DUALITY BETWEEN COALESCENCE TIMES AND EXIT POINTS IN LAST-PASSAGE PERCOLATION MODELS

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Abstract. In this paper we prove a duality relation between coalescence times and exit points in last-passage percolation models with exponential weights. As a consequence, we get lower bounds for coalescence times with scaling exponent 3/2, and we relate its distribution with variational problems involving the Brownian motion process and the Airy2 process.

1. INTRODUCTION

1.1. Coalescence times and exit points. Consider a collection $\omega = \{W_x : x \in \mathbb{Z}^2\}$ of i.i.d. random variables, distributed according to an exponential distribution function of parameter one. In last-passage percolation (LPP) models, each number $W_x$ represents the passage (or percolation) time through vertex $x = (x(1), x(2))$. For $x \leq y$ (i.e. $x(i) \leq y(i), i = 1, 2$) in $\mathbb{Z}^2$ denote $\Gamma(x, y)$ the set of all up-right oriented paths $\gamma = (x_0, x_1, \ldots, x_k)$ from $x$ to $y$, i.e. $x_0 = x$, $x_k = y$ and $x_{j+1} - x_j \in \{e_1, e_2\}$, for $j = 0, \ldots, k - 1$, where $e_1 := (1, 0)$ and $e_2 = (0, 1)$. The passage time of $\gamma = (x_0, x_1, \ldots, x_k) \in \Gamma(x, y)$ is defined as

$$W(\gamma) := \sum_{j=1}^{k} W_{x_j}.$$ 

The last-passage time between $x$ and $y$ is defined as

$$L(x, y) = L_\omega(x, y) := \max_{\gamma \in \Gamma(x, y)} W(\gamma).$$

(The lower index indicates the dependence with the environment.) The geodesic from $x$ to $y$ is the a.s. unique path $\gamma(x, y) = \gamma_\omega(x, y) \in \Gamma(x, y)$ such that

$$L(x, y) = W(\gamma(x, y)).$$

Since the seminal paper of Newman [15] (in the first-passage percolation context) its known that finite geodesics along a fix direction converge to a semi-infinite path. Convergence of a sequence of finite paths $(\gamma_n)_{n \geq 0}$ to a semi-infinite path $\gamma$ means that, for every finite set $K \subseteq \mathbb{Z}^2$, $\gamma_n$ and $\gamma$ will coincide inside $K$, eventually. The study of semi-infinite geodesics in last-passage percolation models with exponential weights was done in [9, 11]. We summarize the results that we use in this paper below.

The semi-infinite geodesic starting at $x$ and along direction $d := (1, 1)$ is the almost surely unique up-right oriented path $\gamma(x) = (x_n)_{n \geq 0}$ which satisfies:

(i) $x_0 = x$ and, for any $m < n$,

$$\gamma(x_m, x_n) = (x_m, \ldots, x_n);$$
(ii) For any sequence of lattice points \((y_n)_{n \geq 1}\) such that
\[ y_n = (y_n(1), y_n(2)) \geq 0 = (0, 0) \] and
\[ \lim_{n \to \infty} \frac{y_n(2)}{y_n(1)} = 1, \]
we have
\[ \lim_{n \to \infty} \gamma(x, x + y_n) = \gamma(x). \]
Another important property of semi-infinite geodesics with the same direction is coalescence. The symbol \(\oplus\) below stands for the concatenation of two paths.

(iii) For any \(x, y \in \mathbb{Z}^2\) there exists \(c \in \mathbb{Z}^2\) such that
\[ \gamma(x) = \gamma(x, c) \oplus \gamma(c) \quad \text{and} \quad \gamma(y) = \gamma(y, c) \oplus \gamma(c). \]

We note that if \(c\) satisfies (iii), and \(c' \in \gamma(c)\), then \(c'\) also satisfies (iii). From now on we denote \(c(x, y)\) the first (in the up-right orientation) coalescence point, in the sense that \(c' \geq c(x, y)\) for every other geodesic point \(c'\) that satisfies (iii). For \(m \geq 1\), denote \(m^h := (m, 0)\) and \(m^v := (0, m)\). Let \(c_m = c(m^h, m^v)\) and define
\[ T_m := \text{the second coordinate of } c_m. \]

By symmetry, it is clear that the first coordinate of \(c_m\) has the same distribution as \(T_m\). We call \(T_m\) the coalescence time.

In the literature, the scaling order of coalescence times was first treated in [19], in the context of the Hammersley LPP model. There, it was proved that the scaling exponent of coalescence times is at least \(3/2 - \epsilon\) (for all \(\epsilon > 0\)). As far as the author is concerned, till now, this is the only available result for the scaling order of coalescence times. The main contribution of this paper resides in bringing new rigorous results on the subject, and also to shed new lights on the scaling scenario of coalescence times.

