ON NON-NEWTONIAN FLUIDS AND PHASE FIELD APPROACH: EXISTENCE AND REGULARITY

RODOLFO SALVI

Abstract. The object of this paper is twofold. Firstly, we study a class of generalized Newtonian fluid related to "power law". For the corresponding non-Newtonian Navier-Stokes problems, the existence of a weak and periodic solutions is proved in the large for a bounded domain in $\mathbb{R}^3$. Further, variational inequalities and local-in-time well-posedness of the initial-boundary value problem are investigated. Secondly, we deduce a generalization of the Graffi-Kazhikhov-Smagulov model based on an advective-diffusion process in the context of multiphase theory. Local in time well-posedness of the initial-boundary value problem is investigated.

1. Introduction

In an Eulerian description of the flow field of a fluid the balance of a transferable property $P$, defined per unit of mass in a unit control volume, reads

$$\partial_t (\rho P) + \nabla \cdot (\rho \mathbf{v} P) = -\nabla \cdot \mathbf{I}_P + F_P. \tag{1.1}$$

Here, and in the following, Cartesian tensor notation will be used, and the summation convention with respect to repeated indices.

$\rho$ stands for the density of the fluid and $\mathbf{v}$ stands for the velocity vector field. The left-hand side shows the change with time of the property (in a unit volume), and the change due to the divergence in a convective transport by the flow through the boundaries of the control volume. The first term on the right-hand side describes the divergence of the transport $\mathbf{I}_P$ through these boundaries by molecular effects. The last term stands for any internal or external process, or source, that contributes to the change of $P$ in the control volume. The expression of the left-hand side is, in general, independent of $P$ and the process concerned.

The first term on the right-hand side depends on the nature of the property, while the last term depends on both the nature of the property and the process considered. If we consider the mass as the transferable property, thereby making no distinction between possible components, then $P$ becomes equal to unit, being the mass per unit of mass, and $\mathbf{I}_P = 0$. If, further, assume no sources or sinks of mass present in the flow, also $F_P = 0$. We then obtain the equation for the conservation of mass

$$\partial_t \rho + \nabla \cdot \rho \mathbf{v} = 0.$$
Concerning the equation of the conservation of the momentum, we may start from the above general balance equation (1.1), where in this case we take for $P$ any of the components $v_i$ of the momentum per unit of mass. The transfer of momentum by molecular effect results in a stress, so that for the balance equation in $x_i$-direction we have $(I_{vi})_{ij} = -\sigma_{ji}$. Here $\sigma := (\sigma_{ji})$ with $1 \leq (j,i) \leq n$. The stress is defined positive if it is directed in the positive direction of $x_i$.

Let $F_i$ an external force working on a unit volume of the fluid in $x_i$-direction. The balance equation for the momentum then reads

$$\partial_t (\rho v_i) + \partial_{x_j} (\rho v_j v_i) = \partial_{x_j} \sigma_{ji} + F_i.$$ (1.2)

The stress tensor $\sigma$ can be divided into a part that corresponds to the average value of the normal stresses for all directions, that is, a spherically-symmetric part, which is invariant under rotation of the coordinate system, and an anti-symmetric part.

The spherically-symmetric part is equal to $1/3\sigma_{ii}$. It contains the thermodynamic pressure $\pi$ and an additional term proportional to $\nabla \cdot v$. So:

$$\frac{1}{3} \sigma_{ii} = -\pi + k \nabla \cdot v,$$

$k$ is referred as volume viscosity. Consequently, the expression of the stress tensor can be written

$$\sigma_{ji} = (-\pi + k \nabla \cdot v) \delta_{ji} + T_{ji},$$

where $T_{ji}$ is the anti-symmetric part, so that $T_{ii} = 0$ and is called the deviator of stress tensor. System (1.2) contains the deviator $T$ of the stress tensor which is not expressed explicitly via the unknowns of the system.

In conclusion, from the balance equations of mass, momentum the motion of fluid is described by the system of equations in the Cauchy form

$$\rho \partial_t v + \rho v \cdot \nabla v - \nabla \cdot (T - (\pi - k \nabla \cdot v) I) = \rho f,$$

$$\partial_t \rho + \nabla \cdot \rho v = 0,$$ (1.3)

where $v$ is the velocity vector of a particle in a point $x$ at time $t$ and $v_1, v_2, \ldots, v_n$ are the components of $v$, $\pi$ is the fluid pressure, $f$ is the density of external force and $I$ is unit matrix and $\rho$ is the mass density. $\nabla \cdot T$ stands for the vector

$$\left( \sum_{j=1}^{n} \partial_{x_j} T_{j1}, \sum_{j=1}^{n} \partial_{x_j} T_{j2}, \ldots, \sum_{j=1}^{n} \partial_{x_j} T_{jn} \right),$$

whose coordinates are the divergence of rows of the matrix $T = (T_{ji})$.

For $n = 3$, the above four scalar equations contain twelve unknown quantities. To achieve a unique solution up to a constant in the pressure, further relations between the velocity field $v$ and the stress tensor $T$ are necessary.

System (1.3) describes flows of all kinds of fluids. But it contains the deviator $T$ of the stress tensor which is not expressed explicitly via the unknowns of the system. Since the stresses and deformations in a fluid (apart external forces) depend on each other it is reasonable related the stress tensor to the spatial deformations, i.e. the symmetric part $D := (d_{ij})$ of the spatial variation and is called deformation tensor.

So, as a rule, to express the deviator of the stress tensor via the unknowns of
the system, one uses relations between the deviator of stress tensor, the rate of deformation tensor or deformation rate

\[ D = (d_{ji})_{j,i=1}^n, \quad d_{ji} = \frac{1}{2} \left( \frac{\partial v_j}{\partial x_i} + \frac{\partial v_i}{\partial x_j} \right) = \frac{1}{2} (\nabla u + (\nabla u)^T)_{j,i}. \]

\((\cdot, \cdot)^T\) is the transpose matrix.

By establishing the connection between the deviator and the deformation rate, we determine the type of fluid.

These relations depending on the considered substance, are called Constitutive or Rheological equations. These relations are hypothesis to be checked out for concrete fluids by experimental data.

A relation, largely used in the last 150 years, is a linear relation

\[ T = 2\mu D - \frac{2}{3} \mu \nabla \cdot v I, \]

and the constant \(\mu\) is referred as the dynamic viscosity.

Such fluids are classically known as the Newtonian fluids. Newtonian fluid has been the main object of mathematical research in hydrodynamics. It has the determining relation

\[ \sigma = 2\mu D + ((k - \frac{2}{3} \mu) \nabla \cdot v + \pi) I. \]

With this expression of \(T\) the system (1.3) assume the following form

\[ \begin{align*}
\partial_t \rho v + \nabla \cdot (\rho v \otimes v) - \mu \Delta v + \nabla \pi - (k + \frac{1}{3} \mu) \nabla \nabla \cdot v &= \rho f, \\
\partial_t \rho + \nabla \cdot \rho v &= 0.
\end{align*} \]

For Newtonian fluids \(k \simeq 0.\)

This system of equations (Navier-Stokes system) describes flows, under moderate velocities, of most viscous fluids.

The Navier-Stokes equations are generally accepted as a right governing equations for compressible or incompressible motion of viscous fluids. The classical Navier-Stokes equations are essentially the simplest equations describing the motion of a fluid.

Physically, it is assumed that the constituent particles of the fluid are too small for their dynamics can interact substantially with macroscopic motion or it makes no difference what the fluids consisted of. But, the Newtonian model is inadequate for fluids having a complex chemical structure. Many substances of industrial significance, especially of multi-phase nature (forms, emulsions, dispersions etc) do not conform to the Newtonian postulate of the linear relationship between \(T\) and \(D\). Accordingly, these fluids are variously known as non-Newtonian, nonlinear, complex or rheologically complex fluids.

In general, there are essentially two possibilities to get constitutive equations; either based on microstructure of the substances or based on the phenomenological rheology. The description of mechanical properties of different materials in different deforming regimes is the subject of rheology. The motion of a body as a whole is not considered, (it is the subject of theoretical mechanics), rather it is the relative motion of particles of a body that is under consideration. Rheology determines the dependence between forces acting on a material body and its deformations. So far, one of the main problem of rheology is the determination of links between stress,
deformation, velocity of deformation, and their derivatives with respect to time. The simplest possible deviation from the Newtonian fluid behavior occurs when the apparent viscosity is not constant. Such fluids are called non-Newtonian.

System for which the value of $T$ at a point within the fluid is determined only by the current value of $D$ or $\dot{\gamma}$ (shear stress) at that point. These substance are variously known as purely viscous, inelastic, time-independent or generalized Newtonian fluids. Their shear behavior can be described by the relation

$$T \sim f(\dot{\gamma}).$$

Depending upon the form of the above function $f$ three possibilities exist.

1. **Shear-Thinning Fluids.** This is perhaps the most widely encountered type of time-independent non-Newtonian fluid behavior in engineering practice. It is characterized by a viscosity which gradually decreases with increasing shear rate. This behavior is due to the progressive orientations of the molecules in the motion direction.

Shear thinning behavior fluids can be characterized by viscosity of the form

$$\mu = m(\dot{\gamma})^{p-1} \quad (1.5)$$
called "Power law" or "Ostwald-de Waele Equation".

Clearly, $0 < p < 1$ will yield $d\mu/d\dot{\gamma} < 0$, i.e. shear-thinning behavior fluids are characterized by a value of $p$ (power-law index) smaller than unity, for commonly used $\mu$.

1.2. **Shear-Thickening or Dilatant Behavior.** For some fluids, the opposite behavior take place; the viscosity increases with the increasing shear rate, and hence the name "Shear-Thickening".

Of the time-independent fluids, this sub-class has generate very little interest. The currently available limited information suggests that it is possible to approximate the behavior for these systems also by the power law model with the power-law index $p$ taking on values greater than unity.

1.3. **Visco-plastic fluid behavior.** This type of non-Newtonian fluid behavior is characterized by the existence of a threshold stress which must be exceeded for the fluid to deform or flow. Essentially such substance will behave like an elastic solid when the externally applied stress is less than the yield stress. Example is the Bingham fluid.

1.4. **Time-Dependent Behavior.** In case of incompressible pure viscous fluids, for constitutive equations on the foundation of differential models, the stress can be expressed as power series involving increasing powers of the deformation tensor. Continuing, this conception leads to the usage of convected derivatives based on the application of the invariance of material properties with respect to the frame reference. Many substances, notably in food, pharmaceutical product manufacturing sectors display flow characteristics which cannot be described by a simple mathematical expression as $f$. This is so because their viscosities are not only functions of the applied shear stress but also of the duration for which the fluid has been subjected to shearing as well as his previous kinematic history. These fluids present two typical features: stress relaxation and creep. The first phenomenon is the progressive rather than instantaneous stress decay when the fluid deformation
suddenly vanishes. The second effect, dual to the former, consists in a non-linear increasing deformation, though the fluid undergoes a constant stress. One of the simplest ways of modifying the constitutive equation of a Newtonian fluid in order to account for the "memory properties" of a given fluid, is to add a term containing a time derivative or some sort of time derivative of the stress. The addition of the term containing the time derivative of the stress makes the equations capable of representing the phenomenon of stress relaxation. This type of model is called differential model. Rate type of constitutive equation containing time derivative of first order only, or of the general form

\[
\frac{d\tau}{dt} = f(\tau, \text{kinematic tensors}).
\]

In general, \(\tau\) is a non-deviatoric extra stress. \(\tau\) and the kinematic tensors are calculated in \(t\). It follows that the stress at any time \(t\) can, in principle, calculate from a knowledge of the stress at previous time \(\hat{t}\). For this fluids the stress is expressed in integral form.

\[
\sigma = \int_{0}^{\infty} f(s) G(t - s) ds.
\]

The function \(G\) is called the stress relaxation modulus. The physical assumption underlying the form is clear: it is assumed that all the deformations which occurred in the past as measured by the Cauchy tensor contribute linearly to the present value of the stress.

The weighing function \(f(s)\) is a material function which completely determines a particular material obeying such a linearity rule.

In this paper we consider two classes of non-Newtonian fluids. The first a generalized Newtonian fluid. The second is a model deduced in the context of multi-phase flow.

The plan for the paper is as follows. Section 2 contains some preliminary results and notations. Section 3 is devoted to the formulation of the problem and contains the proof of the long life existence of a so-called weak solution. In section 4 we study the existence of periodic solution. Section 5 treats the existence problem for variational inequalities. Section 6 a well-posedness problem is discussed. Finally, in section 7 we deduce a Graffi-Kazhikov-Smagulov model and discuss the existence problem.

2. Preliminaries

In the sequel we will assume that \(\Omega\) denotes an open set in \(\mathbb{R}^n\) which is generally assumed to be bounded hence \(\bar{\Omega}\) is compact. \(\Gamma\) denotes the boundary of \(\Omega\). Moreover, it is assumed that \(\Omega\) is a smooth domain of class \(C^k\) with \(k\) a positive integer. Furthermore, we assume that the unit normal vector field \(n(x)\) with \(x \in \Gamma\) is inward pointing on \(\Gamma\). If it is necessary we consider also an extension of \(n\) in a neighborhood of \(\bar{\Omega}\). With symbols \(c, c_0, c_1, \text{etc.}\), we denote generic positive constants. When we wish to emphasize the dependence of \(c\) on some parameter \(r\), we shall write \(c(r)\).

We do not distinguish in our notations whether the functions are \(\mathbb{R}^m\)-valued (vector) or \(\mathbb{R}^m \times \mathbb{R}^m\)-valued (tensor or matrix). So \(v = (v_i)\) denotes a vector with components \(v_i\); \(T = (T_{ji})\) denotes a tensor with components \(T_{ji}\). Furthermore,
\[|u|^2 = \sum_{i=1}^{n} |u_i|^2\] and for a matrix or 2-tensor \(A = (a_{ji})\), \(1 \leq j, i \leq n\) the Frobenius norm is
\[
\|A\|^2 = a : a = a_{ji}a_{ji} = \sum_{j,i=1}^{n} |a_{ji}|^2.
\]

The form of \(\|A\|^p\) is obvious.

We define \(C_0^\infty(\Omega)\) to be the linear space of infinitely many times differentiable functions (vectors, tensors) with compact supports in \(\Omega\). Now let \((C_0^\infty(\Omega))^\prime\) denote the dual space of \(C_0^\infty(\Omega)\), the space of distributions on \(\Omega\). In our notations \(C_0^\infty(\Omega; \mathbb{R}^n)\) coincides with \(C_0^\infty(\Omega) \times C_0^\infty(\Omega) \times \ldots \times C_0^\infty(\Omega)\). In what follows we do not specify this extension. We denote, in general, by \(\langle \cdot, \cdot \rangle\) the duality pairing between \((C_0^\infty(\Omega))^\prime\) and \(C_0^\infty(\Omega)\).

Let \(\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n\) and set \(|\alpha| = \sum_{i=1}^{n} \alpha_i\). We set
\[
\frac{\partial}{\partial x_i} = \partial_i, \quad D^2_\alpha = D^\alpha = \partial^{[\alpha]}_{\alpha_1 \ldots \alpha_n},
\]
so that \(\nabla = (\partial_1, \ldots, \partial_n)\) the gradient operator and \(\nabla \cdot\) the divergence operator.

We denote \(C_0^\infty(\Omega)\) the linear subspace of divergence free functions of \(C_0^\infty(\Omega)\).

For any \(s, q\), \(s \geq 0, q \geq 1\), \(W^s_q(\Omega)\) denotes the usual Slobodeckii-Sobolev space of order \(s\) on \(L_q(\Omega)\). Further, the norm (defined intrinsically involving first order differences of the highest-order derivatives) on \(W^s_q(\Omega)\) is denoted by \(\|\cdot\|_q^s\). If \(s = 1\) we write \(\|\cdot\|_q\) and \(|\cdot|_q\) denotes the norm in \(L_q(\Omega)\)-spaces. When \(q = 2\), \(W^2_q(\Omega)\) is usually denoted by \(H^s(\Omega)\) and we drop the subscript \(q = 2\) when referring to its norm. \(H^s(\Omega)\) \((s \in \mathbb{N})\) is a Hilbert space for the scalar product
\[
((u, v))_s = \sum_{|\alpha| \leq s} \int_{\Omega} D^\alpha u D^\alpha v dx.
\]

In particular, in \(L_q(\Omega)\), we write the \(L_q\)-duality pairing \((u, v)_q = \int_{\Omega} u v dx\) with \(u \in L_q\), and \(v \in L_{q'}\) with \(q' = q/(q - 1)\) and the norm \(|v|_q\).

Further, we define \(W^s_{q,0}(\Omega)\) the closure of \(C_0^\infty(\Omega)\) for the norm \(\|\cdot\|_q^s\).

We denote \(W^{-s}_{q,0}(\Omega)\) the dual space of \(W^s_{q,0}(\Omega)\) and \(\|\cdot\|_{q^{-s}}\) denotes its norm when \(q'\) satisfies \(1/q + 1/q' = 1\).

Moreover \(W^{2-2/q}_{q}(\Omega) \hookrightarrow BUC^1(\Omega)\) with \(q > n + 2\) (\(BUC =\) bounded uniformly continuous).

The embedding
\[
W^1_p(\Omega) \subset L_q(\Omega)
\]
is compact if \(p < n\) and \(q < \frac{np}{n-p}\) or if \(p \geq n\) and \(\infty > q > 1\).

In particular the embedding
\[
L_{q'}(\Omega) \subset W^{-1}_{p'}(\Omega)
\]
is compact.

Let us introduce the following spaces of divergence-free functions. We denote by
\[
V^s_q = \{v|v \in W^s_{q,0}(\Omega), \nabla \cdot v = 0\}.
\]

\(V^s_q\) is the closure of \(C_0^\infty(\Omega)\) (divergence free) for the norm \(\|\cdot\|_q^s\), and it is a closed subspace of \(W^s_{q,0}(\Omega)\).

Moreover in the sequel we use the triplet
\[
V^s_q \subset H \subset \left(V^s_q\right)^\prime.
\]
We set $V_2^s := V^s$, $V^1 := V$ and $V^0 := H$. Moreover, we introduce the projection operator $P_q$ ($P_2 \equiv P$) : $L_q \rightarrow V_q^0$. It is well known that the operator $P_q$ is continuous on $L_q$ and the subspace $V_q^0$ is complemented. Thus, the following decomposition of $L_q$

$$L_q = \text{Range}P_q \oplus \text{Ker}P_q$$

holds true.

It is interesting to observe that $\text{Ker}P_q = \{ \phi \in L_q | \phi = \nabla p_1 + \nabla p_2 \}$ where $p_1, p_2$ are generalized solutions of the problems

$$\Delta p_1 = 0, \ \partial_n p_1 = f \cdot n \text{ on } \Gamma,$$

and

$$\Delta p_2 = \nabla \cdot g, \ p_2 = 0 \text{ on } \Gamma,$$

respectively. Here $g, f \in L_q$ and $f \cdot n \in H^{-1/2}(\Gamma)$ satisfying $<f \cdot n, 1> = 0$.

We recall that analogous decomposition of $L_q$ holds working with the subspace $\tilde{V} = \{ \phi | \phi \in L_q, \nabla \cdot \phi = 0 \}$.

We further define the Stokes operator on $L_q$

$$A_q = -P_0 \Delta,$$

with domain $D(A_q) = W^2_q(\Omega) \cap V^1_q$. If $q = 2$ we set $A_2 := A$.

For any Banach space $X$ and for any $T > 0$ we denote by $L_r(0, T; X)$ the set of $X$-valued functions defined a.e. in $[0, T]$ and $L_r$ summable in sense of Bochner. Frequently we consider $X = W^s_q(\Omega)$. In such cases, for any $\phi \in L_r(0, T; W^s_q(\Omega))$, $\phi$ stands for the function $\phi(t)$ or $\phi(x, t)$.

Throughout the paper we denote $Q_t = (0, t) \times \Omega$ and the parabolic Sobodeckii-Sobolev space $W^{s, r}_q(Q_T)$ of order $s$ in space variable and of order $r$ in time variable on $L_q$. We will denote $\| \cdot \|_{q, r}^s$ the norm in this space. In the following we make use of the inequality, for $q > 3$ ,

$$\sup_{(x, t) \in Q_T} |v| \leq \|v\|_{W^{2, 1}_q(Q_T)}.$$

Moreover, we set

$$Z(T) := W^1_q(0, T; L_q(\Omega)) \cap L_q(0, T; W^2_q(\Omega)) \hookrightarrow C((0, T; W^{2-2/q}_q(\Omega)).$$

In this case the embedding constant can blow up as $t \rightarrow 0^+$ if the functions are different from zero in $t = 0$.

$W^{2-2/q}_q(\Omega)$ can be considered as time-trace of $Z(T)$.

In addition, let us consider the affine space

$$\tilde{H}^k(\Omega) = \{ \phi \in H^k(\Omega), \partial_\nu \phi = 0 \text{ on } \Gamma, \int_\Omega \phi dx = c \}.$$

In this manner, the functions in $\tilde{H}^k$ are uniquely fixed and we can not distinguish the norms $\| \phi \|_{H^2}$ and $|\Delta \phi|_2$, $\| \phi \|_{H^3}$ and $|\nabla \Delta \phi|_2$ in $\tilde{H}^3$. Throughout the paper we shall use the following propositions.

