Abstract

We propose a categorial grammar based on classical multiplicative linear logic.

This can be seen as an extension of abstract categorial grammars (ACG) and is at least as expressive. However, constituents of linear logic grammars (LLG) are not abstract λ-terms, but simply tuples of words with labeled endpoints, we call them multiwords. At least, this gives a concrete and intuitive representation of ACG.

A key observation is that the class of multiwords has a fundamental algebraic structure. Namely, multiwords can be organized in a category, very similar to the category of topological cobordisms. This category is symmetric monoidal closed and compact closed and thus is a model of linear λ-calculus and classical linear logic. We think that this category is interesting on its own right. In particular, it might provide categorical representation for other formalisms.

On the other hand, many models of language semantics are based on commutative logic or, more generally, on symmetric monoidal closed categories. But the category of word cobordisms is a category of language elements, which is itself symmetric monoidal closed and independent of any grammar. Thus, it might prove useful in understanding language semantics as well.

1 Introduction

A prototypical example of categorial grammar is Lambek grammars [17]. These are based on logical Lambek calculus, which is, speaking in modern terms, a noncommutative variant of (intuitionistic) linear logic [12]. It is well known that Lambek grammars generate exactly the same class of languages as context-free grammars [25].

However, it is agreed that context-free grammar are, in general, not sufficient for modeling natural language. Therefore linguists consider various more ex-
pressive formalisms. Lambek calculus is extended to different complicated multimodal, mixed commutative and mixed nonassociative systems, see [21]. Many grammars operate with more complex constituents than just words. For example displacement grammars [24], extending Lambek grammars, operate on discontinuous tuples of words.

Especially interesting (to the author) are abstract categorial grammars (ACG) [10]. Unlike Lambek grammars, these are based on a more intuitive and familiar commutative logic, namely, the implicational fragment of linear logic. Yet their expressive power is much stronger [31]. This, however, comes with a certain drawback. The constituents are, basically, just linear \( \lambda \)-terms. It is not so easy to identify them with any elements of language. We should add also that there exist Hybrid type logical grammars [16], which extend ACG, mixing them with Lambek grammars.

Finally, we note, that, although the list of existing grammars seems sufficiently long, there exists a very interesting unifying approach of [22]. It turns out that many grammatical formalisms can be faithfully represented as fragments of first order multiplicative intuitionistic linear logic \( \text{MILL}_1 \). This provides some common ground on which different systems can be compared. From the author’s point of view it is quite remarkable that a unifying logic is, again, commutative.

In this work we propose one more categorial grammar based on a commutative system, namely on classical linear logic. Linear logic grammars (LLG) of this paper can be seen as an extension of ACG to full multiplicative fragment. Although, as we just noted, the list of different formalisms is already sufficiently long, we think that our work deserves some interest at least for two reasons.

First, unlike the case of ACG, constituents of LLG are very simple. They are tuples of words with labeled endpoints, we call them multiwords. Multiwords are directly identified as basic elements of language, and apparently they are somewhat easier to deal with than abstract \( \lambda \)-terms. ACG embed into LLG, so at least we give a concrete and intuitive representation of ACG. (We don’t know if LLG have stronger expressive power as ACG, or just the same.)

Second, we identify on the class of multiwords a fundamental algebraic structure. This structure is a category (in the mathematical, rather than linguistic sense of the word), which is symmetric monoidal closed and compact closed. It is this categorical structure that allows us representing linear \( \lambda \)-calculus and ACG, as well as classical linear logic. And, apparently, at least some other formalisms can be represented in this setting as well. Possibly, this can give some common reference for different systems.

We now discuss it in a greater detail.

1.1 Algebraic considerations

The algebraic structure underlying linguistic interpretations of Lambek calculus is that of a monoid.

Indeed, the set of words over a given alphabet is a free monoid under concatenation, and Lambek calculus can be interpreted as a logic of the poset of
this monoid subsets (i.e. of formal languages). Typically, the sequent

\[ X_1, \ldots, X_n \vdash X \]

is interpreted as subset inclusion: the concatenation of languages \( X_1, \ldots, X_n \) is a sublanguage of \( X \).

When constituents of a grammar are more complicated, such as word tuples, there is no unique concatenation, since tuples can be glued together in many ways. Thus the algebra is more complex.

We consider tuples of words with labeled endpoints, we call them multiwords. Multiwords can be conveniently represented as very simple directed graphs with labeled edges and vertices. They are glued together along matching labels on vertices.

For example, we have a multiword with two components

\[
\begin{array}{c}
\alpha \\
\uparrow \\
\text{John} \\
\downarrow \\
\beta
\end{array} \quad \begin{array}{c}
\gamma \\
\text{Mary} \\
\delta
\end{array}
\]

and another multiword with one component.

\[
\begin{array}{c}
\beta \\
\uparrow \\
\text{likes} \\
\downarrow \\
\gamma
\end{array}
\]

These glue together and yield the following.

\[
\begin{array}{c}
\alpha \\
\uparrow \\
\text{John likes Mary} \\
\downarrow \\
\delta
\end{array}
\]

The same multiword can be obtained by gluing a three-component multiword

\[
\begin{array}{c}
\alpha \\
\uparrow \\
\text{John} \\
\downarrow \\
\beta
\end{array} \quad \begin{array}{c}
\gamma \\
\text{likes} \\
\mu \\
\nu \\
\text{Mary} \\
\delta
\end{array}
\]

with another multiword

\[
\begin{array}{c}
\beta \\
\uparrow \\
\gamma \\
\mu \\
\nu
\end{array}
\]

whose all components are empty.

Unfortunately, nothing precludes us from gluing words cyclically, and thus obtaining cyclic sequences of letters with no endpoints. Consider gluing a word

\[
\begin{array}{c}
\alpha \\
\uparrow \\
x \\
\downarrow \\
\beta
\end{array}
\]
with a “wrongly oriented” one.

\[ \begin{array}{c}
\beta \\
\alpha \\
y
\end{array} \]

For consistency we have to allow also such cyclic or singular multiwords, which can be represented as closed loops.

Multiwords can be organized in a monoidal category, very similar to the category of topological cobordisms (see [2]). Its objects, boundaries, are sets of vertex labels, and morphisms, word cobordisms, are (equivalence classes of) multiwords, composed by gluing.

Monoidal structure, “tensor product” is just disjoint union.

Thus, we shift from a non-commutative monoid of words to a symmetric monoidal category of word cobordisms. (We find it amusing to abbreviate the latter term as cowordism.)

1.2 Adding logic

The category of cowordisms (over a given alphabet) is not only symmetric monoidal, but also compact closed, just as the category of cobordisms. This makes it a model of classical multiplicative linear logic [27].

When interpreting logic in such a setting, logical consequence does no longer correspond to subset inclusion. A sequent

\[ X_1, \ldots, X_n \vdash X \]

given together with its derivation, is now a particular cowordism of type

\[ X_1 \otimes \ldots \otimes X_n \rightarrow X, \]

which can be explicitly computed from the derivation.

Adding a lexicon, which is a finite set of non-logical axioms, i.e. cowordisms together with their typing specifications, we obtain a linear logic grammar (LLG).

Syntactic derivations from the lexicon directly translate to cowordisms, (which are just tuples of words). This gives us a linear logic grammar; its language consists of all words that can be written as compositions of cowordisms in the lexicon and “natural” cowordisms coming from linear logic proofs.

Speaking more generally, with an LLG we get a subcategory of cowordism types generated by the grammar. This is, in general, no longer compact. It is, however, a categorical model of linear logic and linear \( \lambda \)-calculus.

Comparing with Lambek calculus, we shift from a poset of formal languages to a category of cowordism types.

1.3 Some wishful thinking on categorical semantics

LLG are at least as expressive as abstract categorial grammars (on the string signature). Indeed, ACG are based on a conservative fragment of classical linear
logic, so they have direct translation to our setting. Thus, cowordisms and LLG provide a concrete categorical model of abstract categorial grammar.

In fact, cowordisms are essentially proof-nets, and passage from ACG to LLG is basically, a passage, from λ-terms to proof-nets. Now, forgetting about LLG, it seems reasonable that any formalism admitting some version of proof-nets has a representation in the category of cowordisms. (It does not necessarily mean that such a representation is useful.) Possibly, this might provide some common, syntax-independent ground, i.e. a model, for different systems. This might be compared with representation of different systems in MILL1 in [22].

One of the main features making categorial grammars interesting is that they allow a bridge between language syntax and language semantics (see [23]). Semantics is often modeled by means of a commutative logic, most notably, linear logic as in [9]. But the category of cowordisms itself is a symmetric monoidal category of language elements, which independent of any grammar. It might prove helpful for understanding this bridge.

An interesting approach is that of categorical compositional distributional models of meaning (DisCoCat) [7], [8]. In DisCoCat it is proposed to model and analyze language semantics by a functorial mapping (“quantization”) of syntactic derivations in a categorial grammar to the (symmetric) compact closed category $\text{FDVec}$ of finite-dimensional vector spaces. The approach has been developed so far mainly on the base of Lambek grammars or pregroup grammars (see [18]), which are, from the category-theoretical point of view, non-symmetric monoidal closed. On the other hand, the cowordism category is symmetric and compact closed, and in this sense it is a better mirror of $\text{FDVec}$. Thus it seems a more natural candidate for quantization. Possibly, cowordism representation may help to apply ideas of DisCoCat to LLG or ACG, thus going beyond context-free languages.

1.4 Structure of the paper

The paper is reasonably self-contained. We assume, however, that the reader has some basic acquaintance with categories, in particular, with monoidal categories, see [19] for background.

In the first section we define the category of word cobordisms (cowordisms). In the second section we discuss monoidal closed categories in general, and monoidal closed structures of cowordism categories in particular. Section 3 introduces linear logic, its categorical semantics and, finally, linear logic grammars. In Section 4, as an example, we show that multiple context-free grammars encode in LLG, and that every LLG with a $\otimes$-free lexicon generates a multiple context-free language. This result is similar to (and stronger than) the known result that all second order ACG generate multiple context-free languages [24].

The fifth section is the encoding of ACG to LLG. Finally, in the last section we show how LLG generates an NP-complete language. The purpose of this last piece is mainly illustrative. We try to convince the reader that the geometric language of cowordisms is indeed intuitive and convenient for analysing language generation.
2 Word cobordisms

2.1 Multiwords

Let $T$ be a finite alphabet. We denote the set of all finite words in $T$ as $T^*$. For consistency of definitions we will also have to consider cyclic words.

