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On the regularity of the composition of diffeomorphisms. (English) [Zbl 1293.58004]

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In this paper the group of diffeomorphisms $\mathcal{D}$ of a smooth manifold $M$ is considered. In various settings, the space of diffeomorphisms of a given manifold with prescribed regularity turns out to be an (infinite-dimensional) topological group with the group operation given by the composition. For such a group of diffeomorphisms, in order to be a Lie group, the composition and the inverse map have to be $C^\infty$-smooth. A straightforward formal computation shows that the differential of the left translation $L_\varphi : \varphi \mapsto \psi \circ \varphi$ of a diffeomorphism $\varphi$ by a diffeomorphism $\psi$ in direction $h : M \to TM$ can be formally computed to be $(dL_\varphi)(h)(x) = (d\varphi(x)\psi)(h(x))$, $x \in M$, and hence involves a loss of derivative of $\psi$. As a consequence, for a space of diffeomorphisms of $M$ to be a Lie group, it is necessary that they are $C^\infty$-smooth and hence such a group cannot have the structure of a Banach manifold, but only of a Fréchet manifold. It is well known that the calculus in Fréchet manifolds is quite involved as the classical inverse function theorem does not hold. However, in many situations, one has to consider diffeomorphisms of Sobolev type. Let $\text{Diff}^s(M)$ denotes the set of all orientation-preserving $C^1$ smooth diffeomorphisms of $M$. For any integer $s$ with $s > \frac{n}{2} + 1$ define $\mathcal{D}^s(M) := \{ \varphi \in \text{Diff}^1(M) \mid \varphi \in H^s(M, M) \}$ where $H^s(M) = H^s(M, M)$ denotes the set of all maps $M \to M$ of Sobolev class $H^s$. Then the group $\mathcal{D}^s$ is a smooth Hilbert manifold, but the group operations are not smooth. It is known that the composition $H^s(M) \times \mathcal{D}^s(M) \to H^s(M)$ is continuous, whereas $H^{s+\tau}(M) \times \mathcal{D}^s(M) \to H^s(M)$ and $\text{inv} : \mathcal{D}^{s+\tau}(M) \to \mathcal{D}^s(M)$, $\varphi \mapsto \varphi^{-1}$, are mappings of class $C^\tau$. Various versions of these statements can be found in the literature, however mostly without proofs.

For $M$ a closed manifold or the Euclidean space $\mathbb{R}^n$, the authors present a detailed proof of regularity properties of the composition of $H^s$-regular diffeomorphisms of $M$ for $s > \frac{n}{2} \dim M + 1$:

Theorem 1. For any $r \in \mathbb{Z}_{\geq 0}$ and any integer $s$ with $s > n/2 + 1$, the maps $\mu : H^{s+r}(\mathbb{R}^n, \mathbb{R}^d) \times \mathcal{D}^s(\mathbb{R}^n) \to H^s(\mathbb{R}^n, \mathbb{R}^n)$, $(u, \varphi) \mapsto u \circ \varphi$, and $\text{inv} : \mathcal{D}^{s+\tau}(\mathbb{R}^n) \to \mathcal{D}^s(\mathbb{R}^n)$, $\varphi \mapsto \varphi^{-1}$, are $C^\tau$-maps.

Theorem 2. Let $M$ be a closed oriented manifold of dimension $n$, $N$ a $C^\infty$-manifold, and $s$ an integer satisfying $s > n/2 + 1$. Then for any $r \in \mathbb{Z}_{\geq 0}$, the maps $\mu : H^{s+r}(M, N) \times \mathcal{D}^s(M) \to H^s(M, N)$, $(f, \varphi) \mapsto f \circ \varphi$, and $\text{inv} : \mathcal{D}^{s+\tau}(M) \to \mathcal{D}^s(M)$, $\varphi \mapsto \varphi^{-1}$, are both $C^\tau$-maps.

There is no other proof of Theorem 1 available in the literature. A complete, quite involved proof of the first statement of Theorem 2 can be found in [H. Omori, Infinite-dimensional Lie groups. Providence, RI: American Mathematical Society (1997; Zbl 0871.58007)].

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58B10 Differentiability questions for infinite-dimensional manifolds
58D15 Manifolds of mappings
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