We consider diquark condensation in external chromomagnetic fields at non–zero temperature. The general features of this process are investigated for various field configurations in relation to their symmetry properties and the form of the quark spectrum. According to the fields, there arises dimensional reduction by one or two units. In all cases there exists diquark condensation even at arbitrary weak quark attraction, confirming the idea about universality of this mechanism in a chromomagnetic field. Possible influence of a nonzero chemical potential on the results obtained is also discussed.

1. INTRODUCTION

Nonperturbative effects in QCD at low energies (large distances) can only be studied by approximate methods in the framework of various effective models proposed. Among such nonperturbative effects are the existence of the QCD vacuum with gluon and quark condensates [1] and the hadronization process. One of the possibilities to approximately describe the gluon condensate is to introduce background color fields of certain configurations. One may, in particular, study the influence of external (background) color fields on quarks [2]. In this case it is possible to find expressions for the quark Green’s functions with exact consideration for the gauge field strength. This approach enables one to make analytical calculations in order to obtain estimates of various nonperturbative processes, such as fermion condensate formation in constant non-Abelian fields [3], thermodynamical stabilization of the vacuum state in an SU(2) model of QCD with condensate fields [4], deep inelastic hadron scattering influenced by gluon vacuum fields [5] etc.

As is well known, the physics of light mesons can be described by effective four-fermion models such as the Nambu–Jona–Lasinio (NJL) quark model, which was successfully used to implement the ideas of dynamical chiral symmetry breaking (DχSB) and bosonization (see e.g. [6] and references therein; for a review of (2+1)-dimensional four-quark effective models see [7]). In particular, for a QCD–motivated NJL–model with gluon condensate and finite temperature, it was shown that a weak gluon condensate plays a stabilizing role for the behavior of the constituent quark mass, the quark condensate, meson masses and coupling constants for varying temperature [8]. The influence of temperature, chemical potential [9], and the external magnetic field [10] on the phase structure of various modifications of the Nambu-Jona-Lasinio model was also discussed.

Moreover, it is in the framework of four–fermion models that a constant magnetic field was shown [11] to induce DχSB, as well as the fermion mass generation, even under conditions when the interaction between fermions is weak. Later, this phenomenon, i.e., the effect of magnetic catalysis, was explained basing upon the idea of effective reduction of space dimensionality in the presence of a strong external magnetic field [12] (see also paper [13] and references therein). It was also demonstrated that a strong chromomagnetic (i.e., nonabelian) field catalyzes DχSB [14]. As was shown in [15], this effect can be understood in the framework of the dimensional reduction mechanism as well, and it does not depend on the particular form of the constant chromomagnetic field configuration.

Recently, the effect of diquark condensation and possible color superconductivity (CSC), has attracted much attention and has been discussed in various publications (see e.g. [16] – [19], and also the review paper [20] and references therein). One may expect that, similar to the case of the quark
condensate, the process of diquark condensation can be catalyzed by intensive external (vacuum) gauge fields. For a (2+1)-dimensional model, this was recently discussed in [21].

The purpose of the present paper is to further investigate this possibility, now for a (3+1)-dimensional model including (\bar{q}q)-and (qq)-interactions, for various external chromomagnetic fields like non-abelian axial–symmetric and rotational–symmetric ones, as well as for abelian fields. In particular, we will show that in all cases, even for weak coupling of quarks, the diquark condensation effect induced by external chromomagnetic fields does exist and is related to an effective dimensional reduction. Moreover, we will find a simple relation between symmetry properties of external fields, the degeneracy of quark energy spectra and the phenomenon of dimensional reduction. The latter effect leads to a nonanalytic logarithmic dependence of the diquark condensate on the field strength in the strong field limit. We shall also consider the effect of finite temperature and show that in the strong field limit there exists a finite critical temperature, at which a phase transition takes place and color symmetry is restored in both abelian and non-abelian models of the gluon condensate. In particular, there arises the BCS relation $T_{C1} = C|\delta(0)|$ between the critical temperature and the zero temperature diquark condensate $\delta_0(0)$, with a universal constant $C$ for different fields. Finally, we shortly discuss the influence of a nonzero chemical potential on the results obtained.

2. QUARK AND DIQUARK CONDENSATES IN EXTERNAL FIELDS

2.1 General definitions

Let us consider an NJL model, which describes the interaction of flavored and colored quarks $q_{i,\alpha}$ ($i=1,\ldots,N_f$, $\alpha=1,\ldots,N_c$) with $N_f=2$, $N_C=3$ as numbers of flavors and colors, respectively (for convenience, corresponding indices are sometimes suppressed in what follows), moving in an external chromomagnetic field. The underlying quark Lagrangian is chosen to contain four-quark interaction terms, which later on are shown to be responsible for spontaneous breaking of both chiral and color symmetries. Hence, two types of condensates characterize the ground state of the model: the quark condensate $<\bar{q}q>$ (spontaneous breaking of chiral symmetry), and the diquark condensate $<qq>$ (spontaneous breaking of color symmetry). Upon performing the usual bosonization procedure [22], [6] and introducing meson and diquark fields $\sigma$, $\pi$ and $\Delta^b$, $\Delta^b$, the four-quark terms are replaced by Yukawa interactions of quarks with these fields, and the Lagrangian takes the following form (our notations refer to four-dimensional Euclidean space with $it = x_4$) \[\mathcal{L} = -\bar{q} (i\gamma_\mu \nabla_\mu + i\mu \gamma_0 + \sigma + i\gamma^5 \overline{\sigma} \pi) q - \frac{1}{4G} (\sigma^2 + \pi^2) - \frac{1}{4G_1} \Delta^b \Delta^b - \Delta^b [i\bar{q} \epsilon^b \gamma^5 q] - \Delta^b [i\bar{q} \epsilon^b \gamma^5 C q].\] (1)

