On Generalized Simple Singular AP-Injective Rings

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ABSTRACT
A ring R is said to be generalized right simple singular AP-injective, if for any maximal essential right ideal M of R and for any \( b \in M \), \( bR/bM \) is AP-injective. We shall study the characterization and properties of this class of rings. Some interesting results on these rings are obtained. In particular, conditions under which generalized simple singular AP-injective rings are weakly regular rings, and Von Neumann regular rings.

Key word: AP-injective Rings, weakly continuous rings, socle of R, Von Neumann regular rings.

1. Introduction:
Throughout this paper, R is an associative ring with identity, and R-module is unital. For \( a \in R \), \( r(a) \) and \( l(a) \) denote the right annihilator and the left annihilator of \( a \), respectively. We write \( J(R) \), \( Y(R)(Z(R)) \), \( N(R) \) and \( \text{Soc}(R) \) for the Jacobson radical, the right ( left ) singular ideal, the set of nilpotent elements and right socle of R, respectively. \( X \leq M \) denoted that \( X \) is a submodule of module \( M \).

Recall that a ring R is called right MC2-ring if \( eRa=0 \) implies \( aRe=0 \), where \( a, e \in R \) and \( eR \) is minimal right ideal of \( R \)[8]. A ring R is Von Neumann (weakly) regular provided that for every \( a \in R \) there exists \( b \in R \) (\( b \in RaR \)) such that \( a=aba \) (\( a=ab \) resp.). Recall that a ring R is right (left) weakly continuous if \( J(R)=Y(R)( \ J(R) = Z( R) )) \), \( R / J(R) \) is regular and idempotent can be left module \( J(R) \)[5]. Clearly every regular ring is right (left) weakly continuous. A ring R is called zero commutative (briefly ZC-ring ) if \( ab=0 \) implies \( ba=0 \), \( a,b \in R \)[1]. A right \( R \)-module M is principally injective (briefly P-injective), if for any principal right ideal \( aR \) of R and any right \( R \)-homomorphism of \( aR \) into M can be extended to one of R into M[11]. The ring R is called right P-injective if \( R_R \) is P-injective.

2. Generalized Simple Singular AP-injective Rings
Recall that a module $M_R$ with $S=\text{End}(M_R)$ is said to be **almost principally injective** (briefly AP-injective), if for any $a \in R$, there exists an $S$-submodule $X_a$ of $M$ such that $l_{M_R}(a)=Ma \oplus X_a$ as left $S$-module[6]. AP-injectivity has been studied by many authors (see [9,10]). Actually, Zhao Yu-e [12] investigated some properties of rings whose simple singular right $R$-module is AP-injective. Now, we give a generalized AP-injective.

**Definition 2.1:**
A ring $R$ is called a **generalized right (left) simple singular AP-injective**, if for any maximal essential right (left) ideal $M$ of $R$, any $b \in M$, $bR/bM$ ($Rb/Mb$) is AP-injective.

The following lemma which is due to Zhao Yu-e [12], plays a central role in several of our proofs

**Lemma 2.2:**
Suppose $M$ is a right $R$-module with $S=\text{End}(M_R)$. If $l_{M_R}(a)=Ma \oplus X_a$, where $X_a$ is left $S$-submodule of $M_R$. Set $f: aR \rightarrow M$ is a right $R$-homomorphism, then $f(a)=ma+x$ with $m \in M$, $x \in X_a$.

**Lemma 2.3:**
If $M$ is a maximal right ideal of $R$ and $r(a) \subseteq M$ with $a \in M$, then
1- $aR \neq aM$
2- $R/M \cong aR/aM$.

**Proof:**
(1) If $aR = aM$, then $a = ay$ for some $y$ in $M$, which implies that $1-y \in r(a) \subseteq M$, whence $1 \in M$, contradicting $M \neq R$.
(2) From (1) $aR \neq aM$, then the right $R$-homomorphism $g: R/M \rightarrow aR/aM$ is defined by $g(r+M) = ar+aM$ for all $r \in R$ implies that $R/M \cong aR/aM$. ■

We start this section with the following results.

