Partially Asymmetric Simple Exclusion Model in the Presence of an Impurity on a Ring

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Abstract

We study a generalized two-species model on a ring. The original model [1] describes ordinary particles hopping exclusively in one direction in the presence of an impurity. The impurity hops with a rate different from that of ordinary particles and can be overtaken by them. Here we let the ordinary particles hop also backward with rate $q$. Using the Matrix Product Ansatz (MPA), we obtain the relevant quadratic algebra. A finite dimensional representation of this algebra enables us to compute the stationary bulk density of the ordinary particles, as well as the speed of impurity on a set of special surfaces of the parameter space. We will obtain the phase structure of this model in the accessible region and show how the phase structure of the original model is modified. In the infinite-volume limit this model presents a shock in one of its phases.

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1 Introduction

Recently much attention has been focused on one-dimensional reaction-diffusion processes. These models can describe many physical phenomena such as hopping conductivity, growth processes and traffic flows [2-4]. They are also of interest from the mathematical point of view due to their relation to integrable quantum chain Hamiltonians [5,6]. The simplest model of this kind is the Asymmetric Simple Exclusion Process (ASEP) with open boundaries [7]. This model comprise particles which jump independently to their right with hard core repulsion along a one-dimensional lattice. The open boundary conditions mean that particles are injected at one end of the lattice and are removed at the opposite end. This model exhibits a shock structure in the density profile of particles. In periodic boundary conditions case the microscopic location of the shock can be identified by defining a second class particle (impurity). The ASEP in the presence of an impurity on a ring has been studied in the two following cases:

1) In the first case, the single impurity hops in the opposite direction relative to the ordinary particles [8,9]. In this case a first-order phase transition between a low-density and a traffic jam phase is observed.

2) In the second case, the impurity moves in the same direction as the ordinary particles [1,17]. The phase diagram of this model consists of six distinctive phases (I-VI) in which two of them are symmetric to other phases under a charge conjugation and reflection symmetry. The authors have shown that one phase exists in the system in which the impurity causes a shock.

Another example of such driven diffusive systems is the Partially Asymmetric Simple Exclusion Process (PASEP). In this model the particles are allowed to jump both to their immediate right (with rate $p$) or left (with rate $q$) site, if the target site is not already occupied. This model has been extensively studied both with open boundaries and on a ring [10].

In this paper we will study the effects of the presence of a single impurity on the PASEP on a ring. Here the ordinary particles can hop to their immediate right (left) site, provided that it is empty, with rate 1 ($q \leq 1$). The single impurity can only hop to its immediate right site (if it is not already occupied) with rate $\alpha(\leq 1)$ and can be exchanged from the left with the ordinary particles with rate $\beta(\leq 1)$. For $q = 0$ this model reduces to the model (2) as discussed above. Using Matrix Product Ansatz
(MPA) introduced in [11], we obtain the relevant quadratic algebra which has both finite and infinite dimensional representations. For simplicity, we adopt a finite dimensional representation of the algebra and carry out all the calculations using a grand canonical ensemble in which the population of ordinary particles can fluctuate. By adjusting the fugacities of the ordinary particles and of the holes, one can produce some fixed densities for them. Although the finite dimensional representation restricts us to the region under the surface \( \alpha + \beta + q = 1 \) in the three-dimensional parameter space, nevertheless we shall find the exact phase structure and calculate precisely the density profile of ordinary particles and the speed of impurity in this region. With this exact results, we will show that three phases exist in this region. In two of them, which are symmetric to each other under a charge conjugation and reflection symmetry, the density profile of particles has an exponential behavior. We will determine the relevant correlation lengths in these phases and the critical values of the rates that characterize the divergence of these correlation lengths. In the third phase the density profile of the ordinary particles is linear which is a signature of a shock.

This paper is organized as follows. In section 2 we will describe the model. In section 3 we will write the weights of the configurations in the stationary state in terms of a trace of \( L + 1 \) non-commuting matrices and obtain the quadratic algebra. In section 4 we will introduce all possible representations of the quadratic algebra and discuss different phases in the accessible region of the phase space. In section 5 using one and two-dimensional representations of the algebra we will obtain some exact results for an infinite system. In the last section we will compare our results with those obtained in [1] for \( q = 0 \).

