Loosely trapped surface and dynamically transversely trapping surface in Einstein-Maxwell system

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We study the properties of the loosely trapped surface (LTS) and the dynamically transversely trapping surface (DTTS) in Einstein-Maxwell systems. These concepts of surfaces were proposed by the four of the present authors in order to characterize strong gravity regions. We prove the Penrose-like inequalities for the area of LTSs/DTTSs. Interestingly, although the naively expected upper bound for the area is that of the photon sphere of a Reissner-Nordström black hole with the same mass and charge, the obtained inequalities include corrections represented by the energy density or pressure/tension of electromagnetic fields. Due to this correction, the Penrose-like inequality for the area of LTSs is tighter than the naively expected one. We also evaluate the correction term numerically in the Majumdar-Papapetrou two-black-hole spacetimes.

Subject Index E0, E31, A13

1. Introduction

Since a black hole creates a strong gravitational field, there exists unstable circular orbits for photons. For a spherically symmetric system, the collection of them makes a surface called a photon sphere. In the Schwarzschild spacetime, for example, the photon sphere exists at the surface \( r = 3m \), where \( m \) is the Arnowitt-Deser-Misner (ADM) mass. Furthermore, a generalized concept of a photon sphere, which is called a photon surface, has been proposed [1]. However, the definition of the photon surface requires a highly symmetric spacetime (e.g., [2]). Moreover, the existence of photon surfaces does not necessarily mean that the gravitational field is strong there [1].

A photon sphere is directly related to observational phenomena. The quasinormal modes of black holes are basically determined by the properties of photon spheres [3]. The black hole shadow, whose direct picture has been taken by the recent radio observations of the Event Horizon Telescope Collaboration [4], is also determined by a photon sphere [5].

Motivated by the recent observations, the four of the present authors proposed concepts that characterizes strong gravity regions: a loosely trapped surface (LTS) [6], a transversely
trapping surface (TTS) [7], and a dynamically transversely trapping surface (DTTS) [8, 9] (see also Ref. [10] for an extension of a TTS). For certain cases, we have proved inequalities analogous to the Penrose inequality [11], that is, their areas are equal to or less than $4\pi(3m)^2$, where $m$ is the ADM mass (see also Ref. [12] for an earlier work for a photon sphere). The upper bound is realized for a photon sphere in the Schwarzschild black hole, where $3m$ comes from the areal radius of unstable circular photon orbits. For an LTS, the application is restricted to a spacelike hypersurface with a positive Ricci scalar. This restriction is natural because it is guaranteed by the positivity of the energy density for maximally sliced initial data. For a DTTS, on the other hand, one requires the non-positivity of the radial pressure on a DTTS in addition to the positivity of the Ricci scalar. This requirement of the non-positive pressure does not significantly restrict the situation because the vacuum cases do work. However, it remains a little mystery why the non-positivity of the radial pressure is required.

Therefore, in this paper, we shall discuss non-vacuum cases. As a first typical example, we will focus on Einstein-Maxwell systems. We adopt Jang’s work [13] to show the inequality, that is, we will employ the method of the inverse mean curvature flow [14, 15]. The upper bound is expected to be given by the area of the outermost photon sphere, i.e., the locus of the unstable circular orbits of photons in a spherically symmetric charged black hole spacetime, namely, a Reissner-Nordström spacetime with the same mass and charge. As a consequence, however, we see that the obtained inequalities depend on (the part of) the energy density and the pressure/tension of the electromagnetic fields which give corrections to the naively expected upper bound. This is impressive because the Penrose inequality for apparent horizons does not depend on such quantities.

The rest of this paper is organized as follows. In Sect. 2, we will briefly describe the Maxwell theory in a curved spacetime. Some of the notations will be explained together. In Sect. 3, we will present the definition of the LTS and prove the Penrose-like inequality in the Einstein-Maxwell system. Then, in Sect. 4, we will present the definition of the DTTS and prove the Penrose-like inequality in the Einstein-Maxwell system. In Sect. 5, we will revisit the problem of DTTSs in Majumdar-Papapetrou two-black-hole spacetimes, which was studied in our previous paper [9], from the viewpoint of the current study. We will examine the properties of the correction term of the Penrose-like inequality through numerical calculations. The last section will give a summary and discussions. In Appendix, we will shortly discuss the case of a TTS defined for static/stationary spacetimes. Note that we use following units in which the speed of light $c = 1$, the Newtonian constant of gravitation $G = 1$ and the Coulomb constant $1/(4\pi\varepsilon_0) = 1$, where $\varepsilon_0$ is the permittivity of vacuum.

2. Einstein-Maxwell theory and setup

In this paper, we consider an asymptotically flat spacelike hypersurface $\Sigma$ in a four-dimensional spacetime with a metric $g_{ab}$. We suppose $n^a$ to be the future-directed unit normal to $\Sigma$ and the induced metric of $\Sigma$ is given by $\gamma_{ab} = g_{ab} + n_an_b$. In $\Sigma$, we consider a two-dimensional closed surface, an LTS (denoted by $S_0$) or a DTTS (denoted by $\sigma_0$),

\footnote{The resulted inequality for the minimal surface in Ref. [13] has a lower bound. However, it is known that the lower bound is violated for multi-black-hole systems [16]. The proof for multi-black-hole systems in the Einstein-Maxwell theory is given in Refs. [17, 18].}
with the induced metric $h_{ab}$. The outward unit normal to that surface is $r_a$ (and therefore, $\gamma_{ab} = h_{ab} + r_a r_b$).

We assume the presence of electromagnetic fields on $\Sigma$. The electromagnetic fields are specified by the anti-symmetric tensor, $F_{ab}$, and its Hodge dual, $^*F_{ab} = (1/2)\epsilon_{abcd}F^{cd}$, where $\epsilon_{abcd}$ is the Levi-Civita symbol in a four-dimensional spacetime. The tensors $F_{ab}$ and $^*F_{ab}$ follow Maxwell's equations,

$$\nabla_a F^{ab} = 0, \quad \nabla_a ^*F^{ab} = -4\pi j^b,$$

where $j^b$ is the four-current vector. The electric and magnetic fields, $E^a$ and $B^a$, are defined by

$$E_a := F_{ab} n^b, \quad B_a := -^*F_{ab} n^b,$$

respectively. Obviously, these fields are tangent to $\Sigma$ because $E_a n^a = B_a n^a = 0$ holds. The electric charge density $\rho_e$ and the electric current $(J_e)^a$ are defined by

$$\rho_e := -j_a n^a, \quad (J_e)^a := \gamma_{ab} ^*j^b.$$

In our paper, we require the charge density to vanish (outside an LTS or a DTTS), i.e. $\rho_e = 0$, but do not necessarily require the electric current $(J_e)^a$ to be zero.

