Revisiting the quantum scalar field in spherically symmetric quantum gravity

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Abstract
We extend previous results in spherically symmetric gravitational systems coupled with a massless scalar field within the loop quantum gravity framework. As a starting point, we take the Schwarzschild spacetime. The results presented here rely on the uniform discretization method. We are able to minimize the associated discrete master constraint using a variational method. The trial state for the vacuum consists of a direct product of a Fock vacuum for the matter part and a Gaussian centered around the classical Schwarzschild solution. This paper follows the line of research presented by Gambini et al (2009 Class. Quantum Grav. 26 215011 (arXiv:0906.1774v1)) and a comparison between their result and the one given in this work is made.

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(Some figures may appear in colour only in the online journal)

Introduction

Loop quantum gravity (LQG) is a promising proposal for the canonical quantization of general relativity [2]. It provides a non-perturbative and mathematically rigorous description of the kinematical sector. Nevertheless, as is well known, the treatment of the dynamics of the theory remains an open problem. The main difficulty of this task is due to the fact that the constraints in general relativity do not satisfy a Lie algebra.

Over the last few years, there have been advances in the understanding of the quantization of spherically symmetric gravity coupled to a massless scalar field. The polymer quantization of both the gravitational and matter sectors was studied in [3, 4] and semiclassical states in this quantization scheme were proposed in [5] for the Schwarzschild case and in [6] in a cosmological context. Furthermore, the spherically symmetric gravity was studied also using
the uniform discretizations technique, within the loop representation, both in the vacuum and in the case coupled with matter [1, 7]. More specifically, Gambini et al employed the uniform discretization method using polymer quantization to deal with the fact that the Hamiltonian constraint satisfies a non-Lie Poisson bracket with itself (after a gauge fixing of the diffeomorphism constraint).

The uniform discretization technique [8–10] provides a method to quantize totally constrained systems. It is as the ‘master constraint’ program [11], an approach to find an alternative for the Dirac quantization used in LQG. As in this program, in the uniform discretization framework states are chosen such that they are in the kernel of the so-called master constraint.

In [1], the master constraint operator is constructed using polymer and Fock quantization for the gravitational and matter sectors, respectively. Subsequently, a trial state for the vacuum state is proposed and the expectation value of the aforementioned operator is computed. More specifically, the scalar field starting from the Minkowski spacetime is studied. The authors performed a minimization of the master constraint through a suitable variational method. In this way, one can check whether the theory provides us with the vacuum state of the scalar field centered around a Minkowskian background. In this paper, we extend that work to the Schwarzschild case. Although it is a modest objective, our motivation is twofold. On the one hand, we want to test the methodology presented by Gambini et al in a less trivial case and, on the other hand, we want to compare the final result for the Schwarzschild spacetime with the one in Minkowski. As we are going to show in the paper, although in this case the procedure becomes more involved than the previous one, the final result is, at the leading order, independent of the Schwarzschild radius.

This paper is organized as follows. In section 1, we review the uniform discretization method, that will be the arena for our study. In section 2, we summarize the classical treatment for spherical symmetric gravity coupled with a massless scalar field. More concretely, we will write explicitly the Hamiltonian constraint for this system. In section 3, we construct the trial state (composed of a gravitational part quantized polymerically and the Fock vacuum for the matter part) that we will use later in section 4 to evaluate and minimize the discrete master constraint. In order to make the main text and ideas more clear, we have gathered most of the technical calculations in the appendix and we added the corresponding references where necessary. Of special importance is appendix B, where we comment the solutions of the Sturm–Liouville equation that plays a central role for the Fock quantization of the matter part.

1. Uniform discretizations

The uniform discretization technique [8–10] is a method to quantize totally constrained systems. It is, as the master constraint program [11], an approach to find an alternative for the Dirac quantization used in canonical LQG. Although LQG showed remarkable results, especially concerning microscopic pictures of the early universe and black holes, the dynamics of the full theory still faces problems. One of those is that the quantum constraint algebra fails to portray the classical algebra exactly. Among the other options under research is the uniform discretization method.

The quantization of discretized, totally constrained systems faces the problem that, in general, the constraint algebra fails to close, i.e. the systems turn out to be second class even if the continuum theory is first class. If one now uses the Dirac method to quantize the discrete theory, the degrees of freedom will not match one of the continua. Furthermore, in general,
the symmetries of the theory are not recovered and taking the continuum limit one will not find the quantized version of the theory that one started with.

If one has first class systems, the uniform discretization technique is akin to the Dirac procedure, so we concentrate on second class systems here.

Starting with the study of a classical, totally constrained system with \( N \) configuration variables and \( M \) constraints \( \phi_j \), we consider the discretization of the constraints with a parameter \( \epsilon \), such that \( \phi_j = \lim_{\epsilon \to 0} \phi_j^\epsilon \). As mentioned beforehand, we assume the constraints to be second class in the discrete theory:

\[
\{ \phi_j^\epsilon, \phi_k^\epsilon \} = C^{\epsilon m}_{jk} \phi_m^\epsilon + A^\epsilon_{jk}.
\]

In order to get first class constraints in the continuum theory, we impose \( \lim_{\epsilon \to 0} A^\epsilon_{jk} = 0 \), and for the structure functions of the continuum theory \( C^m_{jk} = \lim_{\epsilon \to 0} C^{\epsilon m}_{jk} \).

We now construct a discrete ‘master constraint’

\[
\mathcal{H}^\epsilon = \frac{1}{2} \sum_j (\phi_j^\epsilon)^2.
\]

We proceed by introducing the time evolution for a discrete variable \( A_n \) as

\[
A_{n+1} = e^{\frac{\epsilon}{\mathcal{H}^\epsilon}} A_n \equiv A_n + \{ A_n, \mathcal{H}^\epsilon \} + \frac{1}{2} \{ \{ A_n, \mathcal{H}^\epsilon \}, \mathcal{H}^\epsilon \} + \cdots.
\]

It is obvious that \( \mathcal{H}^\epsilon \) is a constant of motion under this time evolution. We fix its value to \( \mathcal{H}^\epsilon = \frac{\delta^2}{2} \) and define the quantities \( \lambda^\epsilon_i := \phi_i^\epsilon / \delta \). Here, \( \delta \) could be interpreted as the size of the discrete time step of evolution (3). With these definitions, the evolution of the constraints is given by

\[
\phi_j^\epsilon (n+1) = \phi_j^\epsilon (n) + C^{\epsilon m}_{jk} \phi_m^\epsilon \lambda^\epsilon_k \delta + A^\epsilon_{jk} \lambda^\epsilon_k \delta + O(\delta^2).
\]

Taking the limits \( \delta \to 0 \) and \( \epsilon \to 0 \), we recover the evolution equations for the constraints in the continuum theory:

\[
\dot{\phi}_j = \lim_{\epsilon, \delta \to 0} \frac{\phi_j^\epsilon (n+1) - \phi_j^\epsilon (n)}{\delta} = C^{m}_{jk} \phi_m^\epsilon \lambda^\epsilon_k.
\]

where \( \lambda^\epsilon_i = \lim_{\epsilon, \delta \to 0} \lambda_i^\epsilon \) are Lagrange multipliers. Let us remark that although at first sight the \( \lambda^\epsilon_i \) seem not to be free in the discrete theory, they are only determined by the constraints evaluated on the initial data, and thus can be chosen arbitrarily.

