ON THE GINZBURG-LANDAU ENERGY WITH A MAGNETIC FIELD VANISHING ALONG A CURVE

AYMAN KACHMAR AND MARWA NASRALLAH

Abstract. The energy of a type II superconductor placed in a strong non-uniform, smooth and signed magnetic field is displayed via a universal reference function defined by means of a simplified two dimensional Ginzburg-Landau functional. We study the asymptotic behavior of this functional in a specific asymptotic regime, thereby linking it to a one dimensional functional, using methods developed by Almog-Helffer and Fournais-Helffer devoted to the analysis of surface superconductivity in the presence of a uniform magnetic field. As a result, we obtain an asymptotic formula reminiscent of the one for the surface superconductivity regime, where the zero set of the magnetic field plays the role of the superconductor’s surface.

1. Introduction

During the two past decades, the mathematics of superconductivity has been the subject of intense activity (see [11] for the physical background). One common model used to describe the behavior of a superconductor is the Ginzburg-Landau functional involving a pair \((\psi, A)\), where \(\psi\) is a wave function (called the order parameter) and \(A\) is a vector field (called the magnetic potential), both being defined on an open set \(\Omega \subset \mathbb{R}^2\). The functional is

\[
\mathcal{E}(\psi, A) = \int_\Omega \left[ (|\nabla - i\kappa H A|\psi|^2 - \kappa^2 |\psi|^2 + \frac{\kappa^2}{2} |\psi|^4 \right] dx + \kappa^2 H^2 \int_\Omega |\text{curl} A - B_0|^2 dx. \tag{1.1}
\]

The quantity \(|\psi|^2\) measures the density of superconducting electrons (so that \(\psi = 0\) defines the normal state); \(\text{curl} A\) measures the induced magnetic field; the parameter \(H\) measures the strength of the external magnetic field and the parameter \(\kappa > 0\) is a characteristic of the superconducting material. The function \(B_0\) is a given function and accounts for the profile of an external non-uniform magnetic field. We will assume that \(B_0 \in C^3(\Omega)\).

Of particular physical interest is the ground state energy

\[
E_{gs}(\kappa, H) := \inf \{ \mathcal{E}(\psi, A) : (\psi, A) \in H^1(\Omega; \mathbb{C}) \times H^1(\Omega; \mathbb{R}^2) \}. \tag{1.2}
\]

As the intensity of the magnetic field varies (i.e. the parameter \(H\)), changes in \(E_{gs}(\kappa, H)\) mark various distinct states of the superconductor. That has been fairly understood for type II superconductors in the case where the magnetic field is uniform (i.e. \(B_0 = 1\)) which has allowed to distinguish between three critical values for the intensity of the applied magnetic field, denoted by \(H_{C_1}, H_{C_2}\) and \(H_{C_3}\) whose role can be described as follows (see [13, 24, 9, 8, 10, 15]):

- If \(H < H_{C_1}\), then the whole superconductor is in the perfect superconducting state;
- If \(H_{C_1} < H < H_{C_2}\), the superconductor is in the mixed phase, where both the superconducting and normal states co-exist in the bulk of the sample; the most interesting aspect of the mixed phase is that the region with the normal state appears in the form of a lattice of point defects, covering the whole bulk of the sample [25];
- If \(H_{C_2} < H < H_{C_3}\), superconductivity disappears in the bulk but survives on the surface of the superconductor;
- If \(H > H_{C_3}\), superconductivity is destroyed and the superconductor returns to the normal state.

The case of a non-uniform sign changing magnetic field has been addressed first in [23] then recently in [4, 5, 6, 17, 19]. In the presence of such magnetic fields, the behavior of the superconductor (and the associated critical magnetic fields) differ significantly from the case of a uniform...
applied magnetic field. In particular, the order of the intensity of the third critical field $H_{C_3}$ increases, and in the mixed phase between $H_{C_2}$ and $H_{C_3}$, superconductivity is neither present everywhere in the bulk, nor it is evenly distributed in the form of a lattice. We refer to [17, 19] for more details.

Now we state our assumption on the function $B_0$. These are two conditions that will allow $B_0$ to represent a non-uniform sign changing applied magnetic field. The first condition is on the zero set of $B_0$ and says

$$\Gamma := \{ x \in \overline{\Omega}, \ B_0(x) = 0 \} \neq \emptyset \quad \text{and} \quad \Gamma \cap \partial \Omega \text{ is finite.}$$

The second condition is on the gradient of the function $B_0$ and yields that the function $B_0$ vanishes non-degenerately and changes sign:

$$|B_0| + |\nabla B_0| \neq 0 \quad \text{in} \quad \overline{\Omega}. \quad (1.3)$$

Note that (1.4) yields that $\Gamma$ consists of a finite number of smooth curves that are assumed to intersect $\partial \Omega$ transversely. Such magnetic fields arise naturally in many contexts [2, 7, 22].

Under the assumptions (1.3) and (1.4), the ground state energy $E_{gs}(\kappa, H)$ is estimated for various regimes of $H$ and $\kappa$. Firstly, in light of results in Pan-Kwek [23] and Attar [6], we know that there exists $\overline{\Omega} > 0$ such that, for $H > \overline{\Omega} \kappa^2$ and $\kappa$ sufficiently large, $E_{gs}(\kappa, H) = 0$ and every critical point $(\psi, A)$ of the functional in (1.1) is a normal solution, i.e. $\psi = 0$ everywhere. The meaning of this is that the critical field $H_{C_3}$, the threshold above which superconductivity is lost, is of the order of $\kappa^2$.

In the recent paper [19], the authors write an asymptotic expansion for the ground state energy in the specific regime where $H$ is of order $\kappa^2$ and $\kappa \to +\infty$ (in this case, $H$ is of the order of the third critical field $H_{C_3}$).

The result in [19] reads as follows. There exists a universal function $E(\cdot)$, introduced in Theorem 2.1 below, such that if $0 < M_1 < M_2$, then, for $H \in [M_1 \kappa^2, M_2 \kappa^2]$, the ground state energy satisfies, as $\kappa \to \infty$,

$$E_{gs}(\kappa, H) = \kappa \int_{\Gamma} \left( |\nabla B_0(x)| \frac{H}{\kappa^2} \right)^{1/3} E \left( |\nabla B_0(x)| \frac{H}{\kappa^2} \right) ds(x) + \frac{\kappa^3}{H} o(1), \quad (1.5)$$

where $ds$ denotes the arc-length measure in $\Gamma$.

The asymptotic analysis of $E_{gs}(\kappa, H)$ has been carried for other regimes of the magnetic field strength, down to $H \approx \kappa^{1/3}$, in [4, 5, 19]. The case where the function $B_0$ is only Hölder continuous or a step function has been discussed in [17, 3].

Let us mention a few properties of the function $E(\cdot)$ appearing in (1.5):

- $L \in (0, \infty) \mapsto E(L) \in (-\infty, 0]$ is a continuous function;
- As $L \to 0_+$, the asymptotic behavior of $E(L)$ is analyzed in [18]; in particular, $|E(L)| \approx L^{-4/3}$;
- There exists a universal (spectral) constant $\lambda_0 > 0$ (defined below in (1.6)) such that $E(L) = 0$ for $L \geq \lambda_0^{-3/2}$ and $E(L) < 0$ for $0 < L \leq \lambda_0^{-3/2}$.

The aim of this paper is to analyze the asymptotic behavior of $E(L)$ as $L \to \lambda_0^{-3/2}$ from below (thereby complementing the result in [18] devoted for the regime $L \to 0_+$. To that end, we introduce the following quantities:

- $\lambda_0 > 0$ and $\tau_0 < 0$ are the constants (see Theorem 3.1)

$$\lambda_0 = \inf_{\alpha \in \mathbb{R}} \lambda(\alpha) = \lambda(\tau_0) \quad (1.6)$$

where $\lambda(\alpha)$ is the lowest eigenvalue of the operator $-\frac{d^2}{dr^2} + \left( \frac{t^2}{2} + \alpha \right)^2$.
- $u_0$ is the positive $L^2$-normalized eigenfunction satisfying

$$\left( -\frac{d^2}{dt^2} + \left( \frac{t^2}{2} + \tau_0 \right)^2 \right) u_0 = \lambda_0 u_0 \text{ in } \mathbb{R}.$$
We obtain:

**Theorem 1.1.** As $L \nearrow \lambda_0^{-3/2}$, the following asymptotic formula holds,

$$E(L) = -\frac{L^{2/3} (L^{-2/3} - \lambda_0)^2}{2 \|u_0\|_{L^2}^4} (1 + o(1)).$$

Now we return back to (1.5) and observe that, when $H$ satisfies

$$\left( \min_{x \in \Omega} |\nabla B_0(x)| \right) \frac{H}{\kappa^2} \geq \lambda_0^{-3/2},$$

the leading order term in (1.5) vanishes (so superconductivity disappears in the bulk of the sample). This leads us to introduce the following critical field

$$H_{C_2}(\kappa) = \gamma \kappa^2$$

where

$$\gamma := \lambda_0^{-3/2} c_0^{-1} \quad \text{and} \quad c_0 = \min_{x \in \Gamma} |\nabla B_0(x)|.$$  \hspace{1cm} (1.7)

Then one may ask whether we can refine the formula in (1.5) under the assumption that $H$ is close to and below $H_{C_2}(\kappa)$ (see (1.10) below). Indeed this is possible by using Theorem 1.1 and by working under a rather generic assumption on $B_0$:

**Assumption 1.2.** Suppose that $B_0$ satisfies (1.3) and (1.4). Let $c_0$ be the constant introduced in (1.8) and

$$\Gamma_0 = \{ x \in \Gamma : |\nabla B_0(x)| = c_0 \}$$

be the set of minimum points of the function $\Gamma \ni x \mapsto |\nabla B_0(x)|$.

We assume that one of the following two conditions hold:

- Either $\Gamma_0 = \Gamma$,
- or the set $\Gamma_0$ is finite, $\Gamma_0 \subset \Omega$ and every point of $\Gamma_0$ is a non-degenerate minimum of the function $\Gamma \ni x \mapsto |\nabla B_0(x)|$.

**Remark 1.3.** In the case of the unit disc $\Omega = B(0,1)$, the following two functions

$$(x,y) \mapsto y - x \quad \text{and} \quad (x,y) \mapsto y - x^2$$

serve as two examples of a magnetic field $B_0$ satisfying Assumption 1.2.

**Remark 1.4.** If the set $\Gamma_0$ is finite and there exists $x_0 \in \Gamma_0 \cap \partial \Omega$, then $x_0$ is a non-degenerate minimum if the derivative of the map $x \mapsto \nabla B_0(x)$ at $x_0$ is not zero.

Assumption 1.2 is reminiscent of the assumption by Fournais-Helffer in [12, Assumption 5.1] but with the function $x \mapsto (−|\nabla B_0(x)|)$ here replacing the curvature there. Also, Assumption 1.2 appears in the analysis of magnetic mini-wells by Helffer-Kordyukov-Raymond-Vu Ngoc [20].