Here $\mu$ is the chemical potential, and $G$, $G_1$ are (positive) four-quark coupling constants (this becomes evident when integrating out the bosonic fields). Furthermore, $\nabla_\mu = \partial_\mu - igA^a_\mu \lambda^a / 2$ is the covariant derivative of quark fields in the background field $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + gf_{abc} A^b_\mu A^c_\nu$, determined by

---

1 We consider $\gamma$–matrices in the 4–dimensional Euclidean space with the metric tensor $g_{\mu\nu} = \text{diag}(-1, -1, -1, -1)$, and the relation between the Euclidean and Minkowski time $\sigma^0_{(E)} = i\tau^0_{(M)}$: $\gamma^0_{(E)} = i\tau^0_{(M)}$, $\gamma^i_{(E)} = \gamma^i_{(M)}$. In what follows we denote the Euclidean Dirac matrices as $\gamma_\mu$, suppressing the subscript $(E)$. They have the following basic properties $\gamma^0_\mu = -\gamma_\mu$, $\{\gamma_\mu, \gamma_\nu\} = -2\delta_{\mu\nu}$. The charge conjugation operation for Dirac spinors is defined as $\bar{\psi}(x) = C (\psi(x)^*)^t$ with $C\gamma^\dagger_0 C^{-1} = -\gamma_0$. We choose the standard representation for the Dirac matrices (see [17]). The $\gamma$ has the following properties: $\{\gamma^a, \gamma^b\} = 0$, $\gamma^+_\mu = \gamma_\mu = \gamma_5$. Hence, one finds for the charge-conjugation matrix: $C = \gamma^5$, $C^+ = C^{-1} = C^t = -C$. 

2
the potentials $A^a_\mu (a = 1, \ldots, 8)$, and $\lambda_a/2$ are the generators of the color $SU_c(3)$ group. Finally, $\vec{\tau} \equiv (\tau^1, \tau^2, \tau^3)$ are Pauli matrices in the flavor space, $\varepsilon$ and $\varepsilon^b$ are operators in the flavor and color spaces with matrix elements $(\varepsilon)^{ik} \equiv \varepsilon^{ik}$, $(\varepsilon^b)^{a\beta} \equiv \varepsilon^{a\beta b}$, where $\varepsilon^{ik}$ and $\varepsilon^{a\beta b}$ are totally antisymmetric tensors, and $t$ denotes the transposition operation. Clearly, the Lagrangian (2) is invariant under the color $SU_c(3)$ and the chiral $SU(2)_L \times SU(2)_R$ groups.

In order to investigate the possible generation of quark and diquark condensates in the framework of the initial model (1), let us introduce the partition function $Z$ of the system

$$Z = \int dq dq d\sigma d\pi d\Delta^b d\Delta^{ab} \exp \left[ \int d^4x L \right].$$

(2)

Next, we shall evaluate the functional integral over meson and diquark fields in (2) by using the saddle point approximation, neglecting field fluctuations around the mean-field (classical) values $<\sigma>=\sigma_0$, $<\pi>=\pi_0=0$ and $<\Delta^b>=\Delta_0^b$, $<\Delta^{ab}> = \Delta_0^{ab}$. We then obtain the following gap equations

$$-\frac{1}{2G} \sigma_0 = <\bar{q}q>,$$

$$-\frac{1}{4G_1} \Delta_0^b = <[iq'C \varepsilon \epsilon^{b5}q]>;$$

$$-\frac{1}{4G_1} \Delta_0^{ab} = <[i\bar{q} \varepsilon \epsilon^{b5}C\bar{q}]>. \quad (3)$$

Within this approximation, we obtain the quark contribution to the partition function

$$Z_q = \exp W_E = \int dq d\bar{q} \exp \left[ \int d^4x L_q \right],$$

(4)

where

$$L_q = -\bar{q}(i\gamma_\mu \nabla_\mu + i\mu \gamma_0 + \sigma_0)q - \Delta_0^{ab}[i\bar{q}'C \varepsilon \epsilon^{b5}q] - \Delta_0^b[i\bar{q} \varepsilon \epsilon^{b5}C\bar{q}].$$

(5)

with $W_E$ being the Euclidean effective action, and $L_q$ the quark Lagrangian. It is evident that $L = L_q + L_{scal\text{ar}}$, where $L_{scal\text{ar}}$ is the lagrangian part of the scalar meson and diquark. Due to the fact that the partition function $Z_q$ is invariant under the color gauge transformations, it is sufficient in the following to study only the case with $\Delta_0^3 \neq 0$ and $\Delta_0^{12} = 0$. Hence, in the color superconducting phase, where the diquark condensate is nonzero, color symmetry breaking from $SU_c(3)$ symmetry down to $SU_c(2)$ takes place. Furthermore, we also assume that the only nonvanishing components of the potential are $A^a_\mu \neq 0$, $a = 1, 2, 3$, while others are equal to zero: $A^a_\mu = 0$, $a = 4, \ldots, 8$. This implies that only quarks of two colors $\alpha = 1, 2$ do interact with the background field $A^a_\mu$, corresponding to the residual $SU_c(2)$ symmetry group of the vacuum. In this case, the calculation of the quark partition function (4) is greatly simplified, and we have (for more details see [21])

$$Z_q = \text{Det}_{(1)}(i\gamma_\partial + \sigma_0 + i\mu \gamma_0) \cdot \text{Det}_{(2)}^{1/2} [\left| \delta_0 \right|^2 + (\gamma \nabla + \sigma_0 + i\mu \gamma_0)(\gamma \nabla + \sigma_0 + i\mu \gamma_0)],$$

