FRW-TYPE UNIVERSE WITH VACUUM ENERGY DENSITY

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We have considered a cosmological model with a cosmological constant of the form $\Lambda = 3\alpha \frac{\dot{R}^2}{R^2} + \beta \frac{\ddot{R}}{R}$, $\alpha$, $\beta$ = const. The cosmological constant is found to decrease as $t^{-2}$ and the rate of particle creation is smaller than the Steady State value. We have found that this behavior gives $\frac{\Delta \Lambda}{\Lambda_{Pl}} = 10^{120}$ where $\Lambda_{Pl}$ is the value of $\Lambda$ at Planck time. Solutions with $\beta = 3\alpha$ in the radiation dominated era and $\beta = 6\alpha$ in the matter dominated era are equivalent to the FRW results. We have found an inflationary solution of the de-Sitter type with $\beta = 3 - 3\alpha$. Some problems of the Standard Model may be resolved with the presence of the above cosmological constant in the Einstein’s equation. Since observations suggest a contribution of the vacuum energy density in the range $0.40 < \Omega_{\Lambda} < 0.76$, one gets $4 < \beta < 12$. If $\alpha = 0$ the minimum age of the universe is found to be $H_p^{-1}$ ($H_p$ is the present Hubble constant) with $\beta = \infty$.

KEY WORDS: Cosmology, Variable $\Lambda$ and $G$, Inflation

1. Introduction

Very recent results [1] suggest that the age of the universe is about $11 - 14$ billion years. However results obtained from Standard Model of cosmology are not consistent with these findings. Numerous models have been presented to reconcile these contradictions with observation. Only few of these models have given a satisfactory solution. The presence of a cosmological constant ($\Lambda$) in a given cosmology prolongs the age of the universe. The inflationary paradigm requires the Universe to have a critical density but observations do not support this. One of the motivations for introducing $\Lambda$ term is to reconcile the age parameter and the density parameter of the universe with current observational data [15, and references therein]. In an attempt to resolve these problems with current observations, we suggest a variation law for the cosmological constant of the form $\Lambda = 3\alpha \frac{\dot{R}^2}{R^2} + \beta \frac{\ddot{R}}{R}$, $\alpha$, $\beta$ = const. Observational data indicate that $\Lambda \sim 10^{-55}\text{cm}^{-2}$ while particle physics prediction for $\Lambda$ is greater than this value by a factor of order $10^{120}$. This discrepancy is known as the cosmological constant problem. It is interesting that this decay law helps to resolve this problem. The entropy problem that exists in the Standard Model can be solved by

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the proposed decay law. The success of the inflationary model in extending the Big Bang models
strengthen the theoretical prejudice toward a flat universe. When we allow $G$ to vary with time we
retain the conventional energy conservation law, i.e., the variation of $\Lambda$ is cancelled by the variation
of $G$. Though the proper way is to look for a field theoretic model for the variation of $G$ and $\Lambda$
nevertheless our present approach could be considered as a limiting case of some covariant theory
yet to be discovered. Brans-Dicke theory [14] is one type of these theories but this theory allows
only a decreasing $G$ at the present time. The variation of $G$ would have a significant effect on the
evolution of stellar objects. Thus one way to obtain a restriction on the way in which $G$ vary is
via its impact on the evolution of astrophysical objects.

A very recent discovery suggests that the universe is flat [12]. Some workers have
shown that the universe is accelerating. In section 6, we show that the universe must be accel-
erating if $\Lambda > 0$. This may be due to the fact that if gravity is increasing then the universe has
to increase its expansion rate to escape the future collapse. Or alternatively, the decayed vacuum
energy is given as a kinetic energy to accelerate the expansion of the universe. In this paper we will
consider a flat universe which preserves the spirit of the inflationary scenario. The present model
could resolve many of Standard Model problems with observation and thus could become a viable
candidate as an alternative model.

2. The model

In a Robertson Walker metric, the Einstein’s field equations with variable cosmological and gravi-
tational ‘constants’ and a perfect fluid yield [2]

\[ \frac{3}{2} \frac{\ddot{R}^2}{R^2} + \frac{3}{2} \frac{k}{R^2} = 8\pi G \rho + \Lambda , \]

\[ 2 \frac{\dot{R}}{R} + \frac{\ddot{R}}{R} + \frac{k}{R^2} = -8\pi G p + \Lambda , \]

where $\rho$ is the fluid energy density and $p$ its pressure. The equation of the state is usually given by

\[ p = (\gamma - 1) \rho , \]

where $\gamma$ is a constant. Elimination of $\ddot{R}$ gives

\[ 3(p + \rho) \dot{R} = - \left( \frac{\dot{G}}{G} \rho + \dot{\rho} + \frac{\dot{\Lambda}}{8\pi G} \right) R. \]

