CAUCHY PROBLEM FOR THE BOLTZMANN-BGK MODEL NEAR A GLOBAL MAXWELLIAN

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Abstract. In this paper, we are interested in the Cauchy problem for the Boltzmann-BGK model for a general class of collision frequencies. We prove that the Boltzmann-BGK model linearized around a global Maxwellian admits a unique global smooth solution if the initial perturbation is sufficiently small in a high order energy norm. We also establish an asymptotic decay estimate and uniform $L^2$-stability for nonlinear perturbations.

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1. Introduction

The dynamics of a monatomic, non-ionized gaseous system is known to be governed by the celebrated Boltzmann equation. But the complicated structure of the collision operator has long been a major obstacle in developing efficient numerical methods [4]. In an effort to find a simplified model of the Boltzmann equation, Bhatnagar, Gross, and Krook [2], and independently Walender [28], introduced the Boltzmann-BGK model:

$$\frac{\partial}{\partial t} F + v \cdot \nabla_x F = \nu (\mathcal{M}(F) - F),$$

$$F(x, v, 0) = F_0(x, v),$$

where $F(x, v, t) \in T^3 \times \mathbb{R}^3 \times \mathbb{R}_+$ is the particle distribution function representing the number density of particles in phase space at position $x$, velocity $v$ and time $t$. $T^3$ denotes the 3-dimensional torus $\mathbb{R}^3/\mathbb{Z}^3$. $\mathcal{M}(F)$ is the local Maxwellian defined as

$$\mathcal{M}(F)(x, v, t) = \frac{\rho(x, t)}{(2\pi T(x, t))^{3/2}} \exp \left( -\frac{|v - U(x, t)|^2}{2T(x, t)} \right),$$

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where $\rho$, $U$ and $T$ denote the macroscopic fields constructed from velocity moments of the distribution function:

$$
\rho(x,t) = \int_{\mathbb{R}^d} F(x,v,t) dv,
$$

$$
\rho(x,t)U(x,t) = \int_{\mathbb{R}^d} F(x,v,t) v dv,
$$

$$
3\rho(x,t)T(x,t) = \int_{\mathbb{R}^d} F(x,v,t)|v - U(x,t)|^2 dv.
$$

Throughout this paper, we assume that the collision frequency $\nu$ takes the following form:

$$
\nu = \nu_{\eta,\omega}(\rho, T) \equiv \rho \eta T^\omega,
$$

where we have suppressed the constant to be unity for simplicity. A wide class of non-trivial collision frequencies is encompassed by this model. For example, Aoki et al. [1] studied the collision frequency defined as

$$
\nu_{1,0} = \rho.
$$

On the other hand, the following model was considered in [6, 20, 32]:

$$
\nu_{1,1-\omega} = \rho T^{1-\omega},
$$

where $\omega$ was chosen to be the exponent of the viscosity law of the gas. The constant collision frequency [2, 5, 8, 19, 21, 22, 24, 25] corresponds to

$$
\nu_{0,0} = 1.
$$

The relaxation operator is designed to share important features with Boltzmann collision operator. For example, the relaxation operator satisfies the following cancelation property:

$$
\int_{\mathbb{R}^3} \nu(\mathcal{M}(F) - F) \left( \frac{1}{v} \right) dv = 0,
$$

which implies the conservation laws of mass, total momentum and total energy:

$$
\frac{d}{dt} \int_{\mathbb{T}^3 \times \mathbb{R}^3} F dx dv = 0,
$$

$$
\frac{d}{dt} \int_{\mathbb{T}^3 \times \mathbb{R}^3} F v dx dv = 0,
$$

$$
\frac{d}{dt} \int_{\mathbb{T}^3 \times \mathbb{R}^3} F |v|^2 dx dv = 0.
$$

We also have the following celebrated H-theorem:

$$
\frac{d}{dt} \int_{\mathbb{T}^3 \times \mathbb{R}^3} F \log F dx dv = \int_{\mathbb{T}^3 \times \mathbb{R}^3} \nu(\mathcal{M}(F) - F) \log F dx dv \leq 0.
$$

From the numerical point of view, the BGK model considerably simplifies the situation in that it is sufficient to update the macroscopic fields in each time step. But mathematical analysis is not necessarily easier, because the relaxation operator involves more nonlinearity compared to the bilinear collision operator of the Boltzmann equation. In [21], Perthame et al. established the global existence of weak solutions for the BGK model with constant collision frequency. Regularity and uniqueness was then considered in [17, 22] under the
local existence framework. The result in [17] was employed in [25] to prove the convergence in a weight $L^1$ norm of a semi-Lagrangian scheme developed in [8, 24, 23, 26], which is, as far as the author knows, the first result on strong convergence of a fully discretized scheme for nonlinear collisional kinetic equations. In near-a-global-Maxwellian regime, the global existence in the whole space $\mathbb{R}^3$ was established in [3] employing Ukai’s spectral analysis [30]. Chan [5] studied the global existence in torus using the nonlinear energy method developed by Liu, Yang and Yu [18]. In [5], however, the decay rate is not known, which is expected to be exponentially fast.

The purpose of the present paper is two-fold: first, we obtain the well-posedness of the Boltzmann-BGK model near a global Maxwellian for a wide class of non-trivial collision frequencies. Secondly, we establish the asymptotic decay estimate and uniform $L^2$-stability [15, 16]. The main theoretical tool is the nonlinear energy method developed by Guo [12, 13, 14] to investigate the well-posedness of various important collisional kinetic equations such as the Boltzmann equation or the Vlasov-Maxwell (Poisson)-Boltzmann equation.

Brief comments on possible extensions of our results are in order. Our assumptions on collision frequency do not cover the velocity dependent models proposed in [27, 31], which involves additional technical difficulties. Cauchy problems for relaxation models describing ionized plasma also can be considered by extending the arguments of this paper. We leave these topics for future research [29].

Before we proceed further, we set some notational conventions here.

- $\langle \cdot, \cdot \rangle$ denotes the standard $L^2$ inner product in $\mathbb{T}^d \times \mathbb{R}^d$.
  $$\langle f, g \rangle = \int_{\mathbb{R}^3 \times \mathbb{R}^3} f(x,v)g(x,v)dxdv.$$

- $\| \cdot \|_{L^2_{x,v}}$ and $\| \cdot \|_{L^2_{x,v}}$ denotes $L^2$ norms in $\mathbb{R}^d_v$ and $\mathbb{T}_x^d \times \mathbb{R}^d_v$ respectively.
  $$\|f\|_{L^2_{x,v}} \equiv \left( \int_{\mathbb{R}^3} |f(v)|^2dv \right)^{\frac{1}{2}},$$
  $$\|f\|_{L^2_{x,v}} \equiv \left( \int_{\mathbb{R}^3 \times \mathbb{R}^3} |f(x,v)|^2dxdv \right)^{\frac{1}{2}}.$$

- Multi-indices $\alpha$, $\beta$ are defined by
  $$\alpha = [\alpha_0, \alpha_1, \alpha_2, \alpha_3], \quad \beta = [\beta_1, \beta_2, \beta_3]$$
  and
  $$\partial_\alpha^\beta = \partial_t^{\alpha_0} \partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} \partial_{x_3}^{\alpha_3} \partial_{v_1}^{\beta_1} \partial_{v_2}^{\beta_2} \partial_{v_3}^{\beta_3}.$$

- The energy norm $\|\| \cdot \||$ is defined as follows.
  $$\|\|f(t)\|| \equiv \sum_{|\alpha|+|\beta| \leq N} \|\partial_\alpha^\beta f(t)\|_{L^2_{x,v}},$$
  where $N > 4$.

- We define the high order energy norm for $f$ as
  $$E(t) = \frac{1}{2} \||f(t)\||^2 + \nu_c \int_0^t \|\|f(s)\||^2ds,$$
  where the constant $\nu_c$ is defined in Proposition 2.2.
Throughout this paper, $C_{a,b,\cdots}$ will denote a generic constant depending on $a, b, \cdots$, but not on $x, v, \text{and } t$.

The paper is organized as follows. In section 2, we investigate the linearization procedure of the Boltzmann-BGK model. In section 3, the main theorem is stated. In section 4, we present several important technical lemmas. Section 5 is devoted to establishing the local in time existence and uniqueness of smooth solutions. In section 6, we study the coercive property of the linearized relaxation operator. Finally, in section 7, we combine these results to obtain the global in time existence of the classical solution.

2. LINEARIZED BGK MODEL

In this section, we consider the linearization of the Boltzmann-BGK model around the normalized global Maxwellian:

$$m(v) = \frac{1}{\sqrt{(2\pi)^3}} e^{-\frac{|v|^2}{2}}.$$

We first establish a technical lemma which will be frequently used in the sequel.

**Lemma 2.1.** Let $G$ be a function given by

$$G(x, t) = \rho |U|^2 + 3\rho T - \frac{3\rho}{\sqrt{6}}.$$

Then the Jacobian matrix of the change of variable $(\rho, U, T) \rightarrow (\rho, \rho U, G)$ is given by

$$\frac{\partial (\rho, \rho U, G)}{\partial (\rho, U, T)} = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
\rho U_1 & 0 & 0 & 0 & 0 \\
0 & \rho & 0 & 0 & 0 \\
\rho U_2 & 0 & 0 & \rho & 0 \\
\rho U_3 & 0 & 0 & 0 & \rho \\
\frac{|U|^2 + 3T - 3}{\sqrt{6}} & 2\rho U_1 & 2\rho U_2 & 3\rho U_3 & 3\rho
\end{pmatrix}.$$

and

$$\left(\frac{\partial (\rho, \rho U, T)}{\partial (\rho, \rho U, G)}\right)^{-1} = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
\rho U_1 & 0 & 0 & 0 & 0 \\
\rho U_2 & 0 & 1 & 0 & 0 \\
\rho U_3 & 0 & 0 & 1 & 0 \\
A & B_1 & B_2 & B_3 & C
\end{pmatrix},$$

where

$$A = \frac{2|U|^2 - (|U|^2 + 3T - 3)\left[1 + \frac{U_1 + U_2 + U_3}{\rho}\right]}{3\rho - (|U|^2 + 3T - 3)},$$

$$B_i = \frac{-2\rho U_i - (|U|^2 + 3T - 3)}{\rho(3\rho - (|U|^2 + 3T - 3))},$$

$$C = \frac{\sqrt{6}}{3\rho - (|U|^2 + 3T - 3)}.$$

**Proof.** It can be verified by a straightforward, but very tedious and lengthy calculation. We omit the proof. \qed
Before we proceed to the next lemma, we define an operator which plays an important role in the theory of kinetic equations:

**Definition 2.1.** The macroscopic projection is defined by

\[ P f \equiv \sum_{i=1}^{5} \langle f, e_i \rangle e_i, \]

where \( \{ e_i \} \) is an orthonormal basis for five-dimensional linear space spanned by \( \sqrt{m}, v_1 \sqrt{m}, v_2 \sqrt{m}, v_3 \sqrt{m}, |v|^2 \sqrt{m} \):

\[
\begin{align*}
  e_1 &= \sqrt{m}, \\
  e_2 &= v_1 \sqrt{m}, \\
  e_3 &= v_2 \sqrt{m}, \\
  e_4 &= v_3 \sqrt{m}, \\
  e_5 &= \frac{|v|^2 - 3}{2 \sqrt{m}}.
\end{align*}
\]

