Analytic detection of non-trivial elements in fundamental groups of Riemann surfaces

Ivan Zubov
Dzerzhinsky street, 43, 5, city Zaraysk, Moscow region, Russian Federation, 140601
E-mail: reestr_rr@mail.ru

Abstract. In the paper we consider the detection problem of homotopic non-triviality of loops on Riemann surfaces. Basic methods of the detection based on using of Chen’s iterated integrals of meromorphic 1-forms on such Riemann surfaces. Iterated integrals possess many different properties that allow us to connect its with the multiplication loops and their homotopic classes. For compact Riemann surfaces the existence of cohomological non-trivial meromorphic 1-forms follows from the Riemann-Roch theorem. Such 1-forms permit us to point non-zero homotopic iterated integral periods for loops.

1. Introduction
In the paper we consider the detection problem of homotopic non-triviality of loops on Riemann surfaces. Basic methods of the detection based on using of Chen’s iterated integrals of meromorphic 1-forms on such Riemann surfaces. Iterated integrals possess many different properties that allow us to connect its with the multiplication loops and their homotopic classes. For compact Riemann surfaces the existence of cohomological non-trivial meromorphic 1-forms follows from the Riemann-Roch theorem. Such 1-forms permit us to point non-zero homotopic iterated integral periods for loops.

In section 2 we recall the notion of divisors, divisors of functions, divisors of meromorphic 1-forms. In section 3 we present formulation of Riemann-Roch theorem. In section 4 we describe the properties of Chen’s iterated integrals and in section 5 we prove Riemann bilinear relations by iterated integrals. In section 6, we define the filtration of the fundamental group of the Riemann surface by members of the lower central series. We explain how meromorphic 1-forms and their iterated integrals detect the nontriviality of an arbitrary element of the fundamental group of Riemann surfaces.

2. Divisors on Riemann surfaces
First we recall the notion of a divisor on a Riemann surfaces and related notions.
A divisor $D$ is an element of the free Abelian group $\text{Div} \, C$, generated by points of $C$, i.e. by formal finite integer linear combination of points $C$:

$$D = \sum n_i P_i.$$  

The support of divisor $D$ is the set
Supp \( D = \{ P_i \mid n_i \neq 0 \} \).

The **degree of the divisor** \( D \) is the sum

\[
\deg D = \sum_i n_i.
\]

The set \( \text{Div}^0 C \) of divisors of degree 0 is a subgroup of \( \text{Div} C \).

If the function \( f \) at point \( P \) has zero, we denote \( \nu_f(P) \) the order of zero at that point \( P \). If the function \( f \) at point \( P \) has a pole, we denote \( \nu_f(P) \) minus order of the pole at that point \( P \). If the function \( f \) at a point \( P \) has no zero and has no pole, then \( \nu_f(P) = 0 \). The divisor

\[
(f) = \sum_{P \in C} v_P(f) P
\]

is called *main* divisor. Well-known that on a compact Riemann surfaces \( \deg(f) = 0 \).

Two divisors \( D, D' \) are called **linearly equivalent**: \( D \sim D' \), if \( D = D' + (f) \) for some meromorphic function \( f \). Divisor \( D = \sum n_i P_i \) is called *effective*, if all \( n_i \geq 0 \).

We will write in this case \( D \geq 0 \). We associate with each divisor \( D \) the vector space over complex number

\[
L(D) = \{ 0 \} \cup \{(f) + D \geq 0 \mid f \text{ is a meromorphic function on } C \}.
\]

Note that for an effective divisor \( D = \sum n_i P_i \), the vector space \( L(D) \) consists of meromorphic functions whose pole order at each point \( P_i \) less or equal \( n_i \).

The fundamental result for Riemann surfaces is that if there \( C \) is no special algebraic curve, then the vector space \( L(D) \) is finite dimensional for any divisor of \( D \): \( \dim C L(D) = l(D) < \infty \).

Note that if \( D \sim D' \), then \( L(D) \cong L(D') \) and \( l(D) = l(D') \). In case \( \deg D < 0 \) the space \( L(D) = \{ 0 \} \).

### 3. Riemann-Roch theorem

Further, we consider only Riemann surfaces that are non-singular algebraic curves. For these varieties, we formulate the Riemann-Roch theorem in its classical formulation \([8, 12]\) as well as in the language of the sheaf theory \([4, 5, 6]\) and describe some consequences from the Riemann-Roch theorem.

