Abstract. In this article, we determine quantum dimensions and fusion rules for the orbifold code VOA $V^\tau_{LC \times D}$. As an application, we also construct certain 3-local subgroups inside the automorphism group of the VOA $V^\sharp$, where $V^\sharp$ is a holomorphic VOA obtained by the $\mathbb{Z}_3$-orbifold construction on the Leech lattice VOA.

1. Introduction

The study of vertex operator algebras as modules of Virasoro VOA was first initiated by Dong, Mason and Zhu [DMZ94]. They proved that the Moonshine VOA $V^\natural$ contains 48 mutually orthogonal elements such that each of them will generate a copy of the rational Virasoro VOA $L(\frac{1}{2}, 0)$ inside $V^\natural$ and the sum of these 48 conformal vectors is the Virasoro element of $V^\natural$. This discovery turns out to be very important for the study of the Moonshine VOA [Don94, Miy04]. It also leads to the development of the theory of framed vertex operator algebras (cf. [Miy98, Miy04] and [DGH98]). Roughly speaking, a framed VOA is a simple VOA which contains a full sub VOA $F \cong L(\frac{1}{2}, 0)^{\otimes n}$ such that $\text{rank}(V) = \text{rank}(F) = n/2$. There are many interesting examples, which include the famous Moonshine VOA. Moreover, it is known that if $V$ is a framed VOA with the weight one subspace $V_1 = 0$, then the full automorphism group $\text{Aut}(V)$ is finite [Miy04, GL12]. Therefore, the theory of framed VOAs is very useful for studying certain finite groups such as the Monster.

In [DMZ94], the Virasoro VOA $L(\frac{1}{2}, 0)$ was constructed inside the lattice type VOA $V^+_\alpha$, where $\langle \alpha, \alpha \rangle = 4$. In fact, $V^+_\alpha \cong V^+_\sqrt{2}A_1 \cong L(\frac{1}{2}, 0) \otimes L(\frac{1}{2}, 0)$. Therefore, a framed VOA with integral central charge $k$ may also be considered as an extension of the tensor product of the orbifold VOA $V^+_\sqrt{2}A_1$. In this article, we consider a generalization of framed VOAs. Namely, we replace the VOA $V^+_\sqrt{2}A_1$ by another orbifold VOA $(V^\tau_{\sqrt{2}A_2})$, where $\tau$ is a lift of a fixed point free isometry of order 3 in $O(\sqrt{2}A_2)$, and study certain extensions of the VOA $(V^\tau_{\sqrt{2}A_2})^{\otimes n}$. We first study a certain integral lattice $LC \times D$ that are constructed from an $F_4$-code $C$ and an $F_3$-code $D$ as an extension of the lattice $(\sqrt{2}A_2)^{\otimes n}$. We also study the irreducible modules for the orbifold VOA $V^\tau_{LC \times D}$. As our main result, we determine the quantum dimensions and the fusion rules for all irreducible $V^\tau_{LC \times D}$-modules. In particular, we show that all irreducible $V^\tau_{LC \times D}$-modules are simple current modules if the $F_4$-code.

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\( \mathcal{C} \) is self-dual. Moreover, the fusion ring for \( V^r_{L\mathcal{C} \times \mathcal{D}} \) is isomorphic to a group ring of an elementary abelian 3-group and the set of all inequivalent irreducible \( V^r_{L\mathcal{C} \times \mathcal{D}} \)-modules forms a quadratic space over \( \mathbb{F}_3 \) if \( \mathcal{C} \) is self-dual.

As an application, we study the case when \( \mathcal{C} \) is isomorphic to the Hexacode in detail. In this case, \( L_{\mathcal{C} \times \{0\}} \) is isomorphic to the Coxeter-Todd lattice \( K_{12} \) of rank 12. We show that the full automorphism group of \( V^r_{K_{12}} \) is isomorphic to \( \Omega_5(3).2 \). Several 3-local subgroups of the VOA \( V^r \), obtained from \( \mathbb{Z}_3 \)-orbifold construction using the Leech lattice VOA, are also studied and computed explicitly.

This article is organized as follows. In Section 2, we review some basic properties of the VOA \( V^r_{\sqrt{2}A_2} \) and the notion of quantum dimensions. In Section 3, we review a construction of the integral lattice \( L_{\mathcal{C} \times \mathcal{D}} \) from some \( \mathbb{F}_4 \) and \( \mathbb{F}_3 \)-codes. Some basic facts about the lattice VOA \( V_{L_{\mathcal{C} \times \mathcal{D}}} \) and its \( \mathbb{Z}_3 \)-orbifold \( V^r_{L_{\mathcal{C} \times \mathcal{D}}} \) will also be recalled. In Section 4, we compute the quantum dimensions of the orbifold VOA \( V^r_{L_{\mathcal{C} \times \mathcal{D}}} \). In Section 5, we compute the fusion rules among irreducible \( V^r_{L_{\mathcal{C} \times \mathcal{D}}} \)-modules. As an application, we construct in Section 6 certain 3-local subgroups inside the automorphism group of the VOA \( V^r \), where \( V^r \) is a holomorphic VOA obtained by a \( \mathbb{Z}_3 \)-orbifold construction on the Leech lattice VOA.

2. Preliminaries and basic properties

**The VOAs \( V^r_{\sqrt{2}A_2} \) and \( V^r_{\sqrt{2}A_2} \).** In this section we review some facts about the orbifold VOA \( V^r_{\sqrt{2}A_2} \). For general background concerning lattice VOA, we refer to [FLM88, LL03].

Let \( \alpha_1, \alpha_2 \) be the simple roots of type \( A_2 \) and set \( \alpha_0 = -(\alpha_1 + \alpha_2) \). Then \( \langle \alpha_i, \alpha_i \rangle = 2 \) and \( \langle \alpha_i, \alpha_j \rangle = -1 \) if \( i \neq j \). We set \( \beta_i = \sqrt{2}\alpha_i \) and let \( L = \mathbb{Z}\beta_1 + \mathbb{Z}\beta_2 \) be the lattice spanned by \( \beta_1 \) and \( \beta_2 \). Then \( L \) is isometric to \( \sqrt{2}A_2 \).

Let \( \mathbb{F}_4 = \{0, 1, \omega, \bar{\omega} \} \) denote the Galois field of four elements, where \( \omega \) is a root of \( x^2 + x + 1 = 0 \) over \( \mathbb{F}_2 \). We adopt the similar notation as in [KLY03, DLT+04] and denote the cosets of \( L \) in the dual lattice \( L^\perp \) as follows:

\[
L^0 = L, \quad L^1 = \frac{-\beta_1 + \beta_2}{3} + L, \quad L^2 = \frac{\beta_1 - \beta_2}{3} + L, \quad L^3 = \frac{\beta_1 - \beta_2}{3} + L, \quad L^4 = \frac{-\beta_1 + \beta_2}{3} + L, \quad L^5 = \frac{-\beta_1 + \beta_2}{3} + L,
\]

and

\[
L^{(i,j)} = L_i + L_j,
\]

for \( i = 0, 1, \omega, \bar{\omega} \) and \( j = 0, 1, 2 \). Then, \( L^{(i,j)} \), \( i \in \mathbb{F}_4, j \in \mathbb{Z}_3 = \{0, 1, 2\} \) are all the cosets of \( L \) in \( L^\perp \) and \( L^\perp/L \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \). It is shown in [Don93] that there are exactly 12 isomorphism classes of irreducible \( V_L \)-modules, which are given by \( V_{L^{(i,j)}} \), \( i = 0, 1, \omega, \bar{\omega} \) and \( j = 0, 1, 2 \).

Consider the isometry \( \tau : L \to L \) defined by

\[
\beta_1 \mapsto \beta_2 \mapsto \beta_0 \mapsto \beta_1.
\]
Then \( \tau \) is fixed point free of order three and can be lifted naturally to an automorphism of \( V_L \) by mapping
\[
a^1(-n_1) \cdot a^k(-n_k)e^b \mapsto (\tau a^1)(-n_1) \cdot (\tau a^k)(-n_k)e^{\tau b}.
\]

By abuse of notation, we also use \( \tau \) to denote the lift.

In [KLY03], it was shown that there are exactly three irreducible \( \tau \)-twisted \( V_L \)-modules and three irreducible \( \tau^2 \)-twisted \( V_L \)-modules, up to isomorphism. They are denoted by \( V^{T,j}_L(\tau) \) or \( V^{T,j}_L(\tau^2) \) for \( j = 0, 1, 2 \).

The automorphism \( \tau \) acts on the set of inequivalent irreducible \( V_L \)-modules by \( V^{(i,j)}_L \circ \tau \). Note also that \( V^{(i,j)}_L \circ \tau \cong V^{(\bar{\omega}_{i,j})}_L \). We denote
\[
U[\varepsilon] = \{ u \in U \mid \tau u = \exp(2\pi \sqrt{-1}\varepsilon/3)u \},
\]
for any \( \tau \)-invariant \( V_L \)-module \( U \) and \( \varepsilon = 0, 1, 2 \). Irreducible modules for the orbifold VOA \( V^\tau_L \) are classified by Tanabe and Yamada [TY07] and the following result was proved.

**Proposition 2.1.** [TY07] The VOA \( V^\tau_L \) is a simple, rational, \( C_2 \)-cofinite, and of CFT type. There are exactly 30 inequivalent irreducible \( V^\tau_L \)-modules. They are represented as follows.

(i) \( V^{(0,j)}_L[\varepsilon] \) for \( j, \varepsilon = 0, 1, 2 \).
(ii) \( V^{(\omega,j)}_L \) for \( j = 0, 1, 2 \).
(iii) \( V^{T,j}_L(\tau^i)[\varepsilon] \) for \( i = 1, 2 \) and \( j, \varepsilon = 0, 1, 2 \).

Weights of these modules are given by (see Tanabe and Yamada[TY07, (5,10)]):
\[
\text{wt } V^{(i,j)}_L[\varepsilon] \in \frac{2j^2}{3} + \mathbb{Z},
\]
\[
\text{wt } V^{T,j}_L(\tau^i)[\varepsilon] \in \frac{10 - 3(j^2 + \varepsilon)}{9} + \mathbb{Z},
\]
for \( i = 1, 2, j, \varepsilon \in \mathbb{Z}_3 \).

**Quantum Dimension.** We now review the notion of quantum dimension introduced by Dong et al. [DJX13].

Let \( V \) be a VOA of central charge \( c \) and let \( M = \bigoplus_{n \in \mathbb{Z}_+} M_{\lambda+n} \) be a \( V \)-module, where \( \lambda \) is the lowest conformal weight of \( M \). The **normalized character** of \( M \) is defined as
\[
\text{ch } M(q) := q^{\lambda-c/24} \sum_{n \in \mathbb{Z}_+} \text{dim } M_{\lambda+n}q^n,
\]
where \( q = e^{2\pi \sqrt{-1}z} \) and \( z = x + \sqrt{-1}y \) is in the complex upper half-plane \( \mathbb{H} \).

The following notion of quantum dimension is introduced by Dong et al. [DJX13].
Definition 2.2. Suppose \( \text{ch} V(q) \) and \( \text{ch} M(q) \) exist. The quantum dimension of \( M \) over \( V \) is defined as

\[
\text{qdim}_V M := \lim_{y \to 0^+} \frac{\text{ch} M(\sqrt{-1}y)}{\text{ch} V(\sqrt{-1}y)},
\]

where \( y \) is a positive real number.

From now on, we will omit the variable \( q \) and write the character \( \text{ch} M(q) \) as \( \text{ch} M \) instead. Fundamental properties of quantum dimension are also proved in their paper.

Proposition 2.3. [DJX13] Let \( V \) be a simple, rational, \( C_2 \)-cofinite VOA of CFT-type and \( V \cong V' \). Let \( W, W_1, W_2 \) be \( V \)-modules. Then

(i) \( \text{qdim}_V W \geq 1 \).

(ii) \( \text{qdim}_V \) is multiplicative, that is \( \text{qdim}_V(W_1 \times W_2) = \text{qdim}_V W_1 \cdot \text{qdim}_V W_2 \), where \( W_1 \times W_2 \) denotes the fusion product.

(iii) A \( V \)-module \( W \) is a simple current if and only if \( \text{qdim}_V W^1 = 1 \).

(iv) \( \text{qdim}_V W = \text{qdim}_V W' \), where \( W' \) is the contragredient dual of \( W \).

Remark 2.4. Recall that a simple \( V \)-module \( M \) is a simple current module if and only if for every simple \( V \)-module \( W \), \( M \times W \) exists and is also a simple \( V \)-module.

Quantum dimensions of irreducible \( V^\tau_L \)-modules are computed in [Che13].

Proposition 2.5. [Che13] We have

(i) \( \text{qdim}_{V_L^\tau} V_L^{(0,\jmath)} [\epsilon] = 1 \) for \( j, \epsilon = 0, 1, 2 \).

(ii) \( \text{qdim}_{V_L^\tau} V_L^{(\omega,\jmath)} = 3 \) for \( j = 0, 1, 2 \).

(iii) \( \text{qdim}_{V_L^\tau} V_L^{T,\jmath} (\tau^i) [\epsilon] = 2 \) for \( i = 1, 2 \) and \( j, \epsilon = 0, 1, 2 \).

3. THE VOAS \( V_{L \times D} \) AND \( V_{L \times D}^\tau \)

\( \mathbb{Z}_3 \) and \( \mathbb{F}_4 \)-codes. We first review the coding theory concerned in this paper. All codes mentioned in this paper are linear codes. From now on, we fix \( \ell \in \mathbb{N} \).

Let \( \lambda = (\lambda_1, \ldots, \lambda_\ell) \) be a codeword of length \( \ell \), its support is defined to be \( \text{Supp}(\lambda) = \{ i \mid \lambda_i \neq 0 \} \). The cardinality of \( \text{Supp}(\lambda) \), denoted by \( \text{wt}(\lambda) \), is called the (Hamming) weight of \( \lambda \). A code \( S \) is said to be even if \( \text{wt}(\lambda) \) is even for every \( \lambda \in S \). Let \( S \) be a code of length \( \ell \). The (Hamming) weight enumerator of \( S \) is defined to be

\[
W_S(X, Y) = \sum_{\lambda \in S} X^{\ell - \text{wt}(\lambda)} Y^{\text{wt}(\lambda)},
\]

which is a homogeneous polynomial of degree \( \ell \).

We consider the inner products for codes over \( \mathbb{F}_4 \) and \( \mathbb{Z}_3 \) as follows. For codes over \( \mathbb{F}_4 \), we use the Hermitian inner product, i.e,

\[
x \cdot y := \sum_{i=1}^{\ell} x_i \bar{y}_i
\]
for \( \mathbf{x} = (x_1, \cdots, x_\ell) \), \( \mathbf{y} = (y_1, \cdots, y_\ell) \in \mathbb{F}_4^\ell \), where \( \bar{x} = x^2 \) is the conjugate of \( x \in \mathbb{F}_4 \). For \( \mathbb{Z}_3 \)-codes, we use the usual Euclidean inner product:

\[
\mathbf{x} \cdot \mathbf{y} := \sum_{i=1}^\ell x_i y_i \quad \text{for } \mathbf{x}, \mathbf{y} \in \mathbb{Z}_3^\ell.
\]

Let \( \mathcal{K} = \mathbb{F}_4 \) or \( \mathbb{Z}_3 \). For a \( \mathcal{K} \)-code \( \mathcal{S} \) of length \( \ell \) with inner product given as above, we define its dual code by

\[
\mathcal{S}^\perp = \{ \lambda \in \mathcal{K}^\ell \mid \lambda \cdot \mu = 0 \text{ for all } \mu \in \mathcal{S} \}.
\]

A \( \mathcal{K} \)-code \( \mathcal{S} \) is said to be self-orthogonal if \( \mathcal{S} \subset \mathcal{S}^\perp \) and self-dual if \( \mathcal{S} = \mathcal{S}^\perp \).

**Remark 3.1.** By [HP03, Thm.1.4.10], an \( \mathbb{F}_4 \)-code \( \mathcal{C} \) is even if and only if \( \mathcal{C} \) is Hermitian self-orthogonal. Note that the underlying “additive” group structure of \( \mathbb{F}_4 \) is \( \mathbb{Z}_2 \times \mathbb{Z}_2 \).

Therefore, an even \( \mathbb{F}_4 \)-code \( \mathcal{C} \) is also an even “additive” \( \mathbb{Z}_2 \times \mathbb{Z}_2 \) code. Moreover, \( \mathcal{C} \) is \( \tau \)-invariant since it is \( \mathbb{F}_4 \)-linear.

In Tanabe and Yamada [TY07], even \( \tau \)-invariant \( \mathbb{Z}_2 \times \mathbb{Z}_2 \) codes are used. Instead of the Hermitian inner product, they used the trace Hermitian inner product defined by \( \mathbf{x} \cdot \mathbf{y} := \sum_{i=1}^\ell x_i y_i + \bar{x}_i y_i \).

In the notation of [RS98], our code \( \mathcal{C} \) belongs to the family \( 4^H \), while Tanabe and Yamada’s code belongs to the family \( 4^{H+} \). If the code \( \mathcal{C} \) is also linear, then its dual \( \mathcal{C}^\perp \) in \( 4^{H+} \) coincides with the dual of \( \mathcal{C} \) in \( 4^H \). Therefore, these two notions are essentially the same and almost all theorems we proved in this paper have analogous statements in their setting.

**The lattice** \( L_{\mathcal{C} \times \mathcal{D}} \) **and the VOAs** \( V_{L_{\mathcal{C} \times \mathcal{D}}} \) **and** \( V_{L_{\mathcal{C} \times \mathcal{D}}}^\tau \). In this paper, we use a boldface lowercase letter \( \mathbf{x} \) to denote a vector or a sequence of length \( \ell \) and its \( i \)-th coordinate is denoted by \( x_i \). That is

\[
\mathbf{x} = (x_1, \cdots, x_\ell).
\]

From now on, we let \( \mathcal{C} \) be a self-orthogonal \( \mathbb{F}_4 \)-code of length \( \ell \) and let \( \mathcal{D} \) be a self-orthogonal \( \mathbb{Z}_3 \)-code of the same length. First we review a construction of an even lattice from \( \mathcal{C} \) and \( \mathcal{D} \) [KLY03, TY13].

For \( \lambda = (\lambda_1, \cdots, \lambda_\ell) \in \mathbb{F}_4^\ell \) and \( \delta = (\delta_1, \cdots, \delta_\ell) \in \mathbb{Z}_3^\ell \), we define

\[
L_{\lambda \times \delta} := \{ (x_1, \ldots, x_\ell) \in (L_\perp)^{\oplus \ell} \mid x_i \in L^{(\lambda_i, \delta_i)}, i = 1, \ldots, \ell \}.
\]

For subsets \( \mathcal{P} \subset \mathbb{F}_4^\ell \) and \( \mathcal{Q} \subset \mathbb{Z}_3^\ell \), we define

\[
L_{\mathcal{P} \times \mathcal{Q}} := \bigcup_{\lambda \in \mathcal{P}, \delta \in \mathcal{Q}} L_{\lambda \times \delta} \subset (L_\perp)^{\oplus \ell}.
\]

Let \( \tau \) acts diagonally on \((L_\perp)^{\oplus \ell}\) and hence it induces an action on \( V_{(L_\perp)^{\oplus \ell}} \). The purpose of this paper is to determine quantum dimensions and fusion rules of irreducible \( V_{L_{\mathcal{C} \times \mathcal{D}}}^\tau \)-modules.
Proposition 3.2 ([TY13]). Let $C$ be a self-orthogonal $\mathbb{F}_4$-code of length $\ell$ and $D$ be a self-orthogonal $\mathbb{Z}_3$-code of the same length. Then the subset $L_{C \times D}$ is an even sublattice of $(L^\perp)^{\oplus \ell}$. Moreover, the dual lattice $(L_{C \times D})^\perp = L_{C^\perp \times D^\perp}$.

Proposition 3.3. [Don93, DL96, TY13] Let $C$ be a self-orthogonal $\mathbb{F}_4$-code of length $\ell$ and $D$ be a self-orthogonal $\mathbb{Z}_3$-code of the same length. Let $V_{L_{C \times D}}$ be the lattice VOA associated to $L_{C \times D}$. Then we have the following.

(i) The set of all inequivalent irreducible $V_{L_{C \times D}}$-modules is given by

$$\{V_{L(\lambda + C) \times (\delta + D)} \mid \lambda + C \in C^\perp / C, \delta + D \in D^\perp / D\}.$$  

(ii) We have $V_{L(\lambda + C) \times (\delta + D)} \circ \tau \cong V_{L(\lambda' + C) \times (\delta' + D)}$.

(iii) For $i = 1, 2$, there are exactly $|D^\perp / D|$ inequivalent irreducible $\tau^i$-twisted $V_{L_{C \times D}}$-modules. They are represented by $(V_{L_{C \times D}}^{T, \eta}(\tau^i), Y_{\tau^i})$ for $\eta \in D^\perp \mod D$.

Remark 3.4. As $\tau$-twisted $V_{L \oplus L}$-modules,

$$V_{L_{C \times D}}^{T, \eta}(\tau) \cong \bigoplus_{\gamma \in D} V_{L \oplus L}^{T, \eta - \gamma}(\tau),$$

for $i = 1, 2$. Furthermore, we have the following decomposition of $V_{L \oplus L}^{T, \eta}(\tau)$ into a direct sum of irreducible $(V_{L}^{T, \eta})^{\oplus \ell}$-modules.

$$V_{L \oplus L}^{T, \eta}(\tau) \cong \bigoplus_{(\varepsilon_1, \ldots, \varepsilon_\ell) \in \mathbb{Z}_3^\ell} V_{L}^{T, \eta}(\tau)[\varepsilon_1] \otimes \cdots \otimes V_{L}^{T, \eta}(\tau)[\varepsilon_\ell].$$  

(3-2)

Similar for $\tau^2$-twisted modules.

