ON THE FUNDAMENTAL GROUP OF POLYHEDRAL PRODUCTS

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Abstract. Let $K$ be a finite simplicial complex, and $(X, A)$ be a pair of spaces. The purpose of this article is to study the fundamental group of the polyhedral product denoted $Z_K(X, A)$, which denotes the moment-angle complex of Buchstaber-Panov in the case $(X, A) = (D^2, S^1)$, with extension to arbitrary pairs in [2] as given in Definition 2.2 here.

For the case of a discrete group $G$, we give necessary and sufficient conditions on the abstract simplicial complex $K$ such that the polyhedral product denoted by $Z_K(BG)$ is an Eilenberg-Mac Lane space. The fundamental group of $Z_K(BG)$ is shown to depend only on the 1-skeleton of $K$. Further special examples of polyhedral products are also investigated.

Finally, we use polyhedral products to study an extension problem related to transitively commutative groups, which are given in Definition 5.2.

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1. Introduction

Let $(X, A)$ be a CW-pair with non-degenerate basepoint, and let $K$ be an abstract simplicial complex with a finite number of vertices. The polyhedral product functor is a covariant functor defined on the category of CW-pairs with values in the category of topological spaces. That is, for any pair $(X, A)$ and abstract finite simplicial complex $K$, the polyhedral product $Z_K(X, A)$ is a topological space with the natural choice of basepoint. This functor has been studied extensively and stands at the foundations of toric topology. See for example work of V. Buchstaber and T. Panov [3, 4], work of T. Bahri, M. Bendersky, F. Cohen and S. Gitler [2], and elegant work of G. Denham and A. Suciu [7].

One of the fundamental examples is the Davis-Januszkiewicz space

$$\mathcal{D}j(K) \cong ET^m \times_{TT^m} Z_K(D^2, S^1) \cong Z_K(\mathbb{C}P^\infty, \ast).$$

The cohomology ring of the space $Z_K(\mathbb{C}P^\infty, \ast)$ is isomorphic as an algebra to the face ring of $K$, also known as the Stanley-Reisner ring. The equivalence between $\mathcal{D}j(K)$ and $Z_K(\mathbb{C}P^\infty, \ast)$
was proved by V. Buchstaber and T. Panov [3]. The polyhedral product $Z_K(D^2, S^1)$ is called the moment-angle complex, and is sometimes abbreviated $Z_K$ in the literature.

Seminal work of T. Bahri, M. Bendersky, F. Cohen and S. Gitler [2] set the solid framework necessary to study polyhedral products in broad generality. In particular, they study the stable decomposition of these spaces for any sequence of CW-pairs $(X, A) = \{(X_i, A_i)\}_{i=1}^\infty$. They prove a stable decomposition which has implications for computations in any cohomology theory. Namely, they prove that for a sequence $(X_i, A_i)$ of connected and pointed CW-pairs there is a pointed homotopy equivalence

$$\Sigma(Z_K(X, A)) \to \Sigma(\bigvee_{\sigma \subseteq \{1, \ldots, n\}} Z_{K_\tau}(X_{\tau}, A_{\tau})),$$

where the “hat” stands for smash product (see Definition 2.3) and

$$K_\tau = \{\sigma \cap I | \sigma \in K\}$$

is the restriction of $K$ to $I$ and $(X_{\tau}, A_{\tau})$ is the restriction of the sequence of pairs to $I$, see [2, Theorem 2.10].

In this paper we study the homotopy groups of the polyhedral products $Z_K(X, A)$ with a focus on the fundamental group for specific choices of CW-pairs $(X, A)$.

Let $G$ be a group with the discrete topology and let $BG$ denote its classifying space with $* = B\{1\}$ as the basepoint. Let $EG$ be a contractible topological space on which $G$ acts freely and properly discontinuously. If $G_1, \ldots, G_n$ are discrete groups, then we show that the existence of the higher homotopy groups of the polyhedral product $Z_K(BG)$ depends only on $K$. In particular, if $K$ is a flag complex (see Definition 2.5), then higher homotopy groups of $Z_K(BG)$ vanish, see [6, Example 1.6]. Here we show that the condition that $K$ is a flag complex is necessary.

**Theorem 1.1.** Let $G_1, \ldots, G_n$ be non-trivial discrete groups and $K$ be a simplicial complex with $n$ vertices. Then $Z_K(BG)$ is an Eilenberg-Mac Lane space if and only if $K$ is a flag complex. Equivalently, $Z_K(EG, G)$ is an Eilenberg-Mac Lane space if and only if $K$ is a flag complex.

Let $SK_1$ denote the 1-skeleton of the simplicial complex $K$. In [6] it was shown that if $K$ is a flag complex, then the fundamental group of the polyhedral product $Z_K(BG)$ is the graph product of the groups $G_1, \ldots, G_n$ over the graph $SK_1$ (see Definition 2.6), that is,

$$\pi_1(Z_K(BG)) \cong \prod_{SK_1} G_i.$$

Here we show that this is the case for any abstract simplicial complex $K$ on a finite number of vertices. That is, the fundamental group in this case is determined by the 1-skeleton of $K$. The higher homotopy groups may appear only when $K$ contains $m$-dimensional faces for $m \geq 2$.

**Proposition 1.2.** Let $K$ be a simplicial complex on $n$ vertices and let $SK_1$ be the 1-skeleton of $K$. Let $G_1, \ldots, G_n$ be discrete groups. Then

$$\pi_1(Z_K(BG)) \cong \pi_1(Z_{SK_1}(BG)) \cong \prod_{SK_1} G_i.$$

The fundamental group of the polyhedral products $Z_K(X)$ is also investigated for a 1-connected space $X$. Let $CY$ denote the cone on the topological space $Y$. Then the following theorem holds.

**Theorem 1.3.** Let $X_1, \ldots, X_n$ be 1-connected CW-complexes. Then the polyhedral product $Z_K(OX_1, OX)$ is 1-connected.

Since the polyhedral product $Z_K(OX_1, OX)$ is 1-connected, one can describe the first non-vanishing homotopy group using the Hurewicz isomorphism. For this choice of pairs of spaces
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the first non-vanishing homotopy group depends on the structure of the simplicial complex $K$ as follows.

Let $I = \{i_1, \ldots, i_k\} \subset \{1, \ldots, n\}$ and for a sequence of spaces $X = \{X_1, \ldots, X_n\}$ let $X^I = X_{i_1} \wedge \cdots \wedge X_{i_k}$. Then the first non-vanishing homotopy group of the polyhedral product $Z_K(C\Omega X, \Omega X)$ is given by the following proposition.

Proposition 1.4. Let $X_1, \ldots, X_n$ be 1-connected CW-complexes and $K$ be a simplicial complex with $n$ vertices. If $Z_K(C\Omega X, \Omega X)$ is $(k - 1)$-connected, then the first non-vanishing homotopy group is given by

$$\pi_k(Z_K(C\Omega X, \Omega X)) \cong \bigoplus_{I \subset \{1, \ldots, n\}} H_k(|K_I| \wedge \Omega X^I; \mathbb{Z}).$$

Now assume that the fundamental group of $X$ and $Y$ for the pairs $(X, A)$ and $(Y, A)$ are isomorphic and there are inclusions $A \subset X \subset Y$. Using the colimit definition of polyhedral products and the fact that homotopy colimit and colimit commute in this case, one can restrict to a subspace if necessary, to compute the fundamental group of the polyhedral product.

Lemma 1.5. If $A \subset X \subset Y$ with $A$ discrete, $X$ and $Y$ path-connected, such that the inclusion induces an isomorphism of the fundamental groups $\pi_1(X) \cong \pi_1(Y)$, then there is an isomorphism

$$\pi_1(Z_K(X, A)) \cong \pi_1(Z_K(Y, A)).$$

Suppose that $H$ is a closed subgroup of the Lie group $G$ and consider the pair $(BG, BH)$. There is an inclusion $G/H \to C(G/H)$ which gives an inclusion

$$EG \times G/H \hookrightarrow EG \times C(G/H).$$

So we can regard $G/H$ as a $G$-equivariant subspace of $EG$. The following is an analogue of the Denham-Suciu fibration.

Proposition 1.6. Let $H$ be a closed subgroup of the Lie group $G$. There is a fibration given by

$$Z_K(EG, G/H) \to Z_K(BG, BH) \to (BG)^n.$$

As a consequence of Proposition 1.6, we obtain the following splitting and short exact sequence of groups.

Proposition 1.7. Let $H$ be a closed subgroup of $G$. Then there is a homotopy equivalence

$$\Omega(Z_K(BG, BH)) \simeq G^n \times \Omega(Z_K(EG, G/H))$$

and a short exact sequence of groups

$$1 \to \pi_1(Z_K(EG, G/H)) \to \pi_1(Z_K(BG, BH)) \to G^n \to 1.$$ 

Furthermore, this decomposition may not preserve the group structure.

