Tachyon condensation at one loop

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Abstract

Using boundary string field theory we study the decay of unstable D-branes to lower dimensional D-branes via the tachyon condensation at one loop level. We analyze one loop divergences and use the Fischler-Susskind mechanism to cancel divergences arising at the boundary of moduli space. The tachyon action up to the second derivative is obtained and a logarithmic correction to the tachyon potential is written down explicitly. Multiple D-branes is also considered and the role of the boundary fermions is highlighted.

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1 Introduction

Tachyon condensation on D-branes has drawn a lot of attention during the last few years. It is essentially an off-shell process. Tachyon condensation can be described using the cubic string field theory [1] or the boundary string field theory (BSFT)[2]. Although progress has been made in understanding the decay of unstable D-branes to lower dimensional D-branes there remains many issues that need to be answered. Both the cubic string field theory and BSFT have been formulated on the disk. The off shell actions have been derived only in tree level approximations. It is interesting to study how the theories change when string loop corrections are taken into account. The first loop correction is given by a boundary string field theory on an annulus.

We will use BSFT to describe tachyon condensation. For a quadratic boundary interaction the bosonic BSFT has been studied in [3],[4]. The exact tachyon potential has been calculated along with value of the D-brane tension. The generalization to superstrings was considered in [5]. It turns out that the supersymmetric case is simpler than then bosonic one. The action is proportional to the partition function on the disk. This statement was rigourously proven using Batalin-Vilkovisky formalism in [6].

To include loop corrections one has to generalize the analysis to the annulus. BSFT on an annulus has been considered by several authors [8],[9],[10],[11],[12],[23]. Different authors have chosen different boundary conditions on the boundaries of the annulus. We will considered the case where the boundary conditions on the two boundaries are the same. We argue that other boundary conditions lead to inconsistencies. In this paper we also study the Fischler-Susskind (FS) mechanism in the presence of the tachyon profile. This allows us to cancel divergences in the partition function that arise at the boundary of moduli space.

In the view of the recent development in cosmology [13] connected with Sen’s proposal of the rolling tachyon [14] it is also interesting to look how the string loop corrections modify the tachyon potential.

For the sake of completeness we want to mention that tachyon condensation was first introduced long time ago in the series of papers[15].

Before we start with our calculations let us summarize the assumptions we will make. When all the $\beta$-functions are linear the action on a the disk has the form [2]

$$S = \left( 1 + \beta_i \frac{\partial}{\partial g_i} \right) Z(g_i), \quad (1)$$

where $g_i$’s are coupling constants and $Z(g_i)$ is the partition function on the the disk. For superstrings the tachyon $\beta$-function is zero and Eq. [11] reduces to [5]

$$S = Z. \quad (2)$$

We assume that this relation is still true on the annulus. The new feature in BSFT on the annulus is that we encounter divergences from integration over moduli. The corresponding two dimensional field theory must be renormalizable [2]. We will show that for the quadratic tachyon profile this is the case. To cancel the infinities one has to introduce a counterterm. It contributes to the graviton $\beta$-function and depends on the tachyon coupling. The boundary RG is mixed with closed string excitations and therefore one also needs the closed string field theory. We assume that at least for the quadratic boundary interaction it is possible to restrict to BSFT and ignore closed string field theory.
Bosonic strings

The boundary interaction for a bosonic string on a disk $D$ has the form

$$S_{\text{bndry}} = \frac{1}{2\pi} \int_{\partial D} d\theta \sqrt{h} T(X),$$  \hspace{1cm} (3)

where $T(x) = uX^2$ is the quadratic tachyon profile and $\sqrt{h}d\theta$ is the length element on the boundary. The tachyonic action can be determined using BSFT. The tree level contribution is obtained from a disk amplitude. To include loop corrections one has to consider world sheets with higher Euler characteristic. In this paper we consider the next term in the loop expansion. It corresponds to world sheet with Euler characteristic $\kappa = 0$. This surface has two boundaries and it can be represented either as a cylinder or as an annulus. For a conformally invariant theory it does not matter which surface one uses. However, a non-trivial tachyon profile breaks conformal invariance and the partition functions on the annulus and cylinder do not coincide.

We will start our calculation by comparing the partition function on the annulus to the one on the disk. The boundary of an annulus is two circles, one with the radius 1, and another one with radius $a < 1$. The choice of the tachyon profile on the outer boundary is fixed by the disk BSFT. On the other hand it is not obvious how to fix the tachyon profile on the inner boundary. It is clear that we want a tachyon profile that is quadratic, but this leaves the freedom of having a different normalization for the quadratic term on the two boundaries. The tachyon profile on the inner radius can depend the modulus $a$. The simplest possibility is to take the same tachyon profile on the two boundaries. This case was considered in [12], see also [11].

Another choice was made in [10] where invariance under the interchange of the ends of the of strings $z \to a/z$ was enforced by tachyon profile $T(X) = u/aX^2$ on the inner boundary. The role of extra $a$ here is to cancel the factor $\sqrt{h}$ for the inner boundary. We can recover the tree level contribution by shrinking the inner radius of the annulus to zero and so that the annulus becomes a disk. For the tachyon profile considered in [10] one is still left with a nonzero tachyon profile at the origin after the radius of the inner circle vanishes. This corresponds to inserting a vertex operator $\langle X^2(0) \rangle$ at the origin. There is no doubt that this contributes to the tachyon $\beta$-function. In the supersymmetric case the operator insertion at the origin would break the world sheet SUSY which of course would be disastrous. Furthermore, in the theory with a symmetric tachyon profile it is impossible to implement Fishler-Susskind (FS) mechanism to cancel the tadpole divergence.

A different tachyon profile was used in [8]. The tachyon profile was fixed by imposing that the partition functions on the annulus and on the cylinder are the same. As we mentioned earlier it is unclear why it should be true since the theory is not conformal invariant in the presence of a tachyon profile. Furthermore, the analysis in [8] was only carried out for a constant tachyon profile. If one carries out the calculation for a quadratic profile one finds unphysical poles at some values of $u$ in the integration over the modulus.

From the discussion above it is clear that the boundary terms on the annulus should be of the form

$$S_{\text{bndry}} = \frac{u}{2\alpha'} \oint_{\rho=1} d\theta X^2(\theta) + \frac{au}{2\alpha'} \oint_{\rho=a} d\theta X^2(\theta),$$  \hspace{1cm} (4)$$

$$S_{\text{bulk}} = \frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{\gamma} \partial_a X \partial_b X.$$  \hspace{1cm} (5)
Our goal in this paper is to calculate partition function on an annulus with this boundary interaction. We will follow the methods used for finding the partition function on the disk. We start by determining the Green function $G(\theta, \theta') = \langle X(\theta) X(\theta') \rangle$ with both points lying on the same boundary. As on the disk, we expand field $X$ around its classical solution $X_{cl}$, $X = X_{cl} + \epsilon$ with boundary condition $\epsilon = 0$ on the boundary. The $u$ dependent part of the action is given by $S[X_{cl}]$. Similar approach was used in [9].

To find the Green function one has to integrate over all possible boundary conditions with an $X(\theta)$ insertion. Details of the calculations are presented in Appendix A.

From Appendix A we find that Green functions are given by

$$ G(\theta, 0)|_{\rho = a} = 2^{\alpha'} \sum_{n=1}^{\infty} A_n(u, a) \cos n\theta, $$

$$ G(\theta, 0)|_{\rho = b} = 2^{\alpha'} \sum_{n=1}^{\infty} B_n(u, a) \cos n\theta, $$

where

$$ A_n(u, a) = \frac{nR_n(a) + u}{n^2 + u(1 + a)nR_n(a) + u^2 a}, $$

$$ B_n(u, a) = \frac{nR_n(a) + ua}{n^2 + u(1 + a)nR_n(a) + u^2 a}. $$

Then partition function $Z(u)$ can be written in terms of the Green function as follows

$$ (2\alpha') \frac{d\ln Z(u)}{du} = -G(0)|_{\rho = 1} - aG(0)|_{\rho = a}. $$

Inserting the mode expansion of the Green function we find

$$ \frac{d\ln Z(u)}{du} = -aA_0(u, a) - B_0(u, a) - \sum_{n=1}^{\infty} (aA_n(u, a) + B_n(u, a)), $$

where coefficients $A_n$’s and $B_n$’s are given above. It is easy to see that

$$ A_0(u, a) = \frac{1}{2} \lim_{n \to 0} A_n(u, a) \quad \quad B_0(u, a) = \frac{1}{2} \lim_{n \to 0} B_n(u, a). $$

We will use this limit later in our calculations.

