The fixed point property in every weak homotopy type

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THE FIXED POINT PROPERTY IN EVERY WEAK HOMOTOPY TYPE

By Jonathan Ariel Barmak

Abstract. We prove that for any connected compact CW-complex \( K \) there exists a space \( X \) weak homotopy equivalent to \( K \) which has the fixed point property, that is, every continuous map \( X \to X \) has a fixed point. The result is known to be false if we require \( X \) to be a polyhedron. The space \( X \) we construct is a non-Hausdorff space with finitely many points.

A topological space \( X \) has the fixed point property if every continuous map \( f : X \to X \) has a fixed point. We will prove the following

**THEOREM 1.** Let \( K \) be a connected compact CW-complex. Then there exists a topological space \( X \) weak homotopy equivalent to \( K \) with the fixed point property.

If we require \( X \) to be a polyhedron, the result is known to be false. Though the fixed point property is not a homotopy invariant, every polyhedron homotopy equivalent to a sphere lacks the fixed point property (see [6, Theorem 7.1], [8] or the proof of [11, Theorem]). The space \( X \) we find has finitely many points. Therefore, we are also proving the following result. The homotopy type of any connected compact CW-complex can be realized by the order complex of a finite partially ordered set with the fixed point property.

A simplicial complex \( K \) has the fixed simplex property if for every simplicial map \( f : K \to K \) there exists a simplex \( \sigma \in K \) such that \( f(\sigma) = \sigma \) or, equivalently, if every simplicial endomorphism of \( K \) fixes a point of the realization of \( K \). The spheres \( S^n \) do not have the fixed point property, but they do have triangulations with the fixed simplex property provided that \( n \geq 2 \) (see Proposition 3). We will show that for every simply connected compact polyhedron \( K \) there exists a finite simplicial complex \( L \) homotopy equivalent to \( K \) with the fixed simplex property. Then the finite topological space \( \mathcal{X}(L) \) associated to \( L \) has the fixed point property. This will prove Theorem 1 for simply connected complexes. If \( K \) is not simply connected we will be able to modify the construction above to obtain a finite model of \( K \) with the fixed point property but it will not be the poset of faces of a complex.

We sketch in a few lines the idea of the construction of \( L \) from \( K \) and the main parts of the proof. We first consider integer homology classes \( \alpha_{k,l} \in H_k(K) \)
which are a basis of the rational $k$-homology of $K$ and then realize each $\alpha_{k,l}$ as the image of the fundamental class $[M_{k,l}]$ of a $k$-dimensional oriented pseudomanifold through a map $M_{k,l} \to K$. We construct $L$ as follows. We find a sufficiently fine ($j$-th barycentric) subdivision $K^j$ of $K$ and attach to $K^j$ mapping cylinders of the maps $M_{k,l}^j \to K^j$ and of approximations to the identities $M_{k,l}^r \to M_{k,l}^r$ ($r < j$). In this way we manage to concentrate the homology classes $\alpha_{k,l}$ in “few” simplices, those of $M_{k,l}^r$. The complex $L$ satisfies this singular property: if $c = \sum t_i \sigma_i \in C_k(L)$ is a cycle such that $\sum |t_i|$ is less than or equal to the number of $k$-simplices in $M_{k,l}^r$ and the $(\alpha_{k,l} \otimes 1_Q)$-coordinate of $[c] \in H_k(L; \mathbb{Q})$ is different from zero, then $c$ is, up to sign, the fundamental cycle of $M_{k,l}^r$. Therefore if $f$ is a simplicial endomorphism of $L$, one of the following holds: (1) the matrices of $f_* : H_k(L; \mathbb{Q}) \to H_k(L; \mathbb{Q})$ in the basis $\{\alpha_{k,l} \otimes 1_Q\}_l$ have all the diagonal zero for each $k \geq 2$, in which case the Lefschetz fixed point theorem gives us a fixed simplex, or (2) the map $f$ maps one of the $M_{k,l}^r$ into itself and $f|_{M_{k,l}^r} : M_{k,l}^r \to M_{k,l}^r$ is a simplicial automorphism. The argument is then complete if we show that the pseudomanifolds $M_{k,l}^r$ can be assumed to be asymmetric, in the sense that every automorphism fixes a vertex.

The following notions are a rigid version of Gromov’s simplicial volume and the $\ell^1$-norm. Let $C$ be a finitely generated free $\mathbb{Z}$-module with a fixed basis $\{b_1, b_2, \ldots, b_r\}$. The norm $\|c\|$ of an element $c = \sum t_i b_i \in C$ is $\sum |t_i|$. If $f : C \to C'$ is a morphism between finitely generated free $\mathbb{Z}$-modules (each of them with a chosen basis), the norm of $f$ is $\|f\| = \max_{c \neq 0} \frac{\|f(c)\|}{\|c\|}$. Note that $\|f\|$ is well-defined since $\|f(c)\| \leq \|c\| \max_i \|f(b_i)\|$, where $\{b_i\}_i$ is the chosen basis of $C$. If $C = 0$ define $\|f\| = 0$. For a composition $fg$ we have $\|fg\| \leq \|f\| \|g\|$.

Let $C_*$ be a finitely generated free chain complex with a given basis for each $C_k$. The norm of $\alpha \in H_k(C)$ is $\|\alpha\| = \min \{\|c\| \mid [c] = \alpha\}$. Here $[c]$ denotes the class of a cycle $c$ in homology.

When $K$ is a finite simplicial complex we will always consider the chain complex $C_*(K)$ with the usual basis for $C_k(K)$ given by one oriented $k$-simplex $[v_0, v_1, \ldots, v_k]$ for each $k$-simplex $\{v_0, v_1, \ldots, v_k\} \in K$. We denote by $H_*(K)$ the simplicial homology of $K$ with integer coefficients.

If $M$ is a closed $n$-dimensional oriented pseudomanifold, the norm $\|[M]\| \in H_n(M)$ of its fundamental class is the number of $n$-simplices in $M$.

If $\varphi : K \to L$ is a simplicial map between finite simplicial complexes, $\varphi_* : C_*(K) \to C_*(L)$ maps an oriented $k$-simplex $[v_0, v_1, \ldots, v_k]$ to $[\varphi(v_0), \varphi(v_1), \ldots, \varphi(v_k)]$ if $\varphi(v_i) \neq \varphi(v_j)$ for $i \neq j$ and to 0 otherwise. Therefore $\varphi_k : C_k(K) \to C_k(L)$ has norm at most 1.

If $L$ is a finite simplicial complex and $K$ is a subdivision of $L$, the subdivision operator $\lambda : C_*(L) \to C_*(K)$ is a homotopy inverse to the chain map induced by any simplicial approximation to the identity and maps a $k$-simplex $\sigma \in L$ into a signed sum of all the $k$-simplices of $K$ contained in $\sigma$. Therefore, the norm of $\lambda_k : C_k(L) \to C_k(K)$ is the maximum number of $k$-simplices in which a $k$-simplex of $L$ is subdivided. In particular, when $K$ is the first barycentric subdivision $L'$ of
such that the fixed point property. The following definition is inspired by their example.

There are \( v \neq b(\sigma) \) for any other \( \sigma \), such that \( \| \lambda_k \| = (k+1)! \) if \( \dim L \geq k \). In this case, if \( \alpha \in H_k(L), \| \lambda_\ast(\alpha) \| \leq \| \lambda_k \| \| \alpha \| \leq (k+1)! \| \alpha \| \). In general the equality \( \| \lambda_\ast(\alpha) \| = (k+1)! \| \alpha \| \) does not hold if \( k < \dim(L) \).

