CRITERION FOR CONVERGENCE ALMOST EVERYWHERE,
with applications

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Abstract.

We derive the necessary and sufficient condition for almost sure convergence of
the sequence of measurable functions, and consider some applications in the theory
of Fourier series and in the theory of random fields.

Key words and phrases: Measure, sigma-finiteness, sigma field, convergence al-
most everywhere (almost surely), partition, random variable (r.v.), random process-
es (r.p.) and random fields (r.f.), separable Banach space, functional, sub-linearity,
upper limit, Grand Lebesgue Spaces (GLS), Dirichlet kernel, critical functional, ap-
proximation, kernel of functional, criterion, probability, distribution, Orlicz space
and Orlicz norm convergence.

1 Introduction, Notations. Statement of problem.

Let \((X = \{x\}, M, \mu)\) be measurable space equipped with sigma-finite non zero
measure measure \(\mu, \quad (B = \{b\}, || \cdot || = || \cdot || B)\) be a (complete) separable Banach
space relative the norm function \(|| \cdot || = || \cdot || B, \) not necessary to be separable (or
reflexive).

Let also

\[ F = \{f(x) = f_\infty(x), \{f_n(x)\}, \ n = 1, 2, \ldots; \ x \in X\} \]

be a numbered family of measurable functions from the set \(X\) into the space \(B:\)

\[ f_n : X \to B, \ n \in \{\infty\} \cup \{1, 2, \ldots\}. \]

Our goal in this short article is finding of the necessary and sufficient
condition on the family \(F\) in the integrals terms for almost sure conver-
gence of the sequence \(f_n \to f_\infty\) or simple \(\exists \lim_{n \to \infty} f_n \ a.e. : \)
\[ \mu \{ x, \ x \in X, \ \lim_{n \to \infty} f_n(x) \neq f(x) \} = 0. \quad (1.0) \]

**Remark 1.1.** We imply in the equality (1.0) that if the limit \( \lim_{n \to \infty} f_n(x) \) can not exists, but on the set with zero measure.

Immediate predecessor of general case besides the special cases: martingales, monotone sequences etc. is the preprint [36]. We intend to generalize the results obtained therein.

There are many applications of solution of this problem in the theory of Fourier series (and integrals) [31], [40] and other orthogonal ones [1], theory of Probability, [3], [4], in particular, in the theory of martingales [19], statistics [38] etc.

## 2 Main Result: Convergence of a Sequence of a measurable Functions almost everywhere.

**Lemma 2.1.** There exists a probabilistic measure \( \nu \) defined on at the same sigma-field \( M \) which is equivalent to the source measure \( \mu \) in the Radon-Nikodym sense, i.e. such that the following implication there holds

\[ \forall A \in M \Rightarrow [\mu(A) = 0 \Leftrightarrow \nu(A) = 0]. \quad (2.1) \]

**Proof.** The case of boundedness of the measure \( \mu(\cdot) \) is trivial; suppose therefore \( \mu(X) = \infty \).

As long as the measure \( \mu \) is sigma - finite, there exist a countable family of disjoint measurable sets \( \{X_m\}, \ m = 1, 2, 3, \ldots, X_m \in M, \ l \neq m \Rightarrow X_m \cap X_l = \emptyset \), such that

\[ 0 < \mu(X_m) < \infty, \quad \cup_{m=1}^{\infty} X_m = X, \quad (2.2) \]

so that the family \( \{X_m\} \) forms the partition of whole set \( X \).

Define for any set \( A \in M \)

\[ \nu(A) \overset{\text{def}}{=} \sum_{m=1}^{\infty} \frac{\mu(A \cap X_m)}{2^m \mu(X_m)}. \quad (2.3) \]

Obviously, \( \nu(\cdot) \) is sigma-additive probability \( (\nu(X) = 1) \) measure which is completely equivalent to the initial measure \( \mu \).

As a consequence: the \( \mu \) – complete convergence of the sequence of measurable functions \( \{f_n(x)\} \) is equal to the convergence with \( \nu \) – measure one.

