Calculation of stress near the cracks in welded plates with taking into account the residual deformation

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Abstract. An algorithm for determining the stress state of plates with cracks caused by residual deformations, including taking into account the contact of their edges, is proposed. The residual stresses in the plates are determined by calculation in the experimental method. The study of residual stresses and stresses at cracks in plates of various shapes is performed. It was established that ignoring contact with the edges of cracks can lead to significant errors in the calculation of stresses.

There are stresses due to residual deformations in the elements of the structures. These stresses may be compatible with stresses due to operational loads, and therefore they must be taken into account when calculating strength and durability. To study the stresses at cracks in welded structures, which often occur in the vicinity of welds, it is necessary to pre-determine the residual stresses. The question of determining the residual stresses in the welded elements of structures was considered in [1-5]. The distribution of residual stresses is influenced by the selected welding modes, the thermomechanical characteristics of the materials that need to be determined for a wide range of temperatures, structural transformations, conditions of fixation, etc. In this regard, the determination of residual stresses by theoretical methods can be carried out only for certain classes of problems.

In this work, the calculation-experimental method was used to determine the residual stresses in the plates [6]. The algorithm for determining the stresses at the cracks in the plates due to residual deformations is based on the method of integral equations. It is taken into account that residual stresses can be compressible. For such cases, the research was done taking into account the contact of the edges of the cracks. Other approaches to the study of stresses around cracks in welded structures considered in [7-15].

Determination of residual stresses in continuous plates. Consider the problem of determining the state of the plate, caused by the residual (initial) deformations $\varepsilon_0, \varepsilon_y, \gamma_{xy}$ with the condition that its edge traction-free. To do this we use Hooke's law.

$$
\varepsilon_x = \varepsilon_0^0 + \frac{1}{E}(\sigma_x - v\sigma_y), \quad \varepsilon_y = \varepsilon_0^0 + \frac{1}{E}(\sigma_y - v\sigma_x), \quad \gamma_{xy} = \gamma_{xy}^0 + \frac{1}{G}\tau_{xy},
$$

(1)

where $\varepsilon_x, \varepsilon_y, \gamma_{xy}$ – complete deformations that satisfy the conditions of compatibility of deformations [16]
Tension in the plate $\sigma_x, \sigma_y, \gamma_{xy}$ satisfy the equilibrium equation

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0, \quad \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} = 0. \quad (3)$$

Consider a more detailed case involving plates with a straight weld seam placed along the axis $Oy$. Remaining deformations in this case in the literature are accepted as non-zero at $|x| < d$ as

$$\varepsilon_x = C_x f(x), \quad \varepsilon_y = C_y f(x), \quad \gamma_{xy} = 0, \quad (4)$$

where $f(x)$ – known function, $C_x = \frac{S_x}{E}, \quad C_y = -\frac{S_y}{E}, \quad S_x = k_x \sigma_y, \quad S_y = k_y \sigma_x, \quad k_x, k_y$ – dimensionless parameters, $\sigma_y$ – yield point, $d$ – width of the plastic zone.

Here is a typical welding case, when the residual deformations are negative. At

$$f(x) = 1 + a_0 \left(\frac{x}{d}\right)^4 - (3 + 2a_0) \left(\frac{x}{d}\right)^2 + (2 + a_0) \left(\frac{x}{d}\right)^4, \quad (5)$$

where $a_0$ – constant located on the interval $[0,2]$. In this case, the description of initial deformations includes four parameters $d, a_0, k_x, k_y$, which must be determined so that the calculated and experimental methods of stress found differ little from one another.

The solution of the problem is represented as the sum of the partial and correction components.