The dimension \( l(D) \) of the vector space \( L(D) \) can be calculated by the Riemann-Roch formula. Let \( \omega \) be some meromorphic differential (i.e. meromorphic form) on the curve \( C \), which in the neighbourhood of the point \( P \) is represented as \( \omega = z^n f(z) dz \) for \( f(0) \neq 0, \infty \) (we consider \( z(P) = 0 \)). Then the divisor of the form is defined as

\[
(\omega) = \sum_P n_P P.
\]

It is well known that all divisors of meromorphic 1-forms are linearly equivalent to each other. This class of divisors is called the **canonical class** of the curve \( C \) and it is denoted by \( K_C \).

For a non-singular curve \( C \) genus \( g \)

\[
l(D) - l(K_C - D) = \deg D + 1 - g.
\]

It follows that \( l(D) \geq \deg D + 1 - g \). In case of equality, the divisor \( D \) is called *non-special*, otherwise, it is called *special*. 

Let \( \{ \omega_1, \cdots, \omega_g \} \) be basis of holomorphic 1-forms on the curve. Fix on the curve \( C \) the point \( P_0 \). For point \( P \in C \) let us consider vector of integrals

\[
j_\gamma(P) = \left( \int_{P_0}^P \omega_1, \cdots, \int_{P_0}^P \omega_g \right)
\]

along some path \( \gamma \) from \( P_0 \) to \( P \). This mapping is called the Abel mapping \( j : C \to \text{Jac} C = C^g/\Pi^g \), where \( \Pi^g \) is the lattice of periods of the curve. Let’s extend it to the group of divisors \( \text{Div} C \) by the linearity

\[
j(D) = j \left( \sum_i n_i P_i \right) = \sum_i n_i j(P_i).
\]

Abel Theorem states, that if \( f \) is a meromorphic function on \( C \), then

\[
j((f)) = 0.
\]

And back, if the divisor \( D \) of degree 0 is such that \( j(D) = 0 \), then \( D = (f) \) for a meromorphic function \( f \) on \( C \).

As consequence of this theorem we obtain that the Jacobian of the curve \( C \) isomorphic to the quotient group of the divisor group \( \text{Div}^0 C \) with respect to linear equivalence: \( \text{Div}^0 C/\sim = \text{Jac} (C) \).

The definition of the Jacobian of the curve can also be given in terms of line bundles and sheaves of the rank one. For a non-singular algebraic curve \( C \) there exist a one-to-one correspondence between the classes of linear equivalence of divisors on \( C \), linear holomorphic bundles on \( C \) and sheaves of rank one on \( C \), locally isomorphic to a sheaf of germs \( \mathcal{O}_C \) of holomorphic functions on \( C \).

For a non-singular algebraic curve \( C \) and a divisor \( D \) on \( C \) the number

\[
\chi(D) = h^0(C, \mathcal{O}_C(D)) - h^1(C, \mathcal{O}_C(D))
\]

is called the Euler characteristic of a sheaf \( \mathcal{O}_C(D) \).

Here \( h^0(C, \mathcal{O}_C(D)) = \dim \mathcal{L}(D) \) and \( h^1(C, \mathcal{O}_C(D)) = h^0(K_C - D) \) — space dimension \( \dim \mathcal{L}(K_C - D) \).

**Lemma.** \( \chi(D) = \chi(0) + \deg D \).

The proof is constructed by induction. It is necessary to check the validity of this formula for the divisor \( D + p \) if it holds for the divisor \( D \). Consider the exact sequence of sheaves

\[
0 \to \mathcal{O}_C(D) \to \mathcal{O}_C(D + p) \to \mathcal{Q} \to 0,
\]

where \( \mathcal{Q} \) — sheaf-skyscraper: \( \mathcal{Q}_q = 0 \), if only \( q \neq p \) and \( \dim \mathcal{Q}_p = 1 \). It follows that

\[
h^0(C, \mathcal{Q}) = 1, \quad h^1(C, \mathcal{Q}) = 0.
\]

From a long sequence of cohomology we obtain

\[
h^0(D) - h^0(D + p) + 1 - h^1(D) + h^1(D + p) - 0 = 0,
\]

i.e. \( \chi(D + p) = \chi(D) + 1 \). The right part of the formula, obviously, increases by one during the transition from \( D \) to \( D + p \).

