CLASSIFICATION OF TOPOLOGICAL INVARIANTS RELATED TO CORNER STATES

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ABSTRACT. We discuss some bulk-surfaces gapped Hamiltonians on a lattice with corners, and propose a periodic table for topological invariants related to corner states aimed at studies of higher-order topological insulators. Our table is based on four things: (1) the definition of topological invariants, (2) a proof of their relation with corner states (3) computations of $K$-groups and (4) a construction of explicit examples.

CONTENTS

1. Introduction 1
2. Preliminaries 4
3. $KO$-Groups of $C^*$-algebras Associated with Half-Plane and Quarter-Plane Toeplitz Operators 7
4. Toeplitz Operators Associated with Subsemigroup $(\mathbb{Z}_{\geq 0})^n$ of $\mathbb{Z}^n$ 16
5. Topological invariants and corner states in Altland–Zirnbauer classification 19
Appendix A. $\mathbb{Z}_2$-Spaces of Self-Adjoint/Skew-Adjoint Fredholm Operators and Boersema–Loring’s $K$-theory 35
References 44

1. Introduction

Recent developments in condensed matter physics have greatly generalized the bulk-boundary correspondence for topological insulators to include corner states. Topological insulators have a gapped bulk, which incorporates some topology that do not change unless the spectral gap of the bulk Hamiltonian closes under deformations. Examples include the TKNN number for quantum Hall systems [65] and the Kane-Mele $\mathbb{Z}_2$ index for quantum spin Hall systems [36]. It is known that, corresponding to these bulk invariants, gapless edge states appear, which is called the bulk-boundary correspondence [51]. After Schnyder–Ryu–Furusaki–Ludwig’s classification of topological insulators [60] for ten Altland–Zirnbauer classes [2], Kitaev
noted the role of $K$-theory and Bott periodicity in the classification problem and obtained the famous periodic table \cite{42}. Recently, some (at least bulk) gapped systems possessing in-gap or gapless states localized around a higher codimensional part of the boundary (corners or hinges) are studied \cite{30,12,41}, which are called higher-order topological insulators (HOTIs) \cite{59}. For example, for second-order topological insulators, not only is the bulk gapped but also the codimension-one boundaries (edges, surfaces), and an in-gap or a gapless state appears around codimension-two corners or hinges. In this framework, conventional topological insulators are regarded as first-order topological insulators. HOTIs are now actively studied and the classification of HOTIs has also been proposed \cite{26,40,52}. Generalizing the bulk-boundary correspondence, relations between some gapped topology and corner states are much discussed \cite{66,5,63}.

Initiated by Bellissard, $K$-theory and index theory are known to provide a powerful tool to study topological insulators. Bellissard–van Elst–Schulz-Baldes studied quantum Hall effects by means of noncommutative geometry \cite{10,11}, and Kellendonk–Richter–Schulz-Baldes went on to prove the bulk-boundary correspondence by using index theory for Toeplitz operators \cite{39}. The study of topological insulators, especially regarding its classification and the bulk-boundary correspondence for each of the ten Altland–Zirnbauer classes by using $K$-theory and index theory has been much developed \cite{39,24,64,16,29,48,57,17,38,44,1}. In \cite{32}, three-dimensional (3-D) class A bulk periodic systems are studied on one piece of a lattice cut by two specific hyperplanes, which is a model for systems with corners. Based on the index theory for quarter-plane Toeplitz operators \cite{62,23,54}, a topological invariant is defined assuming the spectral gap both on the bulk Hamiltonian and two half-space compressions of it. This gapped topological invariant is topological in the sense that it does not change under continuous deformation of the bulk Hamiltonians unless the spectral gap of one of the two surfaces closes. It is proved that, corresponding to this topology gapless corner states appear. A construction of nontrivial examples by using two first-order topological insulators (of 2-D class A and 1-D class AIII) is also proposed. Class AIII codimension-two systems are also studied through this method in \cite{33} and, as an application to HOTIs, the appearance of topological corner states in Benalcazar–Bernevig–Hughes’ 2-D model \cite{12} is explained based on the chiral symmetry. The construction of examples in \cite{33} leads to a proposal of second-order semimetallic phase protected by the chiral symmetry \cite{51}.

The purpose of this paper is to expand the results in \cite{32} to all Altland–Zirnbauer classes and systems with corners of arbitrary codimension. Since class A and class AIII systems (with codimension-two corners) were already discussed in \cite{32,33} by using complex $K$-theory, we focus on the remaining eight cases, for which we use real $K$-theory. For our expansion, a basic scheme has already been well developed in the above previous studies, which we mainly follow: some gapped Hamiltonian defines an element of a $KO$-group of a real $C^*$-algebra, and its relation with corner states are clarified by using index theory \cite{39,24,16,29,44,64,17,38}. Although many techniques have already been developed in studies of topological insulators, in our higher-codimensional cases, we still lack some basic results at the level of $K$-theory and index theory; hence, the first half of this paper is devoted to these $K$-theoretic preliminaries, that is, the computation of $KO$-groups for real $C^*$-algebras associated with the quarter-plane Toeplitz extension and the computation
of boundary maps for the 24-term exact sequence of $KO$-theory associated with it, which are carried out in Sect. 3. Since the quarter-plane Toeplitz extension \[54\] is a key tool in our study of codimension-two corners, such a variant for Toeplitz operators associated with higher-codimensional corners should be clarified, which are carried out in Sect. 4. These variants of Toeplitz operators were discussed in \[23, 22\], and the contents in Sect. 4 will be well-known to experts. Since the author could not find an appropriate reference, especially concerning Theorem 4.1 which will play a key role in Sect. 5, the results are included for completeness. Note that the idea there to use tensor products of the ordinary Toeplitz extension for the study of these variants is based on the work of Douglas–Howe \[23\], where these higher-codimensional generalizations are briefly mentioned. The study of some gapped phases for systems with corners in Altland–Zirnbauer’s classification is carried out in Sect. 5. In the framework of the one-particle approximation, we consider $n$-D systems with a codimension $k$ corner and take compressions of the bulk Hamiltonian onto infinite lattices with codimension $k - 1$ corners \[1\] whose intersection makes the codimension $k$ corner. We assume that they are gapped. Note that, under this assumption, bulk, surfaces and corners up to codimension $k - 1$ which constitute the codimension $k$ corner are also gapped. For such a system, we define two topological invariants as elements of some $KO$-groups: one is defined for these gapped Hamiltonians while the other one is related to in-gap or gapless codimension $k$ corner states. We then show a relation between these two which states that topologically protected corner states appear reflecting some gapped topology of the system. We first study codimension-two cases (Sect. 5.1 to Sect. 5.4) and then discuss higher-codimensional cases (Sect. 5.5). This distinction is made because many detailed results have been obtained for codimension-two cases by virtue of previous studies of quarter-plane Toeplitz operators \[54, 55\] (the shape of the corner we discuss is more general than in higher-codimensional cases, and a relation between convex and concave corners is also obtained in \[33\]). Based on these results, we propose a classification table for topological invariants related to corner states (Table 1). Note that the codimension-one case of Table 1 is Kitaev’s table \[12\] and Table 1 is also periodic by the Bott periodicity. In order further to clarify a relation between our invariants and corner states, in Sect. 5.6 we introduce $\mathbb{Z}$ or $\mathbb{Z}_2$-valued numerical corner invariants when the dimension of the corner is zero or one. They are defined by (roughly speaking) counting the number of corner states. A construction of examples is discussed in Sect. 5.7. As in \[32\], this is given by using pairs of Hamiltonians of two lower-order topological insulators. In the real classes, there are 64 pairs of them and the results are collected in Table 12. By using this method, we can construct nontrivial examples of each entry of Table 1, starting from Hamiltonians of first-order topological insulators. The corner invariant for the constructed Hamiltonian is expressed by corner (or edge) invariants of constituent two Hamiltonians. This is given by using an exterior product of some $KO$-groups in general, though, as in \[32, 33\], the formula for numerical invariants introduced in Sect. 5.6 is also included. For computations of $KO$-groups and classification of such gapped systems, we employ Boersema–Loring’s unitary picture for $KO$-theory \[14\] whose definitions are collected in Sect. 2. Basic results for some Toeplitz operators are also included there. In Appendix A we revisit Atiyah–Singer’s study

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1In standard terminologies, they will be called edges, surfaces, hinges or edge of edges depending on $n$ and $k$. In this paper, we may simply call them corners but state its codimensions.
Table 1. Classification of (strong) topological invariants related to corner state in Altland–Zirnbauer (AZ) classification. In this table, \( n \) is the dimension of the bulk, and \( k \) is the codimension of the corner.

| Symmetry | \( n - k \) mod 8 |
|----------|-----------------|
| AZ \( \Theta \) | 0 Z 0 Z 0 Z 0 Z |
| AZ \( \Xi \) | 0 Z 0 Z 0 Z 0 Z |
| AZ \( \Pi \) | 0 Z 0 Z 0 Z 0 Z |
| A | 0 0 0 0 0 0 0 0 0 0 0 0 |
| AIII | 0 0 1 0 0 0 0 0 0 0 0 0 |
| AI | 1 0 0 0 0 0 0 0 0 0 0 0 |
| BDI | 1 1 1 0 0 0 0 0 0 0 0 0 |
| D | 0 1 0 0 0 0 0 0 0 0 0 0 |
| DIII | 0 0 1 0 0 0 0 0 0 0 0 0 |
| AII | 0 0 1 0 0 0 0 0 0 0 0 0 |
| CII | 1 1 0 0 0 0 0 0 0 0 0 0 |
| C | 0 1 0 0 0 0 0 0 0 0 0 0 |
| CI | 0 0 1 0 0 0 0 0 0 0 0 0 |
| AP | 0 0 0 0 0 0 0 0 0 0 0 0 |

of spaces of skew-adjoint Fredholm operators \([9]\) and collect necessary results from the viewpoint of Boersema–Loring’s \( K \)-theory. Definitions of some \( \mathbb{Z}_2 \)-spaces, maps between them, expression of boundary maps of 24-term exact sequences used in this paper are collected there.

Finally, let us point out a relation with our results and the current rapidly developing studies on HOTIs. In \([26]\), the HOTIs are divided into two classes: intrinsic HOTIs, which basically originate from the bulk topology protected by a point group symmetry, and others extrinsic HOTIs. Our study will be for extrinsic HOTIs since no point group symmetry is assumed and our classification table (Table 1) is consistent with that of Table 1 in \([26]\).

2. Preliminaries

In this section, we collect the necessary results and notations.

2.1. Boersema–Loring’s \( KO \)-Groups via Unitary Elements. In this subsection, we collect Boersema–Loring’s definition of \( KO \)-groups by using unitaries satisfying some symmetries \([14]\). The basics of real \( C^* \)-algebras and \( KO \)-theory can be found in \([28, 61]\), for example.

A \( C^{*\tau} \)-algebra is a pair \((A, \tau)\) consisting of (complex) \( C^* \)-algebra \( A \) and an anti-automorphism \( \tau \) of \( A \) satisfying \( \tau^2 = 1 \). We call \( \tau \) the transposition and write \( a^\tau \) for \( \tau(a) \). There is a category equivalence between the category of \( C^{*\tau} \)-algebras and the category of real \( C^* \)-algebras: for a \( C^{*\tau} \)-algebra \((A, \tau)\), the corresponding real \( C^* \)-algebra is \( A^\tau = \{ a \in A \mid a^\tau = a^\ast \} \), and its inverse is given by the complexification. A real structure on a (complex) \( C^* \)-algebra \( A \) is an antilinear \(*\)-automorphism \( \tau \) satisfying \( \tau^2 = 1 \). For a real structure \( \tau \), there is an associated transposition \( \tau \) given by \( \tau(a) = \tau(a^\ast) \), which gives a one-to-one correspondence between transpositions

\(^2\)i.e., a complex linear automorphism of \( A \) that preserves \(*\) and satisfies \( \tau(ab) = \tau(b)\tau(a) \).
and real structures on the $C^*$-algebra. We extend the transposition $\tau$ on $A$ to the transposition (for which we simply write $\tau$) on the matrix algebra $M_n(A)$ by $(a_{ij})^{\tau} = (a_{ji})$ where $a_{ij} \in A$ and $1 \leq i, j \leq n$. This induces a transposition $\tau_K$ on $K \otimes A$ where $K = K(V)$ is the $C^*$-algebra of compact operators on a separable complex Hilbert space $V$. Let $\sharp$ be a transposition on $M_2(A)$ defined by

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}^{\sharp \tau} = \begin{pmatrix} a_{12} & -a_{11} \\ -a_{21} & a_{22} \end{pmatrix}.$$

If we identify the quaternions $\mathbb{H}$ with $\mathbb{C}^2$ by $x + yj \mapsto (x, y)$, the left multiplication by $j$ corresponds to $j(x, y) = (-y, x)$. Then, we have $\sharp \otimes \text{id}$ where $\sharp$ denotes the operation of taking conjugation of matrices and the $C^{*,\tau}$-algebra $(M_2(\mathbb{C}), \sharp \otimes \text{id})$ corresponds to the real $C^*$-algebra $\mathbb{H}$ of quaternions. We extend this transposition to $M_{2n}(A)$ by $(b_{ij})^{\sharp \otimes \tau} = (b_{ji}^{\sharp \otimes \tau})$ where $1 \leq i, j \leq n$ and $b_{ij} \in M_2(A)$. On $M_{2n}(A)$, we also consider a transposition $\sharp \otimes \tau$ defined by

$$\begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}^{\sharp \otimes \tau} = \begin{pmatrix} c_{12} & -c_{11} \\ -c_{21} & c_{22} \end{pmatrix},$$

where $c_{ij} \in M_n(A)$. For an $m \times m$ matrix $X$, we write $X_n$ for the $mn \times mn$ block diagonal matrix $\text{diag}(X, \ldots, X)$. For example, we write $1_n$ for the $n \times n$ diagonal matrix $\text{diag}(1, \ldots, 1)$.

**Definition 2.1** (Boersema–Loring [14]). Let $(A, \tau)$ be a unital $C^{*,\tau}$-algebra. For $i = -1, 0, \ldots, 6$, let $n_i$ be a positive integer, $R_i$ be a relation and $I^{(i)}$ be a matrix, as indicated in Table 2. Let $U_k^{(i)}(A, \tau)$ be the set of all unitaries in $M_{n_k}(A)$ satisfying the relation $R_i$. On the set $U_{\infty}^{(i)}(A, \tau) = \bigcup_{k=1}^{\infty} U_k^{(i)}(A, \tau)$, we consider the equivalence relation $\sim_i$ generated by homotopy and stabilization given by $I^{(i)}$. We define $KO_i(A, \tau) = U_{\infty}^{(i)}(A, \tau)/\sim_i$ which is a group by the binary operation given by $[u] + [v] = [\text{diag}(u, v)]$.

For a nonunital $C^{*,\tau}$-algebra $(A, \tau)$, the $i$-th $KO$-group $KO_i(A, \tau)$ is defined as the kernel of $\lambda_\tau: KO_i(\mathbb{A}, \tau) \to KO_i(\mathbb{C}, \text{id})$, where $\mathbb{A}$ is the unitization of $A$ and $\lambda: A \to \mathbb{C}$ is the natural projection. In [13], they also describe the boundary maps of the 24-term exact sequence for $KO$-theory associated with a short exact sequence of $C^{*,\tau}$-algebras. In Appendix 3 of this paper, we discuss an alternative description for some of them through exponentials.

### 2.2. Toeplitz Operators

In this subsection, we collect the definitions and basic results for some Toeplitz operators used in this paper [22, 54].

Let $T$ be the unit circle in the complex plane $\mathbb{C}$, and let $c$ be the complex conjugation on $\mathbb{C}$, that is, $c(z) = \bar{z}$. Let $n$ be a positive integer. On the $n$-dimensional torus $T^n$, we consider an involution $\zeta$ defined as the $n$-fold product of $c$. This induces a transposition $\tau_{\zeta}$ on $C(T^n)$ by $(\tau_{\zeta} f)(t) = f(\zeta(t))$. Let $\mathbb{Z}_{\geq 0}$ be the set of non-negative integers and $P_n$ be the orthogonal projection of $l^2(\mathbb{Z}^n)$ onto $l^2(\{n \geq 0\} \cap \mathbb{Z})$. For a continuous function $f: T^n \to \mathbb{C}$, let $M_f$ be the multiplication operator on $l^2(\mathbb{Z}^n)$ generated by $f$. We consider the operator $P_n M_f P_n$ on $l^2(\{n \geq 0\} \cap \mathbb{Z})$, which is

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3Boersema–Loring called $\tau$ the real structure in [14]. In this paper, we distinguish these two since the antilinear structure naturally appears in our application. We call $\tau$ the transposition following [14).

4For notations of the transpositions introduced here, we follow [14].
TABLE 2. Boersema–Loring’s unitary picture for KO-theory [4]

| i  | KO-group   | \( n_i \) | \( \mathcal{K}_i \) | \( I^{(s)} \) |
|----|------------|-----------|-----------------|-------------|
| -1 | \( KO_{-1}(\mathcal{A}, \tau) \) | 1         | \( u^\tau = u \) | 1           |
| 0  | \( KO_0(\mathcal{A}, \tau) \)   | 2         | \( u = u^*, u^\tau = u^* \) | diag(1, -1) |
| 1  | \( KO_1(\mathcal{A}, \tau) \)   | 1         | \( u^\tau = u^* \) | 1           |
| 2  | \( KO_2(\mathcal{A}, \tau) \)   | 2         | \( u = u^*, u^\tau = -u \) | \( \begin{pmatrix} 0 & i \cdot 1 \\ -i \cdot 1 & 0 \end{pmatrix} \) |
| 3  | \( KO_3(\mathcal{A}, \tau) \)   | 2         | \( u^\oplus \tau = u \) | 1_2         |
| 4  | \( KO_4(\mathcal{A}, \tau) \)   | 4         | \( u = u^*, u^\oplus \tau = u^* \) | diag(1_2, -1_2) |
| 5  | \( KO_5(\mathcal{A}, \tau) \)   | 2         | \( u^\oplus \tau = u^* \) | 1_2         |
| 6  | \( KO_6(\mathcal{A}, \tau) \)   | 2         | \( u = u^*, u^\oplus \tau = -u \) | \( \begin{pmatrix} 0 & i \cdot 1 \\ -i \cdot 1 & 0 \end{pmatrix} \) |

the Toeplitz operator associated with the subsemigroup \((\mathbb{Z}_{\geq 0})^n\) of \(\mathbb{Z}^n\) of symbol \(f\). We write \(T^n\) for the \(C^*\)-subalgebra of \(B(l^2((\mathbb{Z}_{\geq 0})^n))\) generated by these Toeplitz operators. The algebra \(T^1\) is the ordinary Toeplitz algebra and we simply write \(T\). Note that the algebra \(T^n\) is isomorphic to the \(n\)-fold tensor product of \(T\). The complex conjugation \(c\) on \(\mathbb{C}\) induces an antiunitary operator \(c\) of order two on the Hilbert space \(l^2(\mathbb{Z}^n)\) by the pointwise operation, for which we also write \(c\). This induces a real structure \(c\) on \(B(l^2((\mathbb{Z}_{\geq 0})^n))\) by \(c(a) = \text{Ad}_c(a) = cac^*\). We write \(T^\tau\) for the transposition on \(T^n\) given by its restriction onto \(T^n\).

We next focus on the case of \(n = 2\). We consider the Hilbert space \(l^2(\mathbb{Z}^2)\) and take its orthonormal basis \(\{\delta_{m,n} \mid (m, n) \in \mathbb{Z}^2\}\), where \(\delta_{m,n}\) is the characteristic function of the point \((m, n)\) on \(\mathbb{Z}^2\). When \(f \in C(\mathbb{T}^2)\) is given by \(f(z_1, z_2) = z_1^m z_2^n\), we write \(M_{m,n}\) for the multiplication operator \(M_f\). Let \(\alpha < \beta\) be real numbers, and let \(\mathcal{H}^\alpha\), \(\mathcal{H}^\beta\), \(\mathcal{H}^{\alpha,\beta}\) and \(\mathcal{H}_{\alpha,\beta}\) be closed subspaces of \(l^2(\mathbb{Z}^2)\) spanned by \(\{\delta_{m,n} \mid -\alpha m + n \geq 0\}\), \(\{\delta_{m,n} \mid -\beta m + n \leq 0\}\), \(\{\delta_{m,n} \mid -\alpha m + n \geq 0\}\), \(\{\delta_{m,n} \mid -\beta m + n \leq 0\}\), and \(\{\delta_{m,n} \mid -\alpha \beta \leq 0\}\), respectively. In the following, we may take \(\alpha = -\infty\) or \(\beta = \infty\), but not both. Let \(P^\alpha, P^\beta, \tilde{P}^{\alpha,\beta}\) and \(\tilde{P}_{\alpha,\beta}\) be the orthogonal projection of \(l^2(\mathbb{Z}^2)\) onto \(\mathcal{H}^\alpha, \mathcal{H}^\beta, \mathcal{H}^{\alpha,\beta}\) and \(\mathcal{H}_{\alpha,\beta}\), respectively. For \(f \in C(\mathbb{T}^2)\), the operators \(P^\alpha M_f P^\alpha\) on \(\mathcal{H}^\alpha\) and \(P^\beta M_f P^\beta\) on \(\mathcal{H}^\beta\) are called half-plane Toeplitz operators. The operator \(\tilde{P}^{\alpha,\beta} M_f \tilde{P}^{\alpha,\beta}\) on \(\mathcal{H}^{\alpha,\beta}\) is called the quarter-plane Toeplitz operator, and \(\tilde{P}_{\alpha,\beta} M_f \tilde{P}_{\alpha,\beta}\) on \(\mathcal{H}_{\alpha,\beta}\) is its concave corner analogue. We write \(T^\alpha\) and \(T^\beta\) for \(C^*\)-algebras generated by these half-plane Toeplitz operators and \(T^{\alpha,\beta}\) and \(T^{\alpha,\beta}\) for \(C^*\)-algebras generated by the quarter-plane and concave corner Toeplitz operators, respectively. There are \(*\)-homomorphisms \(\sigma^\alpha: T^\alpha \to C(\mathbb{T}^2)\) and \(\sigma^\beta: T^\beta \to C(\mathbb{T}^2)\), which map \(P^\alpha M_f P^\alpha\) and \(P^\beta M_f P^\beta\) to the symbol \(f\), respectively. We define the \(C^*\)-algebra \(S^{\alpha,\beta}\) as a pullback \(C^*\)-algebra of these two \(*\)-homomorphisms. The real structure \(c\) on \(\mathcal{H} = l^2(\mathbb{Z}^2)\) induces real structures \(c\) on \(T^\alpha, T^\beta, T^{\alpha,\beta}, T^{\alpha,\beta}\), and \(S^{\alpha,\beta}\) and thus induces transpositions \(\tau_\alpha, \tau_\beta, \tau_{\alpha,\beta}, \tau_{\alpha,\beta}\), and \(\tau_S\) on \(T^\alpha, T^\beta, T^{\alpha,\beta}, T^{\alpha,\beta}\), and \(S^{\alpha,\beta}\), respectively. For transpositions, we may simply write \(\tau\) when it is clear from the context. The maps \(\sigma^\alpha\) and \(\sigma^\beta\) preserve the real structures and we

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\(^5\)An operator \(A\) on a complex Hilbert space \(V\) is called the antunitary operator if \(A\) is an antilinear bijection on \(V\) satisfying \(\langle Av, Aw \rangle = \langle v, w \rangle \) for any \(v\) and \(w\) in \(V\).
have the following pull-back diagram:

\[
\begin{array}{ccc}
(S^\alpha, \tau_S) & \xrightarrow{p^\beta} & (T^\beta, \tau_S) \\
p^\alpha \downarrow & & \sigma^\alpha \downarrow \\
(T^\alpha, \tau_\alpha) & \xrightarrow{\sigma^\alpha} & (C(\mathbb{T}^2), \tau_S)
\end{array}
\]

We write \( \sigma \) for the composition \( \sigma^\alpha \circ p^\alpha = \sigma^\beta \circ p^\beta \). Let \( \tilde{\gamma} \) be a \( * \)-homomorphism from \( T^\alpha, \beta \) to \( S^\alpha, \beta \) which maps \( P^\alpha M_f P^\beta \) to the pair \( (P^\alpha M_f P^\alpha, P^\beta M_f P^\beta) \). This map \( \tilde{\gamma} \) preserves the real structures, and there is the following short exact sequence of \( C^*\)-algebras (Park [54]):

\[
0 \to (K(\hat{H}^\alpha, \beta), \tau_K) \to (\hat{T}^\alpha, \beta, \tau_{\hat{T}_\alpha, \beta}) \xrightarrow{\tilde{\gamma}} (S^\alpha, \beta, \tau_S) \to 0,
\]

where the map from \( (K(\hat{H}^\alpha, \beta), \tau_K) \) to \( (\hat{T}^\alpha, \beta, \tau_{\hat{T}_\alpha, \beta}) \) is the inclusion. Its concave corner analogue is studied in [33] and the following exact sequence is obtained:

\[
0 \to (K(\hat{H}^\alpha, \beta), \tau_K) \to (\hat{T}^\alpha, \beta, \tau_{\hat{T}_\alpha, \beta}) \xrightarrow{\tilde{\gamma}} (S^\alpha, \beta, \tau_S) \to 0,
\]

where \( \tilde{\gamma} \) is a \( * \)-homomorphism mapping \( P^\alpha M_f P^\alpha \) to \( (P^\alpha M_f P^\alpha, P^\beta M_f P^\beta) \).

3. \textit{KO-Groups of C*-algebras Associated with Half-Plane and Quarter-Plane Toeplitz Operators}

In this section, the \( KO \)-theory for half-plane and quarter-plane Toeplitz operators are discussed. In Sect. 3.1 \( KO \)-groups of the half-plane Toeplitz algebra is computed. Quarter-plane Toeplitz operators are discussed in the following sections, and the \( KO \)-groups of the \( C^*\)-algebra \( (S^\alpha, \beta, \tau_S) \) are computed in Sect. 3.2. In Sect 3.3 the boundary maps of the 24-term exact sequence for \( KO \)-theory associated with the sequence (2.2) are discussed and the \( KO \)-groups of the quarter-plane Toeplitz algebra \( (\hat{T}^\alpha, \beta, \tau_{\hat{T}_\alpha, \beta}) \) are computed.

3.1. \textit{KO-Groups of \((T^\alpha, \tau_\alpha)\).} We compute the \( KO \)-groups of the \( C^*\)-algebra \( (T^\alpha, \tau_\alpha) \). The discussion is divided into two cases whether \( \alpha \) is rational (or \(-\infty\)) or irrational.