**Proposition 2.4:**
Let $R$ be generalized right simple singular AP-injective ring, then
1- $J(R) \cap Y(R) = 0$
2- $\text{Soc}(R_R) \cap Y(R) = 0$

**Proof:**
(1) Let $a \in J(R) \cap Y(R)$. If $a \neq 0$, then $r(a) \neq R$ and $RaR + r(a)$ is an essential right ideal of $R$. We shall prove that $RaR + r(a) = R$. If not, there exists a maximal essential right ideal $M$ containing $RaR + r(a)$. Since $r(a) \subseteq M$ and $a \in M$, then by Lemma 2.3 $R/M \cong aR/aM$. Therefore, $R/M$ is AP-injective and $l_{R/M}(a)= (R/M)a \oplus X_a$. $X_a \subseteq R/M$. Let $f:aR \rightarrow R/M$ defined by $f(ar) = r +M$ for all $r \in R$. Note that $f$ is a well-defined and by Lemma 2.2 $1 + M = f(a) = ba + M +x$, $b \in R$, $x \in X_a$. Hence $1-ba + M = x \in R/M \cap X_a = 0$, so $1-ba \in M$. Since $a \in J(R)$, then $ba \in J(R) \subseteq M$ and hence $1 \in M$, which is a contradiction. Therefore $J(R) \cap Y(R) = 0$.
(2) Let $k \in \text{Soc}(R_R) \cap Y(R)$. If $k \neq 0$, then $kR$ is a minimal right ideal and $r(k)$ is an essential right ideal of $R$. Since every minimal one-sided ideal of $R$ is either nilpotent or direct summand of $R$ [8], Thus, if $(kR)^2 \neq 0$, then $kR$ is a direct summand and hence $r(k)$ is also direct summand which is a contradiction. If
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\((kR)^2 = 0,\) then \(k^2 = 0\) and \(k \in r(k)\). But \(r(k)\) is maximal essential right ideal of \(R\). Therefore, by Lemma 2.3 \(R/r(k) \cong k(r(k))\). Hence, \(R/r(k)\) is AP-injective, so there exists \(c \in R\) and \(x \in X_a\) as a proof (1) such that \(1-ck \in r(k)\). Since, \(ck \in RkR \subseteq r(k)\), then \(1 \in r(k)\). This is also contradiction, therefore \(\text{Soc}(R_R) \cap Y(R) = 0\). ■

Following [7], for a prime ideal \(P\) of a ring \(R\), we put 
\[ O_P = \{ a \in P : ab = 0 \text{ for some } b \in R \setminus P \}. \]

In general, \(O_P\) not subset of a prime ideal \(P\). as the following example shows.

Example [2]:

Let \(R\) be a ring of 2x2 matrices over a field \(F\). Then, 
\[
\begin{pmatrix}
0 & 0 \\
1 & 0
\end{pmatrix}
\begin{pmatrix}
0 & 0 \\
0 & 0
\end{pmatrix}
= 0.
\]

Then, \(a \in O_P\), but \(a \notin P\) ■

Theorem 2.5:

Let \(P\) be a prime ideal of a generalized right simple singular AP-injective ring with \(O_P \subseteq P\), then \(P\) is maximal.

Proof:

We claim that \(aR + P = R\) for \(a \in R/P\). If not, there exists a maximal ideal \(M\) of \(R\) containing \(aR + P\). Moreover, \(M\) is a maximal right ideal of \(R\). Suppose not, then there exists a maximal right ideal \(K\) of \(R\) such that \(M \subseteq K\). If \(K\) is not essential in \(R\). Then \(K\) is a direct summand of \(R\), so we can write \(K = r(e)\) for some \(0 \neq e = e^2 \in R\). Then, \(ea = 0\), since \(e \notin P\), then \(a \in O_P \subseteq P\). Therefore, \(K\) must be essential right ideal of \(R\). Now, suppose that \(aR = aK\), then \(a = ac\) for some \(c \in K\) that implies \(a(1-c) = 0\).

