Moduli Space of BPS Walls in Supersymmetric Gauge Theories

Norisuke Sakai\textsuperscript{1} \textsuperscript{*} and Yisong Yang\textsuperscript{2} \textsuperscript{†}

\textsuperscript{1}Department of Physics, Tokyo Institute of Technology
Tokyo 152-8551, JAPAN
and

\textsuperscript{2}Department of Mathematics, Polytechnic University
Brooklyn, New York 11201, U.S.A.

Abstract

Existence and uniqueness of the solution are proved for the ‘master equation’ derived from the BPS equation for the vector multiplet scalar in the $U(1)$ gauge theory with $N_F$ charged matter hypermultiplets with eight supercharges. This proof establishes that the solutions of the BPS equations are completely characterized by the moduli matrices divided by the $V$-equivalence relation for the gauge theory at finite gauge couplings. Therefore the moduli space at finite gauge couplings is topologically the same manifold as that at infinite gauge coupling, where the gauged linear sigma model reduces to a nonlinear sigma model. The proof is extended to the $U(N_C)$ gauge theory with $N_F$ hypermultiplets in the fundamental representation, provided the moduli matrix of the domain wall solution is $U(1)$-factorizable. Thus the dimension of the moduli space of $U(N_C)$ gauge theory is bounded from below by the dimension of the $U(1)$-factorizable part of the moduli space. We also obtain sharp estimates of the asymptotic exponential decay which depend on both the gauge coupling and the hypermultiplet mass differences.

\textsuperscript{*}e-mail address: nsakai@th.phys.titech.ac.jp
\textsuperscript{†}e-mail address: yyang@math.poly.edu
1 Introduction

Solitons have been important in understanding nonperturbative effects in field theories \cite{1}. They are also useful to construct models of the brane-world scenario \cite{2}–\cite{4}. The simplest of these solitons is the domain wall separating two domains of discretely different vacua. The supersymmetric theories are useful to obtain realistic unified theories beyond the standard model \cite{5}. If a field configuration preserves a part of supersymmetry, it satisfies the field equation automatically \cite{6}. Such a configuration is called the Bogomol’nyi-Prasad-Sommerfield (BPS) configuration \cite{7}. The BPS domain walls have been much studied in supersymmetric field theories with four supercharges \cite{8}, \cite{9}, and with eight supercharges \cite{10}–\cite{30}. These soliton solutions often contain parameters, which are called moduli. If we promote the moduli parameters as fields on the world volume of the soliton, they give massless fields on the world volume of the soliton \cite{31}. The metric on the moduli space gives a (nonlinear) kinetic term of the Lagrangian of the low-energy effective field theory. Therefore the determination of the moduli space is of vital importance to understand the dynamics of the solitons.

One of the most interesting classes of models possessing domain walls is the gauge theories with eight supercharges \cite{18}–\cite{29}. To allow domain walls, we need discrete vacua. For that purpose, we introduce Fayet-Iliopoulos (FI) terms for a $U(1)$ factor gauge group. As a natural gauge group with the $U(1)$ factor, we choose $U(N_{C})$ gauge theory. For simplicity, we take the matter hypermultiplets in the fundamental representation of $U(N_{C})$. To obtain more than one supersymmetric vacua, we require that the number of flavors of hypermultiplets $N_{F}$ be larger than the number of colors $N_{C}$

$$N_{F} > N_{C} \quad (1.1)$$

With massless hypermultiplets$^{1}$, the vacuum manifold is a hyper-Kähler manifold, the cotangent bundle over the complex Grassmann manifold $T\ast G_{N_{F},N_{C}}$. It reduces to $T\ast CP^{N_{F}-1}$ manifold in the case of the $U(1)$ gauge theory ($N_{C} = 1$). If the nondegenerate hypermultiplet masses are turned on, a potential term is induced and most of the vacua are lifted, allowing wall solutions. In the resulting vacua of the massive $U(N_{C})$ gauge theories, each color component of the hypermultiplets chooses to have a particular flavor (color-flavor-locking) \cite{21}, \cite{22}. Furthermore, a systematic construction of BPS wall solutions has been established \cite{23}, \cite{24}.

If we take the limit of strong gauge coupling $g^{2} \to \infty$ for the $U(1)$ gauge theory, the vector multiplet can be eliminated to give constraints on the hypermultiplet field space, resulting in a supersymmetric massive nonlinear sigma model with the $T\ast CP^{N_{F}-1}$ target space \cite{32}, \cite{33}. Equations for preserving half of supersymmetry are called the 1/2 BPS equations. As boundary conditions for the BPS equations, the field configurations are required to approach one of the discrete vacua. If the vacua at $y = -\infty$ and $+\infty$ happen to be different, the solution represents (multi-) walls. If the vacua at $y = -\infty$ and $+\infty$ happen to be identical, the solution represents one of vacua. Therefore these BPS equations admit vacua (full supersymmetry conserved) besides (multi-) walls as solutions. The physical relation between these different topological sectors are as follows. If we let the position of one of the walls to go to infinity, we obtain a topological sector with one less wall. Therefore the topological sectors with $n - 1$ walls appear as boundaries of the moduli space of a topological sector with $n$ walls. Continuing this process, we eventually arrive at topological sectors with no walls, namely vacua. In this way, we naturally obtain a compactification of moduli space of various topological sectors of multi-walls. The resulting

$^{1}$A common mass can be absorbed into the shift of vector multiplet scalar and is not relevant.
manifold of all the solutions of the 1/2 BPS equations is topologically \( CP^{N_f-1} \) in our case of the \( T^*CP^{N_f-1} \) nonlinear sigma model [10], [13]. The massive \( U(N_C) \) gauge theory reduces to the massive nonlinear sigma model with the \( T^*G_{N_f,N_C} \) target space and with a potential term in the limit of strong gauge coupling [21]. Similarly to the \( U(1) \) case, the space of all solutions of the 1/2 BPS equations for the massive \( T^*G_{N_f,N_C} \) nonlinear sigma model with the vacuum boundary condition at infinity is found to be the complex Grassmann manifold \( G_{N_f,N_C} \), which is the special Lagrangian submanifold of the target space of the nonlinear sigma model [23], [24].

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A number of exact solutions for \( U(1) \) gauge theories have been obtained for particular discrete finite values of gauge coupling [34], [20], [24]. For generic finite gauge couplings of \( U(1) \) as well as \( U(N_C) \) gauge theories, only the general behavior of domain walls have been studied qualitatively [15], [22], [30]. The systematic construction of BPS walls first solves the hypermultiplet BPS equation and yields the moduli matrix \( H_0 \) as integration constants. After taking account of an equivalence relation called the \( V \)-equivalence relation, the independent variables in the moduli matrix \( H_0 \) constitute the complex Grassmann manifold \( G_{N_f,N_C} \). The remaining BPS equation for the vector multiplet scalar can be rewritten into a “master equation”, which is a nonlinear ordinary differential equation for a gauge invariant quantity \( \Omega \). In the limit \( g^2 \to \infty \), this master equation can be solved algebraically without introducing additional moduli, once the moduli matrix \( H_0 \) is given. It has been conjectured that there exists a unique solution of the master equation even at finite gauge coupling, once the moduli matrix \( H_0 \) is given. Based on this conjecture, it has been pointed out that the moduli space of the BPS equations is given by the moduli matrix \( H_0 \) divided by the \( V \)-equivalence relation even at finite gauge couplings [23], [24].

