The Forced Non-Linear Schrödinger Equation with a Potential on the Half-Line *

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Abstract

In this paper we prove that the initial-boundary value problem for the forced non-linear Schrödinger equation with a potential on the half-line is locally and (under stronger conditions) globally well posed, i.e. that there is a unique solution that depends continuously on the force at the boundary and on the initial data. We allow for a large class of unbounded potentials. Actually, for local solutions we have no restriction on the grow at infinity of the positive part of the potential, and for global solutions very mild assumptions that allow, for example, for exponential grow.

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1 Introduction

In this paper we analyse in detail the initial-boundary value problem for the forced non-linear Schrödinger equation with a potential on the half-line (FNLSP),

\[ i \frac{\partial}{\partial t} u(x,t) = -\frac{d^2}{dx^2} u(x,t) + V(x)u(x,t) + F(x,t,u), \quad u(0,t) = f(t), \quad u(x,0) = \phi(x), \quad (1.1) \]

where \( F(x,t,u) \) is a complex-valued function of \( x \in \mathbb{R}^+, t \in \mathbb{R}, u \in \mathbb{C} \). The functions \( \phi, f, \) satisfy the compatibility condition, \( \phi(0) = f(0) \). We solve this problem along the lines of [14], who studied the pure initial value problem for the non-linear Schrödinger equation on \( \mathbb{R}^n, n \geq 1 \). Note, however, that we allow for a much larger class of potentials than in [14]. In particular, we do not need to require that \( \frac{d^2}{dx^2} V(x) \) is bounded, as is the case in [14]. In fact, for local solutions we have no restriction on the growth of the positive part of the potential and for global solutions very mild assumptions that allow, for example, for exponential growth. We consider potentials - that are in general time dependent- that can be decomposed as the sum of two parts. The first one is what we call \( V \) in the FNLSP (1.1); it is independent of time but it can have singularities and it can grow at infinity. The second part is in general time dependent, but -together with its derivatives with respect to \( x \) and \( t \)- it has to be bounded for \( (x,t) \in (\mathbb{R} \times I) \), with \( I \) any bounded set. This second part is included in \( F \).

We consider the following class of potentials,

\[ V := V_1 + V_2 \text{ with } V_j \in L^1_{\text{loc}}(\mathbb{R}^+), j = 1, 2, V_1 \geq 0, \text{ and } \sup_{x \in \mathbb{R}^+} \int_x^{x+1} |V_2(y)| \, dy < \infty. \quad (1.2) \]

By \( W_{l,2}, l = 0, 1, 2, \cdots, \) we denote the standard Sobolev spaces \( \mathbb{H} \) in \( \mathbb{R}^+ \) and by \( W_{l,2}^{(0)} \) the completion of \( C_0^\infty(\mathbb{R}^+) \) in the norm of \( W_{l,2} \). The functions in \( W_{l,2}^{(0)} \) satisfy the homogeneous Dirichlet boundary condition at zero, \( \frac{d^j}{dx^j} u(0) = 0, j = 0, 1, \cdots, l - 1 \). In the case \( l = 0 \) we use the standard notation, \( W_{0,2} = W_{0,2}^{(0)} = L^2 \).

We designate, \( q := \sqrt{V_1} \), and by \( D(q) \) the domain in \( L^2 \) of the operator of multiplication by \( q \). We denote,

\[ \mathcal{H}_1^{(0)} := W_{1,2}^{(0)} \cap D(q) \text{ with norm } \| \phi \|_{\mathcal{H}_1^{(0)}} := \max \left[ \| \phi \|_{W_{1,2}^{(0)}}, \| q \phi \|_{L^2} \right]. \quad (1.3) \]

Let us denote by \( H_0 \) the self-adjoint realization of \( -\frac{d^2}{dx^2} \) with domain \( W_{2,2} \cap W_{1,2}^{(0)} \), i.e., the self-adjoint realization with homogeneous Dirichlet boundary condition at zero. We have that (see Section 2 for details)
the quadratic form,

\[ h(\phi, \psi) := (\dot{\psi}, \dot{\psi}) + (V\phi, \psi), \text{ with domain, } D(h) := \mathcal{H}_1^{(0)}, \quad (1.4) \]

is closed and bounded below. We denote by \( H \) the associated bounded-below, self-adjoint operator (see [16], [13]). Then, \( D(H) = \{ \phi \in \mathcal{H}_1^{(0)} : H_0 \phi + V \phi \in L^2 \} \) and,

\[ H \phi = H_0 \phi + V \phi, \text{ for } \phi \in D(H). \quad (1.5) \]

We designate,

\[ \mathcal{H}_1 := W_{1,2} \cap D(q) \text{ with norm } ||\phi||_{\mathcal{H}_1} := \max \left[ ||\phi||_{W_{1,2}}, ||q \phi||_{L^2} \right], \quad (1.6) \]

and,

\[ \mathcal{H}_2 := \{ \phi \in \mathcal{H}_1 \text{ such that } (-d^2/dx^2 + V) \phi \in L^2 \}, \]

\[ \text{with norm } ||\phi||_{\mathcal{H}_2} := \max \left[ ||\phi||_{\mathcal{H}_1}, \left( \left\| -\frac{d^2}{dx^2} + V \right\| L^2 \right) \phi \right]. \quad (1.7) \]

In Section 2 we study the initial-boundary value problem for the FNLS P (1.1). We first construct local solutions assuming that for each fixed \( x, t \), the non-linearity \( F(x, t, u) \) is \( C^1 \) in the real sense as a function of \( u \). We prove that the FNLS P (1.1) is locally well posed in \( \mathcal{H}_1 \) and in \( \mathcal{H}_2 \) and that there is continuous dependence on the initial and boundary data. In other words, the FNLS P (1.1) forms a dynamical system by generating a continuous local flow (see [14]). Then, we prove that if \( F \) satisfies a sign condition and has a hamiltonian structure the solutions exist for all times. Under these conditions the continuous local flows become global continuous flows, and in this sense the spaces of initial data \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \) are fundamental for the FNLS P (1.1). Note that if \( V_1 \equiv 0, \mathcal{H}_1 = W_{1,2} \). We give in Section 2 sufficient conditions on \( V \) assuring that \( \mathcal{H}_2 = W_{2,2} \cap D(V_1) \) and in particular if \( V_1 \equiv 0, \) that \( \mathcal{H}_2 = W_{2,2} \).

The existence and uniqueness of global solutions in \( W_{2,2} \) to the FNLS P (1.1) with \( V \equiv 0 \) and \( F = \lambda |u|^2 u \) was proven in [7], and the continuous dependence on the initial value and the boundary condition in [5]. For existence and uniqueness of global solutions with \( V \equiv 0 \) and \( F = \lambda |u|^{p-1} u, \lambda > 0, p > 3 \) see [4]. These papers give references for the application of the FNLS P (1.1) to important physical problems. For the solution of the direct and inverse scattering problems for the FNLS P (1.1) see [18] and [19]. The existence of global solutions in \( \mathbb{R}^n, n \geq 2, \) with \( V \equiv 0 \) and \( F = \lambda |u|^{p-1} u, 1 < p < \infty, \lambda > 0, \) was proven in [6].
the integrable case where \([14]\) can be studied with inverse scattering transform methods see \([9]\) and the references quoted there. For the Korteweg-De Vries equation in the half-line see \([2]\) and \([8]\). For general references in non-linear initial value problems see \([17]\), \([15]\), \([11]\) and \([3]\).

### 2 The Initial Boundary-Value Problem

We first prepare results that we need. The Proposition below is well known. We give the simple proof for the reader’s convenience

**PROPOSITION 2.1.** Suppose that

\[
\sup_{x \in \mathbb{R}} \int_{x}^{x+1} |V_2(y)| \, dy < \infty. \tag{2.1}
\]

Then, for any \(\epsilon > 0\) there is a constant, \(K_\epsilon\), such that,

\[
\int_{0}^{\infty} |V_2(x)| |\phi(x)|^2 \, dx \leq \epsilon \|\phi\|_{L^2}^2 + K_\epsilon \|\phi\|_{L^2}^2, \phi \in W_{1,2}. \tag{2.2}
\]

Moreover, if

\[
\sup_{x \in \mathbb{R}} \int_{x}^{x+1} |V_2(y)|^2 \, dy < \infty, \tag{2.3}
\]

for any \(\epsilon > 0\) there is a constant, \(K_\epsilon\), such that,

\[
\|V_2 \phi\|_{L^2}^2 \leq \epsilon \|H_0 \phi\|_{L^2}^2 + K_\epsilon \|\phi\|_{L^2}^2, \phi \in D(H_0). \tag{2.4}
\]

