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Highlights

**Extended shallow water wave equations**

Theodoros P. Horikis, Dimitrios J. Frantzeskakis, Noel F. Smyth

- We revisit the derivation of the extended versions of the KdV, BBM and Camassa-Holm equations in 1D, the extended cylindrical KdV equation in the quasi-1D setting, the extended Kadomtsev-Petviashvili and its cylindrical counterpart in 2D, as well the extended Green-Naghdi equations.

- All the extended equations are valid one order beyond the usual weakly nonlinear, long wave approximation and incorporate all appropriate dispersive and nonlinear terms.

- The connection and applicability of the shallow water wave equations to other physical contexts, including plasmas, nonlinear optics, Bose-Einstein condensation, solid mechanics and others, are discussed.
Extended shallow water wave equations

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Abstract

Extended shallow water wave equations are derived, using the method of asymptotic expansions, from the Euler (or water wave) equations. These extended models are valid one order beyond the usual weakly nonlinear, long wave approximation, incorporating all appropriate dispersive and nonlinear terms. Specifically, first we derive the extended Korteweg-de Vries (KdV) equation, and then proceed with the extended Benjamin-Bona-Mahony and the extended Camassa-Holm equations in (1+1)-dimensions, the extended cylindrical KdV equation in the quasi-one dimensional setting, as well as the extended Kadomtsev-Petviashvili and its cylindrical counterpart in (2 + 1)-dimensions. We conclude with the case of the extended Green-Naghdi equations.

Keywords: Shallow water waves, Nonlinear waves, Euler equations, Asymptotic expansions

1. Introduction

Water waves, forming and propagating on the surface of the ocean, are probably the most commonly observed natural phenomenon. While easy to observe, from a mathematical point of view their modelling and analysis is very demanding as these waves feature significant differences from waves propagating in other media. Indeed, water waves propagate on a free surface, which is determined as part of the solution, so that the water wave equations form a nonlinear free boundary value problem.

The study of water waves and their various ramifications remains central to fluid dynamics, and to the dynamics of the oceans in particular, and plays—in general—a significant role in applied mathematics and physics. The mathematical theory of water waves [1, 2, 3, 4], while being important on its own merit in fluid mechanics, has provided the solid background and impetus for the development of the theory of nonlinear dispersive waves in general, which has made a tremendous impact on numerous disciplines [5, 6]. Indeed, most of the fundamental ideas and results for nonlinear dispersive waves, and particularly of dispersive shock waves (alias undular bores in fluid mechanics), solitary waves and solitons, originated from the investigation of water waves [2, 7, 8].

Traditionally, the water wave problem [1, 2, 3, 4] consists of studying the motion of the free fluid surface and the evolution of the velocity field of the fluid under the following
assumptions: the fluid is ideal, incompressible, irrotational and under the influence of gravity and/or surface tension. As stated, this forms a nonlinear free surface problem for which there are no general solutions. For this reason, various reductions of the full water wave equations in various limits have been widely studied, including linear (small amplitude) waves, weakly nonlinear waves, long waves and shallow water waves [2, 9]. These limiting cases are not only attractive from a mathematical point of view, but they also have wide applicability to the successful modelling of waves in nature [10, 11, 12, 13, 14, 15]. In this work, we shall construct equations which govern waves on the free surface of water, especially in the shallow water regime of weakly nonlinear long waves. This limit results in ubiquitous equations, such as the Korteweg-de Vries (KdV) equation [2, 8]

\[ u_t + 6uu_x + u_{xxx} = 0, \] (1)

where subscripts denote partial derivatives. The KdV equation is applicable to fields outside of water wave theory, for instance plasma physics [5, 16, 17], mechanical and electrical lattices [6], as well as in physical contexts where the defocusing nonlinear Schrödinger (NLS) equation plays a key role, e.g., nonlinear optics [18], nematic liquid crystals [19] and atomic Bose-Einstein condensates (BECs) [20, 21]. In these latter contexts, the fact that the defocusing NLS can be asymptotically reduced to the KdV equation [22] (and, in two dimensions (2D), to the Kadomtsev-Petviashvili equation [23]—see below) allows for an effective description of dark NLS solitons in terms of KdV solitons. This, in turn, enables the study of solitary waves of perturbed NLS models that have the form of KdV solitons (see, e.g., Ref. [24] and references therein). It is also important to mention that the KdV equation is the generic nonlinear dispersive wave equation which is exactly integrable in a Hamiltonian sense, and leads to the field of inverse scattering transform and soliton solutions [7, 2].

The derivation of weakly nonlinear dispersive wave equations, such as the KdV equation and the Boussinesq system [2]—which is the bidirectional equivalent of the KdV equation—is based on an asymptotic expansion of the water wave equations in the small parameters of the wave amplitude to depth ratio and the depth to wavelength ratio. Balancing these two small parameters results in these equations [2]. Truncating the KdV asymptotic expansion at the order of the small amplitude squared results in the standard KdV equation. However, it has been found that, for many applications, higher order terms in this asymptotic expansion are needed to adequately model physical waves. The inclusion of all the next order terms in the KdV asymptotic expansion results in the extended KdV (eKdV) equation [25]

\[ u_t + 6uu_x + u_{xxx} + c_1u^2u_x + c_2uu_{xxx} + c_3uu_xu_{xx} + c_4u_{xxxx} = 0, \] (2)

where \( c_1 = -\alpha, \ c_2 = 23/6\alpha, \ c_3 = 5/3\alpha \) and \( c_4 = 19/60\alpha \) for the water wave eKdV equation, where \( \alpha \) is the amplitude to depth ratio [25].

The eKdV equation has been used to study solitary wave interaction [26, 27], undular bores [28], the resonant flow of a fluid over topography [29], as well as ion-acoustic waves in plasmas [30], and, more recently, dark solitons in weakly nonlocal nonlinear media [31]. In particular, the inclusion of the higher order terms of the eKdV equation was found to be vital to model and obtain agreement with experimental results on undular bores in solid
mechanics, with the KdV equation found to be an inadequate model [32]. Of relevance to the present review, the equations governing nonlinear optical beams in dye doped nematic liquid crystals, which have a defocusing response (i.e., the refractive index decreases with the optical intensity) can be reduced to the eKdV equation [33, 34]. The higher order corrections present in the eKdV equation are vital for the correct description of beam evolution, in particular for dispersive shock waves as these are resonant. The dispersion relation of the KdV equation is convex, so that resonance between an undular bore and dispersive radiation is not possible, while the higher order terms of the eKdV equation result in a non-convex linear dispersion relation, so that resonance between an undular bore and linear dispersive waves is possible [33, 34].

A particular case of the eKdV is the Kawahara equation [35], for which the only higher order correction is a linear 5th-order dispersive term, i.e., \( c_1 = c_2 = c_3 = 0 \). The Kawahara equation arises through the inclusion of capillary effects in the limit of the Bond number being near \( 1/3 \) [36]. It is important to point out that the inclusion of higher order terms, and in particular of 5th-order dispersion, which can lead to a non-convex linear dispersion relation, is vital for new effects not encompassed by the KdV equation. Such effects include resonant solitary waves [37] and resonant undular bores [36], for which the solitary waves or waves of the undular bore are in resonance with dispersive radiation. Solitary wave and undular bore resonance is strong for the Kawahara equation [37, 36] and the resonant wavetrain can destroy the classic undular bore structure, leaving just the resonant wavetrain [36]. While the eKdV equation contains 5th-order dispersion, the strength of the resonance is highly dependent on the relationship between the coefficients of the higher order terms [28]. For the water wave coefficients, the amplitude of the resonant wavetrain is very small [38].

Another special case of the eKdV equation is the Gardner equation for which higher order nonlinearity dominates over higher order dispersion, so that the only non-zero higher order coefficient in the eKdV equation (2) is \( c_1 \). The Gardner equation is integrable, as for the KdV equation, and first arose in the derivation of the infinite number of conservation laws for the KdV equation [39]. Since then it has been widely studied and used to model water waves, for example to study large amplitude surface and internal waves [11, 40, 41], large amplitude undular bores [42] and resonant flow over topography [25, 43, 44]. Furthermore, the Gardner equation has been used in studies in plasma physics [5], while it has recently been proposed as a model describing internal, bright and dark, rogue waves in three layer fluids [45].

In addition to these one dimensional (1D) models, equations for quasi 1D propagation, as, e.g., in the case of radially symmetric wave structures, have been derived and analyzed in the water waves context, as well as in other branches of physics. In the quasi-1D setting, a pertinent model is the cylindrical KdV (cKdV) equation, which was derived in Ref. [46] for water waves, and later was used to describe cylindrical solitary waves in plasmas [47], in nonlinear optical media with a local [48, 49] or a nonlocal [50, 51] nonlinearity, and in atomic BECs [52]. Additionally, higher order variants of the circular Korteweg-de Vries (cKdV) equation have also been proposed and studied in the contexts of water waves [53] and plasma physics [54].
On the other hand, a key model in the 2D setting is the \((2 + 1)\) dimensional equivalent of the KdV equation, the Kadomtsev-Petviashvili (KP) equation \([55]\), which incorporates weak lateral dispersion. Various types of generalized KP equations, bearing various types of nonlinearity or incorporating 5th order dispersion (from the dispersion relation of the water wave equations), have been studied. In particular, theoretical results concerning the local well-posedness of higher order KP equations \([56]\), as well as the existence and non-existence of localized solitary wave solutions of these generalized KP systems \([57, 58]\), have been reported; see the survey \([59]\). Importantly, much like the KdV equation, the KP equation and its variants appear in many contexts—beyond water waves—as an effective model for the study of the transverse dynamics of 1D solitons in a 2D setting, for the description of weakly localized 2D solitons, so-called “lumps” \([7]\), as well as for studies of soliton interactions. Thus, the KP equation has been used in the study of the transverse instability of dark NLS solitons \([23]\) to predict the existence of 2D solitons in plasmas \([5]\), optical media \([60]\), atomic BECs \([61]\) and exciton-polariton superfluids \([62]\), as well as to effectively describe soliton interactions in nonlocal nonlinear media \([63]\).