We say that two words in $T^*$ are cyclically equivalent if they differ by a cyclic permutation of letters. A cyclic word over $T$ is an equivalence class of cyclically equivalent words in $T^*$.

For $w \in T^*$ we denote the corresponding cyclic word as $[w]$. Observe that there exists a perfectly well-defined empty cyclic word.

Definition 1 A regular multiword $M$ over an alphabet $T$ is a finite directed graph with edges labelled by words in $T^*$, such that each vertex is adjacent to exactly one edge (so that it is a perfect matching).

The left, respectively, right boundary of a multiword $M$ is the set of vertices of the underlying graph that are heads, respectively, tails of some edges.

We denote the left boundary of $M$ as $\partial_l M$ and the right boundary, as $\partial_r M$.

The boundary $\partial M$ of $M$ is the set $\partial M = \partial_l M \cup \partial_r M$.

Definition 2 A multiword $M$ over the alphabet $T$ is a pair $M = (M_0, M_c)$, where $M_0$, the regular part, is a regular multiword over $T$, and $M_c$, the singular or cyclic part, is a finite multiset of cyclic words over $T$.

The boundaries $\partial M, \partial_l M, \partial_r M$ of a multiword $M$ are defined as corresponding boundaries of its regular part $M_0$.

The multiword is acyclic or regular if its singular part is empty. Otherwise it is singular.

A multiword $M$ can be pictured geometrically as the edge-labelled graph $M_0$ and a bunch of isolated loops labelled by elements of $M_c$. The underlying geometric object is no longer a graph, but it is a topological space. It is even a manifold with boundary. In fact, we can equivalently define a multiword as a 1-dimensional compact oriented manifold with boundary (up to a boundary fixing homeomorphism), whose connected components are labelled by cyclic words, if they are closed, and by ordinary words otherwise.

2.1.1 Gluing

It should be clear from a geometric representation how to glue multiwords. We now give a boring accurate definition.

First, we define the disjoint union of multiwords in the most obvious way.

If $M = (M_0, M_c), M' = (M'_0, M'_c)$ are multiwords then we define the disjoint union $M \sqcup M'$ as the multiword

$$M \sqcup M' = (M_0 \sqcup M'_0, M_c \sqcup M'_c).$$

Next we define contraction, which corresponds to elementary gluing.
Let \( M \) be a multiword and \( x \in \partial_l M, y \in \partial_r M \).

The contraction \( M/\{x = y\} \) of \( x \) and \( y \) in \( M \) is obtained by identifying \( x \) with \( y \) in the underlying graph and gluing the corresponding edges into one. The words labeling the edges are also glued, i.e. concatenated.

This means the following.

If vertices \( x, y \) are not connected by an edge in \( M_0 \), then let \( t \) be the tail of the unique edge adjacent to \( x \) and \( z \) be the head of the unique edge adjacent to \( y \). Let \( u \) be the word labeling \((x, t)\) and \( v \) be the word labeling \((z, y)\). We construct a new edge-labelled graph \( M'_0 \) by removing \( x \) and \( y \) together with their adjacent edges from \( M_0 \) and drawing an edge from \((z, t)\). The new edge is labelled by the concatenation \( vu \).

We put \( M/\{x = y\} = (M'_0, M_c) \).

If \( x \) and \( y \) are connected by an edge, let \( w \) be its label. We remove \( x \), \( y \) and \((x, y)\) from \( M_0 \), which gives us the new edge-labelled graph \( M'_0 \), and we add to \( M_c \) the cyclic word \([w] \), which gives us the new multiset \( M'_c \). We put \( M/\{x = y\} = (M'_0, M'_c) \).

Note that iterated contractions commute.

\[ \text{Note 1} \] Let \( M \) be a multiword, and \( x_1, x_2 \in \partial_l M, y_1, y_2 \in \partial_r M \). Then
\[ ((M/\{x_1 = y_1\})/\{x_2 = y_2\}) = ((M/\{x_2 = y_2\})/\{x_1 = y_1\}). \]

In view of the above we can define multiple contractions.

\[ \text{Definition 3} \] Let \( M \) be a multiword. Let
\[ X \subseteq \partial_l M, \quad Y \subseteq \partial_r M, \]
and let \( \phi : X \to Y \) be a bijection.

The contraction \( M/\{X \cong Y\} \) of \( X \) and \( Y \) along \( \phi \) in \( M \) is defined by
\[ M/\{X \cong Y\} = \{\ldots(M/\{x_1 = \phi(x_1)\})\ldots)/\{x_n = \phi(x_n)\}, \]
where \( \{x_1, \ldots, x_n\} \) is any enumeration of elements of \( X \).

(We omit the bijection \( \phi \) from notation, because it will be clear from the context.)

Now let two multiwords \( M, M' \) be given.

Assume that we have subsets
\[ X_l \subseteq \partial_l M, \quad X_r \subseteq \partial_r M, \]
\[ X'_l \subseteq \partial_l M', \quad X'_r \subseteq \partial_r M', \]
and two bijections
\[ \phi : X_l \to X'_r, \quad \phi' : X'_l \to X_r. \]

Let \( X, X' \) be the disjoint unions \( X = X_l \sqcup X_r, X' = X'_l \sqcup X'_r \).

The gluing \( M \sqcup_{X \cong X'} M' \) of \( M \) and \( M' \) along \( X \) and \( X' \) is defined as the multiple contraction
\[ M \sqcup_{X, X'} M' = ((M \sqcup M')/\{X_l \cong X'_l\})/\{X'_r \cong X_r\}. \]
2.2 Category of word cobordisms

2.2.1 Cowordisms

We remarked above that multiwords can be represented geometrically as very simple manifolds with boundary. Manifolds with boundary give rise to the category of cobordisms, see [2]. We are now going to define a similar category of word cobordisms. We find it amusing to abbreviate the latter term as cowordism, and we will do so.

**Definition 4** A boundary is a finite set $X$ equipped with a partition $X = X_l \cup X_r$ into two disjoint subsets.

Now, we want to look at a multiword $M$ as a morphism between boundaries. For that, we need to understand which part of $\partial M$ is the input, and which is the output. This leads to the following definition.

**Definition 5** Let $X, Y$ be boundaries. A cowordism

$$\sigma : X \rightarrow Y$$

over an alphabet $T$ from $X$ to $Y$ is a triple

$$\sigma = (M, \phi_l, \phi_r),$$

where $M$ is a multiword over $T$ together with two bijective labeling functions

$$\phi_l : Y_l \sqcup X_r \rightarrow \partial_l M, \quad \phi_r : Y_r \sqcup X_l \rightarrow \partial_r M.$$

A cowordism is regular if its underlying multiword is regular. Otherwise the cowordism is singular.

For our purposes it is necessary to identify cowordisms that differ by inessential relabeling of boundaries. Therefor we supply our definition of cowordism with a definition of cowordism equality.

**Definition 6** Two cowordisms $\sigma = (M, \phi_l, \phi_r)$ and $\sigma' = (M', \phi'_l, \phi'_r)$ are equal, if their singular part coincide,

$$M_c = M'_c,$$

and there is a pair of bijections

$$\psi_l : \partial_l M \rightarrow \partial_l M', \quad \psi_r : \partial_r M \rightarrow \partial_r M'$$

inducing an edge-labeled graph isomorphism of the regular parts, such that

$$\phi'_l = \psi_l \circ \phi_l, \quad \phi'_r = \psi_r \circ \phi_r.$$
In the sequel we will systematically abuse notation and denote a cowordism and its underlying multiword with a same letter.

Note, however, that, generally speaking, a cowordism and a multiword are two different structures. In particular, we can have two different non-equal multiwords representing the same cowordism (see the definition of cowordism equality above).

We are going to organise cowordisms into a compact closed category (to be discussed below). Since cowordisms, by definition, have geometric representation, it is natural to adapt the pictorial language (see [29]) used for such categories.

We can depict an abstract cowordism $\sigma : X \to Y$ schematically as a box with incoming and outgoing wires, like the following.

$$\sigma \quad \begin{array}{c} \downarrow \quad \uparrow \end{array} \quad \begin{array}{c} X_l \quad X_r \quad \quad \quad \quad Y_r \quad Y_l \end{array}$$

Or, using fewer labels on the wires, like the following.

$$\sigma \quad \begin{array}{c} \downarrow \quad \uparrow \end{array} \quad \begin{array}{c} X_l \quad X_r \quad \quad \quad \quad Y \quad Y_l \end{array}$$

(Of course for a concrete $\sigma$ there are as many wires as there are points in the boundaries $X, Y$.)

### 2.2.2 Composition

Cowordisms are composed simply by gluing multiwords along matching boundary parts.

In the pictorial language of boxes and wires, given two cowordisms

$$\sigma : X \to Y, \quad \tau : Y \to Z,$$

the composition $\tau \circ \sigma$ is represented in a most natural way.

$$\begin{array}{c} \downarrow \quad \uparrow \end{array} \quad \begin{array}{c} X_l \quad X_r \quad \quad \quad \quad \quad \quad \quad \quad \quad Y \quad \quad \quad \quad \quad \quad \quad \quad Z \quad Y_l \quad Y_r \end{array}$$

An accurate definition is as follows.

Let $X, Y, Z$ be boundaries, and

$$\sigma = (\sigma, \phi_l, \phi_r), \quad \tau = (\tau, \psi_l, \psi_r)$$

be cowordisms from $X$ to $Y$ and from $Y$ to $Z$ respectively.

Let $\rho = \sigma \sqcup \tau$. 

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We have the injective maps
\[ \xi_l : Y_l \sqcup Y_r \to \partial_l \rho, \quad \xi_r : Y_r \sqcup Y_l \to \partial_r \rho \]
obtained from restrictions of \( \phi_l \sqcup \psi_l, \psi_r \sqcup \phi_r \) respectively.

Denote the image of \( \xi_l \) as \( I_l \) and the image of \( \xi_r \) as \( I_r \).

The composition \( \tau \circ \sigma \) is defined as the gluing of \( \tau \) and \( \sigma \) along \( I_l \) identified with \( I_r \) by means of bijection \( \xi_l^{-1} \circ \xi_r : I_l \cong I_r \), i.e.
\[ \tau \circ \sigma = (\sigma \sqcup \tau) / \{ I_l \cong I_r \}. \]

Restrictions of \( \psi_l \sqcup \phi_l \) to \( Z_l \sqcup X_r \) and of \( \psi_r \sqcup \phi_r \) to \( Z_r \sqcup X_l \) provide necessary bijections
\[ Z_l \sqcup X_r \cong \partial_l (\tau \circ \sigma), \quad Z_r \sqcup X_l \cong \partial_r (\tau \circ \sigma), \]
which makes the constructed multiword a cowordism from \( X \) to \( Z \).