Let \( K \) be a finite simplicial complex. The barycenter of a simplex \( \sigma \in K \) will be denoted by \( b(\sigma) \) or \( \hat{\sigma} \). The simplices of \( K' \) are then the sets \( \{ \hat{\sigma}_0, \hat{\sigma}_1, \ldots, \hat{\sigma}_k \} \) such that \( \sigma_i \subseteq \sigma_{i+1} \) for every \( i \). Given a simplicial map \( \varphi : K \rightarrow L \), we denote by \( \varphi' : K' \rightarrow L' \) the map \( b(\sigma) \mapsto b(\varphi(\sigma)) \) and in general \( \varphi_j : K^j \rightarrow L^j \) is the map induced in the \( j \)-th barycentric subdivisions.

In [1, Example 2.4], Baclawski and Björner construct a finite model of \( S^2 \) with the fixed point property. The following definition is inspired by their example.

**Definition 2.** We will say that a complex \( K \) is asymmetric if there exists a vertex \( v \in K \) which is fixed by every simplicial automorphism of \( K \).

**Proposition 3.** Let \( M \) be an \( n \)-dimensional pseudomanifold with \( n \geq 2 \). Then there exists a subdivision \( L \) of \( M \) such that the \( j \)-th barycentric subdivision \( L^j \) of \( L \) is asymmetric for every \( j \geq 0 \).

**Proof.** Given a complex \( K \) and a simplex \( \sigma \in K \), we denote by \( \deg_K(\sigma) \) the number of maximal simplices of \( K \) containing \( \sigma \). Then \( \deg_K(\sigma) \) is the number of maximal simplices in the link \( \text{lk}_K(\sigma) \) if \( \sigma \) is not maximal, and 1 if \( \sigma \) is maximal in \( K \). Define \( d(K) = \max_{\sigma \in K} \deg_K(\sigma) = \max_{v \in K} \deg_K(v) \) where the second maximum is taken over all the vertices \( v \) of \( K \). It is not hard to see that there exists a subdivision \( L \) of \( M \) which contains a vertex \( v_0 \) such that \( \deg_L(v) < \deg_L(v_0) \) for any other \( v \in L \). Then clearly \( L \) is asymmetric since the degree \( \deg \) is preserved by automorphisms of \( L \) and thus \( v_0 \) is fixed by any such an automorphism. Moreover, we will show that \( \deg_L(v) < \deg_L(v_0) \) for every vertex \( v \neq v_0 \). It follows by induction that \( L^j \) is asymmetric for each \( j \geq 0 \).

Let \( v \in L^j, v = b(\sigma) \) where \( \sigma \) is a simplex of \( L \). Let \( k = \dim \sigma \). A maximal simplex of \( \text{lk}_{L^j}(b(\sigma)) \) is obtained by choosing a chain \( \sigma^0 < \sigma^1 < \cdots < \sigma^{k-1} \) of proper faces of \( \sigma \) and a chain \( \sigma^{k+1} < \sigma^{k+2} < \cdots < \sigma^n \) of simplices containing \( \sigma \). There are \( (k+1) \) possible choices for \( \sigma^k \), \( k \) for \( \sigma^{k-1} \), \( k \) for \( \sigma^0 \). On the other hand there are \( \deg_L(\sigma) \) choices for \( \sigma^n \), \( (n-k) \) for \( \sigma^{n-1} \), \( (n-k-1) \) for \( \sigma^{n-2} \), \( \ldots \), \( 2 \) for \( \sigma^{k+1} \). Therefore

\[
\deg_{L^j}(b(\sigma)) = (k+1)!(n-k)!(n-k)! \deg_L(\sigma).
\]

- If \( k = n \), \( \deg_{L^j}(b(\sigma)) = (n+1)! < n!d(L) \) since \( d(L) = n+1 \) only when the pseudomanifold is isomorphic to the boundary of an \( (n+1) \)-simplex, which is not the case since \( L \) is asymmetric.
- If \( k = n-1 \), \( \deg_{L^j}(b(\sigma)) = n! \deg_L(\sigma) = 2n! < n!d(L) \).
- If \( 1 \leq k \leq n-2 \), \( \binom{n}{k} \geq n \), so \( \deg_{L^j}(b(\sigma)) \leq (n-1)!(k+1) \deg_L(\sigma) < n!d(L) \).
Moreover, we can assume that $barycenters$ of unions of simplices $\sigma$ in them, then it is a face of $\tau$. If $1$ or $2$ holds for every $i,j$, then the convex hull $S$ is comparable with both $\sigma_i$ and $\sigma_j$, by induction it suffices to prove that $\{\sigma,\sigma_0,\sigma_1,\ldots,\sigma_{i-1},\sigma_{i+1},\ldots,\sigma_k\}$ and $\{\sigma,\sigma_0,\sigma_1,\ldots,\sigma_{j-1},\sigma_{j+1},\ldots,\sigma_k\}$ satisfy the hypothesis of the lemma. Let $0 \leq l \leq k$, $i \neq l \neq j$. We have to verify that $\sigma$ and $\sigma_l$ are comparable or disjoint. If $(i,l)$ and $(j,l)$ satisfy (3), then $\sigma$ and $\sigma_l$ are disjoint. Suppose $(i,l)$ satisfies (3) and $\sigma_j$ is comparable with $\sigma_l$. By the choice of $i$ and $j$, $\sigma_j$ cannot be a proper face of $\sigma_l$. Then $\sigma_l \subseteq \sigma_j \subseteq \sigma$ so $\sigma_l$ and $\sigma$ are comparable. By symmetry it only remains to analyze the case that $\sigma_l$ is comparable with both $\sigma_i$ and $\sigma_j$. If $\sigma_l$ is a face of any of them, then it is a face of $\sigma$. If $\sigma_i$ and $\sigma_j$ are faces of $\sigma_l$, then so is $\sigma$. □

Remark 5. In the conditions of Lemma 4, note that if the convex hull $S$ of $\{\hat{\sigma}_0, \hat{\sigma}_1, \ldots, \hat{\sigma}_k\}$ is a $k$-dimensional subcomplex of $\tau'$, then it contains at most $(k+1)!$ many $k$-simplices of $\tau'$. Moreover, if the equality holds then all the pairs $i,j$ satisfy condition (3). The proof of Lemma 4 shows that the vertices of $S \leq \tau'$ are barycenters of unions of simplices $\sigma_i$. Therefore, the $k$-simplices of $\tau'$ contained in $S$ are of the form $\{\hat{\tau}_0, \hat{\tau}_1, \ldots, \hat{\tau}_k\}$ where $\tau_l = \bigcup_{i \in A_l} \sigma_i$, $A_l \subseteq [0,k]$, and $\tau_l \subseteq \tau_{l+1}$. Moreover, we can assume that $A_l \subseteq A_{l+1}$. Since there are only $(k+1)!$ sequences $\emptyset \neq A_0 \subseteq A_1 \subseteq \cdots \subseteq A_k \subseteq [0,k]$, $S$ is decomposed in at most $(k+1)!$ many $k$-simplices. If $\sigma_i \subseteq \sigma_j$ for some $i \neq j$ then half of these sequences do not give a $k$-simplex, so $S$ is decomposed in at most $\frac{(k+1)!}{2}$ many $k$-simplices.