By definiton \( \chi(0) = h^0(C, \mathcal{O}_C) - h^1(C, \mathcal{O}_C) = 1 - h^0(C, \Omega^1_C) \). Finally,
\[ \chi(D) = \deg D + 1 - h^0(C, \Omega^1_C). \]

For a non-singular algebraic curve we obtain already known formulation:

\[ h^0(D) - h^0(K_C - D) = \deg D + 1 - g(C). \]

Let now the curve \( C \) is singular and \( S \) is the set of its singular points. It is known that in this case there is a single non-singular curve \( C^{\text{nor}} \) that it is called a normalization of \( C \), which is birationally isomorphic \( C \). Let \( \pi : C^{\text{nor}} \to C \) be the corresponding projection. For the point \( p \in C \) define the number \( \delta_p = \dim_k(\pi_* \mathcal{O}_{C^{\text{nor}}})_p/\mathcal{O}_{C,p} \). It is zero if and only if the point is not \( p \) non-singular. Let \( \delta(C) = \sum p \in S \delta_p \) and \( g(C) = g(C^{\text{nor}}) \). A calculation similar to the above leads to a more general formulation of the Riemann-Roch theorem.

**Theorem.** Let \( C \) is a singular curve with the set of singular points \( S \). Then for the divisor \( D \) on \( C \), such that \( \text{Supp} \, D \cap S = \emptyset \), we have equality

\[ \chi(D) = \deg D + 1 - (g(C) + \delta(C)). \]

Two numbers associated with a non-singular algebraic curve \( p_\alpha(C) = h^0(C, \Omega^1_C) \) and \( p_\alpha(C) = h^1(D, \mathcal{O}_C) = l(K_C - D) \) coincide. The number \( g(C) = p_\alpha(C) = p_\beta(C) \) is called the genus of a non-singular algebraic curve. If \( C \) is Riemann sphere, then \( g(C) = 0 \). If the non-singular algebraic curve \( C \) has the genus \( g(C) > 0 \). It then follows from the Riemann-Roch theorem that the dimension of the space of holomorphic 1-forms will be equal \( g(C) \).

### 4. Iterated integrals

We recall the definition of iterated integrals from differential 1-forms on smooth manifolds [1, 2, 4]. Let \( \omega_1, \omega_2, \ldots, \omega_r \) differential 1-forms on a manifold \( M \). Let \( P_{x_0}(M) \) be the space of paths \( \gamma : I = [0, 1] \to M \) starting at point \( x_0 = \gamma(0) \). An iterated integral is a function \( \int \omega_1 \ldots \omega_r : P_{x_0}(M) \to \mathbb{R} \) on the space of paths \( P_{x_0}(M) \), which on the path \( \gamma \) take the value

\[ \int_\gamma \omega_1 \ldots \omega_r = \int_\gamma \left( \int_\gamma \omega_1 \ldots \omega_{r-1} \right) \omega_r, \quad \text{where} \quad \gamma^t = \gamma(\tau t), t \in [0, 1]. \]

For complex-value differential 1-forms \( \int \omega_1 \ldots \omega_r \) takes the value in \( \mathbb{C} \). The space of paths \( \gamma(0) = x_0, \gamma(1) = x_1 \) we denote \( P_{x_0}^{x_1}(M) \). The space of iterated integrals \( B_s(M) \) generated by of iterated integrals \( \int_\gamma \omega_1 \ldots \omega_r \) the length \( r \leq s \).

For iterated integrals of 1-forms have on the loop space \( \Omega_{x_0} M = P_{x_0}(M) \) the formula for differentiation

\[
\begin{align*}
&d \int_\gamma \omega_1 \cdots \omega_q = - \sum_{i=1}^q \int_\gamma \omega_1 \cdots \omega_{i-1} d\omega_i \omega_{i+1} \cdots \omega_q - \\
&\quad - \sum_{i=1}^{q-1} (-1)^i \int_\gamma \omega_1 \cdots \omega_{i-1} (\omega_i \wedge \omega_{i+1}) \omega_{i+2} \omega_{i+3} \cdots \omega_q,
\end{align*}
\]

and the Stokes formula

\[
\int_C (d \int \omega_1 \cdots \omega_q) = \int_C \left( \int \omega_1 \cdots \omega_q \right) = \int_{C(1)} \omega_1 \cdots \omega_q - \int_{C(0)} \omega_1 \cdots \omega_q,
\]
where \( C : [0; 1] \rightarrow P^b_a(M) \) is a path considered as a singular simplex in space \( P^b_a(M) \). This simplex defines the homotopy between paths \( \gamma_1 \) and \( \gamma_2 \), \( C(0) = \gamma_1 \) and \( C(1) = \gamma_2 \) in the space of paths \( P^b_a(X_n) \).

Consider another properties of iterated integrals.

1). We prove that the product of iterated integrals is equal to the linear combination of iterated integrals.

**Theorem.** The product of iterated integrals of orders \( k \) and \( l \) is equal to the following sum of iterated integrals of the order \( k+l \)

\[
\int_\gamma \omega_1 \cdots \omega_k \cdot \int_\gamma \omega_{k+1} \cdots \omega_{k+l} = \sum_{\sigma \in S_{k,l}} \int_\gamma \omega_{\sigma(1)} \cdots \omega_{\sigma(k+l)},
\]

where the sum is taken for all shuffles \((k,l)\) in the permutation group \( S_{n+k} \).