**Proof:** If \(\phi \in W_{1,2}\), for any \(n = 0, 1, \cdots\), any \(x, y \in [n, n+1]\) and any \(\delta > 0\), we have that,

\[
|\phi(x)|^2 - |\phi(y)|^2 = 2 \text{Re} \int_{y}^{x} \phi(z) \overline{\phi(z)} \, dz \leq \delta \int_{n}^{n+1} |\phi(z)|^2 \, dz + \frac{1}{\delta} \int_{n}^{n+1} |\phi(z)|^2 \, dz. \tag{2.5}
\]

By the mean value theorem we can choose \(y\) such that, \(|\phi(y)|^2 = \int_{n}^{n+1} |\phi(z)|^2 \, dz\), and it follows that,

\[
|\phi(x)|^2 \leq \delta \int_{n}^{n+1} |\dot{\phi}(z)| \, dz + \left(1 + \frac{1}{\delta}\right) \int_{n}^{n+1} |\phi(z)|^2 \, dz. \tag{2.6}
\]

Let \(C\) be the finite quantity in the left-hand side of (2.1). Then,

\[
\int_{n}^{n+1} |V_2(x)| |\phi(x)|^2 \, dx \leq C \delta \int_{n}^{n+1} |\dot{\phi}(z)| \, dz + C \left(1 + \frac{1}{\delta}\right) \int_{n}^{n+1} |\phi(z)|^2 \, dz. \tag{2.7}
\]
Taking \( \delta \) so small that \( \epsilon = \delta C \), and adding over \( n \) we obtain (2.2). Let us now denote by \( C \) the finite quantity on the left-hand side of (2.3). As \( D(H_0) = W_{2,2} \cap W_{1,2}^{(0)} \), if follows from (2.6) that,

\[
\int_0^{n+1} |V_2(x)|^2 |\phi(x)|^2 \, dx \leq C\delta \int_0^{n+1} |\dot{\phi}(z)|^2 \, dz + C \left( 1 + \frac{1}{\delta} \right) \int_0^{n+1} |\phi(z)|^2 \, dz.
\]

(2.8)

Taking now \( \epsilon = \delta C/2 \), adding over \( n \) and as \( \|\dot{\phi}\|_{L^2}^2 = (H_0\phi, \phi) \leq \|H_0\phi\|_{L^2}^2/2 + \|\phi\|_{L^2}^2/2 \), we obtain (2.4).

Assuming that (1.2) holds, the results about \( h \) and \( H \) stated in the introduction (see (1.4)-(1.5)) follow from (2.2) and [16], [13]. Below we always assume that (2.1) is satisfied.

We study the initial boundary-value problem for the FNLS (1.1) for \( t \geq 0 \), but by changing \( t \) into \(-t\) and taking the complex conjugate of the solution (time reversal) we also obtain the results for \( t \leq 0 \). Let \( F(x, t, z) \) be a complex-valued function of \( x \in \mathbb{R}^+, t \in \mathbb{R}^+, z \in \mathbb{C} \). As we are not assuming analyticity of \( F \) we consider the derivative, \( \dot{F} \), in the real sense. For each \( z \in \mathbb{C} \), \( \dot{F} \) is defined as the real-linear operator on \( \mathbb{C} \), given by,

\[
\dot{F}(x, t, z)v := \left( \frac{\partial}{\partial z} F(x, t, z) \right) v + \left( \frac{\partial}{\partial z} F(x, t, z) \right) \overline{v}, \quad v \in \mathbb{C},
\]

(2.9)

with the standard notation, \( \frac{\partial}{\partial z} := (1/2) \left( \frac{\partial}{\partial a} - i \frac{\partial}{\partial b} \right) \) and \( \frac{\partial}{\partial \overline{z}} := (1/2) \left( \frac{\partial}{\partial a} + i \frac{\partial}{\partial b} \right) \) where \( z = a + ib \). We denote, \( |\dot{F}| := |\frac{\partial}{\partial z} F| + |\frac{\partial}{\partial \overline{z}} F| \). \( \dot{F} \) can be identified (when viewed as a \( 2 \times 2 \) matrix) with the Gateaux derivative in the real sense of the map \( z \in \mathbb{C} \rightarrow F(x, t, z) \in \mathbb{C} \) for each fixed \( x, t \). We say that for each fixed \( x, t, F \) is \( C^1(\mathbb{C}, \mathbb{C}) \) in the real sense if \( \frac{\partial}{\partial z} F(x, t, z) \) and \( \frac{\partial}{\partial \overline{z}} F(x, t, z) \) are continuous functions of \( z \) for each fixed \( x, t \), or equivalently if the map \( z \rightarrow \dot{F}(x, t, z) \) is continuous from \( \mathbb{C} \) into the real-linear operators in \( \mathbb{C} \). For \( T > 0 \) we denote, \( I := [0, T) \) if \( T < \infty \) and \( I := [0, \infty) \) if \( T = \infty \).

**Assumption A**

Suppose that \( F(x, t, z) \) is a function from \( [0, \infty) \times I \times \mathbb{C} \) into \( \mathbb{C} \), that for each fixed \( x \in [0, \infty), t \in I \), is \( C^1 \) in \( z \) in the real sense. Moreover, assume that for each fixed \( t, z, F \) is differentiable in \( x \in \mathbb{R}^+ \), that \( F(x, t, 0) = 0 \) and that for each \( R > 0 \) and each bounded subset, \( I_N \), of \( I \), there is a constant \( C_{R,N} \) such that,

\[
|\dot{F}(x, t, z)| \leq C_{R,N}, \quad \text{for } x \in [0, \infty), t \in I_N, |z| \leq R,
\]

(2.10)
and,
\[
\left| \frac{\partial}{\partial x} F(x,t,z) \right| \leq C_{R,N} |z|, \text{ for } x \in [0, \infty), t \in I_N, |z| \leq R. \tag{2.11}
\]
Furthermore, if the force, \( f \), in (1.1) is not identically zero, suppose that for each fixed \( z \), \( F(0,t,z) \) is differentiable in \( t \) and
\[
\left| \frac{\partial}{\partial t} F(0,t,z) \right| \leq C_{R,N}, t \in I_n, |z| \leq R. \tag{2.12}
\]

Assumption A allows for a large class of non-linearities. For example, the standard single-power non-linearity, \( |u|^{p-1} u, p > 1 \), or more generally any \( F(z) \) that is \( C^1 \) in the real sense, are allowed. Let us denote by \( \mathcal{H}_{-1} \) the dual of \( \mathcal{H}_{1}^{(0)} \) with the pairing given by the scalar product of \( L^2 \). Then, as the quadratic form domain of \( H \) is \( \mathcal{H}_{1}^{(0)} \), \( H \) extends to a bounded operator from \( \mathcal{H}_{1}^{(0)} \) into \( \mathcal{H}_{-1} \). Moreover, \( e^{-itH} \) is a bounded operator from \( \mathcal{H}_{1}^{(0)} \) into \( \mathcal{H}_{1}^{(0)} \cap C^1(I, \mathcal{H}_{-1}) \) and,
\[
i \frac{\partial}{\partial t} e^{-itH} \phi = H e^{-itH} \phi = e^{-itH} H \phi. \tag{2.13}
\]

Suppose that \( u(x,t) \in C(I, \mathcal{H}_1) \) is a solution of (1.1) where \( f \in C^2(I) \). Furthermore, if \( f \) is not identically zero, assume that \( V \in W_{1,2}((0,\delta)) \) for some \( \delta > 0 \). Note that the compatibility condition \( \phi(0) = f(0) \) has to be satisfied if there is a solution to (1.1). Denote, \( v(x,t) := u(x,t) - r(x,t) \) where \( r(x,t) := [f(t) + \frac{1}{2} x^2 (V(0)f(t) + F(0, t, f(t)) - if(t))] g(x) \), with \( g \in C_0^\infty([0,\infty)), g(x) = 1, 0 \leq x \leq \delta/2 \) and with support contained in \([0,\delta)\). Then, \( v(x,t) \in C(I, \mathcal{H}_{1}^{(0)}) \) solves,
\[
i \frac{\partial}{\partial t} v(x,t) = H v(x,t) + F_1(x,t,v), v(0,t) = 0, v(x,0) = v_0(x) := \phi(x) - r(x,0), \tag{2.14}
\]
where,
\[
F_1(x,t,v) := F(x,t,v + r) - i \frac{\partial}{\partial t} r + V(x) r - \frac{\partial^2}{\partial x^2} r. \tag{2.15}
\]
Note that by the compatibility condition, \( v_0 \in \mathcal{H}_{1}^{(0)} \). Clearly, equations (1.1) and (2.14) are equivalent. By Assumption A and Sobolev’s [1] theorem for any \( t, N > 0 \) there is a constant, \( C \), such that, \( \| F_1(x,s,v) \|_{L^2} \leq C \) for all \( 0 \leq s \leq t \) and \( v \in W_{1,2} \) with \( \| v \|_{W_{1,2}} \leq N \). Multiplying both sides of (2.13) (evaluated at \( \tau \)) by \( e^{-i(t-\tau)H} \) and integrating in \( \tau \) from zero to \( t \) we obtain that,
\[
v(t) = e^{-itH} v_0 + \frac{1}{i} G F(v), \tag{2.16}
\]
where we designate by $\mathcal{F}$ the operator $v \rightarrow \mathcal{F}(v) := F_1(x,t,v)$, and

$$(Gf)(t) := \int_0^t e^{-i(t-\tau)H} f(\tau) d\tau, \quad t \in I. \quad (2.17)$$

