The range of a self-similar additive gamma process is a scale invariant Poisson point process

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Abstract

It is shown that for a non-decreasing self-similar stochastic process $T$ with independent increments, the range of $T$ forms a Poisson point process with $\sigma$-finite intensity if and only if the one-dimensional distribution of $T(1)$ is of the gamma type. This follows from a general hold-jump description of such processes $T$, and implies the known result that the spacings between consecutive points of a scale invariant Poisson point process, with intensity $\theta x^{-1}dx$, are the points of another scale invariant Poisson point process with the same intensity.

Key words

scale invariant Poisson spacings, Poisson point processes, self-similar processes, Sato processes, range, records, extremal processes

1 Introduction

A point process $N_X(B) = \#\{z \in \mathbb{Z} : X_z \in B\}$, counting numbers of points in subintervals $B$ of the positive half line $\mathbb{R}_+ := (0, \infty)$, for some indexing of these points $X_z$ by $z$ in the set of integers $\mathbb{Z}$, is called scale invariant if for each $c > 0$ the point process $N_{cX}$, counting the scaled points $\{cX_z : z \in \mathbb{Z}\}$, has the same finite-dimensional distributions as $N_X$, as $B$ varies over subintervals of $\mathbb{R}_+$. Assuming the point process $N_X$ is simple, meaning the points $X_z$ are all distinct, the point process $N_X$ is then regarded as encoding the random countable set

$$\text{range}(X_z, \in \mathbb{Z}) := \{X_z, \in \mathbb{Z}\}. \quad (1.1)$$
So the identity in distribution of simple point processes $N_X$ and $N_{cX}$ may be indicated by the notation

$$\text{range}(X_z, z \in \mathbb{Z}) \overset{d}{=} \text{range}(cX_z, z \in \mathbb{Z}).$$

(1.2)

It is well known that a Poisson point process (PPP) on $\mathbb{R}_+$ is scale invariant if and only if its intensity measure is $\theta x^{-1}dx$ for some $\theta \geq 0$, when the process is called a $\text{PPP}(\theta x^{-1}dx)$, or a scale invariant Poisson point process with rate $\theta$. So the parameter $\theta$ is the intensity of such a point process relative to the scale invariant measure $x^{-1}dx$ on the positive half-line, and for $0 < a < b < \infty$, the number of points in $(a,b)$ has a Poisson distribution with mean $\theta \log(b/a)$.

The following result on scale invariant Poisson spacings arises in various contexts. The case $\theta = 1$ is contained in the theory of records and extremal processes, first developed by Dwass [7, 8, 9] and further studied by Resnick and Rubinovitch [31], and Shorrock [35]. The formulation for general $\theta > 0$ is due to Arratia [1, 2], who sketched a proof which was later detailed by Arratia, Barbour and Tavaré [3, Section 7].

**Theorem 1.1 (Scale invariant Poisson spacings)** Fix $\theta > 0$. Let $(T_z, z \in \mathbb{Z})$, with $T_z < T_{z+1}$ for all $z \in \mathbb{Z}$, be an exhaustive listing of the points of a scale invariant PPP with rate $\theta$. Then

$$\text{range}(T_{z+1} - T_z, z \in \mathbb{Z}) \overset{d}{=} \text{range}(T_z, z \in \mathbb{Z}).$$

(1.3)

Less formally, the theorem states:

- the spacings between consecutive points of a scale invariant Poisson point process on the positive half-line are the points of another scale invariant Poisson point process with the same rate.

This article places Theorem 1.1 in the broader context of stochastic processes $T = (T(s), s > 0)$ which are self-similar additive (SSA), meaning that

- $T$ is self-similar: for all $c > 0$,

$$\text{(T(cs), s > 0)} \overset{d}{=} (cT(s), s > 0);$$

(1.4)

- $T$ is additive: meaning $T$ has independent increments, and $T$ is stochastically continuous with càdlàg paths starting at 0

$$\lim_{s \to 0^+} T(s) = 0.$$  

(1.5)

Such a process $T$ is also called
• a process of class \( L \), as the distribution of \( T(1) \) is of class \( L \), a subclass of infinitely divisible laws studied by Lévy, or

• or a Sato process, and these processes were studied in depth by Sato.

Here we focus on the range \( \text{range}(T) \), the set of all values ever visited by \( T \), regarded as a random subset of \( \mathbb{R}_+ \), for a non-negative, hence non-decreasing SSA process \( T \) with no drift component. As recalled in Section 3, it is known that for such an SSA process \( T \), the Lévy-Khinchine representation of the Laplace transform of \( T(1) \) has the special form

\[
\log \mathbb{E} e^{-uT(1)} = \int_0^\infty (e^{-ux} - 1) \frac{k(x)}{x} dx,
\]

for a uniquely determined right continuous non-increasing key function \( k : \mathbb{R}_+ \to \mathbb{R}_+ \) subject to

\[
\int_0^\infty (x \land 1) \frac{k(x)}{x} dx < \infty.
\]

So the Lévy measure of \( T(1) \) has a density relative to Lebesgue measure \( dx \) of the form \( k(x)/x \) for such a key function \( k(x) \).

Consider now the random countable set of jump times of \( T \):

\[
\{ s > 0 : T(s) > T(s-) \}.
\]

Let \( k(0+) := \lim_{x \to 0^+} k(x) \). It follows easily from self-similarity of \( T \) that

• either \( k(0+) = \infty \) and the set of jump times is dense in \( \mathbb{R}_+ \);

• or \( k(0+) < \infty \) and the jump times are the points of a scale invariant PPP with rate \( \theta = k(0+) \), when we say the SSA process \( T \) has (finite) rate \( \theta \).

The identification of the rate \( \theta = k(0+) \) is part of an explicit hold-jump construction of \( T \) with finite rate, provided later in Theorem 4.1. Taking the jump sizes into consideration, as well as the jump times, the Lévy-Itô representation of jumps, discussed in Section 4, shows that the random countable set of ordered pairs of jump times and jump sizes

\[
\{(s, T(s) - T(s-)) : s > 0, T(s) > T(s-)\}
\]

is the set of points of a scale invariant Poisson point process on \( \mathbb{R}_+^2 \). This leads to the following theorem:

**Theorem 1.2** For a SSA non-decreasing process \( T \), with no drift component and key function \( k(x) \), if \( k(0+) = \theta \) is finite, then

(1) \( \text{range}(T) := \{ T(s) : s > 0 \} \) is a scale invariant point process on \( \mathbb{R}_+ \) with rate \( \theta \);
(II) the set of jump times of $T$ is a scale invariant PPP on $\mathbb{R}_+$ with rate $\theta$;

(III) the set of jump sizes of $T$ is a scale invariant PPP on $\mathbb{R}_+$ with rate $\theta$.

Here, (II) and (III) are just two coordinate projections of the self-similar Poisson point process (1.9) in the positive quadrant. Note that in (I) it is not asserted that $\text{range}(T)$ is Poisson point process, only that this point process is scale-invariant with mean intensity measure $\theta x^{-1} dx$. Beyond that, we know rather little about attributes of $\text{range}(T)$ as a point process on $\mathbb{R}_+$, such as its higher order factorial moments or Janossy measures, except in the special case when $T(1)$ is gamma distributed, and $\text{range}(T)$ turns out to be Poissonian.

To provide a more detailed description of the range of a non-decreasing SSA process $T$ with finite rate, let the scale invariant PPP of jump times of $T$ be indexed by $Z$ in an increasing way, say

$$0 < \cdots < S_{-1} < S_0 < S_1 < \cdots < \infty,$$

(1.10)

and define for each $z \in Z$

$$T_z := T(S_z).$$

(1.11)

So the jump of $T$ at time $s = S_z$ is from $T_{z-1} = T(S_{z-})$ to $T_z = T(S_z)$. See Figure 1 for an illustration. Then the random countable range of $T$ is

$$\text{range}(T(s), s > 0) = \text{range}(T_z, z \in Z)$$

(1.12)

so the points in the range of $T$ are also indexed by $Z$ in increasing order

$$0 < \cdots < T_{-1} < T_0 < T_1 < \cdots < \infty,$$

(1.13)

as in the setup in for scale-invariant Poisson spacings in Theorem 1.1.

Here we are not specific about how $S_0$ is selected from the random set of jump times of $T$, since the choice is not important when considering the simple point process $\{(T_z, T_{z+1} - T_z) : z \in Z\}$ as a random set on $\mathbb{R}_+^2$. It will be more convenient to define $S_0$ either so that $S_0 \leq 1 < S_1$ or so that $T_0 \leq 1 < T_1$.

Thus for a scale-invariant point process constructed as the range of a non-decreasing SSA process $T$ with finite rate $\theta$, comparing the point process of spacings between points, as on the left side of (1.3), and the points themselves on the right side of (1.3),

- the spacings between points are the jump sizes of $T$:

$$\text{range}(T_{z+1} - T_z, z \in Z) = \text{range}(T(s) - T(s-), 0 : s > 0)$$

(1.14)

which forms a scale invariant PPP with rate $\theta$, no matter what the key function $k(x)$ of $T$ with $k(0+) = \theta$;
the points themselves form the range of $T$:

$$\text{range}(T_z, z \in \mathbb{Z}) = \text{range}(T(s), s > 0)$$

(1.15)

which is a scale invariant point process with rate $\theta$, which might or might not be Poissonian, depending on the choice of the key function.

So to establish the spacings theorem 1.1, for a scale-invariant Poisson process with rate $\theta$, it only remains to show that for a suitable choice of the key function $k(x)$ with $k(0+) = \theta$, the range of $T$ is in fact Poissonian. That scale-invariant Poisson feature of $\text{range}(T)$, with rate $\theta = 1$, was established by Dwass in the 1960s for the SSA exponential process $T$, with $k(x) = e^{-\lambda x}$ for some $\lambda > 0$, so $T(1) \sim \text{gamma}(1, \lambda)$ (or $\text{exp}(\lambda)$) has the exponential distribution with rate $\lambda$, with tail probability

$$\mathbb{P}(T(1) > x) = e^{-\lambda x}1(x > 0).$$

(1.16)

Our main point here is that the Poisson spacings theorem for a general rate $\theta > 0$ is a similar consequence of the following result, which we establish in Sections 5 and 6.

