Using $p$-row graphs to study $p$-competition graphs

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Abstract

For a positive integer $p$, the $p$-competition graph of a digraph $D$ is a graph which has the same vertex set as $D$ and an edge between distinct vertices $x$ and $y$ if and only if $x$ and $y$ have at least $p$ common out-neighbors in $D$. A graph is said to be a $p$-competition graph if it is the $p$-competition graph of a digraph. Given a graph $G$, we call the set of positive integers $p$ such that $G$ is a $p$-competition the competition-realizer of a graph $G$. In this paper, we introduce the notion of $p$-row graph of a matrix which generalizes the existing notion of row graph. We call the graph obtained from a graph $G$ by identifying each pair of adjacent vertices which share the same closed neighborhood the condensation of $G$. Using the notions of $p$-row graph and condensation of a graph, we study competition-realizers for various graphs to extend results given by Kim et al. [p-competition graphs, Linear Algebra Appl. 217 (1995) 167–178]. Especially, we find all the elements in the competition-realizer for each caterpillar.

Keywords: $p$-competition graph; $p$-edge clique cover; competition-realizer; $p$-row graph; condensation of a graph; caterpillar

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1 Introduction

Given a digraph $D = (V, A)$, the competition graph of $D$ is a simple graph having the same vertex set as $D$ and having an edge $uv$ if for some vertex $x \in V$, the arcs $(u, x)$ and $(v, x)$ are in $D$. The notion of competition graph is due to Cohen [1] and has arisen from ecology. Competition graphs also have applications in coding, radio transmission, and modelling of complex economic systems (see [13] and [14] for a summary of these applications and [4] for a sample paper on the modelling application). Since Cohen introduced the notion
of competition graph, various variations have been defined and studied by many authors (see the survey articles by Kim [6] and Lundgren [11]). For recent work on this topic, see [3, 5, 9, 10, 15].

Kim et al. [7] introduced \( p \)-competition graphs as a variant of competition graph. For a positive integer \( p \), the \( p \)-competition graph \( C_p(D) \) of a digraph \( D = (V, A) \) is defined to have the vertex set \( V \) with an edge between two distinct vertices \( x \) and \( y \) if and only if, for some distinct vertices \( a_1, \ldots, a_p \) in \( V \), the pairs \( (x, a_1) \), \( (y, a_1) \), \( (x, a_2) \), \( (y, a_2) \), \ldots, \( (x, a_p) \), \( (y, a_p) \) are arcs in \( D \). Note that \( C_1(D) \) is the ordinary competition graph, which implies that the notion of \( p \)-competition graph generalizes that of competition graph. A graph \( G \) is called a \( p \)-competition graph if there exists a digraph \( D \) such that \( G = C_p(D) \).

By definition, it is obvious that if a nonempty graph \( G \) is a \( p \)-competition graph, then \( p \leq |V(G)| \).

Competition graphs are closely related to edge clique covers and the edge clique cover numbers of graphs. A clique of a graph \( G \) is a subset of the vertex set of \( G \) that induces a complete graph. We regard an empty set as a clique of \( G \) for convenience. An edge clique cover of a graph \( G \) is a family of cliques of \( G \) such that the end vertices of each edge of \( G \) are contained in a clique in the family. The minimum size of an edge clique cover of \( G \) is called the edge clique cover number of \( G \), and is denoted by \( \theta_e(G) \). Dutton and Brigham [2] characterized a competition graph in terms of its edge clique cover number.

**Theorem 1.1 ([2]).** A graph \( G \) with \( n \) vertices is a competition graph if and only if \( \theta_e(G) \leq n \).

A \( p \)-competition graph \( G \) can be characterized in terms of the “\( p \)-edge clique cover number” of \( G \). For a positive integer \( p \), a \( p \)-edge clique cover (\( p \)-ECC for short) of a graph \( G \) is defined to be a multifamily \( F = \{ F_1, \ldots, F_r \} \) of subsets of the vertex set of \( G \) satisfying the following:

- For any \( J \in \binom{[r]}{p} \), the set \( \bigcap_{j \in J} F_j \) is a clique of \( G \);
- The collection \( \left\{ \bigcap_{j \in J} F_j \mid J \in \binom{[r]}{p} \right\} \) covers all the edges of \( G \),

where \( \binom{[r]}{p} \) denotes the set of \( p \)-element subsets of the set \( \{1, \ldots, r\} \) for a positive integer \( r \). The minimum size \( r \) of a \( p \)-edge clique cover of \( G \) is called the \( p \)-edge clique cover number of \( G \), and is denoted by \( \theta^p_e(G) \). The following theorem characterizes \( p \)-competition graphs and so generalizes Theorem 1.1.

**Theorem 1.2 ([7]).** A graph \( G \) with \( n \) vertices is a \( p \)-competition graph if and only if \( \theta^p_e(G) \leq n \).

In this paper, we introduce the notion of \( p \)-row graph of a matrix which generalizes the existing notion of row graph of a matrix. We also introduce the notion of the condensation of a graph that is obtained from a graph by identifying each pair of adjacent vertices.
which share the same closed neighborhood. Using these notions, we study competition-
realizers for various graphs to extend results given by Kim et al. [p-competition graphs,
Linear Algebra Appl. 217 (1995) 167–178]. Especially, we find all the elements in the
competition-realizer for each caterpillar.

2 p-row graphs and competition-realizers

In this section, we introduce the notion of p-row graph of a matrix which generalize
the notion of row graph of a matrix and the notion of competition-realizer for a graph.
Then we study competition-realizers for various graphs in terms of p-row graph and the
condensation of a graph. Particularly, we identify the graphs with n vertices whose
competition-realizers contain n and n − 1, respectively.

Definition 2.1. Given a positive integer p and a (0,1)-matrix A, a graph G is called the
p-row graph of A if the vertices of G are the rows of A, and two vertices are adjacent in G
if and only if their corresponding rows have common nonzero entries in at least p columns
of A.

If p = 1, then G is called the row graph of A, which was introduced by Greenberg et
al. [4].

Suppose that a graph G is a p-competition graph with the vertex set \( \{v_1, \ldots, v_n\} \). Then there exists a p-ECC \( \mathcal{F} = \{F_1, \ldots, F_n\} \) of G for a nonnegative integer \( m \leq n \) by
Theorem 1.2. Now we define a square matrix \( A = (a_{ij}) \) of order n by

\[
a_{ij} = \begin{cases} 
1 & \text{if } v_i \in F_j; \\
0 & \text{otherwise.} 
\end{cases}
\] (1)

By the definition of p-ECC, it is easy to see that G is isomorphic to the p-row graph of
A. Conversely, suppose that a graph G with n vertices is isomorphic to the p-row graph
of a square (0,1)-matrix A of order n. Let

\[
F_j = \{v_i \mid a_{ij} = 1\}.
\]

and let \( \mathcal{F} = \{F_1, \ldots, F_n\} \). By the definition of p-row graph, a vertex \( v_s \) and a vertex \( v_t \)
are adjacent if and only if the sth row and the tth row of A have common nonzero entries
in at least p columns, which is equivalent to the statement that the pair of vertices \( v_s \) and
\( v_t \) is contained in the sets in \( \mathcal{F} \) corresponding to those columns. Then, by Theorem 1.2
G is a p-competition graph.

Now we have shown the following statement:

Theorem 2.2. A graph G with n vertices is a p-competition graph if and only if G is
isomorphic to the p-row graph of a square (0,1)-matrix of order n.
For simplicity’s sake, we denote \( J_{m,n} \) for the \((0,1)\)-matrix of size \( m \) by \( n \) such that every entry is 1, \( I_n \) for the identity matrix of order \( n \), and \( O_{m,n} \) for the zero matrix of size \( m \) by \( n \).

For a graph \( G \) with \( n \) vertices, we denote the set

\[ \{ p \in [n] \mid G \text{ is a } p\text{-competition graph} \} \]

by \( \Upsilon(G) \) and call it the *competition-realizer* for \( G \).

We make the following simple but useful observations.

**Proposition 2.3.** Let \( G \) be a graph with \( n \) vertices. If \( G \) is empty or complete, then \( \Upsilon(G) = [n] \).

**Proof.** If \( G \) is empty, then \( G \) is a \( p \)-row graph of \( O_{n,n} \) and so, by Theorem 2.2, is a \( p \)-competition graph for any \( p \in [n] \). If \( G \) is complete, then \( G \) is a \( p \)-row graph of \( J_{n,n} \) and so, by the same theorem, is a \( p \)-competition graph for any \( p \in [n] \).