The partition function can be cast into the form

$$ \ln Z(u) = \frac{1}{2} \ln u - \frac{1}{2} \ln(1 + a - ua \ln a) - \sum_{n=1}^{\infty} \left[ - \ln n^2 + \ln(1 + a^{2n}) - \sum_{n=1}^{\infty} (\ln(1 + u/n) + \ln(1 + au/n) + \ln(1 - a^{2n} g_n(u, a))) \right], $$

where

$$ g_n(u, a) = \frac{1 - u/n}{1 + u/n} \frac{1 - au/n}{1 + au/n}. $$
The expression is quite formal and there are several divergences that we have to regularize. The partition contains several \( \ln \)-terms. The first two \( \ln \)'s arise from zero modes. The first one comes from the constant mode that controls behavior of the partition function for small \( u \). As expected it is equal to the tree level contribution. There is an additional zero mode due to \( \ln \rho \) function on the annulus but it gives only a subleading term. The first \( \ln \) in the sum in (13) can be regularized by the standard zeta function regularization

\[
\prod_{n=0}^{\infty} \frac{1}{n^2} = \frac{1}{2\pi^2}.
\]

Below we will see that this regularized result plays an essential role in relative normalization of different sectors in the supersymmetric case.

The total partition function \( Z_{\text{total}} = Z_A(a)Z(u) \) is a product of the \( u \)-dependent part \( Z(u) \) and the partition function \( Z_A(a) \) for the annulus with no boundary interaction. The second \( \ln \) in the sum in (13) cancels the corresponding part in \( Z_A(a) \). The first two \( \ln \)'s in (13) are identical to tree level contributions and therefore give a factor proportional to the \( Z_D(ua)Z_D(u) \), a product of two disk partition functions. As for the disk case one has to regularize the part of the sum that is proportional to \( u \). This is a source of a nonzero \( \beta \)-function for the bosonic case. For superstrings the infinite contribution from the bosons is cancelled by the similar contribution from the fermions leaving the \( \beta \)-function zero as expected. We will consider these terms in more details when we study the supersymmetric case. The last \( \ln \) in (14) comes from the oscillators. It is finite unless \( a \neq 1 \). Correspondingly, the partition function integrated over the modulus \( a \) diverges at \( a = 1 \). The problem is how to calculate this divergence. The modular properties of the partition functions in the presence of a tachyon profile are broken. It is hard to find the behavior of \( Z(u) \) when \( a \to 1 \) by doing a modular transformation \( \tau \to -1/\tau \). Again the analysis simplifies considerably when we study the supersymmetric case. Therefore we postpone detailed analysis for later sections.

3 Superstrings

Having analyzed the bosonic case we now turn to the supersymmetric tachyon profile. We start with type IIA theory with a single non-BPS D9-brane and consider a tachyon profile only in one spatial direction. This corresponds to the decay of the D9-brane to a BPS D8-brane. The more general case is obtained by a straightforward generalization. The supersymmetric version of the boundary interaction is given by [5,7]

\[
S_{\text{bndry}} = \oint \sqrt{h} d\theta \frac{1}{2\alpha'} \left( \frac{1}{2} T^2(X) - \eta \dot{\eta} - \psi^\mu \eta \partial_\mu T(X) \right),
\]

where the tachyon on both boundaries is of the form \( T(X) = \sqrt{\alpha'}X \). On each boundary we also have an auxiliary fermion \( \eta(\theta) \). This is the same fermion that was introduced in [18] for IIB and in [19] for IIA theory. For zero tachyon profile, the \((-1)^F\)NS sector is absent since non-BPS D-branes do not couple to R-R-fields.

Next let us turn to the fermionic zero modes. They appear when the fermion \( \eta \) is periodic on the boundary. If the interaction term of the form \( \psi^\mu \eta \partial_\mu T(X) \) is absent the integration over this zero mode results in a vanishing functional integral. On the other hand if the interaction term is present the functional integral does not vanish. For antiperiodic fermions \( \eta \) the functional integral with no tachyon profile is \( \sqrt{\alpha'} \). This factor of \( \sqrt{\alpha'} \) is precisely the one that makes difference between non-BPS and BPS D-brane tensions.
To integrate out the auxiliary fermions we rewrite the boundary interaction in a non-local form\cite{5}

\[ S_{\text{bulk}} = \frac{1}{4\pi} \int d^2z \left( \frac{2}{\alpha'} \partial X \partial \bar{X} + \Psi \partial \bar{\Psi} + \bar{\Psi} \partial \Psi \right), \quad (18) \]

\[ S_{\text{bndry}} = u \oint_{\rho=1} \frac{d\theta}{2\pi} \left( \frac{1}{2\alpha'} X^2(\theta) + \psi_0(\theta) \frac{1}{\partial \bar{\theta}} \psi_b(\theta) \right) + ua \oint_{\rho=a} \frac{d\theta}{2\pi} \left( \frac{1}{2\alpha'} X^2(\theta) + \psi_a(\theta) \frac{1}{\partial \bar{\theta}} \psi_a(\theta) \right), \quad (19) \]

where \( \psi_a \) and \( \psi_b \) are boundary fermions.

\[ \psi_{a,b} = i(\sqrt{iz} \Psi \pm i\sqrt{-iz} \bar{\Psi}). \quad (20) \]

The square roots arise from the change of coordinate from \( z \) to \( \theta \) on the boundary. The sign in the above expression depends on the sector considered.

We define the inverse derivative

\[ \frac{1}{\partial \bar{\theta}} \psi(\theta) = \frac{1}{2} \int_{0}^{2\pi} d\theta \epsilon(\theta - \theta') \psi(\theta') \quad (21) \]

with \( \epsilon(x) = 1 \) for \( x > 0 \) and \( \epsilon(x) = -1 \) for \( x < 0 \).

It turns out that in order to determine the partition function we do not need to know the precise form of boundary interaction \cite{10}. The world sheet supersymmetry is quite restrictive. When the boundary interaction is turned off, the \((-1)^F R\) sector contribution to the partition function vanishes. It is reasonable to expect that this is still the case when the boundary interaction is turned on since it does not break world sheet supersymmetry. In fact this is what happens on the disk. For periodic fermions the expectation value \( < S_{\text{bndry}} > \) vanishes. On each boundary the Green function in the \((-1)^F R\) sector is given by

\[ G_f(\theta, \theta') \equiv < \psi \frac{1}{\partial \bar{\theta}} \psi(\theta) > = -< X^2(\theta)/(2\alpha') >. \]

It is easy to see that the function \( G_f \) is periodic as it should since we have inserted a factor \((-1)^F\). In the open string sector the angle variable \( \theta \) corresponds to time. By a simple change of moding from integer to half integer one can go to the R sector in open string channel or to the \((-1)^F NS - NS\) sector in the closed string channel. It is clear that there are no zero mode contributions and the Green function on the inner boundary takes the form

\[ G_{Rf}(\theta, 0) \big|_{\rho=a} = -\sum_{s=1/2}^{\infty} A_s(u, a) \cos s\theta. \quad (22) \]

To obtain the Green function on the outer boundary we have to change the coefficient \( A \) to the coefficient \( B \).