A group $G$ is called transitively commutative if for any $g, h, k \in G$, $[g, h] = 1$ and $[h, k] = 1$ imply $[g, k] = 1$. For $G$ a finite discrete groups and $G_1, \ldots, G_n$ a collection of subgroups of $G$, there is a natural map

$$\varphi : G_1 \ast \cdots \ast G_n \to G.$$ 

There is also a natural map

$$p : \prod_{\Gamma} G_i \to G,$$

where $\Gamma$ is a simplicial graph on $n$ vertices (see Definition 2.6). We investigate the problem when the map $\varphi$ can factor through $p$, that is through the graph product of groups $\prod_{\Gamma} G_i$. This is equivalent to finding the graphs $\Gamma$ such that the following diagram commutes.
G_1 \ast \cdots \ast G_n \xrightarrow{\varphi} G

\prod G_i.

Among other things, answering this question gives information about commuting elements in G. We study this problem for the class of transitively commutative groups. More precisely, the aim is to give topological conditions characterizing transitively commutative groups, using polyhedral products.

Moreover, we define an extension of the group theoretic concept of \textit{transitively commutative} groups, to \textit{k-transitively commutative} groups, which is a group theoretic construction reflected by the natural topology of the spaces in this article. Features of \textit{k-transitively commutative} groups are investigated after they are introduced, in Section 5.

The organization of this article is as follows. In Section 2 polyhedral products are introduced and some of their properties are given. In Section 3 Theorem 1.1 and Proposition 1.2 are proved. In Section 4 we prove Theorem 1.3, Proposition 1.4, Lemma 1.5 and Propositions 1.6 and 1.7. Finally, in Section 5 we study the extension problem for transitively commutative groups.

2. Polyhedral products

Polyhedral products can be regarded as functors from abstract simplicial complexes with values in the category of topological spaces. Alternatively, for fixed K, they can be seen as functors from the category of topological spaces to the category of topological spaces. Here the discussion is confined to pointed CW-functors from the category of topological spaces to the category of topological spaces. Here the natural topology of the spaces in this article. Features of \textit{k-transitively commutative} groups are investigated after they are introduced, in Section 5.

The realization |K| of K is a simplicial subcomplex of the simplex ∆[n−1].

\textbf{Definition 2.1.} An \textit{abstract simplicial complex} (or only \textit{simplicial complex}) K on n vertices is a subset of the power set 2^{[n]} such that, if σ ∈ K and τ ⊆ σ then τ ∈ K. An element σ ∈ K, called a \textit{simplex}, is given by an increasing sequence of integers σ = {i_1, i_2, \ldots, i_q}, where 1 ≤ i_1 < i_2 < \cdots < i_q ≤ n. In particular, the empty set ∅ is an element of K. The \textit{geometric realization} |K| of K is a simplicial subcomplex of the simplex ∆[n−1].

Define a functor D from an abstract simplicial complex K to the category of pointed CW-complexes, D : K → CW, as follows. For any σ ∈ K let

\[ D(σ) = \prod_{i=1}^{n} Y_i = Y_1 \times \cdots \times Y_n \subseteq \prod_{i=1}^{n} X_i, \quad \text{where} \quad Y_i = \begin{cases} A_i & : i \notin σ, \\ X_i & : i \in σ. \end{cases} \]

\textbf{Definition 2.2.} The \textit{polyhedral product} (or \textit{generalized moment-angle complex} [7]) denoted \(Z_K(X,A)\) is the topological space defined by the colimit

\[ Z_K(X,A) = \text{colim}_{σ ∈ K} D(σ) = \bigcup_{σ ∈ K} D(σ) \subseteq \prod_{i=1}^{n} X_i, \]

where the maps are the inclusions and the topology is the subspace topology.

Similarly define a functor \(\hat{D} : K → CW\), so that for any σ ∈ K we have

\[ \hat{D}(σ) = \prod_{i=1}^{n} Y_i = Y_1 \land \cdots \land Y_n \subseteq \prod_{i=1}^{n} X_i, \quad \text{where} \quad Y_i = \begin{cases} A_i & : i \notin σ, \\ X_i & : i \in σ. \end{cases} \]

\textbf{Definition 2.3.} The \textit{smash polyhedral product} \(\hat{Z}_K(X,A)\) is defined to be the image of the polyhedral product \(Z_K(X,A)\) in the smash product \(X_1 \land \cdots \land X_n\). For example, the image of \(D(σ)\) in \(\hat{Z}_K(X,A)\) is given by \(\hat{D}(σ)\).
In other words the polyhedral product is the colimit of the diagram of spaces \( D(\sigma) \). Different notations can be found in the literature to denote a polyhedral product, such as \( Z_K(X_i,A_i) \), \( Z(K; (X,A)) \) and \( (X,A)^K \). Whenever \( A_i \) is the basepoint of \( X_i \), we will write \( Z_K(X) \) to mean \( Z_K(X,\ast) \), or we simply write \( Z_K(X) \) if all the spaces \( X_i \) in the sequence of pairs are the same.

**Example 2.4.** Some classical examples of polyhedral products are the following.

1. Let \( K = \{\{1\}, \ldots, \{n\}\} \), \( X_i = X \) and \( A_i \) be the basepoint \( \ast \in X \). Then
   \[
   Z_K(X,A) = X \vee \cdots \vee X,
   \]
   the \( n \)-fold wedge sum of the space \( X \).
2. Let \( K = 2^{[n]} \), then \( Z_K(X,A) = X_1 \times \cdots \times X_n \).
3. Let \( K = \{\{1\}, \{2\}\} \) and \( (X,A) = (D^n, S^{n-1}) \). Then
   \[
   Z_K(X,A) = D^n \times S^{n-1} \cup S^{n-1} \times D^n = \partial D^{2n} = S^{2n-1}.
   \]
4. If \( |K| \) is the boundary of the simplex \( \Delta[n-1] \), then \( Z_K(X) \) is the fat wedge of the spaces \( X_1, \ldots, X_n \).

Polyhedral products give combinatorial models for some well-known spaces. As mentioned in the introduction, the Davis-Januszkiewicz space \( D\mathcal{J}(K) \) is one such space since it is homotopy equivalent to \( Z_K(\mathbb{C}P^\infty, \ast) \).

**Definition 2.5.** The simplicial complex \( |K| \) is a flag complex if any finite set of vertices, which are pairwise connected by edges, spans a simplex in \( |K| \).

**Definition 2.6.** Given a simplicial graph \( \Gamma \) with vertex set \( S \) and a family of groups \( \{G_s\}_{s \in S} \), their graph product

\[
\prod_{\Gamma} G_s
\]

is the quotient of the free product of the groups \( G_s \) by the relations that elements of \( G_s \) and \( G_t \) commute whenever \( \{s,t\} \) is an edge of \( \Gamma \). Note that if \( \Gamma \) is the complete graph on \( n \) vertices, then

\[
\prod_{\Gamma} G_s \cong \prod_{i=1}^{n} G_i.
\]

Another example of interest is the classifying space of the right-angled Artin group (RAAG). Recall that an RAAG can be given in terms of generators and relations or as a graph product of a finite number of copies of the infinite cyclic group \( \mathbb{Z} \). The fundamental group of the polyhedral product \( Z_K(S^1, \ast) \) is the right-angled Artin group given by the graph product \( \prod_{SK_1} \mathbb{Z} \), where \( SK_1 \) is the 1-skeleton of \( K \). If \( K \) is a flag complex, then \( Z_K(S^1, \ast) \) is the classifying space of the right-angled Artin group, see [6]. For other finite classifying spaces of Artin groups see [5].

In our discussion, \( K \) will frequently mean the geometric realization of \( K \). In what follows, assume that spaces are in the homotopy category of pointed \( CW \)-complexes unless otherwise stated.

### 3. Pairs of classifying spaces of discrete groups

Let \( G \) be a discrete group. In this section we consider polyhedral products constructed from pairs of spaces of the form \( (BG, \ast) \), where \( \ast \) is the natural basepoint, and study their homotopy groups. More precisely, we are interested in describing the fundamental group of \( Z_K(BG) \), where \( (BG, \ast) = \{(BG_i, \ast)\}_{i=1}^n \) and \( G_1, \ldots, G_n \) are discrete groups. Later we find necessary and sufficient conditions for the simplicial complex \( K \) such that \( Z_K(BG) \) is an Eilenberg-Mac Lane space.
In general, for $G$ a topological group and $K$ a simplicial complex on $n$ vertices, there is a fibration due to G. Denham and A. Suciu [7, Lemma 2.3.2].

**Theorem 3.1** (Denham & Suciu). Let $G$ be a topological group and $K$ a simplicial complex on $n$ vertices. Then the following hold.

1. $EG^n \times G^n Z_K(EG, G) \simeq Z_K(BG)$.
2. The homotopy fiber of the inclusion $Z_K(BG) \hookrightarrow BG^n$ is $Z_K(EG, G)$.