Next we assume that $H$ approaches the critical field in (1.7) as follows

$$H = \left( \gamma - \rho(\kappa) \right) \kappa^2,$$  \hspace{1cm} (1.10)

where the constant $\gamma$ is introduced in (1.8) and

$$\rho : (0, \infty) \to (0, \infty) \text{ satisfies } \kappa^{-1/30} \ll \rho(\kappa) \ll 1.$$  \hspace{1cm} (1.11)

Here and in the sequel, we use the following notation. If $a(\kappa)$ and $b(\kappa)$ are two positive valued functions, the notation $a(\kappa) \ll b(\kappa)$ means that $a(\kappa)/b(\kappa) \to 0$ as $\kappa \to \infty$. Also, by writing $a(\kappa) \approx b(\kappa)$ it is meant that there exist constants $\kappa_0, c_1, c_2 > 0$ such that $c_1 b(\kappa) \leq a(\kappa) \leq c_2 b(\kappa)$, for all $\kappa \geq \kappa_0$.  

Clearly, when (1.10), (1.11) and Assumption 1.2 hold, the principal term in (1.5) satisfies
\[
\int_{\Gamma} \left( \frac{\nabla B_0(x)}{\kappa^2} \right)^{1/3} E \left( \frac{\nabla B_0(x)}{\kappa^2} \right)^{1/3} ds(x)
\]
\[
= -\frac{\gamma}{2||u_0||^2} \left( \int_{\Gamma} |\nabla B_0(x)|^4 \left( \left( \frac{H}{\kappa^2} |\nabla B_0(x)| \right)^{-2/3} - \lambda_0 \right)^2 ds(x) \right)^{1/3 + o(1)}
\]
\[
= -\frac{\lambda_0^{-3/2}}{2||u_0||^2} \left( \int_{\Gamma} \left( \frac{H}{\kappa^2} |\nabla B_0(x)| \right)^{-2/3} - \lambda_0 \right)^2 ds(x) \right)^{1/3 + o(1)}.
\]  
(1.12)

The last step follows since \(\gamma = \lambda_0^{-3/2}c_0^{-1}\) and the function on \(\Gamma\), \(\left( \frac{H}{\kappa^2} |\nabla B_0(x)| \right)^{-2/3} - \lambda_0 \), is supported in \(\Gamma_\kappa\), where
\[
\Gamma_\kappa = \{ x \in \Gamma : \frac{H}{\kappa^2} |\nabla B_0(x)| < \lambda_0^{-3/2} \},
\]  
(1.13)

which yields that \(|\nabla B_0(x)| \sim c_0 \) on \(\Gamma_\kappa\).

Under Assumption 1.2, only one of the following two cases may occur:

- Either \(\Gamma_\kappa = \Gamma\), in which case
  \[
  \int_{\Gamma} \left( \frac{H}{\kappa^2} |\nabla B_0(x)| \right)^{-2/3} - \lambda_0 \right)^2 ds(x) = \left( \frac{2}{3} c_0 \lambda_0^{3/2} \rho(\kappa) \right)^2 |\Gamma| + o(1) ;
  \]

- or \(|\Gamma_\kappa| \approx \sqrt{\rho(\kappa)}\) as \(\kappa \to +\infty\), in which case
  \[
  \int_{\Gamma} \left( \frac{H}{\kappa^2} |\nabla B_0(x)| \right)^{-2/3} - \lambda_0 \right)^2 ds(x) \geq c \rho(\kappa)^2 \sqrt{\rho(\kappa)} \approx (\rho(\kappa))^{5/2},
  \]

for some constant \(c > 0\), which depends on the second derivative of the function \(|\nabla B_0(x)|\) at the minimum points.

As an application of the main result of this paper (Theorem 3.1), we are able to prove that

**Theorem 1.5.** Under Assumption 1.2, if (1.10) and (1.11) hold, then as \(\kappa \to +\infty\),
\[
E_{gs}(\kappa, H) = \left[ -\frac{\kappa \lambda_0^{-3/2}}{2||u_0||^2} \int_{\Gamma} \left( \frac{H}{\kappa^2} |\nabla B_0(x)| \right)^{-2/3} - \lambda_0 \right)^2 ds(x) \right] (1 + o(1)) .
\]

The result in Theorem 1.5 is far from optimal. We mention it as a simple application of Theorem 1.1 and the analysis in [19]. To get the optimal regime (for \(\rho(\kappa)\)) where the result in Theorem 1.5 holds, we need a rather detailed analysis of the ground state energy and the corresponding minimizers, that we postpone to a separate work.

The rest of the paper is organized as follows. We introduce in Section 2 a certain simplified Ginzburg-Landau functional from which arises the definition of the limiting function \(E(L)\) appearing in Theorem 1.1 above. We recall in Section 3 spectral facts concerning the family of Montgomery operators. A related family of 1D linear functionals is introduced in Section 4 where we investigate the infimum over all the ground state energies of those functionals. Moreover, we prove in Section 4 a key-ingredient asymptotic formula needed for the proof of the main result. A technical spectral estimate is proved in Section 5. We perform in Section 6 some Fourier analysis to get a good estimate on the energy functional defined on half-cylinders. We conclude with the proof of Theorem 1.1 in Section 7. Finally, in Section 8, we prove Theorem 1.5.

### 2. The simplified Ginzburg-Landau functional

We consider the following magnetic potential,
\[
A_{app}(x) = \left( -\frac{x_2^2}{2}, 0 \right) \quad (x = (x_1, x_2) \in \mathbb{R}^2)
\]  
(2.1)
which generates the magnetic field $\text{curl} A_{\text{app}} = x_2$ that vanishes along the line $x_2 = 0$.

Let $L > 0, b > 0, R > 0$ and $S_R = (-R, R) \times \mathbb{R}$. Consider the functional

$$\mathcal{E}_{R,b}(u) = \int_{S_R} \left( |(\nabla - iA_{\text{app}})u|^2 - b|u|^2 + \frac{b}{2}|u|^4 \right) dx,$$

(2.2)

and the corresponding ground state energy

$$\epsilon(b; R) = \inf \{ \mathcal{E}_{R,b}(u) : (\nabla - iA_{\text{app}})u \in L^2(S_R), \ u \in L^2(S_R) \text{ and } u = 0 \text{ on } \partial S_R \}.$$  

(2.3)

The following theorem was proven in [19, Theorem 3.8].

**Theorem 2.1.** Given $L > 0$, there exists $E(L) \leq 0$ such that,

$$\lim_{R \to \infty} \frac{\epsilon(L^{-2/3}; R)}{2R} = E(L).$$

(2.4)

The function $(0, \infty) \ni L \mapsto E(L) \in (-\infty, 0]$ is continuous, monotone increasing, and

$$E(L) = 0 \quad \text{if and only if} \quad L \geq \lambda_0^{-3/2},$$

where $\lambda_0 > 0$ is the eigenvalue introduced in (1.6).

Furthermore, there exists a constant $C > 0$ such that

$$\forall R \geq 2, \forall L > 0, \quad E(L) \leq \frac{\epsilon(L^{-2/3}; R)}{2R} \leq E(L) + C(1 + L^{-2/3}) R^{-2/3}.$$  

(2.5)

3. **The Montgomery operator**

For $\alpha \in \mathbb{R}$, consider the self-adjoint operator in $L^2(\mathbb{R})$,

$$P(\alpha) = -\frac{d^2}{dt^2} + \left( \frac{t^2}{2} + \alpha \right)^2$$

(3.1)

with domain

$$\text{Dom}(P(\alpha)) = B^2(\mathbb{R}) = \left\{ f \in H^2(\mathbb{R}) : t^4 f \in L^2(\mathbb{R}) \right\}.$$  

(3.2)

The first eigenvalue $\lambda(\alpha)$ of the operator $P(\alpha)$ is expressed by the min-max principle as follows

$$\lambda(\alpha) := \inf_{u \in B^1(\mathbb{R})} \frac{Q_\alpha(u)}{\|u\|_2^2},$$

(3.3)

where

$$Q_\alpha(u) = \int_{\mathbb{R}} \left( |u'(t)|^2 + \left( \frac{t^2}{2} + \alpha \right)^2 |u(t)|^2 \right) dt.$$  

(3.4)

is the quadratic form defined for $u$ in the space

$$B^1(\mathbb{R}) = \{ u \in H^1(\mathbb{R}) : t^2 u \in L^2(\mathbb{R}) \}.$$  

(3.5)

Recall that $\lambda_0 = \inf_{\alpha \in \mathbb{R}} \lambda(\alpha)$ introduced in (1.6). We collect from [16] some important properties of the function $\alpha \mapsto \lambda(\alpha)$.

**Theorem 3.1.**

1. There exists a unique $\tau_0 \in \mathbb{R}$ such that $\lambda_0 = \lambda(\tau_0)$.
2. $\tau_0 < 0$ and $\lambda_0 < \lambda(0) \leq \left( \frac{3}{4} \right)^{\frac{3}{4}} < 1$.
3. $\lim_{\alpha \to \pm \infty} \lambda(\alpha) = +\infty$.
4. The minimum of $\lambda$ at $\tau_0$ is non-degenerate, that is, $\lambda''(\tau_0) > 0$.

**Remark 3.2.** One finds the numerical approximation $\lambda_0 \cong 0.57$ (see. [21, 22]).
As a consequence of Theorem 3.1, we may define two functions $z_1(b), z_2(b)$ satisfying
\[ z_1(b) < \tau_0 < z_2(b), \quad \lambda^{-1}([\lambda_0, b]) = (z_1(b), z_2(b)). \]

For all $\alpha \in \mathbb{R}$, let $\lambda_2(\alpha)$ be the second eigenvalue of the operator $P(\alpha)$ introduced in (3.1). By continuity of the functions $\alpha \mapsto \lambda_n(\alpha)$, for all $n \in \{1, 2\}$, we get

**Lemma 3.3.** Let $\tau_0$ be the value defined in Theorem 3.1. There exists $\varepsilon_0 > 0$ such that, if $\alpha \in (\tau_0 - \varepsilon_0, \tau_0 + \varepsilon_0)$ and $b \in [\lambda_0, \lambda_0 + \varepsilon_0)$, then $b < \lambda_2(\alpha)$.

In the sequel, we consider $\alpha \in (\tau_0 - \varepsilon_0, \tau_0 + \varepsilon_0)$ and $b \in [\lambda_0, \lambda_0 + \varepsilon_0)$, where $\varepsilon_0$ is defined by Lemma 3.3. Let $u_\alpha$ be the positive normalized ground state of the operator $P(\alpha)$, and let $\pi_\alpha$ be the $L^2$ orthogonal projection on Span$(u_\alpha)$. For $\alpha = \tau_0$, we shorten the notation and write $u_0 := u_{\tau_0}$.

