Exponential and Gaussian behavior in the tails of multivariate functions

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Abstract

We observe that approximate copies of the function \( \Lambda_n : \mathbb{R}^n \to (0, \infty) \) defined by

\[
\Lambda_n(x) = \exp \left( -x_1 - \pi \sum_{i=2}^{n} x_i^2 \right)
\]

appear in the tails of a large class of functions, with properties related to coordinate independence, convexity, homotheticity, and homogeneity. The function \( \Lambda_n \) is an entropy maximizer (on a half-space) that is uniquely determined by a homogeneity condition together with rotational invariance about the \( x_1 \) direction and its behavior near the origin.

These results are connected to the limiting Poisson point processes found near the edges of large random samples, as well as the conditioning of random vectors on certain rare events, and can be thought of as variations of Laplace’s method for estimating integrals.

1 Introduction

An overview of the paper is as follows (we refer to Section 2 for more details). Consider any \( n \in \mathbb{N} \) with \( n \geq 2 \) and a function \( f : \mathbb{R}^n \to (0, \infty) \). There are various conditions that will be imposed on \( f \) at different times that involve smoothness, the growth of the derivatives of \(-\log f\), convexity, coordinate independence, homotheticity, and homogeneity. We are given a point \( y \in \mathbb{R}^n \) such that \(|y|\) is large. Sometimes we also insist that \( y_1 \) or \( \min\{|y_i| : 1 \leq i \leq n\} \) is large. Consider the function \( \Lambda_n : \mathbb{R}^n \to (0, \infty) \) defined by

\[
\Lambda_n(x) = \exp \left( -x_1 - \pi \sum_{i=2}^{n} x_i^2 \right)
\]

We wish to find an invertible \( n \times n \) matrix \( T \) (depending on \( y \) and \( f \)) such that for all \( x \in \mathbb{R}^n \),

\[
\frac{f(y + Tx)}{f(y)} = (1 + \delta_y(x)) \Lambda_n(x)
\]
where \( \delta_y(x) \) is an error term, usually given with appropriate quantitative bounds. As it turns out, for a large class of functions it is possible to achieve this approximation with \( \delta_y(\cdot) \to 0 \) uniformly on compact subsets of \( \mathbb{R}^n \), as \( y \to \infty \) in the sense described above.

**Example 1** Consider the function \( f : \mathbb{R}^2 \to (0, \infty) \) defined by \( f(x) = e^{-\left(x_1^4 + x_2^4\right)} \), and the point \( y = (t, t) \), for some \( t \in \mathbb{R} \). Let

\[
T = \begin{bmatrix}
a & -b \\
-1 & b
\end{bmatrix}
\]

where \( a = t^{-3}/8 \) and \( b = t^{-1}\sqrt{\pi/12} \). Then (after a little algebra), \( f(y + Tx) \) can be written as

\[
\exp\left(-2t^4 - x_1 - \pi x_2^2 - \frac{x_1^4}{2048t^{12}} - \frac{\pi x_1^2 x_2^2}{64t^8} - \frac{\pi^2 x_2^4}{72t^4} - \frac{x_1^3}{64t^8} - \frac{\pi x_1 x_2^2}{4t^4} - \frac{3x_1^2}{16t^4}\right)
\]

As \( t \to \infty \), the function \( x \mapsto f(y + Tx)/f(y) \) converges uniformly on compact subsets of \( \mathbb{R}^2 \) to the function \( \exp(-x_1 - \pi x_2^2) \).

1.1 Interpreation: a multivariate extreme value theory

We refer the reader to [1, 3, 5, 6, 14] for background. Classical extreme value theory in a single dimension is concerned with the behavior of a large i.i.d. random sample \((X_i)_1^N\) around its maximum and minimum values. Let \( F(x) = P\{X_1 \leq x\} \). By the Fisher-Tippett-Gnedenko theorem, if a limiting distribution for \( \max(X_i)_1^N \) exists as \( N \to \infty \), then it must be one of 3 fundamental types. More precisely, if there exist sequences \((a_N)_1^\infty \subset (0, \infty)\) and \((b_N)_1^\infty \subset \mathbb{R}\) such that for all \( x \in \mathbb{R},\)

\[
\lim_{N \to \infty} P\left\{\max(X_i)_1^N - b_N \leq x \right\} = \lim_{N \to \infty} F\left(a_N x + b_N\right)_1^N = G(x)
\]

and \( G \) is non-degenerate (i.e. is not the C.D.F. of a Dirac point mass), then the sequences \((a_N)_1^\infty, (b_N)_1^\infty\) can be chosen such that \( G \) is of the form

\[
\Phi_p(x) = \begin{cases}
0 & : x < 0 \\
\exp(-x^{-p}) & : x \geq 0
\end{cases}
\]

\[
\Psi_p(x) = \begin{cases}
\exp(-|x|^p) & : x < 0 \\
1 & : x \geq 0
\end{cases}
\]

\[
\Lambda(x) = \exp(-e^{-x}) : x \in \mathbb{R}
\]

where \( p > 0 \). This can be expressed in the language of point processes. Let \( \delta(x) \) denote the Dirac point mass at a point \( x \) (in whatever space \( x \) exists). The random measure

\[
\nu_N = \sum_{i=1}^N \delta\left(\frac{X_i - b_N}{a_N} - \frac{i}{N}\right)
\]

converges (in an appropriate sense, see for example [3, Definition 7.1, Theorem 7.1] and [14, Section 3.5, Proposition 3.21]) to a non-homogeneous two dimensional Poisson point process \( \nu^* \) on \( \{x \in \mathbb{R} : 0 < G(x) < 1\} \times (0, 1) \) with intensity

\[
E\nu^* \left[(u_1, v_1) \times (u_2, v_2)\right] = (u_2 - v_2) \int_{u_1}^{v_1} \frac{G'(s)}{G(s)} ds
\]
Extreme value theory is intimately linked to the theory of regular variation and \( \Gamma \)-variation. Going back to de-Haan, see e.g. [4, 14], a function \( f : \mathbb{R} \to (0, \infty) \) is of class \( \Gamma \) if there exists \( t : \mathbb{R} \to (0, \infty) \) such that for all \( x \in \mathbb{R} \),

\[
\lim_{y \to \infty} \frac{f(y + t(y)x)}{f(y)} = e^x
\]

Whether one uses \( e^x \) or \( e^{-x} \) is a fairly superficial matter. Convergence of the re-scaled max to a limiting distribution as in (3) can be expressed in terms of regular or \( \Gamma \) variation. A limit exists in (3) with \( G = \Lambda \) if and only if there exists \( t : \mathbb{R} \to (0, \infty) \) such that for all \( x \in \mathbb{R} \),

\[
\lim_{y \to \infty} \frac{1 - F(y + t(y)x)}{1 - F(y)} = e^{-x}
\]

which essentially follows from

\[
F(a_Nx + b_N)^N = \left[ 1 - (1 - F(b_N)) \frac{1 - F(a_Nx + b_N)}{1 - F(b_N)} \right]^N
\]

taking \( b_N \) such that \( N (1 - F(b_N)) \to 1 \) and \( a_N = t(b_N) \). This is implied by \( X_1 \) having a density \( f \) such that

\[
\lim_{y \to \infty} \frac{f(y + t(y)x)}{f(y)} = e^{-x}
\]

To extend extreme value theory to a multivariate setting, the typical approach that has been taken is to study the limiting distribution (after re-scaling) of the coordinatewise maximum,

\[
\max(X_i)_1^N = \left( \max_{1 \leq i \leq N} X_{i,j} \right)_{j=1}^n
\]

where \( (X_i)_1^N \) is a sequence of random vectors in \( \mathbb{R}^n \), and \( X_i = (X_{i,j})_{j=1}^n \). Such a multivariate theory focuses on a classification of possible limiting distributions, their marginals, and the dependence structure that exists within these distributions. Other interpretations of multivariate extreme value theory involve the convex hull, see for example [14, p. ix] and references in [8, 9], as well as vertices of the convex hull and Pareto points, see [6, Section 7.2]. The results of this paper can be interpreted as a multivariate extension of extreme value theory that goes in a different direction, more in line with [4], that we now discuss.

There are at least two advantages of studying the processes \( \nu_N \) as opposed to the re-scaled maximum \( a_N^{-1} \left( \max(X_i)_1^N - b_N \right) \). The first is that the process \( \nu_N \) (and the limiting process \( \nu^* \)) give more explicit and readily available information about other sample points \( X_i < \max(X_i)_1^N \). For example, for any fixed \( k \in \mathbb{N} \),

\[
\lim_{N \to \infty} \mathbb{P} \left\{ \frac{X_{(N-k+1)} - b_N}{a_N} \leq x \right\} = G(x) \sum_{j=0}^{k-1} \frac{(-\log G(x))^j}{j!}
\]

where \( X_{(N-k+1)} \) denotes the \( k^{\text{th}} \) largest of the sample \( (X_i)_1^N \). This comes down to convergence of the probabilities

\[
\lim_{N \to \infty} \mathbb{P} \{ \nu_N(x, \infty) \leq k - 1 \} = \mathbb{P} \{ \nu^*(x, \infty) \leq k - 1 \}
\]
The joint distribution may be calculated similarly. While it is simple enough to calculate the limiting distribution of $X_{(N-k+1)}$ from scratch, the point is that this information is already contained in the behavior of $\nu_N$. The second advantage is that the processes $\nu_N$ make no reference to an ordering. A multivariate theory based on limiting point processes closely parallels the univariate theory: the only difference is that the affine transformation $x \mapsto b_N + a_N x$ and its inverse $x \mapsto a_N^{-1} (x - b_N)$ are now replaced with the transformation $x \mapsto y + Tx$ and its inverse $x \mapsto T^{-1}(x - y)$.

**Example 2 (similar to existing results)** Let $\mu$ be a probability measure and $\eta$ a non-negative Radon measure, both on the Borel subsets of $\mathbb{R}^n \setminus \{0\}$ (with its subspace topology inherited from $\mathbb{R}^n$), such that $\lim_{t \to \infty} t^p \mu(tE) = \eta(E)$ for all Borel sets $A \subset \mathbb{R}^n \setminus \{0\}$ with $\eta(\partial A) = 0$ (i.e. $t^p \mu(t \cdot)$ converges weakly to $\eta$) for some $p > 0$, and let $(X_i)_{i=1}^\infty$ be an i.i.d. sequence of random vectors, each with distribution $\mu$. The sequence of processes

$$\nu_N = \sum_{i=1}^N \delta \left( N^{-1/p} X_i \right)$$

then converges weakly (in the sense of [14, Section 3.5]) to a Poisson point process with intensity $\eta$. This follows from [14, proof of Proposition 3.21], since $N P \{ N^{-1/p} X_1 \in \cdot \} = N \mu \left( N^{1/p} \cdot \right) \to^w \eta$ on $\mathbb{R}^n \setminus \{0\}$. This holds when, for example,

$$\frac{d\mu}{dx} = \prod_{i=1}^n (1 + |x_i|)^{-1-p_i} \quad \frac{d\eta}{dx} = \prod_{i=1}^n |x_i|^{-1-p_i}$$

where $p_i > 0$ and $\sum p_i = p$. See related results in [2, Section 2] [12, Eq. (5.4.15)] [7, Theorem 2] and references therein.

An interpretation of our work is that for large $N$ the random measure

$$\nu_N = \sum_{i=1}^N \delta \left( T^{-1}(X_i - y) \right)$$

where $(X_i)_{i=1}^N$ is now an i.i.d. sequence of random vectors, approximates a Poisson point process with (density of) intensity $\exp \left( -x_1 - \pi \sum_{i=2}^n x_i^2 \right)$. Here $y \in \mathbb{R}^n$ is a point near the edge of the random sample (the same $y$ as before), and $T \in GL(n)$ depends on $y$, and we assume $f(y) \approx (N \det(T))^{-1}$. The last condition is what determines how far out in the tails $y$ should be. We assume that the common distribution of each $X_i$ has a density $f = d\mu/dx$, which is the function to which we apply the results of the paper. This interpretation can be expressed more precisely as follows.

