An equilibrium position in the generalized mathematical model of a system of rigid bodies elastically mounted on an Euler-Bernoulli beam

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Abstract. The paper considers a generalized mathematical model of a class of mechanical systems, which represent systems of rigid bodies elastically mounted on an Euler-Bernoulli beam. This model has the form of a hybrid system of differential equations having a definite structure and describing the process of transfer (transition) of the system within the frames of some coordinated system chosen. On the basis of the generalized mathematical model, a generalized approach to finding the equilibrium position for mechanical systems, which belong to a given class of systems, in a chosen coordinate system has been proposed. The equilibrium position of a mechanical system is understood as the solution of a definite hybrid system of differential equations, furthermore, this solution does not change with time. The equilibrium position of a mechanical system within a given coordinate system allows one to proceed – by replacement of the variables – to consideration of the generalized mathematical model studied earlier and describing transfer of a system with respect to the equilibrium.

1. Introduction

In [1] we have proposed a generalized mathematical model of a system interconnected rigid bodies, elastically (with the aid of elastic links) mounted on an Euler-Bernoulli beam represented in the form of a hybrid system of differential equations (HSDE)

\[
\begin{align*}
A\dot{q} + Cq + \bar{C}(Dq - \bar{u}) &= 0, \\
\frac{\partial^2 u}{\partial t^2}(x,t) + c\frac{\partial^2 u}{\partial x^2}(x,t) &= \sum_{i=1}^{m} k_i(d^{iT}q(t) - u(x,t))\delta(x-a_i),
\end{align*}
\]

where \( x \) is a variable describing the coordinate axis, which coincides with the beam at rest; \( q(t) \) is an \( n \)-dimensional vector-function, which describes transfer of the bodies; \( u(x,t) \) is a scalar function, which describes transversal transfer of the rod’s points; \( \bar{u}(t) \) is an \( m \)-dimensional vector-function with the elements \( u(a_1,t), u(a_2,t),..., u(a_m,t) \); \( A, C \) are given constant \( n \times n \) -matrices; \( \bar{C} \) is a given constant \( n \times m \)-matrix; \( D \) is a given constant \( m \times n \)-matrix; \( d^i \) is an \( n \)-vector composed of the \( i \)-th row of matrix \( D \); \( a, c, a_i, k_i, (i = 1, 2, ..., m) \) are given constants, furthermore, from now on,
0 ≤ aᵢ ≤ l; (·)ᵀ is the operation of transposition.

Investigation of the generalized mathematical model (1), which was based on the mathematical apparatus of generalized functions, has given us an opportunity to develop a general analytical-numerical method of investigations of free vibrations for the class of mechanical systems, representing systems of interconnected rigid bodies, elastically mounted on an Euler-Bernoulli beam [1].

Note, variables \( q(t) \) and \( u(x,t) \) in the generalized mathematical model (1) describe the process of transfer of a mechanical system with respect to the system’s equilibrium position and in the equilibrium position satisfy the following initial conditions:

\[
q(0) = 0, \quad u(x, 0) = 0, \quad 0 \leq x \leq l.
\]

Application of Hamilton’s variation principle in the process of constructing the equations of motion for definite computational schemes of mechanical systems, which represent a system of rigid bodies attached to a rod, implies (and leads to) consideration of HSDE. In many cases [2-11], for some computational schemes, the HSDE constructed structurally coincide with system (1) or easily – with the aid of replacement of the variables – are transformed to the form of system (1). This allows one to employ the analytical-numerical method proposed in [1] in the process of investigation of free vibrations.

In the present paper the authors consider a generalized mathematical model, which describes transitions of a system of rigid bodies mounted on an Euler-Bernoulli beam within some coordinate system chosen. The generalized mathematical model for the systems of interconnected rigid bodies, mounted on an Euler-Bernoulli beam with the aid of springs, when written in the form of HSDE, writes:

\[
\begin{align*}
\begin{cases}
 A\ddot{g} + Cg + \bar{C}(Dg - \bar{u}) = b, \\
 a\frac{\partial^2 v}{\partial t^2} (x,t) + c\frac{\partial^4 v}{\partial x^4} (x,t) = \sum_{i=1}^{m} k_i (d_i^{T} g(t) - v(x,t))\delta(x-a_i) + \sum_{i=1}^{m} P_i \delta(x-a_i),
\end{cases}
\end{align*}
\]

where \( g(t) \) and \( v(x,t) \) are the variables, which describe the transition of, respectively, the bodies and the rod in the coordinate system chosen; \( b \) is a given \( n \)-dimensional vector; \( P_i, (i=1,2,...,m) \) are given numbers.

