CORRELATION BETWEEN THE DEUTERON CHARACTERISTICS AND THE LOW-ENERGY TRIPLET np SCATTERING PARAMETERS

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The correlation relationship between the deuteron asymptotic normalization constant, $A_S$, and the triplet np scattering length, $a_t$, is investigated. It is found that 99.7% of the asymptotic constant $A_S$ is determined by the scattering length $a_t$. It is shown that the linear correlation relationship between the quantities $A_S^2$ and $1/a_t$ provides a good test of correctness of various models of nucleon-nucleon interaction. It is revealed that, for the normalization constant $A_S$ and for the root-mean-square deuteron radius $r_d$, the results obtained with the experimental value recommended at present for the triplet scattering length $a_t$ are exaggerated with respect to their experimental counterparts. By using the latest experimental phase shifts of Arndt et al., we obtain, for the low-energy scattering parameters ($a_t$, $r_t$, $P_t$) and for the deuteron characteristics ($A_S$, $r_d$), results that comply well with experimental data.

1. Basic features of the deuteron — such as the binding energy $\varepsilon_d$; the electric quadrupole moment $Q$; the root-mean-square radius $r_d$; the asymptotic normalization constants for the $S$ and the $D$ wave, $A_S$ and $A_D$; and the corresponding asymptotic $D/S$ ratio $\eta = A_D/A_S$ — play a significant role in constructing realistic models of nucleon-nucleon interaction and are important physical characteristics of nuclear forces. Of equally great importance are low-energy np scattering parameters in the triplet state. This include the scattering length $a_t$; the effective range $r_t$; and the shape parameters $v_2$, $v_3$, $v_4$, ... appearing in the effective-range expansion

$$k \cot \delta_t (k) = -\frac{1}{a_t} + \frac{1}{2} r_t k^2 + v_2 k^4 + v_3 k^6 + v_4 k^8 + \ldots,$$

where $\delta_t (k)$ is the triplet np scattering eigenphase corresponding to the $^3S_1$ state. For this reason, much attention has been given to these quantities both in theoretical and in
experimental studies [1–17]. At the present time, the experimental value of the deuteron binding energy $\varepsilon_d$ is known to a high precision [9]:

$$\varepsilon_d = 2.22458900 \text{ MeV}. \quad (2)$$

The value of the asymptotic $D/S$ ratio, $\eta$, was also determined to a fairly good precision, both theoretically and experimentally (see [4, 7, 15–17]). The majority of the theoretical estimates of this quantity are in good agreement with its experimental value of $\eta = 0.0272$ [15]. At the same time, the values of some characteristics of the deuteron, such as the asymptotic normalization constant $A_S$ and the root-mean-square radius $r_d$, were the subject of controversy. For example, the value obtained for $A_S$ directly from experimental data by analyzing elastic $pd$ scattering [18],

$$A_S = 0.8781 \text{ fm}^{-1/2}, \quad (3)$$

is at odds with the theoretical estimates derived for this quantity for many realistic potentials [19–26], as well as with those found from an analysis of phase shifts [17, 27] and on the basis of the effective-range expansion [4, 8]. Values of the asymptotic constant $A_S$ that are discussed in the literature vary within a rather broad range — from $0.7592 \text{ fm}^{-1/2}$ to $0.9863 \text{ fm}^{-1/2}$ [28, 29].

In a number of studies the result reported by Ericson in [5], $A_S = 0.8802 \text{ fm}^{-1/2}$, is used for an “experimental” value. It should be noted, however, that this value was obtained from an analysis of the linear relationship between the asymptotic constant $A_S$ and the root-mean-square radius $r_d$ rather than from experimental data directly. This linear correlation relationship between $A_S$ and $r_d$ was established empirically for various models of nucleon-nucleon interaction [5, 30]. The procedure that Ericson used to obtain the above value of $A_S = 0.8802 \text{ fm}^{-1/2}$ on the basis of this relationship involved averaging the values of the root-mean-square radius $r_d$ that were found experimentally in [10, 11]. In view of this, the use of Ericson’s result for an experimental value is not quite correct. The value that is presented in (3) and which was derived in [18] on the basis of a direct method for determining this normalization constant from an analysis of data on elastic $pd$ scattering is more justified.

