THE CAUCHY PROBLEM FOR LIE-MINIMAL SURFACES

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March 29, 2022

Abstract

In the present paper we study the Lie sphere geometry of Legendre surfaces by the method of moving frame and we prove an existence theorem for real-analytic Lie-minimal Legendre surfaces.

Introduction

In his analysis \( \mathbb{R} \) of Lie sphere geometry of surfaces W. Blaschke proposed to study the variational problem for the functional

\[
B : M \subset \mathbb{R}^3 \rightarrow \int_M \frac{\partial_1 k_1 \partial_2 k_2}{(k_1 - k_2)^2} du^1 \wedge du^2,
\]

on immersed surfaces \( M \subset \mathbb{R}^3 \) with no umbilical points, where \( k_1 \) and \( k_2 \) are the principal curvatures and where \( (u^1, u^2) \) are curvature line coordinates. He also showed that the functional is invariant under Lie sphere transformations. Recently, E. Ferapontov [9] reconsidered this classical variational problem and showed that the critical points of (1) do admit a spectral deformation. This work was taken up by F. Burstall and U. Hertrich-Jeromin [4], who introduced a Lie-invariant Gauss map \( \mathcal{D} : M \rightarrow \mathcal{D} \) with values in the Dupin manifold \( \mathcal{D} \), that is the symmetric space consisting of all 3-dimensional subspaces of signature (2, 1) in \( \mathbb{R}^{(4,2)} \). They showed that \( M \subset \mathbb{R}^3 \) is a critical point of the functional (1) if and only if its Lie-invariant Gauss map is harmonic. This explains the origin of the spectral deformation discovered by Ferapontov and suggests the existence of a dressing action on the space of the critical points of the Blaschke functional (see [4]). In this paper, we study the Cauchy problem for Lie-minimal surfaces using the invariance by Lie sphere transformations from the outset. From this point of view, the relevant objects of study are the Legendre lifts in the space of contact elements rather than the surfaces itself.

\( ^1 \)This research was partially supported by the MIUR project Proprietà Geometriche delle Varietà Reali e Compatte, by the group GNSAGA of the INdAM, and by the European Contract Human Potential Programme, Research Training Network HPRN-CT-2000-00101 (EDGE).

2000 Mathematics Subject Classification. Primary 53A40, 58A15; Secondary 53D10, 58A17.

Key words and phrases. Lie sphere geometry, Legendre surfaces, Lie minimal surfaces.
In §1 we consider the Kepler manifold $^1 K$, that is the isotropic Grassmannian of the null-planes through $0 \in \mathbb{R}^{(4,2)}$, acted upon transitively by the Lie sphere group $\tilde{G} = SO(4, 2)/\pm I$ of contact transformations. We also consider the Lie quadric $Q$ (i.e. the projectivization of the light cone $\mathcal{L}$ of $\mathbb{R}^{(4,2)}$) and the Dupin manifold $\mathcal{D}$. We apply moving frame to study Legendre surfaces in the Kepler manifold. Any Legendre surface $M$ can be parameterized by a pair of mappings $\phi_0, \phi_1 : M \to Q$ satisfying $\langle \phi_0, \phi_1 \rangle = 0$ and $\langle d\phi_0, \phi_1 \rangle = 0$. Most of the Lie-invariant properties of $M$ are determined by the sheaf $\mathcal{S}$ of quadratic forms spanned by $\langle d\phi_0, d\phi_0 \rangle$ and by $\langle d\phi_1, d\phi_1 \rangle$. If the stalks $\mathcal{S}_m$ are 2-dimensional, for every $m \in M$ then, the tautological bundle $\mathcal{U}(M) = \{(\ell, V) \in M \times \mathbb{R}^{(4,2)} : V \in \ell \}$ → $M$ has a natural splitting into the direct sum of two line sub-bundles $\Sigma_0(M)$ and $\Sigma_1(M)$. The maps $\sigma_0, \sigma_1 : M \to Q$ induced by $\Sigma_0(M)$ and by $\Sigma_1(M)$ are the two curvature sphere mappings of the surface. If $\sigma_0$ and $\sigma_1$ are everywhere of maximal rank, then $M$ is said to be non-degenerate. Geometrically, this condition means that the Euclidean projection of $M$ can not be obtained as the envelope of a 1-parameter family of spheres. In §2 we indicate how to construct on any non-degenerate Legendre surface $M \subset K$ a canonical lift $\mathcal{A} : M \to \tilde{G}$ to the Lie sphere group (the normal frame field along $M$). By the means of the normal frame field we recover the Blaschke co-frame $(\alpha_1, \alpha_2)$ of $M$ and we introduce a complete set of local differential invariants $(q_1, q_2, p_1, p_2, r_1, r_2)$, the invariant functions of the surface. From the structural equations of the group $\tilde{G}$ we deduce the compatibility conditions fulfilled by the normal co-frame and by the invariant functions. In §3 we analyze the Lie-invariant Gauss map and we write the Euler-Lagrange equations of the variational problem in terms of the invariant functions. Subsequently we set up a Pfaffian differential system $(I, \Omega)$ on $P = G \times \mathbb{R}^6$ with the defining property that its integral manifolds are the canonical lifts of Lie-minimal surfaces. In §4 we prove that the differential system $(I, \Omega)$ is in involution and that its general integral manifolds depend on six functions in one variable. In the last part of the paper we prove our main result:

**Theorem.** Let $\Gamma \subset K$ be a real-analytic Legendre curve and let $\mathcal{U}(\Gamma) \to \Gamma$ be the corresponding tautological bundle. Let $L \subset \mathcal{U}(\Gamma)$ be a real-analytic line sub-bundle of $\mathcal{U}(\Gamma)$ and let $h, w : \Gamma \to \mathbb{R}$ be two real-analytic functions. If $\Gamma$ and $L$ are suitably general, then there exist a real-analytic Lie-minimal surface $M \subset K$ containing $\Gamma$ such that

$$\Gamma^* (\alpha_1 + \alpha_2) = 0, \quad L = \Sigma_0(M)|_{\Gamma}, \quad h = -3(q_1 + q_2)|_{\Gamma}, \quad w = \frac{1}{3}(p_1 - p_2)|_{\Gamma}.$$

Moreover, $M$ is unique in the sense that any other Legendre surface with these properties agrees with $M$ on an open neighborhood of $\Gamma$.

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$^1$We adopt the terminology introduced by J.M Souriau in [10] and by Guillemin and Sternberg in [11]
1 Legendre Surfaces

1.1 Lie sphere geometry.

Let us begin with some basic facts. Consider the vector space \( \mathbb{R}^{(4,2)} \) with the inner product of signature \((4,2)\) defined by

\[
\langle V, V \rangle = -2v^0 v^5 - 2v^1 v^4 + (v^2)^2 + (v^3)^2 = g_{IJ} v^I v^J
\]

where \( v^0, ..., v^5 \) are the components of \( V \) with respect to the standard basis \(( \epsilon_0, ..., \epsilon_5 )\).

We let \( G \) be the identity component of the pseudo-orthogonal group of \( \mathbb{R}^{(4,2)} \) and we let \( \mathfrak{g} \) be its Lie algebra. For each \( A \in G \) we denote by \( A_J = A \cdot \epsilon_J \) the \( J \)-th column vector of \( A \). Thus, \((A_0, ..., A_5)\) is a positive-oriented basis of \( \mathbb{R}^{(4,2)} \) such that

\[
\langle A_I, A_J \rangle = g_{IJ}, \quad I, J = 0, ..., 5.
\]

Expressing the exterior derivative \( dA_J \) in terms of the basis \((A_0, ..., A_5)\) we obtain

\[
dA_J = \omega^I_J A_I, \quad J = 0, ..., 5,
\]

where \( \omega = (\omega^I_J) \) is the \( \mathfrak{g} \)-valued Maurer-Cartan form \( A^{-1} dA \) on the group \( G \). Taking the exterior derivative in (3) yields the structure equations

\[
d\omega = -\omega \wedge \omega.
\]

The **Lie quadric** is the space \( Q \subset \mathbb{R}P^5 \) of the isotropic lines through \( 0 \in \mathbb{R}^{(4,2)} \) and the **Lie sphere group** is defined to be the group \( \tilde{G} = G/\{ \pm I \} \) of all projective transformations of \( \mathbb{R}P^5 \) which send \( Q \) into itself. Elements of \( \tilde{G} \) are equivalence classes of matrices \( A \in G \). Given any such matrix \( A \), its equivalence class in \( G \) is denoted by \([A]\). Thus, \([A] = [B]\) iff \( A = \pm B \). Since the Maurer-Cartan form \( \omega = A^{-1} dA \) is bi-invariant under the action of \( \{ \pm I \} \), then we can identify the Lie algebra of \( \tilde{G} \) with \( \mathfrak{g} \) and we may think of \( \omega \) as being the Maurer-Cartan form of the Lie sphere group \( \tilde{G} \).