**Theorem 1.3** Fix $\theta > 0$. For an SSA process $T := (T(s), s > 0)$,

- $\text{range}(T)$ is a scale invariant PPP on $\mathbb{R}_+$ with rate $\theta$,
if and only if

- the distribution of $T(1)$ is gamma($\theta, \lambda$) for some $\lambda > 0$, with probability density

$$\frac{d}{dx} \mathbb{P}(T(1) \leq x) = \frac{\lambda^\theta}{\Gamma(\theta)} x^{\theta-1} e^{-\lambda x} 1(x > 0).$$  \hspace{1cm} (1.17)

Implicit in this result is a well known consequence of the Lévy-Khinchine formula:

the gamma($\theta, \lambda$) distribution of $T(1)$ has key function $k(x) = \theta e^{-\lambda x}$. \hspace{1cm} (1.18)

However, some effort is required to pass from this fact to either of the implications of Theorem 1.3.

The rest of this article is organized as follows:

- Section 2 collects for later use some easy results about a scale invariant PPP.
- Section 3 recalls basic facts about SSA processes and selfdecomposable laws.
- Section 4 characterizes an SSA non-decreasing process $T$ with finite rate by its generic jump distribution, which can be read from the key function in the Lévy-Khinchine representation. The corresponding hold-jump description of $T$ is the basis of proofs in Section 5.
- Section 5 offers proves the “if” part of Theorem 1.3, that the range of an SSA gamma process is a PPP, in two different ways, based on the hold-jump description provided in Section 4.
- Section 6 provides a general uniqueness theorem which includes the converse of Theorem 1.3 under a technical condition, the distribution of the range of a SSA non-decreasing process uniquely determines the distribution of the process itself, up to a scale factor.
- Section 7 briefly discusses three different processes associated with a selfdecomposable law, then introduces a two-parameter process which provides a coupling of SSA processes with stationary independent increments in a second parameter.
- Section 8 offers some historical notes, including how the case $\theta = 1$ of Theorem 1.1 arises in the theory of extremal processes, and how the general case $\theta > 0$ is related to the Ewens sampling formula.
2 Scale invariant point processes

If a point process \( X := \{X_z, z \in \mathbb{Z}\} \) on \( \mathbb{R}_+ \) is scale invariant, i.e.

\[
\text{range}(X) \overset{d}{=} \text{range}(cX), \quad \forall c > 0,
\]

then \( X \) is a simple point process with intensity measure \( \theta x^{-1}dx \) for some \( \theta \), with \( 0 < \theta \leq \infty \). Then we say \( X \) is a scale invariant point process on \( \mathbb{R}_+ \) with rate \( \theta \). We observe that

- the inversion \( 1/X := \{1/X_z, z \in \mathbb{Z}\} \) of a scale invariant point process \( X \) on \( \mathbb{R}_+ \) is a scale invariant point process on \( \mathbb{R}_+ \) with the same rate.

By considering \( L := \log X \), this corresponds to a well known fact about stationary point processes on \( \mathbb{R} \). Just as not all stationary point processes on the line are reversible, not all scale invariant point process are invariant under inversion. However, the distribution of a Poisson point process is entirely determined by its intensity measure, hence:

- if \( X := \{X_z, z \in \mathbb{Z}\} \) is a scale invariant PPP then

\[
\text{range}(1/X) \overset{d}{=} \text{range}(X).
\]

The following lemma provides some useful characterizations of a scale invariant PPP. Recall that for \( U \) with uniform distribution on \([0, 1]\), and \( \theta > 0 \), the distribution of \( U^{1/\theta} \) is the beta(\( \theta, 1 \)) distribution on \([0, 1]\) with probability density

\[
\frac{d}{du} \mathbb{P}(U^{1/\theta} \leq u) = \theta u^{\theta-1}1(0 < u < 1).
\]

**Lemma 2.1** Fix \( x > 0 \) and \( \theta > 0 \). Suppose a scale invariant point process \( X = \{X_z, z \in \mathbb{Z}\} \) is indexed in increasing manner with

\[
0 < \cdots < X_{-2} < X_{-1} < X_0 \leq x < X_1 < X_2 < \cdots < \infty.
\]

Then the following three conditions are equivalent:

- \( X \) is a scale invariant PPP with rate \( \theta \);
- \( x/X_1 \) and \( X_{z-1}/X_z \) for \( z \geq 2 \) are independent and identically distributed (i.i.d.) beta(\( \theta, 1 \)) random variables;
- \( X_0/x \) and \( X_{z-1}/X_z \) for \( z < 0 \) are i.i.d. beta(\( \theta, 1 \)) random variables.

**Proof** Consider \( L := \{L_z = \log(X_z), z \in \mathbb{Z}\} \). Then \( L \) is a stationary PPP with intensity measure \( \theta d\ell \) on \( \mathbb{R} \) if and only if \( X \) is a scale invariant PPP with rate \( \theta \). The statements of the lemma are just transformations of well known characterizations of a stationary PPP on the line. 


3 Selfdecomposable laws and SSA processes

This section recalls some known results about self-similar additive processes and their one-dimensional distributions, known as the selfdecomposable laws. Following Sato [32, 33], we call a process \( T := (T(s), s > 0) \) self-similar with exponent \( H \) if

\[
(T(cs), s > 0) \stackrel{d}{=} (c^H T(s), s > 0) \quad (c > 0),
\]

(3.1)
in the sense of equality of finite-dimensional distributions. If \( T \) is also additive, as defined in Section 1, we say \( T \) is \( H \)-self-similar additive (\( H \)-SSA). We omitted the prefix ‘\( H \)’ when \( H = 1 \) in (1.4), because it has no impact on our discussion of \( \text{range}(T) \) thanks to Lemma 3.2 below.

Easily from the definition, for an \( H \)-SSA process \( T \), the one-dimensional distribution of \( T(1) \) is selfdecomposable, meaning that for each \( 0 < a < 1 \), there is the equality in distribution

\[
T(1) \overset{d}{=} aT(1) + R_a,
\]

(3.2)
for some random variable \( R_a \) independent of \( T(1) \). It is known [33, Theorem 15.3] that the class of selfdecomposable laws is identical to the class \( L \) introduced by Lévy [25] and Khintchine [22]. Lévy showed that such laws are infinitely divisible with a special structure of their Lévy measure. Specifically, from Sato and Yamazato [34], the Lévy-Khinchine representation of a selfdecomposable distribution of \( X \) is

\[
\mathbb{E}e^{iuX} = \exp \left\{ ibu - \frac{1}{2} \sigma^2 u^2 + \int_{\mathbb{R} - \{0\}} \left( e^{iux} - 1 - \frac{iux}{1 + u^2} \right) \frac{k(x)}{x} \, dx \right\},
\]

(3.3)
where \( b \) is real, \( \sigma^2 \geq 0 \) and \( k(x) \) is both

(a) non-negative, non-increasing on \( \mathbb{R}_+ \) and non-positive, non-increasing on \( \mathbb{R}_- \);

(b) subject to the usual requirement for a Lévy density \( k(x)/x \) that

\[
\int_{\mathbb{R} - \{0\}} (x^2 \wedge 1) \frac{k(x)}{x} \, dx < \infty.
\]

(3.4)
Hence, assuming the distribution of \( X \) is infinitely divisible, the distribution is selfdecomposable if and only its Lévy measure has a density of the form \( k(x)/x \) where \( k(x) \) satisfies condition (a) above. Sato [32] gave the following uniqueness theorem for the relationship between selfdecomposable laws and \( H \)-SSA processes.

**Theorem 3.1** For each \( H \)-SSA process \( (T(s), s > 0) \), the marginal distribution of \( T(s) \) is selfdecomposable. And for each selfdecomposable distribution \( \mu \) and each \( H > 0 \), there exists an \( H \)-SSA process \( (T(s), s > 0) \), unique in finite-dimensional distributions, such that \( T(1) \) has distribution \( \mu \).
We restrict the discussion to the case of $H = 1$ for most of the rest of this article, thanks to the following lemma.

**Lemma 3.2** Suppose $(T(s), s > 0)$ is an $H$-SSA non-decreasing process. Then the time change
\[ \tilde{T}(s) = T(s^{1/H}) \quad (s > 0), \] (3.5)
gives a 1-SSA non-decreasing process $(\tilde{T}(s), s \geq 0)$ with range($\tilde{T}$) = range($T$).

We are primarily interested in the case of a 1-SSA process $T$ that is non-decreasing and with no drift. Then (3.3) and (3.4) reduce to the formulas (1.6) and (1.7) for the Laplace transform of $T(1)$.

So the Lévy density of $T(s)$ at $x > 0$ is $k(x/s)/x$. As a result, the distribution of $T$ is fully characterized by the key function $k(x)$, as detailed further in Section 4.

### 4 Hold-jump description

This section presents the relationship between the rate $\theta < \infty$ and the key function $k(x)$ of a 1-SSA non-decreasing process with finite rate $\theta$. The hold-jump description after the jump over 1 will then be introduced as a framework to describe range($T$) ∩ $[1, \infty)$, as required to check whether range($T$) is Poisson using Lemma 2.1.

#### 4.1 Rate and generic jump

**Theorem 4.1** Suppose $T := (T(s), s > 0)$ is a 1-SSA non-decreasing process with no drift, finite rate $\theta$ and key function $k(x)$. Then $k(0+) = \theta$ and
\[ T(s) = \sum_{z \in \mathbb{Z}} S_z J_z 1(S_z \leq s), \quad (s > 0) \] (4.1)
where

- the jump times $(S_z, z \in \mathbb{Z})$ are the points of a scale invariant PPP with rate $\theta$;

Assuming also that these jump times are listed in an order depending only on their point process (1.8), for instance in an increasing order with $S_0$ the time of the first jump after time $s = 1$,

- the normalized jumps $(J_z, z \in \mathbb{Z})$ form a sequence of i.i.d. copies of a positive random variable $J$, called the generic jump of $T$, with tail probability
\[ \mathbb{P}(J > x) = \frac{k(x)}{\theta}, \quad (x \geq 0); \] (4.2)
• the sequence of jump times \((S_z, z \in \mathbb{Z})\) is independent of the sequence of normalized jumps \((J_z, z \in \mathbb{Z})\).