As we shall see next, coalescence times are related to locations of exit-points in a slightly different LPP model, where we introduce boundary conditions as follows. Denote \(\text{Exp}(\rho)\) an exponential random variable with parameter \(\rho\). Take an environment \(\bar{\omega} = \{\bar{W}_z : z \geq 0\}\) mutually independent with the following distribution:
\[ \bar{W}_z \quad \text{dist.} \begin{cases} 0, & \text{if } z = 0; \\ \text{Exp}(1), & \text{if } z > 0; \\ \text{Exp}(1/2), & \text{otherwise} \end{cases} \]

Heuristically speaking, we put i.i.d. exponentials random variables of parameter 1/2 along the horizontal and vertical axes of the first quadrant, and leave its interior with the same distribution as before. We denote
\[ L(x) := L_{\bar{\omega}}(0, x) \]
the last-passage time from \(0\) to \(x\), with respect to the \(\bar{\omega}\) environment. We call the exit-point of the geodesic \(\gamma_{\bar{\omega}}(0, x)\) the last boundary point of the path (following the up-right orientation). To distinguish between exit via the horizontal or the vertical axis, we introduce a non-zero integer-valued random variable \(Z(x) = Z_{\bar{\omega}}(x)\) such that if \(Z(x) > 0\) then the exit-point is \((Z(x), 0)\), while if \(Z(x) < 0\) then the exit-point is \((0, -Z(x))\). The exit-point process is defined as
\[ Z_n := (\zeta_n(z), z \in [-n, \infty)) \in \{0, 1\}^{[-n, \infty)}, \]
where, for fixed \(n \geq 1,\)
\[ \zeta_n(z) := \begin{cases} 1 & \text{if } z = Z(x, n) \text{ for some } x \in [1, \infty), \\ 0 & \text{otherwise} \end{cases} \]
We also define the counting variable
\[ Z_n(A) := \sum_{z \in A} \zeta_n(z), \quad \text{for } A \subseteq [-n, \infty). \]

1.2. Main results. The key result of this article (Theorem 3) is a duality relation between the coalescence time and the exit-point process. It states that,
\[ \mathbb{P}(T_m < n) = \mathbb{P}(Z_n([-m, m]) = 0), \quad \text{for } n > m. \]
This duality relation allows us to use well known results for the scaling behavior of exit-points to get new results for the scaling behavior of the coalescence times.

Let
\[ G(r) := \lim \inf_{m \to \infty} \mathbb{P}\left( \frac{T_m}{2^{5/2}m^{3/2}} > r \right). \]
The reason to put the additional scaling factor $2^{-5/2}$ will became clear in the sequel, and its is related to universality of the expected limiting distribution. As a first application of the duality formula we study the behavior of $G$ close to 0. By using that exit points scale like $n^{2/3}$, together with duality (1.1), we prove (Corollary 1) that there exists a constant $c_0$ such that, for sufficiently small $r$,
\[ G(r) \geq 1 - c_0 r^2. \]
In particular, this shows that
\[ \lim_{r \to 0} G(r) = 1. \]

The fluctuations of last-passage times are related to variational problems involving the Brownian motion and the Airy$_2$ process [17]. The Airy$_2$ process is a one-dimensional stationary process with continuous paths, whose finite dimensional distributions are describe by Fredholm determinants. Duality allows us to link coalescence times with these processes as well. Let
\[ U := \arg\max_{u \in \mathbb{R}} \left\{ \sqrt{2}B(u) + A(u) - u^2 \right\}, \]
where $(B(u), u \in \mathbb{R})$ is a standard two-sided Brownian motion, and $(A(u), u \in \mathbb{R})$ is an independent Airy$_2$ process, and denote
\[ F(s) := \mathbb{P}(U \leq s). \]
The random variable $U$ is the limit in distribution of the rescaled exit point (Theorem 4):
\[ \lim_{n \to \infty} \frac{Z(n,n)}{2^{5/3}n^{2/3}} \overset{\text{dist.}}{=} U. \]
We prove that (Corollary 2)
\[ G(r) \geq F(r^{-2/3}) - F(-r^{-2/3}). \]
This clearly implies that
\[ \lim_{r \to \infty} r^{2/3} G(r) \geq 2f(0), \]
where $f$ is the density of $F$. Although, as far as the author knows, there is no analytical description of $f$. We do expect that $f$ is bell shaped around 0, as in the case of a Brownian motion minus a parabola [13], as well as in the case of an Airy$_2$ process minus a parabola [12]. In particular, we also expect that $f(0) > 0$, which would imply non-integrability of $G$. We do believe that $r^{2/3}G(r)$ converges to a constant, as $r \to \infty$, which should be given by the intensity of the point process composed by locations of maxima of $(\sqrt{2}B(u) + A(u,v) - (u-v)^2, u \in \mathbb{R})$, parametrized by $v \in \mathbb{R}$, where $A(u,v)$ is the so called Airy$_2$ sheet [8]. In the last section we will discuss more on the conjectured scaling scenario of coalescent geodesics and coalescence times.
The proof of (1.1) relies on self-duality of the geodesic tree (Theorem 2), defined as
\[ \mathcal{L} := \{ \gamma(x) : x \in \mathbb{Z}^2 \}. \]
The coalescence property (iii) of semi-infinite geodesics allows us to introduce Busemann functions in the LPP model, which are defined as
\[ B(x, y) := L(y, c) - L(x, c), \text{ for } x, y \in \mathbb{Z}^2, \]
where \( c = c(x, y) \) is coalescence point between \( \gamma(x) \) and \( \gamma(y) \). Busemann functions provide an alternative construction of LPP models with boundary \([5]\). These models enjoys a very special property, named, the Burke property. This property, formulated in terms of Busemann functions (Theorem 1), will lead us to self-duality of the geodesic tree.

**Overview.** In Section 2 we state and prove all results. After that, in Section 3 we discuss the conjectural picture of the scaling limit of coalescence times, and further relations with Brownian motion and the Airy\(_2\) process.

