SOME APPLICATIONS OF TWO COMPLETELY
COPOSITIVE MAPS

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Abstract. A linear map $\Phi : M_n \to M_k$ is called completely copositive if the resulting matrix $[\Phi(A_{i,j})]_{i,j=1}^m$ is positive semidefinite for any integer $m$ and positive semidefinite matrix $[A_{i,j}]_{i,j=1}^m$. In this paper, we present some applications of the completely copositive maps $\Phi(X) = (\text{tr}X)I + X$ and $\Psi(X) = (\text{tr}X)I - X$. Some new extensions about traces inequalities of positive semidefinite $3 \times 3$ block matrices are included.

1. Introduction

The space of $m \times n$ complex matrices is denoted by $M_{m \times n}$. If $m = n$, we use $M_n$ instead of $M_{n \times n}$ and if $n = 1$, we use $C^m$ instead of $M_{m \times 1}$. The identity matrix of $M_n$ is denoted by $I_n$, or simply by $I$ if no confusion is possible. We use $M_m(M_n)$ for the set of $m \times m$ block matrices with each block in $M_n$. Let $X \otimes Y$ denote the Kronecker product of $X, Y$, that is, if $X = [x_{ij}] \in M_m$ and $Y \in M_n$, then $X \otimes Y \in M_m(M_n)$ whose $(i, j)$ block is $x_{ij}Y$. By convention, if $X \in M_n$ is positive semidefinite, we write $X \geq 0$. For two Hermitian matrices $A$ and $B$ of the same size, $A \geq B$ means $A - B \geq 0$.

Now we introduce the definition of the partial transpose and partial traces. For any $A = [A_{i,j}]_{i,j=1}^m \in M_m(M_n)$, the usual transpose of $A$ is defined to be

$$A^T = \begin{bmatrix}
A_{1,1}^T & \cdots & A_{m,1}^T \\
\vdots & \ddots & \vdots \\
A_{1,m}^T & \cdots & A_{m,m}^T
\end{bmatrix}.$$ 

We define the partial transpose of $A$ by

$$A^\tau = \begin{bmatrix}
A_{1,1} & \cdots & A_{m,1} \\
\vdots & \ddots & \vdots \\
A_{1,m} & \cdots & A_{m,m}
\end{bmatrix}.$$
It is clear that $A \geq 0$ does not necessarily imply $A^\tau \geq 0$. If both $A$ and $A^\tau$ are positive semidefinite, then $A$ is said to be the positive partial transpose (or PPT for short).

Given $A = [A_{i,j}]_{i,j=1}^m \in \mathbb{M}_m(\mathbb{M}_n)$, we next introduce the definition of two partial traces $\text{tr}_1A$ and $\text{tr}_2A$. There are several equivalent ways to explain the partial traces, and we recommend the recent monographs [15] and [2, pp.120–121] for a comprehensive survey of the subject. For notational convenience, we define two partial traces of $A$ by

$$\text{tr}_1A = \sum_{i=1}^m A_{i,i},$$
$$\text{tr}_2A = \left[\text{tr}A_{i,j}\right]_{i,j=1}^m,$$
where $\text{tr}X$ stands for the usual trace of $X$.

Recall that a linear map $\Phi : \mathbb{M}_n \to \mathbb{M}_k$ is called positive if it maps positive matrices to positive matrices. A linear map $\Phi : \mathbb{M}_n \to \mathbb{M}_k$ is said to be $m$-positive if for $[A_{i,j}]_{i,j=1}^m \in \mathbb{M}_m(\mathbb{M}_n)$,

$$[A_{i,j}]_{i,j=1}^m \geq 0 \Rightarrow [\Phi(A_{i,j})]_{i,j=1}^m \geq 0. \quad (1)$$

It is said to be completely positive if (1) holds for any integer $m \geq 1$. It is well known that the trace map and the determinant map are both completely positive, see, e.g., [16, p. 221, p. 237]. On the other hand, a linear map $\Phi$ is said to be $m$-copositive if for $[A_{i,j}]_{i,j=1}^m \in \mathbb{M}_m(\mathbb{M}_n)$,

$$[A_{i,j}]_{i,j=1}^m \geq 0 \Rightarrow [\Phi(A_{j,i})]_{i,j=1}^m \geq 0, \quad (2)$$
and $\Phi$ is said to be completely copositive if (2) holds for any positive integer $m \geq 1$. Furthermore, $\Phi$ is called a completely PPT map if it is completely positive and completely copositive. A comprehensive survey of the standard results on completely positive maps can be found in [2, Chapter 3] or [14].