Up to now, the best supporting evidence for this proposal is given by the index theorem, although the evidence is only indirect. The index theorems are proven for \( U(1) \) gauge theories [35], [29], and for \( U(N_C) \) gauge theories [30], respectively. They state that the complex Grassmann manifold contains necessary and sufficient number of moduli parameters. However, it is much more desirable to demonstrate the existence and uniqueness of the solution of the master equation.

The purpose of this paper is to study the master equation for the gauge invariant quantity \( \Omega \). We present a proof of the existence and uniqueness of the solution for the \( U(1) \) gauge theories, and extend the proof to a class of the moduli matrix \( H_0 \) in the \( U(N_C) \) gauge theories. Our proof for the \( U(1) \) case finally establishes that solutions of the 1/2 BPS equations are completely characterized by the moduli matrices divided by the \( V \)-equivalence relation. This moduli space is topologically the same as the moduli space of the BPS equations in the nonlinear sigma model with \( T^*CP^{N_f-1} \) target space, which is obtained in the limit of \( g^2 \to \infty \). Of course the metric on the moduli space at finite gauge coupling is expected to be deformed from that of nonlinear sigma model (infinite gauge coupling). We also obtain estimates of the asymptotic exponential decay which depend on both the hypermultiplet mass differences and the gauge coupling squared multiplied by the FI-parameter. These estimates agree with our previous result based on an iterative approximation scheme [20]. For the non-Abelian \( U(N_C) \) gauge theories, we show that the proof can be applied to the part of the moduli space which is described by the \( U(1) \) factorizable moduli matrix \( H_0 \). This result implies a lower bound of the dimensions of the moduli space for the \( U(N_C) \) gauge theories. We will leave for future publication to extend the proof of the existence and uniqueness of the master equation for the gauge invariant \( \Omega \) to entire moduli space of the BPS walls in the
Interestingly, our one-dimensional nonlinear equation and its variational structure resemble in many ways the two-dimensional Abelian BPS vortex equation which allows us to extend the method developed in Jaffe and Taubes \[36\] to our problem here. There are two major technical differences/difficulties, though, that need to be overcome. The first one is that in one dimension, the ranges of the exponents in the Gagliardo–Nirenberg inequality cannot render as strong an estimate as in two dimensions (a relevant lower bound takes the weaker, sublinear, form, \(\|v\|^{2/3}\) instead of the usual stronger, linear, form, \(\|v\|_2\). See (4.19)). The second one is that, unlike in the Abelian vortex situation in which the vacuum state is uniquely characterized by the asymptotic amplitude of the Higgs field, our domain wall solution needs to interpolate two different vacua at the two infinities of the real line. Hence, the behavior of the solution in a local region is less uniform and the decay rates near the two infinities are also necessarily different.

In section 2, BPS equations and their systematic solutions are introduced. We also describe implication of the existence and uniqueness of the master equation for the gauge invariant. In section 3, we extend our analysis to the \(U(1)\) factorizable case of the \(U(N_C)\) gauge theories, giving a lower bound for the dimension of the BPS wall moduli space. In section 4, we present an analytic proof of the existence and uniqueness of the master equation for the gauge invariant, and give estimates of the asymptotic behavior of the solution.

## 2 Moduli Space of BPS Equations in \(U(1)\) Theories

Let us take a supersymmetric \(U(1)\) gauge theory with eight supercharges in one time and four spatial dimensions\(^2\). The \(U(1)\) vector multiplet contains gauge field \(W_M\), gaugino \(\lambda^i\), a real neutral scalar field \(\Sigma\), and \(SU(2)_R\) triplet of real auxiliary fields \(Y^a\), where \(M, N = 0, 1, \cdots, 4\) denote space-time indices, and \(i = 1, 2\) and \(a = 1, 2, 3\) denote \(SU(2)_R\) doublet and triplet indices, respectively. The hypermultiplet contains two complex scalar fields \(H^iA\), hyperino \(\psi^A\) and complex auxiliary fields \(F^A\), where \(A = 1, \cdots, N_F\) stand for flavors. For simplicity, we assume that these \(N_F\) hypermultiplets have the same \(U(1)\) charge, say, unit charge. Denoting the gauge coupling \(g\), the mass of the \(A\)-th hypermultiplet \(m_A\), and the FI parameters \(\zeta^a\), the bosonic part of the Lagrangian reads \[18\]–\[34\]

\[
\mathcal{L}_{\text{boson}} = -\frac{1}{4g^2}(F_{MN}(W))^2 + \frac{1}{2g^2}[(\partial_M\Sigma)^2 + (\mathcal{D}_M^+ H^i_A)^+ (\mathcal{D}^M H_i^A) - H_i^A(\Sigma - m_A)^2 + F^A_i F_i^A] + \frac{1}{2g^2}(Y^a)^2 - \zeta^a Y^a + H_i^A(\sigma^a Y^a)_j H_j^A + F^A_i F_i^A, \tag{2.1}
\]

where a sum over repeated indices is understood, \(F_{MN}(W) = \partial_M W_N - \partial_N W_M\), covariant derivative is defined as \(\mathcal{D}_M = \partial_M + iW_M\), and our metric is \(\eta_{MN} = (+1, -1, \cdots, -1)\). We assume that the hypermultiplet masses are nondegenerate and are ordered as

\[
m_1 > m_2 > \cdots > m_{N_F}. \tag{2.2}
\]

The auxiliary fields are given by their equations of motion: \(F_i^A = 0\) and

\[
Y^a = g^2[\zeta^a - H_i^A(\sigma^a)_j H_j^A]. \tag{2.3}
\]

\(^2\)Discrete vacua are required for wall solutions and are possible by mass terms for hypermultiplets which are available only in spacetime dimensions equal to or less than five.
By making an $SU(2)_R$ transformation, we can choose the FI parameters to the third direction
\[
\zeta^a = (0, 0, \zeta), \quad \zeta > 0. \tag{2.4}
\]
In this choice, we find $N_F$ discrete SUSY vacua ($A = 1, \cdots, N_F$) as
\[
\Sigma = m_A, \quad |H^{1A}|^2 = \zeta, \quad H^{2A} = 0, \quad H^{1B} = 0, \quad H^{2B} = 0, \quad (B \neq A). \tag{2.5}
\]