Applying the Dirac procedure to the discrete theory would lead us to different degrees of freedom in the continuum theory, due to the fact that the constraints in this case are second class. This can be shown in the following way. Within the Dirac procedure, one would impose the constraints \( \phi_j^\epsilon \) and construct the total Hamiltonian \( H_T = C^\epsilon \phi_j^\epsilon \) with a number \( M \) of Lagrange multipliers \( C_j \). Consistency together with the discrete constraint algebra (1) would require

\[
C^\epsilon A^\epsilon_{jk} = 0.
\]

Therefore, the \( C_j \) would generally not be free but restricted. This implies that one would have more observables than in the continuum theory, i.e. more than \( 2N - 2M \), so the degrees of freedom of the discretized theory would not match the ones of the continuum theory. Thus, the uniform discretization method is better suited for studying a discretized, totally constrained system.

The uniform discretization approach provides a method to handle the fact that the constraints become second class through discretization, without changing the number of degrees of freedom. Moreover, the discrete constraint algebra mimics one of the continua.

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4 In this context, \( \epsilon \) is an abstract discretization parameter, which could, for example, be a spatial lattice separation.

5 In this expression and in the rest of the section, we will consider the sum over repeated indices, with independence of their raised or lowered positions.
2. Spherically symmetric gravity coupled to a scalar field

In this section, we review the treatment of the spherically symmetric gravity in Ashtekar variables. This symmetry reduction was first performed by Bengtsson [12]. Later on, the definition of the invariant connection components was slightly changed to ensure correct transformation properties [13]. We will use the invariant connection formulation based on the deep investigation on the symmetry reduction of connections in [14]. Using the invariant formulation of the basic variables, we find the constraints in the spherical symmetric framework. Furthermore, we will couple a spherical symmetric massless scalar field to gravity, invariant formulation of the basic variables, we find the constraints in the spherical symmetric framework. Since the spherical symmetric theory is manifestly gauge invariant, we are left with the Hamiltonian constraint. The variational method we are using is based on the uniform discretization method and thus we end this section with the discretization of the Hamiltonian constraint.

In order to describe spherically symmetric spacetimes, we use spherical coordinates \((r, \varphi, \theta)\) on a spatial manifold with topology \(\Sigma = \mathbb{R}^+ \times S^2\). The invariant connection, densitized triad and extrinsic curvature can be written as

\[
A = A_r \Lambda_3 dr + (A_1 \Lambda_1 + A_2 \Lambda_2) \, d\theta + ((A_1 \Lambda_2 - A_2 \Lambda_1) \sin \theta + \Lambda_3 \cos \theta) \, d\varphi,
\]

\[
E = E^r \sin \theta \Lambda_3 \, dr + (E^1 \Lambda_1 + E^2 \Lambda_2) \sin \theta \, d\theta + (E^1 \Lambda_2 - E^2 \Lambda_1) \, d\varphi,
\]

\[
K = K_r \Lambda_3 dr + (K_1 \Lambda_1 + K_2 \Lambda_2) \, d\theta + (K_1 \Lambda_2 - K_2 \Lambda_1) \sin \theta \, d\varphi,
\]

where \(A_i, E^i, K_i \ (i = r, 1, 2)\) are arbitrary functions on \(\mathbb{R}^+\) and \(\Lambda_i\) are generators\(^6\) of \(su(2)\) [14, 15].

In order to obtain canonically conjugate variables and suitable fall off conditions, we use the extended ADM phase space variables [15, 16]. For convenience, we introduce the following change of variables:

\[
K_\varphi(r) = \sqrt{K_1(r)^2 + K_2(r)^2}, \quad E^\varphi(r) = \sqrt{E^1(r)^2 + E^2(r)^2},
\]

(10)

that implies the Poisson relations

\[
\{K_\varphi(x), E^\varphi(x')\} = 2G\delta(x - x'),
\]

(11)

\[
\{K_\varphi(x), E^\varphi(x')\} = G\delta(x - x').
\]

(12)

In terms of the canonical conjugated variables \(\{K_\varphi, E^\varphi, K_\varphi, E^\varphi\}\), we can write the constraints as

\[
\mathcal{G} = 0,
\]

(13)

\[
GC = K'_\varphi E^\varphi - \frac{1}{2} K_\varphi (E')',
\]

(14)

\[
GH = \frac{\sqrt{E'}(E')'}{2E^\varphi} - \frac{\sqrt{E'}(E')'(E^\varphi)'}{2E^\varphi} - \sqrt{E'}K_\varphi K_\varphi + \frac{(E')^2}{8E^\varphi \sqrt{E'}} - \frac{E^\varphi K_\varphi^2}{2\sqrt{E'}} - \frac{E^\varphi}{2\sqrt{E'}}.
\]

(15)

where \(\mathcal{G}, C\) and \(H\) stand for the Gauß, diffeomorphism and Hamiltonian constraints, respectively, \(G\) is the Newton constant and \(f'\) denotes the derivative of the function \(f\) with respect to the radial coordinate. Note that the variables are manifestly gauge invariant.

Now we add the constraints corresponding to the matter content (the massless scalar field \(\phi\)) [17]:

\[
C_{\text{mat}} = P^\phi \phi',
\]

(16)

\(^6\) \(\Lambda_J = -i\sigma_J/2\), where \(\sigma_J\) denotes a rigid rotation of the Pauli matrices.
\[ \hat{H}_{\text{mat}} = \frac{1}{2E^\nu \sqrt{|E'|}} \left((E')^2 (\phi')^2 + (P^\phi)^2\right). \] (17)

Then, the Hamiltonian and diffeomorphism constraint of the reduced theory minimally coupled to the massless scalar field are
\[ H = \frac{1}{G} \left[ -\frac{E^\nu}{2\sqrt{|E'|}} K_\nu \sqrt{|E'|} K_\nu - \frac{E^\nu K^2_\nu}{2\sqrt{|E'|}} + \frac{|E'|}{8|E'| E^\nu} - \frac{\sqrt{|E'| |E'|'} (E')'}{2(E^\nu)^2} \right] + \frac{(P^\phi)^2}{2\sqrt{|E'| E^\nu}} + \frac{|E'|^3 (\phi')^2}{2E^\nu}. \] (18)

\[ C = \frac{1}{G} \left( \frac{E^\nu}{2} (K_\nu') K_\nu \right) + P_\nu \phi'. \] (19)

Now we proceed by gauge fixing \( E' = r^2 = (x + a)^2 \) as in [7], where \( x \) is a radial coordinate and \( a \) is a constant that we will relate with the Schwarzschild radius later on. Then, we solve the diffeomorphism constraint, which leads to
\[ K_\nu = \frac{E^\nu (K_\nu')}{r} + G P_\nu \phi'. \] (20)

Rescaling the lapse function \( N \to NG2r/E^\nu \), one obtains for the Hamiltonian constraint
\[ H = H_{\text{vac}} + GH_{\text{mat}}, \] (21)

\[ H_{\text{vac}} = \left(-x - (x + a)K^2_\nu + \frac{(x + a)^3}{(E^\nu)^2}\right)'. \] (22)

\[ H_{\text{mat}} = \frac{P^2_\nu}{(E^\nu)^2} + \frac{(x + a)^4 (\phi')^2}{(E^\nu)^2} - 2\frac{(x + a)K_\nu P_\nu \phi'}{E^\nu}. \] (23)

Now we proceed to discretize and polymerize the Hamiltonian constraint. We consider
\[ r \to r(i), \quad H \to \frac{H(i)}{\epsilon}, \quad \phi(r) \to \phi(i), \quad P^\phi \to \frac{P^\phi(i)}{\epsilon}, \]
\[ E^\nu \to \frac{E^\nu(i)}{\epsilon}, \quad K_\nu \to \frac{\sin(\rho K_\nu(i))}{\rho}, \quad f' \to \frac{f(i+1) - f(i)}{\epsilon}. \]

For the sake of simplicity, we will use a constant parameter \( \rho \) for the polymerization, like in the early loop quantum cosmology (LQC).