It follows from Note 1 and definition of cowordism equality that composition is associative.

### 2.2.3 Identities

In order to construct a category we only need to find identities.

Let \( X \) be a boundary.

The identity cowordism \( \text{id}_X \) is constructed as follows.

Take two copies of \( X \) and then draw a directed edge from each point of \( X_r \) in the first copy to its image in the second copy and from each point of \( X_l \) in the second copy to its image in the first copy. Label every constructed edge with the empty word. This gives us an acyclic multiword with the left and right boundaries isomorphic to \( X_r \sqcup X_l \).

In the pictorial language, \( \text{id}_X \) looks as follows.

\[
\begin{array}{c}
\text{X} \\
\text{X}
\end{array}
\]

It is immediate now that the following is well defined.

**Definition 7** The category \( \text{Cow}_T \) of cowordisms over the alphabet \( T \) has boundaries as objects and cowordisms over \( T \) as morphisms.

### 2.3 Over the empty alphabet

Note that even when the alphabet is empty, the category of cowordisms is non-trivial. In fact, it becomes literally the category of oriented 1-dimensional cobordisms.

In the sequel we will use the term *cobordism* for a cowordism over the empty alphabet, and denote
\[ \text{Cow}_\emptyset = \text{Cob}. \]

Given two boundaries \( X, Y \) and a cowordism \( \sigma : X \to Y \) over some alphabet \( T \), we define the *pattern* of \( \sigma \) as the cobordism from \( X \) to \( Y \) obtained by erasing from \( \sigma \) all letters.
3 Cowordisms and monoidal closed categories

3.1 Structure of cowordisms category

The category of cowordisms has a rich structure (which it inherits, in fact, from the underlying category of cobordisms).

It is a symmetric monoidal closed, ∗-autonomous, and compact closed category, which makes it a model of linear λ-calculus and of classical multiplicative linear logic.

3.1.1 Monoidal structure

First, the operation of disjoint union makes this category monoidal.

The tensor product ⊗ on \textbf{Cow}_T is defined both on objects and morphisms as the disjoint union.

The monoidal unit 1 is the empty boundary,

\[ 1 = 1_r = 1_l = \emptyset. \]

Obviously, tensor product of cowordisms is associative up to a natural transformation.

In order to avoid very cumbersome notations we will, as is quite customary in literature, treat the category of cowordisms as strict monoidal. That is we will write \( X \otimes Y \otimes Z \) without brackets, as if the associativity isomorphisms were strict equalities. Similarly, we will usually identify \( 1 \otimes X \) and \( X \otimes 1 \) with \( X \). This is legitimate, because any monoidal category is equivalent to a strict monoidal category, see [19], Chapter VII for details.

In the pictorial language, given two cowordisms \( \sigma : X \rightarrow Y, \tau : Z \rightarrow T, \) we depict the tensor product \( \sigma \otimes \tau \) as two disjoint boxes.

\[ \begin{array}{c}
\sigma \\
\otimes \\
\tau \\
\end{array} \]

\[ \begin{array}{c}
X \leftrightarrow Y \\
Z \leftrightarrow T \\
\end{array} \]

For an abstract cowordism \( \sigma \) of the form

\[ \sigma : X_1 \otimes \ldots \otimes X_n \rightarrow Y_1 \otimes \ldots \otimes Y_m, \]

it is convenient to depict \( \sigma \) as a box with different slots for different tensor...
factors, as follows.

\[ \sigma_{Y_1 \otimes \cdots \otimes Y_m} \otimes \cdots \otimes X_n. \]

When the cowordism \( \sigma \) is of the form

\[ \sigma : 1 \rightarrow X_1 \otimes \cdots \otimes X_n \]

It is natural to represent it without wires on the left as follows.

\[ \sigma : X_1 \otimes \cdots \otimes X_m. \]

### 3.1.2 Symmetry

The above monoidal structure is also symmetric.

The symmetry transformation

\[ s_{X,Y} : X \otimes Y \rightarrow Y \otimes X \]

is given for any boundaries \( X, Y \) by the following cowordism.

Take a copy of \( X \sqcup Y \) and a copy of \( Y \sqcup X \). For each \( x \in X_r \) draw a directed edge from the image of \( x \) in \( X \sqcup Y \) to the image of \( x \) in \( Y \sqcup X \), similarly for each \( y \in Y_r \). Then for each \( x \in X_l \) draw a directed edge from the image of \( x \) in \( Y \sqcup X \) to the image of \( x \) in \( X \sqcup Y \), similarly for each \( y \in Y_l \). Label each constructed edge with the empty word. This gives an acyclic multiword, which is a cowordism from \( X \otimes Y \) to \( Y \otimes X \) in the obvious way.

In the pictorial language symmetry is the following.

\[ \begin{array}{c}
\begin{array}{c}
X \\
\otimes \\
Y \\
\otimes \\
Y \\
\otimes \\
X
\end{array}
\end{array} \]

**Note 2** The above defined tensor product, monoidal unit and symmetry make \( \text{Cow}_T \) a symmetric monoidal category. \( \square \)
3.1.3 Duality and internal homs

The category of cowordisms also has a well-behaved contravariant duality \( (\cdot)^\perp \), defined by switching left and right.

Let \( X = X_r \cup X_l \) be a boundary.

The dual \( X^\perp \) of \( X \) is defined by

\[
X^\perp = X, \quad (X^\perp)_r = X_l, \quad (X^\perp)_l = X_r.
\]

On morphisms, duality amounts to relabeling boundary points.

Let \( \sigma : X \to Y \) be a cowordism.

By definition \( \sigma \) is a multiword \( \sigma \) together with two labeling functions

\[
\phi_l : Y_l \sqcup X_r \to \partial_l \sigma, \quad \phi_r : Y_r \sqcup X_l \to \partial_r \sigma.
\]

Let

\[
s_{r,l} : X_r \sqcup Y_l \to Y_l \sqcup X_r, \quad s_{l,r} : X_l \sqcup Y_r \to Y_r \sqcup X_l
\]

be the natural bijections.

Then the triple

\[
\sigma^\perp = (\sigma, \phi_r \circ s_{l,r}, \phi_l \circ s_{r,l})
\]

is a cowordism from \( Y^\perp \) to \( X^\perp \).

In the pictorial language, given a cowordism \( \sigma : X \to Y \), the dual cowordism \( \sigma^\perp \) looks as follows.

\[\text{Note 3} \text{ The above defined duality is a contravariant functor commuting with the tensor product: } (X \otimes Y)^\perp \cong X^\perp \otimes Y^\perp. \square \]

Tensor and duality equip \( \text{Cow}_T \) with a very rich categorical structure that we discuss in the next section.
3.2 Zoo of monoidal closed categories

**Definition 8** Monoidal closed category $C$ is a symmetric monoidal category $C$ equipped with a bifunctor $\rightarrow$, contravariant in the first entry and covariant in the second entry, such that there exists a natural bijection

\[
\text{Hom}(X \otimes Y, Z) \cong \text{Hom}(X, Y \rightarrow Z).
\]  

(1)

The functor $\rightarrow$ in the above definition is called internal homs functor.

**Definition 9** $\star$-Autonomous category $C$ is a symmetric monoidal category $C$ equipped with a contravariant functor $(\cdot)_\star$, such that there is a natural isomorphism

\[
A_{\star\star} \cong A
\]

and a natural bijection

\[
\text{Hom}(X \otimes Y, Z) \cong \text{Hom}(X, (Y \otimes Z)_{\star}).
\]

Duality $(\cdot)_{\star}$ equips a $\star$-autonomous category with a second monoidal structure. The cotensor product $\wp$ is defined by

\[
X \wp Y = (X_{\star} \otimes Y)_{\star}.
\]

The neutral object for the cotensor product is

\[
\perp = 1_{\star}.
\]

Any $\star$-autonomous category is monoidal closed. The internal homs functor is defined by

\[
X \rightarrow Y = X_{\star} \wp Y.
\]

Note that we have a natural isomorphism

\[
X_{\star} \cong X \rightarrow \perp.
\]  

(2)

**Definition 10** $\mathcal{C}$ A compact closed or, simply, compact category is a $\star$-autonomous category for which duality commutes with tensor, i.e. such that

\[
X_{\wp} Y \cong X \otimes Y, \quad 1 \cong \perp.
\]

For compact categories it is convenient to define internal homs by

\[
X \rightarrow Y = X_{\star} \otimes Y,
\]

(3)

A prototypical example of a compact category is the category of finite-dimensional vector spaces with the usual tensor product and algebraic duality. Note, however, that in this case, and, in general, in the algebraic setting, duality is denoted as a star $(\cdot)^*$. Another example of a compact category widely used in mathematics and important for our discussion is the category of cobordisms.
Note 4  The category of cowordisms is compact closed (hence monoidal closed and *-autonomous).

Proof exercise. □

Compact structure provides a lot of important maps and constructions. A short and readable introduction into the subject can be found, for example, in [1].

We pick some necessary bits in the next section.

3.2.1 Names

Let $C$ be a monoidal closed category.

For any morphism $\sigma : A \to B$
correspondence (1) together with the isomorphism

$$A \cong 1 \otimes A$$

yields the morphism

$$\gamma \sigma^{-1} : 1 \to A \otimes B,$$

sometimes called the name of $\sigma$.

In the case of cowordisms, the name $\gamma \sigma^{-1} : 1 \to A \otimes B \cong A^\perp \otimes B$ of a cowordism $\sigma : A \rightarrow B$ can be depicted as follows.

![Diagram](image)

3.2.2 Applications

As before, let $C$ be a monoidal closed category.

For any two objects $A, B$, correspondence (1) composed with symmetry applied to $\text{id}_{A \rightarrow B}$ yields the evaluation morphism

$$\text{ev}_{A,B} : A \otimes (A \rightarrow B) \to B.$$  

In a compact closed case, where we have identifications (3), evaluation is especially simple.

We have the natural pairing map

$$\epsilon_A : A \otimes A^\perp \to 1,$$
usually called *counit*, and evaluation can be computed as
\[
ev_{A,B} = \epsilon_A \otimes \text{id}_B.
\]

In the case of cowordisms the pairing \(\epsilon_A\) has the following shape (remember that \(A^\perp_l = A_l\) and \(A^\perp_r = A_r\)).

