A FEW REMARKS ON THE OCTOPUS INEQUALITY AND ALDOUS' SPECTRAL GAP CONJECTURE

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A conjecture by D. Aldous, which can be formulated as a statement about the first nontrivial eigenvalue of the Laplacian of certain Cayley graphs on the symmetric group generated by transpositions, has been recently proven by Caputo, Liggett, and Richthammer. Their proof is a subtle combination of two ingredients: a nonlinear mapping in the group algebra which permits a proof by induction, and a quite hard estimate named the octopus inequality. In this article we present a simpler and more transparent proof of the octopus inequality, which emerges naturally when looking at the Aldous’ conjecture from an algebraic perspective.

Key Words: Aldous’ conjecture; Cayley graph; Kazhdan constant; Spectral group; Symmetric group.

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1. INTRODUCTION

Let $G$ be a finite group with complex group algebra $\mathbb{C}G$. Given a representation $R$ of $G$ on the $d$-dimensional complex vector space $V$, and given an element $w = \sum_{g \in G} w_g g$ of the group algebra $\mathbb{C}G$, we define the representation Laplacian $\Delta_G(w, R)$ as the linear operator on $V$ given by

$$\Delta_G(w, R) := \sum_{g \in G} w_g [I_V - R(g)] = (I_V - R)(w) \quad w_g \in \mathbb{C}, \quad (1.1)$$

where $I_V$ is the identity on $V$ and $I_V$ is the trivial representation given by $I_V(g) = I_V$ for each $g \in G$. The support of $w$ is defined as

$$\text{supp } w := \{ g \in G : w_g \neq 0 \} \quad w \in \mathbb{C}G.$$ 

As a justification for using the term “Laplacian,” we observe that, if $L$ is the left regular representation of $G$ and $w = \sum_{q \in Q} q$, where $Q \subset G$ is a symmetric generating set, then $\Delta_G(w, L)$ is the standard (unnormalized) Laplacian of the Cayley graph of $(G, Q)$. More generally, if $H$ is a subgroup of $G$, and $I_H^G$ is the trivial representation
of $H$ induced to $G$, then $\Delta_G(w, I^G_{\alpha})$ is the Laplacian of the Schreier graph of $(G, H, Q)$.

To the pair $(w, R)$, we can also associate a spectral gap, denoted by $\psi_G(w, R)$, which is essentially the first nontrivial eigenvalue of $\Delta_G(w, R)$ (see Section 2 for a precise definition). We then define the spectral gap of $w$, by minimizing over representations

$$\psi_G(w) := \inf \left\{ \psi_G(w, R) : R \in \text{rep}_f(G) \right\},$$

(1.2)

where $\text{rep}_f(G)$ stands for the set of all the (equivalence classes of) finite-dimensional representations of $G$ over the field of complex numbers.

Around 1992 a conjecture was formulated by David Aldous about a problem which “arose in conversation with Persi Diaconis” [1]. This conjecture, which became known as Aldous’ spectral gap conjecture, originates in a probabilistic framework and it asserts that two distinct Markov chains, namely the random walk and the interchange process have generators with the same spectral gap. The reader is referred to [7] for details on the probabilistic angle. It is not too difficult to realize that this conjecture is actually a statement about the representations of the symmetric group $S_n$ (see [9] or [6]). More precisely, we have the following theorem.

**Theorem 1.1** (Aldous’ spectral gap conjecture, proven in [7]). Let $S_n$ be the symmetric group, and let $w = \sum_{g \in S_n} w_g g \in \mathbb{C}S_n$. Assume as follows:

(a) $w_g \in \mathbb{R}$ and $w_g \geq 0$ for each $g \in S_n$;
(b) supp $w$ is a subset of the set of all transpositions of $S_n$;
(c) supp $w$ generates $S_n$.

Then we have

$$\psi_G(w) = \psi_G(w, D_n),$$

(1.3)

where $D_n$ is the $n$-dimensional defining representation of $S_n$, associated with the natural action of $S_n$ on the set $\{1, 2, \ldots, n\}$.

Another way of stating this result (the spectral graph theory angle) is as follows: let $\mathcal{G}$ be a finite weighted graph with vertex set $\{1, 2, \ldots, n\}$ and symmetric weights $w_{ij} \geq 0$. There is a natural way to associate a weighted Cayley graph $\text{Cay}(\mathcal{G})$ to our original graph $\mathcal{G}$: any edge $e = \{i, j\}$ of $\mathcal{G}$ can be identified with a transposition $(ij)$ of the symmetric group $S_n$. Consider then the Cayley graph with vertex set equal to $S_n$ and such that each edge $\{\pi, (ij)\pi\}$, where $\pi$ is a permutation of $S_n$, carries a weight $w_{ij}$. It is easy to show that the spectrum of the Laplacian of $\mathcal{G}$ is a subset of the spectrum of the Laplacian of $\text{Cay}(\mathcal{G})$. Theorem 1.1 is equivalent to the striking assertion that if $\mathcal{G}$ is connected (by edges with positive weight), then the two Laplacian operators have the same lowest nontrivial eigenvalue (see [6] for details).

Aldous’ conjecture has been proven in several special cases by different authors in a series of articles [2], [6], [9], [10], [12], [17], [18], [22] spanning about 25
Unfortunately none of these papers contains tools which, with maybe some hammering and bending, can be forged to tackle the general case. A proof which works in the general case has required, in fact, a subtle combinations of two new ingredients: (a) a (nonlinear) “reduction mapping” $\vartheta : \mathbb{C}S_n \to \mathbb{C}S_{n-1}$ which enables a proof by induction on $n$; (b) an inequality, named the octopus inequality by the authors of [7], who were “inspired by its tentacular nature.” Quite amazingly both elements (a) and (b) appeared almost simultaneously in the preprint versions of [8] and [7], but the octopus inequality was left as a conjecture in [8] (it was actually proved in some particular cases), while it was proved in [7].

The main objective of this article is to present (Section 4) a simpler and more transparent proof of the octopus inequality which hopefully can unravel its tentacular nature. Our approach emerges naturally if one looks at paper [7] with algebraically tinted glasses. For this reason, we also include in Section 3 a more or less self-contained “algebraic proof” of Aldous’ conjecture, showing how this conjecture follows from the octopus inequality. The approach we follow in Section 3, is not really different from what is done in [7], since it uses the same reduction map $\vartheta$. It is more like the same proof, as it appears from a different perspective. We believe, however, that this perspective may be useful for a better understanding of what is going on and for investigating possible generalizations.

In the last section, we discuss the connection with a result by M. Kassabov [16] where Aldous’ conjecture is proven$, using a completely different approach, for every finite Coxeter group $G$ in the special case of $w = \sum_{q \in Q} q$, where $Q$ is a Coxeter generating set for $G$. Kassabov also shows that, in such a case, the analog of identity (1.3) holds also for the Kazhdan constant. At the end of this section we also publicize an interesting conjecture by P. Caputo, which can be seen as a generalization of Theorem 1.1.

2. THE REPRESENTATION LAPLACIAN AND ITS SPECTRAL GAP

Let $G$ be a finite group. If $R$ is a finite-dimensional representation$\dagger$ of $G$ on the complex vector space $V$, we will sometimes write that $(R, V)$ is a representation of $G$. $R$ extends to a representation of the group algebra $\mathbb{C}G$ by letting, for $w = \sum_{g \in G} w_g g$,

$$R(w) = R\left(\sum_{g \in G} w_g g\right) := \sum_{g \in G} w_g R(g) \quad w_g \in \mathbb{C}.$$  

The degree of $R$ (that is the dimension of $V$) is denoted with $f_R$. We denote with $I$ the one-dimensional trivial representation, so that

$$I(w) = \sum_{g \in G} w_g \in \mathbb{C}.$$  

$\dagger$Some of these articles even predate the “official” formulation of the conjecture, as it is often the case.

$\ddagger$Even if it is not explicitly mentioned.