Their theorem is a result of studying the bundle

$$Z_K(EG, G) \to EG^n \times G^n Z_K(EG, G) \to BG^n,$$

where $EG^n \times G^n Z_K(EG, G)$ is the quotient of $EG^n \times Z_K(EG, G)$ by the diagonal action of $G^n$, and proving that the total space $EG^n \times G^n Z_K(EG, G)$ is homotopy equivalent to $Z_K(BG)$. This theorem can be extended directly to the sequence of CW-pairs $(BG) = \{(BG_i, *)\}_{i=1}^n$, where $G_i$ are topological groups for all $i$. Hence, there is a fibration

$$Z_K(EG, G) \to Z_K(BG) \to BG_1 \times \cdots \times BG_n,$$

where similarly, the total space is homotopy equivalent to the twisted product of spaces

$$(EG_1 \times \cdots \times EG_n) \times (G_1 \times \cdots \times G_n) \simeq Z_K(EG, G).$$

We will refer to fibration (3.1) as the Denham-Suciu fibration.

**Lemma 3.2.** Assume that $G_1, \ldots, G_n$ are discrete groups and $K$ is a simplicial complex on $n$ vertices. The following properties hold:

1. $Z_K(BG)$ is path connected for any $K$.
2. $\pi_q(Z_K(EG, G)) \simeq \pi_q(Z_K(BG))$ for $q \geq 2$.
3. There is a short exact sequence of groups

$$1 \to \pi_1(Z_K(EG, G)) \to \pi_1(Z_K(BG)) \to \pi_1(BG_1 \times \cdots \times BG_n) \to 1.$$

**Proof.** This follows directly from the long exact sequence in homotopy applied to the Denham-Suciu fibration (3.1) and the fact that $BG_1 \times \cdots \times BG_n = B(G_1 \times \cdots \times G_n)$. □

Consider the following definition, which will give a different characterization of flag complexes.

**Definition 3.3.** Let $K$ be a simplicial complex on $n$ vertices. A minimal non-face in $K$ is a subcomplex of $K$, which is the boundary of a full simplex not in $K$ on 3 or more vertices. That is, $\sigma$ is a minimal non-face in $K$ if $\sigma = \partial \tau \subseteq K$ and $\tau = \Delta[q]$ for $2 \leq q \leq n - 1$, but $\tau \notin K$.

**Remark.** Clearly if $K$ has a minimal non-face, then it cannot be a flag complex. Therefore, a flag complex can be redefined to be a simplicial complex with no minimal non-faces.

The fundamental group of the spaces $Z_K(BG)$ is calculated next.

**Proposition 3.4.** Let $K$ be a simplicial complex on $n$ vertices and let $SK_1$ be the 1-skeleton of $K$. If $G_1, \ldots, G_n$ are discrete groups, then

$$\pi_1(Z_K(BG)) \simeq \pi_1(Z_{SK_1}(BG)) \simeq \prod_{SK_1} G_i.$$

**Proof.** If $K$ is a flag complex, then this is true since $Z_K(BG)$ is a $K(\pi, 1)$ with $\pi = \prod_{SK_1} G_i$, see [6, Lemma 1.4].

Now assume that $K$ is not a flag complex. Then $K$ has a minimal non-face $\sigma$ on $k$ vertices, where $k > 2$. Hence, $\sigma$ contains the complete graph $\Gamma$ on its vertices such that $\Gamma \subseteq \sigma = \partial \Delta[k-1] \subset \Delta[k-1]$ on these $k$ vertices, for $3 \leq k \leq n$ (trivial for $k = 1, 2$).

First we claim that

$$\pi_1(Z_\sigma(BG)) \simeq \pi_1(Z_{\Delta[k-1]}(BG)) \simeq \prod_{\Gamma} G_i = \prod_{i=1}^{k} G_i.$$
Note that if $k = 3$ and $G_{j_1}, G_{j_2}, G_{j_3}$ are subgroups of a discrete group $G$ such that they pairwise commute in $G$, then any product of elements commutes in $G$. Hence, if $\sigma$ is the boundary of the 2-simplex, then

$$Z_\sigma(BG) = (BG_{j_1} \times BG_{j_2} \times 1) \cup (BG_{j_1} \times 1 \times BG_{j_3}) \cup (1 \times G_{j_2} \times BG_{j_3})$$

and using the Seifert-van Kampen theorem twice it follows that

$$\pi_1(Z_\sigma(BG)) \cong \pi_1(Z_{\Delta[2]}(BG)) \cong \prod_{\Gamma} G_{j_i} \cong G_{j_1} \times G_{j_2} \times G_{j_3}.$$

Note that for $k \geq 3$, the polyhedral product can be written as follows

$$Z_\sigma(BG) = \bigcup_{1 \leq i \leq k} (BG_{j_1} \times \cdots \times BG_{j_{i-1}} \times 1 \times BG_{j_{i+1}} \times \cdots \times BG_{j_n}).$$

Using the Seifert-van Kampen theorem finitely many times, more exactly $k - 1$ times, it follows that

$$\pi_1(Z_\sigma(BG)) \cong \pi_1(Z_{\Delta[k-1]}(BG)) \cong \prod_{\Gamma} G_{i} \cong G_{j_1} \times \cdots \times G_{j_k}.$$

Now $\Gamma$ is the complete graph on the vertices of $\sigma$. The graph product $\prod_{\Gamma} G_{j_i}$ of the groups $G_{j_1}, \ldots, G_{j_k}$ is actually isomorphic to the product $\prod_{i=1}^k G_{j_i}$, since all the groups pairwise commute. Also note that adding a 2-dimensional face to $\Gamma$ does not introduce a new generator in the fundamental group of $Z_\Gamma(BG)$. Hence, if the groups $G_i$ are subgroups of $G$ and $\Gamma$ is the complete graph on its vertices, the computation above is equivalent to saying that the map

$$Z_\Gamma(BG) \to BG$$

factors through $Z_{\Delta[k-1]}(BG_i) = \prod_{i=1}^k G_{j_i}$, so that we have the following commutative diagram

$$\begin{array}{ccc}
Z_\Gamma(BG) & \to & BG \\
\downarrow & & \downarrow \\
Z_{\Delta[k-1]}(BG) & \to & BG
\end{array}$$

Hence, on the level of fundamental groups the following diagram commutes

$$\begin{array}{ccc}
\pi_1(Z_\Gamma(BG)) & \to & \pi_1(BG) \\
\downarrow & & \downarrow \\
\pi_1(Z_\sigma(BG)) & \to & \pi_1(Z_{\Delta[k-1]}(BG)) \\
\downarrow & & \downarrow \\
\pi_1(Z_{\Delta[k-1]}(BG)) & \to & \pi_1(BG)
\end{array}$$

Therefore,

$$\pi_1(Z_\sigma(BG)) \cong \pi_1(Z_{\Delta[k-1]}(BG)) \cong \prod_{\Gamma} G_{j_i} \cong \prod_{i=1}^k G_{j_i}.$$ 

If $\alpha$ is a simplex in $K$, then $\alpha$ can intersect $\sigma$ at most at a single top dimensional face. Then,

$$Z_\alpha(BG) \cap Z_\alpha(BG) = \prod_{i,j \in \sigma \cap \alpha} BG_{i,j}$$

where $|\sigma \cap \alpha| \leq k - 1$.

Now let $\sigma_1, \ldots, \sigma_p$ be all the minimal non-faces of $K$. For $\sigma_1$ there is a simplicial subcomplex $K_1 \subseteq K$ not containing $\sigma_1$ such that $\sigma_1 \cup K_1 = K$ and $\sigma_1 \cap K_1$ is a flag complex. Since the CW-pairs are $(BG, *)$, it follows that

$$Z_{\sigma_1}(BG) \cup Z_{K_1}(BG) = Z_K(BG)$$
and
\[ Z_{\sigma_1}(BG) \cap Z_{K_1}(BG) = Z_{\sigma_1 \cap K_1}(BG), \]
where \( \sigma_1 \cap K_1 \) is a flag complex. Thus,
\[ \pi_1(Z_{\sigma_1 \cap K_1}(BG)) = \prod_{i \in S[\sigma_1 \cap K_1]} G_i. \]

Hence, using Seifert-van Kampen theorem
\[ \pi_1(Z_{\sigma_1}(BG)) \ast \prod_{i \in S[\sigma_1 \cap K_1]} G_i \pi_1(Z_{K_1}(BG)) = \pi_1(Z_{K}(BG)). \]

Now, to find \( \pi_1(Z_{K_1}(BG)) \) we repeat the same process. Clearly, \( \sigma_2 \subseteq K_1 \). Then, there is
\( K_2 \subseteq K_1 \) not containing \( \sigma_2 \) such that \( \sigma_2 \cup K_2 = K_1 \) and \( \sigma_2 \cap K_2 \) is a flag complex. Similarly,
\[ Z_{\sigma_2}(BG) \cup Z_{K_2}(BG) = Z_{K_1}(BG) \]
and
\[ Z_{\sigma_2}(BG) \cap Z_{K_2}(BG) = Z_{\sigma_2 \cap K_2}(BG), \]
where \( \sigma_2 \cap K_2 \) is a flag complex. Thus,
\[ \pi_1(Z_{\sigma_2 \cap K_2}(BG)) = \prod_{i \in S[\sigma_2 \cap K_2]} G_i. \]

