k Z_s^k(x) + f_s(x) \right] \, dt - \int_t^T Z_s^k(x) \, dB_s^k$$

with $(t, x) \in [0, T] \times D$. Moreover, we have the following backward version Itô’s formula for BSPDE:

$$|V_t|_{0,2}^2 = |\varphi|_{0,2}^2 + \int_t^T \int_D \left[ -2 a^{ij} D_i V_s D_j V_s + 2 \left( b^i - D_j a^{ij} \right) D_i V_s V_s + 2 c V_s^2 \
-2 \sigma^{ik} Z_s^k D_i V_s + 2 \left( \mu^i - D_i \sigma^{ik} \right) Z_s^k V_s + 2 f_s V_s - |Z_s|^2 \right] \, dx \, ds$$

$$-2 \int_t^T \int_D V_s Z_s^k \, dx \, dB_s^k \text{ for all } t \in [0, T] \text{ a.s. in } \Omega.$$

3 A generalized Itô-Kunita-Wentzell’s formula.

Since a strong solution $V$ is only known to belong to $H^{2,2}$, Lemma 2.1 fails to be applied to $V$, and need be generalized. We have
**Theorem 3.1.** Suppose that the random function $V : \Omega \times [0, T] \times \mathbb{R}^d \to \mathbb{R}$ satisfies the following: $V(x)$ is a continuous semimartingale of form

$$V_t(x) = V_0(x) + \int_0^t U_s(x) \, ds + \int_0^t Z_s^k(x) \, dB_s^k \quad a.e. \ x \in \mathbb{R}^d$$

for every $t \in [0, T]$ and almost surely $\omega \in \Omega$, such that $V \in H^2$, $Z \in H^1$, and $U \in H^0$. Let $X$ be a continuous semi-martingale of form (1.1), and Assumptions D1 and D2 be satisfied. Then formula (2.1) still holds.

**Remark 3.1.** The random function $V$ is not well-defined at every point $(w, t, x)$ of $\Omega \times [0, T] \times \mathbb{R}^d$ (since it is defined only in a subset of full measure), and the value of $V_t(X(w))$ is thus not well-defined in general. However, Lemma 3.3 below indicates that $V(X) \triangleq V \cdot (X \cdot)$ is well-defined as a path-continuous stochastic process in $\Omega \times [0, T]$ (see Remark 3.3 below).

**Lemma 3.2.** Let the assumptions in Lemma 2.3 be satisfied. Assume that $F \in \mathbb{M}^p(D)$ with $p > 1$, the domain $D$ is bounded, and the terminal value satisfies the following:

$$\text{ess.sup}_{w \in \Omega} | \varphi |_{0,p} < \infty.$$

Let $(V, Z)$ be the unique strong solution of BSPDE (2.3). Then we have

$$\|V\|_{\mathbb{M}^p} \leq C(p, \kappa, K, T) \left( \|F\|_{\mathbb{M}^p} + \text{ess.sup}_{w \in \Omega} | \varphi |_{0,p} \right).$$

Moreover, if $p \geq 2$, then the above estimate also holds even if $D$ is unbounded.

**Remark 3.2.** Our boundedness assumption on $D$ is used to guarantee $\int_D \Phi_n(0) \, dx < +\infty$ for the case of $p \in (1, 2)$ in the following proof. It can be removed by using bounded domains to approximate $D$ if it is unbounded.

**Proof of Lemma 3.2.** If $1 < p < 2$, we construct the mollification of $|s|^p$ as

$$\Phi_n(s) = \begin{cases} 
|s|^p, & |s| > 1/n; \\
\frac{p(p-2)}{8 n^p} (n^2 s^2 - 1)^2 + \frac{p}{2n^p} (n^2 s^2 - 1) + \frac{1}{n^p}, & |s| \leq 1/n,
\end{cases}$$

which uniformly converges to $|s|^p$. Moreover, $\Phi_n \in C^2(\mathbb{R})$, $\Phi_n'(0) = 0$, and for any $s \in \mathbb{R},$

$$\Phi_n(s) \geq 0, \quad 0 \leq \Phi_n''(s) \leq C(p, n), \quad |\Phi_n'(s)|^2 \leq C(p) \Phi_n''(s) \Phi_n(s),$$

$$|\Phi_n'(s)| \leq C(p) |\Phi_n(s)|^{(p-1)/p}, \quad |\Phi_n'(s) s| \leq C(p) \Phi_n(s).$$
Applying Itô’s formula to $\Phi_n(V_t(x))$ for any $x \in D$ and integrating with respect to $x$, we have

$$\int_D \Phi_n(V_t) \, dx = \int_t^T \int_D \left\{ -\Phi_n''(V_s) \left[ a^{ij} D_i V_s D_j V_s + \sigma^{ik} Z_s^k D_i V_s + \frac{1}{2} |Z_s|^2 \right] \\
+ \Phi_n'(V_s) \left[ (b^i - D_i a^{ij}) D_i V_s + c V_s + F_s + (\mu^k - D_i \sigma^{ik}) Z_s^k \right] \right\} \, dx \, ds$$

$$- \int_t^T \int_D \Phi_n'(V_s) Z_s^k \, dx \, dB_s^k + \int_D \Phi_n(V_T) \, dx$$

$$\leq \int_t^T \int_D \left\{ \Phi_n''(V_s) \left[ \left( -a^{ij} + \left( \frac{1}{2} + \frac{\kappa}{8K} \right) \sigma^{ik} \sigma^{jk} \right) D_i V_s D_j V_s + \frac{\kappa}{4} |DV_s|^2 \\
+ \left( \frac{8K + \kappa}{4(\kappa + 4K)} - \frac{1}{2} \right) |Z_s|^2 \right] + C(p, \kappa, K) (\Phi_n(V_s) + |F_s|^p) \right\} \, dx \, ds$$

$$- \int_t^T \int_D \Phi_n'(V_s) Z_s^k \, dx \, dB_s^k + \int_D \Phi_n(V_s) \, dx$$

$$\leq C(p, \kappa, K) \int_t^T \int_D (\Phi_n(V_s) + |F_s|^p) \, dx \, ds - \int_t^T \int_D \Phi_n'(V_s) Z_s^k \, dx \, dB_s^k$$

$$+ \int_D \Phi_n(V) \, dx.$$
Then, \( f_{\chi P \chi D} \in M^2 \cap M^p \), where \( \chi_A \) is the indicator function of set \( A \) and \( p > d^* + 2 \). Then there exists a constant \( C \) independent of \( f, P, D \) such that

\[
\mathbb{E} \left[ \int_0^{T \wedge \tau_P \wedge \tau_D} |f_t(X_t)| \, dt \right] \leq C \left( \| f_{\chi P \chi D} \|_{\M^p} + \| f_{\chi P \chi D} \|_{\M^2} \right),
\]

where

\[
\tau_P \triangleq \inf\{ t \geq 0 : \chi_P(t) = 0 \}, \quad \tau_D \triangleq \inf\{ t \geq 0 : X_t \not\in D \}.
\]

If \( D \) is bounded and \( g_{\chi P \chi D} \in M^q \) with \( 1 < q \leq p \), then there exists a constant \( C \) independent of \( g, P \) such that

\[
\mathbb{E} \left[ \int_0^{T \wedge \tau_P \wedge \tau_D} \left( \int_t^{T \wedge \tau_P \wedge \tau_D} |g_s(X_s)| \, ds \right)^{q/p} \, dt \right] \leq C \| g_{\chi P \chi D} \|_{\M^q}^{q/p}.
\]

**Proof. Step 1.** We prove the first estimate. Without loss of generalization, we suppose the random field \( f \geq 0 \).

We shall use Itô-Kunita-Wentzell’s formula and the maximum theorem for BSPDE to prove the desired results. In order to apply Itô-Kunita-Wentzell’s formula, we need smooth \( f_{\chi P \chi D} \).

Define

\[
\hat{\zeta}(s) \triangleq \begin{cases} M \exp \left( \frac{1}{s^2 - 1} \right) & \text{if } |s| \leq 1; \\ 0 & \text{if } |s| > 1 \end{cases}
\]

with

\[
M \triangleq \left( \int_{-1}^1 \exp \left( \frac{1}{s^2 - 1} \right) \, ds \right)^{-1}, \quad \zeta(y) = \hat{\zeta}(\{ y \}), \quad \text{and} \quad \zeta_n(y) = n^d \hat{\zeta}(ny),
\]

and the mollification of \( f_{\chi P \chi D} \) as

\[
f^n(w, t, x) \triangleq ((f_{\chi P \chi D}) * \zeta_n)(w, t, x) \triangleq \int_{\mathbb{R}^{d^*}} (f_{\chi P \chi D})(w, t, y) \zeta_n(x - y) \, dy.
\]

Then, \( f^n \in H^{k,2} \) with \( k > d^*/2 + 2 \), and there exists a positive constant \( C \) such that

\[
\| f^n \|_{\M^p} \leq C \| f_{\chi P \chi D} \|_{\M^p} \quad \text{and} \quad \| f^n \|_{\M^2} \leq C \| f_{\chi P \chi D} \|_{\M^2}
\]

for any \( n \in \mathbb{N}_+ \).

Construct \( (V^{n,m}, Z^{n,m}) \) as the solution of the following BSPDE

\[
\begin{cases}
    dV_t^{n,m}(x) = - (L^n V_t^{n,m}(x) + M^n_k Z^{n,m}_{k,t}(x) + f_t^n(x)) \, dt + Z^{n,m}_{k,s}(x) \, dB_t^k, \\
    V_T^{n,m}(x) = 0, \quad x \in \mathbb{R}^{d^*},
\end{cases}
\]

where

\[
L^n \triangleq \frac{1}{2} \left( \gamma_{i1}^m \gamma_{i1}^m + \theta_{ik}^m \theta_{ik}^m \right) D_{ij} + \beta_i^m D_i, \quad M^n_k \triangleq \theta_{ik}^m D_k, \quad \gamma_{i1}^m \triangleq \gamma_{i1} * \zeta_m, \quad \theta_{ik}^m \triangleq \theta_{ik} * \zeta_m, \quad \beta_i^m \triangleq \beta_i * \zeta_m.
\]
Since the coefficients of $L$ are bounded and Lipschitz continuous with respect to $x$, then $\gamma^m, \theta^m,$ and $\beta^m$ converge to $\gamma, \theta, \beta$ uniformly in $x$, respectively. Moreover, we can check that

$$\sum_{1 \leq |\alpha| \leq k} (|D^\alpha \gamma^m| + |D^\alpha \theta^m| + |D^\alpha \beta^m|) \leq M_{k,m} \text{ for any } k, m \in \mathbb{N}_+;$$

$$|D\gamma^m| + |D\theta^m| + |D\beta^m| + |\gamma^m| + |\theta^m| + |\beta^m| \leq CK;$$

$$\gamma^m_{ij} \xi_i \xi_j \geq \kappa |\xi|^2 \text{ for any } \xi \in \mathbb{R}^d.$$
Moreover, there exists a subsequence \( \{ f_{n_k}(X) \}_{k=1}^\infty \) such that \( f_{n_k}(X) \) converges to \( f(X) \) a.e. in \( \Omega \times [0, T] \).

Since \( f \geq 0 \) and \( f^n \) is the mollification of \( f \chi_P \chi_D \), then \( f^n(X) \) is nonnegative a.e. in \( \Omega \times [0, T] \). Hence, we deduce that

\[
\mathbb{E} \left[ \int_0^{T \wedge \tau_T \wedge D} |f_t(X_t)| \, dt \right] \leq \mathbb{E} \left[ \int_0^T |f_t(X_t)| \, dt \right] = \lim_{n \to \infty} \mathbb{E} \left[ \int_0^T |f^n_t(X_t)| \, dt \right] \leq C \left( \|f^n\|_{M^p} + \|f^n\|_{M^2} \right) \leq C \left( \|f \chi_P \chi_D\|_{M^p} + \|f \chi_P \chi_D\|_{M^2} \right).
\]

**Step 2.** We prove the second estimate. Suppose that \( g \geq 0 \) and smooth \( g \chi_P \chi_D \) as the above, then \( g^n \geq 0 \). Define \( v^{n,m} \) as the solution of the following BSPDE:

\[
\begin{cases}
  dv^{n,m}_t(x) = - (L^m v^{n,m}_s(x) + M^m z^{n,m}_{k,s}(x) + g^n_s(x)) \, ds + z^{n,m}_{k,s}(x) \, dB^k_s, \\
  v^{n,m}_t(w, x) = 0, \forall (w, t, x) \in \Omega \times \{T\} \times D \cup [0, T] \times \partial D,
\end{cases}
\]

and define \( v^{n,m}(w, t, x) \equiv 0 \) if \( x \in D \).

