Incomparable copies of a poset in the Boolean lattice

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Abstract
Let $B_n$ be the poset generated by the subsets of $[n]$ with the inclusion as relation and let $P$ be a finite poset. We want to embed $P$ into $B_n$ as many times as possible such that the subsets in different copies are incomparable. The maximum number of such embeddings is asymptotically determined for all finite posets $P$ as $\binom{\lceil \frac{n}{2} \rceil}{M(P)}$, where $M(P)$ denotes the minimal size of the convex hull of a copy of $P$. We discuss both weak and strong (induced) embeddings.

1 Introduction

Definition Let $B_n$ be the Boolean lattice, the poset generated by the subsets of $[n]$ with the inclusion as relation and $P$ be a finite poset. (If $S$ is a set of size $n$ we may also write $B_S$.) $f : P \to B_n$ is an embedding of $P$ into $B_n$ if it is an injective function that satisfies $f(a) \subseteq f(b)$ for all $a <_P b$. $f$ is called an induced embedding if it is an injective function such that $f(a) \subseteq f(b)$ if and only if $a <_P b$.

Definition Let $X$ and $Y$ be two sets of subsets of $[n]$. $X$ and $Y$ are incomparable if there are no sets $x \in X$ and $y \in Y$ such that $x \subseteq y$ or $y \subseteq x$. A family of sets of subsets of $[n]$ is incomparable if its elements are pairwise incomparable.

We investigate the following problem. How many times can we embed a poset into $B_n$ such that the resulting copies form an incomparable family? An asymptotic answer is given in both the induced and the non-induced case. Before we can state our main result, some notations are needed.

Notation Let $F \subseteq B_n$. The convex hull of $F$ is the set
\[
\text{conv}(F) = \{b \in B_n \mid \exists a, c \in F \ a \subseteq b \subseteq c\}.
\] (1)
We use the following notations for the minimal size of the convex hull. For a finite poset $P$
\[
t_1(P) = \min_{f_n} \{|\text{conv}(\text{Im}(f))| \mid f : P \to B_n \text{ is an embedding}\}\]
(2)
\[
t_2(P) = \min_{f_n} \{|\text{conv}(\text{Im}(f))| \mid f : P \to B_n \text{ is an induced embedding}\}\]
(3)
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Theorem 1.1. Let $P$ be a finite poset. Let $M_1(P, n)$ (and $M_2(P, n)$) denote the largest $M$ such that there are embeddings (induced embeddings) $f_1, f_2, \ldots, f_M : P \to B_n$ such that $\{Im(f_i), i = 1, 2, \ldots M\}$ is an incomparable family. Then

$$\lim_{n \to \infty} M_1(P, n) \left(\frac{n}{\lceil n/2 \rceil}\right) = \frac{1}{t_1(P)}$$

(4)

$$\lim_{n \to \infty} M_2(P, n) \left(\frac{n}{\lceil n/2 \rceil}\right) = \frac{1}{t_2(P)}.$$  

(5)

We prove upper and lower bounds for $M_j(P, n)$ in the next two sections (Theorem 2.2 and Theorem 3.3). The two bounds will imply the theorem immediately. Since the proofs are almost identical for $j = 1, 2$, they will be done simultaneously.

Remark Theorem 1.1 was independently proved by A. P. Dove and J. R. Griggs, [5].

The problem discussed in this paper is related to the problem of determining the largest families in $B_n$ avoiding certain configurations of inclusion.

Definition Let $P_1, P_2, \ldots P_k$ be finite posets. $La(n, \{P_1, \ldots P_k\})$ denotes the size of the largest subset $F \subset B_n$ such that none of the posets $P_i$ can be embedded into $F$.

Let $V_k$ denote the $(k+1)$-element poset that has a minimal element contained in the other $k$ unrelated elements. $\Lambda_k$ is obtained from $V_k$ by reversing the relations. Katona and Tarján proved that a subset of $B_n$ containing none of the posets $\{V_2, \Lambda_2\}$ has at most $\left(\frac{n-1}{\lceil n/2 \rceil}\right)$ elements, and this bound is sharp [8]. Such a family consists of pairwise incomparable copies of the one-element poset and the two-element chain.