Let $\varphi : K \to L$ be a simplicial map between finite simplicial complexes. We will work with the following version of the simplicial mapping cylinder $Z_\varphi$ of $\varphi$. First we choose a total ordering in the set of vertices of $K$. The vertex set of $Z_\varphi$ is the disjoint union of the vertex set of $K$ and of $L$. The simplices of the cylinder are the simplices of $L$ together with sets of the form $\{v_0, v_1, \ldots, v_l, \varphi(v_{l+1}), \varphi(v_{l+2}), \ldots, \varphi(v_m)\}$ where $\{v_0, v_1, \ldots, v_m\}$ is a simplex of $K$ and $v_i \leq v_{i+1}$ for every $i$ ($v_l$ could be equal to $v_{l+1}$).
There is a simplicial retraction $p : Z_\varphi \to L$ of the canonical inclusion $j : L \to Z_\varphi$ defined by $p(v) = \varphi(v)$ if $v \in K$. Therefore $pi = \varphi$ where $i$ denotes the canonical inclusion of $K$ into the cylinder. The composition $jp$ lies in the same contiguity class as the identity $1_{Z_\varphi}$, so $p_* : C_*(Z_\varphi) \to C_*(L)$ is a homotopy equivalence [10, p.151].

Suppose $K$ is a subdivision of a complex $L$ and that $\psi : K \to L$ is a simplicial approximation to the identity. In other words, $\psi$ is a vertex map which maps each vertex $v \in K$ to any vertex $w \in L$ of the unique open simplex of $L$ containing $v$. In this case $\psi_* : C_*(K) \to C_*(L)$ is a homotopy equivalence. Since $p_* : C_*(Z_\varphi) \to C_*(L)$ is a homotopy equivalence and $\psi = pi$, $i_* : C_*(K) \to C_*(Z_\varphi)$ is a homotopy equivalence. Since $C_*(K)$ is a subcomplex of $C_*(Z_\varphi)$, it is known that there exists a retraction $r : C_*(Z_\varphi) \to C_*(K)$. However, we need to control the norm $\|r_k\|$ of each $r_* : C_k(Z_\varphi) \to C_k(K)$. We will prove that for barycentric subdivisions $K = L'$ there is a retraction $r$ such that $\|r_k\| \leq (k+1)!$. It is not true that this inequality holds for any retraction $r$.

**Lemma 6.** Let $K$ be a finite simplicial complex. Then there exists an ordering of the vertices of $K'$, a simplicial approximation to the identity $\psi : K' \to K$ and a retraction $r : C_*(Z_\psi) \to C_*(K')$ satisfying the following:

1. If $S$ is a $k$-simplex of $Z_\psi$, then $\|r_k(S)\| \leq (k+1)!$.
2. If $S$ is a $k$-simplex of $Z_\psi$ with $k \geq 1$ such that $\|r_k(S)\| = (k+1)!$, then $S \subset K$.

**Proof.** Order the vertices $\hat{\sigma}$ of $K'$ in such a way that $\hat{\sigma} < \hat{\tau}$ implies $\dim(\sigma) \geq \dim(\tau)$. Let $\psi : K' \to K$ be any approximation to the identity. In other words, if $\hat{\sigma}$ is a vertex of $K'$, then $\psi(\hat{\sigma}) \in \sigma$. A $k$-simplex of $Z_\psi$ is of the form $S = \{\hat{\sigma}_0, \hat{\sigma}_1, \ldots, \hat{\sigma}_l, \psi(\hat{\sigma}_{l+1}), \psi(\hat{\sigma}_{l+2}), \ldots, \psi(\hat{\sigma}_m)\}$ where $l \geq -1$, $m \geq l$ and $\sigma_{i+1} \subset \sigma_i$ for all $0 \leq i \leq m-1$ (the simplices of $K$ are included in these). We can consider $S$ as a set of vertices of $K'$ identifying $v_i = \psi(\hat{\sigma}_i)$ with the barycenter of $\{v_i\}$. The hypothesis of Lemma 4 is satisfied. For any $0 \leq i, j \leq l$, $\sigma_i$ and $\sigma_j$ are comparable; if $l+1 \leq i, j \leq m$, $\{v_i\}$ and $\{v_j\}$ are disjoint or equal; if $0 \leq i \leq m$ and $l+1 \leq j \leq m$, then $\sigma_i \supseteq \sigma_j \supseteq \{v_j\}$. Thus, by Lemma 4, the convex hull of $S$ is a subcomplex $\Phi(S)$ of $K'$. The application $\Phi$ defines an acyclic carrier from $Z_\psi$ to $K'$. Let $r : C_*(Z_\psi) \to C_*(K')$ be a chain map carried by $\Phi$. Note that $r$ is uniquely determined by $\Phi$ since for a $k$-simplex $S \subset Z_\psi$, $\Phi(S)$ is $j$-dimensional with $j \leq k$. Thus, any homotopy $F : C_*(Z_\psi) \to C_{*+1}(K')$ carried by $\Phi$, given by the Acyclic carrier theorem, must be trivial.

If $S \subset Z_\psi$ is a $k$-simplex such that $\dim(\Phi(S)) < k$, then $r(S) \subset C_k(\Phi(S)) = 0$ is trivial. Otherwise $\dim(\Phi(S)) = k$ and then $\Phi(S)$ is a subdivision of $S$ (considered as a set of $k+1$ affinely independent vertices of $K'$). One has then the subdivision operator $\lambda : C_*(S) \to C_*(\Phi(S))$. Since for each $j$-face $\tilde{S}$ of $S$, $\Phi(\tilde{S})$ is a $j$-dimensional subcomplex, the acyclic carrier $\Phi$ when restricted to $C_*(S)$ is the usual subdivision carrier $\Phi(\tilde{S}) = K'(\tilde{S})$. Thus $\lambda, r|_{C_*(S)} : C_*(S) \to C_*(\Phi(S))$ are
carried by the same acyclic carrier $\Phi$ and by the same argument as before, they coincide. Hence $\|r_k(S)\| = \|\lambda_k(S)\|$ is the number of $k$-simplices in $\Phi(S)$ which is at most $(k + 1)!$ by Remark 5.

Clearly $r : C_*(Z_\psi) \to C_*(K)$ is a retraction since for $S \in K'$ we have $\Phi(S) = S$ and $\lambda(S) = S$.

Finally, suppose $S = \{\hat{\sigma}_0, \hat{\sigma}_1, \ldots, \hat{\sigma}_l, \psi(\hat{\sigma}_{l+1}), \psi(\hat{\sigma}_{l+2}), \ldots, \psi(\hat{\psi}_m)\}$ is a $k$-simplex of $Z_\psi$ with $k \geq 1$. If $l \geq 0$, $\sigma_0$ is comparable with each $\sigma_j$ and each $\{v_j\}$. By Remark 5, $S$ is subdivided in less than $(k + 1)!$ $k$-simplices of $K'$ so $\|r_k(S)\| < (k + 1)!$. This proves the second assertion of the lemma. □

Remark 7. It is well known that every singular $k$-homology class $\alpha$ of a space $X$ can be realized by a disjoint union $\bigsqcup M_i$ of closed $k$-dimensional oriented pseudomanifolds, meaning that there is a continuous map $f : \bigsqcup M_i \to X$ such that $\sum f_*([M_i]) = \alpha$ (see [5, p.108] for example). Then for every simplicial homology class $\alpha \in H_k(K)$ of a simplicial complex $K$ there exists a simplicial map $\varphi : \bigsqcup M_i \to K$ from a disjoint union of closed oriented pseudomanifolds such that $\sum \varphi_*([M_i]) = \alpha$.

