Notes on Lynch-Morawska Systems

Daniel S. Hono II
Namrata Galatage
Kimberly A. Gero
Paliath Narendran
Ananya Subburathinam
Abstract

In this paper we investigate convergent term rewriting systems that conform to the criteria set out by Christopher Lynch and Barbara Morawska in their seminal paper "Basic Syntactic Mutation." The equational unification problem modulo such a rewrite system is solvable in polynomial-time. In this paper, we derive properties of such a system which we call an LM-system. We show, in particular, that the rewrite rules in an LM-system have no left- or right-overlaps.

We also show that despite the restricted nature of an LM-system, there are important undecidable problems, such as the deduction problem in cryptographic protocol analysis (also called the cap problem) that remain undecidable for LM-systems.

Keywords: Equational unification, Term rewriting, Polynomial-time complexity, NP-completeness.

1 Introduction

Unification modulo an equational theory $E$ (equational unification or $E$-unification) is an undecidable problem in general. Even in cases where it is decidable, it is often of high complexity. In their seminal paper “Basic Syntactic Mutation” Christopher Lynch and Barbara Morawska present syntactic criteria on equational axioms $E$ that guarantee a polynomial time algorithm for the corresponding $E$-unification problem. As far as we know these are the only purely syntactic criteria that ensure a polynomial-time algorithm for unifiability.

In [5] it was shown that relaxing any of the constraints imposed upon a term-rewriting system $R$ by the conditions given by Lynch and Morawska results in an unification problem that is NP-Hard. Thus, these conditions are tight in the sense that relaxing any of them leads to an intractable unification problem (assuming $P \neq NP$).

In this work, we continue to investigate the consequences of the syntactic criteria given in “Basic Syntactic Mutation”. As in [5] we consider the case where $E$ forms a convergent and forward closed term-rewriting system, which we call LM-Systems. This definition differs from that of [8] in which $E$ is saturated by paramodulation and not necessarily convergent. The criteria of determinism introduced in [8] remains essentially unchanged.

We give a structural characterization of these systems by showing that if $R$ is an LM-System, then there are no overlaps between the left-hand sides of any rules in $R$ and there are no forward-overlaps between any right-hand side and a left-hand side. This characterization shows that LM-Systems form a very restricted subclass of term-rewriting systems. Any term-rewriting system that contains overlaps of these kinds cannot be an LM-System. Using these results, we show that saturation by paramodulation is equivalent to forward-closure when considering convergent term-rewriting systems that satisfy all of the remaining conditions for LM-Systems.

Despite their restrictive character, we show in Section 5 that the cap problem, which is undecidable in general, remains undecidable when restricted to LM-Systems. The cap problem (also called the deduction problem) originates from the field of cryptographic protocol analysis. This result shows that LM-Systems are yet strong enough to encode important undecidable problems. The reduction considered is essentially the same as that given in [5] to show that determining if a term-rewriting system is subterm-collapsing given all the other conditions of LM-Systems is undecidable.

2 Notation and Preliminaries

We assume the reader is familiar with the usual notions and concepts in term rewriting systems [1] and equational unification [2]. We consider rewrite systems over ranked signatures, usually denoted $\Sigma$, and a possibly infinite set of variables, usually denoted $\mathcal{X}$. The set of all terms over $\Sigma$ and $\mathcal{X}$ is denoted as $T(\Sigma, \mathcal{X})$. An equation is an ordered pair of terms $(s, t)$, usually written as $s \approx t$. Here $s$ is the left-hand side and $t$ is the right-hand side of
the equation \([1]\). A rewrite rule is an equation \(s \approx t\) where \(\text{Var}(t) \subseteq \text{Var}(s)\), usually written as \(s \rightarrow t\). A term rewriting system is a set of rewrite rules.

A set of equations \(E\) is subterm-collapsing\([1]\) if and only if there are terms \(t\) and \(u\) such that \(t\) is a proper subterm of \(u\) and \(E \vdash t \approx u\) (or \(t =_E u\)) \([4]\). A set of equations \(E\) is variable-preserving\([2]\) if and only if for every equation \(t \approx u\) in \(E\), \(\text{Var}(t) = \text{Var}(u)\) \([9]\). A term rewriting system is convergent if and only if it is confluent and terminating \([10]\). If \(R\) is a term rewriting system denote by \(\text{IRR}(R)\) the set of all \(R\)-irreducible terms, i.e. the set of all \(R\) normal forms.

A term \(t\) is \(\bar{e}\)-irreducible modulo a rewriting system \(R\) if and only if every proper subterm of \(t\) is irreducible. A term \(t\) is an innermost redex of a rewrite system \(R\) if all proper subterms of \(t\) are irreducible and \(t\) is reducible, i.e., \(t\) is \(\bar{e}\)-irreducible and \(t\) is an instance of the left-hand-side of a rule in \(R\).

The following definition is used in later sections to simplify the exposition. It is related to the above notion of \(\bar{e}\)-irreducible. Specifically, we define a normal form of a term \(t\) that depends also on the reduction path used. More formally, we have:

**Definition 1.** Let \(R\) be a rewrite-system. A term \(t\) is said to be an \(\bar{e}\)-normal-form of a term \(s\) if and only if \(s \rightarrow^*_R t\), no proper subterm of \(t\) is reducible and none of the rewrites in the sequence of reduction steps is at the root.

Given a set of equations \(E\), the set of ground instances of \(E\) is denoted by \(\text{Gr}(E)\). We assume a reduction order \(\prec\) on \(E\) which is total on ground terms. We extend this order to equations as \((s \approx t) \prec (u \approx v)\) iff \(\{s,t\} \prec_{\text{mul}} \{u,v\}\), where \(\prec_{\text{mul}}\) is the multiset order induced by \(\prec\). An equation \(e\) is redundant in \(E\) if and only if every ground instance \(\sigma(e)\) of \(e\) is a consequence of equations in \(\text{Gr}(E)\) which are smaller than \(\sigma(e)\) modulo \(\prec\) \([8]\).

### 2.1 Paramodulation

Lynch and Morawska define paramodulation, which is an extension to the critical pair rule. Since our focus is only on convergent term rewriting systems, this definition can be modified for rewrite rules as the following inference rule:

\[
\frac{u[s'][p] \approx v \quad s \rightarrow t}{\sigma(u[t]_p) \approx \sigma(v)}
\]

where \(\sigma = \text{mgu}(s \equiv v')\) and \(p \in \text{FPos}(u)\). A set of equations \(E\) is saturated by paramodulation if all inferences among equations in \(E\) using the above rule are redundant.

In this work, we consider rewrite systems that are forward-closed as opposed to saturated by paramodulation. A later result of section 4 will show that saturation by paramodulation and forward-closure are equivalent properties when all of the other LM-conditions obtain.

#### 3 Forward Closures

Following Hermann \([7]\) the forward-closure of a convergent term rewriting system \(R\) is defined in terms of the following operation on rules in \(R\): let \(\rho_1 : l_1 \rightarrow r_1\) and \(\rho_2 : l_2 \rightarrow r_2\) be two rules in \(R\) and let \(p \in \text{FPos}(r_1)\). Then

\[
\rho_1 \rightsquigarrow_p \rho_2 = \sigma(l_1 \rightarrow r_1[r_2]_p)
\]

\(^1\)Non-subterm-collapsing theories are called simple theories in \([4]\).

\(^2\)Variable-preserving theories are also called non-erasing or regular theories \([1]\).
where $\sigma = \mgu(r_1 \mid p = ? l_2)$. Given rewrite systems $R_1$ and $R_2$ such that $R_2 \subseteq R_1$, we define $R_1 \rightsquigarrow R_2$ as the rules in:

$$\{ (l_1 \to r_1) \rightsquigarrow_p (l_2 \to r_2) \mid (l_1 \to r_1) \in R_1, (l_2 \to r_2) \in R_2 \text{ and } p \in \mathcal{FPos}(r_1) \}$$

which are not redundant in $R_1$.