Finally, the expression for the discretized Hamiltonian constraint is
\[ H(i) = -\frac{1}{2\Lambda} \epsilon + r(i+1) + \frac{\sin^2(\rho K_\nu(i+1))}{\rho^2} + r(i) + \frac{\sin^2(\rho K_\nu(i))}{\rho^2} + \frac{r(i+1)^3 \epsilon^2}{E^\nu(i+1)^2} \]
\[ - \frac{r(i)^3 \epsilon^2}{E^\nu(i)^2} + \ell_p^2 \left[ \frac{P^\phi(i)^2}{E^\nu(i)^2} + \epsilon \frac{r(i)^4}{E^\nu(i)^2} \frac{P^\phi(i)^2}{E^\nu(i)^2} ight] \]
\[ - 2\frac{r(i)^4 P^\phi(i)^2}{E^\nu(i)^2} \frac{\sin(\rho K_\nu(i))}{\rho} \left( \phi(i+1) - \phi(i) \right) - \epsilon \rho_{\text{vac}} \] (24)
will choose our vacuum to be the one defined in the asymptotic region of the Schwarzschild spacetime) and computing the 'energy of the vacuum', i.e. the expectation value of the matter Hamiltonian. We find that the vacuum energy at the order of interest is
\[ \rho_{\text{vac}} = \frac{\pi}{2\epsilon^2} \].
As pointed out in [1], the introduction of the cosmological constant has some disadvantages. Introducing a cosmological constant before the symmetry reduction implies ending up with a non-constant term in two dimensions, which cannot give rise to the vacuum energy. Because of this, the theory does not stem from a dimensional reduction of the full theory if one introduces the cosmological constant in the way we did. To circumvent this problem, one can use the fact that, unlike in the full four-dimensional theory, there is already a constant term in the Hamiltonian constraint. The cosmological constant at that level can be interpreted as a rescaling of the radial coordinate. However, this rescaling has the disadvantage that the volume of spheres is not \( 4\pi R^2 \) anymore, i.e. the full theory one approximates has topological defects.

3. Construction of quantum trial states

Although the discrete Hamiltonian constraint (24) fails to close a first class algebra, it can be shown that using the uniform discretization approach one can consistently treat the problem by minimizing the resulting Hamiltonian constraint. In order to achieve that, we use the variational technique described in [1].

We have the opportunity to consider the gravitational sector and the matter sector separately, since classically the scalar field vanishes in the vacuum. We thus construct a polymeric Hilbert space \( \mathcal{H}_{\text{poly}} \) for the gravitational variables. The gravitational trial state \( |\Psi_1\rangle \) is then constructed as a Gaussian in phase space with width \( \sigma \) centered around the Schwarzschild solution. With the help of this trial state, we find an effective Hamiltonian for the matter variables. Out of the equations of motion for this Hamiltonian, we construct solutions based on creation and annihilation operators. Thus, we introduce the Fock vacuum state \( |0\rangle \) for the matter sector. In this way, we make contact with the usual treatments of quantum field theory in curved spacetimes.

The trial state used for the minimization of the master constraint is then constructed as a direct product of the gravitational Gaussian state and the Fock vacuum
\[ |\Psi_{\text{trial}}^\text{trial}\rangle = |\Psi_1\rangle \otimes |0\rangle. \] (25)

3.1. Gravitational part of the trial states

In the usual LQC treatment one uses cylindrical functions, which are related with the connection variables through the holonomies (see e.g. [2, 18]). Here, we mimic this method for the spherical symmetric theory. First, we set up a polymeric Hilbert space \( \mathcal{H}_{\text{poly}} \) with a spin network like basis and define the action of the operators \( E^v \) and \( K_\psi \) on this state. We will then proceed by defining the gravitational part of our trial states centered around the classical Schwarzschild solution.

In order to use the uniform discretization method, we consider a discretized setting. The symmetry-reduced spatial manifold in the spherically symmetric case is a dimensional line (the radial direction). In analogy with LQC, we consider the bulk Hilbert space for the gravitational sector as
\[ \mathcal{H}_{\text{poly}} = L^2(\otimes \bar{R}_{\text{Bohr}}, \otimes d\mu_0), \] (26)
where \( \bar{R}_{\text{Bohr}} \) is the Bohr compactification of the real line, \( d\mu_0 \) is the Haar measure and \( N \) is the number of cells in the discretization. In this framework, the basis for a fixed graph \( g \) composed of \( N \) edges \( e_j \) and vertices \( v_j \) (with \( j = 1, \ldots, N \)) is

\[
\langle K_\varphi(j) | \vec{\mu} \rangle = \prod_j \exp(i \mu_j K_\varphi(j)),
\]

with \( \mu_j \in \mathbb{R} \). The variables satisfy the classical Poisson bracket

\[
\{ K_\varphi(i), E_\varphi(j) \} = G \delta_{ij},
\]

which suggests defining

\[
\hat{E}_\varphi(i) = -\ell_p^2 \frac{\partial}{\partial K_\varphi(i)}.
\]

Its action over the elements \( |\vec{\mu}\rangle \) on the basis of the gravitational Hilbert space is defined by

\[
\hat{E}_\varphi(i) |\vec{\mu}\rangle = \ell_p^2 \mu_i |\vec{\mu}\rangle.
\]

We associate the \( K_\varphi \) with point holonomies, and their action is defined by

\[
\exp(i \rho \hat{K}_\varphi(i)) |\vec{\mu}\rangle = |\vec{\mu} + \rho \vec{e}_i\rangle.
\]

We want to remark that the Hilbert space we are using is a direct product of the Hilbert space of LQC for every lattice position \( i \), which enables us to use techniques developed in LQC for our purpose.

Finally, we construct the trial states as

\[
|\Psi_\varphi\rangle = \langle \vec{\mu} | \psi_\varphi \rangle = \prod_i \sqrt{\frac{2}{\pi \sigma(i)}} \exp \left( -\frac{1}{\sigma(i)} \left( \mu_i - \frac{r_1(i) \epsilon}{\ell_p^2} \right)^2 \right),
\]

with \( \mu_i \) centered at \( E_\varphi(i) = \ell_p^2 \mu_i = \epsilon r_1(i) \), where the value of \( r_1(i) \) will be determined in the following.

Since the classical counterpart of the vacuum solution corresponds to vanishing scalar fields, we can ignore the matter part of the Hamiltonian \( H_{\text{mat}} \) and focus on the gravitational part \( H_{\text{vac}} \). Demanding \( H_{\text{vac}} = 0 \) and additionally using the gauge \( K_\varphi = 0 \) one obtains

\[
\left( -x(1 - 2\Lambda) + \frac{(x + a)^3}{(E_\varphi)^2} \right)' = 0 \quad \Rightarrow \quad E_\varphi = \frac{1}{\sqrt{1 - 2\Lambda}} \frac{x + a}{\sqrt{1 - \frac{a - C}{x + a}}},
\]

where \( C \) is an integration constant. In order to recover the Schwarzschild metric for \( \Lambda = 0 \), we set \( C = 0 \). Then, we obtain

\[
r_1(i) = \frac{1}{\sqrt{1 - 2\Lambda}} \frac{r(i)}{\sqrt{1 - \frac{a - C}{r(i)}}}.
\]

We find that, in contrast to \([7]\), \( a \) is not the Schwarzschild radius \( R_S \) but rather

\[
a = R_S \sqrt{1 - 2\Lambda}.
\]

As was commented before, the introduction of the cosmological constant induces a rescaling of the radial coordinate.
3.2. Matter part of the trial states

In order to construct the matter part of the trial states, we define an effective Hamiltonian as the expectation value of the matter Hamiltonian constraint over the gravitational part of the trial states (32). We compute the equations of motion corresponding to this effective Hamiltonian and obtain a differential equation for the matter variables φ and $P^\phi$. Working in the Fourier space, this equation turns out to be a Sturm–Liouville problem. We can construct the associated creation and annihilation operators without knowing the explicit solution to the equation (making use of the tools of the Sturm–Liouville theory). It is thus straightforward to introduce the Fock-vacuum state, which completes the construction of the trial state. We will make extensive use of the results obtained in appendix A in this section.

