THE ONE DIMENSIONAL MATRIX MODEL AND STRING THEORY

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ABSTRACT

We discuss the basic features of the double scaling limit of the one dimensional matrix model and its interpretation as a two dimensional string theory. Using the collective field theory formulation of the model we show how the fluctuations of the collective field can be interpreted as the massless "tachyon" of the two dimensional string in a linear dilaton background. We outline the basic physical properties of the theory and discuss the nature of the S-matrix. Finally we show that the theory admits of another interpretation in which a certain integral transform of the collective field behaves as the massless "tachyon" in the two dimensional string with a blackhole background. We show that both the classical background and the fluctuations are non-singular at the black hole singularity.

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1. Matrix Models and Random surfaces

One of the most fruitful approaches to the study of random surfaces and noncritical strings is to consider a dynamical triangulation of a two dimensional surface and then searching for continuum limits.

Discretized random surfaces may be viewed as Feynman diagrams of field theories of matrix fields, or Matrix models [1]. This really follows from ’t Hooft’s discovery that the large-N expansion of certain matrix models ia a topological expansion [2].

Consider a matrix field theory of a $N \times N$ hermitian matrix $M_{ij}(X^\mu)$ in $d$ dimensions. The action is taken to be of the form

$$S_M = N \left[ \int d^dx \int d^dy \, \text{Tr} \frac{1}{2} M(x) G^{-1}(x,y) M(y) + \int d^d x \, \text{Tr} V(M) \right]$$

(1.1)

Here $V(M)$ is some polynomial potential which is of the form

$$V(M) = \sum_k g_k M^k$$

(1.2)

In (1.1) we have an yet unspecified propagator for the matrix, $G(x, y)$. The above field theory has a global $U(N)$ symmetry $M \rightarrow U M U^{-1}$ for an arbitrary constant $U(N)$ matrix $U$. We shall be interested in computing quantities which are invariant under this symmetry.

We shall study this model in the large-$N$ expansion, i.e. we shall expand all quantities in inverse powers of $N$, considering all the couplings $g_k$ to be of order unity. This means that to every order of $\frac{1}{N}$ we have to sum an infinite number of feynman diagrams which have all possible powers of the couplings, but each of which is of the same order of $\frac{1}{N}$.

One can represent each propagator of the matrix field by a double line, each line corresponding to an index of the matrix $M_{ij}$ (hence called
an index line). The two index lines have arrows attached opposite direction [2]. In terms of double lines a typical feynman diagram looks like a two dimensional surface which is broken up into tiles. Each tile is bounded by a closed index loop and adjacent tiles are attached to each other along the boundaries. This, of course, is a discretization of a surface with the following correspondence

- faces → index loops
- edges → propagators
- vertices → interaction vertices

Vacuum diagrams of the field theory represent closed two dimensional surfaces. For general correlators we will have surfaces with points whose positions are not integrated over. Since $N$ appears as a multiplicative factor in front of the action, each propagator brings a factor of $\frac{1}{N}$, each index loop brings a factor of $N$ since indices have to be summed over for invariant quantities, while each vertex also brings a factor of $N$. Thus the total $N$-dependence in any diagram is given by

$$(N)^{V+T-L} = N^\chi$$

where $V$ denotes the number of vertices, $T$ the number of faces and $L$ the number of edges. $\chi$ is the Euler characteristic. This shows that the $\frac{1}{N}$ expansion is a topological expansion, i.e. for a given order of $\frac{1}{N}$ one has to sum all possible diagrams of a given topology. This, of course, is a feature of string perturbation theory and $\frac{1}{N}$ as a string coupling.

Let us examine a bit more detail the expression for the free energy (just as an example) for our matrix field theory. For simplicity we shall consider an action which has a monomial interaction. For example we could take only the coupling $g_4$ defined above to be non-zero. A given
feynman diagram is evaluated in the standard way by joining together vertices with propagators. The real space expression is

\[ g_4^V \int \prod_i dx_i \prod_{<ij>} G(x_i, x_j) \]

where \( V \) is the number of vertices. The product is over all pairs of points connected by a propagator. The result for the free energy is obtained by a sum over all feynman diagrams and finally a sum over all orders of \( \frac{1}{N} \)

\[ F_{MM} = \sum_\chi (\frac{1}{N})^\chi \sum_{\text{diagrams}(N)} g_4^V \int \prod_i dx_i \prod_{<ij>} G(x_i, x_j) \]  

(1.3)

The sum over feynman diagrams in (1.3) is split in accordance with the \( \frac{1}{N} \) expansion, i.e. first we sum over all diagrams in a given order of \( N \) and then sum over the various orders. Note that since each feynman diagram is a discretization of a two dimensional surface, the sum over all closed feynman diagrams is precisely the sum over all possible discretizations of a surface. The expression (1.3) is precisely of the same form as the partition function of a set of scalar fields \( x^\mu \) on a dynamically discretized random surface. Since the number of vertices is a measure of the area, the cosmological constant of the random surface theory is given by

\[ \mu = -\log g_4 \]

In fact if we choose the propagator to be of the exponential form

\[ G(x_i, x_j) = e^{-(x_i - x_j)^2} \]  

(1.4)

we have precisely the discretized version of the Polyakov string. This is because if one is able to define a continuum limit for this model, the factor
\[ \prod_{<ij>} G(x_i, x_j) \text{ becomes } \exp[-\int d^2 \xi (\partial x)^2] \text{ while the sum over all feynman diagrams becomes a functional integral over the worldsheet metric.} \]

Finally the sum over \( \chi \) is the genus expansion so the **string coupling constant** \( g_{st} = \frac{1}{N} \). It is worth remembering that the equivalence is between the *free energy* of the matrix field theory and the *partition function* of the random surface model. The relationship between the cosmological constant and the couplings of the matrix model shows that the continuum limit of the random surface theory corresponds to singularities of the free energy of the matrix model.

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**The Double Scaling Limit**

The main point about matrix models is that they provide a *non-perturbative* definition of random surface theories. As argued above the \( \frac{1}{N} \) expansion of matrix models gives the genus expansion of the random surface theory, i.e. the string perturbation expansion. However the matrix model exists independently of its large-N expansion. This gives a hope that one could study non-perturbative effects in string theories using matrix model methods.

In fact the continuum treatment of David, Distler and Kawai already indicates what should be done in order to define a continuum limit which does not proceed genus by genus. Recall that for a conformal field theory with central charge \( c \) coupled to two dimensional gravity, the partition function behaves as \( (\mu^\alpha)^\chi \), where \( \alpha \) is determined in terms of \( c \) and positive for \( c \leq 1 \) and \( \chi \) is the euler characteristic. In the matrix model we have seen that \( \mu \) in the above expression corresponds to \( (g - g_c) \). Furthermore, the string coupling constant \( g_{st} \) comes raised to the power \( -\chi \). Thus near the critical point the full partition function as a sum over genus has a
The $\frac{1}{N}$ corrections to the planar limit of several solvable matrix models also led to an identical form for the free energy.

The equation (1.5) shows that it is possible to define a **double scaling limit**

$$ g \to g_c \quad g_{st} \to 0 \quad \frac{(g - g_c)^{-\alpha}}{g_{st}} = \text{fixed} \quad (1.6) $$

In the matrix model $g_{st} = \frac{1}{N}$ so that this double scaling limit is described by both $g \to g_c$ as well as $N \to \infty$. In this limit all orders of genus expansion contribute equally and one clearly has a **non-perturbative definition** of the continuum limit of the random surface theory.

The crucial idea of double scaling limit was discovered in [4]. In these papers the one matrix model, whose genus zero solution was already known from the classic paper of [3], was solved *directly* in this limit. The remarkable feature of their work is that the specific heat of the model satisfies a non-linear integrable differential equation (different for different multicritical points) and the flow between various multicritical points is described by euqations of the KdV hierarchy. We shall not describe this seminal work. Rather we shall directly proceed to describe the double scaling limit of the matrix field theory in *one* dimension.
2. The $d = 1$ Matrix model: Double scaling.