**Remark 1.1** The role of the Lie quadric is to represent the oriented spheres of \( \mathbb{R}^3 \) (including point spheres, oriented planes and the "point at infinity"). Given a point \( p \in \mathbb{R}^3 \) and a real number \( r \) we let \( \sigma(p, r) \) be the oriented sphere with center \( p \) and signed radius \( r \). Similarly, for every \( p \in \mathbb{R}^3 \) and every \( \nu \in S^2 \), we let \( \pi(p, \nu) \) be the oriented plane passing through \( p \) and orthogonal to the unit vector \( \nu \). Then, the correspondence between the points of \( Q \) and oriented spheres is given by

\[
\begin{align*}
\sigma(p, r) &\to \left[ \left( 1, \frac{r+p^1}{\sqrt{2}}, \frac{r+p^2}{\sqrt{2}}, \frac{r+p^3}{\sqrt{2}}, \frac{r+p^4}{\sqrt{2}}, \frac{r+p^5}{\sqrt{2}} \right) \right], \\
\pi(p, \nu) &\to \left[ \left( 0, \frac{1+n^1}{\sqrt{2}}, \frac{n^2}{\sqrt{2}}, \frac{n^3}{\sqrt{2}}, \frac{1-n^1}{\sqrt{2}}, \frac{n^5}{\sqrt{2}} \right) \right], \\
\infty &\to [\epsilon_5]
\end{align*}
\]

**Definition 1.2** The **Kepler manifold** \( \mathcal{K} \) is defined to be the isotropic Grassmannian consisting of all null planes through the origin of \( \mathbb{R}^{(4,2)} \).
Remark 1.3 Two oriented spheres corresponding to \([V], [V'] \in \mathcal{Q}\) are in oriented contact if and only if \(\langle V, V' \rangle = 0\). Geometrically, this means that a null plane \(\ell \in \mathcal{K}\) represents a pencil of oriented spheres in oriented contact. Thus, we may think of \(\mathcal{K}\) as the *space of the parabolic pencils of oriented spheres*. Another classical model of the Kepler manifold is \(\mathbb{R}^3 \times S^2\) with a 2-dimensional sphere \(S^2_\infty\) at the infinity. The sphere \(S^2_\infty\) is the set of the null planes through the isotropic line \([\varepsilon_5]\). The complement \(\mathcal{K}_0\) of \(S^2_\infty\) is identified with \(\mathbb{R}^3 \times S^2\) by

\[
F : (p, \overline{\omega}) \in \mathbb{R}^3 \times S^2 \to [F_0(p) \wedge F_1(p, \overline{\omega})] \in \mathcal{K},
\]

where

\[
\begin{align*}
F_0(p) &= \epsilon_0 + \frac{p^1}{\sqrt{2}}\epsilon_1 + p^2\epsilon_2 + p^3\epsilon_3 - \frac{p^1}{\sqrt{2}}\epsilon_4 + \frac{p^2p^3}{2}\epsilon_5, \\
F_1(p, \overline{\omega}) &= \frac{1+\nu^2}{2}\epsilon_1 + \frac{n^2}{\sqrt{2}}\epsilon_2 + \frac{n^3}{\sqrt{2}}\epsilon_3 + \frac{1-n^2}{2}\epsilon_4 + \frac{v^p}{2}\epsilon_5.
\end{align*}
\]

Let \(G\) act on \(\mathcal{K}\) in the usual way: given a null plane \([V \wedge V']\) spanned by a pair \(V, V'\) of isotropic vectors, and given \(A \in G\), then \(A \cdot [V \wedge V'] = [AV \wedge AV']\). The projection map

\[
\pi_\mathcal{K} : [A] \in G \to [A_0 \wedge A_1] \in \mathcal{K}
\]

makes \(G\) into a \(G_0\)-principal fibre bundle over \(\mathcal{K}\), where

\[
G_0 = \{ [A] \in G : A_0^I = A_1^I = 0, \quad I = 2,\ldots, 5 \}.
\]

Remark 1.4 For later use we observe that the elements of \(G_0\) can be written as

\[
X(D, B, Y, b) = \begin{pmatrix}
D & DY^*JB & Z(D, Y, b) \\
0 & B & Y \\
0 & 0 & (D^*)^{-1}
\end{pmatrix},
\]

where

\[
D \in GL_+(2, \mathbb{R}), \quad B \in SO(2), \quad Y \in \mathfrak{gl}(2, \mathbb{R}), \quad b \in \mathbb{R}
\]

and where

\[
Z(D, Y, b) = \frac{1}{2}DJ \begin{pmatrix}
Y^tY + \begin{pmatrix} 0 & -b \\ b & 0 \end{pmatrix} \\
0 & 1
\end{pmatrix}, \quad J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad X^* = J^tXJ.
\]

The forms \(\{\omega_0^2, \omega_0^3, \omega_1^2, \omega_1^3, \omega_1^4\}\) are linearly independent and span the semi-basic forms for the projection \(\pi_\mathcal{K}\). In particular, \(\omega_0^4\) is well-defined up to a positive multiple on \(\mathcal{K}\) and, from the structure equations, we get

\[
d\omega_0^4 = \omega_0^2 \wedge \omega_1^2 + \omega_0^3 \wedge \omega_1^3 + (\omega_0^0 + \omega_1^1) \wedge \omega_0^4.
\]

From this we infer that \(\omega_0^4\) defines a \(G\)-invariant contact structure on \(\mathcal{K}\).

We let \(\mathcal{D}\) be the *Dupin manifold*, that is the manifold of all 3-dimensional linear subspaces of \(\mathbb{R}^{(4,2)}\) of signature \((2,1)\). Then, \(G\) act transitively on \(\mathcal{D}\) by

\[
A : [V \wedge V' \wedge V''] = [AV \wedge AV' \wedge AV''],
\]
for every $A \in G$ and every $[V \wedge V' \wedge V''] \in \mathcal{D}$. The projection map
\[ \pi_D : A \in G \to [A_0 \wedge A_3 \wedge A_5] \in \mathcal{D} \]
gives on $G$ the structure of a principal fibre bundle with structure group
\[ S((O(2, 1) \times O(2, 1)) \cong \{ A \in G : A \cdot [\epsilon_0 \wedge \epsilon_3 \wedge \epsilon_5] = [\epsilon_0 \wedge \epsilon_3 \wedge \epsilon_5]\}. \]
Thus, $\mathcal{D}$ can be viewed as the pseudo-Riemannian symmetric space
\[ SO(4, 2)/SO(2, 1) \times SO(2, 1). \]
The canonical pseudo-Riemannian metric $g_D$ on $\mathcal{D}$ induced by the fibering $\pi_D$ is represented by the tensorial quadratic form on $G$ given by
\[ \omega_0^1 \omega_0^4 + \omega_0^2 \omega_0^2 + 2\omega_0^1 \omega_0^3 - \frac{1}{2}(\omega_2^3)^2. \]

### 1.2 Legendre Surfaces

**Definition 1.5** An oriented, connected immersed surface $M \subset \mathcal{K}$ is said to be *Legendrian* if it is tangent to the contact distribution on $\mathcal{K}$.

