Abstract. For a directed graph $E$, we compute the $K$-theory of the $C^*$-algebra $C^*(E)$ from the Cuntz-Krieger generators and relations. First we compute the $K$-theory of the crossed product $C^*(E) \times_\gamma T$, and then using duality and the Pimsner-Voiculescu exact sequence we compute the $K$-theory of $C^*(E) \otimes K \cong (C^*(E) \times T) \times Z$. The method relies on the decomposition of $C^*(E)$ as an inductive limit of Toeplitz graph $C^*$-algebras, indexed by the finite subgraphs of $E$. The proof and result require no special assumptions about the graph, and is given in graph-theoretic terms. This can be helpful if the graph is described by pictures rather than by a matrix.

1. Introduction

Since the work of Bratteli in the early 1970’s, graphs have been used as a tool to study a large class of $C^*$-algebras. Bratteli classified AF algebras in terms of their diagrams, later called Bratteli diagrams (3). The current use of directed graphs in $C^*$-algebras goes back to the work of Cuntz and Krieger in [6]. In that work, they associated a $C^*$-algebra to a finite irreducible 0-1 matrix.

Later, it was noticed that if $A = (a_{ij})$ is an $n \times n$ matrix of 0’s and 1’s, then $A$ may be viewed as the incidence matrix of a graph. It then became natural to view Cuntz-Krieger algebras as arising from the graphs. This approach of viewing Cuntz-Krieger algebras as $C^*$-algebras associated to graphs made the construction more visual and communicable.

In [14], Kumjian, Pask, Raeburn and Renault defined the graph groupoid of a countable row-finite directed graph with no sinks, and showed that the $C^*$-algebra of this groupoid coincided with a universal $C^*$-algebra generated by partial isometries satisfying relations naturally generalizing those given in [6]. Since that time, many people have worked on generalizing these results to arbitrary directed graphs (and beyond — for a survey, see [13]). In [15], an approach to the general case is given that results in a direct limit decomposition of the $C^*$-algebra of a general graph, over the directed set of its finite subgraphs. This work motivates the current paper.

Cuntz and Krieger computed the $K$-theory of their $C^*$-algebra associated to an irreducible matrix, and showed that it is an invariant of flow equivalence of the matrix. Since then several proofs have been given for the computation of the $K$-theory of the $C^*$-algebra of a directed graph ([7] 8 9 12 14 16 18 20). Most of these gave the proof for a restricted class of graphs, e.g. row-finite and/or sourceless (or, in the case of [18], for graphs having a finite vertex set). Proofs of the general case occur in [7] 9. In this paper we give a proof is simpler than [9] (that paper treats topological graphs), and does not rely on the row-finite case as does [7]. We follow the general strategy of [12], first computing the $K$-theory of the AF algebra $C^*(E) \times_\gamma T$, where $\gamma$ is the gauge action. We do this by using the decomposition of $C^*(E)$ as a direct limit. Then we give a fairly simple account of the algebra involved in using the Pimsner-Voiuculescu exact sequence to compute the $K$-theory of $C^*(E)$. The formula we give for the $K$-theory of the stable AF core $C^*(E) \times_\gamma T$ is new, we believe, as is its proof. (A different formula was given in [12] in the row-finite case.) We emphasize that in our treatment, no restrictions of any kind are made about row-finiteness, sources and sinks, and cardinality of the graph (the results in [15] do not require countability of the vertex and edge sets).

The outline of the paper is as follows. In section 2 we provide the basic definitions of graph $C^*$-algebras. In section 3 we compute the $K$-theory of $C^*(E) \times_\gamma T$, and in section 4, that of $C^*(E)$.

