On Computing Centroids According to the $p$-Norms of Hamming Distance Vectors

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Abstract

In this paper we consider the $p$-Norm Hamming Centroid problem which asks to determine whether some given binary strings have a centroid with a bound on the $p$-norm of its Hamming distances to the strings. Specifically, given a set of strings $S$ and a real $k$, we consider the problem of determining whether there exists a string $s^*$ with $(\sum_{s \in S} d^p(s^*, s))^{1/p} \leq k$, where $d(\cdot, \cdot)$ denotes the Hamming distance metric. This problem has important applications in data clustering, and is a generalization of the well-known polynomial-time solvable Consensus String ($p = 1$) problem, as well as the NP-hard Closest String ($p = \infty$) problem.

Our main result shows that the problem is NP-hard for all rational $p > 1$, closing the gap for all rational values of $p$ between 1 and $\infty$. Under standard complexity assumptions the reduction also implies that the problem has no $2^{o(n+m)}$-time or $2^{o(k^{1/p})}$-time algorithm, where $m$ denotes the number of input strings and $n$ denotes the length of each string, for any fixed $p > 1$. Both running time lower bounds are tight. In particular, we provide a $2^{k^{1/p + \varepsilon}}$-time algorithm for each fixed $\varepsilon > 0$.

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1 Introduction

The Hamming distance between two strings of equal length is the number of positions at which the corresponding symbols in the strings differ. In other words, it measures the number of substitutions of symbols required to change one string into the other, or the number of errors that could have transformed one string into the other. This is perhaps the most fundamental string metric known, named after Richard Hamming who introduced the concept in 1950 [17].

While Hamming distance has a variety of applications in a plethora of different domains, a common usage for it appears when clustering data of various sorts. Here, one typically wishes to cluster the data into groups that are centered around some centroid, where the notion of centroid varies from application to application. Two prominent examples in this context are:

- **Consensus String**, where the centroid has a bound on the sum of its (Hamming) distance to all strings, and
- **Closest String**, where the centroid has a bound on the maximum distance to all strings.

In functional analysis terms, these two problems can be formalized using the $p$-norms of the Hamming distance vectors associated with the clusters. That is, if $S \subseteq \{0, 1\}^n$ is a cluster and $s^* \in \{0, 1\}^n$ is its centroid, then the $p$-norm of the corresponding Hamming distance vector is defined by

$$\|(s^*, S)\|_p := \left( \sum_{s \in S} d^p(c, s) \right)^{1/p},$$

where $d(s^*, s) = |\{i : s^*[i] \neq s[i], 1 \leq i \leq n\}|$ denotes the Hamming distance between $s^*$ and $s$. Using this notation, we can formulate Consensus String as the problem of finding a centroid $s^*$ with a bound on $\|(s^*, S)\|_1$ for a given set of strings $S$, while the Closest String problem can be formulated as the problem of finding a centroid $s^*$ with a bound on $\|(s^*, S)\|_{\infty}$.

The following cluster $S$ with 5 strings, each of length 7, shows that for different $p$, we indeed obtain different optimal centroids. For each $p \in \{1, 2, \infty\}$, string $s^*_p$ is an optimal $p$-norm centroid which is not an optimal $q$-norm centroid, for $q \in \{1, 2, \infty\} \setminus \{p\}$.

| $S$ : 1111 111 | $p$ | $\| \cdot \|_1$ | $\| \cdot \|_2$ | $\| \cdot \|_{\infty}$ |
|-----------------|-----|----------------|----------------|----------------|
| 1111 000        | $s^*_1 = 0000 000$ | 14              | $\sqrt{68}$     | 7              |
| 0000 100        | $s^*_2 = 0011 000$ | 16              | $\sqrt{56}$     | 5              |
| 0000 001        | $s^*_\infty = 0011 001$ | 17              | $\sqrt{61}$     | 4              |

Moreover, one can verify that $s^*_2$ is the only optimal 2-norm centroid and no optimal $\infty$-norm centroid is an optimal 2-norm centroid.

The notion of $p$-norms for distance vectors is very common in many different research fields [26, 23, 16, 27, 14, 2, 19, 3]. In cluster analysis of data mining and machine learning, one main goal is to partition $m$ observations (i.e. $m$ real vectors of the same dimension) into $K$ groups so that the sum of “distances” of each observation to the nearest center is minimized. Here, two highly prominent clustering methods are $K$-means [25] and $K$-medians [18, 4] clustering, each using a slightly different notion of distance measure. The first method aims to minimize the sum of 2-norms between each observation and the “mean” of its respective group. This results in partitioning the data space into Voronoi cells, typically but not exclusively according to the 2-norm metric. $K$-medians, on the other hand, uses the 1-norm to define the distance metric. Thus, instead of calculating the mean for each group to determine its centroid, one calculates the median. Note, however, that most implementations of $K$-means and $K$-medians, such as R and SPSS, allow specifying the actual value of $p$ used in the underlying $p$-norm metric.

Problem definition, notations, and conventions. Since the Hamming distance is frequently used in clustering applications, *e.g.* in computational biology [28], in information theory, coding
theory and cryptography [17, 6, 29], and since the notion of \( p \)-norm is very prominent in clustering tools [23, 30], where often \( p = 1,2 \) but also larger values of \( p \) are used, it is natural to consider computational problems associated with the \( p \)-norm of the Hamming distance metric. This is the main purpose of this paper. Specifically, we consider the following problem:

\[ \text{p-Norm Hamming Centroid (p-HDC)} \]

\[ \text{Input: A set } S \text{ of strings } s_1, \ldots, s_m \in \{0,1\}^n \text{ and a real } k. \]

\[ \text{Question: Is there a string } s^* \in \{0,1\}^n \text{ such that } \|s^*, S\|_p \leq k? \]

Note that there is nothing special about the binary alphabet used in the definition above, but for ease of presentation we use it throughout the paper. Also, note that when \( p = 1 \), our p-HDC problem is precisely the Consensus String problem, and when \( p = \infty \) it becomes the Closest String problem.

In the following, we list some notations and conventions for reading our paper. For two binary strings \( s \) and \( s' \), let \( s \circ s' \) denote the concatenation of \( s \) and \( s' \). By \( s[j] \) we denote the \( j^{th} \) bit value or the \( j^{th} \) column of string \( s \). By \( \overline{s} = (1 - s[j])_{j \in [s]} \) we denote the complement of string \( s \). Given two integers \( j, j' \in \{1,2,\ldots,|s|\} \) with \( j \leq j' \), we use the notation \( s[j]' \) to denote the substring \( s[j]s[j+1] \cdots s[j'] \). Given a number \( \ell \), we use \( 0_\ell \) and \( 1_\ell \) denote the length-\( \ell \) all-zero string and the length-\( \ell \) all-one string, respectively. Given two binary strings \( s \) and \( s' \), and an integer \( j \) with \( 1 \leq j \leq |s|+1 \), by \( \text{ins}(s,s',j) \) we mean inserting the string \( s' \) into \( s \) at the position \( j \). For instance, \( \text{ins}(0110,00,3) = 010010 \). In particular, \( \text{ins}(s,s',1) = s' \circ s \) and \( \text{ins}(s,s',|s|+1) = s \circ s' \).

Unless stated explicitly, by strings we mean binary strings over alphabet \( \{0,1\} \) and by \( p \)-distance we mean the \( p^{th} \)-power of the Hamming distance.

**Our contributions.** Our main result in this paper is a tight running time bound on the p-HDC problem for all fixed rationals \( p > 1 \). Specifically, we show that the problem can be solved in \( O^*(2^{k/p^{(p+1)+\epsilon}}) \) time for arbitrary small \( \epsilon > 0 \), but cannot be solved in \( 2^{o(k/p^{(p+1)})} \) time unless the Exponential Time Hypothesis (ETH) fails. While the upper bound in this result is not very difficult, the lower bound uses an intricate construction and some delicate arguments to prove its correctness. As another consequence of this construction, we also obtain a \( 2^{o(n+m)} \) lower bound assuming ETH, which shows that the trivial brute-force \( O^*(2^n) \) algorithm for the problem cannot be substantially improved.

In the final part of the paper we present an \( O^*(2^{O(m2^m)}) \) time algorithm for the problem, by first formulating the problem as a Convex Integer Programming, and then applying the algorithm developed by Dadush [10] (see also [9]) for such problems using our own separation oracle for p-HDC. We also show that the problem can be approximated in polynomial time within a factor of 2, using an extension of the well known 2-approximation algorithm for Closest String.

**Related work.** The NP-complete Closest String [12, 21] problem (aka. Minimum Radius) is a special case of our p-HDC with \( p = \infty \). It has been studied extensively under the lens of parameterized complexity and approximation algorithmics. It is solvable in \( O(n \cdot 2^{O(d)}) \) time [24] and in \( m^{O(n^2)} \cdot \log n \) time, where \( m \) and \( n \) denote the number and the length of input strings, respectively, and \( d \) is the Hamming distance bound [20]. The first result (wrt. \( n \)) is achieved by a sophisticated recursive tree algorithm while the latter result is based on \( n \)-fold programming. As for approximability, Closest String admits a PTAS with running time \( O(n^{O(\epsilon^{-2})}) \) [24] but refutes EPTAS unless \( \text{FPT} = \text{W}[1] \) [8].
Our problem is related to the so-called Closest Lattice Vector (CLV) problem [22, 15, 1], which given \(m\) linear independent vectors in \(\mathbb{R}^n\), a target \(t \in \mathbb{R}^n\), and an integer bound \(k\), asks whether there is a linear combination of the vectors that has \(L_p\) distance at most \(k\) to the target \(t\). Herein, Euclidean distance (\(p = 2\)) is usually used. To the best of our knowledge, CLP for \(p = 2\) is NP-hard, while the computational complexity for other fixed \(p > 2\) remains open. Our problem falls into the general framework of convex optimization with binary variables. If a solution is allowed to have fractional values, then the underlying convex optimization can be solved in polynomial time for each fixed value \(p \leq 2\) [27, Chapter 6.3.2].

