Dynamics of stochastic approximation with Markov iterate-dependent noise with the stability of the iterates not ensured

Prasenjit Karmakar, Shalabh Bhatnagar, Arunselvan Ramaswamy
February 14, 2017

Abstract
This paper compiles several aspects of the dynamics of stochastic approximation algorithms with Markov iterate-dependent noise when the iterates are not known to be stable beforehand. We achieve the same by extending the lock-in probability (i.e. the probability of convergence to a specific attractor of the limiting o.d.e. given that the iterates are in its domain of attraction after a sufficiently large number of iterations (say) \(n_0\)) framework to such recursions. Specifically, with the more restrictive assumption of Markov iterate-dependent noise supported on a bounded subset of the Euclidean space we give a lower bound for the lock-in probability. We use these results to prove almost sure convergence of the iterates to the specified attractor when the iterates satisfy an asymptotic tightness condition. This, in turn, is shown to be useful in analyzing the tracking ability of general adaptive algorithms. Additionally, we show that our results can be used to derive a sample complexity estimate of such recursions, which then can be used for step-size selection.

1 Introduction
Stochastic approximation algorithms are sequential non-parametric methods for finding a zero or minimum of a function in cases where only the noisy observations of the function values are available. Stochastic approximation iterates in \(\mathbb{R}^d\) are given by

\[
\theta_{n+1} = \theta_n + a(n)f(\theta_n, Y_{n+1}), n \geq 0,
\]

where \(\theta_0\) is the initial point, \(\{\theta_n\}\) are the iterates, \(\{Y_n\}\) is an \(\mathbb{R}^d\)-valued ‘Markov iterate-dependent’ noise, i.e., satisfies

\[
P[Y_{n+1} \in A|\mathcal{F}_n] = \int_A \Pi_{\theta_n}(Y_n; dx) \text{ a.s.,}
\]

\(^*\)Among the student authors (first and third) the first author did most of the work except Section 3.1.1 which is contributed by the third author.
where $\mathcal{F}_n :=$ the $\sigma$-field generated by all random variables realized till time $n$ and $a(n)$ is the $n$-th step-size.

It is well known that under reasonable assumptions \[15\], \[3\], \[1\] asymptotically tracks the o.d.e.

$$\dot{\theta}(t) = h(\theta(t)),$$

(3)

where $h(\theta) = \int f(\theta, y) \Gamma_{\theta}(dy)$, with $\Gamma_{\theta}$ being the unique stationary distribution of the Markov iterate-dependent process $\{Y_n\}$ for a fixed $\theta$. Among them the most important assumption is the stability of the iterates i.e.

$$\sup_n \|\theta_n\| < \infty \text{ a.s.}$$

(4)

In literature sufficient conditions that guarantee (4) are available (e.g. based on a Lyapunov function \[11\] Chap. 6.7, \[1\] and scaled trajectory \[5\] Chap 6, Theorem 9), etc. As mentioned in \[1\], proving stability of the iterates is a tedious task with the Markovian dynamic due to the noise term $f(\theta_n, Y_{n+1}) - h(\theta_n)$. Further, the stability theorem stated in the second work also requires assumptions such as 1) continuity of the transition kernel, 2) Lipschitz continuity of $f$ in the first component uniformly w.r.t. the second, 3) $f$ is jointly continuous.

In this work, we investigate the dynamics of stochastic approximation with Markov iterate-dependent noise when (4) is not known to be satisfied beforehand. We achieve the same by extending the lock-in probability framework of Borkar \[4\] to such recursion, leading in turn to the following:

1. Let $H$ be an asymptotically stable attractor of (3) and $G$ its domain of attraction. If $\{\theta_n\}$ is asymptotically tight (which is a much weaker condition than (4)) and $\liminf_n P(\theta_n \in G) = 1$, then $P(\theta_n \rightarrow H) = 1$ under reasonable set of assumptions satisfied in application areas such as reinforcement learning \[6\]. To the best of our knowledge this is the first time an almost sure convergence proof for such recursion is presented without assuming the stability of the iterates, however following the classic Poisson equation model of Metivier and Priouret \[15\] for such recursion which is designed keeping in mind the stability of the iterates. Additionally, a simple test for asymptotic tightness is also provided.

2. We show that under some reasonable assumptions for common step-size sequences such as $\{\frac{1}{n^k}\}, \frac{1}{2} < k \leq 1$ and $\frac{1}{n(\log n)^{\frac{1}{k}}}, k \leq 1$, if the iterates belong to some special open set with compact closure in the domain of attraction of the local attractor infinitely often w.p. 1, then the iterates are stable and converge a.s. to the local attractor.

3. We show that our results can be used to analyze the tracking ability of general (not necessarily linear) stochastic approximation driven by another “slowly” varying stochastic approximation process when the iterates are not known to be stable. Such results are useful in the context of adaptive algorithms \[14\] as not much is known about the stability of frameworks with different timescales.
4. We derive a sample complexity estimate (explained later) for such a recursion.

The motivation for lock-in probability comes from a phenomenon noticed by W.B. Arthur in simple urn models ([5, Chap. 1]) of increasing return economics: if occurrences predominantly of one type tend to fetch more occurrences of the same type, then after some initial randomness the process gets locked into that type of (possibly undesirable) occurrence. Moreover, it is known that under reasonable conditions, every asymptotically stable equilibrium will have a positive probability of emerging as \( \lim_{n \to \infty} \theta_n \) [2], while this probability is zero for unstable equilibria under mild conditions on the noise [7,17].

With this picture in mind and to give a quantitative explanation of this phenomenon, Borkar defined lock-in probability [4] for the iterates of the form

\[
\theta_{n+1} = \theta_n + a(n)(h(\theta_n) + M_{n+1}),
\]

where \( \{M_n\} \) is a martingale noise, i.e., a martingale difference sequence, as the probability of convergence of \( \theta_n \) to an asymptotically stable attractor \( H \) of (3) given that the iterate is in a neighbourhood \( B \) thereof after a sufficiently large \( n_0 \), i.e.,

\[
P(\theta_n \to H | \theta_{n_0} \in B)
\]

for a compact \( B \subset G \). He also found a lower bound for this quantity by studying the local behavior of iterates in a neighborhood of the attractor. Clearly, \( n_0 \) depends on the specific \( H \). Specifically, under the assumption \( E[|M_{n+1}|^2 | \mathcal{F}_n] \leq K(1 + \|\theta_n\|^2) \) a.s. the bound obtained is \( 1 - O(\sum_{i \geq n_0} a(i)^2) \) and under the more restrictive condition \( |M_{n+1}| \leq K_0(1 + \|\theta_n\|) \) a.s., a tighter bound of \( 1 - O(e^{-\sum_{i \geq n_0} a(i)^2}) \) has been obtained [4]. There are recent results [12,20] which obtain tighter bound under much weaker assumptions on martingale and step-size sequence.

The fact that lock-in probability is not just a theoretical quantity to explain the lock-in phenomenon of information economics was shown by Kamal [12]. If the iterates are tight then lock-in probability results are used in [12] to prove almost sure convergence of stochastic approximation algorithm (with only martingale difference noise) to the global attractor.

The phenomenon described earlier can be observed in reinforcement learning (RL) applications where the limiting o.d.e. has multiple equilibria, e.g., several instances of stochastic gradient descent in machine learning. We extend in this paper the currently available lock-in probability estimates to the case where the vector field includes a Markov iterate-dependent noise. This is for instance the case with many reinforcement learning algorithms where Markov iterate-dependent noise arises naturally because of the Markov decision process in the background.

Although the recursion [1] covers most of the cases of stochastic approximation with Markov iterate-dependent noise, there are reinforcement learning scenarios where there can be a dependence on both the present and the next
sample of the Markov iterate-dependent noise in the vector field \[13\]. For such scenarios the general recursion is:

$$\theta_{n+1} = \theta_n + a(n) f(\theta_n, Y_n, Y_{n+1}).$$

(6)

One can write (6) as

$$\theta_{n+1} = \theta_n + a(n) \left[ E[f(\theta_n, Y_n, Y_{n+1})|F_n] + M_{n+1} \right],$$

where \(F_n = \sigma(\theta_m, Y_m, m \leq n)\) and \(M_{n+1} = f(\theta_n, Y_n, Y_{n+1}) - E[f(\theta_n, Y_n, Y_{n+1})|F_n]\) is a martingale difference sequence. Therefore, with abuse of notation, the general recursion which takes care of Markov iterate-dependent noise can be described as

$$\theta_{n+1} = \theta_n + a(n) \left[ f(\theta_n, Y_n) + M_{n+1} \right].$$

(7)

In fact, this also covers the situation where both Markov iterate-dependent and martingale difference noise are present. In this work, we give a lower bound on the lock-in probability estimate of iterates of the form (7) using the Poisson equation based analysis as in [3,15]. Under some assumptions in [15] and some further assumptions, we get a lower bound of \(1 - O(\epsilon \sum_{i=n_0}^{\infty} a(i)^2)\) for the recursion (7), and thus also for the special case (1). Therefore, with the more general assumption of Markov iterate-dependent noise, we recover the same bounds available for the setting of martingale noise [5, p. 38] although with some additional assumptions on the Markov iterate-dependent process and step size sequence.