We assume the configuration to depend only on a single coordinate, which we denote as $y \equiv x^4$, and assume the four-dimensional Lorentz invariance in the world volume coordinates $x^\mu = (x^0, \cdots, x^3)$. Let us examine the supersymmetry transformations of fermions: the gaugino $\lambda^i$ and hyperino $\psi^A$ transform as\(^3\)
\[
\delta_\epsilon \lambda^i = \left( \frac{1}{2} \gamma^{MN} F_{MN}(W) + \gamma^M \partial_M \Sigma \right) \epsilon^i + i \left( Y^a \sigma^a \right)^i \epsilon^i, \tag{2.6}
\]
\[
\delta_\epsilon \psi^A = -i \sqrt{2} \left[ \gamma^M \partial_M H^{1A} + i (\Sigma - m_A) H^{1A} \right] \epsilon_{ij} \epsilon^j + \sqrt{2} F^{iA} \epsilon^i. \tag{2.7}
\]

We require the following half of supersymmetry to be preserved
\[
P_+ \epsilon^1 = 0, \quad P_- \epsilon^2 = 0, \tag{2.8}
\]
where $P_{\pm} \equiv (1 \pm \gamma_5)/2$ are the chiral projection operators. Then we obtain the $1/2$ BPS equations
\[
D_y H^{1A} = (m_A - \Sigma) H^{1A}, \quad D_y H^{2A} = (-m_A + \Sigma) H^{2A}, \quad A = 1, \cdots, N. \tag{2.9}
\]
\[
0 = Y^1 + i Y^2 = -2g^2 H^{2A}(H^{1A})^*, \tag{2.10}
\]
\[
\partial_y \Sigma = Y^3 = g^2 \left( \zeta - H^{1A}_1 H^{1A} + H^{1A}_2 H^{2A} \right), \tag{2.11}
\]

We wish to obtain solutions of these BPS equations which interpolate two different vacua in Eq.(2.5). The boundary condition of these two vacua at $y = \pm \infty$ specifies the topological sector. By letting the outer-most wall to infinity, one can obtain topological sectors with one less wall. Therefore we are interested in the maximal topological sector which allows the maximal number of walls and possesses the maximal number of moduli parameters. The boundary conditions for the maximal topological sector are given by
\[
\Sigma(-\infty) = m_{N_F}, \quad \Sigma(\infty) = m_1, \tag{2.12}
\]
\[
H^{1A}(-\infty) = \sqrt{\zeta} \delta^{1A}_{N_F}, \quad H^{1A}(\infty) = \sqrt{\zeta} \delta^{1A}_1, \tag{2.13}
\]
\[
H^{2A}(-\infty) = 0, \quad H^{2A}(\infty) = 0.
\]

Let us define \cite{18, 21, 23} an $GL(1, \mathbb{C})$ group element $S(y)$ that expresses the vector multiplet scalar and gauge field as a pure gauge
\[
\frac{1}{S(y)} \frac{dS(y)}{dy} = \Sigma(y) + iW_4(y). \tag{2.14}
\]

\(^3\)Our gamma matrices are $4 \times 4$ matrices and are defined as: $\{ \gamma^M, \gamma^N \} = 2\eta^{MN}$, $\gamma^{MN} = \frac{1}{2} [\gamma^M, \gamma^N] = \gamma^M \gamma^N$, $\gamma^5 = i \gamma^0 \gamma^1 \gamma^2 \gamma^3 = -i \gamma^4$. 

4
The BPS equation for hypermultiplets (2.9) can be solved in terms of this single complex function $S(y)$ as

$$H^{1A}(y) = S^{-1}(y) H_0^A e^{m_A y},$$

(2.15)

where complex integration constants $H_0^A$ can be assembled into a constant complex vector $H_0 = (H_0^1, \cdots, H_0^{N_F})$. This $N_F$ component vector is nothing but the $N_C = 1$ case of a $N_C \times N_F$ complex constant matrix called the moduli matrix [23]. Since Eq. (2.14) defines the function $S(y)$ only up to a complex multiplicative constant, equivalent descriptions of physical fields ($H^i$ and $\Sigma$) result from two complex vectors $H_0$ which are different by multiplication of a non-vanishing complex constant $V$:

$$H_0 \rightarrow VH_0, \quad S \rightarrow VS,$$

(2.16)

which is called the $V$-equivalence relation [23]. Therefore the moduli matrix $H_0$ in this case is topologically $CP^{N_F-1}$. We define a $U(1)$ local gauge invariant function $\Omega(y)$ as

$$\Omega(y) = S(y) S^*(y).$$

(2.17)

Using the hypermultiplet solution (2.15), the remaining BPS equation for the vector multiplet scalar can be rewritten in terms of $\Omega$

$$\frac{d}{dy} \left( \frac{1}{\Omega(y)} \frac{d\Omega(y)}{dy} \right) = g^2 \zeta \left( 1 - \frac{\Omega_0(y)}{\Omega(y)} \right),$$

(2.18)

$$\Omega_0(y) \equiv \frac{1}{\zeta} \sum_{A=1}^{N_C} H_0^A e^{2m_A y} (H_0^A)^* \equiv e^{2W(y)}.$$

(2.19)

The boundary conditions in Eqs. (2.12) and (2.13) now become the following boundary conditions for the master equation

$$\Omega(y) \rightarrow \Omega_0(y) \rightarrow (e^{2m_1 y}|H_0^1|^2/\zeta, 0, \cdots, 0), \quad y \rightarrow \infty$$

$$\Omega(y) \rightarrow \Omega_0(y) \rightarrow (0, \cdots, 0, e^{2m_{N_F} y}|H_0^{N_F}|^2/\zeta), \quad y \rightarrow -\infty.$$

(2.20)

The non-vanishing left-most element of the moduli matrix $H_0$ specifies the boundary condition at $y = +\infty$, and the non-vanishing right-most element specifies the boundary condition at $y = -\infty$. We can see that the boundary condition is encoded in the choice of the moduli matrix $H_0$. The moduli matrix $H_0$ with more than one non-vanishing elements gives a (multi-)wall solutions, whereas the moduli matrix with a single non-vanishing element gives a vacuum solution. Therefore the space of all possible moduli matrix $H_0$ divided by the $V$-equivalence relation automatically gives all possible solutions of the BPS equation including the vacuum solutions and multi-wall solutions.

Using $W(y)$ in Eq. (2.19), we finally obtain the master equation for the $U(1)$ gauge theory [20], [19]

$$\frac{1}{\zeta g^2} \frac{d^2 \psi}{dy^2} = 1 - e^{-2\psi(y)+2W(y)}, \quad \psi(y) \equiv \frac{1}{2} \log \Omega(y).$$

(2.21)

\footnote{We have changed our notation Re$\psi \rightarrow \psi$ from Ref. [20].}
The boundary conditions (2.20) now become the following boundary conditions for the master equation:

\[
\psi \rightarrow m_1 y + \log \left( \frac{|H_1^0|}{\sqrt{\zeta}} \right), \quad y \rightarrow +\infty, \quad (2.22)
\]

\[
\psi \rightarrow m_{N_F} y + \log \left( \frac{|H_{0}^{N_F}|}{\sqrt{\zeta}} \right), \quad y \rightarrow -\infty. \quad (2.23)
\]

In Ref. [23], it has been conjectured that there exists a unique solution of the master equation (2.21) given the boundary conditions (2.22) and (2.23). We shall prove the conjecture in section 4. With this proved, we can now state that the moduli space of the domain walls in the \( U(1) \) gauge theory with \( N_F \) flavors is given by \( \mathbb{C}P^{N_F-1} \) as described by the moduli matrix \( H_0 \) furnished with the equivalence relation (2.16).