Hence, using Seifert-van Kampen theorem
\[ \pi_1(Z_{\sigma_2}(BG)) \ast \prod_{i \in S[\sigma_2 \cap K_2]} G_i \pi_1(Z_{K_2}(BG)) = \pi_1(Z_{K_1}(BG)). \]

Hence, for \( \sigma_q \) there is
\( K_q \subseteq K_{q-1} \subseteq \cdots \subseteq K_1 \subseteq K \)
not containing \( \sigma_q, \sigma_{q-1}, \ldots, \sigma_1 \) such that \( \sigma_q \cup K_q = K_{q-1} \) and \( \sigma_q \cap K_q \) is a flag complex, for all \( 1 \leq q \leq p \). Similarly,
\[ Z_{\sigma_q}(BG) \cup Z_{K_q}(BG) = Z_{K_{q-1}}(BG) \]
and
\[ Z_{\sigma_q}(BG) \cap Z_{K_q}(BG) = Z_{\sigma_q \cap K_q}(BG), \]
where \( \sigma_q \cap K_2 \) is a flag complex. Thus,
\[ \pi_1(Z_{\sigma_q \cap K_q}(BG)) = \prod_{i \in S[\sigma_q \cap K_q]} G_i. \]

Hence, using Seifert-van Kampen theorem
\[ \pi_1(Z_{\sigma_q}(BG)) \ast \prod_{i \in S[\sigma_q \cap K_q]} G_i \pi_1(Z_{K_q}(BG)) = \pi_1(Z_{K_{q-1}}(BG)). \]

Let us denote \( \prod_{i \in S[\sigma_q \cap K_q]} G_i \) by \( N_q \). Therefore, combining all the steps we get
\[ \pi_1(Z_{K}(BG)) = \pi_1(Z_{\sigma_1}(BG)) \ast_{\prod_{i \in S[\sigma_1 \cap K_1]} G_i} \pi_1(Z_{K_1}(BG)) \ast_{\prod_{i \in S[\sigma_2 \cap K_2]} G_i} \pi_1(Z_{K_2}(BG)) \ast_{\prod_{i \in S[\sigma_3 \cap K_3]} G_i} \cdots \pi_1(Z_{K_p}(BG)). \]

Note that \( K_p \) is a flag complex since it does not contain any of the minimal non-faces \( \sigma_1, \ldots, \sigma_p \), so its fundamental group is \( \prod_{i \in S(K_p)} G_i \). Let us denote
\[ M_q = \pi_1(Z_{\sigma_q}(BG)) = \prod_{i \in S(\sigma_q)} G_i. \]

Hence,
\[ \pi_1(Z_{K}(BG)) = M_1 \ast_{N_1} (M_2 \ast_{N_2} \cdots (M_p \ast_{N_p} \prod_{i \in S(K_p)} G_i)) \cdots). \]

Therefore, it follows by induction that
\[ \pi_1(Z_{K}(BG)) \cong \prod_{S(K)} G_i. \]

□
An alternative proof of Proposition 3.4 is by comparing fibrations.

Second proof of Proposition 3.4.: Consider the following commutative diagram of spaces

\[
\begin{array}{ccc}
F & \longrightarrow & Z_{SK_q}(EG, G) \\
\downarrow & & \downarrow \\
F & \longrightarrow & Z_{SK_q}(BG) \\
\downarrow & & \downarrow \\
* & \longrightarrow & \prod_{i=1}^n BG_i
\end{array}
\]

where \( F \) is the homotopy fibre and \( SK_q \) is the \( q \)-skeleton of \( K \), for \( 1 \leq q \leq n \). Hence, the homotopy fibre \( F \) of the inclusion

\[ Z_{SK_q}(EG, G) \hookrightarrow Z_K(EG, G) \]

is the same as the homotopy fibre \( F \) of the inclusion

\[ Z_{SK_q}(BG) \hookrightarrow Z_K(BG) \]

It suffices to show that on the level of fundamental groups, the induced map

\[ \pi_1(Z_{SK_q}(EG, G)) \longrightarrow \pi_1(Z_K(EG, G)) \]

is an injection. This is true since adding 2-dimensional or higher dimensional faces to the \( q \)-skeleton, for \( q \geq 1 \), adds 3-dimensional or higher dimensional cells to the space \( Z_{SK_q}(EG, G) \), and adding these cells to the space does not introduce new elements in the fundamental group, see [10]. This clearly holds for the case when \( K \) is a flag complex, since in that case \( K \) has no minimal non-faces.

3.1. Zero simplicial complexes. For a fixed abstract simplicial complex \( K \), the polyhedral product is a homotopy functor, which means that for fixed \( K \) the homotopy type of \( Z_K(X, A) \) depends only on the relative homotopy type of the pairs \( (X, A) \). This fact was also observed in [7, Section 2.2]. In particular, this is true for the 0-simplicial complex on \( n \) vertices. First note the following lemma.

Lemma 3.5. Let \( G \) be a finite discrete group of order \( m \). Then there is a relative homotopy equivalence \( (EG, G) \sim ([0, 1], F) \), where \( F \) is a subset of \([0, 1]\) of cardinality \( m \).

Proof. By definition \( EG \) is contractible and the group \( G \) can be identified with the orbit of a point \( x \in EG \), since \( G \) acts freely on \( EG \). This gives an equivalence of CW-pairs \((EG, G) \sim ([0, 1], F) = (I, F)\), where \( F = \{f_1 < \cdots < f_m\} \approx \{(1 = g_1) \cdot x = x, g_2 \cdot x, \ldots, g_m \cdot x\} \) is a finite subset of \( I \) with the same cardinality as \( G \). One can pick a path \( \gamma : [0, 1] \rightarrow EG \) such that \( \gamma(f_i) = g_i \cdot x \).

Hence, there is a homotopy equivalence \( Z_K(EG, G) \simeq Z_K(I, F) \).

Proposition 3.6. Let \( K \) be the zero skeleton of \( \Delta[r - 1] \) and \( F_i \) be finite subsets of \( I = [0, 1] \) with \( |F_i| = m_i \), for \( 1 \leq i \leq r \). Then

\[ Z_K(I, F) \simeq \bigvee_{N_r} S^1, \]

where \( N_r \) is given inductively as follows:

\[ N_2 = (m_1 - 1)(m_2 - 1) \]

\[ N_r = m_r N_{r-1} + (m_r - 1)(\prod_{i=1}^{r-1} m_i - 1), \text{ for } r \geq 3. \]
Proof. Recall that if $T$ is a spanning tree of a connected graph $\Gamma$ with a finite number of vertices, then collapsing $T$ to a point does not change the homotopy type of $\Gamma$. $\Gamma/T$ has only one vertex and has the homotopy type of a finite wedge of circles, and so does $\Gamma$; see [10]. It suffices to find the number of circles $N_r$ and the proof is given by induction on $r$.

- **Figure 1.** $T_2, r = 2$

- **Figure 2.** $T_3, r = 3$

For $r = 2$: $K$ has two vertices and $Z_K(I, E)$ contains a maximal tree, which we denote $T_2$, defined in the following way: It starts at the point $(0, 0) \in I \times I$ and runs parallel to the first coordinate and goes to the next level by using one of the extreme vertical edges. $T_2$ contains all the vertices and has no loops, hence it is a spanning tree. There are $N_2 = (m_1 - 1)(m_2 - 1)$ edges not in $T_2$. Figure (1) shows a version of $I, E$. Assume true for $r = 2$ then collapsing $T_2$ with $n$ having $N_2$ and it needs a vertical edge to jump to the next dimension each time. There are $m_3$ edges, each having $N_2$ and has no loops, hence it is a spanning tree. There are $m_3$ levels, each having $N_2$ edges not in $T_3$, and there are $m_3 - 1$ spaces between levels, each having $\prod_{i=1}^{m_3}(m_i - 1)$ edges not in $T_3$. Therefore there are $N_3 = m_3N_2 + (m_3 - 1)(\prod_{i=1}^{m_3}(m_i) - 1)$.

Assume true for $r = n$, that is, $N_n = m_nN_{n-1} + (m_n - 1)(\prod_{i=1}^{m_n-1}(m_i) - 1)$. Let $K_n$ denote the 0-skeleton with $n$ vertices. For $r = n + 1$ there is an inclusion $Z_{K_{n+1}}(I, E) \subseteq \mathbb{R}^{n+1}$. Set

$$A_n = Z_{K_n}(I, E) \subseteq \mathbb{R}^n,$$

then $A_n \cong \bigvee_{N_n} S^1$. The following diagram (Figure (3)) describes $A_{n+1}$, where between any two consecutive $A_n$’s there are $\prod_{i=1}^{m_n} m_i$ edges. Each $A_n$ has the maximal tree $T_n$ and the number of edges in $A_n$ not in $T_n$ is $N_n$. One edge is used between two consecutive $A_n$’s to complete the graph $T_{n+1}$, so there are $(\prod_{i=1}^{m_n} m_i) - 1$ edges not in $T_{n+1}$. Hence the total number of edges not in $T_{n+1}$ is $N_{n+1} = m_{n+1}N_n + (m_{n+1} - 1)(\prod_{i=1}^{m_n} m_i - 1)$.