The evaluation \(\ev_{A,B}\), accordingly, is pictured as follows.

Now given two morphisms
\[
\tau : 1 \to A, \quad \sigma : A \to B,
\]
we can define the application
\[
(\sigma \cdot \tau)_A : 1 \to B
\]
of \(\sigma\) to \(\tau\) as
\[
(\sigma \cdot \tau)_A = \ev_{A,B} \circ (\tau \otimes \sigma).
\]

The following property holds for any monoidal closed category.

**Note 5** *For any two morphisms*
\[
\tau : 1 \to A, \quad \sigma : A \to B,
\]

*it holds that*
\[
\tau \sigma \cdot \tau = \sigma \circ \tau. \quad \square
\]

In the case of cowordisms, the property is evident from geometric representation.
3.2.3 Partial pairing

Now let $\mathbf{C}$ be a $\ast$-autonomous category.

For any objects $A, B, C, D$ there is a natural linear distributivity morphism

$$
\delta_{A,B,C,D} : (A \otimes B) \otimes (C \otimes D) \to A \otimes (B \otimes C) \otimes D.
$$

(4)

In a compact closed case, where cotensor and tensor can be identified, linear distributivity is just associativity of tensor product.

Using linear distributivity, for any two morphisms

$$\tau : 1 \to A \otimes U, \quad \sigma : 1 \to U \otimes B,$$

we can define the partial pairing

$$\langle \tau, \sigma \rangle_U : 1 \to A \otimes B$$

of $\tau$ and $\sigma$ over $U$ by

$$\langle \tau, \sigma \rangle_U = (\text{id}_A \otimes \epsilon_U \otimes \text{id}_B) \circ \delta_{A,U,U,B} \circ (\tau \otimes \sigma).$$

In the case of cowordisms, given two cowordisms

$$\sigma : 1 \to A \otimes U, \quad \tau : 1 \to U \otimes B,$$

the partial pairing $\langle \tau, \sigma \rangle_U$ has the following shape.

Partial pairing can be understood as a symmetrized composition, as the following observation shows.

**Note 6** For all morphisms

$$\tau : A \to B, \quad \sigma : B \to C$$

it holds that

$$\overline{\gamma \sigma \tau} = \langle \tau, \sigma \rangle_B. \quad \square$$

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3.3 Categories of cowordism types

We now discuss subcategories of \( \text{Cow}_T \), which are no longer compact, but are monoidal closed. They will be helpful for understanding categorial grammars considered in this paper.

**Definition 11** Given a boundary \( X \), a cowordism type over an alphabet \( T \) or, simply, a type on the boundary \( X \) is a set of cowordisms over \( T \) from \( 1 \) to \( X \).

A set of cowordisms over the alphabet \( T \) is a cowordism type or, simply, a type, if it is a type on some boundary.

Given a type \( A \), we denote the corresponding boundary as \( \partial A \).

**Definition 12** Given two cowordism types \( A, B \) over the same alphabet, a cowordism \( \sigma : \partial A \to \partial B \) is a morphism of types \( \sigma : A \to B \) if for any \( \tau \in A \) it holds that \( \sigma \circ \tau \in B \).

Obviously, morphisms of types compose, and identity cowordisms are morphisms of types. So, types over an alphabet \( T \) form a category. We denote it as \( \text{Types}_T \).

Categories of types inherit symmetrical monoidal, and even monoidal closed structure of \( \text{Cow}_T \).

For two types \( A, B \) we define the tensor product type \( A \otimes B \) as the type on the tensor product of boundaries,

\[
\partial (A \otimes B) = \partial A \otimes \partial B,
\]

given by

\[
A \otimes B = \{ \sigma \otimes \tau \mid \sigma \in A, \tau \in B \}.
\]

We define the internal homs type \( A \to B \) as the type on the boundary

\[
\partial (A \to B) = \partial A \to \partial B = (\partial A)^\perp \otimes \partial B
\]

given by

\[
A \to B = \{ \sigma \mid \forall \tau \in A \ \sigma \cdot \tau \in B \}.
\]

Elements of \( A \to B \) are precisely all names of cowordisms which are morphisms of types \( A \) and \( B \).

The unit type \( 1 \) is the type on the empty boundary that contains only the empty cowordism \( \emptyset \).

**Note 7** The category \( \text{Types}_T \) of cowordism types is symmetric monoidal closed.

The forgetful functor

\[
\text{Types}_T \to \text{Cow}_T
\]

which sends each type \( A \) to the boundary \( \partial A \) and is identity on morphisms preserves monoidal closed structure. □
3.3.1 Cowordisms of a formal language

Let $L$ be a formal language in the alphabet $T$. Without loss of generality we assume that the symbol $\star$ is not in $T$. Let

$$T' = T \cup \{\star\}.$$  

We define on the empty boundary the type $\bot$ over $T'$ as the set of cyclic words

$$\bot = \{[w\star] \mid w \in L\},$$

where each cyclic word is seen as a singular cowordism.

Now for any type $A$ over $T'$ we define the dual $A^\perp$ of $A$ (with respect to $L$) as the type

$$A^\perp = A \rightarrow \bot.$$  

We say that the type $A$ is a closed type (of the language $L$) if $A = A^\perp\perp$ (using the identification $(\partial A)^\perp\perp \cong \partial A$ on the level of boundaries).

Closed types of $L$ form a (full) subcategory of $\text{Types}_{T'}$, which we denote as $C\text{Types}_L$.

The category $C\text{Types}_L$ is, in fact, $\ast$-autonomous.

It is easy to see that for all closed types $A, B$, the type $A \rightarrow B$ is closed. Also the types $\bot, 1$ are closed with

$$1 = \bot^\perp.$$  

In general, we have the following.

**Note 8** A type $A$ is closed iff $A \cong B^\perp$ for some type $B$ on the boundary $(\partial A)^\perp$.

There is a contravariant functor

$$(\cdot)^\perp : \text{Types}_T \rightarrow C\text{Types}_L$$

sending a type $A$ to the type $A^\perp$ and a cowordism $\sigma$, to the cowordism $\sigma^\perp$. □

In particular, if $A$ is a type, then we can complete it to the type $Cl(A)$ on the same boundary $\partial A$, defined as

$$Cl(A) = A^\perp\perp$$

with the usual identification

$$\partial A \cong (\partial A)^\perp\perp.$$  

We say that $Cl(A)$ is the closure of $A$ (with respect to $L$).

Then the preceding Note implies the following.

**Corollary 1** Let $A, B \in \text{Types}_T$. Any cowordism $\sigma$ which is a morphism of types

$$\sigma : A \rightarrow B$$

is also a morphism of closed types

$$\sigma : Cl(A) \rightarrow Cl(B).$$
Proof. By the preceding Note, we have a covariant functor
\[(\cdot)^{\perp\perp} : \text{Types}_T \to \text{CTypes}_L.\]
But, under identification (5), it sends any type to its closure and is identity on morphisms. □

For closed types \(A, B\) we define the closed tensor product type \(A \otimes B\) as the closure of the tensor product type,
\[A \otimes B = \text{Cl}(\{\sigma \otimes \tau | \sigma \in A, \tau \in B\}).\]

Note 9. With the above defined tensor product and duality \((\cdot)^{\perp}\), the category \(\text{CTypes}_L\) is ∗-autonomous.
The forgetful functor \(\text{CTypes}_L \to \text{Cow}_T\),
which sends type \(A\) to the boundary \(\partial A\) and is identity on morphisms, preserves ∗-autonomous structure. □

It is useful to observe that the original language \(L\) can be represented as a closed type of \(L\).
Indeed, let \(X\) be some boundary with \(|X_l| = |X_r| = 1\).
Any regular cowordism from \(1\) to \(X\), seen as a graph consists of a single edge. Define \(\text{star}\) as the type on \(X\) consisting of the single regular cowordism whose only edge is labeled with \(\ast\).
Then the closed type \(S = \text{star}^{\perp}\) consists of all regular cowordisms whose only edge is labeled with an element of \(L\). It seems natural to identify \(S\) with the language \(L\).

4 Linear logic grammars

4.1 Linear logic

Strictly speaking, the system discussed below is multiplicative linear logic, a fragment of full linear logic. However, since we do not consider other fragments, the prefix “multiplicative” will be omitted. A more detailed introduction to linear logic can be found in [12], [13].

Given a set \(N\) of positive literals, we define the set \(N^\perp\) of negative literals as
\[N^\perp = \{X^\perp | X \in N\}.\]
Elements of \(N \cup N^\perp\) will be called literals.
The set \(\text{Fm}(N)\) of LL formulas (over the alphabet \(N\)) is defined by the following induction.
• Any \(X \in N \cup N^\perp\) is a formula;
• if \(X, Y\) are formulas, then \(X _0 Y\) and \(X \otimes Y\) are formulas;
Connectives $\otimes$ and $\wp$ are called respectively *times* (also tensor) and *par* (also cotensor).

**Linear negation** $A^\perp$ of a formula $A$ is defined inductively as

$$ (P^\perp)^\perp = P, \text{ for } P \in N, $$

$$ (A \otimes B)^\perp = A^\perp \wp B^\perp, \quad (A \wp B)^\perp = A^\perp \otimes B^\perp. $$

**Linear implication** is defined as

$$ A \to B = A^\perp \wp B. \quad (6) $$

An LL sequent is an expression of the form $\vdash \Gamma$, where $\Gamma$ is a finite sequence of LL formulas.

The **sequent calculus** for LL is given by the following rules:

$$ \vdash X^\perp, X \ (\text{Identity}), \quad \vdash \Gamma, X \vdash X^\perp, \Delta \ (\text{Cut}), $$

$$ \vdash X_1, \ldots, X_n \vdash X_{\pi(1)}, \ldots, X_{\pi(n)} \ (\text{Exchange}), $$

$$ \vdash \Gamma, X, Y \vdash \Gamma, X, Y, \Delta \ (\wp), \quad \vdash \Gamma, X \vdash Y, \Delta \ (\otimes). $$

Linear logic enjoys the fundamental property of **cut-elimination**. Any sequent derivable in LL is derivable also in the cut-free system, i.e., without use of the Cut rule. Moreover, any proof has an essentially unique, up to some permutation of rules, cut-free form, which can be found algorithmically.

This allows computational and categorical interpretations in the proofs-as-programs or proofs-as-functions paradigm.