We prove below that if $v(t) \in C \left( I, \mathcal{H}_1^{(0)} \right)$ is a solution to (2.16), it is also a solution to (2.14). We denote,

$$v_1(x,t) := \frac{1}{i} \int_0^t e^{-i(t-\tau)H} F_1(x,\tau,v(x,\tau)) d\tau. \quad (2.18)$$

It follows from Assumption A that $F_1(x,\tau,v) \in L^{\inf}_{\text{loc}} \left( I, \mathcal{H}_1^{(0)} \right)$. Hence, $v_1 \in C \left( I, \mathcal{H}_1^{(0)} \right) \cap C^1 \left( I, \mathcal{H}_{-1} \right)$, and

$$i \frac{\partial}{\partial t} v_1(t) = Hv_1(t) + F_1(x,t,v). \quad (2.19)$$

Equations (2.16) and (2.19) imply that (2.14) holds. This proves that (2.14) and (2.16) are equivalent.

### Assumption B

Suppose that $V$ can be decomposed as, $V := V_1 + V_2$ with $V_j \in L^1_{\text{loc}}(\mathbb{R}^+), j = 1, 2$, $V_1 \geq 0$, and $V_2$ satisfies (2.11). Moreover, assume that $f \in C^2(I)$, and if $f$ is not identically zero, suppose that $V \in W_{1,2}((0,\delta))$ for some $\delta > 0$.

We designate, $\mathcal{M} := L^\infty \left( I, \mathcal{H}_1^{(0)} \right)$. By Sobolev’s theorem [11], $\mathcal{M} \subset L^\infty \left( \mathbb{R}^+ \times I \right)$ and $\|v(x,t)\|_{L^\infty(\mathbb{R}^+ \times I)} \leq C\|v\|_{\mathcal{M}}$. Moreover, we denote, $B := L^\infty(I,L^2)$ and $\dot{B} := L^1(I,L^2)$.

**THEOREM 2.2.** Suppose that Assumptions A and B are satisfied. Then, for any $\phi \in \mathcal{H}_1$ satisfying $\phi(0) = f(0)$, there is a finite $T_0 \leq T$ such that the FNLSP (1.1) has a unique solution, $u \in C \left( [0,T_0], \mathcal{H}_1 \right)$ with $u(x,0) = \phi$. $T_0$ depends only on $\|\phi\|_{\mathcal{H}_1}$.

**Proof:** we prove the theorem by showing that (2.16) has a unique solution $v \in C \left( [0,T_0], \mathcal{H}_1^{(0)} \right)$ such that, $v(x,0) = v_0(x) := \phi(x) - r(x,0)$. Let us take $I = [0,T_0]$ with $T_0 < \infty$. Let us denote by $\overline{\mathcal{M}}$ the space of bounded and continuous functions from $I$ into $\mathcal{H}_1^{(0)}$. Let $\mathcal{M}_R$ and $\overline{\mathcal{M}}_R$ be, respectively, the closed ball in $\mathcal{M}$, and in $\overline{\mathcal{M}}$, with center zero and radius $R$. Let us prove that $\mathcal{M}_R$ is closed in the norm of $B$. Suppose that $v_n \in \mathcal{M}_R$ converges to $u$ in the norm of $B$. Then, $\lim_{n \to \infty} \|v_n(t) - v(t)\|_{L^2} = 0$ for a.e. $t$. 


But as \( v_n \in \mathcal{M}_R \), \( \|v_n(t)\|_{H^1_0} \leq R \) for a.e. \( t \). In consequence \( \|v(t)\|_{H^1} \leq R \) for a.e. \( t \). Moreover, there is a subsequence - denoted also \( v_n(t) - \) such \( \frac{\partial}{\partial x} v_n \) converges weakly to \( \frac{\partial}{\partial x} v \) in \( L^2 \), for a.e. \( t \), and then, 
\( v(x,t) = \lim_{n \to \infty} v_n(x,t) = \lim_{n \to \infty} \int_0^x \frac{\partial}{\partial y} v_n(y,t) dy = \int_0^x \frac{\partial}{\partial y} v(y,t) dy \), and it follows that \( v(0,t) = 0 \), i.e., \( v \in \mathcal{M}_R \). Hence, \( \mathcal{M}_R \) is a complete metric space in the norm of \( B \).

We define,
\[
P(v) := e^{-itH}v_0 + \frac{1}{i} G \mathcal{F}(v), v \in \mathcal{M}.
\]
(2.20)

As \( D(\sqrt{H+M}) = \mathcal{H}^0_1 \), for \( M \) large enough, we have that the norm \( \|\sqrt{H+M} \phi\|_{L^2} \) is equivalent to the norm of \( \mathcal{H}^0_1 \). Then, by the unitarity of \( e^{-itH} \) in \( L^2 \) and as \( \sqrt{H+M} \) commutes with \( e^{-itH} \),
\[
\|e^{-itH}v_0\|_{\mathcal{H}^0_1} \leq C\|\sqrt{H+M} e^{-itH}v_0\|_{L^2} \leq C\|v_0\|_{\mathcal{H}^0_1},
\]
(2.21)

and moreover,
\[
\|e^{-itH}\|_{B(\mathcal{H}^0_1)} \leq C, \text{ for } t \in \mathbb{R}.
\]
(2.22)

As \( V \in W_{1,2}((0,\delta)) \) if \( f \) is not identically zero, it follows from Assumption A and Sobolev’s theorem that there is a constant \( C_R \) such that for \( v \in \mathcal{M}_R \),
\[
\|\mathcal{F}(v)\|_{\mathcal{H}^0_1} \leq C_R.
\]
(2.23)

Note that \( \mathcal{F}(v)(t) \in \mathcal{H}^0_1 \) if \( v \in \mathcal{M} \). By (2.22) and (2.23), there is a constant \( C_R \) such that,
\[
\|G \mathcal{F}(v)\|_{\mathcal{M}} = \sup_{t \in [0,T_0]} \|G \mathcal{F}(v)\|_{\mathcal{H}^0_1} \leq C_R \int_0^{T_0} dt \|\mathcal{F}(v)\|_{\mathcal{H}^0_1} \leq C_R T_0,
\]
(2.24)

for all \( v \in \mathcal{M}_R \). By (2.21) and (2.24) we can take \( R \) large enough and \( T_0 \) small enough (depending only on \( \|\phi\|_{\mathcal{H}^0_1} \)) such that \( P \) sends \( \mathcal{M}_R \) into \( \overline{\mathcal{M}_R} \). By Assumption A there is a constant \( C \) such that,
\[
\|\mathcal{F}(u) - \mathcal{F}(v)\|_B \leq C \|u - v\|_B, \ u, v \in \mathcal{M}_R.
\]
(2.25)

Then, by the unitarity of \( e^{-itH} \) in \( L^2 \),
\[
\|P(u) - P(v)\|_B \leq C T_0 \|u - v\|_B, \ u, v \in \mathcal{M}_R.
\]
(2.26)

Given \( R \) we can take \( T_0 \) so small that \( P \) is a contraction on the metric of \( B \). By the contraction mapping theorem \( P(u) \) has a unique fixed point that is the only solution to the FNLS in \( \mathcal{M}_R \). If there is another
solution, \( u_1 \in C([0, T_0], \mathcal{H}_1^{(0)}) \), to (1.1), then, \( v_1 := u_1 - r \) has to be a solution to (2.10), but since \( \|GF(v_1(t))\|_{\mathcal{H}_1^{(0)}} \) can be made arbitrarily small by taking \( 0 \leq t \leq T_1, T_1 \leq T_0 \), by the same argument as above we have that \( v(t) = v_1(t), 0 \leq t \leq T_1 \), where \( T_1 \) depends only on \( \|\phi\|_{\mathcal{H}_1} \). By iterating this argument we prove that \( v(t) = v_1(t), 0 \leq t \leq T_0 \).