**Corollary 3.7.** The value of $N_r$ in Proposition 3.6 is

$$N_r = (r - 1)\prod_{i=1}^{r} m_i - \sum_{i=1}^{r} (\prod_{j \neq i} m_j) + 1.$$ 

Proof. The proof follows by induction on $r$. For $r = 2$ then $(m_1 - 1)(m_2 - 1) = m_1m_2 - (m_1 + m_2) + 1$. Now assume this is true for $r = n$, then for $r = n + 1$ it follows from Lemma 3.2 that

$$N_{n+1} = m_{n+1}N_n + (m_{n+1} - 1)(\prod_{i=1}^{n} m_i - 1),$$
where by assumption \( N_n \) equals
\[
N_n = (n - 1) \prod_{i=1}^{n} m_i - \sum_{i=1}^{n} (\prod_{j \neq i} m_j) + 1.
\]
Substituting this value for \( N_n \) and rearranging the terms shows that
\[
N_{n+1} = (n) \prod_{i=1}^{n+1} m_i - \sum_{i=1}^{n+1} (\prod_{j \neq i} m_j) + 1.
\]

**Corollary 3.8.** Let \( G_1, \ldots, G_r \) be finite discrete groups with \( |G_i| = m_i \), for all \( i \). Then there is a homotopy equivalence
\[
Z_K(EG, G) \simeq \bigvee_{N_r} S^1,
\]
where \( N_r \) is given in Corollary 3.7.

**Proof.** Follows from Proposition 3.6 and Corollary 3.7.

Consider the Denham-Suciu fibration (3.1) for \( K \) the 0-simplex, to get the following fibration
\[
Z_K(EG, G) \simeq \bigvee_{N_n} S^1 \rightarrow \bigvee_{1 \leq i \leq n} BG_i \rightarrow BG_1 \times \cdots \times BG_n.
\]
Each of the spaces in the fibration is an Eilenberg-Mac Lane space, hence there is a short exact sequence of groups
\[
1 \rightarrow F[x_1, \ldots, x_{N_n}] \rightarrow G_1 \ast \cdots \ast G_n \rightarrow G_1 \times \cdots \times G_n \rightarrow 1.
\]
The rank of the free group in the kernel is \( N_n \), which is given in Corollary 3.7. This rank was also computed algebraically in an early paper of J. Nielsen [13]. Thus Proposition 3.6 gives a topological proof of that same result.

### 3.2. Flag complexes.
Suppose \( K \) is a flag complex. If \( G \) is a non-trivial discrete group then the space \( Z_K(EG, G) \) is a \( K(\pi, 1) \), see [6, Example 1.6]. By a short argument in Lemma 3.9, the converse of this statement is also true.

**Lemma 3.9.** If \( K \) is not a flag complex, then \( Z_K(EG, G) \) is not an Eilenberg-Mac Lane space.

**Proof.** There is an inclusion of pairs \( ([0, 1], \{0, 1\}) \rightarrow (EG, G) \). If \( K \) is not a flag complex, then \( K \) contains a minimal non-face \( \sigma \subseteq K \) with \( k \) vertices, where \( k \geq 3 \). Then there is an embedding
\[
Z_\sigma([0, 1], \{0, 1\}) \rightarrow Z_K(EG, G).
\]
There is an equivalence \( Z_\sigma([0, 1], \{0, 1\}) \simeq S^{k-1} \). Therefore, \( Z_K(EG, G) \) has non-trivial higher homotopy groups.

**Remark.** If \( K \) is a flag complex, then \( Z_K(EG, G) \) is the classifying space of the kernel of the projection
\[
\prod_{SK_1} G_i \longrightarrow \prod_{i=1}^{n} G_i.
\]
Hence, if \( G_1, \ldots, G_n \) are finite groups, then Lemma 3.9 and 3.5 show that the kernel of this projection is torsion free, since \( Z_K(EG, G) \simeq K(\pi_1(Z_K(EG, G)), 1) \) is of finite type.

The following theorem is an immediate corollary.

**Theorem 3.10.** Let \( G_1, \ldots, G_n \) be non-trivial discrete groups and \( K \) be a simplicial complex with \( n \) vertices. Then \( Z_K(BG) \) is an Eilenberg-Mac Lane space if and only if \( K \) is a flag complex. Equivalently, \( Z_K(EG, G) \) is an Eilenberg-Mac Lane space if and only if \( K \) is a flag complex.

**Proof.** This follows from [6], Proposition 3.9, and the Denham-Suciu fibration.
4. Homotopy groups of other polyhedral products

Suppose $X$ is a path-connected, finite dimensional and pointed $CW$-complex. Then it is a classical result that $X$ is the classifying space of a topological group $G$, which can be described precisely, see for example [11]. Hence, we can write $X \simeq BG$. This gives a homotopy equivalence $\Omega X \simeq G$, which implies an equivalence between the cone of the spaces $CG \simeq C(\Omega X)$. Let $*$ be the basepoint of $EG$. There is a commutative diagram of spaces

$$
\begin{array}{ccc}
EG \times C(G) & \xrightarrow{\simeq} & C(\Omega X) \\
\uparrow & & \uparrow \\
\ast \times G & \xrightarrow{\simeq} & \Omega X.
\end{array}
$$

Hence, there is a homotopy equivalence of pairs $(EG, G) \simeq (C\Omega X, \Omega X)$. The Denham-Suciu fibration (3.1) gives us

$$
Z_K(EG, G) \rightarrow Z_K(BG, \ast) \rightarrow BG^n,
$$

which can also be written as

$$
Z_K(C\Omega X, \Omega X) \rightarrow Z_K(X, \ast) \rightarrow X^n,
$$

because of the equivalence $(EG, G) \simeq (C\Omega X, \Omega X)$. This is an instance of the more general case of a sequence of $CW$-pairs $(X, \ast)$. Hence, there is a fibration

$$
Z_K(C\Omega X, \Omega X) \rightarrow Z_K(X, \ast) \rightarrow \prod_{i=1}^n X_i.
$$

Assume that $G$ is path-connected. That means $\pi_0(G) = 1$ and $\pi_1(X) = \pi_0(\Omega X) = \pi_0(G) = 1$. That is, assume that $X$ is 1-connected. We want to prove that if $X$ is 1-connected, then $Z_K(EG, G)$ is 1-connected. In particular, if $K$ is a shifted complex, then it follows from [9] that if $X$ is 1-connected, then $Z_K(C\Omega X, \Omega X)$ is 1-connected.

**Theorem 4.1.** Let $X_1, \ldots, X_n$ be 1-connected $CW$-complexes. Then the polyhedral product $Z_K(C\Omega X, \Omega X)$ is 1-connected.

**Proof.** We prove this for $X_1 = \cdots = X_n = X$, then the general proof is the same. As mentioned above, if $X$ is a 1-connected $CW$-complex, then $X \simeq BG$ and $G$ is path-connected, see [11]. Moreover, there is an equivalence

$$
Z_K(C\Omega X, \Omega X) \simeq Z_K(EG, G),
$$

where $EG$ and $G$ are path-connected. Hence, $Z_K(EG, G)$ is path-connected. In this proof we will work with $Z_K(EG, G)$.

To show that $\pi_1(Z_K(EG, G)) = 0$, the definition of the polyhedral product will be used. Recall that

$$
Z_K(EG, G) = \colim_{\sigma \in K} D(\sigma),
$$

where $D(\sigma)$ is a product of $EG$’s and $G$’s. Also recall that $G \times \cdots \times G = D(\emptyset) \subset D(\sigma)$, for all $\sigma \in K$. For two simplices $\tau, \sigma \in K$ consider the pushout diagram

$$
\begin{array}{ccc}
D(\tau) \cap D(\sigma) & \xrightarrow{i} & D(\tau) \\
\downarrow j & & \downarrow \\
D(\sigma) & \xrightarrow{} & D(\tau) \cup_{D(\tau) \cap D(\sigma)} D(\sigma),
\end{array}
$$

where $D(\tau) \cup_{D(\tau) \cap D(\sigma)} D(\sigma)$ is the colimit of the two maps emanating from the intersection $D(\tau) \cap D(\sigma)$. Using Seifert-van Kampen theorem for the fundamental group, it follows that

$$
\pi_1(D(\tau) \cup_{D(\tau) \cap D(\sigma)} D(\sigma)) \simeq \pi_1(D(\tau)) \ast_N \pi_1(D(\sigma)),
$$

where $\ast_N$ is the free product with amalgamation along $\pi_1(D(\tau) \cap D(\sigma))$. 

**Theorem 4.2.** The following are equivalent:

1. $Z_K(EG, G)$ is 1-connected.
2. $\pi_1(Z_K(EG, G)) = 0$.
3. $Z_K(EG, G)$ is a 1-connected CW-complex.

