SAGA and Restricted Strong Convexity

Chao Qu\textsuperscript{1}, Yan Li\textsuperscript{2}, and Huan Xu\textsuperscript{2}

\textsuperscript{1}Department of Mechanical Engineering, National University of Singapore
\textsuperscript{2}H. Milton Stewart School of Industrial and Systems Engineering, Georgia Institute of Technology

Abstract

SAGA is a fast incremental gradient method on the finite sum problem and its effectiveness has been tested on a vast of applications. In this paper, we analyze SAGA on a class of non-strongly convex and non-convex statistical problem such as Lasso, group Lasso, Logistic regression with $\ell_1$ regularization, linear regression with SCAD regularization and Correct Lasso. We prove that SAGA enjoys the linear convergence rate up to the statistical estimation accuracy, under the assumption of restricted strong convexity (RSC). It significantly extends the applicability of SAGA in convex and non-convex optimization.

1 Introduction

We study the finite sum problem in the following forms:

- **Convex** $G(\theta)$:

  \[
  \text{Minimize: } G(\theta) \triangleq f(\theta) + \lambda \psi(\theta) \\
  \psi(\theta) \leq \rho \\
  = \frac{1}{n} \sum_{i=1}^{n} f_i(\theta) + \lambda \psi(\theta), \tag{1}
  \]

  where $f_i(\theta)$ is a convex loss such as $f_i(\theta) = \frac{1}{2}(y_i - \theta^T x_i)^2$ and $\psi(\theta)$ is a norm, $\rho$ is some predefined radius. We denote the dual norm of $\psi(\theta)$ as $\psi^*(\theta)$ and assume that each $f_i(\theta)$ is $L$ smooth.

- **Non-convex** $G(\theta)$:

  \[
  \text{Minimize: } G(\theta) \triangleq f(\theta) + g_{\lambda,\mu}(\theta) \\
  g_{\lambda}(\theta) \leq \rho \\
  = \frac{1}{n} \sum_{i=1}^{n} f_i(\theta) + g_{\lambda,\mu}(\theta), \tag{2}
  \]

  where $f_i(\theta)$ is convex and $L$ smooth, $g_{\lambda,\mu}(\theta)$ is some non-convex regularizer, $g_{\lambda}(\theta)$ is close related to $g_{\lambda,\mu}(\theta)$ and we defer the formal definition to the section 2.3.
Such finite sum structure is common for machine learning problems particularly in the empirical risk minimization (ERM) setting. To solve the above problem, the standard Prox-full gradient (FG) method update $\theta^k$ iterative by

$$
\theta^{k+1} = \text{prox}_{\gamma\psi}(\theta^k - \gamma \nabla f(\theta^k)).
$$

It is well known that FG enjoys fast linear convergence under smoothness and strong convexity assumption. However this result may be less appealing when $n$ is large since the cost of calculation of full gradient scales with $n$. Stochastic gradient (SG) method remedies this issue but only possess the sub-linear convergence rate.

Recently, a set of stochastic algorithms including SVRG [Johnson and Zhang 2013, Xiao and Zhang 2014], SAGA [Defazio et al. 2014], SAG [Schmidt et al. 2013], SDCA [Shalev-Shwartz and Zhang 2014] and many others [Harikandeh et al. 2015, Qu et al. 2015, Zhang and Lin 2017] have been proposed to exploit the finite sum structure and enjoy linear rate convergence under smoothness and strong convexity assumption on $f_i(\theta)$. We study SAGA in this paper. From a high level, SAGA is a midpoint between SAG and SVRG, see the discussion in [Defazio et al. 2014] for more details. Different from SVRG, it is a fully incremental gradient method. Comparing with SAG, it uses an unbiased estimator of the gradient, which results in an easier proof among other things. In fact, to the best of our knowledge, the analysis of SAG has not yet been extended to proximal operator version.

A second trendy topic in optimization and statistical estimation is the study of non-convex problems, due to a vast array of applications such as SCAD [Fan and Li 2001], MCP [Zhang and Zhang 2012], robust regression (Corrected Lasso [Loh and Wainwright 2011]) and deep learning [Goodfellow et al. 2016]. Some previous work have established fast convergence for batch gradient methods without assuming strong convexity or even convexity: Xiao and Zhang 2013 proposed a homotopy method to solve Lasso with RIP condition, Agarwal et al. 2010 analyzed the convergence rate of batched composite gradient method on several models, such as Lasso, logistic regression with $l_1$ regularization and noisy matrix decomposition, and showed that the convergence is linear under mild conditions of the solution (sparse or low rank). Loh and Wainwright 2011, 2013 extended the above work to the non-convex case.

These two line of research thus motivate this work to investigate whether SAGA enjoys the linear convergence rate without strong convexity or even in the non-convex problem. Specifically, we prove that under Restricted strong convexity assumption, SAGA converges linearly up to the fundamental statistical precision of the model, which covers five statistical models we mentioned above but not limited to these. In a high level, it is a stochastic counterpart of the work in Loh and Wainwright 2013, albeit with more involved analysis due to the stochastic nature of SAGA.

We list some notable non-strongly convex and non-convex problems in the following. Indeed, our work proves that SAGA converges linearly in all these models. Note that the first three belong to the non-strongly convex category especially when $p > n$ and the last two are non-convex.

1. Lasso: $f_i(\theta) = \frac{1}{2}((\theta, x_i) - y_i)^2$ and $\psi(\theta) = \|\theta\|_1$.
2. Group Lasso: $f_i(\theta) = \frac{1}{2}((\theta, x_i) - y_i)^2$, $\psi(\theta) = \|\theta\|_{1,2}$.
3. Logistic Regression with $l_1$ regularization: $f_i(\theta) = \log(1+\exp(-y_i(\theta, x_i))$ and $\psi(\theta) = \|\theta\|_1$.
4. Corrected Lasso [Loh and Wainwright 2011]: $G(\theta) = \sum_{i=1}^n \frac{1}{2n}((\theta, x_i) - y_i)^2 - \frac{1}{2}\theta^T\Sigma\theta + \lambda\|\theta\|_1$, where $\Sigma$ is some positive definite matrix.
5. Regression with SCAD regularizer [Fan and Li, 2001]:

\[ G(\theta) = \sum_{i=1}^{n} \frac{1}{2n} (\langle \theta, x_i \rangle - y_i)^2 + SCAD(\theta). \]

Very recently, [Qu et al., 2016] explore the similar idea of us called restrict strong convexity condition (RSC) [Negahban et al., 2009] on SVRG and prove that under this condition, a class of ERM problem has the linear convergence even without strongly convex or even the convex assumption. From a high level perspective, our work can be thought as of similar spirit but for SAGA algorithm. We believe analyzing the SAGA algorithm is indeed important as SAGA enjoys certain advantage compared to SVRG. As discussed above, SVRG is not a completely incremental algorithm since it need to calculate the full gradient in every epoch, while SAGA avoids the computation of the full gradient by keeping a table of gradient. Moreover, although in general SAGA costs \( O(np) \) storage (which is inferior to SVRG), in many scenarios the requirement of storage can be reduced to \( O(n) \). For example, many loss function \( f_i \) take the form \( f_i(\theta) = g_i(\theta^T x_i) \) for a vector \( x_i \) and since \( x_i \) is a constant we just need to store the scalar \( \nabla g_i(u^k_i) \) for \( u^k_i = x_i^T \theta^k \) rather than full gradient. When this scenario is possible, SAGA can perform similarly or even better than SVRG. In addition, SVRG has an additional parameter besides step size to tune – the number of iteration \( m \) per inner loop. To conclude, both SVRG and SAGA can be more suitable for some problems, and hence it is useful to understand the performance of SAGA for non-strongly convex or non-convex setups. At last, the proof steps are very different. In particular, we define a Lyapunov function in SAGA and prove it converges geometrically until the optimality gap achieves the statistical tolerance, while [Qu et al., 2016] directly look at evolution of \( G(\theta^k) \).

1.1 Related work

There are a plethora of work on the finite sum problem and we review those most closely related to ours. [Li et al., 2016] consider SVRG on a non-convex sparse linear regression setting different from ours, where \( f_i \) is convex and the non-convexity comes from the hard-thresholding operator. We focus on a non-convex regularizer such as SCAD and corrected Lasso. In addition, we consider a unified framework on SAGA thus our work not only covers the linear sparse model but also the group sparsity and other model satisfying our assumptions. [Karimi et al., 2016], [Reddi et al., 2016], [Hajinezhad et al., 2016] proved global linear convergence of SVRG and SAGA on non-convex problems by revisiting the concept Polyak-Lojasiewicz inequality or its equivalent idea such as error bound. We emphasize that our work looks at the problem from different perspective. In particular, our theory asserts that the algorithm converges faster with sparser \( \theta^* \), while their results are independent of the sparsity \( r \). Empirical observation seems to agree with our theorem. Indeed, when \( r \) is dense enough a phase transition from linear rate to sublinear rate occurs (also observed in [Qu et al., 2016]), which agrees with the prediction of our theorem. Furthermore, their work requires the epigraph of \( \psi(\theta) \) to be a polyhedral set which limits its applicability. For instance, the popular group Lasso does not satisfy such an assumption. Other non-convex stochastic variance reduction works include [Shalev-Shwartz, 2016], [Shamir, 2015] and [Allen-Zhu and Hazan, 2016]: [Shalev-Shwartz, 2016] considers the setting that \( f(\theta) \) is strongly convex but each individual \( f_i(\theta) \) is non-convex. [Shamir, 2015] discusses a projection version of non-convex SVRG and its specific application on PCA. [Allen-Zhu and Hazan, 2016] consider a general non-convex problem, which only achieves a sublinear convergence rate.
2 Preliminaries

2.1 Restricted Strong Convexity

As mentioned in the abstract, Restricted strong convexity (RSC) is the key assumption underlying our results. We therefore define RSC formally. We say a function $f(\theta)$ satisfies RSC w.r.t. to a norm $\psi(\theta)$ with parameter $(\sigma, \tau)$ if the following holds.

$$
\begin{align*}
    f(\theta_2) - f(\theta_1) - \langle \nabla f(\theta_2), \theta_2 - \theta_1 \rangle \\
    \geq \frac{\sigma}{2} \|\theta_2 - \theta_1\|_2^2 - \tau \sigma \psi^2(\theta_2 - \theta_1).
\end{align*}
$$

We remark that we assume $f(\theta) = \frac{1}{n} \sum_{i=1}^{n} f_i(\theta)$ satisfies the RSC rather than individual loss function $f_i(\theta)$. Indeed, $f_i(\theta)$ does not satisfy RSC in practice. Note that when $f(\theta)$ is $\sigma$-strongly convex, obviously we have $\tau = 0$. For more discussions on RSC, we refer reader to Negahban et al. [2009].

2.2 Assumptions for the Convex regularizer $\Psi(\theta)$

2.2.1 Decomposibility of $\Psi(\theta)$

Given a pair of subspaces $M \subseteq \bar{M}$ in $\mathbb{R}^p$, the orthogonal complement of $\bar{M}$ is

$$
\bar{M}^\perp = \{v \in \mathbb{R}^p | \langle u, v \rangle = 0 \text{ for all } u \in \bar{M} \}.
$$

$M$ is known as the model subspace, where $\bar{M}^\perp$ is called the perturbation subspace, representing the deviation from the model subspace. A regularizer $\psi$ is decomposable w.r.t. $(M, \bar{M}^\perp)$ if

$$
\psi(\theta + \beta) = \psi(\theta) + \psi(\beta)
$$

for all $\theta \in M$ and $\beta \in \bar{M}^\perp$. A concrete example is $\ell_1$ regularization for sparse vector supported on subset $S$. We define the subspace pairs with respect to the subset $S \subset \{1, \ldots, p\}$, $M(S) = \{\theta \in \mathbb{R}^p | \theta_j = 0 \text{ for all } j \notin S\}$ and $\bar{M}(S) = M(S)$. The decomposability is thus easy to verify. Other widely used examples include non-overlap group norms such as $\| \cdot \|_{1,2}$, and the nuclear norm $\| \cdot \|_*$. Negahban et al. [2009]. In the rest of the paper, we denote $\theta_M$ as the projection of $\theta$ on the subspace $M$.

2.2.2 Subspace compatibility

Given the regularizer $\psi(\cdot)$, the subspace compatibility $H(\bar{M})$ is given by

$$
H(\bar{M}) = \sup_{\theta \in \bar{M} \setminus \{0\}} \frac{\psi(\theta)}{\|\theta\|_2}.
$$

In other words, it is the Lipschitz constant of the regularizer restricted in $\bar{M}$. For instance, in the above-mentioned sparse vector example with cardinality $r$, $H(\bar{M}) = \sqrt{r}$.

2.3 Assumptions for the Nonconvex regularizer $g_{\lambda, \mu}(\theta)$

In the non-convex case, we consider regularizers that are separable across coordinates, i.e., $g_{\lambda, \mu}(\theta) = \sum_{j=1}^{p} \bar{g}_{\lambda, \mu}(\theta_j)$. Besides the separability, we have additional assumptions on $g_{\lambda, \mu}(\cdot)$. For the univariate function $\bar{g}_{\lambda, \mu}(t)$, we assume
Algorithm 1 SAGA

Input: Step size $\gamma$, number of iterations $K$, and smoothness parameters $L$.

for $k = 1, \ldots, K$ do

Pick a $j$ uniformly at random

1. Take $\phi_{j}^{k+1} = \phi_{j}^{k}$, and store $f_{j}(\phi_{j}^{k+1})$ in the table. All other entries in the table remain unchanged.

2. Update $\theta$ using $f_{j}(\phi_{j}^{k+1})$, $f_{j}(\phi_{j}^{k})$ and the table average:

$$w^{k+1} = \theta^{k} - \gamma[f_{j}(\phi_{j}^{k+1}) - f_{j}(\phi_{j}^{k}) + \frac{1}{n} \sum_{i=1}^{n} f_{i}(\phi_{i}^{k})].$$

$$\theta^{k+1} = \arg \min_{\psi(\theta) \leq \rho} \frac{1}{2} \| \theta - w^{k+1} \|^{2} + \gamma \lambda \psi(\theta).$$

end for

1. $\bar{g}_{\lambda, \mu}(\cdot)$ satisfies $\bar{g}_{\lambda, \mu}(0) = 0$ and is symmetric around zero. That is, $\bar{g}_{\lambda, \mu}(t) = \bar{g}_{\lambda, \mu}(-t)$.

2. On the nonnegative real line, $\bar{g}_{\lambda, \mu}$ is nondecreasing.

3. For $t > 0$, $\frac{\bar{g}_{\lambda, \mu}(t)}{t}$ is nonincreasing in $t$.

4. $\bar{g}_{\lambda, \mu}(t)$ is differentiable at all $t \neq 0$ and subdifferentiable at $t = 0$, with $\lim_{t \to 0^+} \bar{g}_{\lambda, \mu}(t) = \lambda L_{g}$ for a constant $L_{g}$.

5. $\bar{g}_{\lambda}(t) := (\bar{g}_{\lambda, \mu}(t) + \frac{2}{\lambda} t^{2})/\lambda$ is convex.

We provide two examples satisfying the above assumptions.

(1) SCAD$\lambda, \zeta(t) \triangleq$

$$\begin{cases} \lambda |t|, & \text{for } |t| \leq \lambda, \\ -(t^{2} - 2 \zeta \lambda |t| + \lambda^{2})/(2(\zeta - 1)), & \text{for } \lambda < |t| \leq \zeta \lambda, \\ (\zeta + 1) \lambda^{2}/2, & \text{for } |t| > \zeta \lambda, \end{cases}$$

where $\zeta > 2$ is a fixed parameter. It satisfies the assumption with $L_{g} = 1$ and $\mu = \frac{1}{\zeta - 1}$ [Loh and Wainwright 2013].

(2) MCP$\lambda, b(t) \triangleq$ sign$(t) \lambda \int_{0}^{\left|t\right|} \left(1 - \frac{z}{\lambda b}\right)_{+} dz$, where $b > 0$ is a fixed parameter. MCP satisfies the assumption with $L_{g} = 1$ and $\mu = \frac{1}{b}$ [Loh and Wainwright 2013].