**Proof** The self-similarity of \(T\) ensures that there are no jumps at fixed times. Thus, as a non-decreasing additive process, \(T(s)\) has the following almost surely unique Lévy-Itô representation [21, Theorem 16.3]:

\[
T(s) = a_s + \int_0^s \int_0^\infty x \eta(dy \, dx), \quad (s > 0).
\]

where \(a_s\) is a non-decreasing function with \(a_0 = 0\) and \(\eta(dy \, dx)\) is a Poisson point process on \((0, \infty)^2\) satisfying

\[
\int_0^s \int_0^\infty (x \wedge 1) \mathbb{E} \eta(dy \, dx) < \infty. \quad (s > 0)
\]

In fact, \(\eta\) characterizes the jump structure of \(T\):

\[
\eta(\cdot) = \sum_y 1\{ (s, \Delta_y) \in \cdot \}
\]

where the summation extends over all times \(y > 0\) with \(\Delta_y := T(y) - T(y-1) > 0\). Hence

\[
T(s) = a_s + \sum_y \Delta_y 1(y \leq s) \quad (s > 0).
\]

We call this \(\eta\) the underlying Poisson point process of the non-decreasing SSA process \(T\). Taking into account self-similarity of \(T\), it is easy to see that \(a_s \equiv a_1 s\). If we further require \(T\) to be a pure jump process, then \(a_s \equiv 0\). Moreover, as we know the jump times are from a scale invariant PPP with rate \(\theta\), the representation (4.4) follows immediately from self-similarity. To finish the proof, it remains to show (4.2). Thanks again to self-similarity, note that \(\{(S_z, \Delta_{S_z} = S_z J_z), z \in \mathbb{Z}\}\) is an exhaustive listing of points from the underlying Poisson point process \(\eta\) on \(\mathbb{R}_+^2\) with intensity measure

\[
\nu(ds \, dy) := \mathbb{E} \eta(ds \, dy) = \theta s^{-1} ds \, \mathbb{P}(s J \in dy).
\]

Hence, for each Borel set \(B \subset \mathbb{R}_+\) and \(a > 0\),

\[
\int_0^a \theta s^{-1} \mathbb{P}(s J \in B) ds = \int_{y \in \mathbb{R}_+} dF_J(y) \int_{s \in \mathbb{R}_+} \theta s^{-1} 1(sy \in B, s < a) ds
\]

\[
= \int_{y \in \mathbb{R}_+} dF_J(y) \int_{x \in \mathbb{R}_+} \theta x^{-1} 1(x \in B, y > x/a) dx
\]

\[
= \int_{x \in B} \theta \mathbb{P}(J > x/a) dx
\]

\[
\tag{4.10}
\]

\[
\tag{4.11}
\]
But the Lévy measure of $T(1)$ is

$$\frac{k(x)}{x}dx = \int_{s\in(0,1]} \theta s^{-1}P(sJ \in dx)ds,$$

which implies (4.2) by setting $a = 1$ in (4.11).

To illustrate Theorem 4.1, we give two examples.

**Example 4.2** Fix $\theta, \lambda > 0$. A 1-SSA gamma process $(T(s), s > 0)$ with $T(1) \sim \text{gamma}(\theta, \lambda)$ has exponential generic jump $J \sim \exp(\lambda)$ and finite rate $\theta$, thanks to the key function given in (1.18).

**Example 4.3** This example is implicit in the discussion of the least concave majorant of a one-dimensional Brownian motion by Groeneboom [15] and Pitman and Ross [29]. Following the notations in [15], let $\omega$ denote a standard Brownian motion starting at the origin. For $a > 0$, let $\sigma(a)$ be the last time that maximum of $\omega(t) - at$ is attained, and $\tau(a) := \sigma(1/a), a > 0$ and $\tau(0) = 0$. The process $\tau$ is 2-SSA non-decreasing process with no drift. By Lemma 3.2 it can be reduced to fit our 1-SSA framework through the time change

$$T(a) := \tau(\sqrt{a}) = \sigma(1/\sqrt{a}).$$

Then $T$ is a 1-SSA process, whose range is the random set of times of vertices of the least concave majorant of Brownian motion. The underlying Poisson point process driving $T$ has intensity measure $\nu$ which can be read from Groeneboom’s description of the process $\tau$:

$$\nu(ds \ dx) = \frac{1}{s\sqrt{x}}\phi\left(\sqrt{\frac{x}{s}}\right) \frac{1}{2} s^{-\frac{1}{2}} ds dx = |k'|\left(\frac{x}{s}\right) \frac{ds \ dx}{s^2},$$

with $\phi(\cdot)$ the standard normal probability density function, and

$$|k'|(x) = -\frac{dk(x)}{x} = \frac{1}{2} \cdot \frac{1}{\sqrt{2\pi}}x^{-\frac{1}{2}}e^{-\frac{1}{2}x^2} \quad (x > 0),$$

the absolute first derivative of the key function $k(x) = \mathbb{P}(J > x)/2$ for $J \sim \text{gamma}(\frac{1}{2}, \frac{1}{2})$, indicating that $T$ has rate $\frac{1}{2}$. Therefore, the known results that

- range($T$) is a scale invariant point process on $\mathbb{R}_+$ with rate $\frac{1}{2}$ [29, Corollary 9], and

- the set of jump times of $T$ is a scale invariant PPP on $\mathbb{R}_+$ with rate $\frac{1}{2}$ [15, Theorem 2.1]
are the instances of parts I and II of Theorem 1.2 for this particular 1-SSA process T. Because the key function \( k(x) \) of \( T(1) \) is not just an exponential, the “only if” part of Theorem 1.3 shows that range \( (T) \), the set of times of vertices of the least concave majorant of Brownian motion, is not a Poisson point process. See Pitman and Ouaki [28] for a deeper study of Markovian structure in the concave majorant of Brownian motion.

Theorem 4.1 also yields the following hold-jump description of a 1-SSA non-decreasing process.

**Corollary 4.4** For each fixed time \( s > 0 \), and \( t \geq 0 \), the future of \( T \) after time \( s \), conditional on \( T(s) = t \), can be constructed by

- ‘Hold’ - at level \( t \) till the random time \( H_s \triangleq s\beta^{-1} \) for \( \beta \sim \text{beta}(\theta, 1) \), i.e.
  \[
  \frac{d}{dx} P(H_s \leq x) = \theta x^{-\theta-1}s^\theta 1(x > s); \tag{4.16}
  \]

- ‘Jump’ - up by \( H_s J \), where \( J \) is the generic jump and is independent of \( H_s \), i.e.
  \[
  P(T(H_s) - T(H_s-) > y | H_s = s') = P(s'J > Y) = P(J > Y/s'); \tag{4.17}
  \]

- then repeat, conditioning on \( T(s') = t' \) for \( s' = H_s, t' = t + H_s J \).

By setting a fixed starting time \( S_1 = s \), (4.16) and (4.17) specify a homogeneous pure jump-type Markov process \( ((S_n, T_n), n \in \mathbb{Z}_+) \) with state space \( \mathbb{R}_+^2 \), whose entrance law is given by \( T_1 = T(s) \triangleq sT(1) \). Moreover, apart from this entrance law, the same description applies with the fixed time \( s \) replaced by any stopping time \( \sigma \) relative to the filtration of \( T \), on the event \( (\sigma > 0) \).

The following corollary of Theorem 4.1 gives all possible distributions of generic jumps.

**Corollary 4.5** For each 1-SSA non-decreasing process \( T \) with finite rate \( \theta \), its generic jump \( J \) satisfies

\[
\mathbb{E} \log^+(J) < \infty. \tag{4.18}
\]

Conversely, for each \( \theta < \infty \) and each positive random variable \( J \) satisfying (4.18), there exists a unique 1-SSA process \( T \) with no drift, rate \( \theta \) and generic jump \( J \).

**Proof** As per (4.2), each right continuous, non-decreasing function \( k(x) \) uniquely determines \( \theta \) and \( J \), and vice versa. So the only thing to check is the integrable condition of the Lévy density \( k(x)/x \) in (1.7),

\[
\int_0^\infty (x \wedge 1) \frac{k(x)}{x} dx = \theta \left[ \int_0^1 P(J > x) dx + \int_1^\infty P(J > x)x^{-1} dx \right] < \infty, \tag{4.19}
\]
where the integral is restricted on $\mathbb{R}_+$ since $T$ is non-decreasing. The first term is finite no matter what distribution of $J$ is, while

$$
\int_{1}^{\infty} \mathbb{P}(J > x)x^{-1}dx = \int_{1}^{\infty} \mathbb{P}(\log J > \log x)d(\log x)
$$

(4.20)

$$
= \int_{0}^{\infty} \mathbb{P}(\log J > r)dr = \mathbb{E}\log^+ J,
$$

(4.21)

which finishes the proof.

The convergence condition (4.18) appeared first in Vervaat [37, Theorem 1.6b], in the discussion of stochastic difference equations. See also Wolfe [38, Theorem 1].

4.2 The jump over 1

In considering whether or not $\text{range}(T)$ is a Poisson process, Lemma 2.1 shows it is sufficient to examine the ratios of adjacent points in $\text{range}(T) \cap (1, \infty)$ only. The hold-jump description provides us with sufficient information to calculate the ratios as long as we know the joint distribution of where and when the jump of $T$ over 1 is made. That is given by the following lemma.