Furthermore, higher order KP equations, as well versions in cylindrical coordinates—first derived for water waves \([46]\) and then used extensively in plasma physics \([5]\)—have also appeared in many other contexts. Indeed, in higher dimensions, the defocusing nematic equations describing nonlinear optical beam propagation in nematic liquid crystals can be reduced to the KP equation \([64, 63, 65]\) and the cKdV equation \([66]\). While the higher order corrections to these equations have not, as yet, been derived, these are anticipated to be of the same form as the extended KP (eKP) and extended cylindrical KdV (eKdV) equations of the present review. These higher order equations are necessary to model \((2 + 1)\) dimensional dispersive shock waves in nematic liquid crystals. Similar equations apply to optical beams in thermal nonlinear media \([67, 68, 69]\), such as lead glasses \([70, 71, 72]\) and liquids \([69]\), and some photorefractive crystals \([68, 73, 74, 75]\). In this respect, notice that thermal optical media are usually defocusing and so support optical undular bores.

The aim of this work is the derivation and systematisation of various extended, dispersive shallow water wave equations arising as asymptotic approximations from the water wave equations. As is evident from the above discussion, the theme of extended shallow water models is not only important for the water wave problem, but also for many other physically important applications. The fact that water wave models may appear generically as asymptotic reductions of other nonlinear evolution equations appearing, e.g., in plasmas, optical systems, BECs, solid mechanics, etc, makes the methodology for their systematic derivation particularly relevant and important. This is also especially so because the derivation of the higher order corrections to other nonlinear dispersive wave equations, e.g., the NLS equation, is similar to the derivations of the present review \([76]\).

Furthermore, since our focus is on extended shallow water wave equations, novel effects—that can not be captured by their “non-extended” counterparts—are predicted to occur. Such effects are pertinent to the evolution of steeper waves with shorter wavelengths, or pulse-widths as in the case of pulse propagation in nonlinear optical fibres \([76, 77]\), for which higher order corrections have been found to be particularly important. In addition, from a mathematical point of view, the study of extended water wave models is relevant to the
theory of integrable systems. Indeed, the eKdV equation, for instance, stems naturally from the underlying Hamiltonian systems [78, 79], and is related to the first higher-order equation in the KdV hierarchy [80]. In addition, the eKdV equation has been proved to be an important model in the context of asymptotic integrability of weakly dispersive nonlinear wave equations [81, 82]. It is thus anticipated that the derivation and presentation of extended shallow water wave equations will be relevant to—and will inspire new studies in—a variety of physical contexts and applied mathematics.

The organization of our presentation is as follows. Firstly, in Section 2, we present the Euler equations and discuss the asymptotic regimes for which the shallow water wave equations are relevant. In Section 3, we revisit the derivation of the extended KdV equation [25], and then derive other extended models in (1 + 1) dimensions, namely the extended Benjamin-Bona-Mahony (BBM) and Camassa-Holm (CH) equations. In Section 4 we consider a quasi-1D setting for shallow water waves of radial symmetry, and derive the extended circular KdV (cKdV) equation. In Section 5, we deal with waves in (2 + 1) dimensions and present the derivation of extended KP equations in both Cartesian and cylindrical coordinates. Furthermore, for completeness, in Section 6, we revisit the derivation of the extended Green-Naghdi (GN) equations [83] in both (1 + 1) and (2 + 1) dimensions. Finally, in Section 7, we summarize our conclusions.

2. Mathematical formulation and asymptotic regimes

We consider waves on the surface of an incompressible, inviscid fluid of undisturbed depth $h$. The bottom boundary is taken to be flat. The $\tilde{z}$ direction is taken vertically upwards, opposite to the direction of gravity, and the $(\tilde{x}, \tilde{y})$ directions are horizontal. Typical wavelengths in the $\tilde{x}$ and $\tilde{y}$ directions are $\lambda_x$ and $\lambda_y$, respectively, and a typical wave amplitude is $a$. For simplicity, the water wave equations are transformed to non-dimensional form. The space variables $(\tilde{x}, \tilde{y}, \tilde{z})$ are non-dimensionalised in the horizontal directions using the typical wavelengths, $\tilde{x} = \lambda_x x$, $\tilde{y} = \lambda_y y$, and in the vertical direction using the depth, $\tilde{z} = h z$. The velocity potential $\phi$ is made non-dimensional based on the typical value $\lambda_x g a/c_0$, $\phi = (\lambda_x g a/c_0) \phi$, where $g$ is the acceleration due to gravity and $c_0 = \sqrt{gh}$ is the linear long wave speed. Time $\tilde{t}$ is non-dimensionalised on the typical value $\lambda_x/c_0$, $\tilde{t} = (\lambda_x/c_0) t$. Finally, the surface displacement $\tilde{\eta}$ is non-dimensionalised by the typical wave amplitude $a$, $\tilde{\eta} = a \eta$. The bulk motion of the fluid is governed by Laplace’s equation for the velocity potential $\phi$ [2]

$$
\mu^2 (\phi_{xx} + \delta^2 \phi_{yy}) + \phi_{zz} = 0, \quad -1 < z < \varepsilon \eta.
$$

This is solved together with the impenetrable boundary condition at the fluid bottom,

$$
\phi_z = 0, \quad z = -1,
$$

and the dynamic and kinematic boundary conditions at the fluid surface $z = \varepsilon \eta$, respectively,

$$
\phi_t + \frac{1}{2} \varepsilon \left[ \phi_x^2 + \delta^2 \phi_y^2 + \frac{1}{\mu^2} \phi_z^2 \right] + \eta = 0, \quad z = \varepsilon \eta,
$$

$$
\mu^2 \left[ \eta_t + \varepsilon \left( \phi_x \eta_x + \delta^2 \phi_y \eta_y \right) \right] = \phi_z, \quad z = \varepsilon \eta.
$$
The dimensionless parameters appearing in these equations are $\varepsilon = a/h$, which is a measure of nonlinearity, $\delta = \lambda_x/\lambda_y$ which measures the ratio of the wavelengths in the $x$- and $y$-directions, and $\mu = h/\lambda_x$, which measures the strength of dispersion. In the present work we consider weakly nonlinear long waves such that the wavelength is much greater than the water depth, i.e., $\mu \ll 1$, and the wave amplitude is much less than the fluid depth, so that $\varepsilon \ll 1$. Our asymptotic analysis of the water wave equations exploits these small parameters, which has proven to be a powerful tool for deriving shallow water wave models of great utility, see, e.g., Refs. [3, 84]. Other formulations [85] that may also include varying topography and more general boundary conditions will be studied in another communication.

To help put the many approximations to the water wave equations derived in this work in context, the domains of validity of these various approximate equations are illustrated in a schematic fashion in Fig. 1 in the dispersion, measured by $\mu$, and nonlinearity, measured by $\varepsilon$, plane.

The focus of this review is nonlinear, dispersive asymptotic reductions of the water wave equations. The limit of no dispersion results in the shallow water equations [2] and the linear amplitude limit results in classical linear dispersive theory, which are not dealt with in the current review. These two limits are extensively dealt with in standard texts, see [2], for instance. An extension of KdV-type equations to include full dispersion is the Whitham equation [2, 86] and its extensions, as discussed in Sections 3.1 and 3.3. The inclusion of
full dispersion results in solutions of these equations having effects, such as peaking and breaking, not present in KdV-type approximations.

3. (1+1) dimensional equations

3.1. The extended Korteweg-de Vries equation

The eKdV equation will be rederived here from the water wave equations in a slightly different manner to that of [25] in order to motivate the derivation of the other higher order weakly nonlinear, weakly dispersive equations of this work and to put them into context. This same procedure was later used to further extend the theory, which goes beyond the KdV equation [87]. KdV-type approximations lie around the line $\varepsilon = \mu^2$ in Figure 1. To derive the KdV equation from the above Euler system (3)–(6) consider the case $\delta = 0$, so that waves propagate in one spatial dimension and obey the reduced Euler system

$$
\phi_{zz} + \varepsilon \phi_{xx} = 0, \quad -1 < z < \varepsilon \eta, \quad (7a)
$$

$$
\frac{\partial \phi}{\partial z} = 0, \quad z = -1, \quad (7b)
$$

$$
\phi_t + \frac{1}{2} \varepsilon \left( \phi_x^2 + \frac{1}{\varepsilon} \phi_z^2 \right) + \eta = 0, \quad z = \varepsilon \eta, \quad (7c)
$$

$$
\varepsilon (\eta_t + \varepsilon \phi_x \eta_x) = \phi_z, \quad z = \varepsilon \eta. \quad (7d)
$$

We first solve Laplace’s equation with the bottom boundary condition and then substitute into the remaining free surface boundary conditions. As such, we expand the velocity potential $\phi$ as

$$
\phi = \phi_0 + \varepsilon \phi_1 + \varepsilon^2 \phi_2 + \varepsilon^3 \phi_3 + \cdots, \quad (8)
$$

and then substitute this into Laplace’s equation (7a), which results in

$$
\phi_{0zz} + \varepsilon (\phi_{1zz} + \phi_{0xx}) + \varepsilon^2 (\phi_{2zz} + \phi_{1xx}) + \varepsilon^3 (\phi_{3zz} + \phi_{2xx}) + \cdots = 0.
$$

After applying the bottom boundary condition (7b), we find

$$
\phi_0(x, z, t) = A(x, t), \quad \phi_1(x, z, t) = -\frac{(z+1)^2}{2} A_{xx},
$$

$$
\phi_2(x, z, t) = \frac{(z+1)^4}{24} A_{xxxx}, \quad \phi_3(x, z, t) = -\frac{(z+1)^6}{720} A_{xxxxx}.
$$