We first consider the case when \( \alpha \) is a rational number or \(-\infty\). When \( \alpha \in \mathbb{Q} \), we write \( \alpha = \frac{p}{q} \) where \( p \) and \( q \) are relatively prime integers and \( q \) is positive. Let \( m \) and \( n \) be integers such that \(-pm + qn = 1\) and let

\[
\Gamma = \begin{pmatrix} n & -m \\ -p & q \end{pmatrix} \in SL(2, \mathbb{Z}).
\]

Then, the action of \( \Gamma \) on \( \mathbb{Z}^2 \) induces the Hilbert space isomorphism \( H^\alpha \cong H^0 \) and an isomorphism of \( C^*\)-algebras \( (T^\alpha, \tau_\alpha) \cong (T^0, \tau_0) \). Thus, the \( C^*\)-algebra \( (T^\alpha, \tau_\alpha) \) is isomorphic to \( (T, \tau_T) \otimes (C(\mathbb{T}), \tau_T) \), and its \( KO \)-groups are computed as \( KO_i(T^\alpha, \tau_\alpha) \cong KO_i(C(\mathbb{T}), \tau_T) \cong KO_i(C(\mathbb{C}), \text{id}) \oplus KO_{i-1}(C, \text{id}) \). For the first isomorphism, see Proposition 1.5.1 of [61]. Generators of the group \( KO_i(C(\mathbb{T}), \tau_T) \) are obtained in Example 9.2 of [14], and the unital \( * \)-homomorphism \( \iota: \mathbb{C} \to T \) induces an isomorphism \( (\text{id} \otimes \iota)_*: KO_i(C(\mathbb{T}), \tau_T) \to KO_i((C(\mathbb{T}), \tau_T) \otimes (T, \tau_T)) \cong KO_i(T^0, \tau_0) \). Combined with them, \( KO \)-group \( KO_i(T^\alpha, \tau_\alpha) \) and its generators are given as follows.

- \( KO_0(T^\alpha, \tau_\alpha) \cong \mathbb{Z} \) and its generator is [12].
The case of $\alpha = -\infty$ is computed similarly, and its generators are given by replacing $p$ and $q$ above with $-1$ and $0$, respectively.

We next consider the cases of irrational $\alpha$. In this case, complex $K$-groups of $T^\alpha$ are computed by Ji–Kaminker and Xia in [34, 69].

**Lemma 3.1.** For irrational $\alpha$ and for each $i$, we have $KO_i(T^\alpha, \tau_\alpha) \cong KO_i(\mathbb{C}, \text{id})$, where the isomorphism is given by $\lambda^\alpha$.

**Proof.** As for complex $K$-groups, we have $K_0(T^\alpha) = \mathbb{Z}$ and $K_1(T^\alpha) = 0$ by [34, 69]. We consider a split $*$-homomorphism of $C^{*,\tau}$-algebras $\lambda^\alpha: (T^\alpha, \tau_\alpha) \to (\mathbb{C}, \text{id})$ given by the composition of $\phi^\alpha: (T^\alpha, \tau_\alpha) \to (\mathbb{C}(T^2), \tau_\alpha)$ and the pull-back onto a fixed point of the involution $\zeta$ on $T^2$. Let $T^\alpha_0 = \text{Ker} \lambda^\alpha$. By the six-term exact sequence associated with the extension $0 \to T^\alpha_0 \to T^\alpha \xrightarrow{\lambda^\alpha} \mathbb{C} \to 0$, complex $K$-groups of $T^\alpha_0$ are trivial. For a $C^{*,\tau}$-algebra $(A, \tau)$, it follows from Theorem 1.12, Proposition 1.15 and Theorem 1.18 of [13] that $KO_*(A, \tau) = 0$ if and only if $K_*(A) = 0$. Therefore, $KO_*(T^\alpha_0, \tau_\alpha) = 0$. The result follows from the 24-term exact sequence of $KO$-theory for $C^{*,\tau}$-algebras associated with the short exact sequence $0 \to (T^\alpha_0, \tau_\alpha) \xrightarrow{\lambda^\alpha} (T^\alpha, \tau_\alpha) \xrightarrow{\lambda^\alpha} (\mathbb{C}, \text{id}) \to 0$. \hfill $\Box$

### 3.2. $KO$-Groups of $(S^{\alpha,\beta}, \tau_S)$. In this subsection, we compute the $KO$-groups of the $C^{*,\tau}$-algebra $(S^{\alpha,\beta}, \tau_S)$. The basic tool is the following Mayer–Vietoris exact sequence associated with the pull-back diagram (2.2) (see Theorem 1.4.15 of [61], for example):

$$
\begin{array}{c}
\cdots \xrightarrow{\partial_{i+1}} KO_{i+1}(\mathbb{C}(T^2), \tau_\tau) \\
\end{array}
\begin{array}{c}
\xrightarrow{\sigma_1} \xrightarrow{\sigma_2}
\end{array}
\begin{array}{c}
\xrightarrow{\sigma_3} \xrightarrow{\sigma_4}
\end{array}
\begin{array}{c}
\cdots \\
\xrightarrow{\phi_1} \\
\xrightarrow{\phi_2} \\
\xrightarrow{\phi_3} \\
\xrightarrow{\phi_4}
\end{array}
\begin{array}{c}
KO_i(S^{\alpha,\beta}, \tau_S) \xrightarrow{(p^\alpha, p^\beta)} KO_i(T^\alpha, \tau_\alpha) \oplus KO_i(T^\beta, \tau_\beta) \\
\xrightarrow{\partial_1} \xrightarrow{\partial_2}
\end{array}
\begin{array}{c}
\xrightarrow{\partial_3} \xrightarrow{\partial_4}
\end{array}
\begin{array}{c}
\cdots \\
\xrightarrow{\partial_{i-1}} \\
\xrightarrow{\partial_i} \\
\xrightarrow{\partial_{i+1}}
\end{array}
\begin{array}{c}
KO_{i-1}(S^{\alpha,\beta}, \tau_S) \\
\xrightarrow{\phi_1} \\
\xrightarrow{\phi_2} \\
\xrightarrow{\phi_3} \\
\xrightarrow{\phi_4}
\end{array}
\begin{array}{c}
\cdots \\
\xrightarrow{\partial_{i+1}}
\end{array}
\begin{array}{c}
KO_{i+1}(\mathbb{C}(T^2), \tau_\tau) \\
\xrightarrow{\sigma_1} \\
\xrightarrow{\sigma_2}
\end{array}
\begin{array}{c}
\xrightarrow{\sigma_3} \\
\xrightarrow{\sigma_4}
\end{array}
\begin{array}{c}
\cdots
\end{array}
$$

As in [54], the computation of the group $KO_*(S^{\alpha,\beta}, \tau_S)$ is divided into three cases corresponding to whether $\alpha$ and $\beta$ are rational (or $\pm\infty$) or irrational. As in Sect. 3, we have a unital $*$-homomorphism $\lambda^\alpha \circ p^\alpha: (S^{\alpha,\beta}, \tau_S) \to (\mathbb{C}, \text{id})$ which splits. Correspondingly, the $KO$-group $KO_*(S^{\alpha,\beta}, \tau_S)$ have a direct summand corresponding to $KO_*(\mathbb{C}, \text{id})$. Noting this, these $KO$-groups are computed by Lemma 3.1 and the sequence (2.2) when at least one of $\alpha$ and $\beta$ is irrational. The results are collected in Tables 3 and 4. In the rest of this subsection, we focus on the cases when both $\alpha$ and $\beta$ are rational (or $\pm\infty$).
When \( \alpha, \beta \in \mathbb{Q} \), we write \( \alpha = \frac{p}{q} \) and \( \beta = \frac{r}{s} \) by using mutually prime integers where \( q \) and \( s \) are positive. In the following discussion, the case of \( \alpha = -\infty \) corresponds to the case where \( p = -1 \) and \( q = 0 \), and the case of \( \beta = +\infty \) corresponds to the case where \( r = 1 \) and \( s = 0 \). By using the action of \( \Gamma \in SL(2, \mathbb{Z}) \) in (3.1) on \( \mathbb{Z}^2 \), there are isomorphisms \( (T^\alpha, \tau_\alpha) \cong (T^0, \tau_0) \) and \( (T^\beta, \tau_\beta) \cong (T^\gamma, \tau_\gamma) \), where \( \gamma = \frac{1}{s} \) for \( u = ns - mr \) and \( t = -ps + qr \). Note that \( t \) is positive since \( \alpha < \beta \).

We have the following commutative diagram:

\[
\begin{array}{c}
KO_i(T^\alpha, \tau_\alpha) \oplus KO_i(T^\beta, \tau_\beta) & \xrightarrow{\sigma_\alpha^i - \sigma_\beta^i} & KO_i(C(T^2), \tau_T) \\
\cong & & \cong \\
KO_i(T^0, \tau_0) & \oplus KO_i(T^\gamma, \tau_\gamma) & \xrightarrow{\sigma_\gamma^i - \sigma_\alpha^i} KO_i(C(T^2), \tau_T)
\end{array}
\]

where the vertical isomorphisms are induced by the action of \( \Gamma \). In the following, we discuss the lower part of the diagram, which is enough for our purpose since the isomorphism \( KO_i(S^{0,\gamma}, \tau_S) \cong KO_i(S^{0,\gamma}, \tau_S) \) is also induced. We write \( \varphi_i \) for the above map \( \sigma_\gamma^i - \sigma_\alpha^i \). By the exact sequence (3.2), we have the following short exact sequence.

\[
(3.3) \quad 0 \to \text{Coker}(\varphi_{i+1}) \to KO_i(S^{0,\gamma}, \tau_S) \to \text{Ker}(\varphi_i) \to 0.
\]

We first compute kernels and cokernels of \( \varphi_i \). Cases for \( i = -1, 0, 4, 6 \) is easy, thus we consider the other cases.

When \( i = 1 \), groups \( KO_1(T^0, \tau_0) \) and \( KO_1(T^\gamma, \tau_\gamma) \) are both isomorphic to \( \mathbb{Z}_2 \oplus \mathbb{Z} \). The \( \mathbb{Z}_2 \) direct summand is generated by \([1,0] \), and the other \( \mathbb{Z} \) direct summand is generated by \([P^0M_{1,0}P^0]\) and \([P^0M_{1,0}P^0]\), respectively. They map to \([M_{1,0}]\) and \([M_{1,0}]\) in \( KO_1(C(T^2), \tau_T) \) by \( \sigma_\alpha^1 \) and \( \sigma_\gamma^1 \), respectively. We have \( KO_1(C(T^2), \tau_T) \cong \mathbb{Z}_2 \oplus \mathbb{Z}^2 \), where the \( \mathbb{Z}_2 \) direct summand is generated by \([-1]\). For \((m,n) \in \mathbb{Z}^2\), the element \([M_{m,n}] \in KO_1(C(T^2), \tau_T) \) corresponds to \((0,m,n) \in \mathbb{Z}_2 \oplus \mathbb{Z}^2 \). Therefore, \( \text{Ker}(\varphi_1) \cong \mathbb{Z}_2 \) which is generated by \((-1, -1)\), and \( \text{Coker}(\varphi_1) \cong \mathbb{Z}_4 \).

We next consider the case of \( i = 2 \). We have \( KO_2(C(T^2), \tau_T) \cong \mathbb{Z}_2 \oplus (\mathbb{Z}_2)^2 \oplus \mathbb{Z} \), where the first \( \mathbb{Z}_2 \) direct summand is generated by \([-I(2)]\). For \((m,n) \in \mathbb{Z}^2\), the element \( \left( \begin{array}{ll} 0 & iM_{m,n} \\ -iM_{m,n} & 0 \end{array} \right) \) in \( KO_2(C(T^2), \tau_T) \) corresponds to \((0,m \mod 2, n \mod 2, 0) \in \mathbb{Z}_2 \oplus (\mathbb{Z}_2)^2 \oplus \mathbb{Z} \) (Example 9.2 of [14]). The groups \( KO_2(T^0, \tau_0) \) and \( KO_2(T^\gamma, \tau_\gamma) \) and their generators are obtained in Sect. \( \mathbb{Z}_2 \) and we have

\[
\text{Ker}(\varphi_2) \cong \begin{cases} (\mathbb{Z}_2)^2 & \text{when } t \text{ is even,} \\ \mathbb{Z}_2 & \text{when } t \text{ is odd,} \end{cases} \quad \text{Coker}(\varphi_2) \cong \begin{cases} \mathbb{Z}_2 \oplus \mathbb{Z} & \text{when } t \text{ is even,} \\ \mathbb{Z} & \text{when } t \text{ is odd.} \end{cases}
\]

When \( i = 3 \), we have \( KO_3(C(T^2), \tau_T) \cong (\mathbb{Z}_2)^3 \). For \((m,n) \in \mathbb{Z}^2\), the element \([\text{diag}(M_{m,n}, M_{m,n})] \in KO_3(C(T^2), \tau_T) \) corresponds to \((m \mod 2, n \mod 2, 0) \in (\mathbb{Z}_2)^3 \). By Sect. \( \mathbb{Z}_2 \) we have

\[
\text{Ker}(\varphi_3) \cong \begin{cases} \mathbb{Z}_2 & \text{when } t \text{ is even,} \\ 0 & \text{when } t \text{ is odd,} \end{cases} \quad \text{Coker}(\varphi_3) \cong \begin{cases} (\mathbb{Z}_2)^2 & \text{when } t \text{ is even,} \\ \mathbb{Z}_2 & \text{when } t \text{ is odd.} \end{cases}
\]

\(^6\)When \( \alpha = -\infty \) and \( \beta \in \mathbb{Q} \), we have \( \text{Coker}(\varphi_1) \cong \mathbb{Z}_4 \). This is the case when \( p = -1 \) and \( q = 0 \) and \( t = -ps + qr = s \) in this case. A similar remark also holds for \( i = 2, 3, 5 \).
When \( i = 5 \), we have \( KO_2(C(T^2), \tau_T) \cong \mathbb{Z}^2 \). For \((m, n) \in \mathbb{Z}^2\), the element \([\text{diag}(M_{m,n}, M_{m,n})]\) in \( KO_2(C(T^2), \tau_T) \) corresponds to \((m, n) \in \mathbb{Z}^2\). By Sect. 3.2 we have \( \ker(\phi_5) = 0 \) and \( \text{coker}(\phi_5) = \mathbb{Z}_4 \).

Combined with the above computation and the exact sequence \((3.3)\), \( KO_1(S^{\alpha, \beta}, \tau_S) \) are computed, though some complication appears when \( i = 2, 3 \). We discuss these two cases in the following subsections.

### 3.2.1. The Group \( KO_2(S^{\alpha, \beta}, \tau_S) \)

We compute the group \( KO_2(S^{\alpha, \beta}, \tau_S) \), which is isomorphic to \( KO_2(S^{\alpha, \beta}, \tau_S) \). The computation is divided into cases depending on whether \( t \) is even or odd. Note that \( u \) is odd when \( t \) is even since \( r \) and \( s \) are mutually prime.

When \( t \) is odd, \( \ker(\phi_2) \cong \mathbb{Z}_2 \) is generated by \([[-I(2)], [-I(2)]\) and the sequence \((3.3)\) splits. Therefore, \( KO_2(S^{\alpha, \beta}, \tau_S) \cong (\mathbb{Z}_2)^2 \).

We next discuss the cases of even \( t \). In this case, both of the kernel and the cokernel of \( \phi_3 \) are isomorphic to \((\mathbb{Z}_2)^2\). Let \( KO_2(S^{\gamma, \gamma}, \tau_S) \) be the kernel of the map \( \lambda^\alpha \circ p_2^\gamma : KO_2(S^{\alpha, \beta}, \tau_S) \to KO_2(C, \Id) \cong \mathbb{Z}_2 \) which splits. Then, the sequence \((3.3)\) reduces to the following extension:

\[
0 \to (\mathbb{Z}_2)^2 \to KO_2(S^{\gamma, \gamma}, \tau_S) \to \mathbb{Z}_2 \to 0.
\]

In the following, we show that this sequence \((3.4)\) splits. We find a lift of the generator of \( \mathbb{Z}_2 \) in \( KO_2(S^{\gamma, \gamma}, \tau_S) \) and show this lift has order two. For \((m, n) \in \mathbb{Z}^2\) and \( \kappa = 0 \) and \( \gamma \), we write \( T^\kappa_{m,n} \) for \( P^\kappa M_{m,n} P^\kappa \), and let \( Q \) be the projection \( T^\gamma_{0,0} T^\gamma_{0,0} \). Note that \( 1 - Q \) is the projection onto the closed subspace spanned by \( \{ \delta_{m,n} \mid 0 \leq \gamma m - n < t \} \).

For \( j = 1, \ldots, t \), let \( P_j \) be a projection in \( T^\gamma \), defined inductively as follows:

\[
P_1 = (1 - Q) M_{0,-t+1} (1 - Q) M_{0,t-1} (1 - Q),
\]

\[
P_j = (1 - Q) M_{0,-t+j} (1 - Q) M_{0,t-j} (1 - Q) - \sum_{k=1}^{j-1} P_k.
\]

Specifically, \( P_j \) is the orthogonal projection of \( \mathcal{H}^\gamma \) onto the closed subspace spanned by \( \{ \delta_{n,tn-j+1} \}_{n \in \mathbb{Z}} \). Note that \( \sum_{j=1}^{t} P_j = 1 - Q \).

For odd \( j = 1, 3, \ldots, t-1 \), let \( s_j = P_j M_{0,1} P_{j+1} - P_{j+1} M_{0,-1} P_j \) and \( s = \sum_{j=1, \text{odd}} s_j \). The element \( s \) satisfies the relations \((i)\) \( s^* = -s \), \((ii)\) \( s^\gamma = -s \), \((iii)\) \( s^2 = -1 + Q \), \((iv)\) \( Qs = sQ = 0 \) and \((v)\) \( s T^\gamma_{u,0} T^\gamma_{u,0} s = 0 \). Note that \( \sigma^\gamma(s) = 0 \) since \( \sigma^\gamma(1 - Q) = 0 \).

We first consider the following elements:

\[
a = \begin{pmatrix}
0 & i - 1 \tau \\
-i \cdot 1 \tau & 0
\end{pmatrix} \in M_2(T^0), \quad b_\pm = \begin{pmatrix}
\pm is & iQ \\
-iQ & is
\end{pmatrix} \in M_2(T^\gamma),
\]

where the double-sign corresponds. Elements \( a \) and \( b_\pm \) are self-adjoint unitaries satisfying \( a^\gamma = -a \) and \( b_\pm^\gamma = -b_\pm \), and pairs \((a, b_\pm)\) are elements of \( M_2(S^{\alpha, \beta}) \); therefore, they define the elements of \( KO_2(S^{\alpha, \beta}, \tau_S) \).

**Lemma 3.2.** As elements of \( KO_2(S^{\alpha, \beta}, \tau_S) \), we have \( [(a, b_+)] = [(a, b_-)] = 0 \).

**Proof.** We first show that \( [(a, b_\pm)] = 0 \). For \( j = 1, 3, \ldots, t-1 \), let \( r_j = P_j M_{0,1} P_{j+1} + P_{j+1} M_{0,-1} P_j \) and \( r = \sum_{j=1, \text{odd}} r_j \). The element \( r \) satisfies \((i)\) \( r^* = r \), \((ii)\) \( r^\gamma = r \), \((iii)\) \( r^2 = 1 - Q \), \((iv)\) \( Qr = rQ = 0 \), \((v)\) \( r T^\gamma_{u,0} T^\gamma_{u,0} r = 0 \) and \((vi)\) \( r \) anticommutes.
Figure 1. The case of $u = 1$ and $t = 4$. $1 - Q$ is the projection onto the closed subspace corresponding to lattice points in between two lines (lattice points on the line $y = \gamma x$ are included, while that on $y = \gamma(x - 1)$ are not). $P_j$ is the projection onto the closed subspace spanned by $\{\delta_{n,4n-j+1} \mid n \in \mathbb{Z}\}$. $s_j$ interchanges two points in a pair up to the sign.

with $s$. For $0 \leq \theta \leq \frac{\pi}{2}$, let

$$b_\theta = \begin{pmatrix} \cos \theta & i Q + i r \sin \theta \\ -i \cos \theta & i r \sin \theta \end{pmatrix}, \quad d = \begin{pmatrix} 0 & i \cdot 1_{\mathcal{H}} \\ -i \cdot 1_{\mathcal{H}} & 0 \end{pmatrix}.$$  

This $b_\theta$ is a self-adjoint unitary satisfying $b_\tau b_\theta = b_\theta b_\tau$ and $b_0 = b_+$. Therefore, $b_+$ and $b_\pm$ are homotopic in $\mathcal{U}(2)$.

We further discuss $b_\pm$. Let consider lattice points $(m, n) \in \mathbb{Z}^2$ satisfying $0 \leq \gamma m - n < t$, as indicated in Figure 1 for the case where $u = 1$ and $t = 4$. As in Figure 1 we divide these points to $\pm$ pairs of lattice points: for $n \in \mathbb{Z}$ and odd $j = 1, 3, \ldots, t - 1$, a pair consists of $\{(n, tn - j), (n, tn - j + 1)\}$. The action of $b_\pm$ is closed on each pair of lattice points and is expressed by a $4 \times 4$ matrix (acting on $\mathbb{C}^2 \otimes \mathbb{C}^2$; one $\mathbb{C}^2$ corresponds to a pair of lattice points, and the other $\mathbb{C}^2$ corresponds to the $2 \times 2$ matrix we consider). Let $V$ be the following matrix.

$$V = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 \\ -1 & 1 & 1 & 1 \end{pmatrix}.$$  

Then $V \in SO(4)$ and satisfies

$$V \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix} V^* = \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & i \\ -i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix},$$

where the left matrix inside the conjugation is the restriction of $b_{\pm}$ onto the closed subspace spanned by generating functions of these two lattice points tensor $\mathbb{C}^2$ and the right matrix is that of $d$ (note that $Q = 0$ on these lattice points). Let $W$ be the unitary on $\mathcal{H}^2 \otimes \mathbb{C}^2$ defined by applying $V$ to these pair of lattice points satisfying
have a path in $U_1^{(2)}(\gamma, \tau_\gamma)$ from $b_+$ to $d$. By its construction, the pair of the constant path at $a \in M_2(T^0)$ and this path gives a path in $U_1^{(2)}(S^{0, \gamma}, \tau_{S})$ from $(a, b_+)$ to $(a, d)$. Therefore, we have $[(a, b_+)] = [(a, d)] = [I^{(2)}] = 0$ in $KO_2(S^{0, \gamma}, \tau_{S})$.

We next discuss the class $[(a, b_-)]$. For $0 \leq \theta \leq \frac{\pi}{2}$, let

$$b_\theta' = \begin{pmatrix}-is \cos \theta & iQ + i(1 - Q) \sin \theta \\ -iQ - i(1 - Q) \sin \theta & is \cos \theta \end{pmatrix}.$$  

Then, $b_\theta'$ is a self-adjoint unitary satisfying $(b_\theta')^\tau = -b_\theta'$. We have $b_0' = b_-$ and $b_{\frac{\pi}{2}}' = I^{(2)}$. Therefore, $[(a, b_-)] = [(a, b_	heta')] = [I^{(2)}] = 0$ in $KO_2(S^{0, \gamma}, \tau_{S})$.

**Lemma 3.3.** In $KO_2(S^{0, \gamma}, \tau_{S})$, we have $[(v_2, w_+)] = [(v_2, w_-)]$.

**Proof.** For $0 \leq \theta \leq \pi$, let consider the following element in $M_4(T^\tau)$:

$$R_\theta = \begin{pmatrix} is \cos \theta & iT^\gamma_{u,0} & -ir \sin \theta & 0 \\ -iT^\gamma_{u,0} & 0 & 0 & 0 \\ ir \sin \theta & 0 & is \cos \theta & iQ \\ 0 & 0 & -iQ & is \end{pmatrix},$$

Then, we have $R_0 = w_+ \oplus b_-, R_\pi = w_- \oplus b_-$ and $R_\theta$ is a self-adjoint unitary satisfying $R_\theta^\tau = -R_\theta$. Since $\sigma^\gamma(R_\theta) = \sigma^\gamma(v_2 \oplus a)$, the pair $(v_2 \oplus a, R_\theta)$ is contained in $U_2^{(2)}(S^{0, \gamma}, \tau_{S})$ and gives a path from $(v_2 \oplus a, w_+ \oplus b_+)$ to $(v_2 \oplus a, w_- \oplus b_-)$. By using Lemma 3.2, we obtain the following equality in $KO_2(S^{0, \gamma}, \tau_{S})$:

$$[(v_2, w_+)] = [(v_2 \oplus a, w_+ \oplus b_+)] = [(v_2 \oplus a, w_- \oplus b_-)] = [(v_2, w_-)].$$

**Lemma 3.4.** In $KO_2(S^{0, \gamma}, \tau_{S})$, the element $[(v_2, w_+)]$ has order two.

**Proof.** For $0 \leq \theta \leq \frac{\pi}{2}$, let

$$A^0_\theta = \begin{pmatrix} 0 & iT^0_{u,0} \cos \theta & i \sin \theta & 0 \\ -iT^0_{u,0} \cos \theta & 0 & 0 & -i \sin \theta \\ -i \sin \theta & 0 & 0 & iT^0_{u,0} \cos \theta \\ 0 & i \sin \theta & -iT^0_{u,0} \cos \theta & 0 \end{pmatrix} \in M_4(T^0),$$

$$A^\gamma_\theta = \begin{pmatrix} is \cos \theta & iT^\gamma_{u,0} \cos \theta & i \sin \theta & 0 \\ -iT^\gamma_{u,0} \cos \theta & 0 & 0 & -i \sin \theta \\ -i \sin \theta & 0 & -is \cos \theta & iT^\gamma_{u,0} \cos \theta \\ 0 & i \sin \theta & -iT^\gamma_{u,0} \cos \theta & 0 \end{pmatrix} \in M_4(T^\gamma).$$

Then, $A^0_\theta$ and $A^\gamma_\theta$ are self-adjoint unitaries satisfying $(A^0_\theta)^\tau = -A^0_\theta$ and $(A^\gamma_\theta)^\tau = -A^\gamma_\theta$, and their pair $(A^0_\theta, A^\gamma_\theta)$ is contained in $M_4(S^{0, \gamma})$. Note that $(A^0_\theta, A^\gamma_\theta) = \cdots$
\((v_2 \oplus v_2, w_+ \oplus w_-)\). Therefore, by Lemma 3.3 the following equality holds in \(KO_2(S^{0,\gamma}, \tau_S)\):
\[
2 \cdot [(v_2, w_+)] = [(v_2, w_+)] + [(v_2, w_-)] = [(A^0_{u_0}, A^0_{-u_0})] = [(A^0_{\gamma}, A^0_{-\gamma})] = 0. \]

Proposition 3.5. When \(\alpha\) and \(\beta\) are rational numbers and \(t = -ps + qr\) is even, we have \(KO_2(S^{0,\alpha,\beta}, \tau_S) \cong (Z_2)^3\).

Proof. Since \(u\) is odd when \(t\) is even, the pair \([(v_2), [w_+]) \in KO_2(T^0, \tau_0) \oplus KO_2(T^\gamma, \tau_\gamma)\) constitutes a nontrivial element of the right \(Z_2 \subset Ker(\varphi_2)\) in the sequence (3.4). The element \([(v_2, w_+)] \in KO_2(S^{0,\gamma}, \tau_S)\) is a lift of it. Therefore, \([(v_2, w_+)]\) is nontrivial and has order two by Lemma 3.3. This element belongs to \(KO_2(S^{0,\gamma}, \tau_S)\) and, by mapping \(1 \in Z_2\) to \([(v_2, w_+)]\), we obtain a splitting of the the sequence (3.4). Therefore, \(KO_2(S^{0,\gamma}, \tau_S) \cong (Z_2)^3\) and the result follows.

3.2.2. The Group \(KO_3(S^{0,\alpha,\beta}, \tau_S)\). We next compute \(KO_3(S^{0,\alpha,\beta}, \tau_S)\). Note that \(Ker(\varphi_3)\) depends on whether \(t\) is even or odd. When \(t\) is odd, \(Ker(\varphi_3)\) is zero and, from the sequence (3.3), we have \(KO_3(S^{0,\alpha,\beta}, \tau_S) \cong Z_2\).

