Wormhole Effects on Yang-Mills Theory

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Abstract

In this paper wormhole effects on $SO(3)$ YM theory are examined. The wormhole wave functions for the scalar, the vector and the tensor expansion modes are computed assuming a small gauge coupling which leads to an effective decoupling of gravity and YM theory. These results are used to determine the wormhole vertices and the corresponding effective operators for the lowest expansion mode of each type. For the lowest scalar mode we find a renormalization of the gauge coupling from the two point function and the operators $\text{tr}(F^3)$, $\text{tr}(F^2 \tilde{F})$ from the three point function. The two point function for the lowest vector mode contributes to the gauge coupling renormalization only whereas the lowest tensor mode can also generate higher derivative terms.
1 Introduction

Great interest has been devoted to the question of topology change in space time and its effects on low energy field theories. In the framework of Euclidean path integral for gravity wormholes appear as stationary points which can cause a small baby universe to branch off from a large smooth universe. In the semiclassical approximation the effect of these wormholes on the low energy theory can be determined [1] resulting in an infinite number of local operators, one for each type of wormhole. These effective operators are multiplied by unknown parameters $\alpha$ which describe the “wormhole content” of the universe and are unobservable from the large universe point of view. It has been argued that this can introduce a new uncertainty into physics [1, 2] in addition to that of quantum mechanics.

However, using the Hartle–Hawking wave function of the universe [3] Coleman [4] has proposed a method to fix this uncertainty. In his scenario a probability distribution for the $\alpha$’s is computed in an effective theory consisting of the original theory and the wormhole induced operators. He showed that the probability distribution is peaked on a subspace of $\alpha$’s which corresponds to a vanishing cosmological constant. Furthermore, restricting to this subspace one might be able to fix the $\alpha$’s completely and hence all other constants of nature. Such a reasoning can e. g. be used to explain why $\theta_{QCD}$ takes a CP conserving value [4]. Though Coleman’s argumentation has been criticized by several authors [6] it still appears to be very attractive because of its ability to solve long outstanding theoretical problems.

Clearly, its application requires an understanding of the wormhole induced operators and their relation to the wormhole states. Such an analysis has been carried out for a conformally coupled scalar field [1], for fermions [7], for gravitons [8] and for electromagnetism [9]. In this paper we are interested in wormhole effects on YM theory which we study by looking at the simplest gauge group $SO(3)$. We will not touch the questions related to the validity of the semiclassical approximation. Instead we assume a wormhole scale safely below the Planck scale. It should be stressed that the semiclassical approximation for YM theory can differ substantially from electromagnetism. This arises because for YM theory on a space with spatial topology $S^3$ gauge freedom allows for a nontrivial symmetric state around which the expansion is carried out. E. g. in contrast to electrodynamics YM perturbations do not decouple from gravity and in the $SO(3)$ case harmonics up to spin 2 (tensor harmonics) appear in the expansion. However, here we will assume a small gauge coupling which causes an effective decoupling of gravity and YM theory and makes the equations analytically solvable. In this approximation the general wormhole wave function solution of the Wheeler–De Witt equation is determined for the scalar the vector and the tensor perturbations. It is used to compute the wormhole vertex of the lowest mode for each type of perturbation.

Symmetric Einstein-YM (EYM) theory on a space with spatial topology $S^3$ has been constructed in ref. [10]. In ref. [11] this construction has been applied to determine the Hartle–Hawking wave function for the gauge group $SO(3)$. A generalization which includes perturbations has been given in ref. [12]. In the next section we will briefly review some results of these papers which we will need. Our main goal is to find the wormhole
vertex

\[ < 0 | A_{\mu_1}(y_1) \cdots A_{\mu_k}(y_k) | \Psi > = \int d[h] d[A_0] \Psi[h, A_0] \int d[g] d[A] A_{\mu_1}(y_1) \cdots A_{\mu_k}(y_k) \exp(-S_{E}[g, A]) . \] (1)

The second path integral on the RHS of this expression has to be carried out over all 4–geometries (metric \( g \)) which are bounded by a three sphere \( S^3 \) and approximate flat space at infinity. Analogously, the gauge field \( A \) should match its value \( A_0 \) on \( S^3 \) and tend to zero asymptotically. The exponent \( S_E \) is the Euklidean action of these 4–configurations which relate the baby universe state specified by the wave functional \( \Psi \) to flat space. This wave functional will be determined in section 3. In section 4 we describe the semiclassical approximation for the path integral. Wormhole effects for the two- and three-point function induced by the symmetric mode are discussed in section 5 and section 6 will extend this to the lowest vector and tensor mode. We conclude in section 7. Two appendices are also added which fix the geometry of \( S^3 \) as a coset space and the conventions for harmonic expansion on it.

2 Harmonic expansion of \( SO(3) \) YM-theory

In this section we present the basic formulae for the harmonic expansion of \( SO(3) \) YM–theory on a space with spatial geometry \( S^3 \) which have been derived in ref. [12]. Our conventions concerning the geometry of \( S^3 \) and the harmonics are summarized in appendix A.