We now define

$$FC_0(R) = R$$

and

$$FC_{k+1}(R) = FC_k(R) \cup (FC_k(R) \rightsquigarrow R)$$

for all $k \geq 0$. Finally,

$$FC(R) = \bigcup_{i=1}^{\infty} FC_i(R)$$

Note that $FC_j(R) \subseteq FC_{j+1}(R)$ for all $j \geq 0$. A set of rewrite rules $R$ is forward-closed if and only if $FC(R) = R$.

We also define the sets of “new rules”

$$NR_0(R) = R$$

and

$$NR_{k+1}(R) = FC_{k+1}(R) \setminus FC_k(R)$$

Then

$$FC(R) = \bigcup_{i=1}^{\infty} NR_i(R)$$

The following theorem, shown in [3], gives necessary and sufficient conditions for a rewrite-system to be forward-closed. This property will be used repeatedly in the sequel below.

**Theorem 3.1.** [3] A convergent rewrite system $R$ is forward-closed if and only if every innermost redex can be reduced to its $R$-normal form in one step.

We next show that there are ways to reduce a rewrite system $R$ while still maintaining the properties that we are interested in. More precisely, given a convergent, forward-closed rewrite system $R$ we can reduce the right-hand sides of rules in $R$ while maintaining convergence, forward-closure and the equational theory generated by $R$.

Let $R$ be a convergent rewrite system. Following [6] we define

$$R \downarrow = \{ l \to r \downarrow \mid (l \to r) \in R \}$$

**Lemma 3.2.** Let $R$ be a convergent, forward-closed rewrite system. Then $R \downarrow$ is convergent, equivalent to $R$ (i.e., they generate the same congruence), and forward-closed.
Proof. The cases of convergence and equivalence of \( R \downarrow \) are handled in [6]; it remains to show that \( R \downarrow \) is forward-closed. By Theorem 3.1, it suffices to show that every innermost redex modulo \( R \downarrow \) is reducible to its normal form in a single step.

Let \( t \) be an innermost redex of \( R \downarrow \). Since the passage from \( R \) to \( R \downarrow \) preserved the left-hand sides of the rules, \( t \) must also be an innermost redex of \( R \). Thus, \( \exists l \rightarrow r \in R \) such that \( \sigma(r) \in IRR(R) \). Then \( l \rightarrow r \downarrow \in R \downarrow \). Since \( r \rightarrow^* r \downarrow \) and \( \rightarrow \) is closed under substitutions, we have that \( \sigma(r) \rightarrow^* \sigma(r \downarrow) \), but \( \sigma(r) \) is irreducible, thus \( \sigma(r) = \sigma(r \downarrow) \). Therefore, \( t \) reduces to its normal form modulo \( R \downarrow \) in a single step. \( \square \)

A convergent rewrite system \( R \) is right-reduced if and only if \( R = R \downarrow \).

From the above lemma, it is clear that right-reduction does not affect forward-closure. However, full interreduction, where one also deletes rules whose left-hand sides are reducible by other rules, will not preserve forward-closure. The following example illustrates this:

\[
\begin{align*}
f(x, i(x)) & \rightarrow g(x) \\
g(b) & \rightarrow c \\
f(b, i(b)) & \rightarrow c
\end{align*}
\]

The last rule can be deleted since its left-hand side is reducible by the first rule. This will preserve convergence, but forward-closure will be lost since \( f(b, i(b)) \), an innermost redex, cannot be reduced in one step to \( c \) in the absence of the third rule.

However, the following lemma enables us to do a restricted deletion of superfluous rules:

**Lemma 3.3.** Let \( R \) be a convergent, forward-closed, term rewriting system. Let \( l_i \rightarrow r_i \in R \) for \( i \in \{1,2\} \) such that \( \exists p \in FPos(l_1) : p \neq \varepsilon \) and \( l_1 \upharpoonright p = \sigma(l_2) \) for some substitution \( \sigma \). That is, \( l_1 \) contains a proper subterm that is an instance of the left-hand side of another rule in \( R \). Then, \( R' = R \setminus \{ l_1 \rightarrow r_1 \} \) is convergent, forward-closed and equivalent to \( R \).

**Proof.** For the sake of deriving a contradiction, assume that \( R' \) is not forward-closed. Then there must exist some term \( t \) such that \( t \) is an innermost redex modulo \( R' \) and \( t \) is not reducible to its normal form in a single step.

Since \( R \) is forward-closed \( t \) must be reducible to its normal form in a single step modulo \( R \), and since \( R \) differs from \( R' \) by the rule \( l_1 \rightarrow r_1 \), then it must be that \( t = \theta(l_1) \) for some substitution \( \theta \).

However, since \( l_1 \upharpoonright p = \sigma(l_2) \) we have that \( \theta(l_1) \upharpoonright p = \theta(\sigma(l_2)) \), but since \( \theta(l_1) = t \) we also have that \( \theta(l_1) \upharpoonright p = t \upharpoonright p \). Thus, \( t \upharpoonright p = \theta(\sigma(l_2)) \), but this contradicts the assumption that \( t \) is an innermost redex as \( p \neq \varepsilon \). Therefore, \( R' \) must be forward-closed.

The termination of \( R' \) follows from the fact that \( R \) is terminating and \( R' \subset R \). Towards showing confluence and equivalence we show that \( IRR(R') = IRR(R) \). First, it is clear that \( IRR(R) \subseteq IRR(R') \) since \( R' \subseteq R \). For the reverse containment, suppose that \( t \in IRR(R') \) but \( t \notin IRR(R) \). That is, \( t \) must be reducible modulo \( R \). However, since \( R \) and \( R' \) differ by only a single rule, it must be the case that \( t \) is reducible by the rule \( l_1 \rightarrow r_1 \), but \( l_1 \) is reducible by \( l_2 \rightarrow r_2 \), thus \( t \) would also be reducible by \( l_2 \rightarrow r_2 \in R' \). But this contradicts the assumption that \( t \in IRR(R') \).

Now, suppose that \( R' \) is not confluent. Then there must a term \( t \) with two distinct \( R' \)-normal forms \( t' \) and \( t'' \), but \( t' \downharpoonright R t'' \) as \( R \) is confluent. However, this contradicts the above fact that \( IRR(R) = IRR(R') \) as at least one of \( t', t'' \) must be reducible. Thus, \( R' \) is confluent, and thus we have established that \( R' \) is convergent.

Finally, we show that \( R' \) is equivalent to \( R \). Since \( R' \subset R \) we have that \( \leftrightarrow_{R'} \subseteq \leftrightarrow_R \). For the reverse containment, suppose there are two distinct \( R' \)-irreducible terms \( s \) and \( t \) such that \( s \leftrightarrow_R t \), but then since \( IRR(R) = IRR(R') \), \( s = t \), which contradicts the assumption that they are distinct terms. Thus \( R' \) is equivalent to \( R \). \( \square \)
We call systems that have no rules such that the conditions of Lemma 3.3 obtain almost-left-reduced. Note, however, that they are not fully left-reduced as there is still the possibility of overlaps at the root. If a convergent, forward-closed and right-reduced \( R \) is not almost-left-reduced, then the above lemma tells us that we may delete such rules and obtain an equivalent system.

4 Lynch-Morawska Conditions

In this section we define the Lynch-Morawska conditions. We also derive some preliminary results on convergent term rewriting systems that satisfy the Lynch-Morawska conditions.

A new concept introduced by Lynch and Morawska is that of a Right-Hand-Side Critical Pair, defined as follows:

\[
\begin{align*}
s & \approx t \\
s\sigma & \approx u\sigma
\end{align*}
\]

where \( s\sigma \not\approx t\sigma, u\sigma \not\approx v\sigma, \sigma = \text{mgu}(v, t) \) and \( s\sigma \neq u\sigma \)

Since our focus is only on convergent term rewriting systems, this definition can be modified as follows:

\[
\begin{align*}
s & \rightarrow t \\
s\sigma & \approx u\sigma
\end{align*}
\]

where \( \sigma = \text{mgu}(v, t) \) and \( s\sigma \neq u\sigma \)

For instance the right-hand-side critical pair \( f(x, s(y)) \rightarrow s(f(x, y)) \) and \( f(s(x), y) \rightarrow s(f(y, x)) \) is \( f(x, s(x)) \approx f(s(x), x) \). Also note (as pointed out in [8]) that the rule \( f(x, x) \rightarrow 0 \) has a right-hand-side critical pair: \( f(x, x) \approx f(x', x') \).