2.4 Implementation of the algorithm

For the convex $G(\theta)$ case, we directly apply the Algorithm 1. As to the non-convex $G(\theta)$ case, we essentially solve the following equivalent problem

$$\min_{g_{\lambda}(\theta) \leq \rho} \left( f(\theta) - \frac{\lambda}{2} \| \theta \|_{2}^{2} \right) + \lambda g_{\lambda}(\theta).$$

We define $F_{i}(\cdot) = f_{i}(\cdot) - \frac{\lambda}{2} \| \theta \|_{2}^{2}$ and $F(\theta) = f(\theta) - \frac{\lambda}{2} \| \theta \|_{2}^{2}$. To implement Algorithm 1 on non-convex $G(\theta)$, we replace $f_{i}(\cdot)$ and $\psi(\cdot)$ in the algorithm by $F_{i}(\cdot)$ and $g_{\lambda}(\cdot)$. Remark that
according to the assumptions on $g_{\lambda,\mu}(\cdot)$ in Section 2.3, $g_{\lambda}(\cdot)$ is convex thus the proximal step is well-defined. The update rule of proximal operator on several $g_{\lambda,\mu}$ (such as SCAD) can be found in Loh and Wainwright [2013].

3 Main result

In this section, we present the main theoretical results, and some corollaries that instantiate the main results in several well known statistical models.

3.1 Convex $G(\theta)$

We first present the results on convex $G(\theta)$. In particular, we prove a Lyapunov function converges geometrically until $G(\theta^k) - G(\hat{\theta})$ achieves some tolerance. To this end, we first define the Lyapunov function

$$T_k \triangleq \frac{1}{n} \sum_{i=1}^{n} \left( f_i(\phi_i^k) - f_i(\hat{\theta}) - \langle \nabla f_i(\hat{\theta}), \phi_i^k - \hat{\theta} \rangle \right) + (c + \alpha)\|\theta^k - \hat{\theta}\|_2^2 + b(G(\theta^k) - G(\hat{\theta})),$$

where $\hat{\theta}$ is the optimal solution of problem (1), $c$, $\alpha$, $b$ are some positive constant will be specified later in the theorems. Notice our definition is a little different from the one in the original SAGA paper in Defazio et al. [2014]. In particular, we have an additional term $G(\theta^k) - G(\hat{\theta})$ and choose different value of $c$ and $\alpha$, which helps us to utilize the idea of RSC.

We list some notations used in the following theorems and corollaries.

- $\theta^*$ is the unknown true parameter. $\hat{\theta}$ is the optimal solution of (1).
- $\psi^*(\cdot)$ is the dual norm of $\psi(\cdot)$.
- Modified restricted strongly convex parameter:

$$\bar{\sigma} = \sigma - 64\tau_\sigma H^2(\bar{M}).$$

- Tolerance

$$\delta = 24\tau_\sigma \left( 8H(\bar{M})\|\hat{\theta} - \theta^*\|_2 + 8\psi(\theta^*_M) \right)^2$$

**Theorem 1.** Assume each $f_i(\theta)$ is $L$ smooth and convex, $f(\theta)$ satisfies the RSC condition with parameter $(\sigma, \tau_\sigma)$ and $\bar{\sigma} > 0$, $\theta^*$ is feasible, the regularizer is decomposable w.r.t. $(M, \bar{M})$, if we choose the parameter $\lambda \geq \max(2\psi^*(\nabla f(\theta^*)), c_1\tau_\alpha \rho)$ where $c_1$ is some universal positive constant, then with $\gamma = \frac{1}{4L}, c = \frac{9L}{n}, \alpha = \frac{5}{2}, b = 2\alpha\gamma, \frac{1}{\kappa} = \min(\frac{1}{14}, \frac{1}{M})$, we have

$$ET_k \leq \left( 1 - \frac{1}{\kappa} \right)^k T_0,$$

until $G(\theta^k) - G(\hat{\theta}) \leq \delta$, where the expectation is for the randomness of sampling of $j$ in the algorithm.

Some remarks are in order.
• The requirement $\hat{\sigma} > 0$ is easy to satisfy in some popular statistical models. Take Lasso as an example, where $\tau_\sigma = c_2 \frac{\log p}{n}$, $c_2$ are some positive constant, $H^2(M) = r$. Thus $\hat{\sigma} = \sigma - 64c_2 \frac{\log p}{n}$. Hence when $64c_2 \frac{\log p}{n} \leq \frac{1}{2} \sigma$, we have $\hat{\sigma} \geq \frac{\sigma}{2}$.

• Since $\frac{1}{n}$ depends on $\hat{\sigma}/L$, the convergence rate is indeed affected by the sparsity $r$ (Lasso for example) as we mentioned in the introduction. Particularly, sparser $r$ leads to larger $\hat{\sigma}$ and faster convergence rate.

• In some models, we can choose the subspace pair such that $\theta^* \in M$, thus the tolerance $\delta$ is simplified to $\delta = c_3 \tau_\sigma H^2(M) ||\hat{\theta} - \theta^*||^2_2$. In Lasso as we mentioned above, $\delta = c_3 \frac{\log p}{n} ||\hat{\theta} - \theta^*||^2_2$, i.e., the tolerance is dominated by the statistical error $||\hat{\theta} - \theta^*||^2_2$.

• When $G(\theta^k) - G(\hat{\theta}) \leq \delta$, use modified restricted strong convexity (Lemma 5 in the appendix), it is easy to derive $||\theta^k - \hat{\theta}||^2_2 \leq \frac{c_4 \delta}{\hat{\sigma}}$.

Combine all remarks together, the theorem says the Lyapunov function decreases geometrically until $G(\theta^k) - G(\hat{\theta})$ achieves the tolerance $\delta$. This tolerance is dominated by the statistical error $||\theta^k - \hat{\theta}||^2_2$, thus can be ignored from the statistical perspective.

3.1.1 Sparse linear regression

The first model we consider is Lasso, where $f_i(\theta) = \frac{1}{2} \langle \theta, x_i \rangle - y_i^2$ and $\psi(\theta) = ||\theta||_1$. More concretely, we consider a model where each data point $x_i$ is i.i.d. sampled from a zero-mean normal distribution, i.e., $x_i \sim N(0, \Sigma)$. We denote the data matrix by $X \in \mathbb{R}^{n\times p}$ and the smallest eigenvalue of $\Sigma$ by $\sigma_{\text{min}}(\Sigma)$, and let $\nu(\Sigma) \triangleq \max_{i=1,\ldots,p} \Sigma_{ii}$. The observation is generated by $y_i = x_i^T \theta^* + \xi_i$, where $\xi_i$ is a zero mean sub-Gaussian noise with variance $\varsigma^2$.

We use $X_j \in \mathbb{R}^n$ to denote $j$-th column of $X$. Without loss of generality, we require $X$ to be column-normalized, i.e., $\|X_j\|_2 \leq 1$ for all $j = 1, 2, \ldots, p$. Here, the constant 1 is chosen arbitrarily to simplify the exposition, as we can always rescale the data.

**Corollary 1.** Assume $\theta^*$ is the true parameter supported on a subset with cardinality at most $r$, and we choose $\lambda$ such that $\lambda \geq \max(6\varsigma \sqrt{\frac{\log p}{n}}, c_1 \rho \nu(\Sigma) \frac{\log p}{n}, \sigma = \frac{1}{4} \sigma_{\text{min}}(\Sigma) - c_2 \nu(\Sigma) \frac{\log p}{n}$, then with $\gamma = \frac{1}{9L}$, $c = \frac{9L}{\pi}$, $c = \frac{9L}{\pi}$, $b = 2\alpha \gamma$, $\frac{1}{\pi} = \min(\frac{\sigma}{14\gamma}, \frac{1}{9n})$, we have

$$E T_k \leq \left(1 - \frac{1}{\kappa}\right)^k T_0,$$

with probability at least $1 - \exp(-3 \log p) - \exp(-c_3 n)$, until $G(\theta^k) - G(\hat{\theta}) \leq \delta$, where $\delta = c_4 \nu(\Sigma) \frac{\log p}{n} ||\hat{\theta} - \theta^*||^2_2$. Here $c_1, c_2, c_3, c_4$ are some universal positive constants.

We offer some discussions on this corollary.

• The requirement of $\lambda \geq 6\varsigma \sqrt{\frac{\log p}{n}}$ is known to play an important role in proving bounds on the statistical error of Lasso, see Negahban et al. [2009] and reference therein for further details.

• The requirement $\lambda \geq c_1 \rho \nu(\Sigma) \frac{\log p}{n}$ is to guarantee the fast global convergence of the algorithm, which is similar to the requirement in its batch counterpart [Agarwal et al. 2010].

• When $r$ is small and $n$ is large, which is necessary for statistical consistency of Lasso, we obtain $\hat{\sigma} > 0$, which guarantees the existences of $\kappa$. Under this condition we have $\delta = c_4 \nu(\Sigma) \frac{\log p}{n} ||\hat{\theta} - \theta^*||^2_2$, which is dominated by $||\hat{\theta} - \theta^*||^2_2$. 


3.1.2 Group Sparse model

The group sparsity model aims to find a regressors such that predefined groups of covariates are selected into or out of a model together. The most commonly used regularization to encourage group sparsity is $\| \cdot \|_{1,2}$. Formally, we are given a class of disjoint groups of the features, i.e., $G = \{G_1, G_2, ..., G_{N_G}\}$ and $G_i \cap G_j = \emptyset$. The regularization term is $||\theta||_{G, q} \triangleq \sum_{i=1}^{N_G} ||\theta_i||_q$. When $q = 2$, it reduces to the popular group Lasso [Yuan and Lin 2006] while another widely used case is $\| \cdot \|_\infty$.

We define the subspace pair $(M, \bar{M})$ in the group sparsity model. For a subset $S_G \subseteq \{1, ..., N_G\}$ with cardinality $s_G = |S_G|$, we define the subspace

$$M(S_G) = \{\theta|\theta_{G_i} = 0 \text{ for all } i \notin S_G\},$$

and $M = \bar{M}$. The orthogonal complement is

$$M^\perp(S_G) = \{\theta|\theta_{G_i} = 0 \text{ for all } i \in S_G\}.$$

We can easily verify that

$$\|\alpha + \beta\|_{G, q} = \|\alpha\|_{G, q} + \|\beta\|_{G, q},$$

for any $\alpha \in M(S_G)$ and $\beta \in M^\perp(S_G)$.

We mainly focus on the discussion on the group Lasso, as an example. We assume the observation $y_i$ is generated as $y_i = \theta_i^T + \xi_i$, where $x_i \sim N(0, \Sigma)$, and $\xi_i \sim N(0, \xi^2)$.

**Corollary 2.** (Group Lasso) Assume $\theta \in \mathbb{R}^p$ and each group has $m$ parameters, i.e., $p = mN_G$. Denote the cardinality of non-zero group by $s_G$, and we choose parameter $\lambda$ such that

$$\lambda \geq \max\left(4\xi\left(\sqrt{\frac{m}{n}} + \frac{\log N_G}{n}, c_1\rho \sigma_2(\Sigma)\left(\sqrt{\frac{m}{n}} + \sqrt{\frac{3\log N_G}{n}}\right)^2\right),\right)$$

then with $c = \frac{9k}{n}, \alpha = \frac{\xi}{2}, b = 2\alpha\gamma, \frac{1}{\gamma} = \min\left(\frac{\tilde{\sigma}}{m}, \frac{1}{m}\right)$, we have

$$\mathbb{E}T_k \leq (1 - \frac{1}{k})k T_0$$

with probability at least $1 - 2\exp(-2\log N_G) - c_2\exp(-c_3n)$, until $G(\theta^*) - G(\hat{\theta}) \leq \delta$, where $\tilde{\sigma} = \sigma_1(\Sigma) - c_2\sigma_2(\Sigma)s_G\left(\sqrt{\frac{m}{n}} + \sqrt{\frac{3\log N_G}{n}}\right)^2, \sigma_1(\Sigma)$ and $\sigma_2(\Sigma)$ are positive constant depending only on $\Sigma$, $\delta = c_4\sigma_2(\Sigma)s_G\left(\sqrt{\frac{m}{n}} + \sqrt{\frac{3\log N_G}{n}}\right)^2\|\theta - \theta^*\|_2^2$. $c_1, c_2, c_3, c_4$ are some universal positive constants.

We offer some discussions to put above corollary into context.
To satisfy the requirement of $\bar{\sigma} > 0$, it suffices to have $s_G(\sqrt{\frac{m}{n}} + \sqrt{\frac{3\log N}{n}})^2 = o(1)$. It is also the condition to guarantee the statistical consistency of group Lasso [Negahban et al. 2009].

$s_G$ and $m$ affect the speed of the convergence, in particular, smaller $m$ and $s_G$ leads to larger $\bar{\sigma}$ and thus $\bar{\sigma}/L$.

The requirement of $\lambda$ is similar to the batch gradient method in [Agarwal et al. 2010].

### 3.2 Non-convex $G(\theta)$

The definition of Lyapunov function in the non-convex case is same with the convex one, i.e.,

$$T_k \triangleq \frac{1}{n} \sum_{i=1}^{n} \left( f_i(\phi_i^k) - f_i(\hat{\theta}) - \langle \nabla f_i(\hat{\theta}), \phi_i^k - \hat{\theta} \rangle \right) + (c + \alpha)\|\theta^k - \hat{\theta}\|^2_2 + b(G(\theta^k) - G(\hat{\theta})).$$

Note that $\hat{\theta}$ is the global optimum of problem (2), and $f_i(\cdot)$ is convex, thus $T_k$ is always positive. In the non-convex case, we require $f(\theta)$ satisfy the RSC condition with parameter $(\sigma, \tau \log p)$, where $\tau$ is some positive constant.

We list some notations used in the following theorem and corollaries of it.

- $\hat{\theta}$ is the global optimum of problem (2), and $\theta^*$ is the unknown true parameter with cardinality $r$.

- Modified restricted strongly convex parameter:

$$\check{\sigma} = \sigma - 64r\tau \frac{\log p}{n} - \mu.$$ 

Recall $\mu$ is defined in section 2.3 and represent the degree of non-convexity.

- Tolerance $\delta = c_1 r \tau \frac{\log p}{n} \|\hat{\theta} - \theta^*\|^2_2$, where $c_1$ is some universal positive constant.

#### Theorem 2.

Suppose $\theta^*$ is $r$ sparse, $\hat{\theta}$ is the global optimum of Problem (2), each $f_i(\cdot)$ is $L$ smooth and convex, $f(\theta)$ satisfies the RSC condition with $(\sigma, \tau \frac{\log p}{n})$, $\bar{\sigma} > 3\mu$, $L > 3\mu$, $g_{\lambda, \mu}$ satisfies the assumption in Section 2.3, and $\lambda Lg \geq \max\{c_1 \rho \tau \frac{\log p}{n} , 4\|\nabla f(\theta^*)\|_{\infty} \}$, where $c_1$ is some positive constant, then with $\gamma = \frac{1}{24L}$, $c = \frac{24L}{\tau}$, $\alpha = \frac{\tau}{2}$, $b = 2\alpha \gamma$, $\frac{1}{\kappa} = \frac{1}{24} \min\{\frac{2\alpha}{\tau L}, \frac{1}{n}\}$, we have

$$ET_k \leq \left(1 - \frac{1}{\kappa}\right)^k T_0,$$

until $G(\theta^k) - G(\hat{\theta}) \leq \delta$, where the expectation is for the randomness of sampling of $j$ in the algorithm.

Notice that we require $\check{\sigma} > 3\mu$, that is $\sigma - 64r\tau \frac{\log p}{n} - 4\mu > 0$. Thus to satisfy this requirement, the non-convex parameter $\mu$ can not be large.

The tolerance $\delta = c_2 r \tau \frac{\log p}{n} \|\hat{\theta} - \theta^*\|^2_2$ is dominated by the statistical error $\|\hat{\theta} - \theta^*\|^2_2$, when the model is sparse ($r$ is small) and $n$ is large.
• When $G(\theta^k) - G(\hat{\theta}) \leq \delta$, using the modified restricted strong convexity on non-convex $G(\theta)$ (Lemma [10] in the appendix), we obtain $\|\theta^k - \hat{\theta}\|_2 \leq c_3 \delta$.

• The requirement of $\lambda$ is similar to the batched gradient algorithm [Loh and Wainwright 2013].

Again, the theorem says the Lyapunov function decreases geometrically until $G(\theta^k) - G(\hat{\theta})$ achieves the tolerance $\delta$ and this tolerance can be ignored from the statistical perspective.

### 3.2.1 Linear regression with SCAD regularization

The first non-convex model we considered is linear regression with SCAD regularization.