**Theorem 3 (Poissonization)** Let $n \geq 2$ and let $\mu$ be a probability measure on $\mathbb{R}^n$ with density $f = d\mu/dx$. There exists a probability space $(\Omega, \mathcal{F}, P)$ and an i.i.d. sequence of random vectors $(X_i)_{i=1}^\infty$ in $\mathbb{R}^n$ defined on $\Omega$ with common distribution $\mu$ such that the
following holds. Let $N \in \mathbb{N}$ with $N > N_0$ ($N_0$ a universal constant), $y \in \mathbb{R}^n$, $T \in \text{GL}(n)$, $0 < \varepsilon < 1$, and $R > 0$, and consider the random measure

$$\nu_N = \sum_{i=1}^{N} \delta(T^{-1}(X_i - y))$$

Let $B^\circ(0, R) = \{x \in \mathbb{R}^n : |x| < R\}$. Assume that

$$\int_{|x| < R} \left| \exp \left( -x_1 - \pi \sum_{i=2}^{n} x_i^2 \right) - N \det(T) f(y + Tx) \right| \, dx < \varepsilon$$

(7)

Then there exists a Poisson point process $\nu = \nu_\omega (\omega \in \Omega)$ on $B^\circ(0, R)$ with (density of) intensity $\exp(-x_1 - \pi \sum_{i=2}^{n} x_i^2)$ such that with probability at least

$$1 - \varepsilon - 2N^{-1/4} - 3\sqrt{\frac{\log N}{N}} \left( 1 + \int_{|x| < R} \exp \left( -x_1 - \pi \sum_{i=2}^{n} x_i^2 \right) \, dx \right)$$

for all Borel sets $E \subseteq B^\circ(0, R)$,

$$\nu_N(E) = \nu(E)$$

Note that the type of approximation one gets, $\nu_N|_{B^\circ(0, R)} = \nu|_{B^\circ(0, R)}$ with high probability, is beyond what is possible in the setting of weak convergence (where $\nu_N|_{B^\circ(0, R)}$ and $\nu|_{B^\circ(0, R)}$ often have disjoint support). This is facilitated through the assumption (7) on the total variation distance. Another notable difference between our results (interpreted through Theorem 3) and results in the spirit of Example 2 is as follows: in Example 2 the observer is stationed at $y = 0$ and we get a single global view of the entire sample, whereas in our results the observer is stationed in the tails of the probability distribution, near the outer regions of the point cloud $\{X_i\}_{i=1}^{N}$, and we view the cloud locally.

Since the relationship between a large i.i.d. sample of fixed size and a Poisson process is well understood, we don’t include any further discussion about limiting point processes and instead focus entirely on the analysis of the density $f$. Obviously, (2) can be thought of as a multivariate equivalent of de Haan’s $\Gamma$-variation, as in (5) and (6). In this context, both

$$\Lambda_n^\ast(x) = \exp(-x_1) \quad \text{and} \quad \Lambda_n^x(x) = \exp\left(-\sum_{i=1}^{n} x_i \right)$$

could also be thought of as multivariate analogues of $x \mapsto \exp(-x)$, $x \in \mathbb{R}$. These two functions are equivalent up to composition on the right with a linear map and are degenerate versions of $\Lambda_n$. The latter has a richer shape since it contains approximate copies of $\Lambda_n^\ast$ but not vice versa (although this means that copies of the function $\Lambda_n^\ast$ are slightly more ubiquitous), and it models the behavior of a strictly convex point cloud. Furthermore, the scale on which we view the point cloud in Theorem 3 is approximately uniquely determined (up to orthogonal transformations of $x_2 \ldots x_n$), but this is not the case if we use $\Lambda_n^x$ with $n \geq 3$ since it is invariant under arbitrary transformations of $x_2 \ldots x_n$, in particular by elements of $SL(n-1)$. See [14, Section 5.4.2] and [1, Section 8.4] for information on multivariate regular variation and references contained therein (regular variation is the polynomial decay version of $\Gamma$-variation).
1.2 Interpretation: conditioning on rare events

Let \( X \) be a single random vector in \( \mathbb{R}^n \) with coordinates \( (X_i)_{i=1}^n \) (a change in notation from Section 1.1) and joint density function \( f \), and let \( \theta \in S^{n-1} \). Under appropriate assumptions on \( f \) and \( \theta \) (that depend on which of our results is being invoked), if \( L > 0 \) is large and we let \( \tilde{X} \) be the random vector \( X \) conditioned on the event \( \{ \sum_1^n \theta_i X_i \geq L \} \), then we may write

\[
\tilde{X} = y + TW
\]

where \( y \) maximizes \( f \) on \( \{ x \in \mathbb{R}^n : \sum_1^n \theta_i x_i = L \} \) and \( W \) is a random vector in \([0, \infty) \times \mathbb{R}^{n-1}\) with a density \( h \) such that \( |h(x) - \exp(-x_1 - \pi \sum_{i=2}^n x_i^2)| \leq \varepsilon_{f,L,\theta}(x) \) for all \( x \in [0, \infty) \times \mathbb{R}^{n-1} \) and \( \varepsilon_{f,L,\theta}(\cdot) \to 0 \) uniformly on compact sets as \( L \to \infty \) and \( \theta \) remains fixed (i.e. \( W \) is approximately a random vector with independent exponential and normal coordinates). If we apply this interpretation to Theorem 6, for example, then we must impose the condition that \( \theta_i \neq 0 \) for all \( i \), which is not much of an assumption since if some \( \theta_i = 0 \) then we just drop that coordinate entirely from the problem (in this sense we ultimately need two nonzero coordinates). As \( L \) and \( y \) are fixed and \( x \to \infty \), the bound \( \varepsilon_{f,L,\theta}(x) \) may become large, and we must impose some regularity on \( f \) to prevent bad behavior of \( f(y + Tx) \). For example, if we assume that \( f \) has a log-concave density, i.e. \( f = \exp(-g) \) for some convex function \( g \), then \( h \) too will be log-concave. If \( \varepsilon_{f,L,\theta}(x) \to 0 \) uniformly in \( \{ x \in [0, \infty) \times \mathbb{R}^{n-1} : |x| < R \} \), for some \( R > 0 \) with \( R \to \infty \) as \( L \to \infty \), then \( h \) must continue to decay rapidly outside \( \{ x \in [0, \infty) \times \mathbb{R}^{n-1} : |x| < R \} \), and so

\[
\lim_{L \to \infty} \frac{\int_{\{ x \in [0, \infty) \times \mathbb{R}^{n-1} : |x| \geq R \} \cdot h(x) \, dx}{\int_{\{ x \in [0, \infty) \times \mathbb{R}^{n-1} : |x| < R \} \cdot h(x) \, dx} = 0
\]

This is important, otherwise the mass outside \( \{ x \in [0, \infty) \times \mathbb{R}^{n-1} : |x| < R \} \) may dominate the conditioning that defines \( \tilde{X} \).

**Example 4** Let \( X \) be a random vector in \( \mathbb{R}^2 \) with independent coordinates, each with distribution \( \mathbb{P} \{ X_i \leq t \} = \left( 2 \Gamma \left( \frac{5}{4} \right) \right)^{-1} \int_{-\infty}^t \exp(-u^4) \, du \). The joint density of \( X \) is therefore

\[
f(x_1, x_2) = \left( 2 \Gamma \left( \frac{5}{4} \right) \right)^{-2} \exp\left( -x_1^4 - x_2^4 \right)
\]

Let us condition on the event that \( X_1 + 2X_2 \geq L \), and label the resulting random vector \( \tilde{X} \). The density \( f \) is maximized on \( \{ (x_1, x_2) : x_1 + 2x_2 = L \} \) at the point \( y = (y_1, y_2) \) with

\[
y_1 = (1 + 2^{4/3})^{-1} L \quad y_2 = 2^{1/3} \left( 1 + 2^{4/3} \right)^{-1} L
\]

and setting \( g = -\log f, \nabla g(y_1, y_2) = 4 \left( 1 + 2^{4/3} \right)^{-3} L^3 (1, 2) \). Referring to the statement and proof of Theorem 6 the norm \( \| \cdot \|_2 \) becomes

\[
\| u \|_2 = \sqrt{\frac{6}{\pi} \left( y_1^2 u_1^2 + y_2^2 u_2^2 \right)}
\]

and we may take

\[
Q(x_2) = x_2 \sqrt{\frac{\pi}{6 \left( 4y_1^2 + y_2^2 \right)}} \quad (2, -1)
\]
The linear map $T$ then takes the form

$$Tx = \frac{1}{20} (1 + 2^{4/3})^3 L^{-3} x_1 (1, 2) + (1 + 2^{4/3}) L^{-1} x_2 \sqrt{\frac{\pi}{6 (4 + 2^{2/3})}} (2, -1)$$

and $\tilde{X} = (\tilde{X}_1, \tilde{X}_2)$ where

$$\tilde{X}_1 = (1 + 2^{4/3}) L + \frac{1}{20} (1 + 2^{4/3})^3 L^{-3} Y + \frac{(1 + 2^{4/3}) L^{-1}}{\sqrt{3 (4 + 2^{2/3})}} Z$$

$$\tilde{X}_2 = 2^{1/3} (1 + 2^{4/3})^{-1} L + \frac{1}{10} (1 + 2^{4/3})^3 L^{-3} Y - \frac{(1 + 2^{4/3}) L^{-1}}{2 \sqrt{3 (4 + 2^{2/3})}} Z$$

and $(Y, Z)$ converges weakly and in the total variation distance (as $L \to \infty$) to a random vector in $[0, \infty) \times \mathbb{R}$ with independent coordinates, the first standard exponential and the second standard normal.

Example 4 demonstrates a significant difference between the univariate setting and the multivariate setting. In the former, conditioning on $\{X_1 \geq T\}$, there is no normal component, only exponential. In the latter, there are exponential and normal components, and in this particular example the normal components dominate. Conditioning on certain rare events thus facilitates convergence to a normal even with $n = 2$, unlike the classical central limit theorems where $n$ must be large.

1.3 Interpretation: variations of Laplace’s method

One reason for considering the limiting shape of functions in their tails is that it leads to accurate estimates for $\int_E f(z) dz$ when $E \subset \mathbb{R}^n$ is such that (2) holds for all $x \in T^{-1} (E - y)$. Assuming $\text{vol}_n (E) > 0$, such estimates take the form

$$\int_E f(z) dz = (1 + \varepsilon) f(y) |\text{det}(T)| \int_{T^{-1}(E-y)} \exp \left(-x_1 - \pi \sum_{i=2}^n x_i^2 \right) dx$$

(8)

where $y$ depends on $f$ and $E$, and $\varepsilon$ depends on $\delta_y (\cdot)$ from (2). Similar estimates also hold when the dimension of $E$ is less than $n$, for example when $E$ is a hyperplane. An obvious application of (8) is to finding asymptotic $(1 + o (1))$ estimates for tail probabilities and densities of linear and nonlinear functionals of random vectors, as in Example 5 below.

