Free Fermions Violate the Area Law For Entanglement Entropy

Robert C. Helling\textsuperscript{1} and Wolfgang Spitzer\textsuperscript{2},
\textsuperscript{1} Arnold Sommerfeld Center, Ludwig-Maximilians-Universität München
\textsuperscript{2} Institut für Theoretische Physik, Universität Erlangen-Nürnberg

Key words Entanglement entropy, Area Law, LMU-ASC-08-10

We show that the entanglement entropy associated to a region grows faster than the area of its boundary surface. This is done by proving a special case of a conjecture due to Widom that yields a surprisingly simple expression for the leading behaviour of the entanglement entropy.

1 Background

An interesting model for the area law of Bekenstein’s black-hole entropy is a local quantum field theory for which one restricts all observations to the complement of a compact spatial region \( \Omega \). One should think of \( \Omega \) as the interior of the horizon although one does not require \( \Omega \) to have any gravitational relevance. In fact, we will restrict our attention to a region in the \( n \)-dimensional flat Euclidean space \( \mathbb{R}^n \). Even if the quantum field is globally in a pure state (e.g., the vacuum), the state restricted to the complement of \( \Omega \) will in general be mixed with finite von Neumann entropy, called the entanglement entropy, \( S(\Omega) \) \textsuperscript{[1, 2, 3]}. For many types of quantum field theories, it was observed that in the semi-classical limit of large \( \Omega \), the entanglement entropy \( S(\Omega) \) scales like the area of \( \partial \Omega \). In other words, if we rescale \( \Omega \subset \mathbb{R}^n \) by a factor \( R \), the entanglement entropy, \( S(R\Omega) \), should asymptotically scale like \( R^{n-1} \) for large \( R \).

Recently, it was observed by Gioev and Klich \textsuperscript{[4, 5]}, that for free fermions at zero temperature, however, the entanglement entropy of the ground state scales like \( R^{n-1} \log(R) \) for \( R \to \infty \) based on a conjecture by Widom \textsuperscript{[6, 7]}. They provide a lower bound on the entanglement entropy in terms of the trace of a quadratic function of the restricted state. For the latter, we have proved in \textsuperscript{[8]} the leading asymptotic behaviour. This establishes indeed that the entanglement entropy scales at least like \( R^{n-1} \log(R) \), which violates of the beforementioned area law scaling.

For notational clarity we will often write equalities that only hold asymptotically for large \( R \), that is, we will drop subleading terms that are not central to our argument and only mention it in the text.

General arguments imply that for a pure state, the entanglement entropy of this state with respect to \( \Omega \) is the same as for the complement \( \mathbb{R}^n \setminus \Omega \). For simplicity, we will restrict the state to a compact region \( \Omega \). The ground state, \( \rho_G \), of a system of non-interacting fermions in \( \mathbb{R}^n \) is given in terms of the Fermi surface at Fermi energy \( \epsilon_F \): All one-particle states with momentum \( p \in \Gamma = \{ p \in \mathbb{R}^n | E(p) \leq \epsilon_F \} \) and energy \( E(p) \) are occupied. The ground state \( \rho_G \) is then characterised by the one-particle Fermi projector, \( P_{\Gamma} \), defined in momentum space by the kernel \( P_{\Gamma}(p, p') = \chi_{\Gamma}(p) \delta(p - p') \). Here, and in the following, \( \chi_{\Gamma} \) denotes the indicator function of a}

\textsuperscript{1} We will not discuss here assumptions on the dispersion relation \( E(p) \) but one may, of course, think of the example \( E(p) = p^2 \). We require that \( \Gamma \) and \( \Omega \) are compact sets in \( \mathbb{R}^n \) with sufficiently smooth boundaries.

Copyright line will be provided by the publisher
set $A$. In position space, the kernel $P_{\Gamma}(x,x') = \hat{\chi}_{\Gamma}(x-x')$ with $\hat{\chi}_{\Gamma}(x) = (2\pi)^{-n} \int dp \, e^{i \mathbf{x} \cdot \mathbf{p}}$ being the inverse Fourier transform of $\chi_{\Gamma}$.