The electric and magnetic fields satisfy Gauss' laws, $D_a E^a = 4\pi \rho_e$ and $D_a B^a = 0$, where $D_a$ is the covariant derivative with respect to $\gamma_{ab}$. The total electric and magnetic charges are

$$4\pi q_e := \int_{S_\infty} E^a r_a dA, \quad 4\pi q_m := \int_{S_\infty} B^a r_a dA,$$

where $S_\infty$ is a sphere at spacelike infinity and $r_a$ is the outward unit normal to $S_\infty$. It would be important to point out that $q_e$ can have a nonzero value even if the electric charge density is zero throughout the spacetime, as one can understand by imagining a spacelike hypersurface with an Einstein-Rosen bridge and two asymptotically flat regions in the maximally extended Reissner-Nordström spacetime. Similarly, although we assume the absence of magnetic monopoles throughout the paper, the value of $q_m$ can be nonzero. In a spherically symmetric spacetime, if both electric and magnetic fields are present, the total squared charge defined by

$$q^2 := q_e^2 + q_m^2$$

appears in the spacetime metric of the Reissner-Nordström solution. We handle the magnetic charge $q_m$ in the following way. In Sects. 3 and 4, we will assume $q_m = 0$ (and therefore, $q^2 = q_e^2$), and derive the Penrose-like inequalities. In the final section, we will discuss the modifications to those inequalities when $q_m$ is nonzero.

The energy-momentum tensor for the electromagnetic fields is given by

$$T^{(em)}_{ab} = \frac{1}{4\pi} \left( F^a_c F_{bc} - \frac{1}{4} g_{ab} F^{cd} F_{cd} \right).$$

In addition to $T^{(em)}_{ab}$, we consider the presence of ordinary matters whose energy-momentum tensor is $T^{(m)}_{ab}$. The total energy-momentum tensor is given by $T_{ab} = T^{(em)}_{ab} + T^{(m)}_{ab}$. The
relations of particular importance in this paper are the energy density,
\[ 8\pi \rho := 8\pi T_{ab} n^a n^b = E^a E_a + B^a B_a + 8\pi \rho^{(m)}, \]
where \( \rho^{(m)} := T^{(m)}_{ab} n^a n^b \), and the radial pressure,
\[ 8\pi P_r := 8\pi T_{ab} r^a r^b = (E_a E_b + B_a B_b) h^{ab} - (E_a r^a)^2 - (B_a r^a)^2 + 8\pi P_r^{(m)}, \]
where \( P_r^{(m)} := T^{(m)}_{ab} r^a r^b \).

3. Loosely trapped surface in Einstein-Maxwell system

In this section, we review the definition of an LTS following Ref. [6], and show the Penrose-like inequality for it in Einstein-Maxwell systems.

3.1. Definition of an LTS

The definition of an LTS is motivated by the following observation. As an example, we consider a Reissner-Nordström spacetime. The metric is
\[ ds^2 = -f_{\text{RN}}(r) dt^2 + f_{\text{RN}}^{-1}(r) dr^2 + r^2 d\Omega^2, \]
where \( f_{\text{RN}}(r) := 1 - 2m/r + q^2/r^2 \), and \( m \) and \( q \) are the ADM mass and total charge, respectively. \( d\Omega^2 \) is the two-dimensional metric of the unit round sphere. From the behavior of a null geodesic, one can find unstable circular orbits of photons at
\[ r = r_p := \frac{3m + \sqrt{9m^2 - 8q^2}}{2}, \]
where we suppose \( 9m^2 \geq 8q^2 \). Note that a photon sphere exists even if the spacetime possesses a naked singularity at the center for \((9/8)m^2 \geq q^2 > m^2\).

Now we define a similar concept to a photon sphere for general setups in terms of geometry. Here, we recall the fact that an apparent horizon is the minimal surface on time-symmetric initial data. Therefore, one possibility to specify a strong gravity region is to employ the mean curvature, that is, the trace of the extrinsic curvature of two-dimensional surfaces. Therefore, we look at the mean curvature for the Reissner-Nordström spacetime. It is easy to see that the mean curvature of an \( r = \) constant surface on \( t = \) constant hypersurface is given by
\[ k = \frac{2}{r} f_{\text{RN}}^{1/2}(r). \]

From the first derivative of \( k \) with respect to \( r \),
\[ \frac{dk}{dr} = -\frac{2}{r^2} \left( 1 - \frac{3m}{r} + \frac{2q^2}{r^2} \right) f_{\text{RN}}^{-1/2}(r), \]
we find that the maximum value of \( k \) exists at \( r = r_p \). This is exactly the same location with that of unstable circular orbits of photons. In the region between the event horizon and the photon sphere at \( r = r_p \), the mean curvature satisfies \( k \geq 0 \) and \( dk/dr \geq 0 \).

From the above argument, one may adopt the following definition of an LTS [6].

