Recursive Polynomial Remainder Sequence
and its Subresultants

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Abstract

We introduce concepts of “recursive polynomial remainder sequence (PRS)” and “recursive subresultant,” along with investigation of their properties. A recursive PRS is defined as, if there exists the GCD (greatest common divisor) of initial polynomials, a sequence of PRSs calculated “recursively” for the GCD and its derivative until a constant is derived, and recursive subresultants are defined by determinants representing the coefficients in recursive PRS as functions of coefficients of initial polynomials. We give three different constructions of subresultant matrices for recursive subresultants; while the first one is built-up just with previously defined matrices thus the size of the matrix increases fast as the recursion deepens, the last one reduces the size of the matrix drastically by the Gaussian elimination on the second one which has a “nested” expression, i.e. a Sylvester matrix whose elements are themselves determinants.

Key words: polynomial remainder sequence, subresultants, Gaussian elimination, Sylvester’s identity

1 Introduction

The polynomial remainder sequence (PRS) is one of the most fundamental tools in computer algebra. Although the Euclidean algorithm (see Knuth [10]) for calculating PRS is simple, coefficient growth in PRS makes the Euclidean
algorithm often very inefficient. To overcome this problem, the mechanism of coefficient growth has been extensively studied through the theory of subresultants; see Collins [3], Brown and Traub [2], Loos [12], etc. By the theory of subresultant, we can remove extraneous factors of the elements of PRS systematically.

In this paper, we consider a variation of the subresultant. When we calculate PRS for polynomials which have a nontrivial GCD, we usually stop the calculation with the GCD. However, it is sometimes useful to continue the calculation by calculating the PRS for the GCD and its derivative; this is necessary for calculating the number of real zeros including their multiplicities. We call such a PRS a “recursive PRS.”

Although the theory of subresultants has been developed widely, the corresponding theory for recursive PRS is still unknown within the author’s knowledge; this is the main problem which we investigate in this paper. By “recursive subresultants,” we denote determinants which represent elements of recursive PRS as functions of the coefficients of initial polynomials.

We give three different constructions of subresultant matrices to express recursive subresultants in this paper. The first matrix construction recursively builds the matrix by shifting previously defined matrices, similarly as the Sylvester matrix shifts coefficients of the initial polynomials, thus the size of the matrices increases fast as the recursion deepens. The second matrix construction uses “nested” matrices, or a Sylvester matrix whose entries are themselves determinants. Finally, by the Gaussian elimination with the Sylvester’s identity on the second construction, we succeed to give the reduced matrix construction which expresses the coefficients of the polynomials in the recursive PRS as determinants of very small matrices, whose size actually decreases as the recursion deepens.

This paper is organized as follows. In Sect. 2, we introduce the concept of recursive PRS. In Sect. 3, we define recursive subresultant and show its relationship to recursive PRS. In Sect. 4 we define the “nested subresultant,” which is derived from the second construction of subresultant matrix, and show its equivalence to the recursive subresultant. In Sect. 5 we define the “reduced nested subresultant,” whose matrix is derived from the nested subresultant, and show that it is a reduced expression of the recursive subresultant. In Sect. 6 we briefly discuss usage of the reduced nested subresultant in approximate algebraic computation.
2 Recursive Polynomial Remainder Sequence (PRS)

First, we review the PRS, then define the recursive PRS. In the end of this section, we show a recursive Sturm sequence as an example of recursive PRS. We follow definitions and notations by von zur Gathen and L"{u}cking \cite{gathen-luecking}. Throughout this paper, let $R$ be an integral domain and $K$ be its quotient field. We define a polynomial remainder sequence as follows.

**Definition 2.1 (Polynomial Remainder Sequence (PRS))** Let $F$ and $G$ be polynomials in $R[x]$ of degree $m$ and $n$ ($m > n$) respectively. A sequence $(P_1, \ldots, P_l)$ of nonzero polynomials is called a polynomial remainder sequence (PRS) for $F$ and $G$, abbreviated to $\text{prs}(F, G)$, if it satisfies

$$P_1 = F, \quad P_2 = G, \quad \alpha_i P_{i-2} = q_{i-1} P_{i-1} + \beta_i P_i,$$

for $i = 3, \ldots, l$, where $\alpha_3, \ldots, \alpha_l, \beta_3, \ldots, \beta_l$ are elements of $R$ and $\deg(P_{i-1}) > \deg(P_i)$. A sequence $((\alpha_3, \beta_3), \ldots, (\alpha_l, \beta_l))$ is called a division rule for $\text{prs}(F, G)$. If $P_l$ is a constant, then the PRS is called complete. \(\square\)

If $F$ and $G$ are coprime, the last element in the complete PRS for $F$ and $G$ is a constant. Otherwise, it equals the GCD of $F$ and $G$ up to a constant: we have $\text{prs}(F, G) = (P_1 = F, P_2 = G, \ldots, P_l = \gamma \cdot \gcd(F, G))$ for some $\gamma \in R$. Then, we can calculate new PRS, $\text{prs}(P_l, \frac{d}{dx} P_l)$, and if this PRS ends with a non-constant polynomial, then calculate another PRS for the last element, and so on. By repeating this calculation, we can calculate several PRSs “recursively” such that the last polynomial in the last sequence is a constant. Thus, we define “recursive PRS” as follows.

**Definition 2.2 (Recursive PRS)** Let $F$ and $G$ be the same as in Definition 2.1. Then, a sequence

$$(P_1^{(1)}, \ldots, P_l^{(1)}, P_1^{(2)}, \ldots, P_l^{(2)}, \ldots, P_1^{(t)}, \ldots, P_l^{(t)})$$

of nonzero polynomials is called a recursive polynomial remainder sequence (recursive PRS) for $F$ and $G$, abbreviated to $\text{rprs}(F, G)$, if it satisfies

$$P_1^{(1)} = F, \quad P_2^{(1)} = G, \quad P_{l_k}^{(1)} = \gamma_1 \cdot \gcd(P_1^{(1)}, P_2^{(1)}) \text{ with } \gamma_1 \in R,$$

$$(P_1^{(1)}, P_2^{(1)}, \ldots, P_{l_k}^{(1)}) = \text{prs}(P_1^{(1)}, P_2^{(1)}),$$

$$P_1^{(k)} = P_{l_{k-1}}^{(k-1)}, \quad P_2^{(k)} = \frac{d}{dx} P_{l_{k-1}}^{(k-1)}, \quad P_{l_k}^{(k)} = \gamma_k \cdot \gcd(P_1^{(k)}, P_2^{(k)}) \text{ with } \gamma_k \in R,$$

$$(P_1^{(k)}, P_2^{(k)}, \ldots, P_{l_k}^{(k)}) = \text{prs}(P_1^{(k)}, P_2^{(k)}),$$
for \( k = 2, \ldots, t \). If \( \alpha_i^{(k)}, \beta_i^{(k)} \in R \) satisfy
\[
\alpha_i^{(k)} P_i^{(k)} = q_i^{(k)} P_i^{(k)} - \beta_i^{(k)} P_{i-1}^{(k)}
\]
for \( k = 1, \ldots, t \) and \( i = 3, \ldots, l_k \), then a sequence \( ((\alpha_3^{(1)}, \beta_3^{(1)}), \ldots, (\alpha_i^{(l)}, \beta_i^{(l)})) \) is called a division rule for \( \text{prs}(F, G) \). Furthermore, if \( P_i^{(l)} \) is a constant, then the recursive PRS is called complete. \( \square \)

**Remark 2.3** In this paper, we use the following notations unless otherwise defined: for \( k = 1, \ldots, t \) and \( i = 1, \ldots, l_k \), let \( c_i^{(k)} = \text{lc}(P_i^{(k)}) \), \( n_i^{(k)} = \deg(P_i^{(k)}) \) (letters \( \text{lc} \) and \( \deg \) denote the leading coefficient and the degree of the polynomial, respectively), \( j_0 = m \) and \( j_k = n_i^{(k)} \), and let \( d_i^{(k)} = n_i^{(k)} - n_{i+1}^{(k)} \) for \( k = 1, \ldots, t \) and \( i = 1, \ldots, l_k - 1 \). Furthermore, we represent \( P_i^{(k)}(x) \) as
\[
P_i^{(k)}(x) = a_{i,n_i^{(k)}}^{(k)} x^{n_i^{(k)}} + \cdots + a_{i,0}^{(k)} x^0,
\]
and its “coefficient vector” as
\[
p_i^{(k)} = (a_{i,n_i^{(k)}}^{(k)}, \ldots, a_{i,0}^{(k)}).
\]

As an example of recursive PRS, we calculate Sturm sequences recursively for calculating the number of real zeros of univariate polynomials including multiplicities (see Bochnak, Coste and Roy [1]), as follows.

**Example 2.4 (Recursive Sturm Sequence)** Let \( P(x) = (x + 2)^2 \{(x - 3)(x + 1)\}^3 \), and calculate the recursive Sturm sequence of \( P(x) \) as

\[
\text{(complete) } \text{prs}(P(x), \frac{d}{dx} P(x)),
\]
with division rule given by
\[
(\alpha^{(k)}_i, \beta^{(k)}_i) = (1, -1),
\]
for \( k = 1, \ldots, t \) and \( i = 3, \ldots, l_k \).

The first sequence \( L_1 = (P_1^{(1)}, \ldots, P_4^{(1)}) \) has the following elements:

\[
P_1^{(1)} = P(x) = (x + 2)^2 \{(x - 3)(x + 1)\}^3,
\]
\[
P_2^{(1)} = \frac{d}{dx} P(x) = 8x^7 - 14x^6 - 102x^5 + 80x^4 + 460x^3 + 66x^2 - 558x - 324,
\]
\[
P_3^{(1)} = \frac{75}{16} x^6 - \frac{45}{16} x^5 - 60x^4 - \frac{225}{8} x^3 + \frac{3315}{16} x^2 + \frac{4815}{8} x + \frac{945}{4},
\]
\[
P_4^{(1)} = \frac{128}{25} x^5 - \frac{256}{25} x^4 - \frac{256}{5} x^3 + \frac{1024}{25} x^2 + \frac{4224}{25} x + \frac{2304}{25}.
\]
The second sequence \( L_2 = (P^{(2)}_1, \ldots, P^{(2)}_4) \) has the following elements:

\[
P^{(2)}_1 = P^{(1)}_4 = \frac{128}{25} x^5 - \frac{526}{25} x^4 - \frac{526}{5} x^3 + \frac{1024}{25} x^2 + \frac{4224}{25} x + \frac{2304}{25},
\]
\[
P^{(2)}_2 = \frac{d}{dx} P^{(1)}_4 = \frac{128}{5} x^4 - \frac{1024}{25} x^3 - \frac{768}{5} x^2 + \frac{2048}{25} x + \frac{4224}{25},
\]
\[
P^{(2)}_3 = \frac{14848}{625} x^3 - \frac{1536}{125} x^2 - \frac{88576}{625} x - \frac{66048}{625},
\]
\[
P^{(2)}_4 = \frac{12800}{841} x^2 - \frac{25600}{841} x - \frac{38400}{841}.
\]

