On the QCD Phase Transition at Finite Baryon Density

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Abstract

We investigate the QCD chiral phase transition at finite chemical potential $\mu$, using the renormalization group (RG) to characterize the infrared behavior of sigma models constrained by the flavor and spacetime symmetries. The results are similar to those obtained from RG analysis of the finite temperature transition at zero baryon density. When there are more than two massless flavors of quarks, a first order transition is predicted for the entire phase boundary. In the two flavor case, a boundary with first and second order regions separated by a tricritical point seems most likely. We discuss the real-world case with two light quarks and an intermediate mass strange quark. Improved lattice data on the temperature transition will strongly constrain the possibilities for the phase boundary.

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1 Introduction

Due to a technical obstruction, our theoretical understanding of QCD at finite baryon density remains limited. The basic difficulty is that the introduction of a chemical potential leads to a complex effective action after integration over the quark fields. This creates severe difficulties for lattice simulations [1] and precludes the use of rigorous inequalities [2]. While intuitive arguments suggest a phase diagram like that displayed in figure (1), with the quark condensate $\langle \bar{q}q \rangle$ playing the role of the order parameter, in reality only the zero density axis of the diagram has been explored in any systematic way. In particular, little is known about the properties of the phase transition at finite chemical potential $\mu$.

This lack of theoretical results is particularly galling considering the relevance of large quark density to topics such as cosmology, astrophysics (neutron stars) and heavy ion collisions. Recent work using an instanton-inspired model of quark interactions at high density [3] suggests some of the possible exotic phenomena.

In this letter we will apply the renormalization group (RG) and the ideas of universality to the QCD chiral phase transition at finite $\mu$. These methods have been successful in condensed matter physics [4] and have previously been applied to the finite temperature transition in QCD [5]. The basic idea is to assume a second order transition and attempt to find a sigma model description, consistent with the symmetries, of the relevant long-wavelength degrees of freedom. In order that the description be self-consistent the model must possess an infrared stable (IRS) fixed point. Lack of such a fixed point signals an instability, and therefore a first order transition. There are two logical weaknesses of this technique. One is that a non-perturbative fixed point might exist, even when the perturbative beta function does not exhibit one. In this case the transition would be second order, but in some novel universality class which is difficult to analyze. Secondly, even when a (perturbative) IRS fixed point exists in the sigma model, the dynamics of the underlying theory at shorter distances can still cause a first order transition to occur before the basin of attraction of the fixed point is reached. A related point has been raised by Kocic and Kogut [6], who point out that the composite nature of the mesons, which is ignored in this analysis, may affect the results.

Having stated these important caveats, we now proceed with the analysis. Massless QCD at finite chemical potential is described by the partition function

$$Z = \int DA \ e^{-S[A]} \ det D(\mu),$$

where

$$D(\mu) = D + \mu \gamma_4.$$  

In our convention $D$ is hermitian and $\gamma_4$ antihermitian. The eigenvalues $\lambda_i$ of $D(\mu)$ are
Figure 1: Phase diagram of QCD for varying baryon density ($n$), temperature ($T$). The shaded region exhibits chiral symmetry breaking.

generally complex, leading to a complex quark determinant. Despite this, the partition function can be rewritten as a sum over real (albeit possibly negative) terms. One way to see this is to work in $A_0 = 0$ gauge, where direct examination of the Dirac equation shows that the spectrum of eigenvalues $\{\lambda_i\}$ associated with a gauge configuration $A_i(\vec{x},x_4)$ is mapped to its complex conjugate $\{\lambda_i^*\}$ when the gauge configuration is parity-inverted to $-A_i(-\vec{x},x_4)$. (The original eigenspinors $\Psi_i(\vec{x},x_4)$ are mapped to $\gamma_2 \Psi_i^*(-\vec{x},x_4)$.) Since the original gauge field and its parity partner have the same action $S[A]$, we can rewrite (1) as a real integral over pairs of gauge configurations. The boundary conditions on the fields in the time direction (periodic for bosons and anti-periodic for fermions) determine the temperature of the system.