Very few results [8] are available on non-asymptotic rate of convergence of general stochastic approximation iterates (1), see also [18] for stochastic gradient descent. But lock-in probability estimates can be used to calculate an upper bound for the sample complexity estimate of stochastic approximation [5] chap. 4.2, [12]. Given a desired accuracy \(\epsilon > 0\) and confidence \(\gamma\), the sample complexity estimate is defined to be the minimum number of iterations \(N(\epsilon, \gamma)\) after which the iterates are within a certain neighbourhood (which is a function of \(\epsilon\)) of \(H\) with probability at least \(1 - \gamma\). This is slightly different from the sample complexity estimate arising in the context of consistent supervised learning algorithms in statistical learning theory [19]. The reasons are:

1. In the case of statistical learning theory, sample complexity corresponds to the number of i.i.d training samples needed for the algorithm to successfully learn a target function. However, in our case, we have a recursive scheme whose sample complexity depends on the step-size.

2. Ours is a conditional estimate, i.e., the estimate is conditioned on the fact that \(\theta_{n_0} \in B\) where \(B\) is an open subset of the domain of attraction of \(H \subset B\) and has compact closure, and \(n_0\) is sufficiently large.

Another point worth noting is that sample complexity results are much weaker than lock-in probability and do not require existence of Lyapunov function. In
our work, we give a sample complexity estimate for the setting where the recursion is a stochastic fixed point iteration driven by a Markov iterate-dependent noise. This shows a quantitative estimate of large vs. small step size trade-off well known in stochastic approximation literature that is shown to be useful in choosing the optimal step-size.

The organization of the paper is as follows: Section 2 formally defines the problem and provides background and assumptions. Section 3 shows our main lock-in probability results. Section 4 shows how to prove almost sure convergence to a local attractor using our results along with asymptotic tightness of the iterates. Moreover, this section shows that stability of the iterates can be proved using our results. Section 5 analyzes the tracking ability of adaptive algorithms using our results. Section 6 describes the results on sample complexity. Finally, we conclude by providing some future research directions.

2 The Problem and Assumptions

In the following we describe the preliminaries and notation that we use in our proofs. Most of the definitions and notation are from [5,12,15]. The notations used for ordinary differential equation is from [5, Appendix 11.2]. In the following we describe the lock-in probability settings based on the approach in [15]. The main idea is to assume existence of a solution to the Poisson equation (Assumption (M4) from Section III B of [15]), thus converting Markov iterate-dependent noise into a martingale difference sequence and additional additive errors. We refer the readers to [3, Part II, Chap. 2, Theorem 6], [15, Section III D, Appendix A] for details on the existence and properties of solution of Poisson equation for a Markov iterate-dependent process.

In this work we prove almost sure convergence for recursion (7) without assuming stability of the iterates, however following the classic Possion equation model stated above where the assumptions are designed keeping in mind the stability of the iterates. To make up for this we need to strengthen some existing assumptions of [15] (shown next), these are standard assumptions satisfied in application areas such as reinforcement learning.

Let $G \subset \mathbb{R}^d$ be open and let $V : G \to [0, \infty)$ be such that $\langle \nabla V, h \rangle : G \to \mathbb{R}$ is non-positive. We shall assume as in [5] that $H := \{ \theta : V(\theta) = 0 \}$ is equal to the set $\{ \theta : (\nabla V(\theta), h(\theta)) = 0 \}$ and is a compact subset of $G$. Thus $V$ is a strict Lyapunov function. Then $H$ is an asymptotically stable invariant set of the differential equation $\dot{\theta}(t) = h(\theta(t))$. Let there be an open set $B$ with compact closure such that $H \subset B \subset \overline{B} \subset G$. In this setting, the lock-in probability is defined to be the probability that the sequence $\{ \theta_n \}$ is convergent to $H$, conditioned on the event that $\theta_{n_0} \in B$ for some $n_0$ sufficiently large.

Recall that, Theorem 8 of [5, p. 37], shows that for the case of martingale difference noise, $\mathbb{P}[\theta_n \to H | \theta_{n_0} \in B] \geq 1 - O(e^{-\pi^2 s(n_0) \omega})$, where $s(n_0) := \sum_{m=n_0}^\infty a(m)^2$. In this paper we obtain these results when the noise is Markov iterate-dependent under the following assumptions:
(A1) \( \limsup_{n \to \infty} \|Y_n\| < \bar{C} \) a.s. for some \( \bar{C} > 0 \). This is stronger than \( \limsup_{n} E[\|Y_n\|^2] < \infty \) which is implied by (M2) of [15].

(A2) \( \sup_y \|f(\theta, y)\| \leq K(1 + \|\theta\|) \) for all \( \theta \).

**Remark 1** (A2) is a standard assumption satisfied in reinforcement learning scenarios as pointed in [4, p 6]. Clearly, this is stronger than the hypothesis (F) on \( f \) as mentioned in [15, p 143].

(A3) The stepsizes \( \{a(n)\} \) are non-increasing positive scalars satisfying
\[
\sum_n a(n) = \infty, \quad \sum_n a(n)^2 < \infty.
\]

(A4) For every \( \theta \), the Markov chain \( \Pi_\theta \) has a unique invariant probability \( \Gamma_\theta \). ((M1) from [15]). Further, \( h(\theta) = \int f(\theta, y)\Gamma_\theta(dy) \) is Lipschitz continuous in \( \theta \) with Lipschitz constant \( 0 < L < \infty \).

(A5) \( \|M_{n+1}\| \leq K'(1 + \|\theta_n\|) \) a.s. \( \forall n \). Note that in [15] there was no martingale noise.

(A6) For every \( \theta \) the Poisson equation
\[
(1 - \Pi_\theta)v_\theta = f(\theta, \cdot) - \int f(\theta, y)\Gamma_\theta(dy)
\]
has a solution \( v_\theta \). This is (M4) from [15].

(A7) For all \( R > 0 \) there exist constants \( C_R > 0 \) such that
\[
\begin{align*}
&\text{(a) } \sup_{\|\theta\| \leq R} \|v_\theta(x)\| \leq C_R(1 + \|x\|). \\
&\text{(b) } \|v_\theta(x) - v_{\theta'}(x)\| \leq C_R\|\theta - \theta'\|(1 + \|x\|) \text{ for all } \|\theta\| \leq R, \|\theta'\| \leq R.
\end{align*}
\]
This is (M5)\( b,c \) from [15].

Under the above assumptions we show that
\[
P[\theta_n \to H|\theta_{n_0} \in B] \geq 1 - O\left(e^{-n_0/c}\right)
\]
for sufficiently large \( n_0 \).

We provide a more detailed discussion on assumptions (A1) and (A2) as well as possible relaxations of these in Section 3.1.

### 3 Main Results

In this subsection we give a lower bound for \( P[\theta_n \to H|\theta_{n_0} \in B] \) in terms of \( s(n_0) \) when \( n_0 \) is sufficiently large based on the settings described in Section 2. How large \( n_0 \) needs to be will be specified soon. Before proceeding further we
describe our notations and recall some known results. For \( \delta > 0 \), \( N_\delta(A) \) denotes its \( \delta \) -neighborhood \( \{ y : ||y - x|| < \delta \ \forall \ x \in A \} \). Let \( H^a = N_\alpha(H) \).

Fix some \( 0 < \epsilon_1 < \epsilon \) and \( \delta_B > 0 \) such that \( N_{\delta_B}(H^\epsilon) \subset H^\epsilon \subset B \).

Let \( T \) be an upper bound for the time required for a solution of the o.d.e (3) to reach the set \( \theta \). Subsequently the idea is to find an upper bound of \( \bar{\rho} \) for all \( 0 < \theta < \delta \).

We recall here a few key results from [4]. As shown there, if \( \theta_{n_0} \in B, \) and \( \rho_m < \delta_B \) for all \( m \geq 0 \), then \( \theta(\Pi_{n_0}) \) is in \( H^\epsilon \subset B \) for all \( n \geq 1 \). Therefore using discrete Gronwall’s inequality we can show that \( \sup_{t \geq T_{n_0}} \theta(t) < \infty \). It is also known (15, section IIC) that if the sequence of iterates \( \{ \theta_n \} \) remains bounded almost surely on a prescribed set of sample points, and if on this set the iterates belongs to a compact set in the domain of attraction of any local attractor infinitely often then it converges almost surely on this set to that local attractor.