2. **Burke’s property, self-duality and scaling of coalescence times**

2.1. **The last-passage percolation model and the exclusion process.** The LPP model can be seen as a function of the motion of particles in the one-dimensional totally asymmetric simple exclusion process (TASEP). This process is a Markov process \((\eta_t, t \geq 0)\) in the state space \(\{0, 1\}^\mathbb{Z}\) whose elements are particle configurations: \(\eta_t(j) = 1\) indicates a particle at site \(j\) at time \(t\); otherwise \(\eta_t(j) = 0\) (a hole is at site \(j\) at time \(t\)). With rate 1, if there is a particle at site \(j\), it attempts to jump to site \(j + 1\); if there is a hole at \(j + 1\) the jump occurs, otherwise nothing happens. The generator of the process is given by
\[ \mathcal{G} f(\eta) = \sum_{x \in \mathbb{Z}} \eta(x) (1 - \eta(x + 1)) [f(\eta^{x,x+1}) - f(\eta)], \]
where \(\eta^{x,y}(j) = \eta(j) \forall j \notin \{x, y\}, \eta^{x,y}(x) = \eta(y)\) and \(\eta^{x,y}(y) = \eta(x)\). For \(p \in (0, 1)\), let \(\nu_p\) denote the product measure on \(\mathbb{Z}\) with density \(p\). Then \(\nu_p\) is invariant for \(\mathcal{G}\). The reverse process with respect to \(\nu_p\) has generator \(\mathcal{G}^*\) which is also a TASEP with reversed jumps:
\[ \mathcal{G}^* f(\eta) = \sum_{x \in \mathbb{Z}} \eta(x) (1 - \eta(x - 1)) [f(\eta^{x,x-1}) - f(\eta)]. \]
This property is called reversibility of the TASEP.

A construction of the (time) stationary process \(\eta = (\eta_t, t \in \mathbb{R})\) with the marginal distribution \(\nu_\rho\) can be done by choosing a configuration \(\eta\) according to \(\nu_\rho\) and then running the process with generator \(L\) forward in time and the process with generator \(L^*\) backward in time. The reversed process \(\eta^*\) is given by \(\eta^*_t = \eta_{t-}\). The particle jumps of \(\eta\) induce a stationary point process \(S\) in \(\mathbb{Z} \times \mathbb{R}\). Let \(S_x \subseteq \mathbb{R}\) be the (discrete and random) set of times for which a particle of \(\eta\) jumps from \(x\) to \(x + 1\), and \(S = (S_x, x \in \mathbb{Z})\). The map \(\eta \mapsto S\) associates alternate point processes to each trajectory. The law of the process \(S\) is space and time translation invariant. Let \(S^0\) be the Palm version of \(S\), that is, the process with the law of \(S\) conditioned to have a point at \((x, t) = (0, 0)\). In the corresponding process \(\eta^0\) there is a particle jumping from 0 to 1 at time zero. In the reverse process \(\eta^0\) there is a particle jumping from 1 to 0 at time zero.

We now construct a random function \(\mathcal{G} = G(\eta^0)\) as follows [13]: first label the particles of \(\eta^0\) in increasing order, giving label 0 to the particle at site 1. Call \(P_j(0)\) the position of the \(j\)th particle at time zero; we have \(P_0(0) = 1\) and \(P_{j+1}(0) < P_j(0)\) for all \(j \in \mathbb{Z}\). Label the holes of \(\eta^0\) in increasing order...
order, giving the label 0 to the hole at site 0: \( H_0(0) = 0 \) and \( H_{i+1}(0) > H_i(0) \) for all \( i \in \mathbb{Z} \). The position of the \( j \)th particle and the \( i \)th hole at time \( t \) are denoted, respectively, \( P_j(t) \) and \( H_i(t) \). The order is preserved at later and earlier times: \( P_j(t) > P_{j+1}(t) \) and \( H_i(t) < H_{i+1}(t) \), for all \( t \in \mathbb{R}, \ i, j \in \mathbb{Z} \). Let \( G(i, j) \) denote the time the \( i \)th hole and the \( j \)th particle of \( \zeta^0 \) interchange positions; in particular \( G(0, 0) = 0 \). Let

\[
\zeta^0 = G(\eta^0) := \{G(z), \ z \in \mathbb{Z}^2\}.
\]

The LPP model with boundary condition and the TASEP are related by (we take \( p = 1/2 \))

\[
\{\bar{L}(z), \ z \in \mathbb{Z}^2\} \overset{\text{dist.}}{=} \{G(z), \ z \in \mathbb{Z}^2\}.
\]

To construct the analog object for the reversed process, we set \( \hat{\eta}_t^0(j) := \eta_t^*(-j) \). By reversibility, \( \hat{\eta}_t^0 \) is also a stationary TASEP, but now with jumps in the same orientation as before. For this process, there is a particle jumping from \(-1\) to \(0\) at time zero. At this time, we give label 0 to the particle at 0 and label 0 to hole at \(-1\), and construct the interchanging times \( G^*(i, j) \) as before, so that

\[
G^* := \{G^*(z), \ z \in \mathbb{Z}^2\} = \{-G(z), \ z \in \mathbb{Z}^2\}.
\]

As a consequence of reversibility \( G^* \overset{\text{dist.}}{=} \zeta \), and therefore

\[
(2.1) \quad \{-G(-z), \ z \in \mathbb{Z}^2\} \overset{\text{dist.}}{=} \{G(z), \ z \in \mathbb{Z}^2\}.
\]

