Li-Yau Estimates for a Nonlinear Parabolic Equation on Manifolds

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Received: 29 December 2013 / Accepted: 26 June 2014 / Published online: 20 July 2014
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Abstract In this paper, we derive Li-Yau gradient estimates for the positive solution of a nonlinear parabolic equation $u_t = \Delta u - qu - au(\ln u)^\alpha$, where $q$ is a $C^2$ function and $a$, $\alpha$ are constants, on a complete manifold $(M, g)$ with bounded below Ricci curvature. The results generalize classical Li-Yau gradient estimates and some recent works on this direction.

Keywords Nonlinear parabolic equation · Li-Yau estimates

Mathematics Subject Classification (2010) 53C44

1 Introduction

In this paper, we consider a parabolic equation of the type

$$\left(\Delta - q(x, t) - \frac{\partial}{\partial t}\right)u(x, t) = au(x, t) (\ln(u(x, t)))^\alpha,$$

(1.1)
on $M \times (0, \infty)$, where $a, \alpha$ are constants, and $q$ is a $C^2$ function defined on $M \times (0, \infty)$. We sometimes write $u(x, t)$ as $u$ and $q(x, t)$ as $q$, etc, also write $\frac{\partial}{\partial t}$ as $\partial_t$.

Gradient estimates is one of the fundamental tools in studying nonlinear partial differential equations from geometry. Li and Yau [6] obtained a gradient estimate, called Li-Yau estimate, for heat equation

$$
\left( \Delta - \frac{\partial}{\partial t} \right) u(x, t) = 0,
$$

(1.2)

on $M \times (0, \infty)$; that is, the (1.1) with $q = a = 0$. Using gradient estimates, Li and Yau proved the optimal upper and lower bounds for heat kernel. Later, this estimate have been extended to Ricci flow by Hamilton [4], and furthermore, by Perelman [9].

After the fundamental work of Li-Yau, there are variant estimate for heat-type equations. One of them arises from gradient Ricci soliton $(M, g, c, f)$; that is,

$$
\text{Rc}_g = cg + \nabla^2 f,
$$

(1.3)

where $(M, g)$ is an $n$-dimensional Riemannian manifold, $c$ is a constant, and $f$ is a smooth function. Letting $u = e^f$, the (1.3) can be written as (see [8])

$$
\Delta u + 2cu \ln u = (A_0 - cn)u,
$$

(1.4)

for some constant $A_0$. On the other hand, Yang [13] considered the similar equation

$$
\left( \Delta - b - \frac{\partial}{\partial t} \right) u(x, t) = au(x, t) \ln(u(x, t)),
$$

(1.5)

where $a, b \in \mathbf{R}$; moreover Qian [10] and Wu [12] studied the same (1.5) where $a, b$ are functions. Observe that (1.2), (1.4), and (1.5) are special cases of (1.1). For gradient estimates for (1.1) under the Ricci flow, we refer to [5]. Our estimates give more refinement than that in [5]. In a later paper [7], we will consider the gradient estimates for a more general nonlinear parabolic equation under a geometric flow.

Throughout this paper, $M$ is assumed to be an $n$-dimensional complete Riemannian manifold with (possibly empty) boundary $\partial M$. We denoted by $\frac{\partial}{\partial \nu}$ the outward pointing unit normal vector to the boundary $\partial M$, and $\Pi$ the second fundamental form of $\partial M$ with respect to $\frac{\partial}{\partial \nu}$.

We now state our main results in this paper.

**Theorem 1.1** Let $(M, g)$ be a compact manifold with nonnegative Ricci curvature. Suppose that the boundary $\partial M$ of $M$ is convex, i.e., the second fundamental form...
II is nonnegative, whenever \( \partial M \neq \emptyset \). Let \( u(x, t) \) be a positive solution of the equation

\[
(\Delta - \partial_t) u = au \ln u,
\]
on \( M \times (0, \infty) \) for some constant \( a \), with Neumann boundary condition

\[
\frac{\partial u}{\partial \nu} = 0,
\]
on \( \partial M \times (0, \infty) \).

(1) If \( a \leq 0 \), then \( u \) satisfies

\[
\frac{|\nabla u|^2}{u^2} - \frac{u_t}{u} - a \ln u \leq \frac{n}{2t} - \frac{na}{2},
\]
on \( M \times (0, \infty) \).

(2) If \( a \geq 0 \), then \( u \) satisfies

\[
\frac{|\nabla u|^2}{u^2} - \frac{u_t}{u} - a \ln u \leq \frac{n}{2t}.
\]

To the general (1.1), we obtain the following Li-Yau gradient estimate.

**Theorem 1.2** Let \((M, g)\) be a complete manifold with boundary \( \partial M \). Assume that \( p \in M \) and the geodesic ball \( B_p(2R) \) does not intersect \( \partial M \). We denote by \(-K(2R)\) with \( K(2R) \geq 0 \), a lower bound of the Ricci curvature on the ball \( B_p(2R) \). Let \( q(x, t) \) be a function defined on \( M \times [0, T] \) which is \( C^2 \) in the \( x \)-variable and \( C^1 \) in the \( t \)-variable. Assume that

\[
\Delta q \leq \theta(2R), \quad |\nabla q| \leq \gamma(2R),
\]
on \( B_p(2R) \times [0, T] \) for some constants \( \theta(2R) \) and \( \gamma(2R) \). If \( u(x, t) \) is a positive solution of the equation

\[
\left(\Delta - q - \frac{\partial}{\partial t}\right) u = au(\ln u)^\alpha, \quad \alpha > 0,
\]
on \( M \times (0, T) \) for some constant \( a \), then for any \( \beta > 1 \) and \( \epsilon \in (0, 1) \), on \( B_p(R) \), \( u(x, t) \) satisfies the following estimates:

