ON BLOCKING SETS IN PROJECTIVE HJELMSLEV PLANES

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Abstract. A \((k, n)\)-blocking multiset in the projective Hjelmslev plane \(PHG(R_3^m)\) is defined as a multiset \(\mathcal{K}\) with \(\mathcal{K}(\mathcal{P}) = k\), \(\mathcal{K}(l) \geq n\) for any line \(l\) and \(\mathcal{K}(l_0) = n\) for at least one line \(l_0\). In this paper, we investigate blocking sets in projective Hjelmslev planes over chain rings \(R\) with \(|R| = q^m\), \(R/\text{rad} \cong \mathbb{F}_q\), where \(q = p^r\), \(p\) prime. We prove that for a \((k, n)\)-blocking multiset with \(1 \leq n \leq q\), \(k \geq nq^{m-1}(q+1)\). The image of a \((nq^{m-1}(q+1), n)\)-blocking multiset with \(n < \text{char} \, R\) under the canonical map \(\pi(1)\) is “sum of lines”. In particular, the smallest \((k, 1)\)-blocking set is the characteristic function of a line and its cardinality is \(k = q^{m-1}(q+1)\). We prove that if \(R\) has a subring \(S\) with \(\sqrt{|R|}\) elements that is a chain ring such that \(R/\text{rad} \cong \mathbb{F}_q\) then the subplane \(PHG(S_3^m)\) is an irreducible 1-blocking set in \(PHG(R_3^m)\). Corollaries are derived for chain rings with \(|R| = q^2\), \(R/\text{rad} \cong \mathbb{F}_q\).

In case of chain rings \(R\) with \(|R| = q^2\), \(R/\text{rad} \cong \mathbb{F}_q\) and \(n = 1\), we prove that the size of the second smallest irreducible \((k, 1)\)-blocking set is \(q^2 + q + 1\). We classify all blocking sets with this cardinality. It turns out that if \(\text{char} \, R = p\) there exist (up to isomorphism) two such sets; if \(\text{char} \, R = p^2\) the irreducible \((q^2 + q + 1, 1)\)-blocking set is unique. We introduce a class of irreducible \((q^2 + q + s, 1)\) blocking sets for every \(s \in \{1, \ldots, q + 1\}\). Finally, we discuss briefly the codes over \(\mathbb{F}_q\) obtained from certain blocking sets.

1. Introduction

The motivation for this work comes from two sources – coding theory and finite geometry. In the past ten years, a substantial research has been done on linear codes over finite rings. It has been fuelled by the discovery that certain nonlinear codes that perform better than any linear codes over a finite field are in fact images of linear codes over the ring \(\mathbb{Z}_4\) \([8, 21]\). Attempts have been made to obtain a theory of error-correcting codes over a reasonable class of rings (cf. \([7, 21, 25, 28]\)). In \([11]\) and \([13]\), the theory of linear codes over the class of so-called finite chain rings was developed. Various results from \([8, 11, 24, 26, 27]\) show that using “good” linear codes over chain rings one can construct “good” (not necessarily linear) codes over finite fields.

It turns out that many good properties known for linear codes over finite fields still hold for linear codes over finite chain rings. In particular, there is an one-to-one correspondence between the classes of equivalent multisets of points in the projective
Hjelmslev geometries and the classes of semilinearly isomorphic fat linear codes over the chain ring $R$. With each multiset in $\text{PHG}(R^k_3)$, we can associate a linear code over $R$ by taking a matrix with columns the points of this multiset written in homogeneous coordinates as a generator matrix. Conversely, each fat linear code over a finite chain ring $R$ gives a multiset of points in $\text{PHG}(R^k_3)$. Moreover, two multisets are equivalent if and only if the corresponding codes are semilinearly isomorphic. The task of constructing multisets of maximal cardinality in projective Hjelmslev geometries containing no more than a prescribed number of points in each hyperplane very often leads to the construction of linear codes over chain rings with interesting properties.

The problem of constructing optimal multisets of points in certain finite geometries is interesting in its own right and is older than its coding theoretic counterpart. There is a vast literature about sets of points in the classical projective geometries over finite fields (cf. [9, 10] and the references there), but there are almost no results on such sets in finite Hjelmslev geometries. In this paper, we investigate blocking multisets in projective Hjelmslev planes $\text{PHG}(R^3_3)$, where $R$ is a chain ring with $|R| = q^m$ and $R/\text{rad} R \cong \mathbb{F}_q$. We confine ourselves to right Hjelmslev planes $\text{PHG}(R^3_3)$. This is no restriction since every left $R$-module is a right module over the opposite ring $R^{\text{opp}}$, which is also a chain ring with $q^m$ elements and residue field of order $q$.

In section 2, we introduce projective Hjelmslev planes and present some results on the structure of planes obtained from finite chain rings. In section 3, we define arcs and blocking multisets and present a general lower bound on the minimal size of a blocking multiset with given parameters. It is a reformulation of a upper bound known for arcs over chain rings $R$ with $|R| = q^m$, $R/\text{rad} R \cong \mathbb{F}_q$. In Section 4 we prove that for every $(k,n)$-blocking multiset in $\text{PHG}(R^3_3)$, $n = 1, \ldots, q$, $|R| = q^m$, $R/\text{rad} R \cong \mathbb{F}_q$, $k \geq nq^m-1(q+1)$. In case of equality, and $n < \text{char} \mathbb{F}_q$ the image of the blocking set under the canonical map $\pi^{(1)}$ is a “sum of lines” (Theorem 2). Further, we prove that if $R$ has a subring $S$ with $\sqrt{|R|}$ elements that is a chain ring and such that $R$ is free over $S$ then the subplane $\text{PHG}(S^3_3)$ is an irreducible 1-blocking set (Theorem 3). Two corollaries are derived for chain rings with $|R| = q^2$, $R/\text{rad} R \cong \mathbb{F}_q$. In Section 5 we prove that the size of the second smallest irreducible $(k,1)$-blocking set in $\text{PHG}(R^3_3)$, where $|R| = q^2$, $R/\text{rad} R \cong \mathbb{F}_q$, is $q^2 + q + 1$. Moreover, in projective Hjelmslev planes over chain rings of characteristic $p$ there exist (up to isomorphism) two such blocking sets, while in planes over chain rings of characteristic $p^2$ there is only one irreducible $(q^2+q+1,1)$-blocking set (Theorem 4). In section 6 we compute the parameters and the weights of various codes obtained from the linear codes associated with the blocking sets from the previous sections.

2. PROJECTIVE HJELMSLEV PLANES

We start by introducing projective Hjelmslev planes. Let $\Pi = (\mathcal{P}, \mathcal{L}, I)$ be an incidence structure. Here $\mathcal{P}$ and $\mathcal{L}$ are referred to as the sets of points and lines, respectively, and $I \subseteq \mathcal{P} \times \mathcal{L}$ is an incidence relation. A neighbour relation $\sim$ is defined on $\mathcal{P}$ and $\mathcal{L}$ by the following conditions:

(N1) $\forall X, Y \in \mathcal{P} : X \sim Y \iff \exists s, t \in \mathcal{L}, s \neq t : \{(X, s), (X, t), (Y, s), (Y, t)\} \subseteq I$;

(N2) $\forall s, t \in \mathcal{L} : s \sim t \iff \exists X, Y \in \mathcal{P}, X \neq Y : \{(X, s), (X, t), (Y, s), (Y, t)\} \subseteq I$.

A projective Hjelmslev plane is an incidence structure $\Pi = (\mathcal{P}, \mathcal{L}, I)$ with a neighbour relation $\sim$ that satisfies the axioms:
Given a point \( P \) and a line \( l \) there exist exactly \( q^{m-1} \) points on \( l \) that are \( i \)-neighbours to \( P \) and exactly \( q^{m-1} \) lines through \( P \) that are \( i \)-neighbours to \( l \).
(e) if $X \sim_i Y$ and $X \not\sim_{i+1} Y$, $X, Y \in \mathcal{P}$, $i = 0, 1, \ldots, m$, there exist exactly $q^i$ lines incident with both $X$ and $Y$; by duality, if $s \sim t$ and $s \not\sim_{i+1} t$, $s, t \in \mathcal{L}$, there exist exactly $q^i$ points incident with both $s$ and $t$;

(f) the number of $i$-neighbour classes of points (lines) is $q^{2(i-1)}(q^2 + q + 1)$.

