UPPER BOUNDS, COFINITENESS, AND ARTINIANNNESS OF LOCAL COHOMOLOGY MODULES DEFINED BY A PAIR OF IDEALS

M. AGHAPOURNAHR, KH. AHMADI-AMOLI, AND M. Y. SADEGHI

Abstract. Let \( R \) be a commutative noetherian ring, \( I, J \) be two ideals of \( R \), \( M \) be an \( R \)-module, and \( \mathcal{S} \) be a Serre class of \( R \)-modules. A positive answer to the Huneke’s conjecture is given for a noetherian ring \( R \) and minimax \( R \)-module \( M \) of krull dimension less than 3, with respect to \( \mathcal{S} \). There are some results on cofiniteness and artinianness of local cohomology modules with respect to a pair of ideals.

For a \( \mathbb{Z}D \)-module \( M \) of finite krull dimension and an integer \( n \in \mathbb{N} \), if \( H^i_{I,J}(M) \in \mathcal{S} \) for all \( i > n \), then \( H^i_{I,J}(M) / aH^i_{I,J}(M) \in \mathcal{S} \) for any \( a \in \tilde{\mathcal{W}}(I, J) \), all \( i \geq n \), and all \( j \geq 0 \). By introducing the concept of Seree cohomological dimension of \( M \) with respect to \((I, J)\), for an integer \( r \in \mathbb{N}_0 \), \( H^r_{I,J}(R) \in \mathcal{S} \) for all \( j > r \) iff \( H^r_{I,J}(M) \in \mathcal{S} \) for all \( j > r \) and any finite \( R \)-module \( M \).

1. Introduction

As a generalization of the notion of local cohomology modules, R. Takahashi, Y. Yoshino, and T. Yoshizawa [36], introduced local cohomology modules with respect to a pair of ideals. This paper is concerned about this new notion of local cohomology and Serre subcategories. For notations and terminologies not given in this paper, if necessary, the reader is referred to [36] and [1]. Throughout this paper, \( R \) is denoted a commutative noetherian ring with non-zero identity, \( I \), \( J \) are denoted two ideals of \( R \), and \( M \) is denoted an arbitrary \( R \)-module. The \((I,J)\)-torsion submodule \( \Gamma_{I,J}(M) \) of \( M \) is a submodule of \( M \) consists of all elements \( x \) of \( M \) with \( \text{Supp}(Rx) \subseteq W(I,J) \), in which \( W(I,J) = \{ p \in \text{Spec}(R) \mid I^n \subseteq p + J \text{ for an integer } n \geq 1 \} \).

For an integer \( n \), the \( n \)-th local cohomology functor \( H^n_{I,J} \) with respect to \((I,J)\) is the \( n \)-th right derived functor of \( \Gamma_{I,J} \). The \( R \)-module \( H^n_{I,J}(M) \) is called the \( n \)-th local cohomology module of \( M \) with respect to \((I,J)\). In the case \( J = 0 \), \( H^n_{I,J} \) coincides with the ordinary functor \( H^n_I \). Also, we are concerned with the following set of ideals of \( R \):

\[ \tilde{W}(I,J) = \{ a \subseteq R \mid I^n \subseteq a + J \text{ for an integer } n \geq 0 \} \]

A class of \( R \)-modules is called a Serre subcategory (or Serre class) of the category of \( R \)-modules when it is closed under taking submodules, quotients and extensions. Always, \( \mathcal{S} \) stands for a Serre class.

According to the third Huneke’s problem on local cohomology [23, Conjecture 4.3], one of the main problem in commutative algebra is finiteness of the socle of local cohomology modules on a local ring. Solving this problem, gives an answer to the finiteness of the set of associated primes of local cohomology modules.

On this area, some remarkable attempts have been done, e.g. see [24], [28], [29], [31], and [2].

In section 2, we give a positive answer to the Huneke’s conjecture more general for an arbitrary Serre
subcategory $\mathcal{S}$, instead of the category of finitely generated modules. Let $R$ be a noetherian (not necessary local) ring. Let $R/m \in \mathcal{S}$ for all $m \in \text{Max}(R)$. For any minimax $R$-module $M$ of krull dimension less than 3, we show that Ext$_R^i(R/m, H_I^j(M)) \in \mathcal{S}$ for any $m \in \text{Max}(R) \cap V(I)$ and all $i,j \geq 0$. In particular Hom$_R(R/m, H_I^j(M)) \in \mathcal{S}$ for any $m \in \text{Max}(R) \cap V(I)$ and all $i \geq 0$ (see Theorem 2.14).

We get the same result for local cohomology modules with respect to a pair of ideals, but in local case (see Theorem 2.18).

In section 3, we obtain some results on cofiniteness and artinianness of local cohomology with respect to a pair of ideals.

M. Aghapournahr and L. Melkersson in [1], obtained some conditions in which the ordinary local cohomology $H^I_N(M)$ belongs to $\mathcal{S}$ for all $i < n$ (from below). In section 4, as a complement of this work, we study some conditions in which the local cohomology $H^I_N(M)$ belongs to $\mathcal{S}$ for all $i > n$ (from top). For a ZD-module $M$ of finite krull dimension, we show that if the integer $n \in \mathbb{N}$ is such that $H^I_N(M) \in \mathcal{S}$ for all $i > n$, then the modules $H^I_M(M)/a^j H^I_N(M) \in \mathcal{S}$ for any $a \in \hat{W}(I,J)$, all $i \geq n$, and all $j \geq 0$ (see Theorem 4.4). Replacing $\mathcal{S}$ with some familiar Serre subcategories such as zero modules, finite modules, and artinian modules (resp.), we show that a necessary and sufficient condition for $H^I_N(M)$ to be zero, finite, and artinian (resp.), is the existence of an integer $m \in \mathbb{N}_0$ such that $H^I_N(M)$ is zero, finite, and artinian (resp.) (see Corollary 4.6).

More generally, for a finite $R$-module $M$ if $n \in \mathbb{N}_0$ is such that $H^I_N(M)$ belongs to $\mathcal{S}$ for all $i > n$ and $b$ is an ideal of $R$ such that $H^I_N(M/bM)$ belongs to $\mathcal{S}$, then $H^I_N(M/bM) \in \mathcal{S}$ (see Theorem 4.7). As a consequence of this theorem, we obtain a similar result as Corollary 4.6 for the finite module $M$ (see Corollary 4.10). Other consequence of this theorem is concerned about finiteness of $H^I_N(M)$ where $n = \text{cd}(I,J,M)$ or $n = \dim M \geq 1$ (see Corollary 4.11).