(1) for \( a \geq 0 \), we have

\[
|\nabla f|^2 - \beta f - \beta q - \beta af^\alpha \leq \frac{\beta^2}{2(1-\epsilon)} + \frac{(A + \gamma)n\beta^2}{2(1-\epsilon)} + \frac{n^2\beta^4C_1^2}{4\epsilon(1-\epsilon)(\beta - 1)R^2} + \frac{\beta^2[K + a(\beta - 1)|f^{\alpha - 1}|_\infty]}{(1-\epsilon)(\beta - 1)} + \frac{n\beta^3\alpha|\alpha - 1||f^{\alpha - 2}|_\infty}{2(\beta - 1)(1-\epsilon)} + \sqrt{\frac{[\beta\theta + (\beta - 1)\gamma]n\beta^2}{2(1-\epsilon)}}.
\]
(2) for $a \leq 0$, we have

$$|
abla f|^2 - \beta f_t - \beta q - \beta a f^\alpha \leq \frac{n\beta^2}{2(1-\epsilon)t} + \frac{(A + \gamma)n\beta^2}{2(1-\epsilon)} + \frac{n^2\beta^4C_1^2}{4\epsilon(1-\epsilon)(\beta - 1)R^2}$$

$$+ \frac{n\beta^2[K - \frac{q}{2}(\beta - 1)\alpha|f^{\alpha-1}|_\infty]}{(1-\epsilon)(\beta - 1)}$$

$$+ \sqrt{\frac{[\beta\theta + (\beta - 1)\gamma]|n\beta^2}{2(1-\epsilon)}}.$$  

Here $f(x, t) := \log(u(x, t))$, $|f|_\infty := \max_M |f|$, and $A = [2C_1^2 + (n - 1)C_1^2(1 + R\sqrt{K}) + C_2]/R^2$ for some positive constants $C_1, C_2$.

When $a = 1$, the above theorem recovers the main result in [10, 12]. As an application, we prove the gradient estimate for the elliptic equation

$$(\Delta - q)u = au(\ln u)^\alpha, \quad \alpha > 0,$$  

(1.7)

where $u$ is a positive solution.

**Corollary 1.3** Let $(M, g)$ be a complete non-compact $n$-dimensional Riemannian manifold. Suppose that $u(x, t)$ is a positive solution on $M$ of the (1.7). Assume that

(a) the Ricci curvature of $(M, g)$ is bounded from below by $-K$, for some constant $K \geq 0$, and

(b) there exists a constant $\theta$, and a function $\gamma(t)$ such that $|
abla q| \leq \gamma$ and $\Delta q \leq \theta$ on $M$.

Then

(1) for $a \geq 0$, we have

$$\frac{|
abla u|^2}{u^2} - \beta a(\ln u)^\alpha \leq \beta q + \left(\frac{\gamma}{2} + a(\ln u)^{\alpha-1}|_\infty + \frac{a\beta|\alpha - 1||u(\ln u)^{\alpha-2}|_\infty}{2(\beta - 1)}\right)n\beta^2$$

$$+ \frac{n\beta^2 K}{\beta - 1} + \sqrt{\frac{[\beta\theta + (\beta - 1)\gamma]|n\beta^2}{2}}.$$  

on $M$ for all $\beta > 1$.

(2) for $a \leq 0$, we have

$$\frac{|
abla u|^2}{u^2} - \beta a(\ln u)^\alpha \leq \beta q + \left(\frac{\gamma}{2} - \frac{a}{2}|(\ln u)^{\alpha-1}|_\infty\right)n\beta^2$$

$$+ \frac{n\beta^2 K}{\beta - 1} + \sqrt{\frac{[\beta\theta + (\beta - 1)\gamma]|n\beta^2}{2}},$$  

on $M$ for all $\beta > 1$.  

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In particular, if $u$ is a positive solution of the equation $(\Delta - q)u = au \ln u$, then

(1') for $a > 0$, we have a lower bound

$$u \geq \exp \left[ -\frac{q}{a} - \left( 1 + \frac{\gamma}{2a} \right) n\beta - \frac{n\beta K}{(\beta - 1)a} - \frac{1}{a} \left( \frac{[\beta \theta + (\beta - 1)\gamma]n}{2} \right)^{1/2} \right],$$
onumber

on $M$ for all $\beta > 1$.

(2') for $a < 0$, we have an upper bound

$$u \leq \exp \left[ -\frac{q}{a} + \left( \frac{1}{2} - \frac{\gamma}{2a} \right) n\beta - \frac{n\beta K}{(\beta - 1)a} - \frac{1}{a} \left( \frac{[\beta \theta + (\beta - 1)\gamma]n}{2} \right)^{1/2} \right],$$

on $M$ for all $\beta > 1$.

Remark 1.4 When $q$ is a constant, Theorem 1.1 reduces to Theorem 1.1 in [13].

Corollary 1.3 gave a much better bound for a positive solution of (1.7) on $M$ if $q = 0$, $\alpha = 1$ and the Ricci curvature of $M$ is nonnegative (compared with Corollary 1.6 in [10] and Corollary 1.2 in [13]). In fact, in this case, taking $q = \gamma = \theta = K = 0$, we have

$$u \geq e^{-n} (a > 0), \text{ or } u \leq e^{n/2} (a < 0).$$

Note that our constant $a$ is actually the constant $-a$ used in [10, 13].

2 Gradient Estimates

Suppose that $u(x, t)$ is a positive solution of (1.1). Let

$$f(x, t) := \ln(u(x, t)).$$

Then the (1.1) now can be written as $(\Delta - \partial_t) f = -|\nabla f|^2 + q + af$. We would like to consider a more general situation:

$$(\Delta - \partial_t) f = -|\nabla f|^2 + q + af^\alpha,$$

where $\alpha > 0$.

Lemma 2.1 Let $f(x, t)$ be a smooth function on $M \times [0, \infty)$ satisfying (2.2), where $a$ is a constant, $\alpha$ is a positive constant, and $q$ is a $C^2$ function defined on $M \times (0, \infty)$. For any given $\beta \geq 1$, the function

$$F := t \left( |\nabla f|^2 - \beta f_i - \beta q - \beta af^\alpha \right),$$

satisfies the inequality

$$(\Delta - \partial_t) F \geq -2(\nabla f, \nabla F) - \frac{F}{t} - 2Kt|\nabla f|^2 + \frac{2t}{n} \left( |\nabla f|^2 - q - f_i - af^\alpha \right)^2$$

$$- \beta t aq - 2(\beta - 1)t(\nabla f, \nabla q) - 2(\beta - 1)taf^\alpha - |\nabla f|^2$$

$$- \beta ta(\alpha - 1)f^\alpha - 2|\nabla f|^2 - \beta a \alpha f^{\alpha - 1} \left( -|\nabla f|^2 + f_i + q + af^\alpha \right),$$

where $\alpha > 0$. 

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where $-K(x)$, with $K(x) \geq 0$, is a lower bound of the Ricci curvature tensor of $M$ at the point $x \in M$, and $f_t := \partial_t f$.