Due to the usual factor ordering ambiguities in the quantization procedure, we need to choose one prescription for it. Here, we use the factor ordering of [1], namely putting the dependence on the value $\alpha$ symmetrically around $\hat{K}_\nu$ and $\phi$, respectively. In this way, the discretized matter Hamilton constraint operator can be written as

\[
\hat{H}_{\text{mat}}(i) = \epsilon (\hat{P}^\phi(i)^2 + r(i)^4 (\hat{\phi}(i+1) - \hat{\phi}(i))^2) \frac{1}{(\hat{E}^\nu(i))^2} - 2 \frac{r(i)}{\rho} \sqrt{\hat{P}^\phi} (\hat{\phi}(i+1) - \hat{\phi}(i)) \sqrt{\hat{P}^\phi} \frac{1}{\sqrt{\hat{E}^\nu}} \sin(\rho \hat{K}_\nu(i)) \frac{1}{\sqrt{\hat{E}^\nu}} - \epsilon \rho_{\text{vac}}.
\]

(34)

Using the expectation values (A.3) and (A.9), we obtain the effective Hamiltonian constraint

\[
\hat{H}^\text{eff}_{\text{mat}} = \langle \Psi_{\bar{\beta}} | \hat{H}_{\text{mat}} | \Psi_{\bar{\beta}} \rangle = ((\hat{P}^\phi)^2 + r^4(\hat{\phi})^2) f(r) - \rho_{\text{vac}},
\]

(35)

\[
f(r) = \frac{1}{r_1^2} + \frac{1}{r_1^4} \frac{\ell^4}{\epsilon^2} \alpha + O \left( \frac{\ell^4}{\epsilon^2} \right),
\]

\[
\alpha = \left( \frac{5m_\alpha}{24} \rho^2 + \frac{3}{4} \sigma \right), \quad m_\alpha \geq 2,
\]

where we went back to the continuum theory because it is easier to solve the corresponding differential equations than the original difference equations. As is explained in appendix A, the dependence on the value $m_\alpha$ shows an ambiguity which occurs due to the appearance of inverse operators.

The equations of motion for the effective Hamiltonian constraint (35) are

\[
\dot{\phi}(r,t) = \frac{\delta H^\text{eff}_{\text{mat}}}{\delta P^\phi} = 2f(r)P^\phi(r,t),
\]

(36)

\[
P^\phi(r,t) = -\frac{\delta H^\text{eff}_{\text{mat}}}{\delta \phi} = (2f(r)r^4\dot{\phi}(r,t')).
\]

(37)

Now, we consider the Fourier-like transformation

\[
\phi(r,t) = \int_0^\infty \frac{\phi(r,\omega)}{\sqrt{2\omega}} (e^{i\omega t} \tilde{C}(\omega) + e^{-i\omega t} C(\omega)) \, d\omega,
\]

(38)

\[
P^\phi(r,t) = \frac{1}{2f(r)} \frac{\partial \phi(r,t)}{\partial t} = \frac{i}{2f(r)} \int_0^\infty \sqrt{\frac{\omega}{2}} \phi(r,\omega) (e^{i\omega t} \tilde{C}(\omega) - e^{-i\omega t} C(\omega)) \, d\omega,
\]

(39)

where the transformation for the momentum follows from (36). This leads to the following Sturm–Liouville differential equation

\[
(B(r)\phi'(r,\omega))' + \omega^2 A(r)\phi(r,\omega) = 0,
\]

(40)
According to the Sturm–Liouville theory, solutions of such differential equations form a basis of the Hilbert space $L^2(\mathbb{R}, A(r)dr)$. This means that properly normalized functions $\phi(r, \omega)$ satisfy the orthogonality and closure relations:

$$
\int_0^\infty d\omega A(r) \phi(r, \omega) \phi(r', \omega) = \frac{1}{A(r)} \delta(r - r'),
$$

$$
\int_0^\infty \int_0^\infty d\omega \phi(r, \omega) \phi(r', \omega) = \delta(r - r').
$$

Consequently, the field and momentum operators are constructed as

$$
\hat{\phi}(r, t) = \int_0^\infty \frac{\phi(r, \omega)}{\sqrt{2\omega}} (e^{i\omega t} \hat{C}^\dagger(\omega) - e^{-i\omega t} \hat{C}(\omega)) d\omega,
$$

$$
\hat{P}\phi(r, t) = iA(r) \int_0^\infty \sqrt{\omega} \phi(r, \omega) (e^{i\omega t} \hat{C}^\dagger(\omega) - e^{-i\omega t} \hat{C}(\omega)) d\omega.
$$

Considering the commutation relation $[\hat{C}(\omega), \hat{C}^\dagger(\omega')] = \delta(\omega - \omega')$ and making use of the closure relation, we obtain the standard commutator for the field and its conjugate momentum operators:

$$
[\hat{\phi}(r, t), \hat{P}\phi(r', t)] = i\delta(r - r').
$$

As is inferred by the previous commutation relations, the operators $\hat{C}$ and $\hat{C}^\dagger$ act as annihilation and creation operators, respectively.

Finally, we construct the trial state as the direct product of the Gaussian state (32) and the Fock vacuum of the matter part:

$$
|\Psi_{\text{trial}}\rangle = |\Psi_{\sigma}\rangle \otimes |0\rangle,
$$

where $|0\rangle$ is the state annihilated by the annihilation operator $\hat{C}$. In this treatment, by construction, the case of entanglement between matter and gravitational variables is excluded, because the split between the gravitational and the matter sector works only with the vacuum state. For excited states, this requirement should be relaxed.

4. Expectation value of the master constraint

In this section, we use the trial state (46) to calculate the expectation value of the master constraint. First, we construct the discrete master constraint out of (24). We are able to write the principal order of the expectation value in orders of the lattice spacing. It shows that the master constraint is becoming small for bigger lattice spacings. Since the approximations made breakdown at a certain point, one concludes that there is a minimum of the master constraint. However, the minimum value is not reached in the limit $\epsilon \rightarrow 0$. This fact suggests that there is no continuum limit.

4.1. The discrete master constraint

Using the uniform discretization technique, we construct the master constraint (2) as

$$
\mathbb{H} = \frac{1}{2} \int dr \frac{H(r)^2}{r E^2},
$$
where we used our gauge and took into account that the Hamiltonian is a density of weight 1, so we added the determinant of the 3-metric in order to make contact with the continuum theory. Note that the master constraint is proportional to $1/E^\rho$. This implies that the vacuum state corresponds to the zero loop state ($\rho = 0$) which is degenerate in the polymer representation.