We start with the matrix field theory in one dimension which we will identify as time. In other words we are dealing with some matrix quantum mechanics. As seen above the matrix model action which corresponds to the Polyakov action for strings is of the form

$$S = \int dt \text{Tr}[M(t) e^{\partial_t^2 t} M(t) + V(M(t))]$$  \hspace{1cm} (2.1)

For the model to have a smooth large-$N$ limit the couplings in the expansion of the polynomial potential $V(M)$ have to be specific, viz

$$V(M) = \sum_{k=2}^{\infty} \frac{g_k}{N^{k-2}} M^k$$  \hspace{1cm} (2.2)

By rescaling $M$ we can always choose $g_2 = 1$ and we will also rename $g_4 \equiv g$ and define $\beta = \frac{N}{g}$. Field theories with exponential propagators are cumbersome to work with. In one dimension, however, there is a great simplification. This is because in one dimension the behavior of loop integrals do not depend on the behavior of the propagator at small distances. This means that we can keep only the lowest terms in the expansion of $e^{\partial_t^2}$. This brings the action to a conventional form with the standard kinetic term for matrix quantum mechanics.

$$S = \int dt \text{Tr}[(\partial_t M)^2 + V(M)]$$  \hspace{1cm} (2.3)

In the lowest order of the large-$N$ expansion this model was solved in [3]. Its interpretation in terms of random surfaces is discussed for example in [5]. We shall discuss the double scaling limit of this problem by considering
first a potential of the form

$$V(M) = M^2 + \frac{g}{N}M^4$$  \hspace{1cm} (2.4)

Very soon we shall see that only some generic properties of the potential are important in the double scaling limit. First let us rescale the matrix $M$ such that all $N$ dependence appears as an overall factor in the action.

$$S = \beta \int dt \text{Tr}[(\partial_t M)^2 + v(M)]$$  \hspace{1cm} (2.5)

where $v(M)$ denotes the potential $V(M)$ with $g$ being set to be equal to $1$. Since we have a standard kinetic term it is convenient to consider the corresponding Hamiltonian. This is given by

$$\hat{H} = \text{Tr} \left( -\frac{1}{2\beta} \frac{\partial^2}{\partial M^2} + \beta v(M) \right)$$  \hspace{1cm} (2.6)

We shall restrict ourselves to the sector of the theory which is singlet under the symmetry group of $U(N)$ rotations. One can then reformulate the problem in terms of the eigenvalues $\lambda_i$ of the matrix $M$. In the change of variables from the matrices $M$ to the eigenvalues one has a Jacobian which is the Van der Monde determinant [3].

$$\prod_{ij} dM_{ij} = \prod_i d\lambda_i (\prod_{i \neq j} (\lambda_i - \lambda_j))^2$$  \hspace{1cm} (2.7)

and one can redefine the wave functions to absorb the determinant $\prod_{i \neq j} (\lambda_i - \lambda_j)$. The new wave functions are then Slater determinants, i.e. fermionic many body wave functions. The Hamiltonian acting on these fermionic
wavefunctions reads

\[
\hat{H} = \sum_i \left[ \frac{1}{2\beta} \frac{\partial^2}{\partial \lambda_i^2} + \beta v(\lambda_i) \right]
\] (2.8)

All we have are \( N \) fermions living in an external potential \( v(\lambda) \). It is convenient to talk of a reduced hamiltonian \( h \) given by

\[
\hat{H} = \beta h = \beta \sum_i h_i
\]

Let \( e \) denote the energy levels and \( \rho(e) \) the density of states of the reduced single particle hamiltonian \( h_i \). Since we have \( N \) fermions the fermi level \( \mu_F \) is determined by

\[
\int_0^{\mu_F} de \rho(e) = N
\] (2.9)

While the ground state energy of the system is given by

\[
E_{gs} = \beta \int_0^{\mu_F} de e \rho(e)
\] (2.10)

Note that this is the energy of the original system, not the value of the reduced hamiltonian, which explains the factor of \( \beta \) in (2.10). The equations (2.10) and (2.9) determine the ground state energy of the model.

Let us first consider the semiclassical limit. It is clear from the form of the hamiltonian \( \hat{H} \) that the inverse of \( \beta \) acts as a Planck’s constant. Thus the semiclassical limit of the model is given by \( \beta \to \infty \) at fixed value of the coupling \( g \). This is, of course the large-\( N \) limit at fixed coupling, or the planar limit. The form of the reduced potential \( v(\lambda) \) relevant for our case is sketched below
The crucial feature of the potential is the existence of maxima at \( \pm \lambda_c \).

In the lowest order of the semiclassical approximation one can use the Bohr-Sommerfeld quantization rule to determine the energy levels of the Hamiltonian. Thus if \( e_n \) denotes the \( n \)-th energy level of the reduced Hamiltonian \( h \) one has

\[
\int_{-\lambda_c}^{\lambda_c} d\lambda \sqrt{2(e_n - v(\lambda))} = (n - \frac{1}{2}) \frac{1}{\beta}
\]  
(2.11)

The factor of \( \frac{1}{\beta} \) appears on the right hand side because this is the Planck’s constant for our problem. The limits of integration over \( \lambda \) are as usual the endpoints of the classical orbits. The density of states then follows:

\[
\rho(e) = \beta \int \frac{d\lambda}{\sqrt{2(e - v(\lambda))}}
\]  
(2.12)

Criticality is obtained when the fermi energy \( \mu_F \) approaches \( v(\lambda_c) = \mu_{Fc} \), which happens when the cosmological constant \( g \) approaches a critical value \( g_c \). A crucial role is played by the quantity

\[
\mu = \mu_{Fc} - \mu_F
\]  
(2.13)

Near criticality it is now straightforward to find the relationship between \( \mu \) and the quantity \( (g - g_c) \). Then the filling condition (2.9) implies

\[
\frac{dg}{d\mu_F} = \int_{-\lambda_0}^{\lambda_0} \frac{d\lambda}{\sqrt{2(\mu_F - v(\lambda))}}
\]  
(2.14)

Now in the limit where \( \mu_F \to \mu_{Fc} \), the classical particle spends most of its time near the turning point and the integral in (2.14) receives most of
its contribution from the turning point. It is then easy to evaluate the integral by expanding the potential around the point $\lambda_c$ and show that

$$\frac{dg}{d\mu_F} \sim \log \mu \quad (2.15)$$

near the critical limit. Integrating (2.15) one concludes that

$$\Delta g = (g - g_c) \sim \mu \log \mu \quad (2.16)$$

The value of the ground state energy also follows easily:

$$E_{gs}^0 - \frac{1}{4\pi} (\beta \mu)^2 \log \mu \quad (2.17)$$

The expression for the density of states (2.12) shows why the limit $\mu \to 0$ corresponds to a continuum limit. This is because it is fairly easy to check that the density of states at the fermi level $\rho(\mu_F)$ diverges as $\log \mu$ in the critical limit (this is in fact the content of (2.15)). Thus the excitations above the ground state, which are particle hole pairs because of the fixed number of fermions constraint, will have a continuous spectrum of energies - a signature of continuum physics.

It is also possible to the next correction to the ground state energy. We will not detail the procedure here, but simply quote the answer

$$E_{gs}^{(1)} = \frac{1}{12\pi} \log \mu \quad (2.18)$$

The expressions for the ground state energy in the first two orders of the large-$N$ expansion suggests what the double scaling limit could be. In the double scaling limit the energies in every order should be comparable.
Comparing (2.17) and (2.18) we see that this can be almost achieved if we consider the limit

\[ \beta \to \infty, \quad \mu \to 0, \quad (\beta \mu) = \text{fixed} \quad (2.19) \]

The problem is the additional log \( \mu \) piece in (2.18). We shall see that this logarithmic factor contains important physics. For the moment we ignore its presence and declare the double scaling limit of the model to be defined by (2.19).

It is remarkable that in the double scaling limit the problem simplifies considerably. To see this consider solving the single particle Schrödinger equation with the hamiltonian \( h \) for energies near the fermi level. We have to solve

\[ \frac{1}{2\beta^2} \frac{\partial^2}{\partial \lambda^2} + v(\lambda)]\psi(\lambda) = \mu_F \psi(\lambda) \quad (2.20) \]

We now expand the potential around the critical point \( \lambda_c \) and perform a further rescaling by defining the variable \( x \)

\[ x = \sqrt{\beta}(\lambda_c - \lambda) \quad (2.21) \]

Then the above Schrödinger equation for a wavefunction at the fermi level reads

\[ \frac{1}{2\beta^2} \frac{\partial^2}{\partial x^2} + \beta \mu + \frac{1}{2} v''(\lambda_c)x^2 - \frac{1}{6\sqrt{\beta}} v'''(\lambda_c)x^3 + \cdots \psi(x) = 0 \quad (2.22) \]

Since \( \beta \mu \) is fixed in the double scaling limit, it is clear that the terms in the potential involving powers of \( x \) higher than 2 are suppressed by inverse powers of \( \frac{1}{\sqrt{\beta}} \). Furthermore, we shall be interested in the case of the potential having a maximum, i.e. \( v''(\lambda_c) < 0 \). In the double scaling limit,
therefore, one has a problem of fermions living in an inversion harmonic oscillator potential\textsuperscript{*}.

The resulting quantum mechanical problem thus has a Hamiltonian

\[ H_{ds} = -\frac{1}{2} \frac{\partial^2}{\partial x^2} + \beta v(\lambda_c) - \frac{1}{2} x^2 \]  

(2.23)

where we have rescaled fields to have \( v''(\lambda_c) = -1 \). The height of the potential is of order \( \beta \) at the maximum, and it is easily seen that in terms of these scaled coordinates \( x \) the zero of the original potential, which is at \( \lambda = 0 \) is of order \( \sqrt{\beta} \) from the origin \( x = 0 \). In the following we shall measure the energies from the top of the hump of the potential, which means that the term \( \beta v(\lambda_c) \) is not present in the Hamiltonian (2.23).

