Higher-order-in-spin interaction Hamiltonians for binary black holes from Poincaré invariance

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The fulfillment of the space-asymptotic Poincaré algebra is used to derive new higher-order-in-spin interaction Hamiltonians for binary black holes in the Arnowitt-Deser-Misner canonical formalism almost completing the set of the formally 1/c⁴ spin-interaction Hamiltonians involving nonlinear spin terms. To linear order in G, the expressions for the S³p and the S²p² Hamiltonians are completed. It is also shown that there are no quartic nonlinear S⁴ Hamiltonians to linear order in G.

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I. INTRODUCTION

In order to obtain higher accuracy in the templates for analyzing gravitational waves from binary black holes, gravitational spin-interaction terms have to be taken into account beyond the leading order ones which are of the formal order 1/c² (in this counting, spins are treated of the order zero in terms of 1/c²), where c denotes the speed of light. At the formal order of 1/c⁴, several Hamiltonians or Lagrangians have been determined already: The spin-orbit coupling ones, H₃ = H₂₊₁, have been given in [1], also see [2], the spin(1)-spin(2) coupling ones, H₁ₛ₂ + Hₛ₁ᵢ, can be found in [3], also see [4], and the following Hamiltonians H₃ₛ₁ₛ₂₊₁ + H₃ₛ₁ₛ₂₊₂ + H₁ₛ₁ₛ₂₊₂, H₃ₛ₁ₛ₂₊₁, H₁ₛ₁ₛ₂₊₁ have been derived in [5]; recently, the dynamics corresponding to the Hamiltonians H₁ₛ₂₊₁ + H₃ₛ₁ₛ₂₊₂ + H₂ₛ₁ₛ₂₊₂ has been announced, [6]. In this paper the missing S³p-Hamiltonians Hₛ₃₊₁ + Hₛ₂₊₂ and Hₛ₁ₛ₃₊₁ + Hₛ₁ₛ₃₊₂ are calculated in the Arnowitt-Deser-Misner (ADM) canonical formalism by applying the space-asymptotic Poincaré algebra, [7], [8]. These expressions contain post-Newtonian leading order quadrupole deformation effects represented by spin-squared terms (the coefficients in the quadrupole terms tell that the treated bodies are black holes, see, e.g. [9], [10]). Because of the power of the Poincaré algebra in controlling the higher post-Newtonian dynamics we are also able to give reasonable expressions for the spin-squared S²p²-Hamiltonians to linear order in G, i.e. Hₛ₂₊₁ + Hₛ₂₊₂. Unfortunately, only after completion of the calculation of all static G²-terms Hₛ₂ a comparison with the result presented in [6] can be made because of quite different approaches.

Our units are very often c = 1 and also 16πG = 1, where G is the Newtonian gravitational constant. Greek indices will run over 0, 1, 2, 3, Latin from the beginning of the alphabet over 1, 2, 3 (denoting black-hole numbers) and from its midst over 1, 2, 3. For the signature of spacetime we choose +2. r = rᵦᵦ = |xᵦ - xᵦᵦ| will denote the Euclidean distance between black hole a and b, and r nᵦ = rᵦ, nᵦᵦ = xᵦ - xᵦᵦ, (nᵦᵦ) = nᵦ = n.

II. THE POINCARÉ INVARIANCE

The starting point for our approach will be the Poincaré algebra. The Hamiltonians we wish to calculate have to fulfill this algebra using standard Poisson brackets for the fundamental position, linear momentum and spin variables, xᵦ, pᵦ, (pᵦ) = p, and Sᵦ, (Sᵦ) = S, respectively, so the plan is to make the most general ansatz for these Hamiltonians and plug them into the algebra to see how restrictive it is upon them. Of course, the Hamiltonians cannot be determined uniquely because all Hamiltonians which are canonically equivalent fulfill the algebra, so there will be degrees of freedom left which can be fixed only by choosing an appropriate representation. We will choose to work within the ADM canonical formalism using generalized isotropic coordinates, [11]. With the aid of a reasonable ansatz for the source terms in the constraint equations we are able to fix all the coefficients of the Hamiltonians in question.

The Poisson bracket relations defining the Poincaré algebra read, see, e.g. [4],

\{Pᵦ, Pᵦ\} = 0, \quad \{Pᵦ, H\} = 0, \quad \{Jᵦ, H\} = 0, \quad (2.1)

\{Jᵦ, Pᵦ\} = \epsilonᵦᵦ Pᵦ, \quad \{Jᵦ, Jᵦ\} = \epsilonᵦᵦ Jᵦ, \quad \epsilonᵦᵦ = \epsilonᵦᵦ = \epsilonᵦᵦ, \quad (2.2)
\[
\{J_i, G_j\} = \epsilon_{ijk} G_k, \quad (2.3)
\]

\[
\{G_i, H\} = P_i, \quad (2.4)
\]

\[
\{G_i, P_j\} = \frac{1}{c^2} H \delta_{ij}, \quad (2.5)
\]

\[
\{G_i, G_j\} = -\frac{1}{c^2} \epsilon_{ijk} J_k. \quad (2.6)
\]

The total linear momentum \( P_i \) and the total angular momentum \( J_i \) are respectively given by \( P_i = \sum m_i p_{ai} \) and \( J_i = \sum \epsilon_{ijk} x_a^j p_{ak} + S_{ai} \). The crucial equation which needs to be checked and which finally will be used for determining the new Hamiltonians is Eq. (2.4). It describes the time evolution of the center-of-mass vector \( /c \) already in Ref. [10], which is the source paper for [9]. Obviously, the Hamiltonian in Eq. (2.5) needs to be known to one order in \( 1/c^2 \) against 3-dimensional translations and rotations. The second post-Newtonian Hamiltonian (2PN-Hamiltonian) of the contributions is known from [1] and \( H_{SO} \) and \( H_{P N} \), which enters the equations above reads

\[
H = H_N + H_{1PN} + H_{2PN} + H_{SO}^{P N} + H_{3SO}^{P N} + H_{S2} + H_{S3p} + H_{S2p2} + H_{S4} \quad (2.7)
\]

with

\[
H_{S2} = H_{S1S2} + H_{S2}^2 + H_{S2}^3. \quad (2.8)
\]

The Hamiltonians \( H_N, H_{1PN} \) and \( H_{2PN} \) are the point particle ones given in e.g. [9]. The Hamiltonians \( H_{SO}^{P N}, H_{S1S2} \) and \( H_{S2}^2 + H_{S2}^3 \), as far as being of order 1PN, were (re-)calculated in [5], also see [11], [12]. The Hamiltonian \( H_{SO}^{P N} \) is known from [1] and \( H_{S1S2}(p^2 + G) \) has been derived in [3]. The Hamiltonian \( H_{S3p} \) splits into the parts

\[
H_{S3p} = H_{S1p1} + H_{S2p2} + H_{S3p1} + H_{S1S2p1} + H_{S1S2p1} + H_{S1S2p1} + H_{S2S1p2} + H_{S3S1p2}, \quad (2.9)
\]

and \( H_{S2p2} \) is an abbreviation for

\[
H_{S2p2} = H_{S1S2p2} + H_{S2p2} + H_{S2p2}. \quad (2.10)
\]

