B-spline quasi-interpolant representations and sampling recovery of functions with mixed smoothness

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August 17, 2010 -- Version 0.95

Abstract

Let $\xi = \{x_j\}_{j=1}^n$ be a grid of $n$ points in the $d$-cube $I^d := [0,1]^d$, and $\Phi = \{\varphi_j\}_{j=1}^n$ a family of $n$ functions on $I^d$. We define the linear sampling algorithm $L_n(\Phi, \xi, \cdot)$ for an approximate recovery of a continuous function $f$ on $I^d$ from the sampled values $f(x_1), \ldots, f(x_n)$, by

$$L_n(\Phi, \xi, f) := \sum_{j=1}^n f(x_j)\varphi_j.$$ 

For the Besov class $B^\alpha_{p,\theta}$ of mixed smoothness $\alpha$ (defined as the unit ball of the Besov space $MB^\alpha_{p,\theta}$), to study optimality of $L_n(\Phi, \xi, \cdot)$ in $L_q(I^d)$ we use the quantity

$$r_n(B^\alpha_{p,\theta}) := \inf_{\xi, \Phi} \sup_{f \in B^\alpha_{p,\theta}} \|f - L_n(\Phi, \xi, f)\|_q,$$

where the infimum is taken over all grids $\xi = \{x_j\}_{j=1}^n$ and all families $\Phi = \{\varphi_j\}_{j=1}^n$ in $L_q(I^d)$. We explicitly constructed linear sampling algorithms $L_n(\Phi, \xi, \cdot)$ on the grid $\xi = G^d(m) := \{(2^{-k_1}s_1, \ldots, 2^{-k_d}s_d) \in I^d : k_1 + \ldots + k_d \leq m\}$, with $\Phi$ a family of linear combinations of mixed B-splines which are mixed tensor products of either integer or half integer translated dilations of the centered B-spline of order $r$. The grid $G^d(m)$ is of the size $2^m m^{d-1}$ and sparse in comparing with the generating dyadic coordinate cube grid of the size $2^{dm}$. For various $0 < p, q, \theta \leq \infty$ and $1/p < \alpha < r$, we proved upper bounds for the worst case error $\sup_{f \in B^\alpha_{p,\theta}} \|f - L_n(\Phi, \xi, f)\|_q$ which coincide with the asymptotic order of $r_n(B^\alpha_{p,\theta})$ in some cases. A key role in constructing these linear sampling algorithms, plays a quasi-interpolant representation of functions $f \in B^\alpha_{p,\theta}$ by mixed B-spline series with the coefficient functionals which are explicitly constructed as linear combinations of an absolute constant number of values of functions. Moreover, we proved that the quasi-norm of the Besov space $MB^\alpha_{p,\theta}$ is equivalent to a discrete quasi-norm in terms of the coefficient functionals.

Keywords Linear sampling algorithm · Quasi-interpolant · Quasi-interpolant representation · Mixed B-spline · Besov space of mixed smoothness.

Mathematics Subject Classifications (2000) 41A15 · 41A05 · 41A25 · 41A58 · 41A63.
1 Introduction

The aim of the present paper is to investigate linear sampling algorithms for recovery of functions on the unit $d$-cube $I^d := [0,1]^d$, having a mixed smoothness. Let $\xi = \{x^j\}_{j=1}^n$ be a grid of $n$ points in $I^d$, and $\Phi = \{\varphi_j\}_{j=1}^n$ a family of $n$ functions on $I^d$. Then for a continuous function $f$ on $I^d$, we can define the linear sampling algorithm $L_n = L_n(\Phi, \xi, \cdot)$ for approximate recovering $f$ from the sampled values $f(x^1), \ldots, f(x^n)$, by

$$L_n(f) = L_n(\Phi, \xi, f) := \sum_{j=1}^n f(x^j)\varphi_j.$$  

Let $L_q := L_q(I^d), 0 < q \leq \infty$, denote the quasi-normed space of functions on $I^d$ with the $q$th integral quasi-norm $\| \cdot \|_q$ for $0 < q < \infty$, and the ess sup-norm $\| \cdot \|_\infty$ for $q = \infty$. The recovery error will be measured by $\|f - L_n(\Phi, \xi, f)\|_q$.

If $W$ is a class of continuous functions, $\sup_{f \in B_{p,q}^s} \|f - L_n(\Phi, \xi, f)\|_q$ is the worst case error of the recovery of functions $f$ from $W$ by the linear sampling algorithm $L_n(\Phi, \xi, \cdot)$. To study optimality of linear sampling algorithms of the form (1.1) for recovering $f \in W$ from $n$ their values, we will use the quantity

$$r_n(W)_q := \inf_{\xi, \Phi} \sup_{f \in W} \|f - L_n(\Phi, \xi, f)\|_q,$$  

(1.2)

where the infimum is taken over all grids $\xi = \{x^j\}_{j=1}^n$ and all families $\Phi = \{\varphi_j\}_{j=1}^n$ in $L_q$.

A challenging problem in linear sampling recovery of functions from a class $W$ with a given mixed smoothness, is to construct a sampling algorithm $L_n(\Phi, \xi, \cdot)$ with an appropriate sampling grid $\xi = \{x^j\}_{j=1}^n$ and family $\Phi = \{\varphi_j\}_{j=1}^n$ which would be asymptotically optimal in terms of the quantity $r_n(W)_q$.

For periodic functions Smolyak [23] first constructed a specific linear sampling algorithm based on the de la Vallee Poussin kernel and the following dyadic grid in $I^d$

$$G^d(m) := \{(2^{-k_1}s_1, \ldots, 2^{-k_d}s_d) \in I^d: k \in \Delta(m)\} = \{2^{-k}s: k \in \Delta(m), s \in I^d(k)\}.$$  

Here and in what follows, we use the notations: $xy := (x_1y_1, \ldots, x_dy_d); 2x := (2x_1, \ldots, 2x_d); |x| := \sum_{i=1}^d |x_i|$ for $x, y \in \mathbb{R}^d$; $\Delta(m) := \{k \in \mathbb{Z}_+^d : |k| \leq m\}; I^d(k) := \{s \in \mathbb{Z}_+^d : 0 \leq s_i \leq 2k_i, \ i \in N[d]\}$; $N[d]$ denotes the set of all natural numbers from 1 to $d$; $x_i$ denotes the $i$th coordinate of $x \in \mathbb{R}^d$, i.e., $x := (x_1, \ldots, x_d)$. Temlyakov [24, 26, 27] and Dinh Dung [9–11] developed Smolyak’s construction for study the asymptotic order of $r_n(W)_q$ for periodic Sobolev classes $W^\alpha_p$ and Hölder classes $H^\alpha_p$ as well their intersection. In particular, the first asymptotic order

$$r_n(H^\alpha_p)_q \asymp (n^{-1}\log^{d-1}n)^{\alpha - 1/p + 1/q}(\log^{d-1}n)^{1/q}, 1 < p < q \leq 2, \ \alpha > 1/p,$$

was obtained in [9–10]. For non-periodic functions of mixed smoothness $1/p < \alpha \leq 2$, this problem has been recently studied by Sickel and Ullrich [22], using the mixed tensor product of piecewise linear B-splines (of order 2) and the grid $G^d(m)$. It is interesting to notice that the linear sampling algorithms considered by above mentioned authors are interpolating at the grid $G^d(m)$.  

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Naturally, the quantity \( r_n(W)_q \) of optimal linear sampling recovery is related to the problem of optimal linear approximation in terms of the linear \( n \)-width \( \lambda_n(W)_q \) introduced by Tikhomirov [28]:

\[
\lambda_n(W)_q := \inf_{A_n} \sup_{f \in W} \|f - A_n(f)\|_q,
\]

where the infimum is taken over all linear operators \( A_n \) of rank \( n \) in \( L_q \). The linear \( n \)-width \( \lambda_n(W)_q \) was studied in [14], [20], [21], etc. for various classes \( W \) of functions with mixed smoothness. The inequality \( r_n \geq \lambda_n \) is quite useful in investigation of the (asymptotic) optimality of a given linear sampling algorithm. It also allows to establish a lower bound of \( r_n \) via a known lower bound of \( \lambda_n \).

In the present paper, we continue to research this problem. We will take functions to be recovered from the Besov class \( B^\alpha_{p,\theta} \) of functions on \( I^d \), which is defined as the unit ball of the Besov space \( MB^\alpha_{p,\theta} \) having mixed smoothness \( \alpha \). For functions in \( B^\alpha_{p,\theta} \), we will construct linear sampling algorithms \( L_n(\Phi,\xi,\cdot) \) on the grid \( \xi = G^d(m) \) with \( \Phi \) a family of linear combinations of mixed B-splines which are mixed tensor products of either integer or half integer translated dilations of the centered B-spline of order \( r > \alpha \). We will be concerned with the worst case error of the recovery of \( B^\alpha_{p,\theta} \) in the space \( L_q \) by these linear sampling algorithms and their asymptotic optimality in terms of the quantity \( r_n(B^\alpha_{p,\theta})_q \) for various \( 0 < p, q, \theta \leq \infty \) and \( 1/p \leq \alpha < r \). A key role in constructing these linear sampling algorithms, plays a quasi-interpolant representation of functions \( f \in MB^\alpha_{p,\theta} \) by mixed B-spline series which will be explicitly constructed. Let us give a sketch of the main results of the present paper.

We first describe representations by mixed B-spline series constructed on the basis of quasi-interpolants. For a given natural number \( r \), let \( M \) be the centered B-spline of order \( r \) with support \([-r/2, r/2]\) and knots at the points \(-r/2, -r/2+1, ..., r/2-1, r/2,\). We define the integer translated dilation \( M_{k,s} \) of \( M \) by

\[
M_{k,s}(x) := M(2^k x - s), \quad k \in \mathbb{Z}_+, \quad s \in \mathbb{Z},
\]

and the mixed \( d \)-variable B-spline \( M_{k,s} \) by

\[
M_{k,s}(x) := \prod_{i=1}^d M_{k_i,s_i}(x_i), \quad k \in \mathbb{Z}_+, \quad s \in \mathbb{Z}^d, \tag{1.3}
\]

where \( \mathbb{Z}_+ \) is the set of all non-negative integers, \( \mathbb{Z}^d_+ := \{ s \in \mathbb{Z}^d : s_i \geq 0, \quad i \in N[d] \} \). Further, we define the half integer translated dilation \( M^*_{k,s} \) of \( M \) by

\[
M^*_{k,s}(x) := M(2^k x - s/2), \quad k \in \mathbb{Z}_+, \quad s \in \mathbb{Z},
\]

and the mixed \( d \)-variable B-spline \( M^*_{k,s} \) by

\[
M^*_{k,s}(x) := \prod_{i=1}^d M^*_{k_i,s_i}(x_i), \quad k \in \mathbb{Z}_+, \quad s \in \mathbb{Z}^d.
\]

In what follows, the B-spline \( M \) will be fixed. We will denote \( M_{k,s} := M_{k,s} \) if the order \( r \) of \( M \) is even, and \( M^*_{k,s} := M^*_{k,s} \) if the order \( r \) of \( M \) is odd.
Let $0 < p, \theta \leq \infty$, and $1/p < \alpha < \min(r, r - 1 + 1/p)$. Then we prove the following mixed B-spline quasi-interpolant representation of functions $f \in MB_{p, \theta}^\alpha$. Namely, a function $f$ in the Besov space $MB_{p, \theta}^\alpha$ can be represented by the mixed B-spline series

$$
f = \sum_{k \in \mathbb{Z}_+^d} \sum_{s \in J^d_r(k)} c^r_{k,s}(f) M^r_{k,s},$$

(1.4)

converging in the quasi-norm of $MB_{p, \theta}^\alpha$, where $J^d_r(k)$ is the set of $s$ for which $M^r_{k,s}$ do not vanish identically on $\mathbb{I}^d$, and the coefficient functionals $c^r_{k,s}(f)$ explicitly constructed as linear combinations of an absolute constant number of values of $f$ which does not depend on neither $k, s$ nor $f$. Moreover, we prove that the quasi-norm of $MB_{p, \theta}^\alpha$ is equivalent to some discrete quasi-norm in terms of the coefficient functionals $c^r_{k,s}(f)$. B-spline quasi-interpolant representations of functions from the isotropic Besov sapce has been constructed in [12], [13]. Different B-spline quasi-interpolant representations were considered in [7]. Both these representations were constructed on the basic of B-spline quasi-intepolants. The reader can see the books [2], [5] for survey and details on quasi-interpolants.