Define the incidence relation $J^{(i)} \subseteq \mathcal{P}^{(i)} \times \mathcal{L}^{(i)}$ by

$$([X]^{(i)}, [s]^{(i)}) \in J^{(i)} \iff \exists X' \in [X]^{(i)}, \exists s' \in [s]^{(i)} : (X', s') \in I.$$

**Fact 2.** The incidence structure $(\mathcal{P}^{(i)}, \mathcal{L}^{(i)}, J^{(i)})$ is isomorphic to the projective Hjelmslev plane $\text{PHG}((R/R\theta^3)_{R/R\theta^i})$. In particular, $(\mathcal{P}^{(1)}, \mathcal{L}^{(1)}, J^{(1)})$ is isomorphic to $\text{AG}(2, q)$.

Given a projective Hjelmslev plane $\Pi = \text{PHG}(R_{R_R}^3)$, we define the affine Hjelmslev plane $\text{AHG}(R_{R_R}^3)$ as an incidence structure with points – all points not incident with a fixed $1$-neighbourhood class $[l]$ of lines, with lines – all lines not in $[l]$, and with incidence – the one inherited from $\Pi$. Equivalently, we define the points as all pairs $(a, b)$, $a, b \in R$, the lines as all cosets of free rank 1 submodules of $R_{R_R}^3$, and incidence by set-theoretical inclusion. The lines of $\text{AHG}(R_{R_R}^3)$ can be partitioned into $q^{m-1}(q + 1)$ classes of parallel lines.

Let us fix a point $P \in \mathcal{P}$ and denote by $\mathcal{L}^{(i)}(P)$ the set of all lines of $\mathcal{L}$ that are incident with at least one point from $[P]^{(i)}$. For two lines $s, t \in \mathcal{L}^{(i)}(P)$, we write $s \sim t$ if they coincide on $[P]^{(i)}$. Denote by $\mathcal{L}'$ a set of lines from $\mathcal{L}^{(i)}(P)$ that contains exactly one representative from each equivalence class under $\sim$.

**Fact 3.** The structure $([P]^{(i)}, \mathcal{L}', I|_{[P]^{(i)} \times \mathcal{L}'})$ is isomorphic to the affine Hjelmslev plane $\text{AHG}((R/R\theta^{m-i})_{R/R\theta^{m-i-1}})$. In particular, $([P]^{(m-i-1)}, \mathcal{L}', I|_{[P]^{(m-i-1)} \times \mathcal{L}'})$ is isomorphic to $\text{AG}(2, q)$.

For any $X \in \mathcal{P}$ and any $l \in \mathcal{L}$, we write $X \sim_i l$ if there exists a point $Y \in \mathcal{P}$ such that $(Y, l) \in I$ and $X \sim_i Y$. It is clear that in any projective Hjelmslev plane $\Pi = \text{PHG}(R_{R_R}^3)$

$$X \sim_i l \iff \pi^{(i)}(X) \in \pi^{(i)}(l).$$

Let $l$ be a line in $\Pi$ and let $i \in \{0, 1, \ldots, m\}$ be fixed. We define a set of points $\mathfrak{P}^{(i)}$ by

$$\mathfrak{P}^{(i)} = \{ s \cap [X]^{(m-i)} \mid s \in \mathcal{L}, X \sim_i l, s \sim_i l \}.$$

Further define an incidence relation $\mathfrak{J} \subseteq \mathfrak{P}^{(i)} \times \mathcal{L}$ by

$$(s \cap [X]^{(m-i)}, t) \in \mathfrak{J} \iff (s \cap [X]^{(m-i)}) \cap t \neq \emptyset.$$

For two lines $s, t \in \mathcal{L}$, we write $s \sim t$ if they are incident under $\mathfrak{J}$ with the same elements of $\mathfrak{P}^{(i)}$. We denote by $\mathfrak{L}^{(i)}$ a set of lines containing exactly one representative from each equivalence class of $\mathcal{L}$ under $\sim$. Set $\mathfrak{J}^{(i)} = \mathfrak{J}|_{\mathfrak{P}^{(i)} \times \mathfrak{L}^{(i)}}$.

**Fact 4.** The incidence structure $(\mathfrak{P}^{(i)}, \mathfrak{L}^{(i)}, \mathfrak{J}^{(i)})$ can be embedded isomorphically into $\text{PHG}((R/R\theta^{m-i})_{R/R\theta^{m-i-1}})$.

Note that the structure from Fact 4 is actually isomorphic to $\text{PHG}((R/R\theta^{m-i})_{R/R\theta^{m-i}})$ with one neighbour class of points deleted.
3. Multisets of Points in Projective Hjelmslev Planes

Let \( \Pi = (\mathcal{P}, \mathcal{L}, I) \) be a projective Hjelmslev plane. Any mapping from the pointset \( \mathcal{P} \) to the nonnegative integers \( \mathcal{R} : \mathcal{P} \to \mathbb{N}_0 \) is called a multiset in \( \Pi \). The integer \( \mathcal{R}(P), P \in \mathcal{P} \), is called the multiplicity of \( P \). The mapping \( \mathcal{R} \) induces a mapping on the subsets of \( \mathcal{P} \) by

\[
\mathcal{R}(Q) = \sum_{P \in Q} \mathcal{R}(P), \quad Q \subseteq \mathcal{P}.
\]

Here, the induced mapping is denoted (by a slight abuse of notation) again by \( \mathcal{R} \).

The integer \( |\mathcal{R}| = \mathcal{R}(\mathcal{P}) \) is called the cardinality or the size of \( \mathcal{R} \). A set of points \( Q \) with \( \mathcal{R}(Q) = i \) is called an \( i \)-set with respect to \( \mathcal{R} \). In particular, points of multiplicity \( i \) are \( i \)-points and lines of multiplicity \( i \) are \( i \)-lines. The support \( \text{supp} \mathcal{R} \) of a multiset \( \mathcal{R} \) is the set of points of positive multiplicity: \( \text{supp} \mathcal{R} = \{ P \in \mathcal{P} \mid \mathcal{R}(P) > 0 \} \).

Two multisets \( \mathcal{R}' \) and \( \mathcal{R}'' \) in the projective Hjelmslev plane \( \Pi \) are said to be equivalent if there exists a collineation \( \sigma \) in \( \Pi \) such that \( \mathcal{R}'(P) = \mathcal{R}''(\sigma(P)) \) for every point \( P \in \mathcal{P} \).

**Definition 1.** A multiset \( \mathcal{R} \) in \( (\mathcal{P}, \mathcal{L}, I) \) is called a \((k, n)\)-arc if

1. \( \mathcal{R}(\mathcal{P}) = k \);
2. \( \mathcal{R}(\ell) \leq n \) for every line \( \ell \in \mathcal{L} \);
3. there exists at least one line \( \ell_0 \) with \( \mathcal{R}(\ell_0) = n \).

**Definition 2.** A multiset \( \mathcal{R} \) in \( (\mathcal{P}, \mathcal{L}, I) \) is called a \((k, n)\)-blocking multiset if

1. \( \mathcal{R}(\mathcal{P}) = k \);
2. \( \mathcal{R}(\ell) \geq n \) for every line \( \ell \in \mathcal{L} \);
3. there exists at least one line \( \ell_0 \) with \( \mathcal{R}(\ell_0) = n \).

A \((k, n)\)-blocking multiset \( \mathcal{R} \) is called reducible if there exists \((k', n)\)-blocking multiset \( \mathcal{R}' \) with \( k' < k \) and \( \mathcal{R}'(P) \leq \mathcal{R}(P) \) for every point \( P \in \mathcal{P} \). A blocking multiset that is not reducible is called minimal or irreducible.

An arc (resp. a blocking multiset) \( \mathcal{R} \) with \( \mathcal{R}(\mathcal{P}) \in \{0, 1\} \) for every \( P \in \mathcal{P} \) is called a projective arc (respectively, a projective blocking multiset, or simply a projective blocking set). Projective arcs and projective blocking sets can be considered as sets of points by identifying them with their support.

