Emergence of Non-Axisymmetric Vortex in Strong-Coupling Chiral $p$-Wave Superconductor

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Abstract We studied strong-coupling effect upon an isolated vortex in a two-dimensional chiral $p$-wave superconductor. We solved the Eilenberger equation for the quasiclassical Green’s functions and the Eliashberg equation with single mode Einstein boson self-consistently. We calculated the free-energy of each obtained vortex, and found that a non-axisymmetric vortex metastably exists in some situation.

1 Introduction

The Migdal and Eliashberg theory of “strong-coupling superconductivity”[1,2,3] has been very successful in qualitative description of superconductivity in real materials[4,5,6]. For example, it explains the deviation from the universal value $(2\Delta)/(k_B T_c) \approx 3.53$ in Bardeen-Cooper-Schrieffer (BCS) theory, or the dependence of critical magnetic field on the transition temperature. Moreover, in some situation, it is known that the strong-coupling effect is not only a quantitative but also a qualitative effect (e.g., Refs. [7,8]). The strong-coupling effect modifies the spectrum of the quasiparticles. Therefore, one can expect that it may change the structure of low-energy states within the vortices in type-II superconductors. As far as we know, however, there have been only a few studies of the strong-coupling effect on a vortex on the basis of microscopic theories.

Two-dimensional chiral $p$-wave superconductivity is considered to be realized in Sr$_2$RuO$_4$[9,10,11,12]. This state is topologically non-trivial and attracts much attention in these days. Within the vortices of this superconductor, reflected in the topology of this system, there is a zero-energy bound state, which is expected to be very robust against not-so-strong impurities[13,14,15,16,17,18,19,20,21]. Recently, the relationship between this robustness and the odd-frequency pairing also has been discussed[17,21,22].

In the present paper, we calculated the self-consistent Eliashberg equation to study how the strong-coupling feature affects the vortex of a chiral $p$-wave superconductor microscopically. We also calculated the free-energies of the vortices and discuss its stability.
2 Methods

In this study, we consider an isolated vortex in the two-dimensional spinless chiral p-wave superconductor with isotropic Fermi surface. We use quasiclassical theory \cite{23, 24}, we assume that the product of the coherence length of the superconductor \( \xi \) and the Fermi wavevector \( k_F \) is much larger than unity. The quasiclassical Green’s function \( \tilde{g} \) is a 2 \times 2 matrix and obeys the Eilenberger equation

\[
\dot{\tilde{g}} = \left( \begin{array}{cc} \sigma & \Delta \\ -\Delta^* & -\sigma \end{array} \right),
\]

(1)

where \( \epsilon_n = (2n+1)\pi k_B T / h \) are the Matsubara frequencies, \( v_F \) is a Fermi velocity, \( \alpha \) denotes a direction of momentum on the Fermi surface such that \( k = k_F (\cos \alpha, \sin \alpha) \), \( \tau_i \) (\( i = 0, \ldots, 3 \)) are the Pauli matrices, \( q \) is the elementary charge, \( c \) is the speed of light, \( A \) is the vector potential, and

\[
\tilde{g} = \frac{\pi}{\sqrt{-(ih\epsilon_n - \sigma)^2 + |\Delta|^2}} \left( \begin{array}{cc} -ih\epsilon_n + \sigma & \Delta \\ -\Delta^* & ih\epsilon_n - \sigma \end{array} \right).
\]

(2)

To incorporate strong-coupling effect, we use Eliashberg equation to calculate the self-energy \( \Sigma \) from the quasiclassical Green’s function;

\[
\Sigma(i\epsilon_n, \alpha, \mathbf{r}) = N_0 k_B T \sum_{|\epsilon_m| < \epsilon_c} \langle v(i\epsilon_n, \alpha, i\epsilon_m, \alpha') \tilde{g}(i\epsilon_m, \alpha', \mathbf{r}) \rangle_{\alpha'\alpha},
\]

(3)

where \( \epsilon_c \) is the cutoff of the Matsubara frequencies, \( N_0 \) is the density of states on the Fermi level, and \( \langle \cdots \rangle_\alpha \) denotes the average over the Fermi surface and is defined \( \langle A(\alpha) \rangle_\alpha = \int_0^{2\pi} d\alpha A(\alpha) / (2\pi) \). We took 47 equally spaced points in the momentum space. We assume that the interaction between electrons \( v \) has the following form:

\[
v(i\epsilon_n, \alpha, i\epsilon_m, \alpha') = \frac{C\alpha_0^2}{(\epsilon_n - \epsilon_m)^2 + \alpha_0^2} \times 2\cos(\alpha - \alpha'),
\]

