Communication-Optimal Parallel Standard and Karatsuba Integer Multiplication in the Distributed Memory Model

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Abstract

We present COPSIM a parallel implementation of standard integer multiplication for the distributed memory setting, and COPK a parallel implementation of Karatsuba's fast integer multiplication algorithm for a distributed memory setting. When using $P$ processors, each equipped with a local memory, to compute the product of two $n$-digits integer numbers, under mild conditions, our algorithms achieve optimal speedup of the computational time. That is, $O(n^2/P)$ for COPSIM, and $O(n^{\log_2 3}/P)$ for COPK. The total amount of memory required across the processors is $O(n)$, that is, within a constant factor of the minimum space required to store the input values. We rigorously analyze the Input/Output (I/O) cost of the proposed algorithms. We show that their bandwidth cost (i.e., the number of memory words sent or received by at least one processors) matches asymptotically corresponding known I/O lower bounds, and their latency (i.e., the number of messages sent or received in the algorithm's critical execution path) is asymptotically within a multiplicative factor $O(\log_2^2 P)$ of the corresponding known I/O lower bounds. Hence, our algorithms are asymptotically optimal with respect to the bandwidth cost and almost asymptotically optimal with respect to the latency cost.

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1 Introduction

Integer multiplication is a widely used and widely studied basic primitive with many important applications, among which primes factorization is of particular notice due to its impact on the field of cryptography [20, 28]. The importance of integer multiplication can be fully appreciated by noting many computers implement it in hardware. Still, it is also complex enough that in many other very successful cases, it is entirely computed by software.

The standard algorithm (also known as the long multiplication or the schoolbook algorithm) takes \( \Theta(n^2) \) digit operations to multiply two \( n \)-digit numbers. In 1960, Karatsuba [22] showed how to improve the bound to \( \Theta(n^\omega) \), where \( \omega = \log_2 3 \approx 1.585 \). This result has motivated a number of efforts which have led to increasingly faster algorithms. Among these, of particular note are the Toom-Cook algorithmic scheme originally introduced by Andrei Toom [38] for circuits and later adapted by Stephen Cook [14] to software programs, the asymptotically faster \( \Theta(n \log n \log \log n) \) Schönhage-Strassen algorithm [33], and Fürer’s algorithm [16] with complexity \( \Theta(n \log n 2^{O(\log^* n)}) \), where \( \log^* n \) is the iterated logarithm, and, most recently, the algorithm by Harvey and van der Hoven [19] with complexity \( \Theta(n \log n) \). However, due to the, sometimes extremely high, constant multiplicative factors “hidden” by the asymptotic notation, standard-long integer multiplication and Karatsuba’s algorithm actually outperform the other, asymptotically faster, algorithms for a wide range of input sizes up to \( 2^{2^{14}} \) [17]. Hence, both standard and Karasuba’s algorithms are of great practical interest. The problem of improving the performance of integer multiplication algorithms is actively researched, as evidenced by the significant number of publications in this field.

While promising, designing parallel algorithms based on the known fast multiplication algorithms appears challenging due to the apparent “sequential nature” of the integer multiplication algorithms discussed so far, and the necessity to carefully manage communications among the processors participating in the computation around such sequential components.

When designing efficient parallel algorithms, it is important not only to balance the computational effort among processors but also to minimize the time spent by the processors communicating to each other to transfer data and coordinate operations. The communication cost (or I/O cost) is, in many cases, much higher than that due to computation, and, therefore, is the real bottleneck of algorithmic performance. This technological trend [31] appears destined to continue, as physical limitations on minimum device size and maximum message speed lead to inherent costs when moving data, whether across the levels of a hierarchical memory system or between processing elements of a parallel system [7]. Due to these challenges, most parallel algorithms for integer multiplication were proposed for the shared memory model where all processors have access to a shared memory space (among others, [27, 24, 18, 21, 37, 15]). While this model simplifies many of the mentioned challenges related to communication, it is rather unrealistic for modern architectures. In this work, we consider a more realistic parallel distributed-memory model, where each of the \( P \) processor is equipped with a local (non-shared) memory space which can hold up to \( M \) memory words, and data communication among processors occurs only by message exchange.

Other approaches have been presented for specific hardware devices (e.g., FPGA) [30, 32, 2, 36] or for models with limitations in the number of available processors. While some
parallel versions of the standard and Karatsuba algorithms were presented in the literature for the distributed memory model, these contributions focus on specific settings with respect to the number of available processors \[10,11\], or assume unbounded local memory space \[23\]. Further, in all mentioned contributions, the impact of the communication over execution time is evaluated through experimental evaluation of specific implementations rather than a formal theoretical analysis, or a rigorous comparison with theoretical lower bounds.

In recent contributions, De Stefani \[34\], and Bilardi and De Stefani \[8\] presented the first analytical lower bound on the communication cost of, respectively, standard integer multiplication algorithms, and Toom-Cook fast integer multiplication, of which Karatsuba can be seen as a special case. Their results for the distributed-memory parallel model yield lower bounds for both the bandwidth cost (i.e., the number of memory words transmitted by at least one processor) and the latency (i.e., the number of messages exchanged by at least one processor) of parallel integer multiplication algorithms. These works left open the important question of whether it is actually possible to construct algorithms matching these bounds.

In this work, we present COPSIM a parallel algorithm based on the recursive long multiplication algorithm, and COPK a parallel fast integer multiplication algorithm based on Karatsuba’s algorithm. Both our algorithms are designed for the distributed memory model. Under very mild conditions (i.e., \(n \geq P\) and \(M \geq \log_2 P\)), our algorithms achieve optimal speedup of the computation time with respect to their sequential counterpart, asymptotically optimal communication bandwidth cost, and latency within a \(O(\log^2 P)\) multiplicative factor of the corresponding lower bounds \[8, 34\]. Finally, both our algorithms require only \(O(n)\) memory space to be available when combining the size of the local memories of the processors. That is, the total required memory space is within a constant multiplicative factor of the memory space required to store the input. Hence, COPSIM and COPK have asymptotically optimal memory requirement. To the best of our knowledge, ours are the first parallel algorithms for integer multiplication to achieve computational and communication optimality in the distributed memory setting.

**Related work** As discussed in the introduction, various parallel implementation of standard-long integer multiplication algorithms have been presented in the literature for the shared memory model (among others, \[27, 24, 18, 21, 37, 15\]), and for for specific hardware (among others, \[30, 32, 2, 36\]). The analysis of the communication component of these algorithms’ execution time is mostly given as experimental evaluation of specific implementations of the proposed algorithms rather than a formal analysis of their scalability for a range of values if input size, number of available processors, and available memory. In this work, we present a rigorous analysis of the computation time, memory requirement, and communication cost for both our proposed algorithms.

Similarly, parallel versions of the Karatsuba’s algorithm are mostly presented for the shared memory setting \[25\], or focus on the experimental analysis of specific implementations without formally analyzing the scalability and the communication cost of the proposed algorithms \[11\]. In \[10\], Cesari and Maeder introduce three parallel Karatsuba-based algorithms for the distributed memory setting: The first two algorithms have time complexity \(O(n)\), where \(n\) denotes the number of digits of the input integers, when using \(n^{\log_4 3}\) processors. The last one exhibits \(O(n \log n)\) time complexity while using \(n\) processors. Their
approach follows an *master-slave* approach where single processors are assigned recursively-generated subproblems to be solved in parallel, and they may themselves use other, still unused processors to do so. Thus, the scalability of these algorithms is limited by the fact that long integer additions and subtractions need to be computed by single processors. Further, their approach does not account for limitations due to the size of the local memory available to the processors being used, as several processors need to store integer values of size $O(n)$ entirely. In contrast, our COPK algorithm achieves computational time $O\left(n^{\log_2 3}/P\right)$ for any number $P \leq n$ of processors available processors. Both the computational and the communication cost of COPK scales proportionally with $1/P$, thus exhibiting perfect strong scaling. Further, the *cumulative memory space* across the processors required by the algorithm is within a constant factor of that necessary to represent the input factor integers and their product.

The analysis of the communication requirement of algorithms has been studied extensively in the literature both in the sequential and the parallel setting. There have been also numerous efforts to obtain communication efficient parallel algorithms for many important problems among whom the computation of the FFT [13], Cholesky decomposition [5, 29], Matrix Factorization [29], and Matrix Multiplication [3, 4, 6, 12, 9, 1]. In particular, in [8] Ballard et. al presented CAPS a parallel version of Strassen’s algorithm for fast matrix multiplication [35]. Their algorithm achieves optimal speedup and it minimizes the bandwidth cost among all parallel Strassen-based algorithms. This work draws inspiration from the technique used in their work to obtain communication-optimal algorithms for integer multiplication. Doing so requires several major, and challenging, modifications due to differences between matrix and integer multiplication, and, in particular, the apparently sequential nature of components of the latter, which our algorithms overcome by speculatively precalculating some intermediate results of the algorithm.

As mentioned in the introduction, De Stefani [31] and Bilardi and De Stefani [8], presented, respectively, lower bounds on the communication complexity of parallel implementations of standard-long integer multiplication algorithms [34], and of Toom-Cook algorithms [8] in a memory-distributed model. We present these bounds in detail in Section 2.3 and they will serve as a term of comparison when evaluating the performance of our proposed algorithms.

Our contributions We present two parallel integer multiplication algorithms for the distributed memory setting called COPSIM (Communication Optimal Parallel Standard Integer Multiplication) and COPK (Communication Optimal Parallel Karatsuba):

- COPSIM computes the product of two given $n$ digits input integers using $P$ processors (with $n \geq P$) each equipped with a local memory of size $M \geq 24\sqrt{P}$, in $O\left(n^2/P\right)$ parallel computational steps. COPSIM exhibits $O\left(n^2/(MP)\right)$ bandwidth cost and $O\left(n^2/(M^2P)\right)$ latency. Thus, by the known lower communication lower bounds in [34]:

Theorem 1 (Communication optimality of COPSIM). COPSIM achieves optimal computation time speedup and optimal bandwidth cost among all parallel standard integer multiplication algorithms. It also minimizes the latency cost up to a $O\left(\log^2 P\right)$ multiplicative factor.
• COPK computes the product of two given $n$ digits input integers using $P$ processors (with $n \geq P$) each equipped with a local memory of size $M \geq 10P^{(\log_3 3)/2}$, in $O\left(\frac{n^{\log_2 3}}{P}\right)$ parallel computational steps. COPK exhibits $O\left(\left(\frac{n}{M}\right)^{\log_2 3} \frac{M}{P}\right)$ bandwidth cost and $O\left(\left(\frac{n}{M}\right)^{\log_2 3} \frac{1}{P}\right)$ latency. Thus, by the known lower communication lower bounds in [8, 34]:

**Theorem 2** (Communication optimality of COPK). COPK achieves optimal computation time speedup and optimal bandwidth cost among all parallel Karatsuba-based integer multiplication algorithms. It also minimizes the latency cost up to a $O\left(\log^2 P\right)$ multiplicative factor.

Both our algorithms require $O\left(n\right)$ total memory space to be available across all processors. That is, each of the $P$ processors requires a local memory of size only $O\left(n/P\right)$. That is, the required memory space is within a constant factor of the minimum memory space necessary to store the input (and output) values. Both COPSIM and COPK are strongly scaling as both the computation time and the bandwidth cost scale linearly with respect to $P^{-1}$, provided that the size of the local memory of each processor scales accordingly (i.e., it is $O\left(n/P\right)$).

A rigorous analysis of the performance of COPSIM (resp., COPK) is given in Theorem 11 and Theorem 12 in Section 5 (resp., Theorem 14 and Theorem 15 in Section 6). Proof of Theorem 1 is presented in Section 5.3, and proof Theorem 2 is presented in Section 6.3. Further, these results imply that the lower bounds on the bandwidth cost of parallel standard-long inter multiplication algorithms [34] and for parallel Karatsuba’s algorithms [8, 34] are indeed asymptotically tight.

Our methods use a recursive divide-and-conquer approach and speculatively precalculate multiple possible values that may be used in the continuation of the algorithm in order to overcome the challenges related to the apparently sequential nature of integer multiplication algorithms. While this may seem wasteful in computation time and usage of available computational resources, this allows us to exploit the available parallelism while incurring low computational overhead.

Our algorithms are designed with the intent of making the best possible use of the memory space available to each processor. This is achieved by analyzing the recursion tree corresponding to the algorithm’s execution, and by scheduling its traversal using an opportune combination of depth-first and breadth-first steps as discussed in Section 3.

**Paper organization** In Section 2 we present an overview of the notation and the computational model considered in this work. In Section 2.3 we present an overview of the known lower bounds on communication cost of integer multiplications, which will serve as a term of comparison in evaluating the performance of our proposed algorithms. In Section 3 we present the main common strategy used by both our proposed algorithm. In Section 4 we present subroutines for adding, comparing, and subtracting integers using multiple processors in the distributed memory setting. These subroutines are used extensively in our algorithms. We present and fully analyze COPSIM (Communication Optimal Integer Multiplication) in Section 5 and COPK (Communication Optimal Karatsuba) in Section 6.
2 Preliminaries

We discuss algorithms that compute the product of two integers: \( C = A \times B \). We assume the input integers to be expressed as a sequence of \( n \) base-\( s \) digits in positional notation. We further assume that each integer is represented as an unsigned integer with an additional bit to denote the sign. For a given integer \( A \), we denote its expansion in base \( s \) as:

\[
A = (A[n-1], A[n-2], \ldots, A[0])_s,
\]

where \( n \) is the number of digits in the base-\( s \) expansion of \( A \), and its digits are indexed in order from the least significant digit \( A[0] \) to the most significant digit \( A[n-1] \). Further, for \( i \in \{1, 1, \ldots, n\} \), we refer to \( A[i-1] \) (resp., \( A[n-i] \)) as the \( i \)-th least (resp., most) significant digit of \( A \). With a slight abuse of notation, we use \( A \) (resp., \( B \)) to denote both the value being multiplied and the set of input variables to the algorithm.

We refer to the number of digits of the base-\( s \) expansion of an integer \( A \) as its “size”, and we denote it as \(|A|\).

We consider parallel algorithms for integer multiplication in a distributed-memory parallel model where \( P \) processors, each equipped with local (non shared) memory that can hold up to \( M \) memory words, are directly connected to each other by a network. Each processor in the model is identified by an unique code given by an integer value from \( \{0, 1, \ldots, P-1\} \). Processors can exchange point-to-point messages, with every message containing up to \( B_m \) memory words. In the following, we refer to the number of memory words which can be stored in the local memory (resp., that can be transmitted in a single message), as the size of the memory (resp., of the messages). We assume that each processor is equipped with digit-wise product and algebraic sum elementary operations. Further, we assume that the processor is equipped with operations for producing the most and least significant digits of an integer in base \( s \). Unless explicitly stated otherwise, when referring to the “digits” of integers, we mean the digits of their expansion in the base chosen for their representation in memory.

2.1 Data layout

We assume both the input integers and intermediate results to be stored in memory expressed as their base-\( s \) expansion, with \( s \in \mathbb{N}^+ \), and with \( s \) being at most equal to the maximum value which can be maintained in a single memory word plus one. (That is, if a memory word can fit 32 bits, we have \( 2 \leq s \leq 2^{32} - 1 \).) In particular, we assume each digit in the base-\( s \) expansion of a value to be stored in a different memory word.

Given a set of available processors, in this work, we will often consider them organized in ordered sequences. An ordered sequence of processors \( P = (P_{|P|-1}, \ldots, P_0) \), we denote as \( P[i] \) the \( i \)-th processor in the sequence (indexed from the end), for \( 0 \leq i \leq |P|-1 \). That is, if \( P = (P_z, P_y, \ldots, P_b, P_a) \), then \( P[0] = P_a \), \( P[1] = P_b \), and \( P[|P|-1] = P_z \). Such ordered sequences will be used extensively through the presentation to clarify the organization of the processors in the computation, the assignment of digits of the same integer value across the local memory of multiple processors, and the patterns of communication among the processors.

Given an integer \( A \), the digits of its base-\( s \) representation may be stored in the local memories of different processors. Given an ordered sequence of processors \( P \), we say that
an $n$-digit integer $A$ is “partitioned among the processors in $P$ in $n'$ digits” if, for $0 \leq j \leq |P| - 1$, the $n'$ digits of $A$ form the $(jn')$-least significant (if any) to the $((j + 1)n')$-least significant are stored in the local memory of processor $P[j + 1]$. If $n \geq n'|P|$, the remaining digits of $A$ are stored in the local memory of $|P| - 1$. Sometimes we use the shorter expression “$A$ is partitioned in $P$”, which implies $n' = \lceil n/|P| \rceil$. When the digits of an integer $A$ are distributed among multiple processors, we assume that their digits are stored in positional notation in the local memories of each of these processors. In the following, we use the notation $A_{P_j}$ to denote the integer value whose base-$s$ expression corresponds to the digits of $A$ stored in the local memory of $P_j$.

### 2.2 Algorithmic performance metrics

We characterize the performance of the proposed algorithms according to the following metrics:

- The Memory requirement $M(n, P)$, which denotes the memory space used in the local memory available to each processor;
- The Computational cost $T(n, P, M)$, which denotes the number of digit-wise computations executed by during the algorithm’s execution;
- The Bandwidth cost $BW(n, P, M)$, which denotes the number of memory words exchanged during the algorithm’s execution;
- The Latency cost $L(n, P, M)$, which denotes the number of point-to-point words exchanged during the algorithm’s execution;

where $n$ denotes the number of digits of the integers being multiplied, $P$ denotes the number of processors being utilized, and $M$ denotes the size (in terms of memory words) of the local memory available to each processor. We count the number of digit-wise operations, memory words exchanged, and messages exchanged along the critical execution path of the algorithm as defined by Yang and Miller [39]. That is, operations executed in parallel by distinct processors are counted only once. Similarly, messages (and, thus, memory words) exchanged in parallel between distinct pairs of processors are counted only once. We assume that in any execution step of the algorithm a processor may only either send or receive a message to/from another processor but not both.

These metrics can be composed to characterize the execution time of the algorithm. Assume that the processors are homogeneous, that is, time $\alpha$ is required to compute a single digit-wise operation for each processor, and for each pair of processors the communication latency is $\beta$ and $\gamma$ time is required to transmit a memory word. Then the overall execution time of the algorithm can be bound as:

$$\alpha T(n, P, M) + \beta L(n, P, M) + \gamma BW(n, P, M).$$

While the values of the constants $\alpha, \beta$ and $\gamma$ depend on the specific hardware being used, our analysis holds for any device and any network being used.
2.3 Communication lower bounds for integer multiplication algorithms

Lower bounds for parallel standard integer multiplication algorithms In [34], De Stefani introduced the following lower bounds on the communication costs of any parallel standard-long integer multiplication algorithm for a model analogous to that considered in this work and discussed at the beginning of Section 2.

**Theorem 3** ([34][Corollary 8]). Let \( A \) be any standard integer multiplication algorithm which computes \( \Omega \left( n^2 \right) \) digit operations to multiply two integers \( A, B \) represented as \( n \)-digit base-\( s \) numbers using \( P \) processors each equipped with a local memory of size \( M < n \). Then:

\[
\begin{align*}
    BW(n, P, M) &= \Omega \left( \frac{n^2}{M^2P} \right), \\
    L(n, P, M) &= \Omega \left( \frac{n^2}{M^2} \right).
\end{align*}
\]

In the same work, the author also provides memory independent I/O lower bounds which hold under the assumption that the input integers are originally distributed in a balanced way among the processors, but require no assumption of the size of the local memories available to each processor:

**Theorem 4** ([34][Corollary 12]). Let \( A \) be any standard integer multiplication algorithm to multiply two integers \( A, B \) represented as \( n \)-digit base-\( s \) numbers using \( P \) processors each equipped with an unbounded local memory. Assume further that at the beginning of \( A \) no processor has more than \( \alpha n/P \) (where \( \alpha \) is a constant with respect to \( n/P \)) digits of each of the input integers stored in its local memory. Then:

\[
\begin{align*}
    BW(n, P) &\geq \Omega \left( \frac{n}{B^m\sqrt{P}} \right), \\
    L(n, P) &\geq \Omega \left( 1 \right).
\end{align*}
\]

The memory-independent bound is dominant for \( M \geq \Omega \left( \frac{n}{B^m\sqrt{P}} \right) \), while the memory-dependent bound is dominant for \( M \geq \Omega \left( \frac{n}{B^m\sqrt{P}} \right) \).