Finally, substituting for the potential $\phi$ into the dynamic (7c) and kinematic (7d) boundary conditions, we find that up to $O(\varepsilon^2)$

$$
\eta + \varepsilon A_t + \varepsilon^2 \left[ \frac{1}{2} A_{x}^2 - \frac{1}{2} A_{xx} \right] + \varepsilon^2 \left[ \frac{1}{2} A_{xx}^2 - \eta A_{xxx} - \frac{1}{2} A_x A_{xxx} + \frac{1}{24} A_{xxxxx} \right] = 0, \quad (9)
$$

$$
\eta_t + \varepsilon A_{xx} + \varepsilon^2 \left[ \frac{1}{6} (\eta A_x)_x \right] + \varepsilon^2 \left[ -\frac{1}{2} (\eta A_{xxx})_x + \frac{1}{120} A_{xxxxx} \right] = 0. \quad (10)
$$
Differentiating Eq. (9) with respect to $x$ and defining $w = A_x$, Eqs. (9)–(10) can finally be cast into the form

\begin{align*}
w_t + \eta_x + \varepsilon \left[ w w_x - \frac{1}{2} w_{xxx} \right] + \varepsilon^2 \left[ -(\eta w_x)_x + \frac{1}{2} w_x w_{xx} - \frac{1}{2} w w_{xxx} + \frac{1}{24} w_{xxxx} \right] &= 0, \quad (11) \\
\eta_t + w_x + \varepsilon \left( (\eta w)_x - \frac{1}{6} w_{xxx} \right) + \varepsilon^2 \left[ -\frac{1}{2} \eta w_{xx} + \frac{1}{120} w_{xxxx} \right] &= 0, \quad (12)
\end{align*}

up to $O(\varepsilon^2)$. The above equations, (11) and (12), constitute a higher order Boussinesq-type system [2]. In order for the two equations to be compatible, let us define

$$w = \eta + \varepsilon w_1 + \varepsilon^2 w_2 + O(\varepsilon^3),$$

so that now the system (11)–(12) is

\begin{align*}
\eta_t + \eta_x + \varepsilon \left[ w_{1t} + \eta_{1x} - \frac{1}{2} \eta_{ext} \right] \\
+ \varepsilon^2 \left[ w_{2t} + (\eta w_1)_x - \frac{1}{2} w_{1xxx} - (\eta_{1x})_x + \frac{1}{2} \eta_x \eta_{xx} - \frac{1}{2} \eta \eta_{xxx} + \frac{1}{24} \eta_{xxxx} \right] &= 0, \\
\eta_t + \eta_x + \varepsilon \left[ w_{1x} + 2 \eta_{1x} - \frac{1}{6} \eta_{xx} \right] \\
+ \varepsilon^2 \left[ w_{2x} + (\eta w_1)_x - \frac{1}{6} w_{1xxx} - \frac{1}{2} (\eta \eta_{xx})_x + \frac{1}{120} \eta_{xxxx} \right] &= 0.
\end{align*}

Direct comparison of these two equations, and replacing the time derivatives as

$$\eta_t = -\eta_x - \varepsilon \left( \frac{3}{2} \eta_{1x} + \frac{1}{6} \eta_{xxx} \right),$$

results in

$$w = \eta + \varepsilon \left( -\frac{1}{4} \eta^2 + \frac{1}{3} \eta_{xx} \right) + \varepsilon^2 \left( \frac{1}{8} \eta^3 + \frac{3}{16} \eta_x^2 + \frac{1}{2} \eta \eta_{xx} + \frac{1}{10} \eta_{xxx} \right) + O(\varepsilon^3).$$

This finally leads to the extended KdV equation, incorporating higher order dispersive and nonlinear terms at one order beyond the KdV approximation

$$\eta_t + \eta_x + \varepsilon \left( \frac{3}{2} \eta_{1x} + \frac{1}{6} \eta_{xxx} \right) + \varepsilon^2 \left( -\frac{3}{8} \eta^2 \eta_x + \frac{23}{24} \eta_x \eta_{xx} + \frac{5}{12} \eta \eta_{xxx} + \frac{19}{360} \eta_{xxxx} \right) = 0, \quad (13)$$

as found in [25] on a slight rescaling of the equation. This eKdV equation with these coefficients on the higher order $\varepsilon^2$ terms is strictly applicable to water waves. However, it is applicable in the more general form (2) to other weakly nonlinear, weakly dispersive waves, such as internal waves in a stratified fluid, with the coefficients $c_i, i = 1, \ldots, 4$, taking values applicable to the specific application. The higher order terms in the eKdV equation (2) introduce new effects not present for the integrable KdV equation. In particular, fifth order dispersion leads to resonance between solitary waves, undular bores and other structures and the eKdV equation is then not integrable [36].
3.2. The extended Benjamin-Bona-Mahony equation

Another scalar \((1+1)\)-dimensional model associated with weakly nonlinear long wave reductions of the water wave equations and the bidirectional Boussinesq system is the Benjamin-Bona-Mahony (BBM) equation [88]

\[ \eta_t + \eta_x + \epsilon \left( \frac{3}{2} \eta \eta_x - \frac{1}{6} \eta_{xxt} \right) = 0. \]

While this model was originally introduced for the study of undular bores [89] (cf. Ref. [90] for a classification of solutions of the dispersive Riemann problem for the BBM equation), in principle, numerical solutions of the BBM equation have improved stability over those of the KdV equation under the unidirectional assumption in \((1+1)\) dimensions due to its bounded dispersion relation. The issue of the improved numerical stability of the BBM equation over the KdV equation has become less of an issue due to the development of improved numerical methods for the KdV equation, see [91, 92], for instance. Note that, contrary to the infinite number of integrals of motion of the KdV equation, the BBM equation possesses only three conservation laws and is not integrable, so that it has received much less attention than the KdV equation. The properties of surface water waves in a channel governed by the BBM equation have been discussed in Ref. [93].

At leading order the KdV and BBM equations are closely related as they are asymptotically the same. At \(O(\epsilon)\) the change \(\eta_x = -\eta_t\) can be made in the dispersive term of the eKdV equation (13), since this relation holds at \(O(1)\). In a similar fashion, the change

\[ \eta_{xxx} = -\eta_{xxt} - \epsilon \frac{\partial^2}{\partial x^2} \left( \frac{3}{2} \eta \eta_x - \frac{1}{6} \eta_{xxt} \right), \]

in the eKdV Eq. (13) results in the higher order correction of the BBM equation

\[ \eta_t + \eta_x + \epsilon \left( \frac{3}{2} \eta \eta_x - \frac{1}{6} \eta_{xxt} \right) + \epsilon^2 \left( \frac{3}{8} \eta^2 \eta_x + \frac{5}{24} \eta_x \eta_{xx} - \frac{1}{6} \eta_{xxt} - \frac{1}{40} \eta_{xxxx} \right) = 0, \tag{14} \]

which is the extended BBM (eBBM) equation. Alternatively, the compatibility condition for Eqs. (11) and (12) in the derivation of the eKdV equation may be constructed as

\[ w = \eta + \epsilon \left( -\frac{1}{4} \eta^2 - \frac{1}{3} \eta_{xt} \right) + \epsilon^2 \left( \frac{1}{8} \eta^3 - \frac{5}{16} \eta_x^2 - \frac{2}{45} \eta_{xxx} \right) + O(\epsilon^3). \]

Then the change

\[ \eta_{xxt} = -\eta_{xxt} - \epsilon \frac{\partial^2}{\partial x \partial t} \left( \frac{3}{2} \eta \eta_x - \frac{1}{6} \eta_{xxt} \right), \]

also results in the eBBM equation (14).
3.3. The extended Camassa-Holm equation

Both the KdV and BBM equations are globally well posed and, therefore, these models are not able to capture wave breaking (recall that, in both models, dispersion exactly balances nonlinearity, $\epsilon = \mu^2$). Nevertheless, there exist other shallow water wave equations which avoid this drawback by including either short wave effects or stronger nonlinearity. One such model is the so-called Whitham equation [2, 86, 94], which includes short wave effects by replacing the dispersive term $u_{xxx}$ in the KdV equation by a Fourier integral whose kernel is the full water wave dispersion relation, so that the Whitham equation applies in a region to the right of the KdV region in Figure 1 with higher $\mu$. It has been shown that the Whitham equation possesses solutions showing both peaking and breaking, as for real water waves [2, 94]. While Whitham derived this equation on an ad-hoc basis, it has subsequently been shown that it can be rigorously derived from the Hamiltonian formulation of the water wave equations [86]. Another model, characterized by a stronger nonlinearity, so that it lies above the KdV region in Figure 1, is the Camassa-Holm (CH) equation [95]

$$u_T + \frac{10}{19} u_X + \epsilon \left( \frac{3}{2} uu_X - \frac{19}{60} u_{TXX} \right) + \epsilon^2 \left(- \frac{19}{60} u_X u_{XX} - \frac{19}{120} uu_{XXX} \right) = 0,$$

where the variables $X$ and $T$ are defined in a travelling frame. Interestingly, the CH equation was first discovered as a completely integrable equation [96], while its relevance to the description of shallow water waves gained notice much later [95]. As seen by its form, the CH equation captures stronger nonlinear effects than the nonlinear dispersive KdV and BBM equations and can indeed accommodate wave breaking phenomena [97]. Furthermore, the nonlinear dispersive terms $u_X u_{XX}$ and $uu_{XXX}$ in this equation are the same as the higher order nonlinear dispersive terms in the eKdV equation (13).

Below, we shall show that the starting point for the derivation of the CH equation, and its extended form, the eCH equation, is the KdV equation. Indeed, the CH equation cannot be derived directly from the Euler system using the analysis of Section 3.1 for the eKdV equation [98, 99]. On the other hand, we note that, alternatively, the CH equation can be derived by a variational approach in a Lagrangian formalism of the water wave equations [100], or via the Green-Naghdi equations [98].

Before proceeding with the derivation of the extended CH equation, it is useful to make some remarks. The integrable KdV equation emerges at first order in an asymptotic expansion for unidirectional shallow water waves. However, at quadratic order, this asymptotic expansion produces an entire family of shallow water wave equations that are asymptotically equivalent to each other under a group of nonlinear, nonlocal, normal form transformations [101] in combination with the application of the Helmholtz operator. These transformations can be used to present connections between shallow water waves, the integrable fifth-order KdV equation and a generalization of the CH equation that contains an additional integrable case. The linear dispersion relation for the full water wave equations for arbitrary depth and any equation in this family agree to fifth order. The travelling wave solutions of the CH equation have been shown to asymptotically agree with the exact solution of the fifth order KdV equation [102]. The formal analysis for parameter ranges consistent with
the asymptotic derivation of the CH equation can be found in Ref. [103], while it has been shown that the solutions of the higher order KdV equation can be related to solutions of the CH equation [104].