**Proposition 1** (Gagliardo-Nirenberg inequality). Let $\Omega \subset R^n$ bounded and sufficiently regular. The multiplicative inequality,

$$\sum_{|\alpha| = r} |D^\alpha \phi|_q \leq c|\phi|_{q_1}^{1-\theta}(\sum_{|\alpha| = r} |D^\alpha \phi|_q)^{\theta},$$

(2.1)
for $1 \leq q_1, q_2 \leq \infty$, $0 \leq r \leq l$,
\[
\frac{n}{q} - r = (1 - \theta)\frac{n}{q_1} + \theta(\frac{n}{q_2} - l), \quad \frac{r}{l} \leq \theta \leq 1,
\]
holds with the following exceptions:

(a) if $r = 0$, $l < \frac{n}{q_2}$, and $q_1 = \infty$ and $\Omega$ unbounded, we assume in addition that
or $\phi \to 0$ as $x \to \infty$ or $\phi \in L_p$ for some $p > 0$;

(b) if $1 < q_1 < \infty$ and $l - r - \frac{n}{q_2}$ is a non-negative integer, then does not hold for
$\theta = 1$.

Let $\Omega \subset \mathbb{R}^n$ be a open bounded set and $X$ is a topological metric space with
metric $d_X$. We consider the Fréchet space $C(\Omega; X)$ of continuous functions $f : \Omega \to X$
equipped with the metric
\[
d(f, g) = \sup_{y \in \Omega} d_X(f(y), g(y)).
\]
Further we denote $X^*_E$ the dual of $X$ equipped with the weak topology.

**Theorem 2.1** (Abstract Ascoli-Arzelà Theorem). Let $\Omega$ be compact and $X$ is a

topological metric space equipped with a metric $d_X$. Let $\mathcal{F} = \{f_n\}$ be a sequence of

functions in $(\Omega; X)$. We assume that

1. $\mathcal{F}(y) = \{f_n(y) \mid f \in \mathcal{F}\}$ has compact closure in $X$;
2. $\mathcal{F}$ is equi-continuous under $d_X$, i.e.
\[
\sup_{y} d_X(f_n(y + h) - f(y)) \to 0, \text{ for } h \to 0,
\]
uniformly in $n$.

Then $\mathcal{F}$ is precompact in $C(\Omega; X)$.

Moreover, let the set of functions $\langle f_n(y), \phi \rangle$ ($y \in \Omega$) be equi-continuous for
any fixed $\phi$ belonging to a dense subset of $X$.

Then $f_n \in C(\Omega; X^*_E)$ and there exists an $f \in C(\Omega; X^*_E)$ such that

$f_n \to f$,
in $C(\Omega; X^*_E)$ as $n \to \infty$.

The previous theorem has $L_p(\Omega)$-version (see [35], [38], for example).

**Theorem 2.2** (Frechét-Kolmogorov Theorem). Let $X \subset E \subset Y$ be Banach
spaces, the embeddings $X \subset E \subset Y$ be continuous. Let $\mathcal{F} = \{f_n\}$ be a sequence of
functions in $L_p(0, T; X)$ with $1 \leq p \leq \infty$. We assume that

1. $\mathcal{F}(t) = \{f_n(t) \mid f \in \mathcal{F}\}$ has compact closure in $E$;
2. $\mathcal{F}$ is equi-continuous in $L_p(0, T; Y)$ i.e.
\[
\|f_n(t + h) - f(t)\|_{L_p(0, T-h; Y)} \to 0, \text{ for } h \to 0,
\]
uniformly in $n$.

Then $\mathcal{F}$ is precompact in $L_p(0, T; Y)$. If $p = \infty$ then $\mathcal{F}$ belongs to $C(0, T; E)$ and
is relatively compact in this space.

Making use of the derivative in time the theorem assumes the following form.

**Theorem 2.3**. Let $X \subset E \subset Y$ be Banach spaces, the embeddings $X \subset E$, and
$E \subset Y$ be continuous. Let $\mathcal{F} = \{f_n\}$ be a sequence of functions in $L_p(0, T; X)$ with
$1 \leq p \leq \infty$ and $\partial_t f_n \in L_r(0, T; Y)$ with $1 \leq r \leq \infty$. We assume that
Schwarz’s inequality: the Cauchy inequalities in integral inequalities: Schwarz’s inequality and Gram-absurdum argument (see [2]). Assume \( \det A \) is positive for all \( x,y \) for \( p = \infty \) the property continues to hold for \( r > 1 \).

2.1. About Cauchy and Schwarz’s Inequalities. Let \( a = (a_1, a_2, \ldots, a_n) \), \( b = (b_1, b_2, \ldots, b_n) \), be two vectors with \( n \) components. Then

\[
\det \left( \sum_{i=1}^{n} a_i b_i - \sum_{i=1}^{n} a_i^2 \right) > 0 \iff \sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2 - (\sum_{i=1}^{n} a_i b_i)^2 = \frac{1}{2} \sum_{i,j=1}^{n} (a_i b_j - a_j b_i)^2 > 0.
\]

We have written the classical Cauchy inequality in matrix form. An alternative proof is as follows (see [14]). The quadratic form

\[
\sum_{i=1}^{n} (xa_i + yb_i)^2 = x^2 \sum_{i=1}^{n} a_i^2 + 2xy \sum_{i=1}^{n} a_i b_i + y^2 \sum_{i=1}^{n} b_i^2
\]

is positive for all \( x, y \) and therefore has a negative discriminant, unless \( xa_i + yb_i = 0 \) for some \( x, y \) not both zero, and for all \( i \). A generalization of the above inequality is obtained by considering the leading principal minors property of a matrix \( A \) obtained with \( n \) vectors \( a_i \) (linearly independent), \( i = 1, 2, \ldots, n \), with \( n \) real components in the following manner:

\[
\sum_{i=1}^{n} (x_1 a_1^2 + x_2 a_2^2 + \cdots + x_n a_n^2) > 0 \iff
\]

\[
\det \begin{pmatrix}
    a_1 \cdot a_1 & a_1 \cdot a_2 & \cdots & a_1 \cdot a_n \\
    a_2 \cdot a_1 & a_2 \cdot a_2 & \cdots & a_2 \cdot a_n \\
    \cdots & \cdots & \cdots & \cdots \\
    a_n \cdot a_1 & a_n \cdot a_2 & \cdots & a_n \cdot a_n
\end{pmatrix} > 0.
\]

It is also possible to express the determinant as the sum of squares of determinants. A simple proof to obtain \( \det A \neq 0 \) (we need this in what follows) is a reductio ad absurdum argument (see [2]). Assume \( \det A = 0 \), then there exists \( n \) numbers \( \lambda_i \), \( i = 1, 2, \ldots, n \) (not all equal to zero) such that

\[
\sum_{i=1}^{n} \lambda_i a_i \cdot a_j = 0.
\]

Multiplying the above equation by \( \lambda_j \) and summing up with respect to \( j \) we obtain

\[
\sum_{i=1}^{n} \lambda_i a_i \cdot \sum_{i=1}^{n} \lambda_i a_i = 0.
\]

This contradicts the linear independence of \( \{a_i\} \).

Bearing in mind the Leibnitz’s "Calculus Summatorium ", we can translate the Cauchy inequalities in integral inequalities: Schwarz’s inequality and Gram-Schwarz’s inequality:
1) Schwarz’s inequality: let \( f, g \) be two functions defined and continuous in a domain \( \Omega \). Then
\[
\left( \int_{\Omega} f(x)g(x)dx \right)^2 < \int_{\Omega} f^2(x)dx \int_{\Omega} g^2(x)dx.
\]
unless \( af(x) = bg(x) \) with \( a, b \) constants no both zero.

2) Gram-Schwarz’s inequality: let \( f_i(x) \) be \( n \) continuous function in \( \Omega \) then
\[
\det \begin{pmatrix}
\int f_1(x)f_1(x)dx & \int f_1(x)f_2(x)dx & \cdots & \int f_1(x)f_n(x)dx \\
\int f_2(x)f_1(x)dx & \int f_2(x)f_2(x)dx & \cdots & \int f_2(x)f_n(x)dx \\
\cdots & \cdots & \cdots & \cdots \\
\int f_n(x)f_1(x)dx & \int f_n(x)f_2(x)dx & \cdots & \int f_n(x)f_n(x)dx
\end{pmatrix} > 0
\]
unless the functions \( f_i(x) \), \( i=1,2,\ldots,n \), are linearly dependent.

3. Solvability of non-Newtonian Flow Problems

In this section we concentrate on incompressible generalized Newtonian fluids related to power law. The section is devoted to the solvability of system (3.1) in bounded domain \( \Omega \). Most of the results have their origin in the earlier results of the author concerning the theory of non-homogeneous viscous fluids.

3.1. Additional Notation. Set
\[
D := D(u) = (d_{j,i}) = \frac{1}{2} (\nabla u + (\nabla u)^T),
\]
the symmetric part of the tensor \( \nabla u \) with \( d_{j,i} = \frac{1}{2} (\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}) \).

Let us remark, recalling the Frobenius norm, that
\[
\|D\|^2 = D : D = |d_{j,i}|^2 := \sum_{j,i=1}^{n} d_{j,i}d_{j,i}.
\]
We consider as viscous stress tensor a modified power-law, commonly used in practice and in mathematical literature, i.e.
\[
T := T(u) = 2\mu(\|D\|^2)D(u),
\]
with \( \mu \) a suitable function. Then, the \( i^{th} \) entry of \( \nabla \cdot T \) can be written as follows
\[
(\nabla \cdot (2\mu(\|D\|^2)D)_{i} = \sum_{j=1}^{n} \partial_j (2\mu(\|D\|^2)d_{ji}) = 2\mu'(\|D\|^2) \sum_{j=1}^{n} d_{ji}\partial_j \|D\|^2 +
\]
\[
\mu(\|D\|^2) \sum_{j=1}^{n} \partial_j (\partial_j u_i + \partial_i u_j) = \mu(\|D\|^2) \Delta u_i +
\]
\[
4\mu'(\|D\|^2) \sum_{j,k,l=1}^{n} d_{ik}d_{jl}\partial_k\partial_l u_j = \sum_{j,k,l=1}^{n} t_{ij}^{kl}(u)\partial_k\partial_l u_j.
\]
Here
\[
t_{ij}^{kl} = \mu(\|D\|^2)\delta_{kl}\delta_{ij} + 4\mu'(\|D\|^2)d_{ik}(u)d_{jl}(u),
\]
with
\[ \partial_j \|D\|^2 = 2 \sum_{k,l=1}^n d_{kl} \partial_j d_{kl}, \nabla \cdot u = 0. \]

We define the quasi-linear differential operator
\[ T_p := T_p(u) = -\nabla \cdot T(u) = \sum_{k,l=1}^n T^{k,l}(u) \partial_k \partial_l, \]
with the matrix-valued coefficients
\[ T^{k,l}(u) = (t^{k,l}_{ij}). \]

For the viscosity function \( \mu(\cdot) \) several models may be given. For convenience, we tacitly have in mind the model
\[ 2\mu(\|D\|^2)D := \mu_0(1 + \|D\|^2)\frac{p}{2} - 2D, \]
(\( \mu_0 \) is a positive constant) which is often quoted as standard model in the mathematical literature.

If \( p > 1 \) the operator \( T_p \) is strongly elliptic (see [37]).

Thus we consider the system
\[ \begin{align*}
\rho \partial_t u + \rho u \cdot \nabla u - \nabla \cdot (T(u)u - \pi I) &= \rho f, \\
\partial_\rho + \nabla \cdot (\rho u) &= 0, \\
\nabla \cdot u &= 0, \quad u(0) = u_0, \quad u_G = 0, \quad \rho(0) = \rho_0 \geq 0, \quad 1 < p.
\end{align*} \tag{3.2} \]

The question is for which exponent, respectively for which \( p \), does a solution of the system (3.2) exist.

### 3.2. Structure conditions.

The approach to find a solution of system (3.2) is via monotone operator theory and compactness arguments. So we set classical structure condition assumptions. We denote \( M_{sim} \) the set of symmetric matrices of order \( n \). In general, we assume that for a \( p > 1 \) and \( q \in [p-1, p) \) there exist \( \alpha, \beta > 0 \) such that for all \( \eta \in M_{sim} \) the function \( A \) satisfies

1. (continuity) \( A : Q_T \times \mathbb{R}^n \times M_{sim} \to M_{sim} \) is a Carathéodory function, i.e., \( (x,t) \mapsto A(x,t,u,D) \) is measurable for every \( (u,D) \) and \( (u,D) \mapsto A(x,t,u,D) \) is continuous for almost \( (x,t) \in Q_T \);
2. (Growth and coercivity) There exist \( c_1 > 0, c_2 > 0, g_1 \in L^{p'}(Q_T), \)
   \( g_2 \in L^1(Q_T), g_3 \in L_{(p/r)'}(Q_T) \), such that
   \[ |A(x,t,u,D)| \leq g_1(x,t) + c_1(|u|^{p-1} + |D|^{p-1}), \]
   \[ A(x,t,u,D) : D \geq -g_2(x,t) - g_3(x,t)|u|^r + c_2|D|^p; \]
3. (Strict monotonicity) For \( (x,t) \in Q_T \), the map \( (u,D) \mapsto A(x,t,u,D) \) is \( C^1 \)-function and
   \[ (A(x,t,u,D_1) - A(x,t,u,D_2) : (D_1 - D_2)) > 0, \quad \forall D_1 \neq D_2. \]

Now we state the definition of solution which we are seeking.

First, \( T \) satisfies the structure conditions. The form
\[ T_p := -\nabla \cdot (2\mu(\|D\|^2)D(u), \]

defines a continuous operator $T_p$ acting from $W^{1}_{p,0}(\Omega)$ into $(W^{1}_{p,0}(\Omega))^\prime$ and
\[
<T_p u, v> = \int_\Omega \mu(||D||^p)D(u)D(v)dx,
\]
for $u, v \in W^{1}_{p,0}(\Omega)$.

Remark: The operator $T_p$ defines a one-to-one correspondence between $W^{1}_{p,0}(\Omega)$ and $(W^{1}_{p,0}(\Omega))^\prime$ with inverse $(T_p)^{-1}$, monotone, bounded and continuous.

By the Korn-inequality,
\[
|||u|||^p = \int_\Omega |D(u)|^p dx,
\]
is a new equivalent norm on $W^{1}_{p,0}(\Omega)$ and the operator $T_p$ is well defined in this space and conserves the continuity from $W^{1}_{p,0}(\Omega)$ into $(W^{1}_{p,0}(\Omega))^\prime$.

Further, an accurate choice of basis is relevant.

$V^s$ is the adherence of $V$ in $H^s(\Omega)$.

We choose $s$ such that if $v \in H^s(\Omega)$ then $\partial_s v \in L^\infty(\Omega)$, so holds
\[
V^s \subset V_p \subset H \subset (V_p)^\prime \subset V^s)^\prime.
\]

In the case that $p = 2$ we omit the subscript.

In general, we denote $\{w_m\}$ a total sequence, that is, a sequence such that span \{\{w_m : m \in \mathbb{N}\} is dense in $V_p$. This is guaranteed by the assumption that $V_p$ is separable. According to the problem treated, we, tacitly, choose or an arbitrary total sequence $\{w_m\}$ or the set of eigenvectors to the problem
\[
(w_m, \phi)_{V_p^*} = \lambda_i(w_m, \phi),
\]
where $(\cdot, \cdot)$ is the scalar product in $H$.

**Definition 1.** - Let $p > 1$, $u_0 \in H$, $f \in L^p(0,T;(W^{1}_{p,0}(\Omega))^\prime)$ with $p' = \frac{p}{p-1}$, $q$ a Sobolev conjugate and $M$ is a positive constant. Then a couple of functions $(u, \rho)$ is called a weak solution to the problem (3.2) if:

i) $\sqrt{p}u \in L^\infty(0,T;H) \cap L_p(0,T;W^{1}_{p,0}(\Omega))$, $\partial_t(\rho u) \in (L_p(0,T;V_p(\Omega)))^\prime$,

\[\rho \in L^\infty(Q_T), \partial_t \rho \in (L^p(0,T;W^{1,q}_{p,0}(\Omega)))^\prime \cap L^{5p/3}(0,T;W^{-1}_{5p/3}(\Omega)), 0 \leq \rho \leq M,\]

ii) the following integral identities hold for all smooth $\psi$ and divergence free $\phi$, with $\psi(T) = \phi(T) = 0$,

\[
\int_0^T (\rho, \partial_t \psi + u \cdot \nabla \psi) dt = -(\rho_0 u_0, \psi(0)),
\]

\[
\int_0^T ((\rho u, \partial_t \phi) + (\rho u, u \cdot \nabla \phi) - (T(u)u, D(\phi)) + (\rho f, \phi)) dt = -(\rho_0 u_0, \phi(0)).
\]

The problem (3.2) with $\rho$ constant is studied, relatively. The existence of weak solutions for $p \geq \frac{3n+2}{n+2}$ first appeared in [17], [18]. Later, Nečas and his collaborators investigated the existence and regularity of the problem (3.2). The best results are obtained in the case of periodic spatial functions.

We do not want to comment on all of the literature here, but concerning the existence, the regularity problem and the attempts to find optimal exponent $p$, we refer to the survey article [23], and to the paper [3].

The case $\rho \geq c > 0$ is considered in [12].
We prove next the existence of weak solution of the problem (3.2).

**Theorem 3.1.** Let \( u_0 \in H, \rho_0 \in L_\infty(\Omega), 0 \leq \rho_0 \leq M \) and \( f \in L_p(0,T; (W^1_p,0(\Omega))') \), \( p \geq 1 + \frac{2n}{n+1} \). Then there exists a weak solution to the problem (3.2).

**Proof.** The plan of the proof is as follows:

1. construction of a semi-Faedo-Galerkin approximation;
2. a priori estimates;
3. compactness results;
4. passing to the limit in the semi-Faedo-Galerkin approximation.

### 3.3. Semi-Faedo-Galerkin approximation.

Throughout the paper \( P \) denotes the orthogonal projection from \( L_2(\Omega) \) onto \( H \). We shall denote by \( \{ w_k(x) \} \) a total sequence in \( V_p \), sufficiently regular.

Let \( W_m := \text{span}\{ w_k(x) : 1 \leq k \leq m \} \). We denote \( P_m \) the projection from \( H \) onto \( W_m \).

The basic idea of the existence proof is to approximate \( u(x,t) \) by functions \( u^m(x,t) \) in finite-dimensional subspace \( W_m \) of \( H \) of dimension \( m \) and approximate \( \rho(x,t) \) by functions \( \rho^m(x,t) \) in an infinite-dimensional space by solving the transport equation replacing \( u \) for \( u^m \). This approximating procedure is called semi-Faedo-Galerkin approximation. This gives a system of ODE’s for \( u^m \).

For our analysis, we take \( W_m \subset H \) the linear space spanned by the vectors \( (w_1, w_2, \ldots, w_m) \) where \( w_1, w_2, \ldots, w_m \) are the first \( m \)-solutions of the spectral problem

\[
(w_j, v)_{V^*} = \lambda_i(w_j, v),
\]

\( \forall v \in V^* \).

We look for \( (u^m, \rho^m) \) solution of the following system

\[
\begin{align*}
u^m &= \sum_{i=1}^{m} c_i^m(t) w_i(x) \in C(0,T; W_m), \\
\rho^m \partial_t u^m + \rho^m u^m \cdot \nabla u^m - \nabla \cdot Tu^m + \nabla \pi^m &= \rho^m f, \\
\partial_t \rho^m + u^m \cdot \nabla \rho^m &= 0, \\
\nabla \cdot u^m &= 0, \\
\nabla \cdot u^m &= 0, \quad u^m(0) = u_{0}^m, \quad \rho^m(0) = \rho_{0}^m,
\end{align*}
\]

where \( u^m(0) \to u_0 \) in \( H \) and \( \rho_{0}^m \) is a smooth function such that \( \rho_{0}^m \to \rho_0 \) in \( L_q(\Omega) \) for some \( q \geq 1 \) and \( M \geq \rho_{0}^m \geq \frac{1}{m} > 0 \).

### 3.4. Continuity equation.

First, we treat the existence of the continuity equation (3.3).

Assuming \( u^m \in C^1(0,T; H^2(\Omega)) \) is known, the solution of the continuity equation

\[
\begin{align*}
\partial_t \rho^m + u^m \cdot \nabla \rho^m &= 0, \\
\rho^m(0) &= \rho_{0}^m,
\end{align*}
\]

is given by
with \( 0 < \frac{1}{m} \leq \rho_0^m \leq M \) is obtained by the method of the characteristics.

According, the characteristic \( y^m \) to (3.4) are determined as a solution of the system

\[
\frac{dy^m(\tau; t, x)}{d\tau} = u^m(\tau; y^m), \quad y^m(0; t, x) = x.
\]

(3.5)

The Cauchy-Lipschitz theory provides the existence of a solution of (3.5). Consequently, the explicit form for \( \rho^m \) is

\[
\rho^m(x, t) = \rho_0^m(y^m(0, t, x)).
\]

It follows the maximum principle:

\[
\frac{1}{m} \leq \rho^m(x, t) \leq M, \quad \forall (x, t) \in Q_T.
\]

Integrating over \( \Omega \), we get

\[
dt \int_{\Omega} \rho^m(t) dx = 0,
\]

and after integration over interval \((0, t)\), the conservation of mass results:

\[
\int_{\Omega} \rho^m(t) dx = \int_{\Omega} \rho^m(0).
\]

It may be noticed that, at this level, the estimate of the density does not depend on the regularity of \( u^m \).