The comparison theorem for linear BSPDE (see [9]) implies that \( v^{n,m} \geq 0 \). Moreover, Lemma 3.2 implies that

\[
\|v^{n,m}\|_{M^q} \leq C \|g^n\|_{M^q} \leq C \|g \chi_P \chi_D\|_{M^q}, \tag{3.4}
\]

where \( C \) depends on \( \text{diam}(D) \), and is independent of \( g, P, m, n \). Hence, we calculate that

\[
\mathbb{E} \left[ \int_0^{T \wedge \tau_T \wedge D} \left( \int_t^{T \wedge \tau_T \wedge D} g^n_s(X^m_s) \, ds \right) \right] \leq C \left( \|(v^{n,m})^{q/p}\|_{M^p} + \|(v^{n,m})^{q/p}\|_{M^2} \right) \leq C \|g \chi_P \chi_D\|^{q/p}_{M^q}. \tag{by the result in Step 1}
\]

First, taking \( m \to +\infty \) and then letting \( n \to +\infty \), we deduce that

\[
\mathbb{E} \left[ \int_0^{T \wedge \tau_T \wedge D} \left( \int_t^{T \wedge \tau_T \wedge D} |g_s(X_s)| \, ds \right) \right] \leq \liminf_{n \to \infty} \liminf_{m \to \infty} \mathbb{E} \left[ \int_0^{T \wedge \tau_T \wedge D} \left( \int_t^{T \wedge \tau_T \wedge D} |g^n_s(X^m_s)| \, ds \right) \right] \leq C \|g \chi_P \chi_D\|^{q/p}_{M^q}.
\]

\( \Box \)
Remark 3.3. For any $V \in \mathbb{H}^{0,2}$, define

$$P_m \triangleq \left\{(w, t) : \mathbb{E} \left( \int_0^T |V_u|^2_{0,2} \, du \right) < m \right\} \quad \text{and} \quad v \triangleq \text{sign}(V) \, |V|^{2/(d^*+3)}.$$ 

Then $v \chi_{P_m} \in L^2 \cap L^{d^*+3}$ for any $m \in \mathbb{N}_+$. Applying the first estimate in Lemma 3.3, we deduce that $v(X) \chi_{\{ t \leq T \wedge \tau_{P_m} \wedge \tau_{B_m} \}} \in L^1$ and $v(X)$ is well-defined in the set $\{(w, t) : t \leq T \wedge \tau_{P_m}(\omega) \wedge \tau_{B_m}(\omega) \}$ for any $m \in \mathbb{N}_+$. Since $P_m \times B_m \uparrow \Omega \times [0, T] \times \mathbb{R}^d$, then $T \wedge \tau_{P_m} \wedge \tau_{B_m} \uparrow T$ and $v(X)$ is well-defined in $\Omega \times [0, T]$. Hence, the process $V(X) \triangleq \text{sign}(v(X)) \, |v(X)|^{(d^*+3)/2}$ is well-defined.

Proof of Theorem 3.1. Smooth $V$, $U$, and $Z$ as follows

$$V^n \triangleq V * \zeta_n, \quad U^n \triangleq U * \zeta_n, \quad Z^n \triangleq Z * \zeta_n.$$

Then we have that for any $x \in \mathbb{R}^d$,

$$V^n_t(x) = V^n_0(x) + \int_0^t U^n_s(x) \, ds + \int_0^t Z^n_s(x) \, dB_s$$

for any $x \in \mathbb{R}^d$, and $V^n, U^n, Z^n$ satisfy the assumptions in Lemma 2.1 and converge to $V, U,$ and $Z$ in the spaces $\mathbb{H}^{2,2}, \mathbb{H}^{0,2}$, and $\mathbb{H}^{1,2}$, respectively.

Denote $\tau^n \triangleq \tau_{P_m} \wedge \tau_{B_m}$, with $P_m$ waiting to be defined in (3.6). Applying Lemma 2.1 we deduce that

$$V^n_{t \wedge \tau_m}(X_{t \wedge \tau_m}) = V^n_{T \wedge \tau_m}(X_{T \wedge \tau_m}) - \int_{t \wedge \tau_m}^{T \wedge \tau_m} (LV^n_s + M^k Z^n_{k,s} + U^n_s)(X_s) \, ds$$

$$- \int_{t \wedge \tau_m}^{T \wedge \tau_m} (Z^n_{k,s} + M^k V^n_s) \, dB_s - \int_{t \wedge \tau_m}^{T \wedge \tau_m} (N^l V^n_s) \, dW^l_s. \quad (3.5)$$

First, we prove that there is a subsequence (still denoted by itself) such that

$$\lim_{n \to \infty} \int_{t \wedge \tau_m}^{T \wedge \tau_m} f^n_s(X_s) \, ds = \int_{t \wedge \tau_m}^{T \wedge \tau_m} f_s(X_s) \, ds$$

and

$$\lim_{n \to \infty} \int_{t \wedge \tau_m}^{T \wedge \tau_m} g^n_{k,s}(X_s) \, dB_s = \int_{t \wedge \tau_m}^{T \wedge \tau_m} g_{k,s}(X_s) \, dB_s$$

almost everywhere in $\Omega \times (0, T)$, where

$$f^n \triangleq LV^n + M^k Z^n_k + U^n, \quad f \triangleq LV + M^k Z_k + U, \quad g^n \triangleq Z^n_k + M^k V^n, \quad g_k \triangleq Z_k + M^k V.$$

Since $V^n$ converges to $V$ in $\mathbb{H}^{2,2}$, $Z^n$ converges to $Z$ in $\mathbb{H}^{1,2}$ and $U^n$ converges to $U$ in $\mathbb{H}^{0,2}$, then $f^n$ converges to $f$ in $\mathbb{H}^{0,2}$ and $g^n$ converges to $g$ in $\mathbb{H}^{1,2}$. The Sobolev imbedding theorem implies that $g^n$ converges to $g$ also in $L^{2,2}_{\sigma} (H^{0,2})$, where $q = 2$ if $d^* \leq 2$ and $q = \frac{d^*}{d^* - 2}$ if $d^* > 2$. Hence, we have that

$$|f^n - f|_{0,2}^2 + |(g^n - g)|_{0,q}^2 \to 0 \text{ in } \mathcal{L}^1.$$
So, there exists a strictly increasing sequence of numbers \( \{K_n\}_{n=1}^{\infty} \) such that
\[
\mathbb{Q} \left\{ (w, t) \in \Omega \times [0, T] : \left| f^n - f \right|_{0,2} + \left| (g^n - g)^2 \right|_{0,q} \geq \frac{1}{m} \right\} < \frac{1}{2^n} \quad \text{for any } n \geq K_m,
\]
where \( \mathbb{Q} \) is the product measure of \( \mathbb{P} \) and Lebesgue measure on \( [0, T] \).

Define
\[
P_m \triangleq \bigcap_{n=m}^{\infty} \left\{ (w, t) \in \Omega \times [0, T] : \left| f^{K_n} - f \right|_{0,2} + \left| (g^{K_n} - g)^2 \right|_{0,q} < \frac{1}{n} \right\}. \tag{3.6}
\]

We have that
\[
\mathbb{Q}(P_m) \geq 1 - \frac{1}{2^{m-1}} \to 1 \quad \text{as } m \to \infty,
\]
\[
\left| (f^{K_n} - f) \chi_{P_m} \chi_{B_m} \right|_{0,2} + \left| (g^{K_n} - g)^2 \chi_{P_m} \chi_{B_m} \right|_{0,q} \leq \frac{1}{n} \quad \text{for any } n \geq m,
\]
\[
\left\| (f^{K_n} - f) \chi_{P_m} \chi_{B_m} \right\|_{M^2} + \left\| (g^{K_n} - g)^2 \chi_{P_m} \chi_{B_m} \right\|_{M^q} \to 0 \quad \text{as } n \to \infty \quad \text{for any } m \in \mathbb{N}_+.
\]
From Lemma 3.3, we obtain that as \( n \to \infty \),
\[
\mathbb{E} \left[ \int_0^{T \wedge \tau_m} \left| \int_t^{T \wedge \tau_m} \left( f^{K_n} (X_s) - f_s (X_s) \right) \, ds \right|^{\frac{2q}{p}} \, dt \right] \leq C \left\| (f^{K_n} - f) \chi_{P_m} \chi_{B_m} \right\|_{M^2}^{2q/p} \to 0.
\]
So, we deduce that there exists a subsequence (still denoted by itself) such that,
\[
\int_{t \wedge \tau_m}^{T \wedge \tau_m} f^n_s (X_s) \, ds \to \int_{t \wedge \tau_m}^{T \wedge \tau_m} f_s (X_s) \, ds \quad \text{a.e. in } \Omega \times (0, T) \quad \text{as } n \to \infty.
\]
Moreover, as \( n \to \infty \),
\[
\mathbb{E} \left[ \int_0^{T \wedge \tau_m} \left| \int_t^{T \wedge \tau_m} \left( g^{K_n}_{k,s} (X_s) - g_{k,s} (X_s) \right) \, dB_s \right|^{\frac{2q}{p}} \, dt \right] = \mathbb{E} \left\{ \int_0^{T \wedge \tau_m} \mathbb{E} \left[ \left| \int_t^{T \wedge \tau_m} \left( g^{K_n}_{k,s} (X_s) - g_{k,s} (X_s) \right) \, dB_s \right|^{\frac{2q}{p}} \right] \, dt \right\} \leq C \mathbb{E} \left\{ \int_0^{T \wedge \tau_m} \left( \int_t^{T \wedge \tau_m} \left| g^{K_n}_{k,s} (X_s) - g_{k,s} (X_s) \right|^2 \, ds \right)^{\frac{q}{p}} \, dt \right\} \leq C \left\| (g^{K_n} - g)^2 \chi_{P_m} \chi_{B_m} \right\|_{M^q}^{q/p} \to 0.
\]
For any fixed $m$, we pass to the limit in (3.5) as $n \to \infty$ (at least for a subsequence). Using the above method, we easily achieve that a.e. in $\{(w, t) : 0 \leq t \leq \tau_m(w), w \in \Omega\}$,

$$V_{t \wedge \tau_m}(X_{t \wedge \tau_m}) = V_{T \wedge \tau_m}(X_{T \wedge \tau_m}) - \int_{t \wedge \tau_m}^{T \wedge \tau_m} (LV_s + M^k Z_{k, s} + U_s)(X_s) \, ds$$

$$- \int_{t \wedge \tau_m}^{T \wedge \tau_m} (Z_{k, s} + M^k V_s)(X_s) \, dB^k_s - \int_{t \wedge \tau_m}^{T \wedge \tau_m} (N^l V_s)(X_s) \, dW^l_s.$$

Hence, we can choose a version of $w \in \Omega$ to 

Using the above method, we easily achieve that a.e. in $\{(w, t) : 0 \leq t \leq \tau_m(w), w \in \Omega\}$, the terminal value of the Dynkin game are characterized by the strong solution of BSPDVI (1.2).

In this section, we prove the verification theorem that the Nash equilibrium point and the

4 Verification theorem.

In this section, we prove the verification theorem that the Nash equilibrium point and the value of the Dynkin game are characterized by the strong solution of BSPDVI (1.2).

Consider the following assumptions on the free term $f$, the terminal value $\varphi$, and the upper and lower obstacles $\overline{V}$ and $\underline{V}$ in BSPDVI (1.2).

Assumption V3. (Regularity) $f \in H^{0,2}, \varphi \in L^{1,2}$, and $\overline{V}$ and $\underline{V}$ are continuous semimartingales of the following form

$$d\overline{V}_t = -\overline{g}_t \, dt + Z^k_t \, dB^k_t, \quad d\underline{V}_t = -\underline{g}_t \, dt + Z^k_t \, dB^k_t,$$

where $\underline{g}, \overline{g}, Z, \overline{Z} \in H^{0,2}, \underline{V}, \overline{V} \in H^{1,2}$ and there exists a nonnegative random field $h \in H^{0,2}$ such that

$$\mathcal{L}\overline{V} + \mathcal{M}^k Z^k - \underline{g} + h \geq 0 \quad \text{and} \quad \mathcal{L}\underline{V} + \mathcal{M}^k Z^k - \overline{g} - h \leq 0$$

hold in the sense of distribution, that is, for any nonnegative function $\eta \in C^2_0(\mathbb{R}^d)$, we have

$$\mathcal{T}(\underline{V}, Z, \underline{g}, h, \eta) \geq 0 \quad \text{and} \quad \mathcal{T}(\overline{V}, Z, \overline{g}, -h, \eta) \leq 0 \quad \text{a.e. in } \Omega \times [0, T],$$

where

$$\mathcal{T}(\underline{V}, Z, \underline{g}, h, \eta) \triangleq -\int_{\mathbb{R}^d} \left( a^{ij} D_i \underline{V} + \sigma^{jk} Z^k \right) D_j \eta \, dx + \int_{\mathbb{R}^d} (h - \underline{g}) \eta \, dx$$

$$+ \int_{\mathbb{R}^d} \left[ \left( b^i - D_j a^{ij} \right) D_i \underline{V} + c \underline{V} + \left( \mu^k - D_l \sigma^{ik} \right) Z^k \right] \eta \, dx.$$
Assumption V4. (Compatibility) \( V \leq \overline{V}, \overline{V}_T \leq \varphi \leq \overline{V}_T \) and
\[
\left( \mathcal{L} V_t + \mathcal{M}^{k} Z^k_t + f_t - g_t \right) \chi_{\{V = \varphi\}} = 0 \text{ a.e. in } \Omega \times \overline{Q}, \text{ where } Q \triangleq [0,T] \times \mathbb{R}^d.
\]

The following is a stronger version of Assumption V3.

Assumption V3'. \( f \in \mathbb{H}^{0,2}, \varphi \in \mathbb{L}^{1,2}, \text{ and } V \text{ and } \overline{V} \) have the following representation:
\[
dV_t = -g_t dt + Z^k_t dB^k_t, \quad d\overline{V}_t = -\overline{g}_t dt + \overline{Z}^k_t dB^k_t,
\]
with \( V, \overline{V} \in \mathbb{H}^{2,2}, Z, \overline{Z} \in \mathbb{H}^{1,2}, \text{ and } g, \overline{g} \in \mathbb{H}^{0,2}. \)

The following clarifies the relationship between Assumptions V3 and V3'.