The contributions \( H_{S1p1}, H_{S1p2}, H_{S2S1p2} \) and \( H_{S1S2p1} \) were all calculated in [5] \( (S_b = m_b a_b) \),

\[
H_{S1p1} = -G S_1 \cdot (n_{12} \times p_2) \left( \frac{a_1^2}{r_{12}^4} - \frac{5(a_1 \cdot n_{12})^2}{r_{12}^4} \right), \quad (2.11)
\]

\[
H_{S1p1} = H_{S1p2} \ (1 \leftrightarrow 2), \quad (2.12)
\]

\[
H_{S2p2} = -G \frac{3m_2}{4m_1} \left( \frac{3a_1^2 p_2 \cdot (S_2 \times n_{12})}{r_{12}^4} + \frac{6(a_1 \cdot n_{12}) p_2 \cdot (S_2 \times a_1)}{r_{12}^4} - \frac{15(a_1 \cdot n_{12})^2 p_2 \cdot (S_2 \times n_{12})}{r_{12}^4} \right), \quad (2.13)
\]

\[
H_{S1S2p1} = H_{S2S1p2} \ (1 \leftrightarrow 2). \quad (2.14)
\]
The Hamiltonian $H_{S^4}$ splits into the parts

$$H_{S^4} = H_{S_1^2 S_2} + H_{S_1 S_2^2} + H_{S_2^2 S_1} + H_{S_1^2} + H_{S_2^2}. \tag{2.15}$$

The Hamiltonians $H_{S_1^2 S_2} + H_{S_1 S_2^2} + H_{S_2^2 S_1}$ are given in [3] and the Hamiltonians $H_{S_1^2} + H_{S_2^2}$ will be shown to be zero in Sec. VI below. The remaining Hamiltonians will be chosen in the form, with to be determined $\mu$-coefficients,

$$H_{S_1^2 p_1} = \frac{G}{r_{12}^2} \left[ S_1 \cdot (n_{12} \times p_1) \left( \mu_1 S_1^2 + \mu_2 (S_1 \cdot n_{12})^2 \right) \right], \tag{2.16}$$

$$H_{S_2^2 p_2} = H_{S_1^2 p_1} (1 \leftrightarrow 2), \tag{2.17}$$

$$H_{S_1^2 S_2 p_1} = \frac{G}{r_{12}^2} \left[ \mu_3 S_1^2 S_2 \cdot (n_{12} \times p_1) + \mu_4 (S_1 \cdot n_{12}) S_2 \cdot (S_1 \times p_1) + \mu_5 (S_1 \cdot n_{12})^2 S_2 \cdot (n_{12} \times p_1) \\
+ \mu_6 (S_1 \cdot S_2) S_1 \cdot (n_{12} \times p_1) + \mu_7 (S_1 \cdot n_{12}) (S_2 \cdot n_{12}) S_1 \cdot (n_{12} \times p_1) \\
+ n_{12} \cdot (S_1 \times S_2) \left( \mu_8 S_1 \cdot p_1 + \mu_9 (S_1 \cdot n_{12}) (p_1 \cdot n_{12}) \right) \right], \tag{2.18}$$

$$H_{S_2^2 S_1 p_2} = H_{S_2^2 S_1 p_2} (1 \leftrightarrow 2). \tag{2.19}$$

Notice that not all terms are independent of each other. They are connected via the following identity for the mixed product of three vectors in three dimensions,

$$(U_1, U_2, U_3) U = (UU_1) U_2 \times U_3 + (UU_2) U_3 \times U_1 + (UU_3) U_1 \times U_2 \tag{2.20}$$

$$= (U, U_2, U_3) U_1 + (U_1, U, U_3) U_2 + (U_1, U_2, U) U_3 \tag{2.21}$$

with definitions $(U U_1) = U \cdot U_1$ and $(U_1, U_2, U_3) = U_1 \cdot (U_2 \times U_3)$. This will help us later to understand that some of the $\mu$ coefficients must be zero, because some terms can be shifted into others. The ansatz for the $S_1^2 p^2$ Hamiltonian reads, using $\alpha$, $\beta$, and $\gamma$ coefficients,

$$H_{S_1^2 p^2} = \frac{G}{r_{12}^2} \left[ \alpha_1 (p_1 \cdot S_1)^2 + \alpha_2 p_2^2 S_1^2 + \alpha_3 (p_1 \cdot n_{12})^2 S_1^2 + \alpha_4 p_2^2 (S_1 \cdot n_{12})^2 \\
+ \alpha_5 (p_1 \cdot n_{12})^2 (S_1 \cdot n_{12})^2 + \alpha_6 (p_1 \cdot n_{12}) (S_1 \cdot n_{12}) (p_1 \cdot S_1) \\
+ \beta_1 (p_2 \cdot S_1)^2 + \beta_2 p_2^2 S_1^2 + \beta_3 (p_2 \cdot n_{12})^2 S_1^2 + \beta_4 p_2^2 (S_1 \cdot n_{12})^2 \\
+ \beta_5 (p_2 \cdot n_{12})^2 (S_1 \cdot n_{12})^2 + \beta_6 (p_2 \cdot n_{12}) (S_1 \cdot n_{12}) (p_2 \cdot S_1) \\
+ \gamma_1 (p_1 \cdot p_2) S_1^2 + \gamma_2 (p_1 \cdot p_2) (S_1 \cdot n_{12})^2 + \gamma_3 (p_1 \cdot S_1) (p_2 \cdot S_1) \\
+ \gamma_4 (p_1 \cdot n_{12}) (p_2 \cdot S_1) (S_1 \cdot n_{12}) + \gamma_5 (p_2 \cdot n_{12}) (p_1 \cdot S_1) (S_1 \cdot n_{12}) \\
+ \gamma_6 (p_1 \cdot n_{12}) (p_2 \cdot n_{12}) S_1^2 + \gamma_7 (p_1 \cdot n_{12}) (p_2 \cdot n_{12}) (S_1 \cdot n_{12})^2 \right], \tag{2.22}$$

$$H_{S_2^2 p^2} = H_{S_1^2 p^2} (1 \leftrightarrow 2). \tag{2.23}$$

The center-of-mass vector $G$ which enters in Eq. (2.3) is given by
\[ G = G_N + G_{1PN} + G_{2PN} + G_{1PN}^{SO} + G_{2PN}^{SO} + G_{1PN} + G_{2PN}^{2} + G_{S1} + G_{S2} + G_{S1}^{2} + G_{S2}^{2}. \] (2.24)

The point particle expressions \( G_N, G_{1PN} \) and \( G_{2PN} \) are given in [9]. The parts \( G_{1PN}^{SO} \) and \( G_{2PN}^{SO} \) are presented in [1] and \( G_{S1}, G_{S2} \) was calculated in [3]. What is left is to give the expression for \( G_{S1}^{2} \) and \( G_{S2}^{2} \). Here in this section we will also make a general gauge invariant ansatz for them to show how they are involved in the algebra. It reads with coefficients \( \nu_1, \nu_2, \nu_3, \nu_4 \) and \( \nu_5 \)

\[ G_{S1}^{2} = G \frac{m_2}{m_1} \left[ \nu_1 \left( \frac{S_1 \cdot n_{12}}{r_{12}^2} \right) S_1 + \left( \frac{S_1 \cdot n_{12}}{r_{12}^2} \right)^2 (\nu_2 x_1 + \nu_3 x_2) + \frac{S_1^2}{r_{12}^3} (\nu_4 x_1 + \nu_5 x_2) \right] \] (2.25)

plus the expression with \( 1 \leftrightarrow 2 \).

Later we will calculate \( G_{S1}^{2} \) directly by an appropriate covariant source for the Hamilton Constraint, which will fixate all the \( \nu \)-coefficients.