This paper centers on the application of the following result (Theorem 1.1) due to Lin [11, Theorem 1.1] and [13, Proposition 2.1]. We provide an alternatively elegant proof here.

**Theorem 1.1.** (see [11, 13]) The maps $\Phi(X) = (\text{tr}X)I + X$ and $\Psi(X) = (\text{tr}X)I - X$ are completely copositive.

**Proof.** Here we use a standard method from Choi [6]. It is sufficient to show that for any positive integer $m$,

$$[\Phi(E_{j,i})]_{i,j=1}^m \geq 0 \text{ and } [\Psi(E_{j,i})]_{i,j=1}^m \geq 0,$$

where $E_{j,i} \in \mathbb{M}_n$ is the matrix with 1 in the $(j,i)$-th entry and 0 elsewhere. Since both $[\Phi(E_{j,i})]_{i,j=1}^m$ and $[\Psi(E_{j,i})]_{i,j=1}^m$ are Hermitian (symmetric), row diagonally dominant with non-negative diagonal entries, this yields

$$[\Phi(E_{j,i})]_{i,j=1}^m \geq 0 \text{ and } [\Psi(E_{j,i})]_{i,j=1}^m \geq 0.$$

So we complete the proof. \qed
We remark that $\Phi(X) = (\text{tr}X)I + X$ is apparently a completely positive map, therefore $\Phi$ is a completely PPT map. However, the map $\Psi(X) = (\text{tr}X)I - X$ is not completely positive since it is even not 2-positive (see [5]), thus $\Psi$ is not a completely PPT map.

The paper is organized as follows. In Section 2, we show a partial traces inequality about PPT matrices based on the application of Theorem 1.1. Some other recent results are implicitly included in our proof of Theorem 2.3. In Section 3, we provide a proof of a trace inequality that has been applied to quantum information, such as, the subadditivity of $q$-entropies and the separability of mixed states. In the last of the third section, we give some unified extensions of some traces inequalities (Theorem 3.3 and Theorem 3.4).

2. Inequalities related to Partial traces

By the completely copositivity of $\Psi$ in Theorem 1.1, we get the following Theorem 2.1, which is the main result in [3, Theorem 2]. Of course, the proof provided by Choi is quite different and technical.

**Theorem 2.1.** (see [3, Theorem 2]) Let $A = [A_{i,j}]_{i,j=1}^m \in M_m(M_n)$ be positive semidefinite. Then

$$(\text{tr}_2 A^T) \otimes I_n \geq A^T,$$

$I_m \otimes \text{tr}_1 A^T \geq A^T.$

**Proof.** By Theorem 1.1 $\Psi(X) = (\text{tr}X)I - X$ is completely copositive, that is

$$[(\text{tr}A_{j,i}) - A_{j,i}]_{i,j=1}^m \geq 0.$$

Under the above definition, we can write

$$(\text{tr}_2 A^T) \otimes I_n \geq A^T.$$  (3)

We may assume that $A_{i,j} = [a_{i,j}]_{i,j=1}^m$, then we define $\tilde{A} \in M_n(M_m)$ by

$$\tilde{A} = [B_{r,s}]_{r,s=1}^n,$$

where $B_{r,s} = [a_{i,j}]_{i,j=1}^m \in M_m$. By a direct computation, we get

$$\text{tr}_2 \tilde{A} = \left[\text{tr} [a_{i,j}]_{i,j=1}^m \right]_{r,s=1}^n = \sum_{i=1}^m a_{i,i}^r \sum_{s=1}^n [a_{r,s}^i]_{r,s=1}^n = \text{tr}_1 A,$$

and for any $X = [x_{ij}] \in M_m$ and $Y = [y_{rs}] \in M_n$, since

$$X \otimes Y = [x_{ij}y_{rs}]_{i,j=1}^m = [x_{ij}y_{rs}]_{i,j=1}^m,$$

Then, it follows that

$$\overline{X \otimes Y} = [x_{ij}y_{rs}]_{i,j=1}^m = [y_{rs}X]_{r,s=1}^n = Y \otimes X.$$
Replacing $A$ with $\widetilde{A}$ in (3), we get
\[ I_m \otimes \text{tr}_1 A^\tau = I_m \otimes \text{tr}_2 \widetilde{A}^\tau = (\text{tr}_2 \widetilde{A}^\tau) \otimes I_m \geq \widetilde{A}^\tau, \]
that is
\[ I_m \otimes \text{tr}_1 A \geq \text{tr}_1 \widetilde{A}, \]
where we frequently use the fact that $\widetilde{A}^\tau = \widetilde{A}$. □

As a byproduct of our proof, we have the following Corollary 2.2, see [8] for details and references to the physics literature.