**Lemma 4.6** Consider the jump over level $t$ of a 1-SSA non-decreasing process $(T(s), s > 0)$ with no drift and finite rate $\theta$. Suppose the jump is made at time $S(t)$ from $G_t := T(S(t-)) \leq t$ to $D_t := T(S(t)) > t$. Then

$$
\mathbb{P}(S(t) \in ds, G_t \in dg, D_t - G_t > y) = \frac{\theta ds}{s} \mathbb{P}(T(s) \in dg)\mathbb{P}(sJ > y), \quad \forall 0 \leq g \leq t \leq g+y.
$$

(4.22)

In particular, for $t = 1$ and $y = 1 - g$,

$$
\mathbb{P}(S(1) \in ds, G_1 \in dg) = \frac{\theta ds}{s} \mathbb{P}(T(s) \in dg)\mathbb{P}(sJ > 1 - g), \quad \forall 0 \leq g \leq 1.
$$

(4.23)

**Proof** Since $T$ has independent increments and the set of jump times is not dense,

$$
\mathbb{P}(S(t) \in ds, G_t \in dg, D_t - G_t > y) = \mathbb{P}(T(s-)) \in dg)\mathbb{P}(T(s + ds) - T(s-)) > y)
$$

$$
= \mathbb{P}(T(s-)) \in dg) \int_{x \in (y, \infty)} \nu(ds, dx)
$$

$$
= \mathbb{P}(T(s)) \in dg) \int_{x \in (y, \infty)} \frac{\theta ds}{s} F_J \left( \frac{dx}{s} \right)
$$

$$
= \frac{\theta ds}{s} \mathbb{P}(T(s) \in dg)\mathbb{P}(sJ > y), \quad \forall 0 < g \leq t \leq g+y.
$$

(4.24)

Now we give the following theorem as the *hold-jump description after the jump over 1.*
Theorem 4.7 Suppose $T := (T(s), s > 0)$ is a 1-SSA non-decreasing process with no drift and finite rate $\theta$. Let $J$ denote its generic jump. Let $S_1$ be the time when $T$ jumps over 1

\[ T(S_1-) \leq 1 < T(S_1), \]

and $S_1 < S_2 < \cdots$ be the times of successive jumps $s$ of $T$ with $T(s) > T(s-) \text{ and } T(s) > 1$, and define

\[ T_0 := T(S_1-) \leq 1 < T_1 := T(S_1) < T_2 := T(S_2) < \cdots \]

so that $T_1, T_2, \ldots$ is the increasing sequence of values greater than 1 which are attained by $T$ on the successive intervals $[S_1, S_2), [S_2, S_3), \ldots$. Then the joint distribution of the two sequences $(S_n, n \geq 1)$ and $(T_n, n \geq 0)$ is determined as follows:

\[ \mathbb{P}(S_1 \in ds, T_0 \in dt) = \frac{\theta ds}{s} \mathbb{P}(T(s) \in dt) \mathbb{P}(sJ > 1 - t), \]

\[ S_n = S_1 \left( \prod_{i=1}^{n-1} \beta_i \right)^{-1} \quad (n = 2, 3, \ldots), \]

\[ T_n = T_{n-1} + S_n J_n \quad (n = 1, 2, \ldots), \]

where $\beta_1, \beta_2, \ldots$ and $J_1, J_2, \ldots$ are independent random variables, with

- $\beta_1, \beta_2, \ldots$ all with the beta($\theta, 1$) distribution \[2.3\];
- $J_1, J_2, \ldots$ identically distributed copies of the generic jump $J$.

**Proof** Formula (4.27) is just (4.23) with different notation, while (4.28) and (4.29) are due to the hold-jump description presented in Corollary 4.4.

### 4.3 Proof of Theorem 1.2

Suppose that $T$ is a non-decreasing 1-SSA process with no drift and finite rate $\theta$. Then Part (II) holds by definition, while (III) is observed by setting $a = \infty$ in (4.11). To see (I), set $g = t$ and $y = 0$ in (4.22) of Lemma 4.6. Suppose the rate of $\text{range}(T)$ is $\alpha$. 

\[ \text{range}(T) \quad \text{with different notation, while (4.28) and (4.29) are due to the hold-jump description presented in Corollary 4.4.} \]
then for each Borel set $B \subset \mathbb{R}_+$,

\[
\int_{B} \frac{\alpha dt}{t} = \int_{s \in (0, \infty)} \int_{t \in B} \mathbb{P}(S(1) \in ds, G_1 \in dt, D_1 - G_1 > 0) \\
= \int_{s \in (0, \infty)} \frac{\theta ds}{s} \int_{t \in B} \mathbb{P}(T(s) \in dt) \mathbb{P}(sJ > 0) \\
= \int_{s \in (0, \infty)} \frac{\theta ds}{s} \int_{t \in s^{-1}B} \mathbb{P}(T(1) \in dt) \\
= \int_{s \in (0, \infty)} \frac{\theta ds}{s} \int_{t \in s^{-1}B} dF_{T(1)}(t) \\
= \int_{s \in (0, \infty)} \frac{\theta ds}{s} \int_{t \in s^{-1}B} dF_{T(1)}(t) \\
(\text{set } r = st) = \int_{t \in (0, \infty)} \int_{r \in B} \frac{\theta dr}{r} dF_{T(1)}(t) = \int_{r \in B} \frac{\theta dr}{r},
\]

which shows $\alpha = \theta$.

5 Proof that the range of an SSA gamma process is a PPP

This is the “if” part of Theorem 1.3. We offer two different proofs.

5.1 First proof - through the hold-jump description after the jump over 1

This argument shows that the description of $\text{range}(T) \cap (1, \infty)$ implied by Theorem 4.7 matches that required by Lemma 2.1. We follow the notation of Theorem 4.7, which contains a complete description of the joint distribution of the first arrival times and levels $(S_n, T_n)$ at all levels $t > 1$ that are ever attained by the path of $T$. By scaling, it is enough to consider the case $\lambda = 1$. From Example 4.2, the generic jump is $\text{exp}(1)$ distributed: $\mathbb{P}(J > x) = e^{-x}$ for $x > 0$, so (4.27) becomes

\[
\mathbb{P}(S_1 \in ds, T_0 \in dt) = \frac{\theta ds}{s} \frac{1}{s^\theta \Gamma(\theta)} t^{\theta-1} e^{-\frac{t}{s}} dt e^{-(\frac{1}{s} - \frac{1}{r})} = \frac{s^{1-\theta} e^{-\frac{1}{s}} ds}{\Gamma(\theta)} \theta t^{\theta-1} dt.
\]

That means $T_0$ and $S_1$ are independent, with $T_0$ distributed $\text{beta}(\theta, 1)$ and $S_1$ distributed as $\text{inverse gamma}(\theta, 1)$, meaning $S_1^{-1}$ distributed as the $\text{gamma}(\theta, 1)$.

Hence by the memoryless property of exponential random variables, we may assume $T_0 = 1$ without hurting the joint distribution of $(S_n, n \geq 1)$ and $(T_n, n \geq 1)$. The joint
distribution is now re-written as follows:

\[ S_n = \left( \gamma_\theta \prod_{i=1}^{n-1} \beta_i \right)^{-1} \quad (n = 1, 2, \ldots) \quad (5.2) \]

\[ T_n = T_{n-1} + S_n \varepsilon_n \quad (n = 1, 2, \ldots) \quad (5.3) \]

where \( \gamma_\theta, \beta_1, \beta_2, \ldots \) and \( \varepsilon_1, \varepsilon_2, \ldots \) are independent random variables, with

- \( \gamma_\theta \) assigned the \( \text{gamma}(\theta, 1) \) distribution;
- \( \beta_1, \beta_2, \ldots \) all with the \( \text{beta}(\theta, 1) \) distribution;
- \( \varepsilon_1, \varepsilon_2, \ldots \) all with the \( \text{exp}(1) \) distribution.

All the ingredients needed for calculating \( \text{range}(T) \cap (1, \infty) \) are in view. But the description of the levels \( T_n \) is tangled up with the description of the times \( S_n \) in such a way that it is not immediately obvious why the sequence of ratios \( T_{n-1}/T_n \) is also a sequence of i.i.d. copies of \( \text{beta}(\theta, 1) \). However, the argument is completed by the following lemma.

**Lemma 5.1** Suppose that random variables \( S_1 := 1/\gamma_\theta \) and \( S_{n+1} := S_n/\beta_n \) for \( n \geq 1 \) are defined by the recursion \[5.2\], along with \( T_0 := 1 < T_1 < T_2 < \cdots \) by \[5.3\], from independent random variables \( \gamma_\theta, \beta_i \) and \( \varepsilon_i \) as above. Then for each \( n = 1, 2, \ldots \), the \( n + 1 \) ratios

\[ \frac{T_0}{T_1} , \cdots , \frac{T_{n-1}}{T_n} , \frac{T_n}{S_n} \quad (5.4) \]

are independent, with the first \( n \) consecutive \( T \)-ratios all distributed according to the common \( \text{beta}(\theta, 1) \) distribution of all the \( \beta_i \), and with the last of the \( n + 1 \) ratios

\[ \frac{T_n}{S_n} \overset{d}{=} \gamma_{\theta+1}, \quad (5.5) \]

the \( \text{gamma}(\theta + 1, 1) \) distribution.

**Proof**

For \( n = 1 \), with \( T_0 := 1 \)

\[ T_1 = 1 + \frac{\varepsilon_1}{\gamma_\theta} = \frac{\gamma_\theta + \varepsilon_1}{\gamma_\theta} \quad (5.6) \]

and hence

\[ \frac{T_0}{T_1} = \frac{1}{T_1} = \frac{\gamma_\theta}{\gamma_\theta + \varepsilon_1} \overset{d}{=} \beta_{\theta, 1} \quad (5.7) \]

and this variable \( T_0/T_1 \) is independent of

\[ \frac{T_1}{S_1} := T_1 \gamma_\theta = \gamma_\theta + \varepsilon_1 \overset{d}{=} \gamma_{\theta+1} \quad (5.8) \]

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by the beta-gamma algebra mentioned in Lukacs [26]. The case of general \( n = 1, 2, 3, \ldots \) now follows by induction on \( n \), starting from this base case \( n = 1 \). Multiply the recursion (5.3) by \( 1/T_{n+1} = \beta_n/T_n \) to see that

\[
\frac{T_{n+1}}{S_{n+1}} = \frac{T_n}{S_n} \beta_n + \varepsilon_{n+1} \overset{d}{=} \gamma_{\theta} + \varepsilon_1 \overset{d}{=} \gamma_{\theta+1}
\]  

(5.9)

because the gamma(\( \theta + 1 \)) distribution of \( T_n/S_n \) and the independence of this variable and \( \beta_n \) makes their product \( (T_n/S_n)\beta_n \overset{d}{=} \gamma_{\theta} \). Moreover

\[
\frac{T_n}{T_{n+1}} = \frac{T_n/S_{n+1}}{T_{n+1}/S_{n+1}} = \frac{(T_n/S_n)\beta_n}{(T_n/S_n)\beta_n + \varepsilon_{n+1}} \overset{d}{=} \frac{\gamma_{\theta}}{\gamma_{\theta} + \varepsilon_1} = \beta_{\theta,1}
\]

(5.10)

and this ratio \( T_n/T_{n+1} \) is independent of the ratio \( T_{n+1}/S_{n+1} \) in (5.9), again by beta-gamma algebra. Thus \( T_n/T_{n+1} \) and \( T_{n+1}/S_{n+1} \) are independent with the required distributions. By inductive assumption, \( T_n/S_n \) is independent of the \( n \) ratios \( T_{i-1}/T_i \) for \( 1 \leq i \leq n \). The two variables \( T_n/T_{n+1} \) and \( T_{n+1}/S_{n+1} \) are functions of \( T_n/S_n \) and two further independent variables \( \beta_n \) and \( \varepsilon_{n+1} \). Hence the required independence of the \( n + 2 \) variables involved in (5.4) with \( n + 1 \) in place of \( n \).