2.2. Burke’s property for Busemann functions. In Cator and Pimentel [5], it was developed a connection between the LPP model with boundary and Busemann functions. Almost sure existence and coalescence of semi-infinite geodesics along the negative diagonal direction are also true. Let \( \gamma^+(x) = (x_n)_{n \geq 0} \) denote the down-left oriented semi-infinite geodesic starting at \( x \) and along the negative diagonal direction. Thus, \( \gamma^+(x) \) satisfies (i), (ii) and (iii), but now in the down-left orientation. For \( x, y \in \mathbb{Z}^2 \), let \( c^+(x, y) \) denote the coalescence point between \( \gamma^+(x) \) and \( \gamma^+(y) \), and set

\[
B^+(x, y) := L(c^+, y) - L(c^+, x).
\]

The main result in [5] states that

\[
\{\bar{L}(z) : z \in \mathbb{Z}^2_+\} \overset{\text{dist.}}{=} \{B^+(z) : z \in \mathbb{Z}^2_+\},
\]

where \( B^+(z) := B^+(0, z) \). It also follows from the results in [5] that

\[
G(z) : z \in \mathbb{Z}^2 \overset{\text{dist.}}{=} \{B^+(z) : z \in \mathbb{Z}^2\}.
\]

We call (2.3) below the Burke property of Busemann functions.

**Theorem 1.** For the Busemann functions, we have that

\[
B^+(z) : z \in \mathbb{Z}^2 \overset{\text{dist.}}{=} \{-B^+(z) : z \in \mathbb{Z}^2\}.
\]

**Proof.** It follows directly from (2.1) and (2.2). \( \square \)
2.3. **Self-duality of the geodesic tree.** By the coalescence property (iii), the collection of paths

\[ \mathcal{L} := \{ \gamma(\mathbf{x}) : \mathbf{x} \in \mathbb{Z}^2 \} \]

is a.s. an up-right oriented tree, called the geodesic tree. We also consider the collection of down-left oriented semi-infinite geodesics defined as

\[ \mathcal{L}^\downarrow := \{ \gamma^\downarrow(\mathbf{x}) : \mathbf{x} \in \mathbb{Z}^2 \}. \]

It is clear that \( \mathcal{L} \) and \( \mathcal{L}^\downarrow \) have the same law, up to a rotation of 180° degrees.

Let \( \mathbb{Z}^2^* \) denote the dual of \( \mathbb{Z}^2 \). We take as vertices the set \( \{ \mathbf{z}^* = \mathbf{z} + \frac{1}{2}\mathbf{d} : \mathbf{z} \in \mathbb{Z}^2 \} \), where \( \mathbf{d} = (1, 1) \), and we join two such neighboring (distance 1) vertices by a dual edge. Thus each edge of \( \mathbb{Z}^2 \) is bisected by a dual edge of \( \mathbb{Z}^2^* \), and vice-versa, which establishes a bijection (isomorphism) between edges and dual edges. Consider the last-passage percolation tree \( \mathcal{L} \). The dual system \( \mathcal{L}^* \) is defined as follows: in the case that an edge is in \( \mathcal{L} \) then its dual is not in \( \mathcal{L}^* \); in the case that an edge is not in \( \mathcal{L} \) then its dual is in \( \mathcal{L}^* \). Self-duality states that \( \mathcal{L} \) and \( \mathcal{L}^* \) have the same law, up to a rotation of 180° degrees. The proof parallels the ideas in section 4.2 of [10], where duality between geodesics and equilibrium competition interfaces was established.

**Theorem 2.** For the dual system, we have that

\[ \mathcal{L}^* \overset{\text{dist.}}{=} \mathcal{L}^\downarrow. \]

In particular, the dual system \( \mathcal{L}^* \) is a.s. a tree and there is no bi-infinite maximizing path in \( \mathcal{L} \).

**Proof.** For notational convenience, we will prove the equivalent statement that

\[ \mathcal{L}^\downarrow^* \overset{\text{dist.}}{=} \mathcal{L}. \]

To prove that we first notice that the tree \( \mathcal{L}^\downarrow \) is a deterministic function of the Busemann function \( B^\downarrow \). In order to see this we use that

\[ B^\downarrow(\mathbf{x}) = \max \left\{ B^\downarrow(\mathbf{x} - \mathbf{e}_1), B^\downarrow(\mathbf{x} - \mathbf{e}_2) \right\} + W_x. \]

Hence, for any down-left semi-infinite geodesic \( \gamma^\downarrow(\mathbf{x}) = (\mathbf{x}_n)_{n \geq 0} \),

\[ \mathbf{x}_{n+1} = \arg \max \left\{ B^\downarrow(\mathbf{x}_n - \mathbf{e}_1), B^\downarrow(\mathbf{x}_n - \mathbf{e}_2) \right\}. \]

(Notice that a similar property holds for finite geodesics.) By (2.4), the tree \( \mathcal{L}^\downarrow \) can be seen as the set composed of down-left oriented edges \( (\mathbf{x}, \mathbf{e}_x) \) such that \( \mathbf{x} \in \mathbb{Z}^2 \) and

\[ \mathbf{e}_x = \begin{cases} \mathbf{x} - \mathbf{e}_1 & \text{if } B^\downarrow(\mathbf{x} - \mathbf{e}_1) > B^\downarrow(\mathbf{x} - \mathbf{e}_2), \\ \mathbf{x} - \mathbf{e}_2 & \text{if } B^\downarrow(\mathbf{x} - \mathbf{e}_2) > B^\downarrow(\mathbf{x} - \mathbf{e}_1). \end{cases} \]

Therefore,

\[ \mathcal{L}^\downarrow = \Upsilon(\mathcal{L}^\downarrow) \]

is a deterministic function \( \Upsilon \) of \( B^\downarrow = \{ B^\downarrow(\mathbf{x}) : \mathbf{x} \in \mathbb{Z}^2 \} \).