The gauge field \( A \equiv A_0 dt + A_\alpha \omega^\alpha \) is splitted into a symmetric part \( A^{(0)} \) and perturbations \( \tilde{A} \)

\[ A = A^{(0)} + \tilde{A} \]

\[ A^{(0)}_a = [1 + \chi(t)] T_a . \] (2)

Here \( A^{(0)} \) has the property that its change under \( G = SU_L(2) \times SU_R(2) \), the symmetry group of \( S^3 \), can be compensated by a suitable gauge transformation. The expansion for \( \tilde{A} \) reads

\[ \tilde{A}_0 = [\sqrt{2} \alpha_n S^n_b + \beta_n P^n_b] T_b \]

\[ \tilde{A}_a = \frac{1}{\sqrt{6}} \gamma_n \delta_{ab} Q^n T_b + \left[ \frac{1}{\sqrt{2}} \rho_n S^n_c + \frac{1}{\sqrt{6}} \sigma_n P^n_c \right] \varepsilon_{abc} T_b \]

\[ + \mu_n G^n_{ab} T_b + \frac{1}{\sqrt{2}} \nu_n S^n_{ab} T_b + \sqrt{6} \xi_n P^n_{ab} T_b . \] (4)

The Lifshitz-harmonics are denoted by \( Q, P, S \) and \( G \) and we have chosen a simplified notation where all relevant indices including even/odd are hidden in \( n \). All expansion coefficients are constant on the coset and depend only on time. They split into scalar–coefficients (\( \gamma, \xi, \sigma, \beta \)), vector–coefficients (\( \nu, \rho, \alpha \)) and a tensor–coefficient (\( \mu \)). These
three types completely separate from each other when the Lagrangian is computed up to second order in the perturbations. Here we will not reproduce this Lagrangian which has been determined in ref. [12] since in view of the wormhole wave function we are mainly interested in the Hamiltonian. To proceed in an explicit gauge invariant way we study the effect of an infinitesimal gauge transformation

\[ \delta A_\mu = \nabla_\mu U - [U, A_\mu], \quad U = u_a T_a \]

on the perturbations:

\[
\begin{align*}
\delta \gamma_n &= -\frac{1}{3} v_n, \\
\delta \xi_n &= \frac{1}{6} v_n, \\
\delta \sigma_n &= \chi v_n, \\
\delta \beta_n &= \dot{v}_n, \\
\delta \nu_n &= w_n, \\
\delta \alpha_n &= \dot{w}_n, \\
\delta \rho_n &= 2\chi w_n + n\tilde{w}_n, \\
\delta \mu_n &= 0,
\end{align*}
\]

where we have expanded

\[ u_a = \frac{1}{\sqrt{6}} v_n P_a^n + \sqrt{2} w_n S_a^n . \]

These transformations suggest the introduction of gauge invariant variables which are particular useful to formulate explicitly invariant wave functions. A possible choice is

\[
\begin{align*}
\Gamma_n &= \gamma_n + 2\xi_n, \\
B_n &= \beta_n + 3\gamma_n, \\
S_n &= \sigma_n + 3\chi\gamma_n, \\
A_n &= \alpha_n - \dot{\nu}_n, \\
R_n &= \rho_n - 2\chi\nu_n - n\tilde{\nu}_n .
\end{align*}
\]

For the \( n = 2 \) scalar– and vector–coefficients a special treatment is necessary since neither \( \xi_2 \) nor \( \nu_2 \) appears in the Lagrangian. We define

\[
\begin{align*}
s &= \frac{1}{e} \left( \frac{1}{3} \sigma_2 + \chi \gamma_2 \right) , \\
\sigma &= \frac{1}{e} \left( \rho_2^{(e)} - \chi \rho_2^{(o)} \right) ,
\end{align*}
\]

where \( (e)/(o) \) refer to the even/odd component of a perturbation.

For the gravitational part we assume a metric of the form

\[ g = \sigma^2 (-N_0^2 dt^2 + a(t)^2 \sum_b \omega^b \omega^b) , \quad \sigma^2 = \frac{2}{3\pi m_p^2} , \]

i. e. we will neglect gravitational perturbations. Therefore, in addition to the decoupling of the symmetric quantities \( a \) and \( \chi \) by means of conformal invariance of YM theory [10] we demand a decoupling of the corresponding perturbations. According to ref. [12] this arises automatically if the field \( \chi \) is assumed to be close to one of its “classical” values \( \chi = \pm 1 \). This assumption is well justified for a small gauge coupling \( e \) in which case the wave function for \( \chi \) is strongly peaked around these values [16] (see eq. (19) below). Consequently, as long as perturbations are concerned we assume that \( \chi = \tilde{\chi} = \pm 1 \).

3 Hamiltonian and Wormhole Wave Function

Working with the unperturbed metric (11) the Hamiltonian shows the following structure:

\[ H = N_0 (H_{0,Gr} + H_{0,YM} + \sum_n \mathcal{H}_2^{(n)}) + \sum_n (\alpha_n \mathcal{H}_\alpha^{(n)} + \beta_n \mathcal{H}_\beta^{(n)}) . \]
In the gauge $N_0 = a$ the Hamiltonians for the symmetric degrees of freedom $a$ and $\chi$ read
\[ H_{0,Gr} = \frac{1}{2}(-\pi_a^2 - a^2) \]
\[ H_{0,YM} = \frac{e^2}{2} \left( \frac{1}{3} \pi_\chi^2 + 6V \right) \] (12)
with
\[ V = \frac{1}{2e^4}(\chi^2 - 1)^2. \] (13)

