String bits in small radius AdS and weakly coupled $\mathcal{N} = 4$ Super Yang-Mills Theory: I

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Abstract: We study light-cone gauge quantization of IIB strings in AdS$_5 \times$ S$^5$ for small radius in Poincare coordinates. A picture of strings made up of noninteracting bits emerges in the zero radius limit. In this limit, each bit behaves like a superparticle moving in the AdS$_5 \times$ S$^5$ background, carrying appropriate representations of the super conformal group PSU(2,2|4). The standard Hamiltonian operator which causes evolution in the light-cone time has continuous eigenvalues and provides a basis of states which is not suitable for comparing with the dual super Yang-Mills theory. However, there exist operators in the light-cone gauge which have discrete spectra and can be used to label the states. We obtain the spectrum of single bit states and construct multi-bit states in this basis. There are difficulties in the construction of string states from the multi-bit states, which we discuss. A non-zero value of the radius introduces interactions between the bits and the spectrum of multi-bit states gets modified. We compute the leading perturbative corrections at small radius for a few simple cases. Potential divergences in the perturbative corrections, arising from strings near the boundary, cancel. This encourages us to believe that our perturbative treatment could provide a framework for a rigorous and detailed testing of the AdS/CFT conjecture, once the difficulties in the construction of string states are resolved.

Keywords: String theory, Gauge theory.

Dedicated to the memory of Bunji Sakita.
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1. Introduction

The connection between 4-dim. gauge theories and string theory is an idea with a long history that goes back to attempts to discover strings and string dynamics in gauge theories. A part of this history, in connection with the $1/N$ expansion, can be found in [1]. See also [2, 3]. The hope underlying such attempts was to guarantee consistent string dynamics in the context of a theory in 4 spacetime dimensions rather than in 10 or 26 spacetime dimensions. The main motivation for finding a description of gauge theories in terms of a string theory was to solve the problem of quark confinement. This problem remains unsolved even today. For a semi-popular account of this problem see [4].

It took more than two decades to arrive at a precise formulation of this idea in the form of the AdS/CFT correspondence proposed by Maldacena [5]. The most studied case of this correspondence is the duality between $N = 4$ supersymmetric Yang-Mills theory in 4-dimensions and type IIB string theory on $AdS_5 \times S^5$. The clue came from the seemingly unrelated problem of understanding thermal properties of a class of black holes in terms of D-branes.

The AdS/CFT correspondence has been precisely formulated in the limit when the 't Hooft coupling $\lambda = g_{YM}^2 N$ is large [7, 8]. On the AdS side this means that $R^2 = \alpha' \sqrt{\lambda}$ is large and the dual type IIB string theory is approximated by supergravity. A correspondence between local gauge-invariant operators of the gauge theory and local supergravity fields has been extensively discussed in the literature [9]. Besides the local operators, the correspondence has also been studied for gauge-invariant Wilson loop operators [11, 12, 13, 14, 15]. Here one is aided by the fact that $\lambda = g_{YM}^2 N$ is precisely the semi-classical expansion parameter for the loop equations.

In this paper we study AdS/CFT correspondence in the opposite limit viz. $\lambda \rightarrow 0$. In this limit the gauge theory can be analyzed perturbatively but the string theory is more difficult and the standard semi-classical expansion of strings when $R^2 = \alpha' \sqrt{\lambda}$ is large, breaks down. The problem is conceptually similar to the treatment of gauge theories on the lattice [16, 17] where the gluon picture breaks down at strong coupling. The zeroth order solution of the lattice gauge theory is obtained by exactly diagonalizing the kinetic (square of the electric field) term of the Hamiltonian. Gauge invariance (Gauss’ law) then connects the individual string bits on the links of the lattice into long strings which can connect charged sources. The potential energy (magnetic term) is treated perturbatively and it corrects the energy and wave function of the string state of the zeroth order Hamiltonian [18].

We adopt a similar method, for the strongly coupled world sheet string theory, when $\lambda \rightarrow 0$. We assume that $N$ is large enough so that string loops are suppressed.

\[^1\text{see [4] for a recent review of this subject.}\]
and the first quantized picture is an adequate starting point. We work in the light cone gauge which leaves only global reparametrizations of the string as a residual symmetry. We consider the parameter space of the gauge-fixed string as a discrete lattice consisting of $M$ string bits. At $\lambda = 0$, the string bits are non-interacting. On each bit we have a Hilbert space which turns out to consist of the states of a super-particle in $\text{AdS}_5 \times S^5$. The total wave function of the string is a cyclically symmetric direct product of states carried by the bits. We explicitly present these states.

Bit strings have been discussed earlier in the context of strings in flat space \cite{13} and more recently in the pp-wave background \cite{20, 21, 22, 23, 24}. A problem similar to the one treated here is that of bosonic strings in $\text{AdS}_5$ in the small radius limit, which has been addressed in \cite{26}, using somewhat different methods. A sensible perturbative treatment is unlikely to exist in this case since fermionic contributions turn out to be crucial to give finite results in our calculations. The idea of string bits has also appeared in 't Hooft’s paper \cite{25} on planar diagrams.

The organization of this paper is as follows. In the next section we will review the quantization of Green-Schwarz world-sheet type IIB string theory in the light-cone gauge. This gauge-fixing uses Poincare co-ordinates. In section 3 we will discuss a limit of this theory in which the AdS radius vanishes. In this limit the theory reduces to a system of non-interacting bits. We make this more explicit by introducing discrete bits of string. Each bit behaves like a superparticle moving in the $\text{AdS}_5 \times S^5$ background, carrying “appropriate representations” of the superconformal group\(^2\). In the light-cone gauge, the operator that gives evolution in light-cone time has continuous eigenvalues. This basis is not convenient for comparison with the gauge theory at the boundary. For this purpose, it is desirable to work with the basis frequently used in describing the UIR’s of the superconformal group $\text{PSU}(2, 2|4)$. We identify the operators in the light-cone gauge which have this basis as eigenstates\(^3\). In section 4 we discuss the states of a single bit in such a basis. In section 5 we discuss corrections to the multi-bit spectrum due to a small but non-vanishing value of the AdS radius. A non-vanishing value of the radius introduces bit-bit interactions which modifies the spectrum. We compute the leading order corrections to the simplest, but potentially the most singular, multi-bit states. We find that potential divergences from the boundary

\(^2\)In the light cone gauge, some of the generators of the superconformal algebra, which involve $\tilde{x}^-$, the non-zero mode of $x^-$, do not have well-defined action on a single bit and one needs to consider at least two bits to define their action. A single bit carries a representation of the superconformal algebra only in the naive sense that $\tilde{x}^-$, which is defined in terms of gradients of the independent coordinates or momenta, is taken to be zero by fiat. We will discuss the action of these generators on multi-bit states in Appendix D.

\(^3\) One can encounter continuous or discrete eigenvalues in a given physical system depending on which operator is being diagonalized, e.g. in the case of the two-dimensional rotor we have continuous eigenvalues of linear momentum but discrete eigenvalues of angular momentum.
$z = 0$ cancel between fermionic and bosonic contributions, leaving only finite corrections to the anomalous dimensions. In section 6 we make preliminary attempts at constructing string states out of multi-bit states. There are difficulties in doing this, which we discuss in this section. A naive continuum limit, which is arranged so as to give finite energy states, seems to be in conflict with unitarity bounds\footnote{We thank Shiraz Minwalla for discussions on this point.} on the representations of the superconformal group. This motivates us to consider the alternative possibility that discretization is essential. In this case one must include string states made of varying number of bits. This is because splitting and joining of strings with a given number of bits will produce strings with lower as well as higher number of bits. However, we find that the string spectrum made this way does not agree with that of the gauge theory operators. We discuss this in this detail in this section. We end with some discussion and concluding remarks in section 7. Appendix A contains the continuum expressions for superconformal generators as realized in our string theory, and Appendix B the discrete version. Appendix C contains details of perturbation calculation and the cancellation of divergences. Appendix D contains a discussion on stringy features of the superconformal generators and their action on multi-bit states.

In this paper, we will work in the $N \to \infty$ limit and ignore string-string interactions.

2. Type IIB strings in AdS background in the light-cone gauge

In this section we will review type IIB Green-Schwarz superstring in the $\text{AdS}_5 \times S^5$ background geometry in the light-cone gauge \cite{27,28}. This will also serve to set our notations and conventions. We will use Poincare co-ordinates since the discussion of light-cone gauge fixing is simplest in these co-ordinates. Also, in these co-ordinates comparison with the boundary gauge theory is the most direct. We will not address global issues arising out of the use of a Poincare patch in the present work.

The $\text{AdS}_5 \times S^5$ metric in Poincare co-ordinates is

$$ds^2 = \frac{R^2}{z^2} (\eta_{\mu \nu} dx^\mu dx^\nu + dz^a dz^a), \quad (2.1)$$

where $\eta_{\mu \nu} = (-, +, +, +)$ is the Minkowski metric and the six co-ordinates $z^a, a = 1, 2, \cdots, 6$ parametrize the $S^5$ and the AdS “radial” direction. The transformation between the “Cartesian” $z^a$ and the more customary $z, \Omega_5$ is given by

$$z^a = zn^a, \quad n^a n^a = 1,$$
where the six-dimensional unit vector \( n^a \) represents \( S^5 \) angles \( \Omega_5 \). The \( z \to 0 \) limit is called the AdS boundary whereas \( z \to \infty \) is called the Poincare horizon.

The AdS radius \( R \) is related to the 't Hooft coupling \( \lambda \) and the string scale \( \alpha' \) by the standard expression

\[
R^2 = \alpha' \sqrt{\lambda}, \quad \lambda = g_{YM}^2 N
\]  

(2.2)

Note that in the world-sheet action the AdS radius \( R \) naturally appears in the combination \( R^2/\alpha' = \sqrt{\lambda} \). This is the effective tension of the string moving in \( \text{AdS}_5 \times S^5 \). For later convenience we introduce the parameter

\[
T = \frac{\sqrt{\lambda}}{2\pi}
\]  

(2.3)

In addition to the bosonic co-ordinates, appearing in (2.1), the world-sheet superstring action has two left Majorana-Weyl spinors. In the light-cone gauge, the \( \kappa \)-symmetry of the Green-Schwarz action can be used to gauge away half of the fermionic degrees of freedom of each of these two spinors [27]. The remaining 16 physical fermionic degrees of freedom can be rearranged into two fundamental representations of the R-symmetry group \( SU(4) \), \( \theta^i, \eta^i, i = 1, \cdots, 4 \) and their conjugates, \( \theta_i, \eta_i \). The \( \theta^i \) are related to the 8 linear Poincare supersymmetries while the \( \eta^i \) are related to the 8 non-linear “conformal” supersymmetries of the \( psu(2,2|4) \) superconformal algebra. In fact, \( \theta \) appears at most quadratically in the generators of this algebra, while there are also terms quartic in \( \eta \) in these generators. These latter terms reflect the presence of curvature and the five form flux.

On the bosonic side, it turns out [28] that the usual light-cone gauge choice

\[
x^+ \equiv \frac{x^3 + x^0}{\sqrt{2}} = p^+ \tau,
\]

where \( \tau \) is the world-sheet time co-ordinate, cannot be combined with a conformal gauge on the world-sheet metric since that is not consistent with equations of motion of the system. A modified gauge choice for the world-sheet metric, \( g_{\alpha\beta} \), which is consistent with the gauge choice on \( x^+ \), is [28]

\[
\sqrt{-g}g^{\tau\tau} = \frac{-1}{\sqrt{-g}g^{\sigma\sigma}} = -\frac{2\pi z^2}{\sqrt{\lambda}},
\]  

(2.4)

where \( \sigma \) is the world-sheet co-ordinate running along the string\(^5\).

\(^5\)This gauge choice is potentially singular near the boundary, \( z \to 0 \). However, as we shall see later in this paper, physical quantities seem to be regular, so presumably this is just a gauge artifact. Note that \( \lambda \to 0 \) is not a problem since the Hamiltonian (2.5) is well-defined in this limit.
Before gauge fixing, the Green-Schwarz type IIB superstring moving in AdS$_5 \times$ S$^5$ background manifestly has the global isometries of the supergroup PSU(2, 2$|$4). The choice of a light-cone gauge destroys manifest symmetry, but one can still derive expressions for the generators of the symmetry algebra in terms of the light-cone bosonic and fermionic degrees of freedom of the superstring [28]. For example, the “Hamiltonian” which generates evolution in the light-cone time variable $x^+$, and is one of the generators of the superalgebra $\text{psu}(2, 2$|$4)$, is given by $-\int d\sigma \ P^- (\sigma)$, where

$$P^- = -\frac{1}{2p^+} \left( 2p^x p^z - \partial_z^2 + \frac{1}{z^2} [l_i^i l_i^i + 4\eta_i^i \eta_j^j + (\eta_i^i \eta_j^j - 2)^2 - \frac{1}{4}] \right) - \frac{T^2}{2p^+ z^4} (2x' \bar{x}' + (z')^2)$$

$$+ \frac{T}{z^2 p^+} \eta_i^j n^a (\theta_j^i - i \frac{\sqrt{2}}{z} \eta^j x') + \frac{T}{z^2 p^+} \eta_i^j n^a (\theta_j^i + i \frac{\sqrt{2}}{z} \eta_j \bar{x}') ,$$

(2.5)

Here, T is given by (2.3). The notation is as in Appendix A, in particular $l_i^i$ is as in (A.22). In this appendix, for the convenience of the reader, we have also given the light-cone gauge expressions for the basic generators of $\text{psu}(2, 2$|$4)$ algebra, together with their (anti)commutation relations. This algebra, and the light-cone gauge realization of it, will play a central role in what follows.