In order to restrict the state $\rho_{\Gamma}$ to the region $\Omega$ we project the Fermi projector $P_{\Gamma}$ onto $L^2(\Omega)$ with $Q_{\Omega} = \chi_{\Omega}$. This gives the reduced one-particle density matrix $\rho_{\Omega,\Gamma} = Q_{\Omega} P_{\Gamma} Q_{\Omega}$. The entanglement entropy, $S(\Omega, \Gamma)$, of the many particle system in the ground state $\rho_{\Gamma}$ restricted to the region $\Omega$ is then defined as the grand canonical entropy of $\rho_{\Omega,\Gamma}$, that is, $S(\Omega, \Gamma) = \text{tr} \eta(\rho_{\Omega,\Gamma})$ with $\eta(t) = -t \log(t) - (1- t) \log(1-t)$ for $0 < t < 1$. For details see [5 Section 4]. We are interested here in the behaviour of this entropy for fixed $\Gamma$ but large $\Omega$. To this end, we also fix $\Omega$ and study the asymptotic behaviour of $S(R\Omega, \Gamma)$ as $R \to \infty$.

Our main result is the asymptotic computation of $\text{tr}[\rho_{R\Omega,\Gamma}(1 - \rho_{R\Omega,\Gamma})]$ as $R \to \infty$:

\[
\text{tr}[\rho_{R\Omega,\Gamma}(1 - \rho_{R\Omega,\Gamma})] = \left(\frac{R}{2\pi}\right)^{n-1} \frac{\log(R)}{4\pi^2} \int_{\partial\Omega \times \partial\Gamma} dA(x)dA(p) \, |n_x \cdot n_p|, \tag{1}
\]

up to terms that grow slower in $R$. Here, $n_x$ denotes the unit normal vector at $x \in \partial \Omega$, $dA(x)$ is the surface measure on $\partial \Omega$, and similarly for $dA(p)$ and $dA(p)$.

Since $\eta(t) \geq \log(2) 4t(1-t)$ for $0 < t < 1$, we obtain the asymptotic lower bound on the entanglement entropy,

\[
S(R\Omega, \Gamma) \geq \frac{\log(2)}{\pi^2} \left(\frac{R}{2\pi}\right)^{n-1} \log(R) \int_{\partial\Omega \times \partial\Gamma} dA(x)dA(p) \, |n_x \cdot n_p|, \tag{2}
\]

Gioev and Klich conjectured in [3] that the exact scaling of $S(R\Omega, \Gamma)$ is obtained if we replace the factor $\log(2)/\pi^2$ in (2) by $\frac{1}{16}$. This remains an open problem.

## 2 Computation of $\text{tr}[\rho_{R\Omega,\Gamma}(1 - \rho_{R\Omega,\Gamma})]$

The trace of $\rho_{R\Omega,\Gamma}$ is simply equal to $(R/2\pi)^n |\Omega||\Gamma|$, where $| \cdot |$ denotes the $n$-dimensional Lebesgue volume. The trace of $\rho_{R\Omega,\Gamma}^2$ equals

\[
\text{tr}(Q_{R\Omega} P_{\Gamma} Q_{R\Omega} P_{\Gamma}) = \int dx \int_{R\Omega} dx' |\hat{\chi}_{\Gamma}(x-x')|^2 = \int_{R(\Omega-\Omega)} dv |\hat{\chi}_{\Gamma}(v)|^2 |R\Omega \cap (R\Omega - v)|. \tag{3}
\]

In the last line we have changed the variables $x$ and $x'$ to $u = x$ and $v = x - x'$. Then we expand the volume $|R\Omega \cap (R\Omega - v)|$ to first order in $v$ (cf. [3 Theorem 2.1]),

\[
|R\Omega \cap (R\Omega - v)| = R^n |\Omega| + R^{n-1} \int_{\partial\Omega} dA(x) \max(0,v \cdot n_x) + R^{n-2}O(|v|^2). \tag{4}
\]