To obtain the Green function in the NS sector we have to find a way to transform from the R sector to the NS sector. It is easy to see that if we shift the time in the closed string sector by 1 it amounts to switching from R sector to NS sector. This shift can be accomplished by the modular transformation \( a \rightarrow -a \). When performing this modular transformation one has to be careful not to change \( a \) that comes from \( \sqrt{h} \) factor in the boundary interaction. Carrying out the modular transformation we find

\[ G_{Rf}(\theta, 0) \big|_{\rho=a} = -\sum_{s=1/2}^{\infty} A'_s(u, -a) \cos s\theta, \quad (23) \]
where prime in $\mathcal{A}'$ means that the combination $au$ does not change sign. Again to obtain the Green function on the outer boundary one has to replace $\mathcal{A}$ with $\mathcal{B}$. We still need to consider $(-1)^F$ NS sector which is absent when $u = 0$. The Green function $G_f$ in this sector is obtained by changing the moding from half integer to integers. Carrying out this change of moding we find

$$G_f^{(-1)^F}NS(\theta, 0)|_{\rho = a} = -\sum_{n=1}^{\infty} \mathcal{A}'_n(u, -a) \cos n\theta. \quad (24)$$

To find the zero mode contribution we write the fermionic part of the boundary interaction in the form

$$S_{\text{bndry}} = u \oint_{\rho = a} d\theta \frac{1}{2\pi} \left( \psi_b(\theta) \frac{1}{\partial_{\theta}} \psi_b(\theta) + \ldots \right) + v \oint_{\rho = a} d\theta \frac{1}{2\pi} \left( \psi_a(\theta) \frac{1}{\partial_{\theta}} \psi_a(\theta) + \ldots \right). \quad (25)$$

Here $v$ is an arbitrary coupling on the inner radius of the annulus. If we set $v = au$ it reduces to the boundary interaction studied in this paper. Using the property (12) one finds

$$A_0 = -\frac{1}{2v}, \quad B_0 = -\frac{1}{2u}, \quad (26)$$

which gives

$$\frac{1}{Z} \frac{dZ}{du} = \frac{1}{2u} + \ldots, \quad \frac{1}{Z} \frac{dZ}{dv} = \frac{1}{2v} + \ldots \quad (27)$$

After a simpler integration we have

$$Z^{(-1)^F}NS(u, a) \sim \sqrt{uv} = u\sqrt{a}. \quad (28)$$

Note the result is proportional to $\sqrt{a}$. Therefore the partition function $Z^{(-1)^F}NS$ vanishes when $a \to 0$. This is exactly what one would expect since in the limit $a \to 0$ the annulus reduces to a disk and on the disk the partition function $Z^{(-1)^F}NS$ vanishes. There is another reason why the factor $\sqrt{a}$ is important. For large $u$ it will cancel a similar factor arising from bosonic zero modes to give the correct partition function for a BPS D-brane.

When determining the partition function it is convenient to split it into two parts: the zero mode part and the oscillator part. The contribution from the oscillators is proportional to the usual partition function for the annulus $Z_A(u)$. This is just a normalization constant in the integration over $u$. The partition function for the disk without the zero mode contribution is given by

$$Z_D(u) \equiv \frac{1}{\sqrt{2}} 4^u a \frac{\Gamma^2(u)}{\Gamma(2u)}. \quad (29)$$

This function has two important limits. When $u = 0$, we find $Z_D(0) = \sqrt{2}$ which is just the contribution from the integral over the boundary fermion $\eta$. For large $u$, we have $Z_D(u) \sim \sqrt{2\pi u}$.

Let consider the Ramond sector. The sums appearing are now integer moded rather than half integer moded

$$\frac{1}{Z} \frac{dZ}{du} = -(aA_0 + B_0) - \sum_{n=1}^{\infty} (aA_n(u, a) + B_n(u, a)) + \sum_{s=1/2}^{\infty} (aA_s(u, a) + B_s(u, a)). \quad (30)$$

Following the analysis on the disk, we find

$$\frac{1}{Z} \frac{dZ}{du} = -(aA_0 + B_0) - 2 \sum_{n=1}^{\infty} (aA_n(u, a) + B_n(u, a)) + \sum_{n=1}^{\infty} (aA_{n/2}(u, a) + B_{n/2}(u, a)). \quad (31)$$
The zero mode contribution is easy to integrate. It gives the factor
\[
\frac{1}{\sqrt{u(1 + a - ua \ln a)}}. 
\] (32)

The oscillator part can be cast into the form
\[
Z \sim Z_D(u)Z_D(ua) \left( \prod_{n=1}^{\infty} \frac{1}{n^2} \right) \frac{\prod_{n=1}^{\infty} (1 - a^2n)}{\prod_{n=1/2}^{\infty} (1 - a^{2s})} \prod_{n=1}^{\infty} (1 - a^{2n}g_n(u, a)) \right). \] (33)

Again $\zeta$-function regularization gives a factor $1/(2\pi)$ instead of first infinite product. In the NS sector the factor $(1 - a^{2s}g_s(u, a))$ in (33) is replaced by $(1 + a^{2s}g_s(u, a))$. The part that is proportional to the product of the partition functions on the disk remains the same as in the Ramond sector. The zero mode part is also the same since it comes only from bosonic strings. The expression for the partition function in the $(-1)^F$NS-sector is quite different. First of all there is a fermionic contribution proportional to the product of the disk partition function that exactly cancels similar contribution arising from the bosonic sector. The second difference is that the factor $1/(2\pi)$ from the bosonic part is cancelled by the same factor from the fermionic part. Finally we have a contribution from fermionic zero modes.

Let summarize our results. For NS and R sectors the partition function is a product
\[
Z(u, a) = Z^{(0)}(u, a)Z_D(u)Z_D(ua)f^7(0, a)f(u, a), \] (34)

where $Z^{(0)}$ is the zero mode contribution
\[
Z^{(0)}_{NS,R}(u, a) = \frac{1}{2\pi} \frac{1}{\sqrt{u(1 + a - ua \ln a)}}. \] (35)

and
\[
f_{R,NS}(u, a) = \frac{1}{a^{1/8}} \frac{\prod_{n=1}^{\infty} (1 + a^{2s}g_s(u, a))}{\prod_{n=1}^{\infty} (1 - a^{2n}g_n(u, a))}. \] (36)

The plus sign for NS sector and minus sign is for R sector. For $(-1)^F$NS sector the partition function is given by
\[
Z(u, a) = Z^{(0)}(u, a)f^7(0, a)f(u, a), \] (37)

where the zero mode contribution is
\[
Z^{(0)}(u, a) = \frac{\sqrt{ua}}{\sqrt{1 + a - ua \ln a}} \] (38)

and
\[
f^{(-1)^FNS}(u, a) = \sqrt{2} \frac{\prod_{n=1}^{\infty} (1 + a^{2n}g_n(u, a))}{\prod_{n=1}^{\infty} (1 - a^{2n}g_n(u, a))}. \] (39)

The factor $\sqrt{2}$ is a normalization factor that is need in order for the function $f(0, a)$ to reduce to the corresponding function with a vanishing tachyon profile. To obtain the full partition function we need to sum over different sectors and integrate over the modulus $a$
\[
Z_{total} = \int_0^1 \frac{da}{a} (NS(u, a) - (-1)^FNS(u, a) - R(u, a)). \] (40)
Let us consider two limits of this partition function. First one is \( u = 0 \). In this limit the contribution from \((-1)^F\) NS sector vanishes and we are left with

\[
Z_{\text{total}} = \frac{1}{\pi \sqrt{u}} \int_0^1 \frac{da}{a \sqrt{1 + a}} Z_A(a),
\]

(41)

where \( Z_A \) is the partition function for non-BPS D9-branes with no tachyon profile. Note the factor \( 1/\sqrt{1 + a} \) in the integration measure for the partition function. It has a very simple explanation. We can rewrite the boundary action in the form

\[
F^2 + \eta \dot{\eta} + T(X) F + \psi \eta \partial T(X).
\]

(42)

If we eliminate the auxiliary field \( F \) the boundary action reduces to (17). When the tachyon profile vanishes the boundary interaction has the form \( F^2 + \eta \dot{\eta} \). For vanishing \( u \) the integration over the bosonic \( F \) contributes the factor \( 1/\sqrt{1 + a} \) to the partition function.