2). Consider the forms \( \omega_1, \ldots, \omega_r \) defined on \( M^n \) and path \( \gamma : [0,1] \rightarrow M^n \). Denote \( \gamma'(\tau) = \gamma(t(\tau)) \), where \( t(\tau) : [0,1] \rightarrow [0,1] \) there is some variable replacement.

If the function \( t(\tau) \) monotonically increases, the equivalence class of paths with the accuracy of such substitution, the variable is called an oriented curve. There is a property of invariance at differentiable monotonically increasing replacement:

\[
\int_{\gamma'} \omega_1 \cdots \omega_r = \int_\gamma \omega_1 \cdots \omega_r.
\]

3). The product of two ways \( \alpha \cdot \beta : [0,1] \rightarrow M^n \) it is defined as follows. Let the two paths be such that the end of the first path coincides with the beginning of the second, that is

\[
\alpha : [0,1] \rightarrow M^n \quad \text{the first path}
\]
\[
\beta : [0,1] \rightarrow M^n \quad \text{the second path and}
\]
\[
\alpha(1) = \beta(0).
\]

Then the product is determined by the formulas

\[
(\alpha \cdot \beta)(t) = \alpha(2t), \quad 0 \leq t \leq \frac{1}{2};
\]
\[
(\alpha \cdot \beta)(t) = \beta(2t - 1), \quad \frac{1}{2} \leq t \leq 1.
\]

**Theorem.** Let \( \alpha \) and \( \beta \) two paths for which the product is defined \( \gamma = \alpha \cdot \beta : [0,1] \rightarrow M^n \). There is the following formula for the value of the iterated integral on the product of paths:

\[
\int_{\gamma = \alpha \cdot \beta} \omega_1 \cdots \omega_r = \int_{\alpha} \omega_1 \cdots \omega_r + \sum_{k=1}^{r-1} \int_{\alpha} \omega_1 \cdots \omega_k \int_{\beta} \omega_{k+1} \cdots \omega_r + \int_{\beta} \omega_1 \cdots \omega_r.
\]

4). The path \( \gamma^{-1} \) is given by the formula

\[
\gamma^{-1}(t) = \gamma(1-t), \quad t \in [0,1].
\]

A generalization of property 2 is the formula for the value of the iterated integral on the inverse path:

**Theorem.**
\[
\int_{\gamma^{-1}} \omega_1 \cdots \omega_r = (-1)^r \int_{\gamma} \omega_r \cdots \omega_1
\]

5). Definition. The pin is called path \( \alpha = \gamma \gamma^{-1} \).

A pin insert is a representation of a path as a product \( \alpha = \beta_1 \gamma \gamma^{-1} \beta_2 \). Removing a factor \( \gamma \gamma^{-1} \) called the removal of the pin.

Theorem. The iterated integral does not depend from of the insertion or removal of pins.

Proof. To prove the independence of the iterated integral from the insertion or removal of pins, it is sufficient to prove that

\[
\int_{\gamma^{-1}} \omega_1 \cdots \omega_r = 0 \quad \text{for } r \geq 1.
\]

The latter follows from the properties 1, 3 and 4 of the iterated integral and is proved by induction by \( r \).

6). A loop is a path whose beginning and end coincide: \( \gamma(0) = \gamma(1) \).

Let us denote this space \( \Omega_{x_0}(M) \) the space of loops in the point \( x_0 \). If we consider loops up to insertion or removal of pins, it is not difficult to notice that this is the equivalence relation on the loop space.

The set of equivalence classes is a topological space with respect to the quotient topology of the original compact-open topology on the loop space. Let us denote the obtained quotient-space \( \Omega_{x_0}(M) \).

This space is a topological group with respect to the operation induced by the product of loops in the original loop space. In particular, the product of equivalence classes is associative, and in terms of loops we can say that the product of loops considered up to monotone change of variable is an associative operation on loops.

A connected component of the identity of the group \( \Omega_{x_0}(M) \) is a normal divisor in it. Quotient-group is a normal subgroup is isomorphic to the fundamental group of manifolds \( M^n \).

Property 5 shows that iterated integrals are continuous (actually differentiable of the same class of smoothness as the space of differential forms and the space of loops under consideration) functions on the group \( \Omega_{x_0}(M) \).