We next discuss the cases of even \(t\). In this case, the extension (3.3) is of the following form:
\[
(3.5) \quad 0 \to Z_2 \to KO_3(S^{0,\gamma}, \tau_S) \to Z_2 \to 0.
\]

As in Sect. 3.2.1, we show that this sequence splits by finding a lift of the generator of the right \(Z_2 \subset KO_3(S^{0,\gamma}, \tau_S)\) of order two. Let consider the following elements:
\[
v_3 = \begin{pmatrix} T^0_{u_0,0} & 0 \\ 0 & T^0_{-u_0,0} \end{pmatrix} \in M_2(T^0), \quad z_\pm = \begin{pmatrix} T^\gamma_{u_0,0} & \pm s \\ 0 & T^\gamma_{-u_0,0} \end{pmatrix} \in M_2(T^\gamma),
\]
where the double-sign in the second equality corresponds. Pairs \((v_3, z_\pm)\) are units in \(M_2(S^{0,\gamma})\) satisfying \((v_3, z_\pm)^{#@\tau} = (v_3, z_\pm)\) and define elements \([(v_3, z_\pm)]\) of the KO-group \(KO_3(S^{0,\gamma}, \tau_S)\).

Lemma 3.6. In \(KO_3(S^{0,\gamma}, \tau_S)\), we have \([(v_3, z_+)] = [(v_3, z_-)]\).

Proof. For \(0 \leq \theta \leq \pi\), let \(z_\theta = \begin{pmatrix} T^\gamma_{u_0,0} & e^{i\theta}s \\ 0 & T^\gamma_{-u_0,0} \end{pmatrix} \in M_2(T^\gamma)\) which gives a path \(\{z_\theta\}_{0 \leq \theta \leq \pi}\) of unitaries satisfying \((z_\theta)^{#@\tau} = z_\theta\). Its endpoints are \(z_0 = z_+\) and \(z_\pi = z_-\). The pair \((v_3, z_\theta)\) satisfies \((v_3, z_\theta)^{#@\tau} = (v_3, z_\theta)\) and gives a homotopy between \((v_3, z_+)\) and \((v_3, z_-)\) in \(U_1^{(3)}(S^{0,\gamma}, \tau_S)\).

Lemma 3.7. The element \([(v_3, z_+)]\) in \(KO_3(S^{0,\gamma}, \tau_S)\) has order two.

Proof. For \(0 \leq \theta \leq \frac{\pi}{2}\), let
\[
B^0_\theta = \begin{pmatrix} T^0_{u_0,0} \cos \theta & 0 & 0 & \sin \theta \\ 0 & T^0_{-u_0,0} \cos \theta & \sin \theta & 0 \\ 0 & -\sin \theta & T^0_{u_0,0} \cos \theta & 0 \\ -\sin \theta & 0 & 0 & T^0_{-u_0,0} \cos \theta \end{pmatrix},
\]
\[
B^\gamma_\theta = \begin{pmatrix} T^\gamma_{u_0,0} \cos \theta & s \cos \theta & 0 & \sin \theta \\ 0 & T^\gamma_{-u_0,0} \cos \theta & \sin \theta & 0 \\ 0 & -\sin \theta & T^\gamma_{u_0,0} \cos \theta & -s \cos \theta \\ -\sin \theta & 0 & 0 & T^\gamma_{-u_0,0} \cos \theta \end{pmatrix}.
\]
Table 3. $KO_*(S^{\alpha,\beta}, \tau_S)$ when both $\alpha$ and $\beta$ are rational (or $\pm \infty$).

| $i$ | 0 | 1 | 2 |
|-----|---|---|---|
| $t = -ps + qr$ | even | odd | even | odd |
| $KO_i(S^{\alpha,\beta}, \tau_S)$ | $\mathbb{Z} \oplus \mathbb{Z}_4$ | $(\mathbb{Z}_2)^2 \oplus \mathbb{Z}$ | $\mathbb{Z}_2 \oplus \mathbb{Z}$ | $(\mathbb{Z}_2)^4$ | $(\mathbb{Z}_2)^2$ |

| 3 | 4 | 5 | 6 | 7 |

Table 4. $KO_*(S^{\alpha,\beta}, \tau_S)$ when one of $\alpha$ and $\beta$ is rational (or $\pm \infty$) and the other is irrational.

| $i$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|-----|---|---|---|---|---|---|---|---|
| $KO_i(S^{\alpha,\beta}, \tau_S)$ | $\mathbb{Z}$ | $(\mathbb{Z}_2)^2 \oplus \mathbb{Z}$ | $(\mathbb{Z}_2)^4 \oplus \mathbb{Z}$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z} \oplus \mathbb{Z}_4$ | $\mathbb{Z}$ | 0 | 0 |

Table 5. $KO_*(S^{\alpha,\beta}, \tau_S)$ when both $\alpha$ and $\beta$ are irrational.

| $i$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|-----|---|---|---|---|---|---|---|---|
| $KO_i(S^{\alpha,\beta}, \tau_S)$ | $\mathbb{Z}$ | $(\mathbb{Z}_2)^2 \oplus \mathbb{Z}$ | $(\mathbb{Z}_2)^4 \oplus \mathbb{Z}$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | $\mathbb{Z}_4 \oplus \mathbb{Z}_4$ | $\mathbb{Z}$ | 0 | 0 |

For each $\theta$, matrices $B_\theta^0$ and $B_\theta^7$ are unitaries satisfying $(B_\theta^0)^{\# \otimes \tau} = B_\theta^0$ and $(B_\theta^7)^{\# \otimes \tau} = B_\theta^2$. We have $B_\theta^0 = v_3 \oplus v_3$ and $B_\theta^7 = z_+ \oplus z_-$. Note that matrices $B_\theta^0$ and $B_\theta^7$ are contained in $M_4(\mathbb{C})$, where they coincide. Since this unitary satisfies the symmetry of the $KO_3$-group, this is an element of the quaternionic unitary group $U(2, \mathbb{H})$. Since $U(2, \mathbb{H})$ is path-connected, there is a path of unitaries in $U_2(S^{\alpha,\gamma}, \tau_S)$ connecting $(B_\theta^0, B_\theta^7)$ to $(1_S)_4$. By using Lemma 3.6 we obtain the following equality in $KO_3(S^{\alpha,\gamma}, \tau_S)$:

$2 \cdot [(v_3, z_+)] = [(v_3 \oplus v_3, z_+ \oplus z_-)] = [(B_\theta^0, B_\theta^7)] = [(1_S)_4] = 0$. □

Proposition 3.8. When $\alpha$ and $\beta$ are rational numbers and $t = -ps + qr$ is even, we have $KO_3(S^{\alpha,\beta}, \tau_S) \cong (\mathbb{Z}_2)^2$.

Proof. The pair $[(v_3, z_+)] \in KO_3(T^0, \tau_0) \oplus KO_3(T^7, \tau_7)$ is contained in $\text{Ker}(\varphi_3) \cong \mathbb{Z}_2$ and is nontrivial. The element $[(v_3, z_+)] \in KO_3(S^{\alpha,\gamma}, \tau_S)$ is its lift. Therefore, the class $[(v_3, z_+)]$ is nontrivial and has order two by Lemma 3.7. We thus obtain a splitting of the sequence 3.5 and the group $KO_3(S^{\alpha,\gamma}, \tau_S)$ is isomorphic to $(\mathbb{Z}_2)^2$. □

The results in this subsection are summarized in Tables 3, 4 and 5.

3.3. Boundary Maps Associated with Quarter-Plane Toeplitz Extensions and $KO$-Groups of $(T^{\alpha,\beta}, \tau_{\alpha,\beta})$. We next consider the boundary maps of the 24-term exact sequence for $KO$-theory associated with the sequence (2.2):

$$
\hat{\partial}_i: KO_i(S^{\alpha,\beta}, \tau_S) \to KO_{i-1}(K(\mathcal{H}^{\alpha,\beta}), \tau_K).
$$

Proposition 3.9. For each $i$, the boundary map $\hat{\partial}_i$ in (3.6) is surjective.

Proof. When $i = -1, 0, 4, 6$, the group $KO_{i-1}(K(\mathcal{H}^{\alpha,\beta}), \tau_K)$ is trivial and the statement is obvious. We discuss the other cases. The proof is given by constructing explicit elements of the group $KO_i(S^{\alpha,\beta}, \tau_S)$, which maps to a generator of the group $KO_{i-1}(K(\mathcal{H}^{\alpha,\beta}), \tau_K)$. As in 3.5, by using the action of $SL(2, \mathbb{Z})$ on

\footnote{We write $\hat{\partial}_i$ for boundary maps associated with (3.6) and write $\partial_i$ for that with (2.2).}
we assume that \( 0 < \alpha \leq \frac{1}{2} \) and \( 1 \leq \beta < +\infty \) without loss of generality. Let \( \hat{P}_{m,n} = P^{\alpha,\beta}M_{m,n}P^{\alpha,\beta}M_{-m,-n}P^{\alpha,\beta} \). As in [35], we consider the following element in \( T^{\alpha,\beta} \):

\[
(3.7) \quad \hat{A} = \hat{P}_{0,1} + M_{1,1}(1 - \hat{P}_{-1,0}) + M_{1,0}(\hat{P}_{-1,0} - \hat{P}_{0,1}).
\]

The operator \( \hat{A} \) is Fredholm whose kernel is trivial and has one dimensional cokernel [33]. We also have the following.

- \( \hat{\gamma}(\hat{A}) \) is a unitary in \( S^{\alpha,\beta} \).
- \( \hat{A} \) is a real operator, that is \( \tau(\hat{A}) = \hat{A} \), and \( \hat{A}^* = \tau(\hat{A}^*) = \hat{A}^* \) holds.

From these preliminaries, the proof of Proposition 3.4 is parallel to the computation in Example 1.4 of [13]. We summarize the results here.

- Let \( u_1 = \hat{\gamma}(\hat{A}) \). \( u_1 \) is a unitary satisfying \( u_1^* = u_1^\dagger \) and gives an element \( [u_1] \in KO_1(S^{\alpha,\beta}, \tau_S) \). \( \hat{\delta}_1([u_1]) \) is a generator of \( KO_0(H^{\alpha,\beta}, \tau_K) \cong \mathbb{Z} \).

- Let \( u_2 = \begin{pmatrix} 0 & i\hat{\gamma}(\hat{A})^* \\ -i\hat{\gamma}(\hat{A})^* & 0 \end{pmatrix} \). \( u_2 \) is a self-adjoint unitary satisfying \( u_2^* = -u_2 \) and gives \( [u_2] \in KO_2(S^{\alpha,\beta}, \tau_S) \). Its image \( \hat{\delta}_2([u_2]) \) is the generator of the operator \( KO_1(K(H^{\alpha,\beta}, \tau_K) \cong \mathbb{Z}_2 \).

- Let \( u_3 = \text{diag}(\hat{\gamma}(\hat{A}), \hat{\gamma}(\hat{A})^*) \). \( u_3 \) is a unitary satisfying \( u_3^{\dagger} = u_3 \) and gives \( [u_3] \in KO_3(S^{\alpha,\beta}, \tau_S) \). Its image \( \hat{\delta}_3([u_3]) \) is the generator of the group \( KO_2(K(H^{\alpha,\beta}, \tau_K) \cong \mathbb{Z}_2 \).

- Let \( u_5 = \text{diag}(\hat{\gamma}(\hat{A}), \hat{\gamma}(\hat{A})) \). \( u_5 \) is a unitary satisfying \( u_5^{\dagger} = u_5 \) and gives \( [u_5] \in KO_4(S^{\alpha,\beta}, \tau_S) \). Its image \( \hat{\delta}_5([u_5]) \) is a generator of the group \( KO_4(K(H^{\alpha,\beta}, \tau_K) \cong \mathbb{Z} \).

**Remark 3.10.** In the case when \( \alpha, \beta \) are both rational (or \( \pm\infty \)) and \( t = -ps + qr \) is even, the group \( KO_2(S^{0,\gamma}, \tau_S) \cong (\mathbb{Z}_2)^k \) is generated by \( \{ -I^{(2)} \} \), \( \{ [v_2, w_+] \} \), \( \{ [u_2] \} \), \( \{ [w', I_2^{(2)}] \} \), where \( w' = Y_4^{(3)} \text{diag}(1, 1 - 2T_{0,1}^0 T_{0,-1}^0, 2T_{0,1}^0 T_{0,-1}^0 - 1, -1)Y_4^{(3)*} \).

By the map \( \sigma_\gamma : KO_1(S^{\alpha,\beta}, \tau_S) \rightarrow KO_1(C(T^2), \tau_T) \), components generated by \( \{ [v_2, w_+] \} \) (when \( i = 2 \)) and \( \{ [v_3, z_+] \} \) (when \( i = 3 \)) maps injectively.

The \( KO \)-groups of \( T^{\alpha,\beta}, \tau_{\alpha,\beta} \) are computed by the 24-term exact sequence of \( KO \)-theory associated with \( \{ \mathbb{C} \} \) and Proposition 3.9. The results are collected in Tables 6, 7, and 8.

**Remark 3.11.** Similar results in this section also hold for convex corners. Let \( \hat{A} \in T^{\alpha,\beta} \) be an operator defined by replacing \( P^{\alpha,\beta} \) in the definition of \( \hat{A} \) by \( \hat{P}^{\alpha,\beta} \). This \( \hat{A} \) is a Fredholm operator satisfying \( \hat{A}^* = \hat{A}^* \) which have one dimensional kernel and trivial cokernel [33]. As in Proposition 3.9 by using this example, we see that the boundary maps \( \hat{\delta}_i \) of \( KO \)-theory associated with the sequence \( \{ \mathbb{C} \} \) is surjective. The \( KO \)-groups of \( T^{\alpha,\beta}, \tau_{\alpha,\beta} \) is computed by the 24-term exact sequence associated with \( \{ \mathbb{C} \} \), and the results are the same as in Tables 6, 7, and 8. Through the stabilization isomorphism, we have two boundary maps \( \hat{\delta}_i \) and \( \hat{\delta}_i \) from \( KO_1(S^{\alpha,\beta}, \tau_S) \) to \( KO_1(-1)(\mathbb{C}, \text{id}) \) associated with \( \{ \mathbb{C} \} \) and \( \{ \mathbb{C} \} \). Since \( \hat{\gamma}(\hat{A}) = \hat{\gamma}(\hat{A}) \), the relation \( \hat{\delta}_i = -\hat{\delta}_i \) holds, as in Corollary 1 of [33].

---

\[ ^8 \] The matrix \( Y_4^{(3)} \) is introduced in Appendix A.3.
Table 6. $KO_*(T^{α,β}, π_{α,β})$ when both $α$, $β$ are rational (or $±\infty$).

| $t = ps + qr$ | 0 | 1 | 2 | 3 |
|----------------|---|---|---|---|
| $KO_*(T^{α,β}, π_{α,β})$ | $\mathbb{Z} \oplus \mathbb{Z}_4$ | $(\mathbb{Z}_2)^2$ | $\mathbb{Z}_2$ | $(\mathbb{Z}_2)^3$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | 0 |
| $i$ | 4 | 5 | 6 | 7 |
| $i$ | $\mathbb{Z} \oplus \mathbb{Z}_4$ | 0 | 0 | 0 |

Table 7. $KO_*(T^{α,β}, π_{α,β})$ when one of $α$ and $β$ is rational (or $±\infty$) and the other is irrational.

| $i$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|-----|---|---|---|---|---|---|---|---|
| $KO_i(T^{α,β}, π_{α,β})$ | $\mathbb{Z}^2$ | $(\mathbb{Z}_2)^2$ | $(\mathbb{Z}_2)^2$ | 0 | $\mathbb{Z}^2$ | 0 | 0 | 0 |

Table 8. $KO_*(T^{α,β}, π_{α,β})$ when both $α$ and $β$ are irrational.

| $i$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|-----|---|---|---|---|---|---|---|---|
| $KO_i(T^{α,β}, π_{α,β})$ | $\mathbb{Z}^3$ | $(\mathbb{Z}_2)^3$ | $(\mathbb{Z}_2)^3$ | 0 | $\mathbb{Z}^3$ | 0 | 0 | 0 |

4. Toeplitz Operators Associated with Subsemigroup $(\mathbb{Z}_{≥0})^n$ of $\mathbb{Z}^n$

In this section, Toeplitz operators associated with the subsemigroup $(\mathbb{Z}_{≥0})^n$ of $\mathbb{Z}^n$ for $n ≥ 3$ are discussed. They are an $n$-variable generalization of the ordinary Toeplitz and quarter-plane Toeplitz operators and are briefly discussed in [23, 22], where a necessary and sufficient condition for these Toeplitz operators to be Fredholm is obtained. We revisit these operators since, in our application to condensed matter physics, models of higher-codimensional corners are given by using these $n$-variable generalizations. Since the Toeplitz extension (4.1) and the quarter-plane Toeplitz extension (2.3) provide a framework for these applications, we seek this extension for our $n$-variable cases (Theorem 4.1). Note that we consider corners of arbitrary codimension, though of a specific shape compared to the codimension-two case [54]. In this section, let $n$ be a positive integer bigger than or equals to three.

To study such Toeplitz operators, we follow Douglas–Howe’s idea [23] to use the tensor product of the Toeplitz extension,

\begin{equation}
(4.1) \quad 0 \rightarrow K \xrightarrow{i} T \xrightarrow{γ} C(T) \rightarrow 0,
\end{equation}

where $K = K(l^2(\mathbb{Z}_{≥0}))$. There is a linear splitting of the $*$-homomorphism $γ$ given by the compression onto $l^2(\mathbb{Z}_{≥0})$. For a subset $A \subset \{1, \ldots, n\}$, let $T_A^n = A_1 \otimes \cdots \otimes A_n$, where $A_i = C(T)$ when $i \in A$ and is $T$ when $i \notin A$. Note that $T_0^n$ is isomorphic to $T^n$ introduced in Sect. 2.2. For subsets $D \subset R \subset \{1, \ldots, n\}$, let $π^D_{R}: T_D^n \rightarrow T_R^n$ be the $*$-homomorphism induced by $γ$. Specifically, $π^D_{R} = a_1 \otimes \cdots \otimes a_n$, where $a_i$ is id$_{C(T)}$ when $i \in D$, is $γ$ when $i \in R \setminus D$ and is id$_{T}$ otherwise. Note that $π^0_{R}$ is a surjection and $π^0_0 = id$. In the following, we use a subset $A$ of $\{1, \ldots, n\}$ as a label to distinguish $C^*$-algebras and the morphisms between them, which we may abbreviate brackets $\{\}$ in our notation. For example, we write $T_{1,2}^n$ for $T_{\{1,2\}}^n$, $π_i$ for $π^0_((i)$, and $π^1_{1,2}$ for $π^{(1)}_{\{1,2\}}$. For each $A \subset \{1, \ldots, n\}$, the map $π_A$ has a linear splitting $ρ_A : T_A^n \rightarrow T^n$ given by the compression onto $l^2((\mathbb{Z}_{≥0})^n)$. By these preliminaries,
we consider the following $C^*$-subalgebra of $T^*_n \oplus \cdots \oplus T^*_n$.

$$S^n = \left\{ (T_1, \ldots, T_n) \mid \text{For } 1 \leq i \leq n, \ T_i \in T^*_n, \right. \\
\left. \quad \text{For } 1 \leq i < j \leq n, \ \pi^i_j(T_i) = \pi^j_i(T_j) \right\}.$$  

Let $(T_1, \ldots, T_n) \in S^n$. For a nonempty subset $\mathcal{A} \subset \{1, \ldots, n\}$, we take $i \in \mathcal{A}$ and consider the element $\pi^i_A(T_i) \in T^*_A$. This element does not depend on the choice of $i \in \mathcal{A}$, and we write $T_A = \pi^i_A(T_i)$. Let $\rho' : S^n \to T^n$ be a linear map defined by

$$\rho'(T_1, \ldots, T_n) = \sum_{k=1}^{n} \sum_{|A|=k} (-1)^{k+1} \rho_A(T_A)$$

for $(T_1, \ldots, T_n) \in S^n$, where the second summation is taken over all subsets $\mathcal{A} \subset \{1, \ldots, n\}$ consisting of $k$ elements. Let $K^n = K(\mathbb{P}^2(\mathbb{Z}_{\geq 0}^n))$, and let $\gamma_n : T^n \to S^n$ be an $*$-homomorphism given by $\gamma_n(T) = (\pi_1(T), \ldots, \pi_n(T))$. Let $t_n$ be the $n$-fold tensor product of $t$.

**Theorem 4.1.** There is the following short exact sequence of $C^*$-algebras:

$$0 \to K^n \xrightarrow{\iota_n} T^n \xrightarrow{\gamma_n} S^n \to 0,$$

where the map $\gamma_n$ has a linear splitting given by $\rho'$.

**Proof.** The map $\iota_n$ is injective since $\iota$ is injective. We first show the exactness at $T^n$. Since $\gamma \circ \iota = 0$, we have $\gamma_n \circ t_n = 0$, and thus, $\text{Im}(t_n) \subset \ker(\gamma_n)$. Let $T \in \ker(\gamma_n)$. Since $\pi_1(T) = (\gamma \otimes 1 \otimes \cdots \otimes 1)(T) = 0$, there exists some $S_1 \in K \otimes T \otimes \cdots \otimes T$ such that $(\iota \otimes 1 \otimes \cdots \otimes 1)(S_1) = T$. Since

$$0 = (1 \otimes \gamma \otimes 1 \otimes \cdots \otimes 1)(T) = (1 \otimes 1 \otimes \cdots \otimes 1)(\iota \otimes 1 \otimes \cdots \otimes 1)(S_1)$$

and $\iota \otimes 1 \otimes \cdots \otimes 1$ is injective, $(1 \otimes \gamma \otimes 1 \otimes \cdots \otimes 1)(S_1) = 0$. Therefore, there exists some $S_2 \in K \otimes K \otimes T \otimes \cdots \otimes T$ such that $S_1 = (1 \otimes \iota \otimes 1 \otimes \cdots \otimes 1)(S_2)$.

By continuing this argument, we see that there exists some $S_n \in K \otimes \cdots \otimes K \cong K^n$ such that $(\iota \otimes \cdots \otimes \iota)(S_n) = T$. Thus, we have $\ker(\gamma_n) \subset \text{Im}(t_n)$.

For the surjectivity of $\gamma_n$, we see that $\rho$ is a linear splitting of $\gamma_n$, that is, for $(T_1, \ldots, T_n) \in S^n$ and $1 \leq i \leq n$, the relation $\pi_i \circ \rho'(T_1, \ldots, T_n) = T_i$ holds. In the following, we show $\pi_1 \circ \rho'(T_1, \ldots, T_n) = T_1$ and the other case is proved similarly. Note that

$$\pi_1 \circ \rho'(T_1, \ldots, T_n) = \sum_{k=1}^{n} \sum_{|A|=k} (-1)^{k+1} \pi_1 \circ \rho_A(T_A)$$

and that $\pi_1 \circ \rho_1(T_1) = T_1$. Thus, it is sufficient to show that the sum over $\mathcal{A} (\neq \{1\})$ is zero. Note that for $2 \leq i_1 < \cdots < i_{k-1} \leq n$, we have

$$\pi_1 \circ \rho_{i_1, \ldots, i_{k-1}}(T_{i_1, \ldots, i_{k-1}}) = \pi_1 \circ \rho_{i_1, i_{k-1}}(T_{i_1, i_{k-1}, \ldots, i_{k-1}}).$$

By using this relation, we compute the sum on the right-hand side of (4.3). For $k = 1$ and $k = 2$ of the sum, we have the following:

$$\sum_{|A|=1, \ A \neq \{1\}} \pi_1 \circ \rho_A(T_A) - \sum_{|A|=2} \pi_1 \circ \rho_A(T_A)$$

$$= \sum_{i=2}^{n} \pi_1 \circ \rho_i(T_i) - \sum_{1 \leq i < j \leq n} \pi_1 \circ \rho_{i,j}(T_{ij}) = - \sum_{2 \leq i < j \leq n} \pi_1 \circ \rho_{i,j}(T_{ij}).$$
Table 9. $KO$-groups of $(S^n, \tau_S)$.

| $i$ | 0   | 1   | 2   | 3   | 4   | 5   | 6   | 7   |
|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| $KO_i(S^n, \tau_S)$ | $\mathbb{Z}$ | $\mathbb{Z} \oplus \mathbb{Z}/2$ | $(\mathbb{Z}/2)^2$ | $\mathbb{Z}$ | $\mathbb{Z}$ | $0$ | $0$ | $0$ |

For $2 < k < n$, we have

$$(-1)^k \sum_{2 \leq i_1 < \ldots < i_{k-1} \leq n} \pi_1 \circ \rho_{i_1, \ldots, i_{k-1}}(T_{i_1, \ldots, i_{k-1}})$$
$$+ (-1)^{k+1} \sum_{1 \leq i_1 < \ldots < i_k \leq n} \pi_1 \circ \rho_{i_1, \ldots, i_k}(T_{i_1, \ldots, i_k})$$
$$= (-1)^{k+1} \sum_{2 \leq i_1 < \ldots < i_k \leq n} \pi_1 \circ \rho_{i_1, \ldots, i_k}(T_{i_1, \ldots, i_k}),$$

and since

$$(-1)^n \pi_1 \circ \rho_{2, \ldots, n}(T_{2, \ldots, n}) + (-1)^{n+1} \pi_1 \circ \rho_{1, \ldots, n}(T_{1, \ldots, n}) = 0,$$

we have $\pi_1 \circ \rho'(T_1, \ldots, T_n) = T_1$. $\square$

Theorem 4.3 leads to the necessary and sufficient condition for Toeplitz operators associated with these codimension-$n$ corners to be Fredholm.

**Corollary 4.2** (Theorem 18 of [22]). Let $k$ be a positive integer. An operator $T \in M_k(T^n)$ is a Fredholm operator if and only if $\gamma_n(T)$ is invertible in $M_k(S^n)$ or, equivalently, if and only if $\pi_i(T)$ is invertible for any $1 \leq i \leq n$.

As in Sect. 2.2, the real structure on $l^2(Z^n)$ induces real structures on $T^n$ and $S^n$. We write $\tau_S$ for the transposition on $S^n$ associated with this real structure. The map $\gamma_n$ preserves the real structure, and we obtain the following exact sequence of $C^{*,\tau}$-algebras:

$$(4.4) 0 \rightarrow (K^n, \tau_K) \xrightarrow{i} (T^n, \tau_T) \xrightarrow{\gamma_n} (S^n, \tau_S) \rightarrow 0.$$

We next compute the $K$-groups of the $C^{*,\tau}$-algebra $S^n$ and $KO$-groups of the $C^{*,\tau}$-algebra $(S^n, \tau_S)$.

**Proposition 4.3.** $K_i(S^n) \cong \mathbb{Z}$ for $i = 0, 1$.

**Proof.** Note that $K_i(T^n) \cong K_i(C)$. The result follows from the six-term exact sequence of $K$-theory associated with the sequence (1.2) in Theorem 4.1. $\square$

**Proposition 4.4.** For each $i$, we have

$$KO_i(S^n, \tau_S) \cong KO_i(C, \text{id}) \oplus KO_{i-1}(C, \text{id}).$$

The results are collected in Table 9.