For an equational theory \( E \), \( \text{RHS}(E) = \{ e \mid e \text{ is the conclusion of a Right-Hand-Side Critical Pair inference of two members of } E \} \cup E \) [8].

A set of equations \( E \) is quasi-deterministic if and only if

1. No equation in \( E \) has a variable as its left-hand side or right-hand side,
2. No equation in \( E \) is root-stable—i.e., no equation has the same root symbol on its left- and right-hand side, and
3. \( E \) has no root pair repetitions—i.e., no two equations in \( E \) have the same pair of root symbols on their sides.

The following lemma was proved in [5].

**Lemma 4.1.** Suppose \( R \) is a variable-preserving convergent rewrite system and \( R \) is quasi-deterministic. Then \( \text{RHS}(R) \) is not quasi-deterministic if and only if \( \text{RHS}(R) \) has a root pair repetition.

A theory \( E \) is deterministic if and only if it is quasi-deterministic and non-subterm-collapsing.

A Lynch-Morawska term rewriting system or \( LM\)-system is a convergent, almost-left-reduced and right-reduced term rewriting system \( R \) which satisfies the following conditions:
(i) \( R \) is non-subterm-collapsing,
(ii) \( R \) is forward-closed, and
(iii) \( \text{RHS}(R) \) is quasi-deterministic.

The goal of the remainder of this section is to show that, given an \( \text{LM-system} \) \( R \), there can be no overlaps between the left-hand sides of any rules in \( R \) and that there can be no forward-overlaps. These notions are defined precisely below. Further, we use those results to derive the equivalence of forward-closure and saturation by paramodulation when \( R \) is an \( \text{LM-system} \).

These results show that \( \text{LM-systems} \) are a highly restrictive subclass of term-rewriting systems. However, in a later section, we show that there are important decision problems that remain undecidable when restricted to \( \text{LM-systems} \).

The first of these results, Lemma 4.2 and its proof, are used multiple times to prove other results. It concerns that there are only two possible cases, and further, only one of these cases can hold at a time.

**Lemma 4.2.** Let \( R \) be an \( \text{LM-system} \), \( s = f(s_1, \ldots, s_m) \) an innermost redex and \( t = g(t_1, \ldots, t_n) \) an \( \bar e \)-irreducible term such that \( f \neq g \). Then \( s \) and \( t \) are joinable modulo \( R \) if and only if exactly one of the following conditions holds:

(a) there is a unique rule \( l \rightarrow r \) with root pair \( (f, g) \) and \( s \xrightarrow{l\rightarrow r} t \), or

(b) there are unique rules \( l_1 \rightarrow r_1 \) and \( l_2 \rightarrow r_2 \) with root pairs \( (f, h) \) and \( (g, h) \) such that \( s \xrightarrow{l_1\rightarrow r_1} \hat t \) and \( t \xrightarrow{l_2\rightarrow r_2} \hat t \)

for some term \( \hat t \).

**Proof.** \( \Rightarrow \) There are two cases to consider.

(i) \( t = g(t_1, \ldots, t_n) \) is in normal form.

(ii) \( t \) is an innermost redex.

**Case (i).** Since \( s \downarrow_R t \) by assumption, \( s \) must reduce to \( t \) in a single step (as \( s \) is an innermost redex and \( t \) cannot be reduced modulo \( R \) any further). Therefore \( \exists ! \rho = l \rightarrow r \in R \) that will reduce \( s \) to \( t \), and this rule is unique as there can be no root-pair repetitions.

**Case (ii).** Since \( t \) is an innermost redex and \( s \) is \( \bar e \)-irreducible (and therefore also an innermost redex in this case) they both must be reducible to their normal forms in one step and since \( s \downarrow_R t \) these normal forms must be equal. Let \( \hat s \) be the normal form of \( s \) and \( t \). Since there can be no root-stable equations \( \hat s(e) = h \) and \( h \neq f, g \). Thus, \( \exists ! \rho_i = l_i \rightarrow r_i \in R \) for \( i \in \{1, 2\} \) with root pairs \( (f, h) \) and \( (g, h) \) respectively such that \( s \xrightarrow{\rho_1} \hat t \) and \( t \xrightarrow{\rho_2} \hat t \).

It remains to show that both case (a) and case (b) cannot obtain simultaneously. For the sake of deriving a contradiction, assume that both case (a) and case (b) hold. The only way this could occur is with case (ii) above. Without loss of generality assume that \( \text{Var}(\rho_1) \cap \text{Var}(\rho_2) = \emptyset \). Since \( s \) and \( t \) are \( \bar e \)-irreducible we have that \( \sigma_1(r_1) = \hat t = \sigma_2(r_2) \) where \( \sigma_1 = \text{mgw}(l_1 \leq s) \) and \( \sigma_2 = \text{mgw}(l_2 \leq t) \). And so by defining \( \sigma := \sigma_1 \cup \sigma_2 \) we have that \( \sigma(r_1) = \sigma(r_2) \), i.e., \( \sigma \) is a unifier of \( r_1 \) and \( r_2 \).

We can thus perform a RHS inference step using \( \rho_1 \) and \( \rho_2 \), i.e.,

\[
\frac{l_1 \rightarrow r_1}{\theta(l_1) \approx \theta(l_2)}
\]
to get that \( \theta(l_1) \approx \theta(l_2) \in RHS(R) \) where \( \theta = mgu(r_1 = r_2) \). This equation has root-pair \( \{f, g\} \). But since \( RHS(R) \) can have no root-pair repetition, it must be that \( \theta(l_1) \approx \theta(l_2) = l \approx r \), but then \( r \) is reducible by \( l_2 \rightarrow r_2 \) which contradicts the assumption that \( R \) is right-reduced.

\((\Leftarrow)\) If exactly one of the two conditions hold, then \( s \) and \( t \) are joinable by definition.

We present an interesting consequence of the above lemma. Namely, if a term \( t \) such that \( t(\varepsilon) = f \) has as normal form a term \( s \) such that \( s(\varepsilon) = g \) for \( g \) differing from \( f \), then we can establish some information on the rules in \( R \).

**Corollary 4.3.** If \( f(s_1, \ldots, s_m) \rightarrow^1_R g(t_1, \ldots, t_n) \) where \( f \neq g \), then \( (f, g) \) is a root pair in \( R \).

**Proof.** Let \( s = f(s_1, \ldots, s_n) \) and \( t = g(t_1, \ldots, t_m) \). Suppose that \( s \rightarrow^1_R t \). Then by definition \( s \downarrow_R t \). Since \( t \) is a normal form it is also an \( \varepsilon \)-irreducible term modulo \( R \).

Let \( s' = f(s'_1, \ldots, s'_n) \) be the term obtained from \( s \) by reducing all top-level subterms of \( s \) to their normal forms modulo \( R \). Thus we have the following situation: \( s \rightarrow^1_R s' \rightarrow^1_R t \) as \( s' \) must be an \( \varepsilon \)-irreducible term and hence also an innermost redex in this case. Therefore we can apply **case (a) of Lemma 4.2** to conclude that there must be a unique rule \( \rho = l \rightarrow r \in R \) that reduces \( s' \) to \( t \) with root pair \( (f, g) \).

Given Lemma 4.2 we can immediately derive two results that will be useful in proving the main result of this section.

**Corollary 4.4.** Suppose \( l \rightarrow r \in R \) is a rule with root-pair \( (f, g) \), \( s = f(s_1, \ldots, s_m) \) and \( t = g(t_1, \ldots, t_n) \), and \( s \) and \( t \) are \( \varepsilon \)-irreducible. Then, \( s \downarrow_R t \) if and only if \( s \overset{l \rightarrow r}{\longrightarrow} t \).