### 4.2 Semantics

Categorical interpretation of proof theory is based on the idea that formulas should be understood as objects and proofs, as morphisms in a category, while composition of morphisms corresponds to cut-elimination.

In a two-sided sequent calculus, formulas are interpreted as objects in a monoidal category, and a proof of the sequent

$$ X_1, \ldots, X_n \vdash X $$

is interpreted as a morphism of type

$$ X_1 \otimes \ldots \otimes X_n \to X. $$

This includes the case $n = 0$, with the usual convention that the tensor of the empty collection of objects is the monoidal unit 1.
Then the Cut rule corresponds to composition. A crucial requirement is that the interpretation should be invariant with respect to cut-elimination; a proof and its cut-free form are interpreted the same.

In the case of linear logic, whose sequents are one-sided, the appropriate setting for categorical interpretation is *-autonomous categories [27, 20].

In this setting, a proof of the sequent
\[ \vdash X_1, \ldots, X_n \]
is interpreted as a morphism of type
\[ 1 \to X_1 \varphi \ldots \varphi X_n. \]
The Cut rule corresponds to partial pairing, which can be understood as a symmetrized composition.

A special case of *-autonomous categories are compact categories, and, in particular, categories of cowordisms.

Given a *-autonomous category \( C \) and an alphabet \( N \) of positive literals, an interpretation of LL in \( C \) consists in assigning to any positive literal \( A \) an object \( \lbrack A \rbrack \) of \( C \). The assignment of objects extends to all formulas in \( Fm(N) \) by the obvious induction
\[
\lbrack A \otimes B \rbrack = \lbrack A \rbrack \otimes \lbrack B \rbrack, \quad \lbrack A^\perp \rbrack = \lbrack A \rbrack^\perp.
\]

It is quite customary in literature to omit square brackets and denote a formula and its interpretation by the same expression, and we will follow this practice when convenient.

Given interpretation of formulas, proofs are interpreted by induction on the rules.

The axiom \( \vdash A^\perp, A \) is interpreted as the name
\[
\tau \lbrack \text{id}[A] \rbrack^{-1} : 1 \to \lbrack A \rbrack^\perp \varphi[\lbrack A \rbrack]
\]
of the identity.

The Cut rule corresponds to partial pairing, as stated above.

The Exchange rule corresponds to a symmetry transformation.

The (\( \otimes \)) rule does nothing.

The (\( \varphi \)) rule is linear distributivity \([\text{4}]\). In the case of a compact category, in particular the category of cowordisms, the (\( \otimes \)) rule just tensors two morphisms together (up to associativity of tensor product).

Two proofs are equivalent, if they get the same interpretation for any interpretation in any *-autonomous category.

When the category \( C \) is a compact category of cowordisms (over some alphabet), and formulas are interpreted as boundaries, we denote the interpretation of a formula \( A \) as \( \partial A \) and use the convention
\[
(\partial A)_l = \partial_l A, \quad (\partial A)_r = \partial_r A.
\]
Observe that in this, interpretations of proofs do not depend on the alphabet at all. So it would be more honest to say that this is an interpretation in the category \textbf{Cob} of cobordisms. The alphabet comes into play if we add new axioms to the logic, which gives us a \textit{logic grammar}.

4.3 Adding lexicon

An \textbf{LL} grammar is an interpretation of \textbf{LL} in a category of cowordisms supplied with a set of axioms together with cowordisms representing their "proofs". Here is an accurate definition

\textbf{Definition 13 Linear logic grammar (LLG)} \textit{G} is a tuple \( G = (N, T, Lex, S) \), where

- \( N \) is a finite set of positive literals together with an interpretation \( A \mapsto \partial A \) of elements of \( N \) as boundaries;
- \( T \) is a finite alphabet;
- \( Lex \), the \textbf{lexicon}, is a finite set of expressions of the form \( \sigma : F \), where \( F \) is an \textbf{LL} formula, and \( \sigma : 1 \rightarrow \partial F \) is a cowordism;
- \( S \in N \), the \textbf{standard type}, is interpreted as a boundary with \( |\partial S| = 1 \).

Elements of the lexicon \( Lex \) will be often called \textit{axioms}, and elements of \( N \) will be called \textit{atomic types}.

Now let \( A \) be an \textbf{LL} formula, and let \( \rho : 1 \rightarrow \partial A \) be a cowordism.

We say that \( G \) \textit{generates the cowordism} \( \rho \) \textit{of type} \( A \), if there exists axioms \( \tau_1 : A_1, \ldots, \tau_k : A_n \in Lex \) for some \( k \geq 0 \) and a cowordism \( \sigma : 1 \rightarrow \partial A_1 \otimes \cdots \otimes \partial A_n \otimes \partial A \) arising as the interpretation of some \textbf{LL} proof of the sequent

\[ \vdash A_1 \cdots, A_n, A, \]

such that,

\[ \rho = \langle \tau_1 \otimes \cdots \otimes \tau_n, \sigma \rangle_{\partial A_1 \otimes \cdots \otimes \partial A_n}. \]

The \textit{cowordism type} \( A \) \textit{generated by} \( G \), or, simply, the \textit{cowordism type} \( A \) \textit{of} \( G \), is the set of all cowordisms of type \( A \) generated by \( G \).

Now any regular cowordism of the standard type \( S \) is an edge-labeled graph containing a single edge. Thus the set of type \( S \) regular cowordisms can be identified with a set of words.

The \textit{language} \( L(G) \) \textit{generated by} \( G \) is the set of type \( S \) regular cowordisms generated by \( G \).
5 Encoding multiple context-free grammars

In this section, as an example, we establish a relationship between LLG and multiple context-free grammars.

5.1 Multiple context-free grammars

Multiple context-free grammars were introduced in [28]. We follow (with minor variations in notation) the presentation in [14].

Definition 14 A multiple context free grammar (MCFG) $G$ is a tuple $G = (N,T,S,P)$ where

- $N$ is a finite alphabet of nonzero arity predicate symbols called nonterminal symbols or nonterminals;
- $T$ is a finite alphabet of terminal symbols or terminals;
- $S \in N$, the start symbol, is unary;
- $P$ is a finite set of sequents, called productions of the form

$$B_1(x_1^1, \ldots, x_{k_1}^1), \ldots, B_n(x_1^n, \ldots, x_{k_n}^n) \vdash A(s_1, \ldots, s_k), \quad (7)$$

where

(i) $n \geq 0$ and $A, B_1, \ldots, B_n$ are nonterminals with arities $k, k_1, \ldots, k_n$ respectively;
(ii) $\{x_i^j\}$ are pairwise distinct variables not from $T$;
(iii) $s_1, \ldots, s_k$ are words built of terminals and $\{x_i^j\}$;
(iv) each of the variables $x_i^j$ occurs exactly once in exactly one of the words $s_1, \ldots, s_k$.

Remark Productions are often written in the opposite order in literature; with $A$ on the left and $B_1, \ldots, B_n$ on the right.

Also, our “non-erasing” condition (iv) in the definition of a MCFG, namely, that all $x_i^j$ occurring on the left occur exactly once on the right, is too strong compared with original definitions in [28], [14]. Usually it is required only that each $x_i^j$ should occur at most once on the right. However, it is known [28] that adding the non-erasing condition does not change the expressive power of MCFG, in the sense that the class of generated languages (see below) remains the same.

Definition 15 The set of predicate formulas derivable in $G$ is the smallest set satisfying the following.

(i) If a production

$$\vdash A(s_1, \ldots, s_k)$$

is in $P$, then $A(s_1, \ldots, s_k)$ is derivable.
(ii) For every production (7) in $P$, if

- $B_1(s_1^1,\ldots,s_{k_1}^1),\ldots,B_n(s_1^n,\ldots,s_{k_n}^n)$ are derivable,
- $t_m$ is the result of substituting the word $s_i^j$ for every variable $x_i^j$ in $s_m$, for $m = 1,\ldots,k$,

then the formula $A(t_1,\ldots,t_k)$ is derivable.

**Definition 16** The language generated by an MCFG $G$ is the set of words $s$ for which $S(s)$ is derivable in $G$.

Multiple context-free language is a language generated by some MCFG.

When all predicate symbols in $N$ are unary, the above definition reduces to the more familiar case of a context free grammar (CFG).

### 5.2 MCFG productions as cowordisms

Assume that we are given alphabets $N$ and $T$ of nonterminals and terminals respectively, as in Definition 14.

For each $A \in N$ with arity $k$ introduce left vertices

$$l_A^1,\ldots,l_A^k$$

and right vertices

$$r_A^1,\ldots,r_A^k$$.

Denote the set of left vertices as $\partial_l A$, and the set of right vertices, as $\partial_r A$.

Define the boundary $\partial A$ as $\partial A = \partial_l A \cup \partial_r A$.

Now for any production $p$ of the form

$$B_1(x_1^1,\ldots,x_{k_1}^1),\ldots,B_n(x_1^n,\ldots,x_{k_n}^n) \vdash A(s_1,\ldots,s_k)$$

we construct a cowordism $\text{graph}(p)$ over the alphabet $T$ of the type

$$\text{graph}(p) : \partial B_1 \otimes \ldots \otimes \partial B_n \rightarrow \partial A,$$

if $n > 0$, or

$$\text{graph}(p) : 1 \rightarrow \partial A$$

otherwise.

In order to get $\text{graph}(p)$ it is sufficient to construct a multiword with the left boundary

$$\partial_r A \sqcup (\partial_l B_1) \sqcup \ldots \sqcup (\partial_l B_n)$$

and the right boundary

$$\partial_l A \sqcup (\partial_r B_1) \sqcup \ldots \sqcup (\partial_r B_n) .$$

The multiword is constructed as follows.

Let $V$ be the set of all variables $x_i^j$ occurring in $p$. 
For each \( y = x_i^j \in V \) let
\[
h(y) = l_i^{B_j}, \quad t(y) = r_i^{B_j}.
\]
Now each word \( s_m, m = 1, \ldots, k \), on the righthand side of \( p \) is a concatenation of the form
\[
s_m = w_m^0 y_m^1 w_m^1 \ldots y_m^{\alpha_m} w_m^{\alpha_m},
\]
where all
\[
w_m^0, \ldots, w_m^{\alpha_m}
\]
are words in the alphabet \( T \) (possibly empty), and
\[
y_m^1, \ldots, y_m^{\alpha_m}
\]
are variables from \( V \). (With the convention that \( \alpha_m \) may equal zero, in which case \( s_m = w_m^0 \).)