Since, \(a \notin P\), then \(1-c \in O_P \subseteq P \subseteq K\) which is a contradiction. If \(aR \neq aK\), the right \(R\)-homomorphism \(g : R/K \to aR/aK\) is defined by \(g(b+K) = ab + aK\) for all \(b \in R\) which implies that \(R/K \cong aR/aK\). Therefore, \(R/K\) is AP-injective . Let \(f : aR \to R/K\) be defined by \(f(ar) = r+K\) for all \(r \in R\). So by Lemma 2.2 \(f(a) = ca + K, x \in X_a\). Hence, \(1-ca+K = x \in R/K \sqcup X_a \neq 0\), so \(1-ca \in K\) whence \(1 \in K\). Therefore, \(M\) is a maximal essential right ideal of \(R\). So by the same method in the above proof \(P\) is a maximal of \(R\). ■

Recall that \(R\) is called 2-Primal if its prime radical \(P(R)\) concedes with the set \(N(R)\) [7]. Kim and Kwak [3] showed that if \(R\) is a 2-primal, then \(O_P \subseteq P\) for each prime ideal of \(R\).

Corollary 2.6:

Let \(R\) be 2-primal generalized right simple singular AP-injective ring, then every prime ideal of \(R\) is maximal. ■

Proposition 2.7:

Let \(R\) be \(ZC\)-generalized simple singular AP-injective rings, then for any \(a,b \in R\) with \(ab = 0\), then \(r(a) + r(b) = R\).

Proof:

Suppose that \(ab = 0\) and \(r(a) + r(b) \neq R\). Then, there exists a maximal right ideal \(M\) containing \(r(a) + r(b)\). If \(M\) not essential, then there exists \(0 \neq e = e^2 \in R\) such that \(M = r(e)\). Since \(b \in r(a) \subseteq M = r(e) = l(e)\), then \(be = 0\) which implies that \(e \in r(b)\). Therefore, \(M\) must be essential. Since, \(r(a) \subseteq M\) and \(a \in M\), then by Lemma 2.3 \(R/M \cong aR/aM\). Therefore \(R/M\) is AP-injective. Let \(f : aR \to R/M\) is defined by \(f(ar) = r+M\) for all \(r \in R\). Note that \(f\) is
well-defined and by Lemma 2.2  \(1+M = f(a) = ca + M + x, \ c \in R, \ x \in X_a.\) Hence, \(1 - ca + M = x \in R/M \cap X_a = 0,\) so \(1-ca \in M.\) Since, \(a \in r(b)\) and \(R\) is ZC- ring, then \(ca \in r(b) \subseteq M\) whence \(1 \in M\) which is a contradiction. Therefore, \(r(a) + r(b) = R.\) ■

3. The Connection between Generalized Simple Singular AP-injective and Other Rings

In this section, we give the connection between Von Neumann regular rings and generalized simple singular AP-injective rings.

**Theorem 3.1:**

Let \(R\) be right MC2-generalized right simple singular AP-injective, then \(R\) is right weakly regular ring.

**Proof:**

We will show that \(RaR + r(a) = R\) for any \(a \in R.\) Suppose that there exists \(b \in R\) such that \(RbR + r(b) \neq R.\) Then, there exists a maximal right ideal \(M\) of \(R\) containing \(RbR + r(b).\) If \(M\) not essential, then \(M\) is a direct summand of \(R.\) So, we can write \(M = eR\) for some \(0 \neq e = e^2 \in R.\) Thus, \((1-e)Rb = 0,\) since \(R\) is MC2 and \((1-e)R\) is minimal, then \(bR(1-e) = 0.\) Hence, \((1-e) \in r(b) \subseteq M,\) so \(1 \in M.\) It is a contradiction. Therefore, \(M\) must be essential right ideal of \(R.\)

Since, \(r(a) \subseteq M\) and \(a \in M,\) then by Lemma 2.3 \(R/M \cong aR/aM.\) Therefore, \(R/M\) is AP-injective. Let \(f: bR \rightarrow R/M\) defined by \(f(br) = r + M\) for all \(r \in R.\) Note that \(f\) is well-defined and by Lemma 2.2, \(1+M = f(b) = cb + M + x, \ c \in R, \ x \in X_b.\) Hence, \(1 - cb + M = x \in R/M \cap X_b = 0,\) so \(1-cb \in M.\) Since, \(cb \in RbR \subseteq M,\) then \(1 \in M\) which is a contradiction. Therefore, \(RaR + r(a) = R\) for all \(a \in R.\) Hence, \(R\) is a right weakly regular ring. ■

Now, we shall prove the main results of this section.

**Theorem 3.2:**

Let \(R\) be a ring, then the following statements are equivalent:

1. \(R\) is Von Neumann regular.
2. \(R\) is generalized right simple singular AP-injective right weakly continuous.