To derive the higher order, to \(O(\varepsilon^3)\), correction to the KdV equation we use the water wave equations, Eqs. (7), and take into account the relevant additional terms. Following the methodology of Section 3.1, we first derive the higher order Boussinesq system to \(O(\varepsilon^2)\) using an expansion of the form (8), namely

\[
\begin{align*}
  w_t + \eta_x + \varepsilon \left[ w w_x - \frac{1}{2} w_{xxx} \right] + \varepsilon^2 \left[ -\left( \eta w_x \right)_x + \frac{1}{2} w_x w_{xx} - \frac{1}{2} w w_{xxx} + \frac{1}{24} w_{xxxx} \right] \\
  + \varepsilon^3 \left[ -w_x^2 \eta_x - \eta w x w_{xxx} + \frac{1}{2} \eta^2 w_{xxx} - \eta w w_{xxx} + \frac{1}{12} w_{xxx} w_{xx} \right] \\
  + \varepsilon^3 \left[ \frac{1}{6} \eta_x w_{xxx} - \frac{1}{8} w_x w_{xxx} + \frac{1}{6} \eta w_{xxx} + \frac{1}{24} w w_{xxx} - \frac{1}{720} w_{xxxx} w_{xx} \right] = 0, \\

  \eta_t + w_x + \varepsilon \left[ \left( \eta w \right)_x - \frac{1}{6} w_{xxx} \right] + \varepsilon^2 \left[ -\frac{1}{2} \left( \eta w_x \right)_x + \frac{1}{120} w_{xxx} \right] \\
  + \varepsilon^3 \left[ -\eta \eta w_{xxx} - \frac{1}{2} \eta^2 w_{xxx} + \frac{1}{24} \left( \eta w_{xxx} \right)_x - \frac{1}{5040} w_{xxxxxx} \right] = 0.
\end{align*}
\]

To make these two equations compatible, we set

\[
w = \eta + \varepsilon w_1 + \varepsilon^2 w_2 + \varepsilon^3 w_3 + O(\varepsilon^4),
\]

with

\[
\begin{align*}
  w_1 &= -\frac{1}{4} \eta^2 + \frac{1}{3} \eta_{xx}, \\
  w_2 &= \frac{1}{8} \eta^3 + \frac{3}{16} \eta_x^2 + \frac{1}{2} \eta \eta_{xx} + \frac{1}{10} \eta_{xxxx}, \\
  w_3 &= \frac{3}{16} \eta_x \eta_{x}^{-1} (\eta_{x}^3) - \frac{5}{64} \eta^4 + \frac{3}{32} \eta_x^2 \eta_{xx} + \frac{163}{360} \eta_x^2 \eta_{xx} + \frac{1091}{1440} \eta_x \eta_{xxx} \\
  &+ \frac{7}{20} \eta_{xxx} + \frac{61}{1890} \eta_{xxxxx},
\end{align*}
\]

where

\[
\partial_{X}^{-1}(u) = \int_{-\infty}^{X} u(X', T) \, dX'.
\]

In this manner, we derive the higher order, \(O(\varepsilon^3)\), KdV equation

\[
\begin{align*}
  \eta_t + \eta_x + \varepsilon \left( \frac{3}{2} \eta \eta_x + \frac{1}{6} \eta_{xxx} \right) + \varepsilon^2 \left( -\frac{3}{8} \eta^2 \eta_x + \frac{5}{12} \eta \eta_{xxx} + \frac{23}{24} \eta_x \eta_{xx} + \frac{19}{360} \eta_{xxxx} \right) \\
  + \varepsilon^3 \left( \frac{3}{16} \eta^3 \eta_x + \frac{23}{16} \eta \eta_x \eta_{xx} + \frac{5}{16} \eta^2 \eta_{xxx} + \frac{19}{32} \eta^3 \eta_x + \frac{1079}{1440} \eta_x \eta_{xxx} \\
  + \frac{317}{288} \eta_{xx} \eta_{xxx} + \frac{19}{80} \eta_{xxxx} + \frac{55}{3024} \eta_{xxxxxx} \right) = 0.
\end{align*}
\]
To transform the above $O(\varepsilon^3)$ version of the KdV equation into a form which can be rescaled to the CH equation, first we introduce the Galilean transformation

$$X = x - \frac{9}{19}t, \quad T = t,$$

and perform the Kodama transformation [101]

$$\eta(X, T) = u + \varepsilon \left[ \frac{7}{20}u^2 + \frac{1}{30}u_{XX} - \frac{1}{5}u_X \partial_X^{-1}(u) \right],$$

to express the resulting equation in terms of the new dependent variable $u = u(X, T)$. To this end, applying the Helmholtz operator

$$\mathcal{H} = 1 - \varepsilon \frac{19}{60} \partial_X^2$$

to the resulting equation, we find that the third order KdV Eq. (16) becomes the extended CH (eCH) equation

$$u_T + \frac{10}{19}u_X + \varepsilon \left[ \frac{3}{2}uu_X - \frac{19}{60}u_{TXX} \right] + \varepsilon^2 \left[ -\frac{19}{60}u_Xu_{XX} - \frac{19}{120}uu_{XXX} \right] + \varepsilon^3 \left[ \frac{223}{151200}u_{XXXXXX} - \frac{3}{100}u_{XX} \left( u^2 - 2u_X \partial_X^{-1}(u) \right) \partial_X^{-1}(u) \right] + \frac{\varepsilon^3}{2400} \left[ 976uu_Xu_{XX} + 680u^2u_{XXX} + 2765u_X^3 - 48uu_{XXX} \partial_X^{-1}(u) + 48u_X^2 \partial_X^{-1}(u) \right] + \frac{\varepsilon^3}{3600} \left[ 903u_Xu_{XXX} + 316uu_{XXX} + 305u_{XX}u_{XXX} \right] = 0. \quad (18)$$

4. The cylindrical KdV equation

The above derivations of the extended KdV, BBM and CH equations have been for (1+1) dimensional waves, that is waves propagating in one horizontal direction only. It is clear that most water waves in nature are not one dimensional (1D), so the extended equations of the previous Sections will now be extended to (2 + 1) dimensions. We first consider the quasi-1D setting and consider the water wave equations (3)–(6) with radial symmetry. In this case, the water wave equations can be expressed in cylindrical polar coordinates as

$$\phi_{zz} + \varepsilon \left( \phi_{rr} + \frac{1}{r} \phi_r \right) = 0, \quad (19a)$$

$$\phi_z = 0, \quad z = -1, \quad (19b)$$

$$\phi_t + \frac{\varepsilon}{2} \left( \phi_r^2 + \frac{1}{\varepsilon} \phi_z^2 \right) + \eta = 0, \quad z = \varepsilon \eta, \quad (19c)$$

$$\varepsilon(\eta_t + \varepsilon \phi_r \eta_r) = \phi_z, \quad z = \varepsilon \eta. \quad (19d)$$
From this system, Johnson [46] derived (see also [3]) the so-called cylindrical KdV (cKdV) equation

\[ H_T + \frac{3}{2}HH_R + \frac{1}{6}H_{RRR} + \frac{1}{2T}H = 0. \]  

(20)

Here, \( H \) is the rescaled wave amplitude of the free boundary, which depends on the independent slow variables \( R \) and \( T \), defined in a proper travelling frame (see below). It should be noted that the cKdV equation was presented in an earlier work by the same author [105], where it was also shown that the Inverse Scattering Transform for the cKdV can be obtained directly from that for the Kadomtsev-Perviashvili (KP) equation. Since this work, relevant cKdV models have been derived with the addition of surface tension [53], while experiments were performed which compared axisymmetric free surface solitary waves with theoretical and numerical solutions of the cKdV equation, with good agreement found [106].

Following the above mentioned previous work [3, 46], we introduce the new variables

\[ R = \varepsilon(r - t), \quad T = \varepsilon^4t, \quad \phi = \varepsilon\Phi, \quad \eta = \varepsilon^2H, \]

and expand the scaled velocity potential \( \Phi \) asymptotically as

\[ \Phi = \Phi_0 + \varepsilon^3\Phi_1 + \varepsilon^6\Phi_2 + \varepsilon^9\Phi_3 + \cdots. \]

Laplace’s equation (19a) then gives

\[
(R + T/\varepsilon^3)\Phi_{0zz} + T(\Phi_{1zz} + \Phi_{0RR}) + \varepsilon^3[R\Phi_{1zz} + T\Phi_{2zz} + (R\Phi_{0R})_R + T\Phi_{1RR}] + \varepsilon^6[R\Phi_{2zz} + T\Phi_{3zz} + (R\Phi_{1R})_R + T\Phi_{2RR}] = O(\varepsilon^9).
\]

Solving this equation for each order of \( \varepsilon \), and applying the bottom condition Eq. (19b), yields

\[
\Phi_0 = A(R, T), \quad \Phi_1 = -\frac{(z + 1)^2}{2}A_{RR}, \quad \Phi_2 = -\frac{(z + 1)^2}{2T}A_R + \frac{(z + 1)^4}{24}A_{RRRR}, \\
\Phi_3 = \frac{(z + 1)^2R}{2T^2}A_R + \frac{(z + 1)^4}{12T}A_{RRR} - \frac{(z + 1)^6}{720}A_{RRRRR}, 
\]

where any homogeneous solutions that arise in higher order terms are absorbed into the leading-order term \( \Phi_0 \). These solutions are then substituted back into the dynamic, Eq. (19c), and kinematic, Eq. (19d), boundary conditions. It should be noted that, in this work, we keep terms up to \( O(\varepsilon^6) \).