We represent \( p \) as the following multiword graph \( (p) \).

In a verbal language, the multiword graph \( (p) \) is defined as follows.

For each \( m = 1, \ldots, k \), if \( \alpha_m = 0 \) draw a directed edge from \( l_m^A \) to \( r_m^A \) and label it with \( s_m \).

Otherwise
- draw a directed edge from \( l_m^A \) to \( h(y_m^1) \) and label it with \( w_m^0 \),
- draw a directed edge from \( t(y_m^{\alpha_m}) \) to \( r_m^A \) and label it with \( w_m^{\alpha_m} \),
- for each \( \beta = 1, \alpha_m - 1 \) draw a directed edge from \( t(y_m^\beta) \) to \( h(y_m^{\beta+1}) \) and label it with \( w_m^\beta \).

Since each element of \( V \) occurs on the left side of \( p \) exactly once, it follows that the obtained edge-labeled graph is a perfect matching, hence a (regular) multiword, and its boundary satisfies the desired specification.

The constructed cowordism graph \( (p) \) represents the production \( p \) in a very direct sense.

Let us construct, for every nonterminal \( C \in N \) of arity \( \alpha \), an oriented graph on the vertex set \( \partial C \) by drawing for each \( m = 1, \ldots, \alpha \) a directed edge from \( l_m^C \) to \( r_m^C \) as depicted below.

\[
l_m^C \quad \rightarrow \quad r_m^C \quad \rightarrow \quad \cdots \quad \rightarrow \quad l_\alpha^C \quad \rightarrow \quad r_\alpha^C
\]
This graph is a perfect matching. We call it the pattern of $C$ and denote as \( \text{Pat}(C) \).

We will represent a predicate formula

\[
C(s_1,\ldots,s_\alpha),
\]

where $s_1,\ldots,s_\alpha$ are words, as a multiword whose underlying graph is \( \text{Pat}(C) \) in the following obvious way.

We say that the above multiword represents formula (8).

Then the following holds.

**Note 10** Let $\sigma_1,\ldots,\sigma_n$ be cowordisms,

\[
\sigma_j : 1 \to \partial B_j, \quad j = 1,\ldots,n,
\]

such that, seen as multiwords, they represent formulas

\[
B(s_1^1,\ldots,s_{k_1}^1),\ldots,B(s_1^n,\ldots,s_{k_n}^n)
\]

respectively, where $k_j$ is the arity of $B_j$, $j = 1,\ldots,n$.

Let $t_m$ be the result of substituting the word $s_m^j$ for every variable $x_m^j$ in $s_m$, for $m = 1,\ldots,k$.

Then the composition

\[
\text{graph}(p) \circ (\sigma_1 \otimes \cdots \otimes \sigma_n) : 1 \to \partial A
\]

gives the multiword representing the formula

\[
A(t_1,\ldots,t_k). \quad \square
\]

### 5.3 From MCFG to LLG

Any MCFG $G = (N,T,P,S)$ gives rise to an LLG by means of the translation described in Section 5.2.

We treat each nonterminal $A$ as a positive literal and assign to it the boundary $\partial A$ as in Section 5.2. This gives us a set $N$ of positive literals and an interpretation $A \mapsto \partial A$ in the category of cowordisms.

Then, to any production $p \in P$ of form (7) we assign the axiom

\[
\text{graph}(p) \uparrow : \partial B_1 \otimes \cdots \otimes \partial B_n \rightarrow \partial A,
\]

and this gives us the lexicon $\text{Graph}(P)$.

The LLG $G'$ is defined as the tuple $G' = (N,T,\text{Graph}(P),S)$. 

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From Note 10 (using Note 6 on the properties of partial pairing of cowordisms) it is immediate that the language generated by $G$ identifies with a subset of the language generated by $G'$. Let us prove the opposite inclusion.

Let $L(G)$ be the language generated by $G$. Consider the category $\text{CTypes}_{L(G)}$ of closed types of $L(G)$.

For any $A \in N$ of arity $k$ we define the type $\hat{A}$ as the type on $\partial A$ consisting of all multiwords representing formulas

$$A(s_1, \ldots, s_k)$$

derivable in $G$. We then define the closed type $A \in \text{Types}_{L(G)}$ as the closure $A = Cl(\hat{A})$.

(We deliberately abuse notation using the same symbol for an atomic type of $G'$ and the corresponding closed cowordism type.)

Now we refine the interpretation of $LL$ in $\text{Cow}_T$ to an interpretation in $\text{CTypes}_{L(G)}$.

We assign to each literal $A \in N$ the corresponding cowordism type $A \in \text{CTypes}_{L(G)}$ and extend the assignment to all formulas in $Fm(N)$ by induction.

Since the category $\text{CTypes}_{L(G)}$ is $\ast$-autonomous this gives us also a sound interpretation of proofs as morphisms of closed types.

Since the forgetful functor $\text{CTypes}_{L(G)} \rightarrow \text{Cow}_T$ preserves $\ast$-autonomous structure, the two interpretations (in $\text{CTypes}_{L(G)}$ and in $\text{Cow}_T$) coincide on the level of cowordisms. In particular, if $\pi$ is a proof of a sequent

$$\vdash A_1, \ldots, A_n,$$

then its interpretation, the cowordism

$$[\pi] : 1 \rightarrow \partial A_1 \otimes \ldots \otimes \partial A_n$$

is in the type $A_1 \varphi \ldots \varphi A_n$.

Now we have the following.

**Note 11** Elements of the type $S \in \text{CTypes}_{L(G)}$ are all regular cowordisms whose single edge is labeled with a word of $L$.

**Proof** repeats the discussion in the end of Section 3.3.1. □

**Note 12** For any axiom $\sigma : F$ in the lexicon $\text{Graph}(P)$, the cowordism $\sigma$ belongs to the corresponding cowordism type $F \in \text{CTypes}_{L(G)}$. 

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Proof We have that \( \sigma = \text{\textit{graph}}(p) \) is the name of a cowordism representing some production \( p \in P \) of form (7), and

\[
F = B_1 \otimes \ldots \otimes B_n \rightarrow A.
\]

By Note 10, the cowordism \( \text{\textit{graph}}(p) \) is a morphism of types

\[
\text{\textit{graph}}(p) : \tilde{B}_1 \otimes \ldots \otimes \tilde{B}_n \rightarrow \tilde{A}.
\]

By Note 1, it remains a morphism of closed types

\[
\text{\textit{graph}}(p) : B_1 \otimes \ldots \otimes B_n \rightarrow A.
\]

It follows that the name \( \sigma \) of \( \text{\textit{graph}}(p) \) is in the closed type \( F \) of \( L(G) \). □

It follows that \( G' \) generates the language \( L(G) \). Thus we have the following.

Theorem 1 Any multiple context-free language is generated by an LL grammar. □

5.4 From LLG to MCFG

Note that LLG constructed from MCFG in the preceding section have particularly simple lexicons: formulas occurring in such lexicons do not contain \( \otimes \) connective. We call such lexicons \( \otimes \)-free.

We are going to prove the converse of the preceding theorem: any LLG with a \( \otimes \)-free lexicon generates a multiple context-free language.

5.4.1 Extended MCFG grammars

It will be convenient to reformulate (and slightly generalize) MCFG in a more category-theoretic language.

Definition 17 An extended MCFG grammar \( G \) is a tuple \( G = (N, T, P, S) \), where

- \( N \) is a finite set of types together with an interpretation \( A \mapsto \partial A \) of elements of \( N \) as boundaries;
- \( T \) is a finite alphabet of terminal symbols;
- \( P \), is a finite set of rules of the form

\[
\sigma : \partial A_1 \otimes \ldots \otimes \partial A_n \rightarrow \partial A,
\]

where \( A_1, \ldots, A_n, A \) are elements of \( N \), and

\[
\sigma : \partial A_1 \otimes \ldots \otimes \partial A_n \rightarrow \partial A.
\]

is a cowordism;
• \( S \in N \), the **standard type**, is interpreted a boundary with \(|\partial_hS| = |\partial_vS| = 1\).

Elements of \( P \) are called *cowordism productions*.

Now, for any type \( A \in N \), we will define a cowordism type on \( \partial A \), called the **cowordism type** \( A \) generated by \( G \), or, simply, the **cowordism type** \( A \) of \( G \). We will write \( G \vdash \sigma : A \) to express that \( \sigma \) is in the cowordism type \( A \) of \( G \).

The set is defined by induction.

- If a cowordism production \( \sigma : 1 \rightarrow A \) is in \( P \), then \( G \vdash \sigma : A \).
- If a cowordism production

\[
\sigma : A_1 \otimes \ldots \otimes A_n \rightarrow A
\]

is in \( P \), and

\[
G \vdash \tau_i : A_i, \quad i = 1, \ldots, n,
\]

then \( G \vdash \sigma \circ (\tau_1 \otimes \ldots \otimes \tau_n) : A \).

The set of regular cowordisms of type \( S \) is called the **language generated by the extended MCFG** \( G \).

**5.4.2 From extended MCFG to ordinary MCFG**

Let \( G = (N, T, P, S) \) be an extended MCFG.

For each \( A \in N \) and regular cowordism \( \sigma : 1 \rightarrow \partial A \) such that \( G \vdash \sigma : A \) let \( Pat(\sigma) \) be the pattern of \( \sigma \).

We say that \( Pat(\sigma) \) is a possible pattern of \( A \).

We denote the set of possible patterns of \( A \) as \( Patt(A) \). Note that this set is finite.

**Definition 18** The extended MCFG \( G \) is **simple**, if for any type \( A \in N \) the set \( Patt(A) \) contains at most one element.

Quite obviously, any ordinary MCFG, can be seen as a simple extended MCFG.

**Lemma 1** If a language is generated by a simple extended MCFG, then it is also generated by an ordinary MCFG.

**Proof** Let \( P_0 \subseteq P \) be the set of regular cowordism productions that participate in generation of \( L(G) \).

For each element \( p \in P_0 \) we easily write an MCFG production as the inverse of the “graph map” (see Section 5.2). This is left as an exercise to the reader. \( \square \)

Now we generalise the above to arbitrary extended MCFG \( G \).

Since the empty language is obviously multiple context-free, we may assume that \( L(G) \) is nonempty, otherwise there is nothing to prove.
We construct a new extended MCFG $G'$ as follows.

For any type $A$ of $G$ and any possible pattern $\pi$ of $A$ we introduce a new symbol $(A, \pi)$.