**Proof** Differentiating (2.3) we have

$$\nabla_i F = t \left( 2 \nabla^j f \nabla_j f - \beta \nabla_i f - \beta \nabla_i q - \beta a \alpha f^{\alpha - 1} \nabla_i f \right).$$

Then the Laplace of $F$ equals

$$\Delta F = \nabla^i \nabla_i F = t \left[ 2 \left| \nabla^2 f \right|^2 + 2 \langle \nabla f, \Delta \nabla f \rangle - \beta (\Delta f)_t - \beta \Delta q - \beta a \alpha (\alpha - 1) f^{\alpha - 2} |\nabla f|^2 + f^{\alpha - 1} \Delta f \right].$$

Using the Ricci formula yields

$$\Delta \nabla_i f = \nabla_i \Delta f + R_{ij} \nabla^j f,$$

from which the Laplacian of $F$ can be simplified as

$$\Delta F = t \left[ 2 \left| \nabla^2 f \right|^2 + 2 \langle \nabla f, \Delta \nabla f \rangle - \beta (\Delta f)_t - \beta \Delta q - \beta a \alpha (\alpha - 1) f^{\alpha - 2} |\nabla f|^2 + f^{\alpha - 1} \Delta f \right].$$

Since $|\nabla^2 f|^2 \geq \frac{(\Delta f)^2}{n}$. Recall from (2.2) that

$$\Delta f = -|\nabla f|^2 + q + f_t + a f^\alpha = -\frac{F}{t} - (\beta - 1) (q + f_t + a f^\alpha).$$

Therefore,

$$\Delta F \geq \frac{2t}{n} \left( |\nabla f|^2 - q - f_t - a f^\alpha \right)^2 - 2t \left[ \nabla f, \nabla \left( \frac{F}{t} + (\beta - 1) (q + f_t + a f^\alpha) \right) \right] - 2 K t |\nabla f|^2 - t \beta \left( -\frac{F}{t} - (\beta - 1) (q + f_t + a f^\alpha) \right) \Delta f - \beta t \Delta q - \beta a \alpha \left( (\alpha - 1) f^{\alpha - 2} |\nabla f|^2 + f^{\alpha - 1} \Delta f \right).$$

Since

$$\left( \frac{F}{t} + (\beta - 1) (q + f_t + a f^\alpha) \right)_t = \frac{F_t}{t} - \frac{F}{t^2} + (\beta - 1) (q_t + f_{tt} + a \alpha f^{\alpha - 1} f_t),$$
and
\[ F_t = |\nabla f|^2 - \beta f_t - \beta q - \beta a f^\alpha, \]
we obtain
\begin{align*}
\Delta F &\geq \frac{2t}{n} \left( |\nabla f|^2 - q - f_t - a f^\alpha \right)^2 - 2 \langle \nabla f, \nabla F \rangle - (\beta - 1) t \langle \nabla f, \nabla f_t \rangle \\
&\quad - 2(\beta - 1) t \langle \nabla f, \nabla q \rangle - 2(\beta - 1) t a \alpha f^\alpha - 1 |\nabla f|^2 - 2 K t |\nabla f|^2 + \beta F_t \\
&\quad - \beta \left( |\nabla f|^2 - \beta f_t - \beta q - \beta a f^\alpha \right) + \beta (\beta - 1) t q_t + \beta (\beta - 1) t f_{tt} \\
&\quad + t \beta (\beta - 1) a \alpha f^\alpha - 1 f_t - \beta t \Delta q \\
&\quad - \beta a \alpha t (\alpha - 1) f^\alpha - 2 |\nabla f|^2 - \beta a \alpha f^\alpha - 1 \Delta f.
\end{align*}

On the other hand,
\[ F_t = |\nabla f|^2 - \beta f_t - \beta q - \beta a f^\alpha \\
+ t \left( \partial_t |\nabla f|^2 - \beta f_{tt} - \beta q_t - \beta a \alpha f^\alpha - 1 f_t \right). \]

Combining above two formulas we conclude that
\begin{align*}
(\Delta - \partial_t) F &\geq \frac{2t}{n} \left( |\nabla f|^2 - q - f_t - a f^\alpha \right)^2 - 2 \langle \nabla f, \nabla F \rangle - (\beta - 1) t \partial_t |\nabla f|^2 \\
&\quad - 2(\beta - 1) t \langle \nabla f, \nabla q \rangle - 2(\beta - 1) t a \alpha f^\alpha - 1 |\nabla f|^2 - 2 K t |\nabla f|^2 \\
&\quad + (\beta - 1) \left( |\nabla f|^2 - \beta f_t - \beta q - \beta a f^\alpha \right) + t \beta (\beta - 1) a \alpha f^\alpha - 1 f_t \\
&\quad + (\beta - 1) t \left( \partial_t |\nabla f|^2 - \beta f_{tt} - \beta q_t - \beta a \alpha f^\alpha - 1 f_t \right) - \beta t \Delta q \\
&\quad - \beta \left( |\nabla f|^2 - \beta f_t - \beta q - \beta a f^\alpha \right) + \beta (\beta - 1) t q_t + \beta (\beta - 1) t f_{tt} \\
&\quad - \beta a \alpha t (\alpha - 1) f^\alpha - 2 |\nabla f|^2 - \beta a \alpha f^\alpha - 1 \Delta f \\
= -2 \langle \nabla f, \nabla F \rangle - \frac{F}{t} - 2 K t |\nabla f|^2 + \frac{2t}{n} \left( |\nabla f|^2 - q - f_t - a f^\alpha \right)^2 \\
&\quad - \beta t \Delta q - 2(\beta - 1) t \langle \nabla f, \nabla q \rangle - 2(\beta - 1) t a \alpha f^\alpha - 1 |\nabla f|^2 \\
&\quad - \beta a \alpha t (\alpha - 1) f^\alpha - 2 |\nabla f|^2 - \beta a \alpha f^\alpha - 1 \Delta f.
\end{align*}

Now, (2.4) immediately follows from (2.2). \qed

**Theorem 2.2** Let \((M, g)\) be a compact manifold with nonnegative Ricci curvature. Suppose that the boundary \(\partial M\) of \(M\) is convex, i.e., the second fundamental form \(II\) is nonnegative, whenever \(\partial M \neq \emptyset\). Let \(u(x, t)\) be a positive solution of the equation
\[(\Delta - \partial_t) u = a u \ln u,\]
on \(M \times (0, \infty)\) for some constant \(a\), with Neumann boundary condition
\[\frac{\partial u}{\partial \nu} = 0,\]
on \(\partial M \times (0, \infty)\).
(1) If \( a \leq 0 \), then \( u \) satisfies
\[
\frac{|\nabla u|^2}{u^2} - \frac{u_t}{u} - a \ln u \leq \frac{n}{2t} - \frac{na}{2},
\]
on \( M \times (0, \infty) \).