Let us construct linear sampling algorithms $L_n(\Phi, \xi, \cdot)$ on the grid $\xi = G^d(m)$ on the basic of the representation (1.4). For $m \in \mathbb{Z}_+$, let the linear operator $R_m$ be defined for functions $f$ on $\mathbb{I}^d$ by

$$
R_m(f) := \sum_{k \in \Delta^d(m)} \sum_{s \in J^d_r(k)} c^{r}_{k,s}(f) M^r_{k,s}.$$

(1.5)

If $\bar{m}$ is the largest of $m$ such that

$$2^{m} m^{d-1} \leq |G^d(m)| \leq n$$

for a given $n$, where $|A|$ denotes the cardinality of $A$, then the operator $R_{\bar{m}}$ is a linear sampling algorithm of the form (1.1) on the grid $G^d(\bar{m})$. More precisely,

$$R_{\bar{m}}(f) = L_n(\Phi, \xi, f) = \sum_{(k,s) \in G^d(\bar{m})} f(2^{-k} s) \psi_{k,s},$$

where $\psi_{k,s}$ are explicitly constructed as linear combinations of an absolute constant of B-splines $M^r_{k,j}$, which does not depend on neither $k, s$ nor $f$. It is worth to emphasize that the grid $G^d(m)$ is of the size $2^{m} m^{d-1}$ and sparse in comparing with the generating dyadic coordinate cube grid of the size $2^{dm}$. We give now a brief of our results concerning with the worst case error of the recovery of functions $f$ from $B_{p, \theta}^\alpha$ by the linear sampling algorithms $R_{\bar{m}}(f)$ and their asymptotic optimality.

We use the notations: $x_+ := \max(0, x)$ for $x \in \mathbb{R}$; $A_n(f) \ll B_n(f)$ if $A_n(f) \leq CB_n(f)$ with $C$ an absolute constant not depending on $n$ and/or $f \in W$, and $A_n(f) \asymp B_n(f)$ if $A_n(f) \ll B_n(f)$ and $B_n(f) \ll A_n(f)$. Let us introduce the abbreviations:

$$E(m) := \sup_{f \in B_{p, \theta}^\alpha} \| f - R_m(f) \|_q, \quad r_n := r_n(B_{p, \theta}^\alpha)_q.$$

Let $0 < p, q, \theta \leq \infty$ and $1/p < \alpha < r$. Then we have the following upper bound of $r_n$ and $E(\bar{m})$. 4
For $p \geq q$, \( r_n \ll E(\bar{m}) \ll \begin{cases} (n^{-1}\log^{d-1} n)^{\alpha}, & \theta \leq \min(q, 1), \\ (n^{-1}\log^{d-1} n)^{\alpha} (\log^{d-1} n)^{1/q - 1/\theta}, & \theta > \min(q, 1), \quad q \leq 1, \\ (n^{-1}\log^{d-1} n)^{\alpha} (\log^{d-1} n)^{1/\theta}, & \theta > \min(q, 1), \quad q > 1. \end{cases} \) \tag{1.6}

For $p < q$, \( r_n \ll E(\bar{m}) \ll \begin{cases} (n^{-1}\log^{d-1} n)^{\alpha - 1/p + 1/q} (\log^{d-1} n)^{(1/q - 1/\theta) +}, & q < \infty, \\ (n^{-1}\log^{d-1} n)^{\alpha - 1/p} (\log^{d-1} n)^{(1 - 1/\theta) +}, & q = \infty. \end{cases} \) \tag{1.7}

From the embedding of $MB_{\alpha p, \theta}^\alpha$ into the isotropic Besov space of smoothness $d\alpha$ and known asymptotic order of the quantity (1.2) of its unit ball in $L_q$ (see [8], [16], [17], [18], [27]) it follows that for $0 < p, q \leq \infty$, $0 < \theta \leq \infty$ and $\alpha > 1/p$, there always holds the lower bound $r_n \gg n^{-\alpha + (1/p - 1/q) +}$. However, this estimation is too rough and does not lead to the asymptotic order. By use of the inequality $\lambda_n(B_{p, \theta}^\alpha) \geq r_n$ and known results on $\lambda_n(B_{p, \theta}^\alpha)$ [13], [20], from (1.6) and (1.7) we obtain the asymptotic order of $r_n$ for some cases. More precisely, we have the following asymptotic orders of $r_n$ and $E(\bar{m})$ which show the asymptotic optimality of the linear sampling algorithms $R_{\bar{m}}$.

(i) For $p \geq q$ and $\theta \leq 1$,
\[
E(\bar{m}) \asymp r_n \asymp (n^{-1}\log^{d-1} n)^{\alpha}, \quad \begin{cases} 2 \leq q < p < \infty, \\ 1 < p = q \leq \infty. \end{cases} \tag{1.8}
\]

(ii) For $1 < p < q < \infty$,
\[
E(\bar{m}) \asymp r_n \asymp (n^{-1}\log^{d-1} n)^{\alpha - 1/p + 1/q} (\log^{d-1} n)^{(1/q - 1/\theta) +}, \quad \begin{cases} 2 \leq p, \quad 2 \leq \theta \leq q, \\ q \leq 2. \end{cases} \tag{1.9}
\]

The present paper is organized as follows. In Section 2 we give a necessary background of Besov spaces of mixed smoothness, B-spline quasi-interpolants, and prove a theorem on the mixed B-spline quasi-interpolant representation (1.4) and a relevant discrete equivalent quasi-norm for the Besov space of mixed smoothness $MB_{p, \theta}^\alpha$. In Section 3 we prove the upper bounds (1.6)–(1.7) and the asymptotic orders (1.8)–(1.9). In Section 4 we consider interpolant representations by series of the mixed tensor product of piecewise constant or piecewise linear B-splines. In Section 5 we present some auxiliary results.

## 2 B-spline quasi-interpolant representations

Let us introduce Besov spaces of functions with mixed smoothness and give necessary knowledge of them. For univariate functions the $l$th difference operator $\Delta_h^l$ is defined by
\[
\Delta_h^l f(x) := \sum_{j=0}^{l} (-1)^{l-j} \binom{l}{j} f(x + jh).
\]
If \( e \) is any subset of \( N[d] \), for multivariate functions the mixed \((l,e)\)th difference operator \( \Delta^l_e \) is defined by

\[
\Delta^l_e := \prod_{i \in e} \Delta^l_{h_i}, \quad \Delta^0_h = I,
\]

where the univariate operator \( \Delta^l_{h_i} \) is applied to the univariate function \( f \) by considering \( f \) as a function of variable \( x_i \) with the other variables held fixed. For a domain \( \Omega \) in \( \mathbb{R}^d \), denote by \( L_p(\Omega) \) the quasi-normed space of functions on \( \Omega \) with the \( p \)th integral quasi-norm \( \| \cdot \|_{p, \Omega} \) for \( 0 < p < \infty \), and the ess sup-norm \( \| \cdot \|_{\infty, \Omega} \) for \( p = \infty \). Let

\[
\omega^e_l(f, t)_p := \sup_{|\eta_i| < t_i, i \in e} \| \Delta^l_e(f) \|_{p, [I, \eta]}, \quad t \in \mathbb{R}^d,
\]

be the mixed \((l,e)\)th modulus of smoothness of \( f \), where \( \mathbb{R}^d(h, e) := \{ x \in \mathbb{R}^d : x_i + l \eta_i \in \mathbb{I}, \ i \in e \} \) (in particular, \( \omega^e_l(f, t)_p = \| f \|_p \)). We will need the following modified \((l,e)\)th mixed modulus of smoothness

\[
w^e_l(f, t)_p := \left( \prod_{i \in e} t_i^{-1} \int_{U(t, e)} \int_{I^d(h, e)} |\Delta^l_h(f, x)|^p \, dx \, dh \right)^{1/p},
\]

where \( U(t, e) := \{ x \in \mathbb{R}^d : x_i \leq t, \ i \in e \} \). There hold the following inequalities

\[
C_1 w^e_l(f, t)_p \leq \omega^e_l(f, t)_p \leq C_2 w^e_l(f, t)_p
\]

(2.1)

with constants \( C_1, C_2 \) which depend on \( l, p, d \) only. A proof of these inequalities for the univariate modulus of smoothness is given in [19]. They can be proven in a similar way for the multivariate \((l,e)\)th mixed modulus of smoothness.

If \( 0 < p, \theta \leq \infty, \alpha > 0 \) and \( l > \alpha \), we introduce the quasi-semi-norm \( \| f \|_{B^\alpha, \theta}_{p, \theta} \) for functions \( f \in L_p \) by

\[
|f|_{MB^\alpha, \theta}_{p, \theta} := \left\{ \left( \int_{\mathbb{R}^d} \left( \int_{I^d} t_i^{-\alpha_i} \omega^e_l(f, t)_p \right)^\theta \, dt \right)^{1/\theta} \right\}, \quad \theta < \infty,
\]

\[
\| f \|_{B^\alpha, \theta}_{p, \theta} = \sup_{t \in \mathbb{R}^d} \left( \int_{I^d} t_i^{-\alpha_i} \omega^e_l(f, t)_p \right)^{1/\theta}, \quad \theta = \infty
\]

(in particular, \( |f|_{MB^\alpha, \theta}_{p, \theta} = \| f \|_p \)).

For \( 0 < p, \theta \leq \infty \) and \( 0 < \alpha < l \), the Besov space \( MB^\alpha_{p, \theta} \) is defined as the set of functions \( f \in L_p \) for which the Besov quasi-norm \( \| f \|_{MB^\alpha_{p, \theta}} \) is finite. The Besov quasi-norm is defined by

\[
\| f \|_{MB^\alpha_{p, \theta}} := \sum_{e \subset N[d]} |f|_{MB^\alpha, \theta}_{p, \theta}.
\]

We will study the linear sampling recovery of functions from the Besov class

\[
B^\alpha_{p, \theta} := \{ f \in MB^\alpha_{p, \theta} : B(f) \leq 1 \},
\]

with the restriction on the smoothness \( \alpha > 1/p \), which provides the compact embedding of \( MB^\alpha_{p, \theta} \) into \( C(\mathbb{R}^d) \), the space of continuous functions on \( \mathbb{R}^d \) with max-norm. We will also study this
problem for $B_{p,\theta}^\alpha$ with the restrictions $\alpha = 1/p$ and $p \leq \min(1, \theta)$ which is a sufficient condition for the continuous embedding of $MB_{p,\theta}^\alpha$ into $C(\mathbb{I}^d)$. In both these cases, $B_{p,\theta}^\alpha$ can be considered as a subset in $C(\mathbb{I}^d)$.

For any $e \subset N[d]$, put $Z^d_+(e) := \{ s \in Z^d_+ : s_i = 0, \ i \notin e \}$ (in particular, $Z^d_+(\emptyset) = \{0\}$ and $Z^d_+(N[d]) = Z^d_+$). If $\{g_k\}_{k \in Z^d_+(e)}$ is a sequence whose component functions $g_k$ are in $L_p$, for $0 < p, \theta \leq \infty$ and $\beta \geq 0$ we define the $b_{\theta}^{\beta,e}(L_p)$ “quasi-norms”

$$\|\{g_k\}\|_{b_{\theta}^{\beta,e}(L_p)} := \left( \sum_{k \in Z^d_+(e)} \left( 2^{2\beta|k|} \|g_k\|_p \right)^{\theta} \right)^{1/\theta}$$

with the usual change to a supremum when $\theta = \infty$. When $\{g_k\}_{k \in Z^d_+(e)}$ is a positive sequence, we replace $\|g_k\|_p$ by $|g_k|$ and denote the corresponding quasi-norm by $\|\{g_k\}\|_{b_{\theta}^{\beta,e}}$.