Using the linear Hamiltonian constraints enforced by the Lagrange multipliers $\alpha_n$ and $\beta_n$
\[ H^{(n)}_\alpha = 2\pi_{\rho_n} + \frac{1}{3}\pi_{\gamma_n} + \frac{1}{6}\pi_{\xi_n} \]
\[ H^{(n)}_\beta = \chi\pi_{\gamma_n} - \frac{1}{3}\pi_{\gamma_n} + \frac{1}{6}\pi_{\xi_n} \] (14)
it can be shown that the wave function should only depend on the gauge invariant combinations defined in (8) and (9). This fact can be used to rewrite the quadratic Hamiltonians
\[ H^{(2)}(n) = S H^{(n)} + V H^{(n)} + T H^{(n)} \] exclusively in terms of these combinations:
\[ T H^{(n)} = e^2 \frac{\pi_{\mu_n}^2}{2} + \frac{1}{2e^2}[(n^2 + 1)\mu_n^2 + 2n\chi_{\mu_n}\tilde{\mu}_n] \]
\[ V H^{(n)} = \frac{2e^2 n^2}{n^2 - 4} \left[ \frac{1}{2} \pi_{R_n}^2 + \frac{\chi_{R_n}}{n} \right] + \frac{1}{4e^2}[(n^2 R_n^2 - 2n\chi_{R_n}\tilde{R}_n] \]
\[ S H^{(n)} = e^2 \left[ 3(n^2 - 3) \frac{\pi_{\Gamma_n}^2}{2(n^2 - 4)} + \frac{3}{2} (n^2 + 5)\pi_{S_n}^2 + 6\chi_{\pi_{\Gamma_n}}\pi_{S_n} \right] \]
\[ + \frac{1}{6e^2}[(n^2 - 4)\Gamma_n^2 + S^2]. \] (15)

In these expressions a sum over even/odd is implied and the tilde operation is defined by
\[ \tilde{p}_{n}^{(even/odd)} = p_{n}^{(odd/even)} \] for a generic perturbation $p_{n}$. The $n = 2$ vector and scalar part has to be treated separately. In terms of the variables defined in (9) the Hamiltonians take the simple form
\[ V H^{(2)} = \pi_r^2 + r^2, \quad S H^{(2)} = \frac{3}{2}(\pi_s^2 + s^2). \] (16)

We are now ready to compute the wormhole wave function $\Psi$ subject to the Wheeler–De Witt equation
\[ H\Psi = 0 \] (17)
which enters the expression (14) for the wormhole vertex. It can be written as a product $\Psi = \Psi_0(a,\chi)\Psi_2(p)$ with wave functions $\Psi_0$ and $\Psi_2$ for the symmetric fields and the perturbations, respectively. Let us first concentrate on $\Psi_0$. The only nonlinear term in $H$ is contained in the potential (13) for the symmetric field $\chi$. However, in the case of a small gauge coupling which we assume here we can neglect tunneling between the two minima at $\bar{\chi} = \pm 1$. Then the field $\chi$ sits close to one of these minima and approximately feels the linearized potential
\[ V_\pm = \frac{2}{e^4}(\chi \mp 1)^2. \] (18)
Under this assumption we easily find a complete set of harmonic oscillator solutions for (12)

\[ \Psi_{0,\pm kGr kS_0} \sim H_{kGr}(a)H_{kS_0}(\sigma_\pm) \exp \left( -\frac{1}{2}(a^2 + \sigma_\pm^2) \right) \] (19)

\[ \sigma_\pm = \frac{\sqrt{6}}{e} (\pm \chi - 1) , \] (20)

where \( H_k \) are the Hermitian polynomials and we have omitted the normalization constant. The label \( k_0 \) is interpreted as the number of \( \chi \) quanta in the wormhole. Their eigenvalues are given by

\[ E_0 = -k_{Gr} + 2k_{S_0} + \frac{1}{2} . \] (21)

Now we turn to the perturbations. To simplify the notation from now on we will stick to a certain mode and drop the index \( n \). We write the separation ansatz

\[ \Psi_2 = S \Psi(\Gamma, S)^V \Psi(R_e, R_o)^T \Psi(\mu_e, \mu_o) \] (22)

which automatically fulfills the two constraints (14). The diagonalization of the kinetic terms and the potentials in the quadratic Hamiltonians (15) is straightforward. For \( n > 2 \) we find the eigenmodes

\[ \mu_\pm = \frac{1}{e} \sqrt{\frac{n + \bar{\chi}}{2}} (\mu_e \pm \mu_o) \]

\[ R_\pm = \frac{1}{2e} \sqrt{\frac{n^2 - 4}{n + 2\bar{\chi}}} (R_e \pm R_o) \] (23)

\[ \Gamma_\pm = \frac{1}{e} \sqrt{\frac{n + 2}{6n(n \pm 1)}} ((n \mp 2)\Gamma \pm \bar{\chi}S) \]

corresponding to eigenvalues

\[ E_T = (k_{T_+} + \frac{1}{2})(n + \bar{\chi}) + (k_{T_-} + \frac{1}{2})(n - \bar{\chi}) \]

\[ E_V = (k_{V_+} + k_{V_-} + 1)n \] (24)

\[ E_S = (k_{S_+} + \frac{1}{2})(n + 1) + (k_{S_-} + \frac{1}{2})(n - 1) . \]

The wave function is given by harmonic oscillator solutions in the above eigenmodes. The Wheeler-De Witt equation (17) is satisfied if all energies sum up to zero :

\[ k_{Gr} = 2k_{S_0} + k_{T_+}(n + \bar{\chi}) + k_{T_-}(n - \bar{\chi}) + (k_{V_+} + k_{V_-})n + k_{S_+}(n + 1) + k_{S_-}(n - 1) . \] (25)

We have assumed that the ground state energy is canceled by a suitable renormalization.