**Proof.** This result follows from Lemma 4.2 and its proof.

**Corollary 4.5.** Let \( R \) be an LM-System. Suppose \( l \rightarrow r \in R \) is a rule with root-pair \( (f, g) \) and \( s = f(s_1, \ldots, s_m) \) and \( t = g(t_1, \ldots, t_n) \) be terms that are joinable. Let \( \hat{s}_1, \ldots, \hat{s}_m, \hat{t}_1, \ldots, \hat{t}_n \) be respectively the normal forms of \( s_1, \ldots, s_m, t_1, \ldots, t_n \). Then

\[
f(\hat{s}_1, \ldots, \hat{s}_m) \overset{l \rightarrow r}{\longrightarrow} g(\hat{t}_1, \ldots, \hat{t}_n).
\]

(Thus the normal form of \( s \) and \( t \) is an instance of the right-hand side \( r \).)

**Proof.** Since \( s \) and \( t \) are joinable, it must be the case that \( \hat{s} = f(\hat{s}_1, \ldots, \hat{s}_m) \) and \( \hat{t} = g(\hat{t}_1, \ldots, \hat{t}_n) \) are joinable, but since each \( \hat{s}_i \) and \( \hat{t}_i \) are in normal form, \( \hat{s} \) and \( \hat{t} \) must be \( \varepsilon \)-irreducible, therefore we can apply Corollary 4.4.

The above two corollaries, along with Lemma 4.2, allows us to state the first of the results concerning the non-overlapping property of LM-systems. The following establishes that there can be no overlaps between left-hand sides of two rules occurring at the root-position and that there can be no overlaps between a right-hand side of a rule and a left-hand side of another rule at the root position. This is achieved by showing that these terms cannot be unified.

**Corollary 4.6.** Let \( R \) be an LM-System and let \( l_1 \rightarrow r_1 \) and \( l_2 \rightarrow r_2 \) be distinct rules in \( R \). Then

(a) \( l_1 \) and \( l_2 \) are not unifiable, and

(b) \( r_1 \) and \( r_2 \) are not unifiable.
Proof. Suppose that the rule \( l_1 \rightarrow r_1 \) has root pair \((f, g)\) and the rule \( l_2 \rightarrow r_2 \) has root pair \((h, i)\). We show that each case above leads to a contradiction.

Thus, for case (a), towards deriving such a contradiction suppose that \( \theta = \text{mgu}(l_1 =^2 l_2) \). Then, we have that \( \theta(l_1) \rightarrow \theta(r_1) \), and so \( \theta(l_1) \) and \( \theta(r_1) \) are obviously joinable. Note that, since \( l_1 \) is unifiable with \( l_2 \), \( l_1(\varepsilon) = l_2(\varepsilon) \). Thus, \( f = h \) above, and \( g \neq i \) as the contrary would induce a root-pair repetition in \( R \).

By applying Corollary 4.5 to \( \theta(l_1) \) and \( \theta(r_1) \) we can conclude that their normal form must be some instance of \( r_1 \). Likewise, we can apply the same corollary on \( \theta(l_1) \) and \( \theta(r_2) \), i.e. their normal form must be an instance of \( r_2 \).

But since \( R \) is convergent, \( \theta(r_1) \) and \( \theta(r_2) \) must be joinable and thus must have the same normal form, but this is impossible as \( r_1 \) and \( r_2 \) have different root symbols.

For case (b), suppose that \( \beta = \text{mgu}(r_1 =^2 l_2) \). Thus \( \beta(l_1) \rightarrow \beta(r_1) \rightarrow \beta(r_2) \). Hence \( \beta(l_1) \) and \( \beta(r_1) \) are joinable, and \( \beta(l_1) \) and \( \beta(r_2) \) are joinable. The rest of the argument is the same as for the above case. \( \square \)

Next, we work towards showing that the other possible overlaps cannot occur either. The next lemma, and its extension, are used towards this goal. The technical result is used in the proofs of various other lemmas and corollaries.

Lemma 4.7. Let \( R \) be an LM-System, and suppose \( f(s_1, \ldots, s_m) \xrightarrow{\: l \rightarrow r } g(t_1, \ldots, t_n) \). Then the following diagram commutes.

\[
\begin{array}{ccc}
  f(s_1, \ldots, s_m) & \xrightarrow{\: l \rightarrow r } & g(t_1, \ldots, t_n) \\
  \downarrow & & \downarrow \\
  f(s'_1, \ldots, s'_m) & \xrightarrow{\: l \rightarrow r } & g(t'_1, \ldots, t'_n)
\end{array}
\]

where \( s'_1, \ldots, s'_m, t'_1, \ldots, t'_n \) are the normal forms of \( s_1, \ldots, t_n \) respectively.

Proof. We have that \( f(s_1, \ldots, s_m) \xrightarrow{\: \ast \ } f(s'_1, \ldots, s'_m) \) and \( f(s_1, \ldots, s_m) \xrightarrow{\: \ast \ } g(t'_1, \ldots, t'_n) \), and since \( R \) is confluent \( s' = f(s'_1, \ldots, s'_m) \xrightarrow{\: R \ } g(t'_1, \ldots, t'_n) = t' \). Since both \( s' \) and \( t' \) are \( \varepsilon \)-irreducible, they must be joinable in a single step. Let \( \widehat{t} \) be their normal form. There are three cases corresponding to Lemma 4.2. Cases 1 and 2 below correspond to case (a) of Lemma 4.2 and case 3 corresponds to case (b) of the Lemma 4.2. We have:

1. \( s' \rightarrow t' \) by a unique rule in \( R \) with root-pair \((f, g)\),
2. \( t' \rightarrow s' \), by a unique rule in \( R \) with root-pair \((g, f)\),
3. \( s' \rightarrow \widehat{t} \) and \( t' \rightarrow \widehat{t} \), by two unique rules with root-pairs \((f, h)\) and \((g, h)\) for some \( h \).

However, case 2 leads to a contradiction as it would imply that there exists a rule in \( R \) with root pair \((g, f)\) which would be a root pair repetition in \( E \).

Suppose case 3 were true. Let \( \rho_i = l_i \rightarrow r_i \in R \) for \( i \in \{1, 2\} \) be the unique rules with root pairs \((f, h)\) and \((g, h)\) respectively that reduce \( s' \) and \( t' \) to \( \widehat{t} \). Without loss of generality assume that \( \text{Var}(\rho_1) \cap \text{Var}(\rho_2) = \emptyset \). Then, \( r_1 \) and \( r_2 \) are unifiable and therefore we can perform a RHS inference step to get \( \theta(l_1) \Rightarrow \theta(l_2) \). However, there is a root-rewrite step along the path from \( s \) to \( t \). Let \( l \rightarrow r \in R \) be the rule that induces the root-rewrite step.
Thus $\theta(l_1) \approx \theta(l_2) = l \approx r$ as $l \rightarrow r$ must be unique. This implies, however, that $r$ is reducible by $l_2 \rightarrow r_2$ would contradict the assumption that $R$ is right-reduced.

We are then only left with case 1. This rule is unique by Lemma 4.2 and since $l \rightarrow r$ has root pair $(f, g)$ these rules must be the same.

We now extend the previous result by induction.

**Lemma 4.8.** Let $R$ be an LM-System and suppose $f(s_1, \ldots, s_m) \rightarrow_R^+ g(t_1, \ldots, t_n)$ are terms such that $f \neq g$. Then the following diagram commutes:

$$
\begin{array}{c}
\begin{array}{ccc}
f(s_1, \ldots, s_m) & \rightarrow^+ & g(t_1, \ldots, t_n) \\
* & \downarrow & * \\
f(s'_1, \ldots, s'_m) & \rightarrow^* & g(t'_1, \ldots, t'_n)
\end{array}
\end{array}
$$

where $s'_i = s_i \downarrow_R$ and $t'_j = t_j \downarrow_R$ for $1 \leq i \leq n, 1 \leq j \leq m$.