2 The Model

The model proposed here contains two species of particles on a closed ring of \( L + 1 \) sites which are labeled from 1 to \( L + 1 \). We specify each configuration of the system by an \( L + 1 \)-tuple \((\tau_1, \tau_2, ..., \tau_{L+1})\), where \( \tau_i = 1 \) if site \( i \) is occupied with a particle of kind 1, \( \tau_i = 2 \) if it is occupied with a particle of kind 2, and \( \tau_i = 0 \) if site \( i \) is empty. There are \( M \) particles of kind 1, and only one particle of kind 2 on the ring. We will refer to them as ordinary particles and impurity respectively.

The system evolves under stochastic dynamics. The possible exchanges between two adjacent sites during a time interval \( dt \) is as follows:
The space of configuration is connected. Each configuration can evolve into any other and therefore, it has a unique stationary state [18]. As we mentioned above, we will perform all the calculations in the grand canonical ensemble and let the number of ordinary particles fluctuate around a mean value. We will show that in the large $L$ limit the fluctuations in the density of ordinary particles drop to zero so that in this limit, the results of the canonical ensemble agree with those of the grand canonical ensemble. The above process can be mapped onto the one introduced in [12] by interchanging the impurity and vacancies ($2 \leftrightarrow 0$). It can also be considered as a special case of the three-species diffusion problems introduced in [13].

3 The Stationary Measure and the Quadratic Algebra

According to the matrix product formalism, the stationary probability $P(\{\tau\})$ of any configuration $\{\tau\}$ can be written as a trace of a product of non-commuting operators. Since this model is translationally invariant, one can always keep the single impurity at site $L+1$ and write the normalized weight $P(\{\tau\})$ in terms of three operators $D$, $E$ and $A$:

$$P(\{\tau\}) = P(\{\tau_1, \tau_2, ..., \tau_L, \tau_{L+1} = 2\}) = \frac{1}{Z_L} Tr\prod_{i=1}^{L}(x\tau_i D + y(1 - \tau_i)E)A$$  \hspace{1cm} (1)

The non-zero real variables, $x$ and $y$, are the fugacities of ordinary particles and vacancies respectively. In the grand canonical ensemble approach, one must choose the value of the fugacities to fix the density of ordinary particles to be $\rho = \frac{M}{L}$. Although it would be sufficient to introduce only one fugacity (since the total number of sites "$L+1$" is fixed), we use both of them to make the symmetry between ordinary particles and vacancies more apparent. The normalization factor $Z_L$ in the denominator of Eq.(1), which
plays a role analogous to the partition function in equilibrium statistical mechanics, is a fundamental quantity and can be calculated using the fact \( \sum_{\{\tau\}} P(\{\tau_1, ..., \tau_L, \tau_{L+1}\}) = 1 \). Thus one finds

\[
Z_L = \sum_{\text{all configurations}} Tr[\prod_{i=1}^{L}(x\tau_i D + y(1-\tau_i)E)A] = Tr(C^L A)
\]  

in which \( C = xD + yE \). The operators \( D, E \) and \( A \) satisfy the following quadratic algebra:

\[
DE - qED = D + E
\]

\[
\beta DA = A
\]

\[
\alpha AE = A
\]

Assuming that \( A = |V\rangle \langle W| \), Eqs.(3-5) can be written in terms of \( |V\rangle \) and \( \langle W| \) as follows

\[
DE - qED = D + E
\]

\[
D|V\rangle = \frac{1}{\beta}|V\rangle
\]

\[
\langle W|E = \frac{1}{\alpha}\langle W|
\]

With this assumption, the calculation of \( Tr(...) \) in expression (2) reduces to the calculation of a matrix element

\[
Z_L = \langle W|C^L|V\rangle
\]

In the next section we will investigate all possible representations of (6-8).