Finally, these results motivate us to introduce the concept of Serre cohomological dimension of $M$ with respect to $(I,J)$, $\text{cd}_S(I,J,M)$, as the supremum of non-negative integers $i$ such that $H^I_N(M) \not\in \mathcal{S}$. We give some characterizations to $\text{cd}_S(I,J,M)$ (Corollary 4.18) and as a main result of its properties, we show that for an integer $r \in \mathbb{N}_0$, $H^I_N(R) \in \mathcal{S}$ for all $j > r$ iff $H^I_N(M) \in \mathcal{S}$ for all $j > r$ and any finite $R$-module $M$ (see Corollary 4.23).

2. THE MEMBERSHIP OF THE $\text{HOM}_R(-, H^I_N(M))$ IN SERRE CLASSES

In this section, Proposition 2.3 plays a main role to obtain our results. For this purpose we need the following Lemmas.

**Lemma 2.1.** For a Serre class $\mathcal{S}$, we have $\mathcal{S} \neq \emptyset$ if and only if $R/m \in \mathcal{S}$ for some $m \in \text{Max}(R)$.

**Proof.** ($\Rightarrow$) Let $L \in \mathcal{S}$ and $0 \neq x \in L$. Then $(0:_Rx) \subseteq m$ for some $m \in \text{Max}(R)$. Now, since $Rx \in \mathcal{S}$, the assertion follows from the natural epimorphism $Rx \cong R/(0:_Rx) \rightarrow R/m$.

($\Leftarrow$) It is obvious. 

**Lemma 2.2.** Let $\mathcal{FL}$ be the class of finite length $R$-modules. Then $\mathcal{FL} \subseteq \mathcal{S}$ if and only if $R/m \in \mathcal{S}$ for all $m \in \text{Max}(R)$.

**Proof.** ($\Rightarrow$) It is obvious.

($\Leftarrow$) Let $N \in \mathcal{FL}$ and $\ell := \ell_R(N)$. So, consider the following chain of $R$-submodules of $N$:

$$0 = N_0 \subseteq N_1 \subseteq \cdots \subseteq N_\ell = N$$
in which, for all \(1 \leq j \leq \ell\), \(N_j/N_{j-1} \cong R/m\) for some \(m \in \text{Max}(R)\). Now, the assertion can be followed by induction on \(\ell\). \(\Box\)

**Corollary 2.3.** Let \((R,m)\) be a local ring and \(S \neq 0\). Then \(\mathcal{F} \subseteq \mathcal{S}\).

**Example 2.4.** In general case, it not true that \(R/m \in \mathcal{S}\) for all \(m \in \text{Max}(R)\). To see this, let \(R\) be a non-local ring and \(m \in \text{Max}(R)\). Let \(I\) be an ideal of \(R\) such that \(I \not\subseteq m\). Let \(S\) be the class of \(I\)-cofinite minimax \(R\)-modules, (see [34 Corollary 4.4]). Then \(\text{Supp}(R/m) \not\subseteq V(I)\) and so \(R/m \not\in \mathcal{S}\). For example, let \(R := \mathbb{Z}[x]\), \(m := (x-1)R\), and \(I := xR\).

**Proposition 2.5.** For a noetherian ring \(R\), we have

\(\) (i) If \(R/m \in \mathcal{S}\), for any \(m \in \text{Max}(R)\), and \(M\) is a finite or an artinian \(R\)-module, then \(\text{Ext}_R^j(R/m, M) \in \mathcal{S}\) for any \(m \in \text{Max}(R)\) and all \(j \geq 0\).

(ii) If \(R/m \in \mathcal{S}\), for any \(m \in \text{Max}(R)\), and \(M\) is a minimax \(R\)-module, then \(\text{Ext}_R^j(R/m, M) \in \mathcal{S}\) for any \(m \in \text{Max}(R)\) and all \(j \geq 0\).

(iii) If \((R,m)\) be a local ring, \(S \neq 0\), and \(M\) be a minimax \(R\)-module, then \(\text{Ext}_R^j(R/m, M) \in \mathcal{S}\) for all \(j \geq 0\).

**Proof.** (i) Let \(m \in \text{Max}(R)\) and \(j \geq 0\). Since \(\text{Ext}_R^j(R/m, M)\) is finite and is annihilated by \(m\), hence \(\text{Ext}_R^j(R/m, M)\) has finite length. Now, the result follows from Lemma 2.2.

(ii) Since \(M\) is minimax, there exists a short exact sequence

\[ 0 \to N \to M \to A \to 0, \]

where \(N\) is a finite module and \(A\) is an artinian module. This induces the exact sequence

\[ \cdots \to \text{Ext}_R^j(R/m, N) \to \text{Ext}_R^j(R/m, M) \to \text{Ext}_R^j(R/m, A) \to \cdots. \]

Now, the assertion follows from part (i).

(iii) Apply Lemma 2.1 and part (ii). \(\Box\)

**Corollary 2.6.** Let \(R/m \in \mathcal{S}\) for all \(m \in \text{Max}(R)\). Let \(t \in \mathbb{N}_0\) be such that \(H_t^{I,J}(M)\) is a minimax \(R\)-module. Then \(\text{Ext}_R^j(R/m, H_t^{I,J}(M)) \in \mathcal{S}\) for all \(i \geq 0\).

**Lemma 2.7.** Let \(a \in \mathbb{W}(I,J)\). Let \(X\) be an \(R\)-module. Then

\[ (0 :_X a) = (0 :_{\Gamma_{\alpha}(X)} a) = (0 :_{\Gamma_{\alpha,J}(X)} a) = (0 :_{\Gamma_{\alpha,I}(X)} a), \]

in particular, for any ideal \(b\) of \(R\) with \(b \supseteq I\), we have

\(\) (i) \((0 :_X b) = (0 :_{\Gamma_{\alpha}(X)} b) = (0 :_{\Gamma_{\alpha,J}(X)} b) = (0 :_{\Gamma_{\alpha,I}(X)} b) = (0 :_{\Gamma_{I,J}(X)} b)\).

(ii) \((0 :_X b) = (0 :_{\Gamma_{\alpha,J}(X)} b) \subseteq (0 :_{\Gamma_{\alpha,I}(X)} I) \subseteq (0 :_{\Gamma_{I,J}(X)} I) = (0 :_X I)\).

**Proof.** All proofs are easy and we leave them to the reader. \(\Box\)

**Proposition 2.8.** Let \(a \in \mathbb{W}(I,J)\) and \(t \in \mathbb{N}_0\). Consider the natural homomorphism

\[ \psi : \text{Ext}_R^t (R/a, M) \to \text{Hom}_R (R/a, H_t^{I,J}(M)). \]

(i) If \(\text{Ext}_R^{t-j} (R/a, H_t^{I,J}(M)) \in \mathcal{S}\) for all \(j < t\), then \(\text{Ker} \psi \in \mathcal{S}\).

