On the First Eigenvalue of the Degenerate $p$-Laplace Operator in Non-convex Domains

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Abstract. In this paper we obtain lower estimates of the first non-trivial eigenvalues of the degenerate $p$-Laplace operator, $p > 2$, in a large class of non-convex domains. This study is based on applications of the geometric theory of composition operators on Sobolev spaces that permits us to estimate constants of the Poincaré–Sobolev inequalities. On this base we obtain lower estimates of the first non-trivial eigenvalues for Ahlfors-type domains (i.e. quasidiscs). This class of domains includes some snowflake-type domains with fractal boundaries.

Mathematics Subject Classification. 35P15, 46E35, 30C65.

Keywords. Elliptic equations, Sobolev spaces, Composition operators.

1. Introduction

In this article we consider the Neumann eigenvalue problem for the two-dimensional degenerate $p$-Laplace operator ($p > 2$)

$$-\Delta_p u = -\text{div}(|\nabla u|^{p-2}\nabla u).$$

This operator arises in the study of vibrations of nonelastic membranes \cite{7,37}.

The weak statement of the frequency problem for the vibrations of a free nonelastic membrane is equivalent to the following spectral problem: find eigenvalues $\mu_p$ and eigenfunctions $u \in W^1_p(\Omega)$ of the following weak problem:

$$\iint_\Omega (|\nabla u(x,y)|^{p-2}\nabla u(x,y) \cdot \nabla v(x,y)) \, dx \, dy = \mu_p \iint_\Omega |u(x,y)|^{p-2}u(x,y)v(x,y) \, dx \, dy, \quad p > 2,$$

for all $v \in W^1_p(\Omega), \, \Omega \subset \mathbb{R}^2$.

The problem of estimating of the first non-trivial eigenvalue $\mu_p^{(1)}(\Omega)$ is one of the basic problems of modern geometric analysis and its applications to the continuum mechanics. The classical upper estimate for the first nontrivial Neumann eigenvalue of the Laplace operator...
\[
\mu_2^{(1)}(\Omega) \leq \mu_2^{(1)}(\Omega^*) = \frac{p_n/2}{R^2_*}
\]

was proved by Szegö [41] for simply connected planar domains and by Weinberger [45] for domains in \(\mathbb{R}^n\), \(n \geq 3\). In this inequality \(p_n/2\) denotes the first positive zero of the function \((t^{1-n/2}J_{n/2}(t))'\), and \(\Omega^*\) is an \(n\)-ball of the same \(n\)-volume as \(\Omega\) with \(R_*\) as its radius. In particular, if \(n = 2\), we have \(p_1 = j'_{1,1} \approx 1.84118\) where \(j'_{1,1}\) denotes the first positive zero of the derivative of the Bessel function \(J_1\). In recent decades upper estimates of the Laplace eigenvalues, with the help of different techniques, were intensively studied (see, for example, [3–5,12,34]).

The usual approach to lower estimates of the Laplace eigenvalues is based on the integral representation machinery in convex domains. On this base lower estimates of first non-trivial eigenvalues for convex domains were given in terms of the Euclidean diameter of the domains (see, for example, [13, 15,36]). But Nikodim type examples [35] show that in non-convex domains estimates of \(\mu_2^{(1)}(\Omega)\) in terms of Euclidean diameters is not relevant. We suggested in our previous works another type of estimate in terms of integrals of conformal derivatives that can be reformulated in terms of the hyperbolic radii of the domains. So, we can say that hyperbolic geometry represents a natural language for study of the spectral properties of Laplace operators. The integrals of conformal derivatives are not simple for analytical estimates, but if the domain allows a quasiconformal reflection [1,21] we can simplify the problem and obtain a low estimate of the principal frequency \(\mu_2^{(1)}(\Omega)\) in terms of a “quasiconformal” geometry of the domain.

The main result of this paper is:

**Theorem A.** Let \(\Omega \subset \mathbb{R}^2\) be a \(K\)-quasidisc. Then

\[
\mu_2^{(1)}(\Omega) \geq \frac{M_p(K)}{|\Omega|^{\frac{p}{2}}} = \frac{M_p^*(K)}{R^p_*},
\]

where \(R_*\) is a radius of a disc \(\Omega^*\) of the same area as \(\Omega\) and \(M_p^*(K) = M_p(K)\pi^{-p/2}\).

The quantity \(M_p(K)\) in Theorem A depends only on \(p\) and a quasiconformality coefficient \(K\) of \(\Omega\):

\[
M_p(K) = \frac{\pi^{\frac{p}{2}}}{2^{p-2}K^2} \exp \left\{ -\frac{K^2\pi^2(2 + \pi^2)^2}{2 \log 3} \right\} \times \inf_{2 < \alpha < \alpha^*} \inf_{1 \leq q \leq 2} \left\{ \left( \frac{1 - \delta}{1/2 - \delta} \right)^{(\delta-1)p} C_\alpha^{-2} \right\},
\]

\[
C_\alpha = \frac{10^6}{[(\alpha-1)(1-\nu(\alpha))]^{1/\alpha}},
\]

where \(\delta = 1/q - (\alpha-2)/p\alpha\), \(\alpha^* = \min\left(\frac{K^2}{K^2 - 1}, \gamma^*\right)\), and where \(\gamma^*\) is the unique solution of the equation \(\nu(\alpha) := 10^{4\alpha}((\alpha-2)/(\alpha-1))(24\pi^2K^2)^\alpha = 1\).
Remark 1.1. The function $\nu(\alpha)$ is a monotone increasing function. Hence for any $\alpha < \alpha^*$ the number $(1 - \nu(\alpha)) > 0$ and $C_\alpha > 0$.

Remark 1.2. Recall that $K$-quasidiscs are images of the unit disc under $K$-quasiconformal homeomorphisms of the plane $\mathbb{R}^2$. This class includes all Lipschitz simply connected domains but also includes a class of fractal domains (for example, so-called Rohde snowflakes [38]). The Hausdorff dimension of the quasidisc’s boundary can be any number in $[1, 2]$.

In previous works we introduced a concept of conformal $\alpha$-regular domains. Recall that a domain $\Omega \subset \mathbb{R}^2$ is called conformal $\alpha$-regular, $\alpha \in (2, \infty]$, if there exists a conformal mapping $\varphi : \mathbb{D} \to \Omega$ of the unit disc $\mathbb{D}$ onto $\Omega$ such that $\|\varphi' \mid L_\alpha(\mathbb{D})\| < \infty$ [11]. The degree $\alpha$ (by the Riemann Mapping Theorem) depends solely on the hyperbolic metric on $\Omega$. The domain $\Omega$ is a conformal regular domain if it is an $\alpha$-regular domain for some $\alpha > 2$. Note that any $C^2$-smooth simply connected bounded domain is $\infty$-regular (see, for example, [33]).

Theorem A is based on the following theorem, which characterizes the Neumann eigenvalues in the terms of conformal derivatives.