### III. FULFILLMENT OF THE POINCARÉ ALGEBRA

We concentrate on Eq. (2.4) which gives rise to the following equations for the coefficients:

\[ \mu_1 = \frac{m_2}{4 m_1^2}, \] (3.1)
\[ \mu_2 = -\frac{5 m_2}{4 m_1^2}, \] (3.2)
\[ \mu_3 = \frac{3}{m_1^2} - \mu_8, \] (3.3)
\[ \mu_4 = \frac{15}{2 m_1^2} + \mu_9 + \mu_8, \] (3.4)
\[ \mu_5 = -\frac{15}{m_1^2} - \mu_9, \] (3.5)
\[ \mu_6 = -\frac{3}{2 m_1^2} + \mu_8, \] (3.6)
\[ \mu_7 = \frac{15}{2 m_1^2} + \mu_9. \] (3.7)
\begin{align}
\alpha_1 &= \frac{3}{4} m_2 - \frac{m_2}{2m_1} \gamma_3 - \frac{m_2}{2m_1^2} \nu_1, \\
\alpha_2 &= \frac{m_2}{m_1} - \frac{m_2}{2m_1} \gamma_1 + \frac{m_2}{2m_1^2} \nu_5, \\
\alpha_3 &= -\frac{9}{8} m_2 - \frac{m_2}{2m_1} \gamma_6 - \frac{3m_2}{2m_1} \nu_5, \\
\alpha_4 &= -\frac{9}{8} m_2 - \frac{m_2}{2m_1} \gamma_2 - \frac{m_2}{2m_1^2} \nu_2, \\
\alpha_5 &= -\frac{15}{4} m_2 - \frac{m_2}{2m_1} \gamma_7 - \frac{5m_2}{2m_1} \nu_2, \\
\alpha_6 &= -\frac{15}{4} m_2 - \frac{m_2}{m_1} \gamma_4 + \frac{3m_2}{m_1^2} \nu_1, \\
\beta_1 &= \frac{1}{m_1 m_2} - \frac{m_1}{2m_2} \gamma_3 + \frac{\nu_1}{2m_1 m_2}, \\
\beta_2 &= -\frac{1}{m_1 m_2} - \frac{m_1}{2m_2} \gamma_1 - \frac{\nu_5}{2m_1 m_2}, \\
\beta_3 &= \frac{9}{4 m_1 m_2} - \frac{m_1}{2m_2} \gamma_6 + \frac{3}{2m_1 m_2} \nu_5, \\
\beta_4 &= \frac{3}{4 m_1 m_2} - \frac{m_1}{2m_2} \gamma_2 + \frac{\nu_2}{2m_1 m_2}, \\
\beta_5 &= -\frac{m_1}{2m_2} \gamma_7 - \frac{5}{2m_1 m_2} \nu_2, \\
\beta_6 &= -\frac{3}{m_1 m_2} - \frac{m_1}{m_2} \gamma_4 + \frac{2\nu_2}{m_1 m_2}, \\
\gamma_5 &= \gamma_4 - \frac{3\nu_1}{m_1^2} - \frac{2\nu_2}{m_1^2}, \\
\nu_3 &= \frac{3}{2} - \nu_2, \\
\nu_4 &= \frac{1}{2} - \nu_5.
\end{align}

The Poincaré algebra obviously fixes the Hamiltonian \( H_{S_1^3 p_1} \) (and thus also \( H_{S_2^3 p_2} \)). It also restricts five from seven coefficients for \( H_{S_1^3 p_1} \) (and thus also for \( H_{S_2^3 p_2} \)), and two out of five coefficients for the center-of-mass vector \( G_{S_1^3} \). At this stage of our investigations, the coefficients in \( H_{S_1^3 p_1} \) and \( G_{S_1^3} \) can still be changed by the application of a canonical transformation generated by

\[
g_{S_1^3 p_1} = \frac{G}{r_{12}^2} \left( S_1^2 (\sigma_1 p_1 \cdot n_{12} + \sigma_2 p_2 \cdot n_{12}) + S_1 \cdot n_{12} (\sigma_3 S_1 \cdot p_1 + \sigma_4 S_1 \cdot p_2) \right. \\
+ \left. (S_1 \cdot n_{12})^2 (\sigma_5 p_1 \cdot n_{12} + \sigma_6 p_2 \cdot n_{12}) \right)
\]

with coefficients \( \sigma_k, k = 1, 2, 3, 4, 5, 6 \). Later on we shall see that two coefficients of the center-of-mass vector \( G_{S_1^3} \), and thus many other coefficients which are connected with them, can uniquely only be fixed with the aid of the explicit expression for the energy density of a static source. Coming back to the Poincaré algebra, all its other equations are trivially fulfilled and impose no further restrictions on our coefficients.

**IV. THE ADM GENERALIZED ISOTROPIC COORDINATES REPRESENTATION**

To get a unique representation of the Hamiltonians, we now have to fix the coordinates. We will use the ADM formalism and generalized isotropic coordinates. This makes it very easy to calculate interaction Hamiltonians if
the source expressions in the constraint equations are known. Before imposing constraint equations and coordinate conditions, the Hamiltonian in the ADM framework reads \[ \mathcal{H} = \int d^3x (N\mathcal{H} - N^i\dot{\mathcal{H}}_i) + E [\gamma_{ij}], \] (4.1)

where respectively \( N \) and \( N^i \) denote lapse and shift function, which are merely Lagrange multipliers. The super-Hamiltonian \( \mathcal{H} \) and the supermomentum \( \mathcal{H}_i \) densities decompose into gravitational field and matter parts as follows:

\[
\mathcal{H} = \mathcal{H}^{\text{field}} + \mathcal{H}^{\text{matter}}, \quad \mathcal{H}_i = \mathcal{H}_i^{\text{field}} + \mathcal{H}_i^{\text{matter}},
\] (4.2)

where the field parts are given by

\[
\mathcal{H}^{\text{field}} = -\frac{1}{\sqrt{\gamma}} \left[ \gamma R + \frac{1}{2} \left( \gamma_{ij} \pi^{ij} \right)^2 - \gamma_{ij} \gamma_{kl} \pi^{ik} \pi^{jl} \right], \quad \mathcal{H}_i^{\text{field}} = 2\gamma_{ij} \pi^{ijk}.
\] (4.3)

Here \( \gamma \) is the determinant of the 3-metric \( \gamma_{ij} = g_{ij} \) of the spacelike hypersurfaces \( t = \text{const.} \), whereas the determinant of the 4-dimensional metric \( g_{\mu\nu} \) will be denoted \( \gamma \). The canonical conjugate to \( \gamma_{ij} \) is \( \pi^{ij} \). \( R \) is the Ricci scalar of the spacelike hypersurfaces and \( ; \) denotes the 3-dimensional covariant derivative. The expressions for lapse and shift functions in terms of the metric are \( N = (-g^{00})^{-1/2} \) and \( N^i = \gamma^{i\ell}g_{\ell j} \), where \( \gamma^{ij} \) is the inverse 3-metric. As in \[3\], we will assume that the relation between field momentum \( \pi^{ij} \) and extrinsic curvature \( K_{ij} \) is the same as in the vacuum case:

\[
\pi^{ij} = -\sqrt{\gamma}(\gamma^{ik} \gamma^{jl} - \gamma^{ij} \gamma^{kl})K_{kl}.
\] (4.4)

The energy \( E \) is defined by (herein, the comma \( , \) denotes partial 3-dimensional derivative)

\[
E = \int d^2 s_i (\gamma_{ij,j} - \gamma_{jji,i}).
\] (4.5)

\( E \) turns into the ADM Hamiltonian \( H_{\text{ADM}} \) after imposing the constraint equations \( \mathcal{H} = \mathcal{H}_i = 0 \) and appropriate coordinate conditions. Comparing the constraint equations with the Einstein field equations, projected onto the spacelike hypersurfaces, results in

\[
\mathcal{H}^{\text{matter}} = \sqrt{-\gamma}T^{\mu\nu}n_\mu n_\nu = N\sqrt{-g}T^{00}, \quad \mathcal{H}_i^{\text{matter}} = -\sqrt{-\gamma}T^i\nu n_\nu = \sqrt{-g}T_i^0,
\] (4.6)

where \( \sqrt{-g}T^{\mu\nu} \) is the stress-energy tensor density of the matter system. The timelike unit 4-vector \( n_\mu = (-N, 0, 0, 0) \) points orthogonal to the spacelike hypersurfaces.

