New solvable sigma models in plane–parallel wave background

Ladislav Hlavatý∗, Ivo Petr†

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Czech Technical University in Prague, Faculty of Nuclear Sciences and Physical Engineering, Břehová 7, 115 19 Prague 1, Czech Republic

Abstract

We explicitly solve the classical equations of motion for strings in backgrounds obtained as non-abelian T-duals of a homogeneous isotropic plane–parallel wave. To construct the dual backgrounds, semi-abelian Drinfeld doubles are used which contain the isometry group of the homogeneous plane wave metric. The dual solutions are then found by the Poisson–Lie transformation of the explicit solution of the original homogeneous plane wave background. Investigating their Killing vectors, we have found that the dual backgrounds can be transformed to the form of more general plane–parallel waves.

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∗hlavaty@fjfi.cvut.cz
†ivo.petr@fjfi.cvut.cz
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1 Introduction

Sigma models satisfying supplementary conditions can serve as models of string theory in curved and time-dependent backgrounds. Finding the solution of sigma models in such backgrounds is often very complicated, not to say impossible. That is why every solvable case attracts considerable attention. An example of such a solvable model is a string theory in homogeneous plane–parallel wave background investigated in Ref. [1]. The general relativistic statement that any spacetime has a plane wave as a limit [2] was reformulated for string theories in Ref. [3]. The authors of Ref. [1] mention the relevance of the homogeneous plane wave backgrounds as Penrose limits of cosmological and p-brane backgrounds, and study the possibility of string propagation through the singularity of the metric. The classical string is solved there in terms of Bessel functions and subsequently quantized.

Ref. [3] also discusses the relation between Penrose limits and T-duality. Since the homogeneous plane–parallel wave metric investigated in Ref. [1] has rather large group of isometries, dual backgrounds can be constructed via non-abelian T-duality introduced in Ref. [4] and investigated further in the context of WZW models or gravitational instantons in Refs. [5],[6]. Studied originally for the bosonic strings, the non-abelian T-duality was recently extended to superstring backgrounds including the Ramond fluxes [7],[8], with the supersymmetric plane–parallel wave background studied in Ref. [9]. However, other applications of non-abelian duality in cosmology can be found, see e.g. Ref. [10].

The non-abelian T-duality provides us with a prescription how to relate solutions of sigma models with apparently different target space geometries. Knowledge of the solution of the original model gives us in principle the
possibility to solve the equations of motion of strings in dual backgrounds. In the case of the homogeneous plane wave we were able to find the explicit solutions. In the following, we shall understand the non-abelian T-duality as a special case of Poisson–Lie T-duality \([11]\), which solved the problem that the non-abelian dual may have less symmetries than the original model, making it impossible to find reverse non-abelian T-duality transformation by the procedure of Ref. \([4]\).

The backgrounds resulting from the Poisson–Lie T-duality are expressed in the so called group coordinates that may hide their commonly used forms. To find these, one can analyze the the Killing vectors, which provide important information about the symmetries of the dual backgrounds, and meanwhile suggest a way to find the coordinate transformations for transition to the commonly used forms of the dual metrics.

The plan of the paper is the following. In the next section we review the method of the Poisson–Lie T-duality that is used later as a tool for construction of the dual models and their solution. Relevant results concerning homogeneous isotropic plane wave and the solution of classical string equations of motion obtained in \([1]\) are summarized in the third section. Section 4 describes the two particular Drinfeld doubles which are dealt with later in sections 5 and 6, where the corresponding dual backgrounds are solved. In their subsections the symmetries of the backgrounds are studied, and the plane–parallel wave form of the background is revealed through an appropriate coordinate transformation.

2 Elements of Poisson–Lie T-dual sigma models

The basic concept used for construction of mutually dual sigma models is a Drinfeld double – a Lie group with an additional structure. More precisely, there are two equally dimensional subgroups \(G, \tilde{G}\) in the Drinfeld double \(D\), such that Lie subalgebras \(\mathfrak{g}, \mathfrak{g}^\prime\) are isotropic subspaces of the Lie algebra \(\mathfrak{d}\) of the Drinfeld double. The Drinfeld double suitable for a given sigma model living in curved background can sometimes be found from the knowledge of the symmetry group of the metric. In the case that the metric has sufficient number of independent Killing vectors, the subgroups of the isometry group can be taken as one of the subgroups of the Drinfeld double. The other
one then has to be chosen Abelian, in order to satisfy the conditions of dualizability. We shall focus on the case that the isometry subgroup acts freely and transitively on the manifold, which is the case usually referred to as the atomic duality \[12\]. Let us summarize the details.

Let $G$ be a Lie group and $\mathfrak{g}$ its Lie algebra. Sigma model on the group $G$ is given by the classical action

$$ S_F[\phi] = -\int d\sigma_+d\sigma_- (\partial_-\phi^\mu F_{\mu\nu}(\phi)\partial_+\phi^\nu) = $$

$$ = \frac{1}{2} \int d\tau d\sigma \left[ -\partial_\tau \phi^\mu G_{\mu\nu}(\phi)\partial_\tau \phi^\nu + \partial_\sigma \phi^\mu G_{\mu\nu}(\phi)\partial_\sigma \phi^\nu - 2\partial_\tau \phi^\mu B_{\mu\nu}(\phi)\partial_\sigma \phi^\nu \right], $$

where $F$ is a second order tensor field on the Lie group $G$, with the metric and the torsion potential

$$ G_{\mu\nu} = \frac{1}{2}(F_{\mu\nu} + F_{\nu\mu}), \quad B_{\mu\nu} = \frac{1}{2}(F_{\mu\nu} - F_{\nu\mu}), $$

and the worldsheet coordinates

$$ \sigma_+ = \frac{1}{\sqrt{2}}(\tau + \sigma), \quad \sigma_- = \frac{1}{\sqrt{2}}(\tau - \sigma). $$

The functions $\phi^\mu$ are determined by the composition $\phi^\mu(\tau, \sigma) = x^\mu(g(\tau, \sigma))$, where $g : \mathbb{R}^2 \ni (\tau, \sigma) \mapsto g(\tau, \sigma) \in G$ and $x^\mu : U_g \rightarrow \mathbb{R}$ are components of a coordinate map of neighborhood $U_g$ of element $g(\tau, \sigma) \in G$.

The condition of dualizability of sigma models on the level of the Lagrangian is given by the condition \[11\] for the Lie derivative of $F_{\mu\nu}$

$$ \mathcal{L}_{v_i} F_{\mu\nu} = F_{\mu\nu} c_i^j k_{i}^k \zeta_{i}^{kj} F_{\lambda\nu}, $$

where $c_i^{jk}$ are structure coefficients of a dual algebra $\hat{\mathfrak{g}}$ and $v_i$ are left-invariant fields on the Lie group $G$. The Lie algebras $\mathfrak{g}$ and $\hat{\mathfrak{g}}$ then define the Drinfeld double as a connected Lie group whose Lie algebra $\hat{\mathfrak{d}}$ can be decomposed into the pair of subalgebras $\mathfrak{g}, \hat{\mathfrak{g}}$ that are maximally isotropic with respect to a symmetric ad-invariant nondegenerate bilinear form $<.,.>$ on $\mathfrak{d}$.

Drinfeld doubles enable us to construct the tensor $F$ satisfying \[2\]. The general solution of this equation is

$$ F_{\mu\nu}(x) = c^a_\mu(g(x)) E_{ab}(g(x)) c^b_\nu(g(x)), $$

\[3\]
where \( e^a_\mu (g(x)) \) are components of right-invariant forms \( e^a_\mu = ((dg)g^{-1})^a_\mu \) expressed in coordinates \( x^\mu \) on the group \( G \),

\[
E(g) = [E_0^{-1} + \Pi(g)]^{-1},
\]

\( E_0 \) is a constant non-singular matrix, \( \Pi(g) \) is given by the formula

\[
\Pi(g) = b(g).a(g)^{-1} = -\Pi'(g),
\]

and matrices \( a(g), b(g), d(g) \) are given by the adjoint representation of the Lie subgroup \( G \) on the Lie algebra of the Drinfeld double in the mutually dual bases

\[
Ad(g)^T = \left( \begin{array}{cc} a(g) & 0 \\ b(g) & d(g) \end{array} \right).
\]

If \( \tilde{G} \) is Abelian, then \( \Pi(g) = 0 \).