Another example of the relation of the two problems is determining $La(n, V_2)$. (See [4] for asymptotic bounds on $La(n, V_2)$.). A $V_2$-free family consists of pairwise independent copies of the posets $\{\Lambda_0, \Lambda_1, \Lambda_2, \ldots\}$.

The value of $La(n, P)$ is not known for a general poset $P$, but many special cases have been solved. See [2] for posets whose Hasse diagram is a tree. See [6] for diamond and harp posets. [3] provides upper bounds on $La(n, P)$ for all posets $P$.

2 The upper bound

To prove the upper bound for $M_j(P, n)$ we need a lemma about chains. Let $S$ be a set of size $n$. A chain in $S$ is a set of subsets $\emptyset = C_0 \subset C_1 \subset C_2 \subset \cdots \subset C_n = S$, where $|C_m| = m$ for all $m$.

Lemma 2.1. Let $F$ be a family of subsets of $S$, where $|S| = n$ and $|F| = t$. Then the number of chains intersecting at least one member of $F$ is at least

$$\left( t - \frac{\binom{t}{2}}{n} \right) \frac{n}{2} !! \frac{n}{2} !!.$$  

(6)
Proof. We prove the lemma by induction on \( t \). The statement is true for \( t = 1 \), as the number of chains passing through a subset \( F \) is \(|F|!(n-|F|)! \geq |n/2|!\lfloor n/2 \rfloor!\). Now let \( t \geq 2 \), and \( F = \{F_1, F_2, \ldots, F_t\} \). Since taking complements does not change the number of intersecting chains, we may assume that some set of \( F \) has size at most \( \lfloor n/2 \rfloor \). We can also assume that \( F_t \) is one of the smallest subsets.

By induction, the number of chains intersecting \( F \setminus \{F_t\} \) is at least

\[
\left(t - 1 - \frac{\binom{t-1}{2}}{n}\right) \lfloor n/2 \rfloor!\lfloor n/2 \rfloor!.
\]  

(7)

The number of chains through \( F_t \) is \(|F_t|!(n-|F_t|)!\). Assume that \( F_t \subset F_i \) for some \( i \in \{1, \ldots, t-1\} \). The number of chains intersecting both \( F_t \) and \( F_i \) is \(|F_t|!(|F_i| - |F_t|)!)(n-|F_t|)! \leq |F_t|!(n-|F_t|-1)!\). So there are at least

\[
|F_t|!(n-|F_t|)! \left(1 - \frac{t-1}{n-|F_t|}\right) \geq |n/2|!\lfloor n/2 \rfloor! \left(1 - \frac{2(t-1)}{n}\right)
\]  

(8)

chains that intersect \( F \) only in \( F_t \). The statement of the lemma follows after summation:

\[
\left(t - 1 - \frac{\binom{t-1}{2}}{n}\right) + \left(1 - \frac{2(t-1)}{n}\right) = t - \frac{\binom{t}{2}}{n}.
\]  

\(\blacksquare\)

**Theorem 2.2.** For any finite poset \( P \)

\[
M_j(P, n) \leq \frac{\binom{n}{\lfloor n/2 \rfloor}}{t_j(P)} (1 + O(n^{-1}))
\]  

holds for \( j = 1, 2 \).

Proof. Assume that \( f_1, f_2, \ldots, f_k : P \to B_n \) are embeddings (induced if \( j = 2 \)) such that the family \( \{\text{Im}(f_i), \ i = 1, 2, \ldots, k\} \) is incomparable. Then \( \{\text{conv(Im}(f_i)), \ i = 1, 2, \ldots, k\} \) is also an incomparable family. To see that, assume there are sets \( a, b \) such that \( a \subseteq b, a \in \text{conv(Im}(f_i)), b \in \text{conv(Im}(f_j)) \) and \( i \neq j \). Then by the definition of the convex hull there are sets \( a' \in \text{Im}(f_i) \) and \( b' \in \text{Im}(f_j) \) such that \( a' \subseteq a \subseteq b \subseteq b' \). But \( a' \not\subseteq b' \) since \( \{\text{Im}(f_i), i = 1, 2, \ldots, k\} \) is an incomparable family.