(ii) If \(\text{Ext}_R^{t+1-j} (R/a, H_t^{I,J}(M)) \in \mathcal{S}\) for all \(j < t\), then \(\text{Coker} \psi \in \mathcal{S}\).
(iii) If \( \text{Ext}^{n-j}_R(R/a, H^j_{I,J}(M)) \in S \) for \( n = t, t+1 \) and for all \( j < t \), then \( \text{Ker} \psi \) and \( \text{Coker} \psi \) both belong to \( S \). Thus \( \text{Ext}^n_R(R/a, M) \in S \) iff \( \text{Hom}_R(R/a, H^j_{I,J}(M)) \in S \).

**Proof.** Let \( F(\cdot) = \text{Hom}_R(R/a, -) \) and \( G(\cdot) = \Gamma_{I,J}(\cdot) \). By Lemma 2.7 \( FG = F \). Now, the result can be followed by [2] Proposition 3.1. \( \square \)

The next result can be a generalization of main results of \([7], [25], [11], [18], [26], [8], \) and \([6]\).

**Theorem 2.9.** Let \( a \in \hat{W}(I,J) \) and \( t \in \mathbb{N}_0 \) be such that \( \text{Ext}^t_R(R/a, M) \in S \) and \( \text{Ext}^{n-j}_R(R/a, H^j_{I,J}(M)) \in S \) for \( n = t, t+1 \) and all \( j < t \). Then for any submodule \( N \) of \( H^j_{I,J}(M) \) such that \( \text{Ext}^1_R(R/a, N) \in S \), we have

(i) \( \text{Hom}_R(R/a, H^j_{I,J}(M)/N) \in S \).

(ii) \( \text{Hom}_R(L, H^j_{I,J}(M)/N) \in S \) for any finite \( R \)-module \( L \) with \( \text{Supp}(L) \subseteq \text{V}(a) \).

(iii) \( \text{Hom}_R(R/p, H^j_{I,J}(M)/N) \in S \) for any \( p \in \text{V}(a) \).

All the statements are hold for \( a = I \).

**Proof.** (i) Apply Proposition 2.8 and the exact sequence

\[ 0 \to N \to H^j_{I,J}(M) \to H^j_{I,J}(M)/N \to 0. \]

For (ii) and (iii) apply [5] Theorem 2.10. \( \square \)

**Corollary 2.10.** Let \((R, m)\) be a local ring and \( a \in \hat{W}(I,J) \). Let \( t \in \mathbb{N}_0 \) be such that \( \text{Ext}^t_R(R/a, M) \in S \) and \( \text{Ext}^{n-j}_R(R/a, H^j_{I,J}(M)) \in S \) for \( n = t, t+1 \) and all \( j < t \). Then for any submodule \( N \) of \( H^j_{I,J}(M) \) such that \( \text{Ext}^1_R(R/a, N) \in S \), we have \( \text{Hom}_R(R/a, H^j_{I,J}(M)/N) \in S \), specially \( \text{Hom}_R(R/m, H^j_{I,J}(M)/N) \in S \).

**Example 2.11.** In Theorem 2.9 the assumption \( \text{Ext}^1_R(R/a, M) \in S \) is necessary. To see this, let \((R, m)\) be a local Gorenstein ring of positive dimension \( d \), and \( S = 0 \). Then \( H^n_m(R) = 0 \) for \( i < d \). But we have \( \text{Ext}^d_R(R/m, R) \cong \text{Hom}_R(R/m, H^d(R)) \cong \text{Hom}_R(R/m, E(R/m)) \cong R/m \neq 0 \).

One of the main results of this paper is the following theorem which is a generalization of [7] Theorem 2.12

**Theorem 2.12.** Let \( R/m \in S \) for all \( m \in \text{Max}(R) \). Let \( M \) be a minimax \( R \)-module and \( t \in \mathbb{N}_0 \) be such that \( \text{Ext}^{n-j}_R(R/m, H^j_{I,J}(M)) \in S \) for \( n = t, t+1 \), all \( j < t \), and all \( m \in \text{Max}(R) \). Then for any submodule \( N \) of \( H^j_{I,J}(M) \) such that \( \text{Ext}^1_R(R/m, N) \in S \), we have \( \text{Hom}_R(R/m, H^j_{I,J}(M)/N) \in S \), for any \( m \in W_{\text{Max}}(I,J) := \text{Max}(R) \cap W(I,J) \).

**Proof.** The assertion follows from Proposition 2.8 (ii) and Theorem 2.9. \( \square \)

**Corollary 2.13.** Let \((R,m)\) be a local ring, \( S \neq 0 \) and \( M \) be a minimax \( R \)-module. Let \( t \in \mathbb{N}_0 \) be such that \( \text{Ext}^{n-j}_R(R/m, H^j_{I,J}(M)) \in S \) for \( n = t, t+1 \) and all \( j < t \). Then for any minimax submodule \( N \) of \( H^j_{I,J}(M) \), we have \( \text{Hom}_R(R/m, H^j_{I,J}(M)/N) \in S \).

**Proof.** Apply Proposition 2.8 and Theorem 2.12. \( \square \)

The following familiar conjecture is due to Huneke [23].

**Conjecture.** Let \((R,m,k)\) be a regular local ring and \( I \) be an ideal of \( R \). For all \( n \), \( \text{soc}(H^n(R)) \) is finitely generated.
As we mentioned in the introduction, the following theorem which is one of the main result of this section, can be a positive answer to Huneke’s conjecture. In fact the following theorem proves a generalization of the conjecture for an arbitrary noetherian ring $R$ (not necessary regular local one), a minimax $R$-module $M$ of krull dimension less than 3, and a Serre class $S$.

**Theorem 2.14.** Let $R$ be a noetherian ring and $M$ be a minimax $R$-module of krull dimension less than 3. Let $R/m \in S$ for any $m \in \text{Max}(R)$. Then $\text{Ext}_R^i \left( \frac{R}{m}, H^j_I(M) \right) \in S$ for any $m \in \text{Max}(R) \cap V(I)$ and all $i, j \geq 0$. In particular $\text{Hom}_R \left( \frac{R}{m}, H^j_I(M) \right) \in S$ for any $m \in \text{Max}(R) \cap V(I)$ and all $i \geq 0$.