3 \( U(1) \) Factorizable case of \( U(N_C) \) Gauge Theories

Let us now turn our attention to a supersymmetric \( U(N_C) \) gauge theory with \( N_F(>N_C) \) flavors of hypermultiplets in the fundamental representation. We denote gauge fields \( W_M \) and real scalar fields \( \Sigma \) as \( N_C \times N_F \) matrices \( H^i \). The bosonic part of the Lagrangian of the supersymmetric \( U(N_C) \) gauge theory reads

\[
L_{\text{bosonic}} = \text{Tr} \left[ -\frac{1}{2g^2} F_{MN}(W) F^{MN}(W) + \frac{1}{g^2} (D_M \Sigma)^2 + \frac{1}{g^2} (Y^a)^2 \right. \\
\left. + D^M H^i \bar{D}^M H^i - (\Sigma H^i - H^i M)(\Sigma H^i - H^i M)^\dagger + F^i F^{i\dagger} \right. \\
\left. - Y^a (c_a - (\sigma_a)^i H^i H^{i\dagger}) \right] \quad (3.2)
\]

with the Fayet-Iliopoulos parameter \( c_a = (0, 0, c) \) with \( c > 0 \) and the hypermultiplet mass matrix \( M = \text{diag}(m_1, \ldots, m_{N_F}) \). Covariant derivatives are defined as \( D_M \Sigma = \partial_M \Sigma + i [W_M, \Sigma] \), \( D_M H^{rA} = (\partial_M \delta^{r}_{s} + i [W_M]^{r}_{s}) H^{sA} \), and the gauge field strength is \( F_{MN}(W) = -i [D_M, D_N] = \partial_M W_N - \partial_N W_M + i [W_M, W_N] \).

Considering the supersymmetry transformations in Eqs. (2.6) and (2.7), and requiring the half of supersymmetry in Eq. (2.8) to be preserved, we obtain the 1/2 BPS equations for walls with \( 5 \)

We observe that one out of two constants \( \log(|H_0^1|/\sqrt{\zeta}) \) and \( \log(|H_0^{N_F}|/\sqrt{\zeta}) \) can be absorbed into a shift of \( \psi \). This freedom corresponds to the \( V \)-equivalence relation implying that only one out of these two constants in the boundary conditions is the genuine moduli parameter.
profile in $y$

\begin{eqnarray}
D_y H^1 &=& - \Sigma H^1 + H^1 M, \quad (3.3) \\
D_y H^2 &=& \Sigma H^2 - H^2 M, \quad (3.4) \\
D_y \Sigma &=& Y^3 = \frac{g^2}{2} (c A_{N_C} - H^1 H^{1\dagger} + H^2 H^{2\dagger}), \quad (3.5) \\
0 &=& Y^1 + i Y^2 = -g^2 H^2 H^{1\dagger}. \quad (3.6)
\end{eqnarray}

The supersymmetric vacua are characterized by the vanishing of the right-hand side of all of these BPS equations. The supersymmetric vacua have been found to be color-flavor-locked and discrete [21]. For each color component $r$, only one flavor $A_r$ of hypermultiplet $H^{rA}, A = A_r$ should take a non-vanishing value,

$$H^{1rA} = \sqrt{c} \delta^{A_r}, \quad H^{2rA} = 0,$$

and the corresponding color component of the vector multiplet scalar $\Sigma$ should be equal to the mass of that flavor of the hypermultiplets

$$\Sigma = \text{diag.}(m_{A_1}, m_{A_2}, \ldots, m_{A_{N_C}}). \quad (3.8)$$

We denote the above SUSY vacuum as

$$\langle A_1 A_2 \cdots A_{N_C} \rangle. \quad (3.9)$$

Consequently topological sector is labeled by two vacua:

$$\langle A_1, \cdots, A_{N_C} \rangle \leftarrow \langle B_1, \cdots, B_{N_C} \rangle, \quad (3.10)$$

where the first vacuum is at $y = \infty$ and the second at $y = -\infty$. Ideally we wish to solve the BPS equations for each topological sector and we wish to obtain all possible solutions.

Let us consider the maximal topological sector, which is specified in the $U(N_C)$ gauge theory as:

$$\langle 1, \cdots, N_C \rangle \leftarrow \langle N_F - N_C - 1, \cdots, N_F \rangle. \quad (3.11)$$

In this maximal topological sector, the boundary conditions for the hypermultiplet scalar $H^1(y)$ is given by

\begin{equation}
H^1(y) \to \sqrt{c} \begin{pmatrix}
1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
0 & 1 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1 & 0 & \cdots & 0
\end{pmatrix}, \quad y \to \infty \quad (3.12)
\end{equation}

\begin{equation}
H^1(y) \to \sqrt{c} \begin{pmatrix}
0 & 0 & \cdots & 1 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & 0 & \cdots & 1
\end{pmatrix}, \quad y \to -\infty \quad (3.13)
\end{equation}

To solve the BPS equations, we define the following $GL(N_C, \mathbb{C})$ group element $S(y)$

$$\Sigma + i W_y = S^{-1}(y) \partial_y S(y). \quad (3.14)$$
The hypermultiplet scalar BPS equations (3.3) and (3.4) can be solved with this matrix function $S(y)$ as

$$H^1 = S^{-1}(y) H_0 e^{My}, \quad H^2 = 0,$$  

(3.15)

with the constant matrix $H_0$ which is called the moduli matrix. We used the boundary conditions $H^2(y) \to 0$ at $y \to \pm \infty$ to fix $H^2(y)$ in the above solution [24]. Since the matrix function $S(y)$ in Eq. (3.14) is defined up to a constant $GL(N_C, \mathbb{C})$ matrix function $V$, equivalent descriptions for the hypermultiplet scalar $H^1$ and the vector multiplet scalar $\Sigma$ are obtained by two sets of moduli matrices $H_0$ and $S$ that are related by $GL(N_C, \mathbb{C})$ transformations $V$

$$(H_0, S) \sim (H_0', S'), \quad H_0 \to H_0' = VH_0, \quad S \to S' = VS, \quad V \in GL(N_C, \mathbb{C}).$$  

(3.16)

We call this symmetry the $V$-equivalence relation. The space of the moduli matrix divided by the $V$-equivalence relation is found [23] to be the complex Grassmann manifold $G_{N_F, N_C}$

$$\{H_0|H_0 \sim VH_0, V \in GL(N_C, \mathbb{C})\} \simeq G_{N_F, N_C} \simeq \frac{SU(N_F)}{SU(N_C) \times SU(N_C) \times U(1)}.$$  

(3.17)

In place of the gauge variant matrix function $S(y)$, we define a gauge invariant $N_C \times N_C$ matrix function

$$\Omega \equiv SS^\dagger.$$  

(3.18)