Using this fact gives the following estimate on the probability of convergence, conditioned on \( \theta_{n_0} \in B \) (5, Lemma 1, p. 33):

\[
P[\theta(t) \to H | \theta_{n_0} \in B] \geq P[\rho_m < \delta_B \ \forall m \geq 0 | \theta_{n_0} \in B].
\]

Let \( B_m \) denote the event that \( \theta_{n_0} \in B \) and \( \rho_k < \delta_B \) for \( k = 0, 1, \ldots, m \). Clearly, \( B_m \subset F_{n+1} \). The following lower bound for the above probability has been obtained in (6, Lemma 2, p. 33):

\[
P[\rho_m < \delta_B \ \forall m \geq 0 | \theta_{n_0} \in B] \geq 1 - \sum_{m=0}^{\infty} P[\rho_m \geq \delta_B | B_{m-1}].
\]

Subsequently the idea is to find an upper bound of \( \rho_m \) consisting of errors (asymptotically negligible on \( B_{m-1} \)) as well as martingale terms. Then for some large \( n_0 \), one may bound \( P(\rho_m \geq \delta_B | B_{m-1}) \) using a suitable martingale concentration inequality. In the following we describe how to achieve the above in our setting.

Using the Poisson equation one can write the recursion (3) as

\[
\theta_{n+1} = \theta_n + a(n) \Phi_a(\theta_n) + a(n) [v_{\theta_n}(Y_n) - \Pi_{\theta_n} v_{\theta_n}(Y_n) + M_{n+1}]
\]

where \( \Pi_{\theta} \phi(x) = \int \phi(y) \Pi_{\theta}(x; dy) \). Let \( \zeta_{n+1} = v_{\theta_n}(Y_n) - \Pi_{\theta_n} v_{\theta_n}(Y_n) \). We decompose \( \zeta_{n+1} = v_{\theta_n}(Y_{n+1}) - \Pi_{\theta_n} v_{\theta_n}(Y_n) + v_{\theta_n}(Y_n) - v_{\theta_{n+1}}(Y_{n+1}) + v_{\theta_{n+1}}(Y_{n+1}) - v_{\theta_n}(Y_{n+1}) \)
and set

\[ A_n = \sum_{k=0}^{n-1} a(k) \zeta_{k+1}^{(1)}, \quad B_n = \sum_{k=0}^{n-1} a(k) \zeta_{k+1}^{(2)}, \quad C_n = \sum_{k=0}^{n-1} a(k) \zeta_{k+1}^{(3)}, \]

\[ D_n = \sum_{k=0}^{n-1} a(k) M_{k+1}, \quad n \geq 1 \]

where

\[ \zeta_{n+1}^{(1)} = v_{\theta_n}(Y_{n+1}) - \Pi_{\theta_n} v_{\theta_n}(Y_n), \quad \zeta_{n+1}^{(2)} = v_{\theta_n}(Y_n) - v_{\theta_{n+1}}(Y_{n+1}), \]

\[ \zeta_{n+1}^{(3)} = v_{\theta_{n+1}}(Y_{n+1}) - v_{\theta_n}(Y_{n+1}). \]

Then one can easily see that as in the proof of Lemma 3 of [5, p. 34]

\[
\rho_{m} \leq (Ca(n_0) + K_T CLs(n_0)) + K_T \left[ \max_{n_m \leq j \leq n_{m+1}} \|A_j - A_{n_m}\| + \right. \\
\left. \max_{n_m \leq j \leq n_{m+1}} \|B_j - B_{n_m}\| + \max_{n_m \leq j \leq n_{m+1}} \|C_j - C_{n_m}\| + \right. \\
\left. \max_{n_m \leq j \leq n_{m+1}} \|D_j - D_{n_m}\| \right],
\]

\[ \text{(8)} \]

where \( C \) is a bound on \( \|h(\Phi_t(\theta))\| \), with \( \Phi_t \) the time-\( t \) flow map for the o.d.e [3], \( 0 \leq t \leq T + 1 \) and \( \theta \in \bar{B} \). Also, \( K_T = e^{LT} \).

Choose an \( n_0^{(1)} \) such that

\[
(Ca(n_0^{(1)}) + K_T CLs(n_0^{(1)})) < \delta_B/2. \]

The following important lemma shows that \( \forall m \geq 1 \), on \( B_{m-1} \), iterates are stable over \( T \)-length interval with the stability constant independent of \( m \). This is enough for our proofs to go through and justifies the importance of assumptions (A2) and (A5).

**Lemma 3.1** On \( B_{m-1} \), \( \|\theta_j\| \leq K'' \) for any \( n_m \leq j \leq n_{m+1} \) where the constant \( K' \) is independent of \( m \).

**Proof 1** From the definition of \( B_{m-1} \), we know that \( \theta_{n_m} \in B \) on this event. Let \( \|\theta_{n_m}\| \leq \tilde{C} \) \( \forall m \). Clearly, for \( n_m \leq j \leq n_{m+1} \),

\[
\|\theta_{n_m}\| + \sum_{k=n_m}^{j-1} a(k) \|f(\theta_k, Y_k)\| + \|M_{k+1}\| \leq \tilde{C} + \tilde{K} \sum_{k=n_m}^{j-1} a(k)(1 + \|\theta_k\|)
\]

where \( \tilde{K} = \max(K, K') \). As \( \sum_{k=n_m}^{j-1} a(k) \leq T \), discrete Gronwall inequality gives the result.
Lemma 3.2 For sufficiently large $n_m$, $\max_{n_m \leq j \leq n_{m+1}} \|B_j - B_{n_m}\| < \frac{\delta_B}{8K_T} \ a.s.$ on the event $B_{m-1}$.

Proof 2 Now, if we write $B_{n_m} = a(0)v_{\theta_0}(Y_0) + \sum_{k=1}^{n_m-1}(a(k) - a(k-1))v_{\theta_k}(Y_k) - a(n_m-1)v_{\theta_{n_m}}(Y_{n_m})$ we obtain

$$B_j - B_{n_m} = \sum_{k=n_m}^{j-1} (a(k) - a(k-1))v_{\theta_k}(Y_k) + a(n_m-1)v_{\theta_{n_m}}(Y_{n_m}) - a(j-1)v_{\theta_j}(Y_j).$$

As $\|\theta_i\| \leq K'$ on $B_{m-1}$

$$\|B_j - B_{n_m}\| \leq C_R \sum_{k=n_m}^{j-1} (a(k) - a(k-1))(1 + \|Y_k\|) + C_R [a(n_m-1)(1 + \|Y_m\|) + a(j-1)(1 + \|Y_j\|)]$$

using $(A7a)$. Now using $(A1), (A3)$ we see that

$$\|B_j - B_{n_m}\| \leq 2C'^{\prime \prime}_R a(n_m - 1),$$

for some $C'^{\prime \prime}_R > 0$. Now choose $n_0^{(2)}$ such that

$$2C'^{\prime \prime}_R a(n_0^{(2)} - 1) < \frac{\delta_B}{8K_T}. \quad (10)$$

The claim follows $\forall n_m \geq n_0^{(2)}$.

Lemma 3.3 For sufficiently large $n_m$, $\max_{n_m \leq j \leq n_{m+1}} \|C_j - C_{n_m}\| < \frac{\delta_B}{8K_T} \ a.s.$ on the event $B_{m-1}$.

Proof 3 Using $(A7b)$ we see that

$$\|c_{k+1}^{(3)}\| \leq C_R \|\theta_{k+1} - \theta_k\|(1 + \|Y_{k+1}\|).$$

Again using the stability of the iterates in the $T$ length interval on $B_{m-1}$ and the assumptions $(A1)$ and $(A2)$ we see that

$$\|s_{k+1}^{(3)}\| \leq C_R \bar{K} \bar{C} a(k).$$

Therefore

$$\|C_j - C_{n_m}\| \leq C_R \bar{K} \bar{C} \sum_{k=n_m}^{j-1} a(k)^2.$$
Now choose \( n_0^{(3)} \) such that

\[
C_R \bar{K} \tilde{C} \sum_{k=n_0^{(3)}}^{i-1} a(k)^2 < \frac{\delta_B}{8K_T}. \tag{11}
\]

This is possible due to \((A7a)\).