Proposition 4.1. (i) Assumption V3' implies Assumption V3. (ii) If \( V \text{ and } \overline{V} \) satisfy Assumption V3, then there exist two sequences of functions \( \{V_n\}_{n=1}^{\infty} \text{ and } \{\overline{V}_n\}_{n=1}^{\infty} \) such that \( V_n \) and \( \overline{V}_n \) satisfy Assumption V3' and the following
\[
dV_{n,t} = -g_{n,t} dt + Z^k_{n,t} dB^k_t, \quad d\overline{V}_{n,t} = -\overline{g}_{n,t} dt + \overline{Z}^k_{n,t} dB^k_t,
\]
\[
V_n \to V, \quad \overline{V}_n \to \overline{V} \text{ a.e. in } \Omega \times \overline{Q} \text{ and in } \mathbb{H}^{1,2} \cap \mathbb{S}^{0,2}.
\]
Moreover, there exists a sequence of nonnegative random fields \( \{\tilde{h}_n\}_{n=1}^{\infty} \) such that \( \tilde{h}_n \in \mathbb{H}^{0,2} \) and
\[
\mathcal{L} V_n + \mathcal{M}^{k} Z^k_n - g_n \geq -\tilde{h}_n, \quad \mathcal{L} \overline{V}_n + \mathcal{M}^{k} \overline{Z}^k_n - \overline{g}_n \leq \tilde{h}_n \text{ a.e. in } \Omega \times \overline{Q},
\]
\[
\|h_n\|_{0,2} \leq C(K) \left( \|V\|_{1,2} + \|\overline{V}\|_{1,2} + \|Z\|_{0,2} + \|\overline{Z}\|_{0,2} + \|h\|_{0,2} \right). \tag{4.3}
\]

Proof. (i) For the two processes \( V \) and \( \overline{V} \) in Assumption V3', define
\[
h \triangleq \max \left\{ \left( \mathcal{L} V + \mathcal{M}^{k} Z^k - g \right)^-, \left( \mathcal{L} \overline{V} + \mathcal{M}^{k} \overline{Z}^k - \overline{g} \right)^+ \right\},
\]
where \( f^+ \) and \( f^- \) represent the positive and negative parts of \( f \), respectively. We can check that \( V \) and \( \overline{V} \) satisfy Assumption V3.

(ii) For the process \( V \) in Assumption V3, define
\[
V_n = V * \zeta_n, \quad g_n = g * \zeta_n, \quad Z_n = Z * \zeta_n, \quad h_n = h * \zeta_n.
\]
We have
\[
dV_{n,t} = -g_{n,t} dt + Z^k_{n,t} dB^k_t, \quad V_n \in \mathbb{H}^{2,2}, \ g_n \in \mathbb{H}^{0,2}, \ Z_n \in \mathbb{H}^{1,2}, \ h_n \in \mathbb{H}^{0,2}.
\]
Since \( V \in \mathbb{H}^{1,2}, g, Z, h \in \mathbb{H}^{0,2} \) and \( V \) has the special representation (4.1), then we have that
\[
V_n \to V \text{ a.e. in } \Omega \times \overline{Q} \text{ and in } \mathbb{H}^{1,2} \cap \mathbb{S}^{0,2}, \quad g_n \to g, \ Z_n \to Z, \ h_n \to h \text{ in } \mathbb{H}^{0,2}.
\]
So, it is sufficient to prove that there exist a nonnegative random field sequence \( \{\tilde{h}_n\}_{n=1}^{\infty} \) satisfying (4.3).
Let \( \eta(x) = \zeta_n(y - x) \) in (4.2). In the following, we estimate every term in (4.2). At first, we define
\[
I_n^1(t, y) \triangleq - \int_{\mathbb{R}^d} a^{ij}(t, x) D_i V(t, x) D_j \eta(x) \, dx - a^{ij}(t, y) D_{ij} V_n(t, y).
\]
We have
\[
| I_n^1(t, y) | \leq \int_{\mathbb{R}^d} | a^{ij}(t, x) - a^{ij}(t, y) | | D_i V(t, x) n^{d+1} D_j \zeta(n(y - x)) | \, dx
\]
\[
\leq K n^{d+1} \int_{\mathbb{R}^d} | x - y | | D_i V(t, x) D_j \zeta(n(y - x)) | \, dx
\]
\[
= K \int_{B_1} | x | | D_i V(t, x - \frac{x}{n}) D_j \zeta(x) | \, dx
\]
\[
\leq C(K) n^d \int_{D_n(y)} | D_i V(t, x) | \, dx,
\]
where
\[
D_n(y) \triangleq \{(x_1, \ldots, x_d) : |x_i - y_i| \leq \frac{1}{n} \text{ for any } i = 1, \ldots, d^* \}.
\]
Hence, we have the following estimate
\[
\| I_n^1 \|_{0, 2}^2 \leq C(K) E \left[ \int_{Q} \int_{D_n(y)} n^d | D_i V(t, x) |^2 \, dx \, dy \, dt \right] \leq C(K) \| D V \|_{0, 2}^2.
\]
Denote
\[
I_n^2(t, y) \triangleq - \int_{\mathbb{R}^d} (\sigma^{ik} Z^k D_j \eta) (t, x) \, dx - (\sigma^{ik} D_j Z^k_n) (t, y),
\]
\[
I_n^3(t, y) \triangleq \int_{\mathbb{R}^d} \left[ (b^i D_i V + c V + \mu^k Z^k) \eta \right] (t, x) \, dx - \left( b^i D_i V_n + c V_n + \mu^k Z^k_n \right) (t, y),
\]
\[
I_n^4(t, y) \triangleq - \int_{\mathbb{R}^d} \left[ \left( D_j a^{ij} D_i V + D_i \sigma^{ik} Z^k \right) \eta \right] (t, x) \, dx.
\]
Repeating the above argument, we get the following estimate:
\[
\| I_n^2 \|_{0, 2}^2 + \| I_n^3 \|_{0, 2}^2 + \| I_n^4 \|_{0, 2}^2 \leq C(K) \left( \| Z \|_{0, 2}^2 + \| V \|_{1, 2}^2 \right).
\]
Hence, if we denote
\[
\tilde{h}_n = h_n + \| I_n^1 \| + \| I_n^2 \| + \| I_n^3 \| + \| I_n^4 \|,
\]
then the above estimates and \( T(V, Z, \xi, h, \eta) \geq 0 \) imply that
\[
\mathcal{L} V_n + \mathcal{M}^k Z^k_n - g_n + \tilde{h}_n \geq 0, \quad \| \tilde{h}_n \|_{0, 2} \leq C(K) \left( \| Z \|_{0, 2}^2 + \| V \|_{1, 2}^2 + \| h \|_{0, 2}^2 \right).
\]
Using the same method, we can deduce the rest of (4.3). □

A strong solution of BSPDVI (1.2) is defined as follows.
Definition 4.1. If \((V, Z, k^+, k^-) \in H_2 \times H_1 \times H_0 \times H_0\) such that

\[
\begin{aligned}
V_t &= \varphi + \int_0^T (L V_s + M^k Z^k_s + f^k_s + k^+_s - k^-_s) \, ds \\
&- \int_0^T Z^k_s \, dB^k_s, \quad \text{a.e. } x \in \mathbb{R}^d \text{ for all } t \in [0, T] \text{ and a.s. in } \Omega; \\
V &\leq V \leq \bar{V}, \quad k^\pm \geq 0 \quad \text{a.e. in } \Omega \times Q; \\
\int_0^T (V_t - \bar{V}_t) k^+_t \, dt &= \int_0^T (\bar{V}_t - V_t) k^-_t \, dt = 0 \quad \text{a.e. in } \Omega \times \mathbb{R}^d.
\end{aligned}
\]

(4.4)

Then \((V, Z, k^+, k^-)\) is called a strong solution of BSPDVI (1.2).

We have the following verification theorem for Problem \(\mathcal{D}_{tx}\).

Theorem 4.2. (Verification) Let Assumptions V1-V4 be satisfied and \((t, x) \in Q\). Let \(X\) be the solution of SDE (1.1) with the value being \(x\) at the initial time \(t\), and \(\bar{V}(X) \triangleq \bar{V}(\cdot, X)\) and \(\underline{V}(X) \triangleq \underline{V}(\cdot, X)\) are stochastic processes with continuous paths. Assume that the four-tuple \((V, Z, k^+, k^-)\) is a strong solution of BSPDVI (1.2) with

\[
a^{ij} \triangleq \frac{1}{2} \left( \sum_{l=1}^{d_1} \gamma_{il} \gamma_{jl} + \sum_{l=1}^{d_2} \theta_{il} \theta_{jl} \right), \quad b^i \triangleq \beta_i, \quad c \triangleq 0, \quad \sigma^{ik} \triangleq \theta_{ik}, \quad \mu_k \triangleq 0
\]

(4.5)

for any \(i, j = 1, \ldots, d^*\) and \(k = 1, \ldots, d_2\). Then \(V(t, x)\) is the value of Problem \(\mathcal{D}_{tx}\).

Define

\[
\tau^*_1 \triangleq \inf \{ s \in [t, T] : V_s(X_s) = \bar{V}_s(X_s) \} \wedge T
\]

and

\[
\tau^*_2 \triangleq \inf \{ s \in [t, T] : V_s(X_s) = \underline{V}_s(X_s) \} \wedge T.
\]

Then, \((\tau^*_1, \tau^*_2)\) is a Nash equilibrium point of Problem \(\mathcal{D}_{tx}\).

Remark 4.1. Assumptions D1, D2, and (4.5) imply Assumptions V1 and V2. If Assumptions D1, D2, V3', and (4.5) are all satisfied, then the processes \(\bar{V}(X)\) and \(\underline{V}(X)\) are Itô processes and possess path continuous versions by Theorem 3.1. If \(\bar{V}\) and \(\underline{V}\) are continuous with respect to \((t, x)\) a.s. in \(\Omega\), and Assumptions D1 and D2 are satisfied, then \(\bar{V}(X)\) and \(\underline{V}(X)\) are stochastic processes of continuous paths.

Proof of Theorem 4.2. It is sufficient to prove that for any \(\tau_1, \tau_2 \in U_{t,T}\), it holds that

\[
\mathbb{E} \left[ R_t(x; \tau^*_1, \tau_2) \, \mathcal{F}_t \right] \geq V_t(x) \geq \mathbb{E} \left[ R_t(x; \tau_1, \tau^*_2) \, \mathcal{F}_t \right] \text{ a.s. in } \Omega,
\]

with equality in the first inequality if \(\tau_2 = \tau^*_2\) and in the second inequality if \(\tau_1 = \tau^*_1\). In what follows, we only prove the second inequality since the first one can be proved in a symmetric way.
From Theorem 3.1, we deduce that \( V(X) \) is an Itô process and possesses a path continuous version. Hence, \( V(X) - \overline{V}(X) \) and \( \overline{V}(X) - V(X) \) are stochastic processes with continuous paths. So, we have almost everywhere in \( \Omega \times Q \),

\[
\chi_{\{\tau^*_1 < T\}} V_{\tau_1}^* (X_{\tau_1}) = \chi_{\{\tau^*_1 < T\}} \overline{V}_{\tau_1} (X_{\tau_1}) \quad \text{and} \quad \chi_{\{\tau^*_2 < T\}} V_{\tau_2}^* (X_{\tau_2}) = \chi_{\{\tau^*_2 < T\}} \overline{V}_{\tau_2} (X_{\tau_2}),
\]

and a.s. in \( \Omega \),

\[
V_s(X_s) - \overline{V}_s(X_s) > 0 \quad \text{for any} \quad t \leq s < \tau^*_1 \quad \text{and} \quad \overline{V}_s(X_s) - V_s(X_s) > 0 \quad \text{for any} \quad t \leq s < \tau^*_2.
\]

Moreover, the third equality in Definition 4.1 implies that:

\[
k^+ \chi_{\{(w, s, x): (\overline{V} - V)(w, s, x) > 0\}} = 0, \quad k^- \chi_{\{(w, s, x): (\overline{V} - V)(w, s, x) > 0\}} = 0 \quad \text{in} \quad \mathbb{M}^{d+3}.
\]

From Lemma 3.3, we deduce that

\[
k^+_s(X_s) \chi_{\{s < \tau^*_1\}} = 0, \quad k^-_s(X_s) \chi_{\{s < \tau^*_2\}} = 0 \quad \text{a.e. in} \quad \Omega \times (0, T),
\]

and for any \( \tau_1, \tau_2 \in \mathcal{U}_{t, T} \) satisfying \( \tau_1 \leq \tau^*_1 \), \( \tau_2 \leq \tau^*_2 \), the following hold

\[
\int_t^{\tau_1} k^+_s(X_s) \, ds = 0, \quad \text{and} \quad \int_t^{\tau_2} k^-_s(X_s) \, ds = 0, \quad \text{a.s. in} \quad \Omega. \quad (4.6)
\]

On the event \( \{\tau_1 \in \mathcal{U}_{t, T} : \tau_1 \geq \tau^*_2\} \), applying Theorem 3.1, we have

\[
R_t(x; \tau_1, \tau^*_2)
\]

\[
= \int_t^{\tau^*_2} f_u(X^t_{u,x}) \, du + \overline{V}_{\tau^*_2} (X^t_{\tau^*_2,x}) \chi_{\{\tau^*_2 < T\}} + \varphi(X^t_{T-x}) \chi_{\{\tau^*_2 = T\}}
\]

\[
= \int_t^{\tau^*_2} f_u(X^t_{u,x}) \, du + \overline{V}_{\tau^*_2} (X^t_{\tau^*_2,x})
\]

\[
= \int_t^{\tau^*_2} f_u(X^t_{u,x}) \, du + \int_t^{\tau^*_2} \left[ \alpha V_u + M^k \varphi^k - (\alpha V_u + M^k \varphi^k + f_u + k_u^+ - k_u^-) \right] (X^t_{u,x}) \, du
\]

\[
+ M_1(\tau^*_2) + V(t; X^t_{\tau^*_2,x}),
\]

where

\[
M_1(\tau^*_2) \triangleq \int_t^{\tau^*_2} \left[ \int_t^{\tau^*_2} (Z^k_u + M^k V_u)(X^t_{u,x}) \, dB^k_u + \int_t^{\tau^*_2} (N^l V_u)(X^t_{u,x}) \, dW^l_u \right].
\]

Recalling (4.5), (4.6) and \( k^+ \geq 0 \), we have

\[
R_t(x; \tau_1, \tau^*_2) = V_t(x) - \int_t^{\tau^*_2} k^+_u(X^t_{u,x}) \, du + M_1(\tau^*_2) \leq V_t(x) + M_1(\tau^*_2) \quad \text{a.s. in} \quad \Omega,
\]

with equality if \( \tau_1 = \tau^*_1 \), which follows from \( \tau^*_2 \leq \tau^*_1 \) and (4.6).

