Mass and free energy in the thermodynamics of squashed Kaluza–Klein black holes

Yasunari Kurita$^1$ and Hideki Ishihara$^2$

$^1$ Advanced Mathematical Institute, Osaka City University, Osaka 558-8585, Japan
$^2$ Department of Mathematics and Physics, Osaka City University, Osaka 558-8585, Japan

E-mail: kurita@sci.osaka-cu.ac.jp and ishihara@sci.osaka-cu.ac.jp

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Abstract

The Abbott–Deser mass, the Hamiltonian and the Komar mass of the five-dimensional Kaluza–Klein black hole with squashed horizons take different values. Introducing a new couple of thermodynamic variables for the Komar mass, we show that each mass can be interpreted as a thermodynamic potential with its own natural variables, i.e. all masses are related to each other by the Legendre transformations. It is found that the new variables and the gravitational tension represent the squashing of the outer horizon.

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1. Introduction

In recent years, studies on black holes in higher dimensions have attracted much attention. Some of these studies show that they have a much richer and more complicated structure than four-dimensional ones. The study of Kaluza–Klein (KK) black holes is particularly important in association with our apparent four-dimensional spacetime. The Gregory–Laflamme instability [1] (see also [2] and references therein) and black hole/black string phase transition (see e.g. the reviews [3]) have motivated many studies on the thermodynamic aspects of KK black holes. The thermodynamic properties and the first law for asymptotically flat KK black holes have now been widely investigated [4–10].

In five-dimensional Einstein–Maxwell theory, there is an analytic solution representing an electrically charged black hole with squashed horizons [11] as a generalization of the solution given in [12, 13]. Intriguingly, the spacetime far from the black hole is locally a product of the four-dimensional Minkowski spacetime and $S^1$. In this sense, the black hole resides in the KK spacetime and is worth being named a KK black hole.

The black hole has an interesting property that various definitions of mass take different values, which means that the black hole gives an opportunity to investigate the differences in various definitions of mass. In this paper, we show some expressions for the first law of black
hole thermodynamics satisfied by those masses and discuss the differences from the viewpoint of thermodynamics.

2. Kaluza–Klein black hole

Let us review the KK black holes with squashed horizons [11], which is a solution of the five-dimensional Einstein–Maxwell theory. The action is given by

\[ I = \frac{1}{16\pi G} \int_M \sqrt{-g} \left[ R - F_{\mu\nu} F^{\mu\nu} \right] + \frac{1}{8\pi G} \int_{\partial M} K \sqrt{-h} \ d^4 x, \tag{1} \]

where \( G \) is the five-dimensional Newton constant, \( g_{\mu\nu} \) is the metric, \( R \) is the scalar curvature, \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \) is the field strength of the five-dimensional \( U(1) \) gauge field \( A_\mu \). The second term is the so-called Gibbons–Hawking term, in which \( h \) is the determinant of the induced metric and \( K \) is the trace of the extrinsic curvature of the boundary \( \partial M \), respectively.

The metric of the black hole is given by

\[ ds^2 = -V(\rho) \ d\tau^2 + B(\rho) V(\rho) \ d\rho^2 + \rho^2 \ d\Omega^2 + r_\infty^2 \left( d\psi + \cos \theta \ d\phi \right)^2, \tag{2} \]

where \( d\Omega^2 = d\theta^2 + \sin^2 \theta \ d\phi^2 \) is the metric of the unit two-dimensional sphere and

\[ V(\rho) = \left( 1 - \frac{\rho_0}{\rho} \right) \left( 1 - \frac{\rho_-}{\rho} \right), \quad B(\rho) = 1 + \frac{\rho_0}{\rho}, \quad r_\infty = 2 \sqrt{(\rho_+ + \rho_0)(\rho_- + \rho_0)}. \tag{3} \]

Here, the coordinate ranges are \( 0 \leq \theta < 2\pi, 0 \leq \phi < 2\pi, 0 \leq \psi < 4\pi \). The gauge potential is given by

\[ A = \mp \frac{\sqrt{3}}{2} \frac{\rho_0 \rho_-}{\rho_0 \rho_+} \left( 1 + \frac{\rho_0}{\rho} \right) d\tau. \tag{4} \]