Since $\tau$ acts trivially on $D$,

$$V_{L(\lambda + C) \times (\delta + D)} \cong V_{L(\lambda' + C) \times (\delta' + D)}$$

if and only if (1) $\lambda + C$ and $\lambda' + C$ belong to the same $\tau$-orbit of $C^\perp$; and (2) $\delta + D = \delta' + D$ in $D^\perp / D$. Let $C_{\equiv \tau}$ denote the set of all $\tau$-orbits in $C^\perp$. Then

$$\{V_{L_{C} \times (\delta + D)}[\varepsilon], V_{L(\lambda + C) \times (\delta + D)} \mid 0 \neq \lambda + C \in C_{\equiv \tau} \mod C, \delta + D \in D^\perp / D\}$$

is a set of inequivalent irreducible $V_{L_{C \times D}}^{T}$-modules, which are obtained from the irreducible (untwisted) $V_{L_{C \times D}}$-modules.

It is usually very difficult to classify all irreducible modules of an orbifold VOA. Recently, Miyamoto gave a classification in the $\mathbb{Z}_3$-orbifold case.

Proposition 3.5. [Miy13, Thm.A] Let $V$ be a simple VOA of CFT-type. Assume $V \cong V'$, its contragredient dual, and all $V$-modules are completely reducible. If $\sigma$ is an automorphism of $V$ of order three and a fixed point subVOA $V^\sigma$ is $C_2$-cofinite, then all $V^\sigma$-modules are completely reducible. Moreover, every simple $V^\sigma$-module appears as a $V^\sigma$-submodule of some $\sigma^j$-twisted (or ordinary) $V$-module.
By this proposition, we can classify irreducible $V_{L_c \times D}^\tau$-modules.

**Proposition 3.6.** An irreducible $V_{L_c \times D}^\tau$-module must belong to one of the following types.

(i) $V_{L_0 \times (\delta + D)}[\varepsilon]$,  

(ii) $V_{L_1 \times (\delta + D)}[\varepsilon]$,  

(iii) $V_{L_{\gamma_1} \times (\delta + D)}[\varepsilon]$,  

where $i = 1, 2$, $\varepsilon \in \mathbb{Z}_3$, $0 \neq \lambda + C \in C_{=\tau} / \lambda \mod C$, $\eta \in D_{=\tau} / \mod D$ and $\delta + D \in D_{=\tau} / D$. In particular, there is no modules of the second type (ii) if $C$ is self-dual.

In this paper, our calculations depend heavily on the decomposition of the irreducible $V_{L_c \times D}^\tau$-modules as $(V_L^\tau)^{\otimes \ell}$-modules.

**Proposition 3.7 ([KLY03, TY13]).** As modules of $(V_L^\tau)^{\otimes \ell}$, we have the following decomposition.

(i) $V_{L_{0 \times (\delta + D)}}[\varepsilon] \cong V_{L_0 \times (\delta + D)}[\varepsilon] \oplus \bigoplus_{0 \neq \gamma \in C_{=\tau}} V_{L_{\gamma_1} \times (\delta + D)}$; \hspace{1cm} (3-3a)

(ii) $V_{L_{\tau} \times (\delta + D)}[\varepsilon] \cong \bigoplus_{\delta \in D_{=\tau} / \mod 3} \bigoplus_{\ell_1 + \cdots + \ell_\ell \equiv \varepsilon \mod 3} V_{L_{\tau} \times (\delta + D)}[\varepsilon]$, \hspace{1cm} (3-3b)

where $\delta = (\delta_1, \cdots, \delta_\ell) \in \mathbb{Z}_3^\ell$ and $C_{=\tau}$ denote the set of all $\tau$-orbits in $C$.

In order to use properties about quantum dimensions, we need the following proposition.

**Proposition 3.8.** The VOAs $V_{L_c \times D}$ and $V_{L_{\tau} \times D}$ are simple, rational, $C_2$-cofinite VOAs of CFT type and are isomorphic to their contragredient dual, respectively.

Assertions about lattice VOA is well-known. The simplicity and rationality of $V_{L_c \times D}$ are proved in [TY13]. The $C_2$-cofiniteness of $V_{L_c \times D}$ is given in [Miy13, Thm B]. By [Li94, Cor.3.2], $V_{L_{\tau} \times D}$ is self-dual.

4. Quantum Dimensions of Irreducible $V_{L_c \times D}^\tau$-Modules

In this section, we compute the quantum dimensions of irreducible $V_{L_c \times D}^\tau$-modules. We will first consider the case when $D = \{0\}$ is the trivial $\mathbb{Z}_3$-code. Results in this case are summarized in Theorem 4.10. The general case will be considered in Section 4 (see Theorem 4.11). In addition, we verify one conjecture about global dimensions proposed by Dong et. al. for the VOA $V_{L_c \times D}^\tau$ in this section.

**Weight enumerators.** For $\varepsilon = 0, 1, 2$, let

$$S_\varepsilon := \{ x := (x_1, \cdots, x_\ell) \in \mathbb{Z}_3^\ell \mid \sum x_i \equiv \varepsilon \mod 3 \}.$$  

We define the weight enumerator as follows. Let

$$W_{\varepsilon}(X, Y) := \sum_{x \in S_\varepsilon} X^{\varepsilon - \text{wt}(x)} Y^{\text{wt}(x)},$$

(4-1)
where $\text{wt}(\mathbf{x})$ is the number of nonzero coordinates of $\mathbf{x}$. That is,

$$\text{wt}(\mathbf{x}) := \# \{ x_i \mid x_i \neq 0 \text{ for } 1 \leq i \leq \ell \}.$$ 

We also consider a weight enumerator induced from an $\mathbb{F}_4$-code $C$. Let $W_C(X, Y)$ denote the Hamming weight enumerator, we define

$$W'_C(X, Y) := \frac{1}{3} (W_C(X, Y) - X^\ell). \quad (4-2)$$

Note that $W_\epsilon(X, Y), W'_C(X, Y)$ are homogeneous polynomials in $X, Y$ of the same degree $\ell$.

Lemma 4.1. The self-orthogonal $\mathbb{F}_4$-code $C$ is self-dual if and only if $W'_C(1, 1) = \frac{2^\ell - 1}{3}$.

Proof. First we note that $W_C(1, 1) = |C|$ is equal to the number of elements in $C$; hence

$$W'_C(1, 1) = \frac{W_C(1, 1) - 1}{3} = \frac{|C| - 1}{3}.$$ 

Since $C$ is self-orthogonal, we know $C^\perp \supset C$ and $\dim C^\perp + \dim C = \ell$. Therefore, $|C| \leq 2^\ell$ and the equality holds if and only if $C$ is self-dual. The lemma now follows. \qed

The following lemmas explain why we introduce these weight enumerators. Recall that the module $V_{L,C_{x,\delta}}[\epsilon]$ admits a decomposition of $(V_L^\tau)^{\otimes \ell}$-modules as

$$V_{L,C_{x,\delta}}[\epsilon] \cong V_{L,a_{\delta}}[\epsilon] \oplus \bigoplus_{0 \neq \gamma \in C_{x,\epsilon}} V_{L,\gamma,\delta}, \quad (4-3)$$

where $\delta \in \mathbb{Z}_3^\ell$ and $C_{x,\epsilon}$ denotes the set of all orbits of $\tau$ in $C$. In particular, when $\delta = 0$, we have $V_{L,a_0}[0] \cong (V_{L,\tau})^\tau$, which should not be confused with the subVOA $(V_L^\tau)^{\otimes \ell} \subset (V_{L,\tau})^\tau$.

Lemma 4.2. For $\epsilon = 0, 1, 2$, the character of $V_{L,a_0}[\epsilon]$ is given by

$$\text{ch } V_{L,\epsilon}[\epsilon] = W_\epsilon(Z_0(q), Z_1(q)),$$

where

$$Z_0(q) := \text{ch } V_L[0], \quad Z_1(q) := \text{ch } V_L[1] = \text{ch } V_L[2]. \quad (4-4)$$

Proof. For $\epsilon = 0, 1, 2$, we have a decomposition of $(V_L^\tau)^{\otimes \ell}$-modules:

$$V_{L,\epsilon}[\epsilon] = \bigoplus_{\sum r_i \equiv \epsilon \mod 3} V_L[r_1] \otimes \cdots \otimes V_L[r_\ell]. \quad (4-5)$$

We also know

$$\text{ch } (V_L[r_1] \otimes \cdots \otimes V_L[r_\ell]) = \text{ch } V_L[r_1] \times \cdots \times \text{ch } V_L[r_\ell] = Z_0^{\ell-r} Z_1^r.$$
where \( r \) is the weight of \( r := (r_1, \ldots, r_\ell) \in \mathbb{Z}_3^\ell \). Using the definition of weight enumerator given in (4-1), we can rewrite
\[
ch_{V_L \otimes \ell}\left[ \varepsilon \right] = \sum_{\gamma \in C_{\tau \cdot \varepsilon}} \prod ch_{V_L(\gamma_i, 0)} = Y_0^{-\text{wt}(\gamma)} Y_1^\text{wt}(\gamma).
\]

We know the \( \tau \)-orbit of \( \gamma \) is the set \( \{ \gamma, \omega \gamma, \omega^2 \gamma \} \), where \( \omega \gamma := (\omega \gamma_1, \ldots, \omega \gamma_\ell) \). Note that \( \omega \gamma_i = 0 \) if and only if \( \gamma_i = 0 \). This means \( \text{wt} \gamma = \text{wt} \tau \gamma \) and hence
\[
ch_{V_L(\gamma, 0)} = ch_{V_L(\tau \cdot \gamma, 0)} = ch_{V_L(\omega \gamma, 0)}.
\]

Therefore, by the definition of (4-2) we have
\[
ch\left( \bigoplus_{0 \neq \gamma \in C_{\tau \cdot \varepsilon}} V_{L, \gamma \cdot 0} \right) = \frac{1}{3} \sum_{0 \neq \gamma \in C_{\tau \cdot \varepsilon}} ch_{V_L(\gamma, 0)} = \frac{1}{3} \sum_{0 \neq \gamma \in C} Y_0^{-\text{wt}(\gamma)} Y_1^\text{wt}(\gamma) = W'_C(Y_0, Y_1). \tag{4-6}
\]

**Lemma 4.3.** We have the character
\[
ch\left( \bigoplus_{0 \neq \gamma \in C_{\tau \cdot \varepsilon}} V_{L, \gamma \cdot 0} \right) = W'_C(Y_0, Y_1),
\]
where \( Y_0(q) := ch_{V_L(0, 0)} \), and \( Y_1(q) := ch_{V_L(1, 0)} \).

**Proof.** We first note that \( Y_1(q) = ch_{V_L(x, 0)} \) for \( x = 1, \omega, \bar{\omega} \in \mathbb{F}_4 \). Let \( 0 \neq \gamma \in C_{\tau \cdot \varepsilon} \). Then
\[
ch_{V_L(\gamma, 0)} = \prod ch_{V_L(\gamma_i, 0)} = Y_0^{-\text{wt}(\gamma)} Y_1^\text{wt}(\gamma).
\]

We know the \( \tau \)-orbit of \( \gamma \) is the set \( \{ \gamma, \omega \gamma, \omega^2 \gamma \} \), where \( \omega \gamma := (\omega \gamma_1, \ldots, \omega \gamma_\ell) \). Note that \( \omega \gamma_i = 0 \) if and only if \( \gamma_i = 0 \). This means \( \text{wt} \gamma = \text{wt} \tau \gamma \) and hence
\[
ch_{V_L(\gamma, 0)} = ch_{V_L(\tau \cdot \gamma, 0)} = ch_{V_L(\omega \gamma, 0)}.
\]

Therefore, by the definition of (4-2) we have
\[
ch\left( \bigoplus_{0 \neq \gamma \in C_{\tau \cdot \varepsilon}} V_{L, \gamma \cdot 0} \right) = \frac{1}{3} \sum_{0 \neq \gamma \in C} ch_{V_L(\gamma, 0)} = \frac{1}{3} \sum_{0 \neq \gamma \in C} Y_0^{-\text{wt}(\gamma)} Y_1^\text{wt}(\gamma) = W'_C(Y_0, Y_1). \tag{4-6}
\]

**Proposition 4.4.** We have
\[
ch_{V_L(0, 0)}[\varepsilon] = W_\varepsilon(Z_0, Z_1) + W'_C(Y_0, Y_1),
\]
for \( \varepsilon = 0, 1, 2 \). Moreover, we have
\[
W_\varepsilon(1, 1) = 3^{\ell-1}. \tag{4-7}
\]

**Proof.** The first statement on characters follows directly from the above two lemmas. Using a basic combinatorial argument, it is easy to show \( W_i(1, 1) = 3^{\ell-1} \) for all \( 0 \leq i \leq 2 \); note that \( S_\varepsilon = (\varepsilon, \ldots, 0) + S_0 \) for any \( \varepsilon = 1, 2 \).
Quantum dimensions of $V_{L_{\mathcal{C} \times \delta}}^T$-Modules. We first compute the quantum dimensions of irreducible $V_{L_{\mathcal{C} \times \delta}}^T$-modules in the case that the code $\mathcal{D} = \{0\} \in \mathbb{Z}_3^l$ is the trivial code. Note that in this case $\mathcal{D}^\perp = \mathbb{Z}_3^l$.

**Proposition 4.5.** For $\varepsilon \in \mathbb{Z}_3$ and $\delta \in \mathbb{Z}_3^l$, the irreducible $V_{L_{\mathcal{C} \times \delta}}^T[0]$-module $V_{L_{\mathcal{C} \times \delta}}^T[\varepsilon]$ has the quantum dimension one.

**Proof.** Using the same proof as in [Che13, Lemma 3.2], the irreducible $V_{L_{\mathcal{C} \times \delta}}^T$-modules $V_{L_{\mathcal{C} \times \delta}}^T[\varepsilon]$, $\varepsilon = 0, 1, 2$, are simple current and hence have quantum dimension 1. Thus

$$1 = \text{qdim}_{V_{L_{\mathcal{C} \times \delta}}^T[0]} V_{L_{\mathcal{C} \times \delta}}^T[\varepsilon] = \lim_{y \to 0^+} \frac{\text{ch} V_{L_{\mathcal{C} \times \delta}}^T[\varepsilon]}{\text{ch} V_{L_{\mathcal{C} \times \delta}}^T[0]}.$$ 

If $\delta \neq 0$, the computation is similar with some modification.

Fix $0 \leq \varepsilon \leq 2$ and $0 \neq \delta \in \mathcal{D}^\perp$. Let

$$Z(q) := \frac{\text{ch} V_{L_{\mathcal{C} \times \delta}}^T[\varepsilon]}{\text{ch} V_{L_{\mathcal{C} \times \delta}}^T[0]} = \frac{\text{ch} V_{L_{\mathcal{C} \times \delta}}^T[0] + \sum_{0 \neq \gamma \in \mathcal{C} \times \varepsilon} \text{ch} V_{L_{\mathcal{C} \times \delta}}^T}{\text{ch} V_{L_{\mathcal{C} \times \delta}}^T[0]}.$$ 

Dividing both denominator and numerator by $(\text{ch} V_{L[0]}^T)$. We have

$$Z(q) = \frac{\sum_{r \in S_0} \frac{\text{ch} V_{L_{\mathcal{C} \times \delta}}^T[r]}{\text{ch} V_{L[0]}^T} + \sum_{0 \neq \gamma \in \mathcal{C} \times \varepsilon} \frac{\text{ch} V_{L_{\mathcal{C} \times \delta}}^T}{\text{ch} V_{L[0]}^T}}{\sum_{r \in S_0} \frac{\text{ch} V_{L[r]}^T}{\text{ch} V_{L[0]}^T} + \sum_{0 \neq \gamma \in \mathcal{C} \times \varepsilon} \frac{\text{ch} V_{L_{\mathcal{C} \times \delta}}^T}{\text{ch} V_{L[0]}^T}}.$$ 

Recalling the quantum dimensions of $V_L[0]$-modules given in Prop. 2.5, we have

$$\text{qdim}_{V_{L_{\mathcal{C} \times \delta}}^T[0]} V_{L_{\mathcal{C} \times \delta}}^T[\varepsilon] = \lim_{y \to 0^+} Z(q) = \frac{W_1(1, 1) + W_2(1, 1)}{W_0(1, 1) + W_1(1, 1)} = \frac{3^{l-1} + W_2(1, 1)}{3^{l-1} + W_0(1, 1)} = 1$$

as desired. 

**Proposition 4.6.** Let $\varepsilon = 0, 1, 2$ and $\eta \in \mathbb{Z}_3^l$. The irreducible $V_{L_{\mathcal{C} \times \delta}}^T[0]$-module $V_{L_{\mathcal{C} \times \delta}}^{T, \eta}(T^i)[\varepsilon]$ has the quantum dimension $\frac{\ell}{|\mathcal{C}|}$.

**Proof.** We know an irreducible $V_{L_{\mathcal{C} \times \delta}}^T[0]$-module of twisted type admits a decomposition of $(V_L^T)^{\otimes \ell}$-modules:

$$V_{L_{\mathcal{C} \times \delta}}^{T, \eta}(T^i)[\varepsilon] \cong \bigoplus_{\varepsilon \in S_0} V_{L_{\mathcal{C} \times \delta}}^{T, \eta}(T^i)[\varepsilon_1] \otimes \cdots \otimes V_{L_{\mathcal{C} \times \delta}}^{T, \eta}(T^i)[\varepsilon].$$ (4-8)

It was shown in [Che13] that $\text{ch} V_{L}^{T, j}(T^i)[1] = \text{ch} V_{L}^{T, j}(T^k)[2]$ and $\text{ch} V_{L}^{T, j}(T^i)[0] = \text{ch} V_{L}^{T, 0}(T^i)[0]$ for $i, k = 1, 2$ and $j = 0, 1, 2$. Denote $T_0(q) := \text{ch} V_{L}^{T, j}(T^i)[0]$ and $T_1(q) := \text{ch} V_{L}^{T, j}(T^i)[1]$. Similar to the untwisted case, we have

$$\text{ch} V_{L_{\mathcal{C} \times \delta}}^{T, \eta}(T^i)[\varepsilon] = W_\varepsilon(T_0, T_1).$$ (4-9)
Since $W_\varepsilon$ are homogeneous polynomials of degree $\ell$, we have
\[
\text{qdim}_{V_{L,C}^{T,\eta}} V_{L,C}^{T,\eta}(\tau^i)[\varepsilon] = \lim_{y \to 0^+} \frac{W_\varepsilon(T_0, T_1)}{W_0(Z_0, Z_1) + W_\varepsilon(Y_0, Y_1)}
\]
(4.10)
\[
= \frac{W_\varepsilon(2, 2)}{W_0(1, 1) + W_\varepsilon(3, 3)} = \frac{2^\ell W_\varepsilon(1, 1)}{W_0(1, 1) + 3^\ell W_\varepsilon(1, 1)}.
\]

Note that all $V_{L,C}^T$-modules $V_{L,C}^{T,j}(\tau^i)[\varepsilon]$ have quantum dimension 2.

Now by Thm. 4.4 and Lemma 4.1 we know
\[
\text{qdim}_{V_{L,C}^{T,\eta}} V_{L,C}^{T,\eta}(\tau^i)[\varepsilon] = \frac{2^\ell \cdot 3^\ell - 1}{3^\ell - 1 + 3^\ell - 1(|C| - 1)} = \frac{2^\ell}{|C|}.
\]

\[
\square
\]

Remark 4.7. Note that $\frac{2^\ell}{|C|} = \sqrt{|C|/|C|}$ since $|C| \cdot |C| = F_4^\ell = (2^\ell)^2$.

Corollary 4.8. Let $C$ be a self-dual code. Then all irreducible $V_{L,C}^{T,\eta}$-modules are simple current modules.

Proof. If $C$ is self-dual, then $V_{L,C}^{T,\eta}$ has only two types of irreducible modules. Moreover,
\[
\text{qdim}_{V_{L,C}^{T,\eta}} V_{L,C}^{T,\eta}(\tau^i)[\varepsilon] = \frac{2^\ell}{|C|} = 1
\]
by Lemma 4.1. That means all irreducible modules of the type $V_{L,C}^{T,\eta}(\tau^i)[\varepsilon]$ are simple current modules. By Proposition 4.5, the irreducible modules of the type $V_{L,C}^{T,\eta}(\tau^i)[\varepsilon]$ are simple current modules, also.

Now suppose $C$ is self-orthogonal but not self-dual. Then the quantum dimension of the $V_{L,C}^{T,\eta}$-module $V_{L,C}^{T,j}(\tau^i)[\varepsilon]$ is strictly greater than 1. In addition, $V_{L,C}^{T,\eta}$ has irreducible modules of the type $V_{L,(\lambda + C)\times 3}$.