For large \( u \) we find that the partition function is given by

\[
Z_{\text{total}} = \int_0^1 \frac{da}{a \sqrt{-\ln a}} Z_A^{BPS}(a),
\]

(43)

where \( Z_A^{BPS}(a) \) is the partition function of a BPS D-brane. This is a formal expression since the partition for BPS D-branes vanishes. The factor \( 1/\sqrt{-\ln a} \) appearing in eq. (43) is cancelled by a similar factor arising from the integration over the momentum. This can be seen by writing \( a = e^{-\pi/t} \), where \( t \) is the open string time. For large \( u \) we indeed have a single D8-brane. For finite value of \( u \) a more careful analysis is required. There are two issue on has to deal with. First the tachyon field gives rise to a divergence that must be regularized. The second problem is that there is a tadpole divergence. The usual way to deal with tadpole divergence is to analytically continue. In this case the partition function and the tachyon potential become complex. One can think of this as a signal of instability of the system. The tadpole divergence is more crucial. Logarithmic divergences unlike power ones are physical. They could possibly give a contribution to the tachyon \( \beta \)-function or to \( \beta \)-functions of other fields. These issues will be analyzed in the next section in more details.

4 Tadpole divergence.

Having constructed the partition function we are ready to study tadpole divergences. First let us consider a constant tachyon profile \( T = c \). The tachyon potential is

\[
V(c^2) \sim Z(c^2) = e^{-c^2} \int_{1/\Lambda}^1 \frac{da}{a} e^{-c^2 a} Z_A(a).
\]

(44)

Above we have introduced a cutoff \( \Lambda \) in order to regularize our potential. The infinite part of the integral is \( 16 e^{-c^2} \ln(\Lambda/c^2) \). Clearly, for finite \( c \) we need to introduce a counterterm to cancel this divergence. On the other hand if one takes limit \( \Lambda \to \infty \) in such a way that \( \Lambda/c^2 \) remains finite the divergence disappears and there is no need to introduce counterterms. This shows that the theory has two different phases. In the first phase with finite \( c \) a counterterm has to be introduced to cancel the infinity. This counterterm will contribute to the to the graviton \( \beta \)-function. It is important to note that although there are divergences the \( \beta \)-function for the tachyon still vanishes. In the second phase with \( \Lambda/c^2 \) finite the
theory is finite. For a constant tachyon profile one cannot get a BPS D-brane since R-R-sector is missing. Tadpole divergence is usually cancelled by the Fischler-Susskind (FS) mechanism\cite{20}, see also\cite{21}. The insertion of the dilaton vertex operator at zero momentum $g_0^2 \ln \Lambda \Psi^\mu(0) \bar{\Psi}_\mu(0)$ at the origin of the disk makes the total partition function finite even for a constant tachyon profile. The coupling constant $g_0$ is the open string coupling constant.

Next we will consider the linear tachyon profile. For small $u$ the partition function has a $1/\sqrt{u}$ behavior. This arises from a potential that has a $T^2$ dependence after one integrates over the bosonic field $X$. The tachyon potential is determined by a constant tachyon profile.

We are now ready to analyze the tadpole divergence for the linear profile. Again we have to introduce a cutoff $\Lambda$ when we integrate over the modulus $a$. We find that there are two phases depending on the order we take the limits. In the first phase we let $a \to 0$ while $u$ is fixed. The second phase is obtained by first taking the limit $u \to \infty$ and then integrating over the modulus $a$. Since in the end one has to take $\Lambda \to \infty$ limit the second phase corresponds to BPS D-brane and no counter term is needed. Let us now consider the first phase. We keep $u$ finite while $a \to 0$. Since the contribution of R-R sector is proportional to $\sqrt{a}$ there is no divergence in this sector as $a \to 0$. The only divergence comes from NS and $(−1)^F$ NS sectors. The closed string tachyon is cancelled between these two sectors but the contribution from the massless state still remains. We find that the logarithmic divergence has the form

$$\ln \Lambda \frac{\sqrt{2}}{2\pi \sqrt{u}} Z_D(u) 2 \left(7 + \frac{1 - 2u}{1+2u}\right). \quad (45)$$

The $\sqrt{2}$ factor comes from $Z_D(au)$ at $a = 0$. The factor 2 arises because NS and R sectors give identical contributions. The numerical factor 7 in the bracket is the contribution from directions with vanishing tachyon profile whereas the last factor is the contribution from the direction with a non-vanishing tachyon profile. Since we are using the FS mechanism to cancel divergences arising from the boundary of moduli space we need to add an operator at the origin of a disk.

To determine this contribution we need to find the Green function in the bulk of the disk in presence of the tachyon profile $T(X) = \sqrt{u}X$. A simple calculation gives

$$<\Psi(z)\bar{\Psi}(\bar{w})> = \frac{1}{\sqrt{zw}} \left(-\frac{\sqrt{zw}}{1 - zw} + 2u \sum_{s=1/2}^\infty \frac{1}{s + u} (zw)^s\right). \quad (46)$$

At the origin it takes the form

$$<\Psi(0)\bar{\Psi}(0)> = -\frac{1 - 2u}{1 + 2u}. \quad (47)$$

This shows the tadpole can be cancelled by a dilaton vertex operator

$$\frac{2}{g_0} \ln \Lambda e^{-\phi} e^{-\tilde{\phi}} \Psi_\mu \bar{\Psi}^\mu \quad (48)$$

at zero momentum at the origin. Here $\phi$ and $\tilde{\phi}$ are bosonized superghosts. Note that the vertex operator is in $(−1, −1)$ picture. This shows that the FS mechanism works for the linear tachyon profile.

One would like to transform the above vertex operator to $(0, 0)$ picture. When conformal invariance is not broken the dilaton vertex operator in $(0, 0)$ picture is $\partial X^\mu \partial \bar{X}_\mu$ integrated over the disk. Any point in the interior of the disk can be brought to the origin by a conformal transformation. Then the integration over the disk becomes an integration over the volume of the conformal group. The quadratic tachyon
profile on the boundary breaks conformal invariance. Given a vertex operator at a fixed point on the disk there is no simple way to find a corresponding operator in (0, 0) picture. For related discussion see [22]. At the fixed points of RG flow where the conformal invariance is restored it is again possible to write the vertex operator as an integral over the disk in (0, 0) picture.

Let us summarize what has been done so far. If the large $\Lambda$ limit is taken first while $u$ is kept finite we are in the first phase. In this phase the NS-NS divergence can be eliminated by introducing the dilaton vertex operator on the disk. If then let $u$ approach infinity R-R anomaly appears. This indicates that the system is entering the second phase. In the second phase the large $u$ limit is taken first. The total partition function now is just the partition function of a D8-brane. The tadpole is cancelled by R-R anomaly. No dilaton vertex operator is needed. This phase transition happens at $u \sim \Lambda$.

It is easy to see that if the tachyon profile on the inner boundary were as in [10] ($\sqrt{h}$ factor is absent) this divergence would be proportional to $Z^2_d(u)$ which would be impossible to cancel by FS mechanism.

So far we have considered space filling D9-branes that do not have any Dirichlet directions. The discussion for lower dimensional D-branes can proceed along the same lines except one subtle point. Each Dirichlet direction kills one zero mode for the corresponding boson and thus adds the factor $1/\sqrt{-\ln a}$. Denote $d$ as a number Dirichlet directions. The contribution of massless modes is proportional to

$$\int_{1/\Lambda} da \frac{1}{(-\ln a)^{-d/2}}.$$  \hspace{1cm} (49)

For $d = 0, 1, 2$ there is a tadpole and the dilaton vertex operator must be inserted on the disk. For all other lower dimensional D-branes the integral converges in the closed string limit. Therefore the phase transition occurs only for D9, D8 and D7 branes.

Although the factor $1/\sqrt{-\ln a}$ also affects behavior of the integrand in the open string limit $a \to 1$ the main divergence in this case comes from the open string tachyon. We consider this divergence in the next section.

## 5 The fate of the open string tachyon

The open string sector of the theory is determined by the behavior of function $f(u, a)$ in the limit $a \to 1$. To find this limit we can not use modular properties of the function $f(u, a)$ since we do not know how it transforms under the modular transformation $\tau \to -1/\tau$. In Appendix B we show how to determine leading behavior of the function $f(u, a)$ in the limit $a \to 1$.