The loss function is $f_i(w) = \frac{1}{2}(y - \langle \theta, x_i \rangle)^2$, and $g_{\lambda, \mu}(\cdot)$ is SCAD$(\cdot)$ with parameter $\lambda$ and $\zeta$. The data $(x_i, y_i)$ are generated in the similar way as that in Lasso case.

**Corollary 3.** (Linear regression with SCAD regularization) Suppose $\theta^*$ is the true parameter supported on a subset with cardinality at most $r$, $\hat{\theta}$ is the global optimum, $\bar{\sigma} \geq \frac{3}{\varsigma} \frac{1}{1}$, $L > \frac{3}{\varsigma} \frac{1}{1}$ and we choose $\lambda$ such that $\lambda \geq \max \{c_1 \rho \nu(\Sigma) \frac{\log p}{n}, 12 \kappa \sqrt{\frac{\log p}{n}} \}$ then with $\gamma = \frac{1}{2 \kappa}$, $c = \frac{24 L}{n}$, $\alpha = c$, $b = 2 \alpha \gamma$, $\frac{1}{\kappa} = \frac{1}{24} \min \left( \frac{2 \sigma}{\kappa}, \frac{1}{n} \right)$, we have

$$ET_k \leq \left( 1 - \frac{1}{\kappa} \right)^k T_0,$$

with probability at least $1 - \exp(-3 \log p) - \exp(-c_2 n)$, until $G(\theta^k) - G(\hat{\theta}) \leq \delta$, where $\bar{\sigma} = \frac{1}{2} \sigma_{\min}(\Sigma) - c_3 \nu(\Sigma) \frac{\log p}{n} - \frac{1}{\varsigma - 1}$, $\delta = c_4 \nu(\Sigma) \frac{\log p}{n} \| \theta - \theta^* \|_2^2$. Here $c_1, c_2, c_3, c_4$ are some universal positive constants.

We remark that to satisfy the requirement $\bar{\sigma} \geq \frac{3}{\varsigma} \frac{1}{1}$, we need the non-convex parameter $\mu = \frac{1}{\varsigma - 1}$ to be small, the model sparse ($r$ is small) and the number of sample $n$ large.

### 3.2.2 Linear regression with noisy covariates

The **corrected Lasso** is proposed by [Loh and Wainwright 2011]. Suppose data are generated according to a linear model $y_i = x_i^T \theta^* + \xi_i$, where $\xi_i$ is a random zero-mean sub-Gaussian noise with variance $\varsigma^2$. The observation $z_i$ of $x_i$ is corrupted by additive noise, in particular, $z_i = x_i + w_i$, where $w_i \in \mathbb{R}^p$ is a random vector independent of $x_i$, with zero-mean and known covariance matrix $\Sigma_w$. Define $\hat{\Gamma} = \frac{Z^T Z}{n} - \Sigma_w$ and $\hat{\gamma} = \frac{Z^T \mu}{n}$. Our goal is to estimate $\theta^*$ based on $y_i$ and $z_i$ (but not $x_i$ which is not observable), and the corrected Lasso proposes to solve the following:

$$\hat{\theta} \in \arg \min_{\|\theta\|_1 \leq \rho} \frac{1}{2} \theta^T \hat{\Gamma} \theta - \hat{\gamma} \theta + \lambda \|\theta\|_1.$$

Equivalently, it solves

$$\min_{\|\theta\|_1 \leq \rho} \frac{1}{2n} \sum_{i=1}^n (y_i - \theta^T z_i)^2 - \frac{1}{2} \theta^T \Sigma_w \theta + \lambda \|\theta\|_1.$$

Notice that due to the term $-\frac{1}{2} \theta^T \Sigma_w \theta$, the optimization problem is non-convex.
3.3 Corrected Lasso

We consider a model where each data point $x_i$ is i.i.d. sampled from a zero-mean normal distribution, i.e., $x_i \sim N(0, \Sigma)$. We denote the data matrix by $X \in \mathbb{R}^{n \times p}$, the smallest eigenvalue of $\Sigma$ by $\sigma_{\min}(\Sigma)$ and the largest eigenvalue by $\sigma_{\max}(\Sigma)$ and let $\nu(\Sigma) \triangleq \max_{i=1, \ldots, p} \Sigma_{ii}$. We observe $z_i$ which is $x_i$ corrupted by additive noise, i.e., $z_i = x_i + w_i$, where $w_i \in \mathbb{R}^p$ is a random vector independent of $x_i$, with zero-mean and known covariance matrix $\Sigma_w$.

**Corollary 4.** (Corrected Lasso) Suppose we are given i.i.d. observations $\{(z_i, y_i)\}$ from the linear model with additive noise, $\theta^* \approx s$ sparse and $\Sigma_w = \gamma_w I$, $\bar{\sigma} > 3\gamma_w$, $L \geq 3\gamma_w$ where $\bar{\sigma} = \frac{1}{2}\sigma_{\min}(\Sigma) - c_1\sigma_{\min}(\Sigma) \max\left(\frac{\sigma_{\max}(\Sigma) + \gamma_w}{\sigma_{\min}(\Sigma)}, 1\right) \frac{r \log p}{n}$. Let $\hat{\theta}$ be the global optimum.

We choose $\lambda \geq \max\{c_2\rho \frac{\log n}{n}, c_3\varphi \sqrt{\frac{\log n}{p}}\}$ where $\varphi = (\sqrt{\sigma_{\max}(\Sigma)} + \sqrt{\gamma_w})(c + \sqrt{\gamma_w}\|\theta^*\|_2)$, then with $\gamma = \frac{1}{11L}$, $c = \frac{24L}{5\sqrt{5}}$, $\alpha = \frac{2}{7}$, $b = 2\alpha c$, $\frac{1}{n} = \frac{1}{24}\min\left(\frac{2\varphi}{\rho L^2}, \frac{1}{c}\right)$, we have

$$ET_k \leq (1 - \frac{1}{\kappa})^k I_0,$$

with high probability at least $1 - c_4 \exp\left(-c_5 n \min\left(\frac{\sigma_{\min}(\Sigma)}{\sigma_{\max}(\Sigma)}\frac{\rho^2}{\gamma_w^2}, 1\right)\right) - \exp(-c_6 \log p)$ until $G(\theta^k) - G(\hat{\theta}) \leq \delta$, where $\delta = c_7\sigma_{\min}(\Sigma) \max\left(\frac{\sigma_{\max}(\Sigma) + \gamma_w}{\sigma_{\min}(\Sigma)}, 1\right) \frac{r \log p}{n} \|\hat{\theta} - \theta^*\|_2^2$. $c_1$ to $c_7$ are some universal positive constants.

Some remarks are listed below.

- The result can be easily extended to more general $\Sigma_w \preceq \gamma_w I$.
- To satisfy the requirement $\bar{\sigma} > 3\gamma_w$, we need $\gamma \leq \frac{1}{4}\left(\frac{1}{2}\sigma_{\min}(\Sigma) - c_1\sigma_{\min}(\Sigma) \max\left(\frac{\sigma_{\max}(\Sigma) + \gamma_w}{\sigma_{\min}(\Sigma)}, 1\right) \frac{r \log p}{n}\right)$.

Similar requirement is needed in the batch gradient method [Loh and Wainwright 2013].
- The requirement of $\lambda$ is similar to that in batch gradient method [Loh and Wainwright 2013].

3.4 Extension to Generalized linear model

The results on Lasso and group Lasso are readily extended to generalized linear models, where we consider the model

$$\hat{\theta} = \arg\min_{\theta \in \Omega} \left\{\frac{1}{n} \sum_{i=1}^{n} \Phi(\theta, x_i) - y_i(\theta, x_i) + \lambda \|\theta\|_1\right\},$$

with $\Omega' = \Omega \cap B_2(R)$ and $\Omega = \{\theta \|\theta\|_1 \leq \rho\}$, where $R$ is a universal constant [Loh and Wainwright 2013]. This requirement is essential, for instance for the logistic function, the Hessian function $\Phi''(t) = \frac{\exp(t)}{(1+\exp(t))^2}$ approached to zero as its argument diverges. Notice that when $\Phi(t) = t^2/2$, the problem reduces to Lasso. The RSC condition admit the form

$$\frac{1}{n} \sum_{i=1}^{n} \Phi''(\langle \theta_i, x_i \rangle)(x_i, \theta - \theta')^2 \geq \frac{\sigma^2}{2} \|\theta - \theta'\|_2^2 - \sigma\|\theta - \theta'\|_1, \text{for all } \theta, \theta' \in \Omega'.$$
For a board class of log-linear models, the RSC condition holds with \( \tau = C \frac{\log p}{n} \). Therefore, we obtain same results as those of Lasso, modulus change of constants. For more details of RSC conditions in generalized linear model, we refer the readers to Negahban et al. [2009].

4 Empirical Result

We report the experimental results in this section to validate our theorem that SAGA can enjoy the linear convergence rate without strong convexity or even without convexity. We did experiment both on synthetic and real datasets and compare SAGA with several candidate algorithms. The experiment setup is similar to Qu et al. [2016]. Due to space constraints, some addition simulation results are presented in the appendix. The algorithms tested are Prox-SVRG [Xiao and Zhang 2014], Prox-SAG which is a proximal version of the algorithm in Schmidt et al. [2013], proximal dual averaging method (RDA) [Xiao 2010], and the proximal full gradient method (Prox-GD) [Nesterov 2013]. For the algorithms with a constant learning rate (i.e., SAGA,Prox-SAG, Prox-SVRG, Prox-GD), we tune the learning rate from an exponential grid \( \{2, 2^{\frac{1}{2}}, ..., 2^{\frac{12}{12}}\} \) and chose the one with best performance. Below are some remarks on the candidate algorithms.

- The linear convergence of SVRG in our setting has been proved in Qu et al. [2016].
- We adapt SAG to its Prox version. To the best of our knowledge, the convergence of Prox-SAG has not been established. In addition, it is not known whether the Prox-SAG converges or not although it works well in the experiment.
- The step size in Prox-SGD is \( \eta_k = \eta_0 / \sqrt{k} \). The step size for RDA is \( \beta_k = \beta_0 \sqrt{k} \) suggested in Xiao [2010]. \( \beta_0 \) and \( \eta_0 \) are chosen from exponential grid (with power of 10) with the best performance.

4.1 Synthetic data

We report the experimental result on Lasso, Group Lasso, Linear regression with SCAD regularization and Corrected Lasso.

4.1.1 Lasso

The feature vector \( x_i \in \mathbb{R}^p \) are drawn independently from \( N(0, \Sigma) \), where we set \( \Sigma_{ii} = 1 \), for \( i = 1, ..., p \) and \( \Sigma_{ij} = b, \) for \( i \neq j \). The responds \( y_i \) is generated as follows: \( y_i = x_i^T \theta^* + \xi_i \), and \( \theta^* \in \mathbb{R}^p \) is a sparse vector with cardinality \( r \), where the non-zero entries are \( \pm 1 \) drawn from the Bernoulli distribution with probability 0.5. The noise \( \xi_i \) follows the standard normal distribution. The parameter of regularizer is set to be \( \lambda = 0.05 \). We set \( p = 5000, n = 2500 \) and vary the value on \( r \) and \( b \). The results are shown in Figure 1.

Figure 1 demonstrates that SAGA, Prox-SVRG and Prox-SAG enjoy a linear convergence rate in all settings. In the most challenging setup (\( r = 100, b = 0.4 \)), SAGA outperforms Prox-SVRG and Prox-SAG. The batched method, Prox-GD converges linearly when \( b = 0 \) and does not work well when \( b = 0.1 \) and \( b = 0.4 \). It is possibly because the condition number is large when \( b \neq 0 \). We also observe that SAGA with sparser \( r \) converges faster, which matches our Theorem 1. As we discussed in the remarks of Theorem 1, \( \frac{1}{\kappa} \) depends on \( \bar{\sigma} / L \) and smaller \( r \) cause larger \( \bar{\sigma} \) thus faster convergence rate.
after 200 iterations, convergence rate but with a slower ratio. SGD and RDA have a large optimality gap even and Prox-SAG work well and have similar performance. Prox-GD also enjoys the linear choose $N$ is sparse with cardinality $r$ and the Bernoulli distribution with parameter 0.

### 4.1.3 Corrected Lasso

Figure 1: Comparison between six algorithms on Lasso. The x-axis is the number of passes over the dataset, and the y-axis is the objective gap $G(\theta^k) - G(\hat{\theta})$ with a log scale.

In all settings, SAGA, Prox-SVRG and Prox-SAG performs well. In the challenging setup ($m = 20, s_G = 20$), SAGA outperforms the other two. Prox-GD work with slower rate in the setting ($m = 10, s_G = 10, b = 0$), while its performance deteriorates in other three settings. Prox-GD and RDA have large optimality gap even after long time running. We have similar observation as that in Lasso, i.e., smaller $m$ and $s_G$ lead to faster convergence. Again, it can be explained by the dependence of $\bar{\sigma}$ on $m$ and $s_G$.

### 4.1.2 Group Lasso

We generate the observation $y_i = x_i^T \theta^* + \xi_i$ with the feature vectors independently sampled from $N(0, \Sigma)$, where $\Sigma_{ii} = 1$ and $\Sigma_{ij} = b, i \neq j$. The cardinality of non-zero group is $s_G$, and the non-zero entries are sampled uniformly from $[-1, 1]$. We vary the values of $b$, group size $m$ and group sparsity $s_G$ and report the results in Figure 2.

In all settings, SAGA, Prox-SVRG and Prox-SAG performs well. In the challenging setup ($m = 20, s_G = 20$), SAGA outperforms the other two. Prox-GD work with slower rate in the setting ($m = 10, s_G = 10, b = 0$), while its performance deteriorates in other three settings. Prox-GD and RDA have large optimality gap even after long time running. We have similar observation as that in Lasso, i.e., smaller $m$ and $s_G$ lead to faster convergence. Again, it can be explained by the dependence of $\bar{\sigma}$ on $m$ and $s_G$.

### 4.1.3 Corrected Lasso

We generate data as follows: $y_i = x_i^T \theta^* + \xi_i$, where each data point $x_i \in \mathbb{R}^p$ is drawn from normal distribution $N(0, I)$, and the noise $\xi_i$ is drawn from $N(0, 1)$. The coefficient $\theta^*$ is sparse with cardinality $r$, where the non-zero coefficient equals to $\pm 1$ generated from the Bernoulli distribution with parameter 0.5. We set covariance matrix $\Sigma_w = \gamma_w I$. We choose $\lambda = 0.05$ in the formulation.

Figure 3 reports the result on Corrected Lasso. In both settings, SAGA, Prox-SVRG and Prox-SAG work well and have similar performance. Prox-GD also enjoys the linear convergence rate but with a slower ratio. SGD and RDA have a large optimality gap even after 200 iterations,
Figure 3: The x-axis is the number of pass over the dataset. y-axis is the objective gap $G(\theta^k) - G(\bar{\theta})$ with log scale. We try two different settings.

4.1.4 SCAD

The way to generate data is same with Lasso. Here $x_i \in \mathbb{R}^p$ is drawn from normal distribution $N(0, 2I)$ (Here We choose $2I$ to satisfy the requirement of $\bar{\sigma}$ and $\mu$ in our Theorem, although if we choose $N(0, I)$, the algorithm still works. ). $\lambda = 0.05$ in the formulation. We present the result in Figure 4 for two settings on $n, p, r, \zeta$.

Figure 4: The x-axis is the number of pass over the dataset. y-axis is the objective gap $G(\theta^k) - G(\bar{\theta})$ with log scale.

Figure 4 reports the simulation result on linear regression with SCAD regularizer. It is easy to see SAGA, Prox-SVRG and Prox-SAG works well, followed by Prox-GD. RDA and Prox-SGD does not converge well.
4.2 Real datasets

4.2.1 Sparse Classification Problem

In this section, we evaluate the performance of the algorithms when solving the logistic regression with $\ell_1$ regularization:

$$
\min_{\theta} \sum_{i=1}^{n} \log(1 + \exp(-y_i x_i^T \theta)) + \lambda \|\theta\|_1.
$$

We conduct experiments on two real-world data sets: sido0 ($n=12678$, $p=4932$) and rcv1 ($n=20242$, $p=47236$). The regularization parameters are set as $\lambda = 2 \cdot 10^{-5}$ in rcv1 and $\lambda = 10^{-4}$ in sido0, as suggested in Xiao and Zhang [2014].

