Modular structures and extended-modular-group-structures after Hecke pairs

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Abstract. The simplices and the complexes arising form the grading of the fundamental (desymmetrized) domain of arithmetical groups and non-arithmetical groups, as well as their extended (symmetrized) ones are described also for oriented manifolds in $dim > 2$. The conditions for the definition of fibers are summarized after Hamiltonian analysis, the latters can in some cases be reduced to those for sections for graded groups, such as the Picard groups and the Vinberg group. The cases for which modular structures rather than modular-group-structure measures can be analyzed for non-arithmetic groups, i.e. also in the cases for which Gelfand triples (rigged spaces) have to be substituted by Hecke couples, as, for Hecke groups, the existence of intertwining operators after the calculation of the second commutator within the Haar measures for the operators of the correspondingly-generated C$^*$ algebras is straightforward. The results hold also for (also non-abstract) groups with measures on (manifold) boundaries. The Poincaré invariance of the representation of Wigner-Bargmann (spin $1/2$) particles is analyzed within the Fock-space interaction representation. The well-posed-ness of initial conditions and boundary ones for the connected (families of) equations is discussed. As an example, Picard-related equations can be classified according to the genus of the modular curve(s) attached to the solutions(s). From the Hamiltonian analysis, further results in the contraction of the congruence (extended sub-)groups for non-arithmetical groups for the construction of tori is provided as an alternative to the free diffeomorphism group. In addition, the presence of Poincaré complexes is found compatible with non-local interactions, i.e. both lattices interactions or spin-like ones.

1. Introduction

The investigations is aimed at understanding what and if are the hypotheses intertwining operators for the second commutator of the Hecke groups to obtain a von Neumann algebras by studying the orientation-preserving homotopy groups for closed oriented domains on manifolds (i.e. also the domains on which the group are defined) within the functorial equivalence classes ‘borrowed’ from the (also, quantum) groupoid formalism$^1$. Given a holomic module sheaf of a differential operator on a complex manifold $Q,[1]$, the support of the sheaf is a closed homogeneous involution subsvariety of the bundle of the complex manifold, on which for the sheaf $\exists$ an exact functorial type form from the holomic module sheaf of a differential operator of finitely properly-chosen generated $U$-modules. Be therefore $U = U(\mathfrak{g})$ be the enveloping algebra of $\mathfrak{g}$, endowed with center and at least one ideal $[1]$. It is possible to prepare exact sequences of injective applications (from the manifold to the fibre) of the composition of $U(\mathfrak{g}$ for the direct product of holomic modules for direct product of the composition of the to the category generated by the $U(\mathfrak{g}$ on the complex manifold according to their weight. The disjoint union of such injective applications (on appropriately-chosen enclosures) are on a the complex manifold, quotient of a
2. Geometrical study: algebroids

Let $A$ and $B$ be unital algebras. **Def.:** a Hopf algebroid is defined by [5] a structure on $A$, over $B$, consisting of i) the source map: algebra homomorphism $\alpha : B \to A$; ii) the target map: algebra anti-homomorphism $\beta : B \to B$ satisfying $\alpha(a)\beta(b) = \beta(b)\alpha(a)$, for all $a, b \in A$. □

Let $A$ and $B$ be differential graded algebras. **Def.:** a Hopf bialgebroid is defined by $\beta$ a Hopf algebroid is defined by [5] a structure on $A$, over $B$, which admits an algebra anti-isomorphism $\zeta : A \to A$, where for which all the needed maps must both i) be compatible with the differentials; ii) be of degree 0.

The map $\rho$ from the Hopf cyclic cohomology of $H(G)$ s.t. $H(G)$ is equal to the cyclic cohomology of the algebra (i.e. the double complex of sheaves of differential forms on the simplicial manifold) to the groupoid sheaf cohomology of the sheaf $C$ on the fiber either is the identity, or is an isomorphism. □

The proof for the inverse map $\rho^{-1}$ is given in [6].