Let us first look at the contribution of $R^n |\Omega|$ to the trace of $\rho_{R\Omega,\Gamma}^2$. The function $v \mapsto |\hat{\chi}_{\Gamma}(v)|$ decays like $|v|^{-n(n+1)/2}$ for large $|v|$, see [7]. At the cost of an order $R^{n-1}$-term we may therefore extend the $v$-integration to all of $\mathbb{R}^n$. By the Plancherel formula this integral gives $(2\pi)^{-n} |\Gamma|$ and cancels with $\text{tr} \rho_{R\Omega,\Gamma}$. When integrated over $v$, the remainder term $R^{n-2}O(|v|^2)$ is also seen to yield a term of the order $R^{n-1}$ by using again the above mentioned decay of $\hat{\chi}_{\Gamma}$.

Thus, our bound on the entanglement entropy will come from the second term in (4). Here, we write $\max(0,v \cdot n_x) = \chi_{[0,\infty)}(v \cdot n_x) v \cdot n_x$ and use the Gauß Theorem so that

\[
(2\pi)^n v \hat{\chi}_{\Gamma}(v) = -i \int_{\partial\Gamma} dA(p) \, n_p e^{i v \cdot p}. \tag{5}
\]

So it remains to show that for $p \in \partial\Gamma$,

\[
\left| \int_{R(\Omega-\Omega)} dv \chi_{[0,\infty)}(v \cdot n_x) \hat{\chi}_{\Gamma}(-v) e^{i v \cdot p} + (2\pi)^{-1} \text{sgn}(n_x \cdot n_p) \log(R) \right| = o(R). \tag{6}
\]
Let us now consider the function $\mathbf{v} \mapsto \hat{\chi}_r(-\mathbf{v})$ in detail. We use the representation from (6), that is, $(2\pi)^{-n} \hat{\chi}_r(-\mathbf{v}) = \frac{1}{\sqrt{v}} \frac{i}{2} \int_{\partial \Omega} dA(p') \mathbf{p}' e^{-iv \cdot \mathbf{p}'}$. Then we introduce a coordinate system where $\mathbf{v} = \mathbf{v}(0, \ldots, 0, v)$ and where the boundary $\partial \Gamma$ is locally written as the graph of a function $f : U \subset \mathbb{R}^{n-1} \to \mathbb{R}$, that is, $p = (t, f(t))$ and $dA(p') = \sqrt{1 + |\nabla f|^2} dt$. The unit normal vector is $\mathbf{n}_{p'} = \text{sgn}(\mathbf{v} \cdot \mathbf{p}')(-\nabla f, 1)/\sqrt{1 + |\nabla f|^2}$. Then,

$$\int_{\partial \Omega} dA(p') (\mathbf{n}_{p'})_n e^{-iv \cdot \mathbf{p}} = -\frac{1}{v} \int_{\partial \Omega} dt \text{ sgn}(f(t)) e^{-ivf(t)}.$$ 

In order to find the leading asymptotic behaviour of this $t$-integral for large $v$ we apply the method of stationary phase. Let $k_a = k_a(\mathbf{v}) = (t_a(\mathbf{v}), f(t_a(\mathbf{v})))$ be the collection of all stationary points of such local functions $f$, that is, $\nabla f(t_a) = 0$; in other words, the points $k_a \in \partial \Gamma$ are such that the unit normal vector $\mathbf{n}_{k_a}$ at $k_a$ is parallel to $\mathbf{v}$. Thus,

$$\hat{\chi}_r(-\mathbf{v}) = -i(2\pi v)^{-(n+1)/2} \sum_{k_a} \frac{\text{sgn}(\mathbf{v} \cdot k_a)}{\sqrt{|\det f_{ij}(t_a)|}} e^{-iv \cdot k_a - i\frac{v}{2} \text{sgn}(f(t_a))} + o(v^{-(n+1)/2}).$$

(7)

Here, $f_{ij}(t_a)$ denotes the Hessian of $f$ at $t_a$, $\text{sgn}(f_{ij}(t_a))$ the sign of this Hessian. The determinant, $\det f_{ij}(t_a)$, equals the Gaussian curvature of $\partial \Gamma$ at $k_a$.