**Proof.** Let $m \in \text{Max}(R) \cap V(I)$. By Proposition 2.15(ii) and the Grothendieck’s vanishing theorem there is nothing to prove for cases $i = 0$ and $i > 2$. Now, assume that $0 < i \leq 2$. If $\text{dim }M = 2$ and $i = 2$, then the result follows from [19, Corollary 3.3] and Proposition 2.5(i). Also, in the case $i = 1$, the result is obtained from [1, Theorem 2.3], by replacing $s := j$, $t := 1$, and $N := R/m$, and Proposition 2.5(ii). Finally if $\text{dim }M \leq 1$, we can obtain the desired result in similar way. \hfill \Box

**Corollary 2.15.** Let $R$ be a noetherian ring of dimension less than 3 and let $M$ be a minimax $R$-module. Let $R/m \in S$ for any $m \in \text{Max}(R)$. Then $\text{Ext}_R^i \left( \frac{R}{m}, H^j_I(M) \right) \in S$ for any $m \in \text{Max}(R) \cap V(I)$ and all $i, j \geq 0$. In particular for the class of finite $R$-modules.

**Corollary 2.16.** Let $R$ be a noetherian ring. Let $M$ be a minimax $R$-module of krull dimension less than 3. Then the Bass numbers of $H^i_I(M)$ are finite for all $i \geq 0$, in particular it is true when $\text{dim }R \leq 2$.

One can generalize Theorem 2.14 for local cohomology modules with respect to a pair of ideals, but in local case. To do this, we need the following lemma.

**Lemma 2.17.** Let $(R, m)$ be a local ring and $M$ be a minimax $R$-module of finite krull dimension $d$. Then $H^i_{I,J}(M)$ is artinian.

**Proof.** Let $N$ be a finite submodule of $M$ such that $A := M/N$ is artinian. Since $\text{dim }N \leq d$, by [15] Theorem 2.1], $H^i_{I,J}(N)$ is artinian. Also, the exact sequence $0 \rightarrow N \rightarrow M \rightarrow A \rightarrow 0$ induces the following exact sequence

$$0 \rightarrow \Gamma_{I,J}(N) \rightarrow \Gamma_{I,J}(M) \rightarrow \Gamma_{I,J}(A) \rightarrow H^1_{I,J}(N) \rightarrow H^1_{I,J}(M) \rightarrow 0,$$

and $H^i_{I,J}(N) \cong H^i_{I,J}(M)$ for all $i \geq 2$. By the exactness of the above sequence, it is easy to see that $H^i_{I,J}(M)$ is artinian. \hfill \Box

**Theorem 2.18.** Let $(R, m)$ be a local ring and $M$ be a minimax $R$-module of krull dimension less than 3. Let $S \neq 0$. Then $\text{Ext}_R^i \left( \frac{R}{m}, H^j_{I,J}(M) \right) \in S$ for all $i, j \geq 0$.

**Proof.** Apply Lemma 2.17, Proposition 2.5, and the same method of the proof of Theorem 2.14. \hfill \Box

The next results can be useful for finiteness of Bass numbers of local cohomology modules over a noetherian ring and modules of krull dimension 3. Recall that an $R$-module $M$ is called locally minimax if $M_m$ is minimax for any $m \in \text{Max}(R)$ (see [3]).

**Theorem 2.19.** Let $R$ be a noetherian ring and $M$ be an $R$-module of krull dimension 3. Let $R/m \in S$ for any $m \in \text{Max}(R)$. Then $\text{Ext}_R^i \left( \frac{R}{m}, H^j_I(M) \right) \in S$ for any $m \in \text{Max}(R) \cap V(I)$ and all $i, j \geq 0$, if one of the following conditions holds:
(i) $M$ and $H^1_I(M)$ are minimax;
(ii) $M$ is finite and $H^1_I(M)$ is locally minimax;
(iii) $M$ and $I^m H^2_I(M)$ are minimax for some $m \in \mathbb{N}_0$;
(iv) $M$ is minimax, $I^m H^2_I(M)$ is locally minimax, and $\text{Hom}_R \left( R/I, I^m H^2_I(M) \right)$ is finite for some $m \in \mathbb{N}_0$;
(v) $(R, m)$ is local, $M$ is minimax, and $I^m H^2_I(M)$ is locally minimax for some $m \in \mathbb{N}_0$.

Proof. Let $m \in \text{Max}(R) \cap V(I)$.

(i) By [19] Corollary 3.3, $H^2_I(M)$ is artinian. Thus the assertion is true for $i = 0, 1, 3$, by Proposition 2.5. Also, for $i = 2$, we apply [4] Theorem 2.3, by replacing $s := j, t := 2$, and $N := R/m$.

(ii) By [8] Theorem 2.3 and [3] Theorem 2.6, $H^1_I(M)$ is minimax. Now, the assertion follows from part (i).

(iii) By the short exact sequence
$$0 \to I^m H^2_I(M) \to H^2_I(M) \to H^2_I(M)/I^m H^2_I(M) \to 0,$$
and [7] Theorem 3.1, for the class of minimax $R$-modules, we get $H^2_I(M)$ is minimax. Now, for $i = 0, 2, 3$, the claim follows from [19] Corollary 3.3 and Proposition 2.5. In the case $i = 2$, apply [4] Theorem 2.3, by replacing $s := j, t := 1, N := R/m$ and Proposition 2.5.

(iv) Apply [3] Theorem 2.6 and part (iii).

(v) By proof of Lemma 2.17, we may assume that $M$ is finite. Now, the assertion follows from [3] Propositions 3.4, 2.2 and part (iii). □

Corollary 2.20. Let $R$ be a noetherian ring and $M$ be a minimax $R$-module of krull dimension $d \leq 3$. Then the Bass numbers of $H^1_I(M)$ are finite for all $i \geq 0$, if one of the following conditions holds:

(i) $M$ and $H^1_I(M)$ are minimax;
(ii) $M$ is finite and $H^1_I(M)$ is locally minimax;
(iii) $M$ and $I^m H^2_I(M)$ are minimax for some $m \in \mathbb{N}_0$;
(iv) $M$ is minimax, $I^m H^2_I(M)$ is locally minimax, and $\text{Hom}_R \left( R/I, I^m H^2_I(M) \right)$ is finite for some $m \in \mathbb{N}_0$;
(v) $M$ is minimax, and $I^m H^2_I(M)$ is locally minimax, for some $m \in \mathbb{N}_0$.

In particular the statements are true when $\dim R \leq 3$.