Furthermore, if \( u^m \in W_{(m)} \) then \( \partial_t \rho^m \in L_p(0, T; (W_{q}^1(\Omega))') \cap L_r(0, T; H^{-1}(\Omega)) \) with \( r \geq 1 \) and \( q = \frac{3p}{3-p} \) if \( p < 3 \) or \( q > 1 \) if \( p \geq 3 \), uniformly with respect to \( m \).

Now, applying the gradient operator \( \nabla \) to (3.4), we easily get

\[
\partial_t \nabla \rho^m + \nabla u^m \cdot \nabla \rho^m + u^m \cdot \nabla \rho^m = 0.
\]

(3.6)

Multiplying (3.6) by \( |\nabla \rho^m|^p - 2 \nabla \rho^m \) \((p \geq 2)\) and, after integration over \( \Omega \), we get easily

\[
\frac{d}{dt} |\nabla \rho^m|^p \leq c |\nabla \rho^m|^p |\nabla u^m|_{\infty},
\]

and it follows

\[
|\nabla \rho^m(t)|_{p} \leq |\nabla \rho_0^m|_p \exp \int_0^t c |\nabla u^m|_{\infty} d\tau.
\]

It is worth to note that, at first level of regularity of the density, we need \( \nabla u^m \in L_1(0, T; L_{\infty}(\Omega)) \).

(In the case of compressible fluid the above regularity for the divergence of the velocity is required also at zero level. This is the main obstacle to prove the existence of a weak solution in that context. According, to obtain some results on the existence of weak solution for compressible fluids assumption on the summability of the density does seem to be required.)

We continue to prove estimates of the solution of the continuity equation. Multiplying (3.6) by \((\rho^m)^{p-1} \), \((p \geq 2)\) and, after integration over \( \Omega \), we get
\[ \frac{d}{dt}(\rho^m)^p + u^m \cdot \nabla (\rho^m)^p = 0, \]

then

\[ |\rho^m(t)|_p = |\rho_0^m|_p, \]

for every \( t \in [0, T] \).

This simple relation implies the uniqueness of the solution and if \( \rho_0^m \) converges to \( \rho_0 \) in \( L_p(\Omega) \) then

\[ |\rho^m(t)|_p \to |\rho_0|_p. \]

Later, we will discuss the consequences of the above estimate.

Now, we pass to consider the existence problem for the system (momentum equation)

\[ \rho^m \partial_t u^m + \rho^m u^m \cdot \nabla u^m - \nabla \cdot T(u^m) + \nabla \pi^m = \rho^m f, \]

\[ u^m(0) = u_0^m, \]

with \( u^m = \sum_{i=1}^{m} c_i^m(t)w_i^m \), divergence free.

It is sufficient to prove the existence of the coefficient \((c_i^m(t))_{i=1}^{m}\) such that (3.7) holds. For this, we project (3.7) onto \( W(\rho^m) \) and obtain

\[ \rho^m \partial_i u^m, w_i + (\rho^m u^m \cdot \nabla u^m, w_i) + (T(D)u^m, D(w_i)) = (\rho^m f, w_i), \quad i = 1, \ldots, m, \]

\[ u^m(0) = u_0^m. \]

We introduce the matrix \( A = (a_{i,j})_{i,j=1}^{m} \) with components

\[ a_{i,j} = \int_{\Omega} \rho^m w_i w_j dx. \]

The system (3.8) can be written as

\[ A \frac{d}{dt} c^m = H(c^m), \quad c^m(0) = c_0^m, \]

where \( c^m = (c_1^m(t), c_2^m(t), \ldots, c_m^m(t)) \) and \( H \) is easily understood. The existence of a solution of (3.8) follows by standard ODE theory once we prove that \( A \) is invertible. This fact is a consequence of the results in subsection 2.1 and of linear independence of the vectors \( \{\sqrt{\rho^m} w_i\}, \quad i = 1, 2, \ldots, m. \)

(In [2] the invertibility of \( A \) is obtained by ”reductio ad absurdum” ; in other papers it is assumed true).

Consequently, \( A \) is invertible and the system (3.8) can be written

\[ \frac{d}{dt} c^m = A^{-1} H(c^m). \]

Standard ODE theory implies the solvability of (3.8) in \((0, t_m) \subseteq (0, T)\).

By the Carathéodory’s extension theorem, the global solvability of (3.8) derives from the following global a priori estimates for \( u^m \).
3.5. A Priory Estimates. Passing to consider the momentum equation, we multiply (3.10) by $c^m(t)$ and summing the result over $i$ we get

\begin{equation}
\frac{1}{2} \int_{\Omega} (\rho^m \partial_t |u^m|^2 + \rho_m u^m \cdot \nabla |u^m|^2) dx + (Tu^m, D(u^m)) = \int_{\Omega} \rho^m f u^m dx.
\end{equation}

Multiplying the continuity equation by $|u^m|^2/2$, integrating over $\Omega$ and adding the result to (3.9), we get

\begin{equation}
\frac{1}{2} \int_{\Omega} \partial_t (\rho^m |u^m|^2) dx + (Tu^m, D(u^m)) = \int_{\Omega} \rho^m f u^m dx.
\end{equation}

According to the assumptions on $T$ and the Korn inequality, after integration over $(0, t)$ ($t < t_m$), we obtain

\begin{equation}
|\sqrt{\rho^m(t)} u^m(t)|_2^2 + \int_0^t \|u^m\|^p_p d\tau \leq c \int_0^t \|f\|_{(V_p)}^p d\tau + |\sqrt{\rho^m_0 u^m_0}|_2.
\end{equation}

Consequently,

$$\sqrt{\rho^m} u^m \in L_\infty(0, t_m; L_2(\Omega)), \quad u^m \in L_p(0, t_m; V_p),$$

uniformly with respect to $m$.

We notice that $u^m \in L_\infty(0, t_m; L_2(\Omega))$ holds for finite $m$, only.

Thanks to the above uniform estimates,

$$|c^m(t)| \leq C,$$

follows, uniformly with respect to $m$, jointly to the continuity of $c^m(t)$ on $[0, t_m]$. By Carathéodory’s theory we can set $t_m = T$.

We estimate now the time-derivative of the unknowns. First, we recall that $0 < \frac{1}{m} \leq \rho^m \leq M$. Moreover

$$\partial_t \rho^m \in L_q(0, T; H^{-1}(\Omega)) \cap L_p(0, T; W_r^{-1}(\Omega), )$$

for arbitrary $q \geq 1$ and $r$ is a Sobolev conjugate. Since $\rho^m$ is bounded uniformly with respect to $m$, thanks to the $L_q$-(Ascoli-Arzelà) theorem, $\{\rho^m\}$ is a compact set in $L_\infty(0, T; W^{-1}_r(\Omega))$ or in $L_\infty(0, T; H^{-1}(\Omega))$, and, in addition, is a compact set in $C(0, T; (L_q(\Omega))_w)$ with $q \geq r$, for example.

Now, taking into account the continuity equation, (3.7) can assume the form

\begin{equation}
\partial_t (\rho^m u^m) + \nabla \cdot (\rho^m u^m \otimes u^m) - \nabla \cdot Tu^m + \nabla \pi^m = \rho^m f.
\end{equation}

Set

$$b(\rho^m u^m, u^m, w) := \int_{\Omega} \nabla \cdot (\rho^m u^m \otimes u^m) w dx.$$

We note that for $w \in V^s$,

$$|b(\rho^m u^m, u^m, w)| = \int_{\Omega} \nabla \cdot (\rho^m u^m \otimes u^m) w dx = |
\int_{\Omega} (\rho^m u^m \cdot \nabla w) u^m dx| \leq |\sqrt{\rho^m} u^m|_2^2 \|w\|_{V^s},$$
then
\[ b(\rho^m u^m, u^m, w) := < g^m(t), w > , \]
where \( g^m(t) \) belongs to a bounded set in \( L_\infty(0, T; (V^*)') \).

Considering the projection operator \( P_m : H \rightarrow W_{(m)} \), since \( T_p u^m, g^m, f \) belong to a bounded set in \( L_{p'}(0, T, (V^*)') \), uniformly with respect to \( m \), \eqref{3.11} gives
\[ P_m \partial_t (\rho^m u^m) \text{ belongs to a bounded set in } L_{p'}(0, T, (V^*)') , \]
uniformly with respect to \( m \).

### 3.6. Time estimate - compactness results.

The a priori estimates obtained in the last subsection imply
\( \sqrt{\rho^m u^m} \in L_\infty(0, T; L_2(\Omega)) \cap L_p(0, T; L_q(\Omega)) \),
where \( q \) is a Sobolev conjugate.

We deduce that
\( \rho^m u^m \rightharpoonup v \) weakly* in \( L_\infty(0, T; L_2(\Omega)) \).

We prove next that
\( v = \rho u \).

According to the estimates on the density, it holds, for example,
\[ \| \partial_t \rho^m \|_{L_2(0, T; H^{-1}(\Omega))} \leq C. \]

Since \( \{ \rho_m(t) \} \) is a compact set in \( H^{-1}(\Omega) \) for \( t \in (0, T) \) hence
\( \rho^m \rightarrow \rho \) strongly in \( L_2(0, T; H^{-1}(\Omega)) \).

Now, \( \{ u^m \} \) is a bounded set in \( L_p(0, T; W^1_{p,0}(\Omega)) \) with \( p \geq 2 \), then follows
\( u^m \rightharpoonup u \) weakly in \( L_2(0, T; H^1_0(\Omega)) \).

Consequently,
\( \rho^m u^m_i \rightharpoonup \rho u_i \),
in the sense of distributions.

So, we have proved that \( v = \rho u \).

Finally,
\( \rho^m u^m \rightarrow \rho u \) weakly in \( L^2(Q_T) \).

Now, we prove that
\( \rho^m u^m_i u^m_j \rightarrow \rho u_i u_j \).

The estimate of the time derivative of \( \rho^m \) implies the time continuity of \( \rho^m(t) \) in \( (L_2(\Omega))_w \).

At this stage we consider the integral identity that is satisfied by \( \rho \) and \( u \) with smooth \( \psi \) with \( \psi(T) = 0 \), that is
\begin{equation}
(3.12) \quad \int_0^T (\rho, \psi_t + u \cdot \nabla \psi) dt = (\rho_0, \psi(0)).
\end{equation}

Kazhikhov- Smagulov \[15\] extend \( u \) and \( \rho \) by zeros onto the exterior of the domain \( Q_T \), so \eqref{3.12} holds in the region \( (-\infty, +\infty) \times \mathbb{R}^3 \). Using test function in form of an average (Steklov-functions) prove that, for almost \( t_1, t_2 \) in \([0, T]\), is valid
\[ |\rho(t_1)|_q = |\rho(t_2)|_q , \]
for $2 \leq q < \infty$.

It being understood that $(\rho(t_2) - \rho(t_1), \psi) \to 0$ when $t_2 \to t_1$, we obtain that

$\rho(t_2) \to \rho(t_1)$ strongly in $L_2(\Omega)$ and, in virtue of the boundedness of $\rho$, the convergence is valid in $L_q(\Omega)$ for every finite $q \geq 1$. It is immediate that $|\rho(t)|_q = |\rho_0|_q$.

It follows, in particular,

$\rho^m \to \rho$ strongly in $L_q(Q_T)$.

Consequences of the previous estimates - summary.

1. $\rho^m \to \rho$ weak* in $L_\infty(Q_T)$;
2. $\rho^m \to \rho$ strongly in $L_q(0, T; W^{-1}_r(\Omega))$ for arbitrary finite $q > 1$ and $r \in (2, 5)$;
3. $\rho^m \to \rho$ strongly in $L_q(Q_T)$ for arbitrary finite $q \geq 1$;
4. $\rho^m \to \rho$ in $C([0, T]; (L_q(\Omega))_{\text{w}})$ for arbitrary finite $q > 1$;
5. $\rho^m \to \rho$ in $C([0, T]; L_q(\Omega))$ for arbitrary finite $q > 1$;
6. $u^m \to u$ weakly in $L_p(0, T; W^1_p(\Omega))$;
7. $\rho^m u^m_i u^m_j \to \alpha_{ij}$ weakly in $L_r(Q_T)$ for some $r > 1$ if $p > 6/5$.

Now, we prove that $\alpha_{ij} = \rho u_i u_j$ using the method introduced in [27].

To begin with, we recall that

$\partial_t (P_m \rho^m u^m) \text{ is uniformly bounded in } L_p(0, T; (W^1_{p,0}(\Omega))') \text{ with } s > 1$.

In addition, $\{P_m \rho^m u^m(t)\}$ is a compact set in $(W^1_{p,0}(\Omega))'$ for every $t \in (0, T)$.

It follows

$\int_0^T \int_\Omega \rho^m |u^m|^2 dx dt = \int_0^T (\rho^m(t) u^m(t), u^m(t)) dt$

$\int_0^T (P_m \rho^m(t) u^m(t), u^m(t)) dt =
\int_0^T < P_m \rho^m(t) u^m(t), u^m(t) >_{(W^1_{p,0}(\Omega))' \times W^1_{p,0}(\Omega)} dt.$

Then

$\lim_{m \to \infty} \int_0^T \int_0^T < P_m \rho^m(t) u^m(t), u^m(t) >_{(W^1_{p,0}(\Omega))' \times W^1_{p,0}(\Omega)} dt =
\int_0^T < \rho(t) u(t), u(t) >_{(W^1_{p,0}(\Omega))' \times W^1_{p,0}(\Omega)} dt = \int_0^T (\rho(t) u(t), u(t)) dt =
\int_0^T \int_\Omega \rho |u|^2 dx dt.$

Moreover, since

$\rho^m \to \rho$ strongly in $L^q(Q_T)$, $\forall q > 1$,

we have

$\sqrt{\rho^m} \to \sqrt{\rho}$ strongly in $L^q(Q_T)$,

consequently

$\sqrt{\rho^m} u^m \to \sqrt{\rho} u$ weakly in $L^2(Q_T)$.

By the previous results, we can conclude that

$\sqrt{\rho^m} u^m \to \sqrt{\rho} u$ strongly in $L^2(Q_T)$.

This implies that
\( \rho u_i^m u_j^m = \sqrt{\rho^m u_i^m} \sqrt{\rho^m u_j^m} \rightarrow \rho u_i u_j \),
in the sense of distributions and, finally,
\( \alpha_{i,j} = \rho u_i u_j \).

### 3.7. Initial conditions.

The time evolution of the integral average
\[ t \in (0, T) \mapsto \int_{\Omega} \rho(t, x) \psi(x) dx, \]
is governed by equation (3.11). This function, considered as function of \( t \), is absolutely continuous, in other words, by virtue of the estimates proved,
\[ \rho \in C(0, T, L_{q,w}(\Omega)). \]
The instantaneous value of the density is representable by a function \( \rho(t) \in L_q(\Omega) \).
So, the initial datum is assumed in the following sense
\[ \lim_{t \to 0^+} \int_{\Omega} \rho(t) \psi(x) ds := \rho(0^+) = \int_{\Omega} \rho_0(x) \psi(x) dx. \]
Analogously, the function
\[ h(t) = \int_{\Omega} \rho(t) u(t) \phi(x) dx \]
represents the instantaneous value of the momentum and the initial data are assumed in the following sense:
\[ \lim_{t \to 0^+} \int_{\Omega} \rho(t) u(t) \phi(x) dx := \rho(0^+) u(0^+) = \int_{\Omega} \rho_0(x) u_0 \phi(x) dx. \]
Since the density can vanish, there is not information concerning the time weak continuity of the velocity.

### 3.8. Monotony and convergence of stress tensor.

The previous estimates imply that, for \( m \to \infty \),
\[ P T_p u^m = P \nabla \cdot T u^m \rightarrow \chi \text{ weakly in } (L_p(0, T; W_{p,0}^1(\Omega)))'. \]
In this section we formulate the sufficient conditions which permit to represent \( \chi \) as \( P T_p(u) \) (\( P \) is an "projection" on the space of divergence free function). First, we recall that for a fixed \( w_j \), we have, for \( m \to \infty \),
\[ b(\rho^m u^m, u^m, w_j) \rightarrow b(\rho u, u, w_j), \]
in the sense of distributions with respect to \( t \) (for example). In fact, for \( \phi(t) \in C_0^\infty(0, T) \),
\[
\int_0^T b(\rho^m(t) u^m(t), u^m(t), w_j) \phi(t) dt = \\
\int_0^T \phi(t) \int_{\Omega} \nabla \cdot (\rho^m(t) u^m(t) \otimes u^m(t)) w_j dx dt = \\
- \int_0^T \phi(t) \int_{\Omega} (\rho^m u^m_k(t) u^m_k(t)) \partial_k w_i dx dt \rightarrow \\
\int_0^T \phi(t) \int_{\Omega} (\rho u_k u_k) \partial_k w_i dx dt = \int_0^T b(\rho u(t), u(t), w_j(t)) \phi(t) dt.
\]
Consequently, \((3.10)\) implies
\[
(\partial_t (p(t)v), w_j) + (\chi(t), w_j) + b(p(t)v(t), u(t), w_j) = (p(t)f(t), w_j).
\]

It follows
\[
(3.13) \quad (\partial_t (p(t)v), v) + (\chi(t), v) + b(p(t)v(t), u(t), v) = (p(t)f(t), v),
\]
\(\forall v \in V^q\) and a.e. in \((0, T)\).

Since \(W^{1,1}_p(\Omega) \hookrightarrow L_q(\Omega)\) with \(\frac{1}{q} = \frac{1}{p} - \frac{1}{n},\) \((p > 1)\) it follows that the form
\[
b(p(t)v, u(t), v),
\]
is continuous on \(V_p\) provided \(\frac{2}{q} + \frac{1}{p} \leq 1,\) i.e. if \(p \geq \frac{3n}{n+2}\).

Thus, by continuous extension, \((3.13)\) holds \(\forall v \in V_p\).

Now, we analyze the time-summability of \(b(\cdot, \cdot, \cdot)\).

We consider \(\frac{2}{q} + \frac{1}{p} > 0,\) otherwise there is nothing to prove.

The estimates in subsection 3.5, the Sobolev theorem and interpolation theory imply
\[
\sqrt{\rho u} \in L_{\infty}(0, T); L_2(\Omega) \cap L_p(0, T; L_q(\Omega)) \subset L_{\infty}(0, T; L_2(\Omega)),
\]
where
\[
\frac{1}{r} = \frac{1 - \theta}{p} + \frac{\theta}{\infty} = \frac{1 - \theta}{p} + \frac{1 - \theta}{q} + \frac{\theta}{2}.
\]
Choosing \(\theta\) such that \(\frac{1}{r} = \frac{1}{s}\), that is \(\theta = \frac{2}{n+2}\), then
\[
L_{\infty}(0, T); L_2(\Omega) \cap L_p(0, T; L_q(\Omega)) \subset L_s(Q_T).
\]

Finally, the result is implied by \(\frac{2}{q} + \frac{1}{p} \leq 1,\) i.e. \(p \geq \frac{11}{5}\). In conclusion, for such a \(p\) the function
\[
t \mapsto b(p(t)v(t), u(t), v) = -\int_{\Omega} \sqrt{\rho(t)}u(t) \cdot (\sqrt{\rho(t)}u(t) \cdot \nabla v)dx
\]
is in \(L_1(0, T)\) with \(\frac{1}{s} = \frac{p}{(n+2)p} - 1\).