**Proof.** The proof proceeds by induction on the number of rewrites occurring at the root along the chain $s = f(s_1, \ldots, s_m)$ to $t = g(t_1, \ldots, t_n)$. Let $q$ be the number of root rewrite steps occurring as stated above.

**Base Step:** The base step, where $q = 1$, corresponds to Lemma 4.7.

**Inductive Step:** Assume that the result holds for $q = k$. We show that the result is also true for $q = k + 1$. Since there can be no root-pair repetitions in $R$, we must have that there is a sequence of root pairs starting with $f$ and ending with $g$ corresponding to the rules used in the reductions. In the diagram below, the first square commutes by the base case, and the rest of the chains can be filled in to create commuting squares by the induction hypothesis up to the $k + 1$ root rewrite. That is:

$$
\begin{array}{c}
\begin{array}{c}
f(s_1, \ldots, s_m) \rightarrow^+ h_0(u_1, \ldots, u_s) \rightarrow^* \cdots \rightarrow^* h_k(u_1, \ldots, u_j) \rightarrow^+ g(t_1, \ldots, t_n) \\
* \downarrow \quad * \downarrow \\
h_k(u_1', \ldots, u_j') \rightarrow^* g(t_1', \ldots, t_n)
\end{array}
\end{array}
$$

Therefore, we only need to fill in the final square. However, a similar argument as for the proof of Lemma 4.7 applies to the terms $h_k(u_1, \ldots, u_j) \rightarrow^+ g(t_1', \ldots, t_n')$ and $h_k(u_1', \ldots, u_j') \rightarrow^* h_k(u_1', \ldots, u_j')$.

**Corollary 4.9.** Let $f(s_1, \ldots, s_m) \rightarrow g(t_1, \ldots, t_n)$ be a rule in $R$ where $f \neq g$. Then no term with $g$ as its root symbol can be reduced modulo $R$ to a term with $f$ at its root.
Corollary 4.11. Basis.

Proof. The proof is by contradiction. Suppose that there exists a reduction chain \( g(s_1, \ldots, s_m) \rightarrow^+ f(t_1, \ldots, t_n) \) such that \( g \neq f \). Applying Lemma 4.8 we obtain a reduction chain \( s' = g(s'_1, \ldots, s'_m) \rightarrow^+ f(t'_1, \ldots, t'_n) = t' \) where \( s'_i = s_i \downarrow_R \) and \( t'_j = t_j \downarrow_R \) for \( 1 \leq i \leq m \) and \( 1 \leq j \leq n \). Thus, \( s' \) and \( t' \) are joinable, and \( s' \) must be an innermost redex as it is not in normal form, therefore we can apply Lemma 4.2 and get two cases.

Case (a) leads to a contradiction as \((f, g)\) is already a root pair in \( R \) by assumption, thus, \( \{f, g\} \) is a root pair in \( E \) and therefore there cannot be a root pair \((g, f)\) as this would contradict the assumption that \( R \) is an \( LM \)-system.

Therefore, since exactly one of the two cases must obtain, case (b) must be true. Thus there exist unique rules \( \rho_i = l_i \rightarrow r_i \) for \( i \in \{1, 2\} \) such that \( s' \rightarrow_{\rho_i} t \) and \( t' \rightarrow_{\rho_j} t \) with root pairs \((g, h)\) and \((f, h)\) respectively. However, again we have that \( r_i \) for \( i \in \{1, 2\} \) have a common instance and hence are unifiable, and so we can perform an RHS inference to get \( \theta(l_1) \approx \theta(l_2) \in \text{RHS}(R) \) where \( \theta = \text{mgd}(r_1) = r_2 \). Since we assumed that \( l \rightarrow r = f(s_1, \ldots, s_m) \rightarrow g(t_1, \ldots, t_m) \) is a rule in \( R \) this inference step must yield that \( \theta(l_1) \approx \theta(l_2) = l \approx r \) as it has root pair \((f, g)\). This leads to a contradiction however, as it implies that \( r \) is reducible by \( l_2 \rightarrow r_2 \) which contradicts the assumption that \( R \) is right-reduced. \( \square \)

We next establish two corollaries that give us information about where reductions can take place.

Corollary 4.10. Let \( R \) be an \( LM \)-System, then \( s = f(s_1, \ldots, s_n) \rightarrow^*_R f(t_1, \ldots, t_n) = t \) if and only if

\[
\forall i \in \{1, \ldots, n\}: [s_i \rightarrow^*_R t_i].
\]

Proof. The “if” part is obvious. For the “only if” part, suppose \( s \rightarrow^*_R t \) as above and \( s_j \not\rightarrow^*_R t_j \) for some \( 1 \leq j \leq n \). Then some rewrite step in the sequence from \( f(s_1, \ldots, s_n) \) to \( f(t_1, \ldots, t_n) \) must have occurred at the root. Thus there must be a rule with (directed) root pair \((f, h)\) for some \( h \neq f \). However, \( t \) has root symbol \( f \). Thus, a term with \( h \) as the root symbol must be reducible to a term with \( f \) at the root symbol, but this contradicts Corollary 4.9. \( \square \)

Definition 2. Let \( s_1, s_2 \) be terms. A position \( p \in \text{Pos}(s_1) \cup \text{Pos}(s_2) \) is said to be an outermost distinguishing position between \( s_1 \) and \( s_2 \) if and only if \( s_1(p) \neq s_2(p) \) and \( s_1(p') = s_2(p') \) for all proper prefixes \( p' \) of \( p \). The set of all outermost distinguishing positions between two terms \( s \) and \( t \) is denoted by \( \text{ODP}(s, t) \).

Note that \( \text{ODP}(s, s) = \emptyset \).

Corollary 4.11. Let \( R \) be an \( LM \)-System. Then \( s \rightarrow^*_R t \) if and only if \( \forall p \in \text{ODP}(s, t): [s|_p \rightarrow^*_R t|_p] \).

Proof. The “if” direction is straightforward. We prove the “only if” direction by induction on triples (w.r.t. the induced lexicographic order) \( \langle |s|, |t|, |p| \rangle \) where \( s, t \) are terms and \( p \) is a position such that \( s \rightarrow^*_R t \) and \( p \in \text{ODP}(s, t) \).

Basis. The “least” (i.e., lowest in the ordering) such triple is \( \langle 1, 1, 0 \rangle \), corresponding to terms such as \( s = a \) and \( t = b \), i.e., constants. Suppose that \( s \rightarrow^*_R t \). Then, \( \text{ODP}(s, t) = \{\varepsilon\} \) and therefore \( s|_p = s \rightarrow^*_R t = t|_p \), which is exactly the statement of the corollary. Thus, we can conclude that the base case holds.

Inductive Step. Assume that the result is true for all triples \( C < \langle |s'|, |t'|, |p'| \rangle \). We show that the result holds for the triple \( \langle |s'|, |t'|, |p'| \rangle \) itself. Suppose \( s' \rightarrow^*_R t' \) and \( p' \in \text{ODP}(s', t') \). Again, since the result clearly holds for \( p' = \varepsilon \) we assume that \( p' \neq \varepsilon \). Then \( s' = f(s_1, \ldots, s_n) \) and \( t' = f(t_1, \ldots, t_n) \) for some \( f \) and terms \( s_1, \ldots, s_n, t_1, \ldots, t_n \). By Corollary 4.10 it must be the case that \( \forall i \in \{1, \ldots, n\}: [s_i \rightarrow^*_R t_i] \). Since \( |s_i| < |s'| \) and \( |t_i| < |t'| \) we invoke the
indicate that \((\forall i \in \{1, \ldots, n\})(\forall q \in ODP(s_i, t_i)) [s_i|q \rightarrow^* t_i|q]\). But, we then have that 
\(p' = i \cdot q\) for some \(i \in \{1, \ldots, n\}\) and some \(q \in ODP(s_i, t_i)\), and so we have that 
\(s'|i_q = s|p' \rightarrow^* t'|i_q = t'|p'\).