This is precisely the setting of Laplace’s method, where $f : \mathbb{R}^n \to \mathbb{R}$ is of the form

$$f(t, z) = h(z) \exp (\varepsilon t q(z))$$

for $t \in \mathbb{R}$, $z \in \mathbb{R}^{n-1}$, and we identify $\mathbb{R} \times \mathbb{R}^{n-1}$ with $\mathbb{R}^n$. Let us assume for simplicity that $h$ is bounded and continuous, that $q$ is $C^2$ and achieves a strict global minimum at a point $z_0 \in \mathbb{R}^{n-1}$, that the Hessian $H_q(z_0)$ is positive definite, that $q(z) \to \infty$ as $|z| \to \infty$, $\int_{\mathbb{R}^n} \exp (-q(z)) dz < \infty$, and that $h(z_0) \neq 0$. The objects $y$ and $T$ then take the form

$$y = (t, z_0) \quad T(s, u) = \left( \frac{s}{q(z_0)}, \sqrt{\frac{2\pi}{t} H^{-1/2}_q(z_0) u} \right)$$
where \( s \in \mathbb{R} \) and \( u \in \mathbb{R}^{n-1} \). The estimate \(^{[2]} \) then leads to the classical approximation (setting \( m = n - 1 \))

\[
\int_{z \in \mathbb{R}^m} h(z) \exp(-tq(z)) \, dz = \int_{\mathcal{H}_t} f(x) \, dx = (1 + o(1)) \frac{h(z_0) \exp(-tq(z_0))}{\sqrt{\det H_q(z_0)}} \left( \frac{2\pi}{t} \right)^{m/2}
\]

as \( t \to \infty \), where \( \mathcal{H}_t = \{ x \in \mathbb{R}^n : x_1 = t \} \) and integration is performed with respect to \( (n - 1) \)-dimensional Lebesgue measure on \( \mathcal{H}_t \).

**Example 5 (tail probabilities of a nonlinear functional)** Let \( X \) be a random vector in \( \mathbb{R}^n \) with distribution \( \mu \) and density

\[
f(x) = d\mu/dx = (2\Gamma(5/3))^{-n} \exp\left(-\sum_{i=1}^n |x_i|^{3/2}\right)
\]

We estimate the tail probabilities \( \mathbb{P}\{\prod_{i=1}^n X_i \geq t\} \) in \(^{[3]} \) below as \( n \) is fixed and \( t \to \infty \) (obviously in this particular example one can convert to a linear functional by using the logarithm; the technique is nonetheless demonstrated). For all \( s > 0 \) let \( \mathcal{M}_s = \{ x \in (0, \infty)^n : \prod_{i=1}^n x_i = s^n \} \). Note that \( r\mathcal{M}_s = \mathcal{M}_{rs} \) for all \( r > 0 \). By log-concavity, \( f \) is maximized over \( \mathcal{M}_{t^{1/n}} \) at the point \( y = (t^{1/n}, t^{1/n}, \ldots, t^{1/n}) \). Let \( T \in GL(n) \) be as in Theorem \(^{[2]} \) in which case

\[
|\det T| = \frac{(2\pi)^{(n-1)/2}}{\sqrt{3n}} \left( \frac{4}{3} \right)^{n/2} t^{-3/(4n) + 1/4}
\]

As \( t \to \infty \), the curvature of \( \mathcal{M}_{t^{1/n}} \) at \( y \) decreases to zero (at an appropriate rate), and the Hessian matrix of \( g = -\log f \) at \( y \) is the diagonal matrix with entries \((3/4) t^{-1/(2n)}\), so in operator norm

\[
\|H_g(y)^{-1/2}\|_{\ell_2^n \to \ell_2^n} = \frac{2}{\sqrt{3}} t^{1/(4n)} \quad \|T\|_{\ell_2^n \to \ell_2^n} = o\left(t^{1/n}\right)
\]

This means that on the scale that matters, the curvature of \( \mathcal{M}_{t^{1/n}} \) around \( y \) becomes negligible. More precisely, the set \( T^{-1}(E - y) \) increases to \( (0, \infty)^n \times \mathbb{R}^{n-1} \), where \( E = \{ x \in (0, \infty)^n : \prod_{i=1}^n x_i \geq t \} \). By \(^{[3]} \), \( \mathbb{P}\{\prod_{i=1}^n X_i \geq t\} = 2^{n-1} \mu(E) \) can be expressed as

\[
\frac{(1 + o_n(1))}{\sqrt{3n}} \left( 8\pi \right)^{(n-1)/2} \left( \sqrt{3}\Gamma(5/3) \right)^{-n} t^{-3/(4n) + 1/4} \exp\left(-nt^{3/(2n)}\right)
\]

as \( t \to \infty \), where the subscript in \( o_n(1) \) indicates dependence on \( n \).

### 1.4 Further discussion

Returning to the general setting, set \( g = -\log f \) and assume momentarily that \( g \) is convex (in which case \( f \) is said to be log-concave) and \( C^2 \). Let \( H_g(y) \) denote the Hessian of \( g \) at \( y \).
Provided $\nabla g(y) \perp \text{null}H_g(y) = \{0\}$, $f$ resembles the function $\Lambda_n$ in a region surrounding $y$ when viewed in the coordinate structure corresponding to the inner product

$$\langle u, v \rangle_z = \frac{1}{2\pi} \langle u, H_g(y)v \rangle + \langle u, \nabla g(y) \rangle \langle v, \nabla g(y) \rangle$$

More precisely, (2) holds with appropriate quantitative bounds when $T \in GL(n)$ is such that for all $u, v \in \mathbb{R}^n$, $\langle Tu, Tv \rangle_z = \langle u, v \rangle$, and such that $Te_1 = A^{-1} \nabla g(y)$, where

$$A = \frac{1}{2\pi} H_g(y) + \nabla g(y) \otimes \nabla g(y)$$

$A_{i,j} = \frac{1}{2\pi} \partial_{i,j}g(y) + \partial_i g(y) \partial_j g(y)$

In Theorem $10$ we provide a formula for the matrix $T$, provided $f$ is smooth enough (in practice however, we don’t always use this exact $T$). Observe that $\nabla g(y) \perp \text{null}H_g(y)$ can be expressed using the following coordinate-free representations:

$$\nabla g(y) \perp = \left\{ x \in \mathbb{R}^n : \frac{d}{dt}g(y + tx) \bigg|_{t=0} = 0 \right\}$$

$$\text{null}H_g(y) = \left\{ x \in \mathbb{R}^n : \frac{d^2}{dt^2}g(y + tx) \bigg|_{t=0} = 0 \right\}$$

and that these subspaces are orthogonal with respect to $\langle \cdot, \cdot \rangle_z$. The case when $g$ is not convex is a little different. For example setting $g(x_1, x_2) = x_1^2 - x_2^2$, (10) clearly breaks down.

The function $\Lambda_n$ has, in a sense, the most natural possible shape for a function that grows or decays rapidly. It has the homogeneity property that for all $y \in \mathbb{R}^n$ there exists $T \in GL(n)$ such that for all $x \in \mathbb{R}^n$,

$$\frac{\Lambda_n(y + Tx)}{\Lambda_n(y)} = \Lambda_n(x)$$

Furthermore $\Lambda_n$ is uniquely determined by this property, as well as its gradient and Hessian at zero, $C^3$-smoothness, and the fact that $\Lambda_n \circ U = \Lambda_n$ for all $U \in O(n)$ such that $Ue_1 = e_1$ (to shorten the paper we have omitted the proof). It is also worth noting that the function $\Lambda_n$ restricted to $[0, \infty) \times \mathbb{R}^{n-1}$ (which still encodes the full shape of $\Lambda_n$) maximizes differential entropy among probability density functions of random vectors $X$ supported on $[0, \infty) \times \mathbb{R}^{n-1}$ with mean $(1, 0, \ldots, 0)$ and covariance

$$\begin{bmatrix}
1 & 0 & \cdots & 0 \\
0 & (2\pi)^{-1} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & (2\pi)^{-1}
\end{bmatrix}$$

## 2 Main results

Recall that the density of an absolutely continuous $n$-fold product measure on $\mathbb{R}^n$ can always be written in the form $f(x) = \prod_{i=1}^n \exp(-g_i(x_i))$, where each $g_i : \mathbb{R} \to (-\infty, \infty]$. 

Theorem 6 (coordinate independence) Let \( n \geq 2 \) and for each \( 1 \leq i \leq n \) let \( g_i : \mathbb{R} \to \mathbb{R} \) be three times differentiable. Let \( y \in \mathbb{R}^n \) such that for all \( i \), \( g_i'(y_i) \neq 0 \) and \( g_i''(y_i) > 0 \), and consider the function \( f : \mathbb{R}^n \to (0, \infty) \) defined by

\[
f(x) = \prod_{i=1}^{n} \exp(-g_i(x_i))
\]

Set \( g = -\log f \). There exists a linear bijection \( T : \mathbb{R}^n \to \mathbb{R}^n \) such that for all \( x \in \mathbb{R}^n \),

\[
\frac{f(y + Tx)}{f(y)} = \exp \left( -x_1 - \pi \sum_{i=2}^{n} x_i^2 + \delta(x) \right)
\]

where \( |\delta(x)| \) can be bounded above by

\[
\begin{align*}
\frac{1}{2} |x_1|^2 & \sum_{i=1}^{n} \frac{g_i'(y_i)^4}{|\nabla g(y)|^2} \frac{g_i''(y_i)^2}{g_i'(y_i)^2} + \sqrt{2\pi} |x_1| \left( \sum_{i=2}^{n} x_i^2 \right)^{1/2} \left( \sum_{i=1}^{n} \frac{g_i'(y_i)^4}{|\nabla g(y)|^2} \frac{g_i''(y_i)^2}{g_i'(y_i)^2} \right)^{1/2} \\
+ \frac{1}{6} |x_1|^3 & \sum_{i=1}^{n} \frac{g_i'(y_i)^6}{|\nabla g(y)|^6} \frac{g_i''(y_i)^3/2}{g_i'(y_i)^3/2} \frac{g_i'''(y_i + w_i)}{g_i''(y_i)^3/2} \\
+ \sqrt{\frac{\pi}{2}} |x_1|^2 & \left( \sum_{i=2}^{n} x_i^2 \right)^{1/2} \left( \sum_{i=1}^{n} \frac{g_i'(y_i)^8}{|\nabla g(y)|^8} \frac{g_i''(y_i)^2}{g_i'(y_i)^2} \frac{g_i'''(y_i + w_i)^2}{g_i''(y_i)^3} \right)^{1/2} \\
+ \pi |x_1| & \left( \sum_{i=2}^{n} x_i^2 \right)^{1/2} \max_{1 \leq i \leq n} \left\{ \frac{g_i'(y_i)^2}{|\nabla g(y)|^2} \frac{g_i''(y_i)^1/2}{g_i'(y_i)} \frac{g_i'''(y_i + w_i)}{g_i''(y_i)^{3/2}} \right\} \\
+ \sqrt{\frac{2\pi^4}{3}} & \left( \sum_{i=2}^{n} x_i^2 \right)^{3/2} \max_{1 \leq i \leq n} \left\{ \frac{|g_i'''(y_i + w_i)|}{g_i''(y_i)^{3/2}} \right\}
\end{align*}
\]

and \( w \in \mathbb{R}^n \) is such that

\[
\left( \sum_{i=1}^{n} g_i''(y_i) w_i^2 \right)^{1/2} \leq |x_1| \left( \sum_{i=1}^{n} \frac{g_i'(y_i)^4}{|\nabla g(y)|^4} \frac{g_i''(y_i)^2}{g_i'(y_i)^2} \right)^{1/2} + \sqrt{2\pi} \left( \sum_{i=2}^{n} x_i^2 \right)^{1/2}
\]

Furthermore, \( T \) is of the form

\[
Tz = \frac{\nabla g(y)}{|\nabla g(y)|^2} z_1 + Q(z_2, z_3, \ldots, z_n)
\]

where \( Q : \mathbb{R}^{n-1} \to \nabla f(y)^\perp \) and

\[
|\text{Det}T| = (2\pi)^{(n-1)/2} \left( \sum_{i=1}^{n} \frac{g_i'(y_i)^2}{g_i''(y_i)} \right)^{-1/2} \left( \prod_{i=1}^{n} g_i''(y_i) \right)^{-1/2}
\]