$\S$Representations are finite-dimensional throughout this article, with the exception of Section 5.
Since $G$ is finite, any representation $(R, V)$ can be realized as a *unitary* representation. $V^G$ stands for the subspace of all invariant vectors

$$V^G := \{ v \in V : R(g)v = v, \ \forall g \in G \}.$$  

We have that $V^G \neq \{0\}$ if and only if $I \subseteq R$, where, in general, $S \subseteq T$ means that $S$ is a subrepresentation of $R$. An eigenvalue $\lambda$ of the representation Laplacian $\Delta_G(w, R)$, defined in (1.1), will be called *trivial* if its corresponding eigenspace $v \in V^G$. Clearly, if $\lambda$ is trivial, then $\lambda = 0$. The opposite implication will be discussed in Proposition 2.1. We introduce a canonical involution in the group algebra $CG$ as

$$w = \sum_{g \in G} w_g g \rightarrow w^* := \sum_{g \in G} w_g g^{-1},$$

and we denote the set of all symmetric elements as

$$CG^{(s)} := \{ w \in CG : w = w^* \}.$$  

We will be also interested in a subset of $CG^{(s)}$, i.e., the subset of all *positive symmetric* elements, which we denote by

$$CG^{(+)} := \{ w \in CG : w_g \in \mathbb{R}, \ w_g \geq 0, \ w_g^{-1} = w_g^{-1} \ \forall g \in G \}.$$  

In the following proposition, we summarize a few elementary properties of the spectrum of the representation Laplacian.

**Proposition 2.1.** Let $R$ be a representation of $G$ on $V$, and let $w \in CG$. Then:

1. If $w \in CG^{(s)}$, then $\Delta_G(w, R)$ has real eigenvalues;
2. If $w \in CG^{(+)}$, then $\Delta_G(w, R)$ has (real) non-negative eigenvalues;
3. Let $w \in CG^{(s)}$ and assume that supp $w$ generates $G$. If $0$ is an eigenvalue of $\Delta_G(w, R)$, then its eigenspace coincides with $V^G$, and hence it is necessarily trivial.

**Proof.** Let $\langle \cdot, \cdot \rangle$ be an inner product on $V$ which makes $R$ a unitary representation. If $w \in CG^{(s)}$, then we have

$$R(w) = R(w^*) = \sum_{g \in G} w_g R(g^{-1}) = \sum_{g \in G} w_g R(g)^* = R(w)^*,$$

i.e., $R(w)$ is a self-adjoint linear operator, which proves (1).

Since $R$ is unitary and the coefficients $w_g$ are real and non-negative, from (1.1) we obtain for $v \in V$

$$\langle \Delta_G(w, R) v, v \rangle = \sum_{g \in G} w_g \left[ \|v\|^2 - \langle R(g)v, v \rangle \right]$$

$$= \frac{1}{2} \sum_{g \in G} w_g \left[ 2\|v\|^2 - \langle (R(g) + R(g^{-1})v, v \rangle \right]$$
\[ \frac{1}{2} \sum_{g \in G} w_g \left[ 2 \|v\|^2 - \langle R(g)v, v \rangle - \langle v, R(g)v \rangle \right] \]

\[ \frac{1}{2} \sum_{g \in G} w_g \|R(g)v - v\|^2 \geq 0. \] (2.2)

Hence \( \Delta_G(w, R) \) is positive semidefinite as quadratic form, which implies (2).

For proving (3), let \( v \neq 0 \) with \( \Delta_G(w, R)v = 0 \). Then the last inequality in (2.2) is in fact an equality. Thus \( R(g)v = v \) for each \( g \in \text{supp} \, w \). Since \( \text{supp} \, w \) generates \( G \), we conclude that \( v \in V^0 \).

The inverse implication, that is if \( v \) is an invariant vector, then \( v \) belongs to the 0-eigenspace of \( \Delta_G(w, R) \) is trivial. \( \square \)

If \( w \in \mathbb{C}G^{(+)} \), we define the spectral gap of the pair \((w, R)\) as

\[ \psi_G(w, R) := \min \{ \lambda \in \text{spec} \Delta_G(w, R) : \lambda \text{ is nontrivial} \} w \in \mathbb{C}G^{(+)}, \] (2.3)

with the convention that \( \min \emptyset = +\infty \). The spectral gap of \( w \) is defined by minimizing over representations

\[ \psi_G(w) := \inf \{ \psi_G(w, R) : R \in \text{rep}_f(G) \}. \] (2.4)

By Maschke’s complete reducibility theorem, we have for each \( R \in \text{rep}_f(G) \),

\[ R \cong \bigoplus_{T \in \text{Irr}(G)} r(T) T, \] (2.5)

where \( \text{Irr}(G) \) stands for the set of all the (equivalence classes of) irreducible complex representations of \( G \), and \( r(T) \) are suitable nonnegative integers. As a consequence

\[ \text{spec} \Delta_G(w, R) = \bigcup_{T \in \text{Irr}(G) : r(T) > 0} \text{spec} \Delta_G(w, T), \] (2.6)

which implies

\[ \psi_G(w, R) = \min \{ \psi_G(w, T) : T \in \text{Irr}(G) : r(T) > 0 \}. \]

For this reason, in (2.4) we can limit ourselves to consider irreducible representations, that is,

\[ \psi_G(w) = \min \{ \psi_G(w, R) : R \in \text{Irr}(G) \}, \] (2.7)

where we have also replaced \( \inf \) with \( \min \) since \( \text{Irr}(G) \) has finite cardinality. Furthermore, because of our Definition (2.3), we have \( \psi_G(w, I) = +\infty \); hence the conditions \( R \neq I \) and \( I \not\subset R \) be freely added in (2.7) and in (2.4), respectively.

\[ ^4 \text{This happens when } R \text{ is a multiple of } I. \text{ In this case, } 0 \text{ is the only eigenvalue, and it is trivial.} \]
Let $L$ be the left regular representation of $G$. Since $L$ contains all irreducible representations, 

$$L = \bigoplus_{T \in \text{Irr}(G)} f_T T.$$  \hfill (2.8)

$L$ determines the spectral gap of $w$, that is $\psi_G(w) = \psi_G(w, L)$. Statement 3 of Proposition 2.1 implies that, if $w \in \mathbb{C}G^{(+)}$ and $	ext{supp}w$ generates $G$, then $\psi_G(w) > 0$. If $A$ is a self-adjoint linear operator on a finite dimensional vector space, we also define 

$$\lambda^*(A) := \text{max spec } A.$$ 

If $I \not\subset \mathbb{R}$, then $\mathbb{R}$ has no invariant nonzero vectors, and thus $\Delta_G(w, \mathbb{R})$ has no trivial eigenvalue, which, in view of (1.1), implies 

$$\psi_G(w, \mathbb{R}) = \text{min spec } \Delta_G(w, \mathbb{R}) = I(w) - \lambda^*(\mathbb{R}(w)).$$  \hfill (2.9)

3. ALDOUS’ CONJECTURE: AN ALGEBRAIC PERSPECTIVE

In this section, we prove Theorem 1.1, given the octopus inequality whose proof is postponed to Section 4. The fundamental tool is a map $\vartheta : \mathbb{C}S_n \to \mathbb{C}S_{n-1}$ introduced in [7] and [8].

For a self-adjoint linear operator $A$, we write $A \geq 0$ if $\langle \cdot, \cdot \rangle$ is a positive semidefinite bilinear form. For a finite group $G$, we let 

$$\Gamma(G) := \{ w \in \mathbb{C}G^{(o)} : \Delta_G(w, T) \geq 0, \forall T \in \text{Irr}(G) \} = \{ w \in \mathbb{C}G^{(o)} : \Delta_G(w, L) \geq 0 \},$$  \hfill (3.1)

where, as usual, $L$ denotes the left regular representation. The two definitions are equivalent thanks to (2.8). We know the $\mathbb{C}G^{(+)}$ is a subset of $\Gamma(G)$, but we will have to deal with elements $w$ with negative coefficients. A necessary and sufficient condition for $w \in \Gamma(G)$ is, clearly, 

$$\lambda^*(T(w)) \leq I(w) \quad \forall T \in \text{Irr}(G).$$  \hfill (3.2)

If $H$ is a subgroup of $G$, we denote with $T|_H^{\downarrow}$ the restriction of the representation $T$ to $H$.