**Theorem B.** Let $\varphi : \mathbb{D} \to \Omega$ be a conformal mapping from the unit disc $\mathbb{D}$ onto a conformal $\alpha$-regular domain $\Omega$. Then for any $p > 2$ the following inequality holds:

$$
\frac{1}{\mu_p^{(1)}(\Omega)} \leq C_p \cdot |\Omega|^{\frac{p-2}{p}} \cdot \|\varphi' \mid L_\alpha(\mathbb{D})\|^2,
$$

where

$$
C_p = 2^p \pi^{\frac{\alpha-2}{2}} \inf_{q \in [1, 2]} \left( \frac{1 - \delta}{\frac{1}{2} - \delta} \right)^{(1-\delta)p}, \quad \delta = \frac{1 - \alpha - 2}{p \alpha}.
$$

Remark 1.3. As an application of this result we obtain lower estimates of $\mu_p^{(1)}(\Omega)$ in the non-convex domain bounded by an epicycloid of $(n - 1)$ cusps (Example 1).

Remark 1.4. Theorem B can be reformulated in terms of the hyperbolic radii $R(\varphi(z), \Omega)$ [8, 9] of the domain.

Theorem B is based on the existence of the composition operator in Sobolev spaces

$$
\varphi^* : L^1_p(\Omega) \to L^1_q(\mathbb{D}), \quad q < p.
$$

In the case of conformal mappings $\varphi : \mathbb{D} \to \Omega$ and $1 \leq q \leq 2 < p < \infty$, we have

$$
\|\varphi^*\| \leq K_{p,q}(\mathbb{D}) := \left( \iint_{\mathbb{D}} |\varphi'(x, y)|^{\frac{(p-2)q}{p-q}} \, dx dy \right)^{\frac{p-q}{pq}} \leq |\Omega|^{\frac{p-2}{2p}} \cdot \pi^{\frac{2-q}{2}}. \quad (1.1)
$$

We can distinguish three different cases of estimates of norms of the composition operators are generated by conformal mappings. The first case $p = q = 2$ corresponds to the classical Laplace operator. Conformal map-
pings induce isometries of spaces $L^1_2(\Omega)$ and $L^2_2(\mathbb{D})$, and as a result we obtain estimates of a first non-trivial eigenvalue with the help of Lebesgue norms of conformal derivatives in spaces $L^\alpha(\mathbb{D})$ for $\alpha$-regular domains $\Omega$. The case $p < 2$ corresponds to singular $p$-Laplace operators, then $K_{p,q}(\mathbb{D})$ is the singular integral and its convergence and estimates of first non-trivial eigenvalues depends on Brennan’s Conjecture [19,28] for composition operators. The case $p > 2$ corresponds to degenerate $p$-Laplace operators. The proposed approach permits us also to obtain spectral estimates of a degenerate $p$-Laplace Neumann operator in quasidiscs (Theorem A) in terms of the quasiconformal geometry. Theorem A will be illustrated by estimating the first non-trivial eigenvalue of the degenerate $p$-Laplace operator in star-shaped and spiral-shaped domains. By reformulating the notion of quasidiscs in terms of Ahlfors’ three-point condition we obtain Theorem C that gives estimates of the first non-trivial eigenvalues in terms of the bounded turning condition. As a consequence we obtain the spectral estimates in snowflake-like domains.

The suggested machinery is based on the geometric theory of composition operators [42,44] and its applications to the Sobolev type embedding theorems [18,22]. In the recent works we studied composition operators on Sobolev spaces defined on planar domains in connection with conformal mapping theory [23]. This connection leads to weighted Sobolev embeddings [24,25] with the universal conformal weights. Another application of conformal composition operators was given in [11] where the spectral stability problem for conformal regular domains was considered.

2. Composition Operators in $\alpha$-Regular Domains

The Sobolev space $W^1_p(\Omega)$, $1 \leq p < \infty$, is a Banach space of locally integrable weakly differentiable functions $f : \Omega \rightarrow \mathbb{R}$ equipped with the following norm:

$$
\| f \|_{W^1_p(\Omega)} = \left( \int_{\Omega} \left| \nabla f(x,y) \right|^p \, dx dy \right)^{\frac{1}{p}} + \left( \int_{\Omega} \left| f(x,y) \right|^p \, dx dy \right)^{\frac{1}{p}}.
$$

Recall that the Sobolev space $W^1_p(\Omega)$ coincides with the closure of the space of smooth functions $C^\infty(\Omega)$ in the norm of $W^1_p(\Omega)$.

The homogeneous seminormed Sobolev space $L^1_p(\Omega)$, $1 \leq p < \infty$ is a linear space of locally integrable weakly differentiable functions $f : \Omega \rightarrow \mathbb{R}$ equipped with the following seminorm:

$$
\| f \|_{L^1_p(\Omega)} = \left( \int_{\Omega} \left| \nabla f(x,y) \right|^p \, dx dy \right)^{\frac{1}{p}}.
$$

Remark 2.1. By the standard definition elements of $L^1_p(\Omega)$ are lasses of functions that are different up to a set of measure zero. Any such function can be redefined quasi-everywhere i.e. up to a set of $p$-capacity zero. Indeed, every function $f \in L^1_p(\Omega)$ has a unique quasicontinuous representation $\tilde{f} \in L^1_p(\Omega)$. A function $\tilde{f}$ is termed quasicontinuous if for any $\varepsilon > 0$ there is an open set...
$U_\varepsilon$ such that the $p$-capacity of $U_\varepsilon$ is less than $\varepsilon$ and on the set $\Omega \setminus U_\varepsilon$ the function $\tilde{f}$ is continuous (see, for example [30,35]). In this paper we suppose that all functions to be quasicontinuous.

Let a weakly differentiable mapping $\varphi$ maps a domain $\Omega$ onto a domain $\tilde{\Omega}$. The mapping $\varphi$ is the mapping of finite distortion if $|\varphi'(x,y)| = 0$ for almost all $(x,y) \in Z = \{(x,y) \in \Omega : J_\varphi(x,y) = 0\}$. Here $J_\varphi(x,y)$ is a formal Jacobian of $\varphi$.

Let $\varphi : \Omega \rightarrow \tilde{\Omega}$ be a homeomorphism. Then $\varphi$ is called a mapping of bounded $(p, q)$-distortion [43], if $\varphi \in W_{1,\text{loc}}^{1}(\Omega)$, has finite distortion, and

$$K_{p,q}(\Omega) = \left( \int\int_{\Omega} \left( \frac{|\varphi'(x,y)|^p}{|J_\varphi(x,y)|^{\frac{q}{p}}} \right)^{\frac{p-q}{pq}} dxdy \right)^{\frac{1}{p-q}} < \infty.$$ 

Classes of mappings of bounded $(p, q)$-distortion are closely connected with composition operators on Sobolev spaces.

Theorem 2.2. [42,44] A homeomorphism $\varphi : \Omega \rightarrow \tilde{\Omega}$ between two domains $\Omega$ and $\tilde{\Omega}$ induces by a composition rule $\varphi^*(f) = f \circ \varphi$ a bounded operator

$$\varphi^* : L^1_{p}(\tilde{\Omega}) \rightarrow L^1_{q}(\Omega), \quad 1 \leq q < p < \infty,$$

if and only if $\varphi$ is a mapping of bounded $(p, q)$-distortion. The norm of the composition operator $\|\varphi^*\| \leq K_{p,q}(\Omega)$.