Again the case \( n = 2 \) needs a special treatment : The coefficients \( r \) and \( s \) in eq. (16) are already chosen as eigenmodes of the Hamiltonian (16) with energies

\[ E_S = 3(k_S + \frac{1}{2}) , \quad E_V = 2(k_V + \frac{1}{2}) . \] (26)
In the following we will be interested in wormholes which contain quanta from the lowest mode in each the scalar-, the vector- and the tensor sector. For the scalar sector this lowest mode \(n = 1\) is just the symmetric field \(\chi\). The lowest vector- and tensor modes are the ones for \(n = 2\) and \(n = 3\), respectively. For later reference we collect some of the above results to write the properly normalized “lowest \(n\)” wave function:

\[
\Psi^{(Gr)}(a) = N(k_{Gr})(a) \exp \left( -\frac{1}{2}a^2 \right)
\]

\[
\Psi^{(YM)}_{\pm k_S k_V k_+ k_-}(\sigma, r, \mu_+, \mu_-) = NH_{k_S}(\sigma_{\pm})H_{k_V}(r)H_{k_+}(\mu_+)H_{k_-}(\mu_-)
\]

\[
\times \exp \left( -\frac{1}{2}(\sigma^2_{\pm} + r^2 + \mu^2_+ + \mu^2_-) \right)
\]

\[
N = N(k_S)N(k_V)N(k_+)N(k_-)
\]

with the normalization constant

\[
N(k) = \frac{1}{\pi^{1/4}} \frac{1}{\sqrt{2^k k!}}
\]

and \(k_{Gr}\) given by

\[
k_{Gr} = 2k_S + 2k_V + (3 + \bar{\chi})k_+ + (3 - \bar{\chi})k_-.
\]

### 4 Semiclassical Approximation

As a next step in the calculation of the wormhole vertex we have to determine the saddle point of the path integral in eq. (1). For this the Euclidean equations of motion subject to appropriate boundary conditions have to be solved. At conformal time \(\eta = 0\) the solutions should match the geometry and the particle content specified by the wormhole wave function whereas at \(\eta \to \infty\) flat empty space should be approximated. For the scale factor \(a\) and the lowest \(n\) YM modes these boundary conditions and the equations of motion are summarized in table 1. The solutions are

\[
a = a_0 \exp(\eta), \quad \chi_{\pm} = \pm 1 \frac{e}{\sqrt{6}} \frac{\sigma_0}{a^2_0} - \frac{a_0^2}{a^2}, \quad r = r_0 \frac{a_0^2}{a^2}, \quad \mu_{\pm} = \mu_0 \left( a_0 \right)^{3(3+\bar{\chi})}
\]
leading to an Euclidean action of

\[ S_{E,\pm} = \frac{1}{2} (a_0^2 + \sigma_{0\pm}^2 + r^2 + \mu_+^2 + \mu_-^2) . \]  

These solutions can be inserted into the harmonic expansion for the gauge field (34) and (35). For the vector modes we have a gauge freedom from the infinitesimal transformation (6). It allows us to set \( A_0 = 0 \) and to write \( A_n \) in the specific form

\[
A_{\pm,a}(y) = \left[ 1 \pm 1 \pm \frac{e}{\sqrt{6}} \sigma_{0\pm} \frac{a^2}{a^2} \right] T_a + \frac{1}{4} \epsilon_{pq} \frac{a^2}{a^2} \left[ (1 - \bar{\chi}) D_{\epsilon q}^{(110)}(\sigma(y)) + (1 + \bar{\chi}) D_{\epsilon q}^{(110)}(\sigma(y)) \right] \epsilon_{abc} T_b \\
+ e \left[ \frac{\mu_{0+}(pq)}{\sqrt{3 + \chi}} \left( \frac{a_0}{a} \right)^{3+\chi} D_{(ab)(pq)}^{(220)}(\sigma(y)) \right. \\
- \left. \frac{\mu_{0-}(pq)}{\sqrt{3 - \chi}} \left( \frac{a_0}{a} \right)^{3-\chi} D_{(ab)(pq)}^{(220)}(\sigma(y)) \right] T_b .
\]

Here we have used the formulae (A.13) and (A.14) from appendix A. The sphere \( S^3 \) is thought of being embedded in 4-dimensional flat space centered around \( x_0 \) and is parameterized by the four vector \( y : a^2 = (x_0 - y)^2 \). The correspondence between \( y \) and a group element \( g \in SU(2) \cong S^3 \) is fixed in appendix B.

Before we specify eq. (34) for the wormhole vertex we remark that the path integral has to be carried out over all insertion points \( x_0 \). This integration ensures momentum conservation of the result. Furthermore we note that all gauge fields which arise from eq. (33) by a \( SU_L(2) \times SU_R(2) \) rotation are solutions of the equation of motion with the same Euclidean action (32). Therefore an integration over all these configurations has to be performed. In order to remove the asymmetry between \( A_{+,a} \) and \( A_{-,a} \) which differ by the pure gauge background \( 2T_a \) we carry out the gauge transformation \( A_+ \to UA_+U^{-1} - dUU^{-1} \) with \( U = D^1(y) \) (\( D^1 \) is the spin 1 representation of \( SU(2) \)). In addition this transformation causes a change in the matrices \( T_a : T_a \to D_{ab}(y)T_b \). Now we can write