(4)

where \( \alpha_0 \) is a characteristic frequency of a mediated boson, and \( C \) is a constant parameter. We set these parameters so that \( \hbar\alpha_0 = 3k_B T_c \), where \( T_c \) is the critical temperature of the superconductivity. We choose this value so that the strong-coupling effect is very large but not unrealistic.\(^1\) We set the cutoff of the Matsubara frequencies \( \hbar \epsilon_c = 20k_B T_c \), and confirm that this cutoff is considered to be sufficiently large by comparison of the magnitude of the bulk pair-potential with those for \( \hbar \epsilon_c = 10k_B T_c \) and \( 15k_B T_c \). We also define \( \xi_0 = h v_F / (k_B T_c) \) and use it as a characteristic length of the spatial modulation of the self-energy.

The vector potential \( A \) is obtained from the quasiclassical Green’s function as

\[
\nabla \times (\nabla \times A) = \frac{\hbar}{c} N_0 k_B T \sum_{i\epsilon_n} \text{Tr}(v_F \tilde{g})_{11},
\]

(5)

\(^1\) With this parameter, the ratio of the energy gap to the critical temperature (\( 2\Delta / T_c \)) is about 5.6 at \( T = 0.02 T_c \). For example, CeCoIn$_5$ exhibits such a large value (\( \approx 6 \)) \cite{25}.\n
where Tr is a trace over the Nambu space. We define \( \lambda_0 = (N_0 v_F^2 q^2 c^{-2})^{-1/2} \) as a characteristic length of the electromagnetic entities. We set \( \lambda_0/\xi_0 = 2.5 \) in this paper.

To discuss the stability of the isolated vortices, we calculated the free-energy deviation from the normal state \( \Omega_{\text{sn}} \) with the following equation:

\[
\Omega_{\text{sn}} = \int dr \left( N_0 k_B T \sum_{\alpha} \text{Tr} \left\{ \int_0^1 ds \langle \hat{g}_s \hat{E} \rangle - \frac{1}{2} \left\langle \hat{g} \hat{E} \right\rangle + \frac{B^2}{2} \right\} \right),
\]

where \( \mathbf{B} = \nabla \times \mathbf{A} \) is the magnetic field, and \( \hat{g}_s \) is a solution of

\[
i \hbar \mathbf{v}_F \cdot \nabla \hat{g}_s + [i \hbar e_\alpha \hat{r}_3 + (q/c) \mathbf{v}_F \cdot \mathbf{A} \hat{r}_3 - s \hat{E} \cdot \hat{g}_s] = 0, \]

\[
\hat{g}_s^2 = -\pi^2 \hat{\xi}_0.
\]

The above expression of \( \Omega_{\text{sn}} \) is a simple extension of the weak-coupling BCS one\(^{26, 27} \)\(^{26, 27} \). We used the 15-points Gauss-Kronrod quadrature formula to integrate respect to \( s \).

We numerically confirmed that the self-energy for Matsubara-frequencies \( \tilde{\Sigma}(i\epsilon_n, \alpha, \mathbf{r}) \) can be decoupled as

\[
\tilde{\Sigma}(i\epsilon_n, \alpha, \mathbf{r}) = \hbar(i\epsilon_n) \left[ \tilde{\Sigma}_+(\mathbf{r}) e^{+i\alpha} + \tilde{\Sigma}_-(\mathbf{r}) e^{-i\alpha} \right],
\]

and thus we show only the \( \mathbf{r} \)-dependent part \( \tilde{\Sigma}_+(\mathbf{r}) \) and \( \tilde{\Sigma}_-(\mathbf{r}) \) in the following section. At sufficiently far from the vortex, only \( \tilde{\Sigma}_+ \) or \( \tilde{\Sigma}_- \) survives. Hereafter we assume that \( \tilde{\Sigma}_+ \) is a dominant part of the self-energy and survives in the bulk. As we can see in \( [7] \), Cooper pair of chiral \( p \)-wave superconductivity has internal angular momentum (chirality). If there is a vortex, two types of vortices can exist in this system; one type of vortex has vorticity (the angular momentum of vortex) parallel to the chirality, and the other type has vorticity anti-parallel to the chirality. In the present paper, we call the former “parallel vortex” and the latter “anti-parallel vortex”. In the Ginzburg-Landau(GL) theory, an anti-parallel vortex was shown to be more stable than a parallel vortex\(^{28} \).

