Inadequacy of the Shannon entropy in quantifying wave-particle duality for three-path interferometers

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Abstract. Based on the principle that any introduce of entanglement should not improve the limit of simultaneously obtainable information of the particle aspect and the wave aspect, we show that the Shannon entropy is not suitable for quantifying wave-particle duality by giving a counter example in three-path interferometers.

1. Introduction

Wave-particle duality, or the more general Bohr’s Principle of Complementarity [1, 2] has played an important role in the conceptual development of quantum mechanics. Although this phenomenon is already familiar and well-studied, new related experiments, such as Wheeler’s delay choice experiment [3, 4, 5] and the quantum eraser [6, 7, 8], constantly provide us counter-intuitive examples. Moreover, the rise of quantum information technology also stimulates the investigation on complementarity, e.g., the famous BB84 protocol [9] is actually based on complementarity. Therefore a quantitative understanding of wave-particle duality or complementarity is necessary not only for a rigorous foundation of quantum mechanics, but also for many quantum information applications.

The first such attempt was made by Wootters and Zurek in 1979 [10], which was followed by many others [11-20]. Notably Jaeger et. al [13] and Englert [14] summarized the wave-particle duality in terms of the following inequality

\[ D^2 + V^2 \leq 1, \]  

(1)

where \( D \) and \( V \) were quantifiers of the particle aspect and the wave aspect respectively. Relation (1) is now the standard quantitative statement of wave-particle duality for two-path interferometry. There are also efforts in the more general case of \( n \)-path interferometry, such as Dürr [16] and Englert et. al [18]. One would like to generalize the quantifiers \( D \) and \( V \) in relation (1) to similar ones in \( n \)-dimensional Hilbert spaces. A natural choice is to employ the Shannon entropy [21] which provides a standard measure of information. Furthermore, quantifying wave-particle duality using entropic relations relates it directly to many known entropic uncertainty relations [22-25], and hence to Heisenberg’s uncertainty principle[26]. Recently, there are also efforts on relating quantitative wave-particle duality relations to the problem of quantum state discrimination [27-30].

Unfortunately, as already observed by some authors [13, 16], the Shannon entropy may not be a suitable wave-particle duality measure for \( n \)-path interferometers. For example, in Ref. [13] Jaeger et. al gave an example that the Shannon entropy measure of particle aspect failed to be a monotonic function of the measure \( D \) in Eq. (1). But their disapproval of the Shannon entropy measure is based on the assumption that the measure \( D \) in Eq. (1) is the definitive measure of particle behavior and therefore
any other measures which are not monotonic functions of $D$ fail to be proper. On the other hand, the Shannon entropy is well-accepted as a valid measure of information, and there are also many authors [10-19] use it to quantify particle behavior. Hence we would like to analyze the Shannon entropy measure by forgetting the definition of $D$ in Eq. (1), and approve it or disapprove it only by the following principle: Any introduce of entanglement should not improve the amount of simultaneously obtainable information of the particle aspect and the wave aspect. This principle, which implies that the introduce of any which-path detector will not enlarge the upper bound in wave-particle duality relations like Eq. (1), is presented in all widely accepted quantitative wave-particle duality relations known so far.

We describe the abstract setting of an $n$-path interferometer and introduce various related concepts in Sec. 2. Then we analyze the Shannon entropy measure in three-path interferometers, and provide a counter example shows that the wave-particle duality relation can be enlarged and therefore exhibit the inadequacy of the Shannon entropy measure in Sec. 3. Finally, we close this paper with a summary.