Theorem 8. Let $K$ be a finite simplicial complex which is simply connected or, more generally, such that $H_1(K) = 0$. Then there exists a finite simplicial complex $L$ homotopy equivalent to $K$ with the fixed simplex property.

Proof.

First part: Construction of $L$. Let $n = \dim(K)$. For each $2 \leq k \leq n$ let $d_k$ be the rank of $H_k(K; \mathbb{Q})$. Take for each $2 \leq k \leq n$ homology classes $\alpha_{k,1}, \alpha_{k,2}, \ldots, \alpha_{k,d_k} \in H_k(K)$ such that $\{\alpha_{k,1} \otimes 1_\mathbb{Q}, \alpha_{k,2} \otimes 1_\mathbb{Q}, \ldots, \alpha_{k,d_k} \otimes 1_\mathbb{Q}\}$ is a basis of $H_k(K; \mathbb{Q})$. By Remark 7 each $\alpha_{k,l}$ can be realized by a disjoint union of closed oriented pseudomanifolds. Moreover, by changing the $\alpha_{k,l}$’s if needed we can assume that each of them is realized by a single pseudomanifold. For each $k \geq 2$ and $1 \leq l \leq d_k$ let $M_{k,l}$ be a $k$-dimensional oriented pseudomanifold and let $\varphi_{k,l} : M_{k,l} \to K$ be a simplicial map such that $(\varphi_{k,l})_*([M_{k,l}]) = \alpha_{k,l}$. By Proposition 3 we can assume that $M_{k,l}$ is asymmetric for each $k, l$ as well as all their iterated barycentric subdivisions.

We define an increasing sequence $s_1, s_2, s_3, \ldots, s_n$ of non-negative integers as follows. Let $s_1 = 0$. Let $N_{2,l} = \|[M_{2,l}]\|$ be the number of 2-simplices in $M_{2,l}$ for each $1 \leq l \leq d_2$ and let $N_2 = \max_l N_{2,l}$. If $P$ is any finite simplicial complex, the cover $\mathcal{U}$ of $P$ given by the open stars $st_P(v)$ of the vertices of $P$ has a Lebesgue number $\delta > 0$. Therefore, there exists a positive integer $s_2$ such that for each $s \geq s_2$, every connected subcomplex of $P^s$ generated by at most $N_2$ many simplices is contained in an element of $\mathcal{U}$, and in particular in a contractible subcomplex of $P$. We take $s_2$ in such a way that the assertion above holds for $P$ when $P$ is any $M_{k,l}$ with $k \geq 3$ and $1 \leq l \leq d_k$. 


Now let $N_{3,l} = \|[M_{3,l}^{s_2}]\|$ for each $1 \leq l \leq d_3$ and let $N_3 = \max_l N_{3,l}$. Take $s_3 \geq s_2$ such that for each $s \geq s_3$, $k \geq 4$ and $1 \leq l \leq d_k$, every connected subcomplex of $M_{k,l}^s$ generated by at most $N_3$ many simplices is contained in a contractible subcomplex.

In general, suppose $s_2, s_3, \ldots, s_m$ are defined, with $m \leq n - 2$. Then define $N_{m+1,l} = \|[M_{m+1,l}^{s_m}]\|$ for each $1 \leq l \leq d_{m+1}$ and let $N_{m+1} = \max_l N_{m+1,l}$. Take $s_{m+1} \geq s_m$ such that for each $s \geq s_{m+1}$, every connected subcomplex of $M_{k,l}^s$ generated by at most $N_{m+1}$ many simplices is contained in a contractible subcomplex for each $k \geq m + 2$ and $1 \leq l \leq d_k$.

Finally define $N_{n,l} = \|[M_{n,l}^{s_n}]\|$, $N_n = \max_l N_{n,l}$, $N = \max_{2 \leq k \leq n} N_k$ and take $s_n \geq s_{n-1}$ such that for each $s \geq s_n$, any connected subcomplex of $K^s$ generated by at most $N$ simplices is contained in a contractible subcomplex.

We now define for each $k \geq 2$ and $1 \leq l \leq d_k$ a cylinder $C_{k,l}$ which will be attached to $K^{s_n}$. Each $C_{k,l}$ consists of three parts. The first one is $C_{k,l}^a = Z_{\varphi_{k,l}^{s_n}}$, the cylinder of $\varphi_{k,l}^{s_n} : M_{k,l}^{s_n} \to K^{s_n}$ (see Figure 1). The second part $C_{k,l}^b$ is constructed as follows. We glue $N$ cylinders $Z_1 M_{k,l}^{s_n}$ of the identity $1_{M_{k,l}^{s_n}} : M_{k,l}^{s_n} \to M_{k,l}^{s_n}$, the second base of one with the first base of the following, to build a long cylinder $C_{k,l}^b$ with both bases equal to $M_{k,l}^{s_n}$. The last part $C_{k,l}^c$ is the union of $s_n - s_{k-1}$ mapping cylinders. For each $s_{k-1} < m \leq s_n$ there is a simplicial approximation to the identity $\psi_{k,l,m} : M_{k,l}^m \to M_{k,l}^{m-1}$ and a retraction $R_m = R_{k,l,m} : C_*(Z_{\psi_{k,l,m}}) \to C_*(M_{k,l}^m)$ satisfying properties (1) and (2) in the statement of Lemma 6. When we glue the cylinders $Z_{\psi_{k,l,m}}$, identifying a base of one with a base of the following, we obtain a cylinder $C_{k,l}^c$ with one base equal to $M_{k,l}^{s_n}$ and the other equal to $M_{k,l}^{s_{k-1}}$.

Finally we glue $C_{k,l}^a$ with one extreme of $C_{k,l}^b$ and $C_{k,l}^c$ with the other extreme of $C_{k,l}^b$. This is $C_{k,l}$.
Let $L$ be the union of all the cylinders $C_{k,l}$ for $k \geq 2$ and $1 \leq l \leq d_k$, all intersecting in $K^{s_n}$. Each $C^c_{k,l}$ deformation retracts to $M^{s_n}_{k,l}$, $C^b_{k,l}$ deformation retracts to $M^{s_n}_{k,l}$, and $C^a_{k,l}$ deformation retracts to $K^{s_n}$. Therefore $L$ is homotopy equivalent to $K$. We will show that $L$ has the fixed simplex property.

**Second part: $L$ has the fixed simplex property.** Let $i_{k,l} : M^{s_{k-1}}_{k,l} \hookrightarrow L$ be the inclusion map in the free extreme of $C^c_{k,l}$. Note that $(i_{k,l})_* [M^{s_{k-1}}_{k,l}] = i_* (r_{k,l})_* [M^{s_{k-1}}_{k,l}] = i_* \lambda_*(\alpha_{k,l})$, where $i : K^{s_n} \hookrightarrow L$ is the inclusion. Hence, $B_k = \{ (i_{k,l})_* ([M^{s_{k-1}}_{k,l}]) \otimes 1 \mathbb{Q} \}$ is a basis of $H_k(L; \mathbb{Q})$.