**Proposition 2.4.** Given a graph \( G \) with \( n \) vertices, suppose that \( G \) is a \( p \)-row graph of a matrix of size \( n \) by \( m \) for positive integers \( p \) and \( m \leq n \). Then \( \Upsilon(G) \supset \{ p + i \mid i \in [n - m] \cup \{0\} \} \).

**Proof.** Let \( M \) be an \( n \times m \) matrix whose \( p \)-row graph is \( G \). Take \( i \in [n - m] \cup \{0\} \). Then we add \( i \) all-one columns and \( n - m - i \) all-zero columns to \( M \) to obtain a square matrix of order \( n \) whose \((p+i)\)-row graph is \( G \). By Theorem 2.2 \( G \) is a \((p+i)\)-competition graph and so \( p + i \in \Upsilon(G) \).

**Proposition 2.5.** Let \( G \) be a graph and \( G' \) be a graph obtained from \( G \) by adding \( k \) isolated vertices for a nonnegative integer \( k \). Then \( \Upsilon(G') \supset \{ p + i \mid i \in [k] \cup \{0\} \} \) for each \( p \in \Upsilon(G) \).

**Proof.** Let \( |V(G)| = n \) and suppose \( p \in \Upsilon(G) \). By Theorem 2.2 \( G \) is a \( p \)-row graph of a square \((0,1)\)-matrix \( M \) of order \( n \). Fix \( i \in [k] \cup \{0\} \). We define a square \((0,1)\)-matrix \( M_i \) of order \( n + k \) as follows:

\[
M_i = \begin{bmatrix}
M & J_{n,i} & O_{n,k-i} \\
\hline
\hline
O_{k,n+k}
\end{bmatrix}.
\]

Obviously, the \((p+i)\)-row graph of \( M_i \) is \( G \) together with \( k \) isolated vertices. By Theorem 2.2 \( p + i \in \Upsilon(G') \).

For a \( p \)-row graph \( G \) of a matrix \( M \) and a vertex \( u \) of \( G \), we let

\[ \Lambda_M(u) = \{ i \mid \text{the } i\text{th component of the row corresponding to } u \text{ in } M \text{ is } 1 \} \]

A vertex \( v \) of a graph \( G \) is called a *simplicial vertex* if its neighbors form a clique in \( G \).
Proposition 2.6. Let $G$ be a $p$-row graph of a matrix $M$. Then, for a non-isolated non-simplicial vertex $u$, $|\Lambda_M(u)| \geq p + 1$.

Proof. By the condition on $u$, $u$ is adjacent to two nonadjacent vertices $v$ and $w$. Then

$$p \leq |\Lambda_M(u) \cap \Lambda_M(v)| \quad \text{and} \quad p \leq |\Lambda_M(u) \cap \Lambda_M(w)|.$$ 

Suppose that $|\Lambda_M(u)| \leq p$. Then

$$|\Lambda_M(u) \cap \Lambda_M(v)| \leq p \quad \text{and} \quad |\Lambda_M(u) \cap \Lambda_M(w)| \leq p.$$ 

Thus $|\Lambda_M(u) \cap \Lambda_M(v)| = |\Lambda_M(u) \cap \Lambda_M(w)| = p$ and so $\Lambda_M(u) \cap \Lambda_M(v) = \Lambda_M(u) \cap \Lambda_M(w) = \Lambda_M(u)$. Hence $\Lambda_M(u) \subset \Lambda_M(v) \cap \Lambda_M(w)$ and so $|\Lambda_M(v) \cap \Lambda_M(w)| \geq p$, which is a contradiction. $\square$

The following proposition characterizes a graph $G$ with $n$ vertices and $n \in \Upsilon(G)$.

We denote a set of $m$ isolated vertices by $I_m$. Technically, we let $I_0 = \emptyset$ and $K_0 = \emptyset$.

Proposition 2.7. Let $G$ be a graph with $n$ vertices. Then $G$ is an $n$-competition graph if and only if $G \cong K_m \cup I_{n-m}$ for some integer $m \in \{0, \ldots, n\}$.

Proof. By definition, $G$ is an $n$-competition graph if and only if $n \in \Upsilon(G)$. To show the “if” part, suppose that $G \cong K_m \cup I_{n-m}$ for some integer $m$, $0 \leq m \leq n$. By Proposition 2.3, $m \in \Upsilon(K_m)$. By Proposition 2.5, $m + (n - m) \in \Upsilon(G)$. To show the “only if” part, suppose that $G$ is an $n$-competition graph. Then $G$ is isomorphic to the $n$-row graph of a matrix $M$ of order $n$ by Theorem 2.2. Suppose that $u$ is a non-isolated vertex in $G$. Then $u$ is adjacent to a vertex $v$ in $G$, so $|\Lambda_M(u) \cap \Lambda_M(v)| = n$. Thus we may conclude that each row of $M$ corresponding to a non-isolated vertex is the all-one vector in $\mathbb{R}^n$. Thus the subgraph of $G$ induced by non-isolated vertices is a clique. Hence $G = K_m \cup I_{n-m}$ where $m$ is the number of non-isolated vertices in $G$. $\square$

Corollary 2.8. Suppose that a graph $G$ with $n$ vertices has no isolated vertices. Then $G$ is an $n$-competition graph if and only if $G = K_n$.

Corollary 2.9. Let $G$ be a graph with $n$ vertices. Then $\Upsilon(G) = [n]$ if and only if $G \cong K_m \cup I_{n-m}$ for some $m \in \{0, \ldots, n\}$.

Proof. The “only if” part immediately follows by Proposition 2.7. To show the “if” part, suppose that $G \cong K_m \cup I_{n-m}$ for some $m$, $0 \leq m \leq n$. Let $M$ be a square $(0,1)$-matrix of order $n$ such that the first $m$ rows are all-one vector and the other $n-m$ rows are all-zero vector. Then it is easy to check that the $p$-row graph of $M$ is isomorphic to $G$ for each $p \in [n]$. $\square$

Corollary 2.10. If $G$ is a disjoint union of complete graphs with $n$ vertices, then $\Upsilon(G) \supseteq [n-1]$. 

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Proof. Let $X_1, \ldots, X_k$ be the complete components of $G$. Then $k \leq n$. Now fix $i \in [n-1]$. Then $k \leq \binom{n}{i}$. Let $S_1, \ldots, S_k$ be distinct $i$-subsets of $[n]$. Let $M$ be a matrix of order $n$ satisfying $\Lambda_M(v) = [n] \setminus S_j$ for each vertex $v$ in $X_j$ for each $j \in [k]$. Then it is easy to check that $G$ is the $(n-i)$-row graph of $M$ and so $n-i \not\in \Upsilon(G)$ by Theorem 2.2. \hfill \Box

Example 2.11. The competition-realizer may be empty for some graph. For example, for the complete bipartite graph $K_{3,3}$, $\Upsilon(K_{3,3}) = \emptyset$. To see why, we note that the number of vertices of $K_{3,3}$ is 6 and $\theta_K(K_{3,3}) = 9$. Therefore $1 \not\in \Upsilon(K_{3,3})$ by Theorem 1.2. By Proposition 2.6 $5 \not\in \Upsilon(K_{3,3})$ and $6 \not\in \Upsilon(K_{3,3})$. Suppose that $K_{3,3}$ is a $p$-competition graph for some $p \in \{2, 3, 4\}$. Then $G$ is isomorphic to the $p$-row graph of a square $(0,1)$-matrix $M_p$ of order 6 by Theorem 2.2.

Consider the case $p = 4$. Then each row of $M_4$ contains at least five 1s by Proposition 2.6. This implies that any two rows of $M_4$ have at least four common 1s and so $G$ is isomorphic to $K_6$, which is a contradiction. Thus $4 \not\in \Upsilon(K_{3,3})$.

Now consider the case $p = 3$. Then each row of $M_3$ contains at least four 1s by Proposition 2.6. This implies that any two rows of $M_3$ have at least two common 1s. If there is a row containing at least five 1s, then it shares at least three common 1s with each of the other vertices, which is impossible. Thus each row of $M_3$ contains exactly four 1s. Since $K_{3,3}$ has a partite set of size 3, we may assume that $M_3$ contains the following submatrix by permuting columns, if necessary:

$$
\begin{bmatrix}
1 & 1 & 1 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 & 1 & 1 \\
1 & 1 & 0 & 0 & 1 & 1
\end{bmatrix}
$$

Now we take a vertex $u$ in the other partite set. Then $u$ is adjacent to the vertex corresponding to each row of the above submatrix. To have $u$ and the vertex corresponding to the first row of the above submatrix be adjacent, $\Lambda_{M_3}(u) \cap \{1, 2, 3, 4\}$ is one of $\{1, 2, 3\}, \{2, 3, 4\}, \{1, 2, 4\}, \{1, 3, 4\}, \{1, 2, 3, 4\}$. Then, in case of $\{1, 2, 3, 4\}$, $u$ is not adjacent to the vertices corresponding to the second row and third row; in case of $\{1, 2, 3\}$ or $\{1, 2, 4\}$, $u$ is not adjacent to the vertex corresponding to the second row; in case of $\{1, 3, 4\}$ or $\{2, 3, 4\}$, $u$ is not adjacent to the vertex corresponding to the third row. Therefore we reach a contradiction. Thus $3 \not\in \Upsilon(K_{3,3})$.