On the other hand, the dual system \( \mathcal{L}^\downarrow^* \) can be seen as the set composed of up-right oriented edges \( (\mathbf{x}^*, \mathbf{e}_{x^*}) \) such that

\[ \mathbf{e}_{x^*} = \begin{cases} \mathbf{x}^* + \mathbf{e}_1 & \text{if } \mathbf{e}_{x+d} = (\mathbf{x} + \mathbf{d}) - \mathbf{e}_1, \\ \mathbf{x}^* + \mathbf{e}_2 & \text{if } \mathbf{e}_{x+d} = (\mathbf{x} + \mathbf{d}) - \mathbf{e}_2. \end{cases} \]
In other words, the edge in $L^{i*}$ starting at vertex $x^* = x + \frac{1}{2}d$ will point up or right if the edge in $L^i$ starting at $x + d$ points down or left, respectively. Now, by (2.5) and (2.6),

$$e_{x^*} = \begin{cases} x^* + e_1 & \text{if } B^{i*}(x^* + e_1) < B^{i*}(x^* + e_2), \\ x^* + e_2 & \text{if } B^{i*}(x^* + e_2) < B^{i*}(x^* + e_1), \end{cases}$$

where $B^{i*}(x^*) := B^i(x)$. Let $\phi : x \in \mathbb{Z}^2 \mapsto \phi(x) := (-x)^* \in \mathbb{Z}^{2*}$ and set

$$\tilde{B}(x) := -B^{i*}(\phi(x)).$$

Then we have that $\phi^{-1}(L^{i*})$ can be represented as the set composed of down-left oriented edges $(x, e_x)$ such that

$$e_x = \begin{cases} x - e_1 & \text{if } \tilde{B}(x - e_1) > \tilde{B}(x - e_2), \\ x - e_2 & \text{if } \tilde{B}(x - e_2) > \tilde{B}(x - e_1). \end{cases}$$

Or, equivalently,

$$\phi^{-1}(L^{i*}) = \Upsilon(\tilde{B}).$$

By Theorem 1

$$\{\tilde{B}(x) : x \in \mathbb{Z}^2\} \overset{\text{dist.}}{=} \{B^{i*}(x) : x \in \mathbb{Z}^2\}.$$

Hence,

$$\phi^{-1}(L^{i*}) = \Upsilon(\tilde{B}) \overset{\text{dist.}}{=} \Upsilon(B^i) = L^i,$$

and the proof of self-duality is completed.

By self-duality, all almost sure statements for $L$ also hold for $L^*$. Therefore, a.s. $L^*$ is a tree. If, with positive probability, there were a bi-infinite path in $L$, then the dual system $L^*$ would be split into two disjoint parts, which can not happen, since $L^*$ is a.s. a tree.

\[\square\]

2.4. The duality formula. The construction of the LPP model with boundary by using Busemann functions also allows us to interpret exit points as crossing points of semi-infinite geodesics. For $x, n \geq 1$, let $Z^i(x, n)$ denote the first point in $\gamma^i((x, n))$ (following the down-left orientation) that intersects $[1, x] \times \{0\} \cup \{0\} \times [1, n]$. Notice that this intersection has to be transversal to the axis. Again, to distinguish between crossings via the horizontal or the vertical axis, we introduce a non-zero integer-valued random variable $Z^\perp$ such that if $Z^\perp > 0$ then the crossing point is $(Z^\perp, 0)$, while if $Z^\perp < 0$ then the crossing point is $(0, -Z^\perp)$. Define the crossing-point process as

$$Z^\perp_n := (\zeta^\perp_n(z), z \in [-n, \infty)) \in [0, 1][-n, \infty],$$

where, for fixed $n \geq 1$,

$$\zeta^\perp_n(z) = \begin{cases} 1 & \text{if } z = Z^i(x, n) \text{ for some } x \in [1, \infty), \\ 0 & \text{otherwise.} \end{cases}$$

In [5], it was proved that

$$(2.7) \quad Z^\perp_n \overset{\text{dist.}}{=} Z_n.$$ 

A key observation is that the coalescence time $T^{i*}_m := T_m(L^{i*})$ of the dual tree and the crossing point process $Z^\perp_n$ are related by:

$$(2.8) \quad \{T^{i*}_m < n\} = \{Z^\perp_n([-m, m]) = 0\}.$$ 

This is a topological consequence of the fact that, by definition, $L^i$ and $L^{i*}$ do not cross each other. Hence, if $T^{i*}_m < n$, then the dual paths emanating from $m^{i*}$ and $m^{ii*}$ prevent that
$Z^+(x,n) \in [-m,m]$ for any $x \geq 1$, and vice-versa (recall that $Z^+$ is the transversal intersection point). Now we are able to prove the duality formula, given by the theorem below.