To study the limit $a \to 1$ while $u$ is constant we will consider different sectors separately. We start with the NS sector and find

$$Z \sim e^{-\pi u t/3} e^{\pi^2 u/3} \frac{\sinh 2\pi u}{\sinh \pi u} \frac{1}{2\pi} \frac{1}{\sqrt{2u}} \frac{1}{2} e^{\pi t} + O(1/t),$$  \hspace{1cm} (50)

where the factor of 2 is needed in order to have the correct $u \to 0$ limit. In the $(-1)^F$ NS sector we find

$$Z \sim e^{-\pi u t/3} e^{\pi^2 u/3} \sinh \pi u \frac{1}{\sqrt{2u}} \frac{1}{\pi} e^{\pi t} + O(1/t)$$  \hspace{1cm} (51)

and in the R sector we have

$$Z \sim e^{2\pi u t/3} e^{-2\pi^2 u/3} \frac{1}{\sqrt{2u}} \frac{1}{2\pi} 2 e^{2\pi t} + O(1/t),$$  \hspace{1cm} (52)
where again the factor of 2 was inserted to have the correct $u \to 0$ limit. As we mentioned in Appendix B this analysis only works for the supersymmetric case. In the bosonic case one encounters divergences that we do not know how to analyze. In [12], [11] this behavior was investigated by calculating the central charge of strings on the cylinder. But as the conformal invariance is broken the partition functions for the cylinder and for the annulus do not coincide to each other. They are the same only in the two conformal limits. Therefore it is not clear whether the central charge of the theory on the cylinder determines the leading behavior of the partition function on the annulus in the open string limit.

The most interesting conclusion one can derive from the asymptotic behavior considered above is that the tachyon disappears when $u = 3$ in NS and $(−1)^F$ NS sectors. The ground state becomes massive if $u > 3$. In the limit of $u \to \infty$ the contribution from NS and $(−1)^F$ NS sectors are equal and cancel each other as they contribute with an opposite sign. There is a tachyon in the R sector for any non zero $u$ but its contribution is multiplied by a factor that decreases with $u$.

Since there is a tachyon in the open string sector the integral over modulus $a$ is divergent and has to be regularized. If we try to follow the analysis of the close string sector by introducing a cutoff, we find for the open string has power law divergences. These are clearly unphysical.

There is a way to treat this divergence. One can analytically continue the partition function. The price one has to pay for this is that the partition function as well as the action for the tachyon becomes complex. This is a common situation in the field theory when a loop corrected effective potential with a negative second derivative is considered. Since the mass of the tachyon is related to the second derivative of the potential it is very likely that the loop corrected action for the tachyon is also complex. The imaginary part of the action displays the fact that the system is not stable and decays[10]. The imaginary part is proportional to the decay rate[23]. Following [24] the imaginary part of the tachyon action is

\[
\text{Im} \int_R^\infty dt \frac{1}{t} e^{-\beta t} = \frac{\pi}{\Gamma(1 + \alpha)} b^\beta. \tag{53}
\]

Since there is a tachyon in the R sector for any finite $u$ one has to check that in the large $u$ limit the imaginary part of the partition function goes to zero so one would have a stable configuration. It is indeed true since the imaginary part in the R sector is of the form $u^{10} e^{-2\pi^2 u/3}/\sqrt{u}$ as $u$ is large.

If there are some Dirichlet directions the formula (53) is still useful. Since $1/\sqrt{-\ln a} = \sqrt{t/\pi}$ the dimension of the brane affects only the constant $\beta$ in (53).

\section{Tachyon potential and tachyon action}

In this section we study the tachyon potential $V(T^2)$. For large values of the tachyon field $T$ the potential has the following behavior

\[
V(T^2) \sim e^{-T^2} \left( 1 + A g_o \int_{T^2/\Lambda}^\infty \frac{da}{a} e^{-a} \right), \tag{54}
\]

where the first term is the contribution from the disk and the second term is the contribution from the annulus. The coupling constant $g_o$ is the open string coupling constant and $A$ is some constant. As before we find two phases. The first one is obtained when $T^2/\Lambda \to 0$ and $T$ is large but finite. In this limit the potential has the form

\[
V(T^2) \sim e^{-T^2} \left( 1 + A g_o \left( -\gamma - \ln \frac{T^2}{\mu} + \ldots \right) \right), \tag{55}
\]
where $\gamma$ is the Euler constant. Above we have added a counter term to subtract off the infinities following our discussion in the previous chapter. The constant $\mu$ is a finite scale that appears when we regulate the theory and subtract the infinite contribution. The sign of the potential depends on the ratio of $T^2/\mu$ and it can be either negative or positive.

In the second phase $T^2/\Lambda \to \infty$ as $T^2 \to \infty$. In this limit the potential has the leading behavior of the form

$$V(T^2) \sim e^{-T^2} \left(1 + A_{0} e^{-T^2/\Lambda} \left(\frac{\Lambda}{T^2} + \ldots\right)\right).$$

As we discussed before no counter term is required in this limit. In both phases the potential vanished when $u \to \infty$.

In ref. [5] it was shown that the linear tachyon profile gives the correct ratio for different D-brane tensions. Following their analysis we will study the decay of a single D9-brane to a BPS D8-brane. In order to get ratio of the D-brane tensions we have to study tachyon action for both small and large $u$. In the large $u$ limit one obtains the BPS D8-brane. The D8 tension does not receive any loop corrections as the one loop partition function vanishes for large $u$.

The D9-brane tension is obtained from the $u \to 0$ limit. The loop correction to the D9-brane tension does not vanish. Since the $(−1)^F$ NS sector does not contribute to the non BPS D-brane tension, we will not consider it below. Before proceeding with our calculation it is useful to collect some of our results.

The potential receives a contribution at tree level $V^{(0)}(T^2)$ and at one loop $V^{(1)}(T^2)$. The tree level contribution is given by

$$V^{(0)}(T^2) = e^{-T^2}$$

and the one loop contribution is given by

$$V^{(1)}(T^2) = \frac{1}{2} \int_0^1 \frac{da}{a} e^{-(1+a)T^2} (\text{NS}(0,a) - \text{R}(0,a)).$$

The tachyon action has the form

$$S = S_0 \left(\frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{u}} Z_D(u) + g_o \frac{1}{2\sqrt{\alpha'}} \frac{1}{\sqrt{u}} Z_D(u) Z_{NS,R}(u)\right),$$

where

$$Z_{NS,R}(u,a) = \int_0^1 \frac{da}{a} \frac{1}{1 + a - ua \ln a} Z_D(au)(\text{NS}(u,a) - \text{R}(u,a))$$

and $S_0$ is a constant proportional to the D9-brane tension.

The D-brane tension can be determined from the non-derivative terms of the effective action. These are controlled by the zero mode structure. On the disk this amounts to replacing the bosonic zero mode factor $\frac{2}{\sqrt{u}}$ by the following expression [6]

$$\sqrt{\frac{2}{u}} = \int_\infty^{-\infty} \frac{dX}{(\alpha'\pi)^{1/2}} e^{-u/(2\alpha')} X^2.$$

For the contribution due to the annulus we have a similar expression

$$\frac{1}{2} \frac{2}{\sqrt{u}} = N \int_{-\infty}^{\infty} \frac{dX}{\sqrt{\alpha'\pi}} V^{(1)}(T^2) = \frac{1}{2} N \int_{-\infty}^{\infty} \frac{dX}{\sqrt{\alpha'\pi}} \int_0^1 \frac{da}{a} e^{-(1+a)u/(2\alpha')} X^2 (\text{NS}(0,a) - \text{R}(0,a)).$$
The normalization constant $N$ is determined as follows

$$N^{-1} = \frac{1}{\sqrt{2}} \int_0^1 \frac{da}{a} Z_{NS,R}(0).$$  \hspace{1cm} (63)$$

Inserting these expressions to the tachyon action we find

$$S = \frac{S_0}{(\pi\alpha')^5} \int d^{10}x \left( \frac{1}{2\sqrt{2}} V^{(0)}(T^2) Z_D(2\alpha'\dot{T}) + g_o \frac{N}{2\pi} V^{(1)}(T^2) Z_{NS, R}(2\alpha'\dot{T}) \right).$$  \hspace{1cm} (64)$$