**Theorem 3.4** For \( n_0 \) sufficiently large,

\[
P(\tilde{\theta}(t) \to H|\theta_{n_0} \in B) \geq 1 - 2de^{-\frac{K\delta_B^2}{8t(\|n_0\|)}} - 2de^{-\frac{K\delta_B^2}{8(n_0)}}.
\]

**Proof 4** Set

\[
n_0 = \max(n_0^{(1)}, n_0^{(2)}, n_0^{(3)}).
\tag{12}
\]

From \([8]\) we see that for this (large) \( n_0 \)

\[
P(\rho_m \geq \delta_B|B_{m-1}) \leq P(\max_{n_m \leq j \leq n_{m+1}} \|A_j - A_{n_m}\| > \frac{\delta_B}{8K_T}|B_{m-1})
\]

\[
+ P(\max_{n_m \leq j \leq n_{m+1}} \|D_j - D_{n_m}\| > \frac{\delta_B}{8K_T}|B_{m-1}).
\]

Again using the stability of the iterates in the \( T \) length interval on \( B_{m-1} \) and assumption \((A7a)\) we see that \( \gamma_{k+1} \) is bounded a.s. on \( B_{m-1} \) by the constant \( C_0 = 2C_R(1 + C) \) for \( n_m \leq k \leq j - 1 \). Therefore each of the components in this vector is also bounded by the same constant. Therefore,

\[
P(\max_{n_m \leq j \leq n_{m+1}} \|A_j - A_{n_m}\| > \frac{\delta_B}{8K_T}|B_{m-1})
\]

\[
< \sum_{i=1}^d P(\max_{n_m \leq j \leq n_{m+1}} |A_j^i - A_{n_m}^i| > \frac{\delta_B}{8K_T}|B_{m-1})
\]

\[
< \sum_{i=1}^d \exp\left(-\frac{32K_T^2 dC_R^2 |A_{n_m}^i|}{\delta_B^2} \right)
\]

\[
< 2d \exp\left(-\frac{32K_T^2 dC_R^2 \sum_{j=n_m}^{n_{m+1}} a(j)^2}{\delta_B^2} \right)
\]

\[
= 2d \exp\left(-\frac{32K_T^2 dC_R^2 (s(n_m) - s(n_{m+1}))}{\delta_B^2} \right)
\]

In the third inequality above we use the conditional version of the martingale concentration inequality \([7, p. 39, chap. 4]\). We give a proof outline of it in
Appendix. Now it can be shown as in Theorem 11 of [5, Chap. 4] that for sufficiently large $n_0$,

$$P(\rho_n < \delta_B \forall m \geq 0 | \theta_{n_0} \in B) \geq 1 - 2de^{-\frac{K\delta_B^2}{\pi(\sigma_0^2)}} - 2de^{-\frac{C\delta_B^2}{\pi(\sigma_0^2)}}$$

where $\hat{K} = 1/32K^2C^2_0$ and $\hat{C}$ is same as in Theorem 11 [5, p 40].

3.1 Discussion on the assumptions

3.1.1 $Y_n$ unbounded

Even if $Y_n$ is unbounded and iterate-dependent our analysis will go through in the following case by creating functional dependency between $\{Y_n\}$ and $\{\theta_n\}$.

$$(A1)' \text{ For large } n, \|Y_{n+1}\| \leq K_0(1 + \|\theta_n\|) \text{ for some } 0 < K_0 < \infty.$$ 

Such assumption will be satisfied if $Y_{n+1} = \psi(\theta_n, Y_n)$ with $\psi$ roughly growing linearly as a function of $\theta$ alone i.e., $\|\psi(\theta, y)\| \leq K_0(1 + \|\theta\|)$. In other words, $\psi$ is point-wise bounded with respect to $\theta$ alone.

Accordingly we may replace $(A2)$ by the point-wise boundedness of $f \ i.e.,$

$$(A2)' \|f(\theta, y)\| \leq K(1 + \|\theta\| + \|y\|).$$

3.1.2 $Y_n$ pointwise bounded

Our analysis will also go through (with the addition of an error term) for the following relaxation of $(A1)$:

$$(A1)” \text{ lim sup}_n \|Y_n\| < \infty \ \text{a.s..}$$

In this case the lock-in probability statement in Theorem 3.4 will be as follows: For $\nu > 0$, $n_0(\nu)$ sufficiently large,

$$P(\bar{\theta}(t) \to H | \theta_{n_0} \in B) \geq 1 - 2de^{-\frac{\hat{K}(\nu)\delta_B^2}{\pi(\sigma_0^2)}} - 2de^{-\frac{\hat{C}(\nu)\delta_B^2}{\pi(\sigma_0^2)}} - 2\nu.$$ 

The proof will work by selecting a large compact set $C(\nu)$ s.t. $P(\text{lim sup}_n \|Y_n\| < C(\nu)) > 1 - \nu$ and doing the same calculation as in Section 3 on this set with probability at least $1 - \nu$.

4 Proof of almost sure convergence

4.1 Almost sure convergence under asymptotic tightness

Definition 1 A sequence of random variables $\{\theta_n\}$ is called asymptotically tight if for each $\epsilon > 0$ there exists a compact set $K_\epsilon$ such that

$$\limsup_{n \to \infty} P(\theta_n \in K_\epsilon) \geq 1 - \epsilon.$$
Clearly, (13) is a much weaker condition than (4). In the following, we give a sufficient condition to guarantee the above:

**Lemma 4.1** If there is a $g \geq 0$ so that $\phi(\theta) \to \infty$ as $\|\theta\| \to \infty$ and

$$\liminf_{n \to \infty} E[\phi(\theta_n)] < \infty,$$

then $\{\theta_n\}$ is asymptotically tight.

**Proof 5** Proof by contradiction and similar to the proof of sufficient condition for full tightness as given in Theorem 3.2.8 of [9, p. 104].

Next, we show that if the stochastic approximation iterates are asymptotically tight then we can prove almost sure convergence to $H$ under some reasonable assumptions.

**Theorem 4.2** Under $(A1)-(A7)$, if $\{\theta_n\}$ is asymptotically tight and $\liminf_{n \to \infty} P(\theta_n \in G) = 1$ then $P(\theta_n \to H) = 1$.

**Proof 6** Choose an open $B$ with compact closure such that $H, K \cap G \subset B \subset G$. Therefore

$$\limsup_{n_0 \to \infty} P(\theta_{n_0} \in B) \geq \limsup_{n_0 \to \infty} P(\theta_{n_0} \in G \cap K)$$

$$= \limsup_{n_0 \to \infty} [P(\theta_{n_0} \in K) + P(\theta_{n_0} \in G) - P(\theta_{n_0} \in G \cup K)]$$

$$\geq \limsup_{n_0 \to \infty} P(\theta_{n_0} \in K) + \liminf_{n_0 \to \infty} P(\theta_{n_0} \in G) - \limsup_{n_0 \to \infty} P(\theta_{n_0} \in G \cup K)$$

$$\geq 1 - \epsilon + 1 - 1.$$

Thus there exists a subsequence $n_0(k)$ s.t. $P(\theta_{n_0(k)} \in B) > 0$. Now,

$$\lim_{k \to \infty} P(\theta_{n_0(k)} \in B, \theta_n \to H) = \lim_{k \to \infty} P(\theta_{n_0(k)} \in B)P(\theta_n \to H | \theta_{n_0(k)} \in B)$$

$$= \lim_{k \to \infty} P(\theta_{n_0(k)} \in B) \text{ using Theorem 3.4}$$

Therefore,

$$P(\theta_n \to H) \geq \limsup_{n_0 \to \infty} P(\theta_n \to H, \theta_{n_0} \in B)$$

$$= \limsup_{n_0 \to \infty} P(\theta_{n_0} \in B)$$

$$\geq 1 - \epsilon$$

(15)

Now let $\epsilon \to 0$. 

12
Remark 2 We compare Theorem 4.2 to the main convergence result (Kushner-Clark Lemma) from [15, Section II C] where stability of the iterates was assumed. Note that in that case much weaker condition, namely $\theta_n \in A$ infinitely often where $A$ is some compact subset of $G$ was sufficient to draw the conclusion. Here we need a much stronger condition such as $\lim \inf_{n_0} P(\theta_{n_0} \in G) = 1$.

Remark 3 Theorem 4.2 is valid for any ‘local’ attractor $H$ whereas in [12, Theorem 1] $H$ was a ‘global’ attractor.

There are sufficient conditions to guarantee tightness ([11, Chap 6, Theorem 7.4]) of the iterates in literature. In the following we describe another set of sufficient conditions which guarantee (14):

Lemma 4.3 Suppose there exists a $\phi \geq 0$ and $\phi(\theta) \to \infty$ as $\|\theta\| \to \infty$ with the following properties: Outside the unit ball

(S1) $\phi$ is twice differentiable and all second order derivatives are bounded by some constant $c$.