The last sequence \( L_3 = (P^{(3)}_1, \ldots, P^{(3)}_3) \) has the following elements:

\[
P^{(3)}_1 = P^{(2)}_4 = \frac{12800}{841} x^2 - \frac{25600}{841} x - \frac{38400}{841},
\]
\[
P^{(3)}_2 = \frac{d}{dx} P^{(2)}_4 = \frac{25600}{841} x - \frac{25600}{841},
\]
\[
P^{(3)}_3 = \frac{51200}{841}.
\]

For PRS \( L_k, k = 1, 2, 3 \), define sequences of nonzero real numbers \( \lambda(L_k, -\infty) \) and \( \lambda(L_k, +\infty) \) as

\[
\lambda(L_k, -\infty) = ((-1)^{n_1^{(k)}} \text{lc}(P^{(k)}_1), \ldots, (-1)^{n_k^{(k)}} \text{lc}(P^{(k)}_k)),
\]
\[
\lambda(L_k, +\infty) = (\text{lc}(P^{(k)}_1), \ldots, \text{lc}(P^{(k)}_k)),
\]

where \( n_i^{(k)} = \deg(P^{(k)}_i) \) denotes the degree of \( P^{(k)}_i \) and \( \text{lc}(P^{(k)}_i) \) denotes the leading coefficients of \( P^{(k)}_i \). Then, \( \lambda(L_k, -\infty) \) and \( \lambda(L_k, +\infty) \) for \( k = 1, 2, 3 \) are

\[
\lambda(L_1, \pm\infty) = (1, \pm8, \frac{75}{16}, \pm\frac{128}{25}),
\]
\[
\lambda(L_2, \pm\infty) = (\pm\frac{128}{25}, \pm\frac{128}{5}, \pm\frac{18848}{625}, \frac{12800}{841}),
\]
\[
\lambda(L_3, \pm\infty) = (\frac{12800}{841}, \pm\frac{25600}{841}, \frac{51200}{841}).
\]

For a sequence of nonzero real numbers \( L = (a_1, \ldots, a_m) \), let \( V(L) \) be the number of sign variations of the elements of \( L \). Then, we calculate the number of the real zeros of \( P(x) \), including multiplicity, as

\[
\sum_{k=1}^{3} \{ V(\lambda(L_k, -\infty)) - V(\lambda(L_k, +\infty)) \} = 3 + 3 + 2 = 8. \quad \square
\]
To make this paper self-contained and to use notations in our definitions, we first review the fundamental theorem of subresultants, then discuss subresultants for recursive PRS.

Although the theory of subresultants is established for polynomials over an integral domain, in what follows, we handle polynomials over a field for the sake of simplicity. Let $F$ and $G$ be polynomials in $K[x]$ such that

$$
F(x) = f_m x^m + \cdots + f_0 x^0, \quad G(x) = g_n x^n + \cdots + g_0 x^0,
$$

with $m \geq n > 0$. For a square matrix $M$, we denote its determinant by $|M|$.

### 3.1 Fundamental Theorem of Subresultants

**Definition 3.1 (Sylvester Matrix)** Let $F$ and $G$ be as in (3.1). The Sylvester matrix of $F$ and $G$, denoted by $N(F, G)$, is an $(m + n) \times (m + n)$ matrix constructed from the coefficients of $F$ and $G$, such that

$$
N(F, G) = \begin{pmatrix}
    f_m & g_n \\
    \vdots & \vdots \\
    f_0 & f_m & g_0 & g_n \\
    \vdots & \vdots & \vdots & \vdots \\
    f_0 & g_0
\end{pmatrix}.
$$

**Definition 3.2 (Subresultant Matrix)** Let $F$ and $G$ be defined as in (3.1). For $j < n$, the $j$-th subresultant matrix of $F$ and $G$, denoted by $N^{(j)}(F, G)$, is an $(m + n - j) \times (m + n - 2j)$ sub-matrix of $N(F, G)$ obtained by taking the left $n - j$ columns of coefficients of $F$ and the left $m - j$ columns of coefficients
of \(G\), such that

\[
N_i(F, G) = \begin{pmatrix}
  f_m & g_n \\
  \vdots & \vdots \\
  f_0 & f_m g_0 & g_n \\
  \vdots & \vdots & \vdots \\
  f_0 & g_0
\end{pmatrix}_{n-j \times m-j}.
\]

Furthermore, define \(N_U^{(j)}(F, G)\) as a sub-matrix of \(N^{(j)}(F, G)\) by deleting the bottom \(j + 1\) rows. \(\square\)

**Definition 3.3 (Subresultant)** Let \(F\) and \(G\) be defined as in (3.1). For \(j < n\) and \(k = 0, \ldots, j\), let \(N_k^{(j)} = N_k^{(j)}(F, G)\) (distinguish it from \(N_U^{(j)}(F, G)\) in the above) be a sub-matrix of \(N^{(j)}(F, G)\) obtained by taking the top \(m + n - 2j - 1\) rows and the \((m + n - j - k)\)-th row (note that \(N_k^{(j)}(F, G)\) is a square matrix).

Then, the polynomial

\[
S_j(F, G) = |N_j^{(j)}|x^j + \cdots + |N_0^{(j)}|x^0
\]

is called the \(j\)-th subresultant of \(F\) and \(G\). \(\square\)

**Theorem 3.4 (Fundamental Theorem of Subresultants [2])** Let \(F\) and \(G\) be defined as in (3.1), \((P_1, \ldots, P_k) = \text{prs}(F, G)\) be complete PRS, and \(((\alpha_3, \beta_3), \ldots, (\alpha_k, \beta_k))\) be its division rule. Let \(n_i = \deg(P_i)\) and \(c_i = \text{lcm}(P_i)\) for \(i = 1, \ldots, k\), and \(d_i = n_i - n_{i+1}\) for \(i = 1, \ldots, k - 1\). Then, we have

\[
S_j(F, G) = 0 \quad \text{for } 0 \leq j < n_k, \quad (3.2)
\]

\[
S_{n_i}(F, G) = P_i c_i^{d_i - 1} \times \prod_{l=3}^{i} \left\{ \left( \frac{\beta_l}{\alpha_l} \right)^{n_{l-1} - n_i} c_{l-1}^{d_{l-2} + d_{l-1} - 1} (-1)^{(n_{l-2} - n_l)(n_{l-1} - n_i)} \right\}, \quad (3.3)
\]

\[
S_j(F, G) = 0 \quad \text{for } n_i < j < n_{i-1} - 1, \quad (3.4)
\]

\[
S_{n_{i-1}}(F, G) = P_i c_i^{1 - d_i - 1} \prod_{l=3}^{i} \left\{ \left( \frac{\beta_l}{\alpha_l} \right)^{n_{l-1} - n_i + 1} c_{l-1}^{d_{l-2} + d_{l-1} - 1} (-1)^{(n_{l-2} - n_{l-1} + 1)(n_{l-1} - n_i + 1)} \right\}, \quad (3.5)
\]

for \(i = 3, \ldots, k\). \(\square\)

By the Fundamental Theorem of subresultants, we can express coefficients of
PRS by determinants of matrices whose elements are the coefficients of initial polynomials.

### 3.2 Recursive Subresultants

We construct “recursive subresultant matrix” whose determinants represent elements of recursive PRS by the coefficients of initial polynomials. To help the readers, we first show an example of recursive subresultant matrix for the recursive Sturm sequence in Example 2.4.

**Example 3.5 (Recursive Subresultant Matrix)** We express \( P(x) \) and \( \frac{d}{dx} P(x) \) in Example 2.4 by

\[
P(x) = f_8 x^8 + \cdots + f_0 x^0, \quad \frac{d}{dx} P(x) = g_7 x^7 + \cdots + g_0 x^0.
\]

Let \( \tilde{N}^{(1,5)}(F, G) = N^{(5)}(F, G) \), then the matrices \( \tilde{N}_U^{(1,5)}(F, G) \), \( \tilde{N}_L^{(1,5)}(F, G) \) and 
\( \tilde{N}_L^{(1,5)}(F, G) \) are given as

\[
\tilde{N}^{(1,5)}(F, G) = \begin{pmatrix}
\tilde{N}_U^{(1,5)} \\
\tilde{N}_L^{(1,5)}
\end{pmatrix} = \begin{pmatrix}
f_8 & g_7 \\
f_7 & f_8 & g_6 & g_7 \\
f_6 & f_7 & g_5 & g_6 & g_7 \\
f_5 & f_6 & g_4 & g_5 & g_6 \\
f_4 & f_5 & g_3 & g_4 & g_5 \\
f_3 & f_4 & g_2 & g_3 & g_4 \\
f_2 & f_3 & g_1 & g_2 & g_3 \\
f_1 & f_2 & g_0 & g_1 & g_2 \\
f_0 & f_1 & g_0 & g_1 & g_2 \\
f_0 & g_0
\end{pmatrix},
\]

\[
\tilde{N}_L^{(1,5)}(F, G) = \begin{pmatrix}
5f_4 & 5f_5 & 5g_3 & 5g_4 & 5g_5 \\
4f_3 & 4f_4 & 4g_2 & 4g_3 & 4g_4 \\
3f_2 & 3f_3 & 3g_1 & 3g_2 & 3g_3 \\
2f_1 & 2f_2 & 2g_0 & 2g_1 & 2g_2 \\
f_0 & f_1 & g_0 & g_1
\end{pmatrix},
\]

where horizontal lines in matrices divide them into the upper and the lower components. Note that the matrix \( \tilde{N}_L^{(1,5)}(F, G) \) is derived from \( \tilde{N}_L^{(1,5)}(F, G) \) by
multiplying the \(l\)-th row by \(6 - l\) for \(l = 1, \ldots, 5\) and deleting the bottom row.

Then, the \((2, 3)\)-th recursive subresultant matrix \(\bar{N}^{(2,3)}(F,G)\) is constructed as

\[
\bar{N}^{(2,3)}(F,G) = \begin{pmatrix}
\bar{N}_U^{(1,5)} & \bar{N}_U^{(1,5)} & \bar{N}_U^{(1,5)} \\
\bar{N}_U^{(1,5)} & 0 \cdots 0 \\
\bar{N}_L^{(1,5)} & \bar{N}_L^{(1,5)} & 0 \cdots 0 \\
0 \cdots 0 & 0 \cdots 0
\end{pmatrix}. \quad \Box \quad (3.6)
\]

**Definition 3.6 (Recursive Subresultant Matrix)** Let \(F\) and \(G\) be defined as in \[3.1\], and let \((P_1^{(1)}, \ldots, P_1^{(1)}, \ldots, P_t^{(1)}, \ldots, P_t^{(1)}\) be complete recursive PRS for \(F\) and \(G\) as in Definition \[2.2\]. Then, for each pair of numbers \((k, j)\) with \(k = 1, \ldots, t\) and \(j = j_{k-1} - 2, \ldots, 0\), define matrix \(\bar{N}^{(k,j)} = \bar{N}^{(k,j)}(F,G)\) recursively as follows.

1. For \(k = 1\), let \(\bar{N}^{(1,j)}(F,G) = N^{(j)}(F,G)\).
2. For \(k > 1\), let \(\bar{N}^{(k,j)}(F,G)\) consist of the upper and the lower block, defined as follows:
   
   (a) The upper block is partitioned into \((2j_{k-1} - 2j - 1) \times (2j_{k-1} - 2j - 1)\) blocks with the diagonal blocks filled with \(\bar{N}^{(k-1,j_{k-1})}_U\), where \(\bar{N}^{(k-1,j_{k-1})}_U\) is a sub-matrix of \(\bar{N}^{(k-1,j_{k-1})}(F,G)\) obtained by deleting the bottom \(j_{k-1} + 1\) rows.
   