As it stands the problem is not well defined since one has a potential which is bottomless. To define the problem one has to put a wall to bound the potential. Now, in the critical limit a classical particle moving in an orbit with energy equal to the fermi energy spends most of its time near the turning point. In the double scaling limit this region is blown up, which is why only the \( x^2 \) term is important. The higher order terms in the potential are suppressed by powers of \( \frac{1}{\sqrt{\beta}} \) and become important only when \( x \sim O(\sqrt{\beta}) \). Thus one may put a wall at \( |x| = \Lambda \sim \sqrt{\beta} \) without interfering with the universal physics.

The density of states of the problem may be obtained in a number of ways. We will give the treatment of Brezin et al. in [6]. We are interested in solving the Schrödinger equation for energies not far from

\textsuperscript{*} In the above considerations we have assumed that \( v''(\lambda_c) \neq 0 \). This is certainly the situation for generic potentials with a critical point. However, for more special potentials one could have \( \frac{d^{2k}u}{d\lambda^k} = 0 \) for all \( k < m \). The various scaling behaviors of the density of states, the ground state energy etc. are now dependent on \( m \), leading to different exponents\textsuperscript{[43]} . However, the physics of these models are not understood very well. In what follows we shall exclusively with the generic case of \( m = 2 \).
the top of the potential hump. If $\xi$ denotes the energy measured from the
top we are interested in $1 << \xi << \Lambda^2$. Thus there is a large coordinate
range $\sqrt{\xi} << x << \Lambda$ where one is (a) far from the turning point so
that semiclassical approximation for the wave function is valid and (b) far
from the wall so that it is sufficient to retain the quadratic term in the
potential. The wavefunctions of the inverted harmonic oscillator potential
are parabolic cylinder functions which has the asymptotics

$$\psi(x) \sim \frac{1}{\sqrt{x}} \sin \left[ \frac{1}{4} x^2 + \xi \log(x) + \frac{1}{2} \phi(\xi) \right]$$  \hspace{1cm} (2.24)

where

$$\phi(\xi) = -\frac{1}{2} i \log \frac{\Gamma\left(\frac{1}{2} + i \xi\right)}{\Gamma\left(\frac{1}{2} - i \xi\right)} + O(e^{-2\pi \xi})$$  \hspace{1cm} (2.25)

On the other hand far from the turning point, i.e. for $x >> \sqrt{\xi}$ one can
use the WKB wave function

$$\psi(x) \sim \frac{1}{\sqrt{x}} \sin \left[ \frac{1}{4} x^2 + \xi \log\left(\frac{x}{\Lambda}\right) + \frac{1}{2} \Phi\left(\xi, E_0, g_i\right) \right]$$  \hspace{1cm} (2.26)

where the function $\Phi$ depends on the details of the potential (the nature of
the wall, hence on the values of the coefficients of the higher terms in the
Taylor expansion of the potential, $g_i$). $E_0$ is the height of the potential.
Since $\xi << E_0$ in our region of interest we can set $\frac{\xi}{E_0} = 0$ so that the
entire information of the details of the potential far from the hump is
encoded in a single parameter $\Phi(0, g_i) \equiv \Phi_0$ independent of the energy
level. Matching the two wave functions (2.24) and (2.26) one has the
quantization condition

$$\Phi_0 - \xi \log \Lambda - \phi(\xi) = \pi n$$
where $n$ is an integer. The density of states follow immediately

$$
\rho = -\frac{1}{\pi}(\frac{\partial \phi}{\partial \xi} + \log \Lambda)
$$

where $\phi(\xi)$ is given in (2.25). The density of states may be now expanded in inverse powers of $\xi$ and used in the expressions (2.10) and (2.9) to obtain the following expansion for the ground state energy [6]

$$
E_{gs} = -\frac{1}{4\pi}(\beta \mu)^2 \ln \mu + \frac{1}{12\pi} \ln \mu + \sum_{m=1}^{\infty} C_m (\beta \mu)^{-m} \tag{2.27}
$$

where $C_m$ are some numerical coefficients. We also write down the expression for the finite temperature free energy $F$ for the singlet sector at temperature $T$ [7]

$$
\frac{F}{T} = -\frac{1}{4\pi T}(\beta \mu)^2 \ln \mu + \frac{1}{12} \ln \mu (\pi T + \frac{1}{\pi T}) + .... \tag{2.28}
$$

The higher terms are all finite in the double scaling limit and respect the duality symmetry $\pi T \rightarrow \frac{1}{\pi T}$. Note that unlike the situation in the single matrix model, the expressions for $E_{gs}$ and $F$ are not finite in the double scaling limit. Rather the two leading contributions have the well known "logarithmic scaling violations". One of our main aims is to understand these scaling violations.
3. Collective Field Theory

The spacetime interpretation of noncritical string theory hinges on the fact that the liouville mode acts as an extra dimension in target space [11, 12, 13]. Since the one dimensional matrix model becomes, in the critical limit, the Polyakov string with one worldsheet scalar (the target space time) one expects that the true target space dimension must be two. At first sight this appears confusing. In the matrix model representation the integration over the liouville mode is really the sum over all feynman diagrams and some extra dimension does not seem to appear explicitly. Nevertheless the formulation of the theory as a model of \( N \) mutually non-interacting fermions in an external potential immediately shows that there is an extra dimension - viz. the space of eigenvalues which we have been calling \( \lambda \). In fact the double scaled model is by definition equivalent to a \( 1 + 1 \) dimensional field theory of non-relativistic fermions. One might wonder whether the eigenvalue direction \( \lambda \) or its scaled version \( x \) is the additional dimension in target space. We shall now argue that this is not quite true - the ”liouville mode” is related to \( x \), but not \( x \) itself.

Though the model is naturally written in terms of fermionic fields, the excitations are all bosonic. This is because \( N \) is fixed so that the excitations are all bosonic particle-hole pairs. In this section we will show that application of the Collective Field Theory approach leads to a direct physical interpretation of the one dimensionl matrix model [8].

The main idea is to rewrite the problem in terms of invariant variables using the method of collective fields developed in [9]. Correlation functions of any number singlet operators may be written in terms of the collective fields,

\[
\phi(x) = \int \frac{dk}{2\pi} e^{-ikx} \text{Tr} (e^{ikM}) = \sum_i \delta(x - \lambda_i) \tag{3.1}
\]
\( \phi(x) \) is simply the density of eigenvalues \( \lambda_i \).

We now make a change of variables from the fields \( M_{ij} \) to the collective fields \( \phi(x) \). The field \( \phi(x) \) is, of course, constrained to satisfy:

\[
\int dx \, \phi(x) = N \tag{3.2}
\]

The change of variables is made using the procedure of [9]. We also perform the rescalings \( x \to \sqrt{\beta} x, \phi \to \sqrt{\beta} \phi \). One finally gets a Hamiltonian [8]

\[
H_\phi = \int dx \left( \frac{1}{2\beta^2} \partial_x \pi(x) \phi(x) \partial_x \pi(x) + \beta^2 V_0 + \Delta V \right) \tag{3.3}
\]

where \( \pi(x) \) is the momentum conjugate to the field \( \phi(x) \), and

\[
V_0 = \frac{\pi^2}{6} \phi^3 + (v(x) - \mu_F)\phi(x) \tag{3.4}
\]

\[
\Delta V = \frac{1}{2} \int_{y=x} dx \phi(x) \partial_x \partial_y \ln |x - y| \tag{3.5}
\]

In this hamiltonian, the constraint (3.2) which has a \( g \) instead of \( N \) on the right hand side, has been implemented by a lagrange multiplier \( \mu_F \). The lagrangian which follows from the above hamiltonian may be written as

\[
\mathcal{L} = (\beta)^2 \int dx \left\{ \frac{1}{2} \partial_x^{-1} \phi \frac{1}{\phi} \partial_x^{-1} \dot{\phi} - V_0(\phi(x,t)) - \Delta V \right\} \tag{3.6}
\]

**Leading Order**

It is clear that \( \frac{1}{\beta} \) is the bare string coupling constant. In the leading order of the WKB expansion the free energy is dominated by the saddle point solution for \( \phi(x) \) which is given by

\[
\phi_0(x) = \frac{1}{\pi} \sqrt{2(\mu_F - v(x))} \tag{3.7}
\]

while its integral must be equal to \( g \). In the critical limit of small \( \mu \) one may easily show that the energy evaluated at the saddle point coincides with the leading order ground state energy in (2.27).
The Liouville mode and the Spectrum

The lagrangian density (3.6) is clearly the lagrangian of a two dimensional field theory with $x$ as the extra dimension. However $x$ is not quite the liouville mode we are looking for. To find the liouville mode we have to study the fluctuations around the saddle point solution [14]

$$\phi(x) = \phi_0(x) + \frac{1}{\beta} \frac{\partial \eta}{\partial x}$$

(3.8)

It is convenient to introduce the variable $q$ defined by:

$$q = \frac{1}{\pi} \int \frac{dx}{\phi_0(x)}, \quad dq = \frac{dx}{\phi_0}$$

(3.9)

Note that $q$ is simply the time of flight of a classical particle moving in the potential $v(x)$. In the following we will use the specific form of the potential

$$v(x) = x^2 - 2x^4$$

(3.10)

The results are of course independent of the detailed form of the potential provided the maximum is generic, i.e. the second derivative $v''(x_0)$ is non-vanishing.