(6)

where $\delta_0 = 2\Delta_0^3$, and indices (1) and (2) mean that determinants are calculated in the one-dimensional (with $\alpha = 3$) and in the two-dimensional (with $\alpha = 1, 2$) subspaces of the color group, respectively. In principle, the gap is complex and we have two complex conjugated gap equations in (3). However, the partition function is real and depends only on the module squared of the gap. Its phase characterizes just the degeneracy of the vacuum and may be set here equal to zero. For the general case, it is understood that the gap equations and the following determinants are expressed directly in terms of the module $|\delta_0|$, i.e., $\delta_0 \to |\delta_0|$.

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2The vanishing of the pion mean-field is here related to the assumed parity conservation of the ground state.
Let us assume that the background field is constant and homogeneous, $F_{\mu\nu}^a = \text{const}$. Then the Dirac equation

$$(i\gamma \nabla + \sigma_0) \psi = 0$$

for a quark with flavor $i$ has stationary solutions $\psi_{k,i}$ with the energy spectrum $\varepsilon_{k,i}$, where $k$ stands for the quantum numbers of the quark in the background field. In this case we arrive at the following Euclidean effective action:

$$W_E = \frac{1}{2} \int dp_4 \sum_{k(0), i, \kappa} \log \left( p_4^2 + (\varepsilon_{k(0), i} - \kappa \mu)^2 \right) + \sum_{k, i, \kappa} \log \left( p_4^2 + |\delta_0|^2 + (\varepsilon_{k,i} - \kappa \mu)^2 \right).$$  (7)

Here, $\kappa = \pm 1$ corresponds to charge conjugate contributions of quarks, the first term in the sum corresponds to free quarks (not interacting with the color $SU_c(2)$ field) with color $\alpha = 3$ and the spectrum $\varepsilon^{(0)}_{k(0), i} = \sqrt{\sigma_0^2 + p^2}$, and the second term corresponds to quarks with color indices $\alpha = 1, 2$ (included in the quantum number $k$) and the spectrum $\varepsilon_{k,i}$, moving in the background color field $F_{\mu\nu}^a (a = 1, 2, 3)$.

In the case of finite temperature $T = 1/\beta > 0$, the thermodynamic potential $\Omega = -W_E/(\beta L^3)$ is obtained after substituting $p_4 \to \frac{2\pi}{\beta}(l + \frac{1}{2}), l = 0, \pm 1, \pm 2, ...,$

$$\Omega = -\frac{1}{\beta L^3} \sum_{i=1}^{N_f} \sum_{\kappa} \sum_{l=-\infty}^{l=+\infty} \left\{ \sum_{k(0)} \log \left( \frac{2\pi(l + 1/2)}{\beta} \right)^2 + (\varepsilon_{k(0), i} - \kappa \mu)^2 \right\}$$

$$+ \sum_k \log \left( \frac{2\pi(l + 1/2)}{\beta} \right)^2 + |\delta_0|^2 + (\varepsilon_{k,i} - \kappa \mu)^2 \right\}. \quad (8)$$

Next, let us consider the proper time representation

$$\Omega = \frac{1}{\beta L^3} \sum_{l=-\infty}^{l=+\infty} \sum_{i=1}^{N_f} \sum_{\kappa} \int \frac{ds}{s} \exp \left[-s \left( \frac{2\pi(l + 1/2)}{\beta} \right)^2 \right]$$

$$\times \left\{ \sum_{k(0)} \exp[-s(\varepsilon_{k(0), i} - \kappa \mu)^2] + \sum_k \exp[-s(|\delta_0|^2 + (\varepsilon_{k,i} - \kappa \mu)^2)] \right\}, \quad (9)$$

where $\Lambda$ is an ultraviolet cutoff ($\Lambda \gg \sigma_0, |\delta_0|$). According to (2) we then find for the quark condensate

$$\langle \bar{q}q \rangle = \frac{\int d\bar{q}dq \ \bar{q}q \exp \left[ \int d^4x L_q \right]}{\int d\bar{q}dq \exp \left[ \int d^4x L_q \right]} = -\frac{1}{Z_q} \frac{\partial Z_q}{\partial \sigma_0} = \frac{\partial \Omega}{\partial \sigma_0}, \quad (10)$$

which gives (for simplicity, we start with the assumption that $\mu = 0$, and later return to the discussion of the general case of $\mu \neq 0$).

---

3 It is well known that for usual superconductors the instability of the normal-fluid phase occurs in finite density systems with
\[
\langle \bar{q}q \rangle = -\sigma_0 \frac{L^3}{L^3 \sqrt{\pi}} \int_{1/\Lambda^2}^{\infty} \frac{ds}{\sqrt{s}} \left[ 1 + 2 \sum_{l=1}^{\infty} (-1)^l e^{-\frac{b^2 s^2}{4s}} \right] \\
\times \left( \sum_{k(0),i} e^{-s \bar{e}_k^{2,i}} + \sum_{k,i} e^{-s \bar{e}_k^{2,i} + |\delta_0|^2} \right). \tag{11}
\]

Here, the first term in the square brackets corresponds to the \(T = 0\) contribution, while the second term is the finite temperature contribution \((T \neq 0)\).