Under the condition (2), the field equations for the sigma model can be rewritten as the equation

\[
< (\partial_\pm l)^{-1}, \varepsilon^\pm > = 0
\]

for the mapping \( l \) from the worldsheet in \( \mathbb{R}^2 \) into the Drinfeld double \( D \), where the subspaces \( \varepsilon^+ = \text{Span}(T_i + E_{0,ij}\tilde{T}^j), \varepsilon^- = \text{Span}(T_i - E_{0,ij}\tilde{T}^j) \) are orthogonal w.r.t. \( <, > \) and span the whole Lie algebra \( \mathfrak{d} \), and \( \{T_i\}, \{\tilde{T}^j\} \) are the the mutually dual bases of \( \mathfrak{g} \) and \( \tilde{\mathfrak{g}} \).

Due to Drinfeld, there exists a unique decomposition (at least in the vicinity of the unit element of \( D \)) of an arbitrary element \( l \) of \( D \) as a product of elements from \( G \) and \( \tilde{G} \). Solutions of equation (7) and solution of the equations of motion for the sigma model \( \phi^\mu(\tau, \sigma) = x^\mu(g(\tau, \sigma)) \) are related by

\[
l(\tau, \sigma) = g(\tau, \sigma)\tilde{h}(\tau, \sigma) \in D,
\]

where \( \tilde{h} \in \tilde{G} \) fulfills the equations

\[
((\partial_\tau \tilde{h}).\tilde{h}^{-1})_j = -v^\lambda_j [G_{\lambda\nu}\partial_\sigma \phi^\nu + B_{\lambda\nu}\partial_\tau \phi^\nu],
\]

\[
((\partial_\sigma \tilde{h}).\tilde{h}^{-1})_j = -v^\lambda_j [G_{\lambda\nu}\partial_\tau \phi^\nu + B_{\lambda\nu}\partial_\sigma \phi^\nu].
\]

and \( v^\lambda_j \) are components of the left-invariant fields on \( G \).

The dual model can be obtained by the exchange

\[
G \leftrightarrow \tilde{G}, \quad \mathfrak{g} \leftrightarrow \tilde{\mathfrak{g}}, \quad \Pi(g) \leftrightarrow \tilde{\Pi}(\tilde{g}), \quad E_0 \leftrightarrow E_0^{-1}.
\]
The relation between the solution \( \phi^\mu(\tau, \sigma) \) of the equations of motion of the sigma model given by \( F \) and the solution \( \tilde{\phi}^\mu(\tau, \sigma) \) of the model given by \( \tilde{F} \) follows from two possible decompositions of elements \( l \) of the Drinfeld double

\[
g(\tau, \sigma)\tilde{h}(\tau, \sigma) = \tilde{g}(\tau, \sigma)h(\tau, \sigma),
\]

where \( g, h \in G, \tilde{g}, \tilde{h} \in \tilde{G} \). The map \( \tilde{h} : \mathbb{R}^2 \to \tilde{G} \) that we need for this transformation is the solution to the equations (9,10).

The equation (12) then defines the Poisson–Lie transformation between the solution of the equations of motion of the original sigma model and its dual. Its application may be very complicated. To use it for solving the dual model the following three steps must be done:

- One has to know the solution \( \phi^\mu(\tau, \sigma) \) of the sigma model given by \( F \).
- Given \( \phi^\mu(\tau, \sigma) \), one has to find \( \tilde{h}(\tau, \sigma) \) i.e. solve the system of PDEs (9,10).
- Given \( l(\tau, \sigma) = g(\tau, \sigma)\tilde{h}(\tau, \sigma) \in D \), one has to find the dual decomposition \( l(\tau, \sigma) = \tilde{g}(\tau, \sigma)h(\tau, \sigma) \), where \( \tilde{g}(\tau, \sigma) \in \tilde{G}, h(\tau, \sigma) \in G \).

In the following sections we shall apply these three steps of the Poisson–Lie transformation to solve the equations of motion for strings in backgrounds dual to the homogeneous plane–parallel wave metric. We will start with the description of a particular plane wave background and recall the solution of its classical equations of motion.

### 3 Homogeneous plane–parallel wave metrics

Homogeneous plane–parallel wave in \( d + 2 \) dimensions is generally defined by the metric of the following form [1],[13],[14]

\[
ds^2 = 2du dv - K_{ij}(u)x^i x^j du^2 + d\tilde{x}^2,\]

where \( d\tilde{x}^2 \) is the standard metric on Euclidean space \( E^d \) and \( \tilde{x} \in E^d \). The form of this metric seems to be simple, but explicit solution of sigma models can be complicated. Therefore the authors of [1] restricted themselves to the special case of isotropic homogeneous plane wave metric

\[
K_{ij}(u) = \lambda(u)\delta_{ij}, \quad \lambda(u) = \frac{k}{u^2}, \quad k = \nu(1 - \nu) = \text{const.} > 0.
\]

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In the following we shall investigate the case $d = 2$, i.e. the dimension of the spacetime is four. This seems to be the simplest physically interesting background\[1\]. The metric tensor in the (Brinkman) coordinates $(u, v, x, y)$ then has components

$$G_{\mu\nu}(u, v, x, y) = \begin{pmatrix} \frac{-k(x^2+y^2)}{u^2} & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (15)$$

This metric is not flat, but its Gaussian curvature vanishes. It has a singularity in $u = 0$, and does not satisfy the Einstein equations but the conformal invariance condition equations for vanishing of the $\beta$ function

$$0 = R_{\mu\nu} - \nabla_\mu \nabla_\nu \Phi - \frac{1}{4} H_{\mu\kappa\lambda} H^{\nu\kappa\lambda}, \quad (16)$$

$$0 = \nabla^\mu \Phi H_{\mu\kappa\lambda} + \nabla^\mu H_{\mu\kappa\lambda}, \quad (17)$$

$$0 = R - 2 \nabla_\mu \nabla^\mu \Phi - \nabla_\mu \Phi \nabla^\mu \Phi - \frac{1}{12} H_{\mu\kappa\lambda} H^{\mu\kappa\lambda}, \quad (18)$$

where the covariant derivatives $\nabla_k$, the Ricci tensor $R_{\mu\nu}$ and the scalar curvature $R$ are calculated from the metric $G_{\mu\nu}$ that is also used for lowering and raising indices. The torsion $H$ as well as its potential $B$ in this case vanishes, and the dilaton field is

$$\Phi = \Phi_0 - c u + 2\nu(\nu - 1) \ln u. \quad (19)$$

The metric admits symmetries generated by the following Killing vectors

$$K_1 = \partial_v, \quad K_2 = u^\nu \partial_x - \nu u^{\nu-1} x \partial_v, \quad K_3 = u^\nu \partial_y - \nu u^{\nu-1} y \partial_v, \quad (20)$$

$$K_4 = u^{1-\nu} \partial_x - (1 - \nu) u^{-\nu} x \partial_v, \quad K_5 = u^{1-\nu} \partial_y - (1 - \nu) u^{-\nu} y \partial_v, \quad K_6 = u \partial_u - v \partial_v, \quad K_7 = x \partial_y - y \partial_x.$$

\[1\]The study of higher-dimensional cases would mean that we would have to look for $(d + 2)$-dimensional subalgebras of the algebra of Killing vectors of the dimension $2 + \frac{1}{2} d(d + 3)$. 
One can easily check that the Lie algebra spanned by these vectors is the semidirect sum $\mathcal{S} \ltimes \mathcal{N}$ of $\mathcal{S} = \text{Span}[K_6, K_7]$ and an ideal $\mathcal{N} = \text{Span}[K_1, K_2, K_3, K_4, K_5]$. The algebra $\mathcal{S}$ is Abelian and its generators can be interpreted as dilation in $u, v$ and rotation in $x, y$. Generators of the algebra $\mathcal{N}$ commute as the two-dimensional Heisenberg algebra with the center $K_1$.