The range of $q$ is given by $-L < q < L$ where $4L$ is the time period of classical motion. In the critical limit $L \to \infty$ as $L = -\frac{1}{4} \ln \mu$ (as follows from the saddle point solution above). The action quadratic in the fluctuations

$$S_2 = \pi^3 \int dt \int_{-L}^{L} dq \left[ \frac{1}{2} (\partial_t \eta)^2 - \frac{1}{2} (\partial_q \eta)^2 \right]$$

(3.11)

which is the lagrangian of a massless scalar field in two dimensions.
That is exactly what is expected from the continuum theory for $d = 1$. The one dimensional non-critical string may be considered as a two dimensional critical string theory with a linear dilaton background in the liouville direction [11, 12]. The equations of motion for the modes are then obtained in the standard manner by considering the corresponding backgrounds in the worldsheet theory and requiring Weyl invariance. In two dimensions, the scalar is the only full-fledged field. Consider a tachyon background $T(t, \phi)$. Its equation of motion which follows from the weyl invariance condition is

$$B(T) = (-\partial_\phi^2 + 2\sqrt{2}\partial_\phi + \partial_t^2 + 2)T(t, \phi) + V'[T(t, \phi)] = 0 \quad (3.12)$$

The potential $V(T)$ is nonuniversal but starts with a term cubic in $T$. In our case one has a cosmological constant background as well. This means that there is a time-independent tachyon background. The precise nature of this background depends on the non-universal nonlinear pieces in (3.12). However, as argued in [18] the features of the background for $\phi \to \pm\infty$ can be nevertheless obtained.

$$T_0(\phi - \phi_0) \to 1 - a \, e^{(2-\sqrt{2})\phi} \quad \phi \to -\infty$$
$$T_0(\phi - \phi_0) \to b \, \phi \, e^{-\sqrt{2}\phi} \quad \phi \to +\infty \quad (3.13)$$

Here $\phi_0$ is the center of the kink like configuration. The energy of this configuration is infinite, so one needs to put an upper cutoff to $\phi$, which may be taken to be $\phi = 0$. Thus all $\phi < 0$. The value of the field at $\phi = 0$ is taken to be $T(0, t) = \Delta$, where $\Delta$ will be identified with the bare cosmological constant on the world sheet. The aim is now to minimize the energy maintaining the boundary condition at $\phi = 0$. The resultant minimum energy configuration is given by $\bar{T}(\phi, t) \equiv T(\phi - \bar{\phi})$ where $\bar{\phi}$ is
defined by
\[-b \bar{\phi} e^{-\sqrt{2} \bar{T} \phi} = \Delta \] (3.14)

Note that as $\Delta \to 0^+$, $\bar{\phi} \to -\infty$. Comparison with the relationship between the cosmological constant and the fermi energy in the matrix model immediately leads to the identification $\bar{\phi} = \frac{1}{\sqrt{2}} \log \mu$. To find the equation satisfied by the excitation one has to expand around this ground state solution $\bar{T}$. However, for small $\Delta$, the classical configuration is practically zero for all $\bar{\phi} < \phi < 0$ so the linear equation for the fluctuation $\tilde{T} = T - \bar{T}$ is given once again by the linear part of the equation (3.12). One can now make a field redefinition $\tilde{T} \to e^{-\sqrt{2} \phi} \tilde{T}$ so that the linearized equation for the new $\tilde{T}$ is

$$[-\partial_{\phi}^2 + \partial_t^2] \tilde{T} = 0$$ (3.15)

which represents a massless scalar field in two dimensions. However, very close to the ”wall” $\bar{\phi}$ there is an additional term in the linearized equation of motion (3.15) which is $\bar{T} \tilde{T}$. This additional term is proportional to the cosmological constant $\Delta$.

Comparison of the linearized equation of motion of the collective field fluctuation $\eta$ and the tachyon fluctuation $\bar{T}$ far away from the wall suggests the interpretation that the time of flight variable $q$ is the liouville mode and $\eta$ the tachyon field. However the fact that the tachyon fluctuations in the linear dilaton- cosmological constant background has departures from the massless free field behaviour even at the linearized level indicates that this interpretation can be only approximate. A better interpretation has been suggested in [31]. Consider the macroscopic loop operator which reads, in the language of collective field theory

$$W(p, t) = \int_{\sqrt{\mu}}^\infty dx \ e^{-px} \ \phi(x, t)$$ (3.16)
The parameter $p$ may be regarded as the invariant length of a large loop on the worldsheet. It may be easily checked that the classical value of $W(p, t)$, obtained by substituting $\phi_0$ for $\phi(x, t)$ in (3.16) is given by $\frac{\sqrt{p}}{p} K_1(\sqrt{\mu} p)$ where $K_\nu(z)$ stands for the standard modified Bessel function. The fluctuations of $W$ around this classical value may be seen to obey the equation

$$ [(p\partial_p)^2 - \partial_t^2 - \mu p^2] \tilde{W}(p, t) = 0 \quad (3.17) $$

This equation seems to indicate that $\log p$ is the quantity that has to interpreted as the liouville mode and $W(p, t)$ as the tachyon field. This is because (3.17) has the right qualitative features of the equation for the tachyon fluctuation around a linear dilaton - cosmological constant background. Far away from the "wall" this reduces to the standard massless scalar equation, with $\log p \sim \phi$ and close to the wall the departure is proportional to the cosmological constant. One may wonder whether the theory can be written in terms of the $W(p, t)$. In principle it can be done, but the result is nonlocal and cumbersome to work with and for obtaining the consequences of the theory it is much easier to work with the fluctuations of the collective field $\eta$ itself.

The boundary conditions on the fluctuation $\eta(q, t)$ are determined from the time independence of the constraint, i.e. $\frac{d}{dt}(\int dx \phi(x)) = 0$, which leads to Dirichlet boundary conditions on $\eta$ : $\eta(-L, t) = \eta(L, t) = 0$. The eigenfunctions are therefore

$$ \eta_n(q) = \frac{1}{\sqrt{L}} \sin\left(\frac{n\pi q}{L}\right) \quad \text{or} \quad = \frac{1}{\sqrt{L}} \cos\left(n + \frac{1}{2}\right) \frac{\pi q}{L} \quad (3.18) $$

where $n = 0, 1, 2, \cdots$. The frequencies are

$$ \omega_j = \frac{j\pi}{2L} = j \omega_c; \quad j = 0, 1, 2, \cdots \quad (3.19) $$
The propagator is then
\[ D(t - t'; q, q') = \int \frac{dE}{\pi} e^{iE(t - t')} D(E, q, q') \]

where
\[ D(E, q, q') = \sum_j \frac{\eta_j(q)\eta_j(q')}{E^2 - \omega_j^2 + i\epsilon}. \] (3.20)

In the scaling limit we have \( L \to \infty \) and we define continuum momenta \( p = \frac{n\pi}{2L} \). The propagator now becomes the standard massless scalar propagator in two dimensions. The dispersion relation becomes \( E^2 - p^2 = 0 \).

Using the above basic two point function it is straightforward to evaluate the class of two-point functions in the matrix model

\[ <Tr M_p(t) Tr M_q(0)> = \int dx \int dx' x^p x^q \partial_x \partial_{x'} <\eta(x, t)\eta(x', t)> \] (3.21)

Using (3.18) and (3.20) it is easy to check that the result is identical to that obtained from a direct calculation of the above correlator in the matrix model [15].

**One loop free energies**

To obtain the one-loop (torus) free energy at zero temperature we need to calculate the expression

\[ \Delta E_0 = \pi^3 \int dq [\partial_q \partial_{q'} G(q, q')]_{q=q'} \] (3.22)

where \( G(q, q') \) is the standard propagator following from the action (3.11)

\[ G(q, q') = -\frac{1}{2\pi^3} \ln |q - q'| \] (3.23)

One must also add the term \( \Delta V \) in the original collective field lagrangian.

To treat the various singular pieces, we make a change of variables in (3.23)
to the original variable $x$. Using the definition of $q(x)$ one finds that the singularity as $x \to y$ cancels between $\Delta V$ and (3.23), leaving with the finite answer

$$\Delta E_1 = \frac{1}{24\pi} \int \frac{v''(x)dx}{[2(\mu_F - v(x))]^2}$$

(3.24)

Using the form of the potential one can find the singular (as $\mu \to 0$) piece in the one loop free energy density to be

$$\Delta E_1 = \frac{1}{12\pi} \ln \mu$$

(3.25)

which agrees with the corresponding term in (2.27). The expression (3.25) also agrees with the leading WKB correction obtained in [16], and the collective field calculation in [AB].