**THEOREM 2.3.** Suppose that Assumptions A and B are satisfied by \( V, F, f, f_n, n = 1, 2, \cdots \), where we require that \( V \in W_{1,2}((0, \delta)) \) for some \( \delta > 0 \) only if the \( f_n, n = 1, 2, \cdots \) are not all identically zero for \( n \) large enough. Then, the solution \( u \in C([0, T_0], \mathcal{H}_1), u(0) = \phi, T_0 \leq T \), to the FNLS (1.1) depends continuously on the initial value and on the boundary condition. In a precise way, let \( u \in C([0, T_0], \mathcal{H}_1) \) be the solution to (1.1) with \( u(0) = \phi \), let \( \phi_n \to \phi \) in \( \mathcal{H}_1 \) with \( \phi_n(0) = f_n(0) \) and assume that \( f_n(t) \to f(t) \) in \( C^2([0, T_0]) \). Then, for \( n \) large enough the solution \( v_n \in C([0, T_0], \mathcal{H}_1) \) to the FNLS (1.1) with initial condition \( \phi_n \) and boundary condition \( f_n \) exits for \( t \in [0, T_0] \) and \( u_n \to u \) in \( C([0, T_0], \mathcal{H}_1) \).

**Proof:** We first prove a local version for \( T_0 \) small enough. We denote \( v_{0,n}(x) := \phi_n(x) - r_n(x, 0) \), with \( r_n(x, t) := [f_n(t) + \frac{1}{2} x^2 (V(0) f_n(t) + F(0, t, f_n(t)) - i \tilde{f}_n(t))] g(x) \), and,

\[
P_n(v) := e^{-iTH}v_{0,n} + \frac{1}{i} GF_n(v) \in \mathcal{M},
\]

where we designate by \( \mathcal{F}_n \) the operator \( v \to \mathcal{F}_n(v) := F_n(x, t, v(x, t)) \), with,

\[
F_n(x, t, v) := F(x, t, v + r_n) - i \frac{\partial}{\partial t} r_n(x, t) + V(x) r_n(x, t) - \frac{\partial^2}{\partial x^2} r_n(x, t).
\]

As in the proof of Theorem 2.2 we prove that for \( R \) large enough and \( T_0 \) small enough all the \( P_n \) send \( \mathcal{M}_R \) into \( \overline{\mathcal{M}_R} \) and are contractions in the norm of \( B \) with a uniform contraction rate \( \sigma < 1 \) independent of \( n \). Let \( \nu_n \) be the unique fixed point. Then, \( u_n := v_n + r_n \in C([0, T_0], \mathcal{H}_1) \) are the unique solutions to the FNLS (1.1) with initial value \( \phi_n \) and boundary condition \( f_n \). Furthermore,

\[
\|v_n - v\|_B = \left\| e^{-iTH}[v_{0,n} - v_0] + \frac{1}{i} \left(G_{\mathcal{F}_n}(v_{0,n}) - G_{\mathcal{F}}(v)\right) \right\|_B \leq \|v_{0,n} - v_0\|_{L^2} + \sigma \|v_n - v\|_B + C T_0 \|f_n - f\|_{C^2([0, T_0])},
\]

and it follows that \( v_n \to v \) in \( B \). Moreover, by (2.21), (2.22) and denoting, \( v_x := \frac{\partial}{\partial x} v \), and \( v_n,x := \frac{\partial}{\partial x} v_n \),

\[
\|v_n(t) - v(t)\|_{H_1^{(0)}} \leq C \|v_{0,n} - v_0\|_{H_1^{(0)}} + C T_0 \left[ \|\mathcal{F}_n(v_n) - \mathcal{F}(v)\|_B + \|D_n(v_n, v_n,x) - D_n(v_n, v_x)\|_B \right] + \]

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\[ \| D_n(v_n, v_x) - D(v, v_x) \|_B + \| q(F_n(v_n) - F(v)) \|_B \], \quad (2.30) \]

where,

\[ D(v, v_x) := \dot{F}(x, t, v + r)(v_x + \frac{\partial}{\partial x} r) + \left( \frac{\partial}{\partial x} F \right)(x, t, v + r) - i \frac{\partial^2}{\partial x \partial t} r + \dot{V}(x) r + V(x) \frac{\partial}{\partial x} r - \frac{\partial^3}{\partial x^3} r, \quad (2.31) \]

and,

\[ D_n(v_n, v_n, x) := \dot{F}(x, t, v_n + r_n)(v_n, x + \frac{\partial}{\partial x} r_n) + \left( \frac{\partial}{\partial x} F \right)(x, t, v_n + r_n) - \]

\[ - i \frac{\partial^2}{\partial x \partial t} r_n + \dot{V}(x) r_n + V(x) \frac{\partial}{\partial x} r_n - \frac{\partial^3}{\partial x^3} r_n. \quad (2.32) \]

Furthermore,

\[ \| F_n(v_n) - F(v) \|_B + \| q(F_n(v_n) - F(v)) \|_B \leq C[\| v_n - v \|_B + \| q(v_n - v) \|_B + \| f_n - f \|_{C^2([0, T_0])}], \quad (2.33) \]

Also,

\[ \| D_n(v_n, v_n, x) - D_n(v_n, v_x) \|_B \leq C \| v_n, x - v_x \|_B. \quad (2.34) \]

But then,

\[ \| v_n - v \|_{C([0, T_0], H^s_1)} \leq C \| v_{0, n} - v_0 \|_{H^s_1} + CT_0 \left[ \| v_n(t) - v(t) \|_{C([0, T_0], H^s_1)} + \| f_n - f \|_{C^2([0, T_0])} + \right. \]

\[ \left. \| D_n(v_n, v_x) - D(v, v_x) \|_B \right]. \quad (2.35) \]

And it follows that for \( CT_0 < 1/2 \),

\[ \lim_{n \to \infty} \| v_n - v_x \|_{C([0, T_0], H^s_1)} \leq 2 C \lim_{n \to \infty} \left[ \| v_{0, n} - v_0 \|_{H^s_1} + T_0 \left( \| D_n(v_n, v_x) - D(v, v_x) \|_B + \right. \right. \]

\[ \left. \left. \| f_n - f \|_{C^2([0, T_0])} \right] \right] = 0, \quad (2.36) \]

where we used that, as \( v_n \to v \) in \( L^2 \) and \( \| v_n \|_{W^{s,2}_{1,2}} \leq C \), it follows by interpolation \[16\] that \( v_n \to v \) in \( W^{s,2}_{0,1} \), \( 0 < s < 1 \), and Sobolev’s theorem. This proves that \( u_n \to u \) in \( C([0, T_0], H_1) \). In a standard way we extend the result of the theorem -step by step- to the original interval. For this purpose it is essential that the interval of existence given by Theorem 2.2 depends only on the \( H_1 \) norm of \( \phi \).
REMARK 2.4. Suppose that Assumptions A and B are satisfied with $I = [0, \infty)$. Let $T_m$ be the maximal time such that the solution, $u$, given by Theorem 2.2 can be extended to a solution $u \in C ([0, T_m), \mathcal{H}_1)$, to the FNLSP $u(0) = \phi$. Then if $T_m$ is finite we necessarily have that $\lim_{t \uparrow T_m} \| u(t) \|_{\mathcal{H}_1} = \infty$. In other words, the solution exists for all times unless it blows up in the $\mathcal{H}_1$ norm for some finite time.

This result follows from Theorem 2.2, because if $\| u(t) \|_{\mathcal{H}_1}$ remains bounded as $t \uparrow T_m$ we can extend the solution $u$ continuously to $T_m + \epsilon$ for some $\epsilon > 0$, contradicting the definition of $T_m$. Theorem 2.2 implies also that the FNLSP (1.1) has a unique solution, $u \in C (I, \mathcal{H}_1)$, with $u(0) = \phi$. For, suppose that there is another solution, $v \in C (I, \mathcal{H}_1)$, of this problem. Then, by Theorem 2.2 $u(t) = v(t)$, for $t \in I_0 := [0, T_0], 0 < T_0 \leq T$. Let $I_m := [0, T_m) \subset I$ be the maximal interval such that $u(t) = v(t), t \in I_m$. If $T < \infty$ this follows because by Theorem 2.2 if $T_m < \infty$, $u(t) = v(t)$ for $t \leq T_m + \epsilon$, for some $\epsilon > 0$, contradicting the definition of $T_m$. By the same argument if $T = \infty$, $T_m$ can not be finite.

We now consider solutions in $\mathcal{H}_2$.

THEOREM 2.5. Suppose that Assumptions A and B are satisfied. Furthermore, assume that for each fixed $x, z$, $F(x, t, z)$ is differentiable in $t$ and,

$$\left| \left( \frac{\partial}{\partial t} F \right)(x, t, z) \right| \leq C_{R,N} |z|, \text{ for } x \in [0, \infty), t \in I_N, |z| \leq R. \quad (2.37)$$

Then, for any $\phi \in \mathcal{H}_2$ with $\phi(0) = f(0)$ there is a finite $T_0 \leq T$ such that the FNLSP (1.1) has a unique solution $u \in C ([0, T_0], \mathcal{H}_2)$ with $u(x, 0) = \phi$. $T_0$ depends only on $\| \phi \|_{\mathcal{H}_2}$.