The equations of motion for $\phi^\mu(\tau, \sigma) = (U(\tau, \sigma), V(\tau, \sigma), X(\tau, \sigma), Y(\tau, \sigma))$ are given by the sigma model action (1). They read

\[
(\partial_\sigma^2 - \partial_\tau^2)U = 0, \\
(\partial_\sigma^2 - \partial_\tau^2)X + \frac{k}{U^2} \partial_a U \partial^a UX = 0, \\
(\partial_\sigma^2 - \partial_\tau^2)Y + \frac{k}{U^2} \partial_a U \partial^a UY = 0,
\]

\[
(\partial_\sigma^2 - \partial_\tau^2)V + \frac{k}{U^3} \partial_a U \partial^a U(X^2 + Y^2) - \frac{2k}{U^2} \partial_a U(\partial^a XX + \partial^a YY) = 0.
\]

The plane wave metric (15) allows to adopt the light-cone gauge

\[
U(\tau, \sigma) = \kappa \tau, \quad \kappa := 2 \alpha' p^u,
\]

in which the equations of motion simplify and acquire the form

\[
(\partial_\sigma^2 - \partial_\tau^2)X - \frac{k}{\tau^2} X = 0, \quad (22)
\]

\[
(\partial_\sigma^2 - \partial_\tau^2)Y - \frac{k}{\tau^2} Y = 0, \quad (23)
\]

\[
(\partial_\sigma^2 - \partial_\tau^2)V - \frac{k}{\kappa \tau^3} (X^2 + Y^2) + \frac{2k}{\kappa \tau^2} (\partial_\tau XX + \partial_\tau YY) = 0. \quad (24)
\]

Solution of the equations (22), (23) that for $\tau \to \infty$ tends to the free string solution was given in [1] as

\[
X^i(\sigma, \tau) = x_0^i(\tau) + \frac{i}{2} \sqrt{2\alpha'} \sum_{n=1}^{\infty} \frac{1}{n} \left[ X_n^i(\tau, \sigma) - X_n^i(\tau, \sigma) \right], \quad (25)
\]

where $i = 2, 3$, $X = X^2, Y = X^3, \nu = \frac{1}{2} (1 + \sqrt{1 - 4k})$, the zero modes are

\[
x_0^i(\tau) = \frac{1}{\sqrt{2\nu - 1}} (\bar{x}^i \tau^{1-\nu} + 2\alpha' \bar{p}^i \tau^{\nu}), \quad \text{for} \ k \neq \frac{1}{4}, \quad (26)
\]
The higher modes are expanded as

\[ X_n^i(\tau, \sigma) = Z(2n\tau)(\alpha_n^i e^{2i\sigma} + \tilde{\alpha}_n^i e^{-2i\sigma}), \tag{28} \]

and \( H^{(2)}_{\nu-\frac{1}{2}} \) is the Hankel function of the second kind

\[ H^{(2)}_{\nu-\frac{1}{2}}(t) = [J_{\nu-\frac{1}{2}}(t) - i Y_{\nu-\frac{1}{2}}(t)]. \tag{30} \]

Being interested in the string solutions of the sigma model, we have to add supplementary string conditions

\[ \partial_a \Phi^\mu G_{\mu\nu}(\Phi) \partial_b \Phi^\nu = e^{\omega} \eta_{ab} \tag{31} \]

that for \( \eta = \text{diag}(-1, 1) \) and in the light-cone gauge read

\[ \kappa \partial_\sigma V + \partial_\tau X \partial_\sigma X + \partial_\tau Y \partial_\sigma Y = 0, \tag{32} \]

\[ 2\kappa \partial_\tau V - \frac{k}{\tau^2} (X^2 + Y^2) + \partial_\tau X \partial_\sigma X + \partial_\sigma X \partial_\tau X + \partial_\tau Y \partial_\sigma Y + \partial_\sigma Y \partial_\tau Y = 0. \tag{33} \]

Compatibility of these two equations is guaranteed by the equations of motion for \( X \) and \( Y \) \[(22)\] and \[(23)\], so that

\[ V(\tau, \sigma) = v(\tau) - \frac{1}{\kappa} \int d\sigma (\partial_\tau X \partial_\sigma X + \partial_\tau Y \partial_\sigma Y), \tag{34} \]

where \( v(\tau) \) is an arbitrary function. The equation \[(24)\] is solved by \[(34)\] provided that the functions \( X, Y \) satisfy \[(22)\] and \[(23)\].

4 Data for construction of dual backgrounds

As explained in section \[2\] dualizable metrics are constructed by virtue of the Drinfeld double. To get the metric \[(15)\], the Lie algebra of the Drinfeld double can be composed from the four-dimensional Lie subalgebra of the isometry algebra of Killing vectors \[(20)\] and four-dimensional Abelian
algebra. Moreover, the four-dimensional subgroup of isometries must act freely and transitively on the Riemannian manifold $M$ where the metric is defined, so that $M \approx G$. These four-dimensional subgroups were found in [15] as $g_1 = \text{Span}\{K_1, K_2, K_5, K_6\}$ or $g_2 = \text{Span}\{K_1, K_2, K_3, K_6 + \rho K_7\}$ with non-vanishing commutation relations

$$
\begin{align*}
[K_6, K_1] &= K_1, \\
[K_6, K_2] &= \nu K_2, \\
[K_6, K_5] &= (1 - \nu) K_5,
\end{align*}
$$

(35)

and

$$
\begin{align*}
[K_6 + \rho K_7, K_1] &= K_1, \\
[K_6 + \rho K_7, K_2] &= \nu K_2 - \rho K_3, \\
[K_6 + \rho K_7, K_3] &= \nu K_3 + \rho K_2
\end{align*}
$$

(36)

respectively, where the parameter $\nu$ was given in [14] and $\rho$ is an arbitrary real.

In the following we shall find metrics dual to (15) constructed from the Drinfeld doubles $\mathcal{D} = g \oplus a$ where $g$ is $g_1$ or $g_2$ and $a$ is four-dimensional Abelian algebra.

To get the matrix $E_0$, the metric (15) must be transformed to coordinates $x^1, x^2, x^3, x^4$ used for parametrization of the group elements identified with points of the manifold. Choosing the parametrization of the elements of the group $G$ as

$$g(x) = e^{x^1 T_1} e^{x^2 T_2} e^{x^3 T_3} e^{x^4 T_4},$$

(37)

where $T_1, T_2, T_3, T_4$ are generators of the group $G$, the matrix $E_0$ was found in [15] to be

$$E_0 = \begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0
\end{pmatrix}.
$$

(38)

The dual metrics on the group $\tilde{G}$ with elements

$$\tilde{g}(\tilde{x}) = e^{\tilde{x}_1 \tilde{T}_1} e^{\tilde{x}_2 \tilde{T}_2} e^{\tilde{x}_3 \tilde{T}_3} e^{\tilde{x}_4 \tilde{T}_4},$$

(39)

\[2\text{The equation (2) is then trivially fulfilled.}\]
as well as classical solutions of their dual sigma model

\[ \tilde{g}(\tau, \sigma) = e^{\tilde{x}_1(\tau, \sigma) \tilde{T}_1} e^{\tilde{x}_2(\tau, \sigma) \tilde{T}_2} e^{\tilde{x}_3(\tau, \sigma) \tilde{T}_3} e^{\tilde{x}_4(\tau, \sigma) \tilde{T}_4}, \]  

(40)
can be then obtained by the method described in the Sec. 2. We will use it in the following two sections.

5 Classical strings in the dual background obtained from \( \mathfrak{g}_1 \)

Let us first consider the group generated by the Lie algebra \( \mathfrak{g}_1 = \text{Span}[T_1, T_2, T_3, T_4] \) with commutation relations (cf. (35))

\[
\begin{align*}
[T_4, T_1] &= T_1, \\
[T_4, T_2] &= \nu T_2, \\
[T_4, T_3] &= (1 - \nu)T_3.
\end{align*}
\]

(41)
The transformation between group coordinates \( x^1, x^2, x^3, x^4 \) defined by (37) and geometrical coordinates \( u, v, x, y \) on \( M \) can be obtained by comparing the left-invariant vector fields on the group \( G \) and Killing vectors of the metric (15). One gets

\[
\begin{align*}
u &= e^{x^4}, \\
v &= \frac{1}{2} [2x^1 - \nu(x^2)^2 - (1 - \nu)(x^3)^2] e^{-x^4}, \\
x &= x^2, \\
y &= x^3.
\end{align*}
\]

(42)
The metric (15) expressed in the group coordinates then acquires the form

\[
G_{\mu\nu}(x^k) = \begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & 1 & 0 & -\nu x^2 \\
0 & 0 & 1 & (\nu - 1)x^3 \\
1 & -\nu x^2 & (\nu - 1)x^3 & \nu^2(x^2)^2 + (\nu - 1)^2(x^3)^2 - 2x^1
\end{pmatrix}. \]