Some boundary conditions are imposed on function \( v(x,t) \). These boundary conditions correspond to some conditions of fixing the beam’s ends [1]:

\[
\begin{align*}
\Gamma_1 (v(0,t)) = 0, \\
\Gamma_2 (v(l,t)) = 0.
\end{align*}
\]

Variables \( g(t) \) and \( v(x,t) \) in the equilibrium position are defined by the following initial conditions

\[
g(0) = \bar{q}, \quad v(x,0) = V(x), \quad 0 \leq x \leq l,
\]

where \( \bar{q} \) is an \( n \)-vector, which defines the equilibrium position for the system of rigid bodies; \( V(x) \) is a scalar function defining the deflection of the beam in the state of equilibrium (rest). Together \( \bar{q} \) and \( V(x) \) define the equilibrium position for a system of rigid bodies mounted on the Euler-Bernoulli beam in a chosen coordinate system.

On the basis of the generalized mathematical model (3), which describes the transfer of a system of rigid bodies mounted on an Euler-Bernoulli beam, a general approach to finding the equilibrium position for the mechanical systems of a given class in some coordinate system chosen in the scrutinized computational scheme has been proposed. The equilibrium positions found allow one – after the replacement of the variables – to proceed to the consideration of the generalized model (1) studied in [1].
2. A general approach to finding the equilibrium position

Let us define the equilibrium position for a mechanical system as the solution of HSDE (3), which does not vary with time. To this end let us employ the approach applied in the process of deriving the frequency equation in [1]. According to this approach, having substituted \( g(t) = \bar{q} \) and \( v(x,t) = V(x) \) into system (3), we obtain a system of algebraic-differential equations

\[
\begin{align*}
(C + \bar{C}D)\bar{q} - \bar{C}\bar{V} &= b, \\
c \frac{\partial^4 V}{\partial x^4}(x) &= \sum_{i=1}^{m} \left(k_i (d_i^T \bar{q} - V(x) + P_i)\delta(x - a_i)\right),
\end{align*}
\]

where \( \bar{V} \) is an \( m \)-vector with the components \( V(a_1), V(a_2), \ldots, V(a_m) \).

The boundary conditions (4) are imposed on function \( v(x,t) \). Consequently, function \( V(x) \) shall satisfy some corresponding boundary conditions

\[
\gamma_1(V(0)) = 0, \quad \gamma_2(V(l)) = 0.
\]

As far as the functions, which define the boundary conditions (7) are concerned, we can assume that the following property [1]

\[
\gamma_j \left( \sum_{i=1}^{m} \alpha_i v_i(x) \right) = \sum_{i=1}^{m} \alpha_i \gamma_j(v_i(x)), \quad (j = 1, 2),
\]

Holds, where \( \alpha_i \) are constants, \( v_i(x) \) are functions.

**Definition 1.** Vector \( \bar{q} \) , function \( V(x) \) are called the generalized solution of the boundary-value problem (6)-(7), when these satisfy the algebraic equations of system (6), boundary-value conditions (7) and for any main function \( \varphi(x) \) [12] the following identity

\[
\int_0^l \left( c \frac{\partial^4 V}{\partial x^4}(x) - \sum_{i=1}^{m} k_i (d_i^T \bar{q} - V(x) + P_i)\delta(x - a_i) - \sum_{i=1}^{m} P_i \delta(x - a_i) \right) \cdot \varphi(x) dx = 0
\]

holds.