Proposition 4.9. Let $\lambda + C \in C^\perp/C$ and $\delta \in Z_3^\ell$, we have
\[
\text{qdim}_{V_{L,C}^{T,\eta}} V_{L,(\lambda + C)\times 3} = 3.
\]

Proof. By definition,
\[
\text{qdim}_{V_{L,C}^{T,\eta}} V_{L,(\lambda + C)\times 3} = \lim_{y \to 0^+} \frac{\text{ch} V_{L,C}^{T,\eta}(\tau^i)[\varepsilon]}{\text{ch} V_{L,C}^{T,\eta}(\tau^i)[\varepsilon]} = \lim_{y \to 0^+} \frac{\sum_{\mu \in C} \text{ch} V_{L,(\lambda + C)\times 3}^{T,\eta}}{\text{ch} V_{L,C}^{T,\eta}}.
\]
Dividing both the denominator and the numerator by $(\text{ch} V_{L}[0])^\ell$. Since $\text{qdim}_{V_{L,0}} V_{L(v,j)} = 3$ for any $i \in F_4, j \in Z_3$, we have
\[
\text{qdim}_{V_{L,C}^{T,\eta}} V_{L,(\lambda + C)\times 3} = \frac{|C| \cdot 3^\ell}{3^\ell - 1 + 3^\ell - 1(|C| - 1)} = 3
\]
as desired.

To summarize, we have the theorem.
Theorem 4.10. The quantum dimensions for irreducible $V_{Lc \times 0}$-modules are as follows.

(i) $\text{qdim}_{V_{Lc \times 0}} V_{Lc \times \delta}[\varepsilon] = 1$;

(ii) $\text{qdim}_{V_{Lc \times 0}} V_{L(\lambda + \mathcal{C}) \times \delta} = 3$;

(iii) $\text{qdim}_{V_{Lc \times 0}} V_{Lc \times 0}^{T, \eta_i}(\tau^i)[\varepsilon] = \frac{2t}{|\mathcal{C}|}$,

where $i = 1, 2, \varepsilon \in \mathbb{Z}_3, 0 \neq \lambda + \mathcal{C} \in \mathcal{C}_\varepsilon^I \mod \mathcal{C}$ and $\eta, \delta \in \mathbb{Z}_3$.

Quantum dimension of $V_{Lc \times \mathcal{D}}$-modules. We now deal with the general case. Let $\mathcal{D}$ be a self-orthogonal $\mathbb{Z}_3$-code. The basic idea is to express the characters of $V_{Lc \times \mathcal{D}}$-modules in terms of the characters of $V_{Lc \times 0}$-modules.

Theorem 4.11. The quantum dimensions for irreducible $V_{Lc \times \mathcal{D}}$-modules are as follows.

(i) $\text{qdim}_{V_{Lc \times \mathcal{D}}} V_{Lc \times (\delta + \mathcal{D})}[\varepsilon] = 1$;

(ii) $\text{qdim}_{V_{Lc \times \mathcal{D}}} V_{L(\lambda + \mathcal{C}) \times (\delta + \mathcal{D})} = 3$;

(iii) $\text{qdim}_{V_{Lc \times \mathcal{D}}} V_{Lc \times \mathcal{D}}^{T, \eta_i}(\tau^i)[\varepsilon] = \frac{2t}{|\mathcal{C}|}$,

where $i = 1, 2, \varepsilon \in \mathbb{Z}_3, 0 \neq \lambda + \mathcal{C} \in \mathcal{C}_\varepsilon^I \mod \mathcal{C}, \eta \in \mathcal{D}^I \mod \mathcal{D}$ and $\delta + \mathcal{D} \in \mathcal{D}_\varepsilon^I / \mathcal{D}$.

Proof. (i) For the module $V_{Lc \times (\delta + \mathcal{D})}[\varepsilon]$ we have a decomposition of $(V_L^T)^{\otimes \ell}$-modules:

$$V_{Lc \times (\delta + \mathcal{D})}[\varepsilon] \cong V_{L0 \times (\delta + \mathcal{D})}[\varepsilon] \bigoplus_{0 \neq \gamma \in \mathcal{C}_\varepsilon} V_{Lc \times \mathcal{D}}.$$ (4-11)

Although the characters $\text{ch} V_{Lc \times \Delta}$ may vary as $\Delta$ varies in $\mathcal{D}$, we still have

$$\lim_{y \to 0^+} \frac{\text{ch} V_{Lc \times (\delta + \Delta)}}{\text{ch} (V_L^T)^{\otimes \ell}} = \prod_{i=1}^{\ell} \lim_{y \to 0^+} \frac{\text{ch} V_{\gamma_i \times (\delta_i + \Delta_i)}}{\text{ch} V_{Lc \times \mathcal{D}}^T} = \lim_{y \to 0^+} \frac{\text{qdim} V_{\gamma_i \times \delta}}{\text{ch} (V_L^T)^{\otimes \ell}},$$

for all $\Delta \in \mathcal{D}$. This implies

$$\lim_{y \to 0^+} \frac{\text{ch} V_{Lc \times (\delta + \Delta)}}{\text{ch} (V_L^T)^{\otimes \ell}} = \frac{\sum_{\Delta \in \mathcal{D}} \text{ch} V_{Lc \times \Delta}}{\text{ch} (V_L^T)^{\otimes \ell}} = |\mathcal{D}| \lim_{y \to 0^+} \frac{\text{ch} V_{Lc \times \mathcal{D}}}{\text{ch} (V_L^T)^{\otimes \ell}}.$$ (4-12)

Similarly, we have

$$\lim_{y \to 0^+} \frac{\text{ch} V_{L0 \times (\delta + \Delta)}}{\text{ch} (V_L^T)^{\otimes \ell}} = \lim_{y \to 0^+} \frac{\text{ch} V_{L0 \times \delta}}{\text{ch} (V_L^T)^{\otimes \ell}},$$

for all $\Delta \in \mathcal{D}$.

Therefore,

$$\lim_{y \to 0^+} \frac{\text{ch} V_{L0 \times (\delta + \Delta)}}{\text{ch} (V_L^T)^{\otimes \ell}} = \lim_{y \to 0^+} \frac{\sum_{\Delta \in \mathcal{D}} \text{ch} V_{L0 \times (\delta + \Delta)}}{\text{ch} (V_L^T)^{\otimes \ell}} = |\mathcal{D}| \lim_{y \to 0^+} \frac{\text{ch} V_{L0 \times \delta}}{\text{ch} (V_L^T)^{\otimes \ell}}.$$ (4-13)

Thus by (4-11), (4-12) and (4-13) we know

$$\lim_{y \to 0^+} \frac{\text{ch} V_{Lc \times (\delta + \Delta)}}{\text{ch} (V_L^T)^{\otimes \ell}} = \frac{\text{ch} V_{L0 \times (\delta + \Delta)}}{\text{ch} (V_L^T)^{\otimes \ell}} + \frac{\text{ch} V_{L0 \times \delta}}{\text{ch} (V_L^T)^{\otimes \ell}} = \frac{|\mathcal{D}| \text{ch} V_{Lc \times \Delta}}{\text{ch} (V_L^T)^{\otimes \ell}}.$$
Moreover,
\[
\text{qdim}_{V_{\mathcal{L} \times D}} V_{L \times (\delta + \mathcal{D})}[\varepsilon] = \lim_{y \to 0^+} \frac{\text{ch } V_{L \times (\delta + \mathcal{D})}[\varepsilon]}{\text{ch } V_{L \times D}} = \lim_{y \to 0^+} \frac{\frac{1}{\text{ch } V_{L}} \text{ch } V_{L \times (\delta + \mathcal{D})}[\varepsilon]}{\frac{1}{\text{ch } V_{L}} \text{ch } V_{L \times D}}
\]
\[
= \lim_{y \to 0^+} \frac{\mathcal{D}}{\text{ch } V_{L \times D}} = \text{qdim}_{V_{\mathcal{L} \times 0}} V_{L \times D}[\varepsilon] = 1.
\]

(ii) By the similar arguments as (i), we have
\[
\lim_{y \to 0^+} \frac{\text{ch } V_{L(\lambda + \mathcal{C}) \times (\delta + \mathcal{D})}}{\text{ch } (V_L^\otimes)^\otimes} = \lim_{y \to 0^+} \frac{\text{ch } (\bigoplus_{\Delta \in \mathcal{D}} V_{L(\lambda + \mathcal{C}) \times (\delta + \Delta)})}{\text{ch } (V_L^\otimes)^\otimes} = \lim_{y \to 0^+} \frac{\mathcal{D}}{\text{ch } V_{L(\lambda + \mathcal{C}) \times \delta}},
\]
and hence
\[
\text{qdim}_{V_{\mathcal{L} \times D}} V_{L(\lambda + \mathcal{C}) \times (\delta + \mathcal{D})} = \text{qdim}_{V_{\mathcal{L} \times 0}} V_{L(\lambda + \mathcal{C}) \times \delta} = 3.
\]

(iii) For the irreducible \(V_{\mathcal{L} \times D}^\tau\)-modules of the type \(V_{L \times D}^T,\eta (\tau^i)[\varepsilon]\), we have the decomposition of \((V_L^\otimes)^\otimes\)-modules:
\[
V_{L \times D}^T,\eta (\tau^i)[\varepsilon] = \bigoplus_{\gamma \in \mathcal{C} \gamma_1 \cdots \gamma_L} V_{L^\otimes,\eta, i \gamma_1, \cdots, i \gamma_L} (\tau^i)[\varepsilon_1] \otimes \cdots \otimes V_{L^\otimes,\eta, i \gamma_1, \cdots, i \gamma_L} (\tau^i)[\varepsilon].
\]

Fix \(\varepsilon \in \mathbb{Z}^\ell\); the characters \(\text{ch } V_{L \times D}^T,\eta, i \gamma_1, \cdots, i \gamma_L (\tau^i)[\varepsilon_1] \otimes \cdots \otimes V_{L^\otimes,\eta, i \gamma_1, \cdots, i \gamma_L} (\tau^i)[\varepsilon]\) are all the same for any \((\gamma_1, \cdots, \gamma_L) \in \mathcal{D}\). Thus,
\[
\text{ch } V_{L \times D}^T,\eta (\tau^i)[\varepsilon] = \mathcal{D} \bigoplus_{e \in S_\varepsilon} \text{ch } V_{L \times D}^T,\eta, i \gamma_1, \cdots, i \gamma_L (\tau^i)[\varepsilon_1] \otimes \cdots \otimes V_{L^\otimes,\eta, i \gamma_1, \cdots, i \gamma_L} (\tau^i)[\varepsilon] = \mathcal{D} \text{ch } V_{L \times 0} (\tau^i)[\varepsilon].
\]

As before we have
\[
\text{qdim}_{V_{\mathcal{L} \times D}} V_{L \times D}^T,\eta (\tau^i)[\varepsilon] = \text{qdim}_{V_{\mathcal{L} \times 0}} V_{L \times 0} (\tau^i)[\varepsilon],
\]
which is \(2^\ell / |\mathcal{C}|\).

\section*{Global Dimension}
Let \(V\) be a VOA with only finitely many irreducible modules, the \textit{global dimension} of \(V\) \cite{DJX13} is defined as
\[
\text{glob}(V) := \sum_{M \in \text{Irr}(V)} \text{qdim}(M)^2.
\]

Assume \(G\) is a finite subgroup of \(\text{Aut}(V)\), it is conjectured in \cite{DJX13} that
\[
|G|^2 \text{glob}(V) = \text{glob}(V^G).
\]

We will verify this conjecture in our case, \textit{i.e.}, \(V = V_{L \times D}\) and \(G = \langle \tau \rangle\).
Since all irreducible \(V_{L \times D}\)-modules are simple current, we have
\[
\text{glob}(V_{L \times D}) = |\mathcal{C} / \mathcal{C}| \cdot |\mathcal{D} / \mathcal{D}| \cdot 1^2.
\]
The global dimension of \( V_{L,C,D}^\tau \) will be computed below. We count the number of irreducibles that have the same quantum dimensions.

(i) \( \text{qdim}_{V_{L,C,D}^\tau} V_{L,C}^{(\delta+D)}[\varepsilon] = 1 \). There are \( |D^\perp/D| \cdot 3 \) irreducible modules of this type.

(ii) \( \text{qdim}_{V_{L,C,D}^\tau} V_{L,(\lambda+C)\times(\delta+D)}[\varepsilon] = 3 \) if \( 0 \neq \lambda + C \in C_{\equiv \tau} \mod C \). There are \( |D^\perp/D| \cdot \frac{|C^\perp/C|-1}{3} \) irreducible modules of this type.

(iii) \( \text{qdim}_{V_{L,C,D}^\tau} V_{L,C}^{T,n}(\tau^i)[\varepsilon] = 2^i/C \). There are \( |D^\perp/D| \cdot 3 \cdot 2 \) irreducible modules of this type.

Note that \( (2^i/C)^2 = |C^\perp/C| \). Therefore,

\[
glob V_{L,C,D}^\tau = |D^\perp/D| \left( 3 + \frac{|C^\perp/C| - 1}{3} \cdot 3^2 + 6 |C^\perp/C| \right) = 9 |C^\perp/C| |D^\perp/D|.
\]

Hence we have \( \glob(V_{L,C,D}) \cdot 3^2 = \glob(V_{L,C,D}^\tau) \). This verified the conjecture of Dong, Jiao and Xu in this special case.

5. Fusion Rules

In this section, we compute the fusion rules of \( V_{L,C,D}^\tau \)-modules. The next three propositions are crucial to our calculations.

Proposition 5.1 ([TY13, Prop.4.5]). Let \( \varepsilon, \varepsilon_1, \varepsilon_2, j, j_1, j_2, k \in \mathbb{Z}_3 \) and \( i = 1, 2 \). Then

(i) \( V_{L,(0,1)}[\varepsilon] \times V_{L,(0,j)}[\varepsilon_2] = V_{L,(0,j_1+j_2)}[\varepsilon_1 + \varepsilon_2] \);

(ii) \( V_{L,(0,1)}[\varepsilon] \times V_{L,(c,j_2)} = V_{L,(c,j_1+j_2)} \);

(iii) \( V_{L,(0,c)} \times V_{L,(c,j_2)} = \sum_{p=0}^{2} V_{L,(0,j_1+j_2)}[\rho] + 2V_{L,(c,j_1+j_2)} \);

(iv) \( V_{L,(0,\varepsilon)}[\varepsilon_1] \times V_{L}^{T,k}(\tau^i)[\varepsilon_2] = V_{L}^{T,k-i}(\tau^i)[\varepsilon_1 + \varepsilon_2] \);

(v) \( V_{L,(c,j)} \times V_{L}^{T,k}(\tau^i)[\varepsilon] = \sum_{p=0}^{2} V_{L}^{T,k-i}(\tau^i)[\rho] \).

Proposition 5.2. [Che13] We have the following fusion rules among irreducible \( V_{L}^\tau \)-modules of twisted type.

(i) \( V_{L}^{T,j}(\tau^i)[\varepsilon] \times V_{L}^{T,j}(\tau^i)[\varepsilon'] = V_{L}^{T,-(i+j)}(\tau^{2i})[\varepsilon + \varepsilon'] + V_{L}^{T,-(i+j)}(\tau^{2i})[2 - (\varepsilon + \varepsilon')] \);

(ii) \( V_{L}^{T,j}(\tau^i)[\varepsilon] \times V_{L}^{T,j}(\tau^i)[\varepsilon'] = V_{L,(0,i+j)}[\varepsilon + 2\varepsilon'] + V_{L,(c,i+j)} \),

where \( l \in \{1, 2\} \), \( i, j, \varepsilon, \varepsilon' \in \{0, 1, 2\} \).

Proposition 5.3 ([ADL05, Prop.2.9]). Let \( V \) be a vertex operator algebra and let \( M^1 \), \( M^2 \), \( M^3 \) be \( V \)-modules among which \( M^1 \) and \( M^2 \) are irreducible. Suppose that \( U \) is a vertex operator subalgebra of \( V \) (with the same Virasoro element) and that \( N^1 \) and \( N^2 \) are irreducible \( U \)-submodules of \( M^1 \) and \( M^2 \), respectively. Then the restriction map from \( I_V(M^3_{M^1, M^2}) \) to \( I_U(M^3_{N^1, N^2}) \) is injective. In particular,

\[
\dim I_V(M^3_{M^1, M^2}) \leq \dim I_U(M^3_{N^1, N^2}). \tag{5-1}
\]
In our case, we consider the following chain of subVOAs:

\[ V_{L_{\mathcal{C} \times \mathcal{D}}} \supset V_{L_{\mathcal{C} \times \mathcal{D}}}^T \supset V_{L_{\mathcal{C} \times \mathcal{D}}}^T \supset (V_L^T)^{\otimes \ell}. \]

For simplicity, we denote

\[ N_{\mathcal{C} \times \mathcal{D}} \left( \begin{array}{c} - \\ - \\ - \end{array} \right) = \dim I_{V_{L_{\mathcal{C} \times \mathcal{D}}}^T} \left( \begin{array}{c} - \\ - \\ - \end{array} \right), \quad N_{\mathcal{C} \times \mathcal{D}}^T \left( \begin{array}{c} - \\ - \\ - \end{array} \right) = \dim I_{V_{L_{\mathcal{C} \times \mathcal{D}}}^T} \left( \begin{array}{c} - \\ - \\ - \end{array} \right), \]

\[ N_{\otimes} \left( \begin{array}{c} - \\ - \\ - \end{array} \right) = \dim I_{(V_L^T)^{\otimes \ell}} \left( \begin{array}{c} - \\ - \\ - \end{array} \right), \quad N_{\otimes}^T \left( \begin{array}{c} - \\ - \\ - \end{array} \right) = \dim I_{V_L^T} \left( \begin{array}{c} - \\ - \\ - \end{array} \right). \]

The basic idea is to use Proposition 5.3 and the quantum dimensions of \( V_L^T \)-modules to show that many fusion coefficients are zero. This gives some inequalities on fusion rules. Next by using quantum dimensions, we show that these inequalities are actually equalities.

Let \( \lambda + \mathcal{C}, \lambda^1 + \mathcal{C}, \lambda^2 + \mathcal{C} \in \mathcal{C}^+/\mathcal{C}, \delta + \mathcal{D}, \delta^1 + \mathcal{D}, \delta^2 + \mathcal{D} \in \mathcal{D}^+/\mathcal{D}, \eta, \eta^1, \eta^2 \in \mathcal{D}^1 \mod \mathcal{D} \) and \( \varepsilon, \varepsilon^1, \varepsilon^2 \in \mathbb{Z}_3 \). We will compute fusion rules separately in the following cases:

(I) Fusion rules of the form \( V_{L_{\mathcal{C} \times \mathcal{D}}} [\epsilon] \times M \) for any irreducible module \( M \) (see Prop. 5.4);

(II) Fusion rules of the form \( V_{L_{\mathcal{C} \times \mathcal{D}}} \left( \begin{array}{c} \lambda \\ \lambda^1 \\ \lambda^2 \end{array} \right) \times V_{L_{\mathcal{C} \times \mathcal{D}}} \left( \begin{array}{c} \lambda + \mathcal{C} \\ \lambda^1 + \mathcal{C} \\ \lambda^2 + \mathcal{C} \end{array} \right) \) (see Prop. 5.6);

(III) Fusion rules of the form \( V_{L_{\mathcal{C} \times \mathcal{D}}} \left( \begin{array}{c} \lambda + \mathcal{C} \end{array} \right) \times V_{T_{\mathcal{C} \times \mathcal{D}}} \left( \begin{array}{c} \tau^1 \end{array} \right) \) (see Prop. 5.8);

(IV) Fusion rules of the form \( V_{T_{\mathcal{C} \times \mathcal{D}}} \left( \begin{array}{c} \tau^1 \end{array} \right) \times V_{T_{\mathcal{C} \times \mathcal{D}}} \left( \begin{array}{c} \tau^2 \end{array} \right) \) (see Prop. 5.9);

(V) Fusion rules of the form \( V_{T_{\mathcal{C} \times \mathcal{D}}} \left( \begin{array}{c} \tau^1 \end{array} \right) \times V_{T_{\mathcal{C} \times \mathcal{D}}} \left( \begin{array}{c} \tau^2 \end{array} \right) \). In this case, we first determine the fusion coefficients up to a permutation in Prop. 5.10. Then we use modular invariance property of trace functions to get an explicit result in Prop. 5.17.

We start with Case (I).

**Proposition 5.4.** We have the following fusion rules.

(i) \( V_{L_{\mathcal{C} \times \mathcal{D}}} [\epsilon^1] \times V_{L_{\mathcal{C} \times \mathcal{D}}} [\varepsilon^2] = V_{L_{\mathcal{C} \times \mathcal{D}}} [\epsilon^1 + \varepsilon^2]; \) \hspace{1cm} (5-2a)

(ii) \( V_{L_{\mathcal{C} \times \mathcal{D}}} [\epsilon^1] \times V_{L_{\mathcal{C} \times \mathcal{D}}} \left( \begin{array}{c} \lambda + \mathcal{C} \end{array} \right) \) \hspace{1cm} (5-2b)

(iii) \( V_{L_{\mathcal{C} \times \mathcal{D}}} [\epsilon^1] \times V_{L_{\mathcal{C} \times \mathcal{D}}} \left( \begin{array}{c} \tau^1 \end{array} \right) \) \hspace{1cm} (5-2c)

where \( \delta^1 + \mathcal{D}, \delta^2 + \mathcal{D} \in \mathcal{D}^+/\mathcal{D}, 0 \neq \lambda + \mathcal{C} \in \mathcal{C}^1 \ mod \mathcal{C} \) and \( \varepsilon^1, \varepsilon^2 \in \mathbb{Z}_3 \).

**Proof.** (i) Observe that \( \dim (V_{L_{\mathcal{C} \times \mathcal{D}}} [\epsilon^1] \times V_{L_{\mathcal{C} \times \mathcal{D}}} [\varepsilon^2]) = 1 \); therefore the fusion product \( V_{L_{\mathcal{C} \times \mathcal{D}}} [\epsilon^1] \times V_{L_{\mathcal{C} \times \mathcal{D}}} [\varepsilon^2] \) is irreducible.

