Entanglement-like properties in spin–orbit coupled ultra cold atom and violation of Bell-like inequality

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Abstract
We show that the general quantum state of synthetically spin–orbit coupled ultra cold bosonic atoms, whose condensate was experimentally created recently by Lin et al (2011 Nature 471, 83), shows entanglement between motional degrees of freedom (momentum) and internal degrees of freedom (hyperfine spin). We demonstrate the violation of Bell-like inequality (CHSH) for such states that provide a unique opportunity to verify fundamental principles like quantum contextuality for commutating observables which are not spatially separated. We analyze in detail the Rabi oscillation executed by such an atom-laser system and how this influences quantities like entanglement entropy, the violation of Bell-like inequality, etc. We also discuss the implications of our result in testing quantum contextuality and Bell’s Inequality violation by macroscopic quantum object like Bose–Einstein condensate of ultra cold atoms.

Keywords: spin–orbit coupled ultra cold atoms, entanglement, Bell-like inequality, quantum contextuality

(Some figures may appear in colour only in the online journal)

Introduction

Entanglement, a highly enigmatic feature of quantum mechanics, implies a non-local correlation between two (or more) quantum systems such that the description of their states has to be done in reference to each other even when they are separated by a spacelike interval. This is a direct consequence of the fact that a state of a composite quantum system can be expressed as a linear superposition of the tensor product of the states corresponding to its different parts. This was first pointed out by E Schrödinger [1] and around the same time by Einstein, Podolsky and Rosen (EPR) [2] who cited this property to question the compatibility of quantum mechanics with local realism. Almost a half century back in a milestone paper, J S Bell [3] addressed this fundamental question raised by EPR with the help of Bell’s Inequality (BI) whose violation denies the existence of local hidden variable (LHV) theories and validates the existence of EPR like non-local correlation in quantum mechanical states. The demonstration of the violation of BI in experiments [4] thus forms a fundamental test to validate quantum mechanics as the right physical theory.

The entanglement between non-spatially separated observables is of particular interest due to phenomenological reasoning, although the status is quite different from standard non-locality-based entanglement. Subsequent works indicate that locality-based BI has a more general implication such as quantum non-contextuality [5–7], which means that the outcome of a particular measurement is determined independently from previous (or simultaneous) measurement of any set of mutually commutating observable [8, 9]. Only when such a set of observables commute due to their space-like separation, they correspond to the cases for which EPR like non-local correlation is relevant. A true test of quantum contextuality thus requires the exploration of entanglement-like properties and the test for BI in commutating observables that are not spatially separated. Alternatively, this implies that BI or equivalent inequalities need to be tested not only on the product state of two particle systems, but also on the product
state of single particle systems where the observable associated with each part commutes with each other. The availability of such experimental systems exhibiting contextuality is not very common [10].

In this paper, we studied one such quantum macroscopic system. In particular, we show that synthetically spin–orbit coupled (SSOC) ultra cold bosonic atoms, which were recently realized in experiments [11–13] provide a quantum system where bipartite entanglement-like quantum correlation occurs between the motional observable (momentum) and intrinsic observable (spin) of a single atom (and subsequently in interacting SSOC Bose–Einstein condensate (BEC)) and thus provides us a unique opportunity to test the concept of quantum contextuality for the commutative observables of the same object, which are not spatially separated. It must be stressed that ultra cold atomic systems [14–18] are very clean, almost bereft of any thermal fluctuation and highly isolated from the classical environment and thereby are very important and form one of the most ideal quantum systems to test the foundation of quantum mechanics. Furthermore, the SSOC ultra cold atomic system [11, 12, 19–23] is one of the most significant developments in recent times in this direction and is particularly being explored extensively for the possibility of quantum simulating novel topological condensed matter phases [24] with such SSOC ultra cold atoms.

In this context, our work extends the outreach of such ultra cold quantum system in a different direction by showing that it has the potential to testify fundamental questions associated with the quantum world such as contextuality, which to our knowledge has not been done so far. Even though it may look obvious from the term ‘spin–orbit coupling’ that there may be entanglement in the system, the origin of entanglement is deep-rooted coming from coupling with the artificial non-Abelian gauge field, which further leads to contextuality. The purpose of this paper is to make a detailed study of such contextual correlation and analyze how the various parameters affect it to achieve a considerable entanglement in SSOC BEC. Additionally, the experiments on SSOC BEC [12] already suggest that even though the system is interacting, its phase diagram can be understood from a non-interacting single-boson effective Hamiltonian. Given this assertion, we first perform the calculation for non-interacting SSOC BEC, and then extend it to interacting BEC by invoking weak interaction among atoms in order to confirm that the basic conclusions remain unchanged even in the presence of interaction. Therefore, our results rigorously derived for SSOC ultra cold bosonic system in the non-interacting (single particle) limit can also be applied to SSOC BEC, which is a macroscopic quantum object consisting of a large number of weakly interacting bosonic atoms in quantum mechanical ground state. This not only provides an opportunity for studying quantum contextuality, BI violation, and entanglement-like properties for a macroscopic and massive quantum objects, an instance which is not commonly available [25, 26], but also gives a better perspective to understand the effects of various parameters upon the contextual correlations.