**Proof.** This is a direct consequence of the definitions and the Seifert-van Kampen theorem.
where \( N \) is the subgroup generated by the images of the fundamental group of the intersection under the induced maps of \( i \) and \( j \) in \( \pi_1 \). Let the set \( V_{\sigma,\tau} = \{ v_1, \ldots, v_l \} \) be the maximal set of vertices in \( K \) such that \( V_{\sigma,\tau} \cap \sigma = \emptyset \) and \( V_{\sigma,\tau} \cap \tau = \emptyset \). Clearly, \( \pi_1(D(\tau)) \ast_{N_j} \pi_1(D(\sigma)) = \pi_1(G^t) \), since \( i \) and \( j \) induce monomorphisms in \( \pi_1 \) and the images of these two maps do not hit the coordinates corresponding to the vertices in \( V_{\sigma,\tau} \).

\( K \) contains only a finite number of simplices \( \tau_1, \ldots, \tau_k \). Let \( V_{i_1,\ldots,i_l} \) be the maximal set of vertices in \( K \) such that \( V_{i_1,\ldots,i_l} \cap \tau_i = \emptyset \) for \( 1 \leq j \leq l \leq k \). As explained above

\[
\pi_1(D(\tau_1) \cup D(\tau_2) \cap D(\tau_k)) = \pi_1(D(\tau_1)) \ast_{N_{1,\ldots,k-1}} \pi_1(D(\tau_2)) = \pi_1(G^{V_{i_1,\ldots,i_l}}).
\]

To complete the proof, first perform the computation by taking the colimit with more simplices until all simplices \( \tau_1, \ldots, \tau_k \) are used. It follows that

\[
\pi_1(\operatorname{colim}_{1 \leq i \leq k-1} D(\tau_i)) = \pi_1(G^{V_{i_1,\ldots,i_{k-1}}}).
\]

In the last step, the polyhedral product equals

\[
Z_K(EG, G) = \operatorname{colim}_{1 \leq i \leq k-1} D(\tau_i) \cup \operatorname{colim}_{1 \leq i \leq k-1} D(\tau_i) \cap D(\tau_k) D(\tau_k).
\]

Thus, the fundamental group equals

\[
\pi_1(\operatorname{colim}_{1 \leq i \leq k-1} D(\tau_i) \cup \operatorname{colim}_{1 \leq i \leq k-1} D(\tau_i) \cap D(\tau_k) D(\tau_k)) = \pi_1(\operatorname{colim}_{1 \leq i \leq k-1} D(\tau_i)) \ast_{N_{1,\ldots,k-1}} \pi_1(D(\tau_k)).
\]

Since \( V_{1,\ldots,k} = \emptyset \), it follows that \( \pi_1(Z_K(EG, G)) = 0 \).

Now using Hurewicz’s theorem, we can describe the first non-vanishing homotopy group of \( Z_K(COX, \Omega X) \).

**Proposition 4.2.** Let \( X_1, \ldots, X_n \) be 1-connected CW-complexes and \( K \) be a simplicial complex with \( n \) vertices. If \( Z_K(COX, \Omega X) \) is \( (k-1) \)-connected, then the first non-vanishing homotopy group is given by

\[
\pi_k(Z_K(COX, \Omega X)) \cong \bigoplus_{I \subset \{1, \ldots, n\}} H_k([K_I] \wedge \Omega X^I ; \mathbb{Z}).
\]

**Proof.** From Theorem 4.1 we have that \( Z_K(COX, \Omega X) \) is 1-connected. Furthermore assume that the first non-vanishing homotopy group is the \( k \)-th homotopy group. Then from the Hurewicz theorem we have

\[
\pi_k(Z_K(COX, \Omega X)) \cong H_k(Z_K(COX, \Omega X), \mathbb{Z}).
\]

It follows from [2, Theorem 2.10] that there is a homotopy equivalence

\[
\Sigma Z_K(COX, \Omega X) \to \Sigma \left( \bigvee_{I \subset [n]} \tilde{Z}_{K_I}(COX, \Omega X) \right).
\]

Since \( COX_i \) is contractible for all \( i \), it follows from [2, Theorem 2.19] that there is a homotopy equivalence

\[
\tilde{Z}_{K_I}(COX, \Omega X) \to [K_I] \wedge \Omega X^I.
\]

Hence, it follows that there is a homotopy equivalence

\[
\Sigma Z_K(COX, \Omega X) \to \Sigma \left( \bigvee_{I \subset [n]} [K_I] \wedge \Omega X^I \right).
\]

Finally, we have isomorphisms

\[
H_k(Z_K(COX, \Omega X)) \cong H_k \left( \bigvee_{I \subset [n]} [K_I] \wedge \Omega X^I ; \mathbb{Z} \right) \cong \bigoplus_{I \subset [n]} H_k([K_I] \wedge \Omega X^I ; \mathbb{Z}).
\]

\( \square \)
Another question is the case when a topological group $G$ acts freely and properly discontinuously on a CW-complex $Y$ and $p : Y \to X = Y/G$ is a bundle projection. There is a lemma due to Denham and Suciu [7] which describes the fibre when comparing certain fibrations involving polyhedral products.

**Lemma 4.3** (Denham & Suciu). Let $p : (E, E') \to (B, B')$ be a map of pairs, such that both $p : E \to B$ and $p' = p|_{E'} : E' \to B'$ are fibrations with fibres $F$ and $F'$ respectively. Suppose that either $F = F'$ or $B = B'$. Then the product fibration $p \times n : E^n \to B^n$ restricts to a fibration

$$Z_K(F, F') \to Z_K(E, E') \to Z_K(B, B').$$

Moreover, if $(F, F') \to (E, E') \to (B, B')$ is a relative bundle (with structure group $G$), and either $F = F'$ or $B = B'$, then (4.1) is also a bundle (with structure group $G^n$).

Now, consider the relative map $p : (Y, G) \to (Y/G, *)$ with fibres $F = F' = G$. Thus, there is a fibration

$$G^n \to Z_K(Y, G) \to Z_K(Y/G, *).$$

Pushing the fibre to the right by taking its classifying space, one gets a fibration

$$Z_K(Y, G) \to Z_K(Y/G, *) \to BG^n.$$

In this case it is not true in general that $Z_K(Y, G)$ is 1-connected, as shown in the next example.

**Example 4.4.** Let $G = \mathbb{Z}/2\mathbb{Z}$ act on the 2-sphere $S^2$ by the antipodal map. Then the orbit space is $S^2/(\mathbb{Z}/2\mathbb{Z}) = \mathbb{R}P^2$. Hence there is a fibration

$$Z_K(S^2, \mathbb{Z}/2\mathbb{Z}) \to Z_K(\mathbb{R}P^2, *) \to B(\mathbb{Z}/2\mathbb{Z})^n.$$

If $K$ is the 0-skeleton of the 1-simplex, then $Z_K(S^2, \mathbb{Z}/2\mathbb{Z}) = (S^2 \times \mathbb{Z}/2\mathbb{Z}) \cup (\mathbb{Z}/2\mathbb{Z} \times S^2)$ is equivalent to the wedge sum $(\bigvee_4 S^2) \vee S^1$. Hence, $Z_K(S^2, \mathbb{Z}/2\mathbb{Z})$ is not simply connected.

Nevertheless, the following theorem shows that the fundamental group of these polyhedral products can be computed with the information that was obtained from Section 3.

**Lemma 4.5.** If $A \subset X \subset Y$ with $A$ finite discrete, $X$ and $Y$ path-connected, such that the induced map $\pi_i(X) \to \pi_i(Y)$ is an isomorphism for $i = 0, 1$, then

$$\pi_1(Z_K(X, A)) \cong \pi_1(Z_K(Y, A)).$$

**Proof.** Let $D_X(\sigma)$ denote the functor $D$ evaluated on the pair $(X, A)$ and $D_Y(\sigma)$ denote the functor $D$ evaluated on the pair $(Y, A)$. If $\sigma \in K$ is a simplex and if $\pi_i(X) \to \pi_i(Y)$ is an isomorphism for $i = 0, 1$, then the inclusion map $D_X(\sigma) \to D_Y(\sigma)$ induces an isomorphism on the fundamental group. By definition,

$$\pi_1(Z_K(X, A)) = \pi_1(\text{colim}_{\sigma \in K} D_X(\sigma))$$

and

$$\pi_1(Z_K(Y, A)) = \pi_1(\text{colim}_{\sigma \in K} D_Y(\sigma)).$$

Recall that if $X_2$ is the 2-skeleton of $X$, then $\pi_1(X) \cong \pi_1(X_2)$. Similarly, if $Y_2$ is the 2-skeleton of $Y$, then $\pi_1(Y) \cong \pi_1(Y_2)$. So this shows that $\pi_1(X_2) \cong \pi_1(Y_2)$. Similarly, it suffices to work with the 2-skeleton of the spaces $Z_K(X, A)$ and $Z_K(Y, A)$. Denote these two spaces by $Z_K(X, A)_2$ and $Z_K(Y, A)_2$, respectively. It is clear that there are inclusions up to homotopy $Z_K(X, A)_2 \to Z_{SK_1}(X_2, A)$ and $Z_K(Y, A)_2 \to Z_{SK_1}(Y_2, A)$ since adding a simplex $\sigma$ on $k \geq 3$ vertices, does not change the 2-skeleton of the polyhedral product. Both inclusions induce isomorphisms in $\pi_1$. This suffices.