   (b) Let \(\bar{N}^{(k-1,j_{k-1})}_L\) be a sub-matrix of \(\bar{N}^{(k-1,j_{k-1})}(F,G)\) obtained by taking the bottom \(j_{k-1} + 1\) rows, and let \(\bar{N}^{(k-1,j_{k-1})}_L\) be a sub-matrix of \(\bar{N}^{(k-1,j_{k-1})}(F,G)\) by multiplying the \((j_{k-1} + 1 - \tau)\)-th rows by \(\tau\) for \(\tau = j_{k-1}, \ldots, 1\), then by deleting the bottom row. Then, the lower block consists of \(j_{k-1} - j - 1\) blocks of \(\bar{N}^{(k-1,j_{k-1})}_L\) such that the leftmost block is placed at the top row of the container block and the right-side block is placed down by 1 row from the left-side block, then followed by \(j_{k-1} - j\) blocks of \(\bar{N}^{(k-1,j_{k-1})}_L\) placed by the same manner as \(\bar{N}^{(k-1,j_{k-1})}_L\).

As a result, \(\bar{N}^{(k,j)}(F,G)\) becomes as shown in Fig. \[4\]. Then, \(\bar{N}^{(k,j)}(F,G)\) is called the \((k,j)\)-th recursive subresultant matrix of \(F\) and \(G\). \(\Box\)

**Proposition 3.7** The numbers of rows and columns of \(\bar{N}^{(k,j)}(F,G)\), the \((k,j)\)-th recursive subresultant matrix of \(F\) and \(G\), are as follows: for \(k = 1\) and \(j < n\), they are equal to

\[
m + n - j \quad \text{and} \quad m + n - 2j, \quad (3.7)
\]
Definition 3.8 (Recursive Subresultant) Now, we define recursive subresultants. This proves the proposition.

\[(m + n - 2j_1) \left\{ \prod_{l=2}^{k-1} (2j_{l-1} - 2j_l - 1) \right\} (2j_{k-1} - 2j - 1) + j\] (3.8)

and

\[(m + n - 2j_1) \left\{ \prod_{l=2}^{k-1} (2j_{l-1} - 2j_l - 1) \right\} (2j_{k-1} - 2j - 1),\] (3.9)

respectively, with \(j_0 = j_1 + 1\) for \((k, j) = (1, j_1)\).

**PROOF.** By induction on \(k\). For \(k = 1\), (3.7) immediately follows from Case 1 of Definition 3.6 and we also have (3.8) and (3.9) for \((k, j) = (1, j_1)\). Let us assume that we have (3.8) and (3.9) for \(1, \ldots, k - 1\). Then, we calculate the numbers of the rows and columns of \(N^{(k,j)}(F, G)\) as follows.

1. The numbers of rows of \(N^{(k-1,j_{k-1})}_L\) and \(N^{(k-1,j_{k-1})}_L'\) are equal to \(j_{k-1} + 1\) and \(j_{k-1} - 1\), respectively, thus the number of rows a block which consists of \(N^{(k-1,j_{k-1})}_L\) and \(N^{(k-1,j_{k-1})}_L'\) in \(N^{(k,j)}(F, G)\) equals \(2j_{k-1} - j - 1\). (3.10)

On the other hand, the number of rows of \(N^{(k-1,j_{k-1})}_U\) is equal to \((m + n - 2j_1) \left\{ \prod_{l=2}^{k-1} (2j_{l-1} - 2j_l - 1) \right\} - 1\), thus the number of rows of diagonal blocks in \(N^{(k,j)}(F, G)\) is equal to

\[\left\{ \prod_{l=2}^{k-1} (2j_{l-1} - 2j_l - 1) \right\} (2j_{k-1} - 2j - 1).\] (3.11)

By adding (3.10) and (3.11), we obtain (3.8).

2. The number of columns of \(N^{(k-1,j_{k-1})}_L(F, G)\) is equal to \((m + n - 2j_1) \times \left\{ \prod_{l=2}^{k-1} (2j_{l-1} - 2j_l - 1) \right\}\), hence the number of columns of \(N^{(k,j)}(F, G)\) is equal to (3.9).

This proves the proposition. \(\Box\)

Now, we define recursive subresultants.

**Definition 3.8 (Recursive Subresultant)** Let \(F\) and \(G\) be defined as in (3.1), and let \((P_1^{(1)}, \ldots, P_1^{(t)}), \ldots, (P_k^{(1)}, \ldots, P_k^{(t)})\) be complete recursive PRS for \(F\) and \(G\) as in Definition 2.2. For \(j = j_{k-1} - 2, \ldots, 0\) and \(\tau = j, \ldots, 0\), let \(\hat{N}^{(k,j)}(F, G)\) be a sub-matrix of the \((k,j)\)-th recursive subresultant matrix \(N^{(k,j)}(F, G)\) obtained by taking the top \((m + n - 2j_1) \{ \prod_{l=2}^{k-1} (2j_{l-1} - 2j_l - 1)\} \) columns of \(N^{(k,j)}(F, G)\), respectively, at \(\tau = j, \ldots, 0\).

\[
\hat{N}^{(k,j)}(F, G) = \left[ \begin{array}{c} \vdots \\ \vdots \\ \vdots \end{array} \right]_{m \times n - 2j_1} \left[ \begin{array}{c} \vdots \\ \vdots \\ \vdots \end{array} \right]_{m \times n - 2j_1} \]
For $k \geq 0$ rows and the \((m + n - 2j_1)\{\prod_{l=2}^{j_1-1}(2j_l - 1)\}(2j_k - 1 - 2j - 1) - 1 \) rows and the \((m + n - 2j_1)\{\prod_{l=2}^{j_1-1}(2j_l - 1)\}(2j_k - 1 - 2j - 1) + j - \tau \)-th row (note that $N^{(k,j)}_\tau$ is a square matrix). Then, the polynomial

$$S_{k,j}(F,G) = |N^{(k,j)}_j| x^j + \ldots + |N^{(k,j)}_0| x^0$$

is called the \((k,j)\)-th recursive subresultant of $F$ and $G$. □

We show the relationship between recursive subresultants and coefficients in the recursive PRS.

**Lemma 3.9** Let $F$ and $G$ be defined as in (3.1), and let $(P_1^{(1)}, \ldots, P_{t_1}^{(1)}, \ldots, P_1^{(t)}, \ldots, P_{t_t}^{(t)})$ be complete recursive PRS for $F$ and $G$ as in Definition 2.2. For $k = 1, \ldots, t - 1$, define

$$B_k = \left( c^{(k)}_{l_k} \right)^{d^{(k)}_{l_k} - 1} \prod_{l=3}^{l_k} \left\{ \frac{b^{(k)}_{l_k}}{\alpha^{(k)}_{l_k}} \right\}^{n^{(k)}_{l_k} - n^{(k)}_{l_k}} \left( c^{(k)}_{l_k - 1} \right)^{d^{(k)}_{l_k - 2} + d^{(k)}_{l_k - 1}} (-1)^{n^{(k)}_{l_k - 2} - n^{(k)}_{l_k}} \left( n^{(k)}_{l_k - 1} - n^{(k)}_{l_k} \right) \left( n^{(k)}_{l_k} - n^{(k)}_{l_k} \right) \right\}.$$

For $k = 2, \ldots, t$ and $j = j_{k-1} - 2, \ldots, 0$, define

$$u_{k,j} = (m + n - 2j_1) \left\{ \prod_{l=2}^{j_1-1}(2j_l - 1) \right\} (2j_{k-1} - 2j - 1)$$

with $u_k = u_{k,j_k}$ and $u_1 = m + n - 2j_1$, $b_{k,j} = 2j_k - 2j - 1$ with $b_k = b_{k,j_k}$ and $b_{1,j} = 1$ for $j < n$, $r_{k,j} = (-1)^{(u_{k-1,j-1})(1 + 2 + \ldots + (b_{k,j-1} - 1))}$ with $r_k = r_{k,j_k}$ and $r_{1,j} = 1$ for $j < n$, $R_k = (R_{k-1})^{b_k} r_k B_k$ with $R_0 = 1$.

Then, for the \((k,j)\)-th recursive subresultant of $F$ and $G$ with $k = 1, \ldots, t$ and $j = j_{k-1} - 2, \ldots, 0$, we have

$$S_{k,j}(F,G) = (R_{k-1})^{b_{k,j}} r_{k,j} \cdot S_j(P^{(k)}_1, P^{(k)}_2). \tag{3.12}$$

To prove Lemma 3.9, we prove the following lemma.

**Lemma 3.10** For $k = 1, \ldots, t$, $j = j_{k-1} - 2, \ldots, 0$ and $\tau = j, \ldots, 0$, we have

$$|\tilde{N}^{(k,j)}_{\tau}(F,G)| = (R_{k-1})^{b_{k,j}} r_{k,j} |N^{(j)}_{\tau}(P^{(k)}_1, P^{(k)}_2)|.$$

**PROOF.** By induction on $k$. For $k = 1$, it is obvious from Case 1 in Definition 3.6. Let us assume that the lemma is valid for $1, \ldots, k - 1$, then we prove the claim for $k$ by the following steps.
Lemma 3.11 Assume that we have Lemma 3.10 for 1, . . . , k − 1. Then, for k, j = j_{k-1} − 2, . . . , 0 and τ = j, . . . , 0, \( \bar{N}^{(k,j)}(F,G) \) can be transformed by certain eliminations and permutations on its columns into \( M^{(k,j)}(F,G) \) as shown in Fig. 2 satisfying

\[
|\bar{N}^{(k,j)}(F,G)| = ((\bar{R}_{k-2})^{b_{k-1}}_{k-1})^{b_{k,j}} |M^{(k,j)}(F,G)|, \tag{3.13}
\]

where \( M^{(k,j)}(F,G) \) is a sub-matrix of \( M^{(k,j)}(F,G) \) obtained by the same manner as we have obtained \( \bar{N}^{(k,j)}(F,G) \) from \( N^{(k,j)}(F,G) \) in Definition 3.8.