### 3. String bits and the limit of zero radius

We begin by noting that due to the gauge choice (2.4) the AdS radius R enters the expression in (2.3), through T, only in the terms that have $\sigma$-derivatives. This is, in fact, the case with all the generators$^6$, as can be seen from the expressions given in Appendix A. As a consequence of this, in the limit $R \to 0$ all the $\sigma$-derivative terms disappear from these generators and they become pointwise local in $\sigma$. That is, the string “breaks up” into independent bits. Since the finite T terms are apparently singular at the boundary $z \to 0$, one might worry whether it makes any sense to think of a non-zero radius in (2.3) as a perturbation. Later in this paper we will see by explicit calculations that perturbation expansion in T is finite. With this post facto justification, we will just go ahead with the naive zero radius limit.

To make the bit picture manifest, let us discretize $\sigma$ into a lattice of $M$ points with a lattice spacing $\epsilon = l/M$, where $l$ is the total length of the string. As usual, we need to rescale continuum variables by powers of $\epsilon$ to get the lattice variables. Thus, the fact that the total light cone momentum $P^+ = \int_0^1 d\sigma \ p^+_{\text{cont.}} = l p^+_{\text{cont.}}$ must be equated to $M p^+_{\text{latt.}}$ implies that the momentum of each bit must be $p^+_{\text{latt.}} = \epsilon p^+_{\text{cont.}}$. We will permit ourselves the abuse of notation $p^+_{\text{latt.}} = p^+$ when we are discussing bit variables. Similarly, the momentum densities (at any point $\sigma = n \epsilon$ along the string) in the directions $(x^i, z^a), i = 1, 2; a = 1, ..., 6$, are related to the discrete

$^6$except for generators mentioned in footnote (3) on page 3 which we discuss in Appendix D.
momenta of the $n$th bit by relations such as $p_i^\prime = \epsilon p_i^\prime(\sigma)$. This implies that (a) the total momenta reduce to simple sums, e.g. $P^n = \sum_1^M p_n^i$, and (b) since we should not rescale the $x$’s, so that the continuum canonical commutation relation, 
$$[x^i(\sigma), p_j(\sigma^\prime)] = i\delta^i_j \delta(\sigma - \sigma^\prime),$$
translates to the discrete expression $[x^i_n, p_{m,j}] = i\delta^i_j \delta_{mn}$, which is free of $\epsilon$. Here we have used $\delta(\sigma - \sigma^\prime) = (1/\epsilon) \delta_{mn}$. A similar rescaling of the superpartners, $\sqrt{\epsilon} \theta \to \theta, \sqrt{\epsilon} \eta \to \eta$ removes the lattice spacing from their anti-commutation relations. The above rescalings are summarized in (B.1).

In terms of these discrete bit variables, the T-independent part of the Hamiltonian, is given by

$$P_0^- \equiv \sum_{n=1}^M p_n^- = -\sum_{n=1}^M \frac{1}{2p^+} \left( 2p_n^x p_n^x - \partial^2_{z_n} + \frac{1}{z_n^2} \left[ \left( l_{n,j}^j l_{n,i}^i \right) + 4\eta_n^i l_{n,j}^j \eta_n^j + (\eta_n^i \eta_n^j - 2\delta_n^i - 1) \right] \right)$$

The subscript on $P_0^-$ denotes that $T$ has been set to zero. We see that the lattice spacing has completely disappeared from the expression on the right hand side of (3.1). In fact, this continues to be so even when finite $T$ parts are included. This property of the Hamiltonian is actually shared by all the generators of the superconformal algebra.

In the limit of vanishing AdS radius we thus see that the string “falls apart” into non-interacting discrete bits. The first step in the problem of solving for the spectrum of states of the free (i.e. $\lambda = 0$) superstring then reduces to the problem of solving for the spectrum of a single string bit\(^7\). It is to this problem that we will now turn our attention. In Appendix B we have listed discretized expressions for all the generators, including the finite $T$ parts. We have also listed there the canonical commutation relations satisfied by the bits. We will need these expressions in the discussions that follow.

4. States of a single bit

In thinking of the spectrum of states of a single bit, the first difficulty that we face is that the spectrum of the single-bit Hamiltonian $-p_n^-$ is continuous. Clearly the eigenstates of $-p_n^-$ are not a good starting point for building free superstring states since these states cannot be directly compared with the gauge-invariant operators of the boundary theory. This is because the latter have definite integer or half-integer dimensions in the limit of vanishing 't Hooft coupling $\lambda$. It is, however, easy to resolve this problem. The unitary irreducible representations (see, e.g. [26, 30, 31, 32, 33] of the global symmetry group $SO(4,2)$ of AdS$_5$ are well-known to be labeled by three numbers corresponding to the representations of the

\(^7\)Provided that generators which involve $\tilde{x}^-$, such as $K^x$, which a priori do not have any well-defined action on a single bit Hilbert space, act appropriately on these tensor product states. See footnote (2).
maximal compact subgroup $SO(2) \times SO(4) = SO(2) \times SU(2) \times SU(2)$. Here $SO(2)$
denotes rotation in the $(-1, 0)$ directions (generated by $S_{-1,0}$ of (A.1)) while $SO(4)$
denotes rotation in the remaining four directions (see (A.1), (A.2) for notation).
These quantum numbers are denoted as $(E_0, J_1, J_2)$ ([31], see also [34]), where $E_0$
physically means the energy, conjugate to time translations in global co-ordinates, and $J_1$ and $J_2$
denote the angular momentum representations associated with the two $SU(2)$ factors in the $SO(4)$.
(see equations in section A.2 for more detail).
Equations (A.25) and (A.26) express these generators in terms of the generators of
the conformal algebra in $(3 + 1)$-dimensional Minkowski space. It turns out from
these considerations (see Sec 2.2 of [33]) that the Euclidean dimension operator of
the conformal algebra (equivalently the dimension in the boundary gauge theory)
corresponds to $E_0$.

Clearly, the spectrum should be arranged in terms of simultaneous
eigenfunctions of $(E_0, J_1, J_2)$, or equivalently, $(H_+, H_-, J_1 - J_2)$, where $H_+ \equiv E_0 \mp (J_1 + J_2)$
on a suitable highest weight state, see (A.28)). The latter basis
is particularly convenient because these operators have simple expressions in the
light-cone frame, viz.

$$
H_+ = -P^- + K^+, \quad H_- = -K^- + P^+, \quad J_1 - J_2 = J^{xx}.
$$

(4.1)

Thus, we should find simultaneous eigenstates of these operators ⁸ (and not of
$-P^-$) in order to determine which UIR’s of $SO(4,2)$ appear. Representations
of the full superalgebra, which includes additional bosonic generators (of the
$SU(4)$ R-symmetry), are constructed in our case by acting on the abovementioned
eigenstates with supersymmetry “raising operators” (A.29) (cf. [31, 30, 33]).

### 4.1 The spectrum of $h_+$

In this section we will determine the spectrum of the single-bit operator $h_{+,n}$ where

$$
H_+ = \sum_{n=1}^M h_{+,n}
$$

(4.2)
in the limit $\lambda = 0$. In a later section we will determine the perturbative correction
to the spectrum of $H_+$ at small $\lambda$. Since $H_+ = E_0 - (J_1 + J_2)$ on a highest weight
state, as mentioned above, and since $J_1 - J_2$, being an integer or a half-integer, does
not receive perturbative corrections, the perturbative correction to the spectrum
of $E_0$ is the same as that of $H_+$. Henceforth, we will sometimes use the word
“Hamiltonian” for the operator $H_+$.

We will see that it is easy to determine $J^{xx}$ on the eigenfunctions of (4.2). In
order to determine the triple $(E_0, J_1, J_2)$, we also need to evaluate $H_-;$ this is not

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⁸Similar operators have also been considered in [35].
easy because it involves $\bar{x}^-$ and is hence non-local even at $\lambda = 0$. However, we will be able to measure $H_-$ indirectly on states of our interest.

The expression for $h_+$, as defined in (4.2), can be obtained from the equations (B.4) and (B.9) given in Appendix B:

$$h_+ = -\frac{\partial_x \partial_{\bar{x}}}{p^+} + p^+ x \bar{x} - \frac{\partial_x^2}{2p^+} + \frac{p^+}{2z^2} + \frac{1}{2p^+ z^2} [(\eta^i)^2 + 4\eta_i \eta_j \eta^j + (\eta_i \eta^i - 2)^2] - \frac{1}{4}$$ (4.3)

Here and in the rest of this section we will drop the bit index $n$ from single bit operators to simplify the notation. Note that the fermionic terms in the above come entirely from the $-p^-$ part of $h_+$.

4.1.1 States with no fermions: eight bosonic oscillators

To find the purely bosonic states, it is more instructive first to transform from the “polar coordinates” $(z, n_a)$ used in (4.3), to “Cartesian coordinates” $z_a = z n_a, a = 1, ..., 6$. The Hamiltonian appropriate to the Cartesian coordinates, denoted by $\tilde{h}_+$, is given by

$$\tilde{h}_+ = z^{-5/2} h_+ z^{5/2}$$

$$\tilde{h}_+ = \frac{p^x p^\bar{x}}{p^+} + p^+ x \bar{x} + \frac{(p^a)^2}{2p^+} + \frac{p^+}{2} (z^a)^2 + \frac{1}{2p^+ z^2} (4\eta_i \eta_j \eta^j + (\eta_i \eta^i - 4\eta_i \eta^i))$$ (4.4)

Here $h_+$ refers to (4.3). The eigenfunctions of (4.4), $\tilde{\psi}$, are related to eigenfunctions $\psi$, by $\tilde{\psi} = z^{-5/2} \psi$. Here $p^x = -i \partial_x, p^\bar{x} = -i \partial_{\bar{x}}, p^a = -i \partial/\partial z_a$.

On states that do not contain any fermions, the coefficient of $1/z^2$ in (4.4) vanishes. The rest of the terms represent eight harmonic oscillators corresponding to the eight transverse bosonic degrees of freedom of the light-cone gauge-fixed string. The spectrum of $\tilde{h}_+$ (hence of $h_+$) is obviously discrete (in fact, integer$^{10}$), and is given by

$$E = \sum_{i=1}^{8} (n_i + 1/2), \quad n_i = 0, 1, ...$$ (4.5)

The length scale of the harmonic oscillators is set by $p^+$, e.g. $\langle x \bar{x} \rangle = \langle (z^a)^2 \rangle = 1/p^+$, although the spectrum is free of $p^+$. The creation and annihilation operators are defined in Appendix B (see (B.3)).

Note that the existence of the eight bosonic oscillators ties up with the fact that there are precisely eight fermionic oscillators $\eta_i, \theta_i, i = 1, ..., 4$, as we expect from supersymmetry.

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$^9$ We stress that $J_1, J_2$ are both (half-)integers and are immune from perturbative corrections. Since $H_- - H_+ = 2(J_1 + J_2)$, it is enough to consider perturbative correction to $H_+$ which, as argued earlier, represents also the perturbative correction to $E_0$.

$^{10}$ Although each of the triple $(E_0, J_1, J_2)$ can be an integer or half integer, it turns out that $H_+ = E_0 - (J_1 + J_2)$ is always an integer.
4.1.2 General wavefunctions

Let us now consider the more general case when fermionic degrees of freedom are excited. We will first write down the result.

The spectrum of $h_+$ ((4.3)) is given by (see footnote (10))

$$e_+ = \alpha + 2r + s + \bar{s} + 2, \quad \alpha, r, s, \bar{s} \in \mathbb{Z}$$  \hspace{1cm} (4.6)

where the corresponding eigenfunction is given by

$$|\Psi\rangle = |\eta\rangle|\theta\rangle \times \Upsilon^\alpha_{l,s,\bar{s}}(x, \bar{x}, z) \times |l\rangle$$  \hspace{1cm} (4.7)

Here $|\eta\rangle|\theta\rangle$ denotes the part of the wavefunction involving fermionic coordinates. $|\eta\rangle$ is any polynomial in the four variables $\eta_i, i = 1, \ldots, 4$, understood as acting on the fermionic ground state $|0\rangle$. There are 16 such linearly independent polynomials: $1, \eta_i, \eta_i\eta_j, \ldots, \eta_1\eta_2\eta_3\eta_4$. Similarly $|\theta\rangle$ denotes any polynomial in $\theta_i, i = 1, \ldots, 4$. Recall that $\{\theta_i, \theta_j\} = \{\eta_i, \eta_j\} = \delta^i_j$, $\eta^0 = 0 = \theta^0$.