Note that Lemma 4.5 can be proved using the fact that there is a homotopy equivalence

$$Z_K(X, A) = \text{colim}_{\sigma \in K} D_X(\sigma) \simeq \text{hocolim}_{\sigma \in K} D_X(\sigma),$$

where $\text{colim}_{\sigma \in K} D_X(\sigma)$ represents the colimit of the family of functors $D_X(\sigma)$.
see [2], and the fact that
\[ \pi_1(\operatorname{hocolim}_{\sigma \in K} D_X(\sigma)) \cong \colim_{\sigma \in K} \pi_1(D_X(\sigma)), \]
see [8] for details.

Next consider the pair \((BG, BH)\), where \(H\) is a closed subgroup of \(G\). As mentioned in the introduction, there is an inclusion \(G/H \hookrightarrow C(G/H)\) which gives an inclusion \(EG \times G/H \hookrightarrow EG \times C(G/H)\). Hence, \(G/H\) can be regarded as a \(G\)-equivariant subspace of \(EG\). There is a fibration
\[ Z_K(EG, G/H) \longrightarrow (EG)^n \times_{G^n} Z_K(EG, G/H) \longrightarrow (BG)^n. \]
Since \(EG \times G/H \cong G/H\) and \(EG \times_G G/H \cong EG/H \cong BH\), then the space \((EG)^n \times_{G^n} Z_K(EG, G/H)\) is homotopy equivalent to \(Z_K(EG/G, EG \times_G G/H) \cong Z_K(BG, BH)\). This proves the following proposition

**Proposition 4.6.** Let \(H\) be a closed subgroup of the Lie group \(G\). There is a fibration given by \(Z_K(EG, G/H) \longrightarrow Z_K(BG, BH) \longrightarrow (BG)^n\).

**Proposition 4.7.** Let \(H\) be a closed subgroup of \(G\). Then there is a homotopy equivalence
\[ \Omega(Z_K(BG, BH)) \cong G^n \times \Omega(Z_K(EG, G/H)) \]
and a short exact sequence of groups
\[ 1 \longrightarrow \pi_1(Z_K(EG, G/H)) \longrightarrow \pi_1(Z_K(BG, BH)) \longrightarrow G^n \longrightarrow 1. \]
Furthermore, this decomposition may not preserve the group structure.

**Proof.** There is a fibration as follows
\[ Z_K(EG, G/H) \longrightarrow Z_K(BG, BH) \longrightarrow (BG)^n. \]
Taking the loops of this fibration gives a fibration
\[ \Omega(Z_K(EG, G/H)) \longrightarrow \Omega(Z_K(BG, BH)) \longrightarrow \Omega((BG)^n) = G^n. \]
There is a section \(G^n \overset{s}{\longrightarrow} \Omega(Z_K(BG, BH))\) which implies that there is a homotopy equivalence of topological spaces
\[ \Omega(Z_K(BG, BH)) \cong G^n \times \Omega(Z_K(EG, G/H)). \]
The short exact sequence follows from this equivalence. \(\square\)

5. An Extension Problem

In this section we are interested in investigating a certain extension problem that arises from polyhedral products. Assume \(\{G_1, \ldots, G_n\}\) is a family of subgroups of a finite discrete group \(G\). Then there is a natural map
\[ G_1 \ast \cdots \ast G_n \longrightarrow G. \]
A. Adem, F. Cohen and E. Torres Giese [1] introduced the spaces \(B(q, G)\) for integers \(q \geq 2\), which form a sequence of spaces and a filtration of \(BG\)
\[ B(2, G) \subset B(3, G) \subset \cdots \subset B(\infty, G) = BG, \]
In particular, \(B(2, G)\) is defined to be the geometric realization
\[ B(2, G) = \left( \bigsqcup_{k \geq 1} \operatorname{Hom}(Z^k, G) \times \Delta[k] \right)/\sim, \]
where \(\sim\) is generated by the standard face and degeneracy relations, see [12].

A proof of the following lemma can be found in [1].

**Lemma 5.1.** Let \(G\) be a nonabelian group. The following are equivalent
a. If \(g \notin Z(G)\), then \(C(g)\) is abelian.
b. If \([g, h] = 1\), then \(C(g) = C(h)\) whenever \(g, h \notin Z(G)\).

c. If \([g, h] = 1 = [h, k]\), then \([g, k] = 1\) whenever \(h \notin Z(G)\).

d. If \(A, B \leq G\) and \(Z(G) < C_G(A) \leq C_G(B) < G\), then \(C_G(A) \leq C_G(B)\).

**Definition 5.2.** Let \(G\) be a nonabelian group. If \(G\) satisfies any of the equivalent statements in Lemma 5.1, then \(G\) is called a **transitively commutative** group, or simply a \(TC\) group.

In [1] Adem, Cohen and Torres Giese prove the following theorem.

**Theorem 5.3 (Adem, Cohen & Torres Giese).** If \(G\) is a finite \(TC\) group with trivial center, then there is a homotopy equivalence

\[
B(2, G) \simeq \bigvee_{1 \leq i \leq k} \left( \prod_{p | |C_G a_i|} BP \right),
\]

where \(P \in \text{Syl}_p(G)\).

This theorem states that if \(G\) is a \(TC\) group, then

\[
B(2, G) \simeq BG_1 \vee \cdots \vee BG_n,
\]

and this is the polyhedral product \(Z_{K_0}(BG)\). Hence, there is a map

\[
Z_{K_0}(BG) \rightarrow BG.
\]

On the level of the fundamental groups, this brings the discussion back to the homomorphism

\[
G_1 \ast \cdots \ast G_n \xrightarrow{\varphi} G.
\]

It is natural to ask for which abstract simplicial complexes \(K\) on \([n]\), does the map \(BG_1 \vee \cdots \vee BG_n \xrightarrow{B\varphi} BG\) extend to \(Z_K(BG)\), that is for which \(K\) does the following diagram commute

\[
\begin{array}{ccc}
BG_1 \vee \cdots \vee BG_n & \xrightarrow{B\varphi} & BG \\
\downarrow i & & \downarrow \iota_S \\
Z_K(BG) & & G
\end{array}
\]

The importance of this question is that if the map in question extends, then we detect commuting elements in \(G\). One could also pose the same question algebraically. For what simplicial complexes \(K\) does the following diagram commute

\[
\begin{array}{ccc}
G_1 \ast \cdots \ast G_n & \xrightarrow{\varphi} & G \\
\downarrow \iota_S & & \downarrow \iota_S \\
\prod_{SK_1} G_i & & \prod_{SK_1} G_i
\end{array}
\]

The rest of this section will be devoted to understanding this question.

The first example is that of a transitively commutative (TC) group \(G\) with trivial center, and the family of subgroups \(\{G_1, \ldots, G_n\}\) is chosen to consist of the maximal abelian subgroups of \(G\).

**Proposition 5.4.** Let \(G\) be a transitively commutative group and let \(\{A_1, \ldots, A_k\}\) be the distinct maximal abelian subgroups of \(G\). Then the map

\[
B \varphi : BA_1 \vee \cdots \vee BA_k \rightarrow BG
\]

does not extend for any simplicial complex \(K\) such that \(K_0 \subseteq K\).

**Proof.** If it extends for a simplicial complex \(K\), then \(K\) has at least one edge, say \([i, j]\). This implies that \([A_i, A_j] = 1\) for some \(i \neq j\). One can check that the groups \(A_i\) are centralizers of elements in \(G\) not in the center, and that they intersect trivially, yielding a contradiction to the assumption. \(\square\)
Recall that for a group $G$ there is a normal series called the descending central series of $G$ given by

$$G = \Gamma^1(G) \supset \Gamma^2(G) \supset \cdots \supset \Gamma^n(G) \supset \cdots,$$

where $\Gamma^2(G) = [G, G]$ and $\Gamma^{n+1}(G) = [\Gamma^n(G), G]$ for all $n \geq 2$, and $\Gamma^k(G)$ is called the $k$-th stage series.

Next we extend the notion of TC-groups by defining a similar class of groups with similar properties, which eventually include TC-groups as a special case.

**Definition 5.5.** Let $g_l, k_l \in \Gamma^{-l}(G)$. We say that $G$ is an $l$-transitively commutative group, or simply $l$-TC group, if

$$[g_l, h] = 1 = [h, k_l] \implies [g_l, k_l] = 1 \text{ for all } h \in G.$$

Note that a 2-TC group is the same as the ordinary TC group. Also that the condition that $l \geq 2$ is required for the definition.