7). Value of 2-iterated integral \( \int \omega_1 \omega_2 \) on the commutator \( [\alpha, \beta] = \alpha \beta \alpha^{-1} \beta^{-1} \) two loops \( \alpha \) and \( \beta \) it is calculated through the values of 1-iterated integrals by the formula:

\[
\int_{[\alpha, \beta]} \omega_1 \omega_2 = \int_{\alpha} \omega_1 \int_{\beta} \omega_2 - \int_{\beta} \omega_1 \int_{\alpha} \omega_2.
\]

To prove this, we place brackets in the switch of two loops \( \alpha \) and \( \beta \) as follows \( [\alpha, \beta] = (\alpha \beta) (\alpha^{-1} \beta^{-1}) = (\alpha \beta) ((\beta \alpha)^{-1}) \). It was found out above that the iterated integral does not depend on the arrangement of brackets in the products of loops. The required equality is proved by direct calculation using the above considered properties of iterated integrals. So,

\[
\int_{[\alpha, \beta]} \omega_1 \omega_2 = \int_{(\alpha \beta)(\beta \alpha)^{-1}} \omega_1 \omega_2 = \int_{\alpha \beta} \omega_1 \omega_2 + \int_{\beta \alpha} \omega_1 \omega_2 + \int_{\alpha \beta} \omega_1 \int_{\beta \alpha} \omega_2 - \int_{\beta \alpha} \omega_1 \int_{\alpha \beta} \omega_2
\]
\[-(\int_{\alpha} \omega_1 + \int_{\beta} \omega_1)(\int_{\alpha} \omega_2 + \int_{\beta} \omega_2) = \]

\[= \int_{\alpha} \omega_1 \omega_2 + \int_{\beta} \omega_1 \omega_2 + \int_{\alpha} \omega_1 \int_{\beta} \omega_2 + \int_{\beta} \omega_1 \int_{\alpha} \omega_2 + \int_{\alpha} \omega_2 \omega_1 + \int_{\beta} \omega_1 \omega_2 - (\int_{\alpha} \omega_1 + \int_{\beta} \omega_1)(\int_{\alpha} \omega_2 + \int_{\beta} \omega_2) = \]

\[= (\int_{\alpha} \omega_1 \omega_2 + \int_{\beta} \omega_2 \omega_1) + (\int_{\alpha} \omega_1 \omega_2 + \int_{\beta} \omega_2 \omega_1) + \int_{\alpha} \omega_1 \int_{\beta} \omega_2 + \int_{\beta} \omega_1 \int_{\alpha} \omega_2 = \]

\[= \int_{\alpha} \omega_1 \int_{\beta} \omega_2 + \int_{\alpha} \omega_1 \int_{\beta} \omega_2 + \int_{\beta} \omega_1 \int_{\alpha} \omega_2 + \int_{\beta} \omega_1 \int_{\alpha} \omega_2 = \]

\[-(\int_{\alpha} \omega_1 + \int_{\beta} \omega_1)(\int_{\alpha} \omega_2 + \int_{\beta} \omega_2) = \int_{\alpha} \omega_1 \int_{\beta} \omega_2 - \int_{\alpha} \omega_1 \int_{\beta} \omega_2.\]

The property is proved.

8). The value of 2-iterated \(\int \omega_1 \omega_2\) integral for the product of several commutator loops \(\gamma = \prod_{i=1}^{m} [\alpha_i, \beta_i]\) equal to the sum of the values of 2-iterated integrals on the commutator factors, that is:

\[
\int \omega_1 \omega_2 = \sum_{i=1}^{m} \int_{[\alpha_i, \beta_i]} \omega_1 \omega_2.
\]

By virtue of the previous property, the 2-iterated integral is equal to the sum of switch-type expressions

\[
\int \omega_1 \omega_2 = \sum_{i=1}^{m} (\int_{\alpha} \omega_1 \int_{\beta} \omega_2 - \int_{\beta} \omega_1 \int_{\alpha} \omega_2).
\]

The proof of the initial equality is carried out by induction on the number of commutator factors. The proof is carried out for the beginning of induction, that is, for \(m = 2\). The General induction step easily follows from property 3 and the induction assumption. Let \(\gamma = \gamma_1 \gamma_2\), where \(\gamma_1 = [\alpha_1, \beta_1]\) and \(\gamma_2 = [\alpha_2, \beta_2]\). By property 3 we have

\[
\int_{\gamma_1 \gamma_2} \omega_1 \omega_2 = \int_{\gamma_1} \omega_1 \omega_2 + \int_{\gamma_2} \omega_1 \omega_2 + \int_{\gamma_1} \omega_1 \int_{\gamma_2} \omega_2.
\]

From the properties of ordinary curvilinear integrals it follows that for \(i = 1, 2\)
\[ \int_{\gamma_1} \omega_1 = \int_{\alpha_i} \omega_i = \int_{\beta_i} \omega_i = 0. \]

Now use property 3 easy to prove that
\[ \int_{\gamma_1\gamma_2} \omega_1\omega_2 = \int_{\gamma_1} \omega_1\omega_2 + \int_{\gamma_2} \omega_1\omega_2. \]

As noted above, the general case is proved similarly using the same properties and induction assumption.