(43)
The group \( \tilde{G} \) is Abelian and the right-hand sides of the equations (9,10) are invariant w.r.t coordinate transformations. That is why in the Brinkman
coordinates $u, v, x, y$ we can use just $K_1, K_2, K_5, K_6$ as the left-invariant fields on $G$. The equations (9,10) for $\tilde{h}$ then read

\[
\partial_\tau \tilde{h} = - \begin{pmatrix}
0 \\
(\kappa \tau)^\nu \partial_\sigma X \\
(\kappa \tau)^{1-\nu} \partial_\sigma Y \\
\kappa \tau \partial_\sigma V
\end{pmatrix},
\]

\[
\partial_\sigma \tilde{h} = - \begin{pmatrix}
\kappa \\
(\kappa \tau)^\nu (\partial_\tau X - \frac{\nu}{\tau} X) \\
\kappa (\kappa \tau)^{-\nu} (\tau \partial_\sigma Y + (\nu - 1) Y) \\
\nu (\nu - 1)(X^2 + Y^2) + \kappa \tau \partial_\tau V - \kappa V
\end{pmatrix},
\]

and are solved by

\[
\tilde{h}_1 = c_1 - \kappa \sigma, 
\]

\[
\tilde{h}_2 = c_2 - (\kappa \tau)^\nu \int d\sigma (\partial_\tau X - \frac{\nu}{\tau} X),
\]

\[
\tilde{h}_3 = c_3 - (\kappa \tau)^{1-\nu} \int d\sigma (\partial_\sigma Y - \frac{1 - \nu}{\tau} Y),
\]

\[
\tilde{h}_4 = c_4 + \int d\sigma \left[ \nu \ln \left( \frac{1 - \nu}{\tau} \right) (X^2 + Y^2) + \kappa \left( V - \tau \partial_\sigma V \right) \right]
\]

\[
= \int d\sigma \left[ \nu \ln \left( \frac{1 - \nu}{\tau} \right) (X^2 + Y^2) + \frac{\tau}{2} \left[ (\partial_\tau X)^2 + (\partial_\sigma X)^2 + (\partial_\sigma Y)^2 + (\partial_\sigma Y)^2 \right] \right]
\]

\[
- \int d\sigma \int d\sigma' (\partial_\tau X \partial_\sigma X + \partial_\tau Y \partial_\sigma Y) + c_4 + \kappa \sigma v(t),
\]

where $c_i, i = 1, \ldots, 4$ are arbitrary constants.

The dual tensor on the group $\tilde{G}$ is constructed by the procedure explained in the section 2, namely by using (3), (4), (11)

\[
\tilde{F}_{\mu\nu}(\tilde{x}) = \begin{pmatrix}
\nu^2 \tilde{x}_2^2 + (1 - \nu)^2 \tilde{x}_2^2 \\
\nu \tilde{x}_2 \\
1 - \tilde{x}_2 \\
1 - \tilde{x}_2
\end{pmatrix} \begin{pmatrix}
\nu \tilde{x}_3 \\
1 - \tilde{x}_3 \\
(\nu - 1) \tilde{x}_3 \\
\tilde{x}_3 + 1
\end{pmatrix} \begin{pmatrix}
1 \\
0 \\
1 \\
0
\end{pmatrix}.
\]

Even though it is not symmetric, its torsion is zero. It satisfies the conformal invariance conditions (16)–(18) with the dilaton field (15)

\[
\tilde{\Phi} = \tilde{\Phi}_0 + C \ln \left( \frac{\tilde{x}_1 - 1}{\tilde{x}_1 + 1} \right) + (\nu - 1 - \nu^2) \ln \left( 1 - \tilde{x}_1^2 \right)
\]

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that we rederive later.

To obtain the solution of the sigma model on $\tilde{G}$ given by

$$S_\tilde{F}[\tilde{x}] = - \int d\sigma_+ d\sigma_- (\partial_- \tilde{x}^\mu \tilde{F}_{\mu\nu}(\tilde{x}) \partial_+ \tilde{x}^\nu),$$  \hfill (53)

we have to solve the equation (12) for $\tilde{x}_j$, where

$$g = e^{x^1 T_1} e^{x^2 T_2} e^{x^3 T_3} e^{x^4 T_4}, \quad \tilde{h} = e^{\tilde{h}_1 T_1} e^{\tilde{h}_2 T_2} e^{\tilde{h}_3 T_3} e^{\tilde{h}_4 T_4},$$

$$\tilde{g} = e^{\tilde{x}_1 T_1} e^{\tilde{x}_2 T_2} e^{\tilde{x}_3 T_3} e^{\tilde{x}_4 T_4}, \quad h = e^{h_1 T_1} e^{h_2 T_2} e^{h_3 T_3} e^{h_4 T_4}.$$  \hfill (54)

To accomplish this, we can use a representation of the semi-abelian Drinfeld double in the form of block matrices $(\dim g + 1) \times (\dim g + 1)$, such that

$$r(g) = \left( \begin{array}{cc} Ad g & 0 \\ 0 & 1 \end{array} \right), \quad r(\tilde{h}) = \left( \begin{array}{cc} 1 & 0 \\ v(\tilde{h}) & 1 \end{array} \right),$$

where $v(\tilde{h}) = (\tilde{h}_1, \ldots, \tilde{h}_{\dim g})$. From the equation (12) we then get

$$r(l) = r(g\tilde{h}) = \left( \begin{array}{cc} Ad g & 0 \\ v(\tilde{h}) & 1 \end{array} \right) = r(\tilde{gh}) = \left( \begin{array}{cc} Ad h & 0 \\ v(\tilde{g}) \cdot (Ad h) & 1 \end{array} \right).$$  \hfill (57)

If the adjoint representation of the Lie algebra $g$ is faithful, which is the case of (41), then the representation $r$ of the Drinfeld double is faithful as well and the relation (57) gives a system of equations for $\tilde{x}_j$ and $h^j$.

Plugging (54) and (55) into (57) together with the the adjoint representation of the Lie algebra (41) we get the solution of (12) in the form

$$h^j = x^j,$$

$$\tilde{x}_1 = e^{-x^4} \tilde{h}_1,$$

$$\tilde{x}_2 = e^{-\nu x^4} \tilde{h}_2,$$

$$\tilde{x}_3 = e^{(\nu - 1)x^4} \tilde{h}_3,$$

$$\tilde{x}_4 = e^{-x^4} x^1 \tilde{h}_1 + \nu e^{-\nu x^4} x^2 \tilde{h}_2 + (1 - \nu) e^{(\nu - 1)x^4} x^3 \tilde{h}_3 + \tilde{h}_4.$$  \hfill (62)

The expressions for $\tilde{x}_1, \tilde{x}_2, \tilde{x}_3$ are very simple and it can be checked that combining them with (42) they give explicit solution of the equations of motion of the sigma model on $\tilde{G}$ in the background (51). Namely, inserting the light-cone gauge solution (21) into (42) and (59) we get from (46)

$$\tilde{X}_1(\tau, \sigma) = \frac{c_1 - \kappa \sigma}{\kappa \tau}.$$  \hfill (63)

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that solves the dual equation of motion
\[
\frac{\delta S_F}{\delta X_4} = \left( \partial^2_{\tau} \ddot{X}_1 - \partial^2_{\sigma} \ddot{X}_1 \right) \left( 1 - \dddot{X}_1^2 \right) + 2 \dddot{X}_1 \left( (\partial_{\sigma} \dddot{X}_1)^2 - (\partial_{\tau} \dddot{X}_1)^2 \right) = 0. \tag{64}
\]

Other two equations then reduce to
\[
\frac{\delta S_F}{\delta X_2} = (\partial^2_{\tau} - \partial^2_{\sigma}) \dddot{X}_2 + \frac{\nu (1 + \nu)}{\tau^2} \dddot{X}_2 = 0, \tag{65}
\]
\[
\frac{\delta S_F}{\delta X_3} = (\partial^2_{\tau} - \partial^2_{\sigma}) \dddot{X}_3 + \frac{\nu - 2}{\tau^2} (\nu - 1) \dddot{X}_3 = 0, \tag{66}
\]
solved in agreement with (60, 61) and (42, 47, 48) as
\[
\dddot{X}_2(\tau, \sigma) = c_2 (\kappa \tau)^{-\nu} + \int d\sigma \left( \frac{\nu}{\tau} \dddot{X} - \partial_{\tau} \dddot{X} \right),
\]
\[
\dddot{X}_3(\tau, \sigma) = c_3 (\kappa \tau)^{\nu - 1} + \int d\sigma \left( \frac{1 - \nu}{\tau} Y - \partial_{\tau} Y \right).
\]