### 4 Representations of the Quadratic Algebra

In [14] the Fock representation of the most general quadratic algebra has been obtained. It has been shown that in order to have a finite dimensional representation of certain
quadratic algebras, the parameters of the model, say $\alpha, \beta$ and $q$ in (6-8), should satisfy a set of constraints. The algebra (6-8) has a one-dimensional representation for $\alpha, \beta, q < 1$, where the operators $D$, $E$ and $A$ are represented by real numbers. Here the vectors $|V\rangle$ and $\langle W|$ can be discarded and one is dealing with a scalar product state in (9). This representation exists if

$$D = \frac{1}{\beta}, \quad E = \frac{1}{\alpha}, \quad A = 1$$

and the following constraint holds

$$\alpha + \beta + q = 1$$

In Fig.1 we have ploted (11) in the three-dimensional parameters space. One can easily check that the following matrices [15]

$$D = \frac{1}{1 - q} \begin{pmatrix} 1 + a & 0 & 0 & \cdot & \cdot \\ 0 & 1 + aq & 0 & \cdot & \cdot \\ 0 & 0 & 1 + aq^2 & 0 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 0 & 1 + aq^{n-2} & 0 \\ \cdot & \cdot & 0 & 0 & 1 + aq^{n-1} \end{pmatrix}$$

$$E = \frac{1}{1 - q} \begin{pmatrix} 1 + \frac{1}{a} & 0 & 0 & \cdot & \cdot \\ 1 & 1 + \frac{1}{aq} & 0 & \cdot & \cdot \\ 0 & 1 & 1 + \frac{1}{aq^2} & 0 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 & 1 + \frac{1}{aq^{n-2}} & 0 \\ \cdot & \cdot & 0 & 1 & 1 + \frac{1}{aq^{n-1}} \end{pmatrix}$$

and the vectors

5
\[ |V\rangle = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad |W\rangle = \begin{pmatrix} 1 \\ \omega_2 \\ \omega_3 \\ \vdots \\ \omega_n \end{pmatrix} \]

\[ \omega_i = \prod_{j=0}^{i-2} \frac{1}{\alpha} \left( \frac{1}{q^{n-1}} - \frac{1}{q^j} \right), \quad i = 2, \ldots, n \]  \hspace{1cm} (14)

In which

\[ a = \frac{1 - q - \beta}{\beta} = \frac{1}{1 - q - \alpha} q^{n-1} \]  \hspace{1cm} (15)

constitute an \( n \)-dimensional representation of (6-8) for \( \alpha, \beta, q < 1 \), provided that the following constraint between the parameters \( \alpha, \beta \) and \( q \) holds

\[ \left( \frac{1 - q - \beta}{\beta} \right) \left( \frac{1 - q - \alpha}{\alpha} \right) = q^{1-n} \]  \hspace{1cm} (16)

Note that for \( n = 1 \) the above constraint reduces to (11). Using (16) it can be verified that for \( q < 1 \), the region of the phase space which is accessible by the totality of all finite dimensional (\( n \geq 2 \)) representations is

\[ \alpha + \beta + q < 1 \]  \hspace{1cm} (17)

In Fig.2 we have plotted (16) for two values of \( n \). One can see that as the dimension of representation increases from 2, the two-dimensional surface (16) begins to approach towards the \( q \) axis. Therefore the finite dimensional representation of the algebra (6-8) allows us to derive exact results for this model only on some special surfaces given by (16). However using the same conjecture proposed in [15], one can introduce a kind of analytical continuation to obtain results valid in the whole accessible region given by (17).

In what follows we show that in the region specified by (17), only three phases can exist and this result is independent of \( n \) (dimension of the representation). We now find the asymptotic behavior of (9) for large values of \( L \). Note that the fugacity of ordinary particles \( x \) and vacancies \( y \) remain constant in this limit. In the thermodynamic limit \( (L \rightarrow \infty) \), the behavior of \( Z_L \) is governed by the largest eigenvalue of \( C \). The eigenvalues \( \xi_i \) of
\[ C = xD + yE = \frac{1}{1-q} \begin{pmatrix}
(x+y) + xa + \frac{x}{a} & 0 & \cdots & 0 \\
y & (x+y) + xaq + \frac{y}{aq} & 0 & \cdots \\
\cdots & \cdots & \cdots & \cdots \\
0 & \cdots & y(x+y) + xaq^{n-1} + \frac{y}{aq^{n-1}} & \cdots 
\end{pmatrix} \] (18)

can readily be computed

\[ \xi_i = \frac{1}{1-q} \left\{ (x+y) + xaq^{i-1} + \frac{y}{aq^{i-1}} \right\}, \quad i = 1, ..., n \] (19)