Proof. For part (v), apply localization and Theorem 2.14 (v). □

3. COFINITENESS AND ARTINIANNES OF $H^1_{I,J}(M)$

In this section, we need the concept of $(I,J)$-cofinite modules and $\text{cd}(I,J,M)$. An $R$-module $M$ is called $(I,J)$-cofinite if $\text{Supp}(M) \subseteq W(I,J)$ and $\text{Ext}_R^i(R/I,M)$ is a finite $R$-module, for all $i \geq 0$. Also $\text{cd}(I,J,M) = \sup \{ i \in \mathbb{N}_0 \mid H^i_{I,J}(M) \neq 0 \}$ (see [37] and [15], resp.)

Theorem 3.1. Let $(R, m)$ be a local ring and $M$ be a finite $R$-module. Let $t \in \mathbb{N}$ be an integer such that $\text{Supp}(H^i_{I,J}(M)) \subseteq \{ m \}$ for all $i < t$. Then, for all $i < t$, the $R$-module $H^i_{I,J}(M)$ is artinian and $I$-cofinite.

Proof. We do this by induction on $t$. When $t = 1$, it is obvious that $H^0_{I,J}(M)$ is artinian $I$-cofinite $R$-module, since it is a finite module with support in $\{ m \}$. Now, suppose that $t \geq 2$ and the case $t - 1$
is settled. By [36 Corollary 1.13], we may assume that $M$ is $(I, J)$-torsion free, and so $I$-torsion free $R$-module. Therefore, by [12 Lemma 2.1.1], there exists $x \in I \setminus \bigcup_{p \in \text{Ass}(M)} p$ such that $0 \to M \xrightarrow{-t} M \to M/xM \to 0$ is exact. Now, by the exact sequence

$$H_{I,J}^{i-2}(M) \to H_{I,J}^{i-2}(M/xM) \to H_{I,J}^{i-1}(M) \to H_{I,J}^{i-1}(M),$$

we get the following exact sequence

$$H_{I,J}^{i-2}(M) \to H_{I,J}^{i-2}(M/xM) \to (0 : H_{I,J}^{i-1}(M) x) \to 0.$$

Thus by inductive hypothesis $H_{I,J}^{i-2}(M/xM)$ is artinian $I$-cofinite. So $0 : H_{I,J}^{i-1}(M) x$ is artinian $I$-cofinite. As $\text{Supp}(H_{I,J}^{i-1}(M)) \subseteq \{m\}$, hence $H_{I,J}^{i-1}(M)$ is $I$-torsion. Now, the assertion follows from [34 Proposition 4.1].

**Theorem 3.2.** Let $M$ be an $R$-module such that $\text{Ext}_R^i(R/I, M)$ is finite for all $i \geq 0$. Let $t \in \mathbb{N}_0$ be such that $H_{I,J}^i(M)$ is $(I, J)$-cofinite for all $i \neq t$. Then $H_{I,J}^i(M)$ is $(I, J)$-cofinite.

**Proof.** Apply [5 Theorem 3.11].

**Theorem 3.3.** Let $M$ be a finite $R$-module. If $\text{cd}(I, J, M) \leq 1$, then $H_{I,J}^i(M)$ is $(I, J)$-cofinite for all $i \geq 0$.

**Proof.** When $i = 0$ and $i \geq 2$, the claim is true, since $\Gamma_{I,J}(M)$ is finite and $\text{cd}(I, J, M) \leq 1$. For $i = 1$, apply Theorem 3.2.

**Theorem 3.4.** Let $(R, m)$ be a local ring and $M$ be a finite $R$-module with $\dim M = n$. Then $H_{I,J}^p(M)$ is artinian and $I$-cofinite. In fact, $\text{Ext}_R^i(R/I, H_{I,J}^p(M))$ has finite length for all $i$.

**Proof.** The assertion follows from [15 Theorem 2.1], [14 Theorem 2.3] and [16 Theorem 3].

**Theorem 3.5.** Let $(R, m)$ be a local ring and $M$ be a finite $R$-module. If $\dim M \leq 2$, then $H_{I,J}^i(M)$ is $(I, J)$-cofinite for all $i \geq 0$, in particular it is true when $\dim R \leq 2$.

**Proof.** Apply Theorems 3.4, 3.2.

4. **Upper bounds of $H_{I,J}^i(M)$**

In this section, we introduce the concept of Serre cohomological dimension of $M$ with respect to a pair of ideals $(I, J)$, but first, we characterize the membership of $H_{I,J}^i(M)$ in a Serre class from upper bound.

**Theorem 4.1.** Let $n \in \mathbb{N}_0$ and $M$ be a finite $R$-module. Then the following statements are equivalent:

(i) $H_{I,J}^i(M)$ is in $S$ for all $i > n$.

(ii) $H_{I,J}^i(N)$ is in $S$ for all $i > n$ and for any finite $R$-module $N$ such that $\text{Supp}_R(N) \subseteq \text{Supp}_R(M)$.

(iii) $H_{I,J}^i(R/p)$ is in $S$ for all $p \in \text{Supp}_R(M)$ and for all $i > n$.

(iv) $H_{I,J}^i(R/p)$ is in $S$ for all $p \in \text{Min Ass}_R(M)$ and for all $i > n$.

**Proof.** Apply the method of the proof of Theorem 3.1 in [4] to $H_{I,J}^i$.

**Corollary 4.2.** Let $M, N$ be finite $R$-modules such that $\text{Supp}(N) \subseteq \text{Supp}(M)$ and $n \in \mathbb{N}_0$. If $H_{I,J}^i(M) \in S$ for all $i > n$, then $H_{I,J}^i(N) \in S$ for all $i > n$.

**Lemma 4.3.** For any $R$-module $N$, the following statements are fulfilled.
Therefore, the following two sequences are both exact. Since

\[ \text{Tor}_R^n(M/\alpha M) \cong H_{I,J}^n(M/\alpha M) \] 

so, by Lemma 4.3, \( \Gamma \) induces the following exact sequences

\[ \begin{align*}
\Delta(n) &\subseteq \Delta_{a,J}(n) \subseteq \Delta_{I,J}(n) \quad \text{for any } a \in \mathcal{W}(I,J). \\
\Gamma_{I,J}(n) &\equiv 0 \text{ if and only if } \Gamma_{a,J}(n) = 0 \text{ for any } a \in \mathcal{W}(I,J). \\
\Gamma_{I,J}(N) &\equiv N \text{ if and only if there exists } a \in \mathcal{W}(I,J) \text{ such that } \Gamma_{a,J}(N) = N. \\
\text{If there exists } a \in \mathcal{W}(I,J) \text{ such that } \Gamma_a(N) = N, \text{ then } \Gamma_{I,J}(N). \\
\Gamma_{I,J}(N) &\equiv \bigcup_{a \in \mathcal{W}(I,J)} \Gamma_a(N) = \bigcup_{a \in \mathcal{W}(I,J)} \Gamma_{a,J}(N).
\end{align*} \]

