Defining implication relation for classical logic

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Abstract

In classical logic, “P implies Q” is equivalent to “not-P or Q”. It is well known that the equivalence is problematic. Actually, from “P implies Q”, “not-P or Q” can be inferred (“Implication-to-Disjunction” is valid), whereas from “not-P or Q”, “P implies Q” cannot be inferred in general (“Disjunction-to-Implication” is not generally valid), so the equivalence between them is invalid in general. This work aims to remove the incorrect Disjunction-to-Implication from classical logic (CL). The paper proposes a logic (IRL) with the expected properties: (a) CL is obtained by adding Disjunction-to-Implication to IRL, and (b) Disjunction-to-Implication is not derivable in IRL; while (c) fundamental laws in classical logic – including law of excluded middle and principle of double negation, law of non-contradiction and ex contradictione quodlibet, conjunction elimination and disjunction introduction, and hypothetical syllogism and disjunctive syllogism – are all retained in IRL.

Keywords implication, conditional, material implication, material conditional, paradox, semantics, classical logic, propositional logic, first-order logic, proof system, Hilbert system, natural deduction

1 Introduction

In classical logic, “P implies Q” (P → Q) is taken to be logically equivalent to “it cannot be P and not Q” (¬(P ∧ ¬Q)) or “not-P or Q” (¬P ∨ Q) by duality. This equivalence can be viewed as the definition of implication by disjunction and negation (P → Q = ¬P ∨ Q, called material implication), or used to define disjunction by implication and negation, so P → Q and ¬P ∨ Q can be replaced in a proof with one another. It is well known that this definition of implication is problematic in that it leads to counter-intuitive results or “paradoxes”. There are many paradoxes of material implication (see e.g. Bronstein, 1936). For instance, the theorem

(P → Q) ∨ (Q → P)

of classical logic means that any two propositions must be related by implication (total order). This does not agree with a large number of situations in the real world. For example, let P = “Alice is in Athens” and Q = “Alice is in London” – given an instant in the future, then both P → Q and Q → P cannot be true, so (P → Q) ∨ (Q → P) cannot be true.

Many efforts have been made to resolve this problem and important non-classical logics are developed, such as modal logics which introduce the “strict implication” and relevance logics which, as paraconsistent logics, reject ex contradictione quodlibet (ECQ).

However, classical logic exists on its own reason (see e.g. Fulda, 1989). And implication is the kernel concept in logic: “implies” means the same as that “of ordinary inference and proof” (Lewis, 1912). So it is not satisfactory that the uniquely fundamental, simple and useful classical logic has a wrong definition of implication – the material implication.

The paper explains why “Implication-to-Disjunction” ((P → Q) → ¬P ∨ Q) is valid whereas “Disjunction-to-Implication” (¬P ∨ Q → (P → Q)) is not, by describing firstly the original meaning of “implication”. So the goal of this work is a logic in which Disjunction-to-Implication is removed while fundamental laws in classical logic such as law of excluded middle (LEM) and principle of double negation, law of non-contradiction (LNC) and ex contradictione quodlibet (ECQ), conjunction elimination and disjunction introduction, and hypothetical syllogism (transitivity of implication) and disjunctive syllogism, are all retained in IRL.
2 Original meaning of implication

2.1 Four situations

Let $P$ and $Q$ be propositions.

Consider how $P$ affects $Q$. There are four possible situations when $P$ is true:

1. when $P$ is true, $Q$ is necessarily true;  
2. when $P$ is true, $Q$ is coincidentally true;  
3. when $P$ is true, $Q$ is necessarily false;  
4. when $P$ is true, $Q$ is coincidentally false.

“Necessarily” means “with a certain mechanism”;  
“Coincidentally” means “not necessarily”, i.e. “without a certain mechanism”.  
“Certain mechanism” means one of the two:

1. It is certain literally or physically (in some scope, cf. Subsection 4.6), see Example 4.6.2;
2. It is certain formally or logically, for example, “from $(P \lor Q) \land \lnot P$ to $Q$” is a logical mechanism.

2.2 Original meaning of implication

The original meaning of implication $P \rightarrow Q$ is described as follows.

- $P \rightarrow Q$ is true, if and only if whenever $P$ is true, $Q$ is necessarily true, no matter whether $Q$ is true or false when $P$ is false. Equivalently,
- $P \rightarrow Q$ is false, if and only if whenever $P$ is true, $Q$ is not necessarily true (i.e. $Q$ is coincidentally true, necessarily false, or coincidentally false), no matter whether $Q$ is true or false when $P$ is false.

According to this description, implication from propositions $P$ to $Q$ is (can be defined as) a relation determined by a certain mechanism (either physical or logical) from $P$ to $Q$, which guarantees that $Q$ is true whenever $P$ is true. Without such a mechanism, even if $Q$ is true when $P$ is true, $P \rightarrow Q$ is still false since $Q$ is just coincidentally true when $P$ is true.

The original meaning of bi-implication $P \leftrightarrow Q$ is defined similarly as that “$P \leftrightarrow Q$ is true” means “there is a certain mechanism between $P$ and $Q$ in some scope, which guarantees bi-directional truth-preserving”.

2.3 Non-truth-functionality of implication

Syntactically, all the logical connectives $\{\top, \bot, \lnot, \land, \lor, \rightarrow, \leftrightarrow\}$ are functions mapping zero, one or two formulas to a formula. But semantically, implication $\rightarrow$ and bi-implication $\leftrightarrow$ are essentially different from the others. The following is a comparison of implication $\rightarrow$ and conjunction $\land$ as an example.

Let $L$ be the set of all the formulas representing a propositional language. Let $U$ be the set of all valuation functions from $L$ to the set of truth values $\{T,F\}$. Then:

According to the meaning of conjunction, there exists a function $f : \{T,F\}^2 \rightarrow \{T,F\}$ such that for any $v \in U$ and any $\phi, \psi \in L$, $v(\phi \land \psi) = f(v(\phi), v(\psi))$. This is, $\land$ is truth-functional.

In contrast, according to the original meaning of implication, there exists no function $f : \{T,F\}^2 \rightarrow \{T,F\}$ such that for any $v \in U$ and any $\phi, \psi \in L$, $v(\phi \rightarrow \psi) = f(v(\phi), v(\psi))$. This is, $\rightarrow$ is not truth-functional.

For instance, in terms of the meaning of conjunction, if we know that $P$ is true and $Q$ is true, then we know that $P \land Q$ is true for sure; whereas, in terms of the original meaning of implication, even if we know that $P$ is true and $Q$ is true, we are not sure about whether $P \rightarrow Q$ is true or false, because this depends on whether there is a “certain mechanism” from $P$ to $Q$ to guarantee that whenever $P$ is true $Q$ must be true.

From the semantic viewpoint, implication $\rightarrow$ is a binary relation rather than a binary operation like conjunction $\land$ and disjunction $\lor$. Specifically, the implication $\rightarrow$ being a binary relation on $L$, the truth value of $\phi \rightarrow \psi$ is determined by whether $(\phi, \psi)$ belongs to the set of the relation $\rightarrow$ that is a subset of $L \times L$ (semantically, it is the T-set of the implication, cf. Subsection 4.1), rather than by the truth value of $\phi$ and the truth value of $\psi$. We call $\rightarrow$ implication relation when emphasizing this fact.

So is for bi-implication $\leftrightarrow$. 

2
2.4 Incorrectness of material implication as “implication”

Consider the relation between $P \rightarrow Q$ and $\neg P \lor Q$ in terms of the semantics of implication defined above.

$\neg P \lor Q$ is true” consists of three cases: $P$ is true and $Q$ is true, $P$ is false and $Q$ is true, $P$ is false and $Q$ is false. The first case “$P$ is true and $Q$ is true” includes not only the sole situation where $P \rightarrow Q$ is true (i.e. $Q$ is necessarily true when $P$ is true), but also the the situation “$P$ is true and $Q$ is coincidently true” which is not a context where $P \rightarrow Q$ is true. In other words, “$\neg P \lor Q$ is true” contains not only the case where “$P \rightarrow Q$ is true” but also more cases. In symbols, $(P \rightarrow Q) \rightarrow \neg P \lor Q$ holds whereas $\neg P \lor Q \rightarrow (P \rightarrow Q)$ does not.

“$\neg P \lor Q$ is false” contains only one case: $P$ is true and $Q$ is false that includes two sub-cases, i.e. “$P$ is true and $Q$ is necessarily false” and “$P$ is true and $Q$ is coincidently false”. The two sub-cases is a subset of situations where $P \rightarrow Q$ is false. In symbols, $\neg(\neg P \lor Q) \rightarrow \neg(P \rightarrow Q)$ holds whereas $\neg(P \rightarrow Q) \rightarrow \neg(\neg P \lor Q)$ does not. This result is the same as above in terms of contraposition.

In summary, $(P \rightarrow Q) \leftrightarrow \neg P \lor Q$ is generally not a tautology since

\[ (P \rightarrow Q) \rightarrow \neg P \lor Q \text{ (Implication-to-Disjunction) is valid while } \neg P \lor Q \rightarrow (P \rightarrow Q) \text{ (Disjunction-to-Implication) is generally invalid.} \]

Some researchers have addressed this issue with different motivations or explanations, such as MacColl (1880), Bronstein (1936), Woods (1967), and Dale (1974).

2.5 Difference from “strict implication”

The meaning of “strict implication” of modal logics is NOT the same as the original meaning of implication described above. In a classical modal logic:

- $\Box(P \rightarrow Q)$ is true means “necessarily $P$ implies $Q$” is true. Equivalently,
- $\Box(P \rightarrow Q)$ is false, or $\neg \Box(P \rightarrow Q)$ is true, or $\Diamond \neg(P \rightarrow Q)$ is true, means “not necessarily $P$ implies $Q$” is true, or “possibly not $P$ implies $Q$” is true.

The negation of strict implication, “it is possible not that $P$ implies $Q$”, is not excluding the situation “it is possible that $P$ implies $Q$”. This is different from the original meaning of implication defined previously.

In fact, “strict implication” does not define the meaning of the implication itself: let $S = P \rightarrow Q$, the strict implication $\Box(P \rightarrow Q) = \Box S$ says just “necessarily $S$”, nothing about the implication $P \rightarrow Q$ itself.

2.6 Construction of the right logic

The goals of this work in construction of the logic are:

1. Classical scope. The system is “in the classical scope”. For example, it is a two-valued logic and a sub-system of classical logic so that any operators new to classical logic like modal operators are not considered.

2. Not conflicting with the original meaning of implication. The system should not contain invalid theorems in terms of the meaning of operators, especially the original meaning of implication $\rightarrow$. For example, Disjunction-to-Implication $\neg \phi \lor \psi \rightarrow (\phi \rightarrow \psi)$ must be rejected.

3. Strength. Under the above two conditions, the system should be as strong as possible. For example, both LEM and ECQ are not rejected. LEM is irrelevant to the original meaning of implication. ECQ, which is necessary for retaining both disjunction introduction and disjunctive syllogism (cf. Section [9]), does not conflict with the original meaning of implication (cf. Subsection [4.1]). And Implication-to-Disjunction $(\phi \rightarrow \psi) \rightarrow \neg \phi \lor \psi$ is kept since it follows directly from the original meaning of implication.

There are many choices of ways for constructing the system. Two main approaches are:

1. Choose bi-implication $\leftrightarrow$ as the primitive one, then define implication $\rightarrow$ by bi-implication with the rule $\phi \rightarrow \psi \leftrightarrow \phi \land \psi \leftrightarrow \phi$ or $\phi \rightarrow \psi \leftrightarrow \phi \lor \psi \leftrightarrow \psi$. 

3
2. Choose implication $\to$ as the primitive one, then define bi-implication $\leftrightarrow$ by implication with the rule 

$$
\phi \leftrightarrow \psi \equiv (\phi \to \psi) \land (\psi \to \phi).
$$

Because of its symmetry, bi-implication-based approach has an advantage in structure. The logic IRL presented in Section 3 and its algebraic system IRLA in Section 4 are constructed in the bi-implication-based approach.

3 Propositional logic IRL

IRL (Implication-Relation Logic) is used to refer the logic presented in this section.

3.1 Definitions and notations

The following definitions and notations are adopted for the systems proposed in this paper. The description of concepts is general, e.g. “formula” in this subsection may mean not only the “(well-formed) formula” of a propositional language or first-order language, but also a statement like a rule expression or a sentence of some meta-language being used.

Definitions

**Definition 3.1.1 (Rule of inference).** Let $X$ be a (normally non-empty) finite set of formulas (premises) and $y$ be a formula (conclusion).

A (single-conclusion) rule of inference is an expression of the form

$$X \Rightarrow y$$

that is interpreted as that if every element of $X$ holds, then $y$ holds.

Let $Y = \{y_1, ..., y_n\}$ be a non-empty finite set of formulas (conclusions).

A multiple-conclusion rule

$$X \Rightarrow Y$$

is defined as the set of single-conclusion rules \{X \Rightarrow y_1, ..., X \Rightarrow y_n\}.

A bi-inference rule

$$X \leftrightarrow Y$$

is defined as the union of $X \Rightarrow Y$ and $Y \Rightarrow X$ (view a single-conclusion rule as a singleton).

**Remark 3.1.1.**

Empty-premise rule: In Definition 3.1.1 a rule with an empty set of premises represents actually axiom(s) or theorem(s).

Meta rule: If a premise or a conclusion of a rule contains symbol(s) not belonging to the object language, then the rule is a meta rule, e.g. the rule of conditional proof ($\phi \vdash \psi$) $\Rightarrow (\vdash \phi \to \psi)$ is a meta rule.

Domain-specific rule: A domain-specific rule is sometimes called an axiom or a theorem in that domain. For example, in the Boolean lattice, $x \leq y \Leftrightarrow x \land y = x$ is listed as an axiom although it is essentially a (bi-inference) domain-specific rule.

**Definition 3.1.2 (Proof and theorem).** Let $X$ be a (possibly empty) finite set of formulas (premises) and $y$ be a formula (conclusion).

A proof of $y$ from $X$, denoted $X \vdash y$, is a non-empty finite sequence of formulas $(y_1, ..., y_n)$ with $y_n = y$, where $y_k (1 \leq k \leq n)$ is an axiom, a theorem, or a result from a subset of $X \cup \{y_1, ..., y_{k-1}\}$ by applying a rule of inference. The length or steps of a proof is the length of the proof sequence.

An unconditional proof is a proof with an empty set of premises. Formula $y$ is called a theorem if there exists an unconditional proof of $y$, denoted $\vdash y$.

A Conditional proof is a proof with a non-empty set of premises.
Remark 3.1.2. According to Definition 3.1.2,
All axioms are theorems.

In a proof sequence \((y_1, ..., y_n)\) of an unconditional proof, each subsequence of the form \((y_1, ..., y_k)\) \((1 \leq k \leq n)\) is a unconditional proof, so that every member \(y_k\) \((1 \leq k \leq n)\) is a theorem.

Definition 3.1.3 (Derivable rule). Let \(X\) and \(Y\) be non-empty finite sets of formulas.

A rule \(X \Rightarrow Y\) is derivable if \(X \vdash Y\). This can be expressed by the meta rule \((X \vdash Y) \Rightarrow (X \Rightarrow Y)\).

Obviously, any primitive rule is derivable.

Definition 3.1.4 (Uniform substitution). Let \(x_1, ..., x_k\) be variables, \(t_1, ..., t_k\) be terms, and \(E\) be an expression.

A uniform substitution is a set \(\{t_1/x_1, ..., t_k/x_k\}\) and \(E\{t_1/x_1, ..., t_k/x_k\}\) means to (simultaneously) substitute \(t_i\) for every occurrence of each \(x_i\) in \(E\). A member \(x_i/x_1\) in a substitution \(\{\ldots, x_i/x_1, \ldots\}\) can be omitted, e.g. \(\{t_1/x_1, x_2/x_2, t_3/x_3\}\) is the same as \(\{t_1/x_1, t_3/x_3\}\).

The rule of uniform substitution is of the form \(E \Rightarrow E\{t_1/x_1, ..., t_k/x_k\}\), where \(E\) is an axiom, a theorem, a rule, or a meta rule of a formal system.

Remark 3.1.3. The rule of uniform substitution is common to logical systems for instantiating (schemas of) axioms, theorems, rules and meta rules, and it is often used tacitly.

Definition 3.1.5 (Subformula replacement). Let \(x, y, z\) be formulas.

The expression \(z[x \mapsto y]\) denotes any member of the set of all possible results where each result is obtained via replacing one or more arbitrarily selected occurrence(s) of \(x\) by \(y\) in \(z\).

For instance, \((x \land y \rightarrow x \land y)[x \land y \mapsto y]\) represents any result in the set \(\{y \rightarrow x \land y, x \land y \rightarrow y, y \rightarrow y\}\).

Presentation of proofs

The presentation format of proof is as follows.

A unconditional proof \(\vdash \psi\) of length \(n\) is presented in the format:

\[
\begin{align*}
1 & \vdash \psi_1 \quad \text{reason} \\
\vdots & \vdots \quad \vdots \\
n-1 & \vdash \psi_{n-1} \quad \text{reason} \\
n & \vdash \psi \quad \text{reason}
\end{align*}
\]

A conditional proof \(\{\phi_1, ..., \phi_m\} \vdash \psi\) of length \(n\) is presented in the format:

\[
\begin{align*}
1 & \vdash \phi_1 \quad \text{Premise} \\
\vdots & \vdots \quad \vdots \\
m & \vdash \phi_m \quad \text{Premise} \\
m+1 & \vdash \psi_1 \quad \text{reason} \\
\vdots & \vdots \quad \vdots \\
m+n-1 & \vdash \psi_{n-1} \quad \text{reason} \\
m+n & \vdash \psi \quad \text{reason}
\end{align*}
\]

Note: In some expressions, \(\alpha_0 \bowtie \alpha_1 \bowtie \cdots \bowtie \alpha_n\) is a shorthand of \(\alpha_0 \bowtie \alpha_1, \alpha_1 \bowtie \alpha_2, ..., \alpha_{n-1} \bowtie \alpha_n\), so that in a proof the format

\[
\begin{align*}
1 & \bowtie \alpha_1 \\
\vdots & \vdots \\
n & \bowtie \alpha_n
\end{align*}
\]

is a shorthand of the format

\[
\begin{align*}
1 & \alpha_0 \bowtie \alpha_1 \\
\vdots & \vdots \\
n & \alpha_{n-1} \bowtie \alpha_n
\end{align*}
\]

where \(\bowtie\) stands for a binary-relation operator such as \(\leftrightarrow, \rightarrow, \equiv, \Rightarrow, =, \leq\).
A note on nested proof

By definition of proof (Definition 3.1.2), any member of a proof sequence must be either a theorem (or an assumed theorem in its outer layer) or a result from premise(s) in the same layer by a rule (or by an assumed rule in its outer layer), not allowing any other cases. Attention must be paid to this, especially in nested conditional proofs.

Example 3.1.1 (An illegal proof). Conditional proof of \( \phi \rightarrow (\psi \rightarrow \phi) \):

1. \( \phi \)  
   Premise (main)
2. \( \psi \)  
   Premise (nested)
3. \( \phi \)  
   1: Reiteration – Not allowed
4. \( \psi \rightarrow \phi \)  
   2–3: Rule of conditional proof
5. \( \phi \rightarrow (\psi \rightarrow \phi) \)  
   1–4: Rule of conditional proof

Line 3 is not allowed because \( \phi \) in this line does not belong to any of the allowed cases: (a) it is theorem, or it is assumed be a theorem in its outer premises; (b) it is a result from premises of its own layer (that must be \( \psi \) in line 2) by applying a rule.