*Therefore, we can and will assume without loss of generality that the initial measure \( \mu \) is probabilistic, i.e. \( \mu(X) = 1 \).*

Further, we can reduce our problem again without loss of generality passing to the sequence \( g_n = f_n - f_\infty \) to the case when \( f_\infty = 0 \).
A probability language: given the sequence of random variables $\xi_n, \xi, n = 1, 2, \ldots$ with values in certain separable Banach space $B$; find the necessary and sufficient condition for the following convergence with probability one:

$$
P(\lim_{n \to \infty} ||\xi_n - \xi||_B = 0) = 1.
$$

(2.4)

One can assume as before $\xi = 0$.

Let as introduce the Banach space $c_0(B)$, also separable, consisting on all the sequences $y = \{y(n)\}$ with values in the space $B$ and converging to zero in this space:

$$
||y_n||_B \to 0, \ n \to \infty,
$$

with the norm

$$
||y||_{c_0(B)} \overset{\text{def}}{=} \sup_n ||y_n||_B.
$$

The equality (2.4) may be reformulated as follows: under what necessary and sufficient conditions (criterion)

$$
P\left(\{\xi_n\} \in c_0(B)\right) = 1.
$$

(2.5)

On the other words, we raise the question of finding of necessary and sufficient conditions for convergence of random elements in (separable) Banach spaces, in the terms of a famous monograph of V.V.Buldygin [5].

Let us return to the initial notations: $\{f_n(x)\}$, $\nu(\cdot)$ and so one. Suppose $f(x) = f_\infty(x) = 0$.

We need to introduce some new notations.

$$
\tilde{f}_n(x) := \arctan f_n(x), \ \kappa^m_n = \kappa^m_n(F) \overset{\text{def}}{=} \int_X \arctan(\max_{k=n}^m |f_k(x)|) \ \nu(dx) = \\
\int_X \max_{k=n}^m |\tilde{f}_k(x)| \ \nu(dx), \ m \geq n + 1;
$$

(2.6)

$$
\chi(F) \overset{\text{def}}{=} \lim_{n \to \infty} \sup_{m \geq n+1} \kappa^m_n(F).
$$

(2.7)

Definition 2.1. The introduced above functional $\chi(F)$ and the like functionals that will be appear further, are sub-linear and will be named as "critical functional".

Theorem 2.1. In order to $\nu\{x : \lim_{n \to \infty} f_n(x) = 0\} = 1$, is necessary and sufficient that the family $F$ belongs to the kernel of the critical functional $\chi(F)$:

$$
\lim_{n \to \infty} \sup_{m > n} \kappa^m_n(F) = 0
$$

(2.8)

or equally

$$
\chi(F) = 0 \quad \text{or equally} \quad F \in \ker(\chi).
$$

(2.8a)
Proof. Necessity.

Let $\nu\{x : (\lim_{n \to \infty} |f_n(x)| = 0)\} = 1$, then with at the same value of the $\nu$ - measure

$$\lim_{n \to \infty} \sup_{m > n} |f_m(x)| = 0$$

and a fortiori

$$\lim_{n \to \infty} \sup_{m > n} \arctan |f_m(x)| = 0$$

$\nu(\cdot)$ almost surely. We conclude on the basis of the dominated convergence theorem

$$\lim_{n \to \infty} \int X \sup_{m > n} \arctan |f_m(x)| \nu(dx) = 0,$$

which is quite equivalent to the equality (2.8.)

Proof. Sufficiency.

0. We set primarily

$$A = \{x, x \in X : \lim_{n \to \infty} f_n(x) = 0\} = \{x \in X : \lim_{n \to \infty} \tilde{f}_n(x) = 0\},$$

$$A_Q = \{x, x \in X : \forall s = 1, 2, \ldots \exists N = 1, 2, \ldots : \max_{k \in [N, N+Q]} |\tilde{f}_k(x)| < 1/s\}.$$

Obviously,

$$\nu(A) = \lim_{Q \to \infty} \nu(A_Q).$$

1. Note first of all that

$$A = \{x, x \in X : \lim_{n \to \infty} |f_n(x)| = 0\} = \{x : \lim_{n \to \infty} |\tilde{f}_n(x)| = 0\} =$$

$$\cap_s \cup_N \{x : \sup_{n \geq N} |\tilde{f}_n(x)| < 1/s\} \quad (2.9)$$

and correspondingly

$$A_Q := \cap_s \cup_N \{x : \max_{n \in [N, N+Q]} |\tilde{f}_n(x)| < 1/s\}.$$

2. Let the condition (2.8) be satisfied. We consider a supplementary set

$$B := \overline{A} = X \setminus A, \ B_Q = \overline{A}_Q = X \setminus A_Q. \quad (2.10)$$
Elucidation: the set of elementary events $B$ may contains (theoretically) also the points when the limit does not exists.