**Proof.** Note that $KO_i(T^n, \tau_T) \cong KO_i(C, \text{id})$. The result follows from the 24-term exact sequence of $KO$-theory associated with the sequence (4.4). $\square$

A Fredholm Toeplitz operator associated with a codimension-$n$ corner whose Fredholm index is one is constructed as follows.
Example 4.5 (A Fredholm Operator of Index One). Let $T_z$ be the Toeplitz operator whose symbol $z: \mathbb{T} \to \mathbb{C}$ is the inclusion. Its adjoint $T_z^*$ is a Fredholm operator on $l^2(\mathbb{Z}_{\geq 0})$ of index one. Let $p = T_z T_z^*$ and $q = 1 - p$, then $p,q \in \mathcal{T}$ and are projections onto $l^2(\mathbb{Z}_{\geq 1})$ and $\mathbb{C} \delta_0$, respectively, where $\delta_0$ is the characteristic function of the point $0 \in \mathbb{Z}$. For a subset $A = \{1, \ldots, n\}$, let $P^n_A = r_1 \cdots r_n$, where $r_i$ is $p$ if $i \in A$ and is $q$ otherwise. The operator $P^n_A$ is a projection which satisfies $\sum_A P^n_A = 1$. Let $\bar{T} = T_z^* \otimes q \otimes \cdots \otimes q$ and consider the following element in $\mathcal{T}^n$:

$$G = \bar{T} + \sum_{A \neq \{1\}} P^n_A,$$

where the sum is taken over all subsets of $\{1, \ldots, n\}$ except $\{1\}$. Then, we can see that $\text{Ker}(G) \cong \mathbb{C}$ and $\text{Coker}(G) = 0$, that is, $G$ is a Fredholm Toeplitz operator associated with codimension-$n$ corners whose Fredholm index is one.

This example leads to the following result.

**Proposition 4.6.** The boundary maps of the six-term exact sequence for $K$-theory associated with $[12]$ are surjective. Moreover, the boundary maps of the 24-term exact sequence for $KO$-theory associated with $[14]$ are surjective.

**Proof.** The result for complex $K$-theory is immediate from Example 4.5. For $KO$-theory, since the operator $G$ in Example 4.5 satisfies $G^* = G^*$, the result follows as in Proposition 3.9. □

Note that, we have $\gamma_n(G) = (\pi_1(G), 1, \ldots, 1) \in S^n$ by using

$$\pi_1(G) = M_z^* \otimes q \otimes \cdots \otimes q + \sum_{\emptyset \neq A \subset \{1, \ldots, n-1\}} 1_{C(T)} \otimes P^n_A,$$

The element $\gamma_n(G)$ is a unitary and defines an element $[\gamma_n(G)]$ of the group $K_1(S^n)$. Since the Fredholm index of $G$ is one, this gives a generator of $K_1(S^n) \cong \mathbb{Z}$. As in the proof of Proposition 4.4, generators of the $KO$-groups $KO_i(S^n, \tau_S)$ are also obtained by using $G$.

**Remark 4.7.** Let $1 \leq j \leq n$. We have the following $*$-homomorphisms:

$$S^n \rightarrow \mathcal{T}^{n-1} \otimes C(T) \xrightarrow{\gamma_n \otimes 1} S^n \otimes C(T),$$

where the first map maps $(T_1, \ldots, T_n)$ to $T_j$. We write $\sigma^{n,n-1}$ for the composite of the above maps which induces the map $\sigma^{n,n-1}: K_i(S^n) \to K_i(S^{n-1} \otimes C(T))$. When $i = 0$, $K_0(S^n) \cong \mathbb{Z}$ is generated by $[12]$ and $\sigma^{n,n-1}[12] = [12]$. When $i = 1$, the map $\sigma^{n,n-1}$ is zero since, by Example 4.5, the element $[\gamma_n(G)]$ is a generator of $K_1(S^n) \cong \mathbb{Z}$ and $\sigma^{n,n-1}[\gamma_n(G)] = [1] = 0$. A similar observation also holds in real cases. The map $\sigma^{n,n-1}$ from $KO_i(S^n, \tau_S)$ to $KO_i(S^{n-1} \otimes C(T), \tau)$ maps direct summands corresponding to $KO$-groups of a point injectively and the other components to zero.

5. Topological Invariants and Corner States in Altland–Zirnbauer Classification

In this section, some gapped Hamiltonians on a lattice with corners are discussed in each of the Altland–Zirnbauer classes. Since two of them (class A and AIII) are already studied in [32][33], we consider the remaining cases here. The codimension of the corner will be arbitrary, though we mainly discuss codimension-two...
cases, with many detailed results being obtained by [54, 35, 33] and the results in Sect. 5.5. Higher-codimensional cases are discussed in a similar way, whose results are collected in Sect. 5.5.

5.1. Setup. Let $V$ be a finite rank Hermitian vector space of complex rank $N$. Let $n$ be a positive integer. Let $\Theta$ and $\Xi$ be antunitary operators on $V$ whose squares are $+1$ or $-1$. Let $\Pi$ be a unitary operator on $V$ whose square is one. These operators $\Theta$, $\Xi$ and $\Pi$ are naturally extended to the operator on $L^2(\mathbb{Z}^n; V)$ by the fiberwise operation; we also denote them as $\Theta$, $\Xi$ and $\Pi$, respectively. Let $H$ be a continuous map defined by a bounded linear self-adjoint operator $\{H(t)\}_{t \in \mathbb{T}^{n-2}}$ on $\mathcal{H} \otimes V$.

By taking a compression onto $\mathcal{H}^{\alpha} \otimes V$, $\mathcal{H}^{\beta} \otimes V$ and $\mathcal{H}^{\alpha, \beta} \otimes V$, we obtain a family of operators $H^{\alpha}(t)$, $H^{\beta}(t)$ and $H^{\alpha, \beta}(t)$ parametrized by $t = (t_1, \ldots, t_n) \in \mathbb{T}^{n-2}$. $H^{\alpha}(t)$ and $H^{\beta}(t)$ are our models for two surfaces (codimension-one boundaries), and $H^{\alpha, \beta}(t)$ is our model of the corner (codimension-two corner). We assume the following spectral gap condition.

Assumption 5.1 (Spectral Gap Condition). We assume that both $H^{\alpha}$ and $H^{\beta}$ are invertible.

Under this assumption, the bulk Hamiltonian $H$ is also invertible since, when we take a basis of $V$ and identify $V$ with $\mathbb{C}^N$, there is a unital *-homomorphism $M_N(S^{\alpha, \beta} \otimes C(\mathbb{T}^{n-2})) \rightarrow M_N(C(\mathbb{T}^n))$ that maps $(H^{\alpha}, H^{\beta})$ to $H$. In classes AI and AII, we further assume that the spectrum of $H$ is not contained in either $\mathbb{R}_{>0}$ or $\mathbb{R}_{<0}$. Note that in other classes where Hamiltonians preserve PHS or chiral symmetry, this condition follows from Assumption 5.1. Let $h$ be the pair $(H^{\alpha}, H^{\beta})$. Under Assumption 5.1, we set

$$\text{sign}(h) = h|h|^{-1}. \quad (5.1)$$

When the bulk Hamiltonian $H$ satisfies TRS, PHS or chiral symmetry, the operators $H^{\alpha}$, $H^{\beta}$, $H^{\alpha, \beta}$ and $\text{sign}(h)$ also satisfy the symmetry, that is, commutes with $\Theta$ or anticommutes with $\Xi$ or $\Pi$.

5.2. Gapped Topological Invariants. In the following, starting from a Hamiltonian satisfying Assumption 5.1 in each class AI, BDI, D, DIII, AII, CII, C and CI, we construct a unitary and see that this unitary satisfies the relation $R_0$ in Table 2. By using this unitary, we define a topological invariant as an element of some $KO$-group.
In class AI, the Hamiltonian has even TRS. We take an orthonormal basis of $V$ to identify $V$ with $\mathbb{C}^N$ and express $\Theta$ as $\Theta = \text{diag}(c, \ldots, c)$. Under our spectral gap condition, let

\begin{equation}
(5.2) \quad u = \begin{pmatrix}
\text{sign}(h) & 0 \\
0 & 1_N
\end{pmatrix}.
\end{equation}

This $u$ is a self-adjoint unitary satisfying $u^* = \text{Ad}_{\mathbb{C}^\mathbb{C}}(u^*) = u^*$ by the TRS.

In class BDI, the Hamiltonian has both even TRS and even PHS. Note that the chiral symmetry is given by $\Pi = \Theta \Xi$ and commutes with $\Theta$ and $\Xi$. For a Hamiltonian satisfying chiral symmetry and Assumption 5.1 to exist, the even/odd decomposition $V \cong V^0 \oplus V^1$ with respect to $\Pi$ should satisfy $\text{rank}_\mathbb{C} V^0 = \text{rank}_\mathbb{C} V^1$, and we assume that. Then, there is an orthonormal basis of $V$ to identify $V$ with $\mathbb{C}^N$ such that $\Pi$ and $\Theta$ are expressed as follows:

\begin{equation}
(5.3) \quad \Pi = \begin{pmatrix}
1 & 0 \\
0 & -1
\end{pmatrix}, \quad \Theta = \begin{pmatrix}
\mathcal{C} & 0 \\
0 & \mathcal{C}
\end{pmatrix},
\end{equation}

where $\mathcal{C} = \text{diag}(c, \ldots, c)$. Since the Hamiltonian $H$ anticommutes with $\Pi$, the operator $\text{sign}(h)$ in (5.1) is written in the following off-diagonal form:

\begin{equation}
(5.4) \quad \text{sign}(h) = \begin{pmatrix}
0 & u^* \\
u & 0
\end{pmatrix},
\end{equation}

where $u$ is a unitary. By the TRS, we have $u^* = \mathcal{C} u^* \mathcal{C}^* = u^*$.

In class D, the Hamiltonian has even PHS. We take an orthonormal basis of $V$ to identify $V \cong \mathbb{C}^N$ and express $\Xi$ as $\Xi = \text{diag}(c, \ldots, c)$. Let $u = \text{sign}(h)$, then we have $u^* = \Xi u \Xi^* = -u$ by the PHS.

In class DIII, the Hamiltonian has both odd TRS and even PHS. Note that the chiral symmetry is given by $\Pi = i \Theta \Xi$ and anticommutes with $\Theta$ and $\Xi$. For such a Hamiltonian $H$ satisfying Assumption 5.1 to exist, the complex rank of $V$ must be a multiple of 4 since $\text{sign}(H(t))$, $i \Pi$, $\imath$ and $\Theta$ provides a $\text{Cl}_{1,1} \otimes \text{Cl}_{2,0} \cong \mathbb{H}(2)$-module structure on $V$. We assume $\text{rank}_\mathbb{C} V = 4M$ for some positive integer $M$.

**Lemma 5.2.** If a Hamiltonian $H$ satisfying Assumption 5.1 exists, there is an orthonormal basis of $V$ such that $\Pi$ and $\Theta$ are expressed as follows.

\begin{equation}
(5.5) \quad \Pi = \begin{pmatrix}
1 & 0 \\
0 & -1
\end{pmatrix}, \quad \Theta = \begin{pmatrix}
0 & \mathcal{J} \\
\mathcal{J} & 0
\end{pmatrix}.
\end{equation}

We write $\mathcal{J} = \text{diag}(j, \ldots, j)$, where $j$ is the quaternionic structure on $\mathbb{H}$.

**Proof.** By using $\Pi$, we decompose $V \cong V^0 \oplus V^1$. We identify $V^0 \cong V^1 \cong \mathbb{C}^{2M} \cong \mathbb{H}^M$, on which we consider $\mathcal{J} = \text{diag}(j, \ldots, j)$. Let $U = \Theta \begin{pmatrix}
0 & \mathcal{J} \\
\mathcal{J} & 0
\end{pmatrix}$. Since $U$ is a unitary and commutes with $\Pi$, we have $U = \text{diag}(u_0, u_1)$, where $u_0$ and $u_1$ are unitaries on $\mathbb{C}^{2M}$. Since $\Theta^2 = -1$, we have $u_1 = \text{Ad}_\mathcal{J}(u_0^*)$. Let $W = \text{diag}(-u_0, 1)$, then $W \Theta W^* = \begin{pmatrix}
0 & \mathcal{J} \\
\mathcal{J} & 0
\end{pmatrix}$. \hfill $\Box$

We take this basis on $V$ and express $\Pi$ and $\Theta$ as above. By the chiral symmetry, we take $u$ in (5.3). By the TRS, we have $u^{2 \otimes \tau} = \mathcal{J} u^* \mathcal{J}^* = u$.

In class AII, the Hamiltonian has odd TRS. The space $V$ has a quaternionic structure given by $\Theta$, and the complex rank of $V$ is even, for which we write $2M$. There is an orthonormal basis of $V$ for identifying $V$ with $\mathbb{C}^{2M} \cong \mathbb{H}^M$ and
Table 10. \(i(\clubsuit)\) and \(c(\clubsuit)\) for each of the Altland–Zirnbauer classes \(\clubsuit\).

| \(\clubsuit\) | AI | BDI | D | DIII | All | CII | C | CI |
|---|---|---|---|---|---|---|---|---|
| \(i(\clubsuit)\) | 0 | 1 | 2 | 3 | 4 | 5 | 6 | −1 |
| \(c(\clubsuit)\) | 1 | 1 | 1 | 1 | 1 | 1 | i | i |

expressing \(\Theta\) as \(\mathcal{J} = \text{diag}(j, \ldots, j)\). Let \(u\) be a self-adjoint unitary in \(\mathbb{C}^{N}\). By the TRS, we have \(u^{\otimes \tau} = \text{Ad}_{\mathcal{J} \otimes \mathcal{J}}(u^*) = u^*\).

In class CII, the Hamiltonian has both odd TRS and odd PHS. The chiral symmetry is given by \(\Pi = \Theta \Xi\) and commutes with \(\Theta\) and \(\Xi\). As in the class BDI case, we take an orthonormal basis of \(V\) to identify \(V\) with \(\mathbb{C}^{N}\) and express \(\Pi\) and \(\Theta\) as

\[
\Pi = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1 \\
\end{pmatrix}, \quad \Theta = \begin{pmatrix} \mathcal{J} & 0 \\
0 & \mathcal{J} \\
\end{pmatrix},
\]

where \(\mathcal{J} = \text{diag}(j, \ldots, j)\). By the chiral symmetry, we take \(u\) in (5.4). By the TRS, we have \(u^{\otimes \tau} = \mathcal{J} u^* \mathcal{J}^* = u^*\).

In class C, the Hamiltonian has odd PHS. Since \(\Xi\) provides a quaternionic structure on \(V\), its complex rank is even, for which we write \(2M\). We take an orthonormal basis of \(V\) to identify \(V\) with \(\mathbb{C}^{2M} \cong \mathbb{H}^{M}\) and express \(\Xi\) as \(\mathcal{J} = \text{diag}(j, \ldots, j)\). Let \(u = \text{sign}(h)\), then we have \(u^{\otimes \tau} = \mathcal{J} u^* \mathcal{J}^* = -u\) by the PHS.

In class CI, the Hamiltonian has both even TRS and odd PHS. The chiral symmetry is given by \(\Pi = i \Theta \Xi\) and anticommutes with \(\Theta\) and \(\Xi\). As in Lemma 5.2, we take an orthonormal basis of \(V\) to express,

\[
\Pi = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1 \\
\end{pmatrix}, \quad \Theta = \begin{pmatrix} 0 & C \\
C & 0 \\
\end{pmatrix},
\]

where \(C = \text{diag}(c, \ldots, c)\). By the chiral symmetry, we take \(u\) in (5.4). By the TRS, we have \(u^* = C u^* C^* = u\).

**Definition 5.3.** For a Hamiltonian in class \(\clubsuit\) = AI, BDI, D, DIII, All, CII, C or CI satisfying Assumption 5.1, let \(u\) be the unitary defined as above. As we have seen, this unitary \(u\) satisfies the relation \(R_i(\clubsuit)\) where \(i(\clubsuit)\) is as indicated in Table 10. We denote its class \([u]\) in the \(KO\)-group \(KO_i(\mathbb{S}^\alpha, \mathbb{S}^\beta \otimes C(T^n-2), \tau)\) by \(T_{\text{Gapped}}^{\mathbb{S}^\alpha, \mathbb{S}^\beta}(H)\).

**Remark 5.4.** We expressed the symmetry operators in a specific way, though we may choose another one. In class DIII, for example, the operator \(\Theta\) can also be expressed as \(\begin{pmatrix} 0 & -C \\
C & 0 \\
\end{pmatrix}\), where \(C = \text{diag}(c, \ldots, c)\). Then, we obtain unitaries satisfying \(u^* = -u\), which are treated in [38].

**5.3. Gapless Topological Invariants.** We next define another topological invariant by using our model for the corner \(H^{\alpha, \beta}\). By Assumption 5.1 and Theorem 2.6 in [51], \(\{H^{\alpha, \beta}(t)\}_{t \in T^n-2}\) is a continuous family of self-adjoint Fredholm operators. Corresponding to its Altland–Zirnbauer classes, this family provides a \(\mathbb{Z}_2\)-map from \((T^n-2, \zeta)\) to some \(\mathbb{Z}_2\)-spaces of self-adjoint or skew-adjoint Fredholm operators introduced in Appendix A as follows.

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9We simply write \(\tau\) in place of \(\tau_{\mathbb{S}^\alpha} \otimes \tau_{\mathbb{S}^\beta}\). In the following, these abbreviations for tensor products of transpositions are employed, though the meaning will be clear from the context.
Here, we write \( r \) for the \( \mathbb{Z}_2 \)-spaces of self-adjoint or skew-adjoint Fredholm operators is isomorphic to the \( KO \)-group \( KO_i(C(T^{n-2}), \tau_T) \) of some degree \( i \).

**Definition 5.5.** For \( \clubsuit \) = AI, BDI, D, DIII, AII, CII, C or CI, let \( i(\clubsuit), c(\clubsuit) \) and \( \text{Fred}^{\clubsuit} \) be numbers and the \( \mathbb{Z}_2 \)-space as in Tables 10 and 11. For a Hamiltonian \( H \) in class \( \clubsuit \) satisfying Assumption 5.1, we denote \( T^{n,2}_e(\clubsuit)(H) \) for the class \( [c(\clubsuit)\tilde{H}^{\alpha,\beta}] \) in \( KO_{i(\clubsuit)}+1(C(T^{n-2}), \tau_T) \). We call \( T^{n,2}_e(\clubsuit)(H) \) the gapless corner invariant.

If the gapless corner invariant is nontrivial, zero is contained in the spectrum of \( \tilde{H}^{\alpha,\beta} \). In Sect. 5.6, we discuss more refined relations between the gapless corner invariant and corner states when \( k = n - 1 \) and \( n \).

**Table 11.** In each Altland–Zirnbauer class \( \clubsuit \), gapped invariants and gapless invariants are defined as elements of some \( K \)- and \( KO \)-groups of some degree, as indicated in this table. Classifying spaces for topological \( K \)- and \( KR \)-groups through self-adjoint or skew-adjoint Fredholm operators and unitaries are also included. \( (\mathbb{Z}_2) \)-spaces \( \text{Fred}^{\clubsuit} \) and \( U^{\clubsuit} \) are introduced in Appendix A.

| AZ | Gapped | Gapless | U^{\clubsuit} |
|----|--------|---------|--------------|
|    | \( K_0 \) | \( K_1 \) | \( (\text{Fred}^{0,\bot}_* \), \( r_\Theta \) ) | \( U^{\text{cpt}} \) |
| A | KO_0 | KO_1 | \( (\text{Fred}^{0,\bot}_* \), \( r_\Theta \) ) | \( U^{\text{cpt}} \) |
| AI | KO_1 | KO_0 | \( (\text{Fred}^{0,\bot}_* \), \( r_\Theta \) ) | \( U^{\text{cpt}} \) |
| BDI | KO_2 | KO_3 | \( (\text{Fred}^{0,\bot}_* \), \( r_\Theta \) ) | \( U^{\text{cpt}} \) |
| D | KO_3 | KO_2 | \( (\text{Fred}^{0,\bot}_* \), \( r_\Theta \) ) | \( U^{\text{cpt}} \) |
| DIII | KO_0 | KO_4 | \( (\text{Fred}^{0,\bot}_* \), \( r_\Theta \) ) | \( U^{\text{cpt}} \) |
| AII | KO_5 | KO_6 | \( (\text{Fred}^{0,\bot}_* \), \( r_\Theta \) ) | \( U^{\text{cpt}} \) |
| CII | KO_0 | KO_5 | \( (\text{Fred}^{0,\bot}_* \), \( r_\Theta \) ) | \( U^{\text{cpt}} \) |
| C | KO_6 | KO_0 | \( (\text{Fred}^{0,\bot}_* \), \( r_\Theta \) ) | \( U^{\text{cpt}} \) |

**5.4. Correspondence.** By taking a tensor product of the extension \( 2.2 \) and \( (C(T^{n-2}), \tau_T) \), we have the following short exact sequence of \( C^* \)-algebras,

\[
0 \rightarrow (\mathcal{K} \otimes C(T^{n-2}), \tau) \rightarrow (T^{\alpha,\beta} \otimes C(T^{n-2}), \tau) \rightarrow (S^{\alpha,\beta} \otimes C(T^{n-2}), \tau) \rightarrow 0.
\]
Let consider the following diagram containing the boundary map of 24-term exact sequence for $KO$-theory associated with this sequence:

$$KO_i(\{S^{\alpha,\beta} \otimes C(\mathbb{T}^{n-2}), \tau\}) \xrightarrow{\partial_i(\bullet)} KO_i(\{\mathcal{K}(\hat{H}^{\alpha,\beta}) \otimes C(\mathbb{T}^{n-2}), \tau\})$$

$$\xrightarrow{\exp} [(\mathbb{T}^{n-2}, \zeta), \text{Fred}^\bullet]_{\mathbb{Z}_2} \cong [(\mathbb{T}^{n-2}, \zeta), \text{Fred}^\bullet]_{\mathbb{Z}_2}$$

where $F^\bullet$ is the $\mathbb{Z}_2$-subspace of Fred$^\bullet$ as in Appendix A whose inclusion $F^\bullet \hookrightarrow \text{Fred}^\bullet$ is the $\mathbb{Z}_2$-homotopy equivalence. Maps $L$ and $\exp$ are as follows.

- When $i(\bullet) = 0, 4$, for $[u] \in KO_i(\{S^{\alpha,\beta} \otimes C(\mathbb{T}^{n-2}), \tau\)$, we take a self-adjoint lift $a$ of $u$ as in Definition A.13 and set $L([u]) = [A]$. The map $\exp$ is defined as in Definition A.13.
- When $i(\bullet) = 2, 6$, for $[u] \in KO_i(\{S^{\alpha,\beta} \otimes C(\mathbb{T}^{n-2}), \tau\)$, we take a self-adjoint lift $a$ of $u$ as in Definition 8.3 of [14] and set $L([u]) = [a]$. The map $\exp$ is defined by $\exp([a']) = [-\exp(\pi a')]$.

In each case, the map $\exp$ is an isomorphism by Proposition A.7 and Sect. A.2. For boundary maps $\partial_i(\bullet)$, we use its expressions through exponentials (see [14] for even $i(\bullet)$ and Appendix A.3 for odd $i(\bullet)$) and the diagram commutes. Note that, by Proposition 5.9 the boundary map $\partial_i(\bullet)$ is surjective. The following is the main result of this section.

**Theorem 5.6.** $\partial_i(\bullet)(\mathcal{I}_{\text{Gapped}}^{n,2}(H)) = \mathcal{I}_{\text{Gapless}}^{n,2}(H)$.

**Proof.** The operator $\hat{H}^{\alpha,\beta}$ is a self-adjoint lift of $(H^\alpha, H^\beta)$ and preserves the symmetries of the class $\bullet$. Therefore, we have $L(\mathcal{I}_{\text{Gapped}}^{n,2}(H)) = [c(\bullet)H^{\alpha,\beta}]$ and the results follows from the commutativity of the above diagram. \(\square\)

**Remark 5.7 (Relation with bulk weak invariants).** Under Assumption 5.1 the bulk Hamiltonian $H$ is also invertible. When we take $H$ in place of $h = (H^\alpha, H^\beta)$ and define the unitary $u'$ as in Sect. 5.2 this unitary defines an element $[u']$ in $KO_i(\{C(\mathbb{T}^n), \tau_T\)$, which classifies bulk invariants in class $\bullet$. A relation between our gapped invariants $\mathcal{I}_{\text{Gapped}}^{n,2}(H)$ and these bulk invariants can be discussed through the map $(\sigma \otimes 1)_*: KO_i(\{S^{\alpha,\beta} \otimes C(\mathbb{T}^{n-2}), \tau\) \rightarrow KO_i(\{C(\mathbb{T}^{n-2}), \tau_T\)$, which maps $[u]$ to $[u']$ and we briefly mention its consequences here. Our gapped invariant $\mathcal{I}_{\text{Gapped}}^{n,2}(H)$ has no relation with bulk invariants in the sense that, under Assumption 5.1 bulk invariants are trivial except for a component corresponding to $KO$-groups of a point$^{10}$ and for the cases when $\alpha, \beta$ are both rational (or $\pm \infty$) and $t = -ps + qr$ is even. By Remark 5.10 when $\alpha$ and $\beta$ takes these values, some bulk weak invariants can be nontrivial, though they have no relation with $\mathcal{I}_{\text{Gapless}}^{n,2}(H)$, which can be seen by comparing the above map $(\sigma \otimes 1)_*$ and the boundary map $\partial_i(\bullet)$.

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10This component maps to zero by the boundary map $\partial_i(\bullet)$ and has no relation with gapless corner invariants.
Definition 5.11.  

Remak 5.8 (Convex and concave corners). When we fix $\alpha$ and $\beta$, there exist two models of corners: convex and concave corners $(\mathcal{H}^{\alpha,\beta}$ and $\mathcal{H}^{\alpha,\beta})$. We have discussed convex corners though, as in $[33]$, similar results also hold for concave corners by using $(2.2)$ in place of $(2.2)$ in our discussion. By Remark 5.11, the gapless invariants of these two are related by the factor $-1$.

5.5. Higher-Codimensional Cases. Let $n$ and $k$ be positive integers satisfying $3 \leq k \leq n$. In this subsection, we consider $n$-D system with a codimension-$k$ corner. Let $d = n-k$. We consider a continuous map $T^n \to \text{Herm}(V)$ and the bounded linear self-adjoint operator $H$ on $l^2(Z^n)$ generated by this map, which is our model of the bulk. We next introduce models of corners of codimension $k-1$ whose intersection makes a codimension $k$ corner. For this, we choose $d$ variables $t_{i1}, \ldots, t_{id}$ in $t_1, t_2, \ldots, t_n$ and consider the partial Fourier transform in these $d$ variables to obtain a continuous family of self-adjoint operators $(H(t))_{t \in T^d}$ on $l^2(Z^k; V)$. On the Hilbert space $l^2(Z^k; V) = (l^2(Z) \otimes \cdots \otimes l^2(Z)) \otimes V$, we consider projections $P_i = (P_{\geq 0} \cdots \otimes P_{\geq 0}) \otimes 1_V$, and $P_{k,i} = (P_{\geq 0} \cdots \otimes 1 \cdots \otimes P_{\geq 0}) \otimes 1_V$ for $1 \leq i \leq k$ where inside the brackets is the tensor products of $P_{\geq 0}$ except for the $i$-th tensor product replaced by the identity. By using these projections, we define the following operators:

$$H^c(t) = P_i H(t) P_{k,i}, \quad H_i(t) = P_{k,i} H(t) P_{k,i},$$

for $1 \leq i \leq k$ and for $t \in T^d$. These two are our model for a codimension $k$ corner and codimension $k-1$ corners, respectively. When we fix a basis on $V$, we have $(H_1(t), \ldots, H_k(t)) \in M_N(S^k)$ by the construction. We assume the following condition in this subsection.