We first demonstrate that the general quantum mechanical states of such SSOC ultra cold bosons satisfy the criterion of bipartite entanglement and then we quantify such entanglement with well known entanglement measure like entanglement entropy and concurrence. We particularly show how the dressed states of such atoms exhibit a Rabi-like oscillation between two different energy minima in the momentum space and similarly followed by entanglement entropy and concurrence. The discussion is also extended for the interacting SSOC BEC, where we consider a spin symmetric interaction. In the last part of the paper, we show how such entanglement between motional and spin degrees of freedom can lead to BI violation and emphasize the experimental realization in the laboratory. Since the availability of such systems featuring contextuality is not very rich regardless of their importance, and the fact that SSOC BEC is a very important experimental realizable quantum system, the study of entanglement between non-spatially separated degrees of freedom of SSOC BEC is legitimate.

Basic formalism: Hamiltonian and time evolution

Non-interacting system

The SSOC system considered here was experimentally realized in the NIST experiment [12] and consists of ultra cold $^8\text{Rb}$ atoms in two internal states, $|m_F\rangle = |0\rangle$, $|-1\rangle$ available in $F = 1$ hyperfine state of $5S^2\text{F}$ electronic levels. A pair of counter-propagating Raman lasers couple these states and the non-interacting/single-atom Hamiltonian for the corresponding system can be written as $H_{SO} = \hbar^2(k_x + k_y)^2 + \frac{\hbar c}{2m}|\sigma_z|^2 + \frac{\omega_e}{\hbar}k_x + \frac{\omega_e}{m}\sigma_z + \frac{\omega_e}{m}\sigma_x$. After performing a suitable pseudo-spin rotation on the above Hamiltonian, it can be rewritten as [27]

$$H_{\text{SO}} = \begin{bmatrix} \frac{\hbar^2}{2m}(k_x + k_y)^2 + \frac{\Omega}{2} & \frac{\hbar c}{2m}(k_x - k_y) \frac{\Omega}{2} \\ \frac{\hbar c}{2m}(k_x - k_y) \frac{\Omega}{2} & \frac{\hbar^2}{2m}(k_x + k_y)^2 - \frac{\delta}{2} \end{bmatrix}. \tag{1}$$

Here $k_x$ and $k_y$ are respectively the atom and photon linear momentum, $\Omega$ is the Rabi coupling strength and $\delta = h(\Delta\omega_2 - \omega_2)$, where $\Delta\omega_2$ is the detuning between the Raman lasers and $\omega_2$ is the Zeeman frequency. It is found that the properties of the SSOC ultra cold BEC can be understood through this non-interacting Hamiltonian (1) [12]; as a result, our subsequent discussion in this section will be based on this Hamiltonian. However, to better understand a more realistic system, we will discuss the effects of interaction in the next subsection. Diagonalization of (1) gives two energy eigenvalues $E_{\pm} = E \pm A$. Following the standard approach [12, 28, 29] (see appendix), we can write the energy eigenstate (dressed states) in terms of the bare states $\{|\rangle, k_x - k_y\rangle$ and $\{|\rangle, k_x + k_y\rangle$, which reads as

$$|g\rangle = -C_+|\rangle, k_x + k_y\rangle + C_+|\rangle, k_x - k_y\rangle,$$

$$|e\rangle = C_+|\rangle, k_x + k_y\rangle + C_+|\rangle, k_x - k_y\rangle,$$

with $C_\pm = \frac{1}{\sqrt{2A}} \left(A \pm \sqrt{A^2 - \frac{\Omega^2}{4}}\right)^{\pm}$. 

2
Here the suffixes $g,e$ respectively denote the ground and excited state of the dressed atom (dressed states), with $E_g = \frac{\delta^2 k^2}{2m}$ and $E_e = \frac{\delta^2 k^2}{2m}$, such that

$$E = E_g + E_e,$$

$$A = 2E_0 \sqrt{\frac{E_g}{E_0}} \left( 1 + \frac{\delta}{4\sqrt{E_g E_0}} \right)^2 + \frac{\Omega^2}{4E_0}.$$

(3)