The generalized isotropic coordinates \[7\], also called the ADMTT gauge, are the most often used and best adapted coordinate conditions for explicit calculations and they are defined by

\[
\gamma_{ij} = \psi^4 \delta_{ij} + h_{ij}^{\text{TT}} \quad \text{with} \quad \psi = 1 + \frac{1}{8} \phi, \quad \text{or} \quad 3\gamma_{ij,j} - \gamma_{jji,i} = 0, \quad \text{and} \quad \pi^{ii} = 0, \quad \pi^{ij} = \tilde{\pi}^{ij} + \pi_{TT}^{ij}.
\] (4.7)

\( h_{ij}^{\text{TT}} \) has the properties \( h_{ij}^{\text{TT}} = h_{ji}^{\text{TT}} = 0 \) (transverse and traceless) and the same holds for \( \pi_{TT}^{ij} \) which is the canonically conjugate to \( h_{ij}^{\text{TT}} \). \( \tilde{\pi}^{ij} \) denotes the longitudinal part of \( \pi^{ij} \).

The ADM Hamiltonian and the center-of-mass vector read

\[
E = H_{\text{ADM}} = -\int d^3x [\Delta], \quad G_i = -\int d^3x [\Delta i],
\] (4.8)

where \( \phi \) is expressed in terms of the canonical matter variables and the canonical field variables \( h_{ij}^{\text{TT}} \) and \( \pi_{TT}^{ij} \) of the field-reduced phase space. Herewith the Poincaré algebra can be verified. The most elegant way for the obtention of an autonomous conservative Hamiltonian in the matter variables is via the Routhian approach, see, e.g. \[13\]. However, to the order of our investigations \( \pi_{TT}^{ij} \) does not play any role so we may stay on the Hamiltonian level for the field variables.

The constraint equations explicitly read
\[
\frac{1}{\sqrt{\gamma}} \left[ \gamma R + \frac{1}{2} (\gamma_{ij} \pi^{ij})^2 - \gamma_{ij} \gamma_{kl} \pi^{ij} \pi^{kl} \right] = \mathcal{H}_{\text{matter}}^r, \\
-2 \gamma_{ij} \pi_{ik}^j = \mathcal{H}_{\text{matter}}^i.
\] (4.9) (4.10)

To include momentum squared terms interacting with spin-squared terms we have to make an ansatz for the sources generalizing the point particle terms without spin. The most general ansatz for our purpose reads for the source in the Hamilton constraint, cf. [5],

\[
\mathcal{H}_{\text{matter}} = \sum_b \left[ -\frac{m_b}{2} Q_b^{ij} \partial_i \partial_j - \frac{1}{2} p_b \cdot (a_b \times \partial) + (\gamma_{ij} p_b p_b + m_b^2) \right] + \lambda_1 \frac{p_b^2}{2 m_b} Q_b^{ij} \partial_i \partial_j \\
+ \frac{\lambda_2}{m_b} (p_b \cdot \partial) Q_b^{ij} p_b \partial_j + \frac{\lambda_3}{m_b} a_b^2 (p_b \cdot \partial)^2 - \lambda_8 p_b \cdot (a_b \times \partial) Q_b^{ij} \partial_i \partial_j \right] \delta_b
\] (4.11)

with the quadrupole tensor

\[
Q_b^{ij} = a_b^i a_b^j - \frac{1}{3} a_b^k \delta_{ij},
\] (4.12)

where by definition \(a_b = S_b/m_b\) holds and \(\delta_b \equiv \delta(x^i - x_b^i)\) with \(\int d^3x \delta_b = 1\) and \(\partial = (\partial_i), \partial_i = \frac{\partial}{\partial x^i}\). Notice that purely Laplacian source terms, \(\Delta \delta_b\), apart from the ones appearing via \(Q^{ij}\), are left out because they would lead to a distributional Hamiltonian (cf. paragraph on Breit’s equation in [14]) which is of no interest here.

The ansatz for the source of the momentum constraint reads, also not taking into account terms of the form \(\Delta \delta_b\) apart from the ones appearing with \(Q^{ij}\) (taking into account those terms would lead to no effects on our Hamiltonians when controlled by Poincaré algebra), cf. [3],

\[
\mathcal{H}_{\text{matter}}^i = -2 \sum_b \left[ Q_b^{kl} \left( \lambda_5 p_b \partial_k \partial_l + \lambda_6 p_b \partial_l \partial_k + \lambda_7 (p_b \cdot \partial) \delta_{kl} \partial_l \partial_k \right) + \lambda_4 a_b^2 (p_b \cdot \partial) \partial_i \right] \\
+ \frac{m_b}{4} (a_b \times \partial) \left( 1 - \frac{1}{6} Q_b^{kl} \partial_k \partial_l \right) - \frac{1}{2} p_b \partial_i \right] \delta_b.
\] (4.13)

This ansatz is not 3-dimensional general covariant, but it is of sufficient general form that will lead to all the searched for contributions of the \(H_{S^3P}\) and \(H_{S^2P}\) Hamiltonians. It also correctly holds \(\int d^3x \mathcal{H}_{\text{matter}}^i = P_i\) and \(\int d^3x \epsilon_{ijk} x^j \mathcal{H}_{\text{matter}}^i = J_i\). To the order needed for a 2PN-Hamiltonian for self-spin interaction, the Hamilton constraint expands as

\[
-\Delta \phi_2 = \mathcal{H}_{\text{matter}}^r, \\
-\Delta \phi_4 = \mathcal{H}_{\text{matter}}^r - \frac{1}{8} \mathcal{H}_{\text{matter}} \phi_2, \\
-\Delta \phi_6 = \mathcal{H}_{\text{matter}}^r - \frac{1}{8} \mathcal{H}_{\text{matter}} \phi_2 + \mathcal{H}_{\text{matter}}^r \phi_4 + \frac{1}{64} \mathcal{H}_{\text{matter}} \phi_2^2 + \left[ (\pi_{(3)}^{ij})^2 - \frac{1}{2} \phi_2 \phi_4 \right],
\] (4.14) (4.15)

\[
-\Delta \phi_8 = \mathcal{H}_{\text{matter}}^r - \frac{1}{8} \mathcal{H}_{\text{matter}} \phi_2 + \mathcal{H}_{\text{matter}}^r \phi_4 + \mathcal{H}_{\text{matter}}^r \phi_6 + \frac{1}{64} \mathcal{H}_{\text{matter}}^r \phi_2^2 + \frac{1}{16} \phi_2 \phi_4 \phi_6 + \frac{1}{16} \phi_2 \phi_4 \phi_6 + \frac{1}{4} \left( h^{TT}_{(4)} \phi_2 \phi_4 \phi_6 \right) + (\text{td}),
\] (4.16)

(“td” means total derivative) and the momentum constraint can be written as

\[
\pi^{ij}_{(3),j} = -\frac{1}{2} \mathcal{H}_{\text{matter}}^{(3)ij},
\]

\[
\pi^{ij}_{(5),j} = -\frac{1}{2} \mathcal{H}_{\text{matter}}^{(5)ij} - \frac{1}{2} (\phi_2 \pi^{ij}_{(3)})_{,j}.
\] (4.17) (4.18)

The integral over \(-\Delta \phi_8\) is the one leading to the new Hamiltonians to linear order in G. The \(h^{TT}_{ij}\) term will drop out from the calculation of the Hamiltonians in question. The expressions needed for this integral read
\[ H^{\text{matter}}_{(2)} = \sum_b m_b \delta_b, \quad (4.19) \]

\[ H^{\text{matter}}_{(4)} = \sum_b \left[ -\frac{m_b}{2} Q_b^{ij} \partial_i \partial_j + \frac{p_b^2}{2m_b} - \frac{1}{2} p_b \cdot (a_b \times \partial) \right] \delta_b, \quad (4.20) \]

\[ H^{\text{matter}}_{(6)} = \sum_b \left( -\frac{1}{4} \phi^{(2)}_b \frac{p_b^2}{m_b} - \lambda_8 p_b \cdot (a_b \times \partial) Q_b^{ij} \partial_i \partial_j + \lambda_1 \frac{p_b^2}{2m_b} Q_b^{ij} \partial_i \partial_j \right. \]
\[ \left. + \lambda_2 \frac{m_b}{m_b} (p_b \cdot \partial) Q_b^{ij} p_b \partial_j \right) \delta_b, \quad (4.21) \]