**Remark.** If $G$ is an $m$-TC group then it is an $n$-TC group for all $n \geq m$. This follows from the structure of the descending central series. Hence, if $G$ is a TC group, then it is a $k$-TC group for all $k \geq 2$. By convention, let finite simple groups be called $1$-TC groups.

**Example 5.6.** A group which is 3-TC but not 2-TC is the quaternion group, $Q_8$. This group has a presentation as follows

$$Q_8 = \{ \pm 1, \pm i, \pm j, \pm k \mid i^2 = j^2 = k^2 = 1, ij = k, jk = i, ki = j \}.$$

First we note that $[i, -1] = 1 = [-1, j]$ but $[i, j] = -1$, so $Q_8$ is not 2-TC. Now $[i, j] = [j, k] = [k, i] = -1$ and $[[i, j], h] = 1$ for any $h \in Q_8$. Therefore $Q_8$ is 3-TC. One can also compute the descending central series for $Q_8$. Its commutator subgroup is $\Gamma^2(Q_8)[Q_8, Q_8] = \{ \pm 1 \} \cong \mathbb{Z}/2\mathbb{Z}$ and $\Gamma^3(Q_8) = [Q_8, Q_8, Q_8] = 1$. Therefore, the descending central series for $Q_8$ is

$$1 < \mathbb{Z}/2\mathbb{Z} < Q_8.$$

**Lemma 5.7.** If $G$ is nilpotent of nilpotency class $m$, then $G$ is an $m$-TC group.

**Proof.** $G$ is nilpotent of nilpotency class $m$ means $\Gamma^m(G) = 1$, that is, $[g_m, h] = 1$ for all $g_m \in \Gamma^{m-1}(G), h \in G$. In particular, $[g_m, k_m] = 1$ for all $g_m, k_m \in \Gamma^{m-1}(G)$, hence $G$ is $m$-TC. \hfill $\square$

**Definition 5.8.** Let $g \in G$. The $l$-stage centralizer of $g \in G$, denoted $C^l_G(g)$, is the subgroup

$$C^l_G(g) := \{ g_{l+1} \in \Gamma^l(G) \mid g_{l+1} g_l^{-1} g_{l+1} = g \}.$$

We can also write $C^l_G(g) = C_G(g) \cap \Gamma^l(G)$.

**Lemma 5.9.** Let $G$ be an $(l+1)$-TC group with trivial center. Then $C^l_G(g)$ are abelian subgroups of $G$ for all $g \in G$. Moreover, if $Z(G) = 1$, then $C^l_G(g)$ are distinct and either intersect trivially or coincide.

**Proof.** It is clear that $C^l_G(g)$ are abelian. If $C^l_G(g) \cap C^l_G(h)$ is non-trivial then there is $1 \neq k \in C^l_G(g) \cap C^l_G(h)$ and thus, the two coincide by definition of an $(l+1)$-TC group. \hfill $\square$

**Lemma 5.10.** Let $\{C^l_G(g)\}_{g \in A}$ be the family of distinct $l$-stage centralizers of elements in $G$. If $G$ is an $(l+1)$-TC group with $Z(G) = 1$, the map

$$\bigvee_{g \in A} BC^l_G(g) \to BG$$

does not extend for any $K$ on $[m]$ where $m = |\{C^l_G(g)\}_{g \in A}|$.

**Proof.** This follows from the previous lemma. \hfill $\square$
Corollary 5.11. Let $G$ be a finite $(k+1)$-TC group with trivial center. Let $G_1, G_2$ be two subgroups of $G$ such that $C_G^l(g_1) \leq G_1$ and $C_G^l(g_2) \leq G_2$, where $C_G^k(g_1) \cap C_G^k(g_2) = 1$. Then the map

$$BG_1 \lor BG_2 \to BG$$

does not extend to $BG_1 \times BG_2$.

Proof. The proof follows from Lemma 5.10. □

Using this corollary the following theorem follows easily. This theorem applies to TC groups in particular.

Theorem 5.12. Let $\{G_i\}_{i=1}^m$ be a family of subgroups of a finite $(k+1)$-TC group $G$ with trivial center. If $G_i$ contain distinct $C_G^l(g_i)$, for $i = 1, 2, \ldots, m$ and $l \geq k$, then the map

$$\bigvee_{1 \leq i \leq m} BG_i \to BG$$

does not extend to $Z_K(BG)$ for any $K$ on $[m]$, other than the 0-skeleton.

Proof. From the definition we have $C_G^l(g_i) = C_G(g_i) \cap \Gamma^l(G)$ which gives $C_G^l(g_i) \geq C_G^{l+1}(g_i)$. The rest of the proof follows from Corollary 5.11. □

Below we give a result for when this extension actually exists.

Definition 5.13. Let $K$ be a simplicial complex. The flag complex of $K$ is the simplicial complex Flag($K$) obtained from $K$ by completing the minimal non-faces to faces.

Note that an edge $\{i, j\}$ is in $K$ if and only if it is an edge in Flag($K$). Now let $\{H_1, \ldots, H_n\}$ be a collection of subgroups of a group $G$. Let $\Gamma$ be a graph on $n$ vertices that records the commutativity relations of the set $\{H_1, \ldots, H_n\}$. That means, $\{i, j\}$ is an edge in $\Gamma$ if and only if $[H_i, H_j] = 1$. Let Flag($\Gamma$) be flag complex of $\Gamma$.

Lemma 5.14. Let $\{H_1, \ldots, H_n\}$ be a collection of subgroups of a finite group $G$ with $Z(G) = 1$. If $\Gamma$ is the graph described above, then the map

$$BH_1 \lor \cdots \lor BH_n \to BG$$

extends to $Z_{\text{Flag}(\Gamma)}(BH)$.

Proof. By definition, Flag($\Gamma$) is a flag complex. Hence, the polyhedral product $Z_{\text{Flag}(\Gamma)}(BH)$ is an Eilenberg-Mac Lane space with fundamental group $\prod_\Gamma H_i$. So there is a commutative diagram of groups

$$H_1 \star \cdots \star H_n \xrightarrow{\varphi} G$$

$$\downarrow^{i \#} \quad \downarrow^{id}$$

$$\prod_\Gamma H_i \xrightarrow{\tilde{\varphi}} G.$$ 

Hence, there is a commutative diagram of spaces

$$BH_1 \lor \cdots \lor BH_n \xrightarrow{B\varphi} BG$$

$$\downarrow^{B i \#} \quad \downarrow^{id}$$

$$Z_{\text{Flag}(\Gamma)}(BH) \xrightarrow{B \tilde{\varphi}} BG.$$ 

□

This lemma actually holds for any simplicial complex $K$ such that $\Gamma \subseteq K \subseteq \text{Flag}(\Gamma)$, even though $K$ is not a flag complex.
Theorem 5.15. Let \( \{H_1, \ldots, H_n\} \) be a collection of subgroups of a finite group \( G \) with \( Z(G) = 1 \). If \( \Gamma \) is the graph described above and \( K \) is a simplicial complex such that \( \Gamma \subseteq K \subseteq \text{Flag}(\Gamma) \), then the map
\[
BH_1 \vee \cdots \vee BH_n \longrightarrow BG
\]
events to \( Z_K(BH) \).

Proof. The proof follows by naturality and Lemma 5.14. First, the following diagram of inclusions commutes
\[
\begin{array}{ccc}
BH_1 \vee \cdots \vee BH_n & \overset{\text{id}}{\longrightarrow} & Z_{\text{Flag}(\Gamma)}(BH) \\
\downarrow i_1 & & \downarrow i_2 \\
Z_K(BH) & & Z_{\text{Flag}(\Gamma)}(BH)
\end{array}
\]
Combining this diagram with the diagram of spaces in Lemma 5.14, it follows that the diagram
\[
\begin{array}{ccc}
BH_1 \vee \cdots \vee BH_n & \overset{\text{id}}{\longrightarrow} & BH_1 \vee \cdots \vee BH_n \\
\downarrow i_1 & \overset{\text{id}}{\longrightarrow} & \downarrow i_2 \\
Z_K(BH) & \overset{B_i \varphi}{\longrightarrow} & Z_{\text{Flag}(\Gamma)}(BH) \\
\downarrow i_2 & \overset{\text{id}}{\longrightarrow} & \downarrow i_2 \\
& \overset{B_i \varphi}{\longrightarrow} & BG
\end{array}
\]
commutes. Hence, the map \( BH_1 \vee \cdots \vee HB_n \overset{B_i \varphi}{\longrightarrow} BG \) extends to \( Z_K(BH) \). That is, the following diagram of spaces commutes
\[
\begin{array}{ccc}
BH_1 \vee \cdots \vee BH_n & \overset{B_i \varphi}{\longrightarrow} & BG \\
\downarrow i_1 & \overset{B_i \varphi \circ i_2}{\longrightarrow} & \downarrow i_2 \\
Z_K(BH) & \overset{B_i \varphi \circ i_2}{\longrightarrow} & BG
\end{array}
\]
\[ \Box \]

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