**Definition 1.** A loosely trapped surface (LTS), \( S_0 \), is defined as a compact two-surface in a spacelike hypersurface \( \Sigma \), and has the mean curvature \( k \) for the outward spacelike normal vector such that \( k|_{S_0} > 0 \) and \( k' \big| S_0 \geq 0 \), where \( ' \) is the derivative along the outward spacelike normal vector.
3.2. Penrose-like inequality for an LTS

In this section, we present the inequality for the area of an LTS in Einstein-Maxwell systems. Our theorem is as follows:

**Theorem 1.** Let $\Sigma$ be an asymptotically flat spacelike hypersurface with the Ricci scalar $(3) R \geq 2(E^a E_a + B^a B_a) + 16\pi \rho^{(m)}$, where $\rho^{(m)}$ is the non-negative energy density for other matters. We assume that $\Sigma$ is foliated by the inverse mean curvature flow, and a slice of the foliation parameterized by $y$, $S_y$, has topology $S^2$. We also suppose the electric charge density $\rho_e$ to vanish outside the LTS, $S_0$. Then, the areal radius $r_0 = (A_0/4\pi)^{1/2}$ of the LTS, $S_0$, in $\Sigma$ satisfies the inequality

$$m \geq \frac{1}{3} \left( 1 + \Phi_0^+ \right) r_0 + \frac{2q^2}{3r_0}, \quad (13)$$

where $m$ is the ADM mass and $q$ is the total charge. $\Phi_0^+$ is defined by

$$\Phi_0^+ := \frac{1}{8\pi} \int_{S_0} [(B_a r^a)^2 + (E_a E_b + B_a B_b) h^{ab}] dA, \quad (14)$$

where $h^{ab}$ is the induced metric of $S_0$ and $r^a$ is the outward-directed unit normal vector to $S_0$ in $\Sigma$.

**Proof.** On $\Sigma$, the derivative of the mean curvature $k$ along $r^a$ is given by

$$r^a D_a k = -\varphi^{-1} D^2 \varphi - \frac{1}{2} (3) R + \frac{1}{2} (2) R - \frac{1}{2} (k^2 + k_{ab} k^{ab}), \quad (15)$$

where $D_a$ is the covariant derivative of $\Sigma$, $D_a$ is the covariant derivative of $S_0$, $(2) R$ is the Ricci scalar of $S_0$, $(3) R$ is the Ricci scalar of $\Sigma$, $k_{ab}$ is the extrinsic curvature of $S_0$ and $\varphi$ is the lapse function for $y$, that is, $r_a = \varphi D_a y$. Then, the integration of Eq. (15) over $S_0$ gives us

$$\frac{1}{2} \int_{S_0} (2) R dA = \int_{S_0} \left[ r^a D_a k + \varphi^{-2} (D \varphi)^2 + \frac{1}{2} (3) R + \frac{1}{2} k_{ab} k^{ab} + \frac{3}{4} k^2 \right]$$

$$\geq \frac{3}{4} \int_{S_0} \left[ k^2 + \frac{4}{3} (E_a E_a + B^a B_a) \right] dA$$

$$= \frac{3}{4} \int_{S_0} \left[ k^2 + \frac{4}{3} (E_a r^a)^2 \right] dA + 8\pi \Phi_0^+, \quad (16)$$

where $k_{ab} := k_{ab} - (1/2) h_{ab} k$. Note that $\Phi_0^+ \geq 0$. Using the Gauss-Bonnet theorem and Cauchy-Schwarz inequality, we can derive the following inequality for the mean curvature

$$\int_{S_0} k^2 dA \leq \frac{16\pi}{3} - \frac{4}{3} (4\pi q)^2 - \frac{32}{3} \pi \Phi_0^+, \quad (17)$$

where we used Gauss' law for the electric field, $\int_{S_0} E_a r^a dA = \int_{S_\infty} E_a r^a dA = 4\pi q$. Here, $S_\infty$ denotes the two-sphere at spacelike infinity.

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2 From Eq. (7) and the Hamiltonian constraint of the Einstein equations, this condition is equivalent to $K_{ab} K^{ab} \geq K^2$, where $K_{ab}$ is the extrinsic curvature of $\Sigma$. This condition is obviously satisfied by maximally sliced hypersurfaces, on which $K = 0$ holds.
Let us consider Geroch’s quasilocal energy \[13-15\]
\[E(y) := \frac{A^{1/2}(y)}{64\pi^{3/2}} \int_{S_y} (R - k^2)dA, \tag{18}\]
where \(A(y)\) is the area of \(S_y\). Here, we suppose that the surfaces \(y = 0\) and \(y = \infty\) correspond to the LTS and a sphere at spacelike infinity, respectively. Under the inverse mean curvature flow generated by the condition \(k\varphi = 1\), the first derivative of \(E(y)\) is computed as
\[
\frac{dE(y)}{dy} = \frac{A^{1/2}(y)}{64\pi^{3/2}} \int_{S_y} \left[ 2\varphi^{-2}(D\varphi)^2 + (3) R \right] dA. \tag{19}\]
Using \((3) R \geq 2(E^a E_a + B^a B_a)\), we can derive
\[
\frac{dE(y)}{dy} \geq \frac{A^{1/2}(y)}{32\pi^{3/2}} \left[ \frac{(4\pi q)^2}{A(y)} + 8\pi \Phi_y^+ \right], \tag{20}\]
with the same procedure as the derivation of the inequality of Eq. (17), where \(8\pi \Phi_y^+ = \int_{S_y} [(B_a r^a)^2 + (E_a E_b + B_a B_b) h^{ab}]dA\).

The integration of the inequality of Eq. (20) over \(y\) in the range \(0 < y < \infty\) implies us
\[
m \geq \frac{A_0^{1/2}}{4\pi^{1/2}} \left( 1 - \frac{1}{16\pi} \int_{S_0} k^2 dA \right) + \frac{\pi^{1/2} q^2}{A_0^{1/2}} + \frac{1}{4\pi^{1/2}} \int_0^\infty \Phi_y^+ A^{1/2}(y) dy
\geq \frac{A_0^{1/2}}{6\pi^{1/2}} + \frac{4\pi^{1/2} q^2}{3 A_0^{1/2}} + \frac{A_0^{1/2}}{6\pi^{1/2}} \Phi_0^+. \tag{21}\]

In the above, we have used the well-known relation \(A(y) = A_0 \exp(y)\) that holds in the inverse mean curvature flow at the first step, and the non-negativity of \(\Phi_y^+\) and the inequality of Eq. (17) in the second step. Then, we find the inequality of Eq. (13).

There are four remarks. First, the minimum value of the right-hand side of the inequality of Eq. (13) implies
\[
m \geq \frac{2\sqrt{3}}{3} \sqrt{1 + \Phi_0^+} |q|. \tag{22}\]
Setting \(\Phi_0^+ = 0\), this inequality is reduced to \(m \geq \frac{2\sqrt{3}}{3} |q|\) which corresponds to the condition for the existence of a photon sphere in the Reissner-Nordström solution [see Eq. (10)].