Example 3.1.2 (A legal proof). Conditional proof of \( (\top \rightarrow \phi) \rightarrow (\psi \rightarrow \phi) \):

1. \( \top \rightarrow \phi \)  
   Premise (main)
2. \( \psi \)  
   Premise (nested)
3. \( \top \)  
   A theorem (\( \top \) is a theorem)
4. \( \psi \rightarrow \top \)  
   2–3: Rule of conditional proof
5. \( \psi \rightarrow \phi \)  
   4,1: Rule of transitivity
6. \( (\top \rightarrow \phi) \rightarrow (\psi \rightarrow \phi) \)  
   1–5: Rule of conditional proof

The following details a meta proof of a meta rule.

Example 3.1.3 (A legal meta proof). Proof of \( (\vdash \psi) \Rightarrow (\vdash \phi \rightarrow \psi) \):

1. \( \vdash \psi \)  
   Premise (meta proof)
2. \( \phi \)  
   Premise (proof)
3. \( \psi \)  
   1 (\( \psi \) is assumed in line 1 be a theorem)
4. \( \phi \vdash \psi \)  
   2–3: Definition of conditional proof
5. \( \vdash \phi \rightarrow \psi \)  
   4: Rule of conditional proof
6. \( (\vdash \psi) \vdash (\vdash \phi \rightarrow \psi) \)  
   1–5: Definition of conditional proof
7. \( (\vdash \psi) \Rightarrow (\vdash \phi \rightarrow \psi) \)  
   6: Definition of derivable rule

3.2 Logic IRL

Let \( \{\top, \bot, \neg, \land, \lor, \rightarrow, \leftrightarrow\} \) be the set of logical connectives where constants \( \top \) and \( \bot \) are nullary operators.

Let \( \{A, B, \ldots\} \) be a countable (finite or countably infinite) set of propositional constants representing atomic or primitive propositions.

Let \( \phi, \psi, \ldots \) denote (well-formed) formulas built as usual from propositional constants \( \{A, B, \ldots\} \) using operations \( \{\top, \bot, \neg, \land, \lor, \rightarrow, \leftrightarrow\} \) finite times.

Let \( \eta = \eta(\phi_1, \ldots, \phi_n), \zeta = \zeta(\phi_1, \ldots, \phi_n), \ldots \) denote formula schemas built from variables \( \{\phi_1, \ldots, \phi_n\} \) using operations \( \{\top, \bot, \neg, \land, \lor, \rightarrow, \leftrightarrow\} \) finite times.

The set of all formulas is countably infinite as a subset of the set of finite-length strings over the countable alphabet \( \{A, B, \ldots\} \cup \{\top, \bot, \neg, \land, \lor, \rightarrow, \leftrightarrow\} \); and the set of all schemas is also countably infinite as a subset of the set of finite-length strings over the countable alphabet \( \{\phi_1, \phi_2, \ldots\} \cup \{\top, \bot, \neg, \land, \lor, \rightarrow, \leftrightarrow\} \).

The propositional logic IRL is defined by the following axioms and primitive rules of inference, where \( \phi, \psi, \chi, \ldots \) are arbitrary formulas for which one can substitute any schemas.

**Axioms**

1. \( \phi \leftrightarrow \phi \) (Reflexivity of bi-implication). Symmetry and transitivity can be derived.
2. \(\phi \land \psi \leftrightarrow \psi \land \phi\), \(\phi \lor \psi \leftrightarrow \psi \lor \phi\) (Commutativity).

3. \((\phi \land \psi) \land \chi \leftrightarrow \phi \land (\psi \land \chi)\), \((\phi \lor \psi) \lor \chi \leftrightarrow \phi \lor (\psi \lor \chi)\) (Associativity). Redundant.

4. \(\phi \land (\psi \lor \chi) \leftrightarrow (\phi \land \psi) \lor (\phi \land \chi)\), \(\phi \lor (\psi \land \chi) \leftrightarrow (\phi \lor \psi) \land (\phi \lor \chi)\) (Distributivity).

5. \(\phi \land \top \leftrightarrow \phi\), \(\phi \lor \bot \leftrightarrow \phi\) (Identity elements, top and bottom).

6. \(\phi \land \neg \phi \leftrightarrow \bot\), \(\phi \lor \neg \phi \leftrightarrow \top\) (Complement/Negation, LNC and LEM).

7. \((\phi \rightarrow \psi) \leftrightarrow (\phi \land \psi) \lor (\phi \land \chi)\), \(\phi \lor (\psi \land \chi) \leftrightarrow (\phi \lor \psi) \land (\phi \lor \chi)\) (Distributivity).

Primitive rules of inference

1. \(\{\phi, \psi\} \Rightarrow \phi \land \psi\) (Conjunction introduction).

2. \(\phi \land \psi \Rightarrow \phi\) (Conjunction elimination).

3. \(\{\phi \leftrightarrow \psi, \chi\} \Rightarrow \chi\) (Replacement property of bi-implication).

4. \((\phi \vdash \psi) \Rightarrow (\vdash \phi \rightarrow \psi)\) (Rule of conditional proof). Meta rule.

**Remark 3.2.1** (Axioms and primitive rules of IRL).

- Symmetry and transitivity of bi-implication are proved in the next subsection.
- Associativity can be proved following a procedure similar to that by Huntington (1904).
- In a proof, by definition, any member of its proof sequence must be either a theorem (or an assumed theorem) or a result from the same-layer premise(s) by a rule (or by an assumed rule), not allowing any other cases (cf. Subsection 3.1). It is important to pay attention to this, especially in nested conditional proofs.
- Rule of conditional proof of the general form \((\Gamma \cup \{\phi\} \vdash \psi) \Rightarrow (\vdash \phi \rightarrow \psi)\) does not hold for IRL: If so, from \(\{\phi, \psi\} \vdash \phi\) it follows \(\vdash \phi \rightarrow \psi\), hence \(\vdash \phi \rightarrow (\psi \rightarrow \phi)\), contrary to that \(\nabla_{IRL} \phi \rightarrow (\psi \rightarrow \phi)\) (cf. Subsection 5.3).

The logic IRL is consistent as a sub-system of classical logic (cf. Subsection 5.3).

### 3.3 Basic derived results

The results in this subsection include mainly derived rules and meta rules.

Symmetry and transitivity of bi-implication

**Rule 3.3.1** (Symmetry of bi-implication).

\[\phi \leftrightarrow \psi \Rightarrow \psi \leftrightarrow \phi.\]

**Proof.**

1. \(\phi \leftrightarrow \psi\) Premise
2. \(\phi \leftrightarrow \phi\) Reflexivity
3. \((\phi \leftrightarrow \phi)[\phi \rightarrow \psi]\) 1.2: Replacement
4. \(\phi[\phi \rightarrow \psi] \leftrightarrow \phi\) 3: Selected replacement
5. \(\psi \leftrightarrow \phi\) 4: Result of replacement
6. \(\phi \leftrightarrow \psi \vdash \psi \leftrightarrow \phi\) 1-5: Definition of conditional proof
7. \(\phi \leftrightarrow \psi \Rightarrow \psi \leftrightarrow \phi\) 6: Definition of derivable rule
Note: The definitions of conditional proof and derivable rule can be used tacitly in a conditional proof as usual. For example, the last two lines in the above proof can be omitted.

**Rule 3.3.2** (Transitivity of bi-implication).

\{φ ↔ ψ, ψ ↔ χ\} ⇒ φ ↔ χ.

**Proof.**
1. φ ↔ ψ  Premise
2. ψ ↔ χ  Premise
3. (φ ↔ ψ)[ψ ↔ χ]  2,1: Replacement
4. φ ↔ χ  3: Result of Replacement

Leibniz’s replacement

**Rule 3.3.3** (Leibniz’s replacement).

φ ↔ ψ ⇒ χ ↔ χ[φ ↔ ψ].

**Proof.**
1. φ ↔ ψ  Premise
2. χ ↔ χ  Reflexivity
3. (χ ↔ χ)[φ ↔ ψ]  1,2: Replacement
4. χ ↔ χ[φ ↔ ψ]  3: Selected replacement

Theorems to rules

**Meta Rule 3.3.1.**

(⊢ φ ↔ ψ) ⇒ (φ ⇒ ψ).

**Proof.**
1. ⊢ φ ↔ ψ  Premise (begin meta proof)
2. φ  Premise (begin proof)
3. φ ↔ ψ  1 (assumed theorem)
4. φ[φ ↔ ψ]  3,2: Replacement
5. ψ  4: Result of replacement (end proof)
6. φ ⊢ ψ  2–5: Definition of conditional proof
7. φ ⇒ ψ  6: Definition of derivable rule (end meta proof)

By this meta rule, from axioms, for example, φ ∧ ψ ↔ ψ ∧ φ, (φ ∧ ψ) ∧ χ ↔ φ ∧ (ψ ∧ χ) and (φ → ψ) ↔ (φ ∧ ψ → φ), we obtain rules φ ∧ ψ ⇒ ψ ∧ φ, (φ ∧ ψ) ∧ χ ⇒ φ ∧ (ψ ∧ χ) and φ → ψ ⇒ φ ∧ ψ ↔ φ, respectively.

**Tacit use of symmetry properties**

Any formula of the form (φ ◦ ψ) ↔ (ψ ◦ φ) (where ◦ stands for ↔, ∧ or ∨) is a theorem, so (ψ ◦ φ) ↔ (φ ◦ ψ) is also a theorem by symmetry of ↔, hence η(φ ◦ ψ) ↔ η(ψ ◦ φ) and η(ψ ◦ φ) ↔ η(φ ◦ ψ) are theorems by Leibniz’s replacement, and from these we obtain the bi-inference rule η(φ ◦ ψ) ↔ η(ψ ◦ φ). Thus, the symmetry properties can be used tacitly in a proof, viewing η(φ ◦ ψ) and η(ψ ◦ φ) as the same thing.

**Modus ponens**

**Rule 3.3.4** (Modus ponens).

{φ → ψ, φ} ⇒ ψ.
Proof.
1 \( \phi \rightarrow \psi \) Premise
2 \( \phi \) Premise
3 \( \phi \leftrightarrow \phi \land \psi \) 1: Definition of implication
4 \( \phi[\phi \rightarrow \phi \land \psi] \) 3,2: Replacement
5 \( \phi \land \psi \) 4: Result of replacement
6 \( \psi \) 5: Conjunction elimination

Properties of theorems

Theorem 3.3.1 (Top is a theorem).

\( \top \).

Proof.
1 \( \phi \leftrightarrow \phi \) Reflexivity of bi-implication
2 \( (\phi \leftrightarrow \phi) \land \top \) 1: Identity element
3 \( \top \) 2: Conjunction elimination

Meta Rule 3.3.2 (Theorem like top).

\( \vdash \psi \Rightarrow (\vdash \phi \rightarrow \psi) \leftrightarrow (\vdash \phi \land \psi \leftrightarrow \phi) \).

Proof.
1 \( \vdash \psi \) Premise (meta proof)
2 \( \phi \) Premise (proof)
3 \( \psi \) 1 (assumed theorem)
4 \( \vdash \phi \rightarrow \psi \) 2–3: Conditional proof

The second part \( (\vdash \phi \rightarrow \psi) \leftrightarrow (\vdash \phi \land \psi \leftrightarrow \phi) \) follows from the definition of implication.

A theorem functions like the top \( \top \).

Rule 3.3.5 (Theorem equivalence).

\( \{\vdash \phi, \vdash \psi\} \Rightarrow (\vdash \phi \leftrightarrow \psi) \).

Proof.
1 \( \vdash \phi \) Premise
2 \( \vdash \psi \) Premise
3 \( \phi \) 1 (assumed theorem)
4 \( \psi \) 2 (assumed theorem)
5 \( \phi \land \psi \leftrightarrow \psi \) 3: Theorem like top
6 \( \phi \land \psi \leftrightarrow \phi \) 4: Theorem like top
7 \( \phi \leftrightarrow \psi \) 5,6: Transitivity of \( \leftrightarrow \)
3 \( \vdash \phi \leftrightarrow \psi \) 3–7: Unconditional proof

Meta Rule 3.3.3 (Theorem representation).

\( (\vdash \phi \leftrightarrow \top) \leftrightarrow (\vdash \phi) \).

Proof.
Equivalence of the three forms

With the rules of commutativity and associativity for conjunction, \( \phi_1 \land \cdots \land \phi_n \) is well defined for any \( n \geq 1 \), so that the rules of conjunction introduction and elimination can be generalized to \( \{ \phi_1, \ldots, \phi_n \} \Rightarrow \phi_1 \land \cdots \land \phi_n \) and \( \phi_1 \land \cdots \land \phi_n \Rightarrow \phi_i \) respectively.

**Meta Rule 3.3.4** (Rule of conditional proof with multiple premises).

\[
(\{ \phi_1, \ldots, \phi_n \} \Rightarrow \psi) \Rightarrow (\vdash \phi_1 \land \cdots \land \phi_n \rightarrow \psi).
\]

**Proof.** Suppose \( \{ \phi_1, \ldots, \phi_n \} \vdash \psi \) has a proof sequence \((\psi_1, \ldots, \psi)\), then \((\phi_1, \ldots, \phi_n, \psi_1, \ldots, \psi)\) is a proof sequence of \( \phi_1 \land \cdots \land \phi_n \vdash \psi \) as shown below:

\[
\begin{align*}
0 & \quad \phi_1 \land \cdots \land \phi_n \quad \text{Premise} \\
1 & \quad \phi_1 \quad \text{0: Conjunction elimination} \\
\vdots & \quad \vdots \\
n & \quad \phi_n \quad \text{0: Conjunction elimination} \\
1-n & \quad \psi_1 \quad 1-n: \text{First proof-sequence member of } \{ \phi_1, \ldots, \phi_n \} \vdash \psi \\
\vdots & \quad \vdots \\
1-n & \quad \psi \quad 1-n: \text{Last proof-sequence member of } \{ \phi_1, \ldots, \phi_n \} \vdash \psi
\end{align*}
\]

Thus, from \( \phi_1 \land \cdots \land \phi_n \vdash \psi \) we obtain \( \vdash \phi_1 \land \cdots \land \phi_n \rightarrow \psi \) by the rule of conditional proof with single premise.

**Lemma 3.3.1.** (Two meta rules)

\[
(\{ \phi_1, \ldots, \phi_n \} \Rightarrow \psi) \Rightarrow (\{ \phi_1, \ldots, \phi_n \} \vdash \psi), (\vdash \phi_1 \land \cdots \land \phi_n \rightarrow \psi) \Rightarrow (\{ \phi_1, \ldots, \phi_n \} \vdash \psi).
\]

**Proof.**

1. \((\{ \phi_1, \ldots, \phi_n \} \Rightarrow \psi) \Rightarrow (\{ \phi_1, \ldots, \phi_n \} \vdash \psi)\):
   \[
   \begin{align*}
   1 & \quad \{ \phi_1, \ldots, \phi_n \} \Rightarrow \psi \quad \text{Premise (meta proof)} \\
   2 & \quad \phi_1 \quad \text{Premise (proof)} \\
   \vdots & \quad \vdots \\
   3 & \quad \phi_n \quad \text{Premise (proof)} \\
   4 & \quad \psi \quad 2...3: 1 (assumed rule) \\
   5 & \quad \{ \phi_1, \ldots, \phi_n \} \vdash \psi \quad 2-4: \text{Definition of conditional proof}
   \end{align*}
   \]
2. \((\vdash \phi_1 \land \cdots \land \phi_n \rightarrow \psi) \Rightarrow (\{\phi_1, \ldots, \phi_n\} \vdash \psi)\):  
1 \(\vdash \phi_1 \land \cdots \land \phi_n \rightarrow \psi\)  
2 \(\phi_1\)  
\vdots \(\vdots\)  
3 \(\phi_n\)  
4 \(\phi_1 \land \cdots \land \phi_n\)  
5 \(\phi_1 \land \cdots \land \phi_n \rightarrow \psi\) (assumed theorem)  
6 \(\psi\)  
7 \(\{\phi_1, \ldots, \phi_n\} \vdash \psi\)  
2–6: Definition of conditional proof  

Meta Rule 3.3.5 (Equivalence of the three forms).

\[ (\vdash \phi_1 \land \cdots \land \phi_n \rightarrow \psi) \iff (\{\phi_1, \ldots, \phi_n\} \vdash \psi) \iff (\{\phi_1, \ldots, \phi_n\} \Rightarrow \psi). \]

Proof. It follows from the definition of derivable rule, Meta Rule 3.3.4 and Lemma 3.3.1.

If we have any one of the three forms, we get the other two. We just need to find either an unconditional proof \(\vdash \phi_1 \land \cdots \land \phi_n \rightarrow \psi\) or a conditional proof \(\{\phi_1, \ldots, \phi_n\} \vdash \psi\) to obtain both the theorem \(\phi_1 \land \cdots \land \phi_n \rightarrow \psi\) and the rule \(\{\phi_1, \ldots, \phi_n\} \Rightarrow \psi\).

3.4 Some theorems of IRL

This subsection demonstrates proofs of some theorems of IRL.

Basic theorems

Theorem 3.4.1 (Complementarity of top and bottom).

\[ \neg \top \leftrightarrow \bot, \ \neg \bot \leftrightarrow \top. \]

Proof.

1 \(\neg \top \leftrightarrow \neg \top \land \top\)  
2 \(\neg \top \land \top \leftrightarrow \bot\)  
3 \(\neg \top \leftrightarrow \bot\)  
1–2: Transitivity of bi-implication

The second is proved dually.

Theorem 3.4.2 (Double negation).

\[ \neg \neg \phi \leftrightarrow \phi. \]

Proof.

1 \(\neg \neg \phi \leftrightarrow \neg \neg \phi \land \top\)  
2 \(\neg \neg \phi \land \top \leftrightarrow \neg \neg \phi \land \phi\)  
3 \(\neg \neg \phi \land \phi \leftrightarrow \neg \neg \phi \land \neg \neg \phi \land \phi\)  
4 \(\neg \neg \phi \land \neg \neg \phi \land \phi \leftrightarrow \neg \neg \phi \land \neg \neg \phi \land \phi\)  
5 \(\neg \neg \phi \land \neg \neg \phi \land \phi \leftrightarrow \neg \neg \phi \land \neg \neg \phi \land \phi\)  
6 \(\neg \neg \phi \land \neg \neg \phi \land \phi \leftrightarrow \neg \neg \phi \land \neg \neg \phi \land \phi\)  
7 \(\neg \neg \phi \land \neg \neg \phi \land \phi \leftrightarrow \neg \neg \phi \land \neg \neg \phi \land \phi\)  
8 \(\neg \neg \phi \land \neg \neg \phi \land \phi \leftrightarrow \neg \neg \phi \land \neg \neg \phi \land \phi\)  
9 \(\neg \neg \phi \land \neg \neg \phi \land \phi \leftrightarrow \neg \neg \phi \land \neg \neg \phi \land \phi\)  
1–8: Transitivity of bi-implication

Note: Leibniz’s replacement may be used tacitly, e.g. “LEM: Leibniz’s replacement” can be written as just “LEM".