The (scalar isoscalar) diquark condensate can be obtained in a similar way

\[
\langle qq \rangle \equiv \langle iq^t C \varepsilon \gamma^5 q \rangle = 2 \frac{\partial \Omega}{\partial \delta_0^*}. \tag{12}
\]

Hence, we have

\[
\langle qq \rangle = -\delta_0 \frac{N_f}{L^3 \sqrt{\pi}} \int_{1/\Lambda^2}^{\infty} \frac{ds}{\sqrt{s}} \left[ 1 + 2 \sum_{l=1}^{\infty} (-1)^l e^{-\frac{b^2 s^2}{4s}} \right] \\
\times \sum_{k,i} e^{-s \bar{e}_k^{2,i} + |\delta_0|^2}. \tag{13}
\]

Clearly, in the case of a vanishing external field \((F_{\mu \nu}^a = 0)\), we have \(\bar{e}_k^2 = \bar{p}^2 + \sigma_0^2\). Then, at \(T = 0\) one obtains for the quark condensate

\[
\langle \bar{q}q \rangle = -\sigma_0 \frac{N_f}{2\pi^2} \int_{\frac{1}{\Lambda^2}}^{\infty} \frac{ds}{s^2} \left( e^{-s \sigma_0^2} + 2 e^{-s (\sigma_0^2 + |\delta_0|^2)} \right), \tag{14}
\]

and for the diquark condensate

\[
\langle qq \rangle = -\delta_0 \frac{N_f}{\pi^2} \int_{\frac{1}{\Lambda^2}}^{\infty} \frac{ds}{s^2} e^{-s (\sigma_0^2 + |\delta_0|^2)}. \tag{15}
\]

For subsequent discussion, this result can be easily generalized for the case of a space-time of arbitrary dimensionality \(D\):

\[
\langle qq \rangle = -\frac{4\delta_0 N_f}{2^{D-2} \pi^{D/2}} \int_{\frac{1}{\Lambda^2}}^{\infty} \frac{ds}{s^{D/2}} e^{-s (\sigma_0^2 + |\delta_0|^2)}. \tag{16}
\]

In what follows, we shall analyze three special cases of external chromomagnetic fields.

**Case 1):**

\(\mu \neq 0\). Notice that the possibility for vacuum CSC at \(\mu = 0\) was recently studied in several papers [21,23–25]. Since the chemical potential is a factor promoting the appearance of CSC, we put it first equal to zero in order to get a better understanding of the role just played by the external chromomagnetic field in the CSC formation. In this case the Fermi surface is replaced by the “zero energy surface” \(E = 0\) of the system. Due to the attraction between \(\bar{q}q\)- or \(qq\)-pairs, the zero-energy surface might become unstable and \(\bar{q}q\)- or \(qq\)-pairs or both are produced depending on the values of \(G, G_1\). Then the true vacuum is the coherent superposition of the \(\bar{q}q\)- and Cooper-pairs.
Rotational–symmetric non–abelian chromomagnetic field

$$A_1^2 = A_2^2 = A_3^3 = \sqrt{\frac{H}{g}}, \quad H_i^a = \delta_i^a H(i = 1, 2, 3), \quad (17)$$

with all other components of $A_i^a$ vanishing.

The energy spectrum has six branches, two of which correspond to quarks that do not interact with the chromomagnetic field

$$\varepsilon_{1,2}^2 = \vec{p}^2 + \sigma_0^2, \quad (18)$$

and the other four are given as follows

$$\varepsilon_{3,4}^2 = \sigma_0^2 + (\sqrt{a} \pm \sqrt{\vec{p}^2})^2, \quad \varepsilon_{5,6}^2 = \sigma_0^2 + (\sqrt{4a + \vec{p}^2})^2, \quad (19)$$

where $a = gH/4$.

**Case ii):**
Axial–symmetric non–abelian chromomagnetic field

$$A_1^1 = A_2^2 = \sqrt{\frac{H}{g}}, H_i^a = \delta_{3}^a \delta_{3}^h H, \quad (20)$$

with all other components of the potential vanishing.

The branches of the quark energy spectrum are besides (18) as follows

$$\varepsilon_{3,4,5,6}^2 = \sigma_0^2 + 2a \pm \sqrt{4a^2 + 4ap^2_\perp + p_3^2 + p_1^2} = \sigma_0^2 + p_3^2 + (\sqrt{a + p_1^2} \pm \sqrt{a})^2. \quad (21)$$

**Case iii):**
Abelian chromomagnetic field

$$A_\mu = \delta_{3}^a \delta_{\mu \perp} x_1 H. \quad (22)$$

This time only two color degrees of freedom of quarks with ”charges” $\pm g/2$ interact with the external field. The energy spectrum of quarks is now given by

$$\varepsilon_{3,4,5,6}^2 = \varepsilon_{n,\zeta, p_3}^2 = gH(n + \frac{1}{2} + \zeta + p_3^2 + \sigma_0^2), \quad (23)$$

where $\zeta = \pm 1$ is the spin projection on the external field direction, $p_3$ is the longitudinal component of the quark momentum ($-\infty < p_3 < \infty$),

$$p_\perp^2 = gH(n + \frac{1}{2}) \quad (24)$$

is the transversal component squared of the quark momentum, and $n = 0, 1, 2, ...$ is the Landau quantum number. As can be seen from (11) and (13), the form of the spectrum is essential for the quark condensate formation. Using the above three expressions of energy spectra for field configurations i), ii) and iii), we shall next study the corresponding three types of quark and diquark condensates in the strong field limit.
2.2. Asymptotic estimates for strong fields $gH \gg |\delta_0|^2, \sigma_0^2$.