Now, integrating \((3.13)\) over \((0, t)\), we get
\[
(3.14) \quad (p(t)v(t), v) - (\rho_0 v_0, v) =
\]
\[
-\int_0^t (b(p(\tau)v(\tau), u(\tau), v) + (\chi(\tau), v) + (\rho f(\tau), v))d\tau.
\]

Passing to the limit for \(t \to 0^+\) in \((3.14)\), we get
\[
(3.15) \quad \lim_{t \to 0^+} (p(t)v(t), v) - (\rho_0 v_0, v) = 0.
\]

Recalling
\[
\sqrt{\rho(t)}u(t) \to \sqrt{\rho(t)}u(t),
\]
strongly in \(L_2(\Omega)\) for almost all \(t \in [0, T]\) and letting \(m \to \infty\) in \((3.10)\) we get for \(\alpha > 0\)
\[
(3.16) \quad (p(t)v(t), u(t)) \leq (\rho_0 v_0, u_0) + ct^\alpha.
\]
Now, since
\[ |\sqrt{\rho(t)(u(t) - u_0)}|^2 = (\rho(t)u(t), u(t)) - 2(rho(t)u(t), u_0) + (rho(t)u_0, u_0), \]
taking into account \(3.10\) and the properties of \(\rho(t)\), we conclude
\[ \lim_{t \to 0^+} |\sqrt{\rho(t)(u(t) - u_0)}|^2 = 0. \]
Thus, the previous results imply
\[ \lim_{t \to 0^+} (\rho(t)u(t), u(t)) = (\rho_0u_0, u_0). \] (3.17)

**Proposition 2.** - The following formula holds for almost all \(t_0, t_1 \in [0, T]\) \((t_0 < t_1)\),
\[ \int_{t_0}^{t_1} \partial_t (\rho u(t), u(t)) dt = \frac{1}{2} |\sqrt{\rho(t_1)u(t_1)}|^2 - \frac{1}{2} |\sqrt{\rho(t_0)u(t_0)}|^2 - \int_{t_0}^{t_1} (\nabla \cdot (\rho u(t) \otimes u(t)), u(t)) dt. \] (3.18)

**Proof.** Using convolution product or averaging, it is possible prove that \(3.18\) holds true.
In fact, let \(\phi(t)\) be a regularizing kernel, i.e.
\[ \phi_\epsilon := \frac{1}{\epsilon} \phi(\frac{t}{\epsilon}) := \frac{1}{\epsilon} \phi(\frac{\tau - t}{\epsilon}), \]
with a even mollifier \(\phi \in D_+(\mathbb{R})\) and \(\int_{\mathbb{R}_+} \phi(t) dt = 1\).
Set
\[ u_\epsilon(t) = u * \phi_\epsilon(t) = \frac{1}{\epsilon} \int_{|\tau - t| < \epsilon} u(\tau) \phi_\epsilon(\frac{\tau - t}{\epsilon}) d\tau = \int_{|z| < 1} u(t + \epsilon z) \phi(z) dz. \]
Notice
\[ |\partial_t u_\epsilon|_p \leq \frac{C}{\epsilon} |u|_p. \]

We analyze
\[ \theta_\epsilon(t_0, t_1) := \int_{t_0}^{t_1} (\partial_t (\rho(t)u(t)), u * \phi_\epsilon(t)) dt, \] (3.19)
with \(t_0 > \epsilon\) and \(t_1 < T - \epsilon\).
By integration by parts and by the properties of the convolution, we have
\[ \theta_\epsilon(t_0, t_1) = (\rho(t_1)u(t_1), u * \phi_\epsilon(t_1)) - (\rho(t_0)u(t_0), u * \phi_\epsilon(t_0)) - \int_{t_0}^{t_1} ((\rho u) * \phi_\epsilon(t), \partial_t u * \phi_\epsilon(t)) dt. \] (3.20)
We consider the last term of the right hand side of (3.20).
\[
(3.21) \quad \int_{t_0}^{t_1} \left( (\rho u) \phi_\epsilon(t), \partial_t u \phi_\epsilon(t) \right) dt = \int_{t_0}^{t_1} \left( \rho(t)(u \phi_\epsilon(t)), \partial_t u \phi_\epsilon(t) \right) dt + \nabla \cdot \int_{t_0}^{t_1} \left( \rho(t) (u \phi_\epsilon(t)) \right) dt + \frac{1}{2} \left( (\rho(t_1), |u \phi_\epsilon(t_1)|^2) - (\rho(t_0), |u \phi_\epsilon(t_0)|^2) \right) + \frac{1}{2} \int_{t_0}^{t_1} \left( \nabla \cdot (\rho(t) u(t)), |u \phi_\epsilon(t)|^2 \right) dt + \frac{1}{\epsilon} \int_{t_0}^{t_1} \left( (\rho u) \phi_\epsilon(t) - \rho(t)(u \phi_\epsilon(t)), \epsilon \partial_t u \phi_\epsilon(t) \right) dt.
\]

Now, setting $\tau - t = \epsilon z$ and thank to the continuity equation, we obtain as $\epsilon \to 0$
\[
\int_{t_0}^{t_1} \frac{1}{\epsilon} \left( (\rho u) \phi_\epsilon(t) - \rho(t)(u \phi_\epsilon(t)), \epsilon \partial_t u \phi_\epsilon(t) \right) dt = - \int_{t_0}^{t_1} \int_{|z|<1} \frac{1}{\epsilon} \left( \int_t^{t+\epsilon z} \nabla \cdot (\rho u(s)) ds \right) u(t + \epsilon z) \phi(z) dz = - \epsilon \partial_t \int_{|z|<1} u(t + \epsilon z) \phi(z) dz dt \to 0.
\]

Indeed, making use of the strong convergence of translations in $L_\rho$ and the strong convergence of the Steklov function, we have for $\epsilon \to 0$
\[
\int_{|z|<1} z \phi(z) dz = 0.
\]

Similarly,
\[
\epsilon \partial_t \int_{|z|<1} u(t + \epsilon z) \phi(z) dz \to u(t) \int_{|z|<1} \partial_z \phi(z) dz = 0,
\]
a.e. in $(0,T)$.

In conclusion, as $\epsilon \to 0$, we get
\[
\theta_\epsilon(t_0, t_1) \to \int_{t_0}^{t_1} \left( \partial_t (\rho(t) u(t)), u(t) \right) dt = \frac{1}{2} \left( \sqrt{\rho(t_1)} u(t_1) \right)^2 - \frac{1}{2} \left( \sqrt{\rho(t_0)} u(t_0) \right)^2 - \int_{t_0}^{t_1} \left( \nabla \cdot (\rho u(t) \otimes u(t)), u(t) \right) dt.
\]

Thus (3.18) is proved. \qed
Now, using (3.18) and setting \( u = v \) in the relation
\[
(\partial_t (\rho(t)u(t), \nu(t)) + (\chi(t), \nu(t)) + (\nabla \cdot (\rho(t)u(t) \otimes u(t)), \nu(t)) = \\
(\rho(t)f(t), \nu(t)) ,
\]
we get
\[
(3.23) \quad \frac{1}{2} (|\rho(t_1)|^2 - |\rho_0|^2) + \int_0^{t_1} (\chi(t), u(t)) dt = \\
\int_0^{t_1} (\rho(t)f(t), u(t)) dt.
\]

3.9. Conclusion. Now we prove that \( \chi = PT_p u \).

We use the relation
\[
(3.24) \quad \frac{1}{2} \rho \left( \int_0^t \rho f(t) dt + \frac{1}{2} \int_0^t (\chi(t), u(t)) dt \right) \geq \int_0^t (\rho f(t), u(t)) dt + \frac{1}{2} \| \sqrt{\rho u}(s) \|^2.
\]

We introduce, for \( \phi \in L_p(0, T; V_p) \),
\[
X^s_m = \int_0^s (T_p u^m(t) - T_p \phi(t), u^m(t) - \phi(t)) dt + \frac{1}{2} \| \sqrt{\rho u}(s) \|^2
\]
a.e. in \((0, T)\).

Thanks to the monotony of the operator \( T_p \), we deduce that
\[
(3.25) \quad \lim_{m \to \infty} X^s_m \geq \frac{1}{2} \| \sqrt{\rho u}(s) \|^2.
\]

(3.19) implies
\[
(3.26) \quad X^s_m = \int_0^s (\rho(t)f(t), u^m(t)) dt + \frac{1}{2} \| \sqrt{\rho u}(0) \|^2 - \\
\int_0^s (T_p \phi(t), u^m(t) - \phi(t)) dt \to X^s,
\]
with
\[
X^s = \int_0^s (\rho f(t), u(t)) dt + \frac{1}{2} \| \sqrt{\rho u}(0) \|^2 - \int_0^s (\chi(t), \phi(t)) dt - \int_0^s (T_p \phi(t), u(t) - \phi(t)) dt,
\]
and by (3.26) we get
\[
\int_0^s (\rho f(t), u(t)) dt + \frac{1}{2} \| \sqrt{\rho u}(0) \|^2 - \int_0^s (\chi(t), \phi(t)) dt - \\
\int_0^s (T_p \phi(t), u(t) - \phi(t)) dt \geq \frac{1}{2} \| \sqrt{\rho u}(s) \|^2,
\]
and, finally, (3.24) implies
\[ \int_0^s (\chi(t) - T_p \phi(t), u(t) - \phi(t))dt \geq 0, \]
for almost s.

Since \( T_p \) is a monotone and hemicontinuous operator, we have
\[ \chi = P T_p u. \]

The couple \((\rho, u)\) is a weak solution of system (8). Theorem 2 is proved.

\[ \square \]

4. Periodic problem

This section is devoted to the existence of periodic solution of problem (3.2), i.e. we look for a solution of
\[
\begin{align*}
\rho &\partial_t u + u \cdot \nabla u - \nabla \cdot (T u - \pi I) = \rho f, \\
\partial_t \rho + \nabla \cdot (\rho u) &= 0, \\
\nabla \cdot u &= 0, \\
u(0) &= u(T), \\
u_T &= 0, \\
\rho(0) &= \rho(T).
\end{align*}
\]

**Theorem 4.1.** Assume the hypotheses of Theorem 3.1. Further \( f(x, t) \) is time periodic with period \( T \) in \( L_2(Q_T) \), for convenience, \( 0 < \alpha \leq \rho_0 \leq \beta \) and \( p \geq \frac{11}{5} \). Then there exists a weak solution of system (4.1).

**Proof.** To prove the existence of a weak solution of (4.1) we proceed as in Theorem 3.1. We follow the next scheme: first, we assign the velocity and consider a parabolic approximation of the continuity equation (a diffusion equation) proving the existence of a periodic solution and associated a priori estimates. Next, we pass to consider the existence of a linearized momentum equation and then we conclude the scheme passing to the limit.

Since we have treated, in subsection 3.3, with the semi-Faedo-Galerkin approximation, we start with
\[
\begin{align*}
\partial_t \rho^m(t) + u^{m-1}(t) &\cdot \nabla \rho^m(t) - \frac{1}{m} \Delta \rho^m(t) + \rho^m(t) = \rho^{m-1}(t), \\
\partial_n \rho(x, t) &= 0 \text{ on } \Gamma, \\
\rho^m(0) &= \rho_0^m.
\end{align*}
\]

\( u^m \) is defined in subsection 3.3. We assume \( 0 < \alpha \leq \rho_0^m \leq \beta \) (\( \alpha, \beta \) real positive numbers) and \( u^{m-1}, \rho^{m-1} \) are assigned periodic functions and \( \alpha \leq \rho^{m-1} \leq \beta \). The existence of a solution of problem (4.2) is well known in literature. We need some estimates of the solution \( \rho^m \).

First, we prove that \( \alpha \leq \rho^m \leq \beta \) (maximum principle).

In fact, multiplying (4.2) by \((\rho^m - \alpha)^- := \min(0, \rho^m - \alpha)\), after integration by parts, we have
\[
\begin{align*}
d_t [((\rho^m(t) - \alpha)^-)^2] + \frac{2}{m} |\nabla(\rho^m(t) - \alpha)^-|^2 + 2((\rho^m(t) - \alpha)^-)^2 &= 2((\rho^{m-1}(t) - \alpha), (\rho^m(t) - \alpha)^-) \leq 0.
\end{align*}
\]
Consequently, $\rho^m(t) \geq \alpha$ for all $t$.

Analogously, multiplying (4.2) by $(\rho^m - \beta)^+ := \sup(0, \rho^m - \beta)$ and, after integration by parts, we get

$$d_t(\rho^m(t) - \beta)^+|_2^2 + \frac{2}{m} |\nabla(\rho^m(t) - \beta)^+|_2^2 + 2((\rho^m(t) - \beta)^+|_2^2 = 2((\rho^{m-1}(t) - \beta), (\rho^m(t) - \beta)^+) \leq 0,$$

thus $\rho^m \leq \beta$ for all $t$.

Now, following the procedure of subsection 3.3, we get $H^2$-regularity; multiplying (4.2) by $-\frac{1}{m}\Delta\rho^m$, then integrating over $Q_t$ and recalling $|\nabla\rho^m|^2 \leq c|\rho^m|\Delta\rho^m|_2$, we obtain

$$\frac{1}{m}|\nabla\rho^m(t)|^2_2 + \frac{1}{m} \int_0^t |\Delta\rho^m(\tau)|^2_2 d\tau + \frac{1}{m} \int_0^t |\nabla\rho^m(\tau)|^2_2 d\tau \leq$$

$$\frac{2}{m}|\nabla\rho_0^m|_2^2 + c \int_0^t (|u^{m-1}(\tau)|^2 + |\rho^{m-1}(\tau)|^2_2) d\tau.$$

Similarly, we can obtain $H^3$-estimate of $\rho^m$ but we omit details. By the previous estimates, $\partial_\tau\rho^m \in L_2(Q_T)$ . The existence of a solution of (4.2) permits to define a map $S : L_2(\Omega) \to L_2(\Omega)$,

$$S\rho^m(0) = \rho^m(T).$$

$S$ is a continuous map in $L_2(\Omega)$.

In fact, let $\rho^m_1, \rho^m_2$ be solutions of (4.2) corresponding to initial conditions $\rho^m_1(0), \rho^m_2(0)$, respectively ( with bounds $\alpha, \beta$ ).

By (4.2) we get

$$|S\rho^m_1(0) - S\rho^m_2(0)|^2_2 = |\rho^m_1(T) - \rho^m_2(T)|^2_2 \leq |\rho^m_1(0) - \rho^m_2(0)|^2_2 e^{-\epsilon_m T},$$

consequently, $S$ is a $L_2$ – contraction map.

Then, the fixed point of $S$ yields periodic solution of (4.2).

Briefly, we show that the fixed point is in $H^1(\Omega)$.

Indeed, $\rho^m \in L_2(0, T; H^1(\Omega))$ then in $t = \epsilon > 0$, $\rho^m(\epsilon, x) \in H^1(\Omega)$. Considering the equation (4.2) in the interval $[\epsilon, T]$, we can show that $\rho^m \in C(\epsilon, T; H^1(\Omega))$ and follows $\rho^m(T) \in H^1(\Omega)$, consequently $\rho^m(0) \in H^1(\Omega)$.

Now, we pass to consider the existence of a periodic solution of the momentum equation.

We assume, appealing to section 3, the existence of a solution of the following system

$$\rho^m(\partial_t u^m + u^{m-1} \cdot \nabla u^m - f) + \frac{1}{2}(\frac{1}{m}\Delta\rho^m - \rho^m + \rho^{m-1})u^m -$$

$$\nabla \cdot T u^m + \nabla \pi^m = 0,$$

$$\nabla \cdot u^m = 0, \ u^m(0) = u_0^m.$$

Now, multiplying (4.3) by $u^m$, integrating by parts, applying the
Poincaré’s inequality and taking into account the diffusion equation, we have
\[
|\sqrt{\rho^m(t)} u^m(t)|^2 \leq e^{-ct} (|\sqrt{\rho^m(0)} u^m(0)|^2 + \int_0^t e^{cs} |\rho^m|^2 ds).
\]

We consider now the map
\[
Z : \sqrt{\rho^m(0)} u^m(0) \rightarrow \sqrt{\rho^m(T)} u^m(T).
\]
We prove that \(Z\) is a contraction on \(L^2(\Omega)\).

First, let \(B(R)\) be a ball with radius \(R\).

If \(R \geq (1 - e^{-ct})^{-1} \int_0^T |\rho f|^2 dt\) we can prove easily
\[
ZB(R) \subseteq B(R).
\]
Let \(u^m_1, u^m_2\) be solutions of problem (4.4) with initial conditions \(u^m_1(0), u^m_2(0)\), respectively. Thus, \(U^m = u^m_1 - u^m_2\) satisfies
\[
\begin{align*}
\rho^m (\partial_t U^m + u^m_1 \cdot \nabla U^m) &- (\nabla \cdot (T(u^m_2) u^m_2) + \nabla \cdot (T(u^m_1) u^m_1)) - \\
&\frac{1}{2} \rho^m U^m + \frac{1}{2} \rho^m u_1 u_2 + \frac{1}{2} \Delta \rho^m (\nabla \bar{\pi}^m + \nabla \pi^m) = 0.
\end{align*}
\]
Multiplying (4.5) by \(U^m\), integrating the result over \(\Omega\), taking into account the diffusion equation (4.2) and the structure conditions, the Gronwall’s lemma implies
\[
|\sqrt{\rho^m(T)} U^m(T)|^2 \leq e^{-ct} |\sqrt{\rho^m(0)} U^m(0)|^2.
\]
Thus the map \(Z\) is a \(L^2\)-contraction and then the existence of a periodic solution of (4.4) follows as the fixed point of \(Z\). From now on the proof of the existence of periodic weak solution proceeds as in Theorem 3.1. \(\square\)

In the case of unbounded domains it is not possible to simply extend the methods used for the bounded domains, since these involve, in general, tools such as the Poicaré’s inequality, compact embedding, etc., that no longer holds for unbounded domains, in general. Consequently, it is necessary to resort different arguments. In [31] the author solved the open problem of the existence of weak and strong periodic solutions for the Navier-Stokes equations in exterior domains using an alternative method. For the time being, this is the only result for three-dimensional case in exterior domains.

5. Variational inequality

The differential inequalities appear in the presence of additional constraints imposed on the unknowns of the problem in order to describe particular physical situations. These type of problems have been studied for instance in [4], [7], [10], [16], [20], [21], [22], [25], [26]. In [20] the existence of a global weak solution of the Navier-Stokes equations in a convex set is obtained under some additional conditions. In [7] the existence problem was proved for two dimensional case using a method of non linear semigroup. In [26] the existence of a global weak solution was solved, in three dimension and in the general case, for density-dependent Navier-Stokes equations, applying the compactness method. For the time being, this is the only result for three-dimensional case.
The lines of the proof of the existence for variational inequality are traced in subsection 3.3; we use semi-Faedo-Galerkin approximation jointly with a penalization operator.

5.1. **Formulation of the problem.** The functional spaces are those of the section 3. In addition, for convenience, let \( K \subset H \) be an arbitrary closed convex set (independent of time) with \( 0 \leq \rho_0 \leq M \), the external force \( f \in L_2(Q_T) \) (for convenience).

We consider the following system for \( u, v \in K \),

(5.1) \[
(\rho(t)\partial_t u(t), v(t) - u(t)) + (\rho(t)u(t) \cdot \nabla u(t), v(t) - u(t)) - \\
(\nabla \cdot Tu(t), v(t) - u(t)) \geq (\rho(t)f(t), v(t) - u(t)), \\
\partial_t \rho + \nabla \cdot (\rho u) = 0, \\
\nabla \cdot u = 0, u(0) = u_0, u_T = 0, \rho(0) = \rho_0.
\]

Then, we give the definition of weak solution of problem (5.1). In what follows \( \Omega \) is a bounded domain in \( \mathbb{R}^3 \).

**Definition 2.** \( (u, \rho) \) is a weak solution of (5.1) if hold

(5.2) \[
\int_0^T ((\rho \partial_t v, v - u) + (\rho u \cdot \nabla u, v - u) - (\nabla \cdot Tu, v - u)) - \\
(\rho f, v - u)) dt \geq -|\sqrt{\rho(0)}(v(0) - u(0))|^2, \\
\partial_t \rho + \nabla \cdot (\rho u) = 0, \text{ in the sense of distributions,} \\
\nabla \cdot u = 0, u(0) = u_0, u_T = 0, \rho(0) = \rho_0,
\]

with \( \sqrt{\rho} u \in L_{\infty}((0,T;L_2(\Omega))); u \in L_p(0,T;V^1_p \cap K), v \in L_p(0,T;V^1_p \cap K), \)

\( 0 \leq \rho \leq M, \partial_t \rho \in L_{p'}(0,T;L'_{p'}(\Omega)). \)

Briefly, we deduce (5.2) from (5.1).

Multiplying (5.1) by \( u \cdot (v - u) \), integrating over \( \Omega \) and adding the result to (5.1), integration in \( t \) gives

(5.3) \[
\int_0^T ((\partial_t (\rho u(t)), v(t) - u(t)) - (\rho u(t) \otimes u(t), \nabla (v(t) - u(t))) - \\
(\nabla \cdot Tu(t), v(t) - u(t)) dt \geq \int_0^T (\rho f(t), v(t) - u(t)) dt,
\]

and, easily follows

(5.4) \[
\int_0^T ((\partial_t (\rho v(t)), v(t) - u(t)) - (\rho u(t) \otimes u(t), \nabla (v(t) - u(t))) - \\
(\nabla \cdot Tu(t), v(t) - u(t)) - (\rho f(t), v(t) - u(t))) dt \geq \\
\int_0^T (\partial_t (\rho (v(t) - u(t))), v(t) - u(t)) dt.
\]
Notice
\[ \int_0^T (v(t) \partial_t \rho(t), v(t) - u(t)) dt = - \int_0^T (\nabla \cdot (\rho u(t)) v(t), v(t) - u(t)) dt, \]
and
\[ \int_0^T ((v(t) - u(t)) \partial_t \rho(t), v(t) - u(t)) dt =
- \int_0^T (\nabla \cdot (\rho u(t))(v(t) - u(t)), v(t) - u(t)) dt. \]

In addition,
\[ - (\nabla \cdot (\rho u(t))v(t), v(t) - u(t)) + \frac{1}{2}(\nabla \cdot (\rho u(t))(v(t) - u(t)), v(t) - u(t)) -
(\rho u(t) \cdot \nabla (v(t) - u(t)), u(t)) = (\nabla \cdot (\rho u(t))v(t), v(t) - u(t)) -
(\rho u(t) \cdot \nabla (v(t) - u(t)), v(t)) = (\rho u(t) \cdot \nabla v(t), v(t) - u(t)), \]

thus \(5.1\) assumes the following form
\[ \int_0^T ((\rho(t) \partial_t v(t), v(t) - u(t)) + (\rho(t) u(t) \cdot \nabla v(t), v(t) - u(t)) -
(\nabla \cdot T u(t), v(t) - u(t)) - (\rho f(t), v(t) - u(t))) dt \geq
\frac{1}{2}(|\sqrt{\rho(T)}(v(T) - u(T))|^2 - |\sqrt{\rho(0)}(v(0) - u(0))|^2 \geq -|\sqrt{\rho(0)}(v(0) - u(0))|^2. \]

We notice that in the last right hand side the term \((\rho(T)(v(T) - u(T)), v(t) - u(T))\) is dropped. The reason is clear. We are not able to prove that, for Galerkin approximations \(\rho^n, u^n, \{\rho^n u^n(t), u^n(t)\}\) is a compact set in \(C(0, T)\) (see Theorem 3).