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2. Preliminaries

The paper [15] is a reference for the remarks in this section. (The survey [13] is excellent. However, unlike that survey, we follow the original convention for graph algebras: the vertex at the tip of an arrow corresponds to the initial projection of the partial isometry corresponding to that arrow.) A directed graph $E = (E^0, E^1, o, t)$ consists of sets $E^0$ of vertices and $E^1$ of edges, and maps $o, t : E^1 \to E^0$ identifying the origin and terminus of an edge (when an edge is pictured as an arrow between two vertices, the terminus is the vertex to which it points). A vertex $x$ is called a sink if $o^{-1}(x) = \emptyset$, a source if $t^{-1}(x) = \emptyset$, and non-singular if $o^{-1}(v)$ is a finite nonempty set. A path is a sequence $e_1 e_2 \cdots e_n$ of edges satisfying $t(e_i) = o(e_{i+1})$ for each $i = 1, \ldots, n - 1$. For a path $\mu = e_1 e_2 \cdots e_n$, we define $o(\mu) = o(e_1)$, $t(\mu) = t(e_n)$, and the length, $\ell$, of $\mu$ by $\ell(\mu) = n$. We regard vertexes as paths of length zero. Let $E^j$ denote the set of paths of length $j$, and put $E^\ast = \bigcup_{j=0}^\infty E^j$, the path space of the graph. For $x, y \in E^0$, let $x E^j$, $E^j y$, and $x E^j y$ denote the sets of paths of length $j$ with origin $x$, with terminus $y$, or both, respectively.

Let $E$ be a directed graph. A Cuntz-Krieger $E$-family consists of mutually orthogonal projections $\{s_v : v \in E^0\}$, and partial isometries $\{s_e : e \in E^1\}$, satisfying

1. $s_{t(e)} = s_e^* s_e$ for all $e \in E^1$.
2. $\sum_{e \in F} s_e s_e^* \leq s_v$ for any $v \in E^0$ and finite subset $F \subseteq v E^1$.
3. $\sum_{e \in v E^1} s_e s_e^* = s_v$ for each non-singular vertex $v \in E^0$.

The graph $C^\ast$-algebra is the $C^\ast$-algebra generated by a universal Cuntz-Krieger $E$-family. For a path $\mu = e_1 \cdots e_n$ we write $s_\mu = s_{e_1} \cdots s_{e_n}$. One easily checks from the relations that $s_{\mu}^* s_\mu = s_{t(\mu)}$, $s_\mu s_{\nu}^* \leq s_{o(\mu)}$ and that $s_{\mu}^* s_\mu = 0$ unless one of $\mu, \nu$ extends the other. In this case, e.g. if $\mu = \nu \alpha$, we have $s_{\mu}^* s_\mu = s_{\alpha}$. Therefore we find that

$$C^\ast(E) = \overline{\operatorname{span}}\{s_\mu s_\nu^* : \mu, \nu \in E^\ast \text{ and } t(\mu) = t(\nu)\}.$$

Our methods rely crucially on the $C^\ast$-subalgebras of $C^\ast(E)$ determined by subgraphs of $E$. These are termed relative Toeplitz graph algebras in [15], and we describe them here. Let $F$ be a subgraph of $E$; that is, $F^0 \subseteq E^0$, $F^1 \subseteq E^1$, and the origin and terminus maps for $F$ are the restrictions of those for $E$. We let $S_F$ denote the set of vertices $v$ of $F$ such that

1. $v$ is non-singular as a vertex of $E$.
2. $x F^1 = x E^1$.

The relative Toeplitz Cuntz-Krieger relations for $F$ and $S_F$ are the same as the Cuntz-Krieger relations for $F$ except that (3) is imposed only at vertices in $S_F$. The (relative) Toeplitz graph algebra, $TC^\ast(F)$, of $F$ is the $C^\ast$-algebra universal for the relative Toeplitz Cuntz-Krieger relations. It is shown in [15] that $TC^\ast(F) \subseteq C^\ast(E)$ in the obvious way. (We should indicate the dependence of the Toeplitz algebra on the choice of subset $S_F \subseteq F^0$, as in [15]; we omit it in this article.)

Given a directed graph $E$, let $\gamma : \mathbb{T} \to \operatorname{Aut}(C^\ast(E))$ be defined on the generators by $\gamma(z)(s_e) = zs_e$, $e \in E^1$. (Since $\{zs_e : e \in E^1\}$ is a Cuntz-Krieger $E$-family, this does define an automorphism.) Then we see that $\gamma(z)(s_\mu s_\nu^*) = z^{\ell(\mu) - \ell(\nu)} s_\mu s_\nu^*$ for any $\mu, \nu \in E^\ast$. $\gamma$ is called the gauge action, and $(C^\ast(E), \mathbb{T}, \gamma)$ is a $C^\ast$-dynamical system. It is a standard fact (see, e.g. [15]) that the crossed product algebra $C^\ast(E) \times_{\gamma} \mathbb{T}$ is AF. In the next section we compute the $K$-theory of $C^\ast(E) \times_{\gamma} \mathbb{T}$. In the last section, we use the Pimsner-Voiculescu exact sequence to compute the $K$-theory of $C^\ast(E) \otimes \mathcal{K} \cong (C^\ast(E) \times_{\gamma} \mathbb{T} \times_{\hat{\gamma}})$, where $\hat{\gamma}$ is the dual action.