Other values sometimes used for the experimental asymptotic normalization constant include $A_S = 0.8846, 0.8848,$ and $0.8883 \text{ fm}^{-1/2}$ (see [4], [30, 31], and [8], respectively). In
just the same way as Ericson’s result, they can hardly be treated, however, as correct experimental values, since they were found in the effective-range approximation with allowance for some corrections associated with the form of interaction; moreover, the low-energy triplet \( np \) scattering parameters that were employed in doing this are not determined from experimental data unambiguously. By way of example, we indicate that, in [1–3, 12–14, 32], values between 5.377 \( fm \) [1] and 5.424 \( fm \) [32] are given for experimental values of the triplet scattering length \( a_t \), but the asymptotic normalization constant \( A_S \) greatly depends on \( a_t \). As will be shown below, 99.7% of the normalization constant \( A_S \) is determined by the triplet scattering length \( a_t \).

In the following, the value in (3) from [18] will be used for the experimental value of the normalization constant \( A_S \). It is close to the value of \( A_S = 0.8771 \ fm^{-1/2} \), which corresponds to the vertex-constant value of \( G^2_d = 0.427 \ fm \) for the \( d \rightarrow n + p \) vertex and which was obtained earlier in [33]. As we have already indicated, the authors of [18, 33] employed a direct method for determining the constants \( A_S \) and \( G^2_d \) that relies on extrapolating the experimental cross sections for elastic \( pd \) scattering to the exchange-singularity point.

In just the same way as the constant \( A_S \), the root-mean-square radius \( r_d \) of the deuteron is determined, to great extent, by the triplet scattering length \( a_t \). A linear correlation relationship between the quantities \( r_d \) and \( a_t \) was established empirically in [6, 30]. To a high precision, this relationship can be approximated as follows:

\[
    r_d = 0.4a_t - 0.1985 \ (fm) .
\]  

(4)

At present, the following experimental values are used in the literature for the root-mean-square radius of the deuteron:

\[
    r_d = 1.9635 \ fm \ [10] \quad (B), \\
    r_d = 1.9560 \ fm \ [11] \quad (S), \\
    r_d = 1.950 \ fm \ [30] \quad (K). 
\]  

(5a)  

(5b)  

(5c)

According to Eq. (4), the following values of the scattering length \( a_t \) correspond to the values of the radius \( r_d \) in (5):

\[
    a_t = 5.4050 \ fm \quad (B),
\]  

(6a)
a_t = 5.3863 \text{ fm} \quad \text{(S)}, \quad (6b)

a_t = 5.3713 \text{ fm} \quad \text{(K)}, \quad (6c)

On the other hand, the values of the triplet scattering length that were calculated for many realistic nucleon-nucleon potentials [19–21, 24, 25] are close to the experimental value [32]

\[ a_t = 5.424 \text{ fm,} \quad (7) \]

which is recommended at present, but which, in accordance with Eq. (4), leads to the root-mean-square radius exaggerated in relation to the experimental values in (5a)–(5c); that is,

\[ r_d = 1.9711 \text{ fm.} \quad (8) \]

Thus, we can see that, frequently, values obtained and used in various studies for the characteristics of the deuteron and for the triplet low-energy np scattering parameters are contradictory and deviate from experimental results.

2. In accordance with [8], the asymptotic normalization constant \( A_S \) for the deuteron can be represented in the form

\[ A_S^2 = \frac{2\alpha}{1 - \alpha \rho_d}, \quad (9) \]

where \( \alpha \) is the deuteron wave number defined according to the relation \( \varepsilon_d = \hbar^2 \alpha^2 / m_N \) and \( \rho_d \equiv \rho (-\varepsilon_d, -\varepsilon_d) \) is the deuteron effective range corresponding to \( S \)-wave interaction. The definition and properties of the radius \( \rho_d \) and of the function \( \rho(E_1, E_2) \) are discussed in detail elsewhere [1]. The quantity \( \rho_d \) appears in the expansion of the function \( k \cot \delta_t(k) \) at the point \( k^2 = -\alpha^2 \) — that is, at the energy value equal to the deuteron binding energy. The expansion at the point \( k^2 = -\alpha^2 \) is similar to the expansion in (1), which is performed at the origin, involving the ordinary effective range \( r_t \equiv \rho(0,0) \) of scattering theory — that is, the effective range at zero energy. It can easily be found that the quantities \( \rho_d \) and \( r_t \) are expanded in powers of the parameter \( \alpha^2 \) as

\[ \rho_d = \rho_m - 2v_2\alpha^2 + 4v_3\alpha^4 - 6v_4\alpha^6 + \ldots, \quad (10) \]

\[ r_t = \rho_m + 2v_2\alpha^2 - 2v_3\alpha^4 + 2v_4\alpha^6 - \ldots, \quad (11) \]
where $\rho_m \equiv \rho(0, -\varepsilon_d)$ is the so-called mixed effective range [1], for which the following relation holds:

$$\rho_m = \frac{2}{\alpha} \left(1 - \frac{1}{\alpha a_t}\right).$$

(12)