We are interested in the effective, long-wavelength description of the theory as we approach the phase boundary in the ($\mu, T$) plane. Let us discuss the symmetry constraints on this effective description. If we approach from the direction of the phase with chiral symmetry breaking, the effective theory must exhibit spontaneous breaking of $SU(N) \times SU(N)$ symmetry (here $N$ is the number of massless quark flavors) with the corresponding Goldstone degrees of freedom. Actually, the issue is somewhat more complicated than this because of
the anomalous $U_A(1)$ symmetry, which we discuss in the next section. As far as spacetime symmetries, the chemical potential term breaks the Euclidean $O(4)$ symmetry down to $O(3)$. This allows a more general form for the ‘time’ derivative parts of the kinetic energy term. In other words $\text{tr}(\partial \Phi)\dagger(\partial \Phi)$ becomes

$$\text{tr}(\partial \Phi)\dagger(\partial \Phi) + a(\mu, T) \text{tr}(\partial \Phi)\dagger(\partial \Phi) + b_1(\mu, T) \text{tr} \Phi^4 \partial_\mu^4 \Phi + b_2(\mu, T) \text{tr} \Phi \partial_\mu^4 \Phi\dagger,$$

where $a(\mu = 0, T) = 1$ and $b_1(\mu = 0, T) = 0$. Physical arguments suggest that $a(\mu, T)$ remains positive. Otherwise, $x_4$-dependent fluctuations of $\Phi$ would grow without bound. It is fortuitous, and somewhat surprising, that the unknown coefficients in (3) are largely irrelevant to the RG analysis, as we discuss below.

At finite temperature the boundary conditions lead to a discretization of the $x_4$ derivatives. Usually one assumes that only the static mode is relevant to the RG analysis, leading to an effectively three dimensional problem. However, this assumption needs to be reconsidered here, if only because we might be interested in the transition at zero temperature and finite $\mu$. When the temperature is exactly zero the RG analysis must be performed in $d = 4$ dimensions. It is clear that the introduction of the $a, b_i$ parameters will affect the manner in which the non-static modes decouple from the IR analysis. This can alter the dynamics of the model as the IR region is approached. However, for any non-zero temperature (as long as $a, b_i$ are not precisely zero), one eventually reaches a scale at which the non-static modes are suppressed and the dimensionality is effectively $d = 3$. Unless $T = 0$, the test of the self-consistency of a second order transition remains the same. We will return to the role of the $a, b_i$ coefficients in section 3.

## 2 Anomalous $U_A(1)$ and eta meson

To determine the precise flavor symmetry at the phase transition we must understand the fate of the $U_A(1)$ axial symmetry. Towards this goal, we consider the two-point functions for particles which are in the same $U(N) \times U(N)$ multiplet but not in the same $SU(N) \times SU(N)$ multiplet. For $N > 2$ it is possible to show that the respective correlation functions become identical as the chiral symmetry is restored. This was originally demonstrated in [7] (see also [3]) with the high temperature phase in mind, but the proof applies equally to high densities. The only difference in the expressions is that the Dirac eigenvalues become complex quantities. The result implies a degeneracy within the entire $U(N) \times U(N)$ multiplet, and a corresponding restoration of the $U_A(1)$ symmetry at the phase transition.

The two-point correlation functions for the $\pi$ and $\eta'$ are given by

$$\langle \eta'(x) \eta'(0) \rangle = \langle \bar{\psi}_i \gamma_5 \psi_i(x) \bar{\psi}_j \gamma_5 \psi_j(0) \rangle$$

(4)
\[ \langle \pi^a(x)\pi^a(0) \rangle = \langle \bar{\psi} \tau^a \gamma_5 \psi(x) \bar{\psi} \tau^a \gamma_5 \psi(0) \rangle \]

We can express these correlators in terms of exact quark propagators
\[ S_A(x,y) = \sum_k \frac{\Psi_k^\dagger(x) \Psi_k(y)}{\lambda_k - im_q}, \]
where \( A \) denotes the background gauge field in which the eigenvalues and eigenfunctions are computed. One finds two types of contributions: a disconnected contribution
\[ \frac{1}{Z} \int DA \ e^{-S[A]} \det \mathcal{P}_{(\mu)} \ \text{tr}[\Gamma S_A(x,x)\text{tr}[\Gamma S_A(0,0)]] \]
and a connected part
\[ \frac{1}{Z} \int DA \ e^{-S[A]} \det \mathcal{P}_{(\mu)} \ \text{tr}[S_A(x,0)\Gamma S_A(x,0)\Gamma]. \]

Here \( \Gamma = \gamma_5 \) for the \( \eta' \) and \( \Gamma = \tau^a \gamma_5 \) for the \( \pi^a \). The connected parts (8) are identical since \([\tau^a,S_A] = 0\). For the pion, the disconnected part is zero since \( \text{tr}[^{\tau^a}] = 0 \). Any \( \eta' - \pi^a \) splitting is the result of (7) for the \( \eta' \).