\[ H^{\text{matter}}_{(8)} = \sum_b \left( -\frac{1}{4} \phi^{(4)}_b \frac{p_b^2}{m_b} + \frac{5}{64} \phi^{(2)}_b \frac{p_b^2}{m_b} \right) \delta_b, \quad (4.22) \]

\[ H^{\text{matter}}_{(3)i} = \sum_b \left( p_{bi} - \frac{1}{2} m_b (a_b \times \partial) \right) \delta_b, \quad (4.23) \]

\[ H^{\text{matter}}_{(3)i} = -2 \sum_b \left[ Q_b^{ij} \left( \lambda_5 p_{kk} \partial_i \partial_l + \lambda_6 p_{bi} \partial_k \partial_l + \lambda_7 (p_b \cdot \partial) \delta_i \partial_k \right) \right. \]
\[ \left. + \lambda_2 \alpha^2_b (p_b \cdot \partial) \delta_i + \frac{m_b}{24} (a_b \times \partial)_i Q_b^{kl} \partial_k \partial_l \right] \delta_b, \quad (4.24) \]

and

\[ \phi^{(2)} = 4G \sum_b \frac{m_b}{r_b}, \quad (4.26) \]

\[ \phi^{(4)} = 4G \sum_b \left[ -\frac{m_b}{2} Q_b^{ij} \partial_i \partial_j + \frac{p_b^2}{2m_b} + \frac{1}{2} p_b \cdot (a_b \times \partial) \right] \frac{1}{r_b} - 2G^2 \sum_{a \neq b} \frac{m_a m_b}{r_{ab} r_b}, \quad (4.27) \]

\[ \phi_{(6)}^{I} = -\Delta^{-1} \phi_{(6)}^{I} \]
\[ = 4G \sum_b \left( \lambda_8 p_b \cdot (a_b \times \partial) Q_b^{ij} \partial_i \partial_b j + \lambda_1 \frac{p_b^2}{2m_b} Q_b^{ij} \partial_i \partial_b j + \lambda_2 \frac{m_b}{m_b} (p_b \cdot \partial) Q_b^{ij} p_b \partial_j \right) \frac{1}{r_b} \]
\[ - \sum_{a \neq b} 4G^2 \frac{m_a}{m_b} \frac{p_b^2}{r_{ab} r_b}, \quad (4.28) \]

\[ \phi_{(6)}^{II} = -\Delta^{-1} \left[ -\frac{1}{8} \left( H^{\text{matter}}_{(4)} \phi^{(2)} + H^{\text{matter}}_{(2)} \phi^{(4)} \right) \right], \quad (4.29) \]

\[ \phi_{(6)}^{III} = -\Delta^{-1} \left( \frac{1}{64} H^{\text{matter}}_{(2)} \phi^{(4)} \right), \quad (4.30) \]

\[ \phi_{(6)}^{IV} = -\Delta^{-1} \left( \tilde{\pi}_{(3)}^{ij} \right)^2, \quad (4.31) \]

\[ \tilde{\pi}_{(3)}^{ij} = \Theta_{(5)}^{ij} \left( -\frac{1}{2} H_{(5)k} \right), \quad (4.32) \]

\[ \tilde{\pi}_{(5)}^{ij} = \Theta_{(5)}^{ij} \left( -\frac{1}{2} H_{(5)(3)}^{ij} - \frac{1}{2} \phi^{(2)} \tilde{\pi}_{(3)}^{kl} \right) \quad \text{with} \]
\[ \Theta_{(5)}^{ij} = \left( -\frac{1}{2} \delta_{i} \partial_k \partial_b  + \delta_{ik} \partial_j + \delta_{jk} \partial_i - \frac{1}{2} \partial_i \partial_j \partial_k \Delta^{-1} \right) \Delta^{-1}. \quad (4.34) \]

We do not need to know \( \tilde{\pi}_{(3)}^{ij} \) and \( \tilde{\pi}_{(5)}^{ij} \) in full detail as we can apply partial integrations within our used analytical regularization procedure. The integrals are calculated with the aid of the same methods and regularization formulas as outlined in, e.g. [3]. The results are (the symbol \( \simeq \) indicates that only the relevant contributions are given).
\[ H_{(8)} = \int \mathcal{H}_{(8)} \, d^3 x \simeq G \sum_{a \neq b} \frac{1}{2} m_a p_b^2 Q^{ij} \partial_a \partial_{aj} \frac{1}{r_{ab}}, \]  
(4.35)

\[ H_{(8)2} = \int -\frac{1}{8} \mathcal{H}_{(6)} \phi_{(2)} \, d^3 x \]
\[ \simeq -\frac{G}{2} \sum_{a \neq b} m_a \left( \lambda_a p_b \cdot (a_b \times \partial_b) \right) Q^{ij} \partial_{ai} \partial_{bj} + \lambda_1 \frac{p_b^2}{2m_b} Q^{ij}_{ab} \partial_{ai} \partial_{bj} + \frac{\lambda_2}{m_b} (p_b \cdot \partial_b) Q^{ij}_{ab} \partial_{ai} \partial_{bj} \]
\[ + \frac{\lambda_3}{m_b} a^2_b (p_b \cdot \partial_b)^2 \frac{1}{r_{ab}}, \]  
(4.36)

\[ H_{(8)3} = \int -\frac{1}{8} \mathcal{H}_{(2)} \phi_{(6)} \, d^3 x = \int -\frac{1}{8} \mathcal{H}_{(2)} \left( \phi^I_{(6)} + \phi^II_{(6)} + \phi^III_{(6)} + \phi^IV_{(6)} \right) \, d^3 x, \]  
(4.37)

\[ H^I_{(8)3} = H_{(8)2}, \]  
(4.38)

\[ H^II_{(8)3} = \mathcal{O} \left( G^2 \right), \]  
(4.39)

\[ H^III_{(8)3} = \mathcal{O} \left( G^2 \right), \]  
(4.40)

\[ H^IV_{(8)3} = \int \frac{1}{8} \Delta \phi_{(2)} \phi_{(6)} \, d^3 x = \int \frac{1}{8} \phi_{(2)} \left( \pi_{(3)}^{ij} \right)^2 \, d^3 x, \]  
(4.41)

\[ H_{(8)4} = \int -\frac{1}{8} \mathcal{H}_{(4)} \phi_{(4)} \, d^3 x \]
\[ \simeq \frac{1}{4} \sum_{a \neq b} \frac{m_a}{m_b} p_b^2 Q^{ij}_{a} \partial_{ai} \partial_{aj} \frac{1}{r_{ab}} + \mathcal{O} \left( G^2 \right), \]  
(4.42)

\[ H_{(8)5} \simeq \int \frac{1}{64} \mathcal{H}_{(4)} \phi_{(2)}^2 \, d^3 x = \mathcal{O} \left( G^2 \right), \]  
(4.43)

\[ H_{(8)s} = \int \frac{1}{8} \phi_{(2)} \left( \pi_{(3)}^{ij} \right)^2 \, d^3 x = -H^IV_{(8)3}, \]  
(4.44)
\[ H_{(8)9} = \int 2 \pi \varpi^3 \varpi^{(3)} \tilde{z}^{ij} d^3 x \]