The shape parameters $v_n$ in expansions (1), (10), and (11) are dimensional quantities. Instead of them, one often introduces the dimensionless shape parameters $P_t$, $Q_t$, ... related to the parameters $v_n$ by the equations

$$v_2 = -P_t r_t^3, \quad v_3 = Q_t r_t^5, \ldots .$$

(13)

From the expansions in (10) and (11), it follows that the quantities $r_t$, $\rho_d$, and $\rho_m$ are related as

$$\rho_d + r_t = 2\rho_m + 2v_3\alpha^4 - 4v_4\alpha^6 + \ldots ,$$

(14)

$$\rho_d + 2r_t = 3\rho_m + 2v_2\alpha^2 - 2v_4\alpha^6 + \ldots .$$

(15)

The asymptotic normalization constant $A_S$ for the deuteron is directly expressed in terms of the residue of the $S$ matrix $S(k)$ at the pole $k = i\alpha$ corresponding to a bound state of the two-nucleon system; that is,

$$A^2_S = i \text{ Res}_{k=i\alpha} S(k).$$

(16)

Along with the constant $A_S$, other physical quantities, such as the nuclear vertex constant $G_d$ and the dimensionless asymptotic normalization constant $C_d$, are frequently used in the literature [28, 29]. These two quantities are directly related to the constant $A_S$ by the equations

$$G_d^2 = \pi \bar{\lambda}^2 A^2_S,$$

(17)

$$C_d^2 = A^2_S / 2\alpha,$$

(18)

where $\bar{\lambda} = 2\bar{\lambda}_N$, with $\bar{\lambda}_N \equiv \hbar / m_N c$ being the Compton wavelength of the nucleon.

In the approximation where there is no dependence on the form of interaction ($v_2 = v_3 = \ldots = 0$),

$$k \cot \delta_t(k) = -\frac{1}{a_t} + \frac{1}{2} r_t k^2,$$

(19)

the quantities $r_t$, $\rho_d$, and $\rho_m$ are equal to each other, as follows from Eqs. (10) and (11):

$$r_t = \rho_d = \rho_m .$$

(20)
In this approximation, the use of relations (12) and (20) makes it possible to recast Eq. (9) into the more convenient form

\[ C_d^{-2} = -1 + 2 \frac{R_d}{a_t}, \]

(21)

where the quantity

\[ R_d \equiv \frac{1}{\bar{\alpha}}, \]

(22)

which characterizes the spatial dimensions of the deuteron, is referred to as the deuteron radius [2]. By using the value in (2) for the deuteron binding energy, one can easily obtain the following numerical value for the deuteron radius:

\[ R_d = 4.317688 \text{ fm}. \]

(23)

In the effective-range approximation (19) the low-energy triplet np scattering parameters \( a_t \) and \( r_t \) are expressed in terms of the bound-state parameters (deuteron radius \( R_d \) and normalization constant \( C_d \)) as

\[ a_t = \frac{2R_d}{1 + C_d^{-2}}, \]

(24)

\[ r_t = R_d \left(1 - C_d^{-2}\right). \]

(25)

With the aid of the deuteron parameters, the behavior of the phase shift at low energies can be predicted in the effective-range approximation (19) with allowance for expressions (24) and (25).

3. As was indicated above, a great number of studies have been devoted to exploring and calculating the asymptotic normalization constant \( A_S \). The values of the constant \( A_S \) (and of the quantities \( \varepsilon_d \) and \( a_t \)) for some realistic potentials [19–25, 27, 34–36] are quoted in Table 1, along with the values of \( A_S \) that were found from the analysis of phase shifts in [17, 27] and on the basis of the effective-range expansion in [4, 8]. Also given in the same table are the values of the constant \( A_S \) that were calculated in the present study by formulas (18) and (21), which correspond to the effective-range approximation. In addition, Table 1 presents the absolute (\( \Delta \)) and the relative (\( \delta \)) error in the calculation of \( A_S \) in this approximation.