Now we establish a connection between conformal $\alpha$-regular domains and the composition operators on Sobolev spaces.

Theorem 2.3. Let $\Omega \subset \mathbb{R}^2$ be a simply connected domain. Then $\Omega$ is a conformal $\alpha$-regular domain if and only if any conformal mapping $\varphi : \mathbb{D} \rightarrow \Omega$ generates a bounded composition operator

$$\varphi^* : L^1_{p}(\Omega) \rightarrow L^1_{q}(\mathbb{D})$$

for any $p \in (2, +\infty)$ and $q = p\alpha/(p + \alpha - 2)$.

Proof. By Theorem 2.2

$$K_{p,q}(\mathbb{D}) = \left( \int\int_{\mathbb{D}} \left( \frac{|\varphi'(x,y)|^p}{|J_\varphi(x,y)|^{\frac{q}{p}}} \right)^{\frac{p-q}{pq}} dxdy \right)^{\frac{q}{p-q}} < \infty$$

if and only if a homeomorphism $\varphi : \mathbb{D} \rightarrow \Omega$ has finite distortion and induces a bounded composition operator

$$\varphi^* : L^1_{p}(\Omega) \rightarrow L^1_{q}(\mathbb{D}), \quad 1 \leq q < p < \infty.$$ 

Let $\Omega$ be a conformal $\alpha$-regular domain. Then any conformal mapping $\varphi : \mathbb{D} \rightarrow \Omega$ satisfies the inequality:

$$\int\int_{\mathbb{D}} |\varphi'(x,y)|^\alpha dxdy < \infty \quad \text{for some} \quad \alpha > 2.$$
Using the conformal equality \(|\varphi'(x, y)|^2 = J_\varphi(x, y) > 0\), we obtain

\[
K_{p,q}^{\frac{p-\alpha}{p-q}}(\D) = \iint_{\D} \left( \frac{|\varphi'(x, y)|^p}{J_\varphi(x, y)} \right)^{\frac{q}{p-q}} \, dxdy
\]

\[
= \iint_{\D} |\varphi'(x, y)|^{\frac{(p-\alpha)q}{p-q}} \, dxdy = \iint_{\D} |\varphi'(x, y)|^{\alpha} \, dxdy < \infty
\]

for \(\alpha = (p-2)q/(p-q)\). Hence we have a bounded composition operator

\[
\varphi^*: L^1_p(\Omega) \to L^1_q(\D)
\]

for any \(p \in (2, +\infty)\) and \(q = p\alpha/(p + \alpha - 2)\).

Let us check that \(q < p\). Because \(p > 2\) implies that \(p + \alpha - 2 > \alpha > 2\) and so \(\alpha/(p + \alpha - 2) < 1\). Hence we obtain \(q < p\).

Suppose that the composition operator

\[
\varphi^*: L^1_p(\Omega) \to L^1_q(\D)
\]

is bounded for any \(p \in (2, +\infty)\) and \(q = p\alpha/(p + \alpha - 2)\). Then

\[
\iint_{\D} |\varphi'(x, y)|^\alpha \, dxdy = \iint_{\D} |\varphi'(x, y)|^{\frac{(p-\alpha)q}{p-q}} \, dxdy
\]

\[
= \iint_{\D} \left( \frac{|\varphi'(x, y)|^p}{J_\varphi(x, y)} \right)^{\frac{q}{p-q}} \, dxdy = K_{p,q}^{\frac{p-\alpha}{p-q}}(\D) < \infty.
\]

\(\square\)

If \(\Omega \subset \mathbb{R}^2\) is a conformal \(\alpha\)-regular domain, then by the Sobolev embedding theorem a conformal mapping \(\varphi: \D \to \Omega\) belongs to the Hölder class \(H^\gamma(\D)\), \(\gamma = (\alpha - 2)/\alpha\). Hence, the class of conformal regular domains allows description in terms of \(\gamma\)-hyperbolic boundary condition [10]:

\[
\rho_\Omega \leq \frac{1}{\gamma} \log \frac{\text{dist}(z_0, \partial \Omega)}{\text{dist}(z, \partial \Omega)} + C_0, \quad z = (x, y),
\]

where \(\rho_\Omega\) is the hyperbolic metric in \(\Omega\).

Note, that if \(\Omega\) is a conformal \(\alpha\)-regular domain, then it is a domain with \(\gamma\)-hyperbolic boundary condition for \(\gamma = (\alpha - 2)/\alpha\). Inversely, if \(\Omega\) is a domain with \(\gamma\)-hyperbolic boundary condition, then \(\Omega\) is a conformal regular domain for some \(\alpha\), but the calculation of \(\gamma\) in terms of \(\alpha\) is a non solved problem. For our study we need the exact value of \(\alpha\).

Theorem 2.3 implies:

**Corollary 2.4.** Let \(\Omega \subset \mathbb{R}^2\) be a simply connected domain. Then \(\Omega\) satisfies a \(\gamma\)-hyperbolic boundary condition if and only if any conformal mapping \(\varphi: \D \to \Omega\) generates a bounded composition operator

\[
\varphi^*: L^1_p(\Omega) \to L^1_q(\D)
\]

for any \(p \in (2, +\infty)\) and some \(q = q(p, \gamma) > 2\).
We define the geodesic diameter $\text{diam}_G(\Omega)$ of a domain $\Omega \subset \mathbb{R}^n$ as 

$$\text{diam}_G(\Omega) = \sup_{x,y \in \Omega} \text{dist}_\Omega(x,y).$$

Here $\text{dist}_\Omega(x,y)$ is the intrinsic geodesic distance:

$$\text{dist}_\Omega(x,y) = \inf_{\gamma \in \Omega} \int_0^1 |\gamma'(t)| \, dt$$

where the infimum is taken over all rectifiable curves $\gamma \in \Omega$ such that $\gamma(0) = x$ and $\gamma(1) = y$.

Using [26] and Corollary 2.4 we obtain a simple necessary geometric condition for domains with $\gamma$-hyperbolic boundary condition.

**Theorem 2.5.** Let $\Omega \subset \mathbb{R}^2$ be a simply connected domain. If $\Omega$ satisfies a $\gamma$-hyperbolic boundary condition, then $\Omega$ has a finite geodesic diameter.

Note, that this theorem gives a simple proof that ”maze-like” domain [32] does not satisfy the $\gamma$-hyperbolic boundary condition, because this domain obviously has an infinite geodesic diameter.

### 3. Poincaré–Sobolev Inequalities

**Two-weight Poincaré–Sobolev inequalities.** Let $\Omega \subset \mathbb{R}^2$ be a planar domain and let $h : \Omega \to \mathbb{R}$ be a real valued locally integrable function such that $h(x) > 0$ a.e. in $\Omega$. We consider the weighted Lebesgue space $L^p(\Omega, h)$, $1 \leq p < \infty$, as the space of measurable functions $f : \Omega \to \mathbb{R}$ with the finite norm

$$\|f|_{L^p(\Omega, h)} := \left( \int_{\Omega} \int |f(x,y)|^p h(x,y) \, dxdy \right)^{\frac{1}{p}} < \infty.$$ 

It is a Banach space for the norm $\|f|_{L^p(\Omega, h)}$.