$\Gamma(G)$ plays a central role in the strategy for estimating the spectral gap. The idea can be roughly described as follows: suppose we are interested in a lower bound on $\psi_G(w)$. Let then $H$ be a subgroup of $G$, and let $z \in \mathbb{C}H^{(+)}$. If $w - z \in \Gamma(G)$, then it turns out that $\psi_G(w)$ can “almost” be estimated in terms of $\psi_H(z)$, where “almost” means that one has still to worry about irreducible representations of $G$ which, when restricted to $H$, contain the trivial representation. This idea is implemented in the next two results.
Lemma 3.1. Let \( G \) be a finite group, and let \( H \) be a subgroup of \( G \). If \( w \in \mathbb{C}^G(\epsilon) \) and \( z \in \mathbb{C}^H(\epsilon) \) are such that \( w-z \in \Gamma(G) \), then, for any irreducible representation \( T \) of \( G \) such that \( I \not\subset T_{\downarrow_H}^G \) we have

\[
\psi_G(w, T) \geq \min \left\{ \psi_H(z, S) : S \in \text{Irr}(H), \, S \subset T_{\downarrow_H}^G \right\}.
\]  
(3.3)

**Proof.** Let \( y := w-z \in \Gamma(G) \), and let \( T \) be an irreducible representation of \( G \). Since \( z \in \mathbb{C}^H(\epsilon) \), we have

\[
\lambda^*(T(z)) = \lambda^* \left( T_{\downarrow_H}^G(z) \right) = \max \left\{ \lambda^*(S(z)) : S \in \text{Irr}(H), \, S \subset T_{\downarrow_H}^G \right\}.
\]

Since \( \lambda^*(A+B) \leq \lambda^*(A) + \lambda^*(B) \) for each pair of self-adjoint linear operators \( A, B \), and using the fact that \( y \in \Gamma(G) \), so that (3.2) applies to \( y \), we get

\[
\lambda^*(T(w)) = \lambda^*(T(z) + T(y)) \leq \lambda^*(T(z)) + \lambda^*(T(y)) \leq \lambda^*(T(z)) + I(y)
\]

\[
\leq \max \left\{ \lambda^*(S(z)) : S \in \text{Irr}(H), \, S \subset T_{\downarrow_H}^G \right\} + I(w) - I(z).
\]

Since, by hypothesis, no subrepresentation \( S \) of \( T_{\downarrow_H}^G \) is equal to the trivial one, we have that (3.3) follows from (3.4) and (2.9). \( \square \)

As a consequence, we get the following “semirecursive” result.

**Proposition 3.2.** Let \( w \) and \( z \) be as in Lemma 3.1. Then

\[
\psi_G(w) \geq \min \left\{ \psi_H(z), \, \min \left\{ \psi_G(w, T) : T \in \text{Irr}(G), \, T_{\downarrow_H}^G \supset I \right\} \right\}.
\]

**Proof.** Let us write the set \( \text{Irr}(G) \) as a disjoint union \( \text{Irr}(G) = \mathcal{A} \cup \mathcal{B} \), where

\[
\mathcal{A} := \left\{ T \in \text{Irr}(G) : T_{\downarrow_H}^G \supset I \right\}
\]

\[
\mathcal{B} := \left\{ T \in \text{Irr}(G) : T_{\downarrow_H}^G \not\supset I \right\}.
\]

We can now apply Lemma 3.1 to each \( \psi_G(w, T) \) when \( T \in \mathcal{B} \). In this way, we obtain

\[
\min_{T \in \mathcal{B}} \psi_G(w, T) \geq \min_{S \in \text{Irr}(H), \, S \neq I} \psi_H(z, S) =: \psi_H(z),
\]

and the proposition follows. \( \square \)

**The Case of the Symmetric Group.** Let now consider the special case \( G = S_n \). We write \( \pi \vdash n \) if \( \pi = (x_1, x_2, \ldots) \) is a (proper) partition of \( n \). The equivalence classes of irreducible representations of \( S_n \) are in one-to-one correspondence with the set of all partitions of \( n \). We denote with \( [\pi] \) the equivalence class corresponding to partition \( \pi \). The defining representation \( D_n \) of the symmetric group \( S_n \) decomposes as

\[
D_n = I \oplus [n-1, 1]\]

(3.5)
as a direct sum of irreducible factors, and hence
\[ \psi_{S_n}(w, D_n) = \psi_{S_n}(w, [n - 1, 1]). \]

If \( G = S_n \) and \( H = S_{n-1} \), the only irreducible representations of \( S_n \) which contain the trivial representation when restricted to \( S_{n-1} \) are \( \mathbf{1} \) and \([n - 1, 1]\) (see (4.2)). Hence, in this particular case, Proposition 3.2 becomes the following proposition.

**Proposition 3.3.** Let \( w \in \mathbb{C}S_n^{(+)} \), \( z \in \mathbb{C}S_{n-1}^{(+)} \) be such that \( w - z \in \Gamma(S_n) \). Then
\[ \psi_{S_n}(w) \geq \min \{ \psi_{S_{n-1}}(z), \psi_{S_n}(w, D_n) \}. \]

### 3.1. Proof of Aldous’ Conjecture

Imagine that we want to prove (1.3) for all \( w \) in a certain subset \( \mathcal{A}_n \subset \mathbb{C}S_n^{(+)} \), for all \( n \). The strategy is as follows: suppose that, for all \( n \geq 3 \), we find a map \( \vartheta : \mathcal{A}_n \rightarrow \mathcal{A}_{n-1} \) such that, for all \( w \in \mathcal{A}_n \), we have as follows:

(a) \( w - \vartheta(w) \in \Gamma(S_n) \);
(b) \( \psi_{S_n}(w, D_n) \leq \psi_{S_{n-1}}(\vartheta(w), D_{n-1}) \).

Then the proof can be done by induction (assuming that we have a good starting point at \( n = 2 \)): in fact, if (1.3) holds for \( n = k - 1 \), then
\[ \psi_{S_{k-1}}(z) = \psi_{S_{k-1}}(z, D_{k-1}) \quad \forall z \in \mathcal{A}_{k-1}, \tag{3.6} \]

From Proposition 3.3 and (3.6) with \( z = \vartheta(w) \), and from property (b) of the map \( \vartheta \), it follows that, for all \( w \in \mathcal{A}_k \), we have
\[ \psi_{S_k}(w) \geq \min \{ \psi_{S_{k-1}}(\vartheta(w), D_{k-1}), \psi_{S_k}(w, D_k) \} \]
\[ = \psi_{S_k}(w, D_k), \]

which implies \( \psi_{S_k}(w) = \psi_{S_k}(w, D_k) \). Hence the induction is completed.

Let now \( \mathcal{A}_n \) be the set considered in Theorem 1.1, i.e., the set of all \( w \in \mathbb{C}T_n^{(+)} \) such that \( \text{suppu}w \) generates \( S_n \). The starting point of the induction is trivial because the only irreducible representations of \( S_2 \) are \([2]\) and \([1, 1]\). The map \( \vartheta \) which does the trick was found in [7] and [8] and is given by
\[ \vartheta : w = \sum_{(ik) \in T_k} w_{ik} (ik) \mapsto \sum_{(ik) \in T_{k-1}} \left[ w_{ik} + \frac{w_{ik} w_{ka}}{w_{1a} + \cdots + w_{n-1,a}} \right](ik). \]

All is left is proving properties (a) and (b) for this map. Let, for brevity,
\[ \Delta_n := \Delta_{S_k}(w, D_k) \quad \Delta_n^\vartheta := \Delta_{S_k}(\vartheta(w), D_n). \]

It is straightforward to check that the matrix elements of \( \Delta_n - \Delta_n^\vartheta \) are given by
\[ ([\Delta_n - \Delta_n^\vartheta]_{ij} = \frac{\delta_i \delta_j}{\delta_n}. \]
where $\delta_i = -w_{in}$ for $i = 1, \ldots, n - 1$ and $\delta_n = \sum_{i=1}^{n-1} w_{in}$. Following [8], we realize that $\Delta_n - \Delta_n^\vartheta$ is a positive semidefinite rank-1 matrix, so by standard linear algebra results as [11, Cor. 4.3.3 and Thm. 4.3.4], one obtains the following interlacing property of the eigenvalues: let $\lambda_{n,k}$ and $\hat{\lambda}_{n,k}$ denote the $k$th lowest eigenvalue of respectively $\Delta_n$ and $\Delta_n^\vartheta$. Then

$$\lambda_{n,1} \leq \hat{\lambda}_{n,1} \leq \lambda_{n,2} \leq \cdots \leq \hat{\lambda}_{n,n} \leq \lambda_{n,n}. \quad (3.7)$$

Since $\vartheta(w) \in \mathbb{C}S_{n-1}$, the last row and the last column of $\Delta_n^\vartheta$ are zero. More precisely, we have

$$\Delta_n^\vartheta = \Delta_{n-1}^\vartheta \oplus [0]_{1 \times 1},$$

where $[0]_{1 \times 1}$ is the $1 \times 1$ zero matrix. Thus

$$\hat{\lambda}_{n,k} = \lambda_{n-1,k-1} \quad k = 1, \ldots, n.$$ 

This implies that the spectral gap of $\Delta_{n-1}^\vartheta$ corresponds to the third lowest eigenvalue of $\Delta_n^\vartheta$. Combining with (3.7), we get

$$\psi_{\vartheta_n}(w, D_n) = \hat{\lambda}_{n,2} = \hat{\lambda}_{n,3} = \lambda_{n-1,2} = \psi_{\vartheta_{n-1}}(\vartheta(w), D_{n-1});$$

thus property (b) holds.