**Corollary 2.2.** (see [8]) Let $A = [A_{ij}]_{i,j=1}^m \in M_m(M_n)$ be PPT. Then
\[ I_m \otimes \text{tr}_1 A \geq A, \]
\[ (\text{tr}_2 A) \otimes I_n \geq A. \]

**Proof.** Since $A$ and $A^\tau$ are positive semidefinite, by replacing $A$ with $A^\tau$ in Theorem 2.1, we get the desired results. □

By combining Theorem 2.1 and Corollary 2.2, we immediately obtain the following partial traces inequality.

**Corollary 2.3.** Let $A = [A_{ij}]_{i,j=1}^m \in M_m(M_n)$ be PPT. Then
\[ I_m \otimes (\text{tr}_1 A) + (\text{tr}_2 A) \otimes I_n \geq 2A, \]
and
\[ I_m \otimes (\text{tr}_1 A^\tau) + (\text{tr}_2 A^\tau) \otimes I_n \geq 2A^\tau. \]

**Proposition 2.4.** Let $A \in M_2(M_n)$ be positive semidefinite. Then
\[ I_2 \otimes (\text{tr}_1 A) + (\text{tr}_2 A) \otimes I_n \leq A + (\text{tr}A)I_2n. \]

**Proof.** We may assume that
\[ A = \begin{bmatrix} B & C \\ C^* & D \end{bmatrix}, \]
where $B, C, D \in M_n$. The desired inequality is
\[
\begin{bmatrix}
B + D & 0 \\
0 & B + D
\end{bmatrix} + 
\begin{bmatrix}
(\text{tr}B)I_n & (\text{tr}C)I_n \\
(\text{tr}C^*)I_n & (\text{tr}D)I_n
\end{bmatrix}
\leq 
\begin{bmatrix}
B & C \\
C^* & D
\end{bmatrix} + 
\begin{bmatrix}
(\text{tr}A)I_n & 0 \\
0 & (\text{tr}A)I_n
\end{bmatrix},
\]
which is equivalent to (note that $\text{tr}A = \text{tr}B + \text{tr}D$)
\[ G := 
\begin{bmatrix}
(\text{tr}D)I_n - D & C - (\text{tr}C)I_n \\
C^* - (\text{tr}C^*)I_n & (\text{tr}B)I_n - B
\end{bmatrix} \geq 0. \]

By Theorem 1.1 the map $\Psi(X) = (\text{tr}X)I - X$ is completely copositive,
\[
\begin{bmatrix}
(\text{tr}B)I_n - B & (\text{tr}C^*)I_n - C^* \\
(\text{tr}C)I_n - C & (\text{tr}D)I_n - D
\end{bmatrix} \geq 0,
\]
and then
\[
G = 
\begin{bmatrix}
0 & -I_n \\
I_n & 0
\end{bmatrix} \begin{bmatrix}
(\text{tr}B)I_n - B & (\text{tr}C^*)I_n - C^* \\
(\text{tr}C)I_n - C & (\text{tr}D)I_n - D
\end{bmatrix} \begin{bmatrix}
0 & I_n \\
-I_n & 0
\end{bmatrix} \geq 0.
\]
Hence we complete the proof. □

We remark that if $A = [A_{i,j}]_{i,j=1}^m \in M_m(M_n)$ is positive semidefinite, by induction and the 2-copositivity of $\Psi(X) = (\text{tr}X)I - X$, one can show that

$$I_m \otimes (\text{tr}_1A) + (\text{tr}_2A) \otimes I_n \leq A + (\text{tr}A)I_m \otimes I_n,$$

which was proved by Ando [1] and independently by Lin [13].