The $K$ theory of the corresponding $C^*$ algebra of operators is defined at least on a subgroup of the (manifold comprehending the) invariant(s) of these transformations which defines the space spanned by the maps, the antipode map and the co-unit map. It is based on its the (manifold comprehending the) invariant(s) of these transformations which defines the space $\rho$.

Looking for the solution on the fiber

Given a KW section,

- the map $\rho$ between the composition of the action of $B$ on Lie Groups on $H$ by the K-theory of the reduced $C^*$ algebra of operation on Lie groups is an isomorphism;
- the map $\iota$ between the $C^*$ algebra of the action of $\Gamma$ on Lie Groups on $H$ and that on $G$ is an isomorphism, i.e. after the definitions of gradings for Vinberg groups induced by the same homomorphism on cyclic cohomology for which:
  - $\exists N \in \mathbb{N}$ s.t. the $N$-th iteration of $\alpha$, $\alpha^n$ is trivial (idempotent) $\forall n < N$ independently of $\iota$,
  - $\exists$ an assembly injective map(s) from idempotent transformations on groupoid transformations which is an isomorphism,
  - $\exists$ and injective map from the fibration of these transformations to these transformations themselves.

The intertwining operators for the mentioned $C^*$ algebras (i.e. double affine Hecke algebras) are the Dunkl operators $D_j$ for $\beta$ non-negative integer s.t. $i$ $D_j$ preserve the space of polynomials of variables $x_1, \ldots, x_N$; ii) $D_j = \frac{\partial}{\partial x_j} + \beta \sum_{k \neq j, j=1}^{N} \frac{1}{x_j - x_k}$ elements of the symmetric semisimple Lie group and a semisimple Lie algebra [1].

Indeed, the proper covariant representations of a pair generate normal $*$-representations of an enveloping von Neumann algebra compares the covariance $C^*$-algebra of the pair [2].

2 These domains can be arbitrarily extended with the restrictions of [1] imposed on [4], and compared by means of the free (known) Hamiltonian flow for the so-defined algebraic structures (whose eigenvalues have been nevertheless at least conjectured) identifying the homotopy invariance(s) classes) of the second commutator for the considered Hecke (sub-) group. The definition(s) for the fiber in any number of dimensions and also in the presence of Poincaré complexes allows to extend the analysis also for non-arithmetical groups.
group $S(\mathbb{N})$; $ii_b$ act on functions of $x_1, \ldots, x_N$ as operators which permute the arguments $x_i$ and $x_j$. □

Properties of quantum groupoids Quantum groupoids are based on the degenerate double affine Hecke algebra $^3$. It is expected that the results given here extend to the non-degenerate case, i.e. to the $q$-deformed case.

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2.2. Geometrical study: after decomposable groups- Extended Hecke groups, Hecke groups, Picard groups and commutator subgroups

Extended Hecke groups, Hecke groups, Picard groups and commutator subgroups Def.: are subgroups realized by a correspondence that arises $^{[1]}$ between a subgroup of a group of symmetries which defines the tiling of the subdomains, and the transitivity (under the (sub-group)pal) action of the tiling.

Hecke groups $H(\lambda) \in PSL(2;R)$ Def.: are orientation-preserving groups, whose generators are $T(z) = -1/z, U(z) = z + \lambda$, s.t. $S \equiv TU, S(z) = -\frac{1}{z+\lambda}$. A Hecke group $H(\lambda)$ is discrete iff $\lambda = \lambda_0 = 2\cos(\frac{\pi}{3})$, or Def.: a discrete Hecke pair $(G, H)$ is given, when $G$ is a discrete group, and $H$ a subgroup.