Recall the fusion rules of \( V_{L_{\mathcal{C} \times \mathcal{D}}} \)-modules:

\[ 1 = N_{\mathcal{C} \times \mathcal{D}} \left( V_{L_{\mathcal{C} \times \mathcal{D}}} \left( \begin{array}{c} \delta^1 + \mathcal{D} \\ \delta^2 + \mathcal{D} \end{array} \right) \right). \]
By restricting to $V_{L \times D}^\tau$-modules and using Prop. 5.3, we have
\[
1 \leq N_{C \times D}^\tau \left( \frac{V_{L \times (\delta^1 + \delta^2 + D)} \otimes V_{L \times (\delta^1 + \delta^2 + D)} \otimes \cdots \otimes V_{L \times (\delta^1 + \delta^2 + D)}}{N_{L \times (\delta^1 + \delta^2 + D)} \otimes N_{L \times (\delta^1 + \delta^2 + D)} \otimes \cdots \otimes N_{L \times (\delta^1 + \delta^2 + D)}} \right).
\]

Therefore, we know
\[
V_{L \times (\delta^1 + D)} \otimes V_{L \times (\delta^2 + D)} \otimes \cdots \otimes V_{L \times (\delta^1 + \delta^2 + D)} \subseteq V_{L \times (\delta^1 + \delta^2 + D)} \otimes \cdots \otimes V_{L \times (\delta^1 + \delta^2 + D)},
\]
for some $\varepsilon \in \mathbb{Z}_3$. For simplicity, we let $M^i := V_{L \times (\delta^1 + D)} \otimes \cdots \otimes V_{L \times (\delta^1 + \delta^2 + D)}$ and $M := V_{L \times (\delta^1 + \delta^2 + D)} \otimes \cdots \otimes V_{L \times (\delta^1 + \delta^2 + D)}$.

Recall the decompositions
\[
V_{L \times (\delta + D)} \otimes V_{L \times (\delta + D)} \otimes \cdots \otimes V_{L \times (\delta + D)} \otimes V_{L \times (\delta + D)} = \bigoplus_{0 \neq \gamma \in C_{\mathbb{F}_r}} V_{L \times (\delta + D)}; \quad \bigoplus_{0 \neq \gamma \in C_{\mathbb{F}_r}} V_{L \times (\delta + D)}.
\]

Now fix an irreducible $(V_L^\tau)^{i, \delta}$-submodule
\[
M^i \subseteq V_{N \times (\delta^1 + \delta^2 + D)} \otimes \cdots \otimes V_{N \times (\delta^1 + \delta^2 + D)} \subseteq M \subseteq V_{N \times (\delta^1 + \delta^2 + D)}.
\]

for some $\varepsilon \in \mathbb{Z}_3$. Since
\[
M := V_{L \times (\delta^1 + \delta^2 + D)} \otimes \cdots \otimes V_{L \times (\delta^1 + \delta^2 + D)} = \bigoplus_{0 \neq \gamma \in C_{\mathbb{F}_r}} V_{L \times (\delta + D)},
\]
we have the fusion coefficient
\[
1 = N_{C \times D}^\tau \left( \frac{M}{M_1, M_2} \right) \leq N_{c} \left( \frac{V_{L \times (\delta^1 + \delta^2 + D)} \otimes \cdots \otimes V_{L \times (\delta^1 + \delta^2 + D)} \otimes \cdots \otimes V_{L \times (\delta^1 + \delta^2 + D)}}{N_{1, N_2}} \right) + \sum_{0 \neq \gamma \in C_{\mathbb{F}_r}} N_{c} \left( \frac{V_{L \times (\delta + D)} \otimes \cdots \otimes V_{L \times (\delta + D)} \otimes \cdots \otimes V_{L \times (\delta + D)}}{N_{1, N_2}} \right).
\]

We claim that
\[
N_{c} \left( \frac{V_{L \times (\delta + D)} \otimes \cdots \otimes V_{L \times (\delta + D)}}{N_{1, N_2}} \right) = 0 \quad \text{for all } 0 \neq \gamma \in C_{\mathbb{F}_r}.
\]

We have
\[
N_{c} \left( \frac{V_{L \times (\delta + D)} \otimes \cdots \otimes V_{L \times (\delta + D)}}{N_{1, N_2}} \right) = \sum_{\Delta \in D} \prod_{k=1}^{\ell} N_{c} \left( \frac{V_{L \times (\delta + D)} \otimes \cdots \otimes V_{L \times (\delta + D)}}{N_{1, N_2}} \right).
\]

Now, since $\gamma \neq 0$ we have $\gamma_k \neq 0$ for some $1 \leq h \leq \ell$ and hence
\[
N_{c} \left( \frac{V_{L \times (\delta + D)} \otimes \cdots \otimes V_{L \times (\delta + D)}}{N_{1, N_2}} \right) = 0.
\]

This proves our claim and equation (5-4) becomes
\[
1 \leq N_{c} \left( \frac{V_{L \times (\delta^1 + \delta^2 + D)} \otimes \cdots \otimes V_{L \times (\delta^1 + \delta^2 + D)}}{N_{1, N_2}} \right).
\]
Now, we set \((e_i^1, \cdots, e_i^2) = (\varepsilon^1, 0, \cdots, 0)\) for \(i = 1, 2\). Then we have

\[
1 \leq N_{\otimes}\left( \frac{V_{L_{0}\otimes(\delta^1+\delta^2+D)}[\varepsilon]}{N_1, N_2} \right) = \sum_{\Delta \in D} N_{\otimes}\left( \frac{V_{L_{0}\otimes(\delta^1+\delta^2+\Delta)}[\varepsilon]}{N_1, N_2} \right)
\]

\[
= \sum_{\Delta \in D} N_{\otimes}\left( \sum_{r \in S_r} V_{L_{[0,\delta_1]}^1}[r_1] \otimes \cdots \otimes V_{L_{[0,\delta_1]}^2}[r_2] \right)
\]

\[
= \sum_{r \in S_r} \left( N_{\otimes}^T \left( V_{L_{[0,\delta_1]}^1}[r_1], V_{L_{[0,\delta_1]}^2}[\varepsilon^2] \right) \prod_{k=2}^{\ell} N_{\otimes}^T \left( V_{L_{[0,\delta_1]}^1}[0], V_{L_{[0,\delta_1]}^2}[0] \right) \right).
\]

By Prop. 5.1 we know if \((r_2, \cdots, r_\ell) \neq (0, \cdots, 0)\) then

\[
N_{\otimes}^T \left( V_{L_{[0,\delta_1]}^1}[r_1], V_{L_{[0,\delta_1]}^2}[\varepsilon^2] \right) \prod_{k=2}^{\ell} N_{\otimes}^T \left( V_{L_{[0,\delta_1]}^1}[0], V_{L_{[0,\delta_1]}^2}[0] \right) = 0.
\]

Thus only \(r = (r_1, 0, \cdots, 0) \in S_\varepsilon\) contributes a nonzero summand. Therefore

\[
1 \leq N_{\otimes}\left( \frac{V_{L_{0}\otimes(\delta^1+\delta^2+D)}[\varepsilon]}{N_1, N_2} \right) = \sum_{\Delta \in D} \left( N_{\otimes}^T \left( V_{L_{[0,\delta_1]}^1}[r_1], V_{L_{[0,\delta_1]}^2}[\varepsilon^2] \right) \prod_{k=2}^{\ell} N_{\otimes}^T \left( V_{L_{[0,\delta_1]}^1}[0], V_{L_{[0,\delta_1]}^2}[0] \right) \right).
\]

Since \(r \in S_\varepsilon\), we must have \(r_1 = \varepsilon = \varepsilon^1 + \varepsilon^2\). This proves (i).

(ii) We know the fusion coefficient of \(V_{L_{C\times D}}\)-modules:

\[
1 = N_{C\times D} \left( \frac{V_{L_{(\lambda+C)\times(\delta^1+\delta^2+D)}}}{V_{L_{C\times(\delta^1+\delta^2+D)}}, V_{L_{(\lambda+C)\times(\delta^1+\delta^2+D)}}} \right).
\]

By restricting to \(V_{L_{C\times D}}\)-modules, we have

\[
1 \leq N_{C\times D}^T \left( \frac{V_{L_{(\lambda+C)\times(\delta^1+\delta^2+D)}}}{V_{L_{C\times(\delta^1+\delta^2+D)}}, V_{L_{(\lambda+C)\times(\delta^1+\delta^2+D)}}} \right).
\]

Since \(\text{qdim} V_{L_{(\lambda+C)\times(\delta^1+\delta^2+D)}} = \text{qdim} (V_{L_{C\times(\delta^1+\delta^2+D)}}[\varepsilon^1] \times V_{L_{(\lambda+C)\times(\delta^1+\delta^2+D)}})\), we prove (ii).

(iii) Since \(V_{L_{C\times(\delta^1+\delta^2+D)}}[\varepsilon^1]\) is simple current and \(\text{qdim} (V_{L_{C\times(\delta^1+\delta^2+D)}}[\varepsilon^1] \times V_{L_{C\times D}}[\tau^j][\varepsilon^2]) = \frac{2^j}{|C|}\), we know the fusion product \(V_{L_{C\times(\delta^1+\delta^2+D)}}[\varepsilon^1] \times V_{L_{C\times D}}[\tau^j][\varepsilon^2]\) is either \(V_{L_{C\times D}}[\tau^j][\varepsilon]\) for some \(\delta + C \in C \perp /C, \varepsilon \in \mathbb{Z}_3\) and \(j = 1, 2\) or \(V_{L_{C\times(\delta^1+\delta^2+D)}}[\varepsilon]\) if \(|C| = 2^\ell\).

Assume

\[
V_{L_{C\times(\delta^1+\delta^2+D)}}[\varepsilon^1] \times V_{L_{C\times D}}[\tau^j][\varepsilon^2] = V_{L_{C\times(\delta^1+\delta^2+D)}}[\varepsilon].
\]

Then we have

\[
V_{L_{C\times(-\delta^1+\delta^2+D)}}[-\varepsilon^1] \times V_{L_{C\times(\delta^1+\delta^2+D)}}[\varepsilon^1] \times V_{L_{C\times D}}[\tau^j][\varepsilon^2] = V_{L_{C\times(-\delta^1+\delta^2+D)}}[-\varepsilon^1] \times V_{L_{C\times(\delta^1+\delta^2+D)}}[\varepsilon],
\]

\[
V_{L_{C\times(-\delta^1+\delta^2+D)}}[-\varepsilon^1] \times V_{L_{C\times(\delta^1+\delta^2+D)}}[\varepsilon] \times V_{L_{C\times D}}[\tau^j][\varepsilon]\]
and hence by \((5-2a)\)
\[
V_{L' \times D}^{T, \delta^2}(\tau^i)[\varepsilon^2] = V_{L' \times D}^{T, \delta^2}(\tau^i)[\varepsilon^2] = V_{L' \times (\delta - \delta^1 + D)}[\varepsilon - \varepsilon^1],
\]
a contradiction. Therefore,
\[
V_{L' \times (\delta - \delta^1 + D)}[\varepsilon^1] \times V_{L' \times D}^{T, \delta^2}(\tau^i)[\varepsilon^2] = V_{L' \times D}^{T, \delta^3}(\tau^j)[\varepsilon^3],
\]
for some \(j=1,2,\delta + h \in \mathcal{C} / \mathcal{C}\), \(\varepsilon \in \mathbb{Z}_3\), for \(h=1,2,3\).

Similar to (i), we pick the following irreducible \((V_L^{T}) \otimes \ell\)-modules
\[
V_{L^{(0, \delta^i)}_1}[\varepsilon^1] \otimes \cdots \otimes V_{L^{(0, \delta^i)}_1}[\varepsilon^j] \subset V_{L \times (\delta^1 + D)}[\varepsilon^1];
\]
\[
V_{L}^{T, \delta^2}(\tau^i)[\varepsilon^1] \otimes \cdots \otimes V_{L}^{T, \delta^2}(\tau^i)[\varepsilon^j] \subset V_{L' \times D}^{T, \delta^3}(\tau^j)[\varepsilon^j];
\]
of \(M^i\) for some \(\varepsilon^h := (\varepsilon^1, \cdots, \varepsilon^h) \in S_{e^h}, h=1,2,3\).

Prop. 5.3 suggests that
\[
1 = N_{L' \times D}^T \left( V_{L' \times (\delta^1 + D)}^{T, \delta^3}(\tau^j)[\varepsilon^3], V_{L' \times D}^{T, \delta^2}(\tau^j)[\varepsilon^2] \right)
\]
\[
\leq N_0^T \left( \sum_{\varepsilon^3 \in S_{e^3}} V_{L^{(0, \delta^i)}_1}[\varepsilon^1] \otimes \cdots \otimes V_{L^{(0, \delta^i)}_1}[\varepsilon^j], V_{L}^{T, \delta^2}(\tau^i)[\varepsilon^1] \otimes \cdots \otimes V_{L}^{T, \delta^2}(\tau^i)[\varepsilon^j] \right)
\]
\[
= \sum_{\varepsilon^3 \in S_{e^3}} \prod_{k=1}^{\ell} N_0^T \left( V_{L^{(0, \delta^i)}_1}[\varepsilon^1] \otimes \cdots \otimes V_{L^{(0, \delta^i)}_1}[\varepsilon^j], V_{L}^{T, \delta^2}(\tau^i)[\varepsilon^1] \otimes \cdots \otimes V_{L}^{T, \delta^2}(\tau^i)[\varepsilon^j] \right).
\]
If \(j \neq i\), then Prop. 5.1 gives \(1 \leq \ell\), a contradiction. Therefore \(j = i\). If there exists \(1 \leq k \leq \ell\) such that \(\delta^3_k \neq \delta^2_k - i\delta^1_k\) or \(\varepsilon_3^i \neq i\varepsilon_1^k + \varepsilon_2^k\), again Prop. 5.1 gives \(1 \leq \ell\), a contradiction.

Therefore, we must have \(\delta^3_k = \delta^2_k - i\delta^1_k\) and \(\varepsilon_3^i = i\varepsilon_1^k + \varepsilon_2^k\) for all \(k\). This gives \(\delta^3 = \delta^2 - i\delta^1\) and \(\varepsilon_3 \equiv \sum_{k=1}^{\ell} \varepsilon_3^i = \sum_{k=1}^{\ell} i\varepsilon_1^k + \varepsilon_2^k \equiv i\varepsilon + \varepsilon^2 \mod 3\). This completes the proof.

Using the above proposition, we can find the contragredient dual of irreducible modules. Recall there are natural isomorphisms between the following fusion rules:

\[
N \left( \begin{array}{c} C \\ A \\ B \end{array} \right) = N \left( \begin{array}{c} C \\ B \\ A \end{array} \right) = N \left( \begin{array}{c} B' \\ A \\ C' \end{array} \right),
\]
for every \(V\)-modules \(A, B\) and \(C\).

**Proposition 5.5.** The contragredient dual of irreducible \(V_{L' \times D}^T\)-modules is listed below.

(i) \((V_{L' \times (\delta + D)}[\varepsilon])') = V_{L' \times (-\delta + D)}[-\varepsilon]);

(ii) \((V_{L \times (\delta + D)})[\varepsilon']) = V_{L \times (-\delta + D)} = V_{L \times (-\delta + D)};

(iii) \((V_{L' \times D}^{T, \eta}(\tau^i)[\varepsilon])') = V_{L' \times D}^{T, \eta}(\tau^{2i})[\varepsilon].\)
Proof. It is discussed in Prop. 3.8 that \( (V_{L_{\mathcal{C} \times \mathcal{D}}}(0))' \cong V_{L_{\mathcal{C} \times \mathcal{D}}}[0] \) is self-dual. We know the fusion rule:

\[
1 = N_{\mathcal{C} \times \mathcal{D}}^T \left( \frac{V_{L_{\mathcal{C} \times (\delta' + \mathcal{D})}}[\varepsilon]}{V_{L_{\mathcal{C} \times \mathcal{D}}}[0], V_{L_{\mathcal{C} \times (\delta' + \mathcal{D})}}[\varepsilon]} \right) = N_{\mathcal{C} \times \mathcal{D}}^T \left( \frac{(V_{L_{\mathcal{C} \times (\delta' + \mathcal{D})}}[\varepsilon])'}{(V_{L_{\mathcal{C} \times (\delta' + \mathcal{D})}}[\varepsilon])', V_{L_{\mathcal{C} \times (\delta' + \mathcal{D})}}[\varepsilon]} \right).
\]

Since \( \text{qdim } M = \text{qdim } M' \) for any module \( M \), by Prop. 5.4 we may assume \( (V_{L_{\mathcal{C} \times (\delta' + \mathcal{D})}}[\varepsilon])' \cong V_{L_{\mathcal{C} \times (\delta' + \mathcal{D})}}[\varepsilon'] \) for some \( \delta', \varepsilon' \). Now using equation (5-2a) we must have

\[
(V_{L_{\mathcal{C} \times (\delta + \mathcal{D})}}[\varepsilon])' = V_{L_{\mathcal{C} \times (\delta + \mathcal{D})}}[-\varepsilon].
\]

This proves (i). Similarly, using equation (5-2b) we have (ii).

(iii) We take a different approach. We first consider the contragredient dual of the irreducible \( V_L^T \)-modules of twisted type. We know \( V_L^T \) is self-dual. Let \( i, \varepsilon \in \mathbb{Z}_3 \), then

\[
1 = N_{\mathcal{C}}^\tau \left( \frac{V_{L_{\mathcal{C} \times \mathcal{D}}}(\tau^i)[\varepsilon]}{V_L, V_{L_{\mathcal{C} \times \mathcal{D}}}(\tau^i)[\varepsilon]} \right) = N_{\mathcal{C}}^\tau \left( \frac{V_{L_{\mathcal{C} \times \mathcal{D}}}(\tau^i)[\varepsilon]}{V_L, (V_{L_{\mathcal{C} \times \mathcal{D}}}(\tau^i)[\varepsilon])'} \right).
\]

By quantum dimensions and fusion rules of \( V_L^T \)-modules, we must have

\[
(V_{L_{\mathcal{C} \times \mathcal{D}}}(\tau^i)[\varepsilon])' = V_{L_{\mathcal{C} \times \mathcal{D}}}(\tau^{2i})[\varepsilon].
\]

Now, consider the decomposition of \( (V_L^T) \otimes \ell \)-modules:

\[
V_{L_{\mathcal{C} \times \mathcal{D}}}(\tau^i)[\varepsilon] \cong \bigoplus_{\delta \in \mathcal{D}} \bigoplus_{\ell \in \mathbb{Z}_3} V_{L_{\mathcal{C} \times \mathcal{D}}}(\tau^i)[\varepsilon] \otimes V_{L_{\mathcal{C} \times \mathcal{D}}}(\tau^{2i})[\varepsilon] = V_{L_{\mathcal{C} \times \mathcal{D}}}(\tau^{2i})[\varepsilon].
\]

Taking contragredient dual as \( (V_L^T) \otimes \ell \)-modules, we have

\[
(V_{L_{\mathcal{C} \times \mathcal{D}}}(\tau^i)[\varepsilon])' \cong \bigoplus_{\delta \in \mathcal{D}} \bigoplus_{\ell \in \mathbb{Z}_3} V_{L_{\mathcal{C} \times \mathcal{D}}}(\tau^i)[\varepsilon] \otimes V_{L_{\mathcal{C} \times \mathcal{D}}}(\tau^{2i})[\varepsilon] = V_{L_{\mathcal{C} \times \mathcal{D}}}(\tau^{2i})[\varepsilon].
\]

Case (II): \( V_{L_{(\lambda + \mathcal{C}) \times (\delta + \mathcal{D})}} \times V_{L_{(\lambda + 2\mathcal{C}) \times (\delta + 2\mathcal{D})}} \)

Proposition 5.6. We have the fusion rules

\[
V_{L_{(\lambda + \mathcal{C}) \times (\delta + \mathcal{D})}} \times V_{L_{(\lambda + 2\mathcal{C}) \times (\delta + 2\mathcal{D})}} = \bigoplus_{h=0}^{2} V_{L_{(\lambda + h\lambda \mathcal{C} \times (\delta + h\delta \mathcal{D}}).}
\]

Proof. Fix \( 0 \leq h \leq 2 \) we have the fusion rules of \( V_{L_{\mathcal{C} \times \mathcal{D}}} \)-modules:

\[
1 = N_{\mathcal{C} \times \mathcal{D}} \left( \frac{V_{L_{(\lambda + h\lambda \mathcal{C}) \times (\delta + h\delta \mathcal{D})}}}{V_{L_{(\lambda + \mathcal{C}) \times (\delta + \mathcal{D})}}, V_{L_{(\lambda + 2\mathcal{C}) \times (\delta + 2\mathcal{D})}} \right) \leq N_{\mathcal{C} \times \mathcal{D}} \left( \frac{V_{L_{(\lambda + h\lambda \mathcal{C}) \times (\delta + h\delta \mathcal{D})}}}{V_{L_{(\lambda + \mathcal{C}) \times (\delta + \mathcal{D})}}, V_{L_{(\lambda + 2\mathcal{C}) \times (\delta + 2\mathcal{D})}} \right).
\]
Since $\omega^h\lambda^2 + C$, $0 \leq h \leq 2$, are identical in $C_{\equiv \tau} \mod C$, there is an isomorphism of $V_{L_{C \times D}}^\tau$-modules
\[ V_{L(\lambda^2 + C) \times (\delta^2 + D)} \cong V_{L(\omega^h\lambda^2 + C) \times (\delta^2 + D)} \cong V_{L(\omega^2\lambda^2 + C) \times (\delta^2 + D)}. \]
Therefore, we can write
\[ 1 \leq N_{C \times D}^\tau \left( \begin{array}{c}
V_{L(\lambda^2 + C) \times (\delta^2 + D)}, \\
V_{L(\lambda^2 + C) \times (\delta^2 + D)}
\end{array} \right), \]
for all $h$. Since $\lambda^1 + \omega^h\lambda^2 + C$, $0 \leq h \leq 2$, are distinct in $C_{\equiv \tau} \mod C$, by counting quantum dimensions, we prove
\[ V_{L(\lambda^1 + C) \times (\delta^1 + D)} \times V_{L(\lambda^2 + C) \times (\delta^2 + D)} = \bigoplus_{h=0}^2 V_{L(\lambda^1 + \omega^h\lambda^2 + C) \times (\delta^1 + \delta^2 + D)}. \]
This completes the proof. \hfill \Box

Remark 5.7. Note that if $\lambda^1 + \omega^h\lambda^2 = 0$ for some $h$, then the module $V_{L(\lambda^1 + \omega^h\lambda^2 + C) \times (\delta^1 + \delta^2 + D)}$ is not irreducible and admits a decomposition of irreducible modules of $V_{L_{C \times D}}^\tau$-modules:
\[ V_{L(\lambda^1 + \omega^h\lambda^2 + C) \times (\delta^1 + \delta^2 + D)} = \sum_{\varepsilon=0}^2 V_{L(\lambda^1 + \omega^h\lambda^2 + C) \times (\delta^1 + \delta^2 + D)}[\varepsilon]. \]

Case (III): $V_{L(\gamma + C) \times (\delta^1 + D)} \times V_{L_{C \times D}^T}^{(\tau^i)}[\varepsilon]$

Proposition 5.8. We have
\[ V_{L(\gamma + C) \times (\delta^1 + D)} \times V_{L_{C \times D}^T}^{(\tau^i)}[0] = \bigoplus_{\rho=0}^{2} V_{L_{C \times D}^T}^{(\tau^i)}[\rho]; \] (5.7a)
\[ V_{L(\gamma + C) \times (\delta^1 + D)} \times V_{L_{C \times D}^T}^{(\tau^i)}[\varepsilon] = \bigoplus_{\rho=0}^{2} V_{L_{C \times D}^T}^{(\tau^i)}[\rho]; \] (5.7b)
where $\delta^1 + D, \delta^2 + D \in D^\perp / D, 0 \neq \lambda + C \in C_{\equiv \tau} \mod C$.