Proof: We designate,

$$\mathcal{H}_2^{(0)} := \{ \phi \in \mathcal{H}_2 : \phi(0) = 0 \}, \text{ and } \mathcal{N} := \left\{ v \in L^\infty ([0, T_0], \mathcal{H}_2^{(0)}) : \frac{\partial}{\partial t} v(t) \in B \right\}, \quad (2.38)$$

with norm

$$\| v \|_{\mathcal{N}} := \max \left[ \| v \|_{L^\infty ([0, T_0], \mathcal{H}_2^{(0)})}, \left\| \frac{\partial}{\partial t} v \right\|_B \right]. \quad (2.39)$$
We define \( \mathcal{N} \) as in (2.38) but replacing \( L^\infty \) with continuous. Note that if \( v \in B \) and \( \frac{\partial}{\partial t} v(t) \in \dot{B} \), it follows that \( v(t) \) is a absolutely continuous function of \( t \in [0,T_0] \), with values in \( L^2 \). In consequence, \( v(0) \in L^2 \) exists and,

\[
\| v(0) \|_{L^2} \leq \| v \|_B.
\]  

(2.40)

We use the designation,

\[
\mathcal{N}_R := \{ v \in \mathcal{N} : \| v \|_\mathcal{N} \leq R, \text{ and } v(0) = v_0 \}.
\]  

(2.41)

We first prove that \( \mathcal{N}_R \) is a complete metric space in the norm of \( B \). It is enough to prove that it is a closed subset of \( B \). Suppose that \( v_n \in \mathcal{N}_R \) converges to \( v \in B \) in the norm of \( B \). We have to prove that \( v \in \mathcal{N}_R \). We have that \( \lim_{n \to \infty} \| v_n(t) - v(t) \|_{L^2} = 0 \), for a.e. \( t \). But as \( v_n \in \mathcal{N}_R \), \( \| v_n(t) \|_{\mathcal{N}} \leq R \) for a.e. \( t \). In consequence, \( \max [\| v \|_{L^\infty([0,T_0], H^2)}, \| \frac{\partial}{\partial t} v \|_B] \leq R \) for a.e. \( t \). We prove that \( v(0,t) = 0 \) as in the proof of Theorem 2.2. Moreover, we have that (eventually passing to a subsequence) \( \frac{\partial}{\partial t} v_n \to \frac{\partial}{\partial t} v \) weakly. Then, as \( v_n(t) = v_0 + \int_0^t \frac{\partial}{\partial s} v_n(s) \, ds \), we obtain that \( v(0) = v_0 \). Hence, \( v \in \mathcal{N}_R \).

Let \( P \) be defined as in (2.20). Let us prove that we can take \( R \) so large and \( T_0 \) so small (depending only on \( \| \phi \|_{H^2} \)) that \( P \) sends \( \mathcal{N}_R \) into \( \mathcal{N}_R \). As \( D(H) = H^2(0) \), and since \( H \) commutes with \( e^{-itH} \) and \( i \frac{\partial}{\partial t} e^{-itH} \phi = He^{-itH} \phi \), \( e^{-itH} \) is bounded from \( H^2(0) \) into \( \mathcal{N} \) with operator norm independent of \( T_0 \). Furthermore, suppose that \( w \in B \) and that \( \frac{\partial}{\partial t} w(t) \in \dot{B} \), with \( w(0) = \psi \in L^2 \). Then,

\[
\frac{\partial}{\partial t} Gw = G \frac{\partial}{\partial t} w + e^{-itH} \psi.
\]  

(2.42)

We write, \( Gw = e^{-itH} w_1(t) \), with \( w_1(t) := \int_0^t e^{i\tau H} w(\tau) \, d\tau \). Then,

\[
\frac{\partial}{\partial s} e^{isH} w_1(t)|_{s=0} = e^{itH} w(t) - \psi - \int_0^t e^{i\tau H} \dot{w}(\tau) \, d\tau.
\]  

(2.43)

As the right-hand side of (2.43) belongs to \( L^2 \) for a.e. \( t \), it follows that \( w_1(t) \in D(H) = H^2(0) \) for a.e. \( t \). Then, \( Gw = e^{-itH} w_1(t) \in H^2(0) \) for a.e. \( t \), and

\[
HGw = i \frac{\partial}{\partial t} Gw - iw.
\]  

(2.44)

By (2.42) and (2.44) \( Gw \in \mathcal{N} \) and,

\[
\| Gw \|_{\mathcal{N}} \leq C \left[ \| w \|_B + \left\| \frac{\partial}{\partial t} w \right\|_B \right].
\]  

(2.45)
For $v \in \mathcal{N}_R$ we write

$$\begin{aligned}
P(v) &= e^{-itH}v_0 - \frac{1}{i}G\mathcal{F}(v_0) + \frac{1}{i}G[\mathcal{F}(v) - \mathcal{F}(v_0)] \tag{2.46}
\end{aligned}$$

We take $R$ so large and $T_0$ so small that $\|e^{-itH}v_0 - \frac{1}{i}G\mathcal{F}(v_0)\|_{\mathcal{N}} \leq R/2$. Here we take $w = \mathcal{F}(v_0)$ in the estimates above. We now put $w(t) := \mathcal{F}(v)(t) - \mathcal{F}(v_0)$. By Assumption A, $w \in B$ and $\frac{\partial}{\partial t} w \in \dot{B}$. Then, as $w(0) = 0$, by (2.45) given $R$ we can take $T_0$ so small that,

$$\begin{aligned}
\left\|\frac{1}{i}G[\mathcal{F}(v) - \mathcal{F}(v_0)]\right\|_{\mathcal{N}_R} \leq \frac{R}{2} \tag{2.47}
\end{aligned}$$

With this choice of $R$ and $T_0$, $P$ sends $\mathcal{N}_R$ into $\overline{\mathcal{N}}_R$. We already know -see the proof of Theorem 2.2- that $P$ is a contraction in the norm of $B$. The unique fixed point is the only solution to the FNLS (1.1) in $\overline{\mathcal{N}}_R$. We complete the proof of the theorem as in the proof of Theorem 2.2.

**THEOREM 2.6.** Suppose that the assumptions of Theorem 2.5 are satisfied for $V, F, f, f_n, n = 1, 2, \ldots$, where we require that $V \in W_{1,2}((0, \delta))$ for some $\delta > 0$ only if the $f_n, n = 1, 2, \ldots$ are not all identically zero for $n$ large enough. Moreover, assume that for each fixed $x, t$, $(\frac{\partial}{\partial t} F)(x, t, z)$ is $C^1$ in the real sense and that,

$$\begin{aligned}
\left|\left(\frac{\partial}{\partial t} F\right)(x, t, z)\right| \leq C_{R,N} \text{ for } x \in [0, \infty), t \in I_N, |z| \leq R. \tag{2.48}
\end{aligned}$$

Note that (2.48) implies (2.37). Then, the solution $u \in C([0, T_0], \mathcal{H}_2)$, $u(0) = \phi$, $T_0 \leq T$, to the FNLS (1.1) depends continuously on the initial value and on the boundary condition. In a precise way, let $u \in C([0, T_0], \mathcal{H}_2)$ be the solution to the FNLS (1.1) with $u(0) = \phi$. Let $\phi_n \to \phi$ in $\mathcal{H}_2$ satisfy, $\phi_n(0) = f_n(0)$ and assume that $f, f_n \in C^3$ and that $f_n(t) \to f(t)$ in $C^3([0, T_0])$. Moreover, if all the $f_n$ are not identically zero for $n$ large enough, suppose that $\frac{\partial^2}{\partial t^2} F(0, t, z), \frac{\partial^2}{\partial t \partial z} F(0, t, z)$ and $\hat{F}(0, t, z)$ are continuous in $z$, uniformly in $t$. Then, for $n$ large enough the solution $u_n \in C([0, T_0], \mathcal{H}_2)$ to the FNLS (1.1) with initial value $\phi_n$ and boundary condition $f_n$ exists for $t \in [0, T_0]$ and $u_n \to u$ in $C([0, T_0], \mathcal{H}_2)$.