We can write
\[
B = \bigcup_s \cap_N \{ x : \sup_{n \geq N} |\tilde{f}_n(x)| \geq 1/s \}, \tag{2.11}
\]
\[
B_Q = \bigcup_s \cap_N \{ x : \max_{n \in [N,N+Q]} |\tilde{f}_n(x)| \geq 1/s \} = \cup_s C_{s,Q}, \tag{2.12}
\]
where
\[
C_{s,Q} = \cap_N \{ x : \max_{n \in [N,N+Q]} |\tilde{f}_n(x)| \geq 1/s \} = \cap_N D_{s,Q}^{(N)}, \tag{2.13}
\]
\[
D_{s,Q}^{(N)} := \{ x, x \in X : \max_{n \in [N,N+Q]} |\tilde{f}_n(x)| \geq 1/s \}. \tag{2.14}
\]

3. We obtain using the Tchebychev’s inequality:
\[
\nu\{ D_{s,Q}^{(N)} \} \leq \frac{\kappa_{N+Q}}{\arctan(1/s)} \to 0, \ N \to \infty,
\]
therefore for all the natural values $s, Q$
\[
\nu( C_{s,Q} ) = 0,
\]
following
\[
\forall Q = 1,2,\ldots \Rightarrow \nu( B_Q ) = 0. \tag{2.15}
\]

4. We find
\[
\nu(B) = \lim_{Q \to \infty} \nu( B_Q ) = 0, \tag{2.16}
\]
and ultimately $\nu(A) = 1$, Q.E.D.

Remark 2.1.

We will consider here the case of the Banach space $c$ consisting on all the numerical sequences $\{ x(n) \}$ with existing the limit
\[
\exists \lim_{n \to \infty} x(n) =: x(\infty)
\]
with at the same norm as above. As before, we consider the classical problem: let $\xi = \{ \xi(n) \}$ be a random sequence; find the conditions (necessary conditions and sufficient conditions) under which $P(\exists \lim_{n \to \infty} \xi(n)) = 1$ or equally $P(\{ \xi \} \in c) = 1$.

Notations:
\[
\tilde{\xi}(n) := \arctan(\xi(n)), \ \gamma_n^m = \gamma_n^m(\xi) \overset{\text{def}}{=} E \arctan( \max_{k=n} |\xi(k) - \xi(n)| ).
\]
\[
\gamma(\xi) := \lim_{n \to \infty} \sup_{m \geq n+1} \gamma_n^m(\xi).
\]
**Proposition 2.1.** We find analogously to the theorem 2.1: $P(\{\xi \in c\}) = 1$ if and only if
\[
\lim_{n \to \infty} \sup_{m \geq n + 1} \gamma^m_n(\xi) = 0,
\]
(2.17)
or briefly
\[
\varpi(\xi) = 0, \quad \text{or equally} \quad \xi(\cdot) \in \ker \varpi.
\]  
(2.17a)

**Remark 2.2.**

It is known in the theory of martingales, see e.g. [6], [7], [19], chapters 2,3, [39] that the estimation of the maximum distribution play a very important role for the investigation of limit theorems, non asymptotical estimations etc.

It follows from our considerations that at the same is true in more general case of non-martingale processes and sequences.

**Remark 2.3.** Roughly speaking, the result of theorem 2.1 may be reformulated as follows. Let again $\{\xi(n)\}$ be a sequence of a r.v., $\tilde{\xi}(n) = \arctan(|\xi(n)|)$, and
\[
\eta(n) = \sup_{m \geq n} |\tilde{\xi}(m)|.
\]
Then
\[
\{\omega : \xi(n) \to 0\} = \{\omega : \eta(n) \to 0\}.
\]

But the random sequence $\{\eta(n)\}$ is monotonically non-increasing, therefore the sequence $\{\eta(n)\}$ tends to zero with probability one iff this sequence tends to zero in probability, or equally
\[
\lim_{n \to \infty} E\eta(n) = 0,
\]
(2.18)
because the variables $\eta(n)$ are uniformly bounded.