We notice that all the eigenvalues of \( C \) lie on the curve [15]

\[ z \rightarrow \frac{1}{1-q}((x+y) + xz + \frac{y}{z}) \]

in which \( z = a, aq, ..., aq^{n-1} \). Therefore apart from the values of the fugacities, the largest eigenvalue \( \xi_{\text{max}} \) of \( C \) takes one of the following values:

1) \( \xi_{\text{max}} = \xi_1 \), if \( \xi_1 > \xi_n \) or equivalently \( x(\frac{1-q-\beta}{\beta}) > y(\frac{1-q-\alpha}{\alpha}) \)

2) \( \xi_{\text{max}} = \xi_n \), if \( \xi_n > \xi_1 \) or equivalently \( x(\frac{1-q-\beta}{\beta}) < y(\frac{1-q-\alpha}{\alpha}) \)

3) \( \xi_{\text{max}} = \xi_1 = \xi_n \), if \( \xi_1 = \xi_n \) or equivalently \( x(\frac{1-q-\beta}{\beta}) = y(\frac{1-q-\alpha}{\alpha}) \)

These three different cases correspond to different phases. The process is invariant when the direction of motion is reversed and the following transformations are applied

\[
\begin{align*}
\text{Particle of kind 1} & \quad \rightarrow \quad \text{Particle of kind 0 (Vacancy)} \\
\text{Site number } i & \quad \rightarrow \quad \text{Site number } L+1-i \\
\rho & \quad \rightarrow \quad 1-\rho \\
\alpha & \quad \rightarrow \quad \beta
\end{align*}
\]

Now considering that the interchange of \( x \) and \( y \) is equivalent to the exchange of the density of ordinary particles and vacancies, \((I)\) is symmetric to \((II)\) under these trans-
formations.

By using (18) one can see that in the regions (I) and (II), $C$ is diagonalizable, so we expect that all correlation functions of form $\langle \tau_{i_1}...\tau_{i_{L+1}} \rangle$ depend exponentially on the distances involved. In the region (III) the eigenvalues of $C$ coincide

$$\xi_k = \xi_{n-k+1} = \frac{1}{1 - q} \{(x + y) + y\left(\frac{1 - q - \alpha}{\alpha}\right)(q^{k-1} + q^{n-k})\}, \quad k = 1, ..., n$$

In this case, since $C$ has an off-diagonal line (see 18), it is not diagonalizable. Hence it implies an algebraic behavior of all correlation functions (see discussions in [16]). In the next section we will show this explicitly for a two-dimensional representation.

The algebra (6-8) also has infinite dimensional representations [13] and by using them one has access to the entire phase space without any constraints on the parameters. Since for these representations the calculations seem to be very difficult, we shall adopt one and two-dimensional representations to study the general behavior of some interesting quantities, such as the density profile of ordinary particles and the speed of impurity in different phases.

5 One and Two Dimensional Representations. Exact Results

As mentioned in the previous section, using $n$-dimensional representations $(n \geq 2)$, only the region $\alpha + \beta + q < 1$ is accessible, where three phases exist. One-dimensional representation (10) limits us to that part of phase space given by the constraint (11). On this two dimensional surface, the partition function (9) has a simple form

$$Z_L = \left(\frac{x}{\beta} + \frac{y}{\alpha}\right)^L$$

The mean value of density of the ordinary particles which is defined by
\[ \rho = \frac{x}{L} \frac{d}{dx} \ln Z_L \]  

(22)

can easily be evaluated

\[ \rho = \frac{x}{\beta} \left( \frac{y}{\alpha} \right) \]  

(23)

If \( \rho_i \) denotes the expectation that site \( i \) is occupied with an ordinary particle in the stationary state, knowing that the only impurity is at site \( L + 1 \), we have

\[ \rho_i = x \frac{Tr(C^{i-1} D C^{L-i} A)}{Tr(C^L A)} = x \frac{\langle W | C^{i-1} D C^{L-i} | V \rangle}{\langle W | C^L | V \rangle} \]  

(24)