Proof. All these statements follow easily from the definitions. We will only prove the statement (v). Since \( \Gamma_b(N) \subseteq \Gamma_{b,J}(N) \), for any ideal \( b \) of \( R \), so by [36, Theorem 3.2] and part(i), we get \( \Gamma_{I,J}(N) \subseteq \bigcup_{a \in \mathcal{W}(I,J)} \Gamma_a(N) \subseteq \bigcup_{a \in \mathcal{W}(I,J)} \Gamma_{a,J}(N) \subseteq \Gamma_{I,J}(N). \quad \square \]

In [19], authors introduced the concept of ZD-modules. An \( R \)-module \( M \) is said to be a ZD-module (zero-divisor module) if for every submodule \( N \) of \( M \), the set of zero divisors of \( M/N \) is a union of finitely many prime ideals in \( \text{Ass}_R(M/N) \). By [19, Example 2.2], the class of ZD-modules contains modules with finite support, finitely generated, Laskerian, weakly Laskerian, linearly compact, Matlis reflexive and minimax \( R \)-modules.

Now, we are in position to prove two other main results of this paper (Theorem 4.4 and Theorem 4.7), which the first one can be considered as a generalization of [7, Theorem 3.1].

**Theorem 4.4.** Let \( M \) be a ZD-module of finite Krull dimension. Let \( n \in \mathbb{N} \) be such that \( H_{I,J}^n(M) \in S \) for all \( i > n \). Then \( H_{I,J}^i(M)/aH_{I,J}^i(M) \in S \) for any \( a \in \mathcal{W}(I,J) \), all \( i \geq n \), and all \( j \geq 0 \).

Proof. It is enough to verify the assertion for just \( i = n \) and \( j = 1 \). To do this, we use induction on \( d := \dim M \). When \( d = 0 \), the result follows from [36, Theorem 3.2] and Grothendieck’s Vanishing theorem. Next, we assume that \( d > 0 \) and the claim is true for all \( R \)-modules of dimension less than \( d \). By [36, Theorem 1.3 (4)], we have \( H_{I,J}^i(M) \cong H_{I,J}^i(M/IJ) \) for all \( j > 0 \). Also, \( M/IJ \) has dimension not exceeding \( d \), and is an \((I,J)\)-torsion-free \( R \)-module. Therefore we may assume that \( \Gamma_{I,J}(M) = 0 \) and so, by Lemma 4.3, \( \Gamma_a(M) = 0 \). By [19, Lemma 2.4], \( x \in a \setminus \bigcup_{p \in \text{Ass}(M)} p \). Now, the \( R \)-module \( M/xM \) is ZD-module of dimension \( d - 1 \). Considering the exact sequence

\[ 0 \to M \to M/xM \to 0 \]

induces a long exact sequence of local cohomology modules, which shows that \( H_{I,J}^i(M/xM) \in S \) for all \( i > n \). Thus by inductive hypothesis \( H_{I,J}^n(M/xM)/aH_{I,J}^n(M/xM) \in S \). Now, the exact sequence

\[ H_{I,J}^n(M) \xrightarrow{\alpha} H_{I,J}^n(M/xM) \xrightarrow{\beta} H_{I,J}^{n+1}(M), \]

induces the following exact sequences

\[ H_{I,J}^n(M) \xrightarrow{\alpha} H_{I,J}^n(M) \to N := \text{Im } \alpha \to 0, \]

\[ 0 \to N \to H_{I,J}^n(M/xM) \to K := \text{Im } \beta \to 0. \]

Therefore, the following two sequences

\[ \begin{align*}
(\ast) &\quad H_{I,J}^n(M)/aH_{I,J}^n(M) \xrightarrow{\alpha} H_{I,J}^n(M)/aH_{I,J}^n(M) \to N/aN \to 0, \\
(\ast\ast) &\quad \text{Tor}_R^n(R/a, K) \to N/aN \to H_{I,J}^n(M/xM)/aH_{I,J}^n(M/xM) \to K/aK \to 0
\end{align*} \]

are both exact. Since \( x \in a \) and from the exact sequence (\( \ast \)), we get \( N/aN \cong H_{I,J}^n(M)/aH_{I,J}^n(M) \). On the other hand, by [7, Lemma 2.1], we have \( \text{Tor}_R^n(R/a, K) \in S \). Therefore \( N/aN \in S \), by \( \ast\ast \), as required. \quad \square
The following result is an immediate consequence of the above theorem.

**Corollary 4.5.** Let $M$ be a ZD-module of finite Krull dimension. Let $n \in \mathbb{N}$ be such that $H^n_{I,J}(M) \in S$ for all $i > n$. Then $H^n_{I,J}(M) \in S$ if and only if there exist $a \in \hat{W}(I,J)$ and $m \in \mathbb{N}_0$ such that $a^m H^n_{I,J}(M) \in S$.

**Proof.** $(\Rightarrow)$ It is obvious.

$(\Leftarrow)$ Apply the short exact sequence $0 \to a^m H^n_{I,J}(M) \to H^n_{I,J}(M) \to H^n_{I,J}(M)/a^m H^n_{I,J}(M) \to 0$ and Theorem 4.3.

Applying Corollary 4.5 for some familiar Serre classes of modules, we get some results as follows.

**Corollary 4.6.** Let $M$ be a ZD-module of finite Krull dimension and let $n \in \mathbb{N}$. Then the following statements are fulfilled.

(i) If $H^n_{I,J}(M)$ is finite for all $i > n$, then $H^n_{I,J}(M)$ is finite if and only if there exist $a \in \hat{W}(I,J)$ and $m \in \mathbb{N}_0$ such that $a^m H^n_{I,J}(M)$ is finite if and only if there exist $a \in \hat{W}(I,J)$ and $m \in \mathbb{N}_0$ such that $H^n_{I,J}(M)/(0:H^n_{I,J}(M) a^m)$ is finite.

(ii) If $H^n_{I,J}(M)$ is artinian for all $i > n$, then $H^n_{I,J}(M)$ is artinian if and only if there exist $a \in \hat{W}(I,J)$ and $m \in \mathbb{N}_0$ such that $a^m H^n_{I,J}(M)$ is artinian.

(iii) If $H^n_{I,J}(M) = 0$ for all $i > n$, then $H^n_{I,J}(M)$ is $a^j H^n_{I,J}(M)$ for any $a \in \hat{W}(I,J)$ and all $j \geq 0$.

