PRELIMINARY VERSION

GAP PROBABILITIES FOR THE CARDINAL SINE

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ABSTRACT. We study the zero set of random analytic functions generated by
a sum of the cardinal sine functions that form an orthogonal basis for the Paley-
Wiener space. As a model case, we consider real-valued Gaussian coefficients.
It is shown that the asymptotic probability that there is no zero in a bounded
interval decays exponentially as a function of the length.

1. INTRODUCTION

A simple point process in \( \mathbb{R} \) is a random integer-valued positive Radon measure
on \( \mathbb{R} \) that almost surely assigns at most measure 1 to singletons. Simple point
processes can be identified with random discrete subsets of \( \mathbb{R} \). In this paper, we
study ‘gap probabilities’ of the simple point process in \( \mathbb{R} \) given by the zeros of the
random function
\[
f(z) = \sum_{n \in \mathbb{Z}} a_n \frac{\sin \pi(z - n)}{\pi(z - n)},
\]
where \( a_n \) are i.i.d. random variables with zero mean and unit variance. Kol-
mogorov’s inequality shows that this sum is almost surely pointwise convergent.
In fact, since
\[
\sum_{n \in \mathbb{Z}} \left| \frac{\sin \pi(z - n)}{\pi(z - n)} \right|^2
\]
converges uniformly on compact subsets of the plane, this series almost surely
defines an entire function. If we take \( a_n \) to be Gaussian random variables then \( f \)
is a Gaussian analytic function (GAF) (see [HKPV, Lemma 2.2.3] for details).

We shall be chiefly concerned with the functions given by taking \( a_n \) to be real
Gaussian random variables. We denote by \( n_f \) the counting measure on the set of
zeros of \( f \). These functions are an example of a stationary symmetric GAF and
Feldheim [F] has shown that the density of zeros is given by
\[
\mathbb{E}[n_f(z)] = S(y)m(x,y) + \frac{1}{2\sqrt{3}}\mu(x),
\]
where \( z = x + iy \), \( m \) denotes the planar Lebesgue measure, \( \mu \) is the singular
measure with respect to \( m \) supported on \( \mathbb{R} \) and identical to Lebesgue measure.

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there, and
\[ S\left(\frac{y}{2\pi}\right) = \pi \left| \frac{d}{dy} \left( \frac{\cosh y - \frac{\sinh y}{y}}{\sqrt{\sinh^2 y - y^2}} \right) \right|. \]
(Here \( S \) is defined only for \( y \neq 0 \), in fact the atom appearing in \( [1] \) is the distributional derivative at 0.) We observe that since \( S(y) = O(y) \) as \( y \) approaches zero there are almost surely zeros on the real line, but that they are sparse close by. Moreover the zero set is on average uniformly distributed on the real line. We are interested in the ‘gap probability’, that is the probability that there are no zeros in a large interval on the line. Our result is the following asymptotic estimate.

**Theorem 1.** Let \( f \) be the symmetric GAF given by the almost surely convergent series
\[
\sum_{n \in \mathbb{Z}} a_n \sin \frac{\pi (z - n)}{\pi (z - n)},
\]
where \( a_n \) are i.i.d. real Gaussian variables with mean 0 and variance 1. Then, there exist constants \( c, C > 0 \) such that for all \( r \geq 1 \),
\[
e^{-cr} \leq \mathbb{P}(\#(Z(f) \cap (-r, r)) = 0) \leq e^{-Cr}.
\]

**Remark 1.** If instead of considering intervals we consider the rectangle \( D_r = (-r, r) \times (-a, a) \) for some fixed \( a > 0 \), then we compute a similar exponential decay for \( \mathbb{P}(\#(Z(f) \cap D_r) = 0) \)

**Remark 2.** Suppose that the \( a_n \) are i.i.d. Rademacher distributed. I.e., each \( a_n \) is equal to either \(-1\) or \(1\) with equal probability. Since \( f(n) = a_n \) for \( n \in \mathbb{N} \), it follows that if not all \( a_n \) for \( |n| \leq N \) are of equal sign, then by the mean value theorem, \( f \) has to have a zero in \((-N, N)\). As is shown below, the remaining two choices of the \( a_n \) for \( |n| \leq N \) each yield an \( f \) without zeroes in \((-N, N)\). Clearly, this gives a probability of \( 2(1/2)^{2N} \) for \( f \) to be without zeroes there.

**Remark 3.** The Cauchy distribution is given by the density
\[
p(x) = \frac{1}{\pi} \frac{1}{x^2 + 1}.
\]
Whereas the Rademacher distribution is in some sense a simplified Gaussian, the Cauchy distribution is very different: It has neither an expectation, nor a standard deviation. If we suppose that the \( a_n \) are i.i.d. Cauchy distributed, it is not hard to see that with probability one the sum \( \sum a_n/n \) diverges, whence the related random function diverges everywhere.