(2.5)

**Proposition 2.1.** Let \( F = m + \sqrt{m} f \). Then the local Maxwellian \( M(F) \) can be linearized around a global Maxwellian \( m \) as follows

\[
M(F) = m + P f \sqrt{m} + \sum_{1 \leq i, j \leq 3} \left( \int_0^1 \{ D_{(\rho_{\theta}, \rho_{\theta} U_{\theta}, G_{\theta})} M(\theta) \} (1 - \theta)^2 d\theta \right) \langle f, e_i \rangle \langle f, e_j \rangle.
\]

Here \( M(\theta) \) denotes

\[
M(\theta) = \frac{\rho_{\theta}}{\sqrt{(2\pi T_{\theta})^3}} e^{-\frac{|v - U_{\theta}|^2}{2\rho_{\theta}}},
\]

where \( \rho_{\theta}, U_{\theta}, T_{\theta} \) are defined by the following relations:

\[
\begin{align*}
  \rho_{\theta} &= \theta \rho + (1 - \theta)1, \\
  \rho_{\theta} U_{\theta} &= \theta \rho U, \\
  \frac{\rho_{\theta}|U_{\theta}|^2 + 3\rho_{\theta} T_{\theta}}{2} - \frac{3}{2} \rho_{\theta} &= \theta \left( \frac{\rho|U|^2 + 3\rho T}{2} - \frac{3}{2} \rho \right).
\end{align*}
\]

(2.6)

**Proof.** We define \( f(\theta) \) as follows

\[
\begin{align*}
  f(\theta) &\equiv M\left( \rho, \rho U, \frac{\rho |U|^2 + 3\rho T}{\sqrt{6}} - \frac{3\rho}{\sqrt{6}} \right) + (1 - \theta)(1, 0, 0) \\
  &\equiv M(\rho_{\theta}, \rho_{\theta} U_{\theta}, G_{\theta}) \\
  &\equiv M(\theta).
\end{align*}
\]

We note that \( f \) represents the transition from the global Maxwellian \( m \) to the local Maxwellian \( M \):

\[
(2.7) \quad f(1) = \frac{\rho}{\sqrt{(2\pi T)^3}} e^{-\frac{|v - U|^2}{2\rho}} \quad \text{and} \quad f(0) = \frac{1}{\sqrt{(2\pi)^3}} e^{-\frac{|v|^2}{2}}.
\]

We then apply Taylor’s theorem around \( \theta = 0 \) to see

\[
(2.8) \quad f(1) = f(0) + f'(0) + \int_0^1 f''(\theta)(1 - \theta)^2 d\theta.
\]
(i) $f'(0)$: We have from Lemma 2.1 and the chain rule

$$f'(0) = \left. \frac{d}{d\theta} M(\theta(\rho, \rho U, G) + (1 - \theta)(1, 0, 0)) \right|_{\theta=0}$$

$$\quad = (\rho - 1, \rho U, G) \left. \left( \frac{\partial (\rho \theta, \rho \theta U, \theta G)}{\partial (\rho \theta, \theta T)} \right)^{-1} \left( \nabla (\rho \theta U, \theta T) \right)^T \right|_{\theta=0}$$

$$\quad = (\rho - 1, \rho U, G) \left( \begin{array}{ccccc} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{2}/3 & 0 \\ \end{array} \right) \left( \begin{array}{c} 1 \\ v_1 \\ v_2 \\ v_3 \\ \sqrt{v^2 - 3}/2 \\ \end{array} \right) m$$

$$\quad = (\rho - 1)m + (\rho U)v m + G\left|v^2 - 3\right|\sqrt{6}/m$$

$$\quad = \left( \int f \sqrt{m} dv \right) m + \left( \int f \sqrt{m} dv \right) v m + \left( \int f \left|v^2 - 3\right| \sqrt{m} dv \right) \left|v^2 - 3\right| m$$

$$\quad = Pf \sqrt{m}.$$

(ii) $\int_0^1 f''(\theta)(1 - \theta)^2 d\theta$: We have from the chain rule

$$f''(\theta) = \frac{d^2}{d\theta^2} M(\theta(\rho - 1, \rho U, G) + (1 - \theta)(1, 0, 0))$$

$$\quad = (\rho - 1, \rho U, G) \left. \left\{ D^2_{(\rho \theta, \rho \theta U, \theta G)} M(\theta) \right\} (\rho - 1, \rho U, G)^T \right|_{\theta=0}.$$

We then substitute (i) and (ii) into (2.8) to obtain the desired result. 

We now consider the linearization of the collision frequency.

**Proposition 2.2.** The collision frequency can be linearized around the normalized global Maxwellian as follows.

$$\nu = \nu_c + \nu_p,$$

where

$$\nu_c = \left( \frac{3}{2} \right) \omega$$

and

$$\nu_p = \sum_i \langle f, e_i \rangle \int_0^1 D_{(\rho \theta, \rho \theta U, \theta G)} \nu(\theta)(1 - \theta) d\theta.$$

Here $\nu(\theta)$ denotes

$$\nu(\theta) = \rho_\theta^0 T_\theta^\omega,$$

where $\rho_\theta, U_\theta, T_\theta$ are defined as in the previous proposition.

**Proof.** Since the proof is almost identical to the previous one, We omit it. 

Since the exact form of $D^2_{(\rho \theta, \rho \theta U, \theta G)} M(\theta)$ is too complicated to be written down and manipulated explicitly, we introduce generic notations which considerably simplifies the
argument. We first note from the chain rule

\[
D^2_{(\rho, \rho U, G)} \mathcal{M}(\theta) = \nabla_{(\rho_0, \rho_0 U_0, G_0)} \left( \frac{1}{\rho_0} + B \cdot \frac{v-T_\theta}{T_\theta} + C \frac{|v-U_\theta|^2 - 3T_\theta}{2T_\theta^2} \right) \mathcal{M}(\theta),
\]

where \( A, B, C \) are rational functions of macroscopic fields defined in Lemma 2.1. Therefore, we can deduce from Lemma 2.1 that there exist polynomials \( P^{\mathcal{M}}_{i,j} \), \( R^{\mathcal{M}}_{i,j} \) such that

\[
D^2_{(\rho_0, \rho_0 U_0, G_0)} \mathcal{M}(\theta) = \sum_{i,j} P^{\mathcal{M}}_{i,j}(\rho_0, v - U_\theta, U_\theta, T_\theta) e^{-\frac{|v-u|^2}{2T_\theta}},
\]

where

\[
G_\theta \equiv 3 + 3\rho_\theta - |U_\theta|^2 - 3T_\theta \rho_\theta
\]

and \( P^{\mathcal{M}}_{i,j}(x_1, \ldots, x_n) \) and \( R^{\mathcal{M}}_{i,j}(x_1, \ldots, x_n) \) satisfy the following structural assumptions (\( \mathcal{H}_\mathcal{M} \)):

1. \((\mathcal{H}_\mathcal{M}1)\) \( P^{\mathcal{M}}_{i,j} \) is a polynomial such that \( P^{\mathcal{M}}_{i,j}(0, 0, \ldots, 0) = 0 \).
2. \((\mathcal{H}_\mathcal{M}2)\) \( R^{\mathcal{M}}_{i,j} \) is a monomial.

More precisely, we have for a multi-index \( m = (m_1, m_2, \ldots, m_3) \):

1. \((\mathcal{H}_\mathcal{M}1)\) \( P^{\mathcal{M}}_{i,j}(x_1, \ldots, x_n) = \sum m a_m x_1^{m_1} x_2^{m_2} \ldots x_n^{m_n} \), where \( a_0 = 0 \),
2. \((\mathcal{H}_\mathcal{M}2)\) \( R^{\mathcal{M}}_{i,j}(x_1, \ldots, x_n) = a_m x_1^{m_1} x_2^{m_2} \ldots x_n^{m_n} \).

From now on, we assume that \( P^{\mathcal{M}}_{i,j} \) and \( R^{\mathcal{M}}_{i,j} \) are defined generically, which means the exact form may change from line to line. These generic notations simplify the calculation drastically and cause no problems if we only keep in mind the structural assumption (\( \mathcal{H}_\mathcal{M} \)) at each step. We now simplify the notation further by defining

\[
Q^{\mathcal{M}}_{i,j} = \frac{P^{\mathcal{M}}_{i,j}(\rho_0, U_\theta, T_\theta, v - U_\theta)}{R^{\mathcal{M}}_{i,j}(\rho_0, T_\theta, G_\theta)}
\]

and

\[
Q^{\mathcal{M}}_{i,j} = \int_0^1 Q^{\mathcal{M}}_{i,j}(\rho_0, U_\theta, T_\theta, v - U_\theta, T_\theta) e^{-\frac{|v-u|^2}{2T_\theta}} e^{\frac{|v|^2}{4} (1 - \theta)^2} d\theta
\]

to see from Proposition 2.1

\[
(2.9) \quad \mathcal{M}(F) = \frac{m}{\sqrt{m}} Pf + \frac{m}{\sqrt{m}} \sum_{i,j} Q^{\mathcal{M}}_{i,j}(f, e_i) \langle f, e_j \rangle.
\]
By the exactly same argument, we write the collision frequency as follows:

\[ \nu = \nu_c + \sum_i Q_i^\nu \langle f, e_i \rangle, \]

where

\[ Q_i^\nu = \sum_i \langle f, e_i \rangle \int_0^1 Q_i^\nu(1-\theta)d\theta \]

\[ = \sum_i \langle f, e_i \rangle \int_0^1 D_{(\rho_0, \rho\theta U_\theta, G_\theta)} \nu(1-\theta)d\theta. \]

We again introduce generic polynomials \( P_i^\nu(x_1, \ldots, x_n) \) and \( R_i^\nu(x_1, \ldots, x_n) \) such that:

\[ D_{(\rho_0, \rho\theta U_\theta, G_\theta)} \nu(\theta) = P_i^\nu(x_1, \ldots, x_n), \]

\[ R_i^\nu(x_1, \ldots, x_n) = C x_1^{m_1} x_2^{m_2} \cdots x_n^{m_n}. \]

We summarize the argument so far in the following proposition.