(S2) for every $\theta, K \subset \mathbb{R}^k$ compact, $\langle \nabla \phi(\theta), f(\theta, y) \rangle \leq 0$ for all $y \in K$.

Then for the step size sequences of the form $a(n) = \frac{1}{n \log n}^p$ with $0 < p \leq 1$, we have [14].

Proof 7 Following similar steps as in [12, Theorem 3] and (S2) we get

$$E[\phi(\theta_{n+1})|F_n] \leq \phi(\theta_n) + ca(n)^2(1 + \|\theta_n\|^2) \ \text{a.s.} \quad (16)$$

Now we know that, for $n \geq 1$

$$\|\theta_n\| \leq \|\theta_0\| + \sum_{k=0}^{n-1} a(k) \left(\|f(\theta_k, Y_k)\| + \|M_{k+1}\|\right)$$

$$\leq \|\theta_0\| + \tilde{K} \sum_{k=0}^{n-1} a(k) + \tilde{K} \sum_{k=0}^{n-1} a(k)\|\theta_k\|.$$

Therefore using a general version of discrete Gronwall inequality (See Appendix) and the fact that $\|\theta_0\| + \tilde{K} \sum_{k=0}^{n-1} a(k)$ is an increasing function of $n$, we get that

$$\|\theta_n\| \leq \left[\|\theta_0\| + \tilde{K} \sum_{k=0}^{n-1} a(k)\right] \exp(\tilde{K} \sum_{k=0}^{n-1} a(k)).$$

Therefore

$$\lim \inf_{n} E[\phi(\theta_n)] < \phi(\theta_0) + ca(0)^2(1 + \|\theta_0\|^2) + c \sum_{n=1}^{\infty} a(n)^2 +$$

$$c \sum_{n=1}^{\infty} a(n)^2[\|\theta_0\| + \tilde{K} \sum_{k=0}^{n-1} a(k)^2] \exp(2\tilde{K} \sum_{k=0}^{n-1} a(k)). \quad (17)$$
In the following, we show that for the mentioned step-size sequence the R.H.S converges. Assume $0 < p < 1$. Then

$$\sum_{i=2}^{n-1} a(i) \leq \int_{1}^{n-1} \frac{1}{i \log(i)^p} di \leq \frac{1}{1 - p} (\log n)^{1-p}.$$  

Then,

$$\sum_{n=2}^{\infty} \frac{(\log n)^{2(1-p)}}{n^2 (\log n)^{2p}} \exp \left[ \frac{2\tilde{K}}{1-p} (\log n)^{1-p} \right] = \sum_{n=2}^{\infty} \frac{(\log n)^{2-4p}}{n^{2-4p} (\log n)^{2p}}.$$  

This is a convergent series for $0 < p < 1$ as there exists an $\epsilon > 0$ such that for large $n$

$$(\log n)^{2-4p} \leq n^{1 - \frac{2\tilde{K}}{1-p} (\log n)^{1-p} - \epsilon}.$$  

Also, the following series converges

$$\sum_{n=2}^{\infty} \frac{\exp \left[ \frac{2\tilde{K}}{1-p} (\log n)^{1-p} \right]}{n^2 (\log n)^{2p}}.$$  

(18)

Moreover, it is easy to check that the above arguments also hold for $p = 1$.

Thus we show that (A5) in Theorem 3 in [12] is not required for the step size sequence of the form $a(n) = \frac{1}{n \log n^p}$ with $0 < p \leq 1$ which is clearly a divergent series but $\sum_n a(n)^2 < \infty$.

\textbf{Remark 4} Theorem 3 of [12] imposes assumptions on the strict Lyapunov function $V(.)$ for the attractor $H$ to ensure tightness of the iterates. For that reason $H$ is required to be a global attractor there. However, we observe that $\phi(.)$ can be different from $V(.)$ because we only require properties like (S2) to ensure tightness of the iterates.

\textbf{Remark 5} Note that the series in R.H.S of (17) won’t converge if $a(n) = \frac{1}{n^p}$ with $1/2 < k \leq 1$. In such a case (A5) from [12] will be required.
4.2 Proof of stability and a.s. convergence using our results

Note that if the iterates belong to some arbitrary compact set (depending on the sample point) infinitely often, it may not imply stability if the time interval between successively visiting it runs to infinity. We show that this does not happen if the compact set and the step-size have special properties. Using the lock-in probability results from Section 3, we prove stability and therefore convergence of the iterates on the set \( \{ \theta_n \in B \text{ i.o.} \} \) when the step-size is \( a(n) = \frac{1}{n^k}, \frac{1}{2} < k \leq 1 \).

Consider the settings described in Section 3. Let \( A = \{ \omega : \exists m \geq 0 \text{ s.t. } \rho_m(\omega) \geq \delta \} \). Then Theorem 3.4 shows that for sufficiently large \( n_0 \),

\[
P(A|\theta_{n_0} \in B) < 4de^{-\frac{c}{n_0}}
\]

\[
\implies P(A \cap \{ \theta_{n_0} \in B \}) < 4de^{-\frac{c}{n_0}}
\]

\[
\implies \sum_{n_0=1}^{\infty} P(A \cap \{ \theta_{n_0} \in B \}) < \sum_{n_0=1}^{\infty} 4de^{-\frac{c}{n_0}}.
\]

(19)

Now, for \( n \geq 2 \)

\[
s(n) = \sum_{i=n}^{\infty} \frac{1}{i^{2k}} < \int_{i=n-1}^{\infty} \frac{1}{i^{2k}} di
\]

\[
= \frac{1}{(2k-1)(n-1)^{2k-1}}
\]

\[
\leq \frac{1}{(2k-1)(\frac{n}{2})^{2k-1}}
\]

Now, for large \( n \),

\[
e^{(2k-1)(\frac{n}{2})^{2k-1}} > n^2.
\]

Therefore R.H.S in (19) is finite for the mentioned step-size. The same argument follows for the step-size \( \frac{1}{n(log(n))^k}, k \leq 1 \) as for large \( n \), \((log(n))^{2k} \geq 1 \).

Therefore,

\[
E[\sum_{n_0=1}^{\infty} I_{A \cap \{ \theta_{n_0} \in B \}}] < \infty
\]

\[
\implies I_A \sum_{n_0=1}^{\infty} I_{\{ \theta_{n_0} \in B \}} < \infty \text{ a.s.}
\]

Therefore on the event \( \{ \theta_{n_0} \in B \text{ i.o.} \} \), \( I_A = 0 \) a.s. which is nothing but \( \sup_n \| \theta_n \| < \infty \text{ a.s.} \). The result can be summarized as follows:

**Corollary 4.4** Under the assumptions made in Section 3 and the following assumptions:
∀N ∃n ≥ N s.t. P(θ_n ∈ B) > 0 where B is chosen as in Section 2

(W2) \[ \sum_{n=1}^{\infty} P(θ_n \in B|F_{n-1}) = \infty \] a.s.,

we have

\[ \sup_n \|θ_n\| < \infty \] a.s. and \[ θ_n \to H \] a.s.

for the step-size sequence of the form \[ a(n) = \frac{1}{n^k}, 0.5 < k \leq 1 \text{ and } \frac{1}{n(logn)^k}, k \leq 1. \]

5 On the tracking ability of “general” adaptive algorithms using lock-in probability

In this section we investigate the tracking ability of algorithms of the type:

\[ w_{n+1} = w_n + b(n) \left[ g(θ_n, w_n, Z_n^{(2)}) + M_{n+1}^{(2)} \right], \quad (20) \]

that are driven by a “slowly” varying single timescale stochastic approximation process:

\[ θ_{n+1} = θ_n + a(n) \left[ h(θ_n, Z_n^{(1)}) + M_{n+1}^{(1)} \right], \quad (21) \]

when none of the iterates are known to be stable. Here, \( θ_n \in \mathbb{R}^d, w_n \in \mathbb{R}^k, Z_n^{(1)} \in \mathbb{R}^l, Z_n^{(2)} \in \mathbb{R}^m \). Note that there is a unilateral coupling between (20) and (21) in that (20) depends on (21) but not the other way. Suppose \( w_n \) converges to a function \( λ(θ) \) in case \( θ_n \) is kept constant at \( θ \), then an interesting question is that if \( θ_n \) changes slowly can \( w_n \) track the changes in \( θ_n \) i.e. what can we say about the quantity \( \|w_n - λ(θ)\| \) in the limit. As mentioned in [14] such algorithms may arise in the context of adaptive algorithms. However, in that work tracking was proved under the restrictive assumption that the stochastic approximation driven by the slowly varying process is linear (see (1) in the same paper) and the underlying Markov process in the faster iterate is driven by only the slow iterate. Using the lock-in probability results of Section 3 we prove convergence as well as tracking ability of much general algorithms such as (20)-(21) under the following assumptions (we also give a detailed comparison with the assumptions of [14]):

(B1) \( h, Z_n^{(1)} \) and \( M_{n+1}^{(1)} \) satisfy the same assumptions satisfied by similar quantities \((f, Y_n, M_n \text{ respectively})\) of Section 2; \( g \) satisfies the following assumption: \( \sup_z \|g(θ, w, z)\| \leq K_1(1 + \|θ\| + \|w\| + \|z\|) \) for all \( θ, w, z \) where \( K_1 > 0 \). Additionally, \( \hat{g}(θ, w) = \int g(θ, w, z)Γ_{θ, w}^{(2)}(dz) \) is Lipschitz continuous, \( Γ_{θ, w}^{(2)} \) being the unique stationary distribution of \( Z_n^{(2)} \) for a fixed \((θ, w)\).
Remark 6 In (1) of [14], the vector field in the faster iterate is linear in the faster iterate variable. Also, the slower iterate is not a stochastic approximation iteration there.