On the event \( \{\tau \in \mathcal{U}_{t, T} : \tau_1 < \tau^*_2\} \), in a similar way, we have

\[
R_t(x; \tau_1, \tau^*_2) = \int_t^{\tau^*_1} f_s(X^t_{s,x}) \, ds + \overline{V}_{\tau_1} (X^t_{\tau_1,x}) \leq \int_t^{\tau^*_1} f_s(X^t_{s,x}) \, ds + V_{\tau_1}(X^t_{\tau_1,x})
\]

\[
\leq V_t(x) + M_1(\tau_1) \quad \text{a.s. in} \quad \Omega,
\]

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with equality if \( \tau_1 = \tau_1^* \). So, we obtain that
\[
R_t(x; \tau_1, \tau_2^*) \leq V_t(x) + M_t(\tau_1 \wedge \tau_2^*),
\]
almost surely with equality if \( \tau_1 = \tau_1^* \). Taking conditional expectations with respect to \( \mathcal{F}_t \), we have
\[
\mathbb{E} \left[ R_t(x; \tau_1, \tau_2^*) \mid \mathcal{F}_t \right] \leq V_t(x) \text{ for any } \tau_1 \in \mathcal{U}_t \text{ a.s. in } \Omega
\]
with the equality holding if \( \tau_1 = \tau_1^* \). The proof is then complete. \( \square \)

5 Strong solution of BSPDVI (1.2): existence and uniqueness, and comparison theorem.

In this section, we use the penalty method to prove the existence, and establish a comparison theorem to obtain the uniqueness.

We first deduce some estimates about BSPDE (2.3), which are crucial to the proof of the main results in this paper.

**Lemma 5.1.** Let the assumptions in Lemma 2.2 be satisfied. Define \( f(\cdot) \triangleq F(\cdot, V(\cdot)) \). Then the strong solution \((V, Z)\) of BSPDE (2.3) satisfies the following:
\[
\|V\|_{1,2}^2 + ||V||_{0,2}^2 + \|Z\chi_{\{V \leq 0\}}\|_{0,2}^2 \leq C(\kappa, K, T) \mathbb{E} \left( |\varphi|_{0,2}^2 + \int_0^T \int_D f_s^- V_s^- \, dx \, ds \right), \tag{5.1}
\]
\[
||V||_{1,2}^2 + ||V||_{0,2}^2 + \|Z\chi_{\{V \geq 0\}}\|_{0,2}^2 \leq C(\kappa, K, T) \mathbb{E} \left( |\varphi|_{0,2}^2 + \int_0^T \int_D f_s^+ V_s^+ \, dx \, ds \right), \tag{5.2}
\]
\[
\|V\|_{1,2}^2 + ||V||_{0,2}^2 + \|Z\|_{0,2}^2 \leq C(\kappa, K, T) \mathbb{E} \left( |\varphi|_{0,2}^2 + \int_0^T \int_D (f_s V_s)^+ \, dx \, ds \right), \tag{5.3}
\]
\[
\mathbb{E} \left( \int_0^T \int_D f_s V_s^- \, dx \, ds \right) \leq C(\kappa, K) \left( \|V^-\|_{0,2}^2 + \mathbb{E} |\varphi^-|_{0,2}^2 \right), \tag{5.4}
\]
\[
\mathbb{E} \left( \int_0^T \int_D -f_s V_s^+ \, dx \, ds \right) \leq C(\kappa, K) \left( \|V^+\|_{0,2}^2 + \mathbb{E} |\varphi^+|_{0,2}^2 \right). \tag{5.5}
\]

**Proof.** Define the auxiliary functions:
\[
\xi(r) = \begin{cases} 
  r^2, & r \leq 0; \\
  r^2 (1 - r)^3, & 0 \leq r \leq 1; \\
  0, & r \geq 1
\end{cases}
\]

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and
\[ \xi_n(r) = \frac{1}{n^2} \xi(nr). \]
we have \( \xi_n \in C^2(\mathbb{R}) \), \(|\xi''_n| \leq C\),
\[ \lim_{n \to \infty} \xi_n(r) = (r^-)^2 \text{ uniformly, } \lim_{n \to \infty} \xi'_n(r) = -2r^- \text{ uniformly,} \]
and
\[ \lim_{n \to \infty} \xi''_n(r) = \begin{cases} 2, & r \leq 0; \\ 0, & r > 0. \end{cases} \]

Applying Itô’s formula for Hilbert-valued semimartingales (see e.g. in [27]) to \( \xi_n(V) \), we deduce that \( J^1_n = J^2_n - J^3_n \) a.s. in \( \Omega \), where
\[
J^1_n \triangleq \int_D \xi_n(V_t) \, dx - \int_D \xi_n(\varphi) \, dx,
\]
\[
J^2_n \triangleq \int_D \int_0^T \left[ \xi''_n(V_s) D_i(a^{ij} D_j V_s + \sigma^{ik} Z^k_s) + \xi'_n(V_s) \left( (b^i - D_j a^{ij}) D_i V_s + c V_s + (\mu^k - D_i \sigma^{ik}) Z^k_s + f_s \right) \right] \, ds \, dx
\]
\[
\quad + \xi''_n(V_s) \left( 2 a^{ij} D_i V_s D_j V_s + 2 \sigma^{ik} Z^k_s D_i V_s + \frac{1}{2} |Z_s|^2 \right) \, ds \, dx
\]
\[
J^3_n \triangleq \int_D \int_0^T \xi'_n(V_s) Z^k_s \, d B^k_s \, dx.
\]

Taking \( n \to \infty \) in the above equalities, we have that
\[
J^1_n \to J_1 \triangleq \int_D (V^-_t)^2 \, dx - \int_D (\varphi^-)^2 \, dx,
\]
\[
J^2_n \to J_2 \triangleq \int_D \int_0^T \chi_{\{V_s \leq 0\}} \left[ -2 a^{ij} D_i V_s D_j V_s + 2 \sigma^{ik} Z^k_s D_i V_s + |Z_s|^2 \right] \, ds \, dx
\]
\[
\quad - 2 V^{-}_s \left( (b^i - D_j a^{ij}) D_i V_s + c V_s + (\mu^k - D_i \sigma^{ik}) Z^k_s + f_s \right) \, ds \, dx
\]
\[
J^3_n \to J_3 \triangleq -2 \int_D \int_0^T V^{-}_s Z^k_s \, d B^k_s \, dx.
\]
Hence, \( J_1 = J_2 - J_3 \) a.s. in \( \Omega \). Denote
\[
\|V\|_{k,2}^{2,t} \triangleq E \left( \sup_{s \in [t,T]} |V_s|^2_{k,2} \right) \quad \text{and} \quad \|V\|_{k,2}^{2,t} \triangleq E \left( \int_t^T |V_s|^2_{k,2} \, ds \right).
\]

Since
\[
J_2 \leq \int_D \int_t^T \int_{\chi \{ V \leq 0 \}} \left[ -\frac{\kappa}{2} |DV_s|^2 + C(\kappa, K) |V_s|^2 - \frac{\kappa}{2(\kappa + 4K)} |Z_s|^2 - 2V_s^- f_s \right] \, ds \, dx,
\]
taking expectation on both sides of \( J_1 = J_2 - J_3 \), we have
\[
E|V_t^-|^2_{0,2} - E|\varphi^-|^2_{0,2} \leq -\frac{\kappa}{2} \|V^-\|_{1,2}^2 + \frac{\kappa}{2(\kappa + 4K)} \|Z \chi_{\{ V \leq 0 \}}\|_{0,2}^2 + C(\kappa, K) \|V^-\|_{0,2}^2 + 2E \left( \int_t^T \int_D V_s^- f_s \, dx \, ds \right). \tag{5.6}
\]
Hence,
\[
\sup_{s \in [t,T]} E|V_s^-|^2_{0,2} + \|V^-\|_{1,2}^2 + \|Z \chi_{\{ V \leq 0 \}}\|_{0,2}^2 \leq C(\kappa, K) \left\{ E(\|\varphi^-\|_{0,2}^2) + \|V^-\|_{0,2}^2 + E \left( \int_t^T \int_D f_s^- V_s^- \, dx \, ds \right) \right\}.
\]
This along with the Gronwall inequality yields that
\[
\sup_{t \in [0,T]} E|V_t^-|^2_{0,2} + \|V^-\|_{1,2}^2 + \|Z \chi_{\{ V \leq 0 \}}\|_{0,2}^2 \leq C(\kappa, K, T) E \left( \|\varphi^-\|_{0,2}^2 + \int_0^T \int_D f_s^- V_s^- \, dx \, ds \right).
\]
Using the BDG inequality, we have
\[
\|V^-\|_{0,2}^2 + \|V^-\|_{1,2}^2 + \|Z \chi_{\{ V \leq 0 \}}\|_{0,2}^2 \leq \frac{1}{2} \|V^-\|_{0,2}^2 + C(\kappa, K, T) \left[ \|Z \chi_{\{ V \leq 0 \}}\|_{0,2}^2 + E \left( \|\varphi^-\|_{0,2}^2 + \int_0^T \int_D f_s^- V_s^- \, dx \, ds \right) \right].
\]
So, the Gronwall inequality implies (5.1).

We prove (5.2) in the same way. Since
\[
|V| + |DV| = V^+ + V^- + |DV^+| + |DV^-|, \quad (fV)^+ = f^+ V^+ + f^- V^-,
\]
we derive (5.3) from (5.1) and (5.2).

On the other hand, taking \( t = 0 \) in (5.6), we have (5.4). Applying (5.4) to the BSPDE for \( -V, -Z \), we have (5.5). \( \Box \)

We have the following comparison theorem, which implies the uniqueness of strong solution of BSPDVI (1.2).
Theorem 5.2. Let Assumptions V1 and V2 be satisfied. Let \((V_1, Z_i, k_i^+, k_i^-)\) be the strong solution to BSPDVI \((1.2)\) with \((f, \varphi, \nabla, V) \triangleq (f_i, \varphi_i, \nabla_i, V_i)\) for \(i = 1, 2\). If \(f_1 \geq f_2, \varphi_1 \geq \varphi_2, \nabla_1 = \nabla_2,\) and \(V_1 \geq V_2,\) then \(V_1 \geq V_2\) a.e. in \(\Omega \times Q.\)

Remark 5.1. In general, we have no comparison on the reflection parts \(k^\pm.\) Here is an illustration. Take \(d^* = 1, \mathcal{L} = D_x^2, \mathcal{M} = D_x, \) and \(T = 1.\)

(i) For the following parameters:
\[
V_1 = (1 + x^2)^{-1}, \quad V_2 = 0, \quad \varphi_1 = (1 + x^2)^{-1}, \quad \varphi_2 = 0, \quad f_1 = 3 (1 + x^2)^{-3};
\]
\[
V_2 = (1 + x^2)^{-1}, \quad V_2 = 0, \quad \varphi_2 = 0, \quad f_2 = -(1 + x^2)^{-3},
\]
we have
\[
V_1 = (1 + x^2)^{-1}, \quad Z_1 = 0, \quad k_1^+ = 0, \quad k_1^- = (1 + 6x^2) (1 + x^2)^{-3};
\]
\[
V_2 = 0, \quad Z_2 = 0, \quad k_2^+ = (1 + x^2)^{-3}, \quad k_2^- = 0,
\]
with
\[k_1^+ < k_2^+ \quad \text{and} \quad k_1^- > k_2^-.
\]