A multiset \( \mathcal{R} \) in \( \Pi = \text{PHG}(R^3_k) \) induces a multiset \( \mathcal{R}^{(i)} \) in the projective Hjelmslev plane \( \Pi^{(i)} = \pi^{(i)}(\Pi) = \text{PHG}((R/R^q)\hat{3}_{R/R^0}) \) by

\[
\mathcal{R}^{(i)} : \begin{cases} 
\mathcal{P}^{(i)} \to \mathbb{N}_0, \\
[P]^{(i)} \to \mathcal{R}([P]^{(i)}). 
\end{cases}
\]

The line \( l \) in \( \text{PHG}(R^3_k) \) is said to be of type \((i_1, i_2, \ldots, i_m)\), if \( \mathcal{R}^{(j)}(l) = i_j+\ldots+i_m, \ j = 1 \ldots m \). Here \( i_j \) denotes the number of points \( X \) for which there exists a point \( Y \) on \( l \) with \( X \sim_j Y \), but for which there is no point \( Z \) on \( l \) with \( X \sim_{j+1} Z \). The sequence \( \mathbf{a} = (a(i_1, i_2, \ldots, i_m)), (i_1, i_2, \ldots, i_m) \in \mathbb{N}^m \), where \( a(i_1, i_2, \ldots, i_m) \) is the number of lines of type \((i_1, i_2, \ldots, i_m)\), is called the spectrum of \( \mathcal{R} \).

For any \( n \in \mathbb{N} \), we denote by \( \kappa_n(R^3_R) \) the minimal value of \( k \) for which there exists a \((k, n)\)-blocking multiset in \( \text{PHG}(R^3_R) \). For chain rings with \( |R| = q^2, R/\text{rad} R \cong \mathbb{Z}_q \),
\( F_q \), we can reformulate the upper bound for arcs from [14] to get

\[
\kappa_n(R^1_R) \geq \min_{1 \leq u \leq q^2} \max \{ u(q^2 + q + 1), q^2(n - 1) + q(n - u) + u, q(q + 1)(n - \lfloor u/q \rfloor) + u \}.
\]

If \( \mathcal{R} \) is a projective \((k, n)\)-arc then \( \mathcal{R}' = 1 - \mathcal{R} \) is a projective blocking set with parameters \((q^2(q^2 + q + 1) - k, q(q + 1) - n)\). Thus results about arcs can be translated into results about blocking sets an vice versa. Traditionally, we consider blocking sets if \( n \) is “big”, i.e. if \( n \) is “close” to the line size \( q(q + 1) \).

4. General Results on Blocking Sets in Projective Hjelmslev Planes

Until the end of this section, \( R \) will be a chain ring with \( |R| = q^m \) and \( R/\text{rad} R \cong F_q \). As before \( \text{rad} R = R\theta \) for some \( \theta \) in \( \text{rad} R \setminus \text{rad}^2 R \). The following theorem is based on the nested structure of projective Hjelmslev planes and provides a large class of blocking sets.

**Theorem 1.** Let \( R \) be a chain ring. Let there exist blocking sets with parameters \((k_1, n_1)\) in \( \text{PHG}((R/R\theta)^3_{R/R\theta^2}) \) and \((k_2, n_2)\) in \( \text{AHG}((R/R\theta^{m-1})^3_{R/R\theta^{m-2}}) \), \( i \in \{1, \ldots, m-1\} \). Then there exists a \((k_1k_2, n_1n_2)\)-blocking set in \( \text{PHG}(R^1_R) \).

Such blocking sets are not minimal in general. Henceforth, they are considered as trivial.

It turns out that for \( n \leq q \) we can find the minimal size of a projective \((k, n)\)-blocking set and provide a characterization for the blocking sets with \( n < p = \text{char} F_q \). This result is known for projective Hjelmslev planes over chain rings \( R \) with \( |R| = q^2 \), \( R/\text{rad} R \cong F_q \) (see e.g. [15]). Below we prove a generalization for arbitrary chain rings. We start with a lemma.

**Lemma 1.** Let \( R \) be a chain ring with \( |R| = q^m \), \( R/\text{rad} R \cong F_q \). Let \( \mathcal{R} \) be a \((k, n)\)-blocking multiset with \( 1 \leq n \leq q^{m-1} \) in \( \Pi = \text{PHG}(R^1_R) \). Then for every neighbour class on points \([P] = [P]^{(1)}\) with \( \mathcal{R}([P]) = a < q^{m-1} \) and every neighbour class on lines \([l] = [l]^{(1)}\) incident with \([P] \) in \( \Pi^{(1)} \), we have

\[\mathcal{R}([l]) \geq a + nq^{m-1}.\]

**Proof.** We use induction on \( m \). The case \( m = 1 \) is trivial. Assume that \( m = 2 \). The structure induced on \([P]\) is isomorphic to AG(2, q) (cf. Fact 3). Consider the parallel class in this plane having the direction of \( l \) (i.e. lines of the form \( l' \cap [P] \) where \( l' \in [l] \)). Since \( \mathcal{R}([P]) < q \), we have an empty line in this class. Therefore there exist \( q \) lines \( l_i \in [l] \) such that \( \mathcal{R}(l_i \cap [P]) = 0 \). These \( q \) lines coincide on \([P]\) and partition the points of \([l] \setminus [P] \). Hence

\[
\mathcal{R}([l]) = \mathcal{R}([P]) + \sum_{i=1}^{q} \mathcal{R}(l_i \setminus [P]) = \mathcal{R}([P]) + \sum_{i=1}^{q} \mathcal{R}(l_i) \geq a + nq.
\]

Now assume that our assertion has been proved for every \( s \leq m - 1 \) and every \((k', n')\)-blocking multiset with \( n' \leq q^{s-1} \) in a plane \( \text{PHG}(S^3_{S}) \), where \( |S| = q^s \), \( S/\text{rad} S \cong F_q \). Further assume that \( R \) is a chain ring with \( q^m \) elements and residue field of order \( q \). Set \( \Pi = \text{PHG}(R^3_R) \) and let \( \mathcal{R} \) be a \((k, n)\)-blocking multiset in \( \Pi \) with \( n \leq q^{m-1} \). Consider the incidence structure having as points all 2-neighbour classes contained in \([P]^{(1)}\), as lines the 2-neighbour classes of lines containing points from \([P]^{(1)}\) and the incidence inherited from \((P^{2}, \mathcal{L}^{(2)}, J^{(2)})\). This structure is...
isomorphic to $\text{AG}(2, q)$ by Facts 2 and 3. The class of parallel lines in this affine geometry that have the direction of $l$ is of cardinality $q$ and, as above, at least one of these lines has multiplicity less than $q^{m-2}$. Let this line be incident with the points

$$[Q_1]^{(2)}, [Q_2]^{(2)}, \ldots, [Q_q]^{(2)} \subset [P]^{(1)}.$$ 

There exist $q$ 2-neighbour line classes in $[l]$, say $[l_i]^{(2)}$, $i = 1, \ldots, q$, that meet $[P]$ in the points $P$. These line classes partition the points of $[l] \setminus [P]$ and coincide on $[P]$. By Fact 4, the class $[l]$ can be embedded isomorphically in the projective Hjelmslev plane $\Pi = \text{PHG}((R/R\theta^{m-1})_{R/R^{\theta^{m-1}}})$. The set $\{[Q_j]^{(2)}, j = 1, \ldots, q\}$ is a 1-neighbour class of points in $\Pi$, say $[Q]$, and each class of lines $[l_i]^{(2)}$ is a 1-neighbour class of lines through $[Q]$. By the induction hypothesis,

$$\mathcal{R}([l_i]^{(2)} \setminus [Q]) \geq nq^{m-2},$$

whence

$$\mathcal{R}([l]) = \mathcal{R}([P]) + \sum_{i=1}^{q} \mathcal{R}([l_i]^{(2)} \setminus [Q]) \geq a + nq^{m-1}.$$

\[\square\]

**Lemma 2.** Let $\Pi = (\mathcal{P}, \mathcal{L}, \mathcal{I}) = \text{PHG}(R_{\theta}^q)$, where $R$ is a chain ring with $|R| = q^m$, $R/radR \cong \mathbb{F}_q$. Let $\mathcal{R}$ be a $(nq^{m-1}(q+1), n)$-blocking set with $n \leq q$. Then there exists a neighbour class on points $[P]$ with $\mathcal{R}([P]) = 0$.