\[
\simeq \frac{G}{m_1^2 r_{12}^3} \left( \lambda_1 - \frac{1}{3} \lambda_5 - \frac{1}{3} \lambda_7 \right) \left[ 6 S_1^2 (p_1 \cdot p_2) - 18 S_1^2 (p_1 \cdot n_{12}) (p_2 \cdot n_{12}) \right] \\
+ \frac{G \lambda_5}{m_1^2 r_{12}^3} \left[ 6 (S_1 \cdot p_1) (S_1 \cdot p_2) - 18 (S_1 \cdot n_{12}) (p_2 \cdot n_{12}) (S_1 \cdot p_1) \right] \\
+ \frac{G \lambda_6}{m_1^2 r_{12}^3} \left[ -15 (p_1 \cdot n_{12}) (p_2 \cdot n_{12}) (S_1 \cdot n_{12})^2 - 21 (p_1 \cdot p_2) (S_1 \cdot n_{12})^2 \\
+ 6 (p_2 \cdot n_{12}) (S_1 \cdot n_{12}) (p_1 \cdot S_1) + 6 (p_1 \cdot n_{12}) (S_1 \cdot n_{12}) (p_2 \cdot S_1) \\
- 2 (p_1 \cdot S_1) (p_2 \cdot S_1) + 3 (p_1 \cdot n_{12}) (p_2 \cdot n_{12}) S_1^2 + 7 (p_1 \cdot p_2) S_1^2 \right] \\
+ \frac{G \lambda_7}{m_1^2 r_{12}^3} \left[ -15 (p_1 \cdot n_{12}) (p_2 \cdot n_{12}) (S_1 \cdot n_{12})^2 + 3 (p_1 \cdot p_2) (S_1 \cdot n_{12})^2 \\
+ 6 (p_2 \cdot n_{12}) (S_1 \cdot n_{12}) (p_1 \cdot S_1) - 18 (p_1 \cdot n_{12}) (S_1 \cdot n_{12}) (p_2 \cdot S_1) \\
+ 6 (p_1 \cdot S_1) (p_2 \cdot S_1) + 3 (p_1 \cdot n_{12}) (p_2 \cdot n_{12}) S_1^2 - (p_1 \cdot p_2) S_1^2 \right] \\
- \frac{G \lambda_6}{3m_1^2 r_{11}^3} \left[ -2 S_1^2 (p_1 \cdot p_2) + 6 S_1^2 (p_1 \cdot n_{12}) (p_2 \cdot n_{12}) \right] \\
+ \frac{G \lambda_6}{m_1^2 r_{12}^3} \left( -60 (S_1 \cdot n_{12})^2 n_{12} \cdot (p_1 \times S_2) + 12 S_1^2 n_{12} \cdot (p_1 \times S_2) + 24 (S_1 \cdot n_{12}) p_1 \cdot (S_2 \times S_1) \right) \\
+ \frac{G \lambda_7}{m_1^2 r_{12}^3} \left( -60 (p_1 \cdot n_{12}) (S_1 \cdot n_{12}) n_{12} \cdot (S_1 \times S_2) + 12 (p_1 \cdot S_1) n_{12} \cdot (S_1 \times S_2) \right. \\
\left. + 12 (S_1 \cdot n_{12}) p_1 \cdot (S_1 \times S_2) \right) \\
+ (1 \leftrightarrow 2).
\]

We conclude

\[ H_{(8)1} + H_{(8)4} = \frac{3 G}{4 m_1 m_2} \rho_2^2 \left( 3 (S_1 \cdot n_{12})^2 - S_1^2 \right) \right] + (1 \leftrightarrow 2), \]

\[ H_{(8)2} + H_{(8)3}^{I} = -\frac{G m_2}{m_1} \left[ \lambda_1 \frac{p_1^2}{2} (S_1 \cdot n_{12})^2 - S_1^2 \right] + (1 \leftrightarrow 2), \]

\[ H_{(8)3}^{IV} + H_{(8)8} = 0. \]

The coefficient equations following from our considerations read
\[ \alpha_1 = \frac{m_2}{m_1^2} \lambda_2, \quad \alpha_2 = \frac{m_2}{2m_1^3} \left( \lambda_1 + 2 \lambda_3 - \frac{2}{3} \lambda_2 \right), \quad \alpha_3 = -3 \frac{m_2}{m_1^2} \left( \lambda_3 - \frac{1}{3} \lambda_2 \right), \]

(4.50)

\[ \alpha_4 = -\frac{3}{2} \lambda_1 \frac{m_2}{m_1^3}, \quad \alpha_5 = 0, \quad \alpha_6 = -3 \frac{m_2}{m_1^2}, \]

(4.51)

\[ \beta_1 = \beta_3 = \beta_5 = \beta_6 = 0, \quad \beta_2 = -\frac{3}{4m_1 m_2}, \quad \beta_4 = \frac{9}{4m_1 m_2}. \]

(4.52)

\[ \gamma_1 = \frac{1}{m_1^2} \left( 6 \lambda_4 - 2 \lambda_5 - 3 \lambda_7 + \frac{23}{3} \lambda_6 \right), \]

(4.53)

\[ \gamma_2 = \frac{1}{m_1^2} (-21 \lambda_6 + 3 \lambda_7), \]

(4.54)

\[ \gamma_3 = \frac{1}{m_1^2} (6 \lambda_5 - 2 \lambda_6 + 6 \lambda_7), \]

(4.55)

\[ \gamma_4 = \frac{1}{m_1^2} (6 \lambda_6 - 18 \lambda_7), \]

(4.56)

\[ \gamma_5 = \frac{1}{m_1^2} (-18 \lambda_5 + 6 \lambda_6 + 6 \lambda_7), \]

(4.57)

\[ \gamma_6 = \frac{1}{m_1^2} (-18 \lambda_4 + 6 \lambda_5 + 9 \lambda_7 + \lambda_6), \]

(4.58)

\[ \gamma_7 = \frac{1}{m_1^2} (-15 \lambda_6 - 15 \lambda_7), \]

(4.59)

\[ \mu_1 = -3 \frac{m_2}{m_1^2} \lambda_8, \quad \mu_2 = 15 \frac{m_2}{m_1^2} \lambda_8, \quad \mu_3 = \frac{12}{m_1^2} \lambda_6, \]

(4.60)

\[ \mu_4 = \frac{1}{m_1^2} (24 \lambda_6 - 12 \lambda_7), \quad \mu_5 = -\frac{60}{m_1^2} \lambda_6, \quad \mu_8 = \frac{12}{m_1^2} \lambda_7, \quad \mu_9 = -\frac{60}{m_1^2} \lambda_7, \]

(4.61)

\[ \mu_6 = \mu_7 = 0. \]

(4.62)

Now we make use of the Eqs. (3.1) to (3.22) and end up with the solution for all the coefficients still having 2 degrees of freedom left parametrized by \( \nu_1 \) and \( \nu_3 \):

\[ \alpha_1 = \frac{m_2}{m_1^2} \left( -\frac{7}{4} - \nu_1 \right), \quad \alpha_2 = \frac{m_2}{m_1^2} \left( \frac{5}{4} + \nu_5 \right), \quad \alpha_3 = \frac{m_2}{m_1^2} \left( -\frac{27}{8} - 3 \nu_5 \right), \quad \alpha_4 = \frac{3 m_2}{8 m_1^2}, \quad \alpha_5 = 0, \]

(4.63)

\[ \alpha_6 = \frac{m_2}{m_1^2} \left( \frac{21}{4} + 3 \nu_1 \right), \]

(4.64)

\[ \beta_1 = \beta_3 = \beta_5 = \beta_6 = 0, \quad \beta_2 = -\frac{3}{4m_1 m_2}, \quad \beta_4 = \frac{9}{4m_1 m_2}. \]

(4.65)

\[ \gamma_1 = \frac{1}{m_1^2} (-1 - \nu_5), \quad \gamma_2 = -\frac{9}{4m_1^2}, \quad \gamma_3 = \frac{1}{m_1^2} (2 + \nu_1), \quad \gamma_4 = -\frac{3}{2m_1^2}, \quad \gamma_5 = \frac{1}{m_1^2} (-3 - 3 \nu_1), \]

(4.66)

\[ \gamma_6 = \frac{1}{m_1^2} \left( \frac{9}{2} + 3 \nu_5 \right), \quad \gamma_7 = -\frac{15}{4m_1^2}, \]

(4.67)

\[ \mu_1 = \frac{m_2}{4m_1}, \quad \mu_2 = -\frac{5 m_2}{4 m_1}, \quad \mu_3 = \frac{3}{2 m_1}, \quad \mu_4 = \frac{3}{2 m_1}, \quad \mu_5 = -\frac{15}{2 m_1}, \quad \mu_8 = \frac{3}{2 m_1}, \quad \mu_9 = -\frac{15}{2 m_1}, \]