(2) If \( a \geq 0 \), then \( u \) satisfies
\[
\frac{|\nabla u|^2}{u^2} - \frac{u_t}{u} - a \ln u \leq \frac{n}{2t}.
\]

Proof Setting \( q = 0, \alpha = \beta = 1, \) and \( K = 0 \) in Lemma 2.1 yields
\[
(\Delta - \partial_t) F \geq -2\langle \nabla f, \nabla F \rangle - \frac{F}{t} + \frac{2t}{n} \left( |\nabla f|^2 - f_t - af \right)^2 + aF
\]
\[
= -2\langle \nabla f, \nabla F \rangle - \frac{F}{t} + \frac{2F^2}{nt} + aF
\]
\[
= -2\langle \nabla f, \nabla F \rangle + \frac{2F}{nt} \left( F - \frac{n}{2} + \frac{ant}{2} \right),
\]
where \( F = t \left( |\nabla f|^2 - f_t - af \right) \).

(1) \( a \leq 0 \). In this case we claim that \( F \leq \frac{n}{2} - \frac{nt}{2} \). If not at the maximum point \((x_0, t_0)\) of \( F \) on \( M \times [0, T] \) for some \( T > 0 \), we have
\[
F(x_0, t_0) > \frac{n}{2} - \frac{nt}{2} \geq \frac{n}{2} > 0.
\]
Consequently, \( t_0 > 0 \). If \( x_0 \) is an interior point of \( M \), we conclude from \((x_0, t_0)\) being a maximum point of \( F \) in \( M \times [0, T] \) that
\[
\Delta F(x_0, t_0) \leq 0, \quad \nabla F(x_0, t_0) = 0, \quad F_t(x_0, t_0) \geq 0.
\]
Together with the proved inequality \( (\Delta - \partial_t) F \geq -2\langle \nabla f, \nabla F \rangle + \frac{2F}{nt} \left( F - \frac{n}{2} + \frac{ant}{2} \right) \), we arrive at
\[
0 \geq \frac{2}{nt_0} F(x_0, t_0) \left[ F(x_0, t_0) - \frac{n}{2} + \frac{ant_0}{2} \right].
\]
By the assumption, it implies that \( F(x_0, t_0) \leq \frac{n}{2} - \frac{ant_0}{2} \), a contradiction.

Therefore we proved that \( x_0 \) is on the boundary of \( M \). Now the strong maximum principle tells us
\[
\frac{\partial F}{\partial v}(x_0, t_0) > 0.
\]
Let \( e_1, \ldots, e_n \), where \( e_n := \partial/\partial v \), be an orthonormal frame field on \( M \), and \( f_j \) means the covariant differentiation in the \( e_i \) direction. Calculate
\[
F_v = t \left[ \sum_{1 \leq j \leq n} f_j f_{jv} - (f_i)_v - af_v \right] = 2t \sum_{1 \leq j \leq n-1} f_j f_{jv} + 2tf_v f_{vv} - (f_i)_v - af_v.
\]
Since \( u_v = 0 \) on \( \partial M \), it follows that \( f_v = 0 \) on \( \partial M \) and hence
\[
F_v = 2t \sum_{1 \leq j \leq n-1} f_j f_{jv} = -2t \sum_{1 \leq j,k \leq n-1} h_{j,k} f_j f_k - 2t \Pi(\nabla f, \nabla f),
\]
because of \( f_{j
u} = - \sum_{1 \leq k \leq n-1} h_{jk} f_k \), where \( h_{jk} \) are components of the second fundamental form of \( \partial M \). Evaluating at the point \((x_0, t_0)\), we get
\[
\Pi(\nabla f, \nabla f)(x_0, t_0) < 0,
\]
which contradicts the convexity of \( \partial M \). Hence, \( F \leq \frac{n}{2} - \frac{at}{2} \).

(2) \( a \geq 0 \). Since the right side of (2.6) is positive, we may assume without loss of generality that \( F \geq 0 \). In this case we obtain
\[
(\Delta - \partial_t) F \geq -2\langle \nabla f, \nabla F \rangle + \frac{2F}{nt} \left( F - \frac{n}{2} \right),
\]
which reduces to the case in [6] and by the same computation we conclude that \( F \leq \frac{n}{2} \).

**Theorem 2.3** Let \((M, g)\) be a complete manifold with boundary \( \partial M \). Assume that \( p \in M \) and the geodesic ball \( B_p(R) \) does not intersect \( \partial M \). We denote by \(-K(2R)\) with \( K(2R) \geq 0 \), a lower bound of the Ricci curvature on the ball \( B_p(2R) \).

Let \( q(x, t) \) be a function defined on \( M \times [0, T] \) which is \( C^2 \) in the \( x \)-variable and \( C^1 \) in the \( t \)-variable. Assume that
\[
\Delta q \leq \theta(2R), \quad |\nabla q| \leq \gamma(2R),
\]
on \( B_p(2R) \times [0, T] \) for some constants \( \theta(2R) \) and \( \gamma(2R) \). If \( u(x, t) \) is a positive solution of the equation
\[
(\Delta - q - \partial_t) u = a u (\ln u)^\alpha, \quad \alpha > 0,
\]
on \( M \times (0, T) \) for some constant \( a \), then for any \( \beta > 1 \) and \( \epsilon \in (0, 1) \), on \( B_p(R) \), \( u(x, t) \) satisfies the following estimates:

(1) for \( a \geq 0 \), we have
\[
|\nabla f|^2 - \beta f_t - \beta q - \beta af^\alpha \leq \frac{n\beta^2}{2(1-\epsilon)t} + \frac{(A + \gamma)\beta^2}{2(1-\epsilon)} + \frac{n^2 \beta^4 C_1^2}{16\epsilon(1-\epsilon)(\beta - 1)R^2} \left( 1 - \epsilon \right) \alpha [f^{\alpha - 1}|_\infty] \]
\[
+ \frac{n\beta^2 [K + a(\beta - 1)|f^{\alpha - 1}|_\infty]}{(1-\epsilon)(\beta - 1)} \]
\[
+ \frac{n\beta^3 a\alpha [f^{\alpha - 2}|_\infty]}{2(\beta - 1)(1-\epsilon)} + \sqrt{\frac{[\beta \theta + (\beta - 1)\gamma n\beta^2]}{2(1-\epsilon)}}.
\]