The remaining BPS equation (3.5) for the vector multiplet scalar $\Sigma$ can be rewritten in terms of $\Omega$ as a master equation for the $U(N_C)$ gauge theory [23, 24]

$$\partial_y (\Omega^{-1} \partial_y \Omega) = g^2 c (\mathbb{1}_C - \Omega^{-1} \Omega_0),$$  

(3.19)

where the source term is defined in terms of the moduli matrix $H_0$ as

$$\Omega_0 \equiv c^{-1} H_0 e^{2My} H_0^\dagger.$$  

(3.20)

We need to specify the boundary conditions for $\Omega(y)$. For the maximal topological sector, the following quantity reduces to the unit matrix for the first $N_C \times N_C$ diagonal part in the limit of $y \to \infty$

$$H^1(y) H^1(y) = e^{My} H_0^\dagger \Omega(y) H_0 e^{My} \to c \begin{pmatrix} 1 & 0 & \ldots & 0 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 1 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 0 & 0 & \ldots & 0 \end{pmatrix}, \quad y \to \infty,$$  

(3.21)

and to the unit matrix for the last $N_C \times N_C$ diagonal part in the limit of $y \to -\infty$

$$H^1(y) H^1(y) = e^{My} H_0^\dagger \Omega(y) H_0 e^{My} \to c \begin{pmatrix} 0 & \ldots & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \ldots & 0 & 0 & \ldots & 0 \\ 0 & \ldots & 0 & 1 & 0 & \ldots & 0 \\ 0 & \ldots & 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \ldots & 0 & 0 & 0 & \ldots & 1 \end{pmatrix}, \quad y \to -\infty.$$  

(3.22)
It has been conjectured [23] that there exists a unique solution of the master equation (3.19) given the above boundary conditions (3.21) and (3.22). If this is proved, the solutions of the BPS equations (3.3)–(3.6) in the $U(N_C)$ gauge theory with $N_F$ flavors are completely characterized by the moduli matrices $H_0$ divided by the $V$-equivalence relation (3.16).

For the generic moduli matrix $H_0$, we cannot give a proof of the existence and uniqueness of the solution $\Omega(y)$ of the master equation (3.19) at present. However, we can exploit our proof for the $U(1)$ gauge theory if the moduli matrix $H_0$ is of a restricted form, as we describe now. Let us note that the master equation (3.19) transforms covariantly under the world-volume transformation (3.16), where the matrix $H_0 e^{2My}H_0^\dagger$ transforms with multiplication of constant matrices $V$ and $V^\dagger$ from both sides of this matrix. This $V$-equivalence relation $V$ allows us to diagonalize the matrix $H_0 e^{2My}H_0^\dagger$ at one point of the extra dimension, say, $y = y_0$. If the matrix $H_0 e^{2My}H_0^\dagger$ with this gauge fixing remains diagonal at every other points in the extra dimension $y \neq y_0$, we obtain

$$H_0 e^{2My}H_0^\dagger = c \text{diag.} (\mathcal{W}_1(y), \mathcal{W}_2(y), \cdots, \mathcal{W}_{N_C}(y)). \quad (3.23)$$

If this is valid, we call that moduli matrix $H_0$ as $U(1)$-factorizable [24]. Whether a given moduli matrix is $U(1)$ factorizable or not is an inherent characteristic of each moduli matrix $H_0$, and is independent of the choice of the initial coordinate $y_0$. Thus the $U(1)$-factorizability is a property attached to each point on the moduli space. If the moduli matrix is $U(1)$-factorizable, off-diagonal components of the matrix $H_0 e^{2My}H_0^\dagger$ vanishes at any point of the extra dimension $y$ by definition. Therefore each coefficient of $e^{2m_Ay}$ in the off-diagonal components must vanish. Since we consider the case of non-degenerate masses, the condition for the $U(1)$-factorizability can be rewritten for each flavor $A$ as

$$(H_0)^r_A ((H_0)^s_A)^* = 0, \quad \text{for } r \neq s, \quad (3.24)$$

where the flavor index $A$ is not summed. Namely, $(H_0)^r_A$ can be non-vanishing in only one color component $r$ for each flavor $A$.

To solve the master equation (3.19) for the gauge invariant matrix function $\Omega$ in the non-Abelian $U(N_C)$ gauge theory with the $U(1)$-factorizable moduli, we are allowed to take an ansatz where only the diagonal components of the matrix $\Omega$ survive

$$\Omega = \text{diag.} (e^{2\psi_1}, e^{2\psi_2}, \cdots, e^{2\psi_{N_C}}), \quad (3.25)$$

where $\psi_r(y)$’s are real functions. With this ansatz, the master equation (3.19) for the $U(1)$-factorizable moduli with the condition (3.23) reduces to a set of the master equations for the Abelian gauge theory [23]

$$\frac{d^2\psi_r}{dy^2} = \frac{g^2c}{2} \left(1 - e^{-2\psi_r} \mathcal{W}_r\right), \quad \text{for } r = 1, 2, \cdots N_C, \quad (3.26)$$

where the functions $\mathcal{W}_r(y)$ defined in (3.23) are given by

$$\mathcal{W}_r = \sum_{A \in \mathcal{A}_r} e^{2m_Ay} \frac{|H_0^A|^2}{\zeta}. \quad (3.27)$$

$\mathcal{A}_r$ is a set of flavors of the hypermultiplet scalars whose $r$-th color component is non-vanishing. Note that the condition (3.24) of the $U(1)$-factorizability can be rewritten as $\mathcal{A}_r \cap \mathcal{A}_s = \emptyset$ for
In this case, the vector multiplet scalars $\Sigma$ and the hypermultiplet scalars $H^{1rA}$ are given by

$$\Sigma = \text{diag.}(\partial_y \psi_1, \partial_y \psi_2, \cdots, \partial_y \psi_{N_C}),$$

(3.28)

$$H^{1rA} = e^{-\psi_r(y) + m_A y} H_0^A,$$

(3.29)

with a gauge choice of $W_y = 0$ and a phase choice of $\text{Im} H^{1rA} = 0$ at $y = \pm \infty$. Since the moduli parameters contained in the master equation (3.26) for each $\psi_r$ are independent of each other, we find that our system of BPS equations for walls becomes a decoupled set of $N_C$ systems of BPS equations of $U(1)$ gauge theories. This fact enables us to apply our method of proof for $U(1)$ gauge theories to this $U(1)$-factorizable case of the $U(N_C)$ gauge theory.

Let us count the dimensions of the part of the moduli space with the $U(1)$-factorizable property. For each color component $r$, we denote the number of non-vanishing hypermultiplets in $A_r$ as $f_r$. For the $r$-th color component, we obtain a $U(1)$ gauge theory with $f_r$ flavors. In the maximal topological sector, all the flavors should participate: $\sum_r f_r = N_F$. Since every color component has to appear at vacua of both infinities, we obtain the complex dimension of the $U(1)$-factorizable part of the moduli space in the maximal topological sector to be

$$\text{dim}_{\mathbb{C}} \mathcal{M}_{U(1)\text{fact}} = \sum_{r=1}^{N_C} (f_r - 1) = N_F - N_C.$$  

(3.30)

We thus obtain a rather weak lower bound for the dimension of the moduli space for the $U(N_C)$ gauge theory:

$$\text{dim}_{\mathbb{C}} \mathcal{M}_{U(N_C)} \geq N_F - N_C.$$  

(3.31)

### 4 Proof of Existence and Uniqueness of Solution

In this section, we prove the existence and uniqueness of the BPS wall solutions to the $U(1)$ and $U(1)$-factorizable models. We first state our results in suitably renormalized parameters. We then carry out the proof using the method of calculus of variations and functional analysis.