**Proposition 2.3.** The relaxation operator is linearized around \( m \) as follows.

\[ \frac{\nu}{\sqrt{m}} (\mathcal{M}(F) - F) = (\nu_c + \nu_p) \left\{ \left( P f - f \right) + \sum_{i,j} Q_i^M \langle f, e_i \rangle \langle f, e_j \rangle \right\}. \]

We now substitute the standard perturbation \( F = m + \sqrt{m} f \) into (1.1) and apply Proposition 2.1, 2.2 and 2.3 to obtain the perturbed Boltzmann-BGK model:

\[ \frac{\partial f}{\partial t} + v \cdot \nabla_x f = Lf + \Gamma f, \]

\[ f(x, v, 0) = f_0(x, v), \quad (x, v, t) \in T^3 \times \mathbb{R}^3 \times \mathbb{R}_+, \]

where \( f_0(x, v) = \frac{f_0 - m}{\sqrt{m}} \). The linearized relaxation operator \( L \) is given by

\[ Lf = \nu_c (P f - f), \]

and the nonlinear perturbation \( \Gamma(f) \) is defined as follows:

\[ \Gamma(f) = \nu_p L f + (\nu_c + \nu_p) \sum_{1 \leq i,j \leq 5} Q_i^M \langle f, e_i \rangle \langle f, e_j \rangle, \]

\[ = \Gamma_1(f, f) - \Gamma_2(f, f) + \Gamma_3(f, f) + \Gamma_4(f, f, f), \]

where

\[ \Gamma_1(f, g) = \nu_p L f = \sum_{i,j} Q_i^\nu \langle f, e_i \rangle \langle g, e_j \rangle, \]

\[ \Gamma_2(f, g) = \nu_p f = \sum_{i,j} Q_i^\nu \langle f, e_j \rangle g, \]

\[ \Gamma_3(f, g) = \nu_c \sum_{i,j} Q_i^M \langle f, e_i \rangle \langle g, e_j \rangle, \]
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\[ \Gamma_{4}(f, g, h) = \nu c \sum_{i,j,k} Q_i^\nu Q_j^M f(e_i) g(e_j) h(e_k). \]

First we note that the conservation laws \((2.13)\) now take the following form:

**Lemma 2.2.** Suppose \( f \) is a smooth solution of \((2.12)\). Then \( f \) satisfies the following conservation laws.

\[
\begin{align*}
\int_{T^3 \times \mathbb{R}^3} f \sqrt{m} dx \, dv &= 0, \\
\int_{T^3 \times \mathbb{R}^3} fv \sqrt{m} dx \, dv &= 0, \\
\int_{T^3 \times \mathbb{R}^3} f|v|^2 \sqrt{m} dx \, dv &= 0.
\end{align*}
\]

(2.13)

We close this section by recalling the following important properties of the linearized Boltzmann-BGK model.

**Lemma 2.3.** The macroscopic projection

\[ Pf = \sum_{i=1}^{5} \langle f, e_i \rangle e_i \]

is a compact operator from \( L^2 \) into \( L^2 \).

**Proof.** It follows directly from the fact that the kernel of each integral operator lies in \( L^2(T^3 \times \mathbb{R}^3) \).

**Lemma 2.4.** \( L \) satisfies the following coercivity property.

\[ \langle Lf, f \rangle = -\nu c \| (I - P) f \|_{L^2_{x,v}}^2 \]

**Proof.** By \( \langle Pf, (I - P)f \rangle = 0 \), we have

\[ \langle Pf, f \rangle = \langle Pf, Pf \rangle = \| Pf \|_{L^2_{x,v}}^2, \]

which yields

\[ \langle Lf, f \rangle = \nu c \langle Pf, f \rangle - \nu c \| f \|_{L^2_{x,v}}^2 \]

\[ = -\nu c \| (I - P) f \|_{L^2_{x,v}}^2. \]

3. Main Result

We are finally in a position to state our main result.

**Theorem 3.1.** Let \( N > 4 \) and \( F_0 = m + \sqrt{mf_0} \geq 0 \). Suppose that \( f_0 \) satisfies the conservation laws \((2.13)\). Then there exist positive constants \( C, M, \delta^* \) and \( \delta \), such that if \( E(0) < M \), then there exists a unique global solution \( f(x, v, t) \) to \((2.12)\) such that

(1) The high order energy norm is uniformly bounded:

\[ E(t) \leq CE(0). \]

(2) The perturbation decays exponentially fast:

\[ ||| f(t) |||(t) \leq e^{-\delta^* t} \sqrt{E(0)}. \]
if \( \bar{f} \) denotes another solution corresponding to initial data \( \bar{f}_0 \) satisfying the same assumptions, then we have the following uniform \( L^2 \)-stability estimate:

\[
\| f(t) - \bar{f}(t) \|_{L^2_{x,v}} \leq e^{-\delta_* t} \| f_0 - \bar{f}_0 \|_{L^2_{x,v}}.
\]

**Remark 3.1.** Extension of these results to collision frequencies of the following form is straightforward.

\[
\nu_{\eta, \mu}(\rho, T) = \left\{ \sum_{i=0}^{m_1} a_i \rho^{\eta_i}(x, t) \right\} \left\{ \sum_j b_j T^{\mu_j}(x, t) \right\}.
\]

## 4. Preliminary estimates

In this section, we present several estimates on macroscopic fields which are crucial to develop the argument further.

**Lemma 4.1.** Let \(|\alpha| \geq 1\). Suppose \( E(t) \) is sufficiently small. Then we have the following upper and lower bounds for macroscopic fields:

1. \( 1 - \sqrt{E(t)} \leq \rho(x, t) \leq 1 + \sqrt{E(t)} \),
2. \( |U(x, t)| \leq 3 \sqrt{E(t)} \),
3. \( \frac{1}{2} \leq T(x, t) \leq \frac{3}{2} \),
4. \( |\partial^\alpha \rho(x, t)| \leq \sqrt{E(t)} \),
5. \( |\partial^\alpha U(x, t)| \leq C_{|\alpha|} \sqrt{E(t)} \),
6. \( |\partial^\alpha T(x, t)| \leq C_{|\alpha|} \sqrt{E(t)} \),

for some positive constant \( C_{|\alpha|} \).

**Proof.** (1) We have from Hölder inequality

\[
\rho = 1 + \int f \sqrt{m} dv \leq 1 + \| f \|_2 \leq 1 + \sqrt{E(t)}.
\]

Similarly, we have

\[
\rho \geq 1 - \int f \sqrt{m} dv \geq 1 - \| f \|_2 \geq 1 - \sqrt{E(t)}.
\]

(2) Since \( \int m dv = 0 \), we have by Hölder inequality

\[
U = \frac{\int (m + \sqrt{mf}) dv}{\rho} = \frac{\int f \sqrt{md} v}{\rho} \leq \frac{3}{2 \| f \|_2^2} \leq \frac{3 \sqrt{E(t)}}{2 \sqrt{1 - \sqrt{E(t)}}} \leq 3 \sqrt{E(t)}.
\]

(3) The estimate of \( T \) can be treated similarly as follows:

\[
T = \frac{3 \rho}{\int f |v|^2 dv - \rho |U|^2} \leq \frac{3 \rho}{3 \rho} = 3 + 240 \sqrt{2 \pi^3 \| f \|_2^2}.
\]
\[
\begin{align*}
\leq & \quad 1 + 80\sqrt{2\pi^3} \sqrt{E(t)} \\
\leq & \quad \frac{3}{2},
\end{align*}
\]
where we used the smallness assumption on \(E(t)\) and
\[
\int |v|^4 e^{-\frac{|v|^2}{2}} dv \leq 240\sqrt{2\pi^3}.
\]
The lower bound can be estimated analogously as follows:
\[
T = \frac{\int (m + \sqrt{m} f) |v|^2 dv - \rho |U|^2}{\rho}
\geq \frac{3 - \int f |v|^2 \sqrt{mdv} - \rho |U|^2}{\rho}
\geq \frac{3 - 15\sqrt{\pi} \sqrt{E(t)} - (1 - \sqrt{E(t)}) \sqrt{3E(t)}}{3(1 - \sqrt{E(t)})}
\geq \frac{1 - (5\sqrt{\pi} + 3) \sqrt{E(t)}}{1 - \sqrt{E(t)}}
\geq \frac{1}{2}.
\]
We now turn to the derivatives of the macroscopic fields.
(4) follows directly by the same argument as in (1) noting that
\[
\partial^\alpha \rho = \partial^\alpha \left( \int m + f \sqrt{mdv} \right) = \int \partial^\alpha f \sqrt{mdv}.
\]
(5) We observe from \(U = \frac{\int f \sqrt{mdv}}{\rho}\) that
\[
|\partial^\alpha U| \leq C_{|\alpha|} \left\{ \sum_{1 \leq i \leq |\alpha|} |\rho|^{2i} \right\} \left\{ \sum_{1 \leq \gamma \leq |\alpha|} |\partial \rho|^{2i} \right\} \ldots \left\{ \sum_{1 \leq \gamma \leq |\alpha|} |\partial^\alpha \rho|^{2i} \right\} \rho^{-2|\alpha|}.
\]
We now employ (1) and (4) to see
\[
|\partial^\alpha U| \leq \frac{C_{|\alpha|} E(t)}{(1 - \sqrt{E(t)})^{2|\alpha|}},
\]
where we used for \(i \geq 1\)
\[
E^i(t) \leq E(t).
\]
(6) Similarly, we observe that
\[
|\partial^\alpha T| \leq C_{|\alpha|} \left\{ \sum_{1 \leq i \leq |\alpha|} |\rho|^{2i} \right\} \left\{ \sum_{1 \leq \gamma \leq |\alpha|} |\partial \rho|^{2i} \right\} \ldots \left\{ \sum_{1 \leq \gamma \leq |\alpha|} |\partial^\alpha \rho|^{2i} \right\} \times \left\{ \sum_{1 \leq i \leq |\alpha|} |U|^{2i} \right\} \left\{ \sum_{1 \leq \gamma \leq |\alpha|} |\partial U|^{2i} \right\} \ldots \left\{ \sum_{1 \leq \gamma \leq |\alpha|} |\partial^\alpha U|^{2i} \right\} \rho^{-2|\alpha|}.
\]
This gives by (1), (2), (4) and (5)
\[
|\partial^\alpha T| \leq \frac{C_{|\alpha|} E(t)}{(1 - \sqrt{E(t)})^{2|\alpha|}}.
\]
\[\square\]
Lemma 4.2. Let $|\alpha| \geq 1$. Suppose $E(t)$ is sufficiently small. Then we have

\begin{align*}
(1) \quad 1 - \sqrt{E(t)} & \leq \rho_\theta(x,t) \leq 1 + \sqrt{E(t)}, \\
(2) \quad |U_\theta(x,t)| & \leq 3\sqrt{E(t)}, \\
(3) \quad \frac{1}{2} & \leq T_\theta(x,t) \leq \frac{3}{2}, \\
(4) \quad |\partial^\alpha \rho_\theta(x,t)| & \leq \sqrt{E(t)}, \\
(5) \quad |\partial^\alpha U_\theta(x,t)| & \leq C_{|\alpha|} E(t), \\
(6) \quad |\partial^\alpha T_\theta(x,t)| & \leq C_{|\alpha|} E(t),
\end{align*}

for some positive constant $C_{|\alpha|}$.

Having established the preceding estimates for the macroscopic fields, we can now prove the following crucial proposition for the nonlinear perturbation $\Gamma(f)$.