(B2) \(\{a(n)\}\) is as in (A3). \(\{b(n)\}\) satisfies the similar assumptions as \(\{a(n)\}\). Additionally, \(a(n) < b(n)\) for all \(n\) and \(\frac{a(n)}{b(n)} \to 0\).

Remark 7 The latter is much weaker than Assumption 4 of [14].

(B3) The dynamics of \(Z_{n}^{(2)}\) is specified by

\[
P(Z_{n+1}^{(2)} \in B | Z_{m}^{(2)}, \theta_{m}, w_{m}, m \leq n) = \int_{B} \Pi^{(2)}_{\theta_{n}, w_{n}}(Z_{n}^{(2)}; dz), \text{ a.s. } n \geq 0,
\]

for \(B\) Borel in \(\mathbb{R}^{m}\). Assumptions similar to (A1), (A4), (A6) and (A7) will be true in case of \(Z_{n}^{(2)}\) also with the exception that now \(\theta\) will be replaced by the tuple \((\theta, w)\).

Remark 8 In [14], the Markov process depends on only the slow parameter.

(B4) \(\{M_{n}^{(i)}\}, i = 1, 2\) are martingale difference sequences w.r.t increasing \(\sigma\)-fields

\[
\mathcal{F}_{n} = \sigma(\theta_{m}, w_{m}, M_{m}^{(i)}, Z_{m}^{(j)}, m \leq n, i = 1, 2), n \geq 0,
\]

where \(M_{n}^{(2)}\) satisfies the following:

\[
\|M_{n+1}^{(2)}\| \leq K_{2}(1 + \|\theta_{n}\| + \|w_{n}\|), K_{2} > 0
\]

Remark 9 Our assumptions on martingale difference noise is stronger than the same in [14](See Assumption 5).

(B5) The o.d.e

\[
\dot{\psi}(t) = \hat{\psi}(\theta, w(t))
\]

has a global attractor \(\lambda(\theta)\) with \(\lambda : \mathbb{R}^{d} \to \mathbb{R}^{k}\) Lipschitz continuous.

The o.d.e

\[
\dot{\theta}(t) = \hat{h}(\theta(t)) \tag{22}
\]

has an asymptotically stable set \(H^{s}\) with domain of attraction \(G^{s}\) where \(\hat{h}(\theta) = \int h(\theta, y)\Gamma^{(1)}_{\theta}(dy)\) is Lipschitz continuous with \(\Gamma^{(1)}_{\theta}\) same as \(\Gamma_{\theta}\) in (A4).

For every compact set \(C_{1} \subset \mathbb{R}^{d}\) the set \(\{\theta, \lambda(\theta) : \theta \in C_{1}\}\) is Lyapunov stable.
(B6) The iterates \( \{\theta_n, w_n\} \) are asymptotically tight (for which a sufficient condition is stated latter).

**Remark 10** In [14] one important step in the proof is the proof of the stability of the iterates.

Let there be an open set \( B_1 \) with compact closure such that \( H^s \subset B_1 \subset \bar{B}_1 \subset G^s \). From the results of Section 3 we can find a \( T^s \) such that any trajectory for the o.d.e (22) starting in \( \bar{B}_1 \) will be within some \( \epsilon_1 \) neighborhood of \( H^s \) after time \( T^s \). Let, \( S_1 = \sup_{\theta \in \bar{B}_1} ||\theta|| + K \) and \( C_1 = \{\theta : ||\theta|| \leq S_1\} \). Let there be an open set \( B_2 \) with compact closures such that \( \lambda(C_1) \subset B_2 \subset \bar{B}_2 \subset \mathbb{R}^k \). Choose \( \delta_{B_1} \) in the same way \( \delta_{B_2} \) is chosen in Section 3. Choose \( \delta_{B_2}, 0 < \epsilon'_1 < \epsilon'' \) such that \( N_{\delta_{B_2} + \epsilon'_1}(\lambda(C_1)) \subset N_{\epsilon''}(\lambda(C_1)) \subset B_2 \). If the coupled o.d.e starts at a point such that its \( \theta \) and \( w \) co-ordinates are in \( C_1 \) and \( B_2 \) respectively then as in Section 3 one can find a \( T^f > 0 \) (independent of the starting point) such that after that time the o.d.e will be in the \( \epsilon'' \) neighbourhood of \( \{(\theta, \lambda(\theta)) : \theta \in C_1\} \).

Now, let \( T^c = \max(T^f, T^s + 1) \) and for \( m \geq 1 \) define,

\[
n^c_0 = n^s_0 = n_0.
\]

\[
t^c(n) = \sum_{i=0}^{n-1} b(i), n^c_m = \min\{n : t^c(n) \geq t^c(n^c_{m-1}) + T^c\}.
\]

\[
t^s(n) = \sum_{i=0}^{n-1} a(i), n^s_m = \min\{n : t^s(n) \geq t^s(n^s_{m-1}) + T^s\}.
\]

Similarly, for \( m \geq 0 \) define

\[
T^c_m = t^c(n^c_m), I^c_m = [T^c_m, T^c_{m+1}],
\]

\[
T^s_m = t^s(n^s_m), I^s_m = [T^s_m, T^s_{m+1}],
\]

\[
l_m = \max(k : t^s(n_k^s) \leq t^c(n^c_m)).
\]

Now define,

\[
\rho^c_m := \sup_{t \in I^c_m} ||\vec{\alpha}(t) - \alpha^T_m(t)||
\]

where \( \vec{\alpha}(\cdot) \) is the interpolated trajectory for the coupled iterate

\[
\alpha_{n+1} = \alpha_n + b(n) \left[ G(\alpha_n, Z_n^{(2)}) + \epsilon'_n + M_{n+1}^{(4)} \right]
\]

where \( \alpha_n = (\theta_n, w_n) \), \( \epsilon_n = \frac{a(n)}{m_n} h(\theta_n, Z_n^{(1)}) \) and \( M_{n+1}^{(3)} = \frac{a(n)}{m_n} M_{n+1}^{(1)} \) for \( n \geq 0 \). Let \( \alpha = (\theta, w) \in \mathbb{R}^{d+k} \), \( G(\alpha, z) = (0, g(\alpha, z)) \), \( \epsilon'_n = (\epsilon_n, 0) \), \( M_{n+1}^{(4)} = (M_{n+1}^{(3)}, M_{n+1}^{(2)}) \), and \( \alpha^T_m(\cdot) \) is the solution of the o.d.e

\[
\dot{w}(t) = g(\theta(t), w(t)), \dot{\theta}(t) = 0,
\]

18
on \( I_m^c \) with the initial point \( \alpha^{T_m^c}(T_m^c) = \tilde{\alpha}(T_m^c) \). Also, define

\[
\rho_m^{c*} := \sup_{t \in I_m^c} ||\tilde{\theta}(t) - \theta^{T_m^c}(t)||
\]

where \( \theta^{T_m^c}(.) \) denotes the solution of the o.d.e (22) on \( I_m^c \) with the initial point \( \theta(T_m^c) = \tilde{\theta}(T_m^c) \). Let us assume for the moment that \( \theta_n \in B_1, w_n \in B_2 \), and that \( \rho_m^{c*} < \delta_{B_1} \) and \( \rho_m^{s*} < \delta_{B_2} \). Then using similar arguments as in Section 3 one can show that \( \sup_{t \geq T_0} (\theta(t), \tilde{\theta}(t)) < \infty \) a.s. Further, \( (\theta_n, w_n) \) infinitely often visits the compact set \( C_1 \times B_2 \) which is in the domain of attraction \( C_1 \times \mathbb{R}^d \) of the set \( \{(\theta, \lambda(\theta)) : \theta \in C_1\} \). Therefore,

\[
(\theta_n, w_n) \rightarrow \{(\theta, \lambda(\theta)) : \theta \in \mathbb{R}^d \} \text{ a.s.}
\]