It can be seen from Table 1 that, for the majority of the models, the relative error in the \( A_S \) value calculated in the effective-range approximation does not exceed 0.3%, while the...
absolute error is not greater than 0.003 \, fm^{-1/2}, as a rule. This is not so only for some early models of the Bonn potential (lines 15–17 in Table 1), in which case the relative error in the approximate value of the constant $A_S$ is 2 to 3%. For the same potentials, the relative error in the approximate values of the deuteron mean-square radius $r_d$ that were obtained by formula (4) is also overly large. For example, the relative error in $r_d$ is 2.52% for the HM-2 potential and 1.49% for the Bonn F potential. At the same time, this error is small for more correct models of the Bonn potential, Bonn R and Bonn Q (0.081 and 0.137%, respectively). For the Paris potential, the error in question is 0.035%.

Thus, we can see from Table 1 that the values of the asymptotic normalization constant $A_S$ for realistic potentials are strongly correlated with the values of the triplet $np$ scattering length $a_l$. The same is also true for the values of $A_S$ that were found from the analysis of phase shifts (lines 18, 19) and on the basis of the effective-range expansion in [4] (line 20). As to the value of $A_S = 0.8883 \, fm^{-1/2}$ (line 21), which was calculated in [8] on the basis of the effective-range expansion, it is strongly overestimated in relation to the value of $A_S = 0.88191 \, fm^{-1/2}$, which was found in the present study in the effective-range approximation. In that case, the relative error in the approximate value was 0.725%. An incorrect choice of the shape parameter $P_t$ in [8] is the reason for this discrepancy — namely, the following data from [37] on the low-energy scattering parameters were used in [8] to calculate the constant $A_S$: $a_t = 5.412 \, fm$, and $r_t = 1.733 \, fm$; however, the value of $P_t = -0.0188$, chosen there for the shape parameter, corresponded to the Paris potential, for which the scattering length and the effective range take values ($a_t = 5.427 \, fm$, $r_t = 1.766 \, fm$) that exceed considerably those that were employed in [8].

Considering that the deuteron binding energy has been determined to a high degree of precision and is taken to have an approximately the same value in all of the calculations, one can conclude on the basis of the results in Table 1 that 99.7% of the asymptotic normalization constant $A_S$ is determined by the triplet scattering length $a_l$. The inverse also holds: knowing the values of the constant $A_S$, one can determine the triplet scattering length $a_l$ to a high degree of precision.

Taking the aforesaid into consideration, we will investigate the asymptotic constant $A_S$
as a function of the triplet scattering length $a_t$. It is convenient to perform this investigation for the dimensionless quantity $C_d^{-2}$, which, in the effective-range approximation, is a linear function of the dimensionless quantity $R_d/a_t$ [see Eq. (21)]. The $R_d/a_t$ dependence of $C_d^{-2}$ is shown in the figure. The straight line represents the results of the calculation based on the approximate formula (21), while the points in the figure correspond to $\varepsilon_d$, $a_t$, and $A_S$ values computed by various authors and quoted in Table 1. As can be seen from the figure, there is a linear relationship between the quantities $C_d^{-2}$ and $R_d/a_t$. Points corresponding to their values lie in the close proximity of the straight line specified by Eq. (21). As was mentioned above, this is not so only for points corresponding to early models of the Bonn potential and the point corresponding to the values of the quantities in question from [8]. Points that represent values of $C_d^{-2}$ and $R_d/a_t$ for more correct versions of the Bonn potential model (Bonn R, Bonn Q) lie near the straight line specified by Eq. (21) (points 13, 14).

Thus, it follows from Table 1 and from the figure that the asymptotic normalization constant $A_S$ and the triplet scattering length $a_t$ are well correlated quantities, so that any of these can be determined to a high degree of precision if the other is known. For the experimental value presented in (3) for the constant $A_S$, the corresponding value of the scattering length $a_t$ can easily be determined in the effective-range approximation by formulas (18) and (21). The result is

$$a_t = 5.395 \text{ fm}.$$ (26)

At the same time, the currently recommended experimental value of the triplet scattering length in (7) leads, in the effective-range approximation, to the asymptotic-normalization-constant value

$$A_S = 0.88451 \text{ fm}^{-1/2},$$ (27)

which is well above the experimental value of this quantity in (3). As was indicated above, the experimental scattering-length value in (7) also leads to the exaggerated value in (8) for the root-mean-square radius $r_d$ of the deuteron.

Thus, it can be concluded from the above analysis that the currently recommended experimental value of the triplet $np$ scattering length in (7) does not comply with the experimental values of the asymptotic normalization constant $A_S$ for the deuteron and its
root-mean-square radius \( r_d \) in (3) and (5a)–(5c), respectively. Therefore, it is of paramount importance to determine, for the characteristics of the deuteron and for the low-energy triplet \( np \) scattering parameters, such values that would be consistent with one another, on one hand, and which would be compatible with experimental data on the other hand.