For \( \nu \neq \frac{1}{2} \) we have
\[
\dddot{X}_2(\tau, \sigma) = (c_2 \kappa^{-\nu} + \sqrt{2\nu - 1} \dddot{X}) \tau^{-\nu} + \Sigma^2_2(\tau, \sigma), \tag{67}
\]
\[
\dddot{X}_3(\tau, \sigma) = (c_3 \kappa^{\nu - 1} - \sqrt{2\nu - 1} \dddot{X}) \tau^{\nu - 1} + \Sigma^3_2(\tau, \sigma), \tag{68}
\]
where
\[
\Sigma^2_2(\tau, \sigma) = \frac{1}{2} \sqrt{2\alpha} \sum_{n=1}^{\infty} \frac{1}{n} \left[ W(2n\tau) (\alpha_n^j e^{2i\sigma} - \bar{\alpha}_n^j e^{-2i\sigma}) + W^*(2n\tau) (\bar{\alpha}_n^j e^{-2i\sigma} - \alpha_n^j e^{2i\sigma}) \right], \tag{69}
\]
\[
\Sigma^3_2(\tau, \sigma) = \frac{1}{2} \sqrt{2\alpha} \sum_{n=1}^{\infty} \frac{1}{n} \left[ \bar{W}(2n\tau) (\alpha_n^j e^{2i\sigma} - \bar{\alpha}_n^j e^{-2i\sigma}) + \bar{W}^*(2n\tau) (\bar{\alpha}_n^j e^{-2i\sigma} - \alpha_n^j e^{2i\sigma}) \right], \tag{70}
\]
and
\[
W(2n\tau) = \frac{1}{2n} \left( \frac{\nu}{\tau} Z(2n\tau) - \partial_{\tau} Z(2n\tau) \right) = e^{-\frac{i\pi\nu}{4} \sqrt{n\pi}} H^{(2)}_{\nu + \frac{1}{2}}(2n\tau),
\]
\[
\bar{W}(2n\tau) = \frac{1}{2n} \left( \frac{1 - \nu}{\tau} Z(2n\tau) - \partial_{\tau} Z(2n\tau) \right) = -e^{-\frac{i\pi\nu}{4} \sqrt{n\pi}} H^{(2)}_{\nu - \frac{1}{2}}(2n\tau).
\]
For $\nu = \frac{1}{2}$ the expressions are a bit different and can be derived from (27).

Finally, from the expression (62) we get

$$\tilde{X}_4(\tau, \sigma) = c_4 - \frac{1}{2} \frac{\sigma + c_1}{\tau} \left(2\kappa \tau V + \nu X^2 + (1 - \nu)Y^2\right) + \nu X \tilde{X}_2 + (1 - \nu)Y \tilde{X}_3$$

$$+ \int d\sigma \left[\nu \frac{(1 - \nu)}{\tau} (X^2 + Y^2) + \kappa (V - \tau \partial_\tau V)\right]$$

that solves the last equation of the dual sigma model.

### 5.1 Killing vectors for the dual metric and its plane–parallel wave form

The purpose of this section is to show that the metric corresponding to the dual tensor (51) is again a plane–parallel wave.

In this case the relevant part of the dual tensor is only its symmetric part because the torsion $H = dB$ vanishes and does not influence neither the equations of motion nor the $\beta$ equations (16)–(18). The dual metric calculated from (51) is

$$\tilde{G}_{\mu\nu}(\tilde{x}) = \begin{pmatrix}
\frac{\nu^2 \tilde{x}_2^2 + (1 - \nu)^2 \tilde{x}_3^2}{\tilde{x}_1^2} & \frac{\nu \tilde{x}_1 \tilde{x}_2}{\tilde{x}_1^2} & \frac{(1 - \nu)^2 \tilde{x}_1 \tilde{x}_3}{\tilde{x}_1^2} & \frac{1}{\tilde{x}_1^2}
\end{pmatrix}.

(71)$$

This metric for $\nu \neq 1, 0$, i.e. $k \neq 0$ has just five-dimensional algebra generated by Killing vectors

$$\tilde{K}_1 = P_{\nu}(\tilde{x}_1) \frac{\partial}{\partial \tilde{x}_2} + \tilde{x}_2[(\nu + 1)P_{\nu + 1}(\tilde{x}_1) - \tilde{x}_1(1 + 2\nu)P_{\nu}(\tilde{x}_1)] \frac{\partial}{\partial \tilde{x}_4},

(72)$$

$$\tilde{K}_2 = Q_{\nu}(\tilde{x}_1) \frac{\partial}{\partial \tilde{x}_2} + \tilde{x}_2[(\nu + 1)Q_{\nu + 1}(\tilde{x}_1) - \tilde{x}_1(1 + 2\nu)Q_{\nu}(\tilde{x}_1)] \frac{\partial}{\partial \tilde{x}_4},

(73)$$

$$\tilde{K}_3 = P_{\nu - 2}(\tilde{x}_1) \frac{\partial}{\partial \tilde{x}_3} + \tilde{x}_3(\nu - 1)P_{\nu - 1}(\tilde{x}_1) \frac{\partial}{\partial \tilde{x}_4},

(74)$$

$$\tilde{K}_4 = Q_{\nu - 2}(\tilde{x}_1) \frac{\partial}{\partial \tilde{x}_3} + \tilde{x}_3(\nu - 1)Q_{\nu - 1}(\tilde{x}_1) \frac{\partial}{\partial \tilde{x}_4},

(75)$$

$$\tilde{K}_5 = -\frac{\partial}{\partial \tilde{x}_4},

(76)$$
where \( P_\nu \) and \( Q_\nu \) are Legendre functions of the first and second kind. The commutators close to the Heisenberg algebra with the central element \( \tilde{K}_5 \)
\[
[\tilde{K}_1, \tilde{K}_2] = \tilde{K}_5, \quad [\tilde{K}_3, \tilde{K}_4] = \tilde{K}_5
\]
due to the identity
\[
P_{1+\nu}(z)Q_\nu(z) - Q_{1+\nu}(z)P_\nu(z) = 1/(1 + \nu). \tag{78}
\]

The number of the Killing vectors as well as the fact that they close to the Heisenberg algebra suggests (cf. [13]) that this metric might be brought to the form of plane–parallel wave. Trying to rewrite the Killing vectors (72)–(76) to new (Rosen) coordinates \( z, w, y^1, y^2 \) such that the Killing vectors acquire the form [14]
\[
e_+ = \frac{\partial}{\partial w}, \quad e_i = \frac{\partial}{\partial y^i}, \quad e_i^* = y^i \frac{\partial}{\partial w} - \Gamma^i_{ij}(z) \frac{\partial}{\partial y^j}, \quad i, j \in \{1, 2\}, \tag{79}
\]
we can get the transformation
\[
\tilde{x}_1 = z, \\
\tilde{x}_2 = y^1 P_\nu(z), \\
\tilde{x}_3 = y^2 P_{\nu-2}(z), \\
\tilde{x}_4 = -w + \frac{1}{2} \left[ (y^1)^2 P_\nu(z) \left( (\nu + 1) P_{\nu+1}(z) - z (1 + 2\nu) P_\nu(z) \right) \\
+ (y^2)^2 (\nu - 1) P_{\nu-2}(z) P_{\nu-1}(z) \right], \tag{80}
\]
which leads to
\[
\Gamma^{11}(z) = -\frac{Q_\nu(z)}{P_\nu(z)}, \quad \Gamma^{12}(z) = \Gamma^{21}(z) = 0, \quad \Gamma^{22}(z) = -\frac{Q_{\nu-2}(z)}{P_{\nu-2}(z)}. \tag{81}
\]

In the coordinates \( z, w, y^1, y^2 \) the dual metric (71) reads
\[
ds^2 = \frac{2}{z^2 - 1} dz \, dw + (P_\nu(z))^2 (dy^1)^2 + (P_{\nu-2}(z))^2 (dy^2)^2. \tag{82}
\]