For \(p = 2\), maximizing (instead of minimizing) the \(p\)-norm reduces to Mirkin Minimization (MM) in consensus clustering with input and output restricted to two-clusters, which was shown to be Turing NP-hard [11]. Recently, Chen et al. [5] provided a many-one reduction showing that the simple \(2^n\)-algorithm by brute-force searching all possible outcome solutions is essentially tight under ETH. They also provided some efficient algorithms and showed that the problem admits an FPTAS by a simple rounding technique.

\section{NP-hardness for the \(p\)-norm Hamming distances}

In this section, we give a reduction from 3-COLORING showing that \(p\)-HDC is NP-hard for each fixed rational number \(p > 1\) and that algorithms with running time \(2^{o(n+m)}\) or \(2^{o(k^{(1/p)+1})}\) would contradict the Exponential Time Hypothesis. The basic idea is to construct a gadget (see Lemma 1) that enforces a prescribed number of ones in the solution distributed across some fixed set of columns. These columns later encode the colors to be assigned to the vertices. We then adjoin this gadget to a set of strings which encode the graph using vertex strings and edge strings. The crucial part are the edge strings; for each edge, we will introduce six strings which will induce this gadget to a set of strings which encode the graph using vertex strings and edge strings. The columns. These columns later encode the colors to be assigned to the vertices. We then adjoin that enforces a prescribed number of ones in the solution distributed across some fixed set of columns.

As mentioned, we first show how to construct a set of strings to enforce some structure on the optimal solution, that is, a binary string with minimum sum of the \(p\)-distances.

\begin{lemma}
Let \(p\) be a fixed rational number and let \(a\) and \(b\) be two coprime fixed integers with \(p = a/b\), and let \(S\) consist of one string of the form \(1_{(2^b+1)\hat{n}}0_p\) and \(2^{a^p}\) strings of the form \(0_{(2^b+2)\hat{n}}\). Let \(s^*\) be an arbitrary length-\((2^b + 2)\hat{n}\) string. Then, the following holds.

1. If \(s^*\) satisfies \(d(s^*, 0_{(2^b+2)\hat{n}}) = \hat{n}\) and \(hs(s^*, 0_{(2^b+2)\hat{n}}) \subseteq [(2^b + 1)\hat{n}]\), then \(\sum_{s \in S} d^p(s^*, s) = (2^a + 2^{a-b}) \cdot \hat{n}^p\).

2. If \(d(s^*, 0_{(2^b+2)\hat{n}}) \neq \hat{n}\), then \(\sum_{s \in S} d^p(s^*, s) > (2^a + 2^{a-b}) \cdot \hat{n}^p\).

\end{lemma}

\begin{proof}
The first statement is straightforward to see by a simple calculation.

We now prove the second statement. Let \(y\) equal the number of ones in \(s^*\) in the first \((2^b + 1)\cdot \hat{n}\) columns. Then, \(\sum_{s \in S} d^p(s^*, s) \geq ((2^b + 1)\cdot \hat{n} - y)^p + 2^{a-b} \cdot y^p\). We define a function \(f : [0, (2^b + 1)\cdot \hat{n}] \rightarrow \mathbb{Z}\) with \(f(y) := ((2^b + 1)\cdot \hat{n} - y)^p + 2^{a-b} \cdot y^p\), and show that this function attains its sole minimum over \([0, (2^b + 1)\cdot \hat{n}]\) at \(y = \hat{n}\). Note that this function is a lower bound on the sum of \(p\)-distances.
of $s^*$. The first derivative of $f$ with respect to $y$ is
\[
\frac{df}{dy} = -p((2^b + 1)\hat{n} - y)^{p-1} + p \cdot 2^{a-b} y^{p-1} \\
= -p((2^b + 1)\hat{n} - y)^{a-b} + p(2\sqrt{y})^{a-b} \\
= p(2y^{1/b} - ((2^b + 1) \cdot \hat{n} - y)^{1/b}) \left( \sum_{j=0}^{a-b-1} (2^b y)^j \cdot ((2^b + 1) \cdot \hat{n} - y)^{a-b-1-j} \right).
\]
(1)

Herein, the last equality can be seen by using the following simple refactorization:
\[
r^n - s^n = (r - s) \sum_{j=0}^{n-1} r^j \cdot s^{n-1-j}.
\]
(2)

Observe that the third factor in (1) is positive and non-zero over $[0, (2^b + 1) \cdot \hat{n}]$ since $a - b - 1 \geq 0$, and hence, over $[0, (2^b + 1) \cdot \hat{n}]$, $\frac{df}{dy}$ is zero only at $y = \hat{n}$, it is negative for $y < \hat{n}$ and otherwise positive. Hence, indeed, the sole minimum of $f(y)$ over $[0, (2^b + 1) \cdot \hat{n}]$ is attained at $y = \hat{n}$. □

In the reduction we make crucial use of pairs of strings whose mutual Hamming distance has some lower bound. They will enforce local structure in some columns of the solution, while being somewhat immune to changes elsewhere. For the sake of the reduction we derive the following lower bound on the $p$-distance of an arbitrary string to a pair of strings who are quite far from each other, in terms of Hamming distances.

**Lemma 2.** Let $s_1$ and $s_2$ be two strings of the same length $R$ such that the Hamming distance between $s_1$ and $s_2$ is $d(s_1, s_2) = 2L$. Then, for each solution string $\hat{s}$ the following holds. 1. $d^p(\hat{s}, s_1) + d^p(\hat{s}, s_2) \geq 2 \cdot L^p$. 2. $d^p(\hat{s}, s_1) + d^p(\hat{s}, s_2) = 2 \cdot L^p$ if and only if $d(\hat{s}, s_1) = d(\hat{s}, s_2) = L$ and $h\bar{s}(\hat{s}, s_1) \cup h\bar{s}(\hat{s}, s_2) = h\bar{s}(s_1, s_2)$.

**Proof.** Again, let $a$ and $b$ be two fixed integers such that $p = a/b$. To simplify the notation, let $d(\hat{s}, s_1) = L + x$ with $x \in \{-L, -L + 1, \ldots, R - L\}$. Then, $d(\hat{s}, s_2) \geq d(s_1, s_2) - d(\hat{s}, s_1) = L - x$ because of triangle inequality of the Hamming distances. Let $f(x) = (L + x)^p$ and $g(x) = (L - x)^p$. We derive the following.
\[
d^p(\hat{s}, s_1) + d^p(\hat{s}, s_2) \geq (L + x)^p + (L - x)^p = f(x) + g(x) \\
= f(0) + g(0) + \sum_{j=1}^{\infty} \frac{f^{(j)}(0) + g^{(j)}(0)}{j!} \cdot x^j \\
\geq 2 \cdot L^p,
\]
where $f^{(j)}(\cdot)$ and $g^{(j)}(\cdot)$ represent the $j^{th}$ derivatives. The equality in the second line follows from Taylor expansion. The last inequality holds since for each $j$ we have
\[
f^{(j)}(0) + g^{(j)}(0) = p(p - 1) \cdots (p - j + 1) \cdot L^{p-j} \cdot (1 + (-1)^j) \geq 0
\]

To show the second statement, we first observe that $d^p(\hat{s}, s_1) + d^p(\hat{s}, s_2) = 2 \cdot L^p$ if $d(\hat{s}, s_1) = d(\hat{s}, s_2) = L$. This completes the proof for the “if” direction. As for the “only if” direction, assume that $d(\hat{s}, s_1) \neq L$ or $h\bar{s}(\hat{s}, s_1) \cup h\bar{s}(\hat{s}, s_2) = h\bar{s}(s_1, s_2)$, meaning that $x \neq 0$. Then $\frac{f^{(2)}(0) + f^{(2)}(0)}{2} \cdot x^2 > 0$. By the above Taylor expansion (3), we have $d^p(\hat{s}, s_1) + d^p(\hat{s}, s_2) > 2 \cdot L^p$. □
Using Lemmas 1 and 2, we can show NP-hardness of p-HDC for each fixed \( p > 1 \). We reduce from the NP-hard 3-COLORING problem [13], which given an undirected graph \( G = (V, E) \) asks whether there is a proper vertex coloring \( \text{col}: V \to \{0, 1, 2\} \), that is, no two adjacent vertices \( u, u' \) receive the same color. Lemma 1 gives us a way to ensure that our solution string corresponds to some proper coloring. The idea is to introduce two vertex strings that are complement to each other so as to force each vertex to have exactly one color and introduce edge strings so as to exclude any two adjacent vertices from having the same color.

**Theorem 1.** For each fixed rational number \( p > 1 \), p-HDC is NP-hard.

**Proof.** First of all, let \( a \) and \( b \) be two fixed integers such that \( p = a/b \). To show the hardness result, we reduce from the NP-hard 3-COLORING problem [13] defined above. Let the undirected graph \( G = (V, E) \) be an instance of 3-COLORING. Let \( n \) be the number of vertices in \( G \) and \( m \) the number of edges. Denote \( V = \{v_1, v_2, \ldots, v_n\} \) and \( E = \{e_1, e_2, \ldots, e_m\} \).

We introduce three groups of strings of length \( (2^b+2) \cdot \hat{n} \) each, where \( \hat{n} = n + m \). The first group ensures that each optimal solution string must have exactly \( \hat{n} \) ones which appear in the first \( 3\hat{n} \) columns, the second group ensures that each vertex has exactly one of the three colors, and the third group, combined with the second group, ensures that no two adjacent vertices obtain the same color.