In the following theorem we obtain the estimate of the norm of the composition operator on Sobolev spaces in any simply connected domain with finite measure.

**Theorem 3.1.** Let $\Omega \subset \mathbb{R}^2$ be a simply connected domain with finite measure. Then the conformal mapping $\varphi : \mathbb{D} \to \Omega$ generates a bounded composition operator

$$\varphi^* : L^1_p(\Omega) \to L^1_q(\mathbb{D})$$

for any $p \in (2, +\infty)$ and $q \in [1, 2]$ with $K_{p,q}(\mathbb{D}) \leq |\Omega|^{\frac{p-2}{2p}} \cdot \pi^{\frac{2-q}{2q}}$.

**Proof.** By Theorem 2.2 a homeomorphism $\varphi : \mathbb{D} \to \Omega$ induces a bounded composition operator

$$\varphi^* : L^1_p(\mathbb{D}) \to L^1_q(\Omega), \ 1 \leq q < p < \infty,$$
if and only if \( \varphi \in W^1_{1,\text{loc}}(\Omega) \), has finite distortion and
\[
K_{p,q}(\mathbb{D}) = \left( \iint_{\mathbb{D}} \left( \frac{|\varphi'(x,y)|^p}{J_\varphi(x,y)} \right)^{\frac{q}{p-q}} \, dx \, dy \right)^{\frac{p-q}{pq}} < \infty.
\]

Because \( \varphi \) is a conformal mapping, then \( \varphi \) has finite distortion. Using the conformal equality \( |\varphi'(x,y)|^2 = J_\varphi(x,y) > 0 \) we obtain
\[
K_{p,q}(\mathbb{D}) = \left( \iint_{\mathbb{D}} \left( \frac{|\varphi'(x,y)|^p}{J_\varphi(x,y)} \right)^{\frac{q}{p-q}} \, dx \, dy \right)^{\frac{p-q}{pq}}.
\]

Note that if \( q \leq 2 \) then the quantity \( (p-2)q/(p-q) \leq 2 \). Hence applying H"older inequality to the last integral we get
\[
\left( \iint_{\mathbb{D}} |\varphi'(x,y)|^{\frac{(p-2)q}{p-q}} \, dx \, dy \right)^{\frac{p-q}{pq}} \leq \left( \left( \int_{\mathbb{D}} |\varphi'(x,y)|^2 \, dx \, dy \right)^{\frac{(p-2)q}{2(p-q)}} \right)^{\frac{p-q}{pq}} \cdot \left( \int_{\mathbb{D}} \, dx \, dy \right)^{\frac{(2-q)p}{2(p-q)}}^{\frac{p-q}{pq}}.
\]

By the condition of the theorem, the domain \( \Omega \) is simply connected with finite measure and therefore
\[
K_{p,q}(\mathbb{D}) \leq |\Omega|^{\frac{p-2}{2p}} \cdot \pi^{\frac{2-q}{2q}} < \infty.
\]

We proved that a composition operator
\( \varphi^* : L^1_p(\Omega) \to L^1_q(\mathbb{D}) \)
is bounded for any \( p \in (2, +\infty) \) and \( q \in [1, 2] \).

On the basis of this theorem we prove the existence of universal two-weight Poincaré–Sobolev inequalities in any simply connected domain \( \Omega \subset \mathbb{R}^2 \) with finite measure.

**Theorem 3.2.** Let \( \Omega \subset \mathbb{R}^2 \) be a simply connected domain with finite measure and \( h(u,v) = J_{\varphi^{-1}}(u,v) \) is the conformal weight defined by a conformal mapping \( \varphi : \mathbb{D} \to \Omega \). Then for every function \( f \in W^1_p(\Omega) \), \( p > 2 \), the inequality
\[
\inf_{c \in \mathbb{R}} \left( \iint_{\Omega} |f(u,v) - c|^r h(u,v) \, dudv \right)^{\frac{1}{r}} \leq B_{r,p}(\Omega, h) \left( \iint_{\Omega} |\nabla f(u,v)|^p \, dudv \right)^{\frac{1}{p}}
\]
holds for any \( r \geq 1 \) with the constant
\[
B_{r,p}(\Omega, h) \leq \inf_{q \in [1, 2]} \left\{ B_{r,q}(\mathbb{D}) \cdot \pi^{\frac{2-q}{2q}} \right\} \cdot |\Omega|^{\frac{p-2}{2p}}.
\]
Here $B_{r,q}(D)$ is the best constant in the (non-weighted) Poincaré– Sobolev inequality for the unit disc $D \subset \mathbb{R}^2$.

**Proof.** Let $r \geq 1$. By the Riemann Mapping Theorem there exists a conformal mapping $\varphi : D \rightarrow \Omega$. Denote by $h(u,v) := J_{\varphi^{-1}}(u,v)$ the conformal weight in $\Omega$.

Using the change of variable formula for conformal mapping, the classical Poincaré–Sobolev inequality for the unit disc $D \subset \mathbb{R}^2$,

$$\inf_{c \in \mathbb{R}} \left( \iint_D |g(x,y) - c|^r \, dx \, dy \right)^{\frac{1}{r}} \leq B_{r,q}(D) \left( \iint_D |\nabla g(x,y)|^q \, dx \, dy \right)^{\frac{1}{q}}$$

and Theorem 2.3, we get for a smooth function $g \in W^1_p(\Omega)$

$$\inf_{c \in \mathbb{R}} \left( \iint_{\Omega} |f(u,v) - c|^r h(u,v) \, dudv \right)^{\frac{1}{r}}$$

$$= \inf_{c \in \mathbb{R}} \left( \iint_{\Omega} |f(u,v) - c|^r J_{\varphi^{-1}}(u,v) \, dudv \right)^{\frac{1}{r}}$$

$$= \inf_{c \in \mathbb{R}} \left( \iint_D |g(x,y) - c|^r \, dx \, dy \right)^{\frac{1}{r}} \leq B_{r,q}(D) \left( \iint_D |\nabla g(x,y)|^q \, dx \, dy \right)^{\frac{1}{q}}$$

$$\leq B_{r,q}(D) \cdot \pi^{\frac{2-q}{2q}} \cdot |\Omega|^{\frac{p-2}{2p}} \left( \iint_{\Omega} |\nabla f(u,v)|^p \, dudv \right)^{\frac{1}{p}}.$$

Approximating an arbitrary function $f \in W^1_p(\Omega)$ by smooth functions we have

$$\inf_{c \in \mathbb{R}} \left( \iint_{\Omega} |f(u,v) - c|^r h(u,v) \, dudv \right)^{\frac{1}{r}} \leq B_{r,p}(\Omega, h) \left( \iint_{\Omega} |\nabla f(u,v)|^p \, dudv \right)^{\frac{1}{p}}$$

with the constant

$$B_{r,p}(\Omega, h) \leq \inf_{q \in [1,2]} \left\{ B_{r,q}(D) \cdot \pi^{\frac{2-q}{2q}} \right\} \cdot |\Omega|^{\frac{p-2}{2p}}.$$