The definition of the iterated integral and the listed properties, except property 4, are generalized to the case of matrix-valued differential 1-forms. Let us write in the framework of such a generalization of the formula for differentiation of iterated integrals of differential 1-forms and are closely related to the Stokes formula. Iterated integrals from 1-forms are functions on the loop space \( \Omega_{x_0}(M^n) \) with a given starting point \( x_0 \) in the manifold \( M^n \) or, in other words, are differential 0-forms on \( \Omega_{x_0}(M^n) \). Then have:

\[ d \int \omega_1 \ldots \omega_r = \]
\[ = - \sum_{k=1}^{r} \int \omega_1 \ldots d\omega_k \ldots \omega_r - \sum_{k=1}^{r-1} (-1)^k \int \omega_1 \ldots (\omega_k \wedge \omega_{k+1}) \ldots \omega_r. \]

The left and right parts of the written equation are treated as differential 1-forms on the loop space \( \Omega_{x_0}(M^n) \).

Let \( h : I = [0, 1] \to \Omega_{x_0}(M^n) \) a path in space considered as a singular 1-simplex in this loop space. Then
\[ \int_{h} \left( d \int \omega_1 \ldots \omega_r \right) = \int_{h(1)} \left( \int \omega_1 \ldots \omega_r \right) = \int_{h(1)} \omega_1 \ldots \omega_r - \int_{h(0)} \omega_1 \ldots \omega_r. \]

Denote the vector space of iterated integrals on \( M \) of length \( \leq s \) by \( B_s(M) \). Denote the constant path at the point \( x \) of \( M \) by \( x_\cdot \). (That is, \( x_\cdot(t) = x \) for all \( t \).) If \( r \geq 1 \), then
\[ \left\langle \int \omega_1 \ldots \omega_r, \eta_x \right\rangle = 0 \]
for all \( x \in M \). Thus evaluating at a constant path \( \eta_x \) defines a linear functional
\[ \varepsilon : B_s(M) \to R, \]
\[ I \to \langle I, \eta_x \rangle \]
that is independent of \( x \). If
\[ I = \lambda + \sum a_i \int \omega_i + \sum a_{ij} \int \omega_i \omega_j + \cdots, \]
then \( \varepsilon(I) = \lambda \). Denote the kernel of \( \varepsilon \) by \( B_\varepsilon(M) \). There are the iterated integrals of length \( \leq s \) with zero constant term. Since there is a natural inclusion \( i : R \to B_s(M) \) such that \( \varepsilon \circ i = id \), we have a natural direct sum decomposition
\[ B_s(M) \cong R \oplus \mathcal{F}_s(M). \]

For loops \( \alpha, \beta \in PM \) based at \( x \), we can form the commutator \([\alpha, \beta] = \alpha\beta^{-1}\alpha^{-1}\beta^{-1}\). Often we will denote the constant loop \( \eta_x \) at \( x \) by 1. Since the paths \( \alpha\eta_x \) and \( \eta_x\alpha \) differ from the loop \( \alpha \) by a reparametrization. If \( I \) is an iterated integral, then

\[ I(\alpha) = I(\alpha\eta_x) = I(\eta_x\alpha). \]

Recall that the classical line integral satisfies

\[ \left\langle \int \omega, [\alpha, \beta] \right\rangle = 0 \quad \text{and} \quad \left\langle \int \omega, (\alpha - 1)(\beta - 1) \right\rangle = 0, \]

where \( \alpha \) and \( \beta \) are loops based at \( x \). The following lemma is a generalization of this fact.

**Lemma.** Suppose that \( \omega_1, \ldots, \omega_r \in E^1(M) \) and that \( x \in M \). Suppose that \( \alpha_1, \alpha_2, \ldots, \alpha_s \) are loops in \( M \) based at \( x \).

(a) If \( I \in B_r \) and \( r < s \), then

\[ \langle I, (1 - \alpha_1)(1 - \alpha_2) \cdots (1 - \alpha_s) \rangle = 0, \]

where 1 denotes \( \eta_x \), the constant path at \( x \).