Using (4) we return to the oscillatory integral in (3), and employ once more the method of stationary phase. The composite phase from (5) and (7) is equal to

$$\int_{\partial \Omega} dA(p') (\mathbf{n}_{p'})_n e^{-iv \cdot \mathbf{p}}.$$ 

Using (7) we return to the oscillatory integral in (6), and employ once more the method of stationary phase. The composite phase from (5) and (7) is equal to

$$\int_{\partial \Omega} dA(p') (\mathbf{n}_{p'})_n e^{-iv \cdot \mathbf{p}}.$$ 

Using (4) we return to the oscillatory integral in (3), and employ once more the method of stationary phase. The composite phase from (5) and (7) is equal to $\mathbf{v} \cdot (\mathbf{p} - k_a(\mathbf{v}))$. Next, we introduce generalised spherical coordinates for $\mathbf{v}$ as $\mathbf{v} = \rho(\mathbf{u}, h(\mathbf{u}))$, where the map $h : V \subset \mathbb{R}^{n-1} \to \mathbb{R}$ locally parametrises the boundary $\partial (\Omega - \Omega') \ni (\mathbf{u}, h(\mathbf{u}))$ and $\rho \in [0, R]$ is a radial coordinate. The stationary point $k_a$ is now a function of $\mathbf{u}$. We have the freedom to assume that $\mathbf{n}_p = (0, \ldots, 0, 1)$ such that $k_a((0, h(0))) = \mathbf{p}$ for one index $a$ and that $h$ has its extremum at the origin $\mathbf{u} = 0$. Instead of integrating $\rho$ from 0 to $R$ we will actually integrate $\rho$ only over $[1, R]$ thereby making an irrelevant error independent of $R$.

To express $k_a(\mathbf{v})$ as a function of $\mathbf{u}$ we equate $\mathbf{v} / \mathbf{u} = \mathbf{n}_{k_a(\mathbf{v})}$, that is,

$$\frac{(\mathbf{u}, h(\mathbf{u}))}{\sqrt{\mathbf{u}^2 + h(\mathbf{u})^2}} = \text{sgn}(\mathbf{v} \cdot \mathbf{n}_{k_a}) \frac{(-\nabla f, 1)}{\sqrt{1 + |\nabla f|^2}}.$$ 

Taking derivatives and evaluating at $\mathbf{u} = 0$, we find $\frac{\partial}{\partial u_i}(0) = -f_{ij}^{-1}(k_a(0)) / h(0)$, where $k_a(0)$ is short for $k_a(0, h(0))$ and $f_{ij}^{-1}$ is the matrix inverse of the Hessian of $f$. With this, we can expand the phase of the remaining $\mathbf{v}$-integral to second order as

$$\mathbf{v} \cdot (\mathbf{p} - k_a(\mathbf{v})) = \rho h(0)(\mathbf{p} - k_a(0))_n + \rho f_{ij}^{-1}(k_a(0)) u_i u_j.$$ 

(9)

The volume element is given by $d\mathbf{v} = \rho^{n-1} h(\mathbf{u}) d\rho d\mathbf{u}$. The stationary phase integral over the $n - 1$ coordinates $\mathbf{u}$ yields $(2h(0)/\rho)^{(n-1)/2} \sqrt{|\det f_{ij}^{-1}|} \exp(i\rho h(0)(\mathbf{p} - k_a(0)))_n + i\frac{v}{2} \text{sgn}(f_{ij}(t_a))$.

This, surprisingly, reduces the remaining $\rho$-integral to $\int_0^R d\rho e^{i\rho h(0)(\mathbf{p} - k_a(0))_n} / \rho$. As noted above, for one index $a$, the exponent vanishes as $\mathbf{p} = k_a(0)$ and the integral is the desired $\log(R)$. If, however, $\mathbf{p} \neq k_a(0)$ the integral is bounded for large $R$ and does not contribute to leading order.

Collecting all terms we have thus proved the lower bound (4) which grows faster than the area law scaling $R^{n-1}$ by a factor of $\log(R)$. Note that the stationary phase integrals localises $\mathbf{v}$ and $\mathbf{p}'$ such that there are only contributions from $\mathbf{p}$, $\mathbf{n}_v$, $\mathbf{n}_{p'}$ all being parallel.