Nevertheless, the Hamiltonian constraint can be rescaled by a scalar function of the canonical variables without changing the first class nature of the constraint algebra, so we can solve this problem by rescaling $H(\chi) \rightarrow H(\chi)\sqrt{E^\rho/(E^\rho)}$. Then, in the discrete theory, we obtain

$$\mathbb{H}^i = \sum_j H(i) = \sum_j \frac{1}{4} \frac{H(i)^2}{\epsilon r(i)^2}.$$  (48)

For computational purposes, we proceed to separate ‘matter’ and ‘gravity’ operators acting on their corresponding Hilbert spaces, i.e. we write the Hamiltonian constraint (24) as

$$H(i) = H_{\text{mat}}^{(1)}(i) c_1(i) + H_{\text{mat}}^{(2)}(i) c_2(i) + H_{\text{mat}}^{(3)}(i) c_3(i) + c_4(i),$$  (49)

where

$$H_{\text{mat}}^{(1)}(i) = \ell_p^2 (\epsilon P^\rho(i))^2 + \epsilon r(i)^4 (\phi(i + 1) - \phi(i))^2,$$

$$H_{\text{mat}}^{(2)}(i) = \ell_p^2 (-2r(i)P^\rho(\phi(i + 1) - \phi(i))),$$

$$H_{\text{mat}}^{(3)}(i) = \ell_p^2 \rho_{\text{vac}}.$$  (50)

and

$$c_1(i) = \frac{1}{E^\rho(i)^2}, \quad c_2(i) = \frac{\sin(\rho K_\nu(i))}{E^\rho(i)\rho}, \quad c_3(i) = -1,$$

$$c_4(i) = -(1 - 2\Lambda)\epsilon - \left( r(i + 1) \frac{\sin^2(\rho K_\nu(i + 1))\rho^2}{\rho^2} - r(i) \frac{\sin^2(\rho K_\nu(i))\rho^2}{\rho^2} \right) + \frac{r(i + 1)^2\epsilon^2}{E^\rho(i + 1)^2} - \frac{r(i)^3\epsilon^2}{E^\rho(i)^2}. \quad (53)$$

In terms of these operators, we can write the expression for the discretized master constraint in the following form:

$$\mathbb{H}^i = \frac{1}{4\epsilon r(i)^2} \left( c_{11}(i) (H_{\text{mat}}^{(1)}(i))^2 + c_{22}(i) (H_{\text{mat}}^{(2)}(i))^2 + c_{33}(i) (H_{\text{mat}}^{(3)}(i))^2 \right.$$

$$\left. + 2c_{12}(i) H_{\text{mat}}^{(1)}(i) H_{\text{mat}}^{(2)}(i) + 2c_{13}(i) H_{\text{mat}}^{(1)}(i) H_{\text{mat}}^{(3)}(i) + 2c_{23}(i) H_{\text{mat}}^{(2)}(i) H_{\text{mat}}^{(3)}(i) \right)$$

$$+ 2c_{14}(i) H_{\text{mat}}^{(1)}(i) H_{\text{mat}}^{(4)}(i) + 2c_{24}(i) H_{\text{mat}}^{(2)}(i) H_{\text{mat}}^{(4)}(i) + 2c_{34}(i) H_{\text{mat}}^{(3)}(i) H_{\text{mat}}^{(4)}(i) + c_4(i). \quad (54)$$

where $c_{jk}(i) = c_j(i) \cdot c_k(i)$.

As the matter and the gravitational sector are not entangled, i.e. $\langle 0 | \hat{c}_j(i) | 0 \rangle = \hat{c}_j(i)$ and $\langle \Psi_\sigma | \hat{H}_{\text{mat}}^{(j)}(i) | \Psi_\sigma \rangle = \hat{H}_{\text{mat}}^{(j)}(i)$, we can accomplish the computation of the expectation values separately.

### 4.2. Minimizing the master constraint

In this part, we treat the problem of minimizing the master constraint. All the intermediate calculations are included in the appendices. For the sake of simplicity, we go back to the continuum theory and assume that $\sigma$ is independent of the lattice position, which means

$$r(i) \rightarrow r, \quad r(i + 1) \rightarrow r + \epsilon,$$

$$\sigma(i) \rightarrow \sigma, \quad \sigma(i + 1) \rightarrow \sigma.$$  (55)

Because of the independence of $\sigma$ with respect to the position, we also set $\sigma = \sigma_0 r^\rho$. 

---

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As was explained in [1], the approximation we considered in order to handle the expressions (neglecting higher powers of $\epsilon$) is inadequate for large values of $\epsilon$. More specifically, the approximation is valid up to values of $\epsilon \approx 10^{-23}\text{cm}$. In this range, it is convenient to sort the terms in orders of $O((\ell_p/\epsilon)^2 \epsilon)$. Then, the non-vanishing components of the master constraint (55) up to the principal order are

$$
\langle c_{11} (H^{(1)}_{\text{matter}})^2 \rangle = (3A(r)^4 I_1^2 + 3r^8 I_2^2 + 2A(r)r^4 (2I_1^2 + I_1 I_2)) \left[ -\frac{\ell^2 p^2}{r_1^2} + O\left(\frac{\ell^p}{\epsilon^2}\right) \right],
$$

$$
\langle c_{13} H^{(1)}_{\text{matter}} H^{(3)}_{\text{matter}} \rangle = (A(r)^2 I_1^2 + r^4 I_2) \left[ -\frac{\ell^2 p^2}{r_1^2} \Lambda + O\left(\frac{\ell^p}{\epsilon^2}\right) \right],
$$

$$
\langle c_{14} H^{(1)}_{\text{matter}} \rangle = (\Lambda(r)^2 I_1^2 + r^4 I_2) \left[ \left( \frac{1}{2p r_1^2} \right) \left( 1 - \exp\left( \frac{-2\sigma^2}{\sigma_0 \epsilon^n} \right) + \frac{3r^2}{r_1^2} \right) \ell^2 p^2 + O\left(\frac{\ell^p}{\epsilon^2}\right) \right].
$$

$$
\langle c_{22} (H^{(2)}_{\text{matter}})^2 \rangle = r^2 A(r)^2 (2I_1^2 + I_1 I_2) \left[ \frac{1}{2p^2} \left( 1 - \exp\left( \frac{-2\sigma^2}{\sigma_0 \epsilon^n} \right) \right) \ell^2 p^2 + O\left(\frac{\ell^p}{\epsilon^2}\right) \right],
$$

$$
\langle c_{33} (H^{(3)}_{\text{matter}})^2 \rangle = 4\Lambda^2 \epsilon^2,
$$

$$
\langle c_{43} H^{(3)}_{\text{matter}} H^{(4)}_{\text{matter}} \rangle = \frac{1}{2p^2} \left( 1 - \exp\left( \frac{-2\sigma^2}{\sigma_0 \epsilon^n} \right) \right) 2\Lambda^2 \epsilon^2 + O\left(\frac{\ell^p}{\epsilon^2}\right),
$$

$$
\langle c_{44} \rangle = \frac{r^2}{4p^4} \left( 1 - 2 \exp\left( \frac{-4\sigma^2}{\sigma_0 \epsilon^n} \right) + \exp\left( \frac{-8\sigma^2}{\sigma_0 \epsilon^n} \right) \right) + O\left(\frac{\ell^p}{\epsilon^2}\right),
$$

where $I_1, I_2$ and $I_3$ are integrals coming from the computation of the expectation value of the Hamiltonian matter constraint and are given by

$$
I_1 = \int_0^\infty d\omega \omega \phi(r, \omega)^2, \quad I_2 = \int_0^\infty d\omega \frac{1}{\omega} (\phi'(r, \omega))^2, \quad I_3 = \int_0^\infty d\omega \phi(r, \omega) \phi'(r, \omega),
$$

(63)

where $\phi(r, \omega)$ is the solution of the Sturm–Liouville equation (40). In appendix C, we give expressions (C.25) of these integrals for an approximate solution $\phi_0(r, \omega)$ of the zeroth order of the Sturm–Liouville equation (B.5).

Finally, the main order of the expectation value of the integrand of the master constraint takes the form

$$
\langle \Xi(r) \rangle = \frac{1}{\epsilon} \frac{1}{16p^4} \left( 1 - 2 \exp\left( \frac{-4\sigma^2}{\sigma_0 \epsilon^n} \right) + \exp\left( \frac{-8\sigma^2}{\sigma_0 \epsilon^n} \right) \right) + O\left(\frac{\ell^p}{\epsilon^2}\right).
$$

(64)

Note that the leading order term does not depend on the variable $a$. Also, since the equations are lengthy and they do not provide any further conclusions, we leave away higher order corrections.