Proof. (i) Similar to Prop. 5.4(iii), we quickly have
\[ 0 = N_{C \times D}^\tau \left( \begin{array}{c}
V_{L(\gamma + C) \times D}^{T,\delta} (\tau^j) \\
V_{L(\gamma + C) \times D}^{T,0} (\tau^i)[0]
\end{array} \right), \]
when (1) $i = j$ and $\delta \neq 0$ or (2) $i \neq j$. Also, by Prop. 5.5, Prop. 5.4 and Prop. 5.6 we have
\[ N_{C \times D}^\tau \left( \begin{array}{c}
V_{L(\gamma + C) \times D}, \\
V_{L_{C \times D}^T}^{(\tau^i)}[\varepsilon]
\end{array} \right) = N_{C \times D}^\tau \left( \begin{array}{c}
V_{L(\gamma + C) \times D}, \\
V_{L_{C \times D}^T}^{(\tau^i)}[\varepsilon]
\end{array} \right) = 0, \]
\[ N_{C \times D}^\tau \left( \begin{array}{c}
V_{L_{C \times D}^T}^{(\tau^i)}[\varepsilon] \\
V_{L_{C \times D}^T}^{(\tau^i)}[\varepsilon]
\end{array} \right) = N_{C \times D}^\tau \left( \begin{array}{c}
V_{L_{C \times D}^T}^{(\tau^i)}[\varepsilon] \\
V_{L_{C \times D}^T}^{(\tau^i)}[\varepsilon]
\end{array} \right) = 0. \]
Therefore

\[ V_{L(\gamma+c) \times D} \times V_{L_c \times D}^{T,0}(\tau^i)[0] = \bigoplus_{\rho=0}^{2} n_{\rho} V_{L_c \times D}^{T,0}(\tau^i)[\rho], \quad (5-8) \]

for some \( n_{\rho} \in \mathbb{N} \). Multiply the equation (5-8) by \( V_{L_c \times D}[h], h = 1, 2 \), we have

\[ (V_{L_c \times D}[h] \times V_{L(\gamma+c) \times D}) \times V_{L_c \times D}^{T,0}(\tau^i)[0] = V_{L_c \times D}[h] \times \bigoplus_{\rho=0}^{2} n_{\rho} V_{L_c \times D}^{T,0}(\tau^i)[\rho]. \]

By Prop. 5.4, the left hand side is equal to

\[ V_{L(\gamma+c) \times D} \times V_{L_c \times D}^{T,0}(\tau^i)[0] = \bigoplus_{\rho=0}^{2} n_{\rho} V_{L_c \times D}^{T,0}(\tau^i)[\rho] \]

while the right hand side is \( \bigoplus_{\rho=0}^{2} n_{\rho} V_{L_c \times D}^{T,0}(\tau^i)[\rho + h] \); thus, we have

\[ \bigoplus_{\rho=0}^{2} n_{\rho} V_{L_c \times D}^{T,0}(\tau^i)[\rho] = \bigoplus_{\rho=0}^{2} n_{\rho} V_{L_c \times D}^{T,0}(\tau^i)[\rho + h], \]

for all \( 0 \leq h \leq 2 \). This gives \( n_0 = n_1 = n_2 \). Finally, by comparing the quantum dimensions of both sides of (5-8), we have \( 3(2^\ell/|\mathcal{C}|) = (n_0 + n_1 + n_2)(2^\ell/|\mathcal{C}|) \) and hence \( n_0 = n_1 = n_2 = 1 \). This proves (i).

(ii) By Prop. 5.4 we have

\[ V_{L_c \times D}^{T,\delta^2}(\tau^i)[\varepsilon] = V_{L_c \times (\delta^{-1}+D)}^{T,\delta^2}((-1)^{i+1}\varepsilon] \times V_{L_c \times D}^{T,0}(\tau^i)[0]. \]

Therefore,

\[ V_{L(\gamma+c) \times (\delta^{-1}+D)} \times V_{L_c \times D}^{T,\delta^2}(\tau^i)[\varepsilon] = V_{L(\gamma+c) \times (\delta^{-1}+D)} \times (V_{L_c \times ((-1)^{\delta}+D)}^{T,\delta^2}((-1)^{i+1}\varepsilon] \times V_{L_c \times D}^{T,0}(\tau^i)[0]) \]

\[ = V_{L_c \times ((-1)^{\delta}+D)}((-1)^{i+1}\varepsilon] \times (V_{L_c \times (\gamma+c) \times (\delta^{-1}+D)}^{T,\delta^2}(\tau^i)[0]) \]

\[ = V_{L_c \times ((-1)^{\delta}+D)}((-1)^{i+1}\varepsilon] \times \bigoplus_{\rho=0}^{2} V_{L_c \times D}^{T,0}(\tau^i)[\rho] \]

\[ = \bigoplus_{\rho=0}^{2} V_{L_c \times D}^{T,-(\delta^{-1})}(\tau^i)[(-1)^{i+1}\varepsilon + \rho] = \bigoplus_{\rho=0}^{2} V_{L_c \times D}^{T,\delta^2}(\tau^i)[\rho - \varepsilon] = \bigoplus_{\rho=0}^{2} V_{L_c \times D}^{T,\delta^2}(\tau^i)[\rho]. \]

This completes the proof. \( \square \)

**Case (IV):** \( V_{L_c \times D}^{T,\eta^1}(\tau^i)[\varepsilon] \times V_{L_c \times D}^{T,\eta^2}(\tau^i)[\bar{\varepsilon}] \)
Proposition 5.9. We have the fusion rules:

\[(i) \quad V_{L_{\tau}^C}(\tau)[0] \times V_{L_{\tau}^C}(\tau^2)[0] = V_{L_{(\gamma+C)\times D}}[0] \oplus \bigoplus_{0 \neq \gamma \in \mathcal{C}_{\equiv r}^\perp \text{mod } \mathcal{C}} V_{L(\gamma+C)\times D} \quad (5-9a)\]

\[(ii) \quad V_{L_{\tau}^C}(\tau)[\varepsilon^1] \times V_{L_{\tau}^C}(\tau^2)[\varepsilon^2] = V_{L_{(\gamma-C)\times D}}[\varepsilon^1 - \varepsilon^2] \oplus \bigoplus_{0 \neq \gamma \in \mathcal{C}_{\equiv r}^\perp \text{mod } \mathcal{C}} V_{L(\gamma+C)\times D}\cdot (5-9b)\]

In particular, if \(\mathcal{C}\) is self-dual, we have

\[V_{L_{\tau}^C}(\tau)[\varepsilon^1] \times V_{L_{\tau}^C}(\tau^2)[\varepsilon^2] = V_{L_{(\gamma-C)\times D}}[\varepsilon^1 - \varepsilon^2].\]

Proof. (i) By Prop. 5.5 we have

\[N_{\tau}^T \begin{pmatrix} V_{L_{\tau}^C}[\varepsilon] \\ V_{L_{\tau}^C}(\tau)[0], V_{L_{\tau}^C}(\tau^2)[0] \end{pmatrix} = N_{\tau}^T \begin{pmatrix} V_{L_{\tau}^C}(\tau)[0] \\ V_{L_{\tau}^C}(\tau)[0], V_{L_{\tau}^C}[2\varepsilon] \end{pmatrix}.\]

By Prop. 5.4 we have

\[N_{\tau}^T \begin{pmatrix} V_{L_{\tau}^C}[\varepsilon] \\ V_{L_{\tau}^C}(\tau)[0], V_{L_{\tau}^C}(\tau^2)[0] \end{pmatrix} = \begin{cases} 1 & \text{if } \varepsilon = 0; \\ 0 & \text{if } \varepsilon = 1, 2. \end{cases}\]

Similarly, by Prop. 5.5 and Prop. 5.8 we have

\[N_{\tau}^T \begin{pmatrix} V_{L_{(\gamma+C)\times D}}[\varepsilon] \\ V_{L_{(\gamma+C)\times D}}(\tau)[0], V_{L_{(\gamma+C)\times D}}(\tau^2)[0] \end{pmatrix} = N_{\tau}^T \begin{pmatrix} V_{L_{(\gamma+C)\times D}}(\tau)[0] \\ V_{L_{(\gamma+C)\times D}}(\tau)[0], V_{L_{(-\gamma+C)\times D}} \end{pmatrix} = 1.\]

Therefore,

\[V_{L_{\tau}^C}(\tau)[0] \times V_{L_{\tau}^C}(\tau^2)[0] \geq V_{L_{\tau}^C}[0] \oplus \bigoplus_{0 \neq \gamma \in \mathcal{C}_{\equiv r}^\perp \text{mod } \mathcal{C}} V_{L(\gamma+C)\times D}.\]

Recall that

\[\text{qdim} \left( V_{L_{\tau}^C}(\tau)[0] \times V_{L_{\tau}^C}(\tau^2)[0] \right) = \left( \frac{2^\ell}{|\mathcal{C}|} \right)^2 = \frac{|\mathcal{C}^\perp/\mathcal{C}|}{|\mathcal{C}|};\]

\[\text{qdim} V_{L_{\tau}^C}[0] = 1.\]

Moreover,

\[\text{qdim} \left( \bigoplus_{0 \neq \gamma \in \mathcal{C}_{\equiv r}^\perp \text{mod } \mathcal{C}} V_{L(\gamma+C)\times D} \right) = \#\{0 \neq \gamma \in \mathcal{C}_{\equiv r}^\perp \text{mod } \mathcal{C} \} \cdot 3.\]

Since

\[\#\{0 \neq \gamma \in \mathcal{C}_{\equiv r}^\perp \text{mod } \mathcal{C} \} = \frac{1}{3}(|\mathcal{C}^\perp/\mathcal{C}| - 1),\]
we know
\[
q\dim \left( \frac{V_{L_{\mathcal{C} \times \mathcal{D}}}[0] \bigoplus \bigoplus_{0 \neq \gamma \in \mathbb{C}^{\mathbb{Z}_r}_{\equiv r} \mod \mathcal{C}} V_{L_{(\gamma \mathcal{C}) \times \mathcal{D}}} }{ } \right) = |\mathcal{C}/\mathcal{C}| \\
= q\dim \left( V_{L_{\mathcal{C} \times \mathcal{D}}}[0] \times V_{L_{\mathcal{C} \times \mathcal{D}}}[\tau^2][0] \right).
\]
This proves (i).

(ii) By Prop. 5.4 we have
\[
V_{L_{\mathcal{C} \times \mathcal{D}}}^{T, \eta^i}(\tau)[\varepsilon^i] = V_{L_{\mathcal{C} \times \mathcal{D}}}^{(-1)^i \eta^i}[(-1)^{i+1} \varepsilon^i] \times V_{L_{\mathcal{C} \times \mathcal{D}}}[0].
\]
Therefore,
\[
V_{L_{\mathcal{C} \times \mathcal{D}}}^{T, \eta^i}(\tau)[\varepsilon^i] \times V_{L_{\mathcal{C} \times \mathcal{D}}}^{T, \eta^2}(\tau)[\varepsilon^2] = \frac{V_{L_{\mathcal{C} \times \mathcal{D}}}^{(-1)^i \eta^i}[(-1)^{i+1} \varepsilon^i]}{V_{L_{\mathcal{C} \times \mathcal{D}}}^{T, 0}(\tau)[0]} \times V_{L_{\mathcal{C} \times \mathcal{D}}}^{T, 0}(\tau^2)[0] = V_{L_{\mathcal{C} \times \mathcal{D}}}^{T, 0}(\tau)[0].
\]
This proves (ii).

\textbf{Case (V): } $V_{L_{\mathcal{C} \times \mathcal{D}}}^{T, \delta^1}(\tau)[\varepsilon^1] \times V_{L_{\mathcal{C} \times \mathcal{D}}}^{T, \delta^2}(\tau)[\varepsilon^2]
\]
We first consider the case $\delta^1 = \delta^2 = 0$ and $\varepsilon^1 = \varepsilon^2 = 0$. By the similar analysis as in the previous few cases, we can show quickly that many fusion coefficients are zero. Assume
\[
V_{L_{\mathcal{C} \times \mathcal{D}}}^{T, 0}(\tau)[0] \times V_{L_{\mathcal{C} \times \mathcal{D}}}^{T, 0}(\tau)[0] = xV_{L_{\mathcal{C} \times \mathcal{D}}}^{T, 0}(\tau^2)[0] + yV_{L_{\mathcal{C} \times \mathcal{D}}}^{T, 0}(\tau^2)[1] + zV_{L_{\mathcal{C} \times \mathcal{D}}}^{T, 0}(\tau^2)[2],
\]
for some $x, y, z \in \mathbb{Z}_{\geq 0}$.

For simplicity, we denote
\[
T[j] := V_{L_{\mathcal{C} \times \mathcal{D}}}^{T, 0}(\tau)[j]; \quad \bar{T}[j] := V_{L_{\mathcal{C} \times \mathcal{D}}}^{T, 0}(\tau^2)[j]; \quad S[j] := V_{L_{\mathcal{C} \times \mathcal{D}}}^{T, 0}(\tau^3)[j].
\]
Equation (5-10) then becomes
\[
T[0] \times T[0] = x \bar{T}[0] + y \bar{T}[1] + z \bar{T}[2] = \bar{T}[0] \times (xS[0] + yS[2] + zS[1]).
\]
We multiply this equation by $\bar{T}[0]$ and get
\[
\bar{T}[0] \times T[0] \times T[0] = \bar{T}[0] \times \bar{T}[0] \times (xS[0] + yS[2] + zS[1]).
\]
On the left hand side of (5-11), by Prop. 5.4 and Prop. 5.8, we have

\[
\tilde{T}[0] \times T[0] \times T[0] = (\tilde{T}[0] \times T[0]) \times T[0] = \left( S[0] \oplus \bigoplus_{0 \neq \gamma \in C^\perp \mod C} V_{L(\tau_i \gamma) \times \mathbb{D}} \right) \times T[0]
\]

\[= T[0] + Q(T[0] + T[1] + T[2]),\]  

(5-12)

where \( Q = \# \{ 0 \neq \gamma \in C^\perp \mod C \} = \frac{4^{\ell-2d} - 1}{3} \) and \( d = \text{dim}\, C \).

By symmetry between automorphisms \( \tau \) and \( \tau^2 \), we can rewrite (5-10) as

\[
(\tilde{T}[0] \times \tilde{T}[0]) \times (xS[0] + yS[2] + zS[1])
\]

\[= (x^2 + y^2 + z^2)T[0] + (xy + yz + zx)T[1] + (xy + yz + zx)T[2].\]  

(5-13)

Comparing (5-12) and (5-13) we have

\[
x + y + z = q\text{dim}\, T[0] = 2^{\ell-2d}; \quad (5-14a)
\]

\[
xy + yz + zx = \frac{4^{\ell-2d} - 1}{3}; \quad (5-14b)
\]

\[
x^2 + y^2 + z^2 = xy + yz + zx + 1 \quad (5-14c)
\]

From (5-14c) we know

\[(x - y)^2 + (y - z)^2 + (z - x)^2 = 2.\]

Assuming \( x \geq y \geq z \geq 0 \), then we have \( x - z = 1 \) and thus \( z + 1 \geq y \geq z \). Therefore,

\[
x = y = \frac{2^{\ell-2d} + 1}{3}, \quad z = \frac{2^{\ell-2d} - 2}{3}, \quad \text{if } 2^{\ell-2d} \equiv 2 \mod 3;
\]

\[
x = \frac{2^{\ell-2d} + 2}{3}, \quad y = z = \frac{2^{\ell-2d} - 1}{3}, \quad \text{if } 2^{\ell-2d} \equiv 1 \mod 3,
\]

where \( d = \text{dim}\, C \). Note that \( 2^{\ell-2d} \equiv 2^\ell \mod 3 \). As a summary, we have the proposition.

**Proposition 5.10.** We have fusion rules:

\[
V_{L(\tau_i \gamma) \times \mathbb{D}}^{T,0} (\tau^i)[0] \times V_{L(\tau_i \gamma) \times \mathbb{D}}^{T,0} (\tau^i)[0]
\]

\[= x'V_{L(\tau^2i) \times \mathbb{D}}^{T,0} (\tau^2i)[0] + y'V_{L(\tau^2i) \times \mathbb{D}}^{T,0} (\tau^2i)[1] + z'V_{L(\tau^2i) \times \mathbb{D}}^{T,0} (\tau^2i)[2],\]  

(5-15)

for some \( x', y', z' \in \mathbb{N} \).
If \((x \geq y \geq z)\) is the decreasing reordering of \((x', y', z')\), then we have

\[
x = y = \frac{2^{\ell-2d} + 1}{3}, \quad z = \frac{2^{\ell-2d} - 2}{3}, \quad \text{if } \ell \text{ is odd},
\]
\[
x = \frac{2^{\ell-2d} + 2}{3}, \quad y = z = \frac{2^{\ell-2d} - 1}{3}, \quad \text{if } \ell \text{ is even},
\]

where \(d = \dim \mathcal{C}\). Note that \(2^{\ell-2d} = \sqrt{|\mathcal{C}^\perp / \mathcal{C}|}\).

**S-matrix and Verlinde formula.** Next we compute the exact fusion rules using Verlinde formula and S-matrix. First we review Dong-Li-Mason's theory on trace functions [DLM00].

Let \(V\) be a rational VOA and \(g, h \in \text{Aut} V\) be commuting automorphisms of finite orders. Let \(M\) be a \(g\)-twisted \(h\)-stable \(V\)-module. There exists a linear isomorphism \(\varphi(h)\) of \(M\) such that

\[
\varphi(h)Y_M(u, z) = Y_M(hu, z)\varphi(h).
\]

For a homogeneous \(v \in V\) with \(L(1)v = 0\) we define the trace function

\[
T_M(v, g, h; z) := \text{tr}_M \varphi(h) o(v) q^{-c/24} = q^{\lambda - c/24} \sum_{n \in \frac{1}{|g|} \mathbb{Z}_+} \text{tr}_{M_{\lambda + n}} o(v) \varphi(h) q^n,
\]

where \(o(v)\) is the degree zero operator of \(v\), \(\lambda\) is the conformal weight of \(M\), \(c\) is the central charge of \(V\) and \(q = e^{2\pi \sqrt{-1} z}\).