**Proof.** Assume otherwise. Let $[P]$ be a neighbour class on points of minimal multiplicity $\mathcal{R}([P]) = a > 0$. If $a \geq q^{m-1}$ we have

$$|\mathcal{R}| \geq q^{m-1}(q^2 + q + 1) > q^{m-1}(q^2 + q) \geq nq^{m-1}(q + 1).$$

If $a < q^{m-1}$ we count the multiplicities of all line classes $[l]$ through $[P]$ in $\Pi^{(1)}$. By Lemma 3

$$\mathcal{R} = \sum_{[l]: [l] \in [P]} \mathcal{R}([l]) - q \cdot \mathcal{R}([P])$$

$$= (q + 1)(a + nq^{m-1}) - qa$$

$$= a + nq^{m-1}(q + 1) > nq^{m-1}(q + 1),$$

again a contradiction. \[\square\]

**Lemma 3.** Let $\mathcal{R}$ be a multiset in $\text{AG}(2, q)$ such that $\mathcal{R}(l) = n < p = \text{char} \mathbb{F}_q$ for the lines of all but one parallel class. Then $\mathcal{R}$ is the sum of $n$ (not necessarily different) lines of the missing class.

**Proof.** Extend $\text{AG}(2, q)$ to $\text{PG}(2, q)$ and let $P_0$ be the point on the infinite line that corresponds to the missing parallel class. Define $\mathcal{R}$ by

$$\mathcal{R}(P) = \begin{cases} 
\mathcal{R}(P) & \text{if } P \text{ is in } \text{AG}(2, q), \\
n & \text{if } P = P_0, \\
0 & \text{otherwise}.
\end{cases}$$

Clearly $\mathcal{R}$ is a $(n(q + 1), n)$-blocking multiset in $\text{PG}(2, q)$ and the result follows by Theorem 5 from [15]. \[\square\]
Lemma 4. Let \( \Pi = (P, \mathcal{L}, I) = \mathrm{PHG}(R/R_\theta), \) where \( R \) is a chain ring with \( |R| = q^n, \) \( R/\operatorname{rad} R \cong \mathbb{F}_q. \) Let \( \mathcal{R} \) be a \((nq^{m-1}(q + 1), n)\)-blocking set \( n < p = \text{char} \mathbb{F}_q. \) Then \( q^{m-1} \) divides \( \mathcal{R}([Q]^{(i)}) \) for every point \( Q \in \mathcal{P} \) and every \( i \in \{1, \ldots, m-1\}. \)

Proof. By Lemma 2 there exists an empty neighbour class of points, \( [P] \) say. An easy counting argument shows that for every line \( l \) with \( l \cap [P] \neq \emptyset, \) we have \( \mathcal{R}(l) = n. \) Let \( Q \) be a nonempty point with \( Q \neq P. \) First we are going to prove the following statement: if \( l = (P, Q) \) then all the numbers \( \mathcal{R}(t \cap [Q]^{(j)}) \) are constant, where \( t \) is any line in \( [l] \) meeting \( [Q]^{(j)} \) and \( j = 0, 1, \ldots, m-1. \)

We prove the lemma by induction on \( m \) and \( j. \) For \( m = 2, j = 1, \) the statement is contained in Theorem 6 from [15]. For \( j = 0 \) and any \( m \) the statement is equivalent to the fact that that any line in \( [l] \) has the same multiplicity \( n. \) Now assume that the statement is true for any \((nq^s-1(q + 1), n)\)-blocking set in any projective Hjelmslev plane over a chain ring \( S \) with \(|S| = q^s, s \leq m - 1\) \( \operatorname{rad} S \cong \mathbb{F}_q \) for all \( j, \) as well as for such a blocking set in \( \Pi = (P, \mathcal{L}, I) = \mathrm{PHG}(R/R_\theta) \) where \( R \) is a chain ring with \(|R| = q^m, \) \( R/\operatorname{rad} R \cong \mathbb{F}_q, \) and every \( 0 \leq j < j_0 \leq m - 1. \) We have to prove that \( \mathcal{R}(t \cap [Q]^{(j)}) \) is a constant. Here \( t \in [l] \) and \( t \) meets \([Q]^{(j)}. \)

Let \( \mathcal{R} \) be an \((nq^{s-1}(q + 1), n)\)-blocking set in \( \Pi. \) Denote by \( \Pi^{(i)} = (Q^{(i)}, L^{(i)}, \mathcal{J}^{(i)}) \) the factor geometries induced by the line \( l = (P, Q). \) By Fact 8 it is isomorphic to \( \mathrm{PHG}((R/R_\theta^{m-i}) |_{R/R_\theta^{m-i}}) \) with a 1-neighbour class of points deleted.

Consider the geometry \( \Pi^{(j_0-1)}. \) It is isomorphic to \( \mathrm{PHG}((R/R_\theta^{m-j_0}) |_{R/R_\theta^{m-j_0}}). \) By the induction hypothesis, the segments \( t_1 \cap [Q]^{(j_0-1)} \) and \( t_2 \cap [Q]^{(j_0-1)} \) have the same multiplicity \( 0 < a \leq n \) if \( t_1 \subset t_2. \) In fact, these segments have the same multiplicity even if \( t_1 \subset t_2. \) To see this, we have to count in two ways the multiplicities of the points on all lines from \([t_1]^{(j_0)} \) in \([Q]^{(j_0-2)} \) not in and the points on all lines from \([t_2]^{(j_0)} \) in \([Q]^{(j_0-2)} \). Consider an 1-neighbour class of points (with respect to the induced neighbourhood in \( \Pi^{(j_0)} \)). It contains the points of \( q^{j_0} \) neighbour point classes (with respect to the neighbour in \( \Pi \), \([Q_1]^{(j_0)}, \ldots, [Q_{q^{j_0}}]^{(j_0)} \) say. Now the (nonempty line segments \( t \cap [Q_i]^{(j_0)} \) form an affine plane isomorphic to \( \mathrm{AG}(2, q). \) The restriction of \( \mathcal{R} \) to this affine plane has the property that all lines in all but one direction are blocked exactly \( n' \) times. By Lemma 3 this restriction is the sum of \( n' \) parallel lines having the exceptional direction. The lines in this parallel class are the sets of ‘points’ \( t \cap [Q_i]^{(j_0)}, t \in [l], \) \( i = 1, \ldots, q. \) This shows that the integers \( \mathcal{R}(t \cap [Q]^{(j_0)}) \) do not depend on \( t. \)

Now it is clear that \( \mathcal{R}(Q)^{(m-1)} = \sum t \mathcal{R}(t \cap [Q]^{(m-1)}) \). The sum on the right contains \( q \) constant terms and is, therefore, divisible by \( q. \) To complete the proof, we use once again induction on \( m. \) Consider the geometry \( (P^{(m-1)}, L^{(m-1)}, I^{(m-1)}) \cong \mathrm{PHG}((R/R_\theta^{m-j_0}) |_{R/R_\theta^{m-j_0}}), \) and the multiset \( \mathcal{R} = \frac{1}{q} \mathcal{R}^{(m-1)}. \) Since the class \([l]^{(m-1)} \) contains \( q^2 \) lines and since every point (incident with lines in this class) is on exactly \( q \) of them, we have

\[ \mathcal{R}([l]^{(m-1)}) = \frac{1}{q^2} \sum_{t \in [l]^{(m-1)}} \mathcal{R}(t) \geq n \]

it is a \((nq^{m-2}(q+1), n)\)-blocking multiset and by the induction hypothesis \( \mathcal{R}(Q)^{(i)} = q \mathcal{R}^{(i)} \) is divisible by \( q^{m-1}. \)

Theorem 2. Let \( R \) be a chain ring with \(|R| = q^n, R/\operatorname{rad} R \cong \mathbb{F}_q \) and let \( \mathcal{R} \) be a \((k, n)\)-blocking multiset with \( 1 \leq n \leq q, \) in \( \Pi = \mathrm{PHG}(R/R_\theta). \) Then \( k \geq nq^{m-1}(q + 1). \)
Equality is achieved for every \( n = 1, 2, \ldots, q \). If \( \mathcal{R} \) is a \((k, n)\)-blocking multiset with \( k = nq^{m-1}(q+1) \), \( n < p = \text{char } \mathbb{F}_q \) there exist lines, \( l_1, l_2, \ldots, l_n \) say, such that

\[
\mathcal{R}^{(1)} = q^{m-1}\{j \mid j \in \{1, \ldots, n\}, ([P]_j) \in J\}.
\]