It is somewhat typical that in Theorem 6 the error term \( \delta(\cdot) \to 0 \) uniformly on compact subsets of \( \mathbb{R}^n \) as \( \min \{|y_i|\} \to \infty \). As Proposition 7 below suggests, we can often expect terms such as

\[
\frac{g_i''(y_i)}{g_i'(y_i)^2} \frac{|g_i'''(y_i + w_i)|}{g_i''(y_i)^{3/2}}
\]
to be small when $|y_i|$ is large. This may then be combined with the fact that

$$\sum_{i=1}^{n} \frac{g_i'(y_i)^2}{|\nabla g(y)|^2} = 1$$

**Proposition 7** Let $\varepsilon > 0$ and let $\omega : (0, \infty) \to \mathbb{R}$ be any differentiable function such that $\lim_{t \to \infty} t^{1/\varepsilon} \omega(t) = \infty$. Then

$$\liminf_{t \to \infty} \left| \frac{\omega'(t)}{\omega(t)^{1+\varepsilon}} \right| = 0 \quad (12)$$

Furthermore, if

$$\lim_{t \to \infty} \left| \frac{\omega'(t)}{\omega(t)^{1+\varepsilon}} \right| = 0$$

then for all $r > 0$,

$$\limsup_{t \to \infty} \left\{ \left| \frac{\omega'(t + s)}{\omega(t)^{1+\varepsilon}} \right| : s \in \mathbb{R}, |s| \leq \frac{r}{\omega(t)^\varepsilon} \right\} = 0 \quad (13)$$

A function $f : \mathbb{R}^n \to \mathbb{R}$ is called homothetic if it is of the form $f(x) = \psi(h(x))$, where $\psi : \mathbb{R} \to \mathbb{R}$ is strictly increasing and $h : \mathbb{R}^n \to \mathbb{R}$ is positively 1-homogeneous, i.e. $h(\alpha x) = \alpha h(x)$ for all $x \in \mathbb{R}^n$ and $\alpha > 0$, see e.g. [15]. For our purposes, we do not require $\psi$ to be increasing. In the case where $f$ is the density of a probability measure, and we assume that $\psi$ and $h$ are measurable and that $h$ is non-negative, and we consider a random vector $X$ with density $f$, then $h(X)$ and $X/|X|$ are independent.

Let $\mathcal{M}$ be a $C^3$ differentiable manifold of dimension $n-1$ in $\mathbb{R}^n$ (by this, let us take as our definition that $\mathcal{M}$ is locally the graph of a $C^3$ function in the appropriate coordinate system, i.e. for all $\theta \in \mathcal{M}$ there exists an open set $U_\theta \subset \mathbb{R}^n$, a $C^3$ function $f_\theta : \mathbb{R}^{n-1} \to \mathbb{R}$, and an affine isometry $I_\theta : \mathbb{R}^n \to \mathbb{R}^n$ such that $I_\theta(\theta) \in U_\theta$ and $U_\theta \cap I_\theta(\mathcal{M}) = U_\theta \cap f_\theta$, where we identify $f_\theta$ as a subset of $\mathbb{R}^n$). Consider any $\theta \in \mathcal{M}$, and let $n(\theta)$ denote one of the two unit normal vectors associated to $\mathcal{M}$ at $\theta$. Then there exists an affine isometry $J : \mathbb{R}^n \to \mathbb{R}^n$ such that $J(\theta) = 0$, $J(n(\theta)) = (0, 0, \ldots, 0, 1)$, and $J(\mathcal{M})$ coincides with the graph of a $C^3$ function $\psi : \mathbb{R}^{n-1} \to \mathbb{R}$ in a neighbourhood of $0 \in \mathbb{R}^n$, with $\nabla \psi(0) = 0$. By the spectral theorem applied to the Hessian $H_\psi(0)$ we may choose $J$ so that $H_\psi(0)$ is diagonal. The second order Taylor expansion then takes the form

$$\psi(z) = \frac{1}{2} \sum_{i=1}^{n-1} \kappa_i z_i^2 + R(z)$$

where $R = O(|z|^3)$ as $z \to 0$. The sequence of values $(\kappa_i)_{1}^{n-1}$ are known as the principal curvatures of $\mathcal{M}$ at $\theta$ with respect to $n(\theta)$, and $(J^{-1}e_i - \theta)_{1}^{n-1}$ is called a sequence of principal directions. By the Gauss-Kronecker curvature of $\mathcal{M}$ at $\theta$, we mean

$$\kappa(\theta) = \det H_\psi(0) = \prod_{i=1}^{n-1} \kappa_i$$
which is independent of the choice of J (but the sign of \( \kappa(\theta) \) depends on the choice of \( n(\theta) \) when \( n - 1 \) is odd).

These definitions, as well as Proposition 7, are relevant to the following result. We label the principal curvatures \( (\kappa_i)_{\rho, M} \) instead of \( (\kappa_i)_{1}^{n-1} \).

**Theorem 8 (homotheticity)** Let \( n \geq 2 \) and let \( h : \mathbb{R}^n \to \mathbb{R} \) be a continuous function that is not identically zero, such that for all \( x \in \mathbb{R}^n \) and all \( \alpha > 0 \), \( h(\alpha x) = \alpha h(x) \). Let \( \mathcal{M} = \{ x \in \mathbb{R}^n : |h(x)| = 1 \} \) and \( \theta \in \mathcal{M} \) be such that \( \mathcal{M} \) is \( C^3 \) is a neighbourhood of \( \theta \) and such that the Gauss-Kronecker curvature of \( \mathcal{M} \) at \( \theta \), denoted \( \kappa(\theta) \), is nonzero. Let \( (\kappa_i)_{\rho} \) be the principal curvatures of \( \mathcal{M} \) at \( \theta \) (in any order) with respect to the inward pointing normal vector \( n(\theta) = -h(\theta)|\nabla h(\theta)|^{-1} \nabla h(\theta) \). Let \( \rho : \mathbb{R} \to \mathbb{R} \) be any \( C^2 \) function and let \( f : \mathbb{R}^n \to (0, \infty) \) be defined as \( f(x) = \exp(-\rho(h(x))) \). Then there exists \( C > 0 \) and an injective linear map \( Q : \mathbb{R}^{n-1} \to \mathbb{R}^n \) such that for all \( x \in \mathbb{R}^n \) and all \( t > 0 \) satisfying \( \rho'(th(\theta)) \neq 0 \),

\[
\frac{f(t\theta + T_i x)}{f(t\theta)} = \exp\left(-x_1 - \pi \sum_{i=2}^{n} \varepsilon_i x_i^2 + \delta(x, t)\right)
\]

where

\[
\varepsilon_i = \frac{h(\theta) \rho'(th(\theta)) \kappa_i}{|\rho'(th(\theta))| \kappa_i}
\]

\[
T_i x = \frac{h(\theta)x_1}{\rho'(th(\theta))} \theta + \sqrt{\frac{t}{|\rho'(th(\theta))|} Q(x_2, x_3, \ldots, x_n)}
\]

and \(|\delta(x, t)|\) can be bounded above by

\[
C \frac{|\rho''(th(\theta) + s)|}{|\rho'(th(\theta))|^2} \left[ |x_1|^2 + \left( \sum_{i=2}^{n} |x_i|^2 \right)^2 + \min \{ I^2, II^2 \} \right] + C \min \{ I, II \}
\]

where

\[
I = \frac{|x_1|^2}{|t \rho'(th(\theta))|} + \sum_{i=2}^{n} |x_i|^2 \quad \quad \quad \quad \quad II = \frac{|t \rho'(th(\theta))|^{-3/2} |x_1|^3 + \left( \sum_{i=2}^{n} |x_i|^2 \right)^{3/2}}{|t \rho'(th(\theta))|^{1/2}}
\]

and

\[
|s| \leq C |\rho'(th(\theta))|^{-1} \left( |x_1| + \sum_{i=2}^{n} |x_i|^2 + \min \{ I, II \} \right)
\]

Lastly

\[
|\text{Det}(T_i)| = |\kappa(\theta)|^{-1/2} (2\pi t)^{(n-1)/2} (|\nabla h(\theta)| \rho'(th(\theta)))^{-(n+1)/2}
\]

In the special case where \( h \) is a norm and \( \rho \) is increasing on \([0, \infty)\), \( \varepsilon_i = 1 \) for all \( 2 \leq i \leq n \), and if (in addition) \( \mathcal{M} \) is \( C^3 \) everywhere with \( \kappa(\theta) \neq 0 \) for all \( \theta \in \mathcal{M} \), then the value of \( C \) may be taken independently of \( \theta \).
Theorem 9 (mixed homogeneity) Let \( n, m \geq 2 \) and for each \( 1 \leq i \leq m \) let \( p(i) \in (0, \infty) \) and let \( q_i : \mathbb{R}^n \to \mathbb{R} \) be a continuous function that is \( C^3 \) on \( \mathbb{R}^n \setminus \{0\} \) and not identically zero, such that for all \( x \in \mathbb{R}^n \) and all \( \alpha \geq 0 \), \( q_i(\alpha x) = \alpha^{p(i)} q_i(x) \). Assume that \( p(1) > \max_{2 \leq i \leq m} p(i) \). Define \( f : \mathbb{R}^n \to (0, \infty) \) as
\[
f(x) = \exp \left( - \sum_{i=1}^m q_i(x) \right).
\]
Let \( \mathcal{M} = \{ \theta \in \mathbb{R}^n : |q_1(\theta)| = 1 \} \) and consider any \( \theta \in \mathcal{M} \) such that \( \kappa(\theta) \neq 0 \). Then there exists \( (\varepsilon_i)^n \in \{ \pm 1 \}^{n-1} \) and a function \( t \mapsto T_t \) from \([1, \infty)\) into \( GL(n) \) such that the function
\[
x \mapsto f(t \theta + T_t x) / f(t \theta)
\]
converges uniformly on compact subsets of \( \mathbb{R}^n \) to the function \( \exp (-x_1 - \pi \sum_{i=2}^n \varepsilon_i x_i^2) \) as \( t \to \infty \).