**Proposition 5.11.** [DLM00] Let \(C_1(g, h)\) be the \(\mathbb{C}\)-vector space

\[
C_1(g, h) := \text{Span}_\mathbb{C}\{T_M(v, g, h; z) \mid M \text{ is a } g\text{-twisted } h\text{-stable } V\text{-module}\}.
\]

Then

(i) \(C_1(g, h)\) has a basis:

\[
\{T_M(v, g, h; z) \mid M \text{ is an irreducible } g\text{-twisted } h\text{-stable } V\text{-module}\}.
\]

(ii) Modular invariance: Let \(T_M(v, g, h; z) \in C_1(g, h)\) and \(\Gamma = (a \ b \ c \ d) \in \text{SL}_2(\mathbb{Z})\). Then we have \(T_M(v, g, h; \Gamma \circ z) \in C_1(g, h) \circ \Gamma\) in the sense that

\[
T_M(v, g, h; \frac{az + b}{cz + d}) \in C_1(g^a h^c, g^b h^d).
\]

In fact, if \(M\) is a \(g\)-twisted \(h\)-stable \(V\)-module, then

\[
T_M(v, g, h; \frac{az + b}{cz + d}) = \sum S^{(g, h)}_N T_N(v, g, h; z),
\]

where \(N\) runs over irreducible \(g^a h^c\)-twisted \(g^b h^d\)-stable \(V\)-module, and the coefficients \(S^{(g, h)}_N\) are independent of \(v\).
In particular, when \( g = h = \text{id} \), \( \Gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \) and \( V = M^0, \ldots, M^d \) are all inequivalent irreducible \( V \)-modules, we have

\[
T_{M^i}(v, \text{id}, \text{id}; -\frac{1}{z}) = \sum_{j=0}^{d} S_{i,j} T_{M^j}(z). \tag{5-16}
\]

For simplicity, we denote
\[
M(g, h; z) = Z_M(g, h; z) := T_M(\mathbb{1}, g, h; z),
\]
and
\[
M(z) := Z_M(\text{id}, \text{id}; z) = \text{ch} M(z).
\]

**Definition 5.12.** The matrix \( S = (S_{i,j}) \) defined in equation (5-16) is called the \( S \)-matrix.

**Theorem 5.13.** [Hua08] Let \( V \) be a rational and \( C_2 \)-cofinite simple VOA of CFT type and assume \( V \cong V' \). Let \( S = (S_{i,j})_{i,j=0}^{d} \) be the \( S \)-matrix as defined in (5-16). Then

(i) \( (S^{-1})_{i,j} = S_{i,j'} = S_{j', j} \), and \( S_{i,j'} = S_{i,j} \);

(ii) \( S \) is symmetric and \( S^2 = (\delta_{i,j}) \);

(iii) \( N^k_{i,j} = \sum_{s=0}^{d} \frac{S_{j,s} S_{s,k}}{S_{0,s}} \);

(iv) The \( S \)-matrix diagonalizes the fusion matrix \( N(i) = (N^k_{i,j})_{j,k=0}^{d} \) with diagonal entries \( \frac{S_{i,s}}{S_{0,s}} \), for \( i, s = 0, \ldots, d \). More explicitly, \( S^{-1} N(i) S = \text{diag}(\frac{S_{i,s}}{S_{0,s}})_{s=0}^{d} \). In particular, \( S_{0,s} \neq 0 \) for \( s = 0, \ldots, d \).

**Proposition 5.14.** [DJX13] Let \( V \) be a simple, rational and \( C_2 \)-cofinite VOA of CFT type. Let \( M^0, M^1, \ldots, M^d \) be as before with the corresponding conformal weights \( \lambda_i > 0 \) for \( 0 < i \leq d \). Then \( 0 < \text{qdim}_V M^i < \infty \) for any \( 0 \leq i \leq d \). In fact, we have

\[
\text{qdim}_V M^i = \frac{S_{i,0}}{S_{0,0}}. \tag{5-17}
\]

**The case that \( \mathcal{D} \) is self-dual.** Denote \( \xi = e^{(2\pi \sqrt{-1})/3} \) a primitive cubic root of unity.

We define a function \( \Xi : \mathbb{Z} \rightarrow \{-1, 2\} \) by

\[
\Xi(n) := \xi^n + \xi^{2n}, \quad \text{for } n \in \mathbb{Z}.
\]

Note that

\[
\Xi(n) = \begin{cases} 
2 & \text{if } n \equiv 0 \mod 3, \\
-1 & \text{otherwise}.
\end{cases}
\]

**Proposition 5.15.** Let \( \mathcal{D} \) be a self-dual \( \mathbb{Z}_3 \)-code of length \( \ell \), then we have

\[
T[0] \times T[0] = \frac{2^\ell - 2d}{3} \Xi(\ell) \hat{T}[0] + \frac{2^\ell - 2d}{3} \Xi(\ell + 2) \hat{T}[1] + \frac{2^\ell - 2d}{3} \Xi(\ell + 1) \hat{T}[2]
\]

\[
= \sum_{\varepsilon=0,1,2} \frac{2^\ell - 2d}{3} \Xi(\ell - \varepsilon) \hat{T}[\varepsilon].
\]
Proof. We mimic the proof of [Miy13, Lemma 18].

Let \( V \) denote the lattice VOA \( V_{L \times D} \). Since \( D \) is self-dual, we know \( V \) is the unique \( \tau \)-stable irreducible module. Thus, \( V \) has exactly one \( \tau^i \)-twisted module for each \( i = 1, 2 \). We denoted them by \( T \) and \( \bar{T} \), respectively. Let

\[
M^i := V[i], \quad M^{3+i} := T[i], \quad M^{6+i} := \bar{T}[i],
\]

for \( i = 0, 1, 2 \). Then we know \( M^j, (j = 0, \cdots, 8) \) are irreducible \( V^\tau \)-modules. Note that there are also irreducible \( V^\tau \)-modules which are not \( \tau \)-stable, but we won’t need in the proof.

Denote \( C_1(g, h) \) the vector space generated by trace functions of \( g \)-twisted and \( h \)-stable \( V \)-modules. By [DLM00] we know the modular transformation \( \Gamma : z \mapsto \frac{1}{z} \) maps \( C_1(g, h) \) to \( C_1(h, g^{-1}) \). In particular, \( \Gamma \) sends \( C_1(\tau, \tau^j) \) to \( C_1(\tau^j, \tau^2) \) for \( j = 0, 1, 2 \).

First, we know \( Z_T(\tau, 1; \frac{-1}{z}) \in C_1(1, \tau^2) \) which is spanned by \( Z_V(1, 1; z) \). Therefore, we can write

\[
Z_T(\tau, 1; \frac{-1}{z}) = \lambda_1 Z_V(1, 1; z),
\]

for some \( \lambda_1 \in \mathbb{C} \).

Denote \( W^i(g, h, z) = Z_M(g, h; z) \) for any \( i \). Then we have

\[
W^3(\frac{-1}{z}) + W^4(\frac{-1}{z}) + W^5(\frac{-1}{z}) = \lambda_1 \left( W^0(z) + W^1(z) + W^2(z) \right). \tag{5-18}
\]

Similarly, using \( Z_T(\tau, \tau^j; \frac{-1}{z}) \in C_1(\tau^j, \tau^2) \) for \( j = 1, 2 \), we can write

\[
W^3(1, \tau; \frac{-1}{z}) + W^4(1, \tau; \frac{-1}{z}) + W^5(1, \tau; \frac{-1}{z}) = \mu_1 \left( W^3(1, \tau^2; z) + W^4(1, \tau^2; z) + W^5(1, \tau^2; z) \right),
\]

and

\[
W^3(1, \tau^2; \frac{-1}{z}) + W^4(1, \tau^2; \frac{-1}{z}) + W^5(1, \tau^2; \frac{-1}{z}) = \mu_2 \left( W^6(1, \tau^2; z) + W^7(1, \tau^2; z) + W^8(1, \tau^2; z) \right), \tag{5-19}
\]

for some \( \mu_1, \mu_2 \in \mathbb{C} \).

We can define a linear isomorphism \( \varphi(\tau^j) \) as following:

\[
\varphi(\tau^j) = \xi^j \text{ on } M^{3+i} \text{ and } M^{6+i}.
\]

Therefore we can rewrite the equation (5-19) as

\[
W^3(\tau, 1; \frac{-1}{z}) + \xi W^4(\tau, 1; \frac{-1}{z}) + \xi^2 W^5(\tau, 1; \frac{-1}{z}) = \mu_1 \left( W^3(\tau, 1; z) + \xi^2 W^4(\tau, 1; z) + \xi W^5(\tau, 1; z) \right),
\]

\[
W^3(\tau, 1; \frac{-1}{z}) + \xi^2 W^4(\tau, 1; \frac{-1}{z}) + \xi W^5(\tau, 1; \frac{-1}{z}) = \mu_2 \left( W^6(\tau^2, 1; z) + \xi^2 W^7(\tau^2, 1; z) + \xi W^8(\tau^2, 1; z) \right). \tag{5-20}
\]
Solving equations (5-18) and (5-20), we know
\[ W^3\left(-\frac{1}{z}\right) = \frac{\lambda_1}{3} (W^0(z) + \xi^2 W^1(z) + \xi W^2(z)) + \frac{\mu_1}{3} (W^3(z) + \xi^2 W^4(z) + \xi W^5(z)) + \frac{\mu_2}{3} (W^6(z) + \xi W^7(z) + \xi^2 W^8(z)), \]
\[ W^4\left(-\frac{1}{z}\right) = \frac{\lambda_1}{3} (W^0(z) + \xi^2 W^1(z) + \xi W^2(z)) + \frac{\mu_1}{3} (\xi^2 W^3(z) + \xi W^4(z) + W^5(z)) + \frac{\mu_2}{3} (\xi W^6(z) + \xi^2 W^7(z) + W^8(z)), \]
\[ W^5\left(-\frac{1}{z}\right) = \frac{\lambda_1}{3} (W^0(z) + \xi^2 W^1(z) + \xi W^2(z)) + \frac{\mu_1}{3} (\xi W^3(z) + W^4(z) + \xi^2 W^5(z)) + \frac{\mu_2}{3} (\xi^2 W^6(z) + W^7(z) + \xi W^8(z)). \]
In other words, the rows \( S_{i,j} \) for \( i = 3, 4, 5 \) are given by
\[
\frac{1}{3} \begin{pmatrix}
\lambda_1 & \xi^2 \lambda_1 & \xi \lambda_1 & \xi \lambda_1 & \mu_1 & \xi^2 \mu_1 & \xi \mu_1 & \mu_1 & \xi \mu_2 & \xi^2 \mu_2 & \xi^2 \mu_2 & \mu_2 & \xi \mu_2 & 0 & \cdots & 0 \\
\lambda_1 & \xi^2 \lambda_1 & \xi \lambda_1 & \xi \lambda_1 & \mu_1 & \xi^2 \mu_1 & \xi \mu_1 & \mu_1 & \xi \mu_2 & \xi^2 \mu_2 & \xi^2 \mu_2 & \mu_2 & \xi \mu_2 & 0 & \cdots & 0 \\
\lambda_1 & \xi^2 \lambda_1 & \xi \lambda_1 & \xi \lambda_1 & \mu_1 & \xi^2 \mu_1 & \xi \mu_1 & \mu_1 & \xi \mu_2 & \xi^2 \mu_2 & \xi^2 \mu_2 & \mu_2 & \xi \mu_2 & 0 & \cdots & 0
\end{pmatrix}.
\]
Since \( S_{0,0}^{2} \text{glob}(V^\tau) = 1 \), we know
\[ S_{0,0}^{2} \cdot 9 \left| \mathcal{C}^\perp / \mathcal{C} \right| \left| \mathcal{D}^\perp / \mathcal{D} \right| = 1, \]
\[ S_{0,i}/S_{0,0} = \text{qdim} M^i, \]
and
\[ \text{qdim} M^i = \begin{cases} 1, & \text{if } i = 0, 1, 2 \\ \frac{2^{d}}{C}, & \text{if } 3 \leq i \leq 8. \end{cases} \]
This gives \( S_{0,0} = S_{0,1} = S_{0,2} = \frac{\pm 2^{2d-\ell}}{3} \) and \( \lambda_1 = 3S_{0,h} = \pm 1 \) for \( 3 \leq h \leq 8 \).
By Verlinde formula, we know fusion rules are given by
\[ N_{3,3}^6 = \frac{3 \cdot (\lambda_1/3)^3}{S_{0,0}} + \frac{3((\mu_1/3)^3 + (\mu_2/3)^3)}{(\lambda_1/3)} = \frac{\pm (2^{\ell-2d} + \mu_1^3 + \mu_2^3)}{3}, \]
\[ N_{3,3}^7 = \frac{\pm (2^{\ell-2d} + \xi^2 \mu_1^3 + \xi \mu_2^3)}{3}, \]
\[ N_{3,3}^8 = \frac{\pm (2^{\ell-2d} + \mu_1^3 + \xi^2 \mu_2^3)}{3}. \]
Since
\[ N_{3,3}^6 + N_{3,3}^7 + N_{3,3}^8 = 2^{\ell-2d}, \]
we know
\[ S_{0,0} = 2^{2d-\ell}, \]
and
\[
N_{3,3}^6 = \frac{2^{\ell-2d} + \mu_1^3 + \mu_2^3}{3},
\]
\[
N_{3,3}^7 = \frac{2^{\ell-2d} + \xi^2\mu_1^3 + \xi\mu_2^3}{3},
\]
\[
N_{3,3}^8 = \frac{2^{\ell-2d} + \xi\mu_1^3 + \xi^2\mu_2^3}{3}.
\]

By \( S_{3,6} = 1 \) we have
\[
9 = 3\lambda_1^2 + 6\mu_1\mu_2,
\]
and hence
\[
\mu_1\mu_2 = 1.
\]

Notice that the weights of irreducible \( V^\tau \)-modules of twisted type are
\[
\text{wt } V^{T,0}_{L_{\mathcal{C} \times \mathcal{D}}} (\tau^i)[\varepsilon] \in \ell/9 + 1/3(\sum_{e \in S_\varepsilon} e_i) + \mathbb{Z} = \varepsilon/3 + \ell/9 + \mathbb{Z},
\]
for \( i = 1, 2 \). By considering the characters, we have from the above \( S \)-matrix that
\[
Z_V(1, \tau; z) = \text{ch}(W^0) + \xi \text{ch}(W^1) + \xi^2 \text{ch}(W^2),
\]
\[
Z_V(1, \tau; -1/z) = \lambda_1 \{ \text{ch}(W^3) + \text{ch}(W^4) + \text{ch}(W^5) \},
\]
\[
Z_V(1, \tau; -1/(z + 1)) = e^{2\pi\sqrt{-1N/24}} \cdot e^{2\pi\sqrt{-1\ell/9}} \lambda_1 \{ \text{ch}(W^3) + \xi \text{ch}(W^4) + \xi^2 \text{ch}(W^5) \},
\]
\[
Z_V(1, \tau; -1/((-1/z) + 1)) = e^{2\pi\sqrt{-1N/24}} \cdot e^{2\pi\sqrt{-1\ell/9}} \lambda_1 \{ \text{ch}(W^3) + \xi^2 \text{ch}(W^4) + \xi \text{ch}(W^5) \},
\]
where \( N = 2\ell \) is the rank of the lattice \( L_{\mathcal{C} \times \mathcal{D}} \). On the other hand, since
\[
Z_V(1, \tau; -1/((-1/z) + 1)) = Z_V(1, \tau; -1 - \frac{1}{z - 1}) = e^{-2\pi\sqrt{-1N/24}} Z_V(1, \tau, -1/(z - 1)) = e^{-4\pi\sqrt{-1N/24}} \cdot e^{-2\pi\sqrt{-1\ell/9}} \lambda_1 \{ \text{ch}(W^3) + \xi^2 \text{ch}(W^4) + \xi \text{ch}(W^5) \}
\]
we have \( \mu_1 \cdot e^{6\pi\sqrt{-1N/24}} \cdot e^{4\pi\sqrt{-1\ell/9}} = 1 \). Since \( N = 2\ell \) and \( \ell \) is a multiple of 4, we know \( 8|N \) and \( \mu_1 = e^{-4\pi\sqrt{-1\ell/9}} \). Using \( \mu_1\mu_2 = 1 \), we have \( \mu_1^3 = \xi^\ell \) and \( \mu_2^3 = \xi^{2\ell} \). This gives
\[
T[0] \times T[0]
= \frac{2^{\ell-2d} + \xi^\ell + \xi^{2\ell}}{3} T[0] + \frac{2^{\ell-2d} + \xi^{\ell+2} + \xi^{2\ell+1}}{3} T[1] + \frac{2^{\ell-2d} + \xi^{\ell+1} + \xi^{2\ell+2}}{3} T[2]
= \frac{2^{\ell-2d} + 3\ell}{3} T[0] + \frac{2^{\ell-2d} + 3(\ell + 2)}{3} T[1] + \frac{2^{\ell-2d} + 3(\ell + 1)}{3} T[2],
\]
and completes the proof.
General Case. Recall the decomposition
\[ V_{L \times D}(\tau)[\varepsilon] \cong \bigoplus_{\gamma \in D} V_{L \times 0}(\tau\gamma)[\varepsilon]. \]  
(5-21)

In the following, we will denote
\[ T_{C \times D}[\varepsilon] := V_{L \times D}(\tau)[\varepsilon], \]
\[ \tilde{T}_{C \times D}[\varepsilon] := V_{L \times D}(\tau^2)[\varepsilon], \]
in addition, we let
\[ T_{C \times D}[\varepsilon] := V_{L \times D}(\tau)[\varepsilon], \]
\[ \tilde{T}_{C \times D}[\varepsilon] := V_{L \times D}(\tau^2)[\varepsilon]. \]

We also let \( \mathbf{0} \) be the trivial \( \mathbb{Z}_3 \)-code of various length depending on context.

Proposition 5.16. Let \( \mathcal{B} \) be a self-dual \( \mathbb{F}_4 \)-code of length 2. Then
\[ T_{\mathcal{B} \times 0}[0] \times T_{\mathcal{B} \times 0}[0] = \tilde{T}_{\mathcal{B} \times 0}[2]. \]

Proof. In this case, all irreducible modules are simple current. It suffices to find the nonzero fusion rules.

Let \( \mathcal{B}^2 := \mathcal{B} \oplus \mathcal{B} \) be a self-dual code of length 4 and let \( \mathcal{S} \) be a self-dual \( \mathbb{Z}_3 \)-code of length 4. By Prop. 5.15, we know
\[ T_{\mathcal{B}^2 \times \mathcal{S}}[0] \times T_{\mathcal{B}^2 \times \mathcal{S}}[0] = \tilde{T}_{\mathcal{B}^2 \times \mathcal{S}}[1]. \]

Considering the sub VOA \( V_{L \mathcal{B}^2 \times 0} \subset V_{L \mathcal{B}^2 \times \mathcal{S}} \), we have the decomposition of \( V_{L \mathcal{B}^2 \times 0} \)-modules
\[ T_{\mathcal{B}^2 \times \mathcal{S}}[\varepsilon] = \bigoplus_{\eta \in \mathcal{S}} T_{\mathcal{B}^2 \times 0}[\varepsilon], \]
\[ \tilde{T}_{\mathcal{B}^2 \times \mathcal{S}}[\varepsilon] = \bigoplus_{\eta \in \mathcal{S}} \tilde{T}_{\mathcal{B}^2 \times 0}[\varepsilon]. \]

By Prop. 5.3 we have
\[ 1 = N\bigg( \frac{\tilde{T}_{\mathcal{B}^2 \times \mathcal{S}}[1]}{T_{\mathcal{B}^2 \times \mathcal{S}}[0], \ T_{\mathcal{B}^2 \times \mathcal{S}}[0]} \bigg) \leq N\bigg( \bigoplus_{\eta \in \mathcal{S}} T_{\mathcal{B}^2 \times 0}[\varepsilon], \ T_{\mathcal{B}^2 \times 0}[\varepsilon] \bigg) = N\bigg( \frac{\tilde{T}_{\mathcal{B}^2 \times 0}[1]}{T_{\mathcal{B}^2 \times 0}[0], \ T_{\mathcal{B}^2 \times 0}[0]} \bigg), \]
where the last equality follows from Prop. 5.10. Since \( \mathcal{B}^2 \) is self-dual, then we have
\[ T_{\mathcal{B}^2 \times 0}[0] \times T_{\mathcal{B}^2 \times 0}[0] = \tilde{T}_{\mathcal{B}^2 \times 0}[1]. \]

Now consider the subVOA \( V_{\mathcal{B} \times 0} \otimes V_{\mathcal{B} \times 0} \subset V_{L \mathcal{B}^2 \times 0} \) and the decomposition
\[ T_{\mathcal{B}^2 \times 0}[\varepsilon] = \bigoplus_{\varepsilon_0=0,1,2} T_{\mathcal{B} \times 0}[\varepsilon_0] \otimes T_{\mathcal{B} \times 0}[\varepsilon - \varepsilon_0] \]
of \( V_{\mathcal{B} \times 0} \otimes V_{\mathcal{B} \times 0} \)-modules.
Proof.

(i) First we assume

\[ N\left(\frac{\hat{T}_{B^2 \times 0}^n[1]}{T_{B^2 \times 0}[0], T_{B^2 \times 0}[0]}\right) \leq N\left(\sum_{\varepsilon = 0,1,2} \frac{\tilde{T}_{B \times 0}[\varepsilon_0] \otimes \tilde{T}_{B \times 0}[1-\varepsilon_0]}{T_{B \times 0}[0] \otimes T_{B \times 0}[0], T_{B \times 0}[0] \otimes T_{B \times 0}[0]}\right) \]

\[ = \sum_{\varepsilon = 0,1,2} N\left(\frac{\tilde{T}_{B \times 0}[\varepsilon_0]}{T_{B \times 0}[0], T_{B \times 0}[0]}\right) N\left(\frac{\tilde{T}_{B \times 0}[1-\varepsilon_0]}{T_{B \times 0}[0], T_{B \times 0}[0]}\right). \tag{5-22} \]

Now since \( B \) is self-dual, only one of the fusion rules \( N(\tilde{T}_{B \times 0}[\varepsilon_0]), (\varepsilon = 0,1,2) \) is nonzero. To have the above inequality (5-22), we must have \( N(\tilde{T}_{B \times 0}[\varepsilon_0]) = \delta_{\varepsilon_0,2}. \) This completes the proof.