It may be pointed out that even if the interaction is included in the mean field approximation, the condensate wavefunction can be expressed as a linear combination of these bare spin–orbit eigenstates [30]. In the BEC state, at sufficiently low temperature all atoms occupy the minimum energy state with momentum $k$, which can be obtained through energy minimization. It is found that the energy dispersion profile of $E_g$ has two interesting single and double well-like structures for $\Omega \gtrsim 4E_0$ and $\Omega < 4E_0$, respectively, with single well minima at $k_0$ and double well minima at $k_0 = \pm k_0, \sqrt{1 - \left( \frac{\Omega}{4E_0} \right)^2}$; at $\Omega = 4E_0$ the ground state shows a phase transition from double well structure to single well (cf figure 1). These dispersions are shown schematically (black line), for $\delta = 0$ in figure 1(a) and for $\delta \neq 0$ in figures 1(b), (c) and (d). The time evolution under (1) is given by

$$i\hbar \frac{d\Psi(t)}{dt} = H_0\Psi(t),$$

and a general time-dependent normalized state can be written as

$$|\Psi(t)\rangle = a \exp\left( -\frac{iE_g t}{\hbar} \right) |g\rangle + b \exp\left( -\frac{iE_e t}{\hbar} \right) |e\rangle,$$

$$= C_1(t)|\uparrow, k_e + k_0\rangle + C_2(t)|\downarrow, k_e + k_0\rangle,$$

(4)

where $\sigma_\uparrow, \sigma_\downarrow$ are Pauli metrics, and the coefficients read as

$$\begin{bmatrix} C_1(t) \\ C_2(t) \end{bmatrix} = e^{-\frac{i\tau}{2A}} \begin{bmatrix} b \exp\left( \frac{i\Omega t}{\hbar} \right) \sigma_\uparrow + a \exp\left( -\frac{i\Omega t}{\hbar} \right) \sigma_\downarrow \\ \sigma_\downarrow \end{bmatrix} |\psi\rangle,$$

(5)

which have to be determined from the initial condition. For example, if atoms are prepared in pseudo-spin (up) and momentum state $|\uparrow, k_e + k_0\rangle$, then $|\psi(0)\rangle = |\uparrow, k_e + k_0\rangle \Rightarrow C_1(0) = 0, C_2(0) = 1$ which gives $b = C_\uparrow, a = -C_\downarrow$.

The corresponding time evolved state at $t \geq 0$ becomes

$$|\psi(t)\rangle = \begin{bmatrix} \cos \frac{\pi t}{T} - i \frac{\sqrt{A^2 - \Omega^2}}{A} \sin \frac{\pi t}{T} \end{bmatrix} e^{-\frac{i\Omega t}{2}} |\uparrow, k_e + k_0\rangle - i \Omega \frac{\sin \frac{\pi t}{T}}{2A} e^{-\frac{i\Omega t}{2}} |\downarrow, k_e - k_0\rangle.$$

(6)

The occupation probability $|C_2(t)|^2$ is plotted as a function of $t$ and $\Omega/E_0$ with this initial condition in figures 1(a) and (b) respectively for zero and finite $\delta$. The change in the ground state energy dispersion around $\Omega/E_0 = 4$ is also indicated schematically in each plot. From expression (6) the time period of oscillation for $|C_2(t)|^2$ is $T = \pi\hbar/A$. At $t = nT, n \in \mathbb{Z}$ with $|\psi(nT)\rangle = \cos(n\pi)e^{-\frac{i\Omega t}{2}} |\downarrow, k_e + k_0\rangle \Rightarrow |C_2(t)|^2 = 1$. For $nT \leq t \leq (n + 1)T$, the amplitude get transferred to $|\downarrow, k_e - k_0\rangle$ state.

The time period $T$ depends on $E_0$ and $\Omega/E_0$ for a given value of $\delta$, through equation (3). The occupation probability amplitude changes continuously with $\Omega/E_0$, however, an
abrupt change occurs at $\Omega = 4E_0$, which accounts for the phase transition from double well to single well. This can be easily noticed through figures 1(a) and (b). When the coupling ceases $\Omega \rightarrow 0$, the system stays almost always in the initial state with $C_2 = 1$, $C_1 = 0$ as can be seen in figures 1(a) and (b). The transition probability to $|\downarrow, k_s - k_0\rangle$, $|C(t)|^2$ increases with increasing $\Omega$. It also depends on whether the detuning $\delta$ is zero, positive or negative. For a better understanding, we plotted $|C_2(t)|^2$ with different initial condition $C_1(0) = C_2(0)$ in figures 1(c) and (d), respectively with $\delta > 0$ and $\delta < 0$.