Further analysis involves the careful consideration of contributions to (7) from different sectors of the gauge field configuration space with topological charge \( \nu \). Only the non-zero \( \nu \) sectors can contribute to a splitting between the \( \eta' \) and the pions. Working in the chirally restored phase, it can be shown that the contributions to the partition function from the sectors with non-trivial topology vanish like \( m_q^{\nu|N} \) as the quark mass \( m_q \) approaches zero. (This is essentially a consequence of the index theorem.) The zero-mode part of the two quark propagators in (7) absorbs exactly two powers of \( m_q \), which implies that for \( N > 2 \) this contribution vanishes entirely when the quarks are exactly massless. This result leads to the following conclusions:

- \( N = 2 \): The \( SU(2) \times SU(2) \) global symmetry is restored in the high density/temperature phase. The \( \eta - \pi^a \) splitting is non-zero, but decreases smoothly to zero with temperature and density as asymptotic freedom suppresses topological fluctuations. It remains an open dynamical question whether the \( \eta \) plays a role in the phase transition, and the relevant sigma model has either a \( U(2) \times U(2) \) or \( O(4) \) flavor symmetry.

- \( N > 2 \): The \( U(N) \times U(N) \) global symmetry is effectively restored (up to high-dimension operators which are probably irrelevant in the IR limit) in the high density/temperature phase. If the transition is continuous, the \( \eta' \) becomes degenerate with the \( \pi^a \)'s at the phase boundary. The effective models of this chiral phase transition must incorporate a \( U(N) \times U(N) \) global symmetry. (Note that the large-\( N_c \) limit with any number of flavors falls into this class.)

Note that this analysis applies only in the phase without chiral symmetry breaking. In the broken phase the limit \( m_q \to 0 \) is more subtle.
### 3 Sigma models and RG flows

Having identified the relevant symmetries constraining our sigma models we can now proceed with the analysis of RG evolution in the infrared. The critical behavior of the $U(N) \times U(N)$ linear sigma model has been studied in $4 - \epsilon$ dimensions \cite{5,6}. The most general renormalizable potential consistent with the symmetries is

$$U(\Phi) = \frac{1}{2} m_\Phi^2 \text{tr} \Phi \Phi^\dagger + g_1 (\text{tr} \Phi \Phi^\dagger) + g_2 \text{tr} (\Phi \Phi^\dagger)^2 .$$  \hfill (9)

The one loop $\beta$-functions for $g_1$ and $g_2$ are

$$\beta_1 = - \epsilon g_1 + \frac{N^2 + 4}{3} g_1^2 + \frac{4N}{3} g_1 g_2 + g_2^2 ,$$

$$\beta_2 = - \epsilon g_2 + 2 g_1 g_2 + \frac{2N}{3} g_2^2 .$$  \hfill (10)

The stability of a fixed point $g^*$ (zero of (10)) is determined by the presence of real and positive eigenvalues for the matrix $w_{ij} = \partial \beta_i / \partial g_j$ at $g = g^*$. The corresponding analysis has been done before for finite temperature ($T \neq 0$) and zero baryon density ($\mu = 0$), where the effective dimensionality is $d = 3$ ($\epsilon = 1$).

For $N > \sqrt{3}$ there is no infrared-stable fixed point with $g_1, g_2 \sim O(\epsilon)$. For example, when $g_2 = 0$ the system effectively becomes the $O(2N)$ linear sigma model. But when both couplings are present the fixed point with $g_2^* = 0$ is unstable in $g_2$ direction. Therefore the phase transition is predicted to be of the first order.

The case with $N = 2$ is more complicated because of the status of the eta meson. If the eta meson becomes massless at the transition, there is a $U(2) \times U(2)$ symmetry. The RG equations are those of (10), and for $N = 2$ they have no IRS fixed point. Otherwise, the relevant model is the $O(4)$ sigma model (n=4 isotropic Heisenberg magnet) with only one coupling. The RG analysis exhibits an IRS fixed point and the possibility of a second order transition.

Now, consider non-zero $\mu$. If the temperature is also non-zero, the effective dimensionality is $d = 3$, and the $a, b$ terms in (3) play no role, as they only affect non-static modes. The analysis in this case is therefore identical to that already performed. In other words, the universality classes available to describe the phase transition along the entire boundary in figure (1), except near zero temperature, are precisely the same as for the transition on the zero density line.

The case of zero temperature is distinct, as there is no discretization of the $x_4$ modes. We must therefore retain the $a, b$ terms and examine their effect on the $d = 4$ beta functions. Simple calculation (at one loop) shows that the new beta functions will be identical, up to
an overall rescaling, to those of (10) with $\epsilon$ set to 0. The one loop analysis is straightforward, requiring only the evaluation of the “fish” diagram. The logarithmic divergence in that graph is changed by an overall constant proportional to $\frac{1}{\sqrt{a}} \arctan\left(\frac{1}{\sqrt{a}}\right)$, but independent of $b_i$. For $N > 2$ there is still no IRS fixed point, and the transition is therefore predicted to be first order on the entire boundary. There is also the possibility of massless fermionic modes which are relevant to the transition. The possibility of massless fermions here is unlikely, as one cannot satisfy the 'tHooft anomaly matching conditions with massless color singlet fermions when $N > 2$ [11]. In any case, the addition of massless baryons coupled to $\Phi$ does not stabilize any fixed point at the one loop level.