Assumption 5.9 (Spectral Gap Condition). We assume that our models for codimension $k-1$ corners $H_1, \ldots, H_k$ are invertible.

Under this assumption, the model for the bulk, surfaces and corners of codimension less than $k$, whose intersection makes our codimension-$k$ corner, are invertible. As in Sect. 5.1 let $h = (H_1, \ldots, H_k)$.

Definition 5.10. For a Hamiltonian in class $\bullet = \text{AI}, \text{BDI}, \text{DIII}, \text{AII}, \text{CII}, \text{C}$ or CI satisfying Assumption 5.9, let $u$ be a unitary defined by using this $h$ in place of that in Sect. 5.2. As in Sect. 5.2, this unitary $u$ satisfies the relation $R_i(\bullet)$ where $i(\bullet)$ is as indicated in Table 10. We denote its class $[u]$ in the $KO$-group $KO_i([\bullet])(S^k \otimes C(T^d), \tau)$ by $T^{n,k}_{\text{Gapless}}(H)$.

The $KO$-groups $KO_i(S^k \otimes C(T^d), \tau)$ are computed by using Proposition 4.3. For each $t \in T^d$, the operator $H^c(t)$ is Fredholm by Corollary 1.2.

Definition 5.11. For $\bullet = \text{AI}, \text{BDI}, \text{DIII}, \text{AII}, \text{CII}, \text{C}$ or CI, let $i(\bullet)$, $c(\bullet)$ and $Fred^\bullet$ be numbers and the $Z_2$-space as in Tables 10 and 11. For a Hamiltonian $H$ in class $\bullet$ satisfying Assumption 5.9, we denote $T^{n,k}_{\text{Gapless}}(H)$ for the class $[c(\bullet)H^c]$ in $KO_i([\bullet])[-1](C(T^d), \tau_\tau)$. We call $T^{n,k}_{\text{Gapless}}(H)$ the gapless corner invariant.

We next discuss a relation between these two topological invariants. As in Sect. 5.4 we consider a tensor product of the extension $(4.1)$ and $(C(T^d), \tau_\tau)$ and let $\partial_i(\bullet) : KO_i([\bullet])(S^k \otimes C(T^d), \tau) \to KO_i([-1](k^k \otimes C(T^d), \tau)$ be the boundary map associated with it expressed through exponentials. Since $H^c$ is a self-adjoint lift of $(H_1, \ldots, H_k)$, the following relation holds, as in Theorem 5.6.
Theorem 5.12. \( \partial |\bullet\rangle (T_{\text{Gapped}}^n, k, \bullet (H)) = T_{\text{Gapped}}^n, k, \bullet (H) \).

Remark 5.13. As in Remark 5.7 under Assumption 5.9 some gapped invariants related to corner states for corners of codimension \( k \) are also defined, though, by Remark 4.7 they are trivial except for a component corresponding to \( KO \)-groups of a point.

Remark 5.14. Gapless corner invariants for each systems are elements of the group \( KO_1(C(S^d)), \tau_0) \cong \bigoplus_{j=0}^d (T_j^d)KO_i(-j)(\mathbb{C}, id) \). As in the case of (first-order) topological insulators \( \mathbb{C} \), we call the component \( KO_{1-d}(\mathbb{C}, id) \) strong and others weak.

Complex cases can also be discussed in a similar way. For class A systems with a codimension \( \geq 3 \) corner, under the Assumption 5.11 we define gapped and gapless invariants as elements of \( KO_0(S^k \otimes C(T^d)) \) and \( KO_0(C(T^d)) \), respectively, and the boundary map \( \partial_0 : KO_0(S^k \otimes C(T^d)) \rightarrow KO_0(C(T^d)) \) associated with (1.3) relates these two, which is surjective by Proposition 4.6. In class AIII systems, we use \( \partial_1 : KO_0(S^k \otimes C(T^d)) \rightarrow KO_0(S^k \otimes C(T^d)) \) instead. Gapless corner invariants takes value in \( KO_0(C(T^d)) \cong \bigoplus_{j=0}^d (T_j^d)K_{0-j}(\mathbb{C}) \), and we call the component \( K_{1-d}(\mathbb{C}) \) strong and others weak.

Strong invariants for each system are classified in Table 1.

5.6. Numerical Corner Invariants. Our gapless corner invariants are defined as elements of some \( KO \)-group. In this subsection, we introduce \( \mathbb{Z} \) - and \( \mathbb{Z}_2 \)-valued numerical corner invariants for our systems in cases where \( k = n \) and \( k = n - 1 \) to make the relation between our gapped invariants and corner states more explicit.

From Table 1 we discuss Hamiltonians in classes BDI, D, DIII, and CII when \( k = n \) and D, DIII, AII and CII when \( k = n - 1 \) satisfying our spectral gap condition.

5.6.1. Case of \( k = n \). In this case, our model of the corner \( H_c \) is a self-adjoint Fredholm operator which has some symmetry corresponding to its Altland--Zirnbauer class. An appropriate definition of numerical topological invariants is introduced in \( \mathbb{Z} \) and we put them in our framework.

In class BDI, the operator \( H_c \) is an element of the fixed point set \( (\text{Fred}^{(0, \mathbb{C})}_{\bullet})_{\tau_0} \) of the involution \( \tau_0 \), where the Clifford action of \( C_{1,2} \) on the Hilbert space is given by \( \epsilon_1 = \Pi \) (see also Lemma A.10 and Remark A.11). We express \( \Pi \) and \( \Theta \) as in (5.10) and express \( H_c \) as follows.

\[
H_c = \begin{pmatrix}
0 & (h^c)^* \\
h^c & 0
\end{pmatrix}.
\]

The operator \( h^c \) is a Fredholm operator that commutes with \( \mathcal{C} \) and thus is a real Fredholm operator. Its Fredholm index is

\[
\text{ind}(h^c) = \text{rank}_\mathbb{C} \text{Ker}(h^c) - \text{rank}_\mathbb{C} \text{Coker}(h^c) = \text{Tr}(\Pi|_{\text{Ker}(H_c)}),
\]

where the right-hand side is the trace of \( \Pi \) restricted to \( \text{Ker}(H_c) \). The Fredholm index induces an isomorphism \( \text{ind}^{\text{BDI}} : \{\text{pt}, \text{id}, (\text{Fred}^{(0, \mathbb{C})}_{\bullet}, \tau_0)\}|_{\mathbb{Z}_2} \rightarrow \mathbb{Z} \).

\footnote{In \[32, 33\], there is a mistake in the computations of the group \( KO_0(S^\alpha, \beta) \) in the case where \( \alpha \) and \( \beta \) are rational numbers (there is a torsion part in general, as in \( KO_0(S^\alpha, \beta, \tau_S) \) computed in Sect. 4), which is correctly stated in \[32\]. The author would like to thank Guo Chuan Thiang for pointing this mistake out.}

\footnote{In what follows, we also write \( H^c \) for \( H^0, \beta \) in \( k = 2 \) case.}
In class D, \(iH^c\) commutes with the real structure \(\Xi\) and is a real skew-adjoint Fredholm operator. Its mod 2 index \([9]\) is
\[
\text{ind}_1(iH^c) = \text{rank}_{\mathbb{C}} \text{Ker}(H^c) \mod 2,
\]
which induces the isomorphism \(\text{ind}^D : [(\text{pt}, \text{id}), (\text{Fred}^{(1,0)}, \mathbb{C}^2)]_{\mathbb{Z}_2} \to \mathbb{Z}_2\).

In class DIII, \(H^c\) is an element of \((\text{Fred}^{(1,0)})^{{\mathbb{C}^2}},\) where the action of \(C_{1,0}\) is given by \(e_1 = i\). The operator \(iH^c\) and \(e_1\) commute with the real structure \(\Xi;\) thus, \(iH^c\) is a real skew-adjoint Fredholm operator that anticommutes with \(e_1\). Its mod 2 index \([9]\) is
\[
\text{ind}_2(iH^c) = \frac{1}{2} \text{rank}_{\mathbb{C}} \text{Ker}(H^c) \mod 2,
\]
which induces the isomorphism \(\text{ind}^{\text{DIII}} : [(\text{pt}, \text{id}), (\text{Fred}^{(1,0)}, \mathbb{C}^2)]_{\mathbb{Z}_2} \to \mathbb{Z}_2\).

In class CII, the operator \(H^c\) is an element of \((\text{Fred}^{(0,2)})^{{\mathbb{C}^2}},\) where the Clifford action of \(C_{0,1}\) is given by \(e_1 = \Pi\). We express \(\Theta\) and \(\Pi\) as in \((5.5)\) and express \(H^c\) as in \((5.6)\). The operator \(h^c\) commutes with \(J\) and is a quaternionic Fredholm operator. Its Fredholm index \(\text{ind}(h^c)\) is an even integer that induces an isomorphism \(\text{ind}^{\text{CII}} : [(\text{pt}, \text{id}), (\text{Fred}^{(0,2)}, \mathbb{Q}_2)]_{\mathbb{Z}_2} \to \mathbb{Z}_2\).

**Definition 5.15.** For \(n\)-D systems with codimension-\(n\) corners in classes BDI, D, DIII and CII, we define the **numerical corner invariant** as follows.

- In class BDI, let \(N_{\text{Gapless}}^{n,n,\text{BDI}}(H) = \text{ind}(h^c) \in \mathbb{Z}\).
- In class D, let \(N_{\text{Gapless}}^{n,n,\text{D}}(H) = \text{ind}_1(iH^c) \in \mathbb{Z}_2\).
- In class DIII, let \(N_{\text{Gapless}}^{n,n,\text{DIII}}(H) = \text{ind}_2(iH^c) \in \mathbb{Z}_2\).
- In class CII, let \(N_{\text{Gapless}}^{n,n,\text{CII}}(H) = \text{ind}(h^c) \in \mathbb{Z}_2\).

Note that by these definitions, they are images of gapless corner invariants \(T_{\text{Gapless}}^{n,n,\text{BDI}}(H)\) for each class \(\blacklozenge = \text{BDI}, \text{D}, \text{DIII} \text{and CII}\) through the isomorphism \(\text{ind}^{\blacklozenge}\). In each case, the numerical corner invariant is computed through \(\text{Ker}(H^c)\) and is related to the number of corner states.

5.6.2. **Case of** \(k = n - 1\). In this case, \(\{H^c(t)\}_{t \in \mathbb{T}}\) is a continuous family of self-adjoint Fredholm operators preserving some symmetry. The numerical corner invariants are given by using \((\mathbb{Z}\text{-valued})\) spectral flow \([8]\) and its \(\mathbb{Z}_2\)-valued variants \([19, 18]\). We first review \(\mathbb{Z}\)- and \(\mathbb{Z}_2\)-valued spectral flow.

Spectral flow is, roughly speaking, the net number of crossing points of eigenvalues of a continuous family of self-adjoint Fredholm operators with zero \([8]\). The following definition of spectral flow is due to Phillips \([50]\).

**Definition 5.16** (Spectral flow). Let \(A : [-1, 1] \to \text{Fred}^{(0,1)}\) be a continuous map. We choose a partition \(-1 = s_0 < s_1 < \cdots < s_n = 1\) and positive numbers \(c_1, c_2, \ldots, c_n\) so that for each \(i = 1, 2, \ldots, n\), the function \(t \mapsto \chi_{[-c_i, c_i]}(A_t)\) is continuous and finite rank on \([s_{i-1}, s_i]\), where \(\chi_{[a,b]}\) is the characteristic function of \([a,b]\). We define the **spectral flow** of \(A\) as follows.
\[
\text{sf}(A) = \sum_{i=1}^{n} (\text{rank}_c(\chi_{[0,c_i]}(A_{s_i})) - \text{rank}_c(\chi_{[0,c_i]}(A_{s_{i-1}}))) \in \mathbb{Z}.
\]

Spectral flow is independent of the choice made and depends only on the homotopy class of the path \(A\) leaving the endpoints fixed. Thus the spectral flow induces a map \(\text{sf} : [\mathbb{T}, \text{Fred}^{(0,1)}] \to \mathbb{Z}\) which is a group isomorphism.
We next discuss $\mathbb{Z}_2$-valued spectral flow. Let $\zeta_0$ be an involution on the interval $[-1, 1]$ given by $\zeta_0(s) = -s$. Let $A$ be a $\mathbb{Z}_2$-map from $([-1, 1], \zeta_0)$ to $(\text{Fred}^{(0, \Delta)}_1, q)$. Then, the spectrum $\text{sp}(A_s)$ of $A_s$ is symmetric with respect to $\zeta_0$, and roughly speaking, $\mathbb{Z}_2$-valued spectral flow counts the mod 2 of the net number of pairs of crossing points of $\text{sp}(A_s)$ with zero. $\mathbb{Z}_2$-valued spectral flow is studied in [19, 18, 21, 15] and we give one definition following [56, 19].

**Definition 5.17 ($\mathbb{Z}_2$-Valued Spectral Flow).** Let $A: ([1, 0], \zeta_0) \to (\text{Fred}^{(0, \Delta)}_1, q)$ be a $\mathbb{Z}_2$-map. We choose a partition $0 = s_0 < s_1 < \cdots < s_n = 1$ of $[0, 1]$ and positive numbers $c_1, c_2, \ldots, c_n$ so that for each $i = 1, 2, \ldots, n$, the map $t \mapsto \chi_{[-c_i, c_i]}(A_s)$ is continuous and finite rank on $[s_{i-1}, s_i]$. We define the $\mathbb{Z}_2$-valued spectral flow $sf_2(A)$ of $A$ as follows.

$$sf_2(A) = \sum_{i=1}^n (\text{rank}_C(\chi_{[0, c_i]}(A_{s_i})) + \text{rank}_C(\chi_{[0, c_i]}(A_{s_{i-1}}))) \mod 2 \in \mathbb{Z}_2.$$  

$\mathbb{Z}_2$-valued spectral flow is independent of the choice made and depends only on the $\mathbb{Z}_2$-homotopy class of the $\mathbb{Z}_2$-map $A$ leaving the endpoints fixed or leaving these points in the $\mathbb{Z}_2$-fixed point set $(\text{Fred}^{(0, \Delta)}_1)^A$. Thus $\mathbb{Z}_2$-valued spectral flow induces a group homomorphism $sf_2: [T, \zeta_0], (\text{Fred}^{(0, \Delta)}_1, q)]_{\mathbb{Z}_2} \to \mathbb{Z}_2$. By Appendix [A] the $\mathbb{Z}_2$-homotopy classes $[(T, \zeta_0), (\text{Fred}^{(0, \Delta)}_1, q)]_{\mathbb{Z}_2}$ is isomorphic to $KO_3(C(T), \tau_T) \cong \mathbb{Z}_2$.

**Example 5.18.** On $\mathbb{C}^2$, let consider a family of self-adjoint operators given by $B_s = \text{diag}(s, -s)$ for $s \in [-1, 1]$, and an antiunitary $j$ given by $j(x, y) = (\bar{y}, \bar{x})$. Then, we have a $\mathbb{Z}_2$-map $B: ([1, 0], \zeta_0) \to (M_2(\mathbb{C}), \text{Ad}_j)$ whose $\mathbb{Z}_2$-valued spectral flow $sf_2(B)$ is one. We extend this finite-dimensional example to an infinite-dimensional one to give an example of a family parametrized by the circle of nontrivial $\mathbb{Z}_2$-valued spectral flow. Let $\mathcal{V}$ be a separable infinite-dimensional complex Hilbert space equipped with a quaternionic structure $q$. On $\mathcal{V}' = \mathbb{C}^2 \oplus \mathcal{V} \oplus \mathcal{V}$, we consider a quaternionic structure $q' = j \oplus q \oplus q$ and a family self-adjoint Fredholm operators given by $C_s = \text{diag}(B_s, 1_V, -1_V)$. Let $U_*^{(0, \Delta)}(\mathcal{V}')$ the space of unitaries on $\mathcal{V}'$ whose spectrum is $\{ \pm 1 \}$ equipped with the norm topology. Then, its endpoints $C_{\pm 1}$ are contained in $U_*^{(0, \Delta)}(\mathcal{V}')$. Through an identification $(\mathcal{V} \oplus \mathcal{V}, q \oplus q) \cong (\mathcal{V}', q')$, the operator $\text{diag}(1_V, -1_V)$ gives an element $v_0 \in U_*^{(0, \Delta)}(\mathcal{V}')$ which satisfies $\text{Ad}_{q'}(v_0) = v_0$. The space $U_*^{(0, \Delta)}(\mathcal{V}')$ is homeomorphic to the homogeneous space $U(\mathcal{V}')/U(\mathbb{C} \oplus \mathcal{V}) \times U(\mathcal{V} \oplus \mathcal{V})$, which is contractible by Kuiper’s theorem [45]. Thus, there is a path $l: [0, 1] \to U_*^{(0, \Delta)}(\mathcal{V}')$ whose endpoints are $l(0) = v_0$ and $l(1) = C_1$. We extend $l$ to a $\mathbb{Z}_2$-map $l': ([1, 0], \zeta_0) \to (U^{(0, \Delta)}_1, \text{Ad}_j)$ by $l'(s) = \text{Ad}_{q'}(l(-s))$ for $s \in [-1, 0]$. Since $l'(\pm 1) = C_{\pm 1}$, we connect the endpoints of $C$ and $l'$ to construct a $\mathbb{Z}_2$-map $C': (T, \zeta) \to (\text{Fred}^{(0, \Delta)}_1, q')$, where $q' = \text{Ad}_{q'}$. Then, $sf_2(C') = sf_2(B) = 1$.

In class D, we have a $\mathbb{Z}_2$-map $iH^c: (T, \zeta) \to (\text{Fred}^{(0, \Delta)}_1, r_{\mathbb{Z}})$. The $\mathbb{Z}_2$-homotopy classes $[(T, \zeta), (\text{Fred}^{(0, \Delta)}_1, r_{\mathbb{Z}})]_{\mathbb{Z}_2}$ is isomorphic to $KO_1(C(T), \tau_T) \cong \mathbb{Z}_2 \oplus \mathbb{Z}$. By forgetting the $\mathbb{Z}_2$-actions and multiplying by $-i$, there is a map from $[(T, \zeta), (\text{Fred}^{(0, \Delta)}_1, r_{\mathbb{Z}})]_{\mathbb{Z}_2}$ to $[T, \text{Fred}^{(0, \Delta)}_1]$. Combined with this map and the spectral flow $sf: [T, \text{Fred}^{(0, \Delta)}_1] \to \mathbb{Z}$, we obtain a homomorphism

$$sf^D: [(T, \zeta), (\text{Fred}^{(0, \Delta)}_1, r_{\mathbb{Z}})]_{\mathbb{Z}_2} \to \mathbb{Z}, \quad [A] \mapsto sf(-iA).$$

**Example 5.19.** Let $B': ([1, 0], \zeta_0) \to (\mathbb{C}, \text{Ad}_c)$ be a $\mathbb{Z}_2$-map defined by $B_s = is$. Then, $sf^D(B')$ is defined and $sf^D(B') = 1$. 


In class DIII, $H^c: (\mathbb{T}, \zeta) \to (\text{Fred}_{1,1}^0, q_\Theta)$ is a $\mathbb{Z}_2$-map, where the action of the Clifford algebra on the right-hand side is given by $e_1 = i\Pi$. The $\mathbb{Z}_2$-homotopy classes $[(\mathbb{T}, \zeta), (\text{Fred}_{1,1}^0, q_\Theta)]_{\mathbb{Z}_2}$ is isomorphic to $KO_2(C(\mathbb{T}), \tau_\gamma) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$. Since $(\text{Fred}_{1,1}^0, q_\Theta)$ is a $\mathbb{Z}_2$-subspace of $(\text{Fred}^0_{0,1}, q_\Theta)$, the inclusion induces a map from $[(\mathbb{T}, \zeta), (\text{Fred}_{1,1}^0, q_\Theta)]_{\mathbb{Z}_2}$ to $[(\mathbb{T}, \zeta), (\text{Fred}^0_{0,1}, q_\Theta)]_{\mathbb{Z}_2}$. Combined with the $\mathbb{Z}_2$-valued spectral flow, we obtain the following map:

$$\text{sf}^\text{DIII}: [(\mathbb{T}, \zeta), (\text{Fred}_{1,1}^0, q_\Theta)]_{\mathbb{Z}_2} \to \mathbb{Z}_2, \quad [A] \mapsto \text{sf}_2(A).$$

For $b = 1$ or $-1$, let $i_b$ be the inclusion $\{b\} \hookrightarrow \mathbb{T}$, and let $w_b$ be the composite of the following maps:

$$[(\mathbb{T}, \zeta), (\text{Fred}_{1,1}^0, q_\Theta)]_{\mathbb{Z}_2} \xrightarrow{i_b} [(\{\pm 1\}, \text{id}), (\text{Fred}_{1,1}^0, q_\Theta)]_{\mathbb{Z}_2} \xrightarrow{\text{id}_2} \mathbb{Z}_2.$$

**Example 5.20.** Let $j, \mathcal{V}, \mathcal{V}', q, q', B_s$ and $C_s$ be as in Example 5.18. Let $e_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$ and $e'_1 = \begin{pmatrix} 0 & -1 \nu \\ 1 \nu & 0 \end{pmatrix}$ which gives a $\mathbb{C}l_{1,0}$-module structure on $\mathbb{C}^2$ and $\mathcal{V} \oplus \mathcal{V}'$, respectively. Then, $C_s = \text{diag}(B_s, 1, -1)$ gives a $\mathbb{Z}_2$-map from $([-1, 1], \zeta_0)$ to $(\text{Fred}_{1,1}^0, q')$. The operator $C_1$ is contained in the space of self-adjoint unitaries on $\mathcal{V}'$ that anticommutes with $e_1 \oplus e'_1$. As in [9], this space of unitaries is contractible by Kuiper’s theorem. We embed $[-1, 1]$ into $\mathbb{T}$ by $s \mapsto \exp(\frac{\pi i s}{2})$ and, as in Example 5.18 extend $C$ onto $\mathbb{T}$ through this contractible space of unitaries to obtain a $\mathbb{Z}_2$-map $D: (\mathbb{T}, \zeta) \to (\text{Fred}_{1,1}^0, q')$. For this example, we have $\text{sf}^\text{DIII}(D) = \text{sf}^\text{DIII}(B) = 1, w_+(D) = 1$ and $w_-(D) = 0$. If we take $D'$ as $D'_b = D_{-b}$, then $D'$ is also such a $\mathbb{Z}_2$-map and its invariants are $\text{sf}^\text{DIII}(D') = 1$, $w_+(D') = 0$ and $w_-(D') = 1$.

In class AII, $H^c: (\mathbb{T}, \zeta) \to (\text{Fred}^0_{1,1}, q_\Theta)$ is a $\mathbb{Z}_2$-map and its $\mathbb{Z}_2$-valued spectral flow is defined. We denote $\text{sf}^\text{AII}$ for $\text{sf}_2$.

In class C, we have a $\mathbb{Z}_2$-map $iH^c: (\mathbb{T}, \zeta) \to (\text{Fred}^0_{1,1}, q_\Xi)$. Note that the $\mathbb{Z}_2$-homotopy classes $[(\mathbb{T}, \zeta), (\text{Fred}^0_{1,1}, q_\Xi)]_{\mathbb{Z}_2}$ is isomorphic to $KO_2(C(\mathbb{T}), \tau_\gamma) \cong \mathbb{Z}$. By forgetting the $\mathbb{Z}_2$-actions and multiplying by $-i$, there is a map from $[(\mathbb{T}, \zeta), (\text{Fred}^0_{1,1}, q_\Xi)]_{\mathbb{Z}_2}$ to $[\mathbb{T}, \text{Fred}^0_{1,1}]$. Combined with the spectral flow, we obtain a homomorphism

$$\text{sf}^C: [(\mathbb{T}, \zeta), (\text{Fred}^0_{1,1}, q_\Xi)]_{\mathbb{Z}_2} \to 2\mathbb{Z}, \quad [A] \mapsto \text{sf}(-iA).$$

Note that image the image of $\text{sf}^C$ are even integers since each eigenspace corresponding to the crossing points of the spectrum of $-iA_i$ with zero has a quaternionic vector space structure given by $\Xi$.

**Example 5.21.** For $s \in [-1, 1]$, let $B'_s = \text{diag}(is, is)$, and let $j$ be the quaternionic structure in Example 5.18. Then, $B': (-[1, 1], \zeta) \to (M_2(\mathbb{C}), \text{Ad}_j)$ is a $\mathbb{Z}_2$-map, and we have $\text{sf}^C(B') = \text{sf}(-iB') = 2$.

**Lemma 5.22.**

1. $\text{sf}^\text{AII}: [(\mathbb{T}, \zeta), (\text{Fred}^0_{1,1}, q)]_{\mathbb{Z}_2} \to \mathbb{Z}_2$ is an isomorphism.
2. $\text{sf}^\text{B}: [(\mathbb{T}, \zeta), (\text{Fred}^0_{1,1}, q)]_{\mathbb{Z}_2} \to \mathbb{Z}$ is surjective.
3. $\text{sf}^\text{DIII}, w_+, w_-: [(\mathbb{T}, \zeta), (\text{Fred}^0_{1,1}, q)]_{\mathbb{Z}_2} \to \mathbb{Z}_2$ are surjective.
4. $\text{sf}^C: [(\mathbb{T}, \zeta), (\text{Fred}^0_{1,1}, q)]_{\mathbb{Z}_2} \to 2\mathbb{Z}$ is an isomorphism.

**Proof.** It is sufficient to find examples of $\mathbb{Z}_2$-maps which maps to generators of $\mathbb{Z}$, $\mathbb{Z}_2$ and $2\mathbb{Z}$. Therefore, (1) and (3) follows from Example 5.18 and Example 5.20 For
(2) and (4), we can construct such examples from Example 5.19 and Example 5.21 as in Example 5.18.

In class DIII cases, we have three surjections $sf^{\text{DIII}}$, $w_+$ and $w_-$ from $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ to $\mathbb{Z}_2$. There is the following relation between them.

**Lemma 5.23.** $sf^{\text{DIII}} = w_+ + w_-$.  

**Proof.** Let $D$ and $D'$ be $\mathbb{Z}_2$-maps in Example 5.20. Let $D'' = D \oplus D'$, then we have $sf^{\text{DIII}}(D'') = 0$, $w_+(D'') = 1$ and $w_-(D'') = 1$. Invariants $sf^{\text{DIII}}$, $w_-$ and $w_+$ for $D$, $D'$ and $D''$ tell that non-trivial three elements in the group $([T, \zeta], (\text{Fred}^{[1, \mathcal{U}], q})\mathbb{Z}_2)$ consists of classes of $D$, $D'$ and $D''$. Therefore, we computed three maps $sf^{\text{DIII}}$, $w_-$ and $w_+$, from which the result follows. □

**Remark 5.24.** For our class DIII systems, $\mathbb{Z}_2$-valued spectral flow counts the strong invariant. This corresponds to one direct summand of $\mathbb{Z}_2 \oplus \mathbb{Z}_2$, while the other corresponds to a weak invariant. When $w_+ \neq w_-$, the strong invariant is nonzero. When $w_+ = w_- = 1$, the strong invariant is zero and the weak invariant is nonzero. When $w_+ = w_- = 0$, both of them are zero.