All Hecke groups are subgroups of $PSL(2;Z), Z \in \mathbb{Z}[\lambda_0]$. □

A degenerate double affine Hecke algebra is Def.: the triple $\{s_j, x_j^\pm, \hat{D}_j^\pm\}$ with $s_j \equiv s_{j,j+1}$ ($j = 1, \ldots, n1$) (simple) translations, (isomorphic to the) degenerate affine Hecke algebra. Intertwiners between two representations decompose as direct products on different (among each set and among the two sets- orthogonal) polynomials for mapping on algebra representations after $sl(2)$-induced algebra automorphisms. □

The map $\sigma$, the structure form $A$ to $\sigma$, the algebra $B$ are applicable to non-symmetric case. As a consequence, it is possible to construct raising operators and shift operators for Jack polynomials $P^\alpha_{\kappa}$, where $P^\alpha_{\kappa} = <p_{\alpha}, p_{\kappa}>_{\alpha} = \delta^{(\alpha)}_{\kappa}\delta_{\lambda,\mu}$. □

Let $q$ be a root of unity of order $d > 1$, with $d$ odd integer; let $A$ be a Hopf algebra with two generators, $E, K$, connected with $\theta, \epsilon$, s.t. $S(\kappa) = \kappa^d, S(\bar{\kappa}) = -\kappa^d$, as in $[7]$. The $sl(2)$ induced algebra automorphism is therefore $\theta = q^2 S^2, \theta(\kappa) = q^d \kappa, \theta(\bar{\kappa}) = \bar{\kappa}$ as in $[8]$, where $S(K) = K^d, S(E) = EK^d, \epsilon(K) = 1, \epsilon(E) = 0$. □
2.2.1. \( \theta \) groups. Let \( G \) be a reflection group over the \( n \)-dim Hermitian space \( V \), defined on \( \mathcal{R} \) reflections: given \( k : G \rightarrow \mathbb{C}, \exists a \text{ Def.: } G \)-invariant parameter family functions, with \( H(G,k) \) the rational Chernikov algebra, isomorphic to \( \mathcal{P} \otimes \mathbb{G} \otimes \mathbb{S} \) \[12\].

The corresponding Hecke-Drinfeld algebra \( HD \) is defined as \( HD \equiv \mathcal{P} \otimes \mathbb{G} \) \[13\].

The evaluation of the second commutant \[14\], \[15\] (after the evaluation of the first commutator group \[16\]) coincides with the results for the extended modular group; for groups generated by three reflections, it is possible compare the congruence subgroups of the \( PGL(2,\mathbb{Z}) \) group (a.k.a. the extended modular group) and also, in particular, \( \Gamma_{2} \) (\( PGL(2,\mathbb{Z}) \)) group, also in the cases \[17\] \[18\] wrt the graded Hecke algebra[1], [23] [24].

3. Connections types for Vinberg’s \( \theta \) groups and period-grading for Vinberg groups

Let \( G \) be a simple complex group of adjoint type. \textbf{Def.:} \( \forall \theta \)-group \((G_{0}, g_{1})\) and a vector \( X \subset \) its element \( g_{1} \) there \( g \) a flat \( G \)-covariant-derivative \( \nabla \) \text{ locally-monodromic connection} \[25\] \( \Rightarrow \) the operator \( \nabla^{X} \) is on the trivial bundle \[25\]: for \( m \neq 0 \) in the (Kurosh) decomposition Footnote 3, it is possible to choose \( m \in M_{0} \), with \( M(\lambda) \) both a \( T(V) \)-module and a \( g(V) \) element.

3.0.1. Period grading for Vinberg groups. A Vinberg group can be examined as a \( \theta \)-group after imposition\(^5\) of period grading, as implying by Kostant-Weierstrass sections (see Appendix

\(^4\) \text{Kurosh theorem} \[19\] for the product of free groups can be compared as a generalization of \[20\]. Be \( G \) a group with associative multiplication law, and \( g \) one of its elements, \( g \in G \), s.t. \( i) g = h_{1} h_{2} \ldots \), representation of products of two elements \( g,g' \in G \), \( g = h_{1} h_{2} \ldots \), \( g' = h_{1}' h_{2}' \ldots \); \( ii) g^{g'} = h_{1} h_{2} \ldots h_{n} h_{n} \ldots \) invertible associative multiplication law; \( iii) g^{-1} = h_{n}^{-1} h_{n}^{-1} \ldots \) from the explicit representation \( n = f(g) \), but \( h_{k+1} \neq h_{n+1}^{-1} \forall 0 \geq k \geq \min(n,m) \Rightarrow \exists ! \) the (injective and surjective) bijective relation with \( h_{n-k} \) of different elements for each component.