Let $f : L \to L$ be a simplicial map. We study for each $k$ the matrix of $f_* : H_k(L; \mathbb{Q}) \to H_k(L; \mathbb{Q})$ in the basis $B_k$. Since $M^{s_{k-1}}_{k,l}$ has $N_k$ simplices, $f(M^{s_{k-1}}_{k,l})$ lies in a connected subcomplex of $L$ generated by at most $N_k$ simplices. Since each $C^b_{k',l'}$ is constructed gluing $N \geq N_k$ cylinders then one of the following holds:

1. $f(M^{s_{k-1}}_{k,l}) \subseteq \bigcup_{k',l'} (C^c_{k',l'} \cup C^b_{k',l'})$ or
2. $f(M^{s_{k-1}}_{k,l}) \subseteq C^b_{k',l'} \cup C^c_{k',l'}$ for some $k',l'$.

In the first case, call $C^{ab} = \bigcup_{k',l'} (C^a_{k',l'} \cup C^b_{k',l'})$. Just as $L$, the complex $C^{ab}$ deformation retracts to $K^{s_n}$, but in contrast to $L$, for $C^{ab}$ the retraction $r^{ab} : C^{ab} \to K^{s_n}$ may be taken simplicial. Since we are assuming $f i_{k,l} : M^{s_{k-1}}_{k,l} \to C^{ab}$, $r^{ab} f i_{k,l} (M^{s_{k-1}}_{k,l})$ is contained in a connected subcomplex of $K^{s_n}$ generated by $N_k$, $l \leq N_k$ simplices. By the choice of $s_n$, this connected subcomplex lies in a contractible subcomplex of $K^{s_n}$, and then $r^{ab}_* (f i_{k,l})_* [M^{s_{k-1}}_{k,l}] = 0 \in H_k(K^{s_n})$. Since $r^{ab}$ is a homotopy equivalence $(f i_{k,l})_* [M^{s_{k-1}}_{k,l}] = 0 \in H_k(C^{ab})$, and then $f_* (i_{k,l})_* [M^{s_{k-1}}_{k,l}] = 0 \in H_k(L)$. In this case the $l$-th column of the matrix of $f_*$ is zero.

Assume then that (2) holds. Call $C^{bc}_{k',l'} = C^b_{k',l'} \cup C^c_{k',l'}$ and suppose that $f i_{k,l} (M^{s_{k-1}}_{k,l}) \subseteq C^{bc}_{k',l'}$. The complex $C^{bc}_{k',l'}$ deformation retracts to $M^{s_{k-1}}_{k',l'}$ by a simplicial retraction $r = r^{bc}_{k',l'} : C^{bc}_{k',l'} \to M^{s_{k-1}}_{k',l'}$. Since $r f i_{k,l} (M^{s_{k-1}}_{k,l})$ is contained in a connected subcomplex of $M^{s_{k-1}}_{k',l'}$ generated by $N_{k,l} \leq N_k$ simplices, $r_* (f i_{k,l})_* [M^{s_{k-1}}_{k,l}] = 0 \in H_k(M^{s_{k-1}}_{k',l'})$ if $k' > k$, by the choices of $s_k$ and $s_{k-1}$. If $k' < k$, then $H_k(M^{s_{k-1}}_{k',l'}) = 0$ since $\dim M^{s_{k-1}}_{k',l'} = k'$. Therefore in this case we also have $r_* (f i_{k,l})_* [M^{s_{k-1}}_{k,l}] = 0 \in H_k(M^{s_{k-1}}_{k',l'})$. Since $r$ is a homotopy equivalence we conclude that if $k' \neq k$, then $(f i_{k,l})_* [M^{s_{k-1}}_{k,l}] = 0 \in H_k(C^{bc}_{k',l'})$ and therefore $f_* (i_{k,l})_* [M^{s_{k-1}}_{k,l}] = 0 \in H_k(L)$. Hence the $l$-th column of $f_*$ is also zero. It remains to analyze the case $k' = k$. In that case $r_* (f i_{k,l})_* [M^{s_{k-1}}_{k,l}] = 0 \in H_k(M^{s_{k-1}}_{k',l'})$ is an integer multiple of the fundamental class $[M^{s_{k-1}}_{k',l'}]$, and then $f_* (i_{k,l})_* [M^{s_{k-1}}_{k,l}] = (i_{k,l})_* r_* (f i_{k,l})_* [M^{s_{k-1}}_{k,l}]$ is an integer multiple of $(i_{k,l})_* [M^{s_{k-1}}_{k',l'}]$. If $l' \neq l$, the $l$-th column of $f_*$ has a zero in the $l$-th entry. The last and most important case is $l' = l$.

If for each $k,l$ the case (1) occurs or the case (2) for $(k',l') \neq (k,l)$, then the trace of the matrix of $f_*$ in each positive degree is zero and by the Lefschetz fixed
point theorem, \( f \) must fix a simplex. We can assume then that there exists one pair \( k, l \) such that (2) holds for \((k',l') = (k,l)\) and that \((fi_{k,l})*[M_{k,l}^{s_{k-1}}] \in H_k(C_{k,l}^{bc})\) is non-trivial.

The simplicial projection \( C_{k,l}^b \to M_{k,l}^{sn} \) of the cylinder into the extreme in contact with \( C_{k,l}^c \) extends to a simplicial retraction \( p : C_{k,l}^{bc} \to C_{k,l}^c \). On the other hand Lemma 6 provides retractions \( R_m : C_*\left(\mathbb{Z}_{\psi_{k,l},m}\right) \to C_*(M_{k,l}^{m})\) for each \( s_{k-1} < m \leq s_n \). Each of them extends to a retraction

\[
\tilde{R}_m : C_*\left(\bigcup_{q=m}^{s_n} \mathbb{Z}_{\psi_{k,l},q}\right) \to C_*\left(\bigcup_{q=m+1}^{s_n} \mathbb{Z}_{\psi_{k,l},q}\right).
\]

When \( m = s_n \), \( \tilde{R}_m \) is just another notation for \( R_{sn} \).

By Lemma 6, the norm of the map \( R_m \) in degree \( k \) is \( ||R_m|| \leq (k+1)! \) for each \( m \), so \( ||\tilde{R}_m|| \leq (k+1)! \).

Let \( c \in Z_k(M_{k,l}^{s_{k-1}}) \) be the fundamental cycle of \( M_{k,l}^{s_{k-1}} \). Then \( ||c|| \) is the number of \( k \)-simplices of \( M_{k,l}^{s_{k-1}} \) and \( p_*(fi_{k,l})*c \in Z_k(C_{k,l}^c) \) is a \( k \)-cycle in \( C_{k,l}^c \). Thus

\[
\tilde{c} = \tilde{R}_{sn} \tilde{R}_{sn-1} \cdots \tilde{R}_{s_{k-1}+2} \tilde{R}_{s_{k-1}+1} p_*(fi_{k,l})*c \in Z_k(M_{k,l}^{sn})
\]

is a cycle with norm at most \(((k+1)!)^{s_{n-s_{k-1}}||c||} \). But this number is exactly the number of \( k \)-simplices in the pseudomanifold \( M_{k,l}^{sn} \). If \( ||\tilde{c}|| < ((k+1)!)^{s_{n-s_{k-1}}||c||} \), the cycle \( \tilde{c} \) is carried by a proper subcomplex of \( M_{k,l}^{sn} \) and then it is trivial in homology. Since each \( \tilde{R}_m \) induces an isomorphism in homology, \( f_*(i_{k,l})*[M_{k,l}^{s_{k-1}}] = 0 \in H_k(L) \). This contradicts the assumption. Therefore, \( ||\tilde{c}|| = ((k+1)!)^{s_{n-s_{k-1}}} ||c|| \).