Locally, there exist two smooth mappings $\phi_0, \phi_1 : U \subset M \to \mathbb{R}^{(4,2)}$ such that $\ell = [\phi_0(\ell) \wedge \phi_1(\ell)]$, for every $\ell \in U$, and that
\[ \|\phi_0\|^2 = \|\phi_1\|^2 = \langle \phi_0, \phi_1 \rangle = 0, \quad \langle d\phi_0, d\phi_1 \rangle = 0. \]
Then, $\langle d\phi_0, d\phi_0 \rangle$ and $\langle d\phi_1, d\phi_1 \rangle$ span a sheaf $\mathcal{S}$ of quadratic forms on $M$. Throughout the paper we shall assume that the fiber $S_m$ of $\mathcal{S}$ is two-dimensional, for every point $m \in M$. We consider
\[ \mathcal{F}_0(M) = \{ (\ell, A) \in M \times G \mid \ell = [A_0 \wedge A_1]\} \]
the pull back of the fiber bundle $\pi_{\mathcal{K}} : G \to \mathcal{K}$ to the surface $M$. The local cross sections of $\mathcal{F}_0(M)$ are called *local frame fields* along $M$. They can be considered as smooth maps $A : U \to G$, where $U$ is an open subset of $M$, such that $\ell = [A_0(\ell) \wedge A_1(\ell)]$, for every $\ell \in U$. For every local frame field $A : U \to G$ we let $\alpha = (\alpha^j_A)$ be the pull-back of the Maurer-Cartan form. Any other local frame field $\tilde{A}$ on $U$ is given by $\tilde{A} = A \cdot X$, where $X = U \to G_0$ is a smooth map. Thus, the 1-forms $\alpha$ and $\tilde{\alpha}$ are related by the gauge transformation
\[ \tilde{\alpha} = X^{-1}dX + X^{-1}X. \]
A frame field $A : U \to G$ is of *first order* if, with respect to it
\[ \alpha^3_0 \wedge \alpha^2_0 > 0, \quad \alpha^2_0 = \alpha^3_1 = 0. \]
From [10] it follows that first order frames exist on a neighborhood of any point of $M$. The totality of first order frames is a principal $G_1$-bundle $\mathcal{F}_1(M) \to M$ where
\[ G_1 = \left\{ X(D, B, Y, b) \in G_1 : B = \epsilon 1d_{2 \times 2}, \quad D = \begin{pmatrix} r & 0 \\ 0 & s \end{pmatrix}, \quad \epsilon = \pm 1, rs > 0 \right\}. \]
Notation. The elements of \( G_1 \) will be denoted by \( Y_\epsilon(r,s,Y,b) \), where \( \epsilon = \pm 1 \), \( Y \in \mathfrak{gl}(2, \mathbb{R}) \), \( b, r, s \in \mathbb{R} \) and \( rs > 0 \).

We let \( \sigma_0, \sigma_1 : M \to \mathcal{Q} \) be defined by \( \sigma_0|_U = [A_0] \) and by \( \sigma_1|_U = [A_1] \), for every first order frame \( A : U \to G \). We follow the classical terminology and we call \( \sigma_0 \) and \( \sigma_1 \) the curvature sphere maps of the surface. We remark that \( \sigma_0 \) and \( \sigma_1 \) define a splitting of the tautological vector bundle
\[
\mathcal{U}(M) = \{ (\ell, V) \in M \times \mathbb{R}^{(4,2)} : \ell \in M, V \in \ell \}
\]
into the direct sum \( \Sigma_0(M) \oplus \Sigma_1(M) \) of the line sub-bundles
\[
\Sigma_a(M) = \{ (\ell, V) \in M \times \mathbb{R}^{(4,2)} : \ell \in M, V \in \sigma_a(\ell) \}, \quad a = 0, 1.
\]

**Definition 1.6** We say that \( M \) is non-degenerate if \( \sigma_0 \) and \( \sigma_1 \) are immersions of \( M \) into the Lie quadric.

**Theorem 1.7** Let \( M \subset \mathcal{K} \) be a non-degenerate Legendre surface. Then there exist a unique lift \( \tilde{A} : M \to \tilde{G} \) to the group \( \tilde{G} \) satisfying the Pfaffian equations
\[
\alpha_0^4 = \alpha_0^2 = \alpha_1^2 = \alpha_2^2 = \alpha_0^6 = \alpha_1^6 = \alpha_2^6 = 0,
\]
and the independence condition
\[
\alpha_0^3 \wedge \alpha_1^2 > 0.
\]

**Proof.** If \( A : U \to G \) is a first order frame then, the linear differential forms \( \alpha^1 = \alpha_0^3 \) and \( \alpha^2 = \alpha_1^2 \) give a positive-oriented co-framing on \( U \) so that we may write
\[
\alpha = P_1 \alpha^1 + P_2 \alpha^2,
\]
where \( P_1, P_2 : U \to \mathfrak{g} \) are smooth maps. The components of \( P_a \) are denoted by \( P_a^{I_J} \), where \( a = 1, 2 \) and where \( I, J = 0, \ldots, 5 \). If \( A, \tilde{A} : U \to G \) are first order frames on \( U \) and if the corresponding transition function is of the form \( Y_\epsilon(r,s,Y,b) : U \to G_1 \), then
\[
\tilde{\alpha}^1 = \epsilon \alpha^1, \quad \tilde{\alpha}^2 = \epsilon \alpha^2, \quad \tilde{\alpha}^0_1 = r \left( s^{-1} \alpha^1_0 - Y^2 \alpha^1_1 \right), \quad \tilde{\alpha}^0_2 = s \left( r^{-1} \alpha^0_1 - Y^1 \alpha^2 \right).
\]
This implies
\[
\begin{align*}
\tilde{P}^{01}_{01} &= \epsilon (s^{-1} P^{01}_{01} - Y^2), & \tilde{P}^{02}_{02} &= \epsilon rs^{-2} P^{01}_{02}, \\
\tilde{P}^{02}_{12} &= -\epsilon (r^{-1} P^{02}_{12} + Y^1), & \tilde{P}^{00}_{11} &= \epsilon rs^{-2} P^{00}_{11}.
\end{align*}
\]
From (15) we see that for every point \( \ell \in M \) there exist a first order frame field \( \tilde{A} : \tilde{U} \to \tilde{G} \) defined on an open neighborhood \( \tilde{U} \) of \( \ell \) with respect to which \( P^{01}_{01} = P^{02}_{12} = 0 \). Such first order frame fields are said to be of second order. In addition, any other second order frame field on \( U \) is of the form \( \tilde{A} = A \cdot X \), where \( X : U \to G_2 \) is a smooth map and
\[
G_2 = \{ Y_\epsilon(r,s,Y,b) \in G_1 : X = \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix}, \quad p, q \in \mathbb{R} \}.
\]
Notation. The elements of $G_2$ will be denoted by $Y_\epsilon(r, s, p, q, b)$, where $\epsilon = \pm 1$, $p, q, r, s, b \in \mathbb{R}$ and $rs > 0$.

Second order frame fields are the cross sections of a reduced sub-bundle $\mathcal{F}_2(M)$ of $\mathcal{F}_1(M)$ with structural group $G_2$. Differentiating $\alpha_0^2 = \alpha_1^2 = 0$ and applying the structure equations and Cartan’s Lemma, we have that $\alpha_3^2 = 0$, for every second order frame field $A$. Taking the exterior derivative of $\alpha_3^2 = 0$ and using again the structure equations and the Cartan’s lemma we have $\alpha_2^0 \wedge \alpha_1^1 \wedge \alpha_3^2 = 0$. This implies $P_{22}^0 = -P_{31}^1$ and hence we may write

$$
\alpha_2^0 = P_{21}^0 \alpha_1^1 + P_{22}^0 \alpha_2^2, \quad \alpha_3^1 = -P_{22}^0 \alpha_1^1 + P_{32}^1 \alpha_2^2. \tag{16}
$$

If $A$ and $\tilde{A}$ are second order frame fields on $U \subset M$ and if $Y_\epsilon(r, s, p, q, b) : U \to G_2$ is the corresponding transition function, we then have

$$
\tilde{\alpha}_2^0 = er^{-1}spa_0^0 + er^{-1}a_2^0 - sb\alpha_2^1, \quad \tilde{\alpha}_3^1 = es^{-1}rq\alpha_0^1 + es^{-1}\alpha_3^1 + rb\alpha_1^1. \tag{17}
$$

From this we obtain

$$
P_{21}^0 = r^{-2} \left( P_{21}^0 + spP_{11}^0 \right), \quad P_{22}^0 = \frac{1}{rs} P_{22}^0 - \epsilon b, \quad P_{32}^1 = s^{-2} \left( P_{32}^1 + rqP_{02}^1 \right). \tag{18}
$$