Figure 5a shows the results of the algorithms on the sido0 Guyon [2008] dataset. On this dataset, SAGA performs best and then followed by Prox-SAG (some part are overlapped with Prox-SVRG) and then Prox-SVRG. The performance of Prox-GD is even worse than Prox-SGD. RDA converges the slowest. In Figure 5b, we report the performance of different algorithms on rcv1 dataset Lewis et al. [2004]. In this problem, Prox-SVRG performs best, and followed by SAGA, and then Prox-SAG. We observe that Prox-GD converges much slower, albeit in theory it should converge with a linear rate Agarwal et al. [2010], possibly because its contraction factor is close to one in this case. Prox-SGD and RDA converge slowly due to the variance in the stochastic gradient. The objective gaps of them remain significant even after 1000 passes of the whole dataset.

4.2.2 Sparse Regression Problem

In this section, we consider regression problem on three different problems, namely Lasso, linear regression with SCAD regularization and Group Lasso and report the results in Figure 6. For Lasso and linear regression with SCAD regularization and Group Lasso and report the results in Figure 6. For Lasso and linear regression with SCAD regularization and Group Lasso and report the results in Figure 6. For Lasso and linear regression with SCAD regularization and Group Lasso and report the results in Figure 6.

We test all algorithms on IJCNN1 dataset ($n=49990$, $p=22$) Prokhorov [2001]. In particular, we set $\lambda = 0.02$ in Lasso formulation and $\lambda = 0.02$ and $\zeta = 5$ in linear regression with SCAD regularization. As to the group sparse regression problem, we conduct the experiment the Boston Housing dataset ($n=506$, $p=13$) Harrison and Rubinfeld [2013]. As suggested in
Figure 6: Six different algorithms on Lasso (left), linear regression with SCAD regularization (middle) and Group Lasso (right). The x-axis is the number of pass over the dataset, y-axis is the objective gap in the log-scale.

Swirszcz et al. [2009], Xiang et al. [2014], to take into account the non-linear relationship between variables and response, up to third-degree polynomial expansion is applied on each feature. In particular, terms $x$, $x^2$ and $x^3$ are grouped together. We consider group Lasso model on this problem with $\lambda = 0.1$.

It is easy to see that for the Lasso problem, SAGA, Prox-SAG and Prox-SVRG have almost identical performance, and Prox-GD converges with linear rate but with slower rate. As to linear regression with SCAD regularization, SAGA performs best in this dataset and then followed by Prox-SVRG, Prox-SAG and Prox-GD. For both problems, Prox-SGD converges faster at the beginning but quickly slows down and eventually has a large optimality gap, possibly due to the variance in the gradient estimation. RDA seems does not work (for both Lasso and SCAD) in this dataset. In Group Lasso, SAGA, Prox-SVRG and prox-SAG have almost same performance. RDA and Prox-GD converge slowly. Prox-SGD does not converge and its value oscillates between 0.1 and 1.

5 Conclusion and future work

In this paper, we analyze SAGA on a class of non-strongly convex and non-convex problem and provide linear convergence analysis under the RSC condition.

A Proofs

In this section, we give the proof to all theorems and corollaries

A.1 SAGA with convex objective function

We start the proof with some technical Lemmas.

The following lemma is the theorem 2.1.5 in Nesterov [1998].

**Lemma 1.** if $f(\theta)$ is convex and $L$ smooth, then we have

$$\|\nabla f(\theta_1) - \nabla f(\theta_2)\|^2 \leq 2L[f(\theta_1) - f(\theta_2) - \langle \nabla f(\theta_2), \theta_1 - \theta_2 \rangle]$$

The next lemma is a simple extension of a standard property proximal operator with a constraint $\Omega$. It is indeed the Lemma 5 in Ou et al. [2010], and we present here for completeness.
Lemma 2. Define $\text{prox}_{h,\Omega}(x) = \arg\min_{z \in \Omega} h(z) + \frac{1}{2}\|z - x\|^2$, where $\Omega$ is a convex compact set, then $\|\text{prox}_{h,\Omega}(x) - \text{prox}_{h,\Omega}(y)\| \leq \|x - y\|$. The following two lemmas are similar to its batched counterpart in Agarwal et al. [2010].

Lemma 3. Suppose that $f(\theta)$ is convex and $\psi(\theta)$ is decomposable with respect to $(M, \bar{M})$, if we choose $\lambda \geq 2\psi^*(\nabla f(\theta^*))$, $\psi(\theta^*) \leq \rho$, define the error $\Delta^* = \hat{\theta} - \theta^*$, then we have the following condition holds,

$$\psi(\Delta^*_{\perp}) \leq 3\psi(\Delta^*_M) + 4\psi(\theta^*_{M\perp}),$$

which further implies $\psi(\Delta^*) \leq \psi(\Delta^*_{M\perp}) + \psi(\Delta^*_M) \leq 4\psi(\Delta^*_M) + 4\psi(\theta^*_{M\perp})$.

Proof. Using the optimality of $\hat{\theta}$, we have

$$f(\hat{\theta}) + \lambda \psi(\hat{\theta}) - f(\theta^*) - \lambda \psi(\theta^*) \leq 0.$$

So we have

$$\lambda \psi(\theta^*) - \lambda \psi(\hat{\theta}) \geq f(\hat{\theta}) - f(\theta^*) \geq \langle \nabla f(\theta^*), \hat{\theta} - \theta^* \rangle \geq -\psi^*(\nabla f(\theta^*))\psi(\Delta^*),$$

where the second inequality holds from the convexity of $f(\theta)$, and the third holds using Holder inequality.

Using triangle inequality, we have

$$\psi(\Delta^*) \leq \psi(\Delta^*_M) + \psi(\Delta^*_{M\perp}).$$

So

$$\lambda \psi(\theta^*) - \lambda \psi(\hat{\theta}) \geq -\psi^*(\nabla f(\theta^*))\psi(\Delta^*_M) + \psi(\Delta^*_{M\perp}) \quad (4)$$

Notice

$$\hat{\theta} = \theta^* + \Delta^* = \theta^*_M + \theta^*_M + \Delta^*_M + \Delta^*_{M\perp},$$

which leads to

$$\psi(\Delta^*) \leq \psi(\Delta^*_M) + \psi(\Delta^*_{M\perp}).$$

where (a) and (c) holds from the triangle inequality, (b) uses the decomposability of $\psi(\cdot)$.

Substitute left hand side of (4) by above result and use the assumption that $\lambda \geq 2\psi^*(\nabla f(\theta^*))$, we have

$$-\frac{\lambda}{2}(\psi(\Delta^*_M) + \psi(\Delta^*_{M\perp})) + \lambda(\psi(\Delta^*_M) - 2\psi(\theta^*_M) - \psi(\Delta^*_M)) \leq 0$$

which implies

$$\psi(\Delta^*_{M\perp}) \leq 3\psi(\Delta^*_M) + 4\psi(\theta^*_{M\perp}).$$
Lemma 4. \( f(\theta) \) is convex and \( \psi(\theta) \) is decomposable with respect to \((M, \hat{M})\), if we choose \( \lambda \geq 2\psi^*(\nabla f(\theta^*)) \), \( \psi(\theta^*) \leq \rho \) and suppose there exist a time step \( K > 0 \) and a given tolerance \( \epsilon \) such that for all \( k > K \), \( G(\theta^k) - G(\hat{\theta}) \leq \epsilon \) holds, then for the error \( \Delta^k = \theta^k - \theta^* \) we have

\[
\psi(\Delta^k_{M\perp}) \leq 3\psi(\Delta^k_M) + 4\psi(\theta^*_{M\perp}) + 2\min\left\{ \frac{\epsilon}{\lambda}, \rho \right\},
\]

which implies

\[
\psi(\Delta^k) \leq 4\psi(\Delta^k_M) + 4\psi(\theta^*_{M\perp}) + 2\min\left\{ \frac{\epsilon}{\lambda}, \rho \right\}.
\]

Proof. First notice \( G(\theta^k) - G(\theta^*) \leq \epsilon \) holds by assumption since \( G(\theta^*) \geq G(\hat{\theta}) \). So we have

\[
f(\theta^k) + \lambda \psi(\theta^k) - f(\theta^*) - \lambda \psi(\theta^*) \leq \epsilon.
\]

Follow same steps in the proof of Lemma 3 we have

\[
\psi(\Delta^k_{M\perp}) \leq 3\psi(\Delta^k_M) + 4\psi(\theta^*_{M\perp}) + 2\frac{\epsilon}{\lambda}.
\]

Notice \( \Delta^k = \Delta^k_{M\perp} + \Delta^k_M \) so \( \psi(\Delta^k_{M\perp}) \leq \psi(\Delta^k_M) + \psi(\Delta^k) \) using the triangle inequality. Then use the fact that \( \psi(\Delta^k) \leq \psi(\theta^*) + \psi(\theta^k) \leq 2\rho \), we establish

\[
\psi(\Delta^k_{M\perp}) \leq 3\psi(\Delta^k_M) + 4\psi(\theta^*_{M\perp}) + 2\min\left\{ \frac{\epsilon}{\lambda}, \rho \right\}.
\]

The second statement follows immediately from \( \psi(\Delta^k) \leq \psi(\Delta^k_{M\perp}) + \psi(\Delta^k_M) \). \( \square \)

Using the above two lemmas we now prove modified restricted convexity.

Lemma 5. Under the same assumptions of Lemma 4, we have

\[

\begin{align*}
\langle \nabla f(\theta^k) - \nabla f(\hat{\theta}), \theta^k - \hat{\theta} \rangle & \geq (\sigma - 64\tau_\sigma H^2(\hat{M})) \| \Delta^k_{M\perp} \|_2^2 - 2\epsilon^2(\theta^*, M, \hat{M}) \quad (6) \\
G(\theta^k) - G(\hat{\theta}) & \geq \left( \frac{\sigma}{2} - 32\tau_\sigma H^2(\hat{M}) \right) \| \Delta^k \|_2^2 - \epsilon^2(\theta^*, M, \hat{M}), \quad (7)
\end{align*}

\]

where \( \epsilon^2(\theta^*, M, \hat{M}) = 2\tau_\sigma(\delta_{stat} + \delta)^2, \delta = 2\min\{\frac{\epsilon}{\lambda}, \rho\}, \) and \( \delta_{stat} = 8H(\hat{M})\| \Delta^* \|_2 + 8\psi(\theta^*_{M\perp}) \).

Proof. At the beginning of the proof, we show a simple fact on \( \hat{\Delta}^k = \theta^k - \hat{\theta} \). Notice the conclusion in Lemma 4 is on \( \Delta^k \), we need transfer it to \( \hat{\Delta}^k \).

\[
\psi(\hat{\Delta}^k) \leq \psi(\Delta^k) + \psi(\theta^*)
\]

\[
\leq 4\psi(\Delta^k_M) + 4\psi(\theta^*_{M\perp}) + 2\min\left\{ \frac{\epsilon}{\lambda}, \rho \right\} + 4\psi(\Delta^k_M) + 4\psi(\theta^*_{M\perp})
\]

\[
\leq 4H(\hat{M})\| \Delta^k \|_2^2 + 8H(\hat{M})\| \Delta^* \|_2 + 8\psi(\theta^*_{M\perp}) + 2\min\left\{ \frac{\epsilon}{\lambda}, \rho \right\},
\]

where the first inequality holds from the triangle inequality, the second inequality uses Lemma 3 and 4, the third holds because of the definition of subspace compatibility.

We now use the above result to rewrite the RSC condition. We know

\[
f(\theta^k) - f(\hat{\theta}) - \langle \nabla f(\hat{\theta}), \hat{\Delta}^k \rangle \geq \frac{\sigma}{2} \| \hat{\Delta}^k \|_2^2 - \tau_\sigma \psi^2(\hat{\Delta}^k)
\]
and
\[ f(\hat{\theta}) - f(\theta^k) - \langle \nabla f(\theta^k), -\hat{\Delta}^k \rangle \geq \frac{\sigma}{2} \| \hat{\Delta}^k \|_2^2 - \tau_\sigma \psi^2(\hat{\Delta}^k). \]

Add above two together, we get
\[ \langle \nabla f(\theta^k) - \nabla f(\hat{\theta}), \theta^k - \hat{\theta} \rangle \geq \sigma \| \hat{\Delta}^k \|_2^2 - 2\tau_\sigma \psi^2(\hat{\Delta}^k) \tag{10} \]

Notice that
\[
\psi(\hat{\Delta}^k) \leq 4H(\tilde{M})\|\hat{\Delta}^k\|_2 + 4H(\tilde{M})\|\Delta^*\|_2 + 8\psi(\theta^*_{M^\perp}) + 2\min\{\frac{\epsilon}{\lambda}, \rho\}
\]
\[
\leq 4H(\tilde{M})\|\hat{\Delta}^k\|_2 + 8H(\tilde{M})\|\Delta^*\|_2 + 8\psi(\theta^*_{M^\perp}) + 2\min\{\frac{\epsilon}{\lambda}, \rho\},
\tag{11}
\]

where the second inequality uses the triangle inequality. Now use the inequality \((a+b)^2 \leq 2a^2 + 2b^2\), we upper bound \(\psi^2(\hat{\Delta}^k)\) with
\[
\psi^2(\hat{\Delta}^k) \leq 32H^2(\tilde{M})\|\hat{\Delta}^k\|_2^2 + 2 \left(8H(\tilde{M})\|\Delta^*\|_2 + 8\psi(\theta^*_{M^\perp}) + 2\min\{\frac{\epsilon}{\lambda}, \rho\}\right)^2.
\]

Substitute this upper bound into Equation (10), we have
\[
\langle \nabla f(\theta^k) - \nabla f(\hat{\theta}), \theta^k - \hat{\theta} \rangle \geq \left(\sigma - 64\tau_\sigma H^2(\tilde{M})\right)\|\hat{\Delta}^k\|_2^2 - 2\epsilon^2(\Delta^*, M, \tilde{M}) \tag{12}
\]

Notice that by \(\delta = 2\min\{\frac{\epsilon}{\lambda}, \rho\}, \delta_{stat} = 8H(\tilde{M})\|\Delta^*\|_2 + 8\psi(\theta^*_{M^\perp}),\) and \(\epsilon^2(\Delta^*, M, \tilde{M}) = 2\tau_\sigma (\delta_{stat} + \delta)^2\), we have
\[
\epsilon^2(\Delta^*, M, \tilde{M}) = 2\tau_\sigma \left(8H(\tilde{M})\|\Delta^*\|_2 + 8\psi(\theta^*_{M^\perp}) + 2\min\{\frac{\epsilon}{\lambda}, \rho\}\right)^2.
\]

We thus establish the result.

\[ \langle \nabla f(\theta^k) - \nabla f(\hat{\theta}), \theta^k - \hat{\theta} \rangle \geq \sigma - 64\tau_\sigma H^2(\tilde{M}) \| \hat{\Delta}^k \|_2^2 - 2\epsilon^2(\Delta^*, M, \tilde{M}) \]  

Using Equation (3) and the fact that \(\hat{\theta}\) is the optimal solution and \(\phi(\cdot)\) is convex, we obtain \(G(\theta^k) - G(\hat{\theta}) \geq \frac{\sigma}{2} \| \hat{\Delta}^k \|_2^2 - \tau_\sigma \psi^2(\hat{\Delta}^k)\). We substitute the upper bound of \(\psi^2(\hat{\Delta}^k)\), and get
\[ G(\theta^k) - G(\hat{\theta}) \geq \left(\frac{\sigma}{2} - 32\tau_\sigma H^2(\tilde{M})\right) \| \hat{\Delta}^k \|_2^2 - 2\tau_\sigma \left(8H(\tilde{M})\|\Delta^*\|_2 + 8\psi(\theta^*_{M^\perp}) + 2\min\{\frac{\epsilon}{\lambda}, \rho\}\right)^2.
\]

That is
\[ G(\theta^k) - G(\hat{\theta}) \geq \left(\frac{\sigma}{2} - 32\tau_\sigma H^2(\tilde{M})\right) \| \hat{\Delta}^k \|_2^2 - \epsilon^2(\Delta^*, M, \tilde{M}). \tag{13} \]

\[ \Box \]

**Lemma 6.** Under the same assumption of Lemma 4, we have

\[ \langle \nabla f(\theta^k) - \nabla f(\hat{\theta}), \theta^k - \hat{\theta} \rangle \geq \frac{1}{2} \| f(\theta^k) - f(\hat{\theta}) - \langle \nabla f(\hat{\theta}), \theta^k - \hat{\theta} \rangle \|_2^2 + \frac{1}{4L} \| \nabla f(\theta^k) - \nabla f(\hat{\theta}) \|_2^2 - \epsilon^2(\Delta^*, M, \tilde{M}). \tag{14} \]
Proof.

\[
\langle \nabla f(\theta^k) - \nabla f(\hat{\theta}), \theta^k - \hat{\theta} \rangle - \frac{1}{2} [f(\theta^k) - f(\hat{\theta}) - \langle \nabla f(\hat{\theta}), \theta^k - \hat{\theta} \rangle] \\
= \frac{1}{2} \langle \nabla f(\theta^k) - \nabla f(\hat{\theta}), \theta^k - \hat{\theta} \rangle + \frac{1}{2} [f(\hat{\theta}) - f(\theta^k) - \langle \nabla f(\theta^k), \hat{\theta} - \theta^k \rangle].
\] (15)

We use the modified RSC condition and the smoothness of \( f \) in the proof, in particular, we have following holds from Lemma 3 and Lemma 1

\[
\langle \nabla f(\theta^k) - \nabla f(\hat{\theta}), \theta^k - \hat{\theta} \rangle \geq \bar{\sigma} \| \theta^k - \hat{\theta} \|_2^2 - 2c^2(\Delta^*, M, M),
\]

\[
\| \nabla f(\theta^k) - \nabla f(\hat{\theta}) \|_2^2 \leq 2L[f(\hat{\theta}) - f(\theta^k) - \langle \nabla f(\theta^k), \hat{\theta} - \theta^k \rangle].
\]

Substitute above bound in the right hand side of (15) we establish the result. \( \square \)

The following Lemma is indeed Lemma 7. We present here for the completeness.