(4.68)

\[ \mu_6 = \mu_7 = 0, \]

(4.69)

\[ \nu_2 = \frac{3}{4}, \quad \nu_3 = \frac{3}{4}, \quad \nu_4 = -\frac{1}{2} - \nu_5 \]

(4.70)

\[ \lambda_1 = \frac{1}{4}, \quad \lambda_2 = -\frac{7}{4} - \nu_1, \quad \lambda_3 = \frac{13}{24} - \frac{\nu_1}{3} + \nu_5, \quad \lambda_4 = -\frac{7}{72} + \frac{\nu_1}{18} - \frac{\nu_5}{6}, \quad \lambda_5 = \frac{1}{4} + \frac{\nu_1}{6}, \quad \lambda_6 = \frac{1}{8}, \]

(4.71)

\[ \lambda_7 = \frac{1}{8}, \quad \lambda_8 = -\frac{1}{12}. \]

(4.72)
It is worthy to point out, that the free parameters $\nu_1$ and $\nu_5$ are related to a canonical transformation which is given by the generator

$$g^{(\nu_1)} = \frac{Gm_2}{m_1 r_1^2} \left( -\nu_5 S_1^2 (p_1 \cdot n_{12}) + \nu_1 (S_1 \cdot n_{12}) (p_1 \cdot S_1) \right).$$

(4.73)

Refering to Eq. (4.8), this means that we may choose $\sigma_1 = -\frac{m_2}{m_1} \nu_5$, $\sigma_3 = \frac{m_2}{m_1} \nu_1$, and $\sigma_2 = \sigma_4 = \sigma_5 = \sigma_6 = 0$ to generate the terms related with these coefficients. So all Hamiltonians parametrized by $\nu_1$ and $\nu_5$ are canonically equivalent and we are free to give them any value. To fixate them we need to calculate $G_{S_1^2}$ explicitly. We use Eq. (4.8) and as a matter source for the static case we adopt the result from [15], Eq. (4), where a static source expression for a black-hole binary has been derived for the next-to-leading order spin-squared terms:

$$H_{S_1^2, \text{matter}} = -\frac{1}{2m_1} \left( P_{ij} \delta_i \right)_{ij} + \frac{c_3}{m_1^2} S_1^2 \left( \gamma^{ij} \delta_i \right)_{ij} + \frac{1}{8m_1^2} g_{mn} \gamma^{ij} \gamma^{kl} \gamma^{jm} \gamma^{in} \delta_i \delta_j \delta_k \delta_l$$

$$+ \frac{1}{4m_1} \left( \gamma^{ij} \gamma^{mn} \gamma^{kl} S_1^1 \delta_i \delta_k \right)_{ij}$$

(4.74)

with, in the present approximation, $P_{ij} = m_1^2 \gamma^{ij} Q_1^{ij}$ and $S_1^{ij} = \epsilon_{ijk} S_1^{jk}$. Notice that contributions arising from the $c_3$ source term cancel each other, which is very nice, because $c_3$ could not be determined and does also not contribute to the $G_{S_1^2}$ Hamiltonian. The explicit calculation then yields $\nu_1 = -2$, $\nu_2 = 3/4$, $\nu_3 = 3/4$, $\nu_4 = 3/4$, and $\nu_5 = -5/4$. This result is fully consistent with the equations (4.70), which were independently obtained by the Poincaré algebra. Now all of the coefficients of the $H_{S_1^2, \text{matter}}$ Hamiltonian and of the source terms have been fixed. They read

$$\alpha_1 = \frac{m_2}{4m_1^2}, \ \alpha_2 = 0, \ \alpha_3 = \frac{3m_2}{8m_1^2}, \ \alpha_4 = -\frac{3m_2}{8m_1^2}, \ \alpha_5 = 0, \ \alpha_6 = -\frac{3m_2}{4m_1^2},$$

$$\beta_1 = \beta_3 = \beta_5 = \beta_6 = 0, \ \beta_2 = -\frac{3}{4m_1 m_2}, \ \beta_4 = \frac{9}{4m_1 m_2},$$

$$\gamma_1 = \frac{3}{4m_1}, \ \gamma_2 = -\frac{9}{4m_1}, \ \gamma_3 = 0, \ \gamma_4 = -\frac{3}{2m_1}, \ \gamma_5 = \frac{3}{m_1}, \ \gamma_6 = \frac{3}{4m_1}, \ \gamma_7 = -\frac{15}{4m_1}.$$ 

(4.75)

(4.76)

(4.77)

$$\lambda_1 = \frac{1}{4}, \ \lambda_2 = \frac{1}{4}, \ \lambda_3 = -\frac{1}{24}, \ \lambda_4 = 0, \ \lambda_5 = -\frac{1}{12}, \ \lambda_6 = \frac{1}{8}, \ \lambda_7 = \frac{1}{8}, \ \lambda_8 = -\frac{1}{12}.$$ 

(4.78)

It may be interesting to mention that in the ADMTT representation the $\lambda_4$ source term is not present. Such a term emerges in [16] with the values $\pm \frac{1}{12}$ and with $a^2$ replaced by the square of the radius of the throat of a nonrotating black hole resulting from inversion symmetry at the throat. In Ref. [17] it was shown that a factor $a^2$ in the $\lambda_4$ term can also be generated by a specific deviation from the point-mass structure of a spherically symmetric body.

V. THE TEST-PARTICLE LIMIT

Next we will consider the test-particle limit of the energy of a black-hole binary up to the 4th order in $1/r_{12}$. To do this we plug into Eq. (5.2) of [18] the static ADM-gauged Kerr metric from Ref. [8] labeled by particle number 1. The mass and momentum explicitly appearing in this equation will then get the (test-)particle number 2:

$$-p_0 = -\gamma^{ij} N_{1i} p_{2j} + N_1 \left( m_2^2 + \gamma^{ij} p_{2i} p_{2j} \right)^{1/2}.$$ 

(5.1)

The inverse 3-metric $\gamma^{ij}$ in ADMTT gauge up to the order $1/r^4$ reads (G=1)

$$\gamma^{ij} = \left( 1 - \frac{2m}{r} + \frac{5m^2}{2r^2} - \frac{5m^3}{2r^3} + \frac{35m^4}{16r^4} - \frac{m a^2 - 3m (a \cdot n)^2}{r^3} + \frac{7m^2 a^2}{2r^4} - \frac{9m^2 (a \cdot n)^2}{r^4} \right) \delta_{ij} - h^{TT}_{ij}. $$

(5.2)

The result of this operation is
\[-p_0 = m_2 + \frac{p_2^2}{2m_2} - \frac{m_1 m_2}{r_{12}} - \frac{3}{2} \frac{m_1 p_2^2}{m_2 r_{12}} + \frac{1}{2r_{12}^2} \left( m_2 m_2^2 + 5 \frac{m_1^2}{m_2} p_2^2 - 4 m_1 a_1 \cdot (n_{12} \times p_2) \right) + \frac{1}{r_{12}^2} \left( -\frac{1}{4} m_2 m_1^3 - \frac{25 m_1^3}{8 m_2} p_2^2 + \frac{3}{2} m_1 m_2 (a_1 \cdot n_{12})^2 + \frac{9 m_1}{4 m_2} p_2^2 (a_1 \cdot n_{12})^2 - \frac{1}{2} m_1 m_2 a_1^2 - \frac{3}{2} \frac{m_1 p_2^2}{m_2} a_1^2 \right. \\
+ 6 m_1^2 a_1 \cdot (n_{12} \times p_2) \right) + \frac{1}{8} \frac{m_1^4}{m_2^2} + \frac{105 m_1^4}{32 m_2} p_2^2 - \frac{9}{2} \frac{m_2 m_1^2 (a_1 \cdot n_{12})^2}{m_2} + \frac{21}{2} \frac{m_1^3}{m_2} (p_2 \cdot n_{12})^2 (a_1 \cdot n_{12})^2 - \frac{53 m_1^2}{4 m_2} p_2^2 (a_1 \cdot n_{12})^2 - \frac{7}{4} m_2 m_1^2 a_1^2 - \frac{7}{2} m_2 (p_2 \cdot n_{12})^2 a_1^2 + \frac{23}{4} m_1^3 p_2^2 a_1^2 + \frac{21}{2} m_1^3 a_1 \cdot (n_{12} \times p_2) \\
+ 5 m_1 (a_1 \cdot n_{12})^2 a_1 \cdot (n_{12} \times p_2) - m_1 a_1^2 a_1 \cdot (n_{12} \times p_2) \right) \right).
\[
(5.3)
\]

Now, from this limit we can also read off the coefficients \( \beta_2 \) and \( \beta_4 \) of Eq. (2.22) in the ADM scheme,
\[
\beta_2 = -\frac{3}{4m_1 m_2},
\]
\[
(5.4)
\]
\[
\beta_4 = \frac{9}{4m_1 m_2},
\]
\[
(5.5)
\]
in confirmation with the previously found ones. This is very strong evidence that we are on the right track with our method.