For the Besov space $MB_{p,\theta}^\alpha$, from the definition and properties of the mixed $(l,e)$th modulus of smoothness it is easy to verify that there is the following quasi-norm equivalence

$$B(f) \simeq B_1(f) := \sum_{e \subset N[d]} \|\{\omega_f^e(l,2^{-k})\}\|_{b_{\theta}^{\alpha,e}}.$$

Let $\Lambda = \{\lambda(s)\}_{j \in P(\mu)}$ be a finite even sequence, i.e., $\lambda(-j) = \lambda(j)$, where $P(\mu) := \{ j \in \mathbb{Z} : |j| \leq \mu \}$ and $\mu \geq r/2 - 1$. We define the linear operator $Q$ for functions $f$ on $\mathbb{R}$ by

$$Q(f, x) := \sum_{s \in \mathbb{Z}} \Lambda(f, s) M(x - s), \quad (2.2)$$

where

$$\Lambda(f, s) := \sum_{j \in P(\mu)} \lambda(j) f(s - j). \quad (2.3)$$

The operator $Q$ is bounded in $C(\mathbb{R})$ and

$$\|Q(f)\|_{C(\mathbb{R})} \leq \|\Lambda\| \|f\|_{C(\mathbb{R})}$$

for each $f \in C(\mathbb{R})$, where

$$\|\Lambda\| = \sum_{j \in P(\mu)} |\lambda(j)|.$$

Moreover, $Q$ is local in the following sense. There is a positive number $\delta > 0$ such that for any $f \in C(\mathbb{R})$ and $x \in \mathbb{R}$, $Q(f, x)$ depends only on the value $f(y)$ at an absolute constant number of points $y$ with $|y - x| \leq \delta$. We will require $Q$ to reproduce the space $P_{r-1}$ of polynomials of order at most $r - 1$, that is, $Q(p) = p$, $p \in P_{r-1}$. An operator $Q$ of the form (2.2)–(2.3) reproducing $P_{r-1}$, is called a quasi-interpolant in $C(\mathbb{R})$.

There are many ways to construct quasi-interpolants. A method of construction via Neumann series was suggested by Chui and Diamond [3] (see also [2, p. 100–109]). A necessary and sufficient
condition of reproducing $\mathcal{P}_{r-1}$ for operators $Q$ of the form (2.2)–(2.3) with even $r$ and $\mu \geq r/2$, was established in [1]. De Bore and Fix [4] introduced another quasi-interpolant based on the values of derivatives.

Let us give some examples of quasi-interpolants. The simplest example is a piecewise constant quasi-interpolant which is defined for $r=1$ by

$$Q(f, x) := \sum_{s \in \mathbb{Z}} f(s)M(x-s),$$

where $M$ is the symmetric piecewise constant B-spline with support $[-1/2, 1/2]$ and knots at the half integer points $-1/2, 1/2$. A piecewise linear quasi-interpolant is defined for $r=2$ by

$$Q(f, x) := \sum_{s \in \mathbb{Z}} f(s)M(x-s),$$

where $M$ is the symmetric piecewise linear B-spline with support $[-1, 1]$ and knots at the integer points $-1, 0, 1$. This quasi-interpolant is also called nodal and directly related to the classical Faber-Schauder basic. We will revisit it in Section 4. A quadric quasi-interpolant is defined for $r=3$ by

$$Q(f, x) := \sum_{s \in \mathbb{Z}} \frac{1}{8} \{f(s-1) - 10f(s) + f(s+1)\}M(x-s),$$

where $M$ is the symmetric quadric B-spline with support $[-3/2, 3/2]$ and knots at the half integer points $-3/2, -1/2, 1/2, 3/2$. Another example is the cubic quasi-interpolant defined for $r=4$ by

$$Q(f, x) := \sum_{s \in \mathbb{Z}} \frac{1}{6} \{f(s-1) - 8f(s) + f(s+1)\}M(x-s),$$

where $M$ is the symmetric cubic B-spline with support $[-2, 2]$ and knots at the integer points $-2, -1, 0, 1, 2$.

If $Q$ is a quasi-interpolant of the form (2.2)–(2.3), for $h>0$ and a function $f$ on $\mathbb{R}$, we define the operator $Q(\cdot; h)$ by

$$Q(f; h) = \sigma_h \circ Q \circ \sigma_{1/h}(f),$$

where $\sigma_h(f, x) = f(x/h)$. By definition it is easy to see that

$$Q(f, x; h) = \sum_k \Lambda(f, k; h)M(h^{-1}x-k),$$

where

$$\Lambda(f, k; h) := \sum_{j \in \mathcal{P}(\mu)} \lambda(j)f(h(k-j)).$$

The operator $Q(\cdot; h)$ has the same properties as $Q$: it is a local bounded linear operator in $C(\mathbb{R})$ and reproduces the polynomials from $\mathcal{P}_{r-1}$. Moreover, it gives a good approximation for smooth functions [5, p. 63–65]. We will also call it a quasi-interpolant for $C(\mathbb{R})$. However, the
quasi-interpolant $Q(\cdot; h)$ is not defined for a function $f$ on $\mathbb{I}$, and therefore, not appropriate for an approximate sampling recovery of $f$ from its sampled values at points in $\mathbb{I}$. An approach to construct a quasi-interpolant for functions on $\mathbb{I}$ is to extend it by interpolation Lagrange polynomials. This approach has been proposed in [12] for the univariate case. Let us recall it.

For a non-negative integer $k$, we put $x_j = j2^{-k}, j \in \mathbb{Z}$. If $f$ is a function on $\mathbb{I}$, let

$$U_k(f, x) := f(x_0) + \sum_{s=1}^{r-1} \frac{2^s \Delta_{2^{-s}k} f(x_0)}{s!} \prod_{j=0}^{s-1} (x - x_j),$$

$$V_k(f, x) := f(x_{2^k - r+1}) + \sum_{s=1}^{r-1} \frac{2^s \Delta_{2^{-s}k} f(x_{2^k - r+1})}{s!} \prod_{j=0}^{s-1} (x - x_{2^k - r+1 + j})$$

be the $(r-1)$th Lagrange polynomials interpolating $f$ at the $r$ left end points $x_0, x_1, \ldots, x_{r-1}$, and $r$ right end points $x_{2^k - r+1}, x_{2^k - r+3}, \ldots, x_{2^k}$, of the interval $\mathbb{I}$, respectively. The function $f_k$ is defined as an extension of $f$ on $\mathbb{R}$ by the formula

$$f_k(x) := \begin{cases} U_k(f, x), & x < 0, \\ f(x), & 0 \leq x \leq 1, \\ V_k(f, x), & x > 1. \end{cases}$$

Obviously, if $f$ is continuous on $\mathbb{I}$, then $f_k$ is a continuous function on $\mathbb{R}$. Let $Q$ be a quasi-interpolant of the form (2.2)–(2.3) in $C(\mathbb{R})$. Put $\mathbb{Z}_+ := \{k \in \mathbb{Z} : k \geq -1\}$. If $k \in \mathbb{Z}_+$, we introduce the operator $Q_k$ by

$$Q_k(f, x) = Q(f_k, x; 2^{-k}), \text{ and } Q_{-1}(f, x) := 0, \ x \in \mathbb{I},$$

for a function $f$ on $\mathbb{I}$. We have for $k \in \mathbb{Z}_+$,

$$Q_k(f, x) = \sum_{s \in J(k)} a_{k,s}(f) M_{k,s}(x), \ \forall x \in \mathbb{I}, \quad (2.5)$$

where $J(k) := \{s \in \mathbb{Z} : -r/2 < s < 2^k + r/2\}$ is the set of $s$ for which $M_{k,s}$ do not vanish identically on $\mathbb{I}$, and the coefficient functional $a_{k,s}$ is defined by

$$a_{k,s}(f) := \Lambda(f_k, s; 2^{-k}) = \sum_{|j| \leq \mu} \lambda(j) f_k(2^{-k}(s - j)).$$

Put $\mathbb{Z}_+^d := \{k \in \mathbb{Z}_+^d : k_i \geq -1, \ i \in N[d]\}$. For $k \in \mathbb{Z}_+^d$, let the mixed operator $Q_k$ be defined by

$$Q_k := \prod_{i=1}^{d} Q_{k_i}, \quad (2.6)$$

where the univariate operator $Q_{k_i}$ is applied to the univariate function $f$ by considering $f$ as a function of variable $x_i$ with the other variables held fixed.
We have
\[ Q_k(f, x) = \sum_{s \in J^d(k)} a_{k,s}(f) M_{k,s}(x), \quad \forall x \in \mathbb{I}^d, \]
where \( M_{k,s} \) is the mixed B-spline defined in (1.3), \( J^d(k) := \{ s \in \mathbb{Z}^d : -r/2 < s_i < 2^k_i + r/2, \ i \in N[d] \} \) is the set of \( s \) for which \( M_{k,s} \) do not vanish identically on \( \mathbb{I}^d \),
\[ a_{k,s}(f) := a_{k_1,s_1}(a_{k_2,s_2}(...a_{k_d,s_d}(f))), \quad (2.7) \]
and the univariate coefficient functional \( a_{k_i,s_i} \) is applied to the univariate function \( f \) by considering \( f \) as a function of variable \( x_i \) with the other variables held fixed.

The operator \( Q_k \) is a local bounded linear mapping in \( C(\mathbb{I}^d) \) and reproducing \( \mathcal{P}_{r-1}^d \) the space of polynomials of order at most \( r-1 \) in each variable \( x_i \). More precisely, there is a positive number \( \delta > 0 \) such that for any \( f \in C(\mathbb{I}^d) \) and \( x \in \mathbb{I}^d \), \( Q(f,x) \) depends only on the value \( f(y) \) at an absolute constant number of points \( y \) with \( |y_i - x_i| \leq \delta 2^{-k_i}, \ i \in N[d] \);
\[ \|Q_k(f)\|_{C(\mathbb{I}^d)} \leq C \|\Lambda\|^d \|f\|_{C(\mathbb{I}^d)} \quad (2.8) \]
for each \( f \in C(\mathbb{I}^d) \) with a constant \( C \) not depending on \( k \); and,
\[ Q_k(p^*) = p, \ p \in \mathcal{P}_{r-1}^d, \quad (2.9) \]
where \( p^* \) is the restriction of \( p \) on \( \mathbb{I}^d \). The multivariate \( Q_k \) is called a mixed quasi-interpolant in \( C(\mathbb{I}^d) \).

From (2.8) and (2.9) we can see that
\[ \|f - Q_k(f)\|_{C(\mathbb{I}^d)} \to 0, \ k \to \infty. \quad (2.10) \]
(Here and in what follows, \( k \to \infty \) means that \( k_i \to \infty \) for \( i \in N[d] \)).

If \( k \in \mathbb{Z}_+^d \), we define \( T_k := I - Q_k \) for the univariate operator \( Q_k \), where \( I \) is the identity operator. If \( k \in \mathbb{Z}_+^d \), we define the mixed operator \( T_k \) in the manner of the definition (2.6) by
\[ T_k := \prod_{i=1}^d T_{k_i}. \]
For any \( e \subset N[d] \), put \( \mathbb{Z}_+^d(e) := \{ s \in \mathbb{Z}_+^d : s_i > -1, \ i \in e, \ s_i = -1, \ i \notin e \} \) (in particular, \( \mathbb{Z}_+^d(\emptyset) = \{(-1,-1,...,-1)\} \) and \( \mathbb{Z}_+^d(N[d]) = \mathbb{Z}_+^d \)). We have \( \mathbb{Z}_+^d(u) \cap \mathbb{Z}_+^d(v) = \emptyset \) if \( u \neq v \), and the following decomposition of \( \mathbb{Z}_+^d \):
\[ \mathbb{Z}_+^d = \bigcup_{e \subset N[d]} \mathbb{Z}_+^d(e). \]

If \( \tau \) is a number such that \( 0 < \tau \leq \min(p,1) \), then for any sequence of functions \( \{g_k\} \) there is the inequality
\[ \left\| \sum g_k \right\|_p^\tau \leq \sum \left\|g_k\right\|_p^\tau. \quad (2.11) \]
Lemma 2.1 Let $0 < p \leq \infty$ and $\tau \leq \min(p, 1)$. Then for any $f \in C(\mathbb{Z}^d)$ and $k \in \mathbb{Z}_+^d(e)$, there holds the inequality

$$
\|T_k(f)\|_p \leq C \left( \sum_{s \in \mathbb{Z}_+^d(e), s \geq k} \left\{ 2^{s-k|1|/p} \omega(f, 2^{-s}) \right\}^{\tau} \right)^{1/\tau}
$$

(2.12)

with some constant $C$ depending at most on $r, \mu, p, d$ and $\|\Lambda\|$, whenever the sum in the right-hand side is finite.

Proof. Notice that $\mathbb{Z}_+^d(\emptyset) = \{-1, -1, ..., -1\}$ and consequently, the inequality (2.12) is trivial for $e = \emptyset$: $\|f\|_p \leq C \omega(f, 1)_p = C\|f\|_p$. Consider the case where $e \neq \emptyset$. For simplicity we prove the lemma for $d = 2$ and $e = \{1, 2\}$, i.e., $\mathbb{Z}_+^d(e) = \mathbb{Z}_+^2$. This lemma has proven in [12, 13] for univariate functions ($d = 1$) and even $r$. It can be proven for univariate functions and odd $r$ in a completely similar way. Therefore, by (2.1) there holds the inequality

$$
\|T_{h_i}(f)\|_p \ll \left( \sum_{s_i \geq k_1} 2^{(s_i-k_1)/p} \left( 2^{-s_i} \int_{U(2^{-s_i})} \int_{I(h_i)} |\Delta_{h_i}^i(f, x)|^p \, dx \, dh \right)^{1/p} \right)^{1/\tau}, \quad i = 1, 2,
$$

(2.13)

where the norm $\|T_{h_i}(f)\|_p$ is applied to the univariate function $f$ by considering $f$ as a function of variable $x_i$ with the other variable held fixed.