The various parts of the wavefunction depending on the bosonic coordinates are as follows.

$$\Upsilon^\alpha_{l,s,\bar{s}}(x, \bar{x}, z) := h^\alpha_{s,\bar{s}}(x, \bar{x}) \psi^\alpha_r(z)$$  \hspace{1cm} (4.8)

Here $h^\alpha_{s,\bar{s}}(x, \bar{x})$ is the standard harmonic oscillator wavefunction of a pair of complex coordinates (see (4.4) and Appendix B) at level $s, \bar{s}$ respectively, and

$$\psi^\alpha_r(z) = e^{-p^+z^2/2}z^{\alpha + \frac{1}{2}}L^\alpha_r(p^+z^2)^{\frac{1}{2}}(r!\alpha!)^{\frac{1}{2}},$$  \hspace{1cm} (4.9)

where $L^\alpha_r$ are generalized Laguerre functions. $\alpha$ is an integer defined below and $\int_{-\infty}^{\infty} dz \psi^\alpha_r(z)^2 = 1$ for $p^+ = 1$.

The wavefunction $|l\rangle$ depends on the $S^5$ angles, equivalently on the $n_a, a = 1, \ldots, 6$, and is given by the spherical harmonic

$$|l\rangle = C_{a_1 \ldots a_l} n^{a_1} \ldots n^{a_l}$$

where $C_{a_1 \ldots a_l}$ is a symmetric, traceless, rank $l$ tensor. $|l\rangle$ satisfies $l^{ab}l^{ba}|l\rangle = l(l+4)|l\rangle$ ($l^{ab}$ is defined in (A.22)).

Definition of $\alpha$

The integer $\alpha$ appearing in (4.6) and (4.7) and onwards, depends on the $S^5$ angular momentum $l$ and the fermion wavefunction $|\eta\rangle$, as follows [36].

Case $l = 0$: $\alpha = |2 - F_\eta|$ where $F_\eta$ is the number of $\eta$’s in the wavefunction $|\eta\rangle$.

Case $l \neq 0$:

1. $F_\eta = 0$ or $F_\eta = 4$: $\alpha = l + 2.$
2. $F_\eta = 1$: the canonical choices for the wavefunction $|\eta\rangle$, viz. $\eta_i, i = 1, \ldots, 4$ do not diagonalize $h_+$. The operator $h_+$ splits the 4-dimensional space $V_1 = \text{Span}\{\eta_i\} \equiv$ complex linear combinations of $\{\eta_i\}$, into two 2-dimensional spaces given by $P_1V_1$ and $Q_1V_1$, where $P_1, Q_1$ are orthogonal projection operators,

\[ (P_1)_j^i = \frac{l + 4}{2l + 4} \left( \delta_j^i - \frac{2}{l + 4} \delta_j^i \right), \quad (Q_1)_j^i = \frac{l}{2l + 4} \left( \delta_j^i + \frac{2}{l + 4} \delta_j^i \right) \]  

(4.10)

If $|\eta\rangle$ belongs to $P_1V_1$, then $\alpha = l + 1$, if $|\eta\rangle$ belongs to $Q_1V_1$, then $\alpha = l + 2$.

3. $F_\eta = 3$: Once again the four canonical basis vectors $\eta_i, \eta_j, \eta_k$ do not diagonalize $h_+$. The operator $h_+$ splits the 4-dimensional space $V_3 = \text{Span}\{\eta_i, \eta_j, \eta_k\}$ into two 2-dimensional spaces given by $P_3V_3$ and $Q_3V_3$, where $P_3, Q_3$ are orthogonal projection operators,

\[ (P_3)_{jmn}^{ijk} = \frac{l + 4}{2l + 4} \left( \delta_j^i \delta_m^k - \frac{6}{l + 4} \delta_j^i \delta_m^k \right), \quad (Q_3)_{jmn}^{ijk} = \frac{l}{2l + 4} \left( \delta_j^i \delta_m^k + \frac{6}{l + 4} \delta_j^i \delta_m^k \right) \]  

(4.11)

Here $[,]$ on the subscripts denote antisymmetrization with weight unity. If $|\eta\rangle$ belongs to $P_3V_3$, then $\alpha = l + 1$, if $|\eta\rangle$ belongs to $Q_3V_3$, then $\alpha = l + 3$.

4. $F_\eta = 2$: Here the operator $h_+$ splits the 6-dimensional space $V_2 = \text{Span}\{\eta_i, \eta_j\}$ into three 2-dimensional spaces given by $P_2V_2$, $Q_2V_2$ and $R_2V_2$, where $P_2, Q_2, R_2$ are orthogonal projection operators,

\[ (P_2)_j^{mn} = \frac{l + 4}{4(l + 1)} \left( \delta_j^m \delta_n^i - \frac{4(l + 3)}{(l + 2)(l + 4)} \delta_j^m \delta_n^i \right), \quad (Q_2)_j^{mn} = \frac{l(l + 4)}{2(l + 1)(l + 3)} \left( \delta_j^m \delta_n^i + \frac{8}{l(l + 4)} \delta_j^m \delta_n^i - \frac{4}{l(l + 4)} l^m \delta_j^i \right) \]  

\[ (R_2)_j^{mn} = \frac{l}{4(l + 3)} \left( \delta_j^m \delta_n^i + \frac{4(l + 1)}{l(l + 2)} \delta_j^m \delta_n^i + \frac{4}{l(l + 2)} l^m \delta_j^i \right) \]  

(4.12)

If $|\eta\rangle$ belongs to $P_2V_2$, then $\alpha = l$, if $|\eta\rangle$ belongs to $Q_2V_2$, then $\alpha = l + 2$, if $|\eta\rangle$ belongs to $R_2V_2$, then $\alpha = l + 4$.

The value of $J^{xx}$ (4.7) is also an eigenfunction of $J^{xx}$ (see (A.14)), with eigenvalue

\[ J^{xx} = -s + \bar{s} + \frac{1}{2} (F_\eta - F_\theta) \]

A note about normalizability
Note that in (4.7) we have considered only normalizable solutions\textsuperscript{11}. This is appropriate for constructing the space of states of the superstring. Alternatively, this can be seen from requiring that the solution be regular at $z \to \infty$. This condition uniquely picks the normalizable solution because of the harmonic oscillator potential piece in (4.3).

### 4.1.3 Summary of this subsection

The spectrum of states for a bit that we have obtained here is identical to that of the type IIB supergravity multiplet [28, 37]. For instance, the 20-plet of states

$$|A\rangle = A^{ij,kl} \eta_{ij} \theta_k \theta_l |0\rangle \times \Upsilon^{0}_{0,0,0}(p^+, x, \bar{x}, z) |l = 0\rangle,$$

(4.13)

where the tensor $A^{ij,kl}$ belongs to the $20'$, i.e. the Dynkin label $(0, 2, 0)$ of SU(4), corresponds to the part of the supergravity multiplet characterized by $(2, 0, 0|20')$. The fact that the $\text{so}(4, 2)$ quantum numbers $(E_0, J_1, J_2)$ are given by $(2, 0, 0)$ follows from the fact that $h_+ = 2, j^{x\bar{x}} = 0$ and, as we discuss in Appendix D, $h_- = 2$.

Note that since the states of a single bit just constitute the supergravity multiplet these states are all part of a short multiplet. This is consistent with the non-renormalization theorems, at least in the limit in which string interactions are ignored, since a single bit cannot receive corrections due to bit-bit interactions.

### 5. Multi-bit spectrum

The first step in the construction of the states of a free string from the states of a bit is to construct multi-bit states. To do this one needs to note the following points:

1. The phase space of the string is given by the canonically conjugate pairs in (A.24). This shows that $p^+$ does not depend on $\sigma$; consequently all bit wavefunctions in the multi-bit state must have the same value of $p^+$. The final multi-bit wavefunction for $m$ bits contains a function of $p^+$ in addition to the plane wave $\exp(i m p^+ x_0^-)$. This is because multi-bit wavefunctions do not have a definite value of $p^+$ in the basis in which $H_-$ is diagonal.

2. Physical string states must satisfy the constraint coming from the residual global reparametrization symmetry in the light-cone gauge. In the discrete version what this means is that to get a physical state of a string from a multi-bit state with $m$ bits, one should cyclically symmetrize a tensor product state

\textsuperscript{11}In the light-cone gauge, the time evolution of the states of a bit is governed by a Schrodinger type evolution equation in flat space with a potential. The appropriate norm in this case is the usual Schrodinger norm.
of \( m \) bits. Thus, we should only consider multi-bit states of the following form:

\[
|p^+; \psi_1, \Psi_2, \cdots \rangle = \sum_{\sigma \in P} |p^+; \psi_{\sigma(1)} \rangle \otimes |p^+; \psi_{\sigma(2)} \rangle \otimes \cdots \otimes |p^+; \psi_{\sigma(m)} \rangle.
\] (5.1)

Here \( P \) is the group of cyclic permutations of \( m \) variables. We have inserted the value of \( p^+ \) as an explicit reminder of point (1) above. Henceforth, in future references to (5.1) the \( p^+ \) factors will not be explicitly written.

In the rest of this section we will first discuss the multi-bit states at zero radius and then compute corrections to the spectrum of these states due to a non-zero, but small, value of the radius.

5.1 Multi-bit spectrum at \( \lambda = 0 \)

For the state (5.1), the value of \( H_+ \) at \( \lambda = 0 \) is clearly given by

\[
E_{+,0} = \sum_{n=1}^{m} e_{+,n}
\] (5.2)

where \( e_{+,n} \) denotes the value (4.6) for the state \( |\psi_n \rangle \). As an example, we can construct a state (5.1), where we take each wavefunction to be of the type given by (4.13). Explicitly, the state will look like

\[
|\Psi_0 \rangle = A_{i_1,j_1,\ldots,k_m,l_m} \prod_{n=1}^{m} \eta_{i_n} \eta_{j_n} \theta_{k_n} \theta_{l_n} |0 \rangle \mathcal{Y}^0_{0,0,0,0}(x_n, \bar{x}_n, z_n)|l = 0 \rangle.
\] (5.3)

Here we have allowed for a general polarization \( A \) rather than a simple cyclically symmetrized product of individual polarizations \( A^{(n)}_{i_n,j_n,k_n,l_n} \). If we take the polarization \( A \) to correspond to a traceless symmetric product of the individual polarizations \( A^{(n)} \)'s, then the state becomes

\[
|\Psi_{\text{sym}} \rangle = A^{\text{sym}}_{i_1,j_1,\ldots,k_m,l_m} \prod_{n=1}^{m} \eta_{i_n} \eta_{j_n} \theta_{k_n} \theta_{l_n} |0 \rangle \mathcal{Y}^0_{0,0,0,0}(x_n, \bar{x}_n, z_n)|l = 0 \rangle.
\] (5.4)

Under \( psu(2,2|4) \), this state transforms as \( (2m,0,0|0,2m,0) \) which is a BPS state made out of \( m \) bits (this can be seen by noting that, in the notation of Gunaydin et al [31], this corresponds to the multiplet based on the Fock space vacuum \( |0 \rangle \) with \( p = 2m \) oscillators).

5.2 Finite radius correction to multi-bit spectrum

In this section we will compute corrections to the free spectrum, (5.2), of multi-bit states due to bit-bit interactions introduced by turning on a small \( T = \sqrt{\lambda}/(2\pi) \) (see (2.3)). Let us write

\[
H_+ = H^0_+ + TV_1 + T^2V_2
\]
where $H_+^0, V_1, V_2$ are defined in (C.1). The leading correction to (5.2) is given by $T^2 \Delta E_+$, where
\[
\Delta E_+ = \langle \Psi_0 | V_2 | \Psi_0 \rangle - \sum_{\text{int}} \frac{\langle |\Psi_0| V_1 | \Psi_{\text{int}} \rangle^2}{E_{+, \text{int}} - E_{+, 0}}.
\]  
(5.5)

In Appendix C we present a calculation of $\Delta E_+$ for the state $|\Psi_0\rangle$ given in (5.3). Here we present the main points.

5.2.1 Potential divergence from the region $z \to 0$

Consider the $V_2$ term in (5.5). Let us look at the term involving $(x_{n+1} - x_n)^2/z_{z_n}^2$ in $V_2$ (see (C.1)). The numerator and the denominator separate out in the expectation value: the first gives a simple factor of $1/p^+$ and the second an integral of the kind
\[
\int_0^\infty \frac{dz_n}{z_n^4} |\psi_{r=0}^{\alpha=0}(z_n)|^2
\]

Using $\psi_0^0(z) \sim \sqrt{z}$ near $z = 0$ (see (4.9)) the above integral, summed over $M$ bits, gives a leading divergence
\[
M \int_{z_{\text{min}}}^\infty \frac{dz}{z^3} \sim M \frac{1}{z_{\text{min}}^2}
\]  
(5.6)

Here we have put an infra-red cut-off $z_{\text{min}}$ near the boundary $z = 0$.