Since the equality holds, we have in particular \( ||\tilde{R}_{s_{k-1}+1} p_*(fi_{k,l})*c|| = (k+1)! ||c|| \). Then for every \( k \)-simplex \( \sigma \in M_{k,l}^{s_{k-1}} \), \( S = pf_{i_{k,l}}(\sigma) \) is a \( k \)-simplex of \( C_{k,l}^c \) and \( ||\tilde{R}_{s_{k-1}+1}(S)|| = (k+1)! ||c|| \). By Lemma 6, \( S \in M_{k,l}^{s_{k-1}} \). We conclude then that \( pf_{i_{k,l}}(M_{k,l}^{s_{k-1}}) \subseteq M_{k,l}^{s_{k-1}} \) and therefore \( fi_{k,l}(M_{k,l}^{s_{k-1}}) \subseteq M_{k,l}^{s_{k-1}} \). If \( fi_{k,l}(M_{k,l}^{s_{k-1}}) \) is contained in a proper subcomplex of \( M_{k,l}^{s_{k-1}} \), \( (fi_{k,l})*[M_{k,l}^{s_{k-1}}] = 0 \) and we have a contradiction. Then \( f|M_{k,l}^{s_{k-1}} : M_{k,l}^{s_{k-1}} \to M_{k,l}^{s_{k-1}} \) is an automorphism and the asymmetry of \( M_{k,l}^{s_{k-1}} \) gives the desired fixed simplex. \( \square \)

The poset of simplices of a finite simplicial complex \( K \) is denoted by \( \mathcal{X}(K) \). Recall that a finite poset \( X \) can be regarded as a topological space with finitely many points in which open sets are those subsets \( U \subseteq X \) such that any \( x \in X \) which is smaller than or equal to an element of \( U \) is itself in \( U \). This space satisfies the \( T_0 \) separation axiom and in fact any finite \( T_0 \)-space is a poset in this sense. Order preserving maps correspond to continuous maps and comparable maps are homotopic ([2]). For every finite simplicial complex \( K \) there is a weak homotopy equivalence \( K \to \mathcal{X}(K) \) (see [9]). The simplicial complex of chains of a poset \( X \)
is denoted by $\mathcal{K}(X)$. There is a weak homotopy equivalence $\mathcal{K}(X) \to X$. A simplicial map $\varphi : K \to L$ and a continuous map $f : X \to Y$ between finite $T_0$-spaces induce maps $\mathcal{X}(\varphi)$ and $\mathcal{K}(f)$ in the obvious way and one has the following commutative diagrams up to homotopy where the vertical maps are the weak homotopy equivalences mentioned above.

\[
\begin{array}{ccc}
K & \xrightarrow{\varphi} & L \\
\downarrow & & \downarrow \\
\mathcal{X}(K) & \xrightarrow{\mathcal{X}(\varphi)} & \mathcal{X}(L) \\
\end{array} \quad \begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\uparrow & & \uparrow \\
\mathcal{K}(X) & \xrightarrow{\mathcal{K}(f)} & \mathcal{K}(Y). \\
\end{array}
\]

**Corollary 9.** Let $K$ be a simply connected compact CW-complex. Then there exists a topological space $X$ weak homotopy equivalent to $K$ which has the fixed point property.

**Proof.** By the previous theorem there is a finite simplicial complex $L$ homotopy equivalent to $K$ with the fixed simplex property. We claim that the associated finite space $\mathcal{X}(L)$ has the fixed point property. Let $f : \mathcal{X}(L) \to \mathcal{X}(L)$ be a continuous map. For every $v \in L$ choose a vertex $g(v)$ of $L$ such that $g(v) \leq f(v)$. The vertex map $g : L \to L$ is simplicial since $f$ maps a bounded set of minimal points into a bounded set. Then $g$ fixes some simplex $\sigma \in L$, so $\mathcal{X}(g) : \mathcal{X}(L) \to \mathcal{X}(L)$ fixes $\sigma$. Since $f \geq \mathcal{X}(g)$, $f(\sigma) \geq \sigma$ and then $f^i(\sigma) = f \circ f \circ \ldots \circ f(\sigma)$ is a fixed point of $f$ for $i$ large enough. \hfill \Box

In order to extend the last corollary to non-simply connected complexes, we need to modify the construction of the space $\mathcal{X}(L)$. The idea we used in the proof of Theorem 8 fails if $H_1(K) \neq 0$ since no 1-dimensional pseudomanifold is asymmetric. We will adapt the proofs of Theorem 8 and Corollary 9 to the general case using the rigidity of finite spaces. Recall from [2, 3] that the non-Hausdorff mapping cylinder $B_f$ of an order preserving map $f : X \to Y$ between finite $T_0$-spaces is the set $X \sqcup Y$ keeping the given ordering within $X$ and $Y$ and setting $x < y$ for $x \in X$ and $y \in Y$ if $f(x) \leq y$. The cylinder $B_f$ deformation retracts to $Y$ so $\mathcal{K}(B_f)$ deformation retracts to $\mathcal{K}(Y)$ by a simplicial retraction. If $f$ is a weak homotopy equivalence, $\mathcal{K}(B_f)$ deformation retracts to $\mathcal{K}(X)$. If $X$ is a finite $T_0$-space, a point $x \in X$ such that $X_{<x}$ or $X_{>x}$ is contractible is called a weak point. In this case the inclusion $X \smallsetminus \{x\} \hookrightarrow X$ is a weak homotopy equivalence. In other words $\mathcal{K}(X)$ deformation retracts to $\mathcal{K}(X \smallsetminus \{x\})$ (see [2, 3] for more details).

**Lemma 10.** There exists a topological space weak homotopy equivalent to $S^1$ with the fixed point property.

**Proof.** Consider the space $\mathcal{X}$ of 14 points in Figure 2. This space is the core of a space considered by G. Kun in [7, Remark 38] in a different context. It is
constructed by gluing two non-Hausdorff mapping cylinders of 1 and 2-degree maps from an 8-point model of $S^1$ to a 4-point model.

Since $x$ and $y$ are weak points of $\mathcal{R}$, $\mathcal{R} \setminus \{x, y\} \hookrightarrow \mathcal{R}$ is a weak homotopy equivalence. The space $\mathcal{R} \setminus \{x, y\}$ is a non-Hausdorff mapping cylinder and then it deformation retracts to $\{x', y', z', w'\}$. Therefore $\mathcal{R}$ is weak homotopy equivalent to $S^1$. We show that $\mathcal{R}$ has the fixed point property.

We will prove the following assertion: there is, up to sign, a unique 1-cycle of norm at most 4 in the order complex $K(\mathcal{R})$ which represents the double of a generator of $H_1(K(\mathcal{R})) \simeq \mathbb{Z}$.

An easy way to prove this assertion is by using colorings (see [4]). The $\mathbb{Z}$-coloring of $\mathcal{R}$ which colors the solid edges with the identity and the dotted edges with a generator $a$ of $\mathbb{Z}$ is connected and admissible, so it is the standard coloring of $\mathcal{R}$. To each directed edge $vw$ of the complex $K(\mathcal{R})$ we assign a weight $\omega(vw)$ which is the sum of the colors of the edges in any increasing path from $v$ to $w$ if $v < w$. If $v > w$, $\omega(vw) = -\omega(wv)$. For example $\omega(zx') = a$, $\omega(w'x') = 0$, $\omega(p_4w) = -a$. The map $H_1(K(\mathcal{R})) \to \mathbb{Z}$ which maps the class of a 1-cycle $\sum v_iw_i$ to $\sum \omega(v_iw_i)$ is a well defined isomorphism (see [10, p.208] and [4]). It is now easy to check that $c = zx + xw + wy + yz$ is the unique cycle of $K(\mathcal{R})$ with norm at most 4 which corresponds to $2a \in \mathbb{Z}$.