We can therefore conclude that the result must hold for all triples. \(\square\)

**Definition 3.** (non-overlay superpositions) Let \(R\) be a rewrite-system, then

\[
NOSUP(R) := \{ \sigma(l_1[l_2]_p) \mid p \in FPos(l_1) \setminus \{e\} \text{ and } \sigma = \text{mgu}(l_1|\_p =^? l_2), l_1 \rightarrow r_1, l_2 \rightarrow r_2 \in R \}
\]

The previous results of this section are now brought together to prove the following main results about LM-systems concerning the status of overlaps. Namely, we show that there are no non-overlay superpositions and no forward-overlaps.

We first establish that there are no superpositions occurring between the left-hand sides of two distinct rules in \(R\). Formally, this amounts to showing that \(NOSUP(R) = \emptyset\). The main idea of the proof is that we show that such superpositions would induce critical pairs, and then these critical pairs cannot be joinable, which would contradict the confluence of our system \((R\ is\ convergent\ by\ definition)\).

**Corollary 4.12.** \(NOSUP(R) = \emptyset\ for all LM-systems R.\)

**Proof.** We prove the following: if \(R\) is an LM-System, \(l_1 \rightarrow r_1\ and \( l_2 \rightarrow r_2\ rules in \( R,\ p \in FPos(l_1)\) a non-root position, and \(\sigma(l_1[l_2]_p)\) a superposition, then the critical pair \((\sigma(l_1[r_2]_p), \sigma(r_1))\) is not joinable modulo \(R\).

Assume towards deriving a contradiction that \(s = \sigma(l_1[r_2]_p)\) and \(t = \sigma(r_1)\) are joinable modulo \(R\). Since \(l_1 \rightarrow r_1 \in R\) there must be distinct function symbols \(f\ and \(g\) such that \(s(e) = f\ and \(t(e) = g\) as \(R\ is an LM-System, i.e., l_1 \rightarrow r_1\ must have root-pair \((f, g)\) for \(f \neq g\).

Then, it must be that \(s = f(s_1, \ldots, s_m)\) and \(t = g(t_1, \ldots, t_n)\). Since \(s\ and \(t\ are assumed to be joinable, \(l_1 \rightarrow r_1\ is a rule in \(R with root pair \((f, g)\ we can apply Corollary 4.5 to get that:

\[
f(s_1, \ldots, s_m) \xrightarrow{l_1 \rightarrow r_1} g(t_1, \ldots, t_n)
\]

where \(s_1, \ldots, s_m, t_1, \ldots, t_n)\ are the \(R\-normal forms of \(s_1, \ldots, s_m, t_1, \ldots, t_n\ resp. However, since \(s(s_1, \ldots, s_m)\) is \(\bar{e}\-irreducible, the rule must be applied at the root, which gives us that \(f(s_1, \ldots, s_m) = \theta(l_1)\) for some substitution \(\theta\). Putting this all together we then get the following reduction:

\[
\sigma(l_1[r_2]_p) \rightarrow^* R \theta(l_1)
\]

Thus, Corollary 4.11 applies and \(\forall q \in ODP(\sigma(l_1[r_2]_p), \theta(l_1)) : [\sigma(l_1[r_2]_p)|q \rightarrow^* \theta(l_1)|q] \). Since \(p\ belongs to \(ODP(\sigma(l_1[r_2]_p), \theta(l_1))\), it follows that \(\sigma(r_2) \rightarrow^* \theta(l_1)|p = \theta(l_1)|_p\). But since \(l_1|p\ is unifiable with \(l_2\), we have \(l_1|p(e) = l_1(p) = l_2(e)\ and this contradicts Corollary 4.9. \(\square\)

We now turn to the case of forward-overlaps as defined in the section on forward-closure.

**Lemma 4.13.** Let \(R\ be an LM-System, then \(R \leadsto R = \emptyset\).

**Proof.** Recall: \(R \leadsto R = \{(l_1 \rightarrow r_1) \leadsto p \mid (l_2 \rightarrow r_2) \mid l_1 \rightarrow r_1, l_2 \rightarrow r_2 \in R \wedge p \in FPos(r_1)\}\) which are not redundant in \(R\). The proof proceeds by contradiction. Suppose that the above set is non-empty. By Corollary 4.6, no forward overlap can occur at \(p = e\).

Thus, there exists at least two rules, \(l_1 \rightarrow r_1\ and \( l_2 \rightarrow r_2\ in \( R, such that \(p \in FPos(r_1)\) is a non-root position, and \(\sigma = \text{mgu}(r_1|p =^? l_2)\ exists. Suppose that \(l_2 \rightarrow r_2\ has root-pair \((f, g)\). Forward closure gives us the
rule $\sigma(l_1) \rightarrow \sigma(r_1|r_2)_p$. By Corollary 4.5 the normal form of $\sigma(l_1)$ and $\sigma(r_1)$ must be an instance of $r_1$, i.e., $\beta(r_1)$ for some substitution $\beta$. The normal form of $\sigma(r_1|r_2)_p$ must also be the same $\beta(r_1)$. But note that $p \in ODP(\sigma(r_1|r_2)_p, \beta(r_1))$ since $r_1(p) = l_2(\epsilon) = f \neq r_2(\epsilon) = g$. Thus $\sigma(r_2) \rightarrow^{\cdot} \beta(r_1)_p$. But, as mentioned, the root symbol of $r_1|p$ (and hence that of $\beta(r_1|p)$) is the same as the root symbol of $l_2$. This contradicts Corollary 4.9.

We can now state the following lemma, which follows easily from the above results concerning overlaps.

First we introduce the following definition:

**Definition 4.** A term-rewriting system $R$ is said to be non-overlapping if and only if there are no left-hand side superpositions and no forward-overlaps.

**Lemma 4.14.** Every LM-system is non-overlapping.

*Proof.* Let $R$ be an LM-system. Suppose $l_i \rightarrow r_i \in R$ for $i \in \{1, 2\}$ are distinct rules. Then, there can be no overlaps between $l_1$ and $l_2$ at position $p = \epsilon$ by lemma 4.6. Lemma 4.12 establishes that $l_1$ and $l_2$ cannot overlap at position $p \neq \epsilon$. Finally, by Lemma 4.13 no overlaps can occur between $r_1$ and $l_2$. □

Finally, using the results derived above about LM-systems, we show that every LM-system is saturated by paramodulation. It is clear that every rewrite system saturated by paramodulation is also forward-closed. The next result establishes that for LM-systems, these two concepts are equivalent. Specifically, an LM-system is trivially saturated by paramodulation as there can be no overlaps into the left-hand side of an equation nor the right-hand side of an equation.

**Corollary 4.15.** Every LM-System is saturated by paramodulation.

*Proof.* Let $R$ be an LM-System and $E$ be the set of equations obtained from $R$. Suppose $u[s]_p \approx v$ in $E$ and $s \rightarrow t$ in $R$ induces a paramodulation inference. There are two cases depending on whether $u[s]_p$ is the lhs or rhs of some rule in $R$. If it is the lhs, then this would contradict Corollary 4.12. If $u[s]_p$ is the rhs of some rule in $R$, then this would contradict Lemma 4.13.

Thus, each case leads to a contradiction, and so no paramodulation inference steps can be performed. □

## 5 The Cap Problem Modulo LM-Systems

In this section we prove that although LM-systems are a restrictive subclass of term-rewriting systems there are still important problems that are undecidable when restricted to LM-systems. Specifically, we show that the cap problem\(^3\), which has important applications in cryptographic protocol analysis, is undecidable even when the rewrite system $R$ is an LM-system.

The cap problem is defined as follows:

**Instance:** A LM-System $R$, a set $S$ of ground terms representing the intruder knowledge, and a ground term $M$.

**Question:** Does there exist a cap term $C(\diamond_1, \ldots, \diamond_n)$ such that $C[\diamond_1 := s_{i_1}, \ldots, \diamond_n := s_{i_n}] \rightarrow^{\cdot} R M$?