We define the set $N'$ of types of $G'$ as

$$N' = \{(A, \pi) | A \in N, \pi \in Patt(X)\}.$$ 

Interpretation of types as boundaries is given by

$$\partial(A, \pi) = \partial A.$$ 

For any cowordism production

$$\sigma : A_1 \otimes \ldots \otimes A_n \rightarrow A$$

of $G$ we consider all possible cowordism productions of the form

$$\sigma' : (A_1, \pi_1) \otimes \ldots \otimes (A_n, \pi_n) \rightarrow (A, \pi), \quad (10)$$

where

$$\pi_i \in Patt(A_i), \ i = 1, \ldots, n,$$

and $\pi \in Patt(A)$ is constructed as the composition

$$\tau = Pat(\sigma) \circ (\pi_1 \otimes \ldots \otimes \pi_n).$$

The set $P'$ of productions for $G'$ consists of all cowordism productions of form (10). Again, there are only finitely many of them.

Since the set $L(G)$ is assumed nonempty, the set $Patt(S)$ is a singleton. We denote $S' = (S, e)$, where $e$ is the only element of $Patt(S)$.

We define $G'$ as $G' = (N', T', L', S')$.

It is immediate that $G'$ is simple and generates the same extended language as $G$.

Combining the above with the preceding lemma, we obtain the following.

**Lemma 2** A language is generated by an MCFG iff it is generated by an extended MCFG. □

### 5.4.3 From $\otimes$-free lexicon to extended MCFG

We start with some simple technical developments.

For a sequent $\Theta$ of the form

$$\Theta = A, A^\bot \otimes B^\bot, B, \quad (11)$$

we have a proof

$$\vdash A, A^\bot \vdash B^\bot, B \ (\otimes).$$

We call this proof the *standard proof* of $\Theta$. 31
Now let \( \Phi \) be a finite set of \( \otimes \)-free LL formulas, which is closed under subformulas. Let \( \Phi^\perp \) be the set
\[
\Phi^\perp = \{ F^\perp | F \in \Phi \}.
\]

Let \( \Pi_0(\Phi) \) be the set of all standard proofs of sequents of form (11) where \( A^\perp, B^\perp, A^\otimes B \in \Phi \). Let \( \Pi(\Phi) \) be the closure of \( \Pi_0(\Phi) \) under the Exchange rule.

**Lemma 3** Let \( \Gamma \) be a sequent all whose formulas are in \( \Phi^\perp \).

Then any proof of \( \Gamma \) is equivalent to a proof obtained from elements of \( \Pi(\Phi) \) using only axioms and the Cut rule.

**Proof** by induction on a cut-free proof. \( \square \)

Now let \( G = (N, T, Lex, S) \) be an LLG with a \( \otimes \)-free lexicon.

We construct a cowordism grammar \( G' \) using Lemma 3 as follows.

Let \( \Phi \) be the set of all subformulas occurring in \( L \).

For every formula \( F \) in \( \Phi \cup \Phi^\perp \) we introduce a fresh symbol \([F]\) and assign to \([F]\) the same interpretation as to \( F \),
\[
\partial[F] = \partial F.
\]

We put
\[
N' = \{ [F] | F \in \Phi \cup \Phi^\perp \}, \quad S' = [S].
\]

Now in order to define an extended MCFG we only need productions.

Let \( P_0 \) be the set of all cowordism productions of the form
\[
\sigma : [F_1] \otimes [F_2] \to [F],
\]
where \( \sigma \) is the interpretation of some proof in \( \Pi(\Phi) \) having the sequent
\[
\vdash F_1^\perp, F_2^\perp, F
\]
as the conclusion.

Let \( P_1 \) be the set of all cowordism productions
\[
\sigma : 1 \to [F]
\]
where \( \sigma : F \in Lex \).

We define the set of productions \( P' \) as \( P' = P_0 \cup P'_1 \).

The extended MCFG \( G' \) is defined as \( G' = (N', T, P', S') \).

Lemma 3 easily yields the following.

**Note 13** For any formula \( F \in \Phi^\perp \) the cowordism type \([F]\) generated by \( G' \) coincides with the cowordism type \( F \) generated by \( G \).

**Proof** Exercise. \( \square \)

We leave it as an exercise to the reader to prove that if \( G \) generates a nonempty language then \( S^\perp \) occurs as a subformula in \( Lex \), hence \( S \in \Phi^\perp \).

Then the above Note implies that the language of \( G' \) coincides with the language of \( G \).

We summarize in the following.
Lemma 4  For any LLG $G$ with a $\otimes$-free lexicon there exists a cowordism grammar $G'$ generating the same extended language. □

Putting Lemmas 3 and 2 together we obtain the following.

Theorem 2  A language is multiple context-free iff it is generated by an LLG with a $\otimes$-free lexicon. □

6  Encoding abstract categorial grammars

Abstract categorial grammars (ACG) were introduced in [10]. They are based on the purely implicational fragment of linear logic, and LL grammars of this paper can be seen as a representation and extension of ACG (over string signature).

In this section we assume that the reader is familiar with basic notions of $\lambda$-calculus, see [3] for a reference.

6.1  Linear $\lambda$-calculus

Linear $\lambda$-terms are $\lambda$-terms where each variable occurs exactly once.

More accurately, given a set $X$ of variables and a set $C$ of constants, with $C \cap X = \emptyset$, the set $\Lambda(X, C)$ of linear $\lambda$-terms is defined by the following.

- Any $a \in X \cup C$ is in $\Lambda(X, C)$;
- if $t, s \in \Lambda(X, C)$ are linear $\lambda$-terms whose sets of free variables are disjoint then $(ts) \in \Lambda(X, C)$;
- if $t \in \Lambda(X, C)$, and $x \in X$ occurs freely in $t$ exactly once then $(\lambda x.t) \in \Lambda(X, C)$.

We type linear terms using linear implicational types.

Given a set $N$ of atomic types, the set $Tp(N)$ of linear implicational types is defined by induction.

- Any $A \in N$ is in $Tp(N)$;
- if $A, B \in Tp(N)$, then $(A \rightarrow B) \in Tp(N)$.

Definition 19  A higher order linear signature, or, simply, a signature, $\Sigma$ is a triple $\Sigma = (N, C, \tau)$, where $N$ is a finite set of atomic types, $C$ is a finite set of constants and $\tau$ is a function assigning to each constant a linear implicational type.

Given a signature $\Sigma = (N, C, \tau)$ and a countable set $X$ of variables, a typing judgement is a sequent of the form

$$x_1 : A_1, \ldots, x_n : A_n \vdash \Sigma t : A,$$
where $x_1, \ldots, x_n \in X$ are pairwise distinct ($n$ may be zero), $t \in \Lambda(X, C)$, and $A_1, \ldots, A_n, A \in Tp(N)$.

Typing judgements are derived from the following type inference rules.

- $\Gamma \vdash \Sigma \ c : \tau(c)$ (const),
- $x : A \vdash \Sigma \ x : A$ (var),
- $\Gamma \vdash \Sigma \ s : A \quad \Delta \vdash \Sigma \ t : A \rightarrow B$ (app),
- $\Gamma, \Delta \vdash \Sigma \ (\lambda x. t) : A \rightarrow B$ (abstr).

We say that a term $t$ is typeable in $\Sigma$ if there is a type $A$ such that $\vdash \Sigma \ t : A$. In this case we say that $A$ is the type of $t$ in $\Sigma$.

### 6.1.1 Semantics

Let $C$ be a symmetric monoidal category, and $\Sigma = (N, C, \tau)$ a signature.

An interpretation of signature $\Sigma$ types in $C$ consists in assigning to each atomic type $A \in N$ an object $[A] \in C$. This is extended to all types in $Tp(N)$ by the obvious induction:

$[A \rightarrow B] = [A] \rightarrow [B]$.

In the following we omit square brackets and denote a type $A \in Tp(N)$ and its interpretation the same.

An interpretation of $\Sigma$ in $C$ consists of an interpretation of types and a function $c \mapsto [c]$ assigning to each constant $c \in C$ a morphism

$[c] : 1 \rightarrow \tau(c)$.

The interpretation extends to all typeable terms and derivable typing judgements.

To each derivable typing judgement $\sigma$ of the form

$x_1 : A_1, \ldots, x_n : A_m \vdash \Sigma \ A$

we assign a $C$-morphism

$[\sigma] : A_1 \otimes \ldots \otimes A_n \rightarrow A,$

if $n > 0$, or

$[\sigma] : 1 \rightarrow A,$

if $n = 0$, by induction on type inference rules.

If the judgement $\sigma$ is $\vdash \Sigma \ c : \tau(c)$ obtained by the (const) rule, then $[\sigma] = [c]$.

If $\sigma$ is $x : A \vdash \Sigma \ x : A$ obtained by the (var) rule, then $[\sigma] = \text{id}_A$.

If $\sigma$ is obtained from a derivable judgement $\sigma'$ by the (abstr) rule, then $[\sigma]$ is obtained from $[\sigma']$ using symmetry and correspondence $[\text{1}]$.

If $\sigma$ is obtained from derivable judgements

$\sigma_1 = \Gamma_1 \vdash \Sigma \ s : A, \quad \sigma_2 = \Gamma_2 \vdash \Sigma \ t : A \rightarrow B$
by the (app) rule, then

$$[\sigma] = \text{ev}_{A,B} \circ ([\sigma_1] \otimes [\sigma_2]).$$

Finally, for a typeable term $t$ of type $A$ we have a derivable typing judgement

$$\vdash_\Sigma t : A,$$

and we put $[t] = [\sigma]$.

**Lemma 5** With notation as above we have:

- if typeable terms $t, s$ are $\beta\eta$-equivalent, then $[t] = [s]$;
- if $\vdash_\Sigma s : A, \vdash_\Sigma t : A \rightarrow B$, then $[ts] = [t] \cdot [s]$.

**Proof** Exercise or see [5]. □

### 6.1.2 String signature

Let $T$ be a finite alphabet.

The *string signature* $\text{Str}_T$ over $T$ has a single atomic type $O$, the alphabet $T$ as the set of constants and a typing assignment

$$\tau(c) = O \twoheadrightarrow O \forall c \in T.$$  

We denote the type $O \twoheadrightarrow O$ as $\text{str}$.

Terms typeable in $\text{Str}_T$ with the type $\text{str}$ are called *string terms*.