This, in turn, implies that \( ||w_n - \lambda(\theta_n)|| \rightarrow 0 \text{ a.s.} \) which implies that \( (\theta_n, w_n) \rightarrow \bigcup_{\theta \in H^*} (\theta, \lambda(\theta)) \). Let \( B_m^s \) denote the event that \( \theta_n \in B_1, w_n \in B_2 \) and \( \rho_m^{s*} < \delta_{B_1} \) for \( k = 0, 1, \ldots, m \). Also, let \( B_m^{s*,k} \) denote the event that \( \theta_n \in B_1, w_n \in B_2, \rho_j^s < \delta_{B_2} \) for \( j = 0, 1, \ldots, m \) and \( \rho_j^s < \delta_{B_1} \) for \( j = 0, 1, \ldots, k \). Therefore,

\[
P((\theta_n, w_n) \rightarrow \bigcup_{\theta \in H^*} (\theta, \lambda(\theta))| \{\theta_n \in B_1, w_n \in B_2\})
\]

\[
\geq P[\rho_m^{c*} < \delta_{B_2} \forall m \geq 0, \rho_m^{s*} < \delta_{B_1} \forall m \geq 0| \theta_n \in B_1, w_n \in B_2]
\]

\[
\geq P[\rho_m^{s*} < \delta_{B_1} \forall m \geq 0| \theta_n \in B_1, w_n \in B_2] P[\rho_m^{c*} < \delta_{B_2} \forall m \geq 0| \theta_n \in B_1, w_n \in B_2, \rho_m^{s*} < \delta_{B_1} \forall m \geq 0]
\]

\[
\geq \left[1 - \sum_{m=0}^{\infty} P(\rho_m^{s*} > \delta_{B_1}| (B_m^{s*})^c)\right] P(\rho_m^{c*} < \delta_{B_2} | \forall m \geq 0| \theta_n \in B_1, w_n \in B_2, \rho_m^{s*} < \delta_{B_1} \forall m \geq 0).
\]

(26)

Now, using the simple fact that \( P(A|BC) \leq \frac{P(A|B)}{P(C|B)} \),

\[
P[\rho_m^{c*} < \delta_{B_2} \forall m \geq 0| \theta_n \in B_1, w_n \in B_2, \rho_m^{s*} < \delta_{B_1} \forall m \geq 0]
\]

\[
\geq \left[1 - \sum_{m=0}^{\infty} \frac{P(\rho_m^{c*} > \delta_{B_2} | (B_m^{s*})^c)}{1 - f(m) - g(m)}\right]
\]

(27)

where \( f(m) = P(\rho_m^{s*} > \delta_{B_1} | (B_m^{s*})^c) \)

and \( g(m) = \sum_{k=m+1}^{\infty} P(\rho_k^{s*} > \delta_{B_1} | (B_m^{s*})^c) \).

Clearly, \( B_{m-1,l_m}^{s*} \in F_{n_{m_{l_m}}} \) and \( B_{m-1,k-1}^{s*} \in F_{n_{m_{l_m}}} \) for all \( k \geq l_m + 1 \). However, \( B_{m-1,l_m}^{s*} \not\in F_{l_m} \). Therefore, the tedious task is to calculate upper bound of
We describe the procedure in detail. Now, due to the way $T^c$ is chosen
\begin{align}
f(m) & \leq \frac{P(\rho_{m}^s > \delta_B | B_{m-2, l_m-1}^{'})}{1 - P(\rho_{m-1}^s > \delta_B | B_{m-2, l_m-1}^{'})} \\
& \leq o(S_1(n_0)) \left( 1 - o(S_2(n_0)) \right)
\end{align}

where
\[ h(k) = P(\rho_k^s > \delta_B | B_{m-2, k-1}^{'}) \]

Let $S_1(n_0) = \sum_{i=n_0}^{\infty} a(i)^2$ and $S_2(n_0) = \sum_{i=n_0}^{\infty} b(i)^2$.

From (28) we can see that
\[ f(m) \leq o(S_1(n_0)) \left( 1 - o(S_2(n_0)) \right). \]

One can recursively calculate the expression. At the bottom level one calculates the following expression:
\[ 1 - P(\rho_{i-1}^s > \delta_B | B_{i-2}^{'}) \]

Using the fact that $S_1(n_0) < S_2(n_0)$ we see from the above that for all $m \geq 0$, $f(m) \leq o(S_2(n_0))$. One can easily show using the technique of Section 3 that for all $m \geq 0$, $g(m) \leq o(S_1(n_0))$.

**Theorem 5.1** Under the above assumptions, for sufficiently large $n_0$,
\[ P((\theta_n, w_n) \to \bigcup_{\theta \in H^c} \{ (\theta, \lambda(\theta)) | \theta_{n_0} \in B_1, w_{n_0} \in B_2 \}) \geq \left( 1 - o(S_1(n_0)) \right) \left( 1 - \frac{o(S_2(n_0))}{1 - o(S_1(n_0)) - o(S_2(n_0))} \right) \]

**Remark 11** For the case of Section 2 i.e. $\theta_n = \theta$ for all $n$, either 1) $a(n) = 0$ or 2) $h(\theta_n, Z_n^{(1)}) + M_n^{(1)} = 0$ for all $n$. Further, all the assumptions (B1) – (B6) are satisfied and we can recover the results of Section 3 by observing that either 1) $S_1(n_0) = 0$ or 2) $M_n^{(1)} = 0$ for all $n$ (follows from the fact that $\{M_n^{(1)}\}$ is a martingale difference sequence).

**Proof 8 (Proof Outline)** Handling the first term in the last inequality of (26) is exactly same as in Section 3. The numerator of the term inside the summation in (27) can also be handled in a similar manner except the fact that the additional error $\epsilon_n$ in (27) can be made negligible on $B_{m-1, l_m-1}$ using the stability of the iterates there over $T^c$ length intervals (the latter can be proved as in Lemma 3.1). $n_0$ will be the maximum of its versions arising to handle these two parts.

From this one can easily prove almost sure convergence under tightness.
**Theorem 5.2** Under (B1)-(B6), if \( \{a_n\} \) is asymptotically tight and \( \liminf_n P(\theta_n \in G^c) = 1 \) then \( P(\theta_n, w_n) \rightarrow \bigcup_{\theta \in H}(\theta, \lambda(\theta)) = 1 \) i.e. \( \|w_n - \lambda(\theta_n)\| \rightarrow 0 \) a.s.

The sufficient conditions for tightness can be derived in the exact similar way as in Section 4.

**Lemma 5.3** Suppose there exists a \( V' : \mathbb{R}^{d+k} \rightarrow [0, \infty) \) and \( V'(\alpha) \rightarrow \infty \) as \( \|\alpha\| \rightarrow \infty \) with the following properties: Outside the unit ball

(S1) \( V' \) is twice differentiable and all second order derivatives are bounded by some constant \( c \).

(S2) for every \( \alpha \), \( K \subset \mathbb{R}^l \) compact, \( \langle (\nabla V'(\alpha))_{1\ldots d}, h((\alpha)_{1\ldots d}, z) \rangle \leq 0 \) for all \( z \in K \).

(S3) for every \( \alpha \), \( K \subset \mathbb{R}^m \) compact, \( \langle (\nabla V'(\alpha))_{d+1\ldots d+k}, g(\alpha, z) \rangle \leq 0 \) for all \( z \in K \).

where the notation \((v)_{m\ldots n}\) is the vector \((v_m, \ldots, v_n)\) with \( v = (v_1, v_2, \ldots, v_{d+k}) \in \mathbb{R}^{d+k} \).

Then for the step size sequences of the form \( b(n) = \frac{1}{n (\log n)^p} \) with \( 0 < p \leq 1 \), the iterate \( \{a_n\} \) is asymptotically tight.