3. Inequalities relating to trace

Recently, Choi established the following partial trace inequalities [Corollary 3.1], which is the key result in [4, Theorem 2]. Here we shall demonstrate that Corollary 3.1 is actually a well application of the completely copositive $\Phi(X) = (\text{tr}X)I + X$. In the sequel, we first give an alternative proof of Corollary 3.1 based on Theorem 1.1. The Corollary 3.2 can be found in [10, Theorem 2.2], here we provide the proof for completeness using the completely copositivity of $\Psi(X) = (\text{tr}X)I - X$. Some interesting consequences about trace are included.

**Corollary 3.1.** (see [4]) Let $A \in M_m(M_n)$ be positive semidefinite. Then

$$(\text{tr}_2A^\tau) \otimes I_n \geq -A^\tau,$$

$$I_m \otimes \text{tr}_1A^\tau \geq -A^\tau.$$

**Proof.** In view of symmetry of definitions of $\text{tr}_1$ and $\text{tr}_2$, we only prove

$$(\text{tr}_2A^\tau) \otimes I_n \geq -A^\tau.$$

By Theorem 1.1 the map $\Phi(X) = (\text{tr}X)I + X$ is completely copositive, then

$$[[\text{tr}A_{i,j}]I_n + A_{i,j}]_{i,j=1}^m \geq 0,$$

which can be rewrite as $(\text{tr}_2A^\tau) \otimes I_n \geq -A^\tau$. □

**Corollary 3.2.** (see [10]) Let $\begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \in M_2(M_n)$ be positive semidefinite. Then

$$\begin{bmatrix} (\text{tr}C)A - BB^* & (\text{tr}B^*)B - AC \\ (\text{tr}B)B^* - CA & (\text{tr}A)C - B^*B \end{bmatrix} \geq 0.$$  \hspace{1cm} (5)

Consequently,

$$\text{tr}(AC) + \text{tr}(B^*B) \leq \text{tr}A\text{tr}C + |\text{tr}B|^2,$$  \hspace{1cm} (6)

$$\text{tr}(B^*B) - \text{tr}(AC) \leq \text{tr}A\text{tr}C - |\text{tr}B|^2.$$  \hspace{1cm} (7)

**Proof.** Since $\begin{bmatrix} Y^*Y & Y^*X \\ X^*Y & X^*X \end{bmatrix}$ is positive semidefinite for any $p \times q$ matrices $X,Y$, by Theorem 1.1 the completely copositivity of $\Psi(X) = (\text{tr}X)I - X$ yields

$$\begin{bmatrix} (\text{tr}Y^*Y)I - Y^*Y & (\text{tr}X^*Y)I - X^*Y \\ (\text{tr}Y^*X)I - Y^*X & (\text{tr}X^*X)I - X^*X \end{bmatrix} \geq 0.$$

Now since \[ \begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \] is positive semidefinite, we may write
\[
\begin{bmatrix} A & B \\ B^* & C \end{bmatrix} = \begin{bmatrix} XX^* & XY^* \\ YX^* & YY^* \end{bmatrix},
\]
for some \( X, Y \in \mathbb{M}_{n \times 2n} \). We observe that
\[
\begin{bmatrix} (\text{tr}C)A - BB^* & (\text{tr}B^*)B - AC \\ (\text{tr}B)B^* - CA & (\text{tr}A)C - B^*B \end{bmatrix}
\]
\[
= \begin{bmatrix} \text{tr}(YY^*)XX^* - XY^*X^* & \text{tr}(XY^*)XY^* - XX^*YY^* \\ \text{tr}(X^*Y^*)XY^* - YY^*X^* & \text{tr}(XX^*)YY^* - YX^*XY^* \end{bmatrix}
\]
\[
= \begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix} \begin{bmatrix} (\text{tr}Y^*Y)I - Y^*Y & (\text{tr}X^*Y)I - X^*Y \\ (\text{tr}X^*Y)I - Y^*X & (\text{tr}X^*X)I - X^*X \end{bmatrix} \begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix}^*.
\]
Therefore, (5) follows. Then
\[
(\text{tr}C)A - BB^* + (\text{tr}A)C - B^*B \geq \pm \left( (\text{tr}B^*)B - AC + (\text{tr}B)B^* - CA \right).
\]
By taking trace on both sides, it yields (6) and (7). \( \square \)

Positive semidefinite \( 2 \times 2 \) block matrices are well studied, such a partition leads to versatile and elegant theoretical inequalities, see \([7, 11, 12, 10]\) for details. However, an analogous partition into \( 3 \times 3 \) blocks matrices seems not to be extensively investigated. At the end of the paper, we will present several results related to positive semidefinite \( 3 \times 3 \) block matrices.