For convenience, we relabel the variables and parameters of our master equations (2.21) and (3.26) so that they are of the equivalent form

$$u'' = \lambda(M(y)e^u - 1),$$

(4.1)

subject to the boundary conditions

$$u(y) \to -\omega_1 y - r_1, \quad y \to \infty,$$

(4.2)

$$u(y) \to -\omega_{N_F} y - r_{N_F}, \quad y \to -\infty,$$

(4.3)

For the $U(1)$ model, we denote $\lambda \equiv 2g^2\zeta$, $\omega_A \equiv 2m_A$, $u(y) \equiv -2\psi(y)$, $r_A \equiv \log(|H_0^A|^2/\zeta)$, and $M(y) \equiv e^{2W(y)} = \Omega_0(y) = \sum_A e^{2m_A y |H_0^A|^2/\zeta}$. 

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where \( u' \) stands for \( du/dy \) and

\[
M(y) = \sum_{A=1}^{N_F} e^{\omega_A y + r_A}, \tag{4.4}
\]

\( \lambda > 0, \omega_A, r_A \) are real constants so that \( \omega_A (A = 1, 2, \cdots, N_F) \) satisfy the nondegeneracy condition

\[
\omega_1 > \omega_2 > \cdots > \omega_{N_F}. \tag{4.5}
\]

For the equation (4.1) subject to the boundary conditions (4.2) and (4.3), we have

**Theorem 4.1.** The problem (4.1)–(4.3) has a unique solution. Moreover, this solution also enjoys the estimates of the asymptotic exponential decay at \( y \to \pm \infty \)

\[
u(y) + (\omega_1 y + r_1) = O(e^{-\lambda_1(1-\epsilon)y}) \quad \text{as} \quad y \to \infty, \tag{4.6}
\]

\[
u(y) + (\omega_{N_F} y + r_{N_F}) = O(e^{\lambda_2(1-\epsilon)y}) \quad \text{as} \quad y \to -\infty, \tag{4.7}
\]

where \( \epsilon > 0 \) can be taken to be arbitrarily small and \( \lambda_1 \) and \( \lambda_2 \) are positive parameters defined by

\[
\lambda_1 = \min\{\sqrt{\lambda}, \omega_1 - \omega_2\}, \tag{4.8}
\]

\[
\lambda_2 = \min\{\sqrt{\lambda}, \omega_{N_F} - 1 - \omega_{N_F}\}. \tag{4.9}
\]

We now proceed with the proof, adapting the main ideas from [36].

Note that the conditions (4.2), (4.3), (4.5) ensure that the right-hand side of (4.1) vanishes at \( y = \pm \infty \) because

\[
M(y) = e^{\omega_1 y + r_1}(1 + e^{(\omega_2 - \omega_1)y + r_2 - r_1} + \cdots + e^{(\omega_{N_F} - \omega_1)y + r_{N_F} - r_1}), \quad y > 0,
\]

\[
M(y) = e^{\omega_{N_F} y + r_{N_F}}(1 + e^{(\omega_1 - \omega_{N_F})y + r_1 - r_{N_F}} + \cdots + e^{(\omega_{N_F} - 1 - \omega_{N_F})y + r_{N_F} - 1 - r_{N_F}}), \quad y < 0.
\]

To take into account of the boundary asymptotics, we introduce a translation, \( u = u_0 + v \), where \( u_0(y) \) has continuous second-order derivative and satisfies

\[
u_0(y) = -\omega_1 y - r_1 \quad \text{if} \quad y > y_0, \quad u_0(y) = -\omega_{N_F} y - r_{N_F} \quad \text{if} \quad y < -y_0, \tag{4.10}
\]

where \( y_0 > 0 \) is a suitable constant to be determined shortly. Then the equation (4.1) becomes

\[
v'' = \lambda(Q(y)e^v - 1) + h(y), \tag{4.11}
\]

where \( h(y) = -u_0''(y) \) is of compact support and \( Q(y) = e^{u_0(y)}M(y) \) has the representations

\[
Q(y) = 1 + e^{(\omega_2 - \omega_1)y + r_2 - r_1} + \cdots + e^{(\omega_{N_F} - \omega_1)y + r_{N_F} - r_1}, \quad y > y_0, \tag{4.12}
\]

\[
Q(y) = 1 + e^{(\omega_1 - \omega_{N_F})y + r_1 - r_{N_F}} + \cdots + e^{(\omega_{N_F} - 1 - \omega_{N_F})y + r_{N_F} - 1 - r_{N_F}}, \quad y < -y_0. \tag{4.13}
\]

Of course, \( Q(y) > 0 \) everywhere, \( Q(\pm \infty) = 1 \), and the boundary conditions (4.2) and (4.3) become the standard one,

\[
v = 0 \quad \text{at} \quad y = \pm \infty. \tag{4.14}
\]

---

\(^7\)The bound agrees with the previous result of the iterative approximation [20].
Since \( u'_0(y) = -\omega_1 \) for \( y > y_0 \) and \( u'_0(y) = -\omega_{Np} \) for \( y < -y_0 \), it is clear that, when \( y_0 > 0 \) is sufficiently large, we can define \( u'_0(y) \) for \( -y_0 \leq y \leq y_0 \) to make \( |u''_0(y)| \) as small as we please. In particular, we may achieve (say)

\[
|h(y)| = |u''_0(y)| < \frac{\lambda}{2} \quad \text{for all } y.
\]

This assumption will be observed in the subsequent analysis.

It is clear that (4.11) is the Euler–Lagrange equation of the action functional

\[
I(v) = \int \left\{ \frac{1}{2}(v')^2 + \lambda Q(y)(e^v - v - 1) + \lambda(Q(y) - 1)v + hv \right\}, \tag{4.16}
\]

where and in the sequel, we use \( \int \) to stand for the Lebesgue integral over the whole real line \((-\infty, \infty)\) and we omit writing out the measure \( dy \) when no risk of confusion arises.