As for the $V_1$ contribution in (5.5) (see Appendix C for details), the matrix element $\langle \Psi_0 | V_1 | \Psi_{\text{int}} \rangle$ has finite $z$-integrals of the type
\[
\sum_n \int_0^\infty \frac{dz_n}{z_n^3} \psi_{r=0}^{\alpha=0}(z_n)\psi_r^{\alpha=2}(z_n) = MC_1(r)
\]  
(5.7)

where $C_1(r)$ also includes the finite angular and fermion matrix elements. The $z$-integral is finite (esp. at $z \to 0$), being $\sim \int dz_z z^{1/2+7/2}/z^3$, using the fact that for $z \to 0$, $\psi_r^\alpha(z) \sim z^{\alpha+1/2}$ (see (4.9)). The initial state has $\alpha = 0, r = 0$. The reason for the appearance of $\alpha = 2$ in the intermediate state is that $V_1$ contains (i) a $n^a$ which converts the initial $l = 0$ state to a $l = 1$ state, and (ii) an $\eta_i\eta_j$ term (or its conjugate), which converts the initial $F_\eta = 2$ state to a $F_\eta = 4$ (or 0, respectively). For such fermion states, (see definition of $\alpha$ following (4.9)), we have $\alpha = l + 2$ which = 3 in this case.

However, although we do not have a divergent matrix element, we have a sum over an infinite number of intermediate states, given by
\[
-M \sum_{r=0}^\infty \frac{C_1^2(r)}{E_1(r)} = -M \sum_r \left( A_1 \frac{B_1}{r} + O\left(\frac{1}{r^2}\right)\right) \sim -M(A_1 r_{\text{max}} + B_1 \ln r_{\text{max}} + \text{finite})
\]  
(5.8)

since $E_1(r) \equiv E_{\text{int}}(r) - E_0 \sim r, C_1(r) \sim \sqrt{r}$ for large $r$. 

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5.2.2 Cancellation between $V_1$ and $V_2$ contributions

The two divergences (5.6) and (5.8) are hard to compare since the cut-off’s are different. There is a way, however, to recast the $V_2$ calculation using the $r_{\text{max}}$ cut-off. By using (see (A.9)) \{ $Q_i^-, Q_j^-$ \} = $-P^{-} \delta_i^j$ we can write the $V_2$ term in (5.5) as

$$\frac{1}{4} \sum_{\text{int}} |\langle \Psi_0 \sum_n q_{n+1,n}^i | \Psi_{\text{int}} \rangle|^2$$

In a manner similar to (5.8), the matrix element is now finite, say $C_2(r)$ (which includes a finite $z$-integral !)

$$M \sum_{r=0}^{\infty} C_2^2(r) = M \sum_r r_{\text{max}} (A_2 + \frac{B_2}{r} + O(\frac{1}{r^2})) \sim M (A_2 r_{\text{max}} + B_2 \ln r_{\text{max}} + \text{finite}) \quad (5.9)$$

As we will find in Appendix C, $A_1 = A_2, B_1 = B_2$.

This implies that the divergent contributions from $V_2$ and $V_1$ in (5.3) cancel, leaving a finite result \footnote{It might seem surprising at first that there is a cancellation of divergences between two rather different looking terms. This is, however, standard in supersymmetric theories. E.g. in a theory with one scalar superfield with $\Phi^3$ interaction the mass corrections cancel between a second order perturbation coming from $V_1 = g\bar{\psi}\psi\phi$ and a first order perturbation coming from $V_2 = g^2\phi^4$.}. Since for the $V_2$ contribution, we have calculated the same quantity using the $z_{\text{min}}$ cut-off as well as the $r_{\text{max}}$ cut-off, these two cut-offs are clearly related

$$\frac{1}{z_{\text{min}}^2} \sim r_{\text{max}}$$

reflecting a UV-IR relation. This in particular implies that the divergence from both $V_1$ and $V_2$ can be interpreted as coming from strings near $z = 0$.

5.2.3 Other states

In the above we have considered as initial state a tensor product of specific single-bit states. Our choice was prompted by the maximally divergent $z_{\text{min}}$ behaviour in the $V_2$ contribution. For instance, instead of an $F_\eta = 2$ state, if we take $F_\eta = 1$ or $F_\eta = 3$ states, we will have a $\ln z_{\text{min}}$ divergence. In fact these two cases are the only ones with a potential divergence. None of the infinite number of other states, with additional $s, \bar{s}$ or $r$ excitations, or $F_\eta = 0$ or $F_\eta = 4$, have any divergent $z$-integration. The methods we have employed work pretty much the same way for the other divergent initial states. Calculations to check the cancellation of divergences in these remaining cases are in progress and will be reported elsewhere.

5.2.4 Summary of this subsection

The fact that the $z \to 0$ divergences cancel implies that our postulate of treating the string world sheet theory perturbatively around $\lambda = 0$ is the correct one. It is clear that in a purely bosonic model without fermionic terms such a cancellation
would not have occurred. We believe that the cancellation will persist to all orders of perturbation theory. It is clearly important to explicitly check this.

6. String spectrum and AdS/CFT

In this section we will try to construct string states out of the multi-bit states discussed above. Our attempts to do so and to compare with the boundary gauge theory will turn out to be unsuccessful, for reasons which are discussed below.

The only multi-bit states that are relevant in a straightforward continuum limit are the ones with an infinite number of bits. All such states have infinite energy. Consider, for example, the state in (5.3). For a large number of bits \( m \), it is a highly degenerate state, with the energy diverging as \( 2^m \). A simple “normal ordering” prescription that gets rid of this divergence does not work. This is because these states satisfy a unitarity bound, which is saturated by the symmetric state in (5.4) which transforms as \((2^m, 0, 0 | 0, 2^m, 0)\) under the superconformal group. For example, subtracting out the divergence from the energy will violate the unitarity bound for this state. In fact, it is easy to see that this is generically the case. One needs a much more sophisticated normal ordering prescription which depends on the \(SO(6)\) quantum numbers of the state in such a way that the finite part in energy left after the subtraction continues to satisfy the unitarity bound. This requirement is equivalent to preserving the superconformal algebra in the normal ordered theory since the unitarity bounds follow from this algebra. It is not obvious how to do this. At any rate, we have so far not succeeded in finding such a normal ordering prescription.

This difficulty in the construction of continuum strings motivates us to consider the alternative possibility of an essential discreteness to strings in AdS at zero radius. Thus we may consider the possibility of defining string states as the multi-bit states constructed in the previous section with a finite number of bits. In this case strings with any number of bits must be considered since even if to start with all strings consist of the same number of bits, their splitting and joining will give rise to other strings with smaller as well as larger number of bits. Thus we must consider strings made up of \(1, 2, 3, \cdots, \infty\) bits. In the following we will consider such string states and compare their properties with the corresponding operators in the boundary gauge theory. Like in the previous section, the states/operators will be labeled by their energy/dimension (\(E_0\) value). As discussed below, it turns out that the spectrum of the multi-bit states does not agree with the spectrum of operators in the boundary gauge theory.

1. **Dimension 1**: There are no gauge invariant operators of dimension 1 (e.g. \( \text{Tr } F_{\mu \nu} = 0 \), since we are dealing with \(SU(N)\) gauge theory). On the string theory side too, there is no state of dimension 1.
2. **Dimension 2:** String states of dimension 2 are provided by single bit strings (which correspond to \( m = 1 \) in the notation of point (2) in section 6). As we have remarked in section 4.1.3, such states simply form the supergravity multiplet in \( \text{AdS}_5 \times S^5 \) background. The lowest dimension state of this multiplet (4.13) corresponds to dimension 2. The states in the supergravity multiplet have been extensively studied \([7]\) and shown to be in one-to-one correspondence with the local, gauge-invariant, single-trace chiral operators of the boundary gauge theory, establishing a complete equivalence of such Yang-Mills operators with single-bit string states in our theory. In particular, the state (4.13) corresponds to the dimension 2, traceless, symmetric tensor \( \text{Tr} \phi^a \phi^b \).

**Missing non-BPS states in the string description:** Besides the chiral operators, however, there are non-chiral gauge invariant operators such as \( \text{Tr} \phi^a \phi^a \) \textsuperscript{13}, which fall in the \( \text{SU}(2,2|4) \) multiplet \((2,0,0 \rightarrow 0,0,0)\). Unfortunately, we do not have any such normalizable state in the string spectrum \textsuperscript{14}.

3. **Dimension 3:** We found that at dimension 2, the BPS counting matches between string theory and gauge theory. How about dimension 3? We find that the matching continues there: Consider the gauge theory operator \( O_3 = A_{(abc)} \text{Tr}(\phi^a \phi^b \phi^c) \), which has the \( \text{PSU}(2,2|4) \) quantum numbers \((3,0,0|50)\). With these quantum numbers this is the only gauge invariant operator, since in a \( \text{SU}(N) \) YM theory, we cannot for example have \( O_{1,2} = A_{(abc)} \text{Tr}(\phi^a)\text{Tr}(\phi^b \phi^c) \).

Interestingly the story is similar in our string construction. As we remarked above, corresponding to \( O_3 = (3,0,0|50) \), there is a single-bit string state. It turns out that this is the only state, since there cannot be a multi-bit state because even for \( m = 2 \) bits, the minimum value of \( E_0 \) will be 4, one higher than that of \( O_3 \).

**Non-BPS:** The mismatch vis-a-vis non-BPS states continues at dimension 3, however. For instance, there is no string state corresponding to \( \text{Tr} \phi^a F^a_{\mu \nu} \).

4. **Dimension 4 and above—extra states in string theory:** At dimension 4, we have the following two chiral gauge theory operators in gauge theory \( O_4 = A_{(abcd)} \text{Tr}(\phi^a \phi^b \phi^c \phi^d) \) and \( O_{2,2} = A_{(abcd)} \text{Tr}(\phi^a \phi^b)\text{Tr}(\phi^c \phi^d) \), both carrying quantum numbers \((E_0, J_1, J_2|\text{SU}(4) \text{ repr.}) = (4,0,0|105)\). A dimension 4 state in the string theory can arise in the following ways:

\textsuperscript{13}In the usual discussion of AdS supergravity vs CFT at large 'tHooft coupling, one normally ignores these non-chiral operators because they have large anomalous dimensions which corresponds to large curvature corrections in supergravity and hence such states are not considered accessible in AdS supergravity which is concerned with small curvature. We are, however, dealing with strings at small radius AdS or small \( \lambda \) for which the anomalous dimensions of non-chiral operators become small; thus we must address these states.

\textsuperscript{14}There is a non-normalizable state carrying these quantum numbers.
(a) one 2-bit dimension 4 state (see \( m = 2 \) in (5.4): this has a dimension 2, \( l = 0 \) excitation at each bit)

(b) tensor product of two 1-bit, \( l = 0 \) string states (this is a multiple-string state)

(c) an \( l = 1 \) excited state of a 1-bit string state.

Clearly we have extra states in the string theory. Since the states (a) and (b) appear to be in natural correspondence with the operators \( O_4 \) and \( O_{2,2} \) respectively, the state (c) appears to be a misfit. Indeed, the state (c) is part of a “Kaluza-Klein tower” over a single bit. Ideally one would have liked to have one state per bit and a string description to arise after combining various bits, but this is not the situation we have. In our formalism each bit behaves like a free superparticle in the AdS background with all the corresponding states available to it.

Generalization to operators of higher dimensions is straightforward. At dimension 5, we have \( O_5 = \text{Tr} \phi^5, O_{2,3} = \text{Tr} \phi^2 \text{Tr} \phi^3 \), and there are again three string states, (a) a 2-bit string state (a dimension 3 excitation in one bit, and a dimension 2 in the other), (b) a 2-string state, each with one bit, where the bits are similar to (a), and (c) a single bit state of dimension 5. Again (c), belonging to the tower over a string state, appears as extra.

We conclude that the spectrum of string states considered as multi-bit states with finite number of bits does not agree with the spectrum of the operators in the boundary gauge theory.

7. Discussion and concluding remarks

In this paper we have begun the program of quantizing string theory in AdS at small radius. We make essential use of the light cone gauge. In this gauge, strings at zero radius seem to fall apart into discrete bits. We have computed the free spectrum of multi-bit states and demonstrated that the leading corrections to this spectrum, coming from a non-zero value of the radius, are finite. Potential divergences which arise from strings near the boundary \( z = 0 \) cancel because of bose-fermi cancellations.

Our central aim is to establish a working framework of string theory in AdS at small radius as a perturbative expansion around zero radius. Although we have made some progress towards this goal, the most important problem of obtaining the spectrum of free strings at zero radius remains unsolved. The importance of a resolution of this issue cannot be overstated. One immediate application of these ideas would then be to test the strong (arbitrary \( \lambda \)) version
of Maldacena conjecture. The strong version of the conjecture allows definition of nonperturbative string theory (in AdS) in terms of gauge theory.

We also have here the beginnings of a calculable framework of strings coupled to a noncompact space whose gravity description approaches a singular limit. This example is more non-trivial than that of a string theory in backgrounds where the compact part of the spacetime includes an orbifold or a conifold singularity. The singularity here is not only in a finite codimensional submanifold in space, but the entire spacetime collapses in the sense that the (constant) curvature blows up uniformly all over spacetime. Calculation of string theory correlation functions in such a background should be interesting.