**PROOF.** By the induction hypothesis, for \( \tau' = j_{k-1}, \ldots, 0 \), we have

\[
|\bar{N}^{(k-1,j_{k-1})}(F,G)| = (\bar{R}_{k-2})^{b_{k-1}}_{k-1} |N^{(j_{k-1})}(P_{1}^{(k-1)}, P_{2}^{(k-1)})|.
\]

Let \( \bar{N}^{(k,j)}(F,G) \) be a matrix defined as

\[
\bar{N}^{(k,j)}(F,G) = \begin{pmatrix} \bar{N}^{(k,j)}_U \\ \bar{N}^{(k,j)}_L \end{pmatrix},
\]

where \( \bar{N}^{(k,j)}_U \) and \( \bar{N}^{(k,j)}_L \) are defined as in Definition 3.6. Furthermore, let \( N^{(j_{k-1})}(P_{1}^{(k-1)}, P_{2}^{(k-1)}) \) be defined as \( N^{(j_{k-1})}(P_{1}^{(k-1)}, P_{2}^{(k-1)}) \) with the \( (j_{k-2} + 1 - \tau) \)-th row multiplied by \( \tau \) for \( \tau = j_{k-1}, \ldots, 1 \), then by deleting the bottom row, \( \bar{N}^{(j_{k-1})}(P_{1}^{(k-1)}, P_{2}^{(k-1)}) \) be a sub-matrix of \( N^{(j_{k-1})}(P_{1}^{(k-1)}, P_{2}^{(k-1)}) \) and \( \bar{N}^{(j_{k-1})}(P_{1}^{(k-1)}, P_{2}^{(k-1)}) \) by taking the top \( j_{k-2} - j_{k-1} \) rows, and \( \bar{N}^{(j_{k-1})}(P_{1}^{(k-1)}, P_{2}^{(k-1)}) \) and \( \bar{N}^{(j_{k-1})}(P_{1}^{(k-1)}, P_{2}^{(k-1)}) \) be sub-matrices of \( N^{(j_{k-1})}(P_{1}^{(k-1)}, P_{2}^{(k-1)}) \) and \( N^{(j_{k-1})}(P_{1}^{(k-1)}, P_{2}^{(k-1)}) \), respectively, by eliminating the top \( j_{k-2} - j_{k-1} \) rows.

Then, by certain eliminations and exchanges on columns, we can transform \( \bar{N}^{(k-1,j_{k-1})}(F,G) \) and \( \bar{N}^{(k-1,j_{k-1})}(F,G) \) to

\[
D^{(k-1,j_{k-1})}(F,G)
= \begin{pmatrix} W_{k-1} & 0 \\ * & N^{(j_{k-1})}(P_{1}^{(k-1)}, P_{2}^{(k-1)}) \end{pmatrix}
= \begin{pmatrix} W_{k-1} & 0 \\ * & N^{(j_{k-1})}(P_{1}^{(k-1)}, P_{2}^{(k-1)}) \end{pmatrix},
\]

\[
D^{(k-1,j_{k-1})}(F,G)
= \begin{pmatrix} W_{k-1} & 0 \\ * & N^{(j_{k-1})}(P_{1}^{(k-1)}, P_{2}^{(k-1)}) \end{pmatrix}
= \begin{pmatrix} W_{k-1} & 0 \\ * & N^{(j_{k-1})}(P_{1}^{(k-1)}, P_{2}^{(k-1)}) \end{pmatrix}.
\]

(3.14)
respectively, satisfying

\[ |W_{k-1}| = 1, \]
\[ |D'_{\tau'}^{(k-1,j_{k-1})}(F,G)| = (R_{k-2})^{b_{k-1}r_{k-1}}|N_{\tau'}^{(j_{k-1})}(P_{1}^{(k-1)}, P_{2}^{(k-1)})|, \]
\[ |D''_{\tau''}^{(k-1,j_{k-1})}(F,G)| = (R_{k-2})^{b_{k-1}r_{k-1}}|N_{\tau''}^{(j_{k-1})}(P_{1}^{(k-1)}, P_{2}^{(k-1)})|, \]

where \( D_{\tau'}^{(k-1,j_{k-1})}(F,G) \) and \( D''_{\tau''}^{(k-1,j_{k-1})}(F,G) \) are sub-matrices of \( D^{(k-1,j_{k-1})}(F,G) \) and \( D''^{(k-1,j_{k-1})}(F,G) \), respectively, obtained by the same manner as we have obtained \( \tilde{N}_{\tau'}^{(k-1,j_{k-1})}(F,G) \) from \( \tilde{N}^{(k-1,j_{k-1})}(F,G) \) (see Definition 3.3).

Therefore, by the above transformations on the columns in each column blocks in \( \tilde{N}^{(k,j)}(F,G) \) as shown in Fig. 1, we obtain \( M_{\tau}^{(k,j)}(F,G) \) as shown in Fig. 2, where \( N_{U}^{(j_{k-1})}(P_{1}^{(k-1)}, P_{2}^{(k-1)}) \), \( N_{L}^{(j_{k-1})}(P_{1}^{(k-1)}, P_{2}^{(k-1)}) \) and \( N_{L}^{(j_{k-1})}(P_{1}^{(k-1)}, P_{2}^{(k-1)}) \) are abbreviated to \( N_{U}^{(j_{k-1})}, N_{L}^{(j_{k-1})} \) and \( N_{L}^{(j_{k-1})} \), respectively, satisfying (3.13) because \( M_{\tau}^{(k,j)}(F,G) \) has \( b_{k,j} = 2j_{k-1} - 2j - 1 \) column blocks that have \( D^{(k-1,j_{k-1})}(F,G) \) or \( D''^{(k-1,j_{k-1})}(F,G) \). This proves the lemma. \( \square \)

Lemma 3.12 For \( k, j = j_{k-1} - 2, \ldots, 0 \) and \( \tau = j, \ldots, 0 \), \( M_{\tau}^{(k,j)}(F,G) \) can be transformed by certain eliminations and permutations on its columns into \( \tilde{M}_{\tau}^{(k,j)}(F,G) \) as shown in Fig. 3, satisfying

\[ |M_{\tau}^{(k,j)}(F,G)| = (B_{k-1})^{b_{k,j}}|\tilde{M}_{\tau}^{(k,j)}(F,G)|, \]

where \( \tilde{M}_{\tau}^{(k,j)} \) is a sub-matrix of \( \tilde{M}_{\tau}^{(k,j)} \) obtained by the same manner as we have obtained \( \tilde{N}_{\tau'}^{(k-1,j_{k-1})} \) from \( \tilde{N}^{(k-1,j_{k-1})} \) in Definition 3.8.

PROOF. For \( N_{\tau''}^{(j_{k-1})}(P_{1}^{(k-1)}, P_{2}^{(k-1)}) \) (defined as in the proof of Lemma 3.11) and \( \tau'' = j_{k-1} - 1, \ldots, 0 \), let \( N_{\tau''}^{(j_{k-1})}(P_{1}^{(k-1)}, P_{2}^{(k-1)}) \) be a sub-matrix of \( N_{\tau''}^{(j_{k-1})}(P_{1}^{(k-1)}, P_{2}^{(k-1)}) \) obtained by taking the top \( 2(n_{1}^{(k-1)} - j_{k-1} - 1) \) rows and the \( (2n_{1}^{(k-1)} - 2 - j_{k-1} - \tau'') \)-th row. Then, by the Fundamental Theorem of subresultants (Theorem 3.3), we have

\[ |N_{\tau''}^{(j_{k-1})}(P_{1}^{(k-1)}, P_{2}^{(k-1)})|x^{j_{k-1} - 1} + \cdots + |N_{0}^{(j_{k-1})}(P_{1}^{(k-1)}, P_{2}^{(k-1)})|x^{0} \]
\[ = S_{j_{k-1}}(P_{1}^{(k-1)}, P_{2}^{(k-1)}) = B_{k-1}P_{k-1}^{(k)} = B_{k-1}P_{1}^{(k)}, \]
\[ |N_{\tau''}^{(j_{k-1})}(P_{1}^{(k-1)}, P_{2}^{(k-1)})|x^{j_{k-1} - 1} + \cdots + |N_{0}^{(j_{k-1})}(P_{1}^{(k-1)}, P_{2}^{(k-1)})|x^{0} \]
\[ = \frac{d}{dx} S_{j_{k-1}}(P_{1}^{(k-1)}, P_{2}^{(k-1)}) = B_{k-1}\frac{d}{dx}P_{k-1}^{(k)} = B_{k-1}P_{2}^{(k)}, \]

\[ = \frac{d}{dx} S_{j_{k-1}}(P_{1}^{(k-1)}, P_{2}^{(k-1)}) = B_{k-1}\frac{d}{dx}P_{k-1}^{(k)} = B_{k-1}P_{2}^{(k)}, \]
hence, for \( \tau' = j_{k-1}, \ldots, 0 \) and \( \tau'' = j_{k-1} - 1, \ldots, 0 \), we have

\[
|N_{\tau'}^{(j_{k-1})}(P_1^{(k-1)}, P_2^{(k-1)})| = B_{k-1}d_{1, \tau'}, \\
|N_{\tau''}^{(j_{k-1})}(P_1^{(k-1)}, P_2^{(k-1)})| = B_{k-1}d_{2, \tau''},
\]

where \( d_{i,j}^{(k)} \) represents the coefficient of degree \( j \) of \( P_i^{(k)} \) (see Remark 2.3). Therefore, by certain eliminations and exchanges on columns, we can transform \( N^{(j_{k-1})}(P_1^{(k-1)}, P_2^{(k-1)}) \) and \( N''^{(j_{k-1})}(P_1^{(k-1)}, P_2^{(k-1)}) \) into

\[
\begin{pmatrix}
\tilde{N}^{(j_{k-1})}_U & 0 \\
* & P_1^{(k)}
\end{pmatrix}
\text{ and }
\begin{pmatrix}
\tilde{N}^{(j_{k-1})}_U & 0 \\
* & P_2^{(k)}
\end{pmatrix},
\]

respectively, satisfying \( |\tilde{N}^{(j_{k-1})}_U| = 1 \) (see Remark 2.3 for the notation of “coefficient vectors”). By these transformations, we can transform \( D^{(k-1,j_{k-1})}(F, G) \) and \( D''^{(k-1,j_{k-1})}(F, G) \) in (3.14) to

\[
\tilde{D}^{(k-1,j_{k-1})}(F, G) = \begin{pmatrix}
W_{k-1} & 0 \\
\tilde{N}^{(j_{k-1})}_U & * \\
* & P_1^{(k)}
\end{pmatrix},
\]

\[
\tilde{D}''^{(k-1,j_{k-1})}(F, G) = \begin{pmatrix}
W_{k-1} & 0 \\
\tilde{N}''^{(j_{k-1})}_U & * \\
* & P_2^{(k)}
\end{pmatrix},
\]

respectively, satisfying

\[
|\tilde{N}^{(j_{k-1})}_U| = 1, \\
|\tilde{D}^{(k-1,j_{k-1})}_{\tau'}(F, G)| = B_{k-1}|D^{(k-1,j_{k-1})}_{\tau'}(F, G)|, \\
|\tilde{D}''^{(k-1,j_{k-1})}_{\tau'}(F, G)| = B_{k-1}|D''^{(k-1,j_{k-1})}_{\tau'}(F, G)|,
\]

where \( D^{(k-1,j_{k-1})}_{\tau'}, D''^{(k-1,j_{k-1})}_{\tau'}, \tilde{D}^{(k-1,j_{k-1})}_{\tau'} \) and \( \tilde{D}''^{(k-1,j_{k-1})}_{\tau'} \) are sub-matrices of \( D^{(k-1,j_{k-1})}, D''^{(k-1,j_{k-1})}, \tilde{D}^{(k-1,j_{k-1})} \) and \( \tilde{D}''^{(k-1,j_{k-1})} \), respectively, obtained by the same manner as we have obtained \( \tilde{N}^{(k-1,j_{k-1})}_{\tau'} \) from \( \tilde{N}^{(k-1,j_{k-1})} \). Therefore, by the above eliminations on the columns in each column blocks, we can transform \( M^{(k,j)}(F, G) \) to \( \tilde{M}^{(k,j)}(F, G) \) as shown in Fig. 3 satisfying (3.15) because \( M^{(k,j)}(F, G) \) has \( b_{k,j} = 2j_{k-1} - 2j - 1 \) column blocks that have \( D^{(k-1,j_{k-1})}(F, G) \) or \( D''^{(k-1,j_{k-1})}(F, G) \). This proves the lemma. \( \square \)
Proof of Lemma 3.10 (continued). By exchanges on column blocks, we can transform \( \tilde{M}^{(k,j)}(F,G) \) to \( \tilde{N}^{(k,j)}(F,G) \) as shown in Fig. 4 with