Since the family \( \{\text{conv(Im}(f_i)), i = 1, 2, \ldots, k\} \) is incomparable, every chain intersects at most one of its members. By Lemma 2.1, each \( \text{conv(Im}(f_i)) \) intersects at least \( t_j(P)\lfloor n/2 \rfloor!\lfloor n/2 \rfloor!(1 - O(n^{-1})) \) chains. Since the total number of chains is \( n! \),

\[
k \leq \frac{n!}{t_j(P)\lfloor n/2 \rfloor!\lfloor n/2 \rfloor!(1 - O(n^{-1}))} = \frac{1}{t_j(P)} \binom{n}{\lfloor n/2 \rfloor} (1 + O(n^{-1})).
\]  

(11)

\(\blacksquare\)

### 3 The lower bound

In this section our aim is to prove a lower bound on \( M_j(P, n) \) by embedding many copies of \( P \) to \( B_n \). We need the following lemmas for the construction.
Lemma 3.1. Let \( P \) be a finite poset, and let \( f : P \rightarrow B_m \) be an embedding. Then we can label the elements of \( B_m \) with the numbers \( 1, 2, \ldots 2^m \) such that all the sets get a higher number than any of their subsets, and the numbers assigned to the elements of \( \text{conv}(Im(f)) \) form an interval in \( [1, 2^m] \).

Proof. We divide the elements of \( B_m \) into three groups:

\[
\begin{align*}
\mathcal{F}_1 &= \{ b \in B_m \mid \exists c \in Im(f) \ b \subset c, \ \exists a \in Im(f) \ a \subset b \}, \\
\mathcal{F}_2 &= \text{conv}(Im(f)) = \{ b \in B_m \mid \exists a, c \in Im(f) \ a \subset b \subset c \} \\
\text{and } \mathcal{F}_3 &= B_m \setminus (\mathcal{F}_1 \cup \mathcal{F}_2) = \{ b \in B_m \mid \not\exists c \in Im(f) \ b \subset c \}.
\end{align*}
\]

We use the numbers of \([1, |\mathcal{F}_1|] \) for the sets of \( \mathcal{F}_1 \), the numbers of \([|\mathcal{F}_1| + 1, |\mathcal{F}_1| + |\mathcal{F}_2|] \) for the sets of \( \mathcal{F}_2 \) and the numbers \([|\mathcal{F}_1| + |\mathcal{F}_2| + 1, 2^m] \) for the sets of \( \mathcal{F}_3 \). In the groups we assign numbers such that the elements representing larger subsets get larger numbers.

We have to check that if that if \( x, y \in B_m \) and \( y \) got a larger number than \( x \), then \( y \not\subset x \).

If \( x \) and \( y \) are in the same group, than \(|x| \leq |y|\), so \( y \not\subset x \). If \( x \in \mathcal{F}_1 \) and \( y \in \mathcal{F}_2 \), then \( y \not\subset x \), because \( y \) contains an element of \( Im(f) \) while \( x \) does not. If \( x \in \mathcal{F}_1 \cup \mathcal{F}_2 \) and \( y \in \mathcal{F}_3 \), then \( y \not\subset x \), because \( x \) is the subset of an element of \( Im(f) \) while \( y \) is not.

\[\square\]

Lemma 3.2. Let \( P \) be a finite poset and let \( \varepsilon' > 0 \) be fixed. Let \( j \in \{1, 2\} \). Then there are integers \( N, K \) and functions \( f_1, f_2, \ldots f_K : P \rightarrow B_N \) such that

(i) For all \( i \in [1, K] \), \( f_i \) is an embedding if \( j = 1 \), and an induced embedding if \( j = 2 \).

(ii) \( K \geq \frac{2^N(1-\varepsilon')}{t_j(P)} \).