Thus, for every point $\ell \in M$ there exist a second order frame field $U \to G$ defined on an open neighborhood of $\ell$ with respect to which

$$
P_{22}^0 = P_{31}^1 = 0. \tag{19}
$$

Such frame fields are said to be of third order. Now, $P_{21}^0 = 0$ implies that these frame fields are the local cross sections of a reduced sub-bundle $\mathcal{F}_3(M)$ of $\mathcal{F}_2(M)$. The structure group of $\mathcal{F}_3(\phi)$ is

$$
G_3 = \{ Y_\epsilon(r, s, p, q, b) \in G_2 : b = 0 \}. \tag{20}
$$

Since $M$ is non-degenerate, then the functions $P_{02}^1$ and $P_{11}^0$ are nowhere vanishing. Thus, from (16) we infer that for every $\ell \in M$ there exist a third order frame field $A : U \to G$ defined on an open neighborhood $U$ of $\ell$ such that

$$
\alpha_1^0 = \alpha_1^1, \quad \alpha_0^1 = \alpha_2^2. \tag{20}
$$

A third order frame field satisfying (20) is said to be of fourth order. If $A$ is a fourth order frame field on $U$, then any other is given by $\tilde{A} = AX$, where $X : U \to G_4$ and

$$
G_4 = \{ Y_\epsilon(r, s, p, q, 0) \in G_3 : r = s = \epsilon \}. \tag{20}
$$

The elements of $G_4$ are denoted by $Y_\epsilon(p, q)$, where $p, q \in \mathbb{R}$. From this we immediately see that the third order frame fields define a $G_4$ sub-bundle $\mathcal{F}_4(M)$ of $\mathcal{F}_3(M)$. Now, using (16) we see that for every $\ell \in M$ there exist a fourth order frame $A : U \to G$ in an open neighborhood $U$ of $\ell$ satisfying

$$
\alpha_2^0 = \alpha_3^1 = 0. \tag{20}
$$

Such frame fields are of fifth order. Notice that fifth order frame satisfy (12) and (13). Moreover, if $A$ and $\tilde{A}$ are fifth order frame fields on an open neighborhood $U$, then $\tilde{A} = \epsilon A$, where $\epsilon = \pm 1$. This implies that the canonical lift $\tilde{A} : M \to \tilde{G}$ is defined by $\tilde{A}|_U = [A]$, for every fifth order frame field $A : U \to G$. \hfill \nabla
Definition 1.8 The map $A : M \to \tilde{G}$ is said to be the normal frame field along $M$.

We then consider the linearly independent 1-forms $\alpha^1 = \alpha_0^1$ and $\alpha^2 = \alpha_0^2$ and we call $(\alpha^1, \alpha^2)$ the canonical co-frame of the Legendre surface.

Remark 1.9 If $M$ is the contact lift of $f : M \to \mathbb{R}^3$ and if $(u^1, u^2)$ are curvature lines coordinates then $\alpha^1$ and $\alpha^2$ coincide with the Blaschke’s differentials

\[
\left\{ \begin{array}{l}
\alpha^1 = \frac{1}{k_1 - k_2} \left( \sqrt{\frac{g_{11}}{g_{22}}} (\partial_1 k_1)^2 \partial_2 k_2 \right) \frac{1}{2} du^1, \\
\alpha^2 = \frac{-1}{k_1 - k_2} \left( \sqrt{\frac{g_{11}}{g_{22}}} (\partial_1 k_1)^2 \partial_2 k_2 \right)^{1/2} du^2.
\end{array} \right.
\] (21)

Taking the exterior derivatives of (21) and using the Maurer-Cartan equations it follows that there exist smooth functions $q_1, q_2, p_1, p_2$ and $r_1, r_2$ such that

\[
\left\{ \begin{array}{l}
\alpha_0^0 = -2q_1 \alpha^1 + q_2 \alpha^2, \\
\alpha_0^1 = -q_1 \alpha^1 + 2q_2 \alpha^2, \\
\alpha_3^0 = r_1 \alpha^1 + p_2 \alpha^2, \\
\alpha_3^1 = p_1 \alpha^1 + r_2 \alpha^2, \\
\alpha_4^0 = -r_2 \alpha^1 + r_1 \alpha^2.
\end{array} \right.
\] (22)

We call $q_1, q_2, p_1, p_2$ and $r_1, r_2$ the invariant functions of the Legendre surface. Using once more the Maurer-Cartan equations we obtain

\[
\begin{align*}
\text{d} \alpha^1 &= \alpha_0^0 \wedge \alpha^1, \\
\text{d} \alpha^2 &= \alpha_1^0 \wedge \alpha^2, \\
\text{d} \alpha_2^1 &= -\alpha_1^1 \wedge \alpha_2^1, \\
\text{d} \alpha_3^0 &= -\alpha_0^0 \wedge \alpha_3^0, \\
\text{d} \alpha_4^0 &= (\alpha^2 - \alpha_3^0) \wedge \alpha^1, \\
\text{d} \alpha_3^1 &= (\alpha^1 \wedge \alpha_2^1) \wedge \alpha^2, \\
\text{d} \alpha_4^1 &= -\alpha_0^0 \wedge \alpha_1^1 \wedge \alpha_4^0.
\end{align*}
\] (23)

We may rewrite these equations in terms of the invariant functions

\[
\begin{align*}
\text{d} \alpha^1 &= -q_2 \alpha^1 \wedge \alpha^2, \\
\text{d} \alpha^2 &= -q_1 \alpha^1 \wedge \alpha^2.
\end{align*}
\] (25)

\[
\begin{align*}
-2dq_1 \wedge \alpha^1 + dq_2 \wedge \alpha^2 &= (p_2 - q_1 q_2 - 1) \alpha^1 \wedge \alpha^2, \\
dq_1 \wedge \alpha^1 + 2dq_2 \wedge \alpha^2 &= (-p_1 q_1 - q_1 q_2 + 1) \alpha^1 \wedge \alpha^2.
\end{align*}
\] (26)

\[
\begin{align*}
-dr_1 \wedge \alpha^1 + dp_2 \wedge \alpha^2 &= (2q_2 r_1 + 3q_1 p_2) \alpha^1 \wedge \alpha^2, \\
dp_1 \wedge \alpha^1 + dr_2 \wedge \alpha^2 &= (2q_1 r_2 + 3q_2 p_1) \alpha^1 \wedge \alpha^2, \\
-dr_2 \wedge \alpha^1 + dr_1 \wedge \alpha^2 &= 4(q_1 r_1 - q_2 r_2) \alpha^1 \wedge \alpha^2.
\end{align*}
\] (27)

2 Lie minimal surfaces

2.1 The Gauss map and the Euler-Lagrange equations

Definition 2.1 The Gauss map of a generic Legendre surface $M \subset \mathcal{K}$ is defined by

\[
\mathcal{D} : \ell \in M \to [A_0(\ell) \wedge A_3(\ell) \wedge A_5(\ell)], \quad \forall \ell \in M,
\]

where $A = [A]$ is the normal frame field along $M$. 
Remark 2.2 If we consider on $M$ the quadratic form $\Phi = \alpha^1 \alpha^2$, then $[9]$ and $[12]$ imply that the Gauss map $D: (M, \Phi) \to (\mathfrak{D}, g_D)$ is an isometric immersion.

Definition 2.3 A non-degenerate Legendre surface $M \subset \mathcal{K}$ is said to be Lie-minimal if it is a critical point of the functional

$$M \subset \mathcal{K} \to \int_M \alpha^1 \wedge \alpha^2$$

with respect to compactly supported variations.

Remark 2.4 It is known (cfr. $[4]$) that Lie-minimal surfaces are characterized by the harmonicity of the Gauss map.

Theorem 2.5 Let $M \subset \mathcal{K}$ be a non-degenerate Legendre surface. Then, $M$ is Lie-minimal if and only if

$$\begin{align*}
\{ & dr_1 \wedge \alpha^2 - 4q_1 r_1 \alpha^1 \wedge \alpha^2 = 0, \\
& dr_2 \wedge \alpha^1 - 4q_2 r_2 \alpha^1 \wedge \alpha^2 = 0.
\end{align*} \quad (28)$$