**Definition 5.25.** For $n$-D systems with codimension $n - 1$ corners in classes D, DIII, AI and C, we define the numerical corner invariant as follows.

- In class D, let $N^{\text{Gapless}}_{n,n-1,D}(H) = sf(H^c) \in \mathbb{Z}$.
- In class DIII, let $N^{\text{Gapless}}_{n,n-1,\text{DIII}}(H) = sf_2(H^c) \in \mathbb{Z}_2$.
- In class AI, let $N^{\text{Gapless}}_{n,n-1,\text{AI}}(H) = sf_2(H^c) \in \mathbb{Z}_2$.
- In class C, let $N^{\text{Gapless}}_{n,n-1,C}(H) = sf(H^c) \in 2\mathbb{Z}$.

For each of the above classes $\spadesuit$, the numerical invariant $N^{\text{Gapless}}_{n,n-1,\spadesuit}(H)$ is the image of the gapless corner invariant $T^{n,n-1,\spadesuit}_{\text{Gapless}}(H)$ through the map $sf_{\spadesuit}$. These numerical invariants account for strong invariants.

**Remark 5.26.** In Definition 5.25, the numerical corner invariants for both class DIII and class AI are defined by using $\mathbb{Z}_2$-valued spectral flow, though these two $\mathbb{Z}_2$ are different from the viewpoint of index theory in the sense that they sit in different Bott clock. A similar remark holds for, e.g., cases of $n = k$ in classes BDI and CII, where both of these numerical corner invariants are defined as Fredholm indices.

### 5.7. Product Formula

In Sect. 4 of [32], a construction of the second-order topological insulators of 3-D class A systems is proposed, which is given by using the Hamiltonians of 2-D class A and 1-D class AIII topological insulators. In this subsection, we generalize this construction to other pairs in the Altland–Zirnbauer classification. This provides a way to construct nontrivial examples of each entry in Table 1 from the Hamiltonians of (first-order) topological insulators. For this purpose, we use an exterior product of topological KR-groups [4].

For $j = 1, 2$, let $H_j$ be a bulk Hamiltonian of an $n_j$-D $k_j$-th order topological insulator in real Altland–Zirnbauer class $\bullet_j$ (AI, BDI, D, DIII, AI, CII, C or CI). Let $n = n_1 + n_2$, $k = k_1 + k_2$ and $d_j = n_j - k_j$ for $j = 1, 2$, and let $d = 13$For the case of $k = 2$, the construction is restricted to $\alpha = 0$ and $\beta = \infty$ case.

14that in Sect. 5.1 satisfying Assumption D72 when $k_j = 2$ or that in Sect. 5.23 satisfying Assumption D95 when $k_j \geq 3$. When $k_j = 1$, the bulk Hamiltonian is assumed to be gapped. When $k_j = 2$, we consider the case of $\alpha = 0$ and $\beta = \infty$.  


Corresponding to the class in the Altland–Zirnbauer classification (for which we write $\spadesuit$) to which the Hamiltonian belongs, it preserves the symmetries as (even/odd) TRS, (even/odd) PHS or chiral symmetry. We write $\Theta_j$, $\Xi_j$ and $\Pi_j$ for the symmetry operator for $H_j$. As in Sect. 5 the models of corners $H^c_j$ lead to a continuous family of self-adjoint or skew-adjoint Fredholm operators and defines an element of the $KO$-group $KO_{i'(\spadesuit)}(C(\mathbb{T}^{d_j}), \tau)$ where $i'(\spadesuit) = i(\spadesuit) - 1$. As in Appendix A.1, we have an exterior product of $KO$-groups

$$KO_{i'(\spadesuit)}(C(\mathbb{T}^{d_1}), \tau_1) \times KO_{i'(\spadesuit)}(C(\mathbb{T}^{d_2}), \tau_2) \to KO_{i'(\spadesuit) + i'(\spadesuit)}(C(\mathbb{T}^d), \tau),$$

described through these Fredholm operators. By using this form of the product, we obtain an explicit form of the gapless invariants of $H_1$ and $H_2$. As a result, we can write down a bulk Hamiltonian $H$ of an $n$-D $k$-th order topological insulator of class $\spadesuit$. The lattice on which we consider $H^c$ as a model of the codimension-$k$ corner is introduced as the product of that of $H^c_1$ and that of $H^c_2$. By this construction, we have the following relation between gapless invariants.

**Theorem 5.27.** For the Hamiltonian $H$ indicated in Table 12 we have

$$T_{\text{Gapless}}^{n,k,\spadesuit}(H_1) \cdot T_{\text{Gapless}}^{n,k,\spadesuit}(H_2) = T_{\text{Gapless}}^{n,k,\spadesuit}(H),$$

where $\cdot$ denotes the exterior product of elements of the $KO$-groups.

Note that Theorem 5.27 is the product formula at the level of $KO$-group elements and accounts for both strong and weak invariants. In order to show this theorem, we need to write down the explicit form of $H$. In the following, we discuss them for some classes.

Let us consider the case where $\spadesuit_1 = BDI$ and $\spadesuit_2 = BDI$. In this case, each $H_j$ has even TRS, even PHS and chiral symmetry. We now consider the following $n$-dimensional Hamiltonian:

$$H = H_1 \otimes 1 + \Pi_1 \otimes H_2,$$

which satisfies even TRS given by $\Theta = \Theta_1 \otimes \Theta_2$, even PHS given by $\Xi = \Xi_1 \otimes \Xi_2$ and the chiral symmetry given by $\Pi = \Pi_1 \otimes \Pi_2$. Thus, the Hamiltonian $H$ belongs to the class $\spadesuit = BDI$. The model of the codimension-$k$ corner $H^c$ of $H$ is written by using the model $H^c_j$ of the codimension-$k_j$ corner as follows:

$$H^c(t_1, t_2) = H^c_j(t_1) \otimes 1 + \Pi_1 \otimes H^c_k(t_2),$$

where $t_j$ is an element of the $d_j$-dimensional torus (momentum space) corresponding to a direction parallel to the corner of $H^c_j$ for $j = 1, 2$. Note that $(t_1, t_2)$ constitute the parameter of the $d$-dimensional momentum space in a direction parallel to the corner of $H^c$. By our assumption, $H^c_j(t_j)$ is an element of the space $\text{Fred}^{(0,2)}_*$ and gives a $\mathbb{Z}_2$-map $(\mathbb{T}^{d_j}, \zeta) \to (\text{Fred}^{(0,2)}_*, \tau_{\Theta_j})$. The operator $H^c(t_1, t_2)$ is the image of the pair $(H^c_j, H^c_k)$ through the map,

$$(\text{Fred}^{(0,2)}_*, \tau_{\Theta_1}) \times (\text{Fred}^{(0,2)}_*, \tau_{\Theta_2}) \to (\text{Fred}^{(0,2)}_*, \tau_{\Theta}).$$

$^{15}$When $k_j = 1$, the lattice is $\mathbb{Z}_{\geq 0} \times \mathbb{Z}^{d_j}$, where $H^c_j$ is the compression of the bulk Hamiltonian onto this half-space. Topological invariants for them are the one discussed in topological insulators. To clarify our sign choices, we mention that they are obtained by applying the discussion in Sect. 5 to the Toeplitz extension (1.1) in place of (2.2) or (3.2).
in (A.2), where the action of $Cl_{0,1}$ to define the left-hand side is given by $\epsilon_j = \Pi_j$
and that for the right-hand side is given by $\epsilon = \epsilon_1 \otimes \epsilon_2 = \Pi$. Since this map induces
the exterior product of $KO$-groups (Appendix A.1),

$$KO_0(C(T^{\mathbf{d}}), \tau_T) \times KO_0(C(T^{\mathbf{d}}), \tau_T) \to KO_0(C(T^{\mathbf{d}}), \tau_T),$$

we obtain Theorem 5.27 in this case.

We next consider the case where $\clubsuit_1 = \text{BDI}$ and $\clubsuit_2 = \text{D}$. In this case, $H_1$
has odd TRS, even PHS and the chiral symmetry, and $H_2$ has even PHS. As in
Sect. 5.3, $H^\prime_1(t_1)$ belongs to (Fred$^{(1)}_1$, $q_{\Theta_1}$), and $iH^\prime_2(t_2)$ belongs to (Fred$^{(0)}_2$, $r_{\Xi_2}$).
By using Proposition A.4, we identify (Fred$^{(1)}_1$, $q_{\Theta_1}$) with (Fred$^{(2)}_2$, $q_{\Theta_2 \otimes \Theta_1}$) and
(Fred$^{(0)}_2$, $r_{\Xi_2}$) with (Fred$^{(0)}_2$, $r_{\Xi_2 \oplus \Xi_2}$). We then use the map (A.2) of the form

$$(\text{Fred}^{(2)}_2, q_{\Theta_2 \oplus \Theta_1}) \times (\text{Fred}^{(0)}_2, r_{\Xi_2 \oplus \Xi_2}) \to (\text{Fred}^{(2)}_2, q'),$$

where $q'$ is the conjugation of the fourfold direct sum of $\Theta_1 \otimes \Xi_2$. By Proposition A.3
we have the $\mathbb{Z}_2$-homeomorphism (Fred$^{(2)}_2$, $q'$) $\cong$ (Fred$^{(0)}_2$, $q_{\Theta_2 \oplus \Theta_1}$).
Thus, we obtain a $\mathbb{Z}_2$-map $H^\prime$: $(T^3, C) \to (\text{Fred}^{(0)}_2, q_{\Theta_2 \oplus \Theta_1})$ from $H^\prime_1$ and $H^\prime_2$ which is a model
for the codimension-$k$ corner in class $\clubsuit = \text{DII}$. Its bulk Hamiltonian $H$
(resp. odd TRS operator $\Theta$) is expressed as [57, 33] and $\Theta = \Theta_1 \otimes \Xi_2$, respectively.
Note that $\Pi_1 = i\theta_1 \Xi_1$ in class DIII and $\Theta$ commutes with $H$. Since the map (A.2)
induces the exterior product of $KO$-groups, Theorem 5.27 holds for this class DII
Hamiltonian $H$.

The other cases are computed in a similar way, and the results are summarized
in Table 12, where we write

$$H_{\clubsuit} = \begin{pmatrix}
0 & H_1 \otimes 1 - i \otimes H_2 \\
H_1 \otimes 1 + i \otimes H_2 & 0
\end{pmatrix}, \quad \Theta_{\clubsuit} = \begin{pmatrix}
\Theta_1 \oplus \Xi_2 & 0 \\
0 & \Theta_1 \oplus \Xi_2
\end{pmatrix},$$

$$H_{\heartsuit} = \begin{pmatrix}
0 & -H_1 \otimes i - 1 \otimes H_2 \\
H_1 \otimes i - 1 \otimes H_2 & 0
\end{pmatrix}, \quad \Theta_{\heartsuit} = \begin{pmatrix}
\Xi_1 \otimes \Theta_2 & 0 \\
0 & \Xi_1 \otimes \Theta_2
\end{pmatrix},$$

$$\Theta_{\spadesuit} = \begin{pmatrix}
0 & -\Xi_1 \otimes \Xi_2 \\
\Xi_1 \otimes \Xi_2 & 0
\end{pmatrix} \quad \text{and} \quad \Theta_{\blacklozenge} = \begin{pmatrix}
\Theta_1 \otimes \Theta_2 & 0 \\
0 & \Theta_1 \otimes \Theta_2
\end{pmatrix}.$$

Table 12: The forms of the Hamiltonians and symmetry operators in class $\clubsuit$ constructed from two pairs of Hamiltonians and symmetry operators in classes $\clubsuit_1$ and $\clubsuit_2$. Complex cases are also included [32, 33].

| $\clubsuit_1$ | $\clubsuit_2$ | $\clubsuit$ | Hamiltonian ($H$) | TRS ($\Theta$) | PHS ($\Xi$) | Chiral ($\Pi$) |
|--------------|--------------|-------------|------------------|--------------|--------------|--------------|
| Al           | Al           | Al          | $H_{\clubsuit}$  | $\Theta_{\heartsuit}$ | $\Theta_{\spadesuit}$ | $\Xi$ = $\Theta_{\heartsuit}$ |
| Al           | BDI          | Al          | $H_{\clubsuit}$  | $\Theta_{\heartsuit}$ | $\Theta_{\spadesuit}$ | $\Xi$ = $\Theta_{\heartsuit}$ |
| Al           | D            | BDI         | $H_{\clubsuit}$  | $\Theta_{\heartsuit}$ | $\Theta_{\spadesuit}$ | $\Xi$ = $\Theta_{\heartsuit}$ |
| Al           | DIII         | D           | $H_{\clubsuit}$  | $\Theta_{\heartsuit}$ | $\Theta_{\spadesuit}$ | $\Xi$ = $\Theta_{\heartsuit}$ |
| Al           | DIII         | D           | $H_{\clubsuit}$  | $\Theta_{\heartsuit}$ | $\Theta_{\spadesuit}$ | $\Xi$ = $\Theta_{\heartsuit}$ |
| Al           | C            | C            | $H_{\clubsuit}$  | $\Theta_{\heartsuit}$ | $\Theta_{\spadesuit}$ | $\Xi$ = $\Theta_{\heartsuit}$ |
| BDI          | Al           | Al          | $H_{\clubsuit}$  | $\Theta_{\heartsuit}$ | $\Theta_{\spadesuit}$ | $\Xi$ = $\Theta_{\heartsuit}$ |
| BDI          | BDI          | BDI         | $H_{\clubsuit}$  | $\Theta_{\heartsuit}$ | $\Theta_{\spadesuit}$ | $\Xi$ = $\Theta_{\heartsuit}$ |
| BDI          | D            | D           | $H_{\clubsuit}$  | $\Theta_{\heartsuit}$ | $\Theta_{\spadesuit}$ | $\Xi$ = $\Theta_{\heartsuit}$ |
| BDI | DIII | DIII | $H_1 \otimes 1 + \Pi_1 \otimes H_2$ | $\Theta_1 \otimes \Theta_2$ | $\Xi_1 \otimes \Xi_2$ | $\Pi_1 \otimes \Pi_2$ |
|-----|-----|-----|-------------------------------|----------------|----------------|----------------|
| BDI | All | All | $H_1 \otimes 1 + \Pi_1 \otimes H_2$ | $\Theta_1 \otimes \Theta_2$ | $\Xi_1 \otimes \Xi_2$ | $\Pi_1 \otimes \Pi_2$ |
| BDI | CII | CII | $H_1 \otimes 1 + \Pi_1 \otimes H_2$ | $\Theta_1 \otimes \Theta_2$ | $\Xi_1 \otimes \Xi_2$ | $\Pi_1 \otimes \Pi_2$ |
| BDI | ClI | ClI | $H_1 \otimes 1 + \Pi_1 \otimes H_2$ | $\Theta_1 \otimes \Theta_2$ | $\Xi_1 \otimes \Xi_2$ | $\Pi_1 \otimes \Pi_2$ |
| D   | BDI | D   | $H_1 \otimes H_2 + 1 \otimes H_2$ | $\Theta_1 \otimes \Theta_2$ | $\Xi_1 \otimes \Xi_2$ | $\Pi_1 \otimes \Pi_2$ |
| D   | DIII | DIII | $H_1 \otimes H_2 + 1 \otimes H_2$ | $\Theta_1 \otimes \Theta_2$ | $\Xi_1 \otimes \Xi_2$ | $\Pi_1 \otimes \Pi_2$ |
| D   | DIII | All | $H_1 \otimes H_2 + 1 \otimes H_2$ | $\Theta_1 \otimes \Theta_2$ | $\Xi_1 \otimes \Xi_2$ | $\Pi_1 \otimes \Pi_2$ |
| D   | All | All | $H_1 \otimes 1 + \Pi_1 \otimes H_2$ | $\Theta_1 \otimes \Theta_2$ | $\Xi_1 \otimes \Xi_2$ | $\Pi_1 \otimes \Pi_2$ |
| D   | CII | CII | $H_1 \otimes 1 + \Pi_1 \otimes H_2$ | $\Theta_1 \otimes \Theta_2$ | $\Xi_1 \otimes \Xi_2$ | $\Pi_1 \otimes \Pi_2$ |
| D   | ClI | ClI | $H_1 \otimes 1 + \Pi_1 \otimes H_2$ | $\Theta_1 \otimes \Theta_2$ | $\Xi_1 \otimes \Xi_2$ | $\Pi_1 \otimes \Pi_2$ |
| D   | All | ClI | $H_1 \otimes 1 + \Pi_1 \otimes H_2$ | $\Theta_1 \otimes \Theta_2$ | $\Xi_1 \otimes \Xi_2$ | $\Pi_1 \otimes \Pi_2$ |
| All | AlI | AlI | $H_1 \otimes H_2 + 1 \otimes H_2$ | $\Theta_1 \otimes \Theta_2$ | $\Xi_1 \otimes \Xi_2$ | $\Pi_1 \otimes \Pi_2$ |
| All | BDI | BDI | $H_1 \otimes H_2 + 1 \otimes H_2$ | $\Theta_1 \otimes \Theta_2$ | $\Xi_1 \otimes \Xi_2$ | $\Pi_1 \otimes \Pi_2$ |
| All | D   | ClI | $H_1 \otimes 1 + \Pi_1 \otimes H_2$ | $\Theta_1 \otimes \Theta_2$ | $\Xi_1 \otimes \Xi_2$ | $\Pi_1 \otimes \Pi_2$ |
| All | ClI | ClI | $H_1 \otimes 1 + \Pi_1 \otimes H_2$ | $\Theta_1 \otimes \Theta_2$ | $\Xi_1 \otimes \Xi_2$ | $\Pi_1 \otimes \Pi_2$ |
| All | BDI | CII | $H_1 \otimes H_2 + 1 \otimes H_2$ | $\Theta_1 \otimes \Theta_2$ | $\Xi_1 \otimes \Xi_2$ | $\Pi_1 \otimes \Pi_2$ |
| All | ClI | ClI | $H_1 \otimes 1 + \Pi_1 \otimes H_2$ | $\Theta_1 \otimes \Theta_2$ | $\Xi_1 \otimes \Xi_2$ | $\Pi_1 \otimes \Pi_2$ |
| All | BDI | BDI | $H_1 \otimes 1 + \Pi_1 \otimes H_2$ | $\Theta_1 \otimes \Theta_2$ | $\Xi_1 \otimes \Xi_2$ | $\Pi_1 \otimes \Pi_2$ |
| All | ClI | ClI | $H_1 \otimes 1 + \Pi_1 \otimes H_2$ | $\Theta_1 \otimes \Theta_2$ | $\Xi_1 \otimes \Xi_2$ | $\Pi_1 \otimes \Pi_2$ |
| All | BDI | CII | $H_1 \otimes 1 + \Pi_1 \otimes H_2$ | $\Theta_1 \otimes \Theta_2$ | $\Xi_1 \otimes \Xi_2$ | $\Pi_1 \otimes \Pi_2$ |
| All | ClI | ClI | $H_1 \otimes 1 + \Pi_1 \otimes H_2$ | $\Theta_1 \otimes \Theta_2$ | $\Xi_1 \otimes \Xi_2$ | $\Pi_1 \otimes \Pi_2$ |
| All | BDI | D   | $H_1 \otimes H_2 + 1 \otimes H_2$ | $\Theta_1 \otimes \Theta_2$ | $\Xi_1 \otimes \Xi_2$ | $\Pi_1 \otimes \Pi_2$ |
| All | ClI | ClI | $H_1 \otimes 1 + \Pi_1 \otimes H_2$ | $\Theta_1 \otimes \Theta_2$ | $\Xi_1 \otimes \Xi_2$ | $\Pi_1 \otimes \Pi_2$ |
| All | BDI | BDI | $H_1 \otimes 1 + \Pi_1 \otimes H_2$ | $\Theta_1 \otimes \Theta_2$ | $\Xi_1 \otimes \Xi_2$ | $\Pi_1 \otimes \Pi_2$ |
| All | ClI | ClI | $H_1 \otimes 1 + \Pi_1 \otimes H_2$ | $\Theta_1 \otimes \Theta_2$ | $\Xi_1 \otimes \Xi_2$ | $\Pi_1 \otimes \Pi_2$ |
| All | BDI | CII | $H_1 \otimes 1 + \Pi_1 \otimes H_2$ | $\Theta_1 \otimes \Theta_2$ | $\Xi_1 \otimes \Xi_2$ | $\Pi_1 \otimes \Pi_2$ |
| All | ClI | ClI | $H_1 \otimes 1 + \Pi_1 \otimes H_2$ | $\Theta_1 \otimes \Theta_2$ | $\Xi_1 \otimes \Xi_2$ | $\Pi_1 \otimes \Pi_2$ |
Our product formula (Theorem 5.29) and the graded ring structure of $KO_*(\mathbb{C}, \text{id})$ (Theorem 6.9 of [7]) lead to the following product formula for numerical corner invariants. We collect the results here where the form of $H$ is as indicated in Table 12.

**Corollary 5.28** (Cases of $n = k$). The case of $k_1 = n_1$ and $k_2 = n_2$.
- $\text{BDI} \times \text{BDI} \rightarrow \text{BDI}$, $\mathcal{N}_{n_1,n_1,\text{BDI}}(H_1) \cdot \mathcal{N}_{n_2,n_2,\text{BDI}}(H_2) = \mathcal{N}_{n,n,\text{BDI}}(H)$.
- $\text{BDI} \times D \rightarrow D$, $\mathcal{N}_{n_1,n_1,\text{BDI}}(H_1) \cdot \mathcal{N}_{n_2,n_2,\text{D}}(H_2) = \mathcal{N}_{n,n,\text{D}}(H)$.
- $\text{BDI} \times \text{DIII} \rightarrow \text{DIII}$, $(\mathcal{N}_{n_1,n_1,\text{BDI}}(H_1) \mod 2) \cdot \mathcal{N}_{n_2,n_2,\text{DIII}}(H_2) = \mathcal{N}_{n,n,\text{DIII}}(H)$.
- $\text{BDI} \times \text{CII} \rightarrow \text{CII}$, $\mathcal{N}_{n_1,n_1,\text{BDI}}(H_1) \cdot \mathcal{N}_{n_2,n_2,\text{CII}}(H_2) = \mathcal{N}_{n,n,\text{CII}}(H)$.
- $D \times D \rightarrow \text{DIII}$, $\mathcal{N}_{n_1,n_1,\text{D}}(H_1) \cdot \mathcal{N}_{n_2,n_2,\text{DII}}(H_2) = \mathcal{N}_{n,n,\text{DII}}(H)$.
- $\text{CII} \times \text{CII} \rightarrow \text{BDI}$, $\mathcal{N}_{n_1,n_1,\text{CII}}(H_1) \cdot \mathcal{N}_{n_2,n_2,\text{BDI}}(H_2) = \mathcal{N}_{n,n,\text{BDI}}(H)$.

**Corollary 5.29** (Cases of $n = k - 1$). The case of $k_1 = n_1$ and $k_2 = n_2 - 1$.
- $\text{BDI} \times D \rightarrow D$, $\mathcal{N}_{n_1,n_1,\text{BDI}}(H_1) \cdot \mathcal{N}_{n_2,n_2-1,\text{D}}(H_2) = \mathcal{N}_{n,n-1,\text{D}}(H)$.
- $\text{BDI} \times \text{DIII} \rightarrow \text{DIII}$, $(\mathcal{N}_{n_1,n_1,\text{BDI}}(H_1) \mod 2) \cdot \mathcal{N}_{n_2,n_2-1,\text{DIII}}(H_2) = \mathcal{N}_{n,n-1,\text{DIII}}(H)$.
- $\text{BDI} \times \text{All} \rightarrow \text{All}$, $(\mathcal{N}_{n_1,n_1,\text{BDI}}(H_1) \mod 2) \cdot \mathcal{N}_{n_2,n_2-1,\text{All}}(H_2) = \mathcal{N}_{n,n-1,\text{All}}(H)$.
- $\text{BDI} \times C \rightarrow C$, $\mathcal{N}_{n_1,n_1,\text{BDI}}(H_1) \cdot \mathcal{N}_{n_2,n_2-1,\text{C}}(H_2) = \mathcal{N}_{n,n-1,\text{C}}(H)$.
- $D \times D \rightarrow \text{DIII}$, $\mathcal{N}_{n_1,n_1,\text{D}}(H_1) \cdot (\mathcal{N}_{n_2,n_2-1,\text{D}}(H_2) \mod 2) = \mathcal{N}_{n,n-1,\text{DIII}}(H)$.
- $D \times \text{DIII} \rightarrow \text{All}$, $\mathcal{N}_{n_1,n_1,\text{D}}(H_1) \cdot \mathcal{N}_{n_2,n_2-1,\text{DIII}}(H_2) = \mathcal{N}_{n,n-1,\text{All}}(H)$.
- $\text{DIII} \times D \rightarrow \text{All}$, $\mathcal{N}_{n_1,n_1,\text{DIII}}(H_1) \cdot (\mathcal{N}_{n_2,n_2-1,\text{D}}(H_2) \mod 2) = \mathcal{N}_{n,n-1,\text{All}}(H)$.
- $\text{CII} \times D \rightarrow C$, $\mathcal{N}_{n_1,n_1,\text{CII}}(H_1) \cdot \mathcal{N}_{n_2,n_2-1,\text{D}}(H_2) = \mathcal{N}_{n,n-1,\text{C}}(H)$.
- $\text{CII} \times C \rightarrow D$, $\mathcal{N}_{n_1,n_1,\text{CII}}(H_1) \cdot \mathcal{N}_{n_2,n_2-1,\text{C}}(H_2) = \mathcal{N}_{n,n-1,\text{D}}(H)$.

We also have a similar formula by exchanging $H_1$ and $H_2$ (e.g., pairs like $D \times \text{BDI} \rightarrow D$). Note that in the case of $\text{CII} \times \text{CII} \rightarrow \text{BDI}$ in Corollary 5.28, we take the product of two even integers, which is necessarily a multiple of four. A similar remark also holds in the case of $\text{CII} \times C \rightarrow D$ in Corollary 5.29.
Appendix A. Z₂-Spaces of Self-Adjoint/Skew-Adjoint Fredholm Operators and Boersema–Loring’s K-theory

In this Appendix, we collect necessary results and notations used in this paper. The results have been developed in much generality [6, 68, 9, 37, 43, 27, 15], and we contain minimal background for this paper focusing on their relation with Boersema–Loring’s K-theory [14]. In Appendix A.1, we introduce some Z₂-spaces of self-adjoint and skew-adjoint Fredholm operators following [9]. Some proofs for known results are contained simply to fix isomorphisms used in this paper (e.g. the derivation of Table [12]). In Appendix A.2, we discuss its relation with Boersema–Loring’s K-theory. In Appendix A.3, inspired by exponential maps in [68, 9], we write the boundary maps of the 24-term exact sequence of KO-theory in Boersema–Loring’s unitary picture through exponentials. Some of them are already expressed by exponentials in [14]; thus, we consider the remaining cases. This form of boundary maps is useful when we discuss a relation between our gapped invariants and gapless invariants through boundary maps [14, 14].

A.1. Z₂-Spaces of Self-Adjoint/Skew-Adjoint Fredholm Operators. For non-negative integers k and l, let Clk,l be the Clifford algebra that is an associative algebra with unit over \( \mathbb{R} \) generated by \( k+l \) elements \( e_1, \ldots, e_k \) and \( \epsilon_1, \ldots, \epsilon_l \), which satisfy \( e_i^2 = -1 \) (\( i = 1, \ldots, k \)) and \( \epsilon_j^2 = 1 \) (\( j = 1, \ldots, l \)) and anticommute with each other. The following are well-known Clifford algebra isomorphisms [47].