**Interacting system**

As stated earlier, the SSOC single particle effective Hamiltonian in the non-interacting limit mimics the phase diagram of interacting SSOC BEC with very high accuracy [12]. Thus it is expected that the features drawn from the previous section should not be altered significantly if a weak interaction is introduced among the atoms that represent the real system. To test this assertion, we now consider weakly interacting SSOC BEC. Several numerical methods have been proposed to study SSOC weakly interacting BEC [31]. The dynamic of such systems in the mean field approximation is governed by the non-linear and time-dependent Gross–Pitaevskii equation [32, 33]

$$i\hbar \frac{\partial \psi(x, t)}{\partial t} = \left[ H_{SO} + (V_{Trap}) + g(|\psi(x, t)|^2) \right] \psi(x, t),$$

(7)

where $H_{SO}$ is the $(2 \times 2)$ Hamiltonian metric of single SSOC atom given by equation (1) and $I$ is the $(2 \times 2)$ identity matrix. Here $\psi(x, t)$ is doublet in the form

$$\psi(x, t) = \begin{pmatrix} \psi_1(x, t) \\ \psi_2(x, t) \end{pmatrix},$$

$$|\psi(x, t)|^2 = |\psi_1(x, t)|^2 + |\psi_2(x, t)|^2.$$  

(8)

$\psi_{1,2}$ are condensate wavefunctions corresponding to two opposite spin components. For the interacting BEC, without loss of generality, we can safely assume that all atoms condensed on a superposition state of the two degenerate single-particle ground states $|g_1\rangle$ and $|g_2\rangle$, which read as

$$|g_1\rangle = \begin{pmatrix} C_+ \\ -C_- \end{pmatrix}, \quad |g_2\rangle = \begin{pmatrix} C_- \\ -C_+ \end{pmatrix},$$

where states are defined in terms of basis $|\downarrow, k_s - k_0\rangle$ and $|\uparrow, k_s + k_0\rangle$ respectively. Therefore, the condensate wavefunction can be written as [30, 34]

$$\begin{pmatrix} \psi_1(x, t) \\ \psi_2(x, t) \end{pmatrix} = \sqrt{\rho_0} \begin{pmatrix} D_1(t)e^{ikx} \\ C_+ \\ D_2(t)e^{-ikx} \\ -C_- \end{pmatrix}.$$  

(9)

$D_{1,2}(t)$ are superposition coefficients. The normalization condition leads to $|D_1|^2 + |D_2|^2 = 1$ or

$$\int (|\psi_1|^2 + |\psi_2|^2) dx = N.$$  

(10)

Here we are considering that the interaction between spin-up and spin-down components ($g_{\uparrow\downarrow}$) are same as the one between same spin components ($g_{\uparrow\uparrow}$ or $g_{\downarrow\downarrow}$). Having this SU(2) invariant interaction under spin rotation is firmly supported from the fact that any asymmetry in the coupling strength can be easily compensated by appropriately choosing the detuning ($\delta$) [34]. This spin symmetric interaction ($g_{\uparrow\downarrow} = g_{\downarrow\uparrow} = g_{\uparrow\uparrow}$) is also relevant for the experimental realization with Rb atoms because the difference in the interaction strength for intra- and inter-spin components is less than 1%. This comes as a natural consequence of the fact that the scattering length $a$ for like and opposite spin atoms are the same. The interaction strength is given by

$$g = \frac{4\pi\hbar^2a}{m},$$

and $n_0$ is the average number density of atoms in system. Let us neglect the effect of trapping potential ($V_{Trap} = 0$). Solving the time-dependent GP equation for this system leads to the pair of coupled equations

$$i\hbar \frac{\partial \psi_{1,2}}{\partial t} = H_{SO} \psi_{1,2} + g(|\psi_1|^2 + |\psi_2|^2) \psi_{1,2}. $$  

(11)

We solved these equations numerically, assuming that initially at $t = 0$ all atoms were condensed into a spin-up state, viz $\psi(t = 0) = 0$.

From the above results, we see that qualitatively the behaviour of entanglement entropy in an interacting system under mean field approximation does not differ significantly from a non-interacting system. For better clarity, we therefore continue our subsequent discussion with non-interacting bosons.

**Entanglement and its measure**

In this section, we explicitly show entanglement in the SSOC system and make a quantitative study by studying the effects of various parameters on the entanglement. The density matrix for the state (4)

$$\rho_{SO} = |\Psi_{SO}\rangle \langle \Psi_{SO}| = \begin{pmatrix} |C_1(t)|^2 & 0 & 0 & C_1(t)C_2(t)^* \\ 0 & 0 & 0 & 0 \\ C_1(t)^*C_2(t) & 0 & 0 & |C_2(t)|^2 \end{pmatrix},$$

(12)

The metric elements of (12) satisfy the condition $\rho_{SO\mu} = \rho_{SO\mu}$ criterion for a pure state. Now we investigate the entanglement-like properties for this state (4) and show that they satisfy various criterion for a bipartite entangled state, where now the entanglement is not between the same degrees of freedom of a spatially separated two subsystem, but rather between two different degrees of freedom of the same system. The basis states in the pseudo-spin space $H_s$ and the momentum (orbital) space $H_k$ are defined as

$$|n_1\rangle = |\uparrow\rangle, |n_2\rangle = |\downarrow\rangle, |v_{1,2}\rangle = |k_s \pm k_0\rangle.$$  

(13)