We introduce the regularized resolvent of $P(\alpha)$ by
\[ R_{\alpha,b} := (P(\alpha) - b)^{-1}(1 - \pi_\alpha). \tag{3.6} \]

The following lemma is straightforward (see [13, Lem. 14.2.6]):

**Lemma 3.4.** The regularized resolvent $R_{\alpha,b}$ maps $L^2(\mathbb{R})$ into $B^2(\mathbb{R})$. Moreover, there exist $\varepsilon, C > 0$ such that for all $(\alpha, b) \in (-\tau_0 - \varepsilon, \tau_0 + \varepsilon) \times [\lambda_0, \lambda_0 + \varepsilon)$,
\[ \|R_{\alpha,b}u\|_{B^2(\mathbb{R})} \leq C \|u\|_{L^2(\mathbb{R})}. \]

4. A FAMILY OF 1D NON-LINEAR FUNCTIONALS

Let $b > 0$ and $\alpha \in \mathbb{R}$. Consider the functional
\[ B^1(\mathbb{R}) \ni f \mapsto E_{\alpha,b}(f) = \int_{\mathbb{R}} \left( |f'(t)|^2 + \left( \frac{t^2}{2} + \alpha \right)^2 |f(t)|^2 - b|f(t)|^2 + \frac{b}{2} |f(t)|^4 \right) dt, \tag{4.1} \]
along with the ground state energy
\[ b(\alpha, b) = \inf \{ E_{\alpha,b}(f) : f \in B^1(\mathbb{R}) \}, \tag{4.2} \]
where $B^1(\mathbb{R})$ is the space introduced in (3.5). We continue to work under the assumptions made in Theorem 3.1 and afterwards.

Our objective is to prove

**Theorem 4.1.** There exists $b_* > 0$ such that, if $\lambda_0 < b \leq b_*$, then there exists a unique $\xi(b) \in (z_1(b), z_2(b))$ satisfying
\[ b(\xi(b), b) = \inf_{\alpha \in \mathbb{R}} b(\alpha, b). \]

Furthermore,
- the function $b \mapsto \xi(b)$ is a $C^\infty$ function on $(\lambda_0, b_*)$ with $\xi(\lambda_0) = \tau_0$;
- $\lambda_0 \searrow b$, $\inf_{\alpha} b(\alpha, b) = -\frac{1}{2b} \frac{(b - \lambda_0)^2}{\|u_0\|^4_4}(1 + o(1))$.

The starting point is the following preliminary result:

**Theorem 4.2.** Let $b > 0$ and $\alpha \in \mathbb{R}$. Then the following hold:

1. The functional $E_{\alpha,b}$ has a strictly positive minimizer $f_{\alpha,b}$ in the space $B^1(\mathbb{R})$ if and only if $\lambda(\alpha) < b$. Furthermore, the minimizer satisfies the Euler-Lagrange equation
\[ -f''_{\alpha,b} + \left( \frac{t^2}{2} + \alpha \right)^2 f_{\alpha,b} = bf_{\alpha,b}(1 - |f_{\alpha,b}|^2), \tag{4.3} \]
and the inequality
\[ \|f_{\alpha,b}\|_{\infty} \leq 1. \]
(2) The ground state energy in (4.2) satisfies
\[ b(\alpha, b) = -\frac{b}{2} \| f_{\alpha, b} \|_4^4. \]

(3) There exists \( \alpha_0 \in (z_1(b), z_2(b)) \) such that,
\[ b(\alpha_0, b) = \inf_{\alpha \in \mathbb{R}} b(\alpha, b). \]

(4) If \( b < \lambda(0) \), then \( \alpha_0 < 0 \).
(5) If \( b > 0 \). The map \( \alpha \in (z_1(b), z_2(b)) \mapsto f_{\alpha, b} \in B^1(\mathbb{R}) \) is \( C^\infty \).

(6) (Feynman-Hellmann)
\[ \int_\mathbb{R} \left( \frac{t^2}{2} + \alpha_0 \right) |f_{\alpha_0}(t)|^2 dt = 0. \]

The proof of Theorem 4.2 is obtained by adapting the same analysis of [13, Section 14.2] devoted to the functional
\[ \mathcal{F}_{\alpha, b}(f) = \int_\mathbb{R} \left( |f'(t)|^2 + (t + \alpha)^2 |f(t)|^2 - b|f(t)|^2 + \frac{b}{2} |f(t)|^4 \right) dt. \]

Remark 4.3. The existing results on the functional in (4.6) suggest that Theorem 4.1 holds for all \( b > \lambda_0 \) (see [9, 8, 10]). However, in the new functional (4.1), the presence of the non-translation invariant potential term \( \left( \frac{t^2}{2} + \alpha \right) \) causes technical difficulties that prevent the application of the method of [9, 8, 10].

According to Theorem 4.2, we observe that the functional \( E_{\alpha, b} \) has non-trivial minimizers if and only if \( \alpha \in (z_1(b), z_2(b)) \). Furthermore, as \( b \searrow \lambda_0, \ z_1(b), z_2(b) \to \tau_0 \) and consequently, \( \alpha_0 \to \tau_0 \). So, if \( b \) is sufficiently close to \( \lambda_0 \), the minimum points of the function \( \alpha \mapsto b(\alpha, b) \) are localized in a neighborhood of \( \tau_0 \).

In the sequel, we assume that the pair \((\alpha, b)\) lives in a sufficiently small neighborhood of \((\tau_0, \lambda_0)\) so that the results in Section 3 hold.

**Lemma 4.4.** Let
\[ \delta = \langle f_{\alpha, b}, u_\alpha \rangle. \]
Then
\[ (b - \lambda(\alpha))\delta = b\langle f_{\alpha, b}^3, u_\alpha \rangle, \]
and
\[ f_{\alpha, b} + bR_{\alpha, b}(f_{\alpha, b}^3) = \delta u_\alpha. \]

**Proof.** The formula in (4.8) results from (4.3) because \( P(\alpha)u_\alpha = \lambda(\alpha)u_\alpha \). Next we prove (4.9). Note that \( \pi_\alpha(P(\alpha) - b)f_{\alpha, b} = \delta(P(\alpha) - b)u_\alpha \). We may write (4.3) as \( bf_{\alpha, b}^3 = -(P(\alpha) - b)f_{\alpha, b} \). Consequently,
\[ R_{\alpha, b}(bf_{\alpha, b}^3) = (P(\alpha) - b)^{-1}(bf_{\alpha, b}^3 - \pi_\alpha(bf_{\alpha, b}^3)) \]
\[ = -f_{\alpha, b} + \delta u_\alpha. \]

Here is the identity in (4.9). \( \Box \)

Since \( B^1(\mathbb{R}) \) is embedded in \( L^\infty(\mathbb{R}) \), we can define the following map
\[ B^1(\mathbb{R}) \ni u \mapsto G_{\alpha, b}(u) = -bR_{\alpha, b}(u^3). \]

As a consequence of Lemma 3.4, we find

**Lemma 4.5.** There exist a neighborhood \( N_0 = (\tau_0 - \varepsilon, \tau_0 + \varepsilon) \times [\lambda_0, \lambda_0 + \varepsilon] \) and a constant \( C > 0 \) such that, for all \((\alpha, b) \in N_0\), the map \( G_{\alpha, b} \) maps \( B^1(\mathbb{R}) \) to itself, and for all \( u \in B^1(\mathbb{R}) \),
\[ \|G_{\alpha, b}(u)\|_{B^1(\mathbb{R})} \leq C\|u\|_{B^1(\mathbb{R})}^3. \]
With Lemma 4.5 in hand, we can invert equation \((I - G_{\alpha,\beta})(u) = f\) when the pair \((\alpha, b)\) lives in the neighborhood \(N_0\), and the norm of \(u\) is sufficiently small. We state this as follows.

**Lemma 4.6.** There exists a constant \(c_* > 0\) such that, for all \((\alpha, b) \in N_0\) and \(u \in B^1(\mathbb{R})\) satisfying \(\|u\|_{B^1(\mathbb{R})} \leq c_*\), the series

\[
t(u) = \sum_{j=0}^{\infty} G_{\alpha,b}^j(u)
\]

is absolutely convergent. Furthermore,

\[
t\left(u - G_{\alpha,b}(u)\right) = u.
\]

Now we return back to (4.9) and observe that it can be expressed in the following form

\[
(I - G_{\alpha,b})(f_{\alpha,b}) = \delta u_{\alpha}.
\]

We will apply Lemma 4.6 to invert the formula (4.11), but we have to prove first that \(\|f_{\alpha,b}\|_{B^1(\mathbb{R})}\) is sufficiently small, which is our next task.

**Lemma 4.7.** There exists a constant \(C > 0\) such that, for all \(\alpha \in \mathbb{R}\) and \(b \geq \lambda_0\), we have

\[
\|f_{\alpha,b}\|_{B^1(\mathbb{R})} \leq C b^{3/2} \sqrt{b - \lambda_0}.
\]

**Proof.** We can find a constant \(C_1 > 0\) such that, for all \(f \in B^1(\mathbb{R})\) and \((\alpha, b)\),

\[
\|f\|_{B^1(\mathbb{R})}^2 \leq C_1 \int_{\mathbb{R}} \left[|f'(t)|^2 + \left(\alpha + \frac{t^2}{2}\right)^2 |f(t)|^2\right] dt
\]

\[
= C_1 \left\{\mathcal{E}_{\alpha,b}(f) + \int_{\mathbb{R}} \left(b |f|^2 - \frac{b}{2} |f|^4\right) dx\right\},
\]

where \(\mathcal{E}_{\alpha,b}(\cdot)\) is the functional introduced in (4.1).

Now we choose \(f = f_{\alpha,b}\). Consequently \(\mathcal{E}_{\alpha,b}(f) \leq \mathcal{E}_{\alpha,b}(0) = 0\). So we can drop the term \(\mathcal{E}_{\alpha,b}(f)\) from (4.12) and get the following two inequalities,

\[
\|f\|_{B^1(\mathbb{R})}^2 \leq C_1 b \|f\|_2^2,
\]

and

\[
\frac{b}{2} \|f\|_4^2 \leq \|f\|_2^2 - \int_{\mathbb{R}} \left[|f'(t)|^2 + \left(\alpha + \frac{t^2}{2}\right)^2 |f(t)|^2\right] dt
\]

\[
\leq (b - \lambda(\alpha)) \|f\|_2^2 \leq (b - \lambda_0) \|f\|_{B^1(\mathbb{R})}^2 \quad \text{since} \quad \lambda(\alpha) \geq \lambda_0.
\]

On the other hand, using Hölder’s inequality, we write

\[
\|f\|_2 = \int_{\mathbb{R}} |f(t)|(1 + t^2)^{1/2}|f(t)|(1 + t^2)^{-1/2} dt
\]

\[
\leq \|f\|_4(1 + t^2)^{1/2}\|f\|_2((1 + t^2)^{-1/2}) \leq C_2 \|f\|_4 \|f\|_{B^1(\mathbb{R})},
\]

for some constant \(C_2\) independent of \((\alpha, b)\). Combining (4.13)-(4.15) gives, for \(C_3 = 2^{1/4}C_1C_2\)

\[
\|f\|_{B^1(\mathbb{R})} \leq C_3 b^{3/4}(b - \lambda_0)^{1/4} \|f\|_{B^1(\mathbb{R})}^{3/2}.
\]

This yields the conclusion in Lemma 4.7 with \(C = C_3^2\).