(2) for \( a \leq 0 \), we have
\[
|\nabla f|^2 - \beta f_t - \beta q - \beta af^\alpha \leq \frac{n\beta^2}{2(1-\epsilon)t} + \frac{(A + \gamma)\beta^2}{2(1-\epsilon)} + \frac{n^2 \beta^4 C_1^2}{16\epsilon(1-\epsilon)(\beta - 1)R^2} \left( 1 - \epsilon \right) \alpha [f^{\alpha - 1}|_\infty] \]
\[
+ \frac{n\beta^2 [K - \frac{a}{2}(\beta - 1)|f^{\alpha - 1}|_\infty]}{(1-\epsilon)(\beta - 1)} \]
\[
+ \sqrt{\frac{[\beta \theta + (\beta - 1)\gamma n\beta^2]}{2(1-\epsilon)}}.
\]
Here $f(x, t) := \log(u(x, t))$, $|f|_{\infty} := \max_M |f|$, and $A = [2C_1^2 + (n - 1)C_1^2(1 + R\sqrt{K}) + C_2]/R^2$ for some positive constants $C_1, C_2$.

Proof As before, we set $f = \log u$ and $F = t(|\nabla f|^2 - \beta f_t - \beta q - \beta af^{\alpha})$. As in [2, 6, 8, 13], we let $\tilde{\varphi}(r)$ be a $C^2$ function defined on $[0, \infty)$ such that

$$\tilde{\varphi}(r) = \begin{cases} 1, & r \in [0, 1], \\ 0, & r \in [2, \infty), \end{cases}$$

and

$$-C_1 \leq \tilde{\varphi}'(r)\tilde{\varphi}^{-1/2}(r) \leq 0, \quad \tilde{\varphi}(r) \geq -C_2,$$

for some positive constants $C_1, C_2$. If $r(x) := \text{dist}(p, x)$ denotes the distance between $p$ and $x$, we set

$$\varphi(x) := \tilde{\varphi}\left(\frac{r(x)}{R}\right).$$

Using Calabi’s argument (see, e.g., [1, 3, 11]), we may assume without loss of generality that $\varphi(x)$ is smooth in the ball $B_p(2R)$. Then by the Laplacian comparison theorem (see [11]) we have

$$\frac{|\nabla \varphi|^2}{\varphi} \leq \frac{C_1^2}{R^2}, \quad \Delta \varphi \geq -\frac{(n - 1)C_1^2(1 + R\sqrt{K}) + C_2}{R^2}.$$

Combining Lemma 2.1 with $\Delta(\varphi F) = \Delta \varphi \cdot F + 2(\nabla \varphi, \nabla F) + \varphi \cdot \Delta F$ yields

$$\Delta(\varphi F) \geq F \left[-\frac{(n - 1)C_1^2(1 + R\sqrt{K}) + C_2}{R^2}\right] + 2\left(\nabla \varphi, \nabla \left(\frac{\varphi F}{\varphi}\right)\right)$$

$$+ \varphi \left[F_t - 2(\nabla f, \nabla F) - \frac{F}{t} - 2Kt|\nabla f|^2 + \frac{2t}{n}\left(|\nabla f|^2 - f_t - q - af^{\alpha}\right)^2\right]$$

$$- \beta t\Delta q - 2(\beta - 1)t(\nabla f, \nabla q) - 2(\beta - 1)ta_\alpha f^{\alpha - 1}|\nabla f|^2$$

$$\leq -F \left[-\frac{(n - 1)C_1^2(1 + R\sqrt{K}) + C_2}{R^2}\right] + 2\left(\nabla \varphi, \nabla (\varphi F)\right) - \frac{2F|\nabla \varphi|^2}{\varphi}$$

$$+ \varphi \left[F_t - 2(\nabla f, \nabla F) - \frac{F}{t} - 2Kt|\nabla f|^2 + \frac{2t}{n}\left(|\nabla f|^2 - f_t - q - af^{\alpha}\right)^2\right]$$

$$- \beta t\Delta q - 2(\beta - 1)t(\nabla f, \nabla q) - 2(\beta - 1)ta_\alpha f^{\alpha - 1}|\nabla f|^2$$

Fix a $T' \leq T$. Let $(x_0, t_0)$ be a point in $M \times [0, T']$ where $\varphi F$ achieves its maximum. We may assume that $(\varphi F)(x_0, t_0) > 0$ (so that $t_0 > 0$), otherwise it is clear. Ay $(x_0, t_0)$, we have

$$\nabla(\varphi F)(x_0, t_0) = 0, \quad (\varphi F)_t(x_0, t_0) \geq 0, \quad \Delta(\varphi F)(x_0, t_0) \leq 0.$$
An obvious consequence is $\nabla \varphi \cdot F + \varphi \cdot \nabla F = 0$ at the point $(x_0, t_0)$. From the inequality $|\nabla \varphi|^2 / \varphi \leq C_1^2 / R^2$ and introducing a constant

$$A := \frac{2C_1^2 + (n - 1)C_1^2(1 + R \sqrt{K}) + C_2}{R^2},$$

we obtain the following inequality

$$0 \geq -AF + 2F(\nabla f, \nabla \varphi) + \frac{2t_0}{n} \varphi \left( |\nabla f|^2 - f_t - q - af^\alpha \right)^2 \frac{\varphi F}{t_0} - 2Kt_0 \varphi |\nabla f|^2 - \beta t_0 \varphi \Delta q - 2(\beta - 1)t_0 \varphi (\nabla f, \nabla q)$$

$$- 2(\beta - 1)t_0 a \varphi f^{\alpha - 1}|\nabla f|^2 - \beta t_0 a \varphi (\alpha - 1)f^{\alpha - 2}|\nabla f|^2 + \beta a t_0 \varphi f^{\alpha - 1} \left( |\nabla f|^2 - f_t - q - af^\alpha \right),$$

at $(x_0, t_0)$. Set (see [2, 13])

$$\mu := \frac{|\nabla f|^2(x_0, t_0)}{F(x_0, t_0)} \geq 0.$$

We calculate

$$|\nabla f|^2 - f_t - q - af^\alpha = F \left( \mu - \frac{\mu t_0 - 1}{\beta t_0} \right),$$