Proposition 5.17. Let \( C \) and \( D \) be self-orthogonal codes of length \( \ell. \)

(i) If the length \( \ell \) is even, then we have

\[ T[0] \times T[0] = \sum_{\varepsilon = 0,1,2} \frac{2^{\ell-2d} + \Xi(\ell - \varepsilon)}{3} \tilde{T}[\varepsilon]. \]

(ii) If the length \( \ell \) is odd, then we have

\[ T[0] \times T[0] = \sum_{\varepsilon = 0,1,2} \frac{2^{\ell-2d} - \Xi(\ell - \varepsilon)}{3} \tilde{T}[\varepsilon]. \]

As a summary, we have

\[ T[0] \times T[0] = \sum_{\varepsilon = 0,1,2} \frac{2^{\ell-2d} + (-1)^{\ell} \Xi(\ell - \varepsilon)}{3} \tilde{T}[\varepsilon]. \]

It also implies

\[ V_{L_{C \times D}}^{T,\eta_1}(\tau^i)[\varepsilon_1] \times V_{L_{C \times D}}^{T,\eta_2}(\tau^i)[\varepsilon_2] = \sum_{\varepsilon = 0,1,2} \frac{2^{\ell-2d} + (-1)^{\ell} \Xi(\ell - \varepsilon)}{3} V_{L_{C \times D}}^{T,-(\eta_1+\eta_2)(\tau^i)}[\varepsilon - \varepsilon_1 - \varepsilon_2]. \]

Proof. (i) First we assume \( \ell \) is a multiple of 4. Then there exists a self-dual code \( S \) of

length \( \ell. \)

Restricting to \( V_{L_{C \times 0}}^T \)-modules, we know

\[ N\left(\frac{\hat{T}_{C \times S}[\varepsilon]}{T_{C \times S}[0], T_{C \times S}[0]}\right) \leq N\left(\hat{T}_{C \times 0}[\varepsilon], T_{C \times 0}[0], T_{C \times 0}[0]\right) \]

\[ = N\left(\frac{\hat{T}_{C \times 0}[\varepsilon]}{T_{C \times 0}[0], T_{C \times 0}[0]}\right). \tag{5-23} \]

On the other hand, we know from (5-14a) that

\[ \sum_{\varepsilon = 0,1,2} N\left(\frac{\hat{T}_{C \times S}[\varepsilon]}{T_{C \times S}[0], T_{C \times S}[0]}\right) = \sum_{\varepsilon = 0,1,2} N\left(\frac{\hat{T}_{C \times 0}[\varepsilon]}{T_{C \times 0}[0], T_{C \times 0}[0]}\right). \]

Therefore, the inequality in (5-23) must attain equality and we prove (i) when \( D \) is the

trivial code of length divisible by 4.
Now let $\mathbf{D}$ be a self-orthogonal code of length $\ell$. Similarly, we have
\[
N\left( \frac{\tilde{T}_{\mathbf{C} \times \mathbf{D}}[\varepsilon]}{T_{\mathbf{C} \times \mathbf{D}}[0], T_{\mathbf{C} \times \mathbf{D}}[0]} \right) \leq N\left( \frac{T_{\mathbf{C} \times 0}[\varepsilon]}{T_{\mathbf{C} \times 0}[0], T_{\mathbf{C} \times 0}[0]} \right).
\]
The same argument as in the case for $\mathbf{D} = 0$ shows
\[
N\left( \frac{\tilde{T}_{\mathbf{C} \times \mathbf{D}}[\varepsilon]}{T_{\mathbf{C} \times \mathbf{D}}[0], T_{\mathbf{C} \times \mathbf{D}}[0]} \right) = N\left( \frac{T_{\mathbf{C} \times 0}[\varepsilon]}{T_{\mathbf{C} \times 0}[0], T_{\mathbf{C} \times 0}[0]} \right).
\]
This implies
\[
N\left( \frac{\tilde{T}_{\mathbf{C} \times \mathbf{D}}[\varepsilon]}{T_{\mathbf{C} \times \mathbf{D}}[0], T_{\mathbf{C} \times \mathbf{D}}[0]} \right) = N\left( \frac{\tilde{T}_{\mathbf{C} \times \mathbf{S}}[\varepsilon]}{T_{\mathbf{C} \times \mathbf{S}}[0], T_{\mathbf{C} \times \mathbf{S}}[0]} \right),
\]
and proves (i) by Prop. 5.15 when $\ell$ is a multiple of 4.

Now assume $\ell \equiv 2 \mod 4$. Let $\mathbf{B}$ be a self-dual $\mathbb{F}_4$-code of length 2. Then $\mathbf{C} \oplus \mathbf{B}$ is a self-orthogonal code of length divisible by 4 and $(\mathbf{D} \oplus 0)$ is a self-orthogonal code of the same length.

Restricting to
\[
V_{L_{\mathbf{C} \times \mathbf{D}}}^T \otimes V_{L_{\mathbf{B} \times 0}}^T \subset V_{L_{(\mathbf{C} \oplus \mathbf{B}) \times (\mathbf{D} \oplus 0)}}^T;
\]
we know
\[
T_{(\mathbf{C} \oplus \mathbf{B}) \times (\mathbf{D} \oplus 0)}[\varepsilon] = \bigoplus_{\varepsilon_0 = 0, 1, 2} T_{\mathbf{C} \times \mathbf{D}}[\varepsilon_0] \otimes T_{\mathbf{B} \times 0}[\varepsilon - \varepsilon_0];
\]
moreover
\[
N\left( \frac{\tilde{T}_{(\mathbf{C} \oplus \mathbf{B}) \times (\mathbf{D} \oplus 0)}[\varepsilon]}{T_{(\mathbf{C} \oplus \mathbf{B}) \times (\mathbf{D} \oplus 0)}[0], T_{(\mathbf{C} \oplus \mathbf{B}) \times (\mathbf{D} \oplus 0)}[0]} \right) \leq \bigoplus_{\varepsilon_0 = 0, 1, 2} \left( N\left( \frac{\tilde{T}_{\mathbf{C} \times \mathbf{D}}[\varepsilon - \varepsilon_0]}{T_{\mathbf{C} \times \mathbf{D}}[0], T_{\mathbf{C} \times \mathbf{D}}[0]} \right) N\left( \frac{\tilde{T}_{\mathbf{B} \times 0}[\varepsilon_0]}{T_{\mathbf{B} \times 0}[0], T_{\mathbf{B} \times 0}[0]} \right) \right).
\]
By Prop. 5.16 we know
\[
T_{\mathbf{B} \times 0}[0] \times T_{\mathbf{B} \times 0}[0] = \tilde{T}_{\mathbf{B} \times 0}[2];
\]
therefore the above inequality becomes
\[
N\left( \frac{\tilde{T}_{(\mathbf{C} \oplus \mathbf{B}) \times (\mathbf{D} \oplus 0)}[\varepsilon]}{T_{(\mathbf{C} \oplus \mathbf{B}) \times (\mathbf{D} \oplus 0)}[0], T_{(\mathbf{C} \oplus \mathbf{B}) \times (\mathbf{D} \oplus 0)}[0]} \right) \leq N\left( \frac{\tilde{T}_{\mathbf{C} \times \mathbf{D}}[\varepsilon - 2]}{T_{\mathbf{C} \times \mathbf{D}}[0], T_{\mathbf{C} \times \mathbf{D}}[0]} \right).
\]
We know $\sqrt{\frac{\mathbf{C} \oplus \mathbf{B}}{\mathbf{C} \oplus \mathbf{B}}} = \sqrt{\frac{\mathbf{C} \oplus \mathbf{B}}{\mathbf{C} \oplus \mathbf{B}}}$ and hence by (5-14a)
\[
\sum_{\varepsilon=0,1,2} N\left( \frac{\tilde{T}_{(\mathbf{C} \oplus \mathbf{B}) \times (\mathbf{D} \oplus 0)}[\varepsilon]}{T_{(\mathbf{C} \oplus \mathbf{B}) \times (\mathbf{D} \oplus 0)}[0], T_{(\mathbf{C} \oplus \mathbf{B}) \times (\mathbf{D} \oplus 0)}[0]} \right) = \sum_{\varepsilon=0,1,2} N\left( \frac{\tilde{T}_{\mathbf{C} \times \mathbf{D}}[\varepsilon]}{T_{\mathbf{C} \times \mathbf{D}}[0], T_{\mathbf{C} \times \mathbf{D}}[0]} \right).
Therefore, we must have

\[ N\left( \frac{\tilde{T}(C \oplus B) \times (D \oplus 0)[\varepsilon]}{T(C \oplus B) \times (D \oplus 0)[0], T(C \oplus B) \times (D \oplus 0)[0]} \right) = N\left( \frac{\tilde{T}(C \oplus D)[\varepsilon - 2]}{T(C \oplus D)[0], T(C \oplus D)[0]} \right). \]

Note that \( C \oplus B \) has length \( \ell + 2 \), thus we have

\[ N\left( \frac{\tilde{T}(C \times D)[\varepsilon]}{T(C \times D)[0], T(C \times D)[0]} \right) = N\left( \frac{\tilde{T}(C \oplus B) \times (D \oplus 0)[\varepsilon + 2]}{T(C \oplus B) \times (D \oplus 0)[0], T(C \oplus B) \times (D \oplus 0)[0]} \right) = \frac{2^{\ell - 2d} + \Xi(\ell + 2 - \varepsilon - 2)}{3} = \frac{2^{\ell - 2d} + \Xi(\ell - \varepsilon)}{3}. \]

This proves (i) when \( \ell \equiv 2 \pmod{4} \).

Now assume \( \ell \) is odd, let \( C_e := C \oplus 0 \) and \( D_e := D \oplus 0 \) be self-orthogonal codes of even length \( \ell + 1 \). Restricting to the subVOA \( V_{L_e}^T \otimes V_{L_e}^\tau \), we have decomposition of \( V_{L_e}^T \otimes V_{0_0^0}^\tau \)-modules

\[ T_{C_e \times D_e}[0] = \bigoplus_{\varepsilon = 0, 1, 2} T_{C \times D}[\varepsilon_0] \otimes T_{0 \times 0}[\varepsilon_0]. \]

We know that \( V_{0_0^0}^\tau = V_{L_e}^\tau \). Recall the fusion rules of \( V_{L_e}^\tau \):

\[ T_{0 \times 0}[0] \times T_{0 \times 0}[0] = T_{0 \times 0}[0] + T_{0 \times 0}[2]. \]

Therefore,

\[ N\left( \frac{\tilde{T}(C_e \times D_e)[0]}{T(C_e \times D_e)[0], T(C_e \times D_e)[0]} \right) \leq N\left( \frac{\tilde{T}(C \times D)[0]}{T(C \times D)[0], T(C \times D)[0]} \right) + N\left( \frac{\tilde{T}(C \times D)[1]}{T(C \times D)[0], T(C \times D)[0]} \right), \]

\[ N\left( \frac{\tilde{T}(C_e \times D_e)[1]}{T(C_e \times D_e)[0], T(C_e \times D_e)[0]} \right) \leq N\left( \frac{\tilde{T}(C \times D)[1]}{T(C \times D)[0], T(C \times D)[0]} \right) + N\left( \frac{\tilde{T}(C \times D)[2]}{T(C \times D)[0], T(C \times D)[0]} \right), \]

\[ N\left( \frac{\tilde{T}(C_e \times D_e)[2]}{T(C_e \times D_e)[0], T(C_e \times D_e)[0]} \right) \leq N\left( \frac{\tilde{T}(C \times D)[2]}{T(C \times D)[0], T(C \times D)[0]} \right) + N\left( \frac{\tilde{T}(C \times D)[0]}{T(C \times D)[0], T(C \times D)[0]} \right). \]

This gives

\[ \sum_{\varepsilon = 0, 1, 2} N\left( \frac{\tilde{T}(C_e \times D_e)[\varepsilon]}{T(C_e \times D_e)[0], T(C_e \times D_e)[0]} \right) \leq 2 \sum_{\varepsilon = 0, 1, 2} N\left( \frac{\tilde{T}(C \times D)[\varepsilon]}{T(C \times D)[0], T(C \times D)[0]} \right). \]

From (5-14a) we know

\[ \sum_{\varepsilon = 0, 1, 2} N\left( \frac{\tilde{T}(C_e \times D_e)[\varepsilon]}{T(C_e \times D_e)[0], T(C_e \times D_e)[0]} \right) = 2^{\ell + 1 - 2d} = 2 \cdot 2^{\ell - 2d} = 2 \sum_{\varepsilon = 0, 1, 2} N\left( \frac{\tilde{T}(C \times D)[\varepsilon]}{T(C \times D)[0], T(C \times D)[0]} \right). \]
Therefore inequalities in (5.24) must attain equalities. This gives
\[
N\left(\frac{\tilde{T}_{C \times D}[0]}{T_{C \times D}[0], T_{C \times D}[0]}\right) = 2^{\ell - 2d} - N\left(\frac{\tilde{T}_{C \times D}[1]}{T_{C \times D}[0], T_{C \times D}[0]}\right) + N\left(\frac{\tilde{T}_{C \times D}[2]}{T_{C \times D}[0], T_{C \times D}[0]}\right) = 2^{\ell - 2d} - \frac{2^{\ell + 1 - 2d} + \Xi(\ell + 1 - 1)}{3}.
\]

Similarly, we have
\[
N\left(\frac{\tilde{T}_{C \times D}[1]}{T_{C \times D}[0], T_{C \times D}[0]}\right) = 2^{\ell - 2d} - N\left(\frac{\tilde{T}_{C \times D}[2]}{T_{C \times D}[0], T_{C \times D}[0]}\right) = \frac{2^{\ell - 2d} - \Xi(\ell + 1 - 2)}{3},
\]
\[
N\left(\frac{\tilde{T}_{C \times D}[2]}{T_{C \times D}[0], T_{C \times D}[0]}\right) = 2^{\ell - 2d} - N\left(\frac{\tilde{T}_{C \times D}[1]}{T_{C \times D}[0], T_{C \times D}[0]}\right) = \frac{2^{\ell - 2d} - \Xi(\ell + 1)}{3}.
\]

This proves (ii). The final statement follows immediately from Prop. 5.4.

6. Z₃-orbifold construction and the Monster group

In this section, we discuss an application of Corollary 4.8 and Proposition 5.17. The main purpose is to construct certain 3-local subgroups of the Monster simple group.

Z₃-orbifold of the Leech lattice VOA. Let \( \Lambda \) denote the Leech lattice and let \( \tau \) be a fixed point free isometry of \( \Lambda \). It is well-known [DLM00] that the lattice VOA \( V_\Lambda \) has exactly one irreducible \( \tau \)-twisted module for \( i = 1, 2 \). We denote these twisted modules by \( V_\Lambda^{T,1} \) and \( V_\Lambda^{T,2} \), respectively. We define the \( V_\Lambda[0]-\)module
\[
V^i := V_\Lambda[0] \oplus (V_\Lambda^{T,1})_\mathbb{Z} \oplus (V_\Lambda^{T,2})_\mathbb{Z},
\]
where \( (V_\Lambda^{T,i})_\mathbb{Z} \) is the subspace of \( V_\Lambda^{T,i} \) of integral weights.

The next proposition is proved in [Miy13].

**Proposition 6.1** ([Miy13]). *The module \( V^i \) has a natural VOA structure. Moreover, it is a \( \mathbb{Z}_3 \) simple current extension of the VOA \( V_\Lambda \).*

**Remark 6.2.** That \( V^i \) has a natural VOA structure was first announced by Dong and Mason [DM94]. They also claimed that the full automorphism group of \( V^i \) is isomorphic to the Monster and \( V^i \cong V^2 \) as a VOA. However, the complete proof has not been published.

In [SS10], a 3-local characterization of the Monster simple group has been obtained.

**Theorem 6.3.** [SS10, Thm. 1.1] *Let \( G \) be a finite group and \( S \in \text{Syl}_3(G) \). Let \( H_1 \) and \( H_2 \) be subgroups of \( G \) containing \( S \) such that*
M1: $H_1 = N_G(Z(O_3(H_1)))$, $O_3(H_1)$ is extraspecial group of order $3^{13}$, $H_1/O_3(H_1) \cong 2.Suz : 2$ and $\text{Cent}_{H_1}(O_3(H_1)) = Z(O_3(H_1))$.

M2: $H_2/O_3(H_2) \cong \Omega^-_8(3)$, $O_3(H_2)$ is elementary abelian of order $3^8$ and $O_3(H_2)$ is a natural $H_2/O_3(H_2)$-module.

M3: $(H_1 \cap H_2)/O_3(H_2)$ is an extension of an elementary abelian group of order $3^6$ by $2 \cdot PSU_4(3) : 2^2$.

Then $G$ is isomorphic to the largest sporadic simple group, the Monster.

In this section, we will construct explicitly certain subgroups $H_1$ and $H_2$ of $\text{Aut}(V^2)$ such that $H_1$ and $H_2$ satisfy the hypotheses [M1] to [M3] above.

**Simple Current extension.** Let $V(0)$ be a simple rational $C_2$-cofinite VOA of CFT type and let $\{V(\alpha) \mid \alpha \in A\}$ be a set of inequivalent irreducible $V(0)$-modules indexed by an abelian group $A$. A simple VOA $V = \bigoplus_{\alpha \in A} V(\alpha)$ is called a $A$-graded extension of $V(0)$ if $V(0)$ is a full sub VOA of $V$ and $V$ carries a $A$-grading, i.e., $Y(x^\alpha, z) x^\beta \in V^{\alpha+\beta}$ for any $x^\alpha \in V^\alpha$, $x^\beta \in V^\beta$. In this case, the group $A^*$ of all irreducible characters of $A$ acts naturally on $V$: for $\chi \in A^*$, $\chi(v) = \chi(\alpha)v$, $v \in V(\alpha)$. In other words, $V(\alpha)$ is an eigenspace of $A^*$ for all $\alpha \in A$ and $V(0)$ is the fixed points of $A^*$.

If all $V^\alpha$, $\alpha \in A$, are simple current $V(0)$-modules, then $V$ is referred to as a $A$-graded simple current extension of $V(0)$. The abelian group $A$ is automatically finite since $V(0)$ is rational.

**Theorem 6.4 ([Shi07, SY03]).** Let $V = \bigoplus_{\alpha \in A} V(\alpha)$ and $V' = \bigoplus_{\alpha \in A} V'(\alpha)$ be simple VOAs graded by a finite abelian group $A$. Suppose that $V(0) = V'(0)$ is a simple rational $C_2$-cofinite VOA of CFT type and $V(\alpha)$ and $V'(\alpha)$ are simple current $V(0)$-modules for all $\alpha \in A$. Let $g$ be an automorphism of $V(0)$ which maps the set of isomorphism classes of $\{V(\alpha) \mid \alpha \in A\}$ to those of $\{V'(\alpha) \mid \alpha \in A\}$. Then there exists an isomorphism $\tilde{g}$ from $V'$ to $V$ such that $\tilde{g}|_{V(0)} = g$.

**Notation 6.5.** Let $S_A$ be the set of the isomorphism classes of the irreducible $V(0)$-modules $V(\alpha)$, $(\alpha \in A)$. For an automorphism $g$ of $V(0)$, we set $S_A \circ g = \{W \circ g \mid W \in S_A\}$, where $W \circ g$ denotes the $g$-conjugate of $W$, i.e., $W \circ g = W$ as a vector space and the vertex operator $Y_{W \circ g}(u, z) = Y_W(gu, z)$, for $u \in V$.

Define

$$H^N_A = \{ h \in \text{Aut}(V(0)) \mid S_A \circ h = S_A \},$$

$$H^C_A = \{ h \in \text{Aut}(V(0)) \mid W \circ h = W \text{ for all } W \in S_A \}.$$
Then we obtain the restriction homomorphisms
\[
\Phi_A^N : N_{\text{Aut}(V)}(A^*) \to H_A^N,
\]
\[
\Phi_A^C : C_{\text{Aut}(V)}(A^*) \to H_A^C.
\]
Applying Theorem to the case \(V = V'\), we show that \(\Phi_A^N\) is surjective. Since each \(V(\alpha)\) is irreducible, \(\text{Ker} \Phi_A^N = A^*\). By similar arguments, \(\Phi_A^C\) is surjective, and \(\text{Ker} \Phi_A^C = A^*\).

**Theorem 6.6** ([Shi07] (cf. [SY03])). Let \(V = \bigoplus_{\alpha \in A} V(\alpha)\) be a simple VOA graded by a finite abelian group \(A\). Suppose that the fusion rule \(V(\alpha) \times V(\beta) = V(\alpha + \beta)\) holds for all \(\alpha, \beta \in A^*\). Then the restriction homomorphism \(\Phi_A^N\) and \(\Phi_A^C\) are surjective and \(\text{Ker} \Phi_A^C = \text{Ker} \Phi_A^N = A^*\). That is, we have short exact sequences
\[
0 \to A^* \to N_{\text{Aut}(V)}(A^*) \to H_A^N \to 0,
\]
\[
0 \to A^* \to C_{\text{Aut}(V)}(A^*) \to H_A^C \to 0,
\]

A subgroup of the shape \(3^{1+12}(2 \text{ Suz : 2})\). Next we will construct a subgroup of the shape \(3^{1+12}(2 \text{ Suz : 2})\) in \(\text{Aut}(V^2)\). First we recall a theorem from [LY13].