\[
|\tilde{M}^{(k,j)}(F,G)| = |\tilde{N}^{(k,j)}(F,G)|,
\]

where \( \tilde{M}^{(k,j)} \) is a sub-matrix of \( \tilde{N}^{(k,j)} \) obtained by the same manner as we have obtained \( \tilde{N}^{(k,j)} \) from \( N^{(k,j)} \), because the \((u_{k,j} - (l - 1)u_{k-1})\)-th column in \( \tilde{M}^{(k,j)}(F,G) \) was moved to the \((u_{k,j} - (l - 1))\)-th column in \( \tilde{M}^{(k,j)}(F,G) \) for \( l = 1, \ldots, b_{k,j} \). (Note that \( \tilde{M}^{(k,j)}(F,G) \) is a block lower triangular matrix.) Then, we have

\[
|\tilde{M}^{(k,j)}(F,G)| = |N^{(j)}(P_1^{(k)}, P_2^{(k)})|,
\]

because we have \(|W_{k-1}| = |\tilde{N}^{(j_k-1)}| = 1 \) and the lower-right block of \( P_1^{(k)} \)s and \( P_2^{(k)} \)s in \( \tilde{M}^{(k,j)}(F,G) \) is equal to \( N^{(j)}(P_1^{(k)}, P_2^{(k)}) \).

Finally, from (3.13), (3.15), (3.16) and (3.17), we have

\[
|\tilde{N}^{(k,j)}(F,G)| = ((\tilde{R}_{k-2})^{b_{j-1}-1}_{b_{j-1}-1} |N^{(j)}(P_1^{(k)}, P_2^{(k)})|)
\]

which proves the lemma. \( \square \)

Theorem 3.13 With the same conditions as in Lemma 3.9 and for \( k = 1, \ldots, t \) and \( i = 3, 4, \ldots, l_k \), we have

\[
\tilde{S}_{k,j}(F,G) = 0 \quad \text{for} \quad 0 \leq j < n_{i_k}^{(k)}, \quad (3.18)
\]

\[
\tilde{S}_{k,n_i^{(k)}}(F,G) = P_i^{(k)}(c_i^{(k)})^{d_i^{(k)}-1}(R_{k-1})^{b_{k,n_i^{(k)}}}_{b_{k,n_i^{(k)}}} \times \prod_{l=3}^{i} \left\{ \frac{\beta_l^{(k)}}{\alpha_l^{(k)}} \right\}^{n_{i-1}^{(k)}-n_i^{(k)}} (c_{i-1}^{(k)})^{d_{i-2}^{(k)}+d_{i-1}^{(k)}}(-1)^{(n_{i-2}^{(k)}-n_i^{(k)})n_{i-1}^{(k)}-n_i^{(k)}} \right\}, \quad (3.19)
\]

\[
\tilde{S}_{k,j}(F,G) = 0 \quad \text{for} \quad n_i^{(k)} < j < n_{i-1}^{(k)}-1, \quad (3.20)
\]

\[
\tilde{S}_{k,n_i^{(k)}-1}(F,G) = P_i^{(k)}(c_i^{(k)})^{d_i^{(k)}-1}(R_{k-1})^{b_{k,n_i^{(k)}-1}}_{b_{k,n_i^{(k)}-1}} \times \prod_{l=3}^{i} \left\{ \frac{\beta_l^{(k)}}{\alpha_l^{(k)}} \right\}^{n_{i-1}^{(k)}-n_i^{(k)}+1} \times (c_{i-1}^{(k)})^{d_{i-2}^{(k)}+d_{i-1}^{(k)}}(-1)^{(n_{i-2}^{(k)}-n_i^{(k)}+1)(n_{i-1}^{(k)}-n_i^{(k)+1})} \right\}. \quad (3.21)
\]

PROOF. By substituting \( S_j(P_1^{(k)}, P_2^{(k)}) \) in (3.12) by (3.2)–(3.5), we obtain (3.18)–(3.21), respectively. \( \square \)
We show an example of the proof of Lemma 3.9 for the recursive subresultant matrix in Example 3.5.

Example 3.14 (Continued from Example 3.5.) Since we have $\tilde{N}^{(1,5)}(F, G) = N^{(5)}(F, G)$, we can regard $\tilde{N}^{(1,5)}(F, G)$ and $\tilde{N}'^{(1,5)}(F, G)$ as $D^{(1,5)}(F, G)$ and $D'^{(1,5)}(F, G)$ in (3.14), respectively. Then, by eliminations and exchanges of columns as shown in Lemma 3.12, we can transform $\tilde{N}^{(1,5)}(F, G) = \begin{pmatrix} \tilde{N}_U^{(1,5)} & 0 \\ \ast & \mathbf{p}_1^{(2)} \end{pmatrix}$ and $\tilde{N}'^{(1,5)}(F, G) = \begin{pmatrix} \tilde{N}_U^{(1,5)} & 0 \\ \ast & \mathbf{p}_2^{(2)} \end{pmatrix}$ in (3.6) to $\tilde{D}^{(1,5)}(F, G)$ and $\tilde{D}'^{(1,5)}(F, G)$, respectively, as

$$\tilde{D}^{(1,5)}(F, G) = \begin{pmatrix} \tilde{N}_U^{(5)} & 0 \\ \ast & \mathbf{p}_1^{(2)} \end{pmatrix}, \quad \tilde{D}'^{(1,5)}(F, G) = \begin{pmatrix} \tilde{N}_U^{(5)} & 0 \\ \ast & \mathbf{p}_2^{(2)} \end{pmatrix},$$

with $|\tilde{N}_U^{(5)}| = 1$ and $B_1 = (a_{2,7}^{(1)})^2(a_{3,6}^{(1)})^2$. Therefore, by the above transformations of columns in each column blocks in $\tilde{N}^{(2,1)}(F, G)$, we have

$$\tilde{M}^{(2,3)}(F, G) = \begin{pmatrix} \tilde{N}_U^{(5)} & 0 \\ \ast & \tilde{N}_U^{(5)} \end{pmatrix},$$

satisfying $|\tilde{N}_U^{(2,3)}(F, G)| = (B_1)^3 |\tilde{M}^{(2,3)}(F, G)|$ for $\tau = 3, \ldots, 0$. Furthermore,
by exchanges of columns, we can transform $\tilde{M}^{(2,3)}(F, G)$ to $\hat{M}^{(2,3)}(F, G)$ as

$$
\begin{bmatrix}
\tilde{N}_U^{(5)} \\
\tilde{N}_U^{(5)} \\
0 \\
* \\
* \\
0 \\
0 \\
0 \\
\end{bmatrix}
= 
\begin{bmatrix}
\hat{N}_U^{(5)} \\
\hat{N}_U^{(5)} \\
\hat{N}_U^{(5)} \\
P_1^{(2)} \\
P_2^{(2)} \\
0 \\
0 \\
0 \\
\end{bmatrix}.
$$

satisfying $|\tilde{M}_\tau^{(2,3)}(F, G)| = r_{2,3} |\hat{M}_\tau^{(2,3)}(F, G)| = r_{2,3} |N^{(3)}(P_1^{(2)}, P_2^{(2)})|$. Therefore, we have

$$
|\tilde{N}_\tau^{(2,3)}(F, G)| = (B_1)^3 r_{2,3} |N^{(3)}(P_1^{(2)}, P_2^{(2)})| = (R_1)^3 r_{2,3} |N^{(3)}(P_1^{(2)}, P_2^{(2)})|,
$$

for $\tau = 3, \ldots , 0$, and we have

$$
\bar{S}_{2,3}(F, G) = (R_1)^3 r_{2,3} \cdot S_3(P_1^{(2)}, P_2^{(2)}) = \{(a_{2,7})^2 (a_{3,6})^2\}^3 (a_{2,4})^2 P_3^{(2)}.
$$

4 Nested Subresultants

As we have seen in the above, the recursive subresultant can represent the coefficients of the elements in recursive PRS. However, the size of the recursive subresultant matrix becomes larger rapidly as the recursion of the recursive PRS deepens, thus making use of the recursive subresultant matrix become inefficient.

To overcome this problem, we should introduce other representations for the subresultant that are equivalent to the recursive subresultant, and more suitable for efficient computations. The nested subresultant matrix is a subresultant matrix whose elements are again determinants of certain subresultant matrices (or even the nested subresultant matrices), and the nested subresultant is a subresultant whose coefficients are determinants of the nested subresultant matrices.

Note that the nested subresultant is mainly used to show the relationship between the recursive subresultant and the reduced nested subresultant that will be defined in the next section.
We show an example of a nested subresultant matrix.

**Example 4.1** Let \( F(x) \) and \( G(x) \) be defined as

\[
\begin{align*}
F(x) &= a_6 x^6 + a_5 x^5 + \cdots + a_0, \quad a_6 \neq 0, \\
G(x) &= b_5 x^5 + b_4 x^4 + \cdots + b_0, \quad b_5 \neq 0.
\end{align*}
\]

Let \( \text{prs}(F, G) = (P_1^{(1)} = F, \ P_2^{(1)} = G, \ P_3^{(1)} = \gcd(F, G)) \) with \( \deg(P_3^{(1)}) = 4 \), and let us consider recursive PRS for \( F \) and \( G \).

Let \( P_1^{(2)} = P_3^{(1)} \), \( P_2^{(2)} = \frac{d}{dx}P_3^{(1)} \), and calculate a subresultant of degree 1, which corresponds to \( P_4^{(2)} \). By the Fundamental Theorem of subresultants (Theorem 3.4), we have

\[
S_4(F, G) = A_4 x^4 + A_3 x^3 + A_2 x^2 + A_1 x + A_0,
\]

\[
\frac{d}{dx}S_4(F, G) = 4A_4 x^3 + 3A_3 x^2 + 2A_2 x + A_1,
\]

where

\[
A_j = |N_j^{(4)}(F, G)| \quad \text{(4.1)}
\]

for \( j = 0, \ldots, 4 \) with \( N_j^{(4)}(F, G) \) as in Definition 3.2.