The unit element \( 1 \) does not correspond to the associative law. For \( F \) a subgroup of \( G, F \subset G \), \textbf{Def.:} for an \( F \equiv F_{1} \cap F_{2} \), \( F \) is called the \textit{mean sub-group} for the sub-groups \( F_{1} \) and \( F_{2} \). The use of decomposable groups is aimed at trying to avoid the need for truncated quantum group algebra(s) \[21\].

\textit{Hypergroups} \( \tilde{K} \) are defined as \textbf{Def.:} \( \tilde{K} = \tilde{K} * K \), with \( K \) any copy of \( K \).

The corresponding Hecke algebra is the \( \ast \)-algebra \( M(\tilde{K}) \) of bounded measures on \( \tilde{K} \).

Let \( M(\tilde{K}) \) be endowed with a \( \delta \) Dirac distributional measure (on a Hilbert space \( H \)) \[22\].

The definition of \textit{Gelfand pairs for Hecke groups} also extends given \( M(\tilde{K}) \) endowed with a \( \delta \) Dirac distributional measure (on a Hilbert space \( H \)) \[22\]. \textbf{Def.:} \( (G, K) \) is a \textit{Gelfand pair} if \( G \) locally compact group (endowed with neutral element, where the definition is not defined by idempotence under iterations and the Haar measure on \( G \) is unimodular), \( K \) a compact subgroup.

\textbf{Def.:} \( (G,K) \) is a \textit{generalized discrete Hecke pair} if \( \nu \) a non-degenerate representation of \( \gamma_{0}(K) \) on a \( (Hilbert space) H \), and \( V \) be a \( \ast \)-representation of \( M_{0}(K) \). Furthermore, \textbf{Def.:} \( \exists ! \) intertwiners \( \nu \otimes 1 \Rightarrow V \) is a representation of \( M(\tilde{K}) \) such that \( (\nu, V) \) is a \textit{covariant pair} \( \Leftrightarrow (\nu, V) \) is a covariant pair if \( \exists ! \) a Hilbert space unitarily equivalent to the covariant representation for which \( \nu \) is equivalent to a multiple of \( M \). □

\(^5\) For the sake of the definition of the following von Neumann algebra(s), a few remarks are in order

\textit{Periodically graded semisimple Lie algebras} or, equivalently, \textit{periodic gradings of} \( g \) (element of a free-composition, also, Kurosh, group) and \( \theta \), being \( \theta \) the corresponding \( m - \theta \) order of the automorphism, define the \textit{little Weyl group} \( W(\mathcal{C}, \theta) \) in \( c \). \textbf{Def.:} the little Weyl group of a graded Lie algebra is a subspace \( c \in g_{1} \) and a finite reflection group \( W(c, \theta) \subset c \); for a given \( \theta \in \text{ Aut}(g) \mid \text{ idg } \), an sl2-triple \( \{ e, h, f \} \) is \( \theta \)-adapted , if \( \theta(e) = \epsilon e, \theta(h) = h, \theta(f) = \epsilon^{-1} f \). □

The \textit{regularity of the grading(s)} is classified as: \textbf{Def.:} the grading is \textit{\( N \)-regular} if \( g_{1} \) contains a regular nilpotent element of \( g \).