**Proof.** The case \( m=1 \) was done in Theorem 5 of [15], so we assume \( m \geq 2 \). Assume there exists a point class \([P]^{(1)}\) with \( \mathcal{R}([P]^{(1)}) = a \), where \( 0 \leq a < q^{m-1} \). By Lemma 1 counting the multiplicities of the classes of lines through \([P]^{(1)}\) in \( \Pi^{(1)} \), we get

\[
k \geq a + nq^{m-1}(q+1) \geq nq^{m-1}(q+1).
\]

If for all point classes \([P]^{(1)}\), \( \mathcal{R}([P]^{(1)}) \geq q^{m-1} \), we have

\[
k \geq q^{m-1}(q^2 + q + 1) > q^m(q+1) \geq nq^{m-1}(q+1).
\]

In order to demonstrate the existence of \((nq^{m-1}(q+1), n)\)-blocking multisets for every \( n = 1, \ldots, q \), fix a line \( l \) in \( \text{PHG}(R^3_{1}) \). Assign multiplicity 1 to the following points:

- all points on \( l \);
- for every \([P]^{(1)}\) having a nonempty intersection with \( l \), the points of \( n-1 \) lines from the affine Hjelmslev plane induced on \([P]^{(1)}\) that are parallel to \([P]^{(1)} \cap l \).

Clearly, every line from \([l]^{(1)}\) meets \( l \) in \( q \geq n \) points. Every line not in \([l]^{(1)}\) meets some point class \([X] \) on \([l] \), and therefore meets \([l] \) in a single point and each one of the additional \( n-1 \) lines in the affine Hjelmslev plane induced on \([X]^{(1)} \) (cf. Fact 3).

Let \( \mathcal{R} \) be a \((k, n)\)-blocking multiset in \( \text{PHG}(R^3_{1}) \) with \( k = nq^{m-1}(q+1) \) for some \( n < p = \text{char } \mathbb{F}_q \). By Lemma 1 and 2, we can assume that if for the point class \([P]^{(1)} \) one has \( \mathcal{R}([P]^{(1)}) > 0 \) then \( \mathcal{R}([P]^{(1)}) \geq q^{m-1} \). Hence there exists a neighbour class of points, \([X]^{(1)} \) say, with \( \mathcal{R}([X]^{(1)}) = 0 \). By 2, \( \mathcal{R}([l]^{(1)}) = nq^{m-1} \) for every 1-neighbour class of lines \([l]^{(1)} \) through \([X]^{(1)} \). Moreover, \( \mathcal{R}(s) = n \) for every line \( s \) with \( s \cap [X]^{(1)} \neq \emptyset \).

By Lemma 4 we can define a multiset \( \mathcal{J} \) in \((\mathcal{P}^{(1)}, \mathcal{L}^{(1)}, J^{(1)}) \cong \text{PHG}((R/R\theta)^3_{R/R\theta})\) by

\[
\mathcal{J} : \begin{cases} \mathcal{P}^{(1)} & \to N_0 \\ [X]^{(1)} & \to \mathcal{R}([X]^{(1)})/q^{m-1}. \end{cases}
\]

For every 1-neighbour class of lines \([l]^{(1)} \) in \( \text{PHG}(R^3_{1}) \), we have

\[
\mathcal{J}([l]^{(1)}) = \frac{1}{q^{2(m-1)}} \sum_{l' \in [l]^{(1)}} \mathcal{R}(l'),
\]

since every point is incident with \( q^{m-1} \) lines in \([l]^{(1)} \) and thus 3 implies \( \mathcal{J}([l]^{(1)}) \geq n \). This gives in turn that \( \mathcal{J} \) is an \((n(q+1), n)\)-blocking set in the projective plane \((\mathcal{P}^{(1)}, \mathcal{L}^{(1)}, J^{(1)}) \cong \text{PG}(2, q) \). We complete the proof by applying once again Theorem 5 from [15].

**Remark 1.** It is impossible to replace the statement in this theorem that \( \mathcal{R}^{(1)}/q^{m-1} \) is the sum of lines by the stronger \( \mathcal{R}^{(i)}/q^{m-i} \) is the sum of lines for some \( i > 1 \). For instance, consider the following example in \( \Pi = \text{PHG}((Z_p)^3_{Z_p}) \) which is easily generalized to the geometry over chain rings of arbitrarily high index of nilpotency. Let \( \mathcal{R} \) be the \((24, 2)\)-blocking set containing the following points \((x_1, x_2, x_3)\):

- all points from the line \( x_1 = 0 \);
- all points on the line $x_2 = 0$ that are not in the point class $[P]^{(1)}$, where $P = (0, 0, 1)$;
- all points on the line $X_1 + 2x_2 + 2x_3 = 0$ that are in the point class $[P]^{(1)}$.

It is a straightforward check that $\frac{1}{2} \mathbb{R}^{(2)}$ is not the sum of two lines in $\Pi^{(2)}$.

**Corollary 1.** Let $\mathbb{R}$ be a $(q^{m-1}(q + 1), 1)$-blocking set in $\Pi = \text{PHG}(R_{2}^{3})$, where $R$ is a chain ring with $|R| = q^{m}$, $R/\text{rad} R \cong \mathbb{F}_{q}$. Then $\text{supp} \mathbb{R}$ is a line in $\Pi$.

**Proof.** As in Lemma 4, the multiset $\mathbb{R} = \frac{1}{q} \mathbb{R}^{(m-1)}$ is a $(q^{m-2}(q + 1), 1)$-blocking set and hence its support is a line in $\Pi^{(m-1)}$, $[\mathbb{L}]^{(m-1)}$ say. The nonempty sets $\text{supp} \mathbb{R} \cap [P]^{(m-1)}$ are line segments having the same direction as $l$. Finally, considering the factor geometry $(\mathbb{P}(1), \mathbb{L}(1), \mathbb{J}(1))$ defined for $l$ (Fact 2), we prove by induction on $m$ that these line segments are collinear with the same line in $\Pi$. □

**Theorem 3.** Let $R$ be a chain ring with $|R| = q^{m}$, $R/\text{rad} R \cong \mathbb{F}_{q}$, where $q^{m}$ is a perfect square. Let there exist a subring $S$ of $R$ that is a chain ring with $|S| = \sqrt{q^{m}}$ and such that $R$ is free over $S$. Then the multiset $\mathbb{R}$ defined by

$$\mathbb{R}(P) = \begin{cases} 1 & \text{if } P \text{ is a point from } \text{PHG}(S_{2}^{3}), \\ 0 & \text{otherwise}, \end{cases}$$

is a blocking set in $\text{PHG}(R_{R}^{3})$.

**Proof.** Define $\xi$ by $R = S \oplus \xi S$. This is possible by the fact that $R$ is free over $S$. Every line in $\text{PHG}(R_{R}^{3})$ can be considered as the set of points $(x_1, x_2, x_3)R$ that are solutions to:

$$ax_1 + bx_2 + cx_3 = 0,$$

where $a, b, c \in R$ and at least one of them is a unit. We have to show that there exists a solution to (4) with $x_1, x_2, x_3 \in S$ such that at least one of the $x_i$’s is a unit. Without loss of generality, we set $a = 1$, $b = s_1 + \xi t_1$, $c = s_2 + \xi t_2$. By (4), we get $x_1 + s_1x_2 + s_2x_3 + \xi(t_1x_2 + t_2x_3) = 0$, which is equivalent to

$$x_1 + s_1x_2 + s_2x_3 = 0$$

$$t_1x_2 + t_2x_3 = 0$$

If we assume that $Rt_1 = R\theta^i$, $Rt_2 = R\theta^j$ with $0 \leq i \leq j$, then the submodule of $R^3$ generated by $(1, s_1, s_2)$ and $(0, t_1, t_2)$ is of shape $(m, m - i, 0)$. In other words, the submodule in question is isomorphic to $R/N^m \oplus R/N^{m-i} \oplus R/N^0$, where $N = \text{rad} R$ and $N^0 = R$. The dual submodule is of shape $(m, i, 0)$ and, therefore, has a free rank 1 submodule (cf. 13). This implies the existence of a solution with the required property. Another possibility is to set $x_3 = 1$ (note that $j \geq i$) and compute $x_1$ and $x_2$ from the linear system above. □

**Corollary 2.** Let $R$ be a chain ring with $|R| = q^2$, $R/\text{rad} R \cong \mathbb{F}_{q}$, that contains a subring $S$ isomorphic to the residue field $\mathbb{F}_{q}$. Then $\text{PHG}(R_{R}^{3})$ contains a subplane $\Gamma$ isomorphic to $\text{PG}(2, q)$ and the projective multiset $\mathbb{R}$ defined by $\text{supp} \mathbb{R} = \Gamma$ is an irreducible $(q^2 + q + 1, 1)$-blocking set.