4. Fixing the features \( \varepsilon_d, A_S, \) and \( r_d \) of the deuteron, we will study the behavior of the \( S \)-wave phase shift at low energies in the approximation that takes into account the shape parameter \( P_t \) in the effective-range expansion; that is,

\[
k \cot \delta_t (k) = -\frac{1}{a_t} + \frac{1}{2} r_t k^2 - P_t r_t^3 k^4 . \tag{28}\]

For the scattering length \( a_t \), use is made here of the values in (6a)–(6c), which correspond to the values of the root-mean-square-radius \( r_d \) of the deuteron in (5a)–(5c), while, in accordance with (10), (13), and (14), the effective range \( r_t \) and the shape parameter \( P_t \) are given by

\[
r_t = 2\rho_m - \rho_d , \tag{29}\]

\[
P_t = \frac{\rho_d - \rho_m}{2r_t^3 \alpha^2} , \tag{30}\]

where the effective radius \( \rho_d \) of the deuteron and the mixed effective range \( \rho_m \) are determined from Eqs. (9) and (12), respectively. The parameters \( a_t, r_t, P_t, \rho_m, \) and \( \rho_d \) calculated in this approximation on the basis of the experimental values of the deuteron binding energy \( \varepsilon_d \) in (2), the asymptotic normalization constant \( A_S \) in (3), and the root-mean-square radius \( r_d \) of the deuteron in (5a)–(5c) are given in Table 2 (B, S, K), along with the values calculated for these parameters in the effective-range approximation (ER) on the basis of the experimental values of \( \varepsilon_d \) and \( A_S \), as well as values found in the present study from the latest partial-wave analysis (PWA) of the \( NN \) scattering by Arndt et al. [38], which, as can be seen from this table, are in very good agreement with the corresponding values for version B, where the features of the deuteron were set to \( \varepsilon_d = 2.22458900 \, MeV \) [9], \( A_S = 0.8781 \, fm^{-1/2} \) [18], and \( r_d = 1.9635 \, fm \) [10].

The quantity \( \delta \rho \) defined as the difference of the effective radius \( \rho_d \) of the deuteron and the mixed effective range \( \rho_m \),

\[
\delta \rho = \rho_d - \rho_m . \tag{31}\]
is often discussed in the literature. According to the estimates obtained by Noyes in [3] on the basis of dispersion relations, this difference arises owing to one-pion exchange and is positive, its magnitude being 0.016 fm. For many potential models, the difference $\delta \rho$ is also positive; as was established in [5, 6, 31], it well correlates with the triplet scattering length $a_t$. In our case, this difference is given by

$$\delta \rho = 2P_t r_t^3 \alpha^2. \quad (32)$$

For the parameter values used in the S and K versions from Table 2, it is positive, taking the values of 0.011 and 0.030 fm, respectively. However, for the cases of B and PWA in Table 2, the difference $\delta \rho$ is negative, its values being $-0.013$ and $-0.015$ fm. For this reason, the problem of the sign and magnitude of the difference of the effective radius of the deuteron $\rho_d$ and the mixed effective range $\rho_m$ calls for a further investigation.

For the low-energy parameters given in Table 2, we have calculated the triplet phase shift $\delta_t(k)$. The results are displayed in Table 3, along with the latest experimental data of Arndt et al. [38] on the triplet phase shift. Table 4 presents the energy dependence of the difference

$$\Delta = \delta_{\text{exp}} - \delta_{\text{theor}} \quad (33)$$

of the experimental value of the phase shift and its theoretical counterparts calculated by formula (28) and quoted in Table 3. As can be seen from Tables 3 and 4, all sets of low-energy parameters $a_t$, $r_t$, and $P_t$ from Table 2 describe well experimental data up to an energy value of 5 $MeV$ (the absolute error being less than 1°). Nonetheless, the distinction between the sets of low-energy parameters in describing experimental data becomes noticeable at an energy as low as 1 $MeV$, and we can see that preference should be given to the parameter sets employed in the B and PWA versions. These sets provide a nearly precision description (with a relative error of about 0.005%) of experimental phase shifts up to an energy value of 5 $MeV$. Thus, the accuracy of existing experimental data that is available at present is quite sufficient for removing ambiguities that arise in determining the scattering length $a_t$ and the effective range $r_t$ [1] and which are associated with the form of the potential. In view of this, the problem of deducing the scattering length, the effective range, and parameters of higher order (shape parameters) in expansion (1) directly from experimental data is pressing. It
should be noted that the B and PWA sets describe well, in contrast to other parameter sets, experimental phase shifts up to an energy value of 50\,MeV. For the B and PWA sets, the absolute error is about 0.5° at $E_{lab} = 30\,MeV$ and about 1° at $E_{lab} = 50\,MeV$. From Table 4, it can be seen that, for other parameter sets, the absolute error is much greater.