### 5.2 Second proof - to exploit symmetry

To see the symmetry, consider the time-inversion \( \tilde{T} := (\tilde{T}(s) := T(s^{-1}), s > 0) \) of \( T \). It is obvious that \( \tilde{T} \) has a non-increasing staircase path, which is fully determined by its corners, i.e. points on the left end of each flat of the path.

To describe the corners, we restate, in a time-reversed manner, the backward hold-jump description of \( \tilde{T} \). Conditional on \( \tilde{T}(s) = t \), the past of \( \tilde{T} \) before time \( s \) can be fully constructed by

- ‘Hold’- at level \( t \) till the random time \( H_s \overset{d}{=} s\beta \) (this is going backward in time), where \( \beta \) has the common beta(\( \theta, 1 \)) distribution;

- ‘Jump’- up by \( H_s^{-1}J \), where \( J \) is the generic jump and is independent of \( H_s \);

- then repeat, conditioning on \( \tilde{T}(s') = t' \) for \( s' = H_s, t' = t + H_s^{-1}J \).

**Lemma 5.2** Suppose \( T \) is a 1-SSA gamma process with \( T(1) \sim \text{gamma}(\theta, 1) \) and \( \tilde{T} := (\tilde{T}(s) := T(s^{-1}), s > 0) \) is the time-inversion of \( T \). Then the set \( \mathcal{C} \) of corners of the path of \( \tilde{T} \) is symmetric about the bisectrix.

Theorem 1.3 then follows since

\[
\text{range}(T) = \text{range}(\tilde{T}) = \pi_t(\mathcal{C}) \overset{d}{=} \pi_s(\mathcal{C}),
\]

(5.11)
where $\pi_t$ and $\pi_s$ are projections onto the $t$- and $s$-axes, and $\pi_s(C)$ is a scale invariant PPP on $\mathbb{R}_+$ with rate $\theta$ thanks the invariance under inversion (2.2).

**Proof (Lemma 5.2)**

Consider the joint density $p_m(a_1, \cdots, a_m)$ of the event that there are $N$ consecutive points $a_i = (s_i, t_i) \in C$ for $i = 1, 2, \cdots, N$ indexed decreasingly in $s$-coordinate

$$s_1 > s_2 > \cdots > s_N,$$

hence is indexed increasingly in $t$-coordinate. Knowing $J \sim \text{gamma}(1,1)$,

$$p_N(a_1, \cdots, a_N) = \frac{s_1^\theta}{\Gamma(\theta)} (t_0)^{\theta-1} e^{-s_1 t_1} \cdot \frac{\theta}{s_1} \cdot \left[ \prod_{n=1}^{N-1} \theta s_{i+1}^{\theta-1} s_i^{-\theta} \right] \cdot \left[ \prod_{n=1}^{N-1} s_i e^{-s_i (t_{i+1} - t_i)} \right]$$

$$= \frac{\theta^N}{\Gamma(\theta)} (t_1 s_N)^{\theta-1} \exp \left\{ \sum_{n=2}^{N-1} s_n t_n - \sum_{n=1}^{N-1} s_n t_{n+1} \right\}.$$  

Expression (5.14) is invariant under the substitution

$$(\tilde{s}_1, \tilde{s}_2, \cdots, \tilde{s}_N) \leftrightarrow (\tilde{t}_N, \tilde{t}_{N-1}, \cdots, \tilde{t}_1),$$

which proved the symmetry as desired.

This proof is inspired by Gneden [13, Equation (5)] where a similar symmetry was shown for a different setup of corners constructed from a PPP on $\mathbb{R}_+^2$ with unit intensity. We also remark that the proof of the Poisson spacing theorem by Arratia, Barbour and Tavaré [3, Lemma 7.1] is also done by checking the density of consecutive points. However, we manage to avoid the brutal integration in their proof by exploiting symmetry.

### 6 Uniqueness of the distribution of the range

In this section, we establish the following uniqueness theorem, from which the “if” part of Theorem 1.3 follows immediately.

**Theorem 6.1** Suppose $T$ and $\tilde{T}$ are two 1-SSA non-decreasing processes with no drift and the same finite rate $\theta$. Then

$$\text{range}(T) \overset{d}{=} \text{range}(\tilde{T}) \quad \text{implies} \quad T \overset{d}{=} c\tilde{T} \quad \text{for some} \quad c > 0.$$  

(6.1)
Proof Following (the results and also the notation of) Theorem 4.7, we may write $S$ in terms of $T$ and $J$ in (4.28), then (4.29) becomes
\[
\frac{T_n - T_{n-1}}{T_{n+1} - T_n} = \frac{\beta_n J_{n-1}}{J_n}, \quad \forall n \geq 2,
\] (6.2)
where (still, as in Theorem 4.7) $1 < T_1 < T_2 < \cdots$ is an exhaustive ordered listing of points of $T$ on $(1, \infty)$, $\beta_n$ are i.i.d. Beta($\theta$, 1)’s and $J_n$ are i.i.d. copies of the generic jump $J$.

Similarly, for another 1-SSA process $\tilde{T}$ with rate $\theta$ whose range is equal in distribution as the one of $T$, we have
\[
\frac{\tilde{T}_n - \tilde{T}_{n-1}}{\tilde{T}_{n+1} - \tilde{T}_n} = \frac{\tilde{\beta}_n \tilde{J}_{n-1}}{\tilde{J}_n}, \quad \forall n \geq 2,
\] (6.3)
where every random variable is similarly defined as in (6.2) for $\tilde{T}$ instead of $T$.

By assumption, $T$ and $\tilde{T}$ are both 1-SSA with the same finite rate $\theta$. So it is sufficient to show $J \overset{d}{=} \tilde{J}$.

Note that $\text{range}(T) \overset{d}{=} \text{range}(\tilde{T})$ implies
\[
(T_n, n \geq 1) \overset{d}{=} \left(\tilde{T}_n, n \geq 1\right).
\] (6.4)

Therefore, by (6.2) and (6.3),
\[
\left(\frac{\beta_n J_{n-1}}{J_n}, n \geq 2\right) \overset{d}{=} \left(\frac{\tilde{\beta}_n \tilde{J}_{n-1}}{\tilde{J}_n}, n \geq 2\right).
\] (6.5)

By Lemma 6.2 below, (6.5) can be simplified to
\[
\left(\frac{J_{n-1}}{J_n}, n \geq 2\right) \overset{d}{=} \left(\frac{\tilde{J}_{n-1}}{\tilde{J}_n}, n \geq 2\right)
\] (6.6)
which implies $J \overset{d}{=} c\tilde{J}$ thanks to Lemma 6.3.

This Lemma 6.2 is a multivariate extension of a simplified version of exercise 1.13.1 in Chaumont and Yor [6], since we only need the case when all coordinates are strictly positive.

Recall $\mathbb{R}_+ := (0, \infty)$. Suppose $Y = (Y_1, Y_2, \cdots, Y_n)$ is a $\mathbb{R}_+^n$-valued random variable. Let $\Phi_{\log(Y)}$ denote the characteristic function of $\log Y := (\log Y_1, \log Y_2, \cdots, \log Y_n)$
\[
\Phi_{\log(Y)}(\lambda) := \mathbb{E} \exp\{i\lambda \cdot \log(Y)\}, \quad \lambda \in \mathbb{R}^n,
\] (6.7)
Lemma 6.2 (Multivariate simplifiable random variables) If the non-zero set of the characteristic function of \((\log Y)\), i.e. \(\{\lambda : \Phi_{\log Y}(\lambda) \neq 0\}\), is dense in \(\mathbb{R}^n\), then \(Y\) is multivariate simplifiable, i.e. for all \(\mathbb{R}^n\)-valued random variables \(X, Z\) independent of \(Y\),

\[
X \times Y \overset{d}{=} Z \times Y \quad \text{implies} \quad X \overset{d}{=} Z, \tag{6.8}
\]

where \(\times\) denotes the entry-wise product \(X \times Y := (X_1Y_1, X_2Y_2, \ldots, X_nY_n)\).

In particular, for each \(\theta > 0\), if \(Y_1, Y_2, \ldots, Y_n\) are i.i.d. copies of \(\beta \sim \text{beta}(\theta, 1)\), \(Y\) is multivariate simplifiable.

Proof

\[
\Phi_{\log(X \times Y)}(\lambda) = \mathbb{E} \exp[i \lambda \cdot \log(X \times Y)] = \mathbb{E} \exp[i \lambda \cdot (\log X + \log Y)]
= \Phi_{\log X}(\lambda)\Phi_{\log Y}(\lambda), \quad \forall \lambda \in \mathbb{R}^n. \tag{6.9}
\]

and similarly \(\Phi_{\log(Z \times Y)}(\lambda) = \Phi_{\log Z}(\lambda)\Phi_{\log Y}(\lambda)\). Hence, \(X \times Y \overset{d}{=} Z \times Y\) implies

\[
\Phi_{\log X}(\lambda)\Phi_{\log Y}(\lambda) = \Phi_{\log Z}(\lambda)\Phi_{\log Y}(\lambda), \quad \lambda \in \mathbb{R}^n. \tag{6.10}
\]

Then the cancellation of \(\Phi_{\log Y}(\lambda)\) on a dense subset \(\{\lambda : \Phi_{\log Y}(\lambda) \neq 0\}\) of \(\mathbb{R}^n\) shows

\[
\log X \overset{d}{=} \log Z \tag{6.11}
\]

which easily implies \(X \overset{d}{=} Z\).