In the sequel, we assume the additional condition \(Cb^{3/2} \sqrt{b - \lambda_0} < c_*\), where \(c_*\) is the constant in Lemma 4.6. Now, Lemma 4.7 and the identity (4.11) yield:
Lemma 4.8. There exists $\varepsilon > 0$ such that, for all $(\alpha, b) \in (\tau_0 - \varepsilon, \tau_0 + \varepsilon) \times [\lambda_0, \lambda_0 + \varepsilon)$, the function $f_{\alpha, b}$ satisfies,

$$f_{\alpha, b} = \sum_{j=0}^{\infty} \delta^j G^j_{\alpha, b}(u_\alpha),$$

(4.16)

where $\delta$ is introduced in (4.7).

Proof of Theorem 4.1.

**Step 1: A spectral expression for $f_{\alpha, b}$.**

The definition of $\delta$ in (4.7) and Lemma 4.7 yield

$$0 \leq \delta \leq C b^{3/2} \sqrt{b - \lambda_0}.$$ 

Assuming $b - \lambda_0$ is sufficiently small, we get $0 \leq \delta < 1$. Consequently, the series

$$S(\delta, \alpha, b) := \sum_{j=0}^{\infty} \delta^j G^j_{\alpha, b}(u_\alpha),$$

(4.17)

is normally convergent in the space $B^1(\mathbb{R})$ and depends smoothly on the parameters $(\delta, \alpha, b)$.

Later, it will be convenient to write

$$S(\delta, \alpha, b) = \delta T(\delta^2, \alpha, b),$$

where, for $\epsilon > 0$,

$$T(\epsilon, \alpha, b) = \sum_{j=0}^{\infty} \epsilon^{j-1} G^j_{\alpha, b}(u_\alpha).$$

(4.18)

Now Lemma 4.8 reads

$$f_{\alpha, b} = \delta T(\delta^2, \alpha, b).$$

(4.19)

The advantage of (4.19) is that $f_{\alpha, b}$ is expressed in terms of the spectral quantity $T(\delta^2, \alpha, b)$ and the value $\delta = (f_{\alpha, b}, u_\alpha)$. We will use (4.19) to write a non-trivial relation between the parameters $\alpha, b, \delta$ which will allow us to select the optimal $\alpha$ which minimizes the ground state energy $b(\alpha, b)$ (see (4.2)). Indeed, there exists a smooth function $n(z_1, z_2, z_3)$ defined in a neighborhood of $(0, \tau_0, \lambda_0)$ such that $n(0, \alpha, b) > 0$ for $(\alpha, b) \neq (\tau_0, \lambda_0)$, and (see [13, Lem. 14.2.9, Eq. (14.46)])

$$\delta = \delta(\alpha, b) = \sqrt{b^{-1}(b - \lambda(\alpha))n(b - \lambda(\alpha), \alpha, b)}.$$ 

(4.20)

So we can write $f_{\alpha, b}$ in the form (using (4.19))

$$f_{\alpha, b} = \delta T(\epsilon(\alpha, b), \alpha, b),$$

(4.21)

with $\epsilon(\alpha, b) = \delta(\alpha, b)^2$. This proves that $f_{\alpha, b}$ depends smoothly on $(\alpha, b)$ near $(\tau_0, \lambda_0)$.

**Step 2: Uniqueness of $\xi(b)$.**

By Theorem 4.2, we know that a minimum $\alpha_0$ for the function $\alpha \mapsto b(\alpha, b)$ exists, and if $b$ is selected sufficiently close to $\lambda_0$, $\alpha_0$ is localized near $\tau_0$. In this case, it is enough to consider $\alpha$ varying in a neighborhood of $\tau_0$. In particular, we may assume that (4.20) holds.

We will prove that any minimum $\alpha_0$, when close enough to $\tau_0$, is unique and depends smoothly on $b$. Using (4.5) and (4.21), we have

$$0 = \int_{\mathbb{R}} \left( \frac{t^2}{2} + \alpha_0 \right) |f_{\alpha_0, b}(t)|^2 dt = \delta(\alpha_0, b)^2 \int_{\mathbb{R}} \left( \frac{t^2}{2} + \alpha_0 \right) |T(\epsilon(\alpha_0, b), \alpha_0, b)|^2 dt.$$ 

(4.22)
By the Feynman-Hellman formula for the eigenvalue $\lambda(\alpha)$, we write
\[
\lambda'(\alpha_0) = 2 \int_{\mathbb{R}} \left( \frac{t^2}{2} + \alpha_0 \right) |u_{\alpha_0}(t)|^2 dt = 2 \int_{\mathbb{R}} \left( \frac{t^2}{2} + \alpha_0 \right) |T(0, \alpha_0, b)|^2 dt
\]
\[
= -2 \int_{\mathbb{R}} \left( \frac{t^2}{2} + \alpha_0 \right) \left( |T(e(\alpha_0, b), \alpha_0, b)|^2 - |T(0, \alpha_0, b)|^2 \right) dt,
\]
where we have used (4.22) in the step.

By (4.18), we see that
\[
\lambda'(\alpha_0) = \delta(\alpha_0, b)^2 a(\alpha_0, b).
\]
where $a(\alpha, b)$ is a smooth function, thanks to (4.20).

Using the expression of $\delta(\alpha_0, b)$ in (4.20), we see that $\alpha_0$ is a solution of the following equation
\[
\lambda'(\alpha_0) = (b - \lambda(\alpha_0)) \tilde{a}(\alpha_0, b)
\]
for a new smooth function $\tilde{a}$.

Now, the function
\[
(\alpha, b) \mapsto h(\alpha, b) := \lambda'(\alpha) - (b - \lambda(\alpha)) \tilde{a}(\alpha, b).
\]
satisfies $h(\tau_0, \lambda_0) = 0$ since $\lambda'(\tau_0) = 0$ and $\lambda(\tau_0) = \lambda_0$. Furthermore,
\[
\frac{\partial h}{\partial \tau}(\tau_0, \lambda_0) = \lambda''(\tau_0) > 0.
\]
By the implicit function theorem, there exists a neighborhood $N_0$ of $(\tau_0, \lambda_0)$ such that, in this neighborhood, the equation $h(\alpha, b) = 0$ has a unique solution given by $\alpha = \xi(b)$, where $\xi$ is a smooth function of $b$.

By selecting $b$ sufficiently close to $\lambda_0$, we get that $(\alpha_0, b) \in N_0$ and satisfies $h(\alpha_0, b) = 0$. Consequently, $\alpha_0 = \xi(b)$.

**Step 3: Asymptotic behavior of the ground state energy.**

We will prove that, as $b \searrow \lambda_0$,
\[
\|f_{\xi(b), b}\|_4^4 = b^{-2} \frac{(b - \lambda_0)^2}{\|u_0\|_4^4} (1 + o(1))
\]
which in turn yields, by Theorem 4.2, the desired asymptotic expansion for the ground state energy $b(\xi(b), b)$. Recall that, for the ease of the notation, we write $u_0 = u_{\tau_0}$.

By the series representation (4.16) of $f_{\alpha, b}$ in the $B^1$-norm (and therefore in the $L^4$-norm) we get
\[
\|f_{\xi(b), b}\|_4^4 = |\delta|^4 \|u_{\xi(b)}\|_4^4 + O(|\delta|^6).
\]

By smoothness of the function $b \mapsto \xi(b)$ and $\alpha \mapsto u_\alpha$, we get $\|u_{\xi(b)}\|_4 = \|u_{\tau_0}\|_4^4 (1 + o(1))$, which in turn yields (4.24).

5. **The spectral estimate**

Let $b$ and $\xi(b)$ be as in Theorem 4.1, and let $\beta \in \mathbb{R}$. We introduce $\gamma(\beta, b)$ to be the infimum of the spectrum of the self-adjoint operator associated with the quadratic form
\[
Q_{\beta,b}(u) = \int_{\mathbb{R}} \left( |u'(t)|^2 + \left( \frac{t^2}{2} + \xi(b) + \beta \right)^2 |u(t)|^2 - b(1 - |f_{\xi(b), b}|^2)|u(t)|^2 \right) dt.
\]

More precisely, using the min-max principle,
\[
\gamma(\beta, b) := \inf_{u \in H^1(\mathbb{R})} \frac{Q_{\beta,b}(u)}{\int_{\mathbb{R}} |u|^2 dt}.
\]

The eigenvalue $\gamma(\beta, b)$ is simple, and by analytic perturbation theory, $\beta \mapsto \gamma(\beta, b)$ is an analytic function. Furthermore, if $u_{\beta, b}$ is a normalized ground state of $\gamma(\beta, b)$, then it depends analytically on $\beta$ as well.
Proposition 5.2. We have:

(1) \( \gamma(0, b) = 0 \) and \( \gamma_\beta(0, b) = 0 \), for all \( b > \lambda_0 \).

(2) \( \lim_{b \searrow \lambda_0} \gamma_\beta(0, b) = \lambda''(\tau_0) \). (5.3)

Proof. Let \( u_{\beta, b} \) denote the unique positive normalized ground state of \( \gamma(\beta, b) \). The function \( u_{\beta, b} \) satisfies the eigenvalue equation

\[
- u_{\beta, b}'' + \left( \frac{t^2}{2} + \xi(b) + \beta \right)^2 u_{\beta, b} - b(1 - |f_{\xi(b), b}|^2)u_{\beta, b} = \gamma(\beta, b)u_{\beta, b}. \tag{5.4}
\]

We set \( \beta = 0 \) and multiply the above equation by \( f_{\xi(b), b} \), then we integrate over \( \mathbb{R} \) to get

\[
\gamma(0, b) \int_{\mathbb{R}} f_{\xi(b), b}(t)u_{\beta, b}(t)dt = 0.
\]

Since \( f_{\xi(b), b} \) and \( u_{\beta, b} \) are positive, \( \int_{\mathbb{R}} f_{\xi(b), b}(t)u_{\beta, b}(t)dt \neq 0 \). Thus \( \gamma(0, b) = 0 \) and it follows from (4.3) that

\[
u_{0, b} = \frac{f_{\xi(b), b}}{\|f_{\xi(b), b}\|_2}.
\]

To prove the statement on the derivative of \( \gamma \), we write the Hellmann-Feynman formula

\[
\frac{\partial \gamma}{\partial \beta}(\beta, b) = 2 \int_{\mathbb{R}} \left( \frac{t^2}{2} + \xi(b) + \beta \right) |u_{\beta, b}(t)|^2 dt.
\]

For \( \beta = 0 \), \( u_{\beta, b} = f_{\xi(b), b}/\|f_{\xi(b), b}\|_2 \) and we obtain

\[
\frac{\partial \gamma}{\partial \beta}(0, b) = \frac{2}{\|f_{\xi(b), b}\|_2^2} \int_{\mathbb{R}} \left( \frac{t^2}{2} + \xi(b) \right) |f_{\xi(b), b}|^2 dt = 0 \text{ by (4.5).}
\]

It remains to prove (2). Note that \( z_1(b), z_2(b) \to \tau_0 \) as \( b \to \lambda_0 \). Since \( z_1(b) < \xi(b) < z_2(b) \),

\[
\xi(b) \to \tau_0 \quad \text{as} \quad b \to \lambda_0. \tag{5.5}
\]

It follows from Corollary 4.8 that

\[
\lim_{b \to \lambda_0} \|f_{\xi(b), b}\|_{B^1(\mathbb{R})} = 0.
\]

By the continuous embedding \( B^1(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R}) \), we infer that

\[
\lim_{b \to \lambda_0} \|f_{\xi(b), b}\|_{L^\infty(\mathbb{R})} = 0. \tag{5.6}
\]

Note that, for all \( u \in B^1(\mathbb{R}) \),

\[
Q_{\beta, b}(u) = Q_{\xi(b)+\beta}(u) - b \int_{\mathbb{R}} (1 - |f_{\xi(b), b}|^2)|u|^2 dt,
\]
where \((\alpha, a) \mapsto Q_{\alpha, a}(\cdot)\) is the quadratic form defined in (5.1), and \(\alpha \mapsto Q_\alpha(\cdot)\) is the quadratic form introduced in (3.4).