Thus, $np$ scattering in the triplet state can be described rather well within the B set up to an energy value of 50\,MeV, the experimental values used in this description for the characteristics of the deuteron being that in (2) for the binding energy, that in (3) for the asymptotic normalization constant, and that in (5a) for the root-mean-square radius. On the other hand, parameters that characterize the neutron-proton bound state (deuteron) can be determined from experimental data on $np$ scattering. By using the values

$$a_t = 5.4030\,fm, \quad r_t = 1.7495\,fm, \quad P_t = -0.0259,$$

which we found here for the low-energy scattering parameters from an analysis of the latest data on phase shifts [38], we will now determine the asymptotic normalization constant $A_S$ and the root-mean-square radius $r_d$ for the deuteron. In accordance with Eqs. (4), (9), (10), (12), and (13), we obtain

$$A_S = 0.8774\,fm^{-1/2},$$

$$r_d = 1.9627\,fm.$$

As might have been expected, these results are in very good agreement with the experimental values of $A_S = 0.8781\,fm^{-1/2}$ [18] and $r_d = 1.9635\,fm$ [10].

5. To summarize, we will formulate our basic results and conclusions. We have investigated the correlation relationship between the asymptotic normalization constant for the deuteron, $A_S$, and the triplet $np$ scattering length $a_t$. It has been established that 99.7\% of the asymptotic constant $A_S$ is determined by the triplet scattering length $a_t$. It has been shown that, in the effective-range approximation, the linear correlation relationship between the quantities $2\alpha/A_S^2$ and $R_d/a_t$ provides a good test of correctness of various potential models and methods that are used in studying nucleon-nucleon interaction.

It has been found that evaluating the asymptotic normalization constant for the deuteron and its root-mean-square radius with the currently recommended triplet-scattering-length
value of $a_t = 5.424 \text{ fm}$ [32] leads to results, $A_S \simeq 0.8845 \text{ fm}^{-1/2}$ and $r_d = 1.9711 \text{ fm}$, that are exaggerated in relation to the corresponding experimental values in (3) and (5a)–(5c).

By using the experimental values of $\varepsilon_d = 2.22458900 \text{ MeV}$ [9], $A_S = 0.8781 \text{ fm}^{-1/2}$ [18], and $r_d = 1.9635 \text{ fm}$ [1] for, respectively, the binding energy of the deuteron, its asymptotic normalization constant, and its root-mean-square radius, we have obtained the following results for the low-energy scattering parameters: $a_t = 5.4050 \text{ fm}$, $r_t = 1.7505 \text{ fm}$, and $P_t = -0.0231$. It turned out that, with these parameter values, the respective experimental phase shift is faithfully reproduced up to an energy value of $50 \text{ MeV}$. The absolute error in the phase shift at this energy value is about $1^\circ$. As the energy decreases, the absolute error becomes smaller, taking the value of $0.06^\circ$ at an energy of $1 \text{ MeV}$. If the root-mean-square radius of the deuteron is set to the experimental value of $r_d = 1.9560 \text{ fm}$ from [11] or the experimental value of $r_d = 1.950 \text{ fm}$ from [3], the corresponding low-energy scattering parameters lead to a much poorer description of the experimental phase shift in the shape-parameter approximation. Even at an energy of $10 \text{ MeV}$, the absolute error exceeds $1^\circ$ in this case.

The values found for the low-energy scattering parameters with the aid of the latest experimental results of Arndt et al. [38] for the phase shifts are $a_t = 5.4030 \text{ fm}$, $r_t = 1.7495 \text{ fm}$, and $P_t = -0.0259$. They are in good agreement with the parameter values of $a_t = 5.4050 \text{ fm}$, $r_t = 1.7505 \text{ fm}$, and $P_t = -0.0231$, which were obtained on the basis of the experimental values of the characteristics of the deuteron. Both sets of these parameters make it possible to describe well, in the shape-parameter approximation, the experimental phase shift up to an energy of $50 \text{ MeV}$, this indicating that the parameter $Q_t$ and parameters of higher order in the effective-range expansion are small.