**Partial solution.** For a partial solution we will accept: $\sigma_x = 0, \quad \varepsilon_y = 0, \quad \tau_{xy} = 0, \quad \gamma_{xy} = 0$. Then from the second formula (1) we get $\sigma_y = -E \varepsilon_y$, and then we determined from the first formula $\varepsilon_x = \varepsilon_x + \nu \varepsilon_y$. Thus, we obtain a partial solution for stresses and deformations in the form

$$\sigma_x = 0, \quad \sigma_y = -E \varepsilon_y = -E C_x f(x), \quad \tau_{xy} = 0, \quad (6)$$

$$\varepsilon_x = (C_x + \nu C_y) f(x), \quad \varepsilon_y = 0, \quad \gamma_{xy} = 0. \quad (7)$$

Inspection makes sure that tension (6) satisfying the equilibrium equation (3), and deformation (7) – compatibility equation (2).

Stress vector $(X_0, Y_0)$ on an arbitrary plane, which corresponds to a partial solution, is determined by the formulas

$$X_0 = 0, \quad Y_0 = \sigma_y \sin \gamma, \quad (8)$$

where $\gamma$ – the angle between the axis $Ox$ and perpendicular to the plane.

**Correction solution.** To determined it it is necessary to consider the problem of the theory of elasticity for a given plate, to the boundary of which the above-identified forces with the opposite sign are applied $X = 0, Y = -S_y f(x) \sin \gamma$.

Let's accept, that the well-known expressions for Muskhelishvili's complex potentials for plates with a free limit loaded by a system of forces $(X_k, Y_k)$, applied at points $z_k, \quad k = 1,\ldots,K$ (For a limited plate of force, the forces must be balanced themselves). We will write down these potentials in the form

$$\Phi(z) = \sum_{k=1}^{K} \Phi_0(z, z_k, X_k, Y_k), \quad \Psi(z) = \sum_{k=1}^{K} \Psi_0(z, z_k, X_k, Y_k), \quad (9)$$

where $\Phi_0(z, z_k, X_k, Y_k), \quad \Psi_0(z, z_k, X_k, Y_k)$ – known functions.
Consider the case when the forces applied to the boundary are distributed on the arc $AB$ and are equal to $X(t), Y(t)$. Then the complex potentials corresponding to this load will be

$$
\Phi(z) = \int_{AB} \Phi_0(z,t,X(t),Y(t)) ds, \quad \Psi(z) = \int_{AB} \Psi_0(z,t,X(t),Y(t)) ds,
$$

(10)

Taking into account that $X = 0, Y = -S_y f(x) \sin \gamma$, we get

$$
\Phi(z) = S_y \int_{AB} \Phi_0(z,t,0,1) f(\xi) d\xi, \quad \Psi(z) = S_y \int_{AB} \Psi_0(z,t,0,1) f(\xi) d\xi,
$$

(11)

where $\xi = \text{Re} t$. Functions $\Phi_0(z,t,X,Y), \Psi_0(z,t,X,Y)$ are linear relatively to $X, Y$ and $\sin \gamma = -dx / ds$.

These potentials in the inner points of the plate for which the sub-integral functions are smooth can be calculated using known quadrature formulas. In particular, we assume that the boundary equation is written in a parametric form $z = \omega(\theta), 0 < \theta < 2\pi$. Using the quadrature formula of rectangles, which, for periodic smooth functions, with increased accuracy, we have

$$
\Phi(z) = h S_y \sum_{n=1}^{M} \Phi_0(z,t_n,0,1) f(x_n) x'(\theta_n),
$$

(12)

$$
\Psi(z) = h S_y \sum_{n=1}^{M} \Psi_0(z,t_n,0,1) f(x_n) x'(\theta_n),
$$

(13)

where $h = 2\pi / M$, $\theta_n = n h$, $t_n = \omega(\theta_n)$, $x_n = \text{Re} t_n, x'(\theta) = \text{Re}(\omega'(\theta))$.

When determining hoop stress $\sigma_\theta$ on the edge of the plate just enough to calculate the function $\Phi$, as here $\sigma_\theta + N = 4 \text{Re} \Phi(z)$, where $N + i T = (X + iY) ds / dt$. Integral function at the point $t$ at limit of concentrated force $(X,Y)$ is

$$
\Phi_0(z,t,X,Y) \sim \frac{(X + iY)}{2\pi(t - z)},
$$

(14)

Therefore, we shall represent $\Phi_0(z,t,0,1) = -\frac{1}{2\pi i(t - z)} + F(t,z)$, where $F$ - regular function.