Recall the definitions of \(\gamma\) and \(\lambda\) from (5.2) and (3.3) respectively. Using the min-max principle we get

\[
\lambda(\xi(b) + \beta) - b \leq \gamma(\beta, b) \leq \lambda(\xi(b) + \beta) - b + \|f_{\xi(b), b}\|_{L^\infty}.
\]

It follows from (5.5) and (5.6) that

\[
\gamma(\beta, b) \xrightarrow{b \to \lambda_0} \lambda(\tau_0 + \beta) - \lambda_0,
\]

where the convergence is uniform (with respect to \(\beta\)) on every bounded interval in \(\mathbb{R}\).

Since \(\gamma\) is holomorphic in \(\beta\), the derivatives must converge uniformly as well, hence

\[
\gamma_{\beta \beta}(\beta, b) \to \gamma''(\tau_0 + \beta),
\]

from which (2) follows simply upon taking \(\beta = 0\). \(\square\)

**Proof of Theorem 5.1.** Using a Taylor expansion of \(\gamma(\beta, b)\) near \(\beta = 0\), it follows from Proposition 5.2 that there exist \(\gamma_0 > 0\) and \(\epsilon_1 > 0\) such that

\[
\lambda_0 \leq b < \lambda_0 + \epsilon_1 \& |\beta| \leq \gamma_0 \Rightarrow \gamma(\beta, b) > 0.
\]

From the definition of \(\gamma\) in (5.2) and the min-max principle, we get

\[
\gamma(\beta, b) \geq \lambda(\xi(b) + \beta) - b.
\]

Since \(\lambda''(\tau_0) > 0\), we get by Taylor’s formula the existence of \(\epsilon_2 \in (0, \epsilon_1)\) and \(\delta \in (0, \frac{\beta_0}{2})\) such that

\[
z \notin (\tau_0 - \delta, \tau_0 + \delta) \implies \lambda(z) \geq \lambda_0 + \epsilon_2.
\]

Since \(\xi(b) \to \tau_0\) as \(b \to \lambda_0\), there exists \(\epsilon_3 \in (0, \epsilon_2)\) such that

\[
\lambda_0 \leq b \leq \lambda_0 + \epsilon_3 \implies |\xi(b) - \tau_0| \leq \frac{\beta_0}{2}.
\]

It is easy to see that, for \(b \in [\lambda_0, \lambda_0 + \epsilon_3]\) and \(|\beta| \geq \frac{\beta_0}{2}\), \(\xi(b) + \beta \notin (\tau_0 - \delta, \tau_0 + \delta)\), and consequently

\[
\gamma_{\beta \beta}(\beta, b) \geq \lambda(\xi(b) + \beta) - b \geq \lambda_0 + \epsilon_2 - b \geq 0.
\]

This combined with (5.8) finishes the proof of Theorem 5.1. \(\square\)

**6. The Model on a Half Cylinder**

Recall that \(S_R = (-R, R) \times \mathbb{R}\) and \(A_{\text{app}}\) is the magnetic potential introduced in (2.1). We introduce the space

\[
D^{\text{per}} = \left\{ u \in L^2_{\text{loc}}(\mathbb{R}^2) : (\nabla - iA_{\text{app}})u \in L^2(S_R), \exists z \in \mathbb{R}, \ u(x_1 + 2R, x_2) = e^{2iz}u(x_1, x_2) \right\},
\]

and the ground state energy,

\[
e^{\text{per}}(b; R) = \inf \{ E_{R,b}(u) : u \in D^{\text{per}} \},
\]

where \(E_{R,b}\) is the functional in (2.2).

For every \(b > 0\), let \(\xi(b)\) be as defined in Theorem 4.1 and define the function

\[
\mathbb{R}^2 \ni (x_1, x_2) \mapsto \psi_b(x_1, x_2) = e^{i\xi(b)x_1}f_{\xi(b), b}(x_2).
\]

We will prove

**Theorem 6.1.** There exists \(\epsilon > 0\) such that, for all \(b \in [\lambda_0, \lambda_0 + \epsilon]\) and \(\psi \in D^{\text{per}},
\]

\[
E_{R,b}(\psi) \geq E_{R,b}(\psi_b).
\]
Remark 6.2. It is easy to see that
\[ \psi_b(x_1 + 2R, x_2) = e^{2i\xi(b)R}e^{i\xi(b)x_1}f_{\xi(b),b}(x_2) = e^{2i\xi(b)R}\psi_b(x_1, x_2). \]
Thus \( \psi_b \in D_{\text{per}} \) (take \( z = \xi(b) \)). Consequently, we infer from Theorem 6.1 that \( \psi_b \) is the minimizer of \( E_{R,b} \) in \( D_{\text{per}} \). By (4.4) and invoking Theorem 4.1, the minimal energy is:
\[ e^{\text{per}}(b; R) = E_{R,b}(\psi_b) = -bR\|f_{\xi(b),b}\|_4^4 = -b^{-1}R(b - \lambda_0)^2\|u_0\|_4^4(1 + g(b)), \quad (6.5) \]
where \( g(b) \) is independent of \( R \) and satisfies \( g(b) \to 0 \) as \( b \searrow \lambda_0 \).

Proof of Theorem 6.1. We follow the proof of Almgren-Helffer [1] devoted to the potential term \((t + \xi(b))^2\). Firstly, let us notice that the space
\[ D_0 = \{ \psi \in D_{\text{per}} \cap C^\infty(\mathbb{R}^2) : \exists M > 0, \supp \psi \subset \mathbb{R} \times [-M, M] \} \quad (6.6) \]
is dense in \( D_{\text{per}} \), the space in (6.1), relative to the norm \( \|u\|_{D_{\text{per}}} := \|u\|_{L^2(S_R)} + \|\nabla - iA_{\text{app}}\)u\|_{L^2(S_R)} \). So it is enough to prove (6.4) for \( \psi \in D_0 \). The proof consists of four steps. Since \( f_{\xi(b),b} > 0 \) in \( \mathbb{R}_+ \), we can represent the space \( D_0 \) in the following useful form
\[ D_0 = \{ e^{ix_1}f_{\xi(b),b}(x_2)v(x_1, x_2) : z \in \mathbb{R}, \ v \in C^\infty(\mathbb{R}^2) \text{ is } 2R\text{-periodic in the variable } x_1 \ \\
&\quad \text{ & } \ \exists M > 0, \supp v \subset \mathbb{R} \times [-M, M] \}. \quad (6.7) \]

Step 1.
Choose \( b \in [\lambda_0, \lambda_0 + \epsilon] \) so that Theorem 5.1 holds. Pick \( \psi \in D_0 \) in the form (see (6.7))
\[ (x_1, x_2) \mapsto \psi(x_1, x_2) := e^{i\xi(b)x_1}f_{\xi(b),b}(x_2)v, \quad (6.8) \]
where \( v(x_1, x_2) \) is smooth, vanishes for \( |x_2| \) large enough, and periodic with respect to the first variable, i.e. \( v(x_1, x_2) = v(x_1 + 2R, x_2) \).

The following formula will allow us to compare the energies of \( \psi \) and \( \psi_b \) (see [1, Thm. 3.1, Eqs. (3.5)-(3.7)] for the detailed computations):
\[ E_{R,b}(\psi) - E_{R,b}(\psi_b) = \int \int e^{i\xi(b)x_1}f_{\xi(b),b}(x_2) \left( f_{\xi(b),b}^4(\nabla v)^2 + 2\left( \frac{x_1^2}{2} + \xi(b) \right) f_{\xi(b),b}^2(\nabla v) \right) dx_1 dx_2 \]
\[ + \frac{b}{2} \int \int e^{i\xi(b)x_1}f_{\xi(b),b}(x_2) \left( 1 - |v|^2 \right)^2 dx_1 dx_2. \quad (6.9) \]
By periodicity we can expand \( v \) in a Fourier series as follows
\[ v(x_1, x_2) = \sum_{n=-\infty}^{\infty} v_n(x_2)e^{in\frac{\pi}{R}x_1} \]
where
\[ v_n(x_2) = \frac{1}{2R} \int v(x_1, x_2)e^{-in\frac{\pi}{R}x_1} dx_1. \quad (6.10) \]
So, we can rewrite
\[ \psi(x_1, x_2) = \sum_{n=-\infty}^{\infty} e^{in\frac{\pi}{R}x_1}e^{i\xi(b)x_1}(v_n f_{\xi(b),b})(x_2). \]
Thus, the equation (6.9) reads as follows
\[ E_{R,b}(\psi) - E_{R,b}(\psi_b) = \sum_{n=-\infty}^{\infty} J(v_n) \frac{n\pi}{R} + \frac{b}{2} \int \int e^{i\xi(b)x_1}(1 - |v|^2)^2 dx_1 dx_2, \quad (6.11) \]
where
\[ J(v_n; \beta) = \int \left| f_{\xi(b),b}^n \right|^2 \left[ |v_n|^2 + \frac{\beta^2 + 2\beta\left( \frac{x_1^2}{2} + \xi(b) \right)}{4} \right] dx_2. \]
It results from (6.10) that \(v_n(x_2)\) is a smooth function with compact support (since \(v(x_1, x_2)\) is smooth and vanishes for \(x_2\) large enough). Let \(w_n(x_2) = f_{\xi(b), b}(x_2)v_n(x_2)\). It is easy to see that

\[
\int_{\mathbb{R}} |f_{\xi(b), b}|^2 |v_n'|^2 \, dx_2 = \int_{\mathbb{R}} \left[ -\left( \frac{w_n^2 f_{\xi(b), b}'}{f_{\xi(b), b}} \right)' + \frac{w_n^2 f_{\xi(b), b}'}{f_{\xi(b), b}} + |w_n'|^2 \right] \, dx_2,
\]

where, after an integration by parts,

\[
\int_{\mathbb{R}} \left( \frac{w_n^2 f_{\xi(b), b}'}{f_{\xi(b), b}} \right)' \, dx_2 = 0.
\]

Consequently, using the equation satisfied by \(f_{\xi(b), b}\) in (4.3), we get

\[
\int_{\mathbb{R}} |f_{\xi(b), b}|^2 |v_n'|^2 \, dx_2 = \int_{\mathbb{R}} \left[ |w_n'|^2 + \left( \left( \frac{x_2^2}{2} + \xi(b) \right)^2 - b(1 - f_{\xi(b), b}^2) \right) |w_n|^2 \right] \, dx_2.
\]