Lemma A.1. (1) \( Cl_{k,l+1} \cong Cl_{l,k+1} \).
(2) \( Cl_{k,l} \otimes Cl_{1,1} \cong Cl_{k+1,l+1} \).
(3) \( Cl_{k,l} \otimes Cl_{4,0} \cong Cl_{k+4,l} \) and \( Cl_{k,l} \otimes Cl_{0,4} \cong Cl_{k,l+4} \).
(4) \( Cl_{k,l} \otimes Cl_{8,0} \cong Cl_{k+8,l} \) and \( Cl_{k,l} \otimes Cl_{0,8} \cong Cl_{k+l+8} \).

Proof. (1) Let \( e_1, \ldots, e_k \) and \( \epsilon_1, \ldots, \epsilon_{l+1} \) be generators of the Clifford algebra \( Cl_{k,l+1} \). Let \( \tilde{e}_i = \epsilon_{i+1} e_i \) (\( i = 1, \ldots, l \)), \( \tilde{\epsilon}_1 = \epsilon_1 \) and \( \tilde{\epsilon}_i = e_{i-1} \epsilon_i \) (\( i = 2, \ldots, k+1 \)). Then, \( \tilde{e}_1, \ldots, \tilde{e}_l \) and \( \tilde{\epsilon}_1, \ldots, \tilde{\epsilon}_{l+1} \) correspond to generators of the Clifford algebra \( Cl_{l,k+1} \).

(2) Let \( e_1, \ldots, e_k \) and \( \epsilon_1, \ldots, \epsilon_l \) be generators of the Clifford algebra \( Cl_{k,l} \), and let \( e_1 \) and \( \epsilon_1 \) be those of \( Cl_{1,1} \). We write \( \omega_{1,1} \) for \( e_i \epsilon_i' \in Cl_{1,1} \). Then, \( \tilde{e}_i = e_i \otimes \omega_{1,1} \) (\( i = 1, \ldots, k \)), \( \tilde{\epsilon}_{k+1} = 1 \otimes e_i' \), \( \tilde{e}_i = e_i \otimes \omega_{1,1} \) (\( i = 1, \ldots, l \)) and \( \tilde{\epsilon}_i = 1 \otimes e_i' \) correspond to generators of the Clifford algebra \( Cl_{k+l+1} \).

(3) We show that \( Cl_{k,l} \otimes Cl_{0,4} \cong Cl_{k,l+4} \): the other is proved similarly. Let \( e_1, \ldots, e_k \) and \( \epsilon_1, \ldots, \epsilon_l \) be generators of the Clifford algebra \( Cl_{k,l} \), and let \( e_1', e_2', e_3', \) and \( e_4' \) be those of \( Cl_{0,4} \). We write \( \omega_{0,4} \) for \( -e_1' e_2' e_3' e_4' \in Cl_{0,4} \). Then, \( \tilde{e}_i = e_i \otimes \omega_{0,4} \) (\( i = 1, \ldots, k \)), \( \tilde{\epsilon}_i = e_i \otimes \omega_{0,4} \) (\( i = 1, \ldots, l \)) and \( \tilde{\epsilon}_i = 1 \otimes e_i' \) (\( i = 1, \ldots, 4 \)) correspond to generators of the algebra \( Cl_{k+4,l} \).

(4) We show that \( Cl_{k,l} \otimes Cl_{0,8} \cong Cl_{k+l+8} \): the other is proved similarly. Let \( e_1, \ldots, e_k \) and \( \epsilon_1, \ldots, \epsilon_l \) be generators of \( Cl_{k,l} \), and let \( e_1', \ldots, e_8' \) be those of \( Cl_{0,8} \). We write \( \omega_{0,8} \) for \( e_1' \cdots e_8' \in Cl_{0,8} \). Then, \( \tilde{e}_i = e_i \otimes \omega_{0,8} \) (\( i = 1, \ldots, k \)), \( \tilde{\epsilon}_i = e_i \otimes \omega_{0,8} \) (\( i = 1, \ldots, l \)) and \( \tilde{\epsilon}_i = 1 \otimes e_i' \) (\( i = 1, \ldots, 8 \)) correspond to generators of the algebra \( Cl_{k+8,l} \). \( \square \)

Let \( W \) be a (ungraded) complex left \( Cl_{k,l} \)-module. We say that \( W \) is a (ungraded) real (resp. quaternionic) \( Cl_{k,l} \)-module\(^\text{16}\) if \( W \) is equipped with an antilinear map
$r : W \to W$ (resp. $q : W \to W$), which commutes with the $Cl_{k,l}$-action and satisfies $r^2 = 1$ (resp. $q^2 = -1$). We call this $r$ (resp. $q$) the real (resp. quaternionic) structure on the Clifford module. Since a real (resp. quaternionic) $Cl_{k,l}$-module is the same thing as a module of $Cl_{k,l} \otimes Cl_{1,1}$ \cong $Cl_{k+1,l+1}$ (resp. $Cl_{k,l} \otimes Cl_{2,0} \cong Cl_{l+2}$) over $\mathbb{R}$, the algebra $Cl_{k,l}$ has one inequivalent irreducible real or quaternionic module when $k - l \not\equiv 3 \mod 4$ and has two when $k - l \equiv 3 \mod 4$.

**Lemma A.2.**

(1) Let $\Delta_{1,1}$ be a complex irreducible representation of $Cl_{1,1}$. There exists a real structure $r_{1,1}$ on $\Delta_{1,1}$ that commutes with the Clifford action.

(2) Let $\Delta_{0,4}$ (resp. $\Delta_{4,0}$) be a complex irreducible representation of $Cl_{0,4}$ (resp. $Cl_{4,0}$). There exists a quaternionic structure $q_{0,4}$ (resp. $q_{4,0}$) on $\Delta_{0,4}$ (resp. $\Delta_{4,0}$) that commutes with the Clifford action.

(3) Let $\Delta_{0,8}$ (resp. $\Delta_{8,0}$) be a complex irreducible representation of $Cl_{0,8}$ (resp. $Cl_{8,0}$). There exists a real structure $r_{0,8}$ (resp. $r_{8,0}$) on $\Delta_{0,8}$ (resp. $\Delta_{8,0}$) that commutes with the Clifford action.

For the proof of this lemma, see [25], for example. For a $\mathbb{Z}_2$-space $(X, \zeta)$ with two $\mathbb{Z}_2$-fixed points $x_0, x_1 \in X$, we write $P(X; x_0, x_1)$ for the path space starting from $x_0$ and ending at $x_1$, that is, the space of continuous maps $f : [0, 1] \to X$ satisfying $f(0) = x_0$ and $f(1) = x_1$ equipped with the compact-open topology. On this space, we consider an involution, for which we also write $\zeta$ by abuse of notation, defined as $(\zeta(f))(t) = \zeta(f(t))$ for $t$ in $[0, 1]$, and obtain a $\mathbb{Z}_2$-space $(P(X; x_0, x_1), \zeta)$. When $x_0 = x_1$, we write $\Omega X$ for $P(X; x_0, x_0)$, which is the based loop space of $X$ with the base point $x_0$.

**Remark A.3.** Banach $\mathbb{Z}_2$-spaces and its open $\mathbb{Z}_2$-subspaces are $\mathbb{Z}_2$-absolute neighborhood retracts [3], and have the homotopy type of $\mathbb{Z}_2$-CW complexes [10]. The path spaces and loop spaces we discuss in the following also have the homotopy type of $\mathbb{Z}_2$-CW complexes [67]. By the equivariant Whitehead theorem, weak $\mathbb{Z}_2$-homotopy equivalences between these spaces are $\mathbb{Z}_2$-homotopy equivalences [50, 4].

Let $\mathcal{V}$ be a separable infinite-dimensional complex Hilbert space. Let $B(\mathcal{V})$ be the space of bounded complex linear operators on $\mathcal{V}$ equipped with the norm topology. Let $GL(\mathcal{V})$, $U(\mathcal{V})$, Fred($\mathcal{V}$) and $K(\mathcal{V})$ be subspaces of $B(\mathcal{V})$ consisting of invertible, unitary, Fredholm and compact operators on $\mathcal{V}$, respectively. We assume that our Hilbert space $\mathcal{V}$ has a real structure $r$ or a quaternionic structure $q$, that is, an antilinear operator on $\mathcal{V}$ satisfying $r^2 = 1$ or $q^2 = -1$, respectively. Correspondingly, the space $B(\mathcal{V})$ has an (antilinear) involution $r = Ad_r$ or $q = Ad_q$. These involutions induce involutions on $GL(\mathcal{V})$, $U(\mathcal{V})$, Fred($\mathcal{V}$) and $K(\mathcal{V})$, for which we also write $r$ or $q$. We write $a$ for $r$ or $q$ and $a$ for $r$ or $q$. We also assume that there is a complex linear action of the Clifford algebra $Cl_{k,l}$ on the Hilbert space $\mathcal{V}$ that commutes with the real or the quaternionic structure. For an element $v \in Cl_{k,l}$, we also write $v$ for its action on $\mathcal{V}$, for simplicity. When $k - l \equiv 3 \mod 4$, we further assume that each of the two inequivalent irreducible real or quaternionic representations of $Cl_{k,l}$ has infinite multiplicity. In the following, we discuss the subspaces of $B(\mathcal{V})$; we may abbreviate the Hilbert space $\mathcal{V}$ from its notation when it is clear from the context. When the Hilbert space $\mathcal{V}$ is such a $Cl_{k,l}$-module, let $B_a(k,l)$ (resp. $B_b(k,l)$) be the subspace of $B(\mathcal{V})$ consisting of skew-adjoint (resp. self-adjoint) operators $A$ on $\mathcal{V}$ satisfying $e_i A = -A e_i$ for $i = 1, \ldots, k$ and $e_j A = -A e_j$ for $j = 1, \ldots, l$. Let
Fred\(_{(k,l)}^{(k,l)}\) = Fred \(\cap B_{\text{sk}}^{(k,l)}\) and Fred\(_{(k,l)}^{(k,l)}\) = Fred \(\cap B_{\text{sa}}^{(k,l)}\). The involution \(a\) on \(B(V)\) induces involutions on Fred\(_{(k,l)}^{(k,l)}\) and Fred\(_{(a)}^{(k,l)}\) for which we also write \(a\). Consider the space Fred\(_{(k,l)}^{(k,l)}\) and let \(\Upsilon = e_1 \cdots e_{k-1} \epsilon_1 \cdots \epsilon_l\). When \(k-l\) is odd, the space Fred\(_{(k,l)}^{(k,l)}\) is decomposed into three components Fred\(_{(k,l)}^{(k,l)}\), Fred\(_{(k,l)}^{(k,l)}\) and Fred\(_{(k,l)}^{(k,l)}\) corresponding to whether the following element is essentially positive, essentially negative or neither: \(i^{-1} \Upsilon A\) when \(k-l \equiv 1 \mod 4\) and \(\Upsilon A\) when \(k-l \equiv 3 \mod 4\) for \(A \in \text{Fred}_{\text{sk}}^{(k,l)}\).

In [9], each of these three components is nonempty. When \(k-l \equiv 1 \mod 4\), the involution \(a\) maps Fred\(_{(k,l)}^{(k,l)}\) to Fred\(_{(k,l)}^{(k,l)}\) (double-sign corresponds), and Fred\(_{(k,l)}^{(k,l)}\) is closed under the action of \(a\). When \(k-l \equiv 3 \mod 4\), each of the three components is closed under the action of \(a\). The space Fred\(_{(k,l)}^{(k,l)}\) is also decomposed into three components in the same way, except that we take \(e_1 \cdots e_k e_1 \cdots e_l \Upsilon\) for \(\Upsilon\) in this case, and we define the space Fred\(_{(k,l)}^{(k,l)}\) when \(k-l\) is odd. When \(k-l\) is even, we set Fred\(_{(k,l)}^{(k,l)}\) = Fred\(_{(k,l)}^{(k,l)}\) and Fred\(_{(k,l)}^{(k,l)}\) = Fred\(_{(k,l)}^{(k,l)}\). Summarizing, we have the following \(\mathbb{Z}_2\)-spaces:

\[
(A.1) \quad (\text{Fred}_{\text{sk}}^{(k,l)}, \tau), \quad (\text{Fred}_{\text{sk}}^{(k,l)}, \tau), \quad (\text{Fred}_{\text{sa}}^{(k,l)}, \tau), \quad (\text{Fred}_{\text{sa}}^{(k,l)}, \tau),
\]

**Proposition A.4.** The following \(\mathbb{Z}_2\)-homeomorphisms exist.

1. (Fred\(_{(k,l)}^{(k,l)}\), \(a\)) \(\cong\) (Fred\(_{(k+1,l+1)}^{(k+1,l+1)}\), \(a\)) and (Fred\(_{(k,l)}^{(k,l)}\), \(a\)) \(\cong\) (Fred\(_{(k+1,l+1)}^{(k+1,l+1)}\), \(a\)),
2. (Fred\(_{(k,l)}^{(k,l)}\), \(a\)) \(\cong\) (Fred\(_{(k+4,l)}^{(k+4,l)}\), \(a\)) and (Fred\(_{(k,l)}^{(k,l)}\), \(a\)) \(\cong\) (Fred\(_{(k+4,l+2)}^{(k+4,l+2)}\), \(a\)),
3. (Fred\(_{(k,l)}^{(k,l)}\), \(a\)) \(\cong\) (Fred\(_{(k,l+4)}^{(k,l+4)}\), \(a\)) and (Fred\(_{(k,l)}^{(k,l)}\), \(a\)) \(\cong\) (Fred\(_{(k+2,l+2)}^{(k+2,l+2)}\), \(a\)),
4. (Fred\(_{(k,l)}^{(k,l)}\), \(a\)) \(\cong\) (Fred\(_{(k+2,l)}^{(k+2,l)}\), \(a\)) and (Fred\(_{(k,l)}^{(k,l)}\), \(a\)) \(\cong\) (Fred\(_{(k+2,l+2)}^{(k+2,l+2)}\), \(a\)),
5. (Fred\(_{(k,l)}^{(k,l)}\), \(a\)) \(\cong\) (Fred\(_{(k,l+8)}^{(k,l+8)}\), \(a\)) and (Fred\(_{(k,l)}^{(k,l)}\), \(a\)) \(\cong\) (Fred\(_{(k+8,l+8)}^{(k+8,l+8)}\), \(a\)),
6. (Fred\(_{(k+1,l+1)}^{(k+1,l+1)}\), \(a\)) \(\cong\) (Fred\(_{(k+2,l+2)}^{(k+2,l+2)}\), \(a\)),

where \(\tilde{a} = q\) when \(a = \tau\) and \(\tilde{a} = r\) when \(a = q\).

**Proof.** Once the Clifford module structure on the left-hand side of these homeomorphisms is fixed, that on the right-hand side is given following the isomorphisms of Clifford algebras in Lemma A.1. By using Lemma A.2, the \(\mathbb{Z}_2\)-homeomorphisms are given as follows.

1. The map (Fred\(_{(k,l)}^{(k,l)}\)(\(V\), Ad\(_a\)) \(\rightarrow\) (Fred\(_{(k+1,l+1)}^{(k+1,l+1)}\)(\(V \otimes \Delta_{1,1}\), Ad\(_a\otimes_{\text{tr}}\)), Ad\(_a\otimes_{\text{tr}}\)) given by \(A \mapsto A \otimes \omega_{1,1}\) is a \(\mathbb{Z}_2\)-homeomorphism. The other one is proved similarly.
2. The map (Fred\(_{(k,l)}^{(k,l)}\)(\(V\), Ad\(_a\)) \(\rightarrow\) (Fred\(_{(k,l+2)}^{(k,l+2)}\)(\(V \otimes \Delta_{0,4}\), Ad\(_a\otimes_{\text{gr}}\)), Ad\(_a\otimes_{\text{gr}}\)) given by \(A \mapsto A \otimes \omega_{0,4}\) is a \(\mathbb{Z}_2\)-homeomorphism. The other one and (2), (4) and (5) follow in a similar way.
3. The map (Fred\(_{(k+1,l+1)}^{(k+1,l+1)}\)(\(V\), Ad\(_a\)) \(\rightarrow\) (Fred\(_{(k+2,l+2)}^{(k+2,l+2)}\)(\(V\), Ad\(_a\)), Ad\(_a\)) given by \(A \mapsto A \epsilon_{1}\) is a \(\mathbb{Z}_2\)-homeomorphism. \(\square\)

**Proposition A.5.** The following \(\mathbb{Z}_2\)-homotopy equivalences exist.

1. (Fred\(_{(k+1,l+1)}^{(k+1,l+1)}\), \(a\)) \(\simeq\) (\(\Omega_{\epsilon_{1}}\text{Fred}_{\text{sk}}^{(k,l)}\), \(a\)), for \(k \geq 1\) and \(l \geq 0\).
2. (Fred\(_{(k+1,l+1)}^{(k+1,l+1)}\), \(a\)) \(\simeq\) (\(\Omega_{\epsilon_{1}}\text{Fred}_{\text{sa}}^{(k,l)}\), \(a\)), for \(k \geq 0\) and \(l \geq 1\).
3. (Fred\(_{(k,l)}^{(k,l)}\), \(a\)) \(\simeq\) (\(\Omega_{1}\text{Fred}_{\text{sk}}^{(k,l)}\), \(a\)).

Proposition A.5 is proved as in [9]. In what follows, we outline its proof since some spaces introduced there are of our interest.

**Proposition A.6.** The following maps are \(\mathbb{Z}_2\)-homotopy equivalences.
Corollary A.9. The following $Z_2$-homotopy equivalences exist.

(1) $(\text{Fred}_{k,l}^0, \alpha) \simeq (\Omega^k \text{Fred}, \alpha)$.

(2) $(\text{Fred}_{k,l}^0, \alpha) \simeq (\Omega^k \text{Fred}, \alpha)$.

(3) $(\text{Fred}_{k,l}^0, \alpha) \simeq (\Omega^k \text{Fred}, \alpha)$. 

Proof. By Remark A.3 it is sufficient to show that these maps are weak $Z_2$-homotopy equivalences. Equivalently, to show that $p_i$ and its restriction to the $Z_2$-fixed point sets (the map $p_i^{Fred}: (\text{Fred}_{k,l}^0)^a \rightarrow (-U_{\text{cpt}}^1)^a$ in the case of (1)) are weak homotopy equivalences. They are proved by using quasifibrations on some dense subspaces of contractible fibers as in [9].

Lemma A.8. There is a $Z_2$-homeomorphism $(\text{Fred}, \alpha) \cong (\text{Fred}_{k,l}^0, \alpha)$.

Proof. This is given by a $Z_2$-map $(\text{Fred}(\mathcal{V}), \alpha) \rightarrow (\text{Fred}_{k,l}^0(\mathcal{V} \oplus \mathcal{V}), a \oplus a)$, $A \mapsto \begin{pmatrix} 0 & A^* \\ A & 0 \end{pmatrix}$, where the action of $\text{Cl}_{0,1}$ on $\mathcal{V} \oplus \mathcal{V}$ is given by $\epsilon_1 = \text{diag}(1, -1)$. 

Proposition A.4, Proposition A.5 and Lemma A.8 lead to the following.
(4) \((\text{Fred}^{(k,l)}_*, q) \simeq (\Omega^{1-k+2} \text{Fred}, r)\).

When the subscript \(m\) on \(\Omega^m\) is negative, this should be replaced by \(m+8n\) by taking a sufficiently large integer \(n\) to make the subscript non-negative.

Note that when \(k\) and \(l\) are relatively small, we further have the following \(\mathbb{Z}_2\)-homeomorphisms.

Lemma A.10. Multiplication by the imaginary unit \(i = \sqrt{-1}\) induces the following \(\mathbb{Z}_2\)-homeomorphisms:

1. \((\text{Fred}^{(0,2)}_*, \text{Ad}_r) \to (\text{Fred}^{(1,1)}_*, \text{Ad}_r), \text{ where } \tilde{r} = r e_1.\)
2. \((\text{Fred}^{(0,2)}_*, \text{Ad}_q) \to (\text{Fred}^{(1,1)}_*, \text{Ad}_q), \text{ where } \tilde{q} = q e_1.\)
3. \((\text{Fred}^{(1,1)}_*, \text{Ad}_r) \to (\text{Fred}^{(2,0)}_*, \text{Ad}_r), \text{ where } \tilde{r} = q e_1.\)
4. \((\text{Fred}^{(1,1)}_*, \text{Ad}_r) \to (\text{Fred}^{(2,0)}_*, \text{Ad}_q), \text{ where } \tilde{q} = -r e_1.\)

Remark A.11. The \(\mathbb{Z}_2\)-spaces in Lemma A.10 appear in the study of topological insulators. Specifically, Table 11 is obtained by taking the quantum symmetries as real or quaternionic Clifford module structures as follows.

- In class BDI, we put \(r = \Theta\) and \(e_1 = \Pi\) in (1); then, \(\tilde{r} = \Xi.\)
- In class CII, we put \(q = \Theta\) and \(e_1 = \Pi\) in (2); then, \(\tilde{q} = \Xi.\)
- In class DIII, we put \(q = \Theta\) and \(e_1 = i \Pi\) in (3); then, \(\tilde{r} = \Xi.\)
- In class CI, we put \(r = \Theta\) and \(e_1 = 1\Pi\) in (4); then, \(\tilde{q} = \Xi.\)

For \(l \geq 1,\) let us consider the map

\[
(\text{Fred}^{(k,l+1)}_*, a) \times (\text{Fred}^{(k',l'+1)}_*, a') \to (\text{Fred}^{(k+k',l+l')}_*, a \otimes a')
\]

defined by \((A, B) \to A \otimes 1 + \epsilon_l \otimes B,\) where the Clifford action to define \(\text{Fred}^{(k+k',l+l')}_*\) is generated by \(\tilde{e}_i = e_i \otimes 1 (i = 1, \ldots, k), \tilde{e}_{k+i} = \epsilon_l \otimes e_i (i = 1, \ldots, k'), \tilde{e}_i = e_i \otimes 1 (i = 1, \ldots, l - 1)\) and \(\epsilon_{l+i-1} = \epsilon_l \otimes e_i (i = 1, \ldots, l').\) This map induces the exterior product of topological \(KR\)-groups as in [9].

A.2. Relation with Boersema–Loring’s Unitary Picture. In this subsection, we discuss a relation between these \(\mathbb{Z}_2\)-spaces of self-adjoint/skew-adjoint Fredholm operators and Boersema–Loring’s \(K\)-theory.

Let \(\{W_i\}_{i \in I}\) be the set of mutually inequivalent irreducible real (resp. quaternionic) representations of \(C_k, l\) with hermitian inner-products which \(\{W_i\}_{i \in I}\) consists of one or two elements corresponding to \(k\) and \(l.\) Let \(W = \oplus_{i \in I} W_i\) and \(V = l^2(\mathbb{Z}_{\geq 0}) \otimes W\) which has a real (resp. quaternionic) \(C_k, l\)-module structure induced by that of \(\{W_i\}_{i \in I}.\) We take a complete orthonormal basis \(\{\delta_j\}_{j \in \mathbb{Z}_{\geq 0}}\) of \(l^2(\mathbb{Z}_{\geq 0})\) given by generating functions of each points in \(\mathbb{Z}_{\geq 0}.\) Let \(V_n\) be the subspace of \(V\) spanned by \(\{\delta_j \otimes w \mid 0 \leq j \leq n, w \in W\},\) which is a real (resp. quaternionic) \(C_k, l\)-module. Let \(GL_{\text{cpt}}\) be the space of invertible operators on \(V\) of the form \(e_k + T\) for some compact operator \(T.\) Let \(GL^{(k,l)}_n = GL_{\text{cpt}} \cap B^{(k,l)}_n(V)\) and \(GL^{(k,l)} = GL_{\text{cpt}} \cap B^{(k,l)}(V).\) Let \(GL_n^{(k,l)}\) (resp. \(GL'_n^{(k,l)}\)) be the subspace of \(B^{(k,l)}_n(V_n)\) (resp. \(B^{(k,l)}(V_n)\)) consisting of invertible operators, and let \(U_n^{(k,l)}\) (resp. \(U'_n\)) be its subspace of unitaries. We have an injection \(GL_n^{(k,l)} \to GL_{n+1}\) (resp. \(GL_n^{(k,l)} \to GL^{(k,l)}_{n+1}\)) given by mapping \(A\) to \(A \oplus e_k\) (resp. \(A \oplus e_l\)), and let \(GL_n^{(k,l)}\) (resp. \(GL'_n\)) be the inductive limit \(\lim GL_n^{(k,l)}\) (resp. \(\lim GL'_n\)). We also define \(U_n^{(k,l)}\) and \(U_n'\) for unitaries in the same way. The space \(GL_n^{(k,l)}\) (resp. \(GL'_n\))
is identified with the subspace of $GL_{\text{cpt}}^{(k,l)}$ (resp. $GL_{\text{cpt}}^{(k,l)}$) consisting of operators of the form $e_k + T$ (resp. $e_l + T$), where $T \in B(V_n)$, and we have an injective $\mathbb{Z}_2$-map $(GL_{\infty}^{(k,l)}, a) \to (GL_{\text{cpt}}^{(k,l)}, a)$ (resp. $(GL_{\infty}^{(k,l)}, a) \to (GL_{\text{cpt}}^{(k,l)}, a)$). As in [53], the following holds:

**Proposition A.12.** The map $(GL_{\infty}^{(k,l)}, a) \to (GL_{\text{cpt}}^{(k,l)}, a)$ and the map $(GL_{\infty}^{(k,l)}, a) \to (GL_{\text{cpt}}^{(k,l)}, a)$ are $\mathbb{Z}_2$-homotopy equivalences.

By using a deformation of invertibles to unitaries, $(U_{\infty}^{(k,l)}, a)$ and $(U_{\infty}^{(k,l)}, a)$ are $\mathbb{Z}_2$-homotopy equivalent to $(U_{\text{cpt}}^{(k,l)}, a)$ and $(U_{\text{cpt}}^{(k,l)}, a)$, respectively. We denote $U_{\bullet}$ for these subspaces of $U_{\bullet}$ as indicated in Table 11.

These $\mathbb{Z}_2$-spaces of unitaries appears in Boersema–Loring’s $KO$-theory [14]. Let $(X, \zeta)$ be a compact Hausdorff $\mathbb{Z}_2$-space, and consider a $C^*\tau$-algebra $(C(X), \tau_{\zeta})$ of continuous functions on $X$, whose transposition $\tau_{\zeta}$ is given by $f^\tau_{\zeta}(x) = f(\zeta(x))$. Then, the $\mathbb{Z}_2$-homotopy classes $[(X, \zeta), U_{\bullet}]_{\mathbb{Z}_2}$ can be identified with the group $KO_{1}(\bullet)^{-1}(C(X), \tau_{\zeta})$ where $i(\bullet)$ is as indicated in Table 10. In the following, we discuss two of eight $KO$-groups and the others are discussed in a similar way.

As for the $KO_{1}$-group, an element of the set $[(X, \zeta), (U_{\infty}, \tau \circ *)]_{\mathbb{Z}_2}$ is represented by a $\mathbb{Z}_2$-map $f: (X, \zeta) \to (U_{\text{cpt}}, \tau \circ *)$. This $f$ is a unitary element of $M_{\text{cpt}}(C(X))$ satisfying $f(\zeta(x)) = \tau(f(x))\tau$ which is the same as the relation $f^\tau = f$ to define $KO_{1}$-groups. Thus, the set $[(X, \zeta), (U_{\infty}, \tau \circ *)]_{\mathbb{Z}_2}$ is the same as $KO_{1}(C(X), \tau_{\zeta})$ by the definition of Boersema–Loring’s $KO_{1}$-group.