Now consider the case $p = 2$. Then each row of $M_2$ contains at least three 1s by Proposition 2.6. Suppose that there is a row $r_1$ containing at least four 1s. Let $v_1$ be the vertex corresponding to $r_1$ and $v_2$ and $v_3$ be the other vertices in the partite set to which $v_1$ belongs. We may assume that $\Lambda_{M_2}(v_1) \supset \{1, 2, 3, 4\}$. Since $v_1$ and $v_2$ are not adjacent, $r_1$ has exactly four 1s and we may assume that the row $r_2$ corresponding to $v_2$ has 1 in the fourth component through the sixth component. Then the row corresponding to $v_3$ must share at least two 1s with $r_1$ or $r_2$ and we reach a contradiction. Thus each row of $M_2$ contains exactly three 1s. If there are two vertices $w_1$ and $w_2$ in a partite set $W$ such that their corresponding rows do not share 1s, then the row corresponding to the remaining
vertex in $W$ must share at least two 1s with one of the rows corresponding to $w_1$ and $w_2$, and we reach a contradiction. Therefore the rows corresponding to two vertices in the same partite set share exactly one 1. Thus we may assume that $M_2$ contains the following submatrix by permuting columns, if necessary:

$$
\begin{bmatrix}
1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 1 & 0 \\
1 & 0 & 0 & 1 & 1 & 0
\end{bmatrix}.
$$

Now we take a vertex $x$ in the other partite set $X$. Then $x$ is adjacent to each of the vertices corresponding to the rows of the above submatrix. Therefore $\Lambda_{M_2}(x) = \{1, 3, 5\}$. Since $x$ is arbitrarily chosen, the rows corresponding to the other two vertices in $X$ also have 1 in the first, the third, and the fifth component, which is impossible. Hence we have shown that $\Upsilon(K_{3, 3}) = \emptyset$.

Now we introduce the notion of condensation of a graph. Let $G$ be a graph with $n$ vertices. Two vertices $u$ and $v$ of $G$ are said to be homogeneous, denoted by $u \sim v$, if they have the same closed neighborhood. Clearly $\sim$ is an equivalence relation on $V(G)$. We denote the equivalence class containing a vertex $u$ of $G$ by $[u]$. Then we define a new simple graph $G/\sim$ for $G$ by

$$
V(G/\sim) = \{[u] \mid u \in V(G)\} \quad \text{and} \quad E(G/\sim) = \{[u][v] \mid u, v \in V(G) \text{ and } uv \in E(G)\}.
$$

See Figure 1 for an illustration. We call $G/\sim$ the condensation of $G$ for a graph $G$.

We note the following:

- Two vertices $u$ and $v$ are adjacent in $G$ $\iff$ $v \in N_G[u]$
- $v \in N_G[u']$ for any $u' \in [u]$ $\iff$ $u' \in N_G[v]$ for any $u' \in [u]$
- $u' \in N_G[v']$ for any $u' \in [u]$ and $v' \in [v]$
- $u'$ and $v'$ are adjacent in $G$ for any $u' \in [u]$ and $v' \in [v]$.

Therefore $G/\sim$ is well-defined.

It is obvious that

![Figure 1: $G_1 \cong G_2/\sim \ (in \ G_2, \ u \sim v)$](image-url)
for each isolated vertex in $G$, its equivalence class is isolated in $G/\sim$.

The notion of $p$-row graph provides a way of getting information on the competition-realizer for a graph $G$ from the competition-realizer for a simpler graph $G/\sim$ as seen in the following results.

**Proposition 2.12.** The condensation of $G$ has the same diameter as $G$ for a connected non-complete graph $G$.

**Proof.** Let $m$ be the diameter of $G$. Since $G$ is not complete, $m \geq 2$. Then there exists an induced $(u, v)$-path of length $m$ for some vertices $u$ and $v$ and there is no induced $(u, v)$-path of length $l$ for any $l < m$. Thus, by the definition of condensation of a graph, there exists an induced $([u], [v])$-path of length $m$ and there is no induced $([u], [v])$-path of length $l$ for any $2 \leq l < m$ in $G/\sim$. If there exists a $([u], [v])$-path of length 1, then $u$ and $v$ are adjacent, which contradicts the choice of $u$ and $v$ so that $d_{G}(u, v) = m \geq 2$. Therefore the diameter of $G/\sim$ is greater than or equal to $m$.

Let $m'$ be the diameter of $G/\sim$ for a nonnegative integer $m'$. Since $G$ is non-complete, $m' \geq 1$. Then there exists an induced $([x], [y])$-path of length $m'$ in $G/\sim$ and there is no induced $([x], [y])$-path of length $l$ for any $l < m'$ and some vertices $x$ and $y$ in $G$. Therefore, by the definition of condensation of a graph, there is an induced $(x, y)$-path of length $m'$ and there is no induce $(x, y)$-path of length $l$ for any $l < m'$ in $G$. Thus the diameter of $G/\sim$ is less than or equal to $m$ and this completes the proof.

**Proposition 2.13.** A graph $G$ is a $p$-competition graph if and only if there exists a square matrix $M$ such that $G$ is a $p$-row graph of $M$ and the rows corresponding to two homogeneous vertices are identical.

**Proof.** The “if” part is obvious. To show the “only if” part, suppose that a graph $G$ is a $p$-competition graph for some positive integer $p$. Then, by Theorem 2.2, there exists a square matrix $M'$ such that $G$ is a $p$-row graph of $M'$. If there are at least two homogeneous vertices, then we fix one row among the rows corresponding to the homogeneous vertices and replace the remaining rows with the fixed row. We denote by $M$ the matrix obtained by repeatedly applying the above procedure. It is easy to see that $G$ is a $p$-row graph of $M$.

**Proposition 2.14.** Given a graph $G$ with $n$ vertices, suppose that $G/\sim$ is a $p$-row graph of a matrix $M$ satisfying the property that $M$ has $m$ columns for a positive integer $m \leq n$ and every row of $M$ has at least $p$ 1s. Then $\Upsilon(G) \supset \{p + i \mid i \in \{0, \ldots, n - m\}\}$.

**Proof.** Let $n_j$ be the size of equivalence class under $\sim$ corresponding to the $j$th row of $M$ for each $j$, $1 \leq j \leq m$. We replace the $j$th row of $M$ with $n_j$ copies of it to obtain the matrix $M^*$ which contains $M$ as a submatrix. We note that the size of $M^*$ is $n \times m$. Suppose that $u$ and $v$ are two distinct vertices in $G$. Then let $r_u$ and $r_v$ be the rows of $M^*$ corresponding to $u$ and $v$, respectively. If $u$ and $v$ are not homogenous, then they belong to distinct equivalence classes under $\sim$ and the following are true:

(⋆) for each isolated vertex in $G$, its equivalence class is isolated in $G/\sim$. 
Two vertices $u$ and $v$ are adjacent in $G$
$\iff [u]$ and $[v]$ are adjacent in $G/\sim$
$\iff$ the row corresponding to $[u]$ and the row corresponding to $[v]$ have at least $p$ common 1s in $M$
$\iff u$ and $v$ are adjacent in the $p$-row graph of $M^*$.

Suppose that $u$ and $v$ are homogenous. Then the rows $r_u$ and $r_v$ are identical. By the hypothesis, every row of $M$ has at least $p$ 1s. Thus $r_u$ and $r_v$ have at least $p$ common 1s and so $u$ and $v$ are adjacent in the $p$-row graph of $M^*$. Hence $G$ is the $p$-row graph of $M^*$ and so, by Proposition 2.4, $G$ is a $(p+i)$-competition graph, that is, $p+i \in \Upsilon(G)$ for any $i \in \{0, \ldots, n-m\}$. \qed

For a positive integer $p$ and the $p$-row graph $G$ of a matrix $M$, each non-isolated vertex in $G$ has at least $p$ 1s in the row of $M$ corresponding to it and so the following corollary is immediately true by the above proposition.