A general state in $H_s$ and in $H_k$ can be written as $|\Psi_S\rangle = a|\uparrow\rangle + b|\downarrow\rangle$, $|\Psi_m\rangle = c|k_s - k_0\rangle + d|k_s + k_0\rangle$. It follows directly that for $a, b, c, d \neq 0$, the state defined in equation (4) cannot be written as the tensor product between
spin and momentum state,
\[ |\Psi_{SO}(t)\rangle = |\Psi_{S}\rangle \otimes |\Psi_{m}\rangle. \] (14)

The above conclusion for state (4) can also be verified from the Schmidt decomposition criterion [35] of a pure state that allows one to write it as an expansion of biorthogonal terms \[ |u_i\rangle \otimes |v_i\rangle \] \((i = 1, 2)\), so that \[ |\Psi_{SO}\rangle = \sum g_{ij} |u_i\rangle \otimes |v_i\rangle \] with \[ \sum |g_{ij}|^2 = 1. \] Here \(g_{ij}\) are the Schmidt coefficient and the number of such \(g\) are called the Schmidt rank. The criterion for bipartite entanglement is that the Schmidt rank must be greater than 1. Now from (13) we can directly check that the Schmidt rank for the state \(|\Psi_{SO}\rangle\) is 2. Therefore the state satisfies the property of an entangled state. Additionally, from equation (12), the reduced density matrix for (pseudo) spin and momentum are respectively
\[ \rho_s = T_{\text{fm}} \rho_{SO}, \] (15)

It is clear from (12) and (15) that \(\rho_{SO} \propto \rho_s \otimes \rho_m\), and this also confirms the entanglement-like behavior in the system. The entanglement of the state in equation (4) can also be verified through the PPT (positive partial transposition) criterion given by Peres and Horodecki [36, 37], which is a necessary and sufficient condition for separability for dim(\(H_1 \otimes H_2\)) \(\leq 6\). For the present case, suffixes 1 and 2 refers to spin and orbital (momentum) degrees of freedom and the dimension of the product space is 4. The partial transposition (block-wise transposition) of matrix \(\rho_{SO}\) given in (12) becomes
\[ \rho^\rho_{SO} = \begin{bmatrix} |C_1(t)|^2 & 0 & 0 & 0 \\ 0 & 0 & C_1(t) C_2(t)^* & 0 \\ 0 & C_1(t)^* C_2(t) & 0 & 0 \\ 0 & 0 & 0 & |C_2(t)|^2 \end{bmatrix}. \] (16)

The eigenvalues of metric (16) are given by \(\sqrt{|C_1(t)|^2 |C_2(t)|^2}, -\sqrt{|C_1(t)|^2 |C_2(t)|^2}, |C_1(t)|^2, |C_2(t)|^2\). Since \(\rho^\rho_{SO}\) has at least one negative eigenvalue, according to the PPT criterion it has entanglement between spin and momentum degrees of freedom.

As we have demonstrated that the state defined in equation (4) exhibits entanglement between spin and the motional degree of freedom of each atom, we can now proceed for suitable quantitative measures of entanglement, such as the Von Neuman entropy (denoted as \(S\)) [38]. It is known that for any given entangled pure state, entanglement entropy is zero \((S = 0)\), whereas the entropy of the subsystems is greater than zero [39], because of the fact that the state corresponding to each subsystem of this bipartite system is a mixed state. The above definition for the Von Neuman entropy can be established for the system under consideration. By definition [39], \(S(\rho) = -Tr(\rho \log \rho) = -\sum \lambda_i \log(\lambda_i)\), where \(\lambda_i\) are eigenvalue of the density matrix \(\rho\). For the density matrix given in (12) the eigenvalues are respectively \((1, 0, 0, 0)\), which readily gives \(S(\rho) = 0\) as expected. Similarly, from the density metric in (15), the entropy \(S\) for the subsystem can be determined as \(S(\rho_s) = S(\rho_m) \equiv S\), with
\[ S = -|C_1(t)|^2 \log |C_1(t)|^2 - |C_2(t)|^2 \log |C_2(t)|^2 \] (17)

which for the interacting BEC takes the following form
\[ S = -|\psi_1(x, t)|^2 \log |\psi_1(x, t)|^2 - |\psi_2(x, t)|^2 \log |\psi_2(x, t)|^2. \] (18)

The entanglement entropy of the system can be defined as the Von Neuman entropy of either of subsystem \((S_i)\), whereas the numerical value of entanglement entropy quantifies the entanglement \((0 \leq S_i \leq 1)\). In figures 2(a), (b) and (d) \(S_i\) is plotted along the color axis for different initial conditions and different values of the detuning parameter \(\delta\). By (17) \(S_i\) is entirely determined by \(|C_2(t)|^2\). Therefore the variation of \(S_i\) is very similar to the corresponding variation of \(|C_2(t)|^2\) given in figure 1.