### 3 Discussion

Above, we presented a lower bound for the entanglement entropy which violates an $R^{n-1}$ area law scaling. The expression (2) is in terms of integrals over the boundaries $\partial \Omega$ and $\partial \Gamma$ and is thus still a “boundary effect”.

Copyright line will be provided by the publisher.
We find it most curious that although we have used stationary phase methods, which in general depend on (the existence of) second order derivatives at the stationary points, all these curvature terms involving $f_{ij}$ and $h_{ij}$ eventually cancel out and the integrand in the bound does not contain derivatives. However, one expects \cite{4, 10} that a fractal boundary $\partial \Omega$ of dimension $n - 1 + \alpha$ with $0 < \alpha < 1$ leads to a scaling of the entanglement entropy of at least of the order $R^{n-1+\alpha}$ and, presumably, without a $\log(R)$ correction.

For $n = 1$, if $\Omega$ and $\Gamma$ are a disjoint union of $k$ and $\ell$ compact intervals of finite length respectively, then our lower bound \cite{2} for the entanglement entropy gives $\log(R) \log(2) 4k\ell / \pi^2$. In this one-dimensional case, the precise scaling has been proved, namely, $\log(R) k\ell / 3$. If we then consider the hypercubes, say $\Omega = \Gamma = [0, 1]^n$, it is not difficult to derive the exact asymptotic scaling of $S(R\Omega, \Gamma)$ to be $(R/(2\pi))^{n-1} \log(R)n^2/3$, which is in agreement with the conjecture by Gioev and Klich mentioned at the end of Section 1. Our method of proof requires that the surfaces $\partial \Omega$ and $\partial \Gamma$ are $C^3$. Hence, hypercubes are not included. For a $C^3$ surface, $\partial \Gamma$, it was crucial that $\hat{\chi}(v)$ behaves like $|v|^{-(n+1)/2}$ for large $|v|$. This is not the case for a non-smooth surface such as the hypercube, where $\hat{\chi}[0,1]^n(v) = \prod_{i=1}^n \frac{\sin(\pi v_i)}{\pi v_i}$.

It should be noted as well that the discontinuity of $\chi$ is crucial for the decay of its Fourier transform. For example, for the equilibrium state at positive temperature, the entanglement entropy scales like the volume $R^n$, see \cite{5}. We did not have to introduce an ultraviolet regulator since momentum integrations are limited to the compact region $\Gamma$.

Acknowledgments

We are grateful to Hajo Leschke with whom the work in \cite{8} was performed on which is text is based. Furthermore, RCH would like to thank the Elitenetwork of Bavaria for financial support and Jacobs University Bremen, where this work was started. We also thank Urs Frauenfelder for discussions.

References

\begin{enumerate}
\item M. Srednicki, Phys. Rev. Lett. 71(5), 666–669 (1993).
\item L. Bombelli, R. K. Koul, J. Lee, and R. D. Sorkin, Phys. Rev. D 34(2), 373–383 (1986).
\item J. Eisert, M. Cramer, and M. B. Plenio, Area laws for the entanglement entropy - a review, 2008, arXiv.org:0808.3773.
\item D. Gioev, Int. Math. Res. Not. 2006(O95181), 95181–23 (2006).
\item D. Gioev and I. Klich, Phys. Rev. Lett. 96(10), 100503 (2006).
\item H. Widom, Trans. AMS 94(1), 170–180 (1960).
\item H. Widom, J. Funct. Anal. 88, 166–193 (1990).
\item R. C. Helling, H. Leschke, and W. L. Spitzer, A special case of a conjecture by Widom with implications to fermionic entanglement entropy, 2009, arXiv.org:0906.4946.
\item R. Roccaforte, Trans. AMS 285(2), 581–602 (1984).
\item M. Fannes, B. Haegeman, and M. Mosonyi, J. Math. Phys. 44(12), 6005–6019 (2003).
\end{enumerate}