3. Particle creation

3.1 Matter-dominated universe (MDU)

For a pressure-less MDU ($p = 0$) and for a constant gravitational constant ($G$), eq.(4) reads

\[ \frac{d(\rho R^3)}{dt} = - \frac{R^3}{8\pi G} \frac{d\Lambda}{dt} . \]
In this paper we will consider a decay law of the form \[3,4\]

\[ \Lambda = 3\alpha \frac{\dot{R}^2}{R^2} + \beta \frac{\ddot{R}}{R} , \tag{6} \]

where \(\alpha\) and \(\beta\) are dimension-less constants. We suggest from the linear relationship between \(\Lambda\) and the Ricci scalar in the Einstein field equation that \(\Lambda\) has a general variation which resembles this scalar. For flat space one gets the above variation. For a flat universe \((k = 0)\), eqs.(2) and (6) yield

\[ (2 - \beta)\dddot{R}R = (3\alpha - 1)\dot{R}^2 , \tag{7} \]

which can be solved to give

\[ R(t) = \left[ \frac{A(3 - 3\alpha - \beta)}{(2 - \beta)} \right] t^{(2-\beta)/(3-3\alpha-\beta)} , \tag{8} \]

where \(A = \text{constant}\). Using eq.(8), eq.(6) becomes

\[ \Lambda(t) = \frac{(2 - \beta)(6\alpha - \beta)}{(3 - 3\alpha - \beta)^2} \frac{1}{t^2} . \tag{9} \]

From eqs.(1), (6) and (8) the energy density takes the form

\[ \rho(t) = \frac{(2 - \beta)}{4\pi G(3 - 3\alpha - \beta)} \frac{1}{t^2} . \tag{10} \]

The vacuum energy density \((\rho_v)\) is given by

\[ \rho_v(t) = \frac{\Lambda}{8\pi G} = \frac{(2 - \beta)(6\alpha - \beta)}{8\pi G(3 - 3\alpha - \beta)^2} \frac{1}{t^2} . \tag{11} \]

The deceleration parameter \((q)\) is defined as

\[ q = -\frac{\dddot{R}R}{\dot{R}^2} = \frac{1 - 3\alpha}{2 - \beta} , \beta \neq 2 . \tag{12} \]

The density parameter of the universe \((\Omega^m)\) is given by

\[ \Omega^m = \frac{\rho}{\rho_c} = \frac{2(3 - 3\alpha - \beta)}{3(2 - \beta)} , \beta \neq 2 , \tag{13} \]

where \(\rho_c = \frac{3H^2}{8\pi G}\) is the critical energy density of the universe and \(H = \frac{\dot{R}}{R}\) is the Hubble constant. We notice that the Standard Model formula \(\Omega^m = 2q\) is now replaced by \(\Omega^m = \frac{2}{3}q + \frac{2}{3}\). However, both models give \(q = \frac{1}{2}\) for a critical density. The density parameter due to vacuum contribution is defined as \(\Omega^\Lambda = \frac{\Lambda}{3H^2}\). Using eqs.(8) and (9) this yields

\[ \Omega^\Lambda = \frac{(6\alpha - \beta)}{3(2 - \beta)} , \beta \neq 2 . \tag{14} \]

We shall define \(\Omega_{\text{total}}\) as

\[ \Omega_{\text{total}} = \Omega^m + \Omega^\Lambda , \text{ and } \rho_{\text{total}} = \rho + \rho_v . \tag{15} \]
Hence eqs. (13), (14) and (15) give $\Omega_{\text{total}} = 1$. This situation is favored by the inflationary scenario. 

$$t_p = \frac{(2 - \beta)}{(3 - 3\alpha - \beta)} H^{-1}, \quad \Omega^m_p = \frac{2}{3} \frac{(3 - 3\alpha - \beta)}{(2 - \beta)}, \quad \Lambda_p = \frac{(6\alpha - \beta)}{(2 - \beta)} H^2, \quad \beta \neq 2, \quad (16)$$

(hereafter the subscript ‘p’ denotes the present value of the quantity). For ages larger than the Standard Model one requires $\beta < 6\alpha$ and for $t_p > 0$, $\beta < 2$ and $\alpha > 1/3$. This constraint indicates that $\Lambda$ is positive. The precise value of $\alpha$ and $\beta$ has to be determined from observational data.