**Theorem 1.** For any values of \( \bar{q} \) of the generalized solution \( V(x) \) of the differential equation of system (6), which satisfies the boundary conditions (7), valid is the following representation

\[
V(x) = \sum_{i=1}^{m} \eta_i(x - a_i) \left(k_i (d_i^T \bar{q} - V(a_i) + P_i)\right),
\]

where functions \( \eta_i(x), (i = 1, 2, \ldots, m) \) are generalized solutions of equation

\[
c \frac{d^4 \eta_i}{dx^4} = \delta(x),
\]

furthermore, the following boundary conditions are satisfied

\[
\gamma_1(\eta_i(-a_i)) = 0, \quad \gamma_2(\eta_i(l - a_i)) = 0,
\]

**Proof:** As regards the function \( V(x) \), which satisfies representation (10), the validity of the fact of satisfaction of the boundary conditions (7) follows directly from the boundary conditions (12) for the functions \( \eta_i(x), (i = 1, 2, \ldots, m) \) due to property (8) of the functions, which define the boundary conditions (7).

It follows from identity (9) that if function \( V(x) \) represents a generalized solution of the differential equation of system (6), then for any main function \( \varphi(x) \) the following equation is valid.

\[
\int_0^l \left( c \frac{\partial^4 V}{\partial x^4}(x) \cdot \varphi(x) dx = \sum_{i=1}^{m} \left( k_i (d_i^T \bar{q} - V(a_i) + P_i) \right) \varphi(a_i) \right.
\]
Let us rewrite (10) in the following form
\[ V(x) = \sum_{i=1}^{m} \eta_i (x - \xi_i) \left( k_i (d^T \bar{q} - V(\xi_i)) + P_i \right) \delta(\xi - a_i) \, d\xi. \] (14)

Let us substitute (14) into the left-hand side of relation (13). Having conducted sequential transformation, on account of (11), and next, changing the order of integration, we obtain the following expression
\[
\int_0^l \left[ \sum_{i=1}^{m} \int_0^l c \frac{d^4 \eta_i (x - \xi)}{dx^4} \left( k_i (d^T \bar{q} - V(\xi)) + P_i \right) \delta(\xi - a_i) \, d\xi \right] \varphi(x) \, dx = \\
= \int_0^l \left[ \sum_{i=1}^{m} \int_0^l \delta(x - \xi) \left( k_i (d^T \bar{q} - V(\xi)) + P_i \right) \delta(\xi - a_i) \, d\xi \right] \varphi(x) \, dx = \\
= \sum_{i=1}^{m} \int_0^l \left( k_i (d^T \bar{q} - V(\xi)) + P_i \right) \delta(\xi - a_i) \cdot \int_0^l \varphi(x) \delta(x - \xi) \, dx \, d\xi = \\
= \sum_{i=1}^{m} \left[ \int_0^l \left( k_i (d^T \bar{q} - V(\xi)) + P_i \right) \varphi(\xi) \delta(\xi - a_i) \, d\xi = \sum_{i=1}^{m} \left( k_i (d^T \bar{q} - V(a_i)) + P_i \right) \varphi(a_i), \right]
\]
whose right-hand side coincides with the right-hand side of (13).

Therefore, we have proved the validity of representation (10) for the generalized solution \( V(x) \) of the differential equation of system (6). *This proves the theorem.*

To find the equilibrium position for a system of rigid bodies, mounted on an Euler-Bernoulli beam in a chosen coordinate system, in the beginning, having substituted the values \( x = a_i, (i = 1, 2, ..., m) \), sequentially into relation (10), we can obtain the following system of linear algebraic equations with respect to \( V(a_1), V(a_2), ..., V(a_m) \)
\[
\sum_{i=1}^{m} \eta_i (a_i - a_j) k_i d^T \bar{q} - (1 - \eta_i(0) k_i) V(a_j) - \sum_{i=1, i \neq j}^{m} \eta_i (a_j - a_i) k_i V(a_i) = \\
= \sum_{i=1}^{m} \eta_i (a_i - a_j) P_i, \quad (j = 1, ..., m). \] (15)