**Corollary 2.15.** Given a graph $G$ with $n$ vertices, suppose that $G/\sim$ has no isolated vertices and is a $p$-row graph of a matrix $M$ having $m$ columns for a positive integer $m \leq n$. Then \( \Upsilon(G) \supset \{p+i \mid i \in \{0, \ldots, n-m\}\} \).

**Corollary 2.16.** Given a graph $G$ with $n$ vertices, suppose that the number of non-isolated vertices in $G/\sim$ is $m$ for a positive integer $m \leq n$. Then

\[ \Upsilon(G) \supset \{p+i \mid p \in \Upsilon(G/\sim), i \in \{0, \ldots, n-m\}\}. \]

**Proof.** Suppose $p \in \Upsilon(G)$. Then, by Theorem 2.2, $G/\sim$ is a $p$-row graph of a matrix $M$ having $m$ columns. Thus by Corollary 2.15, $p+i \in \Upsilon(G)$ for any $i \in \{0, \ldots, n-m\}$. \qed

**Remark 2.17.** Even if $G$ is a $p$-competition graph, $G/\sim$ may not be a $(p-i)$-competition graph for some $i \in [n-m] \cup \{0\}$ where $n = |V(G)|$ and $m = |V(G/\sim)|$. For example, the graph $G_2$ in Figure 1 is a 2-competition graph. To see why, we note that $G_1$ is the 2-row graph of the matrix

\[
\begin{bmatrix}
1 & 1 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 1 & 1 \\
0 & 1 & 0 & 1 & 0 & 1 \\
1 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 1 & 1 & 0 & 1
\end{bmatrix},
\]

so $G_1$ is a 2-competition graph by Theorem 2.2. Since $G_1$ is isomorphic to $G_2/\sim$ and has no isolated vertices, $G_2$ is a 2-competition graph by Corollary 2.16. Yet, $G_1$, which is isomorphic to $G_2/\sim$, is not a 1-competition graph by Theorem 1.1 since $|V(G_1)| = 6 < 7 = |E(G_1)| = \theta_e(G_1)$. \[9\]
A \textit{union} $G\cup H$ of two graphs $G$ and $H$ is the graph having its vertex set $V(G)\cup V(H)$ and edge set $E(G)\cup E(H)$. In this paper, the union of $G$ and $H$ means their disjoint union which has an additional condition $V(G)\cap V(H) = \emptyset$. A \textit{join} $G \lor H$ of two vertex-disjoint graphs $G$ and $H$ is the graph having its vertex set $V(G)\cup V(H)$ and edge set $E(G)\cup E(H) \cup \{uv \mid u \in V(G), v \in V(H)\}$.

For a positive integer $n$, a nonnegative integer $k \leq n$, and the power set $\mathcal{P}([n])$ of $[n]$, we denote by $\Psi_{n,k}$ the simple graph with the vertex set $\mathcal{P}([n])$ and the edge set

$$\{ST \mid S, T \subset [n], |S \cup T| \leq k\}.$$

\textbf{Theorem 2.18.} Let $G$ be a connected graph with $n$ vertices and $k$ be a nonnegative integer less than $n$. Then $G$ is an $(n-k)$-competition graph if and only if $G/\sim$ is isomorphic to an induced subgraph of $\Psi_{n,k}$.

\textit{Proof.} To show the “only if” part, suppose that $G$ is an $(n-k)$-competition graph. Then, by Proposition 2.13 there exists a square matrix $M$ of order $n$ such that $G$ is an $(n-k)$-row graph of $M$ and the rows corresponding to two homogeneous vertices are identical. Let $M'$ be a submatrix of $M$ obtained by taking all the distinct rows of $M$. Then obviously $G/\sim$ is an $(n-k)$-row graph of $M'$. Therefore two distinct vertices $[x]$ and $[y]$ are adjacent in $G/\sim$ if and only if $|\Lambda_{M'}([x]) \cap \Lambda_{M'}([y])| \geq n-k$ if and only if $|(n \setminus \Lambda_{M'}([x])) \cup (n \setminus \Lambda_{M'}([y]))| \leq k$ if and only if $[n] \setminus \Lambda_{M'}([x])$ and $[n] \setminus \Lambda_{M'}([y])$ are adjacent in $\Psi_{n,k}$. Hence we have shown that $G/\sim$ is isomorphic to an induced subgraph of $\Psi_{n,k}$.

To show the “if” part, suppose that $G/\sim$ is isomorphic to an induced subgraph of $\Psi_{n,k}$. Then each vertex $[v]$ of $G/\sim$ is assigned a subset $S_v$ of $[n]$ so that two distinct vertices $[v]$ and $[w]$ are adjacent in $G/\sim$ if and only if $|S_v \cup S_w| \leq k$. If $|V(G/\sim)| = 1$, then $G$ is complete and so $\mathcal{Y}(G) = [n]$ by Proposition 2.3. Thus, if $|V(G/\sim)| = 1$, then $n-k \in \mathcal{Y}(G)$. Now suppose that $|V(G/\sim)| \geq 2$. Since $G$ is connected by the hypothesis, $G/\sim$ is connected and so every vertex $[v]$ in $G/\sim$ is adjacent to a vertex, which implies $|S_v| \leq k$. We denote by $M''$ the matrix with $n$ columns and with each row corresponds to a vertex of $G/\sim$ in such a way that $[n] \setminus \Lambda_{M''}([v]) = S_v$. Then it is easy to see that $G/\sim$ is an $(n-k)$-row graph of $M''$. By Corollary 2.16 we can conclude that $n-k \in \mathcal{Y}(G)$. \qed

\textbf{Lemma 2.19.} The star graph $K_{1,n}$ is an $n$-competition graph for a positive integer $n$.

\textit{Proof.} It is obvious that $K_{1,n}$ is the $n$-row graph of the following square matrix of order $n+1$:

$$M = \begin{bmatrix}
1 & 1 & 1 & \cdots & 1 & 1 & 1 \\
0 & 1 & 1 & \cdots & 1 & 1 & 1 \\
1 & 0 & 1 & \cdots & 1 & 1 & 1 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
1 & 1 & 1 & \cdots & 0 & 1 & 1 \\
1 & 1 & 1 & \cdots & 1 & 0 & 1 \\
\end{bmatrix}.$$
The following proposition characterizes a graph $G$ with $n$ vertices and $n - 1 \in \Upsilon(G)$.

**Proposition 2.20.** Let $G$ be a graph with $n$ vertices. Then $G$ is an $(n-1)$-competition graph if and only if $G \cong (K_{n_0} \lor (K_{n_1} \lor \cdots \lor K_{n_k})) \lor I_m$ for some nonnegative integers $k, n_0, n_1, \ldots, n_k$, and $m$ satisfying $m + \sum_{i=0}^{k} n_i = n$.

**Proof.** We show the “if” part. If $G$ is empty, then, by Proposition 2.3, $G$ is an $(n-1)$-competition graph. Now suppose that $G$ is a nonempty graph. Let $G'$ be the subgraph of $G$ resulting from deleting all the isolated vertices in $G$. It is easy to check that $G'\sim$ is an empty graph or a star graph by the hypothesis. Suppose that $|V(G'\sim)| = 1$. Then $G'$ is a complete graph and so $\Upsilon(G') = \{1, \ldots, |V(G')|\}$. By Proposition 2.5, $\Upsilon(G) = [n]$ and so $G$ is an $(n-1)$-competition graph. Now suppose that $|V(G'\sim)| \geq 2$. If $G'\sim$ is an empty graph, then $G'$ is a disjoint union of complete graphs and so, by Corollary 2.10 and Proposition 2.5, $G$ is an $(n-1)$-competition graph. Suppose that $G'\sim$ is not an empty graph. Then, by Proposition 2.3 and Lemma 2.19, $|V(G'\sim)| - 1 \in \Upsilon(G'\sim)$. Then $|V(G')| - 1 \in \Upsilon(G')$ by Corollary 2.16. By Proposition 2.5, $G$ is an $(n-1)$-competition graph.

Now we show the “only if” part. Suppose that $G$ is an $(n-1)$-competition graph. Then $G$ is isomorphic to the $(n-1)$-row graph of a matrix $M$ of order $n$ by Theorem 2.2. Suppose that there exists a row, say $r$, of $M$ such that $r$ contains at most $n-2$ 1s. Then the vertex in $G$ corresponding to $r$ is an isolated vertex, so $G$ is still the $(n-1)$-row graph of the matrix resulting from replacing $r$ with all-zero row. Therefore we may assume that $M$ contains the rows of exactly three types:

1. the row with $n$ 1s;
2. the row with $n-1$ 1s;
3. the row with 0 1s.