Next, under the condition given by Eq. (22), the rearrangement of the inequality of Eq. (13) gives us
\[
4\pi r_{\text{LTS}}^2 - A_0 \leq 4\pi r_{\text{LTS}}^2, \tag{23}\]
where
\[
r_{\text{LTS}} = \frac{3m \pm \sqrt{9m^2 - 8(1 + \Phi_0^+) q^2}}{2(1 + \Phi_0^+)} \tag{24}\]
This inequality must be interpreted carefully in the sense that the lower bound would not hold in general. In the case of the ordinary Riemannian Penrose inequality \[16\], it has been pointed out that the lower bound is expected to be incorrect for multi-black-hole systems.
We consider this also may be the case for an LTS with multiple components. This is not a contradiction to our proof since an LTS with multiple components is out of the application of our theorem. Therefore, in general, we would have only the upper bound for the area,

\[ A_0 \leq 4\pi r_0^2. \tag{25} \]

Note that if we restrict our attention on an LTS with a single component, the lower bound must hold true. The physical reason is as follows. Let us consider a Reissner-Nordström spacetime in the parameter region \( (9/8)m^2 > q^2 > m^2 \) which possesses both a naked singularity and two photon spheres. The singularity of a Reissner-Nordström spacetime is known to be repulsive. This is because the energy of electromagnetic fields outside a small sphere near the singularity (e.g., in the sense of the Komar integral) exceeds the ADM mass \( m \), and hence, the gravitational field is generated by negative energy in that region. The repulsive gravitational field is, of course, not strong. This is the reason why an LTS with a single component cannot exist in the vicinity of a naked singularity with an electric charge and a lower bound exists for its area in the Einstein-Maxwell theory.

The third remark is that there is no contribution from \( \Phi_0^+ \) in the Riemannian Penrose inequality for the Einstein-Maxwell system \[13\], whereas in our theorem it appears. This is because the Riemannian Penrose inequality discusses the minimal surface with \( k = 0 \), for which the inequality of Eq. (17) is unnecessary.

Finally, the presence of \( \Phi_0^+ \) makes the inequality tighter than the case of \( \Phi_0^+ = 0 \). From Eq. (15), one can see that the quantity \( \Phi_0^+ \) appears from (the part of) the energy density in the Hamiltonian constraint. The electromagnetic energy density increases if \( \Phi_0^+ \) is turned on. Since Eq. (15) shows that the increase in \( (3) \mathcal{R} \) makes the value of \( k'|_{S_0} \) decrease, the condition \( k'|_{S_0} \geq 0 \) for an LTS \( S_0 \) is inclined to be broken, that is, the positive energy density tends to make the formation of an LTS more difficult. This means that the area of \( S_0 \) will become smaller.

4. Dynamically transversely trapping surface in Einstein-Maxwell system

In this section, we first explain the observation that motivates the definition of a DTTS, and introduce the definition of a DTTS. Then we prove an inequality for its area in Einstein-Maxwell systems.

4.1. Definition of a DTTS

The concept of a DTTS is inspired by the induced geometry of a photon surface in a spherically symmetric spacetime. The photon surface is defined as a timelike hypersurface \( S \) such that any photon emitted tangentially to \( S \) at an arbitrary point of \( S \) remains in \( S \) forever \[1\]. Let us consider spherically symmetric spacetimes with the metric,

\[ ds^2 = -f(r)dt^2 + f^{-1}(r)dr^2 + r^2d\Omega_2^2. \tag{26} \]

Solving the null geodesic equations, we can find that \( S \) satisfies \( dr/dt = \pm f(1 - b^2f/r^2)^{1/2} \), where \( b \) is the impact parameter. The induced metric of the photon surface \( S \) is obtained as

\[ ds^2|_S = -\alpha^2(r)dt^2 + r^2d\Omega_2^2, \tag{27} \]

where \( \alpha(r) := bf(r)/r \). The mean curvature of \( t = \text{constant} \) surface \( \sigma_t \) in \( S \) is given by

\[ \bar{k} = \frac{2}{bf(r)} \frac{dr}{dt}. \tag{28} \]
and the Lie derivative along $\bar{n}^a$ for $\bar{k}$ is computed as

$$
(3) \bar{\mathcal{L}}_{\bar{n}} \bar{k} = \frac{2}{r^2} \left( f - \frac{1}{2} r f' \right),
$$

where $\bar{n}^a$ is the future-directed unit normal to $\sigma_t$ in $S$. For the Reissner-Nordström black hole, it becomes

$$
(3) \bar{\mathcal{L}}_{\bar{n}} \bar{k} = \frac{2}{r^2} \left( 1 - \frac{3m}{r} + \frac{2q^2}{r^2} \right).
$$

Thus, $(3) \bar{\mathcal{L}}_{\bar{n}} \bar{k}$ is negative and positive in the inside and outside regions of the photon sphere, respectively. Hence, the non-positivity of $(3) \bar{\mathcal{L}}_{\bar{n}} \bar{k}$ is expected to indicate the strong gravity.

We now give the definition of a DTTS [8].

**Definition 2.** A closed orientable two-dimensional surface $\sigma_0$ in a smooth spacelike hypersurface $\Sigma$ is a dynamically transversely trapping surface (DTTS) if and only if there exists a timelike hypersurface $S$ that intersects $\Sigma$ at $\sigma_0$ and satisfies $\bar{k} = 0$, $\max(K_{ab}k^ak^b) = 0$, and $(3) \bar{\mathcal{L}}_{\bar{n}} \bar{k} \leq 0$ at every point in $\sigma_0$, where $\bar{k}$ is the mean curvature of $\sigma_0$ in $S$, $K_{ab}$ is the extrinsic curvature of $S$, $\bar{n}^a$ is the unit normal vector of $\sigma_0$ in $S$, $(3) \bar{\mathcal{L}}_{\bar{n}}$ is the Lie derivative associated with $S$ and $k^a$ is arbitrary future-directed null vectors tangent to $S$.

Since the emitted photons do not form a photon sphere in general without spherical symmetry, here we emit photons in the transverse direction (to satisfy the condition $\bar{k} = 0$), and adopt the location of the outermost photons as the surface $S$ [the condition $\max(K_{ab}k^ak^b) = 0$]. Then, we judge that the surface $\sigma_0$ exists in a strong gravity region if the condition $(3) \bar{\mathcal{L}}_{\bar{n}} \bar{k} \leq 0$ is satisfied. See our previous papers [8, 9] for more details.