\textbf{Def.:} the grading is \textit{locally free} if \( \exists x \in g_{1} s. t. \ (z \cap g_{0} = \{ 0 \}) \); it is a centraliser of \( \zeta(x) \in g \) if there exists \( x \in g_{1} s. t. \ (\zeta(x)) = \{ 0 \} \).

Let \( G \) be a finite unitary reflection group on a complex vector space \( V \) and \( g \in G s. t. \gamma \)-eigenspace of the elements of \( G, \gamma(g) \). \textbf{Def.:} the centralizer \( Z(g) \) of \( g \) in \( G \) is a stabilizer for the eigenspace \( \mathcal{E} \) of \( g \).

The same terminology applies also to the element of a \( \theta \) group, \( \theta_{1}, \theta_{2} \); if the corresponding periodic (also, Kurosh, composition) gradings are \( S \)-regular and locally-free \( \Rightarrow \theta_{1}, \theta_{2} \) are conjugated by means of an element of the normal complete intersection \( \pi^{-1} \pi \subset g(\text{Int } g) \).
(Appendix A.0.1)) for $g_0$ semisimple and $m > 3$: with $\theta \in \text{Aut}$ any automorphism of order $m$, if $\theta$ is S-regular and locally free, then the corresponding $\theta$-group admits a KW-section.

If $\theta$ is not locally free, only one of the two subgroups, say $\theta_1$, is admitted for the KW section.

If both $\theta_1$ and $\theta_2$ are S-regular but not locally free, they are not conjugate.

If $\theta \in \text{Aut} g$ is N-regular, there are two automorphisms given by the suitable restrictions.

If $\theta$ is N-regular, its ideal in $\kappa[g]$ is generated by the basic invariants $\pi^1(\pi(c))$ i.e. a complete intersection.

Complex reflections: The Weyl group contains complex transformations, i.e. complex reflections.

Figure 3. $n = 1$ grading for the Vinberg group $x_{\beta} = \frac{\sqrt{5}}{3}$. The domain of the subgrouppal structure for the $n = 1$ grading of the Vinberg (non-arithmetical) group characterized by

- $\beta = \frac{1}{3} \pi$, i.e. $x_{\beta} \simeq \frac{\sqrt{5}}{3}$;
- $0 \leq x \left( \Gamma_0 \left( x_{\beta} = \frac{\sqrt{5}}{3} \right) \right) \leq \frac{\sqrt{5}}{3}, 0 \leq y \leq \frac{1}{2}$;

the complex at $x = \tilde{m} \frac{\sqrt{5}}{3}, \tilde{m} \in (Z)$ is sketched.

Figure 4. $n = 1$ Grading for the subgrouppal structure $\Gamma_0 (\text{Vinberg} \left( x_{\beta} > \frac{1}{2} \right))$. The domain of the subgrouppal structure for the $n = 1$ grading of the subgrouppal structure $\Gamma_0 (\text{Vinberg} \left( x_{\beta} > \frac{1}{2} \right))$;

the simplex $v = 0 \pm \frac{m'}{2}$ is sketched, $m' = 2m + 1$.

4. Outlook: Hamiltonian analysis, Results and Remarks

In the Hamiltonian analysis, cuspidal datum and initial cuspidal data for non-trivial remainders and virtually surjective pairs on subgroups can be compared, i.e. also in the case of complexes. The existence of the invariants (1) allows one for the analysis of initial conditions and boundary conditions.

As the closable involutive operator(s) of the von Neumann algebra involves non-trivial implications, further invariants can be defined: Def: the $\eta-$ invariants are defined as [24]

$$\eta_{\pm} = \lim_{r \rightarrow 1 \pm} \eta, \quad \eta(\nabla_i) = \pm(\ast \nabla_i - \nabla_i \ast), \quad \eta_+ = \eta_0 + \sum_{i \geq 1} \sigma_i, \quad \eta_- = \eta_0 + \sum_{i \geq 1} (-1)^i \sigma_i \quad (1)$$

the domains of the arithmetic groups, as well as those of the non-arithmetical ones, are delimited by surfaces of equal probabilities (as (integral) functions of the composition of two invariants).