Let $V_1$, $V_2$, and $V_3$ be the sets of vertices corresponding to rows of Type $i$ for each $i = 1, 2, 3$. Then the vertex set of $G$ is partitioned into $V_1$, $V_2$, and $V_3$. Obviously, each vertex in $V_1$ is adjacent to each vertex of $G$ not in $V_3$ and each vertex in $V_3$ is isolated. We note that two rows of Type 2 are identical if and only if their corresponding vertices in $V_2$ are adjacent in $G$. Therefore each vertex in $V_2$ is a simplicial vertex, that is, a vertex whose neighbors form a clique. Let $n_0 = |V_1|$ and $m = |V_3|$. If $V_2 = \emptyset$, then $G \cong K_{n_0} \lor I_m$ by the above observation. Now suppose that $V_2 \neq \emptyset$. Then the subgraph of $G$ induced by $V_2$ is a disjoint union of cliques. Let $W_1, W_2, \ldots, W_k$ be the vertex sets of those cliques and let $|W_i| = n_i$ for $1 \leq i \leq k$. Then $m + \sum_{i=0}^{k} n_i = n$. By the above observation again, $G \cong (K_{n_0} \lor (K_{n_1} \lor \cdots \lor K_{n_k})) \lor I_m$. □
Theorem 2.21. Let $G$ be a graph with $n$ vertices. Then $\Upsilon(G) = [n - 1]$ if and only if $G \cong H \cup \mathcal{I}_m$ for some integer $m$, $0 \leq m \leq n - 3$ and some graph $H$ for which $H/\sim$ is an induced subgraph of a star graph and has more than one vertex.

Proof. To show the “if” part, suppose that $G \cong H \cup \mathcal{I}_m$ for some integer $m$, $0 \leq m \leq n - 3$ and some graph $H$ for which $H/\sim$ is an induced subgraph of a star graph $Q$ with more than one vertex. We denote the number of vertices in $H/\sim$ by $t$. Take $p \in [t - 1]$. We construct a square $(0, 1)$-matrix $M$ of order $t$ in the following way. If $H/\sim$ contains a center of $Q$, then the row of $M$ corresponding to it is the all-one vector. The rows of $M$ corresponding to the vertices in $H/\sim$ which are not a center of $Q$ are mutually distinct, and the number of $1$s in each of them is $p$. Such a matrix $M$ exists since $\binom{t}{1} \geq t$. It is easy to check that $H/\sim$ is isomorphic to the $p$-row graph of $M$. Thus $[t - 1] \subset \Upsilon(H/\sim)$ by Theorem 2.2. By Proposition 2.14, $[n - m - 1] \subset \Upsilon(H)$. Now, by Proposition 2.8, $[n - 1] \subset \Upsilon(G)$. By Proposition 2.7, $n \notin \Upsilon(G)$ and so $\Upsilon(G) = [n - 1]$.

To show the “only if” part, suppose that $\Upsilon(G) = [n - 1]$. Then, by Proposition 2.20, $G \cong (K_{n_0} \cup (K_{n_1} \cup \cdots \cup K_{n_k})) \cup \mathcal{I}_m$ for some nonnegative integers $k, n_0, n_1, \ldots, n_k$, and $m$ satisfying $m + \sum_{i=0}^{k} n_i = n$. If there is at most one nonzero integer among $n_1, n_2, \ldots, n_k$, then $G \cong K_{n-l_i} \cup \mathcal{I}_l$ for some integer $l_i$, $0 \leq l_i \leq n$ and so, by Corollary 2.9, $\Upsilon(G) = [n]$, which is a contradiction. Therefore there are at least two nonzero integers among $n_1, n_2, \ldots, n_k$ and $H := K_{n_0} \cup (K_{n_1} \cup \cdots \cup K_{n_k})$ is an induced subgraph of $G$. It is easy to check that $H/\sim$ is an induced subgraph of a star graph and has more than one vertex. Finally we show that $m \leq n - 3$. Suppose that there are exactly two nonzero integers among $n_0, n_1, n_2, \ldots, n_k$. If $n_0$ and $n_1$ are nonzero integers for some $i$, $1 \leq i \leq k$, then $G \cong K_{n_0+n_1} \cup \mathcal{I}_2$ for some integer $l_2$, $0 \leq l_2 \leq n$ and we reach a contradiction as before. If $n_i$ and $n_j$ are nonzero integers for some $i$ and $j$, $1 \leq i < j \leq k$, then $G \cong K_{n_i} \cup K_{n_j} \cup \mathcal{I}_3$ for some integer $l_3$, $0 \leq l_3 \leq n$, which implies $n_i \geq 2$ and $n_j \geq 2$. Therefore $m \leq n - 4$ if there are exactly two nonzero integers among $n_0, n_1, n_2, \ldots, n_k$. If there are at least three nonzero integers among $n_0, n_1, n_2, \ldots, n_k$, then it is obvious that $m \leq n - 3$. \hfill $\Box$

A hole of a graph is a cycle of length greater than or equal to 4 which is an induced subgraph of the graph. A graph without holes is said to be chordal.

For $p = n$ or $n - 1$, a $p$-competition graph is a chordal graph by Propositions 2.7 and 2.20. As a matter of fact, an $(n - 2)$-competition graph is also chordal by Corollary 2.24.

An induced path of a graph means a path as an induced subgraph of the graph.

Theorem 2.22. If a graph $G$ with $n$ vertices contains two internally disjoint induced paths of length 2 (whose internal vertices are nonadjacent), then $\Upsilon(G) \subset [n - 3]$.

Proof. Let $G$ be a graph with $n$ vertices containing two internally disjoint induced paths $uvw$ and $xyz$ of length 2 with $v$ and $y$ nonadjacent (see Figure 2). Then, by Propositions 2.7 and 2.20, $\Upsilon(G) \subset [n - 2]$. Suppose, to the contrary, that $n - 2 \in \Upsilon(G)$. By Theorem 2.2, $G$ is isomorphic to the $(n - 2)$-row graph of a square matrix $M$ of order
However, interestingly, the diameter of a connected graph with \( n \) vertices can be arbitrarily large, which will be shown by Proposition 3.4.

Proposition 2.25. For a graph \( G \) with \( \theta_e(G) \leq |V(G)|, |V(G)| - \theta_e(G) + 1 \) \( \subseteq \Upsilon(G) \).

Proof. Let \( |V(G)| = n \) and \( V(G) = \{v_1, v_2, \ldots, v_n\} \). There is an edge clique cover \( C := \{C_1, C_2, \ldots, C_{\theta_e(G)}\} \) of \( G \) as \( \theta_e(G) \) is the edge clique number of \( G \). We define an \( n \times \theta_e(G) \) matrix \( M = (m_{ij}) \) as follows:

\[
m_{ij} = \begin{cases} 
1 & \text{if } v_i \in C_j \\
0 & \text{if } v_i \notin C_j
\end{cases}
\]
Then $G$ is isomorphic to the 1-row graph of $M$. Therefore the statement is true by Proposition 2.4.

It is a well-known fact that any graph $G$ can be made into the competition graph of an acyclic digraph as long as it is allowed to add new isolated vertices to $G$. The smallest among such numbers is called the competition number of $G$ and denoted by $k(G)$. Opsut [12] showed that

$$k(G) \geq \theta_e(G) - |V(G)| + 2.$$  \hspace{1cm} (2)

**Corollary 2.26.** Let $G$ be a graph with $\omega$ components. If each component of $G$ has competition number one, then $[\omega + 1] \subset \Upsilon(G)$.

**Proof.** Let $G_1, G_2, \ldots, G_\omega$ be the components of $G$. Then, by (2), $|V(G_i)| - \theta_e(G_i) \geq 2 - k(G_i) = 1$ for any $1 \leq i \leq \omega$. Since $|V(G)| = \sum_{i=1}^\omega |V(G_i)|$ and $\theta_e(G) = \sum_{i=1}^\omega \theta_e(G_i)$, $|V(G)| - \theta_e(G) + 1 \geq \omega + 1$. Thus, by Proposition 2.25, the corollary statement is true. \hfill $\Box$

It is known that a chordal graph and a forest both have the competition number at most 1. Since any graph without isolated vertex has competition number at least 1, the following corollaries immediately follow from Corollary 2.26.

**Corollary 2.27.** For a chordal graph $G$ having $\omega$ components none of which is trivial, $[\omega + 1] \subset \Upsilon(G)$.

**Corollary 2.28.** For a forest $G$ having $\omega$ components none of which is trivial, $[\omega + 1] \subset \Upsilon(G)$.