The property of the conformal $\alpha$-regularity implies the integrability of the Jacobian of conformal mappings and therefore for any conformal $\alpha$-regular domain we have the embedding of weighted Lebesgue spaces $L_r(\Omega, h)$ into non-weighted Lebesgue spaces $L_s(\Omega)$ for $s = \frac{\alpha-2}{\alpha} r$: 

\[ B_{r,p}(\Omega, h) \leq \inf_{q \in [1,2]} \left\{ B_{r,q}(D) \cdot \pi^{\frac{2-q}{2q}} \right\} \cdot |\Omega|^{\frac{p-2}{2p}}. \]
Lemma 3.3. Let $\Omega$ be a conformal $\alpha$-regular domain. Then for any function $f \in L^r(\Omega, h)$, $\alpha/(\alpha - 2) \leq r < \infty$, the inequality

$$\|f \|_{L^s(\Omega)} \leq \left( \int \int_D |\varphi'(x, y)|^\alpha \, dx \, dy \right)^{\frac{2}{\alpha} \cdot \frac{1}{s}} \|f \|_{L^r(\Omega, h)}$$

holds for $s = \frac{\alpha - 2}{\alpha} \cdot r$.

Proof. Because $\Omega$ is a conformal $\alpha$-regular domain then for any conformal mapping $\varphi : D \rightarrow \Omega$ we have

$$\left( \int \int_D J_{\varphi}^{-\frac{r}{s}}(x, y) \, dx \, dy \right)^{\frac{r - \frac{s}{s}}{r}} = \left( \int \int_D |\varphi'(x, y)|^\alpha \, dx \, dy \right)^{\frac{2}{\alpha} \cdot \frac{1}{s}} < +\infty,$$

for $s = \frac{\alpha - 2}{\alpha} \cdot r$. Using the change of variable formula for conformal mappings, Hölder’s inequality with exponents $(r/s, r/(r - s))$ and the conformal weight $h(u, v) := J_{\varphi^{-1}}(u, v)$, we get

$$\|f \|_{L^s(\Omega)}$$

$$= \left( \int \int_{\Omega} |f(u, v)|^s \, du \, dv \right)^{\frac{1}{s}}$$

$$= \left( \int \int_{\Omega} |f(u, v)|^s J_{\varphi^{-1}}^s(u, v) J_{\varphi^{-1}}(u, v) \, du \, dv \right)^{\frac{1}{s}}$$

$$\leq \left( \int \int_{\Omega} |f(u, v)|^r J_{\varphi^{-1}}(u, v) \, du \, dv \right)^{\frac{1}{r}} \left( \int \int_{\Omega} J_{\varphi^{-1}}^{-\frac{r}{s}}(u, v) \, du \, dv \right)^{\frac{r - \frac{s}{s}}{r}}$$

$$\leq \left( \int \int_{\Omega} |f(u, v)|^r h(u, v) \, du \, dv \right)^{\frac{1}{r}} \left( \int \int_D J_{\varphi^{-1}}^r(x, y) \, dx \, dy \right)^{\frac{r - \frac{s}{s}}{r}}$$

$$= \left( \int \int_{\Omega} |f(u, v)|^r h(u, v) \, du \, dv \right)^{\frac{1}{r}} \left( \int \int_D |\varphi'(x, y)|^\alpha \, dx \, dy \right)^{\frac{2}{\alpha} \cdot \frac{1}{s}}.$$

The following theorem gives the upper estimate of the Poincaré–Sobolev constant as an application of Theorem 3.2 and Lemma 3.3:

**Theorem 3.4.** Let $\Omega \subset \mathbb{R}^2$ be a conformal $\alpha$-regular domain. Then for any function $f \in W^{1,p}_p(\Omega)$, $p > 2$, the Poincaré–Sobolev inequality

$$\inf_{c \in \mathbb{R}} \left( \int \int_{\Omega} |f(u, v) - c|^s \, du \, dv \right)^{\frac{1}{s}} \leq B_{s,p}(\Omega) \left( \int \int_{\Omega} |\nabla f(u, v)|^p \, du \, dv \right)^{\frac{1}{p}}$$
holds for any \( s \geq 1 \) with the constant

\[
B_{s,p}(\Omega) \leq \left( \frac{\alpha}{2} \right)^{\frac{1}{s}} \int_D |\varphi'(x,y)|^\alpha \, dx dy \] \( B_{r,p}(\Omega, h) \)

\[
\leq \inf_{q \in [1,2]} \left\{ B_{\frac{\alpha r}{\alpha-2}, q}(\mathbb{D}) \cdot \pi^{\frac{2-2q}{2q}} \cdot |\Omega|^{\frac{p-2}{2p}} \cdot \|\varphi'\|_{L^\infty(\mathbb{D})} \right\}.
\]

Proof. Let \( f \in W^1_p(\Omega), p > 2 \). Then by Theorem 3.2 and Lemma 3.3 we get

\[
\inf_{c \in \mathbb{R}} \left( \int_\Omega |f(u,v) - c|^s \, dudv \right)^{\frac{1}{s}} \leq \left( \int_D |\varphi'(x,y)|^\alpha \, dx dy \right)^{\frac{2}{\alpha}} \left( \int_\Omega |f(u,v) - c|^r \, dudv \right)^{\frac{1}{r}} \leq B_{r,p}(\Omega, h) \left( \int_D |\varphi'(x,y)|^\alpha \, dx dy \right)^{\frac{2}{\alpha}} \left( \int_\Omega |\nabla f(u,v)|^p \, dudv \right)^{\frac{1}{p}}
\]

if \( s \geq 1 \).

But by Lemma 3.3 \( s = \frac{\alpha-2}{\alpha} r \) and by Theorem 3.2 \( r \geq 1 \). Hence \( s \geq 1 \) and the theorem proved. \( \square \)

By the generalized version of Rellich-Kondrachov compactness theorem (see, for example, [35] or [29]) and the \((r,p)\)-Sobolev–Poincaré inequality for \( r > p \), it follows that the embedding operator

\( i : W^1_p(\Omega) \hookrightarrow L_p(\Omega) \)

is compact in conformal \( \alpha \)-regular domains.

Hence, the first non-trivial Neumann eigenvalue \( \mu^{(1)}_p(\Omega) \) can be characterized [13] as

\[
\mu^{(1)}_p(\Omega) = \min \left\{ \iint_\Omega |\nabla u(x,y)|^p \, dx dy : u \in W^1_p(\Omega) \setminus \{0\}, \iint_\Omega |u|^{p-2} u \, dx dy = 0 \right\}.
\]

Moreover, \( \mu^{(1)}_p(\Omega)^{-\frac{1}{p}} \) is the best constant \( B_{p,p}(\Omega) \) (see, for example, [28]) in the following Poincaré–Sobolev inequality

\[
\inf_{c \in \mathbb{R}} \left( \int_\Omega |f(x,y) - c|^p \, dx dy \right)^{\frac{1}{p}} \leq B_{p,p}(\Omega) \left( \int_\Omega |\nabla f(x,y)|^p \, dx dy \right)^{\frac{1}{p}}, \quad f \in W^1_p(\Omega).
\]
In the case \( s = p \), taking into account estimates constants in \((r, q)\)-Poincaré–Sobolev inequalities in the unit disc \([16, 27]\), Theorem 3.4 implies the lower estimates of the first non-trivial eigenvalue of the degenerate \( p \)-Laplace Neumann operator in conformal \(\alpha \)-regular domains \(\Omega \subset \mathbb{R}^2\).