The proof of property (a) is definitely harder, and it corresponds to proving the octopus inequality which we do in the next section, Theorem 4.2. □

4. THE OCTOPUS

In this section, we prove property (a) of the map $\vartheta$ defined in the previous section, i.e., we show that for each $n \geq 3$ we have

$$\Delta_{\vartheta_n}(w - \vartheta(w), L) \geq 0 \quad \forall w \in \mathbb{C}T_n^{(+)}.$$

It follows from the definition of the left regular representation $L$ that this is equivalent to the octopus inequality as stated in [7, Thm. 2.3] or [8, Conjecture 1].

We start by recalling a few well-known facts about $S_n$ that we need later. The symmetric group $S_n$ is the (disjoint) union of its conjugacy classes

$$S_n = \bigcup_{\pi \vdash n} C^\pi,$$

where, for $\pi = (\pi_1, \pi_2, \ldots) \vdash n$, $C^\pi$ is the set of all permutations $\pi$ which are product of disjoint cycles of lengths $\pi_1, \pi_2, \ldots$. The partition $\pi$ is called the cycle partition of $\pi$. The cardinality of these conjugacy classes is given [21, Ch. 1] by

$$|C^\pi| = n! \left[ \prod_{k=1,2,\ldots}^{\pi_k} k^{\pi_k} (\pi_k)! \right]^{-1}, \quad (4.1)$$
where we denote with $x^k$ the content of $x$, given by
\[ x^k := |\{ i : x_i = k\}| \quad k = 1, 2, \ldots. \]

For instance, if $x = (3, 3, 2, 1, 1)$, we have $x^k = (3, 1, 2)$ and
\[ |C^n| = \frac{11!}{1! 3! 2! 1! 3! 2!}. \]

An irreducible representation $[x]$ of $S_n$ is no longer (in general) irreducible when restricted to a subgroup $H$ of $S_n$. If $H = S_{n-1}$, there is a simple branching rule [21, Section 2.8] for computing the coefficient of the decomposition
\[ [x] \mid_{S_{n-1}} = \bigoplus_{\beta \in \kappa} [\beta] \quad \kappa \vdash n \]
where, if $x = (x_1, \ldots, x_r)$, $x^{-}$ is defined as the collection of all sequences of the form
\[ (x_1, \ldots, x_{i-1}, x_i - 1, x_{i+1}, \ldots, x_r) \]
which are partitions of $n - 1$. For example,
\[ [6, 5, 5, 3, 1] \mid_{S_{n-1}} = [5, 5, 5, 3, 1] \oplus [6, 5, 4, 3, 1] \oplus [6, 5, 5, 2, 1] \oplus [6, 5, 5, 3]. \]

For $x \vdash n$, we define the following element in the group algebra of $S_n$:
\[ \mathfrak{C}S_n \ni J^x := \sum_{\pi \in C^n} \pi. \]  
(4.3)

If $A$ is a subset of $\{1, \ldots, n\}$ with $|A| = m$, and $x \vdash m$, we also let
\[ J^x_A := \sum_{\pi \in C^x_A} \pi, \]
(4.4)

where $C^x_A$ is the set of all permutations $\pi \in S_A$ whose cycle partition is equal to $x$, and $S_A$ stands for the group of permutations of $A$. Since $\mathfrak{C}S_A$ is naturally embedded in $\mathfrak{C}S_n$, we can regard $J^x_A$ as an element of $\mathfrak{C}S_n$ if useful.

For $x, \beta \vdash n$, we denote with $\chi^\beta(x)$ the value taken by the irreducible character $\chi^\beta$ on the conjugacy class $C^x$, while $f_\beta$ stands for the degree of the irreducible representation $\beta$.

A more or less standard application of Schur’s Lemma implies that the image of $J^x$ under any irreducible representation of $S_n$ is a multiple of the identity. More precisely, we have the following lemma.

**Lemma 4.1.** If $x, \beta$ are two partitions of $n$, then
\[ T^\beta(J^x) = \frac{|C^n|}{f_\beta} \chi^\beta(x) I_{f_\beta}. \]
**Proof.** Since $\pi J^\pi = J^\pi \pi$ for each $\pi \in S_n$, by Schur’s Lemma $T^\theta (J^\pi) = c I_{f_\theta}$ for some $c \in \mathbb{C}$. Taking the trace of both sides produces the result. \qed

**Theorem 4.2.** For each $w \in \mathbb{C} T_n^{(+)}$, we have $w - \vartheta (w) \in \Gamma(S_n)$.

**Proof.** We start with the elementary observation that

$$\Gamma(S_n) \text{ is closed under linear combinations}$$

with nonnegative coefficients. \hfill (4.5)

Let $w \in \mathbb{C} T_n^{(+)}$, and thus

$$w = \sum_{(ij) \in I_n} w_{ij} (ij) \quad w_{ij} \geq 0.$$ 

Let $\hat{w} := I(w) (w - \vartheta w)$. Letting, for simplicity, $x_i := w_{in}$ for $i = 1, \ldots, n - 1$, we have

$$\hat{w} = \left( \sum_{i=1}^{n-1} x_i \right) \sum_{j=1}^{n-1} x_j (jn) - \frac{1}{2} \sum_{i,j=1 \atop i \neq j}^{n-1} x_i x_j (ij)$$

$$= \sum_{i=1}^{n-1} x_i^2 (in) + \sum_{i,j=1 \atop i \neq j}^{n-1} x_i x_j [(in) + (jn) - (ij)]. \hfill (4.6)$$

The idea of the proof is as follows: $\hat{w}^2$ is a homogeneous quartic polynomial in the variables $(x_i)_{i=1}^{n-1}$ with coefficients in the group algebra $\mathbb{C} S_n$

$$\hat{w}^2 = \sum_{\mu \in \mathbb{N}^{n-1} : |\mu| = 4} B_{\mu}^{(n)} x^\mu \quad B_{\mu}^{(n)} \in \mathbb{C} S_n,$$

where $\mu$ is a multi-index, $|\mu| = \sum_i \mu_i$, and $x^\mu = \prod_i x_i^{\mu_i}$. If we can show that

for each $\mu \in \mathbb{N}^{n-1}$ with $|\mu| = 4$ we have $B_{\mu}^{(n)} \in \Gamma(S_n)$, \hfill (4.7)

then the theorem is proved. In fact, thanks to (4.5), since $x^\mu \geq 0$, statement (4.7) implies that $\hat{w}^2 \in \Gamma(S_n)$. Thus, by (3.2), we obtain

$$T^\theta(\hat{w})^2 = T^\theta(\hat{w}^2) I_{f_\theta} = I(\hat{w})^2 I_{f_\theta} \quad \forall x \vdash n.$$ 

Since $I(w) > 0$, it follows that each eigenvalue $\lambda$ of $T^\theta(\hat{w})$ satisfies $|\lambda| \leq I(\hat{w})$, and hence $\hat{w} \in \Gamma(S_n)$. Since $w - \vartheta (w) = c \hat{w}$ with $c > 0$, we conclude that $w - \vartheta (w) \in \Gamma(S_n)$.