Differentiating the dynamic boundary condition (19c) with respect to \( R \) produces

\[
H_R - w_R + \varepsilon^3\left[w_T + ww_R + \frac{1}{2}w_{RRR}\right] + \varepsilon^6\left[\frac{1}{2T}w_{RR} + H_{RR} + \frac{1}{2}w_{R}w_{RR} - \frac{1}{2}w_{RRT} + H_{R}w_{RRR} - \frac{1}{24}w_{RRRRR}\right] = 0, 
\]

(21)
while the kinematic boundary condition Eq. (19d) reads

\[-H_R + w_R + \varepsilon^3 \left[ \frac{1}{T} w + H_T + (Hw)_R - \frac{1}{6} w_{RRR} \right] + \varepsilon^6 \left[ - \frac{R}{T^2} w + \frac{1}{T} H w - \frac{1}{3T} w_{RR} - \frac{1}{2} (Hw)_R + \frac{1}{120} w_{RRRR} \right] = 0, (22)\]

to \(O(\varepsilon^6)\), where we have introduced \(w = A_R\). The above equations, (21) and (22), now need to be consistent. To enforce this we set

\[w = H + \varepsilon^3 w_1 + \varepsilon^6 w_2 + O(\varepsilon^9).\]

Compatibility then gives \(w_1\) and \(w_2\) as

\[w_1 = -\frac{1}{4} H^2 + \frac{1}{3} H_{RR} - \frac{1}{2T} (\partial_R^{-1} H),\]

\[w_2 = \frac{1}{8} H^3 + \frac{3}{16} H_R^2 + \frac{1}{2} H H_{RR} + \frac{1}{10} H_{RRR} + \frac{1}{T} \left[ \frac{1}{6} H_R - \frac{1}{16} (\partial_R^{-1} H^2) \right] + \frac{1}{T^2} \left[ \frac{1}{2} (\partial_R^{-1} RH) + \frac{5}{8} (\partial_R^{-2} H) \right].\]

With this compatibility enforced either of Eqs. (21) or (22) give the extended cylindrical KdV (ecKdV) equation

\[H_T + \frac{3}{2} H H_R + \frac{1}{6} H_{RRR} + \frac{1}{2T} H + \varepsilon^3 \left[ -\frac{3}{8} H^2 H_R + \frac{23}{24} H_R H_{RR} + \frac{5}{12} H H_{RRR} + \frac{19}{360} H_{RRRR} \right] + \varepsilon^3 \left[ \frac{3}{16} H^2 + \frac{1}{4} H_{RRR} - \frac{1}{2} H_R \partial_R^{-1} (H) \right] + \varepsilon^3 \left[ -\frac{R}{2} H + \frac{1}{8} \partial_R^{-1} (H) \right] = 0, (23)\]

where

\[\partial_R^{-1} H = \int_0^R H(R', T) \, dR'.\]

A restricted form of the ecKdV equation with only higher order fifth order dispersion, \(H_{RRRRR}\), was derived by Huang and Dai [107].

What is of interest here is that if the terms involving \(1/T\) and \(1/T^2\) are ignored, then the eKdV equation (13) is recovered. This is expected, since as time increases the waves propagate in \(R\), which means that their radii of curvature decrease with the wavefronts become increasingly flat, so that the waves become \((1+1)\) dimensional. While the water wave equations (19) have no dependence on the radial angle \(\theta\), they still govern the propagation of radially symmetric waves, which is not a 1D effect. Thus, the higher dimensionality involved in radially symmetric waves introduces nonlocal terms, being encompassed by the operator \(\partial_R^{-1}\). This induced nonlocality in higher dimensions will become more apparent in the case
of the KP equation (see Section 5.1). Solutions of the 2D wave equation also show such nonlocality—see Section 7.4 of Ref. [2].

We conclude this Section with an important remark on the connection between the ecKdV equation (23) and its Cartesian counterpart. For the regular cKdV equation and integrable KdV equation case, there exists a transformation that links one equation to the other [105, 108]. There also exists such a transformation which maps the ecKdV equation (23) to the form of a perturbed KdV equation with non-constant coefficients. Indeed, upon defining

\[ H = \frac{R}{3T} - \frac{1}{2T} u(\xi, \tau) - \frac{4\varepsilon^3}{3} \xi^2 \tau^2 \log \tau, \quad R = \frac{\xi}{\tau}, \quad T = -\frac{1}{2\tau^2}, \]

(24)

one may express the ecKdV equation (23) as

\[ u_\tau + \frac{3}{2} uu_\xi + \frac{1}{6} u_{\xi\xi\xi} + \varepsilon^3 \left[ -\frac{\xi}{6} (11 + 24 \log \tau) u - \frac{\xi^2}{2} (1 + 4 \log \tau) u_\xi - \frac{1}{6} \partial_\xi^{-1}(u) \right] \]

\[ + \varepsilon^3 \tau \left[ -\frac{1}{8} u^2 + \frac{1}{2} \xi uu_\xi - \frac{41}{36} u_{\xi\xi} - \frac{5}{18} \xi u_{\xi\xi\xi} + u_\xi \partial_\xi^{-1}(u) \right] \]

\[ + \varepsilon^3 \tau^2 \left[ -\frac{3}{8} u^2 u_\xi + \frac{23}{24} u_\xi u_{\xi\xi} + \frac{5}{12} uu_{\xi\xi\xi} + \frac{19}{360} u_{\xi\xi\xi\xi} \right] = O(\varepsilon^6). \]

(25)

Note that here the \(O(\varepsilon^3)\) correction to the original transformation for \(H\) in (24) serves to cancel any inhomogeneous terms produced up to that order.

5. (2+1) dimensional equations

5.1. The extended Kadomtsev-Petviashvili equation

Here, we derive the full extended KP equation from the water wave equations, as originally derived by [109, 110], which incorporates all possible higher order correction terms at the next order of approximation beyond the standard KP equation, as was done for the KdV equation in Section 3.1. The water wave equations will again be approximated using an asymptotic expansion. The parameter \(\delta = \lambda_x/\lambda_y\), the ratio of the wavelengths in the two horizontal directions, is taken as \(\delta^2 \mapsto \varepsilon \delta^2\) in Laplace’s equation (3). While this parameter can be absorbed via a change in the coordinates, we opt to keep it in the wave equations so as to act as a measure of the dimensionality contribution. KP-type equations are valid in same the KdV-type region of Figure 1.

As for the previous (1 + 1) dimensional equations we expand the velocity potential as in Eq. (8), and substitute back into Laplace’s equation (3) to obtain

\[ \phi_{0zz} + \varepsilon (\phi_{1zz} + \phi_{0xx} + \delta^2 \phi_{0yy}) + \varepsilon^2 (\phi_{2zz} + \phi_{1xx} + \delta^2 \phi_{1yy}) + \varepsilon^3 (\phi_{3zz} + \phi_{2xx} + \delta^2 \phi_{2yy}) = O(\varepsilon^4). \]

Solving this differential equation at each order of \(\varepsilon\), and applying the bottom boundary
condition (4), gives

\[
\phi_0(x, y, z, t) = A(x, y, t), \quad \phi_1(x, y, z, t) = -\frac{(z + 1)^2}{2}(A_{xx} + \delta^2 A_{yy}),
\]

\[
\phi_2(x, y, z, t) = \frac{(z + 1)^4}{24}(A_{xxxx} + 2\delta^2 A_{xyxy} + \delta^4 A_{yyyy}),
\]

\[
\phi_3(x, y, z, t) = -\frac{(z + 1)^6}{720}(A_{xxxxxx} + 3\delta^2 A_{xyxyy} + 3\delta^4 A_{yyyyyy} + 6A_{yyyyyy}).
\]

Next, differentiating the dynamic boundary condition (5) with respect to \( x \), substituting the above components of the velocity potential and introducing \( w = A_x \), casts the dynamic (5) and kinematic (6) boundary conditions into the form

\[
w_t + \eta_x + \varepsilon \left( w w_x - \frac{1}{2} w_{xx} \right) + \varepsilon^2 \left[ \delta^2 w_y \partial_x^{-1}(w_y) - \frac{1}{2} \delta^2 w_{yy} - \eta_x w_xt + \frac{1}{2} w_x w_{xx} - \eta w_{xxt} - \frac{1}{2} w w_{xxx} + \frac{1}{24} w_{xxxx} \right] = 0,
\]

\[
\eta_t + w_x + \varepsilon \left[ \delta^2 \partial_x^{-1}(w_{yy}) + (\eta w)_x - \frac{1}{6} w_{xxx} \right] = 0,
\]

\[
\varepsilon^2 \left[ \delta^2 \eta \partial_x^{-1}(w_{yy}) + \delta^2 \eta_y \partial_x^{-1}(w_y) - \frac{1}{3} \delta^2 w_{xyy} - \frac{1}{2} (\eta w_{xx})_x + \frac{1}{120} w_{xxxx} \right] = 0.
\]

where, as before, \( \partial_x^{-1}(u) = \int_{-\infty}^u u(x', y, t) dx' \). In order for these two equations to be consistent, we again set

\[
w = \eta + \varepsilon (w_1 + \delta^2 w_{12}) + \varepsilon^2 (w_2 + \delta^2 w_{21}) + O(\varepsilon^3),
\]

where we isolate different corrections to emphasize the role of dimensionality. Then,

\[
w_1 = -\frac{1}{4} \eta^2 + \frac{1}{3} \eta_{xx}, \quad w_{12} = -\frac{1}{2} \partial_x^{-2}(\eta_{yy}), \quad w_2 = \frac{1}{8} \eta^3 + \frac{3}{16} \eta_x^2 + \frac{1}{2} \eta_{xx} + \frac{1}{10} \eta_{xxxx}, \quad w_{21} = \frac{1}{6} \eta_{yy} - \frac{3}{8} \partial_x^{-1}(\eta \partial_x^{-1}(\eta_{yy}))(\eta_{yy}) + \frac{5}{8} \partial_x^{-2}(\eta_{yy} + \eta_y^2) + \frac{3 \delta^2}{8} \partial_x^{-4}(\eta_{yyyy}).
\]

Note that \( w_1 \) and \( w_2 \) have already been derived in Ref. [25], in which the derivation of the extended KdV equation was presented. In this manner, under unidirectional propagation, the extended KP (eKP) equation is now obtained

\[
(\eta_t + \eta_x)_x + \varepsilon \left[ \frac{3}{2} \eta \eta_x + \frac{1}{6} \eta_{xxx} \right] + \varepsilon \delta^2 \left[ \frac{1}{2} \eta_{yy} \right] + \varepsilon^2 \partial_x \left[ \frac{3}{8} \eta^2 \eta_x + \frac{23}{24} \eta_x \eta_{xx} + \frac{5}{12} \eta_{xxx} + \frac{19}{360} \eta_{xxxx} \right] + \varepsilon^2 \delta^2 \left[ \frac{9}{8} \eta_y^2 + \frac{1}{4} \eta \eta_{yy} + \frac{1}{4} \eta_{xyy} - \frac{1}{2} \eta_{xx} \partial_x^{-2}(\eta_{yy}) \right] + \frac{3}{8} \eta_x \partial_x^{-1}(\eta_{yy}) + \eta_{xy} \partial_x^{-1}(\eta_y) - \delta^2 \partial_x^{-2}(\eta_{yyyy}) = 0. \tag{26}
\]
It can be seen that, in contrast to the \((1+1)\) dimensional case of the eKdV equation (13), the
effect of dimensionality is now more pronounced. Indeed, the additional terms appearing,
measured by the parameter \(\delta\), are highly nonlocal with the operator \(\partial_x^{-1}\) being applied
multiple times. As for the eKdV equation (13) this eKP equation has been derived in the
specific case of water waves with the higher order \(\epsilon^2\) coefficients taking specific values for
this application. For other applications, such as, again, internal waves in a stratified fluid,
the higher order coefficients take other values specific to the physical application. As for the
eKdV equation, a restricted form of the eKP equation with only higher order fifth order
dispersion was derived by [107]. It can be seen that the full eKP equation is much more
extensive than just fifth order dispersion.