Lemma 7. Define \( \Delta = -\frac{1}{\gamma} (w^{k+1} - \theta^k) - \nabla f(\theta^k) \), It holds that for any \( \phi^k_i, \hat{\theta}, \theta^k \) and \( \beta > 0 \), we have

\[
E \| w^{k+1} - \theta^k - \gamma \nabla f(\hat{\theta}) \|_2^2 \leq \gamma^2 (1 + \beta^{-1}) E \| \nabla f_j(\phi^k_i) - \nabla f_j(\hat{\theta}) \|_2^2 \\
+ \gamma^2 (1 + \beta) E \| \nabla f_j(\theta^k) - \nabla f_j(\hat{\theta}) \|_2^2 - \gamma^2 \beta \| \nabla f(\theta^k) - \nabla f(\hat{\theta}) \|_2^2.
\]

and

\[
E \| \Delta \|_2^2 \leq (1 + \beta^{-1}) E \| \nabla f_j(\phi^k_i) - \nabla f_j(\hat{\theta}) \|_2^2 + (1 + \beta) E \| \nabla f_j(\theta^k) - \nabla f_j(\hat{\theta}) \|_2^2.
\]

Now we are ready to prove Theorem 1.

Proof of Theorem Recall that we aim to prove that Lyapunov function \( T_k \) converges geometrically until \( G(\theta^k) - G(\hat{\theta}) \) achieves tolerance related to statistical error. Recall the definition of \( T_k \) is

\[
T_k \triangleq \frac{1}{n} \sum_{i=1}^{n} \left( f_i(\phi^k_i) - f_i(\hat{\theta}) - \langle \nabla f_i(\hat{\theta}), \phi^k_i - \hat{\theta} \rangle \right) + (c + \alpha) \| \theta^k - \hat{\theta} \|_2^2 + b(G(\theta^k) - G(\hat{\theta})).
\]

Now we bound \( E T_{k+1} \).

1. The first term on the right hand side of \( E T_{k+1} \)

It is easy to obtain

\[
E \left[ \frac{1}{n} \sum_{i=1}^{n} f_i(\phi^{k+1}_i) \right] = \frac{1}{n} f(\theta^k) + (1 - \frac{1}{n}) \frac{1}{n} \sum_{i=1}^{n} f_i(\phi^k_i).
\]

\[
E \left[ \frac{1}{n} \sum_{i=1}^{n} \langle \nabla f_i(\hat{\theta}), \phi^{k+1}_i - \theta \rangle \right] = \frac{1}{n} \langle \nabla f(\hat{\theta}), \theta^k - \hat{\theta} \rangle - (1 - \frac{1}{n}) \frac{1}{n} \sum_{i=1}^{n} \langle \nabla f_i(\hat{\theta}), \phi^k_i - \hat{\theta} \rangle.
\]

2. The second term \( (c + \alpha) E \| \theta^{k+1} - \hat{\theta} \|_2^2 \)

Notice we bound the term \( c E \| \theta^{k+1} - \hat{\theta} \|_2^2 \) and \( \alpha E \| \theta^{k+1} - \hat{\theta} \|_2^2 \) in different ways.
As for the term $cE\|\theta^{k+1} - \hat{\theta}\|^2$, we have following bound.

$$E\|\theta^{k+1} - \hat{\theta}\|^2 \leq E\|\text{prox}_{\psi, \Omega}(w^{k+1}) - \text{prox}_{\psi, \Omega}(\hat{\theta} - \gamma \nabla f(\hat{\theta}))\|^2$$

$$(a) \leq E\|w^{k+1} - \hat{\theta} + \gamma \nabla f(\hat{\theta})\|^2$$

$$= \|w^{k+1} - \hat{\theta}\|^2 + 2\E[(w^{k+1} - \theta^k + \gamma \nabla f(\hat{\theta}), \theta^k - \hat{\theta})] + \E\|w^{k+1} - \theta^k + \gamma \nabla f(\hat{\theta})\|^2$$

$$(b) \leq \|w^{k+1} - \theta^k + \gamma \nabla f(\hat{\theta})\|^2$$

$$+ (1 + \beta^{-1})\gamma^2 E\|\nabla f_j(\phi^k_i) - \nabla f_j(\hat{\theta})\|^2 + (1 + \beta)\gamma^2 E\|\nabla f_j(\hat{\theta}) - \nabla f_j(\hat{\theta})\|^2,$$

where (a) holds from the non-expansiveness of the proximal operator, i.e., Lemma 2 and (b) holds from the fact that $E[w^{k+1}] = \theta^k - \gamma \nabla f(\theta^k)$, (c) uses Lemma 7.

Now we apply Lemma 10 on $\langle \nabla f(\theta^k) - \nabla f(\hat{\theta}), \theta^k - \hat{\theta} \rangle$ and obtain

$$E\|\theta^{k+1} - \hat{\theta}\|^2 \leq (1 - \gamma \alpha)\|\theta^k - \hat{\theta}\|^2 - 2\alpha\gamma^2 \E\|\nabla f_j(\phi^k_i) - \nabla f_j(\hat{\theta})\|^2 + (1 + \beta^{-1})\gamma^2 E\|\nabla f_j(\phi^k_i) - \nabla f_j(\hat{\theta})\|^2$$

$$+ (1 + \beta)\gamma^2 E\|\nabla f_j(\hat{\theta}) - \nabla f_j(\hat{\theta})\|^2 + 2\gamma \epsilon^2 (\nabla f(\hat{\theta}), \nabla f(\hat{\theta}))$$

$$\leq (1 - \gamma \alpha)\|\theta^k - \hat{\theta}\|^2 + (1 + \beta^{-1})\gamma^2 E\|\nabla f_j(\phi^k_i) - \nabla f_j(\hat{\theta})\|^2 + 2\gamma \epsilon^2 (\nabla f(\hat{\theta}), \nabla f(\hat{\theta}))$$

$$- \gamma [f(\theta^k) - f(\hat{\theta}) - \langle \nabla f(\hat{\theta}), \theta^k - \hat{\theta} \rangle] + 2(1 + \beta^{-1})\gamma^2 L\frac{1}{n} \sum_{i=1}^{n} f_i(\phi^k_i) - f(\hat{\theta}) - \frac{1}{n} \sum_{i=1}^{n} \langle \nabla f_i(\hat{\theta}), \phi^k_i - \hat{\theta} \rangle$$

(18)

Then we bound the term $\alpha E\|\theta^{k+1} - \hat{\theta}\|^2$. Define $\Delta = -\frac{1}{2} (w^{k+1} - \theta^k) - \nabla f(\theta^k)$.

The following equation can be obtained from second equation on pg. 12 in Xiao and Zhang [2014].

$$\alpha E\|\theta^{k+1} - \hat{\theta}\|^2 \leq \alpha\|\theta^k - \hat{\theta}\|^2 - 2\alpha\gamma E[\nabla f(\theta^{k+1}) - \nabla f(\hat{\theta})] + 2\alpha\gamma^2 \E\|\Delta\|^2,$$

(19)

Notice although the definition of $\Delta$ is different, they only use the property $E(\Delta) = 0$ to prove above equation.

We apply Lemma 7 on $E\|\Delta\|^2$ and get

$$E\|\Delta\|^2 \leq (1 + \beta^{-1})\E\|\nabla f_j(\phi^k_i) - \nabla f_j(\hat{\theta})\|^2 + (1 + \beta)\E\|\nabla f_j(\theta^k) - \nabla f_j(\hat{\theta})\|^2.$$  

Then

$$E\|\theta^{k+1} - \hat{\theta}\|^2 \leq \alpha\|\theta^k - \hat{\theta}\|^2 - 2\alpha\gamma E[\nabla f(\theta^{k+1}) - \nabla f(\hat{\theta})]$$

$$+ 2\alpha(1 + \beta^{-1})\gamma^2 E\|\nabla f_j(\phi^k_i) - \nabla f_j(\hat{\theta})\|^2 + 2\alpha(1 + \beta)\gamma^2 \E\|\nabla f_j(\theta^k) - \nabla f_j(\hat{\theta})\|^2.$$

(20)

Combine the result (18) and (20) together and apply Lemma 10 on $E\|\nabla f_j(\phi^k_i) - \nabla f_j(\hat{\theta})\|^2$ we obtain

21
\[(\alpha + c)\mathbb{E}[\|\theta^{k+1} - \bar{\theta}\|^2] \leq (c + \alpha - c\gamma\sigma)\|\theta^k - \bar{\theta}\|^2 + ((c + 2\alpha)(1 + \beta)\gamma^2 - \frac{c\gamma}{2L})\mathbb{E}\|\nabla f_j(\theta^k) - \nabla f_j(\hat{\theta})\|^2 + 2(c + 2\alpha)(1 + \beta^{-1})\gamma^2 L \left[ \frac{1}{n} \sum_{i=1}^{n} f_i(\phi_i^k) - f(\hat{\theta}) - \frac{1}{n} \sum_{i=1}^{n} \langle \nabla f_i(\hat{\theta}), \phi_i^k - \hat{\theta} \rangle \right] + 2c\gamma^2(\Delta^*, M, \bar{M}) - c\gamma[f(\theta^k) - f(\hat{\theta}) - \langle \nabla f(\hat{\theta}), \theta^k - \hat{\theta} \rangle] - 2\alpha\gamma\mathbb{E}[G(\theta^{k+1}) - G(\hat{\theta})].\]

Combine all pieces together, we obtain

\[\mathbb{E}T_{k+1} - T_k \leq -\frac{1}{\kappa} T_k + \left( \frac{c + \alpha}{\kappa} - c\gamma\sigma \right)\|\theta^k - \bar{\theta}\|^2 + \left( \frac{1}{\kappa} + 2(c + 2\alpha)(1 + \beta^{-1})\gamma^2 L - \frac{1}{n} \right) \left[ \frac{1}{n} \sum_{i=1}^{n} f_i(\phi_i^k) - f(\hat{\theta}) - \frac{1}{n} \sum_{i=1}^{n} \langle \nabla f_i(\hat{\theta}), \phi_i^k - \hat{\theta} \rangle \right] + \left( \frac{1}{\kappa} - c\gamma \right) [f(\theta^k) - f(\hat{\theta}) - \langle \nabla f(\hat{\theta}), \theta^k - \hat{\theta} \rangle] + \left( (c + 2\alpha)(1 + \beta)\gamma^2 - \frac{c\gamma}{2L} \right) \mathbb{E}\|\nabla f_j(\theta^k) - \nabla f_j(\hat{\theta})\|^2 + 2c\gamma^2(\Delta^*, M, \bar{M}) - 2\alpha\gamma\mathbb{E}[G(\theta^{k+1}) - G(\hat{\theta})] + b\mathbb{E}(G(\theta^{k+1}) - G(\hat{\theta})) - (1 - \frac{1}{\kappa})b[G(\theta^k) - G(\hat{\theta})].\]

Recall that we choose \(c = 2\alpha, b = 2\alpha\gamma, \beta = 2, \gamma = \frac{1}{12L}, \frac{1}{\kappa} = \min\{\frac{\rho}{14\gamma}, \frac{1}{12L}\}, \) so that the coefficient \(\frac{\frac{1}{\kappa}}{\kappa} - c\gamma\sigma, \frac{1}{\kappa} + 2(c + 2\alpha)(1 + \beta^{-1})\gamma^2 L - \frac{1}{n}, \frac{1}{\kappa} - c\gamma, (c + 2\alpha)(1 + \beta)\gamma^2 - \frac{c\gamma}{2L}\)
are all non-positive.

Thus, we obtain

\[\mathbb{E}T_{k+1} - T_k \leq -\frac{1}{\kappa} T_k + 2c\gamma^2(\Delta^*, M, \bar{M}) - (1 - \frac{1}{\kappa})c\gamma[G(\theta^k) - G(\hat{\theta})].\]

3. The geometrical convergence of \(T_k\)

Next we prove the Lyapunov function decreases geometrically until \(G(\theta^k) - G(\hat{\theta})\) achieves the tolerance \(\delta\). In high level, we divide the time steps \(k = 1, 2, \ldots\) into several epochs, i.e., \([T_0, T_1), (T_1, T_2), \ldots\). At the end of each epoch \(j\), we prove that \(T_k\) decreases with linear rate until the optimality gap \(G(\theta^k) - G(\hat{\theta})\) decreases to some tolerance \(\xi_j\). We then prove that \((\xi_1, \xi_2, \xi_3, \ldots)\) is a decreasing sequence and finish the proof.

Now we analyze the progress of \(T_k\) across different epochs. Suppose time step \(k\) is in the epoch \(j\), and we have \(G(\theta^k) - G(\hat{\theta}) \leq \xi_{j-1}\). Then we apply Lemma \[10\] and have

\[\epsilon_j^2(\Delta^*, M, \bar{M}) = 2\tau_\sigma (\delta_{\text{stat}} + \delta_{j-1})^2,\]

with \(\delta_{j-1} = 2\min\{\frac{\xi_{j-1}}{14\gamma}, \rho\}\), and \(\delta_{\text{stat}} = 8H(\bar{M})\|\Delta^*\|_2 + 8\psi(\theta^*_{M, \perp})\).

Now we start the induction step. Although we do not know \(\xi_0\), we can choose \(\delta_0 = 2\rho\). In this case, \(\epsilon_1^2(\Delta^*, M, \bar{M}) = 2\tau_\sigma (\delta_{\text{stat}} + 2\rho)^2\).

We choose \(T_1\) such that

\[\left(1 - \frac{1}{\kappa}\right) \left( G(\theta^{T_1}) - G(\hat{\theta}) \right) \geq 2\epsilon_1^2(\Delta^*, M, \bar{M}) \]

and

\[\left(1 - \frac{1}{\kappa}\right) \left( G(\theta^{T_1}) - G(\hat{\theta}) \right) \leq 2\epsilon_1^2(\Delta^*, M, \bar{M}).\]
Notice such $T_1$ must exist, otherwise we have $(1 - \frac{1}{\kappa}) \left( G(\theta^k) - G(\hat{\theta}) \right) \geq 2\epsilon_1^2(\Delta^*, M, \bar{M})$ and $ET_{k+1} \leq (1 - \frac{1}{\kappa})T_k$ holds for every $k$, i.e., $T_k$ converges geometrically, which is a contradiction with $(1 - \frac{1}{\kappa}) \left( G(\theta^k) - G(\hat{\theta}) \right) \geq 2\epsilon_1^2(\Delta^*, M, \bar{M})$.