**Remark 4.3.** A somehow similar approach is pursued in [7, Lemma 3.1, 3.2], where it is proved that $\Delta_{\frac{1}{2}}(\hat{w}^2, L) = - \sum_{\mu \in \mathbb{N}^n : |\mu| = 4} A_\mu^{(n)} x^\mu$, where $A_\mu^{(n)}$ are positive semidefinite matrices. The difference is that this expansion is with respect to the $n$ dependent variables $x_0, x_1, \ldots, x_{n-1}$, where $x_0 := - \sum_{i=1}^{n-1} x_i$. As a consequence, the monomials $x^\mu$ have no definite sign, and thus one cannot simply get rid of them. The required extra work is done in Lemma 3.3 and Lemma 3.4 of [7].
We are thus left with the proof of (4.7). The first step is to find manageable expressions for the coefficients $B^n_\mu$. The next lemma does not require the coefficients $x_i$ to be non-negative.

**Lemma 4.4.** If $w \in \mathcal{C}T_n$ and $\tilde{w} = I(w)(w - \delta w)$, then we have\(^5\)

\[
\tilde{w}^2 = \left( \sum_{i=1}^{n-1} x_i^2 + 2 \sum_{i,j \neq k} x_i^2 x_j + 3 \sum_{i,j \neq k} x_i^2 x_j \right) \cdot 1_{S_n} + \sum_{i,j \neq k} x_i^2 x_j \left( X_{[i,j,k,n]} + 2 \cdot 1_{S_n} \right) + 2 \sum_{i \leq j \leq k < \ell} x_i x_j x_k x_\ell Y_{i,j,k,\ell}^n, \quad (4.8)
\]

where $1_{S_n}$ is the identity in $S_n$ and, remembering (4.4),

\[
X_{[i,j,k,n]} := J^{(3,1)}_{[i,j,k,n]} - 2J^{(2,2)}_{[i,j,k,n]}, \quad (4.9)
\]

\[
Y_{i,j,k,\ell}^n := J^{(3,1)}_{[i,j,k,\ell]} - J^{(2,2,1)}_{[i,j,k,\ell]} - X_{[i,j,k,\ell]} - X_{[i,j,k,\ell]}. \quad (4.10)
\]

We postpone the proof of this lemma, and we observe that, thanks to (4.5), in order to show that $\tilde{w}^2 \in \Gamma(S_n)$, and thus to conclude the proof of Theorem 4.2, it is sufficient to show that both $X_{[i,j,k,n]}$ and $Y_{i,j,k,\ell}^n$ are in $\Gamma(S_n)$. We also claim that it is sufficient to consider two special cases

\[
X_{[1,2,3,4]} \in \Gamma(S_4) \quad \text{and} \quad Y_{[1,2,3,4]}^5 \in \Gamma(S_5). \quad (4.11)
\]

Let in fact $\sigma, \pi$ be 2 permutations of $S_n$. If $\sigma$ is written as a product of disjoint cycles, then $\pi \sigma \pi^{-1}$ is the permutation which is obtained from $\sigma$ by replacing in each cycle $i$ with $\pi(i)$. Thus, if $\pi$ sends $(i, j, k, n)$ to $(1, 2, 3, 4)$, then

\[
\pi X_{[i,j,k,n]} \pi^{-1} = X_{[1,2,3,4]}. 
\]

As a consequence, for any $R \in \text{rep}_\sigma(S_n)$, the two matrices $R(X_{[i,j,k,n]}^\pi)$ and $R(X_{[1,2,3,4]}^\pi)$ are equivalent. We then observe that, in order to prove that $X_{[1,2,3,4]} \in \Gamma(S_4)$, it is sufficient to consider the irreducible representations of $S_4$. Let in fact $\pi \vdash n$, and let $T^\pi$ be the corresponding irreducible representation of $S_n$. Then we have

\[
T^\pi |_{S_4} = \bigoplus_{b \vdash 4} v_b T_b
\]

for some non-negative integral coefficients $v_b$. Since $X_{[1,2,3,4]}$ is an element of $S_4$, we get

\[
T^\pi (X_{[1,2,3,4]}) = \bigoplus_{b \vdash 4} v_b T_b^\pi (X_{[1,2,3,4]}),
\]

\(^5\)If $n < 5$, one or more of the following terms are missing, which is all right.
and thus the spectrum of $T^\beta(X_{[1,2,3,4]})$ is the union of the spectra of those $T^\alpha(X_{[1,2,3,4]})$, where $\beta$ is a partition of 4 for which $\nu_\beta > 0$. This shows that if $X_{[1,2,3,4]}$ belongs to $\Gamma(S_4)$ it necessarily belongs to $\Gamma(S_n)$ for all $n \geq 4$.

A similar argument applies to $Y_{[i,j,k,l]}$. In that follows, we let for simplicity

$$\hat{X} := X_{[1,2,3,4]} \quad \hat{Y} := Y_{[1,2,3,4]^5}$$

So the theorem is proved if we show that (4.11) hold, i.e., if we prove that

$$\lambda^* \left[ T^\alpha(\hat{X}) \right] \leq I(\hat{X}) \quad \forall \lambda \vdash 4 \quad (4.12)$$
$$\lambda^* \left[ T^\alpha(\hat{Y}) \right] \leq I(\hat{Y}) \quad \forall \lambda \vdash 5 \quad (4.13)$$

It is well known [15, 2.1.8] that if $z \in \mathfrak{S} \alpha$ is a linear combination of even permutations and $\chi'$ if the conjugate partition of $\alpha'$, then $T^\alpha(z) \cong T^\chi(z)$, so we must only check previous inequalities for (roughly) half the irreducible representations of $S_4$ and $S_5$. Thanks to Lemma 4.1 and to (4.1), we have

$$T^\alpha(\hat{X}) = f_\lambda^{-1} \left[ [C^{(3,1)}] \chi^\alpha(3,1) - 2[C^{(2,2)}] \chi^\alpha(2,2) \right] \cdot I_{f_\alpha}$$

$$= f_\lambda^{-1} \left[ 8 \chi^\alpha(3,1) - 6 \chi^\alpha(2,2) \right] \cdot I_{f_\alpha}.$$ 

Hence $T^\alpha(\hat{X})$ is a multiple of the identity, and, in particular,

$$X^* := \lambda^*(T^\alpha(\hat{X})) = f_\lambda^{-1} \left[ 8 \chi^\alpha(3,1) - 6 \chi^\alpha(2,2) \right]. \quad (4.14)$$

The relevant characters can be quickly computed with the Murnaghan-Nakayama rule. Moreover,

$$I(\hat{X}) = |C^{(3,1)}| - 2|C^{(2,2)}| = 2.$$ 

Table 1 shows that (4.12) holds.

| $\lambda$ | $f_\lambda$ | $\chi^\alpha(3,1)$ | $\chi^\alpha(2,2)$ | $X^*$ |
|-----------|--------------|---------------------|---------------------|-------|
| (4)       | 1            | 1                   | 1                   | 2     |
| (3,1)     | 3            | 0                   | -1                  | 2     |
| (2,2)     | 2            | -1                  | 2                   | -10   |

6This means that the Young diagram of $\chi'$ is obtained from the Young diagram of $\chi$ by interchanging rows and columns.

7Or, even more quickly, one can look them up on the internet.

8Partitions $\lambda = (2,1,1)$ and $\lambda = (1^4)$, absent in this table, are conjugate to (3,1) and (4), respectively.
For \( \bar{Y} \) things are slightly more complicated. If \( x \vdash 5 \), since \( \hat{X} \in S_4 \), we have
\[
T^x(\hat{X}) = T^x \downarrow_{S_4} (\hat{X}) = \bigoplus_{\beta \in \bar{X}} T^\beta(\hat{X}).
\]

Since \( |C^{(3,1,1)}| = 20 \) and \( |C^{(2,2,1)}| = 15 \), from (4.10) and (4.2), we obtain
\[
T^x(\bar{Y}) = f_x^{-1} [20 \chi^x(3, 1, 1) - 15 \chi^x(2, 2, 1)] \cdot I_{f_x} - \bigoplus_{\beta \in \bar{X}} T^\beta(\hat{X}),
\]
which, thanks to (4.14), implies
\[
Y^x := \max_{\beta \in \bar{X}} F_{x\beta},
\]
where
\[
F_{x\beta} := f_x^{-1} [20 \chi^x(3, 1, 1) - 15 \chi^x(2, 2, 1)] - X^\beta.
\]

On the other hand, we have
\[
I(\bar{Y}) = |C^{(3,1,1)}| - |C^{(2,2,1)}| - I(\hat{X}) = 3.
\]

We collect all relevant quantities in Table 2, which shows that (4.13) holds.