Any word $a_1 \ldots a_n$ in the alphabet $T$ can be represented as the string term

$$/a_1 \ldots a_n/ = (\lambda x.a_1(\ldots(a_n(x))\ldots)).$$

It is not hard to see that, if we identify $\beta\eta$-equivalent terms, the map $w \mapsto /w/$ has an inverse.

**Note 14** Any $\beta$-normal term $t$ typeable in $\text{Str}_T$ with the type $\text{str}$ is $\beta\eta$-equivalent to the term $/w/$ for some $w \in T^*$.  

**Proof**

(i) There is no typeable term of type $O$ (for example, because any derivable typing judgement has an even number of $O$ occurrences).

(ii) Using (i), we prove by induction on type inference that any $\beta$-normal term $t$ typeable in $\text{Str}_T$ is either a constant $t \in T$, or an abstraction, $t = (\lambda x.t')$ for some variable $x$ and term $t'$.

(iii) Using (ii), we prove by induction on type inference that for any derivable typing judgement $x : O \vdash_{\text{Str}_T} t : O$, where $t$ is a $\beta$-normal term, it holds that $t = c_1(\ldots(c_n(t))\ldots)$ for some constants $c_1, \ldots, c_n \in T$.  

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Now if \( \vdash_{Str} t : O \to O \), then either \( t \) is a constant, hence \( \beta\eta \)-equivalent to \( /t/ \), or its typing was obtained by the (abstr) rule. In the latter case the claim follows from (iii). \( \square \)

Thus we have a map from typeable string terms to words over \( T \). It turns out that this map extends to all typeable terms as a map to cowordisms.

Let us choose an interpretation of the atomic type \( O \) as a one-point boundary

\[ \partial O = \partial_l O \cup \partial_r O \]

with \( |\partial_r O| = 1, \partial_l O = \emptyset \).

By induction this gives us an interpretation \( A \mapsto \partial A \) of all types in \( Tp(O) \) as boundaries.

We extend this to an interpretation of the string signature in the category \( \text{Types}_T \) by defining the cowordism type \( O \) on the boundary \( \partial O \) as the empty set.

Any regular cowordism \( \sigma : \partial O \to \partial O \) which is a morphism of types \( \sigma : O \to O \), is a graph consisting of a single edge labeled with some word \( w \in T^* \). We denote this cowordism as \( \text{graph}(w) \).

We interpret each constant \( c \in T \) as the corresponding regular cowordism \( \text{graph}(c) : O \to O \).

This gives us an interpretation of the signature \( \text{Str}_T \).

We denote the interpretation of a typeable term \( t \in \Lambda(X,C) \) as \( \text{graph}(t) \).

We call an interpretation of the above form a standard interpretation of the string signature.

### 6.2 Abstract categorial grammars

Given two signatures \( \Sigma_i = (N_i, C_i, \tau_i), i = 1, 2 \), a map of signatures

\[ \phi : \Sigma_1 \to \Sigma_2 \]

is a pair \( \phi = (F, G) \), where

- \( F : Tp(\Sigma_1) \to Tp(\Sigma_2) \) is a function satisfying the homomorphism property
  \[ F(A \to B) = F(A) \to F(B) \],

- \( G : C_1 \to \Lambda(X, C_2) \) is a function such that for any \( c \in C_1 \) it holds that \( \vdash_{\Sigma_2} G(c) : F(\tau(c)) \).

The map \( G \) above extends inductively to a map

\[ G : \Lambda(X, C_1) \to \Lambda(X, C_1) \]

by

\[ G(x) = x, \ x \in X, \]
\[ G(ts) = (G(t)G(s)), \ G(\lambda x.t) = (\lambda x.G(t)). \]

For economy of notation, we write \( \phi(A) \) for \( F(A) \) when \( A \in Tp(C_1) \), and we write \( \phi(t) \) for \( G(t) \) when \( t \in \Lambda(X, C_1) \).
Definition 20 A string abstract categorial grammar (string ACG) $G$ is a tuple $G = (\Sigma, T, \phi, S)$, where

- $\Sigma$, is a signature;
- $T$ is a finite alphabet
- $\phi : \Sigma \rightarrow Str_T$, the lexicon, is a map of signatures;
- $S$, the standard type, is an atomic type of $\Sigma$, such that $\phi(S) = str$.

The string language $L(G)$ generated by $G$ is the set of words over $T$ given by

\[ L(G) = \{ w \in T^* | \exists t \vdash \Sigma t : S \text{ and } \phi(t) = /w/ \} \]

Equivalently

\[ L(G) = \{ w \in T^* | \exists t \vdash \Sigma t : S \text{ and } graph(t) = graph(w) \} \]

6.3 Encoding

Let $G = (\Sigma, T, \phi, S)$ be a string ACG.

Choose some standard interpretation of $Str_T$ in $\text{Types}_T$. This yields us an interpretation of the signature $\Sigma$ defined as follows.

To any type $A \in Tp(\Sigma)$ we assign the boundary

$\partial A = \partial(\phi(A))$

and the cowordism type $A \in \text{Types}_T$ given by

$A = \{ graph(\phi(t)) | t \vdash \Sigma t : A \}$.

To any term $t$ typeable in $\Sigma$ we assign the cowordism

$graph(t) = graph(\phi(t))$.

It is immediate from definitions that the interpretation is sound, i.e. we have the following.

Note 15 If $t \vdash \Sigma t : A$ then $graph(t) \in A$. $\square$

Now treating the set of atomic types of $\Sigma$ as literals and types of $\Sigma$ as LL formulas we construct an LLG $G'$ encoding $G$.

Let $N, C$ be the sets of, respectively, atomic types and constants of $\Sigma$. We already have the assignment $A \mapsto \partial A$ of elements of $N$ to boundaries.

We define the set of axioms

$Lex = \{ graph(c) : \tau(c) | c \in C \}$.

The LLG $G'$ is defined as $G' = (N, T, Lex, S)$. 37
Now, by induction on type inference rules using Note 15 we prove that the language \( L(G) \) generated by \( G \) is a subset of language of \( G' \).

Proof of the opposite inclusion repeats the argument in Section 5.3 where we consider encoding of MCFG. We consider the category \( \text{CTypes}_{L(G)} \) of closed types of \( L(G) \) and observe that any cowordism type \( A \) of \( G' \) is a subset of the corresponding closed type of \( \text{CTypes}_{L(G)} \).

We summarise.

**Theorem 3** If a language is generated by a string ACG then it is also generated by an LLG. □

It seems an interesting question whether the converse is true or not.

**Remark** Since MCFG embed into string ACG [11], Theorem 3 on encoding ACG in LLG grammars implies that MCFG embed into LLG. However it does not imply the converse statement (Theorem 2 that any \( \otimes \)-free lexicon gives rise to an MCFG).

On the other hand it is not hard to see that Theorem 2 together with Theorem 3 do imply the known result [26] that any second order string ACG generates a multiple context-free language. Thus we gave another, more “category-theoretic” proof of this result.

## 7 Encoding backpack problem

It is known that ACG, in general, can generate NP-complete languages. In view of Theorem 3 it is no wonder that LLG can generate NP-complete languages as well. In this last section we show how an LLG can generate solutions of the backpack problem. Our purpose here is mainly illustrative. We try to convince the reader that the geometric language of cowordisms is indeed intuitive and convenient for analysing language generation.

We consider backpack problem in the form of the subset sum problem.

**Definition 21** Subset sum problem (SSP): Given a finite sequence \( s \) of integers, determine if there is a subsequence \( s' \subseteq s \) such that \( \sum_{z \in s'} z = 0 \).

SSP is known to be NP-complete, see [30].

We now define a language representing solutions of SSP.

We represent integers as words in the alphabet \( \{+,-\} \), we call them numerals. An integer \( z \) is represented (non-uniquely) as a word for which the number of \( + \) occurrences minus the number of \( - \) occurrences is \( n \).

We represent finite sequences of integers as words in the alphabet \( T = \{+,-,\bullet\} \), with \( \bullet \) interpreted as a separation sign. Thus a word in this alphabet should be read as a list of numerals separated by bullets.

Let \( L_0 \subset T^* \) be the set of words representing sequences having a subsequent summing to zero. Obviously \( L_0 \) is NP-complete.

We now construct a system of cowordisms over \( T \) which (together with symmetry transformations) generates \( L_0 \).
We will use three atomic boundaries $E, B, S$, each of them having one point in the left boundary and one point in the right boundary.

First we construct a system which generates all lists of numerals, i.e., all words in $T^*$.

We define four cowordisms

$$
\text{cons} : E \otimes E \to E, \quad \text{close} : 1 \to E,
$$

$$
\text{push}_+ : E \to E, \quad \text{push}_- : E \to E
$$

in the graphical language as follows.

\[ 
\text{cons} : \quad \begin{array}{c}
E \\
\otimes \\
E
\end{array} \rightarrow
\begin{array}{c}
E
\end{array}
\]

\[ 
\text{push}_+ : \quad E \leftarrow E \rightarrow E
\]

\[ 
\text{push}_- : \quad E \leftarrow E \rightarrow E
\]

\[ 
\text{close} : \quad \begin{array}{c}
E
\end{array} \rightarrow
\begin{array}{c}
E
\end{array}
\]

The cowordism \text{cons}, by iterated compositions with itself, generates lists with arbitrary many slots, then \text{push}_+ and \text{push}_- fill the slots with pluses and minuses, and \text{close} closes them.

Now, in order to generate $L_0$ we need lists with secret slots, which will contain elements summing to zero. Secret slots will be represented by the boundary $B$.

We define cowordisms

$$
\text{conv}_B : B \to E, \quad \text{push}_B : B \otimes B \to B \otimes B, \quad \text{close}_B : 1 \to B
$$

as follows.

\[ 
\text{conv}_B : \quad B \leftarrow E \rightarrow E
\]
The cowordism $cons_B$ converts some slots in the list to secret ones, $push_B$ fills secret slots, and $close_B$ closes them (always in pairs).

Obviously, any cowordism $\sigma : 1 \to E$ generated by the above system and symmetry transformations will be labeled with a word from $L_0$ iff the secret converter $conv_B$ participates in generation of $\sigma$.

Now to make sure that $conv_B$ participates we add the final cowordism

$$fin : E \otimes B \otimes E \to S$$

defined as follows.

Then the set of cowordisms from $1$ to $S$ generated by the above system and symmetry transformations identifies with $L_0$.

It remains to show that if we define an LLG by a lexicon consisting of names of the above cowordisms it will generate the same language. This is a technical and not difficult exercise in multiplicative linear logic proof-search.

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