To obtain the free energy at finite temperature one has to add to $\Delta E_1$ the contribution of the thermal free energy of a massless scalar field in $(1 + 1)$ dimensions having a massless dispersion relation. At temperature $T$ the free energy is easily seen to be

$$F/T = \frac{\pi LT}{3} = -\frac{\pi T}{12} \ln \mu$$

(3.26)

The total thermal free energy is obtained by adding (3.25) and (3.26)

$$\frac{F}{T} = \frac{1}{12} \ln \mu (\pi T + \frac{1}{\pi T})$$

which displays duality and agrees with (2.28).

Thus the genus zero and one free energies are proportional to $\ln \mu$ simply because the effective field theory lives in a box of length $L \sim \ln \mu$.  

**Interactions**
To obtain the full structure of the interactions in the theory it is best to work in the hamiltonian framework. We thus work with the collective hamiltonian and perform perturbations around the classical solution. It is convenient to introduce a field \( \xi \) and its canonically conjugate momentum \( \Pi_\xi \) and rescale \( x \)

\[
x \to \left(\frac{1}{2}\mu\right)^{\frac{1}{2}} x \quad \phi = \left(\frac{1}{2}\mu\right)^{-\frac{1}{2}} \partial_x \xi \quad \partial_x \Pi_\phi = -\left(\frac{1}{2}\mu\right)^{-\frac{3}{2}} \Pi_\xi \tag{3.27}
\]

and introduce

\[
P_\pm = -g_{st}^2 \Pi_\xi (x, t) \pm \pi \partial_x \xi (x, t) \tag{3.28}
\]

The hamiltonian (3.3) then acquires a simple form

\[
H = \frac{1}{2\pi g_{st}^2} \int dx \left\{ \frac{1}{6}(P_3^+ - P_3^-) - \frac{1}{2}x^2(P_+ - P_-) + \mu(P_+ - P_-) \right\} \tag{3.29}
\]

In (3.29) we have imposed the constraint (3.2) by means of a lagrange multiplier \( \mu \). The fluctuations are then carried out by writing

\[
P_\pm = \pm \phi_0 + \frac{g_{st}}{\phi_0} \eta_\pm \tag{3.30}
\]

where

\[
\eta_\pm = -\Pi_\eta \pm \partial_\tau \eta \tag{3.31}
\]

The fluctuation hamiltonian is then

\[
H = \frac{1}{2\pi} \int dq \{ [(\eta_+)^2 + (\eta_-)^2] + \frac{g_{st}}{3\phi_0^2}[(\eta_+)^3 - (\eta_-)^3] \} \tag{3.32}
\]

The discussion of the one loop free energy indicates that normal ordering of this hamiltonian removes the singularities already present in \( H \).
The normal ordered form of the Hamiltonian turns out to be

\[ H = \int dq \left[ \frac{1}{2} : \Pi_\eta^2 + (\partial_q \eta)^2 : + \frac{1}{2} \beta \phi_0^2 \right] : \Pi_\eta (\partial_q \eta) \Pi_\eta + \frac{1}{3} (\partial_q \eta)^3 : + V' \]  \hspace{1cm} (3.33)

where the extra finite term

\[ V' = -\frac{1}{12} \left[ \frac{\phi''_0}{\phi_0} - \frac{1}{2} \left( \frac{\phi'_0}{\phi_0} \right)^2 (\partial_q \eta) + \frac{1}{24} \left[ \frac{\phi'_0}{\phi_0} \right]^2 \right] \]  \hspace{1cm} (3.34)

replaces the infinite term \( \Delta V \). The interactions in the Hamiltonian are purely cubic and there is a finite tadpole. The finite field independent term in (3.34) is of course the one loop ground state energy.

This exact form of the Hamiltonian has been derived from the fermionic field theory of the matrix model [10]. The remarkable feature of the Hamiltonian is that the interactions are strong only at the boundary. For our potential \( \phi_0(q) \) may be obtained in terms of standard elliptic functions, and in the critical limit, one can show that

\[ \frac{1}{\phi_0^2} \sim \cosh^4(q) \]  \hspace{1cm} (3.35)

The coupling has the exponential dependence on the liouville direction as has been found in the continuum theory [11, 18]. The coupling grows near the boundaries. In fact in the double scaling limit the coupling at the boundary is held fixed

\[ g_{st} = \frac{1}{\beta} \exp(4L) = \frac{1}{\beta \mu} = \text{fixed.} \]  \hspace{1cm} (3.36)

and equal to the string coupling constant \( g_{st} \). This immediately provides a qualitative explanation of why there are no \( \ln \mu \) factors in front of the higher genus contributions to the free energy. The higher genus contributions are purely due to interactions which fall off exponentially away from the boundary; therefore no overall volume factor is present.
In several recent papers Demeterfi, Jevicki and Rodrigues [22] have developed a systematic hamiltonian perturbation theory based on (3.33) and (3.34) and have calculated the two loop ground state energy and the two loop finite temperature free energy. The results are completely finite and in exact agreement with the results of the matrix model. However the calculations seem to depend on the type of regularization used. In [21] the two point function of the eigenvalue density was calculated to one loop using a different regularization and the answer was found to be divergent. This indicates the presence of counterterms which are both finite and infinite in this regularization scheme. The bosonization technique employed in [19] also seems to indicate the presence of higher terms in the hamiltonian. In [27] the fermionic theory has been bosonized in terms of the quantum distribution function in phase space and it has been concluded that the collective field theory results only if some approximations are made. However, the fact that the prescription for handling infinities used in [22] leads to the correct expression of the thermodynamic free energy at two loops indicates that the cubic hamiltonian is sufficient to calculate such quantities, at least in string perturbation theory. It is, however, possible that the non-perturbative extension defined by the collective field theory may be defined by the non-perturbative extension defined by the fermionic field theory.

The Tree Level S-Matrix

The tree level S-Matrix of the theory has been obtained from many different approaches. The most direct approach is to use the standard perturbation theory techniques in the collective field theory [21, 22] or directly use the fermionic field theory [23]. In this section we shall summarize the approach of [26]. The main motivation to explain this approach is that it uses the fermi fluid picture and provides a useful intuition about the scattering process, and in formulating an exact bosonization of the system
[27].

Our model is a model of $N$ fermions which do not interact with one another living in an external inverted harmonic oscillator potential. The second quantized form of the action is therefore

$$S = \frac{1}{g_{st}} \int dt \; dx \; \psi^\dagger(x,t)[i\partial_t + \frac{1}{2}\partial_x^2 + \frac{1}{2}x^2]\psi(x,t)$$  \hspace{1cm} (3.37)

The collective field theory action has an overall factor of $\frac{1}{g_{st}}$ whereas the fermionic action has an overall factor of $\frac{1}{g_{st}}$ (recall that $g_{st} = \frac{1}{\beta\mu}$). This implies that the classical limit of the collective field theory corresponds to the classical single particle limit in the fermionic theory, viz. to the classical motion of particles in the external inverted harmonic oscillator potential.

The ground state of the many-fermion system corresponds to the filled fermi sea - this corresponds to the classical ground state of the collective field theory. In the two dimensional phase space $(p, x)$ of the single particle the fermi surface is a hyperbola $\frac{1}{2}(p^2 - x^2) = \beta\mu$. An excited state means an excitation of a particle hole pair. This would correspond to a deformation of the hyperbola. In the collective field theory this represents a general field configuration with energy higher than the ground state. Thus a general state of the bosonic field is in one to one correspondence with a profile of the fermi surface.

To look for classical solutions of the collective field theory one has to, therefore, look for deformations of the fermi surface. Let $\sigma$ be a parameter which denotes a point on the fermi surface. The motion of the point in phase space may be obtained directly from the Hamiltonian evolution equations:

$$x(\sigma, t) = -a(\sigma) \cosh (t - b(\sigma)) \hspace{1cm} p(\sigma, t) = -a(\sigma) \sinh (t - b(\sigma))$$  \hspace{1cm} (3.38)

We can use the freedom of reparametrization of the fermi surface to set
\[ b(\sigma) = \sigma \text{ so that the general solution is} \]
\[ x(\sigma, t) = -a(\sigma) \cosh (t - \sigma) \quad p(\sigma, t) = -a(\sigma) \sinh (t - \sigma) \quad (3.39) \]

There is a direct translation of the quantities in the single particle phase space and the collective field theory. The density of eigenvalues of the matrix is the collective field \( \phi(x, t) = \frac{\partial_x \xi}{g_{st}} \). At the classical level the density of points in phase space \( u(x, p, t) \) is given by (in our normalization)

\[ u(x, p, t) = \frac{1}{2\pi g_{st}} \theta(-\beta \mu - \frac{1}{2}(p^2 - x^2)) \quad (3.40) \]

(Recall that we are measuring energies from the top of the hump). This means that the density of fermions should be \( \psi^\dagger \psi = \int_{p_-^+ (x, t)}^{p_+ (x, t)} \frac{dp}{2\pi g_{st}} = p_+ (x, t) - p_-(x, t) \) where \( p_\pm \) are the upper (lower) edge of the fermi surface for a given value of \( x \). The momentum density in the collective field theory is simply \( \Pi_\xi \partial_x \xi(x, t) \), which is given by \( \int_{p_-^+ (x, t)}^{p_+ (x, t)} \frac{dp}{2\pi g_{st}} = \frac{p_+^2 (x, t) - p_-^2 (x, t)}{4\pi g_{st}} \).