5.2. Surface tension

As noted in the Introduction, the addition of capillary effects to the KdV equation can
introduce fundamental physical and mathematical changes in that the resulting incorpora-
tion of fifth order dispersion can lead to resonance between solitary waves and undular bores
and dispersive radiation if the linear dispersion relation is non-convex [36, 37]. To include
capillary effects, only Bernoulli’s equation, the dynamic boundary condition, Eq. (5), needs
to be altered through a curvature dependent surface tension term [2, 8]

\[
\phi_t + \frac{\epsilon}{2} \left( \phi_x^2 + \delta^2 \phi_y^2 + \frac{1}{\mu^2} \phi_z^2 \right) + \eta
= T \frac{\epsilon^2 \eta_{xx}(1 + \epsilon^2 \mu^2 \delta^2 \eta_y^2) + \mu^2 \eta_{yy}(1 + \epsilon^2 \mu^2 \eta_x^2) - 2 \epsilon^2 \mu^2 \delta^2 \eta_{xy} \eta_x \eta_y}{(1 + \epsilon^2 \eta_x^2 + \epsilon^2 \mu^2 \delta^2 \eta_y^2)^{3/2}}
\]

As such, it can be assumed that surface tension is a linear effect in the derivation of the KP
equation, as keeping terms that will affect the Euler equations at our level of approximation
requires the extra term \(T(\eta_{xx} + \eta_{yy})\epsilon^2 + O(\epsilon^4)\) in dimensionless form [7, 8, 111]. The easiest
manner in which to incorporate these capillary effects in the KP equation is via the linear
dispersion relation. The linear dispersion relation for gravity-capillary waves is [2]

\[
k_x \omega = k_x \sqrt{(k + T k^3) \tanh(k)} = \left( k_x^2 + \frac{-1 + 3T}{6} k_x^4 + \frac{19 - 30T - 45T^2}{360} k_x^6 \right)
+ \delta^2 \left( \frac{1}{2} - \frac{1 - 3T}{4} k_x^2 - \frac{-19 + 30T + 45T^2}{144} k_x^4 \right) k_y^2
+ \delta^4 \left( -\frac{1}{8} k_x^2 - \frac{1 - 3T}{16} - \frac{-19 + 30T + 45T^2}{192} k_x^4 \right) k_y^4.
\]

Then, using the usual connections \(\omega \rightarrow i \partial_t, k_x \rightarrow -i \partial_x\) and \(k_y \rightarrow -i \partial_y\) for linear dispersion relations [2], one may convert the dispersion relation (28) to the linearized version of the
extended KP equation, which is (26) in the absence of surface tension under the KP weak
transverse dispersion assumption \(|k_y/k_x| \ll 1\). Hence, we can deduce that the fully nonlinear version of the extended KP equation, incorporating surface tension, is

\[
(\eta_t + \eta_x)_x + \varepsilon \left[ \frac{3}{2} \eta \eta_x + \frac{1}{6} (1 - 3T) \eta_{xxx} \right]_x + \varepsilon \delta^2 \left[ \frac{1}{2} \eta_{yy} \right]_x \\
+ \varepsilon^2 \left[ -\frac{3}{8} \eta^2 \eta_x + \frac{23}{24} \eta_x \eta_{xx} + \frac{5}{12} \eta \eta_{xxx} + \frac{19 - 30T - 45T^2}{360} \eta_{xxxx} \right]_x \\
+ \varepsilon^2 \delta^2 \left[ \frac{9}{8} \eta^2_y + \frac{1}{4} \eta \eta_{yy} + \frac{1}{4} (1 - 3T) \eta_{xyy} - \frac{1}{2} \eta_{xx} \delta_x^{-2} (\eta_{yy}) - \frac{3}{8} \eta_x \delta_x^{-1} (\eta_{yy}) \\
+ \eta_{xy} \delta_x^{-1} (\eta_y) - \frac{\delta^2}{8} \delta_x^{-2} (\eta_{yyyy}) \right] = 0.
\]

(29)

Note that other linear terms stemming from the dispersion relation (28) and involving surface tension do not appear in the eKP equation (29), but they would appear at a higher order of approximation. In a similar manner, one may incorporate surface tension to the equations studied in Section 3.

5.3. The extended cylindrical Kadomtsev-Petviashvili equation

The water wave equations (3)–(6) can also be set in cylindrical polar coordinates, namely

\[
\phi_{zz} + \varepsilon \left( \phi_{rr} + \frac{1}{r} \phi_r + \frac{\delta^2}{r^2} \phi_{\theta\theta} \right) = 0,
\]

(30a)

\[
\phi_z = 0, \quad z = -1,
\]

(30b)

\[
\phi_t + \varepsilon \left( \phi_{r}^2 + \frac{\delta^2}{r^2} \phi_{\theta}^2 + \frac{1}{\varepsilon} \phi_{r}^2 \right) + \eta = 0, \quad z = \varepsilon \eta,
\]

(30c)

\[
\varepsilon \left[ \eta_t + \varepsilon \left( \phi_r \eta_r + \frac{\delta^2}{r^2} \phi_{\theta} \eta_{\theta} \right) \right] = \phi_z, \quad z = \varepsilon \eta,
\]

(30d)

in order to derive the cylindrical KP (cKP) equation, also known as Johnson’s equation [46],

\[
\frac{\partial}{\partial R} \left( H_T + \frac{3}{2} H H_R + \frac{1}{6} H_{RRR} + \frac{1}{2T} H \right) + \frac{\delta^2}{2T^2} H_{\Theta\Theta} = 0.
\]

Using the same approach, the extended cKP equation can be derived from the polar coordinate water wave equations (30a)–(30d).

We start by defining the new scaled and travelling wave variables

\[
R = \varepsilon (r - t), \quad T = \varepsilon^4 t, \quad \Theta = \theta/\varepsilon^{3/2}, \quad \phi = \varepsilon \Phi, \quad \eta = \varepsilon^2 H,
\]

and expand the scaled velocity potential \(\Phi\) as

\[
\Phi = \Phi_0 + \varepsilon^3 \Phi_1 + \varepsilon^6 \Phi_2 + \varepsilon^9 \Phi_3 + \cdots
\]

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This perturbation expansion can be used to solve Laplace’s equation (30a). Substituting this expansion into Laplace’s equation and satisfying the bottom boundary condition (30b) gives

\[ \Phi_0 = A(R, \Theta, T), \quad \Phi_1 = \frac{(z+1)^2}{2} A_{RR}, \]

\[ \Phi_2 = -\frac{(z+1)^2}{2T} A_R + \frac{(z+1)^4}{24} A_{RRRR} - \frac{\delta^2(z+1)^2}{2T} A_{R\Theta}, \]

\[ \Phi_3 = \frac{(z+1)^2 R}{2T^2} A_R + \frac{(z+1)^4}{12T} A_{RR} - \frac{(z+1)^6}{720} A_{RRRRRR} \]

\[ + \frac{\delta^2(z+1)^2 R}{T^3} A_{\Theta\Theta} + \frac{\delta^2(z+1)^4}{12T^2} A_{RR\Theta\Theta}. \]

Next, we define \( w = A_R \) and substitute into the dynamic boundary condition (30c) (after differentiating with respect to \( R \)) and the kinematic boundary condition (30d). This yields, to \( O(\varepsilon^6) \), the following equations

\[ H_R - w_R + \varepsilon^3 \left[ \frac{2}{T} H - \frac{2}{T} w + \frac{2R}{T} H_R - \frac{2R}{T} w_R + w_T + w w_R + \frac{1}{2} w_{RRR} \right] \]

\[ + \varepsilon^6 \left[ \frac{1}{T} w^2 + \frac{2R}{T} w w_R + \frac{3}{2T} w_{RR} + \frac{R}{T} w_{RRR} + \frac{2R}{T} w_T + \frac{2}{T} \partial^{-1}_R (w_T) \right] \]

\[ + \frac{2R}{T^2} H - \frac{2R}{T^2} w + \frac{R^2}{T^2} H_R - \frac{R^2}{T^2} w_R + \frac{\delta^2}{T^2} w_{R\Theta} - \frac{\delta^2}{2T^2} w_{R\Theta\Theta} \]

\[ + H_R w_{RR} + \frac{1}{2} w_{Rw_{RR}} - \frac{1}{2} w_{RRT} + H w_{RRR} - \frac{1}{2} w_{w_{RR}} - \frac{1}{24} w_{w_{RRR}} \right] = 0, \tag{31} \]

and

\[ -H_R + w_R + \varepsilon^3 \left[ \frac{1}{T} w - \frac{2R}{T} H_R + \frac{2R}{T} w_R + w_T + (H w)_R - \frac{1}{6} w_{RRR} + \frac{\delta^2}{T^2} \partial^{-1}_R (w_{\Theta}) \right] \]

\[ + \varepsilon^6 \left[ \frac{1}{T} H w + \frac{2R}{T} H_T + \frac{2R}{T} H_{RT} + \frac{3}{2T} w_{RR} + \frac{R}{T} w_{RRR} - \frac{1}{3T} w_{w_R} - \frac{R}{3T} w_{RR} \right] \]

\[ + \frac{R}{T^2} w - \frac{R^2}{T^2} H_R + \frac{R^2}{T^2} w_R + \frac{\delta^2}{T^2} (H \partial^{-1}_R (w_{\Theta}))_{\Theta} - \frac{\delta^2}{3T^2} w_{R\Theta} \]

\[ - \frac{1}{2} H_R w_{RR} - \frac{1}{2} H w_{RRR} + \frac{1}{120} w_{w_{RRR}} \right] = 0. \tag{32} \]