Finally, the success of the light-cone gauge, at least perturbatively, encourages its use in other backgrounds which have physical gauge theory duals, especially those which describe a confining gauge theory like [13].

We end by mentioning that it will clearly be important to eventually extend the present framework to take into account string-string interactions, which we have ignored in this paper.

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Dedication

This work is dedicated to the memory of Professor Bunji Sakita, who passed away on 31st August 2002. Prof. Sakita made many pioneering contributions to high energy physics and in particular to string theory. His ‘Reminiscences’ [44] are a striking summary of an exciting period in high energy physics. He was a great mentor and a deeply humble and generous man. He also contributed significantly to the high energy physics effort at the Tata Institute of Fundamental Research in Mumbai. His obituary can be found at http://www.cerncourier.com/main/article/42/9/20/3.
A. Generators of the superconformal algebra in the light-cone gauge

The bosonic part of the \( \text{psu}(2,2|4) \) superalgebra is the \( \text{so}(4,2) \) algebra. In terms of the standard hermitian rotation generators, \( S_{AB} \), this algebra is

\[
[S_{AB}, S_{CD}] = -i(\eta_{BC}S_{AB} - \eta_{BD}S_{AC} + \eta_{AD}S_{BC} - \eta_{AC}S_{BD}),
\]

where \( A, B, \cdots = -1, 0, 1, \cdots, 4 \) and the metric \( \eta_{AB} \) is \( \text{diag} \, [-1, -1, 1, 1, 1] \). The \( S_{AB} \) acts on AdS\(_5\), viewed as the hyperbola

\[
\eta_{AB}Y^A Y^B = -R^2 \quad (A.2)
\]

Eqn. (A.1) can be recast into the standard conformal algebra in (3+1)-dimensions by the redefinitions (see, for example, [33])

\[
\tilde{D} = S_{-14}, \quad S_{\mu-1} = \frac{1}{2}(\tilde{P}_\mu + \tilde{K}_\mu), \quad S_{\mu4} = \frac{1}{2}(\tilde{P}_\mu - \tilde{K}_\mu), \quad S_{\mu\nu} = \tilde{J}_{\mu\nu}, \quad (A.3)
\]

where \( \mu, \nu = 0, 1, 2, 3 \). We will actually be working with a slightly different set of generators for the conformal algebra. We will use anti-hermitian generators for the dilations and rotations, while retaining hermitian generators for translations and special conformal transformations. That is, we will set

\[
\tilde{D} = iD, \quad \tilde{J}_{\mu\nu} = -iJ_{\mu\nu}. \quad (A.4)
\]

It will also be convenient to set

\[
\tilde{P}_\mu = -\sqrt{2}P_\mu, \quad \tilde{K}_\mu = -\sqrt{2}K_\mu. \quad (A.5)
\]

In terms of these generators the conformal algebra is

\[
[D, P_\mu] = -P_\mu, \quad [D, K_\mu] = K_\mu, \quad [P_\mu, K_\nu] = \eta_{\mu\nu}D - J_{\mu\nu},
\]

\[
[J_{\mu\nu}, P_\gamma] = \eta_{\nu\gamma}P_\mu - \eta_{\mu\gamma}P_\nu, \quad [J_{\mu\nu}, K_\gamma] = \eta_{\nu\gamma}K_\mu - \eta_{\mu\gamma}K_\nu,
\]

\[
[J_{\mu\nu}, J_{\gamma\tau}] = \eta_{\nu\gamma}J_{\mu\tau} - \eta_{\mu\gamma}J_{\nu\tau} + \eta_{\mu\tau}J_{\nu\gamma} - \eta_{\nu\tau}J_{\mu\gamma}. \quad (A.6)
\]

To fix conventions for our light-cone frame, we define the light-cone variables as follows.

\[
x^\pm \equiv \frac{(x^3 \pm x^0)}{\sqrt{2}}, \quad x = \frac{(x^1 + ix^2)}{\sqrt{2}}, \quad \bar{x} = \frac{(x^1 - ix^2)}{\sqrt{2}}.
\]

Similarly

\[
p^\pm \equiv \frac{(p_3 \pm p_0)}{\sqrt{2}}, \quad p^x = \frac{(p_1 + ip_2)}{\sqrt{2}}, \quad p^{\bar{x}} = \frac{(p_1 - ip_2)}{\sqrt{2}}.
\]

The corresponding operator statements are in (A.10). The metric in the light-cone frame is

\[
\eta_{++} = \eta_{--} = \eta_{xx} = \eta_{\bar{x}\bar{x}} = 1.
\]
The algebra in (A.6) needs to be supplemented by the supersymmetry generators, $Q_i^\pm$ and $S_i^\pm$ in the light-cone gauge, to get the full superconformal algebra. The additional commutators are

$$[D, Q_i^\pm] = -\frac{1}{2}Q_i^\pm, \quad [D, S_i^\pm] = \frac{1}{2}S_i^\pm,$$

$$[J^{+-}, Q_i^\pm] = \pm\frac{1}{2}Q_i^\pm, \quad [J^{+-}, S_i^\pm] = \mp\frac{1}{2}S_i^\pm, \quad [J^{xx}, Q_i^\pm] = \pm\frac{1}{2}Q_i^\pm, \quad [J^{xx}, S_i^\pm] = \mp\frac{1}{2}S_i^\pm,$$

$$[Q_i^\pm, J^j_k] = \delta^j_i Q_k^\pm - \frac{1}{4}\delta^j_i Q^\pm_k, \quad [S_i^\pm, J^j_k] = \delta^j_i S_k^\pm - \frac{1}{4}\delta^j_i S^\pm_k,$$

$$[J^{+x}, Q_i^-] = Q_i^{+i}, \quad [J^{-x}, Q_i^+] = -Q_i^{-i}, \quad [J^{+x}, S_i^-] = -S_i^+ - i, \quad [J^{-x}, S_i^+] = S_i^- + i,$$

$$[S_i^\pm, P^\pm] = Q_i^\pm, \quad [S_i^\mp, P^\mp] = Q_i^\mp, \quad [S_i^+, P^+] = -Q_i^+, \quad [S_i^-, P^-] = Q_i^-,$$

$$[Q_i^\pm, K^\pm] = -S_i^\pm, \quad [Q_i^+, K^x] = -S_i^-, \quad [Q_i^-, K^x] = S_i^+.$$  \hspace{1cm} (A.7)

Here $J^j_k$ are the generators of su(4) algebra in hermitian basis, $(J^j_j)^\dagger = J^j_j$,

$$[J^j_j, J^k_k] = \delta^j_j J^k_k - \delta^k_k J^j_j.$$  \hspace{1cm} (A.8)

The anticommutators are

$$\{Q_i^{+i}, Q_j^{-i}\} = \pm P^{\pm j}_i, \quad \{Q_i^{+i}, Q_j^{+j}\} = P^{\pm j}_i,$$

$$\{S_i^{+i}, S_j^{-i}\} = \pm K^{\pm j}_i, \quad \{S_i^{+i}, S_j^{+j}\} = K^{\pm j}_i,$$

$$\{Q_i^{+i}, S_j^{+j}\} = -J^{+x} \delta^j_i, \quad \{Q_i^{-i}, S_j^{-j}\} = -J^{-x} \delta^j_i,$$

$$\{Q_i^{+i}, S_j^{-j}\} = \frac{1}{2}(J^{+-} + J^{xx} \mp D) \delta^j_i \mp J^j_j.$$  \hspace{1cm} (A.9)

The rest of the (anti)commutation relations are obtained by using the hermiticity conditions

$$P^{\pm j} = P_j^{\pm j}, \quad P^{\pm j} = P_j^{\pm j}, \quad K^{\pm j} = K_j^{\pm j}, \quad K^{\pm j} = K_j^{\pm j}, \quad (Q_i^{+j})^\dagger = Q_i^{+j}, \quad (S_i^{+j})^\dagger = S_i^{+j},$$

$$\quad (J^{+x})^\dagger = -J'^{+x}, \quad (J^{-x})^\dagger = -J'^{-x}, \quad (J^{xx})^\dagger = J'^{xx}, \quad D^\dagger = -D.$$  \hspace{1cm} (A.10)

### A.1 String realization of the superconformal generators

Expressions for many of the generators have been obtained in \[28\] \[15\]. All the generators may be written as integrals over the closed string of the corresponding density,

$$G = \int_0^l d\sigma \, G(\sigma),$$

where $l$ is the length of the string. Below we reproduce the light-cone gauge expressions obtained in \[28\] for these densities. These are written in terms of

\[15\] Expressions for the rest can be obtained from these by using the (anti)commutation relations given above.
the bosonic transverse co-ordinates, \(x(\sigma), \bar{x}(\sigma), z^a(\sigma)\), their momentum conjugates \(p^x(\sigma), p^z(\sigma), p^a(\sigma)\), the superpartners \(\theta^i(\sigma), \eta^i(\sigma)\) and their conjugates.

\[
\mathcal{P}^- = -\frac{1}{2p^+} \left( 2p^x p^x + (p^z)^2 + \frac{1}{z^2} \delta^{ij} l^i \theta^j \eta^i + (\eta^i \eta^i - 2) - \frac{1}{4} \right) - \frac{T^2}{2p^+ z^2} (2x' x' + (z')^2) \]
\[
+ \frac{T}{z^2 p^+} \eta^i \rho^a_{ij} \eta^a (\theta^j - i \sqrt{2} \eta^i x'), \quad \mathcal{J}^{++} = -ix p^+, \]
\[
\mathcal{J}^{+} = -ix p^+ + 2, \quad \mathcal{J}^{xx} = i(x p^x - \bar{x} p^x) + \frac{1}{2} (\theta^i \theta_i - \eta^i \eta_i), \]
\[
\mathcal{D} = i(x^+ p^x + x p^x) + ip^z - \frac{1}{2}, \quad \mathcal{K}^{+} = \frac{1}{2} (z^2 + 2x \bar{x}) p^+, \]
\[
\mathcal{K}^x = \frac{1}{2} z^2 p^x - x(x^+ p^x + x p^x) + \frac{i}{2} (\theta^i \theta_i + \eta^i \eta_i) + \frac{\theta^i}{\sqrt{p^z}} S^+_i, \]
\[
\mathcal{J}^{i} = l^i + \theta^i \theta_j + \eta^i \eta_j - \frac{1}{4} \delta^i_j (\theta^k \theta_k + \eta^k \eta_k), \quad Q^+_i = \sqrt{p^+} \theta_i, \quad S^+_i = \sqrt{\frac{p^+}{2}} z \eta_i + i \sqrt{p^+} x \theta_i, \]
\[
Q^-_i = \frac{1}{\sqrt{2p^+}} (\sqrt{2} \eta^i x_i - i \eta p^x - \frac{1}{z} (-3 \eta_3/2 + \eta^i \eta_i - 2(\eta_i) i)) \]
\[
+ \frac{T}{\sqrt{2p^+} z^2} \rho^a_{ij} z^a (\theta^j - i \sqrt{2} \eta^i x'). \quad (A.20)
\]

Notation used above: \(\rho^a_{ij}\) and \(\rho^{aij}\) are SU(4) Clebsh-Gordon coefficients (SO(6) Dirac matrices in the chiral representation). Some useful identities involving them are

\[
\rho^a_{ij} \rho^{bijk} + \rho^b_{ij} \rho^{aijk} = 2 \delta^{ab} \delta^j_i, \]
\[
\rho^a_{ij} = -\rho^a_{ji}, \quad \rho^{aij} = (\rho^{aij})^*, \quad \rho^{aij} \rho^{km} = 2 (\delta^i_m \delta^j_k - \delta^i_k \delta^j_m) \]
\[
\rho^a_{ij} = \frac{1}{2} \epsilon_{ijkl} \rho^{aijkl}. \quad (A.21)
\]

Also, \(l^i_j\) is given by

\[
l^i_j = \frac{i}{8} [\rho^a, \rho^b]_{ij} l^a, \quad l^{ab} = n^a p^b - n^b p^a, \quad (A.22)
\]

and satisfies the identities

\[
l^i m l^m_j = \frac{1}{4} p^q l^p q^i \delta^j = 2 l^i_j, \quad [l^i_j, l^k_m] = \delta^k_j l^i_m - \delta^i_m l^k_j. \quad (A.23)
\]

In the above formulae, \(p^z\) is the momentum conjugate to \(z\) in the spherical polar co-ordinates and \(P^a\) is the corresponding quantity for \(n^a\). Also, note that \(n^a P^a = 0.\)
Canonical (anti)commutation relations:

In order to evaluate the above generators (A.13)-(A.20) on wavefunctions we need the (anti)commutation relations between the fundamental variables, which are as follows:

\[ [x_0, p^+] = i, \]
\[ [x(\sigma), p^x(\sigma')] = [\bar{x}(\sigma), p^x(\sigma')] = [z(\sigma), p^z(\sigma')] = i\delta(\sigma - \sigma'), \]
\[ [n^a(\sigma), \mathcal{P}^b(\sigma')] = i(\delta^{ab} - n^{a}n^{b})\delta(\sigma - \sigma'), \]
\[ [\mathcal{P}^a(\sigma), \mathcal{P}^b(\sigma')] = -i(n^a\mathcal{P}^b - n^b\mathcal{P}^a)\delta(\sigma - \sigma'), \]
\[ \{\eta^i(\sigma), \eta_j(\sigma')\} = \delta^i_j\delta(\sigma - \sigma') \] (A.24)