Then, we can express the subresultant matrix \( N^{(2)}(S_4(F, G), \frac{d}{dx}S_4(F, G)) \) as

\[
N^{(2)}(S_4(F, G), \frac{d}{dx}S_4(F, G)) = \begin{pmatrix}
A_4 & 4A_4 \\
A_3 & 3A_3 & 4A_4 \\
A_2 & 2A_2 & 3A_3 \\
A_1 & A_1 & 2A_2 \\
A_0 & A_1
\end{pmatrix}, \quad \text{(4.2)}
\]

and the subresultant \( S_2(S_4(F, G), \frac{d}{dx}S_4(F, G)) \) as

\[
S_2(S_4(F, G), \frac{d}{dx}S_4(F, G)) = \begin{vmatrix}
A_4 & 4A_4 \\
A_3 & 3A_3 & 4A_4 \\
A_2 & 2A_2 & 3A_3 \\
A_1 & A_1 & 2A_2 \\
A_0 & A_1
\end{vmatrix} x^2 + \begin{vmatrix}
A_4 & 4A_4 \\
A_3 & 3A_3 & 4A_4 \\
A_2 & 2A_2 & 3A_3 \\
A_1 & A_1 & 2A_2 \\
A_0 & A_1
\end{vmatrix} x + \begin{vmatrix}
A_4 & 4A_4 \\
A_3 & 3A_3 & 4A_4 \\
A_2 & 2A_2 & 3A_3 \\
A_1 & A_1 & 2A_2 \\
A_0 & A_1
\end{vmatrix}, \quad \text{(4.3)}
\]

respectively, with \( A_j \) as in (4.1). We see that the elements in (4.2) are minors of subresultant matrix, hence the coefficients in (4.3) is “nested” expression of determinants. □
Definition 4.2 (Nested Subresultant Matrix) Let $F$ and $G$ be defined as in (3.1), and let $(P_1^{(1)}, \ldots, P_{t_1}^{(1)}, \ldots, P_1^{(t)}, \ldots, P_{t_t}^{(t)})$ be complete recursive PRS for $F$ and $G$ as in Definition 2.2. Then, for each pair of numbers $(k,j)$ with $k = 1, \ldots, t$ and $j = j_{k-1} - 2, \ldots, 0$, define matrix $\tilde{N}^{(k,j)}(F,G)$ recursively as follows.

1. For $k = 1$, let $\tilde{N}^{(1,j)}(F,G) = N^{(j)}(F,G)$.
2. For $k > 1$, let

$$
\tilde{N}^{(k,j)}(F,G) = N^{(j)}\left(\tilde{S}_{k-1,jk-1}(F,G), \frac{d}{dx} \tilde{S}_{k-1,jk-1}(F,G)\right),
$$

where $\tilde{S}_{k-1,jk-1}(F,G)$ is defined by Definition 4.3. Then, $\tilde{N}^{(k,j)}(F,G)$ is called the $(k,j)$-th nested subresultant matrix of $F$ and $G$. \(\square\)

Definition 4.3 (Nested Subresultant) Let $F$ and $G$ be defined as in (3.1), and let $(P_1^{(1)}, \ldots, P_{t_1}^{(1)}, \ldots, P_1^{(t)}, \ldots, P_{t_t}^{(t)})$ be complete recursive PRS for $F$ and $G$ as in Definition 2.2. For $j = j_{k-1} - 2, \ldots, 0$ and $\tau = j, \ldots, 0$, let $\tilde{N}^{(k,j)}(F,G)$ be a sub-matrix of the $(k,j)$-th nested subresultant matrix $N^{(k,j)}(F,G)$ obtained by taking the top $n_1^{(k)} + n_2^{(k)} - 2j - 1$ rows and the $(n_1^{(k)} + n_2^{(k)} - j - \tau)$-th row (note that $N^{(k,j)}(F,G)$ is a square matrix). Then, the polynomial

$$
\tilde{S}_{k,j}(F,G) = |N_j^{(k,j)}|x^j + \cdots + |N_0^{(k,j)}|x^0
$$

is called the $(k,j)$-th nested subresultant of $F$ and $G$. \(\square\)

We show the relationship between the nested subresultant and the recursive subresultant.

Lemma 4.4 Let $F$ and $G$ be defined as in (3.1), and let $(P_1^{(1)}, \ldots, P_{t_1}^{(1)}, \ldots, P_1^{(t)}, \ldots, P_{t_t}^{(t)})$ be complete recursive PRS for $F$ and $G$ as in Definition 2.2. For $k = 1, \ldots, t-1$, define $B_k$, and for $k = 2, \ldots, t$ and $j = j_{k-1} - 2, \ldots, 0$, define $b_{k,j}$ as in Lemma 3.3. Furthermore, for $k = 2, \ldots, t-1$, define $\tilde{R}_k = (\tilde{R}_{k-1})^{b_{k,j}}B_k$ with $\tilde{R}_0 = \tilde{R}_1 = 1$. Then, we have

$$
\tilde{S}_{k,j}(F,G) = (\tilde{R}_{k-1})^{b_{k,j}} \cdot S_j(P_1^{(k)}, P_2^{(k)}).
$$

PROOF. By induction on $k$. For $k = 1$, it is obvious by the definition of the nested subresultant. Assume that (4.3) is valid for $1, \ldots, k - 1$. Then, by the Fundamental Theorem of subresultants (Theorem 3.1), we have

$$
\frac{d}{dx} \left(\tilde{S}_{k-1,jk-1}(F,G)\right) = (\tilde{R}_{k-1})\frac{d}{dx} \left(P_1^{(k)}\right) = (\tilde{R}_{k-1})P_2^{(k)}.
$$

$$
\frac{d}{dx} \left(\tilde{S}_{k-1,jk-1}(F,G)\right) = (\tilde{R}_{k-1})P_2^{(k)}.
$$
Then, we have

\[
\tilde{N}^{(k,j)}(F, G) = (\tilde{R}_{k-1} N^{(j)}(P_1^{(k)}, P_2^{(k)}), \quad \text{for } \tau = j, \ldots, 0.
\]

Therefore, we have (4.4), which proves the lemma.

**Theorem 4.5** Let \( F \) and \( G \) be defined as in (3.1), and let \((P_1^{(1)}, \ldots, P_1^{(t)}, \ldots, P_1^{(t)})\) be complete recursive PRS for \( F \) and \( G \) as in Definition 2.2. For \( k = 2, \ldots, t \) and \( j = j_{k-1} - 2, \ldots, 0 \), define \( u_{k,j}, b_{k,j}, r_{k,j} \) as in Lemma 3.9 and \( R'_k = (R'_{k-1})^{b_k} \) with \( R'_0 = R'_1 = 1 \). Then, we have

\[
\tilde{S}_{k,j}(F, G) = (R'_{k-1})^{b_{k,j}} \tilde{S}_{k,j}(F, G).
\]

**PROOF.** By induction on \( k \). For \( k = 1 \), it is obvious by the definitions of the recursive and the nested subresultants. We first show that \( \tilde{R}_k = \tilde{R}_k \cdot R'_k \) for \( k = 0, \ldots, t - 1 \). It is obvious for \( k = 0 \) and 1. Let us assume \( \tilde{R}_{k-1} = \tilde{R}_{k-1} \cdot R'_{k-1} \). Then, we have

\[
\tilde{R}_k = (\tilde{R}_{k-1})^{b_k} \tilde{R}_k B_k = (\tilde{R}_{k-1} \cdot R'_{k-1})^{b_k} \tilde{R}_k B_k = (\tilde{R}_{k-1})^{b_k} \tilde{R}_k B_k \cdot (R'_{k-1})^{b_k} \cdot R'_k,
\]

Now, by Lemma 3.9, we have \( \tilde{S}_{k,j}(F, G) = (\tilde{R}_{k-1})^{b_{k,j}} r_{k,j} \cdot S_j(P_1^{(k)}, P_2^{(k)}) \), then, by Lemma 4.4, we have

\[
\tilde{S}_{k,j}(F, G) = (R'_{k-1})^{b_{k,j}} r_{k,j} \cdot S_j(P_1^{(k)}, P_2^{(k)})
\]

which proves the theorem.

**Remark 4.6** Since \( r_{k,j} = \pm 1 \), we see that \( R'_{k} = \pm 1 \) hence the nested subresultant is equal to the recursive subresultant up to a sign.

5 Reduced Nested Subresultants

The nested subresultant matrix has “nested” representation of subresultant matrices, which makes practical use difficult. However, in some cases, we can reduce the representation of the nested subresultant matrix to a “flat” representation, or a representation without nested determinants by the Gaussian elimination; this is the reduced nested subresultant (matrix). As we will see,
the size of the reduced nested subresultant matrix becomes much smaller than that of the recursive subresultant matrix, with reasonable computing time.

First, we illustrate the idea of reduction of the nested subresultant matrix with an example.

**Example 5.1** Let \( F(x) \) and \( G(x) \) be defined as

\[
F(x) = a_6 x^6 + a_5 x^5 + \cdots + a_0, \quad a_6 \neq 0, \\
G(x) = b_5 x^5 + b_4 x^4 + \cdots + b_0, \quad b_5 \neq 0,
\]

with vectors of coefficients \((a_6, a_5)\) and \((b_5, b_4)\) are linearly independent as vectors over \( K \). Assume that \( \text{prs}(F, G) = (F^{(1)}_1 = F, F^{(1)}_2 = G, P^{(1)}_3 = \gcd(F, G)) \) with \( \deg(P^{(1)}_3) = 4 \). Consider the \((2, 2)\)-th nested subresultant; its matrix \( \tilde{N}^{(2,2)}(F, G) \) is defined as

\[
\tilde{N}^{(2,2)}(F, G) = \begin{pmatrix} A_4 & 4A_4 \\ A_3 & 3A_3 & 4A_4 \\ A_2 & 2A_2 & 3A_3 \\ A_1 & A_1 & 2A_2 \\ A_0 & A_1 \end{pmatrix}, \quad A_j = \begin{vmatrix} a_6 & b_5 \\ a_5 & b_4 & b_5 \\ a_j & b_{j-1} & b_j \end{vmatrix}, \quad (5.1)
\]

for \( j \leq 4 \) with \( b_j = 0 \) for \( j < 0 \). Now, let us calculate the leading coefficient of \( \tilde{S}_{2,2}(F, G) \) as

\[
|\tilde{N}^{(2,2)}_2| = \begin{vmatrix} A_4 & 4A_4 \\ A_3 & 3A_3 & 4A_4 \\ A_2 & 2A_2 & 3A_3 \end{vmatrix} = \begin{vmatrix} a_6 & b_5 \\ a_5 & b_4 & b_5 \\ a_4 & b_3 & b_4 \\ a_3 & b_2 & b_3 \\ a_2 & b_1 & b_2 \end{vmatrix} = \begin{vmatrix} a_6 & b_5 \\ a_5 & b_4 & b_5 \\ a_4 & b_3 & b_4 \\ a_3 & b_2 & b_3 \end{vmatrix}.
\]

Then, we make the \((3, 1)\) and the \((3, 2)\) elements in \( H_{p,q} \) \((p, q = 1, 2, 3)\) equal to 0 by adding the first and the second rows, multiplied by certain numbers, to
the third row. For example, in $H_{1,1}$, calculate $x_{11}$ and $y_{11}$ by solving a system of linear equations

$$\begin{cases} a_6 x_{11} + a_5 y_{11} = -a_4, \\ b_5 x_{11} + b_4 y_{11} = -b_3, \end{cases} \tag{5.3}$$

(Note that (5.3) has a solution in $K$ by the assumption), followed by adding the first row multiplied by $x_{11}$ and the second row multiplied by $y_{11}$, respectively, to the third row. Then, we have

$$H_{1,1} = \begin{vmatrix} a_6 & b_5 \\ a_5 & b_4 & b_5 \\ 0 & 0 & h_{11} \end{vmatrix} = \begin{vmatrix} a_6 & b_5 \\ a_5 & b_4 \end{vmatrix} h_{11} \quad \text{with} \quad h_{11} = b_4 + y_{11} b_5. \tag{5.4}$$

Doing similar calculations for the other $H_{p,q}$, we calculate $h_{p,q}$ ($p, q = 1, 2, 3$) for $H_{p,q}$ similarly as in (5.3). Finally, by putting such new representations of $H_{p,q}$ into (5.2), we have

$$|\hat{N}_2^{(2,2)}| = \begin{vmatrix} a_6 & b_5 \\ a_5 & b_4 \end{vmatrix}^3 \begin{vmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{vmatrix} = \begin{vmatrix} a_6 & b_5 \\ a_5 & b_4 \end{vmatrix}^3 |\hat{N}_2^{(2,2)}|, \tag{5.5}$$

note that we have derived $\hat{N}_2^{(2,2)}$ as a reduced form of $N_2^{(2,2)}$. □

As (5.5) shows, we derive a “reduced” form of the nested subresultant matrix by the Gaussian elimination for solving certain systems of linear equations. We define the reduced nested subresultant (matrix), as follows.