### 4.2. Penrose-like inequality for a DTTS

We present the following theorem on a Penrose-like inequality for a DTTS:

**Theorem 2.** Consider a spacetime which satisfies the Einstein equations with the energy-momentum tensor $T_{ab}$ composed of the Maxwell field part, $T_{ab}^{(\text{em})}$ given by Eq. (6), and the ordinal matter part, $T_{ab}^{(\text{m})}$. We suppose that an asymptotically flat spacelike hypersurface $\Sigma$ is time-symmetric and foliated by the inverse mean curvature flow. We also assume that a slice of the foliation parameterized by $y$, where each of the $y = \text{constant}$ surfaces, $\sigma_y$, has topology $S^2$, and $\sigma_0$ is a convex DTTS. We further assume the electric charge density $\rho_e$ to vanish outside $\sigma_0$. Then, if $\rho^{(m)} := T_{ab}^{(m)} n^a n^b \geq 0$ in the outside region of $\sigma_0$ and $P_r^{(m)} := T_{ab}^{(m)} r^a r^b \leq 0$ on $\sigma_0$, where $n^a$ is the future-directed timelike unit normal to $\Sigma$ and $r^a$ is the outward spacelike unit normal to $\sigma_y$ in $\Sigma$, the areal radius $r_0 = (A_0/4\pi)^{1/2}$ of the convex DTTS $\sigma_0$ satisfies the inequality

$$
m \geq \frac{1}{3} \left( 1 + \Phi_0^- \right) r_0 + \frac{2q^2}{3r_0},
$$

where $m$ is the ADM mass and $q$ is the total charge. Here, $\Phi_0^-$ is defined by

$$
\Phi_0^- := \frac{1}{8\pi} \int_{\sigma_0} \left[ (B_a r^a)^2 - (E_a E_b + B_a B_b) h^{ab} \right] dA,
$$

where $h_{ab}$ is the induced metric of $\sigma_0$.  

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Proof. On $\sigma_0$, the Lie derivative of the mean curvature $\bar{k}$ along $n^a$ is given by [8]

$$(3) \bar{\mathcal{L}}_n \bar{k} = -\frac{1}{2} (2) R - 8\pi P_r + \frac{1}{2} (2kk_L + k^2 - k_{ab}k^{ab}),$$

where $(2) R$ is the Ricci scalar of $\sigma_0$, $k_{ab}$ is the extrinsic curvature of $\sigma_0$ in $\Sigma$, that is, $k_{ab} = (1/2)(3)L_h h_{ab}$, $k$ is its trace, $k_L$ is the largest value of the eigenvalues of $k_{ab}$ and $P_r$ is the radial pressure defined in Eq. (8). Then, the condition $(3) \bar{\mathcal{L}}_n \bar{k}|_{\sigma_0} \leq 0$ gives us

$$\frac{3}{4} k^2 \leq \frac{1}{2} (2) R + (E_a E_b + B_a B_b) h^{ab} - (E_a r^a)^2 - (B_a r^a)^2 + 8\pi P_r^{(m)}$$

on $\sigma_0$, where we used Eq. (8) and the inequality for a convex DTTS [8], $2kk_L + k^2 - k_{ab}k^{ab} \geq (3/2)k^2$. With the condition $P_r^{(m)} \leq 0$, the integration of the above over $\sigma_0$ gives us

$$\frac{3}{4} \int_{\sigma_0} k^2 dA \leq \frac{1}{2} \int_{\sigma_0} (2) R dA - 8\pi \Phi_0^- - \int_{\sigma_0} (E_a r^a)^2 dA$$

$$\leq 4\pi - \frac{(4\pi q)^2}{A_0} - 8\pi \Phi_0^-,$$

where we used the Gauss-Bonnet theorem, the Cauchy-Schwarz inequality and Gauss’ law as in the proof for Theorem 1.

We now consider the inverse mean curvature flow in which the foliation is given by $y =$ constant surfaces. Similarly to the proof for Theorem 1, the surfaces $y = 0$ and $y = \infty$ are set to be the DTTS and a sphere at spacelike infinity, respectively. With the same procedure, one can derive the inequality of Eq. (20) again. Then, the integration of the inequality of Eq. (20) over the range $0 \leq y < \infty$ shows us

$$m \geq \frac{A_0^{1/2}}{4\pi^{1/2}} \left(1 - \frac{1}{16\pi} \int_{\sigma_0} k^2 dA \right) + \frac{\pi^{1/2} q^2}{A_0^{1/2}}$$

$$\geq \frac{A_0^{1/2}}{6\pi^{1/2}} + \frac{4\pi^{1/2} q^2}{3 A_0^{1/2}} + \frac{A_0^{1/2}}{6\pi^{1/2}} \Phi_0^-,$$

where we used the inequality of Eq. (35) at the last step. Then, we arrive at the inequality given by Eq. (31).

There are four remarks. Similarly to Theorem 1, in general, the minimum value of the right-hand side of the inequality of Eq. (31) implies us the lower bound for $m$. However, unlike $\Phi_0^+$ in the case of an LTS, the quantity $\Phi_0^-$ does not have a definite signature. For

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3 Note that the origin for $\Phi_0^-$ in the case of a DTTS is different from that for $\Phi_0^+$ in the case of an LTS. As we can see from the derivation, $\Phi_0^+$ come from the energy density and the radial pressure of electromagnetic fields in the cases of the LTS and the DTTS, respectively. This is essential reason why we have different results for the LTS and the DTTS.
\( \Phi_0 < -1 \), there is no such restriction for \( m \), whereas, for \( \Phi_0 \geq -1 \), \( m \) has a lower bound,

\[
m \geq \frac{2\sqrt{2}}{3} \sqrt{1 + \Phi_0} |q|.
\]

Next, under the condition of the inequality of Eq. (37), a short rearrangement of the inequality of Eq. (36) gives

\[
4\pi r_{\text{DTTS}}^2 \leq A_0 \leq 4\pi r_{\text{DTTS}+}^2,
\]

where

\[
r_{\text{DTTS}+} := \frac{3m \pm \sqrt{9m^2 - 8(1 + \Phi_0) q^2}}{2(1 + \Phi_0^2)}.
\]

However, from the same reason to the remark in Theorem 1, we expect that the lower bound is not correct for a DTTS with multiple components, and just the inequality

\[
A_0 \leq 4\pi r_{\text{DTTS}+}^2
\]

would hold in a general context. For a DTTS with a single component, the lower bound must hold true with the same physical reason as the one given in Sect. 3.2.