Thus $H^n_{I,J}(M) = 0$ if and only if there exist $a \in \hat{W}(I,J)$ and $m \in \mathbb{N}_0$ such that $a^m H^n_{I,J}(M) = 0$.

In particular for $n = \dim M$ and $cd(I,J,M)$

**Proof.** Apply Corollary 4.5 and [3] Theorem 3.1.

Although the following theorem is seemed to be similar to Theorem 4.3, but it is more useful and general than 4.4 for finite R-modules.

**Theorem 4.7.** Let $M$ be a finite $R$ module and $n \in \mathbb{N}_0$ be such that $H^n_{I,J}(M)$ belongs to $S$ for all $i > n$. If $b$ is an ideal of $R$ such that $H^n_{I,J}(M/bM)$ belongs to $S$, then the module $H^n_{I,J}(M)/b H^n_{I,J}(M)$ belongs to $S$.

**Proof.** Let $b = (b_1, \ldots, b_r)$ and consider the map $f : M^r \to M$, defined by $f(x_1, \ldots, x_r) = \sum b_i x_i$. Then $\text{Im } f = bM$ and $\text{Coker } f = M/bM$. Since $H^n_{I,J}(M)$ is in $S$ for all $i > n$ and $\text{Supp}(\text{Ker } f) \subseteq \text{Supp}(M)$, it follows from Theorem 4.3 that $H^n_{I,J}(\text{Ker } f)$ is also in $S$. By hypothesis $H^n_{I,J}(\text{Coker } f)$ belongs to $S$. Hence by [3] Corollary 3.2 $\text{Coker } H^n_{I,J}(f)$, which equals to $H^n_{I,J}(M)/b H^n_{I,J}(M)$, is in $S$.

**Corollary 4.8.** Let $M$ be a finite $R$-module and $n \in \mathbb{N}$ be such that $H^n_{I,J}(M) \in S$ for all $i > n$. Then $H^n_{I,J}(M)/a H^n_{I,J}(M) \in S$ for any $a \in \hat{W}(I,J)$, in particular for $a = 1$.

**Proof.** Let $a \in \hat{W}(I,J)$. Since $M/\text{aM}$ is $a$-torsion $R$-module, thus the assertion follows from Lemma 4.3 (iv), [3] Corollary 1.13 and Theorem 4.7.

**Corollary 4.9.** Let $M$ be a finite $R$-module. Let $n \in \mathbb{N}$ be such that $H^n_{I,J}(M) \in S$ for all $i > n$. Then $H^n_{I,J}(M) \in S$ if and only if there exist $m \in \mathbb{N}_0$ such that $a^m H^n_{I,J}(M) \in S$.

**Corollary 4.10.** Let $M$ be a finite $R$-module. Let $a \in \hat{W}(I,J)$ and $n \in \mathbb{N}$. 

(i) If $H^i_{I,J}(M) = 0$ for all $i > n$, then $H^i_{I,J}(M)$ is finite if and only if there exist $a \in \hat{W}(I,J)$ and $m \in \mathbb{N}_0$ such that $a^m H^i_{I,J}(M)$ is finite if and only if there exist $a \in \hat{W}(I,J)$ and $m \in \mathbb{N}_0$ such that $H^i_{I,J}(M)/(0 : H^i_{I,J}(M) \cdot a^m)$ is finite.

(ii) If $H^i_{I,J}(M)$ is artinian for all $i > n$, then $H^i_{I,J}(M)$ is artinian if and only if there exist $m \in \mathbb{N}_0$ such that $a^m H^i_{I,J}(M)$ is artinian.

(iii) If $H^i_{I,J}(M) = 0$ for all $i > n$, then $H^i_{I,J}(M) = a^j H^i_{I,J}(M)$ for all $j \in \mathbb{N}_0$. Thus $H^i_{I,J}(M) = 0$ if and only if there exists $m \in \mathbb{N}_0$ such that $a^m H^i_{I,J}(M) = 0$, in particular for $n = \dim M$ and $cd(I,J,M)$.

(iv) If $R$ is local and $H^i_{I,J}(M)$ is finite for all $i > n$, then $H^j_{I,J}(M) = a^j H^j_{I,J}(M)$ for all $j \geq 0$.

Proof. Apply Corollary 4.3 [27 Proposition 1], and [3 Theorem 3.1]. □

The following result is more useful whenever $R$ is a local ring and $I$ is a proper ideal. (see [10, Lemma 2.1])

Corollary 4.11. Let $M$ be a non-zero ZD-module with $d := \dim M$ and $t := cd(I,J,M)$. Let $a \in \hat{W}(I,J)$ be such that $a^m \subseteq \text{Jac}(R)$ for some $m \in \mathbb{N}_0$.

(i) If $d \geq 1$, then $a^d H^i_{I,J}(M)$ is not finite for all $j \geq 0$.

(ii) If $d \geq 1$, then $H^i_{I,J}(M)$ is finite if and only if $H_{I,J}(M) = 0$.

(iii) If $d \geq 2$ and $H^i_{I,J}(M)$ is finite, then $H^i_{I,J}(M)/a^j H^i_{I,J}(M)$ has finite length for all $j \geq 0$.

In particular when $R$ is local ring and $I \neq R$.

Proof. (i), (ii) Apply Corollary 4.3 (iii) and Nakayama’s Lemma.

(iii) Apply part (ii) and Theorem 4.3 for the class of finite length. □

Corollary 4.12. Let $M$ be a non-zero finite $R$-module and set $t := \sup\{i \geq 1 \mid H^i_{I,J}(M) \text{ is not finite}\}$, $n := cd(I,J,M)$, and $r := \dim M/IM$.

(i) If $a \in \hat{W}(I,J)$ is such that $a^m \subseteq \text{Jac}(R)$ for some $m \in \mathbb{N}_0$, then $n \geq 1$ if and only if $n = t$.

(ii) Let $(R,m)$ be a local ring and $r \geq 1$ be an integer such that $H^r_{I,J}(M)$ is finite. Let $a \in \hat{W}(I,J)$ be such that $a^m + J \subseteq \mathfrak{m}$ for some $m \in \mathbb{N}_0$. Then $H^r_{I,J}(M) = 0$.

(iii) If $(R,m)$ is a local ring and $I + J$ is an $m$-primary ideal, then $r = n = t$.

Proof. Apply Corollaries 4.9 4.10 and 36 Theorems 4.3,4.5]. □

Corollary 4.13. Let $M$ be a finite $R$-module and $n \in \mathbb{N}$. If $H^i_{I,J}(M)$ is artinian for all $i > n$. Then $H^i_{I,J}(M)/aH^i_{I,J}(M)$ is artinian for any $a \in \hat{W}(I,J)$.