**Group 1.** Construct one string of the form \( 1_{(2^b+1)\hat{n}} \circ 0_{\hat{n}} \) and \( 2^{a-b} \) strings of the form \( 0_{(2^b+2)\hat{n}} \).

**Group 2.** For each vertex \( v_i \in V \) let \( u_i = \text{ins}(0_{3\hat{n}-3}, 111, 3i-2) \) and \( \overline{u}_i \) be the complement of \( u_i \). Deriving from \( u_i \), we construct two vertex strings \( s_i \) and \( r_i \) with \( s_i = u_i \circ 0_{(2^b-2)\hat{n}} \circ 0 \circ 1_{\hat{n}-1} \) and \( r_i = \overline{u}_i \circ 0_{(2^b-2)\hat{n}} \circ 1 \circ 0_{\hat{n}-1} \). Note that \( d(s_i, r_i) = 4\hat{n} \).

For an example, the strings \( s_2 \) and \( r_2 \) corresponding to the vertex \( v_2 \) has the form

\[
\begin{align*}
s_2 &= 000111 \circ 0_{3\hat{n}-5} \circ 0_{(2^b-2)\hat{n}} \circ 1_{\hat{n}-1}, \\
r_2 &= 111000 \circ 1_{3\hat{n}-5} \circ 0_{(2^b-2)\hat{n}} \circ 0_{\hat{n}-1}.
\end{align*}
\]

**Group 3.** For each edge \( e_j \in E \) let \( e^{(0)}_j \), \( e^{(1)}_j \), and \( e^{(2)}_j \) denote three strings, each of length \( 3\hat{n} \), that indicate which color one of its two endpoints has:

\[
\forall \ell \in \{1, 2, \ldots, \hat{n}\}: e^{(0)}_j[3\ell - 2, 3\ell - 1, 3\ell] = \begin{cases} 100, & 1 \leq \ell \leq n \text{ with } v_\ell \in e_j, \text{ or } \ell = j + n, \\ 000, & \text{otherwise}. \end{cases}
\]

\[
e^{(1)}_j[3\ell - 2, 3\ell - 1, 3\ell] = \begin{cases} 010, & 1 \leq \ell \leq n \text{ with } v_\ell \in e_j, \text{ or } \ell = j + n, \\ 000, & \text{otherwise}. \end{cases}
\]

\[
e^{(2)}_j[3\ell - 2, 3\ell - 1, 3\ell] = \begin{cases} 001, & 1 \leq \ell \leq n \text{ with } v_\ell \in e_j, \text{ or } \ell = j + n, \\ 000, & \text{otherwise}. \end{cases}
\]

Now, we construct the following six edge strings for edge \( e_j \):

\[
\forall z \in \{0, 1, 2\}: e^{(z)}_j = e^{(z)}_j \circ 0_{(2^b+1)\hat{n}} \circ 0 \circ 1_{\hat{n}-1} \\
w^{(z)}_j = \overline{e}^{(z)}_j \circ 0_{(2^b+1)\hat{n}} \circ 1 \circ 0_{\hat{n}-1}.
\]
For an example, assume that \( a = 3, b = 2, n = 3, \) and \( m = 2, \) and there is an edge of the form \( e_2 = \{v_1, v_3\}. \) Then, the triple pairs of strings that we construct for \( e_2 \) are

\[
\begin{align*}
t_j^{(0)} &= 100 000 100 000 100 000 000 000 000 01111, \\
w_j^{(0)} &= 011 111 011 111 011 000 000 000 000 000, \\
t_j^{(1)} &= 010 000 010 000 010 000 000 000 000 01111, \\
w_j^{(1)} &= 101 111 101 111 101 000 000 000 000 000, \\
t_j^{(2)} &= 001 000 001 000 001 000 000 000 01111, \\
w_j^{(2)} &= 110 111 110 111 110 000 000 000 000 000.
\end{align*}
\]

The instance \( I' \) of \( p\)-HDC consists of the following strings, each of length \((2^b+2)\hat{n} = (2^b+2)(n+m)\):

1. Add the \( 2^{a-b} + 1 \) strings in group 1 to \( I' \).
2. For each vertex \( v_i \in V \), add the vertex strings \( s_i \) and \( r_i \) to \( I' \).
3. For each edge \( e_j \in E \), add two triple \( t_j^{(0)}, t_j^{(1)}, t_j^{(2)} \) and \( w_j^{(0)}, w_j^{(1)}, w_j^{(2)} \) to \( I' \).

See Figure 1 for an illustration.

Figure 1: Illustration of the reduction used in Theorem 1. The left part depicts a graph \( G \) with four vertices and five edges. This graph \( G \) admits a proper vertex coloring \( \text{col} \) (see the labels on the vertices). For instance, vertex \( v_1 \) has color 0, i.e. \( \text{col}(v_1) = 0 \). The right part shows the crucial part of an instance of \( p\)-HDC with \( p = 2 \) that we will construct according to the proof for Theorem 1. A solution string \( s^* \) that corresponds to the coloring \( \text{col} \) is depicted at the bottom of the right figure.

Finally, let \( k = 2\hat{n} \cdot \sqrt[2]{2^{a-b}} + 2^{a-b-p} + 2(n+3m) \). This completes the construction, which can clearly be done in polynomial time.

Before we show the correctness of our construction, we define a notion and make some observations. Let \( s \) and \( s' \) be two strings of equal length. We say that \( s \) covers \( s' \) exactly once if there is exactly one integer \( \ell \in \{1, 2, \ldots, |s|\} \) with \( s[\ell] + s'[\ell] = 2 \).
Claim 1. Let $s$ and $s^*$ be two strings, both of length $4\hat{n}$, that fulfill the following properties.
(1) In $s$, the first $3\hat{n}$ columns have exactly 3 ones and the last $\hat{n}$ columns have the form $0 \circ_1 1_{\hat{n}-1}$.
(2) $s^*$ has exactly $\hat{n}$ ones and each of them is in the first $3\hat{n}$ columns.
If $s^*$ covers $s$ exactly once, then $d^p(s^*, s) + d^p(s^*, \overline{s}) = 2 \cdot (2\hat{n})^p$; else $d^p(s^*, s) + d^p(s^*, \overline{s}) > 2 \cdot (2\hat{n})^p$.

Proof. Assume that $s^*$ covers $s$ exactly once and let $\ell \in \{1, 2, \ldots, 4\hat{n}\}$ be an integer with $s^*_{[\ell]} = s_{[\ell]} = 1$. Since $s^*[4\hat{n}+1] = 0$, it follows that $\ell \in \{1, 2, 3\hat{n}\}$. By the property of $s$, we have that $d(s^*, s) = d(s^*[3\hat{n}], s[3\hat{n}]) + d(s^*[4\hat{n}], s[4\hat{n}]) = (2 + \hat{n} - 1) + (\hat{n} - 1) = 2\hat{n}$ (note that $s$ consists only of ones in the last $\hat{n} - 1$ columns). Thus, $d(s^*, \overline{s}) = 4\hat{n} - d(s^*, s) = 2\hat{n}$. In summary, $d^p(s^*, s) + d^p(s^*, \overline{s}) = 2 \cdot (2\hat{n})^p$.

Assume that $s^*$ does not cover $s_i$ exactly once. Next, we claim that $d(s^*, s_i) \neq 2\hat{n}$. Since $s$ has exactly 3 ones in the first $3\hat{n}$ columns, there are three cases to consider.

Case 1: For each $\ell \in \{1, 2, \ldots, 3\hat{n}\}$, it holds that $s^*_{[\ell]} + s_{[\ell]} \leq 1$. Consider the values in the first $3\hat{n}$ columns of $s$ and $s^*$: since $s$ contains exactly 3 ones and $s^*$ contains exactly $\hat{n}$ ones, it follows that $d(s^*, s) = d(s^*[3\hat{n}], s[3\hat{n}])$.

Case 2: There are two distinct integers $\ell, \ell' \in \{1, 2, \ldots, 3\hat{n}\}$ with $s^*_{[\ell]} = s^*_{[\ell']} = s_{[\ell]} = s_{[\ell']} = 1$. Thus, $d(s^*, s) = d(s^*[3\hat{n}], s[3\hat{n}])$.

Case 3: For each integer $\ell \in \{1, 2, \ldots, 3\hat{n}\}$ with $s^*_{[\ell]} = s_{[\ell]} = 1$. By assumption, $s^*$ has exactly $\hat{n}$ ones in the first $3\hat{n}$ columns, and $s$ has exactly 3 ones in the first $3\hat{n}$ columns. Therefore, $d(s^*, s) = d(s^*[3\hat{n}], s[3\hat{n}])$.

Together with the second statement in Lemma 2, we obtain that $d^p(s^*, s) + d^p(s^*, \overline{s}) > 2 \cdot (2\hat{n})^p$ (of Claim 1).

We show that $G$ has a proper coloring if and only if there is a string $s^*$ such that the sum of the $p$-distances from $s^*$ to all strings in $I'$ is at most $k^p = (2^a + 2^{a-b}) \cdot \hat{n}^p + 2(n + 3m) \cdot (2\hat{n})^p$.

For the “if” direction, let $s^*$ be a string which has a sum of $p$-distances of at most $k^p$ to all strings in $I'$. By Lemma 2, the sum of $p$-distances to the second and the third group of strings is at least $2 \cdot (2\hat{n})^p \cdot (n + 3m)$ since these groups consist of $n + 3m$ pairs of strings that each have Hamming distance at least $4\hat{n}$ to each other. Thus, $s^*$ can have a sum of $p$-distances of at most $(2^a + 2^{a-b}) \cdot \hat{n}^p$ to the first group of strings. By the second statement of Lemma 1 string $s^*$ has exactly $\hat{n}$ ones, which all appear in the first $(2^6 + 1)\hat{n}$ columns.