\[
< 0 \mid \text{tr}(A_{\mu_1}(y_1) \cdots A_{\mu_k}(y_k)) \pm k_S k_V k_+ k_- > = \\
\int d\sigma_0 d\sigma_0 dr_0 d\mu_0 d\mu_- \Psi^{(Gr)}(a_0) \Psi^{(YM)}_{\pm k_S k_V k_+ k_-} (\sigma_{0\pm}, r_0, \mu_0, \mu_-) \\
\times \int dx_0 d\mu(g_L) d\mu(g_R) \prod_{i=1,k} P_{a_i \mu_i}(x_0, y_i) D_{a_i b_i}^1(g_L^{-1}) \\
\times \text{tr}(A_{\pm, b_i}(gx_1) \cdots A_{\pm, b_k}(gx_k)) \exp(-S_{E,\pm}) ,
\]

with \( g = (g_L, g_R) \) and the wormhole wave function as given in (27), (28). The matrices \( P_{a \mu} \) rotate from a coordinate system specified by the one forms \( \omega^a \) on \( S^3 \) to Euclidean coordinates in the flat 4-dimensional embedding space and are specified in appendix B. Rotating by \( D^1(g_L^{-1}) \) is necessary because of the transformation property (A.1) of \( \omega^a \).
In the following we will evaluate this expression for the various modes. This allows us to determine the effective interactions $L_{\pm k_S k_V k^+ k^-}$ in the asymptotic flat space which satisfy

\[ <0|\text{tr}(A_{\mu_1}(y_1) \cdots A_{\mu_k}(y_k))| \pm k_S k_V k^+ k^- > \sim \int d^4 x_0 <0|\text{tr}(A_{\mu_1}(y_1) \cdots A_{\mu_k}(y_k))L_{\pm k_S k_V k^+ k^-}(x_0)|0 > . \] (35)

The gauge choice in eq. (33) guarantees that the path integral is performed over fields with $\partial_\mu A^b_\mu = \partial_\mu P_a A^b_a = 0$. The flat space amplitude on the RHS of eq. (35) should therefore be evaluated in a covariant gauge with a gauge fixing term $(\partial_\mu A^b_\mu)^2$. Clearly, the requirement $\partial_\mu A^b_\mu = 0$ does not eliminate the whole gauge freedom. For simplicity this remaining freedom has been chosen such that $A_0$ vanishes.

5. Wormhole Vertices from the Symmetric Mode $\chi$

An evaluation of eq. (34) for a wormhole state containing only quanta in the symmetric mode $\chi$ leads to

\[ <0|\text{tr}(A_{\mu_1}(y_1) \cdots A_{\mu_k}(y_k))| \pm k_S = k > = \]

\[ (\pm)^k C(k) \int d x_0 \text{tr}(T_{a_1} \cdots T_{a_k}) \prod_{i=1,k} \frac{P^\pm_{a_i\mu_i}(p_i)}{p_i^2} \] (36)

with $p_i = x_0 - y_i$ and the numerical constants $C(k)$ which result from the first integration in eq. (34) over the wormhole degrees of freedom. The matrices $P^\pm$ are defined in appendix B. We can concentrate on the case with equal number of quanta and insertion points ($k_S = k$). If $k_S > k$ the wormhole vertex vanishes by means of integration over the wormhole wave function. All quanta of the wormhole state have to be created or annihilated in the asymptotically flat region $[1]$. In the opposite case contractions in flat space appear which we are not interested in.

To calculate the integrand of eq. (36) for the 2- and 3-point function let us define the states

\[ |e/o, k_S > = \frac{1}{\sqrt{2}}(| + k_S > \pm |- k_S > ) . \] (37)

They are even ($e$) and odd ($o$) with respect to the parity $\hat{P}_2$ on $S^3$. Asymptotically this parity approaches the ordinary parity in flat space. With the help of eq. (B.3) and (B.6) we get

\[ <0|\text{tr}(A_\mu(x)A_\nu(y))|e/o, k_S = 2 > = \frac{e^2 V^2_{S^3}}{\sqrt{2}} C(2) \left\{ \begin{array}{ll} I_2(x, y) & (e) \\ 0 & (o) \end{array} \right\} \] (38)

\[ I_2(x, y) = \int d x_0 \frac{1}{p^2 q^2} (p.q \delta_{\mu \nu} - p_{\mu} q_{\nu}) \] (39)
and

\[ < 0 | \text{tr}(A_\mu(x)A_\nu(y)A_\rho(z)) | e/o, k_S = 3 > = \]

\[ \frac{e^2 V^2}{8} C(3) \int dx_0 \frac{1}{p^2 q^2 s^2} \times \left\{ \begin{array}{ll}
(p.s q_\mu - p.q s_\mu) & (e) \\
(\epsilon_{\mu\nu\rho\sigma} q_\nu q_\rho s_\sigma) & (o) \end{array} \right. . \] (40)

The effective interactions which reproduce these amplitudes are

\[ L_{e,k_S=2} \sim \text{tr}(F_{\mu\nu} F_{\mu\nu}) \]
\[ L_{e,k_S=3} \sim \text{tr}(F_{\mu\nu} F_{\nu\rho} F_{\rho\mu}) \] \hspace{1cm} (41)
\[ L_{o,k_S=3} \sim \text{tr}(F_{\mu\nu} F_{\nu\rho} \tilde{F}_{\rho\mu}) . \]

The parity even lowest scalar mode causes a renormalization of the gauge coupling \( e \). It shows the same behaviour as the lowest modes of electrodynamics [9]. From the 3-point function we find the two higher operators which can be written in terms of the field strength only. Their behaviour under parity reflects the parity of the corresponding wormhole wave function.