**Remark 2.4.** Let’s turn our attention to the properties of introduced above critical functional $\varpi(F’)$ and $\varpi(\xi)$ and so one. Naturally and obviously, the kernels of these functionals are closed linear subspaces; if for definiteness $\varpi(\xi_1) = 0$ and $\varpi(\xi_2) = 0$, then
\[
\varpi(c_1\xi_1 + c_2\xi_2) = 0, \quad c_1, c_2 = \text{const}.
\]  
(2.19)

3 Almost everywhere convergence of Fourier series.

Let $T$ be a segment $T = [0, 2\pi]$ and $\mu$ is customary renormed Lebesgue measure $d\mu = dx/(2\pi)$. 
Let also $g = g(x), \ x \in R$ be measurable and integrable $(2\pi)$ - periodical numerical function. Denote by $s_n(x) = s_n[g](x)$ the its $n^{th}$ partial sum of ordinary Fourier (trigonometrical) series, which may be written through $(2\pi)$ - periodical convolution with Dirichlet kernel $D_n(x)$:

$$s_n[g](x) = g \ast D_n(x) = \int_0^{2\pi} D_n(x - y) g(y) \, dy,$$

where

$$D_n(x) = \frac{\sin[(n + 1/2)x]}{2\pi \sin(x/2)}.$$

The problem of finding (sufficient) conditions under the function $g(\cdot)$ for the almost everywhere convergence

$$s_n[g](x) \mu - a.e. \rightarrow g(x)$$

is named ordinary as Luzin’s problem and has a long history; see for example [2], [8], [18], [20], [31], [40], [49].

The following famous result belongs to N.Y.Antonov [2]: if

$$\int_T | f(x) | \cdot \ln | f(x) | \cdot \ln | \ln | f(x) | \, dx < \infty,$$

where

$$\ln^+ z = \max(e, \ln z), \quad z > 0,$$

or equally $f(\cdot) \in L \ln L \ln \ln \ln L$, then the convergence (3.3) holds true.

Define also a difference Dirichlet kernel $D_{m,n}(x) = D_m(x) - D_{n1}(x), \ m \geq n + 1, \ n \geq 2$, then

$$D_{m,n}(x) = \pi^{-1} \frac{\sin[(m - n)x/2] \cdot \cos[(m + n + 1)x/2]}{\sin(x/2)},$$

and introduce the difference of the Fourier sums

$$s_{m,n}(x) := [D_{m,n} \ast g](x) = s_m[g](x) - s_n[g](x).$$

Denote also

$$\theta^n_m[g] \overset{df}{=} \int_{[0,2\pi]} \max_{k \in [n+1,m]} \arctan \max_{x \in [0,2\pi]} |s_{k,n}(x)| \, dx,$$

$$\bar{\theta}[g] \overset{df}{=} \lim_{m \to \infty} \sup_{m \geq n+1} \theta^n_m[g].$$

The functional $g \rightarrow \bar{\theta}[g]$ is now the critical sub-linear functional for considered here problem.

It follows immediately from Theorem 2.1 and Proposition 2.1.
**Proposition 3.1.** The Fourier sums \( s_n[g](x) \) for the function \( g = g(x) \) converges almost surely to the function \( g = g(x) \) (3.3) if and only if

\[
\lim_{n \to \infty} \sup_{m \geq n+1} \theta_n^m[g] = 0. \quad (3.6)
\]

or for brevity

\[
\overline{\theta}(g) = 0 \quad \text{or equally} \quad g \in \ker(\overline{\theta}). \quad (3.6a)
\]

4 About trial function.

We used in the second section a trial function \( x \to \arctan(|x|) \). Evidently, it can apply some another functions.