Proposition 4.1. Suppose $E(t)$ is sufficiently small such that estimates in Lemma 4.1 and Lemma 4.2 are valid. Then we have

\begin{align*}
(1) \quad & \left| \int \partial^\alpha_3 \Gamma(f,f,f) dv \right| \leq C \sum_{|\alpha|+|\alpha_2|\leq|\alpha|} \|\partial^\alpha_1 f\|_{L^2_x} \|\partial^\alpha_2 f\|_{L^2_x} \|r\|_{L^2_x} \\
& \quad + C \sum_{|\alpha_1|+|\alpha_2|\leq|\alpha|, |\beta_2|\leq|\alpha|} \|\partial^\alpha_1 f\|_{L^2_x} \|\partial^\alpha_{\beta_2} f\|_{L^2_x} \|r\|_{L^2_x} \\
& \quad + C \sum_{|\alpha_1|+|\alpha_2|+|\alpha_3|\leq|\alpha|} C \|\partial^\alpha_1 f\|_{L^2_x} \|\partial^\alpha_2 f\|_{L^2_x} \|\partial^\alpha_3 f\|_{L^2_x} \|r\|_{L^2_x},
(2) \quad & |\Gamma_{1,2,3}(f,g,f)| + |\Gamma_{1,2,3}(g,f,f)| \leq C \sup_x \|g\|_{L^2_x} \|f\|_{L^2_x}^2, \\
& |\Gamma_4(f,g,h,f)| + |\Gamma_4(g,f,h,f)| \leq C \sup_x \|g\|_{L^2_x} \sup_x \|h\|_{L^2_x} \|f\|_{L^2_x}^2, \\
(3) \quad & |\Gamma_{1,2,3}(f,g)+\Gamma_{1,2,3}(g,f)+r\|_{L^2_{x,v}}^2 \leq C \sup_{x,v} |r| \sup_x \|f\|_{L^2_x} \|g\|_{L^2_x}, \\
& |\Gamma_{4}(f,g,h)+\Gamma_4(g,f,h)+r\|_{L^2_{x,v}}^2 \leq C \sup_{x,v} |r| \sup_x \|f\|_{L^2_x} \|g\|_{L^2_x} \|h\|_{L^2_{x,v}}.
\end{align*}

Remark 4.1. Note that, unlike the case of the Boltzmann equation, we need to impose the smallness condition on the high order energy to prove the estimates.

Proof. (1) To prove (1), we should consider $\Gamma_i(f)$ ($1 \leq i \leq 4$) separately. But for simplicity we only present the proof for $\Gamma_3(f)$. Other estimates can be obtained in an almost identical manner.

The estimate of $\Gamma_3(f)$: We first prove the following claim:

Claim: There exists a positive constant $\varepsilon = \varepsilon(\alpha, \beta)$ such that

\[ |\partial^\alpha_3 \left\{ Q_{ij}^M e^{-\frac{|v-u|^2}{2T_\theta}} + \frac{|u|^2}{T_\theta} \right\} | \leq C e^{-\frac{|v-u|^2}{2(x+\varepsilon)T_\theta} + \frac{|u|^2}{T_\theta}}. \]
Then the induction argument gives the desired result. We now employ the claim and use H"older inequality again to see

\[ \partial \left\{ Q_{ij}^M e^{-\frac{|v - U_\theta|^2 + |v|^2}{2T_\theta}} \right\} = \partial \left\{ Q_{ij}^M e^{-\frac{|v - U_\theta|^2 + |v|^2}{2T_\theta}} \right\} + Q_{ij}^M \partial \left\{ e^{-\frac{|v - U_\theta|^2 + |v|^2}{2T_\theta}} \right\} \]

\[ = Q_{ij}^M e^{-\frac{|v - U_\theta|^2 + |v|^2}{2T_\theta}} + Q_{ij}^M \left\{ \partial T \left( \frac{|v - U_\theta|^2}{2T_\theta^2} + \partial U \cdot \frac{v - U}{2T} - \frac{v}{2} \right) \right\} e^{-\frac{|v - U_\theta|^2 + |v|^2}{2T_\theta}} \]

\[ = \frac{P_{ij}^M (\rho_\theta, U_\theta, T_\theta, v - U_\theta, v)}{R_{ij}^M (\rho_\theta, U_\theta, T_\theta, \theta)} e^{-\frac{|v - U_\theta|^2 + |v|^2}{2T_\theta}} \]

\[ \leq C e^{-\frac{|v - U_\theta|^2 + |v|^2}{2T_\theta}}, \]

where we used the upper and lower bounds of Lemma 4.2 with

\[ G_\theta = 3 + 3\rho_\theta - |U_\theta|^2 - 3T_\theta \]

\[ \geq 3 + 3(1 - \sqrt{E(t)}) - (3\sqrt{E(t)})^2 - \frac{9}{2} \]

\[ = \frac{3}{2} - 3\sqrt{E(t)} - 3E(t) \]

\[ \geq 1 \]

and

\[ |v - U_\theta|^r e^{-\frac{|v - U_\theta|^2}{(2+\epsilon)T_\theta}} < C \{(2 + \epsilon)T_\theta \}^\frac{r}{2} < \infty, \]

\[ |v|^r e^{-\frac{|v|^2}{(2+\epsilon)T_\theta}} \leq C_r \{(v - U_\theta)^r + |U_\theta|^r\} e^{-\frac{|v - U_\theta|^2}{(2+\epsilon)T_\theta}} < \infty. \]

Then the induction argument gives the desired result. We now employ the claim and use Hölder inequality to see

\[ \int \left| \partial_\beta^\alpha \Gamma_3(f, f) r \right| dv \]

\[ \leq \sum_{|\alpha_0| + |\alpha_1| + |\alpha_2| = |\alpha|} \int \left| \partial_\beta^\alpha \left\{ Q_{ij}^M e^{-\frac{|v - U_\theta|^2 + |v|^2}{2T_\theta}} \right\} \right| \left\{ \partial^{\alpha_1} f, e_i \right\} \left\{ \partial^{\alpha_2} f, e_j \right\} r dv \]

\[ \leq C \sum_{|\alpha_0| + |\alpha_1| + |\alpha_2| = |\alpha|} \int e^{-\frac{|v - U_\theta|^2 + |v|^2}{(2+\epsilon)T_\theta}} \left\{ \partial^{\alpha_1} f, e_i \right\} \left\{ \partial^{\alpha_2} f, e_j \right\} r dv \]

\[ = C \sum_{|\alpha_0| + |\alpha_1| + |\alpha_2| = |\alpha|} \left\| \partial^{\alpha_1} f \right\|_{L^2_\theta} \left\| \partial^{\alpha_2} f \right\|_{L^2_\theta} \int e^{-\frac{|v - U_\theta|^2 + |v|^2}{(2+\epsilon)T_\theta}} r dv. \]

We apply Hölder inequality again to see

\[ \int e^{-\frac{|v - U_\theta|^2 + |v|^2}{(2+\epsilon)T_\theta}} h dv \leq C \left\| e^{-\frac{|v - U_\theta|^2 + |v|^2}{(2+\epsilon)T_\theta}} \right\|_{L^2_\theta} \left\| r \right\|_{L^2_\theta} \leq C \left\| r \right\|_{L^2_\theta}, \]
where we used
\[ \left\| e^{-\frac{|v-U\theta|}{2T\theta}} \right\|_2^2 = \int \exp \left( -\left[ \frac{2}{(2+\varepsilon)T\theta} - \frac{1}{2} \right] |v - \frac{4}{4-(2+\varepsilon)T\theta} U\theta|^2 - \frac{(2+\varepsilon)T\theta}{2-(2+\varepsilon)T\theta} \right) dv \]
\[ \leq C \int \exp \left( -\left[ \frac{1}{(2+\varepsilon)T\theta} - \frac{1}{2} \right] |v + \frac{2}{4-(2+\varepsilon)T\theta} U\theta|^2 \right) dv \]
\[ = C \sqrt{\left( \frac{4-(2+\varepsilon)T\theta}{2} \right)^3} < \infty. \]

In the last line, we used Lemma 4.2. We now substitute (4.15) into (4.14) to obtain the desired result.

(2) In (1), we set \(\alpha=\beta=0\). Then we have from (4.14)
\[ \int \Gamma_3(f,g) f \, dx dv \leq C \int \|f\|_{L^2_v} \|g\|_{L^2_v} \left( \int e^{-\frac{|v-U\theta|}{2T\theta}} \frac{|v|^2}{4} f \, dv \right) dx \]
\[ \leq C \int \|f\|_{L^2_v} \|g\|_{L^2_v} \|f\|_{L^2_v} dx \]
\[ \leq C \sup_x \|g\|_{L^2_v} \|f\|_{L^2_{x,v} \, v}. \]

We now take \(L^2\) norms with respect to spatial variables to obtain the desired result.

(3) Let \(\phi \in L^2\). Then we have from the same argument used in (1)
\[ \langle \Gamma_3(f,g), r, \phi \rangle \leq C \int \|f\|_{L^2_v} \|g\|_{L^2_v} \|r\|_{L^2_v} dx \]
\[ \leq C \sup_{x,v} |r| \int \|f\|_{L^2_v} \|g\|_{L^2_v} \|\phi\|_{L^2_v} dx \]
\[ \leq C \sup_{x,v} |r| \left( \sqrt{\int \|f\|_{L^2_v}^2 \|g\|_{L^2_v}^2 dx} \right) \|\phi\|_{L^2_{x,v}}. \]

Therefore, the duality argument gives
\[ \|\Gamma(f,g) r\| \leq C \sup_{x,v} |r| \sqrt{\int \|f\|_{L^2_v}^2 \|g\|_{L^2_v}^2 dx} \]
\[ \leq C \sup_{x,v} |r| \sup_x \|f\|_{L^2_v} \|g\|_{L^2_{x,v}}. \]

\[ \square \]

5. Local existence

In this section, we establish the local in time existence of classical solutions under the assumption that the high order energy \(E(t)\) is sufficiently small. This local solution will be extended to the global solution in the last section by combining the coercivity estimate of \(L\) and a refined energy estimate.

**Theorem 5.1.** Let \(F_0 = m + \sqrt{m} f_0 \geq 0\). Suppose \(f_0\) satisfies the conservation laws (2.13). Then there exist \(M_0 > 0\), \(T^* > 0\), such that if \(T^* \leq \frac{M_0}{2}\) and \(E(0) \leq \frac{M_0}{2}\), there is a unique solution \(f(x,v,t)\) to the Boltzmann-BGK (2.12) such that

1. The high order energy \(E(t)\) is continuous in \([0,T^*)\) and uniformly bounded:
\[ \sup_{0 \leq t \leq T^*} E(t) \leq M_0. \]
(2) The distribution function remains positive in $[0, T_*)$:

$$F(x, v, t) = m + \sqrt{m} f(x, v, t) \geq 0.$$  

(3) The conservation laws (2.13) hold for all $[0, T_*]$.

Proof. We consider the following iteration sequence:

$$\partial_t F^{n+1} + v \cdot \nabla_x F^{n+1} = \nu^n (\mathcal{M}(F^n) - F^{n+1}),$$

$$F^{n+1}(x, v, 0) = F_0(x, v),$$

which is equivalent to

$$\{ \partial_t + v \cdot \nabla_x + \nu_c \} f^{n+1} = \nu_c P f^n + \Gamma(f^n),$$

$$f^{n+1}(x, v, 0) = f_0(x, v).$$

Now the theorem follows easily once we establish the following lemma.

Lemma 5.1. There exist $M_0 > 0$ and $T_* > 0$ such that if $E(f_0) < M_0$ then $E(f^n(t)) < M_0$ implies $E(f^{n+1}(t)) < M_0$ for $t \in [0, T_*]$.

Proof. We take $\partial_\beta$ derivatives of (5.17) to obtain

$$\{ \partial_t + v \cdot \nabla_x + \nu_c \} \partial_\beta f^{n+1} = - \sum_{\beta \neq 0} \{ \partial_\beta v \cdot \nabla_x \} \partial_\alpha f^{n+1} + \nu_c \partial_\beta P \partial_\alpha f^n + \partial_\beta \Gamma(f^n).$$

We take the inner product of (5.18) with $\partial_\beta f^{n+1}$ and estimate each term separately.