To solve \( [1] \), we used so-called Riccati-parametrization method\(^{29, 30} \) and solved the parametrized differential equation with a 4th- and 5th-order adaptive Runge-Kutta method. We used the cylindrical coordinate system and took 48 equally spaced points on the azimuthal coordinates. To improve the accuracy of numerical integration of the free-energy, we used the composite Gauss-Lobatto quadrature to choose discrete points \( r_i \) on the radial line. We divided the closed interval \( [0, 2.4] \) into 16 subintervals, applied the 7-points Gauss-Lobatto formula to each subinterval and obtained 97 discrete points \( x_i \) and changed the variable as \( r_i = x_i^2/2 + x_i^4/3 + x_i^6/4 \) in order to make the sampling points denser near the center and more sparse far from the vortex. We calculated the self-energy from the quasiclassical Green’s functions via \( [1] \) and iterated the above until the self-energy sufficiently converged. After obtaining converged solution, we calculated the free-energy of vortices using \( [7] \). We changed initial profiles of the vortices so that the initial dominant- and induced-vortices were at separate positions, and repeated the same procedure as the above. Finally, we compared the resultant profiles and their free-energies to discuss stability.

3 Results and Discussion

As for the anti-parallel vortices, we only obtained circular axisymmetric vortices for all temperature that we studied \( (T/T_c = 0.1, 0.2, 0.3, 0.4, \text{ and } 0.5) \), regardless of the initial profiles; in this case, the strong-coupling effect just modifies the shape of the vortex.
Fig. 1 Profile of the off-diagonal part of the self-energy of the parallel vortex at $T = 0.3 T_c$ and $\hbar \epsilon_n = \pi k_B T$. Left-top: amplitude of dominant part ($h(i \epsilon_n)(\Sigma_+)_12 / T_c$), right-top: phase of dominant part (arg($\Sigma_+)_12$), left-down: amplitude of induced part ($h(i \epsilon_n)(\Sigma_-)_12 / T_c$), right-down: phase of induced part (arg($\Sigma_-)_12$). (Color figure online)

Fig. 2 Electromagnetic quantities around the non-axisymmetric parallel vortex. Left: magnetic field, right: electric current density. (Color figure online)

On the other hand, at moderately low temperatures, a non-axisymmetric solution emerges for parallel vortices, when the initial vortex sufficiently breaks the axisymmetry. Figure 1 shows the non-axisymmetric profile of dominant and induced parts of off-diagonal self-energy at $T / T_c = 0.3$ and $\hbar \epsilon_n = \pi k_B T$. There the vortex of dominant component forms triangle and those of the induced component split into three. Figure 2 shows the current density around the vortex. We can confirm that the electromagnetic quantities also break the axisymmetry.
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When we calculated parallel vortices at $T/T_c = 0.5$, we found only an axisymmetric vortex: both circular and non-circular initial configurations of self-energy produce the same result. We thus conclude that unusual parallel vortices may not exist at high temperatures.

In Figure 3, we plot the free-energy of each vortex. We can see that at low temperatures, the non-axisymmetric vortex is more stable than the symmetric one. We note that the symmetric anti-parallel vortex is more stable than the non-axisymmetric parallel vortex, at least under the parameters in this study.

There are many studies of non-axisymmetric vortices in spin triplet superfluids or superconductors with an isotropic Fermi surface. However, many of them have targeted vortices in the superfluid $^3$He-B\cite{31,32,33,34,35,36,37}, or an $f$-wave superconductor similar to the $^3$He-B\cite{38}; these studies therefore cannot be compared with our work directly.

Tokuyasu, et al. have studied two-dimensional chiral $p$-wave superconductor within the GL theory in the weak- to strong-coupling regimes\cite{41}. They have reported that non-axisymmetric vortices can emerge in some non-weak-coupling coefficients. However, the coefficients of the GL-functional of our target system fall into the same ones that we obtain in the weak-coupling limit (the $\beta$ parameter in Ref. \cite{41} is 0.5 in our system). Thus, the origin of non-axisymmetric vortices in our work is different from that of the previous work. This is also consistent with the fact that non-axisymmetric vortices only exist at low temperatures in the present work. Aoyama and Ikeda have reported that a vortex of $^3$He-A can be non-axisymmetric under the existence of anisotropic scatterers\cite{39,40}. Their model is different from ours, and the relationship between their and our results considered an important but remaining issue.

4 Conclusion

In this study, we numerically found that a non-axisymmetric vortex metastably exists in strong-coupling chiral $p$-wave superconductors. This anomalous vortex is more stable than the axisymmetric parallel one at sufficiently low temperatures, but symmetric anti-parallel vortex is still most stable. The emergence of this anomalous vortex is a consequence of the strong-coupling effect because we did not obtain such a vortex with the conventional weak-coupling gap equation. To clarify the underlying energetics that makes the non-axisymmetric
vortex metastable is an interesting issue. The total phase diagram of this system is also left as a future issue.

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