If $1 \leq p < \infty$, we have by (2.13) applied for $i = 1$,

$$
\|T_{k_1}T_{k_2}(f)\|_p \ll \left( \int_{\mathbb{I}^1} \left\{ \sum_{s_1 \geq k_1} 2^{(s_1-k_1)/p} \left( 2^{-s_1} \int_{U(2^{-s_1})} \int_{I(h_1)} |\Delta_{h_1}^i((T_{k_2}^1 f), x)|^p \, dx \, dh \right)^{1/p} \right\} \, dx_2 \right)^{1/p}
$$

$$
\ll \sum_{s_1 \geq k_1} 2^{(s_1-k_1)/p} \left( 2^{-s_1} \int_{\mathbb{I}^1} \int_{U(2^{-s_1})} \int_{I(h_1)} |\Delta_{h_1}^i((T_{k_2}^1 f), x)|^p \, dx \, dh \, dx_2 \right)^{1/p}
$$

$$
= \sum_{s_1 \geq k_1} 2^{(s_1-k_1)/p} \left( 2^{-s_1} \int_{U(2^{-s_1})} \int_{I(h_1)} \left\{ \int_{\mathbb{I}^1} |(T_{k_2}(\Delta_{h_1}^i f), x)|^p \, dx_2 \right\} \, dx_1 \, dh_1 \right)^{1/p}.
$$

Hence, applying (2.13) with $i = 2$ gives

$$
\|T_{k_1}T_{k_2}(f)\|_p \ll \sum_{s \geq k} 2^{s-k|1|/p} \left( 2^{-|s|} \int_{U(2^{-s})} \int_{I^2(h)} |\Delta_{h}^i(f, x)|^p \, dx \, dh \right)^{1/p}
$$

$$
\ll \sum_{s \geq k} 2^{s-k|1|/p} \omega_r(f, 2^{-k}) \ll \sum_{s \geq k} 2^{s-k|1|/p} \omega_r(f, 2^{-k})_p.
$$

Thus, the lemma has proven when $1 \leq p < \infty$. The cases $0 < p < 1$ and $p = \infty$ can be proven in a similar way. \[\square\]
Let $J_d^r(k) := J_d(k)$ if $r$ is even, and $J_d^r(k) := \{s \in \mathbb{Z}^d : -r < s_i < 2^{k_i+1} + r, i \in \mathbb{N}[d]\}$ if $r$ is odd. Notice that $J_d^r(k)$ is the set of $s$ for which $M_{k,s}^r$ do not vanish identically on $\mathbb{I}^d$. Denote by $\Sigma_d^r(k)$ the span of the B-splines $M_{k,s}^r$, $s \in J_d^r(k)$. If $0 < p \leq \infty$, for all $k \in \mathbb{Z}^d_+$ and all $g \in \Sigma_d^r(k)$ such that

$$
g = \sum_{s \in J_d^r(k)} a_s M_{k,s}^r, \quad (2.14)$$

there is the quasi-norm equivalence

$$
\|g\|_p \asymp 2^{-|k|_1/p} \|\{a_s\}\|_{p,k}, \quad (2.15)
$$

where

$$
\|\{a_s\}\|_{p,k} := \left( \sum_{s \in J_d^r(k)} |a_s|^p \right)^{1/p}
$$

with the corresponding change when $p = \infty$.

Let the mixed operator $q_k$, $k \in \mathbb{Z}^d_+$, be defined in the manner of the definition (2.16) by

$$
q_k := \prod_{i=1}^d (Q_{k_i} - Q_{k_i-1}). \quad (2.16)
$$

We have

$$
Q_k = \sum_{k' \leq k} q_{k'}. \quad (2.17)
$$

Here and in what follows, for $k, k' \in \mathbb{Z}^d$ the inequality $k' \leq k$ means $k'_i \leq k_i$, $i \in \mathbb{N}[d]$. From (2.17) and (2.10) it is easy to see that a continuous function $f$ has the decomposition

$$
f = \sum_{k \in \mathbb{Z}^d_+} q_k(f) \quad (2.18)
$$

with the convergence in the norm of $C(\mathbb{I}^d)$.

From the definition (2.16) and the refinement equation for the B-spline $M$, we can represent the component functions $q_k(f)$ as

$$
q_k(f) = \sum_{s \in J_d^r(k)} c^r_{k,s}(f) M_{k,s}^r, \quad (2.19)
$$

where $c^r_{k,s}$ are certain coefficient functionals of $f$, which are defined as follows. We first consider the univariate case. We have

$$
q_k(f) = \sum_{s \in J(k)} a_{k,s}(f) M_{k,s} - \sum_{s \in J(k-1)} a_{k-1,s}(f) M_{k-1,s}. \quad (2.20)
$$

If the order $r$ of the B-spline $M$ is even, by using the refinement equation

$$
M(x) = 2^{-r+1} \sum_{j=0}^r \binom{r}{j} M(2x - j + r/2), \quad (2.21)
$$
from (2.20) we obtain

\[ q_k(f) = \sum_{s \in J_r(k)} c^r_{k,s}(f) M^r_{k,s}, \tag{2.22} \]

where

\[ c^r_{k,s}(f) := a_{k,s}(f) - a'_{k,s}(f), \quad k > 0, \tag{2.23} \]

\[ a'_{k,s}(f) := 2^{-r+1} \sum_{(m,j) \in C(k,s)} \binom{r}{j} a_{k-1,m}(f), \quad k > 0, \quad a'_{0,s}(f) := 0. \]

and

\[ C_r(k,s) := \{(m,j) : 2m + j - r/2 = s, \ m \in J(k-1), \ 0 \leq j \leq r\}, \quad k > 0, \quad C_r(0,s) := \{0\}. \]

If the order \( r \) of the B-spline \( M \) is odd, by using (2.21) from (2.20) we get (2.22) with

\[ c^r_{k,s}(f) := \begin{cases} 0, & k = 0, \\ a_{k,s/2}(f), & k > 0, \ s \text{ even}, \\ 2^{-r+1} \sum_{(m,j) \in C_r(k,s)} \binom{r}{j} a_{k-1,m}(f), & k > 0, \ s \text{ odd}, \end{cases} \]

where

\[ C_r(k,s) := \{(m,j) : 4m + 2j - r = s, \ m \in J(k-1), \ 0 \leq j \leq r\}, \quad k > 0, \quad C_r(0,s) := \{0\}. \]

In the multivariate case, the representation (2.19) holds true with the \( c^r_{k,s} \) which are defined in the manner of the definition (2.7) by

\[ c^r_{k,s}(f) = c^r_{k_1,s_1}((c^r_{k_2,s_2}(\ldots c^r_{k_d,s_d}(f)))). \tag{2.24} \]

Let us use the notations: \( 1 := (1,\ldots,1) \in \mathbb{R}^d; \ x_+ := ((x_1)_+,...,(x_d)_+) \) for \( x \in \mathbb{R}^d; \ N^d(e) := \{s \in \mathbb{Z}_+^d : s_i > 0, \ i \in e, \ s_i = 0, i \notin e\} \) for \( e \subset N[d] \) (in particular, \( N^d(\emptyset) = \{0\} \) and \( N^d(N[d]) = N^d \)). We have \( N^d(u) \cap N^d(v) = \emptyset \) if \( u \neq v \), and the following decomposition of \( \mathbb{Z}_+^d \):

\[ \mathbb{Z}_+^d = \bigcup_{e \subset N[d]} N^d(e). \]

**Lemma 2.2** Let \( 0 < p \leq \infty \) and \( \tau \leq \min(p,1) \). Then for any \( f \in C(\mathbb{N}^d) \) and \( k \in \mathbb{N}^d(e) \), there holds the inequality

\[ \|q_k(f)\|_p \leq C \left( \sum_{n \geq k} \sum_{s \in \mathbb{Z}_+^d(v), \ s \geq k} \left\{ 2^{s-k_1/p} \omega^*_\nu(f, 2^{-s}) \right\}^r \right)^{1/r} \]

with some constant \( C \) depending at most on \( r, \mu, p, d \) and \( \|A\| \), whenever the sum in the right-hand side is finite.
Proof. From the equality
\[ q_k = \prod_{i=1}^{d} (T_{k_i}^i - T_{k_i}^i), \]
it follows that
\[ q_k = \sum_{u \subseteq N[d]} (-1)^{|u|} \prod_{i \in u} T_{k_i}^i \prod_{i \notin u} T_{k_i}^i = \sum_{u \subseteq N[d]} (-1)^{|u|} T_{k^u}, \]
where \( k^u \) is defined by \( k^u_i = k_i \) if \( i \in u \), and \( k^u_i = k_i - 1 \) if \( i \notin u \). Hence,
\[ \| q_k(f) \|_p \ll \sum_{u \subseteq N[d]} \| T_{k^u}(f) \|_p. \] (2.25)
Notice that \( k^u \in \mathbb{Z}^d_+(v) \) for some \( v \supset e \), and \( 0 \leq k - k^u \leq k - k^u \leq 1 \). Moreover, for \( s \in \mathbb{Z}^d_+(v) \), \( s \geq k^u \) if only if \( s \geq k^u + 1 \). Hence, by Lemma 2.1 and properties of mixed modulus smoothness we have
\[ \| T_{k^u}(f) \|_p \ll \left( \sum_{s \in \mathbb{Z}^d_+(v), s \geq k^u} \left\{ 2^{-|s-k^u|/1/p} \omega^v_p(f, 2^{-s}) \right\}^\tau \right)^{1/\tau} \]
\[ \ll \left( \sum_{s \in \mathbb{Z}^d_+(v), s \geq k^u} \left\{ 2^{-|s-k^u|/1/p} \omega^v_p(f, 2^{-s}) \right\}^\tau \right)^{1/\tau} \]
\[ = \left( \sum_{s' \in \mathbb{Z}^d_+(v), s' \geq k} \left\{ 2^{-|s'-k|/1/p} \omega^v_p(f, 2^{-(s'-k+k^u)}) \right\}^\tau \right)^{1/\tau} \]
\[ \ll \left( \sum_{s \in \mathbb{Z}^d_+(v), s \geq k} \left\{ 2^{-|s-k|/1/p} \omega^v_p(f, 2^{-s}) \right\}^\tau \right)^{1/\tau}. \]
The last inequality together with (2.25) proves the lemma. \( \square \)

Lemma 2.3 Let \( 0 < p \leq \infty \), \( 0 < \tau \leq \min(p, 1) \), \( \delta = \min(r, r - 1 + 1/p) \). Then for any \( f \in C(\mathbb{I}^d) \) and \( k \in \mathbb{Z}^d_+(c) \), there holds the inequality
\[ \omega^v_p(f, 2^{-k}) \leq C \left( \sum_{s \in \mathbb{Z}^d_+(v), s \geq k} \left\{ 2^{-|s-k|/1/p} \| q_s(f) \|_p \right\}^\tau \right)^{1/\tau} \]
with some constant \( C \) depending at most on \( r, \mu, p, d \) and \( \| \Lambda \| \), whenever the sum in the right-hand side is finite.
Proof. For simplicity we prove the lemma for \( e = N[d], \) i.e., \( \mathbb{Z}^d_+ \). Let \( f \in C(\mathbb{I}^d) \) and \( k \in \mathbb{Z}^d_+ \). From (2.18) and (2.11) we obtain

\[
\| \Delta^r_h(f) \|_p \leq C \left( \sum_{s \in \mathbb{Z}^d_+} \| \Delta^r_h(q_s(f)) \|_p^p \right)^{1/p}.
\] (2.26)

Further, by (2.19) we get

\[
\Delta^r_h(q_k(f)) = \sum_{j \in \mathcal{J}^d(s)} c_{s,j}^r(f) \Delta^r_h(M^r_{s,j}).
\]

Notice that for any \( x \), the number of non-zero B-spines in (2.19) is an absolute constant depending on \( r, d \) only. Thus, we have

\[
|\Delta^r_h(q_s(f), x)|^p \ll \sum_{j \in \mathcal{J}^d(s)} |c_{s,j}^r(f)|^p |\Delta^r_h(M^r_{s,j}, x)|^p, \quad x \in \mathbb{I}^d.
\] (2.27)