Theorem 10 (finding \( T \)) Let \( n \geq 2, f : \mathbb{R}^n \to (0, \infty) \), \( g = -\log f \), and let \( \xi : \mathbb{R}^n \to \mathbb{R} \) be \( C^2 \) in a neighbourhood of 0 with
\[
\nabla \xi(0) = e_1, \quad H_{\xi}(0) = \begin{bmatrix} 0 & 0 & \ldots & 0 \\ 0 & 2\pi & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 2\pi \end{bmatrix}
\]
Let \( y \in \mathbb{R}^n \), and let \( T \) be an invertible \( n \times n \) matrix such that for all \( x \in \mathbb{R}^n \),
\[
\frac{f(y + T x)}{f(y)} = \exp (-\xi(x)) \tag{15}
\]
Then \( T \) can be expressed as
\[
T = A^{-1/2} F^T \tag{16}
\]
where \( A \) is the positive definite \( n \times n \) matrix defined by
\[
A_{ij} = \frac{1}{2\pi} \partial_{ij}g(y) + \partial_i g(y) \partial_j g(y)
\]
\( A^{-1/2} \) is the principal square root of \( A^{-1} \), and \( F \in O(n) \) such that
\[
FA^{-1/2} \nabla g(y) = e_1 \tag{17}
\]
This also implies, by \( (16) \) and \( (17) \) that
\[
T e_1 = A^{-1} \nabla g(y) \tag{18}
\]
Furthermore, in the special case where \( \xi(x) = x_1 + \pi \sum_{i=2}^n x_i^2 \), for any \( G \in O(n) \) such that
\[
G A^{-1/2} \nabla g(y) = e_1 \tag{19}
\]
it follows that for all \( x \in \mathbb{R}^n \),
\[
\frac{f(y + A^{-1/2} G^T x)}{f(y)} = \exp \left( -x_1 - \pi \sum_{i=2}^n x_i^2 \right)
\]

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Theorem 11 (homogeneity of $\Lambda_n$) Let $n \geq 2$. Then for all $y \in \mathbb{R}^n$ there exists an invertible $n \times n$ matrix $T$ with the following property. For all $x \in \mathbb{R}^n,$

$$\frac{\Lambda_n(y + Tx)}{\Lambda_n(y)} = \Lambda_n(x)$$

3 Proofs

Proof of Theorem 3. The proof is based on the well known concept of Poissonization: the empirical distribution of an i.i.d. random sample of random size is a Poisson point process, as long as the size of the sample follows a Poisson distribution and is independent of the sample points themselves. Since the underlying probability space $\Omega$ is not arbitrary, we may assume that it is rich enough so that we may introduce new independent random objects as necessary. Let us assume, as we may, that the random vectors $(X_i)_{i=1}^\infty$ are generated as follows: Note that the set $\{(x,t) \in \mathbb{R}^n \times [0,\infty) : t \leq f(x)\}$ has Lebesgue measure 1, and let $(X_i^*)_{i=1}^\infty$ denote an i.i.d. sequence of random vectors uniformly distributed in this set. Then set $X_i = PX_i^*$, where $P : \mathbb{R}^n \times [0,\infty) \rightarrow \mathbb{R}^n$ is the natural projection defined by $P(x,t) = x$. This construction is useful because it will allow us to decompose a certain Poisson process into the sum of two (particular) independent Poisson processes. For each $i$ the random vector $T^{-1}(X_i - y)$ has a distribution with density $\det(T)f(y + Tx)$.

Defining $Q : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}^n \times \mathbb{R}$ as

$$Q(x,t) = (T^{-1}(x - y), N \det(T)t)$$

we see that each $QX_i^*$ is uniformly distributed in

$$\{(x,t) \in \mathbb{R}^n \times [0,\infty) : 0 \leq t \leq N \det(T)f(y + Tx)\}$$

and that $PQX_i^* = T^{-1}(X_i - y)$. Now define $W_1$ as

$$\{(x,t) \in B^o(0,R) \times [0,\infty) : 0 \leq t \leq \min\{N \det(T)f(y + Tx), \Lambda(x)\}\}$$

and

$$W_2 = \{(x,t) \in B^o(0,R) \times [0,\infty) : \Lambda(x) < t \leq N \det(T)f(y + Tx)\}$$

$$W_3 = \{(x,t) \in B^o(0,R) \times [0,\infty) : N \det(T)f(y + Tx) < t \leq \Lambda(x)\}$$

Consider a Poisson distributed random variable $M \sim \text{Pois}(N)$ independent of the sequence $(X_i^*)_{i=1}^\infty$. The random measures on $\mathbb{R}^{n+1}$ defined by

$$\nu_1 = \sum_{i=1}^M \delta(QX_i^*) \mathbb{I}\{QX_i^* \in W_1\}$$

$$\nu_2 = \sum_{i=1}^M \delta(QX_i^*) \mathbb{I}\{QX_i^* \in W_2\}$$

are then Poisson point processes with a constant (density of) intensity of 1 in $W_1$ and $W_2$ respectively, where $\mathbb{I}(A)$ denotes the indicator function of an event $A$. Now let $\nu_3$
Lemma 12 Let $H$ be a real symmetric positive definite matrix and $\mathcal{E} = H^{-1}(B_2^n)$. Then for all $\theta \in S^{n-1}$,

$$\text{vol}_{n-1} (\mathcal{E} \cap \theta^\perp) = \frac{\text{vol}_{n-1}(B_2^{n-1})}{|H^{-1}\theta| \det(H)}$$

**Proof.** Let $\phi_n$ denote the parallel section function of $B_2^n$ (in any direction),

$$\phi_n(t) = \text{vol}_{n-1} \{ x \in B_2^n : x_1 = t \}$$

and let $\psi$ denote the parallel section function of $\mathcal{E}$ in the direction of $\theta$,

$$\psi(t) = \text{vol}_{n-1} (\mathcal{E} \cap (\theta^\perp + t\theta))$$

Since a non-degenerate ellipsoid is a Euclidean ball when viewed in the correct coordinate system, and all parallel section functions of the Euclidean ball are the same, $\psi(t) = a\phi_n(bt)$ for some $a, b > 0$. Recall that the dual Minkowski functional of $\mathcal{E}$ is defined as $\|y\|_{\mathcal{E}} = \sup \{ \langle x, y \rangle : x \in \mathcal{E} \}$. Since $\psi(0) = \text{vol}_{n-1} (\mathcal{E} \cap \theta^\perp)$ and the support of $\psi$ is the interval $[-\|\theta\|_{\mathcal{E}}, \|\theta\|_{\mathcal{E}}]$ whereas the support of $\phi_n$ is $[-1, 1]$, it follows that

$$a = \frac{\text{vol}_{n-1} (\mathcal{E} \cap \theta^\perp)}{\text{vol}_{n-1}(B_2^{n-1})}$$
and \( b = \|\theta\|_{E^0}^{-1} \). Therefore

\[
\forall_n (E) = \int_{-\|\theta\|_{E^0}}^{\|\theta\|_{E^0}} \psi(t) dt = \|\theta\|_{E^0} \times \frac{\forall_{n-1}(E \cap \theta^\perp)}{\forall_{n-1}(B_n^{B_2^2})} \times \forall_n (B_2^n)
\]

The result now follows because

\[
\|\theta\|_{E^0} = \sup \{ \langle \theta, H^{-1} x \rangle : x \in B_n^n \} = \sup \{ \langle H^{-1} \theta, x \rangle : x \in B_n^n \} = |H^{-1}\theta|
\]

**Proof of Theorem 6.** Consider the following inner product and corresponding norm defined on \( \mathbb{R}^n \),

\[
\langle u, v \rangle_2 = \frac{1}{2\pi} \sum_{i=1}^{n} g''(y_i) u_i v_i \quad \| u \|_2 = \left( \frac{1}{2\pi} \sum_{i=1}^{n} g''(y_i) u_i^2 \right)^{1/2}
\]

Since any two real Hilbert spaces of dimension \( n - 1 \) are linearly isometric, there exists a linear map \( Q : \mathbb{R}^{n-1} \to \nabla f(y)^\perp \) such that \( \|Qz\|_2 = |z| \) for all \( z \in \mathbb{R}^{n-1} \). Define \( T : \mathbb{R}^n \to \mathbb{R}^n \) by

\[
Tx = \frac{\nabla g(y)}{|\nabla g(y)|^2} x_1 + Q(x_2, x_3, \ldots x_n)
\]

Now fix any \( x \in \mathbb{R}^n \) and define

\[
\alpha = f(y)^{-1} \quad u = \frac{\nabla g(y)}{|\nabla g(y)|^2} \quad v = Q(x_2, x_3, \ldots x_n)
\]

Introduce the parameter \( s \in \mathbb{R} \) and define

\[
\psi(s) = -\log \alpha f(y + sTx) = \log f(y) + \sum_{i=1}^{n} g_i(y_i + su_i x_1 + sv_i)
\]

\[
\eta(s) = -\log \Lambda_n(sx) = sx_1 + \pi s^2 \sum_{i=2}^{n} x_i^2
\]

Our goal is to estimate

\[
\left| \frac{\log f(y + Tx)}{f(y)\Lambda_n(x)} \right| = |\psi(1) - \eta(1)|
\]

which by Taylor’s theorem can be bounded above by

\[
|\psi(0) - \eta(0)| + |\psi'(0) - \eta'(0)| + \frac{1}{2} |\psi''(0) - \eta''(0)| + \frac{1}{6} |\psi'''(\xi) - \eta'''(\xi)|
\]

for some \( \xi \in (0, 1) \). Since \( v \in Q(\mathbb{R}^{n-1}) = \nabla g(y)^\perp \),

\[
\psi(s) = \sum_{i=1}^{n} g_i'(y_i + su_i x_1 + sv_i)(u_i x_1 + v_i)
\]

\[
\psi'(0) = x_1 \langle \nabla g(y), u \rangle + \langle \nabla g(y), v \rangle = x_1
\]
and

\[ \psi''(s) = \sum_{i=1}^{n} g_i''(y_i + s u_i x_1 + s v_i) (u_i x_1 + v_i)^2 \]

\[ \psi''(0) = x_1^2 \sum_{i=1}^{n} g_i''(y_i) \frac{g_i'(y_i)^2}{|\nabla g(y)|^4} + 2x_1 \sum_{i=1}^{n} g_i''(y_i) \frac{g_i'(y_i)^2}{|\nabla g(y)|^2} v_i + \sum_{i=1}^{n} g_i''(y_i) v_i^2 \]

\[ = x_1^2 \sum_{i=1}^{n} g_i''(y_i) \frac{g_i'(y_i)^4}{g_i'(y_i)^2 |\nabla g(y)|^4} + 2x_1 \sum_{i=1}^{n} g_i''(y_i)^{1/2} \frac{g_i'(y_i)^2}{g_i'(y_i)^2 |\nabla g(y)|^2} g_i''(y_i)^{1/2} v_i + \]

\[ + \sum_{i=1}^{n} g_i''(y_i) v_i^2 \]

where the last term may be written as

\[ \sum_{i=1}^{n} g_i''(y_i) v_i^2 = 2\pi ||v||^2 = 2\pi \sum_{i=2}^{n} x_i^2 = \eta''(0) \]

and therefore

\[ \frac{1}{2} |\psi''(0) - \eta''(0)| \]

\[ \leq \frac{1}{2} |x_1|^2 \sum_{i=1}^{n} \frac{g_i''(y_i)}{g_i'(y_i)^2 |\nabla g(y)|^4} + \sqrt{2\pi} |x_1| \left( \sum_{i=2}^{n} x_i^2 \right)^{1/2} \left( \sum_{i=1}^{n} \frac{g_i''(y_i)}{g_i'(y_i)^2 |\nabla g(y)|^4} \right)^{1/2} \]