### 3 Competition-realizers for trees

In this section, we study $p$-competition trees. Especially, we completely characterize the competition-realizers for caterpillars.

Let $G$ be a $p$-competition graph. Then $G$ is isomorphic to the $p$-row graph of a matrix $M = (m_{ij})$. If $|\Lambda_M(v)| \leq p - 1$, then $v$ is an isolated vertex in $G$, and so $G$ is still the $p$-row graph of the resulting matrix even if the row corresponding to $v$ is replaced by the row with $p - 1$ 1s. Thus we may conclude as follows:

\(\S\) If a $p$-competition graph $G$ is isomorphic to the $p$-row graph of a matrix $M$, then we may assume that $|\Lambda_M(v)| \geq p - 1$ for each vertex $v$ in $G$.

Adding a pendant vertex $v$ to a graph $G$ means obtaining a graph $G'$ such that $v \notin V(G)$, $V(G') = V(G) \cup \{v\}$, and $E(G') = E(G) \cup \{vu\}$ for a vertex $u$ in $G$.

**Theorem 3.1.** If $G$ is a graph obtained from a $p$-competition graph by adding a pendant vertex, then $p \in \Upsilon(G)$. 

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Proof. Let $G$ be a graph obtained from a $p$-competition graph $G'$ with $n$ vertices by adding a pendant vertex $u$ at vertex $v$ in $G'$. Since $G'$ is a $p$-competition graph, $G'$ is isomorphic to the $p$-row graph of a square matrix $M' = (m'_{ij})$ of order $n$. By (§), we may assume that $|\Lambda_{M'}(v)| \geq p - 1$. Without loss of generality, we may assume that the row corresponding to $v$ is located at the bottom of $M'$ and $\Lambda_{M'}(v) = \{1, 2, \ldots, |\Lambda_{M'}(v)| \}$.

Now we define a matrix $M = (m_{ij})$ of order $n + 1$ by

$$
M = \begin{bmatrix}
0 & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & 1 \\
M' \\
1 & \cdots & 1 & 0 & 0 & \cdots & 1 \\
\vdots & \ddots & \vdots & \ddots & \ddots & \ddots & \ddots \\
1 & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
1 & 1 & 1 & \cdots & 1 & 0 & \cdots & 0 \\
0 & 1 & 1 & \cdots & 1 & 1 & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 & 1 & 1 & 1 & \cdots & 1
\end{bmatrix},
$$

It is easy to check that $G$ is the $p$-row graph of $M$. By Theorem 2.2 $G$ is a $p$-competition graph.

Kim et al. [8] specified the length of a cycle which is a $p$-competition graph in terms of $p$.

**Theorem 3.2** [8]. Let $C_n$ be a cycle with $n$ vertices for a positive integer $n \geq 4$. Then $\Upsilon(C_n) = [n - 3]$.

In the proof of Theorem 3.2, $\mathcal{F} := \{S_0, \ldots, S_{n-1}\}$, where, for each $i = 0, \ldots, n - 1$, $S_i := \{v_i, v_{i+1}, \ldots, v_{i+p}\}$, is given as a $p$-edge clique cover of $C_n = v_0v_1 \cdots v_{n-1}v_0$ for which all the subscripts are reduced modulo $n$. The following square matrix of order $n$ is obtained from $\mathcal{F}$ by (1):

$$
M_{p,n} := 
\begin{bmatrix}
1 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 1 & 1 & \cdots & 1 \\
1 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 & 1 & \cdots & 1 \\
1 & 1 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 & 1 & \cdots & 1 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
1 & 1 & 1 & 1 & \cdots & 1 & 1 & 0 & 0 & 0 & \cdots & 0 \\
0 & 1 & 1 & 1 & \cdots & 1 & 1 & 1 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 0 & 1 & 1 & 1 & 1 & \cdots & 1
\end{bmatrix},
$$

where ith row (resp. column) is corresponding to $v_{i-1}$ (resp. $S_{i-1}$) and the $(i + 1)$st row (we identify the $(n + 1)$st row with the first row) is obtained by cyclically shifting the ith row by 1 to the right for each $i$, $1 \leq i \leq n$. Therefore the $p$-row graph of $M_{p,n}$ is isomorphic to $C_n$ and the following proposition is immediately true.
Lemma 3.3. Let \( n \) be an integer greater than or equal to 4 and \( p \) be a positive integer less than or equal to \( n - 3 \). Then, for the matrix \( M_{p,n} \) given in the equation (3), the following are true:

1. the \( k \)th row contains exactly \( p + 1 \) 1s for each integer \( k \), \( 1 \leq k \leq n \);
2. the \( k \)th row and the \((k + 1)\)st row have common 1s in exactly \( p \) columns for each integer \( k \), \( 1 \leq k \leq n \) (we identify the \((n + 1)\)st row with the first row);
3. the \( k \)th row and the \( l \)th row have common 1s in at most \( p - 1 \) columns for integers \( k \) and \( l \) satisfying \( 1 \leq k, l \leq n \) and \( 2 \leq |k - l| \).

We denote the path graph with \( n \) vertices by \( P_n \).

Proposition 3.4. For an integer \( n \geq 3 \),

\[
\Upsilon(P_n) = \begin{cases} 
\{1, 2\} & \text{if } n \in \{3, 4\}; \\
[n - 3] & \text{if } n \geq 5.
\end{cases}
\]

Proof. For \( n \geq 5 \), then \( \Upsilon(P_n) \subset [n - 3] \) by Corollary 2.28. By Corollary 2.8, \( \Upsilon(P_3) \subset \{1, 2\} \). By Corollary 2.8 and Proposition 2.20, \( \Upsilon(P_4) \subset \{1, 2\} \).

Now we show the other direction containment. If \( n = 3 \) and \( p \leq 2 \), then \( \{1, 2\} \subset \Upsilon(P_n) \) by Corollary 2.28. It is easy to check that \( P_4 \) is isomorphic to the 2-row graph of the following matrix:

\[
M^*_{2,4} := \begin{bmatrix}
1 & 1 & 0 & 0 \\
1 & 1 & 1 & 0 \\
0 & 1 & 1 & 1 \\
0 & 0 & 1 & 1
\end{bmatrix}.
\]

Thus \( 2 \in \Upsilon(P_4) \). Now suppose that \( n \geq 4 \) and \( p \leq n - 3 \). In \( M_{p,n} \) given in (3), we replace 1 in the \((1, n - p + 1)\)-entry with 0 to obtain a square matrix \( M^*_{p,n} \) of order \( n \). Let \( G' \) be the \( p \)-row graph of \( M^*_{p,n} \). Then the first row and the second row of \( M^*_{p,n} \) still share \( p \) 1s. Yet the first row and the \( n \)th row of \( M^*_{p,n} \) share only \( p - 1 \) 1s. Thus, by (2) and (3) of Lemma 3.3, \( G' \) is isomorphic to a path graph with \( n \) vertices. Hence \([n - 3] \subset \Upsilon(P_n)\) and this completes the proof.

Proposition 3.5. Let \( T \) be a tree with the diameter \( m \) for some integer \( m \), \( m \geq 3 \). Then \( \Upsilon(T) \supset [m - 2] \).

Proof. Take \( p \in [m - 2] \). Since the diameter of \( T \) is \( m \), there exists an induced path of length \( m \). Since \( m \geq p + 2 \), we may take a section \( P \) of this path which has length \( p + 2 \). Then \( P \) is a \( p \)-competition graph by Proposition 3.4. Since \( T \) can be obtained from \( P \) by adding pendant vertices sequentially, \( G \) is a \( p \)-competition graph by Theorem 3.1.
Corollary 3.6. Given a graph $G$ with $n$ vertices and diameter $m$, if $G/\sim$ is a tree with $n'$ vertices, then $\Upsilon(G) \supset [n-n'+m-2]$.

Proof. Suppose that $G/\sim$ is a tree with $n'$ vertices. Then, by Proposition 2.12, $G/\sim$ has diameter $m$. Thus, by Proposition 3.5, $\Upsilon(G/\sim) \supset [m-2]$. By Corollary 2.16,

$$\Upsilon(G) \supset \{p+i \mid p \in [m-2] \text{ and } i \in [n-n'] \cup \{0\} \} = [n-n'+m-2].$$

A caterpillar is a tree with at least 3 vertices the removal of whose pendant vertices produces a path. A spine of a caterpillar is the longest path of the caterpillar. In the following, for a caterpillar $T$ with $n$ vertices, we shall find all the positive integers $p$ such that $T$ is a $p$-competition graph in terms of $n$. To do so, we need the following lemmas.