**Theorem 5.1.** Assume the following hypotheses:
\[ f \in L_2(Q_T), \; u_0 \in L_p(\Omega) \cap K, \]
\[ \rho_0 \in L_\infty(\Omega), \; 0 \leq \rho_0 \leq M < \infty, \; p \geq \frac{11}{5}. \]

Then, there exists a weak solution \((u, \rho)\) of the problem \(5.2\).

**Proof.** To study the existence of weak solutions to problem \(5.2\) we use the compactness method and penalization argument. First, we introduce the penalty operator \(\beta(\cdot)\) related to \(K\).
Let \(P_K : V \to K\) be the projection operator on the convex \(K \in L^2(\Omega)\).
Clearly, \(P_K V \subset K\) and \((v - P_K v, u) \geq 0 \; \forall u \in K\).

We define \(\beta\) as
\[ \beta : V \to (I - P_K) V, \]
and
\[ \beta(v) = v - P_K v, \; \forall v \in V. \]
\( \beta \) is a monotone and hemi-continuous operator; moreover, the following relations hold true:

\[
(\beta(v), P_K v) \geq 0,
\]
\[
(\beta(v), P_K v - z) \geq 0, \quad \forall z \in K.
\]

5.2. **Semi-Faedo-Galerkin approximation.** We choose a basis as in section 3, and we treat the system (5.2) including the penalty term, namely

\[
\begin{align*}
\rho_m \frac{\partial u_m}{\partial t} + \rho_m u_m \cdot \nabla u_m - \nabla \cdot T u_m + m \beta(u_m) + \nabla \pi_m - \rho_m f &= 0, \quad i = 1, \ldots, m, \\
\partial_t \rho_m + u_m \cdot \nabla \rho_m &= 0,
\end{align*}
\]
\[
\begin{align*}
u_m(0) &= u_m^0, \quad \mu = 0, \quad \rho_m(0) &= \rho_0^m.
\end{align*}
\]

The local existence of (5.5) is proved in subsection 3.3. The global existence is a consequence of the estimates in subsection 3.5 and of

\[
m \int_0^T (\beta(u_m(t)), u_m(t)) \, dt = \int_0^T (\beta(u_m(t)), P_K u_m(t)) \, dt \leq c.
\]

Moreover, follows

\( \beta(u_m) \to 0 \) strongly in \( L^2(Q_T) \).

Now, we consider the estimate that permits to apply the compactness method to variational inequalities.

5.3. **Time-translation estimate.** We use the time-translation estimate instead of the time-derivative estimate, in other words, we appeal to the \( L_p \)-Ascoli-Arzelà theorem.

Indeed, we will prove

\[
\int_0^{T-h} \| \sqrt{\rho_m}(t + h)(u_m(t + h) - u_m(t)) \|^2 \, dt \leq c h^\alpha,
\]

\( \forall h \in \mathbb{R} \) with \( 0 < h < T \) and \( \alpha > 0 \).
Let $u^n_i(t) = \frac{1}{h} \int_{h-i}^{h-i+1} u^n(s) ds$ be a test function. Multiplying (5.5) by $u^n_i(t)$ and, after integration, we get

$$
(\partial_t \rho^m u^n(t), u^n_i(t)) + ((\rho^m u^n(t), u^n_i(t) \cdot \nabla u^n_i(t)) + m(\partial_t \rho^m u^n(t), u^n_i(t)) - (\nabla \cdot T u^n(t), u^n_i(t)) - (\rho^m(t) f(t), u^n_i(t))) dt = I_1 + I_2 + I_3 + I_4 + I_5 = 0.
$$

Now, we estimate $I_i$ with $i = 1, 2, 3, 4, 5$.

$$
I_1 = \int^T_h (\partial_t \rho^m u^n(t), u^n_i(t)) dt = (\rho^m(T) u^n(T), u^n_i(T)) - \rho^m(h) u^n_i(h) - \frac{1}{h} \int^T_h (\rho^m(t) u^n_i(t), u^n(t) - u^n(t-h)) dt \leq c \frac{1}{\sqrt{h}} \left( \int^T_0 \|u^n(t)\|^2 dt \right)^{1/2} - \frac{1}{h} \int^T_h (\rho^m(t) u^n_i(t), u^n(t) - u^n(t-h)) dt.
$$

We analyze the last term in $I_1$.

$$
- \frac{1}{h} \int^T_h (\rho^m(t) u^n_i(t), u^n(t) - u^n(t-h)) dt =
- \frac{1}{h} \int^T_h (\rho^m(t) u^n_i(t), u^n(t)) dt + \frac{1}{h} \int^T_h (\rho^m(t) u^n_i(t), u^n(t-h)) dt =
- \frac{1}{2h} \int^T_h |\sqrt{\rho^m(t)} u^n_i(t)|^2 dt + \frac{1}{2h} \int^T_h |\sqrt{\rho^m(t)} u^n_i(t-h)|^2 dt -
\frac{1}{2h} \int^T_h |\sqrt{\rho^m(t)} u^n_i(t-h)|^2 dt =
- \frac{1}{2h} \int^T_h |\sqrt{\rho^m(t)} u^n_i(t)|^2 dt + \frac{1}{2h} \int^T_h |\sqrt{\rho^m(t-h)} u^n_i(t-h)|^2 dt +
\frac{1}{2h} \int^T_h ((\rho^m(t) - \rho^m(t-h)) u^n_i(t-h), u^n_i(t-h)) dt -
\frac{1}{2h} \int^T_h |\sqrt{\rho^m(t) u^n_i(t) - u^n(t-h)}|^2 dt.
$$

Thanks to the continuity equation, we get

$$
\left| \frac{1}{2h} \int^T_h ((\rho^m(t) - \rho^m(t-h)) u^n_i(t-h), u^n(t-h)) dt \right| \leq
\frac{1}{h} \int^T_h \left( \int_{t-i}^{t-i+1} \rho(s) u^n(s) ds \cdot \nabla u^n_i(t-h), u^n_i(t-h)) dt \right| \leq c \frac{1}{\sqrt{h}} \left( \int^T_h \|u^n(t)\|^2 dt \right)^{3/2}.
$$
In conclusion,

\[ I_1 = \int_h^T (\partial_t \rho^m u^m(t), u^m_h(t)) dt \leq c_1 + \frac{c_2}{\sqrt{h}} \]
\[ \frac{1}{h} \int_h^T |\sqrt{\rho^m(t)}(u^m(t) - h - u^m(t))| dt. \]

As before,

\[ I_2 = \int_h^T |(\rho^m u^m(t) u^m(t), \nabla u^m_h(t))| dt \leq \]
\[ c \int_h^T |u^m(t)||u^m(t)||u^m_h(t)|| dt \leq \]
\[ \frac{c}{\sqrt{h}} \int_h^T \|u^m(t)\|^2 dt (\int_h^T \|u^m_h(t)\|^2)^{1/2} \leq \frac{c}{\sqrt{h}} \]
\[ I_3 = \int_h^T (\nabla \cdot T u^m(t), u^m_h(t)) dt \leq \frac{c}{h^2} \int_0^T \|u^m(t)\|_p^2 dt, \gamma > 0. \]

Now, we estimate the crucial term \( I_4 \).

\[ I_4 = m \int_h^T (u^m(t) - P_K u^m(t), u^m_h(t)) dt = \]
\[ m \int_h^T (u^m(t) - P_K u^m(t), \frac{1}{h} \int_{t-h}^t (u^m(s) - P_K u^m(s)) ds) dt + \]
\[ m \int_h^T (u^m(t) - P_K u^m(t), \frac{1}{h} \int_{t-h}^t (P_K u^m(s) ds - P_K u^m(t)) dt + \]
\[ m \int_h^T (u^m(t) - P_K u^m(t), P_K u^m(t)) dt. \]

Since \( \frac{1}{h} \int_{t-h}^t P_K u^m(s) ds \in K \) and bearing in mind the properties of \( \beta(u^m) \), it follows that, in the right hand side, the second term is negative and the last term is bounded. Finally,

\[ m \int_h^T (u^m(t) - P_K u^m(t), \frac{1}{h} \int_{t-h}^t (u^m(s) - P_K u^m(s)) ds) dt \leq \]
\[ \frac{m}{\sqrt{h}} \int_h^T |u^m(t) - P_K u^m(t)| \int_{t-h}^t |u^m(s) - P_K u^m(s)|^2 ds)^{1/2} dt \leq \]
\[ \frac{c}{\sqrt{h}} \int_0^T |\beta(u^m(t))|^2 dt \leq \frac{1}{\sqrt{h}}, \]

In conclusion,

\[ I_4 = m \int_h^T (u^m(t) - P_K u^m(t), u^m_h(t)) dt \leq c + \frac{c}{\sqrt{h}}. \]
Finally,

$$I_5 = \int_h^T (\rho^m f(t), u_h^m(t)) dt \leq \int_0^T |\rho^m f(t)|^2 |u_h^m(t)|^2 dt \leq c \frac{1}{\sqrt{h}}$$

Thanks to the previous estimates, it follows

\begin{equation}
\frac{1}{h} \int_0^{T-h} |\sqrt{\rho^m(t+h)}(u^m(t+h) - u^m(t))|^2 dt \leq ch^a.
\end{equation}

[5.7] is proved.

Since \(\rho^m \in L_\infty(Q_T)\) uniformly in \(m\), thus

\begin{equation}
\int_0^{T-h} |\rho^m(t+h)u^m(t+h) - \rho^m(t)u^m(t)|^2 dt \leq ch^a,
\end{equation}

with \(\alpha < 1\).

Moreover, the continuity equation and the energy estimate imply

\begin{equation}
\|\rho^m(t+h) - \rho^m(t)\|_{H^{-1}(\Omega)} \leq ch.
\end{equation}

Since \(H^{-1}(\Omega) \times H_0^1(\Omega) \rightarrow W_r^{-1}(\Omega)\) with \(r < 3/2\), plainly follows

\begin{equation}
\int_0^{T-h} \|\rho^m(t+h)u^m(t+h) - \rho(t)u^m(t)\|_{W_r^{-1}(\Omega)} \leq ch^a,
\end{equation}

so \(\{\rho^m u^m\}\) is a compact set in \(L_2(0,T;H^{-1}(\Omega))\). In virtue of the estimates in subsection 3.5, we have

\[
\rho^m u^m \rightarrow \rho u \text{ strongly in } L_2(0,T;H^{-1}(\Omega)).
\]

But \(u^m \rightharpoonup u\) weakly in \(L_p(0,T;V_p^1(\Omega))\), it follows

\[
\rho^m u_i^m u_j^m \rightharpoonup \rho u_i u_j
\]

in the sense of distributions.

As in section 3, it remains to prove

\[
\chi = PT_p u.
\]

We set \(c_m(s) = m \int_0^s (\beta(u^m(t)), u^m(t)) dt \in \mathbb{R}\), thanks to the Bolzano-Weirstrass theorem, \(c_m(s) \rightarrow c(s) \geq 0\) as \(m \rightarrow \infty\) for fixed \(s\). We use the relation

\begin{equation}
\frac{1}{2} |\sqrt{\rho u}(s)|^2 + c(s) + \int_0^s (\chi(t), u(t)) dt = \int_0^s (\rho f(t), u(t)) dt + \frac{1}{2} |\sqrt{\rho u_0}|^2.
\end{equation}

Now, we introduce, for \(\phi \in L_p(0,T;V_p^1(\Omega))\),

\[
X_m^s = \int_0^s (T_p u^m(t) - T_p \phi(t), u^m(t) - \phi(t)) dt + \frac{1}{2} |\sqrt{\rho u^m}(s)|^2 + c_m(s),
\]

a.e. in \((0,T)\). x
Thanks to the monotony of the operator $T_p$, we derive
\begin{equation}
\liminf_{m \to \infty} X^s_m \geq c(s) + \frac{1}{2}\|\sqrt{\rho}u(s)\|^2.
\end{equation}

(5.14) implies, for $m \to +\infty$,
\begin{align*}
X^s_m &= \int_0^s (\rho(t)f(t), u^m(t)) dt + \frac{1}{2}\|\sqrt{\rho}u^m_0\|^2 - \int_0^s (T_p u^m(t), \phi(t)) dt - \int_0^s (T_p \phi(t), u^m(t) - \phi(t)) dt \to X^s,
\end{align*}
with
\begin{align*}
X^s &= \int_0^s (\rho f(t), u(t)) dt + \frac{1}{2}\|\sqrt{\rho}u_0\|^2 - \int_0^s (\chi(t), \phi(t)) dt - \int_0^s (T_p \phi(t), u(t) - \phi(t)) dt.
\end{align*}

Consequently, with (5.13), we get
\begin{align*}
\int_0^s (\rho f(t), u(t)) dt + \frac{1}{2}\|\sqrt{\rho}u_0\|^2 - \int_0^s (\chi(t), \phi(t)) dt - \int_0^s (T_p \phi(t), u(t) - \phi(t)) dt &\geq c(s) + \frac{1}{2}\|\sqrt{\rho}u(s)\|^2, \\
\end{align*}
and, finally, (5.13) implies
\begin{align*}
\int_0^s (\chi(t) - T_p \phi(t), u(t) - \phi(t)) dt &\geq 0,
\end{align*}
for almost $s$. Using the classical procedure, i.e. using $\phi = u + \lambda \psi$ with arbitrary $\lambda \in \mathbb{R}$ and $\psi \in L_p(0, T; V_p)$ in the last relation follows
\begin{align*}
\chi = PT_p u.
\end{align*}

At this stage all terms in (5.5) are convergent. We show that the functions $(\rho, u)$ satisfy (5.2).

First, since $T_p$ is monotone,
\begin{align*}
(T_p u^m(t), u^m(t)) &= (T_p u^m(t), u^m(t) - u(t)) + (T_p u^m(t), u(t)) \\
&\geq (T_p u(t), u^m(t) - u(t)) + (T_p u^m(t), u(t)),
\end{align*}
thus
\begin{align*}
\liminf_{m \to \infty} (T_p u^m(t), u^m(t)) &\geq (T_p u(t), u(t)), \\
\end{align*}
Therefore,
\begin{align*}
\liminf_{m \to \infty} (T_p u^m(t), u^m(t) - v(t)) &\geq (T_p u(t), u(t) - v(t)),
\end{align*}
for $\forall v \in K$.

Let $v(t)$ be an arbitrary regular function such that $v(t) \in V^1_p(\Omega) \cap K$ for all $t \in (0, T)$ and $v(T) = 0$. Let $v^m(t) \in W_m$ such that
\begin{align*}
v^m(t) \to v(t) \text{ strongly in } V^1_p(\Omega), \\
\text{with } v^m(T) = 0 \text{ and } v^m(t) \in K \text{ for all } m \geq m_0 \text{ (for some finite } m_0).
Let \( m > \bar{m} > m_0 \), thus
\[
(\beta(u^m(t)), v^m(t) - u^m(t)) \leq 0.
\]

By taking \( v^\bar{m} - u^m \) as a test function in the momentum equation (5.2), and taking into account the last inequality, we get
\[
\int_0^T ((\partial_t \rho(u^m(t)) + \nabla \cdot (\rho(u^m(t) \otimes u^m(t)), v^m(t) - u^m(t)) + (T_p u^m(t), v^m(t) - u^m(t))) dt \geq \int_0^T (\rho f(t), v^m(t) - u^m(t)) dt.
\]

Using the procedure to obtain (5.2), we get
\[
\int_0^T ((\rho(u^m(t)) \partial_t v^\bar{m}(t), v^\bar{m}(t) - u^m(t)) + (T_p u^m(t), v^\bar{m}(t) - u^m(t)) + (\rho(u^m(t)) \cdot \nabla u^m(t), v^\bar{m}(t) - u^m(t)) - (\rho f(t), v^\bar{m}(t) - u^m(t)) dt \geq 0.
\]

Passing to the limit \( m \to +\infty \) in (5.16), it is plain that \((u, \rho)\) is a weak solution of the problem (5.2).

Theorem 5.1 is completely proved.

\[\square\]

Remark: The theorem continues to hold for convex \( K(t) \) which depends monotonically on \( t \), i.e. \( K(t_1) \subseteq K(t_2) \) for \( t_1 \leq t_2 \).

The proof of the theorem is the same once we perform the change \( P_K \to P_{K(t)} \) and observing that \( \frac{1}{h} \int_{t-h}^{t} P_{K(s)} u^m(s) ds \in K(t) \) with \( h \geq 0 \).

6. LOCAL WELL-POSEDNESS PROBLEM

We recall that the quasi-linear differential operator \( T_p(u, D) \) is defined as
\[
T_p(u, D) := \sum_{k,l=1}^{k,j} \tau_{ij}^{k,l}(x,t) \partial_k \partial_l,
\]
in section 3.

We use the following notations.

Set \( X_0 := L_q(Q_T) \), \( X_2 := W^2_q(\Omega) \) and \( Y_q := W^{2-2/q}_q(\Omega) \) equipped with the standard norms that we denote \( | \cdot |_q \), \( || \cdot ||_2 \) and \( || \cdot ||_{(q)} \), respectively; assume \( q > n + 2 \).

\( T_p(u) \) satisfies
\[
T_p : Y_q \to B(X_2, X_0)
\]
is continuous (\( B \) stands for bounded operator set) and
\[
|T_p(v)u - T_p(\bar{v})u_q| \leq c(||v||_{(q)}||v - \bar{v}||_{(q)}||u||_2,
\]
if \( v, \bar{v} \in Y_q \) and \( u \in X_2 \), \( q > n + 2 \) and \( c(\cdot) \) is continuous positive function. Furthermore,
\[
Z := W^{1}_q(0, T; L_q(\Omega)) \cap L_q(0, T; W^2_q(\Omega)) \to C(0, T; W^{2-2/q}_q(\Omega)) \to C((0, T]; C^1(\Omega)).
\]
The embedding constant depends on \( T \) and can blow up as \( T \to 0^+ \), in general, if the initial datum is different from zero. In general, \( W^{2-2/q}_q(\Omega) \) is considered as a time-trace space. Now, we consider the well-posedness of the following problem,

\[
\rho \partial_t u + T_p(u)u + \rho u \cdot \nabla u + \nabla \pi = \rho f,
\]
\[
\partial_t \rho + u \cdot \nabla \rho = 0,
\]
\[
\nabla \cdot u = 0, \quad u(0) = 0, \quad \rho(0) = \rho_0, \quad u = 0 \text{ on } \Gamma.
\]

The main result of the section is the following,

**Theorem 6.1.** Let \( \Omega \subset \mathbb{R}^3 \) be a domain with compact boundary \( \Gamma \) of class \( C^3 \); let \( 5 < q < \infty \), and assume that \( \mu \in C^2(\mathbb{R}^+ \setminus 0) \) is such that \( \mu(s) \geq c > 0 \) for every \( s \geq 0 \). Then, for each \( u_0 \in W^{2-2/q}_q(\Omega) \cap V \) and \( \rho_0 \in W^{1}_q(\Omega) \), \( f \in L_q(Q_T) \), and \( 0 < m \leq \rho_0 \leq 1 \), there exists a \( T > 0 \) such that there is a unique solution \((u, \rho, \pi)\) of (6.1) on the time interval \([0, T]\) such that

\[
u \in W^{1}_q(0, T; V_q^0(\Omega)) \cap L_q(0, T; W^{2}_q(\Omega) \cap V), \quad \pi \in L_q(0, T; W^{1}_q(\Omega))/\mathbb{R},
\]

\[
\rho \in L_\infty(Q_T) \cap L_\infty(0, T; W^{1}_q(\Omega)), \quad \partial_t \rho \in L_\infty(0, T; L_q(\Omega)).
\]

We prove theorem 6.1 assuming \( u_0 = 0 \) for simplicity of exposition. The non-homogeneous case is treated using a translation. Now, we recall some results concerning the solvability of the generalized Stokes equations with variable coefficients. In [3, 30, 37] is considered the following problem

\[
\partial_t u + A(x, t, \partial_x)u(t) + \nabla \pi = f,
\]
\[
\nabla \cdot u = 0, \quad u(0) = u_0, \quad u = 0 \text{ on } \Gamma,
\]

where \( A(x, t, D) = \sum_{k,l=1}^{3} A^{k,l}(x, t)\partial_k \partial_l \) is a matrix elliptic-type differential operator and \( A^{k,l}(x, t) \) are regular functions with real coefficients depending on \( x, t \). Assuming the coefficients of the operator \( A \) are bounded, and the coefficients of the operator \( A_0 \) (principal part of \( A(x, t, D) \)) are continuous with respect to \((x, t)\) and belong to \( W^{1}_q(\Omega) \), \( 1/r < 1/n + \min(1/q; (q - 1)/q; 1/n) \), for all \( t \), \( f \in L_q(Q_T) \), \( u_0 \in W^{2-2/q}_q(\Omega) \), the following theorem is proved.