As a third remark, in a similar way to the case of an LTS, the obtained inequality depends on the electromagnetic field. Interestingly, if \( \Phi_0 \) is negative, the contribution from \( \Phi_0^- \) makes the inequality weaker than the cases of \( \Phi_0^- = 0 \). Furthermore, for the case of \( \Phi_0^- \leq -1 \), the upper bound disappears. Let us discuss the effect of \( \Phi_0^- \) physically. It is known that there are two kinds of pressure for magnetic fields. One is the negative pressure in the direction of magnetic field lines, called the magnetic tension. The other is repulsive interaction (i.e., positive pressure) between two neighboring magnetic field lines, called the magnetic pressure. A similar thing happens also to electric field lines (say, the electric tension and the electric pressure). We recall the formula for \( 8\pi P_r \), Eq. (8). In that formula, \( -(E_a r^a) - (B_a r^a)^2 \) is the contribution of the electric/magnetic tension, while \( (E_a E_b + B_a B_b)h^{ab} \) is the contribution of the electric/magnetic pressure. Then, Eq. (33) tells that the electric/magnetic tension has a tendency to break the condition, \( \Phi_0^- \) is negative, the upper bound for the DTTS becomes weaker and the DTTS can exist at farther outside. However, \( \Phi_0^- \) depends on the position of the DTTS and we naively expect that it is sharply decreasing according to the distance from the center, if the electromagnetic field is intrinsic to the compact object; namely, monopole or multi-pole fields. Therefore, when we take a farther surface, \( \Phi_0^- \) becomes immediately negligible. Then, the area of the DTTS cannot be large. On the other hand, \( \Phi_0^- \) could be large at some point by extrinsic effects, such as, external fields and/or dynamical generation of fields.

The final remark is on the relation between an LTS and a convex DTTS \( \sigma_0 \) on time-symmetric initial data. Recall Proposition 1 in Ref. [8], that is, a convex DTTS with \( k > 0 \).
in time-symmetric initial data is an LTS as well if $\rho + P_r = 0$ is satisfied on $\sigma_0$. Since

$$8\pi \left(\rho^{\text{em}} + P_r^{\text{em}}\right) = 2(E_aE_b + B_aB_b)h^{ab},$$  \hspace{1cm} (41)$$
the presence of $\Phi_0^+$ disturbs the equivalence between an LTS and a convex DTTS in general. This feature is reflected in the two inequalities obtained in this paper.

5. Numerical examination of the Majumdar-Papapetrou spacetime

In Sect. 4.2, we have obtained the Penrose-like inequality for a DTTS. There, the quantity $\Phi_0^+$ appears, and this quantity depends on the configuration of electromagnetic fields. The purpose of this section is to examine the values of $\Phi_0^+$ in an explicit example. Specifically, in our previous paper [8], we numerically solved for marginally DTTSs in systems of two equal-mass black holes adopting the Majumdar-Papapetrou solution. We revisit this problem from the viewpoint of our current work.

A Majumdar-Papapetrou spacetime is a static electrovacuum spacetime. The metric is

$$ds^2 = -U^{-2}dt^2 + U^2[\tilde{r}^2 + \tilde{r}^2(d\tilde{\theta}^2 + \sin^2\tilde{\theta}d\tilde{\phi}^2)],$$  \hspace{1cm} (42)$$
where the spatial structure is conformally flat, and we span the spherical-polar coordinates here. The electromagnetic four-potential is

$$A_a = U^{-1}(dt)_a = -n_a.$$  \hspace{1cm} (43)$$
Any solution to the Laplace equation $\nabla^2 U = 0$ gives an exact solution, where $\nabla^2$ is the flat space Laplacian. In this situation, $E^a$ and $B^a$ are calculated as

$$E^a = -D^aU,$$ $$B^a = 0.$$  \hspace{1cm} (44)$$
Setting $E_\parallel^2 := h_{ab}E^aE^b$, we have $8\pi\Phi_0^- = -\int_{\sigma_0} E_\parallel^2 dA$. Therefore, the value of $\Phi_0^-$ is non-positive.

In our previous paper [8], we chose the solution

$$U = 1 + \frac{m/2}{\sqrt{\tilde{r}^2 + z_0^2 - 2\tilde{r}z_0\cos\tilde{\theta}}} + \frac{m/2}{\sqrt{\tilde{r}^2 + z_0^2 + 2\tilde{r}z_0\cos\tilde{\theta}}}.$$  \hspace{1cm} (45)$$
that represents the system in which two extremal black holes with the same charge are located with the coordinate distance $2z_0$. Then, assuming the functional form $\tilde{r} = h(\tilde{\theta})$, we numerically solved for a marginally DTTS that surrounds both black holes for each value of $z_0$. The solution was found in the range $0 \leq z_0/m \leq 0.79353$. We refer readers to our previous paper [8] for explicit shapes of the obtained solutions.

We examine the value of $\Phi_0^-$. The left panel of Fig. 1 presents the behavior of $E_\parallel^2$ as a function of $\tilde{\theta}$ on a marginally DTTS for $z_0/m = 0.25, 0.375, 0.5, 0.625, 0.75, \text{ and } 0.79353$. The value of $E_\parallel^2$ is generally nonzero, but is less than $10^{-3}$. The right panel of Fig. 1 plots the value of $\Phi_0^-$ as a function of $z_0$. It is negative and its absolute value is less than $5 \times 10^{-4}$.