2. Predictability, Distinguishability and Visibility

In an $n$-path interferometer, as shown in Figure 1, a light beam is first splitted into $n$ paths by a general path-splitter. The computational basis $\{|k\rangle, k = 1, \ldots, n\}$ is fixed by the states taking different paths (i.e., denote the quantum state of taking the $k$-th path as $|k\rangle$), then Predictability $P$, which is related to the path knowledge, and hence quantifies the particlelike property, depends only on the diagonal entries of the density matrix $\rho$ expressed in the above computational basis. For example, the standard Predictability $P$ for two-path interferometers is defined [12-15] as

$$P(\rho) = |\rho_{11} - \rho_{22}|,$$

(2)

so that it is a function of the diagonal entries of the density matrix $\rho$. More path knowledge may be obtained by applying a which-path detector, i.e., entangling the quantum state with other quantum degrees of freedom (e.g., the micromaser used in Ref. [31]), so that extra path knowledge may be gained by performing measurement on the other quantum degrees of freedom, as discussed in Ref. [31]. This process is discussed in detail by Englert and Bergou in Ref. [15], where they show mathematically that the use of a which-path detector amounts to a sorting of the quantum state $\rho$ into sub-ensembles $\{\rho_i\}$, or more explicitly

$$\rho \rightarrow \{\rho_i, p_i\} \text{ with } \rho = \sum_i p_i \rho_i,$$

(3)

where $\{p_i\}$ is the probability distribution produced by a measurement on the other quantum degrees of freedom. We define the maximum average path knowledge that one may obtain with the help of a which-path detector to be Distinguishability $D$ [15] as

$$D(\rho) = \max \left\{ \sum_i p_i P(\rho_i) \right\},$$

(4)

where the maximum is taking over all possible measurements which produce different probability distributions $\{p_i\}$ and subensembles $\{\rho_i\}$, and hence different average Predictibilities. Note that Predictability $P$ is the quantifier of the original particle aspect knowledge without any use of which-path detector; whereas Distinguishability $D$ quantifies the maximal knowledge of the particle aspect that one may obtain by any means. Therefore by definition, one should have

$$D(\rho) \geq P(\rho),$$

(5)

or in other words, by Eq. (4), Predictability $P$ should be a convex function. Indeed, all known measures of the particle aspect, i.e. Predictability, are convex function, which implies that after applying a which-path detector, more path knowledge is obtained.

As shown in Figure 1, then the light beams undergo some Hadamard transform, i.e., a transform that is expressed as an $n \times n$ complex Hadamard matrix $H$ in the computational basis. For example, in the well-known Mach-Zehnder interferometer, when the light beams meet the second half-silvered mirror, the
Figure 1. Scheme of an \(n\)-path interferometer. An incoming beam enters a beam splitter which splits the beam with different intensities into \(n\) paths. By a which-path detector, the original degrees of freedom are entangled with other quantum degrees of freedom which may be used to extract path information.

We obtain information of the wave aspect by applying a Hadamard transform, and detecting the corresponding intensities of interference by the rightmost detector \(D_i\). We would like to quantify the limit of the information that are obtainable simultaneously about the path and the interference strength.

Net effect can be described abstractly by a Hadamard matrix depending on the phase change along each path. The states \(\{H|i\}\) form another basis of the \(n\)-dimensional Hilbert space, so that we have another matrix representing the same quantum state

\[
\tilde{\rho} = H \rho H^\dagger.
\]  

The \(i\)-th diagonal entry of \(\tilde{\rho}\) is just the probability \(I_i\) of the click of the \(i\)-th detector in Figure 1. Explicitly,

\[
I_i(H) = \frac{1}{n} \left( 1 + \sum_j \sum_{k \neq j} |\rho_{jk}| \cos(\phi_{ij} - \phi_{ik} + \arg \rho_{jk}) \right),
\]

where \(e^{i\phi_{ij}}/\sqrt{n}\) is the \(ij\)-th entry of the Hadamard matrix \(H\). Note that the quantity \(I_i\) depends only on the off-diagonal entries of \(\rho\), and is proportional to the intensity of the interference with the fixed phase difference \(\{\phi_{ij}, j = 1, \ldots, n\}\). As in the familiar case of two-path interferometry, Visibility \(V\), which quantifies the wavelike property, is defined [12-15] as

\[
V = \max_H |I_1(H) - I_2(H)|,
\]

whereas the maximum is taking over all \(2 \times 2\) Hadamard matrices. For two-path interferometers, this is proportional to the maximal intensity of interference minus the minimal intensity of interference, so that \(V\) is an observable quantity. But for \(n\)-path interferometers, it is more difficult to associate the quantities \(\{I_i(H), i = 1, \ldots, n\}\) with clear physical significance, except for the greatest possible one

\[
I_{\text{max}} = \max_H \max_i \{I_i(H)\},
\]

where the maximization is over all possible Hadamard transforms and all the path labels. Since Eq. (9) corresponds to the greatest intensity of interference, it can be observed in experiments.