We now turn to calculate the rate of particle creation (annihilation) $n$, which is defined as [5]

$$n = \frac{1}{R^3} \frac{d(\rho R^3)}{dt} |_{p}. \quad (17)$$

Using eqs. (5), (8), (9) and (16) one obtains

$$n_p = \frac{(6\alpha - \beta)}{(2 - \beta)} \rho_p H_p, \quad \beta \neq 2. \quad (18)$$

We remark that this rate is less than that of the Steady State model ($= 3\rho_0 H_p$). If $\beta = 6\alpha$ then $\Lambda = 0$, $n_p = 0$, $t_p = \frac{2}{3} H^{-1}$ and $\Omega^m_p = 1$. This case is equivalent to the Standard Model result.

We observe that when $\alpha = \frac{1}{3}$, $q = 0$ and $\beta$ decouples from all cosmological parameters.

3.2 Radiation-dominated universe (RDU)

This is characterized by the equation of the state $p = \frac{1}{3} \rho$ ($\gamma = 4/3$). In this case, for a flat universe ($k = 0$), eqs. (1), (2) and (6) yield

$$3(1 - 2\alpha) \frac{\dot{R}^2}{R^2} + (3 - 2\beta) \frac{\ddot{R}}{R} = 0. \quad (19)$$

This can be solved to give

$$R = \left[ \frac{(3 - 2\beta) D}{(3 - 3\alpha - \beta)} \right]^{(3-2\beta)/2(3-3\alpha-\beta)} t, \quad D = \text{const.} \quad (20)$$

For an expanding universe $\beta < 3/2$ and $\alpha > 1/2$. Eqs. (1), (4), (6) and (20) give

$$\Lambda = \frac{3(3-2\beta)(3\alpha-\beta)}{4(3-3\alpha-\beta)^2} \frac{1}{t^2} = \frac{3(3\alpha-\beta)}{(3-2\beta)} H^2, \quad (21)$$

$$\rho = \frac{3}{32\pi G (3-3\alpha-\beta)} \frac{1}{t^2}, \quad (22)$$

Thus an expanding universe ($\beta < 3/2, \alpha > 1/2$) demands that the cosmological constant to be positive. It is evident that if $\beta = 3\alpha$ the FRW results will be recovered viz. $R \propto t^{1/2}$, $\rho \propto t^{-2}$, $\Lambda = 0$.

4. No particle creation

We now consider a model in which both $G$ and $\Lambda$ vary with time in such a way the usual energy conservation law holds.
4.1 Matter-dominated universe

Equation (4) can be split to give [2,6]

$$\dot{\rho} + 3H\rho = 0 ,$$

and

$$\dot{\Lambda} + 8\piG\rho = 0 .$$

Using eqs. (8) and (9), eqs. (23) and (24) yield

$$R(t) = \left[\frac{A(3-3\alpha-\beta)}{(2-\beta)}t^{(2-\beta)/(3-3\alpha-\beta)}\right],$$

where $A = \text{constant}.$

$$\Lambda(t) = \frac{(2-\beta)(6\alpha-\beta)}{(3-3\alpha-\beta)^2} \frac{1}{t^2} .$$

$$\rho(t) = Ft^{-3(2-\beta)/(3-3\alpha-\beta)} , \ F = \text{const}.$$ (27)

and

$$G(t) = \left[\frac{(2-\beta)}{4\pi F(3-3\alpha-\beta)}\right]t^{(6\alpha-\beta)/(3-3\alpha-\beta)} .$$

Equation (24) represents a coupling between vacuum and gravity and that the vacuum decays to strengthen the gravitation interaction that will induce an acceleration of the expansion of the universe. Hence as long as gravity is increasing the expansion of the universe will continue. The variation of $G$ could have been overwhelming in the early universe. This big gravitational force might have been the cause for stopping the rapid expansion during inflationary period and later assist in making the universe matter dominated. This because the increasing gravity forces smaller particles to form bigger ones.