**Proof:** As in the proof of Theorem 2.3 it is enough to prove a local version for $T_0$ small enough. We define $v_{0,n}$ and $P_n$ as in the proof of Theorem 2.3. As in the proof of Theorem 2.5 we prove that for $R$ large enough and $T_0$ small enough all the $P_n$ send $\mathcal{N}_R$ (where we now require that $v(0) = v_{0,n}$) into $\overline{\mathcal{N}}_R$ and are contractions in the norm of $B$ with a uniform contraction rate $\sigma < 1$ independent of $n$. Let $v_n$ be the
unique fixed point. Then, \( u_n := v_n + r_n \in C([0, T_0], \mathcal{H}_2) \) are the unique solutions to the FNLS (1.1) with initial value \( \phi_n \) and boundary condition \( f_n \). Furthermore,

\[
\|v_n - v\|_B = \left\| e^{-itH}[v_{0,n} - v_0] + \frac{1}{i}(GF_n(v_n) - GF(v)) \right\|_B \leq \|v_{0,n} - v_0\|_{L^2} + \sigma \|v_n - v\|_B + C T_0 \|f_n - f\|_{C^2([0,T_0])}. \tag{2.49}
\]

In consequence, \( v_n \to v \) in \( B \). By taking the derivative of (2.16) with respect to \( t \) we obtain that,

\[
i \frac{\partial}{\partial t} v = e^{-itH}[HV_0 + F(v_0)] + GE, \tag{2.50}
\]

where,

\[
E(v, \frac{\partial}{\partial t} v) := \mathcal{F}(x, t, v + r) \left( \frac{\partial}{\partial t} v + \frac{\partial}{\partial t} r \right) + \left( \frac{\partial}{\partial t} F \right)(x, t, v + r) - i \frac{\partial^2 r}{\partial t^2} + \frac{\partial r}{\partial t} V(x) - \frac{\partial^3}{\partial t \partial x^2} r. \tag{2.51}
\]

We similarly prove that \( \frac{\partial}{\partial t} v_n \) satisfies,

\[
i \frac{\partial}{\partial t} v_n = e^{-itH}[HV_{0,n} + F_n(v_{0,n})] + GE_n, \tag{2.52}
\]

with,

\[
E_n(v_n, \frac{\partial}{\partial t} v_n) := \mathcal{F}(x, t, v_n + r_n) \left( \frac{\partial}{\partial t} v_n + \frac{\partial}{\partial t} r_n \right) + \left( \frac{\partial}{\partial t} F \right)(x, t, v_n + r_n) - i \frac{\partial^2 r_n}{\partial t^2} + \frac{\partial r_n}{\partial t} V(x) - \frac{\partial^3}{\partial t \partial x^2} r_n. \tag{2.53}
\]

By (2.50) and (2.52),

\[
i \frac{\partial}{\partial t} v_n(t) - i \frac{\partial}{\partial t} v(t) = e^{-itH}[HV_{0,n} + F_n(v_{0,n}) - HV_0 - F(v_0)] + GE_n \left( v_n, \frac{\partial}{\partial t} v_n \right) - GE \left( v, \frac{\partial}{\partial t} v \right) + GE_n \left( v_n, \frac{\partial}{\partial t} v \right) - GE \left( v, \frac{\partial}{\partial t} v \right). \tag{2.54}
\]

Furthermore,

\[
\left\| G \left[ E_n \left( v_n, \frac{\partial}{\partial t} v_n \right) - E_n \left( v_n, \frac{\partial}{\partial t} v \right) \right] \right\|_B \leq C T_0 \left\| \frac{\partial}{\partial t} v_n - \frac{\partial}{\partial t} v \right\|_B. \tag{2.55}
\]

Hence, by (2.54) and (2.55) if \( C T_0 < 1/2 \),

\[
\lim_{n \to \infty} \left\| \frac{\partial}{\partial t} v_n - \frac{\partial}{\partial t} v \right\|_B \leq 2 \lim_{n \to \infty} \left\| HV_{0,n} + F_n(v_{0,n}) - HV_0 - F(v_0) \right\|_{L^2} + \left\| G \left[ E_n \left( v_n, \frac{\partial}{\partial t} v \right) - \right. \right.
\]

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\[ E \left( v, \frac{\partial}{\partial t} v \right) \bigg|_{B_1} = 0, \]  

(2.56)

where we used that as \( v_n \to v \) in \( L^2 \) and \( \|v_n\|_{W^{0,1}_{1,2}} \leq C \), it follows by interpolation [16] that \( v_n \to v \) in \( W_{s,2}, 0 < s < 1 \), and Sobolev’s theorem. Then, by the FNLS (1.1) \( Hv_n \to Hv \) in \( L^2 \), and since \( v_n \to v \) in \( L^2 \), we have that \( v_n \to v \) in the norm of \( H^{(0)}_1 = D \left( \sqrt{H + M} \right) \). It follows that \( v_n \) converges to \( v \) in the norm of \( H^{(0)}_2 \), and then \( u_n \to u \) in \( C([0, T_0], H_2) \). In a standard way we extend the result of the theorem -step by step- to the original interval. For this purpose it is essential that the interval of existence given by Theorem 2.5 depends only on the \( H_2 \) norm of \( \phi \).

**Remark 2.7.** We prove as in Remark 2.4 that if \( I = [0, \infty) \) the solution in \( H_2 \) exits for all times unless it blows up in the \( H_2 \) norm for some finite time, and that Theorem 2.5 implies that the FNLS (1.1) has a unique solution \( u \in C(I, H_2) \), with \( u(0) = \phi \in H_2 \).

If the assumptions of Theorem 2.5 are satisfied, for any \( \phi \in H_2 \) the FNLS (1.1) has a unique solution in \( H_1 \) and a unique solution in \( H_2 \) both with \( u(0) = \phi \). In the proposition below we prove that it is impossible that the \( H_2 \) solution blows up before the \( H_1 \) solution does.

**Proposition 2.8.** (Regularity) Suppose that the assumptions of Theorem 2.5 are satisfied. Let \( u \in C([0, T], H_1) \) be a solution to the FNLS (1.1) with \( u(0) = \phi \in H_2 \). Then \( u \in C([0, T], H_2) \).

*Proof:* By Theorem 2.5 there is \( T_m \leq T \) such that \( u(t) \in C([0, T_m], H_2) \) and \( \frac{\partial}{\partial t} u \in C([0, T_m], L^2) \). Furthermore, \( \frac{\partial}{\partial t} v \) is a solution of the real-linear equation (where \( v \) is now fixed) (2.50). Applying the contraction mapping theorem - step by step- to this equation we prove that \( \frac{\partial}{\partial t} v \in C([0, T], L^2) \), and then, it follows from equation (2.14) that \( u = v + r \in C([0, T], H_2) \).

We impose now further restrictions on \( F \) that will allow us to derive an a-priori bound on the \( H_1 \) norm of the solutions, and then, by Remark 2.4 that the solutions exist for all times. We say that \( F \) satisfies the sign condition if

\[ \text{Im} \bar{z} F(x, t, z) = 0, \quad x, t \in \mathbb{R}^+, z \in \mathbb{C}, \]  

(2.57)
and we say that there is a hamiltonian structure if there is a function \( h(x, t, z) \), such that for each fixed \( x, t \in \mathbb{R}^+ \), \( h \) is in \( C^2 (\mathbb{C}, \mathbb{R}) \) in the real sense, \( h(x, t, 0) = 0 \) and,

\[
F(x, t, z) = 2 \frac{\partial}{\partial z} h(x, t, z).
\] (2.58)

If Assumption A is satisfied we have that,

\[
|h(x, t, z)| + \left| \left( \frac{\partial}{\partial x} h \right) (x, t, z) \right| \leq C_{R,N} |z|^2, \text{ for } x \in [0, \infty), t \in I_N, |z| \leq R.
\] (2.59)

Remark that as \( h(x, t, 0) = 0 \), equation (2.37) implies that \( h(x, t, z) \) is differentiable in \( t \), and that for each \( R > 0 \) and each bounded subset, \( I_N \), of \( I \), there is a constant \( C_{R,N} \) such that,

\[
\left| \left( \frac{\partial}{\partial t} h \right) (x, t, z) \right| \leq C_{R,N} |z|^2, \text{ for } x \in \mathbb{R}^+, t \in I_N, |z| \leq R.
\] (2.60)

Note that if (2.58) is satisfied, then, (2.57) is true if and only if \( h \) depends only on \( |z| \), i.e., if \( h(x, t, z) = h(x, t, |z|) \). 14.

Below we always assume that \( H_2 \subset W_{2,2} \).