**Proof of Lemma 4.4.** The proof is a more or less straightforward computation requiring a little care to avoid double counting. In order to avoid excessively cluttered formulas, all variables appearing below the summation symbol are always summed from 1 to \( n-1 \). Moreover, if two or more variable appear in brackets below the summation, they are assumed to be *all distinct*. For instance,
\[
\sum_{j<k \atop \{j\neq k\}} \text{ means } \sum_{i,j,k,l=1 \atop i,j,k,l \neq \{j\neq k\}}
\]

From (4.6), letting \( A_{ij} := (in) + (jn) - (ij) \), we get
\[
\hat{w} = \sum_i \chi^x(i) + \frac{1}{2} \sum_{\{ij\}} x_i x_j A_{ij},
\]

| \( x \) | \( \beta \) | \( f_x \) | \( \chi^x(3,1,1) \) | \( \chi^x(2,2,1) \) | \( X^\beta \) | \( F_{x\beta} \) | \( Y^x \) |
|-----|-----|-----|-----|-----|-----|-----|-----|
| (5)  | (4) | 1   | 1   | 1   | 2   | 3   | 3   |
| (4,1)| (4) | 4   | 1   | 0   | 2   | 3   | 3   |
| (4,1)| (3.1)| 4   | 1   | 0   | 2   | 3   | 3   |
| (3,2)| (3,1)| 5   | -1  | 1   | 2   | -9  | 3   |
| (3,2)| (2,2)| 5   | -1  | 1   | -10 | 3   | 3   |
| (3,1,1)| (3,1)| 6   | 0   | -2  | 2   | 3   | 3   |
| (3,1,1)| (2,1,1)| 6   | 0   | -2  | 2   | 3   | 3   |
Given two elements $a, b$ of the group algebra, we denote with $\{a, b\} = ab + ba$ the anticommutator of $a$ and $b$. Thus

$$\hat{w}^2 = \frac{u}{2} + \frac{v}{2} + \frac{z}{8},$$

where

$$u := \sum_{i,j} x_i^2 x_j^2 \{(in), (jn)\} \quad v := \sum_{i,[k]} x_i^2 x_j x_k \{(in), A_k\}$$

and

$$z := \sum_{[i][j][k][\ell]} x_i x_j x_k x_\ell \{A_ij, A_{k\ell}\}.$$

If $i, j, k, \ell$ are distinct integers in $\{1, \ldots, n - 1\}$, we have the following identities:

$$\{(ij), (k\ell)\} = 2(ij)(k\ell),$$

$$\{(in), (jn)\} = (jn) + (jn),$$

$$\{(in), A_k\} = (jn) + (nk) + (kn) - 2(in)(jk),$$

$$\{(in), A_{jk}\} = 2 \cdot 1_{S_n},$$

$$\{A_{ij}, A_{ij}\} = 2A_{ij} = 2(3 \cdot 1_{S_n} - (in) - (jn)).$$

Let us start with the $u$ term:

$$u = 2 \sum_i x_i^4 1_{S_n} + 2 \sum_{[i]} x_i^3 x_j^2 \{(in)\}.$$

For $v$, we obtain

$$v = \sum_{i,[k]} x_i^2 x_j x_k \{(in), A_k\} + 2 \sum_{i,[k]} x_i^3 x_k \{(jn), A_{jk}\} = \sum_{i,[k]} x_i^2 x_j x_k \{(in) + (jn) + (nk) + (kn) - 2(in)(jk)\} + 4 \sum_{[i]} x_i^3 x_k 1_{S_n}.$$

We split $z$ as a sum $z = z_1 + z_2 + z_3$, according to the cardinality of $\{i, j\} \cap \{k, \ell\}$, as follows

$$z := 2 \sum_{[i]} x_i^2 x_j^2 \{A_{ij}, A_{ij}\} + 4 \sum_{[i]} x_i^3 x_k x_\ell \{A_{ij}, A_{k\ell}\}$$

$$+ \sum_{[i][j][k][\ell]} x_i x_j x_k x_\ell \{A_{ij}, A_{k\ell}\} = z_1 + z_2 + z_3.$$
We have

\[ z_1 := \sum_{i \neq j} x_i^2 x_j^2 \left( 12 \cdot 1_{S_n} - 8 (inj) \right). \]

Hence

\[
\frac{u}{2} + \frac{z_1}{8} = \sum_i x_i^4 1_{S_n} + 3 \sum_{i < j} x_i^2 x_j^2 1_{S_n}.
\] (4.19)

Next, using the anticommutation relations we found above, we find

\[
\frac{v}{2} + \frac{z_2}{8} := \frac{1}{2} \sum_{[ijk]} x_i^2 x_j x_k \left[ 2 \cdot 1_{S_n} + J_{(i,j,k)}^{(3,1)} - 2 J_{(i,j,k,n)}^{(2,2)} \right] + 4 \sum_{[jk]} x_j^3 x_k 1_{S_n}
\]

\[ = \frac{1}{2} \sum_{[ijk]} x_i^2 x_j x_k \left[ 2 \cdot 1_{S_n} + X_{(i,j,k)} \right] + 4 \sum_{[jk]} x_j^3 x_k 1_{S_n}. \] (4.20)

Thanks to the symmetry of the four indices, the term \( z_3 \) can be written as

\[
z_3 = 2 \sum_{[ijkl]} x_i x_j x_k x_l A_{ij} A_{kl}
\]

\[ = \frac{2}{3} \sum_{[ijkl]} x_i x_j x_k x_l \left( A_{ij} A_{kl} + A_{ik} A_{jl} + A_{il} A_{jk} \right)
\]

\[ = 16 \sum_{i < j < k < \ell} x_i x_j x_k x_l \left( A_{ij} A_{kl} + A_{ik} A_{jl} + A_{il} A_{jk} \right). \] (4.21)

The term \( A_{ij} A_{kl} + A_{ik} A_{jl} + A_{il} A_{jk} \) contains: (a) all 3-cycles \((inj)\), with \(1 \leq i, j \leq n - 1\), with coefficient +1; (b) all double transpositions \((in)(jk)\) with \(1 \leq i, j, k \leq n - 1\), with coefficient −1; (c) all double transpositions \((ij)(k\ell)\) with \(1 \leq i, j, k, \ell \leq n - 1\), with coefficient +1. Hence we have

\[
A_{ij} A_{kl} + A_{ik} A_{jl} + A_{il} A_{jk} = J_{(i,j,k)}^{(3,1)} - J_{(i,j,k)}^{(3,1)} - J_{(i,j,k,\ell)}^{(2,2)} + 2 J_{(i,j,k)}^{(2,2)}
\]

\[ = Y_n^{ij} (i,j,k,\ell). \] (4.22)

The proof of the lemma now follows from (4.15), (4.18), (4.19), (4.20), (4.21), and (4.22). □

5. THE KAZHDAN CONSTANT AND (IM)POSSIBLE GENERALIZATIONS

Let \( G \) be a finite group, let \( G^* \) be the set of all (not necessarily finite-dimensional) unitary representations of \( G \), and let \( G_0^* \) be the set of all \( R \in G^* \) with no invariant nonzero vector. That is,

\[ G_0^* := \{ R \in G^* : I \not\subset R \}. \]
If \((R, V) \in G_0^*\) and \(Q\) is a generating set of \(G\), the Kazhdan constant associated to the pair \((Q, R)\) is defined as

\[
\kappa_G(Q, R) := \inf_{v \in V, |v| = 1} \max_{q \in Q} \|R(q)v - v\|.
\] (5.1)

By minimizing over \(R \in G_0^*\), we obtain the Kazhdan constant of \(Q\)

\[
\kappa_G(Q) := \inf_{R \in G_0^*} \kappa_G(Q, R).
\]

We can always assume that \(Q\) is a symmetric generating set, that is \(Q = Q^{-1}\), since adding the inverse of an element to \(Q\) does not change the value of the Kazhdan constant. The Kazhdan constant can be defined in the much more general framework of a topological group \(G\). We refer the reader to [4] for a review on the Kazhdan constant and Kazhdan’s property (T) in this more general setting.