Proof. Without loss of generality we assume that $\mathcal{D}$ is an embedding and we identify $M$ with its image in $\mathfrak{D}$. We extend the normal frame field $\mathcal{A}$ to a local cross section $\bar{\mathcal{A}} : \mathcal{U} \to G$ of $\pi_D : G \to \mathfrak{D}$ defined on an open neighborhood $\mathcal{U} \subset \mathfrak{D}$ of $M$. If we set $\bar{\alpha} = \bar{\mathcal{A}}^{-1} d\bar{\mathcal{A}}$ then

$$\begin{align*}
\{ & \beta^1 = \bar{\alpha}^0_1, \quad \beta^2 = \bar{\alpha}^0_2, \quad \beta^3 = \bar{\alpha}^0_3, \quad \beta^4 = \bar{\alpha}^0_4, \quad \beta^5 = \bar{\alpha}^0_5 \\
& \beta^6 = \bar{\alpha}^3_0, \quad \beta^7 = \bar{\alpha}^3_1, \quad \beta^8 = \bar{\alpha}^3_2, \quad \beta^9 = \bar{\alpha}^3_3
\end{align*} \quad (29)$$

is a co-frame on $\mathcal{U}$. We let $B_1, ..., B_9$ be the local trivialization of $T(\mathfrak{D})$ dual to $(\beta^1, ..., \beta^9)$. Then, $[2]$ implies that

$$X_1 = B_2|_M - r_2 B_3|_M, \quad X_2 = B_1|_M + r_1 B_3|_M \quad (30)$$

is the trivialization\(^2\) of $T(M)$ dual to the canonical co-frame $(\alpha^1, \alpha^2)$ and that

$$\bar{B}_3 = B_3|_M \quad \bar{B}_4 = B_4|_M + r_2 B_1|_M - r_1 B_2|_M, \quad \bar{B}_5 = B_5|_M, ..., \bar{B}_9 = B_9|_M \quad (31)$$

is a trivialization of the normal bundle $\mathcal{N} \to M$. Taking the exterior derivative of $[2]$ and using the Maurer-Cartan equations we compute the covariant derivatives of the vector fields $B_1, ..., B_9$. We then have

$$\begin{align*}
\nabla B_1 &= (\bar{\alpha}^1 - \bar{\alpha}^1_0) B_1 + \bar{\alpha}^3_2 B_5 - \bar{\alpha}^3_3 B_7, \\
\nabla B_2 &= (\bar{\alpha}^3_0 - \bar{\alpha}^1_1) B_2 - \bar{\alpha}^3_3 B_6 + \bar{\alpha}^3_3 B_8, \\
\nabla B_3 &= (\bar{\alpha}^3_0 + \bar{\alpha}^1_1) B_3 - \bar{\alpha}^3_3 B_6 + \bar{\alpha}^3_3 B_7
\end{align*} \quad (32)$$

\(^2\)Here $T(M)$ is viewed as a sub-bundle of $D^* (T(\mathfrak{D}))$.
and

\[
\begin{align*}
\nabla B_1 &= -(\bar{\alpha}_0^1 + \bar{\alpha}_1^1)B_4 + \bar{\alpha}_1^1B_5 - \bar{\alpha}_3^0B_8, \\
\nabla B_5 &= \bar{\alpha}_1^2B_4 + \bar{\alpha}_1^2B_1 - \bar{\alpha}_1^0B_5 - \bar{\alpha}_0^0B_9, \\
\nabla B_6 &= -\bar{\alpha}_1^2B_2 - \bar{\alpha}_3^0B_3 + \bar{\alpha}_0^0B_6 + \bar{\alpha}_0^0B_9, \\
\nabla B_7 &= -\bar{\alpha}_0^3B_1 - \bar{\alpha}_3^0B_3 + \bar{\alpha}_1^1B_7 - \bar{\alpha}_2^2B_9, \\
\nabla B_8 &= \bar{\alpha}_0^3B_2 - \bar{\alpha}_0^3B_4 - \bar{\alpha}_1^1B_8 - \bar{\alpha}_1^1B_9, \\
\n\end{align*}
\]

(33)

Thus, from (30), (31), (32) and (33) we infer that the shape operator

\[ S \in \Gamma(M, \text{Hom}(TM, \Omega^1(M) \otimes \mathcal{N})) \]

of \( M \subset \mathcal{D} \) is given by

\[
\begin{align*}
\{ & S(X_1) = -(dr_2 + 2r_2(-q_1\alpha^1 + 2q_2\alpha^2)) \overline{\mathcal{B}_3} + \alpha^1 \left(-p_1\mathcal{B}_6 - r_2\mathcal{B}_7 + \mathcal{B}_8\right), \\
& S(X_2) = (dr_1 + 2r_1(-2q_1\alpha^1 + q_2\alpha^2)) \overline{\mathcal{B}_3} + \alpha^2 \left(\mathcal{B}_5 - r_1\mathcal{B}_6 - p_2\mathcal{B}_7\right). \quad (34)
\end{align*}
\]

In particular, we obtain the following formula for the mean curvature vector

\[ H = \frac{1}{2} (S(X_1)(X_2) + S(X_2)(X_1)) = (-dr_2(X_2) - 4r_2q_2 + dr_1(X_1) - 4p_1q_1) \overline{\mathcal{B}_3} \]

(35)

From this we deduce that \( \mathcal{D} : (M, \Phi) \to (\mathcal{D}, g_D) \) is harmonic if and only if

\[ -dr_2 \wedge \alpha^1 + 4q_2r_2\alpha^1 \wedge \alpha^2 + dr_1 \wedge \alpha^2 - 4r_1q_1\alpha^1 \wedge \alpha^2 = 0. \]

(36)

From (27) and (36) we get the required result. \( \nabla \)

2.2 The differential system of Lie-minimal surfaces

Let \( P \) be the configuration space \( G \times \mathbb{R}^6 \) and let denote by \((q_1, q_2, p_1, p_2, r_1, r_2)\) the coordinates on \( \mathbb{R}^6 \). On \( P \) we consider the Pfaffian ideal \( \mathcal{I} \subset \Omega^*(P) \) generated (as a differential ideal) by the linear differential forms

\[
\begin{align*}
\eta^1 &= \omega^1_0, \quad \eta^2 = \omega^2_0, \quad \eta^3 = \omega^3_1, \\
\eta^4 &= \omega^2_2, \quad \eta^5 = \omega^1_0 - \omega^2, \quad \eta^6 = \omega^1_0 - \omega^1, \\
\eta^7 &= \omega^2_3, \quad \eta^8 = \omega^3_3,
\end{align*}
\]

(37)

\[
\begin{align*}
\eta^9 &= \omega^0_0 + 2q_1\omega^1 - q_2\omega^2, \\
\eta^{10} &= \omega^1_1 + q_1\omega^1 - 2q_2\omega^2, \\
\eta^{11} &= \omega^0_2 - r_1\omega^1 - p_2\omega^2, \\
\eta^{12} &= \omega^2_2 - p_1\omega^1 - r_2\omega^2, \\
\eta^{13} &= \omega^2_3 + r_2\omega^1 - r_1\omega^2,
\end{align*}
\]

(38)

and by the exterior differential 2-forms

\[ \Theta^1 = dr_1 \wedge \alpha^2 - 4q_1r_1\alpha^1 \wedge \alpha^2, \quad \Theta^2 = dr_2 \wedge \alpha^1 - 4q_2r_2\alpha^1 \wedge \alpha^2. \]

(40)
together with the independence condition \( \Omega = \omega^1 \wedge \omega^2 \), where \( \omega^1 = \omega_0^3 \) and \( \omega^2 = \omega_1^2 \).

Taking the exterior derivatives of \( q^1, \ldots, q^8 \) and using the structural equations of \( G \) we obtain the quadratic equations

\[
d\eta^1 \equiv \ldots \equiv d\eta^8 \equiv d\eta^{13} \equiv 0, \quad (41)
\]

and

\[
\begin{align*}
d\eta^9 &\equiv 2\pi^1 \wedge \omega^1 - \pi^2 \wedge \omega^2 + (-1 + p_2 - q_1 q_2)\omega^1 \wedge \omega^2, \\
d\eta^{10} &\equiv \pi^1 \wedge \omega^1 - 2\pi^2 \wedge \omega^2 + (1 - p_1 + q_1 q_2)\omega^1 \wedge \omega^2, \\
d\eta^{11} &\equiv \zeta^1 \wedge \omega^1 + \nu^2 \wedge \omega^2 - (2r_1 q_2 + 3q_1 p_2)\omega^1 \wedge \omega^2, \\
d\eta^{12} &\equiv \nu^1 \wedge \omega^1 + \zeta^2 \wedge \omega^2 - (3p_1 q_2 + 2r_2 q_1)\omega^1 \wedge \omega^2,
\end{align*} \quad (42)
\]

where \( \equiv \) denotes equality up to the algebraic ideal generated by \( \eta^1, \ldots, \eta^{13}, \Theta^1, \Theta^2 \) and where

\[
\pi^i = dq_i, \quad \nu^i = dp_i, \quad \zeta^i = dr_i, \quad i = 1, 2.
\]