The main motivation for our work comes from the ‘hole theorem’ proved by Sodin and Tsirelson in \([ST]\) for point-processes uniformly distributed in the plane. The authors consider the GAF defined by
\[
F(z) = \sum_{n=0}^{\infty} a_n \frac{z^n}{\sqrt{n!}},
\]
where $a_n$ are i.i.d. standard complex normal variables. For this function, the density of zeros is proportional to the planar Lebesgue measure and the authors compute the asymptotic probability that there are no zeros in a disc of radius $r$ to be $e^{-cr^4}$, where $c > 0$. The analogy with the GAF we consider becomes clear if we note that $(z_n^{1/2})_{n=0}^{\infty}$ constitutes an orthonormal basis for the Bargmann-Fock space

$$\mathcal{F} = \{ f \in H(\mathbb{C}) : \|f\|_\mathcal{F}^2 = \int_{\mathbb{C}} |f(z)|^2 e^{-2|z|^2} \frac{dm(z)}{\pi} < +\infty \},$$

where $m$ is the planar Lebesgue measure. We are replacing these functions with the sinc functions, which constitute a basis for the Paley-Wiener space. An important caveat is that, though $f$ is constructed from an orthonormal basis, $f$ is almost surely not in the Paley-Wiener space, since the sequence of coefficients $a_n$ is almost surely not in $\ell^2(\mathbb{Z})$. It is not hard to see however that $f$ belongs almost surely to the Cartwright class.

2. Proof of Theorem 1

2.1. Upper bound. We want to compute the probability of an event that contains the event of not having any zeroes on $(-N, N)$, for $N \in \mathbb{N}$. One such event is that the values $f(n)$ have the same sign for $|n| \leq N$. The probability of this event is

$$\mathbb{P}\left(a_n > 0 \text{ for } |n| \leq N \text{ or } a_n < 0 \text{ for } |n| \leq N \right) = 2(1/2)^{2N+1} = e^{-CN},$$

for some constant $C > 0$.

Remark 4. The same upper bound holds when $a_n$ are i.i.d. random variables with $0 < \mathbb{P}(a_n > 0) < 1$ for which the random function $\sum_{n \in \mathbb{Z}} a_n \text{sinc}(x - n)$ converges.

2.2. Lower bound. To compute the lower hole probability, we use the following scheme. First, we introduce the deterministic function

$$f_0(x) = \sum_{n=-2N}^{2N} \text{sinc}(x - n).$$

We show in Lemma 1 that it has no zeroes on $(-N, N)$, and we find an explicit lower bound on $(-N, N)$ for it. This lower bound does not depend on $N$. Second, we consider the functions

$$f_1(x) = \sum_{n=-2N}^{2N} (a_n - 1) \text{sinc}(x - n) \quad \text{and} \quad f_2(x) = \sum_{|n| > 2N} a_n \text{sinc}(x - n),$$

which induce the splitting

$$f = f_0 + f_1 + f_2.$$
We show that for all $x \in [-N, N]$ we have $|f_1(x)| \leq \epsilon$ with probability at least $e^{-cN}$ for large $N$ and some constant $c > 0$. Moreover, we show that

$$\mathbb{P}\left(\sup_{x \in [-N, N]} |f_2(x)| \leq \epsilon\right)$$

is larger than, say, $1/2$ for big enough $N$. As the events on $f_1$ and $f_2$ are clearly independent, the lower bound now follows by choosing $\epsilon$ small enough.

We turn to the first part of the proof.

**Lemma 1.** Given $N \in \mathbb{N}$ and

$$f_0(x) = \sum_{n=-2N}^{2N} \text{sinc}(x - n) = \sin \pi x \sum_{n=-2N}^{2N} \frac{(-1)^n}{\pi(x - n)}. \hspace{1cm} (2)$$

Then, there exists a constant $C > 0$ such that, for $N$ big enough,

$$1 - \frac{C}{N} \leq \inf_{|x| \leq N} f_0(x) \leq \sup_{|x| \leq N} f_0(x) \leq 1 + \frac{C}{N}.$$

**Proof.** Let $R = R(N)$ be the rectangle of length $4N + 1$ and height $4N$, centered at $x = 1/2$. By the residue theorem, it holds that

$$\frac{1}{2\pi i} \oint_R \frac{dz}{(z - x) \sin \pi z} = \frac{1}{\pi} \sum_{n=-2N+1}^{N} \frac{(-1)^n}{n - x} + \frac{1}{\sin \pi x}.$$ 

Observe that if we shift around the terms, this yields

$$\sin \pi x \sum_{n=-2N+1}^{2N} \frac{(-1)^n}{x - n} = 1 + \frac{\sin \pi x}{2\pi i} \oint_R \frac{dz}{(x - z) \sin \pi z}.$$ 

Now, given $-N \leq x \leq N$, it is easy to bound this last integral by $C/N$. $\square$

**Remark 5.** The same bound holds for all points $z$ in a strip with fixed height $[-N, N] \times [-C, C]$ for some $C > 0$. We observe though, that the function $f_0(z)$ in (2) is close to zero around $\text{Im } z = \log N$. Indeed, it is smaller than $e^{-cN}$ there, for some $c > 0$ independent of $N$.