Consequently, using (10) and (23), we are able to evaluate the density profile of ordinary particles in this case

\[ \rho_i = \rho \]  

(25)

In Fig.3 we have plotted the phase diagram of the model for a constant \( q \). The accessible region (17) lies under the line \( \alpha + \beta = 1 - q \). On this line each configuration has the same stationary probability and mean field results become exact. In the stationary state, the speed of impurity \( v \) can be expressed as

\[ v = y\alpha \frac{Tr(EC^{L-1} A)}{Tr(C^L)} - x\beta \frac{Tr(C^{L-1} DA)}{Tr(C^L)} = (y - x) \frac{Z_{L-1}}{Z_L} \]  

(26)

when use has been made of the algebraic relations (6-8). Using (21), (23) and (26) the speed of impurity is found to be

\[ v = \alpha - (1 - q)\rho \]  

(27)

One may also note that for \( \alpha > (1 - q)\rho \), (27) becomes negative i.e. the impurity starts moving backward. This can easily be understood from the fact that whenever an ordinary particle appears behind the impurity, it overtakes the impurity while pushing it
backward (See section 2). Thus, when the density of ordinary particles exceeds a critical value \( \rho_c = \frac{\alpha}{1-q} \), a macroscopic negative current of impurity evolves in the system.

Now we will consider a two-dimensional representation. The following matrices

\[
D = \begin{pmatrix}
\frac{1}{\beta} & i \sqrt{\frac{q}{\beta}} \\
0 & \frac{\alpha}{\sqrt{q}}
\end{pmatrix}, \quad E = \begin{pmatrix}
\frac{1}{\alpha} & 0 \\
\frac{\alpha + q}{\alpha} & \frac{\alpha + q}{\alpha}
\end{pmatrix}
\] (28)

with the vectors

\[
|V\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |W\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}
\] (29)

satisfy (6-8), if the following relation holds

\[
(q + \alpha)(q + \beta) = q \quad (30)
\]

For \( n = 2 \), this representation is equivalent to (12-14) and (30) coincides with (16). As we saw, the properties of the matrix

\[
C = \begin{pmatrix}
\frac{x}{\beta} + \frac{y}{\alpha} & \frac{xi}{\sqrt{q}} \\
\frac{yi}{\sqrt{q}} & x(\frac{\beta + q}{\beta}) + y(\frac{\alpha + q}{\alpha})
\end{pmatrix}
\] (31)

is of prime importance in determining the phase structure of the model. The eigenvalues of \( C \) can easily be computed

\[
\xi_1 = \frac{x}{\beta} + (1 + \frac{q}{\alpha})y, \quad \xi_2 = (1 + \frac{q}{\beta})x + \frac{y}{\alpha}
\]

These eigenvalues also coincide with those obtained from (19) for \( n = 2 \). For \( \xi_1 \neq \xi_2 \), one can introduce

\[
U = \frac{1}{\sqrt{\frac{i}{\sqrt{q}}(y(\frac{1-q}{\alpha}) - x(\frac{1-q}{\beta}))}} \begin{pmatrix}
\frac{xi}{\sqrt{q}} & -y\frac{1-q}{\alpha} \\
\frac{yi}{\sqrt{q}} & -\frac{1-q}{\beta}
\end{pmatrix}
\] (32)
to diagonalize $C$

$$UCU^{-1} = \begin{pmatrix} \xi_1 & 0 \\ 0 & \xi_2 \end{pmatrix}$$

and using (9) and (29) obtain the following expression for $Z_L$

$$Z_L = \frac{x(\frac{1-q-\beta}{\beta})\xi_1^L - y(\frac{1-q-\alpha}{\alpha})\xi_2^L}{x(\frac{1-q-\beta}{\beta}) - y(\frac{1-q-\alpha}{\alpha})} \quad (33)$$

In the thermodynamic limit, two cases can be distinguished. The first case, corresponding to the phase I, is specified with $\xi_1 > \xi_2$, where the first term in (33) becomes dominant. The mean density of ordinary particles (22) can be evaluated as

$$\rho = \frac{\frac{x}{\beta}}{\frac{x}{\beta} + y(1 + \frac{\alpha}{\alpha})} \quad (34)$$