For $n \in \mathbb{Z}$ we let $\zeta_n : \mathbb{T} \to \mathbb{T}$ be the $n$th character of $\mathbb{T}$: $\zeta_n(z) = z^n$. We note some basic computations in $C^\ast(E) \times_{\gamma} \mathbb{T}$. First, for $a \in C^\ast(E)$ we write $\zeta_n a$ for the element of $C^\ast(\mathbb{T}, C^\ast(E)) \subseteq C^\ast(E) \times_{\gamma} \mathbb{T}$ given by

$$\zeta_n a(z) = \zeta_n(z) a.$$

Thus $\{\zeta_n s_\mu s_\nu^*\}$ is a total set in $C^\ast(E) \times_{\gamma} \mathbb{T}$. We use $\cdot$ for multiplication in $C^\ast(E) \times_{\gamma} \mathbb{T}$. Thus, if $\mu, \nu, p, q \in E^\ast$, and $m, n \in \mathbb{Z}$, then

$$\zeta_n s_\mu s_p^* s_q^* \zeta_m s_\nu s_p s_q^* = \delta_{n,m+\ell(q)-\ell(p)} \zeta_m s_\mu s_p^* s_q^* s_\nu,$$

$$\zeta_m s_p s_q^* s_\nu = \zeta_{m+\ell(q)-\ell(p)} s_q^* s_p^* s_\nu.$$

The dual action of $\mathbb{Z}$ on $C^\ast(E) \times_{\gamma} \mathbb{T}$ is generated by $\hat{\gamma} \in \operatorname{Aut} (C^\ast(E) \times_{\gamma} \mathbb{T})$, where

$$\hat{\gamma}(\zeta_n s_\mu s_\nu^*) = \zeta_{n+1} s_p s_q^*.$$
3. The $K$-theory of $C^*(E) \times \gamma \mathbb{T}$

Let $M$ be the incidence matrix of $E$. Thus $M : E^0 \times E^0 \to \mathbb{N} \cup \{\infty\}$ is defined by requiring that $M(x, y)$ equal the cardinality of $x E^1 y$. We let $S$ be the set of non-singular vertices of $E$: $S = \{x \in E^0 : x E^1$ is finite and nonempty $\}$. For a subgraph $F$ of $E$ we let $M_F$ denote the incidence matrix of $F$, and we let $S_F = \{x \in S \cap F^0 : x E^1 = x E^1\}$.

**Definition 3.1.** We define two maps, $\alpha$ and $\beta$, as follows. Let $V = C_c(E^0 \times \mathbb{Z}, \mathbb{Z})$ and $W = C_c(S \times \mathbb{Z}, \mathbb{Z})$. Then $\alpha : V \to V$ is given by

$$(\alpha f)(x, n) = f(x, n - 1), \ f \in V,$$

and $\beta : W \to V$ is given by

$$(\beta f)(x, n) = \sum_{y \in S} M(y, x) f(y, n), \ f \in W.$$ 

Equivalently, we may write (for $x \in S$)

$$\beta(\delta_{x,n}) = \sum_{e \in x E^1} \delta_{l(e),n}.$$ 

Thus we may describe $\beta$ loosely by $\beta f = M^t f$. Note that $\alpha$ is an isomorphism of $V$, $\alpha(W) = W$, and $\alpha \circ \beta = \beta \circ \alpha$. We define $\Phi : V \to K_0(C^*(E) \times \mathbb{T})$ by $\Phi(\delta_{x,n}) = [\zeta_n s_x]$ (this defines $\Phi$ on a basis for $V$, and we extend to all of $V$ by linearity). Let $I = (1 - \alpha \beta)(W)$.