Similarly,

\[ \psi'''(s) = \sum_{i=1}^{n} g_i''(y_i + s u_i x_1 + s v_i) (u_i x_1 + v_i)^3 \]

\[ = x_1^3 \sum_{i=1}^{n} g_i''(y_i + s u_i x_1 + s v_i) \frac{g_i'(y_i)^6}{g_i'(y_i)^3 |\nabla g(y)|^6} \frac{g_i''(y_i)^3}{g_i'(y_i)^3} \]

\[ + 3x_1^2 \sum_{i=1}^{n} \frac{g_i''(y_i + s u_i x_1 + s v_i)}{g_i'(y_i)^3/2} g_i''(y_i)^{1/2} v_i \frac{g_i'(y_i)^4}{|\nabla g(y)|^4} g_i''(y_i)^{1/2} \]

\[ + 3x_1 \sum_{i=1}^{n} \frac{g_i''(y_i + s u_i x_1 + s v_i)}{g_i'(y_i)^3/2} g_i''(y_i)^{1/2} v_i^2 \frac{g_i'(y_i)^2}{|\nabla g(y)|^2} g_i''(y_i)^{1/2} \]

\[ + \sum_{i=1}^{n} \frac{g_i''(y_i + s u_i x_1 + s v_i)}{g_i'(y_i)^3/2} (g_i''(y_i)^{1/2} v_i)^3 \]

which can be bounded above as before. Lastly, using Lemma 12 with \( \theta = |\nabla g(y)|^{-1} \nabla g(y) \) and \( H = (2\pi)^{-1/2} H g(y)^{1/2} \), and observing that \( Q(B_2^{-1}) = E \cap \nabla g(y)^{-1} \),

\[ |\text{Det}(T)| = |\nabla g(y)|^{-1} \times \frac{\text{vol}_{n-1}(Q(B_2^{-1}))}{\text{vol}_{n-1}(B_2^{-1})} = \frac{1}{|H^{-1} \nabla g(y)| \cdot \text{Det}(H)} \]
Proof of Proposition 7. Let $\epsilon$ and $\omega$ be as in the statement of the proposition, and suppose that the $\lim \inf$ in (12) is nonzero. Then there exists $t_0, c > 0$ such that for all $t \geq t_0$, $\omega(t) > 0$ and $|\omega'(t)| > c \omega(t)^{1+\epsilon}$. In particular, $\omega'(t) \neq 0$ and $\omega$ is either strictly increasing on $[t_0, \infty)$, or strictly decreasing. Hence the sign of $\omega'$ is constant on $[t_0, \infty)$. It also follows that $\omega$ is injective on $[t_0, \infty)$ and therefore satisfies an autonomous differential equation $\omega'(t) = \Theta(\omega(t))$, where $\Theta(s) = \omega'(\omega^{-1}(s))$ and $|\Theta(s)| > cs^{1+\epsilon}$ for all $s \in \{\omega(t) : t \in [t_0, \infty)\}$. In what follows it may help the reader to consider two cases separately: $\omega(t_0) < 0$ and $\omega(t_0) > 0$. Consider the interval

$$I = \left\{ t \in [t_0, \infty) : \omega(t_0) - \frac{c\epsilon \omega'(t_0)(t - t_0)}{|\omega'(t_0)|} > 0 \right\}$$

We now compare $\omega$ to the function

$$u(t) = \left( \omega(t_0) - \frac{c\epsilon \omega'(t_0)(t - t_0)}{|\omega'(t_0)|} \right)^{-1/\epsilon}, \quad t \in I$$

which is the solution to the initial value problem

$$u(t_0) = \omega(t_0) \quad u'(t) = \frac{\omega'(t_0)u(t)^{1+\epsilon}}{|\omega'(t_0)|}, \quad t \in I$$

The functions $\omega$ and $u$ therefore satisfy

$$\int_{\omega(t_0)}^{\omega(t)} \frac{1}{\Theta(s)} ds = t - t_0 \quad \int_{\omega(t_0)}^{u(t)} \frac{\omega'(t_0)}{c|\omega'(t_0)| s^{1+\epsilon}} ds = t - t_0$$

(just differentiate both sides), where we use a standard convention that $\int_a^b = -\int_b^a$. This implies that on $I$, $\omega'(t_0)(\omega(t) - u(t)) \geq 0$, for if $\omega(a) = u(b)$ then $a \leq b$. This fact, which is a standard monotonicity property of autonomous ODEs, means that $\omega$ outruns $u$. If $\omega'(t_0) < 0$ then $\omega(t) \leq u(t)$ on $[t_0, \infty)$, which contradicts our assumption that $t^{1/\epsilon} \omega(t) \to \infty$. If $\omega'(t_0) > 0$ then $\omega(t) \geq u(t)$ on $I$ which is also a contradiction since in this case $u$ explodes to infinity in finite time. We now move on to the second part of the proposition. Since $t^{1/\epsilon} \omega(t) \to \infty$ as $t \to \infty$, there exists $t_0 > 0$ such that $\omega(t) > 0$ for all $t \geq t_0$. Fix any $r, \delta > 0$ and consider the initial value problem

$$\psi(0) = t_0 \quad \psi'(x) = \omega(\psi(x))^{-\epsilon}, \quad x > 0$$

A unique solution to this differential equation exists on $[0, \infty)$, by the basic theory of autonomous ODEs, and

$$\frac{d}{dx} \log \psi'(x) = \frac{\epsilon |\omega'(\psi(x))|}{\omega(\psi(x))^{1+\epsilon}}$$

It also follows that $\psi$ is strictly increasing and $\lim_{t \to \infty} \psi(t) = \infty$ (if not then $\lim_{t \to \infty} \psi(t) = a < \infty$ and $\lim_{t \to \infty} \psi(t) = \omega(a)^{-\epsilon} > 0$ which in turn implies that $\lim_{t \to \infty} \psi(t) = \infty$). By assumption, there exists $x_0 > t_0$ such that for all $x > x_0$,

$$\frac{|\omega'(x)|}{\omega(x)^{1+\epsilon}} < \min\{1, \epsilon^{-1}\} \times \min\left\{1, \exp\left(-\frac{1+\epsilon}{50\epsilon}\right)\right\} \times \min\left\{\frac{1}{100r}, \delta\right\}$$
and there exists $t_2 > x_0$ such that for all $t > t_2$, $\psi'(\psi^{-1}(t) - 2r) > x_0$. Now consider any $t > t_2$ and $s \in \mathbb{R}$ such that $|s| \leq r/\omega(t)^{\epsilon}$. It follows that for all $x \geq \psi^{-1}(t) - 2r$, 

$$\left| \frac{d}{dx} \log \psi'(x) \right| < \frac{1}{100r}$$

which implies that $|\log \psi'(x) - \log \psi'(\psi^{-1}(t))| < |x - \psi^{-1}(t)|/(100r)$, which can be rewritten as

$$\exp\left(\frac{-1}{100r} |x - \psi^{-1}(t)|\right) \psi'(\psi^{-1}(t)) < \psi'(x) < \exp\left(\frac{1}{100r} |x - \psi^{-1}(t)|\right) \psi'(\psi^{-1}(t))$$

(21)

Since $\exp(1/50) \approx 1.02$,

$$\psi(\psi^{-1}(t) - 2r) \leq \psi(\psi^{-1}(t)) - 1.8r\psi'(\psi^{-1}(t)) = t - 1.8\frac{r}{\omega(t)^{\epsilon}}$$

$$\psi(\psi^{-1}(t) + 2r) \geq \psi(\psi^{-1}(t)) + 1.8r\psi'(\psi^{-1}(t)) = t + 1.8\frac{r}{\omega(t)^{\epsilon}}$$

which implies

$$\psi(\psi^{-1}(t) - 2r) \leq t + s \leq \psi(\psi^{-1}(t) + 2r)$$

$$\psi^{-1}(t) - 2r \leq \psi^{-1}(t + s) \leq \psi^{-1}(t) + 2r$$

Lastly, by (20) and (21),

$$\frac{|\omega'(t + s)|}{\omega(t)^{1+\epsilon}} = \frac{|\omega'(t + s)|}{\omega(t + s)^{1+\epsilon}} \left( \frac{\omega'(\psi^{-1}(t + s))}{\omega'(\psi^{-1}(t))} \right)^{-(1+\epsilon)/\epsilon} \leq \frac{\delta}{100}$$

Since this holds for all such $s$ and $\delta$, (13) follows. ■

**Proof of Theorem 8.** The first part of the proof is to establish (25) with error term $|\mathcal{R}(x)| \leq C \min \{|x|^2, |x|^3\}$. It may happen that such an expansion exists without the assumption that $\mathcal{M}$ is $C^3$ near $\theta$, and in that case the conclusion of the theorem still holds. The existence of a Taylor expansion of $h$ around $\theta$ follows immediately from the $C^3$ condition. Our goal is to show that it takes a particular form, as in (25), i.e. essentially to compute the Hessian $H_h(\theta)$. It is our preference to give full detail here. This first part of the proof is more of a technicality, and the reader may choose to skip directly to (25) onwards to see how the approximation (14) comes about. By homogeneity and continuity of $h$, $\mathcal{M}$ is homeomorphic to a subset $V \subseteq S^{n-1}$ (open in the subspace topology of $S^{n-1}$). This homeomorphism $\varphi : \mathcal{M} \rightarrow V$ and its inverse are given by

$$\varphi(x) = |x|^{-1} x \quad \varphi^{-1}(x) = |h(x)|^{-1} x$$

Since $\mathcal{M}$ is assumed to be $C^3$ in an $(n-1)$-dimensional neighbourhood of $\theta \in \mathcal{M}$, the radial function $rad : V \rightarrow (0, \infty)$ defined by $rad(x) = |\varphi^{-1}(x)| = |h(x)|^{-1}$ is $C^3$ in an $(n-1)$-dimensional neighbourhood of $|\theta|^{-1} \theta \in V$, and therefore $h(x) = |x| h\left(|x|^{-1} x\right) = |x| \left(rad\left(|x|^{-1} x\right)\right)^{-1}$ is $C^3$ in an $n$-dimensional neighbourhood of $\theta \in \mathbb{R}^n$. The homogeneity
condition and the fact that \( h(\theta) \neq 0 \) ensure that \( \nabla h(\theta) \neq 0 \). Consider the inward-pointing normal vector \( n(\theta) = -h(\theta)\nabla h(\theta)/|\nabla h(\theta)| \). Let \((v_i)_2^n \) denote a sequence of principal directions of \( \mathcal{M} \) at \( \theta \) and \((\kappa_i)_2^n \) the corresponding principal curvatures (defined with respect to \( n(\theta) \)). Let \( W : \mathbb{R}^{n-1} \to \nabla h(\theta)^\perp \) be the linear isometry (with respect to the standard Euclidean metric on both spaces) such that \( Wv_i = v_i+1 \) (\( 1 \leq i \leq n - 1 \)). It follows that there exists a connected neighbourhood \( U \) of \( 0 \in \mathbb{R}^{n-1} \) and a continuous function \( \psi : U \to \mathbb{R} \) of the form

\[
\psi(z) = \frac{1}{2} \sum_{i=1}^{n-1} \kappa_{i+1} z_i^2 + R_1(z)
\]

where \( R_1(z) = O(|z|^3) \) as \( z \to 0 \), such that for all \( z \in U , \theta + W(z) + \psi(z)n(\theta) \in \mathcal{M} \). We may assume (after possibly choosing a smaller \( U \)), that \( h(\theta + W(z))/h(\theta) > 0 \) for all \( z \in U \). Let

\[
\begin{align*}
\eta(z) &= 1 - \frac{h(\theta)}{h(\theta + W(z))} \\
a &= \theta + W(z) + \psi(z)n(\theta) \\
b &= (1 - \eta(z))(\theta + W(z))
\end{align*}
\]

By the mean value theorem there exists \( \lambda \in (0, 1) \) such that, setting \( \xi = \lambda a + (1 - \lambda)b \),

\[
h(b) = h(a) + \langle b - a, \nabla h(\xi) \rangle
\]

By homogeneity of \( h \) and by definition of \( \eta(\cdot) \), \( h(b) = h(\theta) \). Since \( a \in \mathcal{M} \), \( h(a) \in \{ \pm 1 \} \). However since \( \psi \) and \( h \) are continuous and \( U \) is connected, and \( \psi(0) = 0 \), it follows that \( h(a) = h(\theta) \). Therefore (22) can be rewritten as

\[
\eta(z) \langle \theta + W(z), \nabla h(\xi) \rangle = \psi(z)h(\theta) \left( \frac{\nabla h(\theta)}{|\nabla h(\theta)|}, \nabla h(\xi) \right)
\]