and

$$\langle \nabla f, \nabla \varphi \rangle \leq |\nabla f| |\nabla \varphi| \leq \frac{C_1}{R} \varphi^{1/2}|\nabla f|,$$

at the point $(x_0, t_0)$. Simplifying (2.9) at $(x_0, t_0)$ yields

$$0 \geq -AF - \frac{2C_1^2}{R} \varphi^{1/2} \mu^{1/2} F^{3/2} + \frac{2t_0}{n} |\nabla f|^2 - \frac{2Kt_0 \varphi F}{t_0} - 2(\beta - 1)t_0 a \varphi f^{\alpha - 1} \mu F$$

$$- \beta t_0 \varphi \theta - 2(\beta - 1)t_0 \varphi \gamma F^{1/2} \mu^{1/2} - 2(\beta - 1)t_0 a \varphi f^{\alpha - 1} \mu F$$

$$- \beta t_0 a \varphi (\alpha - 1)f^{\alpha - 2} \mu F + a \varphi f^{\alpha - 1} [1 + (\beta - 1)\mu t_0] F.$$

Multiplying by $\varphi t_0$ on both sides, we have

$$AFt_0 \varphi \geq -\frac{2C_1t_0}{R} \varphi^{3/2} \mu^{1/2} F^{3/2} - \varphi^2 F + \frac{2\varphi^2}{n\beta^2} |1 + (\beta - 1)\mu t_0|^2 F^2$$

$$- 2t_0 \varphi^2 [K + a(\beta - 1)af^{\alpha - 1}] \mu F + at_0 \varphi^2 a f^{\alpha - 1} [1 + (\beta - 1)\mu t_0] F$$

$$- \beta t_0 \varphi^2 \theta - 2(\beta - 1)t_0 \varphi^2 \gamma (\mu F)^{1/2} - \beta t_0 \varphi^2 a \varphi (\alpha - 1)f^{\alpha - 2} \mu F. (2.10)$$

If we set $G := \varphi F$, then at the point $(x_0, t_0)$ the inequality (2.10) becomes

$$A_0 G \geq -\frac{2C_1t_0}{R} \mu^{1/2} G^{3/2} - \varphi G + \frac{2\varphi^2}{n\beta^2} |1 + (\beta - 1)\mu t_0|^2 G^2$$

$$- 2\varphi t_0^2 [K + a(\beta - 1)af^{\alpha - 1}] \mu G + at_0 \varphi G G^{\alpha - 1} [1 + (\beta - 1)\mu t_0] G$$

$$- \beta (\varphi t_0)^2 \theta - 2(\beta - 1)t_0 \varphi^2 \gamma (\mu G)^{1/2} - \beta t_0 \varphi^2 \varphi (\alpha - 1)f^{\alpha - 2} \mu G. (2.11)$$

Using the inequalities, where $0 < \epsilon < 1$,

$$\frac{2C_1t_0}{R} \mu^{1/2} G^{3/2} \leq \frac{2\epsilon}{n\beta^2} |1 + (\beta - 1)\mu t_0|^2 G^2 + \frac{n\beta^2 C_1^2 t_0^2 \mu G}{2\epsilon R^2 [1 + (\beta - 1)\mu t_0]^2},$$

$$2\mu^{1/2} G^{1/2} \leq 1 + \mu G,$$
we simplify (2.11) as the following inequality

$$A_{t_0}G \geq \frac{2(1 - \epsilon)}{n\beta^2} \left[ 1 + (\beta - 1)\mu t_0 \right]^2 G^2 - \varphi G - \frac{n\beta^2 C^2 \mu}{2\epsilon R^2 [1 + (\beta - 1)\mu t_0]^2} G$$

$$- 2\varphi t_0^2 [K + a(\beta - 1)\alpha f^\alpha - 1] \mu G + a\varphi t_0 \alpha f^\alpha - 1 [1 + (\beta - 1)\mu t_0] G$$

$$- \beta \varphi^2 t_0^2 \theta - (\beta - 1)\varphi^2 \gamma G - \beta t_0^2 \varphi \alpha(\alpha - 1) f^\alpha - 2 \mu G,$$

or equivalently,

$$\frac{2(1 - \epsilon) [1 + (\beta - 1)\mu t_0]^2 G^2}{n\beta^2} \leq \left[ A_{t_0} + \varphi + \frac{n\beta^2 C^2 \mu}{2\epsilon R^2 [1 + (\beta - 1)\mu t_0]^2} G + 2\varphi t_0^2 [K + a(\beta - 1)\alpha f^\alpha - 1] \mu$$

$$- a\varphi t_0 \alpha f^\alpha - 1 [1 + (\beta - 1)\mu t_0] + (\beta - 1)\varphi^2 \gamma G$$

$$+ \beta t_0^2 \varphi \alpha(\alpha - 1) f^\alpha - 2 \mu \right] t_0^2.$$

Note that $0 \leq \varphi \leq 1$ and $1 + (\beta - 1)\mu t_0 \geq 1$. Therefore

$$2(1 - \epsilon) G^2 \leq \left[ A_{t_0} + 1 + \frac{n\beta^2 C^2 \mu}{2\epsilon R^2 [1 + (\beta - 1)\mu t_0]^2} G + 2\varphi t_0^2 [K + a(\beta - 1)\alpha f^\alpha - 1] \mu$$

$$- a\varphi t_0 \alpha f^\alpha - 1 [1 + (\beta - 1)\mu t_0] + (\beta - 1)\varphi^2 \gamma G$$

$$+ \beta t_0^2 \varphi \alpha(\alpha - 1) f^\alpha - 2 \mu \right] t_0^2.$$

Before completing the proof, we recall a fact: if $x^2 \leq ax + b$ for some $b, x \geq 0$ and $a \in \mathbb{R}$, then

$$x \leq \frac{a}{2} + \sqrt{b + \left(\frac{a}{2}\right)^2} \leq \frac{a}{2} + \sqrt{b} + \frac{a}{2} = a + \sqrt{b}. \quad (2.13)$$