From properties of the B-spline \( M \) it is easy to prove the following estimate

\[
\int_{\mathbb{I}^{d(h)}} |\Delta^r_h(M^r_{s,j}, x)|^p dx \ll 2^{-|s|_1 - \delta p (-\log |h| - s)_+ |1},
\]

where we used the abbreviation \( \log |h| := (\log |h_1|, \ldots, \log |h_d|) \). Hence, by (2.27) we obtain

\[
\| \Delta^r_h(q_s(f)) \|_p \ll 2^{-\delta (-\log |h| - s)_+ |1} \left( \sum_{j \in \mathcal{J}^d(s)} |c_{s,j}^r(f)|^p \right)^{1/p}
\ll 2^{-\delta (-\log |h| - s)_+ |1} \| q_s(f) \|_p.
\]

By (2.26) we have

\[
\| \Delta^r_h(f) \|_p \ll \left( \sum_{s \in \mathbb{Z}^d_+} \left\{ 2^{-\delta (-\log |h| - s)_+ |1} \| q_s(f) \|_p \right\}^p \right)^{1/p}.
\]

From the last inequality we prove the lemma. \( \square \)

For functions \( f \) on \( \mathbb{I}^d \), we introduce the following quasi-norms:

\[
B_2(f) := \| \{ q_k(f) \} \|_{b_0^\delta(L_p)};
B_3(f) := \left( \sum_{k=0}^{\infty} 2^{(a-d/p)k} \| \{ c_{k,s}^r(f) \} \|_{p,k} \right)^{1/\theta}.
\]

We will need the following discrete Hardy inequality. Let \( \{ a_k \}_{k \in \mathbb{Z}^d_+} \) and \( \{ b_k \}_{k \in \mathbb{Z}^d_+} \) be two positive sequences and let for some \( M > 0, \tau > 0 \)

\[
b_k \leq M \left( \sum_{s \in \mathbb{Z}^d_+} 2^{\delta (k-s)_+ |1} a_s \right)^{1/\tau}.
\] (2.28)
Then for any $0 < \beta < \delta$, $\theta > 0$,

$$\|\{b_k\}\|_{b^\beta} \leq CM\|\{a_k\}\|_{b^\beta}$$

(2.29)

with $C = C(\beta, \theta, d)$. For a proof of this inequality for the univariate case see, e.g., [6]. In the general case it can be proven by induction based on the univariate case.

**Theorem 2.1** Let $0 < p, \theta \leq \infty$ and $1/p < \alpha < r$. Then the hold the following assertions.

(i) A function $f \in MB^\alpha_{p,\theta}$ can be represented by the mixed B-spline series

$$f = \sum_{k \in \mathbb{Z}^d_+} q_k(f) = \sum_{k \in \mathbb{Z}^d_+} \sum_{s \in J^d_+(k)} c^\tau_{k,s}(f) M^r_{k,s},$$

(2.30)

satisfying the convergence condition

$$B_2(f) \asymp B_3(f) \ll B(f),$$

where the coefficient functionals $c^\tau_{k,s}(f)$ are explicitly constructed by formula (2.23)–(2.24) as linear combinations of an absolute constant number of values of $f$ which does not depend on neither $k, s$ nor $f$.

(ii) If in addition, $\alpha < \min(r, r - 1 + 1/p)$, then a continuous function $f$ on $\mathbb{R}^d$ belongs to the Besov space $MB^\alpha_{p,\theta}$ if and only if $f$ can be represented by the series (2.30). Moreover, the Besov quasi-norm $B(f)$ is equivalent to one of the quasi-norms $B_2(f)$ and $B_3(f)$.

**Proof.** Since by (2.15) the quasi-norms $B_2(f)$ and $B_3(f)$ are equivalent, it is enough to prove (i) and (ii) for $B_3(f)$. Fix a number $0 < \tau \leq \min(p, 1)$.

**Assertion (i):** For $k \in \mathbb{Z}^d_+$, put

$$b_k := 2^{k|1/p} \|q_k(f)\|_p, \quad a_k := \left(\sum_{v \ni e} \left\{2^{k|1/p} \omega^v_{p,f}(2^{-k})\right\}^\tau\right)^{1/\tau}$$

if $k \in \mathbb{N}^d(e)$. By Lemma 2.2 we have for $k \in \mathbb{Z}^d_+$,

$$b_k \leq C \left(\sum_{s \geq k} a^\tau_s\right)^{1/\tau}.$$

Then applying the mixed discrete Hardy inequality (2.28)–(2.29) with $\beta = \alpha - 1/p$, gives

$$B_3(f) = \|\{b_k\}\|_{b^\beta} \leq C\|\{a_k\}\|_{b^\beta} \asymp B_1(f) \asymp B(f).$$

**Assertion (ii):** Let in addition, $\alpha < \min(r, r - 1 + 1/p)$. For $k \in \mathbb{Z}^d_+$, put

$$b_k := \left(\sum_{v \ni e} \left\{\omega^v_r(f, 2^{-k})\right\}^\tau\right)^{1/\tau} \quad a_k := \|q_k(f)\|_p.$$

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if $k \in \mathbb{N}^d(e)$. By Lemma 2.3 we have for $k \in \mathbb{Z}_+^d$

$$b_k \leq C \left( \sum_{s \in \mathbb{Z}_+^d} \left( 2^\delta |(k-s)+1| a_s \right)^\tau \right)^{1/\tau},$$

where $\delta = \min(r, r - 1 + 1/p)$. Then applying the mixed discrete Hardy inequality (2.28)–(2.29) with $\beta = \alpha$, gives

$$B(f) \asymp B_1(f) \asymp \| \{b_k\}\|_{b^\delta} \leq C \| \{a_k\}\|_{b^\delta} = B_3(f).$$

The assertion (ii) is proven.

**Remark** From (2.23)–(2.24) we can see that if $r$ is even, for each pair $k, s$ the coefficient $c_{k,s}(f)$ is a linear combination of the values $f(2^{-k}(s-j))$, and $f(2^{-k+1}(s'-j))$, $j \in P^d(\mu)$, $s' \in C_r(k, s)$. The number of these values does not exceed the fixed number $(2\mu + 1)^d((r+1)^d + 1)$. If $r$ is odd, we can say similarly about the coefficient $c_{k,s}(f)$.

### 3 Sampling recovery

Recall that the linear operator $R_m, m \in \mathbb{Z}_+$, is defined for functions on $\mathbb{T}^d$ in (1.5) as follows.

$$R_m(f) = \sum_{k \in \Delta(m)} q_k(f) = \sum_{k \in \Delta(m)} \sum_{s \in P^d(k)} c_{k,s}(f) M_{r,s}^k.$$

**Lemma 3.1** For functions $f$ on $\mathbb{T}^d$, $R_m$ defines a linear sampling algorithm of the form (1.1) on the grid $G^d(m)$. More precisely,

$$R_m(f) = L_n(f) = \sum_{(k,s) \in G^d(m)} f(2^{-k}j) \psi_{k,j},$$

where

$$n := |G^d(m)| = \sum_{k \in \Delta(m)} |I^d(k)| \asymp 2m^{-d-1};$$

$\psi_{k,j}$ are explicitly constructed as linear combinations of at most $(4\mu + r + 5)^d$ B-splines $M_{r,s}^k \in M_r^d(m)$ for even $r$, and $(12\mu + 2r + 13)^d$ B-splines $M_{r,s}^k \in M_r^d(m)$ for odd $r$; $M_r^d(m) := \{M_{k',s}^r : k' \in \Delta(m), s' \in J_r^d(k')\}$.

**Proof.** Let us prove the lemma for even $r$. For odd $r$ it can be proven in a similar way. For univariate functions the coefficient functionals $a_{k,s}(f)$ can be rewritten as

$$a_{k,s}(f) = \sum_{|s-j| \leq \mu} \lambda(s-j) f_k(2^{-k}j) = \sum_{j \in P(k,s)} \lambda_{k,s}(j) f(2^{-k}j),$$

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where \( \lambda_{k,s}(j) := \lambda(s - j) \) and \( P(k, s) = P_s(\mu) := \{ j \in \{0, 2^k\} : s - j \in P(\mu) \} \) for \( \mu \leq s \leq 2^k - \mu; \lambda_{k,s}(j) \) is a linear combination of no more than \( \max(r, 2\mu + 1) \leq 2\mu + 2 \) coefficients \( \lambda(j), j \in P(\mu), \) for \( s < \mu \) or \( s > 2^k - \mu \) and

\[
P(k, s) \subseteq \begin{cases} P_s(\mu) \cup \{0, r - 1\}, & s < \mu, \\ P_s(\mu) \cup \{2^k - r + 1, 2^k\}, & s > 2^k - \mu. \end{cases}
\]

Further, for univariate functions we have

\[
c_{k,s}^r(f) = \sum_{j \in P(k, s)} \lambda_{k,s}(j) f(2^{-k} j) - 2^{-r+1} \sum_{(m, \nu) \in C_r(k, s)} \binom{r}{\nu} \sum_{j \in P(k-1, m)} \lambda_{k-1,m}(j) f(2^{-k} (2j))
\]

\[
= \sum_{j \in G(k, s)} \lambda_{k,s}(j) f(2^{-k} j),
\]

where \( G(k, s) := P(k, s) \cup \{2j : j \in P(k-1, m), (m, \nu) \in C(k, s)\} \). If \( j \in P(k, s) \), we have \( |j - s| \leq \max(r, 2\mu + 1) \leq 2\mu + 2 \). If \( j \in P(k-1, m), (m, \nu) \in C(k, s) \), we have \( |2j - s| = |2j - 2m - \nu + r/2| \leq 2|j - m| + |\nu - r/2| \leq 2 \max(r, 2\mu + 1) + r + 1 \leq 4\mu + r + 5 =: \bar{\mu} \). Therefore, \( G(k, s) \subset P_s(\bar{\mu}) \), and we can rewrite the coefficient functionals \( c_{k,s}^r(f) \) in the form

\[
c_{k,s}^r(f) = \sum_{j-s \in P(\bar{\mu})} \lambda_{k,s}(j) f(2^{-k} j)
\]

with zero coefficients \( \lambda_{k,s}(j) \) for \( j \notin G(k, s) \). Therefore, we have

\[
q_k(f) = \sum_{s \in J_r(k)} c_{k,s}^r(f) M_{k,s}^r = \sum_{s \in J_r(k)} \sum_{j-s \in P(\bar{\mu})} \lambda_{k,s}(j) f(2^{-k} j) M_{k,s}^r
\]

\[
= \sum_{j \in I(k)} f(2^{-k} j) \sum_{s-j \in P(\bar{\mu})} \gamma_{k,j}(s) M_{k,s}^r
\]

for certain coefficients \( \gamma_{k,j}(s) \). Thus, the univariate \( q_k(f) \) is of the form

\[
q_k(f) = \sum_{j \in I(k)} f(2^{-k} j) \psi_{k,j},
\]

where

\[
\psi_{k,j} := \sum_{s-j \in P(\bar{\mu})} \gamma_{k,j}(s) M_{k,s},
\]

are a linear combination of no more than the absolute number \( 4\mu + r + 5 \) of B-splines \( M_{k,s}^r \), and the size \( |I(k)| \) is \( 2^k \). Hence, the multivariate \( q_k(f) \) is of the form

\[
q_k(f) = \sum_{j \in I(k)} f(2^{-k} j) \psi_{k,j},
\]

where

\[
\psi_{k,j} := \prod_{i=1}^d \psi_{k_{i,j_i}}
\]
are a linear combination of no more than the absolute number \((4\mu + r + 5)^d\) of B-splines \(M_{k,s} \in M^d(m)\), and the size \(|I^d(k)|\) is \(2^{[k]}\). From (3.1) we can see that \(R_m(f)\) is of the form (1.1) with \(n\) as in (3.2).

**Theorem 3.1** Let \(0 < p, q, \theta \leq \infty\) and \(1/p < \alpha < r\). Then we have the following upper bound of \(E(m)\).