**Proposition 3.2.** $\ker(\Phi) = I$, and $\Phi$ is onto. (Thus $K_0(C^*(E) \times \mathbb{T}) \cong V/I$.)

**Proof.** First we show the equality. ($\supseteq$): Let $x \in S$. For $e \in x E^1$ we have

$$(\zeta_n s_x^*)_e \cdot \zeta_n s_x^e = \zeta_n s_x s_x^e,$$

and hence

$$\sum_{e \in x E^1} [\zeta_n s_{x+1} s_{l(e)}] = \sum_{e \in x E^1} [\zeta_n s_x s_x^e] = [\zeta_n s_x].$$

Therefore

$$\Phi(1 - \alpha \beta)(\delta_{x,n}) = [\zeta_n s_x] - \sum_{e \in x E^1} [\zeta_{n+1} s_{l(e)}] = 0.$$ 

($\subseteq$): Let $f \in \ker \Phi$. Note that $f \in I$ if and only if $\alpha(f) \in I$. Also $\Phi(f) = 0$ if and only if $\hat{\phi}(\Phi(f)) = 0$, i.e. if and only if $\Phi(\alpha(f)) = 0$. Thus we may assume that $f(x, i) = 0$ whenever $i < 0$. We intend to use this simplification to push $\Phi(f)$ into $K_0(C^*(E)^\gamma)$, since the AF structure of the fixed-point algebra is easier to deal with than that of $C^*(E) \times \mathbb{T}$. There is one more adjustment necessary for this.

Let $x \in E^0$, $i \geq 0$ be such that $f(x, i) \neq 0$. Recall that $[\zeta_i s_x] = [s_\mu s_{\mu}^e]$ for any path $\mu \in E^1 x$ (for such a path $\mu$, let $W = s_{\mu} \in C_c(T, C^*(E))$; then $W^*W = s_{\mu} s_{\mu}^* \in C_0(S)$, and $W^*W = [\zeta_i]_{K_0(T)}$). However, if $E$ has sources, there might not exist such a path. To get around this problem, consider a source $y \in E^0$. Let $D$ be the graph with $D^0 = E^0 \cup \{\omega\}$ and $D^1 = E^1 \cup \{\theta\}$, where $\omega \notin E^0$, $\phi(\theta) = \omega$, and $t(\omega) = y$. Then $C^*(E)$ is a full corner in $C^*(D)$, and hence the two algebras have the same $K$-theory. (This is easily seen by observing that $C^*(D) = C^*(E) + s_0 C^*(E) + C^*(E)s_0 + C_0$. Moreover, the same observation lets one deduce that $C^*(E) \times \mathbb{T}$ is a full corner in $C^*(D) \times \mathbb{T}$ as well. Thus we may replace $E$ by $D$ in our situation. Iterating this process allows us to assume that for any $(x, i) \in \text{supp}(f)$ there is a path $\mu \in E^1$ such that $[\zeta_i s_x] = [s_\mu s_{\mu}^e]$. (The removal of sources and sinks has a long history in the literature of graph $C^*$-algebras. One may add an infinite path leading to a source rather than just a few edges as we have done here.)

Next we choose a finite dimensional subalgebra of $C^*(E)^\gamma$ to work in. Let $F$ be a finite subgraph of $E$ with the following properties:

1. $\text{supp}(f) \subseteq F^0 \times \mathbb{Z}$.
2. For all $(x, i) \in \text{supp}(f)$ there is a path $\mu \in F^1 x$.
3. $\text{supp}(f) \cap (S \times \mathbb{Z}) \subseteq S_F \times \mathbb{Z}$.
4. $\sum_{x,i} f(x, i) [\zeta_i s_x]_{K_0((T C^*(F)) \times \mathbb{T})} = 0$. 


Hence, replacing the sum on \( \nu \) by the sum on \( y = t(\nu) \),

\[
\begin{align*}
\sum_{\nu \in \{t(\mu) : \nu \in F^0 \}} s_{\mu} s_{\mu}^* &= \sum_{j=0}^{k-\ell(\mu) - 1} \sum_{\nu \in F^0} M^j_F(t(\mu), y)[\xi_{t(\mu) + j} y] + \sum_{y \in F^0} M^{k-\ell(\mu)}_F(t(\mu), y)[\xi_k y] \\
&= \sum_{j=0}^{k-1} \sum_{\nu \in F^0} M^j_F(t(\mu), y)[\xi_j y] + \sum_{y \in F^0} M^{k-\ell(\mu)}_F(t(\mu), y)[\xi_k y]
\end{align*}
\]