for reflections, the definition of the direct sum of character sheaves as a character sheaf follows [1], the algebra of invariants of any of its $\theta$-group is free [26], [1].
4.0.1. Particles representations and eigenfunctions

In the orthonormal directions of the the embedded space the target map $\rho$ ensures a Wigner-Bargman representation via the $\eta$-invariants (1) as

$$\eta^N(\rho(a)) = \eta^N(\rho(\alpha)(a))$$

for any $N$-th iteration of the invariants on $r \in \Lambda$, with $\Lambda$ Poincaré transformations: any Hamiltonian system implying a continuous Hamiltonian flow must contain at least a second order differential operator in the complex $v$ direction (i.e. in the direction along which the grading is not considered).

The presence of simplices and complexes does not imply interaction [33], [27] also as from the 7 Fock-representation-space $K_F$.

The Hamiltonian formalism proved one with the simplest invariant structure (1); i) $\forall$ Poincaré

7 Be $\mathcal{S}$ the corresponding closable involutive operator of the von Neumann algebra. The Fock space is spanned by the operators $\Xi \equiv \text{CPT}$, for which the composition $\mathcal{S} \equiv \Xi \Xi$ holds, with $\Xi_0$, modular conjugation; here, eigenfunctions (physical states) $K_F$ of operators which are CPT-invariant $2\pi$ rotations (after the definition of Hecke algebras of type $D$) (Appendix A.1.1)

$$K_F = \sum_n e^{\pi m^2} P_n,$$

8, with, the spin-statistics-theorem, either $s \in \mathbb{N}$, or $s = j/2 . \in \mathbb{N}$, as summarized in [34], [35]. (At least) holomorphic functions are eigenfunctions both of the Hamiltonian and of the $\eta$- invariant (1); after grading, the symmetry group is still a free diffeomorphism group.

The following cases are distinguished: • $\mathcal{S} = 1$ trivial; • $\mathcal{S} \neq 1$ non-trivial restrictions: • interaction or • issues concerning the manifolds and/or (Haar) measure [8], [9], [10]
complex \( \exists \) the tangent fibration up to equivalence classes (i.e. of fibrewise homotopy):

ii) ∀ Poincaré (-complex) embedding, the normal fibration is the tangent fibration of the Poincaré complex [36] (the tangent fibration of a smooth manifold is an invariant of such a manifold up to the fibration equivalence): \( \Rightarrow \) there is one and only one class of pairs that trivialize the neighborhood (of a point on the complex) [37]. Indeed, a manifold \( V \) can be foliated covariantly to a holonomy groupoid \( G \) on which can apply Lie groups, with an induced map \( H^*(V) = H^*(BG) \): the grading of modular structures (for desymmetrized group domains) provides a measure for the Hamiltonian flow. The analysis of initial-values-conditions for the evaluation of the invariants includes non-trivial results.

Starting from the analyses [19] and [38], in [40], the case elliptic modular equations, which applies also for Picard-Fuchs equations, are studied to generate elliptic curves of genus zero congruence subgroup(s) [41] of \( \Gamma_0(N) \), which admit a covering map \( X_0(N) \to X(1) \), and associated modular forms vanishing only at an infinite simplices or at an infinite complexes. The examinations can be pursued the analysis of [42] for curves of genus one. This analysis is subsequent to the fact that the pertinent equations and the fibration of the pertinent manifolds are not from a Banach space [43], [44], but are at least on Hecke pencils [45], [46].

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Appendix A. A description of the fiber

Kostant-Weierstrass sections are a linear subvariety \( \tilde{v} \) which induces an isomorphism of affine varieties. There exists a grading [47] \( g = g(0) \otimes g(1) \) of the the symmetric space decomposition for \( m = 2 \).
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Figure 8. The desymmetrized Picard group. The domain of the (arithmetical) Picard group, i.e. \( x_B = \frac{1}{2} \).