(1) $L$: l.h.s can be calculated directly as follows:

$$\langle L, \partial_\beta f^{n+1} \rangle = \frac{1}{2} \frac{d}{dt} \| \partial_\beta f^{n+1}(t) \|_{L^2_{x,v}}^2 + \nu_c \| \partial_\beta f(t) \|_{L^2_{x,v}}^2.$$  

(2) $R_1$: We see from the following observation

$$\partial_t v \cdot \nabla_x \partial_\alpha f^{n+1} = \partial_x \partial_\alpha f^{n+1}$$

that

$$\langle R_1, \partial_\beta f \rangle \leq \sum_{\beta \neq 0} \| \{ \partial_\beta v \cdot \nabla_x \} \partial_\alpha f^{n+1} \|_{L^2_{x,v}} \| \partial_\beta f^{n+1} \|_{L^2_{x,v}}$$

$$\leq C \| f^{n+1}(t) \|^2$$

$$\leq C E_{n+1}(t).$$

(3) $R_2$: We first note that

$$\| \partial_\beta P \partial_\alpha f \|_{L^2_{x,v}} \leq C_{\beta} \| \partial_\alpha f \|_{L^2_{x,v}}.$$  

Therefore, we have from Hölder inequality and Young’s inequality

$$\langle R_2, \partial_\beta f \rangle \leq \leq C \| \partial_\alpha f^n \|_{L^2_{x,v}}^2 + \| \partial_\alpha f^{n+1} \|_{L^2_{x,v}}^2$$

$$\leq C (\| f^n \|^2 + \| f^{n+1} \|^2)$$

$$\leq C (E_n(t) + E_{n+1}(t)).$$
We now turn to the estimate of the nonlinear term.

(4) $R_3$: We have from Lemma 4.11

$$\langle R_4, \partial_\beta^2 f \rangle \leq C \sum_{|\alpha_1+|\alpha_2|\leq |\alpha|} \int_{\mathbb{R}^3} \|\partial^{\alpha_1} f^n\|_{L^2_{x,v}} \|\partial^{\alpha_2} f^n\|_{L^2_{x,v}} \|\partial_\beta^2 f^{n+1}\|_{L^2_{x,v}} dx$$

$$+ C \sum_{|\alpha_1+|\alpha_2|\leq |\alpha|, |\beta_2| \leq |\beta|} \int_{\mathbb{R}^3} \|\partial^{\alpha_1} f\|_{L^2_{x,v}} \|\partial^{\alpha_2} f^n\|_{L^2_{x,v}} \|\partial_\beta^3 f^{n+1}\|_{L^2_{x,v}} dx$$

$$+ C \sum_{|\alpha_1+|\alpha_2|+|\alpha_3| \leq |\alpha|, |\beta_2| \leq |\beta|} \int_{\mathbb{R}^3} \|\partial^{\alpha_1} f\|_{L^2_{x,v}} \|\partial^{\alpha_2} f^n\|_{L^2_{x,v}} \|\partial_\beta^3 f^n\|_{L^2_{x,v}} \|\partial_\beta^3 f^{n+1}\|_{L^2_{x,v}} dx$$

$$\leq C \sum_{|\alpha_1+|\alpha_2| \leq |\alpha|} \left( \sup_x \|\partial^{\alpha_1} f^n\|_{L^2_{x,v}} + \sup_x \|\partial^{\alpha_1} f^n\|_{L^2_{x,v}}^2 \right) \int_{\mathbb{R}^3} \|\partial^{\alpha_2} f^n\|_{L^2_{x,v}} \|\partial_\beta^3 f^{n+1}\|_{L^2_{x,v}} dx$$

$$\leq C \sum_{|\alpha_1+|\alpha_2| \leq |\alpha|} \left( \sup_x \|\partial^{\alpha_1} f^n\|_{L^2_{x,v}} + \sup_x \|\partial^{\alpha_1} f^n\|_{L^2_{x,v}}^2 \right) \left( \|\partial^{\alpha_2} f^n\|_{L^2_{x,v}}^2 + \|\partial_\beta^3 f^{n+1}\|_{L^2_{x,v}}^2 \right)$$

$$\leq C \left( E_n^2(t) + E_n^2(t) + \sqrt{E_n(t)E_{n+1}(t)} + E_n(t)E_{n+1}(t) \right),$$

where we assumed $\alpha_1$ to be the smallest index without loss of generality and used the following Sobolev embedding

\[(5.19) \quad H^2(\mathbb{T}^3) \subseteq L^\infty(\mathbb{T}^3).\]

We substitute all these ingredients into \[(5.18)\] to obtain

$$\frac{1}{2} \frac{d}{dt} \|\partial_\beta^2 f^{n+1}(t)\|_{L^2_{x,v}}^2 + \nu_c \|\partial_\beta^2 f^{n+1}(t)\|_{L^2_{x,v}}^2$$

$$\leq C \left\{ E_{n+1}(t) + \left( E_n(t) \right)^{\frac{3}{2}} + \left( E_n(t) \right)^2 + \left( E_n(t) \right)^{\frac{1}{2}} E_{n+1}(t) + E_n(t)E_{n+1}(t) \right\}.$$ 

We then sum over $\alpha$ and $\beta$ and integrate in time to see

$$E_{n+1}(t) \leq E_{n+1}(0)$$

$$+ C \int_0^t \left\{ E_{n+1}(t) + \left( E_n(t) \right)^{\frac{3}{2}} + \left( E_n(t) \right)^2 + \left( E_n(t) \right)^{\frac{1}{2}} E_{n+1}(t) + E_n(t)E_{n+1}(t) \right\} ds$$

$$\leq \frac{M_0}{2} + C \left\{ T_\ast \sup_{0 \leq t \leq T_\ast} E_{n+1}(t) + T_\ast \sup_{0 \leq t \leq T_\ast} E_n + T_\ast \left( \sup_{0 \leq t \leq T_\ast} E_n \right)^{\frac{3}{2}} + T_\ast \left( \sup_{0 \leq t \leq T_\ast} E_n \right)^2 \right\}$$

$$+ T_\ast \left( \sup_{0 \leq t \leq T_\ast} E_n \right)^{\frac{1}{2}} \sup_{0 \leq t \leq T_\ast} E_{n+1}(t) + T_\ast \left( \sup_{0 \leq t \leq T_\ast} E_n \right) \sup_{0 \leq t \leq T_\ast} E^{n+1}(t) \right\},$$

which yields

$$\left( 1 - CT_\ast - CT_\ast \sqrt{M_0 - CT_\ast M_0} \right) \sup_{0 \leq t \leq T_\ast} E_{n+1}(t) \leq \left( \frac{1}{2} + CT_\ast + CT_\ast \sqrt{M_0 + CT_\ast M_0} \right) M_0.$$ 

This gives the desired result for sufficiently small $M_0$ and $T_\ast$.

We now go back to the proof of the theorem and let $n \to \infty$ to establish the local in time existence of a smooth solution. To prove uniqueness, we assume that $g$ is another local
solution corresponding to the same initial data \( f_0 \). We then have
\[
(5.20) \quad \{ \partial_t + v \cdot \nabla + \nu_c \}(f - g) = P(f - g) + \Gamma_{1,2,3}(f - g, f) + \Gamma_{1,2,3}(g, f - g) \\
+ \Gamma_4(f - g, f, f) + \Gamma_4(g, f - g, f) + \Gamma_4(g, g, f - g).
\]
We recall from Proposition 4.1 and (5.19)
\[
(\Gamma_{1,2,3}(f - g, f) + \Gamma_{1,2,3}(g, f - g), f - g) \\
+ (\Gamma_4(f - g, f, f) + \Gamma_4(g, f - g, f) + \Gamma_4(g, g, f - g), f - g)
\]
\[
\leq \sum_{|\alpha| \leq 2} (\| \partial^\alpha f \|^2_{L^2_{x,v}} + \| \partial^\alpha g \|^2_{L^2_{x,v}} + \| \partial^\alpha f \|_{L^2_{x,v}} + \| \partial^\alpha g \|_{L^2_{x,v}}) \| f - g \|^2_{L^2_{x,v}}.
\]
We now multiply \( f - g \) to both sides of (5.20), integrate with respect to \( x, v, t \) and use the above estimate to see
\[
\| f(t) - g(t) \|^2_{L^2_{x,v}} + \int_0^t \| f(s) - g(s) \|^2_{L^2_{x,v}} \, ds
\]
\[
\leq C \sum_{|\alpha| \leq 2} \sup_{0 \leq s \leq T} (\sqrt{E_f(t)} + \sqrt{E_g(t)} + E_f(t) + E_g(t) + 1) \int_0^t \| f(s) - g(s) \|^2_{L^2_{x,v}} \, ds.
\]
Therefore, for sufficiently small \( E_f(0) \) and \( E_g(0) \), the uniqueness follows from Gronwall’s theorem. We now turn to the continuity of \( E(t) \). Let \( f \) be the smooth local solution constructed above:
\[
\partial_t f + v \cdot \nabla f + \nu_c f = Pf + \Gamma(f).
\]
We multiply \( f \) and integrate over \( x, v \) and then over \([s, t]\) to see.
\[
|E(t) - E(s)| \leq C \left[ 1 + \sqrt{E(t)} + E(t) \right] \int_s^t \sum_{|\alpha| \leq 2} \| \partial^\alpha f \|_{L^2_{x,v}} \, d\tau \to 0.
\]
The positivity of \( m + \sqrt{m} f \) can be verified iteratively from the positivity of \( F_0 \) using (5.16). Finally, since the local solution is smooth, the conservation laws can be obtained straightforwardly.

6. Coercivity of \( L \)

The coercivity estimate in Lemma 2.4 involving only microscopic components, is not strong enough to play as a good term in the energy method. In this section, we show that the full coercivity \( L \) can be recovered as long as the energy \( E(t) \) remains sufficiently small. We first set for simplicity
\[
a(x, t) = \int f \sqrt{m} dv, \quad b(x, t) = \int fv \sqrt{m} dv \quad \text{and} \quad c(x, t) = \int f |v|^2 \sqrt{m} dv.
\]
Recall that \( f \) can be divided into its hydrodynamic part \( \tilde{P} f \) and microscopic part \( (I - \tilde{P}) f \):
\[
(6.21) \quad f = \tilde{P} f + (I - \tilde{P}) f,
\]
where
\[
\tilde{P} f = a \sqrt{m} + b \cdot v \sqrt{m} + c |v|^2 \sqrt{m}.
\]
We observe that there exists constants \( C \) such that
\[
(6.22) \quad \frac{1}{C} \|(I - P) f\|_{L^2_{x,v}} \leq \|(I - \tilde{P}) f\|_{L^2_{x,v}} \leq C \|(I - P) f\|_{L^2_{x,v}}.
\]
Now, we substitute (6.21) into the BGK model (2.12) to obtain
(6.23) \[ \{ \partial_t + v \cdot \nabla \} \tilde{P} f = \ell \{ (I - \tilde{P}) f \} + h(f), \]
where
\[ \ell \{ (I - \tilde{P}) f \} \equiv \{ -\partial_t - v \cdot \nabla_x + L \} \{ I - \tilde{P} \} f, \]
\[ h(f) \equiv \Gamma(f). \]

The l.h.s of (6.23) is calculated as follows
\[
\sum_i \left\{ v_i \delta^i c |v|^2 + (\partial_t c + \partial^i b_i) v_i^2 + \sum_{j > i} (\partial^i b_j + \partial^j b_i) v_i v_j + (\partial b + \partial^i a) v_i + \partial t a \right\} \sqrt{m},
\]
where \( \partial^i = \partial_x^i \). We then expand the l.h.s of (6.23) with respect to the basis:
\[
\sqrt{m}, \ v_i \sqrt{m}, \ v_i v_j \sqrt{m}, \ v_i^2 \sqrt{m}, \ v_i |v|^2 \sqrt{m} \quad (1 \leq i, j \leq 3).
\]
Equating both sides of the above identity, we obtain the following result.