**Theorem B.** Let \( \varphi : \mathbb{D} \to \Omega \) be a conformal mapping from the unit disc \(\mathbb{D}\) onto a conformal \(\alpha\)-regular domain \(\Omega\). Then for any \( p > 2 \) the following inequality holds

\[
\frac{1}{\mu_p^{(1)}(\Omega)} \leq C_p \cdot |\Omega|^{\frac{p-2}{p}} \cdot \|\varphi'\|_{L^p(\mathbb{D})}^2,
\]

where

\[
C_p = 2^p \pi^{\frac{\alpha-2}{p}} \inf_{q \in [1,2]} \left( \frac{1 - \delta}{1/2 - \delta} \right)^{(1-\delta)p}, \quad \delta = \frac{1}{q} - \frac{\alpha - 2}{p\alpha}.
\]

In the case of conformal \(\infty\)-regular domains, we have the following assertion:

**Corollary 3.5.** Let \( \varphi : \mathbb{D} \to \Omega \) be a conformal mapping from the unit disc \(\mathbb{D}\) onto conformal \(\infty\)-regular domain \(\Omega\). Then for any \( p > 2 \) the following inequality holds

\[
\frac{1}{\mu_p^{(1)}(\Omega)} \leq C_p \cdot |\Omega|^{\frac{p-2}{p}} \cdot \|\varphi'\|_{L^\infty(\mathbb{D})}^2,
\]

where

\[
C_p = 2^p \pi^{1-\frac{2}{p}} \inf_{q \in [1,2]} \left( \frac{1 - \delta}{1/2 - \delta} \right)^{(1-\delta)p}, \quad \delta = \frac{1}{q} - \frac{1}{p}.
\]

As examples, we consider the domains bounded by an epicycloid. Since the domains bounded by an epicycloid are conformal \(\infty\)-regular domains, we can apply Corollary 3.5, i.e.:

**Example 1.** For \( n \in \mathbb{N} \), the diffeomorphism

\[
\varphi(z) = z + \frac{1}{n} z^n, \quad z = x + iy,
\]

is conformal and maps the unit disc \(\mathbb{D}\) onto the domain \(\Omega_n\) bounded by an epicycloid of \((n - 1)\) cusps, inscribed in the circle \(|w| = (n + 1)/n\). The image of \(\varphi\) for \( n = 2, n = 5 \) and \( n = 8 \) is illustrated in Fig. 1.

Now we estimate the norm of the complex derivative \(\varphi'\) in \(L^\infty(\mathbb{D})\) and the area of domain \(\Omega_n\). A straightforward calculation yields

\[
\|\varphi'\|_{L^\infty(\mathbb{D})} = \text{ess sup}_{|z| \leq 1}(|1 + z^{n-1}|) \leq 2
\]

and

\[
|\Omega_n| \leq \pi \left( \frac{n + 1}{n} \right)^2.
\]

Then by Corollary 3.5 we have
Figure 1. Image of $\mathbb{D}$ under $\varphi(z)$

\[
\frac{1}{\mu_p^{(1)}(\Omega_n)} \leq 2^{p+2} \left( \frac{n+1}{n} \right)^{p-2} \inf_{q \in [1,2]} \left( \frac{1-\delta}{1/2-\delta} \right)^{(1-\delta)p},
\]

where $\delta = 1/q - 1/p$.

4. Spectral Estimates in Quasidiscs

In this section we obtain estimates of integrals of conformal derivatives in quasidiscs.

Following [2], a homeomorphism $\varphi : \Omega \to \Omega'$ between planar domains is called $K$-quasiconformal if it preserves orientation, belongs to the Sobolev class $W_{2,\text{loc}}^1(\Omega)$ and its directional derivatives $\partial_{\xi}$ satisfy the distortion inequality

\[\max_{\xi} |\partial_{\xi}\varphi| \leq K \min_{\xi} |\partial_{\xi}\varphi| \text{ a.e. in } \Omega.\]

For any planar $K$-quasiconformal homeomorphism $\varphi : \Omega \to \Omega'$ the following sharp result is known: $J(z,\varphi) \in L_{p,\text{loc}}(\Omega)$ for any $1 \leq p < \frac{K}{K-1}$ ([6,17]). Hence for any conformal mapping $\varphi : \mathbb{D} \to \Omega$ of the unit disc $\mathbb{D}$ onto a $K$-quasidisc $\Omega$ its derivatives $\varphi' \in L_p(\mathbb{D})$ for any $1 \leq p < \frac{2K^2}{K^2-1}$ [27].

Using the integrability of conformal derivatives on the base of the weak inverse Hölder inequality and the measure doubling condition [20] we obtain an estimate of the constant in the inverse Hölder inequality for Jacobians of quasiconformal mappings. The following theorem was proved but not formulated in [20].

**Theorem 4.1.** Let $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ be a $K$-quasiconformal mapping. Then for every disc $\mathbb{D} \subset \mathbb{R}^2$ and for any $1 < \kappa < \frac{K}{K-1}$ the inverse Hölder inequality

\[
\left( \iint_{\mathbb{D}} |J_{\varphi}(x,y)|^\kappa \, dx dy \right)^{\frac{1}{\kappa}} \leq C^2_K \frac{K \pi^{\frac{1}{\kappa}-1}}{4} \exp \left\{ \frac{K \pi^2 (2+\pi^2)^2}{2 \log 3} \right\} \iint_{\mathbb{D}} |J_{\varphi}(x,y)| \, dx dy.
\]
holds. Here
\[ C_\kappa = \frac{10^6}{[(2\kappa - 1)(1 - \nu)]^{1/2\kappa}}, \quad \nu = 10^8\kappa \frac{2\kappa - 2}{2\kappa - 1} (24\pi^2 K)^{2\kappa} < 1. \]

If \( \Omega \) is a \( K \)-quasidisc, then a conformal mapping \( \varphi : \mathbb{D} \to \Omega \) allows \( K^2 \)-quasiconformal reflection [1]. Hence, by Theorem 4.1 we obtain the following integral estimates of complex derivatives of conformal mapping \( \varphi : \mathbb{D} \to \Omega \) of the unit disc onto a \( K \)-quasidisc \( \Omega \):