(iii) If \( i_1 < i_2 \), \( a \in \text{Im}(f_{i_1}) \) and \( b \in \text{Im}(f_{i_2}) \), then \( b \not\leq a \).

Proof. Let \( P \) be a finite fixed poset. There is embedding (or induced embedding, if \( j = 2 \)) \( f : P \rightarrow B_m \) for some \( m \) such that \(|\text{conv}(Im(f))| = t_j(P)\). Fix \( m \) and \( f \). Choose \( k \in \mathbb{N} \) such that \( \left(1 - \frac{t_j(P)}{2^m}\right)^k \leq \varepsilon' \), and let \( N = km \). Let \( S_1, S_2, \ldots S_k \) be disjoint sets of size \( m \) and let \( S = \bigcup_{i=1}^k S_i \). Consider the elements of \( B_N \) as the subsets of \( S \).

Let \( g_i : P \rightarrow B_{S_i} \) (\( i = 1, 2, \ldots k \)) be embeddings that map the elements of \( P \) to \( m \)-element sets the same way as \( f \) does. Assign the numbers \( 1, 2, \ldots 2^m \) to the subsets of \( S_i \) as in Lemma 3.1. The elements of \( \text{conv}(Im(g_i)) \) will get the numbers of the interval \( I = [p, p + t_j(P) - 1] \) for all \( i \).

We call an embedding \( g : P \rightarrow B_S \) good if there is an index \( i \in [1, k] \) and there are \( k \) disjoint sets \( A_1 \subseteq S_1, A_2 \subseteq S_2, \ldots A_k \subseteq S_k \) such that none of the numbers assigned to \( A_1, A_2, \ldots A_k \) is in \( I \), and for any \( x \in P \), \( g(x) \cap S_i = g_i(x) \), and \( g(x) \cap (S \setminus S_i) = \bigcup_{r \in [n \setminus \{i\}]} A_r \).

The number of good functions is

\[
\sum_{i=1}^{k} (2^m - t_j(P))^{i-1} \cdot (2^m)^{k-i} = 2^{N-m} \sum_{i=1}^{k} \left(1 - \frac{t_j(P)}{2^m}\right)^{i-1}
\]

\[
2^{N-m} \frac{1 - \left(1 - \frac{t_j(P)}{2^m}\right)^k}{t_j(P)\left(1 - \left(1 - \frac{t_j(P)}{2^m}\right)^k\right)} \geq \frac{2^N(1-\varepsilon')}{t_j(P)}.
\]
be the first coordinate where the codes of $f$ is strictly larger than that of $f_k$ last numbers assigned to $f_k$.

A (contradicting conv numbers assigned to the elements of $g$.

Choose $(j,n)$ obviously an embedding induced if $j > 2$.

Let $A_n$ be a finite poset, $\varepsilon > 0$ and $j \in \{1, 2\}$. Then for all large enough $n$

$$M_j(P, n) \geq \frac{1}{t_j(P)} \left(\frac{n}{|n/2|}\right)(1 - \varepsilon).$$

(13)

Proof. Choose $N, K$, and $f_1, f_2, \ldots, f_K : P \to B_N$ as in Lemma 3.2 (Use $\varepsilon' = \frac{\varepsilon}{2}$). Consider the elements of $B_N$ as the subsets of a set $S$ of size $N$. Let $R$ be a set such that $S \subset R$ and $|R| = n$. Let $Q = R \setminus S$.

$$Q = \left\{ T \subset Q \mid \frac{n - N}{2} - K \leq |T| \leq \frac{n - N}{2} - 1 \right\}.$$ (14)

If $n$ is large enough, then the following inequality is true:

$$\sum_{i=1}^K \left(\frac{n - N}{2} - i\right) \geq K \cdot \left(\frac{n - N}{2}\right) \left(1 - \frac{\varepsilon}{2}\right).$$

(15)

Then

$$|Q| \geq K \cdot \left(\frac{n - N}{2}\right) \left(1 - \frac{\varepsilon}{2}\right) \geq 2^N \left(\frac{n - N}{2}\right) \left(1 - \frac{\varepsilon}{2}\right) \geq \frac{1}{t_j(P)} \left(\frac{n}{|n/2|}\right)(1 - \varepsilon).$$

(16)

We used that $2^N \left(\frac{n - N}{2}\right) \geq \left(\frac{n}{2}\right)$. It can be verified easily by induction on $N$.