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Theorem 3.4.3 (Idempotence). \(\phi \land \phi \leftrightarrow \phi, \phi \lor \phi \leftrightarrow \phi\).

Proof.
\[
\begin{array}{ll}
\phi \land \phi \\
1 & \leftrightarrow (\phi \land \phi) \lor \bot \\
2 & \leftrightarrow (\phi \land \phi) \lor (\phi \land \neg \phi) \\
3 & \leftrightarrow \phi \land (\phi \lor \neg \phi) \\
4 & \leftrightarrow \phi \land \top \\
5 & \leftrightarrow \phi \\
6 & \phi \land \phi \leftrightarrow \phi \\
\end{array}
\]
1–5: Transitivity

The second is proved similarly.

Theorem 3.4.4 (Annihilator). \(\bot \land \phi \leftrightarrow \bot, \top \lor \phi \leftrightarrow \top\).

Proof.
\[
\begin{array}{ll}
\bot \land \phi \\
1 & \leftrightarrow (\bot \land \phi) \lor \bot \\
2 & \leftrightarrow (\bot \land \phi) \lor (\neg \phi \land \phi) \\
3 & \leftrightarrow (\bot \lor \neg \phi) \land \phi \\
4 & \leftrightarrow \neg \phi \land \phi \\
5 & \leftrightarrow \bot \\
6 & \bot \land \phi \leftrightarrow \bot \\
\end{array}
\]
1–5: Transitivity

The second is proved similarly.

Theorem 3.4.5 (Absorption laws). \(\phi \land (\phi \lor \psi) \leftrightarrow \phi, \phi \lor (\phi \land \psi) \leftrightarrow \phi\).

Proof.
\[
\begin{array}{ll}
\phi \land (\phi \lor \psi) \\
1 & \leftrightarrow (\phi \lor \bot) \land (\phi \lor \psi) \\
2 & \leftrightarrow \phi \lor (\bot \land \psi) \\
3 & \leftrightarrow \phi \lor \bot \\
4 & \leftrightarrow \phi \\
5 & \phi \land (\phi \lor \psi) \leftrightarrow \phi \\
\end{array}
\]
1–4: Transitivity

The second is proved similarly.

Basic properties of implication

Theorem 3.4.6 (Reflexivity of implication). \(\phi \rightarrow \phi\).

Proof.
\[
\begin{array}{ll}
\phi \land \phi \\
1 & \leftrightarrow \phi \land \phi \\
2 & \phi \rightarrow \phi \\
\end{array}
\]
1: Definition of implication

Theorem 3.4.7 (Antisymmetry of implication / Bi-implication introduction).
\((\phi \rightarrow \psi) \land (\psi \rightarrow \phi) \rightarrow (\phi \leftrightarrow \psi)\).

Proof.
\[
\begin{array}{ll}
\phi \rightarrow \psi \\
1 & \text{Premise} \\
\psi \rightarrow \phi \\
2 & \text{Premise} \\
\phi \land \psi \leftrightarrow \phi \\
3 & \text{1: Definition of implication} \\
\psi \land \phi \leftrightarrow \psi \\
4 & \text{2: Definition of implication} \\
\phi \leftrightarrow \psi \\
5 & \text{3,4: Transitivity of bi-implication} \\
\end{array}
\]
**Theorem 3.4.8** (Transitivity of implication).

\[(\phi \rightarrow \psi) \land (\psi \rightarrow \chi) \rightarrow (\phi \rightarrow \chi).\]

**Proof.**

1. \(\phi \rightarrow \psi\)  
   Premise
2. \(\psi \rightarrow \chi\)  
   Premise
3. \(\phi \land \psi \leftrightarrow \phi\)  
   1: Definition of implication
4. \(\psi \land \chi \leftrightarrow \psi\)  
   2: Definition of implication
5. \(\phi \land \chi \leftrightarrow \phi \land \psi \land \chi\)  
   3: Replacement
6. \(\phi \land \chi \leftrightarrow \phi \land \psi\)  
   4: Replacement
7. \(\phi \land \chi \leftrightarrow \phi\)  
   3: Replacement
8. \(\phi \land \chi \leftrightarrow \phi\)  
   5–7: Transitivity of bi-implication
9. \(\phi \land \chi \leftrightarrow \phi\)  
   8: Definition of implication

**Equivalent definitions of implication**

**Lemma 3.4.1.**

\[\phi \land (\neg \phi \lor \psi) \leftrightarrow \phi \land \psi, \phi \lor (\neg \phi \land \psi) \leftrightarrow \phi \lor \psi.\]

**Proof.**

1. \(\phi \land (\neg \phi \lor \psi)\)  
   \(\leftrightarrow (\phi \land \neg \phi) \lor (\phi \land \psi)\)  
   Distributivity
2. \(\leftrightarrow \bot \lor (\phi \land \psi)\)  
   LNC
3. \(\leftrightarrow \phi \land \psi\)  
   Identity
4. \(\phi \land (\neg \phi \lor \psi) \leftrightarrow \phi \land \psi\)  
   1–3: Transitivity

The second is proved dually.

**Theorem 3.4.9** (Equivalent definitions of implication).

\[(\phi \rightarrow \psi) \leftrightarrow (\phi \land \psi \leftrightarrow \phi) \leftrightarrow (\phi \lor \psi \leftrightarrow \psi) \leftrightarrow (\phi \land \neg \psi \leftrightarrow \bot) \leftrightarrow (\neg \phi \lor \psi \leftrightarrow \top).\]

**Proof.**

1. \(\phi \rightarrow \psi\)  
   Premise
2. \(\phi \land \psi \leftrightarrow \phi\)  
   1: Definition of implication
3. \(\phi \land \psi \leftrightarrow \phi\)  
   Premise
4. \(\phi \lor \psi \leftrightarrow \psi\)  
   3: Replacement
5. \(\leftrightarrow (\phi \land \psi) \lor \psi\)  
   1: Replacement
6. \(\leftrightarrow \psi\)  
   Absorption
7. \(\phi \lor \psi \leftrightarrow \psi\)  
   2–3: Transitivity
1 $\phi \lor \psi \leftrightarrow \psi$  
   $\phi \land \neg \psi$  
   Premise
2 $\leftrightarrow (\phi \land \neg \psi) \lor \bot$  
   Identity
3 $\leftrightarrow (\phi \land \neg \psi) \lor (\psi \land \neg \psi)$  
   LNC
4 $\leftrightarrow (\phi \lor \psi) \land \neg \psi$  
   Distributivity
5 $\leftrightarrow \psi \land \neg \psi$  
   1: Replacement
6 $\leftrightarrow \bot$  
   LNC
7 $\phi \land \neg \psi \leftrightarrow \bot$  
   2–6: Transitivity

1 $\phi \land \neg \psi \leftrightarrow \bot$  
   Premise
2 $\neg \phi \lor \bot \lor \psi$  
   Identity
3 $\leftrightarrow \neg \phi \lor (\phi \land \neg \psi) \lor \psi$  
   1: Replacement
4 $\leftrightarrow \neg \phi \lor \phi \lor \psi$  
   Lemma 3.4.1
5 $\leftrightarrow \top \lor \psi$  
   LEM
6 $\leftrightarrow \top$  
   Annihilator
7 $\neg \phi \lor \psi \leftrightarrow \top$  
   2–6: Transitivity

1 $\neg \phi \lor \psi \leftrightarrow \top$  
   Premise
2 $\phi \land \psi$  
   Identity
3 $\leftrightarrow \bot \lor (\phi \land \psi)$  
   Identity
4 $\leftrightarrow (\phi \land \neg \phi) \lor (\phi \land \psi)$  
   LNC
5 $\leftrightarrow \phi \land (\neg \phi \lor \psi)$  
   Distributivity
6 $\leftrightarrow \phi$  
   Identity
7 $\phi \land \psi \leftrightarrow \phi$  
   2–6: Transitivity

1 $\phi \land \psi \leftrightarrow \phi$  
   Premise
2 $\phi \rightarrow \psi$  
   1: Definition of implication

Hereinafter, any one of the four is simply called the definition of implication (by bi-implication).

**De Morgan’s laws**

**Theorem 3.4.10** (De Morgan’s laws).

\[
\neg (\phi \land \psi) \leftrightarrow (\neg \phi \lor \neg \psi), \quad \neg (\phi \lor \psi) \leftrightarrow (\neg \phi \land \neg \psi).
\]

**Proof.**
The second is proved dually.

Lattice-like properties

**Theorem 3.4.11** (Conjunction elimination and disjunction introduction).

\( \phi \land \psi \rightarrow \phi, \phi \rightarrow \phi \lor \psi. \)

*Proof.* In terms of equivalent definitions of implication, conjunction elimination and disjunction introduction are equivalent to absorption laws \( \phi \lor (\phi \land \psi) \leftrightarrow \phi \) and \( \phi \land (\phi \lor \psi) \leftrightarrow \phi \) respectively.

**Theorem 3.4.12** (One-to-two and Two-to-one).

\( (\chi \rightarrow \phi) \land (\chi \rightarrow \psi) \leftrightarrow (\chi \rightarrow \phi \land \psi), (\phi \rightarrow \chi) \land (\psi \rightarrow \chi) \leftrightarrow (\phi \lor \psi \rightarrow \chi). \)

*Proof.*

1. \( \chi \rightarrow \phi \) \hspace{1cm} Premise
2. \( \chi \rightarrow \psi \) \hspace{1cm} Premise
3. \( \chi \land \phi \leftrightarrow \chi \) \hspace{1cm} 1: Definition of implication
4. \( \chi \land \psi \leftrightarrow \chi \) \hspace{1cm} 2: Definition of implication
5. \( \chi \land \phi \land \psi \leftrightarrow \chi \) \hspace{1cm} 3,4: Replacement \( [\chi \rightarrow \chi \land \phi] \)
6. \( \chi \rightarrow \phi \land \psi \) \hspace{1cm} 5: Definition of implication
7. \( (\chi \rightarrow \phi) \land (\chi \rightarrow \psi) \leftrightarrow (\chi \rightarrow \phi \land \psi) \) \hspace{1cm} 1–5: Conditional proof
8. \( \chi \rightarrow \phi \land \psi \) \hspace{1cm} Premise
9. \( \phi \land \psi \rightarrow \phi \) \hspace{1cm} Conjunction elimination
10. \( \chi \rightarrow \phi \) \hspace{1cm} 8–9: Transitivity
11. \( \chi \rightarrow \psi \) \hspace{1cm} 8–10: similarly
12. \( (\chi \rightarrow \phi) \land (\chi \rightarrow \psi) \) \hspace{1cm} 10,11: Conjunction introduction
13. \( (\chi \rightarrow \phi \land \psi) \rightarrow (\chi \rightarrow \phi) \land (\chi \rightarrow \psi) \) \hspace{1cm} 8–12: Conditional proof
14. \( (\chi \rightarrow \phi) \land (\chi \rightarrow \psi) \leftrightarrow (\chi \rightarrow \phi \land \psi) \) \hspace{1cm} 7,13: Bi-implication introduction

The second is proved dually.

**Theorem 3.4.13** (Strengthening).

\( (\phi \rightarrow \psi) \rightarrow (\phi \rightarrow \phi \land \psi), (\phi \rightarrow \psi) \rightarrow (\phi \lor \psi \rightarrow \psi). \)

*Proof.*

1. \( \phi \rightarrow \psi \) \hspace{1cm} Premise
2. \( \phi \rightarrow \phi \) \hspace{1cm} Reflexivity
3. \( \phi \rightarrow \phi \land \psi \) \hspace{1cm} 1,2: One-to-two

The second is proved dually.
Theorem 3.4.14 (Weakening).

\[(\phi \rightarrow \psi) \rightarrow (\phi \land \chi \rightarrow \psi), (\phi \rightarrow \psi) \rightarrow (\phi \rightarrow \psi \lor \chi).\]

Proof.
1. \(\phi \rightarrow \psi\)  Premise
2. \(\phi \land \chi \rightarrow \phi\)  Conjunction elimination
3. \(\phi \land \chi \rightarrow \psi\)  2,1: Transitivity
The second is proved dually.

Theorem 3.4.15 (Compatibility of implication with conjunction and disjunction).

\[(\phi \rightarrow \psi) \rightarrow (\phi \land \chi \rightarrow \psi \land \chi), (\phi \rightarrow \psi) \rightarrow (\phi \lor \chi \rightarrow \psi \lor \chi).\]

Proof.
1. \(\phi \rightarrow \psi\)  Premise
2. \(\phi \lor \chi \rightarrow \psi\)  1: Weakening
3. \(\phi \land \chi \rightarrow \chi\)  Conjunction elimination
4. \(\phi \land \chi \rightarrow \psi \land \chi\)  2,3: One-to-two
The second is proved dually.

Theorem 3.4.16 (Monotonicity of conjunction and disjunction w.r.t. implication).

\[(\phi_1 \rightarrow \psi_1) \land (\phi_2 \rightarrow \psi_2) \rightarrow (\phi_1 \land \phi_2 \rightarrow \psi_1 \land \psi_2),\]
\[(\phi_1 \rightarrow \psi_1) \land (\phi_2 \rightarrow \psi_2) \rightarrow (\phi_1 \lor \phi_2 \rightarrow \psi_1 \lor \psi_2).\]

Proof.
1. \(\phi_1 \rightarrow \psi_1\)  Premise
2. \(\phi_2 \rightarrow \psi_2\)  Premise
3. \(\phi_1 \land \phi_2 \rightarrow \psi_1\)  1: Weakening
4. \(\phi_1 \land \phi_2 \rightarrow \psi_2\)  2: Weakening
5. \(\phi_1 \land \phi_2 \rightarrow \psi_1 \land \psi_2\)  3,4: One-to-two
The second is proved dually.

Corollary 3.4.1 (Constructive dilemma).

\[(\phi_1 \lor \phi_2) \land (\phi_1 \rightarrow \psi_1) \land (\phi_2 \rightarrow \psi_2) \rightarrow \psi_1 \lor \psi_2.\]

Proof.
1. \(\phi_1 \lor \phi_2\)  Premise
2. \(\phi_1 \rightarrow \psi_1\)  Premise
3. \(\phi_2 \rightarrow \psi_2\)  Premise
4. \(\phi_1 \lor \phi_2 \rightarrow \psi_1 \lor \psi_2\)  2,3: Monotonicity of \(\lor\) w.r.t. \(\rightarrow\)
5. \(\psi_1 \lor \psi_2\)  1,4: Modus penens

Theorem 3.4.17 (Contraposition).

\[(\phi \rightarrow \psi) \leftrightarrow (\neg \psi \rightarrow \neg \phi).\]

Proof.
1. \((\phi \land \neg \psi \leftrightarrow \bot)\)  Definition of implication
2. \((\neg \psi \land \neg \phi \leftrightarrow \bot)\)  Commutativity, Double negation
3. \((\neg \psi \rightarrow \neg \phi)\)  Definition of implication

Theorem 3.4.18 (ECQ and its dual).

\[\bot \rightarrow \phi, \phi \rightarrow \bot.\]

Proof. In terms of equivalent definitions of implication, ECQ and its dual are equivalent to Annihilators \(\bot \land \phi \leftrightarrow \bot\) and \(\bot \lor \phi \leftrightarrow \bot\) respectively.
Some properties of bi-implication

**Theorem 3.4.19** (Bi-implication elimination).

\[(\phi \leftrightarrow \psi) \rightarrow (\phi \rightarrow \psi).\]

**Proof.**
1. \(\phi \leftrightarrow \psi\) Premise
2. \(\phi \rightarrow \phi\) Reflexivity of implications
3. \(\phi \rightarrow \psi\) 1,2: Replacement

**Theorem 3.4.20** (Bi-implication by implications).

\[(\phi \leftrightarrow \psi) \leftrightarrow (\phi \rightarrow \psi) \land (\psi \rightarrow \phi).\]

**Proof.**
1. \(\phi \leftrightarrow \psi\) Premise
2. \(\phi \rightarrow \psi\) 1: Bi-implication elimination
3. \(\psi \rightarrow \phi\) 1: Bi-implication elimination
4. \((\phi \rightarrow \psi) \land (\psi \rightarrow \phi)\) 2,3: Conjunction introduction

1. \((\phi \rightarrow \psi) \land (\psi \rightarrow \phi)\) Premise
2. \(\phi \leftrightarrow \psi\) 1: Bi-implication introduction

**Theorem 3.4.21** (Bi-implication of negations).

\[(\phi \leftrightarrow \psi) \leftrightarrow (\neg \phi \leftrightarrow \neg \psi).\]

**Proof.**
1. \(\phi \leftrightarrow \psi\) Bi-implication by implications
2. \((\phi \rightarrow \psi) \land (\psi \rightarrow \phi)\) Bi-implication by implications
3. \((\neg \psi \rightarrow \neg \phi) \land (\neg \phi \rightarrow \neg \psi)\) Contraposition
4. \((\neg \phi \leftrightarrow \neg \psi)\) Bi-implication by implications

**Bounds of implication**

**Theorem 3.4.22** (Implication-to-Disjunction / Maximum of implication).

\[(\phi \rightarrow \psi) \rightarrow \neg \phi \lor \psi.\]

**Proof.**
1. \(\phi \rightarrow \psi\) Premise
2. \(\neg \phi \lor \psi \leftrightarrow \top\) 1: Definition of implication
3. \(\top\) Top is a theorem
4. \(\neg \phi \lor \psi\) 2,3: Replacement \([\top \leftrightarrow \neg \phi \lor \psi]\)

**More properties of implication**

**Theorem 3.4.23** (Generalized ECQ of implication).

\[(\phi \rightarrow \psi) \rightarrow (\chi \rightarrow (\phi \rightarrow \psi)).\]
Proof.
1. \( \phi \rightarrow \psi \)  
   Premise
2. \( \phi \land \psi \leftrightarrow \phi \)  
   1: Definition of implication
3. \( \phi \land \psi \rightarrow \psi \)  
   Conjunction elimination
4. \( \chi \rightarrow (\phi \land \psi \rightarrow \psi) \)  
   3: Theorem like top
5. \( \chi \rightarrow (\phi \rightarrow \psi) \)  
   2,4: Replacement \([\phi \land \psi \mapsto \phi]\)

Note: Generalized ECQ \( \psi \rightarrow (\phi \rightarrow \psi) \) does not hold for IRL (cf. Subsection 5.3).

**Theorem 3.4.24** (Safe generalized ECQ).

\[(\psi \leftrightarrow \top) \rightarrow (\phi \rightarrow \psi), (\phi \leftrightarrow \bot) \rightarrow (\phi \rightarrow \psi).\]

Proof.
1. \( \psi \leftrightarrow \top \)  
   Premise
2. \( \phi \rightarrow \top \)  
   Dual of ECQ
3. \( \phi \rightarrow \psi \)  
   1,2: Replacement \([\top \mapsto \psi]\)

The second is proved dually.

**Theorem 3.4.25** (Importation).

\[(\phi \rightarrow (\psi \rightarrow \chi)) \rightarrow (\phi \land \psi \rightarrow \chi).\]

Proof.
1. \( \phi \rightarrow (\psi \rightarrow \chi) \)  
   Premise
2. \( \phi \land \psi \rightarrow (\psi \rightarrow \chi) \)  
   1: Compatibility of \( \rightarrow \) with \( \land \)
3. \( \phi \rightarrow \chi \)  
   Modus ponens
4. \( \phi \land \psi \rightarrow \chi \)  
   2–3: Transitivity

**Corollary 3.4.2** (Contraction). \( (\phi \rightarrow (\phi \rightarrow \psi)) \rightarrow (\phi \rightarrow \psi) \).

Proof. It follows from the above theorem and Idempotence.