### 6 Effects of the Lowest Vector– and Tensor Mode

Finally, we are interested in the 2-point functions which result from the lowest vector mode \( r_0 \) and the lowest tensor modes \( \mu_0^+, \mu_0^- \).

The integration over \( SU_L(2) \times SU_R(2) \) in eq. (34) causes a contraction of the non-coset indices of the harmonics and the additional indices of the perturbations, e. g. leading to \( \sum_{q=1,2,3} r_{0q} r_{0q} \). Therefore, if the two quanta needed for the 2-point function are excited in different components of \( r_{0q} \) (or \( \mu_{0+q}, \mu_{0-q} \)) the wormhole vertex vanishes by means of orthogonality of the Hermitian polynomials. Taking both quanta in the same component we obtain for the vector mode

\[ < 0|\text{tr}(A_\mu(x)A_\nu(y))|e/o, k_V = 2 > = \frac{e^2}{\sqrt{2}} C(2) \left\{ \begin{array}{ll}
I_2(x, y) & (e) \\
0 & (o) \end{array} \right. , \] (42)

where we have defined eigenfunctions of the parity \( \hat{P}_2 \)

\[ |e/o, k_V > = \frac{1}{\sqrt{2}} (| + k_V > \pm | - k_V >) \] (43)

as we did in the scalar case. To arrive at this result we have used the explicit form of the harmonics [A.13] and appendix B. Again only the parity even modes contribute to the renormalization of the gauge coupling. For a different (but covariant) gauge choice as that fixed by eq. (33) we could have generated an additional contribution to the gauge fixing term \( (\partial_\mu A_\mu)^2 \), i. e. a renormalization of the gauge fixing parameter. The simplicity of this result is due to the fact that for \( \bar{\chi} = 1 \) (\( \bar{\chi} = -1 \)) only the pure right handed (left handed) vector harmonics appear in the expansion (33). If deviations of \( \bar{\chi} \) from ±1 are taken into account higher dimensional corrections to this result can be expected.
Even with out such generalizations the tensor modes show a less trivial behaviour. The states $|\pm, k_\pm = 2, k_\mp = 0 >$ will project the terms with $(x_0 - y)^{-4}$ out of the expansion and lead to higher derivative operators. The other states

$$|e/o, k_T = 2 > = \frac{1}{\sqrt{2}}(| + k_+ = 2 k_- = 0 > | - k_+ = 2 k_- = 0 >),$$

however, show the familiar behaviour

$$<0|\text{tr}(A_\mu(x)A_\nu(y))|e/o, k_T = 2 > = \frac{5e^2}{3\sqrt{2}}C(2) \begin{pmatrix} I_2(x, y) & (e) \\ 0 & (o) \end{pmatrix}.$$ 

which can be seen by using (A.15) and appendix B.

7 Conclusion

In this paper we have computed the wormhole wave function for $SO(3)$ YM theory in the case of a small gauge coupling. In that case we found it to be well approximated by harmonic oscillator wave functions.

This wave function has been used to determine the wormhole vertex and the corresponding flat space effective interaction for the lowest mode of each type of perturbation. We found the parity behaviour of these effective interactions to reflect the $S^3$ parity behaviour of the wave function. In particular, only the parity even wave functions generate a renormalization of the gauge coupling from the 2–point function. While this renormalization exhausts the effects from the lowest scalar and vector 2–point function the lowest tensor mode can also contribute to higher dimensional operators with additional derivatives. In addition, we calculated the 3–point function for the lowest scalar coefficient (the symmetric mode). We found the operators $\text{tr}(F^3)$ and $\text{tr}(F^2\tilde{F})$ from the parity even and parity odd wave function, respectively.

The minisuperspace wave function of the universe for $SO(3)$ YM theory as it has been proposed in ref. [11] shows a degeneracy in the two minima $\chi = \pm 1$ of the symmetric mode $\chi$ which results from parity invariance. This might give rise to an indeterminacy in fixing the constants of nature in the sense of Coleman. However, this fixing has to be performed in an effective theory taking into account all wormhole induced interactions. One can hope that due to the parity violation of the second operator above such a degeneracy can be lifted.

A departure from a small gauge coupling would complicate the situation substantially since gravity and YM expansion modes would be coupled. We have not addressed this situation here. Clearly, the wave function would be much less simpler in such a case. Concerning the wormhole vertex one can expect effective operators coupling gravity and YM with each other which might be an interesting aspect for future studies.
The work was partially supported by the Deutsche Forschungsgemeinschaft and the EC under contract no. SC1-CT92-0789 and the CEC Science Program no. SC1-CT91-0729.

Acknowledgement

Appendices

A Coset Geometry and Harmonic Expansion on $S^3$

In this appendix we will briefly develop the geometry of $S^3$ as a coset space and discuss harmonic expansion on this space. In particular the relation between the various types of harmonics used in the literature \[13, 14, 15\] will become transparent.

To follow the formalism of ref. \[13\] we identify $S^3$ with the coset $G/H$ where $G = SU_L(2) \times SU_R(2) = \{g = (g_L, g_R) | g_L, g_R \in SU(2)\}$ and $H = SU_D(2) = \{(h, h) | h \in SU(2)\}$. The representatives of the coset elements are fixed by the map $\sigma: (g_L, g_R) H \rightarrow (e, g_R g L^{-1})$. This definition immediately shows how to identify $S^3$ with $SU(2)$. We will use this identification in the following to denote a coset element represented by $(e, y)$ just as $y \in SU(2)$. An element $g = (g_L, g_R) \in G$ acts on the coset by left multiplication which we write as $\rho_g$ or sometimes simply as $g$. A change of the representatives induced by such a multiplication can be compensated by a function $F$ defined by $g \sigma(y) = \sigma(gy) F(g, h)$. For our particular choice of $\sigma$ we find $F(g, y) = (g_L, g_L)$.