As \( z \to 0 \), \( W(z) \to 0 \) and \( \nabla h(\xi) \to \nabla h(\theta) \) (since \( h \) is \( C^1 \) at \( \theta \)). By definition of \( W \), \( \langle W(z), \nabla h(\theta) \rangle = 0 \), and by homogeneity \( \langle \theta, \nabla h(\theta) \rangle = h(\theta) \). Therefore as \( z \to 0 \),

\[
\eta(z) = (1 + o(1))\psi(z)|\nabla h(\theta)|
\]

and

\[
\begin{align*}
\frac{h(\theta + W(z))}{1 - \eta(z)} &= h(\theta) (1 + (1 + o(1))\eta(z)) \\
&= h(\theta) + \frac{1}{2} h(\theta) |\nabla h(\theta)| \sum_{i=1}^{n-1} \kappa_{i+1} z_i^2 + R_2(z)
\end{align*}
\]

where \( R_2(z) = o(|z|^3) \) as \( z \to 0 \) (the error term will be improved to \( O(|z|^3) \) in a moment). Consider the \((n-1) \times (n-1)\) diagonal matrix \( E \) defined by

\[
E_{i,i} = \sqrt{\frac{2\pi}{|\nabla h(\theta)| \cdot |\kappa_{i+1}|}}
\]
and set $Q = WE$, in which case (23) transforms to

$$h(\theta + Qz) = h(\theta) + \pi \sum_{i=1}^{n-1} \varepsilon_i x_i^2 + R_3(z)$$

(24)

where $\varepsilon_i = h(\theta)\kappa_i/|\kappa_i|$. This Taylor approximation in $(n - 1)$ variables extends to an approximation in $n$ variables by homogeneity, as follows. Recycling the variable $z$, set

$$z = \theta + x_1 h(\theta) + Q(x_2, x_3, \ldots x_n)$$

Assuming without loss of generality that $|x_1| < 1/2$, set $s = (1 + x_1 h(\theta))^{-1}$. By homogeneity of $h$, this definition of $z$, and using (24),

$$h(z) = (1 + x_1 h(\theta)) h((1 + x_1 h(\theta))^{-1} z)$$

$$= (1 + x_1 h(\theta)) h(\theta + Q(s x_2, s x_3, \ldots s x_n))$$

$$= (1 + x_1 h(\theta)) \left( h(\theta) + \pi s^2 \sum_{i=2}^{n} \varepsilon_i x_i^2 + R_3(s x_2, s x_3, \ldots s x_n) \right)$$

$$= h(\theta) + x_1 + \pi \sum_{i=2}^{n} \varepsilon_i x_i^2 + R(x)$$

where $R(x) = o(|x|^2)$ as $x \to 0$. The would-be factor of $s$ next to $\pi$ can be deleted since the difference gets absorbed into $R(x)$. What we have just proved is that,

$$h(\theta + x_1 h(\theta) + Q(x_2, x_3, \ldots x_n)) = h(\theta) + x_1 + \pi \sum_{i=2}^{n} \varepsilon_i x_i^2 + R(x)$$

(25)

Since the left side of (25) is a $C^3$ function of $x$ (in a neighborhood of 0), we have in fact that $R(x) = O(|x|^3)$ as $x \to 0$. But clearly $R(x) = O(|x|^2)$ as $x \to \infty$ and therefore $|R(x)| \leq C \min \{|x|^2, |x|^3\}$ for all $x \in \mathbb{R}^n$. Note: in the special case where $h$ is a norm and $M$ is everywhere $C^3$, it follows that $h$ is $C^3$ on $\mathbb{R}^n \setminus \{0\}$ and the estimate $|R_\theta(x)| \leq C \min \{|x|^2, |x|^3\}$ holds for all $\theta \in M$ and $x \in \mathbb{R}^n$ with a single $C > 0$ independent of $\theta$. Using linearity of $Q$ and (25),

$$h\left( t\theta + \frac{x_1 h(\theta)}{\rho'(t h(\theta))} x_1 + \sqrt{\frac{t}{|\rho'(t h(\theta))|}} Q(x_2, \ldots x_n) \right)$$

$$= t h\left( \theta + \frac{x_1 h(\theta)}{t \rho'(t h(\theta))} x_1 + \sqrt{\frac{1}{t |\rho'(t h(\theta))|}} Q(x_2, \ldots x_n) \right)$$

$$= t h(\theta) + \frac{x_1}{\rho'(t h(\theta))} + \frac{\pi}{|\rho'(t h(\theta))|} \sum_{i=2}^{n} \varepsilon_i x_i^2$$

$$+ t R\left( \frac{x_1}{t \rho'(t h(\theta))}, \sqrt{\frac{1}{t |\rho'(t h(\theta))|}} x_2, \ldots, \sqrt{\frac{1}{t |\rho'(t h(\theta))|}} x_n \right)$$

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Distributing $\rho$ and applying the triangle inequality, (26) implies that

$$\left| \rho(a) - \rho(b) - (a - b)\rho'(b) \right| = \frac{1}{2} (a - b)^2 |\rho''(\xi)|$$

$$\leq \frac{1}{2} (a - b)^2 \frac{\rho'(b)^2}{\rho'(b)} |\rho''(b + (\xi - b))|$$

By Taylor’s theorem there exists $\xi$ between $a$ and $b$ such that

$$\left| \rho(a) - \rho(b) - (a - b)\rho'(b) \right| = \frac{1}{2} (a - b)^2 |\rho''(\xi)|$$

$$\leq \frac{1}{2} (a - b)^2 \frac{\rho'(b)^2}{\rho'(b)} |\rho''(b + (\xi - b))|$$

Distributing $\rho'(b)$ into the following expression for $a - b$,

$$\frac{x_1}{\rho'(th(\theta))} + \frac{\pi}{|\rho'(th(\theta))|} \sum_{i=2}^{n} \epsilon_i x_i^2$$

$$+ tR \left( \frac{x_1}{t\rho'(th(\theta))}, \sqrt{\frac{1}{t|\rho'(th(\theta))|}} x_2, \cdots, \sqrt{\frac{1}{t|\rho'(th(\theta))|}} x_n \right)$$

and applying the triangle inequality, (26) implies that

$$\left| \rho(a) - \rho(b) - x_1 - \pi \sum_{i=2}^{n} \epsilon_i x_i^2 \right|$$

is bounded above by

$$\left| \rho'(th(\theta)) tR \left( \frac{x_1}{t\rho'(th(\theta))}, \sqrt{\frac{1}{t|\rho'(th(\theta))|}} x_2, \cdots, \sqrt{\frac{1}{t|\rho'(th(\theta))|}} x_n \right) \right|$$

$$+ \frac{1}{2} (a - b)^2 \frac{\rho'(b)^2}{\rho'(b)} |\rho''(b + (\xi - b))|$$

Now $|\xi - b| \leq |a - b|$, which is bounded above by

$$\frac{\pi}{|\rho'(th(\theta))|} \left( |x_1| + \sum_{i=2}^{n} x_i^2 \right)$$

$$+ tR \left( \frac{x_1}{t\rho'(th(\theta))}, \sqrt{\frac{1}{t|\rho'(th(\theta))|}} x_2, \cdots, \sqrt{\frac{1}{t|\rho'(th(\theta))|}} x_n \right)$$

and

$$\left| tR \left( \frac{x_1}{t\rho'(th(\theta))}, \sqrt{\frac{1}{t|\rho'(th(\theta))|}} x_2, \cdots, \sqrt{\frac{1}{t|\rho'(th(\theta))|}} x_n \right) \right|$$

$$\leq Ct \min \left\{ \frac{|x_1|^2}{|t\rho'(th(\theta))|^2} + \sum_{i=2}^{n} \frac{|x_i|^2}{|t\rho'(th(\theta))|^2}, \frac{|x_1|^3}{|t\rho'(th(\theta))|^3}, \frac{|x_1|^3}{|t\rho'(th(\theta))|^3}, \left( \sum_{i=2}^{n} |x_i|^2 \right)^{3/2} \right\}$$

22
The main estimate then follows from the definition \( f(x) = \exp(-\rho(h(x))) \). Lastly,

\[
|\text{Det}(T_i)| = \frac{|\langle \theta, \nabla h(\theta) \rangle|}{|\rho'(th(\theta)) \cdot |\nabla h(\theta)||} \times \frac{\text{vol}_{n-1}(QB_2^{n-1})}{\text{vol}_{n-1}(B_2^{n-1})} \times \left( \frac{t}{|\rho'(th(\theta))|} \right)^{(n-1)/2}
\]

\[
= \frac{1}{|\rho'(th(\theta)) \cdot |\nabla h(\theta)||} \times \text{Det}(E) \times \left( \frac{t}{|\rho'(th(\theta))|} \right)^{(n-1)/2}
\]

\[
= |\kappa(\theta)|^{-1/2} \frac{1}{|\nabla h(\theta)| \cdot |\rho'(th(\theta))|} \left( \frac{2\pi t}{|\nabla h(\theta)| \cdot |\rho'(th(\theta))|} \right)^{(n-1)/2}
\]

\[\square \]

**Proof of Theorem 9.** Note that the function

\[
h_1(x) = \begin{cases} 
q_1(x) |q_1(x)|^{-1 + 1/p(1)} & : q_1(x) \neq 0 \\
0 & : q_1(x) = 0
\end{cases}
\]

has the property that \( h_1(\alpha x) = \alpha h_1(x) \) for all \( x \in \mathbb{R}^n \) and all \( \alpha \geq 0 \), and

\[
q_1(x) = \begin{cases} 
h_1(x) |h_1(x)|^{-1 + p(1)} & : h_1(x) \neq 0 \\
0 & : h_1(x) = 0
\end{cases} \tag{28}
\]

Consider the variable \( t > 0 \) large enough so that

\[tp(1) > \left| \sum_{i=2}^{m} t^{p(i)} q_i(\theta) \right|\]

and set

\[s = \left| \sum_{i=1}^{m} q_i(t\theta) \right|^{1/p(1)} = (1 + o(1))t\]

\[q(x) = q_1(x) + \sum_{i=2}^{m} s^{p(i)-p(1)} q_i(x) \tag{29}\]

\[\omega = ts^{-1} \theta\]

Then

\[q(\omega) = s^{-p(1)} \sum_{i=1}^{m} t^{p(i)} q_i(\theta) = \left| \sum_{i=1}^{m} q_i(t\theta) \right|^{-1} \sum_{i=1}^{m} q_i(t\theta) = q_1(\theta)\]