To make the above equations, (31) and (32), compatible we expand \( w \) as

\[ w = H + \varepsilon^3 (w_1 + \delta^2 w_{12}) + \varepsilon^6 (w_2 + \delta^2 w_{22}) + \ldots, \]

where \( w_1, w_2, \ldots \) are functions of \( R, \Theta, T \).
and then find that

\[ w_1 = -\frac{1}{4}H^2 + \frac{1}{3}H_{RR} - \frac{1}{2T} \partial_R^{-1}(H), \quad w_{12} = -\frac{1}{2T^2} \partial_R^{-2}(H_{\Theta \Theta}) \]

\[ w_2 = \frac{1}{8}H^3 + \frac{3}{16}H_{RR}^2 + \frac{1}{2}HH_{RR} + \frac{1}{10}H_{RRRR} + \frac{1}{T} \left[ \frac{1}{6}H_R - \frac{1}{16} \partial_R^{-1}(H^2) \right] \]

\[ + \frac{1}{T^2} \left[ \frac{1}{2} \partial_R^{-1}(RH) + \frac{5}{8} \partial_R^{-2}(H) \right], \]

\[ w_{22} = \frac{1}{2T^2} \left[ \frac{1}{3}H_{\Theta \Theta} + \frac{1}{4} \partial_R^{-2}(5H_{\Theta}^2 + 2HH_{\Theta \Theta} + 3H_R \partial_R^{-1}(H_{\Theta \Theta})) \right] \]

\[ + \frac{1}{T^3} \left[ \partial_R^{-2}(RH_{\Theta \Theta}) + \frac{9}{4} \partial_R^{-3}(H_{\Theta \Theta}) \right] + \frac{3\delta^2}{8T^4} \partial_R^{-4}(H_{\Theta \Theta \Theta}). \]

Finally, substituting this expansion for \( w \) into either equation (31) or (32) gives the extended cKP (ecKP) equation

\[ \left[ H_T + \frac{3}{2}HH_R + \frac{1}{6}H_{RRR} + \frac{1}{2T}H \right]_R + \frac{\delta^2}{2T^2} H_{\Theta \Theta} \]

\[ + \varepsilon^3 \frac{\partial}{\partial R} \left[ -\frac{3}{8}H^2H_R + \frac{23}{24}H_RH_{RR} + \frac{5}{12}HH_{RRR} + \frac{19}{360}H_{RRRRR} \right] \]

\[ + \frac{\varepsilon^3}{T} \left[ \frac{3}{16}H^2 + \frac{1}{4}H_{RRR} - \frac{1}{2}H_R \partial_R^{-1}(H) \right] + \frac{\varepsilon^3}{T^2} \frac{\partial}{\partial R} \left[ -\frac{R}{2}H + \frac{1}{8} \partial_R^{-1}(H) \right] \]

\[ + \varepsilon^3 \frac{\delta^2}{T^2} \left[ \frac{9}{8}H_{\Theta}^2 + \frac{1}{4}HH_{\Theta \Theta} - \frac{3}{8}H_R \partial_R^{-1}(H_{\Theta \Theta}) + H_{R \Theta \Theta} \partial_R^{-1}(H_{\Theta}) - \frac{1}{2}H_{RRR} \partial_R^{-2}(H_{\Theta \Theta}) + \frac{1}{4}H_{RRR \Theta \Theta} \right] \]

\[ - \varepsilon^3 \frac{\delta^2}{T^3} \left[ \frac{3}{4} \partial_R^{-1}(H_{\Theta \Theta}) + RH_{\Theta \Theta} \right] - \varepsilon^3 \frac{\delta^4}{T^4} \partial_R^{-2}(H_{\Theta \Theta \Theta \Theta}) = 0. \] (33)

Similarly to the cases of the extended cKdV and KP equations, the extended cKP equation is also nonlocal due to the presence of the integral terms arising from the operators \( \partial_R^{-1} \) and \( \partial_R^{-2} \). In addition, as for the extended cKdV equation, as \( T \) increases and the wave propagates outwards, this equation approaches the cKdV equation. This is again expected since the expanding wavefront becomes locally flat for large \( T \).

6. The Green-Naghdi equations

Alternative approximations to the water wave equations to KdV-type weakly nonlinear, long wave approximations are the Green-Naghdi (GN) and related equations. The GN equations provide a depth averaged description of shallow water motion with a free surface under gravity. They are an extension of the shallow water equations, which fully include the effects of finite fluid depth in \( \mu \), but are weakly dispersive. In the simplest case, the model was first derived by Serre [112], with extensions to two dimensions and the inclusion of higher order dispersion subsequently derived [113, 114]. The system of GN equations, as a
bidirectional, nonlinear dispersive wave model, has been proved to be a close approximation to the 2D full water wave problem in that if the initial conditions for the water wave equations and the GN equations are close, then the solutions of the two sets of equations will remain close [115]. In addition, it has been proved that if the dispersion parameter $\mu$ is small enough, solutions of the water wave equations and GN equations exist for the same finite time.

We now focus on the higher order corrections to the Green-Naghdi equations. The higher order correction to the GN system, while preserving the full nonlinearity of the original system, along with its solitary wave solutions, was first obtained in Ref. [83]. The 2D Green-Naghdi shallow water model for surface gravity waves was extended to incorporate arbitrary higher order dispersive effects by [113, 116]. In addition, it was shown that the extended GN system has a Hamiltonian formulation [83, 113], mirroring the full water wave equations [117]. Due to full nonlinearity, GN-type equations are valid above the KdV region of Figure 1, that of higher nonlinearity $\varepsilon$ and small dispersion $\mu$.

Let us first consider (1+1) dimensional waves. We introduce the mean horizontal velocity $\bar{u} = \bar{u}(x, t)$ by

$$\bar{u}(x, t) = \frac{1}{\bar{h}} \int_{-1}^{\eta} \phi_x(x, z, t) dz, \quad \bar{h} = 1 + \varepsilon \eta, \quad \text{(34)}$$

where $\bar{h}$ is the total depth of the fluid, including the wave displacement. The horizontal and vertical components of the surface velocity $u$ and $v$, respectively, are given in terms of the velocity potential as

$$u(x, t) = \phi_x(x, z, t) \quad \text{at} \quad z = \varepsilon \eta, \quad v(x, t) = \phi_z(x, z, t) \quad \text{at} \quad z = \varepsilon \eta. \quad \text{(35)}$$

Multiplying the mean velocity expression (34) by $\bar{h}$, differentiating the result with respect to $x$, and then using Laplace’s equation (3) in the fluid bulk, the bottom boundary condition (4) and the velocity components (35), we obtain

$$\frac{\partial \bar{h} \bar{u}}{\partial x} = \varepsilon \frac{\partial \phi_x}{\partial x} u - v/\mu^2. \quad \text{(36)}$$

Furthermore, as $\varepsilon \eta_x = \partial \bar{h}/\partial x$, we have

$$v = \mu^2 \left[ -\frac{\partial}{\partial x} (\bar{h} \bar{u}) + \frac{\partial \bar{h}}{\partial x} u \right]. \quad \text{(36)}$$

Substitution of the expression (36) for the vertical velocity into the kinematic boundary condition (6) yields the evolution equation for the total depth $\bar{h}$

$$\frac{\partial \bar{h}}{\partial t} + \varepsilon \frac{\partial}{\partial x} (\bar{h} \bar{u}) = 0. \quad \text{(37)}$$

An advantage of choosing the total depth $\bar{h}$ and the averaged horizontal velocity $\bar{u}$ as the dependent variables is that the mass conservation equation (37) is exact, meaning that it does not involve any approximation.
Next, we differentiate the velocity components (35) with respect to \( x \) and \( t \) to obtain

\[
\begin{align*}
  u_x &= \phi_{xx} + \varepsilon \phi_{xz} \eta_x, \\
  v_x &= \phi_{xy} + \varepsilon \phi_{zz} \eta_x, \\
  u_t &= \phi_{xt} + \varepsilon \phi_{zt} \eta_t, \\
  v_t &= \phi_{yt} + \varepsilon \phi_{zt} \eta_t,
\end{align*}
\]

with all the derivatives evaluated at the surface \( z = \varepsilon \eta \). Similarly,

\[
\frac{\partial}{\partial x} \left( \frac{\partial}{\partial t} \phi(x, \varepsilon \eta) \right) = \phi_{xt} + \epsilon \phi_{zt} \eta_x.
\]

Then, using Eqs. (38) to eliminate \( \phi_{xt} \) and \( \phi_{zt} \), it is found that Eq. (39) reads

\[
\frac{\partial}{\partial x} \left( \frac{\partial}{\partial t} \phi(x, \varepsilon \eta) \right) = u_t + v_t \bar{h}_x - v_x \bar{h}_t.
\]

The final evolution equation for \( u \) can now be obtained by differentiating the dynamic boundary condition (5) with respect to \( x \) and then using Eqs. (36), (37) and (40), to give

\[
\frac{\partial u}{\partial t} + \frac{\partial v}{\partial t} \frac{\partial \bar{h}}{\partial x} + \varepsilon \frac{\partial u}{\partial x} + \varepsilon \frac{\partial \bar{h}}{\partial x} \frac{\partial v}{\partial x} + \frac{\partial \eta}{\partial x} = 0.
\]

Using the mass equation Eq. (37) we may re-express this velocity equation as

\[
\frac{\partial}{\partial t} \left( \bar{h} \left( u + v \frac{\partial \bar{h}}{\partial x} \right) \right) + \varepsilon \frac{\partial}{\partial x} \left( \bar{h} \left( u + v \frac{\partial \bar{h}}{\partial x} \right) \right) + \bar{h} (u - \bar{u}) \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial x} \right) + \bar{h} \frac{\partial \eta}{\partial x} = 0.
\]

The system of equations (37) and (41)–(42) is equivalent to the basic water wave equations (3)–(6). To obtain the extended GN equations, one needs to express the velocities \( u \) and \( v \) in terms of the total depth \( \bar{h} \) and the mean horizontal velocity \( \bar{u} \). To enable this change of variable, we set

\[
\bar{u}_t = \sum_{n=0}^{\infty} \mu^{2n} K_n,
\]

where \( K_n \) are polynomials in \( \bar{h} \). If one retains terms up to \( O(\mu^{2n}) \), the extended GN equation, accurate up to \( O(\mu^{2n}) \), is obtained.