Proof. Note that $H^i_{I,J}(M/aM) = 0$ for any $a \in \hat{W}(I,J)$ and all $n \geq 1$. Now, apply Theorem 4.7 for the class of artinian modules. □

Remark 4.14. In [1,13] we have to assume that $n \geq 1$. Take an ideal $I$ in a ring $R$ such that $R/I$ is not artinian. Let $J = 0$ and $M = R/I$. Then $H^i_{I,J}(M) = 0$ for $i \geq 1$, and $\Gamma_I(M) = M$. On the other hand $M/IM \cong M$. Thus $H^i_{I,J}(M)/I H^i_{I,J}(M)$ is not artinian.

Properties of Serre classes of modules and the previous results motivate us to introduce the following definition as a generalization of the concept of cohomological dimension. (see [15]).
Definition 4.15. Let $I$, $J$ be two ideals of $R$ and let $M$ be an $R$-module. For a Serre subcategory $S$ of the category of $R$-modules, we define Serre cohomological dimension of $M$ with respect to $(I, J)$, by
\[ cd_S(I, J, M) = \sup \{ i \in \mathbb{N}_0 \mid H^j_{I, J}(M) \notin S \}, \]
if this supremum exists, and $\infty$ otherwise.
It is easy to see that $cd_S(I, J, M) = \inf \{ n \in \mathbb{N}_0 \mid H^j_{I, J}(M) \in S \text{ for all } i > n \}.$

Remark 4.16. For an arbitrary Serre class $S$, we have $cd_S(I, J, M) \leq cd(I, J, M)$, and if in Definition 4.16, we let $S := 0$ then we have
\[ cd_S(I, J, M) = \sup \{ i \in \mathbb{N}_0 \mid H^j_{I, J}(M) \neq 0 \} = cd(I, J, M). \]
Also, if $S$ is the class of Artinian $R$-modules, we get
\[ cd_S(I, J, M) = \sup \{ i \in \mathbb{N}_0 \mid H^j_{I, J}(M) \text{ is not Artinian } R\text{-module} \}. \]
We denote it by $q(I, J, M)$.

Proposition 4.17. Let $M$ be a $ZD$-module of finite Krull dimension or finite $R$-module. Let $S$ be a Serre class and $n := cd_S(I, J, M) \geq 1$. Then $a^j H^j_{I, J}(M) \notin S$ for any $a \in \hat{W}(I, J)$ and all $j \geq 0$, in particular $a^j H^j_{I, J}(M) \neq 0$ for any $a \in \hat{W}(I, J)$ and all $j \geq 0$.

Proof. Apply Corollaries 4.5, 4.9.

It is well known that if $a$ is an ideal of $R$ and $M$, $N$ are finite $R$-modules with $\text{Supp}(N) \subseteq \text{Supp}(M)$, then $cd(a, N) \leq cd(a, M)$. The next result is a generalization of this fact.

Proposition 4.18. Let $M$, $N$ be finite $R$-modules such that $\text{Supp}(N) \subseteq \text{Supp}(M)$. Then $cd_S(I, J, N) \leq cd_S(I, J, M)$.

Proof. Apply Corollary 4.2.

Corollary 4.19. Let $M$, $N$ be finite $R$-modules such that $\text{Supp}(N) \subseteq \text{Supp}(M)$. Then $cd(I, J, N) \leq cd(I, J, M)$.

Corollary 4.20. For a finite $R$-module $M$, there exist the following equalities.
\[
\begin{align*}
\text{eq1} & : cd_S(I, J, M) = \max \left\{ cd_S(I, J, R/p) \mid p \in \text{Ass}(M) \right\} \\
\text{eq2} & : = \max \left\{ cd_S(I, J, R/p) \mid p \in \text{Min Ass}(M) \right\} \\
\text{eq3} & : = \max \left\{ cd_S(I, J, R/p) \mid p \in \text{Supp}(M) \right\} \\
\text{eq4} & : = \max \left\{ cd_S(I, J, R/p) \mid p \in \text{Min Supp}(M) \right\} \\
\text{eq5} & : = \max \left\{ cd_S(I, J, N) \mid N \text{ is a finite submodule of } M \right\} \\
\text{eq6} & : = \max \left\{ i \geq 0 \mid H^i_{I, J}(R/p) \notin S \text{ for some } p \in \text{Ass}(M) \right\} \\
\text{eq7} & : = \max \left\{ n \geq 0 \mid H^i_{I, J}(R/p) \in S \text{ for all } i > n \text{ and all } i > n \right\}.
\end{align*}
\]

Proof. Apply Corollary 4.1 and Proposition 4.19.

Corollary 4.21. Let $M$, $N$ be finite $R$-modules such that $\text{Supp}(N) \subseteq \text{Supp}(M)$. Then $q(I, J, N) \leq q(I, J, M)$. 

**Proposition 4.22.** Let $0 \to L \to M \to N \to 0$ be an exact sequence of finite $R$-modules. Then $\text{cd}_S(I,J,M) = \max \left\{ \text{cd}_S(I,J,L), \text{cd}_S(I,J,N) \right\}$

Proof. Let $t := \text{cd}_S(I,J,M)$ and $s := \max \left\{ \text{cd}_S(I,J,L), \text{cd}_S(I,J,N) \right\}$. By Corollary 4.2, we have $t \geq s$. Now, let $t > s$. Then by the following exact sequence

$$\cdots \to H^j_I(J)(L) \to H^j_I(J)(M) \to H^j_I(J)(N) \to \cdots,$$

we get $H^j_I(J)(M) \in S$ which is a contradiction with $\text{cd}_S(I,J,M) = t$. □

**Corollary 4.23.** For a noetherian ring $R$ there exists the following equality.

$$\text{cd}_S(I,J,R) = \sup \left\{ \text{cd}_S(I,J,N) \mid N \text{ is a finite } R\text{-module} \right\}.$$

In particular, for $r \in \mathbb{N}_0$ the following statements are equivalent:

(i) $H^j_I(J)(R) \in S$ for all $j > r$.

(ii) $H^j_I(J)(M) \in S$ for all $j > r$ and all finite $R$-module $M$.

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1 Department of Mathematics, Faculty of Science, Arak University, Arak, 38156-8-8349, Iran.

E-mail address: m-aghapour@araku.ac.ir

2 Department of Mathematics, Payame Noor University, Tehran, 19395-3697, Iran.

E-mail address: kahmadi@pnu.ac.ir

3 Department of Mathematics, Payame Noor University, Tehran, 19395-3697, Iran.

E-mail address: m.sadeghi@phd.pnu.ac.ir