We have observed in Section 2 that, being \(\psi_G(w)\) a spectral quantity, in its definition we can either minimize over all representations or just the irreducible ones getting the same result. This is false for the Kazhdan constant. If we define

\[
\kappa^I_G(Q) := \inf_{R \in G_0^* \cap \text{Irr}(G)} \kappa_G(Q, R),
\]

then we have [3]

\[
|Q|^{-1/2} \kappa^I_G(Q) \leq \kappa_G(Q) \leq \kappa^I_G(Q),
\]

and explicit examples where the second inequality is strict are presented in [3].

In order to illustrate the connection between the Kazhdan constant and the spectral gap \(\psi\), we start by introducing a slight abuse of notation, by identifying the generating set \(Q\) with the group algebra element \(\hat{Q} := \sum_{q \in Q} q\), and we observe that, if \(Q\) is symmetric, then \(\hat{Q} \in \mathbb{C}G^{(+)}\), so (3) of Proposition 2.1 applies to \(\hat{Q}\). Thus if \(I \not\subset R\), then the associated representation Laplacian \(\Delta_G(Q, R)\) has no trivial eigenvalues. As a consequence, in accord with (2.3), we have

\[
\psi_G(Q, R) := \psi_G(\hat{Q}, R) = \min \text{spec} \Delta_G(Q, R) = \min \text{spec} \left(\sum_{q \in Q} (I - R(q))\right).
\] (5.2)

From (2.2) it follows that, if \((R, V) \in G_0^*\), then

\[
\langle \Delta_G(Q, R)v, v \rangle = \frac{1}{2} \sum_{q \in Q} \|R(q)v - v\|^2 \quad \text{for each } v \in V.
\] (5.3)

Since \(\Delta_G(Q, R)\) is self-adjoint, we get

\[
\min \text{spec} \Delta_G(Q, R) = \inf_{v \in V, |v| = 1} \langle \Delta_G(Q, R)v, v \rangle.
\]
In this way, we obtain a formula for the spectral gap which resembles the definition of the Kazhdan constant

\[ \psi_G(Q, R) = \frac{1}{2} \inf_{v \in V: \|v\|^2 = 1} \|R(q)v - v\|^2. \]  

(5.4)

Comparing with (5.1), we get, for any \( R \in G_0^* \),

\[ \kappa_G^2(Q, R) \leq 2\psi_G(Q, R) \leq |Q| \kappa_G^2(Q, R) \]  

(5.5)

\[ \kappa_G^2(Q) \leq 2\psi_G(Q) \leq |Q| \kappa_G^2(Q). \]  

(5.6)

In [16] Kassabov, using an approach completely different from [7] proved that the “minimality” property (1.3) of the defining representation for computing the spectral gap and the Kazhdan constant also holds for finite Coxeter systems \( (G, Q) \).

A Coxeter group \( G \) generated by \( Q = \{s_1, \ldots, s_n\} \) is defined by integral numbers \( m_{\alpha\beta} = m_{\beta\alpha} \) such that \( m_{\alpha\alpha} = 1 \) and \( m_{\alpha\beta} \geq 2 \) if \( \alpha \neq \beta \). \( G \) has presentation (see [13] for background)

\[ G \cong \langle s_1, \ldots, s_n | (s_\alpha s_\beta)^{m_{\alpha\beta}} = 1, \forall \alpha, \beta \rangle. \]

It is known that \( G \) has a defining representation on a \( n \)-dimensional vector space \( V \) such that each generator \( s_\alpha \) acts as a reflection with respect to a hyperplane \( V_\alpha \). Moreover, if \( G \) is finite the angle between the hyperplanes \( V_\alpha \) and \( V_\beta \) is equal to \( \pi/m_{\alpha\beta} \).

In [16, Thm. 1.3], the following result has been proved.

**Theorem 5.1.** Let \( G \) be a finite Coxeter group with Coxeter generating set \( Q \). Then

\[ \psi_G(Q) = \psi_G(Q, D') \quad \kappa_G(Q) = \kappa_G(Q, D'), \]  

(5.7)

where \( D' \) is the defining representation of \( G \).

(Harmless) Notation Ambiguity. When dealing with the symmetric group \( S_n \), one usually calls defining representation the reducible \( n \)-dimensional representation \( D \) associated with the natural action of \( S_n \) on the set \( \{1, 2, \ldots, n\} \), which decomposes as in (3.5). On the other hand, \( S_n \), when generated by \( n - 1 \) adjacent transpositions \( Q = \{(1, 2), (2, 3), \ldots, (n - 1, n)\} \) corresponds to Coxeter group \( A_{n-1} \) and, in this context, its defining representation is the \( (n - 1) \)-dimensional irreducible representation \( D' \cong [n - 1, 1] \). This is really harmless, since \( \psi_G(Q, D) = \psi_G(Q, D') \), so the left half of (5.7) coincides with (1.3).

Kassabov’s theorem is more general in some respects and less general in others than in the case of Theorem 1.1. It is less general because it applies (essentially) to only one element of the group algebra, namely \( w = \sum q \in Q \), where \( Q \) is a Coxeter generator for \( G \). Thus, for instance, in the case of the symmetric group,
it holds for $\psi_{S_n}(w)$ when $w$ is a sum of distinct adjacent transpositions, and hence (modulo permutations of the indices)

$$w = (1, 2) + (2, 3) + \cdots + (n-1, n).$$

It is more general because it holds for any finite Coxeter group and it applies to the Kazhdan constant. These considerations prompt at least two questions:

(1) Can Kassabov’s approach be used for dealing with more general elements $w$ of the group algebra or more general generating sets?

(2) Does the analog of Theorem 1.1 hold for the Kazhdan constant?

The answers are (1) not likely and (2) no.

The first answer can be explained as follows: A key ingredient in the proof of Theorem 5.1 is that the symmetric matrix $A$ with elements

$$A_{\alpha\beta} = -\cos \frac{\pi}{m_{\alpha\beta}}$$

is (strictly) positive definite as a quadratic form. The problem is that positive definite matrices of the form (5.8) have been classified (see, for instance, [13]) and correspond exactly to finite Coxeter systems $(G, Q)$, so there is little hope to go beyond Coxeter systems with this approach. To consider a concrete example, let us take $S_4$ with non-Coxeter generating set

$$Q = \{s_1 = (1, 2), \ s_2 = (1, 3), \ s_3 = (1, 4)\}.$$

Then it is easy to check that $(s_\mu s_\nu)^w = 1$, where $m_{s_\alpha s_\beta} = m_{s_\delta s_\gamma} = 3$. Thus we have

$$A = \begin{pmatrix} 1 & -1/2 & -1/2 \\ -1/2 & 1 & -1/2 \\ -1/2 & -1/2 & 1 \end{pmatrix} \quad \text{det } A = 0.$$

Hence $A$ is positive semidefinite, but not positive definite and the approach of Theorem 5.1 will not work in this case, even if the left half of (5.7) holds, thanks to Theorem 1.1.

We now present a family of counterexamples which justify our negative answer to question (2).

**Proposition 5.2.** Let $G = S_n$ with $n \geq 4$, and let $T_n$ be the set of all transpositions of $S_n$. Then, if $D' \cong [n-1, 1]$ is the defining representation, we have

$$\kappa_{S_n}(T_n, D') < \kappa_{S_n}(T_n) = \frac{2}{\sqrt{n-1}}.$$
This is quite well known. It can be easily deduced, for instance, from [9, Cor. 4], taking into account their different normalization. Since the proof is a simple combination of facts previously discussed in this paper, we include it for the sake of completeness. From Theorem 1.1, (2.9), and Lemma 4.1 applied to the conjugacy class of transpositions with cycle partition $\alpha = (2, 1^{n-2})$, we get

$$
\psi_{S_n}(T_n) = \psi_{S_n}(T_n, D) = \psi_{S_n}(T_n, [n-1, 1])
$$

$$
= |T_n| - \lambda^* \left[ T^{(n-1, 1)}(J^{2,1^{n-2}}) \right]
$$

$$
= |T_n| - \frac{\lambda^{(n-1, 1)}((2, 1^{n-2})) |T_n|}{f_{(n-1,1)}}.
$$

(5.9)

In the case of transpositions, Frobenius formulas for the irreducible characters take the simple form [14]

$$
\frac{\chi^\beta((2, 1^{n-2})) |T_n|}{\bar{\kappa}_G} = \frac{1}{2} \sum_{i=1}^{r} \beta_i [\beta_i - (2i - 1)],
$$

(5.10)

where $r$ is the length (number of terms) of the partition $\beta$. The lemma follows form (5.9) end (5.10) with $\beta = (n-1, 1)$. $\square$

The second result shows that if, the second inequality in (5.5) is saturated, then we are in a very special situation.