Now we insert this into the expression of \(J(v_n; \beta)\) then use the min-max principle and get

\[
J(v_n; \frac{n\pi}{R}) = \int_{\mathbb{R}} \left[ |w_n'|^2 + \left( \left( \frac{x_2^2}{2} + \xi(b) + \frac{n\pi}{R} \right)^2 - b(1 - f_{\xi(b), b}^2) \right) |w_n|^2 \right] \, dx_2 
\geq \gamma \left( \frac{n\pi}{R}, b \right) \int_{\mathbb{R}} |w_n|^2 \, dx_2,
\]

where \(\gamma(\cdot, b)\) was introduced in (5.2). Note that \(\gamma(\cdot, b) \geq 0\) by Theorem 5.1. Inserting this into (6.12), we obtain

\[
\mathcal{E}_{R,b}(\psi) - \mathcal{E}_{R,b}(\psi_b) 
\geq 2R \gamma \left( \frac{n\pi}{R}, b \right) \sum_{n=-\infty}^{\infty} \int_{\mathbb{R}} |f_{\xi(b), b}(x_2)v_n(x_2)|^2 \, dx_2 + b \int_{\mathbb{R}} \int_{-R}^{R} f_{\xi(b), b}^2(1 - |v|^2)^2 \, dx_1 \, dx_2 
\geq 0. \quad (6.12)
\]

**Step 2.**

Now we consider an arbitrary function \(\psi \in \mathcal{D}_0\) which can be expressed in the form (see (6.7))

\[
\psi(x_1, x_2) = e^{ixx_1} f_{\xi(b), b}(x_2)v(x_1, x_2).
\]

Note that in (6.8), we handled the special case \(z = \xi(b)\). Here we assume that:

\[
\frac{R}{\pi}(z - \xi(b)) = \frac{r}{s},
\]

for some \((r, s) \in \mathbb{Z} \times \mathbb{N}\). We can rewrite \(\psi\) as

\[
\psi(x_1, x_2) = e^{i\xi(b)x_1} f_{\xi(b), b}(x_2)v^{\text{per}}(x_1, x_2),
\]

where \(v^{\text{per}}(x_1, x_2) := e^{i(z-\xi(b))x_1}v(x_1, x_2)\).

The function \(v^{\text{per}}\) is \(2sR\)-periodic with respect to the first variable. Thus \(\psi\) falls in the case studied in Step 1 but with \(R\) replaced by \(sR\) and \(s \in \mathbb{N}\). We apply the conclusion in Step 1 and write

\[
\mathcal{E}_{sR,b}(\psi) \geq \mathcal{E}_{sR,b}(\psi_b).
\]

Next we observe that, for \(s \in \mathbb{N}\),

\[
\mathcal{E}_{sR,b}(\psi) = s\mathcal{E}_{R,b}(\psi) \quad \text{and} \quad \mathcal{E}_{sR,b}(\psi_b) = s\mathcal{E}_{R,b}(\psi_b).
\]

So we deduce that

\[
\mathcal{E}_{R,b}(\psi) \geq \mathcal{E}_{R,b}(\psi_b),
\]

for all \(\psi \in \mathcal{D}_0\) but under the condition in (6.14).

**Step 3.**
The general result follows from the density of rational numbers in \( \mathbb{R} \). We present the details for the sake of convenience. Pick \( z \in \mathbb{R} \) and an arbitrary smooth function \( \psi(\cdot; z) \in \mathcal{D}_0 \) having the form (see (6.7))

\[
\psi(x_1, x_2; z) := e^{izx_1}f_{\xi(b), b}(x_2)v(x_1, x_2).
\]

We will prove that

\[
\mathcal{E}_{R,b}(\psi(\cdot; z)) \geq \mathcal{E}_{R,b}(\psi_b),
\]

which yields the desired result.

Define \( \alpha \in \mathbb{R} \) as follows

\[
\frac{R}{\pi}(z - \xi(b)) = \alpha \in \mathbb{R}.
\]

Let \( \alpha_n = \left[\frac{\alpha}{n}\right] \in \mathbb{Q} \), where \([\cdot]\) denotes the integer part. It is clear that \( \alpha_n \to \alpha \) in \( \mathbb{R} \). Define the sequence \( z_n \) as follows

\[
\frac{R}{\pi}(z_n - \xi(b)) = \alpha_n \in \mathbb{Q}.
\]

We apply the conclusion in Step 2 with \( z_n \), it follows that

\[
\mathcal{E}_{R,b}(\psi(\cdot; z_n)) \geq \mathcal{E}_{R,b}(\psi_b).
\]

(6.16)

It is clear that \( z_n \to z \). From this, we deduce that \( \mathcal{E}_{R,b}(\psi(\cdot; z_n)) \to \mathcal{E}_{R,b}(\psi(\cdot; z)) \). Since \( \mathcal{E}_{R,b}(\psi_b) \) is independent of \( z \), taking the limit in (6.16) yields (6.15). \( \square \)

7. Proof of Theorem 1.1

Recall the ground state energies \( \epsilon \) and \( \epsilon_{\text{per}} \) from (2.3) and (6.2) respectively. We decompose the proof of Theorem 1.1 into two steps.

**Step 1: Lower bound.**

Since every function in \( H_0^1(S_R) \) can be extended by periodicity to a function in the domain \( \mathcal{D}_{\text{per}} \), we get immediately that, for all \( L, R > 0 \),

\[
\epsilon(L^{-3/2}; R) \geq \epsilon_{\text{per}}(L^{-3/2}; R).
\]

(7.1)

Now, Theorem 6.1 and the formula in (6.5) give us, for all \( L, R > 0 \),

\[
\frac{\epsilon_{\text{per}}(L^{-3/2}; R)}{2R} \geq \frac{L^{2/3}(L^{-2/3} - \lambda_0)^2}{\|u_0\|_1^2} (1 + g(L)),
\]

where \( g(L) \) is independent of \( R \) and tends to 0 as \( L \rightarrow \lambda_0^{-3/2} \). Thus (7.1) yields

\[
\frac{\epsilon(L^{-3/2}; R)}{2R} \geq \frac{L^{2/3}(L^{-2/3} - \lambda_0)^2}{\|u_0\|_1^2} (1 + g(L)).
\]

In light of Theorem 2.1, we get the desired lower bound upon taking \( R \to \infty \).

**Step 2: Upper bound.**

To get an upper bound, we need to use a suitable test configuration. Let \( \theta_R \in C_c^\infty(\mathbb{R}) \) be a function satisfying,

\[
supp \theta_R \subset (-R, R), \quad 0 \leq \theta_R \leq 1, \quad \theta_R = 1 \quad \text{in} \quad (-R + 1, R - 1),
\]

and

\[
|\theta_R'| \leq C,
\]

where \( C > 0 \) is a universal constant.

We introduce

\[
\psi(x_1, x_2) = e^{i\xi(L^{-2/3})x_1}f_L(x_2)\theta_R(x_1).
\]

where

\[
f_L(x_2) := f_{\xi(L^{-2/3}), b}(x_2).
\]

Here, we recall \( \xi(b) \) and \( f_{\xi(b), b} \) from Theorems 4.1 and 4.2 respectively.
We start by estimating
\[ E_{R,L^{-2/3}}(\psi) = \int_{S_R} \left( |(\nabla - iA_{\text{app}})\psi|^2 dx - L^{-2/3}|\psi|^2 dx + \frac{L^{-2/3}}{2}|\psi|^4 \right) dx. \] (7.2)

An integration by parts yields,
\[ \int_{S_R} |(\nabla - iA_{\text{app}})\psi|^2 dx = \left\langle \theta^2_R(x_1) f_L(x_2), -(\nabla - iA_{\text{app}})^2 e^{i\xi(L^{-2/3})x_1} f_L(x_2) \right\rangle \]
\[ + \int_{S_R} |f_L(x_2)\theta^2_R(x_1)|^2 dx. \] (7.3)

Note that
\[ \left\langle \theta^2_R(x_1) f_L(x_2), -(\nabla - iA_{\text{app}})^2 e^{i\xi(L^{-2/3})x_1} f_L(x_2) \right\rangle \]
\[ = \int_{S_R} \theta^2_R(x_1) \left( |f'_L(x_2)|^2 + \left( \frac{x^2}{2} + \xi(L^{-2/3}) \right)^2 |f_L(x_2)|^2 \right) dx_1 dx_2 \]
\[ \leq 2R \int_{S_R} \left( |f'_L(x_2)|^2 + \left( \frac{x^2}{2} + \xi(L^{-2/3}) \right)^2 |f_L(x_2)|^2 \right) dx_2. \] (7.4)

By the construction of \( \theta_R \), we have that supp \( \theta^2_R \subset [-R + 1, R - 1] \) and \( |\theta^2_R| \leq C \). Thus
\[ \int_{S_R} |f_L(x_2)\theta^2_R(x_1)|^2 dx_1 dx_2 = \int_{-R+1}^{R-1} |\theta^2_R(x_1)|^2 dx_1 \int_{S_R} |f_L(x_2)|^2 dx_2 \leq C\|f_L\|^2_2. \] (7.5)

Here \( \|f_L\|_2 < \infty \) but depends on \( L \). Substituting (7.4) and (7.5) in (7.3), we find
\[ \int_{S_R} |(\nabla - iA_{\text{app}})\psi|^2 dx \leq 2R \left( \int_{S_R} |f'_L(x_2)|^2 + \left( \frac{x^2}{2} + \xi(L^{-2/3}) \right)^2 |f_L(x_2)|^2 \right) dx_2 + C\|f_L\|^2_2. \] (7.6)

We have the following decomposition,
\[ \int_{S_R} |\psi|^2 dx = \int_{S_R} \theta^2_R(x_1) |f_L(x_2)|^2 dx_1 dx_2 \]
\[ = 2R \int_{R} |f_L(x_2)|^2 dx_2 - \int_{S_R} (1 - \theta^2_R(x_1)) |f_L(x_2)|^2 dx_1 dx_2. \]

Again, the assumption on the support of \( \theta_R \) yields
\[ \int_{S_R} (1 - \theta^2_R(x_1)) |f_L(x_2)|^2 dx_1 dx_2 \leq 2\|f_L\|^2_2. \] (7.7)

Consequently, we obtain, for all \( R > 2 \),
\[ \epsilon(L^{-2/3}; R) \leq E_{R,L^{-2/3}}(\psi) \]
\[ \leq 2R \int_{R} \left( |f'_L|^2 + \left( \frac{x^2}{2} + \xi(L^{-2/3}) \right)^2 |f_L|^2 - L^{-2/3}|f_L|^2 + \frac{L^{-2/3}}{2}|f_L|^4 \right) dx_2 \]
\[ + \max(C,2)\|f_L\|^3_2. \] (7.8)

Since \( f_L \) is a minimizer of the functional (4.1) for \( (\alpha, b) = (\xi(L^{-2/3}), L^{-2/3}) \), (7.8) reads
\[ \epsilon(L^{-2/3}; R) \leq 2R b (\xi(L^{-2/3}), L^{-2/3}) + \max(C,2)\|f_L\|^3_2, \] (7.9)

where \( b \) was introduced in (4.2).