\(^3\)Also known as the deduction problem
We show that the above problem is undecidable by a many-one reduction from the halting problem for reversible deterministic 2-counter Minsky machines (which are known to be equivalent to Turing machines). The construction below is extremely similar to the one given in [5]. Originally, the construction was used to show the undecidability of the subterm-collapse problem for LM-Systems. Here it is modified slightly to account for the cap problem, however the majority of the rules remain unchanged.

A reversible deterministic 2-counter Minsky machine (henceforth a Minsky machine) is described as a tuple \(N = (Q, \delta, q_0, q_L)\) where \(Q\) is a finite non-empty set of states, \(q_0, q_L \in Q\) are the initial and final states respectively and \(\delta\) is the transition relation. The elements of the transition relation \(\delta\) are represented as 4-tuples of the following form:

\[
[q_i, j, k, q_f] \text{ or } [q_i, j, d, q_f]
\]

where \(q_i, q_f \in Q, j \in \{1, 2\}, k \in \{Z, P\}, d \in \{0, +, -\}\). Tuples of the first form represent that the machine is in state \(q_i\), checks if counter \(j\) is zero (\(Z\)) or positive (\(P\)) and transitions to state \(q_f\). Tuples of the second form represent that the machine is in state \(q_i\), and either decrements (\(-\)), increments (\(+\)), or does nothing (\(0\)) to counter \(j\) and transitions to state \(q_f\).

Each configuration of machine \(N\) is written as \((q_i, C_1, C_2)\) where \(q_i \in Q\) is the current state of the machine, and \(C_1, C_2\) are the values of the counters. We encode such configurations as terms. The initial and final configurations of \(N\) are encoded by: \(c(q_0, s^k(0), s^p(0), 0)\) and \(c(q_L, s^k(0), s^p(0), 0)\) respectively. The fourth argument of \(c\) corresponds to the number of steps the machine has taken.

We need the fact that \(N\) is deterministic and reversible in the sequel to establish the results that the construction provided actually produces an LM-System. Namely, we need the following fact: For every pair of tuples in \(\delta\), \([q_{i_1}, j_1, x_1, q_{f_1}]\) and \([q_{i_2}, j_2, x_2, q_{f_2}]\) we have that

\[(i_1 = i_2) \lor (i_1' = i_2') \Rightarrow (j_1 = j_2 \land \{x_1, x_2\} = \{Z, P\})\]

This means that \(N\) can leave or enter the same state on two different transitions only when the same counter is being checked and different checks are being performed on that counter.

Let \(N\) be a Minsky machine. We construct a term-rewriting system \(R_N\) over the signature

\[\Sigma = \bigcup_{i=1}^L \{f_i, f'_i, q_i\} \cup \{c, s, 0, g, g', e\}\]

and show that the resulting TRS is an LM-System such that if \(N\) starts in the initial configuration and halts in a final configuration then there exists a cap term \(C\) such that the ground term \(c(e, 0, 0, 0)\) (playing the role of \(M\) in the description of the problem above) can be deduced. We begin the construction by initializing \(R_N\) with the following rules:

\[f_L(c(q_L, s^k(0), s^p(0), z)) \rightarrow g(c(e, 0, 0, z))\]
\[g'(g(c(e, 0, 0, s(z))) \rightarrow c(e, 0, 0, z)\]

We then add the following rules to \(R_N\), each of which encodes a possible move of the machine. That is, each rule represents an element of \(\delta\).

(a1) \([q_i, 1, P, q_j] : R_N := R_N \cup \{f_l(c(q_i, s(x), y, z)) \rightarrow c(q_j, x, y, s(z))\}\]

(a2) \([q_i, 2, P, q_j] : R_N := R_N \cup \{f_l(c(q_i, x, s(y), z)) \rightarrow c(q_j, x, s(y), s(z))\}\]
We now state the following theorem, and hold off on providing a proof until various claims have been shown to hold. The result will then following as an easy corollary.

**Theorem 5.1.** The cap problem modulo LM-Systems is undecidable.

We first begin by showing that given a reversible deterministic 2-counter Minsky machine, $N$, the rewrite system constructed above, $R_N$, is convergent.

**Claim 5.2.** Let $N$ be a Minsky machine, then the TRS, $R_N$, is convergent.

**Proof.** Define $\triangleright$ on $\Sigma$ as follows: $f_i > f'_i > f_L > g > g' > c > s > q_j > e \triangleright 0$. Then termination of $R_N$ is apparent by applying a recursive path ordering induced by $\triangleright$. We show that $R_N$ is confluent by showing that $R_N$ has no critical pairs. By construction, there are clearly no superpositions that can occur at positions other than $e$. However, by the determinism of $N$, if the index $i$ occurs more than once as the subscript of the lhs of a rule with an $f$ as its root symbol, then it could only ever occur again as the index of a term with an $f'_i$ as the root symbol of the lhs. Thus, there can be no critical pairs. \hfill $\square$

The next claims establish that $R_N$ is also forward closed and that $RHS(R_N)$ is quasi-deterministic. Each of which is a condition of a rewriting system to be an LM-System.

**Claim 5.3.** Let $N$ be a Minsky machine, then $R_N$ is forward-closed.

**Proof.** No root-symbol of the lhs of any rule in $R_N$ occurs in the rhs of any other rule. That is, there is no way to unify a subterm of the rhs of any rule with the lhs of any other rule. Thus, there can be no forward-overlaps and therefore $R_N$ is forward closed. \hfill $\square$

**Claim 5.4.** Let $N$ be a Minsky machine, then $RHS(R_N)$ is quasi-deterministic.

**Proof.** We first show that there are no RHS overlaps in $R_N$. Suppose $l_i \rightarrow r_i \in R_N$ for $i \in \{1, 2\}$ induces a RHS overlap. By construction of $R_N$, it must be that $r_1(\varepsilon) = c = r_2(\varepsilon)$ and $r_1(1) = q_i = r_2(1)$. Since $N$ is deterministic, the only way this could occur is between a rule from set $(a)$ and a rule from set $(b)$, but then $r_1$ and $r_2$ are not unifiable as there would be a function clash between “$s$” and “$0$”. Thus, there are no RHS overlaps.
It then suffices to show that \( R_N \) itself is quasi-deterministic. Clearly, no rule contains a variable as its left-hand side or its right-hand side, and no rule is root-stable. It remains to show that there are no root-pair repetitions. Suppose \( l_i \to r_i \in R_N \) for \( i \in \{1,2\} \) induces a root-pair repetition. Suppose \( (h,c) \) is the repeated root-pair. This implies that there are some pair of 4-tuples \( x,y \in \delta \) such that the first coordinates of \( x \) and \( y \) where the same. Let \( x = [q_\alpha, f_1, m, q_p] \) and \( y = [q_\alpha, f_2, n, q_k] \) be such tuples. Since \( N \) is deterministic and reversible, it must be that \( j_1 = j_2 \) and \( \{m,n\} = \{Z,P\} \), but then \( l_1 \to r_1 \) would have root-pair \( (f_\alpha, c) \) and \( l_2 \to r_2 \) would have root pair \( (f'_\alpha, c) \).

Thus, assuming there is a root-pair repetition contradictions the definition of the construction of \( R_N \) from \( \delta \). Therefore, there are no root-pair repetitions in \( R_N \), and therefore \( \text{RHS}(R_N) \) is quasi-deterministic.

Finally, the claim below, along with the claims above, establish that \( R_N \) is an LM-System. Namely, we prove that \( R_N \) is non-subterm collapsing, and thus we can conclude that \( \text{RHS}(R_N) \) is deterministic. Then, we establish that \( N \) halts if and only if there exists a cap-term that allows the deduction of \( M \) modulo \( R_N \). Thus, putting it all together we establish a many-one reduction and provide a proof of Theorem 5.1.

Claim 5.5. Let \( N \) be a Minsky machine, then \( R_N \) is non-subterm-collapsing.