(ii) For the following parameters:
\[
V_1 = e^4 (1 + x^2)^{-1}, \quad V_1 = e^{3-3t} (1 + x^2)^{-1}, \quad \varphi_1 = e^3 (1 + x^2)^{-1}, \quad f_1 = 0;
\]
\[
V_2 = e^{3-3t} (1 + x^2)^{-1}, \quad V_2 = 0, \quad \varphi_2 = e^{-3} (1 + x^2)^{-1}, \quad f_2 = 0,
\]
we have
\[
V_1 = e^{3-3t} (1 + x^2)^{-1}, \quad Z_1 = 0, \quad k_1^+ = e^{3-3t} (3x^4 + 5) (1 + x^2)^{-3}, \quad k_1^- = 0;
\]
\[
V_2 = e^{3-3t} (1 + x^2)^{-1}, \quad Z_2 = 0, \quad k_2^+ = 0, \quad k_2^- = e^{3-3t} (3x^4 + 12x^2 + 1) (1 + x^2)^{-3},
\]
with
\[k_1^+ > k_2^+ \quad \text{and} \quad k_1^- < k_2^-.
\]

Proof of Theorem 5.2. Define
\[
\Delta V \triangleq V_1 - V_2, \quad \Delta Z \triangleq Z_1 - Z_2, \quad \Delta k^+ \triangleq k_1^+ - k_2^+,
\]
\[
\Delta k^- \triangleq k_1^- - k_2^-, \quad \Delta f \triangleq f_1 - f_2, \quad \Delta \varphi \triangleq \varphi_1 - \varphi_2.
\]
Then \(\Delta f \geq 0, \Delta \varphi \geq 0\) and \(\Delta V\) satisfies the following BSPDE:
\[
\Delta V_t = \Delta \varphi + \int_0^T (\mathcal{L} \Delta V_s + \mathcal{M} k^\Delta V^k_s + \Delta f_s + \Delta k^+_s - \Delta k^-_s) \, ds - \int_0^T \Delta Z^k_s \, dB^k_s.
\]

In view of Lemma 5.1, since \(\Delta f \geq 0, \Delta \varphi \geq 0, k_1^+ \geq 0, k_2^+ \geq 0,\) we have that
\[
\| (\Delta V)^- \|^2_{1,2} + \| (\Delta V)^- \|^2_{0,2} + \| Z \chi_{\{ \Delta V \leq 0 \}} \|^2_{0,2}
\]
\[
\leq C \mathbb{E} \left( \| (\Delta \varphi)^- \|^2_{0,2} + \int_0^T \int_\mathbb{R} (\Delta f_s + \Delta k^+_s - \Delta k^-_s) (\Delta V_s)^- \, dx \, ds \right)
\]
\[
\leq C \mathbb{E} \left( \int_\mathbb{R} \int_0^T \chi_{\{ V_1 < V_2 \}} \left( k^-_{1,s} + k^+_{2,s} \right) (V_2 - V_1) \, ds \, dx \right). \tag{5.7}
\]
On the other hand, in the domain \( \{ V_1 < V_2 \} \), we have
\[
V_2 \leq V_1 \leq V_1 < V_2 \leq V_1.
\]
Hence, we deduce that
\[
\int_0^T \chi_{\{ V_1 < V_2 \}} (k_{1,s}^- + k_{2,s}^+) (V_{2,s} - V_{1,s}) \, ds
= \int_0^T \chi_{\{ V_1 < V_2 \}} \left[ (V_{2,s} - V_1,s) k_{1,s}^- - (V_1,s - V_{1,s}) k_{1,s}^- \right] \, ds
+ \int_0^T \chi_{\{ V_1 < V_2 \}} \left[ (V_{2,s} - V_{2,s}) k_{2,s}^+ - (V_1,s - V_{2,s}) k_{2,s}^+ \right] \, ds
\leq \int_0^T \chi_{\{ V_1 < V_2 \}} \left[ - (V_1,s - V_{1,s}) k_{1,s}^- + (V_{2,s} - V_{2,s}) k_{2,s}^+ \right] \, ds. \tag{5.8}
\]
Since \((V_1 - V_1)_s \leq 0\) and \((V_2 - V_2)_s \geq 0\) a.e. in \( \Omega \times Q \), then the fourth equality in \( (4.4) \) implies that
\[
\int_0^T \chi_{\{ V_1 < V_2 \}} (V_{1,s} - V_{1,s}) k_{1,s}^- \, ds = \int_0^T \chi_{\{ V_1 < V_2 \}} (V_{2,s} - V_{2,s}) k_{2,s}^+ \, ds = 0. \tag{5.9}
\]
From \( (5.7)-(5.9) \), we conclude that
\[
\| (\Delta V)^- \|_{1,2}^2 + ||(\Delta V)^-||_{0,2}^2 + \| Z \chi_{\{ \Delta V \leq 0 \}} \|_{0,2}^2 = 0,
\]
which means that \( \Delta V \geq 0 \), i.e., \( V_1 \geq V_2 \).

The main result of this section is stated as follows.

**Theorem 5.3.** Let Assumptions \( V1-V4 \) be satisfied. Then BSPDVI \( (1.2) \) has a unique strong solution \( (V,Z,k^+,k^-) \) such that
\[
\| V \|_{2,2} + \| V \|_{1,2} + || Z ||_{1,2} + \| k^+ \|_{0,2} + \| k^- \|_{0,2}
\leq C(\kappa, K, T) \left( \mathbb{E} [ \| \varphi \|_{1,2} ] + \| f \|_{0,2} + \| V \|_{1,2} + \| \nabla V \|_{1,2}
+ \| Z \|_{0,2} + \| \nabla Z \|_{0,2} + \| h \|_{0,2} \right). \tag{5.10}
\]
Moreover, if \( \nabla (X) \) and \( \nabla (X) \) are stochastic processes of continuous paths, then the strong solution of BSPDVI \( (1.2) \) coincides with the value of Problem \( \mathcal{Z}_{tx} \).

**Proof.** The uniqueness of strong solutions is an immediate consequence of Theorem 5.2. The rest of the proof is divided into the four steps.

**Step 1. Penalty.** We construct the following penalized problem to approximate BSPDE \( (1.2) \):
\[
\begin{cases}
dV_{n,t} = - \left[ \mathcal{L} V_{n,t} + \mathcal{M}^k Z_{n,t}^k + f_t + n (V_{n,t} - V_{n,t})^+ - n (V_{n,t} - V_{n,t})^- \right] \, dt \\
+ Z_{n,t}^k dB_t^k,
\end{cases}
\tag{5.11}
\]
\( V_{n,T}(x) = \varphi_n(x) \),
where $V_n$ and $\overline{V}_n$ are defined in Proposition 4.1 and $\varphi_n$ is the mollification of $\varphi$, i.e., $\varphi_n = \varphi * \zeta$, which is defined in (3.1). It is clear that $V_n \leq \varphi_n \leq \overline{V}_n$ and $\varphi_n$ converges to $\varphi$ in $L^{1,2}$.

Denote

$$ F(w, t, x, u) \triangleq f_t(w, x) + n (V_{n, t}(w, x) - u)^+ - n (u - \overline{V}_{n, t}(w, x))^+. $$

Then $F(\cdot, 0)$ is $\mathcal{P} \times \mathcal{B}(\mathbb{R}^d)$-measurable for any $u \in \mathbb{R}$ and

$$ F(\cdot, 0) = f + n V^+_n - n \overline{V}^-_n \in \mathbb{H}^{0,2}, \quad |F(w, t, x, u) - F(w, t, x, v)| \leq 2n|u - v|. $$

According to Lemma 2.2, BSPDE (5.11) has a unique strong solution denoted by $(V_n, Z_n)$.

**Step 2. The sequence $\{V_n\}$ is bounded in $\mathbb{H}^{2,2}$.** Define

$$ \Delta V_n \triangleq \sqrt{n}(V_n - \overline{V}_n), \quad \Delta Z_n \triangleq \sqrt{n}(Z_n - Z_n), \quad \Delta V \triangleq \sqrt{n}(V_n - \overline{V}_n). $$

Then $(\Delta V_n, \Delta Z_n)$ is a strong solution to the following BSPDE:

$$
\begin{aligned}
&
d \Delta V_n, t = - (L \Delta V_n, t + \mathcal{M}^k \Delta Z_n, t + \sqrt{n} f_{n, t} + n (\Delta V_{n, t})^- - n (\Delta V_{n, t})^+) dt \\
&\quad + \Delta Z_n^{k, t} dB_t^{k, t},
\end{aligned}
\begin{aligned}
&\Delta V_{n, t}(x) = \varphi_n(x)
\end{aligned}
$$

with

$$ f_{n} \triangleq f + L V_n + \mathcal{M}^k Z_n - g_n \in \mathbb{H}^{0,2} \quad \text{and} \quad \varphi_n \triangleq \sqrt{n}(\varphi_n - \overline{V}_{n, T}) \geq 0. $$

From Lemma 2.5 we have $\varphi_n \in \mathbb{H}^{1,2}$. In view of (4.3), we have

$$ f_{n, 0} \geq f_n \geq f - h_n, \quad f_{n} \leq f_n \leq f + h_n, $$

$$ \|f_{n, 0}\|_{0,2} \leq C(K) \left( \|f\|_{0,2} + \|V\|_{1,2} + \|V\|_{1,2} + \|Z\|_{0,2} + \|Z\|_{0,2} + \|h\|_{0,2} \right). $$

Since

$$ \left( \sqrt{n} f_{n} + n (\Delta V_n)^- - n (\Delta V_n)^+ \right)^- \leq \sqrt{n} f_{n} + n (\Delta V_n)^+ - (\Delta V_n)^+ (\Delta V_n)^- = 0, $$

we have from (5.1) in Lemma 5.1 that

$$
\begin{aligned}
&\|\Delta V_n\|_{1,2}^2 + \|\Delta V_n\|_{0,2}^2 + \|\Delta Z_n\|_{0,2}^2 \leq C(\kappa, K, T) \mathbb{E} \left[ \int_0^T \int_{\mathbb{R}^d} \left( \sqrt{n} f_{n, s} + n (\Delta V_{n, s})^+ \right)^- (\Delta V_{n, s})^- dx ds \right] \\
&= C(\kappa, K, T) \mathbb{E} \left[ \int_0^T \int_{\mathbb{R}^d} f_{n, s}^- (\sqrt{n} \Delta V_{n, s})^- dx ds \right] \\
&\leq \frac{n}{4 C(\kappa, K)} \|\Delta V_n\|_{0,2}^2 + C(\kappa, K, T) C(\kappa, K) \|f_{n, 0}\|_{0,2}^2.
\end{aligned}
$$

(5.12)
Recalling \((\Delta \overline{V}_n)^+ (\Delta \overline{V}_n)^- = 0\), and applying (5.1) in Lemma 5.1 to BSPDE of \((\Delta \overline{V}_n, \Delta \overline{Z}_n)\), we have that
\[
\mathbb{E} \left[ \int_0^T \int_{\mathbb{R}^d} \sqrt{n} f_{n,s} (\Delta V_{n,s})^- + n \left( (\Delta V_{n,s})^- \right)^2 \, dx \, ds \right]
\]
\[
= \mathbb{E} \left[ \int_0^T \int_{\mathbb{R}^d} \left( \sqrt{n} f_{n,s} + n (\Delta V_{n,s})^- - n (\Delta \overline{V}_n)^+ \right) (\Delta V_{n,s})^- \, dx \, ds \right]
\]
\[
\leq C(\kappa, K) \left( \| (\Delta \overline{V}_n)^- \|_{0,2}^2 + \mathbb{E} \left[ \| \varphi_n^- \|_{0,2}^2 \right] \right) \leq C(\kappa, K) \| (\Delta \overline{V}_n)^- \|_{1,2}^2.
\]
Hence, we deduce that
\[
n^2 \| (V_n - \overline{V}_n)^- \|_{0,2}^2 = \mathbb{E} \left[ \int_0^T \int_{\mathbb{R}^d} n \left( (\Delta V_{n,s})^- \right)^2 \, dx \, ds \right]
\]
\[
\leq C(\kappa, K) \| (\Delta \overline{V}_n)^- \|_{1,2}^2 - \mathbb{E} \left[ \int_0^T \int_{\mathbb{R}^d} f_{n,s} (\sqrt{n} \Delta V_{n,s})^- \, dx \, ds \right]
\]
\[
\leq \frac{n^2}{2} \| (V_n - \overline{V}_n)^- \|_{0,2}^2 + C(\kappa, K, T) \| f_n^- \|_{0,2}^2. \tag{by (5.12)}
\]
Hence, we have
\[
\| k^+_n \|_{0,2}^2 \leq C(\kappa, K, T) \| f_n^- \|_{0,2}^2
\]
with
\[
k^+_n \triangleq n (V_n - \overline{V}_n)^+.
\]
In a similar way, we have
\[
\|k^-_n\|_{0,2}^2 \leq C(\kappa, K, T) \| f + \mathcal{L} \overline{V}_n + \mathcal{M}^k \overline{Z}_n - \varphi_n \|_{0,2}^2
\]
\[
\leq C(\kappa, K, T) \left( \| \overline{V} \|_{1,2}^2 + \| \nabla \overline{V} \|_{1,2}^2 + \| \overline{Z} \|_{0,2}^2 + \| \nabla \overline{Z} \|_{0,2}^2 + \| h \|_{0,2}^2 + \| \varphi \|_{0,2}^2 \right)
\]
with
\[
k^-_n \triangleq n (V_n - \overline{V}_n)^+.
\]
From Lemma 2.22 we have the following estimate:
\[
\| V_n \|_{2,2} + \| V_n \|_{1,2} + \| Z_n \|_{1,2} + \| k^+_n \|_{0,2}^2 + \| k^-_n \|_{0,2}^2
\]
\[
\leq C(\kappa, K, T) \left( \mathbb{E} \left[ \| \varphi \|_{1,2}^2 \right] + \| f \|_{0,2}^2 + \| k^+_n \|_{0,2}^2 + \| k^-_n \|_{0,2}^2 \right)
\]
\[
\leq C(\kappa, K, T) \left( \mathbb{E} \left[ \| \varphi \|_{1,2}^2 \right] + \| f \|_{0,2}^2 + \| \overline{V} \|_{1,2}^2 + \| \nabla \overline{V} \|_{1,2}^2 \right.
\]
\[
\left. + \| \overline{Z} \|_{0,2}^2 + \| \nabla \overline{Z} \|_{0,2}^2 + \| h \|_{0,2}^2 \right). \tag{5.13}
\]
Hence, there exists a subsequence of \(\{(V_n, Z_n, k^+_n, k^-_n)\}\), still denoted by itself, such that \(V_n, Z_n, k^+_n, k^-_n\) converges to \(V, Z, k^+,\) and \(k^-\) weakly in \(H^{2,2}, H^{1,2}, H^{0,2},\) and \(H^{0,2},\) respectively. Letting \(n\) tend to \(\infty\) in (5.13), we have (5.10).
Step 3. $V_n$ converges to $V$ strongly in $\mathbb{H}^{1,2}$. It is sufficient to show that $\{V_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $\mathbb{H}^{1,2}$. Define