Finally, we discuss the $KO_{6}$-group. By the multiplication of $-i$, we have a $\mathbb{Z}_2$-homeomorphism $(U_{\infty}^{(1,0)}, q) \to (U_{\infty}^{(0,1)}, -q)$. A $\mathbb{Z}_2$-continuous map $f: (X, \zeta) \to (U_{\infty}^{(1,0)}, q)$ is a self-adjoint unitary in $M_{\text{cpt}}(C(X))$ satisfying $f^{\tau} = -f^* = -f$. The Clifford algebra $Cl_{1,0}^{1,0}$ has just one irreducible quaternionic representation up to equivalence, which is constructed as follows. On $W = \mathbb{C}^{2}$, we consider the action $\rho$ of $Cl_{1,0}^{1,0} \otimes Cl_{2,0}$ defined as follows:

\[
\rho(1 \otimes e_1) = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}, \quad \rho(1 \otimes e_2) = \begin{pmatrix} 0 & -c \\ c & 0 \end{pmatrix}, \quad \rho(e_1 \otimes 1) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},
\]

where $c$ is the complex conjugation on $\mathbb{C}$. The space $U_{\infty}^{(1,0)}$ is defined as the inductive limit of maps $U_{n}^{(1,0)} \to U_{n+1}^{(1,0)}$, $A \mapsto A \oplus I$ where $I = \rho(e_1 \otimes 1)$ and the space $U_{\infty}^{(0,1)}$ is defined as that of maps $A \mapsto A \oplus -iI$ where $-iI = I^{(0)}$.

A.3. Boersema–Loring’s $K$-Theory and Exponential Maps. We describe boundary maps of the 24-term exact sequence of $KO$-theory (which we denote as $\partial_{i}^{\text{BL}}$ in this section) in Boersema–Loring’s unitary picture through exponential maps. The map $\partial_{i}^{\text{BL}}$ for even $i$ has already been expressed as an exponential map in [14]; thus, we focus on $\partial_{i}^{\text{BL}}$ for odd $i$. A clue is the exponential maps given in Proposition A.7.

For a short exact sequence of $C^*\tau$-algebras,

\[
(A.3) \quad 0 \to (\mathcal{I}, \tau) \to (\mathcal{A}, \tau) \xrightarrow{\gamma} (\mathcal{B}, \tau) \to 0,
\]

and for each odd $i$, we construct a group homomorphism

\[
(A.4) \quad \partial_{i}^{\exp}: KO_{i}(\mathcal{B}, \tau) \to KO_{i-1}(\mathcal{I}, \tau)
\]
and show they coincide with $\partial_1^{\text{BL}}$ up to a factor of $-1$. Let $W_{2n} \in M_{2n}(\mathbb{C})$ and $Q_{4n} \in M_{4n}(\mathbb{C})$ be the following matrices:

$$W_{2n} = \frac{1}{\sqrt{2}} \begin{pmatrix} i \cdot 1_n & 1_n \\ 1_n & i \cdot 1_n \end{pmatrix}, \quad Q_{4n} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1_{2n} & -I_{2n}^{(2)} \\ I_{2n}^{(2)} & 1_{2n} \end{pmatrix},$$

and let $V_{2n} \in M_{2n}(\mathbb{R})$ and $X_{4n} \in M_{4n}(\mathbb{R})$ be the permutation matrices satisfying

$$V_{2n} \text{diag}(x_1, \ldots, x_{2n})V_{2n}^* = \text{diag}(x_1, x_{n+1}, x_2, x_{n+2}, \ldots, x_n, x_{2n}),$$

$$X_{4n} \text{diag}(x_1, \ldots, x_{4n})X_{4n}^* = \text{diag}(x_1, x_2, x_{2n+1}, x_{2n+2}, x_3, x_4, x_{2n+3}, x_{2n+4}, \ldots, x_{4n}).$$

As in [14], let $Y_{2n}^{(-1)} = V_{2n}W_{2n}, \ Y_{2n}^{(1)} = V_{2n}, \ Y_{4n}^{(3)} = V_{4n}Q_{4n}W_{4n}$ and $Y_{4n}^{(5)} = X_{4n}$.

**Definition A.13.** Suppose we have a short exact sequence of $C^*$-$\tau$-algebras as in (A.3). We assume $\mathcal{I} = \text{Ker}(\phi)$ and identify the unit in $\mathcal{I}$ with that of $\mathcal{A}$.

For $i \in \{-1, 1, 3, 5\}$, suppose $[u] \in KO_i(\mathcal{B}, \tau)$, where $u \in M_{n_i}(\mathcal{B})$ is a unitary satisfying the relation $\mathcal{R}_i$ and $\lambda(u) = I_n^{(i)}$, for which $n_i, \mathcal{R}_i$ and $I^{(i)}$ are as in Table 2. Let $a$ in $M_{n_i}(\mathcal{A})$ be a lift of $u$ satisfying the relation $\mathcal{R}_i$ and $\|a\| \leq 1$. Then, define

$$\partial_i^{\text{exp}}([u]) = [-Y_{2n_i}^{(-1)}(\epsilon_1 \exp(\pi \epsilon_1 A))]Y_{2n_i}^{(1)*}] \in KO_i(\mathcal{I}, \tau),$$

where $A = \begin{pmatrix} 0 & a^* \\ a & 0 \end{pmatrix}$ and $\epsilon_1 = \text{diag}(1_{n_i}, -1_{n_i})$.

**Lemma A.14.** $\partial_i^{\text{exp}}$ for odd $i$ are well-defined group homomorphisms.

**Proof.** We need to show that (a) the unitaries constructed all satisfy the correct relation, (b) the choice of lift is not important, (c) some lift is always available, (d) homotopy is respected, (e) compatible with respect to the stabilization by $I^{(i)}$ and (f) the addition is respected. (c), (b) and (d) are proved in the same way as in Lemma 8.2 of [14] and we discuss the other parts. For convenience, let $C(a) = -\epsilon_1 \exp(\pi \epsilon_1 A)$ and $C'(a) = Y_{2n_i}^{(1)*}C(a)Y_{2n_i}^{(1)*}$.

1. We first consider the case of $i = 1$. Let $u \in M_n(\mathcal{B})$ be a unitary satisfying $u^\tau = u^*$ and $\lambda_n(u) = I_n^{(1)}$. We take a lift $a \in M_n(\mathcal{A})$ of $u$ such that $\|a\| \leq 1$ and $a^\tau = a^*$. Since $\varphi(C'(a)) = V_{2n} \epsilon_1 V_{2n}^* = I_n^{(0)}$, we have $C'(a) \in M_n(\mathcal{I})$ and $\lambda(C'(a)) = I_n^{(0)}$. Since $A^\tau = A$, we have

$$C(a)^\tau = -\exp(\pi A)^\tau \epsilon_1^* = -\epsilon_1^* \exp(\pi \epsilon_1 A)^\tau \epsilon_1 = -\epsilon_1 \exp(\pi \epsilon_1 A \tau) \epsilon_1^* = C(a).$$

Since $Y_{2n}^{(1)} = V_{2n}$ is the orthogonal matrix, $(Y_{2n}^{(1)})^\tau = Y_{2n}^{(1)*}$, and thus, $C'(a)^\tau = C'(a)$ holds. When $u = 1$, we can take $a = 1$ and $C'(1) = I^{(1)}$ in this case. Combined with this, the proof is completed once we have checked that $\partial_i^{\text{exp}}$ preserves the addition. Let $u \in M_n(\mathcal{B})$ and $v \in M_n(\mathcal{B})$.

We take their lift $a$ and $b$ such that $a^\tau = a^*$ and $b^\tau = b^*$. Then, we have $C'(\text{diag}(a, b)) = \text{diag}(C'(a), C'(b))$ since

$$C\left(\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\right) = -\begin{pmatrix} 1_{m+n} & 0 \\ 0 & -1_{m+n} \end{pmatrix} \exp\left(\pi\begin{pmatrix} 0 & \text{diag}(-a^*, -b^*) \\ \text{diag}(a, b) & 0 \end{pmatrix}\right).$$

---

18Matrices $W_{2n}, \ Q_{4n}, \ V_{2n}$ and $X_{4n}$ are what we borrowed from Sect. 8 of [14]. Some of the basic formulas that they satisfy can be found there.
\[
\text{Ad}_{V_{2m+n}} \exp \left( \pi \begin{pmatrix}
0 & \text{diag}(-a^*,-b^*) \\
\text{diag}(a,b) & 0
\end{pmatrix} \right) = \exp \left( \pi \cdot \text{diag} \left( \text{Ad}_{V_{2m}} \left( \begin{pmatrix}
0 & -a^* \\
a & 0
\end{pmatrix}, \text{Ad}_{V_{2n}} \left( \begin{pmatrix}
0 & -b^* \\
b & 0
\end{pmatrix}\right) \right) \right),
\]

which shows that \( \partial^\text{exp}_1([u] + [v]) = \partial^\text{exp}_1([u]) + \partial^\text{exp}_1([v]) \).

(2) We next consider the case of \( i = -1 \). Let \( u \in M_n(\mathcal{B}) \) be a unitary satisfying \( u^\tau = u \) and \( \lambda_n(u) = I_n^{(1)^{-1}} \). We take a lift \( a \in M_n(\tilde{\mathcal{A}}) \) of \( u \) such that \( \|a\| \leq 1 \) and \( a^\tau = a \). Since \( \varphi(C'(a)) = I_n^{(6)} \), we have \( C'(a) \in M_n(\tilde{T}) \) and \( \lambda(C'(a)) = I_n^{(6)} \). Since \( A_1^{\otimes \tau} = -A \), we have

\[
C(a)\tilde{T} = -\exp(\pi A_1)\tilde{T} = \exp(\pi \epsilon_1 A_{-1}) = \exp(\pi \epsilon_1 A) = -C(a).
\]

Since \( (V_{2n} x V_{2n}^*) \tilde{T} = V_{2n} x V_{2n}^* \tilde{T} \), and \( W_{2n}^{\otimes \tau} = -W_{2n}^\tau \), we have \( C'(a) \tilde{T} = -C'(a) \). For \( u = 1 \), we take \( a = 1 \) and \( C'(1) = I_n^{(6)} \) holds. Therefore, as in (1), all we have to show is the additivity of \( \partial^\text{exp}_1 \). Let \( a \in M_m(\tilde{\mathcal{A}}) \) and \( b \in M_m(\tilde{T}) \) be lifts of the unitaries \( u \) and \( v \). Then, \( C'(\text{diag}(a,b)) = \text{diag}(C'(a), C'(b)) \) follows from

\[
\left( V_{2m+2n} W_{2m+2n} \begin{pmatrix}
0 & \text{diag}(-a^*,-b^*) \\
\text{diag}(a,b) & 0
\end{pmatrix} \right) \left( W_{2m+2n}^* V_{2m+2n}^* \right) = \text{diag} \left( V_{2m} W_{2m} \begin{pmatrix}
0 & -a^* \\
a & 0
\end{pmatrix} W_{2m}^* V_{2m}^*, V_{2n} W_{2n} \begin{pmatrix}
0 & -b^* \\
b & 0
\end{pmatrix} W_{2n}^* V_{2n}^* \right).
\]

(3) Let us consider the case of \( i = 5 \). Let \( u \in M_{2n}(\tilde{\mathcal{B}}) \) be a unitary satisfying \( u^{\tilde{T}} = u^* \) and \( \lambda_{2n}(u) = I_{2n}^{(5)} = 1_{2n} \). We take a lift \( a \in M_{2n}(\tilde{\mathcal{A}}) \) of \( u \) such that \( \|a\| \leq 1 \) and \( a^{\tilde{T}} = a^* \). Since \( (X_{4n} x X_{4n})^{\tilde{T}} = X_{4n} x X_{4n} \), \( A_2^{\tilde{T}} = A \) and \( \epsilon_1^{\tilde{T}} = \epsilon_1 \), the relation

\[
C'(a)^{\tilde{T}} = -X_{4n} \exp(\pi \epsilon_1 A) = -X_{4n} \exp(\pi \epsilon_1 A) \epsilon_1 X_{4n}^* = C'(a)
\]

holds. We have \( C'(1_2) = I_n^{(4)} \), and for the additivity of \( \partial^\text{exp}_5 \), note that

\[
X_{4m+4n} \begin{pmatrix}
0 & \text{diag}(-a^*,-b^*) \\
\text{diag}(a,b) & 0
\end{pmatrix} X_{4m+4n}^* = \text{diag} \left( X_{4m} \begin{pmatrix}
0 & -a^* \\
a & 0
\end{pmatrix} X_{4m}^*, X_{4n} \begin{pmatrix}
0 & -b^* \\
b & 0
\end{pmatrix} X_{4n}^* \right).
\]

(4) Consider the case of \( i = 3 \). Let \( u \in M_{2n}(\tilde{\mathcal{B}}) \) be a unitary satisfying \( u^{\tilde{T}} = u \) and \( \lambda_{2n}(u) = I_{2n}^{(1)} = 1_{2n} \). We take a lift \( a \in M_{2n}(\tilde{\mathcal{A}}) \) of \( u \) such that \( \|a\| \leq 1 \) and \( a^{\tilde{T}} = a \). Since \( A_1^{\tilde{T}} = -A \) and \( \epsilon_1^{\tilde{T}} = -\epsilon_1 \), the relation \( C(a)^{\tilde{T}} = -C(a) \) holds. Since \( (Q_{4n} x Q_{4n})^{\tilde{T}} = Q_{4n} x \tilde{T} Q_{4n} \) and \( W_{4n}^{\tilde{T}} = -W_{4n}^\tau \), we have \( C'(a) = -C'(a) \). For the remaining part, we
Then, we have \( \iota \circ \phi \) each map. We assume \( \ker(\iota) \).

**Proof.** As in Lemma 8 of [14], this lemma is proved by following the definition of each map. We assume \( \ker(\iota) \).

**Lemma A.15.** Each \( \partial_i^{\exp} \) is natural with respect to the morphisms of short exact sequences of \( C^* \)-algebras. That is, suppose we have the following commutative diagram of exact lows:

\[
\begin{array}{ccc}
0 & \longrightarrow & (\mathcal{I}_1, \tau) \\
\downarrow & & \downarrow \alpha \\
(\mathcal{I}_2, \tau) & \longrightarrow & (\mathcal{A}_1, \tau) \\
\downarrow & & \downarrow \beta \\
(\mathcal{A}_2, \tau) & \longrightarrow & (\mathcal{B}_1, \tau) \\
\downarrow & & \downarrow \phi_2 \\
0 & \longrightarrow & (\mathcal{B}_2, \tau)
\end{array}
\]

Then, we have \( \iota_* \circ \partial_i^{\exp} = \partial_i^{\exp} \circ \beta_* \).

**Proof.** As in Lemma 8.5 of [14], this lemma is proved by following the definition of each map. We assume \( \ker(\iota) \).

**Proposition A.16.** \( \partial_i^{BL} = -\partial_i^{exp} \) for odd \( i \).

As in the proof of Theorem 8.9 of [14], we can reduce the proof to the complex case, and Proposition A.10 follows from the lemma below. In the complex case, the boundary map \( \partial_i^{exp}: K_1(B) \to K_0(I) \) is defined by forgetting the real structure in the case of \( i = 1 \) of Definition A.13

**Lemma A.17.** The boundary maps \( \partial_i^{BL} \) and \( \partial_i^{exp} \) from \( K_1(B) \) to \( K_0(I) \) satisfy the relation \( \partial_i^{BL} = -\partial_i^{exp} \).

**Proof.** Suppose that \( [u] \in K_1(B) \) where \( u \in M_n(\hat{B}) \) and \( \lambda(u) = 1_n \). We take a lift \( \alpha \) of \( u \) in \( M_n(\hat{A}) \) satisfying \( ||\alpha|| \leq 1 \). Consider the partial isometry \( V = \begin{pmatrix} \alpha & 0 \\ \sqrt{1 - \alpha^* \alpha} & 0 \end{pmatrix} \) and let \( V = \begin{pmatrix} 0 & v^* \\ v & 0 \end{pmatrix} \). \( \partial_1^{exp}([u]) \) is computed as

\[
\partial_1^{exp}([u]) = \begin{pmatrix} -Y_{4n}^{(1)}(\epsilon_1 \exp(\pi V \epsilon_1))Y_{4n}^{(1)*} \\ [Y_{4n}^{(1)}(1 - 2v^* v)]Y_{4n}^{(1)*} \end{pmatrix} - [Y_{4n}^{(1)}(1 - 2v^* v)]Y_{4n}^{(1)*}.
\]

As in [14], \( \partial_1^{BL}([u]) \) is also expressed by using \( v \), which is \( -\partial_1^{exp}([u]) \).

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References

[1] Alldridge, A., Max, C., Zirnbauer, M.R.: Bulk-boundary correspondence for disordered free-fermion topological phases. Comm. Math. Phys. 377(3), 1761–1821 (2020).

[2] Altland, A., Zirnbauer, M.R.: Nonstandard symmetry classes in mesoscopic normal-superconduction hybrid structures. Phys. Rev. B 55(2), 1142–1161 (1997).

[3] Antonian, S.: Equivariant embeddings into $G$-ARs. Glas. Mat. Ser. III 22(2), 503–533 (1987).

[4] Antonyan, S.A., Elfving, E.: The equivariant homotopy type of $G$-ANR’s for proper actions of locally compact groups. In: Algebraic topology—old and new, Banach Center Publ., vol. 85, pp. 155–178. Polish Acad. Sci. Inst. Math., Warsaw (2009).

[5] Araki, H., Mizoguchi, T., Hatsugai, Y.: $Z_Q$ Berry phase for higher-order symmetry-protected topological phases. Phys. Rev. Research 2, 012009 (2020).

[6] Atiyah, M.F.: $K$-theory and reality. Quart. J. Math. Oxford Ser. (2) 17, 367–386 (1966).

[7] Atiyah, M.F., Bott, R., Shapiro, A.: Clifford modules. Topology 3, 3–38 (1964).

[8] Atiyah, M.F., Patodi, V.K., Singer, I.M.: Spectral asymmetry and Riemannian geometry. III. Math. Proc. Cambridge Philos. Soc. 79(1), 71–99 (1976).

[9] Bellissard, J.: The quantum Hall effect in K-theory. J. Math. Phys. 35(10), 5373–5451 (1994).

[10] Benalcazar, W.A., Bernevig, B.A., Hughes, T.L.: Quantized electric multipole insulators. Science 357, 61–66 (2017).

[11] Boersema, J.L.: Real $C^*$-algebras, united $K$-theory, and the Künneth formula. $K$-Theory 26(4), 345–402 (2002).

[12] Bourne, C., Carey, A.L., Lesch, M., Rennie, A.: The KO-valued spectral flow for skew-adjoint Fredholm operators. J. Topol. Anal (2020)

[13] Bourne, C., Carey, A.L., Lesch, M., Rennie, A.: The KO-valued spectral flow for skew-adjoint Fredholm operators. J. Topol. Anal (2020)

[14] Bourne, C., Kellendonk, J., Rennie, A.: The $K$-theoretic bulk-edge correspondence for topological insulators. Ann. Henri Poincaré 18(5), 1833–1866 (2017).

[15] Douglas, R.G.: Banach algebra techniques in the theory of Toeplitz operators. CBMS Regional Conference Series in Mathematics, no. 15. American Mathematical Society, Providence, R.I. (1973).

[16] Douglas, R.G., Howe, R.: On the $C^*$-algebra of Toeplitz operators on the quarterplane. Trans. Amer. Math. Soc. 158, 203–217 (1971).

[17] Freed, D.S., Moore, G.W.: Twisted equivariant matter. Ann. Henri Poincaré 14(8), 1927–2023 (2013).

[18] Friedrich, T.: Dirac Operators in Riemannian Geometry. Graduate Studies in Mathematics, vol. 25. American Mathematical Society, Providence, RI (2000). Translated from the 1997 German original by Andreas Nestke

[19] Geier, M., Trifunovic, L., Hoskam, M., Brouwer, P.W.: Second-order topological insulators and superconductors with an order-two crystalline symmetry. Phys. Rev. B 97, 205135 (2018).

[20] Goodearl, K.R.: Notes on Real and Complex $C^*$-Algebras. Shiva Mathematics Series, vol. 5. Shiva Publishing Ltd., Nantwich (1982).
[29] Großmann, J., Schulz-Baldes, H.: Index pairings in presence of symmetries with applications to topological insulators. Comm. Math. Phys. 343(2), 477–513 (2016).
[30] Hashimoto, K., Wu, X., Kimura, T.: Edge states at an intersection of edges of a topological material. Phys. Rev. B 95, 165443 (2017).
[31] Hatsugai, Y.: Chern number and edge states in the integer quantum hall effect. Phys. Rev. Lett. 71(22), 3697–3700 (1993).
[32] Hatsugai, Y.: Topological invariants and corner states for Hamiltonians on a three-dimensional lattice. Comm. Math. Phys. 364(1), 343–356 (2018).
[33] Hayashi, S.: Toeplitz operators on concave corners and topologically protected corner states. Lett. Math. Phys. 109(10), 2223–2254 (2019).
[34] Ji, R., Kaminker, J.: The $K$-theory of Toeplitz extensions. J. Operator Theory 19(2), 347–354 (1988).
[35] Jiang, X.: On Fredholm operators in quarter-plane Toeplitz algebras. Proc. Amer. Math. Soc. 123(9), 2823–2830 (1995).
[36] Kane, C.L., Mele, E.J.: $Z_2$ topological order and the quantum spin Hall effect. Phys. Rev. Lett. 95, 146802 (2005).
[37] Karoubi, M.: Espaces classifiants en $K$-théorie. Trans. Amer. Math. Soc. 147, 75–115 (1970).
[38] Kellendonk, J.: On the $C^*$-algebraic approach to topological phases for insulators. Ann. Henri Poincaré 18(7), 2251–2300 (2017).
[39] Kellendonk, J., Richter, T., Schulz-Baldes, H.: Edge current channels and Chern numbers in the integer quantum Hall effect. Rev. Math. Phys. 14(1), 87–119 (2002).
[40] Khalaf, E.: Higher-order topological insulators and superconductors protected by inversion symmetry. Phys. Rev. B 97, 205136 (2018).
[41] Khalaf, E., Po, H.C., Vishwanath, A., Watanabe, H.: Symmetry indicators and anomalous surface states of topological crystalline insulators. Phys. Rev. X 8, 031070 (2018).
[42] Kitaev, A.: Periodic table for topological insulators and superconductors. AIP Conf. Proc. 1134(1), 22–30 (2009).
[43] Kubota, Y.: Notes on twisted equivariant $K$-theory for $C^*$-algebras. Internat. J. Math. 27(6): 1650058 (2016).
[44] Kubota, Y.: Controlled topological phases and bulk-edge correspondence. Comm. Math. Phys. 349(2), 493–525 (2017).
[45] Kuiper, N.H.: The homotopy type of the unitary group of Hilbert space. Topology 3, 19–30 (1965).
[46] Kwasik, S.: On the homotopy type of $G$-manifolds and $G$-ANRs. Bull. Acad. Polon. Sci. Sér. Sci. Math. 28(9-10), 509–515 (1981).
[47] Lawson Jr., H.B., Michelsohn, M.L.: Spin Geometry. Princeton Mathematical Series, vol. 38. Princeton University Press, Princeton, NJ (1989).
[48] Mathai, V., Thiang, G.C.: T-Duality Simplifies Bulk-Boundary Correspondence. Comm. Math. Phys. 345(2), 675–701 (2016).
[49] Matumoto, T.: Equivariant $K$-theory and Fredholm operators. J. Fac. Sci. Univ. Tokyo Sect. I A Math. 18, 109–125 (1971).
[50] Matumoto, T.: On $G$-CW complexes and a theorem of J. H. C. Whitehead. J. Fac. Sci. Univ. Tokyo Sect. IA Math. 18, 363–374 (1971).
[51] Okugawa, R., Hayashi, S., Nakanishi, T.: Second-order topological phases protected by chiral symmetry. Phys. Rev. B 100, 235302 (2019).
[52] Okuma, N., Sato, M., Shiozaki, K.: Topological classification under nonmagnetic and magnetic point group symmetry: Application of real-space Atiyah-Hirzebruch spectral sequence to higher-order topology. Phys. Rev. B 99, 085127 (2019).
[53] Palais, R.S.: On the homotopy type of certain groups of operators. Topology 3, 271–279 (1965).
[54] Park, E.: Index theory and Toeplitz algebras on certain cones in $Z^2$. J. Operator Theory 23(1), 125–146 (1990).
[55] Park, E.L.: The index theory of Toeplitz operators on the skew quarter plane. Ph.D. thesis, State University of New York at Stony Brook (1988).
[56] Phillips, J.: Self-adjoint Fredholm operators and spectral flow. Canad. Math. Bull. 39(4), 460–467 (1996).
[57] Prodan, E., Schulz-Baldes, H.: Bulk and Boundary Invariants for Complex Topological Insulators: From $K$-theory to physics. Mathematical Physics Studies. Springer, Berlin (2016).
[58] Rørdam, M., Larsen, F., Laustsen, N.: An Introduction to $K$-Theory for $C^*$-Algebras, London Mathematical Society Student Texts, vol. 49. Cambridge University Press, Cambridge (2000).

[59] Schindler, F., Cook, A.M., Vergniory, M.G., Wang, Z., Parkin, S.S.P., Bernevig, B.A., Neuquert, T.: Higher-order topological insulators. Sci. Adv. 4(6), eaat0346 (2018).

[60] Schnyder, A.P., Ryu, S., Furusaki, A., Ludwig, A.W.W.: Classification of topological insulators and superconductors in three spatial dimensions. Phys. Rev. B 78, 195125 (2008).

[61] Schröder, H.: $K$-Theory for Real $C^*$-Algebras and Applications. Pitman Research Notes in Mathematics Series, vol. 290. Longman Scientific & Technical, Harlow; copublished in the United States with John Wiley & Sons, New York (1993).

[62] Simonenko, I.B.: Operators of convolution type in cones. Mat. Sb. (N.S.) 74(116), 298–313 (1967).

[63] Takahashi, R., Tanaka, Y., Murakami, S.: Bulk-edge and bulk-hinge correspondence in inversion-symmetric insulators. Phys. Rev. Research 2, 013300 (2020).

[64] Thiang, G.C.: On the $K$-theoretic classification of topological phases of matter. Ann. Henri Poincaré 17(4), 757–794 (2016).

[65] Thouless, D.J., Kohmoto, M., Nightingale, M.P., den Nijs, M.: Quantized Hall conductance in a two-dimensional periodic potential. Phys. Rev. Lett. 49, 405–408 (1982).

[66] Trifunovic, L., Brouwer, P.W.: Higher-order bulk-boundary correspondence for topological crystalline phases. Phys. Rev. X 9, 011012 (2019).

[67] Waner, S.: Equivariant homotopy theory and Milnor’s theorem. Trans. Amer. Math. Soc. 258(2), 351–368 (1980).

[68] Wood, R.: Banach algebras and Bott periodicity. Topology 4, 371–389 (1966).

[69] Xia, J.: The $K$-theory and the invertibility of almost periodic Toeplitz operators. Integral Equations Operator Theory 11(2), 267–286 (1988).

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