A.2 Useful linear combinations

In addition to the above generators, some special linear combinations turn out to be useful (see (1.1). We have already used \( H_\pm, E_0 \) which are given by:

\[ H_+ = -P^- + K^+, \quad H_- = P^+ - K^-, \quad E_0 = \frac{1}{2}(H_+ + H_-) \] (A.25)

We list below some more. Consider the \( so(4) = su(2) \times su(2) \) subalgebra of \( so(4,2) \); let us call its generators \( \mathbf{I}_i, \mathbf{K}_i \). We write these in terms of the above generators as

\[ \mathbf{I}_3 = \frac{1}{2}J^{xx} + \frac{1}{4}(H_+ - H_-), \quad \mathbf{K}_3 = \frac{1}{2}J^{xx} - \frac{1}{4}(H_+ - H_-) \]
\[ \mathbf{I}_+ = \frac{1}{2}(J^{++} + J^{--} - P^z + K^z), \quad \mathbf{K}_+ = \frac{1}{2}(J^{++} + J^{--} + P^z - K^z), \]
\[ \mathbf{I}_- = \frac{1}{2}(-J^{++} - J^{--} - P^z + K^z), \quad \mathbf{K}_- = \frac{1}{2}(-J^{++} - J^{--} + P^z - K^z) \] (A.26)

Using the above we get

\[ J^{xx} = \mathbf{K}_3 + \mathbf{I}_3, \quad H_\pm = E_0 \mp (\mathbf{K}_3 - \mathbf{I}_3) \] (A.27)

On a state annihilated by \( \mathbf{I}_-, \mathbf{K}_+ \), we have \( \mathbf{K}_3 = J_1, \mathbf{I}_3 = -J_2 \), therefore

\[ H_\pm = E_0 \mp (J_1 + J_2), \quad J^{xx} = J_1 - J_2 \] (A.28)

We also list here some special linear combinations of the odd generators which are particularly useful:

\[ \bar{Q}_i^\pm = Q_i^\pm \mp S_i^{\mp}, \quad \bar{Q}_i^{\mp i} = Q_i^{\pm i} \pm S_i^{\mp i}, \] (A.29)
\[ \bar{S}_i^\pm = Q_i^\pm \pm S_i^{\mp}, \quad \bar{S}_i^{\mp i} = Q_i^{\mp i} \mp S_i^{\mp i} \] (A.30)

The first line raises the value of \( E_0 \), while the second lowers the value of \( E_0 \), by 1/2. The bottom of a supermultiplet is defined by annihilation of the \( \bar{S} \) operators (cf. these are similar to the \( Q', S' \) operators discussed in [33].)
B. Discretized expressions for the generators

In this appendix we list the discretized versions of the expressions for the generators given above. As discussed in the main text, section 3, the lattice spacing $\epsilon$ completely disappears from these expressions after appropriately rescaling the momenta and the fermionic variables to have finite canonical commutation relations among the bit variables. These rescalings are

$$(p^+_n, p^x_n, p^z_n, p^a_n, P^+_n, P^x_n, P^z_n, P^a_n) = \epsilon \left( p^+_n(\sigma), p^x_n(\sigma), p^z_n(\sigma), p^a_n(\sigma), P^+_n(\sigma), P^x_n(\sigma), P^z_n(\sigma), P^a_n(\sigma) \right), \quad \sigma = n\epsilon$$

$$(x^0_n, x_n, \bar{x}_n, z_n, n^a_n) = \left( x^-(\sigma), x(\sigma), \bar{x}(\sigma), z(\sigma), n^a(\sigma) \right),$$

$$(\eta^i_n, \eta_j, \theta^i_n, \theta_j) = \epsilon^{1/2} \left( \eta^i(\sigma), \eta_j(\sigma), \theta^i(\sigma), \theta_j(\sigma) \right)$$

The discrete version of the canonical (anti)commutation relations (A.24) is

$$[x_n, p^a_m] = i\delta_{nm}, \quad [\bar{x}_n, p^a_m] = i\delta_{nm}, \quad [z_n, p^a_m] = i\delta_{nm},$$

$$[n^a_m, P^b_m] = i(\delta^{ab} - n^a_m n^b_m)\delta_{nm},$$

$$[P^a_m, P^b_m] = -i(n^a_m P^b_m - n^b_m P^a_m)\delta_{nm},$$

$$\{\eta^i_n, \eta_j\} = \delta_{nm}\delta_j^i, \quad \{\theta^i_n, \theta_j\} = \delta_{nm}\delta^j_i.$$ (B.2)

The oscillators for the two transverse directions $x, \bar{x}$, parallel to the boundary, are defined as follows:

$$a = \frac{p^x - ip^+ x}{\sqrt{2p^+}}, \quad a^\dagger = \frac{p^x + ip^+ \bar{x}}{\sqrt{2p^+}}, \quad [a, a^\dagger] = 1,$$

$$\bar{a} = \frac{p^\bar{x} - ip^+ \bar{x}}{\sqrt{2p^+}}, \quad \bar{a}^\dagger = \frac{p^\bar{x} + ip^+ x}{\sqrt{2p^+}}, \quad [\bar{a}, \bar{a}^\dagger] = 1.$$ (B.3)

The general form of the discrete version of the generators, in the limit $T \to 0$ indicated by the subscript '0' on the generators, is $G_0 = \sum_{n=1}^M g_{0n}$, where $g_{0n}$ is the corresponding generator for the $n$th bit. We list the latter below, but omit the bit index for ease of notation.

$$p^+_0 = -\frac{1}{2p^+} \left( 2p^x p^\bar{x} + (p^+)^2 + \frac{1}{z^2} \left( (l^i_j l^i_j) + 4\eta_i l^i_j \eta^j + (\eta_i \eta^j - 2) - \frac{1}{4} \right) \right),$$

$$j^0_{0x} = -ixp^+,$$ (B.5)

$$j^0_{0y} = -ixp^+ + 2,$$ (B.6)

$$j^0_{0z} = i(xp^x - \bar{x}p^\bar{x}) + \frac{1}{2}(\eta_i \eta^i - \theta_i \theta^i),$$

$$d_0 = i(x^+ p^+ + xp^+ + \bar{x}p^\bar{x} + izp^z - \frac{1}{2},$$

$$k^+_0 = \frac{1}{2}(z^2 + 2x\bar{x})p^+,$$ (B.9)
We assume the string to consist of $H$ bits. Perturbation of $H$ can be independently obtained from (B.15) using (A.9):

$$k_0^x = \frac{1}{2}z^2 p^x - x \left( x^{-} p^+ + x p^{-} + z p^+ + i \frac{1}{2} \left( \theta^i \partial_i + \eta^i \eta_i \right) \right) + \frac{1}{\sqrt{p^+}} \theta^i s^+_0,$$

(B.10)

$$j_0^i = l_0^i + \theta^i \partial_i + \eta^i \eta_i - \frac{1}{4} \delta^i_j \left( \theta^k \theta_k + \eta^k \eta_k \right),$$

(B.11)

$$q_{0i}^+ = \sqrt{p^+} \theta_i, \quad s_0^+ = \sqrt{p^+} z \eta_i + i \sqrt{p^+} x \theta_i,$$

(B.12)

$$q_{0i}^- = \frac{1}{\sqrt{2p^+}} \left( \sqrt{2p^+} \theta_i - i \eta_i p^z - \frac{1}{2} \left( -3 \eta_i / 2 + \eta^j \eta_j - 2(\eta l)_i \right) \right).$$

(B.13)

Others can be obtained using the (anti)commutation relations.

When $T$ is non-zero some of the generators get modified by terms that involve bit-bit interactions. We write these generators as $G = G_0 + G_1$, where $G_1 = \sum_n g_1(n+1,n)$ vanishes for $T = 0$. We list $g_1(n+1,n)$ terms for some of the generators that get modified.

$$-p_{1(n+1,n)}^- = \frac{T^2}{2p^+} \left( \frac{2| x_{n+1} - x_n |^2}{z_n^2} + \left( \frac{z_{n+1} - z_n}{z_n} \right)^2 \right) - \frac{T}{z_n^2} \eta_i \rho_j \theta^a \left( \left( n_{n+1} - n_n \right) - \frac{1}{2} \left( \theta^i_{n+1} - \theta^i_n \right) \right) - \frac{i \sqrt{2}}{z_n} \eta_j (x_{n+1} - x_n),$$

(B.14)

$$q_{1(n+1,n)}^i = \frac{T}{\sqrt{2p^+}} \theta^a \left( \frac{n_{n+1}^a - n_n^a}{z_n} (\theta^i_{n+1} - \theta^i_n) \right) - i \sqrt{2} \frac{n_{n+1}^a}{z_n^2} \eta^a_j (x_{n+1} - x_n).$$

(B.15)

These expressions are obtained by a straightforward discretization of the corresponding continuum expressions in (A.11) and (A.20). There is of course no a’ priori unique discretized expression. We make a natural choice for a minimal set and then fix the rest by demanding that the algebra be satisfied. Thus, e.g., (B.14) can be independently obtained from (B.13) using (A.9): $\{ Q^{-i}, Q^- \} = - P^- \delta^i_j$.

C. Perturbation of $H_+$ spectrum and cancellation of $z \to 0$ divergence

We assume the string to consist of $M$ bits. We define $H_+ = H_0^0 + TV_1 + T^2 V_2$.

$$H_0^0 = \sum_{n=1}^{M} \left[ - \frac{\partial_{x_n} \partial_{x_n}}{p^+} + p^+ x_n \bar{x}_n - \frac{\partial_{x_n}^2}{2p^+} + \frac{p^+}{z_n^2} + \frac{1}{2p^+ z_n^2} ((l^i_{n,j})^2 + 4 \eta_{n,i} l^i_{n,j} \eta^j_n + (\eta_{n,i} \eta^j_n - 2)^2 - \frac{1}{4}) \right]$$

$$V_2 = \sum_{n=1}^{M} \left[ \frac{1}{2p^+} \left( \frac{2| x_{n+1} - x_n |^2}{z_n^4} + \left( \frac{z_{n+1} - z_n}{z_n} \right)^2 \right) \right]$$

16These are $P^-, K^-, J^{-\bar{x}}, J^{-\bar{x}}, Q^{-i}$ and $S^{-i}$, the so-called dynamical generators. Note that they also receive corrections from string-string interactions, which we are ignoring here.
\[ V_1 = \sum_{n=1}^{M} v_{1(n+1,n)} = 2 \sum_{n=1}^{M} \left[ -\frac{1}{2z_n^2} \eta_n^i \rho_{ij}^n \eta_n^o \left( (\theta_{n+1}^j - \theta_n^j) - i\frac{\sqrt{2}}{z_n} \eta_n^i (x_{n+1} - x_n) \right) \right. \\
+ \left. \frac{1}{2z_n^2} \eta_n^i \rho_{ij}^n \eta_n^o \left( (\theta_{n+1}^j - \theta_{n+1}^j) + i\frac{\sqrt{2}}{z_n} \eta_n^i (\bar{x}_{n+1} - \bar{x}_n) \right) \right] \quad\tag{C.1} \]

C.1 Calculation of \( \langle \Psi_0 | V_2 | \Psi_0 \rangle \)

We will calculate this quantity as

\[
\langle \Psi_0 | V_2 | \Psi_0 \rangle = \frac{1}{4} \langle \Psi_0 | \{ Q_{1,i}^- , Q_{1,i}^- \} | \Psi_0 \rangle \\
= \frac{1}{4} \sum_{n,m} \sum_{\text{int}} \left[ \langle \Psi_0 | q_{(n+1,n)}^- | \Psi_{\text{int}} \rangle \langle \Psi_{\text{int}} | q_{(n+1,n)}^- | \Psi_0 \rangle \\
+ \langle \Psi_0 | q_{(n+1,n)}^- | \Psi_{\text{int}} \rangle \langle \Psi_{\text{int}} | q_{(n+1,n)}^- | \Psi_0 \rangle \right] \quad\tag{C.2} \]

The first equality can be derived by considering the \( T^2 \) term in the equality \(-P^- = \frac{1}{4} \{ Q_{1,i}^- , Q_{1,i}^- \} \). Here,

\[
q_{(n+1,n)}^- = \frac{1}{\sqrt{2z_n}} (\rho_n n_{ij}) \left[ (\theta_{n+1}^j - \theta_n^j) + \frac{1}{z_n} \eta_n^i \left( (a_{n+1} - a_{n+1}^t) - (a_n - a_n^t) \right) \right] \quad\tag{C.3} \]

This expression is as in \( (B.13) \) except that we have rescaled \( \sqrt{p^+} z_n \rightarrow z_n, \sqrt{p^+} x_n \rightarrow x_n \) and have used \(-i\sqrt{2}x_n = a_n - a_n^t\).