We define an embedding $f_T : P \to B_R$ if $j = 2$ for every $T \in Q$ such that $\{Im(f_T) \mid T \in Q\}$ is an incomparable family. For any $x \in P$ let $f_T(x) \cap Q = T$ and $f_T(x) \cap S = f_{\lfloor n - N/(n/2)\rfloor - |T|}(x)$. Then $f_T$ is obviously an embedding if $j = 2$.

Now we check that the family $\{Im(f_T) \mid T \in Q\}$ is incomparable. Let $T_1, T_2 \in Q$ be different sets. Assume that $A_1 \in Im(f_{T_1})$, $A_2 \in Im(f_{T_2})$ and $A_1 \subseteq A_2$. Then $T_1 = A_1 \cap Q \subseteq A_2 \cap Q = T_2$. Since $T_1 \neq T_2$, $|T_1| < |T_2|$ holds. Since $A_1 \cap S \in Im(f_{\lfloor n - N/(n/2)\rfloor})$ and $A_2 \cap S \in Im(f_{\lfloor n - N/(n/2)\rfloor})$, Lemma 3.2 (3) implies $|A_1 \cap S| \subseteq |A_2 \cap S|$. It contradicts $A_1 \subseteq A_2$, so the family is indeed incomparable.

We found at least $\frac{n}{t_j(P)} \left(\frac{n}{|n/2|}\right)(1 - \varepsilon)$ different embeddings $(induced if j = 2)$ of $P$ to $B_R$, where $|R| = n$, such that the resulting copies form an incomparable family. It proves the theorem. □

Theorem 3.3. Let $P$ be a finite poset, $\varepsilon > 0$ and $j \in \{1, 2\}$. Then for all large enough $n$
4 Remarks

In this section we exactly determine the maximum number of incomparable copies for certain posets. The problem has already been solved for the path posets.

**Theorem 4.1.** (Griggs, Stahl, Trotter) \[7\] Let \( P^{h+1} \) be the path poset with \( h+1 \) elements. Then for all \( n \geq h \)

\[
M_1(P^{h+1}, n) = \left( n - h \right) \left\lfloor \frac{n}{2} \right\rfloor.
\]

(17)

We include an alternative proof for the sake of completeness. The following theorem will be used.

**Theorem 4.2.** (Bollobás) \[1\] Let \( (A_i, B_i) \) \((1 \leq i \leq m)\) be a family of disjoint subsets \((A_i \cap B_i = \emptyset)\), where \( A_i \cap B_j \neq \emptyset \) holds for \( i \neq j \) \((1 \leq i, j \leq m)\). Then

\[
\sum_{i=1}^{m} \frac{1}{\left( |A_i| + |B_i| \right)} \leq 1.
\]

(18)

**Proof.** (Theorem 4.1.) Consider an embedding of \( P^{h+1} \) into \( B_n \). Let its maximal and minimal elements embedded into \( C_i \) and \( D_i \) respectively. \( C_i \supset D_i \) implies \( C_i \cap D_i = \emptyset \). On the other hand, choosing these sets for all \( i = 1, \ldots, m \), the incomparability conditions imply \( C_i \cap D_j \neq \emptyset \). The theorem of Bollobás can be applied for the pairs \((C_i, D_i)\):

\[
\sum_{i=1}^{m} \frac{1}{\left( |C_i| + |B_i| \right)} \leq 1.
\]

(19)

\(|C_i - D_i| \geq h\) results in \(|C_i| + |D_i| \leq n - h\). Therefore the left hand side of (19) can be decreased in the following way.

\[
\frac{m}{\left( \left\lfloor \frac{n-h}{2} \right\rfloor \right)} = \sum_{i=1}^{m} \frac{1}{\left( \left\lfloor \frac{n-h}{2} \right\rfloor \right)} \leq \sum_{i=1}^{m} \frac{1}{\left( \left\lfloor \frac{n-h}{2} \right\rfloor \right)} \leq 1
\]

(20)

holds, proving the upper bound in the theorem.