**Corollary 4.2.** Let \( \Omega \subset \mathbb{R}^2 \) be a \( K \)-quasidisc and \( \varphi : \mathbb{D} \to \Omega \) be a conformal mapping. Suppose that \( 2 < \lambda < \frac{2K^2}{K^2 - 1} \). Then
\[
\left( \iint_{\mathbb{D}} |\varphi'(x, y)|^\lambda \, dx \, dy \right)^{\frac{1}{\lambda}} \leq \frac{C_\lambda K \pi^{\frac{2-\lambda}{2}}}{2} \exp \left\{ \frac{K^2 \pi^2 (2 + \pi^2)^2}{4 \log 3} \right\} \cdot |\Omega|^{\frac{1}{2}},
\]
where
\[ C_\lambda = \frac{10^6}{[(\lambda - 1)(1 - \nu)]^{1/\lambda}}, \quad \nu = 10^4\lambda \frac{\lambda - 2}{\lambda - 1} (24\pi^2 K^2)^{\lambda} < 1. \]

Combining Theorem B and Corollary 4.2 we obtain spectral estimates of the degenerate \( p \)-Laplace operator with the Neumann boundary condition:

**Theorem A.** Let \( \Omega \subset \mathbb{R}^2 \) be a \( K \)-quasidisc. Then
\[
\mu_p^{(1)}(\Omega) \geq \frac{M_p(K)}{|\Omega|^{\frac{p}{2}}} = \frac{M^*_p(K)}{R^*_p},
\]
where \( R_* \) is a radius of a disc \( \Omega^* \) of the same area as \( \Omega \) and \( M^*_p(K) = M_p(K)\pi^{-p/2} \).

**Proof.** Quasidiscs are conformal \( \alpha \)-regular domains for \( 2 < \alpha < \frac{2K^2}{K^2 - 1} \) [27]. Then by Theorem B for any \( 2 < \alpha < \frac{2K^2}{K^2 - 1} \) we have
\[
\frac{1}{\mu_p^{(1)}(\Omega)} \leq C_p \cdot |\Omega|^{\frac{p-2}{2}} \cdot \|\varphi'|_{L_\alpha(\mathbb{D})}\|^2,
\]
where
\[ C_p = 2^p \pi^{\frac{\alpha - 2}{2} - \frac{p}{2}} \inf_{q \in [1, 2]} \left( \frac{1 - \delta}{1/2 - \delta} \right)^{(1-\delta)p}, \quad \delta = \frac{1 - \alpha - 2}{p\alpha}. \]

Now we estimate the integral from the right-hand side of this inequality. According to Corollary 4.2 we obtain
\[
\|\varphi'|_{L_\alpha(\mathbb{D})}\|^2 = \left( \iint_{\mathbb{D}} |\varphi'(x, y)|^\alpha \, dx \, dy \right)^{\frac{1}{\alpha - 2}} \leq \frac{C_\alpha^2 K^2 \pi^{\frac{\alpha - 1}{2}}}{4} \exp \left\{ \frac{K^2 \pi^2 (2 + \pi^2)^2}{2 \log 3} \right\} \cdot |\Omega|.
\]
Combining inequalities (4.2) and (4.3) we get the required inequality. □
As an application of Theorem A we obtain the lower estimates of the first non-trivial eigenvalue on the Neumann eigenvalue problem for the degenerate \( p \)-Laplace operator in star-shaped and spiral-shaped domains.

**Star-shaped domains.** We say that a domain \( \Omega^* \) is \( \beta \)-star-shaped (with respect to \( z_0 = 0 \)) if the function \( \varphi(z), \varphi(0) = 0 \), conformally maps a unit disc \( \mathbb{D} \) onto \( \Omega^* \) and the condition satisfies \([14,40]\):

\[
\left| \arg \frac{z \varphi'(z)}{\varphi(z)} \right| \leq \beta \pi/2, \quad 0 \leq \beta < 1, \quad |z| < 1.
\]

In \([14]\) it is proved that: the boundary of the \( \beta \)-star-shaped domain \( \Omega^* \) is a \( K \)-quasicircle with \( K = \cot^2(1 - \beta)\pi/4 \).

Then by Theorem A we have

\[
\frac{1}{\mu_p^{(1)}(\Omega^*)} \leq \frac{10^6}{\left( \alpha - 1 \right)^{\alpha - \nu}} \left( \frac{1 - \delta}{1/2 - \delta} \right)^{(1-\delta)p} \exp \left\{ \frac{\pi^2 (2 + \pi^2)^2 \cot^4(1 - \beta) \frac{\pi}{4}}{2 \log 3} \right\} \cdot |\Omega^*|^{\frac{p}{2}},
\]

where \( \delta = 1/q - (\alpha - 2)/p\alpha \),

\[
C_\alpha = \frac{10^6}{\left( \alpha - 1 \right)^{\alpha - \nu}} \left( \frac{1 - \delta}{1/2 - \delta} \right)^{(1-\delta)p} \exp \left\{ \frac{\pi^2 (2 + \pi^2)^2 \cot^4(1 - \beta) \frac{\pi}{4}}{2 \log 3} \right\} \cdot |\Omega^*|^{\frac{p}{2}},
\]

\[
\nu = 10^4 \frac{\alpha - 2}{\alpha - 1} (24 \pi^2 \cot^4(1 - \beta) \pi/4)^\alpha < 1.
\]

**Spiral-shaped domains.** We say that a domain \( \Omega_s \) is \( \beta \)-spiral-shaped (with respect to \( z_0 = 0 \)) if the function \( \varphi(z), \varphi(0) = 0 \), conformally maps a unit disc \( \mathbb{D} \) onto \( \Omega_s \) and the condition satisfies \([39,40]\):

\[
\left| \arg e^{i\gamma} \frac{z \varphi'(z)}{\varphi(z)} \right| \leq \beta \pi/2, \quad 0 \leq \beta < 1, \quad |\gamma| < \beta \pi/2, \quad |z| < 1.
\]

In \([39]\) it is proved that: the boundary of the \( \beta \)-spiral-shaped domain \( \Omega_s \) is a \( K \)-quasicircle with \( K = \cot^2(1 - \beta)\pi/4 \).

Then by Theorem A we have

\[
\frac{1}{\mu_p^{(1)}(\Omega_s)} \leq \frac{10^6}{\left( \alpha - 1 \right)^{\alpha - \nu}} \left( \frac{1 - \delta}{1/2 - \delta} \right)^{(1-\delta)p} \exp \left\{ \frac{\pi^2 (2 + \pi^2)^2 \cot^4(1 - \beta) \frac{\pi}{4}}{2 \log 3} \right\} \cdot |\Omega_s|^{\frac{p}{2}},
\]

where \( \delta = 1/q - (\alpha - 2)/p\alpha \),

\[
C_\alpha = \frac{10^6}{\left( \alpha - 1 \right)^{\alpha - \nu}} \left( \frac{1 - \delta}{1/2 - \delta} \right)^{(1-\delta)p} \exp \left\{ \frac{\pi^2 (2 + \pi^2)^2 \cot^4(1 - \beta) \frac{\pi}{4}}{2 \log 3} \right\} \cdot |\Omega_s|^{\frac{p}{2}},
\]

\[
\nu = 10^4 \frac{\alpha - 2}{\alpha - 1} (24 \pi^2 \cot^4(1 - \beta) \pi/4)^\alpha < 1.
\]
In order to obtain the lower estimates of the first non-trivial eigenvalue on the Neumann eigenvalue problem for the degenerate $p$-Laplace operator in fractal type domains we use the description of quasidiscs in the terms of Ahlfors’ 3-point condition: if a Jordan curve $\Gamma$ satisfies the Ahlfors’ 3-point condition: there exists a constant $C$ such that

$$|\zeta_3 - \zeta_1| \leq C|\zeta_2 - \zeta_1|, \quad C \geq 1$$

for any three points on $\Gamma$, where $\zeta_3$ is between $\zeta_1$ and $\zeta_2$.