Proof. Suppose \( t \to^* t' \) such that \( t' \) is a proper subterm of \( t \). By construction of \( R_N \), \( t|_p(\varepsilon) = c \) for some position \( p \). Since rules \((a1) - (e2)\) and the second rule in the initialization of \( R_N \) produce a “c” term with the last argument with an extra \( s \) added or removed, the only rule that could have been used to produce the collapse was the first rule of the initialization of \( R_N \). But in this case, the \( q_L \) is replaced with an \( e \) and thus the resulting term in the reduction could not be a proper subterm of \( t \). Thus we have a contradiction and can conclude that \( R_N \) is non-subterm-collapsing.

Claim 5.6. Let \( N \) be a Minsky machine. Then the TRS \( R_N \) is an LM-System.

Proof. Claim 5.2 shows that \( R_N \) is convergent and claim 5.3 shows that \( R_N \) is forward-closed. Claims 5.4 and 5.5 show that \( \text{RHS}(R_N) \) is deterministic. Therefore, \( R_N \) is an LM-System.

Claim 5.7. Let \( N \) be a Minsky machine. Then starting in the initial configuration \((q_0,k, p)\) \( N \) halts in \((q_L,k', p')\) iff there exists a cap term \( C(\varepsilon) \) such that if \( M = c(e,0,0,0) \) and \( S = \{c(q_0,s^{d}(0),s^{p}(0),0)\} \) then \( C(\varepsilon) = c(q_0,s^{d}(0),s^{p}(0),0) \to^*_{R_N} M. \)

Proof. \((\Rightarrow)\) Suppose \( N \) halts, then it does so in a finite number of steps. Let the number of steps \( N \) takes to be \( n \in \mathbb{N} \). The proof then proceeds similarly to that of Lemma 4.4 in [3]. That is, let \( (\tau_i)_{i=1}^n \) be the sequence of transitions that \( N \) passes through on its computation run. Then, \( \tau_i = [q_0, j, d, q_1] \) and \( \tau_n = [q_L, j, d, q_L] \). Define \( f^*_j \) as follows:

\[
f^*_j = \begin{cases} f_j & \text{if } d = Z \\ f_j & \text{otherwise} \end{cases}
\]

We can then construct the cap term \( C(\varepsilon) \) as follows:

\[
C(\varepsilon) = (g' \circ g)^{n-1}(g'(f_L ((f^*_n \circ \cdots \circ f^*_1)(\varepsilon))))
\]

Since the rules of \( R_N \) simulate the sequence of configurations of \( N \) and \( N \) halts, we can set \( C(\varepsilon) = c(q_0,s^{d}(0),s^{p}(0),0) \) and use the rules of \( R_N \) to reduce this term to

\[
C(\varepsilon) \to^*_{R_N} (g' \circ g)^{n-1}(g'(c(e,0,0,s^{p}(0))))
\]
which can then be reduced using the second rule of $R_N$ to get the term $(g' \circ g)^{n-1}(c(e,0,0,s^{n-1}(0)))$. This can then be reduced by further applications of the same rule to get the term $g'(g(c(e,0,0,s(0))))$. At this point, one more application of the second rule will yield $c(e,0,0,0) = M$. Thus, $M$ can be deduced.

$(\Leftarrow)$ Suppose that there exists a cap $C(\circ)$ such that $C[\circ := c(q_0,s^0(0),s^0(0),0)] \rightarrow_{R_N} M$. Let $t$ be such a cap term. Thus, the above says that there is a reduction chain starting with $t$ and ending in the term $M$. That is, $t \rightarrow t_1 \rightarrow t_2 \rightarrow \cdots \rightarrow t_k = c(e,0,0,0)$. Since the only rule that can introduce an “$e$” term is the first rule, there must be a sub-chain of reductions $t \rightarrow t_1 \rightarrow \cdots \rightarrow t_l$ such that $t_l = \sigma(f_L(c(q_L,s^0(0),s^0(0),0),z))$.

By design of the rewrite system $R_N$ from $N$, one can see that the chain above corresponds to a sequence of configurations of $N$ that eventually leads to the configuration $(q_L,k',p')$. Therefore, $N$, when started in $(q_0,k,p)$, halts in configuration $(q_L,k',p')$. \qed

We can now state the proof of Theorem 5.1

**Proof.** Given a Minsky machine $N$, let $G$ be the function such that $G(N) \rightarrow R_N$, i.e. the function that takes as input a Minsky machine and produces the corresponding system $R_N$ as described above. Then $G$ is clearly a total recursive function. Claim 5.7 says then that $G$ is in fact a many-one reduction. Thus, since the halting problem for Minsky machines is undecidable, then so is the cap problem modulo LM-Systems. \qed

**References**

[1] Franz Baader and Tobias Nipkow. *Term rewriting and all that*. Cambridge university press, 1999.

[2] Franz Baader and Wayne Snyder. Unification theory. *Handbook of automated reasoning*, 1:445–532, 2001.

[3] Christopher Bouchard, Kimberly A. Gero, Christopher Lynch, and Paliath Narendran. On forward closure and the finite variant property. In Pascal Fontaine, Christophe Ringeissen, and Renate A. Schmidt, editors, *FroCos*, volume 8152 of *Lecture Notes in Computer Science*, pages 327–342. Springer, 2013.

[4] Hans-Jürgen Bürckert, Alexander Herold, and Manfred Schmidt-Schauß. On equational theories, unification, and (un) decidability. *Journal of Symbolic Computation*, 8(1-2):3–49, 1989.

[5] Kimberly Gero, Chris Bouchard, and Paliath Narendran. Some notes on basic syntactic mutation. In Santiago Escobar, Konstantin Korovin, and Vladimir Rybakov, editors, *UNIF 2012 Post-Worshop Proceedings. The 26th International Workshop on Unification*, volume 24 of *EPiC Series in Computing*, pages 17–27. EasyChair, 2014.

[6] Bernhard Gramlich. On interreduction of semi-complete term rewriting systems. *Theoretical Computer Science*, 258(1):435–451, 2001.

[7] Miki Hermann. Chain properties of rule closures. *Formal Aspects of Computing*, 2(1):207–225, 1990.

[8] Christopher Lynch and Barbara Morawska. Basic syntactic mutation. In *Automated Deduction (CADE-18)*, pages 471–485. Springer, 2002.

[9] Enno Ohlebusch. Termination is not modular for confluent variable-preserving term rewriting systems. *Information processing letters*, 53(4):223–228, 1995.
6 Appendix

Lemma 6.1. *It is not the case that the equational theory of every LM-system is finite (i.e., all congruence classes are finite).*

Proof. The system

\[ f(g(h(x))) \rightarrow g(x) \]

is an LM-system, but the congruence class of \( g(y) \) is clearly infinite. \( \square \)

Lemma 6.2. *Every LM-system is free over the signature.*

Proof. Let \( R \) be an LM-system and \( f \in \Sigma^{(n)} \). Let \( s = f(s_1, \ldots, s_n) \) and \( t = f(t_1, \ldots, t_n) \) be \( \bar{\varepsilon} \)-irreducible terms that are joinable modulo \( R \). (Thus \( s_1, \ldots, s_n, t_1, \ldots, t_n \) are in normal form.) Then either \( t \) is in normal form and \( s \rightarrow t \), or \( s \rightarrow \hat{t} \) and \( t \rightarrow \hat{t} \) where \( \hat{t} \) is the normal form of \( s \) and \( t \). (Note that since \( R \) is forward-closed, reduction to a normal form will take only one step.) But the former is impossible because no rule in \( R \) can have the same root symbol on both the left-hand side and the right-hand side. The latter is ruled out because no two rules can have the same root symbols at their sides, i.e., root-pair repetitions are not allowed. \( \square \)

Lemma 6.3. *Let \( R \) be an LM-system and \( s = f(s_1, \ldots, s_m) \) and \( t = g(t_1, \ldots, t_n) \) be \( \bar{\varepsilon} \)-irreducible terms such that \( f \neq g \). Then \( s \) and \( t \) are joinable modulo \( R \) if and only if there is a unique equation \( e_1 \approx e_2 \in \text{RHS}(R) \) with root pair \( (f, g) \) such that \( s \rightarrow e_1 \rightarrow e_2 t \).*

Proof. The result follows from Lemma 4.2 and its proof. \( \square \)