Figure 9. The desymmetrized Vinberg group. The domain of a (non-arithmetical) Vinberg group, for which \( x_B = \frac{\sqrt{2}}{2} \):
- 0 \leq x_B \leq \frac{\sqrt{2}}{2}, 0 \leq y \leq \frac{1}{2};
- \text{with } \bar{a} \in \mathbb{R}^n, \text{i.e. } a = (1,0), \text{and the trivial automorphism } A = 1 \text{ wrt the } x \text{ direction.}

Appendix A.0.1. Subalgebra isomorphisms and canonical basis elements Let \( H \) be a unital associative algebra isomorphic to \( H_A \). \( H \) restricts to an isomorphism between the \( A \) subalgebras of \( H \), and \( H_A \) generated by \( f(n) \): \( F \in H \) is a unique mixed structure such that the \( H \) elements in \( H_C \) provide with the identification \( U^A \) both \( \subset H^+ \) and \( \subset H_A \); this is a subalgebra isomorphisms.

Canonical basis elements are given as a mixed structure \( \Rightarrow \) the product of canonical basis elements \( b_1 \cdot b_2 \) expanded into a linear combination of canonical basis elements \( \sum c_{b_1}^{b_2} \) of \( (v) \) in \( H \) corresponds to \( v \) under (for) the subalgebra isomorphisms identification, as in [48], [49], [50]: be \( A \) a finite dimensional \( K \)-algebra and \( A \) is of infinite representation type, with \( I \) is a twosided ideal of \( A \), in which case, \( A/I \) has infinite representation type: \( \Rightarrow \) \( A \) is of infinite representation type; be \( B \) a direct summand of \( A \) as a \( (B,B) \)-bimodule \( \Rightarrow \) if \( B \) is of infinite representation type, then so is \( A \Leftrightarrow \) so is \( B \). \( \Box \)

As in [30], [31] be \( A \) be a finite dimensional \( K \)-algebra, and be \( P_1,...P_l \) the complete set of projective indecomposable \( A \)-modules, up to isomorphism: i) if \( \text{End } A(P_j) \) has infinite representation type for some \( 1 \leq j \leq l \), then \( A \) is of infinite representation type; ii) for each \( 1 \leq j \leq l \), the algebra \( \text{End } A(P_j) \) has finite representation type if \( \text{End } A(P_j) \simeq K[x]/\langle x^m \rangle \) for some integer \( m \geq 0 \) (which depends on \( j \), \( m \equiv m(j), m \in \mathbb{N}/0 \) : for a \( A \)-module \( M \), under suitable assumptions, for a finite-dimensional K-algebra there might exist \( P_1,...,P_l \) a set of projective indecomposable \( A \)-modules, up to isomorphism \( \Box \); for \( m \geq 2 \), and \( G \) a complex reductive group, whose elements \( g = \text{Lie}(G) \) are generated by the algebra \( \text{Lie}(G(0)) = g(0), \text{with } G(0) \text{ the normalizer for } g(1) \Rightarrow \text{Def. } d\theta \) is the automorphism of order \( m \) of \( G \).

Appendix A.1. Induced gradings

For the \( \theta \)-induced grading\(^{10} \), a Kostant-Weierstrass (KW)-section for the pair \( (G, \theta) \) is a linear subvariety \( v \) of \( g(1) \) for which the restriction of functions \( k[g(1)]G(0) \to \kappa[i] \) is an isomorphism.

\(^9\) According to the Dipper-James criterion, there is a direct summand in the \( q \) permutation \( Y^A \) s.t. \( Y^A \) contains a unique submodule \( S^A \) : Def.: the fundamental domain for the action of its indecomposable groups is the closure of the affine Weyl group; for \( F_M \) the boundary of a manifold \( M \) and \( V \) a direct sum of indecomposable modules with vertices (or, also, vertex) not conjugated on \( F_M \); by \( m_i \), any indecomposable summand is the direct sum of the indecomposable summands with the \( S^A \) pullback on the direct \( p \) map.