$$\Delta V_{n,m} \triangleq V_n - V_m, \ \Delta V_{n,m} \triangleq \nabla V_n - \nabla V_m, \ \Delta V_{n,m} \triangleq V_n - V_m, \ \Delta Z_{n,m} \triangleq Z_n - Z_m,$$
and $\Delta k_{n,m}^{\pm} \triangleq k_n^{\pm} - k_m^{\pm}$ with $m \geq n$. Then $(\Delta V_{n,m}, \Delta Z_{n,m})$ satisfies the following BSPDE:

$$\Delta V_{n,m} = \int_t^T (\mathcal{L} \Delta V_{n,m,s} + \mathcal{M}^k \Delta Z_{n,m,s} + \Delta k_{n,m,s}^{+} - \Delta k_{n,m,s}^{-}) \, ds - \int_t^T \Delta Z_{n,m,s} \, dB_s^{k}.$$

From (5.3) in Lemma 5.1, we have

$$\Delta V_{n,m} \cong \int_0^T \int_{\mathbb{R}^d_r} \left[ \Delta V_{n,m,s} \left( \Delta k_{n,m,s}^{+} - \Delta k_{n,m,s}^{-} \right) \right] \, dx \, ds.$$ 

Moreover, we have

$$\Delta V_{n,m} \Delta k_{n,m}^{+} = n \left[ (V_n - \nabla V_n) - (V_m - \nabla V_m) + \Delta V_{n,m} \right] (V_n - V_n)^{+}$$

and

$$\Delta V_{n,m} \Delta k_{n,m}^{-} = n \left[ (V_n - \nabla V_n) - (V_m - \nabla V_m) + \Delta V_{n,m} \right] (V_n - \nabla V_n)^{+}$$

Combining (5.13) and (5.14), we derive that as $n \to \infty$,

$$\|\Delta V_{n,m}\|_{1,2}^2 + |||\Delta V_{n,m}|||_{0,2}^2 + \|\Delta Z_{n,m}\|_{0,2}^2 \leq \frac{C}{n} + C (\|\Delta V_{n,m}\|_{0,2}^2 + |||\Delta V_{n,m}|||_{0,2}^2) 
\to \|\mathbb{H}^{0,2}\|_{0,2}^2, \quad \text{since} \quad \{V_n\}_{n=1}^{\infty} \quad \text{and} \quad \{V_n\}_{n=1}^{\infty}$$

we derive that $(V_n - V_n)^{+}$ converges to $0$ in $\mathbb{H}^{0,2}$. On the other hand, $V_n - V_n$ converges to $V - V$ in $\mathbb{H}^{0,2}$. Hence, we deduce that $(V - V)^{+} = 0$, i.e., $V \geq \overline{V}$ a.e. in $\Omega \times Q$. In a similar way, we have

$$V \leq \overline{V}, \quad \text{and} \quad \text{a.e. in} \ \Omega \times Q,$$
which is the second and third inequalities in (4.4).

**Step 4.** It remains to show that \((V, Z, k^+, k^-)\) satisfies the first and forth equalities in (4.4). Let \(\xi\) be an arbitrary \(P^B\)-predictable element of \(L^2\), and \(\forall \eta \in H^{2,2}\). We have \(\eta \xi \in H^{2,2}\).

Rewrite (5.11) into the integrated form:

\[
V_{n,t} = \varphi + \int_t^T (\mathcal{L}V_{n,s} + \mathcal{M}^kZ_{n,s}^k + f_s + k_{n,s}^+ - k_{n,s}^-) \, ds - \int_t^T Z_{n,s}^k dB_s^k.
\]

Multiplying by \(\eta \xi\) both sides of the last equality, and integrating, we have

\[
\mathbb{E} \left( \int_0^T \int_{\mathbb{R}^d*} V_{n,t}(x) \eta(x) \xi_t \, dx \, dt \right) = I_1 + I_2,
\]

with

\[
I_1 \equiv \mathbb{E} \left( \int_0^T \int_{\mathbb{R}^d*} \varphi(x) \eta(x) \xi_t \, dx \, dt \right) - \mathbb{E} \left( \int_0^T \int_{\mathbb{R}^d*} \int_t^T \eta(x) \xi_t Z_{n,s}^k(x) \, dB_s^k \, dx \, dt \right),
\]

\[
I_2 \equiv \mathbb{E} \left[ \int_0^T \int_{\mathbb{R}^d*} \int_t^T \eta(x) \xi_t \left( \mathcal{L}V_{n,s} + \mathcal{M}^kZ_{n,s}^k + f_s + k_{n,s}^+ - k_{n,s}^- \right)(x) \, dx \, ds \, dt \right].
\]

Evidently, we have

\[
\lim_{n \to \infty} \mathbb{E} \left( \int_0^T \int_{\mathbb{R}^d*} V_{n,t}(x) \eta(x) \xi_t \, dx \, dt \right) = \mathbb{E} \left( \int_0^T \xi_t \int_{\mathbb{R}^d*} V_t(x) \eta(x) \, dx \, dt \right).
\]

Since \(Z_n\) converges to \(Z\) weakly in \(H^{1,2}\), from a known result (see [31] Theorem 4, page 63), we have

\[
\lim_{n \to \infty} \mathbb{E} \left( \xi_t \int_t^T \int_{\mathbb{R}^d*} \eta(x) \, dx \, dB_s^k \right) = \mathbb{E} \left( \xi_t \int_t^T \int_{\mathbb{R}^d*} \eta(x) \, dB_s^k \right)
\]

for any \(t \in [0, T]\). Moreover, for every \(t \in [0, T]\),

\[
\left| \mathbb{E} \left( \xi_t \int_t^T \int_{\mathbb{R}^d*} \eta(x) \, dx \, dB_s^k \right) \right|
\]

\[
= \int_{\mathbb{R}^d*} |\eta(x)| \left| \mathbb{E} \left( \xi_t \int_t^T Z_{n,s}^k(x) \, dB_s^k \right) \right| dx
\]

\[
\leq |\eta|_{0,2} \left\{ \int_{\mathbb{R}^d*} \mathbb{E} \left( \xi_t^2 \right) \mathbb{E} \left[ \left| \int_t^T Z_{n,s}^k(x) \, dB_s^k \right|^2 \right] dx \right\}^{\frac{1}{2}}
\]

\[
\leq |\eta|_{0,2} \left\{ \int_{\mathbb{R}^d*} \mathbb{E} \left( \xi_t^2 \right) \mathbb{E} \left[ \int_t^T |Z_{n,s}^k(x)|^2 \, ds \right] dx \right\}^{\frac{1}{2}}
\]

\[
\leq |\eta|_{0,2} \left[ \mathbb{E} \left( \xi_t^2 \right) \right]^{1/2} \|Z_n\|_{0,2} \leq C |\eta|_{0,2} \left[ \mathbb{E} \left( \xi_t^2 \right) \right]^{1/2}.
\]
Hence, using Lebesgue’s dominant convergence theorem, we have
\[
\mathbb{E} \left( \int_0^T \int_{\mathbb{R}^d} \int_t^T \eta(x) \xi_t Z_{n,s}^k(x) \, dB_s^k \, dx \, dt \right) \to \mathbb{E} \left( \int_0^T \xi_t \int_t^T \int_{\mathbb{R}^d} \eta(x) Z_{s}^k(x) \, dx \, dB_s^k \, dt \right).
\]
In a similar way, we have
\[
I_2 \to \mathbb{E} \left\{ \int_0^T \xi_t \int_t^T \int_{\mathbb{R}^d} \eta(x) \left[ \mathcal{L} V_s(x) + \mathcal{M}^k Z_s^k(x) + f_s(x) + k^+_s(x) - k^-_s(x) \right] \, dx \, ds \, dt \right\}.
\]
Hence, we show that for any \( \eta \in H^{2,2} \) and \( \mathcal{P}^R \)-predictable stochastic process \( \xi \) belonging to \( \mathcal{L}^2 \), it holds that
\[
\mathbb{E} \left[ \int_0^T \xi_t \int_{\mathbb{R}^d} V_t(x) \eta(x) \, dx \, dt \right] = \mathbb{E} \left[ \int_0^T \xi_t \left( \int_{\mathbb{R}^d} \varphi(x) \eta(x) \, dx - \int_t^T \int_{\mathbb{R}^d} \eta(x) Z_s^k(x) \, dx \, dB_s^k \right) \, dt \right] + \mathbb{E} \left[ \int_0^T \xi_t \int_t^T \int_{\mathbb{R}^d} \eta(x) \left( \mathcal{L} V_s(x) + \mathcal{M}^k Z_s^k(x) + f_s(x) + k^+_s(x) - k^-_s(x) \right) \, dx \, ds \, dt \right].
\]
Since \( \xi \) is arbitrary, we see that for any \( \eta \in H^{2,2} \), a.e. in \( \Omega \times [0,T] \), it holds that
\[
\int_{\mathbb{R}^d} \eta V_t \, dx = \int_{\mathbb{R}^d} \eta \varphi \, dx + \int_t^T \int_{\mathbb{R}^d} \eta \left( \mathcal{L} v_s + \mathcal{M}^k Z_s^k + f_s + k^+_s - k^-_s \right) \, dx \, ds - \int_t^T \int_{\mathbb{R}^d} \eta Z_s^k \, dx \, dB_s^k.
\]
From Lemma 2.5, we deduce that \( V \in H^{2,2} \cap \mathbb{S}^{1,2} \), and \( (V, Z, k^+, k^-) \) satisfies the first equality in (4.4).

Now, we prove the forth equality in (4.4). We have
\[
V \geq \underline{V}, \quad k^+ \geq 0, \quad \int_0^T (V_t - \underline{V}_t) k^+_t \, dt \geq 0.
\]
On the other hand, since \( k^+_n \) converges to \( k^+ \) weakly in \( H^{0,2} \), \( V_n - \underline{V}_n \) converges to \( V - \underline{V} \) strongly in \( H^{0,2} \), and
\[
\int_0^T (V_{n,t} - \underline{V}_{n,t}) k^+_n \, dt = n \int_0^T (V_{n,t} - \underline{V}_{n,t}) (\underline{V}_{n,t} - V_{n,t})^+ \, dt \leq 0.
\]
Therefore, we have
\[
\mathbb{E} \left[ \int_{\mathbb{R}^d} \int_0^T (V_t - \underline{V}_t) k^+_t \, dx \, dt \right] \leq 0,
\]
which implies that
\[
\int_0^T (V_t - \underline{V}_t) k^+_t \, dt = 0 \quad \text{a.e. in } \Omega \times \mathbb{R}^d.
\]
In a similar way, we have
\[
\int_0^T (\overline{V}_t - V_t) k^-_t \, dt = 0 \quad \text{a.e. in } \Omega \times \mathbb{R}^d.
\]
6 Further properties on the strong solution, and the free boundary of BSPDVI (1.2).

In this section, we use the comparison theorem for BSPDVI to derive properties of strong solutions and define the stochastic free boundary of BSPDVI (1.2) with extra conditions. For this purpose, consider the following further assumptions.

**Assumption V5(i).** The coefficient functions $a, b, c, \sigma,$ and $\mu$ are independent of the variable $x_i$ for $i \in \{1, 2, \cdots, d^*\}$. Moreover, the dimension of the state space $d^* < 4$.

**Assumption V6(i).** The functions $\underline{f}$, $\Delta V$, and $\underline{\varphi}$ are increasing (resp. decreasing) in $x_i$, where

$$f \triangleq f + \mathcal{L}V + \mathcal{M}^k Z^k - g, \quad \Delta V \triangleq V - V, \quad \varphi \triangleq \varphi - V_T, \quad i \in \{1, 2, \cdots, d^*\}.$$ 

**Assumption V7(i).** The functions $\overline{f}$, $-\Delta V$, and $\overline{\varphi}$ are increasing (resp. decreasing) in $x_i$, where

$$\overline{f} \triangleq f + \mathcal{L}V + \mathcal{M}^k Z^k - \overline{g}, \quad \overline{\varphi} \triangleq \varphi - \overline{V}_T, \quad i \in \{1, 2, \cdots, d^*\}.$$ 

We have

**Theorem 6.1.** Let Assumptions V1, V2, V3', V4, V5(i), and V6(i) be satisfied. Let $(V, Z, k^+, k^-)$ be the unique strong solution of BSPDVI (1.2).