If we set

\[
\begin{align*}
\Omega^1 &\equiv 2\pi^1 \wedge \omega^1 - \pi^2 \wedge \omega^2 + (-1 + p_2 - q_1 q_2)\omega^1 \wedge \omega^2, \\
\Omega^2 &\equiv \pi^1 \wedge \omega^1 - 2\pi^2 \wedge \omega^2 + (1 - p_1 + q_1 q_2)\omega^1 \wedge \omega^2, \\
\Omega^3 &\equiv \nu^1 \wedge \omega^1 + \nu^2 \wedge \omega^2 - (2r_1 q_2 + 3q_1 p_2)\omega^1 \wedge \omega^2, \\
\Omega^4 &\equiv \nu^1 \wedge \omega^1 + \zeta^2 \wedge \omega^2 - (3p_1 q_2 + 2r_2 q_1)\omega^1 \wedge \omega^2,
\end{align*} \quad (43)
\]

then

\[
\{\eta^1, \ldots, \eta^{13}, \Theta^1, \Theta^2, \Omega^1, \Omega^2, \Omega^3, \Omega^4, d\Theta^1, d\Theta^2\} \quad (44)
\]

is a set of algebraic generators of the differential ideal \( \mathcal{I} \). The integral manifolds of this system are two-dimensional submanifolds \( \tilde{M} \subset P \) such that

\[
\eta^a = 0, \quad \Theta^1 = \Theta^2 = \Omega^1 = \Omega^2 = \Omega^3 = 0 = \Omega^4, \quad \Omega \neq 0.
\]

Thus, the map

\[
\phi : ([A], q_1, q_2, p_1, p_2, r_1, r_2) \in \tilde{M} \rightarrow [A_0 \wedge A_1] \in \mathcal{K} \quad (45)
\]

is a non-degenerate Legendre immersion. Since our arguments are local, we identify \( \tilde{M} \) with its image \( M = \phi(\tilde{M}) \subset \mathcal{K} \). Thus, \( M \) a Lie-minimal surface with normal frame field

\[
\mathcal{A} : ([A], q_1, q_2, p_1, p_2, r_1, r_2) \in M \rightarrow [A] \in \tilde{G}.
\]

Conversely, if \( M \) is a Lie-minimal surface with normal frame field \( \mathcal{A} : M \rightarrow G \) and with invariant functions \( q_1, \ldots, r_2 \), then the map

\[
\ell \in M \rightarrow \{\mathcal{A}(\ell), q_1(\ell), q_2(\ell), p_1(\ell), p_2(\ell), r_1(\ell), r_2(\ell)\} \in P, \quad \forall \ell \in M \quad (46)
\]

defines an integral manifold of the differential system \( (\mathcal{I}, \Omega) \). To summarize :

**Proposition 2.6** Lie-minimal surfaces \( M \subset \mathcal{K} \) may be regarded as being the integral submanifolds of the differential system \( (\mathcal{I}, \Omega) \) on \( P \).
3 The Cauchy problem

3.1 Involutivity of the differential system

On $P$ we consider the parallelization
\[
\left( \frac{\partial}{\partial \omega^i}, \frac{\partial}{\partial \eta^a}, \frac{\partial}{\partial \pi^i}, \frac{\partial}{\partial v^i}, \frac{\partial}{\partial \zeta^i} \right), \quad i = 1, 2, a = 1, \ldots, 13
\]
dual to the co-frame $(\omega^i, \eta^a, \pi^i, v^i, \zeta^i)$. We define
\[
\begin{align*}
V_1(\Omega) &= \{(3, E_1) \in G_1(T(P)) : ((\omega^1)^2 + (\omega^2)^2) |_{E_1} \neq 0\}, \\
V_2(\Omega) &= \{(3, E_2) \in G_2(T(P)) : \Omega |_{E_2} \neq 0\}
\end{align*}
\]
and we set
\[
\begin{align*}
V_1 &= S^1 \times \mathbb{R}^{16}, \\
V_2 &= \mathbb{R}(13, 2) \oplus \mathbb{R}(2, 2) \oplus \mathbb{R}(2, 2) \oplus \mathbb{R}(2, 2) \cong \mathbb{R}^{38},
\end{align*}
\]
with coordinates
\[
z = (\cos(\theta), \sin(\theta), x^a, y^i, u^i, v^i), \quad Z = (X^a_j, Y^i_j, U^i_j, V^i_j), \quad a = 1, \ldots, 13, i, j = 1, 2.
\]
Then, we identify $V_1(\Omega)$ with $P \times V_1$ and $V_2(\Omega)$ with $P \times V_2$ by the means of
\[
\begin{align*}
\begin{cases}
(3, z) \in P \times V_1 & \rightarrow (3, E_1(3, z)) \in V_1(\Omega), \\
(3, Z) \in P \times V_2 & \rightarrow (3, E_2(3, Z)) \in V_2(\Omega),
\end{cases}
\end{align*}
\]
where
\[
E_1(3, z) = \left[ \cos(\theta) \frac{\partial}{\partial \omega^1} |_{3} + \sin(\theta) \frac{\partial}{\partial \omega^2} |_{3} + x^a \frac{\partial}{\partial \eta^a} |_{3} + y^i \frac{\partial}{\partial \pi^i} |_{3} + u^i \frac{\partial}{\partial v^i} |_{3} + v^i \frac{\partial}{\partial \zeta^i} |_{3} \right]
\]
and where
\[
\begin{align*}
E_2(3, Z) &= \left[ T_1(3, Z) \wedge T_2(3, Z) \right], \\
T_1(3, Z) &= \frac{\partial}{\partial \omega^1} |_{3} + X_1^a \frac{\partial}{\partial \eta^a} |_{3} + Y_1^i \frac{\partial}{\partial \pi^i} |_{3} + U_1^i \frac{\partial}{\partial v^i} |_{3} + V_1^i \frac{\partial}{\partial \zeta^i} |_{3}, \\
T_2(3, Z) &= \frac{\partial}{\partial \pi^1} |_{3} + X_2^a \frac{\partial}{\partial \eta^a} |_{3} + Y_2^i \frac{\partial}{\partial \pi^i} |_{3} + U_2^i \frac{\partial}{\partial v^i} |_{3} + V_2^i \frac{\partial}{\partial \zeta^i} |_{3}.
\end{align*}
\]
Thus, the space $V_1(\zeta, \Omega)$ consisting of the 1-dimensional integral elements can be identified with the submanifold of $P \times V_1$ defined by the linear equations $x^1 = \ldots = x^{13} = 0$. Similarly, the space $V_2(\zeta, \Omega)$ consisting of all 2-dimensional integral elements is identified with the submanifold of $P \times V_2$ defined by
\[
\begin{align*}
X^a_i = 0, \quad a = 1, \ldots, 13, i = 1, 2, \\
2Y^2_1 + Y^2_1 + (1 - p_2 + q_1 q_2) = Y^2_1 + 2Y^2_1 - (1 - p_1 + q_1 q_2) = 0, \\
U^1_1 - V^2_2 - (2r_1 q_2 + 3q_1 p_2) = U^1_2 - V^2_2 + (3p_1 q_2 + 2r_1 q_2) = 0, \\
V^1_1 - 4q_1 r_1 = V^2_2 - 4q_2 r_2 = 0.
\end{align*}
\]
From this we infer that all the integral elements of the system are K-ordinary. Let $(3, E_1)$ be a 1-dimensional integral element such that
\[
E_1 = \left[ a^1 \frac{\partial}{\partial \omega^1} |_{3} + a^2 \frac{\partial}{\partial \omega^2} |_{3} + y^i \frac{\partial}{\partial \pi^i} |_{3} + u^i \frac{\partial}{\partial v^i} |_{3} + v^i \frac{\partial}{\partial \zeta^i} |_{3} \right].
\]
Contracting $\Theta^1, \Theta^2$ and $\Omega^1, ..., \Omega^4$ with $E_1$, we obtain the polar equations of the integral element $E_1$

\[
\begin{align*}
\eta^a &= 0, \quad a = 1, ..., 13, \\
2a^1\pi^1 - a^2\pi^2 - (2y^1 + a^2(1 - p_2 + q_1q_2)) \omega^1 + (y^2 + a^1(1 - p_2 + q_1q_2)) \omega^2 &= 0, \\
a^1\pi^1 - 2a^2\pi^2 - (y^1 - a^2(1 - p_1 + q_1q_2)) \omega^1 + (2y^2 - a^1(1 - p_1 + q_1q_2)) \omega^2 &= 0, \\
a^1v^1 + a^2\zeta^2 - (u^1 + a^2(3p_1q_2 + 2r_2q_1)) \omega^1 - (v^2 - a^1(3p_1q_2 + 2r_2q_1)) \omega^2 &= 0, \\
a^2v^2 + a^1\zeta^1 - (v^1 + a^2(2r_1q_2 + 3q_1p_2)) \omega^1 - (u^2 - a^1(2r_1q_2 + 3q_1p_2)) \omega^2 &= 0, \\
a^2\zeta^1 - 4a^2q_1r_1\omega^1 + (4a^1q_1r_1 + v^1)\omega^2 &= 0, \\
a^1\zeta^2 - (4a^2q_2r_2 + v^2)\omega^1 + 4a^1q_2r_2\omega^2 &= 0.
\end{align*}
\]

This shows that the polar space $H(3, E_1)$ of a non-characteristic integral element\(^3\) is two-dimensional. From the Cartan-Kaehler theorem we deduce

**Proposition 3.1** If $\hat{\Gamma} \subset P$ is a non-characteristic real-analytic integral curve of $(I, \Omega)$ then there exist a unique real-analytic integral manifold $\hat{M} \subset P$ such that $\hat{\Gamma} \subset \hat{M}$.