The inequality $\xi_1 > \xi_2$ can be written in terms of $\rho$ as

$$\alpha > (1 - q)(1 - \rho)$$

which with $\alpha + \beta + q < 1$ define the boundaries of the phase (I) (See Fig.3). In the $q \to 0$ limit, this phase corresponds to the phase (VI) of the model studied in [1]. Using (24), (28-30), (32) and (34) we obtain

$$\rho_1 = 1 - \frac{1}{q + \alpha}(1 - \rho) , \quad \rho_L = \rho \quad (35)$$

The density profile decreases exponentially from $\rho_1$ to $\rho_L$. One can define a characteristic length measuring the range of the effect of the impurity

$$\xi^{-1} = \ln \frac{q + \alpha}{1 + \rho(q - 1)} \quad (36)$$

Note that the correlation length diverges as $\rho \to 1 - \frac{\alpha}{1-q}$. For $L >> \xi$ the density profile of the ordinary particles is of the form
\[ \rho_i = c_1 e^{\frac{i}{\xi}} + c_2 e^{\frac{iL}{\xi}} + c_3 \]

in which

\[ c_1 = \frac{-xy(\frac{1-q-\alpha}{\alpha} - \frac{1-q-\beta}{\beta})}{(y\frac{1-q-\alpha}{\alpha} - x\frac{1-q-\beta}{\beta})\xi_2} \]
\[ c_2 = \frac{x(y - x)}{q(y\frac{1-q-\alpha}{\alpha} - x\frac{1-q-\beta}{\beta})\xi_1} \]
\[ c_3 = \frac{x(\frac{1}{q} - \frac{1}{\beta} - \frac{1-q-\beta}{\beta}) + y(1-q-\alpha)(\frac{\beta+q}{\beta})}{q(y\frac{1-q-\alpha}{\alpha} - x\frac{1-q-\beta}{\beta})\xi_1} \]

Using (26), (33) and (34) we find the following expression for the speed of impurity in phase \( (I) \)

\[ v = \frac{\alpha}{q + \alpha} - (1 - q)\rho \] (38)

One observes that for \( \rho > \frac{1-q-\beta}{1-q} \), the average speed of the single impurity again becomes negative. It can easily be checked that in \( q \to 0 \) limit the values of \( \rho_1, \rho_L \) and \( v \) approach to their corresponding values in the phase (VI) of [1]. Next we will examine the second phase which is characterized with \( \xi_2 > \xi_1 \). This phase is symmetric to phase \( (I) \) and all the results can be obtained using the transformations introduced in section 4. Finally the third phase \( (III) \) occurs when \( \xi_1 = \xi_2 \), where \( C \) is not diagonalizable (See 32). Nevertheless, \( Z_L \) can be computed noting that in this case \( C \) can be written as

\[
C = \begin{pmatrix}
\frac{x}{\beta} + \frac{y}{\alpha} & x \left( \frac{\beta+q}{\beta} \right) + y \left( \frac{\alpha+q}{\alpha} \right) \\
\frac{y}{\sqrt{q}} & \frac{x}{\sqrt{q}}
\end{pmatrix}
\]

\[
(\frac{x}{\beta} + \frac{y}{\alpha} - x(\frac{1-q-\beta}{\beta}))I + \begin{pmatrix}
x(\frac{1-q-\beta}{\beta}) & \frac{x}{\sqrt{q}} \\
\frac{y}{\sqrt{q}} & -y(\frac{1-q-\alpha}{\alpha})
\end{pmatrix}
= (\frac{x}{\beta} + \frac{y}{\alpha} - x(\frac{1-q-\beta}{\beta}))I + S \] (39)

in which

\[ S^2 = 0 \] (40)
Using this property with (9) and (29) yield

\[ Z_L = \langle W | C^L | V \rangle = (x \frac{q + \beta}{\beta}) + y \frac{q + \beta}{\alpha} + L x \frac{1 - q - \beta}{\beta} \] \quad (41) 

and the mean value of density of ordinary particles (22) is found to be

\[ \rho = \frac{\sqrt{q x} + \frac{1}{2}(1 + q)\sqrt{x y}}{\sqrt{q x} + \sqrt{q y} + (1 + q)\sqrt{x y}} \] \quad (42) 