Alternatively, in order to prove the assertion, the reader not familiar with colorings may consider the order complex $K(X)$ of the poset $X$ given by the solid edges of $\mathcal{R}$. This complex is contractible and $K(\mathcal{R})$ is obtained from $K(X)$ by adding seven 1-simplices and six 2-simplices. Moreover, $K(\mathcal{R})$ collapses to $K(X) \cup \{x'z'\}$ and then the homology of the 1-cycles of $K(\mathcal{R})$ of norm 4 is easy to understand.

We use now the assertion to prove the fixed point property. Suppose $f : \mathcal{R} \to \mathcal{R}$ is a fixed point free map. Then $K(f)$ has no fixed point and by the Lefschetz fixed point theorem $K(f)_* : H_1(K(\mathcal{R})) \to H_1(K(\mathcal{R}))$ is the identity. Thus $K(f)_*(c) = c$ and then $f$ maps $\{x, y, z, w\}$ into itself, so $f(x) = y, f(y) = x, f(z) = w$ and $f(w) = z$. In particular the set of points greater than $w$ and $z$ is mapped to itself, so $f(\{x', y', z'\}) \subseteq \{x', y', z'\} \cup \{x\} \cup \{y\}$. If the connected subspace $\{x', y', z'\}$ is
mapped into the point \(x\) or into \(y\), then the generating cycle \(z'x' + x'w' + w'y' + y'z'\) is mapped to 0. Therefore \(\{x', y', z'\}\) is mapped into itself and then \(f\) has a fixed point, a contradiction.

**Proof of Theorem 1.** We may suppose that \(K\) is a finite simplicial complex. We begin with the construction of \(L\) performed in the proof of Theorem 8, except that this time we consider also integer homology classes \(\alpha_{1,1}, \alpha_{1,2}, \ldots, \alpha_{1,d_1}\) which are a basis for \(H_1(K; \mathbb{Q})\) and 1-pseudomanifolds \(M_{1,l}\) along with simplicial maps \(\varphi_{1,l} : M_{1,l} \to K\) which map \([M_{1,l}]\) to \(\alpha_{1,l}\). The pseudomanifolds \(M_{k,l}\) will be assumed to be asymmetric for \(k \geq 2\), but of course this is not possible for \(k = 1\).

The numbers \(s_1, N_{k,l}, N_k, s_k\) for \(k \geq 2\) are defined as before. Let \(s_0 = 0, N_1 = 1\) and \(N = \max_{1 \leq k \leq n} N_k\). The cylinders \(C_{k,l}\) are built just as before, except that \(C_{k,l}'\) will be constructed by gluing not \(N\) cylinders but \((n + 1)!N\) cylinders of the identity. Also, we include now the \(C_{1,l}'\)'s. The complex \(L\) is the union of all these cylinders and it is homotopy equivalent to \(K\). The unique difference was the incorporation of the 1-dimensional manifolds with their cylinders and that we increased the length of the cylinders \(C_{k,l}'\).

The space \(X\) will contain \(\mathcal{X}(L)\) as a subspace. For each \(1 \leq l \leq d_1\) consider a weak homotopy equivalence \(\mathcal{X}(M_{1,l}) \to \{x', y', z', w'\}\) where the codomain is the subspace of \(\mathcal{X}\) defined in Lemma 10. Since \(\{x', y', z', w'\}\) is a weak equivalence, the composition \(h_l : \mathcal{X}(M_{1,l}) \to \mathcal{X}\) is a weak equivalence. We take a different copy of \(\mathcal{X}\) for each \(1 \leq l \leq d_1\), so the non-Hausdorff mapping cylinders \(B_{h_l}\) are disjoint. Let \(X = \mathcal{X}(L) \cup \bigcup_{l=1}^{d_1} B_{h_l}\). Since \(h_l\) is a weak equivalence, \(\mathcal{K}(B_{h_l})\) deformation retracts to \(\mathcal{K}(\mathcal{X}(M_{1,l})) = M_{1,l}'\) and then \(\mathcal{K}(X)\) deformation retracts to \(L'\). Therefore \(X\) is weak homotopy equivalent to \(K\). We prove that \(X\) has the fixed point property.

Let \(f : X \to X\) be a continuous map. Then \(\mathcal{K}(f) : \mathcal{K}(X) \to \mathcal{K}(X)\) is a simplicial map. If \(\mathcal{K}(f)\) fixes a simplex, then \(f\) fixes a chain, so \(f\) fixes all the elements of the chain. For each \(k \geq 1\) we consider the basis \(B_k = \{(i'_{k,l})_*([M_{k,l}^{s_{k,l}}] \otimes 1_{\mathbb{Q}})\}_l\) of \(H_k(\mathcal{K}(X); \mathbb{Q})\). Let \(k \geq 2\) and \(1 \leq l \leq d_k\). Then \(\mathcal{K}(f)(M_{k,l}^{s_{k,l}})\) lies in a complex generated by \((k + 1)!||[M_{k,l}^{s_{k,l}}]|| \leq (k + 1)!N_k \leq (n + 1)!N\) simplices. We consider now three cases in order to take into account the 1-dimensional part:

1. \(\mathcal{K}(f)(M_{k,l}^{s_{k,l}}) \subseteq (C_{a,b})'\) or
2. \(\mathcal{K}(f)(M_{k,l}^{s_{k,l}}) \subseteq (C_{a,b}')'\) for some \(k' \geq 2\) and some \(l'\) or
3. \(\mathcal{K}(f)(M_{k,l}^{s_{k,l}}) \subseteq \mathcal{K}(\mathcal{X}(C_{a,b}') \cup B_{h_{l'}})\) for some \(l'\).

In the third case, since \(\mathcal{K}(\mathcal{X}(C_{a,b}') \cup B_{h_{l'}})\) deformation retracts to \((C_{a,b}')'\) which in turn deformation retracts into the 1-dimensional complex \(M_{1,l}'\), then \(\mathcal{K}(f)_{*}(i'_{k,l})_*[M_{k,l}^{s_{k,l}}] = 0 \in H_k(\mathcal{K}(X))\).