Now one can establish wave-particle duality relation using the pair \((P,V)\) or \((D,V)\), the first pair corresponds to the situation that no which-path detectors are presented, or no efforts are made in order to obtain more information of the particle aspect; whereas the second pair corresponds to the situation that the maximal knowledge of the particle aspect is extracted, hence accordingly, one inevitably lose some wavelike knowledge by Bohr’s principle of complementarity. The question is that whether these two wave-particle relations have the same upper bound, or put differently, if one draws the \(V\) vs. \(P\) graph and the \(V\) vs. \(D\) graph, are the outer borders of these two graphs the same? Actually for any
widely accepted wave-particle duality relation, such as Eq. (1) [13, 14] and the Dürr’s measure [16], these two outer borders are always the same.

The definitions in Eq. (2) and Eq. (7) rely on the fact that we are working in a two-dimensional space. In the general case, there are different ways to define Predictability \( P \) and Visibility \( V \) (note that Distinguishability \( D \) is defined immediately as \( P \) is defined) [16-20]. In the following sections, we will focus on the Shannon entropy measure \( P_{\text{ent}} \), \( V_{\text{ent}} \), and examine carefully the outer borders of \( V_{\text{ent}} \) vs. \( P_{\text{ent}} \) graph and \( V_{\text{ent}} \) vs. \( D_{\text{ent}} \) graph for three-path interferometers.

3. The Shannon entropy measure for three-path interferometry

The Shannon entropy measure is actually the first quantitative measure of wave-particle duality used by Wootters and Zurek in [10]. Define Predictability \( P \) as

\[
P_{\text{ent}}(\rho) = \frac{1}{\log n} \sum_{k=1}^{n} \rho_{kk} \log(n \rho_{kk}), \tag{10}
\]

and accordingly, Visibility \( V \) as

\[
V_{\text{ent}}(\rho) = \max_{H} \left\{ P_{\text{ent}} \left( H \rho H^\dagger \right) \right\}, \tag{11}
\]

where the maximum is taking over all the \( n \times n \) Hadamard matrices \( H \). Then the well-known Maassen-Uffink inequality [23] implies directly the relation

\[
P_{\text{ent}} + V_{\text{ent}} \leq 1. \tag{12}
\]

Hence, the \((P_{\text{ent}}, V_{\text{ent}})\) outer border manifests the upper bound in the Maassen-Uffink inequality for unbiased measurement. Since the Shannon entropy, i.e., the sum of all \(-p \log p\) for a probability distribution \( \{p_i\} \), is a standard measure of information in the classical world, Eq. (12) is considered as a proper measure of wave-particle duality in two-path interferometers. On the other hand, it is well-known that the Maassen-Uffink bound is not saturated by all pure states [23, 27], which implies that only at extreme cases that the upper bound 1 in Eq. (11) is reached; for other cases, the upper bound is smaller than one. It has been shown in [18] that the \((P_{\text{ent}}, V_{\text{ent}})\) outer border in three-path interferometers is traced out by the following family of states with \( n = 3 \):

\[
|\alpha\rangle = \begin{pmatrix} \sqrt{\frac{p}{2}} \\ \sqrt{\frac{q}{2}} \\ \cdots \\ \sqrt{\frac{q}{2}} \end{pmatrix}, \quad q = \frac{1-p}{n-1}. \tag{13}
\]