The blocking sets described in Corollary 2 were introduced in the paper 2 in a slightly different context. They are defined as the orbit of a fixed point with coordinates from the residue field under a Singer cycle of $\text{PG}(2, q)$. It turns out that the linear codes associated with these multisets can be mapped (cf. 11) to...
two weight linear codes over \( \mathbb{F}_q \). These in turn give rise to a family of strongly regular graphs with parameters
\[
v = q^6, k = q^4 - q, \lambda = q^3 + q^2 - 3q, \mu = q^2 - q.
\]

The geometric structure of these blocking sets can be easily deduced from their definition. Every point \( X \) from \( \text{supp} \mathcal{R} \) is incident with \( q + 1 \) lines of multiplicity \( q + 1 \) and on \( g(q + 1) - (q + 1) = q^2 - 1 \) lines of multiplicity 1. The number of all lines blocked by points from \( \text{supp} \mathcal{R} \) is
\[
\frac{(q^2 + q + 1)(q + 1)}{q + 1} + (q^2 + q + 1)(q^2 - 1) = q^2(q^2 + q + 1),
\]
i.e. all lines of \( \text{PHG}(R^1_R) \) are blocked by points of \( \text{supp} \mathcal{R} \). This argument provides a combinatorial proof for Corollary 2.

**Corollary 3.** Let \( R \) be a chain ring with \( |R| = p^{4r}, R/\text{rad} R \cong \mathbb{F}_{p^2r}, p \) prime. Then \( R \) has a subring \( S \) with \( |S| = p^{2r} \) and the projective multiset \( \mathcal{R} \) with \( \text{supp} \mathcal{R} = \text{PHG}(S^2S) \) is an irreducible \( (p^{2r}(p^{2r} + p + 1), 1) \) blocking set in \( \text{PHG}(R^1_R) \).

**Proof.** Set \( q = p^{2r} \). The rings \( R \) with \( |R| = q^2 \) and \( R/\text{rad} R \cong \mathbb{F}_q \) are either
- \( \mathbb{F}_q[X;\sigma]/(X^2), \sigma \in \text{Aut} \mathbb{F}_q \), or
- \( \text{GR}(q^2, p^2) \).

In the first case, \( S \) is the set of all elements of the form \( a + bX \), where \( a, b \in \mathbb{F}_{\sqrt{q}} \).
In the second case, \( S = \text{GR}(q, p^2) \). The rest follows by Theorem 2.

**Remark 2.** The blocking set \( \mathcal{R} \) from Corollary 3 is not uniquely determined by its parameters. In fact, we can construct a nonisomorphic blocking set \( \mathcal{I} \) with the same parameters as \( \mathcal{R} \) and having \( \mathcal{R}^{(1)} = \mathcal{I}^{(1)} \). We can do this as follows. Take as \( \text{supp} \mathcal{R}^{(1)} \) the points of the Baer subplane of \( (\mathcal{P}^{(1)}, \mathcal{L}^{(1)}, J^{(1)}) \). Denote the points (resp. the lines) of the Baer subplane by \( [X_i] \) (resp. \( [\ell_i] \)) \( i = 1, \ldots, q^2 + q + 1 \). We can index the points and lines of the Baer subplane in such way that \( (X_i, \ell_i) \in I \).

Now define the blocking set \( \mathcal{I} \) by
\[
\mathcal{I}(P) = \begin{cases} 
1 & \text{if } P \in [X_i] \cap \ell_i \text{ for some } i \in \{1, \ldots, q^2 + q + 1\}, \\
0 & \text{otherwise}.
\end{cases}
\]

Now it is a straightforward check that \( \mathcal{I} \) is indeed a blocking set and that every point from \( \text{supp} \mathcal{I} \) is incident with a 1-line, i.e. the blocking set is irreducible.

5. **Blocking sets with \( n = 1 \).**

In this section, we consider chain rings \( R \) with \( |R| = q^2 \), \( R/\text{rad} R \cong \mathbb{F}_q \) for some prime power \( q \). It has been pointed out already that the classification of such chain rings is known. If \( q = p^r \) there exist exactly \( r + 1 \) isomorphism classes of such rings (cf. [23], [24]). These are the Galois rings \( \text{GR}(q^2, p^2) \) of characteristic \( p^2 \) and the truncated skew polynomial rings \( \mathbb{F}_q[X;\sigma]/(X^2) \), \( \sigma \in \text{Aut} \mathbb{F}_q \) (cf. [23], [24]). The commutative rings among these are \( \text{GR}(q^2, p^2) \) and \( \mathbb{F}_q[X;id]/(X^2) \).

The blocking sets we described in the previous section have a simple structure in the sense that their image under \( \pi^{(1)} \) is a trivial blocking set in \( \text{PG}(2, q) \). On the other hand, these blocking sets are not necessarily “sums of lines”. By Corollary 1 we have that the minimal size of a blocking set with \( n = 1 \) is \( k = q(q + 1) \) and the support of such blocking set contains the points of a single line. A natural question is what is the size of the second smallest irreducible blocking set. Corollary 2 gives
a family of \((q^2 + q + 1, 1)\)-blocking sets for projective Hjelmslev planes over rings that contain a subring isomorphic to their residue field.

It is possible to construct another class of irreducible blocking sets with \(k = q^2 + q + 1, n = 1\). Consider two lines \(\ell\) and \(\ell_0\) with \(\ell \subseteq \ell_0\) and a point \(X \in \ell \setminus \ell_0\). Set

\[
\mathfrak{R}(P) = \begin{cases} 1 & \text{if } P \in (\ell_0 \setminus \{X\}) \cup \{X\} \text{ or } P \in \ell_1 \cap \{X\} \\ 0 & \text{otherwise.} \end{cases}
\]

It is readily checked that \(\mathfrak{R}\) is indeed a blocking set and that it is irreducible. This construction works for every chain ring \(R\). It turns out that there exist no other irreducible blocking sets of size \(q^2 + q + 1\) in \(\text{PHG}(R_{1\alpha})\). Moreover the blocking sets from (6) are the only blocking sets when the characteristic of the ring \(R\) is the Galois ring \(\text{GR}(q^2, p^2)\).

**Theorem 4.** Let \(\mathfrak{R}\) be an irreducible \((q^2 + q + 1, 1)\)-blocking set in \(\text{PHG}(R_{1\alpha})\), \(|R| = q^2, R/\text{rad} R \cong \mathbb{F}_q\). Then either \(\text{supp}\mathfrak{R}\) is a projective plane of order \(q\) or else \(\mathfrak{R}\) is a blocking set of the type described in (6). If \(R = \text{GR}(q^2, p^2)\), then \(\mathfrak{R}\) is of the type described in (6).