For $N = 2$ there is an IRS Gaussian fixed point which could model the second order transition. This leaves open the possibility that the entire phase boundary is second order. For this to be the case there would be a smooth interpolation from the $\epsilon = 1$ to the $\epsilon = 0$ critical behavior as the temperature approaches zero. This seems implausible to us, as it would require a new family of universality classes with critical behavior intermediate between the Gaussian and $O(4)$ fixed points. However, we have no solid evidence to rule out this possibility. In the two flavor case the anomaly matching conditions allow massless fermionic degrees of freedom (parity-doubled baryons) which are relevant to the transition. Including these degrees of freedom leads to a Higgs-Yukawa model like that of the standard model with zero gauge couplings. Triviality of this system again implies a Gaussian fixed point in the IR [12].

4 Discussion

It is somewhat surprising that the introduction of a chemical potential has little effect on the types of sigma models that could govern the QCD phase transition. We should clarify that this is far from a statement that finite baryon density has no effect on the phase transition. In most cases the transition is predicted to be first order, with characteristics such as latent heat and size of discontinuity in order parameter which are presumably strongly dependent on $\mu$. In the case of two flavors, however, there is the intriguing (though implausible) possibility that the transition remains second order at high baryon density. Current lattice data is consistent with a $\mu = 0$, finite $T$ transition which is second order and in the universality class of the O(4) sigma model [10]. This critical behavior could persist along the entire boundary in figure (1), except at the $T = 0$ endpoint, where the fixed point becomes Gaussian.

More likely in the two flavor case is that the transition switches from second to first order at some point along the $(\mu, T)$ boundary. This could occur if, for example, the eta meson becomes light enough to change the behavior from $O(4)$ to $U(2) \times U(2)$. A better understand-
ing of the behavior of the topological susceptibility, and hence the eta mass, at finite baryon density might help to decide this issue. Recent model calculations using instanton-induced interactions [13] and random matrix techniques [14] suggest that the $T = 0$ transition is first order. A transition between first and second order behavior along the boundary in figure 1 would imply a tricritical point. The inclusion of non-zero light quark masses would presumably smooth the second order transition to a crossover, while the first order boundary would remain qualitatively the same. The tricritical point would then become a critical point (second order phase transition) at which the line of first order transitions terminates. Near this critical point the system would exhibit large fluctuations.

We now turn to a discussion of real-world QCD. An important input into this discussion is the character of the real-world finite temperature transition. Unfortunately, there is disagreement on this issue between lattice groups using Kogut-Susskind fermions (Columbia) [15] and Wilson fermions (JLQCD) [16], with the former predicting a smooth crossover and the latter a first order transition. In the most recent simulations of the two flavor case the Wilson method is seen to reproduce the $O(4)$ critical exponents, while the Kogut-Susskind method does not [17]. In what follows we will discuss the implications of both possibilities.

Consider how the nature of the phase boundary changes as we increase the strange quark mass from zero to infinity, as shown in figure 2. In this diagram the vertical axis is strange quark mass and the horizontal axis is the phase boundary itself as a function of baryon density $n_B$ (ie the projection of the boundary from figure 1 onto the $n_B$ axis). At $m_s = 0$ we have a three flavor model and the boundary is predicted to be entirely first order. However, as we increase $m_s$ the small-$n_B$ part of the boundary must disappear, replaced by a smooth crossover. For this to happen, at some intermediate value of $m_s$ a critical point must appear on the boundary, separating the first order and crossover behaviors. This critical point presumably first appears at zero $n_B$ and migrates to larger $n_B$ as $m_s$ is increased. If the massless two flavor boundary has a tricritical point, the line of critical points (heavy line) in figure 2 will terminate at the point $A$. Otherwise the endpoint is at some $A'$ on the far right of the diagram, signalling the absence of any phase boundary above some critical $m_s$. The position of the real-world value of $m_s$ on this diagram is currently unknown, pending better lattice simulations. If the Columbia group is correct, and the zero density transition is a crossover, then the value of $m_s$ is as drawn in figure 2 and the real-world phase boundary is likely to have a critical point. If JLQCD is correct, and the zero density transition is first order, the $m_s = 150 \text{ MeV}$ line is much lower on the diagram and the entire phase boundary is first order.

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Figure 2: Nature of QCD phase boundary for varying strange quark mass \( (m_s) \) and baryon density \( (n_B) \). The heavy line terminating at A is a line of second order transitions (critical points).

clarify the current status of lattice simulations. This work was supported in part under DOE contracts DE-FG02-91ER40676 and DE-FG06-85ER40224.

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