**Lemma 6.1.** \( a, b, c \) satisfy the following relations.

1. \( \nabla c = \ell_c + h_c \),
2. \( \partial_t c + \partial^i b_i = \ell_i + h_i \),
3. \( \partial^i b_j + \partial^j b_i = \ell_{ij} + h_{ij} \),
4. \( \partial b_i + \partial^i a = \ell_{bi} + h_{bi} \),
5. \( \partial t a = \ell_a + h_a \).

where \( \ell_c, \ell_i, \ell_{ij}, \ell_{bi}, \ell_a \) are coefficients of the expansion of \( \ell \) with respect to the preceding basis. Similarly, \( h_c, h_i, h_{ij}, h_{bi}, h_a \) denotes the corresponding coefficients of the expansion of \( h \).

For brevity, we define \( \tilde{\ell} \) and \( \tilde{h} \) as
\[
\tilde{\ell} \equiv \ell_c + \sum_i \ell_i + \sum_{i,j} \ell_{ij} + \sum_i \ell_{bi} + \ell_a,
\]
\[
\tilde{h} \equiv h_c + \sum_i h_i + \sum_{i,j} h_{ij} + \sum_i h_{bi} + h_a.
\]

**Lemma 6.2.** Let \(|\alpha| \leq N - 1 \). Then we have

1. \[ \| \nabla_x \partial^\alpha b_i \|_{L^2_x} + \| \partial_t \partial^\alpha b_i \|_{L^2_x} \leq \sum_{|\alpha| \leq N-1} \| \partial^\alpha \tilde{\ell} \|_{L^2_x} + \sum_{|\alpha| \leq N-1} \| \partial^\alpha \tilde{h} \|_{L^2_x}. \]
2. \[ \| \partial^\alpha b_i \|_{L^2_x} \leq \sum_{|\alpha| \leq N-1} \| \partial^\alpha \tilde{\ell} \|_{L^2_x} + \sum_{|\alpha| \leq N-1} \| \partial^\alpha \tilde{h} \|_{L^2_x}. \]

**Proof.** (1) Following [12, 13, 14], we observe that
\[
-\Delta b_i - \partial_t \partial b_i = -\sum_{j \neq i} \partial^j (\partial^i b_i) - 2\partial^i \partial^j b_i
\]
\[
= -\sum_{j \neq i} \partial^j (-\partial^i b_i - \ell_{ij} - h_{ij}) - 2\partial^i (-\partial c + \ell_i + h_i)
\]
( by Lemma 6.1 (3) and (2))
\[
= \sum_{j \neq i} (\partial^j \partial^i b_j + \partial^j \partial c) + \sum_{j \neq i} (\partial^j \ell_{ij} - \partial^j h_{ij}) - 2\partial^i (\ell_i + h_i)
\]
We then multiply Lemma 6.4.

(2) By Lemma 6.1 (2), we have

\[ \sum_{j \neq i} (\partial_i (-\partial_j \ell - \ell_i - h_i) + \partial^j \partial_i c) + \sum_{j \neq i} (\partial^j \ell_{ij} - \partial^j h_{ij}) - 2 \partial^i (\ell_i + h_i) = \sum_{j \neq i} (\partial_i \ell_i + \partial^i h_i) + \sum_{j \neq i} (\partial^j \ell_{ij} - \partial^j h_{ij}) - 2 \partial^i (\ell_i + h_i). \]

Then the result follows from the standard elliptic estimate.

(2) By Poincaré inequality and Lemma 6.2, we have

\[ \| \partial_t^N b \|_{L^2} \leq \| \nabla_x \partial_t^N b \|_{L^2} \leq C \left( \| \partial_t^N \tilde{\ell} \|_{L^2} + \| \partial_t^N \tilde{h} \|_{L^2} \right). \]

\[ \square \]

Lemma 6.3. For \( |\alpha| \leq N - 1 \), we have

\( \| \partial_\ell \partial^\alpha c \|_{L^2} \leq C \left( \| \partial_t^N \tilde{\ell} \|_{L^2} + \| \partial_t^N \tilde{h} \|_{L^2} \right) \).

Proof. (1) By Poincaré inequality and Lemma 6.1 (1), we have

\[ \| c \|_{L^2} \leq \| \nabla c \|_{L^2} \leq C \| \ell_c + h_c \|_{L^2} \leq C \left( \| \tilde{\ell} \|_{L^2} + \| \tilde{h} \|_{L^2} \right), \]

where we used the conservation of energy:

\[ \int c(x) \, dx = 0. \]

(2) By (2) in Lemma 6.1, we have

\[ \| \partial_t c \|_{L^2} = \| - \nabla \cdot b + \ell + h \|_{L^2} \leq C \left( \| \tilde{\ell} \|_{L^2} + \| \tilde{h} \|_{L^2} \right). \]

(3) follows directly from (1).

\[ \square \]

Lemma 6.4. Let \( |\alpha| \leq N - 1 \). For (2), we assume further that \( \alpha \) is purely spatial: \( \alpha = [0, \alpha_1, \alpha_2, \alpha_3] \neq 0 \). Then we have

\( \| \partial_\ell \partial^\alpha a \|_{L^2} \leq C \left( \| \partial^\alpha \tilde{\ell} \|_{L^2} + \| \partial^\alpha \tilde{h} \|_{L^2} \right) \).

Proof. (1) This follows directly from Lemma 6.1 (5).

(2) By Lemma 6.1 (4), we have

\[ \triangle \partial^\alpha a = \nabla \cdot \nabla \partial^\alpha a = \nabla \cdot (-\partial^\alpha \partial_t b + \partial^\alpha \ell_b + \partial^\alpha h_b) = -\nabla \partial^\alpha \partial_t b + \nabla \partial^\alpha \ell_b + \nabla \partial^\alpha h_b. \]

We then multiply \( \nabla a \) to both sides and use integrate by parts to see

\[ \| \nabla \partial^\alpha a \|_{L^2} \leq \| \partial^\alpha \partial_t b \|_{L^2} + \| \partial^\alpha \ell_b \| + \| \partial^\alpha h_b \|_{L^2}. \]
\[
\| \partial_x \partial_t \partial^\alpha b \|_{L^2_x} + \| \partial^\alpha \ell_b \| + \| \partial^\alpha h_b \|_{L^2_x} \\
\leq \sum_{|\alpha| \leq N-1} \| \partial^\alpha \ell_i \|_{L^2_x} + \sum_{|\alpha| \leq N-1} \| \partial^\alpha \ell_i \|_{L^2_x}.
\]

In the last line, we employed Lemma 6.2. (3) follows from Poincare inequality combined with (2) and the conservation of mass:
\[
\| a \|_{L^2_x} \leq \| \nabla a \|_{L^2_x}.
\]

**Lemma 6.5.** For \(|\alpha| \leq N - 1\), we have
\[
(1) \quad \sum_{|\alpha| \leq N-1} \| \partial^\alpha \ell \|_{L^2_x} \leq C \sum_{|\gamma| \leq N} \| (I - \tilde{P}) \partial^\alpha f \|_{L^2_x,v},
\]
\[
(2) \quad \sum_{|\alpha| \leq N} \| \partial^\alpha h \|_{L^2_x} \leq C(\sqrt{M_0} + M_0) \sum_{|\alpha| \leq N} \| \partial^\alpha f \|_{L^2_x,v}.
\]

**Proof.** (1) Note that there exists constants \(\lambda_n\) such that \(\partial^\alpha \ell\) takes the following form
\[
\sum_{n=1}^{13} \lambda_n \int_{\mathbb{R}^3} \partial^\alpha \ell \{ (I - \tilde{P}) f \} \cdot \varepsilon_n(v) dv,
\]
where \(\varepsilon_n\) denotes the orthogonal basis for the 13 dimensional space spanned by
\[
\{ \sqrt{m}, v_1 \sqrt{m}, v_1^2 \sqrt{m}, v_i v_j \sqrt{m}, v_i^2 v_j \sqrt{m} | 1 \leq i, j \leq 3 \}.
\]
We then observe that
\[
\| \partial^\alpha \ell \{ (I - \tilde{P}) f \} \cdot \varepsilon_n(v) dv \|_{L^2_x}^2
\]
\[
= \left\| \int \{ (\partial_t + v \cdot \nabla + L)(I - \tilde{P}) \partial^\alpha f \} \cdot \varepsilon_n(v) dv \right\|_{L^2_x}^2
\]
\[
\leq \int \| \varepsilon_n(v) \| dv \times
\]
\[
\int \| \varepsilon_n(v) \| \left\{ \| (I - \tilde{P}) \partial^\alpha \partial^\alpha f \|^2 + \| \partial^\alpha f \|^2 \nabla \partial^\alpha f \|^2 + \| (L(I - \tilde{P}) \partial^\alpha f \|^2 \right\} dx dv
\]
\[
\leq C \left\{ \| (I - \tilde{P}) \partial^\alpha \partial^\alpha f \|_{L^2_x,v} + \| (I - \tilde{P}) \nabla \partial^\alpha f \|_{L^2_x,v} + \| (I - \tilde{P}) \partial^\alpha f \|_{L^2_x,v} \right\}^2.
\]
This completes the proof of (1). (2) As in (1), terms in \(\partial^\alpha h\) can be presented as
\[
\sum_{n=1}^{13} \lambda_n \int_{\mathbb{R}^3} \partial^\alpha \ell \{ (I - \tilde{P}) f \} \cdot \varepsilon_n(v) dv
\]
for some constants \(\tilde{\lambda}_n\) We now apply Lemma 4.1 (3) to get
\[
\left\| \int_{\mathbb{R}^3} \partial^\alpha \ell \{ (I - \tilde{P}) f \} \cdot \varepsilon_n(v) dv \right\|_{L^2_x} \leq \sum_{|\alpha_1| + |\alpha_2| \leq |\alpha|} \left\| \int_{\mathbb{R}^3} \Gamma_{1,2,3}(\partial^{\alpha_1} f, \partial^{\alpha_2} f) \cdot \varepsilon_n(v) dv \right\|_{L^2_x}
\]
\[
+ \sum_{|\alpha_1| + |\alpha_2| + |\alpha_3| \leq |\alpha|} \left\| \int_{\mathbb{R}^3} \Gamma_4(\partial^{\alpha_1} f, \partial^{\alpha_2} f, \partial^{\alpha_3} f) \cdot \varepsilon_n(v) dv \right\|_{L^2_x}
\]
\[
\leq C(\sqrt{M} + M) \sum_{|\alpha| \leq N} \| \partial^\alpha f \|_{L^2_x,v}.
\]
This completes the proof. □

We can now prove the main theorem of this section.