The boundaries of the phase (III) (See Fig.3) are limited to

\[ \alpha < (1 - \rho)(1 - q) \]
\[ \beta < \rho(1 - q) \]

which can be distinguished for \( q < 1 \) using the fact that \( x \frac{1 - q - \beta}{\beta} = y \frac{1 - q - \alpha}{\alpha} \), (30) and (42). The density profile of ordinary particles, as mentioned above, has an algebraic behavior in this phase and increases linearly from \( \rho_1 = \frac{\beta}{1 - q} \) to \( \rho_L = 1 - \frac{\alpha}{1 - q} \) according to

\[ \rho(z) = \frac{\beta}{1 - q} + (1 - \frac{\beta + \alpha}{1 - q})z \], \( 0 \leq z \leq 1 \) \quad (43) 

For \( q = 0 \) this phase corresponds to phase (V) in [1] where a shock structure has been observed for certain values of the density \( \rho \). The linear profile which is the consequence of a fluctuating shock front, has also been observed in ASEP and PASEP with open boundaries [11,14,15]. Here the relation \( x \frac{1 - q - \beta}{\beta} = y \frac{1 - q - \alpha}{\alpha} \) prevents fixing the fugacities \( x \) and \( y \) and we cannot adjust the density \( \rho \) given by (42) to the desired value. Therefore we cannot see a real shock structure in the grand canonical ensemble. Nevertheless one can observe it in this phase using the canonical ensemble [19]. The speed of impurity (26) in this phase is of the form

\[ v = \alpha - \beta \] \quad (44) 

which is independent of \( \rho \). Note that for \( \beta > \alpha \) the speed of impurity again becomes negative. One can check that in this phase the values of \( \rho_1, \rho_L \) and \( v \) approach their
corresponding values in the phase (V) in $q \to 0$ limit.
The fluctuation in the density of ordinary particles [17] which is given by
\[
\left[ \frac{x}{L} \frac{d}{dx} \left( \frac{x}{L} \frac{d}{dx} \ln Z_L \right) \right]^{1/2}
\]
(45)
can also be calculated in each phase. Using (33) and (41) one can easily check that in
large $L$ limit these fluctuations drop to zero like $\frac{1}{\sqrt{L}}$ therefore the results of the grand
canonical ensemble agree with those obtained from the canonical one.

6 Comparison and Concluding Remarks

As we mentioned in section 2, this model can be considered as a simple generalization
of the model studied in [1]. Here we compare the results obtained there (we call it the
model A) with those in our model (the model B).

For $q = 0$, the quadratic algebra of model B reduces to the one of model A. Using an
infinite dimensional representation of this algebra, the author has solved model A exactly
and shown that it possesses six distinctive phases (I-VI). Since we have used a finite di-

dimensional representation of the quadratic algebra (6-8), only the region $\alpha + \beta + q < 1$
lies
in the accessible region where three phases exist. These phases (I-III) correspond to the
phases (VI), (IV) and (V) in model A respectively. In model A, the density profile of the
ordinary particles has an exponential behaviour in the phases (VI) and (VI) but it has a
shock structure in phase (V). In phases (I) and (II) of model B, the density profile has
also an exponential behaviour but with a modified correlation length. In the region (III)
we have obtained a linear profile for the density of the ordinary particles in the grand
canonical ensemble which indicates the presence of a shock in this phase. From the phase
structure point of view, the backward hopping of the ordinary particles does not change
the number of phases at least in the accessible region and only the co-existance lines are
shifted.

In this paper we have used a finite dimensional representation of the algebra and done all
the calculations in grand canonical ensemble which make the calculations rather simple.
It would be interesting to study the region $\alpha + \beta + q > 1$, especially the shock structure
in the phase (III), using an infinit-dimensional representation of (6-8) in the canonical
ensemble.
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Figure Captions

Fig. 1 Plot of the equation (11).

Fig. 2 Plot of the equation (16) for n=2 (Left) and n=5 (Right).

Fig. 3 The phase diagram.
Figure 1. Plot of the equation (11).
Figure 2. Plot of the equation (16) for $n=2$ (Left) and $n=5$ (Right).
Figure 3. The phase diagram