**Definition 4.1.** We will denote by \( KB \) the class of all numerical functions \( \{\phi\}, \phi : R \to R_+ \) satisfying the following conditions:

- **A.** \( \phi(x) \geq 0; \phi(x) = 0 \iff x = 0 \), the condition of positivity;

- **B.** \( 0 < x < y \Rightarrow \phi(x) < \phi(y) \), the strong monotonicity on the right-hand real axis;

- **C.** Continuity: function \( x \to \phi(x) \) is continuous on the whole axis \( R \).

- **D.** The function \( \phi(x) \) is even: \( \forall x \in R \Rightarrow \phi(-x) = \phi(x) \).

- **E.** \( \sup_{x \in R} \phi(x) < \infty \), the condition of boundedness.

For example:

\[
\phi(x) = \arctan |x|, \quad \phi(x) = \frac{|x|}{1 + |x|}, \quad \phi(x) = \frac{x^2}{1 + x^2}
\]

and so one.

The assertion of theorem 2.1 may be rewritten as follows. Denote as before for any function \( \phi \) from the set \( KB \)

\[
\kappa^m_n(\phi) = \kappa^m_n(\phi, F) \overset{def}{=} \int_X \phi(\max_{k=m} |f_k(x)|) \nu(dx), \quad m \geq n + 1, \quad (4.1)
\]
κ(φ) = \lim_{n \to \infty} \sup_{m \geq n+1} \kappa_n^m(\phi, F). \quad (4.1a)

**Theorem 4.1.** In order to \( \nu\{ x : \lim_{n \to \infty} f_n(x) = 0 \} = 1 \), it is sufficient that for some function \( \phi(\cdot) \) from the class \( KB \)

\[
\lim_{n \to \infty} \sup_{m \geq n+1} \kappa_n^m(\phi, F) = 0,
\]

or equally

\[
\{ F \} \in \ker(\tau(\phi)). \quad (4.2a)
\]

and is necessary that for arbitrary function \( \phi \) belonging to at the same set \( KB \) the relation (4.2) or (4.2a) holds true.

**Definition 4.2.** We will denote by \( K \) the class of all numerical functions \{\phi\}, \( \phi : R \to R_+ \) satisfying the foregoing conditions at this section A, B, C, D, i.e. all the conditions except for the latter condition of boundedness E. For instance, let \( \phi_p(x) := |x|^p \), \( p = \text{const} > 0 \); then \( \phi_p(\cdot) \in K \).

In particular, arbitrary Young-Orlicz function \( \phi(x) \) belongs to the set \( K \).

**Theorem 4.2.** In order to \( \nu\{ x : \lim_{n \to \infty} f_n(x) = 0 \} = 1 \), it is sufficient that for some function \( \phi(\cdot) \) from the class \( K \)

\[
\lim_{n \to \infty} \sup_{m \geq n+1} \kappa_n^m(\phi, F) = 0,
\]

and herewith

\[
\lim_{n \to \infty} \int_X \phi( ||f_n(x)||B ) \nu(dx) = 0; \quad (4.4a)
\]

\[
\sup_n \int_X \phi( ||f_n(x)||B ) \nu(dx) \leq \sup_n \sup_{m \geq n+1} \kappa_n^m(\phi, F). \quad (4.4b)
\]

The relation (4.4a) implies on the language of the theory of Orlicz’s function the so-called moment, or weak convergence \( f_n \to 0 \) in the Orlicz norm \( ||| \cdot |||L\phi \); the last equality (4.4b) denotes the uniform boundedness of the considered sequence \{\phi_n(\cdot)\} in this space.

If in addition this function \( \phi = \phi(z) \) is Young-Orlicz function satisfying the so-called \( \Delta_2 \) condition, then the sequence of functions \( f_n(\cdot) \) convergent to zero also in the Orlicz’s norm \( ||| \cdot |||L\phi \):

\[
\lim_{n \to \infty} |||f_n|||L\phi = 0. \quad (4.5)
\]

Let us show another approach which is closely to the so-called Grand Lebesgue Spaces (GLS), see e.g. [14], [22], [27], [30], [33], [37].