Figure 2 shows the relation between the area $A$ of the marginally DTTS and $z_0$. We normalize the value of $A$ in two ways: One is $A/4\pi r_{\text{DTTS}}^2$ (a red solid curve), where $r_{\text{DTTS}}$ is defined in Eq. (39), and the other is $A/4\pi r_p^2$ (a green dotted curve), where $4\pi r_p^2$ is the area of a photon sphere with the same mass and charge [see Eq. (10)]. Because the value of
Fig. 1  The quantities related to $\Phi_0^-$ of a marginally DTTS in a Majumdar-Papapetrou two-black-hole spacetime. Left panel: The value of $E_\parallel^2$ as a function of $\tilde{\theta}$ on a marginally DTTS for $z_0/m = 0.25, 0.375, 0.5, 0.625, 0.75,$ and $0.79353$. Right panel: The value of $\Phi_0^-$ for a marginally DTTS as a function of $z_0/m$.

Fig. 2  The area $A$ of a common marginally DTTS in a Majumdar-Papapetrou spacetime as a function of $z_0/m$. The cases of two kinds of normalizations are shown. One is $A/4\pi r_{\text{DTTS}+}^2$ and the other is $A/4\pi r_p^2$. See text for details.

$\Phi_0^-$ is small, the difference is scarcely visible. Both of these values are in agreement with the Penrose-like inequalities.

The reason why the value of $\Phi_0^-$ is so small is that the electric field is approximately perpendicular to the marginally DTTS. From the formula for $E^a$ given in Eq. (44), this means that the marginally DTTS approximately coincides with a contour surface of $U$. Figure 3 confirms this feature for $z_0/m = 0.79353$. Here, the blue curve depicts the marginally DTTS, and the red curve shows the contour surface of $U = 2.14$. They agree well.

The lesson from this numerical experiment is that if a spacetime is static, the quantity $\Phi_0^-$ is small and does not play an important role in the Penrose-like inequality for a DTTS. Of course, it is expected that the absolute value of $\Phi_0^-$ may become large if dynamical situations are considered. For example, if two black holes have opposite charges, the contribution of the electric pressure would become important, although such a situation is more difficult to study. Exploring such issues is left as a remaining problem.
Fig. 3 The marginally DTTS for $z_0/m = 0.79353$ (the blue curve) and the contour surfaces of $U$ in the Majumdar-Papapetrou two-black-hole spacetime (gray and red curves). The red curve depicts the contour surface of $U = 2.14$.

6. Summary and discussion

In this paper, we have examined the properties of LTSs and DTTSs for Einstein-Maxwell systems, particularly focusing on the derivation of Penrose-like inequalities on their area. Similarly to the Riemannian Penrose inequality for charged cases, the electric charge comes into the inequalities, but there are additional contributions from the density or the pressure/tension of electromagnetic fields in general. This is a rather interesting result because one naively expects that the upper bound for the area of an LTS and a DTTS is that of the photon sphere $4\pi r_p^2$, where $r_p$ is the radius of an unstable circular orbit of a photon in the Reissner-Nordström spacetime given in Eq. (10). For an LTS, we have a tighter inequality than the naive one. For a DTTS, the obtained inequality can become both stronger and weaker depending on the configuration of electromagnetic fields. We have numerically examined the value of the correction term, represented by $\Phi^{-}_0$ in Eq. (32), for a Majumdar-Papapetrou two-black-hole spacetime. Although the correction term makes the inequality weaker, we have checked that the value of $\Phi^{-}_0$ is very small in that situation.

Up to now, we have assumed that the magnetic charge $q_m$ vanishes. Here, we consider what happens to our inequalities when nonzero $q_m$ is present. We first consider the case of an LTS. Since the total squared charge $q^2 := q_e^2 + q_m^2$ appears in the metric of the Reissner-Nordström solution when both the electric and magnetic fields are present, we would like to present the Penrose-like inequality in terms of $q^2$. For this reason, we have to use the inequality

$$\int_{S_0} [(E_a r^a)^2 + (B_a r^a)^2] \, dA \geq A_0^{-1} \left[ (4\pi q_e)^2 + (4\pi q_m)^2 \right] = \frac{(4\pi q)^2}{A_0}$$

(46)

in the calculations of Eqs. (16) and (17). As a result, we must introduce the quantity

$$\Phi_0 := \frac{1}{8\pi} \int_{S_0} \left[ (E_a E_b + B_a B_b)h^{ab} \right] \, dA$$

(47)

instead of $\Phi_0^+$ of Eq. (14). The resultant inequality is the one of Eq. (13) but $\Phi_0^+$ being replaced by $\Phi_0$. Next, we consider the case of a DTTS. Similarly to the case of an
LTS, we must use the inequality of Eq. (46) (but $S_0$ being replaced by $\Phi_0$) in the calculations of Eqs. (34) and (35). As a result, instead of $\Phi_0$, we must introduce $\Phi_0' = -(1/8\pi) \int_{S_0} [(E_aE_b + B_aB_b)h^{ab}] \, dA$. The resultant inequality is the same as the one of Eq. (31), but $\Phi_0'$ being replaced by $-\Phi_0$. These results are summarized as follows:

**Corollary 1.** In the presence of nonzero $q_m$, Theorems 1 and 2 hold by changing from $\Phi_0^\pm$ to $\pm \Phi_0$, where $\Phi_0$ is defined by Eq. (47).

In the main article of this paper, we have not considered a TTS for the static and stationary spacetimes defined in our previous paper [7]. We note that the concepts of a TTS and a DTTS are related but independent of each other in the sense that no inclusion relationship can be found [8]. In Appendix A, we present a theorem on the Penrose-like inequality for a TTS in a static spacetime, which is very similar to Theorem 2.

Throughout this study, we have not used the property of Maxwell’s equations except for Gauss’ law. The information from Maxwell’s equation may further restrict the properties of LTSs, DTTSs, and TTSs, especially for static/stationary spacetimes with static/stationary electromagnetic fields.

**Acknowledgements**

T. S. and K. I. are supported by Grant-Aid for Scientific Research from Ministry of Education, Science, Sports and Culture of Japan (No. 17H01091). H.Y. is supported by the Grant-in-Aid for Scientific Research (C) (No. JP18K03654) from Japan Society for the Promotion of Science (JSPS). The work of H.Y. is partly supported by Osaka City University Advanced Mathematical Institute (MEXT Joint Usage/Research Center on Mathematics and Theoretical Physics JPMXP0619217849).