We now proceed to derive the extended GN equation explicitly for the case \( n = 2 \) by truncating the system of equations (37) and (42) at \( O(\mu^4) \). As for the preceding extended equations, we first solve Laplace’s equation for the velocity potential \( \phi \) subject to the bottom boundary condition, and then find the mean velocity (34) and velocity components (35) accordingly as

\[
\phi = \sum_{n=0}^{\infty} (-1)^n \mu^{2n} \frac{(z + 1)^{2n} \partial^2 A}{(2n)! \partial x^{2n}}, \quad \bar{u} = \sum_{n=0}^{\infty} (-1)^n \mu^{2n} \frac{\bar{h}^{2n} \partial^{2n+1} A}{(2n + 1)! \partial x^{2n+1}},
\]

\[
u = \sum_{n=1}^{\infty} (-1)^n \mu^{2n} \frac{\bar{h}^{2n-1} \partial^{2n} A}{(2n - 1)! \partial x^{2n}}.
\]
where, as above, $A = A(x, t)$. Retaining terms up to $O(\mu^4)$ for the mean horizontal velocity, we have

$$
\bar{u} = A_x - \frac{\mu^2}{6} \bar{h}^2 A_{xxx} + \frac{\mu^4}{120} \bar{h}^4 A_{xxxx} + O(\mu^6).
$$

This series may be inverted to obtain

$$
A_x = \bar{u} + \frac{\mu^2}{6} \bar{h}^2 \bar{u}_{xx} + \mu^4 \left[ \frac{\bar{h}^2}{36} \frac{\partial^2}{\partial x^2} \left( \bar{h}^2 \bar{u}_{xx} \right) - \frac{\bar{h}^4}{120} \bar{u}_{xxxx} \right] + O(\mu^6).
$$

Similarly,

$$
u = A_x - \frac{\mu^2}{2} \bar{h}^2 A_{xxx} + \frac{\mu^4}{24} \bar{h}^4 A_{xxxx} + O(\mu^6),$$

and after eliminating $A$ we obtain

$$
u = \bar{u} - \frac{\mu^2}{3} \bar{h}^2 \bar{u}_{xx} - \mu^4 \left[ \frac{1}{45} \bar{h}^4 \bar{u}_{xxxx} + \frac{2}{9} \bar{h}^2 \frac{\partial \bar{h}}{\partial x} \bar{u}_{xx} + \frac{1}{18} \bar{h}^2 \frac{\partial^2 \bar{h}^2}{\partial x^2} \frac{\partial \bar{u}}{\partial x} \right] + O(\mu^6).
$$

Using this expression for the horizontal velocity, the expression for $v$, Eq. (36), can be written as

$$
v = -\mu^2 \bar{h} \bar{u} - \frac{\mu^4}{3} \bar{h}^2 \bar{u}_{xx} + O(\mu^6).
$$

The evolution equation for $\bar{u}$ follows by substituting these expressions for $u$ and $v$ into Eq. (41) and then using the mass equation Eq. (37) to replace $\partial \bar{h}/\partial t$. After some manipulations, in this manner we arrive at the following compact equation for $\bar{u}$

$$
\bar{u}_t + \varepsilon \bar{u} \bar{u}_x + \eta_x = \frac{\mu^2}{3 \bar{h}} \frac{\partial}{\partial x} \bar{h}^3 (\bar{u}_{xx} + \varepsilon \bar{u} \bar{u}_{xx} - \varepsilon \bar{u}_x^2)
$$

$$+ \frac{\mu^4}{45 \bar{h}} \frac{\partial}{\partial x} \left[ \frac{\partial}{\partial x} \bar{h}^5 (\bar{u}_{xx} + \varepsilon \bar{u} \bar{u}_{xx} - 5 \varepsilon \bar{u}_x \bar{u}_{xx}) - 3 \varepsilon \bar{h}^5 \bar{u}_{xx}^2 \right] + O(\mu^6). \quad (43)
$$

The final system of equations (37) and (43) is the extended version of the GN equations, accurate up to $O(\mu^4)$.

Finally, these results can be extended to the two dimensional case and the 2D GN system reads

$$
\frac{\partial \bar{h}}{\partial t} + \varepsilon \nabla \cdot (\bar{h} \bar{u}) = 0,
$$

$$
\bar{u}_t + \varepsilon (\bar{u} \cdot \nabla) \bar{u} + \nabla \eta = \mu^2 R_1 + \mu^4 R_2 + O(\mu^6),
$$

where

$$
R_1 = \frac{1}{3 \bar{h}} \nabla \left[ \bar{h}^5 \{ \nabla \cdot \bar{u}_t + \varepsilon (\bar{u} \cdot \nabla)(\nabla \cdot \bar{u}) - \varepsilon (\nabla \cdot \bar{u})^2 \} \right],
$$

$$
R_2 = \frac{1}{45 \bar{h}} \nabla \left[ \nabla \cdot \left( \bar{h}^5 \nabla \cdot \bar{u}_t + \varepsilon \bar{h}^5 (\nabla^2 \cdot \bar{u}_t) \bar{u} - 5 \varepsilon \bar{h}^5 (\nabla \cdot \bar{u}) \nabla (\nabla \cdot \bar{u}) 
\right.ight.

$$

$$
\left. \left. + \varepsilon \nabla \bar{h}^5 \times (\bar{u} \times \nabla (\nabla \cdot \bar{u})) \right) - 2 \varepsilon \bar{h}^5 \{ \nabla (\nabla \cdot \bar{u}) \}^2 \right] - \frac{\varepsilon}{45 \bar{h}} \left[ \nabla \cdot \left( \bar{h}^5 \nabla (\nabla \cdot \bar{u}) \right) \nabla (\nabla \cdot \bar{u}) + \frac{1}{2} \bar{h}^5 \{ \nabla (\nabla \cdot \bar{u}) \}^2 \right].
$$

Note here we have taken $\delta = 1$ in Eq. (3).
7. Conclusions

The standard weakly nonlinear, long wave approximations to the water wave equations, the Boussinesq system and the KdV equation in (1 + 1) dimensions and the KP equation in (2 + 1) dimensions, are asymptotic approximations one order beyond linear waves for which weak nonlinearity is in balance with weak dispersion. As outlined in the Introduction, these equations appear generically as asymptotic reductions of other models (such as the NLS equation, discrete dynamical lattices, etc) that play a key role in a wide range of physical contexts. Moreover, in a variety of applications, it has been found that it is necessary to extend these asymptotic equations to the next order, so that higher amplitude and steeper waves can be modelled. In addition, it has been found that such extensions are necessary in order to capture effects not included at the KdV order, such as resonance. While the extended equations derived here, in particular the eKdV, ecKdV and eKP equations, have been derived from the water wave equations, they have much a much wider applicability, as noted in the Introduction.

In the present work, extended weakly nonlinear, weakly dispersive reductions of the water wave equations in both (1 + 1) and (2 + 1) dimensions have been derived, as well as extended versions of the fully nonlinear, weakly dispersive GN equations. These extended equations have been derived using asymptotic analyses which highlight the connections between the various equations. From a mathematical point of view, the extended equations are quite relevant to important notions, such as the Hamiltonian structure, as well as the integrability and asymptotic integrability, of weakly nonlinear dispersive wave equations. It is hoped that the newly derived equations will prove to be useful in theoretical contexts as, in existing work, the additional higher order terms were added on an ad hoc basis. Furthermore, it is anticipated that this derivation and review of higher order weakly dispersive, weakly nonlinear equations will prove useful for modelling waves in fluids, plasmas, nonlinear optics, and other application areas. It is anticipated that the newly derived equations will prove to be useful in theoretical contexts as, in existing work, the additional higher order terms were added on an ad hoc basis.

From a mathematical point of view, important themes for future studies concern the properties and the solutions of the extended equations. In that regard, an interesting question is if the extended equations have a Hamiltonian structure, as is the case for the extended GN equations [83, 113]. For instance, the eKdV equation seems to lack the Hamiltonian property as an exact energy conservation equation for it has not been found. Furthermore, resonance effects [25] strongly suggest that a Hamiltonian formulation of the eKdV equation cannot be constructed. More generally, the existence of conserved quantities for the extended equations, which is a “stepping stone” to complete integrability, is an important problem. A promising way to investigate these issues is through the notion of asymptotic integrability [28]. In this context, there exists a map, much like the one connecting the functions \( w \) and \( \eta \) of Sections 3.1 and 5.1, which is constructed to eliminate the higher order contributions (by moving them to higher than the required order) [28]. In this manner, the higher order equation can be reduced to its integrable counterpart at a given asymptotic order. Then, the solution of the integrable equation is used to construct, up to the required
order in ε, the solution of original higher-order equation.

It is hoped that this work will inspire theoretical studies in this direction, which may also concern the asymptotic integrability of other higher order weakly nonlinear dispersive wave equations. Furthermore, it is anticipated that this derivation and review of higher order weakly dispersive, weakly nonlinear equations will prove useful for modelling waves in fluids, plasmas, nonlinear optics, and other application areas.

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