For \(|\Psi_{SO}\rangle = |\uparrow, k_i + k_o\rangle\), the state is a product state \((C_2 = 1, C_1 = 0)\), located at the north pole of the Bloch sphere in figure 6(a). From figures 1(a), (b) and figures 2(a), (b), for such a case the entanglement entropy is \(S_i = 0\), whereas the figures show that for a maximally entangled state such as \(|\Psi_{SO}\rangle = \frac{|\uparrow, k_i + k_o\rangle + |\uparrow, k_i - k_o\rangle}{\sqrt{2}}\) \((C_1 = C_2 = \frac{1}{2})\) (indicated in the equatorial plane of the Bloch sphere in figure 6(a)), the entanglement entropy attains its maximum value \((S_i = 1)\).

Another well known measure of entanglement in a system is concurrence, which was originally formulated to measure entanglement in mixed quantum state. Later, Hill and Wootters [40, 41] introduce concurrence for bipartite pure state as \(C = \sqrt{2(1 - Tr\rho^\rho)}\). The non-zero value of concurrence \(C\) confirmed the entanglement in a system where \(\rho\) is reduce density matrix of the system. For the pure state under consideration, \(\rho = \rho_s \otimes \rho_m\). This straightforwardly gives \(C = 2|C_1(t) C_2(t)|\) which is non-negative for a general state (4). As expected, \(C\) also shows a similar profile with time as \(S_i\) and \(|C_2(t)|^2\) in figure 2(c).

Probability variation \(|\psi|^2\) for SSOC interacting BEC is plotted in figure 3(a). Likewise, the probability variation of \(|C_2(t)|^2\) for the case of non-interacting SSOC bosons studied in the earlier section, here \(|\psi|^2\) also varies between its minimum and maximum value in a periodic manner. Recalling equation (9), the order parameter for interacting BEC is a function of both \(x\) and \(t\) so does the entanglement entropy as described by equation (18) (cf figure 4). The effect of interaction upon entanglement is studied further in figure 3(b). Parameters for interacting \(^{87}\)Rb SSOC BEC are taken from ongoing experimental work [28, 42]. These results clearly indicate that the entanglement properties only go through quantitative changes in the presence of weak interaction, which does not alter our conclusions based on the non-interacting model.

We can compare the entanglement entropy for SSOC BEC in the non-interacting limit with that of a weakly interacting BEC by comparing figures 2(a) and 5, respectively, and can make some more quantitative observations. It is evident from figures 3(b) and 5 that entanglement persists in the weakly interacting BEC even when the coupling \(\Omega/E_0\) is close to zero, whereas in the non-interacting limit it
Figure 2. Figure (a), (b) plot the entanglement entropy ($S_o$) along the color axis as a function of time and $\frac{\Omega}{E_0}$ for initial condition $C_1 = 0$, $C_2 = 1$ with all other conditions same as figures 1(a) and (b). Figure (d) is a similar plot but with initial condition $C_1 = C_2 = \frac{1}{2}$ and other conditions are the same as figure (b). Figure (c) gives a comparative plot for the time evolution of $|C_2(t)|^2$ (black thick solid line), $S_o$ (green dashed line) and concurrence $C$ (red thin solid line) for initial condition $C_1 = 0$, $C_2 = 1$.

Figure 3. Figure (a) plots the $|\psi|^2$ for interacting SSOC BEC along the color axis as a function of time(ms) and position $x$, where $x$ is in unit of laser wavelength $\lambda_o$ with parameter $\delta = 0$, $\Omega = 3E_0$, $n_o = 10^{15}$ m$^{-3}$, $N = 10^4$, $a = 10^{-9}$ m. Figure (b) plots entanglement entropy along color axis as function of time (ms) and position $x$.

Figure 4. A comparative plot for the time evolution of $|\psi_1(x, t)|^2$ (black line), $S_o$ (green line) and $|\psi_2(x, t)|^2$ (red line). (Left) as a function of $x$ in unit of $\lambda_o$ at time $t = 2$ms. (Right) as a function of $t$ at position $x = 2\lambda_o$. 

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vanishes. This can be attributed to the presence of inter-particle interaction. Additionally, the interaction present in the BEC affects the entanglement possessed by the system and limits the entanglement entropy contained in a BEC state by preventing the condensate wavefunction to reach the maximally entangled state \( S \rightarrow 1 \), which was achievable in the non-interacting case.