It is easy to see that the apparent singularity at \( \rho_+ \) corresponds to the outer horizon of the black hole. The inner horizon \( \rho_- \) is analogous to that of the Reissner–Nordström (RN) black holes. The spatial infinity corresponds to a limit \( \rho \to \infty \). It should be noted that the shape of the horizon is a squashed sphere as discussed in [11]. From the metric (2), it is seen that the \( S^1 \) circle parametrized by a coordinate \( \psi \) has finite size even at the spatial infinity. The non-trivial twisting of the \( S^1 \) circle fibrated over the \( S^2 \) base space leads to a four-dimensional \( U(1) \) gauge field by the KK reduction. Actually, in no horizon limit \( \rho_+, \rho_- \to 0 \), the black hole spacetime becomes the KK monopole spacetime [14, 15]. In the limit \( r_\infty \to \infty \), the KK monopole becomes five-dimensional Minkowski spacetime and the black hole reduces to the five-dimensional RN black hole. We term this limit the spherically symmetric limit.

Given the metric (2), we can calculate various physical quantities. The surface gravity is calculated as

\[ \kappa_+ = \frac{\rho_+ - \rho_-}{2 \sqrt{\rho_+ (\rho_+ + \rho_0)}}, \tag{5} \]

which gives the Hawking temperature of the black hole \( T = \kappa_+/2\pi \) [16]. We assume that the entropy of the black hole is given by the Bekenstein–Hawking formula

\[ S = \frac{A_+}{4G} = \frac{4\pi^2}{G} \rho_+ (\rho_+ + \rho_0) \sqrt{\rho_+ (\rho_+ + \rho_0)}, \tag{6} \]

which is consistent with the Wald entropy formula [17, 18]. The electric charge and electrostatic potential of the black hole are also calculated as [11]

\[ Q = \pm \frac{\sqrt{3}\pi}{G} r_\infty \sqrt{\rho_+ \rho_-}, \quad \Phi = \pm \frac{\sqrt{3}}{2} \sqrt{\rho_+ / \rho_-}. \tag{7} \]
3. Mass and free energy

There are several definitions of mass for the black hole spacetime. Cai et al [19] discussed the mass of the black hole defined by the counter-term method for asymptotically locally flat spacetime [20–22]. Using the counter-term mass $M_{ct}$, they investigated the first law of black hole thermodynamics and suggested the existence of a new-work term in the first law. The direct calculation reveals

$$M_{ct} = M_{AD} = \frac{\pi}{2G} r_{\infty} (2\rho_+ + 2\rho_- + \rho_0),$$

(8)

where $M_{AD}$ is the Abbott–Deser (AD) definition of mass [23] evaluated on a product $S^1$ bundle over four-dimensional Minkowski spacetime, which is a completely flat spacetime. Hereafter, we term this spacetime flat background, shortly. Therefore, $M_{AD}$ satisfies the same equations as $M_{ct}$. The Komar mass is also meaningful as a mass of black holes which possess a timelike Killing vector. Using the timelike Killing vector $\partial/\partial \tau$ normalized at the spatial infinity, we can calculate the Komar mass for the black hole (2) as

$$M_K = \frac{3\pi}{4G} r_{\infty} (\rho_+ - \rho_-),$$

(9)

where the integral is taken over the three-dimensional sphere at the spatial infinity. The Smarr-type formula was shown generally in [24] for the Komar mass

$$M_K - Q/\Phi_1 = \frac{3}{2} TS,$$

(10)

which is sometimes called the integrated expression for the first law. This implies that we would have a differential expression for the first law using the Komar mass. Clearly, since $M_K$ does not equal $M_{ct}$, then the expressions for the first law satisfied by these masses should be different.