If (1) holds, \(f(\mathcal{X}(M_{k,l}^{s_{k,l}})) \subseteq \mathcal{X}(C_{a,b})\). As in the proof of Corollary 9, \(f : \mathcal{X}(M_{k,l}^{s_{k,l}}) \to \mathcal{X}(C_{a,b})\) induces a simplicial map \(g : M_{k,l}^{s_{k,l}} \to C_{a,b}\) which is zero in \(H_k\) by the proof of Theorem 8. Then \(\mathcal{X}(g) : \mathcal{X}(M_{k,l}^{s_{k,l}}) \to \mathcal{X}(C_{a,b})\) is zero in
$k$-homology and the same is true for $f: \mathcal{X}(M_{k,l}^{s_{k-1}}) \to \mathcal{X}(C^{ab}_0)$ since this restriction of $f$ is homotopic to $\mathcal{X}(g)$. Then $\mathcal{K}(f): M_{k,l}^{s_{k-1}+1} \to (C^{ab}_0)'$ induces the trivial map so $\mathcal{K}(f)_*(i_{k,l}')_*[M_{k,l}^{s_{k-1}+1}] = 0$. This idea works also to adapt the proof of Theorem 8 to the cases (2) for $k' < k$, (2) for $k' > k$, (2) for $k' = k$ and $l' \neq l$. Also, if we are in the case (2) with $k' = k, l' = l$ then $f: \mathcal{X}(M_{k,l}^{s_{k-1}}) \to \mathcal{X}(C^{bc}_{k,l})$ induces $g: M_{k,l}^{s_{k-1}} \to C^{bc}_{k,l}$. If $\mathcal{K}(f)_*(i_{k,l}')_*[M_{k,l}^{s_{k-1}+1}] \neq 0$, $g_*[M_{k,l}^{s_{k-1}}] \neq 0$ and then by the proof of Theorem 8 $g$ fixes a simplex, so $\mathcal{X}(g)$ fixes a point $\sigma \in \mathcal{X}(M_{k,l}^{s_{k-1}})$. Then $f(\sigma) \geq \mathcal{X}(g)(\sigma) = \sigma$ and $f: X \to X$ has a fixed point.

We can then assume that the trace of $\mathcal{K}(f)_*: H_k(\mathcal{X}(X); \mathbb{Q}) \to H_k(\mathcal{X}(X); \mathbb{Q})$ is zero for each $k \geq 2$.

Now we study the 1-dimensional component. Let $1 \leq l \leq d_1$. Then $\{x, y, z, w\} \subseteq B_{h_l}$ and if $j_l: \{x, y, z, w\} \rightarrow X$ denotes the inclusion which factors through $B_{h_l}$ and $\beta_l = [\mathcal{K}(\{x, y, z, w\})]$ denotes the fundamental class of $\mathcal{K}(\{x, y, z, w\})$, then $\mathcal{K}(j_l)_*(\beta_l) = 2(i_{l_1}')_*[M'_{1,l}] \in H_1(\mathcal{K}(X)).$ Thus $\mathcal{K}(j_l)_*(\beta_l) \otimes 1_{\mathbb{Q}}$ is twice an element of $B_1$, the chosen basis for $H_1(\mathcal{X}(X); \mathbb{Q})$.

Since $(n + 1)!N \geq 1$, for each $1 \leq l \leq d_1$ the $l$-th copy of $\mathcal{K}(\{x, y, z, w\})$ is mapped by $\mathcal{K}(f)$ into $L'$ or into $\mathcal{X}(C^{bc}_{1,l'}) \cup B_{h_{l'}}$ for some $l'$. If $\mathcal{K}(f)\mathcal{K}(\{x, y, z, w\})) \subseteq L'$, then it is contained in a contractible subcomplex since any closed edge-path of four edges in a barycentric subdivision is contained in the star of a vertex, and then $\mathcal{K}(f)_\mathcal{K}(j_l)_*(\beta_l) = 0 \in H_1(\mathcal{K}(X))$. If $\mathcal{K}(f)(\mathcal{K}(\{x, y, z, w\}))$ is contained in $\mathcal{X}(C^{bc}_{1,l'}) \cup B_{h_{l'}}$ for some $l' \neq l$, then $\mathcal{K}(f_j_1') : \mathcal{K}(\{x, y, z, w\}) \rightarrow \mathcal{X}(C^{bc}_{1,l'}) \cup B_{h_{l'}}$.

The codomain of this map deformation retracts to $M'_{1,l'}$ by a retraction $r$ (which is not simplicial). Then $i_{1,l'}'.r\mathcal{K}(f_j_1') : \mathcal{K}(\{x, y, z, w\}) \rightarrow \mathcal{K}(X)$ is homotopic to $\mathcal{K}(f)\mathcal{K}(j_l)$ and therefore $\mathcal{K}(f)_\mathcal{K}(j_l)_*(\beta_l)$ is an integer multiple of $(i_{l_1}')_*[M'_{1,l}]$. In any of the cases considered so far, the matrix of $\mathcal{K}(f)_*$ in the basis $B_1$ has a zero in the entry $(l, l)$. Suppose then that $\mathcal{K}(f)(\mathcal{K}(\{x, y, z, w\}))$ is contained in $\mathcal{X}(C^{bc}_{1,l'}) \cup B_{h_{l'}}$. Since $C^{bc}_{1,l'}$ deformation retracts to $M_{1,l}$ by a simplicial retraction, $\mathcal{K}(\mathcal{X}(C^{bc}_{1,l'}) \cup B_{h_{l'}})$ deformation retracts to $\mathcal{K}(B_{h_{l'}})$ by a simplicial retraction. Moreover, $\mathcal{K}(B_{h_{l'}})$ deformation retracts to $\mathcal{K}(\mathcal{R})$ by a simplicial retraction which maps $M_{1,l}$ into $\mathcal{K}(\{x', y', z', w'\})$.

Then $\mathcal{K}(\mathcal{X}(C^{bc}_{1,l'}) \cup B_{h_{l'}})$ deformation retracts to $\mathcal{K}(\mathcal{R})$ by a simplicial retraction $R : \mathcal{X}(C^{bc}_{1,l'}) \cup B_{h_{l'}} \rightarrow \mathcal{K}(\mathcal{R})$ which maps $C^{bc}_{1,l'})_l'$ into $\mathcal{K}(\{x', y', z', w'\})$. Thus, $\mathcal{K}(f)_\mathcal{K}(j_l)_*(\beta_l) = i_*R_*\mathcal{K}(f_j_1)_*(\beta_l)$ where $i : \mathcal{K}(\mathcal{R}) \rightarrow \mathcal{K}(X)$ denotes the inclusion. Then the homology class $\mathcal{K}(j_l)_*(\beta_l) = 2(i_{l_1}')_*[M'_{1,l}]$ is mapped by $\mathcal{K}(f)$ to an integer multiple of the generator $(i_{l_1}')_*[M'_{1,l}]$ of the image of $i_*$. Therefore, $\mathcal{K}(j_l)_*(\beta_l)$ is mapped to an integer multiple of itself. Since $f$ is a self map of a finite set, the powers of $f$ induce only finite morphisms in homology, so $\mathcal{K}(j_l)_*(\beta_l)$ is mapped to 0, $\mathcal{K}(j_l)_*(\beta_l)$ or $-\mathcal{K}(j_l)_*(\beta_l)$. By the Lefschetz fixed point theorem we can assume that for some $1 \leq l \leq d_1$, $\mathcal{K}(j_l)_*(\beta_l)$ is mapped to itself. Then $R_*\mathcal{K}(f_j_1)_*(xx + xw + w+y + yz) \in Z_1(\mathcal{K}(\mathcal{R}))$ is a 1-cycle of norm at most 4 which represents the double of the generator of $H_1(\mathcal{K}(\mathcal{R}))$. By the
assertion in Lemma 10 there is a unique cycle satisfying these conditions, which is \( zx + xw + wy + yz \). Therefore \( RK(fj) \) maps \( \{x, y, z, w\} \) into itself and then \( f(\{x, y, z, w\}) = \{x, y, z, w\} \). Thus, the set of points smaller than one of those four points, \( \{x, y, z, w, p_1, p_2, \ldots, p_6\} \), is mapped also to itself and then \( f \) maps \( \mathcal{F} \) into \( \mathcal{F} \). By Lemma 10, \( f \) has a fixed point. □

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