We are now in conditions to study the minimum of this master constraint. In figure 1, we plot the expectation value (for values $\sigma_0 = 10, n = 2$) with respect to the lattice spacing $\epsilon$ in the region of interest. We observe that the master constraint drops very fast for lattice spacings larger than the Planck scale. As in the case studied in [1], because of the breakdown of the...
Figure 1. Expectation value of the integrand of the master constraint for $\sigma_0 = 10$, $n = 2$ and $\rho = 1$ as a function of the lattice spacing $\epsilon$ in the region near $\ell_p$.

approximation for $\epsilon > 10^{-23}\text{cm}$, we expect the master constraint to increase again. So, we can conclude that there is a minimum around $\epsilon \approx 10^{-23}\text{cm}$.

We find that varying the variable $a$, which is connected to the Schwarzschild radius of the classical solution where the trial states are centered, has influence on the master constraint only at the subleading orders. So we can conclude that, as in the Minkowski case studied in [1], we can construct the vacuum state for our minimally coupled system.

5. Conclusions

In this paper, we worked out an approximation for the vacuum state of a scalar field coupled to gravity with spherical symmetry. We focused on the Schwarzschild spacetime for the gravitational sector and employed the method presented in [1].

The vacuum for the coupled system is given by the direct product of the Fock vacuum state for the scalar field and a Gaussian centered around the classical Schwarzschild solution for the gravitational sector. In order to deal with the dynamics, the uniform discretization technique is used. This setting allows us to develop a minimization of the (discrete) master constraint of the system using a variational method. In order to accomplish this, we need to construct the Fock vacuum of the matter sector. We obtain this vacuum solving the Hamilton equations for an effective Hamiltonian, which turn out to be a Sturm–Liouville differential equation. It has to be pointed out that we did not manage to find an exact solution for this equation in the case we are interested in. Nevertheless, a suitable approximate solution is provided. Once the trial state is established, it is possible to perform the variational method in order to minimize the master constraint. Finally, we obtain an expression for the principal order of the expectation value of the master constraint with respect to the lattice spacing $\epsilon$ and the Planck length $\ell_p$.

We found that, at least at the main order, the expectation value of the master constraint does not depend on the classical Schwarzschild radius.

At this point, it is worth comparing our results with the ones of Gambini et al. First, we have, at the leading order in the expectation value of the master constraint, the same situation as in the cited paper, where the problem was worked out around a flat Minkowskian spacetime. As in that case, we also have a diverging master constraint in the continuum limit, but the theory gives a good approximation to general relativity for small values of the lattice
separation. The reason for that, clearly explained in [1], is due to the gauge fixing procedure employed to get rid of the diffeomorphism constraint at the classical level.

This work can be extended in several ways. For example, performing a polymeric quantization for the coupled system, or trying to work out this computation avoiding the gauge fixing of the diffeomorphism constraint. Nevertheless, these options would clearly rise the technical complexity of the problem. Anyway, in our opinion, this is a promising research line. A deeper understanding of this setting will provide us with a suitable vacuum state in this context, opening the possibility of performing a detailed study of the Hawking radiation within the LQG framework. We left this problem for future investigation.

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Appendix A. Computation of expectation values for basic operators

In this appendix, we give the expressions for several basic expectation values that are used along the paper.

First, we compute the expectation value of the operator $\hat{E}^\rho(i)$:

$$\langle \Psi_\vec{\mu} | \hat{E}^\rho(i) | \Psi_{\vec{\mu}} \rangle = \int_{-\infty}^{\infty} d\vec{\mu} \vec{\mu} c_\rho \prod_j \sqrt{\frac{2}{\pi \sigma(j)}} \exp \left( -\frac{2}{\sigma(j)} \left( \mu_j - \frac{r_1(j) e}{\ell_p^2} \right)^2 \right) = \epsilon r_1(i),$$

where $| \Psi_{\vec{\mu}} \rangle$ is the gravitational part of the trial state given by equation (32).

For the expectation value of trigonometric functions, we proceed in the same way, and find

$$\langle \Psi_\vec{\mu} | \cos(2\rho \hat{K}_\rho(i)) | \Psi_{\vec{\mu}} \rangle = \exp \left( -\frac{2\rho^2}{\sigma(i)} \right), \quad (A.1)$$

$$\langle \Psi_\vec{\mu} | \cos(4\rho \hat{K}_\rho(i)) | \Psi_{\vec{\mu}} \rangle = \exp \left( -\frac{8\rho^2}{\sigma(i)} \right), \quad (A.2)$$

$$\langle \Psi_\vec{\mu} | \frac{1}{\sqrt{\hat{E}^\rho(i)}} \sin(\rho \hat{K}_\rho(i)) \frac{1}{\sqrt{\hat{E}^\rho(i)}} | \Psi_{\vec{\mu}} \rangle = 0, \quad (A.3)$$

$$\langle \Psi_\vec{\mu} | \frac{1}{(\hat{E}^\rho(i))^{3/2}} \sin(3\rho \hat{K}_\rho(i)) \frac{1}{(\hat{E}^\rho(i))^{3/2}} | \Psi_{\vec{\mu}} \rangle = 0, \quad (A.4)$$

$$\langle \Psi_\vec{\mu} | \frac{1}{\sqrt{\hat{E}^\rho(i)}} \sin(3\rho \hat{K}_\rho(i)) \frac{1}{\sqrt{\hat{E}^\rho(i)}} | \Psi_{\vec{\mu}} \rangle = 0. \quad (A.5)$$

Now we use the inverse $(\hat{E}^\rho)^{-3/2}$ operator, following the prescription given for LQC in [19, 20]. We obtain for the eigenvalue of $| \vec{\mu} \rangle$:

$$(\hat{E}^\rho(i))^{-3/2} | \vec{\mu} \rangle = \epsilon_p^2 \left( \frac{2}{3\rho} \right)^6 \left( (\mu_i + \rho)^{3/4} - (\mu_i - \rho)^{3/4} \right)^6 | \vec{\mu} \rangle. \quad (A.6)$$
This can be used to calculate expectation values of operators involving the inverse of $\hat{E}^\rho$:

$$
\langle \Psi_\sigma | \frac{1}{(\hat{E}^\rho(i))^k} | \Psi_\sigma \rangle = \langle \Psi_\sigma | (\hat{E}^\rho(i))^{m-k} \left( \frac{1}{(\hat{E}^\rho(i))^{3/2}} \right)^{2m/3} | \Psi_\sigma \rangle,
$$

(A.7)

where $m \geq k > 0$ can be chosen arbitrarily, depending on the prescription taken for applying Thiemann’s trick. In fact, there is the same kind of ambiguity in LQC for the expression of inverse volume operators.