**Theorem 6.2.** Let \( \Gamma \in C^3 \), \( f \in L_q(\Omega) \), \( u_0 \in W^{2-2/q}_q(\Omega) \cap V \). Then there exists a unique solution \((u, \pi)\) of (6.2) such that

\[
u \in W^{1}_q(0, T; V) \cap L_q(0, T; W^{2}_q(\Omega) \cap V), \quad \pi \in L_q(0, T; W^{1}_q(\Omega))/\mathbb{R}
\]

such that

\[
\|u\|_{W^{2-1}_q(Q_T)} + \|\nabla \pi\|_{L_q(Q_T)} \leq \tilde{c}(\|f\|_{L_q(Q_T)} + \|u_0\|^q_{(q)}).
\]

The method of the proof consists of frozen techniques, Schauder’s type estimates and of the construction of a “regularizer.”

If the coefficients \( A^{k,l}(x, t) \) are constant or perturbation of a constant, the operator \( A \) has the so-called ”maximal regularity” property.
Proposition 1 provides strong convergence in the intermediate spaces we need.

\[ \partial_t u + A(v(t))u(t) + \nabla \pi = f(t), \]
\[ \nabla \cdot u = 0, \ u(0) = u_0, \ u = 0 \text{ on } \Gamma. \]

with \( v \) and \( u \) belong to the spaces occurring in theorem 6.1.

**Proof.** We prove theorem 6.1. Let us consider the set
\[ B = \{ \phi|\phi(0) = 0; \sup(\|\phi\|_{L_q(0,T;W^2_2(\Omega))}, \|\phi\|_{W^1_2(0,T;L_q(\Omega))}) \leq r \}, \]
with \( r \in \mathbb{R}_+. \)

The existence proof of a solution of (6.1) is done by formulate the existence as a fixed point in \( B \) of the map \( G : v \to \rho \to u \) where \( u \) and \( \rho \) are solutions of the following problem for arbitrary \( v \in B \),

\[ \begin{align*}
\partial_t u(t) + T_p(v(t))u(t) + \nabla \pi(t) = \\
(1 - \rho(t))\partial_t v(t) - \rho(t)v(t) \cdot \nabla v(t) + \rho(t)f(t), \\
\partial_t \rho(t) + v(t) \cdot \nabla \rho(t) = 0, \\
\nabla \cdot u(t) = 0, \ u(0) = 0, \ \rho(0) = \rho_0, \\
0 < m \leq \rho_0 \leq 1, \ 2\bar{c}(1 - \rho_0) < 1, \ u = 0 \text{ on } \Gamma.
\end{align*} \]

(6.4)

According, we prove \( G(B) \subseteq B \) for suitable \( \bar{T} \).

We notice that

\[ \|u\|_{W^2_2(\Omega, Q_T)} \leq c(\|f\|_{L_q(\Omega, Q_T)} + \|F(v, \rho)\|_{L_q(\Omega, Q_T)}). \]

Here \( F(v) := (1 - \rho(t))\partial_t v(t) - \rho(t)v(t) \cdot \nabla v(t). \)

Clearly, B is a compact set in \( L_2(Q_T) \) framework. As we are going to use a fixed point argument, we have to show \( G(B) \subseteq B \), and \( G \) is continuous in \( L_2(Q_T) \), for example.

According, we prove \( G(B) \subseteq B \) for suitable \( \bar{T} \).

We notice that

\[ \|\rho v \cdot \nabla v\|_{L^q(\Omega, Q_T)} \leq c\|v\|_{L^\infty(Q_T)} \|v\|_{L^\infty(0,T;W^1_2(\Omega))}T^{1/q}. \]

Moreover, thanks to the assumptions, we get

\[ \|\partial_t v\|_{L^q(T)} \leq \frac{1}{2c} \|\partial_t v\|_{L^q(T)}. \]

In conclusion, for suitable \( T := \bar{T} \), we get

\[ GB \subseteq B. \]

Now, we prove the continuity of \( G \) in \( L_2(Q_T) \).

First, we observe that if \( \{v^n\} \subset B \) there exists a subsequence ( denoted again \( \{v^n\} \) ) such that as \( n \to +\infty, v^n \to v \) weakly in \( L_q(0, \bar{T}; W^2_2(\Omega)) \), weakly* in \( L_\infty(0, \bar{T}; W^1_2(\Omega)) \), and \( \partial v^n \to \partial v \) weakly in \( L_q(Q_T) \).

Let \( \{v^n\} \subset B \) such that \( v^n \to v \) strongly in \( L_2(Q_T) \). Clearly, \( v \in B \) and Proposition 1 provides strong convergence in the intermediate spaces we need.
Let \( \rho^n \) and \( \rho \) be the solutions of
\[
\partial_t \rho^n + \mathbf{v}^n \cdot \nabla \rho^n = 0, \quad \rho^n(0) = \rho_0,
\]
and
\[
\partial_t \rho + \mathbf{v} \cdot \nabla \rho = 0, \quad \rho(0) = \rho_0,
\]
respectively.

The regularity of \( \rho^n \) and \( \rho \) is proved in section 3.

Now, \( \tilde{\rho}^n = \rho^n - \rho \) satisfies
\[
(6.5) \quad \partial_t \tilde{\rho}^n + (\mathbf{v}^n - \mathbf{v}) \cdot \nabla \rho + \mathbf{v}^n \cdot \nabla \tilde{\rho}^n = 0, \quad \tilde{\rho}^n(0) = 0.
\]

Multiplying (6.5) by \( \tilde{\rho}^n \) and after integration over \( Q_T \) we have
\[
\int_0^t |\tilde{\rho}^n(t)|^2 \leq e^{ct} \int_0^t |(\mathbf{v}^n - \mathbf{v}) \cdot \nabla \rho|^2 dt.
\]

This implies \( \rho^n \to \rho \) strongly in \( L_\infty(0, T; L_2(\Omega)) \). Thanks to \( \tilde{\rho}^n \in L_\infty(Q_T) \) it follows \( \rho^n \to \rho \) strongly in \( L_\infty(0, T; L_r(\Omega)) \) for every finite \( r > 1 \).

Now, let \( \mathbf{u}^n \) and \( \mathbf{u} \) be the solutions of
\[
(6.6) \quad \partial_t \mathbf{u}^n + \mathbf{T}_p(\mathbf{v}^n) \mathbf{u}^n + \nabla \pi^n = (1 - \rho^n) \partial_t \mathbf{v}^n - \rho^n \mathbf{v}^n \cdot \nabla \mathbf{v}^n + \rho^n \mathbf{f},
\]
\[
\partial_t \rho^n + \mathbf{v}^n \cdot \nabla \rho^n = 0,
\]
\[
\nabla \cdot \mathbf{u}^n = 0, \quad \mathbf{u}^n(0) = 0, \quad \rho^n(0) = \rho_0, \quad \mathbf{u}^n = 0 \text{ on } \Gamma,
\]
and
\[
(6.7) \quad \partial_t \mathbf{u} + \mathbf{T}_p(\mathbf{v}) \mathbf{u} + \nabla \pi = (1 - \rho) \partial_t \mathbf{v} - \rho \mathbf{v} \cdot \nabla \mathbf{v} + \rho \mathbf{f},
\]
\[
\partial_t \rho + \mathbf{v} \cdot \nabla \rho = 0,
\]
\[
\nabla \cdot \mathbf{u} = 0, \quad \mathbf{u}(0) = 0, \quad \rho(0) = \rho_0, \quad \mathbf{u} = 0 \text{ on } \Gamma,
\]
respectively.

Subtracting (6.6) and (6.7) we obtain
\[
(6.8) \quad \partial_t (\mathbf{u}^n - \mathbf{u}) + (\mathbf{T}_p(\mathbf{v}^n) - \mathbf{T}_p(\mathbf{v})) \mathbf{u}^n + \mathbf{T}_p(\mathbf{v})(\mathbf{u}^n - \mathbf{u}) + \nabla(\pi^n - \pi) = (1 - \rho^n) \partial_t \mathbf{v}^n - \rho^n \partial_t \mathbf{v} - \rho^n \mathbf{v}^n \cdot \nabla \mathbf{v} + \rho \mathbf{v} + \tilde{\rho} \mathbf{f}.
\]

Denoting \( \mathbf{V}^n = \mathbf{v}^n - \mathbf{v} \), \( \mathbf{U}^n = \mathbf{u}^n - \mathbf{u} \), we get
\[
(6.9) \quad \partial_t \mathbf{U}^n(t) + \mathbf{T}_p(\mathbf{v}) \mathbf{U}^n(t) + (\mathbf{T}_p(\mathbf{v}^n(t)) - \mathbf{T}_p(\mathbf{v}(t))) \mathbf{u}^n(t) + \nabla(\pi^n - \pi) = (1 - \rho^n) \partial_t \mathbf{V}^n(t) - \tilde{\rho}^n(t) \partial_t \mathbf{v}(t) - \rho^n(t) \mathbf{V}^n(t) \cdot \nabla \mathbf{v}^n(t) - \tilde{\rho}^n(t) \mathbf{v}(t) \cdot \nabla \mathbf{v}^n(t) - \rho(t) \mathbf{v}(t) \cdot \nabla \mathbf{v}^n(t) + \tilde{\rho}^n(t) \mathbf{f}(t).
\]
Multiply (6.9) by \(U^n(t)\) and integrating over \(\Omega\) we get
\[
(6.10) \quad d_t |U^n(t)|^2 + \|U^n(t)\|^2 \leq -\int_{\Omega} \bar{\rho}^n \nabla U^n(t) \nabla v(t) dx + \\
\partial_t \int_{\Omega} (1 - \rho^n(t)) U^n(t) V^n(t) dx - \int_{\Omega} U^n(t) V^n(t) \partial_t (1 - \rho^n(t)) dx - \\
\int_{\Omega} (1 - \rho^n(t)) V^n(t) \partial_t U^n(t) dx + |\rho^n(t)|_\infty |\nabla \nabla^n(t)|_2 |V^n(t)|_3 |U^n(t)|_a + \\
|\rho v(t)|_2 |\nabla V^n(t)|_2 |U^n(t)|_6 + |\bar{\rho}^n(t)|_6 |v(t)|_\infty |\nabla V^n(t)|_3 |U^n(t)|_2 + \\
|\bar{\rho}^n(t)|_2 |f(t)|_2 |U^n(t)|_\infty + \int_{\Omega} (T_p(v^n(t)) - T_p(v(t))) u^n(t) U^n(t) dx.
\]

Integration (6.10) in \(t\) gives
\[
(6.11) \quad |U^n(s)|^2 \leq |1 - \rho^n(s)|_\infty |U^n(s)|_2 |V^n(s)|_2 + \\
c \int_0^s |U^n(t)|_\infty (|V^n(t)|_2 |\partial_t \rho^n(t)|_2 + |\bar{\rho}^n \partial_t v(t)|_2) dt + \\
(6.12) \quad c(r) \int_0^s (|V^n(t)|_2 + |\nabla V^n(t)|_2 + |\bar{\rho}^n(t)|_2 |U^n(t)|_2) dt + \\
\int_0^s (T_p(v^n(t)) - T_p(v(t))) u^n(t), U^n(t)) dt.
\]

Bearing in mind that \(\mu(\cdot)\) is a \(C^2\)-function of its argument and
\[
sup(|v^n|_{L^\infty(Q_T)}, |\nabla V^n|_{L^\infty(Q_T)}) \leq c(r),
\]
follows
\[
\int (T_p(v^n(t)) - T_p(v(t))) u^n(t), U^n(t)) \leq c(r) |\nabla V^n(t)|_\infty |\|u^n(t)\|_2 |U^n(t)|_2.
\]

Besides, by the multiplicative inequality (proposition 1), it is routine matter to prove that \(U^n \to 0\) strongly in \(L^2(Q_T)\).

Consequently, the map \(G\) is continuous in \(L^2(Q_T)\), and the existence of a local solution is completely proved. The proof of the uniqueness runs like that the continuity of \(G\).

Let \((\rho, u)\) and \((\bar{\rho}, \bar{u})\) be two solutions of (6.1) and let \(\tilde{\rho} = \rho - \bar{\rho}\) and \(U = u - \bar{u}\). Then \(\tilde{\rho} = \rho - \bar{\rho}\) and \(U\) satisfy the equation.
\[
(6.13) \quad \partial_t \tilde{\rho} + U \cdot \nabla \tilde{\rho} + u \cdot \nabla \tilde{\rho} = 0, \quad \tilde{\rho}(0) = 0.
\]

Multiplying (6.13) by \(\tilde{\rho}\) and after integration over \(Q_T\) we have
\[
|\tilde{\rho}(t)|^2 \leq e^{ct} \int_0^t |U \cdot \nabla \tilde{\rho}|_2^2 dt.
\]

Moreover, \(\tilde{\rho}\) and \(U\) satisfy
\[
(6.14) \quad \rho \partial_t U(t) + T_p(u(t)) U(t) + (T_p(u(t)) - T_p(\bar{u}(t))) \bar{u}(t) + \nabla \Pi = \\
\tilde{\rho} \partial_t \bar{u}(t) - \rho u(t) \cdot \nabla u(t) + \bar{\rho} u(t) \cdot \nabla \bar{u}(t) + \tilde{\rho} f.
\]
Therefore it is easy to derive the inequality

\[
\frac{1}{2} \frac{d}{dt} (|\sqrt{\rho} U|^2 + |\bar{\rho}|^2) + c \| U \|^2 \leq \\
\omega(t) (|\sqrt{\rho} U|^2 + |\bar{\rho}|^2) + \delta |\nabla U|^2,
\]

where

\[
\omega(t) = c(|\partial_t \bar{u}|^2 + |u \cdot \nabla \bar{u}|^2 + |\nabla \bar{\rho}|^2 + |f|^2 + |\nabla \bar{\rho}|^2 + |\nabla \rho|^2 + |\bar{\rho}|^2).
\]

Now, integrating the differential inequality (6.15), we get, for suitable \( \delta \),

\[
|\sqrt{\rho} U|^2 + |\bar{\rho}|^2 = 0 \text{ a.e. in } (0, \bar{T}),
\]

\[
\int_0^\bar{T} |\nabla U|^2 dt = 0.
\]

Hence \( U = 0 \) and \( \bar{\rho} = 0 \) a.e. in \( Q_{\bar{T}} \).

The theorem is completely proved.

\[ \square \]

7. Multiphase problem

Up to now, we have considered problems in fixed domains. Many physical problems deal with unsteady fluid-fluid, fluid-vacuum or fluid-structure interaction phenomena. These phenomena are of major importance for aerospace, mechanical or biomedical applications. The problem is to describe the evolution of a viscous fluid coupled with a moving structure (solid or liquid). Several conditions determine the coupling between the media at the surface of separation or interface. If the fluid domain varies under an assigned law the domain is called ”non-cylindrical domain” in the other cases it is called ”free boundary domain”.

The fluid-fluid or fluid-vacuum problems are well posed in the theory of multiphase systems used in fluid mechanics. There, the equations of conservation of mass, momentum, energy and chemical species are written separately for each phase, assuming that temperature, pressure, density and composition of each phase are equal to their equilibrium values. Accordingly, these equations are supplemented by boundary conditions at interface, namely

\[
\sigma|_+ = C t n - (I - n \otimes n) \cdot \nabla t,
\]

\[
v|_+ = 0, \quad \theta|_+ = 0,
\]

with \( n \) denoting the normal at interface, stating that the jump of the stress tensor \( \sigma \) at the interface is related to the curvature \( C \), the surface tension \( t \) and its gradient, while the velocity \( v \) and the temperature \( \theta \) are continuous. (see \([1], [19]\)).

Similar boundary conditions exist for the transport of energy and mass

\[
\mathbf{J}_q|_+ \cdot n = 0, \quad \mathbf{J}_i|_+ \cdot n = 0
\]

\[
\rho^i|_+ = (k - 1) \rho^i|_+^{\tau}
\]
stating that standard (Fourier) heat flux $J_q$, and the diffusive flux of any chemical species $i$, $J_i$ are continuous across the interface (assuming no phase transition and no surface reactions) while the concentration $\rho^i$ can undergo a jump, depending on a partition coefficient $k$, given by thermodynamics.

Mathematically, the model presents serious difficulties. A solution is obtained, in long time, in the context of Caccioppoli domains. In this case the inner normal and the interface are understood in a measure theoretic sense and non in topological sense (see [33]).

Any way, the free-boundary description is an effective model in a wide range of situations. However, there are important instances where it breaks down, i.e. the interfacial thickness is comparable to the length scale of the phenomenon. So an other approach was proposed by Rayleigh and Van der Waals who assumed that the interface has a non-zero thickness, i.e. it is diffuse. Diffuse-interface models provide an alternative description of interface motion (also in the case of fluid-rigid body problem). Quantities that in the free-boundary formulation are localized in the interfacial surface are distributed throughout interfacial region. The main characteristic of the diffuse interface model is the use of an order parameter which undergoes a rapid but continuous variation across the interphase boundaries, while varies smoothly in each bulk phase. In view of the arbitrary choice of "order parameter " instead of "diffuse model " we write "field phase model". Phase-field methods are based on models of fluid free energy (see [1]). The simplest model of free energy density goes back to Van der Waals. Cahn and Hilliard [5],[6] extended Van der Waals hypothesis to time-dependent situations by approximating interfacial diffusion fluxes as being proportional to chemical potential gradients. The Cahn-Hilliard equation is

$$\partial_t C = \lambda \Delta \phi = -\lambda \Delta (\Delta C - \psi'(C)),$$

where $C$ is order parameter, $\phi$ is the chemical potential, $\psi(C)$ is the bulk energy density that models the fluid components immiscibility and $\lambda$ is a diffusion parameter. Mathematical models are based on coupling of a Cahn-Hilliard equation, incompressible homogeneous Navier-Stokes equations and heuristic speculative formulations. Accordingly, the incompressible Navier-Stokes equations are modified by the addition of the continuum forcing related to chemical potential (Korteweg-type stress tensor ). The density (concentration) is variable that distinguishes the bulk fluids and the intervening of interface.

The fluid-model obtained has two characteristics: 1) is compressible, 2) the order parameter is constrained in the interval $[-1,1]$. These two conditions make complicate the mathematical treatment of the problem.

We quote the paper [4] as a reference for the construction of an incompressible model. In appendix, we adapt the procedure suggested in [2], [8] to deduce the following incompressible model including the effect of advective-diffusion process.

\begin{align}
\rho \partial_t \mathbf{u} + \rho \mathbf{u} \cdot \nabla \mathbf{u} &- \lambda \nabla \theta \cdot \nabla \mathbf{u} - \lambda \mathbf{u} \cdot \nabla \nabla \theta + \nabla \pi - \mu \Delta \mathbf{u} + \\
\lambda^2 \nabla \cdot \left( \frac{\nabla \theta \otimes \nabla \theta}{\rho} \right) & = \rho \mathbf{f},
\end{align}

\begin{align}
\partial_t \rho + \mathbf{u} \cdot \nabla \rho - \Delta \rho = 0, \quad \nabla \cdot \mathbf{u} = 0.
\end{align}
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Here $\theta = \rho - D\Delta\rho$, $\rho$ is the density, $\lambda$ is the diffusion coefficient and $D$ is the mobility coefficient. In the next section we discuss the existence for the system (7.3).

7.1. Existence problem for the diffusion equation and a priori estimates.

We deduce a priori estimates of the solution of the Neumann problem

$$\partial_t \rho + \psi \cdot \nabla \rho - \lambda \Delta (\rho - D\Delta\rho) = 0,$$

$$\rho(0) = \rho_0, \quad \partial_n \rho = \partial_n \Delta \rho = 0 \text{ on } \Gamma.$$  

Here $\psi$ is a smooth divergence free function vanishing on $\Gamma$.

For the mathematical setting of the problem, we introduce

$$\mathcal{H} := \{ \phi | \phi \in H^2(\Omega), \partial_n \phi = 0 \},$$

$$\mathcal{H}^2 := \{ \phi | \phi \in H^4(\Omega), \partial_n \phi = \partial_n \Delta \phi = 0 \}.$$

$\mathcal{H}$ is closed subspace of $H^2(\Omega)$ and is endowed with the scalar product and the norm

$$((\phi, \chi)) = \int_\Omega (\Delta \phi, \Delta \chi) dx, \quad ||\phi||^2_2 = ((\Delta \phi, \Delta \phi)).$$

$$a(\phi, \chi) := ((\phi, \chi)) = \int_\Omega (\Delta \phi, \Delta \chi) dx$$

is a bilinear form on $H^2$. By the Poincare’s inequality and regularity of elliptic problem, $||\phi||_s + c||\phi||_2$ or $||\phi||_s + c\int_\Omega \phi dx$ are norms in $\mathcal{H}$ equivalent to the norm induced by $H^2(\Omega)$. In general, denoted $m(\phi) = \frac{1}{|\Omega|} \int_\Omega \phi(x) dx$, $s \geq 0$,

$$||\phi - m(\phi)||_1 \leq c||\nabla \phi||_2, \quad ||\phi - m(\phi)||_s \leq c||\Delta \phi||_s.$$

For convenience, we set $\lambda = D = 1$. The existence for the diffusion equation (7.4) can be performed through Faedo-Galerkin method.

This procedure is well known in literature so we omit details. We prove now a priori estimates, formally.