With the use of matrix denotations system (15) may be rewritten in the form
\[
N_i \bar{q} - M_i \bar{V} = b_i, \] (16)
where \( M_i \) is a matrix of dimension \( m \times m \) :
\[
M_i = \begin{pmatrix}
1 + \eta_i(0) k_i & \eta_2 (a_i - a_2) k_2 & \cdots & \eta_m (a_i - a_m) k_m \\
\eta_1 (a_2 - a_1) k_1 & 1 + \eta_2(0) k_2 & \cdots & \eta_m (a_2 - a_m) k_m \\
\vdots & \vdots & \ddots & \vdots \\
\eta_1 (a_m - a_1) k_1 & \eta_2 (a_m - a_2) k_2 & \cdots & 1 + \eta_m(0) k_m
\end{pmatrix}.
\]

\( N_i \) is a matrix of dimension \( m \times n \) :
Hence having joined the first equation of system (6) with system (16) we obtain a system of linear algebraic equations with respect to $q, V$

$$
\begin{align*}
\begin{cases}
(C + \bar{C} D)\bar{q} - \bar{C}V &= b,
N_i\bar{q} - M_i\bar{V} &= b_i
\end{cases}
\end{align*}
$$

As soon as defined are the generalized solutions $\eta_i(x), (i = 1, 2, ..., m)$ of equation (11), which satisfy the boundary-value conditions (12), from the solution of system (17) we can find the values of vectors $\bar{q}$ and $\bar{V}$, which allow us to find the deflection of the beam, which takes place in the state of equilibrium according to representation (10).

**Remark.** Functions $\eta_1(x), \eta_2(x), ..., \eta_m(x)$, which enter into the representation (10) may be defined by solving $m$ boundary-value problems for the equation

$$
\frac{d^4 \eta(x)}{dx^4} = \delta(x),
$$

with boundary conditions (12).

The general generalized solution $\eta(x, \bar{c}_1, \bar{c}_2, \bar{c}_3, \bar{c}_4)$ of equation (18) may be found in the form of a sum comprised by a general solution of the homogeneous equation $\eta_0(x, \bar{c}_1, \bar{c}_2, \bar{c}_3, \bar{c}_4)$, corresponding to (18), where $\bar{c}_1, \bar{c}_2, \bar{c}_3, \bar{c}_4$ are arbitrary constants, and a generalized solution $\bar{\eta}(x)$ of non-homogeneous equation (18)

$$
\eta(x, \bar{c}_1, \bar{c}_2, \bar{c}_3, \bar{c}_4) = \eta_0(x, \bar{c}_1, \bar{c}_2, \bar{c}_3, \bar{c}_4) + \bar{\eta}(x).
$$

General solution of the homogeneous equation $\eta_0(x, \bar{c}_1, \bar{c}_2, \bar{c}_3, \bar{c}_4)$ may be found and has the following form

$$
\eta_0(x, \bar{c}_1, \bar{c}_2, \bar{c}_3, \bar{c}_4) = \bar{c}_1 + \bar{c}_2x + \bar{c}_3x^2 + \bar{c}_4x^3.
$$

In accordance with [12], the partial generalized solution $\bar{\eta}(x)$ of equation (18) can be found to have the following form

$$
\bar{\eta}(x) = \sum_{i=1}^{m} \eta_i(x) = \sum_{i=1}^{m} \eta_i(x, a_i, a_i, a_i, a_i).
$$
\[ \tilde{\eta}(x) = g(x)\theta(x), \tag{20} \]

where

\[ g(x) = \frac{1}{6c}x^3 \]

is the solution of equation

\[ \frac{d^4g(x)}{dx^4} = 0, \]

which satisfies the following initial conditions

\[ g(0) = 0, \quad \frac{dg}{dx}(0) = 0, \quad \frac{d^2g}{dx^2}(0) = 0, \quad \frac{d^3g}{dx^3}(0) = \frac{1}{c}; \]

\[ \theta(x) \]

is the classical Heaviside function.