(i) Then $V - \underline{V}$ is continuous and increasing (resp. decreasing) with respect to $x_i$ for any $x_i \in \mathbb{R}^{d-1}$, $a.e.$ in $\Omega \times [0, T]$.

(ii) Define the lower free boundary as

$$S_i(w, t, x_f) \triangleq \sup \{x_i : (V - \underline{V})(w, t, x) = 0\}$$

(resp. $\underline{S}_i(w, t, x_f) \triangleq \inf \{x_i : (V - \underline{V})(w, t, x) = 0\}$) with $x_f \triangleq (x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_{d^*}) \in \mathbb{R}^{d-1}$, and the convention that $\sup \mathcal{O} = -\infty$ and $\inf \mathcal{O} = \infty$. Then, we have

$$V > \underline{V} \quad a.e. \text{ in } \{x_i > \underline{S}_i(w, t, x_f)\} \quad \text{ and } \quad V = \underline{V} \quad a.e. \text{ in } \{x_i \leq \underline{S}_i(w, t, x_f)\}.$$  

(resp. $V > \underline{V} \quad a.e. \text{ in } \{x_i < S_i(w, t, x_f)\} \quad \text{ and } \quad V = \underline{V} \quad a.e. \text{ in } \{x_i \geq S_i(w, t, x_f)\}$.)

(iii) If Assumptions V5(j) and V6(j) with $j \neq i$ are further satisfied, the free boundary $S_i$ is monotone in $x_j$ for any $x_{k \neq f} \in \mathbb{R}^{d-2}$, $a.e.$ in $\Omega \times [0, T]$.

**Remark 6.1.** The lower free boundary is the interface between $\{V = \underline{V}\}$ and $\{\underline{V} < V < V\}$, and the upper free boundary in Theorem 6.1 is the interface between $\{V = \overline{V}\}$ and $\{\overline{V} < V < \overline{V}\}$.

**Proof.** Denote $\delta V \triangleq V - \underline{V}$ and $\delta Z \triangleq Z - \underline{Z}$. Then $(\delta V, \delta Z, k^+, k^-)$ is the strong solution of the following BSPDVI:

$$\begin{cases}
        d\delta V_t = - (\mathcal{L}\delta V_t + \mathcal{M}^k \delta Z^k_t + \underline{f}_t) dt + \delta Z^k_t dB^k_t, \quad & \text{if } 0 < \delta V_t < \Delta V_t; \\
        d\delta V_t \leq - (\mathcal{L}\delta V_t + \mathcal{M}^k \delta Z^k_t + \underline{f}_t) dt + \delta Z^k_t dB^k_t, \quad & \text{if } \delta V_t = 0; \\
        d\delta V_t \geq - (\mathcal{L}\delta V_t + \mathcal{M}^k \delta Z^k_t + \underline{f}_t) dt + \delta Z^k_t dB^k_t, \quad & \text{if } \delta V_t = \Delta V_t; \\
        \delta V_T(x) = \underline{\varphi}(x).
    \end{cases}$$
For any fixed $\varepsilon > 0$, denote
\[
\tilde{\delta}(w, t, x) \triangleq \delta V(w, t, x + \varepsilon e_i), \quad \tilde{Z}(w, t, x) \triangleq \delta Z(w, t, x + \varepsilon e_i),
\]
\[
\tilde{k}^\pm(w, t, x) \triangleq k^\pm(w, t, x + \varepsilon e_i), \quad \tilde{f}(w, t, x) \triangleq f(w, t, x + \varepsilon e_i),
\]
\[
\tilde{\Delta}(w, t, x) \triangleq \Delta V(w, t, x + \varepsilon e_i), \quad \tilde{\varphi}(w, t, x) \triangleq \varphi(w, t, x + \varepsilon e_i),
\]
where $e_i \triangleq (0, \cdots, 0, 1, 0 \cdots, 0)$ is the $i$-th standard coordinate vector. So, Assumption V5(i) implies that $(\tilde{\delta}, \tilde{Z}, \tilde{k}^+, \tilde{k}^-)$ is the strong solution of the following BSPDVI:

\[
\begin{cases}
    d\tilde{\delta}V_t = - (L\tilde{\delta}V_t + M^k \tilde{Z}_t^k + \tilde{\varphi}) dt + \tilde{Z}_t^k dB_t^k, & \text{if } 0 < \tilde{\delta}V_t < \tilde{\Delta}V_t; \\
    d\tilde{\delta}V_t \leq - (L\tilde{\delta}V_t + M^k \tilde{Z}_t^k + \tilde{\varphi}) dt + \tilde{Z}_t^k dB_t^k, & \text{if } \tilde{\delta}V_t = 0; \\
    d\tilde{\delta}V_t \geq - (L\tilde{\delta}V_t + M^k \tilde{Z}_t^k + \tilde{\varphi}) dt + \tilde{Z}_t^k dB_t^k, & \text{if } \tilde{\delta}V_t = \tilde{\Delta}V_t; \\
    \tilde{\delta}V_T(x) = \tilde{\varphi}(x).
\end{cases}
\]

Moreover, Assumption V6(i) implies that
\[
\tilde{f} \geq f, \quad \tilde{\Delta} \geq \Delta, \quad \tilde{\varphi} \geq \varphi. \quad (\text{resp. } \tilde{f} \leq f, \quad \tilde{\Delta} \leq \Delta, \quad \tilde{\varphi} \leq \varphi.)
\]

In view of Theorem 5.2, we deduce that $\tilde{\delta} \geq (\text{resp. } \leq) \delta V$ a.e. in $\Omega \times [0, T] \times \mathbb{R}^{d^*}$, which means that for any $\varepsilon > 0$, $(V - \underline{V})(w, t, x + \varepsilon e_i) \geq (\text{resp. } \leq) (V - \overline{V})(w, t, x)$ a.e. in $\Omega \times [0, T] \times \mathbb{R}^{d^*}$.

Since $V, \underline{V} \in \mathbb{H}^{2,2}$ and $d^* < 4$, then the Sobolev imbedding theorem implies that $V$ and $\underline{V}$ have continuous versions such that they are continuous with respect to $x$ a.e. in $\Omega \times [0, T]$. Hence, Conclusion (i) has been proved.

Since $V - \underline{V} \geq 0$, Conclusion (ii) is clear. In the following, we prove the last result.

Without loss of generality, we suppose $d = 2, i = 1, j = 2$ and $\tilde{f}, \tilde{\Delta}V, \text{ and } \varphi$ are increasing with respect to $x_1$ and $x_2$. Moreover, we fix $(w, t) \in \Omega \times (0, T)$. Then it is sufficient to prove that the free boundary $\underline{S}_1(w, t, x_2)$ is decreasing in $x_2$.

According to Conclusion (ii), for any fixed $x_2^0 \in \mathbb{R}$,
\[
V(w, t, x_1, x_2^0) - \underline{V}(w, t, x_1, x_2^0) > 0, \quad \forall x_1 > \underline{S}_1(w, t, x_2^0).
\]

Moreover, since $V - \underline{V}$ is increasing in $x_2$, we have
\[
V(w, t, x_1, x_2) - \underline{V}(w, t, x_1, x_2) > 0, \quad \forall x_1 > \underline{S}_1(w, t, x_2^0), \quad x_2 > x_2^0.
\]

In view of the definition of $\underline{S}_1$, we have that for any $x_2 > x_2^0$,
\[
\underline{S}_1(w, t, x_2) \triangleq \sup \{ x_1 : (V - \underline{V})(w, t, x_1, x_2) = 0 \} \leq \underline{S}_1(w, t, x_2^0).
\]

Then the proof is complete. \hfill \Box

In a similar way, we have
Theorem 6.2. Let Assumptions V1, V2, V3', V4, V5(i), and V7(i) be satisfied. Let 
\( (V, Z, k^+, k^-) \) be the unique strong solution of BSPDVI \((\ref{12})\).

(i) Then \( V = \nabla V \) is continuous and increasing (resp. decreasing) with respect to \( x_i \) a.e. in \( \Omega \times [0, T] \times \mathbb{R}^{d-1} \).

(ii) Define the upper free boundary as 
\[
\overline{S}_i(w, t, x_j) \triangleq \inf \{ x_i : (V - \nabla V)(w, t, x) = 0 \}.
\]
Then we have 
\[
V = \nabla V \quad {\text{a.e. in}} \quad \{ x_i \geq \overline{S}_i(w, t, x_j) \} \quad {\text{and}} \quad V < \nabla V \quad {\text{a.e. in}} \quad \{ x_i < \overline{S}_i(w, t, x_j) \}.
\]
(resp. \( V = \nabla V \) a.e. in \( \{ x_i \geq \overline{S}_i(w, t, x_j) \} \) and \( V < \nabla V \) a.e. in \( \{ x_i > \overline{S}_i(w, t, x_j) \} \).)

(iii) If Assumptions V5(j) and V7(j) with \( j \neq i \) are further satisfied, the free boundary \( \overline{S}_i \) is monotone in \( x_j \) for any \( x_{k,i} \in \mathbb{R}^{d-2} \), a.e. in \( \Omega \times [0, T] \).

7 The optimal stopping time problem as an extreme case of a Dynkin’s game.

In this section, we consider an optimal stopping time problem (denoted by Problem \( \mathcal{O} \) hereafter), which involves only one choice variable of stopping times. We show that Problem \( \mathcal{O} \) is a special case of Dynkin games under suitable conditions and identify the corresponding results about Problem \( \mathcal{O} \) and BSPDVI with one obstacle.

The state \( X \) is governed by SDE \((\ref{11})\). The payoff is defined by
\[
P_t(x; \tau) = \int_t^\tau f_u(X_{u}^{t,x}) \, du + \nabla \tau(X_{\tau}^{t,x}) \chi_{\{\tau < T\}} + \varphi(X_{T}^{t,x}) \chi_{\{\tau \geq T\}}, \quad \tau \in \mathcal{U}_{t,T}.
\]

The optimal stopping problem \( \mathcal{O}_{tx} \), associated to the initial data \((t, x)\), is to find a stopping time \( \tau^* \in \mathcal{U}_{t,T} \) such that
\[
\mathbb{E} \left[ P_t(x; \tau^*) \bigg| \mathcal{F}_t \right] = V_t(x) \triangleq \text{ess.sup} \mathbb{E} \left[ P_t(x; \tau) \bigg| \mathcal{F}_t \right].
\]
The random variable \( V(t, x) \) is called the value of Problem \( \mathcal{O}_{tx} \).

Consider the following two assumptions on the cost functions \( f, \nabla V \), and \( \varphi \).

Assumption O1. (Regularity) \( f \in \mathbb{H}^{0,2}, \varphi \in \mathbb{L}^{1,2} \) and \( \nabla V \) is in the form of
\[
dV_t = -g_t \, dt + Z_t^{k} \, dB_t^{k},
\]
where \( \nabla V \in \mathbb{H}^{2,2}, Z \in \mathbb{H}^{1,2}, \) and \( g \in \mathbb{H}^{0,2} \).

Assumption O2. (Compatibility) \( \nabla V_T \leq \varphi \).

The HJB equation for Problem \( \mathcal{O}_{tx} \) is the following BSPDVI with one obstacle:
\[
\begin{cases}
  dV_t = -(\mathcal{L}V_t + \mathcal{M}^{k}Z_t^k + f_t) \, dt + Z_t^k \, dB_t^k, & \text{if } V_t > \nabla V_t; \\
  dV_t \leq -(\mathcal{L}V_t + \mathcal{M}^{k}Z_t^k + f_t) \, dt + Z_t^k \, dB_t^k, & \text{if } V_t = \nabla V_t; \\
  V_T(x) = \varphi(x),
\end{cases}
\]
where the operators \( \mathcal{L} \) and \( \mathcal{M} \) are defined by \((\ref{13})\).
Definition 7.1. A triplet \((V, Z, k^+) \in \mathbb{H}^{2,2} \times \mathbb{H}^{1,2} \times \mathbb{H}^{0,2}\) is called a strong solution of BSPDVI (7.1) if it satisfies the following:

\[
\begin{cases}
V_t = \varphi + \int_t^T (L V_s + M^k Z_s^k + f_s + k_s^+) \, ds - \int_t^T Z_s^k \, dB_s^k \text{ a.e. in } \mathbb{R}^{d^*} \text{ for all } t \text{ a.s. in } \Omega; \\
V \geq \underline{V}, \quad k^+ \geq 0 \text{ a.e. in } \Omega \times \overline{Q}; \\
\int_0^T (V_t - \underline{V}_t) \, k_t^+ \, dt = 0 \text{ a.e. in } \Omega \times \mathbb{R}^{d^*}.
\end{cases}
\]

Identical to the proof of Theorem 5.2, we have the following comparison theorem.

Theorem 7.1. Let Assumptions V1 and V2 be satisfied. Let \((V_i, Z_i, k_i^+)\) be the strong solution of BSPDVI (7.1) associated with \((f_i, \varphi_i, \underline{V}_i)\) for \(i = 1, 2\). If \(f_1 \geq f_2, \varphi_1 \geq \varphi_2, \) and \(\underline{V}_1 \geq \underline{V}_2\), then \(V_1 \geq V_2\) a.e. in \(\Omega \times Q\).