Lower bounds for parallel Karatsuba-based algorithms In [8], Bilardi and De Stefani introduced lower bounds on the communication costs of Toom-Cook integer multiplication algorithms for a model analogous to that considered in this work and discussed at the beginning of Section 2. As argued in their work, Karatsuba’s algorithm can be seen as a special case of Toom-2. Hence, here we present a bound on the communication cost of Karatsuba’s algorithm based on their more general result for Toom-Cook:

**Theorem 5** ([8][Theorem 4.2]). Let \( A \) be an algorithm which uses the Karatsuba recursive strategy to compute the product of two integers \( A, B \) whose base-\( s \) expansions have \( n \) digits using \( P \) processors each equipped with a local memory of size \( M \). Then:

\[
\begin{align*}
    BW(n, P, M) &= \Omega \left( \frac{n^{\log_2 3}}{M} \right), \\
    L(n, P, M) &= \Omega \left( \frac{n^{\log_2 3}}{P} \right).
\end{align*}
\]

7
In [34], De Stefani provides memory independent I/O lower bounds for parallel Toom-Cook algorithm which hold under the assumption that the input integers are originally distributed in a balanced way among the processors, but require no assumption of the size of the local memories available to each processor. Once again, here we present a bound on the communication cost of Karatsuba’s algorithm based on the more general result for Toom-Cook in [34]:

**Theorem 6 ([34][Corollary 12]).** Let $A$ be any Karatsuba-based integer multiplication algorithm to multiply two integers $A, B$ represented as $n$-digit base-$s$ numbers using $P$ processors each equipped with an unbounded local memory. Assume further that at the beginning of $A$ no processor has more than $\alpha n / P$ (where $\alpha$ is a constant with respect to $n / P$) digits of each of the input integers stored in its local memory. Then:

$$BW(n, P) \geq \Omega \left( \frac{n}{P^{10/\log_3 2}} \right),$$

$$L(n, P) \geq \Omega (1).$$

The memory-independent bound is dominant for $M \geq \Omega \left( \frac{n}{P^{\log_3 2}} \right)$, while the memory-dependent bound is dominant for $M \geq O \left( \frac{n}{P^{\log_3 2}} \right)$.

## 3 Overview on algorithm strategy

In this section, we outline the main design principle shared by both our proposed integer multiplication algorithms. We will then delve in the details of COPSIM in Section 5 and of COPK in Section 6. All the proposed algorithms follow a recursive strategy: sub-problems are recursively generated and opportunistically assigned to the processors being used depending on the input size, the number of available processors, and the size of the local memory assigned to each processor. At the bottom of the recursion, sub-problems are assigned to single processors, which then compute their solution locally without any further communication.

Let us consider the recursion tree corresponding to the execution of the algorithm. Each node in the tree corresponds to an invocation of COPSIM (resp., COPK). The root of the tree corresponds to the initial invocation. The descendants of a node correspond to the recursive invocations of COPSIM (resp., COPK) used to compute the four (resp., three) sub-problems generated at each recursive level. At each recursion level, the algorithms may choose to schedule the execution of the recursive calls in two possible ways:

- **Breadth First:** In a Breadth First Step (BFS), the available $P$ processors are divided into disjoint subsets of the same cardinality which are each assigned to compute in parallel one of the recursive subproblems.

- **Depth First:** In a Depth First Step (DFS), all processors are assigned to solve, each subproblem, in sequence one at a time.

A BFS incurs a lower communication and computation cost than a DFS, as multiple recursive branches of the execution tree can be continued in parallel. BF steps, however,
require a higher available memory than DFS. Therefore, to minimize communication and computation cost, we pursue a strategy based on the opportune composition of BF and DF steps. Such a strategy is ultimately aimed to make the best possible use of the available memory space.

Let $\mathcal{P}$ denote the number of available processors. In each BFS, the number of processors assigned to each sub-problem is reduced by a factor equal to the number of sub-problems being generated (i.e., 4 for COPSIM, and 3 for COPK). Hence, after $O(\log \mathcal{P})$ BFS, each of the generated sub-problem will be assigned to a single processor to be computed locally. Using DFS allows generating sub-problems of smaller input size whose result is computed using all the available processors.

All our algorithms operate in two execution modes:

- **Memory-independent execution mode (MI):** the algorithm executes $O(\log \mathcal{P})$ consecutive BFS after which each of the sub-problems generated at the $\ell_{BF}$-th level is assigned to a single processor, which computes the assigned sub-problem locally with no further communication. This traversal scheme is only possible if the total combined available memory of the processors being used is sufficient given the input size (i.e., the number of digits in the base-$s$ expansion of the input integers). The denomination “memory independent” highlights the fact that, provided that enough local memory space is available, the behavior of the algorithm while in the MI execution mode does not depend on the actual size of the local memories, which can then be assumed to be unlimited, but rather only on the number of available processors.

- **Main execution mode:** given as input $n$-digit integers, in the main execution mode, the algorithm proceeds by executing $\ell_{DF}$ consecutive DFS steps and then computes each of the sub-problems generated at the $\ell_{DF}$-th level in the MI execution mode. $\ell_{DF}$ is chosen as the minimum value be such that the size of the sub-problems generated at the $\ell_{DF}$-th level of recursion allows for them to be computed according to the MI scheme. As for both our algorithms the size of the sub-problems is reduced by half at every level of recursion, $O(\log_2 n/n_0)$ DF steps are executed in the main execution mode. This execution mode is also referred as the “Limited Memory execution mode”, as, contrary to the MI execution mode, the amount of available memory across the processors heavily affects the execution of the algorithm.

The values $\ell_{BF}$, $\ell_{DF}$ and $n_0$ mentioned above depend on the specific algorithm considered, its memory requirement, the available computation resources (processors and memory), and the input size. In Section 5 (resp., Section 6) we present the details of the proposed algorithms and their execution in both execution modes. Note that while it is possible to consider alternative strategies in which BFS and DFS are interleaved, we will show that our algorithms achieve optimal computation speedup and minimize the bandwidth communication cost.

4 Parallel Algorithmic components

In this section, we present subroutines for parallel addition, comparison, and subtraction of integer numbers, used in our algorithms for integer multiplications.
4.1 Parallel Sum with Distributed Memory

Let $\mathbf{P}$ be a sequence of processors each equipped with a local memory of size $M$, and let $A,B$ be two $n$-digits integer numbers partitioned into $\mathbf{P}$ in $n/|\mathbf{P}|$ digits. That is

$$A = (A(\lfloor|\mathbf{P}|/2\rfloor - 1) \ldots A(\lfloor|\mathbf{P}|\rfloor))_{s_1^{\mathbb{P}_1}},$$
$$B = (B(\lfloor|\mathbf{P}|/2\rfloor - 1) \ldots B(\lfloor|\mathbf{P}|\rfloor))_{s_1^{\mathbb{P}_1}}.$$

The parallel subroutine $\text{SUM}(\mathbf{P},A,B,n/|\mathbf{P}|)$ computes $C = A + B$ in parallel, with $C$ being partitioned in $\mathbf{P}$ in $n/|\mathbf{P}|$ digits, with $\mathbf{P}[|\mathbf{P}| - 1]$ holding the most significant of the $n+1$ digits of $C$. Further, all processors in $\mathbf{P}$ hold a value $v \in \{0,1\}$ which denotes the most significant digit of $C$. In the following, we assume $n$ and $|\mathbf{P}|$ to be integer powers of two. If that is not the case, our algorithm may be applied with minor adjustments (e.g., padding).

The algorithm follows a recursive strategy. When $\text{SUM}(\mathbf{P},A,B,,n/|\mathbf{P}|)$ is invoked, if $|\mathbf{P}| = 1$, the single processor $\mathbf{P}[0]$ computes $C$ and $v$ locally. If $|\mathbf{P}| > 1$, the algorithm executes the following operations:

1. The sequence of available processors is divided in subsequence:

$$\mathbf{P}' = [\mathbf{P}[\lfloor|\mathbf{P}|/2\rfloor - 1], \ldots, \mathbf{P}[0]]$$
$$\mathbf{P}'' = [\mathbf{P}[|\mathbf{P}| - 1], \ldots, \mathbf{P}[|\mathbf{P}|/2]] \quad (1)$$

Correspondingly, let:

$$A_0 = (A(\lfloor|\mathbf{P}|/2\rfloor - 1) \ldots A(\lfloor|\mathbf{P}|\rfloor))_{s_1^{\mathbb{P}_1}},$$
$$A_1 = (A(\lfloor|\mathbf{P}| - 1\rfloor) \ldots A(\lfloor|\mathbf{P}|/2\rfloor))_{s_1^{\mathbb{P}_1}},$$
$$B_0 = (B(\lfloor|\mathbf{P}|/2\rfloor - 1) \ldots B(\lfloor|\mathbf{P}|\rfloor))_{s_1^{\mathbb{P}_1}},$$
$$B_1 = (B(\lfloor|\mathbf{P}| - 1\rfloor) \ldots B(\lfloor|\mathbf{P}|/2\rfloor))_{s_1^{\mathbb{P}_1}} \quad (2)$$

where $A_0$ and $B_0$ (resp., $A_1$ and $B_1$) are partitioned in $\mathbf{P}'$ (resp. $\mathbf{P}''$) in $n/|\mathbf{P}|$ digits.

2. $\text{SUM}$ then invokes $\text{SUM}(\mathbf{P}',A_0,B_0,n/|\mathbf{P}|)$ and the auxiliary subroutine $\text{SUMA}(\mathbf{P}',A_1,B_1,n/|\mathbf{P}|)$ to be executed in parallel. $\text{SUMA}(\mathbf{P}'',\{A',B'\},n)$ computes

$$C_{00}' = (A' + B') \mod s^{|\mathbf{P}'|/|\mathbf{P}|},$$
$$u_0 = [(A'' + B'')/s^{|\mathbf{P}'|/|\mathbf{P}|}]$$
$$C_{10}' = (A' + B' + 1) \mod s^{|\mathbf{P}'|/|\mathbf{P}|},$$
$$u_1 = [(A'' + B'' + 1)/s^{|\mathbf{P}'|/|\mathbf{P}|}]$$

That is, $C_{00}'$ (resp., $u_i$), for $i \in \{0,1\}$, denotes the value corresponding to the $n|\mathbf{P}'|/|\mathbf{P}|$ least significant digits (resp., the most significant digit) of the sum $A' + B' + i$. Such values are used to $\text{speculatively precalculate}$ the value of the digits of the final output $C$ to be partitioned in $\mathbf{P}''$ for the two possible values of the carryover (i.e., $v$) of the sum $A_0 + B_0$ computed by $P_0$. All the computed $C_{00}'$ are partitioned in $\mathbf{P}''$ in $n/|\mathbf{P}|$ digits, and each processor in $\mathbf{P}''$ holds a copy of the $u_i$'s.
3. Let us denote as \( C' = (A_0 + B_0) \mod s^{n/2} \) and \( v' = [(A_0 + B_0)/s^{n/2}] \) the output values of the subroutine \( \text{SUM}(P', A_0, B_0, n/|P|) \). Clearly, \( C' \) corresponds the the value of the \( n/2 \) least significant digits of \( A+B \). In parallel, each processor \( P'[j] \), for \( 0 \leq j \leq |P|/2 - 1 \), sends to \( P''[j] \) the value \( v' \), and then removes it from its local memory.

4. Upon receipt each \( P''[j] \) assigns \( C(P''[j]) = C_{v'}(P''[j]) \) and \( v = u_{v'} \). In parallel, each \( P'[j] \) assigns \( C(P'[j]) = C'(P'[j]) \). At the end of this step, we have:

\[
C = (v, C(P''([|P|/2 - 1])), C(P''([|P|/2 - 2])), \ldots, C(P''[0]), C(P'([|P|/2 - 1]), \ldots, C(P'(0)))_{s^{n/|P|}}.
\]

That is, \( (C \mod s^n) \) is partitioned in \( P \) in \( n/|P| \) digits, and all processors in \( P'' \) have information on the most significant digit of \( C \). Then, each \( P'[j] \) (resp., \( P''[j] \)) removes from its memory the temporary value \( C'(P'[j]) \) (resp., \( C_{v'}(P''[j]) \), \( C_1'(P''[j]) \), \( u_0 \) and \( u_1 \)).

5. In parallel, each processor \( P''[j] \), sends to \( P'[j] \) a copy of \( v \).

After the completion of the initial invocation of \( \text{SUM} \), it easily possible to reconstruct the full \( C = A+B \) by having \( P([|P|/2 - 1]) \) append \( v \) as the most significant digit of \( C(P([|P|/2 - 1])) \). Once \( C \) is computed, all processors in \( P \) may remove \( v \) from their local cache.

To complete the description of the algorithm, we present the details of the auxiliary procedure \( \text{SUMA} \): If \( |P| = 1 \), \( \text{SUMA} (P, \{A, B\}, n') \) computes locally the following values:

\[
C_0 = (A + B) \mod s^{n'} \quad C_1 = (A + B + 1) \mod s^{n'}
\]

\[
u_0 = [(A + B)/s^{n'}] \quad u_1 = [(A + B + 1)/s^{n'}]
\]

where \( n' \) denotes the number of digits of \( A \) and \( B \) stored in the local memory of each processor.

Instead, if \( |P| > 1 \), the algorithm executes the following operations:

1. As done in point (2) of the description of \( \text{ADD} \), the sequence of available processors is divided in the two subsequences \( P' \) and \( P'' \) (1), and the input \( A \) (resp., \( B \)) is partitioned in \( A_0 \) and \( A_1 \) (resp., \( A_1 \) and \( B_1 \)) (2).

2. \( \text{SUMA} \) then recursively invokes \( \text{SUMA}(P', A_0, B_0, n') \) and \( \text{SUMA}(P'', A_1, B_1, n') \) to be executed in parallel. In the following we denote as \( C_0', C_1', u_0' \) and \( u_1' \) (resp., \( C_0'', C_1'', u_0'' \) and \( u_1'' \)) the output values of the former (resp., latter) call.

3. In parallel, each processor \( P'[j] \), for \( j = 0, 1, \ldots, |P|/2 - 1 \), sends to \( P''[j] \) the values \( u_0' \) and \( u_1' \), and then removes them from its local memory. Upon receipt each \( P''[j] \) assigns

\[
C_0(P''[j]) = C_{u_0'}(P''[j]) \quad C_1(P''[j]) = C_{u_1'}(P''[j])
\]

\[
u_0 = u_0'' \quad u_1 = u_1''
\]
In parallel, each $P'[j]$ assigns $C_0(P'[j]) = C'_0(P'[j])$ and $C_1(P'[j]) = C'_1(P'[j])$. Then each $P'[j]$ (resp., $P''[j]$) removes the values $C'_i, u'_i$ (resp., $C''_i, u'_i$) from its local memory, for $i = 0, 1$.

4. In parallel, each processor $P''[j]$, sends to $P'[j]$ a copy of $u_0$ and $u_1$.

The following lemma characterizes the memory requirement, the computational time and I/O cost of SUM:

**Lemma 7.** Let $A, B$ be two $n$-digit integers initially partitioned in a sequence of processors $P$ in $n/|P|$ digits. If each processor in $P$ is equipped with a local memory of size $4(n/|P| + 1)$, algorithm SUM computes the sum $C = A + B$. We have:

$$T_{SUM} (n, |P|) \leq 6n/|P| + 4 \log_2 |P|$$
$$B_{SUM} (n, |P|) \leq 4 \log_2 |P|$$
$$L_{SUM} (n, |P|) \leq 2 \log_2 |P|$$

**Proof.** SUM correctly computes $C = A + B$ by inspection. We focus on the analysis of the computational and I/O requirement of the auxiliary subroutine SUMA, as this subsumes the analysis of SUM. At any point during the computation each processor must maintain in memory at most $n/|P|$ digits of the input integers $A$ and $B$, $n/|P|$ digits for each the values $C_0$ and $C_1$ obtained from the last recursive call, two values $u_0$ and $u_1$ returned by the previous recursive call, and at most two copies of $u'_0$ and $u'_1$ received from another processor and not yet deleted. Hence, SUM can be executed if each processor is equipped with a local memory of size $4(n/|P| + 1)$. For $|P| > 1$, SUMA recursively invokes two instances to be executed in parallel. In step (3) and (4) of the algorithm, half of the processors send two memory words to distinct processors in the remaining half. Finally, at step (3), up to four comparisons are necessary to assign the values $C_0, C_1, u_0$ and $u_1$. Hence, we have that the computational cost, the bandwidth cost and the latency of SUMA satisfy:

$$T_{SUMA} (n, |P|) \leq T_{SUMA} (n/2, |P|/2) + 4$$
$$B_{SUMA} (n, |P|) \leq B_{SUMA} (n/2, |P|/2) + 4$$
$$L_{SUMA} (n, |P|) \leq L_{SUMA} (n/2, |P|/2) + 2$$

In the base case, for $|P| = 1$, SUMA computes the sums $C_0$ and $C_1$ locally without any further communication (i.e., $B_{SUMA}(n/|P|, 1) = L_{SUMA}(n/|P|, 1) = 0$). As the numbers being added have at most $n/|P|$ digits, each value can be computed using at most $3n/|P|$ elementary operations (i.e., $T_{SUMA}(n/|P|, 1, M) < 6n$). Thus, from (3) we have

$$T_{SUMA} (n, |P|) \leq T_{SUMA} (n/|P|, 1) + \sum_{i=1}^{\log_2 |P|} 4$$
$$B_{SUMA} (n, |P|) \leq B_{SUMA} (n/|P|, 1) + \sum_{i=1}^{\log_2 |P|} 4$$
$$L_{SUMA} (n, |P|) \leq L_{SUMA} (n/|P|, 1) + \sum_{i=1}^{\log_2 |P|} 2$$

The lemma follows $\square$
When summing $n$-digits integers using $|P|$ processors, such that $|P| \log_2 |P| \in \mathcal{O}(n)$, SUM achieves optimal speedup. While the presentation discussed here focuses on the sum of two integers, the procedure can be easily extended to more addends. The computation and I/O cost scales linearly with the number of addends.

### 4.2 Parallel Comparison with Distributed Memory

In this subsection we describe how given two $n$-digit input integers $A, B$ partitioned in a sequence of processors $P$ in $n/|P|$ digits, is it possible to efficiently determinate whether $A \geq B$. Our algorithm $\text{COMPARE}(P, A, B)$ achieves asymptotically computational speedup for $n \geq \Omega(|P| \log_2 |P|)$.