In order to deduce the work term suggested by Cai et al, we note the fact that, far from the black hole, the geometry locally looks like the black string. It is known that, in the case of the black p-branes or black string without twisting, the so-called gravitational tension and the size of the extra-dimension contribute to the first law [4–6, 25]. One may think that, also in the case of the squashed KK black hole, these quantities contribute to the first law as a work term. The gravitational tension which can be applied to the non-asymptotically flat spacetime was defined by using the Hamiltonian formalism to a foliation of the spacetime along the asymptotically translationally-invariant spatial direction [26]. The definition requires some reference spacetime in order to give finite gravitational tension. As a reference background, we choose the flat background. Then, using the definition given in [26], we can calculate the gravitational tension associated with the direction $\partial \psi$ as

$$T = \frac{1}{4G} (\rho_+ + \rho_- + 2\rho_0).$$

(11)

The conjugate variable to $T$ is the size of the extra-dimension at the spatial infinity, $L := 2\pi r_{\infty}$. With these quantities, $M_{AD}$ is related to the entropy and the electric charge via the following expression for the first law:

$$dM_{AD} = T dS + \Phi dQ + T dL.$$  

(12)

Thus, the last term $T dL$ is thought of as a work term suggested by Cai et al [19]. The expression for the first law (12) shows that $M_{AD}$ is a thermodynamic potential as a function of

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3 Cai et al showed that $M_{ct}$ equals the AD mass evaluated on a ‘twisted’ $S^1$ bundle over the four-dimensional Minkowski spacetime. However, this background spacetime is not a solution of the vacuum Einstein equation and we cannot obtain finite free energy and the Hamiltonian of the black hole on it. Thus, we consider the flat background.
It means that $M_{AD}$ is relevant to the thermodynamic system with natural variables $(S, Q, L)$. The same holds true for $M_{ct}$, because of $M_{ct} = M_{AD}$.

The free energy of the black hole is obtained from the classical Euclidean action $I_E$ as

$$F = T I_E = \frac{\pi}{2G} r_\infty (\rho_+ + \rho_-),$$

(13)

which was calculated by the traditional background subtraction method with the flat background [27]. The free energy $F$ equals one evaluated by the counter-term method. Thus, in this case, the counter-term method is equivalent to background subtraction method with the flat background. In the calculation of the free energy by the background subtraction method, we fixed the temperature, the electro-static potential and the size of the extra-dimension at the boundary of the spacetime. It follows that the free energy satisfies the following differential relation:

$$dF = -S dT - Q d\Phi + T dL.$$  

(14)

Thus, the free energy $F$ has natural variables $(T, \Phi, L)$. Equations (12) and (14) suggest the following relation with the AD mass,

$$F = M_{AD} - Q/\Phi_1 - TS,$$  

(15)

which can be easily shown by direct calculation. Relation (15) is nothing but the Legendre transformation between $F$ and $M_{AD}$.

For asymptotically flat electrically charged black holes, it is known that the Hamiltonian does not equal ADM mass and these two quantities differ by $Q/\Phi$ [28]. The black hole we consider here is not asymptotically flat but asymptotically locally flat. In spite of the difference in the asymptotic structure, if we regard the AD mass as a counterpart of the ADM mass, the same is true for the case of the squashed black hole; the Hamiltonian of the black hole evaluated on the flat background can be related to the AD mass as [29]

$$H = \frac{\pi}{2G} r_\infty (2\rho_+ - \rho_- + \rho_0) = M_{AD} - Q/\Phi.$$  

(16)

Often, the Hamiltonian for the solution without electric charge is called Hawking–Horowitz (HH) mass [30]. In the case of $Q = 0$, the HH mass of the black hole is equal to the AD mass as is in the case of the asymptotically flat black hole. From the relation (16), the first law can take the form with the Hamiltonian as

$$dH = T dS - Q d\Phi + T dL.$$  

(17)

Equivalently, relation (16) is a Legendre transformation between the Hamiltonian and the AD mass. Equation (17) shows that the Hamiltonian is the thermodynamic potential with natural variables $(S, \Phi, L)$. The Hamiltonian is also the Legendre transform of the free energy with respect to $TS$; $F = H - TS$. Therefore, the AD mass and the Hamiltonian are related to the free energy $F$ via Legendre transformations and can be interpreted as different thermodynamic potentials.