Now it is left to show that \(\{\lambda : \Phi_{\log Y}(\lambda) \neq 0\}\) is dense if \(Y_1, Y_2, \ldots, Y_n\) are i.i.d. copies of \(\beta \sim \text{beta}(\theta, 1)\).

Observe that the characteristic function of \(\log \beta\) is non-zero on \(\mathbb{R}\)

\[
\phi_{\log \beta}(\lambda) = \mathbb{E} \exp[i \lambda \log(\beta)] = \mathbb{E} \beta_i^\lambda = \int_0^1 x^\lambda \theta x^{\theta-1} dx = \frac{\theta}{\theta+i\lambda}, \quad \forall \lambda \in \mathbb{R}. \tag{6.12}
\]

Hence the characteristic function of \(\log Y\) is non-zero on \(\mathbb{R}^n\).

Lemma 6.3 Suppose \(X\) and \(Y\) are two positive random variables, with i.i.d. copies \(X_n, Y_n, n = 1, 2, \ldots\), respectively. If for each \(n\),

\[
\left(\frac{X_1}{X_2}, \frac{X_2}{X_3}, \ldots, \frac{X_{n-1}}{X_n}\right) \overset{d}{=} \left(\frac{Y_1}{Y_2}, \frac{Y_2}{Y_3}, \ldots, \frac{Y_{n-1}}{Y_n}\right), \tag{6.13}
\]

then \(Y \overset{d}{=} cX\) for some constant \(c\).
Proof Identity (6.13) is equivalent to

\[
\left( \frac{X_1}{X_2}, \frac{X_1}{X_3}, \cdots, \frac{X_1}{X_n} \right) \overset{d}{=} \left( \frac{Y_1}{Y_2}, \frac{Y_1}{Y_3}, \cdots, \frac{Y_1}{Y_n} \right),
\]

which implies

\[
\left( \frac{X_1}{X_2}, \frac{X_1}{X_3}, \cdots, \frac{X_1}{X_n} \right) \overset{d}{=} \left( \frac{Y_1}{Y_2}, \frac{Y_1}{Y_3}, \cdots, \frac{Y_1}{Y_n} \right),
\]

where \( Z_n, n = 2, 3, \cdots \) are i.i.d. copies of \( Z \sim \exp(1) \) and are independent of the \( X \)- and \( Y \)-sequences.

The extra randomization \( Z \) is only to ensure that the cumulative distribution function of \( Z/X \) is continuous on \( \mathbb{R}^+ \). When \( n := 2k \to \infty \), the \((k+1)\)-th order statistic of the left sequence of (6.15) converges almost surely to \( m_x X_1 \) where \( m_x > 0 \) is the median of \( Z/X \). Similarly, the \((k+1)\)-th order statistic of the right sequence converges almost surely to \( m_y Y_1 \) where \( m_y > 0 \) is the median of \( Z/Y \).

Hence

\[
m_x X_1 \overset{d}{=} m_y Y_2,
\]

which implies

\[
X \overset{d}{=} cY \quad \text{with} \quad c = \frac{m_y}{m_x}. \tag{6.17}
\]

7 Processes associated with selfdecomposable laws

In this section, we first introduce three different kinds of one-parameter processes associated with selfdecomposable laws, which are known in the previous study related to selfdecomposable laws. Then we introduce a two-parameter process with selfdecomposable margins that behaves differently along its two parameters.

7.1 One-parameter processes

Let \( X \) be a selfdecomposable random variable. According to the result of Sato \[32\] presented in Theorem 3.1

\begin{itemize}
  \item for each \( H > 0 \), there is a unique distribution of an \( H \)-SSA process \( T^{(H)} := (T^{(H)}(s), s > 0) \) such that \( T^{(H)}(1) \overset{d}{=} X \).
\end{itemize}

On the other hand, since \( X \) is infinitely divisible,
• there is a unique Lévy process \( U := (U(s), s > 0) \) with \( U(1) \overset{d}{=} X \).

See Sato [32, Section 4] for a comparison between these two processes, where it is mentioned that \( T^{(H)} \overset{d}{=} U \), in the sense of equality in finite-dimensional distributions, if and only if \( X \) is a constant variable 0 or strictly stable with index \( 1/H \).

A third process associated with selfdecomposable \( X \) is known as the background driving Lévy process (BDLP) first discussed by Wolfe [38] and Jurek and Vervaat [20], and named by Jurek [18]. A Lévy process \( Y := (Y(r), r > 0) \) is called the BDLP of \( X \) if

\[
X \overset{d}{=} \int_0^\infty e^{-r}dY(r). \tag{7.1}
\]

The relationship of \( T^{(H)} \) and \( Y \) is given by

\[
Y(r) = \int_{e^{-r}}^1 s^{-H}dT^{(H)}(s), \quad r > 0, \tag{7.2}
\]

and

\[
T^{(H)}(s) = \int_{-\log(s)}^\infty e^{rH}dY(r), \quad 0 < s < 1. \tag{7.3}
\]

To illustrate the differences among these three processes, we observe if \( k(x)/x \) is the Lévy density of \( X \), then

• the Lévy density of \( T^{(H)}(s) \overset{d}{=} T^{(1)}(s^H) \) is given by \( k(s^{-H}x)/x \);

• the Lévy density of \( U(s) \) is given by \( sk(x)/x \);

• the Lévy density of \( Y(r) \) is given by \( rk(x) \).

The BDLP, denoted \( Y \), can be easily extended for \( s \leq 0 \) by setting \( Y(0) = 0 \) and \( (-Y(-s), s > 0) \) an independent copy of \( (Y(s), s > 0) \), whence (7.2) holds for \( r \in \mathbb{R} \) and (7.3) holds for \( s > 0 \). See Jeanblanc, Pitman and Yor [17, Theorem 1].

\section*{7.2 A two-parameter process}

For simplicity, suppose the selfdecomposable random variable \( X \) is also non-negative with Lévy triple \((0, 0, k(x)x^{-1}dx)\) where \( \theta := k(0+) < \infty \). So the associated 1-SSA process \((T(s), s > 0)\) with \( T(1) \overset{d}{=} X \) is non-decreasing with no drift, finite rate \( \theta \) and generic jump \( J \) whose distribution is determined by

\[
\mathbb{P}(J > x) = k(x)/\theta. \tag{7.4}
\]
Moreover, $T(s)$ has the following Lévy-Itô representation

$$T(s) = \int_0^s \int_0^\infty x\eta(dy \, dx) \quad (s > 0),$$

(7.5)

where $\eta$ is a Poisson point process on $\mathbb{R}_+^2$ satisfying (4.4), with intensity measure

$$\nu(dy \, dx) := \mathbb{E}\eta(dy \, dx) = \theta y^{-1}dy \mathbb{P}(sJ \in dx).$$

(7.6)

Based on $\eta$, define a Poisson point process on $\mathbb{R}_+^3$, say $\eta^*$, by setting its intensity measure

$$\nu^* := \mathbb{E}\eta^* = \nu \otimes \lambda,$$

where $\lambda$ is the ordinary Lebesgue measure on $\mathbb{R}_+$. Now define a two-parameter process $(T(s,w), s > 0, w > 0)$ by setting

$$T(s,w) := \int_{z \in (0,w]} \int_{y \in (0,s]} \int_{x > 0} x\eta^*(dz \, dy \, dx) \quad (s, w > 0).$$

(7.7)

Then the following properties of this two-parameter process are easily checked:

- $T(s,w)$ is non-decreasing in both $s$ and $t$;
- for each fixed positive integer $w$, $(T(s,w), s > 0)$ is a 1-SSA process obtained by adding $w$ independent copies of $T$;
- for each fixed $w > 0$, $T(w) := T(\cdot, w) = (T(s,w), s > 0)$ is a 1-SSA process with finite rate $\theta w$ and generic jump $J$;
- for each fixed $s > 0$, $T(s) := T(s, \cdot) = (T(s,\theta), \theta > 0)$ is a subordinator;
- the family $(T(w), w > 0)$ is coupled such that it is a “subordinator of 1-SSA processes”, i.e. for fixed $u > v > 0$, the increment $T(w) - T(v) \overset{d}{=} T(w-v)$ is a 1-SSA non-decreasing process independent of $(T(s,w), s > 0, 0 < w < v)$;
- the family $(T(s), s > 0)$ is coupled such that it is an “1-SSA family of subordinators”, i.e. for fixed $s > 0$, $T(s) \overset{d}{=} sT^{(1)}$ and for fixed $u > v > 0$, the increment $T(u) - T(v)$ is a subordinator independent of $(T(s,w), 0 < s < v, w > 0)$;
- for each finite interval $I$, when $T$ is not identically 0, the jumps of $T(s) := (T(s)(w) = T(s,w), w > 0)$ as a subordinator are almost surely dense on $I$;
- however, for fixed $u > v > 0$, the number of jumps of $T(u) - T(v)$ is almost surely finite on $I$.

We only discussed above the case when $T$ is non-decreasing and with finite rate. But it is easy to see that the bivariate process can be defined more generally. In particular, it is easy to add a deterministic or Brownian component. But the case when $T$ has infinite rate and jumps of both signs would require more care.
8 Historical remarks

Selfdecomposable laws and self-similar processes  The class of selfdecomposable laws was first studied by Levy [25] and Khintchine [22], as an extension to stable laws as the limit distributions for sums of identically distribute random variables. Self-similar processes were first studied Lamperti [23] as a generalization to stable processes, which he called semi-stable processes.

Theorem 8.1 (Lamperti [23]) Let \((T(s), s > 0)\) be an additive process. Suppose \((T(s), s > 0)\) is also semi-stable, that is, for every \(c > 0\), there exists a constant \(b(c)\) such that

\[
(T(cs), s \geq 0) \overset{d}{=} (b(c)T(s), s \geq 0),
\]

then for some \(H \geq 0\)

\[
b(c) = c^H \quad (c > 0).
\]

Note that if \(H = 0\), the process is trivial; otherwise, it is a \(H\)-SSA process.