Lemma 3.7. Let $n$ and $t$ be positive integers such that either $(n, t) = (4, 2)$ or $n \geq 4$ and $t \leq n - 3$. Then, for any nonnegative integer $k$ and a path graph $P$ of length $n - 1$, a caterpillar $T$ obtained by adding $k$ new vertices to $P$ in such a way that the added vertices are pendant vertices of $T$ adjacent to interior vertices of $P$ is a $(t+k)$-competition graph.

Proof. Since either $(n, t) = (4, 2)$ or $n \geq 4$ and $t \leq n - 3$, $P$ is a $t$-competition graph by Proposition 3.4 which is actually the $t$-row graph of $M^*_{t,n}$ where $M^*_{t,n}$ is the matrix defined in the proof of the same lemma. Thus the statement is true for $k = 0$. Now we assume that $k$ is a positive integer. Let $P = x_1x_2 \cdots x_n$ and $y_1, \ldots, y_k$ be the vertices added to $P$ as described in the theorem statement. There exists a map $\phi : [k] \to [n]$ such that $x_{\phi(i)}$ is a vertex on $P$ adjacent to $y_i$ for each $i$, $1 \leq i \leq k$. By the way that $y_1, \ldots, y_k$ were added, $\phi$ is well-defined. We define a $k \times n$ $(0,1)$-matrix $A$ so that the $i$th row of $A$ is the same as the row of $M^*_{t,n}$ corresponding to $x_{\phi(i)}$ for each $i$, $1 \leq i \leq k$.

Now we consider the matrix $M$ defined as follows:

$$M := \begin{pmatrix}
\begin{array}{cc}
M^*_{t,n} & J_{n,k} \\
A & J_{k,k} - I_k
\end{array}
\end{pmatrix}.$$ 

For an example, see Figure 3. Then, for each $i$ and $j$, $1 \leq i < j \leq k$,

(M-1) if $y_i$ and $y_j$ are adjacent to the same vertex on $P$, then the $i$th row and the $j$th row of the $(2,1)$-block of $M$ are identical and have exactly $t+1$ common 1s; otherwise, the $i$th row and the $j$th row of the $(2,1)$-block of $M$ have at most $t$ common 1s;
Figure 3: A caterpillar $T$ and a matrix whose 8-row graph is isomorphic to $T$ where the row labeled with $w$ corresponds to the vertex $w$ in $T$.

(M-2) if $y_i$ and $x_l$ are adjacent in $T$ for some $l \in [n]$, then the $l$th row of the $(1, 1)$-block of $M$ and the $i$th row of the $(2, 1)$-block of $M$ have exactly $t + 1$ common 1s; otherwise, the $l$th row of the $(1, 1)$-block of $M$ and the $i$th row of the $(2, 1)$-block of $M$ have at most $t$ common 1s.

Let $G$ be the $(t + k)$-row graph of $M$. We denote the row of $M$ containing the row of $M_{t,n}^*$ corresponding to $x_i$ by $\mathbf{x}_i$, for each $i, 1 \leq i \leq n$, the row of $M$ containing the $i$th row of the $(2, 1)$-block of $M$ by $\mathbf{y}_i$ for each $i, 1 \leq i \leq k$.

By the definition of $M_{t,n}^*$, the row of $M_{t,n}^*$ corresponding to $x_i$ and the row of $M_{t,n}^*$ corresponding to $x_j$ have at most $t - 1$ common 1s if and only if $|j - i| \geq 2$. Thus $\mathbf{x}_i$ and $\mathbf{x}_j$ have at most $t + k - 1$ common 1s if and only if $|j - i| \geq 2$ and therefore $P$ is an induced subgraph of $G$.

We note that the $i$th row and the $j$th row of the $(2, 2)$-block of $M$ have exactly $k - 2$ common 1s for each $i$ and $j$, $1 \leq i < j \leq k$. Thus, by (M-1), $\mathbf{y}_i$ and $\mathbf{y}_j$ have at most $t + k - 1$ common 1s for each $i$ and $j$, $1 \leq i < j \leq k$. Thus $\mathbf{y}_i$ and $\mathbf{y}_j$ are not adjacent in $G$ for each $i$ and $j$, $1 \leq i < j \leq k$.

We note that the $l$th row of the $(1, 2)$-block of $M$ and the $i$th row of the $(2, 2)$-block of $M$ have exactly $k - 1$ common 1s for each $i, 1 \leq i \leq k$ and each $l, 1 \leq l \leq n$. Thus, by (M-2), $y_i$ and $x_l$ are adjacent in $T$ if and only if $\mathbf{y}_i$ and $\mathbf{x}_l$ have at least $t + k$ common 1s. Thus we have shown that $T$ is isomorphic to $G$. Hence $T$ is a $(t + k)$-competition graph by Theorem 2.2.

\[ \begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \ 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \ 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \ 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \ 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \ 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \ 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \ 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \ 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \ \end{bmatrix} \]

Lemma 3.8. Given an integer $n \geq 2$ and the star graph $K_{1,n}$, $\Upsilon(K_{1,n}) = [n]$.

\[ n + 1 \not\in \Upsilon(K_{1,n}), \text{ so } \Upsilon(K_{1,n}) \subset [n]. \]

Now we show the converse containment. By Corollary 2.2, $\{1, 2\} \subset \Upsilon(K_{1,n})$. By Lemma 2.19, $n \in \Upsilon(K_{1,n})$. Therefore $[n] \subset \Upsilon(K_{1,n})$ for $n = 2, 3$. Now suppose $n \geq 4$. Then $\{1, 2, n\} \subset \Upsilon(K_{1,n})$ by the above argument. Thus it is sufficient to show that $p \in \Upsilon(K_{1,n})$ for $3 \leq p \leq n - 1$. Now we consider the following matrix $M$ of order $n + 1$:
Let \( M \) be the \( p \)-row graph of \( M \). In addition, let \( r_i \) denote the \( i \)th row of \( M \) and \( v_i \) be the vertex of \( G \) corresponding to \( r_i \) for each \( i \), \( 1 \leq i \leq n + 1 \).

For each \( i = 1, \ldots, n \), the \( i \)th row of \( M_{p-2,n} \) contains exactly \( p-1 \) 1s by Lemma 3.3(1). Thus \( r_i \) contains exactly \( p \) 1s in \( M \) and \( v_i \) and \( v_{n+1} \) are adjacent in \( G \) for each \( i = 1, \ldots, n \).

By (2) and (3) of Lemma 3.3, the \( i \)th row and the \( j \)th row of \( M_{p-2,n} \) have at most \( p-2 \) common 1s for distinct \( i \) and \( j \) in \([n]\). Therefore \( r_i \) and \( r_j \) have at most \( p-1 \) common 1s and so \( v_i \) and \( v_j \) are not adjacent in \( G \) for distinct \( i \) and \( j \) in \([n]\). Thus \( G \) is isomorphic to \( K_{1,n} \). Hence \( p \in \Upsilon(K_{1,n}) \) and so \([n] \subset \Upsilon(K_{1,n})\). \(\blacksquare\)

**Theorem 3.9.** For a caterpillar \( T \) with \( n \) vertices,

\[
\Upsilon(T) = \begin{cases} 
[n-1] & \text{if } d(T) = 2; \\
[n-2] & \text{if } d(T) = 3; \\
[n-3] & \text{if } d(T) \geq 4
\end{cases}
\]

where \( d(T) \) denotes the diameter of \( T \).

**Proof.** If \( d(T) = 2 \), then \( T \cong K_{1,n-1} \) and, by Lemma 3.3, \( \Upsilon(T) = [n-1] \).

Suppose \( d(T) \geq 3 \). If \( d(T) = 3 \), \( \Upsilon(T) \subset [n-2] \) by Corollary 2.8 and Proposition 2.20.

If \( d(T) \geq 4 \), \( \Upsilon(T) \subset [n-3] \) by Corollary 2.23.

To show the converse containment, let \( k(T) \) denote the number of vertices which are attached to the spine of \( T \). Now take a positive integer \( p \in [n-t] \) where \( t = 2 \) if \( d(T) = 3 \) and \( t = 3 \) if \( d(T) \geq 4 \).

Since \( d(T) \) is the length of the spine of \( T \), \( n = d(T) + 1 + k(T) \). Thus \( p \leq (d(T) + 1 + k(T)) - t \) or

\[
p - d(T) + t - 1 \leq k(T).
\]

If either \( d(T) = 3 \) and \( p \leq 2 \) or \( d(T) \geq 4 \) and \( p \leq d(T) - 2 \), then the spine of \( T \) is a \( p \)-competition graph by Proposition 3.4 and so \( T \) is a \( p \)-competition graph by Theorem 3.1.