**Theorem 3.** The coalescence time $T_m$ and the exit-point process $Z_n$ are related by the formula

$$P(T_m < n) = P(Z_n([-m,m]) = 0), \text{ for } n > m.$$  

**Proof.** Recall that $T_m = T_m(\mathcal{L})$ (it is a deterministic function of the tree $\mathcal{L}$). By self-duality, we have that

$$T_m(\mathcal{L}) \overset{\text{dist.}}{=} T_m(\mathcal{L}^{\ast}).$$

Therefore, by (2.9), (2.8) and (2.7) (in this order),

$$P(T_m < n) = P(T_m^{\ast} < n) = P(Z_n^+[-m,m] = 0) = P(Z_n[-m,m] = 0),$$

and the proof of Theorem 3 is finished.

\[\square\]

2.5. Lower bounds for the tail distribution.

**Corollary 1.** Let

$$G(r) := \liminf_{m \to \infty} P\left( \frac{T_m}{2^{-5/2}m^{3/2}} > r \right).$$

Then there exist constants $c_0, r_0 > 0$ such that for all $r \in [0,r_0]$ we have

$$G(r) \geq 1 - c_0r^2.$$  

In particular,

$$\lim_{r \to 0} G(r) = 1.$$  

**Proof.** Denote $Z_n := Z(n,n)$. By Theorem 2.2 in [3], there exists a constant $c > 0$ such that if

$$\lim_{m \to \infty} n/m^{3/2} = r > 0$$

(where $n = n(m)$), then

$$\limsup_{m \to \infty} P(|Z_n| \geq m) \leq cr^2.$$  

On the other hand,

$$P(\mathcal{Z}_{n+1}[-m,m] \geq 1) \geq P(\mathcal{Z}_{n+1} \in [-m,m]).$$

Together with the duality formula, this yields to

$$P(T_m > n) \geq P(\mathcal{Z}_{n+1} \in [-m,m]),$$

and hence

$$\liminf_{m \to \infty} P(T_m > n) \geq 1 - \limsup_{m \to \infty} P(|Z_{n+1}| > m) \geq 1 - cr^2$$

as soon as $n/m^{3/2} \to r$.

\[\square\]

\[1\text{Notice that } |Z_n| = Z_{n+} + Z_{n-} \text{ and that } Z_{n+} \overset{\text{dist.}}{=} Z_{n-}.\]
The last-passage time $\bar{L}$ has a variational representation given by
\begin{equation}
\bar{L}(x, n) = \max_{z \in [-n, x]} \{ M(z) + L_z(x, n) \}, \text{ for } x, n \geq 1,
\end{equation}
where $M(z)$ is the sum of the (i.i.d. Exp(1/2)) passage times along the boundary,
\begin{equation*}
M(z) := \begin{cases} 
0, & \text{if } z = 0; \\
\sum_{k=1}^{z} W(k, 0), & \text{if } z > 0; \\
\sum_{k=1}^{-z} W(0, k), & \text{if } z < 0,
\end{cases}
\end{equation*}
and
\begin{equation*}
L_z(x, n) := \begin{cases} 
L(0, (x, n)), & \text{if } z = 0; \\
L((z, 0), (x, n)), & \text{if } z > 0; \\
L((0, -z), (x, n)), & \text{if } z < 0.
\end{cases}
\end{equation*}
Therefore
\begin{equation}
\bar{L}(x, n) = M(Z) + L_Z(x, n),
\end{equation}
or, in other words, exit-points of geodesics are locations of maxima:
\begin{equation}
Z(x, n) = \arg \max_{z \in [-n, x]} \{ M(z) + L_z(x, n) \}.
\end{equation}
This variational representation for exit points, together with the scaling limit of last-passage times, implies a limit theorem for $Z_n$.

**Theorem 4.** Define
\begin{equation*}
U_n := \frac{Z_n}{2^{5/3} n^{2/3}},
\end{equation*}
and
\begin{equation*}
U = \arg \max_{u \in \mathbb{R}} \left\{ \sqrt{2} B(u) + A(u) - u^2 \right\},
\end{equation*}
where $B$ is a two-sided standard Brownian motion and $A$ is an independent Airy2 process. Then $U$ is a well defined random variable (the location of maxima is a.s. unique) and
\begin{equation*}
\lim_{n \to \infty} U_n \overset{\text{dist.}}{=} U.
\end{equation*}

**Proof.** We present a sketch of the proof and leave further details to the reader. It follows a similar structure as in the proof of convergence of the location of maxima in the point to line LPP model, developed in [14]. The first ingredient is the following functional limit result
\begin{equation}
\lim_{n \to \infty} A_n(u) \overset{\text{dist.}}{=} A(u),
\end{equation}
where
\begin{equation*}
A_n(u) := \frac{L^{2^{5/3} n^{2/3}}(n, n) - (4n - 2^{8/3} un^{2/3}) + 2^{4/3} u^2 n^{1/3}}{2^{4/3} n^{1/3}}.
\end{equation*}
For finite dimensional convergence see [14], and for tightness see [6]. By the functional central limit theorem, we have that
\begin{equation}
\lim_{n \to \infty} B_n(u) \overset{\text{dist.}}{=} \sqrt{2} B(u),
\end{equation}
where
\begin{equation*}
B_n(u) := \frac{M(2^{5/3} un^{2/3}) - 2^{8/3} un^{2/3}}{2^{4/3} n^{1/3}}.
\end{equation*}
Let
\[ C_n := \bar{L}(n, n) - 4n \frac{1}{2^{5/3}n^{1/3}}. \]

By (2.10), we have that (for \( c = 2^{-5/3} \))
\[ C_n = \max_{u \leq cn^{1/3}} \{ B_n(u) + A_n(u) - u^2 \}, \]
and hence (notice that \( A_n \) and \( B_n \) are independent),
\[ \lim_{n \to \infty} C_n \overset{\text{dist.}}{=} \max_{u \in \mathbb{R}} \left\{ \sqrt{2}B(u) + A(u) - u^2 \right\}. \]

See [2] for a description of the limit law of \( C_n \), and [17] for more details on variational problems involving the Airy process and the Brownian motion.