**Definition 5.2 (Reduced Nested Subresultant Matrix)** Let $F$ and $G$ be defined as in (3.1), and let $(P_1^{(1)}, \ldots, P_{l_1}^{(1)}, \ldots, P_{l_t}^{(t)}, \ldots, P_{l_t}^{(t)})$ be complete recursive PRS for $F$ and $G$ as in Definition 2.2. Then, for each pair of numbers $(k, j)$ with $k = 1, \ldots, t$ and $j = j_{k-1} - 2, \ldots, 0$, define matrix $\hat{N}^{(k,j)}(F,G)$ recursively as follows.

(1) For $k = 1$, let $\hat{N}^{(1,j)}(F,G) = N^{(j)}(F,G)$.

(2) For $k > 1$, let $\hat{N}_U^{(k-1,j_{k-1})}(F,G)$ be a sub-matrix of $\hat{N}^{(k-1,j_{k-1})}(F,G)$ by deleting the bottom $j_{k-1} + 1$ rows, and $\hat{N}_L^{(k-1,j_{k-1})}(F,G)$ be a sub-matrix of $\hat{N}^{(k-1,j_{k-1})}(F,G)$ by taking the bottom $j_{k-1} + 1$ rows, respectively. For $\tau = j_{k-1}, \ldots, 0$ let $\hat{N}_e^{(k-1,j_{k-1})}(F,G)$ be a sub-matrix of $\hat{N}^{(k-1,j_{k-1})}(F,G)$ by putting $\hat{N}_U^{(k-1,j_{k-1})}(F,G)$ on the top and the $(j_{k-1} - \tau + 1)$-th row of $\hat{N}_L^{(k-1,j_{k-1})}(F,G)$ in the bottom row. Let $\hat{A}^{(k-1)}(x) = |\hat{N}_e^{(k-1,j_{k-1})}|$ and construct a matrix $H^{(k,j)}$ as

$$H^{(k,j)} = \left( H_{p,q}^{(k,j)} \right) = N^{(j)} \left( \hat{A}^{(k-1)}(x), \frac{d}{dx} \hat{A}^{(k-1)}(x) \right), \tag{5.6}$$

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where
\[ \hat{A}^{(k-1)}(x) = \hat{A}_{j-k}^{(k-1)} x^{j-1} + \cdots + \hat{A}_0^{(k-1)} x^0. \]

Since \( \hat{N}_\tau^{(k-1,j-k-1)} \) consists of \( \hat{N}_U^{(k-1,j-k-1)} \) and a row vector in the bottom, we express \( \hat{N}_U^{(k-1,j-k-1)} = \begin{pmatrix} U(k) & v(k) \end{pmatrix} \), where \( U(k) \) is a square matrix and \( v(k) \) is a column vector, and the row vector in the bottom by \( \begin{pmatrix} b_{p,q}^{(k,j)} & g_{p,q}^{(k,j)} \end{pmatrix} \), where \( b_{p,q}^{(k,j)} \) is a row vector and \( g_{p,q}^{(k,j)} \) is a number, respectively, such that

\[ H_{p,q}^{(k,j)} = \begin{vmatrix} U(k) & v(k) \\ b_{p,q}^{(k,j)} & g_{p,q}^{(k,j)} \end{vmatrix}, \quad (5.7) \]

with \( b_{p,q}^{(k,j)} = 0 \) and \( g_{p,q}^{(k,j)} = 0 \) for \( H_{p,q}^{(k,j)} = 0 \). Furthermore, we assume that \( U(k) \) is not singular. Then, for \( p = 1, \ldots, n_1^{(k)} + n_2^{(k)} - j \) and \( q = 2, \ldots, n_1^{(k)} + n_2^{(k)} - j \), calculate a row vector \( x_{p,q}^{(k)} \) as a solution of the equation

\[ x_{p,q}^{(k)} U(k) = b_{p,q}^{(k,j)}, \quad (5.8) \]

and define \( h_{p,q}^{(k,j)} \) as

\[ h_{p,q}^{(k,j)} = x_{p,q}^{(k)} v^{(k,j)}. \quad (5.9) \]

Note that we have

\[ H_{p,q}^{(k,j)} = \begin{vmatrix} U(k) & v(k) \\ 0 & h_{p,q}^{(k,j)} \end{vmatrix} = \begin{vmatrix} U(k) & h_{p,q}^{(k,j)} \end{vmatrix}. \quad (5.10) \]

Finally, define \( \hat{N}^{(k,j)}(F, G) \) as

\[ \hat{N}^{(k,j)}(F, G) = \begin{pmatrix} h_{1,1}^{(k,j)} & h_{1,2}^{(k,j)} & \cdots & h_{1,j-1}^{(k,j)} \\ h_{2,1}^{(k,j)} & h_{2,2}^{(k,j)} & \cdots & h_{2,j-1}^{(k,j)} \\ \vdots & \vdots & \ddots & \vdots \\ h_{I_{k,j}}^{(k,j)} & h_{I_{k,j}}^{(k,j)} & \cdots & h_{I_{k,j}}^{(k,j)} \end{pmatrix}, \quad (5.11) \]

where

\[ I_{k,j} = n_1^{(k)} + n_2^{(k)} - j = (2j_{k-1} - 2j - 1) + j, \quad (5.12) \]

Then, \( \hat{N}^{(k,j)}(F, G) \) is called the \((k, j)\)-th reduced nested subresultant matrix of \( F \) and \( G \). \( \square \)

**Remark 5.3** Definition 5.2 shows that, For \( k = 1, \ldots, t \) and \( j < j_{k-1} - 1 \), the numbers of rows and columns of the \((k, j)\)-th reduced nested subresultant
matrix $\hat{N}^{(k,j)}(F,G)$ are $I_{k,j}$ and $J_{k,j}$ in (5.12), respectively, which are much smaller than those of the recursive subresultant matrix of the corresponding degree (see Proposition 3.7). □

**Definition 5.4 (Reduced Nested Subresultant)** Let $F$ and $G$ be defined as in (3.1), and let $(P^{(1)}_1, \ldots, P^{(1)}_t, \ldots, P^{(t)}_1, \ldots, P^{(t)}_t)$ be complete recursive PRS for $F$ and $G$ as in Definition 2.2. For $j = j_{k-1} - 2, \ldots, 0$ and $\tau = j, \ldots, 0$, let $\hat{N}^{(k,j)}_\tau = \hat{N}^{(k,j)}(F,G)$ be a sub-matrix of the $(k,j)$-th reduced nested subresultant matrix $\hat{N}^{(k,j)}(F,G)$ obtained by the top $n^{(k)}_1 + n^{(k)}_2 - 2j - 1$ rows and the $(n^{(k)}_1 + n^{(k)}_2 - j - \tau)$-th row (note that $\hat{N}^{(k,j)}(F,G)$ is a square matrix). Then, the polynomial

$$\hat{S}_{k,j}(F,G) = |\hat{N}^{(k,j)}_j(F,G)|x^j + \cdots + |\hat{N}^{(k,j)}_0(F,G)|x^0$$

is called the $(k,j)$-th reduced nested subresultant of $F$ and $G$. □

Now, we derive the relationship between the nested and the reduced nested subresultants.

**Theorem 5.5** Let $F$ and $G$ be defined as in (3.1), and let $(P^{(1)}_1, \ldots, P^{(1)}_t, \ldots, P^{(t)}_1, \ldots, P^{(t)}_t)$ be complete recursive PRS for $F$ and $G$ as in Definition 2.2. For $k = 2, \ldots, t$, $j = j_{k-1} - 2, \ldots, 0$ with $J_{k,j}$ as in (5.14), define $\hat{B}_{k,j}$ and $\hat{R}_k$ as

$$\hat{B}_{k,j} = |U^{(k)}|^{J_{k,j}}$$

with $\hat{B}_k = \hat{B}_{k,j_k}$ and $\hat{B}_1 = \hat{B}_2 = 1$, and

$$\hat{R}_k = (\hat{R}_{k-1} \cdot \hat{B}_{k-1})^{J_{k,j_k}}$$

with $\hat{R}_1 = \hat{R}_2 = 1$, respectively. Then, we have

$$\hat{S}_{k,j}(F,G) = (\hat{R}_{k-1} \cdot \hat{B}_{k-1})^{J_{k,j}} \hat{B}_{k,j} \cdot \hat{S}_{k,j}(F,G).$$

To prove Theorem 5.5 we prove the following lemma.