If $a \geq 0$ in (2.12), then from (2.12) we deduce that

$$G^2 \leq \left[ \frac{An\beta^2 t_o}{2(1 - \epsilon)} + \frac{n\beta^2}{2(1 - \epsilon)} + \frac{n^2\beta^2 C^2 t_0^2}{4\epsilon(1 - \epsilon) R^2 (\beta - 1)} + \frac{n\beta^2 \alpha \alpha + 1 f^\alpha - 2 t_0}{(\beta - 1)(1 - \epsilon)} + \frac{g}{2(1 - \epsilon)} t_0^2 \right] G + \frac{\beta \theta + (\beta - 1)\gamma n\beta^2 t_0^2}{2(1 - \epsilon)}. \quad (2.14)$$

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Applying (2.13) to the inequality (2.14), we get an upper bound for $G$:

$$G \leq \left[ \frac{(A + \gamma)n\beta^2}{2(1 - \epsilon)} + \frac{n^2\beta^4C_1^2}{4\epsilon(1 - \epsilon)(\beta - 1)R^2} + \frac{n\beta^2[K + a(\beta - 1)\alpha|f|^{\alpha - 1}]}{(1 - \epsilon)(\beta - 1)} \right] T'$$

$$+ \frac{n\beta^3 a\alpha|\alpha - 1||f|^{\alpha - 2}}{2(\beta - 1)(1 - \epsilon)} T' + \sqrt{\frac{[\beta\theta + (\beta - 1)\gamma]n\beta^2}{2(1 - \epsilon)}} T' + \frac{n\beta^2}{2(1 - \epsilon)},$$

since $t_0 \leq T'$. By the construction of $\varphi$, we have

$$\sup_{B_\rho(R)} F(x, t) \leq \sup_{B_\rho(R)} (\varphi(x) F(x, t)) \leq G(x_0, t_0),$$

for all $t \in [0, T')$. Because $T'$ is arbitrary, it follows that

$$|\nabla f|^2 - \beta f_t - \beta q - \beta a f^\alpha \leq \frac{n\beta^2}{2(1 - \epsilon)t} + \frac{(A + \gamma)n\beta^2}{2(1 - \epsilon)} + \frac{n^2\beta^4C_1^2}{4\epsilon(1 - \epsilon)(\beta - 1)R^2}$$

$$+ \frac{n\beta^2[K + a(\beta - 1)\alpha|f|^{\alpha - 1}]}{(1 - \epsilon)(\beta - 1)}$$

$$+ \frac{n\beta^3 a\alpha|\alpha - 1||f|^{\alpha - 2} G}{2(\beta - 1)(1 - \epsilon)} + \sqrt{\frac{[\beta\theta + (\beta - 1)\gamma]n\beta^2}{2(1 - \epsilon)}},$$

where $|f|_\infty := \max_M |f|$. Similarly, when $a \leq 0$, we have

$$G^2 \leq \left[ \frac{(A + \gamma)n\beta^2 t_0}{2(1 - \epsilon)} + \frac{n\beta^2}{2(1 - \epsilon)} + \frac{n^2\beta^4C_1^2 t_0}{4\epsilon(1 - \epsilon)R^2(\beta - 1)} + \frac{n\beta^2 K t_0}{(1 - \epsilon)(\beta - 1)} \right] G$$

$$- \frac{n\beta^2 a t_0 \alpha|f|^{\alpha - 1}}{2(1 - \epsilon)} + \frac{[\beta\theta + (\beta - 1)\gamma]n\beta^2 t_0^2}{2(1 - \epsilon)}. \tag{2.15}$$

From (2.13), (2.15), and above argument, an upper bound for desired quantity in this case is

$$|\nabla f|^2 - \beta f_t - \beta q - \beta a f^\alpha \leq \frac{n\beta^2}{2(1 - \epsilon)t} + \frac{(A + \gamma)n\beta^2}{2(1 - \epsilon)} + \frac{n^2\beta^4C_1^2}{4\epsilon(1 - \epsilon)(\beta - 1)R^2}$$

$$+ \frac{n\beta^2[K - \frac{a}{2}(\beta - 1)\alpha|f|^{\alpha - 1}]}{(1 - \epsilon)(\beta - 1)}$$

$$+ \sqrt{\frac{[\beta\theta + (\beta - 1)\gamma]n\beta^2}{2(1 - \epsilon)}}.$$

Hence, we complete the proof. \qed

When $\alpha = 1$, the above theorem reduces the main result in [10, 12]. Letting $R \to \infty$ and then $\epsilon \to 0$, we have the following

**Corollary 2.4** Let $(M, g)$ be a complete non-compact $n$-dimensional Riemannian manifold. Suppose that $u(x, t)$ is a positive solution on $M \times (0, T]$ of the (2.7). Assume that
(a) the Ricci curvature of \((M, g)\) is bounded from below by \(-K\), for some constant \(K \geq 0\), and

(b) there exists a constant \(\theta\), and a function \(\gamma(t)\) such that

\[|\nabla q|(x, t) \leq \gamma(t), \quad \Delta q(x, t) \leq \theta,\]

for any \((x, t) \in M \times (0, T]\).

Then

(1) for \(a \geq 0\), we have

\[
\frac{|\nabla u|^2}{u^2} - \beta \frac{u_t}{u} - \beta a \langle \ln u \rangle^\alpha \leq \beta q + \frac{\alpha \beta^2}{2t} + \left( \frac{\gamma}{2} + a \alpha |f^{\alpha-1}|_\infty \right) n \beta^2 + \frac{n \beta^2 K}{\beta - 1} n \beta^2 + \frac{\beta \theta}{2(\beta - 1)} + \sqrt{\frac{\beta \theta + (\beta - 1) \gamma}{2} n \beta^2},
\]

on \(M \times (0, T]\) for all \(\beta > 1\).

(2) for \(a \leq 0\), we have

\[
\frac{|\nabla u|^2}{u^2} - \beta \frac{u_t}{u} - \beta a \langle \ln u \rangle^\alpha \leq \beta q + \frac{\alpha \beta^2}{2t} + \left( \frac{\gamma}{2} - a \alpha |f^{\alpha-1}|_\infty \right) n \beta^2 + \frac{n \beta^2 K}{\beta - 1} n \beta^2 + \sqrt{\frac{\beta \theta + (\beta - 1) \gamma}{2} n \beta^2},
\]

on \(M \times (0, T]\) for all \(\beta > 1\).

We now apply Corollary 2.4 to the elliptic equation

\[(\Delta - q)u = au\langle \ln u \rangle^\alpha, \quad (2.16)\]

where \(u\) is a \(C^2\) function on \(M\), by letting \(T \to \infty\).