Comparison with (3.28) immediately shows that we should identify \( p_\pm \) with \( P_\pm \). In fact integration of the single particle hamiltonian over phase space and use of this identification immediately leads to the classical collective field hamiltonian.

It is now clear that the general solution to the collective field theory may be obtained directly from (3.39). In fact (3.39) is a parametrized form of the fermi surface which may be alternatively written in terms of \( p_\pm (x, t) \) and hence \( P_\pm (x, t) \).

To obtain the tree level \( S \)-matrix we restrict our attention to the left portion of the potential hump, i.e. \( x < 0 \). For the inverted harmonic oscillator potential the time of flight variable \( q \) is

\[ q = -\log (-x) \quad (3.41) \]

Consider now a point on the deformed fermi surface with some value of \( \sigma = \sigma_0 \). At some time \( t_1 \) this corresponds to some value of \( x(\sigma_0, t_1) = \ldots \)
given by the equation (3.39) and denotes a particle moving with the momentum \( p_0 \) given in (3.39) at the position \( x_0 \) at time \( t_1 \). We will take the time \( t_1 \) such that this corresponds to a particle moving towards the potential hump, so that \( x_0 < 0, p_0 > 0 \). Then eliminating \( p \) from (3.39) one has

\[
x_0 - \sqrt{x_0^2 - a^2} = -a e^{t_1 - \sigma_0}
\]

We will be interested in the large \(|x|\) region, i.e. \( q \rightarrow -\infty \). In this region one can solve for \( \sigma_0 \) in terms of \( q_0 \) and \( t_1 \) from (3.42) to obtain

\[
\sigma_0 = t_1 - q_0 - \frac{1}{2} \log \left( \frac{1}{4a^2} \right)
\]

The particle will travel to the potential and bounce back. Eventually at some later time \( t_2 \) it will again cross the point \( x_0 \), or equivalently \( q_0 \) with a momentum \(-p_0\) since it is now travelling away from the potential. From (3.39) this means

\[
t_2 - \sigma_0 = -(t_1 - \sigma_0)
\]

Using (3.43) we thus have the time difference between successive crossings of the same spatial point as

\[
t_2 - t_1 = -2q_0 - \log \left( \frac{1}{4a^2} \right)
\]

Now in the asymptotic region \( x \rightarrow -\infty \) one has \( p_+ \rightarrow |x| \) and \( p_- \rightarrow -|x| \). It is therefore natural to define new objects \( \epsilon_\pm \) by

\[
p_\pm = \pm e^{-q} \mp e^q \epsilon_\pm
\]

It is easy to check that

\[
\epsilon_+ = (p_+ + x) x
\]

Using the parametrized solutions (3.39), the solution for \( \sigma \) in the asymp-
totic region and (3.47) one gets

\[ a^2 = 2(p_+ + x) \ x = 2\epsilon_+(t_1 - q_0) \quad (3.48) \]

In the process we are considering, i.e. a particle coming in and bouncing back, with momentum \( p_0 \) at the position \( x_0 \) while going in and a momentum \( -p_0 \) at the same position while going back we have

\[ p_0 = e^{-q_0} - e^{q_0} \epsilon_+(q_0, t_1) \quad -p_0 = -e^{-q_0} + e^{q_0} \epsilon_-(q_0, t_2) \quad (3.49) \]

which means

\[ \epsilon_+(t_1, q_0) = \epsilon_-(t_2, q_0) \quad (3.50) \]

The edges of the fermi sea, \( p_\pm(x, t) \) satisfy the equation

\[ \partial_t p_\pm = x - p_\pm \partial_x p_\pm \quad (3.51) \]

From the definition of the \( \epsilon_\pm \) it may be easily checked that in the asymptotic region \( q \to -\infty \) they satisfy the equation

\[ [\partial^2_t - \partial^2_q] \epsilon_\pm = 0 \quad (3.52) \]

These are purely right or left moving waves. Clearly for the incoming particles, one has \( \epsilon_+(t-q) \) whereas for the bounced wave one has \( \epsilon_-(t+q) \). Thus we have from (3.51), using (3.45) and (3.48)

\[ \epsilon(t_1 - q_0) = \epsilon_-[t_1 - q_0 + \log 2 - \log \epsilon_+(t_1 - q_0)] \quad (3.53) \]

Since this equation is valid for all values of \( t_1 \) and \( q_0 \) we might as well remove the subscripts on these quantities.
To see what the equation (3.53) means in terms of the scattering of the fundamental excitations of the theory we have to recall that the quantities \( p_{\pm} \) are nothing but the chiral combinations of momenta and fields of the collective field theory, \( P_{\pm}(x, t) \). We thus express the quantities \( \epsilon_{\pm} \) in terms of the field \( S(q, t) \) and its canonically conjugate momentum \( \Pi_S(q, t) \) as

\[
\frac{1}{\sqrt{\pi g_{st}}} \epsilon_{\pm}(q, t) = \pm \Pi_S(q, t) - \partial_q S(q, t) \quad (3.54)
\]

Note that \( S(q, t) \) is not the same as the fluctuation of the collective field \( \eta(q, t) \) used earlier. In fact the vacuum values are \( \epsilon_{\pm} = \frac{1}{2} \) which corresponds to \( S(q, t) = -\frac{q}{2\sqrt{\pi g_{st}}} \). We thus make the mode decomposition of the field \( S(q, t) \) as

\[
S(q, t) = -\frac{q}{2\sqrt{\pi g_{st}}} + i \int_{-\infty}^{+\infty} \frac{dk}{2\pi} \left[ \alpha_+(k) e^{ik(t-q)} + \alpha_-(k) e^{ik(t+q)} \right] \quad (3.55)
\]

The \( \alpha(k) \) are annihilation operators for right moving modes, while \( \alpha_-(k) \) are annihilation operators for the left moving modes.

The relation (3.53) thus provides a relation between the left and right moving modes. The crucial point is that the equation (3.53) is a non-linear equation so that the relation between the left and right moving modes is nonlinear as well. This is the key to the question as to why there could be non-trivial scattering of the bosons while the fermions from which they come from are free. To find out the relation between the \( \alpha(k) \) and \( \alpha_-(k) \) one has to expand \( \epsilon_{\pm} \) around their classical value \( \frac{1}{2} \)

\[
\epsilon_{\pm}(t \mp q) = \frac{1}{2} + \tilde{\epsilon}_{\pm}(t \mp q \mp \log 2)
\]

Then (3.53) may be used to solve for \( \tilde{\epsilon}_{\pm} \) as a power series in \( \tilde{\epsilon}_- \), which in
turn leads to the following relation between $\alpha_{\pm}(k)$

\[
\alpha_+(k) = \alpha_-(k) - i k \sqrt{2\pi g_{st}} \int_{-\infty}^{+\infty} \frac{dk_1}{2\pi} \alpha_-(k_1) \alpha_-(k-k_1) + \\
\frac{(ik - k^2) 4\pi}{3} g^2_{st} \int \int \frac{dk_1 dk_2}{4\pi^2} \alpha_-(k_1) \alpha_-(k_2) \alpha_-(k-k_1-k_2) + \cdots
\]

(3.56)

It is now easy to read out the S-matrix from (3.56). The main point is that since we are working on a half-line, incoming modes are right moving while outgoing modes are left moving. For example, when we have two particles with momenta $k_1$ and $k_2$ coming in and two particles with momenta $k_3$ and $k_4$ going out one has to simply evaluate the quantity

\[
<0|\alpha_-(k_3) \alpha_-(k_4) \alpha_+(k_1) \alpha_+(k_2)|0>
\]

by using (3.56). The result for the T matrix is

\[
T = -\frac{ig^2_{st}}{16\pi} \sqrt{k_1 k_2 k_1' k_2'} (|k_1 + k_2| + |k_1 - k_1'| + |k_1 - k_2'| - 4i)
\]

(3.57)

The same S-matrix has been obtained in [21- 23]. More significantly the same S-matrix has been reproduced from a continuum calculation in [24].

Note the absence of any momentum conserving delta function in the result. This is because the interactions break translation invariance. It has been suggested by Polyakov [20] that the general form of the S-Matrix in general non-critical string theories is

\[
S = R(k, k') + \frac{A(k, k')}{\left(\sum k_i - \sum k_i'\right)}
\]

(3.58)

where $k$ and $k'$ denote incoming and outgoing momenta. If the momenta are conserved then the second term in (3.58) is proportional to the volume of the spatial direction. In the $d = 1$ theory $A(k, k')$ is identically zero ! In fact this follows that the two chiralities are decoupled in the collective field hamiltonian [25]. For some further understanding of the S-Matrix see [28].
4. The $W_\infty$ Symmetry

The model we have been discussing has a remarkable set of global symmetries which form an infinite dimensional algebra \([29- 32]\). The symmetries are best described in the fermionic formulation since it is only in this formulation the \textit{exact} symmetries are known \([30]\).