Exportation (converse of importation) \( (\phi \land \psi \rightarrow \chi) \rightarrow (\phi \rightarrow (\psi \rightarrow \chi)) \) does not hold for IRL except for some special forms.

**Theorem 3.4.26** (Exported antisymmetry of implication / bi-implication introduction).

\[(\phi \rightarrow \psi) 
\rightarrow ((\psi \rightarrow \phi) \rightarrow (\phi \leftrightarrow \psi)).\]

Proof.
1. \( \phi \rightarrow \psi \)  
   Premise
2. \( (\psi \rightarrow \phi) \rightarrow (\phi \rightarrow \psi) \)  
   1: Generalized ECQ of implication
3. \( (\psi \rightarrow \phi) \rightarrow (\phi \rightarrow \psi) \land (\psi \rightarrow \phi) \)  
   2: Strengthening
4. \( (\phi \rightarrow \psi) \land (\psi \rightarrow \phi) \rightarrow (\phi \leftrightarrow \psi) \)  
   Bi-implication introduction
5. \( (\psi \rightarrow \phi) \rightarrow (\phi \leftrightarrow \psi) \)  
   3,4: Transitivity

**Theorem 3.4.27** (Exported transitivity of implication).

\[(\phi \rightarrow \psi) \rightarrow ((\psi \rightarrow \chi) \rightarrow (\phi \rightarrow \chi)), (\phi \rightarrow \psi) \rightarrow ((\chi \rightarrow \phi) \rightarrow (\chi \rightarrow \psi)).\]
Proof.
1. \( \phi \rightarrow \psi \)  \hspace{1cm} \text{Premise}
2. \( (\psi \rightarrow \chi) \)  \hspace{1cm} 1: Generalized ECQ of implication
3. \( (\phi \rightarrow \psi) \land (\psi \rightarrow \chi) \)  \hspace{.5cm} 2: Strengthening
4. \( \phi \rightarrow (\phi \rightarrow \chi) \)  \hspace{1cm} Transitivity of implication
5. \( (\psi \rightarrow \chi) \rightarrow (\phi \rightarrow \chi) \)  \hspace{.5cm} 2–4: Transitivity of implication

The second is obtained via contraposition.

\[ \square \]

Theorem 3.4.28 (Exported one-to-two and two-to-one).

\[(\chi \rightarrow \phi) \rightarrow (((\chi \rightarrow \psi) \rightarrow (\chi \rightarrow \phi \land \psi)), (\phi \rightarrow \chi) \rightarrow ((\psi \rightarrow \chi) \rightarrow (\phi \lor \psi \rightarrow \chi))).\]

Proof.
1. \( \chi \rightarrow \phi \)  \hspace{1cm} \text{Premise}
2. \( (\chi \rightarrow \phi) \rightarrow (\chi \rightarrow \phi) \)  \hspace{1cm} 1: Generalized ECQ of implication
3. \( (\chi \rightarrow \phi) \rightarrow (\chi \rightarrow \phi) \land (\chi \rightarrow \psi) \)  \hspace{.5cm} 2: Strengthening
4. \( (\chi \rightarrow \phi) \rightarrow (\chi \rightarrow \phi \land \psi) \)  \hspace{1cm} One-to-two
5. \( (\chi \rightarrow \psi) \rightarrow (\chi \rightarrow \phi \land \psi)) \)  \hspace{.5cm} 2–4: Transitivity

The second is obtained via contraposition.

\[ \square \]

Theorem 3.4.29 (Self-distributivity of implication).

\[(\chi \rightarrow (\phi \rightarrow \psi)) \rightarrow ((\chi \rightarrow \phi) \rightarrow (\chi \rightarrow \psi)).\]

Proof.
1. \( \chi \rightarrow (\phi \rightarrow \psi) \)  \hspace{1cm} \text{Premise}
2. \( (\chi \rightarrow \phi) \rightarrow (\chi \rightarrow \phi \land \phi) \)  \hspace{1cm} 1: Importation
3. \( (\chi \rightarrow \phi) \rightarrow (\chi \rightarrow \phi \land \phi) \land (\chi \rightarrow \phi \rightarrow \psi) \)  \hspace{.5cm} 2: Generalized ECQ of implication
4. \( (\chi \rightarrow \phi) \rightarrow (\chi \rightarrow \phi \land \phi) \land (\chi \rightarrow \phi \rightarrow \psi) \)  \hspace{1cm} Strengthening
5. \( (\chi \rightarrow \phi) \rightarrow (\chi \rightarrow \phi \land \phi) \land (\chi \rightarrow \phi \rightarrow \psi) \rightarrow (\chi \rightarrow \psi) \)  \hspace{1cm} Transitivity
6. \( (\chi \rightarrow \phi) \rightarrow (\chi \rightarrow \psi) \)  \hspace{1cm} 5,6: Transitivity

\[ \square \]

Implication with one or no variable

Theorem 3.4.30 (Univariate implication expressions).

\[ (\phi \rightarrow \neg \phi) \leftrightarrow (\phi \leftrightarrow \bot) \leftrightarrow (\phi \rightarrow \bot); \]
\[ (\neg \phi \rightarrow \phi) \leftrightarrow (\phi \leftrightarrow \top) \leftrightarrow (\top \rightarrow \phi). \]

Proof.
1. \( (\phi \rightarrow \neg \phi) \leftrightarrow (\phi \leftrightarrow \bot) \leftrightarrow (\phi \rightarrow \bot) \)  \hspace{1cm} Definition of implication
2. \( (\neg \phi \rightarrow \phi) \leftrightarrow (\phi \leftrightarrow \top) \leftrightarrow (\top \rightarrow \phi) \)  \hspace{1cm} Double negation
3. \( (\phi \leftrightarrow \bot) \leftrightarrow (\phi \leftrightarrow \bot) \)  \hspace{1cm} Idempotence
4. \( (\phi \rightarrow \bot) \leftrightarrow (\bot \rightarrow \phi) \leftrightarrow (\phi \rightarrow \bot) \)  \hspace{1cm} Bi-implication by implication
5. \( (\phi \rightarrow \bot) \leftrightarrow (\bot \rightarrow \phi) \leftrightarrow (\phi \rightarrow \bot) \)  \hspace{1cm} Theorem (ECQ) like top
6. \( (\phi \rightarrow \neg \phi) \leftrightarrow (\phi \leftrightarrow \bot) \leftrightarrow (\phi \rightarrow \bot) \leftrightarrow (\phi \rightarrow \bot) \)  \hspace{1cm} 1–3–5: Transitivity

The second is proved dually.

\[ \square \]

Theorem 3.4.31 (Maximum of univariate implication).

\[(\top \rightarrow \phi) \rightarrow \phi, (\phi \rightarrow \bot) \rightarrow \neg \phi.\]
Proof. They follow from Theorem 3.4.22 by uniform substitution, and properties of top and bottom.

Corollary 3.4.3 (Contradictory implication).

\((\top \to \bot) \to \bot\).

Proof. It follows from Theorem 3.4.31.

Proof methods related

Theorem 3.4.32 (Disjunctive syllogism).

\((\phi \lor \psi) \land \lnot \phi \to \psi\).

Proof.

1 \((\phi \lor \psi) \land \lnot \phi\) Premise
2 \((\phi \land \lnot \phi) \lor (\psi \land \lnot \phi)\) 1: Distributivity
3 \(\bot \lor (\psi \land \lnot \phi)\) 2: LNC
4 \(\psi \land \lnot \phi\) 3: Identity element
5 \(\psi\) 4: Conjunction elimination

Theorem 3.4.33 (Proof by exhaustion).

\((\phi \lor \psi) \land (\phi \to \chi) \land (\psi \to \chi) \to \chi\).

Proof.

1 \(\phi \lor \psi\) Premise
2 \((\phi \to \chi) \land (\psi \to \chi)\) Premise
3 \(\phi \lor \psi \to \chi\) 2: Two-to-one
4 \(\chi\) 1,3: Modus ponens

Theorem 3.4.34 (RAA (reductio ad absurdum)).

\((\phi \to \psi) \land (\phi \to \lnot \psi) \to \lnot \phi\).

Proof.

1 \((\phi \to \psi) \land (\phi \to \lnot \psi)\) Premise
2 \(\phi \to \psi \land \lnot \psi\) 1: One-to-two
3 \(\phi \to \bot\) LNC,2: Replacement
4 \(\lnot \phi\) 3: Maximum of univariate implication

All of \((\phi \to \psi) \land (\phi \to \lnot \psi) \to \lnot \phi\), \((\phi \to \lnot \phi) \to \lnot \phi\) and \((\phi \to \bot) \to \phi\) are forms of RAA, while \((\lnot \phi \to \psi) \land (\lnot \phi \to \lnot \psi) \to \phi\), \((\lnot \phi \to \phi) \to \phi\) (consequentia mirabilis) and \((\lnot \phi \to \bot) \to \phi\) are forms of proof by contradiction.

Lemma 3.4.2.

\((\chi \land \phi \to \psi) \land (\lnot \chi \land \phi \to \psi) \to (\phi \to \psi)\).

Proof.

1 \((\chi \land \phi \to \psi) \land (\lnot \chi \land \phi \to \psi)\) Premise
2 \((\chi \land \phi) \lor (\lnot \chi \land \phi) \to \psi\) 1: Two-to-one
3 \((\chi \lor \lnot \chi) \land \phi \to \psi\) Distributivity,2: Replacement
4 \(\phi \to \psi\) Theorem (LEM) like top,3: Replacement

Theorem 3.4.35 (Resolution).

\((\phi \lor \chi) \land (\psi \lor \lnot \chi) \to \phi \lor \psi\).
Proof.
1. \( \neg \chi \land (\phi \lor \chi) \rightarrow \phi \)  
   Disjunctive syllogism
2. \( \neg \chi \land (\phi \lor \chi) \land (\psi \lor \neg \chi) \rightarrow \phi \)  
   1: Weakening
3. \( \neg \chi \land (\phi \lor \chi) \land (\psi \lor \neg \chi) \rightarrow \phi \lor \psi \)  
   2: Strengthening
4. \( \chi \land (\phi \lor \chi) \land (\psi \lor \neg \chi) \rightarrow \phi \lor \psi \)  
   1–3: similarly
5. \( (\phi \lor \chi) \land (\psi \lor \neg \chi) \rightarrow \phi \lor \psi \)  
   3,4: Lemma \[3.4.2\]

Theorem 3.4.36 (Cut).
\[ (\phi_1 \rightarrow \chi) \land (\phi_2 \land \chi \rightarrow \psi) \rightarrow (\phi_1 \land \phi_2 \rightarrow \psi). \]

Proof.
1. \( \phi_1 \rightarrow \chi \)  
   Premise
2. \( \phi_2 \land \chi \rightarrow \psi \)  
   Premise
3. \( \phi_1 \land \phi_2 \rightarrow \chi \)  
   1: Weakening
4. \( \phi_1 \land \phi_2 \rightarrow \phi_1 \land \phi_2 \land \chi \)  
   3: Strengthening
5. \( \phi_1 \land \phi_2 \land \chi \rightarrow \psi \)  
   2: Weakening
6. \( \phi_1 \land \phi_2 \rightarrow \psi \)  
   4,5: Transitivity

4 Semantics and the algebra IRL

4.1 Semantics of IRL

An interpretation of a formal language assigns a meaning (semantic value) to each expression within that language. However, logic is fundamentally concerned with the truth values of the statements in its language, rather than their specific content. So interpretations that assign the same truth value (even if they assign different contents) to a statement can be viewed as the same in a logic. Hence, an interpretation of a propositional language can be represented by just a truth assignment or valuation function.

Let \( \{T, F\} \) be the set of truth values for two-valued logics.
Let \( L \) be the set of all formulas in the language of IRL.
Let \( U = \{T, F\}^L \), the set of all possible valuation functions for \( L \).

The set \( L \) is countably infinite (as the set of primitive propositions is finite or countably infinite) and \( |U| = |\{T, F\}^L| = 2^{|L|} = 2^{2^{\aleph_0}} = \mathfrak{c} \), so \( U \) is generally uncountable. (In classical logic, interpretations of formulas are reduced to that of primitive propositions as all the operations are truth-functional, so there are only \( 2^n \) interpretations where \( n \) is the number of primitive propositions.)

Definition 4.1.1 (T-set). The T-set of a formula \( \phi \) is defined as
\[ V_\phi = \{v \in U \mid v(\phi) = T\}. \]

In terms of T-set, the usual definitions of validity/tautology, semantic consequence and equivalence, are defined below.

- A formula \( \phi \) is valid or is a tautology, denoted \( \vdash \phi \), if and only if \( V_\phi = U \).
- A formula \( \psi \) is a semantic consequence of a formula \( \phi \), denoted \( \phi \vdash \psi \), if and only if \( V_\phi \subseteq V_\psi \).
- Formulas \( \phi \) and \( \psi \) are semantically equivalent if \( \phi \vdash \psi \) and \( \psi \vdash \phi \), which is equivalent to \( V_\phi = V_\psi \) by the definition of semantic consequence and the algebra of sets.

Semantics of IRL

The semantics of IRL is defined according to the meaning of operations, via T-set, as follows.

1. For any primitive proposition \( A \): \( V_A \) is given (may be unknown, so undervisible).
2. For the truth-functional operations in \{\top, \bot, \neg, \land, \lor\}:

\[ V_\top = U, \ V_\bot = \emptyset. \]
\[ V_\neg\phi = V_\emptyset. \]
\[ V_{\phi \land \psi} = V_\phi \cap V_\psi, \ V_{\phi \lor \psi} = V_\phi \cup V_\psi. \]

3. For the non-truth-functional bi-implication \(\leftrightarrow\):

\[ V_{\phi \leftrightarrow \phi} = U, \ V_{\phi \leftrightarrow \psi} = V_{\psi \leftrightarrow \phi}, \ V_{\phi \leftrightarrow \psi} \cap V_{\psi \leftrightarrow \chi} \subseteq V_{\phi \leftrightarrow \chi}; \]
\[ V_{\phi \leftrightarrow \psi} \cap V_{\chi} \subseteq V_{\chi[\phi \rightarrow \psi]}; \]
\[ V_{\phi \leftrightarrow \psi} = U \text{ if and only if } V_\phi = V_\psi. \]

4. For the non-truth-functional implication \(\rightarrow\): \(V_{\phi \rightarrow \psi} = V_{\phi \land \psi \rightarrow \phi}.\)

Semantics of bi-implication is given by a collection of characterizing conditions as it is not truth-functional. Implication is defined by bi-implication.

More properties of the operations can be derived from the definition of the semantics. An example is of the tautological implication:

\[ V_{\phi \rightarrow \psi} = U \text{ if and only if } V_\phi \subseteq V_\psi, \]

since \(V_{\phi \land \psi \rightarrow \phi} = U\) is equivalent to \(V_\phi \land \psi = V_\phi \cap V_\psi\), and \(V_\phi \land V_\psi = V_\phi\) is equivalent to \(V_\phi \subseteq V_\psi\) with the algebra of sets. In terms of the original meaning of implication, this result indicates that \(V_\phi \subseteq V_\psi\) is just the “certain mechanism” of the tautological implication from \(\phi\) to \(\psi\) (the scope of the mechanism is \(U\)).

Two special cases of \(V_{\phi \subseteq V_\psi}\) are \(\phi = \bot\) and \(\psi = \top\). Since \(\emptyset \subseteq V_\psi\) and \(V_\phi \subseteq U\), we have \(V_{\bot \rightarrow \psi} = U\) and \(V_{\phi \rightarrow \top} = U\). This indicates that ECQ and its dual do not conflict with the original meaning of implication.

The algebraic system IRLA is developed (cf. Subsection 4.2) with a language more concise than that of the semantic system in T-set notation. Essentially, IRLA is just a representation of the semantic system obtained by replacing T-set notation with a more concise notation, as defined in Table 1.
Table 1: T-set and IRLA representation

| IRL         | T-set       | IRLA                          |
|-------------|-------------|-------------------------------|
| Primitive propositions: | $A, B, ...$  | $V_A, V_B, ...$ | $a, b, ...$ |
| Formulas:   | $\phi, \psi, ...$ | $V_\phi, V_\psi, ...$ | $x, y, ...$ |
| Formula schema: | $\eta(\phi, \psi, ...)$ | $t(x, y, ...)$ |
| Operations other than $\leftrightarrow, \rightarrow$: | $\top, \bot, \neg, \land, \lor, \cup, \setminus$ | $1, 0, \neg, \land, \lor$ |

4.2 Algebra IRLA

The algebra presented in this subsection, called **IRLA**, is an extension of Boolean lattice that is an extension of Boolean algebra.

Let $X$ be the underlying set, $a, b, ...$ constants, $x, y, ...$ variables, and $s, t, ...$ terms of the algebraic language.

**Boolean algebra**

A Boolean algebra $(X, 1, 0, \neg, \land, \lor)$ is defined by the following axioms and rules of inference.

- **Axioms:** commutativity, associativity and distributivity for $\land$ and $\lor$; identity elements for 1 and 0; complement element for $\neg$. Where associativity is redundant [Huntington (1904)].

- **Rules of inference:**
  
  **Primitive:**
  
  1. $s = s$. (Reflexivity)
  2. $\{s \mapsto t, u = v\} \Rightarrow (u = v)[s \mapsto t]$. (Replacement property)

  **Derived:**
  
  4. $s = t \leftrightarrow t = x$. (Symmetry)
  5. $\{s = t, t = u\} \Rightarrow s = u$. (Transitivity)
Remark 4.2.1. Although these rules are actually default for (a variety) of algebras such as Boolean algebra so that they are usually not listed, it is best to specify the set of logical rules especially for an algebraic system that represents a logical system (e.g. Boolean algebra and Boolean lattice), even if these rules may be used tacitly.

Boolean lattice

A Boolean lattice \((X, 1, 0, \neg, \land, \lor, \leq)\) is an extension of Boolean algebra \((X, 1, 0, \neg, \land, \lor)\) by adding the definition of partial order \((\leq)\): \(x \leq y \iff x \land y = x\). The following can be derived:

1. \(s \leq s\).
2. \(\{s \leq t, t \leq s\} \Rightarrow s = t\).
3. \(\{s \leq t, t \leq u\} \Rightarrow s \leq u, \{s = t, t \leq u\} \Rightarrow s \leq u\).
4. \(\{s = t, u \leq v\} \Rightarrow (u \leq v)[s \mapsto v]\).
5. \(s = t \Rightarrow u[s \mapsto t] \leq u, s = t \Rightarrow u \leq u[s \mapsto t]\).

Definition of IRLA

Definition 4.2.1 (Algebra IRLA). The algebraic system \(\text{IRLA}(X, 1, 0, \neg, \land, \lor, \leftrightarrow, \rightarrow, \leq)\) is the extension of Boolean lattice \(\text{BL}(X, 1, 0, \neg, \land, \lor, \leq)\) by defining the new operations \((\leftrightarrow, \rightarrow)\) with the axioms below:

1. \((x \leftrightarrow y) \land z \leq z[x \mapsto y]\). (Replacement property)
2. \(x = y \Rightarrow x \leftrightarrow y = 1\). (Tautological bi-implication)
3. \(x \rightarrow y = x \land y \leftrightarrow x\). (Definition of implication)

4.3 Some theorems of IRLA

Theorem 4.3.1 (Reflexivity of bi-implication in IRLA). \(x \leftrightarrow x = 1\).