**Lemma 5.4.** Let $G$ be a finite group. If $Q$ is a generating set for $G$ and $(R, V) \in G_0^*$ is finite-dimensional and such that

$$
|Q|\kappa_G^2(Q, R) = 2\psi_G(Q, R),
$$

(5.11)

then there exists a unit vector $u \in V$ such that

$$
\kappa_G(Q, R) = \|R(q)u - u\| \quad \forall q \in Q.
$$

(5.12)

**Proof.** Since $R$ is finite-dimensional, in (5.1) we are minimizing a continuous function on a compact set, and hence there exists a unit vector $u \in V$ such that

$$
\max_{q \in Q} \|R(q)u - u\| = \kappa_G(Q, R).
$$

(5.13)

If (5.11) holds, we have

$$
2\psi_G(Q, R) \leq \sum_{q \in Q} \|R(q)u - u\|^2 \leq |Q| \max_{q \in Q} \|R(q)u - u\|^2
$$

$$
= |Q|\kappa_G^2(Q, R) = 2\psi_G(Q, R).
$$

(5.14)

Hence the second (as well as the first) inequality above must be an equality, which implies the lemma. $\square$
Proof of Proposition 5.2. Since $T_n$ is a conjugacy class, the group of all inner automorphisms of $S_n$ leaves $T_n$ invariant and acts transitively on $T_n$. Hence we can apply Proposition 1 of [20] which, in our notation, states that the second inequality in (5.6) is saturated. Thus, using Lemma 5.3, we have

$$\kappa^2_{S_n}(T_n) = \frac{2}{|T_n|} \psi_{S_n}(T_n) = \frac{4}{n-1}. \quad (5.15)$$

We intend to show that the assumption

$$\kappa_{S_n}(T_n) = \kappa_{S_n}(T_n, D') \quad (5.16)$$

leads to a contradiction. In fact by Theorem 1.1 we have $\psi_{S_n}(T_n) = \psi_{S_n}(T_n, D')$, so, if (5.16) holds, thanks to (5.15) we can apply Lemma 5.4 to the pair $(T_n, D')$, and we could conclude that, for some unit vector $u$ in the representation space of $D'$, we have

$$\|D'(q)u - u\|^2 = \kappa^2_G(T_n, D') = \frac{4}{n-1} \quad \forall q \in T_n. \quad (5.17)$$

But this eventuality can be easily ruled out. Let $V$ be an $n$-dimensional complex vector space with orthonormal basis $(e_k)_{k=1}^n$, and let $D$ be the defining $n$-dimensional representation of $S_n$ acting on $V$ as

$$D(\pi) \left( \sum_{k=1}^n u_k e_k \right) = \sum_{k=1}^n u_k e_{\pi(k)} \quad u_k \in \mathbb{C}, \ \pi \in S_n.$$ 

The representation $D'$ can be realized by restricting $D$ to the invariant subspace

$$V' := \left\{ v \in V : \left\langle v, \sum_{k=1}^n e_k \right\rangle = 0 \right\}.$$ 

Let then $u = \sum_{k=1}^n u_k e_k$ be a unit vector in $V'$ such that (5.17) holds. Then, by explicit computation, for each transposition $(i, j)$, we have

$$\|D'((i, j))u - u\|^2 = 2|u_i - u_j|^2 = \frac{4}{n-1} \quad \forall i \neq j.$$ 

This implies, in particular, that there are $n$ points $u_1, \ldots, u_n$ in the complex plane which are equidistant from each other, which is impossible if $n \geq 4$. □

Remark 5.5. The condition $n \geq 4$ cannot be omitted in Proposition 5.2. For $n = 3$ we have in fact $\kappa_{S_n}(T_n) = \kappa_{S_n}(T_n, D')$. Proceed as in the previous proof and take $u = (1, e^{i\pi/3}, e^{-i\pi/3})/\sqrt{3}$.

Remark 5.6. A minimizing representation $R$, for which we have $\kappa_{S_n}(T_n) = \kappa_{S_n}(T_n, R)$ is actually a direct sum of a certain number of copies of $(D', V)$.
Proceeding more or less as in the proof of Proposition 1 of [20] (see also the proof of Theorem 1.2 of [19]), let
\[ R := |T_n| \cdot D' = D' \oplus \cdots \oplus D' \quad (|T_n| \text{ terms}) . \]

Let \( u \in V \) be a unit minimizing vector in (5.4) for \( \psi_{S_n}(T_n, D') \), so that
\[ \psi_{S_n}(T_n, D') = \frac{1}{2} \sum_{t \in T_n} \| D'(t)u - u \|^2 . \]

Let
\[ \hat{u} = (D'(t)u)_{t \in T_n} \in \hat{V} := V \oplus \cdots \oplus V . \]

Then, since \( T_n \) is a conjugacy class and \( D' \) is unitary, we get
\[
\| R(q) \hat{u} - \hat{u} \|^2 = \sum_{t \in T_n} \| D'(qt)u - D'(t)u \|^2 \]
\[ = \sum_{t \in T_n} \| D'(t)u - u \|^2 . \]

Hence the LHS is independent of \( q \), which implies
\[
\kappa_{S_n}^2(T_n, R) \leq \frac{\| R(q) \hat{u} - \hat{u} \|^2}{\| \hat{u} \|^2} = \frac{1}{|T_n|} \sum_{t \in T_n} \| D'(t)u - u \|^2 = \frac{2}{|T_n|} \psi_{S_n}(T_n, D') = \frac{4}{n - 1} .
\]

On the other side, by (5.15), we know that \( \kappa_{S_n}^2(T_n) = 4/(n - 1) \), and hence we have \( \kappa_{S_n}(T_n, R) = \kappa_{S_n}(T_n) \).

We conclude with a few words about the possibility of generalizing Theorem 1.1 to elements \( w \) of the group algebra which are no longer combinations of just transpositions. In [5], we have proved (1.3) when \( w \) is the sum of all initial reversals
\[ w = r_1 + r_2 + \cdots + r_n , \]

where \( r_k \in S_n \) is the permutation which reverses the order of the first \( k \) positive integers. General results, however, do not appear easy to obtain. Even in the very special (and simple) case of \( w = J^z \) which is the sum of all elements of the conjugacy class \( C^z \) (remember (4.3)), identity (1.3) in general fails. If \( \alpha \) is a class of even permutations, the conjugacy class \( C^\alpha \) does not generate \( S_n \) and we have, trivially,
\[ \psi_{S_n}(J^z) = \psi_{S_n}(J^z, [1^z]) = 0 . \]
A counterexample for an odd class is given by \( \alpha = (4, 1) \), the set of all 4-cycles in \( S_5 \). It follows from Lemma 4.1 that

\[
\psi_{S_5}(J^{(4,1)}) = \psi_{S_5}(J^{(4,1)}, [2, 2, 1]) = 24 < 30 = \psi_{S_5}(J^{(4,1)}, D).
\]

There is one conjecture though, by P. Caputo, which is defying our attempts to either prove it, or disprove it with numerical experiments. For each \( A \subset \{1, \ldots, n\} \), let \( S_{n,A} \) be the set of all \( \pi \in S_n \) such that \( \pi(i) = i \) for each \( i \in A^c \), and let

\[
J_{n,A} := \sum_{\pi \in S_{n,A}} \pi.
\]

In other words, \( J_{n,A} \) is the sum of all possible shuffles inside \( A \) which leave each element outside invariant. The conjecture is as follows.

**The \( \alpha \)-shuffles Conjecture (P. Caputo, private communication).** Let \( n \) be a positive integer, \( n \geq 3 \). For each \( A \subset \{1, \ldots, n\} \), let \( x_A \geq 0 \), and let

\[
w = \sum_{A \subset \{1, \ldots, n\}} x_A J_{n,A}.
\]

If \( \text{supp} w \) generates \( S_n \), then \( \psi_{S_n}(w) = \psi_{S_n}(w, D) \).

This may be considered a rather natural generalization of Theorem 1.1, since, by imposing the extra condition \( |A| = 2 \), one recovers Aldous' conjecture.

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