By Theorem 2.2 in [3], \( (U_n)_{n \geq 1} \) is tight and, by (2.11),
\[ U_n = \arg \max_{u \leq cn^{1/3}} \{ B_n(u) + A_n(u) - u^2 \}. \]

Therefore, Theorem 4 will follow as soon as the location of maxima of the limit process is a.s. unique (to have continuity of the arg max functional). The method of proof developed in [16] to show uniqueness of the location of maxima for a two-sided Brownian motion minus a parabola (Theorem 3 [16]), and also for a stationary process minus a parabola (Theorem 4 [16]), can be used mutatis mutandis to prove the same result in our context as well. We leave further details for the reader.

□

Now, we apply Theorem 4 to lower bound \( G \).

**Corollary 2.** Let
\[ F(s) := \mathbb{P}(U \leq s). \]

Then
\[ G(r) \geq F(r^{-2/3}) - F(-r^{-2/3}). \]

**Proof.** As we saw in the proof of the previous corollary,
\[ \mathbb{P}(T_m > n) \geq \mathbb{P}(Z_{n+1} \in (-m, m]), \]
and hence
\[ \mathbb{P}\left( \frac{T_m}{2^{-5/2}m^{3/2}} > \frac{n}{2^{-5/2}m^{3/2}} \right) \geq \mathbb{P}\left( U_{n+1} \in \left( -\frac{m}{2^{5/3}n^{2/3}}, \frac{m}{2^{5/3}n^{2/3}} \right) \right). \]
If we take \( m, n \) such that \( n/2^{-5/2}m^{3/2} \to r \), then \( m/2^{5/3}n^{2/3} \to r^{-2/3} \). Thus, by Theorem 4
\[ G(r) \geq \mathbb{P}\left( U \in (-r^{-2/3}, r^{-2/3}] \right). \]

□
3. Final Comments

3.1. Duality in the scaling limit. Define the rescaled processes
\[ A_n(u, v) := \frac{L_{2^{5/3}n^{2/3}}(n + 2^{5/3}vn^{2/3}, n) - (4n + 2^{8/3}(v-u)n^{2/3}) + 2^{4/3}(u-v)^2n^{1/3}}{2^{4/3}n^{1/3}}, \]

and
\[ C_n(v) := \frac{\bar{L}(n, n + 2^{5/3}vn^{2/3}) - (4n + 2^{8/3}vn^{2/3})}{2^{4/3}n^{1/3}}. \]

By (2.10), we have that
\[ C_n(v) = \max_{u \leq n^{1/3}} \left\{ B_n(u) + A_n(u, v) - (u-v)^2 \right\}. \]

The process \( C_n(v) \) has a limit \( \lim_{n \to \infty} C_n(v) \) dist. \( C(v) \),
whose finite dimensional distributions are also expressed in terms of Fredholm determinants. It is known that the sequence \( (A_n)_{n \geq 1} \) is tight (in the space of two parameter continuous processes), although no rigorous result on the convergence of finite dimensional distributions is available \( \cite{6, 8} \). For fixed \( u \), it is not hard to see that, for fixed \( v \in \mathbb{R} \),
\[ \lim_{n \to \infty} A_n(u, v) \] dist. \( A(u-v) \),
as a process in \( u \in \mathbb{R} \). It is conjectured that \( A_n(u, v) \) indeed converges to a two parameter process \( (A(u,v), (u,v) \in \mathbb{R}^2) \), called the Airy_2 sheet \( \cite{5} \). The Airy_2 sheet is symmetric and stationary process with continuous paths. These limit processes are related to each other by the variational relation
\[ C(v) - C(0) \] dist. \( \sqrt{2}B(v) \), for \( v \in \mathbb{R} \) (as process)
where
\[ C(v) := \max_{u \in \mathbb{R}} \left\{ \sqrt{2}B(u) + A(u,v) - (u-v)^2 \right\}. \]

Consider the jump process \( (U(v), v \in \mathbb{R}) \) which runs through the (right-most) locations of maxima:
\[ U(v) := \sup \arg \max_{u \in \mathbb{R}} \left\{ \sqrt{2}B(u) + A(u,v) - (u-v)^2 \right\}. \]

By stationarity of the Airy_2 sheet and shift invariance of the two-sided Brownian motion, the process \( (U(v) - v, v \in \mathbb{R}) \) will be stationary. It is also known that \( \mathbb{E}U(0) = 0 \), and hence
\[ \mathbb{E}U(v) = v. \]

Define the counting process \( U := (\zeta(u), u \in \mathbb{R}) \) induced by the locations of maxima:
\[ \zeta(u) = \begin{cases} 1 & \text{if } u = U(v) \text{ for some } v \in \mathbb{R}, \\ 0 & \text{otherwise}. \end{cases} \]

and
\[ U(A) := \sum_{u \in A} \zeta(u). \]