For any solution \( u \in C (I, H_2) \) to the FNLSP (1.1) the following identities hold. If (2.57) is satisfied,

\[
\frac{d}{dt} \left\| u(t) \right\|^2_{L^2} = 2 \text{Im} P(t) \bar{f}(t),
\] (2.61)

where we denote, \( P(t) := (\frac{\partial}{\partial x} u)(0, t) \). Observe that in the case where there is no external force, \( f \equiv 0 \), this is the conservation of the \( L^2 \) norm. Moreover, let \( W(t) \) be the Hamiltonian,

\[
W(t) := \frac{1}{2} \left\| \frac{\partial}{\partial x} u(t) \right\|^2_{L^2} + \int_{\mathbb{R}^+} \left( \frac{1}{2} V(x) |u(x, t)|^2 + h(x, t, u) \right) dx.
\] (2.62)

Then, if (2.58) is true,

\[
\frac{d}{dt} W(t) = -\text{Re} \dot{f}(t) \overline{P(t)} + \int_{\mathbb{R}^+} \left( \frac{\partial}{\partial t} h \right) (x, t, u) dx.
\] (2.63)

In the case where there is no external force and \( h \) is independent of time this identity is the conservation of energy. Furthermore, if (2.58) is satisfied,

\[
\frac{d}{dt} (u, u_x) = -i |P(t)|^2 + 2i h(0, t, f(t)) - f(t) \overline{f(t)} - 2i \text{Re} (Vu, u_x) + 2i \int_{\mathbb{R}^+} \left( \frac{\partial}{\partial x} h \right) (x, t, u) dx.
\] (2.64)
Note that if $V$ is differentiable,

$$-2\text{Re}(Vu, u_x) = V(0)|f(0)|^2 + \int_{\mathbb{R}_+} \dot{V}(x)|u(x, t)|^2 \, dx.$$  \hfill (2.65)

The identity (2.64) is analogous to the conservation of momentum in the pure initial value problem in $\mathbb{R}$, c.f., [11]. Remark, however, that even in the case without external force and with potential, $V$, and $\frac{\partial}{\partial x}h(x, t, u)$ both identically zero it is not a conservation law. This is to be expected because our problem is not translation invariant. The identities (2.61), (2.63) and (2.64) where proven in the case $V \equiv 0$ and with $F$ a single power, $F = \lambda|u|^{p-1}u$ in [3] and [7] (see also [6] for the multidimensional case) for suitable smooth solutions. For the reader’s convenience, we briefly give below the details that show that the proof extends to our case, and that it holds for solutions $u \in C(I, \mathcal{H}_2)$. As $u(t) \in W_{2,2}, \lim_{x \to \infty} u(x, t) = \lim_{x \to \infty} \frac{\partial}{\partial x} u(x, t) = 0$. Then, by (1.1), (2.58) and integrating by parts,

$$\frac{d}{dt}\|u(t)\|_{L_2}^2 = 2\text{Re} \left( \frac{\partial}{\partial t} u(t), u(t) \right) = 2\text{Im} \left( -\frac{d^2}{dx^2} u(t), u(t) \right) = 2\text{Im} \frac{u(0, t)}{0}\frac{\partial}{\partial x} u(0, t),$$  \hfill (2.66)

and (2.61) holds. Moreover, denoting $u_x := \frac{\partial}{\partial x} u$ and $u_{xx} = \frac{\partial^2}{\partial x^2} u$, we have that,

$$\frac{\partial}{\partial t} \frac{1}{2}(u_x(t), u_x(t)) = \lim_{\delta \to 0} \frac{1}{2\delta} [(u_x(t+\delta) - u_x(t), u_x(t+\delta)) + (u_x(t), u_x(t) - u_x(t))] =$$

$$-\text{Re} \left( \frac{\partial}{\partial t} u(t), u_{xx}(t) \right) - \text{Re} \dot{f}(t) \overline{P(t)},$$  \hfill (2.67)

where we integrated by parts before taking the limit $\delta \to 0$.

Hence, by (1.1) and (2.58),

$$\frac{d}{dt} W(t) = \text{Re} \left( \frac{\partial}{\partial t} u(t), i \frac{\partial}{\partial t} u(t) \right) - \text{Re} \dot{f}(t) \overline{P(t)} + \int_{\mathbb{R}_+} \left( \frac{\partial}{\partial t} h \right)(x, t, u) \, dx =$$

$$-\text{Re} \dot{f}(t) \overline{P(t)} + \int_{\mathbb{R}_+} \left( \frac{\partial}{\partial t} h \right)(x, t, u) \, dx,$$  \hfill (2.68)

and (2.63) holds. Finally, integrating by parts, and using (1.1),

$$\frac{d}{dt} (u, u_x) = \lim_{\delta \to 0} \frac{1}{\delta} [(u(t+\delta) - u(t), u_x(t+\delta)) + (u(t), u_x(t+\delta) - u_x(t))] = 2i\text{Im} \left( \frac{\partial}{\partial t} u(t), u_x \right) - f(t) \overline{\dot{f}(t)} =$$

$$-2i\text{Re}(Hu + F(x, t, u), u_x) - f(t) \overline{\dot{f}(t)} = -i|P(t)|^2 + 2ih(0, t, f(t)) - f(t) \overline{f(t)}$$
and \(2.64\) holds. For any function \(f\) we denote by \(f_+\) its positive part and by \(f_-\) its negative part, i.e., \(f = f_+ - f_-\), \(f_\pm \geq 0\). Below we denote by \(\dot{V}\) the derivative of \(V\) in distribution sense.

**THEOREM 2.9.** Suppose that the assumptions of Theorem 2.5 are satisfied with \(I = [0, \infty)\), that \(\mathcal{H}_2 \subset W_{2,2}\) and that \(\dot{V}\) is a function with,

\[
(\dot{V})_+ \leq CV_1 + Q, \text{ and } (\dot{V})_- \in L^1_{\text{loc}}(\mathbb{R}^+),
\]

where \(Q\) satisfies \(2.7\). Furthermore, assume that \(2.57\), and \(2.58\) hold, where for each fixed \(x, t \in \mathbb{R}^+\), \(h\) is in \(C^2(\mathbb{C}, \mathbb{R})\), in the real sense, and \(h(x, t, 0) = 0\). Moreover, assume that for each bounded subset \(I_N\) of \(I\) there is a constant \(C_N\) such that,

\[
\left(\frac{\partial}{\partial x}h\right)_+ (x, t, z) \leq C_N |z|^2, \text{ for } x \in \mathbb{R}^+, t \in I_N, z \in \mathbb{C},
\]

and that for some \(1 < p \leq 3\),

\[
\left(\frac{\partial}{\partial t}h\right)_+ (x, t, z) \leq C_N \left(|z|^2 + |z|^{p+1}\right), \text{ for } x \in \mathbb{R}^+, t \in I_N, z \in \mathbb{C},
\]

and,

\[
h(x, t, z) \geq -C_N(|z|^2 + |z|^{p+1}), \text{ for } x \in \mathbb{R}^+, t \in I_N, z \in \mathbb{C}.
\]

Then, the solutions in \(\mathcal{H}_1\) and in \(\mathcal{H}_2\) to the FNLS \(1.1\) given, respectively, by Theorems 2.2 and 2.5 exist for all time \(t \in [0, \infty)\).

**Proof:** In view of Remark 2.4 and of Proposition 2.8 it is enough to prove that for any finite time interval \([0, T]\), the solution \(u \in C([0, T), \mathcal{H}_1)\) remains bounded in the norm of \(\mathcal{H}_1\), as \(t \to T\). Suppose first that the solution \(u \in C([0, T), \mathcal{H}_2)\). For \(0 \leq t_1 \leq t < T\) we denote, \(a(t) := (\int_{t_1}^t |P(\tau)|^2 d\tau)\). In the estimates below we designate by \(C_T\) any constant that depends only on \(T\) and \(f\), and by \(C_{T,1}\) any constant that depends on \(T\), \(f\), and on the norm \(\|u(t_1)\|_{\mathcal{H}_1}\). We denote,

\[
b(\phi) := \max[\|\phi\|_{L^2}, \|q\phi\|_{L^2}].
\]
By (2.61)
\[ \|u(t)\|_{L^2}^2 \leq \|u(t_1)\|_{L^2}^2 + C_T a(t), t_1 \leq t < T. \]  
(2.75)

We denote,
\[ \alpha(t) := \sup_{t_1 \leq s \leq t} b(u(s)). \]  
(2.76)

Integrating (2.64) from \( t_1 \) to \( t \), using (2.2), (2.65), (2.70), (2.71), (2.75) and the estimate \(|(u, u_x)| \leq 1/2\|u\|_{L^2}^2 + 1/2\|u_x\|_{L^2}^2\), we prove that,
\[ a(t) \leq C_T \alpha(t) + C_T t_1, t \in [t_1, T), \]  
(2.77)

where we used that \( a(t) \) is a non-decreasing function. Integrating again (2.64) from \( t_1 \) to \( t \), using now (2.2), (2.65), (2.70), (2.71), (2.75), (2.77) and as \(|(u, u_x)| \leq \|u\|_{L^2} \|u_x\|_{L^2}\) we obtain that,
\[ a(t) \leq \sqrt{\|u(t)\|_{L^2} \|u_x(t)\|_{L^2}} + C_T(T - t_1) \sqrt{\alpha(t)}, t_1 \leq t < T. \]  
(2.78)

We denote, \( g(t_1, t) := (\int_{t_1}^t |f(\tau)|^2 d\tau)^{1/2} \). Now we integrate (2.64) from \( t_1 \) to \( t \), and using (2.78) we prove that,
\[ \|u(t)\|_{L^2}^2 \leq \|u(t_1)\|_{L^2}^2 + 2g(t_1, t) \left[ \sqrt{\|u(t)\|_{L^2} \|u_x(t)\|_{L^2}} + C_T(T - t_1) \sqrt{\alpha(t)} \right], t_1 \leq t < T. \]  
(2.79)

By (2.79) for some constant \( C \),
\[ \|u(t)\|_{L^2}^2 \leq C \left[ \|u(t_1)\|_{L^2}^{8/3} + 2^{4/3} (g(t_1, t))^{4/3} \left( \|u_x(t)\|_{L^2}^{2/3} + C_T(T - t_1)^{4/3} \alpha(t)^{2/3} + C_{T,1} \right) \right] + 1, t_1 \leq t < T. \]  
(2.80)

Here we consider first the case, \( \|u(t)\|_{L^2} \leq 1 \), where the estimate is trivial, and then the case \( \|u(t)\|_{L^2} \geq 1 \).