Now we know \( \left( G(\theta^{T_1}) - G(\hat{\theta}) \right) \leq \frac{2}{1 - 1/\kappa} \epsilon_1^2(\Delta^*, M, \bar{M}) \), thus we choose $\xi_1 = \frac{4}{1 - 1/\kappa} \tau(\delta_{stat} + 2\rho)^2$. It is time to follow the same argument in the second epoch. Recall we have

$$\epsilon_2^2(\Delta^*, M, \bar{M}) = 2\tau(\delta_{stat} + \delta_1)^2,$$

where $\delta_1 = 2 \min\{\frac{\delta}{\kappa}, \rho\}$.

We choose $T_2$ such that

\[
(1 - \frac{1}{\kappa}) \left( G(\theta^{T_2-1}) - G(\hat{\theta}) \right) \geq 2\epsilon_2^2(\Delta^*, M, \bar{M})
\]

and

\[
(1 - \frac{1}{\kappa}) \left( G(\theta^{T_2}) - G(\hat{\theta}) \right) \leq 2\epsilon_2^2(\Delta^*, M, \bar{M}),
\]

and $\xi_2 = \frac{4}{1 - 1/\kappa} \tau(\delta_{stat} + \delta_1)^2$.

Similarly, in epoch $j$, we choose $T_j$ such that

\[
(1 - \frac{1}{\kappa}) \left( G(\theta^{T_j-1}) - G(\hat{\theta}) \right) \geq 2\epsilon_j^2(\Delta^*, M, \bar{M})
\]

and

\[
(1 - \frac{1}{\kappa}) \left( G(\theta^{T_j}) - G(\hat{\theta}) \right) \leq 2\epsilon_j^2(\Delta^*, M, \bar{M}),
\]

and $\xi_j = \frac{4}{1 - 1/\kappa} \tau(\delta_{stat} + \delta_{j-1})^2$.

In this way, we arrive at recursive equalities of the tolerance $\{\xi_j\}_{j=1}^\infty$ where $\xi_j = \frac{4}{1 - 1/\kappa} \tau(\delta_{stat} + \delta_{j-1})^2$ and $\delta_{j-1} = 2 \min\{\frac{\delta}{\kappa}, \rho\}$.

We claim that following holds, until $\delta_i = \delta_{stat}$.

\[
(I) \quad \xi_{k+1} \leq \xi_k/(4^{2k-1})
\]

and \( (II) \quad \frac{\xi_{k+1}}{\lambda} \leq \frac{\rho}{4^k} \) for $k = 1, 2, 3, ...$ (23)

Assume $\frac{1}{\kappa} \leq \frac{1}{3}$ (Notice it is safe to do so since $\frac{1}{\kappa} = \min\{\frac{2\delta}{\lambda}, \frac{1}{2}\}$, and when $n > 2$ it holds), we have $\xi_j \leq 6\tau(\delta_{stat} + \delta_{j-1})^2$.

The proof of Equation (23) is same with Equation (60) in [Agarwal et al., 2010], which we present here for completeness.

We assume $\delta_0 \geq \delta_{stat}$ (otherwise the statement is true trivially), so $\xi_1 \leq 96\tau(\delta_{stat} + \delta_{j-1})^2$. We assume that $\lambda \geq 384\tau\rho$, so $\frac{\delta_j}{\lambda} \leq \frac{\rho}{\lambda}$ and $\xi_1 \leq \xi_0$.

In the second epoch we have

$$\xi_2 \leq 12\tau(\delta_{stat} + \delta_1)^2 \leq 24\tau\delta_1^2 \leq \frac{96\tau\lambda}{\lambda^2} \xi_1^2 \leq \frac{96\tau\xi_1^2}{4\lambda} \leq \frac{\xi_1}{4},$$

where (1) holds from the fact that $(a + b)^2 \leq 2a^2 + 2b^2$, (2) holds using $\frac{\delta_1}{\lambda} \leq \frac{\rho}{\lambda}$, (3) uses the assumption on $\lambda$. Thus,

$$\frac{\xi_2}{\lambda} \leq \frac{\xi_1}{4\lambda} \leq \frac{\rho}{16}.$$ 

In $i + 1$th step, with similar argument, and by induction assumption we have

$$\xi_{j+1} \leq \frac{96\tau\xi_j^2}{\lambda^2} \leq \frac{96\tau\xi_j^2}{4^2\lambda} \leq \frac{\xi_j}{4^{2k-1}}.$$
and
\[ \frac{\xi_{j+1}}{\lambda} \leq \frac{\xi_j}{4^{2^{j-1}}} \leq \frac{\rho}{4^2}. \]

Thus we know \( \xi_j \) is a decreasing sequence, and \( \mathbb{E} T_{k+1} \leq (1 - \frac{1}{\kappa}) T_k \) holds until \( G(\theta^k) - G(\hat{\theta}) \leq 6\tau_\sigma (2\delta_{stat})^2 \). We establish the result. \( \square \)

### A.2 SAGA with non-convex objective function

We start with some technical Lemmas. The following lemma is Lemma 5 extract from [Loh and Wainwright 2013], we present here for the completeness.

**Lemma 8.** For any vector \( \theta \in \mathbb{R}^p \), let \( A \) denote the index set of its \( r \) largest elements in magnitude, under assumption on \( g_{\lambda,\mu} \) in Section 2.3, we have
\[
g_{\lambda,\mu}(\theta_A) - g_{\lambda,\mu}(\theta_{A^c}) \leq \lambda L_g (\|\theta_A\|_1 - \|\theta_{A^c}\|_1).
\]

Moreover, for an arbitrary vector \( \theta \in \mathbb{R}^p \), we have
\[
g_{\lambda,\mu}(\theta^*) - g_{\lambda,\mu}(\theta) \leq \lambda L_g (\|\nu_A\|_1 - \|\nu_{A^c}\|_1),
\]
where \( \nu = \theta - \theta^* \) and \( \theta^* \) is \( r \) sparse.

The next lemma is a non-convex counterpart of Lemma 3 and Lemma 4.

**Lemma 9.** Suppose \( g_{\lambda,\mu}(\cdot) \) satisfies the assumptions in section 2.3, \( \lambda L_g \geq 8\rho \tau \frac{\log p}{n} \), \( \lambda \geq \frac{1}{L_g} \|\nabla f(\theta^*)\|_\infty \), \( \theta^* \) is feasible, and there exists a pair \( (\epsilon, K) \) such that
\[
G(\theta^k) - G(\hat{\theta}) \leq \epsilon, \forall k \geq K.
\]
Then for any iteration \( k \geq K \), we have
\[
\|\theta^k - \hat{\theta}\|_1 \leq 4\sqrt{r} \|\theta^k - \hat{\theta}\|_2 + 8\sqrt{r} \|\theta^* - \hat{\theta}\|_2 + 2 \min(\frac{\epsilon}{L_g}, \rho).
\]

**Proof.** Fix an arbitrary feasible \( \theta \), Define \( \Delta = \theta - \theta^* \). Suppose \( G(\theta) - G(\hat{\theta}) \leq \epsilon \), since we know \( G(\hat{\theta}) \leq G(\theta^*) \) so we have \( G(\theta) \leq G(\theta^*) + \epsilon \), which implies
\[
f(\theta^* + \Delta) + g_{\lambda,\mu}(\theta^* + \Delta) \leq f(\theta^*) + g_{\lambda,\mu}(\theta^*) + \epsilon.
\]
Subtract \( \langle \nabla f(\theta^*), \Delta \rangle \) and use the RSC condition we have
\[
\frac{\sigma}{2} \|\Delta\|_2^2 - \tau \frac{\log p}{n} \|\Delta\|_1^2 + g_{\lambda,\mu}(\theta^* + \Delta) - g_{\lambda,\mu}(\theta^*) \\
\leq \epsilon - \langle \nabla f(\theta^*), \Delta \rangle \\
\leq \epsilon + \|\nabla f(\theta^*)\|_\infty \|\Delta\|_1
\]
where the last inequality holds from Holder’s inequality. Rearrange terms and use the fact that \( \|\Delta\|_1 \leq 2\rho \) (by feasibility of \( \theta \) and \( \theta^* \)) and the assumptions \( \lambda L_g \geq 8\rho \tau \frac{\log p}{n} \), \( \lambda \geq \frac{1}{L_g} \|\nabla f(\theta^*)\|_\infty \), we obtain
\[
\epsilon + \frac{1}{2} \lambda L_g \|\Delta\|_1 + g_{\lambda,\mu}(\theta^*) - g_{\lambda,\mu}(\theta^* + \Delta) \geq \frac{\sigma}{2} \|\Delta\|_2^2 \geq 0.
\]
By Lemma 8 we have
\[ g_{\lambda,\mu}(\theta^*) - g_{\lambda,\mu}(\theta) \leq \lambda L_g(\|\Delta A\|_1 - \|\Delta A^r\|_1), \]
where \( A \) indexes the top \( r \) components of \( \Delta \) in magnitude. So we have
\[ \frac{3\lambda L_g}{2}\|\Delta A\|_1 - \frac{\lambda L_g}{2}\|\Delta A^r\|_1 + \epsilon \geq 0, \]
and consequently
\[ \|\Delta\|_1 \leq \|\Delta A\|_1 + \|\Delta A^r\|_1 \leq 4\|\Delta A\|_1 + \frac{2\epsilon}{\lambda L_g} \leq 4\sqrt{r}\|\Delta\|_2 + \frac{2\epsilon}{\lambda L_g}. \]
Combining this with \( \|\Delta\|_1 \leq 2\rho \) leads to
\[ \|\Delta\|_1 \leq 4\sqrt{r}\|\Delta\|_2 + 2\min\{\frac{\epsilon}{\lambda L_g}, \rho\}. \]
Since this holds for any feasible \( \theta \), we have \( \|\theta^k - \theta^*\|_1 \leq 4\sqrt{r}\|\theta^k - \theta^*\|_2 + 2\min\{\frac{\epsilon}{\lambda L_g}, \rho\}. \)

Notice \( G(\hat{\theta}) - G(\theta^*) \leq 0 \), so following same derivation as above and set \( \epsilon = 0 \) we have
\[ \|\hat{\theta} - \theta^*\|_1 \leq 4\sqrt{r}\|\hat{\theta} - \theta^*\|_2. \]
Combining the two, we have
\[ \|\theta^k - \hat{\theta}\|_1 \leq \|\theta^k - \theta^*\|_1 + \|\theta^* - \hat{\theta}\|_1 \leq 4\sqrt{r}\|\theta^k - \hat{\theta}\|_2 + 8\sqrt{r}\|\theta^* - \hat{\theta}\|_2 + 2\min\{\frac{\epsilon}{\lambda L_g}, \rho\}. \]

\[ \Box \]

Now we provide a counterpart of Lemma 5 in the non-convex case. Notice the main difference from the convex case is the coefficient in front of \( \|\theta^k - \hat{\theta}\|_2^2 \).

**Lemma 10.** Under the same assumption of Lemma 5 we have
\[ \langle \nabla F(\theta^k) - \nabla F(\hat{\theta}), \theta^k - \hat{\theta} \rangle \geq (\sigma - \mu - 64\tau_\sigma n)\|\hat{\Delta}^k\|_2^2 - 2\epsilon^2(\Delta^*, r) \] (25)
and
\[ G(\theta^k) - G(\hat{\theta}) \geq (\frac{\sigma - \mu}{2} - 32\tau_\sigma)\|\theta^k - \hat{\theta}\|_2^2 - \epsilon^2(\Delta^*, r), \]
where \( \Delta^* = \hat{\theta} - \theta^* \), \( \hat{\Delta}^k = \theta^k - \hat{\theta} \) and \( \epsilon^2(\Delta^*, r) = 2\tau_\sigma(8\sqrt{r}\|\hat{\theta} - \theta^*\|_2 + 2\min\{\frac{\epsilon}{\lambda L_g}, \rho\})^2, \)
\( \tau_\sigma = \frac{\tau_0}{n} \).

**Proof.** We have the following:
\[ F(\theta^k) - F(\hat{\theta}) - \langle \nabla F(\hat{\theta}), \theta^k - \hat{\theta} \rangle \]
\[ = f(\theta^k) - \frac{\mu}{2}\|\theta^k\|_2^2 - f(\hat{\theta}) + \frac{\mu}{2}\|\hat{\theta}\|_2^2 - \langle \nabla f(\hat{\theta}) - \mu\hat{\theta}, \theta^k - \hat{\theta} \rangle \]
\[ = f(\theta^k) - f(\hat{\theta}) - \langle \nabla f(\hat{\theta}), \theta^k - \hat{\theta} \rangle - \frac{\mu}{2}\|\theta^k - \hat{\theta}\|_2^2 \]
\[ \geq \frac{\sigma - \mu}{2}\|\theta^k - \hat{\theta}\|_2^2 - \tau_\sigma \frac{\log p}{n}\|\theta^k - \hat{\theta}\|_1^2, \] (26)
where the inequality uses the RSC condition.
By Lemma [1] we have
\[
\|\theta^k - \hat{\theta}\|_1^2 \leq (4\sqrt{r}\|\theta^k - \hat{\theta}\|_2 + 8\sqrt{r}\|\theta^* - \hat{\theta}\|_2 + 2\min(\frac{\epsilon}{\lambda L_g}, \rho))^2 \\
\leq 32r\|\theta^k - \hat{\theta}\|_2^2 + 2(8\sqrt{r}\|\hat{\theta} - \theta^*\|_2 + 2\min(\frac{\epsilon}{\lambda L_g}, \rho))^2.
\]
(27)

Substitute this into Equation (26), we obtain
\[
F(\theta^k) - F(\hat{\theta}) - \langle \nabla F(\hat{\theta}), \theta^k - \hat{\theta}\rangle \geq (\frac{\sigma - \mu}{2} - 32r\tau_\sigma)\|\theta^k - \hat{\theta}\|_2^2 - 2\tau_\sigma(8\sqrt{r}\|\hat{\theta} - \theta^*\|_2 + 2\min(\frac{\epsilon}{\lambda L_g}, \rho))^2
\]
Similarly, we have
\[
F(\hat{\theta}) - F(\theta^k) - \langle \nabla F(\theta^k), \hat{\theta} - \theta^k\rangle \geq (\frac{\sigma - \mu}{2} - 32r\tau_\sigma)\|\theta^k - \hat{\theta}\|_2^2 - 2\tau_\sigma(8\sqrt{r}\|\hat{\theta} - \theta^*\|_2 + 2\min(\frac{\epsilon}{\lambda L_g}, \rho))^2
\]
Add above two equations together, we establish the result
\[
\langle \nabla F(\theta^k) - \nabla F(\hat{\theta}), \theta^k - \hat{\theta}\rangle \geq (\sigma - \mu - 64\tau_\sigma r) \|\Delta^k\|_2^2 - 2\epsilon(\Delta^*, r).
\]
(28)

Next we bound \(G(\theta^k) - G(\hat{\theta})\).
\[
G(\theta^k) - G(\hat{\theta}) = f(\theta^k) - f(\hat{\theta}) - \frac{\mu}{2}\|\theta^k\|_2^2 + \frac{\mu}{2}\|\hat{\theta}\|_2^2 + \lambda g_\lambda(\theta^k) - \lambda g_\lambda(\hat{\theta}) \\
\geq \langle \nabla f(\hat{\theta}), \theta^k - \hat{\theta}\rangle + \frac{\sigma}{2}\|\theta^k - \hat{\theta}\|_2^2 - \langle \mu \hat{\theta}, \theta^k - \hat{\theta}\rangle - \frac{\mu}{2}\|\theta^k - \hat{\theta}\|_2^2 \\
+ \lambda g_\lambda(\theta^k) - \lambda g_\lambda(\hat{\theta}) - \tau \frac{\log p}{n}\|\theta^k - \hat{\theta}\|_1^2
\]
(29)
where the first inequality uses the RSC condition, the second inequality uses the convexity of \(g_\lambda(\hat{\theta})\), and the last equality holds from the optimality condition of \(\hat{\theta}\).