The lower bound can be seen by an easy construction. Let \( G \subset \{h+1, h+2, \ldots n\} \) be a subset of size \( \left\lfloor \frac{n-h}{2} \right\rfloor \). Then \( P^{h+1} \) can be embedded to the sets \( G, \{1\} \cup G, \{1, 2\} \cup G, \ldots, \{1, 2, \ldots, h\} \cup G \). We have \( \left( \left\lfloor \frac{n-h}{2} \right\rfloor \right) \) such embeddings and the resulting copies form an incomparable family. This proves the lower bound.

**Definition** Let \( h(P) \) be the height of the poset \( P \), that is the number of elements in a longest chain in \( P \) minus 1. We say that \( P \) is thin if it can be embedded into \( B_{h(P)} \). \( P \) is called slim if it has an induced embedding into \( B_{h(P)} \).

**Theorem 4.3.** If \( P \) is a thin poset, then

\[
M_1(P, n) = \left( n - h \right) \left\lfloor \frac{n}{2} \right\rfloor.
\]

(21)

If \( P \) is slim, then

\[
M_1(P, n) = M_2(P, n) = \left( n - h \right) \left\lfloor \frac{n}{2} \right\rfloor.
\]

(22)
Proof. Since $P^{h+1}$ is a subposet of $P$,
\[
M_2(P,n) \leq M_1(P,n) \leq M_1(P^{h+1},n). \tag{23}
\]

Now consider $M_1(P^{h+1},n)$ incomparable copies of $P^{h+1}$ in $B_n$ as defined in Theorem 4.1. Their convex hulls are isomorphic to $B_h$, so we can embed $P$ to them (in an induced way if $P$ is slim). It proves $M_1(P,n) \geq M_1(P^{h+1},n)$ for thin posets, and $M_2(P,n) \geq M_1(P^{h+1},n)$ for slim posets.

We already determined the value of $M_1(P^{h+1},n)$ in Lemma 4.1 so the proof is completed.

Of course Theorem 4.3 does not contradict Theorem 1.1 since $t_1(P) = 2^h$ and
\[
\frac{1}{2^h} \left( \left\lceil \frac{n}{2} \right\rceil \right) \sim \left( \left\lceil \frac{n-h}{h} \right\rceil \right). \tag{24}
\]

The smallest non-thin poset is $V$ with three elements, $a, b, c$ and the relations $a < b, a < c$. Now we give a large set of incomparable copies for all $n$. Fix the parameter $i (1 \leq i \leq \frac{n+2}{2})$. Choose an element
\[
F \in \left( \left\lfloor \frac{n-2i}{2} \right\rfloor - 2i + 1 \right). \tag{25}
\]

Then the sets
\[
F \cup \{n-(2i-3), \ldots, n\}, F \cup \{n-(2i-3), \ldots, n \cup \{n-(2i-1)\}, F \cup \{n-(2i-3), \ldots, n \cup \{n-(2i-2)\}
\]
form an embedding of the poset $V$. Let $\mathcal{P}_i$ denote the set of all such copies. It is trivial that the copies in $\mathcal{P}_i$ are incomparable. But not much more difficult to check that two copies chosen from $\mathcal{P}_i$ and $\mathcal{P}_j (1 \leq i < j \leq \frac{n+2}{2})$, respectively, are also incomparable. Therefore
\[
\bigcup_{i=1}^{\frac{n+2}{2}} \mathcal{P}_i \tag{26}
\]
is a collection of incomparable embeddings of $V$. We conjecture that this is the largest one.

Conjecture
\[
M_1(V,n) = \sum_{i=1}^{\frac{n+2}{2}} \left( \left\lfloor \frac{n}{2} \right\rfloor - 2i + 1 \right). \tag{27}
\]

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