(i) For \(p \geq q\),

\[
E(m) \ll \begin{cases} 
2^{-\alpha m}, & \theta \leq \min(q, 1), \\
2^{-\alpha m} m^{(d-1)(1/q-1/\theta)}, & \theta > \min(q, 1), q \leq 1, \\
2^{-\alpha m} m^{(d-1)(1-1/\theta)}, & \theta > \min(q, 1), q > 1.
\end{cases}
\]

(ii) For \(p < q\),

\[
E(m) \ll \begin{cases} 
2^{-(\alpha-1/p+1/q)m} m^{(d-1)(1/q-1/\theta)+}, & q < \infty, \\
2^{-(\alpha-1/p)m} m^{(d-1)(1-1/\theta)+}, & q = \infty.
\end{cases}
\]

**Proof.**

Case (i): \(p \geq q\). For an arbitrary \(f \in B^\infty_{p,\theta}\), by the representation (2.30) and (2.11) we have

\[
\|f - R_m(f)\|_q \ll \sum_{|k| > m} \|q_k(f)\|_q^\tau
\]

with any \(\tau \leq \min(q, 1)\). Therefore, if \(\theta \leq \min(q, 1)\), then by the inequality \(\|q_k(f)\|_q \leq \|q_k(f)\|_p\) we get

\[
\|f - R_m(f)\|_q \ll \left( \sum_{|k| > m} \|q_k(f)\|_q^\theta \right)^{1/\theta} \\
\leq 2^{-\alpha m} \left( \sum_{|k| > m} \{2^{\alpha|k|} \|q_k(f)\|_p\}^\theta \right)^{1/\theta} \\
\ll 2^{-\alpha m} B_2(f) \ll 2^{-\alpha m}.
\]

Further, if \(\theta > \min(q, 1)\), then

\[
\|f - R_m(f)\|_q^\tau \ll \sum_{|k| > m} \|q_k(f)\|_q^\tau \ll \sum_{|k| > m} \{2^{\alpha|k|} \|q_k(f)\|_q\}^{\tau} \{2^{-\alpha|k|}\} \|q\|_	au,
\]

where \(q^\tau = \min(q, 1)\). Putting \(\nu = \theta/q^\tau\), by Hölder’s inequality and the inequality \(\|q_k(f)\|_q \leq \|q_k(f)\|_p\) we obtain

\[
\|f - R_m(f)\|_q^{q^\tau} \ll \left( \sum_{|k| > m} \{2^{\alpha|k|} \|q_k(f)\|_q\}^{q^\tau}\nu \right)^{1/\nu} \left( \sum_{|k| > m} \{2^{-\alpha|k|}\}^{q^\tau}\nu \right)^{1/\nu} \\
\ll \{B_2(f)\}^{q^\tau}\{2^{-\alpha m} m^{(d-1)(1/q^\tau-1/\theta)}\}^{q^\tau} \ll \{2^{-\alpha m} m^{(d-1)(1/q^\tau-1/\theta)}\}^{q^\tau}. \tag{3.3}
\]

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This proves the Case (i).

**Case (ii):** \( p < q \). We first assume \( q < \infty \). For an arbitrary \( f \in B^{\alpha}_{p,\theta} \), by the representation (2.30) and Lemma 5.3 we have

\[
\| f - R_m(f) \|_q \ll \sum_{|k|_1 > m} \left\{ 2^{(1/p - 1/q)\|k_1\|} \|q_k(f)\|_p \right\}^q.
\]

Therefore, if \( \theta \leq q \), then

\[
\| f - R_m(f) \|_q \ll \left( \sum_{|k|_1 > m} \left\{ 2^{(1/p - 1/q)\|k_1\|} \|q_k(f)\|_p \right\}^\theta \right)^{1/\theta}
\ll 2^{-(\alpha - 1/p + 1/q)m} B_2(f) \ll 2^{-(\alpha - 1/p + 1/q)m}.
\]

Further, if \( \theta > q \), then

\[
\| f - R_m(f) \|_q^q \ll \sum_{|k|_1 > m} \left\{ 2^{(1/p - 1/q)\|k_1\|} \|q_k(f)\|_p \right\}^q
\ll \sum_{|k|_1 > m} \left\{ 2^{\alpha\|k_1\|} \|q_k(f)\|_p \right\}^q \left\{ 2^{-(\alpha - 1/p + 1/q)|k_1|} \right\}^q.
\]

Hence, similarly to (3.3), we get

\[
E_q^q(m) \ll \left\{ 2^{-(\alpha - 1/p + 1/q)m(d-1)(1/q-1/\theta)} \right\}^q.
\]

When \( q = \infty \), the Case (ii) can be proven analogously by use the inequality

\[
\| f - R_m(f) \|_\infty \ll \sum_{|k|_1 > m} 2^{\|k_1\|/p} \|q_k(f)\|_p.
\]

\[\square\]

The following theorem for the case \( \alpha = 1/p \) can be proven by use of Lemmas 2.2 and 5.3 and the inequality (2.11).

**Theorem 3.2** Let \( 0 < p, q < \infty \), \( 0 < \theta \leq \min(p, 1) \) and \( \alpha = 1/p < r \). Then we have the following upper bound of \( E(m) \).

\[\begin{align*}
(i) \quad & \text{For } p \geq q, \\
& E(m) \ll \begin{cases} 2^{-m/p} m^{(d-1)/p}, & p \geq 1, \\
2^{-m/p} m^{(d-1)/p}, & p < 1. \end{cases}
\end{align*}\]

\[\begin{align*}
(ii) \quad & \text{For } p < q, \\
& E(m) \ll 2^{-m/q} m^{(d-1)/q}.
\end{align*}\]
The following two theorems are a direct corollary of Lemma 3.1 and Theorems 3.1 and 3.2.

**Theorem 3.3** Let \(0 < p, q, \theta \leq \infty\) and \(1/p < \alpha < r\). If \(\bar{m}\) is the largest integer of \(m\) such that
\[
2^m \sum_{k \in \Delta(m)} |I(k)| \leq n,
\]
then we have the following upper bound of \(r_n\) and \(E(\bar{m})\).

(i) For \(p \geq q\),
\[
 r_n \ll E(\bar{m}) \ll \begin{cases} 
 (n^{-1} \log^{d-1} n)^\alpha, & \theta \leq \min(q, 1), \\
 (n^{-1} \log^{d-1} n)^\alpha (\log^{d-1} n)^{1/q - 1/\theta}, & \theta > \min(q, 1), q \leq 1, \\
 (n^{-1} \log^{d-1} n)^\alpha (\log^{d-1} n)^{1-1/\theta}, & \theta > \min(q, 1), q > 1.
\end{cases}
\]

(ii) For \(p < q\),
\[
 r_n \ll E(\bar{m}) \ll \begin{cases} 
 (n^{-1} \log^{d-1} n)^\alpha - 1/p + 1/q (\log^{d-1} n)^{(1/q - 1/\theta)_+}, & q < \infty, \\
 (n^{-1} \log^{d-1} n)^\alpha - 1/p (\log^{d-1} n)^{(1-1/\theta)_+}, & q = \infty.
\end{cases}
\]

**Theorem 3.4** Let \(0 < p, q < \infty\), \(0 < \theta \leq \min(p, 1)\) and \(\alpha = 1/p < r\). If \(\bar{m}\) is the largest integer of \(m\) such that
\[
2^m \sum_{k \in \Delta(m)} |I(k)| \leq n,
\]
then we have the following upper bound of \(r_n\) and \(E(\bar{m})\).

(i) For \(p \geq q\) and \(\theta \leq 1\),
\[
 E(\bar{m}) \asymp r_n \asymp (n^{-1} \log^{d-1} n)^\alpha, \quad \begin{cases} 
 2 \leq q < \infty, \\
 1 < p = q \leq \infty.
\end{cases}
\]

From Theorem 3.3 and Lemma 5.1 we obtain the following theorem.

**Theorem 3.5** Let \(1 \leq p, q \leq \infty\), \(0 < \theta \leq \infty\) and \(1/p < \alpha < r\). If \(\bar{m}\) is the largest integer of \(m\) such that
\[
2^m \sum_{k \in \Delta(m)} |I(k)| \leq n,
\]
then we have the following asymptotic order of \(r_n\) and \(E(\bar{m})\).

(i) For \(p \geq q\) and \(\theta \leq 1\),
\[
 E(\bar{m}) \asymp r_n \asymp (n^{-1} \log^{d-1} n)^\alpha, \quad \begin{cases} 
 2 \leq q < \infty, \\
 1 < p = q \leq \infty.
\end{cases}
\]
(ii) For $1 < p < q < \infty$,

$$E(\tilde{m}) \times r_n \times (n^{-1}\log^{d-1} n)^{\alpha-1/p+1/q}(\log^{d-1} n)^{(1/q-1/\theta)+}, \begin{cases} 2 \leq p, & 2 \leq \theta \leq q, \\ q \leq 2. \end{cases}$$

4 Interpolant representations and sampling recovery

We first consider a piecewise constant interpolant representation. Let $\chi_{[0,1)}$ and $\chi_{[0,1]}$ be the characteristic functions of the half opened and closed intervals $[0,1)$ and $[0,1]$, respectively. For $k \in \mathbb{Z}_+$ and $s = 0, 1, \ldots, 2^k - 1$, we define the system of functions $N_{k,s}$ on $\mathbb{I}$ by

$$N_{k,s}(x) := \begin{cases} \chi_{[0,1)}(2^k x - s), & 0 \leq s < 2^k - 1, \\ \chi_{[0,1]}(2^k x - s), & s = 2^k - 1. \end{cases}$$

(In particular, $N_{0,0} = \chi_{[0,1]}$). Obviously, we have for $k > 0$ and $s = 0, 1, \ldots, 2^k - 1$,

$$N_{k-1,s} = N_{k,2s} + N_{k,2s+1}.$$ 

We let the operator $\Pi_k$ be defined for functions $f$ on $\mathbb{I}$, for $k \in \mathbb{Z}_+$, by

$$\Pi_k(f) := \sum_{s=0}^{2^k-1} f(2^{-k}s) N_{k,s}, \text{ and } \Pi_{-1}(f) = 0.$$ 

Clearly, the linear operator $\Pi_k$ is bounded in $L_\infty(\mathbb{I})$, reproduces constant functions and for any continuous function $f$,

$$\|f - \Pi_k(f)\|_\infty \leq \omega_1(f,2^{-k})_\infty,$$

and consequently, $\|f - \Pi_k(f)\|_\infty \to 0$, when $k \to \infty$. Moreover, for any $x \in \mathbb{I}$, $\Pi_k(f,x) = f(2^{-k}s)$ if $x$ is in either the interval $[2^{-k}s,2^{-k}(s+1)]$ for $s = 0, 1, \ldots, 2^k - 2$ or the interval $[2^{-k}s,1]$ for $s = 2^{-k} - 1$, i.e., $\Pi_k$ possesses a local property. In particular, $\Pi_k(f)$ interpolates $f$ at the points $2^{-k}s$, $s = \{0, 1, \ldots, 2^k - 1\}$, that is,

$$\Pi_k(f,2^{-k}s) = f(2^{-k}s), \text{ } s = 0, 1, \ldots, 2^k - 1. \quad (4.1)$$

Further, we define for $k \in \mathbb{Z}_+$,

$$\pi_k(f) := \Pi_k(f) - \Pi_{k-1}(f).$$

From the definition it is easy to check that

$$\pi_k(f) = \sum_{s \in Z_1(k)} \lambda_{k,s}^1(f) \varphi_{k,s}^1,$$

where $Z_1(0) := \{0\}$, $Z_1(k) := \{0, 1, \ldots, 2^{k-1} - 1\}$ for $k > 0$,

$$\varphi_{k,s}^1(x) := N_{k,2s+1}(x), \text{ } k > 0, \text{ and } \varphi_{0,0}^1(x) := N_{0,0}(x).$$
and
\[ \lambda_{k,s}^1(f) := \Delta_{2-k}^1(f, 2^{-k+1}s), \quad k > 0, \text{ and } \lambda_{0,0}^1(f) := f(0). \]

We now can see that every \( f \in C(\mathbb{I}) \) is represented by the series
\[ f = \sum_{k \in \mathbb{Z}_+} \sum_{s \in Z_1(k)} \lambda_{k,s}^1(f) \varphi_{k,s}^1, \quad (4.2) \]
converging in the norm of \( L_\infty(\mathbb{I}) \).