(b) If \( I \in \overline{B}_r \) and \( r < s \), then

\[ \langle I, [\alpha_1[\alpha_2[\ldots[\alpha_{s-1}]]]] \rangle = 0. \]

5. **Riemann bilinear relations**

Let \( \omega, \omega' \) be holomorphic 1-forms on a non-singular algebraic curve \( C \) genus \( g \) and \( A_i = \int_{a_i} \omega, A'_i = \int_{a_{i+g}} \omega', i = 1, \ldots, g \) be periods of these forms. The Riemann bilinear relations have the type:

\[ \sum_{i=1}^{g} (A_i \cdot A'_{i+g} - A_{i+g} \cdot A'_i) = \int_{C} \omega \wedge \omega' = \int_{C} \omega = 0, \quad (1) \]

\[ \gamma = \prod_{i=1}^{g} [\alpha_i, \alpha_i+g], \]

\[ \sum_{i=1}^{g} (A_i \cdot A'_{i+g} - A_{i+g} \cdot A'_i) = \int_{C} \omega \wedge \omega' = 0. \quad (2) \]

We consider the surface \( C \) as a homotopy \( h : I = [0, 1] \rightarrow \Omega_{x_0}C \) between \( \gamma = \prod_{i=1}^{g} [\alpha_i, \alpha_i+g] \) and trivial loop \( \eta_{x_0} \) in the space of loops \( \Omega_{x_0}C \).

By Stokes theorem for the 2-iterated integrals we obtain the equality

\[ \int_{\gamma} \omega \wedge \omega' = \int_{h} \int_{\eta_{x_0}} \omega \wedge \omega' = \int_{C} \omega \wedge \omega' = 0. \]

Because \( \omega \wedge \omega' \)-holomorphic 2-form, then \( \omega \wedge \omega' \equiv 0 \) on \( C \). Use the equality

\[ \int_{\eta_{x_0}} \omega \cdot \omega' = 0 \]
and the properties of the 2-iterated integral, we obtain the first bilinear Riemann relation
\[ \int \omega \cdot \omega' = \sum_{i=1}^{g} (A_i \cdot A_{i+g} - A_{i+g} \cdot A_i) = \int \omega \wedge \omega' = 0. \]

If we take the holomorphic form \( \omega \) and antiholomorphic form \( \overline{\omega} \), then similar reasoning gives a second bilinear Riemann relation. Let \( \omega' = \omega = \overline{\omega} \), then the value \( \int_C \omega \wedge \overline{\omega} \) be a positive number, i.e. \( \int_C \omega \wedge \overline{\omega} > 0. \)

6. Detection of homotopical non-trivial loops on Riemann surfaces

In this section, we consider a non-singular algebraic curve as a Riemann surface and consider on it holomorphic and meromorphic differential 1-forms. The homotopy nontriviality of a closed loop on a Riemann surface is detected by nonzero values of homotopy periods on this loop. Under homotopy periods we understand iterated integrals that depend only on the homotopy class of loops. Thus the homotopy period is correctly defined on the element of the fundamental group represented by a given loop. Homotopy periods define functions on the fundamental group of the Riemann surface. We consider Riemann surfaces whose fundamental group has a presentation with a finite number of generators and finite number of relations (finitely presented groups). We describe some properties of the fundamental group of such Riemann surfaces.

**Proposition.** The intersection of the members of the lower central series of the finitely presented fundamental group of the Riemann surface is a unit group.

**Proof.** 1. If the Riemann surface \( C \) is non-compact, then its fundamental group is a free group \( \pi_1(C, x_0) = F_n, n > 0 \) with a finite number of generators. As is well known [9]

\[ \bigcap_{k=1}^{\infty} \Gamma_k F_n = \{ e \}. \]

2. If the Riemann surface is compact, then its fundamental group is either the trivial group \( G = \{ e \} \) or a group with a finite number of generators and one relation. Put out one point from Riemann surface \( C \), i.e. \( X = C \setminus \{ x \} \). Then the fundamental group of the surface \( X \) will be a free group \( F_{2g} \) where \( g \) is the genus of the surface \( C \). The embedding \( i : X \rightarrow C \) induces an epimorphism \( \pi_1(X) \rightarrow \pi_1(C) \rightarrow 1 \) and an epimorphism their lower central series \( \Gamma_k \pi_1(X) \rightarrow \Gamma_k \pi_1(C) \rightarrow 1, k = 1, 2, \ldots \). From the fact that the intersection \( \bigcap_{k=1}^{\infty} \Gamma_k \pi_1(X) = \{ e \} \) follows that \( \bigcap_{k=1}^{\infty} \Gamma_k \pi_1(C) = \{ e \} \). Thus the finitely presented fundamental group of the Riemann surface \( \pi_1(C) \) is a trivial intersection of the lower central series members, i.e. residual nilpotence group [10, 11, 7].