However, it can be shown that the Komar mass does not have $T$ or $L$ as a natural variable. Now, let us obtain the differential expression for the first law by the use of the Komar mass. In order to do so, we require that the Komar mass should be related to the free energy via Legendre transformations. Instead of $(L, T)$, we introduce a couple of new variables $(\epsilon, \Sigma)$ satisfying

$$F = M_K - TS - Q\Phi + \epsilon \Sigma, \quad dM_K = T dS + \Phi dQ - \epsilon d\Sigma.$$  

(18)

Then, $(\epsilon, \Sigma)$ is determined up to a constant $C$ as

$$\epsilon = C (2\pi r_\infty)^2 = CL^2, \quad \Sigma = \frac{1}{16\pi GCr_\infty} (\rho_+ + \rho_- + 2\rho_0) = \frac{T}{2CL}.$$  

(19)
Thus, the Komar mass has this quantity $\Sigma$ as a natural variable as well as $S$ and $Q$. The pair of variables $(\epsilon, \Sigma)$ contributes not only to the differential relation for the Komar mass but also to the following:

$$dF = -S \,dT - Q \,d\Phi + \Sigma \,d\epsilon, \quad dH = T \,dS - Q \,d\Phi + \Sigma \,d\epsilon,$$

$$dM_{AD} = T \,dS + \Phi \,dQ + \Sigma \,d\epsilon. \quad \tag{20}$$

Therefore, the expression for the first law including the free energy, the Hamiltonian or the AD mass is not unique. These expressions are consistent with the interpretation that $H$ or $M_{AD}$ is the thermodynamic potential with natural variables $(S, \Phi, L)$ or $(S, Q, L)$, because the system with fixed $L$ is equivalent to that with fixed $\epsilon$ due to the relation $\epsilon \propto L^2$.

In order to clarify the relation with the case of the five-dimensional RN black hole, let us consider the spherically symmetric (SS) limit $r_\infty \to \infty$. However, the free energy, the Hamiltonian and the AD mass evaluated on the flat background diverge in the limit. As an alternative background, we consider the KK monopole spacetime. The free energy and the Hamiltonian on the KK monopole background are calculated as

$$\tilde{F} = \frac{\pi}{2G} r_\infty \left( \rho_+ + \rho_0 - \frac{r_\infty}{2} \right), \quad \tilde{H} = \frac{\pi}{2G} r_\infty \left( 2\rho_+ - \rho_- + \rho_0 - \frac{r_\infty}{2} \right). \quad \tag{21}$$

It is easily checked that the difference between $F$ and $\tilde{F}$ is the free energy of the KK monopole on the flat background; $F - \tilde{F} = \frac{\pi}{2G} r_\infty^2$, which equals the free energy calculated by the counter-term method [31]. The gravitational tension calculated on the KK monopole background is

$$\tilde{T} = \frac{1}{4G} (\rho_+ + \rho_- + 2\rho_0 - r_\infty). \quad \tag{22}$$

With this tension, the free energy and the Hamiltonian satisfy

$$d\tilde{F} = -S \,dT - Q \,d\Phi + \tilde{T} \,dL, \quad d\tilde{H} = T \,dS - Q \,d\Phi + \tilde{T} \,dL. \quad \tag{23}$$

As before, $\tilde{T}$ or $L$ can not be a natural variable for the Komar mass. As in the previous case, we can obtain a couple of thermodynamic variables $(\epsilon, \tilde{\Sigma})$ satisfying

$$\tilde{F} = M_K - TS - Q\Phi + \epsilon \tilde{\Sigma}, \quad dM_K = T \,dS + \Phi \,dQ - \epsilon \,d\tilde{\Sigma}. \quad \tag{24}$$

The result is

$$\epsilon = C (2\pi r_\infty)^2 = CL^2, \quad \tilde{\Sigma} = \frac{1}{16\pi G C r_\infty} (\rho_+ + \rho_- + 2\rho_0 - r_\infty) = \frac{\tilde{T}}{2CL}, \quad \tag{25}$$

where $\tilde{\Sigma}$ is different from $\Sigma$ by a constant $(16\pi GC)^{-1}$. Therefore, the two quantities $\Sigma$ and $\tilde{\Sigma}$ are essentially the same, and the differential expressions (18) and (24) are equivalent. This is consistent with the fact that the Komar mass does not depend on the choice of the background spacetime. In the SS limit, the products $\epsilon \tilde{\Sigma}$ and $L \tilde{T}$ become zero, and $M_K$, $\tilde{F}$ and $\tilde{H}$ become those of the five-dimensional RN black hole evaluated on the five-dimensional Minkowski background. Thus, this formulation for the squashed black hole includes the usual thermodynamic formulation for the five-dimensional RN black hole as a limit. In this sense, it is a generalized formulation of thermodynamics for five-dimensional electrically charged static black holes.