Proof.
1. \(x = x\) Reflexivity of \(\equiv\)
2. \(x \leftrightarrow x = 1\) 1: Tautological bi-implication (Axiom 2)

Theorem 4.3.2 (Symmetry of bi-implication in IRLA). \(x \leftrightarrow y = y \leftrightarrow x\).

Proof.
1. \(x \leftrightarrow x = 1\) Reflexivity of \(\leftrightarrow\)
2. \(x \leftrightarrow y = (x \leftrightarrow y) \land 1\) Identity element
3. \(= (x \leftrightarrow y) \land (x \leftrightarrow x)\) 1: Leibniz’s replacement of \(=\)
4. \(\leq (x \leftrightarrow x)[x \mapsto y]\) Replacement property of \(\leftrightarrow\) (Axiom 1)
5. \(= x[x \mapsto y] \leftrightarrow x\) Selected replacement
6. \(= y \leftrightarrow x\) Result of replacement
7. \(x \leftrightarrow y \leq y \leftrightarrow x\) 1–6: Transitivity of \(=\) and \(\leq\)
8. \(y \leftrightarrow x \leq x \leftrightarrow y\) 1–7: similarly
9. \(x \leftrightarrow y = y \leftrightarrow x\) 7,8: Antisymmetry of \(\leq\)

Theorem 4.3.3 (Transitivity of bi-implication in IRLA).

\((x \leftrightarrow y) \land (y \leftrightarrow z) \leq x \leftrightarrow z\).
Proof.
\[(x \leftrightarrow y) \land (y \leftrightarrow z)\]
1. \(= (y \leftrightarrow z) \land (x \leftrightarrow y)\) Commutativity of \(\land\)
2. \(\leq (x \leftrightarrow y)[y \mapsto z]\) Replacement property of \(\leftrightarrow\) (Axiom 1)
3. \(= (x \leftrightarrow z)\) Result of replacement
4. \((x \leftrightarrow y) \land (y \leftrightarrow z) \leq x \leftrightarrow z\) 1–3: Transitivity of = and \(\leq\)

Theorem 4.3.4 (Leibniz's replacement for bi-implication in IRLA).
\(x \leftrightarrow y \leq z \leftrightarrow z[x \mapsto y]\).

Proof.
1. \(z \leftrightarrow z = 1\) Reflexivity
2. \(= (x \leftrightarrow y) \land 1\) Identity element
3. \(= (x \leftrightarrow y) \land (z \leftrightarrow z)\) 1: Replacement property of =
4. \(\leq (z \leftrightarrow z)[x \mapsto y]\) Replacement property of \(\leftrightarrow\) (Axiom 1)
5. \(= z \leftrightarrow z[x \mapsto y]\) Selected replacement
6. \(x \leftrightarrow y \leq z \leftrightarrow z[x \mapsto y]\) 2–5: Transitivity of = and \(\leq\)

Theorem 4.3.5 (Tautological bi-implication and equality).
\(x \leftrightarrow y = 1 \leftrightarrow x \equiv y\).

Proof.
1. \(x \leftrightarrow y = 1\) Premise
2. \(= 1 \land x\) Identity element
3. \(= (x \leftrightarrow y) \land x\) 1: Replacement property of =
4. \(\leq y\) Replacement property of \(\leftrightarrow\) (Axiom 1)
5. \(x \leq y\) 2–4: Transitivity of = and \(\leq\)
6. \(y \leftrightarrow x = x \leftrightarrow y\) Symmetry of \(\leftrightarrow\)
7. \(y \leftrightarrow x = 1\) 6,1: Transitivity of =
8. \(y \leq x\) 7,1–5: similarly
9. \(x = y\) 5,8: Antisymmetry of \(\leq\)
Combined with \(x = y \Rightarrow x \leftrightarrow y = 1\) (Axiom 2) we get \(x \leftrightarrow y = 1 \leftrightarrow x = y\)

Corollary 4.3.1 (Tautological implication and inequality).
\(x \rightarrow y = 1 \leftrightarrow x \leq y\).

Proof. By definition of \(\rightarrow\) (Axiom 3), \(x \rightarrow y = 1 \leftrightarrow x \land y \leftrightarrow x = 1\), by definition of \(\leq\), \(x \leq y \leftrightarrow x \land y = x\). In terms of the above theorem, \(x \land y \leftrightarrow x = 1 \leftrightarrow x \land y = x\), so \(x \rightarrow y = 1 \leftrightarrow x \leq y\).

In terms of Theorem 4.3.5 and Corollary 4.3.1 an IRLA theorem of the form \(s \leftrightarrow t = 1\) or \(s \rightarrow t = 1\) follows immediately from a corresponding theorem of Boolean lattice (BL) of the form \(s = t\) or \(s \leq t\). Examples of such theorems of IRLA and BL are listed in Table 2.
| Name                          | Theorem of IRLA | Theorem of BL |
|-------------------------------|-----------------|--------------|
| Idempotence of $\land$       | $x \land x \leftrightarrow x = 1$ | $x \land x = x$ |
| Idempotence of $\lor$        | $x \lor x \leftrightarrow x = 1$ | $x \lor x = x$ |
| Commutativity of $\land$     | $x \land y \leftrightarrow y \land x = 1$ | $x \land y = y \land x$ |
| Commutativity of $\lor$      | $x \lor y \leftrightarrow y \lor x = 1$ | $x \lor y = y \lor x$ |
| De Morgan’s law 1            | $\neg(x \land y) \leftrightarrow \neg x \lor \neg y = 1$ | $\neg(x \land y) = \neg x \lor \neg y$ |
| De Morgan’s law 2            | $\neg(x \lor y) \leftrightarrow \neg x \land \neg y = 1$ | $\neg(x \lor y) = \neg x \land \neg y$ |
| Reflexivity of implication   | $x \rightarrow x = 1$ | $x \leq x$ |
| Conjunction elimination      | $x \land y \rightarrow x = 1$ | $x \land y \leq x$ |
| Disjunction introduction     | $x \rightarrow x \lor y = 1$ | $x \leq x \lor y$ |
| Boundedness of $\rightarrow$ | $0 \rightarrow x = 1, x \rightarrow 1 = 1$ | $0 \leq x, x \leq 1$ |

**Theorem 4.3.6** (Bi-implication and implication).

$$x \leftrightarrow y = (x \rightarrow y) \land (y \rightarrow x).$$

**Proof.**

1. $x \leftrightarrow y = (x \leftrightarrow y) \land (x \rightarrow x)$
2. $\leq x \rightarrow y$ Replacement property of $\leftrightarrow$ (Axiom 1)
3. $x \leftrightarrow y \leq x \rightarrow y$ 1–2: Transitivity of $=$ and $\leq$
4. $x \leftrightarrow y \leq y \rightarrow x$ 1–3: similarly
5. $x \leftrightarrow y \leq (x \rightarrow y) \land (y \rightarrow x)$ 3,4: One-to-two of $\leq$
6. $(x \rightarrow y) \land (y \rightarrow x)$ Definition of $\rightarrow$
7. $\leq x \leftrightarrow y$ Transitivity of $\leftrightarrow$
8. $(x \rightarrow y) \land (y \rightarrow x) \leq x \leftrightarrow y$ 6–7: Transitivity of $=$ and $\leq$
9. $x \leftrightarrow y = (x \rightarrow y) \land (y \rightarrow x)$ 5,8 Antisymmetry of $\leq$

**Corollary 4.3.2** (Bi-implication elimination in IRLA).

$$x \leftrightarrow y \leq x \rightarrow y.$$

**Proof.** Because $x \leftrightarrow y = (x \rightarrow y) \land (y \rightarrow x) \leq x \rightarrow y$ by conjunction elimination.

**Theorem 4.3.7** (Modus ponens in IRLA).

$$(x \rightarrow y) \land x \leq y.$$

**Proof.**

1. $(x \rightarrow y) \land x = (x \leftrightarrow y) \land (x \rightarrow x)$ Definition of $\rightarrow$
2. $\leq x \land y$ Replacement property of $\leftrightarrow$
3. $\leq y$ Conjunction elimination
4. $(x \rightarrow y) \land x \leq y$ Transitivity of $=$ and $\leq$

**Theorem 4.3.8** (Antisymmetry of implication in IRLA).

$$(x \rightarrow y) \land (y \rightarrow x) \leq x \leftrightarrow y.$$

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Proof.

\[(x \to y) \land (y \to x)\]

1 \(= (x \land y \leftrightarrow x) \land (x \land y \leftrightarrow y)\) Definition of \(\to\)

2 \(\leq x \leftrightarrow y\) Transitivity of \(\leftrightarrow\)

\[\Box\]

**Theorem 4.3.9** (Transitivity of implication in IRLA).

\[(x \to y) \land (y \to z) \leq x \to z.\]

Proof.

\[(x \to y) \land (y \to z)\]

1 \(= (x \land y \leftrightarrow x) \land (x \land y \leftrightarrow y)\) Definition of \(\to\)

2 \(= (x \land y \leftrightarrow x) \land (x \land y \leftrightarrow x) \land (y \land z \leftrightarrow y)\) Idempotence

3 \(\leq (x \land y \leftrightarrow x) \land (x \land (y \land z) \leftrightarrow x)\) Replacement of \(\leftrightarrow\)

4 \(= (x \land y \leftrightarrow x) \land ((x \land y) \land z \leftrightarrow x)\) Associativity

5 \(\leq x \land z \leftrightarrow x\) Replacement of \(\leftrightarrow\)

6 \(= x \to z\) Definition of \(\to\)

7 \((x \to y) \land (y \to z) \leq x \to z\) 1–6: Transitivity of = and \(\leq\)

\[\Box\]

**Theorem 4.3.10** (One-to-two and Two-to-one in IRLA).

\[(z \to x) \land (z \to y) = z \to x \land y.\]

\[(x \to z) \land (y \to z) = x \lor y \to z.\]

Proof.

\[(z \to x) \land (z \to y)\]

1 \(= (x \land z \leftrightarrow z) \land (y \land z \to z)\) Definition of \(\to\)

2 \(\leq x \land y \land z \leftrightarrow z\) Replacement

3 \(= z \to x \land y\) Definition of \(\to\)

4 \((z \to x) \land (z \to y) \leq z \to x \land y\) 1–3: Transitivity of = and \(\leq\)

5 \(z \to x \land y\)

6 \(\leq z \to x\) Transitivity of \(\to\)

7 \(z \to x \land y \leq z \to x\) 5–6: Transitivity of = and \(\leq\)

8 \(z \to x \land y \leq z \to y\) 5–7: similarly

9 \(z \to x \land y \leq (z \to x) \land (z \to y)\) 7,8: One-to-two of \(\leq\)

10 \((z \to x) \land (z \to y) = z \to x \land y\) 4,9: Antisymmetry of \(\leq\)

The second is proved dually.

\[\Box\]

**Theorem 4.3.11** (Maximum of implication).

\[x \to y \leq \neg x \lor y.\]

Proof.

1 \((x \to y) \land x \leq y\) Replacement property of \(\leftrightarrow\)

2 \(x \to y \leq \neg x \lor y\) 1: Law in BL \((x \land y \leq z \leftrightarrow x \leq \neg y \lor z)\)

\[\Box\]

**Theorem 4.3.12** (Tautological implication and disjunction).

\[x \to y = 1 \leftrightarrow \neg x \lor y = 1.\]

Proof. It follows from \(x \to y = 1 \leftrightarrow x \leq y\) of IRLA and \(x \leq y \leftrightarrow \neg x \lor y = 1\) of BL.

\[\Box\]

**Theorem 4.3.13** (Contradictory implication).

\[1 \to 0 = 0.\]

Proof. It follows from \(1 \to 0 \leq \neg1 \lor 0 = 0 \lor 0 = 0\) by **Theorem 4.3.11** and boundedness \(0 \leq 1 \to 0\).

\[\Box\]
4.4 Soundness and completeness

By definition (cf. Subsection 4.1), the T-set of IRL formula schema \( \eta(\phi, \psi, ...) \) is the IRLA term \( t(x, y, ...) \); and \( \models \phi \) is the same as \( x = 1 \), \( \models \phi \leftrightarrow \psi \) is the same as \( x = y \) or \( x \leftrightarrow y = 1 \), and \( \models \phi \rightarrow \psi \) is the same as \( x \leq y \) or \( x \rightarrow y = 1 \).

Soundness of IRL

Meta Theorem 4.4.1 (Axioms are tautologies). All axioms of IRL are tautologies.

Sketch of proof. For each axiom schema \( \eta(\phi, \psi, ...) \) of IRL, verify \( t(x, y, ...) = 1 \) in IRLA. For example, \( \phi \leftrightarrow \phi \) is a tautology since \( x \leftrightarrow x = 1 \) in IRLA.

Meta Theorem 4.4.2 (Primitive rules preserve tautologousness). The three primitive rules of IRL – Conjunction introduction \( \{\phi, \psi\} \Rightarrow \phi \land \psi \), Conjunction elimination \( \phi \land \psi \Rightarrow \phi \) and Replacement property \( \{\phi \leftrightarrow \psi, \chi\} \Rightarrow \chi[\phi \mapsto \psi] \), are tautologousness-preserving, that is, if their premises are tautologies, so are their conclusions.

Proof.

Conjunction introduction \( \{\phi, \psi\} \Rightarrow \phi \land \psi \):  
1. \( \models \phi \) Known  
2. \( \models \psi \) Known  
3. \( x = 1 \) 1: Definition  
4. \( y = 1 \) 2: Definition  
5. \( x \land y = 1 \) 3,4: Replacement  
6. \( = 1 \) Identity element  
7. \( x \land y = 1 \) 5,6: Transitivity  
8. \( \models \phi \land \psi \) 7: Definition

Conjunction elimination \( \phi \land \psi \Rightarrow \phi \):  
1. \( \models \phi \land \psi \) Known  
2. \( x \land y = 1 \) 1: Definition  
3. \( = x \land y \) 2: Symmetry  
4. \( \leq x \) Conjunction elimination  
5. \( 1 \leq x \) 3,4: Transitivity of \( = \) and \( \leq \)  
6. \( x \leq 1 \) Boundedness  
7. \( x = 1 \) 5,6: Antisymmetry of \( \leq \)  
8. \( \models \phi \) 7: Definition

Replacement property \( \{\phi \leftrightarrow \psi, \chi\} \Rightarrow \chi[\phi \mapsto \psi] \):  
1. \( \models \phi \leftrightarrow \psi \) Known  
2. \( \models \chi \) Known  
3. \( x \leftrightarrow y = 1 \) 1: Definition  
4. \( z = 1 \) 2: Definition  
5. \( = 1 \) Identity element  
6. \( = (x \leftrightarrow y) \land z \) 3,4: Replacement  
7. \( \leq z[x \mapsto y] \) Replacement property of \( \leftrightarrow \)  
8. \( \leq 1 \) Boundedness  
9. \( z[x \mapsto y] = 1 \) 5–7–8: Transitivity of \( = \) and \( \leq \), and Antisymmetry of \( \leq \)  
10. \( \models \chi[\phi \mapsto \psi] \) 9: Definition

Meta Theorem 4.4.3 (CP rule preserves tautologousness). Suppose a conditional proof \( \phi \vdash \psi \) of IRL satisfies that if a member of its proof sequence is a theorem then it is a tautology, then \( \phi \rightarrow \psi \) is a tautology. That is, the rule of conditional proof is tautologousness-preserving.
Proof. We prove by induction that, for any integer \( n \geq 1 \), for a conditional proof \( \phi \vdash \psi \) of length \( n \) satisfying that if a member of its proof sequence is a theorem then it is a tautology (let it be called “theorem-is-tautology” condition below), then \( \phi \rightarrow \psi \) is a tautology.

Only the three primitive rules need considered in the meta proof: Conjunction introduction \( \{ \phi, \psi \} \Rightarrow \phi \land \psi \), Conjunction elimination \( \phi \land \psi \Rightarrow \phi \), Replacement property \( \{ \phi \leftrightarrow \psi, \chi \} \Rightarrow \chi[\phi \rightarrow \psi] \).

Let \( \alpha, \beta \) and \( \gamma \) be IRL formulas that exist in the cases discussed below, and their IRLA counterparts be \( p, q \) and \( r \) respectively.

1. For \( n = 1 \): The proof sequence of \( \phi \vdash \psi \) of length 1 is just \((\psi)\). There are following two cases for \( \psi \).
   - Case 1: \( \psi \) is a theorem. In this case, \( \psi \) is a tautology, so \( \phi \rightarrow \psi \) must be a tautology as shown below:
     1. \( \vdash \psi \)  
     2. \( y = 1 \)  
     3. \( x \leq 1 \)  
     4. \( x \leq y \)  
     5. \( \vdash \phi \rightarrow \psi \)
   - Case 2: \( \psi \) is obtained by a one-premise rule. In this case, \( \psi \) can only be obtained by the rule \( \psi \land \alpha \Rightarrow \psi \) where \( \psi \land \alpha = \phi \), so \( \phi \rightarrow \psi \) must be a tautology as shown below:
     1. \( \psi \land \alpha = \phi \)  
     2. \( y \land p = x \)  
     3. \( y \land p \leq y \)  
     4. \( x \leq y \)  
     5. \( \vdash \phi \rightarrow \psi \)

Thus, if \( \phi \vdash \psi \) of length 1 satisfies the “theorem-is-tautology” condition, then \( \phi \rightarrow \psi \) is a tautology.

2. For any \( n \geq 1 \): Suppose that, for any \( k = 1, \ldots, n \), if \( \phi \vdash \psi \) of length \( k \) satisfies the “theorem-is-tautology” condition, then \( \phi \rightarrow \psi \) is a tautology; it need to show that if \( \phi \vdash \psi \) of length \( n + 1 \) satisfies the “theorem-is-tautology” condition, then \( \phi \rightarrow \psi \) is a tautology.

Let \( (\psi_1, \ldots, \psi_n, \psi) \) be a proof sequence of the length-\((n + 1)\) conditional proof \( \phi \vdash \psi \). Then, for any \( \chi \in \{ \phi, \psi_1, \ldots, \psi_n \} \), \( \phi \rightarrow \chi \) must be a tautology as shown below:

Thus this result is used in the following five cases of \( \psi \).