One can get rid of the denominator of the first term by substitution \( z = -\tanh \tilde{z} \) and bring the metric to the diagonal Rosen form.
Transition to the Brinkman coordinates is obtained by virtue of
the matrices $C(\tilde{z})$ and $Q(\tilde{z})$, where $C(\tilde{z})$ is given by
the Rosen form of the matrix; in our case
\[
C(\tilde{z}) = \begin{pmatrix}
(P_\nu(-\tanh \tilde{z}))^2 & 0 \\
0 & (P_{\nu-2}(-\tanh \tilde{z}))^2
\end{pmatrix},
\]
and the matrix $Q(\tilde{z})$ is a solution of equations
\[
Q(\tilde{z}) \cdot C(\tilde{z}) \cdot Q(\tilde{z}) = I,
\]
\[
\dot{Q}(\tilde{z}) \cdot C(\tilde{z}) \cdot Q(\tilde{z}) = Q(\tilde{z}) \cdot C(\tilde{z}) \cdot \dot{Q}(\tilde{z}).
\]
In our case, the solution can be chosen as
\[
Q(\tilde{z}) = \begin{pmatrix}
(P_\nu(-\tanh \tilde{z}))^{-1} & 0 \\
0 & (P_{\nu-2}(-\tanh \tilde{z}))^{-1}
\end{pmatrix}.
\]
The Brinkman coordinates can be written in terms of the Rosen coordinates as
\[
x^+ = \tilde{z},
\]
\[
x^- = w - \frac{1}{2}(\nu + 1)P_\nu(-\tanh \tilde{z})\left[P_{\nu+1}(-\tanh \tilde{z}) + \tanh \tilde{z}P_\nu(-\tanh \tilde{z})\right](y^1)^2
- \frac{1}{2}(\nu - 1)P_{\nu-2}(-\tanh \tilde{z})\left[P_{\nu-1}(-\tanh \tilde{z}) + \tanh \tilde{z}P_{\nu-2}(-\tanh \tilde{z})\right](y^2)^2,
\]
\[
z^1 = y^1P_\nu(-\tanh \tilde{z}),
\]
\[
z^2 = y^2P_{\nu-2}(-\tanh \tilde{z}).
\]
Using moreover the inverse of the transformation of coordinates we get
\[
x^+ = -\text{arctanh}(\tilde{x}_1),
\]
\[
x^- = -\tilde{x}_4 + \frac{1}{2}\tilde{x}_1(\tilde{x}_3^2(\nu - 1) - \tilde{x}_2^2\nu),
\]
\[
z^1 = \tilde{x}_2,
\]
\[
z^2 = \tilde{x}_3.
\]
The dual metric is then transformed to the form of the plane–parallel wave, where
\[
K_{ij}(x^+) = \frac{1}{(\cosh x^+)^2} \begin{pmatrix}
\nu(\nu + 1) & 0 \\
0 & (\nu - 2)(\nu - 1)
\end{pmatrix}.
\]
It means that the non-abelian T-duality transforms the plane–parallel wave metric (15) to another plane–parallel wave metric that is again solvable. The solutions of classical equations of motion in the dual background can be obtained by the Ansatz
\[ X^+(\tau, \sigma) = -\text{arctanh}\left(\frac{c_1 - \kappa \sigma}{\kappa \tau}\right) \]
(88)
that follows from the duality and coordinate transformation of the light-cone gauge of the original background. From the transformation (86) and (67,68) one can see that the transversal components \( Z^1(\tau, \sigma), Z^2(\tau, \sigma) \) of the classical solutions of the dual model are again expressed in terms of the Hankel functions.

The conformal invariance conditions (16)–(18) written in the Brinkman coordinates result in very simple equation for the dilaton
\[ \ddot{\Phi}''(x^+) = \frac{2(1 - \nu + \nu^2)}{\cosh^2(x^+)}, \]
(89)
solved in agreement with (52) by
\[ \Phi(x^+) = C_0 + C_1 x^+ + 2(1 - \nu + \nu^2) \log(\cosh x^+). \]
(90)

6 Strings in the dual background obtained from \( g_2 \)

Let us now consider the group generated by the Lie algebra \( g_2 = \text{Span}[T_1, T_2, T_3, T_4] \) with commutation relations (cf. (36))
\[ [T_4, T_1] = T_1, \]
\[ [T_4, T_2] = \nu T_2 - \rho T_3, \]
\[ [T_4, T_3] = \nu T_3 + \rho T_2. \]
(91)
The transformation between group coordinates \( x^1, x^2, x^3, x^4 \) and geometrical coordinates \( u, v, x, y \) is
\[ u = e^{x^4}, \]
\[ v = \left[ -\frac{1}{2}\nu((x^2)^2 + (x^3)^2) + x^1\right]e^{-x^4}, \]
(92)
\[ x = x^2 \cos(\rho x^4) - x^3 \sin(\rho x^4), \]
\[ y = x^3 \cos(\rho x^4) + x^2 \sin(\rho x^4). \]
The metric \((15)\) is then transformed into the form

\[
G_{\mu\nu}(x) = \begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & 1 & 0 & -\nu x^2 - \rho x^3 \\
0 & 0 & 1 & -\nu x^3 + \rho x^2 \\
1 & -\nu x^2 - \rho x^3 & -\nu x^3 + \rho x^2 & -2x^1 + (\nu^2 + \rho^2)((x^2)^2 + (x^3)^2)
\end{pmatrix}.
\]

The equations \((9,10)\) for \(\tilde{h}\) now read

\[
\partial_\tau \tilde{h} = -\begin{pmatrix}
0 \\
(\kappa \tau)^\nu \partial_\sigma X \\
(\kappa \tau)^\nu \partial_\sigma Y \\
\kappa \tau \partial_\sigma V - \rho Y \partial_\sigma X + \rho X \partial_\sigma Y
\end{pmatrix},
\]

\[
\partial_\sigma \tilde{h} = -\begin{pmatrix}
\kappa \\
(\kappa \tau)^\nu (\partial_\tau X - \frac{\nu}{\tau} X) \\
(\kappa \tau)^\nu (\partial_\tau Y - \frac{\nu}{\tau} Y) \\
\kappa \tau \partial_\tau V - \kappa V - \frac{\nu(1-\nu)}{\tau}(X^2 + Y^2) - \rho Y \partial_\tau X + \rho X \partial_\tau Y
\end{pmatrix},
\]

and are solved by

\[
\tilde{h}_1 = c_1 - \kappa \sigma, \quad (96)
\]

\[
\tilde{h}_2 = c_2 - (\kappa \tau)^\nu \int d\sigma (\partial_\tau X - \frac{\nu}{\tau} X), \quad (97)
\]

\[
\tilde{h}_3 = c_3 - (\kappa \tau)^\nu \int d\sigma (\partial_\tau Y - \frac{\nu}{\tau} Y), \quad (98)
\]

\[
\tilde{h}_4 = c_4 + \int d\sigma \left[ \frac{\nu(1-\nu)}{\tau}(X^2 + Y^2) + \kappa (V - \tau \partial_\tau V) + \rho Y \partial_\tau X - \rho X \partial_\tau Y \right], \quad (99)
\]

where \(c_i, i = 1, \ldots, 4\) are arbitrary constants.