Let again \( F = \{ f_n(x) \} \), \( x \in X \) be as before in the first section be the sequence of measurable functions. Define a new function
\[
\psi(p) \overset{\text{def}}{=} \sup_n \left[ \int_X |f_n(x)|^p \nu(dx) \right]^{1/p},
\]

the so-called \textit{natural} function for the sequence \( F = \{f_n(x)\}, \ x \in X, \) and suppose its finiteness for certain interval of the form \( 1 \leq p < R, \) where \( 1 < R = \text{const} \leq \infty. \)

This function \( \psi = \psi(p) \) generated the so-called Grand Lebesgue Space (GLS) \( G\psi \) consisting on all the numerical measurable functions \( h = h(x), \ x \in X \) with finite norm

\[
||h(\cdot)||_{G\psi} := \sup_{p \in (1, R)} \left[ \int_X |h(x)|^p \nu(dx) \right]^{1/p},
\]

where as usually

\[
|h|_p := \left[ \int_X |h(x)|^p \nu(dx) \right]^{1/p}.
\]

These spaces are complete rearrangement invariant Banach spaces which are detail investigated in [27], [30], [33] and so one.

Define the following critical functions

\[
\lambda_n^m(F) = || \max_{k=n+1}^m |f_k(\cdot)| ||_{G\psi},
\]

\[
\bar{\lambda}(F) = \lim_{n \to \infty} \sup_{m \geq n+1} \lambda_n^m(F).
\]

We conclude as before:

**Theorem 4.2.** If \( \bar{\lambda}(F) = 0, \) then the sequence \( f_n(x) \) converges to zero almost everywhere and in the Grand Lebesgue Space norm \( G\psi. \)

As long as the classical Lebesgue-Riesz spaces \( L_p(X), \ p = \text{const} \geq 1 \) are the extremal case for GLS spaces, and also the particular cases of the classical Orlicz spaces, we conclude denoting again for \( f = \{f_n(x)\}, \ x \in X \)

\[
\overline{\lambda}_p(F) = \lim_{n \to \infty} \sup_{m \geq n+1} |f_n - f_m|_p.
\]

**Theorem 4.3.** If \( \overline{\lambda}_p(F) = 0, \) then the sequence \( f_n(x) \) converges almost everywhere as well as in the Lebesgue-Riesz norm \( L_p(X). \)

5 Convergence of random elements in separable Banach spaces.

We return to the raised before the question of finding of necessary and sufficient conditions for convergence of random elements in separable Banach spaces, see [5].
The particular case of this problem is the problem of continuity of random processes and fields, is considered in the articles and books [10], [11], [12], [13], [16], [17], [21], [24], [26], [27], [28], [29], [32], [33], [36], [41], [42], [43], [44], [45], [46], [48] etc.

This problem may be easily reduced to the problem of uniform convergence of random numerical functions, for instance, the problem of uniform convergence of expression of the series by Franklin orthogonal system, see [9], [15], [36].

We consider here the problem of convergence of random elements with values in Banach space.

In detail: let $\zeta = \{\zeta(n)\}$ be a random sequence with values in the separable Banach space $B$; find the conditions (necessary conditions and sufficient conditions) under which $P(\exists \lim_{n \to \infty} \zeta(n)) = 1$ or equally $P(\{\zeta\} \in c(B)) = 1$.

Notations:

\[ \tilde{\zeta}(n) := \arctan \zeta(n), \quad \tau_n^m = \tau^m_n(\xi) \overset{def}{=} \mathbb{E} \arctan \left( \max_{k=n} \| \zeta(k) - \zeta(n) \|_B \right), \quad (5.1) \]

\[ \tau(\zeta) := \lim_{n \to \infty} \sup_{m \geq n+1} \tau^m_n(\zeta). \quad (5.1a) \]

**Proposition 5.1.** We conclude analogously to the theorem 2.1: $P(\{\zeta\} \in c) = 1$ if and only if

\[ \lim_{n \to \infty} \sup_{m \geq n+1} \tau^m_n(\xi) = 0, \quad (5.2) \]

or briefly

\[ \tau(\zeta) = 0, \quad \text{or equally} \quad \zeta(\cdot) \in \ker \tau. \quad (5.3) \]

6 Concluding remarks.

I. The case of metric (linear) space $B$.

It is not hard to generalize obtained above results on the case when the Banach space $B$ is replaced by certain separable linear metric space $L$ equipped with translation invariant metric function $\rho = \rho(x - y)$.