We now concentrate on lattice spacings $r \epsilon \gg \ell_p^2$. In this regime, we obtain the following approximation:

$$
\langle \Psi_\sigma | \frac{1}{(\hat{E}^\rho(i))^k} | \Psi_\sigma \rangle \approx \frac{1}{r^k} + \frac{k(k+1)}{8} \sigma(i) + \frac{5m}{24} \rho^2 \epsilon_p^k.
$$

(A.8)

As commented before, the dependence of (A.8) on $m$ shows an ambiguity which occurs due to the use of the inverse operators. More specifically, for the inverse operators needed in our case, we obtain

$$
\langle \Psi_\sigma | \frac{1}{(\hat{E}^\rho(i))^2} | \Psi_\sigma \rangle = \frac{1}{r^2} + \frac{3}{4} \sigma(i) + O \left( \frac{\ell_p^2}{r} \right),
$$

(A.9)

$$
\langle \Psi_\sigma | \frac{1}{(\hat{E}^\rho(i))^4} | \Psi_\sigma \rangle = \frac{1}{r^4} + \frac{3}{4} \sigma(i) + O \left( \frac{\ell_p^4}{r^2} \right),
$$

(A.10)

with

$$
\alpha(i) = \left( \frac{5m}{24} \rho^2 + \frac{3}{4} \sigma(i) \right), \quad \beta(i) = \left( \frac{5m}{24} \rho^2 + \frac{5}{2} \sigma(i) \right),
$$

where $m_a \geq 2$ and $m_\rho \geq 4$. Analogously one obtains

$$
\langle \Psi_\sigma | \frac{1}{\hat{E}^\rho(i)} \cos(2\rho \hat{K}_\rho(i)) \frac{1}{\hat{E}^\rho(i)} | \Psi_\sigma \rangle = \exp \left( -\frac{2\rho^2}{\sigma(i)} \right) \left( \frac{1}{r^2} + \frac{3}{4} \sigma(i) \right) + O \left( \frac{\ell_p^4}{r^2} \right),
$$

(A.11)

with

$$
\gamma(i) = \left( 1 + \frac{5m}{12} \right) \rho^2 + \frac{3}{4} \sigma(i), \quad m_\rho \geq 1.
$$

Appendix B. Solution to the Sturm–Liouville problem

We consider the Sturm–Liouville problem (40) for the leading order $\phi_0(r, \omega)$ in the expansion of the solution in powers of $\epsilon_p^3/r^2$:

$$
\left( 2L \left( 1 - \frac{a}{r} \right) r^2 \hat{\phi}_0(r, \omega) \right)' + \frac{\omega^2}{2L} \frac{r^2}{1 - \frac{a}{r}} \hat{\phi}_0(r, \omega) = 0,
$$

(B.1)

where $L = (1 - 2\Lambda)$. We were able to find the function

$$
\hat{\phi}_0(r, \omega) = \frac{1}{r} \sin \left( \frac{\omega}{2L} r^* \right),
$$

(B.2)

where

$$
r^* = r + a \log \left( \frac{r}{a} - 1 \right)
$$
Figure B1. Comparison of the approximate solution $\tilde{\phi}_0(r, \omega)$ (solid line) given by equation (B.2) with the numerical one (dashed line) for $\omega = L/a$. Note that they overlap completely, showing the quality of the approximation.

Figure B2. Comparison of the Minkowskian solution $\tilde{\phi}_m(r, \omega)$ (solid line) given by equation (B.4) with the numerical one (dashed line) for $\omega = L/a$. Note that $\tilde{\phi}_m(r, \omega)$ is not a suitable approximation.

is the usual tortoise coordinate of the external region of the Schwarzschild metric. The approximate solution (B.2) fulfils

$$\left(2L \left(1 - \frac{a}{r}\right) r^2 \tilde{\phi}_0(r, \omega)\right)^{\prime} + \frac{\omega^2}{2L} \frac{r^2}{1 - \frac{a}{r}} \tilde{\phi}_0(r, \omega) = -\frac{2a}{r} L \tilde{\phi}_0(r, \omega),$$

which for $r \gg a$ properly approximates the Sturm–Liouville equation (B.1). We can observe in figure B1 that indeed the approximation is very accurate (it overlaps completely the exact numerical solution). Another possible option could be motivated by the fact that in the regime $r \gg a$, equation (B.1) becomes the Sturm–Liouville problem of the Minkowski case, studied in [1], with the exact solution

$$\tilde{\phi}_m(r, \omega) = \frac{1}{r} \sin \left(\frac{\omega}{2L} r\right).$$

However, this Minkowskian solution does not succeed in approximating the exact numerical solution, as it is illustrated by figure B2. We will therefore work with $\tilde{\phi}_0(r, \omega)$. As mentioned before, solutions of (B.1) should fulfill the orthogonality relation (42). We can use this fact to normalize our solution, which is then given by

$$\tilde{\phi}_0(r, \omega) = \sqrt{\frac{2}{\pi}} \frac{1}{r} \sin \left(\frac{\omega}{2L} r\right).$$
Appendix C. Coefficient operators and expectation values

C.1. Coefficient operators

The operator form of the coefficients of the discrete master constraint (24) is

\[
\hat{c}_{11} = \frac{1}{E^\nu(i)^4},
\]

\[
\hat{c}_{12} = \frac{1}{E^\nu(i)^3} \left( \frac{\sin(\rho \hat{K}_\nu(i))}{\rho} - \frac{1}{E^\nu(i)^3/2} \right),
\]

\[
\hat{c}_{13} = -\frac{1}{E^\nu(i)^2},
\]

\[
\hat{c}_{14} = \left( r(i) \frac{1}{2E^\nu(i)} - r(i+1) \frac{1 - \cos(2\rho \hat{K}_\nu(i+1))}{2\rho^2} - (1 - 2\Lambda)\epsilon \right) \frac{1}{E^\nu(i)^2}
\]

\[
\hat{c}_{22} = \frac{1}{2\rho^2} \left( \frac{1}{E^\nu(i)^2} - \frac{1}{E^\nu(i)^3/2} \cos(2\rho \hat{K}_\nu(i)) \hat{K}_\nu(i) - \frac{1}{E^\nu(i)} \right),
\]

\[
\hat{c}_{23} = -\frac{1}{(E^\nu(i)^1/2)^2} \left( \frac{1}{E^\nu(i)^1/2} \frac{\sin(\rho \hat{K}_\nu(i))}{\rho} - \frac{1}{E^\nu(i)^1/2} \right),
\]

\[
\hat{c}_{33} = 1,
\]

\[
\hat{c}_{34} = r(i+1) \frac{1 - \cos(2\rho \hat{K}_\nu(i+1))}{2\rho^2} - r(i) \frac{1 - \cos(2\rho \hat{K}_\nu(i))}{2\rho^2}
\]

\[
\hat{c}_{44} = (1 - 2\Lambda)^2\epsilon^2 + 2(1 - 2\Lambda)\epsilon \left( r(i+1) \frac{1 - \cos(2\rho \hat{K}_\nu(i+1))}{2\rho^2} - r(i) \frac{1 - \cos(2\rho \hat{K}_\nu(i))}{2\rho^2} \right)
\]

\[
\quad + \frac{r(i+1)^3\epsilon^2 - 4\cos(2\rho \hat{K}_\nu(i+1)) + \cos(4\rho \hat{K}_\nu(i+1))}{8\rho^4}.
\]
The expectation value of (C.14) becomes

\[
-2r(i)r(i+1) \frac{1 - \cos(2\rho \hat{K}_\varphi(i))}{2\rho^2} \frac{1 - \cos(2\rho \hat{K}_\varphi(i+1))}{2\rho^2} + r(i)^2 \frac{3 - 4 \cos(2\rho \hat{K}_\varphi(i)) + \cos(4\rho \hat{K}_\varphi(i))}{8\rho^4} 
\]

\[
+ 2 \left( r(i) \frac{1 - \cos(2\rho \hat{K}_\varphi(i))}{2\rho^2} - \frac{r(i+1)}{2\rho^2} - (1 - 2\Lambda)\epsilon \right) \frac{r(i+1)^3 \epsilon^2}{E^\varphi(i+1)^2} 
\]

\[
+ 2 \left( r(i+1) \frac{1 - \cos(2\rho \hat{K}_\varphi(i+1))}{2\rho^2} - \frac{r(i)}{2\rho^2} + (1 - 2\Lambda)\epsilon \right) \frac{r(i)^3 \epsilon^2}{E^\varphi(i)^2} 
\]

\[
+ 2r(i)^4 \epsilon^2 \frac{1}{E^\varphi(i)} \frac{\cos(2\rho \hat{K}_\varphi(i))}{2\rho^2} \frac{1}{E^\varphi(i)} + \frac{r(i+1)^6 \epsilon^4}{E^\varphi(i)^4} 
\]

\[
- 2r(i)^2 \frac{r(i+1)^3 \epsilon^2}{E^\varphi(i)^2 E^\varphi(i+1)^2} + \frac{r(i)^6 \epsilon^4}{E^\varphi(i)^4}.
\]