The tensor dual to \((93)\) is \([15]\)

\[
\tilde{F}_{\mu\nu}(\tilde{x}) = \begin{pmatrix}
\frac{(\nu^2 + \rho^2)(\tilde{x}_2^2 + \tilde{x}_3^2)}{\tilde{x}_1^2 - 1} & \nu \tilde{x}_2 - \rho \tilde{x}_3 & \nu \tilde{x}_3 + \rho \tilde{x}_2 & 1 \\
\frac{\tilde{x}_2 + \rho \tilde{x}_3}{1-\tilde{x}_1} & 1 & 0 & 0 \\
\frac{\tilde{x}_3 + \rho \tilde{x}_2}{1-\tilde{x}_1} & 1 & 0 & 0 \\
\frac{\tilde{x}_1 + 1}{\tilde{x}_1 + 1} & 0 & 1 & 0 \\
\frac{\tilde{x}_1 + 1}{\tilde{x}_1 + 1} & 0 & 0 & 0
\end{pmatrix}.
\]
It has nontrivial antisymmetric part (torsion potential)

\[ \tilde{B}_{\mu\nu} = \frac{1}{2}(\tilde{F}_{\mu\nu} - \tilde{F}_{\nu\mu}) \]  \hspace{1cm} (101)

with torsion \( \tilde{H} = d\tilde{B} \) equal to

\[ \tilde{H} = \frac{2\rho}{x_1^2 - 1} d\tilde{x}_1 \wedge d\tilde{x}_2 \wedge d\tilde{x}_3. \] \hspace{1cm} (102)

The dual metric, which is the symmetric part of (100), does not solve the Einstein equations but satisfies the conformal invariance conditions (16)–(18) with the dilaton field

\[ \tilde{\Phi} = C_0 + C_1 \ln \left( \frac{\tilde{x}_1 - 1}{\tilde{x}_1 + 1} \right) - \nu(\nu + 1) \ln(1 - \tilde{x}_1^2). \] \hspace{1cm} (103)

The equations (12) are solved using the same faithful representation of the Drinfeld double as in the previous section. From (57) we get the relation between the solutions of the dual sigma models as

\[ h^j = x^j, \] \hspace{1cm} (104)

\[ \tilde{x}_1 = e^{-x^4} \tilde{h}_1, \] \hspace{1cm} (105)

\[ \tilde{x}_2 = e^{-\nu x^4}(\tilde{h}_3 \sin(\rho x^4) + \tilde{h}_2 \cos(\rho x^4)), \] \hspace{1cm} (106)

\[ \tilde{x}_3 = e^{-\nu x^4}(\tilde{h}_3 \cos(\rho x^4) - \tilde{h}_2 \sin(\rho x^4)), \] \hspace{1cm} (107)

\[ \tilde{x}_4 = e^{-\nu x^4}(\nu x^3 - \rho x^2)(\tilde{h}_3 \cos(\rho x^4) - \tilde{h}_2 \sin(\rho x^4)) \]
\[ + e^{-\nu x^4}(\nu x^2 + \rho x^3)(\tilde{h}_3 \sin(\rho x^4) + \tilde{h}_2 \cos(\rho x^4)) \]
\[ + e^{-x^4} x^4 \tilde{h}_1 + \tilde{h}_4. \] \hspace{1cm} (108)

The equation of motion

\[ \frac{\delta S_{\tilde{F}}}{\delta \tilde{X}_4} = 0 \]

has the same form (64) as in the previous section, and from (96, 105) we get again its solution

\[ \tilde{X}_1(\tau, \sigma) = \frac{c_1 - \kappa \sigma}{\kappa \tau}. \] \hspace{1cm} (109)

The other two equations then reduce to

\[ (\partial_{\sigma}^2 - \partial_{\tau}^2) \tilde{X}_2 + \frac{2\rho}{\tau} \partial_{\tau} \tilde{X}_3 + \frac{1}{\tau^2} \left[ (\nu + \nu^2 + \rho^2) \tilde{X}_2 - \rho \tilde{X}_3 \right] = 0, \] \hspace{1cm} (110)

\[ (\partial_{\sigma}^2 - \partial_{\tau}^2) \tilde{X}_3 - \frac{2\rho}{\tau} \partial_{\tau} \tilde{X}_2 + \frac{1}{\tau^2} \left[ (\nu + \nu^2 + \rho^2) \tilde{X}_3 + \rho \tilde{X}_2 \right] = 0. \] \hspace{1cm} (111)
Their solution for $\nu \neq \frac{1}{2}$ follows from (106), (107) and (97), (98) in the form

\[
\tilde{X}_2(\tau, \sigma) = \cos(\rho \log(\kappa \tau)) \left[ (c_2 \kappa^{-\nu} + \sqrt{2\nu - 1} \hat{x} \sigma) \tau^{-\nu} + \Sigma_2^x(\tau, \sigma) \right] + \\
\sin(\rho \log(\kappa \tau)) \left[ (c_3 \kappa^{-\nu} - \sqrt{2\nu - 1} \hat{y} \sigma) \tau^{-\nu} + \Sigma_2^y(\tau, \sigma) \right],
\]

(112)

\[
\tilde{X}_3(\tau, \sigma) = -\sin(\rho \log(\kappa \tau)) \left[ (c_2 \kappa^{-\nu} + \sqrt{2\nu - 1} \hat{x} \sigma) \tau^{-\nu} + \Sigma_2^x(\tau, \sigma) \right] + \\
\cos(\rho \log(\kappa \tau)) \left[ (c_3 \kappa^{-\nu} - \sqrt{2\nu - 1} \hat{y} \sigma) \tau^{-\nu} + \Sigma_2^y(\tau, \sigma) \right],
\]

(113)

where $\Sigma^j_2$ are given by (69). The last equation of motion is solved by

\[
\tilde{X}_4(\tau, \sigma) = c_4 - \frac{\sigma + c_1}{\tau} x^1 + (\nu x^2 + \rho x^3) \tilde{X}_2 + (\nu x^3 - \rho x^2) \tilde{X}_3 \\
+ \int d\sigma \left[ \frac{\nu(1 - \nu)}{\tau} (X^2 + Y^2) + \kappa(V - \tau \partial_\tau V) + \rho Y \partial_\tau X - \rho X \partial_\tau Y \right].
\]

(114)

It is worth mentioning that the duality is not the only option here. It is possible to use another decomposition [16] of $D$ into groups $\hat{G}, \bar{G}$ with algebra $\mathfrak{d} = \{\hat{e}_1, \ldots, \hat{e}_4, \bar{e}_1, \ldots, \bar{e}_4\} = \{T_1 + T_4, \bar{T}^1 - \bar{T}^4, T_2, T_3, \frac{1}{2}(\bar{T}^1 + \bar{T}^4), \frac{1}{2}(T_1 - T_4), \bar{T}^2, \bar{T}^3\}$ and commutation relations of the basis elements

\[
[\hat{e}_1, \hat{e}_2] = -\hat{e}_2, \quad [\bar{e}_1, \bar{e}_2] = -\frac{1}{2} \bar{e}_1, \\
[\hat{e}_1, \hat{e}_3] = \nu \hat{e}_3 - \rho \hat{e}_4, \quad [\bar{e}_2, \bar{e}_3] = \frac{1}{2} \nu \bar{e}_3 + \frac{1}{2} \rho \bar{e}_4, \\
[\hat{e}_1, \hat{e}_4] = \rho \hat{e}_3 + \nu \hat{e}_4, \quad [\bar{e}_2, \bar{e}_4] = -\frac{1}{2} \rho \bar{e}_3 + \frac{1}{2} \nu \bar{e}_4.
\]

Then it is possible to construct sigma models on the groups $\hat{G}$ or $\bar{G}$ and in principle solve their equations of motion by the Poisson–Lie T-plurality transformation [17], which relates the solutions of the sigma models on $G$ and $\hat{G}$ or $\bar{G}$ respectively. However, all the calculations get very complicated as none of the algebras is Abelian.
6.1 Killing vectors for the dual metric and its plane–parallel wave form

In general there are only two linearly independent Killing vectors of the tensor $\tilde{F}$ satisfying $\mathcal{L}_{\tilde{K}} \tilde{F} = 0$, namely

$$
\tilde{K}_1 = \tilde{x}_2 \frac{\partial}{\partial \tilde{x}_3} - \tilde{x}_3 \frac{\partial}{\partial \tilde{x}_2},
$$

(115)

$$
\tilde{K}_2 = - \frac{\partial}{\partial \tilde{x}_4}.
$$

(116)

However, for $\rho = 0$ the torsion vanishes and both the equations of motion and the $\beta$ equations for the sigma model are equivalent to those calculated from the symmetric part of $F$

$$
\tilde{G}_{\mu \nu} (\tilde{x}) = \begin{pmatrix}
\frac{(\tilde{x}_2^2 + \tilde{x}_3^2) \nu^2}{\nu \tilde{x}_2^2} & \frac{\nu \tilde{x}_2 \tilde{x}_3}{1 - \tilde{x}_2^2} & \frac{\nu \tilde{x}_2 \tilde{x}_3}{1 - \tilde{x}_3^2} & \frac{1}{1 - \tilde{x}_2^2} \\
\frac{\nu \tilde{x}_2 \tilde{x}_3}{1 - \tilde{x}_2^2} & \frac{\nu \tilde{x}_2 \tilde{x}_3}{1 - \tilde{x}_3^2} & 0 & 0 \\
\frac{\nu \tilde{x}_2 \tilde{x}_3}{1 - \tilde{x}_2^2} & 0 & \frac{1}{1 - \tilde{x}_2^2} & 0 \\
\frac{1}{1 - \tilde{x}_2^2} & 0 & 0 & 0 
\end{pmatrix}.
$$