**Corollary 2.5** Let \((M, g)\) be a complete non-compact \(n\)-dimensional Riemannian manifold. Suppose that \(u(x, t)\) is a positive solution on \(M\) of the equation (2.16). Assume that

(a) the Ricci curvature of \((M, g)\) is bounded from below by \(-K\), for some constant \(K \geq 0\), and

(b) there exists a constant \(\theta\), and a function \(\gamma(t)\) such that \(|\nabla q| \leq \gamma\) and \(\Delta q \leq \theta\) on \(M\).

Then

(1) for \(a \geq 0\), we have

\[
\frac{|\nabla u|^2}{u^2} - \beta a \langle \ln u \rangle^\alpha \leq \beta q + \frac{\gamma}{2} + a \alpha |f^{\alpha-1}|_\infty + \frac{a \beta a |\alpha - 1| \langle \ln u \rangle^{\alpha-2}|_\infty}{2(\beta - 1)} n \beta^2
\]

\[
+ \frac{n \beta^2 K}{\beta - 1} + \sqrt{\frac{\beta \theta + (\beta - 1) \gamma}{2} n \beta^2},
\]

on \(M\) for all \(\beta > 1\).
(2) for $a \leq 0$, we have
\[ \frac{|\nabla u|^2}{u^2} - \beta a (\ln u)^\alpha \leq \beta q + \left( \frac{\gamma}{2} - \frac{a}{2} (\ln u)^{\alpha-1} \right) n \beta^2 \]
\[ + \frac{n \beta^2 K}{\beta - 1} + \sqrt{\frac{[\beta + (\beta - 1) \gamma] n \beta^2}{2}}, \]
on $M$ for all $\beta > 1$.

In particular, if $u$ is a positive solution of the equation $(\Delta - q)u = au \ln u$, then

(1') for $a > 0$, we have a lower bound
\[ u \geq \exp \left[ -\frac{q}{a} - \left( 1 + \frac{\gamma}{2a} \right) n \beta - \frac{n \beta K}{(\beta - 1)a} - \frac{1}{a} \left( \frac{[\beta + (\beta - 1) \gamma] n}{2} \right)^{1/2} \right], \]
on $M$ for all $\beta > 1$.

(2') for $a < 0$, we have an upper bound
\[ u \leq \exp \left[ -\frac{q}{a} + \left( \frac{1}{2} - \frac{\gamma}{2a} \right) n \beta - \frac{n \beta K}{(\beta - 1)a} - \frac{1}{a} \left( \frac{[\beta + (\beta - 1) \gamma] n}{2} \right)^{1/2} \right], \]
on $M$ for all $\beta > 1$.

**Remark 2.6** When $q$ is a constant, Theorem 2.3 reduces to Theorem 1.1 in [13]. Corollary 2.5 gives a much better bound for a positive solution of (2.16) on $M$ if $q = 0$ and the Ricci curvature of $M$ is nonnegative (compared with Corollary 1.6 in [10] and Corollary 1.2 in [13]). In fact, in this case, taking $q = \gamma = \theta = K = 0$, we have
\[ u \geq e^{-n} \quad (a > 0), \quad \text{or} \quad u \leq e^{n/2} \quad (a < 0). \]

Note that our constant $a$ is actually the constant $-a$ used in [10, 13].

**Acknowledgments** Yi Li is partially supported by Shanghai YangFan Project (grant) No. 14YF1401400. Xiaorui Zhu is partially supported by CPSF (grant) No. 2014M551721 and Zhejiang Province Natural Science Foundation of China (grant) No. Q14A010002. Yi Li would like to thank Shanghai Center for Mathematical Sciences, where part of their work was done during the visit, for their hospitality.

**References**

1. Calabi, E.: An extension of E. Hopf’s maximum principle with application to Riemannian geometry. Duck Math. J. 25, 45–46 (1958). MR0092069 (19, 1056e)
2. Chen, L., Chen, W.: Gradient estimates for a nonlinear parabolic equation on complete non-compact Riemannian manifolds. Ann. Glob. Anal. Geom. 35(4), 397–404 (2009). MR2506242 (2010k: 35501)
3. Cheng, S.-Y., Yau, S.-T.: Differential equations on Riemannian manifolds and their geometric applications. Comm. Pure. Appl. Math. 28(3), 333–354 (1975). MR0385749 (52 # 6608)
4. Hamilton, R.S.: The Harnack estimate for the Ricci flow. J. Differential Geom. 37(1), 225–243 (1993). MR1198607 (93k: 58052)
5. Hsu, S.: Gradient estimates for a nonlinear parabolic equation under Ricci flow. Differ. Integr. Eq. 24(7–8), 645–652 (2011). MR2830313 (2012h: 53150)
6. Li, P., Yau, S.-T.: On the parabolic kernel of the Schrödinger operator. Acta Math. 156(3–4), 153–201 (1986). MR0834612 (87f: 58156)
7. Li, Y., Zhu, X.: Preparation
8. Ma, L.: Gradient estimates for a simple elliptic equation on complete non-compact Riemannian manifolds. J. Funct. Anal. 241(1), 374–382 (2006). MR2264255 (2007e: 53034)
9. Peleman, G.: The entropy formula for the Ricci flow and its geometric applications, arXiv:math.DG/0211159
10. Qian, B.: Hamilton-type gradient estimates for a nonlinear parabolic equation on Riemannian manifolds. Acta. Math. Sin. (Engl. Ser.) 27(6), 1071–1078 (2011). MR2795355 (2012c: 35438)
11. Schoen, R., Yau, S.-T.: Lectures on differential geometry, Conference Proceedings and Lecture notes prepared by Wei Yue Ding, Kung Ching Chang [Gong Qing Zhang], Jia Qing Zhong and Yi Chao Xu. Translated from the Chinese by Ding and S. Y. Cheng. Preface translated from the Chinese by Kaising Tso. In: Conference Proceedings and Lecture Notes in Geometry and Topology, I, pp. v+235 International Press, Cambridge, MA. ISBN: 1-57146-012-8 MR1333601 (97d: 53001) (1994)
12. Wu, J.-Y.: Li-Yau type estimates for a nonlinear parabolic equation on complete manifolds. J. Math. Anal. Appl. 369(1), 400–407 (2010). MR2643878 (2011b: 35432)
13. Yang, Y.: Gradient estimates for a nonlinear parabolic equation on Riemannian manifolds. Proc. Amer. Math. Soc. 136(11), 4095–4102 (2008). MR2425752 (2009d: 58048)