Now, we claim that the ones in the solution $s^*$ indeed all appear in the first $3\hat{n}$ columns, that is, $\text{hs}(s^*, 0_{(2^b+2)\hat{n}}) \subseteq [3\hat{n}]$. Suppose, for the sake of contradiction, that solution $s^*$ contains $x$ ones which appear in columns between $3\hat{n} + 1$ and $(2^b + 1)\hat{n}$ with $x > 0$. Equivalently, this means that $|\text{hs}(s^*, 0_{(2^b+2)\hat{n}}) \cap \{3\hat{n} + 1, \ldots, (2^b + 1)\hat{n}\}| = x > 0$; recall that we have just shown that all ones in $s^*$ appear in the first $(2^b + 1)\hat{n}$ columns. This means that one string in the pair has Hamming distance more than $2\hat{n}$ from $s^*$; recall that the pairs are constructed in such a way that they have Hamming distance $4\hat{n}$ to each other, but have only zeros in the columns between $3\hat{n} + 1$ and $(2^b + 1)\hat{n}$. However, by the second statement in Lemma 2, this means that the sum of $p$-distances from $s^*$ to each of these pairs exceeds $2 \cdot (2\hat{n})^p$, a contradiction to our reasoning above for group 2 and group 3. Thus, indeed $\text{hs}(s^*, 0_{(2^b+2)\hat{n}}) \subseteq [3\hat{n}]$.

This implies that, when determining the $p$-distance of $s^*$ to the strings from group 2 and group 3, we can ignore the values in the columns in the $3\hat{n} + 1$ and $(2^b + 1)\hat{n}$ in each string, including the solution $s^*$, because $s^*$ has only zeros in these columns. We will hence treat these columns as if they do not exist. In this way, we obtain strings of length $4\hat{n}$. By Claim 1, the $p$-distance from
Claim 1.

One of these three places. To prove this, we consider the sub-
strings

Case 1: $s$ with two endpoints $t$ in
the remaining columns. Thus, by

Theorem of

Now, we focus on strings from group 2 and group 3. Since the solution $s^*$ and each string in these groups have only zeros in the columns between $3n + 1$ and $(2^b + 1)\hat{n}$, we can simply ignore the values in these columns and assume from now on that the strings have length $4n\hat{n}$. Thus, for each string $s_i$ from group 2, $s^*$ and $s_i$ fulfill the properties stated in Claim 1. Moreover, by definition, $s^*$ covers $s_i$ exactly once. Thus, by the same claim, we have that the sum of the $p$-distances from $s^*$ to all strings in group 2 is $n \cdot 2 \cdot (2n)^p$.

Analogously, consider a string $t_j^{(z)}$ from group 3, $j \in \{1, 2, \ldots, m\}$ and $z \in \{0, 1, 2\}$. Recall that $t_j^{(z)}$ corresponds to the edge $e_j$, and let $v_i$ and $v_i'$ be the two endpoints of edge $e_j$. We claim that $s^*$ covers $t_j^{(z)}$ exactly once. Observe that $t_j^{(z)}$ has exactly 3 ones in the first $3\hat{n}$ columns, namely at column $3i - 2 + z$, $3i' - 2 + z$, and $3n + 3j - 2 + z$. That is, we need to show that $s^*$ has an in exactly one of these three places. To prove this, we consider the sub-
strings $t_j^{(z)}|_{3n+3j-2}$ and $s^*|_{3n+3j-2}$.

Case 1: $s^*|_{3n+3j-2} = t_j^{(z)}|_{3n+3j-2}$. By the definition of $s^*$, this implies that $\text{col}(e_j) = \{0, 1, 2\} \setminus \{z\}$.

In addition, for both $y \in \{i, i'\}$ and each $z' \in \text{col}(e_j)$, $s^*[3y - 2 + z'] = 1$ and $t_j^{(z)}[3y - 2 + z'] = 0$, by the respective definition. Thus, $3n + 3j - z$ is the only column in which both $s^*$ and $t_j^{(z)}$ have a one, and thus $s^*$ covers $t_j^{(z)}$ exactly once.

Case 2: $s^*|_{3n+3j-2} \neq t_j^{(z)}|_{3n+3j-2}$. This means that $s^*[3n + 3j - 2 + z] = 0$, and by the definition of $s^*$, there exists another integer $x \in \{0, 1, 2\} \setminus \{z\}$ such that $s^*[3n + 3j - 2 + x] = 0$. (Note

$s^*$ to each pair of strings is indeed equal to $2 \cdot (2n)^p$. By the same claim, it follows that $s^*$ covers each string $s_i$ (resp. $t_j^{(z)}$) from the second (resp. the third) group exactly once.

Let $\text{col}: V \to \{0, 1, 2\}$ be a mapping defined as follows. For each $v \in V$, set $\text{col}(v) = z$ where $z \in \{0, 1, 2\}$ such that $s^*|_{3i - 2 + z} = 1$. Note that, since $s^*$ covers $s_i$ exactly once and since $s_i$ has exactly three ones in the columns $3i - 2, 3i - 1$, and $3i$, there is indeed such a $z$ for $\text{col}(v_i)$.

We claim that $\text{col}$ is a proper coloring for $G$. Suppose, towards a contradiction, that there is an edge $e_j = \{v_i, v_i'\} \in E$ such that $v_i$ and $v_i'$ have the same color from $\text{col}$, say $z \in \{0, 1, 2\}$. By the definition of $\text{col}$, this means that $s^*|_{3i - 2 + z} = s^*|_{3i' - 2 + z} = 1$. However, by the definition of the string $t_j^{(z)}$ which corresponds to the edge $e_j$, we also have that $t_{j}^{(z)}[3i - 2 + z] = t_{j}^{(z)}[3i' - 2 + z] = 1$. This implies that $t_{j}^{(z)}$ is not covered by $s^*$ by exactly once—a contradiction to our reasoning above that $s^*$ covers each string from the third group exactly once.

For the “only if” direction, let $\text{col}: V \to \{0, 1, 2\}$ be a proper coloring for $G$. For an edge $e \in E$ with two endpoints $v_i, v_i'$, let $\text{col}(e) = \{\text{col}(v_i), \text{col}(v_i')\}$. We claim that the string $s^*$ defined as follows has the desired $p$-distance to all strings of $I'$.

$$s^*|_{3n+3j-2} := 0.$$
that $s^*$ and $t_j^{(z)}$ have each exactly 1 one in the columns $3n + 3j - 2$, $3n + 3j - 1$, and $3n + 3j$.

Again, by the definition of $s^*$, this implies that $s^*[3i - 2 + x] = s^*[3i' - 2 + z] = 1$ and $s^*[3i - 2 + z] = 0$ (note that col is a proper coloring). Again, since $t_j^{(z)}$ has exactly three ones in the first $3n$ columns, namely at columns $3i - 2 + z$, $3i' - 2 + z$, and $3n + 3j - 2 + z$, it follows that $s^*$ covers $t_j^{(z)}$ exactly once.

We have just shown that $s^*$ covers $t_j^{(z)}$ exactly once. Since $s^*$ and $t_j^{(z)}$ fulfill the property stated in Claim 1, it follows from the same claim that the sum of $p$-distances from $s^*$ to $t_j^{(z)}$ and to $T_j^{(z)}$ is $2 \cdot (2n)^p$. There are $3m$ pairs in this group. So, the sum of the $p$-distances from $s^*$ to all strings of this group is $3m \cdot 2 \cdot (2n)^p$.

In total, the sum of the $p$-distances from $s^*$ to all strings of $I'$ is $(2a + 2a^b) \cdot \hat{n}^p + 2 \cdot (2n)^p \cdot (n+3m) = k^p$, as required.

Our NP-hardness reduction implies the following running time lower bounds.

**Corollary 1.** For each fixed rational number $p \geq 1$, unless the Exponential Time Hypothesis fails, no $2^{o(n+\hat{n})} \cdot |I'|^{O(1)}$-time or $2^{o(k^p/(p+1))} \cdot |I'|^{O(1)}$-time algorithm exists that decides any given instance $I'$ of $p$-HDC where $\hat{n}$ is the length of the input strings, $\hat{n}$ is the number of input strings, and $k$ is the $p$-norm bound.

**Proof.** Let $a$ and $b$ be two fixed integers such that $p = a/b$. To show our statement, note that we have constructed $2^{a-b} + 1 + 2(n + 3m)$ strings for our $p$-NORM HAMMING CENTROID problem in the proof for Theorem 1, each of which has length $(2^b + 2) \cdot (n + m)$, where $n$ and $m$ are the number of vertices and the number of input strings, respectively. The $p$-norm bound $k$ was set to $2\hat{n} \cdot \sqrt{2^{a-b} + 2^{a-b} - p + 2(n + 3m)}$ which is upper bounded by $((1 + 6 \cdot 2^p) \cdot (n + m))^{\frac{p}{p+1}}$ since $a$ and $b$ are fixed integers. Thus, a $2^{a(n+\hat{n})} \cdot |I'|^{O(1)}$-time or a $2^{a(k^p/(p+1))}$-time algorithm for $p$-NORM HAMMING CENTROID implies a $2^{o(n+m)} \cdot (n \cdot m)^{O(1)}$-time algorithm for 3-COLORING, which is unlikely unless the Exponential Time Hypothesis fails [7, Theorem 14.6].

Using a slight modification of the construction, we can show that our results are not idiosyncratic to instances which contain some strings multiple times. (Recall that the gadget from Lemma 1 in the construction contains $2^{a-b}$ copies of the all-zero string.)