Since the intermediate states \( | \Psi_{\text{int}} \rangle \) involved for various terms in \( (C.3) \) are different, we can separately treat the contribution of each term in \( (C.3) \) to \( (C.2) \). Let us consider, e.g., the term

\[
\frac{1}{\sqrt{2z_n}} (\rho_n n_{ij}) \theta_{n+1}^j \quad\tag{C.4} \]

The intermediate state that will click here is represented in the figure below.

\[ | l = 1, 2\eta, \alpha, r \rangle \]

The wavefunction in the picture differs from the initial state only in the \( n \)-th and the \( n + 1 \)-th bits (the solid circles) and only in the quantum numbers exhibited \(^{17}\) (\( | 2\eta \rangle \) denotes \( | F_\eta = 2 \rangle \), etc). Explanation for the \( n \)-th bit: \( l = 1 \) appears because \( n_\alpha^l \) carries \( l = 1 \), the values of \( \alpha \) can be \( l, l + 2 \) or \( l + 4 \), i.e. 1, 3 or 5, depending on whether \( |2\eta\rangle \) belongs to \( P_2V_2, Q_2V_2 \) or \( R_2V_2 \) (see \( (1.14) \) and below). Only the quantum number \( r \) remains unspecified, any \( r = 0, 1, 2... \) clicks. The corresponding matrix element of the \( z \)-wavefunctions is (cf. \( C_2(r) \) in \( (1.9) \))

\[
C_{2\eta,\alpha}^{\eta,\alpha}(r) = \int_0^\infty dz \psi_0^\eta(z) \frac{1}{\sqrt{2z}} \psi_r^\alpha(z) = \frac{1}{\sqrt{2}} \frac{(r + \frac{\alpha - 1}{2})!}{r!} \sqrt{\frac{r!}{(r + \alpha)!}} \quad\tag{C.5} \]

\(^{17}\)The full \( | \Psi_{\text{int}} \rangle \) is a cyclic permutation of such wavefunctions.
Having done the $z$- (and the trivial $x, \bar{x}$-) integration let us now leave the $S^5$ and the fermion matrix elements unevaluated. This amounts to finding an effective interaction in the finite dimensional $S^5$ and fermionic sector, after integrating out the bosonic degrees of freedom $z, x, \bar{x}$. After carrying out the sum $\sum_r$ involved in the Eqn. (C.2) for each $\alpha = 1, 3$ or 5, we get the following contribution of the interaction (C.4) to (C.2):

\[ \frac{1}{4} \sum_{n,m} \sum_{\alpha=1,3,5} C^{2\eta,\alpha}_2 \langle \tilde{\Psi}_0 | (\rho.n)_ij \theta^i_{n+1} | \tilde{\Psi}_{\text{int},(2\eta,\alpha)} \rangle \langle \tilde{\Psi}_{\text{int},(2\eta,\alpha)} | (\rho.n)_k \theta^k_{m+1,k} | \tilde{\Psi}_0 \rangle \] (C.6)

where

\[ C^{2\eta,\alpha}_2 = \sum_{r=0}^{\infty} \left( C^{2\eta,\alpha}_2 (r) \right)^2 \] (C.7)

In the above, the tildes on the wavefunctions denote that they depend now only on the $S^5$ and the fermionic coordinates. The $(2\eta, \alpha)$ tag on the $\tilde{\Psi}_{\text{int}}$ reminds us that we must choose the appropriate projection ($P_2, Q_2$ or $R_2$) on the wavefunction $|l = 1, 2\eta \rangle$ on whichever bit it appears.

What remains now is to compute the equivalent of (C.6) for all the other interaction terms. We will write here only one other term in (C.3) in some detail, viz.

\[ \frac{1}{\sqrt{2}z_n} (\rho.n)_ij \eta^i_{\alpha} a_{n+1} \] (C.8)

The pictorial representation of the intermediate state now is

\[ \begin{array}{c}
\circ & \circ \\
|l = 1, 3\eta, \alpha, r \rangle & |s = 1 \rangle \\
n & n+1
\end{array} \]

Once again only the changed quantum numbers are shown. $s = 1$ means a harmonic oscillator excitation of the 'a' type (see (4.9)). The $\alpha$ value of $|l = 1, 3\eta \rangle$ is determined ($\alpha = l + 1 = 2$ or $\alpha = l + 3 = 4$), depending on whether the $F_{\eta} = 3$ state belongs to $P_3V_3$ or $Q_3V_3$. The $z, x, \bar{x}$ integration gives

\[ C^{3\eta,\alpha}_2 (r) = \int_0^\alpha dz \psi^0_0(z) \frac{1}{\sqrt{2}z} \psi^a_r(z) = \frac{\sqrt{2}}{\alpha} \frac{(r + \frac{\alpha}{2})!}{\sqrt{r!(r + \alpha)!}} \]

The contribution to (C.2) in terms of $S^5$ and fermions is, in this case,

\[ \frac{1}{4} \sum_{n,m} \sum_{\alpha=2,4} C^{3\eta,\alpha}_2 \langle \tilde{\Psi}_0 | (\rho.n)_ij \eta^i_{m+1} a_{n+1} | \tilde{\Psi}_{\text{int},(3\eta,\alpha)} \rangle \langle \tilde{\Psi}_{\text{int},(3\eta,\alpha)} | (\rho.n)_k \eta^k_{m+1,k} | \tilde{\Psi}_0 \rangle \] (C.9)

where

\[ C^{3\eta,\alpha}_2 = \sum_{r=0}^{\infty} \left( C^{3\eta,\alpha}_2 \right)^2 \] (C.10)
From these examples, it is clear how to compute equations like (C.9) and (C.10) for all the other terms in (C.2).

**Divergence:**
If we explicitly evaluate $C_2^{2\eta,\alpha}, C_2^{3\eta,\alpha}$ using their definitions, we find that

\[
C_2^{2\eta,\alpha} = \frac{1}{2} \sum_{r=1}^{\infty} \frac{1}{r} + F_2(\alpha)
\]

\[
C_2^{3\eta,\alpha} = A_2(\alpha) \sum_{r=1}^{\infty} \frac{1}{r} + B_2'(\alpha) \sum_{r=1}^{\infty} \frac{1}{r} + F_2'(\alpha)
\]

(C.11)

Thus the two types of divergences that appear here are $\sum_{r=1}^{r_{\text{max}}} \frac{1}{r}$ and $\sum_{r=1}^{r_{\text{max}}} \frac{1}{r}$, where we have replaced the upper limit $r = \infty$ by a cut-off $r_{\text{max}}$. The coefficients of such divergences in (C.2) are calculated by combining equations like (C.6) and (C.9).

**C.2 The $V_1$ contribution**

This term has an expression

\[
-\frac{1}{4} \sum_{n,m} \sum_{\text{int}} \frac{1}{E_{\text{int}} - E_0} \left[ \langle \Psi_0 | v_{1(n+1,n)} | \Psi_{\text{int}} \rangle \langle \Psi_{\text{int}} | v_{1(n+1,n)} | \Psi_0 \rangle \right]
\]

(C.12)

which is similar to (C.2), except that it has an additional factor due to the energy denominator.

Like in the case of the $V_2$ term in the previous subsection, we will consider each term of $v_{1(n+1,n)}$ in turn (see (C.1)), consider the intermediate states that click in each case, and “integrate out” the $z, x, \bar{x}$ variables, leaving matrix elements depending on $S^5$ and fermions.

As an example we consider

\[
\frac{1}{z^2 p^+} \eta_n^i (\rho, n)_{i,j} \theta_{n+1}^j
\]

The intermediate state is represented by the diagram

The values of $\alpha$ here are $l + 1 = 2$ or $l + 3 = 4$. The $z$-integral involved in the matrix element is $C_1^{3\eta,\alpha}(r)$, which is identical to $C_2^{3\eta,\alpha}(r)$. After performing the $r$-sum we get the following contribution

\[
\frac{1}{4} \sum_{n,m} \sum_{\alpha=2,4} C_1^{3\eta,\alpha} \langle \tilde{\Psi}_0 | \eta_n^i (\rho, n)_{i,j} \theta_{n+1}^j | \tilde{\Psi}_{\text{int},(3\eta,\alpha)} \rangle \langle \tilde{\Psi}_{\text{int},(3\eta,\alpha)} | (\rho, n)_{m,k} \theta_{m+1,k} \theta_{n,l} | \tilde{\Psi}_0 \rangle
\]

(C.13)
where
\[ C_1^{3\eta,\alpha} = \sum_r \frac{[C_2^{3\eta,\alpha}(r)]^2}{(2r + \alpha + 2) - 2} \] (C.14)

The divergence of this term is of the type \( \sum_r r_{\text{max}} 1/r \).

### C.3 Cancellation of Divergence

We now collect the coefficient of \( \sum_r r_{\text{max}} 1 \) and \( \sum_r r_{\text{max}} (1/r) \) of all the terms in (C.2) and (C.12). We will explicitly write here only the coefficient of \( \sum_r 1 \). This is given by

\[
\frac{1}{4} \sum_{n,m} \left[ \langle \tilde{\Psi}_0 | (\rho_n)^{ij} \eta^i_n \tilde{\Psi}_{\text{int},(3\eta,\alpha=2)} | (\rho_m)^{ki} \eta^k_m | \tilde{\Psi}_0 \rangle 
+ \frac{1}{4} \langle \tilde{\Psi}_0 | (\rho_n)^{ij} \eta^i_n \tilde{\Psi}_{\text{int},(3\eta,\alpha=4)} | (\rho_m)^{ki} \eta^k_m | \tilde{\Psi}_0 \rangle \right] 
\]

\[
\frac{1}{4} \sum_{n,m} \left[ \langle \tilde{\Psi}_0 | (\rho_n)^{ij} \eta^i_n \tilde{\Psi}_{\text{int},(1\eta,\alpha=2)} | (\rho_m)^{jk} \eta^j_m | \tilde{\Psi}_0 \rangle 
+ \frac{1}{4} \langle \tilde{\Psi}_0 | (\rho_n)^{ij} \eta^i_n \tilde{\Psi}_{\text{int},(1\eta,\alpha=4)} | (\rho_m)^{jk} \eta^j_m | \tilde{\Psi}_0 \rangle \right] 
\]

\[-\frac{1}{4} \sum_{n,m} \langle \tilde{\Psi}_0 | \eta^i_n \rho_n^{ij} \eta^j_n \tilde{\Psi}_{\text{int},(4\eta,\alpha=3)} | \eta_m^k \rho_m^{kl} \eta^l_m | \tilde{\Psi}_0 \rangle 
- \frac{1}{4} \sum_{n,m} \langle \tilde{\Psi}_0 | \eta_{n,i} \rho_n^{ij} \eta^i_n \tilde{\Psi}_{\text{int},(0\eta,\alpha=3)} | \eta_m^k \rho_m^{kl} \eta^l_m | \tilde{\Psi}_0 \rangle \]

(C.15)

By repeatedly using (a) the fundamental anticommutation relations of the fermions (B.2), (b) the explicit form of the projection operators \( P_1, Q_1, P_3 \) and \( Q_3 \) (see (4.10), (4.11)), and (c) \( n^a \)-space vev’s such as

\[ \langle n^a n^b \rangle = \frac{1}{6} \delta^{ab}, \quad \langle n^a n^b \rangle = \frac{1}{24} [\rho^a, \rho^b]^i_j \]

we find that the above term (C.15) indeed vanishes. Note that the first two terms (with positive sign) arise from \( V_2 \) the last two (negative sign) from \( V_1 \).

The coefficient of \( \sum_r r_{\text{max}} 1/r \) vanishes by a similar, but somewhat lengthier computation.

### D. Nonlocal features of stringy representation in the light cone gauge

Although many generators of the superconformal algebra involve only the zero mode \( x_0^- \) of \( x^-(\sigma) \), some of the generators, like \( K^x \), also involve non-zero modes of \( x^-(\sigma) \). These generators have expressions which are non-local in \( \sigma \). In general,
therefore, even at $T = 0$, a representation of the stringy realization of the superconformal algebra may be achieved only by taking linear combinations (over and above the cyclic sum in (5.1)) of naive tensor product of single-bit states. In the following we will consider the simplest possible example of such a state (5.4) where a linear combination (a la Clebsch-Gordon) is already in place because we have taken a symmetric and traceless tensor product of the $SU(4)$ polarizations.

In order to see if such states carry a representation of the superconformal algebra, including the non-local generators, let us consider the example of the $so(4)$ subalgebra of the AdS$_5$ group $so(4,2)$. As shown in (A.20), the generators of this algebra involve $K^x$ which contains $\tilde{x}^-$ and the action of $so(4)$ therefore constitutes a non-trivial example.