For $\beta = 0, \alpha = 0, G=\text{const.}$ and $\rho = Dt^{-2}$ and $R = [\frac{3}{2}At]^{2/3},$ which is the familiar FRW result. Moreover, the case $\beta = 6\alpha$ is equivalent to the Standard Model result. Clearly for $\beta < 2, \ \alpha > 1/3$ the gravitational constant increases with time. In an earlier work [7] we have considered the effect of bulk viscosity in variable G and $\Lambda$ models. We have shown that many of non-viscous models are equivalent to viscous models. The present model is equivalent to a viscous model with bulk viscosity ($\eta$) varying as $\eta \propto t^{-n/(1-n)}$ where $\eta \propto \rho^n$ with $n = \frac{(2\beta-\alpha-1)}{(\beta-2)}.$ It has been shown that the development of the large-scale anisotropy is given by the ratio of the shear $\sigma$ to the volume expansion ($\theta = 3\dot{R}/R$) which evolves as [8]

$$\frac{\sigma}{\theta} \propto t^{-(3+3\alpha-2\beta)/(3-3\alpha-\beta)} .$$

The present observed isotropy of the Universe requires this anisotropy to be decreasing as the universe expands. Thus an increasing $G$ guarantees an isotropic universe. For example, if $2\beta = 3 + 3\alpha$ then $G \propto t^{-1}, \ R \propto t^{1/3}, \ \rho \propto t^{-1}.$ This variation of $G$ was considered by Dirac in his Large Number Hypothesis (LNH) model [9]. According to Dirac model one has a constant anisotropy and this may be a problem if the universe was not born isotropic because the anisotropy would not decay with time.
4.2 Radiation-dominated universe

Equation (4) now reads
\[ \dot{\rho} + 4H \rho = 0, \]  
(30)
and
\[ \dot{\Lambda} + 8\pi G \dot{\rho} = 0. \]  
(31)
Employing eqs.(20) and (21), eqs.(30) and (31) yield
\[ \rho = Ct^{2(3-2\beta)/(3-3\alpha-\beta)}, \quad G = \left[ \frac{3(3-2\beta)}{32\pi C(3-3\alpha-\beta)} \right]^{2(3\alpha-\beta)/(3-3\alpha-\beta)}, \quad C = \text{const}. \]  
(32)
We observe that when \( \beta = 3\alpha \) the familiar FRW model is recovered. Abdel Rahman [2] has recently considered a closed universe model with a critical energy density where both \( G \) and \( \Lambda \) are variable. He found that \( R \propto t, \quad G \propto t^2, \quad \rho \propto t^{-4} \) in the radiation era. His solution corresponds to \( \alpha = 1/2 \) and a free \( \beta \). Thus both model, albeit different, evolve in the a similar way in the early universe. The large-scale anisotropy (see eq.(29)) becomes
\[ \sigma \propto t^{-3(3-2\beta)/(3-3\alpha-\beta)} \propto \rho^{3/2}. \]  
(33)
Once again this anisotropy decreases with time as long as \( \rho \) decreases with expansion.

4.3 Static solutions

A static solution can be obtained for both matter and radiation dominated universes with \( \beta = 2 \) and \( \beta = 3/2 \), respectively. Thus
\[ R = \text{const.}, \quad \Lambda = 0, \quad \rho_{\text{total}} = 0, \quad n = 0. \]  
(34)
It has been claimed by Kalligas et al. [10] that they have obtained a static universe with variable \( G \) and \( \Lambda \). In fact, their solution is nothing but the above solution, since with \( R = \text{const.} \) eqs.(1) and (2) give \( \Lambda = 0 \) so that \( G = \text{const.} \). Thus their claim of a static solution with variable \( G \) and \( \Lambda \) can not be true with \( p \neq -\rho \).

5. An inflationary solution

This solution is obtained if we set \( H = \text{const.} \). Thus eqs.(1) and (2) give \( \beta = 3 - 3\alpha \) so that \( \Lambda = 3H^2 \). This can be integrated to give
\[ R = \text{const} \exp(\sqrt{\Lambda/3t}), \quad \rho_{\text{total}} = \rho_v, \quad G = \text{const.}. \]  
(35)
This is the familiar de-Sitter inflationary solution.

6. An accelerating universe

Now consider the case \( \alpha = 0 \), i.e., \( \Lambda = \beta \frac{R}{R} \). Moreover, from eqs.(1)-(3) one finds
\[ \dot{R} = \frac{8\pi G}{3} (1 - \frac{3}{2} \gamma) \rho R + \frac{\Lambda}{3} R. \]  
(36)
Thus writing \( \Lambda \) in the above form is equivalent to considering the universe to be filled with a fluid whose equation of state is given by \( p = -\frac{1}{3} \rho \). Moreover, one may attribute the cosmic acceleration of the universe to the decay of the vacuum energy density (or \( \Lambda \)). It follows from eqs.(8)-(14) that in the matter dominated Universe with \( G \) constant