\(^{10}\) The little Weyl group \( W_c \) is generated by pseudoreflections, and the associated isomorphism \( \kappa \) is a polynomial ring.
The KW-section $\exists$ if $g(0)$ is semisimple, for N-regular $\theta$, for all classical graded Lie algebras in zero or good positive characteristic $p$, $p$ not dividing $m$; from [51], for $p > 3$ and $\theta$ an outer automorphism of $G$, KW-sections exist for all classical graded Lie algebras in zero or odd positive characteristic $\Box$.

Appendix A.1.1. Hecke algebra of the type D A Hecke algebra of the type D is defined as Def.: orthogonal group s.t. $i_4$ needed classification of the isomorphism classes of the indecomposable modules; $i_4$ admits a canonical basis; $i_4$ for $D^\lambda \neq 0$, is defined as disjoint union of $KBP(n)$; $i_4$ for $S^\lambda : D^\mu$ the canonical basis in a combinatorial canonical basis, which plays the role of a Fock (occupation number); $i_4$ the characteristic of $F$ is odd [52], [53]. $\Box$

Appendix A.1.2. (KW)-Section for the Vinberg group. The section $W_1$ for the subgroup $\theta_1$, $W_1 \equiv W_1(\theta_1)$ (not necessarily the same as $W^\sigma$ or $W(c, \theta_1)$, a reflection group in $c$, with $\sigma = \theta \mid_x$) defined only if $m_j$ is a co-exponent of $W(c, \theta_1)$, s.t. the identical element is $\epsilon_1 c^d_1 = I, i.e$ composed by the stabilizer $\zeta(x) \in g$ s.t. $\epsilon_1 (i = 1, ..., l)$, which depend only on the connected component(s) of $Aut \ g$ containing $\theta_1$, s.t. $\kappa_i = \kappa_i(\theta, m), \forall (i \in \mathbb{Z}_m))$.

$\Rightarrow \theta \in Aut \ g$ is N-regular; the triple $\{e, h, f\}$ is a $\theta$-adapted regular sl2-triple. $\Box$

The following cases distinguish for $m_j$ an exponent of $W(c, \theta)$ iff $\epsilon_j \zeta^{m_j} = \zeta^{-1}$: if $\theta$ is also S-regular, $m_j$ is a co-exponent of $W(c, \theta)$ iff $\epsilon_j \zeta^{m_j} = \zeta$; if $\theta 2$ is not N-regular, it might happen that $\exists$ a finite reflection group (multiple of $m$) because $\theta 2$ is S-regular; if $\theta$ is neither N-regular nor S-regular, still $\dim \ g 1 \ G 0 = k_1$.

Figure A1. Grading for the subgroup structure $n = 1$ (Vinberg – $x_\beta > \frac{1}{2}$). The domain of the subgroup structure for the grading of the subgroup structure $\Gamma_0(Vinberg - x_\beta > \frac{1}{2})$, the complex $u = 0 \pm 2x_\beta$ is sketched.

Figure A2. $n = 1$ Grading for the Vinberg group $x_\beta = \frac{\sqrt{r_\beta}}{2}$. The domain of the subgroup structure for the $n = 1$ grading of the Vinberg (non-arithmetical) group characterized by

- $\beta = \frac{1}{2} \pi$, i.e. $x_\beta \approx \frac{\sqrt{r_\beta}}{2}$,
- $0 \leq x \left( \Gamma_0 \left( x_\beta = \frac{\sqrt{r_\beta}}{2} \right) \right) \leq \frac{\sqrt{r_\beta}}{2}$, $0 \leq y \leq \frac{1}{2}$,

the complex at $x = \tilde{m} \frac{\sqrt{r_\beta}}{2}$, $\tilde{m} \in (Z)$ is sketched.
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