First, we notice that the maximum principle does not hold, in general. But the average value is conserved:

$$\int_\Omega \rho(t) dx = \int_\Omega \rho_0.$$

Moreover, since $\rho$ is a solution of the continuity equation (see appendix (9.3)) $\rho(t) \geq 0$ if $\rho_0 \geq 0$. Indeed, $\rho$ satisfies

$$(\inf \rho_0) \exp(- \int_0^t |\nabla \cdot w(\tau)|_{\infty}) d\tau \leq \rho(x, t).$$

Now, we prove five levels of regularity for $\rho$.

Multiplying (7.4) by $\rho$ and integrating by parts in $\Omega$, we get

$$d_t |\rho|^2_2 + 2|\nabla \rho|^2_2 + 2|\Delta \rho|^2_2 = 0,$$

then

$$|\rho(t)|^2_2 + \int_0^t ((|\nabla \rho(\tau)|^2_2 + |\Delta \rho(\tau)|^2_2) d\tau \leq |\rho_0|^2_2.$$  

The estimate (7.5) is independent of $\psi$. 


Now, multiplying (7.4) by \(-\Delta \rho\) and after integration by parts on \(\Omega\), we obtain
\[
\frac{1}{2} \partial_t |\nabla \rho|^2 + |\Delta \rho|^2 + |\nabla \Delta \rho|^2 = (\psi \cdot \nabla \rho, \Delta \rho).
\]
In virtue of
\[
|\psi(t) \cdot \nabla \rho(t), \Delta \rho(t)| \leq c|\psi(t)\rho(t)|^2 + \frac{1}{2}|\nabla \Delta \rho(t)|^2 \leq
\]
\[
c|\psi(t)|^2 \Delta \rho(t)^{3/2} |\rho(t)|^{1/2} + \frac{1}{2}|\nabla \Delta \rho(t)|^2,
\]
thus we get
\[
(7.6) \quad |\nabla \rho(t)|^2 + \int_0^t (|\Delta \rho(\tau)|^2 + |\nabla \Delta \rho(\tau)|^2) d\tau \leq
\]
\[
c \sup_{0 \leq \tau \leq t} |\psi(\tau)|^2 \sqrt{t} + |\nabla \rho(0)|^2 := \Psi_1(t).
\]
Notice that the above estimate requires \(\psi \in L_{\infty}(0, T; L_2(\Omega))\), only. Now, we deduce \(H^1\)-estimate for \(\rho\).

Multiplying (7.4) by \(\Delta^2 \rho\) and after integration by parts (bearing in mind the boundary terms vanish), we deduce
\[
\frac{1}{2} \partial_t |\Delta \rho|^2 + |\nabla \rho|^2 + |\Delta^2 \rho|^2 = -(\psi \cdot \nabla \rho, \Delta^2 \rho).
\]
In virtue of
\[
|\psi(t) \cdot \nabla \rho(t), \Delta^2 \rho(t)| \leq c|\psi(t)\nabla \rho(t)|_\infty |\Delta^2 \rho(t)|_2 \leq
\]
\[
c|\psi(t)|^2 |\nabla \rho(t)|^{1/2} |\nabla \Delta \rho(t)|^{3/2} + \frac{1}{2}|\Delta^2 \rho(t)|^2,
\]
thus we obtain
\[
(7.7) \quad |\Delta \rho(t)|^2 + \int_0^t (|\nabla \Delta \rho(\tau)|^2 + |\Delta^2 \rho(\tau)|^2) d\tau \leq
\]
\[
c \sqrt{t} (\sup_{0 \leq \tau \leq t} |\psi(\tau)|^2) \Psi_1(t) + |\Delta \rho(0)|^2 := \Psi_2(t).
\]
Next, we prove \(H^2\)-estimate. First, we apply the \(\nabla\) operator to (7.4) and then we multiply the result by \(\nabla \Delta^2 \rho\), after integration by parts, we obtain
\[
\frac{1}{2} \partial_t |\nabla \rho|^2 + |\nabla \Delta^2 \rho|^2 + |\Delta^2 \rho|^2 = -(\psi \cdot \nabla \rho, \nabla \Delta^2 \rho) - (\nabla \psi \cdot \nabla \rho, \nabla \Delta^2 \rho) \leq
\]
\[
c (|\nabla|_6 |\nabla \rho|^2 + |\nabla \rho|_\infty^2) + \frac{1}{2} |\Delta^2 \rho|^2 \leq c|\nabla \rho|^2 |\rho|_6 |\nabla \rho|_2 + \frac{1}{2} |\Delta^2 \rho|^2.
\]
We obtain
\begin{equation}
(7.8) \quad |\nabla \Delta \rho(t)|_2^2 + \int_0^t (|\nabla \Delta^2 \rho(\tau)|_2^2 + |\Delta^2 \rho(\tau)|_2^2) d\tau \leq \\
\sqrt{7} \sup_{0 \leq \tau \leq t} |\nabla \psi(\tau)|_2^2 \Psi_2(t) + |\nabla \Delta \rho(0)|_2^2 := \Psi_3(t).
\end{equation}

According to the above estimates, if $\psi \in L^2(0, T; H^1_0(\Omega))$, we have
\[ \partial_t \rho \in L^2(0, T; H^1(\Omega)). \]

In addition, if we assume $\partial_t \psi \in L^2(Q_T)$, we derive analogous estimates for $\partial_t \rho$.
Indeed, setting $\eta = \partial_t \rho$ we get
\begin{equation}
(7.9) \quad \partial_t \eta + \psi \cdot \nabla \eta - \Delta (\eta - \Delta \eta) = -\partial_t \psi \cdot \nabla \rho.
\end{equation}
Similarly, we obtain $H^4$-estimates for $\eta$ considering $-\partial_t \psi \cdot \nabla \rho$ as a given right hand side, namely adding to the above estimates the $L^2(Q_T)$ norm of $-\partial_t \psi \cdot \nabla \rho$ with $\eta(0) \in H^3(\Omega)$.

Consequences of the previous estimates and proposition 1 are:
\begin{equation}
(7.10) \quad 1. \quad \int_0^t |\nabla \rho(\tau)|_2^2 d\tau \leq c \sqrt{7} \Psi_1(t);
2. \quad \sup_{0 \leq \tau \leq t} |\Delta \rho(\tau)|_2^2 \leq c \Psi_2(t);
3. \quad \int_0^t |\Delta^2 \rho(\tau)|_2^2 d\tau \leq c \Psi_2(t);
4. \quad \int_0^t |\nabla^2 \rho(\tau)|_2^2 d\tau \leq c \sqrt{7} (\sup_{0 \leq \tau \leq \tau} |\nabla \psi(\tau)|_2^2) \Psi_2(t) + |\nabla \rho(0)|_2^2 \Psi_3(t);
5. \quad \sup_{0 \leq \tau \leq t} |\Delta \nabla \rho(\tau)|_2^2 \leq c \sqrt{7} (\sup_{0 \leq \tau \leq \tau} |\nabla \psi(\tau)|_2^2) \Psi_2(t) + |\Delta \rho(0)|_2^2 \Psi_3(t);
6. \quad \int_0^t |\partial_t \Delta \rho(\tau)|_2^2 d\tau \leq c \sqrt{7} (\sup_{0 \leq \tau \leq \tau} |\nabla \psi(\tau)|_2^2) \Psi_2(t) + \\
|\Delta \nabla \rho(0)|_2^2 \int_0^t |\partial_t \psi|_2^2 d\tau + \int_0^t |\psi \cdot \nabla \partial_t \rho(\tau)|_2^2 d\tau + |\Delta \partial_t \rho(0)|_2^2 \leq \\
c \Psi_3(t) \int_0^t |\partial_t \psi(\tau)|_2^2 d\tau + \sqrt{7} \sup_{0 \leq \tau \leq t} |\nabla \psi(\tau)|_2^2 \Psi_2(t) \int_0^t |\partial_t \psi(\tau)|_2^2 d\tau + \\
|\partial_t \rho(0)|_2^2 + |\Delta \partial_t \rho(0)|_2^2 := \Psi_4(t).
\end{equation}

7.2. Graffi’s type model. In this section we discuss a simplified model instead of (7.8). We notice that the term
\[ \lambda (\nabla \cdot \nabla \theta) \]
in \(7.3\) is a serious obstacle to prove the existence of a sort of solution (weak or strong). If \(\theta = \rho\) (see \([2]\)) , the above term is efficiently estimated by using the maximum principle, \(|\nabla \rho|_4 \leq c|\rho|_\infty |\Delta \rho|_2\), and a constrain on \(\lambda, \mu\) and the maximum of \(\rho\).

For future investigations, in this section we study the system omitting the terms \(\lambda u \cdot \nabla \theta\) and \(O(\lambda^2)\). Graffi in \([13]\) considered a system of equations type \(7.3\) but discarding the two \(O(\lambda)\) and \(O(\lambda^2)\) terms in \(7.3_1\). In other words we consider the following simplified model \((\lambda = 1)\)

\[
\begin{align*}
\rho \partial_t u + \rho u \cdot \nabla u - \nabla \theta \cdot \nabla u - \nabla \pi - \mu \Delta u &= \rho f, \\
\partial_t \rho + u \cdot \nabla \rho - \Delta (\rho - \Delta \rho) &= 0, \\
\nabla \cdot u &= 0, \quad \rho(0) = \rho_0, \quad u(0) = u_0.
\end{align*}
\]

We will derive essential a priori estimates, formally. The complete proof can be performed through a Faedo-Galerkin process. Now, we prove a priori estimates which allow to deduce the existence of a weak solution of \(7.11\). We consider the approximate system

\[
\begin{align*}
\rho \partial_t u^n + \rho^n u^{n-1} \cdot \nabla u^n - \nabla \theta^n \cdot \nabla u^n - \nabla \pi^n - \mu \Delta u^n &= \rho^n f, \\
\partial_t \rho^n + u^{n-1} \cdot \nabla \rho^n - \Delta (\rho^n - \Delta \rho^n) &= 0, \\
\nabla \cdot u^n &= 0, \quad \rho^n(0) = \rho^n_0, \quad u^n(0) = u^n_0,
\end{align*}
\]

where \(u^{n-1}\) is a suitable solenoidal function. Moreover, we make use of the results in section 3 and of the usual initial-boundary conditions. Multiplying \(7.12_1\) by \(u^n\) and after integration by parts we get

\[
\begin{align*}
dt |\sqrt{\rho^n u^n}|_2 + \mu |\nabla u^n|_2^2 + \frac{1}{2}(u^n \partial_t \rho^n, u^n) + (\rho^n u^{n-1} \cdot \nabla u^n, u^n) + \\
\frac{1}{2}(\Delta \theta^n u^n, u^n) - (\rho^n f, u^n) &= 0.
\end{align*}
\]

Multiplying \(7.12_2\) by \(\frac{|u^n|^2}{2}\) and summing the result to \(7.13\) we get

\[
\begin{align*}
dt |\sqrt{\rho u^n}|_2^2 + \mu |\nabla u^n|_2^2 = (\rho f, u^n),
\end{align*}
\]

and easily we obtain the standard energy estimate

\[
\begin{align*}
|\sqrt{\rho u^n(t)}|_2^2 + \mu \int_0^t |\nabla u^n(\tau)|_2^2 d\tau &\leq |\sqrt{\rho(0)^n u^n(0)}|_2^2 + \int_0^t (|\rho^n f(\tau)|_2^2 d\tau.
\end{align*}
\]

Notice that the right-hand side in \(7.15\) does not depend on \(u^{n-1}\).
Consequently, there exists a subsequence \( \{u^n, \rho^n\} \) such that

\[
\frac{\partial}{\partial t} u^n \rightarrow u \text{ weakly in } L^2(0, T; V), \\
\rho^n \rightarrow \rho \text{ weakly in } L^2(0, T; H^1(\Omega)), \quad \text{strongly in } L^2(0, T; H^2(\Omega)), \\
u^n \rightarrow u, \quad u^n \cdot \nabla \rho^n \rightarrow \alpha \text{ in the sense of distributions.}
\]

To complete the existence proof of a weak solution we have to show \( \alpha_{ij} = \rho u_i u_j \). For this we estimate the time derivative of \( u^n \).}

7.3. Time derivative estimates and compactness result. Now, let \( \phi \) be a smooth solenoidal function vanishing on \( \Gamma \). Multiplying \( (7.12) \) by \( \phi \) and after integration by parts we get

\[
\int_0^T ((\partial_t \rho^n u^n(t), \phi(t)) + (\rho^n(t) u^n - 1(t), u^n(t) \cdot \nabla \phi(t)) - \mu(\nabla u^n(t), \nabla \phi(t)) - ((\nabla \theta^n(t) \cdot \nabla) \phi(t), u^n(t)) + (\rho^n f(t), \phi(t)) dt = 0.
\]

The estimates \( (7.11), (7.15) \) show that \( \partial_t (P\rho^n u^n(t)) \) is bounded in \( L^2(0, T; H^{-2}(\Omega)) \), uniformly with respect to \( n \), while \( \rho^n u^n \) and thus \( P\rho^n u^n \) are bounded in \( L^\infty(0, T; L^2(\Omega)) \), uniformly with respect to \( n \).

Hence, by classical compactness theorems, \( \{P\rho^n u^n\} \) is a compact set in \( L^2(0, T; H^{-1}(\Omega)) \).

In particular, since (subsequence) \( \{\rho^n u^n\} \) converges weakly to \( \rho u \), \( \{P\rho^n u^n\} \) converges to \( P\rho u \) in \( L^2(0, T; H^{-1}(\Omega)) \).

Now, making use of the procedure in theorem 4, we get the strong convergence in \( L^2(Q_T) \) of \( \sqrt{\rho^n u^n} \) to \( \sqrt{\rho u} \). This convergence implies that \( \alpha_{ij} = \rho u_i u_j \).

The existence proof of a weak solution of \( (7.11) \) is thus complete.

8. Local existence problem for the system \( (7.13) \).

In this section we prove the following theorem

**Theorem 8.1.** Let \( \Omega \subset \mathbb{R}^3 \) be a domain with compact boundary \( \Gamma \) of class \( C^3 \).

Assume \( u_0 \in V, (\rho_0, \partial_i \rho(0)) \in (H^3(\Omega), H^2(\Omega)) \) and \( f \in L^\infty(0, T; L^2(\Omega)) \). Then there exists a \( T > 0 \) such that there is a unique solution \( (u, \rho, \pi) \) of

\[
\begin{align*}
\partial_t \rho u - \rho \Delta u + \rho u \cdot \nabla u - \lambda (u \cdot \nabla \theta + (\nabla \theta \cdot \nabla) u) + \\
\frac{\lambda^2}{\rho} ((\nabla \theta \cdot \nabla) \nabla \theta - \frac{1}{\rho} (\nabla \rho \cdot \nabla \theta) \nabla \theta + \Delta \theta \nabla \theta) + \nabla \pi - \rho \mathbf{f} &= 0, \\
\partial_i \rho + u \cdot \nabla \rho - \lambda \Delta \theta &= 0, \\
\nabla \cdot u &= 0, \quad u(0) = u_0, \quad \rho(0) = \rho_0, \quad \partial_n \rho = \partial_n \Delta \rho = 0 \quad \text{on } \Gamma.
\end{align*}
\]
Theorem 6.1. Let

\[ u \in H^1(0, \bar{T}, V^0(\Omega)) \cap L_2(0, \bar{T}, H^2(\Omega) \cap V), \pi \in L_2(0, \bar{T}; H^1(\Omega))/\mathbb{R}, \]
\[ \rho \in L_2(0, T; H^5(\Omega)), \partial_t \rho \in L_2(0, T; H^4(\Omega)). \]

Here \( \bar{\rho} = \rho + m \) with \( m \) a positive number.

Proof. We prove Theorem 8.1 by fixed point argument following the scheme of the

Theorem 6.1. Let

\[ B(r) = \{ \phi \mid \sup_{t \in [0, T]}(\| \phi \|_{L_2(0, T; H^2(\Omega))}, \| \phi \|_{H^1(0, T; L_2(\Omega))}) \leq r \}. \]

We fix a function \( v \in B(r) \) and subsection 7.1 gives the solution of diffusion equation

(7.4) with \( \psi \equiv v \) and relative estimates on \( \rho \) up to the \( H^5 \)-regularity.

Next, we consider the linear problem with fixed \( v \in B(r) \),

(8.2) \[ \bar{\rho}_t u - \mu \Delta u + \bar{\rho} \nabla v \cdot \nabla - \lambda((v \cdot \nabla) \nabla\theta + (\nabla \theta \cdot \nabla) v) + \frac{\lambda^2}{\rho}((\nabla \theta \cdot \nabla) \nabla\theta - \frac{1}{\rho}(\nabla \bar{\rho} \cdot \nabla) \nabla\theta + \Delta \theta \nabla\theta) + \nabla \pi - \bar{\rho} f = 0, \]

complemented with the usual initial-boundary conditions. The existence of a solution of system (8.2) is established in [31] making use of Lax-Milgram theorem.

8.1. A priori estimates. For convenience we set \( \lambda = \mu = 1 \). Moreover we make use of the inequalities

\[ |\phi|_\infty \leq c|\phi|_6|\nabla \nabla \phi|_2, |\phi|_\infty \leq c|\phi|^{1/4}_2|\nabla \nabla \phi|^{3/4}_2. \]

We multiply now by \( u \) (8.2) and after integration by parts, integration in \( t \) gives

(8.3) \[ |\sqrt{\rho} u(t)|^2 + \int_0^t |\nabla u(\tau)|^2 d\tau \leq |\sqrt{\rho} u(0)|^2 + \int_0^t |\nabla \nabla \theta(\tau)|^2 d\tau + \sqrt{t} \sup_{0 \leq \tau \leq t} |\nabla v(\tau)|^2 \Psi_2(t) \left( \int_0^t |A v(\tau)|^2 d\tau \right)^{1/2} + t \rho(0) \sup_{0 \leq \tau \leq t} |f(\tau)|^2 + \int_0^t (u(\tau) \partial_\tau \rho(\tau), u(\tau)) d\tau. \]

Now, we multiply (8.2) by \( \partial_\tau u \), integrate over \( \Omega \), and obtain

(8.4) \[ |\sqrt{\rho(t)} \partial_\tau u(t)|^2 + d_1 |\nabla u(t)|^2 \leq c|\bar{\rho}(t)|^2 \| v(t) \|^2 |\nabla v(t)|^2 + |v(t)|^2 \| \nabla \nabla \theta(t) \|^2 + |\nabla \nabla \theta(t)|^2 \| \nabla v(t) \|^2 + c|\bar{\rho}(t)|^2 \| v(t) \|^2 \| A v(t) \|^2 + |\nabla v(t)|^2 \| v(t) \|^2 + \| v(t) \| \| A v(t) \| \| \Delta \theta(t) \|^2 + |\Delta \theta(t)|^2 \|| \theta(t) \| + |\rho(t)|^2 \| f(t) \| + \frac{1}{2} |\partial_\tau u|^2. \]
Then, integrating (8.4) with respect to $t$, we get

\begin{align*}
\int_0^t \sqrt{\rho}(t) \partial_t u(t) \|e_2\|_2^2 + \|\nabla u(t)\|_2^2 \leq |\nabla u(0)|_2^2 + \\
\sqrt{r}\Psi_2(t) \sup_{0 \leq r \leq t} |\nabla v(\tau)|_2^2 \left( \int_0^t |A v(\tau)|_2^2 dt \right)^{1/2} + \\
\sqrt{r}\Psi_4(t) \sup_{0 \leq r \leq t} |\nabla v(\tau)|_2^2 \left( \int_0^t |A v(\tau)|_2^2 dt \right)^{1/2} + \\
c\sqrt{r}\Psi_3(t) \Psi_4(t) + \sqrt{r} \rho_0 \|e_2\|_2^2 \sup_{0 \leq r \leq t} |f(\tau)|_2^2.
\end{align*}

Now, we consider the Stokes problem

\begin{align*}
Au &= -P(\rho \partial_t u + \bar{\rho} v \cdot \nabla v - (v \cdot \nabla) \nabla \theta + (\nabla \theta \cdot \nabla)v) + \\
\nabla \cdot (\frac{1}{\rho}(\nabla \theta \otimes \nabla \theta)) - \bar{\rho} f.
\end{align*}

From the theory of the Stokes problem there exists a constant $c$ such that

\begin{align*}
\int_0^t |A u(\tau)|_2^2 d\tau \leq |\nabla u(0)|_2^2 + c\sqrt{r} \sup_{0 \leq r \leq t} |\nabla v(\tau)|_2^2 \Psi_4(t) + c\sqrt{r}\Psi_3(t) \Psi_4(t) + \\
+ c\sqrt{r}\Psi_2(t) \sup_{0 \leq r \leq t} |\nabla v(\tau)|_2^2 \left( \int_0^t |A v(\tau)|_2^2 dt \right)^{1/2} + \\
\sqrt{r}\Psi_4(t) \sup_{0 \leq r \leq t} |\nabla v|_2 \left( \int_0^t |A v(\tau)|_2^2 dt \right)^{1/2} + \sqrt{r} \rho_0 \|e_2\|_2^2 \sup_{0 \leq r \leq t} |f(\tau)|_2^2.
\end{align*}

Choosing $r^2 = c(|\nabla u_0|_2^2 + |\Delta \nabla \rho_0|_2^2 + |\Delta \partial_t \bar{\rho}(0)|^2 + \sup_{0 \leq r \leq T} |f(\tau)|_2^2)$ with $c$ big enough, (7.10), (8.6), (8.7) yield

\begin{align*}
\int_0^t \left( |\nabla e_2(\tau)|_2^2 + |A e_2(\tau)|_2^2 \right) d\tau + \sup_{0 \leq r \leq t} |\nabla u(\tau)|_2^2 \leq \\
c(|\nabla u(0)|_2^2 + c\sqrt{r} \tau^3 \left( \int_0^t |A e_2(\tau)|_2^2 dt \right)^{1/2} + \\
\sqrt{r}\sup_{0 \leq r \leq t} |\Delta \theta(\tau)|_2^2 \left( \int_0^t |A e_2(\tau)|_2^2 dt \right)^{1/2} + \\
\sqrt{r}\sup_{0 \leq r \leq t} |\Delta \theta(\tau)|_2^2 \left( \int_0^t |\theta(\tau)|_2^2 dt \right)^{1/2} + \rho(0) \sup_{0 \leq r \leq t} |f(\tau)|_2^2 \leq \\
c(|\nabla u(0)|_2^2 + |\Delta \nabla \rho_0|_2^2 + |\Delta \partial_t \rho(0)|_2^2 + \sqrt{r} \tau^4 + t \rho^2(0) \sup_{0 \leq r \leq t} |f(\tau)|_2^2) \leq r^2.
\end{align*}

for $t = \bar{T}$ small enough.