Thus

\[
0 = \Phi(f) = \sum_{i=0}^{k} \sum_{x \in F^0} f(x, i)[\xi_i x]
\]

\[
= \sum_{i=0}^{k} \sum_{x \in F^0} f(x, i) \left( \sum_{j=1}^{k-1} \sum_{y \in F^0} M^{j-i}_F(x, y)[\xi_j y] + \sum_{y \in F^0} M^{k-i}_F(x, y)[\xi_k y] \right)
\]

\[
= \sum_{i=0}^{k-1} \sum_{y \in F^0} \left( \sum_{x \in F^0} f(x, i) \right)[\xi_j y] + \sum_{y \in F^0} \left( \sum_{i=0}^{k} \sum_{x \in F^0} M^{k-i}_F(x, y)f(x, i) \right)[\xi_k y].
\]

We pause in the proof to introduce some definitions.

**Definition 3.3.** For \( g \in V \) let \( g_i \in C_c(E^0, \mathbb{Z}) \) be defined by \( g_i(x) = g(x, i) \), and let \( A = M^i_F \).

Now we may write:

\[
\sum_{x \in F^0} M^j_F(x, y)f(x, i) = (A^j f_i)(y).
\]

We then have

\[
(5) \quad \sum_{i=0}^{j} (A^{j-i} f_i)(y) = 0, \text{ for all } y \in F^0 \setminus S_F \text{ and } 0 \leq j < k,
\]

\[
(6) \quad \sum_{k} (A^{k-i} f_i)(y) = 0, \text{ for all } y \in F^0.
\]

We pause the proof once more to introduce new notation.

**Definition 3.4.** Let \( V_0 = C_c(E^0, \mathbb{Z}), W_0 = C_c(S, \mathbb{Z}) \), and let \( \beta_0 : W_0 \to V_0 \) be defined by \( \beta_0 = \beta|_{W_0} \). (We may also describe \( \beta_0 \) by \( \beta_0(s_x) = \sum_{x \in E^0} \delta_{t(x), x} \), \( x \in S \) (compare [9], Proposition 6.11).
Then $\beta_0$ agrees with $A$ on $C_c(S_F, \mathbb{Z})$. Using (5), (6), (3), and (1), we find that

$$f_0 \in W_0,$$

$$f_1 + Af_0 \in W_0,$$

$$\ldots$$

$$f_{k-1} + A(f_{k-2} + A(\cdots + A(f_1 + Af_0))) \in W_0,$$

$$f_k + A(f_{k-1} + A(\cdots + Af_0)) = 0.$$  

Thus $A$ can be replaced by $\beta_0$ in these formulas. Let us define $h \in W$ by

$$h_i = \begin{cases} f_0, & \text{if } i = 0 \\ f_i + \beta_0 h_{i-1}, & \text{if } 0 < i < k \\ 0, & \text{if } i < 0 \text{ or } i \geq k. \end{cases}$$

Then it is immediate that $(1 - \alpha \beta)h = f$, proving that $f \in I$.

Finally, we show that $\Phi$ is onto. We have already seen that the classes of minimal projections in $C_k(F, S_F)$ are in the range of $\Phi$. Since the images of these under $\tilde{\gamma}_* = \alpha$ generate $K_0(C^*(E) \times T)$, it follows that $\Phi$ is onto. \hfill \Box

4. The $K$-theory of $C^*(E)$

The rest of our argument consists of algebraic manipulations. We first give some notation.

**Definition 4.1.** Define maps $e_i : V \to V$ by

$$e_i(f)_j = \begin{cases} f_i, & \text{if } j = i \\ 0, & \text{if } j \neq i. \end{cases}$$

Let $q_i : V \to V$ be defined by $q_i = \sum_{j \leq i} e_j$ (note that the sum is finite on elements of $V$). We note that $e_i$ and $q_i$ commute with $\beta$, and that $\alpha^i \circ e_i = e_{i+j} \circ \alpha^i$ (and similarly for $q_j$). We define $E : V \to V_0$ by $E(f) = \sum_i f_i$, and $\varphi : V_0 \to V$ by

$$\varphi(x)_j = \begin{cases} x, & \text{if } j = 0 \\ 0, & \text{if } j \neq 0. \end{cases}$$

Then $E \circ \alpha = E$, $E \circ \beta = \beta_0 \circ E$ and $\varphi \circ \beta_0 = \beta \circ \varphi$.