At the end of an invocation of $\text{COMPARE}(P, A, B)$ each processor in $P$ holds a flag $f$ such that $f = 0$ if $A = B$, $f = 1$ if $A > B$, and $f = -1$ if $B > A$. The algorithms employs a recursive strategy. If $|P| = 1$, the single processor in $P$ computes the value of the flag $f$ locally. If that is not the case, the following operations are executed:

1. Divide the sequence of available processors in the two subsequences $P'$ and $P''$ as in (1), and the input $A$ (resp., $B$) in $A_0$ and $A_1$ (resp., $A_1$ and $B_1$) as in (2).
2. Recursively invoke $\text{COMPARE}(P', A_0, B_0)$ and $\text{COMPARE}(P'', A_1, B_1)$ to be executed in parallel.
3. Let $f'$ (resp., $f''$) denote the flag computed by $\text{COMPARE}(P', A_0, B_0)$ (resp., $\text{COMPARE}(P'', A_1, B_1)$). In parallel, each processor $P'[i]$, for $0 \leq i \leq |P|/2 - 1$, sends to $P''[i]$ the flag $f'$ and then removes them from its cache.
4. Upon receipt, the $P''[i]$’s, compute the new flag $f$:
   $$f = \begin{cases} f' & \text{if } f' \neq 0 \\ f'' & \text{if } f' = 0 \end{cases},$$
   and then remove $f'$ and $f''$ from their local cache.
5. In parallel, each processor $P''[i]$, for $0 \leq i \leq |P|/2 - 1$, sends to $P'[i]$ a copy of the flag $f$.

**Lemma 8.** Using algorithm $\text{COMPARE}$ to compare two $n$-digits integers $A$ and $B$ using $|P|$ processors requires each processor being equipped with a local memory of size $2n/|P| + 2$.

Further:

$$T_{\text{COMPARE}}(n, |P|) \leq \frac{n}{|P|} + \log_2 |P|,$$

$$B_{\text{COMPARE}}(n, |P|) \leq \log_2 |P|,$$

$$L_{\text{COMPARE}}(n, |P|) \leq \log_2 |P|.$$

**Proof.** At any time during the computation each processor needs to maintain in its local cache at most $n/|P|$ digits of $A$ and $B$, the value of the flag $f$ returned by the last invocation of $\text{COMPARE}$, and the value of a second flag received by another processor. Hence the memory requirement is bounded by $2(n/|P| + 1)$. The analysis of the computation and I/O cost is analogous to that discussed in the proof of Lemma 7 for SUM. \[\square\]
4.3 Parallel Difference with Distributed Memory

Let \( A, B \) be two positive integer numbers whose base-\( s \) expansion has at most \( n \) digits, and assume them to be initially partitioned in \( P \) in \( n/|P| \) digits. The parallel subroutine \( \operatorname{DIFF}(P, A, B, n/|P|) \) computes \( C = |A - B| \), with \( C \) being partitioned in \( P \) in \( n/|P| \) digits, and a flag \( f \) such that \( f = 0 \) if \( A = B \), \( f = 1 \) if \( A > B \), and \( f = -1 \) if \( B > A \). In the following, we assume \( n \) and \( |P| \) to be integer powers of two. If that is not the case, our algorithm may be applied with minor adjustments (e.g., padding).

When \( \operatorname{DIFF}(P, A, B, n/|P|) \) is invoked, if \( |P| = 1 \) the single processor \( P[0] \), computes \( |A - B| \) and the value \( f \) locally. If \( |P| > 1 \), the following operations are executed:

1. \( \operatorname{COMPARE}(P, A, B) \) is invoked to set the value \( f \). If \( f = 0 \) then each processor \( P[i] \) sets \( C(P[i]) = 0 \) and no further operation is executed. Instead, if \( f = 1 \) (resp., \( f = -1 \)), the algorithm computes \( A - B \) (resp., \( B - A \)) using a recursive divide and conquer approach. For the sake of simplicity, in the following we assume \( A > B \). The case \( B > A \) follows by swapping \( A \) and \( B \) in the following.

2. Let \( P'[i] \) and \( P''[i] \) be defined as in (1), and let \( A_0, A_1, B_0 \) and \( B_1 \) defined as in (2). \( \operatorname{DIFF} \) proceeds by invoking, in parallel, two, slightly different, recursive subroutines:

   - \( \operatorname{DIFFL}(P', A_0, B_0, n/|P|) \) computes \( C' = (A + s^{n/|P'|/|P|}B) \mod s^{n/|P'|/|P|} \), and partitions it in \( P' \) in \( n/|P'|/|P| \) digits. \( C' \) corresponds to the \( n/2 \) least significant digits of \( |A - B| \). Further, \( \operatorname{DIFFL} \) computes \( b' \) such that \( b' = 0 \) if \( A \geq B \) and \( b' = 1 \) otherwise. \( b' \) is used to denote whether when computing \( A_0 - B_0 \) it will be necessary to “borrow” from the \( (n/2) \)-th digit of \( A \) (i.e., if yes \( b' = 1 \), if no \( b' = 0 \)). At the end of \( \operatorname{DIFFL} \), all processors in \( P \) have a copy of \( b' \) in their local cache.

   - \( \operatorname{DIFFR}(P'', A_1, B_1, n/|P|) \) computes
     
     \[
     C_0'' = \left( A_1 + s^{n/|P''|/|P|} - B_1 \right) \mod s^{n/|P''|/|P|} \\
     b_0'' = 1(A_1 \geq B_1) \\
     C_1'' = \left( A_1 + s^{n/|P''|/|P|} - B_1 \right) - 1 \mod s^{n/|P''|/|P|} \\
     b_1'' = 1(A_1 - 1 \geq B_1)
     \]

   The values \( C_i'' \)'s are used to speculatively calculate the \( n/2 \) most significant digits of \( A - B \) depending whether it will necessary to borrow from \( A \) when computing \( A_0 - B_0 \) (i.e., \( i = 0 \)) or not (i.e., \( i = 1 \)). Similarly, \( f_i'' \)'s are used to speculatively calculate whether when computing the difference \( A_1 - B_1 - i \) it will be necessary to borrow, depending on whether it will necessary to borrow. The computed \( C_i'' \)'s are partitioned in \( P'' \) in \( n/|P| \) digits, and each processor in \( P'' \) holds a copy of the \( b_i'' \)'s.

3. Each processor \( P'[j] \), for \( j = 0, 1, \ldots, |P|/2 - 1 \), sends to \( P''[j] \), the value \( b' \) and then removes it from its cache. Upon receipt, each \( P''[j] \) assigns \( C(P''[j]) = C''[P''[j]] \), and then removes \( b', C'_0''[P''[j]], C'_1''[P''[j]], b_0'', \) and \( b_1'' \) from its local memory. In parallel, each \( P'[j] \) assigns \( C(P'[j]) = C(P'[j]) \), and removes \( C(P'[j]) \) from its local memory.
As desired, the value $C = |A - B|$ is partitioned in $P$ in $n/|P|$ digits.

To conclude the description of DIFF, we now present the details of the recursive subroutines DIFFL and DIFFR:

When DIFFL $(P, \{A, B\}, n/|P|)$ is invoked, if $|P| = 1$ the single processor $P[0]$, computes $|A - B|$ and the value $b$ locally. If $|P| > 1$, the following operations are executed:

1. The sequence of available processors is divided in the two subsequences $P'$ and $P''$ (1), and the input $A$ (resp., $B$) is partitioned in $A_0$ and $A_1$ (resp., $A_1$ and $B_1$) (2).

2. DIFFL recursively invokes DIFFL $(P', A_0, B_0, n/|P|)$ and DIFFR $(P'', A_1, B_1, n/|P|)$ to be executed in parallel. Let $C'$ and $b'$ (resp., $C''_0, C''_1, b'_0, b'_1$) denote the output of the former (resp., latter) call.

3. DIFFL proceeds to compute $|A - B|$ following the operations discussed in step (3) of the main procedure DIFF.

4. Additionally, when the processor $P''[j]$’s receive $b'$ from $P'[j]$, it assigns $b = b''_0$, removes $b', b''_0$ and $b'_1$ from its local memory, and then sends a copy of $b$ to $P'[j]$.

Finally, when DIFFR $(P, A, B, n/|P|)$ is invoked, if $|P| = 1$ the single processor $P[0]$, computes $C_0 = |A - B|$, $C_1 = |A - B - 1|$, $b_0 = 1(A \geq B)$ and $b_1 = 1(A - 1 \geq B)$ locally. If $|P| > 1$, the following operations are executed:

1. The sequence of available processors is divided in the two subsequences $P'$ and $P''$ (1), and the input $A$ (resp., $B$) is partitioned in $A_0$ and $A_1$ (resp., $A_1$ and $B_1$) (2).

2. DIFFR recursively invokes DIFFR $(P', A_0, B_0, n/|P|)$ and DIFFR $(P'', A_1, B_1, n/|P|)$ to be executed in parallel. Let $C'_0, C'_1, b'_0$ and $b'_1$ (resp., $C''_0, C''_1, b''_0, b''_1$) denote the output of the former (resp., latter) call.

3. Each processor $P'[j]$, for $i = 0, 1, \ldots, |P|/2 - 1$, sends to $P''[j]$, the values $b'_0$ and $b'_1$, and then removes them from its cache.

4. Upon receipt, each $P''[i]$ assigns $C_0[P''[j]] = C''_0[P''[j]]$ (resp., $C_1[P''[j]] = C''_1[P''[j]]$). Further, it assigns $b_0 = b''_0$ (resp., $b_1 = b''_1$), and then removes $C''_0, C''_1, b'_0, b'_1, b''_0$ and $b''_1$ from its local memory. In parallel, each $P'[j]$ assigns $C_0(P'[j]) = C_0(P'[j])$ (resp., $C_1(P'[j]) = C_1(P'[j])$), and then removes $C''_0$ and $C''_1$ from its local memory.

5. In parallel, each $P''[j]$ sends to $P[j]$ a copy of $b_0$ and $b_1$.

At the end of these operations, the values $C_0, C_1$ are partitioned in $P$ as desired.

The following lemma characterizes the memory requirement, the computational time and I/O cost of DIFF:

**Lemma 9.** Using algorithm DIFF to compute the difference of two $n$-digit integers $A - B$ using a sequence of processors $P$, requires each processor to be equipped with a local memory of size at least $4n/|P| + 5$. We have:

\[
T_{\text{DIFF}}(n, |P|) \leq 7n/|P| + 5 \log_2 |P|
\]
\[
B_{\text{DIFF}}(n, |P|) \leq 5 \log_2 |P|
\]
\[
L_{\text{DIFF}}(n, |P|) \leq 3 \log_2 |P|
\]
Proof. DIFF correctly computes $|A - B|$ and the sign of $A - B$ by inspection.

By Lemma 8 the initial invocation of COMPARE($P, A, B$), requires each processor in $P$ to be equipped with a memory of size at least $2(n/|P| + 1)$. For $j = 0, 1, \ldots, |P| - 1$, the locations used to store the digits of $A(P[j])$ and $B(P[j])$ are reused in the remainder of the computation. At any point during the computation each processor must maintain in memory at most $n/|P|$ digits of the input integers $A(P[j])$ and $B(P[j])$, $2n/|P|$ (resp., 2) digits for the values $C_0$ and $C_1$ (resp., $b_0$ and $b_1$) returned by the last recursive call, at most two copies of $b'_0$ and $b'_1$ received from another processor and not yet deleted, and the flag $f$ computed by the initial call to COMPARE. Hence, DIFF can be executed if each processor is equipped with a local memory of size $4n/|P| + 5$. We focus on the analysis of the computational and I/O requirement of the auxiliary subroutine DIFFR, as this subsumes the analysis of DIFFL. For $|P| > 1$, DIFFR recursively invokes two instances to be executed in parallel. In step (3) and (5) of the algorithm, half of processors send two memory words to distinct processors in the remaining half. Finally, at step (4), up to four comparisons are necessary to assign the values $C_0, C_1, b_0$ and $b_1$. Hence, we have that the computational cost, the bandwidth cost and the latency of DIFFR satisfy:

$$T_{DIFFR}(n, |P|) \leq T_{DIFFR}(n/2, |P|/2) + 4$$
$$B_{DIFFR}(n, |P|) \leq B_{DIFFR}(n/2, |P|/2) + 4$$
$$L_{DIFFR}(n, |P|) \leq L_{DIFFR}(n/2, |P|/2) + 2$$

(4)

In the base case, for $|P| = 1$, DIFFR computes the differences $C_0$ and $C_1$ locally without any further communication (i.e., $B_{SUMA}(n/|P|, 1, M) = B_{DIFFR}(n/|P|, 1, M) = 0$). As the numbers being added have at most $n/|P|$ digits, each value can be computed using at most $3n/|P|$ elementary operations (i.e., $T_{DIFFR}(n/|P|, 1, M) < 6n$). Thus, from (1) we have

$$T_{DIFFR}(n, |P|) \leq T_{DIFFR}(n/|P|, 1) + \sum_{i=1}^{\log_2 |P| - 1} 4$$
$$B_{DIFFR}(n, |P|) \leq B_{DIFFR}(n/|P|, 1) + \sum_{i=1}^{\log_2 |P|} 4$$
$$L_{DIFFR}(n, |P|) \leq L_{DIFFR}(n/|P|, 1) + \sum_{i=1}^{\log_2 |P| - 1} 2$$

The lemma follows by summing the computation time (resp., I/O cost) of DIFFR with that of the initial invocation of COMPARE (according to Lemma 8).

When computing the difference of $n$-digits integers using $|P|$ processors, such that $|P|\log_2 |P| \in O(n)$, DIFF achieves optimal speedup. As for SUM, while the presentation discussed here focuses on the difference of two integers, the procedure can be easily extended to more inputs. The computation and I/O cost scales linearly with the number of input integers.
5 COPSIM: Communication-optimal Parallel Standard Integer Multiplication

In this section we present our first algorithm, COPSIM (Communication Optimal Parallel Standard Integer Multiplication), a parallel implementation of the sequential recursive long multiplication algorithm, which given two input $n$-digit integers computes their product by executing $O(n^2)$ operations. The sequential algorithm, henceforth referred to as SLIM (Sequential Long Integer Multiplication), follows a simple recursive strategy: If $n = 1$, the product $C = A \times B$ is computed directly. Otherwise, let:

$$A_0 = (A[[n/2] - 1],\ldots,A[0])_s$$
$$B_0 = (B[[n/2] - 1],\ldots,B[0])_s$$
$$A_1 = (A[n - 1],\ldots,A[[n/2]])_s$$
$$B_1 = (B[n - 1],\ldots,B[[n/2]])_s$$

SLIM is then recursively invoked to compute the products $C_0 = A_0 \times B_0$, $C_1 = A_0 \times B_1$, $C_2 = A_1 \times B_0$ and $C_3 = A_1 \times B_1$. Finally, $C$ is computed as $C = C_0 + s^{n/4}(C_1 + C_2) + s^{n/2}C_3$.

**Fact 10.** Algorithm SLIM computes the product of two $n$-digit integers using at most $8n^2$ digit wise operations and memory space of size at most $8n$.

Our parallel algorithm COPSIM implements the recursive scheme of SLIM to take advantage of multiple available processors following the general outline discussed in Section 3. We assume that the input factor $n$-digit integers to be partitioned in a sequence of processors $P$ in $n/|P|$ digits at the beginning of the computation. Further, we assume each processor to be equipped with a local memory of size $M \geq 80n/|P|$. Finally, to simplify our analysis, we assume $n$ to be an integer power of two, and $|P|$ to be an integer power of four, with $n \geq |P|$. If that is not the case, the input can be padded with dummy digits, and/or some of the available processors may not be used. The following description and analysis remain correct in these cases with small corrections of the constant factors.

If $n \leq M \sqrt{|P|}/12$, then the product is computed by COPSIM in the MI execution mode (i.e.,COPSIM$_{MI}$) in $\log_4 |P|$ breadth-first recursive steps. If that is not the case, COPSIM proceeds in the main execution mode by executing up to $\lceil \log_2 n/(24M\sqrt{|P|}) \rceil$ depth-first recursive steps. The sub-problems generated after such steps will have input size which, given the size of the available memory, allows for their solution to be calculated in the MI execution mode. COPSIM uses the sequential algorithm SLIM to compute the products of integers locally. Clearly, any sequential algorithm can be used in place of it. In the following, we present in detail and analyze COPSIM’s execution in the MI and main execution mode.

5.1 COPSIM in the MI execution mode

We denote the operations of COPSIM in the MI execution mode as COPSIM$_{MI}$. Consider an invocation of COPSIM$_{MI}$ to multiply two $n$-digits integers using a sequence of processors $P$. If $|P| = 1$, then the product $C = A \times B$ is computed by the single available processor using the sequential algorithm SLIM. If $|P| > 1$, COPSIM$_{MI}$ proceeds as follows:

1. **Splitting:** Let $P'$ and $P''$ be two subsequences of $P$ defined as in (1), and let $A_0$ and $A_1$ ($B_0$ and $B_1$) be defined as 2. $A_0$ (resp., $B_0$) is partitioned in $P'$ and $A_1$ (resp.,
$B_1$) is partitioned in $P'$ in $n/|P|$ digits. COPSIM divides the available $P$ processors into four sub-sequences each with $|P|/4$ processors:

\[
\begin{align*}
P_0 &= [P[([P]/2) - 2], \ldots, P[2], P[0]] \\
P_1 &= [P[([P]/2) - 1], \ldots, P[3], P[1]] \\
P_2 &= [P([|P| - 2], \ldots, P[|P|/2 + 2], P[|P|/2]] \\
P_3 &= [P([|P| - 1], \ldots, P[|P|/2 + 3], P[|P|/2 + 1]]
\end{align*}
\]

The sub-sequence $P_0$ (resp., $P_1$) is assigned the even-index (resp., odd-index) processors in the first half of $P$ (i.e., $P'$), while $P_2$ (resp., $P_3$) is assigned the even-index (resp., odd-index) processors in the second half of $P$ (i.e., $P''$). In a BF step, COPSIM assigns each subsequence of the available processors to one of the four sub-problems which are recursively invoked to compute the product $A \times B$: $P_0$ computes $A_0 \times B_0$, $P_1$ computes $A_0 \times B_1$, $P_2$ computes $A_1 \times B_0$ and $P_3$ computes $A_1 \times B_1$. COPSIM transfers to each sequence $P_1$ the input integers for the corresponding sub-problem. This is achieved in the two following parallel communication steps:

(a) In parallel, each odd-index (resp., even-index) processor $P[i]$ (resp., $P[j]$) for $i = 1, 3, \ldots, |P|/2 - 1$ (resp., $j = |P|/2, |P|/2 + 2, \ldots, |P| - 2$) sends to $P[i - 1]$ (resp., $P[j + 1]$) a copy of the $n/|P|$ digits of $A(P[i])$ and $B(P[i])$ (resp., $A(P[j])$ and $B(P[j])$). At the end of this step $A_0$ and $B_0$ (resp., $A_1$ and $B_1$) are partitioned in $P_0$ (resp., $P_2$) in $2n/|P|$ digits.

(b) In parallel, all processor $P_0[i]$ (resp., $P_3[i]$), for $i \in \{0, 1, \ldots, |P|/4\}$, send to $P_1[i]$ (resp., $P_2[i]$) a copy of the digits of $A_0(P_0[i])$ (resp., $A_1(P_2[i])$). Ad the end of this step a copy of $A_0$ and (resp., $A_1$) is partitioned in $P_1$ (resp., $P_3$) in $2n/|P|$ digits.

(c) In parallel, all processor $P_0[i]$ (resp., $P_3[i]$), for $i \in \{0, 1, \ldots, |P|/4\}$, send to $P_2[i]$ (resp., $P_1[i]$) a copy of the digits of $B_0(P_0[i])$ (resp., $B_1(P_2[i])$). Ad the end of this step a copy of $B_0$ (resp., $B_1$) is partitioned in $P_3$ (resp., $P_1$) in $2n/|P|$ digits.

2. **Recursive multiplication**: The four sub-products are computed in parallel by recursively invoking COPSIM$_{MI}$: $C_0 =$ COIM$_{MI}(P_0, A_0, B_0, n/|P|)$, $C_1 =$ COIM$_{MI}(P_1, A_0, B_1, n/|P|)$, $C_2 =$ COIM$_{MI}(P_2, A_1, B_0, n/|P|)$ and $C_3 =$ COIM$_{MI}(P_3, A_1, B_1, n/|P|)$. At the end of these recursive calls, $C_i$, for $i = 0, 1, 2, 3$, is partitioned in $P_1$ in $4n/|P|$ digits.