So, according to (19)-(20), the general generalized solution \( \eta(x, \bar{c}_1, \bar{c}_2, \bar{c}_3, \bar{c}_4) \) of equation (18), may be rewritten as follows

\[ \eta(x, \bar{c}_1, \bar{c}_2, \bar{c}_3, \bar{c}_4) = \bar{c}_1 + \bar{c}_2x + \bar{c}_3x^2 + \bar{c}_4x^3 + \frac{1}{6c}x^3\theta(x). \tag{21} \]

After computing the values of arbitrary constants \( \bar{c}_1, \bar{c}_2, \bar{c}_3, \bar{c}_4 \), under the condition of satisfaction of the boundary conditions (12), we can find the generalized solutions \( \eta_1(x), \eta_2(x), \ldots, \eta_m(x) \).

3. An illustrative example

Consider a mechanical system (Fig. 1) composed of a rigid body having mass \( m \), mounted on an Euler-Bernoulli beam with the aid of two rigidity springs \( c_1 \) and \( c_2 \). Both ends of the beam are fixed by stiff attaching (staying) at the rod’s ends.

Let us introduce the two coordinate systems: (i) a stationary (immovable) coordinate system \( Oxz \), whose center (origin) coincides with the left end of the rod, axis \( Ox \) being oriented along the beam’s axis; (ii) a movable coordinate system \( O'x'z' \) bound up with the rigid body. Under the state of equilibrium, the respective coordinate axes \( Oxz \) and \( O'x'z' \) are parallel. The body can transfer (translational motion) in the direction of axis \( Oz \) and perform angular deviations \( \varphi \). The transfer of the beam’s points in the direction of axis \( Oz \) may be described by function \( u(x, t) \). The rigidity springs \( c_1 \) and \( c_2 \) are fixed on the rod at points, respectively, \( x = a_1 \) and \( x = a_2 \).

\[ \text{Figure 1. The computational scheme of the mechanical system «a rigid body mounted on an Euler-Bernoulli beam»} \]

For the purpose of deriving the equations of motion for the given mechanical system we have used
the Hamilton-Ostrogradskii variation principle, which is valid both for the systems with concentrated parameters and for the systems with distributed parameters. According to the Hamilton-Ostrogradskii variation principle, in case of a conservative system, the variation of the operation integral turns zero.

\[ \delta J = \delta \int_{t_0}^{t_1} (T - U) dt = 0. \]  

where \( T \) is the system’s kinetic energy, \( U \) is the system’s potential energy.

The potential energy of the scrutinized system is the sum of the energy of the springs and the energy of the rod; while the system’s kinetic energy is the sum of the body energy and the rod energy:

\[ U = U_1 + U_2, \quad T = T_1 + T_2 \]

where \( U_1, T_1 \) are, respectively, the rod’s potential energy and the rod’s kinetic energy; \( U_2, T_2 \) are, respectively, the potential energy of the springs and the kinetic energy of the rigid body.

According to the technical theory of rods, the expressions for the potential energy and the kinetic energy of the Euler-Bernoulli beam write as follows

\[ U_1 = \frac{1}{2} \int_0^l EI \left( \frac{\partial^2 u(x,t)}{\partial x^2} \right)^2 dx, \quad T_1 = \frac{1}{2} \int_0^l \rho F \left( \frac{\partial u(x,t)}{\partial t} \right)^2 dx, \]

where \( E \) is the Young modulus; \( I \) is the inertia moment for the transversal section of the rod with respect to the neutral axis of the section; \( u(x,t) \) is the transversal shift of the beam’s points with the coordinate \( x \) at the time moment \( t \) in the direction of axis \( 0z \); \( \rho \) is the plain of the rod; \( F \) is the square of the transversal section perpendicular to the plane of vibrations.

The kinetic energy of the rigid body may be written as the sum of the kinetic energy of translational motion and the kinetic energy of rotational motion:

\[ T_2 = \frac{m \dot{z}^2}{2} + I_{\varphi} \dot{\varphi}^2, \]

where \( I_{\varphi} \) is the inertia moment of the rigid body with respect to the mass center.

Note, according to the theory of flat motion of a rigid body, notion of body’s point defined by the coordinates \( x', z' \) in the coordinate system \( 0'x'z' \) satisfies, in the general case, the following system

\[ \begin{cases} \ddot{x} = x' \cos \varphi - z' \sin \varphi, \\ \ddot{z} = z' \sin \varphi + z' \cos \varphi, \end{cases} \]

where \( x, z \) is the transition of the origin of the coordinate system \( 0'x'z' \) in a non-movable coordinate system of \( 0xz \).