**Lemma 4.2.** Let $g \in V$ and $h \in W$ be such that

$$(1 - \alpha^{-1})g = (1 - \alpha \beta)h.$$  

Then $E(h) \in \ker(1 - \beta_0)$, and $g + \varphi \circ E(h) \in I$.

**Proof.** Applying $E$ to equation (1) gives $0 = E \circ (1 - \alpha \beta)h = (1 - \beta_0)(E(h))$. Next, applying $e_i$ to equation (1) gives

$$e_i(g) - \alpha^{-1} e_{i+1}(g) = e_i(h) - \alpha \beta e_{i-1}(h)$$

$$= e_i(h) - e_{i-1}(h) + (1 - \alpha \beta)e_{i-1}(h).$$

Adding equations (2) for $i \leq j$ gives

$$q_j(g) - \alpha^{-1} q_{j+1}(g) = e_j(h) + (1 - \alpha \beta)q_{j-1}(h).$$

Applying $\alpha^{-j}$ to equation (3) gives

$$\alpha^{-j} q_j(g) - \alpha^{-(j+1)} q_{j+1}(g) = \alpha^{-j} e_j(h) + (1 - \alpha \beta)\alpha^{-j} q_{j-1}(h).$$

For $m < n$, we add equations (4) for $m \leq j < n$ to get

$$\alpha^{-m} q_m(g) - \alpha^{-n} q_n(g) = \sum_{j=m}^{n-1} \alpha^{-j} e_j(h) + (1 - \alpha \beta) \sum_{j=m}^{n-1} \alpha^{-j} q_{j-1}(h).$$

$$\sum_{j=m}^{n-1} \alpha^{-j} e_j(h) + (1 - \alpha \beta) \sum_{j=m}^{n-1} \alpha^{-j} q_{j-1}(h).$$
 Choose $m$ and $n$ so that $g_i = h_i = 0$ for $i \leq m$ and $i \geq n$. Then $q_n(g) = g$, $q_m(g) = 0$, and $\sum_{j=m}^{n-1} \alpha^{-j} e_j(h) = \varphi(E(h))$. Thus from equation (5) we obtain

$$g + \alpha^n \circ \varphi \circ E(h) \in I. \tag{6}$$

But for any $j$ we have

$$\alpha^j \circ \varphi \circ E(h) - \alpha^{j+1} \circ \varphi \circ E(h) = \alpha^j \circ \varphi \circ E(h) - \alpha^j \circ \varphi \circ \beta_0 \circ E(h),$$

since $E(h) \in \ker(1 - \beta_0)$,

$$= \alpha^j \circ \varphi \circ E(h) - \alpha^{j+1} \circ \beta \circ \varphi \circ E(h) = (1 - \alpha \beta) \circ \alpha^j \circ \varphi \circ E(h) \in I. \tag{7}$$

From this and equation (6) we have that these maps are isomorphisms.

Similarly, we have

$$\sum_{j<0} \alpha^{-j} q_j = \sum_{j<0} \sum_{i \leq j} \alpha^{-j} e_i = \sum_{i<0} \sum_{j=1}^{i-1} \alpha^{-j} e_i = \sum_{i<0} \sum_{j=1}^{i-1} \alpha^j e_i. \tag{8}$$

Hence

$$(1 - \alpha^{-1}) \sum_{j<0} \alpha^{-j} q_j = \sum_{i<0} (\alpha^{-i} - 1)e_i. \tag{9}$$

Finally, combining equations (7) and (8), we find that in the statement of the lemma, the right-hand side of the equation equals

$$\sum_{i \neq 0} e_i - \sum_{i \neq 0} \alpha^{-i} e_i = \sum_{i} e_i - \sum_{i} \alpha^{-i} e_i = 1 - \varphi \circ E. \tag{10}$$

Now we will compute the $K$-theory of $C^*(E)$. Let $\widetilde{V} = V/I$. Since $\alpha(I) = I$, $\alpha$ descends to an automorphism $\widetilde{\alpha}$ of $\widetilde{V}$. Under the isomorphism of $K_0(C^*(E) \times \mathbb{T})$ with $\widetilde{V}$, $\widetilde{\alpha}$ corresponds to the dual action of $\mathbb{T}$. So by the Pimsner-Voiculescu exact sequence, we must identify the kernel and cokernel of $1 - \widetilde{\alpha}^{-1}$. We will show that the kernel and cokernel of $1 - \widetilde{\alpha}^{-1}$ are isomorphic to those of $1 - \beta_0$. We let $\tilde{\varphi}$ denote the composition of $\varphi$ with the quotient map of $V$ onto $\widetilde{V}$.