3. **Recomposition**: COPSIM$_{MI}$ composes the values $C_0, C_1, C_2$ and $C_3$ to obtain $C$. First, the outputs of the sub-problems are opportunely redistributed among the processors:

(a) In parallel, each processor $P_0[j]$ (resp., $P_3[j]$) for $j = 1, 3, \ldots, |P_0| = |P_3| = |P|/4$, sends to $P_1[j]$ (resp., $P_2[j]$) the $2n/|P|$ most (resp., least) significant digits of $C_0(P_1[j])$ (resp., $C_3(P_3[j])$), and then it deletes them from its local memory. At the end of this step $C_0$ (resp., $C_3$) is partitioned in $P[|P|/2 − 1.0]$ (resp., $P[|P| − 1.0]/2]$) in $2n/|P|$ digits.
Proof. Let $M$ processors in $n$ shift $P$ the subsequence $P$ can be computed with three consecutive invocations of the We have most significant digit of $C$ whose base-
After these five parallel communication steps, $C_0$ is partitioned in $P[|P|/2 - 1..|P|/2]$ in $2n/|P|$ digits. We have $C(P[j]) = C_0(P[j])$ for $j = 0, 1, \ldots, |P|/4 - 1$. Let $C'_0$ denote the integer whose base-$s$ expansion corresponds to the $n/2$ most significant digits of $C_0$, and let $C'_3 = C_3 * s^{n/2}$ (i.e., the integer whose base-$s$ expansion correspond to the digits of $C_3$ shifted $n/4$ times). By construction, $C'_0 C'_3$ (as well as $C_1$ and $C_2$) are partitioned in the subsequence $P^* = [P[|P| - 1], \ldots, 3P/4 + 1, 3P/4 - 1]$, and then it deletes them from its local memory. At the end of this step the $n/2$ most significant digits of $C_1$ are partitioned in $P[3P/4 - 1..|P|/2]$ in $2n/|P|$ digits.

(e) In parallel, each processor $P[j]$ (resp., $P[j + 3P/4]$) for $j = 0, 1, \ldots, |P|/4 - 1$, sends to $P[j + P/2]$ the $2n/|P|$ digits of $C_3(P[j + P/4])$, and then it deletes them from its local memory. At the end of this step the $n/2$ least significant digits of $C_1$ are partitioned in $P[|P|/2 - 1..|P|/4]$ in $2n/|P|$ digits. After these five parallel communication steps, $C_0$ is partitioned in $P[|P|/2 - 1], \ldots, P[0]$ in $2n/|P|$ digits, $C_1$ and $C_2$ are partitioned in $P[3P/4 - 1], \ldots, P[|P|/4]$ in $2n/|P|$ digits, and $C_3$ is partitioned in $P[|P| - 1], \ldots, P[|P|/2]$ in $2n/|P|$ digits. We have $C(P[j]) = C_0(P[j])$ for $j = 0, 1, \ldots, |P|/4 - 1$. Let $C'_0$ denote the integer whose base-$s$ expansion corresponds to the number of $C_0$, and let $C'_3 = C_3 * s^{n/2}$ (i.e., the integer whose base-$s$ expansion correspond to the digits of $C_3$ shifted $n/4$ times). By construction, $C'_0 C'_3$ (as well as $C_1$ and $C_2$) are partitioned in the subsequence $P^* = [P[|P| - 1], \ldots, 3P/4 + 1, 3P/4 - 1]$ in $2n/|P|$ digits. The $3n/2$ most significant digit of $C$ correspond to those of the sum of $C'_0, C_1, C_2$ and $C'_3$, which can be computed with three consecutive invocations of the SUM subroutine discussed in Section [4] using the sequence $P^*$. The $3n/2$ digits of such sum are partitioned in $P^*$ in $2n/|P|$ digits, and, hence, $C$ is partitioned in $P$ in $2n/|P|$ digits.

The following theorem characterizes memory utilization, computation cost, and I/O cost for COPSIM$_{M1}$:

**Theorem 11.** Let $A$ and $B$ be two $n$-digit integers portioned in a sequence of processors $P$, with $n \geq |P|$, in $n/|P|$ digits. COPSIM$_{M1}$ computes the product $C = A \times B$ using the processors in $P$, provided that each processor is equipped with a local memory of size at least $M_{COPSIM_{M1}}(n, |P|) = 12n/\sqrt{|P|}$. We have:

$$ T_{COPSIM_{M1}}(n, |P|) \leq 38 \frac{n^2}{|P|} + 3 \log_2^2 |P| $$

$$ BW_{COPSIM_{M1}}(n, |P|) \leq 14 \frac{n}{\sqrt{|P|}} + 6 \log_2^2 |P| $$

$$ L_{COPSIM_{M1}}(n, |P|) \leq 3 \log_2^2 |P| $$

**Proof.** COPSIM$_{M1}$ correctly computes $C = A \times B$ by inspection.
If \(|P| = 1\), the product \(C\) is computed locally by the single available processor using algorithm SLIM. By Lemma 10 we thus have \(M_{COPSIM_{MI}}(n, 1) \leq 8n\), \(T_{COPSIM_{MI}}(n, 1) \leq 8n^2\), \(BW_{COPSIM_{MI}}(n, 1) = 0\) and \(L_{COPSIM_{MI}}(n, 1) = 0\).

In the following, we assume \(|P| > 1\). By construction, during the \(i\)-th recursion step of \(COPSIM_{MI}\) for \(1 \leq i < \log_4 |P|\), each processor needs to hold in its local memory at most \(2 \times 4^{i-1} n/|P|2^{i-1} = n2^i/|P|\) digits of the inputs one of the sub-problems being computed at the \(i\)-th level, and \(n2^{i+1}/|P|\) digits of the input of the sub-problem being generated to whom the processor will be assigned. Hence, \(M_{COPSIM_{MI}}(n, |P|) \geq 3n/\sqrt{|P|}\).

Similarly, once the output of the sub-problems generated at the \(i\)-th recursion step of \(COPSIM_{MI}\), for \(1 < i \leq \log_4 |P|\), have been computed, each processor needs to hold in its local memory at most \(3 \times n2^i/|P|\) digits of the outputs of one of the sub-problems computed at the \(i\)-th level, and \(n2^{i-1}/|P|\) digits of the output of one the sub-problem at the \((i-1)\)-th level. Hence, \(M_{COPSIM_{MI}}(n, |P|) \geq 4n/\sqrt{|P|}\). Further, each processor must be equipped with enough memory to sum \(C_0, C_1, C_2\) and \(C_3\) using SUM, as discussed at end of phase 3 of \(COPSIM_{MI}\)’s description. In the \(i\)-th recursion level, for \(1 \leq i < \log_4 |P| - 1\), SUM is used to sum \((3n2^{i+1}/|P|)\)-digit integers initially partitioned in \(3|P|/4^{i+1}\) processors (which are used to compute the sum) in \(2n/|P|\) digits. Hence, from Lemma 7, \(M_{COPSIM_{MI}}(n, |P|) \geq 4\left(2^{i+1}n/|P| + 1\right) \leq 12n/\sqrt{|P|}\). Finally, each processor must be equipped with enough memory to compute the product of two \((n/\sqrt{|P|})\)-digit integers using algorithm SLIM. Hence, by Lemma 10 \(M_{COPSIM_{MI}}(n, |P|) \geq 8n/\sqrt{|P|}\). We can thus conclude that all such requirement are met if each processor is equipped with a local memory of size \(M_{COPSIM_{MI}}(n, |P|) = 12n/\sqrt{|P|}\).

In the “Recomposition” step \(COPSIM_{MI}\) computes the sum of \(C_0, C_1, C_2, C_3\) using the SUM parallel subroutine. By construction, each such values has at most \(3n/2\) digits and it is partitioned in \(P^*\) in \(2n/|P|\) digits. Their sum can be computed with three consecutive applications of SUM using the subsequence \(P^*\). As no other computation is executed outside the recursive calls to \(COPSIM\), by Lemma 7 and as \(|P^*| = 3|P|/4\), we have:

\[
T_{COPSIM_{MI}}(n, |P|) \leq T_{COPSIM_{MI}}(n/2, |P|/4) + 3T_{SUM}(3n/2, |P^*|)
\]
\[
\leq T_{COPSIM_{MI}}(n/2, |P|/4) + 3((12n/|P|) + 4\log_2(3|P|/4 - 1))
\]

In phase (1a), each of the \(|P|\) processor either sends or receives \(2n/|P|\). In each of the phases (1b) and (1c), each processor which communicates either sends or receives \(n/|P|\) digits. During each of the phases from (3a) to (3e) every processor which communicates either sends or it receives \(2n/|P|\) memory words to/from other processors. As the four generated subproblems, each with input size at most \(n/2\), are computed in parallel using \(|P|/4\) processors each, by Lemma 7 and as \(|P^*| = 3|P|/4\), we have:

\[
BW_{COIM_{MI}}(n, |P|) \leq BW_{COM_{MI}}(n/2, |P|/4) + 14n/|P| + 3BW_{SUM}(3n/2, |P^*|)
\]
\[
\leq BW_{COM_{MI}}(n/2, |P|/4) + 14n/|P| + 12(\log_2 3|P|/4 - 1)
\]
\[
L_{COIM_{MI}}(n, |P|) \leq L_{COM_{MI}}(n/2, |P|/4) + 8 + 3L_{SUM}(3n/2, |P^*|)
\]
\[
\leq L_{COM_{MI}}(n/2, |P|/4) + 8 + 6(\log_2 3|P|/4 - 1)
\]

After \(\log_4 |P|\) recursive breadth-first levels, each processor is assigned a single subproblem with input size \(n/2^{\log_4 |P|} = n/\sqrt{|P|}\), which is computed locally without any further communication as discussed for the base case (i.e., \(BW_{COIM_{MI}}(n/|P|, 1) = L_{COIM_{MI}}(n/|P|, 1) = \))
0) using a sequential long integer multiplication algorithm (e.g., algorithm SLIM). Hence, by Lemma 10, \( T_{\text{COIM}_M}(n/|P|, 1) \leq 2n^2/|P| \). As, by assumption, \( n \geq |P| \) we have:

\[
T_{\text{COPSIM}_M}(n, |P|) \leq T_{\text{COPSIM}_M}(n/\sqrt{|P|}, 1) + 36 \frac{n}{|P|} \log_4 |P| - 1 + 6 \sum_{i=1}^{\log_4 |P| - 1} 2^i + 6 \sum_{i=1}^{\log_4 |P| - 1} \log_2 \frac{3|P|}{4^i} - 1
\]

\[
< 2 \frac{n^2}{|P|} + 36 \frac{n}{\sqrt{|P|}} + 3 \log_2^2 |P|
\]

\[
= 38 \frac{n^2}{|P|} + 3 \log_2^2 |P|
\]

\[
BW_{\text{COIM}_M}(n, |P|) \leq BW_{\text{COIM}_M}(n/\sqrt{|P|}, 1) + 14 \frac{n}{|P|} \log_4 |P| - 1 + 12 \sum_{i=1}^{\log_4 |P| - 1} (\log_2 \frac{3|P|}{4^i} - 1)
\]

\[
< 14 \frac{n}{\sqrt{|P|}} + 6 \log_2^2 |P|
\]

\[
L_{\text{COIM}_M}(n, |P|) \leq L_{\text{COIM}_M}(n/\sqrt{|P|}, 1) + \log_4 |P| - 1 + \sum_{i=1}^{\log_4 |P| - 1} 8 + 6 \sum_{i=1}^{\log_4 |P| - 1} (\log_2 \frac{3|P|}{4^i} - 1)
\]

\[
< 3 \log_2^2 |P|
\]

\[
\square
\]

5.2 COPSIM in the main execution mode

An important consequence of Theorem [11] is that to execute COPSIM\(_M\) to multiply two \( n \)-digit integers each processor must be equipped with a memory of size at least \( 12n/\sqrt{|P|} \). Considering the aggregate memory available in \( |P| \) processors, this implies that COPSIM\(_M\) can only be used to multiply \( n \)-digit integers where \( n \in \mathcal{O}(M\sqrt{|P|}) \). In this section, we describe how to combine the MI execution mode COPSIM\(_M\) with a depth-first scheduling of the subproblems, as outlined in the general framework in Section 3. Our algorithm COPSIM allows to compute the product of two \( n \)-digit integers provided that (i) \( M \geq 80n/|P| \) and (ii) \( M \geq \log_2 |P| \). By (i), COPSIM can thus be used to multiply \( n \)-digit integers where \( n \in \mathcal{O}(M|P|) \). That is, its memory requirement corresponds asymptotically to the amount of overall memory required to store the input integers and/or the product. Requirement (ii) is also very reasonable in practice, as it is generally more sensible and cost/effective to have each processor equipped with a memory of respectable size rather than exceed the number of available processors.

In the main execution mode, COPSIM proceeds in a sequence of at most \( \mathcal{O}\left(\log_2 n/(M\sqrt{|P|})\right) \) recursive depth-first steps, where \( M \) denotes the size of the memory available to each processor, until the size of the generated sub-problems allows them to be computed in the MI execution mode COPSIM\(_M\).

In the main execution mode, when COPSIM \( (n, A, B, P, M, n/|P|) \) is invoked, if \( M \geq 12n\sqrt{|P|} \), that is if the size of the memory available to each processor allows for it, the product is computed by invoking COPSIM\(_M\) \( (n, A, B, P, n/|P|) \). Otherwise COPSIM
will proceed with a *depth-first step*, by generating four subproblems invoking itself on each of the subproblems one at a time. The memory available to the recursive calls is reduced by the amount of memory space required to maintain the data used in the depth first step, including, for each processor, \(2n/|P|\) digits of the input, up to \(3n/|P|\) digits of the outputs of the sub-problems, and the memory space required for an invocation of SUM to sum \((3n/2)\)-digits integers partitioned in \(3|P|/4\) processors (i.e., by Lemma 7 4(2n/|P| + 1)). Hence, the available memory is reduced by \(20n/|P|\). In the following, we denote \(A_0, B_0, A_1\) and \(B_1\) as defined in [2].

The operations being executed are slightly different depending on the subproblems:

- **C\(_0\) = A_0 \times B_0**: In parallel, each processor \(P[j]\) for \(j = 0, 1, \ldots, |P|/2 - 1\) sends to \(P[j + |P|/2]\) a copy of the \(n/(2|P|)\) most significant digits of \(A(P[j])\) and \(B(P[j])\). At the end of this step \(A_0\) and \(B_0\) are partitioned in the sequence

\[
P' = [P[|P| - 1], P[|P|/2 - 1], \ldots, P[|P|/2 + 1, P[1], P[|P|/2], P[0]]
\]

in \(n/(2|P|)\) digits (i.e., the even/odd index processors in the sequence \(P'\) are the processors in the first/second half of the sequence \(P\)). COPSIM\(_n,A_0,B_0,P,M - 20n/|P|\) is then recursively invoked to compute \(C_0 = A_0 \times B_0\) using the sequence \(P'\). Once computed, \(C_0\) is partitioned in \(P'\) in \(n/|P|\) digits.

Before proceeding in the computation of the second subproblem, COPSIM rearranges the digits of the output \(C_0\) so that it is partitioned in the first half of the processors sequence \(P\), that is \([P[|P|/2 - 1], \ldots, P[0]]\) in \(n/(2|P|)\) digits. This is accomplished in a single communication step during which in parallel, each odd-index processor of \(P'[j]\), for \(j = 1, 3, \ldots, |P'|/2 - 1\), sends to \(P'[j - 1]\) the \(n/|P|\) digits of \(C_0(P'[j])\), and then it deletes them from its local memory.

- **C\(_1\) = A_0 \times B_1**: In parallel, each processor \(P[j]\) for \(j = 0, 1, \ldots, |P|/2 - 1\) sends to \(P[j + |P|/2]\) a copy of the \(n/(2|P|)\) least significant digits of \(B(P[j + |P|/2])\). At the end of these two steps, \(A_0\) and \(B_1\) are partitioned in the sequence \(P'\) in \(n/(2|P|)\) digits. COPSIM\(_n,A_0,B_1,P,M - 20n/|P|\) is then recursively invoked to compute \(C_1 = A_0 \times B_1\) using the sequence \(P'\). Once computed, \(C_1\) is partitioned in \(P'\) in \(n/|P|\) digits.

Before proceeding in the computation of the third subproblem, COPSIM rearranges the digits of the output \(C_1\) so that it is partitioned in the sub-sequence of \(P\) composed of the \(|P|/2\) processors in the “middle” of \(P\), \([P[3|P|/4 - 1], \ldots, P[|P|/4]]\), in \(2n/|P|\) digits. In parallel, each odd-index (resp., even-index) processor \(P'[j]\) (resp., \(P'[j + |P'|/2 - 1]\)) for \(j = 1, 3, \ldots, |P'|/2 - 1\) sends to \(P'[j - 1]\) (resp., \(P[j + |P'|/2]\)) a copy of the \(n/|P|\) digits of \(C_1(P[j])\) (resp., \(C_1(P[j + |P'|/2])\)), and then deletes them from its local memory. After this step, the \(n/2\) least (resp., most) significant digits of \(C_1\) are partitioned in the sub-sequence \([P[|P|/4 - 1], \ldots, P[0]]\) (resp., \([P[|P| - 1], \ldots, P[3|P|/4]]\)) in \(2n/|P|\) digits. In a following parallel communication step, in parallel, each processor \(P[j]\) (resp., \(P[j + 3|P|/4]\)) for \(j = 0, 1, \ldots, |P|/4 - 1\), send to \(P[j + |P|/4]\) (resp., \(P[j + |P|/2]\)) the \(2n/|P|\) digits of \(C_1(P[j])\) (resp., \(C_1(P[j + 3|P|/4])\)), and then removes them from its local memory.
\[ C_2 = A_1 \times B_0 : \text{The operations executed for this sub-problem closely follow those discussed in the previous one. A detailed description can be obtained by replacing } B_0 \text{ with } A_0, A_1 \text{ with } B_1, \text{and } C_2 \text{ with } C_1. \]

\[ C_3 = A_1 \times B_1 : \text{In parallel, each processor } P[j + \lfloor n/2 \rfloor ] \text{ for } j = 0, 1, \ldots, \lfloor n/2 \rfloor \text{ sends to } P[j] \text{ a copy of the } n/(2|P|) \text{ least significant digits of } A(P[i]) \text{ and } B(P[i]). \text{At the end of this step } A_1 \text{ and } B_1 \text{ are partitioned in the sequence } P' \text{ in } n/(2|P|) \text{ digits.} \]

COPSIM is then recursively invoked to compute \( C_3 = A_1 \times B_1 \text{ using the sequence } P' \). Once computed, \( C_3 \) is partitioned in \( P' \) in \( n/|P| \) digits.

Before proceeding, COPSIM rearranges the digits of the output \( C_3 \) so that it is partitioned in the second half of the processors sequence \( P \), that is \( \lfloor P[|P|−1], \ldots, P[(|P|]/2) \rfloor \).

This is accomplished in a single communication step during which in parallel, each even-index processor of \( P'[j] \), for \( j = 0, 2, \ldots, |P'|/2−2 \), sends to \( P'[j + 1] \) the \( n/|P| \) digits of \( C_0(P'[j]) \), and then it deletes them from its local memory.

After the four sub-problems have been computed, and their respective outputs rearranged in \( P \), COPSIM completes the computation of \( C \) following the same steps presented for the MI execution mode memory setting at the end of phase (3). Once computed, \( C \) is correctly partitioned in \( C \) in \( 2n/|P| \) digits.