For any \( 1 \leq p < 5 \) there is a constant \( C \) such that for any \( u \in W_{1,2} \) and any \( \epsilon > 0 \),
\[ \|u\|_{L^{p+1}}^{p+1} \leq C\epsilon \|u_x\|_{L^2}^2 + \frac{C}{\epsilon^{(\nu-1)/2}} \|u\|_{L^2}^{2\nu}, \]  
(2.81)

where \( \nu := 1 + \frac{2(p-1)}{5-p} \). We give the proof of (2.81) below. Integrating (2.63) from \( t_1 \) to \( t \), and by (2.2) with \( \epsilon = 1/4, (2.72), (2.73), (2.75), (2.77), (2.80), (2.81) \) and as \( \nu \leq 3 \),
\[ \frac{1}{4} b(u(t))^2 \leq W(t_1) + C_{T,1} + C_T \alpha(t) + C_T (T - t_1) \left( \epsilon \alpha(t)^2 + \frac{1}{\epsilon^{(\nu-1)/2}} (C_{T,1} + g(t_1, t)^{4\nu/3} \alpha(t)^2) \right), t_1 \leq t < T. \] (2.82)

Pick any \( \epsilon \) and \( \Delta \) such that, \( C_T \left[ \epsilon + \frac{1}{\epsilon^{(\nu-1)/2}} \| f \|_{L^\infty([0,T])} \Delta^{2\nu/3} \right] \leq 1/8. \) Then, by (2.82),

\[ \alpha(t)^2 \leq 8 \left\{ W(t_1) + C_{T,1} + C_T \alpha(t) + \frac{C_{T,1}}{\epsilon^{(\nu-1)/2}} \right\}, t_1 \leq t < \min[t_1 + \Delta, T]. \] (2.83)

As by Theorem 2.3 and Proposition 2.8 we can approximate solutions in \( H_1 \) by solutions in \( H_2 \), equations (2.79), (2.80), (2.82) and (2.83) hold also if \( u \in C([0,T], H_1) \).

Suppose now that we are given a solution \( u \in C([0,T_m], H_1) \) to the FNLS (1.1) where \( T_m \) is the maximal time of existence. Then, we must have \( T_m = \infty \), because if \( T_m < \infty \) we can take \( t_1 = T_m - \Delta \), and then by (2.75), (2.77) and (2.83),

\[ \| u(t) \|_{H_1} \leq C_{T,1}, \text{ for } T_m - \Delta \leq t < T_m, \] (2.84)

and by Remark 2.4 we can continue \( u(t) \) to \( t > T_m \), in contradiction with the definition of \( T_m \).

We now prove (2.81). By the Sobolev-Gagliardo-Nirenberg inequality [10]

\[ \| u \|_{p+1} \leq C \| u_x \|_L^a \| u \|_L^{(p+1)(1-a)}, a := \frac{1}{2} - \frac{1}{p+1}. \] (2.85)

Inequality (2.85) is stated in [10] for \( u \in C_0^\infty(\mathbb{R}) \), but by continuity it applies to \( u \in W_{1,2}(\mathbb{R}) \) and extending \( u \in W_{1,2} \) as an even function in \( W_{1,2}(\mathbb{R}) \) it also holds for \( u \in W_{1,2} \). Denote, \( k := \frac{5-\nu}{4} \). Then, by (2.85),

\[ \| u \|_{p+1} \leq C \| u_x \|_L^{2(1-k)} \| u \|_L^{2k} \leq C \| u_x \|_L^2 + \frac{C}{\epsilon^{(\nu-1)/2}} \| u \|_L^{2\nu}, \] (2.86)

where we used the inequality, \( a^{1-k} b^k \leq \epsilon a + \frac{1}{\epsilon (1-k)} b, a, b \geq 0, \epsilon > 0, 0 < k \leq 1. \)
REMARK 2.10. In the case where \( f \equiv 0 \) we prove that the solutions are global under weaker assumptions because we do not need to use identity (2.64). Suppose that assumptions A and B are satisfied with \( I = [0, \infty) \), and with \( f \equiv 0 \), that (2.61), (2.62), and (2.63) hold, where for each fixed \( x, t \in \mathbb{R}^+ \), \( h \) is in \( C^2(\mathbb{C}, \mathbb{R}) \) in the real sense, and \( h(x, t, 0) = 0 \). Moreover, assume that \( \mathcal{H}_2^{(0)} \subset W_{2,2} \), that (2.72) and (2.73) hold with \( 1 < p < 5 \). Then, the conclusions of Theorem 2.9 are true. The proof is much simpler now because by (2.61) \( \|u(t)\| \leq C \), and then by (2.63), (2.72), (2.73) and (2.81), \( b(u(t)) \leq C \). We complete the proof as in Theorem 2.9.

REMARK 2.11. Recall that our results in local solutions given in Theorems 2.2, 2.3, 2.5 and 2.6, in Remarks 2.4, 2.7 and in Proposition 2.8 hold without any restriction on the grow of \( V_1 \) at infinity. Our results in global solutions given in Theorem 2.9 and in Remark 2.10 require that \( \mathcal{H}_2 \subset W_{2,2} \). We give now a sufficient condition for this to hold. We denote by \( \text{Lip} \) the set of all continuous and bounded functions, \( f \), defined on \([0, \infty)\) that are globally Lipschitz, i.e. such that,

\[
\text{Lip}(f) := \sup_{x, y \in [0, \infty), x \neq y} \frac{|f(x) - f(y)|}{|x - y|} < \infty. \tag{2.87}
\]

Note that if \( f \in \text{Lip} \) then \( f \) is differentiable for a.e. \( x \) with \( \dot{f} \in L^\infty \) and \( \text{Lip}(f) := \|\dot{f}\|_{L^\infty} \). Suppose as above that \( V = V_1 + V_2, V_j \in L^1_{\text{loc}}(\mathbb{R}^+) \), \( j = 1, 2, V_1 \geq 0 \) and \( V_2 \) satisfies (2.3). Remark that, eventually adding 1 to \( V_1 \) and substracting it from \( V_2 \), we can assume that \( V_1 \geq 1 \). Suppose that \( g := (V_1)^{-1/2} \in \text{Lip} \). For \( c > 0 \) denote \( g_c := (V_1 + c)^{-1/2} \). Observe that, \( \dot{g}_c := (1 + c V_1)^{-3/2} \dot{g} \). Then, \( g_c \in \text{Lip} \) and \( \text{Lip}(g_c) \) decreases monotonically as \( c \to \infty \). It follows from Theorem 7.1 of [12] (this paper considers the case in the whole line, but the proof in our case is the same) that if \( \lim_{c \to \infty} \text{Lip}g_c < 1 \), then \( H_1 := H_0 + V_1 \) is selfadjoint in the domain, \( D(H_1) = W_{2,2} \cap W^{(0)}_{1,2} \cap D(V_1) \). Hence, by (2.4) and Kato-Rellich’s theorem, \( D(H) = W_{2,2} \cap W^{(0)}_{1,2} \cap D(V_1) \). Assume moreover, that \( V_1 \in L^2_{\text{loc}}([0, \infty)) \). Let us take any \( h \in C_0^\infty([0, \infty)) \), satisfying \( h(x) = 1, 0 \leq x \leq 1 \). We decompose any \( \phi \in \mathcal{H}_2 \) as \( \phi = \phi_1 + \phi_2 \), with \( \phi_1 := \phi - \phi(0)h, \phi_2 := \phi(0)h \). Then, under the assumptions above, \( \phi \in \mathcal{H}_2 \leftrightarrow \phi_1 \in D(H) \), and it follows that in this case \( \mathcal{H}_2 = W_{2,2} \cap D(V_1) \).

Note that \( V_1 \) can be any positive polynomial, \( p(x) \), or \( \exp(p(x)) \), \( \exp \exp(p(x)) \), \( \cdots \). Moreover, (2.70) is satisfied, for example, if \( V_1 \) any positive polynomial, or \( V_1 = e^x \), and \( (V_2)_+ \) fulfills (2.1). Finally, note that in the case of Remark 2.10 where the force is identically zero we do not need that \( V_1 \in L^2_{\text{loc}}([0, \infty)) \). In this case
we can admit, for example, $V_1 = \frac{1}{x}, 0 < x \leq 1, V_1 = 1, x \geq 1, k \geq 2$ (for the case $k=2$ see Example 7.4 of [12]).

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