Remind we have
\[
\|\theta^k - \hat{\theta}\|_1^2 \leq 32r\|\theta^k - \hat{\theta}\|_2^2 + 2(8\sqrt{r}\|\hat{\theta} - \theta^*\|_2 + 2\min(\frac{\epsilon}{\lambda L_g}, \rho))^2.
\]
(30)
Substitute this into Equation (29) we obtain
\[
G(\theta^k) - G(\hat{\theta}) \geq (\frac{\sigma - \mu}{2} - 32r\tau_\sigma)\|\theta^k - \hat{\theta}\|_2^2 - 2\tau_\sigma(8\sqrt{r}\|\hat{\theta} - \theta^*\|_2 + 2\min(\frac{\epsilon}{\lambda L_g}, \rho))^2.
\]
Lemma 11. Under the same assumption of Lemma 10, we have

\[ \langle \nabla F(\theta^k) - \nabla F(\hat{\theta}), \theta^k - \hat{\theta} \rangle \geq \frac{1}{2} [F(\theta^k) - F(\hat{\theta}) - \langle \nabla F(\hat{\theta}), \theta^k - \hat{\theta} \rangle] + \left( \frac{\alpha}{2} - \frac{\mu}{4} - \frac{\mu^2}{4L} \right) \|\theta^k - \hat{\theta}\|_2^2 \]

\[ + \frac{1}{8L} \|\nabla F(\theta^k) - \nabla F(\hat{\theta})\|_2^2 - \epsilon^2(\Delta^*, r). \tag{31} \]

Proof.

\[ \langle \nabla F(\theta^k) - \nabla F(\hat{\theta}), \theta^k - \hat{\theta} \rangle - \frac{1}{2} [F(\theta^k) - F(\hat{\theta}) - \langle \nabla F(\hat{\theta}), \theta^k - \hat{\theta} \rangle] \]

\[ = \frac{1}{2} [F(\theta^k) - F(\hat{\theta}) - \langle \nabla F(\theta^k), \hat{\theta} - \theta^k \rangle] + \frac{1}{2} [F(\hat{\theta}) - F(\theta^k) - \langle \nabla F(\theta^k), \hat{\theta} - \theta^k \rangle] \tag{32} \]

Then we bound the right hand side of above equation. Notice

\[ \|\nabla F(\theta^k) - \nabla F(\hat{\theta})\|_2^2 \leq 2\|\nabla f(\theta^k) - \nabla f(\hat{\theta})\|_2^2 + 2\mu^2\|\theta^k - \hat{\theta}\|_2^2 \]

\[ \leq 4L[f(\hat{\theta}) - f(\theta^k) - \langle \nabla f(\theta^k), \hat{\theta} - \theta^k \rangle] + 2\mu^2\|\theta^k - \hat{\theta}\|_2^2 \]

\[ \leq 4L[F(\hat{\theta}) - F(\theta^k) - \langle \nabla F(\theta^k), \hat{\theta} - \theta^k \rangle] + \frac{\mu}{2} \|\theta^k - \hat{\theta}\|_2^2 + 2\mu^2\|\theta^k - \hat{\theta}\|_2^2. \tag{33} \]

We then apply Lemma 10 and recall our definition on $\sigma$ on non-convex case, we have

\[ \langle \nabla F(\theta^k) - \nabla F(\hat{\theta}), \theta^k - \hat{\theta} \rangle \geq \sigma \|\theta^k - \hat{\theta}\|_2^2 - 2\epsilon^2(\Delta^*, r), \]

Substitute above bound in the right hand side of Equation (32) we establish the result. \hfill \Box

We are now ready to prove the main Theorem on non-convex $G(\theta)$, i.e., Theorem 2

Proof of Theorem 2 Recall the definition of $F_i(\theta)$ is $F_i(\theta) = f_i(\theta) - \frac{\mu}{2} \|\theta\|_2^2$ and the Lyapunov function

\[ T_k \triangleq \frac{1}{n} \sum_{i=1}^{n} \left( f_i(\phi_i^k) - f_i(\hat{\theta}) - \langle \nabla f_i(\hat{\theta}), \phi_i^k - \hat{\theta} \rangle \right) + (c + \alpha)\|\theta^k - \hat{\theta}\|_2^2 + b(G(\theta^k) - G(\hat{\theta})) \]

\[ = \frac{1}{n} \sum_{i=1}^{n} \left( F_i(\phi_i^k) - F_i(\hat{\theta}) - \langle \nabla F_i(\hat{\theta}), \phi_i^k - \hat{\theta} \rangle + \frac{\mu}{2} \|\phi_i^k - \hat{\theta}\|_2^2 \right) + (c + \alpha)\|\theta^k - \hat{\theta}\|_2^2 + b(G(\theta^k) - G(\hat{\theta})). \tag{34} \]

1. Bound the first term on the right hand side fo $\mathbb{E}[T_{k+1}]$.

Following similar steps in the convex case, we obtain

\[ \mathbb{E}\left[ \frac{1}{n} \sum_{i=1}^{n} F_i(\phi_i^{k+1}) \right] = \frac{1}{n} F(\theta^k) + (1 - \frac{1}{n}) \frac{1}{n} \sum_{i=1}^{n} F_i(\phi_i^k) \]

\[ \mathbb{E}\left[ \frac{1}{n} \sum_{i=1}^{n} \langle \nabla F_i(\hat{\theta}), \phi_i^{k+1} - \hat{\theta} \rangle \right] = -\frac{1}{n} \langle \nabla F(\hat{\theta}), \theta^k - \theta^* \rangle - (1 - \frac{1}{n}) \frac{1}{n} \sum_{i=1}^{n} \langle \nabla F_i(\hat{\theta}), \phi_i^k - \hat{\theta} \rangle. \]
\[ \frac{\mu}{2n} \mathbb{E} \sum_{i=1}^{n} \| \phi_i^{k+1} - \hat{\theta} \|^2 = \frac{\mu}{2} \frac{1}{n} \| \theta^k - \hat{\theta} \|^2 + \left( 1 - \frac{1}{n} \right) \sum_{i=1}^{n} \frac{1}{n} \| \phi_i^k - \hat{\theta} \|^2. \]

2. Bound \((c + \alpha) \mathbb{E} \| \theta^{k+1} - \hat{\theta} \|^2\)

In the following, we provide a upper bound on \(c \mathbb{E} \| \theta^{k+1} - \hat{\theta} \|^2\). Notice Equation (17) and the proof of Lemma 7 does not use convexity (The proof of Lemma 7 just use the fact \(\mathbb{E} \| X - \mathbb{E} X \|^2 = \mathbb{E} \| X \|^2 - \| \mathbb{E} X \|^2\), see Defazio \textit{et al.} 2014 for detail), thus replace \(f_i(\theta)\) by \(F_i(\theta)\) we obtain the bound

\[
\mathbb{E} \| \theta^{k+1} - \hat{\theta} \|^2 \\
\leq \| \theta^k - \hat{\theta} \|^2 + 2\gamma(\nabla F(\theta^k) - \nabla F(\hat{\theta}), \theta^k - \hat{\theta}) - \gamma^2 \| \nabla F(\theta^k) - \nabla F(\hat{\theta}) \|^2 \\
+ (1 + \beta^{-1}) \gamma^2 \mathbb{E} \| \nabla F_j(\phi_j^k) - \nabla F_j(\hat{\theta}) \|^2 + (1 + \beta) \gamma^2 \mathbb{E} \| \nabla F_j(\theta^k) - \nabla F_j(\hat{\theta}) \|^2 \\
\leq (1 - \gamma(\sigma - \frac{\mu}{2} - \frac{\mu^2}{2L})) \| \theta^k - \hat{\theta} \|^2 + 2\gamma \epsilon^2(\Delta^*, r) - \frac{\gamma}{4L} \mathbb{E} \| \nabla F_j(\theta^k) - \nabla F_j(\hat{\theta}) \|^2 + (1 + \beta) \gamma^2 \mathbb{E} \| \nabla F_j(\theta^k) - \nabla F_j(\hat{\theta}) \|^2,
\]

where the second inequality uses Lemma 14.

Then we bound the term \(\alpha \mathbb{E} \| \theta^{k+1} - \hat{\theta} \|^2\). Using the Equation (30) in Qu \textit{et al.} 2016, we obtain

\[
\alpha \mathbb{E} \| \theta^{k+1} - \hat{\theta} \|^2 \leq \alpha(1 + \gamma \mu) \| \theta^k - \hat{\theta} \|^2 - 2\alpha \gamma \mathbb{E}[G(\theta^{k+1}) - G(\hat{\theta})] + 2\alpha \gamma^2 \mathbb{E} \| \Delta \|^2.
\]

Same with convex case, we apply Lemma 7 on \(\mathbb{E} \| \Delta \|^2\) and obtain

\[
\mathbb{E} \alpha \| \theta^{k+1} - \hat{\theta} \|^2 \leq \alpha(1 + \gamma \mu) \| \theta^k - \hat{\theta} \|^2 - 2\alpha \gamma \mathbb{E}[G(\theta^{k+1}) - G(\hat{\theta})] \\
+ 2\alpha(1 + \beta^{-1}) \gamma^2 \mathbb{E} \| \nabla F_j(\phi_j^k) - \nabla F_j(\hat{\theta}) \|^2 + 2\alpha(1 + \beta) \gamma^2 \mathbb{E} \| F_j(\theta^k) - F_j(\hat{\theta}) \|^2.
\]

Combine (35) and (37) together, we obtain

\[
(\alpha + c) \mathbb{E} \| \theta^{k+1} - \hat{\theta} \|^2 \leq \left( c - c\gamma(\sigma - \mu/2 - \frac{\mu^2}{2L}) + \alpha(1 + \gamma \mu) \right) \| \theta^k - \hat{\theta} \|^2 + 2c\gamma^2 \epsilon^2(\Delta^*, r) \\
+ ((c + 2\alpha)(1 + \beta) \gamma^2 - \frac{c^2\gamma^2}{4L}) \mathbb{E} \| \nabla F_j(\theta^k) - \nabla F_j(\hat{\theta}) \|^2 \\
+ (c + 2\alpha)(1 + \beta^{-1}) \gamma^2 \mathbb{E} \| \nabla F_j(\phi_j^k) - \nabla F_j(\hat{\theta}) \|^2 - 2\alpha \gamma \mathbb{E}[G(\theta^{k+1}) - G(\hat{\theta})] \\
-c\gamma[F(\theta^k) - F(\hat{\theta}) - \langle \nabla F(\hat{\theta}), \theta^k - \hat{\theta} \rangle]
\]

We then bound \(\mathbb{E} \| \nabla F_j(\phi_j^k) - \nabla F_j(\hat{\theta}) \|^2\).
\[ \mathbb{E}\|\nabla F_j(\phi^k_{\hat{i}}) - \nabla F_j(\hat{\theta})\|^2 \]
\[ = \mathbb{E}\|\nabla f_j(\phi^k_{\hat{i}}) - \nabla f_j(\hat{\theta}) - u(\phi^k_{\hat{i}} - \hat{\theta})\|^2 \]
\[ \leq 2\mathbb{E}\|\nabla f_j(\phi^k_{\hat{i}}) - \nabla f_j(\hat{\theta})\|^2 + 2\mu^2\mathbb{E}\|\phi^k_{\hat{i}} - \hat{\theta}\|^2 \]
\[ \leq \frac{4L}{n} \sum_{i=1}^n [F_i(\phi^k_{\hat{i}}) - F_i(\hat{\theta}) - (\nabla f_i(\phi^k_{\hat{i}}), \phi^k_{\hat{i}} - \hat{\theta})] + \frac{2\mu^2}{n} \sum_{i=1}^n \|\phi^k_{\hat{i}} - \hat{\theta}\|^2 \]
\[ = \frac{4L}{n} \sum_{i=1}^n [F_i(\phi^k_{\hat{i}}) - F_i(\hat{\theta}) - (\nabla f_i(\phi^k_{\hat{i}}), \phi^k_{\hat{i}} - \hat{\theta}) + \frac{\mu}{2} \|\phi^k_{\hat{i}} - \hat{\theta}\|^2] + \frac{2\mu^2}{n} \sum_{i=1}^n \|\phi^k_{\hat{i}} - \hat{\theta}\|^2 \]
\[ \leq \frac{4L}{n} \sum_{i=1}^n [F_i(\phi^k_{\hat{i}}) - F_i(\hat{\theta}) - (\nabla f_i(\phi^k_{\hat{i}}), \phi^k_{\hat{i}} - \hat{\theta})] + \frac{2\mu(\mu + \mu)}{n} \sum_{i=1}^n \|\phi^k_{\hat{i}} - \hat{\theta}\|^2. \tag{39} \]

where the first inequality holds from the fact \((a + b)^2 \leq 2a^2 + 2b^2\), the second one uses the convexity and smoothness of \(f_i(\theta)\).

Substitute above bound in Equation \((28)\), we obtain

\[
\begin{align*}
& (\alpha + c)\mathbb{E}\|\theta^{k+1} - \hat{\theta}\|^2 \leq \left( c - c\gamma(\hat{\sigma} - \mu/2 - \frac{\mu^2}{2L}) + \alpha(1 + \gamma\mu) \right) \|\theta^k - \hat{\theta}\|^2 + 2c\gamma^2(\Delta^*, r) \\
& + ((c + 2\alpha)(1 + \beta^{-1})\gamma^2 - \frac{c\gamma}{4L})\mathbb{E}\|\nabla F_j(\theta^k) - \nabla F_j(\hat{\theta})\|^2 \\
& + (c + 2\alpha)(1 + \beta^{-1})\gamma^2 \left( \frac{4L}{n} \sum_{i=1}^n [F_i(\phi^k_{\hat{i}}) - F_i(\hat{\theta}) - (\nabla f_i(\phi^k_{\hat{i}}), \phi^k_{\hat{i}} - \hat{\theta})] + \frac{2\mu(\mu + \mu)}{n} \sum_{i=1}^n \|\phi^k_{\hat{i}} - \hat{\theta}\|^2 \right) \\
& - 2\alpha\gamma\mathbb{E}[G(\theta^{k+1}) - G(\hat{\theta})] - c\gamma[F(\theta^k) - F(\hat{\theta}) - (\nabla F(\hat{\theta}), \theta^k - \hat{\theta})]. \tag{40} \end{align*}
\]

3. Relate \(ET_{k+1}\) to \(T_k\)

Combine all above together, we obtain

\[
\begin{align*}
\mathbb{E}T_{k+1} - T_k & \leq -\frac{1}{\kappa}T_k + \left( \frac{c + \alpha}{\kappa} - c\gamma(\hat{\sigma} - \frac{\mu}{2} - \frac{\mu^2}{2L}) + \alpha\gamma\mu \right)\|\theta^k - \hat{\theta}\|^2 \\
& + \left( \frac{1}{\kappa} + 4(c + 2\alpha)(1 + \beta^{-1})\gamma^2 L - \frac{1}{\kappa} \right) \left\{ \frac{1}{n} \sum_{i=1}^n [F_i(\phi^k_{\hat{i}}) - F_i(\hat{\theta}) - \frac{1}{n} \sum_{j=1}^n (\nabla f_i(\hat{\theta}), \phi^k_{\hat{i}} - \hat{\theta})] \\
& + \left( \frac{1}{n} - c\gamma \right) [F(\theta^k) - F(\hat{\theta}) - (\nabla F(\hat{\theta}), \theta^k - \hat{\theta})] + \left( (c + 2\alpha)(1 + \beta^{-1})\gamma^2 - \frac{c\gamma}{4L} \right) \mathbb{E}\|\nabla f_j(\theta^k) - \nabla f_j(\hat{\theta})\|^2 \\
& + 2c\gamma^2(\Delta^*, r) - 2\alpha\gamma\mathbb{E}[G(\theta^{k+1}) - G(\hat{\theta})] + b\mathbb{E}[G(\theta^{k+1}) - G(\theta)] - (1 - \frac{1}{\kappa})b[G(\theta^k) - G(\hat{\theta})] \\
& + \left( \frac{\mu}{2\kappa} - \frac{\mu}{2n} + 2(c + 2\alpha)(1 + \beta^{-1})\gamma^2 \mu(\mu + L) \right) \frac{1}{n} \sum_{i=1}^n \|\phi^k_{\hat{i}} - \hat{\theta}\|^2 \right\} \\
& + \left( \frac{\mu}{2\kappa} - \frac{\mu}{2n} + 2(c + 2\alpha)(1 + \beta^{-1})\gamma^2 \mu(\mu + L) \right) \frac{1}{n} \sum_{i=1}^n \|\phi^k_{\hat{i}} - \hat{\theta}\|^2. \tag{41} \end{align*}
\]

We choose \(\beta = 2, \gamma = \frac{1}{24L}, c = 2\alpha, c = \frac{24L}{n}, b = 2\alpha\gamma, \frac{1}{\kappa} = \frac{1}{24} \min\{\frac{2\sigma}{\Delta^*}, \frac{1}{n}\}\) and recall our assumption that \(\mu \leq \frac{\sigma}{3}\) and \(\mu \leq \frac{L}{3}\), we obtain

\[
\mathbb{E}T_{k+1} - T_k \leq -\frac{1}{\kappa}T_k + 2c\gamma^2(\Delta^*, r) - (1 - \frac{1}{\kappa})c\gamma[G(\theta^k) - G(\hat{\theta})].
\]
The following argument is identical to the convex $G(\theta)$, and the only difference is to replace $\lambda$ by $\lambda L_g$. Thus we have $E T_{k+1} \leq (1 - \frac{1}{r})T_k$ holds until $G(\theta^k) - G(\hat{\theta}) \leq 6\tau_\sigma(2\delta_{stat})^2$. Thus we establish the result.

\[\square\]

A.3 Proof of corollaries

We now prove the corollaries instantiating our main theorems to different statistical estimators.