**Theorem 12.** Let \( A \) and \( B \) be two \( n \)-digit integers portioned in a sequence of processors \( P \), in \( n/|P| \) digits. COPSIM\(_{M1} \) computes the product \( C = A \times B \) using the processors in \( P \), provided that each processor is equipped with a local memory of size at least \( M_{COPSIM_{M1}}(n, |P|) \geq \max\{80n/|P|, \log_2 |P|\} \). We have:

\[
T_{COPSIM}(n, |P|, M) \leq 196 \frac{n^2}{|P|}
\]

\[
BW_{COPSIM}(n, |P|, M) \leq 3530 \frac{n^2}{M|P|}
\]

\[
L_{COPSIM}(n, |P|, M) \leq 7012 \frac{n^2 \log_2 |P|}{M^2|P|}
\]

**Proof.** COPSIM correctly computes \( C = A \times B \) by inspection.

If \( n \leq M \sqrt{|P|}/12 \), the statement follows from Theorem 11. In the following we assume \( M|P|/80 \geq n > M \sqrt{|P|}/12 \). COPSIM then proceeds by executing \( \ell \) consecutive depth-first steps, where \( 0 < \ell \leq \lfloor \log_2 n/(24M \sqrt{|P|}) \rfloor \). In each recursion step, in addition to the space required for the recursive invocation to COPSIM, each processor must maintain \( 2 \times n/(|P|) \) digits for the input of the problem being computed, at most \( 6n/|P| \) digits of the outputs of the recursive subproblems, and the space required for the invocation of SUM used to combine the outputs of the subproblems. Hence:

\[
M_{COPSIM}(n, |P|) \leq M_{COPSIM} \left( \frac{n}{2}, |P| \right) + 2 \frac{n}{|P|} + 6 \frac{n}{|P|} + M_{SUM}(3n/2, 3|P|/4)
\]

\[
\leq M_{COPSIM} \left( \frac{n}{2}, |P| \right) + 8 \frac{n}{|P|} + 4 \left( 2 \frac{n}{|P|} + 1 \right)
\]

\[
\leq M_{COPSIM} \left( \frac{n}{2}, |P| \right) + 8 \frac{n}{|P|} + 12 \frac{n}{|P|}
\]

(5)
\[
\leq M_{COPSIM} \left( \frac{M \sqrt{|P|}}{24}, |P| \right) + 20 \frac{n}{|P|} \sum_{\ell=1}^{\left\lceil \log_2 \frac{24n}{M \sqrt{|P|}} \right\rceil - 1} 2^{-1} \\
< M_{COPSIM,MI} \left( \frac{M \sqrt{|P|}}{24}, |P| \right) + 40 \frac{n}{|P|} \\
\leq \frac{M}{2} + 40 \frac{n}{|P|}
\]

where (5) follows from Lemma 7 and (6) follows from Theorem 11. By construction, after \( \ell \) depth-first steps, the available memory is reduced by at most \( 40 \frac{n}{|P|} \). As, by assumption, \( M \geq 80 \frac{n}{|P|} \), at least half of the space \( M \) originally assigned to each of the processors is still available. After at most \( \left\lceil \log_2 \frac{24n}{(M \sqrt{|P|})} \right\rceil \) recursive steps, the generated subproblems will have size at most \( M \frac{\sqrt{|P|}}{24} \). Thus, by Theorem 11, they can be computed using \( COPSIM_{MI} \) using at most \( \frac{M}{2} \) memory locations. This concludes the proof of the memory requirement for \( COPSIM \).

The computation time required in a depth-first recursion level of \( COPSIM \) execution is bounded by the time required for the sequential computation of \( COPSIM \)'s recursive invocation on the generated subproblems plus the time required to combine the outputs of the sub-problems to compute the product itself using three invocations of \( SUM \). As the numbers being summed have at most \( 3n/(2|P|) \) each and are partitioned in the \( 3|P|/4 \) processors used to sum them in \( 2n/|P| \) digits, by Lemma 7, we have:

\[
T_{COPSIM}(n, |P|, M) \leq 4T_{COPSIM} \left( \frac{n}{2}, |P|, M \right) + 3T_{SUM} \left( \frac{3n}{2}, \frac{3|P|}{4} \right) \\
\leq 4T_{COPSIM} \left( \frac{n}{2}, |P|, M \right) + 3 \left( \frac{2n}{|P|} + 4 \log_2 |P| \right)
\]

Further, in a depth-first recursion level the I/O cost of \( COPSIM \) (both bandwidth and latency) can be bound by that of the four consecutive invocations of \( COPSIM \) used to compute the four subproblems, the cost of redistributing the input (resp., the output) of such subproblems, and the I/O cost of the three invocations of \( SUM \) used to combine the outputs of the three subproblems. By Lemma 7, we have:

\[
BW_{COPSIM}(n, |P|, M) \leq 4BW_{COPSIM} \left( \frac{n}{2}, |P|, M \right) + 3 \frac{n}{|P|} + 8 \frac{n}{|P|} + 3BW_{SUM} \left( \frac{3n}{2}, 3|P| \right) \\
\leq 4BW_{COPSIM} \left( \frac{n}{2}, |P|, M \right) + 11 \frac{n}{|P|} + 12 \log_2 |P| \\
L_{COPSIM}(n, |P|, M) \leq 4L_{COPSIM} \left( \frac{n}{2}, |P|, M \right) + 12 + 3L_{SUM} \left( \frac{3n}{2}, \frac{3|P|}{4} \right) \\
\leq 4L_{COPSIM} \left( \frac{n}{2}, |P|, M \right) + 12 + 6 \log_2 |P|
\]

After \( 1 \leq \ell \leq \log_2 \frac{24n}{(M \sqrt{|P|})} \) depth-first steps, the generated sub-problems have input size at most \( M \frac{\sqrt{|P|}}{24} \), and \( COPSIM \) switches to the MI execution mode by invoking
COPSIM$_{MI}$. Hence, by Theorem 12 and by the assumptions $n \geq |P|$ and $M \geq \log_2 |P|$ we have:

\[
T_{\text{COPSIM}}(n, |P|, M) \leq 4^{\log_2 \frac{24n}{M\sqrt{|P|}}} T_{\text{COPSIM}} \left( \frac{M\sqrt{|P|}}{24}, |P|, M \right) + \frac{6n}{|P|} \sum_{\ell=1}^{\lceil \log_2 \frac{24n}{M\sqrt{|P|}} \rceil - 1} 2^{-\ell} \\
+ \left( \log_2 \frac{24n}{M\sqrt{|P|}} \right) 4 \log_2 |P|
\]

\[
< 4 \times \frac{24^2 n^2}{M^2 |P|} T_{\text{COPSIM}_{MI}} \left( \frac{M\sqrt{|P|}}{24}, |P| \right) + \frac{12n}{|P|} + \left( \log_2 \frac{24n}{M\sqrt{|P|}} \right) 4 \log_2 |P|
\]

\[
\leq 4 \times \frac{24^2 n^2}{M^2 |P|} \left( \frac{38M^2}{24^2} + 3 \log_2 |P| \right) + \frac{12n}{|P|} + \left( \log_2 \frac{24n}{M\sqrt{|P|}} \right) 4 \log_2 |P|
\]

\[
\leq 164 \frac{n^2}{|P|} + \frac{12n}{|P|} + \left( \log_2 \frac{24n}{M\sqrt{|P|}} \right) 4 \log_2 |P|
\]

\[
< 196 \frac{n^2}{|P|}
\]

\[
BW_{\text{COPSIM}}(n, |P|, M) \leq 4^{\log_2 \frac{24n}{M\sqrt{|P|}}} BW_{\text{COPSIM}} \left( \frac{M\sqrt{|P|}}{24}, |P|, M \right) + \frac{n}{|P|} \sum_{\ell=0}^{\lceil \log_2 \frac{24n}{M\sqrt{|P|}} \rceil - 1} 2^{-\ell}
\]

\[
+ 22 \log_2 |P| \left( \log_2 \frac{24n}{M\sqrt{|P|}} \right) - 1
\]

\[
< 4 \times \frac{24^2 n^2}{M^2 |P|} BW_{\text{COPSIM}MI} \left( \frac{M\sqrt{|P|}}{24}, |P| \right) + 22 \frac{n}{|P|}
\]

\[
+ 12 \log_2 |P| \left( \log_2 \frac{24n}{M\sqrt{|P|}} \right) - 1
\]

\[
\leq 4 \times \frac{24^2 n^2}{M^2 |P|} \left( \frac{14M}{24^2} + 6 \log_2 |P| \right) + 22 \frac{n}{|P|}
\]

\[
+ 12 \log_2 |P| \left( \log_2 \frac{24n}{M\sqrt{|P|}} \right) - 1
\]

\[
\leq 4 \times \frac{24^2 n^2}{M^2 |P|} \left( \frac{14 + 6 \times 24^2}{24^2} + 22 \log_2 |P| \left( \log_2 \frac{24n}{M\sqrt{|P|}} \right) - 1 \right)
\]

\[
\leq 3470 \frac{n^2}{M|P|} + \frac{22n}{|P|} + 12 \log_2 |P| \left( \log_2 \frac{24n}{M\sqrt{|P|}} \right) - 1
\]

\[
< 3530 \frac{n^2}{M|P|}
\]
\[ L_{\text{COPSIM}}(n, |P|, M) \leq 4^{\left\lceil \frac{24n}{M\sqrt{|P|}} \right\rceil} L_{\text{COPSIM}} \left( \frac{M\sqrt{|P|}}{24}, |P|, M \right) \]

\[ + \left( \left\lceil \log_2 \frac{24n}{M\sqrt{|P|}} \right\rceil - 1 \right) (12 + 6 \log_2 |P|) \]

\[ < 4 \times \frac{24^2 n^2}{M^2 |P|} L_{\text{COPSIM}_M} \left( \frac{M\sqrt{|P|}}{24}, |P| \right) \]

\[ + \left( \left\lceil \log_2 \frac{24n}{M\sqrt{|P|}} \right\rceil - 1 \right) (12 + 6 \log_2 |P|) \]

\[ \leq 6912 \frac{n^2 \log_2^2 |P|}{M^2 |P|} + \left( \left\lceil \log_2 \frac{24n}{M\sqrt{|P|}} \right\rceil - 1 \right) (12 + 6 \log_2 |P|) \quad (8) \]

\[ \leq 7012 \frac{n^2 \log_2^2 |P|}{M^2 |P|} \]

where (7) and (8) where the last passage follows as, by assumption, \( n \geq |P| \) and \( M \geq \log_2 |P| \), and as we are considering the case \( n \geq M\sqrt{|P|}/12 \). If that was not the case the product would have been computed using \( \text{COPSIM}_M \).

5.3 Comparison with communication lower bounds

Based on the analysis of COPSIM performance presented in Theorem 11 and Theorem 12, we have:

**Theorem 1.** COPSIM achieves optimal computation time speedup and optimal bandwidth cost among all parallel standard integer multiplication algorithms. It also minimizes the latency cost up to a \( O(\log^2 P) \) multiplicative factor.

**Proof.** Let \( \mathcal{P} \) denote the number of processors used in the computation. By Theorem 11, for \( M \geq 12n/\sqrt{|P|} \), the product \( C = A \times B \) can be computed using \( \text{COPSIM}_M \). Under the assumptions \( n \geq \mathcal{P} \) and \( M \geq \log_2 |P| \), the bandwidth cost of \( \text{COPSIM}_M \) asymptotically matches, the memory-independent lower bound in Theorem 4 and its latency latency is within a \( O(\log^2 \mathcal{P}) \) factor of the corresponding lower bound. Note that the initial distribution of the input values among the processors used in \( \text{COPSIM}_M \) satisfies the balanced input distribution assumption used to derive Theorem 4.

For \( 12n/\sqrt{|P|} > M \geq 80n/\mathcal{P} \), by Theorem 12, the product \( C = A \times B \) can be computed using COPSIM. For \( n \geq \mathcal{P} \) and \( M \geq 24\sqrt{|P|} \), the bandwidth cost of \( \text{COPSIM}_M \) asymptotically matches, the memory-dependent lower bound in Theorem 3 and its latency latency is within a \( O(\log^2 P) \) factor of the corresponding lower bound. The total memory space required across the available processors for the execution of COPSIM is \( O(n) \), that is, within a constant factor of the minimum space required to store the input (and output) values. Finally, in both cases, COPSIM achieves optimal computational time speedup \( O(n^2/\mathcal{P}) \). 

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6 Communication-optimal Parallel Karatsuba Multiplication

In this section, we present COPK (Communication Optimal Parallel Karatsuba), our parallel implementation of Karatsuba’s algorithm in the distributed memory setting which achieves optimal speedup, optimal bandwidth cost, and whose latency is within $O(\log_2 |P|)$ of the corresponding lower bound. Compared to standard long integer multiplication algorithms (e.g., SLIM), Karatsuba’s algorithm asymptotically reduces the computation time of integer multiplication by cleverly decreasing the number of sub-problems being generated at each recursion level from four to three.

While several different variations of the algorithm have been discussed in the literature, in this work we consider the following sequential implementation, henceforth referred to as SKIM (Sequential Karatsuba Integer Multiplication). The algorithm follows a simple recursive strategy. Let $A, B$ be $n$-digit integers, if $n = 1$, the product $C = A \times B$ is computed directly. Otherwise, let:

$$
A_0 = (A[\lceil n/2 \rceil - 1], \ldots, A[0])_s \\
B_0 = (B[\lceil n/2 \rceil - 1], \ldots, B[0])_s \\
A_1 = (A[n - 1], \ldots, A[\lceil n/2 \rceil])_s \\
B_1 = (B[n - 1], \ldots, B[\lceil n/2 \rceil])_s
$$

SKIM is then recursively invoked to compute the products $C_0 = A_0 \times B_0$, $C' = (A_0 - A_1) \times (B_1 - B_0)$, $C_2 = A_1 \times B_0$ and $C_3 = A_1 \times B_1$. Then, let $C_1 = C' + C_0 + C_2$. Finally, $C$ is computed as $C = C_0 + s^{n/4}(C_1) + s^{n/2}C_2$.

The following characterization of the time and memory requirements of SKIM can be obtained by inspection:

**Fact 13.** Algorithm SKIM computes the product of two $n$-digit integers using at most $16n^{\log_3 3}$ digit wise operations and a memory space of size at most $8n$.

Our parallel algorithm COPK extends the recursive scheme of SKIM to take advantage of multiple available processors following the general outline discussed in Section 3. We assume that the input factor $n$-digit integers are partitioned in a sequence of processors $P$ in $n/|P|$ digits at the beginning of the computation. Further, we assume that each processor is equipped with a local memory of size $M \geq$. Finally, in order to simplify our analysis, we assume $|P| = 4 \times 3^i$, and $n = |P| \times 2^j$, for $i, j \in \mathbb{N}$. If that is not the case, the input integers can be padded with dummy digits, and/or some of the available processors may not be used. The following description and analysis remain correct in these cases with small corrections of the constant factors.

If $n \leq M|P|^{\log_3 2}/10$, then the product is computed by COPK in the MI execution mode (i.e., COPK$_{MI}$) in $\log_3 3|P|/4$ breadth-first recursive steps. If that is not the case, COPK proceeds by executing up to $\log_2 |P|$ depth-first recursive steps. The sub-problems generated after such steps will have input size which allows their solution to be computed in the MI execution mode. COPK uses the sequential algorithm SKIM to compute the products of integers locally. Clearly, any sequential algorithm can be used in place of it. In the following, we present in detail and analyze COPK’s execution in the UM and main execution mode.
6.1 COPK in the MI execution mode

When multiplying input integers with \( |P| = 1 \), COPK_{MI} computes the product \( C \) using the sequential Karatsuba’s algorithm SKIM discussed in the previous section. If \( |P| = 4 \), then COPK_{MI} proceeds as follows:

1. In parallel, \( P[2] \) and \( P[3] \) send to, respectively, \( P[1] \) and \( P[0] \) a copy of the \( n/4 \) digits of \( A(P[3]) \) (resp., \( A(P[2]) \)).

2. In parallel, \( P[0] \) and \( P[1] \) send to, respectively, \( P[2] \) and \( P[3] \) a copy of the \( n/4 \) digits of \( B(P[0]) \) (resp., \( A(P[1]) \)).

3. After the two previous steps, \( A_0 \) and a copy of \( A_1 \) (resp., a copy of \( B_0 \) and \( B_1 \)) are partitioned in \( [P[1], P[0]] \) (resp., \( [P[3], P[2]] \)) in \( n/4 \) digits. In parallel \( [P[1], P[0]] \) (resp., \( [P[3], P[2]] \)) invoke the parallel subroutine DIFF to compute the flag \( A' = |A_0 - A - 1| \) (resp., \( B' = |B_1 - B_0| \)) and the flag \( f_A \) (resp., \( f_B \)) which equals zero if \( A_0 = A_1 \) (resp., \( B_0 = B_1 \)), 1 if \( A_0 > A_1 \) (resp., \( B_1 > B_0 \)), and \(-1\) if \( A_0 < A_1 \) (resp., \( B_1 < B_0 \)). Once computed \( A' \) (resp., \( B' \)) is partitioned in \( [P[1], P[0]] \) (resp., \( [P[3], P[2]] \)) in \( n/4 \) digits, and each processor in the subsequence holds a copy of \( f_A \) (resp., \( f_B \)). Before proceeding processors in \( [P[1], P[0]] \) (resp., \( [P[3], P[2]] \)) remove the digits of the copy of \( A_1 \) (resp., \( B_0 \)) from their local memory.

4. \( P[3] \) sends \( A(P[3]), B(P[3]) \) and \( B'(P[3]) \) to \( P[2] \). Then it removes these values, as well as \( f_B \), from its local memory. In parallel, \( P[1] \) sends \( A(P[1]) \) and \( B(P[1]) \) to \( P[0] \), and then it removes them from it local memory. After this step, \( P[2] \) (resp., \( P[0] \)) holds \( A_1, B_1 \) and \( B' \) (resp., \( A_0, B_0 \)) in its local memory.

5. \( P[0] \) sends \( A'P[0] \) to \( P[1] \), and then removes it and \( f_A \) from its local memory.

6. \( P[2] \) sends \( B' \) and \( f_B \) to \( P[1] \) and then removes them from its local memory.

7. In parallel, \( P[0] \) computes \( C_0 = A_0 \times B_0 \), \( P[1] \) computes \( C'_0 = f_A A' \times f_B B' \), and \( P[2] \) computes \( C_2 = A_2 \times B_2 \) using the sequential Karatsuba’s algorithm SKIM;

8. In parallel, \( P[0] \) sends to \( P[1] \) a copy of \( C_0 \), and \( P[2] \) sends to \( P[3] \) a copy of \( C_2 \).

9. In parallel, \( P[0] \) (resp., \( P[3] \)) sends to \( P[2] \) (resp., \( P[1] \)) a copy of the \( n/2 \) most (resp., least) significant digits of \( C_0 \) (resp., \( C_2 \)), and then removes them from its local cache.

10. \( P[1] \) sends to \( P[2] \) a copy of the \( n/2 \) most significant digits of \( C' \) and then removes them from its local cache.