Because $\tilde{\Sigma}$ and $\tilde{T}$ vanish when the outer horizon is a perfectly-round three-sphere, one may think that the thermodynamic variable $\tilde{\Sigma}$ or $\tilde{T}$ can be interpreted as quantities representing the deformation in the shape of the horizon. In fact, the variables $\tilde{\Sigma}$ and $\tilde{T}$ can be rewritten as

$$\tilde{\Sigma} = \frac{1}{32\pi G C} \frac{1}{R_+ \ell_+} (\ell_+ - R_+)^2, \quad \tilde{T} = \frac{1}{8G} \frac{r_\infty}{R_+ \ell_+} (\ell_+ - R_+)^2, \quad \tag{26}$$

where we have defined new parameters $R_+$ and $\ell_+$ as follows:

$$R_+ := \sqrt{\rho_+ (\rho_+ + \rho_0)}, \quad \ell_+ := \sqrt{\rho_- (\rho_- + \rho_0)}. \quad \tag{27}$$
which denote the circumference radius of the $S^2$ base space at the outer horizon and that of the twisted $S^1$ fibre there, respectively. In this way, $\tilde{\Sigma}$ and $\tilde{T}$ measure the squashing of the outer horizon.

4. Summary

As shown in this paper, the Abbott–Deser mass which equals the counter-term mass, the Komar mass and the Hamiltonian contribute to different expressions for the first law and are related to each other by the Legendre transformations. Each mass can be interpreted as a thermodynamic potential with its own natural variables. The consistent set of natural variables for each mass has been revealed, and we have obtained a more general thermodynamic formulation for electrically charged black holes in the five-dimensional Einstein–Maxwell system.

Now, we discuss the relation between $(L, T)$ and the pair of new quantities $(\epsilon, \Sigma)$. Let us begin by considering the case of the free energy. In the evaluation of the free energy, the size of the extra-dimension at spatial infinity $L$ was fixed, so that $L$ is a natural variable of the free energy. In general, $L$ can be replaced by any monotonic function of $L$, say $f(L)$, as a thermodynamic variable. This may be trivial because the thermodynamic environment characterized by fixing the size $L$ is equivalent to that by fixing $f(L)$. One may write the differential relation for the free energy as

$$dF = -SdT - Qd\Phi + Wdf(L),$$

(28)

where $W$ is conjugate to $f(L)$. Since the last work term can be rewritten as $Wdf(L) = Wf'(L)dL$, relation (28) is equivalent to (14) if the conjugate variable $W$ satisfies

$$W = \frac{T}{f'(L)}.$$  

(29)

In this way, the work term with the form $TdL$ can be replaced by $Wdf(L)$. The first law (28) is equivalent to (20), if $f(L) = CL^2 = \epsilon$, and the relation between $\Sigma$ and $T$ is given as

$$\Sigma = \frac{TL}{2\epsilon} \propto \frac{T}{2L}.$$  

(30)

The Hamiltonian and the AD mass are Legendre transforms of the free energy with respect to $TS$ or $TS + Q\Phi$ respectively, so that the expressions in (20) are equivalent to (12) and (17). Therefore, the work term in the first law for the Hamiltonian and the AD mass is not unique.

However, the quantities $T$ and $\Sigma$, which are conjugate to $L$ and $\epsilon$, are different in the sense of thermodynamics: the thermodynamic environment characterized by fixing $\Sigma$ is one by fixing $T/2L$, as shown in (30). Thus, the pair $(\epsilon, \Sigma)$ is thermodynamically different from $(L, T)$. It is natural that thermodynamic potentials suitable for different environments are different. The Komar mass is a thermodynamic potential for an environment characterized by $(S, Q, \Sigma)$, while the AD mass is a thermodynamic potential for that by $(S, Q, \epsilon)$ or $(S, Q, L)$. In this way, we can interpret the difference of masses from the thermodynamical viewpoint.

It will be interesting to investigate the thermodynamic properties and stability of the black hole in each environment and to compare the black hole with black string. This will be reported in a future publication.

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