Based on the variational representation (2.11) of exit points, we conjecture that the exit-point process \( Z_n \) converges to \( U \), after rescaling space by \( 2^{5/3}n^{2/3} \). Therefore, by duality (1.1), if this conjecture is true, one gets the existence of the limit coalescence time,
\[ T \] dist. \( \lim_{m \to \infty} \frac{T_m}{2^{-5/2}m^{3/2}}, \]
and
\[ P(T \leq r) = P\left( U\left( (-r^{-2/3}, r^{-2/3}) \right) = 0 \right). \]

We also expect that
\[ \lim_{r \to \infty} \frac{G(r)}{r^{2/3}} = 2\lambda. \]

where
\[ \lambda := \lim_{\delta \to 0^+} \frac{1 - \mathbb{P}(U((0, \delta]) = 0)}{\delta}. \]

A natural guess for \( \lambda \) is \( \lambda = f(0) \), where \( f \) is the density function of \( U \). (Notice that \( \lambda \geq f(0) \).)

The Airy2 sheet can also be seen as a space-time parameter process \( A(s, u; t, v) \), where \( A(u, v) = A(0, u; 1, v) \). This space-time process is conjectured to be the space-time scaling limit of last-passage percolation models, and also of solutions to the Kadar-Parisi-Zhang equation [8]. It induces a random semi-group \( T_{s,t} \), acting on functions \( f \) by the variational formula
\[ C_{s,t}(f)(v) := \max_{u \in \mathbb{R}} \left\{ f(u) + A(s, u; t, v) - \frac{(u - v)^2}{(t - s)} \right\}. \]

The two-sided Brownian motion is a fixed point in the sense that
\[ C_{0,t}(B)(v) - C_{0,t}(B)(0) \overset{\text{dist.}}{=} B(v) \text{ for all } t \geq 0. \]

In this context, one could consider the counting process \( U_t \) induced by the locations of maxima of \( C_{0,t}(B) \):
\[ U(v, t) := \sup \arg \max_{u \in \mathbb{R}} \left\{ \sqrt{2}B(u) + A(0, u; t, v) - \frac{(u - v)^2}{t} \right\}. \]

It seems natural to think in terms of the trajectories of the locations, and that these locations will coalesce as time passes. We conjectured that this collection of coalescing trajectories is the scaling limit of the geodesic tree.

### 3.2. Coalescence times and local equilibrium.

The LPP model can be represented as a discrete time Markov interacting system \( (M_n : n \geq 0) \) on \([0, \infty)^\mathbb{Z} \) [5]. (See also [1] for the Hammersley LPP model and its particle system interpretation.) At time zero we start with a collection of non-negative weights \( \{W_i : i \in \mathbb{Z}\} \). We define the weight (or mass) of the interval \((a, b]\) at time zero as
\[ M(a, b) = M_0(a, b) := \sum_{i=a+1}^{b} W_i. \]

At time \( n \geq 1 \), we define the weight of the interval \((a, b]\) as
\[ M_n(a, b) := \tilde{L}_M(b, n) - \tilde{L}_M(a, n) \]

where\(^2\)
\[ \tilde{L}_M(x, n) := \max_{z \leq x} \{ M(z) + L_z(x, n) \}, \]

and
\[ M(z) := \begin{cases} 0, & \text{if } z = 0; \\ \sum_{k=1}^{z} W_k, & \text{if } z > 0; \\ -\sum_{k=1}^{-z} W_k, & \text{if } z < 0. \end{cases} \]

\(^2\)We assume that \( \lim \inf_{k \to \infty} \frac{\sum_{i=-k}^{-1} W_i}{k} > 1 \) so that the maxima is indeed attained on a compact set.
The last-passage time $L$ can be recovered by choosing an initial weight configuration with infinite mass at negative sites, and with i.i.d exponential weights of parameter one at non-negative sites. Time stationary ergodic measures on $[0, \infty)^\mathbb{Z}$ for this Markov system are represented by i.i.d. collections of exponentials random weights of parameter $\rho \in (0, 1)$: if $\{W_i : i \in \mathbb{Z}\}$ is distributed according to an i.i.d. collection of $\text{Exp}(\rho)$ random variables, then

$$M_n \overset{\text{dist.}}{=} M_0, \text{ for all } n \geq 0.$$ 

Local equilibrium of the LPP interacting system is described by Busemann functions \[5\]:

$$\lim_{n \to \infty} L(0, (n+h, n)) - L(0, (n, n)) \overset{\text{dist.}}{=} B\downarrow(0, (0, h)) = M_0(0, h),$$

with $\rho = 1/2$. If one moves the origin to $-n := -(n, n)$ (and starts the system at time $-n$) then the convergence becomes a.s.:

$$\lim_{n \to \infty} L(-n, (h, 0)) - L(-n, 0) \overset{\text{a.s.}}{=} L(c_h, (0, h)) - L(c_h, 0) = B\downarrow(0, (0, h)),$$

where $c_h := c\downarrow(0, (0, h))$. Thus, the coalescence time also describes how far in the past one needs to start the process to see local equilibrium in the present. In this sense, it would be interesting to analyze coalescence times and duality \[1.1\] in the framework of relaxation and mixing times for Markov processes.

**Acknowledgments.** The author would like to say thanks to Timo Seppäläinen for helpful comments on a previous version of this manuscript.

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