\[
R = [\frac{A(\beta - 3)}{(\beta - 2)} t^{(\beta - 2)/ (\beta - 3)}], \quad \Lambda = \frac{\beta(\beta - 2) 1}{(\beta - 3)^2} t^2, \quad \rho = \frac{1}{4\pi G} \frac{(\beta - 2) 1}{(\beta - 3)} t^2, \quad \beta \neq 3, \beta \neq 2. \quad (37)
\]

\[
\rho_v = \frac{\Lambda}{8\pi G} = \frac{1}{8\pi G} \frac{\beta(\beta - 2) 1}{(\beta - 3)^2} t^2, \quad \beta \neq 3.
\]

\[
q_p = -\frac{\dot{R} R}{R^2} = \frac{1}{2 - \beta}, \quad t_p = \frac{(\beta - 2)}{(\beta - 3)} H_p^{-1}, \quad \Omega_p^m = \frac{2(\beta - 3)}{3(\beta - 2)}, \quad \beta \neq 3, \beta \neq 2
\]

\[
n_p = \frac{\beta}{(\beta - 2)} \rho_p H_p^{-2}, \quad \Omega_p^\Lambda = \frac{\beta}{3(\beta - 2)} H_p^{-2}, \quad \Lambda_p = \frac{\beta}{(\beta - 2)} H_p^{-2}, \quad \beta \neq 2
\]

If \( G \) is changing with time one obtains from eqs.(28) and (29)

\[
G(t) = \left[ \frac{(\beta - 2)}{4\pi A(\beta - 3)} \right] t^{(\beta - 3)/ (\beta - 3)}, \quad \frac{\sigma}{\theta} \propto t^{(3-2\beta)/ (\beta - 3)}, \quad \beta \neq 3, \beta \neq 2
\]

We observe that for \( \Omega_p^\Lambda > 0, \beta > 2 \) and for \( \Omega_p^m > 0, \beta > 3 \). This implies that \( q_p < 0 \) and thus the universe must be accelerating at the present era. We also notice that \( G \) is an increasing function of time. Thus one may attribute the accelerating universe to the increasing gravity, i.e., the universe has to accelerate in order to overcome the ever increasing gravity. Others ascribed this to a scalar field that is still rolling down its potential “quintessence” to justify the acceleration of the universe [13]. We see that all the cosmological parameters shown above are functions of \( \beta \). The minimum age of the universe is \( H_p^{-1} \) and corresponds to \( \beta = \infty \). We restore the Einstein-de Sitter model when \( \beta = 0 \). We also note that \( \frac{\Lambda_p}{\Lambda_t} = (\frac{10^{17}}{10^{17}})^2 = 10^{120} \), where \( \Lambda_t \) is the value of \( \Lambda \) at Planck time. This is evident from eqs.(9) and (21). This may help in solving the cosmological constant problem. Thus the cosmological constant is very small today because the universe is very old. We then have a model with one parameter that can be obtained from observation. Recent constrains on cosmological parameters from MAXIMA-1 [11] suggest that \( 0.40 < \Omega^\Lambda < 0.76 \), this implies that \( 4 < \beta < 12 \). These results are found to be consistent with a flat universe. This constraint on \( \beta \) imposes a stringent constraint on all cosmological parameters so far known. The case \( \beta = 3 \) corresponds to an inflationary solution in the matter dominated universe. If \( G \) is increasing with time the presently observed isotropy of the universe could be one of its consequences, as is evident from eq.(41). The case \( \beta = 2 \) represents a static universe, which is nonsensical for the present universe.

7. Conclusion

We have considered the decay law for \( \Lambda \) of the form \( \Lambda = 3\alpha R c + \beta R \), \( \alpha, \beta = \text{const.} \). The cosmological consequences of the model are shown to be very interesting. Many models in the literature can be retrieved from this model with particular choice of \( \alpha \) and \( \beta \). The exact values of these constants still require more observational data to be determined. We note that the cosmological
problems of the standard model, namely, the age problem, cosmological constant problem and the isotropy and homogeneity of universe, which are still unresolved could be tackled with our present approach. We have found that when $\alpha = 0$ the minimum age of the universe is $H_p^{-1}$ (corresponds to $\beta = \infty$). As present estimates give $\Omega_p^A = 2/3$ and $\Omega_p^m = 1/3$, we suggest a model of $\alpha = 0$ and $\beta = 4$, which gives an age of $2H_p^{-1}$ and $q_p = -0.5$, as describing our present universe. The inflationary solution, which solves the Standard Model problems, is shown to be built in the model. A model with $\alpha = 0$ gives rise to an accelerated expansion of the universe. This model is free from a lot of cosmological problems and could fit well with the present observational data. This model contains one free parameters viz., $\beta$ that can be obtained from the present observational data.

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