**Theorem 6.1.** Let $f$ be a classical solution of (1.1). Then there exists $M$ and $\delta = \delta(M)$ such that if

$$
\sum_{|\alpha| \leq N} \|\partial^\alpha f(t)\|_{L^2_{x,v}}^2 \leq M,
$$

then there exists $\delta > 0$ such that

$$
\sum_{|\alpha| \leq N} \langle L\partial^\alpha f, \partial^\alpha f \rangle \leq -\delta \sum_{|\alpha| \leq N} \|\partial^\alpha f\|_{L^2_{x,v}}^2.
$$

**Proof.** By Lemma 6.2 - 6.4 we have

$$
\sum_{|\alpha| \leq N} \|\partial^\alpha a\|_{L^2_x} + \|\partial^\alpha b\|_{L^2_x} + \|\partial^\alpha c\|_{L^2_x} \leq \sum_{|\alpha| \leq N-1} \|\partial^\alpha \ell\|_{L^2_{x,v}} + \sum_{|\alpha| \leq N} \|\partial^\alpha h\|_{L^2_{x,v}}.
$$

We then apply Lemma 6.5 to see

$$
\sum_{|\alpha| \leq N} \|\partial^\alpha \tilde{P} f\|_{L^2_{x,v}} \leq C \sum_{|\alpha| \leq N} \|\partial^\alpha \ell\|_{L^2_{x,v}} + C \sqrt{M_0} \sum_{|\alpha| \leq N} \|\partial^\alpha f\|_{L^2_{x,v}}.
$$

Hence we have from Lemma 2.4 and the equivalence estimate (6.22)

$$
\sum_{|\alpha| \leq N} \langle L\partial^\alpha f, \partial^\alpha f \rangle = -\nu_c \sum_{|\alpha| \leq N} \|(I - \tilde{P})\partial^\alpha f\|_{L^2_{x,v}}^2
\leq -\nu_c C \sum_{|\alpha| \leq N} \|(I - \tilde{P})\partial^\alpha f\|_{L^2_{x,v}}^2
\leq -\nu_c C_1 \left\{ \sum_{|\alpha| \leq N} \|\partial^\alpha \tilde{P} f\|_{L^2_{x,v}}^2 - C_2 \sqrt{M} \sum_{|\alpha| \leq N} \|\partial^\alpha f\|_{L^2_{x,v}}^2 \right\}
\leq -\frac{\min\{\nu_c, \nu_c C_1\}}{2} \left\{ \sum_{|\alpha| \leq N} \|\partial^\alpha \tilde{P} f\|_{L^2_{x,v}}^2 + \|\partial^\alpha (1 - \tilde{P}) f\|_{L^2_{x,v}}^2 \right\}
+ \frac{1}{2} \nu_c C_2 \sqrt{M} \sum_{|\alpha| \leq N} \|\partial^\alpha f\|_{L^2_{x,v}}^2
\leq -\nu_c \left\{ \min\{1, C_1\} - C_2 \sqrt{M} \right\} \sum_{|\alpha| \leq N} \|\partial^\alpha f\|_{L^2_{x,v}}^2.
$$

We then choose $M$ sufficiently small such that

$$
\min\{1, C_1\} > C_2 \sqrt{M}
$$

to obtain the desired result. □
7. Proof of the main theorem

In this section, we derive a refined energy estimate for the Boltzmann-BGK model and establish the main result. Let \( f \) be the unique smooth solution constructed in Theorem 5.1. First we take \( \partial^\alpha \) on both sides of \((2.12)\) to have

\[
[\partial_t + v \cdot \nabla + L] \partial^\alpha f = \partial^\alpha \Gamma(f).
\]

We multiply \( \partial^\alpha f \), integrate over \( T^d \times \mathbb{R}^d \) and apply Lemma 4.1 (1) and Theorem 6.1 to obtain

\[
E^\alpha : \frac{1}{2} \frac{d}{dt} \| \partial^\alpha f \|^2_{L^2_{x,v}} + \delta \| \partial^\alpha f \|^2_{L^2_{x,v}} \leq C \sqrt{E(t)} \| f \|^2.
\]

For \( \beta \neq 0 \), we take \( \partial^\alpha \beta \) to obtain

\[
\partial^\alpha \beta (v \cdot \nabla_x f) = \partial^\alpha \beta \left\{ \sum_{1 \leq i \leq 3} v_i \partial^\alpha f_i \right\}
\]

\[
= v \cdot \nabla_x \partial^\alpha \beta f + \sum_i \sum_{\beta \neq 0} \partial^\alpha_{\beta - \epsilon_i} f \partial^\alpha_{\epsilon_i} f
\]

\[
= v \cdot \nabla_x \partial^\alpha \beta f + \sum_i \partial^\alpha_{\beta - \epsilon_i} f.
\]

Multiplying \( \partial^\alpha \beta f \) to both sides of \((7.24)\) and integrating over \( T^d \times \mathbb{R}^d \), we obtain by an almost identical manner as in the local existence case

\[
\frac{1}{2} \frac{d}{dt} \| \partial^\alpha \beta f \|^2_{L^2_{x,v}} + \nu_c \| \partial^\alpha \beta f \|^2_{L^2_{x,v}}
\]

\[
\leq - \sum_i \langle \partial^\alpha_{\beta - \epsilon_i} f, \partial^\alpha \beta f \rangle + \nu_c \langle \partial_{\beta} P \partial^\alpha f, \partial^\alpha \beta f \rangle + \langle \partial^\alpha \beta \Gamma(f), \partial^\alpha \beta f \rangle
\]

\[
\leq \sum_i C \epsilon \| \partial^\alpha_{\beta - \epsilon_i} f \|^2_{L^2_{x,v}} + \epsilon \| \partial^\alpha \beta f \|^2_{L^2_{x,v}} + C \epsilon \| \partial^\alpha f \|^2_{L^2_{x,v}} + \epsilon \| \partial^\alpha \beta f \|^2_{L^2_{x,v}}
\]

\[
+C \sum_{|\alpha_1| + |\alpha_2| \leq |\alpha|} \int_{\mathbb{R}^d} \| \partial^{\alpha_1} f \|_{L^2_{x,v}} \| \partial^{\alpha_2} f \|_{L^2_{x,v}} \| \partial^\alpha \beta f \|_{L^2_{x,v}} dx
\]

\[
+C \sum_{|\alpha_1| + |\alpha_2| \leq |\alpha|} \sum_{|\beta| \leq |\beta|} \int_{\mathbb{R}^d} \| \partial^{\alpha_1} f \|_{L^2_{x,v}} \| \partial^{\alpha_2} f \|_{L^2_{x,v}} \| \partial^\alpha \beta f \|_{L^2_{x,v}} dx
\]

\[
+C \sum_{|\alpha_1| + |\alpha_2| + |\alpha_3| \leq |\alpha|} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \| \partial^{\alpha_1} f \|_{L^2_{x,v}} \| \partial^{\alpha_2} f \|_{L^2_{x,v}} \| \partial^\alpha \beta f \|_{L^2_{x,v}} dx.
\]

Therefore, we have for sufficiently small \( \epsilon \)

\[
\frac{d}{dt} \| \partial^\alpha \beta f \|^2_{L^2_{x,v}} + \nu_c \| \partial^\alpha f \|^2_{L^2_{x,v}} \leq C \epsilon \sum_i \| \partial^\alpha_{\beta - \epsilon_i} f \|^2_{L^2_{x,v}} + C \epsilon \| \partial^\alpha f \|^2_{L^2_{x,v}}
\]

The proof is completed.
\[ + C \sum_{|\alpha_1| + |\alpha_2| \leq |\alpha|} \int_{\mathbb{R}^d} \| \partial^{\alpha_1} f \|_{L^2_x} \| \partial^{\alpha_2} f \|_{L^2_x} \| \partial^2 f \|_{L^2_x} \, dx \\
+ C \sum_{|\alpha_1| + |\alpha_2| \leq |\alpha|} \int_{\mathbb{R}^d} \| \partial^{\alpha_1} f \|_{L^2_x} \| \partial^{\alpha_2} f \|_{L^2_x} \| \partial^2 f \|_{L^2_x} \, dx \\
+ C \sum_{|\alpha_1| + |\alpha_2| + |\alpha_3| \leq |\alpha|} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \! \int_{\mathbb{R}^d} \| \partial^{\alpha_1} f \|_{L^2_x} \| \partial^{\alpha_2} f \|_{L^2_x} \| \partial^{\alpha_3} f \|_{L^2_x} \| \partial^3 f \|_{L^2_x} \, dx \\
eq C_\varepsilon \sum_i \| \partial^+ \beta_i^+ f \|_{L^2_x}^2 + C_\varepsilon \| \partial^\alpha f \|_{L^2_x}^2 + I_{np}. \]

The estimates for \( I \) can be treated almost identically as in the proof of Theorem 5.1 to obtain

\[ I_{np} \leq C \{ \sqrt{E(t)} + E(t) \} \| f \| \| f \|, \]

where we used the Sobolev embedding \( H^s \hookrightarrow L^\infty \). This yields

\[ E_\beta^0 : \quad \frac{1}{2} \frac{d}{dt} \| \partial^\beta f \|_{L^2_x}^2 + \frac{\nu}{2} \| \partial^\beta f \|_{L^2_x}^2 \leq C_\varepsilon \sum_i \| \partial^\beta \beta_i^\epsilon f \|_{L^2_x}^2 + C_\varepsilon \| \partial^\alpha f \|_{L^2_x}^2 + C \{ \sqrt{E(t)} + E(t) \} \| f \| \| f \|. \]

From the above inequality, we observe that the bad terms of \( \sum_{|\beta| = m+1} E_\beta^0 \), that is

\[ \sum_{|\beta| = m+1} \left\{ C_\varepsilon \sum_i \| \partial^\beta \beta_i^\epsilon f \|_{L^2_x}^2 + C_\varepsilon \| \partial^\alpha f \|_{L^2_x}^2 \right\}, \]

can be absorbed in the good terms of \( C_\alpha \sum_{|\beta| = m} E_\beta^0 + C_\alpha E^0 \) if \( C_\alpha \) is sufficiently large. Therefore, by an induction argument, we can find constants \( C_m \) and \( \delta_m \) such that

\[ \sum_{|\alpha| + |\beta| \leq N, \, |\beta| \leq m} \left\{ C_m \frac{d}{dt} \| \partial^\beta f \|_{L^2_x}^2 + \delta_m \| \partial^\beta f \|_{L^2_x}^2 \right\} \leq C_N \{ \sqrt{E(t)} + E(t) \} \| f \| \| f \|. \]

We now suppose \( E < 1 \) without loss of generality and set \( m = N \) to obtain

\[ \sum_{|\alpha| + |\beta| \leq N} \left\{ C_N \frac{d}{dt} \| \partial^\beta f \|_{L^2_x}^2 + \delta_N \| \partial^\beta f \|_{L^2_x}^2 \right\} \leq C_N \{ \sqrt{E(t)} \} \| f \| \| f \|. \]