II. Recall that for the convergence in probability (measure) of the sequence of Banach space valued r.v. $\eta_n$ the necessary and sufficient condition is following

\[ \lim_{n,m \to \infty} \mathbb{E} \arctan \| \eta_n - \eta_m \|_B = 0. \]

III. It is interest by our opinion to investigate in the spirit of this article the case of the non-sequential convergence; as well as to obtain the criterion for a.e. convergence for multiple sequences, especially multiple Fourier series.
References

[1] Alexits G. *Convergence Problems of Orthogonal Series*. International Series of Monographs on Pure and Applied Mathematics, Vol. 20. Pergamon Press, New YorkOxfordParis, 1961.

[2] Antonov N.Y. *Convergence of Fourier series*. East J. Approx., 2: 187-196, 1996.

[3] Billingsley P. *Probability and measure*. Wiley, 1979, London, New York.

[4] Billingsley P. *Convergence of probability measures*. Wiley, (1968), London, New York.

[5] Buldygin V.V. *Convergence of random elements in topological spaces*. Naukova Dumka, Kiev, (1980), (in Russian).

[6] Burkholder D.L. and R. F. Gundy. *Extrapolation and interpolation of quasi-linear operators on martingales*, Acta Math. 124, (1970), 249-304. MR 55:13567

[7] Burkholder D.L. *The best constant in the Davis inequality for the expectation of the martingale square function*. Transactions of the American Mathematical Society, (2001), V. 354 No 1 105-131.

[8] Carleson L. *On convergence and growth of partial sums of Fourier series*. Acta Math., 116: 135-157, 1966.

[9] Ciesielski Z. *Properties of the orthogonal Franklin system*. Studia Math., 23 : 2 (1963) pp. 141-157.

[10] Dudley R.M. *Uniform Central Limit Theorem*. Cambridge University Press, 1999.

[11] Fernique X. (1975). *Regularite des trajectoires des function aleatoires gaussiennes*. Ecole de Probablete de Saint-Flour, IV-1974, Lecture Notes in Mathematic. 480, 1-96, Springer Verlag, Berlin.

[12] Fernique X. *Caracterisation de processus de trajectoires majore ou continues*. Seminaire de Probabilitits XII. Lecture Notes in Math. 649, (1978), 691-706, Springer, Berlin.

[13] Fernique X. *Regularite de fonctions aleatoires non gaussiennes*. Ecolee de Ete de Probabiltis de Saint-Flour XI-1981. Lecture Notes in Math. 976, (1983), 1-74, Springer, Berlin.

[14] Fiorenza A., and Karadzhov G.E. *Grand and small Lebesgue spaces and their analogs*. Consiglio Nationale Delle Ricerche, Instituto per le Applicazioni del Calcolo Mauro Picone, Sezione di Napoli, Rapporto tecnico n. 272/03, (2005).
[15] Franklin P. A set of continuous orthogonal functions. Math. Ann., 100 (1928), pp. 522-529.

[16] Garsia, A. M.; Rodemich, E.; and Rumsey, H., Jr. A real variable lemma and the continuity of paths of some Gaussian processes. Indiana Univ. Math. J. 20 (1970/1971), 565-578.

[17] Gikhman I.I., Skorokhod A.V. Introduction to the theory of random processes. Nauka, GIFML, Moscow, (1965), (in Russian).

[18] Grafakos L. Classical and Modern Fourier Analysis. Pearson Education, New Jersey, 2004.

[19] Hall P., Heyde C.C. Martingale Limit Theory and Applications. Academic Press, New York. (1980)

[20] Hunt R.A. On the convergence of Fourier series. In: Orthogonal Expansions and their Continuous Analogues, Proc. Conf. Edwardsville, Ill., 1967, pages 235-255. Illinois University.

[21] Ibragimov I.A. Properties of sample functions of stochastic processes and embedding theorems. Theory Probab. Appl., 18:3 (1973), 468-480.

[22] Iwaniec T., P. Koskela P., and Onninen J. Mapping of finite distortion: Monotonicity and Continuity. Invent. Math. 144 (2001), 507-531.

[23] Ito K., Nisio M. On the convergence of sums of independent Banach space valued random variables. Osaka J. Math., (1968), 5, No 1, 35-48.