- Case 1: \( \psi \) is a theorem. In this case, we proceed as for \( n = 1 \).
- Case 2: \( \psi \) is obtained by the rule \( \{ \alpha, \beta \} \Rightarrow \alpha \land \beta \) where \( \alpha, \beta \in \{ \phi, \psi_1, \ldots, \psi_n \} \) and \( \alpha \land \beta = \psi \). In this case, \( \phi \rightarrow \alpha \) and \( \phi \rightarrow \beta \) are tautologies, so \( \phi \rightarrow \psi \) must be a tautology as shown below:
1. \( \vdash \phi \rightarrow \alpha \) Known  
2. \( \vdash \phi \rightarrow \beta \) Known  
3. \( \alpha \land \beta = \psi \) Known  
4. \( x \leq p \) \hspace{1cm} 1: Definition  
5. \( x \leq q \) \hspace{1cm} 2: Definition  
6. \( p \land q = y \) \hspace{1cm} 3: Definition  
7. \( x \leq p \land q \) \hspace{1cm} 4,5: One-to-two  
8. \( x \leq y \) \hspace{1cm} 6,7: Replacement  
9. \( \vdash \phi \rightarrow \psi \) \hspace{1cm} 8: Definition  

- Case 3: \( \psi \) is obtained by the rule \( \psi \land \alpha \Rightarrow \psi \) where \( \psi \land \alpha \in \{ \phi, \psi_1, ..., \psi_n \} \). In this case, \( \phi \rightarrow \psi \land \alpha \) is a tautology, so \( \phi \rightarrow \psi \) must be a tautology as shown below:
  1. \( \vdash \phi \rightarrow \psi \land \alpha \) Known  
  2. \( x \leq y \land p \) \hspace{1cm} 1: Definition  
  3. \( y \land p \leq y \) \hspace{1cm} Conjunction elimination  
  4. \( x \leq y \) \hspace{1cm} 2,3: Transitivity of \( \leq \)  
  5. \( \vdash \phi \rightarrow \psi \) \hspace{1cm} 4: Definition  

- Case 4: \( \psi \) is obtained by the rule \( \{ \alpha \leftrightarrow \beta, \gamma \} \Rightarrow \psi \) where \( \alpha \leftrightarrow \beta, \gamma \in \{ \phi, \psi_1, ..., \psi_n \} \) and \( \gamma[\alpha \rightarrow \beta] = \psi \). In this case, \( \phi \rightarrow (\alpha \leftrightarrow \beta) \) and \( \phi \rightarrow \gamma \) are tautologies, so \( \phi \rightarrow \psi \) must be a tautology as shown below:
  1. \( \vdash \phi \rightarrow (\alpha \leftrightarrow \beta) \) Known  
  2. \( \vdash \phi \rightarrow \gamma \) Known  
  3. \( \gamma[\alpha \rightarrow \beta] = \psi \) Known  
  4. \( x \leq p \leftrightarrow q \) \hspace{1cm} 1: Definition  
  5. \( x \leq r \) \hspace{1cm} 2: Definition  
  6. \( r[p \leftrightarrow q] = y \) \hspace{1cm} 3: Definition  
  7. \( x \leq (p \leftrightarrow q) \land r \) \hspace{1cm} 4,5: One-to-two  
  8. \( (p \leftrightarrow q) \land r \leq r[p \rightarrow q] \) Replacement property of \( \leftrightarrow \)  
  9. \( x \leq r[p \rightarrow q] \) \hspace{1cm} 7,8: Transitivity of \( \leq \)  
 10. \( x \leq y \) \hspace{1cm} 6,9: Replacement property of \( = \)  
 11. \( \vdash \phi \rightarrow \psi \) \hspace{1cm} 10: Definition  

Thus, for all \( n \geq 1 \), suppose that for all \( k = 1, ..., n \), any conditional proof \( \phi \vdash \psi \) of length \( k \) satisfying the “theorem-is-tautology” condition results in tautological \( \phi \rightarrow \psi \), then any length-(\( n + 1 \)) conditional proof \( \phi \vdash \psi \) leads to tautological \( \phi \rightarrow \psi \) as well, under the same condition.

Therefore, from 1 and 2 by induction, we have that for any \( n \geq 1 \), if a conditional proof \( \phi \vdash \psi \) of length \( n \) satisfies the “theorem-is-tautology” condition, then \( \phi \rightarrow \psi \) is a tautology. \( \square \)

From Meta Theorem 4.4.1, Meta Theorem 4.4.2 and Meta Theorem 4.4.3, it can be concluded that IRL is sound with regard to its semantics specified by IRLA.

**Completeness of IRL**

In the following two meta theorems, IRLA axiom \( x = y \Rightarrow x \leftrightarrow y = 1 \) (one of the two defining axioms for \( \leftrightarrow \)) is considered to be a primitive rule of IRLA, and the universal rule \( s = s \) (one of the two primitive universal rules in IRLA) is considered to be an axiom of IRLA.

**Meta Theorem 4.4.4.** Any axiom in IRLA has a corresponding theorem in IRL.

**Sketch of proof.**

1. All axioms of Boolean lattice have their counterparts in IRL. For example, IRLA axiom \( x \land y = y \land x \) corresponds to IRL axiom \( \phi \land \phi \leftrightarrow \psi \land \phi \).

2. The IRLA axiom \( (x \leftrightarrow y) \land z \leq z[x \leftrightarrow y] \) (one of the two conditions for defining \( \leftrightarrow \) in IRLA) corresponds to IRL theorem \( (\phi \leftrightarrow \psi) \land \chi \rightarrow \chi[\phi \rightarrow \psi] \) (from primitive rule \( \{ \phi \leftrightarrow \psi, \chi \} \Rightarrow \chi[\phi \rightarrow \psi] \)).
3. The IRLA axiom \( x \rightarrow y = x \land y \leftrightarrow x \) (defining axiom for \( \rightarrow \)) corresponds to IRL axiom \( (\phi \rightarrow \psi) 
leftrightarrow (\phi \land \psi \leftrightarrow \phi) \).

4. The universal one \( s = s \) corresponds to IRL axiom \( \phi \leftrightarrow \phi \).

**Meta Theorem 4.4.5.** For all the primitive rules of IRLA, i.e. \( x = y \Rightarrow x \leftrightarrow y = 1 \) and \( \{ s = t, u = v \} \Rightarrow (u = v)[s \mapsto t] \), if their premises have corresponding theorems in IRL, so are their conclusions.

**Proof.**

1. \( x = y \Rightarrow x \leftrightarrow y = 1 \): For the premise \( x = y \), if \( \phi \leftrightarrow \psi \) is a theorem, then \( (\phi \leftrightarrow \psi) \leftrightarrow \top \) is a theorem by the rule of theorem representation in IRL, and the latter is just the counterpart in IRL for the conclusion \( x \leftrightarrow y = 1 \).

2. \( \{ s = t, u = v \} \Rightarrow (u = v)[s \mapsto t] \): For the premises \( s = t \) and \( u = v \), if \( \eta \leftrightarrow \zeta \) and \( \theta \leftrightarrow \kappa \) are theorems, then \( (\theta \leftrightarrow \kappa)[\eta \mapsto \zeta] \), the counterpart in IRL for the conclusion \( (u = v)[s \mapsto t] \) in IRLA, must be a theorem as shown below.

| Step | Expression | Status |
|------|------------|--------|
| 1    | \( \vdash \eta \leftrightarrow \zeta \) | Known |
| 2    | \( \vdash \theta \leftrightarrow \kappa \) | Known |
| 3    | \( \eta \leftrightarrow \zeta \) | 1 (assumed theorem) |
| 4    | \( \theta \leftrightarrow \kappa \) | 2 (assumed theorem) |
| 5    | \( (\theta \leftrightarrow \kappa)[\eta \mapsto \zeta] \) | 3–5: Replacement |
| 6    | \( \vdash (\theta \leftrightarrow \kappa)[\eta \mapsto \zeta] \) | 3-5: Unconditional proof |

From Meta Theorem 4.4.4 and Meta Theorem 4.4.5, it can be concluded that IRL is complete with regard to its semantics specified by IRLA.

### 4.5 Isomorphism between IRL and IRLA

**Definition 4.5.1** (Isomorphism between two systems). Two proof systems \( A \) and \( B \) (possibly for different formal languages) are isomorphic (to each other), denoted as \( A \cong B \), if the following conditions are met:

1. Expressions of the two systems can be translated to each other. Specifically, there is a bijection between expressions of \( A \) and expressions of \( B \), where “expressions” include not only formulas of object languages but also expressions of meta languages such as that for rules and meta rules.

2. Axioms and primitive rules of \( A \) (after translated to expressions of \( B \)) can be derived in \( B \), and vice versa.

Isomorphism between two systems can be equivalently expressed as they are interpretable in each other in terms of the concept of interpretability (Tarski, 1953).

Two systems \( A \) and \( B \) have the following properties related to isomorphism:

1. If \( A \) and \( B \) are isomorphic, then they are equiconsistent and equidecidable.

2. Suppose \( B \) is the semantic system of \( A \). Then \( A \) and \( B \) are isomorphic if and only if \( A \) is sound and complete with regard to \( B \).

Since IRL is sound and complete with regard to IRLA, there is a natural isomorphism between them as shown in Table 3.
4.6 Truth values of implication

The truth values of an implication \( \phi \rightarrow \psi \) is represented by its T-set \( V_{\phi \rightarrow \psi} \), i.e. \( x \rightarrow y \) in IRL.

Define \( x < y \) as \( x \leq y \) and \( x \neq y \). Then \( 0 < x < 1 \) means that \( x \) (represents) a contingency.

By Theorem 4.3.11 in any cases we have

\[
0 \leq x \rightarrow y \leq \neg x \lor y.
\]

Combined with Theorem 4.3.12 there are three specific cases for the truth values of an implication:

\[
\begin{align*}
\text{if and only if} & & \neg x \lor y = 1 \\
\text{if} & & \neg x \lor y = 0 \text{ (i.e. } x = 1 \text{ and } y = 0) \\
0 \leq x \rightarrow y & & 0 < \neg x \lor y < 1
\end{align*}
\]

Thus, generally \( V_{\phi \rightarrow \psi} \) is a subset (that can be an empty set) of \( V_{\neg \phi \lor \psi} \). Especially in the case where \( \neg \phi \lor \psi \) is contingent, unlike classical logic that always adopts the equality \( V_{\phi \rightarrow \psi} = V_{\neg \phi \lor \psi} \), \( V_{\phi \rightarrow \psi} \) can be any subset of \( V_{\neg \phi \lor \psi} \), i.e. \( V_{\phi \rightarrow \psi} \in 2^{V_{\neg \phi \lor \psi}} \), so there are \( 2^{2^{V_{\neg \phi \lor \psi}}} \) subsets, of which only one is “correct” for \( \phi \rightarrow \psi \). The correct \( V_{\phi \rightarrow \psi} \) is determined by the original meaning of implication – a “certain mechanism” (cf. Subsection 2.2) corresponding to the implication \( \phi \rightarrow \psi \) considered.

**Example 4.6.1.** Let \( A \) and \( B \) are primitive propositions having 4 possible truth-value combinations, i.e. \( U = \{TT, TF, FT, FF\} \). Then \( \emptyset \subset V_{\neg A \lor B} = \{TT, FT, FF\} \subset U \) (so \( \neg A \lor B \) is contingent), therefore \( V_{A \rightarrow B} \in 2^{V_{\neg A \lor B}} = 2^{\{TT, FT, FF\}} \), thus there are \( 2^{2^{\{TT, FT, FF\}}} = 2^3 = 8 \) possibilities and only one of them is correct for \( V_{A \rightarrow B} \).

The next example shows which should be the “correct one” for the T-set of implication \( \phi \rightarrow \psi \) as a subset of the T-set of \( \neg \phi \lor \psi \).

**Example 4.6.2.** Let \( O \) = “in the open air”, \( P \) = “it rains”, and \( Q \) = “the ground gets wet”. Consider the truth values of \( P \rightarrow Q \) = “if it rains, then the ground gets wet”. Suppose: (a) In the open air, whenever it
rains the ground must get wet; (b) “it rains” is not the only cause of “the ground gets wet”, no matter it is in
the open air or not, e.g., if someone sprinkles water coincidentally on the ground (or the floor), the ground
(or the floor) gets wet as well. Then, we obtain the truth values of \( P \rightarrow Q \) in all 7 possible cases (there is
one impossible case that must be excluded: “in the open air, it rains and the ground does not get wet”), as
shown in Table 4.

| O | P | Q | \( \neg P \lor Q \) \(^a\) | \( P \rightarrow Q \) \(^b\) |
|---|---|---|----------------|-----------------|
| T | T | T | T | T |
| T | F | T | T | T |
| T | F | F | T | T |
| F | T | T | T | F |
| F | T | F | F | F |
| F | F | T | T | F |
| F | F | F | T | F |

\(^a\) T-set of \( \neg P \lor Q \) is the set of rows \( \{r_1, r_2, r_3, r_4, r_6, r_7\} \), indicating that it is contin-
gent.

\(^b\) T-set of \( P \rightarrow Q \), determined by \( O \) rather than \( P \) and \( Q \), is \( \{r_1, r_2, r_3\} \), a proper
subset of that of \( \neg P \lor Q \).

In this example, it is the certain mechanism of “in the open air” that corresponds to the inevitable
implication relation of “if it rains, then the ground gets wet”. This is a t ypical example showing that it is
not uncommon that \( \neg \phi \lor \psi \rightarrow (\phi \rightarrow \psi) \) is invalid. The example can explain why the classical logic, with its
implication unreasonably defined by \((\phi \rightarrow \psi) \leftrightarrow \neg \phi \lor \psi\), is doomed to fail in a large number of real situations.

5 Relationship between IRL and CL

5.1 Definitions and notations

In the following, view a proof system as the set of its axioms and primitive rules, and use “statement” to
mean formula or rule expression of a system; view an object and the singleton of the object the same thing.

**Definition 5.1.1.** Let \( A \) be a system. Let \( X \) be a set of statements of the system. Denote with \( \vdash_A X \) that
each member \( x \) of \( X \) is derivable in \( A \) (i.e. \( x \) is a theorem if it is a formula or \( x \) is a derivable rule if it is a
rule expression).

**Definition 5.1.2 (Supper-system and sub-system).** Let \( A \) and \( B \) be systems for a same language.

\( A \) is a supper-system of \( B \), or \( B \) is a sub-system of \( A \), if \( \vdash_A B \).

\( A \) and \( B \) are (deductively) equivalent if \( \vdash_A B \) and \( \vdash_B A \), denoted as \( A \equiv B \).

Note: For systems \( A \) and \( B \) for a same language, \( A \equiv B \) and \( A = B \) are equivalent.

**Definition 5.1.3 (System extension and restriction).** Let \( A \) be a system. Let \( X \) be a set of statements.

An extension of \( A \) with \( X \), denoted as \( A + X \), is the system obtained from \( A \) by adding each member of
\( X \) as an axiom or a primitive rule. And \( A \) is a restriction of \( A + X \) in which \( X \) has been removed.

**Definition 5.1.4 (Provable consequence and equivalence).** Let \( A \) be a system. Let \( X \) and \( Y \) be sets of
statements.

Then \( Y \) is called a provable consequence of \( X \) in \( A \), if \( \vdash_{A+X} Y \).

\( X \) and \( Y \) are provably equivalent in \( A \), if \( \vdash_{A+X} Y \) and \( \vdash_{A+Y} X \).

**Definition 5.1.5 (Provable-equivalence class).** Let \( A \) be a system and \( x \) be a statement.

The provable-equivalence class of \( x \) in \( A \) is defined as \([x]_A = \{ y \mid y \text{ is provably equivalent to } x \text{ in } A \} \).

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Remark 5.1.1. In terms of the above definitions: an extension of a system is a super-system of that system; the deductive equivalent relation “\(\models\)” can be used in a usual way (i.e. it has the properties of reflexivity, symmetry, transitivity and replacement); the extension operator “\(+\)” is commutative and associative. Let A, B and C be sets of statements, any of which can be a system, then:

- A = B + C implies \(\vdash_A B\) and \(\vdash_A C\);
- A = A + B is equivalent to \(\vdash_A B\).

5.2 Two super-systems of IRLA

Definition 5.2.1 (BL0 and BL1). Let BL(\(X, 1, 0, \neg, \wedge, \vee\)) be the Boolean lattice described in Subsection 4.2. BL0(\(X, 1, 0, \neg, \wedge, \vee, \rightarrow, \leftrightarrow, \leq\)) and BL1(\(X, 1, 0, \neg, \wedge, \vee, \rightarrow, \leftrightarrow, \leq\)) are two extensions of BL defined as

\[
BL0 = BL + \{x \rightarrow y = \mu(x, y)\} \text{ where } \mu(x, y) = \begin{cases} 1, & x \leq y \\ 0, & x \not\leq y \end{cases}
\]

\[
BL1 = BL + \{x \rightarrow y = \neg x \vee y\}
\]

respectively, with \(\leftrightarrow\) defined as usual by \(x \leftrightarrow y = (x \rightarrow y) \wedge (y \rightarrow x)\) for both of them.

Remark 5.2.1 (Logics BL0 and BL1 isomorphic to).

- BL0 is isomorphic to classical (propositional) logic (CL), i.e. \(BL0 \cong CL\) in symbols.
- BL0 is isomorphic to a “contra-classical” logic, denoted as CL0, obtained by translating BL0 to a logic in terms of Table 3 so that \(BL0 \cong CL0\). The logic CL0 is contra-classical because the definitions of implication \(x \rightarrow y = \mu(x, y)\) in BL0 and \(x \rightarrow y = \neg x \vee y\) in BL1 are contradictory in the general case (cf. Subsection 4.2.3), hence \(BL0 + BL1 (CL0 + CL)\) is inconsistent.

Lemma 5.2.1 (Consistency of BL0 and BL1). BL0 and BL1 are consistent.

Proof.

- Consistency of BL0. The expression \(\mu(x, y)\) is (indeed) a binary operation in BL, so that new operations \(\rightarrow\) and \(\leftrightarrow\) are well-defined. Therefore, BL0 is an extension by definition, hence a conservative extension of BL, so that BL0 is consistent (relative to BL).

- Consistency of BL1. BL1 and CL are equiconsistent since \(BL1 \cong CL\).

\[
\Delta = \{(x \leftrightarrow y) \wedge z \leq z[x \rightarrow y], \; x = y \Rightarrow x \leftrightarrow y = 1, \; x \rightarrow y = x \wedge y \leftrightarrow x\},
\]

then

\[
IRLA = BL + \Delta
\]

by definition (cf. Subsection 4.2).

Lemma 5.2.2 (BL0 is a super-system of IRLA).

\[\vdash_{BL0} IRLA,\]

i.e. BL0 is a super-system of IRLA.

Proof. Since \(\vdash_{BL0} BL\) and IRLA = BL + \(\Delta\), we need only to prove \(\vdash_{BL0} \Delta\), which is shown below.

1. Proof of \(x = y \Rightarrow x \leftrightarrow y = 1\) and more. By definition:

   If \(x = y\), then \(x \leq y\) and \(y \leq x\), so \(x \rightarrow y = 1\) and \(y \rightarrow x = 1\), thus \(x \leftrightarrow y = (x \rightarrow y) \wedge (y \rightarrow x) = 1 \wedge 1 = 1\).

   If \(x \not= y\), then \(x \not\leq y\) or \(y \not\leq x\), so \(x \rightarrow y = 0\) or \(y \rightarrow x = 0\), thus, let’s say \(y \rightarrow x = 0\), \(x \leftrightarrow y = (x \rightarrow y) \wedge (y \rightarrow x) = (x \rightarrow y) \wedge 0 = 0\).
2. Proof of $x \rightarrow y = x \land y \leftrightarrow x$.

If $x \leq y$ i.e. $x \land y = x$ by definition of $\leq$, then $x \rightarrow y = 1$ and $x \land y \leftrightarrow x = 1$, so it holds.

If $x \not< y$ i.e. $x \land y \neq x$, then $x \rightarrow y = 0$ and $x \land y \leftrightarrow x = 0$, so it also holds.