The group \( \pi_1(C) \) is the residually nilpotence group. Thus for any nontrivial element \( g \) of the fundamental group \( \pi_1(C) \), there exists a maximum natural number \( r \) for which \( g \) has a nonzero image in the quotient group \( \Gamma_r(C)/\Gamma_{r+1}(C) \).

Choose a system of canonical loops \( a_1, \ldots, a_g, b_1, \ldots, b_g \) on \( C \), the cuts along which turn \( C \) into a 2g-polygon. We can choose a canonical system of loops starting at one point \( x_0 \in C \) and representing the generators in the fundamental group \( \pi_1(C, x_0) \). We will denote them the same letters. Any iterated integral \( \int \omega_1 \cdots \omega_r \) of length \( r \geq 1 \) from holomorphic 1-forms on a Riemann surface is a homotopy period. Indeed, let two loops \( \gamma_1, \gamma_2 \in \Omega \pi_0(C) \) be homotopy, i.e. exist a mapping \( h : [0, 1] \rightarrow \Omega \pi_0(C) \), \( h(0) = \gamma_1, h(1) = \gamma_2 \), then from the properties of iterated integrals we have equalities

\[ \int_{\gamma_2} \omega_1 \cdots \omega_r - \int_{\gamma_1} \omega_1 \cdots \omega_r = \int \frac{\partial}{\partial h} \omega_1 \cdots \omega_r = \int h^* \frac{\partial}{\partial h} \omega_1 \cdots \omega_r = \int h^* \omega_1 \cdots \omega_r = \]
IOP Conf. Series: Journal of Physics: Conf. Series 1203 (2019) 012099

-form detects the homotopy nontriviality of the generators a
holomorphic form Ω!0 forms), which on the loop r there exists an
holomorphic 1-forms, it is possible to formulate the assumption that for any element b
commutator [a]

We can for commutators of loops [a, b] select a pair of 1-forms from the specified set such
that the 2-iterated integral \[ \int_{[a,b]} \omega \] is a complex number with a nonzero imaginary part, which proves that the commutator [a, b] does not homotope constant loop.

By analyzing the properties of iterated integrals and the properties of period matrices of
holomorphic 1-forms, it is possible to formulate the assumption that for any element γ ∈ Γ_r π_1(C) there exists an r-iterated integral only of holomorphic 1-forms (or only of antiholomorphic 1-forms), which on the loop γ is not equal to zero, i.e. \[ \int_γ \omega_1 \cdots \omega_r \neq 0 \]. This fact indicates that γ is not homotopy to a constant loop.
From the above property of the residual nilpotence and the previously formulated assumption follows that if for $γ ∈ π_1(C)$ all iterated integrals $\int_γ \omega_1 \cdots \omega_r = 0, r > 0$, then we obtain that $γ$ is homotopy to a constant loop.

References

[1] Chen K 1973 Iterated integrals of differential forms and loop space homology. Ann. Math. 97 217–246
[2] Chen K 1977 Iterated path integrals Bull. Ams. Soc. vol 83 Num 5 831–879
[3] Dubrovin B 1982 The Kadomtsev-Petviashvili equation and the relations between the periods of holomorphic differentials on Riemann surfaces Math. USSR-Izv. 19:2 285–296
[4] Hain R 1987 The Geometry of the Mixed Hodge Structure on the Fundamental Group Proceedings of Symp. in Pure Mathematics vol 46 247–282
[5] Hain R 2018 Hodge theory of the Goldman bracket Richard Hain arXiv:1710.06053v2 [math.QA]
[6] Arapura D, Dimca A and Hain R 2014 On the fundamental groups of normal varieties arXiv:1412.1483v2 [math.AG]
[7] Leksin V 2006 The Lappo-Danilevsky method and the triviality of the intersection of radicals of members of the lower central series of some fundamental groups. Math. notes 79:45 577–580
[8] Lang S 1972 Introduction to algebraic and abelian functions Addison-Wesley Publishing Company, Inc. Reading, Massachusetts
[9] Magnus W, Karrass A and Solitar D 1966 Combinatorial Group theory: Presentations of Groups in Terms of Generators and Relations Interscience Publishers A division of John Wiley & Sons, New York, London, Sydney
[10] Marin I 2011 Residual nilpotence for generalizations of pure Braid groups arXiv:1111.5601v1 [math.GR]
[11] Marin I 2016 Homology computations for complex braid groups II arXiv:1605.06953v1 [math.GR]
[12] Springer G 1957 Introduction to Riemann surfaces Addison-Wesley Publishing Company, Inc. Reading, Massachusetts