The factor $\sqrt{2}$ from each $Z_D$ was incorporated in the potential. That is the origin of $1/\sqrt{2}$ in the tree level part (one auxiliary fermion $\eta$) and $1/2$ in the one loop part (two auxiliary fermions). Since the zero mode normalization is fixed by the disk action the extra factor $\sqrt{2}$ in $N$ is needed. Again the contribution from $(-1)^F$ NS sector of the theory is omitted since its presence has no affect to the D9-brane tension. Finally one can look at the behavior of the action in small $u$ limit leaving only the part proportional to the potential

$$S \sim \frac{S_0}{(\pi\alpha')^5} \left( \frac{1}{\sqrt{2}\pi} \int d^{10}x V^{(0)}(T^2) + g_o \sqrt{2} \frac{1}{2\pi} \int d^{10}x V^{(1)}(T^2) \right).$$  \hspace{1cm} (65)$$

The tachyon action must be of the form

$$S = \int d^{10}x (T_9^{(0)} V^{(0)} + g_o T_9^{(1)} V^{(1)}).$$  \hspace{1cm} (66)$$

Comparing it with our expression for the action it is easy to find that

$$\frac{T_9^{(0)}}{T_9^{(1)}} = \sqrt{\pi},$$  \hspace{1cm} (67)$$

which has a clear explanation. All one loop behavior is accumulated in the potential $V^{(1)}(T^2)$. The only difference from the disk case is factors $1/\sqrt{2}\pi$ and $\sqrt{2}$. The first comes from the extra set of boundary modes for the annulus, the second is from the extra auxiliary fermion compare to the disk case. To get an expression for the loop corrected D-brane tension one has to set $T = 0$

$$T_9 = T_9^{(0)} \left( 1 + \frac{1}{\sqrt{\pi}} g_o \int_0^1 \frac{da}{a} (NS(0, a) - R(0, a)) \right),$$  \hspace{1cm} (68)$$

which is as expected proportional to the partition function of non BPS D9-branes\[25\].

7 Multiple D-branes

So far we have considered tachyon profiles with a single D-brane. In this section we will generalize our discussion by including several unstable D-branes. The tachyon profile for these configurations will be represented in terms of a hermitian matrix. The supersymmetric boundary interaction involves several boundary fermions $\eta^I$’s. The index $I$ labels the Chan-Patton (CP) factor and for each CP-factor we have a boundary fermion.

Let us concentrate on the R-R sector. The boundary fermions play an important role in determining whether the R-R sector is present. To see how this works let us first consider a single D-brane with
a tachyon profile $T_a(X) = y_a X$ on the inner boundary and $T_b(X) = y_b X$ on the outer boundary. In the R-R sector the boundary fermions $\eta_{a,b}$ are periodic and hence have zero modes. For a vanishing tachyon profile the functional integral over the boundary fermions vanishes because of the zero modes. When the tachyon profile is non-vanishing the interaction term $\psi \eta \partial T$ will soak up the zero modes. The functional integral over the boundary fermions gives a term proportional to $y^a y^b$. Next let us consider a more complicated example of two non BPS D9 branes in the type IIB theory. These D9 branes can decay into a single D7 brane that then can decay into a BPS D6-brane. In this case we have three CP factors and the tachyon profile has the form \[ T(X) = \sum_{I=1}^{3} y_I X^I \gamma^I = \sum_{I=1}^{3} T^I(X) \gamma^I, \] where $\gamma^I$'s are hermitian and they satisfy a Clifford algebra \{$\gamma^I, \gamma^J$\} = $2\delta^{IJ}$. The boundary interaction has now the form \[ -\eta^I \dot{\eta}^I - \psi^\mu \partial_\mu T^I(X) \eta^I. \]

On the annulus we have two sets of $\eta^I$'s, one for each boundary. In order to get a non-zero result when integrating over boundary fermions the product of the $\eta^I$'s has to appear. The functional integral will then give a term proportional to $y_1^2 y_2^2 y_3^2$. From this it is obvious that the partition function in the R-R sector vanishes if one of the $y_1^I$'s is zero.

If we set $y_3 = 0$ and take $y_1$ and $y_2$ to infinity this describes the decay of a D9 brane to a non-BPS D7 brane. The final configuration is non BPS so the R-R sector is absent and there is no coupling to the R-R fields.

The non BPS D-branes are very similar to D-\bar{D} systems. In this case tachyon is complex. Consider tachyon condensation in a system of D8-\bar{D}8 branes in type IIA. The system decays to a single D6-brane. For this system we have two CP factors and the tachyon profile is given by

\[ T(X) = \begin{pmatrix} 0 & \sqrt{u} X^9 + i \sqrt{v} X^8 \\ \sqrt{u} X^9 - i \sqrt{v} X^8 & 0 \end{pmatrix} = T_1(X) \sigma_1 + T_2(X) \sigma_2. \]

There are no off diagonal entries in this tachyon profile. The boundary interaction is almost the same as in \[ (70) \]. The only difference is that instead of three boundary fermions we now have only two boundary fermions $\eta^I$'s on each boundary. A BPS D6 brane is obtained in the limit $u \to \infty$ and $v \to \infty$. The R-R sector is non-vanishing and the D6 brane couples to the R-R fields. If we set $v = 0$ and let $u \to \infty$, the R-R sector is absent and the final state describes a non-BPS D7-brane.

The discussion can be easily generalized to any number of D-branes and anti D-branes. The only difference is that the particular form of the boundary interaction \[ (70) \] is different \[ (17) \].

8 Tachyon condensation for codimension four

In this section we will revisit the phase transition we encountered previously, this time in codimension four. Recall that the first phase occurs when $u$ is small compared to the cutoff $\Lambda$. In this phase divergences are cancelled by a dilaton vertex operator at the origin of the disk amplitude. In the R-R sector the partition diverges when $u \to \infty$. This infinity is known as R-R anomaly. For a BPS state the R-R anomaly is cancelled by a tadpole divergence in the NS-NS sector. In the first phase tadpole divergences
are cancelled by the FS mechanism. One can see that there is a phase transition since for large $u$ the two point function $\langle \Psi \bar{\Psi} \rangle$ is non-vanishing for a single D-brane.

In the second phase we first let $u \to \infty$ for finite $\Lambda$. In this phase there are no divergences and hence no need for FS mechanism. In this case we have an almost BPS state and the NS-NS and R-R divergences cancel each other.

The dilaton vertex operator gives a non-zero contribution in the large $u$ limit, except in codimension four. To see this let us consider a tachyon profile in $p$ spatial directions. The NS-NS divergence is proportional to

$$8 - p + \sum_{i=1}^{p} \frac{1 - 2u_i}{1 + 2u_i}.$$  \hfill (72)

For large $u_i$'s and $p = 4$ we have

$$\langle \Psi^{\mu}(0) \bar{\Psi}_{\mu}(0) \rangle = 0.$$  \hfill (73)

This means that for large $u_i$'s we approach a BPS state with no contribution from the dilaton operator.

Next let us consider an example with $p = 4$. In the type IIA theory $2(D8-\bar{D}8)$ system will decay into a stable BPS D4-brane. We have a dilaton vertex operator at the origin. As we take the limit $u_i \to \infty$ in all four directions the contribution from the dilaton vanishes and there is no phase transition.

Similar statements hold true for a decay of $2(D9-\bar{D}9)$ and $2(D7-\bar{D}7)$ in type IIB. As mentioned above D-branes with number of Dirichlet directions greater than two do not have tadpole divergence.

9 Conclusion

In this paper we have analyzed boundary string field theory at the one loop level. We have argued that the tachyon profiles on the inner and the outer circles of the annulus must be the same. We find that this is consistent with large $u$ limit and the FS mechanism. We have analyzed tadpole divergences in superstring theory and showed that for a linear tachyon profile the tadpole divergence is cancelled by the FS mechanism. For D9, D8 and D7 branes we find two different phases. One phase has a finite scale and an insertion of the dilaton operator at the origin of the disk amplitude. This phase is an almost non-BPS phase. The other phase is an almost BPS phase. The phase does not have any scale or any divergences and no counterterms are required. For branes of codimension bigger then 2 there is no tadpole divergence and there is no phase transition. The tachyon condensation for D9, D8 and D7 branes can also be viewed as a smooth process if the number of dimensions being reduced is equal to four.