3. Proof of $(x \leftrightarrow y) \land z \leq z[x \leftrightarrow y]$.

If $x = y$, then $(x \leftrightarrow y) \land z \leq z[x \leftrightarrow y]$ becomes $1 \land z \leq z[x \rightarrow y]$, i.e. $z \leq z$, so it holds.

If $x \neq y$, then $(x \leftrightarrow y) \land z \leq z[x \leftrightarrow y]$ becomes $0 \land z \leq z[x \rightarrow y]$, i.e. $0 \leq z[x \leftrightarrow y]$, so it also holds.

\[ \square \]

**Lemma 5.2.3** (BL1 is a super-system of IRLA).

\[ \vdash_{BL1} IRLA, \]

i.e. BL1 is a super-system of IRLA.

**Proof.** Since $\vdash_{BL1} BL$ and $IRLA = BL + \Delta$, we need only to prove $\vdash_{BL1} \Delta$, which is shown below.

1. Proof of $x = y \Rightarrow x \leftrightarrow y = 1$. If $x = y$, then $x \leftrightarrow y = x \leftrightarrow x = (x \rightarrow x) \land (x \leftarrow x) = x \rightarrow x = \neg x \lor x = 1$.

2. Proof of $x \rightarrow y = x \land y \leftrightarrow x$. We have $x \land y \leftrightarrow x = (x \land y \rightarrow x) \land (x \rightarrow x \land y) = (\neg (x \land y) \lor x) \land (\neg x \lor (x \land y)) = 1 \land (\neg x \lor y) = \neg x \lor y = x \rightarrow y$.

3. Proof sketch of $(x \leftrightarrow y) \land z \leq z[x \leftrightarrow y]$.

   1. Prove $x \leftrightarrow y \leq \neg x \leftrightarrow \neg y$ and $x \leftrightarrow y \leq z \circ x \leftrightarrow z \circ y$ where $\circ \in \{\land, \lor, \leftrightarrow, \rightarrow\}$;
   2. Prove $x \leftrightarrow y \leq z \leftrightarrow z[x \leftrightarrow y]$ from 1 by induction;
   3. Prove $(x \leftrightarrow y) \land x \leq y$;
   4. Prove $(x \leftrightarrow y) \land z \leq z[x \leftrightarrow y]$ from 2 and 3.

\[ \square \]

**IRLA-BL0 and IRLA-BL1 relations**

**Meta Theorem 5.2.1** (IRLA-BL0 and IRLA-BL1 relations).

\[
\begin{align*}
IRLA + \{x \not< y \Rightarrow x \rightarrow y = 0\} &= BL0; \\
IRLA + \{x < y \land x \rightarrow y = y \Rightarrow x \rightarrow y = 0\} &= BL1.
\end{align*}
\]

**Proof.** By definition, $IRLA = BL + \Delta$, $BL0 = BL + \{x \rightarrow y = \mu(x, y)\}$ where $\{x \rightarrow y = \mu(x, y)\} = \{x \leq y \Rightarrow x \rightarrow y = 1, x \not< y \Rightarrow x \rightarrow y = 0\}$, and $BL1 = BL + \{x \rightarrow y = \neg x \lor y\}$ where $\{x \rightarrow y = \neg x \lor y\} = \{x \rightarrow y \leq \neg x \lor y, \neg x \lor y \leq x \rightarrow y\}$.

1. IRLA-BL0 relation. IRLA contains $x \leq y \Rightarrow x \rightarrow y = 1$, i.e. $IRLA = IRLA + \{x \leq y \Rightarrow x \rightarrow y = 1\}$; $\vdash_{BL0} IRLA$, so $BL0 + \Delta = BL0$. Thus

\[
\begin{align*}
IRLA + \{x \not< y \Rightarrow x \rightarrow y = 0\} \\
= IRLA + \{x \leq y \Rightarrow x \rightarrow y = 1\} + \{x \not< y \Rightarrow x \rightarrow y = 0\} \\
= IRLA + \{x \leq y \Rightarrow x \rightarrow y = 1, x \not< y \Rightarrow x \rightarrow y = 0\} \\
= IRLA + \{x \rightarrow y = \mu(x, y)\} \\
= BL + \Delta + \{x \rightarrow y = \mu(x, y)\} \\
= BL0 + \Delta \\
= BL0.
\end{align*}
\]

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2. IRL-Bl1 relation. IRLA contains \( \{x \rightarrow y \leq \neg x \vee y\} \), i.e. \( IRLA = IRLA + \{x \rightarrow y \leq \neg x \vee y\}; \) \( \vdash_{BL1} IRLA \), so \( BL1 + \Delta = BL1 \). Thus

\[
IRLA + \{\neg x \vee y \leq x \rightarrow y\} = IRLA + \{x \rightarrow y \leq \neg x \vee y\} + \{\neg x \vee y \leq x \rightarrow y\} = IRLA + \{x \rightarrow y \leq \neg x \vee y\,\neg x \vee y \leq x \rightarrow y\}
\]

In contrast, Heyting lattice \((HL)\) contains \( \neg x \vee y \leq x \rightarrow y \) but not \( x \rightarrow y \leq \neg x \vee y \), as a matter of fact \( HL + \{x \rightarrow y \leq \neg x \vee y\} = BL1 \).

### 5.3 IRL-CL relationship

**Disjunction-to-Implication** (D2I) is the formula schema \( \neg \phi \lor \psi \rightarrow (\phi \rightarrow \psi) \) in IRL, whose translation is the term \( \neg x \lor y \rightarrow (x \rightarrow y) \) in IRL.

From the facts \( IRL \cong IRLA \), \( CL \cong BL1 \) (cf. Subsection 5.2) and \( IRLA + \{\neg x \lor y \leq x \rightarrow y\} = BL1 \) (cf. Subsection 5.3.1), we have \( IRL + \{\neg \phi \lor \psi \rightarrow (\phi \rightarrow \psi)\} = CL \).

It is shown that \( D2I \) is generally independent of IRL in this section. Since \( D2I \) is a theorem of the consistent CL, \( \neg D2I \) is not derivable in CL, so \( \neg D2I \) is not derivable in IRL either, as \( \vdash_{CL} IRL \). Thus, it needs only to show that \( D2I \) is undervisible in IRL.

**Disjunction-to-Implication is undervisible in IRL**

**Lemma 5.3.1** (D2I is not provable in BL0). In \( BL0(X,1,0,\land,\lor,\rightarrow,\leftrightarrow) \), if there exists \( a \in X \) such that \( a \neq 1 \) and \( a \neq 0 \), then \( \neg x \lor y \leq x \rightarrow y \) is not a theorem of \( BL0 \).

**Proof.** From \( a \neq 1 \), \( 1 \notin a \), so \( 1 \rightarrow a = 0 \) by definition; similarly, from \( a \neq 0 \), \( a \rightarrow 0 = 0 \). Let \( t(x,y) = \neg x \lor y \rightarrow (x \rightarrow y) \). Then \( t(1,a) = \neg 1 \lor a \rightarrow (1 \rightarrow a) = 0 \lor a \rightarrow 0 = a \rightarrow 0 = 0 \neq 1 \), so \( t(x,y) = 1 \) is not a theorem since its instance \( t(1,a) \neq 1 \).

**Lemma 5.3.2** (Condition for D2I being a theorem of IRLA). In \( IRLA(X,1,0,\land,\lor,\leftrightarrow,\rightarrow) \), \( \neg x \lor y \leq (x \rightarrow y) \) is a theorem if and only if there exists \( a \in X \) such that \( a \neq 1 \) and \( a \neq 0 \).

**Proof.** Let \( t(x,y) = \neg x \lor y \rightarrow (x \rightarrow y) \).

Necessity. If there exists \( a \in X \) such that \( a \neq 1 \) and \( a \neq 0 \), then \( t(x,y) = 1 \) is not a theorem of \( BL0 \) in terms of Lemma 5.3.1, neither is it a theorem of IRLA, since \( \vdash_{BL0} IRLA \).

Sufficiency. If there exists \( a \in X \) such that \( a \neq 1 \) and \( a \neq 0 \), then for all \( x,y \in X \), \( x,y \in \{1,0\} \).

In IRLA, \( t(1,1) = \neg 1 \lor 1 \rightarrow (1 \rightarrow 1) = 1 \rightarrow 1 = 1 \), \( t(1,0) = \neg 1 \lor 0 \rightarrow (1 \rightarrow 0) = 0 \rightarrow 0 = 1 \), \( t(0,1) = \neg 0 \lor 1 \rightarrow (0 \rightarrow 1) = 1 \rightarrow 1 = 1 \), and \( t(0,0) = \neg 0 \lor 0 \rightarrow (0 \rightarrow 0) = 1 \rightarrow 1 = 1 \). Thus, for all \( x,y \in X \), \( \neg x \lor y \rightarrow (x \rightarrow y) = 1 \).

### The general case and the no-contingency case

Propositions are in the three types: tautology, contradiction and contingency.

There are two cases according to what types of propositional formulas the system contains, as described below.

**The general case** The system contains all the three types of propositional formulas.

**The no-contingency case** The system contains no contingencies.
Both IRL and CL always contain tautologies (e.g. $\phi \lor \neg \phi$) and contradictions (e.g. $\phi \land \neg \phi$). So the only difference between the general case and the no-contingency is whether or not the system contains a contingency.

For either IRL or CL: if it contains a primitive proposition that is contingent, then, of course, it contains a contingency; if it contains no primitive contingency (i.e. every primitive proposition is either a tautology or a contradiction), then it does not contain any contingencies because the set \{1, 0\} (representing all tautologies and contradictions) is closed under all the operations $\{\neg, \land, \lor, \rightarrow, \leftrightarrow\}$ (e.g., all $1 \rightarrow 1 = 1$, $1 \rightarrow 0 = 0$, $0 \rightarrow 1 = 1$ and $0 \rightarrow 0 = 1$ hold in IRLA).

Thus, for IRL and CL, it is the general case if and only if there exists a primitive contingency in the system, and it is the no-contingency case if and only if there exists no primitive contingency so that any formula is either tautological or contradictory.

For example, suppose the set of primitive propositional symbols (the signature of propositional language) is just the singleton \{A\}, and let $S_A$ be a system of IRL or CL with this signature. Then: (1) $S_A$ contains all the three types of propositions if neither $A$ nor $\neg A$ is an axiom; (2) $S_A$ contains no contingencies (each formula is either a tautology or a contradiction) if either $A$ or $\neg A$ is an axiom.

When not specifying a case in context, we mean it is in the general case.

Summary

Based on the previous results, the logic IRL has the relationship to classical logic (CL) as summarized below.

In any cases (no matter the general case or the no-contingency case):

\[
IRL + \{\neg \phi \lor \psi \rightarrow (\phi \rightarrow \psi)\} = CL,
\]

thus IRL is a sub-system of CL so that IRL is consistent (relative to CL), and the negation of Disjunction-to-Implication $\neg(\neg \phi \lor \psi \rightarrow (\phi \rightarrow \psi))$ is not a theorem of IRL.

- In the general case (i.e. there is a primitive contingency):

\[
IRL + \{\neg \phi \lor \psi \rightarrow (\phi \rightarrow \psi)\} = CL,\quad \text{and} \quad \neg \phi \lor \psi \rightarrow (\phi \rightarrow \psi) \text{ is independent of IRL.}
\]

So IRL is a proper sub-system of CL in the general case.

- In the no-contingency case (i.e. there is no primitive contingency):

\[
IRL = CL, \quad \text{and} \quad \neg \phi \lor \psi \rightarrow (\phi \rightarrow \psi) \text{ is a theorem of IRL.}
\]

Remark 5.3.1. In the no-contingency case, any formula is either tautological or contradictory. However, it does not mean that the system is syntactically complete in this case, because we only know that every primitive proposition is a tautology or contradiction, but we do not know whether it is a tautology or a contradiction, and this is undecidable.

5.4 Provable-equivalence class of Disjunction-to-Implication

Meta Theorem 5.4.1 (Some members of the provable-equivalence class of Disjunction-to-Implication). The following formulas are provably equivalent in IRL.

1. $\phi \rightarrow (\top \rightarrow \phi)$, $\neg \phi \rightarrow (\phi \rightarrow \bot)$, (Minimum of univariate implication)
2. $\phi \rightarrow (\psi \rightarrow \phi)$, $\neg \phi \rightarrow (\phi \rightarrow \psi)$, (Generalized ECQ)
3. $\neg \phi \lor \psi \rightarrow (\phi \rightarrow \psi)$, (Disjunction-to-Implication):
4. $(\phi \land \psi \rightarrow \chi) \rightarrow (\phi \rightarrow (\psi \rightarrow \chi))$, (Exportation)
5. $(\phi \rightarrow (\psi \rightarrow \chi)) \leftrightarrow (\phi \rightarrow (\psi \rightarrow \chi))$, (Commutativity of antecedents)
6. \( \phi \rightarrow ((\phi \rightarrow \psi) \rightarrow \psi) \). (Assertion)

7. \( (\phi \rightarrow \psi) \lor (\psi \rightarrow \phi) \). (Totality)

8. \( (\phi \rightarrow \psi) \lor (\phi \rightarrow \neg \psi) \). (Conditional excluded middle)

9. \( (\top \rightarrow \phi) \lor (\phi \rightarrow \bot) \). (Top-or-bottom)

Note: Each of (1) and (2) is a pair of dual formulas that are provably equivalent in IRL.

Proof. Structure of proofs: 1→2→3→4→5→6→1; 2→7→9→1; 2→8→9.

First group 1→2→3→4→5→6→1:
(1) Minimum of univariate implication → (2) Generalized ECQ:
1 \( \phi \) Premise
2 \( \top \rightarrow \phi \) 1: Minimum of univariate implication
3 \( \phi \leftrightarrow \top \) Dual of ECQ
4 \( \phi \leftrightarrow \top \) 2,3: Bi-implication introduction
6 \( \psi \rightarrow \phi \) 4: Safe generalized ECQ
(2) Generalized ECQ → (3) Disjunction-to-Implication:
1 \( \neg \phi \rightarrow (\phi \rightarrow \psi) \) Generalized ECQ
2 \( \psi \rightarrow (\phi \rightarrow \psi) \) Generalized ECQ
3 \( \neg \phi \lor \psi \rightarrow (\phi \rightarrow \psi) \) 1,2: Two-to-one
(3) Disjunction-to-Implication → (4) Exportation (with D2I, material implication \( \phi \rightarrow \psi \leftrightarrow \neg \phi \lor \psi \) holds):
1 \( \phi \land \psi \rightarrow \chi \) Premise
2 \( \neg (\phi \land \psi) \lor \chi \) 1: Material implication
3 \( \neg (\phi \lor \neg \psi) \lor \chi \) 2: De Morgan’s law
4 \( \neg \phi \lor (\neg \psi \lor \chi) \) 3: Associativity
5 \( \neg \phi \lor (\psi \rightarrow \chi) \) 4: Material implication
6 \( \phi \rightarrow (\psi \rightarrow \chi) \) 5: Material implication
(4) Exportation → (5) Commutativity of antecedents:
1 \( \phi \rightarrow (\psi \rightarrow \chi) \) Premise
2 \( \phi \land \psi \rightarrow \chi \) 1: Importation
3 \( \psi \land \phi \rightarrow \chi \) 2: Commutativity
4 \( \psi \rightarrow (\phi \rightarrow \chi) \) 3: Exportation
(5) Commutativity of antecedents → (6) Assertion:
1 \( (\phi \rightarrow \psi) \rightarrow (\phi \rightarrow \psi) \) Reflexivity
2 \( \phi \rightarrow ((\phi \rightarrow \psi) \rightarrow \psi) \) 1: Commutativity of antecedents
(6) Assertion → (1) Minimum of univariate implication:
1 \( (\phi \rightarrow \phi) \leftrightarrow \top \) Reflexivity
2 \( \phi \rightarrow ((\phi \rightarrow \psi) \rightarrow \phi) \) Assertion
3 \( \phi \rightarrow (\top \rightarrow \phi) \) 1,2: Replacement

Second group 2→7→9→1:
(2) Generalized ECQ → (7) Totality:
1 \( \psi \rightarrow (\phi \rightarrow \psi) \) Generalized ECQ
2 \( \neg \psi \rightarrow (\psi \rightarrow \phi) \) Generalized ECQ
3 \( \psi \lor \neg \psi \rightarrow ((\phi \rightarrow \psi) \lor (\psi \rightarrow \phi) \) 1,2: Monotonicity of \( \lor \) w.r.t \( \rightarrow \)
4 \( \psi \lor \neg \psi \) LEM
5 \( (\phi \rightarrow \psi) \lor (\psi \rightarrow \phi) \) 3,4: Modus ponens
(7) Totality → (9) Top-or-bottom:
1 \( (\neg \phi \lor \phi) \leftrightarrow (\top \rightarrow \phi) \) Theorem 3.4.30
2 \( (\phi \rightarrow \neg \phi) \leftrightarrow (\phi \rightarrow \bot) \) Theorem 3.4.30
3 \( (\neg \phi \lor \phi) \lor (\phi \rightarrow \neg \phi) \) Totality
4 \( (\top \rightarrow \phi) \lor (\phi \rightarrow \bot) \) (1,2),3: Replacement
(9) Top-or-bottom → (1) Minimum of univariate implication:
1. \((\phi \rightarrow \bot) \rightarrow \neg \phi\) \hspace{1cm} Maximum of univariate implication
2. \((\phi \rightarrow \bot) \lor (\top \rightarrow \phi) \rightarrow \neg \phi \lor (\top \rightarrow \phi)\)
1: Compatibility of \(\rightarrow\) with \(\lor\)
3. \((\phi \rightarrow \bot) \lor (\top \rightarrow \phi) \leftrightarrow \top\) \hspace{1cm} Top-or-bottom
4. \(\top \rightarrow \neg \phi \lor (\top \rightarrow \phi)\)
3.2: Replacement
5. \(\phi \rightarrow (\top \rightarrow \phi)\)
4: Definition of implication

Third group 2 \(\rightarrow\) 8 \(\rightarrow\) 9:

(2) Generalized ECQ \(\rightarrow\) (8) Conditional excluded middle:
1. \(\psi \rightarrow (\phi \rightarrow \psi)\) \hspace{1cm} Generalized ECQ
2. \(\neg \psi \rightarrow (\phi \rightarrow \neg \psi)\) \hspace{1cm} Generalized ECQ
3. \(\psi \lor \neg \psi \rightarrow (\phi \rightarrow \psi) \lor (\phi \rightarrow \neg \psi)\)
1.2: Monotonicity of \(\lor\) w.r.t. \(\rightarrow\)
4. \(\psi \lor \neg \psi\) \hspace{1cm} LEM
5. \((\phi \rightarrow \psi) \lor (\phi \rightarrow \neg \psi)\)
3.4: Modus ponens

(8) Conditional excluded middle \(\rightarrow\) (9) Top-or-bottom:
1. \((\top \rightarrow \phi) \lor (\top \rightarrow \neg \phi)\) \hspace{1cm} Conditional excluded middle
2. \((\top \rightarrow \phi) \lor (\phi \rightarrow \bot)\)
1: Contraposition

From known members of the provable-equivalence class of Disjunction-to-Implication \([\neg \phi \lor \psi \rightarrow (\phi \rightarrow \psi)]_{IRL}\) such as formulas (1)–(9) in Meta Theorem 5.4.1, new member(s) of \([\neg \phi \lor \psi \rightarrow (\phi \rightarrow \psi)]_{IRL}\) can be decided by a chain of proofs like known1 \(\rightarrow\) new1 \(\rightarrow\) new2 \(\rightarrow\) known2. The following meta theorem is an example.