Next, let us revisit the univariate piecewise linear (nodal) quasi-interpolant for functions on \( \mathbb{R} \) defined in (2.4) with \( M(x) = (1 - |x|)_+ \) (\( r = 2 \)). Consider the generated from it by the formula (2.5) quasi-interpolant for functions on \( \mathbb{I} \)
\[ Q_k(f, x) = \sum_{s \in J(k)} f(2^{-k}s) M_{k,s}(x), \quad (4.3) \]
and the related quasi-interpolant representation
\[ f = \sum_{k \in \mathbb{Z}_+} q_k(f) = \sum_{k \in \mathbb{Z}_+} \sum_{s \in J(k)} c_{k,s}(f) M_{k,s}, \quad (4.4) \]
where we recall that \( J(k) := \{ s \in \mathbb{Z} : 0 \leq s \leq 2^k \} \) is the set of \( s \) for which \( M_{k,s} \) do not vanish identically on \( \mathbb{I} \). From the equality \( M_{k,s}(2^{-k}s') = \delta_{s,s'} \) one can see that \( Q_k(f) \) interpolates \( s \) at the dyadic points \( 2^{-k}s, \ s \in J(k) \), i.e.
\[ Q_k(f, 2^{-k}s) = f(2^{-k}s), \ s \in J(k). \quad (4.5) \]

Because of the interpolation property (4.1) and (4.5), the operators \( \Pi_k \) and \( Q_k \) are interpolants. Therefore, the representations (4.2) and (4.4) are interpolant representations. We will see that the interpolant representation (4.4) coincides with the classical Faber-Schauder series. The univariate Faber-Schauder system of functions is defined by
\[ \mathcal{F} := \{ \varphi_{k,s}^2 : s \in Z_2(k), \ k \in \mathbb{Z}_+ \}, \]
where \( Z_2(0) := \{0, 1\} \) and \( Z_2(k) := \{0, 1, \ldots, 2^k - 1\} \) for \( k > 0 \),
\[ \varphi_{0,0}^2(x) := M_{0,0}(x), \ \varphi_{0,1}^2(x) := M_{0,1}(x), \ x \in \mathbb{I}, \]
(an alternative choice is \( \varphi_{0,1}(x) := 1 \)), and for \( k > 0 \) and \( s \in Z(k) \)
\[ \varphi_{k,s}^2(x) := M_{k,2s+1}(x), \ x \in \mathbb{I}. \]

It is known that \( \mathcal{F} \) is a basis in \( C(\mathbb{I}) \). (See [15] for details about the Faber-Schauder system.)

By a direct computation we have for the component functions \( q_k(f) \) in the piecewise linear quasi-interpolant representation (4.4):
\[ q_k(f) = \sum_{s \in Z_2(k)} \lambda_{k,s}^2(f) \varphi_{k,s}^2(x). \quad (4.6) \]
where
\[ \lambda_{k,s}^2(f) := -\frac{1}{2} \Delta_{2^{-k}s}^2 f(2^{-k+1}s), \quad k > 0, \quad \text{and} \quad \lambda_{0,s}^2(f) := f(s). \]

Hence, the interpolant representation (4.4) can be rewritten as the Faber-Schauder series:
\[ f = \sum_{k \in \mathbb{Z}_+} q_k(f) = \sum_{k \in \mathbb{Z}_+} \sum_{s \in \mathbb{Z}_2(k)} \lambda_{k,s}^2(f) \varphi_{k,s}^2, \]
and for any continuous function \( f \) on \( I \),
\[ \|f - Q_k(f)\|_\infty \leq \omega_2(f, 2^{-k})_\infty. \]

Put \( Z_r^d(k) := \prod_{i=1}^d Z_r(k_i), \quad r = 1, 2 \). For \( k \in \mathbb{Z}_d^+ \), \( s \in Z_r^d(k) \), define
\[ \varphi_{k,s}^r(x) := \prod_{i=1}^d \varphi_{k_i,s_i}(x_i), \]
and \( \lambda_{k,s}^r(f) \) in the manner of the definition (2.7) by
\[ \lambda_{k,s}^r(f) := \lambda_{k_1,s_1}^r((\lambda_{k_2,s_2}^r(...\lambda_{k_d,s_d}^r(f))). \]

**Theorem 4.1** Let \( r = 1, 2, \quad 0 < p, \theta \leq \infty \) and \( 1/p < \alpha < r \). Then there hold the following assertions.

(i) A function \( f \in MB_{p,\theta}^\alpha \) can be represented by the series
\[ f = \sum_{k \in \mathbb{Z}_d^+} \sum_{s \in Z_r^d(k)} \lambda_{k,s}^r(f) \varphi_{k,s}^r, \quad (4.7) \]
converging in the quasi-norm of \( MB_{p,\theta}^\alpha \). Moreover, we have
\[ B^*(f) := \left( \sum_{k \in \mathbb{Z}_d^+} \left\{ 2^{(\alpha - 1/p)|k|} \left( \sum_{s \in Z_r^d(k)} |\lambda_{k,s}^r(f)|^p \right)^{1/p} \right\}^{1/\theta} \right)^{1/\theta} \leq CB(f). \]

(ii) If in addition, \( r = 2 \) and \( \alpha < \min(2, 1 + 1/p) \), then a continuous function \( f \) on \( I \) belongs to the Besov space \( MB_{p,\theta}^\alpha \) if and only if \( f \) can be represented by the series (4.7). Moreover, the Besov quasi-norm \( B(f) \) is equivalent to the discrete quasi-norm \( B^*(f) \).

**Proof.** If \( r = 2 \), from the definition and (4.6) we can derive that for functions on \( I \) and \( k \in \mathbb{Z}_d^+ \), the component function \( q_k(f) \) in the interpolant representation (2.30) related to the interpolant (4.3), can be rewritten as
\[ q_k(f) = \sum_{s \in Z_r^d(k)} \lambda_{k,s}^2(f) \varphi_{k,s}^2(x). \quad (4.8) \]
Therefore, Theorem 4.1 is the rewritten Theorem 2.1. This does not hold for the case \( r = 1 \). However, the last case can be proven in a way completely similar to the proof of Theorem 2.1 by using the above mentioned properties of the functions \( \varphi_{k,s}^1 \) and operators \( \Pi_k \).

For \( m \in \mathbb{Z}_+ \), we have by (4.8)

\[
R_m^r(f) = R_m(f) = \sum_{k \in \Delta(m)} \sum_{s \in Z^d_r(k)} \lambda_{k,s}^r(f) \varphi_{k,s}^r.
\]

For functions \( f \) on \( I^d \), \( R_m^r \) defines a linear sampling algorithm of the form (1.1) on the grid \( G^d_r(m) \) where \( G^d_r(m) := \{2^{-k} : k \in \Delta(m), s \in I^d_r(k)\} \), \( I^d_1(k) := \{s \in \mathbb{Z}^d_+ : 0 \leq s_i \leq 2^{k_i} - 1, \ i \in N[d]\} \), \( I^d_2(k) := I^d(k) \). More precisely,

\[
R_m^r(f) = L_n^r(f) = \sum_{k \in \Delta(m)} \sum_{j \in I^d_r(k)} f(2^{-k}j) \psi_{k,j}^r,
\]

where

\[
n := \sum_{k \in \Delta(m)} |I^d_r(k)| \asymp 2^m m^{d-1};
\]

\[
\psi_{k,s}^r(x) = \prod_{i=1}^d \psi_{k_i,s_i}^r(x_i), \ k \in \mathbb{Z}^d_+, \ s \in I^d_r(k),
\]

and the univariate functions \( \psi_{k,s}^r \) are defined by

\[
\psi_{k,s}^1 = \begin{cases} 
\varphi_{k,s}^1, & k = 0, \ s = 0, \\
\varphi_{k,j}^1, & k > 0, \ s = 2j + 1, \\
-\varphi_{k,j}^1, & k > 0, \ s = 2j,
\end{cases}
\]

and

\[
\psi_{k,s}^2 = \begin{cases} 
\varphi_{k,s}^2, & k = 0, \\
-\frac{1}{2}\varphi_{k,0}, & k > 0, \ s = 0, \\
\varphi_{k,j}, & k > 0, \ s = 2j + 1, \\
-\frac{1}{2}(\varphi_{k,j}^2 + \varphi_{k,j-1}^2), & k > 0, \ s = 2j, \\
-\frac{1}{2}\varphi_{k,2j-1}, & k > 0, \ s = 2^k.
\end{cases}
\]

From the interpolation properties (4.1) and (4.5), the equality \( \varphi_{k,s}^r(2^{-k}s') = \delta_{s,s'} \) one can easily verify that \( R_m^r(f) \) interpolates \( f \) at the grid \( G^d_r(m) \), i.e.,

\[
R_m^r(f, x) = f(x), \ x \in G^d_r(m).
\]

**Theorem 4.2** Let \( r = 2, \ 0 < p, q, \theta \leq \infty, \) and \( 1/p < \alpha < \min(2, 1 + 1/p) \). Then we have

(i) For \( p \geq q \),

\[
E(m) \gg 2^{-\alpha/m} m^{(d-1)(1-1/\theta)1}.
\]

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(ii) For $p < q$, 
\[ E(m) \gg 2^{-(\alpha-1/p+1/q)m}m^{(d-1)(1/q-1/\theta)+}. \]

Proof. Put $\Gamma(m) := \{ k \in \mathbb{N}^d : |k|_1 = m + 1 \}$. Let the half-open $d$-cube $I(k, s)$ be defined by $I(k, s) := \prod_{i=1}^d [s_i 2^{-(k_i-1)} , (s_i + 1) 2^{-(k_i-1)} )$. Notice that $I(k, s) \subset \mathbb{R}^d$ and $I(k, s) \cap I(k, s') = \emptyset$ for $s \neq s'$. Moreover, if $0 < \nu \leq \infty$, for $k \in \Gamma(m), s \in \mathbb{Z}^d(k)$,

\[ \| \varphi_{k,s}^2 \|_\nu \ = \left( \int_{I(k,s)} |\varphi_{k,s}^2(x)|^{\nu} \, dx \right)^{1/\nu} \asymp 2^{-m/\nu}, \quad (4.9) \]

with the change to sup when $\nu = \infty$, and

\[ \left\| \sum_{s \in \mathbb{Z}^d(k)} \varphi_{k,s}^2 \right\|_\nu \asymp 1. \quad (4.10) \]

Case (i): For an integer $m \geq 1$, we take the functions

\[ g_1 := C_1 2^{-\alpha m} \sum_{s \in \mathbb{Z}^d(\bar{k})} \varphi_{\bar{k},s}^2 \quad (4.11) \]

with some $\bar{k} \in \Gamma(m)$, and

\[ g_2 := C_2 2^{-\alpha m} m^{-(d-1)/\theta} \sum_{k \in \Gamma(m)} \sum_{s \in \mathbb{Z}^d(k)} \varphi_{k,s}^2. \quad (4.12) \]

Notice that the right side of (4.11) and (4.12) defines the series (1.7) of $g_i, i = 1, 2$. By Theorem 4.1 and (4.10) we can choose constants $C_i$ so that $g_i \in B_{p,\theta}^{\alpha}$ for all $m \geq 1$ and $i = 1, 2$. It is easy to verify that $g_i - R_m^2(g_i) = g_i, i = 1, 2$. We have by (4.10)

\[ E(m) \geq \| g_1 \|_q \gg 2^{-\alpha m} \]

if $\theta \leq 1$, and

\[ E(m) \geq \| g_2 \|_q \geq \| g_2 \|_{q^*} \gg 2^{-\alpha m} m^{(d-1)(1-1/\theta)} \]

if $\theta > 1$, where $q^* := \min(q, 1)$.