Below we shall refer to functions \( R_i(\cdot, \cdot) \). The first variable may be \( z \in \mathbb{R}^{n-1} \) or \( x \in \mathbb{R}^n \) (this will always be clear from the context). Each has the property that \( \forall t > t_0, \)

\[\lim_{|z| \to 0} |z|^{-2} R_i(z, t) = 0\]

i.e. \( R_i(z, t) = o(|z|^2) \) as \( z \to 0 \). However the rate of convergence may (possibly) depend on \( t \). We shall also refer to functions \( \delta, \delta_i, \delta_{ij}, \gamma_i \) and \( \gamma_{ij} \). These quantities (coefficients) are functions of \( t \) that do not depend on \( z \in \mathbb{R}^{n-1} \) or \( x \in \mathbb{R}^n \), such that as \( t \to \infty \) we have
\[ \delta \to 0 \text{ and } \gamma \to 1. \] They may also denote different functions from one appearance to the next. Let \((v_i)_{i=1}^{n-1}\) denote the principal directions of \(\mathcal{M}\) at \(\theta\), and \((\kappa_i)_{i=1}^{n-1}\) the corresponding principal curvatures. Let \(W: \mathbb{R}^{n-1} \to \nabla q_1(\theta)\) be the linear isometry such that \(W e_i = v_i\) (1 \(\leq i \leq n - 1\)). Let \(U \in SO(n)\) such that
\[
U \left( \frac{\nabla q_1(\theta)}{|\nabla q_1(\theta)|} \right) = \frac{\nabla q(\omega)}{|\nabla q(\omega)|}
\]
and such that \(Ux = x\) for all \(x \in \{\nabla q_1(\theta), \nabla q(\omega)\}\). As \(t \to \infty\), \(\omega \to \theta\) and since \(q_1\) is \(C^1\) at \(\theta\), \(\nabla q_1(\omega) \to \nabla q_1(\theta)\). By (29) and the fact that \(q_i\) (1 \(\leq i \leq m\)) are \(C^1\) at \(\theta\) and do not depend on \(t\), it follows that \(\nabla q(\omega) \to \nabla q_1(\theta)\), and therefore if \(t\) is sufficiently large \(\nabla q(\omega) \neq 0\) and \(U\) is indeed well defined. Furthermore, \(U \to I_n\) (the identity matrix) in the standard topology on \(\mathbb{R}^{n \times n}\). It follows as in the proof of Theorem 3 see in particular (24), that
\[
h_1(\theta + W(z)) = h_1(\theta) + \frac{1}{2} h_1(\theta) |\nabla h_1(\theta)| \sum_{i=1}^{n-1} \kappa_i z_i^2 + R_1(z) \quad (30)
\]
Since \((1 + \varepsilon)^p = 1 + p\varepsilon + o(\varepsilon)\) as \(\varepsilon \to 0\), (30) implies that
\[
q_1(\theta + W(z)) = q_1(\theta) + \sum_{i=1}^{n-1} \beta_i z_i^2 + R_2(z, t)
\]
where \(\beta_i \neq 0\) (1 \(\leq i \leq n - 1\)). Since each \(q_i\) is \(C^2\) at \(\theta\), \(H_q(\omega) \to H_{q_1}(\theta)\). Using this and the fact that \(|x - U(x)| \leq \delta(t) |x|\) for an appropriate \(\delta(\cdot)\), and the fact that \(Range(UW) = \nabla q(\omega)^\perp\),
\[
q(\omega + UW(z)) = q(\omega) + \sum_{i=1}^{n-1} \beta_i \gamma_i z_i^2 + \sum_{1 \leq i < j \leq n-1} \delta_{i,j} z_i z_j + \hat{R}_2(z, t)
\]
In particular, the Hessian matrix of the function \(z \mapsto q(\omega + UW(z))\) is invertible when evaluated at \(z = 0\), and there exists an injective linear map \(Q : \mathbb{R}^{n-1} \to \nabla q(\omega)^\perp\), such that
\[
q(\omega + Q(z)) = q(\omega) + \pi \sum_{i=1}^{n-1} \varepsilon_{i+1} z_i^2 + R_3(z, t) \quad (31)
\]
The vector \((\varepsilon_i)_{i=1}^n \in \{\pm 1\}^{n-1}\) is indexed by \{2, \ldots, n\}, hence the need for the subscript \(i+1\) in the summation. The second order Taylor expansion in \(n\) variables can be written as
\[
q(\omega + x_1 \omega + Q(x_2, \ldots, x_n))
\]
\[
= q(\omega) + \langle \nabla q(\omega), \omega \rangle x_1 + \frac{1}{2} x_1^2 \langle \omega, H_q(\omega) \omega \rangle + x_1 \langle \omega, H_q(\omega) Q(x_2, \ldots, x_n) \rangle
\]
\[
+ \frac{1}{2} \langle Q(x_2, \ldots, x_n), H_q(\omega) Q(x_2, \ldots, x_n) \rangle + R_5(x, t)
\]
Write \( \langle \omega, H_q(\omega)Q(x_2, \ldots, x_n) \rangle = \sum_{i=2}^{n} a_i x_i \). \( Q \) can be chosen so that as \( t \to \infty \), \( Q = Q(t) \) converges in \( \mathbb{R}^{n \times (n-1)} \) and therefore so does each \( a_i \). Combining this with (31) gives

\[
q(\omega + x_1 \omega + Q(x_2, \ldots, x_n)) = q(\omega) + \langle \nabla q(\omega), \omega \rangle x_1 + \pi \sum_{i=2}^{n} \varepsilon_i x_i^2 + \frac{1}{2} \langle \omega, H_q(\omega) \rangle x_1^2 + x_1 \sum_{i=2}^{n} a_i x_i + R_4(x, t)
\]

Using Taylor’s theorem, the \( C^3 \) condition, and (29), it follows that we may take \( |R_4(x, t)| \leq c |x|^3 \) provided \( |x| < \varepsilon \), where \( c, \varepsilon > 0 \) do not depend on \( t \). Note that \( \langle \omega, \nabla q(\omega) \rangle \to \langle \theta, \nabla q_1(\theta) \rangle \neq 0 \). Using linearity of \( Q \) and recalling that \( t \theta = s \omega \),

\[
\sum_{i=1}^{m} q_i \left( t \theta + \frac{x_1}{s \theta(1-1)} \langle \omega, \nabla q(\omega) \rangle \omega + \frac{1}{s \theta(1-2)} Q(x_2, \ldots, x_n) \right)
\]

\[
= s^{p(1)} q \left( \omega + \frac{x_1}{s \theta(1)} \langle \omega, \nabla q(\omega) \rangle \omega + \frac{1}{s \theta(1)} Q(x_2, \ldots, x_n) \right)
\]

\[
= s^{p(1)} \left[ q(\omega) + \frac{x_1}{s \theta(1)} \langle \omega, \nabla q(\omega) \rangle \omega + \frac{1}{s \theta(1)} Q(x_2, \ldots, x_n) \right]
\]

\[
+ s^{p(1)} R_4 \left( \frac{x_1}{s \theta(1)} \langle \omega, \nabla q(\omega) \rangle \omega, s^{-p(1)} x_2, s^{-p(1)} x_3, \ldots, s^{-p(1)} x_n, t \right)
\]

\[
= \sum_{i=1}^{m} q_i (t \theta) + x_1 + \pi \sum_{i=2}^{n} \varepsilon_i x_i^2 + \frac{1}{2} \langle \omega, H_q(\omega) \rangle x_1^2 + \frac{x_1 \sum_{i=2}^{n} a_i x_i}{s^{p(1)}} \langle \omega, \nabla q(\omega) \rangle \omega
\]

\[
+ s^{p(1)} R_4 \left( \frac{x_1}{s \theta(1)} \langle \omega, \nabla q(\omega) \rangle \omega, s^{-p(1)} x_2, s^{-p(1)} x_3, \ldots, s^{-p(1)} x_n, t \right)
\]

The result then follows from the definition of \( f \).

**Proof of Theorem 10.** Consider the bilinear form on \( \mathbb{R}^n \) defined by

\[
\langle x, z \rangle_{f,y} = \frac{1}{2\pi} \langle x, H_g(y)z \rangle + \langle \nabla g(y), x \rangle \cdot \langle \nabla g(y), z \rangle = \langle x, Az \rangle = \langle Bx, Bz \rangle
\]

where \( g = -\log f \) and \( B = A^{1/2} \) (in a moment we will see that \( A \) is positive definite). This bilinear form has two key properties. The first property is that it does not depend on the underlying coordinate structure of \( \mathbb{R}^n \). If \( W : \mathbb{R}^n \to \mathbb{R}^n \) is any linear bijection and \( u \in \mathbb{R}^n \) then

\[
\langle x, z \rangle_{f,y} = \langle Wx, Wz \rangle_{\tilde{f},u+Wy}
\]

where \( \tilde{f}(x) = f(W^{-1}(x-u)) \). This follows since

\[
\nabla \tilde{g}(x) = (W^{-1})^T \nabla g (W^{-1}(x-u))
\]

\[
H_{\tilde{g}}(x) = (W^{-1})^T H_g (W^{-1}(x-u)) W^{-1}
\]

where \( \tilde{g}(x) = g(W^{-1}(x-u)) \). The second is that

\[
\langle \cdot, \cdot \rangle_{\exp(-\xi),0} = \langle \cdot, \cdot \rangle
\]
i.e. when \( f(x) = \exp(-\xi(x)) \) and \( y = 0 \), it reduces to the standard Euclidean inner product. Therefore, setting \( W = T^{-1} \) and \( u = -Wy \),

\[
\langle x, z \rangle_f(y) = \exp(-\xi(z)) = \exp(-\xi(y)) = \langle T^{-1}x, T^{-1}z \rangle_{\exp(-\xi)} = \langle T^{-1}x, T^{-1}z \rangle_{\exp(-\xi)}
\]

Thus \( \langle \cdot, \cdot \rangle_f(y) \) is an inner product and \( A \) is a symmetric positive definite matrix. This implies that \( A \) and \( B \) are invertible. Since \( \langle Bx, Bz \rangle = \langle T^{-1}x, T^{-1}z \rangle \) for all \( x, z \in \mathbb{R}^n \), \( T = B^{-1}U \) for some \( U \in O(n) \). Since \( (15) \) can be written as

\[
g(y + Tx) - g(y) = \xi(x)
\]

it follows that

\[
\nabla_x g(y + Tx) = \nabla \xi(x)
\]

where \( \nabla_x g(y + Tx) \) denotes the gradient of the function \( x \mapsto g(y + Tx) \). However by the chain rule

\[
\nabla_x g(y + Tx) = T^T \nabla g(y + Tx)
\]

Setting \( x = 0 \) and equating the right sides of \( (32) \) and \( (33) \), \( \epsilon_1 = UTB^{-1} \nabla g(y) \) and the first result follows by setting \( F = U^T \). We now consider the special case \( \xi(x) = x_1 + \pi \sum_{i=2}^{n} x_i^2 \) and a matrix \( G \in O(n) \) such that \( GA^{-1/2} \nabla g(y) = \epsilon_1 \). Now \( FG^T \in O(n) \) and by \( (17) \) and \( (19) \),

\[
FG^Te_1 = FG^TG^{-1/2} \nabla g(y) = FA^{-1/2} \nabla g(y) = \epsilon_1
\]

It then follows by rotational invariance of \( \Lambda_n \) about the \( x_1 \) direction that

\[
\frac{f(y + A^{-1/2} G^T x)}{f(y)} = \frac{f(y + A^{-1/2} FTG^T x)}{f(y)} = \Lambda_n(FG^T x) = \Lambda_n(x)
\]

Proof of Theorem (11) Consider the Hilbert space \( \mathcal{H} = \nabla \Lambda(y)^\perp = (\nabla \log \Lambda(y))^\perp \) endowed with the inner product and corresponding norm

\[
\langle x, y \rangle_{\mathcal{H}} = \sum_{i=2}^{n} x_i y_i \quad ||x||_{\mathcal{H}} = \left( \sum_{i=2}^{n} x_i^2 \right)^{1/2}
\]

The function \( ||\cdot||_{\mathcal{H}} \) is indeed a norm because if \( ||x||_{\mathcal{H}} = 0 \) then \( x = te_1 \) for some \( t \in \mathbb{R} \), and it follows from the definition of \( \mathcal{H} \) that \( t = 0 \). Consider a linear map \( U : \mathbb{R}^{n-1} \rightarrow \mathcal{H} \) such that for all \( x \in \mathbb{R}^{n-1} \), \( ||Ux||_{\mathcal{H}} = ||x|| \). Defining \( Tx = x_1 e_1 + U(x_2, \ldots, x_n) \), we then have

\[
\log \frac{\Lambda_n(y + Tx)}{\Lambda_n(y)} = -x_1 - \pi \sum_{i=2}^{n} x_i^2 - \epsilon_1^2(U(x_2, \ldots, x_n))
\]

\[
-2\pi \sum_{i=2}^{n} y_i \epsilon_i^2(U(x_2, \ldots, x_n))
\]

\[
= \log \Lambda_n(x) + \langle U(x_2, \ldots, x_n), \nabla \log \Lambda_n(y) \rangle = \log \Lambda_n(x)
\]

where \( \epsilon_i^2(z) = z_i \) is the \( i^{th} \) coordinate functional. ■
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