The following lemma gives the relationship between Problems \(\mathcal{D}_{tx}\) and \(\mathcal{O}_{tx}\), and between BSPDVIs (1.2) and (7.1).

Lemma 7.2. Let Assumptions D1 and D2 (resp. V1 and V2), O1 and O2 be satisfied. Then there exists a stochastic fields \(\underline{V}\) such that Assumptions V3' and V4 are satisfied. Moreover, Problems \(\mathcal{O}_{tx}\) and \(\mathcal{D}_{tx}\) (resp. BSPDVIs (7.1) and (1.2)) are equivalent. And we have the following estimate

\[
\|\underline{V}\|_{2,2} + \|Z\|_{1,2} + \|g\|_{0,2} \leq C(\kappa, K, T) \left( \mathbb{E} [ \|\underline{V}\|_{1,2} + \|f\|_{0,2} + \|\underline{V}\|_{2,2} + \|Z\|_{1,2} + \|g\|_{0,2} + 1 ] \right). \tag{7.2}
\]

Proof. Let Assumptions D1, D2, O1, and O2 be satisfied. Let \((\tilde{V}, \tilde{Z})\) be the strong solution of the following BSPDE:

\[
\begin{cases}
d\tilde{V}_t = - (L \tilde{V}_t + M^k \tilde{Z}_t^k + \tilde{f}_t) \, dt + \tilde{Z}_t^k \, dB_t^k; \\
\tilde{V}_T = \varphi^+,
\end{cases}
\tag{7.3}
\]

where \(L\) and \(M\) are defined by (2.2) and

\[
\tilde{f} \triangleq \max \{ f, 0, g - L \underline{V} - M^k \underline{Z} \} \in \mathbb{H}^{0,2}.
\]

According to Theorem 2.2 in [8], BSPDE (7.3) has a strong solution \((\tilde{V}, \tilde{Z}) \in \mathbb{H}^{2,2} \times \mathbb{H}^{1,2}\). Moreover, the comparison theorem for linear BSPDE in [8] implies \(\tilde{V} \geq 0\).

Since \(\underline{V}\) satisfies

\[
\begin{cases}
d\underline{V}_t = - [L \underline{V}_t + M^k \underline{Z}_t^k + (g_t - L \underline{V}_t - M^k \underline{Z}_t^k)] \, dt + \underline{Z}_t^k \, dB_t^k; \\
\underline{V}_T \leq \varphi \leq \varphi^+,
\end{cases}
\]

then the comparison theorem for linear BSPDE in [8] implies that \(\tilde{V} \geq \underline{V}\).

Define

\[
\underline{V} \triangleq \tilde{V} + (1 + |x|^{d^*+1})^{-1}.
\]

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Then $\overline{V} \in \mathbb{H}^{2,2}$, $\overline{V} > V^+$, $\overline{V}_T > \varphi \geq \overline{V}_T$, and

$$d\overline{V}_t = -\overline{g}_t \, dt + \overline{Z}_t \, dB^k_t$$

with $\overline{g} = L\overline{V}_t + M^k \overline{Z}^k_t + \overline{f}_t \in \mathbb{H}^{0,2}$ and $\overline{Z} = \overline{Z} \in \mathbb{H}^{1,2}$. Hence, $\overline{V}$, $\overline{V}$, and $\varphi$ satisfy Assumptions V3' and V4. The estimate (7.2) follows from Lemma 2.2.

In the following, we prove that Problems $\mathcal{G}_{tx}$ and $\mathcal{G}_{tx}$ are equivalent. We firstly claim

$$\mathbb{E} \left[ R_t(x; \tau_1, \tau_2) \mid \mathcal{F}_t \right] \geq \mathbb{E} \left[ P_t(x; \tau_1) \mid \mathcal{F}_t \right], \quad \forall \, \tau_1, \tau_2 \in \mathcal{U}_{t,T}.$$  \hspace{1cm} (7.4)

In fact, on the event of $\{\tau_1 < \tau_2\}$, it is clear that

$$P_t(x; \tau_1) = R_t(x; \tau_1, \tau_2).$$

On the event of $\{\tau_1 \geq \tau_2\}$, applying Theorem 3.1 and repeating the method in the proof of Theorem 4.2 we deduce that

$$R_t(x; \tau_1, \tau_2) = \int_t^{\tau_2} f_u(X^{t,x}_u) \, du + \varphi(X^{t,x}_T) \chi_{\{\tau_2 \geq T\}} + \overline{V}_{\tau_2}(X^{t,x}_{\tau_2}) \chi_{\{\tau_2 < T\}}$$

$$\geq \int_t^{\tau_2} f_u(X^{t,x}_u) \, du + \varphi(X^{t,x}_T) \chi_{\{\tau_2 \geq T\}} + \overline{V}_{\tau_2}(X^{t,x}_{\tau_2}) \chi_{\{\tau_2 < T\}}$$

$$= \int_t^{\tau_2} f_u(X^{t,x}_u) \, du + \varphi(X^{t,x}_T) \chi_{\{\tau_2 \geq T\}} + \overline{V}_{\tau_1}(X^{t,x}_{\tau_1}) \chi_{\{\tau_2 < T\}} + \int_{\tau_2}^{\tau_1} \tilde{f}_u(X^{t,x}_u) \, du$$

$$- \int_{\tau_2}^{\tau_1} (\tilde{Z}^k_u + M^k \tilde{V}_u)(X^{t,x}_u) \, dB^k_u - \int_{\tau_2}^{\tau_1} (N^l \tilde{V}_u)(X^{t,x}_u) \, dW^l_u$$

$$\geq P_t(x; \tau_1) - \int_{\tau_2}^{\tau_1} (\tilde{Z}^k_u + M^k \tilde{V}_u)(X^{t,x}_u) \, dB^k_u - \int_{\tau_2}^{\tau_1} (N^l \tilde{V}_u)(X^{t,x}_u) \, dW^l_u.$$

Hence, we obtain

$$R_t(x; \tau_1, \tau_2) \geq P_t(x; \tau_1) - \int_{\tau_2}^{\tau_1} (\tilde{Z}^k_u + M^k \tilde{V}_u)(X^{t,x}_u) \, dB^k_u - \int_{\tau_2}^{\tau_1} (N^l \tilde{V}_u)(X^{t,x}_u) \, dW^l_u.$$

Taking the condition expectation in the above inequality, we have (7.4).

If Problem $\mathcal{G}_{tx}$ has a saddle point $(\tau^*_1, \tau^*_2)$, then we have that

$$\mathbb{E} \left[ P_t(x; \tau^*_1) \mid \mathcal{F}_t \right] = \mathbb{E} \left[ R_t(x; \tau^*_1, T) \mid \mathcal{F}_t \right] \geq \mathbb{E} \left[ R_t(x; \tau^*_1, \tau^*_2) \mid \mathcal{F}_t \right]$$

$$\geq \mathbb{E} \left[ R_t(x; \tau^*_1, \tau^*_2) \mid \mathcal{F}_t \right] \geq \mathbb{E} \left[ P_t(x; \tau_1) \mid \mathcal{F}_t \right],$$

where $\tau_1$ is an arbitrary stopping time in $\mathcal{U}_{t,T}$ and we have used (7.4) in the last inequality. Hence, Problem $\mathcal{G}_{tx}$ has an optimal stopping time $\tau^*_1 \in \mathcal{U}_{t,T}$.

Suppose that Problem $\mathcal{G}_{tx}$ has an optimal stopping time $\tau^*_1 \in \mathcal{U}_{t,T}$. Then we choose $\tau^*_2 = T$. We see that for any $\tau_1 \in \mathcal{U}_{t,T}$,

$$R_t(x; \tau_1, \tau^*_2) = P_t(x; \tau_1)$$

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Moreover, we have

\[ E \left[ R_t(x; \tau_1^*, \tau_2^*) \mid F_t \right] = E \left[ P_t(x; \tau_1^*) \mid F_t \right] \geq E \left[ P_t(x; \tau_1) \mid F_t \right] = E \left[ R_t(x; \tau_1, \tau_2^*) \mid F_t \right]. \]

On the other hand, according to (7.4), we have that

\[ \hat{V} \left( \tau_1^*, \tau_2^* \right) \geq \hat{V} \left( \tau_1^* \right) \geq \hat{V} \left( \tau_1 \right) = \hat{V} \left( \tau_1, \tau_2^* \right) \]

Hence, \((\tau_1^*, \tau_2^*)\) is a saddle point of Problem \(D_{tx}^\ast\). Until now, we have proved that Problems \(D_{tx}^\ast\) and \(D_{tx}\) are equivalent.

Let Assumptions V1, V2, O1, and O2 are satisfied. Denote by \((\hat{V}, \hat{Z})\) the solution of the following BSPDE:

\[
\begin{align*}
\left\{ 
\begin{array}{ll}
d\hat{V}_t & = -(\mathcal{L}\hat{V}_t + \mathcal{M}^k \hat{Z}^k_t + \hat{f}_t) dt + \hat{Z}^k_t dB_t^k; \\
\hat{V}_T(x) & = \varphi^+,
\end{array}
\right.
\end{align*}
\]

(7.5)

where \(\mathcal{L}\) and \(\mathcal{M}\) are defined in (1.3) and \(\hat{f}\) is defined as

\[ \hat{f} = \max\{ f, 0, g - \mathcal{L} \hat{V} - \mathcal{M}^k \hat{Z}^k \}. \]

Moreover, we define

\[ \hat{\nabla} = \hat{V} + (1 + |x|^{d+1})^{-1}. \]

Repeating the above argument, we derive that BSPDE (7.5) has a strong solution \((\hat{V}, \hat{Z}) \in \mathbb{H}^{2,2} \times \mathbb{H}^{1,2}\). Moreover, we have \(\hat{V} \geq \hat{V}^+, \hat{V} > \hat{V}^-\) and \(\hat{\nabla}, \mathcal{M}, \varphi\) satisfy Assumptions V3’ and V4. The estimate (7.2) follows from Lemma 2.2.

So, BSPDVI (1.2) has a unique strong solution \((V, Z, k^+, k^-)\) by Theorem 5.3.

On the other hand, since \(V \leq \hat{V} < \hat{V}^-\) and \((\hat{V}, \hat{Z})\) is the strong solution of BSPDE (7.5), then \((\hat{V}, \hat{Z}, 0, 0)\) is the strong solution of the following BSPDVI:

\[
\begin{align*}
\left\{ 
\begin{array}{ll}
d\tilde{V}_t & = -(\mathcal{L}\tilde{V}_t + \mathcal{M}^k \tilde{Z}^k_t + \tilde{f}_t) dt + \tilde{Z}^k_t dB_t^k, & \text{if } \hat{V}_t < \tilde{V}_t < \hat{V}_t; \\
d\tilde{V}_t & \leq -(\mathcal{L}\tilde{V}_t + \mathcal{M}^k \tilde{Z}^k_t + \tilde{f}_t) dt + \tilde{Z}^k_t dB_t^k, & \text{if } \tilde{V}_t = \hat{V}_t; \\
d\tilde{V}_t & \geq -(\mathcal{L}\tilde{V}_t + \mathcal{M}^k \tilde{Z}^k_t + \tilde{f}_t) dt + \tilde{Z}^k_t dB_t^k, & \text{if } \tilde{V}_t = \hat{\nabla}_t; \\
\tilde{V}_T(x) & = \varphi^+(x).
\end{array}
\right.
\end{align*}
\]

In view of Theorem 5.2, \(\tilde{V} \geq V\) and \(\hat{\nabla} > V\). So, we deduce that \(k^- = 0\) a.e. in \(\Omega \times Q\) and \((V, Z, k^+)\) is the strong solution of BSPDVI (1.1).

On the other hand, in view of Theorem 7.1, the strong solution of BSPDVI (1.1) is unique. So, the unique strong solutions of BSPDVI (1.1) and (1.2) coincide. \(\square\)

Recalling Lemma 7.2 and Remark 4.1 in Section 4, and Theorem 5.3 in Sections 5, we have
Theorem 7.3. Let Assumptions V1, V2, O1, and O2 be satisfied. Then BSPDVI (7.1) has a unique strong solution \((V, Z, k^+)\) such that
\[
\|V\|_{2,2} + \|V\|_{1,2} + \|Z\|_{1,2} + \|k^+\|_{0,2} \\
\leq C(\kappa, K, T) \left( \mathbb{E} [ |\varphi|_{1,2} ] + \|f\|_{0,2} + \|V\|_{2,2} + \|Z\|_{1,2} + \|g\|_{0,2} \right).
\]
Moreover, the strong solution of BSPDVI (7.1) coincides with the value of Problem \(\mathcal{O}\) if (4.3) holds.

Identically as in the proof of Theorem 6.1, we have

Theorem 7.4. Let Assumptions V1, V2, V5(i), O1, and O2 be satisfied, and the functions \(f\) and \(\varphi\) be increasing (resp. decreasing) in \(x_i\), with \(f\) and \(\varphi\) be defined in Assumption V6(i). Then assertions (i) and (ii) in Theorem 6.1 hold.

Moreover, if Assumption V5(j) with \(i \neq j\) is satisfied, and \(f\) and \(\varphi\) are monotone in \(x_j\), then the free boundary \(S_i\) is monotone in \(x_j\) for any \(x_{F,j} \in \mathbb{R}^{d-2}\), a.e. in \(\Omega \times [0, T]\).

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