Dividing by \( 2R \), we get
\[ \frac{\epsilon(L^{-2/3}; R)}{2R} \leq b (\xi(L^{-2/3}), L^{-2/3}) + \frac{\max(C,2)\|f_L\|^3_2}{R}. \] (7.10)
Taking \( \limsup_{R \to \infty} \) on both sides and invoking Theorem 2.1, we infer that, for all \( L > 0 \),
\[
E(L) = \limsup_{R \to \infty} \frac{\varepsilon(L^{-2/3}; R)}{2R} \leq b\left(\xi(L^{-2/3}), L^{-2/3}\right).
\]
(7.11)

In view of Theorem 4.1, we see that, as \( L \gg \lambda_0^{-3/2} \),
\[
b\left(\xi(L^{-2/3}, L^{-2/3})\right) = -\frac{L^{2/3}}{2} \left(\frac{L^{-2/3} - \lambda_0}{\|u_0\|_4^4}\right)(1 + o(1)).
\]
Inserting this into (7.11), we get, as \( L \gg \lambda_0^{-3/2} \),
\[
E(L) \leq \frac{L^{2/3}}{2} \left(\frac{L^{-2/3} - \lambda_0}{\|u_0\|_4^4}\right)(1 + o(1)).
\]

8. Proof of Theorem 1.5

We will improve the estimate in (1.5) by providing an explicit control of the remainder term. We will do this by carefully examining the upper and lower bounds obtained in [19].

To simplify the presentation, we will assume that the set \( \Gamma \) (introduced in (1.3)) consists of a single smooth curve. When \( \Gamma \) consists of a finite number of components, we can apply the analysis in this section to each component separately and sum up the results.

We will use the following notation:
- \( ds \) denotes the arc-length measure on \( \Gamma \);
- \( |\Gamma| = \int_\Gamma ds(x) \) denotes the arc-length measure of \( \Gamma \);
- \( \text{dist}_\Gamma : \Gamma \times \Gamma \to [0, \infty) \) denotes the arc-length distance in \( \Gamma \).

We begin with the following geometric lemma.

Lemma 8.1. There exist two positive constants \( C \) and \( \ell_0 \) (which depend on the domain \( \Omega \), the function \( B_0 \) and the set \( \Gamma \) in (1.3)) such that, for all \( a \in \Gamma \) and \( \ell \in (0, \ell_0) \) satisfying
\[
\overline{D(a, \ell)} \subset \Omega
\]
then
\[
\left| \int_{\overline{D(a, \ell)} \cap \Gamma} ds(x) - 2\ell \right| \leq C\ell^2.
\]

Proof. Let \( a \in \Gamma \) and \( \ell > 0 \) such that \( \overline{D(a, \ell)} \subset \Omega \). By a translation, we may assume that \( a = (0, 0) \). We can select an interval \( I_a \), a \( C^2 \) function \( u_a : I_a \to \mathbb{R} \), and a constant \( \tilde{C} > 0 \) such that
\[
\overline{D(a, \ell)} \cap \Gamma \subset \{(s, u_a(s)) : s \in I_a\}, \quad 0 \in I_a, \quad (0, u_a(0)) = 0,
\]
and
\[
\forall s \in I_a, \quad |u_a(s)| + |u_a'(s)| + |u_a''(s)| \leq \tilde{C}.
\]
Furthermore, by the compactness of the set \( \Gamma \), we may assume that the constant \( \tilde{C} \) is independent of \( a \) and \( \ell \), for \( \ell \) sufficiently small.

Define the function \( f(s) = s^2 + (u_a(s))^2 - \ell^2 \). Using Taylor’s formula for the function \( u_a \) near 0, we can prove the following, for \( \ell \) sufficiently small:
- There exist \( s_1 \in (-2\ell, 0) \) and \( s_2 \in (0, 2\ell) \) such that \( f(s_1) = f(s_2) = 0 \) (by the intermediate value theorem);
- \( f'(s) > 0 \) on \((-2\ell, 2\ell)\);
- \( s_1 \) and \( s_2 \) are the unique zeros of the function \( f \) on the interval \((-2\ell, 2\ell)\);
- \( s_1 \) and \( s_2 \) satisfy
\[
s_1 = \frac{-\ell}{\sqrt{1 + |u_a'(0)|^2}} + O(\ell^2) \quad \text{and} \quad s_2 = \frac{\ell}{\sqrt{1 + |u_a'(0)|^2}} + O(\ell^2).
\]
Therefore, we deduce that \( D(a, \ell) \cap \Gamma = \{(s, u_a(s)) : s_1 \leq s \leq s_2\} \) and

\[
\int_{D(a, \ell) \cap \Gamma} ds(x) = \int_{s_1}^{s_2} \sqrt{1 + |u'_a(s)|^2}\, ds = 2\ell + O(\ell^2) \quad \text{as } \ell \to 0_+ .
\]

With Lemma 8.1 in hand, we can construct a covering of \( \Gamma \) by disks with disjoint interior.

**Lemma 8.2.** There exist two positive constants \( C \) and \( \ell_0 \) such that, for all \( \ell \in (0, \ell_0) \), there exist \( N \in \mathbb{N} \) and a collection of points \( (a_j)_{1 \leq j \leq N} \) on \( \Gamma \) such that

\[
\forall j, \quad \left| \text{distr}(a_j, a_{j+1}) - 2\ell \right| \leq C\ell^2 \quad \& \quad D(a_j, \ell) \subset \Omega ,
\]

\[
D(a_j, \ell) \cap D(a_{j'}, \ell) = \emptyset \quad \text{for } j \neq j',
\]

\[
\left| N - \frac{\left| \Gamma \right|}{2\ell} \right| \leq C .
\]

**Proof.** For all \( \ell \in (0, 1) \), let \( n \) be the unique natural number satisfying

\[
\frac{\left| \Gamma \right|}{2\ell} \left( 1 + \frac{\ell}{2} \right)^{-1} - 1 \leq n < \frac{\left| \Gamma \right|}{2\ell} \left( 1 + \frac{\ell}{2} \right)^{-1} .
\]

We select a collection of points \( (b_j)_{1 \leq j \leq n} \subset \Gamma \) such that \( \text{distr}(b_j, b_{j+1}) = \frac{\left| \Gamma \right|}{n} \). For all \( j \), let \( e_j = |b_{j+1} - b_j| \) be the Euclidean distance between the points \( b_{j+1} \) and \( b_j \). We define the number \( N \) as follows

\[
N = \text{Card}\, J \quad \text{where } J = \{ j : D(b_j, e_j) \subset \Omega \} .
\]

For \( \ell \) sufficiently small, we get that \( J = \{ j_0 + k : 1 \leq k \leq N \} \) for some \( j_0 \in \{ 1, \ldots, n \} \). Now, for all \( k \in \{ 1, \ldots, N \} \), we set \( a_k = b_{j_0+k} \).

The points \( (a_k) \) and the number \( N \) satisfy the properties mentioned in Lemma 8.2. The details can be found in [19, Proof of Lemma 5.2, Step 2].

In Lemma 8.3 below, \( \mathbf{F} \) denotes the unique vector field satisfying

\[
\text{curl} \mathbf{F} = B_0 , \quad \text{div} \mathbf{F} = 0 \quad \text{in } \Omega , \quad \nu \cdot \mathbf{F} = 0 \quad \text{on } \partial \Omega , \quad \text{(8.1)}
\]

where \( \nu \) is the unit normal vector of the boundary of \( \Omega \). Also, we introduce the following local Ginzburg-Landau energy

\[
\mathcal{E}_0(u, A; U) = \int_U \left( |(\nabla - i\kappa H A)u|^2 - \kappa^2|u|^2 + \frac{\kappa^2}{2}|u|^4 \right)\, dx , \quad \text{(8.2)}
\]

where \( U \) is an open subset of \( \mathbb{R}^2 \).

**Lemma 8.3.** Let \( 0 < M_1 < M_2 \). There exist two positive constants \( C \) and \( \kappa_0 \) such that the following is true.

Assume that

- \( \kappa \geq \kappa_0 \) and \( M_1 \kappa^2 \leq H \leq M_2 \kappa^2 \);
- \( \ell = \kappa^{-7/5} \), \( \mathbf{a} \in \Gamma \) and \( D(\mathbf{a}, \ell) \subset \Omega \);
- \( \mathbf{r} \in \overline{D(\mathbf{a}, \ell)} \cap \Gamma \) and \( L_\mathbf{a} = |\nabla B_0(\mathbf{r})|_H \).

Then there exists a function \( w_{\mathbf{a}, \mathbf{r}} \in H^1_0(D(\mathbf{a}, \ell)) \) such that

\[
\mathcal{E}_0(w_{\mathbf{a}, \mathbf{r}}, \mathbf{F}; D(\mathbf{a}, \ell)) \leq \left( 2L_{\mathbf{a}}^{1/3} E(L_{\mathbf{a}}) + C\kappa^{-1/16} \right) \kappa \ell ,
\]

where the function \( E(\cdot) \) is introduced in (2.4).
Proof. We will skip the reference to the points $a$ and $x$ by writing $L = L_1$ and $w = w_{a,x}$. Define $a = A(\kappa \ell)^{-1}$ and $R = L^{1/3} \kappa \ell$, where $A$ is a constant selected such that, for $\kappa$ sufficiently large, we have

$$R \geq 4 \max(a^{-1/2} L^{-2/3}, 1).$$

(8.3)

Then we take $w$ as in [19, Eq. (5.11)]. Since $R$ satisfies (8.3), then the function $w_a$ satisfies (see [19, Eq. (5.15)]), for some constant $C > 0$ and for all $\delta > 0$,

$$\mathcal{E}_0(w, F; D(a, \ell)) \leq 2(1 + \delta)(1 - a) R E(L) + \frac{\tilde{C}}{\delta} \left( (1 + L^{-2/3})^{1/3} + a^{-1/2} + (1 + a^{-1} L^{-2/3} R^{-2}) + (\delta \kappa^2 + \delta^{-1} \kappa^2 H^2 \ell^2 \ell^2) \right).$$

For $\delta = \kappa^{-3/8}$, $\ell = \kappa^{-7/8}$, $a \approx (\kappa \ell)^{-1}$ and $H \approx \kappa^2$, we get the upper bound in Lemma 8.3, for some constant $C > \tilde{C}$. □

Now we can prove the

**Proposition 8.4.** Let $0 < M_1 < M_2$. There exist two positive constants $C$ and $\kappa_0$ such that, for all $\kappa \geq \kappa_0$ and $M_1 \kappa^2 \leq H \leq M_2 \kappa^2$, the ground state energy in (1.2) satisfies

$$E_{gs}(\kappa, H) \leq \kappa \int \left( \frac{H}{\kappa^2} |\nabla B_0(x)| \right)^{1/3} E \left( \frac{H}{\kappa^2} |\nabla B_0(x)| \right) \frac{ds(x)}{16} + C \kappa^{15/16}.$$

Proof. Let $\ell = \kappa^{-7/8}$ and $(D(a_j, \ell))_{1 \leq j \leq N}$ be the collection of the pairwise disjoint disks constructed in Lemma 8.2, for $\kappa$ sufficiently large. For all $j$, choose the point $x_j$ such that

$$\min_{x \in D(a_j, \ell)} \left( \frac{H}{\kappa^2} |\nabla B_0(x)| \right)^{1/3} E \left( \frac{H}{\kappa^2} |\nabla B_0(x)| \right) = \left( \frac{H}{\kappa^2} |\nabla B_0(x_j)| \right)^{1/3} E \left( \frac{H}{\kappa^2} |\nabla B_0(x_j)| \right).$$