Now assume that one of the following: \( d(T) = 3 \) and \( p > 2 \); \( d(T) \geq 4 \) and \( p > d(T) - 2 \).

Let

\[
\alpha = \begin{cases} 
p - 2 & \text{if } d(T) = 3 \text{ and } p > 2, \\
p - d(T) + 2 & \text{if } d(T) \geq 4 \text{ and } p > d(T) - 2.
\end{cases}
\]
Then \( \alpha \geq 1 \) and, by (4), we have \( \alpha \leq k(T) \). Let \( T' \) be a caterpillar obtained from \( T \) by deleting some pendent vertices of \( T \) so that \( d(T') = d(T) \) and \( k(T') = \alpha \). Then, by Lemma 3.7, \( T' \) is a \( p \)-competition graph and so, by Theorem 3.1, \( T \) is a \( p \)-competition graph.

**Corollary 3.10.** Let \( G \) be a graph with \( n \) vertices such that \( G/\sim \) is a caterpillar. Then

\[
\Upsilon(G) = \begin{cases} 
[n - 1] & \text{if } d(G) = 2; \\
[n - 2] & \text{if } d(G) = 3; \\
[n - 3] & \text{if } d(G) \geq 4 
\end{cases}
\]

where \( d(G) \) denotes the diameter of \( G \).

**Proof.** By Theorem 3.9,

\[
\Upsilon(G/\sim) = \begin{cases} 
[m - 1] & \text{if } d(G/\sim) = 2; \\
[m - 2] & \text{if } d(G/\sim) = 3; \\
[m - 3] & \text{if } d(G/\sim) \geq 4,
\end{cases}
\]

where \( m = |V(G/\sim)| \) and \( d(G/\sim) \) denotes the diameter of \( G/\sim \). Since \( G/\sim \) has no isolated vertices,

\[
\Upsilon(G) \supset \begin{cases} 
[n - 1] & \text{if } d(G) = 2; \\
[n - 2] & \text{if } d(G) = 3; \\
[n - 3] & \text{if } d(G) \geq 4,
\end{cases}
\]

by Proposition 2.12 and Corollary 2.16. By Corollary 2.8, \( n \not\in \Upsilon(G) \). Therefore \( \Upsilon(G) = [n - 1] \) if \( d(G) = 2 \). By Proposition 2.20, \( n - 1 \not\in \Upsilon(G) \) if \( d(G) \geq 3 \). Thus \( \Upsilon(G) = [n - 2] \) if \( d(G) = 3 \). By Corollary 2.23, \( \Upsilon(G) \subset [n - 3] \) if \( d(G) \geq 4 \). Hence \( \Upsilon(G) = [n - 3] \) if \( d(G) \geq 4 \).

**Lemma 3.11.** Given a \( p \)-competition graph \( G \) with \( n \) vertices, suppose that \( 2^r+1 \) neighbors of a vertex \( v \) of \( G \) form an independent set for some positive integer \( r \). Then \( p \geq n - r \) implies that there are two nonadjacent neighbors \( x \) and \( y \) of \( v \) with \( |\Lambda_M(x)| < |\Lambda_M(y)| < |\Lambda_M(v)| \) for any square matrix \( M \) of order \( n \) whose \( p \)-row graph is isomorphic to \( G \).

**Proof.** Let \( M \) be a square matrix \( M \) of order \( n \) whose \( p \)-row graph is isomorphic to \( G \). Since \( v \) is not isolated, \( p \leq |\Lambda_M(v)| \leq n \). For notational convenience, we let \( \overline{\Lambda}_M(v) = [n] \setminus \Lambda_M(v) \). Now suppose \( p \geq n - r \). Then \( 0 \leq |\overline{\Lambda}_M(v)| \leq n - p \leq r \). Thus the number of subsets of \( \overline{\Lambda}_M(v) \) is less than \( 2^r + 1 \). For each neighbor \( w \) of \( v \), \( \overline{\Lambda}_M(v) \cap \overline{\Lambda}_M(w) \) is a subset of \( \overline{\Lambda}_M(v) \). Since \( v \) has \( 2^r + 1 \) neighbors which form an independent set by the hypothesis, there are two nonadjacent neighbors \( x \) and \( y \) of \( v \) such that \( \overline{\Lambda}_M(v) \cap \overline{\Lambda}_M(x) = \overline{\Lambda}_M(v) \cap \overline{\Lambda}_M(y) \).
by the Pigeonhole principle. Since $\overline{M}(v) \cap \overline{M}(x) \cap \overline{M}(y)$ are subsets of $\overline{M}(x)$ and $\overline{M}(y)$, respectively, we have

$$\overline{M}(v) \cap \overline{M}(x) = \overline{M}(v) \cap \overline{M}(y) \subset \overline{M}(x) \cap \overline{M}(y).$$

(5)

Since $v$ is adjacent to $x$ and $y$, $|\overline{M}(v) \cup \overline{M}(x)| \leq n - p$ and $|\overline{M}(v) \cup \overline{M}(y)| \leq n - p$. Since $x$ and $y$ are not adjacent, $|\overline{M}(x) \cup \overline{M}(y)| > n - p$. Thus

$$|\overline{M}(v)| + |\overline{M}(x)| - |\overline{M}(v) \cap \overline{M}(x)| = |\overline{M}(v) \cup \overline{M}(x)| \leq n - p$$

$$< |\overline{M}(x) \cup \overline{M}(y)| = |\overline{M}(x)| + |\overline{M}(y)| - |\overline{M}(x) \cap \overline{M}(y)|$$

Then, by (5), $|\overline{M}(y)| > |\overline{M}(v)|$. By the same argument, one can show that $|\overline{M}(x)| > |\overline{M}(v)|$ and we complete the proof.

By Proposition 3.5, $\Upsilon(T) \neq \emptyset$ for a tree $T$ with the diameter at least 3. It is easy to see that $\Upsilon(T) \neq \emptyset$ for a tree $T$ with the diameter at most two. Thus $\max \Upsilon(T)$ exists for any tree $T$. We have shown that $|V(T)| - \max \Upsilon(T) \leq 3$ for a caterpillar $T$. One might think by this result that there exists a positive integer $t$ such that $|V(T)| - \max \Upsilon(T) \leq t$ for any tree $T$, yet it is not true by the following theorem.

Let $k$ be a positive integer. A $k$-ary tree is a rooted tree in which each vertex has no more than $k$ children. A full $k$-ary tree is a rooted tree exactly $k$ children or no children. A perfect $k$-ary tree is a full $k$-ary tree in which all pendant vertices are at the same depth. The depth of a vertex in a rooted tree is the distance between the vertex and the root. The height of a rooted tree is the number of edges on the longest path between its root and a pendant vertex.

**Theorem 3.12.** For any positive integer $r$, there is a tree $T$ with $|V(T)| - \max \Upsilon(T) > r$.

**Proof.** Let $T$ be a perfect $(2^r + 1)$-ary tree with height $r + 1$ and a root $x_0$. Then by the definition of $\Upsilon(G)$ for a graph $G$, $T$ is a $(\max \Upsilon(T))$-competition graph. Then $T$ is a $(\max \Upsilon(T))$-row graph of a matrix $M$. Suppose that $|V(T)| - \max \Upsilon(T) \leq r$. Then, by Lemma 3.11 $|\Lambda_M(x_1)| < |\Lambda_M(x_0)| \leq |V(T)|$ for some children $x_1$ of $x_0$. Then, by the same lemma again, $|\Lambda_M(x_2)| < |\Lambda_M(x_1)| \leq |V(T)| - 1$ for some children $x_2$ of $x_1$. We apply the lemma repeatedly to have $|\Lambda_M(x_{r+1})| < |V(T)| - r$. Since $x_{r+1}$ is non-isolated, $|\Lambda_M(x_{r+1})| \geq \max \Upsilon(T)$. Thus $|V(T)| - \max \Upsilon(T) > r$ and we reach a contradiction. Hence $|V(T)| - \max \Upsilon(T) > r$. \qed

4 Closing Remarks

We have shown that $\Upsilon(K_{3,3}) = \emptyset$. We would like to know whether or not $\Upsilon(K_{n,n}) = \emptyset$ for any $n \geq 4$. We have characterized the graphs with $n$ vertices and the competition-realizer $[n]$ and $[n - 1]$, respectively. It would be interesting to characterize the graphs with $n$ vertices and competition-realizer $[n - 2]$. Finally we suggest to find the competition-realizer for a Lobster to extend our result which gives every element in the competition-realizer for a caterpillar.
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