Case (ii): Let $s(k) \in \mathbb{Z}^d_+$ be defined by $s(k)_i = \sum_{j=1}^{k_i-1} 2^j$ if $k_i > 2$, and $s(k)_i = 0$ if $k_i = 2$ for $i = 1, \ldots, d$, and $\Gamma^*(m) := \{ k \in \Gamma(m) : k_i \geq 2, \ i = 1, \ldots, d \}$. For an integer $m \geq 2$, we take the functions

\[ g_3 = C_3 2^{-(\alpha-1/p)m} \varphi_{k^*,s(k^*)}^2 \quad (4.13) \]

with some $k^* \in \Gamma^*(m)$, and

\[ g_4 = C_4 2^{-(\alpha-1/p)m} m^{-(d-1)/\theta} \sum_{k \in \Gamma^*(m)} \varphi_{k,s(k)}^2. \quad (4.14) \]
Similarly to the functions \( g_i, \ i = 1, 2, \) the right side of (4.13) and (4.14) defines the series (4.7) of \( g_i, \ i = 3, 4, \) and we can choose constants \( C_i \) so that \( g_i \in B_{p, \theta}^\alpha \) for all \( m \geq 2 \) and \( i = 3, 4. \) Obviously, \( g_i - R_m^2(g_i) = g_i, \ i = 3, 4. \) We have by (4.9)

\[
E(m) \geq \|g_3\|_q \gg 2^{-(\alpha-1/p+1/q)m}
\]

if \( \theta \geq q, \) and

\[
E(m) \geq \|g_4\|_q \gg 2^{-(\alpha-1/p+1/q)m(d-1)(1/q-1/\theta)}
\]

if \( \theta < q. \)

From Theorems 3.1 and 4.2 we obtain

**Theorem 4.3** Let \( r = 2, \ 0 < p, q, \theta \leq \infty, \) and \( 1/p < \alpha < \min(2, 1+1/p). \) Then we have

(i) For \( p \geq q, \)

\[
E(m) \asymp \begin{cases} 
2^{-\alpha m}, & \theta \leq \min(q, 1), \\
2^{-\alpha m}m(d-1)(1-1/\theta), & \theta > 1, q \geq 1.
\end{cases}
\]

(ii) For \( p < q < \infty, \)

\[
E(m) \asymp 2^{-(\alpha-1/p+1/q)m}m(d-1)(1/q-1/\theta^+).
\]

Notice that Theorem 4.3(i) has been proven in [22] for the \( 1 \leq p = q = \theta \leq \infty. \)

5 Appendix

**Lemma 5.1** Let \( 1 \leq p, q \leq \infty, \ 0 < \theta \leq \infty \) and \( \alpha > (1/p - 1/q)^+. \) Then we have the following asymptotic order of \( \lambda_n(B_{p, \theta}^\alpha)_q. \)

(i) For \( p \geq q, \)

\[
\lambda_n(B_{p, \theta}^\alpha)_q \asymp \begin{cases} 
(n^{-1} \log^{d-1} n)^\alpha, & 2 \leq q \leq p < \infty, \\
(n^{-1} \log^{d-1} n)^\alpha, & \theta \leq 1, p = q = \infty, \\
(n^{-1} \log^{d-1} n)^\alpha, & 1 < p = q \leq 2, \ \theta \leq q, \\
(n^{-1} \log^{d-1} n)^\alpha(\log^{d-1} n)^{1/q-1/\theta}, & 1 < p = q \leq 2, \ \theta > q \\
(n^{-1} \log^{d-1} n)^\alpha(\log^{d-1} n)^{1/2-1/\theta}, & \theta > 2.
\end{cases}
\]

(ii) For \( 1 < p < q < \infty, \)

\[
\lambda_n(B_{p, \theta}^\alpha)_q \asymp \begin{cases} 
(n^{-1} \log^{d-1} n)^{\alpha-1/p+1/q}, & 2 \leq p, 2 \leq \theta \leq q, \\
(n^{-1} \log^{d-1} n)^{\alpha-1/p+1/q}(\log^{d-1} n)^{1/q-1/\theta^+}, & q \leq 2.
\end{cases}
\]
Proof. This lemma was proven in [14], [20] except the cases \( \theta < 2 \leq q \leq p < \infty \) and \( \theta \leq 1 \), \( p = q = \infty \) which can be obtained from the asymptotic order [20]

\[
\lambda_n(B_{p,q}^\alpha) \times (n^{-1} \log^{d-1} n)^\alpha, \quad \begin{cases} 1 \leq \theta \leq 2 \leq q \leq p < \infty, \\ \theta = 1, \ p = q = \infty, \end{cases}
\]

and the equalities \( \lambda_n(W)_q = \lambda_n(coW)_q \) and \( coB_{p,\theta}^\alpha = B_{p,\max(\theta,1)}^\alpha \), where \( coW \) denotes the convex hull of \( W \). \( \square \)

For \( p = (p_1, \ldots, p_d) \in (0, \infty)^d \), we defined the mixed integral quasi-norm \( \| \cdot \|_p \) for functions on \( \mathbb{R}^d \) as follows

\[
\|f\|_p := \left( \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} |f(x)|^{p_1} \, dx_1 \right)^{p_2/p_1} \, dx_2 \cdots \right)^{1/p_d},
\]

and put \( 1/p := (1/p_1, \ldots, 1/p_d) \). If \( p, q \in (0, \infty)^d \) and \( p \leq q \), then there holds Nikol’skii’s inequality for any \( f \in \Sigma_r^d(k) \),

\[
\|f\|_q \leq C 2^{(1/p - 1/q)k_1} \|f\|_p \tag{5.1}
\]

with constant \( C \) depending on \( p, q, d \) only. This inequality can be proven by a generalization of the Jensen’s inequality for mixed norms and the following equivalences of the mixed integral quasi-norm \( \| \cdot \|_p \). For all \( k \in \mathbb{Z}_+^d \) and all \( f \in \Sigma_r^d(k) \) of the form (2.14),

\[
\|f\|_p \asymp \prod_{i=1}^d 2^{-k_i/p_i} \|\{a_s\}\|_{p,k},
\]

where

\[
\|\{a_s\}\|_{p,k} := \left( \sum_{s_d \in J(k_d)} \cdots \sum_{s_2 \in J(k_2)} \left( \sum_{s_1 \in J(k_1)} |a_s|^{p_1} \right)^{p_2/p_1} \cdots \right)^{1/p_d}.
\]

**Lemma 5.2** Let \( 0 < p < q < \infty \), \( \delta = 1/2 - p/(p + q) \). If \( k, s \in \mathbb{Z}_+^d \), then for any \( \varphi_k \in \Sigma_r^d(k) \) and \( \varphi_s \in \Sigma_r^d(s) \), there holds the inequality

\[
\int_{\mathbb{R}^d} |\varphi_k(x) \varphi_s(x)|^{q/2} \, dx \leq C A_k A_s 2^{-\delta |k-s|_1},
\]

with some constant \( C \) depending at most on \( p, q, d \), where

\[
A_k := \left( 2^{(1/p - 1/q)k_1} \|\varphi_k\|_p \right)^{q/2}.
\]

**Proof.** Put \( \nu := (p + q)/p \). Then \( \delta = 1/2 - 1/\nu \) and \( 2 < \nu < \infty \). Let \( \nu' \) be given by \( 1/\nu + 1/\nu' = 1 \). Then \( 1 < \nu' < 2 \). Let \( u, v \in (0, \infty)^d \) be defined by \( u := qv/2 \) and \( v_i = \nu \) if \( k_i \geq s_i \) and \( v_i = \nu' \) if \( k_i < s_i \) for \( i = 1, \ldots, d \). Let \( u' \) and \( v' \) be given by \( 1/u + 1/u' = 1 \) and \( 1/v + 1/v' = 1 \), respectively.
Notice that $v \in (1, \infty)^d$. Applying Hölder’s inequality for the mixed norm $\| \cdot \|_v$ to the functions $|\varphi_k|^{q/2}$ and $|\varphi_s|^{q/2}$, we obtain

$$
\int_{\mathbb{R}^d} |\varphi_k(x)\varphi_s(x)|^{q/2}dx \leq \|\varphi_k|^{q/2}\|_v\|\varphi_s|^{q/2}\|_v = \|\varphi_k|^{q/2}\|_u\|\varphi_s|^{q/2}\|_{u'} = \|\varphi_k\|_u^{q/2}\|\varphi_s\|_{u'}^{q/2}.
$$

(5.2)

Since $u > p1$ and $u' > p1$, by the inequality (5.1) we have

$$
\|\varphi_k\|_u \leq 2^{(1/p-1/u)|k|_1}\|\varphi_k\|_p, \quad \|\varphi_s\|_{u'} \leq 2^{(1/p-1/u')|s|_1}\|\varphi_s\|_p.
$$

(5.3)

From (5.2) and (5.3) we prove the lemma.

**Lemma 5.3** Let $0 < p < q < \infty$ and $g \in L_q$ be represented by the series

$$
g = \sum_{k \in \mathbb{Z}^+_d} g_k, \quad g_k \in \Sigma^d_r(k).
$$

Then there holds the inequality

$$
\|g\|_q \leq C \left( \sum_{k \in \mathbb{Z}^+_d} 2^{(1/p-1/q)|k|_1}\|g_k\|_p^q \right)^{1/q},
$$

(5.4)

with some constant $C$ depending at most on $p,d$, whenever the right side is finite.

**Proof.** It is enough to prove the inequality (5.4) for $g$ of the form

$$
g = \sum_{k \leq m} g_k, \quad g_k \in \Sigma^d_r(k),
$$

for any $m \in \mathbb{Z}^+_d$.

Put $n := \lceil q \rceil + 1$. Then $0 < q/n \leq 1$. By Jensen’s inequality we have

$$
\left| \sum_{k \leq m} g_k(x) \right|^q = \left( \sum_{k \leq m} g_k(x) \right)^{q/n} \leq \left( \sum_{k \leq m} |g_k(x)|^{q/n} \right)^n = \sum_{k^i \leq m} \cdots \sum_{k^n \leq m} \prod_{j=1}^n |g_{k^j}(x)|^{q/n}.
$$

consequently,

$$
\|g\|_q^q \leq \sum_{k^1 \leq m} \cdots \sum_{k^n \leq m} \int_{\mathbb{R}^d} \prod_{j=1}^n |g_{k^j}(x)|^{q/n}dx.
$$

(5.5)

By use of the identity

$$
\prod_{j=1}^n a_j = \left( \prod_{i \neq j} a_i a_j \right)^{1/(n-1)}
$$

(5.6)
Lemma 5.2 and (5.6) gives

\[ n \text{ for non-negative numbers } a_1, ..., a_n, \text{ we get} \]

\[ J := \int \prod_{j=1}^{n} |g_{k_j}(x)|^{q/n} \, dx = \int \prod_{i \neq j} |g_{k_i}(x)|^{q/2(n-1)} \, dx. \]

Hence, applying Hölder’s inequality to \( n(n-1) \) functions in the right side of the last equality, Lemma 5.2 and (5.6) gives

\[
J \leq \prod_{i \neq j} \left( \int |g_{k_i}(x)|^{q/2} \, dx \right)^{1/n(n-1)} \leq \prod_{i \neq j} \left( A_{k_i} A_{k_j} 2^{-\delta|m-k^i_k|_1} \right)^{1/2(n-1)}
\]

\[
= \prod_{i \neq j} \left( A_{k_i} A_{k_j} \right)^{1/2(n-1)} \left( \prod_{i \neq j} \left( \prod_{j'=1}^{n} 2^{-\delta|m-k^i_k|_1} \prod_{j'=1}^{n} 2^{-\delta|m-k^j_k|_1} \right)^{1/n(n-1)} \right)^{1/2(n-1)}
\]

\[
= \prod_{j=1}^{n} \frac{A_{k_j}^{2/n} \left( \prod_{i=1}^{n} 2^{-\delta|m-k^i_k|_1} \right)^{1/n(n-1)}}{\left( \prod_{j=1}^{n} A_{k_j}^{n} \prod_{i=1}^{n} 2^{-\lambda\delta|m-k^i_k|_1} \right)^{1/n}} =: \prod_{j=1}^{n} B_j.
\]

where \( \lambda := \delta/(n - 1) > 0 \). Therefore, from (5.5) and Hölder’s inequality we obtain

\[
\|g\|^q \leq \sum_{k^1 \leq m} \cdots \sum_{k^n \leq m} \left( \prod_{j=1}^{n} A_{k_j}^{n} \prod_{i=1}^{n} 2^{-\lambda\delta|m-k^i_k|_1} \right)^{1/n} \leq \prod_{j=1}^{n} \left( \sum_{k^1 \leq m} \cdots \sum_{k^n \leq m} A_{k_j}^{n} \prod_{j=1}^{n} 2^{-\lambda\delta|m-k^j_k|_1} \right)^{1/n} =: \prod_{j=1}^{n} B_j.
\]

We have

\[
B_j = \sum_{k^1 \leq m} A_{k^1}^{k^2} \sum_{k^1 \leq m} \cdots \sum_{k^{j-1} \leq m} \sum_{k^{j+1} \leq m} \cdots \sum_{k^n \leq m} \prod_{i=1}^{n-1} 2^{-\lambda\delta|m-k^i_k|_1}
\]

\[
= \sum_{k^j \leq m} A_{k^j}^{k^2} \left( \sum_{s \leq m} 2^{-\lambda\delta|m-k^j_k|_1} \right)^{n-1} \leq C \sum_{k^j \leq m} A_{k^j}^{k^2}.
\]

Using this estimate for \( B_j \), we can continue (5.7) and finish the estimation of \( \|g\|^q \) as follows.

\[
\|g\|^q \leq \prod_{j=1}^{n} B_j^{1/n} \leq C \sum_{k \leq m} A_{k}^{k^2} \leq C \sum_{k \leq m} \|2^{(1/p-1/q)|k|_1} g_k\|^q.
\]
Thus, the proof of the lemma is completed. □

**Remark** A trigonometric polynomial version of Lemma 5.3 was proven in [25] for $1 \leq p < q < \infty$.

**Acknowledgments.** This work is supported by Vietnam National Foundation for Science and Technology Development (NAFOSTED).

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