In this section, we consider the special cases of strong field limits for the above configurations of background fields. Our goal is here to demonstrate that the field is a catalyzing agent for dynamical symmetry breaking, leading to possible chiral breaking and color breaking (CSC) phases. The fields are assumed to be strong as compared to the values of condensates that may be rather small. In this sense, the expected values of fields simulating the presence of a gluon condensate $gH = 0.4–0.6$ GeV$^2$ may be considered to be strong $^4$.

Case $i$):

According to (18), (19) we have

$$
\langle \bar{q}q \rangle = -\frac{\sigma_0 N_f 4\pi a}{\sqrt{\pi}(2\pi)^3} \int_0^{\Lambda^2} \frac{dt}{\sqrt{t}} e^{-\frac{tm_0^2}{a}} \int_0^{\Lambda^2} \frac{dxx^2}{t} \left( e^{-t(1-x)^2} + e^{-t(1+x)^2} + e^{-t(1+\sqrt{x^2+4})t} + e^{-t(1-\sqrt{x^2+4})t} \right) \left( 1 + 2 \sum_{l=1}^{\infty} (-1)^l e^{-\frac{\beta^2 l^2 a}{4t}} \right),
$$

and

$$
\langle qq \rangle = -\frac{\delta_0 N_f 4\pi a}{\sqrt{\pi}(2\pi)^3} \int_0^{\Lambda^2} \frac{dt}{\sqrt{t}} e^{-\frac{tm_0^2}{a}} \int_0^{\Lambda^2} \frac{dxx^2}{t} \left( e^{-t(1-x)^2} + e^{-t(1+x)^2} + e^{-t(1+\sqrt{x^2+4})t} + e^{-t(1-\sqrt{x^2+4})t} \right) \left( 1 + 2 \sum_{l=1}^{\infty} (-1)^l e^{-\frac{\beta^2 l^2 a}{4t}} \right),
$$

where $m_0^2 = |\delta_0|^2 + \sigma_0^2$. Taking the $T = 0$ term in (25), (26), we see that the first term in the square brackets that corresponds to the branch of the spectrum

$$
\varepsilon_4^2 = \sigma_0^2 + \sqrt{a} - \sqrt{\beta^2},
$$

plays the main role, when $h = gH/m_0^2 = 4a/m_0^2 \gg 1$. In this case the following asymptotics are obtained

$$
\langle \bar{q}q \rangle = -\frac{\sigma_0 N_f}{4\pi^2} \left[ 3\Lambda^2 - 2m_0^2 \log \frac{\Lambda^2}{m_0^2} - \sigma_0^2 \log \frac{\Lambda^2}{\sigma_0^2} + m_0^2 \left( \frac{h}{2} \log(C_1h) - hI_1(\beta m_0) \right) \right],
$$

and

$$
\langle qq \rangle = -\frac{\delta_0 m_0^2 N_f}{4\pi^2} \left[ 2 \left( \frac{\Lambda^2}{m_0^2} - \log \frac{\Lambda^2}{m_0^2} \right) + \frac{h}{2} \log(C_1h) - hI_1(\beta m_0) \right].
$$

Here

$^4$Notice that the value $gH = 0.5$ GeV$^2$ corresponds to a chromomagnetic field of order $O(10^{19})$ Gauss which is much larger than the strong magnetic fields of order $O(10^{15})$ Gauss recently found in neutron stars, “magnetars” $^2$.
\[ I_1(\beta m_*) = -\sum_{l=1}^{\infty} (-1)^l \int_0^\infty \frac{dx}{x} \exp \left[ - \left( x + \frac{l^2 m_*^2 \beta^2}{4x} \right) \right] = -2 \sum_{l=1}^{\infty} (-1)^l K_0(\beta m_* l), \]

where \( K_0(y) \) is the Macdonald’s function and \( C_1 \) is a certain numerical constant.

It is well-known that the order parameter of \( D\chi SB \) is the quark condensate which is the origin of dynamical quark masses. At the same time, color superconductivity takes place, when the corresponding order parameter, the diquark condensate, takes nonzero values.

The corresponding order parameters and underlying mechanisms of DSB are studied in this paper for different chromomagnetic background fields (modelling the gluon condensate) on the basis of an extended NJL model given in a quark-meson representation by the above Lagrangian \( \mathcal{L} \). In the one–loop approximation the gap equations (3) can be rewritten according to (27), (28) in the form

\[ \sigma_0 = \frac{\sigma_0 N_f}{4\pi^2} \left[ 3 \Lambda^2 - 2 m_*^2 \log \frac{\Lambda^2}{m_*^2} - \frac{\sigma_0^2}{\beta_0^2} \log \frac{\Lambda^2}{m_*^2} + m_*^2 \left( \frac{h}{2} \log(C_1 h) - h I_1(\beta m_*) \right) \right], \tag{29} \]

and

\[ \delta_0 = \frac{\delta_0 m_*^2 N_f}{8 G_1} \left[ 2 \left( \frac{\Lambda^2}{m_*^2} - \log \frac{\Lambda^2}{m_*^2} \right) + \frac{h}{2} \log(C_1 h) - h I_1(\beta m_*) \right]. \tag{30} \]