Proof of Corollary on Lasso, i.e., corollary \([7]\). We begin the proof, by presenting the below lemma of the RSC, proved in \([\text{Raskutti et al.} 2010]\), and we then use it in the case of Lasso.

Lemma 12. if each data point $x_i$ is i.i.d random sampled from the distribution $N(0, \Sigma)$, then there are some universal constants $c_0$ and $c_1$ such that

$$\frac{\|X \Delta\|_2^2}{n} \geq \frac{1}{2} \|\Sigma^{1/2} \Delta\|_2^2 - c_1 \nu(\Sigma) \log p \|\Delta\|_1^2,$$

for all $\Delta \in \mathbb{R}^p$, with probability at least $1 - \exp(-c_0 n)$. Here, $X$ is the data matrix where each row is data point $x_i$.

Since $\theta^*$ is support on a subset $S$ with cardinality $r$, we choose $\bar{M}(S) := \{ \theta \in \mathbb{R}^p | \theta_j = 0 \text{ for all } j \notin S \}$. It is straightforward to choose $M(S) = \bar{M}(S)$ and notice that $\theta^* \in M(S)$. In Lasso formulation, $f(\theta) = \frac{1}{2n} \|y - X\theta\|_2^2$, and hence it is easy to verify that

$$f(\theta + \Delta) - f(\theta) - \langle \nabla f(\theta), \Delta \rangle \geq \frac{1}{2n} \|X \Delta\|_2^2 \geq \frac{1}{4} \|\Sigma^{1/2} \Delta\|_2^2 - \frac{c_1}{2} \nu(\Sigma) \log p \|\Delta\|_1^2.$$

Also, $\psi(\cdot)$ is $\|\cdot\|_1$ in Lasso, and hence $H(\bar{M}) = \sup_{\theta \in \bar{M} \setminus \{0\}} \frac{\|\theta\|_1}{\|\theta\|_2} = \sqrt{r}$. Thus we have

$$\bar{\sigma} = \frac{1}{2} \sigma_{\min}(\Sigma) - 64 c_1 \nu(\Sigma) \frac{r \log p}{n}.$$

On the other hand, the tolerance is

$$\delta = 24 \tau_\sigma (8H(\bar{M}) \|\Delta^*\|_2 + 8 \psi(\theta^*_{M^\perp}))^2 = c_2 \nu(\Sigma) \frac{r \log p}{n} \|\Delta^*\|_2^2,$$

where we use the fact that $\theta^* \in M(S)$, which implies $\psi(\theta^*_{M^\perp}) = 0$.

Recall we require $\lambda$ to satisfy $\lambda \geq 2 \psi^*(\nabla f(\theta^*))$. In Lasso we have $\psi^*(\cdot) = \|\cdot\|_{\infty}$. Using the fact that $y_i = x_i^T \theta^* + \xi_i$, we have $\lambda \geq \frac{2}{n} \|X^T \xi\|_{\infty}$. Then we apply the tail bound on the Gaussian variable and use union bound to obtain that

$$\frac{2}{n} \|X^T \xi\|_{\infty} \leq 6\sqrt{\frac{\log p}{n}},$$

holds with probability at least $1 - \exp(-3 \log p)$. 

\[\square\]
Proof of Corollary on Group Lasso, i.e., corollary 2. We use the following fact on the RSC condition of Group Lasso Negahban et al. [2009,2012]: if each data point $x_i$ is i.i.d. randomly sampled from the distribution $N(0, \Sigma)$, then there exists strictly positive constant $(\sigma_1(\Sigma), \sigma_2(\Sigma))$ which only depends on $\Sigma$ such that
\[
\frac{\|X\Delta\|^2}{2n} \geq \sigma_1(\Sigma)\|\Delta\|^2 - \sigma_2(\Sigma)(\sqrt{\frac{m}{n}} + \sqrt{\frac{3\log N\bar{G}}{n}})^2\|\Delta\|_{G,2},
\]
for all $\Delta \in \mathbb{R}^p$, with probability at least $1 - c_3 \exp(-c_4 n)$.

Remind we define the subspace
\[
\bar{M}(S_G) = M(S_G) = \{\theta | \theta_{G_i} = 0 \text{ for all } i \notin S_G\}
\]
where $S_G$ corresponds to non-zero group of $\theta^*$. The subspace compatibility can be computed by
\[
H(\bar{M}) = \sup_{\theta \in \bar{M}\setminus\{0\}} \frac{\|\theta\|_{G,2}}{\|\theta\|_2} = \sqrt{s_G}.
\]
Thus, the modified RSC parameter
\[
\bar{\sigma} = \sigma_1(\Sigma) - c\sigma_2(\Sigma)s_g(\sqrt{\frac{m}{n}} + \sqrt{\frac{3\log N\bar{G}}{n}})^2.
\]

We then bound the value of $\lambda$. As the regularizer in Group Lasso is $\ell_1$ grouped norm, its dual norm is $(\infty, 2)$ grouped norm. So it suffices to have any $\lambda$ such that
\[
\lambda \geq 2 \max_{i=1,\ldots,N_G} \frac{1}{n} \|X^T \xi_G_i\|_2.
\]
Using Lemma 5 in Negahban et al. [2009], we know
\[
\max_{i=1,\ldots,N_G} \frac{1}{n} \|X^T \xi_G_i\|_2 \leq 2\varsigma(\sqrt{\frac{m}{n}} + \sqrt{\frac{\log N\bar{G}}{n}})
\]
with probability at least $1 - 2 \exp(-2 \log N\bar{G})$. Thus it suffices to choose $\lambda = 4\varsigma(\sqrt{\frac{m}{n}} + \sqrt{\frac{\log N\bar{G}}{n}})$.

The tolerance is given by,
\[
\delta = 24\tau_\sigma(8H(\bar{M})\|\Delta^*\|_2 + 8\psi(\theta^*_{M,\perp}))^2
\]
\[
= c_2\sigma_2(\Sigma)s_g(\sqrt{\frac{m}{n}} + \sqrt{\frac{3\log N\bar{G}}{n}})^2\|\Delta^*\|_{G,2},
\]
where we use the fact $\psi(\theta^*_{M,\perp}) = 0$.

Proof of Corollary on SCAD, i.e., corollary 3. The proof is very similar to that of Lasso. In the proof of results for Lasso, we established
\[
\|\nabla f(\theta^*)\|_\infty = \frac{1}{n} \|X^T \xi\|_\infty \leq 3\varsigma \sqrt{\frac{\log p}{n}}
\]
and the RSC condition
\[
\frac{\|X\Delta\|^2}{n} \geq \frac{1}{2} \|\Sigma^{1/2}\Delta\|^2 - c_1\nu(\Sigma)\frac{\log p}{n} \|\Delta\|^2.
\]
Recall that $\mu = \frac{1}{\zeta - 1}$ and $L_g = 1$, we establish the corollary.
First notice
\[ \| \nabla f(\theta^*) \|_\infty = \| \hat{\Gamma} \theta^* - \hat{\gamma} \|_\infty = \| \hat{\gamma} - \Sigma \theta^* + (\Sigma - \hat{\Gamma}) \theta^* \|_\infty \leq \| \hat{\gamma} - \Sigma \theta^* \|_\infty + \| (\Sigma - \hat{\Gamma}) \theta^* \|_\infty. \]

As shown in literature (Lemma 2 in Loh and Wainwright [2011]), both terms on the right hand side can be bounded by
\[ c_1 \varphi \sqrt{\log p} n, \]
where \( \varphi \) is defined as \( (\sqrt{\sigma_{\text{max}}(\Sigma)} + \sqrt{\gamma_w})(\varsigma + \sqrt{\gamma_w}\|\theta^*\|_2) \), with probability at least \( 1 - c_2 \exp(-c_3 \log p) \).

To obtain the RSC condition, we apply Lemma 1 in Loh and Wainwright [2011], to get
\[ \frac{1}{n} \Delta^T \hat{\Gamma} \Delta \geq \frac{\sigma_{\text{min}}(\Sigma)}{2} \| \Delta \|_2^2 - c_3 \sigma_{\text{min}}(\Sigma) \max \left( \frac{\sigma_{\text{max}}(\Sigma) + \gamma_w}{\sigma_{\text{min}}(\Sigma)} \right)^2, 1 \right) \log p \| \Delta \|_1^2, \]
with probability at least \( 1 - c_4 \exp(-c_5 n \min \left( \frac{\sigma_{\text{max}}(\Sigma)}{\sigma_{\text{min}}(\Sigma) + \gamma_w \|\theta^*\|_2^2}, 1 \right) \). Combine these together, we establish the corollary.

References

Alekh Agarwal, Sahand Negahban, and Martin J Wainwright. Fast global convergence rates of gradient methods for high-dimensional statistical recovery. In Advances in Neural Information Processing Systems, pages 37–45, 2010.

Zeyuan Allen-Zhu and Elad Hazan. Variance reduction for faster non-convex optimization. arXiv preprint arXiv:1603.05643, 2016.

Aaron Defazio, Francis Bach, and Simon Lacoste-Julien. Saga: A fast incremental gradient method with support for non-strongly convex composite objectives. In Advances in Neural Information Processing Systems, pages 1646–1654, 2014.

Jianqing Fan and Runze Li. Variable selection via nonconcave penalized likelihood and its oracle properties. Journal of the American Statistical Association, 96(456):1348–1360, 2001.

Ian Goodfellow, Yoshua Bengio, and Aaron Courville. Deep Learning. MIT Press, 2016. http://www.deeplearningbook.org

I. Guyon. A pharmacology dataset, 06 2008. URL http://www.causality.inf.ethz.ch/data/SIDO.html

Davood Hajinezhad, Mingyi Hong, Tuo Zhao, and Zhaoran Wang. Nestt: A nonconvex primal-dual splitting method for distributed and stochastic optimization. In Advances in Neural Information Processing Systems, pages 3207–3215, 2016.

Reza Harikandeh, Mohamed Osama Ahmed, Alim Virani, Mark Schmidt, Jakub Konečný, and Scott Sallinen. Stopwasting my gradients: Practical svrg. In Advances in Neural Information Processing Systems, pages 2251–2259, 2015.

D. Harrison and D.L. Rubinfeld. UCI machine learning repository, 2013. URL http://archive.ics.uci.edu/ml

Rie Johnson and Tong Zhang. Accelerating stochastic gradient descent using predictive variance reduction. In Advances in Neural Information Processing Systems, pages 315–323, 2013.
Hamed Karimi, Julie Nutini, and Mark Schmidt. Linear convergence of gradient and proximal-gradient methods under the polyak-lojasiewicz condition. In *Joint European Conference on Machine Learning and Knowledge Discovery in Databases*, pages 795–811. Springer, 2016.

David D Lewis, Yiming Yang, Tony G Rose, and Fan Li. Rcv1: A new benchmark collection for text categorization research. *Journal of machine learning research*, 5 (Apr):361–397, 2004.

Xingguo Li, Tuo Zhao, Raman Arora, Han Liu, and Jarvis Haupt. Stochastic variance reduced optimization for nonconvex sparse learning. *arXiv preprint arXiv:1605.02711*, 2016.

Po-Ling Loh and Martin J Wainwright. High-dimensional regression with noisy and missing data: Provable guarantees with non-convexity. In *Advances in Neural Information Processing Systems*, pages 2726–2734, 2011.

Po-Ling Loh and Martin J Wainwright. Regularized m-estimators with nonconvexity: Statistical and algorithmic theory for local optima. In *Advances in Neural Information Processing Systems*, pages 476–484, 2013.

S Negahban, P Ravikumar, MJ Wainwright, and B Yu. Supplement to a unified framework for high-dimensional analysis of m-estimators with decomposable regularizers., 2012.

Sahand Negahban, Bin Yu, Martin J Wainwright, and Pradeep K Ravikumar. A unified framework for high-dimensional analysis of m-estimators with decomposable regularizers. In *Advances in Neural Information Processing Systems*, pages 1348–1356, 2009.

Yu Nesterov. Introductory lectures on convex programming volume i: Basic course. *Lecture notes*, 1998.

Yurii Nesterov. *Introductory lectures on convex optimization: A basic course*, volume 87. Springer Science & Business Media, 2013.

Danil Prokhorov. Ijcnn 2001 neural network competition, 2001.

Chao Qu, Yan Li, and Huan Xu. Linear convergence of svrg in statistical estimation. *arXiv preprint arXiv:1611.01957*, 2016.

Zheng Qu, Peter Richtárik, and Tong Zhang. Quartz: Randomized dual coordinate ascent with arbitrary sampling. In *Advances in neural information processing systems*, pages 865–873, 2015.

Ariadna Quattoni, Xavier Carreras, Michael Collins, and Trevor Darrell. An efficient projection for l1, infinity regularization. In *Proceedings of the 26th Annual International Conference on Machine Learning*, pages 857–864. ACM, 2009.

Garvesh Raskutti, Martin J Wainwright, and Bin Yu. Restricted eigenvalue properties for correlated gaussian designs. *Journal of Machine Learning Research*, 11(Aug):2241–2259, 2010.

Sashank J Reddi, Suvrit Sra, Barnabas Poczos, and Alex Smola. Fast stochastic methods for nonsmooth nonconvex optimization. *arXiv preprint arXiv:1605.06900*, 2016.
Mark Schmidt, Nicolas Le Roux, and Francis Bach. Minimizing finite sums with the stochastic average gradient. *arXiv preprint arXiv:1309.2388*, 2013.

Shai Shalev-Shwartz. Sdca without duality, regularization, and individual convexity. *arXiv preprint arXiv:1602.01582*, 2016.

Shai Shalev-Shwartz and Tong Zhang. Accelerated proximal stochastic dual coordinate ascent for regularized loss minimization. In *ICML*, pages 64–72, 2014.

Ohad Shamir. Fast stochastic algorithms for svd and pca: Convergence properties and convexity. *arXiv preprint arXiv:1507.08788*, 2015.

Grzegorz Swirszcz, Naoki Abe, and Aurelie C Lozano. Grouped orthogonal matching pursuit for variable selection and prediction. In *Advances in Neural Information Processing Systems*, pages 1150–1158, 2009.

Berwin A Turlach, William N Venables, and Stephen J Wright. Simultaneous variable selection. *Technometrics*, 47(3):349–363, 2005.

Shuo Xiang, Tao Yang, and Jieping Ye. Simultaneous feature and feature group selection through hard thresholding. In *Proceedings of the 20th ACM SIGKDD international conference on Knowledge discovery and data mining*, pages 532–541. ACM, 2014.

Lin Xiao. Dual averaging methods for regularized stochastic learning and online optimization. *Journal of Machine Learning Research*, 11(Oct):2543–2596, 2010.

Lin Xiao and Tong Zhang. A proximal-gradient homotopy method for the sparse least-squares problem. *SIAM Journal on Optimization*, 23(2):1062–1091, 2013.

Lin Xiao and Tong Zhang. A proximal stochastic gradient method with progressive variance reduction. *SIAM Journal on Optimization*, 24(4):2057–2075, 2014.

Ming Yuan and Yi Lin. Model selection and estimation in regression with grouped variables. *Journal of the Royal Statistical Society: Series B (Statistical Methodology)*, 68 (1):49–67, 2006.

Cun-Hui Zhang and Tong Zhang. A general theory of concave regularization for high-dimensional sparse estimation problems. *Statistical Science*, pages 576–593, 2012.

Yuchen Zhang and Xiao Lin. Stochastic primal-dual coordinate method for regularized empirical risk minimization. In *ICML*, pages 353–361, 2015.