VI. THE 2PN QUARTIC SPIN HAMILTONIAN

To see that there is no \( S^4 \) Hamiltonian at the order \( 1/c^4 \), we expand the Kerr metric in ADMTT coordinates up to the orders \( 1/r^4 \) and \( a^4 \) and show that such terms are not present at all and therefore cannot follow from a source with purely \( a^4 \) terms. The strategy is the same as in [6], but now we allow \( a^4 \) terms and take the Kerr metric in quasi-isotropic coordinates of Eq. (43) in [5] up to the aforementioned order and transform to ADMTT coordinates according to the formula
\[
\gamma^{ADM}_{ij} = \gamma^{qiso}_{ij} + \gamma^{qiso}_{ik} s_{kj} + \gamma^{qiso}_{jk} s_{ki} + \gamma^{qiso}_{ij,k} s_k
\]
\[
(6.1)
\]
with the 3-metric \( \gamma^{qiso}_{ij} \)
\[
\gamma^{qiso}_{ij} = \gamma^{(s)}_{ij} + \gamma^{(2)}_{ij} + \gamma^{(3)}_{ij} + \gamma^{(4)}_{ij},
\]
\[
(6.2)
\]
\[
\gamma^{(4)}_{ij} = \gamma^{(s)}_{ij} + \frac{1}{16} \frac{a^4}{r^4} b_{ij},
\]
\[
(6.3)
\]
and the extended transformation vector
\[
\xi^k = -\frac{1}{4} \frac{a^2 n^k}{r} + \frac{1}{2} \frac{(a \cdot n) a^k}{r} - \frac{7}{16} \frac{m^2 a^2 n^k}{r^3} + \frac{7}{4} \frac{m^2 (a \cdot n)^2 n^k}{r^3} - \frac{1}{16} \frac{a^2 n^k}{r^3} + \frac{1}{4} \frac{a^2 (a \cdot n) a^k}{r^3} - \frac{1}{2} \frac{(a \cdot n)^2 a^k}{r^3}.
\]
\[
(6.4)
\]
We end up with a metric being the same as Eqs. (50) and (51) in [3] having all \( a^4 \) terms transformed away. This leads to the conclusion
\[
H_{S_1^4} = H_{S_2^4} = 0
\]
\[
(6.5)
\]
for the quartic nonlinearities in the spin for the Hamiltonians linear in \( G \).
VII. CONCLUSIONS

The Hamiltonians we obtained are summarized as follows:

\[
H_{S_{1}p_1} = \frac{G}{r_{12}^4} \left[ S_1 \cdot (n_{12} \times p_1) \left( \frac{m_2}{4m_1^2} S_1^2 - \frac{5m_2}{4m_1} (S_1 \cdot n_{12})^2 \right) \right],
\]

(7.1)

\[
H_{S_{2}p_1} = \frac{G}{r_{12}^4} \left[ \frac{3}{2m_1^2} S_1 S_2 \cdot (n_{12} \times p_1) + \frac{3}{2m_1^2} (S_1 \cdot n_{12}) S_2 \cdot (S_1 \times p_1) - \frac{15}{2m_1^2} (S_1 \cdot n_{12})^2 S_2 \cdot (n_{12} \times p_1) + n_{12} \cdot (S_1 \times S_2) \left( \frac{3}{2m_1^2} S_1 \cdot p_1 - \frac{15}{2m_1^2} (S_1 \cdot n_{12}) (p_1 \cdot n_{12}) \right) \right],
\]

(7.2)

\[
H_{S_{1}p_2} = \frac{G}{r_{12}^4} \left[ \frac{m_2}{4m_1^3} (p_1 \cdot S_1)^2 + \frac{3m_2}{8m_1^2} (p_1 \cdot n_{12})^2 S_1^2 - \frac{3m_2}{8m_1^2} p_1^2 (S_1 \cdot n_{12})^2
\]

\[- \frac{3m_2}{4m_1^2} (p_1 \cdot n_{12}) (S_1 \cdot n_{12}) (p_1 \cdot S_1) - \frac{3}{4m_1 m_2} p_2^2 S_1^2 + \frac{9}{4m_1 m_2} p_2^2 (S_1 \cdot n_{12})^2
\]

\[+ \frac{3}{4m_1^2} (p_1 \cdot p_2) S_1^2 - \frac{9}{4m_1^2} (p_1 \cdot p_2) (S_1 \cdot n_{12})^2 - \frac{3}{2m_1^2} (p_1 \cdot n_{12}) (p_2 \cdot S_1) (S_1 \cdot n_{12})
\]

(7.3)

\[+ \frac{3}{m_1^2} (p_2 \cdot n_{12}) (p_1 \cdot S_1) (S_1 \cdot n_{12}) + \frac{3}{4m_1^2} (p_1 \cdot n_{12}) (p_2 \cdot n_{12}) S_1^2
\]

\[- \frac{15}{4m_1^2} (p_1 \cdot n_{12}) (p_2 \cdot n_{12}) (S_1 \cdot n_{12})^2 \right],
\]

plus the ones with interchanged indices \((1 \leftrightarrow 2)\), and

\[
H_{S_{1}^4} = H_{S_{2}^4} = 0.
\]

(7.4)

The corresponding sources in the constraint equations take the form

\[
\mathcal{H}_{\text{matter}} = \sum_b \left[ -\frac{m_b}{2} Q_b^{ij} \partial_i \partial_j - \frac{1}{2} p_b \cdot (a_b \times \partial) + \left( \gamma^{ij} p_{bi} p_{bj} + m_b^2 \right)^{1/2} + \frac{1}{8} \frac{p_b^2}{m_b} Q_b^{ij} \partial_i \partial_j
\]

\[+ \frac{1}{4m_b} (p_b \cdot \partial) Q_b^{ij} p_{bi} p_{bj} - \frac{1}{4m_b} a_b^2 (p_b \cdot \partial)^2 + \frac{1}{12} p_b \cdot (a_b \times \partial) Q_b^{ij} \partial_i \partial_j \right] \delta_b,
\]

(7.5)

\[
\mathcal{H}_{i \text{matter}} = -2 \sum_b \left[ Q_b^{ij} \left( -\frac{1}{12} p_{bk} \partial_i \partial_l + \frac{1}{8} p_{bi} \partial_k \partial_l + \frac{1}{8} (p_b \cdot \partial) \delta_{li} \delta_{kj} \right) + \frac{m_b}{4} (a_b \times \partial)_i \left( 1 - \frac{1}{6} Q_b^{ij} \partial_j \partial_k \right) - \frac{1}{5} p_{bi} \right] \delta_b.
\]

(7.6)

The only missing Hamiltonians at the formal order of \(1/c^4\) are the \(G^2\) Hamiltonians \(H_{S_{3}}\) and \(H_{S_{2}}\) resulting from gravitational nonlinearities of the Einstein field equations. These Hamiltonians will be presented in a forthcoming paper [15].
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