On the basis of the parameter values of $a_t = 5.4030 \text{ fm}$, $r_t = 1.7495 \text{ fm}$, and $P_t = -0.0259$, which correspond to the experimental phase shifts obtained by Arndt et al. [38], we have found the asymptotic normalization constant for the deuteron, $A_S = 0.8774 \text{ fm}^{-1/2}$, and its root-mean-square radius, $r_d = 1.9627 \text{ fm}$, these results being in excellent agreement with the experimental values of $A_S = 0.8781 \text{ fm}^{-1/2}$ [18] and $r_d = 1.9635 \text{ fm}$ [10].

In summary, we arrive at the basic conclusion that the latest experimental results of Arndt
et al. [38] for the phase shifts comply very well with the experimental values of parameters that characterize the deuteron, specifically, with the binding energy in (2), the asymptotic normalization constant determined in [18] and given in (3), and the root-mean-square radius of the deuteron as obtained in [10] and presented in (5a).

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Table 1. Asymptotic normalization constant $A_S$ for the deuteron within various models of nucleon- nucleon interaction

| No. | References | $\varepsilon_d$, $MeV$ | $a_t$, $fm$ | $A_S$, $fm^{-1/2}$, precise value | $A_S$, $fm^{-1/2}$, effective-range approximation | $\Delta$, $fm^{-1/2}$ | $\delta$, % |
|-----|------------|-----------------|------------|----------------------------------|----------------------------------|----------------|--------|
| 1.  | RHC [34]   | 2.22464         | 5.397      | 0.88034                          | 0.87867                          | 0.00167        | 0.216  |
| 2.  | RSC [34]   | 2.2246          | 5.390      | 0.87758                          | 0.87711                          | 0.00047        | 0.054  |
| 3.  | Paris [19]  | 2.2249          | 5.427      | 0.8869                           | 0.88528                          | 0.00162        | 0.183  |
| 4.  | Moscow [35] | 2.2246          | 5.413      | 0.8814                           | 0.88211                          | 0.00071        | 0.080  |
| 5.  | Nijm I [21,27] | 2.224575       | 5.418      | 0.8841                           | 0.88272                          | 0.00138        | 0.156  |
| 6.  | Nijm II [21,27] | 2.224575       | 5.420      | 0.8845                           | 0.88316                          | 0.00134        | 0.152  |
| 7.  | Reid 93 [21,27] | 2.224575       | 5.422      | 0.8853                           | 0.88360                          | 0.00170        | 0.193  |
| 8.  | Argonne v18 [25] | 2.224575       | 5.419      | 0.8850                           | 0.88341                          | 0.00159        | 0.180  |
| 9.  | GK-4 [36]   | 2.226           | 5.364      | 0.87462                          | 0.87201                          | 0.00261        | 0.299  |
| 10. | GK-8 [36]   | 2.226           | 5.413      | 0.88434                          | 0.88262                          | 0.00172        | 0.194  |
| 11. | GK-7 [36]   | 2.226           | 5.477      | 0.89776                          | 0.89678                          | 0.00098        | 0.109  |
| 12. | HM-1 [22,23] | 2.224           | 5.50       | 0.901                            | 0.90119                          | 0.00019        | 0.021  |
| 13. | Bonn R [20] | 2.2246          | 5.423      | 0.8860                           | 0.88430                          | 0.00170        | 0.193  |
| 14. | Bonn Q [20] | 2.2246          | 5.424      | 0.8862                           | 0.88452                          | 0.00168        | 0.190  |
| 15. | Bonn F [20] | 2.2246          | 5.427      | 0.9046                           | 0.88517                          | 0.01942        | 2.195  |
| 16. | HM-2 [22,23] | 2.2246          | 5.45       | 0.919                            | 0.89024                          | 0.02876        | 3.230  |
| 17. | M [24]      | 2.22469         | 5.424      | 0.9043                           | 0.8845                           | 0.01975        | 2.233  |
| 18. | SCSS [17]   | 2.224575        | 5.4193     | 0.8838                           | 0.88300                          | 0.00079        | 0.089  |
| 19. | STS [27]    | 2.224575        | 5.4194     | 0.8845                           | 0.88303                          | 0.00147        | 0.166  |
| 20. | ER-C [4]    | 2.224575        | 5.424      | 0.8846                           | 0.88451                          | 0.00009        | 0.011  |
| 21. | KMMA [8]    | 2.224644        | 5.412      | 0.8883                           | 0.88191                          | 0.00639        | 0.725  |
Table 2. Parameters of the effective-range theory that were calculated on the basis of
the experimental values of $\varepsilon_d$, $A_S$, and $r_d$, as well as those found from the analysis of the
experimental $np$ scattering phase shifts