The Lie Algebra of $G$ can be splitted into the Lie Algebra of $H$ spanned by $T_a = T_a^{(L)} + T_a^{(R)}$ and an orthogonal coset part spanned by $T_a^{(c)} = T_a^{(L)} - T_a^{(R)}$. Here $T_a^{(L)}$ and $T_a^{(R)}$ satisfy the usual commutation relations $[T_a^{(L)}, T_b^{(L)}] = \epsilon_{abc} T_c^{(L)}$ and are normalized by $\text{tr}(T_a^{(L,R)} T_b^{(L,R)}) = 2 \delta_{ab}$. An analogous decomposition can be applied to the Maurer-Cartan form of $G$ resulting in the one forms $L^a(g)$ and $L^{(c)a}(g)$ which are mapped to one forms $e^a(y) = \sigma^a L^a(\sigma(y))$, $e^{(c)a}(y) = \sigma^a L^{(c)a}(\sigma(y))$ on the coset. Their properties crucially depend on $\sigma$ and $F$. For the above choice we find that both types of forms are equal $\omega^a(y) := e^{(c)a}(y) = e^a(y)$ and satisfy the Maurer-Cartan equation $d \omega^a + e^a_{bc} \omega^b \wedge \omega^c = 0$. Under left multiplication $\omega^a$ transforms as

$$\rho_g^* \omega^a(y) = D^1(g_L)_{\mu}^{\nu} \omega^b(gy),$$

where $D^j$ denotes the spin $j$ representation of $SU(2)$ and $g = (g_L, g_R)$. Tensors on the coset will be given with respect to the forms $\omega^a$. Finally, the connection on $G/H$ is given by $\omega_{abc} = \epsilon_{abc}$.

In ref. \[12\] the correspondence between Lifshitz-harmonics \[14\] and the harmonics derived from the representation matrices of $G$ \[13\] has been established up to spin $J = 2$. For the scalar harmonics and the purely transverse vector and tensor harmonics it has been found that

$$Q^{(n)}_q(y) = n^2 D_0^{(E)}(\sigma(y))$$

for $n \leq 2$. The form of the arbitrary order harmonics $Q^{(n)}_q$ is not yet fully understood.
\[ S_{a_1 q}^{(n,e/o)}(y) = \sqrt{\frac{n^2 - 1}{6}} \left[ D_{a q}^{(1) \frac{n}{2}, \frac{n}{2} - 1}(\sigma(y)) \pm D_{a q}^{(1) \frac{n}{2} - 1, \frac{n}{2}}(\sigma(y)) \right] \]  (A.3)

\[ C_{a_1 a_2 | q}^{(n,e/o)}(y) = \sqrt{\frac{n^2 - 4}{10}} \sum_{q_1, q_2} P(a_1 a_2 | q_1 q_2)
\times \left[ D_{(q_1 q_2)q}^{(2) \frac{n-2}{2}, \frac{n-2}{2}}(\sigma(y)) \pm D_{(q_1 q_2)q}^{(2) \frac{n-2}{2} - 1, \frac{n-2}{2} - 1}(\sigma(y)) \right]. \]  (A.4)

where the projection operator \( P \)

\[ P(a_1 a_2 | q_1 q_2) \equiv \left[ \frac{1}{2} (\delta_{q_1 a_1} \delta_{q_2 a_2} + \delta_{q_2 a_1} \delta_{q_1 a_2}) - \frac{1}{3} \delta_{q_1 q_2} \delta_{a_1 a_2} \right]. \]  (A.5)

is needed to render the tensor harmonic \( G \) symmetric and traceless in its two coset indices. The matrices \( D_{m|q}^{(j_L,j_R)} \) are \((j_L, j_R)\) representations of \( G \) where the first index \( m \) only runs over the part corresponding to the \( H \) representation \((J) \subset (j_L, j_R)\). The harmonics \( S \) and \( G \) split into even \((e)\) and odd \((o)\) type with respect to the parity \( P_2 \) which corresponds to the ordinary flat space parity. It can be thought of as exchanging \( SU_L(2) \) and \( SU_R(2) \).

The other Lifshitz harmonics with longitudinal directions can be expressed as

\[ P_{a_1 | q}^{(n)}(y) = \frac{1}{n^2 - 1} \nabla_a Q_{q}^{(n)}(y) \]  (A.6)

\[ P_{a_1 a_2 | q}^{(n)}(y) = \frac{1}{n^2 - 1} \nabla_a \nabla_{a_2} Q_{q}^{(n)}(y) + \frac{1}{3} \delta_{a_1 a_2} Q_{q}^{(n)}(y) \]  (A.7)

\[ S_{a_1 a_2 | q}^{(n,e/o)}(y) = \frac{1}{2} \left[ \nabla_a S_{a_2 | q}^{(n,e/o)}(y) + \nabla_{a_2} S_{a_1 | q}^{(n,e/o)}(y) \right]. \]  (A.8)