### 3.2 Legendre Curves.

Let $\Gamma \subset K$ be a smooth Legendre curve. Locally, we have that $\ell = [V_0(\ell) \cap V_1(\ell)]$, for every $\ell \in \Gamma$, where $V_0, V_1 : \Gamma \to \mathbb{R}^{(4,2)}$ are smooth maps such that

\[
\| V_0, \| = \| V_1 \| = \langle V_0, V_1 \rangle = 0, \quad \langle V_0, dV_1 \rangle = 0.
\]

We say that $\Gamma$ is *linearly full* in case\(^4\)

\[
V_0(\ell) \wedge V_1(\ell) \wedge V_0'(\ell) \wedge (V_0'')' \ell) \wedge V_1''(\ell) \wedge V_1'''(\ell) \neq 0 \quad \forall \ell \in N.
\]

We let $U(\Gamma) \to \Gamma$ be the tautological vector bundle of the curve, that is

\[
U(\Gamma) = \{(\ell, V) \in \Gamma \times \mathbb{R}^{(4,2)} : V \in \ell\}.
\]

A cross section of $U(\Gamma)$ is a smooth map $V : \Gamma \to \mathbb{R}^{(4,2)}$ such that $V(\ell) \in \ell$, for every $\ell \in \Gamma$. Accordingly, a line sub-bundle $L \subset U(\Gamma)$ can be viewed as a mapping $\sigma_L : \Gamma \to Q$ such that $\sigma_L(\ell) \subset \ell$, for each $\ell \in \Gamma$. We say that $L \subset U(\Gamma)$ is *fat* if

\[
V(\ell) \wedge V'(\ell) \wedge ... \wedge V^{(v)}(\ell) \neq 0, \quad \forall \ell \in \Gamma,
\]

for every local trivialization $V : U \to \mathbb{R}^{(4,2)}$ of $L$. In this case the osculating space $\delta_L(\ell) = [V(\ell) \wedge V'(\ell) \wedge V''(\ell)] \subset \mathbb{R}^{(4,2)}$ has signature $(2,1)$, for every $\ell \in \Gamma$. The map

\[
\delta_L : \ell \in \Gamma \to \delta_L(\ell) \in \mathcal{D}, \quad \forall \ell \in \Gamma.
\]

is called the *directrix curve* of $L$. If $\delta_L$ is non-isotropic (i.e. $\delta_L^*(g_D)$ is nowhere vanishing) then $L$ is said to be a *polarization* of the curve $\Gamma$.

\(^3\)i.e. an integral element such that $a^1a^2 \neq 0$

\(^4\)We use the notation $dV = V' d\zeta$, where $d\zeta$ is a nowhere vanishing 1-form on $N$. 
Proposition 3.2 Let \((\Gamma, L)\) be a polarized Legendre curve. Then, there exist a unique map \(R : \Gamma \to \tilde{G}\) such that
\[
\ell = [R_0(\ell) \wedge R_1(\ell)], \quad R_0(\ell) \in L|_{\ell}, \quad \forall \ell \in \Gamma
\] (55) and that
\[
R^{-1}dR = \begin{pmatrix}
 k_0 & 1 & 0 & k_1 & k_3 & 0 \\
 -1 & -k_0 & k_2 & 0 & 0 & -k_3 \\
 0 & -1 & 0 & 0 & k_2 & 0 \\
 1 & 0 & 0 & 0 & 0 & k_1 \\
 0 & 0 & -1 & 0 & k_0 & -1 \\
 0 & 0 & 0 & 1 & 1 & -k_0
\end{pmatrix} \mu, \quad (56)
\]
where \(\mu\) is a nowhere vanishing 1-form and where \(k_0, k_1, k_2\) and \(k_3\) are real-valued functions.

**Proof.** We consider the \(G_0\) fiber bundle
\[
\mathcal{R}_0(\Gamma, L) = \{(\ell, R) \in \Gamma \times G : \ell = [R_0 \wedge R_0], \quad R_0 \in L|_{\ell}\}.
\]
The cross-sections of \(\mathcal{R}_0(\Gamma, L)\) are smooth maps \(R : U \to G\) defined on an open subset \(U \subset \Gamma\), such that
\[
\ell = [R_0(\ell) \wedge R_1(\ell)], \quad R_0(\ell) \in L|_{\ell}, \quad \forall \ell \in U.
\]
For each frame field \(R : U \to G\) we let \(\rho\) be the \(g\)-valued 1-form \(R^{-1}dR\). We say that \(R : U \to G\) is of first order if
\[
\rho_0^3 \neq 0, \quad \rho_0^2 = \rho_1^3 = \rho_3^3 + \rho_1^2 = \rho_4^4 = 0. \quad (57)
\]
Since \(\Gamma\) is linearly full then, first order frames do exist near any point of \(\Gamma\) and they define a sub-bundle \(\mathcal{R}_1(\Gamma, L)\) of \(\mathcal{R}_0(\Gamma, L)\) with fiber
\[
H_1 = \{X \in G_0 : X = X(\epsilon I, \epsilon I, Y, b) : \epsilon = \pm 1, r, b \in \mathbb{R}, r \neq 0, Y \in \mathfrak{gl}(2, \mathbb{R})\}.
\]
If \(R\) and \(\tilde{R}\) are first order frames such that \(\tilde{R} = RX(\epsilon I, \epsilon I, Y, b)\) then
\[
\rho_0^0 = \rho_1^0 + rY_1^2\rho_0^0, \quad \rho_1^1 = \rho_0^1 - rY_2^1\rho_0^0, \quad \rho_2^2 = \rho_2^2 + \epsilon r(Y_2^1 + Y_1^2)\rho_0^3. \quad (58)
\]
This shows that near any point of \(\Gamma\) there exist first order frames such that
\[
\rho_0^0 + \rho_1^0 = \rho_2^2 = 0. \quad (59)
\]
Frame fields satisfying (59) are said to be of second order. From (58) it follows that the totality of second order frames defines a fiber bundle \(\mathcal{R}_2(\Gamma, L)\) with fiber
\[
H_2 = \{X = X(\epsilon I, \epsilon I, Y, b) \in H_1 : Y_2^1 = Y_1^2 = 0\}.
\]
Notice that the 1-form $\rho_0^1$ is independent on the choice of the second order frame and hence there exist $\mu \in \Omega^1(\Gamma)$ such that $\mu|_U = \rho_0^1$. At this juncture it is convenient to recall that the pseudo-riemannian metric $g_D$ of $\mathfrak{D}$ is represented by the tensorial quadratic form on $G$ defined by

$$\omega_0^1 \omega_0^1 + \omega_2^0 \omega_0^2 + 2 \omega_3^0 \omega_1^3 - \frac{1}{2}(\omega_2^0)^2.$$  

Thus, $\delta_L^*(g_D) = -\mu^2$ and hence $\mu$ is nowhere vanishing. From (65) it follows that, locally, there exist second order frames such that

$$\rho_3^0 = -\rho_1^0 = -\rho_0^1 = \rho_0^1 = \mu.$$  

Frame fields satisfying (60) are of third order. The totality of third order frames originates a principal fiber bundle $\mathcal{R}_3(\Gamma, L)$ with structural group

$$H_3 = \{X = X(\epsilon \epsilon I, \epsilon I, Y, b) \in H_2 : \epsilon = 1\}.$$  

If $\tilde{R}$ and $R$ are third order frame fields then

$$\tilde{\rho}_0^0 = \rho_0^0 - \epsilon Y_2^2 \rho_0^3, \quad \tilde{\rho}_1^1 = \rho_1^1 + \epsilon Y_1^1 \rho_0^3.$$  

Therefore, near any point of $\Gamma$ there exist a third order frame field $R$ such that

$$\rho_1^1 + \rho_0^0 = 0.$$  

Frame fields satisfying (61) define a reduced sub-bundle $\mathcal{R}_4(\Gamma, L)$ with structure group

$$H_4 = \{X = X(\epsilon \epsilon I, \epsilon I, Y, b) \in H_3 : Y_1^1 = Y_2^2\}.$$  