Then after the limit crossing, using the Plemelj-Sokhotski formula, we obtain

$$
\sigma_\theta = N + 4 \text{Re} \Phi(z), \quad N = -S_y f(x)(dx / ds)^2,
$$

(15)

where $\Phi(z)$ - is equal to the function $\Phi$, in which the Cauchy integral is considered in the sense of the main value. Well-known quadrature formulas can be used to calculate this integral.

Results of calculations. Consider a rectangular plate with half-edges $a$ and $b$ with residual deformations (4), (5) at $b/a = 1.4, d/a = 0.3, a_0 = 0$. Complex potentials $\Phi_0, \Psi_0$ were determined in a closed form on the basis of the Muschelishvili method using the conformal mapping of the area occupied by a plate on a circle of the unit radius [16]. The display function was expanded into a Taylor series, which then held up to one hundred variables. At fig. 1 the distribution of found residual relative stresses is given $\sigma_y / S_y$ in cross-sections $y = 0, 0.5a, a, 1.3a$, and the value is given near the curves $y / a$.

Results of calculations of relative stresses $\sigma_y / S_y$ in cross-sections $x = ma$ at $m = 0, 0.2, 0.5$ depending on the coordinate $y$ see Fig. 1 on the right.

The given graphs practically coincide with the data of calculations, which are obtained in [6] for this task with another method.
Figure 1. The distribution of residual relative stresses $\sigma_y / S_y$ is given in cross-sections $y = 0, 0.5a, a, 1.3a$, and the value is given near the curves $y / a$.

Calculation of residual stresses in a rectangular plate at $b / a = 1.5$ when the distribution of residual deformations has the form $f(x) = \sqrt{1 - x^2 / d^2}$ also showed complete convergence of calculation results with data [6], which are obtained by other methods.

**Residual stresses in the half-plane and the lane.** Consider the half-plane $y < 0$ load-free edge. On the basis of the half-plane, it is expedient to simulate the distribution of residual stresses in a plate of large size in the vicinity of the seam near the limits. Determining a correction solution reduces to the consideration of the half-plane, which is under the influence of the load $p(x) = \sigma_y(x, 0) - i\tau_y(x, 0) = -S_y f(x)$, with $p \neq 0$ at $-d < x < d$. Complex potentials for this case are determined by the formulas [16]

$$
\Phi(z) = \frac{1}{2\pi i} \int_{-d}^{d} \frac{p(t)dt}{z - t}, \quad \Omega(z) = \frac{1}{2\pi i} \int_{-d}^{d} \frac{p(t)dt}{z - t},
$$

and $\Psi(z) = \Omega(z) - z\Phi(z) - \Phi(z)$.

Let $f(x) = 1 - x^2 / d^2$. In this case

$$
\Phi(z) = \frac{C}{2\pi id} \left[ \left( \frac{z^2 - d^2}{z^2 + d^2} \right) L(z, d) + 2dz \right], \quad \Phi'(z) = \frac{C}{2\pi id} \left[ 2zL(z, d) + 4d \right],
$$

$$
\Omega(z) = \frac{C}{2\pi id} \left[ \left( \frac{z^2 - d^2}{z^2 + d^2} \right) L(z, d) + 2dz \right],
$$

where $L(z, d) = \ln \frac{z - d}{z + d}$, $C = -S_y$.

Adding partial and correction components, we obtain the solution of the problem of residual stresses in the half-plane.