### Violation of Bell-like Inequality

Given the fact that the state defined in (4) is entangled in spin and momentum, it is expected that it would violate BI [43]. The relevant inequality in this case is CHSH inequality [44], which is a generalization of BI and referred to as Bell-like inequality. To set the framework for testing BI we first define our basis states for different sets of measurements which are obtained from the basis states defined in (13) by a unitary transformation. They are

\[
\begin{align*}
\mathbf{u}^\alpha &= R(\alpha)\mathbf{u}, \\
\mathbf{v}^\beta &= R(\beta)\mathbf{v},
\end{align*}
\]

with \( R(\theta) = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} \) and \( \mathbf{u} = (|u_1\rangle, |u_2\rangle)^T \), \( \mathbf{v} = (|v_1\rangle, |v_2\rangle)^T \). Here \( \alpha, \beta \) corresponds to the angle made by the detectors that are measuring the mutually commutating observables, pseudo-spin and momentum. In terms of these transformed basis states, the CHSH inequality becomes [44]

\[
\begin{align*}
\beta &= |E(\alpha, \beta) + E(\alpha', \beta) + E(\alpha', \beta') - E(\alpha, \beta')| \leq 2.
\end{align*}
\]  
(19)

The Bell’s coefficient \( \beta \) must be greater than 2 (\( \beta > 2 \)) for the violation of CHSH or Bell-like inequality. Each \( E(\alpha, \beta) \) is the correlation between spin \( |u_\alpha^\alpha\rangle \) and momentum \( |v_\beta^\beta\rangle \) degree of freedom of the ultra cold atom and is given by

\[
E(\alpha, \beta) = P_{11}(\alpha, \beta) - P_{12}(\alpha, \beta) - P_{21}(\alpha, \beta) + P_{22}(\alpha, \beta).
\]  
(20)

Here, \( P_{ij}(\alpha, \beta) \) is the probability of getting the atom in the product state \( |u_\alpha^\alpha\rangle \otimes |v_\beta^\beta\rangle \) and is given by \( P_{ij}(\alpha, \beta) = \langle \psi_{so}(\alpha) | \hat{P}_{ij}^\alpha \hat{P}_{ij}^\beta | \psi_{so}(\beta) \rangle \). Here \( \hat{P}_{ij}^\alpha \) is the projection operator and is given by \( \hat{P}_{ij}^\alpha = |u_\alpha^\alpha\rangle \langle u_\alpha^\alpha| \) and \( \hat{P}_{ij}^\beta = |v_\beta^\beta\rangle \langle v_\beta^\beta| \). From (4) and

\[
E(\alpha, \beta) = \cos(2\alpha)\cos(2\beta) + (C_1 C_2^* + C_2 C_1^*)\sin(2\alpha)\sin(2\beta),
\]  
(21)

which can be used to calculate \( \beta \) in (19) to verify BI for the current problem. It is well known that all choices of \( \alpha \) and \( \beta \) do not satisfy this inequality [45, 46]. We made the choice \( \alpha = 0, \beta = \pi/4, \alpha' = \pi/2, \beta' = 3\pi/8 \) for which \( \beta = \sqrt{2}(C_1 C_2^* + C_2 C_1^*) + 1 \). In figure 6(b) the above expression for the parameter \( \beta \) is plotted as a function of \( \Omega/E_0 \) for the initial conditions mentioned in figures 1(a) and (b). This can be easily noted from figure 6(b) that for a wide range of values of \( \Omega/E_0 \) at different detuning \( \delta \), the inequality is violated, namely \( \beta > 2 \). We must stress that when the state (4) is a maximally entangled state, namely \( C_1 = C_2 = \frac{1}{\sqrt{2}} \), \( \beta \) reaches the maximum bound of the BI violation, that is \( 2\sqrt{2} \) given by Cirelson [47]. On the other hand, when it is a simple product state (i.e., \( C_1 = 0, C_2 = 1 \), \( \beta = \sqrt{2} \), the BI is obeyed.

For the experimental testing of BI we require an ensemble of such atoms. We can calculate the corresponding correlation \( E(\alpha, \beta) \) through statistical measurement given as

\[
E(\alpha, \beta) = \frac{N_{11}(\alpha, \beta) + N_{22}(\alpha, \beta) - N_{12}(\alpha, \beta) - N_{21}(\alpha, \beta)}{N_{so}(\alpha, \beta)}.
\]  
(29)

Here, \( N_{ij}(\alpha, \beta) \) counts the number of pairs in spin \( |u_\alpha^\alpha\rangle \) and momentum \( |v_\beta^\beta\rangle \) states. The Stern–Gerlach (SG) set-up is a unique way of spin detection and was already used for measurement in the case of ultra cold spinorial condensate [29]. Here, we have to make spin measurements for two different settings \( \alpha \) and \( \alpha' \) of SG set-up which could be, for example, the angle from +z axes if we consider the pseudo spin in the +z direction. A schematic of such a set-up is depicted in figure 7. One can further do momentum spectroscopy on the ultra cold atomic cloud to extract the number of atoms collapsing in the state \( |v_\beta^\beta\rangle \) with a given spin orientation. After permitting the system to evolve according to Hamiltonian (1) for sufficient time, we abruptly turned off the Raman laser beam and ceased the coupling, allowing the system to evolve freely after releasing from optical trapping. Consequently, the atomic states will be projected from dressed states to bare momentum states. Now, we can image the atoms using a probe laser through absorption spectra after some fraction of time (few ms) of time-of-flight expansion. This images will reveal the number of atoms distribution in each momentum state. From such joint measurements, the parameter \( \beta \) can be determined.