The integral representation of selfdecomposable laws was studied by Wolfe [38] for \(\mathbb{R}\)-valued random variables and then by Jurek and Vervaat [20] for more general Banach space-valued random variables. Although they did not consider self-similar processes, the integral representation

\[
T(1) = \int_0^\infty e^{-s}Y(ds),
\]

for \(Y(\cdot)\) the background driving Lévy process (BDLP) associated with \(T(1)\), is essentially the same as our integral representation (4.3) by treating \(e^{-s}\) as our index \(s\). See also Jurek [19] for a recent study of selfdecomposable laws and the associated BDLP. Sato [32, 33] investigated in detail and built the connection between the class \(L\) of selfdecomposable distributions and SSA processes as stated in Theorem 3.1. There is also a detailed background and a comprehensive list of references to earlier results on selfdecomposable laws in [32].

Jeanblanc, Pitman and Yor [17] pointed out that either of the two representations by Wolfe [38] and Sato [32] follows easily from the other. That reference also provides further background theory of selfdecomposable laws and their representations. Bertoin [4] treats the entrance law of self-similar processes. Tudor [36] studied the variations of self-similar processes from a stochastic calculus approach.

Summation representation of \(T(1)\)  In the setup of Theorem 4.1, the identity (4.1) can be viewed as a decomposition of the selfdecomposable random variable \(T(s)\). For simplicity, consider only \(T(1)\). Now index \(S_2\) such that

\[
0 < \cdots < S_{-2} < S_{-1} < S_0 \leq 1 < S_1 < S_2 < \cdots < \infty.
\]

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By Lemma 2.1

\[ S_{1-n} = \prod_{k=1}^{n} \beta_k \quad (n \geq 1), \quad (8.5) \]

where \( \beta_k \) are i.i.d. beta(\( \theta \), 1) random variables.

**Corollary 8.2** Suppose \( T(1) \) is a non-negative, selfdecomposable random variable. Then there exists a sequence of i.i.d. non-negative random variables \( J_z \) such that

\[ T(1) \overset{d}{=} \sum_{n=1}^{\infty} \left( \prod_{k=1}^{n} \beta_k \right) J_{1-n}, \quad (8.6) \]

where \( \beta_k \) are i.i.d. beta(\( \theta \), 1) random variables independent of \( J_z \).

The distribution of a random variable \( T(1) \) admitting the representation (8.6) for i.i.d. sequences \( \beta_k \) (not necessarily beta) and \( J_k \) was studied first by Vervaat [37, example 3.8] as the solution of the stochastic difference equation

\[ X \overset{d}{=} A(X + C), \quad (8.7) \]

with \( X, A, C \) independent and \( |A| < 1 \). By iteration,

\[ X \overset{d}{=} \sum_{n=1}^{\infty} \left( \prod_{k=1}^{n} A_k \right) C_n, \quad (8.8) \]

where \( A_k, C_n \) are i.i.d. copies of \( A \) and \( C \), independent of each other. This is identical to (8.6) by setting \( A_n = \beta_n \) and \( C_n = J_n \).

The following corollary of the fact that a SSA gamma process is associated with an exponential generic jump, giving a representation of a gamma distributed random variable, is also given in [37, example 3.8.2].

**Corollary 8.3 (Vervaat [37])** If \( C \sim \exp(\lambda) \) and \( A \sim \text{beta}(\theta, 1) \) for some \( \lambda, \theta > 0 \), then the unique solution to the stochastic difference equation (8.7) is \( X \sim \gamma(\theta, \lambda) \).

However, Vervaat did not make any connection with SSA processes in his work on stochastic difference equations, as he did not treat \( \left( \prod_{k=1}^{n} A_k \right) \) as a time index, and he did not discuss selfdecomposability of \( X \), only infinitely divisibility.
Extremal processes As mentioned in Section 1, the 1-SSA exponential process with rate $\theta = 1$ arises in the theory of extremal process introduced by Dwass [7]. Starting from an i.i.d. sequence $(X_n, n = 1, 2, \ldots)$ of continuous random variables, it is elementary that the record sequence $(M_n := \max_{1 \leq k \leq n} X_k, n = 1, 2, \ldots)$ is a Markov chain with state space $\mathbb{R}$ and transition probabilities specified by
\[
P(M_{n+m} \leq y \mid M_n = x) = F^m(y)1(x \leq y), \quad (8.9)
\]
where $F(t)$ is the $t$-th power of the common cumulative distribution function (c.d.f.) of the $X_i$. It was observed in 1964 by Dwass [7] and Lamperti [24] that the record process can be generalized to a time-homogeneous pure jump-type Markov process, known as the extremal process $M := (M(t), t > 0)$, described by its entrance law
\[
P(M(t) \leq y) = F^t(y), \quad (8.10)
\]
and the hold-jump description: conditional on $M(s) = x$,
- ‘Hold’ - at level $x$ for an exponential time $H_x$ with rate $Q(x) := -\log F(x)$ so that
\[
P(H_x > t) = \mathbb{P}(M(t) \leq x) = F^t(x) = e^{-tQ(x)}; \quad (8.11)
\]
- ‘Jump’ - at time $H_x$ to a state $L_x := M(s + H_x)$ with distribution
\[
P(L_x > b) = \frac{Q(b)}{Q(x)} \quad (b \geq x). \quad (8.12)
\]
See Shorrock [35] or Kallenberg [21, Chapter 13] for more details.

**Proposition 8.4 (Dwass [7] Resnick and Rubinovitch [31] Shorrock [35])** Let $((T_z, M_z), z \in \mathbb{Z})$ be a listing indexed by integers $\mathbb{Z}$ of the times $T_z$ of jumps of $(M(t), t \geq 0)$ and the corresponding record levels $M_z := M(T_z)$, with $T_z < T_{z+1}$. Then
\[
\text{range}((M_z, T_{z+1} - T_z), z \in \mathbb{Z}) \text{ is } \text{PPP}(Q(dm)e^{-Q(m)t}dt). \quad (8.13)
\]
In particular,

(I) the random set of record times $T_z$ is a scale invariant PPP on $\mathbb{R}_+$ with rate 1;

(II) the random set of record levels $\text{range}(M_z, z \in \mathbb{Z})$ is $\text{PPP}(Q(dm)/Q(m))$;

(III) the random set of holding times at these record levels $\text{range}(T_{z+1} - T_z, z \in \mathbb{Z})$ is a scale invariant PPP on $\mathbb{R}_+$ with rate 1.
As indicated by Dwass and Lamperti, the extremal Markov process associated with each continuous distribution \( F \) on the line is essentially the same as that associated with every other continuous distribution \( F' \), via the monotonic transformation
\[
M' = \psi(M), \quad \text{for } \psi \text{ with } F'(\cdot) = F(\psi \in \cdot).
\]

To illustrate the point and see its relation with our Theorems 1.3 and 1.2, consider the case of Gnedenko’s extreme value distribution, that is the distribution of \( \varepsilon - 1 \) for \( \varepsilon \) exponential with mean 1:
\[
F(y) = \exp(-y^{-1}) = \mathbb{P}(\varepsilon^{-1} \leq y) \quad (y > 0),
\]
hence the rate in (8.11) is
\[
Q(x) = -\log F(x) = \frac{1}{x},
\]
and the jump has distribution
\[
\mathbb{P}(J_x > b) = \frac{Q(b)}{Q(x)} = \frac{x}{b} \quad (b \geq x),
\]
namely \( x/J_x \sim \text{beta}(1, 1) \). It is easy to check that the functional inverse (which means the ‘hold’ and ‘jump’ are swapped) coincides with the hold-jump description (5.2) (5.3) of a 1-SSA gamma process \( (T(s), s > 0) \) with rate 1 and \( T(s) \overset{d}{=} s\varepsilon \sim \exp(s^{-1}) \).

Therefore, the path of an extremal process associated with (8.15) has the same distribution as the functional inverse of the path of a standard 1-SSA exponential process. This matches Proposition 8.4(I) with Theorem 1.3 and Proposition 8.4(III) with Theorem 1.2(III). Lastly, in this case the \( \text{PPP}(Q(dm)/Q(m)) \) in Proposition 8.4(II) is a scale invariant PPP on \( \mathbb{R}_+ \) with rate 1, as in Theorem 1.2(II).

It may also be of interest to replace the i.i.d. sequence above with an inhomogeneous Markov sequence \( (X_n, n = 1, 2, \ldots) \) defined as follows:

- The first term \( X_1 \) has c.d.f. \( F^\theta \);
- Conditionally on \( M_n = x \), let \( F_{x+} \) be the c.d.f. of \( X \) given \( X > x \)
  \[
  F_{x+}(y) := (F(y) - F(x))/(1 - F(x)) \quad (y > x),
  \]
  and \( F_{x-} \) be the one given \( X \leq x \)
  \[
  F_{x-}(y) := F(y)/F(x) \quad (y \leq x).
  \]

Then the distribution of \( X_{n+1} \) is the mixture with weights \( 1 - F(x) \) and \( F(x) \) of the distribution of a random variable with c.d.f. \( F^\theta_{x+} \), and distribution of a random variable with c.d.f. \( F^\theta_{x-} \), i.e.
\[
\mathbb{P}(X_{n+1} \leq y \mid M_n = x) = (1 - F(x))F^\theta_{x+}(y) + F(x)F^\theta_{x-}(y).
\]
An example of this sequence is obtained by putting $X \sim \text{Uniform}(0, 1)$. Then $X_1 \sim \text{beta}(1, \theta)$ and conditionally on $M_n = x$, $X_{n+1}$ is the mixture with weights $1-x$ and $x$ of $\text{Uniform}(0, 1-x)$ and $x + (1-x)\beta$ for $\beta \sim \text{beta}(1, \theta)$.

**Theorem 8.5** Suppose $X$ is a real-valued random variable without atoms, and the sequence $(X_n, n = 1, 2, \ldots)$ follows the inductive construction above. Then the maximum indicators $(B_n, n = 1, 2, \ldots)$ form a sequence of independent Bernoulli random variables with $\mathbb{E}B_n = \theta / (\theta + n - 1)$.