### 6 Sample Complexity

It is easy to check that using the results in the previous section one can get a similar probability estimate for sample complexity as in [5, Chap. 4, Corollary 14]. Note that here \( T \) can be any positive real number unlike in the lock-in probability calculation where we need to choose \( T \) appropriately. Therefore we can extend the sample complexity calculation for stochastic fixed point point iteration in the setting of Markov iterate-dependent noise as follows:

Consider the example as shown in [5, p. 43]. Let \( u(\theta) = \int f(\theta, y) \Gamma_{\theta}(dy) \) with \( u \) being a contraction, so that \( \|u(\theta) - u(\theta')\| < \alpha \|\theta - \theta'\| \) for some \( \alpha \in (0, 1) \). \( \theta^* \) be the unique fixed point of \( u(\cdot) \). Let \( T > 0 \). \( B \) can be chosen to be \( \{\theta : \|\theta - \theta^*\| < r\} \) with \( r \geq \frac{3\epsilon}{T} \). For the analysis next choose \( r = \frac{3\epsilon}{T} \). Therefore the sample complexity estimate can be stated as follows:

**Corollary 6.1** Let a desired accuracy \( \epsilon > 0 \) and confidence \( 0 < \gamma < 1 \) be given. Let \( \hat{\theta} \) be the value at iteration \( n_0 \) with \( n_0 \) satisfying:

1. \( n_0 \) sufficiently large as in (12), \( s(n_0) < \frac{\hat{C}\epsilon^2}{T} \) and \( a(n_0) < \frac{\hat{C}\epsilon^2}{T} \) (Theorem 11 of [5, Chap. 4]).

2. \( s(n_0) < \frac{\epsilon^2}{\ln(\frac{4\gamma}{\delta})} \).

(29)
Then on the event \( \{ \theta \in B \} \), one needs

\[
N_0 := \min \left[ n : \sum_{i=n_0+1}^{n} a(i) \geq \frac{(T + 1)}{(1 - e^{-(1-\alpha)T})} - n_0 \right]
\]

more iterates to get within \( 2\epsilon \) of \( \theta^* \) with probability at least \( 1 - \gamma \).

**Remark 12** The results clearly show large vs. small step-size trade-off for non-asymptotic rate of convergence well-known in the stochastic convex optimization literature \([12]\). For large step-size, the algorithm will make fast progress whereas the errors due to noise/discretization will be much higher simultaneously. However, our results show the quantitative estimate of this progress and the error. For large step-size, \( n_0 \) satisfying the hypothesis in Corollary 6.1 will be higher whereas \( N_0 \) will be lower compared to small-step size and the opposite is true for small step-size. Therefore the optimal step-size should be somewhere in between.

However, it is not possible to calculate accurately the threshold \( n_0 \) as the constants such as \( C, \hat{K} \) depend on \( B \) which indeed depends on \( \theta^* \). If we consider some special cases where the range for \( \theta^* \) is given although the actual \( \theta^* \) is unknown, we can replace the terms involving constants in \((12)\) by a single constant \( M \). For those cases the following analysis will be useful.

In the following we state an upper bound \( N_0' \) of \( N_0 + n_0 \) when \( a(n) = \frac{1}{n^k}, \frac{1}{2} < k < 1 \) under the following crucial assumption:

\((T1)\) \( P(\theta_{n_0} \in B) = 1. \)

Let \( \alpha = 0.9 \). Under the assumptions made, the estimates of \( n_0 \) and \( N_0' \) are

\[
n_0 = \max \left( \frac{M}{\epsilon}, \left( \frac{M}{\epsilon(2k-1)} \right)^{\frac{1}{1-k}}, \left( \frac{M}{\epsilon(2k-1)} \right)^{\frac{1}{1-k}}, \left( \frac{2M}{\epsilon^2(2k-1)} \right)^{\frac{1}{1-k}}, \left( \frac{2M}{\epsilon^2(2k-1)} \right)^{\frac{1}{1-k}} \right),
\]

\[
N_0' = \left( (n_0)^{(1-k)} + 15.16(1-k) \right)^{\frac{1}{1-k}}.
\]

Then from \( N_0' \) onwards the iterates will be within \( 2\epsilon \) of \( \theta^* \) with probability at least \( 1 - \gamma \). Note that the minimum value of the quantity \( \frac{2(T+1)}{(1-e^{-(1-\alpha)T})} \) for \( \alpha = 0.9 \) is 15.16.

To understand what should be the optimal step-size i.e. the value of \( k \) for which \( N_0' \) will be minimum, we plot \( N_0' \) as a function of \( k \) for two different values of \( M \) each with two different values of \( \epsilon \) (Fig. 1 and 2).

From the graph it is clear that for large values of \( M \), the optimal \( k \) is biased towards 1 whereas for small values of \( k \) it is biased toward \( \frac{1}{2} \). The reason is that for large \( M \), with large step-size, \( n_0 \) will be much higher although \( N_0' - n_0 \) is small whereas with very small \( M \), even if we use large step-size, \( n_0 \) will not be large, thus one can take advantage of \( N_0' - n_0 \) being small.

### 7 Conclusion

In this paper, we describe asymptotic and non-asymptotic convergence analysis of stochastic approximation recursions with Markov iterate-dependent noise.
Figure 1: Sample complexity vs. step-size parameter; $y : N^0_0, x : k, \gamma = 0.1, M = 1E-07$.

Figure 2: Sample complexity vs. step-size parameter; $y : N^0_0, x : k, \gamma = 0.1, M = 100$.
using the lock-in probability framework. Our results show that we are able to recover the same bound available for lock-in probability in the literature for the much stronger i.i.d noise. Such results are used to calculate sample complexity estimate of such stochastic approximation recursions which are then used for predicting the optimal step size. Moreover, our results are extremely useful to prove almost sure convergence to specific attractors in cases where asymptotic tightness of the iterates can be proved easily. An interesting future direction will be to extend this analysis for two-timescale scenarios, both with and without Markov iterate-dependent noise.

References

[1] C. Andrieu, V. B. Tadic, and M. Vihola. On the stability of some controlled Markov chains and its applications to stochastic approximation with markovian dynamic. Annals of Applied Probability, 25(1):1–45, 2015.

[2] W.B. Arthur. Increasing returns and path dependence in the economy. University of Michigan Press, 1994.

[3] A. Benveniste, M. Metivier, and P. Priouret. Adaptive Algorithms and Stochastic Approximation. Springer Verlag, Berlin - New York, 1990.

[4] V.S. Borkar. On the lock-in probability of stochastic approximation. Combinatorics, Probability and Computing, 11:11–20, 2002.

[5] V.S. Borkar. Stochastic Approximation : A Dynamic Systems Viewpoint. Cambridge University Press, 2008.

[6] V.S. Borkar and S.P. Meyn. The o.d.e. method for convergence of stochastic approximation and reinforcement learning. SIAM Journal on Control and Optimization, 38(2):447–469, 2000.

[7] O. Brandiere. Some pathological traps for stochastic approximation. SIAM Journal on Control and Optimization, 36(4):1293–1314, 1998.

[8] M. Broadie, D. Cicek, and A. Zeevi. General bounds and finite-time improvement for the Kiefer-Wolfowitz stochastic approximation algorithm. Operations Research, 59(5):1211–1224.

[9] R. Durrett. Probability Theory and Examples. Cambridge University Press, 4th edition, 2010.

[10] M. Habib, C. McDiarmid, J. Ramirez-Alfonsin, and B. Reed. 'Concentration', in Probabilistic Methods for Algorithmic Discrete Mathematics. Springer Verla, Berlin-Heidelberg, 1998.

[11] H. J.Kushner and G. Yin. Stochastic Approximation and Recursive Algorithms and Applications. Springer, New York, 2nd edition, 2003.
A Proof of conditional and maximal version of Azuma’s inequality

Let $P_B$ denote probability measure defined by $P_B(A) = \frac{P(A \cap B)}{P(B)}$ where $B \in \mathcal{F}_1$. If we can show that with this new probability measure $\{S_n\}$ is a martingale, then we can follow the steps in [10] (3.30), p 227 to conclude the proof.

Let us denote by $E_B$ the expectation with respect to $P_B$. Clearly, $E_B(X) =$
\[ \int_{B} X dP \] Let \( G \in F_n \). Now,

\[
\int_G E_B[S_{n+1} | F_n] dP_B = E_B[E_B[I_G S_{n+1} | F_n]]
\]

\[
= E_B[I_G S_{n+1}] \frac{E[I_G \cap B S_{n+1}]}{P(B)}
\]

\[
= \frac{E[I_G \cap B E[S_{n+1} | F_n]]}{P(B)}
\]

\[
= \frac{E[I_G \cap B S_n]}{P(B)} = \int_G S_n dP_B.
\]

## B General discrete Gronwall inequality

Let \( \{\theta_n, n \geq 0\} \) (respectively \( \{a_n, n \geq 0\} \)) be non-negative (respectively positive) sequences, \( L \geq 0 \) and \( f(n) \) be an increasing function of \( n \) such that for all \( n \)

\[
\theta_{n+1} \leq f(n) + L \left( \sum_{m=0}^{n} a_m \theta_m \right).
\]

Then for \( T_n = \sum_{m=0}^{n} a_m \),

\[
\theta_{n+1} \leq f(n) e^{LT_n}
\]

**Proof 9** Similar to the proof of Lemma 8 in Appendix B of \([5]\)