**A. Transversely trapping surface**

The four of the present authors also proposed the concept of a TTS [7]. This concept is applicable only to static or stationary spacetimes. The definition is as follows:

**Definition 3.** A static/stationary timelike hypersurface $S$ is a transversely trapping surface (TTS) if and only if arbitrary light rays emitted in arbitrary tangential directions of $S$ from arbitrary points of $S$ propagate on $S$ or toward the inside region of $S$.

The necessary and sufficient condition for a surface $S$ to be a TTS (the TTS condition hereafter) is expressed as $\bar{K}_{ab}k^a k^b \leq 0$, where $\bar{K}_{ab}$ is the extrinsic curvature of $S$ and $k^a$ are arbitrary null vectors tangent to $S$. For a static spacetime, there is the Killing time coordinate $t$ whose basis vector $t^a$ is orthogonal to the $t =$ constant hypersurface, denoted by $\Sigma$. The lapse function $\alpha$ is defined by $t^a = \alpha n^a$, where $n^a$ is a future-directed unit normal to $\Sigma$. We denote the two-dimensional section of $S$ and $\Sigma$ by $\Sigma_0$. As shown in Eq. (15) of Ref. [7], the TTS condition is reexpressed in terms of $\alpha$ as

$$k_L \leq \frac{r^a D_a \alpha}{\alpha},$$  \hfill (A1)

where $k_L$ is the largest value among the two eigenvalues of the extrinsic curvature $k_{ab}$ of $\sigma_0$ in $\Sigma$, $r_a$ is a spacelike unit normal to $S$, and $D_a$ is the covariant derivative with respect to $\Sigma$. In this situation, it is possible to derive the relation

$$(2) R = -16\pi P_r + \frac{2}{\alpha} D^2 \alpha + 2k^a D_a \alpha + k^2 - k_{ab}k^{ab}. \hfill (A2)$$
For a convex TTS, we can derive the inequality
\[ 2k^r D_a \alpha + k^2 - k_{ab} k^{ab} \geq \frac{3}{2} k^2. \] (A3)

These relations are presented as Eqs. (24) and (25) in Ref. [7]. We now consider the Einstein-Maxwell system. Using Eq. (8), we find
\[ \frac{3}{4} k^2 \leq \frac{1}{2} (\gamma) R + (E_a E_b + B_a B_b) h^{ab} - (E_a r^a)^2 - (B_a r^a)^2 + 8 \pi P_r^{(m)} - \frac{1}{\alpha} D^2 \alpha. \] (A4)

Compare this inequality with the one of Eq. (34). Integrating over \( \sigma_0 \), we obtain exactly the same inequality as the one of Eq. (35). Therefore, a TTS in a static spacetime satisfies the same inequality as the Penrose-like inequality for a DTTS in time-symmetric initial data. This result is summarized as the following theorem:

**Theorem 3.** The static time cross section of a convex TTS, \( \sigma_0 \), in an asymptotically flat static spacetime has topology \( S^2 \) and its areal radius \( r_0 = \sqrt{A_0/4\pi} \) satisfies the inequality of Eq. (31) [with \( \Phi_0^- \) defined in Eq. (32)] if \( P_r^{(m)} < 0 \) holds on \( \sigma_0 \), \( k > 0 \) at least at one point on \( \sigma_0 \), and \( \rho^{(m)} \geq 0 \) in the outside region.

As remarked in the final section, this theorem applies to the case that the magnetic charge \( q_m \) is zero. When \( q_m \) is nonzero, \( \Phi_0^- \) must be replaced by \(-\Phi_0 \), where \( \Phi_0 \) is given in Eq. (47).

**References**

[1] C. M. Claudel, K. S. Virbhadra and G. F. R. Ellis, J. Math. Phys. 42, 818 (2001). [gr-qc/0005050].
[2] H. Yoshino, Phys. Rev. D 95, 044047 (2017) [arXiv:1607.07133 [gr-qc]].
[3] V. Cardoso, A. S. Miranda, E. Berti, H. Witek and V. T. Zanchin, Phys. Rev. D 79, 064016 (2009) [arXiv:0812.1806 [hep-th]].
[4] K. Akiyama et al. [Event Horizon Telescope Collaboration], Astrophys. J. 875, L1 (2019) [arXiv:1906.11238 [astro-ph.AG]].
[5] K. S. Virbhadra and G. F. R. Ellis, Phys. Rev. D 62, 084003 (2000). [astro-ph/9904193].
[6] T. Shiromizu, Y. Tomikawa, K. Izumi and H. Yoshino, Prog. Theor. Exp. Phys. 2017, 033E01 (2017) [arXiv:1701.00564 [gr-qc]].
[7] H. Yoshino, K. Izumi, T. Shiromizu and Y. Tomikawa, Prog. Theor. Exp. Phys. 2017, 063E01 (2017) [arXiv:1704.04637 [gr-qc]].
[8] H. Yoshino, K. Izumi, T. Shiromizu and Y. Tomikawa, Prog. Theor. Exp. Phys. 2020, 023E02 (2020) [arXiv:1909.08420 [gr-qc]].
[9] H. Yoshino, K. Izumi, T. Shiromizu and Y. Tomikawa, Prog. Theor. Exp. Phys. 2020, 063E01 (2020) [arXiv:1911.09893 [gr-qc]].
[10] D. V. Gal’tsov and K. V. Kobialko, Phys. Rev. D 99, 084043 (2019) [arXiv:1901.02785 [gr-qc]].
[11] R. Penrose, Annals N. Y. Acad. Sci. 224, 125 (1973).
[12] S. Hod, Phys. Lett. B 727, 345-348 (2013). [arXiv:1701.06587 [gr-qc]].
[13] P. S. Jang, Phys. Rev. D 20, 834 (1979).
[14] R. Geroch, Ann. N. Y. Acad. Sci. 224, 108 (1973).
[15] P. S. Jang and R. M. Wald, J. Math. Phys. 18, 41 (1977).
[16] G. Weinstein and S. Yamada, Commun. Math. Phys. 257, 703-723 (2005) [arXiv:math/0405602 [math.DG]].
[17] M. Khuri, G. Weinstein and S. Yamada, Contemp. Math. 653, 219-226 (2015) [arXiv:1308.3771 [gr-qc]].
[18] M. Khuri, G. Weinstein and S. Yamada, J. Diff. Geom. 106, no.3, 451-498 (2017) [arXiv:1409.3271 [gr-qc]].