At the end of the previous steps, two copies of \( C_0 \) are partitioned in each \( [P[1], P[0]] \) and in \( [P[1], P[0]] \) in \( n/2 \) digits, \( C' \) is partitioned in \( [P[2], P[1]] \) in \( n/2 \) digits, and two copies of \( C_2 \) are partitioned in each \( [P[2], P[1]] \) and \( [P[3], P[2]] \). By construction we have \( C(P[0]) = C_0(P[0]) \). Let \( C'_0 = C_0 \mod s^{n/2} \) and \( C'_2 = C_2 \times s^{n/2} \). \( C'_0, C_0, C_1, C_2 \) and \( C'_2 \) are integers with at most \( 3n/2 \) digits each partitioned in \( [P[3], P[2], P[1]] \) in \( n/2 \) digits. The \( 3n/2 \) most significant digits of \( C \) correspond to those of the algebraic sum \( C'_0 + C' + C_0 + C_1 + C_2 + C'_2 \). If \( C' \geq 0 \) such sum can be computed using four consecutive invocations of the SUM subroutine
discussed in Section 4.1 using the sequence \([P_2, P_1, P_0]\). If instead \(C' < 0\), the sum \(C_0 + C + C_2 + C'_2\) can be computed using three invocations of the SUM and one of DIFF using the sequence \([P_2, P_1, P_0]\). The \(3n/2\) digits of such sum are partitioned in \(P^n\) in \(2n/|P|\) digits, and, hence, \(C\) is partitioned in \(P\) in \(2n/|P|\) digits.

At the end of this procedure, the product \(C\) is partitioned in \(P\) in \(2n/|P|\) digits. We use the operations described for \(|P| = 4\) as the base case for the recursive scheme of \(\text{COPK}_M\).

If \(|P| > 4\), \(\text{COPK}_M\) proceeds following a breadth-first traversal of the recursion tree, similarly to the operations discussed for the case \(|P| = 4\):

1. **Splitting:** Let \(A_0\) and \(B_0\) (resp., \(A_1\) and \(B_1\)) be defined as 2. By assumption, they are partitioned in \(P'\) (resp., \(P''\), that is, the first (resp., second) half of the sequence of \(P\) as defined in 4.

   (a) The values \(f_A, A', f_B\) and \(B'\) are computed following steps analogous to those in the description of the base case \(|P| = 4\) (1-4). Processors in \(P'\) (resp., \(P''\)) operate as those in \([P_1, P_0]\) (resp., \([P_3, P_2]\)), all the communications occur between processors of the same index within the sub-sequence.

Consider the following subsequences of \(P\):

\[
P_0 = [P[(|P|/2) - 1], P[(|P|/2) - 3], \ldots, P[5], P[3], P[2], P[0]]; \]
\[
P_1 = [P[|P| - 2], P[(3|P|)/4 - 2], P[(|P| - 5)], P[3|P|/4 - 5], \ldots, P[3|P|/4 + 1], P[|P|/2 + 1], P[|P|/2 - 2], P[(|P|/4 - 2)], P[|P|/2 - 5], P[|P|/4 - 5], \ldots, P[|P|/4 + 1], P[1]]; \]
\[
P_2 = [P[(|P| - 1), P[(|P| - 3), P[(|P| - 4)], \ldots, P[|P|/2 + 3], P[|P|/2 + 2], P[2|P|/3]]. \]

The sub-sequence \(P_0\) (resp., \(P_2\)) includes the processors in the first (resp., second) half of \(P\) except those of index \(i = 1, 4, 7, \ldots, |P|/2 - 5, |P|/2 - 2\) (resp., \(j = i + |P|/2 + 1\). The sub-sequence \(P_1\) is assigned all the remaining processors rearranged in a way which will reduce communications in the latter phases. \(\text{COPK}_M\) assigns each subsequence of the available processors to one of the three sub-problems which are recursively invoked to computed the product \(A \times B\):

- The product \(A_0 \times B_0\) is computed by \(P_0\);
- The product \((A_0 - A_1) \times (B_1 - B_0)\) is computed by \(P_1\);
- The product \(A_1 \times B_1\) is computed by \(P_2\);

\(\text{COPK}_M\) transfers to each sequence \(P_1\) the input integers for the corresponding sub-problems. This is achieved in the two following parallel communication steps:

(b) In parallel, each processors \(P[i]\) for \(i = 1, 4, \ldots, |P|/2 - 5, |P|/2 - 2\ sends to \(P[i - 1]\) a copy of the \(n/(2|P|)\) least significant digits of \(A(P[i]), B(P[i])\) and either \(A'(P[i])\) if \(i < |P|/2\) or \(B'(P[i])\) if \(i \geq |P|/2\), and then removes them from its local cache.

(c) In parallel, each processors \(P[i]\) for \(i = 1, 4, \ldots, |P|/2 - 5, |P|/2 - 2\ sends to \(P[i + 1]\) a copy of the \(n/(2|P|)\) most significant digits of \(A(P[i]), B(P[i])\) and
either $A'(\mathbf{P}[i])$ if $i < |\mathbf{P}/2|$ or $B'(\mathbf{P}[i])$ if $i \geq |\mathbf{P}/2|$, and then removes them from its local cache. At the end of this step $A_0$, $B_0$ and $A'$ (resp., $A_1$, $B_1$ and $B'$) are partitioned in $\mathbf{P}_0$ (resp., $\mathbf{P}_2$) in $3n/(2|\mathbf{P}|)$ digits.

(d) In parallel, all processors $\mathbf{P}_0[i]$ for $i \in \{0,1,\ldots, |\mathbf{P}|/3 - 1\}$, send the $3n/(2|\mathbf{P}|)$ digits of $A'(\mathbf{P}_0[i])$ to $\mathbf{P}_1[i]$, and then remove them from their local memory.

(e) In parallel, all processors $\mathbf{P}_2[i]$ for $i \in \{0,1,\ldots, |\mathbf{P}|/3 - 1\}$, send the $3n/(2|\mathbf{P}|)$ digits of $B'(\mathbf{P}_0[i])$ to $\mathbf{P}_1[i]$, and then remove them from their local memory.

2. **Recursive multiplication:** The three sub-products are computed in parallel using COPK$_{MI}$: $C_0 = A_0 \times B_0$ is computed by the processors in $\mathbf{P}_0$, $C' = A' \times B'$ is computed by $\mathbf{P}_1$, and $C_2 = A_1 \times B_1$ is computed by $\mathbf{P}_2$. At the end of these recursive calls, each product is partitioned in the subsequence used to compute it in $3n/|\mathbf{P}|$ digits.

3. **Recomposition:** COPK$_{MI}$ combines $C_0$, $C'$ and $C_2$ to obtain the desired $C$. The steps closely follows those discussed for the base case $|\mathbf{P}| = 4$.

(a) In parallel, each processor $\mathbf{P}_0[i]$ (resp., $\mathbf{P}_2[i]$) for $i = 0,3,6,\ldots, |\mathbf{P}_0| - 3$, sends to $\mathbf{P}[i + 1]$ (resp., $\mathbf{P}[i + 1 + |\mathbf{P}|/2]$) the $n/|\mathbf{P}|$ most significant digits of $C_0(\mathbf{P}_0[i])$ (resp., $C_2(\mathbf{P}_2[i])$) and then removes them from its local memory.

(b) In parallel, each processor $\mathbf{P}_0[i]$ (resp., $\mathbf{P}_2[i]$) for $i = 2,5,8,\ldots, |\mathbf{P}_0| - 1$, sends to $\mathbf{P}[i - 1]$ (resp., $\mathbf{P}[i - 1 + |\mathbf{P}|/2]$) the $n/|\mathbf{P}|$ least significant digits of $C_0(\mathbf{P}_0[i])$ (resp., $C_2(\mathbf{P}_2[i])$) and then removes them from its local memory. Note that each of the $\mathbf{P}_0[i]$’s (resp., $\mathbf{P}_2[i]$’s) is communicating is a different processor in $\mathbf{P}_1$. Hence, all such communication can occur in parallel. At the end of these two steps, $C_0$ (resp., $C_2$) is partitioned in the first (resp., second) half of the sequence $\mathbf{P}$, denoted as $\mathbf{P}'$ (resp., $\mathbf{P}''$).

(c) In parallel, each processor $\mathbf{P}_1[i]$: 

- for $i = 0,2,4,\ldots, |\mathbf{P}_1|/2 - 2$, sends to $\mathbf{P}[|\mathbf{P}|/4 + 3i/2]$ the $2n/|\mathbf{P}|$ least significant digits of $C'(\mathbf{P}_1[i])$;
- for $i = 1,3,5,\ldots, |\mathbf{P}_1|/2 - 1$, sends to $\mathbf{P}[|\mathbf{P}|/4 + [3i/2]]$ the $2n/|\mathbf{P}|$ most significant digits of $C'(\mathbf{P}_1[i])$ and then removes them from its local memory;

Further, each processor $\mathbf{P}_1[|\mathbf{P}_1|/2 + i]$: 

- for $i = 0,2,4,\ldots, |\mathbf{P}_1|/2 - 2$, sends to $\mathbf{P}[|\mathbf{P}|/2 + 3i/2]$ the $2n/|\mathbf{P}|$ least significant digits of $C'(\mathbf{P}_1[i])$, and then removes them from its cache;
- for $i = 1,3,5,\ldots, |\mathbf{P}_1|/2 - 1$, sends to $\mathbf{P}[|\mathbf{P}|/2 + [3i/2]]$ the $2n/|\mathbf{P}|$ most significant digits of $C'(\mathbf{P}_1[i])$;

As in these steps each processor in $\mathbf{P}_1$ communicates with a single, distinct, processor in either $\mathbf{P}_0$ or $\mathbf{P}_2$, all these communications may occur in parallel.

(d) In parallel, each processor $\mathbf{P}'[i]$ (resp., $\mathbf{P}''[i + |\mathbf{P}|/4]$) for $i = 0,2,\ldots, |\mathbf{P}|/4 - 2$ sends to $\mathbf{P}'[i + |\mathbf{P}|/4]$ (resp., $\mathbf{P}''[i]$) a copy of the $2n/|\mathbf{P}|$ digits of $C_0(\mathbf{P}'[i])$ (resp., $C_2(\mathbf{P}''[i + |\mathbf{P}|/4])$).

(e) In parallel, each processor $\mathbf{P}'[i + |\mathbf{P}|/4]$, for $i = 0,1,\ldots, |\mathbf{P}|/4 - 1$, sends to $\mathbf{P}''[i]$ a copy of the digits of $2n/|\mathbf{P}|$ of $C_0(\mathbf{P}'[i])$. 

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(f) In parallel, each processor $P''[i]$, for $i = 0, 1, \ldots, |P|/4 - 2$, sends to $P'[i + |P|/4]$ a copy of the $2n/|P|$ of digits of $C_2(P'[i])$.

At the end of these operations, $C_0$ is partitioned in $P'$ in $2n/|P|$ digits, $C_2$ is partitioned in $P''$ in $2n/|P|$ digits. Further, $C'$ and copies of $C_0$ and $C_2$ are partitioned in the subsequence of processors $[P[3|P|/4 - 1], \ldots, P[|P|/4 - 1]]$ in $2n/|P|$ digits. The computation of $C$ is completed following steps analogous to those discussed for the base case $|P| = 4$.

**Theorem 14.** Let $A$ and $B$ be two $n$-digit integers portioned in a sequence of processors $P$, with $n \geq |P|$, in $n/|P|$ digits. COPK$_{MI}$ computes the product $C = A \times B$ using the processors in $P$, provided that each processor is equipped with a local memory of size at least $M_{COPK_{MI}}(n, |P|) = 10n/|P|^{\log_2 3}$. We have:

$$T_{COPK_{MI}}(n, |P|) \leq 173n^{\log_2 3}/|P|$$

$$BW_{COPK_{MI}}(n, |P|) \leq 174n/|P|^{\log_2 3}$$

$$L_{COPK_{MI}}(n, |P|) \leq 25 \log_2^2 |P|$$

**Proof.** COPK$_{MI}$ correctly computes $C = A \times B$ by inspection.

If $|P| = 1$, the product $C = A \times B$ is computed locally by the single available processor using algorithm SKIM. By Lemma 10 we thus have $M_{COPK_{MI}}(n, 1) \leq 8n$, $T_{COPK_{MI}}(n, 1) \leq 8n^2$, $BW_{COPK_{MI}}(n, 1) = 0$ and $L_{COPK_{MI}}(n, 1) = 0$.

For $|P| = 4$, the product $C = A \times B$ is computed using three of the four available processors. By the description, at any time each processor may need to maintain in its local memory at most $3n/2$ digits of the input, $4 \times n/2$ digits of the output of the sub-problems, the data required to invoke DIFF to compute $A'$ and $B'$, the data required to execute SKIM for input integers of size $n/2$, and the data required to invoke SUM and DIFF in the final recombination step. Note that, once the outputs of the subproblems are computed, it is unnecessary to maintain the respective input values in the memory. By Lemma 9 and Lemma 7, $4n^2 + 5 \geq M_{DIFF}(3n/2, 3) \geq M_{SUM}(3n/2, 3)$.

$$M_{COPK_{MI}}(n, 4) \leq \max \left\{ \frac{3n}{4} + M_{DIFF} \left( \frac{n}{2}, 2 \right), M_{SKIM} \left( \frac{n}{2} \right), \frac{3n}{4} + M_{DIFF} \left( \frac{3n}{2}, 3 \right) \right\}$$

$$\leq \max \left\{ \frac{3n}{4} + 4n/4 + 5, 4n, \frac{3n}{2} + 4n/2 + 5 \right\}$$

$$< 4n$$

(9)

where the last passage follows as, by assumption, $n \geq |P| = 4$. The computation time is given by the sum of the time required for computing the differences in step (4), the invocation of SKIM in step (8), and for computing a difference and three sums, as discussed at the end of the presentation of the base case:

$$T_{COPK_{MI}}(n, 4) \leq T_{DIFF} \left( \frac{n}{2}, 2 \right) + T_{SKIM} \left( \frac{n}{2} \right) + T_{DIFF} \left( \frac{3n}{2}, 3 \right) + 3T_{SUM} \left( \frac{3n}{2}, 2 \right)$$

$$< \frac{7n}{4} + 16 \left( \frac{n}{2} \right)^{\log_2 3} + \frac{7n}{2} + 3 \times 6 \frac{n}{2}$$

$$< 12n^{\log_2 3}$$

(10)
where the last passage follows as, by assumption, \( n \geq |P| = 4 \). Similarly, the I/O cost of COPK\(_{MI}\) for the base case is bounded by the cost of operations discussed in the description of the base case as follows:

\[
BW_{COPK_{MI}}(n, 4) \leq \frac{n}{4} + \frac{n}{4} + BW_{DIFF}(\frac{n}{2}, 2) + \frac{3n}{4} + \frac{n}{4} + 1 + BW_{SKIM}(\frac{n}{2}) + n + \frac{n}{2} + \frac{n}{2} + BW_{DIFF}(n, 2) + 3BW_{SUM}(n, 2)
\]

\[
= 4n + BW_{DIFF}(\frac{n}{2}, 2) + BW_{DIFF}(n, 2) + 3BW_{SUM}(n, 2)
\]

\[
= 4n + 5 + 5 + 3 \times 4 + 1
\]

\[
< 10n
\]  

(11)

\[
< 10n
\]  

(12)

\[
L_{COPK_{MI}}(n, 4) \leq 1 + 1 + L_{DIFF}(\frac{n}{2}, 2) + 1 + 1 + L_{SKIM}(\frac{n}{2}) + 1 + 1 + L_{DIFF}(n, 2) + 3L_{SUM}(n, 2)
\]

\[
= 8 + L_{DIFF}(\frac{n}{2}, 2) + L_{DIFF}(n, 2) + 3L_{SUM}(n, 2)
\]

\[
= 8 + 3 + 3 \times 2
\]

(13)

\[
= 20
\]

where (11) and (13) follow from Lemma 7 and Lemma 9, and (12) follows from the assumption \( n \geq |P| = 4 \). This concludes the analysis of the base case.

In the following, we assume \( |P| > 4 \). COPK\(_{MI}\) proceeds to execute \( \log_3 |P|/4 \) breadth-first recursion steps. In each recursion branch, the available processors are divided into three disjoint subsequences and assigned to a different recursive invocation of COPK\(_{MI}\). After \( \log_3 |P|/4 \) recursive breadth-first levels, each of the generated sub-problems is assigned to a subsequence of four processors and COPK\(_{MI}\) proceeds according to the base case.

The analysis largely follows that for the base case \( |P| = 4 \) with only minor adjustments corresponding to the minor difference in the use of the available processors:

\[
M_{COPK_{MI}}(n, |P|) \leq \max \left\{ \frac{3n}{|P|} + M_{DIFF}(\frac{n}{2}, \frac{|P|}{2}), M_{COPK_{MI}}(\frac{n}{2}, \frac{|P|}{3}), \right. \\
3 \frac{2n}{|P|} + M_{DIFF}(n, \frac{|P|}{2}) \}
\]

\[
\leq \max \left\{ 3 \frac{n}{|P|} + 4 \frac{n}{|P|} + 5, M_{COPK_{MI}}(\frac{n}{2}, \frac{|P|}{3}), 6 \frac{n}{|P|} + 4 \frac{2n}{|P|} + 5 \right. \\
\leq \max \left\{ M_{COPK_{MI}}(\frac{n}{2}, \frac{|P|}{3}), 14 \frac{n}{|P|} + 5 \right. \}
\]

\[
\leq \max \left\{ M_{COPK_{MI}}(\frac{n}{2}, \frac{|P|}{3}), 14 \frac{n}{|P|} + 5 \right. \}
\]

\[
\leq \max \left\{ M_{COPK_{MI}}(\frac{n}{2}, \frac{|P|}{3}), 14 \frac{n}{|P|} + 5 \right. \}
\]

\[
\leq \max \left\{ M_{COPK_{MI}}(\frac{n}{2}, \frac{|P|}{3}), 14 \frac{n}{|P|} + 5 \right. \}
\]

\[
\leq \max \left\{ M_{COPK_{MI}}(\frac{n}{2}, \frac{|P|}{3}), 14 \frac{n}{|P|} + 5 \right. \}
\]

\[
\leq \max \left\{ M_{COPK_{MI}}(\frac{n}{2}, \frac{|P|}{3}), 14 \frac{n}{|P|} + 5 \right. \}
\]

\[
\leq \max \left\{ M_{COPK_{MI}}(\frac{n}{2}, \frac{|P|}{3}), 14 \frac{n}{|P|} + 5 \right. \}
\]

\[
\leq \max \left\{ M_{COPK_{MI}}(\frac{n}{2}, \frac{|P|}{3}), 14 \frac{n}{|P|} + 5 \right. \}
\]

\[
\leq \max \left\{ \frac{10n}{|P|^2}, 6 \frac{n}{|P|^2} + 5 \right. \}
\]

\[
= \frac{10n}{|P|^2}.
\]
Hence, by Lemma 10, \( M_{\text{COPK}_{MI}}(n, |P|) \geq 8n/\sqrt{|P|} \). We can thus conclude that all such requirement are met if each processor is equipped with a local memory of size \( M_{\text{COPK}_{MI}}(n, |P|) = 12n/\sqrt{|P|} \).