These equations have trivial solutions \( \sigma_0 = 0 \) and \( \delta_0 = 0 \), as well as nontrivial ones. It is easily seen that the nontrivial condensates satisfy the following gap equations:

\[ \Lambda^2 \left( \frac{1}{\bar{g}} - 1 \right) = -\frac{2}{3} m_*^2 \log \frac{\Lambda^2}{m_*^2} - \frac{1}{3} \sigma_0^2 \log \frac{\Lambda^2}{\sigma_0^2} + m_*^2 \frac{h}{6} \log(C_1 h) - m_*^2 \frac{h}{3} I_1, \tag{31} \]

where \( \bar{g} = \frac{3 \Lambda^2 G}{\pi^2} \) \( (N_f = 2) \), and

\[ \Lambda^2 \left( \frac{1}{\bar{g}_1} - 1 \right) = -m_*^2 \log \frac{\Lambda^2}{m_*^2} + m_*^2 \frac{h}{4} \log(C_1 h) - m_*^2 \frac{h}{2} I_1, \tag{32} \]

where \( \bar{g}_1 = \frac{8 \Lambda^2 G_1}{\pi^2} \). For \( g H \log \frac{g H}{m_*^2} \gg m_*^2 \log \frac{\Lambda^2}{m_*^2} \) \( (g H \ll \Lambda^2) \) we have solutions of (31), (32) even for weak coupling \( \bar{g}, \bar{g}_1 \ll 1 \). It should be noted, however, that in this weak coupling limiting case, the two condensates may simultaneously take nontrivial values only for \( G = 4G_1 - y \), with \( y \) being a small quantity, \( y = O(G_1) \). Otherwise they can exist separately. In what follows, we investigate the case, when the quark and diquark condensates are not simultaneously present. Then the two phases are described by the formulas:

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\(^5\)Indeed, by using this relation and subtracting equation (31) multiplied by a factor \( \frac{3}{\bar{g}} \) from (32), one gets a simpler equation for \( x = \frac{\Lambda^2}{\sigma_0^2} \) alone, which admits a nontrivial solution for \( 0 < y < \frac{4}{3} \bar{g}_1 \). This situation with a narrow region of coexisting phases in the plane of coupling constants is, of course, due to our approximation of very weak couplings that we adopted in our analytical estimates. A wider region of coexisting phases (phase IV) was found in our previous publication \( \mathcal{L} \) in the d=3 model. Preliminary results of numerical calculations show that, in the 4d case considered in the present publication, there also exists a broader region of coexisting phases. It is this region that shrinks to a line \( G \approx 4G_1 \) in the limit of weak couplings. Detailed results of our numerical investigations will be published elsewhere.
\[ \sigma_0(T) = \sqrt{C_1 g H} \exp \left[ -\frac{2\pi^2}{G g H} - I_1(\beta \sigma_0(T)) \right], \]

\[ \delta_0 = 0 \]  

(33)

for the case \( G > 4G_1 \) or

\[ |\delta_0(T)| = \sqrt{C_1 g H} \exp \left[ -\frac{\pi^2}{2G_1 g H} - I_1(\beta |\delta_0(T)|) \right], \]

\[ \sigma_0 = 0 \]  

(34)

in the case \( G < 4G_1 \) (\( G = \frac{8}{3} kG_1, k < \frac{3}{2} \)). In particular, for \( T = 0 \),

\[ \sigma_0(0) = \sqrt{C_1 g H} \exp \left( -\frac{2\pi^2}{G g H} \right). \]

(35)

The critical temperature \( T_c \) can now be found from the condition \( \sigma_0(T_c) = 0 \), which gives (compare with [15])

\[ T_c = \pi^{-1} e^\gamma \sigma_0(0) \simeq 0.5669 \sigma_0(0). \]

(36)

Similarly we obtain

\[ |\delta_0(0)| = \sqrt{C_1 g H} \exp \left( -\frac{\pi^2}{2G_1 g H} \right). \]

(37)

Hence, for the critical temperature \( T_{C1} \) of the phase transition, where \( \delta_0(T_{C1}) \rightarrow 0 \), we have the BCS relation

\[ T_{C1} = \pi^{-1} e^\gamma |\delta_0(0)| \simeq 0.5669 |\delta_0(0)|. \]

(38)

For illustrations, let us quote a rough order of magnitude estimate of \( |\delta_0(0)| \) and \( T_{C1} \). By choosing, e.g. \( \widetilde{g}_1 < 1 \) and \( gH < \Lambda^2 \), we obtain \( |\delta_0(0)| \approx 10 \text{ MeV} \) and \( T_{C1} < 5.7 \text{ MeV} \). Let \( \widetilde{g}_1 = 0.8, \Lambda^2 = 0.64 \text{ GeV}^2 \), then \( |\delta_0(0)| = 7 \text{ MeV}, T_{C1} = 3.5 \text{ MeV} \).