**Proposition 1.** Theorem 1 and Corollary 1 hold even if the input contains only distinct strings.

**Proof.** Again, let $a$ and $b$ be two fixed integers such that $p = a/b$. To show the statement, we modify the instance that we constructed in the proof of Theorem 1 by appending to each string $2^{a-b} + 2^b$ columns. First, observe that it suffices to distinguish all $2^{a-b} \cdot 2^b$ strings in group 1 from one another: All other strings are distinct. We need to preserve, however, the property of the gadget in group 1. To do that, intuitively, we attach an identity matrix to the strings in group 1, and fill up the remaining strings (in group 2 and group 3) with zeros.

More formally, let $g_0, \ldots, g_{2^b-1}$ be the strings in group 1, where $g_0$ is the single string with $(2^b + 1)\hat{n}$ ones. Append to string $g_0$ the string $0_{2^{a-b}} \circ 1_{2^b}$. For each string $g_i, i \in [2^{a-b}]$, append to it the string $0_{i-1} \circ 1 \circ 0_{2^b-i+1}$. Append an all-zero string $0_{2^{a-b}+1}$ to each string from group 2 and group 3, i.e., to each string $s_i, r_i, i \in [n]$ and each string $t_i^{(0)}, t_i^{(1)}, t_i^{(2)}, w_i^{(0)}, w_i^{(1)}, w_i^{(2)}, s_i, r_i, n \cdot 1, i \in [m]$. Finally, we set $k$ to the positive real so that $k^p = (2^a + 2^{a-b}) \cdot (\hat{n} + 1)^p + (n + 3m) \cdot 2 \cdot (2n)^p$; recall that $\hat{n} = n + m$. Note that $k = O\left((n + m)^{\frac{p+1}{p}}\right)$ still holds as $a$ and $b$ are fixed integers.

For ease of notation we use the overloaded symbols $g_0, g_1, \ldots, g_{2^b-1}, s_1, \ldots, s_n, r_1, \ldots, r_n, t_1^{(z)}, t_m^{(z)}, w_1^{(z)}, \ldots, w_m^{(z)}, z \in \{0, 1, 2\}$, to refer to the modified strings.
To show that the construction remains correct, we first claim that an arbitrary solution has sum of \(p\)-distance at least \((2^a + 2^{a-b}) \cdot (\hat{n} + 1)^p\) to the strings of the first group.

**Claim 2.** Let \(s^*\) be an arbitrary solution string, then the sum of \(p\)-distances from \(s^*\) to all strings of group 1 is at least \((2^a + 2^{a-b}) \cdot (\hat{n} + 1)^p\).

**Proof.** Let \(x\) denote the number of ones of solution \(s^*\) in the columns of \(\{1, \ldots, (2^b + 1) \cdot \hat{n}, (2^b + 2) \cdot \hat{n} + 2^{a-b} + 1, \ldots, (2^b + 2) \cdot \hat{n} + 2^{a-b} + 2^b\}\) with \(0 \leq x \leq (2^b + 1) \cdot \hat{n} + 2^b\). To show the statement, we distinguish between two cases, depending on whether \(s^*\) contains a one in the column range \([(2^b + 2) \cdot \hat{n} + 1, (2^b + 2) \cdot \hat{n} + 2^{a-b}]\).

**Case 1:** Solution \(h \in s^* \cap \{2^b+2|\hat{n}+2^{a-b}\}, 0_{2^a-b}\} \neq \emptyset\), that is, \(s^*\) contains a one in the column range \([(2^b + 2) \cdot \hat{n} + 1, (2^b + 2) \cdot \hat{n} + 2^{a-b}]\). In this case, it holds that \(d(s^*, g_0) \geq (2^b + 1) \cdot \hat{n} + 2^b - x + 1\) and for each \(i \in [2^{a-b}]\) it holds that \(d(s^*, g_i) \geq x\). So, the sum of the \(p\)-distances between \(s^*\) and the strings of the first group is at least:

\[
\sum_{i=0}^{2^{a-b}} d^p(g_i, s^*) \geq 2^{a-b} \cdot x^p + ((2^b + 1) \cdot \hat{n} + 2^b - x + 1)^p =: f(x).
\]

To derive a lower bound on the above cost, we use a proof similar to one given for Lemma 1, but utilizing the first and the second derivatives of \(f(x)\):

\[
\frac{df}{dx} = p \cdot 2^{a-b} \cdot x^{p-1} - p \cdot ((2^b + 1) \cdot \hat{n} + 2^b - x + 1)^{p-1}
\]

\[
= p \left[ (2^{\sqrt{x}})^{a-b} - \left( \sqrt{(2^b + 1) \cdot \hat{n} + 2^b - x + 1} \right)^{a-b} \right]
\]

\[
= p \left[ 2^{\sqrt{x}} - b \left( \sqrt{(2^b + 1) \cdot \hat{n} + 2^b - x + 1} \right) \right] \cdot \sum_{j=0}^{a-b-1} (2^{\sqrt{x}})^j \cdot (\sqrt{(2^b + 1) \cdot \hat{n} + 2^b - x + 1})^{a-b-1-j}.
\]

The second equation holds through a simple equivalent reformulation of the exponents. The last equation holds since \(a - b \geq 1\). Moreover, the third component in the last equation is strictly positive for every \(x, 0 \leq x \leq (2^b + 1) \cdot \hat{n} + 2^b\) and hence \(\frac{df}{dx}\) is zero over \([0, (2^b + 1) \cdot \hat{n} + 2^b]\) only when the second component equals zero, that is, \(2^{\sqrt{x}} - b \cdot \sqrt{(2^b + 1) \cdot \hat{n} + 2^b - x + 1} = 0\). Solving the equation, the first derivative is zero when \(x = \hat{n} + 1\). Furthermore, the second derivative \(\frac{d^2f}{dx^2}\) is positive at \(x = \hat{n} + 1\), meaning that \(f(x)\) has a local minimum at this point. The minimum value is thus \(f(\hat{n} + 1) = 2^{a-b} \cdot (\hat{n} + 1)^p + (2^b + 1) \cdot \hat{n} + 2^b\).

**Case 2:** Analogously, we consider the case when \(h \in s^* \cap \{2^b+2|\hat{n}+2^{a-b}\}, 0_{2^a-b}\} = \emptyset\). In this case, it holds that \(d(s^*, g_0) \geq (2^b + 1) \cdot \hat{n} + 2^b - x\) and for each \(i \in [2^{a-b}]\) it holds that \(d(s^*, g_i) \geq x + 1\). So, the sum of the \(p\)-distances between \(s^*\) and the strings of the first group is at least:

\[
\sum_{i=0}^{2^{a-b}} d^p(g_i, s^*) \geq 2^{a-b} \cdot (x + 1)^p + ((2^b + 1) \cdot \hat{n} + 2^b - x)^p =: g(x).
\]

To derive a lower bound on the above cost, we use a proof similar to one given for Lemma 1, but
utilizing the first and the second derivatives of \( g(x) \):

\[
\frac{dg}{dx} = p \cdot 2^{a-b} \cdot (x+1)^{p-1} - p \cdot ((2^b + 1) \hat{n} + 2^b - x)^{p-1}
\]

\[
= p \left[ (2\sqrt{x+1})^{a-b} - (\sqrt{(2^b + 1) \hat{n} + 2^b - x})^{a-b} \right]
\]

\[
\overset{(2)}{=} p (2\sqrt{x+1} - \sqrt{(2^b + 1) \hat{n} + 2^b - x}) \cdot \sum_{j=0}^{a-b-1} (2\sqrt{x+1})^j \cdot (\sqrt{(2^b + 1) \hat{n} + 2^b - x})^{a-b-1-j}.
\]

The second equation holds through a simple equivalent reformulation of the exponents. The last equation holds since \( a - b \geq 1 \). Moreover, the third component in the last equation is strictly positive for every \( x \in [0, (2^b + 1) \hat{n} + 2^b] \) and hence \( \frac{dg}{dx} \) is zero over \( [0, (2^b + 1) \hat{n} + 2^b] \) only when \( 2\sqrt{x+1} = \sqrt{(2^b + 1) \hat{n} + 2^b - x} = 0 \). Solving the equation, the first derivative is zero when \( x = \hat{n} \).

Furthermore, the second derivative \( \frac{d^2g}{dx^2} \) is positive at \( x = \hat{n} \), meaning that \( g(x) \) has a local minimum at this point. The minimum value is thus \( g(\hat{n}) = 2^{a-b}(n+1)p + (2^b(\hat{n}+1))^p = 2^{a-b}(n+1) + 2^b(\hat{n}+1)^p \).

Summarizing, the sum the \( p \)-distances from \( s^* \) to all strings from group 1 is at least \( (2^a + 2^{a-b}) \cdot (\hat{n} + 1)^p \).

(of Claim 2) \(
\)

Now, we prove that any solution string where the last \( 2^{a-b} + 2^b \) columns have at least one one will exceed our cost \( k \).

**Claim 3.** Let \( s^* \) be a solution with \( s^*(2^b+2)\hat{n}+2^{a-b}+2^b \neq 0_{2^{a-b}+2^b} \), then the sum of \( p \)-distances from \( s^* \) to the modified strings is larger than \( k^p \).