Consider two local cross sections $R$ and $\tilde{R}$ of $\mathcal{R}_4(\Gamma, L)$, we then have

$$\tilde{\rho}_2^0 = \rho_2^0 + \epsilon(Y_1^1 + \frac{b}{2})\mu, \quad \tilde{\rho}_3^1 = \rho_3^1 - \epsilon(Y_1^1 - \frac{b}{2})\mu.$$  

This implies that there exist fourth order frame fields with respect to which

$$\rho_2^0 = \rho_3^1 = 0.$$  

Fourth order frame fields satisfying (66) are said to be of fifth order. The totality of fifth order frames generates a reduced sub-bundle $\mathcal{R}_5(\Gamma, L)$ with fiber $\mathbb{Z}_2 = \{\pm I\}$ and henceforth there exist a map $\mathcal{R} : \Gamma \rightarrow \tilde{G}$ such that $\mathcal{R}|_U = [R]$, for every fifth order frame field $R : U \rightarrow G$. From (67), (68), (69), (61) and (63) it follows that $\mathcal{R}$ satisfies the required properties. \(\Box\)

**Definition 3.3** The lift $\mathcal{R} : N \rightarrow \tilde{G}$ is said to be the Frenet frame field of $(\Gamma, L)$. The 1-form $\mu$ is the Lie-invariant line element and the functions $k_0, k_1, k_2$ and $k_3$ are the generalized curvatures of $(\Gamma, L)$.

**Remark 3.4** This proposition shows that polarized Legendre curves are completely determined, up to the action of the Lie sphere group, by the curvatures $k_0, ..., k_3$.  

\[\n\]
3.3 The Cauchy problem

**Theorem 3.5** Let \((\Gamma, L)\) be a real-analytic polarized Legendre curve and let \(h, w : \Gamma \rightarrow \mathbb{R}\) be two real-analytic functions. Then, there exist a real-analytic Lie-minimal surface \(M \subset K\) containing \(\Gamma\) such that

\[
\Gamma^*(\alpha^1 + \alpha^2) = 0, \quad L = \Sigma_0(M)|_{\Gamma}, \quad h = -3(q_1 + q_2)|_{\Gamma}, \quad w = \frac{1}{3}(p_1 - p_2)|_{\Gamma}.
\]

This manifold is unique in the sense that any other Legendre surface with these properties agrees with \(M\) on an open neighborhood of \(\Gamma\).

**Proof.** Let \(\mathcal{R} : \Gamma \rightarrow \tilde{G}\) be the Frenet frame field along \((\Gamma, L)\). We set

\[
X(h) = \begin{pmatrix}
1 & 0 & 0 & -h/2 & h/2 & h^2/8 \\
0 & 1 & h/2 & 0 & h^2/8 & -h/2 \\
0 & 0 & 1 & 0 & h/2 & 0 \\
0 & 0 & 0 & 1 & 0 & h/2 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}.
\]

and we consider the frame field

\[
\tilde{\mathcal{R}} = \mathcal{R}X(h) : \Gamma \rightarrow \tilde{G}.
\]

We then have

\[
\tilde{\mathcal{R}}^{-1} d\tilde{\mathcal{R}} = \begin{pmatrix}
k_0 + h/2 & 1 & 0 & \hat{k}_1 & \hat{k}_3 & 0 \\
-1 & -k_0 + h/2 & \hat{k}_2 & 0 & 0 & -\hat{k}_3 \\
0 & -1 & 0 & 0 & \hat{k}_2 & 0 \\
1 & 0 & 0 & 0 & 0 & \hat{k}_1 \\
0 & 0 & -1 & 0 & k_0 + h/2 & -1 \\
0 & 0 & 0 & 1 & 1 & -k_0 + h/2
\end{pmatrix} \mu, \quad (66)
\]

where

\[
\begin{align*}
\hat{k}_2 &= k_2 + \frac{1}{2} h' - \frac{1}{4} (hk_0 - \frac{1}{3} h^2), \\
\hat{k}_1 &= k_1 - \frac{1}{2} h' - \frac{1}{4} (hk_0 + \frac{1}{3} h^2), \\
\hat{k}_3 &= k_3 - \frac{1}{2} h' - \frac{1}{4} h^2.
\end{align*}
\]

We define \(\bar{\eta}_i, \bar{T}_i, \bar{\tau}_i : \Gamma \rightarrow \mathbb{R}\), \(i = 1, 2\), by

\[
\begin{align*}
\bar{\eta}_1 &= -k_0 - \frac{1}{6} h, \\
\bar{\eta}_2 &= k_0 - \frac{1}{6} h, \\
\bar{T}_1 &= -\frac{1}{6} (k_1 - k_2 - k_3 - 3w), \\
\bar{T}_2 &= -\frac{1}{6} (k_1 - k_2 + k_3 + 3w), \\
\bar{\tau}_1 &= \frac{1}{6} (k_1 + k_2 - k_3 - 3w), \\
\bar{\tau}_2 &= -\frac{1}{6} (k_1 + k_2 + k_3 - 3w).
\end{align*}
\]

Let us now consider the embedding

\[
\bar{\Gamma} = (\tilde{\mathcal{R}}, \bar{\eta}, \bar{T}, \bar{\tau}) : \Gamma \rightarrow P.
\]
From (66) and (68) it follows that $\tilde{\Gamma}$ is a 1-dimensional integral manifold of the differential system $(I, \Omega)$. We set
\[ d\tilde{\sigma}_j = \tilde{\sigma}'_j \mu, \quad d\tilde{\tau}_j = \tau'_j \mu, \quad d\tilde{\tau}'_j = \tau''_j \mu, \quad (70) \]
where $\tilde{\sigma}'_j, \tau'_j, \tau''_j$ are real-analytic functions. From (66) we infer that
\[ \tilde{\Gamma} \ast \left( \frac{\partial}{\partial \mu} \right) = \frac{\partial}{\partial \omega_1} - \frac{\partial}{\partial \omega_2} + \tilde{\sigma}'_i \frac{\partial}{\partial \pi^i} + \tilde{\tau}'_i \frac{\partial}{\partial \upsilon^i} + \tilde{\tau}''_i \frac{\partial}{\partial \zeta^i}. \quad (71) \]
Thus, $\tilde{\Gamma}$ is a non-characteristic $K$-regular integral curve of $(I, \Omega)$. Therefore, there exist a unique 2-dimensional integral manifold $\tilde{M} \subset P$ such that $\tilde{\Gamma} \subset \tilde{M}$. We consider the Legendre immersion
\[ \phi : ([A], q, p, r) \in \tilde{M} \rightarrow [A_0 \wedge A_1] \in \mathcal{K}. \quad (72) \]
Since our arguments are local in nature, we suppose that $\phi$ is one-to-one and we identify $\tilde{M}$ with its image $M = \phi(\tilde{M})$. Then, the map
\[ A : ([A], q, p, r) \in M \rightarrow [A] \in \tilde{G} \quad (73) \]
is the normal frame field along $M$. From this we deduce that $M$ is a Lie-minimal surface. By construction, $\Gamma$ is contained in $M$ and
\[ R = A|_{\Gamma}, \quad \alpha^1|_{\Gamma} = -\alpha^2|_{\Gamma} = \mu, \quad \bar{q}_i = q_i|_{\Gamma}, \quad \bar{p}_i = p_i|_{\Gamma}, \quad \bar{r}_i = r_i|_{\Gamma}, \quad i = 1, 2. \quad (74) \]
In particular, the 1-form $\alpha^1 + \alpha^2$ vanishes identically along $\Gamma$. Combining (68) and (74) we deduce
\[ w = \frac{1}{3}(p_1 - p_2)|_{\Gamma}, \quad f = -3(q_1 + q_2)|_{\Gamma}, \quad \sigma_L = \sigma_0|_{\Gamma}. \]
Let us recall that the curvature sphere mappings $\sigma_0$ and $\sigma_1$ are represented by $[A_0]$ and by $[A_1]$ respectively. On the other hand, $L$ is spanned by the first row vector of the framing $\hat{\mathcal{R}}$ so that $\sigma_L = [\hat{R}_0]$. This implies
\[ \sigma_1|_{\Gamma} = [A_0]|_{\Gamma} = [\hat{R}_0] = \sigma_L. \]
From this we infer that $M$ satisfies the required properties. The uniqueness of $M$ follows from the uniqueness of the real-analytic integral manifold $\tilde{M}$ containing $\tilde{\Gamma}$. $\nabla$

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