[24] Yaozhong Hu and Khoa Le A multiparameter Garsia-Rodemich-Rumsey inequality and some applications. arXiv:1211.6809v1 [math.PR] 29 Nov 2012

[25] Kaczmarz S., Steinhaus H. Theorie der Orthogonalreihen, Chelsea, reprint (1951).

[26] Kolmogorov A.N. On the analitical methods in the probability theory. Soviet Math. Survays, (1938), 5, 5-41. (in Russian).

[27] Kozachenko Yu. V., Ostrovsky E.I. (1985). The Banach Spaces of random Variables of subgaussian type. Theory of Probab. and Math. Stat. (in Russian). Kiev, KSU, 32, 43-57.

[28] Kwapien S. and Rosinsky J. Sample Hölder continuity of stochastic processes and majorizing measures. (2004). Seminar on Stochastic Analysis, Random Fields and Applications IV, Progr. in Probab. 58, 155-163. Birkholser, Basel.

[29] Ledoux M., Talagrand M. (1991) Probability in Banach Spaces. Springer, Berlin, MR 1102015.

[30] Liflyand E., Ostrovsky E., Sirota L. Structural Properties of Bilateral Grand Lebesgue Spaces. Turk. J. Math.; 34 (2010), 207-219.
[31] Lukomskii S.F. A criterion for the almost-everywhere convergence of Fourier-Walsh square partial sums of integrable functions. Mat. Sb., 1995, Volume 186, Number 7, Pages 133-146.

[32] Nisio N. On the continuity of stationary Gaussian processes. Nagoya Math. J., 34, (1969), 89-104.

[33] Ostrovsky E.I. (1999). Exponential estimations for random Fields and its applications (in Russian). Moscow-Obninsk, OINPE.

[34] Ostrovsky E. On the local structure of normal fields. Soviet Math. Doklady, (1970), v. 105 No 1 p. 1425-1428.

[35] Ostrovsky E. Convergence of canonical expression for normal fields. Math Notes, (1973), V. 14 Issue 4 p. 565-572.

[36] Ostrovsky E., Sirota L. Theory of approximation and continuity of random processes. arXiv:1303.3029v1 [math.PR] 12 Mar 2013

[37] Ostrovsky E., Sirota L. Moment Banach spaces: Theory and applications. HAIT Journal of Science and Engineering C, Volume 4, Issues 1-2, pp. 233-262.

[38] Pawlak M. On the almost everywhere properties of the kernel regression estimate. Ann. Inst. Statist. Math., Vol. 43, No. 2, 311-326, (1991).

[39] Peshkir G., Shirjaev A.N. The Khintchine inequalities and martingale expanding sphere of their action. Russian Math. Surveys; 50, 5, 849-904, (1995).

[40] J.A. de Reyna. Pointwiese Convergence Fourier Series. Lect. Notes in Math., New York, 2004.

[41] Slutsky E.E. Some proposals on the theory of random functions. Proceedings of Middle-Asia University. (1949), V 5, 31, 3-15. (in Russian).

[42] Sudakov V.N. A remark on the criterion of continuity of Gaussian sample functions. Lecture Notes in Math., 330, (1973), 444-454.

[43] Talagrand M. (1996). Majorizing measure: The generic chaining. Ann. Probab., 24 1049-1103. MR1825156

[44] Talagrand M. (2005). The Generic Chaining. Upper and Lower Bounds of Stochastic Processes. Springer, Berlin. MR2133757.

[45] Vinkler V. Continuity condition for sample functions of random fields. Probab. Theory Appl., 4, (1959), 439-444.

[46] Wakhaniya N.N., Tarieladze W.I., Chobanjan S.A. Probabilistic Distributions in Banach Spaces. NAUKA, Moscow, (1985), (in Russian).

[47] E. Wagner, W. Wilczyński. Convergence of sequences of measurable functions. Acta Math. Acad. Sci. Hungaricae, 36, (1,2), (1980), 125-128.
[48] Watanabe H. *On the continuity property of Gaussian random fields*. Studia Mathematica, V. XLIX, (1973), 81-90.

[49] Weisz Ferenc. *Summability of Multi-Dimensional Trigonometric Fourier Series*. arXiv:1206.1789v1 [math.CA] 8 Jun 2012