2.2.1. The middle terms. Let $\epsilon > 0$ be given, and consider $N \in \mathbb{N}$ to be fixed. We look at the function

$$f_1(x) = \sum_{n=-2N}^{2N} (a_n - 1)\text{sinc}(x - n) = \frac{\sin \pi x}{\pi} \sum_{n=-2N}^{2N} (a_n - 1) \frac{(-1)^n}{x - n}.$$ 

To simplify the expression, we set $b_n = (a_n - 1)(-1)^n$. We want to compute a lower bound for the probability that, for $x \in [-N, N]$,

$$|f_1(x)| \leq \epsilon.$$

Since, for $|n| \leq N$, we have $f_1(n) = a_n - 1$, the condition

$$|a_n - 1| \leq \epsilon \text{ for } |n| \leq N$$

Now, we want to bound the integral

$$\int_{-N}^{N} |f_1(x)| dx.$$
is necessary.

Define \( B_n = b_{-2N} + \ldots b_n \) for \(|n| \leq 2N\) with \( B_{-2N-1} = 0 \), and suppose that \( x \not\in \mathbb{Z} \). With this, summation by parts yields

\[
\sum_{n=-2N}^{2N} \frac{b_n}{x-n} = -\sum_{n=-2N}^{2N} \frac{B_n}{(x-n)(x-n-1)} + \frac{B_{2N}}{x-2N-1}. \tag{3}
\]

We now claim that under the event \( E = \{ |b_n| \leq \epsilon, |B_n| \leq \epsilon \, \text{ for } |n| \leq 2N \} \), we have

\[
|f(x) - f_0(x)| \leq \epsilon \, \text{ for } |x| \leq N,
\]

with a bound independent of \( N \). Indeed, the second summand at the right hand side of (3) converges almost surely to zero, because

\[
\left| \frac{B_{2N}}{x-2N-1} \right| \leq \frac{\epsilon}{N}.
\]

Suppose that \( x \in (k, k+1) \) and split the first sum in (3) as

\[
\sum_{n=-2N}^{2N} \frac{B_n}{(x-n)(x-n-1)} = \sum_{n=k-1}^{k+1} \frac{B_n}{(x-n)(x-n-1)} + \sum_{n=-2N}^{k-1} \frac{B_n}{(x-n)(x-n-1)} + \sum_{n=k+2}^{2N} \frac{B_n}{(x-n)(x-n-1)}.
\]

Then

\[
\left| \sum_{n=-2N}^{k-1} \frac{B_n}{(x-n)(x-n-1)} \right| \leq \sum_{n=k+2}^{k+1} \frac{\epsilon}{(k+1-n)^2} + \sum_{n=k-2}^{k-1} \frac{\epsilon}{(k-n)^2} \lesssim \epsilon.
\]

For the remaining terms, the function \( \sin \pi x \) comes into play. E.g., suppose that \( |x-k| \leq 1/2 \), then

\[
\left| \frac{\sin \pi x}{(x-k)(x-k-1)} \right| \lesssim \frac{\epsilon}{|x-k-1|} \left| \frac{\sin \pi(x-k)}{\pi(x-k)} \right| \lesssim \epsilon.
\]

The remaining terms are treated in exactly the same way.

What remains is to compute the probability of the event \( E \) defined by (4). We recall that the \( b_n \) were all defined in terms of the real and independent Gaussian variables \( a_n \). So the event \( E \) above defines a set

\[
V = \left\{ (t_{-2N}, \ldots, t_{2N}) \in \mathbb{R}^{4N+1} : |t_n| \leq \epsilon, \right. \left. \sum_{n=-2N}^{n} t_n \leq \epsilon, |n| \leq 2N \right\}
\]

in terms of the values of the \( a_n \). Hence,

\[
\mathbb{P}(E) = c^{2N} \int \cdots \int_{V} \frac{1}{e^{-(t_{-2N}^2+\cdots+t_{2N}^2)/2}} \ dt_{-2N} \cdots dt_{2N}.
\]
Here, $c$ is the normalising constant of the one dimensional Gaussian. Since $|a_n - 1| < \epsilon$, it follows that

$$\Pr(E) \geq e^{2N} e^{-N(1+\epsilon)^2} \int V \prod dt_1 \cdots dt_N = e^{2N} e^{-N(1+\epsilon)^2} \Vol(V).$$

We now seek a lower bound for this euclidean $(4N + 1)$-volume.