The state (5.4) is expected to be a singlet of $so(4)$. Therefore it must be annihilated by $K_\pm, I_\pm$. Since these generators depend on $x^-$, we need to first construct an appropriate action of this operator on the states. Recall that in the light cone gauge the classical Gauss law constraint of $\sigma$-reparametrizations is

$$x^-(\sigma) = x(0) - \int_0^\sigma \frac{d\tilde{\sigma}}{2p^+}(x'p^\pm + \tilde{x}'p^\pm + z'p^\pm + n^{a'}p^a + i(\eta^i\eta_i^\prime - \eta^i\eta_i + \theta^i\theta_i' - \theta^i\theta_i)))(\tilde{\sigma}). \quad (D.1)$$

Here $x(0)$ is related to the zero mode $x_0^\pm \equiv \int_0^1 d\sigma \ x^- (\sigma)$ by

$$x(0) = x_0^- + \int_0^1 d\sigma \int_0^\sigma \frac{d\tilde{\sigma}}{2p^+}(x'p^\pm + \tilde{x}'p^\pm + z'p^\pm + n^{a'}p^a + i(\eta^i\eta_i^\prime - \eta^i\eta_i + \theta^i\theta_i' - \theta^i\theta_i)))(\tilde{\sigma}). \quad (D.2)$$

To discretize a combination of the form $p^i(\tilde{\sigma})x_i^+(\tilde{\sigma})$, we replace the local term $p^i(\tilde{\sigma})$ by $(p_{k+1}^i + p_k^i)/2$, the $\tilde{\sigma}$-derivative by a discrete difference, i.e., $x_i^+(\tilde{\sigma}) \rightarrow \epsilon^{-1}(x_i^+(x_{i,k+1} - x_{i,k}))$ and perform the rescalings (B.1). This leads to

$$x_{n+1}^- = x_0^- - \frac{1}{2p^+}(1 - \frac{1}{M} \sum_{n=1}^M \sum_{k=1}^n [(x_{k+1}^- - x^-_k)(p_{k+1}^x + p_k^x) + (\tilde{x}_{k+1}^- - \tilde{x}_k)(p_{k+1}^\tilde{x} + p_k^\tilde{x}) + (z_{k+1}^- - z_k)(p_{k+1}^z + p_k^z) + (n_{k+1}^a p_k^a - n_k^a p_{k+1}^a) + i\beta(\frac{z_{k+1}}{z_k} - \frac{z_k}{z_{k+1}})) + i\left\{\left(\eta_{i+1}^i \eta_{i,k} - \eta_k^i \eta_{i,k+1}\right) + \left(\theta_{i+1}^i \theta_{i,k} - \theta_k^i \theta_{i,k+1}\right)\right\}] \quad (D.3)$$

In this expression $\beta$ is a constant. This extra term is expected in the quantum theory for the action of $x^-(\sigma)$ on wave-functions $\psi$ in the polar coordinates $z, n^a$.

To evaluate the action of the $so(4)$ generators $K_\pm, I_\pm$ on the state (5.4) we also need a discretized expression for the generator $J^x$. This can be guessed from
the classical continuum expression for it, which may be obtained from the known generators by using the algebra and the basic Poisson brackets. In this way we get

\[
J^{-x} = \sum_{n=1}^{M} \left[ ip_n^x x_n^– - ix_n P_n^- + \frac{1}{\sqrt{p^+}} \theta_i^n Q_i^n \right. \\
\left. - \frac{1}{2} \theta_i^n \theta_i^n + \frac{1}{2} \eta_i^n \eta_i^n \right] \right] p_n^+ \\
(D.4)
\]

Using this one can show that, for an appropriate value of the constant \(\beta\) in (D.3), the \(so(4)\) generators \(K_\pm, I_\pm\) annihilate the state (5.4). Note that in the fermionic part of the wavefunction, it is important to take the traceless symmetric linear combination of the fermion polarizations. For instance, one could consider a state which has only \(\eta_{1,n}\eta_{2,n}\theta_{1,n}\theta_{2,n}\) at each bit \(n\).

One needs to check that the value of \(\beta\) for which the above is true is compatible with the superconformal algebra. In fact, it is an open question whether the discretized algebra is satisfied. It is likely that the specific prescription for discretization plays an important role in this. We do not have a complete understanding of this issue yet and hope to come back to it [45].

Summary:

With the above action of \(x^- (\sigma)\) on \(|\Psi_{\text{sym}}\rangle\), it is easy to see that both \(H_-\) and the lowering operators \(\tilde{S}\) (see (A.30)) annihilate \(|\Psi_{\text{sym}}\rangle\).

References

[1] E. Brezin and S. R. Wadia, “The Large N Expansion In Quantum Field Theory And Statistical Physics: From Spin Systems To Two-Dimensional Gravity,” World Scientific, 1993.

[2] A. M. Polyakov, “Gauge Fields As Rings Of Glue,” Nucl. Phys. B 164, 171 (1980).

[3] A. M. Polyakov, “The wall of the cave,” Int. J. Mod. Phys. A 14, 645 (1999) [arXiv:hep-th/9809057].

[4] E. Witten, “Black holes and quark confinement,” Current Science, Vol 81 No 12, 25th Dec 2001. http://tejas.serc.iisc.ernet.in/currsci/dec252001/contents.htm

[5] J. Maldacena, “The large \(N\) limit of superconformal field theories and supergravity”, Adv. Theor. Math. Phys. 2, 231 (1998) [Int. J. Theor. Phys. 38, 1113 (1998)], hep-th/9711200.

[6] J. R. David, G. Mandal and S. R. Wadia, “Microscopic formulation of black holes in string theory,” Phys. Rept. 369, 549 (2002) [arXiv:hep-th/0203048].

[7] S.S. Gubser, I.R. Klebanov and A.M. Polyakov, “Gauge-theory correlators from non-critical string theory”, Phys. Lett. B428, 105 (1998), hep-th/9802109.
[8] E. Witten, “Anti-de Sitter space and holography”, Adv. Theor. Math. Phys. 2, 253 (1998), hep-th/9802150.

[9] O. Aharony, S.S. Gubser, J. Maldacena, H. Ooguri and Y. Oz, “Large N field theories, string theory and gravity”, Phys. Rept. 323, 183 (2000), hep-th/9905111.

[10] S. J. Rey and J. T. Yee, “Macroscopic strings as heavy quarks in large N gauge theory and anti-de Sitter supergravity,” Eur. Phys. J. C 22, 379 (2001) [arXiv:hep-th/9803001]; S. J. Rey, S. Theisen and J. T. Yee, “Wilson-Polyakov loop at finite temperature in large N gauge theory and anti-de Sitter supergravity,” Nucl. Phys. B 527, 171 (1998) [arXiv:hep-th/9803135].

[11] J. M. Maldacena, “Wilson loops in large N field theories,” Phys. Rev. Lett. 80, 4859 (1998) [arXiv:hep-th/9803002].

[12] A. M. Polyakov and V. S. Rychkov, “Loop dynamics and AdS/CFT correspondence,” Nucl. Phys. B 594, 272 (2001) [arXiv:hep-th/0005173].

[13] N. Drukker, D. J. Gross and H. Ooguri, “Wilson loops and minimal surfaces,” Phys. Rev. D 60, 125006 (1999) [arXiv:hep-th/9904191].

[14] D. Berenstein, R. Corrado, W. Fischler and J. M. Maldacena, “The operator product expansion for Wilson loops and surfaces in the large N limit,” Phys. Rev. D 59, 105023 (1999) [arXiv:hep-th/9809188].

[15] G. W. Semenoff and K. Zarembo, “Wilson loops in SYM theory: From weak to strong coupling,” Nucl. Phys. Proc. Suppl. 108, 106 (2002) [arXiv:hep-th/0205033].

[16] K. G. Wilson, “Confinement Of Quarks,” Phys. Rev. D 10, 2445 (1974).

[17] A. M. Polyakov, “Compact Gauge Fields And The Infrared Catastrophe,” Phys. Lett. B 59, 82 (1975).

[18] J. B. Kogut and L. Susskind, “Hamiltonian Formulation Of Wilson’s Lattice Gauge Theories,” Phys. Rev. D 11, 395 (1975).

[19] R. Giles and C. B. Thorn, “A Lattice Approach To String Theory,” Phys. Rev. D 16, 366 (1977).

[20] D. Bernstein, J. Maldacena and H. Nastase, “Strings in flat space and pp waves from $\mathcal{N} = 4$ Super Yang Mills”, hep-th/0202021.

[21] C. Kristjansen, J. Plefka, G. W. Semenoff and M. Staudacher, “A new double-scaling limit of $\mathcal{N} = 4$ super Yang-Mills theory and PP-wave strings,” Nucl. Phys. B 643, 3 (2002) [arXiv:hep-th/0205033].

[22] N.R. Constable, D.Z. Freedman, M. Headrick, S. Minwalla, L. Motl, A. Postnik and W. Skiba, “PP-wave string interactions from perturbative Yang-Mills theory”, hep-th/0205089.
[23] J. G. Zhou, “pp-wave string interactions from string bit model,” Phys. Rev. D 67, 026010 (2003) [arXiv:hep-th/0208232].

[24] H. Verlinde, “Bits, Matrices and 1/N”, hep-th/0206059; D. Vaman and H. Verlinde, ‘Bit Strings from $\mathcal{N} = 4$ Gauge Theory”, hep-th/0209215.

[25] G. ’t Hooft, “A Planar Diagram Theory For Strong Interactions,” Nucl. Phys. B 72, 461 (1974).

[26] A. Karch, “Lightcone quantization of string theory duals of free field theories,” arXiv:hep-th/0212041.

[27] R.R. Metsaev and A.A. Tseytlin, “Superstring action in $\text{AdS}_5 \times S^5$: $\kappa$-symmetry light cone gauge”, Phys. Rev. D63, 046002 (2001), hep-th/0007036.

[28] R.R. Metsaev, C.B. Thorn and A.A. Tseytlin, “Light-cone superstring in AdS space-time”, Nucl. Phys. B596, 151 (2001), hep-th/0009171.

[29] G. Mack, “All Unitary Ray Representations Of The Conformal Group SU(2,2) With Positive Energy,” Commun. Math. Phys. 55, 1 (1977).

[30] V. K. Dobrev and V. B. Petkova, “All Positive Energy Unitary Irreducible Representations Of Extended Conformal Supersymmetry,” Phys. Lett. B 162, 127 (1985).

[31] M. Gunaydin and N. Marcus, “The spectrum of the $S^5$ compactification of the chiral $\mathcal{N} = 2, D = 10$ supergravity and the unitary supermultiplets of U(2, 2|4)”, Class. Quant. Grav. 2, L11 (1985).

[32] H.J. Kim, L.J. Romans and P. van Nieuwenhuizen, “The mass spectrum of chiral $\mathcal{N} = 2, D = 10$ supergravity on $S^5$”, Phys. Rev. D32, 389 (1985).

[33] S. Minwalla, “Restrictions imposed by superconformal invariance on quantum field theories,” Adv. Theor. Math. Phys. 2, 781 (1998) [arXiv:hep-th/9712074].

[34] S. Ferrara, C. Fronsdal and A. Zaffaroni, “On $\mathcal{N} = 8$ supergravity on AdS(5) and $\mathcal{N} = 4$ superconformal Yang-Mills theory,” Nucl. Phys. B 532, 153 (1998) [arXiv:hep-th/9802203].

[35] R. R. Metsaev, “Massless arbitrary spin fields in AdS(5),” Phys. Lett. B 531, 152 (2002) [arXiv:hep-th/0201226].

[36] R.R. Metsaev, “Light cone gauge formulation of IIB supergravity in AdS$_5 \times S^5$ background and AdS/CFT correspondence”, Phys. lett. B468, 65 (1999), hep-th/9908114.

[37] A. A. Tseytlin, “On limits of superstring in AdS(5) x S**5,” Theor. Math. Phys. 133, 1376 (2002) [Teor. Mat. Fiz. 133, 69 (2002)] [arXiv:hep-th/0201112].
[38] N. Beisert, C. Kristjansen, J. Plefka, G. W. Semenoff and M. Staudacher, “BMN correlators and operator mixing in N = 4 super Yang-Mills theory,” Nucl. Phys. B 650, 125 (2003) [arXiv:hep-th/0208178].

[39] N.R. Constable, D.Z. Freedman, M. Headrick and S. Minwalla, “Operator Mixing and the BMN Correspondence”, hep-th/0209002.

[40] A. M. Polyakov, “Gauge fields and space-time,” Int. J. Mod. Phys. A 17S1, 119 (2002) [arXiv:hep-th/0110196].

[41] B. Sundborg, “The Hagedorn transition, deconfinement and N = 4 SYM theory,” Nucl. Phys. B 573, 349 (2000) [arXiv:hep-th/9908001].

[42] S. Minwalla, ICTP lectures "Stringy Thermodynamics in Large N Gauge Theories", April 2003. (Based on work (to appear) with O. Aharony, J. Marcano, S. Minwalla and M. Van Raamsdonk.)

[43] I. R. Klebanov and M. J. Strassler, “Supergravity and a confining gauge theory: Duality cascades and χ SB-resolution of naked singularities,” JHEP 0008, 052 (2000) [arXiv:hep-th/0007191].

[44] B. Sakita, “A Quest for symmetry: Selected works of Bunji Sakita,” World Scientific Series in 20th Century Physics, Vol. 22, 1999.; “Reminiscences,” arXiv:hep-th/0006083.

[45] A. Dhar, G. Mandal and S. R. Wadia, work in progress.