Since we have a system of \(N\) fermions in an external potential, the problem is completely integrable and has an infinite number of conserved and mutually commuting charges \([10]\) with their conserved currents. It turns out that these charges are the Cartan subalgebra of the full algebra of symmetry generators forming \(W_{1+\infty}\).

Consider the fermionic action (3.37). It is enlightening to regard the argument \(x\) as an index for the fermionic field so that we have

\[
S = \int dt \sum_{x,y} \psi^\dagger_x(t) [i\partial_t \delta(x - y) + \bar{A}_{xy}(t)] \psi_y(t) \quad (4.1)
\]

where the matrix \(\bar{A}_{xy}(t) = \frac{1}{2} (\partial_x^2 + x^2) \delta(x - y)\). This immediately shows that the double scaled matrix model may be viewed as a theory of fermions in the fundamental representation of \(U(\infty)\) living in an \textit{external} gauge field \(\bar{A}_{xy}(t)\). In fact we can consider any general, but fixed, background. Thus the free energy of the system would be unchanged if we consider a different background gauge field related to the original one by the gauge transformation

\[
\bar{A} \to U(t) \bar{A} U^\dagger(t) + iU(t) \partial_t U^\dagger(t) \quad (4.2)
\]

where we have suppressed indices. In (4.2) \(\bar{A}\) is a hermitian matrix in the \((x, y)\) space and \(U\) is a unitary matrix. For arbitrary \(U\) this is \textbf{not} a symmetry of the action (for some given \(\bar{A}\)) since the gauge field is not a dynamical variable. However, there may be some special \(U\)'s for which
the gauge field $\tilde{A}$ does not change: these are then symmetries of the fermionic action. To identify these symmetries consider an infinitesimal transformation $\mathcal{U} = e^\epsilon$ where $\epsilon$ is again a matrix. Then the transformation of the gauge field is $\tilde{A} \rightarrow \tilde{A} + D_t \epsilon$ where $D$ is the covariant derivative in the given background. Thus the symmetries of the theory are given by solutions of

$$D_t \epsilon(t) = 0$$  \hspace{1cm} (4.3)

For our double scaled potential a set of solutions are

$$\epsilon_{xy}(t) = e^{-(r-s)t} \begin{bmatrix} (x - i\partial_x)^r, & (x + i\partial_x)^s \end{bmatrix} \delta(x - y)$$  \hspace{1cm} (4.4)

where $r, s$ are positive integers or zero. The corresponding charges which generate these symmetries on the fermionic fields are

$$W_{rs} = e^{-(r-s)t} \int dx \psi^\dagger (x, t) \begin{bmatrix} (x - i\partial_x)^r, & (x + i\partial_x)^s \end{bmatrix} \psi(x, t)$$  \hspace{1cm} (4.5)

These charges form a closed algebra which is isomorphic to the full $W_{1+\infty}$ algebra. In the classical limit, i.e. for small $g_{st}$ this reduces to the algebra of area preserving diffeomorphisms on the plane. It is this algebra which has been found in the continuum formulation [33]. Note that for $r \neq s$ the charges are time dependent. This means that these charges do not commute with the Hamiltonian, though the action is kept invariant. The commuting set of infinite charges $W_{rr}$ are those found in [10]. The implications of the symmetry and the algebra will be discussed in detail in the lectures of A. Dhar in these proceedings.

We end this section by writing down the symmetries in the collective field formulation. At the classical level, they may be easily obtained by using the correspondence between the single particle operators and collective field theory operators discussed in the previous section. The charges
are, therefore [29]

\[ \omega^{(r,s)} = \int d^2x \int_{P_-}^{P_+} dp \, (p + x)^r (p - x)^s \, e^{-(r-s)t} \]  

(4.6)

It is straightforward to check that the collective field hamiltonian (3.29) is given by \( \omega^{(1,1)} \) while the Poisson bracket algebra

\[ \{\omega^{(r,s)}, \omega^{(r',s')}\}_{PB} = -(r's' - s'r')\omega^{(r+r'-1, s+s'-1)} \]  

(4.7)

which is the \( w_{1+\infty} \) algebra. In a similar way it is straightforward to check that \( \frac{d\omega^{(r,s)}}{dt} = 0 \).

In the quantum theory, one has to define a suitable ordering for the charges defined above. One such definition is given in [29] who conclude that the algebra is unchanged in the quantum theory. This is in contradiction to what we found in the fermionic formulation. It is probable that a different ordering is required which reproduces the correct algebra.

5. The Matrix Model as a Black Hole background

In the above sections we have presented ample evidence that the double scaling limit of the matrix model represents a linear dilaton - cosmological constant background of the two dimensional critical string theory.

In this section we will argue that it is possible to interpret the matrix model rather differently: as a black hole background of the string theory [34]. More precisely, we show that a certain linear integral transform of the fluctuations of the collective field obeys the same linearized classical equation of motion as that of a massless scalar in the black hole background of the two dimensional critical string. Our connection is related to the
observation in [37] that the coset model of [36] may be recast as a liouville
theory by going over to the space of field momenta on the worldsheet,
but rather different from pervious attempts to describe the black hole
background by modifying the collective field theory [42]. To facilitate
comparison with standard conventions of the black hole metric we shall
rescale the fermi level \( \mu \rightarrow 2\mu \) so that the classical solution to the collective
field is \( \phi_0(x) = \frac{\sqrt{2}}{\pi} (x^2 - 4\mu) \frac{1}{2} \).

It may be easily checked that the fluctuation \( \tilde{\phi} \) then satisfies the fol-
lowing equation

\[
\frac{1}{2} \partial^2_t \tilde{\phi} = \tilde{\phi} + 3x \partial_x \tilde{\phi} + (x^2 - 4\mu) \partial^2_x \tilde{\phi}
\]  
(5.1)

Now consider the following transform of the collective field

\[
T(u, v) = \int dp \int dt \ e^{ip[e^t v + e^{-t} u]} \int dx \ e^{-px} \phi(x, t)
\]  
(5.2)

In other words we are considering a transform of the macroscopic loop
operator introduced in (3.16). In an obvious notation we will call the
transform of \( \phi_0 T_0(u, v) \), while the transform of the fluctuation will be
referred to as \( \bar{T}(u, v) \). Using the Dirichlet boundary conditions on the
field \( \eta \) introduced earlier (\( \tilde{\phi} = \partial_x \eta \)) it may be checked that \( \bar{T} \) satisfies the
following equation

\[
[4(uv + \mu) \partial_u \partial_v + 2(u \partial_u + v \partial_v) + 1] \bar{T}(u, v) = 0
\]  
(5.3)

This is precisely the equation of the massless tachyon moving in a black
hole background of two dimensional critical string theory written in Kruskal.
like coordinates. The invariant form of the equation is
\[ \nabla^2 T - 2 \nabla T \cdot \nabla D + \left( \frac{2}{\alpha'} \right) T = 0 \] (5.4)

where \( \nabla \) denotes target space covariant derivative and \( D \) is the dilaton background. \( \alpha' \) is the string tension. In Kruskal like coordinates the black hole solution in the small \( \alpha' \) limit has the metric and dilaton fields [35]
\[ G_{uv} = G_{vu} = \frac{1}{2(\frac{2}{\alpha'} uv + a)} \quad G_{uu} = G_{vv} = 0 \]
\[ D(u, v) = -\frac{1}{2} \log \left( \frac{2}{\alpha'} uv + a \right) \] (5.5)

The parameter \( a \) is the mass of the black hole. Substituting (5.5) in (5.4) and comparing the resulting equation with (5.3) we get for the black hole mass
\[ a = \frac{2}{\alpha'} \mu \] (5.6)

The same relation holds in the connection between \( SL(2, R) \) coset model and liouville theory proposed in [37].