**Proof.** Let \(\mathfrak{R}\) be an irreducible \((q^2 + q + 1, 1)\)-blocking set in \(\text{PHG}(R_{1\alpha})\). By the irreducibility, \(\text{supp}\mathfrak{R}\) does not contain a complete line. Assume \([X]\) is a neighbour class of points with \(0 \leq \mathfrak{R}([X]) < q\). Then each class of parallel lines in the affine plane induced on \([X]\) contains an empty line (cf. Fact 4). Hence each class of neighbour lines through \([X]\) contains at least \(q\) points outside \([X]\). This implies

\[
q^2 + q + 1 = |\mathfrak{R}| = \mathfrak{R}([X]) + \sum_{[\ell]} \mathfrak{R}([\ell] \setminus [X]) 
\geq \mathfrak{R}([X]) + q(q + 1),
\]

and we have \(\mathfrak{R}([X]) \leq 1\). In a similar way, we can prove that \(\mathfrak{R}([X]) \leq q + 1\). Indeed assume that \(\mathfrak{R}([X]) \geq q + 2\). Then there is a class, \([Y]\) say, with \(\mathfrak{R}([Y]) = 0\). Now count the multiplicities of the points in the neighbour classes of lines through \([Y]\), to get the contradiction \(|\mathfrak{R}| \geq q^2 + q + 2\). Thus we have proved that a neighbour class of points in \(\text{PHG}(R_{1\alpha})\) has multiplicity 0, 1, \(q\), or \(q + 1\).

We consider two cases:

1. **There exists a neighbour class of points \([X]\) with \(\mathfrak{R}([X]) = 0\).** Consider a neighbour class of lines, \([\ell]\) say, incident with \([X]\). In order to block all lines in \([\ell]\), we need at least \(q\) points. On the other hand, \(\mathfrak{R}([\ell]) \leq q + 1\). Therefore, every neighbour class of lines through \([X]\) has multiplicity \(q\) except for one class of multiplicity \(q + 1\). Furthermore, using Fact 4 together with the fact that a blocking set with \(q + 1\) or \(q + 2\) points in a projective plane of order \(q\) does necessarily contain a line, we get that the \(q\) points in every line class through \([X]\) are contained in the same neighbour class. For the line class with \(q + 1\) points we can clam that \(q\) of them are neighbours while the \((q + 1)\)st might be in a different class. The \(q + 1\) neighbour classes of points containing at least \(q\) points from \(\text{supp}\mathfrak{R}\) must be collinear in the factor plane. Otherwise, there is a line in the factor plane which is not blocked.

Denote by \([\ell_0]\) the neighbour class of lines incident with the \(q + 1\) point classes of multiplicity at least \(q\). Consider a neighbour class of points \([X]\) with \(\mathfrak{R}([X]) = q\). Assume there exist a point \(X_1\) in \(\text{supp}\mathfrak{R}\cap[X]\) and a point \(X_2\) in \([X]\setminus\text{supp}\mathfrak{R}\) such that the line \((X_1, X_2)\) is in \([\ell_0]\). There exist at least \(q\) lines through \(X_2\) (not in \([\ell_0]\)) that have no points from \(\text{supp}\mathfrak{R}\cap[X]\). This is a contradiction since we cannot block
them by one point outside $[X]$. This implies that the points from $[X] \cap \text{supp } \mathcal{R}$ are collinear and the $q$ lines defined by them are in $[\ell_0]$. Similarly, we see that the same is true for a class with $q + 1$ points: $q$ of them are incident with $q$ neighbour lines from $[\ell_0]$ and the $(q + 1)$-st point is arbitrary.

Now consider incidence structure defined on $[\ell_0]$ (cf. Fact \ref{fact:hyperoval}). It is a isomorphic to the projective plane without a point, which we denote by $P_\infty$. Define a multiset $\mathcal{S}_l$ by

$$
\mathcal{S}_l(P) = \begin{cases} 
1 & \text{if } P = s \cap [X], \mathcal{R}(s \cap [X]) \neq 0, \\
0 & \text{if } P = s \cap [X], \mathcal{R}(s \cap [X]) = 0 \text{ or } P = P_\infty.
\end{cases}
$$

The multiset $\mathcal{S}_l$ is a $(q + 1, 1)$ or $(q + 2, 1)$-blocking set in a projective plane of order $q$ and hence contains a line. This implies that $\text{supp } \mathcal{R}$ either contains a complete line, in which case the blocking set is reducible, or else is of the type described in \ref{fact:hyperoval}.

2. For every neighbour class of points $\mathcal{R}([X]) = 1$. Consider a neighbour class of lines $[\ell]$ and the multiset $\mathcal{S}_l$ from \ref{fact:hyperoval}. Since $\mathcal{S}_l$ is $(q + 1, 1)$ blocking set in a projective plane of order $q$ the points from the classes $[X]$ with $([X], [\ell]) \in J^{(1)}$ are collinear. Therefore the lines have multiplicity 1 or $q + 1$ and the points of $\text{supp } \mathcal{R}$ together with the $(q + 1)$-lines form a projective plane of order $q$.

Now we are going to prove that the projective Hjelmslev plane over $R = \text{GR}(q^2, p^2)$ cannot contain a subplane isomorphic to a projective plane of order $q$. Assume otherwise and denote by $\Delta$ a subplane isomorphic to a projective plane of order $q$ contained in $\Pi = \text{PHG}(R^2_R)$. Without loss of generality $\Delta$ contains the points $(1, 0, 0)$, $(1, 1, 0)$, $(1, 0, 1)$ and $(1, 1, 1)$. Then it contains also $(0, 1, 0)$, $(0, 0, 1)$, and $(0, 1, -1)$. Removing the neighbour class $[\ell]$, where $l$ is given by $x_1 = 0$ we get an affine Hjelmslev plane $\overline{\Pi}$ isomorphic to $\text{AHG}(R^2_R)$. The points of $\Delta \setminus \ell$ form a subplane $\overline{\Xi}$ isomorphic to an affine plane of order $q$. Moreover, the points $(0, 0)$, $(1, 0)$, $(0, 1)$ and $(1, 1)$ are in $\overline{\Xi}$. Denote the two coordinates in $\overline{\Pi}$ by $y$ and $z$. Then the points in $\overline{\Xi}$ on the line $y = 0$ are $(0, a_i), i = 0, \ldots, q - 1$, where $a_0 = 0$, $a_1 = 1$ and $a_i \neq a_j (\text{rad } R)$, for $i \neq j$, in particular all $a_i$ are not invertible. The points on the line $y = 1$ are $S = \{1, a_i\}, i = 0, \ldots, q - 1$. The intersection point of $y = 0$ and the line through $(1, a_j)$ parallel to $(0, a_i)$ is $(0, a_i + a_j)$, i.e. the set $S = \{0, a_1, a_2, \ldots, a_{q-1}\}$ is closed under addition. Similarly, all points on the line $y = z$ are $(a_i, a_i), i = 0, \ldots, q - 1$. The intersection point of $y = 0$ and the line through $(a_j, a_j)$ parallel to $(0, a_i)$ is $(0, a_i a_j)$. Hence $S$ is also closed under multiplication and $S$ is a subring of $R$. But is also a subfield since $S \cap \text{rad } R = \{0\}$. This is a contradiction since $\text{char } R = p^2$.

 Remark 3. The construction from \ref{fact:hyperoval} can be generalized. Fix a line $\ell$ in $\text{PHG}(R^2_R)$ and consider the factor geometry from Fact \ref{fact:hyperoval} defined for the line class $[\ell]$. First assume that $q$ is even. Fix a hyperoval $\{\ell_i \cap [X_i] \mid i = 1, \ldots, q + 1\} \cup P_\infty$, where $[X_i]$ are the points incident with $[\ell]$ in $\Pi^{(1)}$. We can choose $\ell$ to be an external line to this hyperoval and the points $X_i$ to be incident (under $I$) with $\ell$. For every $s = 0, \ldots, q$, define the multiset $\mathcal{R}$ by

$$
\mathcal{R}(P) = \begin{cases} 
1 & \text{if } P \in (\ell \setminus \bigcup_{i=1}^s [X_i]) \cup \{X_0, \ldots, X_s\} \text{ or } P \in \bigcup_{i=0}^s (\ell_i \cap [X_i]) \\
0 & \text{otherwise}.
\end{cases}
$$
The multiset \( \mathcal{K} \) is an irreducible \((q^2 + q + 1 + s, 1)\)-blocking set if every point \( \ell \cap [X_i] \) in the factor geometry is incident with at least two external lines to the hyperoval. This is certainly true for \( q > 2 \).