While taking into consideration the fact that motion of the body in the direction of axis \( 0x \) is absent, and also assuming that the angle of rotation is small (and so replacing the sine with its argument, and the cosine – with a unit), we can rewrite the transition in the direction of axis \( Oz \) for any body’s point having coordinates \( x', z' \) in the following form:

\[ \ddot{z} = z' + x' \varphi. \]

Therefore, according to (24), transitions in the direction of axis \( Oz \) of the points of attaching rigid elements to the body \( \ddot{z}_1, \ddot{z}_2 \) may be written in the form

\[ \ddot{z}_1 = z + z_1 + x_1 \varphi, \quad \ddot{z}_2 = z + z_2 + x_2 \varphi, \]

where \( x_1, z_1 \) and \( x_2, z_2 \) are coordinates of the points of attaching flexible elements to the body in the coordinate system \( 0'x'z' \).
Having taken account of the fact that the spring’s potential energy is proportional to the square of its linear deformation, we can write the potential energy of the springs as follows:

$$U = \frac{c}{2} (z + z_1 + x_1\varphi - u(a_1,t))^2 + c_2 (z + z_2 + x_2\varphi - u(a_2,t))^2.$$  \hfill (26)

Having substituted expressions for the kinetic energy and the potential energy (23), (24) and (26) into the integral of action and having found its variation, according to the Hamilton principle (22), we obtain the following

$$\int_{t_0}^{t_1} \left[ m\dddot{z} + c_1(z + z_1 + x_1\varphi - u(a_1,t)) + c_2(z + z_2 + x_2\varphi - u(a_2,t)) \right] \delta z dt +$$

$$\int_{t_0}^{t_1} \left[ I\dddot{\varphi} + c_1x_1(z + z_1 + x_1\varphi - u(a_1,t)) + c_2x_2(z + z_2 + x_2\varphi - u(a_2,t)) \right] \delta \varphi dt +$$

$$+ \int_{t_0}^{t_1} \left[ EI\dddot{u}(x,t) + \rho F \dddot{u}(x,t) \right] - c_1(z + x_1\varphi - u(x,t))\delta(x - a_1) -$$

$$- c_2(z + x_2\varphi - u(x,t))\delta(x - a_2) \right] \delta u dx dt = 0.$$ \hfill (27)

Owing to independence and arbitrary character of the variations of generalized coordinates $\delta z(t)$, $\delta \varphi(t)$ and $\delta u(x,t)$, due to (27), we have the HSDE

$$\begin{cases} m\dddot{z} + c_1(z + z_1 + x_1\varphi - u(a_1,t)) + c_2(z + z_2 + x_2\varphi - u(a_2,t)) = 0, \\
I\dddot{\varphi} + c_1x_1(z + z_1 + x_1\varphi - u(a_1,t)) + c_2x_2(z + z_2 + x_2\varphi - u(a_2,t)) = 0, \\
EI\dddot{u}(x,t) + \rho F \dddot{u}(x,t) = c_1(z + z_1 + x_1\varphi - u(x,t))\delta(x - a_1) +$$

$$+ c_2(z + z_2 + x_2\varphi - u(x,t))\delta(x - a_2), \hfill (28)$$

If $z_1 = z_2 = z'$, then, while introducing the vector variables

$$q(t) = \begin{pmatrix} z(t) + z' \\ \varphi(t) \end{pmatrix}, \quad \bar{u} = \begin{pmatrix} u(a_1,t) \\ u(a_2,t) \end{pmatrix},$$

it may be readily seen that system (28) – according to its structure – coincides with the generalized model (1). Furthermore, the coefficients and the matrices entering in (1) are defined by the following expressions:

$$a = \rho F, \quad c = EI, \quad k_1 = c_1, \quad k_2 = c_2,$$

$$A = \begin{pmatrix} m & 0 \\ 0 & I_\varphi \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \bar{C} = \begin{pmatrix} c_1 & c_2 \\ c_1x_1 & c_2x_2 \end{pmatrix}, \quad D = \begin{pmatrix} 1 & x_1 \\ 1 & x_2 \end{pmatrix}. \hfill (29)$$