As is seen, the values of \( T_C \) and \( T_{C1} \) are here determined by the values of corresponding condensates at \( T = 0 \). Notice that both condensates depend nonperturbatively on the quantities \( G g H, G_1 g H \). Let us emphasize that the results (27), (28) with the logarithmic term \( \frac{\log h}{2} \) demonstrate the effect of dimensional reduction \( D = 3 + 1 \rightarrow D = 1 + 1 \). Indeed, integration of the main term in (26) gives

\[ \langle qq \rangle \simeq -\frac{\delta_0 N_f a}{\pi^2} \int \frac{ds}{s} e^{-sm^2} \approx -\frac{\delta_0 N_f a}{\pi^2} \log \frac{a}{m^2}, \]

(39)

which, up to a numerical factor, corresponds to [10] with \( D = 2 \) and \( \Lambda^2 \) replaced by \( a \).

Case ii):

In this case we have for the diquark condensate

\[ \langle qq \rangle = -\frac{\delta_0 N_f}{(2\pi)^2} \int \frac{ds}{s} e^{-sm^2} \int dp_\perp p_\perp \times \]

\[ \times \left[ e^{-s(\sqrt{a + p_\perp^2} - \sqrt{a})^2} + \right. \]

\[ \left. + e^{-s(\sqrt{a + p_\perp^2} + \sqrt{a})^2} \right] \left[ 1 + 2 \sum_{l=1}^{\infty} (-1)^l e^{-\frac{a l^2}{4s}} \right]. \]

(40)
The gap equation for $h \gg 1$ now takes the form (when $\sigma_0 = 0$)

$$\Lambda^2 \left( \frac{1}{\tilde{g}_1} - 1 \right) = -|\delta_0|^2 \left( \log \frac{\Lambda^2}{|\delta_0|^2} - \frac{h}{2} - \frac{\sqrt{\pi h}}{2} I_2(\beta|\delta_0|) \right),$$

(41)

where

$$I_2(z) = \sum_{l=1}^{\infty} (-1)^l \frac{dx}{x^{3/2}} \exp \left[ - \left( x + \frac{z^2 l^2}{4x} \right) \right] = 2\sqrt{\frac{\pi}{z}} \sum_{l=1}^{\infty} (-1)^l e^{-\sqrt{zl}/\sqrt{l}}.$$

It is convenient to rewrite (41) in the form

$$|\delta_0| = \Lambda \exp \left[ -\frac{\Lambda^2}{2|\delta_0|^2} \left( 1 - \frac{1}{\tilde{g}_1} \right) - \frac{h}{4} - \frac{\sqrt{\pi h}}{4} I_2(\beta|\delta_0|) \right].$$

(42)

The above solution is valid, when the argument of the exponential function is negative. Thus, for vanishing temperature $\beta \to \infty$, we have the condition

$$\tilde{g}_1 > \frac{1}{1 + (gH/(2\Lambda^2))}.$$

This demonstrates a possibility of color symmetry breaking in a non–abelian chromomagnetic field at $D = 3 + 1$ even for $\tilde{g}_1 < 1$.

The dependence on $h$ in (41) is found from the dominating term in (40) arising from the branch

$$\varepsilon^2 = \sigma_0^2 + p_3^2 + (\sqrt{p_\perp^2 + a^2} - \sqrt{a}).$$

Then we have for $a \to \infty$

$$\langle qq \rangle \sim \int \frac{ds}{s} e^{-sm_*^2} \int d_{\perp}s\_p e^{-sp_\perp^4/4a} \sim \sqrt{a} \int \frac{ds}{s^{3/2}} \sim a$$

corresponding to (43) with $D = 3$, which demonstrates the $3 + 1 \to 2 + 1$ dimensional reduction in this type of the field.

**Case iii):**

For the abelian chromomagnetic field with the spectrum (23) we obtain

$$\langle qq \rangle = -\frac{m_*^2\delta_0 N_f}{4\pi^2} \left\{ h \log \frac{h}{2\pi} + 2 \left( \frac{\Lambda^2}{m_*^2} - \log \frac{\Lambda^2}{m_*^2} \right) - 2hI_1(\beta m_*) \right\},$$

(43)

which is similar to (28), but differs by an overall factor 2 in field–dependent terms. This difference is simply due to the fact, that the main term $h \log h$ is obtained from two colors in the spectrum (23), while in the non–abelian case only one branch of the spectrum contributes to (27). For $gH \log \left( \frac{aH}{|\delta_0|^2} \right) \gg |\delta_0|^2 \log \left( \frac{\Lambda^2}{|\delta_0|^2} \right)$ $(gH \ll \Lambda^2)$ we obtain for $|\delta_0(T)|$, $|\delta_0(0)|$ and $T_{C1}$ the same equations (34), (37), (38) as in the non–abelian case i), but with the obvious replacements $C_1 \to 1/2\pi$ and $4\pi^2 \to 2\pi^2$ in the exponents. The main logarithmic term in (43) is obtained from the $n = 0$, $\zeta = -1$ contribution in the sum over quantum states in (13).
\[ (qq) \sim - \int_1^{\infty} ds \int_{-\infty}^{\infty} d\pi_3 \sum_{n=0}^{\infty} (2 - \delta_{n0}) \exp[-gHns - s|\delta_0|^2 - \pi_3^2 s] \sim \]
\[ \sim - \int_1^{1/gH} ds \exp[-s|\delta_0|^2] \approx - \log \left( \frac{gH}{|\delta_0|^2} \right). \]

Obviously, this corresponds to (16) with \( D = 2 \), which demonstrates the dimensional reduction in this case \( 3 + 1 \rightarrow 1 + 1 \), similar to the non–abelian case \( i \). (The replacement \( \Lambda^2 \rightarrow gH \) follows here from the requirement \( 1/\Lambda^2 \ll 1/gH \ll s \) for the integration region.)