We define the function $w \in H_0^1(\Omega)$ as follows

$$w(x) = \begin{cases} w_{a_j, x_j} : x \in D(a_j, \ell) \\ 0 : x \notin \bigcup_{1 \leq j \leq N} D(a_j, \ell). \end{cases}$$

Let $F$ be the vector field in (8.1). Since $E_{gs}(\kappa, H) \leq \mathcal{E}(w, F) = \sum_{j=1}^N \mathcal{E}_0(w_{a_j, x_j}, F)$, Lemma 8.3 yields

$$E_{gs}(\kappa, H) \leq \sum_{j=1}^N \left( \left( 2L_{x_j}^{1/3} E(L_{x_j}) + C \kappa^{-1/16} \right) \kappa \ell \right) \leq \kappa \sum_{j=1}^N \left( \left( |D(a_j, \ell) \cap \Gamma| L_{x_j}^{1/3} E(L_{x_j}) \right) + C \left( \ell + \kappa^{-1/16} \right) \right) \kappa \text{ by Lemma 8.1}$$

$$\leq \kappa \int_{V_\ell} \left( \frac{H}{\kappa^2} |\nabla B_0(x)| \right)^{1/3} E \left( \frac{H}{\kappa^2} |\nabla B_0(x)| \right) \frac{ds(x)}{16} + C \kappa^{15/16},$$

where $V_\ell = \bigcup_{j=1}^N D(x_j, \ell) \cap \Gamma$. But, by Lemma 8.2, $|\Gamma \setminus V_\ell| \leq C \ell$ which is what we need to obtain the upper bound in Proposition 8.4. □
Proposition 8.5. Let $0 < M_1 < M_2$. There exist two positive constants $C$ and $\kappa_0$ such that, for all $\kappa \geq \kappa_0$ and $M_1 \kappa^2 \leq H \leq M_2 \kappa^2$, the ground state energy in (1.2) satisfies

$$E_{gs}(\kappa, H) \geq \kappa \int_{\Omega} \left( \frac{H}{\kappa^2} |\nabla B_0(x)| \right)^{1/3} E \left( \frac{H}{\kappa^2} |\nabla B_0(x)| \right) \, ds(x) - C \kappa^{11/12}.$$ 

Proof. Let $a > 0$ and $\delta > 0$ be two sufficiently small parameters. Let $\ell = \delta H^{-1/3}$ and define the two domains

$$D_1 = \{ x \in \Omega : \text{dist}(x, \Gamma) < 2\sqrt{a\ell} \} \quad \text{and} \quad D_2 = \{ x \in \Omega : \text{dist}(x, \Gamma) > \sqrt{a\ell} \}.$$ 

There exist two smooth functions $\chi_1$ and $\chi_2$ such that

$$\chi_1^2 + \chi_2^2 = 1, \quad \text{supp}\chi_j \subset D_j, \quad \text{and} \quad |\nabla \chi_j| \leq C(a\ell^2)^{-1},$$

for some positive constant $C$.

Let $(\psi, A)$ be a minimizer of the functional in (1.1). The following holds (see [19, Eq. (7.11)])

$$E_{gs}(\kappa, H) = E(\psi, A) \geq E_0(\psi, A; \Omega) \geq \sum_{j=1}^2 E_0(\chi_j \psi, A; \Omega) - C \frac{1}{\sqrt{a\ell}}$$

where the functionals $E$ and $E_0$ are introduced in (1.1) and (8.2) respectively.

We will select the parameters $a$ and $\delta$ such that $\sqrt{a\ell} \gg \kappa^{-1}$ (recall that $\ell = \delta H^{-1/3}$). By [19, Thm. 6.3], $|\psi|^2$ is exponentially small in $D_2$, hence $E_0(\chi_2 \psi, A; \Omega) \geq -\kappa^{-1}$ for $\kappa$ sufficiently large. Consequently

$$E_{gs}(\kappa, H) \geq E_0(\chi_1 \psi, A; \Omega) - C \left( \frac{1}{\sqrt{a\ell}} + \frac{1}{\kappa} \right). \quad (8.4)$$

Having Lemma 8.1 in hand, we can use the following lower bound (see [19, Eq. (7.19)])

$$E_0(\chi_1 \psi, A; \Omega) \geq \kappa \int_{\Gamma} \left( \frac{H}{\kappa^2} |\nabla B_0(x)| \right)^{1/3} E \left( \frac{H}{\kappa^2} |\nabla B_0(x)| \right) \, ds(x)$$

$$- C(\sqrt{a\ell})^{-1} - C \left( a + \delta + \delta^{2L} H^{-20/3} + \eta \right) \kappa$$

for $\eta = \ell$ and for all $\alpha \in (0, 1)$. We insert this lower bound into (8.4) then we choose $a = \delta = \kappa^{-1/6}$ and $\alpha = 3/4$. This finishes the proof of Proposition 8.5.

Proof of Theorem 1.5. Propositions 8.4 and 8.5 yield that

$$E_{gs}(\kappa, H) = \kappa \int_{\Gamma} \left( \frac{H}{\kappa^2} |\nabla B_0(x)| \right)^{1/3} E \left( \frac{H}{\kappa^2} |\nabla B_0(x)| \right) \, ds(x) + O(\kappa^{11/12}). \quad (8.5)$$

Under the assumption 1.2, the principal term in (8.5) satisfies (1.12) and is of order $|\Gamma_\kappa(\rho(\kappa))|^2 \geq c(\rho(\kappa))^{5/2}$, for some constant $c > 0$. By (1.12) and the assumption $\rho(\kappa) \gg \kappa^{-1/30}$, we get

$$\kappa^{11/12} \ll \left( \frac{\kappa^{3/2}}{2 \|u_0\|_2} \right)^2 \int_{\Gamma} \left( \frac{H}{\kappa^2} |\nabla B_0(x)| \right)^{-2/3} \, ds(x). \quad (8.6)$$

Now, collecting (1.12), (8.6) and (8.5), we finish the proof of Theorem 1.5.

Acknowledgments. The authors would like to thank B. Helffer for his valuable comments on the manuscript, and the anonymous referee for the valuable suggestions. A.K. is supported by a grant from Lebanese University.
References

[1] Y. Almog and B. Helffer. The distribution of surface superconductivity along the boundary: on a conjecture of X. B. Pan. SIAM J. Math. Anal. 38, 1715-1732 (2007).

[2] Y. Almog, B. Helffer and X. B. Pan. Mixed normal-superconducting states in the presence of strong electric currents. Arch. Rational Mech. Anal. 223, 419-462 (2017).

[3] W. Assaad and A. Kachmar. The influence of magnetic steps on bulk superconductivity. Discrete and Continuous Dynamical Systems (A) 36 (12), 6623-6643 (2016).

[4] K. Attar. The ground state energy of the two dimensional Ginzburg-Landau functional with variable magnetic field. Annales de l’Institut Henri Poincaré- Analyse Non-Linéaire 32, 325-345 (2015).

[5] K. Attar. Energy and vorticity of the Ginzburg-Landau model with variable magnetic field. Asymptot. Anal. 93, 75-114 (2015).

[6] K. Attar. Pinning with a variable magnetic field of the two-dimensional Ginzburg-Landau model. Nonlinear Analysis: TMA. 139, 1-54 (2016).

[7] A. Contreras and X. Lamy. Persistence of superconductivity in thin shells beyond $H_{c1}$. Commun. Contemp. Math. 18 article no. 1550047, 21 p, (2016).

[8] M. Correggi and N. Rougerie. Boundary behavior of the Ginzburg-Landau order parameter in the surface superconductivity regime. Arch. Rational Mech. Anal. 219, 553-606 (2015).

[9] M. Correggi and N. Rougerie. Effects of boundary curvature on surface superconductivity. Letters in Mathematical Physics 1-23 (2016).

[10] M. Correggi and Nicolas Rougerie. On the Ginzburg-Landau functional in the surface superconductivity regime. Comm. Math. Phys. 332, 1297-1343 (2014).

[11] P.G. de Gennes. Superconductivity of Metals and Alloys. Benjamin, Amsterdam (1996).

[12] S. Fournais and B. Helffer. Energy asymptotics for type II superconductors. Calc. Var. Partial Differential Equations 24 (3), 341-376 (2005).

[13] S. Fournais and B. Helffer. Spectral methods in surface superconductivity. Progress in Nonlinear Equations and Their Applications, Vol. 77. Birkhäuser Boston Inc., Boston, MA (2010).

[14] S. Fournais and A. Kachmar. The ground state energy of the three dimensional Ginzburg-Landau functional. Part I: Bulk regime. Comm. Partial. Differential Equations 38 (2), 339-383 (2013).

[15] S. Fournais and A. Kachmar. Nucleation of bulk superconductivity close to critical magnetic field. Advan. Math. 226 (2), 1213-1258 (2011).

[16] B. Helffer. The Montgomery operator revisited. Colloquium Mathematicum 118 (2), 391-400 (2011).

[17] B. Helffer and A. Kachmar. Decay of superconductivity away from the magnetic zero set. arXiv:1604.0203e1 (2016).

[18] B. Helffer and A. Kachmar. From constant to non-degenerately vanishing magnetic fields in superconductivity. Annales de l’Institut Henri Poincaré- Analyse Non-Linéaire 34, 423-438 (2017).

[19] B. Helffer and A. Kachmar. The Ginzburg-Landau functional with vanishing magnetic field. Arch. Rational Mech. Anal. 218, 55-122 (2015).

[20] B. Helffer, Y. Kordyukov, N. Raymond, S. Vă Nguc. Magnetic wells in dimension three. Analysis and PDE 9 (7) 1575-1608 (2016).

[21] B. Helffer and A. Mohamed. Semi classical analysis for the ground state energy of a Schrödinger operator with magnetic wells. J. Funct. Anal. 138 (1), 40-81 (1996).

[22] R. Montgomery. Hearing the zero locus of a magnetic field. Commun. Math. Phys. 168 (3), 651-675 (1995).

[23] X. B. Pan and H. Kwek. Schrödinger operators with non-degenerately vanishing magnetic fields in bounded domains. Trans. Am. Math. Soc. 354 (10), 4201-4227 (2002).

[24] E. Sandier and S. Serfaty. Vortices for the magnetic Ginzburg-Landau model. Progress in Nonlinear Differential Equations and their Applications, Vol. 70. Birkhäuser, Basel (2007).

[25] E. Sandier and S. Serfaty. From the Ginzburg-Landau model to vortex lattice problems. Commun. Math. Phys. 313 (3), 635-741 (2012).

Lebanese University, Department of Mathematics, Nabatieh, Lebanon
E-mail address: ayman.kashmar@gmail.com

Lebanese International University, Beirut, Lebanon
& Lebanese University, faculty of Sciences, Section IV, Bekaa, Lebanon
E-mail address: marwa.nasrallah@liu.edu.lb