**Proof:**

(1) \(\Rightarrow\) (2) It is clear.

(2) \(\Rightarrow\) (1) Suppose that \(Y(R) \neq 0.\) Then, there exists a non-zero element \(a \in Y(R)\) such that \(a^2 = 0.\) We claim that \(Y(R) + r(a) = R.\) If not, there exists a maximal essential right ideal \(M\) containing \(Y(R) + r(a).\) Since, \(r(a) \subseteq M\) and \(a \in M,\) then by Lemma 2.3 \(R/M \cong aR/aM.\) Therefore, \(R/M\) is AP-injective and \(1_{R/M}(a) = (R/M)a \oplus X_a, \ X_a \subseteq R/M.\) Let \(f:aR \rightarrow R/M\) be defined by \(f(ar) = r + M\) for all \(r \in R.\) Note that \(f\) is well-defined and by Lemma 2.2, \(1+M = f(a) = ba + M + x, \ b \in R, \ x \in X_a.\) Hence, \(1 - ba + M = x \in R/M \cap X_a = 0,\) so \(1-ba \in M.\) Since, \(a \in Y(R) = J(R)\) implies that \(ca \in J(R) \subseteq M\) and \(1 \in M,\) which is a contradiction. Therefore, \(Y(R) + r(a) = R.\) Thus, we can write \(1 = c + d,\) for some \(c \in Y(R)\) and \(d \in r(a).\) Thus, \(a = ca\) and so \((1-c)a = 0.\) Since \(c \in Y(R) = J(R),\) \(1-c\) is invertible. Thus \(a = 0\) contradicting \(a \neq 0.\) Therefore, \(Y(R) = 0.\) ■

**Lemma 3.3:** [4]

For any \(a \in \text{Cent}(R),\) if \(a = ara\) for some \(r \in R,\) then there exists \(b \in \text{Cent}(R)\) such that \(a = aba\) (where \(\text{Cent}(R)\) is the center of \(R).\)

**Theorem 3.4:**
R is right non-singular generalized right simple singular AP-injective, then Cent(R) is Von Neumann regular ring.

**Proof:**
First, we have to prove Cent(R) is reduced. Let $0 \neq a \in \text{Cent}(R)$ and $a^2 = 0$ implies that $a \in r(a)$. If $r(a)$ is essential, then $a \in Y(R) = 0$ implies that $a = 0$. We are done. If $r(a)$ not essential, there exists a non-zero right ideal $I$ in $R$ such that $r(a) \subseteq I \cap r(a)$ [a $\in$ Cent(R)] but $I \cap r(a) = 0$ implies that $Ia = 0$ and we get $I \subseteq I(a) = r(a)$ so $I = 0$ contradiction. Therefore, $a = 0$, so Cent(R) is a reduced ring. Now, we shall show that $aR + r(a) = R$ for any $a \in \text{Cent}(R)$. If not, there exists a maximal right ideal $M$ of $R$ such that $aR + r(a) \subseteq M$ observe that $M$ is an essential right ideal of $R$. If not, then $M$ is a direct summand of $R$. So, we can write $M = r(e)$ for some $0 \neq e = e^2 \in R$. Since, $a \in M$ and $a \in \text{Cent}(R)$, $ae = ea = 0$. Thus, $e \in r(a) \subseteq M = r(e)$, whence $e = 0$. It is a contradiction. Therefore, $M$ must be an essential right ideal of $R$.

Since, $r(a) \subseteq M$ and $a \in M$, then by Lemma 2.3 $R/M \cong aR/aM$. Therefore, $R/M$ is AP-injective. Let $f : aR \rightarrow R/M$ defined by $f(\text{ar}) = r + M$ for all $r \in R$. Note that $f$ is well-defined and by Lemma 2.2, $1 + M = f(\text{a}) = ca + M + x, c \in R, x \in X_a$. Hence, $1 - ca + M = x \in R/M \cap X_a = 0$, so $1 - ca \in M$ since, $a \in \text{cent}(R)$, then $ca = ac \in M$, and hence $1 \in M$. Therefore, $aR + r(a) = R$ for all $a \in \text{cent}(R)$ and so we have $a = ara$ for some $r \in R$. Applying Lemma 3.3, Cent( R) is Von Neumann regular ring. □

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