Notice that we used \( E(t) \leq \sqrt{E(t)} \) and redefined \( 2C_N \) by \( C_N \). We now define \( y(t) \) as

\[ y(t) = \sum_{|\alpha| + |\beta| \leq N} C_N \frac{d}{dt} \| \partial^\beta f \|_{L^2_x}^2. \]

We choose a constant \( C_1 \) such that

\[ \frac{1}{C_1} \left\{ y(t) + \delta_N \frac{1}{2} \int_0^t \| f(s) \| \| f(s) \| ds \right\} \leq E(t) \leq C_1 \left\{ y(t) + \delta_N \frac{1}{2} \int_0^t \| f(s) \| \| f(s) \| ds \right\} \]

We define

\[ M = \min \left\{ \frac{\delta_N^2}{8C_N^2C_1^2}, \frac{M_0}{2C_2^2} \right\}. \]
and choose the initial data sufficiently small in the sense that

\[ E(0) \leq M < M_0. \]

Let \( T > 0 \) be given as

\[ T = \sup \left\{ t : E(t) \leq 2C_1^2M \right\} > 0, \]

which gives

\[ E(t) \leq 2C_1^2M \leq M_0. \]

We then have for \( 0 \leq t \leq T \)

\[ y'(t) + \delta_N \| f \|^2(t) \leq C_N \sqrt{E(t)} \| f \|^2(t) \leq C_N C_1 \sqrt{2M} \| f \|^2(t) \leq \frac{\delta_N}{2} \| f \|^2(t). \]

Therefore, integration above over \( 0 \leq t \leq T \) yields

\[ E(t) \leq C_1 \left\{ y(t) + \frac{\delta_N}{2} \int_0^t \| f(s) \|^2 ds \right\} \leq C_1 y(0) \leq C_1^2 E(0) \leq C_1^2 M < 2C_1^2 M. \]

This is a contradiction considering the continuity of \( E \) and the definition of \( T \). Hence we have \( T = \infty \). By (7.25) and \( y(t) \leq C \| f(t) \| \) for some constant \( C \), we have

\[ y'(t) + \frac{\delta_N}{2} y(t) \leq y'(t) + \frac{\delta_N}{2} \| f \|^2(t) \leq 0, \]

which gives the exponential decay of the perturbation.

We are now left with the \( L^2 \)-stability estimate. Let \( \bar{f} \) be another solution corresponding to initial data \( \bar{f}_0 \). We subtract the equation for \( \bar{f} \) from the equation for \( f \) to see

\[ \{ \partial_t + v \cdot \nabla \} (f - \bar{f}) = L(f - \bar{f}) + \Gamma_{1,2,3}(f - \bar{f}, f) + \Gamma_{1,2,3}(\bar{f}, f - \bar{f}) + \Gamma_4(f - \bar{f}, f, f) + \Gamma_4(\bar{f}, f - \bar{f}, f) + \Gamma_4(\bar{f}, \bar{f}, f - \bar{f}). \]

We then multiply \( f - \bar{f} \) and integrate over \( x, v \) and \( t \) to have

\[ \frac{1}{2} \frac{d}{dt} \| f(t) - \bar{f}(t) \|_{L^2_{x,v}}^2 = \langle L(f - \bar{f}), f - \bar{f} \rangle + \langle \Gamma_{1,2,3}(f - \bar{f}, f), f - \bar{f} \rangle + \langle \Gamma_{1,2,3}(\bar{f}, f - \bar{f}), f - \bar{f} \rangle + \langle \Gamma_4(f - \bar{f}, f, f), f - \bar{f} \rangle + \langle \Gamma_4(\bar{f}, f - \bar{f}, f), f - \bar{f} \rangle + \langle \Gamma_4(\bar{f}, \bar{f}, f - \bar{f}), f - \bar{f} \rangle. \]

We now apply the coercivity estimate in Theorem 6.1

\[ \langle L(f - \bar{f}), f - \bar{f} \rangle \leq -\delta \| f - \bar{f} \|_{L^2_{x,v}}^2 \]

and the following estimates from Proposition 5.11 and (5.19):

\[ \langle \Gamma_{1,2,3}(f - \bar{f}, f) + \Gamma_{1,2,3}(\bar{f}, f - \bar{f}), f - \bar{f} \rangle \]
\[ + \langle \Gamma_4(f - \bar{f}, f, f) + \Gamma_4(\bar{f}, f - \bar{f}, f) + \Gamma_4(\bar{f}, \bar{f}, f - \bar{f}), f - \bar{f} \rangle \]
\[ \leq C \sum_{|\alpha| \leq 2} (\| \partial^\alpha f \|_{L^2_{x,v}}^2 + \| \partial^\alpha f \|_{L^2_{x,v}}^2 + \| \partial^\alpha \bar{f} \|_{L^2_{x,v}}^2 + \| \partial^\alpha \bar{f} \|_{L^2_{x,v}}^2) \| f - \bar{f} \|_{L^2_{x,v}}^2 \]
\[ \leq C \left\{ \sqrt{E_f(t)} + \sqrt{E_{\bar{f}}(t)} \right\} \| f - \bar{f} \|_{L^2_{x,v}}^2 \]
to get
\[
\frac{1}{2} \frac{d}{dt} \| f(t) - \bar{f}(t) \|^2_{L^2_{x,v}} + \left( \delta - \left\{ \sqrt{E_f(t)} + \sqrt{E_{\bar{f}}(t)} \right\} \right) \| f(t) - \bar{f}(t) \|^2_{L^2_{x,v}} \leq 0,
\]
which gives the desired result for sufficiently small \( E_g(0) \) and \( E_f(0) \).

REFERENCES

[1] Aoki, K., Sone, T. and Yamada, T.: Numerical analysis of gas flows condensing on its plane condensed phase on the basis of kinetic theory, Physics of Fluid A, 2 (1990), 1867-1878.

[2] Bhatnagar, P.L., Gross, E.P. and Krook, M.: A model for collision processes in gases. Small amplitude process in charged and neutral one-component systems, Physical Reviews, 94 (1954), 511-525.

[3] Bellouquid, A.: Global existence and large-time behavior for BGK model for a gas with non-constant cross section, Transport Theory and Statistical Physics, 32 (2003), no. 2, 157-184.

[4] Cercignani, C.: The Boltzmann equation and its applications, Springer, 1987.

[5] Chan, W. M.: An energy method for the BGK model. M.Phil thesis, City University of Hong Kong, 2007.

[6] Chapman, C. and Cowling T. G.: The mathematical theory of non-uniform gases, Cambridge University Press, 1970.

[7] Desvillettes, L. and Villani C.: On the trend to global equilibrium for spatially inhomogeneous kinetic systems: the Boltzmann equation. Invent. Math. 159 (2005), no.2, 245-316.

[8] Filbet. F. and Russo, G.: Semilagrangian schemes applied to moving boundary problems for the BGK model of rarefied gas dynamics, Kinet. Relat. Models 2 (2009), no.1, 231-250.

[9] Gressman, Philip T. and Strain, Robert M.: Classical solutions of the Boltzmann equation with long-range interactions and soft potentials. arXiv: 1002.3639v1 [math. AP].

[10] Gressman, Philip T. and Strain, Robert M.: Global classical solutions of the Boltzmann equation with long-range interactions. Proc. Natl. Acad. Sci. USA 107 (2010), no. 13, 5744-5749.

[11] Guo, Y.: Decay and Continuity of the Boltzmann Equation in Bounded Domains. Arch. Rational Mech. anal. 197, no.3, 713-809.

[12] Guo, Y.: The Boltzmann equation in the whole space. Indiana Univ. Math. J. 53 (2004), no.4, 1081-1094.

[13] Guo, Y.: The Vlasov-Maxwell-Boltzmann system near Maxwellians. Invent. Math. 153 (2003), no.3, 593-630.

[14] Guo, Y.: The Vlasov-Poisson-Boltzmann system near Maxwellians. Comm. Pure. Appl. Math. 55 (2002), no 9, 1004-1135.

[15] Ha, S.-Y., Yang, X. and Yun, S.-B.: \( L^2 \)-stability theory of the Boltzmann equation near a global Maxwellian. Archive Rational mech. Anal. 197 (2010), no.2, 657-688.

[16] Ha, S.-Y., Lee, H., Yang, X. and Yun, S.-B.: Uniform \( L^2 \)-stability estimates for the relativistic Boltzmann equation. J. Hyperbolic. Diff. Equ. 6 (2009), no.2, 295-312.
[17] Issautier, D.: Convergence of a weighted particle method for solving the Boltzmann (B.G.K) Equation, Siam Journal on Numerical Analysis, 33, no.6 (1996), 2099-2199.

[18] Liu, T.-P., Yang, T. and Yu, S.-H.: Energy method for the Boltzmann equation, Physica D 188 (2004) no. 3-4, 178-192.

[19] Mischler, S: Uniqueness for the BGK-equation in $\mathbb{R}^n$ and rate of convergence for a semi-discrete scheme. Differential and Integral equations, 9, no.5 (1996), 1119-1138.

[20] Mieussens, L.: Discrete velocity model and implicit scheme for the BGK equation of rarefied gas dynamics. Mth. Models Methods Appl Sci. 10, 1121-1149.

[21] Perthame, B.: Global existence to the BGK model of Boltzmann equation, Journal of Differential Equations, 82 (1989), 191-205.

[22] Perthame, B. and Pulvirenti, M.: Weighted $L^\infty$ bounds and uniqueness for the Boltzmann B.G.K model, Arch. Rational Mech. anal., 125 (1993), 289-295.

[23] Pieraccini, S., Puppo, G.: Implicit-Explicit Schemes for BGK Kinetic Equations, J. Sci. Comput., 32 , no. 1 (2007), 1 - 28.

[24] Russo, G. and Santagati, P.: A new class of large time step methods for the BGK models of the Boltzmann equation, preprint.

[25] Russo, G. Santagati, P. and Yun, S.-B.: On the convergence of a semi-Lagrangian scheme for the BGK model of the Boltzmann equation. arXiv:1007.2843v1 [math.AP].

[26] Santagati, P.: High order semi-Lagrangian methods for the BGK model of the Boltzmann equation, PhD thesis, University of Catania 2007.

[27] Struchtrup, H.: The BGK-model with velocity-dependent collision frequency Cont. Mech. Thermodyn. 9 (1997) no. 1, 23-32.

[28] Welander, P.: On the temperature jump in a rarefied gas, Ark. Fys. 7 (1954), 507-553.

[29] Yun, S.-B.: Vlasov-Maxwell-BGK model with velocity dependent collision frequency near a global Maxwellian. In progress.

[30] Ukai, S.: On the existence of global solutions of mixed problem for non-linear Boltzmann equation. Proc. Japan Acad. 50 (1974), 179-184.

[31] Zheng, Y. and Struchtrup, H.: Ellipsoidal statistical Bhatnagar-Gross-Krook model with velocity-dependent collision frequency, Phys. of Fluids. 17 (2005) no. 12. 127103.

[32] Yang, J. Y. and Huang, J. C.: Rarefied flow computations using nonlinear model Boltzmann equations. J. Comput. Phys. 125. (1995), 323-339.

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