To simplify notation, we pose this problem as follows. For real variables $x_1, \ldots, x_N$, we wish to compute the euclidean volume of the solid $V_N$ defined by $|x_i| \leq \epsilon$ for $i = 1, \ldots, N$ and

$$|x_1 + x_2| \leq \epsilon,$$
$$|x_1 + x_2 + x_3| \leq \epsilon$$
$$\vdots$$
$$|x_1 + x_2 + \cdots + x_N| \leq \epsilon.$$

One way to do this is as follows. Write $y_N = x_1 + \cdots + x_{N-1}$, then

$$\Vol(V_N) = \int \cdots \int_{V_{N-1}} \left( \int_{\min\{\epsilon, -y_N\}}^{\max\{-\epsilon, -y_N\}} dx_N \right) dx_1 \cdots dx_{N-1}.$$

This is illustrated in Figure 1. Clearly, whenever $y_N < 0$, the upper limit is $\epsilon$, and whenever $y_N > 0$, the lower limit is $-\epsilon$. Hence,

$$\Vol(V_N) \geq \int \cdots \int_{V_{N-1} \cap \{y_N < 0\}} (\int_0^{\epsilon} dx_N) dx_1 \cdots dx_{N-1} + \int \cdots \int_{V_{N-1} \cap \{y_N > 0\}} (\int_0^{-\epsilon} dx_N) dx_1 \cdots dx_{N-1} = \epsilon \Vol(V_{N-1}).$$

Iterating this, we get

$$\Vol(V_N) \geq \epsilon^N.$$
In conclusion,
\[ \mathbb{P}(E) \geq e^{-cN}, \]
which concludes this part of the proof.

2.2.2. The tail. We now turn to the tail term
\[ f_2(x) = \sum_{|n|>2N} a_n \text{sinc}(x - n). \]
Clearly, we need only consider the terms for which \( n \) is positive. Set \( c_n = (-1)^n a_n \).
We factor out the sine factor as above, and apply summation by parts, to get
\[
\sum_{n>2N}^{L} \frac{c_n}{x-n} = \sum_{2N+1}^{L} C_n \frac{-1}{(x-n)(x-n-1)} + \frac{C_L}{x-L-1}. \tag{5}
\]
where
\[ C_n = c_{2N+1} + \cdots + c_n, \quad C_{2N} = 0. \]
We want to take the limit as \( L \to \infty \). It is easy to see that the last term almost surely tends to zero. Indeed, \( C_L \) is a sum of independent normal variables with mean 0 and variance 1, and therefore is itself normal with mean 0 and variance \( L-2N \). Moreover, since
\[
\left| \frac{C_L}{x-L-1} \right| \lesssim \left| \frac{C_L}{N-L} \right|,
\]
and the random variable inside of the absolute values on the right-hand side has variance \( L-2N \), it follows that limit is almost surely equal to 0, whence we are allowed to let \( L \to \infty \) in (5).

We prove the following. With a positive probability, we have
\[
\left| \sum_{2N+1}^{L} C_n \frac{1}{(x-n)(x-n-1)} \right| \leq \epsilon.
\]
As \( n^2 \approx |(x-n)(x-n-1)| \) for \(|x| \leq N \) and \( n > 2N \), it is enough to consider the expression
\[
\sum_{2N+1}^{L} \frac{|C_n|}{n^2}.
\]
The absolute value of a Gaussian random variable has the folded-normal distribution. In particular, if \( X \sim N(0, \sigma^2) \), then
\[
\mathbb{E}(|X|) = \sigma \sqrt{\frac{2}{\pi}}.
\]
Since, in our case, \( \sigma^2 = n-2N \), this yields
\[
\mathbb{E} \left( \sum_{2N+1}^{L} \frac{|C_n|}{n^2} \right) \lesssim \sum_{2N+1}^{L} \frac{\sqrt{n-2N}}{n^2} \lesssim \sum_{1}^{\infty} \frac{1}{(n+2N)^{3/2}} \lesssim \frac{1}{\sqrt{N}}.
\]
Finally, by Chebyshev’s inequality,
\[
P\left( \sum_{2N+1}^{L} \frac{|C_n|}{n^2} \leq \epsilon \right) \geq 1 - \frac{\mathbb{E}(Y)}{\epsilon} \geq 1 - \frac{C}{\epsilon \sqrt{N}}.
\]

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