We have studied the open string limit of the annulus partition function. If the tachyon profile is not zero the partition function for the annulus does not coincide with the partition function for the cylinder, and one cannot use modular properties to obtain the open string limit. Nevertheless, it is possible to find the leading behavior of the superstring partition function in this limit. For finite value of $u$ the open string tachyon disappears in the NS and $(-1)^F$ NS sectors. On the other hand we find an open string tachyon at any non-zero value of $u$ in the R sector. Associated with the tachyon there is a divergence when $a$ is close to the boundary of modula space $a \sim 1$. One can treat this divergence by analytic continuation. The price to be paid is that the action for the tachyon becomes complex.

We find that the loop corrected tachyon potential has a logarithmic term. This might have applications to cosmology of the rolling tachyon since Wick’s rotation of the time direction does not change the form of the potential\cite{footnote1}. We also comment on how the boundary fermions $\eta^I$’s switch off and on the R-R part of the partition function.
Our calculations have shown that tachyon condensation at one loop depends on how many Dirichlet directions are present and how many dimensions are reduced. These differences are related to the one loop divergences and therefore cannot be seen at tree level. It would be interesting to see if these can give rise to a new "loop corrected" K-theory?

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A Calculation of the bosonic Green function

The most general harmonic function on the annulus is

\[ X_{cl}(\rho, \theta) = X_0 + k \ln \rho + \sqrt{\alpha'} \sum_{n=1}^{\infty} \left( \rho^n (C_n e^{i n \theta} + C_{-n} e^{-i n \theta}) + \rho^{-n} (D_n e^{i n \theta} + D_{-n} e^{-i n \theta}) \right). \tag{74} \]

The reality condition gives \( C_n^* = C_{-n}, D_n^* = D_{-n} \). On the boundary harmonic function is given by

\[ X_{cl}(\rho = a) = X_a + \sqrt{\alpha'} \sum_{n=1}^{\infty} (A_n e^{i n \theta} + A_{-n} e^{-i n \theta}), \tag{75} \]

\[ X_{cl}(\rho = 1) = X_b + \sqrt{\alpha'} \sum_{n=1}^{\infty} (B_n e^{i n \theta} + B_{-n} e^{-i n \theta}) \tag{76} \]

with \( A_n^* = A_{-n}, B_n^* = B_{-n} \). Then the relations between \( X_{cl} \) in the bulk and its values on the boundaries are

\[ C_{\pm n} = \frac{B_{\pm n} - A_{\pm n} a^n}{1 - a^{2n}}, \tag{77} \]

\[ D_{\pm n} = a^n \frac{A_{\pm n} - a^n B_{\pm}}{1 - a^{2n}}, \tag{78} \]

\[ k = \frac{-X_b - X_a}{\ln a}, \tag{79} \]

\[ X_0 = X_b. \tag{80} \]

The action can be determined in terms of boundary modes \( A_{\pm n} \)'s and \( B_{\pm n} \)'s. A simple calculation gives

\[ S[X_{cl}] = -\frac{(X_b - X_a)^2}{2a' \ln a} + \frac{u}{2a'} (X_b^2 + a X_a^2) + \]

\[ + \frac{1}{2} \sum_{n=1}^{\infty} [(n R_n(a) + u) B_n B_{-n} + (n R_n(a) + au) A_n A_{-n}] - \]

\[ - \frac{1}{2} \sum_{n=1}^{\infty} n K_n(a) (B_n A_{-n} + A_n B_{-n}), \]
where
\[ R_n(a) = \frac{1 + a^{2n}}{1 - a^{2n}}, \quad (81) \]
\[ K_n(a) = \frac{a^n}{1 - a^{2n}}, \quad (82) \]

The last step is to integrate over the boundary modes and find the Green function on the boundary
\[ G(\theta, 0) = \int \prod_{n=1}^{\infty} dA_n dB_n e^{-S[X_{cl}]} X_{cl}(\theta) X_{cl}(0) / \int \prod_{n=1}^{\infty} dA_n dB_n e^{-S[X_{cl}]} . \quad (83) \]

Consider contribution from zero modes \( X_a \) and \( X_b \)
\[ \frac{1}{2\alpha'} G_0(\rho = 1) \equiv B_0(u, a) + ... = \frac{1}{2u} \left( \frac{1 - ua \ln a}{1 + a - ua \ln a} \right) + ... \quad (84) \]
\[ \frac{1}{2\alpha'} G_0(\rho = a) \equiv A_0(u, a) + ... = \frac{1}{2u} \left( \frac{1 - u \ln a}{1 + a - ua \ln a} \right) + ... \quad (85) \]

The zero mode contribution controls the behavior of the partition function for small values of \( u \). To find the full Green function one has to integrate over all other boundary modes. With zero modes omitted the Green function is
\[ G(\theta, 0)|_{\rho = a} = 2\alpha' \sum_{n=1}^{\infty} A_n(u, a) \cos n\theta, \quad (86) \]
\[ G(\theta, 0)|_{\rho = b} = 2\alpha' \sum_{n=1}^{\infty} B_n(u, a) \cos n\theta, \quad (87) \]

where
\[ A_n(u, a) = \frac{nR_n(a) + u}{n^2 + u(1 + a)nR_n(a) + u^2a}, \quad (88) \]
\[ B_n(u, a) = \frac{nR_n(a) + ua}{n^2 + u(1 + a)nR_n(a) + u^2a}. \quad (89) \]

We can generalize this Green function to the case where we have different boundary conditions on the two boundaries
\[ A_n(u, a) = \frac{nR_n(a) + u}{n^2 + (u + v)nR_n(a) + uv}, \quad (90) \]
\[ B_n(u, a) = \frac{nR_n(a) + v}{n^2 + (u + v)nR_n(a) + uv}. \quad (91) \]

The parameter \( u \) corresponds to the outer boundary and the parameter \( v \) corresponds to the inner boundary. When we have the same tachyon profile on the two boundaries \( v = au \).
B  Open string limit for functions $\ln f(u, a)$

Let us start with the R sector. We want to find the leading behavior of the following function

$$
\sum_{s=1/2}^{\infty} \ln \left(1 - a^{2s} \frac{1 - u/s}{1 + u/s} \frac{1 - au/s}{1 + au/s} \right) - \sum_{n=1}^{\infty} \ln \left(1 - a^{2n} \frac{1 - u/n}{1 + u/n} \frac{1 - au/n}{1 + au/n} \right) - \sum_{s=1/2}^{\infty} \ln(1 - a^{2s}) + \sum_{n=1}^{\infty} \ln(1 - a^{2n})
$$

in the limit $a \to 1$. The sum over half integers can be written in more convenient form

$$
\sum_{s=1/2}^{\infty} \frac{1}{2} \ln \left(1 - a^{2s} \frac{1 - u/s}{1 + u/s} \frac{1 - au/s}{1 + au/s} \right)
$$

Let us try to expand the sum (95) in powers of $u/n$. The zeroth order term of this expansion is cancelled by ln’s expansion from the second line of (95). The second order term is proportional to $1/n^2$. The sum $\sum_n a^{nk} / n^2$ is convergent at $a = 1$. The higher order terms have even better convergence property. Therefore the only divergence in the sum at $a = 1$ comes from the linear $u/n$ term. The sum is of the form

$$
4u(1 + a) \sum_{n=1}^{\infty} \frac{1}{n} \left( \frac{1}{1 - a^n} - \frac{1}{1 - a^{2n}} \right).
$$

When $a$ is close to 1 one can write $a$ as $1 - \epsilon$. Since $a = e^{-\pi/t}$ where $t$ is the open string time, $\epsilon$ corresponds to $\pi/t$. We are interested in divergent and constant parts. Extracting these we find

$$
4u \sum_{n=1}^{\infty} \frac{1}{n} \left( \frac{1}{\epsilon - 1} \right) = 4u \zeta(2) \left( \frac{1}{\epsilon - 1} \right).
$$

To find the finite part we set $a = 1$ in (92) and throw away the divergent contribution leaving us with the finite part

$$
\sum_{n=1}^{\infty} (-2 \ln(1 + 2u/n) + 4 \ln(1 + u/n)) = - \ln Z_{D}^{2}(u).
$$