**Proof.** Suppose, towards a contradiction, that there is a solution \( s^* \) with \( p \)-distances at most \( k^p = (2^a + 2^{a-b}) \cdot \hat{n}^p + 2(n+3m) \cdot (2\hat{n})^p \) to the modified instance. From Claim 2, stating that the sum of \( p \)-distances to the first group is at least \( (2^a + 2^{a-b}) \cdot (\hat{n} + 1)^p \), it follows that the sum of \( p \)-distances to the second and the third group is at most \( (n + 3m) \cdot 2 \cdot (2\hat{n})^p \). Now, consider an arbitrary pair \( s_i \) and \( r_i \) (resp. \( t_j^{(z)} \) and \( w_j^{(z)} \)) of strings, denote them by \( s \) and \( s' \). By construction, \( d(s, s') = 4\hat{n} \) and \( hs(s, s') \subseteq \{1, \ldots, 3\hat{n}, (2^b + 1)\hat{n} + 1, \ldots, (2^b + 2)\hat{n} \} \). Thus, by the first statement of Lemma 2, each of these pairs has sum of \( p \)-distances exactly \( 2(2\hat{n})^p \) to \( s^* \). However, by assumption that \( s^*(2^b+2)\hat{n}+2^{a-b}+2^b \neq 0_{2^{a-b}+2^b} \), we have that at least one of the string from \( \{s, s'\} \) has Hamming distance more than \( 2\hat{n} \) to \( s^* \), a contradiction to the second statement of Lemma 2. (of Claim 3) \(
\)

By Claim 3, we may assume that the last \( 2^{a-b} + 2^b \) columns in a solution (with cost \( k^p \)) contain only zeros, it now follows that the Hamming distance of a solution to each string in the constructed instance in the proof of Theorem 1 remains the same after our modifications—except for those distances that relate to the \( g_i \). It hence remains to show that an analog of Lemma 1 remains valid in which the gadget’s strings are appended with an identity matrix as above and \( s^* \) contains only zeros in the last \( 2^{a-b} + 2^b \) columns.

**Claim 4.** Let \( s^* \in \{0, 1\}^{(2^b+2)\hat{n}+2^{a-b}+2^b} \) be a solution with \( s^*(2^b+2)\hat{n}+2^{a-b}+2^b = 0_{2^{a-b}+2^b} \). Then the following holds.

1. If \( hs(s^*, 0_{(2^b+2)\hat{n}+2^{a-b}+2^b}) \subseteq \{3\hat{n}\} \) and \( d(s^*, 0_{(2^b+2)\hat{n}+2^{a-b}+2^b}) = \hat{n} \). then \( \sum_{i=0}^{2^{a-b}} d^p(s^*, g_i) = (2^a + 2^{a-b}) \cdot (\hat{n} + 1)^p \).

2. Otherwise, \( \sum_{i=0}^{2^{a-b}} d^p(s^*, g_i) > (2^a + 2^{a-b}) \cdot (\hat{n} + 1)^p \).
Proof. The first statement follows by a straight-forward calculation.

The proof for the second statement is analogous to the one given for Claim 2. Again, let $y \in \{0, \ldots, 3\hat{n}\}$ denote the number of ones in the first $3\hat{n}$ columns of $s^*$. Then, $\sum_{i=0}^{2^{a-b}} d^p(s^*, g_i) \geq (2(2^{b}+1)\hat{n} - y+1) p + 2^{a-b} \cdot (y+1) p$; note that by assumption, $s^*$ has zeros in the last $2^{a-b}+2^b$ columns. We define a function $f : [0,3\hat{n}] \rightarrow \mathbb{Z}$ with $f(y) := ((2^{b}+1)\hat{n} - y+1) p + 2^{a-b} \cdot (y+1) p$, and show that this function attains its sole integer minimum over $[0,3\hat{n}]$ at $y = \hat{n}$. Note that this function is a lower bound on the sum of $p$-distances of $s^*$ to the first group of strings. First of all, the first derivative of $f$ with respect to $y$ is

$$
\frac{df}{dy} = -p((2^{b}+1)\hat{n} - y + 2^{b})p^{-1} + p \cdot 2^{a-b} \cdot (y+1)p^{-1}
$$

$$
= \frac{p}{2\sqrt{y+1}} \cdot ((2^{b}+1)\hat{n} + 2^{b}-y)^{a-b} \cdot ((2^{b}+1)\hat{n} + 2^{b}-y)^{a-b-1}. \tag{2}
$$

Herein, the last equality can be seen by a simple refactorization (see (2)). Now, observe that the last factor is positive over $[0,3\hat{n}]$ and hence $\frac{df}{dy}$ is zero over $[0,3\hat{n}]$ only at $y = \hat{n}$. Moreover, the second derivative is

$$
\frac{d^2f}{dy^2} = p(p-1) \cdot [(2^{b}+1)\hat{n} + 2^{b} - y)^{p-2} + 2^{a-b}(y+1)^{p-2}],
$$

which is positive at $y = \hat{n}$ (recall that $p > 1$). Thus, $f(y)$ has a local minimum at $y = \hat{n}$. (of Claim 4) \(\square\)

By the above claim, the correctness of our modified construction follows immediately.

As for the lower bound, no additional string is added to the new construction, and the length of the modified strings is increase by $2^{a-b} + 2^b$, which is a constant. Moreover, as already observed, $k \in O((n + m)^{\frac{p+1}{p}})$. Together, we obtain the same ETH-based lower bounds, even if all input strings are distinct. \(\square\)

3 Algorithmic results

We now turn to our positive results. In Section 3.1 we provide an efficient algorithm when the objective value $k$ is small. Section 3.2 then explains how to derive an integer convex programming formulation to obtain an efficient algorithm for small numbers $m$ of input strings. Finally, we give a simple 2-approximation in Section 3.3.

3.1 A subexponential-time algorithm

In this section, we present an algorithm with running time $2^{k/p^{(p+1)/p}} \cdot |I|^{O(1)}$ for any input instance $I$ with distance bound $k$. By the lower bound result given in Corollary 1, we know that under ETH, the running time of the obtained algorithm is tight.

The algorithm is built on two subcases, distinguishing on a relation between the number $m$ of input strings and the distance bound $k$. In each subcase we use a distinct algorithm that runs in subexponential time when restricted to that subcase. To start with, a dynamic programming algorithm which keeps track of the achievable vector of Hamming distances to each input string after columns 1 to $j \leq n$ has running time $O(n \cdot k^m)$. 

\(13\)
Lemma 3. $p$-HDC can be solved in $O(n \cdot k^m)$ time and space, where $m$ and $n$ are the number and the length of the input strings, respectively, and $k$ is the $p$-norm distance bound.

Proof. Let $I = (S, k)$ be an instance of $p$-HDC with $S = (s_1, \ldots, s_m)$ being the input strings of length $n$ and $k$ being the $p$-norm distance bound. First of all, it is obvious that if $I$ is a yes-instance and $s^*$ is a solution for $I$, meaning that $\sum_{s \in S} d^p(s^*, s) \leq k^p$, then the Hamming distance between $s^*$ and each input string $s_i \in S$ must not exceed $k$. Based on this observation, we can design a dynamic program that keeps track, for each $m$-tuple of Hamming distances, whether there is a partial solution that “fulfills” these Hamming distances. More precisely, our dynamic-programming table $T$ stores for each $m$-tuple $(d_1, \ldots, d_m) \in [k]^m$ and each column index $j$, whether there is a partial solution of length $j$ that has Hamming distance $d_i$ to each input string $s_i$ when restricted to only the first $i$ columns.

For each tuple $D = (d_1, \ldots, d_m) \in [k]^m$, we set $T(D, 1) = true$ if $D = (s_i[1])_{1 \leq i \leq m}$ or $D = (1 - s_i[1])_{1 \leq i \leq m}$ and $T(D, 1) = false$ otherwise. Then, for each column index $j \geq 2$, we set $T(D, j) = T(D, j - 1) \lor T(D, j - 1)$ if there are $D_1, D_2 \in [k]^m$ such that $D_1 = (d_i - s_i[j])_{1 \leq i \leq m}$ and $D_2 = (d_i + 1 - s_i[j])_{1 \leq i \leq m}$ and $T(D, j) = false$ otherwise. Intuitively, $D_1$ (resp. $D_2$) corresponds to setting the $i$th column of a solution to zero (resp. one). Since setting the $i$th column of a solution to zero (resp. one) will increase the Hamming distance of an input string that has a one (resp. a zero) in this column, we should update the Hamming distances accordingly. Finally, our input instance is a yes-instance if and only if there is a tuple $(d_1, \ldots, d_m) \in [k]^m$ with $\sum_{1 \leq i \leq m} d_i^2 \leq k^p$ such that $T(d_1, \ldots, d_m, n) = true$. The running time and space are $O(k^m \cdot n)$ since the dynamic table has $k^m \cdot n$ entries and each entry can be computed in constant time.

The dynamic program given in Lemma 3 is efficient if there is a small number $m$ of input strings only. In particular, if $m$ satisfies $m \leq \frac{k^p(p+1)}{\log k}$, then we immediately obtain an $O(n \cdot 2^{p(p+1)})$-time algorithm. Otherwise, we can use the following result.

Lemma 4. $p$-HDC can be solved in $O(nm^2 \cdot k^{\frac{p}{m}})$ time, where $m$ and $n$ are the number and the length of the input strings, respectively, and $k$ is the $p$-norm distance bound.

Proof. Let $I = (S, k)$ be an instance of $p$-HDC with $S = (s_1, \ldots, s_m)$ being the input strings of length $n$ and $k$ being the $p$-norm distance bound. To show the statement, we first observe that if a column is an all-zero (resp. an all-one) column, then we can simply assume that an optimal solution will also have zero (resp. one) in this column as our objective function is convex. By preprocessing all columns that are either an all-zero or an all-one vector, we obtain an equivalent instance, where each column has at least a zero and at least a one. Thus, for each column, no matter which value a solution has at this column, it will always induce Hamming distance of at least one to some input string. Consequently, if there are more than $k^p$ columns remaining, then we can simply answer “no” as any string will have cost more than $k$ to the input. Otherwise, we have that there are at most $k^p$ columns left.