In order to accommodate the boundary condition (4.14), we work on the standard Sobolev space \( W^{1,2}(\mathbb{R}) = \) the completion of the set of all compactly supported real-valued smooth functions over \( \mathbb{R} \) under the norm

\[
\|f\|_{W^{1,2}(\mathbb{R})}^2 = \|f\|^2_2 + \|f'\|^2_2,
\]

where we use \( \| \cdot \|_p (p \geq 1) \) to denote the integral norm

\[
\|f\|_p = \left( \int |f(y)|^p \right)^{\frac{1}{p}}.
\]

Use \( C(\mathbb{R}) \) to denote the space of continuous functions over \( \mathbb{R} \) vanishing at \( \pm\infty \), equipped with the standard pointwise norm

\[
\|f\|_{C(\mathbb{R})} = \sup_{-\infty < y < \infty} |f(y)|.
\]

Then an immediate application of the Schwartz inequality yields the continuous embedding

\[
W^{1,2}(\mathbb{R}) \subset C(\mathbb{R}) \quad \text{with} \quad \|f\|_{C(\mathbb{R})} \leq \|f\|_{W^{1,2}(\mathbb{R})}, \quad f \in W^{1,2}(\mathbb{R}). \tag{4.17}
\]

Using (4.17), we see that the functional (4.16) is a well-defined, continuously differentiable, functional over \( W^{1,2}(\mathbb{R}) \), which is also strictly convex.

In the following, we show that (4.16) has a critical point in \( W^{1,2}(\mathbb{R}) \) by minimizing (4.16) over \( W^{1,2}(\mathbb{R}) \).

Recall the following one-dimensional Gagliardo–Nirenberg inequality:

\[
\int |f|^{p+1} \leq C(p) \left( \int f^2 \right)^{\frac{p+1}{4}} \left( \int (f')^2 \right)^{\frac{p-1}{4}}, \quad f \in W^{1,2}(\mathbb{R}), \tag{4.18}
\]

where \( p > 1 \) and \( C(p) \) is a positive constant depending only on \( p \).

Note that, in view of the Schwartz inequality and Gagliardo–Nirenberg inequality (4.18) (with
\( p = 3 \), we have

\[
\left( \int v^2 \right)^2 = \left( \int \frac{|v|}{1 + |v|}(1 + |v||v|) \right)^2 \\
\leq \int \frac{v^2}{(1 + |v|)^2} \int (1 + |v|)^2 v^2 \\
\leq 2 \int \frac{v^2}{(1 + |v|)^2} \int (v^2 + v^4) \\
\leq C_1 \int \frac{v^2}{(1 + |v|)^2} \left( \int v^2 + \left( \int v^2 \right)^{\frac{3}{2}} \left( \int (v')^2 \right)^{\frac{1}{2}} \right) \\
\leq \frac{1}{2} \left( \int v^2 \right)^2 + C_2 \left( \int \frac{v^2}{(1 + |v|)^2} \right)^2 + C_3 \left( \left( \int \frac{v^2}{(1 + |v|)^2} \right)^6 + \left( \int (v')^2 \right)^6 \right).
\]

Consequently, we have

\[
\|v\|_2 \leq C_0 \left( 1 + \int \frac{v^2}{1 + |v|} + \int (v')^2 dy \right)^{\frac{4}{7}},
\]

where \( C_0 > 0 \) is a suitable constant.

Define \( J(v) = (DI(v))(v) \) (the Fréchet derivative) for \( v \in W^{1,2}(\mathbb{R}) \). Then

\[
J(v) = \lim_{t \to 0} \frac{I(v + tv) - I(v)}{t} = (DI(v))(v) = \int \left\{ (v')^2 + \lambda Q(y)(e^v - 1)v + q(y)v \right\},
\]

where \( q(y) = \lambda(Q(y) - 1) + h(y) \) vanishes at \( y = \pm \infty \) exponentially fast.

Let \( v^+ \) and \( v^- \) be the positive and negative parts of \( v \) respectively. That is, \( v = v^+ - v^- \) and

\[
v^+ = \max\{v, 0\} = \frac{1}{2}(v + |v|), \quad v^- = \max\{0, -v\} = \frac{1}{2}(|v| - v).
\]

Then the functional \( J(v) \) defined in (4.20) may be rewritten as

\[
J(v) = \|v\|_2^2 + J_1(v) + J_2(v),
\]

where

\[
J_1(v) = \int \left\{ \lambda Q(y)(e^{v^+} - 1)v^+ + q(y)v^+ \right\}, \\
J_2(v) = \int \left\{ \lambda Q(y)(e^{-v^-} - 1)(-v^-) - q(y)v^- \right\}.
\]
Recall that \( Q(y) \geq 1, (e^{v^+} - 1)v^+ \geq (v^+)^2, (e^{-v^-} - 1)(-v^-) \geq (v^-)^2/(1 + |v^-|) \). Hence
\[
J_1(v) \geq \lambda \|v^+\|_2^2 - \|q\|_2 \|v^+\|_2 \\
\geq \frac{\lambda}{2} \|v^+\|_2^2 - \frac{1}{2\lambda} \|q\|_2^2,
\]
(4.22)
\[
J_2(v) \geq \int \lambda Q(y) \frac{(v^-)^2}{1 + |v^-|} - \int q(y) \frac{v^-}{1 + |v^-|} (1 + |v^-|) \\
\geq \int \lambda Q(y) \frac{(v^-)^2}{1 + |v^-|} - \int |q(y)| - \int (\lambda(Q(y) - 1) + h(y)) \frac{(v^-)^2}{1 + |v^-|} \\
\geq \int (\lambda - |h(y)|) \frac{(v^-)^2}{1 + |v^-|} - \int |q(y)| \\
\geq \frac{\lambda}{2} \int \frac{(v^-)^2}{1 + |v^-|} - \int |q(y)|,\]
(4.23)
where we have used (4.15). Inserting (4.22) and (4.23) into (4.21) and applying (4.19), we arrive at
\[
J(v) \geq \|v'\|_2^2 + \frac{\lambda}{2} \int \frac{v^2}{1 + |v|} - C_4,
\]
\[
\geq \frac{1}{2} \|v'\|_2^2 + \min \left\{ \frac{\lambda}{2}, \frac{\lambda}{2} \right\} \left( \|v'\|_2^2 + \int \frac{v^2}{1 + |v|} \right) - C_4 \\
\geq \frac{1}{2} \|v'\|_2^2 + \min \left\{ \frac{\lambda}{2}, \frac{\lambda}{2} \right\} C_6^{-2/3} \|v\|_2^{2/3} - C_5,
\]
where the constants \( C_4, C_5 > 0 \) depend only on \( \lambda \) and \( q(y) \). Note also that \( \|v'\|_2^{2/3} \leq (1/3) \|v'\|_2^2 + (2/3) \). We can rewrite (4.24) more evenly as
\[
J(v) \geq C_6(\|v\|_2^{2/3} + \|v'\|_2^{2/3}) - C_7 \\
\geq C_8 \|v\|_{W^{1,2}(\mathbb{R})}^{2/3} - C_9,
\]
where \( C_6, C_7, C_8, C_9 \) are some positive constants.

With the above estimates, we are now ready to do minimization following a standard path.