In [20] it was proved (Theorem 5.1) that if a domain $\Omega$ is bounded by a Jordan curve $\Gamma$ satisfies the Ahlfors’ 3-point condition, then a conformal mapping $\varphi : \mathbb{D} \to \Omega$ allows a $K^2$-quasiconformal extension $\tilde{\varphi} : \mathbb{R}^2 \to \mathbb{R}^2$ with

$$K < \frac{1}{210} \exp \left\{ \left( 1 + e^{2\pi C^5} \right)^2 \right\}.$$  

(4.5)

Using the estimate (4.5) for the quasiconformal coefficient in Theorem A, we obtain lower estimates of the first non-trivial eigenvalues in domains satisfy the Ahlfors’ 3-point condition.

**Theorem C.** Let a domain $\Omega \subset \mathbb{R}^2$ be bounded by a Jordan curve $\Gamma$ that satisfies the Ahlfors’ 3-point condition. Then

$$\frac{1}{\mu_p^{(1)}(\Omega)} \leq \inf_{q \in [1,2]} \left( \frac{1 - \delta}{1/2 - \delta} \right)^{(1-\delta)p} \frac{2^{p-2}C_2^2 e^{2(1+e^{2\pi C^5})^2}}{\pi \frac{p}{4}} \exp \left\{ \frac{\pi^2 (2 + \pi^2)^2 e^{2(1+e^{2\pi C^5})^2}}{2^{21} \log 3} \right\} \cdot |\Omega|^{\frac{p}{2}}$$

holds for $2 \leq \alpha \leq \min \left( \frac{2K^2}{K^2-1}, \gamma_* \right)$, where $\delta = 1/q - (\alpha - 2)/p\alpha$, $\gamma_*$ is the unique solution of the equation

$$\nu(\alpha) := 10^{4\alpha} \frac{\alpha - 2}{\alpha - 1} (24\pi^2 K^2)^{\alpha} = 1$$

and

$$C_\alpha = \frac{10^6}{[(\alpha - 1)(1 - \nu(\alpha))]^{1/\alpha}}.$$  

(4.6)

Using this theorem we obtain lower estimates of $\mu_p^{(1)}$ for snowflakes.

**Rohde snowflake.** In [38] S. Rohde constructed a collection $S$ of snowflake type planar curves with the intriguing property that each planar quasicircle is bi-Lipschitz equivalent to some curve in $S$. In this article we use the following formulation of Rohde snowflakes [31]:

Rohde’s catalog is

$$S := \bigcup_{1/4 \leq t < 1/2} S_t$$

where $t$ is a snowflake parameter. Each curve in $S_t$ is built in a manner reminiscent of the construction of the von Koch snowflake. Thus, each $S \in S_t$ is the limit of a sequence $S^n$ of polygons where $S^{n+1}$ is obtained from $S^n$ by
Figure 2. Construction of a Rohde-snowflake

using the replacement rule illustrated in Fig. 2: for each of the $4^n$ edges $E$ of $S^n$ we have two choices, either we replace $E$ with the four line segments obtained by dividing $E$ into four arcs of equal diameter, or we replace $E$ by a similarity copy of the polygonal arc $A_t$ pictured at the top right of Fig. 2. In both cases $E$ is replaced by four new segments, each of these with diameter $(1/4)\text{diam}(E)$ in the first case or with diameter $t\text{diam}(E)$ in the second case. The second type of replacement is done so that the ”tip” of the replacement arc points into the exterior of $S^n$. This iterative process starts with $S^1$ being the unit square, and the snowflake parameter, thus the polygon arc $A_t$, is fixed throughout the construction.

The sequence $S^n$ of polygons converges, in the Hausdorff metric, to a planar quasicircle $S$ that is called a **Rohde snowflake** constructed with snowflake parameter $t$. Then $S_t$ is the collection of all Rohde snowflakes that can be constructed with snowflake parameter $t$. In [31] established that each Rohde snowflake $S$ in $S_t$ is $C$-bounded turning with

$$C = C(t) = \frac{16}{1 - 2t}, \quad 1/4 \leq t < 1/2.$$  

Recall that a planar curve $\Gamma$ satisfies the $C$-bounded turning, $C \geq 1$, [31] if for each pair of points $x, y$, on $\Gamma$, the smaller diameter subarc $\Gamma[x, y]$ of $\Gamma$ that joins $x, y$ satisfies

$$\text{diam}(\Gamma[x, y]) \leq C|x - y|. \quad (4.7)$$

The $C$-bounded turning condition (4.7) is equivalent the Ahlfors’ 3-point condition (4.4) with the same constant $C$ [21].

According to Theorem C and by the known fact that any $L$-bi-Lipschitz planar homeomorphism is $K$-quasiconformal with $K = L^2$, we obtain the following lower estimate of the first non-trivial eigenvalue of the degenerate $p$-Laplace Neumann operator in the domain type of a Rohde snowflake:

Let $S_t \subset \mathbb{R}^2$, $1/4 \leq t < 1/2$, be the Rohde snowflake. Then the following inequality

$$\text{Let } S_t \subset \mathbb{R}^2, \quad 1/4 \leq t < 1/2, \quad \text{be the Rohde snowflake. Then the following inequality}$$
\[
\frac{1}{\mu_p^{(1)}(S_t)} \leq \inf_{q \in [1,2]} \left( \frac{1 - \delta}{1/2 - \delta} \right)^{(1-\delta)p}
\]
\[
\times 2^{p-2} C_\alpha^2 e^{4(1+e^{4\pi (16/(1-2t))^5})^2} \pi^{\frac{p}{2}}
\]
\[
\times \exp \left\{ \frac{\pi^2(2 + \pi^2)^2 e^{4(1+e^{4\pi (16/(1-2t))^5})^2}}{2^{21} \log 3} \right\} \cdot |S_t|^\frac{p}{2}
\]
holds for \(2 < \alpha < \frac{2K^2}{R^2-1}\), where \(\delta = 1/q - (\alpha - 2)/p\alpha\),
\[
C_\alpha = \frac{10^6}{[(\alpha - 1)(1 - \nu)]^{1/\alpha}},
\]
\[
\nu = 10^{4\alpha} \frac{\alpha - 2}{\alpha - 1} \left( \frac{3\pi^2}{2^{17}} e^{4(1+e^{4\pi (16/(1-2t))^5})^2} \right)^{\alpha} < 1.
\]

Acknowledgements

The first author was supported by the United States-Israel Binational Science Foundation (BSF Grant No. 2014055). The second author was supported by RFBR Grant No. 18-31-00011.

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Received: January 27, 2018.
Revised: March 9, 2018.