Hence (8.8) implies $G \subset \cdot$.

We prove now the continuity of $G$. 
Let \( \{v^n\} \subset B \) be a sequence such that \( v^n \to v \) in \( L^2(Q_T) \), strongly. We notice that \( v \in B \). Thanks to Proposition 1, we deduce

\[
(8.9) \quad v^n \to v \text{ in } L_2(0, T; L_\infty(\Omega)) \cap L_2(0, T; W_{x2}^1(\Omega)),
\]

\[
\theta^n \to \theta \text{ in } L_2(0, T; L_\infty(\Omega)) \cap L_2(0, T; W_{x2}^1(\Omega)) \cap L_4(0, T; W_{x2}^3(\Omega)) \cap L_\infty(0, T; L_2(\Omega)),
\]

with \( q > 1 \), strongly.

Let \( \rho^n, \rho \) be solutions of

\[
(8.10) \quad \partial_t \rho^n + v^n \cdot \nabla \rho^n - \Delta \theta^n = 0; \quad \rho^n(0) = \rho_0, \quad \partial_n \rho^n = \partial_n \Delta \rho^n = 0 \text{ on } \Gamma,
\]

\[
\partial_t \rho + v \cdot \nabla \rho - \Delta \theta = 0; \quad \rho(0) = \rho_0, \quad \partial_n \rho = \partial_n \Delta \rho = 0 \text{ on } \Gamma,
\]

respectively.

Now \( \tau^n = \rho^n - \rho \) and \( \Theta^n = \theta^n - \theta \) satisfy

\[
(8.11) \quad \partial_t \tau^n + v^n \cdot \nabla \tau^n - \Delta \Theta^n = -(v^n - v) \cdot \nabla \rho,
\]

\[
\tau^n(0) = 0, \quad \partial_n \tau^n = \partial_n \Theta = 0 \text{ on } \Gamma.
\]

The estimates (8.6), (8.10), Gronwall’s lemma and Proposition 1 yield that \( \tau^n \to 0 \) in \( L_2(0, \bar{T}; H^2) \cap L_\infty(0, \bar{T}; H^1) \).

Now, let \( u^n, u \) be the solutions of

\[
(8.12) \quad \bar{\rho}^n \partial_t u^n - \Delta u^n + \bar{\rho}^n v^n \cdot \nabla v^n - (v^n \cdot \nabla) \nabla \theta^n + (\nabla \theta^n \cdot \nabla) v^n +
\]

\[
\nabla \cdot \left( \frac{1}{\rho^n} (\nabla \theta^n \otimes \nabla \theta^n) \right) + \nabla \pi^n - \bar{\rho}^n f = 0,
\]

\[
\bar{\rho} \partial_t u - \Delta u + \bar{\rho} v \cdot \nabla v - (v \cdot \nabla) \nabla \theta + (\nabla \theta \cdot \nabla) v +
\]

\[
\nabla \cdot \left( \frac{1}{\rho} (\nabla \theta \otimes \nabla \theta) \right) + \nabla \pi - \bar{\rho} f = 0,
\]

with \( u^n(0) = u(0) = u_0 \), respectively.

Now \( U^n = u^n - u \) satisfies

\[
(8.13) \quad \bar{\rho} \partial_t U^n - \Delta U^n = H(v^n, v, \rho^n, \rho) - \nabla (\pi^n - \tau) - \tau^n \partial_t u^n.
\]

It is easy to trace \( H(\cdot) \) and to prove that \( H(v^n, v, \rho^n, \rho) \to 0 \) as \( n \to \infty \) in \( L_2(Q_T) \). Indeed,

\[
\bar{\rho}(v^n \cdot \nabla v^n - v \cdot \nabla v) =
\]

\[
\bar{\rho}(v^n \cdot \nabla v^n - v \cdot \nabla v^n) + (v \cdot \nabla v^n - v \cdot \nabla v) \to 0,
\]

thanks to (8.10).

Similarly, we can proceed for every term in \( H(\cdot) \).

Now, multiplying (8.13) by \( \partial_t U^n \), after integration by parts, we get

\[
|\nabla U^n(t)|^2 + \int_0^t |\sqrt{\bar{\rho}} \partial_t U^n(\tau)|^2 d\tau \leq c \int_0^t (H_2^2 + \|\tau^n\|_{L_\infty(Q_T)}^2 |\partial_t u^n|^2) d\tau.
\]
Gronwall’s lemma produces $U^n \to 0$ in $L^2(Q_T)$. Thus the map $G$ is continuous in $L^2(Q_T)$.

The uniqueness can be proved making use of the procedure in theorem 6.1. We omit details. The existence of a local solution is completely proved. \hfill \Box

9. Appendix: Phase field model

A multi-phase flow consists of $n$-fluid constituents which occupy a, possibly time-dependent, region $\Omega$ whereas $\Omega_i \subseteq \Omega$ is the region occupied by constituent $i$; $i = 1, \ldots, n$ denotes the quantities pertaining to the corresponding constituent. The multi-phase flow is quite naturally framed within the scheme of fluid mixtures. In the mixture model the principles of continuum mechanics for a single phase are generalized to several inter-penetrable continua. The basic assumption is that, at any instant of time, all phases are present at every material point. The equations of balance are postulated for mass and momentum conservation. Furthermore, constitutive relations are required to close the system of equations.

Let $\rho_i$ and $r_i$ be the mass density and the mass growth. The balance of mass requires that

$$\partial_t \rho_i + \nabla \cdot \rho_i \mathbf{v}_i = r_i,$$

(9.1)

$\mathbf{v}_i$ is the divergence free velocity.

The mass conservation implies that

$$\sum_{i=1}^{n} r_i = 0.$$

We define the mass density $\rho$ and the mean mass velocity $\mathbf{w}$ of the mixture as

$$\rho := \sum_{i=1}^{n} \rho_i, \quad \mathbf{w} := \frac{1}{\rho} \sum_{i=1}^{n} \rho_i \mathbf{v}_i.$$

Other average velocities can be defined, for example, the volume average velocity. $\mathbf{w}$, not solenoidal, is used in the momentum and energy balance. Consequently, summation of (9.1) over $i$ and account of mass conservation give

$$\partial_t \rho + \nabla \cdot \rho \mathbf{w} = 0.$$

Let $c_i$ and $\mathbf{u}_i$,

$$c_i := \frac{\rho_i}{\rho}, \quad \mathbf{u}_i := \mathbf{v}_i - \mathbf{w},$$

be the concentration and the relative (or diffusion) velocity of constituent $i$, we obtain

$$\partial_t c_i \rho + \nabla \cdot \rho (\mathbf{w} + \mathbf{u}_i) = r_i.$$

By the continuity equation, we get

$$\rho (\partial_t c_i + \nabla \cdot c_i \mathbf{w}) = \tau_i - \nabla \cdot \mathbf{J}_i,$$

where

$$\mathbf{J}_i = \rho_i \mathbf{u}_i.$$
is the diffusion flux of constituent $i$.

By definition, it follows that

$$\sum_{i=1}^{n} J_i = 0.$$ 

The continuity equation (9.1) can also be written

$$\partial_t \rho_i + \nabla \cdot \rho_i \mathbf{w} = r_i - \nabla \cdot \mathbf{J}_i.$$ 

The balance of linear momentum can be written

$$\rho_i \partial_t \mathbf{v}_i + \rho_i \mathbf{v}_i \cdot \nabla \mathbf{v}_i = \nabla \cdot \mathbf{T}_i + \rho_i \mathbf{f}_i + \mathbf{g}_i$$

where $\mathbf{T}_i$ is the Cauchy stress tensor, $\mathbf{f}_i$ the body force, $\mathbf{g}_i$ the supply of linear momentum from the other constituents.

The whole mixture may be viewed as a single body. The balance of linear momentum is written as

(9.2) $$\rho \partial_t \mathbf{w} + \rho \mathbf{w} \cdot \nabla \mathbf{w} = \nabla \cdot \mathbf{T} + \rho \mathbf{f}.$$ 

The balance of the angular momentum results in the symmetry of $\mathbf{T}$.

The continuity equation is, in absence of reaction,

(9.3) $$\partial_t \rho + \nabla \cdot \rho \mathbf{w} = 0.$$ 

In general, $\mathbf{w}$ is not divergence free even $\nabla \cdot \mathbf{v}_i = 0$, $\forall i$.

Now we consider a mixture of two miscible fluids which before mixing are each incompressible. In their unmixed states let the density of fluids (1) and (2) be $\rho_{10}$ and $\rho_{20}$ (constant).

We deduce an incompressible model following the approach in [2], [11].

In the mixture, the densities of the fluids at point $x$ and at time $t$ are denoted $\rho_1(x, t) := \rho_1$, $\rho_2(x, t) := \rho_2$, respectively. Then from volume additivity of the two constituents at the outset,

$$\frac{\rho_1}{\rho_{10}} + \frac{\rho_2}{\rho_{20}} = 1.$$ 

The total density $\rho(x, t)$ in mixture is defined by $\rho = \rho_1 + \rho_2$.

The balance of mass in the mixture gives

(9.4) $$\partial_t \rho_1 + \nabla \cdot \rho_1 \mathbf{v}_1 = r_1,$$

$$\partial_t \rho_2 + \nabla \cdot \rho_2 \mathbf{v}_2 = r_2.$$ 

The mean mass velocity $\mathbf{w}$ is

$$\rho \mathbf{w} = \rho_1 \mathbf{v}_1 + \rho_2 \mathbf{v}_2.$$ 

The continuity equation for the mixture is

$$\partial_t \rho + \nabla \cdot \rho \mathbf{w} = 0.$$
Now divide (9.4) by $\rho_{10}$, and (9.4) by $\rho_{20}$, respectively, and add to obtain

$$
\partial_t \left( \frac{\rho_1}{\rho_{10}} + \frac{\rho_2}{\rho_{20}} \right) + \nabla \cdot \left( \frac{\rho_1}{\rho_{10}} \mathbf{v}_1 + \frac{\rho_2}{\rho_{20}} \mathbf{v}_2 \right) = 0.
$$

(9.5)

Define the mean volume velocity of the mixture

$$
\mathbf{v} = \frac{\rho_1}{\rho_{10}} \mathbf{v}_1 + \frac{\rho_2}{\rho_{20}} \mathbf{v}_2.
$$

From (9.5) we find $\nabla \cdot \mathbf{v} = 0$.

Now, we derive the fundamental relation

$$
\mathbf{w} = \mathbf{v} - \frac{\lambda}{\rho} (\nabla \rho + D \nabla \Delta \rho)
$$

where $\lambda$ is the diffusion coefficient, and $D$ the mobility coefficient, making use of the generalized Fick’s law of diffusion

$$
\mathbf{v}_1 = \mathbf{w} - \lambda \frac{\nabla \omega(c)}{c},
$$

where $\omega(c)$ is the chemical potential and $c = \frac{\rho}{\rho}$ is the mass concentration.

If $D = 0$ we obtain the relation considered by Kazhikhov-Smagulov [14].

This relation is important because expresses the vector $\mathbf{w}$ in terms of a divergence free vector. Inserting this relation in the linear momentum balance we obtain a generalization of the system considered by Kazhikhov-Smagulov. For completeness we report a proof given in [11].

Denote $\alpha = \frac{\rho_1}{\rho_{10}}$ the volume concentration of constituent 1, so $\frac{\rho_2}{\rho_{20}} = 1 - \alpha$, consequently

$$
\mathbf{v} = \alpha \mathbf{v}_1 + (1 - \alpha) \mathbf{v}_2, \quad 1 - c = \frac{\rho - \rho_1}{\rho} = \frac{\rho_2}{\rho},
$$

hence

$$
\mathbf{w} = c \mathbf{v}_1 + (1 - c) \mathbf{v}_2.
$$

Observing that

$$
c = \frac{\alpha \rho_{10}}{\rho}, \quad \rho = \rho_1 + \rho_2 = \alpha \rho_{10} + (1 - \alpha) \rho_{20},
$$

it is

$$
\alpha = \frac{\rho - \rho_{20}}{\rho_{10} - \rho_{20}}.
$$

By differentiation we get

$$
\nabla c = \frac{\rho_{10} \nabla \alpha}{\rho} - \frac{\alpha \rho_{10} \nabla \rho}{\rho^2}, \quad \nabla \rho = (\rho_{10} - \rho_{20}) \nabla \alpha, \quad \nabla \alpha = \frac{\nabla \rho}{\rho_{10} - \rho_{20}}.
$$

In addition

$$
c = \frac{\alpha \rho_{10}}{\rho} = \frac{\rho_{10}}{\rho} \left( \frac{\rho - \rho_{20}}{\rho_{10} - \rho_{20}} \right),
$$
and, from the previous relations, it holds
\[ \nabla c = \frac{\rho_{10}}{\rho} \frac{\nabla \rho}{\rho_{10} - \rho_{20}} - \frac{\rho_{10}}{\rho^2} \nabla \rho \frac{\rho - \rho_{20}}{\rho_{10} - \rho_{20}} = \frac{\rho_{10} \rho_{20} \nabla \rho}{(\rho_{10} - \rho_{20}) \rho^2}. \]
Consequently, upon simplification,
\[ \frac{\nabla c}{c} = \frac{\rho_{20} \nabla \rho}{\rho (\rho - \rho_{20})}. \]  

(9.6)

Next, we eliminate \( v_2 \) from the relations
\[ v = \alpha v_1 + (1 - \alpha) v_2, \quad w = c v_1 + (1 - c) v_2, \]
to find
\[ v_1 \left( \frac{\alpha}{1 - \alpha} - \frac{c}{1 - c} \right) = \frac{1}{1 - \alpha} v - \frac{1}{1 - c} w. \]

(9.7)

By noting
\[ \frac{\alpha}{1 - \alpha} = \frac{\rho_1 \rho_{20}}{\rho_2 \rho_{10}}, \quad \frac{c}{1 - c} = \frac{\rho_1}{\rho_2}, \]
so that
\[ \frac{\alpha}{1 - \alpha} - \frac{c}{1 - c} = \frac{\rho_1 (\rho_{20} - \rho_{10})}{\rho_2 (\rho_{20} - \rho_{10})} = 1, \]
and then, from (9.7),
\[ v_1 = \frac{\rho_2}{\rho_1} \frac{\rho_{10}}{\rho_{20} - \rho_{10}} \left[ \frac{\rho_{20}}{\rho_2} v - \frac{\rho}{\rho_2} w \right]. \]

(9.8)

We now eliminate \( v_1 \) between (9.8) and the generalized Fick’s law, we obtain
\[ w \left[ \frac{\rho_1 (\rho_{20} - \rho_{10}) + \rho_{10}}{\rho_1 (\rho_{20} - \rho_{10})} \right] = \frac{\rho_{10} \rho_{20}}{\rho_1 (\rho_{20} - \rho_{10})} v + \frac{\lambda}{c} \nabla \omega(c). \]

Now the coefficient of \( w \) can be written
\[ \frac{\rho_1 \rho_{20} + \rho_2 \rho_{10}}{\rho_1 (\rho_{20} - \rho_{10})}. \]

(9.9)

Since \( \rho = \rho_1 + \rho_2, \) \( \frac{\rho_{10}}{\rho_{20}} = 1 - \frac{\rho_{10}}{\rho_{20}}, \) we have \( \rho_1 = \frac{\rho_{10}}{\rho_{20}} (\rho_{20} - \rho_2) \) and hence
\[ \rho_1 \rho_{20} + \rho_2 \rho_{10} = \rho_{10} \rho_{20}. \]

(9.10)

Using (9.9) and (9.10) the coefficient of \( w \) simplifies and
\[ w \left[ \frac{\rho_{10} \rho_{20}}{\rho_1 (\rho_{20} - \rho_{10})} \right] = \frac{\rho_{10} \rho_{20}}{\rho_1 (\rho_{20} - \rho_{10})} v + \frac{\lambda}{c} \nabla \omega(c), \]
and finally
\[ w = v + \frac{\lambda}{c} \nabla \omega(c) \frac{\rho_1 (\rho_{20} - \rho_{10})}{\rho_{10} \rho_{20}}. \]

(9.11)

Bearing in mind (9.6) and \( \omega(c) = c - D \Delta c, \) we obtain
\[ \nabla \omega(c) = \frac{\rho_{10} \rho_{20}}{\rho_{10} - \rho_{20}} \nabla \rho - D \left( \frac{\nabla \Delta \rho}{\rho^2} + \nabla \nabla \rho \nabla \frac{1}{\rho^2} + \nabla \cdot (\nabla \rho \nabla \frac{1}{\rho^2}) \right). \]

Discarding the terms \( \rho^{-\gamma} \) with \( \gamma \geq 3 \), follows
\[ (9.12) \quad \nabla \omega(c) = \frac{\rho_{10} \rho_{20}}{\rho_{10} - \rho_{20}} \left( \frac{\nabla \rho}{\rho^2} - D \frac{\nabla \Delta \rho}{\rho^2} \right), \]
and inserting (9.12) in (9.11) we obtain
\[ (9.13) \quad w = v - \frac{\lambda}{\rho} (\nabla \rho - D \nabla \Delta \rho). \]

10. **Derivation of the incompressible model**

We look (9.13) as a Helmholtz-type decomposition of the vector \( w \). Consider the equations governing flow of a binary mixture obtained above
\[ (10.1) \quad \rho \partial_t w + \rho w \cdot \nabla w = \nabla \cdot T + \rho f, \]
\[ \partial_t \rho + \nabla \cdot \rho w = 0. \]

It is straightforward to deduce from (10.1) using (9.12) that the continuity equation becomes
\[ (10.2) \quad \partial_t \rho + \nabla \cdot \rho v - \lambda \Delta (\rho - D \Delta \rho) = 0. \]

Now, upon substituting (9.13) in (10.1) and setting \( \theta = \rho - D \Delta \rho \) we find
\[ \rho \partial_t v + \frac{\lambda}{\rho} \nabla \theta \cdot \partial_t \rho - \lambda \partial_t \nabla \theta + \rho v \cdot \nabla v - \lambda \nabla \theta \cdot \nabla v - \lambda \rho v \cdot \nabla \frac{\nabla \theta}{\rho} + \nabla \pi - \mu \Delta v + \mu \lambda \Delta \frac{\nabla \theta}{\rho} + \eta \nabla \nabla \cdot \nabla \frac{\nabla \theta}{\rho} + \lambda^2 \nabla \theta \cdot \nabla \frac{\nabla \theta}{\rho} = \rho f. \]

Using the continuity equation to substitute for \( \partial_t \rho \) in the above equation we find
\[ \rho \partial_t v + \rho v \cdot \nabla v + \lambda \nabla \theta \cdot \nabla v - \lambda v \cdot \nabla \nabla \theta + \nabla \pi - \mu \Delta v + \lambda v \cdot \nabla \frac{\nabla \theta}{\rho} + \eta \nabla \nabla \cdot \nabla \frac{\nabla \theta}{\rho} = \rho f. \]

where \( \pi = \eta \lambda \nabla \cdot \nabla \theta - \partial_t \theta - \lambda \mu \Delta \log \rho - \mu D \Delta \frac{\rho}{\rho^2} \) and we have discarded the nonlinear term \( \Delta (\nabla \rho^{-1} \Delta \rho) \), for convenience.

Neglecting all terms of \( O(\lambda^2) \) we find a model type Kazhikhov-Smagulov, and it is thus a model for small diffusion.
\[
\rho \partial_t \mathbf{v} + \rho \mathbf{v} \cdot \nabla \mathbf{v} - \lambda \nabla \theta \cdot \nabla \mathbf{v} - \lambda \mathbf{v} \cdot \nabla \nabla \theta + \\
\nabla \pi + \mu \Delta \mathbf{v} = \rho \mathbf{f}.
\]

Instead, if \( D = 0 \) we find the model studied, completely, in [33].

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Rodolfo Salvi
Department of Mathematics Politecnico di Milano
20133 Milan - ITALY

E-mail address: rodolfo.salvi@polimi.it