Our work can lead to interesting possibilities which have both fundamental and applied interest. It can be straightforwardly extended to experimentally realized ultra cold fermions [9, 20] with spin–orbit coupling. By coupling such spin–orbit entangled atoms with each other or with photons it is also possible to create systems with a different type of hybrid entanglement [48]. Since each atom is entangled in spin and momentum degrees of freedom and shows Rabi oscillation, a periodic array of such atoms in an optical lattice can implement SWAP operation in large scale quantum computation [18]. To summarize, this work demonstrated the
and such dressed states lie in the $x-z$ plane. The north and south pole corresponds to bare states which are product states, whereas the maximally entangled state or Bell state lies in the equilateral plane. (b) The Bell’s coefficient $\beta$ (color axis) is shown as a function of time ($x$-axis) and $\frac{1}{E_0}$ ($y$-axis) for conditions corresponding to figure 1(a).

![Schematic diagram for the spin and momentum measurement for testing BI. In the circles we show schematically the transformed basis in the spin and momentum space.](image)

Figure 7. Schematic diagram for the spin and momentum measurement for testing BI. In the circles we show schematically the transformed basis in the spin and momentum space.

exciting possibility of testing certain fundamental issues of quantum mechanics such as quantum contextuality and BI violation with synthetically spin–orbit coupled ultra cold bosons and this is potent with extremely rich possibilities.

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Appendix

The idea of generating SOC in ultra cold atom through Raman coupling has been prodigiously discussed in earlier work [12]. Under the influence of two detuned counter-propagating laser beams, the linear momentum of an ultra cold atom gets shifted by an amount of $\pm k_\omega$, and picks up two independent momentum states $k_x \pm k_\omega$. The two independent internal hyperfine (degenerate) states of the atom are labelled as $\downarrow k_x, \downarrow k_x$. This degeneracy in hyperfine states is lifted up by exploiting a homogeneous magnetic field. The atom absorbs a $\Omega/\sigma^{+/-}$ polarized photon and goes to the virtual excited state; from there it makes a stimulated emission of $\sigma^{+/-}/\Omega$ and comes back to another ground state. In this way a controlled $\hat{d} \cdot \hat{E}$ type interaction between atom and laser fields respecting the energy and angular momentum conservation ensure that these momentum states coupled with two hyperfine states (pseudo-spin states) of atom to form viable bare atomic states $\downarrow k_x, k_x - k_\omega$, $\downarrow k_x, k_x + k_\omega$. Other possible states of composite (atom + laser) system $\downarrow k_x, k_x$, $\downarrow k_x, k_x - k_\omega$, $\downarrow k_x, k_x + k_\omega$ play the role of virtual excited states in transition and do not have any physical contribution otherwise [49]. Before SOC, spin and linear momentum are independent to each other since they belong to commutative observable, however once the SOC is turned up they articulate themselves into bare states [29]. The Hamiltonian describing this two level system is

$$H_{SO} = \begin{pmatrix} \frac{\hbar^2}{2m} k_x^2 + \frac{\hbar^2 k_\omega^2}{\Omega^2} & \Omega \frac{1}{2} \exp(2\imath k_\omega x) \\ \Omega \frac{1}{2} \exp(-2\imath k_\omega x) & \frac{\hbar^2}{2m} k_x^2 - \frac{\hbar^2 k_\omega^2}{\Omega^2} \end{pmatrix}.$$ (22)

The off-diagonal term describes the transition between bare states accompanied by the momentum shift of $2\imath k_\omega$. Applying a unitary transformation to the wavefunction of Hamiltonian (22)

$$U = \begin{pmatrix} \exp(\imath k_\omega x) & 0 \\ 0 & \exp(-\imath k_\omega x) \end{pmatrix}.$$ (23)

one reaches an effective Hamiltonian describing the SOC of ultra cold atom in equation (1). The true eigenstate of Hamiltonian equation (1) is not the bare state; rather, it is the dressed state which is the linear combination of these bare states (for an extensive discussion see [12, 28, 29]). Each minima of energy state is represented by the dressed state, i.e. for particular Raman coupling of $\Omega = 2E_0$, lower energy state has two minima at $k_x = \pm k_\omega$, therefore each minima...
will have state
\[ |g_1\rangle = A_1|\downarrow\rangle, -2k_\alpha + B_1|\uparrow\rangle, 0 \]
\[ |g_2\rangle = A_2|\downarrow\rangle, 0 + B_2|\uparrow\rangle, 2k_\alpha. \]

The \( k_\alpha \) in the dressed state is not a real momentum; rather it is a quasi-momentum. It is replaced by the momentum of atom at that point.

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