If $z_1 \neq z_2$, then, having transformed the system of equations (28) to the following form
\[
\begin{align*}
&\frac{m\ddot{z} + c_1(z + x_1\varphi - u(a_i,t)) + c_2(z + x_2\varphi - u(a_2,t))}{I}\ddot{x} + c_3(z + x_2\varphi - u(a_2,t)) = -c_1z_1 - c_2z_2, \\
&\frac{EI}{2}\dddot{u}(x,t) + \rho F\dddot{u}(x,t) = c_1(z + x_1\varphi - u(x,t))\ddot{\delta}(x - a_i) + \\
&+ c_2(z + x_2\varphi - u(x,t))\ddot{\delta}(x - a_2) + c_1z_1\delta(x - a_1) + c_2z_2\delta(x - a_2)
\end{align*}
\]  

(30)

and having introduced into consideration the vector variables \(g(t), \bar{u}(t)\)

\[
g(t) = \begin{pmatrix} z(t) \\ \varphi(t) \end{pmatrix}, \quad \bar{u} = \begin{pmatrix} u(a_1,t) \\ u(a_2,t) \end{pmatrix},
\]

We can state that system (30) satisfies the form of generalized model (3). Furthermore, the coefficients and the matrices entering in (3) are defined by expressions (29), and the non-homogeneities, which appeared in the right-hand side, are defined by the following relations:

\[
b = \begin{pmatrix} -c_1z_1 - c_2z_2 \\ -c_1X_1z_1 - c_2X_2z_2 \end{pmatrix}, \quad P_1 = c_1z_1, \quad P_2 = c_2z_2.
\]

Therefore, in order to define the equilibrium position in system (3), it is possible to use the approach proposed above.

In the case of stiff attaching (staying) at the beam’s ends, the boundary conditions (12) assume the form:

\[
\eta_i(-a_i) = 0; \quad \eta_i(l - a_i) = 0; \quad \frac{d\eta_i}{dx}(-a_i) = 0; \quad \frac{d\eta_i}{dx}(l - a_i) = 0; \quad (i = 1, 2).
\]

(31)

To the end of finding \(\eta_1(x)\) and \(\eta_2(x)\), while employing relation (21), one can find the integration constants, which provide for satisfaction of the boundary conditions (31). Next, having constructed the linear algebraic system of equations (17), it is possible to find the values of vectors \(\vec{q}\) and \(\vec{V}\), and, on the basis of (1), it is possible to find the deflection of the beam \(V(x)\), which takes place under the state of equilibrium.

4. Conclusion

Application of the Hamilton principle in order to definite systems of rigid bodies attached to the Euler-Bernoulli beam by elastic-damping links (elastically mounted on the Euler-Bernoulli beam) and construct the equations of motion – leads to HSDE, which structurally coincide with generalized mathematical model (3) considered in the present paper. In this connection, the approach proposed in the present paper and bound up with development of an analytical-numerical method of finding the equilibrium position is a universal (general one) for a class of mechanical systems representing various systems of rigid bodies attached to a Euler-Bernoulli beam.

The values of the vectors \(\vec{q}\) and \(\vec{V}\) found from the solution of the system of linear algebraic equations (17), allow one to find the initial deflection of the beam \(V(x)\), while using relation (10). In this case, vector \(\vec{q}\) and the function \(V(x)\), define the equilibrium position for the system of rigid bodies mounted on the Euler-Bernoulli beam in the coordinate system chosen.

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Note, if some replacement of the variables is realized in HSDE (3),

\[
g(t) = q(t) + q, \quad v(x,t) = u(x,t) + V(x),
\]

Then variables \(q(t)\) and \(u(x,t)\) may be interpreted as the variables describing vibrations of the mechanical system with respect to the equilibrium position (state of rest). Furthermore, these satisfy the generalized mathematical model described by HSDE (1), which has been investigated in [1].
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