| Version | $a_t$, fm | $r_t$, fm | $P_t$ | $\rho_m$, fm | $\rho_d$, fm |
|---------|------------|------------|-------|---------------|---------------|
| B       | 5.4050     | 1.75047    | -0.02312 | 1.73716       | 1.72385       |
| S       | 5.3863     | 1.70257    | 0.02010  | 1.71321       | 1.72385       |
| K       | 5.3713     | 1.66391    | 0.06064  | 1.69388       | 1.72385       |
| ER      | 5.395      | 1.72385    | 0       | 1.72385       | 1.72385       |
| PWA     | 5.4030     | 1.7495     | -0.02592 | 1.73461       | 1.7197        |
Table 3. Triplet phase shift calculated for np scattering in the shape-parameter approximation (28) with the parameter values from Table 2 as a function of the laboratory energy $E_{lab}$

| $E_{lab}$, MeV | Experiment [38] | PWA | B | S | K | ER |
|----------------|------------------|-----|---|---|---|----|
| 0.1            | 169.32           | 169.315 | 169.311 | 169.349 | 169.380 | 169.331 |
| 0.5            | 156.63           | 156.645 | 156.637 | 156.728 | 156.802 | 156.686 |
| 1.0            | 147.83           | 147.823 | 147.812 | 147.954 | 148.068 | 147.889 |
| 2.0            | 136.56           | 136.548 | 136.536 | 136.770 | 136.958 | 136.664 |
| 5.0            | 118.23           | 118.236 | 118.228 | 118.750 | 119.168 | 118.516 |
| 10.0           | 102.55           | 102.598 | 102.612 | 103.672 | 104.521 | 103.200 |
| 20.0           | 85.84            | 86.049 | 86.127 | 88.385 | 90.211 | 87.379 |
| 30.0           | 75.61            | 76.043 | 76.195 | 79.702 | 82.592 | 78.131 |
| 40.0           | 68.11            | 68.822 | 69.046 | 73.796 | 77.806 | 71.652 |
| 45.0           | 64.96            | 65.843 | 66.103 | 71.462 | 76.050 | 69.032 |
| 50.0           | 62.11            | 63.171 | 63.466 | 69.424 | 74.602 | 66.709 |
| 100.0          | 42.34            | 45.705 | 46.265 | 57.612 | 69.487 | 52.131 |
Table 4. Difference of the experimental values of the phase shift and its theoretical values calculated in the shape-parameter approximation (28) with the parameter values from Table 2

| $E_{lab}$, MeV | $\Delta$, deg  |
|----------------|----------------|
|                | PWA | B   | S     | K    | ER    |
| 0.1            | 0.005 | 0.009 | −0.029 | −0.060 | −0.011 |
| 0.5            | −0.015 | −0.007 | −0.098 | −0.172 | −0.056 |
| 1.0            | 0.007 | 0.018 | −0.124 | −0.238 | −0.059 |
| 2.0            | 0.012 | 0.024 | −0.210 | −0.398 | −0.104 |
| 5.0            | −0.006 | 0.002 | −0.520 | −0.938 | −0.286 |
| 10.0           | −0.048 | −0.062 | −1.122 | −1.971 | −0.650 |
| 20.0           | −0.209 | −0.287 | −2.545 | −4.371 | −1.539 |
| 30.0           | −0.433 | −0.585 | −4.092 | −6.982 | −2.521 |
| 40.0           | −0.712 | −0.936 | −5.686 | −9.696 | −3.542 |
| 45.0           | −0.883 | −1.143 | −6.502 | −11.090 | −4.072 |
| 50.0           | −1.061 | −1.356 | −7.314 | −12.492 | −4.599 |
| 100.0          | −3.365 | −3.925 | −15.272 | −27.147 | −9.791 |
Correlation relationship between the dimensionless asymptotic normalization constant for the deuteron, $C_d$, and the triplet scattering length $a_t$. Points represent values obtained within various models of nucleon-nucleon interaction in Table 1 (the numerals in the figure correspond to the numbers of the lines in this table), while the straight line was calculated by the approximate formula (21).
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