If $I$ is a yes-instance, meaning that there is a solution $s^*$ for $I$ with $\sum_{s \in S} d^p(s^*, s) \leq k^p$, then there must be an input string $s^{**} \in S$ whose Hamming distance satisfies $d(s^{**}, s^*) \leq \sqrt{\frac{k^p}{m}} = \frac{k}{\sqrt{m}}$. Thus, we iterate over all input strings in $S$, assuming for each string that it is the aforementioned $s^{**}$. For each string $s_i$ that we assume to be the aforementioned $s^{**}$, we go over all strings $\hat{s}$ that differ from $s_i$ by $k'$ columns with $k' \leq \frac{k}{\sqrt{m}}$. We check whether $\sum_{s \in S} d^p(\hat{s}, s) \leq k^p$.

It remains to how the running-time bound. Observe that the preprocessing for all-zero and all-one columns can be done in $O(nm)$ time. After that, for each of the $m$ input strings $s_i$, we search all strings of Hamming distance at most $k' \leq \frac{k}{\sqrt{m}}$ to $s_i$, and their number is $O(n \frac{k}{\sqrt{m}})$. For
Lemma 3, we obtain a subexponential algorithm for the parameter $k$. Combining Lemma 3 with Lemma 4, we obtain a subexponential algorithm for the parameter $k$.

**Theorem 2.** For each fixed positive value $\varepsilon > 0$, $p$-HDC can be solved in $O(nm^2 \cdot 2^{k/p(p+1)+\varepsilon})$ time, where $n$ and $m$ denote the length and the number of input strings, and $k$ is the $p$-norm distance bound with $p > 1$.

**Proof.** Let $I = (S,k)$ be an instance of $p$-HDC with $S = (s_1, \ldots, s_m)$ being the input strings of length $n$ and $k$ being the $p$-norm distance bound. As already discussed, to solve our problem we distinguish between two cases, depending on whether $m \leq \frac{k^{p/(p+1)}}{\log k}$ holds.

If $m \leq \frac{k^{p/(p+1)}}{\log k}$, then $k^m \leq k^{\frac{k^{p/(p+1)}}{\log k}} \leq 2^{k^{p/(p+1)}}$. In this case, we use the dynamic programming approach given in the proof of Lemma 3, which has the desired running time $O(n \cdot k^m) = O(n \cdot 2^{k^{p/(p+1)}})$.

Otherwise, $m > \frac{k^{p/(p+1)}}{\log k}$, meaning that $pk \log k/\sqrt{m} < pk \log k/\sqrt{k^{p/(p+1)}} = p \cdot k^{p/(p+1)} \cdot (\log k)^{(p+1)/p}$.

For each fixed positive $\varepsilon \in \mathbb{R}$ there exists $k_0 = k_0(p,\varepsilon) \in \mathbb{R}$ such that, for each $k \geq k_0$, we have $p \cdot (\log k)^{(p+1)/p} < k^\varepsilon$. If $k < k_0$, then the algorithm in the proof of Lemma 4 runs in $O(nm^2)$. Otherwise $k \geq k_0$, which implies $\frac{k \cdot \log k}{\sqrt{m}} < k^{p/(p+1)+\varepsilon}$. Thus, the algorithm given in the proof of Lemma 4 has a running time of $O(nm^2 \cdot k^{p/(p+1)}) = O(nm^2 \cdot 2^{\sqrt{p(\log k)^{(p+1)/p}}} = O(nm^2 \cdot 2^{k^{p/(p+1)+\varepsilon}}).

Altogether we presented an algorithm which has the desired running time bound. \qed

### 3.2 An integer convex polynomial formulation

In this section, we show that minimizing the sum of the $p$-distances is fixed-parameter tractable for the number $m$ of input strings. The idea is to formulate our problem as an Integer Convex Program (ICP) with $O(2^m)$ integer variables and maximum degree $p$ in each polynomial. We then make use of a Lenstra-type algorithm developed by Dadush [10] (see also [9]). Dadush’s result is stated as follows.

A strong separation oracle for a set $K \subseteq \mathbb{R}^n$ is an algorithm that on input $y \in \mathbb{Q}^n$ returns either yes if $y \in K$, or some vector $c \in \mathbb{Q}^n$ such that for each $x \in K$ we have $\langle c, x \rangle < \langle c, y \rangle$, where $\langle \cdot, \cdot \rangle$ denotes the inner product of two vectors.

**Proposition 2** ([10, Theorem 7.1.1]). Let $K$ be a convex set which is contained in a ball of radius $r \in \mathbb{R}$ and center $w \in \mathbb{R}^n$ and let $O_K$ be a strong separation oracle for $K$. There is an algorithm which on input of $K$ either decides that $K$ does not contain an integer point or returns an integer point in $K$. The algorithm makes $2^{O(n')} \cdot ((n')^{4/3} \text{polylog}(n'))^{(n')} \text{poly}(\ell)$ arithmetic operations and calls to $O_K$, where $\ell$ is the length of the binary encoding of the input, including $c$ and $r$.

To apply Proposition 2, we formulate the set of solutions to $p$-HDC using constraints over integer variables and show that their continuous relaxation defines a convex set. To obtain a bounded number of variables (as required to be able to apply Proposition 2), we group columns with the same “type” together and introduce an integer variable for each column type. To this end, given a set $S = \{s_1, \ldots, s_m\}$ of length-$n$ strings and an integer $j \in [n]$, two columns $j, j' \in [n]$ have the same type if for each $i \in [m]$ it holds that $s_i[j] = s_i[j']$. The type of column $j$ is its equivalence class in the same-type relation. Thus, each type is represented by a vector in $\{0,1\}^m$. Let $n'$ denote the number of different (column) types in $S$. Then, $n' \leq \min(2^m, n)$. Enumerate the $n'$ column
types as \(t_1, \ldots, t_{n'}\). Below we identify a column type with its index for easier notation. Using this, we can encode the set \(S\) succinctly by introducing a constant \(e(j)\) for each column type \(j \in [n']\) that denotes the number of columns with type \(j\). Analogously, given a solution string \(s^*\), we can also encode this string \(s^*\) via an integer vector \(x \in \{0,1,\ldots,n\}^{n'}\), where for each type \(j \in [n']\) we define \(x[j]\) as the number of ones in the solution \(s^*\) whose corresponding columns are of type \(j\).

Note that this encodes all essential information in a solution, since the actual order of the columns is not important.

**Example 1.** For an illustration, let \(S = \{0000, 0001, 1110\}\). The set \(S\) has two different column types, represented by \((0,0,1)^T\), call it type 1, and \((0,1,0)^T\), call it type 2. There are three columns of type 1 and one column of type 2. The solution 0110 for \(S\) can be encoded by two variables \(x[1] = 2\) and \(x[2] = 0\).

**Constraint Formulation.** Using the variables \(x\) that represent a solution \(s^*\), we can reformulate the Hamming distance between the two strings \(s_i\) and \(s^*\) as follows. For the sake of simplicity, we let \(s_i[j] = 1\) if the column type of column \(j\) has one in the \(i\)th row and \(s_i[j] = 0\) if it has zero in the \(i\)th row.

\[
d(s_i, s^*) = \sum_{j=1}^{n'} (s_i[j] \cdot (e(j) - x[j]) + (1 - s_i[j]) \cdot x[j]) = \sum_{j=1}^{n'} (e(j) \cdot s_i[j] + (1 - 2s_i[j]) \cdot x[j]).
\]

Then the \(p\)-distance between \(x\) and \(s_i\) can be formulated as follows, where \(w_i = \sum_{j=1}^{n'} e(j) \cdot s_i[j]\) denotes the number of ones in string \(s_i\), \(c_i[j] = 1 - 2s_i[j] \in \{1,-1\}\), i.e. \(c_i[j] = 1\) if \(s_i[j] = 0\) and \(c_i[j] = -1\) if \(s_i[j] = 1\).

\[
d^p(s_i, s^*) = (w_i + \sum_{j=1}^{n'} x[j] \cdot c_i[j])^p.
\]

For use below, define \(f_i(x) := (w_i + \sum_{j=1}^{n'} x[j] \cdot c_i[j])^p\).

Now, we can formalize the feasible set \(K \subseteq \mathbb{R}^{n'}\) that contains all integer vectors that represent solutions to \(p\)-HDC: A vector \(x \in \mathbb{R}^{n'}\) is in \(K\) if and only if it satisfies the following inequalities:

\[
\sum_{i=1}^{n'} f_i(x) \leq k^p \tag{5}
\]

\[
0 \leq x[j] \leq e[j] \text{ for all } j \in [n'] \tag{6}
\]

Next, we show that the set \(K\) is convex, that is, for each two feasible integer vectors \(x, y \in K\) and each \(\delta \in [0,1]\) we have \(z := \delta x + (1 - \delta) y \in K\). To see that \(z\) satisfies Eq. (6), observe that, because \(x, y \in K\), for each \(j \in [n']\),

\[
0 \leq \delta \cdot x[j] + (1 - \delta) y[j] \leq \delta \cdot e[j] + (1 - \delta) e[j] = e[j].
\]