In view of (4.26), let \( R > 0 \) be such that
\[
\inf \{ J(v) \mid v \in W^{1,2}(\mathbb{R}), \|v\|_{W^{1,2}(\mathbb{R})} = R \} \geq 1,
\]
(4.26)
and consider the minimization problem
\[
\sigma = \inf \{ I(v) \mid \|v\|_{W^{1,2}(\mathbb{R})} \leq R \}.
\]
(4.27)

Let \( \{v_n\} \) be a minimizing sequence of the problem (4.27). Since \( \{v_n\} \) is bounded in \( W^{1,2}(\mathbb{R}) \), by extracting a subsequence if necessary, we may assume that \( \{v_n\} \) is also weakly convergent. Let \( \tilde{v} \) be its weak limit in \( W^{1,2}(\mathbb{R}) \). Since \( I(\cdot) \) is a continuously differentiable functional over \( W^{1,2}(\mathbb{R}) \) and convex, \( I(\cdot) \) is weakly lower semicontinuous. Hence \( I(\tilde{v}) \leq \lim_{n \to \infty} I(v_n) = \sigma \). Of course, \( \|\tilde{v}\|_{W^{1,2}(\mathbb{R})} \leq R \) because the norm of \( W^{1,2}(\mathbb{R}) \) is also weakly lower semicontinuous. Hence \( \tilde{v} \) solves (4.27). That is, \( I(\tilde{v}) = \sigma \). To show that \( \tilde{v} \) is a critical point of \( I(\cdot) \), we need to show that \( \tilde{v} \) is interior, or \( \|\tilde{v}\|_{W^{1,2}(\mathbb{R})} < R \).
Otherwise, if \( \| \tilde{v} \|_{W^{1,2}(\mathbb{R})} = R \), then, by (4.26), we have
\[
\lim_{t \to 0} \frac{I(\tilde{v} - t \tilde{v}) - I(\tilde{v})}{t} = -(DI(\tilde{v}))(\tilde{v}) = -J(\tilde{v}) \leq -1.
\]
In particular, when \( t > 0 \) is sufficiently small, we have \( I(\tilde{v} - t \tilde{v}) < I(\tilde{v}) = \sigma \). On the other hand, \( \| \tilde{v} - t \tilde{v} \|_{W^{1,2}(\mathbb{R})} = (1 - t)R < R \). These two facts violate the definition of \( \sigma \) made in (4.27).

Therefore \( \tilde{v} \) is interior. Consequently, it is a critical point of \( I(\cdot) \) in \( W^{1,2}(\mathbb{R}) \), which is a weak solution of (4.11). The standard elliptic regularity theory shows that this gives rise to a \( C^\infty \)-solution of the original equation (4.1) subject to the boundary conditions (4.2) and (4.3).

The strict convexity of \( I(\cdot) \) already implies that \( I(\cdot) \) can have at most one critical point in \( W^{1,2}(\mathbb{R}) \). Hence \( I(\cdot) \) has exactly one critical point in \( W^{1,2}(\mathbb{R}) \) and the uniqueness of a solution to (4.1)–(4.3) or (4.11) and (4.14) follows. In fact, such a uniqueness result follows from the structure of the equation in a more straightforward way: if \( v_1 \) and \( v_2 \) are two solutions of (4.11) and (4.14), then \( w = v_1 - v_2 \) satisfies \( w'' = \lambda Q(y)e^{\xi(y)}w \) where \( \xi(y) \) lies between \( v_1(y) \) and \( v_2(y) \). Since \( w = 0 \) at \( y = \pm \infty \) and \( Q(y)e^{\xi(y)} > 0 \) for any \( y \), we must have \( w(y) \equiv 0 \).

Finally, we estimate the asymptotic exponential decay rates of the solution of (4.11) and (4.14) near \( y = \pm \infty \). For \( y > y_0 \), we see that (4.11) takes the form
\[
v'' = \lambda Q(y)(e^y - 1) + \lambda Q(y) - 1.
\]
Introduce a comparison function
\[
V = Ce^{-\omega(1-\varepsilon)y}, \quad \omega = \min\{\lambda, (\omega_1 - \omega_2)^2\}, \quad 0 < \varepsilon < 1.
\]
Then \( V \) satisfies \( V'' = \omega(1 - \varepsilon)^2V \). In view of this, (4.28), and (4.29), we have
\[
(v \pm V)'' = \lambda Q(y)e^{\xi(y)}(v \pm V) + \lambda Q(y) - 1 \mp (\lambda Q(y)e^{\xi(y)} - \omega(1 - \varepsilon)^2)Ce^{-\sqrt{\omega}(1-\varepsilon)y},
\]
where \( \xi(y) \) lies between 0 and \( v(y) \). Assume that the constant \( C \) in (4.29) satisfies \( C \geq 1 \) (say). Since \( v(y) \to 0 \) and \( Q(y) - 1 = O(e^{-(\omega_1 - \omega_2)y}) \) as \( y \to \infty \), we can find a sufficiently large \( y_1 > 0 \) so that
\[
\lambda Q(y) - 1 - (\lambda Q(y)e^{\xi(y)} - \omega(1 - \varepsilon)^2)Ce^{-\sqrt{\omega}(1-\varepsilon)y} < 0, \quad y > y_1.
\]
Combining (4.30) and (4.31), we arrive at
\[
(v + V)'' < \lambda Q(y)e^{\xi(y)}(v + V), \quad y > y_1.
\]
Of course, we may choose the constant \( C \geq 1 \) in the definition of the function \( V \) (see (4.29)) sufficiently large so that \( (v + V)(y_1) \geq 0 \). Using this condition, the fact that \( v + V = 0 \) at \( y = \infty \), and (4.32), we get \( (v + V)(y) > 0 \) for all \( y > y_1 \).

On the other hand, since \( Q(y) \geq 1 \) for \( y > y_0 \) (see (4.12)), in view of (4.29) and (4.30) again, we see that there is a sufficiently large \( y_2 > y_0 \) so that
\[
(v - V)'' > \lambda Q(y)e^{\xi(y)}(v - V), \quad y > y_2.
\]
Again, we may choose the constant \( C \geq 1 \) (say) in the definition of the function \( V \) (see (4.29)) sufficiently large so that \( (v - V)(y_2) \leq 0 \). Using this condition, the fact that \( v - V = 0 \) at \( y = \infty \), and (4.33), we get \( (v - V)(y) < 0 \) for all \( y > y_2 \).
In summary, we have obtained the expected asymptotic exponential decay estimate \( |v(y)| < V(y) = Ce^{-\sqrt{\omega}(1-\varepsilon)y} \) for \( y \to \infty \).

A similar argument leads to the exponential decay estimate for \( v(y) \) as \( y \to -\infty \).

The proof of Theorem 4.1 is now complete.

The Acknowledgments

N.S. wishes to thank a fruitful collaboration with Minoru Eto, Youichi Isozumi, Muneto Nitta, Keisuke Ohashi, Kazutoshi Ohta, Yuji Tachikawa, and David Tong. He is also benefitted from useful communications with Jarah Evslin and David Tong on the solvability of the master equations. N.S. is supported in part by Grant-in-Aid for Scientific Research from the Ministry of Education, Culture, Sports, Science and Technology, Japan No.16028203 for the priority area “origin of mass” and No.17540237. Y.Y. was supported in part by NSF grant DMS–0406446.

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