**Lemma 5.6** For $k = 1, \ldots, t$, $j = j_{k-1} - 2, \ldots, 0$ and $\tau = j, \ldots, 0$, we have

$$|\hat{N}^{(k,j)}_\tau(F,G)| = (\hat{R}_{k-1} \cdot \hat{B}_{k-1})^{J_{k,j}} \hat{B}_{k,j} |\hat{N}^{(k,j)}(F,G)|.$$  (5.13)

**PROOF.** By induction on $k$. For $k = 1$, it is obvious from the definitions of the nested and the reduced nested subresultants. Assume that (5.13) is valid for $1, \ldots, k - 1$. Then, for $\tau = j_{k-1}, \ldots, 0$, we have

$$|\hat{N}^{(k-1,j_{k-1})}_\tau(F,G)| = (\hat{R}_{k-2} \cdot \hat{B}_{k-2})^{J_{k-1,j_{k-1}}} \hat{B}_{k-1,j_{k-1}} |\hat{N}^{(k-1,j_{k-1})}(F,G)| = (\hat{R}_{k-1} \cdot \hat{B}_{k-1}) |\hat{N}^{(k-1,j_{k-1})}(F,G)|.$$
Let
\[ \tilde{A}_{\tau}^{(k-1)} = |\tilde{N}_{\tau}^{(k-1,j_{k-1})}(F, G)|, \quad \hat{A}_{\tau}^{(k-1)} = |\hat{N}_{\tau}^{(k-1,j_{k-1})}(F, G)|, \]
and
\[ H_{\tau}^{(k,j)} = (H_{\tau}^{(k,j)})_{r_{p,q}} \]
be a sub-matrix of $H_{\tau}^{(k,j)}$ in (5.6) by taking the top $J_{k,j}$ rows and the $(I_{k,j} - \tau)$-th row, where $I_{k,j}$ and $J_{k,j}$ are defined as in (5.12), respectively. Then, we have

\[
\begin{vmatrix}
\hat{A}_{j_{k-1}}^{(k-1)} & j_{k-1}\hat{A}_{j_{k-1}}^{(k-1)} \\
\vdots & \ddots & \vdots \\
\hat{A}_{j_{k-1}}^{(k-1)} & \hat{A}_{j_{k-1}}^{(k-1)} & j_{k-1}\hat{A}_{j_{k-1}}^{(k-1)} \\
(2j - j_{k-1} + 3)\hat{A}_{j_{k-1}+3}^{(k-1)} & \cdots & (j + 2)\hat{A}_{j_{k-1}+3}^{(k-1)} \\
(j - j_{k-1} + \tau + 2)\tilde{A}_{j_{k-1}+\tau+2}^{(k-1)} & \cdots & (\tau + 1)\tilde{A}_{j_{k-1}+\tau+2}^{(k-1)}
\end{vmatrix},
\]

where $\hat{A}_{l}^{(k-1)} = 0$ for $l < 0$. On the other hand, by the definition of the $(k, j)$-th nested subresultant, we have

\[
\begin{vmatrix}
\tilde{N}_{\tau}^{(k,j)}(F, G) \\
\tilde{A}_{j_{k-1}}^{(k-1)} & \hat{A}_{j_{k-1}}^{(k-1)} \\
\vdots & \ddots & \vdots \\
\tilde{A}_{j_{k-1}}^{(k-1)} & \tilde{A}_{j_{k-1}}^{(k-1)} & \hat{A}_{j_{k-1}}^{(k-1)} \\
(2j - j_{k-1} + 3)\tilde{A}_{j_{k-1}+3}^{(k-1)} & \cdots & (j + 2)\tilde{A}_{j_{k-1}+3}^{(k-1)} \\
(j - j_{k-1} + \tau + 2)\tilde{A}_{j_{k-1}+\tau+2}^{(k-1)} & \cdots & (\tau + 1)\tilde{A}_{j_{k-1}+\tau+2}^{(k-1)}
\end{vmatrix} = (\tilde{R}_{k-1} \cdot \tilde{B}_{k-1})^{J_{k,j}}|H_{\tau}^{(k,j)}|,
\]

(5.14)

where $\tilde{A}_{l}^{(k-1)} = 0$ for $l < 0$. (Note that $\tilde{N}_{\tau}^{(k,j)}$ and $H_{\tau}^{(k,j)}$ are square matrices of order $J_{k,j}$.) By Definition 5.2, we can express $H_{\tau}^{(k,j)}$ as

\[
H_{\tau}^{(k,j)} = \begin{vmatrix}
U^{(k)} \\
B_{p,q}^{(k,j)} \\
g_{p,q}^{(k,j)}
\end{vmatrix},
\]

with $b_{p,q}^{(k,j)} = 0$ and $g_{p,q}^{(k,j)} = 0$ for $H_{\tau}^{(k,j)} = 0$. Note that, for $q = 1, \ldots, J_{k,j}$, we have $b_{p,q}^{(k,j)} = b_{p,q}^{(k,j)}$ and $g_{p,q}^{(k,j)} = g_{p,q}^{(k,j)}$ for $p = 1, \ldots, J_{k,j} - 1$, and $b_{J_{k,j},q}^{(k,j)} = B_{k,j-q}$ and $g_{J_{k,j},q}^{(k,j)} = g_{J_{k,j},q}$, where $b_{p,q}^{(k,j)}$ and $g_{p,q}^{(k,j)}$ are defined as in (5.7), respectively. Thus, by (5.8)–(5.11), we have

\[
|H_{\tau}^{(k,j)}| = |U^{(k)}|J_{k,j}|\tilde{N}_{\tau}^{(k,j)}(F, G)| = \tilde{B}_{k,j}|\tilde{N}_{\tau}^{(k,j)}(F, G)|,
\]

(5.15)
and, by putting (5.15) into (5.14), we prove the lemma. □

**Remark 5.7** We can calculate the \((k, j)\)-th reduced nested subresultant matrix as a sub-matrix of the \((k, 0)\)-th reduced nested subresultant matrix. In (5.6), we see that the matrix \(H^{(k, j)}\) is a sub-matrix of \(N(\hat{A}^{(k-1)}(x), \frac{d}{dx} \hat{A}^{(k-1)}(x))\), and, by the construction of the reduced nested resultant matrix (5.11), we see that \(\hat{N}^{(k, j)}(F, G)\) is a sub-matrix of \(\hat{N}^{(k, 0)}(F, G)\) by taking the left \(n_2^{(k)} - j\) columns from those corresponding to the coefficients of \(\hat{A}^{(k-1)}(x)\) and the left \(n_1^{(k)} - j\) columns from those corresponding to the coefficients of \(\frac{d}{dx} \hat{A}^{(k-1)}(x)\), then taking the top \(n_1^{(k)} + n_2^{(k)} - j\) rows. □

**Remark 5.8** We can estimate arithmetic computing time for the \((k, j)\)-th reduced nested resultant matrix \(\hat{N}^{(k, j)}\) in (5.11), as follows. The computing time for the elements \(h_{p, q}\) is dominated by the time for the Gaussian elimination of \(U^{(k)}\). Since the order of \(U^{(k)}\) \((k = 2, \ldots, t)\) is equal to \(2(j_{k-2} - j_{k-1} - 1)\) (see Remark 2.3), it is bounded by \(O((j_{k-2} - j_{k-1})^3)\) (see Golub and van Loan [9]). As Remark 5.7 shows, we can calculate \(\hat{N}^{(k, j)}(F, G)\) for \(j < j_{k-1} - 2\) by \(\hat{N}^{(k, 0)}(F, G)\). Therefore, total computing time for \(\hat{N}^{(k, j)}\) for entire recursive PRS \((k = 1, \ldots, t)\) is bounded by

\[
\sum_{k=2}^{t} O((j_{k-2} - j_{k-1})^3) = O\left(\sum_{k=2}^{t} (j_{k-2} - j_{k-1})^3\right) = O((j_0 - j_{t-1})^3) = O(m^3),
\]

note that \(j_0 = m\) (see Remark 2.3). See also for the concluding remarks. □

6 Concluding Remarks

In this paper, we have introduced concepts of recursive PRS and recursive subresultants, and investigated constructions of their subresultant matrices to compute the recursive subresultants. Among three different constructions of recursive subresultant matrices, we have shown that the reduced nested subresultant matrix reduces the size of the matrix drastically to at most the order of the degree of initial polynomials in each PRSs, compared with the naive recursive subresultant matrix. We have also shown that we can calculate the reduced nested subresultant matrix by the Gaussian elimination of order at most the sum of the degree of initial polynomials in each PRSs.

From a point of view of computational complexity, the algorithm for the reduced nested subresultant matrix has a cubic complexity bound in terms of the degree of the input polynomials (see Remark 5.8). However, subresultant algorithms which have a quadratic complexity bound in terms of the degree of the input polynomials have been proposed ([6], [11]); those algorithms exploit
the structure of the Sylvester matrix to increase their efficiency with controlling the size of coefficients well. Although, in this paper, we have primarily focused our attention into reducing the structure of the nested subresultant matrix to “flat” representation, development of more efficient algorithms such as exploiting the structure of the Sylvester matrix would be the next problem. Furthermore, the reduced nested subresultant may involve fractions which may be unusual for subresultants, thus more detailed analysis of computational efficiency including comparison with (ordinary and recursive) subresultants would also be necessary.

We expect that the reduced nested subresultants can be used for approximate algebraic computation such as the square-free decomposition of approximate univariate polynomials with approximate GCD computations based on Singular Value Decomposition (SVD) of subresultant matrices ([1], [7]), which motivates the present work. For the approximate square-free decomposition of the given polynomial $P(x)$, we have to calculate the approximate GCDs of $P(x), \ldots, P^{(n)}(x)$ (by $P^{(n)}(x)$ we denote the $n$-th derivative of $P(x)$) or those of the recursive PRS for $P(x)$ and $P'(x)$; we have to find the representation of the subresultant matrices for $P(x), \ldots, P^{(n)}(x)$, or that for the recursive PRS for $P(x)$ and $P'(x)$, respectively. As for the former approach, several algorithms based on different representations of subresultant matrices have been proposed ([2], [13]); our reduced nested subresultant matrix can be used as for the latter approach. To make use of the reduced nested subresultant matrix, we need to reveal the relationship between the structure of the subresultant matrices and their singular values; this is the problem on which we are working now.

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\[ N^{(k,j)}(F, G) = \]

\[
\begin{pmatrix}
N_{1}^{(k-1,jk-1)} & \cdots & \cdots & \cdots & \cdots \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
N_{L}^{(k-1,jk-1)} & \cdots & \cdots & \cdots & \cdots \\
\end{pmatrix}
\]

Fig. 1. Illustration of \( N^{(k,j)}(F, G) \). Note that the number of blocks \( N_{L}^{(k-1,jk-1)} \) is \( j_{k-1} - j - 1 \), whereas that of \( N_{L}^{(k-1,j_{k-1})} \) is \( j_{k-1} - j \); see Definition 3.6 for details.

\[ M^{(k,j)}(F, G) = \]

\[
\begin{pmatrix}
W_{k-1} & 0 & \cdots & \cdots & \cdots & \cdots \\
* & N_{U}^{(j_{k-1})} & \cdots & \cdots & \cdots & \cdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
W_{k-1} & 0 & \cdots & \cdots & \cdots & \cdots \\
* & N_{U}^{(j_{k-1})} & \cdots & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
W_{k-1} & 0 & \cdots & \cdots & \cdots & \cdots \\
* & N_{U}^{(j_{k-1})} & \cdots & \cdots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
W_{k-1} & 0 & \cdots & \cdots & \cdots & \cdots \\
* & N_{U}^{(j_{k-1})} & \cdots & \cdots & \cdots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
W_{k-1} & 0 & \cdots & \cdots & \cdots & \cdots \\
* & N_{U}^{(j_{k-1})} & \cdots & \cdots & \cdots & \cdots \\
\end{pmatrix}
\]

Fig. 2. Illustration of \( M^{(k,j)}(F, G) \). Note that the number of column blocks is equal to \( b_{k,j} = 2j_{k-1} - 2j - 1 \); see Lemma 3.11 for details.
\[ \tilde{M}^{(k,j)}(F, G) = \]

Fig. 3. Illustration of \( \tilde{M}^{(k,j)}(F, G) \); see Lemma 3.12 for details.

\[ \hat{M}^{(k,j)}(F, G) = \]

Fig. 4. Illustration of \( \hat{M}^{(k,j)}(F, G) \). Note that the lower-right block which consists of \( p_1^{(k)} \) and \( p_2^{(k)} \) is equal to \( N_U^{(j)}(P_1^{(k)}, P_2^{(k)}) \) and \( |W_{k-1}| = |\bar{N}_U^{(j,k-1)}| = 1 \); see Lemma 3.10 for details.