(117)

We shall show that this metric is again a plane–parallel wave. It has a six-dimensional algebra of Killing vectors generated by $\tilde{K}_1, \tilde{K}_2$ and

$$
\tilde{K}_3 = P_\nu (\tilde{x}_1) \frac{\partial}{\partial \tilde{x}_2} + \tilde{x}_2 [(\nu + 1) P_{\nu+1} (\tilde{x}_1) - \tilde{x}_1 (1 + 2 \nu) P_\nu (\tilde{x}_1)] \frac{\partial}{\partial \tilde{x}_4},
$$

$$
\tilde{K}_4 = Q_\nu (\tilde{x}_1) \frac{\partial}{\partial \tilde{x}_2} + \tilde{x}_2 [(\nu + 1) Q_{\nu+1} (\tilde{x}_1) - \tilde{x}_1 (1 + 2 \nu) Q_\nu (\tilde{x}_1)] \frac{\partial}{\partial \tilde{x}_4},
$$

$$
\tilde{K}_5 = P_\nu (\tilde{x}_1) \frac{\partial}{\partial \tilde{x}_3} + \tilde{x}_3 [(\nu + 1) P_{\nu+1} (\tilde{x}_1) - \tilde{x}_1 (1 + 2 \nu) P_\nu (\tilde{x}_1)] \frac{\partial}{\partial \tilde{x}_4},
$$

$$
\tilde{K}_6 = Q_\nu (\tilde{x}_1) \frac{\partial}{\partial \tilde{x}_3} + \tilde{x}_3 [(\nu + 1) Q_{\nu+1} (\tilde{x}_1) - \tilde{x}_1 (1 + 2 \nu) Q_\nu (\tilde{x}_1)] \frac{\partial}{\partial \tilde{x}_4}.
$$

(118)

Their nonzero commutation relations are

$$
[\tilde{K}_1, \tilde{K}_3] = - \tilde{K}_5, \quad [\tilde{K}_1, \tilde{K}_4] = - \tilde{K}_6,
$$

$$
[\tilde{K}_1, \tilde{K}_5] = \tilde{K}_3, \quad [\tilde{K}_1, \tilde{K}_6] = \tilde{K}_4,
$$

$$
[\tilde{K}_3, \tilde{K}_4] = \tilde{K}_2, \quad [\tilde{K}_5, \tilde{K}_6] = \tilde{K}_2.
$$

(119)

One can see that the Killing vectors $\tilde{K}_2, \tilde{K}_6$ form the Heisenberg algebra with the central element $\tilde{K}_2$. This opens the possibility that this metric might be
again brought to the form of a plane–parallel wave. The transformation to Rosen coordinates \(z, w, y^1, y^2\)

\[
\begin{align*}
\tilde{x}_1 &= z, \\
\tilde{x}_2 &= y^1 P_\nu(z), \\
\tilde{x}_3 &= y^2 P_\nu(z), \\
\tilde{x}_4 &= -w + \frac{1}{2} \left[(y^1)^2 + (y^2)^2\right] P_\nu(z) \left[(\nu + 1) P_{\nu+1}(z) - z(1 + 2\nu) P_\nu(z)\right] 
\end{align*}
\]

(120)

brings the Killing vectors \(\tilde{K}_2 - \tilde{K}_6\) to the form (79) and the metric to

\[
ds^2 = \frac{2}{z^2 - 1} dz\, dw + (P_\nu(z))^2 (dy^1)^2 + (P_\nu(z))^2 (dy^2)^2.
\]

(121)

Transition to the Brinkman coordinates is obtained similarly as in the section 5.1

\[
\begin{align*}
x^+ &= -\text{arctanh}(\tilde{x}_1), \\
x^- &= -\tilde{x}_4 - \frac{1}{2} \nu \tilde{x}_1 (\tilde{x}_2^2 + \tilde{x}_3^2), \\
z^1 &= \tilde{x}_2, \\
z^2 &= \tilde{x}_3,
\end{align*}
\]

(122)

and the dual metric (117) is transformed to the form of the plane–parallel wave

\[
ds^2 = 2dx^+dx^- - \frac{\nu(\nu + 1)[(z^1)^2 + (z^2)^2]}{(\cosh x^+)^2} (dx^+)^2 + (dz^1)^2 + (dz^2)^2
\]

(123)

that is isotropic in \(z^1, z^2\).

Similarly for \(\rho \neq 0\) we can use a rotated version of (122)

\[
\begin{align*}
x^+ &= -\text{arctanh}(\tilde{x}_1), \\
x^- &= -\tilde{x}_4 - \frac{1}{2} \nu \tilde{x}_1 (\tilde{x}_2^2 + \tilde{x}_3^2), \\
z^1 &= \tilde{x}_2 \cos \Omega - \tilde{x}_3 \sin \Omega, \\
z^2 &= \tilde{x}_2 \sin \Omega + \tilde{x}_3 \cos \Omega,
\end{align*}
\]

(124)

where \(\Omega = \rho \log(\cosh x^+)\), to bring the dual metric derived from (100) to the form of the plane–parallel wave in Brinkman coordinates

\[
ds^2 = -[(z^1)^2 + (z^2)^2] \frac{2\nu(\nu + 1) + \rho^2(1 + \cosh(2x^+))}{2(\cosh x^+)^2} (dx^+)^2 + 2dx^+dx^- + (dz^1)^2 + (dz^2)^2.
\]

(125)
The torsion becomes constant in these coordinates

$$\tilde{H} = 2\rho dx^+ \wedge dz^1 \wedge dz^2,$$

and the vanishing $\beta$ equations (16)–(18) result in the equation for the dilaton

$$\tilde{\Phi}''(x^+) = \frac{2\nu(\nu + 1)}{\cosh^2(x^+)},$$

(126)
solved in agreement with (103) by

$$\tilde{\Phi}(x^+) = C_0 + C_1 x^+ + 2\nu(\nu + 1) \log(\cosh x^+).$$

(127)

Thus we can see that the non-abelian T-duality based on the semi-abelian Drinfeld double given by (36) transforms the plane–parallel wave metric (15) to another plane–parallel wave metric (125) and torsion.

## 7 Conclusions

We have investigated non–Abelian T–duals of homogeneous pp-wave metric (15) that belongs to the class of string backgrounds with metric, B–field and dilaton

$$ds^2 = 2dudv - K_{ij}(u)x^i x^j du^2 + d\vec{x}^2,$$

$$B = \frac{1}{2}H_{ij}(u)x^i du \wedge dx^j, \quad \Phi = \Phi(u)$$

(128)

(129)
analyzed in [18]. We have found classical string solutions for the dual backgrounds and we have also shown that by appropriate coordinate transformations the dual backgrounds can be brought again to the form (128), (129).

The number of Killing vectors of the dual models is less than those of (15), which means that the dual backgrounds are different from the initial one.

The dual metrics (71) and (117), obtained in the group coordinates by the procedure described in the Section 2, have five-dimensional subalgebras of Killing vectors commuting as the two-dimensional Heisenberg algebra. This implies that by a change of coordinates they can be rewritten to the diagonal Rosen form or plane–parallel wave form given by (87) and (123). By virtue of more general coordinate transformation (124) we find that the metric obtained from the dual tensor (100) can be brought to the plane-parallel
wave form (125) even for $\rho \neq 0$. The conformal invariance conditions (16)–(18) for the dual backgrounds in Brinkman coordinates then acquire simple form of solvable ODE [18] for the dilaton

$$\Phi''(u) = K_{ii}(u) + H_{ij}(u)H_{ij}(u),$$

and the dual backgrounds satisfy the conformal invariance as well.

The classical string solutions of the dual models, similarly as in the case of the initial metric (15), are given in terms of Hankel functions. An interesting point is that the equations of dual models may not admit the usual light-cone gauge, but the light-cone gauge (21) is transformed to the solution (63) of the equation (64). On the other hand, notice that the backgrounds (87) and (125) obtained by coordinate transformations of the dual backgrounds do admit the light-cone gauge. However, in these cases it leads to complicated equations for the transversal components. In order to get the solution, the use of Ansatz (88) is crucial.

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