**Proposition 6.7** ([LY13, Thm. 5.15]). Let \(L\) be an even positive definite rootless lattice. Let \(\nu\) be a fixed point free isometry of \(L\) of prime order \(p\) and \(\hat{\nu}\) a lift of \(\nu\) in \(O(\hat{L})\). Then we have an exact sequence
\[
1 \to \text{Hom}(L/(1 - \nu)L, \mathbb{Z}_p) \to \text{Cent}_{\text{Aut}(V_L)}(\hat{\nu}) \xrightarrow{\phi} \text{Cent}_{O(L)}(\nu) \to 1. \tag{6-1}
\]
Recall that
\[
V^2 = V_A[0] \oplus (V_A^{T,1})_Z \oplus (V_A^{T,2})_Z.
\]
There is a natural automorphism \(\tau'\) of order 3 which acts on \(V^2\) as 1 on \(V_A[0]\), as \(\xi\) on \((V_A^{T,1})_Z\), and as \(\xi^2\) on \((V_A^{T,2})_Z\).

**Proposition 6.8.** Let \(\tau'\) be defined as above. Then
\[
N_{\text{Aut}(V^2)}(\langle \tau' \rangle) \cong 3^{1+12}(2 \text{ Suz : 2}).
\]

**Proof.** Let \(S_A := \{V_A[0], (V_A^{T,1})_Z, (V_A^{T,2})_Z\}\) and
\[
H_A^N = \{h \in \text{Aut} V_A[0] \mid S_A \circ h = S_A\}.
\]
Recall that \(V_A[0]\) has exactly nine irreducible modules and only \((V_A^{T,1})_Z\) and \((V_A^{T,2})_Z\) have top weight 2 (see [Miy13]). Therefore, \(S_A\) is invariant under the action of \(\text{Aut} V_A[0]\) and \(H_A^N = \text{Aut} V_A[0]\).

Now consider the simple current extension:
\[
V_A = V_A[0] \oplus V_A[1] \oplus V_A[2],
\]
which is graded by \(\mathbb{Z}_3\).
Let $S_B := \{V_\Lambda[0], V_\Lambda[1], V_\Lambda[2]\}$. Then we have a short exact sequence
\[
0 \to \langle \hat{\tau} \rangle \to N_{\text{Aut}(\Lambda)}(\langle \hat{\tau} \rangle) \xrightarrow{\varphi} N_B^N \to 0,
\]
where
\[
H_B^N := \{ h \in \text{Aut}(V_\Lambda^\tau) \mid S_B \circ h = S_B \}.
\]
Since $V_\Lambda[1]$ and $V_\Lambda[2]$ are the only irreducible modules of $V_\Lambda[0]$ which have the top weight 1, all elements of $\text{Aut}(V_\Lambda^\tau)$ lift to $\text{Aut}(\Lambda)$ and hence
\[
H_B^N = \text{Aut}(V_\Lambda^\tau) \cong N_{\text{Aut}(\Lambda)}(\langle \hat{\tau} \rangle)/\langle \hat{\tau} \rangle \cong 3^{12} \cdot N_{\text{O}(\Lambda)}(\tau)/\langle \tau \rangle.
\]

Finally, we will show that the subgroup $\varphi^{-1}(3^{12})$ is extra-special of order $3^{13}$.

Let $V_{\Lambda,i}^T = S[\tau] \otimes T_i$ be the irreducible $\tau^i$-twisted module of $V_\Lambda$ for $i = 1, 2$. Recall the central extension (see [DL96])
\[
1 \longrightarrow < -\xi > \longrightarrow \hat{\Lambda}_{\tau^i} \longrightarrow \Lambda \longrightarrow 1
\]
associated to the commutator map $c^{\tau^i}(\alpha, \beta) = \xi^{\langle \alpha, \beta \rangle - \langle \tau^i \alpha, \beta \rangle}$, where $\xi^3 = 1$ and $i = 1, 2$.

Set $K = \{a^{-1} \tau^i(a) \mid a \in \hat{\Lambda}_{\tau^i} \}$. Then $\hat{\Lambda}_{\tau^i}/K \cong 2 \times 3^{1+12}$ and $\hat{\Lambda}_{\tau^i}/K$ acts faithfully on $T_i$. By [FLM88, Prop. 5.5.3], there is an exact sequence
\[
1 \to \mathbb{C}^\times \to N_{\text{Aut}(T_i)}(\pi(\hat{\Lambda}_{\tau^i}/K)) \xrightarrow{\text{int}} \text{Aut}(\hat{\Lambda}_{\tau^i}) \to 1,
\]
where $\pi$ is the representation of $\hat{\Lambda}_{\tau^i}/K$ on $T_i$ and $\text{int}(g)(x) = gxg^{-1}$.

Consider a subgroup
\[
X = \{(g, g', g'') \in \hat{\Lambda}/K \times \hat{\Lambda}_{\tau^i}/K \times \hat{\Lambda}_{\tau^2}/K \mid \varphi(g) = \text{int}(g') = \text{int}(g'') \}.
\]
Then $X \cong 2 \times 3^{1+12}$. By the similar argument as in [FLM88, Section 10.4], one can show that $X$ acts on the space $V_\Lambda \oplus V_{\Lambda,1}^T \oplus V_{\Lambda,2}^T$. Moreover, there is a canonical embedding $\nu : X \to N_{\text{Aut}(V_\Lambda)}(\langle \tau^i \rangle)$ such that $\varphi(\nu(3^{1+12})) = \text{Hom}(\Lambda/(1 - \tau)\Lambda, \mathbb{Z}_3)$.

A subgroup of the shape $3^8 \cdot \Omega^{-}(8,3).2$. We will construct a subgroup $H_2$ of $\text{Aut}(V_\Lambda^\tau)$ satisfying the following conditions:

(i) $H_2/O_3(H_2) \cong \Omega_8^-(3) : 2$.
(ii) $O_3(H_2)$ is elementary abelian of order $3^8$.
(iii) $O_3(H_2)$ is a natural $H_2/O_3(H_2)$-module.

Recall that the Coxeter-Todd lattice $K_{12}$ can be constructed by using the Hexacode. Let $\pi : \mathbb{Z}[\xi] \to \mathbb{Z}[\xi]/2\mathbb{Z}[\xi] \cong \mathbb{F}_4$ be the natural quotient, where $\xi$ is the primitive cubic root of unity. We know the lattice $K_{12}$ can be defined as
\[
K_{12} := \{ (x_1, \ldots, x_6) \in (\mathbb{Z}[\xi])^6 \mid (\pi x_1, \ldots, \pi x_6) \in \mathcal{H} \},
\]
where $\mathcal{H}$ is the hexacode.
Since $\mathcal{H}$ is self-dual, every irreducible $V^\tau_{K_{12}}$-module is a simple current module. There are exactly $3^6 \cdot 3 \cdot 3 = 3^8$ of them. These irreducible modules form an abelian group isomorphic to $\mathbb{Z}_3^8$ under the fusion product. Denote $R(V^\tau_{K_{12}})$ the set of irreducible $V^\tau_{K_{12}}$-modules.

As before, we denote irreducible $V^\tau_{K_{12}}$-modules as

$$S^a[x], T^a[x], \text{ and } \check{T}^a[x]$$

for $a \in K_{12}^\ast \mod K_{12}, x \in \mathbb{Z}_3$. If $a = 0$, we will omit this superscript. The fusion rules are given as follows:

1. $S^a[x] + S^b[y] = S^{a+b}[x + y], \quad S^a[x] + T^b[y] = T^{b-a}[x + y]$ \hspace{1cm} (6-2)
2. $S^a[x] + \check{T}^b[y] = T^{a+b}[y - x], \quad T^a[x] + \check{T}^b[y] = S^{b-a}[x - y]$ \hspace{1cm} (6-3)
3. $T^a[x] + T^b[y] = T^{-(a+b)}[-(x + y)], \quad \check{T}^a[x] + \check{T}^b[y] = T^{-(a+b)}[-(x + y)]$. \hspace{1cm} (6-4)

Note that the operation $+$ on the left hand side of the above equations is the fusion product.

Recall that $K_{12}^\ast/K_{12} \cong (\mathcal{H} \times 0)/(\mathcal{H} \times 0) = 0 \times \mathbb{F}_3^6$ as an abelian group. It also forms a non-singular quadratic space of minus type (see for example [CS83]) if we define the quadratic form

$$q_F(a + K_{12}) = 3\langle a, a \rangle \mod 3.$$

**Proposition 6.9.** We define a map $q : R(V^\tau_{K_{12}}) \rightarrow \mathbb{Z}_3$ by

$$q(S^a[x]) = q_F(a), \quad \text{and} \quad q(T^a[x]) = q(\check{T}^a[x]) = q_F(a) + x + 1.$$  

Then $q$ is a quadratic form on $R(V^\tau_{K_{12}})$ and $(R(V^\tau_{K_{12}}), q)$ is non-singular space of minus type.

**Proof.** Let $B(x, y) = \frac{1}{2}(q(x + y) - q(x) - q(y))$. Then $B$ is a symmetric form and we have

(i) $B(S^a[x], S^b[y]) = 3\langle a, b \rangle$;

(ii) $B(S^a[x], T^b[y]) = 6\langle a, b \rangle + 2x$ and $B(S^a[x], \check{T}^b[y]) = 3\langle a, b \rangle + x$;

(iii) $B(T^a[x], T^b[y]) = B(\check{T}^a[x], \check{T}^b[y]) = 3\langle a, b \rangle + 2(x + y - 1)$;

(iv) $B(\check{T}^a[x], T^b[y]) = B(T^a[x], \check{T}^b[y]) = 6\langle a, b \rangle + x + y - 1$.

It is straightforward to check that this form is bilinear with respect to the fusion products and hence $q$ is a quadratic form.

Using the bilinear form, it is clear that

$$R(V^\tau_{K_{12}}) = \{ S^a[0] \mid a \in K_{12}^\ast \mod K_{12} \} \perp \text{Span}\{ S[1], T[1] \}.$$  

Note that $\{ S[1], T[1] \}$ is a hyperbolic pair and $\{ S^a[0] \mid a \in K_{12}^\ast \mod K_{12} \}$ is a quadratic space isometric to $K_{12}^\ast/K_{12} \cong \mathbb{F}_3^6$, which is a non-singular space of minus type. This completes our proof. \hfill \Box
Lemma 6.10. This quadratic form is $\text{Aut} V_{K_{12}}^\tau$-invariant. That is for any $M \in R(V_{K_{12}}^\tau)$ and $g \in \text{Aut} V_{K_{12}}^\tau$, we have $q(M) = q(M \circ g)$.

Proof. Recall that the weights of irreducible $V_L^\tau$-modules are given by (see Tanabe and Yamada[TY07, (5,10)]):

$$\text{wt } V_{L,0,j}[\varepsilon] = \frac{2j^2}{3} + \mathbb{Z},$$

$$\text{wt } V_{L,i,j}^\tau(\tau^j)[\varepsilon] = \frac{10 - 3(j^2 + \varepsilon)}{9} + \mathbb{Z},$$

for $i = 1, 2, j, \varepsilon \in \mathbb{Z}_4$.

Therefore, by the decompositions given in Prop. 3.7, we know that the weights of irreducible $V_{K_{12}}^\tau$-modules are

$$\text{wt } S^a[\varepsilon] = \frac{2(a,a)}{3} + \mathbb{Z},$$

$$\text{wt } T^a[\varepsilon] = \frac{2(1 + \varepsilon + \langle a,a \rangle)}{3} + \mathbb{Z}.$$

Since the action $M \mapsto M \circ g$ preserves weights, we are done. □

Proposition 6.11. Define a map $\phi : \text{Aut} V_{K_{12}}^\tau \rightarrow O(R(V_{K_{12}}^\tau), q)$ by

$$g \mapsto (M \mapsto M \circ g).$$

Then $\phi$ is a group monomorphism and $\text{Im } \phi = \Omega^-_8(3) \cong \Omega^-_8(3).2$.

Proof. That $\phi$ is a group homomorphism follows from that $Y_M q(u, z) = Y_M (gu, z)$.

We now prove that $\phi$ is injective. Let $g \in \ker \phi$ and consider the simple current extension:

$$V_{K_{12}} = V_{K_{12}}^\tau \oplus V_{K_{12}}[1] \oplus V_{K_{12}}[2].$$

Since $M \circ g \cong M$ for $M \in R(V_{K_{12}}^\tau)$, the set $S_B = \{V_{K_{12}}^\tau, V_{K_{12}}[1], V_{K_{12}}[2]\}$ is also fixed by $g$ pointwise. Therefore, $g$ can be lifted to $\text{Cent}_{\text{Aut}(V_{K_{12}}^\tau)}(\langle \tau \rangle)$. By Prop. 6.7, we have the exact sequence

$$0 \rightarrow \text{Hom}(K_{12}/(1-\tau)K_{12}) \rightarrow \text{Cent}_{\text{Aut}(V_{K_{12}}^\tau)}(\langle \tau \rangle) \rightarrow \text{Cent}_{O(K_{12})}(\tau) \rightarrow 0.$$

Recall that $O(K_{12}) \cong 6.\text{PSU}_4(3).2^2 \cong 3.\Omega^-_6(3)$ (see for example [CS83]) and $\text{Cent}_{O(K_{12})}(\tau) \cong 6.\text{PSU}_4(3).2$. Therefore, $\text{Cent}_{O(K_{12})}(\tau)/\langle \tau \rangle \cong 2.\text{PSU}_4(3).2 \cong \Omega^-_6(3)$.

By direct calculations, it is easy to verify that $\text{Cent}_{\text{Aut}(V_{K_{12}}^\tau)}(\langle \tau \rangle)$ acts faithfully on $R(V_{K_{12}}^\tau)$. Hence $\ker \phi = \text{id}$.

Next we determine the image of $\phi$. In [LY13], a subgroup $H \cong +\Omega^-8(8,3)$ is constructed explicitly using $\sigma$-involutions associated to $c = 4/5$ Virasoro vectors (see [LY13, Rmk 5.52, Thm 5.64]). Since $+\Omega^-8(8,3)$ is an index 2 subgroup of the full orthogonal group $O^-_8(3)$, we have $\text{Im } \phi \cong +\Omega^-8(8,3)$ or $O^-_8(3)$. 

Suppose $\text{Im}\phi \cong O^+_8(3)$. In this case, $Z(\text{Aut}(V^r_{K_{12}})) \cong \mathbb{Z}_2$.

Let $h$ be an order 2 element in $Z(\text{Aut}(V^r_{K_{12}}))$. Then $\phi(h)$ is the $-1$ map on $R(V^r_{K_{12}})$. By the fusion rules (see Equation 6.2), we have

$$V_{K_{12}}[1] \circ h = \phi(h)(V_{K_{12}}[1]) \cong V_{K_{12}}[2] \quad \text{and} \quad V_{K_{12}}[2] \circ h = \phi(h)(V_{K_{12}}[2]) \cong V_{K_{12}}[1].$$

Therefore, $h$ lifts to $\text{Aut}(V_{K_{12}})$ and is contained in the subgroup isomorphic to $N_{\text{Aut}V_{K_{12}}}(\hat{\tau})/\langle \hat{\tau} \rangle \cong 3^6 : (2.\text{PSU}_4(3).2^2)$, which is centerless. It is a contradiction and hence $\text{Im}\phi \cong \Omega^-(8,3) \cong \Omega^-_8(3)$. \hfill \Box

For now on, we denote $R(V^r_{K_{12}})$ by $R$ for simplicity. Notice that $(R, -q)$ also forms a non-singular quadratic space of minus type. Therefore, $(R, -q) \cong (R, q)$.

Let $\eta : (R, -q) \to (R, q)$ be a linear isometry and set

$$S_\eta = \{(a, \eta(a)) \in R \times R \mid a \in R\}.$$

**Lemma 6.12.** The set $S_\eta$ is a maximal totally singular subspace of $R \times R$. Moreover, the minimal conformal weight of $S_\eta$ is 2.

**Proof.** It is clear that $S_\eta$ is a vector subspace of $R \times R$ and $\dim_{\mathbb{F}_3} S_\eta = \dim_{\mathbb{F}_3} R = 8$.

By the definition of $\eta$, we also have

$$q(a, \eta(a)) = q(a) + q(\eta(a)) = q(a) - q(a) = 0 \quad \text{for all } a \in R.$$

Therefore, $S_\eta$ is totally singular. It is maximal since $\dim_{\mathbb{F}_3} S_\eta = 1/2 \dim_{\mathbb{F}_3}(R \times R)$.

Recall that the conformal weights of the elements in $R$ are given by

$$\text{wt}(a) = \begin{cases} 0 & \text{if } a = 0, \\ 1 & \text{if } q(a) = 0, a \neq 0, \\ 4/3 & \text{if } q(a) = 1, \\ 2/3 & \text{if } q(a) = 2. \end{cases}$$

Therefore, $\text{wt}(a, \eta(a)) = 2$ if $a \neq 0$. \hfill \Box

**Lemma 6.13.** Let $S$ be a maximal totally singular subspace of $R \times R$ such that the minimal conformal weight of $S$ is $\geq 2$. Then there is a linear isomorphism $\eta : R \to R$ such that $q(\eta(a)) = -q(a)$ for all $a \in R$ and $S = S_\eta$.

**Proof.** Let $p_i : R \times R \to R, i = 1, 2$, be the natural projection to the $i$-th coordinate.

Let $(a, b) \in S$ be a non-zero vector. Then neither $a$ nor $b$ is zero; otherwise, $\text{wt}(a, b) = \text{wt}(a) + \text{wt}(b) = 1$. That means $p_i|_S$ is injective for any $i = 1, 2$ and hence $p_i|_S$ is bijective for any $i = 1, 2$ since $\dim_{\mathbb{F}_3}(S) = \dim_{\mathbb{F}_3}(R)$.

Let $\eta = p_2 \circ (p_1|_S)^{-1}$. Then $\eta : R \to R$ is a linear isomorphism and

$$S = \{(a, \eta(a)) \mid a \in R\}.$$
Since $S$ is totally singular, we have $q(a, \eta(a)) = q(a) + q(\eta(a)) = 0$ for any $a \in R$ and hence $q(\eta(a)) = -q(a)$ for all $a \in R$. \hfill \Box

**Proposition 6.14.** Let $S$ be the set of all maximal totally singular subspaces of $R \times R$ such that the minimal conformal weight is $\geq 2$. Then $S$ is transitive under the action of $O^{-}_8(3) \wr 2$.

**Proof.** This follows immediately from Lemmas 6.12 and 6.13. \hfill \Box

**Lemma 6.15.** Let $S^\sharp$ be a maximal totally isomorphic subspaces of $R \times R$ such that $\bigoplus_{M \in S} M = V^\sharp$. Then

$$Stab_{Aut(V^\sharp)}(S^\sharp) \cong Aut((V^\tau_{K_{12}}) \cong \Omega^{-}_8(3).2. $$

**Proof.** By Lemma 6.13, $S^\sharp = \{(a, \eta(a)) \mid a \in R\}$ for some linear isomorphism $\eta : R \to R$ such that $q(\eta(a)) = -q(a)$.

Note also that for any $g \in O(R, q)$, $\eta g \eta^{-1}$ is also in $O(R, q)$. Moreover, the map

$$\xi : O(R, q) \to O(R, q) \times O(R, q)$$

$$g \mapsto (g, \eta g \eta^{-1})$$

is a group monomorphism. It is also easy to verify that $Stab_{O(R,q)} \wr 2(S^\sharp) = \xi(O(R,q))$. Hence, we have the desired conclusion. \hfill \Box

**Proposition 6.16.** Let $S^\sharp$ be defined as in Lemma 6.15. Let $A$ be the abelian subgroup of $Aut(V^\sharp)$, which acts on $V^\sharp$ as the dual group of $S^\sharp$. Then

$$N_{Aut(V^\sharp)}(A) \cong 3^8.\Omega^{-}_8(3).2.$$ 

**Proof.** This follows from Theorem 6.6 and Lemma 6.15. \hfill \Box

We have constructed two subgroups

$$H_1 = Stab_{Aut(V^\tau)}(V^\tau_\Lambda) = N_{Aut(V^\tau)}(\langle \tau \rangle) \cong 3^{1+12}.(2.\text{Suz} : 2),$$

$$H_2 = Stab_{Aut(V^\tau)}(V^\tau_{K_{12}} \otimes V^\tau_{K_{12}}) = N_{Aut(V^\tau)}(3^8) \cong 3^8.\Omega^{-}_8(3).2.$$ 

**Lemma 6.17.** The intersection of $H_1$ and $H_2$ is the common stabilizer of the subVOAs $V^\tau_\Lambda$ and $V^\tau_{K_{12}} \otimes V^\tau_{K_{12}}$ and $H_1 \cap H_2 = 3^8.(3^6.(2PSU_4(3).2^2))$.

**Proof.** Recall the exact sequence

$$1 \to 3^8 \to H_2 \to \Omega^{-}_8(3).2.$$ 

By definition, it is clear that the normal subgroup $3^8$ stabilizes all irreducible $V^\tau_{K_{12}} \otimes V^\tau_{K_{12}}$ submodules in $S^\sharp$ and hence it stabilizes $V^\tau_\Lambda$, also.

Note that $H_2$ acts on $(R, q)$ as a subgroup of isometries. The subgroup that stabilizes $V^\tau_\Lambda$ will stabilize a 6-dimensional non-singular subspace of $R$ and it has the shape

$$3^6 : (2.PSU_4(3).2^2) \quad \text{(see [CCN+85, page 141])}.$$
Hence, $H_1 \cap H_2 \cong 3^8.\langle 3^6 : (2.PSU_4(3).2^2) \rangle$. \hfill \square

**Remark 6.18.** Unfortunately, we do not have a direct proof that $\text{Aut}(V^\sharp)$ is finite and hence we cannot apply Theorem 6.3 to conclude that $\text{Aut}(V^\sharp)$ is isomorphic to the Monster.

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