**Proposition 4.4.** With the above notation, the kernel and cokernel of $1 - \widetilde{\alpha}^{-1}$ are isomorphic to those of $1 - \beta_0$.

**Proof.** We have for $x \in W_0$,

$$(1 - \tilde{\alpha}^{-1}) \circ \tilde{\varphi}(x) = (1 - \alpha^{-1}) \circ \varphi(x) + I$$

$$\tilde{\varphi} \circ (1 - \beta_0)(x) = (1 - \beta) \circ \varphi(x) + I. \tag{11}$$

Since

$$(1 - \beta) \circ \varphi(x) - (1 - \alpha^{-1}) \circ \varphi(x) = (\alpha^{-1} - \beta) \circ \varphi(x) = (1 - \alpha \beta) \circ \alpha^{-1} \circ \varphi(x) \in I,$$ 

we find that (on $W_0$)

$$\tilde{\varphi} \circ (1 - \beta_0) = (1 - \tilde{\alpha}^{-1}) \circ \tilde{\varphi}. \tag{12}$$

Therefore $\tilde{\varphi}$ defines maps: $\ker(1 - \beta_0) \to \ker(1 - \tilde{\alpha}^{-1})$ and $\text{coker}(1 - \beta_0) \to \text{coker}(1 - \tilde{\alpha}^{-1})$. We will show that these maps are isomorphisms.

First we treat the map on kernels. For surjectivity, let $g + I \in \ker(1 - \tilde{\alpha})$. Then $(1 - \alpha^{-1})g \in I$. Thus there is $h \in W$ such that $(1 - \alpha^{-1})g = (1 - \alpha \beta)h$. By Lemma 4.2 we have that $E(h) \in \ker(1 - \beta_0)$ and that
Example 4.7. Consider the following graph $E$:

Since the graph is transitive and has loops, its $C^*$-algebra is a (UCT) Kirchberg algebra \((\mathbb{C})\). Since there are infinitely many vertices, $C^*(E)$ is non-unital, and hence is stable, by a theorem of Zhang \((\ref{Zhang})\). We use the Remark \((\ref{stable})\) to compute the $K$-theory of $C^*(E)$.

First note that $S = \mathbb{Z} \setminus \{0\}$. Now let $f \in K_1(C^*(E))$. Thus $f \in C_c(S, \mathbb{Z}) \subseteq C_c(E^0, \mathbb{Z})$. In particular, $f(0) = 0$. For $n \geq 1$, we have $f(n) = f(n) + f(n+1) + f(0)$, and hence $f(n+1) = 0$. Similarly, $f(n-1) = 0$ for $n \leq -1$. Finally, $f(0) = f(0) + f(-1) + f(1)$, so that $f(-1) = -f(1)$. We see that $K_1(C^*(E)) = \mathbb{Z} \langle \delta_1 - \delta_{-1} \rangle \cong \mathbb{Z}$.

Now, let $[f]$ denote the class in $K_0$ of an element $f \in C_c(E^0, \mathbb{Z})$. For $n \geq 1$ we have $[\delta_n] = [\delta_n] + [\delta_{n+1}]$, and hence $[\delta_{n-1}] = 0$. Similarly, $[\delta_{n+1}] = 0$ for all $n \leq -1$. Thus $[\delta_n] = 0$ for all $n \in \mathbb{Z}$. It follows that $K_0(C^*(E)) = 0$.

The algebra $C^*(E)$ is the stable form of the Kirchberg algebra denoted $\mathcal{P}_\infty$ by Blackadar \((\ref{Blackadar})\).

Additional interesting examples appear in \([17]\). In the examples computed there, the graphs are more easily presented (and understood) by diagrams rather than by matrices.

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