Now we are ready to provide the following example which shows that the \((D_{\text{ent}}, V_{\text{ent}})\) outer border is larger than that of \((P_{\text{ent}}, V_{\text{ent}})\) in the case of \( n = 3 \). Consider the state

\[
\rho_{\text{qw}} = \frac{1}{2} |a\rangle \langle a| \otimes |+\rangle \langle +| + \frac{1}{2} |b\rangle \langle b| \otimes |-\rangle \langle -|, \tag{14}
\]

where \(|+\rangle, |-\rangle\) are orthonormal states (e.g., may correspond to spin \( s_z = +1/2 \) and \( s_z = -1/2 \)) and

\[
|a\rangle = \begin{pmatrix} \sqrt{\frac{1}{2}} \\ \sqrt{\frac{1}{2}} \\ \sqrt{\frac{1}{2}} \end{pmatrix} \quad \text{and} \quad |b\rangle = \begin{pmatrix} \sqrt{\frac{1}{2}} \\ \sqrt{\frac{1}{2}} \\ 1 \end{pmatrix}. \tag{15}
\]

Then the projective measurement \( \{|+\rangle, |-\rangle\} \) produces

\[
\rho_+ = |a\rangle \langle b| \quad \text{and} \quad \rho_- = |b\rangle \langle b|, \tag{16}
\]

both with probability 1/2. Therefore, we have

\[
D_{\text{ent}} \geq \frac{1}{2} P_{\text{ent}}(\rho_+) + \frac{1}{2} V_{\text{ent}}(\rho_-) \approx 0.094316. \tag{17}
\]
Figure 2. The plot of the second derivative of the composite function $P_{\text{ent}} \circ q(x)$ for $n = 2$ and $n = 3$. It is clear that for $n = 2$, the graph is always below $y = 0$ for $x \in [1/2,1]$, which implies that the function is concave in such domain; whereas for $n = 3$, the graph is above zero near $x = 0.9$, therefore the function is not concave over the whole domain $[1/3,1]$.

On the other hand, let

$$H = \frac{1}{\sqrt{3}} \begin{pmatrix} e^{i \frac{2\pi}{3}} & e^{i \frac{4\pi}{3}} & 1 \\ e^{i \frac{4\pi}{3}} & e^{i \frac{2\pi}{3}} & 1 \\ 1 & 1 & 1 \end{pmatrix},$$

then we have

$$V_{\text{ent}} \geq P_{\text{ent}}(H(\text{Tr}_{\text{wp}} \rho_{\text{wp}})H^\dagger) \approx 0.787677,$$

where $\text{Tr}_{\text{wp}}$ denotes the partial trace of the which-path degrees of freedom. On the $(P_{\text{ent}}, V_{\text{ent}})$ outer border this number corresponds approximately to the number 0.094271, we then conclude that

$$P_{\text{ent}} \leq 0.094271 < D_{\text{ent}}.$$ 

The above example shows that fixing $V_{\text{ent}}$, it is possible that the corresponding $D_{\text{ent}}$ is larger than the corresponding $P_{\text{ent}}$ on the outer border, or the $(D_{\text{ent}}, V_{\text{ent}})$ outer border is larger than that of $(P_{\text{ent}}, V_{\text{ent}})$.

Since we know that the outer borders for both two-path and three-path are traced out by family (13), it is straightforward to show that as long as the composed function $P_{\text{ent}} \circ q(x)$, where

$$q(x) = \frac{1}{n} \left( 1 + 2\sqrt{(n-1)x(1-x)} + (n-2)(1-x) \right)$$

is concave over the whole range $[1/n,1]$ ($n=2$ or 3), the two outer borders are guaranteed to be the same. Then from Figure 2, it is clear that why we can find the preceding example for three-path interferometers.

Hence, we see that the Shannon entropy measure suggests that one may enlarge the natural limit of simultaneous information of particle aspect and wave aspect, at least for three-path interferometers, which is against our basic principle. Besides, its complicated outer borders for general $n$-path makes it a difficult quantifier to use. Therefore, although it is the standard way to quantify information, it is not a suitable measure for wave-particle duality.

4. Conclusion
We have analyzed the Shannon entropy measure for wave-particle duality by considering which-path detectors. We have shown that fixed Visibility $V$, the greatest allowable value of Distinguishability $D$, which quantifies the maximal particlelike information one may obtain by employing which-path
detectors, can be larger than that of Predictability $P$, i.e., the possible particlelike information without any use of which-path detectors, that corresponds to the same amount of wavelike information quantified by the fixed $V$, and therefore show the inadequacy of the Shannon entropy in quantifying wave-particle duality for three-path interferometers.

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