By construction, in each recursion level of the MI execution mode, COPK\(_{MI}\) computes the differences \( A', B' \), and then it recursively invokes itself on three distinct subsequences of available processors to compute the three generated subproblems. Then, COPK\(_{MI}\) combines the output of such subproblems to conclude the computation of \( C \). By Lemma 7 and Lemma 9, we have:

\[
T_{\text{COPK}_{MI}}(n, |P|) \leq T_{\text{DIFF}} \left( \frac{n}{2}, \frac{|P|}{2} \right) + T_{\text{COPK}_{MI}} \left( \frac{n}{2}, \frac{|P|}{3} \right) + T_{\text{DIFF}} \left( \frac{3n}{2}, \frac{3|P|}{4} \right) + 3T_{\text{SUM}} \left( \frac{3n}{2}, \frac{3|P|}{4} \right)
\]

\[
< 7 \frac{n}{|P|} + 5 \log_2 |P| + T_{\text{COPK}_{MI}} \left( \frac{n}{2}, \frac{|P|}{3} \right) + 14 \frac{n}{|P|} + 5 \log_2 \frac{3|P|}{4}
\]

\[
+ 3 \left( 12 \frac{n}{|P|} + 4 \log_2 \frac{3|P|}{4} \right)
\]

\[
< 57 \frac{n}{|P|} + 22 \log_2 |P| + T_{\text{COPK}_{MI}} \left( \frac{n}{2}, \frac{|P|}{3} \right)
\]

\[
\leq T_{\text{COPK}_{MI}} \left( n \left( \frac{4}{|P|} \right)^{\log_3 2}, 4 \right) + 57 \frac{n}{|P|} \sum_{i=0}^{\log_3 |P|/4-1} \left( \frac{3}{2} \right)^i
\]

\[
+ 22 \left( \frac{n}{|P|} + 2 \log_2 \frac{|P|}{3} \right)
\]

\[
< 48 \frac{n^{\log_2 2}}{|P|} + 69 \frac{n}{|P|^3} + 14 \log_2 |P|
\]

\[
< 11 \frac{n^{\log_2 2}}{|P|} + 56 \frac{n}{|P|^3}
\]

where the last passage follows from the fact that, by assumption, \( n > |P| \geq 2 \), and that for such values \( 4 \frac{n^{\log_2 3}}{|P|} > \log_2 |P| \). The I/O operation of COPK\(_{MI}\) have been presented in detail in the description of the algorithm. Here we present the analysis of the recursion, which allows us to characterize both bandwidth and latency cost. To aid readability, in the first line of the analysis of the bandwidth (resp., latency), each term of the sum corresponds to the bandwidth (resp., latency) cost of each phase specified in the description of COPK\(_{MI}\). By Lemma 7 and Lemma 9, we have:

\[
BW_{\text{COPK}_{MI}}(n, |P|) \leq \frac{n}{|P|} + \frac{n}{|P|} + BW_{\text{DIFF}}(n/2, |P|/2, n, |P|) + \frac{n}{|P|} + \frac{n}{|P|} + 2 \times \frac{3n}{2|P|} + 2
\]

\[
+ BW_{\text{COPK}_{MI}} \left( \frac{n}{2}, \frac{|P|}{3} \right) + 2 \times \frac{n}{|P|}
\]

\[
+ 4 \times \frac{2n}{|P|} + BW_{\text{DIFF}} \left( \frac{3n}{2}, \frac{3|P|}{4} \right) + 3BW_{\text{SUM}} \left( \frac{3n}{2}, \frac{3|P|}{4} \right)
\]

\[33\]
\[
\begin{align*}
&\leq 5 \log_2 \left( \frac{|P|}{2} \right) + BW_{COPK_MI} \left( \frac{n}{2}, \left| \frac{|P|}{3} \right| \right) + 5 \log_2 \frac{3|P|}{4} + 3 \times 4 \log_2 \frac{3|P|}{4} \\
&\quad + 20 \frac{n}{|P|} + 2 \\
&\leq BW_{COPK_MI} \left( \frac{n}{2}, \left| \frac{|P|}{3} \right| \right) + 22 \log_2 |P| + 20 \frac{n}{|P|} \\
&\leq BW_{COPK_MI} \left( n \left( \frac{4}{|P|} \right)^{\log_3 2}, 4 \right) + 20 \frac{n}{|P|} \sum_{i=0}^{\log_3 |P|/4-1} \left( \frac{3}{2} \right)^i \\
&\quad + 22 \sum_{i=0}^{\log_3 |P|/4-1} \log_2 \left( \frac{|P|}{3} \right) \\
&< 24 \frac{n}{|P|^{\log_3 2}} + 24 \frac{n}{|P|^{\log_3 2}} + 14 \log_2 |P|. \\
&< 48 \frac{n}{|P|^{\log_3 2}} + 126 \frac{n}{|P|^{\log_3 2}}.
\end{align*}
\]

where (14) follows from the fact that, by assumption, \( n > |P| \geq 2 \), and that for such values \( 9 \frac{n}{|P|^{\log_3 2}} > \log_2 |P| \). Finally,

\[
\begin{align*}
L_{COPK_MI}(n, |P|) &\leq 1 + 1 + L_{DIFF} \left( \frac{n}{2}, \left| \frac{|P|}{2} \right| \right) + 1 + 1 + 1 + L_{COPK_MI} \left( \frac{n}{2}, \left| \frac{|P|}{2} \right| \right) \\
&\quad + 2 + 4 + L_{DIFF} \left( \frac{3n}{2}, \left| \frac{3|P|}{4} \right| \right) + 3L_{SUM} \left( \frac{3n}{2}, \left| \frac{3|P|}{4} \right| \right) \\
&\leq 3 \log_2 \frac{|P|}{2} + L_{COPK_MI} \left( \frac{n}{2}, \left| \frac{|P|}{3} \right| \right) + 3 \log_2 \frac{3|P|}{4} + 3 \times 2 \log_2 \frac{3|P|}{4} + 12 \\
&\leq L_{COPK_MI} \left( \frac{n}{2}, \left| \frac{|P|}{3} \right| \right) + 12 \log_2 |P| \\
&\leq L_{COPK_MI} \left( n \left( \frac{4}{|P|} \right)^{\log_3 2}, 4 \right) + \sum_{i=0}^{\log_3 |P|/4-1} 12 \log_2 |P| \\
&< 20 + 5 \log_2^2 |P| \\
&< 25 \log_2^2 |P|.
\end{align*}
\]

\[\square\]

### 6.2 COPK in the main execution mode

An important consequence of Theorem 11 is that to execute COPK_MI to multiply two \( n \)-digit integers each processor must be equipped with a memory of size at least \( 10n/|P|^{\log_3 2} \). Considering the aggregate memory available in \( |P| \) processors, this implies that COPK_MI can only be used to multiply \( n \)-digit integers where \( n \in O \left( M/|P|^{\log_3 2} \right) \).

As outlined in the general framework in Section 3, in this section, we describe how to combine the MI execution mode in COPK_MI with a depth-first scheduling of the recursively generated subproblems. Our algorithm, COPK, allows to compute the product of two \( n \)-digit integers provided that (i) \( M \geq 40n/|P| \) and (ii) \( M \geq \log_2 |P| \). By (i), COPK can thus
be used to multiply \( n \)-digit integers where \( n \in \mathcal{O}(M|\mathbf{P}|) \). That is, its memory requirement corresponds asymptotically to the amount of overall memory required to store the input integers and/or the product.

In its main execution mode, COPK executes up to \( \log_2 \frac{20n}{M|\mathbf{P}| \log_2 2} \) of depth-first steps, until the size of the subproblems being generated is such that they can be computed by the sequence of processors \( \mathbf{P} \) using COPK according to the steps discussed for the MI execution mode.

When COPK \( (n, A, B, \mathbf{P}, m) \) is invoked, if \( n \leq 12n\sqrt{|\mathbf{P}|} \), the product is computed by invoking COPK_{MI}. Otherwise COPK proceeds as follows:

1. In parallel, each processor \( \mathbf{P}[i] \), for \( i = 0, 1, \ldots, |\mathbf{P}|/2 - 1 \), sends to \( \mathbf{P}[i + |\mathbf{P}|/2] \) a copy of the \( n/(2|\mathbf{P}|) \) most significant digits of \( A(\mathbf{P}[i]) \) and \( B(\mathbf{P}[i]) \), and then remove them from their cache.

2. In parallel, each processor \( \mathbf{P}[|\mathbf{P}|/2 + i] \), for \( i = 0, 1, \ldots, |\mathbf{P}|/2 - 1 \), sends to \( \mathbf{P}[i] \) a copy of the \( n/(2|\mathbf{P}|) \) least significant digits of \( A(\mathbf{P}[|\mathbf{P}|/2 + i]) \) and \( B(\mathbf{P}[|\mathbf{P}|/2 + i]) \), and then remove them from their cache.

At the end of step (1) (resp., (2)), \( A_0 \) and \( B_0 \) (resp., \( A_1 \) and \( B_1 \)) are partitioned in the sequence:

\[
\tilde{\mathbf{P}} = [\mathbf{P}[|\mathbf{P}| - 1], \mathbf{P}[|\mathbf{P}|/2 - 2], \ldots, \mathbf{P}[|\mathbf{P}|/2], \mathbf{P}[0]
\]

in \( n/(2|\mathbf{P}|) \) digits. The even (resp., odd) index processors in the sequence \( \tilde{\mathbf{P}} \) are the processors in the first (resp., second) half of the sequence \( \mathbf{P} \).

3. COPK is then recursively invoked to compute \( C_0 = A_0 \times B_0 \) using the sequence \( \tilde{\mathbf{P}} \). Once computed, \( C_0 \) is partitioned in \( \tilde{\mathbf{P}} \) in \( n/|\mathbf{P}| \) digits.

4. COPK is then recursively invoked to compute \( C_2 = A_1 \times B_1 \) using the sequence \( \tilde{\mathbf{P}} \). Once computed, \( C_2 \) is partitioned in \( \tilde{\mathbf{P}} \) in \( n/|\mathbf{P}| \) digits.

5. DIFF is invoked on \( \tilde{\mathbf{P}} \) to compute \( A' = |A_0 - A_1| \) and the flag \( f_A \) such that \( f_A = 0 \) if \( A_0 = A_1, f_A = 1 \) if \( A_0 > A_1 \), and \( f_A = -1 \) if \( A_0 < A_1 \). Then each processor removes the digits of \( A_0 \) and \( A_1 \) from its local memory.

6. DIFF is invoked on \( \tilde{\mathbf{P}} \) to compute \( B' = |B_1 - A_0| \) and the flag \( f_B \) such that \( f_B = 0 \) if \( B_0 = B_1, f_B = 1 \) if \( B_0 < B_1 \), and \( f_B = -1 \) if \( B_0 > B_1 \). Then each processor removes the digits of \( B_0 \) and \( B_1 \) from its local memory.

7. If \( f_A \times f_B = 0 \), each processor in \( \tilde{\mathbf{P}} \) sets \( C'(\tilde{\mathbf{P}}) = 0 \). Otherwise, COPK is then recursively invoked to compute \( C' = A' \times B' \) using the sequence \( \tilde{\mathbf{P}} \). Once computed, \( C' \) is partitioned in \( \tilde{\mathbf{P}} \) in \( n/|\mathbf{P}| \) digits.

To complete the computation of \( C \), COPK opportunistically redistributes and combine \( C_0, C' \) and \( C_2 \)

8. SUM is invoked on \( \tilde{\mathbf{P}} \) to compute the sum \( C_0 + C_2 \), which, once computed is partitioned in \( \tilde{\mathbf{P}} \) in \( n/|\mathbf{P}| \) digits.
9. If \( f_A \times f_B = 1 \) (resp., \( f_A \times f_B = -1 \)), \( \text{SUM} \) (resp., \( \text{DIFF} \)) is invoked on \( \tilde{P} \) to compute \( C_1 = (f_A \times f_B)C' + C_0 + C_2 \), which, once computed is partitioned in \( \tilde{P} \) in \( n/|P| \) digits.

10. In parallel each processor \( \tilde{P}[i] \), for \( i = 0, 1, \ldots, |P|/2 - 1 \), sends to \( \tilde{P}[i] \) a copy of the \( n/|P| \) digits of \( C_0(\tilde{P}[i] + |P|/2) \), henceforth referred as \( C'_0(\tilde{P}[i] + |P|/2) \), and then removes them from its cache. After these last two steps, its base-

11. In parallel each processor \( \tilde{P}[i] \), for \( i = 0, 1, \ldots, |P|/2 - 1 \), sends to \( \tilde{P}[i] + |P|/2 \) a copy of the \( n/|P| \) digits of \( C_2(\tilde{P}[i] + |P|/2) \), henceforth referred as \( C'_2(\tilde{P}[i] + |P|/2) \), and then removes them from its cache. After these operations, \( \tilde{P}[i] \) is partitioned in \( \tilde{P}[|P|/2 - 1], \ldots, \tilde{P}[|P|/2] \).

12. Let \( C^* = C'_2 \times s^{n/2} + C'_0 \). By construction, \( C^* \) is partitioned in \( \tilde{P} \) in \( n/|P| \) digits. \( C'_1 = C^* + C_1 \) is computed using the parallel subroutine \( \text{SUM} \) on \( \tilde{P} \).

Note that the values \( C_0 + C_2, C_1 \) and \( C'_1 \) may have up to \( n + [3/s] \) non-zero digits in their base-

13. Let \( d = [C'_1/s^n] \). \( \tilde{P}[|P|/2] \) sends \( d \) to \( \tilde{P}[|P|/2] \) and then removes its from its cache.

14. Let \( C''_2 = |C_2/s^n| \). By construction, \( C''_2 \) is partitioned in \( \tilde{P}[|P|/2 - 1], \ldots, \tilde{P}[|P|/2] \) in \( n/|P| \) digits. \( \text{SUM} \) is invoked on this subsequence to compute \( C''_m = C''_2 + d \). As previously mentioned, its base-

15. In parallel each processor \( \tilde{P}[i] \) (resp., \( \tilde{P}[i + |P|/2] \)) for \( i = 1, 3, \ldots, |P|/2 - 1 \), sends to \( \tilde{P}[i - 1] \) (resp., \( \tilde{P}[i + |P|/2] \)) the \( n/|P| \) digits of \( C_0(\tilde{P}[i]) \) (resp., \( C'''(\tilde{P}[i]) \)), and then removes them from its cache. After these operations, \( C''_0 \mod s^{n/2} \) (resp., \( C''' \)) is partitioned in \( |P|/4 - 1], \ldots, [0] \) in \( 2n/|P| \) (resp., \( |P|/2 - 1], \ldots, [3|P|/4] \) in \( 2n/|P| \) digits.

16. In parallel each processor \( \tilde{P}[i] \) (resp., \( \tilde{P}[i + |P|/2] \)) for \( i = 1, 3, 5, \ldots, |P|/2 - 1 \), sends to \( \tilde{P}[i - 1] \) (resp., \( \tilde{P}[i + |P|/2] \)) the \( n/|P| \) digits of \( C''(\tilde{P}[i]) \) (resp., \( C'(\tilde{P}[i + |P|/2 - 1]) \)), and then removes them from its cache.

17. In parallel each processor \( \tilde{P}[i] \) (resp., \( \tilde{P}[i + |P|/2] \)) for \( i = 0, 2, 4, \ldots, |P'|/2 - 2 \), sends to \( \tilde{P}[i + |P|/2] \) (resp., \( \tilde{P}[i + 1] \)) the \( n/|P| \) digits of \( C''(\tilde{P}[i]) \) (resp., \( C'(\tilde{P}[i + |P|/2 - 2]) \)), and then remove them from its cache. After these last two steps, \( C' \) is partitioned in \( |P|/2 - 1], \ldots, [|P|/4] \) in \( 2n/|P| \) digits.
The following theorem rigorously characterizes the performance of COPK:

**Theorem 15.** Let \(A\) and \(B\) be two \(n\)-digit integers portioned in a sequence of processors \(P\), with \(n \geq |P|\), in \(n/|P|\) digits. COPK computes the product \(C = A \times B\) using the processors in \(P\), provided that each processor is equipped with a local memory of size at least \(M_{\text{COPK}}(n, |P|) \geq \max\{40n/|P|, \log_2 |P|\}\). We have:

\[
T_{\text{COPK}}(n, |P|, M) \leq 675 \frac{n \log_2 3}{|P|} M \\
B W_{\text{COPK}}(n, |P|, M) \leq 1708 \left(\frac{n}{M}\right)^{\log_2 3} \frac{M}{|P|} \\
L_{\text{COPK}}(n, |P|, M) \leq 8728 \frac{n \log_2 3}{|P| \log_2 3} \frac{\log_2 2}{|P|}
\]

**Proof.** The algorithm is correct by inspection. If \(n \leq M|P|^{\log_2 3}/10\), the statement follows from Theorem 11. In the following we assume \(M|P|/24 \geq n > M|P|^{\log_2 3}/12\). COPK then proceeds by executing \(\ell\) consecutive depth-first steps, where \(0 < \ell \leq \lceil \log_2 6n/(40M \sqrt{P}) \rceil\).

In each recursion step, in addition to the space required for the recursive invocation to COPK, each processor must maintain \(2 \times n/|P|\) digits for the input of the problem being computed, at most \(4n/|P| + 2\) (including the temporary value \(C_1\)) digits of the outputs of the recursive subproblems, and the space required for the invocation of DIFF and SUM used to combine the outputs of the subproblems. Note that the recursive calls to COPK and the invocations of DIFF and SUM are always executed in distinct steps of the the same recursion level. Hence the memory space used for each can be reused. Further, by Lemma 9 and Lemma 7 the memory requirement of DIFF is higher of that on SUM for same input size and number of available processors, we have:

\[
M_{\text{COPK}}(n, |P|) \leq \frac{4n}{|P|} + 2 + \max\{M_{\text{COPK}}(n, |P|), M_{\text{DIFF}}(n, |P|)\}
\]

\[
\leq \frac{4n}{|P|} + 2 + \max \left\{M_{\text{COPK}}(n, |P|), \frac{4n}{|P|} + 5\right\} \tag{15}
\]

\[
\leq \frac{4n}{|P|} + 2 + M_{\text{COPK}}(n, |P|)
\]

\[
\leq \frac{4n}{|P|} \sum_{i=0}^{\lceil \log_2 \frac{20n}{M|P|^{\log_2 3}} \rceil - 1} 2^{-i} + 2 \left(\lceil \log_2 \frac{20}{M|P|^{\log_2 3}} \rceil - 1\right) + M_{\text{COPK}} \left(\frac{M|P|^{\log_2 3}}{20}, |P|, \frac{M|P|^{\log_2 3}}{20|P|^{1-\log_2 3}}\right)
\]

\[
\leq \frac{8n}{|P|} + 2 \log_2 \frac{20n}{M|P|^{\log_2 3}} + M_{\text{COPK}_{\text{MT}}} \left(\frac{M|P|^{\log_2 3}}{20}, |P|, \frac{M|P|^{\log_2 3}}{20|P|^{1-\log_2 3}}\right)
\]

\[
\leq 20 \frac{n}{|P|} + \frac{M}{2} \tag{16}
\]

where (15) follows from Lemma 9 and (16) from Theorem 14. By construction, after \(\ell\) depth-first steps, the available memory is reduced by at most \(10n/|P|\). As, by assumption,
\( M \geq 40n/|P| \), at least half of the space \( M \) originally assigned to each processor is still available. After at most \( \lceil \log_2 20n/(M|P|^{\log_3 2}) \rceil \) recursive steps, the generated sub-problems will have size at most \( M|P|^{\log_3 2}/20 \). Thus, by Theorem 14, can be computed using COPK using at most \( M/2 \) memory locations. This concludes the proof of the memory requirement for COPK.