By the convexity property of sum of convex functions over the same domains, to show that \(z\) satisfies Eq. (5), it suffices to show that for each fixed rational value \(p > 1\) and each \(i \in [m]\), the multivariate polynomial \(f_i(x) = (w_i + \sum_{j=1}^{n'} x[j] \cdot c_i[j])^p\) is convex over the set of variables \(x\) that satisfy Eq. (6).
In the remainder of the proof, we show the convexty of \( f_i(x) \), that is, \( f_i(\delta \cdot x + (1 - \delta)y) \leq \delta f_i(x) + (1 - \delta)f_i(y) \). Observe the following of the left-hand side of this inequality:

\[
f_i(\delta \cdot x + (1 - \delta)y) = (w_i + \sum_{j=1}^{n'} (\delta \cdot x[j] + (1 - \delta) \cdot y[j]) \cdot c_i[j])^p
\]

\[
= (\delta \cdot w_i + \sum_{j=1}^{n'} \delta \cdot x[j] \cdot c_i[j] + (1 - \delta)w_i + \sum_{j=1}^{n'} (1 - \delta) \cdot y[j] \cdot c_i[j])^p
\]

\[
= (\delta \cdot (w_i + \sum_{j=1}^{n'} x[j] \cdot c_i[j]) + (1 - \delta)(w_i + \sum_{j=1}^{n'} y[j] \cdot c_i[j]))^p \quad (7)
\]

Recall that for each \( \ell > 1 \) the function \( g(z) = z^\ell \) is convex in \([0, \infty]\). Observe that \( w_i + \sum_{j=1}^{n'} x[j] \cdot c_i[j] \geq 0 \) and \( w_i + \sum_{j=1}^{n'} y[j] \cdot c_i[j] \geq 0 \) since \( 0 \leq x[j], y[j] \leq e[j] \). Combined with the convexity of \( g(z) \) for nonnegative \( z \), we obtain the following from (7):

\[
f_i(\delta \cdot x + (1 - \delta)y) \leq \delta \cdot (w_i + \sum_{j=1}^{n'} x[j] \cdot c_i[j])^p + (1 - \delta) \cdot (w_i + \sum_{j=1}^{n'} y[j] \cdot c_i[j])^p
\]

\[
= \delta \cdot f_i(x) + (1 - \delta) \cdot f_i(y).
\]

Thus, indeed, \( f_i(x) \) is convex. We thus derive that \( K \) is convex.

**Separation Oracle.** To apply Proposition 2 we need to obtain an efficient strong separation oracle for \( K \). On an input \( y \in \mathbb{R}^{n'} \) the algorithm works as follows.

First, check whether \( y \) satisfies Eq. (6). If not, then let \( j \in [n'] \) such that Eq. (6) is violated and assume that \( y[j] > e[j] \) (the case \( y[j] < 0 \) is analogous). Return a vector \( c \in \{0, 1\}^{n'} \) with \( c[j] = 1 \) and \( c[j'] = 0 \) for all \( j' \neq j \). It is straight-forward to check that \( c \) satisfies the requirements on the output of the separation oracle. If \( y \) satisfies Eq. (6) and Eq. (5), then return yes. Otherwise, return the gradient \( c := \nabla f(y) \) of the objective function \( f(y) = \sum_{i=1}^n f_i(y) \) at point \( y \); note that \( f(y) \) is a polynomial function and its differentials exist. It remains to show that \( \langle c, y \rangle > \langle c, x \rangle \) for each \( x \in K \). Let \( x \) be an arbitrary integer vector solution with \( x \in K \). Since \( f(x) \) is convex for nonnegative vector \( x \), we have

\[
f(x) \geq f(y) + \langle \nabla f(y), x - y \rangle
\]

\[
f(x) - f(y) \geq \langle \nabla f(y), x - y \rangle
\]

\[
\langle c, y \rangle > \langle c, x \rangle,
\]

the last inequality holds since \( f(x) \leq k^p < f(y) \). Note that \( c := \nabla f(y) \) can be computed with \( O(p \cdot m \cdot n') = O(m \cdot n) \) arithmetic operations and this is the dominating part of the running time of the oracle.

**Conclusion.** From Proposition 2, the formulation as an integer convex program and its above separation oracle, we can now infer the following.

**Theorem 3.** \( p \)-norm Hamming Centroid can be solved in \( 2^{O(m2^n)} \cdot (n \cdot m)^{O(1)} \) time.
3.3 A factor-2 approximation for integer $p$

It is known that by taking an input string that minimizes the largest Hamming distance over all input strings, CLOSEST STRING can be approximated within factor-2. Indeed, using a similar idea, we show that the optimization version of our $p$-HDC problem can also be approximated within factor 2.

Let $S$ be a sequence of $m$ input strings and let $s_1 \in S$ be some input string that minimizes the $p$-distance to the input strings: $s_1 := \arg \min_{s \in S} \sum_{i \in S} d_p(s_i, s)$. We show that $s_1$ is a factor-2 approximate solution, i.e. $\sqrt{\sum_{s \in S} d_p^2(s_1, s)} \leq 2 \| \text{OPT} \|_p$, where $\text{OPT}$ is the $p$-norm of an optimal solution for $S$. To this end, let $s^*$ be an optimal solution for $S$ and let $\| \text{OPT} \|_p = \| s^* \|_p$. Since $S$ has $m$ input strings, it has at least one string, denoted as $\hat{s}$, whose $p$-distance to $s^*$ is at most the arithmetic mean of $\| \text{OPT} \|_p$: $d_p(\hat{s}, s^*) \leq \frac{1}{m} \sum_{s \in S} d_p(s, s^*) = \frac{\| \text{OPT} \|_p}{m}$. This will be important in calculating the relation between the $p$-distance of $s_1$ to $\text{OPT}$ below. Recall that we have selected string $s_1$ with minimum sum of $p$-distances. Thus, the following holds:

$$\sum_{s \in S} d_p(s_1, s) \leq \sum_{s \in S} d_p(\hat{s}, s) \leq \sum_{s \in S} (d_p(\hat{s}, s^*) + d_p(s, s^*))^p. \tag{8}$$

The last inequality holds because $p > 0$ and the Hamming distances fulfill the triangle inequality.

Before we continue with our reasoning, we observe the following:

**Lemma 5.** For each two non-negative integers $x$ and $y$, and for each rational value $p > 1$, it holds that $(x + y)^p \leq 2^{p-1}(x^p + y^p)$.

**Proof.** To show the inequality stated in the lemma, we define a bivariate function $f(x, y) : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}$ with $f(x, y) = 2^{p-1}(x^p + y^p) - (x + y)^p$ and show that $f(x, y) \geq 0$ whenever $x, y \geq 0$. This is equivalent to show that $f(x, y) = 0$ when $x = y \geq 0$ and that for each fixed value $\hat{y} > 0$,

1. the single-variate function $g(x) = f(x, \hat{y}) : \mathbb{R}_0^+ \to \mathbb{R}$ with $g(x) = 2^{p-1}(x^p + \hat{y}^p) - (x + \hat{y})^p$ has a single critical point at $x = \hat{y}$, and
2. the second derivative of $g(x)$ is positive at the point $x = \hat{y}$ with $\hat{y} > 0$.

Since it is straightforward to verify that $f(x, y) = 0$ when $x = y \geq 0$, we only need to show the other two properties. To derive the critical point(s), we have to compute the values $x$ for which the first derivative of $g(x)$ is zero:

$$g'(x) := 2^{p-1} \cdot p \cdot x^{p-1} - p \cdot (x + \hat{y})^{p-1} = p \cdot (2x)^{p-1} - (x + \hat{y})^{p-1}.$$  

$g'(x)$ is zero when $(2x)^{p-1} - (x + \hat{y})^{p-1} = 0$, that is, when $x = \hat{y}$. Thus, the unique critical point of $g(x)$ is at $x = \hat{y}$.

Now, we consider the second derivative of $g(x)$:

$$g''(x) := p \cdot (p - 1) \cdot (2 \cdot (2x)^{p-2} - (x + \hat{y})^{p-2}).$$

The function $g''(x)$ has $g''(\hat{y}) = p \cdot (p - 1) \cdot (2\hat{y})^{p-2} > 0$ when $\hat{y} > 0$; recall that $p > 1$. Summarizing, we have shown that the function $f(x, y)$ has minimum value zero when $x = y$. \qed

By (8) and Lemma 5, we derive that

$$\| s_1, S \|_p \leq (\sum_{s \in S} (d_p(\hat{s}, s^*) + d_p(s, s^*))^p)^{\frac{1}{p}} \leq \frac{\| \text{OPT} \|_p}{m} \sum_{s \in S} d_p(\hat{s}, s^*) + \sum_{s \in S} d_p(s, s^*) \right)^{\frac{1}{p}} \leq (2^{p-1}(m \cdot \frac{\| \text{OPT} \|_p}{m} + \| \text{OPT} \|_p))^{\frac{1}{p}} \leq 2 \cdot \| \text{OPT} \|_p.$$  

Note that the second but last inequality holds since $\hat{s}$ was the string that has $p$-distance at most $\frac{\| \text{OPT} \|_p}{m}$ to the solution $s^*$. 

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4 Conclusion and Outlook

In this paper we analyzed the complexity of \( p \)-NORM HAMMING CENTROID for all cases between \( p = 1 \) and \( p = \infty \). We believe that the running time bounds established in this paper, of essentially \( 2\Theta(k^{\frac{1}{p-1}})(nm)^{O(1)} \), connect the extreme points \( p = 1 \) and \( p = \infty \) in a very satisfying way. We did not consider the non-norm case of \( 0 < p < 1 \), as it does not fit our clustering motivation very well. But this non-convex case might be of independent interest, and may be the subject of future work.

An interesting generalization of CLOSEST STRING is CLOSEST SUBSTRING in which we seek a string \( s^* \) of a certain specified length such that each of the input strings has a substring which is close to \( s^* \) (see, e.g., Ma and Sun [24]). It would be interesting to see how our results carry over to this and other similar variants. Finally, the fact that the simple 2-factor approximation for CLOSEST STRING carries over to \( p \)-HDC may imply that there are similar connections for approximation algorithms. This warrants further investigation into whether \( p \)-HDC admits a PTAS.

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