This is a generalization to Najnudel and Pitman [27, Corollary 1.4] where they proved the same result for $X \sim \text{Uniform}(0, 1)$. This generalization works because the distribution of $X$ has no atoms, hence has no influence on record times.

It is not hard to construct a time-inhomogeneous extremal process $M$ associated with $(X_n, n = 1, 2, \ldots)$ and check that the path of such extremal process has the same distribution as the functional inverse of the path of a 1-SSA gamma process $(T(s), s > 0)$ with $T(1) \sim \text{gamma}(\theta, 1)$. In this case, Theorem 8.5 can be read as a discrete-time analogue of Theorem 1.3 and an analogue of Proposition 8.4 can also be easily given by replacing the rate in (I) and (III) by $\theta$, and replacing the PPP in (II) by $\text{PPP}(\theta Q(dm)/Q(m))$.

**Scale invariant point processes** Scale invariant point processes, and scale invariant random sets, including the scale invariant PPP, were studied in [30, 12] by considering the random partitions of $(0, 1)$ related to the Poisson-Dirichlet distribution. It was observed in [30] that a random closed set is scale invariant if and only if the associated age process is 1-self-similar, while [12] remarked on the relationship between scale invariant PPP and records.

**The scale invariant Poisson spacings lemma for general $\theta \neq 1$** The formulation of the Theorem 1.3 for general $\theta > 0$ was suggested by proofs of the scale invariant Poisson spacings Theorem 1.1, first indicated by Arratia [11, 2], and detailed by Arratia, Barbour and Tavaré [3], where they make use of a Poisson point process on $(0, \infty)^2$ which turns out to be our $\eta$ in (4.3). Based on [3], Gnedin [13, Section 4] pointed out that the scale invariant Poisson spacings theorem for positive integer $\theta$ follows from a specialization of Ignatov’s theorem in the form of [14, Corollary 5.1].

**Feller’s coupling and the Ewens sampling formula** The parameter $\theta$ in the Poisson spacings theorem is related to the Ewens(\theta) distribution [10] as a generalization to the uniform random permutation of $[n]$. Feller [11] provided a coupling between the counts of cycles of various sizes in a uniform random permutation of $[n]$ and the spacings between successes in a sequence of $n$ independent Bernoulli($k^{-1}$) trials at the $k$th trial.
Informally, this sequence of independent Bernoulli trials is a discrete analogue of the scale invariant PPP with rate 1 relative to $x^{-1}dx$.

Ignatov [16] proved that in an infinite sequence of independent Bernoulli($n^{-1}$) trials, as the indicators of record values in an i.i.d. sequence, the numbers of spacings of length $k$ between successes/records are independent Poisson variables with means $k^{-1}$. This Poisson sequence provides another discrete analogue of the scale invariant PPP with rate 1. It is interesting that this discrete result was obtained many years after theory of extremal processes by Dwass [7]. Ignatov’s result was generalized by Arratia, Barbour and Tavaré [3] in the study of cycles of (non-uniform) random permutations governed by the Ewens($\theta$) distribution, i.e. a permutation is weighted $\theta^k$ if there are $k$ cycles. See Najnudel and Pitman [27] for details of the coupling between the random permutations governed by the Ewens($\theta$) distribution and the sequence of inhomogeneous Bernoulli($\theta(\theta + n - 1)^{-1}$) trials, as mentioned in Theorem 8.5.

References

[1] ARRATIA, R. On the central role of scale invariant Poisson processes on $(0, \infty)$. In Microsurveys in discrete probability (Princeton, NJ, 1997), vol. 41 of DIMACS Ser. Discrete Math. Theoret. Comput. Sci. Amer. Math. Soc., Providence, RI, 1998, pp. 21–41.

[2] ARRATIA, R. On the amount of dependence in the prime factorization of a uniform random integer. In Contemporary combinatorics, vol. 10 of Bolyai Soc. Math. Stud. János Bolyai Math. Soc., Budapest, 2002, pp. 29–91.

[3] ARRATIA, R., BARBOUR, A. D., AND TAVARÉ, S. A tale of three couplings: Poisson-Dirichlet and GEM approximations for random permutations. Combin. Probab. Comput. 15, 1-2 (2006), 31–62.

[4] BERTOIN, J., AND YOR, M. The entrance laws of self-similar Markov processes and exponential functionals of Lévy processes. Potential Anal. 17, 4 (2002), 389–400.

[5] CARR, P., GEMAN, H., MADAN, D. B., AND YOR, M. Self-decomposability and option pricing. Math. Finance 17, 1 (2007), 31–57.

[6] CHAUMONT, L., AND YOR, M. Exercises in probability: a guided tour from measure theory to random processes, via conditioning, second ed., vol. 35 of Cambridge Series in Statistical and Probabilistic Mathematics. Cambridge University Press, Cambridge, 2012.

[7] DWASS, M. Extremal processes. Ann. Math. Statist. 35 (1964), 1718–1725.
[8] Dwass, M. Extremal processes. II. *Illinois J. Math.* 10 (1966), 381–391.

[9] Dwass, M. Extremal processes. III. *Bull. Inst. Math. Acad. Sinica* 2 (1974), 255–265. Collection of articles in celebration of the sixtieth birthday of Ky Fan.

[10] Ewens, W. J. The sampling theory of selectively neutral alleles. *Theoret. Population Biol.* 3 (1972).

[11] Feller, W. The fundamental limit theorems in probability. *Bull. Amer. Math. Soc.* 51 (1945), 800–832.

[12] Gneden, A., and Pitman, J. Self-similar and Markov composition structures. *Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI)* 326, Teor. Predst. Din. Sist. Komb. i Algoritm. Metody. 13 (2005), 59–84, 280–281.

[13] Gneden, A. V. Corners and records of the Poisson process in quadrant. *Electron. Commun. Probab.* 13 (2008), 187–193.

[14] Goldie, C. M., and Rogers, L. C. G. The $k$-record processes are i.i.d. *Z. Wahrsch. Verw. Gebiete* 67, 2 (1984), 197–211.

[15] Groeneboom, P. The concave majorant of Brownian motion. *Ann. Probab.* 11, 4 (1983), 1016–1027.

[16] Ignatov, Z. Point processes generated by order statistics and their applications. In *Point processes and queuing problems (Colloq., Keszthely, 1978)*, vol. 24 of *Colloq. Math. Soc. János Bolyai*. North-Holland, Amsterdam-New York, 1981, pp. 109–116.

[17] Jeanblanc, M., Pitman, J., and Yor, M. Self-similar processes with independent increments associated with Lévy and Bessel. *Stochastic Process. Appl.* 100 (2002), 223–231.

[18] Jurek, Z. J. Selfdecomposability: an exception or a rule? *Ann. Univ. Mariae Curie-Skłodowska Sect. A* 51, 1 (1997), 93–107.

[19] Jurek, Z. J. Background driving distribution functions and series representations for log-gamma selfdecomposable random variables. *Teor. Veroyatn. Primen.* 67, 1 (2022), 134–149.

[20] Jurek, Z. J., and Vervaat, W. An integral representation for self-decomposable Banach space valued random variables. *Z. Wahrsch. Verw. Gebiete* 62, 2 (1983), 247–262.
[21] **Kallenberg, O.** *Foundations of modern probability*, vol. 99 of *Probability Theory and Stochastic Modelling*. Springer, Cham, [2021] ©2021. Third edition [of 1464694].

[22] **Khintchine, A. Y.** Limit laws of sums of independent random variables. *ONTI, Moscow,(Russian)* (1938).

[23] **Lamperti, J.** Semi-stable stochastic processes. *Trans. Amer. Math. Soc. 104* (1962), 62–78.

[24] **Lamperti, J.** On extreme order statistics. *Ann. Math. Statist. 35* (1964), 1726–1737.

[25] **Lévy, P.** *Théorie de l’addition des variables aléatoires*. Paris: Gauthier-Villars, 1937.

[26] **Lukacs, E.** A characterization of the gamma distribution. *Ann. Math. Statist. 26* (1955), 319–324.

[27] **Najnudel, J., and Pitman, J.** Feller coupling of cycles of permutations and Poisson spacings in inhomogeneous Bernoulli trials. *Electron. Commun. Probab. 25* (2020), Paper No. 73, 11.

[28] **Ouaki, M., and Pitman, J.** Markovian structure in the concave majorant of brownian motion, 2021.

[29] **Pitman, J., and Ross, N.** The greatest convex minorant of Brownian motion, meander, and bridge. *Probab. Theory Related Fields 153*, 3-4 (2012), 771–807.

[30] **Pitman, J., and Yor, M.** Random discrete distributions derived from self-similar random sets. *Electron. J. Probab. 1* (1996), no. 4, approx. 28 pp.

[31] **Resnick, S. I., and Rubinovitch, M.** The structure of extremal processes. *Advances in Appl. Probability 5* (1973), 287–307.

[32] **Sato, K.-i.** Self-similar processes with independent increments. *Probab. Theory Related Fields 89*, 3 (1991), 285–300.

[33] **Sato, K.-i.** *Lévy processes and infinitely divisible distributions*, vol. 68 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1999. Translated from the 1990 Japanese original, Revised by the author.

[34] **Sato, K.-i., and Yamazato, M.** On distribution functions of class L. *Z. Wahrsch. Verw. Gebiete 43*, 4 (1978), 273–308.
[35] **Shorrock, R. W.** On discrete time extremal processes. *Advances in Appl. Probability* 6 (1974), 580–592.

[36] **Tudor, C. A.** *Analysis of variations for self-similar processes: A stochastic calculus approach.* Probability and its Applications (New York). Springer, Cham, 2013.

[37] **Vervaat, W.** On a stochastic difference equation and a representation of non-negative infinitely divisible random variables. *Adv. in Appl. Probab.* 11, 4 (1979), 750–783.

[38] **Wolfe, S. J.** On a continuous analogue of the stochastic difference equation $X_n = \rho X_{n-1} + B_n$. *Stochastic Process. Appl.* 12, 3 (1982), 301–312.