Consider the lane $-H < y < 0$. The complex potentials for a strip are represented in the form

$$
\Phi(z) = \Phi_p(z) + \Phi_{\Delta}(z), \quad \Omega(z) = \Omega_p(z) + \Omega_{\Delta}(z),
$$

where $\Phi_p(z), \Omega_p(z)$—potentials for the half-plane $y < 0$, loaded with forces applied to the band at $y = 0$. Expressions for potentials $\Phi_{\Delta}(z), \Omega_{\Delta}(z)$, which when $y > -H$ defined by integrals of smooth functions, is given in [5].
For example, calculations of stresses in the band due to residual deformations at 
\[ \varepsilon_y = -S_y (1 - \frac{x^2}{d^2}) / E. \]
Calculated stresses \( \sigma_y, \sigma_x, \tau_{xy} \) divided by \( S_y \) depending on the relative coordinate \( x/d \) at \( d/H=0.05 \) shown in fig. 2. With numbers 0, 1, 2, 3, 4, 5 near the curves correspond to the cross sections \( y=-0.1jd \) at \( j=0, 1, 2, 3, 4, 5. \)

Obtained stresses with \( d/H = 0.1 \) are well consistent with the data for a rectangular plate of fig. 1.

In fig. 3 shows the distribution of relative stresses \( \sigma_x / S_y \) (a) and \( \sigma_y / S_y \) (b) at \( x=0, -H/2 < y < 0 \) depending on the width of the distribution zone of the initial deformations. Here curves 0,….5 correspond to the values \( d/H = 0.05, 0.1, 0.25, 0.5, 0.75, 1. \)
Figure 3. The distribution of relative stresses $\sigma_x / S_x$ (a) and $\sigma_y / S_y$ (b) at $x=0$, $-H/2 < y < 0$.

**Determining in a rectangular plate with residual stresses.** Consider now the case when in a plate with residual stresses a rectilinear cracks a half length $d$ centered at the point $(x_c, y_c)$, which is tilted at an angle $\alpha$ to axis $Ox$. The stresses in such a plate are represented as the sum of stresses in a solid plate with residual deformations and stresses corresponding to the correction solution. Denote the stress vector in the solid plate on the line on which the crack is as $(X_L, Y_L)$. Then, in order to determine the correction solution, we consider the problem of the elasticity theory for a plate with a crack, to the edges of which the effort $(X_L, Y_L)$ is applied. We will also consider the case, when, the edges can contact with each other. Then the above-mentioned efforts will be applied only in sections of the crack, where there is no contact, and in the contact area we assume that there are no tangential stresses (smooth contact) and normal edge displacements are the same. To solve this problem we use an algorithm of work [17].

Calculated relative values $F_{I,II} = K_{I,II} / (S_y \sqrt{\pi d})$ taking into account the contact of the edges for the case $d / a = 0, 2$ at $y_c = 0$, $\alpha = 0$, depending on the location of the center $x_c$ for the left (A) and right tips (B) the crack is shown in Fig. 4. on the left.

**Figure 4.** Calculated relative values $F_{I,II} = K_{I,II} / (S_y \sqrt{\pi d})$ for the left (A) and right tips (B) the crack.
Similar results are parallel to the axis $Ox$ cracks at $y_c / a = 0.5$ shown in fig. 4 on the right.

Figure 5 shows the distribution for an inclined crack depending on the angle $\alpha$ when placing the center of the crack at the points $(0,0)$ and $(0, 2a)$. 

![Figure 5](image)

**Figure 5.** Calculated relative values $F_{\alpha,\beta} = K_{\alpha,\beta} / (S \sqrt{\pi a})$ for the left (A) and right tips (B) the crack.

Based on the calculations made, the following conclusions can be made: when determining SIF for cracks, which are in the field of residual stresses, it is necessary to take into account the contact of the edges of the cracks; the maximum value for cracks perpendicular to the weld seam SIF $K_I$ is achieved if the crack is shifted; the contact of the shores, as a rule, takes place near the peak of the seam.

Conclusion. An algorithm for determining the stress state of plates with cracks, caused by residual deformations is proposed. It has been established that ignoring contact with the edges of the cracks can lead to significant errors in determining SIF under the action of residual stresses. SIF can reach the maximum values for the shifted relative to the center of seam cracks.

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