**Meta Theorem 5.4.2** (Conjunction-to-equivalence). The formula \(\phi \land \psi \rightarrow (\phi \leftrightarrow \psi)\) (Conjunction-to-equivalence) is a member of \([\neg \phi \lor \psi \rightarrow (\phi \rightarrow \psi)]_{IRL}\).

**Proof.** Chain of proofs:
Generalized ECQ (known) \(\rightarrow\) Conjunction-to-equivalence (new) \(\rightarrow\) Minimum of univariate implication (known).

**Generalized ECQ \(\rightarrow\) Conjunction-to-equivalence:**
1. \(\phi \land \psi\) \hspace{1cm} Premise
2. \(\phi\)
1: Conjunction elimination
3. \(\psi \rightarrow \phi\)
2: Generalized ECQ
4. \(\phi \rightarrow \psi\)
1–3: similarly
5. \(\phi \leftrightarrow \psi\)
3.4: Bi-implication introduction

**Conjunction-to-equivalence \(\rightarrow\) Minimum of univariate implication:**
1. \(\phi\)
Premise
2. \(\top \land \phi\)
1: Identity element
3. \(\top \leftrightarrow \phi\)
2: Conjunction-to-equivalence
4. \(\top \rightarrow \phi\)
3: Bi-implication elimination

All members of \([\neg \phi \lor \psi \rightarrow (\phi \rightarrow \psi)]_{IRL}\) are not provable in IRL, unless the extra assumption of no-contingency is made.

6 Discussion

6.1 The decision problem

In this work, neither a decision procedure nor an undecidability proof for IRL is found.

A conjecture: IRL is undecidable. If IRL is indeed undecidable, it is a reflection of the nature of implication from the original meaning of implication relation.

Examination of (in)validity can be done on a case-by-case basis for a specific of formula (schema). The following facts and ideas are useful for the case-by-case approach.
1. If a formula is valid in IRL, it must be valid in both classical logic CL and the contra-classical logic CL0 (cf. Subsection 5.3 and Remark 5.2.1); hence if a formula is invalid in either CL or CL0, it is invalid in IRL.

2. The (in)validity of a formula in CL can be systematically determined by a classical decision procedure such as truth table and truth tree.

3. Since $IRL \cong IRLA$, $CL0 \cong BL0$, $CL \cong BL1$, both syntactic and semantic derivations can be done. For example, examining the invalidity of a schema $\eta(\phi, \psi)$ is equivalent to examining if $t(x, y) \neq 1$ for the corresponding term.

4. A schema is invalid if an instance of it is invalid. Let $\eta(\phi_1, ..., \phi_n)$ be a schema and $t(x_1, ..., x_n)$ its corresponding term. If there exists $(u_1, ..., u_n)$ such that $t(u_1, ..., u_n) \neq 1$ in IRLA, then $\eta$ is invalid in IRL; if there exists $(u_1, ..., u_n)$ such that $t(u_1, ..., u_n) \neq 0$ in IRLA, then $\neg \eta$ is invalid in IRL.

Example 6.1.1 (Invalidity of Peirce’s law). Peirce’s law $((\phi \rightarrow \psi) \rightarrow \phi) \rightarrow \phi$ is invalid in IRL in the general case.

Let $t(x, y) = ((x \rightarrow y) \rightarrow x) \rightarrow x$.

In the general case, there must be a contingency $a$, so $a \neq 1$ and $a \neq 0$.

In $BL0$, since $a \neq 1$ and $a \neq 0$, $1 \rightarrow a = 0$ and $a \rightarrow 0 = 0$. Hence

$$t(a, 0) = ((a \rightarrow 0) \rightarrow a) \rightarrow a$$

$$= (0 \rightarrow a) \rightarrow a$$

$$= 1 \rightarrow a$$

$$= 0.$$

So, $t(x, y) = 1$ is not a theorem of $BL0$ since its instance $t(a, 0) = 0 \neq 1$. Thus, $t(x, y) = 1$ is not a theorem of $IRLA$ either, as $IRLA$ is a sub-system of $BL0$.

Therefore, Peirce’s law is invalid in IRL.

On the other hand, in IRLA, $t(1, 1) = ((1 \rightarrow 1) \rightarrow 1) = (1 \rightarrow 1) = 1 \rightarrow 1 = 1$, indicating that $t(x, y) = 0$ is not a theorem of IRLA since its instance $t(1, 1) = 1 \neq 0$. In fact, as Peirce’s law is a theorem of CL, its negation can not be a theorem of IRL.

Thus, Peirce’s law, as Disjunction-to-Implication, is independent of IRL. It can be shown that Peirce’s law is a provable consequence of Disjunction-to-Implication, but (very likely) not vice versa.

6.2 Interpretation of Generalized ECQ of implication

Generalized ECQ $\psi \rightarrow (\phi \rightarrow \psi)$ means that, for any propositions $\phi$ and $\psi$, if $\psi$ is true, then there exists a “certain mechanism” of implication from $\phi$ to $\psi$ and $\phi \rightarrow \psi$ is true, conflicting with the original meaning of implication by which implication is non-truth functional.

Whereas Generalized ECQ of implication $(\phi \rightarrow \psi) \rightarrow (\chi \rightarrow (\phi \rightarrow \psi))$ means that if there exists the “certain mechanism” of implication from $\phi$ to $\psi$, then this mechanism is irrelevant to any proposition $\chi$. This does not conflict with the original meaning of implication at all.

The “worst” case for generalized ECQ of implication is when $\chi = \neg(\phi \rightarrow \psi)$. From IRL theorems $(\neg \phi \rightarrow \phi) \leftrightarrow (\top \rightarrow \phi)$ and $(\top \rightarrow \phi) \rightarrow \phi$, we have $(\top \rightarrow (\phi \rightarrow \psi)) \leftrightarrow (\phi \rightarrow \psi)$ in the worst case. This can be understood as a way to interpret $\top \rightarrow (\phi \rightarrow \psi)$ as $\phi \rightarrow \psi$. In contrast, since the general form $(\top \rightarrow \phi) \leftrightarrow \phi$ does not hold (as $\phi \rightarrow (\top \rightarrow \phi)$ does not hold), $\top \rightarrow \phi$ can not be interpreted as $\phi$ generally.

6.3 Interpretation of ECQ

A simple conditional proof of ECQ $\phi \land \neg \phi \rightarrow \psi$ is

1. $\phi$ Premise
2. $\neg \phi$ Premise
3. $\phi \lor \psi$ 1: Disjunction introduction
4. $\psi$ 3,2: Disjunctive syllogism
Hence ECQ is necessary for disjunction introduction and disjunctive syllogism – both do not conflict to
the original meaning of implication. Therefore, ECQ contributes to the goal of strength for the system (cf. Subsection 2.6).

Furthermore, ECQ \( \bot \rightarrow \varphi \) and its dual \( \varphi \rightarrow \top \) can be interpreted to intuition as “if an always-false thing is (was) true, then anything is (would be) true” and “in any cases, an always-true thing is true”, respectively, as similarly mentioned, e.g., in \( [\text{Ceniza 1988}] \).

A “standard (set of rules for) interpretation” of ECQ may be explored for practical purpose. For example, given a conditional \( P \rightarrow Q \): if \( P \) is a contradiction (e.g. it is well known that \( P \) is false), then \( P \rightarrow Q \) can be interpreted as “if the false thing like \( P \) were true, anything including \( Q \) must be true”; if \( Q \) is a tautology (e.g. it is a well known fact), then \( P \rightarrow Q \) can be interpreted as “in any cases including \( P \), an always true thing like \( Q \) is true”.

7 Natural deduction

The natural deduction for IRL is given by the set of rules below, which is equivalent to the Hilbert-style
system for IRL defined in Section 3. Verification of the equivalence is not difficult, but omitted here.

\[
1. (\phi \vdash \psi) \Rightarrow (\vdash \phi \rightarrow \psi) \quad \rightarrow I \quad \text{(Restricted, cf. Remark 7.0.1)}
2. \{\phi \rightarrow \psi, \phi\} \Rightarrow \psi \quad \rightarrow E \quad \text{MP}
3. (\phi \rightarrow \psi) \Rightarrow ((\psi \rightarrow \chi) \rightarrow (\phi \rightarrow \chi)) \quad \text{Suffxing}
4. \{\phi, \psi\} \Rightarrow \phi \land \psi \quad \land I \quad \text{Conjunction}
5. \phi \land \psi \Rightarrow \phi, \phi \land \psi \Rightarrow \psi \quad \land E \quad \text{Simplification}
6. \{\chi \rightarrow \phi, \chi \rightarrow \psi\} \Rightarrow (\chi \rightarrow \phi \land \psi) \quad \text{“One-to-two”}
7. \phi \Rightarrow \chi \land \psi \land \chi \Rightarrow (\phi \land \psi) \lor (\phi \land \chi) \quad \lor I \quad \text{Addition}
8. \{\phi \rightarrow \chi, \psi \rightarrow \chi\} \Rightarrow (\phi \lor \psi \rightarrow \chi) \quad \text{“Two-to-one”}
9. \phi \lor \psi \Rightarrow \phi \land \psi \land \chi \Rightarrow (\phi \lor \psi) \land (\phi \lor \chi) \quad \land E \quad \text{Distribution}
10. \{\phi \rightarrow \psi, \phi \rightarrow \neg \psi\} \Rightarrow \neg \phi \quad \neg I \quad \text{RAA}
11. \{\neg \phi \rightarrow \psi, \neg \phi \rightarrow \neg \psi\} \Rightarrow \phi \quad \neg E \quad \text{Proof by contradiction}
12. \bot \Rightarrow \phi \quad \bot I \quad \text{Dual of ECQ}
13. \phi \Rightarrow \top \quad \top I \quad \text{Dual of ECQ}
14. \{\phi \rightarrow \psi, \psi \rightarrow \phi\} \Rightarrow (\phi \leftrightarrow \psi) \quad \leftrightarrow I \quad \text{Bi-implication introduction}
15. (\phi \leftrightarrow \psi) \Rightarrow (\phi \rightarrow \psi), (\phi \leftrightarrow \psi) \Rightarrow (\psi \rightarrow \phi) \quad \leftrightarrow E \quad \text{Bi-implication elimination}
\]

Remark 7.0.1 (Restriction on CP). By the definition of proof in Subsection 3.1, any member of proof sequence must be either a theorem (or an assumed theorem) or a result from the same-layer premise(s) by a rule (or by an assumed rule), not allowing any other cases. Attention should be paid to this restriction, especially in nested conditional proofs.

As an example, the proof of \( (\phi \lor \psi) \land (\phi \rightarrow \chi) \land (\psi \rightarrow \chi) \rightarrow \chi \) in the natural deduction is shown as follows:

\[
1. (\phi \lor \psi) \land (\phi \rightarrow \chi) \land (\psi \rightarrow \chi) \quad \text{Premise}
2. \phi \lor \psi \quad 1: \land E
3. \phi \rightarrow \chi \quad 1: \land E
4. \psi \rightarrow \chi \quad 1: \land E
5. \phi \lor \psi \rightarrow \chi \quad 3,4: “Two-to-one”
6. \chi \quad 5,2: \rightarrow E
\]

From 1–6 the result is obtained by the CP rule.

8 Extension to first-order logic

Extending IRL to first-order logic (FOL) is similar to what is done classically.

Let \( x, y, ..., s, t, ..., P, Q, ... \) and \( \phi, \psi, ... \) be FOL variables, terms, predicates and formulas, respectively.

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Suppose that in a substitution \( \phi[t/x] \) or a replacement \( \phi[s \mapsto t] \), \( s \) and \( t \) are not bound variables in \( \phi \) (if they are, rename them before substitution or replacement).

The extension of the propositional logic IRL to first-order logic is defined as a FOL that contains:

1. All theorems and rules of IRL where propositional formulas are extended to be any FOL formulas.
2. Axioms for equality:
   - Reflexivity: \( s = s \). Symmetry and transitivity can be derived.
   - Replacement: \( (s = t) \land \phi \rightarrow \phi[s \mapsto t] \).
3. Axioms for quantifiers:
   - Universal generalization: \( \phi(y) \rightarrow \forall x(\phi(x)) \) where \( y \) is arbitrary.
   - Universal instantiation: \( \forall x(\phi(x)) \rightarrow \phi[t/x] \).
   - “One-to-many”: \( \forall x(\phi(x) \rightarrow \psi(x)) \rightarrow (\phi \rightarrow \forall x(\psi(x))) \) where \( x \) does not occur free in \( \phi \). Its converse can be derived.
   - Duality: \( \neg \forall x(\phi(x)) \rightarrow \exists x(\neg \phi(x)) \). Another one \( \neg \exists x(\phi(x)) \rightarrow \forall x(\neg \phi(x)) \) can be derived.
   - Existential instantiation: \( \exists x(\phi(x)) \rightarrow \phi\{c/x\} \) where \( c \) is unknown so that it must not occur earlier in the proof and it also must not occur in the conclusion of the proof.
   - Existential generalization (redundant): \( \phi\{t/x\} \rightarrow \exists x(\phi(x)) \).
   - “Many-to-one” (redundant): \( \forall x(\phi(x) \rightarrow \psi(x)) \rightarrow (\exists x(\phi(x)) \rightarrow \psi(x)) \) where \( x \) does not occur free in \( \psi \). Its converse can be derived.

As an example of proofs, consider quantifier switching \( \exists x \forall y(\phi(x, y)) \rightarrow \forall y \exists x(\phi(x, y)) \) that is equivalent to \( \exists x \forall y(\phi(x, y)) \land \neg \forall y \exists x(\phi(x, y)) \rightarrow \bot \). The proof of the latter is shown below.

\[
\begin{align*}
1. & \quad \exists x \forall y(\phi(x, y)) \land \neg \forall y \exists x(\phi(x, y)) & \text{Premise} \\
2. & \quad \exists x \forall y(\phi(x, y)) & \text{1: Conjunction elimination} \\
3. & \quad \forall y(\phi(a, y)) & \text{2: Existential instantiation} \\
4. & \quad \neg \forall y \exists x(\phi(x, y)) & \text{1: Conjunction elimination} \\
5. & \quad \exists y \forall x(\neg \phi(x, y)) & \text{4: Duality} \\
6. & \quad \forall x(\neg \phi(x, b)) & \text{5: Existential instantiation} \\
7. & \quad \phi(a, b) & \text{3: Universal instantiation} \\
8. & \quad \neg \phi(a, b) & \text{6: Universal instantiation} \\
9. & \quad \bot & \text{7,8: LNC}
\end{align*}
\]

It is convenient if including bounded quantifiers or restricted quantifiers in a first-order language.

**Definition 8.0.1 (Bounded quantifiers).** Let \( \chi \) be a unary formula, \( \phi \) be any formula. Let \( X = \{x \mid \chi(x)\} \). Then \( x \in X \) if and only if \( \chi(x) \). The bounded quantifiers \( \exists x \in X \) and \( \forall x \in X \) are defined by the rules:

- \( \exists x \in X(\phi) \iff \exists x(\chi(x) \land \phi) \).
- \( \forall x \in X = \neg(\exists x \in X) \neg, \) or equivalently, \( \forall x \in X(\phi) \iff \forall x(\neg \chi(x) \lor \phi) \).

**Remark 8.0.1.** The universal bounded quantifier \( \forall x \in X \) can not be defined by \( \forall x \in X(\phi) \iff \forall x(\chi(x) \rightarrow \phi) \) since \( \chi(x) \rightarrow \phi \) and \( \neg \chi(x) \lor \phi \) are not equivalent – however, if they are theorems (they are theorems if one of them is a theorem), then they are equivalent.

### 9 Concluding remarks

According to the original meaning of implication, the formula \( \neg \phi \lor \psi \rightarrow (\phi \rightarrow \psi) \) is not a tautology in general, so that the equivalence \( (\phi \rightarrow \psi) \iff \neg \phi \lor \psi \), on which the material implication of classical logic is based, is generally invalid. Defining implication in accordance with its original meaning is the key to avoid invalid results in logic. The logic IRL presented in this work has the expected property that the invalid theorem
\neg \phi \lor \psi \rightarrow (\phi \rightarrow \psi)\) of classical logic is removed while fundamental laws in classical logic, such as LEM and double negation, LNC and ECQ, conjunction elimination and disjunction introduction, and transitivity of implication and disjunctive syllogism, are all retained in IRL.

Based on information provided with IRL, more points are summarized as follows.

1. Minimal functionally complete operator sets. (a) \((\phi \rightarrow \psi) \leftrightarrow \neg \phi \lor \psi\) is invalid, so \(\rightarrow\) and \(\leftrightarrow\) cannot be defined by \{\neg, \land, \lor\}, meaning that one of \(\rightarrow\) and \(\leftrightarrow\) must be primitive; (b) \(\neg \phi \leftrightarrow (\phi \rightarrow \bot)\) is invalid, so \(\neg\) cannot be defined by \{\bot, \rightarrow\}, meaning that \(\neg\) must be primitive; (c) for the same reason as in (a), one of \(\land\) and \(\lor\) must be primitive. Thus, a functionally complete operator set contains at least three members, e.g. \{\neg, \land, \rightarrow\}, \{\neg, \lor, \leftrightarrow\}, \{\neg, \land, \leftrightarrow\}.

2. Truth value of implication. Since implication is not truth functional, valuation of \(\phi \rightarrow \psi\) can not be generally done by valuation of \(\phi\) and \(\psi\), instead, it can be “estimated” via the procedure: (a) \(\phi \rightarrow \psi\) is a tautology, if and only if \(\neg \phi \lor \psi\) is a tautology; (b) \(\phi \rightarrow \psi\) is a contradiction, if \(\neg \phi \lor \psi\) is a contradiction (i.e. \(\phi\) is a tautology and \(\psi\) is a contradiction); (c) the T-set of \(\phi \rightarrow \psi\) is a subset (possibly empty) of the T-set of \(\neg \phi \lor \psi\), if \(\neg \phi \lor \psi\) is a contingency. T-sets of primitive implications, like any primitive propositions, are given (e.g. by physical mechanisms) rather than derived.

3. Classical logic is sound and complete with regard to its own semantics. In the sense that it contains theorems that are not generally valid with regard to semantics based on the original meaning of implication, classical logic is less “sound” than “complete” (actually it is post-complete), while some weaker systems are in the opposite.

4. The logic IRL of this work, in which the generally invalid classical theorems – including especially the class of \(\neg \phi \lor \psi \rightarrow (\phi \rightarrow \psi)\) – are removed, should be used instead of classical logic for the general case (the system contains any types of propositions including especially contingencies). IRL and classical logic are the same if and only if there is no contingent primitive propositions in a system considered.

5. When an interpretation or model is given for a formal system, all statements of that system are not contingent (actually, an interpreted statement is either true or false, irrelevant to “tautology, contradiction or contingency”). It is common that a formal theory has an intended interpretation or standard model or usual model. For example, Tarski’s axioms (for Euclidean geometry) has its standard model of Euclidean plane geometry, and the standard model of Peano axioms is natural numbers. In such situations where there are no contingent primitive sentences, classical logic can be used safely.

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