As it is well known, CSC is expected to appear at nonzero chemical potential (see, e.g., [27] and [28]). Therefore, we now discuss a possible influence of a finite chemical potential on our results. The general case of arbitrary values of \( \mu \) can only be considered by numerical means. Nevertheless, we can estimate its contribution to the critical temperature analytically, when the gauge field is strong. Let us consider the most interesting case of a non-abelian field \( i \). As we see from (9), including a finite \( \mu \) can be made by the replacement \( \varepsilon^2 \rightarrow (\varepsilon \pm \mu)^2 \). As was demonstrated above, the main contribution to the integral in (26) comes from the branch in the energy spectrum \( \varepsilon_4^2 = \sigma_0^2 + (\sqrt{a} - \sqrt{p^2})^2 \). Hence, for \( \sigma_0 = 0 \) we have \( \varepsilon^2 = (\sqrt{a} - |p|)^2 \rightarrow (\sqrt{a} + \mu - |p|)^2 \). Thus, to account for the finite \( \mu \), we have to replace in the final formulas for \( \delta_0 \) and \( T_{C1} \) in (37) and (38): \( a^2 \rightarrow (\sqrt{a} + \mu)^2 \). As a result we obtain for the critical temperature the following estimate:

\[ T_{C1} = \text{const} \left( \sqrt{gH} + 2\mu \right) \exp \left( -\frac{2\pi^2}{4G_1 \left( \sqrt{gH} + 2\mu \right)^2} \right). \] (44)

As follows from the above estimate, the roles of \( \mu \) and the vacuum field \( gH \) are complementary for the diquark condensate formation. It should be mentioned that our result (44) reduces to formula (6) of [28], when the chromomagnetic field vanishes.

3. SUMMARY AND DISCUSSIONS

In the present paper the ability of external chromomagnetic fields to induce dynamical symmetry breaking (DSB) of chiral and color symmetry was studied in the framework of the extended NJL model (1) with attractive quark interactions in \( qq\)- and \( \bar{q}q\)-channels. Particular attention was paid to the CSC generation. In order to understand the genuine role of an external chromomagnetic field for the CSC phenomenon, we have removed from our consideration all other factors which might produce DSB. By this reason, we have put the chemical potential equal to zero, and considered the weak coupling limit of the model (temperature, on the other hand, is taken into account, since this factor only promotes symmetry restoration and never induces the DSB).

As was shown in our paper, the phenomenon of diquark condensation does exist for various non–abelian chromomagnetic field configurations even for the case of weak coupling if \( G < 4G_1 \) (if \( G > 4G_1 \), the external chromomagnetic field only catalyzes the \( D\chi SB \)). This effect is accompanied by an effective lowering of dimensionality in strong chromomagnetic fields, where the number of reduced units of dimensions depends on the concrete type of the field — a conclusion already made in the case of the \( D\chi SB [13] \). It should be mentioned that our result can be justified from the general point of view. Indeed, as the \( \sigma \) and \( \Delta\)–diquark fields appear in a typical combination, the sigma meson and diquark channel are related by an approximate Pauli–Gürsey symmetry (see [29]).
If there appears a catalysis phenomenon in the pure scalar sigma channel, it might be expected to appear also in the combination of sigma and diquark condensates.

The possibility for vacuum CSC at $\mu = 0$ was also studied in the framework of random matrix models on the basis of general symmetry arguments \cite{24}. There it was found a constraint on the coupling constants in $qq$- and $\bar{q}q$-channels, at which the CSC is forbidden. In terms of the NJL model (1) and at $H = 0$ this constraint takes the form $G > 8G_1/3$. We have, in particular, shown that the external chromomagnetic field modifies this constraint and reduces the region of coupling constants, in which the CSC cannot occur. Indeed, in the model (1) at $H \neq 0$ the range of prohibition for CSC is $G > 4G_1$ and is contained in the region $G > 8G_1/3$. Note that for NJL models based on the one-gluon exchange approximation to low energy QCD, a simultaneous Fierz transformation into color-singlet ($\bar{q}q$)- and antitriplet ($qq$)- channels yields the coupling relation $G = 8G_1$ (\cite{22,30}). Clearly, as our analysis shows, for such a case the CSC cannot be produced at $\mu = 0$ even at $H \neq 0$.

Finally, we remark that the chromomagnetic catalysis phenomenon for the diquark condensation is now under further examination, especially with consideration for finite values of the chemical potential and admitting various relations between coupling constants $G$ and $G_1$. A preliminary analysis of the interplay between various condensates at $\mu \neq 0$, which is based on numerical methods and generalizes our present results, is given in \cite{31} for the physically interesting case $G = 16G_1/3$.

**ACKNOWLEDGEMENTS**

We wish to thank V.P. Gusynin, V.A. Miransky and Y. Nambu for fruitful discussions. One of the authors (D.E.) acknowledges the support provided to him by the Ministry of Education and Science and Technology of Japan (Monkasho) for his work at RCNP of Osaka University. V.Ch.Zh. gratefully acknowledges the hospitality of Prof. M. Mueller-Preussker and his colleagues at the particle theory group of the Humboldt University extended to him during his stay there. This work was partially supported by the Deutsche Forschungsgemeinschaft under contract DFG 436 RUS 113/477/4.
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