The computation time required in a depth-first recursion level of COPK’s execution is bounded by the time required for evaluating \( A' \) and \( B' \), the computation steps required for the consecutive recursive invocations of COPK on the generated subproblems, plus the time required to combine the outputs of the sub-problems to compute the product itself using DIFF and three invocations of SUM. By following the description of the algorithm, we have:

\[
T_{\text{COPK}} (n, |P|, M) \leq T_{\text{DIFF}} \left( \frac{n}{2}, \frac{|P|}{2} \right) + 3T_{\text{COPK}} \left( \frac{n}{2}, |P|, M \right) + T_{\text{SUM}} (n, |P|) + T_{\text{DIFF}} (n, |P|) + T_{\text{SUM}} (n, |P|) + 1 + T_{\text{SUM}} \left( n, \frac{|P|}{2} \right)
\]

\[
< \frac{7n}{|P|} + 4 \log_2 \frac{|P|}{2} + 3T_{\text{COPK}} \left( \frac{n}{2}, |P|, M \right) + \frac{6n}{|P|} + 4 \log_2 \frac{|P|}{2} + \frac{7n}{|P|} + 4 \log_2 |P|
\]

\[
+ \frac{6n}{|P|} + 4 \log_2 |P| + 1 + \frac{12n}{|P|} + 4 \log_2 \frac{|P|}{2}
\]

\[
< 3T_{\text{COPK}} \left( \frac{n}{2}, |P|, M \right) + 38 \frac{n}{|P|} + 16 \log_2 |P|
\]

Further, in a depth-first recursion level, the I/O cost of COPK (both bandwidth and latency) can be bound by that of the four consecutive invocations of COPK used to compute the four subproblems, the cost of redistributing the input (resp., the output) of such subproblems, and the I/O cost of the three invocations of SUM used to combine the outputs of the three subproblems. We refer the reader to the detailed description of the algorithm. Here we compose the cost of the various operations step-by-step. By Lemma 7 and Lemma 9:

\[
B_{\text{COPK}} (n, |P|, M) \leq 2 \times \frac{n}{|P|} + B_{\text{DIFF}} \left( \frac{n}{2}, \frac{|P|}{2} \right) + 3B_{\text{COPK}} \left( \frac{n}{2}, |P|, M \right)
\]

\[
+ B_{\text{SUM}} (n, |P|) + B_{\text{DIFF}} (n, |P|) + 2 \times \frac{n}{|P|}
\]

\[
+ B_{\text{SUM}} (n, |P|) + 2 + B_{\text{SUM}} (n, |P|/2) + 3 \times \frac{n}{|P|}
\]

\[
\leq 10 \frac{n}{|P|} + 5 \log_2 \frac{|P|}{2} + 3B_{\text{COPK}} \left( \frac{n}{2}, |P|, M \right)
\]

\[
+ 4 \log_2 |P| + 5 \log_2 |P| + 4 \log_2 |P| + 2 + 4 \log_2 \frac{|P|}{2}
\]

\[
\leq 3B_{\text{COPK}} \left( \frac{n}{2}, |P|, M \right) + 10 \frac{n}{|P|} + 22 \log_2 |P|
\]
\[ L_{\text{COPK}}(n, |P|, M) \leq 2 + L_{\text{DIFF}} \left( \frac{n}{2}, \frac{|P|}{2} \right) + 3L_{\text{COPK}} \left( \frac{n}{2}, |P|, M \right) + L_{\text{SUM}}(n, |P|) \]
\[ + L_{\text{DIFF}}(n, |P|) + 2 + L_{\text{SUM}}(n, |P|) + 1 + L_{\text{SUM}}(n, |P|/2) + 3 \]
\[ \leq 8 + 3\log_2 \frac{|P|}{2} + 3L_{\text{COPK}} \left( \frac{n}{2}, |P|, M \right) + 2\log_2 |P| + 3\log_2 |P| \]
\[ + 2\log_2 |P| + 2\log_2 \frac{|P|}{2} \]
\[ \leq 3L_{\text{COPK}} \left( \frac{n}{2}, |P|, M \right) + 15\log_2 |P| \]

After \( 1 \leq \ell \leq \left\lceil \log_2 \frac{20n}{M|P|^{\log_3 2}} \right\rceil \) depth-first steps, the generated sub-problems have input size at most \( M|P|^{\log_3 2}/20 \), and COPK switches to the MI execution mode by invoking COPK\(_{\text{MI}}\). Hence, by Theorem \[ \text{[4]} \] and by the assumptions \( n \geq |P|, n \geq M|P|^{\log_3 2}/10 \), and \( M \geq \log_2 |P| \) we have:

\[ T_{\text{COPK}}(n, |P|, M) < 3T_{\text{COPK}} \left( \frac{n}{2}, |P|, M \right) + 38 \frac{n}{|P|} + 16\log_2 |P| \]
\[ < 3 \left\lceil \log_2 \frac{20n}{M|P|^{\log_3 2}} \right\rceil T_{\text{COPK}} \left( \frac{M|P|^{\log_3 2}}{20}, |P|, M \right) + 38 \frac{n}{|P|} \sum_{i=0}^{\left\lceil \log_2 \frac{20n}{M|P|^{\log_3 2}} \right\rceil-1} 2^{-i} \]
\[ + 16\log_2 |P| \left( \left\lceil \log_2 \frac{20n}{M|P|^{\log_3 2}} \right\rceil - 1 \right) \]
\[ < 3 \left( \frac{20n}{M|P|^{\log_3 2}} \right)^{\log_2 3} T_{\text{COPK}_{\text{MI}}} \left( \frac{M|P|^{\log_3 2}}{20}, |P| \right) + 76 \frac{n}{|P|} \]
\[ + 16\log_2 |P| \log_2 \frac{20n}{M|P|^{\log_3 2}} \]
\[ < 3 \left( \frac{20n}{M|P|^{\log_3 2}} \right)^{\log_2 3} \sum_{i=0}^{\left\lceil \log_2 \frac{20n}{M|P|^{\log_3 2}} \right\rceil} 173 \frac{M|P|^{\log_3 2}}{20^{\log_2 3}} \frac{|P|}{20^{\log_2 3}} + 16\log_2 |P| \log_2 \frac{20n}{M|P|^{\log_3 2}} \]
\[ < 595 \frac{\log_2 3}{|P|} + 80 \frac{\log_2 3}{|P|} \quad (17) \]

\[ BW_{\text{COPK}}(n, |P|, M) \leq 3BW_{\text{COPK}} \left( \frac{n}{2}, |P|, M \right) + 10 \frac{n}{|P|} + 22\log_2 |P| \]
\[ < 3 \left\lceil \log_2 \frac{20n}{M|P|^{\log_3 2}} \right\rceil BW_{\text{COPK}} \left( \frac{M|P|^{\log_3 2}}{20}, |P|, M \right) + 10 \frac{n}{|P|} \sum_{i=0}^{\left\lceil \log_2 \frac{20n}{M|P|^{\log_3 2}} \right\rceil-1} 2^{-i} \]
\[ + 22\log_2 |P| \log_2 \frac{20n}{M|P|^{\log_3 2}} \]
\[ < 3 \left( \frac{20n}{M|P|^{\log_3 2}} \right)^{\log_2 3} BW_{\text{COPK}_{\text{MI}}} \left( \frac{M|P|^{\log_3 2}}{20}, |P| \right) + 20 \frac{n}{|P|} \]
\[ + 22\log_2 |P| \log_2 \frac{20n}{M|P|^{\log_3 2}} \]
as we are considering the case where (17), (18) and (19) follow as, under the assumptions $O$.

Based on the analysis of COPK performance presented in Theorem 14 and Theorem 15, we have:

**Theorem 2.** COPK achieves optimal computation time speedup and optimal bandwidth cost among all parallel Karatsuba-based integer multiplication algorithms. It also minimizes the latency cost up to a $O(\log^2 \mathcal{P})$ multiplicative factor, where $\mathcal{P}$ denotes the number of processors used in the computation.

**Proof.** By Theorem 14, for $M \geq 10n/|\mathcal{P}|^{\log_2 2}$, the product $C = A \times B$ can be computed using COPK$_{MI}$. Under the assumption $n \geq \mathcal{P}$, the bandwidth cost of COPK$_{MI}$ asymptotically matches the memory-independent lower bound in Theorem 6 and its latency latency is within a $O(\log^2 \mathcal{P})$ factor of the corresponding lower bound. Note that the initial distribution of the input values among the processors used in COPK$_{MI}$ satisfies the balanced input distribution assumption used to derive Theorem 4.

For $10n/|\mathcal{P}|^{\log_2 2} < M \geq 40n/\mathcal{P}$, by Theorem 15, the product $C = A \times B$ can be computed using COPK. For $n \geq \mathcal{P}$ and $M \geq \log_2 \mathcal{P}$, the bandwidth cost of COPK$_{MI}$ asymptotically matches the memory-dependent lower bound in Theorem 5 and its latency latency is within a $O(\log^2 \mathcal{P})$ factor of the corresponding lower bound. The total memory space required across the available processors for the execution of COPK is $O(n)$, that is, within a constant factor of the minimum space required to store the input (and output) values. Finally, in both cases, COPK achieves optimal speedup of the computational time $O(n^{\log_2 3}/\mathcal{P})$. 

6.3 Comparison with communication lower bounds

\begin{align*}
L_{COPK}(n, |\mathcal{P}|, M) &\leq 3L_{COPK} \left(\frac{n}{M}, |\mathcal{P}|, M\right) + 15 \log_2 |\mathcal{P}| \\
&\leq 3 \left[\log_2 \frac{20n}{M|\mathcal{P}|^{\log_2 2}}\right] L_{COPK} \left(\frac{M|\mathcal{P}|^{\log_2 2}}{20}, |\mathcal{P}|, M\right) \\
&\quad + 15 \log_2 |\mathcal{P}| \left(\left\lceil \log_2 \frac{20n}{M|\mathcal{P}|^{\log_2 2}}\right\rceil - 1\right) \\
&< 3 \left(\frac{20n}{|\mathcal{P}|^{\log_2 3}}\right) L_{COPK, MI} \left(\frac{M|\mathcal{P}|^{\log_2 2}}{20}, |\mathcal{P}|\right) + 15 \log_2 |\mathcal{P}| \log_2 \frac{20n}{M|\mathcal{P}|^{\log_2 2}} \\
&\leq 3 \left(\frac{20n}{|\mathcal{P}|^{\log_2 3}}\right) L_{COPK, MI} \left(25 \log_2 |\mathcal{P}| + 15 \log_2 |\mathcal{P}| \log_2 \frac{20n}{M|\mathcal{P}|^{\log_2 2}}\right) \\
&\leq 8728 \left(\frac{n^{\log_2 3}}{|\mathcal{P}|^{\log_2 3}}\right) \log_2 |\mathcal{P}|\tag{19}
\end{align*}

where (17), (18) and (19) follow as, under the assumptions $n \geq |\mathcal{P}|$, $M \geq \log_2 |\mathcal{P}|$, and as we are considering the case $n \geq 10M|\mathcal{P}|^{\log_2 2}$, we have $\frac{n}{|\mathcal{P}|}, \log_2 |\mathcal{P}| \log_2 \frac{20n}{M|\mathcal{P}|^{\log_2 2}} \leq O\left(\left(\frac{n}{M}\right)^{\log_2 3} \frac{M}{|\mathcal{P}|}\right)$. 

For $10n/|\mathcal{P}|^{\log_2 2} < M \geq 40n/\mathcal{P}$, by Theorem 15, the product $C = A \times B$ can be computed using COPK. For $n \geq \mathcal{P}$ and $M \geq \log_2 \mathcal{P}$, the bandwidth cost of COPK$_{MI}$ asymptotically matches the memory-dependent lower bound in Theorem 5 and its latency latency is within a $O(\log^2 \mathcal{P})$ factor of the corresponding lower bound. The total memory space required across the available processors for the execution of COPK is $O(n)$, that is, within a constant factor of the minimum space required to store the input (and output) values. Finally, in both cases, COPK achieves optimal speedup of the computational time $O(n^{\log_2 3}/\mathcal{P})$. 

\[\square\]
7 Conclusion

We presented parallel algorithms for computing the product of integer numbers in the distributed memory setting. Our algorithm COPSIM is based on the recursive long integer multiplication, while COPK is a parallel implementation of Karatsuba’s fast multiplication scheme. Under mild conditions on the input size $n$, the number of available processors $P$, and the size of the local cache available to each of them, our algorithms achieve asymptotically optimal computational speedup and bandwidth cost, while their latency cost is within a $O(\log^2 P)$ multiplicative factor of the respective theoretical lower bounds. Further, our algorithms require that space available across the processors to be within a multiplicative constant factor of the minimum amount required to store the input values. Due to the common underlying strategy used to obtain both COPSIM and COPK, it is possible to combine them seamlessly, thus achieving hybridization of the two algorithmic schemes (as discussed in [34]). Such hybridization is of actual practical interest, as due to the constant factor terms in the complexity characterization of the algorithms (both computational and I/O) COPK allows for overall improved performance over COPSIM for large input size, while when multiplying integers with fewer digits, COPSIM may actually achieve lower execution time.

Due to the large constant factors in the bounds, the presented algorithms are mostly of theoretical interest. While such coefficients can be considerably reduced for particular, and reasonable, values of $n$, $M$, and $P$, the pursuit of improved algorithms to be used successfully in practice is an important natural direction for future research. Further, the $O(\log^2 P)$ multiplicative factor discrepancy between the latency of our proposed algorithms and the corresponding lower bound leaves open the question on whether it is actually possible to obtain algorithms with lower latency, or if instead, it is possible to obtain I/O lower bounds which capture such a higher latency requirement. Finally, we believe that the approach discussed in this work could be used to obtain a communication-optimal parallel version of other integer multiplication algorithms, among whom, in particular, the general Toom-Cook-k algorithm.

References

[1] Ramesh C Agarwal, Susanne M Balle, Fred G Gustavson, Mahesh Joshi, and Prasad Palkar. A three-dimensional approach to parallel matrix multiplication. IBM Journal of Research and Development, 39(5):575–582, 1995.

[2] Hazem M Bahig, Hatem M Bahig, and Khaled A Fathy. Fast and scalable algorithm for product large data on multicore system. Concurrency and Computation: Practice and Experience, page e5259.

[3] G. Ballard, J. Demmel, Olga H., B. Lipshitz, and O. Schwartz. Communication-optimal parallel algorithm for Strassen’s matrix multiplication. In Proc. ACM SPAA, 2012.

[4] G. Ballard, J. Demmel, O. Holtz, and O. Schwartz. Minimizing communication in numerical linear algebra. SIAM Journal on Matrix Analysis and Applications, 32(3):866–901, 2011.
[5] Grey Ballard, James Demmel, Olga Holtz, and Oded Schwartz. Communication-optimal parallel and sequential cholesky decomposition. *SIAM Journal on Scientific Computing*, 32(6):3495–3523, 2010.

[6] Jarle Berntsen. Communication efficient matrix multiplication on hypercubes. *Parallel computing*, 12(3):335–342, 1989.

[7] G. Bilardi and F. P. Preparata. Horizons of parallel computation. *Journal of Parallel and Distributed Computing*, 27(2):172–182, 1995.

[8] Gianfranco Bilardi and Lorenzo De Stefani. The I/O complexity of Toom-Cook integer multiplication. In *Proc. ACM-SIAM SODA*, pages 2034–2052, 2019.

[9] L. E. Cannon. A cellular computer to implement the Kalman filter algorithm. Technical report, DTIC Document, 1969.

[10] Giovanni Cesari and Roman Maeder. Performance analysis of the parallel karatsuba multiplication algorithm for distributed memory architectures. *Journal of Symbolic Computation*, 21(4-6):467–473, 1996.

[11] Bruce Char, Jeremy Johnson, David Saunders, and Andrew P Wack. Some experiments with parallel bignum arithmetic. *Hong [Hon94]*, pages 94–103, 1994.

[12] Jaeyoung Choi, David W Walker, and Jack J Dongarra. Pumma: Parallel universal matrix multiplication algorithms on distributed memory concurrent computers. *Concurrency: Practice and Experience*, 6(7):543–570, 1994.

[13] Eleanor Chu and Alan George. Fft algorithms and their adaptation to parallel processing. *Linear algebra and its applications*, 284(1-3):95–124, 1998.

[14] Stephen A Cook and Stål O Aanderaa. On the minimum computation time of functions. *Transactions of the American Mathematical Society*, 142:291–314, 1969.

[15] Takuya Edamatsu and Daisuke Takahashi. Acceleration of large integer multiplication with intel avx-512 instructions. In *2018 IEEE 20th International Conference on High Performance Computing and Communications; IEEE 16th International Conference on Smart City; IEEE 4th International Conference on Data Science and Systems (HPCC/SmartCity/DSS)*, pages 211–218. IEEE, 2018.

[16] Martin Fürer. Faster integer multiplication. *SIAM Journal on Computing*, 39(3):979–1005, 2009.

[17] L Garcia. Can Schönhage multiplication speed up the RSA encryption or decryption? *University of Technology, Darmstadt*, 2005.

[18] Pascal Giorgi, Laurent Imbert, and Thomas Izard. Parallel modular multiplication on multi-core processors. In *2013 IEEE 21st Symposium on Computer Arithmetic*, pages 135–142. IEEE, 2013.

[19] David Harvey and Joris van der Hoeven. Faster integer multiplication using short lattice vectors. *The Open Book Series*, 2(1):293–310, 2019.
[20] Shahram Jahani, Azman Samsudin, and Kumbakonam Govindarajan Subramanian. Efficient big integer multiplication and squaring algorithms for cryptographic applications. *Journal of Applied Mathematics*, 2014, 2014.

[21] Tudor Jebelean. Using the parallel karatsuba algorithm for long integer multiplication and division. In *European Conference on Parallel Processing*, pages 1169–1172. Springer, 1997.

[22] Anatolii Karatsuba and Yuri Ofman. Multiplication of many-digital numbers by automatic computers. In *Doklady Akad. Nauk SSSR*, volume 145, page 85, 1962.

[23] Donald E Knuth. *The Art of Computer Programming, Volume 2: Seminumerical Algorithms, Part 1*. Addison Wesley, 1981.

[24] M. J. Kronenburg. Toom-cook multiplication: Some theoretical and practical aspects. *ArXiv*, abs/1602.02740, 2016.

[25] Wolfgang Kuechlin, David Lutz, and Nicholas Nevin. Integer multiplication in parsac-2 on stock microprocessors. In *International Symposium on Applied Algebra, Algebraic Algorithms, and Error-Correcting Codes*, pages 206–217. Springer, 1991.

[26] Joseph WH Liu. Computational models and task scheduling for parallel sparse cholesky factorization. *Parallel computing*, 3(4):327–342, 1986.

[27] Farnam Mansouri. On the parallelization of integer polynomial multiplication. 2014.

[28] Jose Maria Bermudo Mera, Angshuman Karmakar, and Ingrid Verbauwhede. Time-memory trade-off in toom-cook multiplication: an application to module-lattice based cryptography. *IACR Transactions on Cryptographic Hardware and Embedded Systems*, pages 222–244, 2020.

[29] Dianne P O’Leary and GW Stewart. Assignment and scheduling in parallel matrix factorization. *Linear Algebra Appl.*, 77:275–300, 1986.

[30] Meitong Pan. Hardware implementation of bit-parallel finite field multipliers based on overlap-free algorithm on fpga. 2019.

[31] C. A. Patterson, M. Snir, and S. L. Graham. *Getting Up to Speed:: The Future of Supercomputing*. National Academies Press, 2005.

[32] R Portugal, CMH Figueiredo, et al. Reversible karatsubas algorithm. *Journal of Universal Computer Science*, 12(5):499–511, 2006.

[33] Arnold Schönhage and Volker Strassen. Schnelle multiplikation grosser zahlen. *Computing*, 7(3-4):281–292, 1971.

[34] Lorenzo De Stefani. On the i/o complexity of hybrid algorithms for integer multiplication, 2019.

[35] Volker Strassen. Gaussian elimination is not optimal. *Numerische Mathematik*, 13(4):354–356, 1969.
[36] Jitendra V Tembturne. Parallel multiplication of big integer on gpu. In International Conference on Next Generation Computing Technologies, pages 276–285. Springer, 2017.

[37] Jitendra V. Tembturne and Shailesh R. Sathe. Performance evaluation of long integer multiplication using openmp and mpi on shared memory architecture. 2014 Seventh International Conference on Contemporary Computing (IC3), pages 283–288, 2014.

[38] Andrei L Toom. The complexity of a scheme of functional elements realizing the multiplication of integers. In Soviet Mathematics Doklady, volume 3, pages 714–716, 1963.

[39] C. Yang and B. P. Miller. Critical path analysis for the execution of parallel and distributed programs. In Distributed Computing Systems, 1988., 8th International Conference on, pages 366–373. IEEE, 1988.