(C.10)

Using (A.1)–(A.5) and (A.9)–(A.11), one can now calculate the expectation values of the coefficients $\langle \hat{c}_{ij} \rangle = \langle \hat{\psi}_d^{\text{null}} | \hat{c}_{ij} | \hat{\psi}_d^{\text{null}} \rangle = \langle \hat{\psi}_d | \hat{c}_{ij} | \hat{\psi}_s \rangle$. We remark that

\[
\langle \hat{c}_{12} \rangle = \langle \hat{c}_{23} \rangle = \langle \hat{c}_{24} \rangle = 0.
\]

(C.11)

C.2. Expectation values of the 'matter Hamiltonians'

Let us focus now on the expectation values of the expressions of the discrete master constraint which contain 'matter Hamiltonians'. Since coefficients (C.11) vanish, we only need to compute $\langle (\hat{H}_\text{mat}(i))^2 \rangle$, $\langle (\hat{H}_\text{mat}(i))^2 \rangle$, and $\langle \hat{H}_\text{mat}(i) \rangle$. The continuum version of (30) can be written as

\[
\hat{H}_\text{mat}^{(1)} = \epsilon^2 \hat{c}_\rho \hat{P}_\rho (r, t)^2 + r^4 \hat{\phi}'(r, t)^2.
\]

(C.12)

Using $\langle \hat{H}_\text{mat}^{(1)}(i) \rangle = \langle 0 | \hat{H}_\text{mat}^{(1)}(i) | 0 \rangle = \langle 0 | \hat{C}(\omega) \hat{C}^\dagger (\omega') | 0 \rangle = \delta(\omega - \omega')$ and that all the other combinations of the $\hat{C}$-operators vanish, one obtains

\[
\langle \hat{H}_\text{mat}^{(1)} \rangle = \frac{\epsilon^2 \hbar^2}{2} \left( A(r)^2 \int_0^\infty d\omega \omega (\tilde{\phi}_0(r, \omega))^2 + r^4 \int_0^\infty d\omega \frac{1}{\omega} (\tilde{\phi}_0'(r, \omega))^2 \right).
\]

(C.13)

For $\langle \hat{H}_\text{mat}^{(1)} \rangle$, with the given factor ordering, we obtain

\[
\langle \hat{H}_\text{mat}^{(1)} \rangle^2 = \epsilon^4 \hat{c}_\rho^2 \hat{P}_\rho (r, t)^4 + 2r^4 \hat{P}_\rho(r, t) \hat{\phi}'(r, t)^2 \hat{P}_\rho(r, t) + r^8 \hat{\phi}'(r, t)^4.
\]

(C.14)

The expectation value of (C.14) becomes

\[
\langle (\hat{H}_\text{mat}^{(1)})^2 \rangle = \frac{\epsilon^4 \hbar^2}{4} \left( 3A(r)^4 I_1^2 + 2A(r)^2 r^4 \left( 2I_2 + I_1 I_2 \right) + 3r^8 I_2^2 \right),
\]

where

\[
I_1 = \int_0^\infty d\omega \omega (\tilde{\phi}_0(r, \omega))^2, \quad I_2 = \int_0^\infty d\omega \frac{1}{\omega} (\tilde{\phi}_0'(r, \omega))^2, \quad I_3 = \int_0^\infty d\omega \hat{\phi}_0(r, \omega) \tilde{\phi}_0(r, \omega).
\]

(C.16)

Now, for $\langle \hat{H}_\text{mat}^{(2)} \rangle$, which following from (51) has the continuum limit

\[
\langle \hat{H}_\text{mat}^{(2)} \rangle = r^2 \epsilon^2 \hat{P}_\rho (r, \omega) \hat{\phi}'(r, \omega)^2 \hat{P}_\rho (r, \omega),
\]

(C.17)
we obtain
\[ \langle \hat{H}^{(2)}_{\text{mat}} \rangle^2 = r^2 A(r) r^2 e^{2 \epsilon} \ell_p^4 (2I_3 + I_2) . \]  
(C.18)

Summarizing and going back to the discrete theory, we obtain the following expectation values for the ‘matter Hamiltonians’:
\[ \langle \hat{H}^{(1)}_{\text{mat}}(i) \rangle = \frac{\ell_p^2}{2} e^3 (A(r(i))^2 I_1(i) + r(i)^4 I_2(i)) , \]  
(C.19)
\[ \langle \langle \hat{H}^{(1)}_{\text{mat}}(i) \rangle \rangle^2 = \frac{\ell_p^4}{4} e^6 (3A(r(i))^4 I_1(i)^2 + 3r(i)^8 I_2(i)^2 + 2A(r(i)) r(i)^4 (2I_3(i)^2 + I_1(i)I_2(i))) , \]  
(C.20)
\[ \langle \hat{H}^{(3)}_{\text{mat}}(i) \rangle = 2\Lambda \epsilon , \]  
(C.21)
\[ \langle \langle \hat{H}^{(3)}_{\text{mat}}(i) \rangle \rangle^2 = 4\Lambda^2 \epsilon^2 , \]  
(C.22)
\[ \langle \hat{H}^{(1)}_{\text{mat}}(i) \hat{H}^{(3)}_{\text{mat}}(i) \rangle = \Lambda \ell_p^4 e^4 (A(r(i))^2 I_1(i) + r(i)^4 I_2(i)) , \]  
(C.23)
where we used \( \rho_{\text{vac}} = 2\Lambda / \ell_p^2 \).

Now all we need to calculate to obtain all the components of the master constraint (55) is the three integrals (C.16). Using the approximate solution (B.5) and taking into account the discretization of the integral one obtains
\[ I_1 = \frac{2}{\pi} \frac{1}{r^2} \left( \frac{\pi^2}{e^2} + \frac{L^2}{2(r^*)^2} - \frac{\cos\left(\frac{2\pi}{e} r^*\right)}{2(r^*)^2} L^2 - \frac{\sin\left(\frac{2\pi}{e} r^*\right)}{e r^*} \pi L \right) , \]  
(C.25)
\[ I_2 = \frac{1}{\pi} \frac{1}{r^3} \left( \gamma - \text{Ci} \left( \frac{2\pi}{e} r^* \right) + \log \left( \frac{2\pi}{e} r^* \right) \right) - \frac{1}{\pi} \frac{1}{r^3} \frac{1}{1 - \frac{2}{r}} \frac{\sin^2\left(\frac{2\pi}{e} r^*\right)}{r^*} \frac{L^2}{2(r^*)^2} + \frac{\cos\left(\frac{2\pi}{e} r^*\right)}{2(r^*)^2} L^2 + \frac{\sin\left(\frac{2\pi}{e} r^*\right)}{e r^*} \pi L \right) , \]  
(C.26)
\[ I_3 = \frac{1}{\pi r^3} \left( \frac{1}{\pi} \frac{1}{r^2} \frac{1}{1 - \frac{2}{r}} \frac{\sin^2\left(\frac{2\pi}{e} r^*\right)}{2(r^*)^2} L^2 - \frac{\cos\left(\frac{2\pi}{e} r^*\right)}{e r^*} \pi L \right) - \frac{1}{\pi} \frac{1}{r^3} \left( \frac{\pi}{\epsilon} - \frac{\sin\left(\frac{2\pi}{e} r^*\right)}{2r^*} L \right) . \]  
(C.27)

where \( \text{Ci}(x) \) is the cosine integral function\(^7\) and \( \gamma \) is Euler’s constant.

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\(^7\) \text{Ci}(x) \equiv \gamma + \log x + \int_0^x \text{d}t (\cos t - 1) .
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