In the case \( q \) odd, we take \( \{\ell_i \cap [X_i] \mid i = 1, \ldots, q\} \cup P_\infty \) to be an oval; \( \ell \) is again an external line to the oval and \( X_i \) are the same as for \( q \) even. The multiset \( \mathcal{K} \) defined by \([5]\) is again an irreducible \((q^2 + q + 1 + s, 1)\)-blocking set if every point \( \ell \cap [X_i] \) in the factor geometry is incident with at least two external lines to the oval, which is true for \( q > 3 \).

Note that irreducible \((8, 1)\)- and \((9, 1)\)-blocking sets for \( q = 2 \) and irreducible \((14, 1)\), \((15, 1)\) and \((16, 1)\)-blocking sets for \( q = 3 \) can easily be constructed with other methods.

6. Codes from Blocking Sets

It is well-known that with every multiset \( \mathcal{K} \) in the projective geometry \( PHG(R^3_{R \ell}) \), \( R \) a chain ring, we can associate a class of isomorphic (left) linear codes over \( R \). Every representative in this class is called a code associated with \( \mathcal{K} \). A code associated with \( \mathcal{K} \) is obtained as the module generated by the rows of a matrix with columns the points of supp \( \mathcal{K} \) taken with the corresponding multiplicities (cf. [11]).

For the sake of simplicity, we consider chain rings \( R \) with \( |R| = q^2, R/\text{rad} R \cong \mathbb{F}_q \).

Let \( \mathcal{K} \) be a multiset in \( PHG(R^3_{\ell}) \) with spectrum \( (a_{i_0, i_1}) \), and let \( C = C_{\mathcal{K}} \) be a code associated with \( \mathcal{K} \). \( C \) can be mapped to a code over a \( q \)-letter alphabet (which we take as \( \mathbb{F}_q \)) in the following way. Let \( T = \{\gamma_0, \gamma_1, \ldots, \gamma_{q-1}\} \) be a set of elements of \( R \) no two of which are congruent modulo \( \text{rad} R = R\theta \). Without loss of generality, \( \gamma_0 = 0, \gamma_1 = 1 \). Every element \( r \in R \) is represented uniquely as \( r = r_0 + r_1 \theta \), where \( r_0, r_1 \in T \).

Set

\[
\varphi: \begin{cases}
T & \to \mathbb{F}_q \\
\gamma & \to (\gamma + \text{rad} R)\sigma
\end{cases},
\]

where \( \sigma : R/\text{rad} R \to \mathbb{F}_q \) is an isomorphism. Every element \( r = r_0 + r_1 \theta \in R \) can be mapped to a \( q \)-tuple over \( \mathbb{F}_q \) by

\[
\psi(r) = (r_1, r_0) \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 0 & 1 & \cdots & \gamma_{q-1} \end{pmatrix}.
\]

This map can be extended to \( \mathbb{F}^k_q \) in a natural way by

\[
\psi((c_1, c_2, \ldots, c_k)) = (\psi(c_1), \psi(c_2), \ldots, \psi(c_k)).
\]

Thus every (right) linear code \( C \) over the chain ring \( R \) can be mapped to a code \( \psi(C) \) over a \( q \)-ary alphabet. Generally, \( \psi(C) \) is not linear, but it can be made linear if \( R \) contains a subring isomorphic to the residue field \( \mathbb{F}_q \) (cf. [11]).

Let \( \mathcal{K} \) be a multiset of cardinality \( k \) in \( PHG(R^3_{R \ell}) \) and let \( C \) be a code associated with \( \mathcal{K} \). It is easily verified that a line \( l \) produces words of the following nonzero weights in \( \psi(C) \):

- if \( l \) is of type \((i_1, i_2), i_1 + i_2 \neq k, \)
  - \( q^2 - q \) words of (Hamming) weight \( qi_1 + (q - 1)(k - i_1 - i_2); \)
  - \( q - 1 \) words of weight \( q(k - i_1 - i_2); \)
- if \( l \) is of type \((i_1, i_2), i_1 + i_2 = k, q - 1 \) words of weight \( qi_1. \)

Below we list the parameters of some codes obtained from the blocking sets defined so far. In the case of codes over finite fields, it is typical to construct codes from the complement of a blocking set. However, in the case of codes over chain rings...
rings, the blocking sets themselves can yield interesting codes. In all examples below, \( R \) is a chain ring with \( q^2 \) elements and residue field of order \( q \). We set also \( \Pi = \text{PHG}(R_{H_1}) \).

1) \( \mathcal{R}_1 \): a projective \((q^2(q + 1), q)\)-blocking set for which \( \mathcal{R}_1^{(1)} \) is \( q^2 \chi_l \), where \( \chi_l \) is the characteristic function of a fixed line from \( \Pi^{(1)} \).

2) \( \mathcal{R}_2 = 1 - \mathcal{R}_1 \).

3) \( \mathcal{R}_3 \): a projective \((q^2(q + 1), q)\)-blocking set for which the support of \( \mathcal{R}_3^{(1)} \) consists of \( q \) different lines having a common point.

4) \( \mathcal{R}_4 = 1 - \mathcal{R}_3 \).

5) \( \mathcal{R}_5 \): the \((q^2 + q + 1, 1)\)-blocking set from Corollary \( \mathcal{R}_6 \).

| \# | type of a line | weight of the codewords |
|---|---|---|
| \( \mathcal{R}_1 \) | \((q^4 + q^3, q^2, q^2 - q^2)\) | \(q^4 + q\) \(q^4 - q\) \(q^2\) \(q^2\) |
| \( \mathcal{R}_2 \) | \((q^2, q^2, q^2 - q^2)\) | \(0\) \(0\) \(q^2\) \(q^2\) |
| \( \mathcal{R}_3 \) | \((q^4 + q^3, q^3, q^2 - q^2)\) | \(q^4 + q\) \(q^4 - q\) \(q^3\) \(q^2\) |
| \( \mathcal{R}_4 \) | \((q^2, q^2, q^2 - q^2)\) | \(0\) \(q^2\) \(q^2\) \(q^2\) |
| \( \mathcal{R}_5 \) | \((q^2, q^2, q^2 - q^2)\) | \(q^2\) \(q^2\) \(q^2\) \(q^2\) |
| \( \mathcal{R}_6 \) | \((q^2 + q^2, q, q^2 - q^2)\) | \(q + 1\) \(0\) \(q^2\) \(q^2\) |
| \( \mathcal{R}_7 \) | \((q^2 + q, q^2, q^2 - q^2)\) | \(1\) \(q\) \(q^2\) \(q^2\) |
| \( \mathcal{R}_8 \) | \((q^2 + q^2 - q^2 - q^2)\) | \(q^2 + q\) \(q^2 - q^2\) \(q^2\) \(q^2 - q^2\) |
| \( \mathcal{R}_9 \) | \((q^2 + q^2 - q^2 - q^2)\) | \(q^2\) \(q^2\) \(q^2\) \(q^2\) |
| \( \mathcal{R}_{10} \) | \((q^2 + q^2 - q^2 - q^2)\) | \(1\) \(q - 1\) \(q^2 + q\) \(q^2 + q^2 - q^2\) |
| \( \mathcal{R}_{11} \) | \((q^2 + q^2 + q, q^2, q^2 - q^2)\) | \(q + q^2\) \(q^2 - q^2\) \(q^2\) \(q^2 - q^2\) |
| \( \mathcal{R}_{12} \) | \((q^2 + q^2 - q^2 - q^2)\) | \(q^2\) \(q^2\) \(q^2\) \(q^2\) |
6) $\mathcal{R}_6 = 1 - \mathcal{R}_5$.
7) $\mathcal{R}_7$: the $(q(q + \sqrt{q} + 1), 1)$-blocking set from Corollary 1.

The codes obtained from $\mathcal{R}_1$ and $\mathcal{R}_2$ are MacDonald codes. For the codes obtained from $\mathcal{R}_1$ this was known from [11]. The codes $\mathcal{R}_3$ (resp. $\mathcal{R}_4$) though obtained from blocking sets with the same parameters as $\mathcal{R}_1$ (resp. $\mathcal{R}_2$) give rise to codes with a smaller minimum distance. The codes $\mathcal{R}_5$ and $\mathcal{R}_6$ lead to 2-weight codes and hence to strongly regular graphs [2]. The blocking set from Corollary 3 has the same spectrum as the blocking set $\mathcal{R}_7$ and therefore yields a code over $\mathbb{F}_q$ with the same parameters.

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