This last step is not quite rigorous since the right hand side contains linear in $u$ terms but this is the natural regularization of the sum on the left hand side. In the R sector the leading contribution is

$$
- \ln Z_{D}(u) + 4u \frac{\pi^2}{6} \left( \frac{1}{\epsilon - 1} \right),
$$

where we have used $\zeta(2) = \pi^2/6$. The same procedure for NS sector gives the leading behavior

$$
- \ln Z_{D}^{2}(u) + \sum_{n=1}^{\infty} \ln(1 + (u/n)^2) - 2u \frac{\pi^2}{6} \left( \frac{1}{\epsilon - 1} \right).
$$
In the $(-1)^F$ NS-sector we find
\[ \sum_{n=1}^{\infty} \ln \frac{1 + (2u/n)^2}{1 + (u/n)^2} - \frac{2\pi^2}{6} \left( \frac{1}{\epsilon} - 1 \right). \] (101)

To express the infinite products that appear in the above formulas one needs
\[ \prod_{n=1}^{\infty} (1 + (u/n)^2) = \frac{\sinh \pi u}{\pi u}. \] (102)

Performing the calculations outlined above it is possible to loose some constant factors that do not depend on either $\epsilon$ or $u$. To fix these terms one has to look at the small $u$ behavior which is easy to analyze in the open string limit.

We want to stress a feature that enables us to calculate the leading behavior. The terms of order $O(\epsilon^0)$ are finite, they still can be summed to $\zeta(2)$. In the case of bosonic strings this is not true. If one is looking at $O(\epsilon^0)$ terms there is a sum of the form $\sum n / n$. For superstring the sums from bosons and fermions cancel against each other.

References

[1] E. Witten, "Noncommutative geometry and string field theory“, Nucl. Phys. B268 (1986) 253.
[2] E. Witten, "On background independent open string field theory“, Phys. Rev. D46 (1992) 5467-5473, "Some Computations in Background Independent Open-String Field Theory“, Phys.Rev. D47 (1993) 3405-3410.
[3] D. Kutasov, M. Marino and G. Moore, "Some Exact Results on Tachyon Condensation in String Field Theory“, JHEP 0010 (2000) 045
[4] A. Gerasimov, S. Shatashvili, "On Exact Tachyon Potential in Open String Field Theory“, JHEP 0010 (2000) 034
[5] D. Kutasov, M. Marino and G. Moore, "Remarks on Tachyon Condensation in Superstring Field Theory“, hep-th/0010108
[6] V. Niarchos and N. Prezas, "Boundary Superstring Field Theory“, Nucl.Phys. B619 (2001) 51, hep-th/0103102
[7] J. A. Harvey, D. Kutasov, E. J. Martinec, "On the relevance of tachyons“, hep-th/0003101
[8] O. Andreev, T. Ott, "On One-Loop Approximation to Tachyon Potentials“, Nucl.Phys. B627 (2002) 330-356, hep-th/0109187
[9] Wung-Hong Huang, "Boundary String Field Theory Approach to High-Temperature Tachyon Potential“, hep-th/0106002
[10] K. Bardakci and A. Konechny, "Tachyon condensation in boundary string field theory at one loop“, hep-th/0105098
[11] T. Suyama, "Tachyon Condensation and Spectrum of Strings on D-branes\textsuperscript{\textcopyright}, hep-th/0102192.
K. S. Viswanathan, Y. Yang, "Tachyon Condensation and Background Independent Superstring Field Theory\textsuperscript{\textcopyright}, hep-th/0104099.
R. Rashkov, K. S. Viswanathan, Y. Yang, "Background Independent Open String Field Theory with Constant B field On the Annulus\textsuperscript{\textcopyright}, hep-th/0101207.
M. Alishahiha, "One-loop Correction of the Tachyon Action in Boundary Superstring Field Theory\textsuperscript{\textcopyright}, hep-th/0104164.

[12] G. Arutyunov, A. Pankiewicz, B. Stefanski, "Boundary Superstring Field Theory Annulus Partition Function in the Presence of Tachyons\textsuperscript{\textcopyright}, JHEP 0106 (2001) 049, hep-th/0105238.

[13] G. Gibbons, "Cosmological Evolution of the Rolling Tachyon\textsuperscript{\textcopyright}, Phys.Lett. B537 (2002) 1-4, hep-th/0204008.
S. Mukohyama, "Brane cosmology driven by the rolling tachyon\textsuperscript{\textcopyright}, Phys.Rev. D66 (2002) 024009, hep-th/0204084.
A. Feinstein, "Power-Law Inflation from the Rolling Tachyon\textsuperscript{\textcopyright}, Phys.Rev. D66 (2002) 063511, hep-th/0204140.

[14] A. Sen, "Rolling Tachyon\textsuperscript{\textcopyright}, JHEP 0204 (2002) 048, hep-th/0203211.
"Tachyon Matter\textsuperscript{\textcopyright}, JHEP 0207 (2002) 065, hep-th/0203265.

[15] K. Bardakci, "Dual Models and Spontaneous Symmetry Breaking\textsuperscript{\textcopyright}, Nucl.Phys. B68 (1974) 331, "Spontaneous Symmetry Breakdown in the Standard Dual String Model\textsuperscript{\textcopyright}, Nucl.Phys.B133 (1978) 297, K.Bardakci and M.B.Halpern, "Explicit Spontaneous Breakdown in a Dual Model\textsuperscript{\textcopyright}, Phys.Rev. D10 (1974) 4230, "Explicit Spontaneous Breakdown in a Dual Model II: N Point Functions\textsuperscript{\textcopyright}, Nucl. Phys. B96 (1975) 285.

[16] S. Shatashvili, "On the Problems with Background Independence in String Theory\textsuperscript{\textcopyright}, Alg.Anal. 6 (1994) 215-226, hep-th/9311177.
"Comment on the Background Independent Open String Theory\textsuperscript{\textcopyright}, Phys.Lett. B311 (1993) 83-86, hep-th/9303143.

[17] P. Kraus and F. Larsen, "Boundary String Field Theory of the DDBar System\textsuperscript{\textcopyright}, Phys.Rev. D63 (2001) 106004, hep-th/0012198.

[18] E. Witten, "D-Branes And K-Theory\textsuperscript{\textcopyright}, JHEP 9812 (1998) 019, hep-th/9810188.

[19] P. Horava, "Type IIA D-Branes, K-Theory, and Matrix Theory\textsuperscript{\textcopyright}, Adv.Theor.Math.Phys. 2 (1999) 1373-1404, hep-th/9812135.

[20] W. Fischler and L. Susskind, "Dilaton Tadpoles, String Condensates and Scale Invariance\textsuperscript{\textcopyright}, Phys. Lett. B171 (1986) 383, "Dilaton Tadpoles, String Condensates and Scale invariance. 2\textsuperscript{\textcopyright}, Phys. Lett. B173 (1986) 262.

[21] Curtis G. Callan,Jr., C. Lovelace, C.R. Nappi, S.A. Yost, "String Loop Corrections to Beta Functions\textsuperscript{\textcopyright}, Nucl. Phys. B288 (1987) 525, "Adding Holes and Crosscaps to the Superstring\textsuperscript{\textcopyright}, Nucl.Phys. B293 (1987) 83, S. R. Das, S-J. Rey, "Dilaton Condensates and Loop Effects in Open and Closed Bosonic Strings\textsuperscript{\textcopyright}, Phys.Lett. B186 (1987) 328.

[22] E. T. Akhmedov, M. Laidlaw, G. W. Semenoff, " On a Modification of the Boundary State Formalism in Off-shell String Theory\textsuperscript{\textcopyright}, JETP Lett. 77 (2003) 1-6, hep-th/0106033.

[23] O. Andreev, "A Note on Nonperturbative Instability in String Theory\textsuperscript{\textcopyright}, Phys.Lett. B534 (2002) 163-166, hep-th/0112088.
[24] N. Marcus, "Unitarity and regularized divergences in string amplitudes", Phys. Lett. 219B 1989, 265

[25] N. D. Lambert and I. Sachs, "String Loop Corrections to Stable Non-BPS Branes", JHEP 0102 (2001) 018, hep-th/0010045