Let \( A \) be an \( n \times n \) complex matrix. For index sets \( \alpha, \beta \subseteq \{1, 2, \ldots, n\} \), we denote by \( A[\alpha, \beta] \) the submatrix of entries that lie in the rows of \( A \) indexed by \( \alpha \) and the columns indexed by \( \beta \). If \( \alpha = \beta \), the submatrix \( A[\alpha, \beta] = A[\alpha] \) is the principal submatrix of \( A \). We denoted by \( |\alpha| \) the cardinality of the index set \( \alpha \).

Recently, Lin and van den Driessche proved a determinantal inequality \([8]\), which is a refinement of the famous Kotelianskii’s inequality (see, e.g., \([9]\)), it states that for any positive semidefinite \( A \in \mathbb{M}_n \) and \( \alpha, \beta \subseteq \{1, 2, \ldots, n\} \) with \( |\alpha| = |\beta| \), then

\[
(\det A[\alpha \cup \beta])(\det A[\alpha \cap \beta]) \leq (\det A[\alpha])(\det A[\beta]) - |\det A[\alpha, \beta]|^2. \tag{8}
\]

The next two results Theorem 3.3 and Theorem 3.4 are extensions of Corollary 3.2 and it also can be regarded as the complement of (8).

**Theorem 3.3.** Let \( A \in \mathbb{M}_n \) be positive semidefinite and let \( \alpha, \beta \subseteq \{1, 2, \ldots, n\} \) such that \( |\alpha| = |\beta| \). Then
\[
\text{tr}(A[\alpha]A[\beta]) + \text{tr}(A^*[\alpha, \beta]A[\alpha, \beta]) \leq (\text{tr}A[\alpha])(\text{tr}A[\beta]) + |\text{tr}A[\alpha, \beta]|^2. \tag{9}
\]
Proof. Without loss of generality, we may assume that $\alpha \cup \beta = \{1, 2, \ldots, n\}$ so that $A[\alpha \cup \beta] = A$ (otherwise, work within the principal submatrix $A[\alpha, \beta]$). By suitable rearrangement of subscripts or by permutational similarity if necessary, we may further assume that

$$A = \begin{bmatrix} A_{1,1} & A_{1,2} & A_{1,3} \\ A_{1,2}^* & A_{2,2} & A_{2,3} \\ A_{1,3}^* & A_{2,3}^* & A_{3,3} \end{bmatrix},$$

and

$$A[\alpha] = \begin{bmatrix} A_{1,1} \\ A_{1,2} \\ 0 \end{bmatrix}, \quad A[\beta] = \begin{bmatrix} A_{2,2} \\ A_{2,3} \\ 0 \end{bmatrix}, \quad A[\alpha, \beta] = \begin{bmatrix} A_{1,2} \\ A_{2,2} \end{bmatrix}.$$

Since $A$ is positive semidefinite, and observe that

$$\begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & -I & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix} \begin{bmatrix} A[\alpha] & A[\alpha, \beta] \\ A^*[\alpha, \beta] & A[\beta] \end{bmatrix} \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & -I & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix}$$

$$= \begin{bmatrix} A_{1,1} & A_{1,2} & 0 & A_{1,3} \\ A_{2,1} & A_{2,2} & 0 & A_{2,3} \\ 0 & 0 & 0 & 0 \\ A_{3,1} & A_{3,2} & 0 & A_{3,3} \end{bmatrix} \geq 0,$$

therefore, we have

$$\begin{bmatrix} A[\alpha] & A[\alpha, \beta] \\ A^*[\alpha, \beta] & A[\beta] \end{bmatrix} \geq 0.$$

By (6) in the previous Corollary 3.2, the desired result (9) now follows. □

Using the same idea in the previous proof and combining [10] Theorem 2.1 or (7), one could also get the following trace inequality.

**Theorem 3.4.** Let $A \in M_n$ be positive semidefinite and let $\alpha, \beta \subseteq \{1, 2, \ldots, n\}$ such that $|\alpha| = |\beta|$. Then

$$|\text{tr}(A[\alpha]A[\beta]) - \text{tr}(A^*[\alpha, \beta]A[\alpha, \beta])| \leq (\text{tr}A[\alpha])(\text{tr}A[\beta]) - |\text{tr}A[\alpha, \beta]|^2.$$

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