To get explicit expressions for the matrices \( D^{(j_L,j_R)}(J) \) which fit to our conventions for the coset geometry we start with a \( G \) representation

\[ D_{(m|p)q}^{(j_L,j_R)}(g_L, g_R) = \left( D_{np}^{j_L}(g_L), D_{sq}^{j_R}(g_R) \right). \]  (A.9)

The spins \((j_L, j_R)\) have to be coupled in the first pair \((n,s)\) of indices. In addition we use the definition of the representatives fixed by the map \( \sigma \) and obtain

\[ D_{m(p|q)}^{(j_L,j_R)}(\sigma(y)) = \sum_{n,s} < Jm | j_L n j_R s > D_{np}^{j_L}(e), D_{sq}^{j_R}(y) \]  (A.10)

\[ = \sum_n < Jm | j_L p j_R n > D_{nq}^{j_R}(y). \]  (A.11)

In this particular form they correspond to the harmonics of ref. [14] which have been used by Dowker [3] to compute wormhole effects on electrodynamics.

Their transformation property under \( G \) can be determined from the above definitions:

\[ D_{m(p|q)}^{(j_L,j_R)}(\sigma(gy)) = D_{m(m'|p')}^{j_L}(g_L) D_{p(p')}^{j_R}(g_R) D_{q(q')}^{(j_L,j_R)}(g(g') \sigma(y)). \]  (A.12)
As a last step to make contact with Lifshitz harmonics one has to rotate back from the Cartan basis for $SU(2)$ representations to a “tensor” basis. We will do this explicitly for the lowest purely transverse vector and tensor harmonics

\[
S_{a|q}^{(2,e/o)}(y) = \frac{1}{\sqrt{2}}(D_{aq}^{(1|10)}(\sigma(y)) \pm D_{aq}^{(1|01)}(\sigma(y))) \tag{A.13}
\]

\[
G_{ab|pq}^{(3,e/o)}(y) = \frac{1}{\sqrt{2}}(D_{(ab)(pq)}^{(2|20)}(\sigma(y)) \pm D_{(ab)(pq)}^{(2|02)}(\sigma(y))) \tag{A.14}
\]

which we will need in our calculation. We find

\[
D_{ap}^{(1|10)}(y) = \delta_{ap}, \\
D_{ap}^{(1|01)}(y) = D_{ap}^1(y), \\
D_{(ab)(pq)}^{(2|02)}(y) = P(ab|pq) \\
D_{(ab)(pq)}^{(2|20)}(y) = \sum_{r,s} P(pq|rs)D_{ar}^1(y)D_{bs}^1(y) . \tag{A.15}
\]

**B  Rotation Matrices and Covariant Expressions**

In the following we will establish the relation between the coordinates of $S^3$ specified by the one forms $\omega^a$ and the Cartesian coordinates of the 4-dimensional embedding space.

A four vector $(p_\mu)$ is mapped to a group element of $SU(2) \approx S^3$ by

\[
(p_\mu) \rightarrow \frac{1}{p^2} \begin{pmatrix}
 p_1 + ip_3 & p_2 + ip_1 \\
 -p_2 + ip_1 & p_4 - ip_3
\end{pmatrix} . \tag{B.1}
\]

The forms $\omega^a$ can be calculated in a particular coordinate system $(\xi^\alpha)$ with $\omega^a = \omega_a^\alpha(\xi)d\xi^\alpha$. If $\Lambda_\mu^\alpha(p, \xi)$ denotes the transformation matrix from $(\xi^\alpha)$ to Cartesian coordinates the desired transformation relating vectors in the basis $\omega^a$ to four vectors is given by $P_\mu^a(p) = \Lambda_\mu^a(p, \xi)\omega_a^\alpha(\xi)$. The calculation has been carried out in ref. [3] and results in

\[
P_{ai}(p) = \frac{1}{p^2} (\epsilon_{ajip_j} + \delta_{ai}p_4) \\
P_{a4}(p) = -\frac{1}{p^2} p_a . \tag{B.2}
\]

The spin 1 representations of $SU(2)$ can be expressed in terms of $(p_\mu)$ using the identification \([B.1]\):

\[
D_{ab}^1(p) = \frac{1}{p^2} (2p_4^2 - p_2^2)\delta_{ab} + 2p_ap_b + 2p_4 \epsilon_{abc}p_c . \tag{B.3}
\]

This allows us to explicitly handle the harmonics of eq. \([A.13]\). With the definitions $P^- = P(p)$ and $P^+ = D^1(p)P(p)$ we obtain

\[
P_{ai}^\pm(p) = \frac{1}{p^2} (\mp \epsilon_{ajip_j} + \delta_{ai}p_4) \\
P_{a4}^\pm(p) = -\frac{1}{p^2} p_a . \tag{B.4}
\]
Contracting the coset indices of the matrices $P^\pm$ in an $SU(2)$ invariant manner should lead to covariant expressions. In particular

$$P^\pm_{a\mu}(p)P^\pm_{a\nu}(q) = \frac{1}{p^2q^2} (p.q\delta_{\mu\nu} - p_\nu q_\mu \mp \epsilon_{\mu\nu\rho\sigma} p_\rho q_\sigma)$$  \hspace{1cm} (B.5)

$$\epsilon_{abc}P^\pm_{a\mu}(p)P^\pm_{b\nu}(q)P^\pm_{c\sigma}(s) = \frac{2}{p^2q^2s^2} (\pm (p.sq_\mu - p.qs_\mu) + \epsilon_{\mu\nu\rho\sigma} p_\rho q_\nu s_\sigma) .$$  \hspace{1cm} (B.6)

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