The field \( T(u, v) \) is defined in the entire two dimensional plane. We will now show explicitly that \( T(u, v) \) thus defined gives the correct solution of (5.3) in all the regions of the \((u, v)\) plane. To do this it is convenient to define coordinates \((r, \theta)\) in the four regions as follows
\[ u = r e^{\theta} \quad v = r e^{-\theta} \quad \text{for} \quad u, v \geq 0 \quad \text{Region I} \]
\[ u = r e^{\theta} \quad v = r e^{-\theta} \quad \text{for} \quad u < 0, v > 0 \quad \text{Region II} \]
\[ u = r e^{\theta} \quad v = -r e^{-\theta} \quad \text{for} \quad u, v < 0 \quad \text{Region III} \]
\[ u = r e^{\theta} \quad v = -r e^{-\theta} \quad \text{for} \quad u > 0, v < 0 \quad \text{Region IV} \] (5.7)

Regions I and III are the exterior regions. Region II contains the future black hole singularity at \( uv = a \) while Region IV contains the past singularity.
Let us now obtain \( \tilde{T}(r, \theta) \) by starting from the definition (5.2). This is done by substituting for the explicit forms of \( \eta(x, t) \) satisfying Dirichlet boundary conditions and explicitly evaluating the transform. One gets

\[
T(r, \theta) = 2e^{-i\nu \theta} \int_0^\infty dp K_{i\nu}(2\sqrt{\mu p}) K_{i\nu}(-2i\nu) \quad \text{Region I}
\]

\[
T(r, \theta) = 2e^{-i\nu \theta} \int_0^\infty dp K_{i\nu}(2\sqrt{\mu p}) K_{i\nu}(2\nu) \quad \text{Region II}
\]

We will not write down formulae for the other two regions since they are trivially related to those in I and II. The \( T(r, \theta) \) are thus given in terms of \( K \)-transforms of the macroscopic loop operator. Both the above integrals may be evaluated explicitly \cite{39}. The final result is

\[
T(r, \theta) = \frac{\pi^2}{4 \cosh \pi \nu} \mu^\frac{i\nu}{2} \ e^{-i\nu \theta} \ (r)^{-\nu} \ F\left(\frac{1}{2} + i\nu, \frac{1}{2}; 1 + \frac{\mu}{r^2}\right) \quad \text{Region I}
\]

\[
T(r, \theta) = \frac{\pi^2}{4 \cosh \pi \nu} \mu^\frac{i\nu}{2} \ e^{-i\nu \theta} \ (r)^{-\nu} \ F\left(\frac{1}{2} + i\nu, \frac{1}{2}; 1 - \frac{\mu}{r^2}\right) \quad \text{Region II}
\]

These are in exact agreement with one of the solutions of (5.3) in each of the regions\(^*\). The second solution of (5.3) is of the form

\[
T'(r) = e^{i\omega \theta} \ (r)^{-\nu} \ F\left(\frac{1}{2} + i\omega, \frac{1}{2}; 1 + \frac{\mu}{r^2}\right) \ F\left(\frac{1}{2} + i\omega, \frac{1}{2}; 1 - \frac{\mu}{r^2}\right) \log (1 + \frac{\mu}{r^2}) + \sum_{m=1}^\infty s_m \frac{\Gamma\left(\frac{1}{2} + m + i\omega\right)\Gamma\left(\frac{1}{2} + m\right)}{(m!)^2 (1 + \frac{\mu}{r^2})^m} \quad \text{Region I}
\]

where \( s_m = \sum_{n=1}^m \frac{1}{n+i\omega} + \frac{1}{n-i\omega} - \frac{2}{n} \). In Region II one has to replace \( \mu \to -\mu \). This solution with a logarithmic singularity at the position of the singularity \( uv = -\mu \) is not seen in the matrix model.

\(^*\) It has been already noted in \cite{37} that the solutions of the liouville model Wheeler de Witt equation may be transformed into solutions of the tachyon fluctuations in a blackhole background. We have seen, however, that the matrix model uniquely picks out one of the solutions of the differential equations.
It is interesting that the $T(r, \theta)$ obtained by evaluating the transform yields one of the solutions of the differential equation satisfied by $T(u, v)$. This is related to the fact that the matrix model always picks out a specific combination of dressings of the continuum theory, as is also manifest in the standard interpretation of the matrix model as a liouville background. Perhaps more significantly the solution in the interior region which is picked out is the one which is regular at the singularity $r^2 = \mu$ in Region II, and not the one which has a logarithmic singularity\(^*\).

To obtain the asymptotic states which satisfies physically interesting boundary conditions in the exterior region it is necessary to rewrite the above solutions using standard relations between hypergeometric functions. For example in Region I

$$T(r, \theta) = \frac{e^{-i\nu \theta}}{r} \left[ (\frac{\mu}{r^2})^{\frac{i\nu}{2}} A(-\nu) F\left(\frac{1}{2} + i\nu, \frac{1}{2} + i\nu + \frac{\mu}{r^2} \right) + 
(-1)^{i\nu} \left( \frac{\mu}{r^2} \right)^{-\frac{i\nu}{2}} A(\nu) F\left(\frac{1}{2} - i\nu, \frac{1}{2} - i\nu, \frac{\mu}{r^2} \right) \right],$$

(5.11)

where $A(\nu) \equiv \frac{\Gamma(i\nu)}{\Gamma(\frac{3}{4} + i\nu)\Gamma(\frac{1}{4})}$ and we have omitted an overall $\nu$-dependent constant. The two terms in (5.11) are in exact agreement with two of the tachyon vertex operators found in the $SL(2, R)/U(1)$ coset model in [40] after a change of variables $r \rightarrow \sinh (r/2)$. These modes vanish on the past and future null infinities respectively and represent left and right moving modes at spatial infinity. The other two modes which vanish at the past and future horizons (and represent left and rightmoving plane waves on the horizon) are given by the two terms of the following rewriting of the

\(^*\)The behavior of the tachyon field near the singularity has been investigated in [35] and [40].
hypergeometric functions in (5.9)

\[ T(r, \theta) = e^{-i\nu \theta} \frac{\mu}{r^2} A(\nu) F\left(\frac{1}{2} - i\nu, \frac{1}{2}; 1 - i\nu; -\frac{r^2}{\mu}\right) + 
\]

\[ (-1)^{i\nu} \left( e^{-i\nu \theta} \frac{\mu}{r^2} A(-\nu) F\left(\frac{1}{2} + i\nu, \frac{1}{2}; 1 + i\nu; -\frac{r^2}{\mu}\right) \right) \]

Thus the transform of the macroscopic loop operator is a linear combination of the asymptotic solutions in the black hole geometry. It may be easily checked that the solutions at \( r = 0 \) approached from Region I differ from that approached from Region III by a factor \( e^{-\pi \nu} \). This is a signature of the fact that the black hole is in equilibrium with a thermal bath at temperature \( T_{BH} = \frac{1}{6\pi \sqrt{\alpha'}} \).

So far we have been considering the fluctuation \( \tilde{T} \). What is the behaviour of the transform of the classical solution, \( T_0(u,v) \)? It is easier to evaluate the derivative \( \partial_\mu T_0 \). Using the form of the classical solution for the collective field it may be easily checked that

\[ \partial_\mu W_0(p) \equiv \partial_\mu \int_{2\sqrt{p}}^{\infty} dx \ e^{-px} \phi_0(x) = 2K_0(2\sqrt{p}) \quad (5.13) \]

Let us now evaluate \( \partial_\mu T_0 \) in the region containing the future singularity, Region II. Using the appropriate \((r, \theta)\) coordinates one gets

\[ \partial_\mu T_0(r, \theta) = \int_{0}^{\infty} dp \int_{-\infty}^{\infty} dt \ e^{2ipr} \sinh t \ K_0(2\sqrt{p}) \quad (5.14) \]

The integral over \( t \) can be now performed, yielding another modified Bessel function \( K_0(2pr) \) and finally the integral over \( p \) can be performed using
standard K-transforms [39], yielding finally

\[ \partial_\mu T_0(r, \theta) = \frac{\pi}{2r} F\left(\frac{1}{2}, \frac{1}{2}; 1; 1 - \frac{\mu}{r^2}\right) \quad \text{RegionII} \quad (5.15) \]

with similar expressions in region I. As expected the result is independent of \( \theta \). Furthermore the background is completely non-singular at the location of the "singularity" at \( r = \sqrt{\mu} \).

We have thus shown that the transform of both the background and the fluctuations are well behaved and nonsingular at the "singularity" already at the classical level. The implications of this fact is unclear. In [41] the above correspondence between the matrix model and the black hole has been discussed in the framework of the bosonization of the fermionic field theory in terms of the quantum distribution function in phase space \( u(x, p, t) \). Consider the bilocal operator of fermions defined as

\[ W(\alpha, \beta, t) = \frac{1}{2} \int_{-\infty}^{\infty} dx \, e^{i\alpha x} \psi^\dagger(x + \beta/2, t) \psi(x - \beta/2, t) \quad (5.16) \]

In [41] one then considers the transform

\[ \tilde{T}(u, v) = \int_{-\infty}^{+\infty} dt \int_{-\infty}^{+\infty} d\alpha [e^{i\alpha(ue^{-t}-ve^{-t})} W(\alpha, 0, t) + e^{i\alpha(ue^{-t}+ve^{-t})} W(0, \alpha, t)] \]

(5.17)

The first term is related to our transform with imaginary values of \( \alpha \). It has been shown that the background value of \( \tilde{T} \) defined above is singular at the black hole singularity at the classical level but the singularity disappears when the exact (all orders) result is used. In our transform the singularity is not present in the classical level to begin with. The singularity of \( \tilde{T} \) at the classical level may be traced to the fact that it involves macroscopic loops with imaginary loop lengths.
The crucial point about the transform we defined is that it is **linear**. This means that the $W_\infty$ symmetry discussed in the previous section is also present in the black hole background. In fact it has been indeed argued that the coset model does have a $W_\infty$ symmetry [38].

The above relationship between the matrix model and black holes is rather intriguing. More insight will surely come from the structure of the interaction terms. In terms of the transformed fields these will be nonlocal. A better understanding will hopefully allow us to use the exact results of the matrix model to resolve vexing issues in quantum aspects of black holes.

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