Integrable nonlinear equations on a circle

Metin Gürses\textsuperscript{a}, Ismagil Habibullin\textsuperscript{b} and Kostyantyn Zheltukhin\textsuperscript{c}

\textsuperscript{a}Department of Mathematics, Faculty of Sciences
Bilkent University, 06800 Ankara, Turkey
e-mail gurses@fen.bilkent.edu.tr

\textsuperscript{b}Department of Mathematics, Faculty of Sciences
Bilkent University, 06800 Ankara, Turkey
e-mail habib@fen.bilkent.edu.tr

\textsuperscript{c}Department of Mathematics, Faculty of Sciences
Middle East Technical University 06531 Ankara, Turkey
e-mail zheltukh@metu.edu.tr

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Abstract

The concept of integrable boundary value problems for soliton equations on $\mathbb{R}$ and $\mathbb{R}_+$ is extended to bounded regions enclosed by smooth curves. Classes of integrable boundary conditions on a circle for the Toda lattice and its reductions are found.

\textbf{PASC:}

\textbf{Keywords:} integrable boundary conditions, Toda lattice, Dirichle and Neumann problems, Lax pair, Liouville equation.

\footnote{e-mail: habibullin_i@mail.rb.ru, (On leave from Ufa Institute of Mathematics, Russian Academy of Science, Chernyshevskii Str., 112, Ufa, 450077, Russia)}
1 Introduction

The inverse scattering transform method (ISM) discovered in 1967 has proved to be a powerful tool to construct exact solutions and to solve the Cauchy problem for a large variety of nonlinear integrable models of mathematical physics. But real physical applications are usually related to mathematical models with boundary conditions. For this reason the problem of adopting the ISM to boundary value problem as well as to initial boundary value (mixed) problem is very important. During the last two decades this field of research has been intensively studied. It becomes clear that only special kinds of boundary conditions preserve the integrability property of the equation given. Different approaches were worked out to look for such classes of boundary conditions based on Hamiltonian structures [1], on higher symmetries [2], [3], [4], and the Lax representation [5], [6]. Integrable initial boundary value problems on a half-line (in 1+1 case) or a half-plane (in 1+2 case) for soliton equations nowadays is a rather studied subject. Analytical aspects have been developed in [7], [8], [9], [10], [11] where large classes of solutions were constructed. However, boundary value problem for the elliptic soliton equations or initial boundary value problem for regions with more complicated boundary are still much less investigated (see, [12], [13], [14]).

If the boundary conditions are not consistent with the integrability property of the equation then the standard version of the inverse scattering transform method cannot be applied to the corresponding boundary value problem. The method requires very essential modification. Various ideas to extend the ISM to the initial boundary value problems are suggested in [12], [15], [16], [17].

In [5], [6] an effective tool to search integrable boundary conditions has been proposed based on some special involutions of the auxiliary linear problem. This method (below for the sake of convenience we refer it as method of involutions) can be applied to integrable equations in both 1+1 and 1+2-dimensional cases. In this article we show that the method of involutions allows one to extend the concept of integrability to boundary value problems on bounded regions enclosed by any closed smooth curve.

Let us explain briefly the approach we use. We call boundary value problem integrable if it admits a Lax pair. Because of this reason we look for boundary condition simultaneously with its Lax representation. The starting point is to make a correct assumption about the possible form of the Lax pair of the boundary value problem. Actually this Lax pair is made
up from several different Lax pairs of the original equation itself by gluing the eigenfunctions along the boundary by properly chosen additional boundary conditions. As examples we take Liouville equation and two-dimensional Toda lattice equation. To generate new Lax pairs we use point symmetries (involutions) which leave invariant the nonlinear equation under consideration but change its Lax pair.

In the next section, as a trial example, we consider the Liouville equation. We remind the definition of integrable boundary conditions and find an example of integrable boundary conditions on a circle. It is shown that the following nonhomogeneous Neumann problem

$$u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 8 e^u, \quad (1)$$
$$u_r|_{r=a} = -\frac{2}{a}$$

is integrable, i.e. it admits a Lax pair (see the list at the end of the second section).

In the third section we study the two-dimensional Toda lattice equation on a circular cylinder: $r < a$, $0 \leq \theta \leq 2\pi$, $-\infty < n < \infty$. Several types of integrable boundary value problems for this lattice are found by using the method of involutions. Let $\omega(n) = \exp\{u(n) - u(n + 1)\}$. It is shown that the following boundary values problems are integrable:

$$u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = \omega(n - 1) - \omega(n), \quad (2)$$
$$u(n)|_{r=a} = 2i\tilde{\theta} + g(\theta) + k(n),$$

$$u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = \omega(n - 1) - \omega(n), \quad (3)$$
$$u_r(n)|_{r=a} = \frac{2n}{a} + g(\theta),$$

$$u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = \omega(n - 1) - \omega(n), \quad (4)$$
$$(u_r(n) + \frac{i}{a} u_\theta(n))|_{r=a} = g(\theta),$$

It is remarkable that the boundary conditions contain arbitrary functions $g(\theta)$ and $k(n)$. The Lax pairs for the above integrable boundary value problems are given in the list at the end of the third section.
In the fourth section we consider periodicity closure constraints reducing Toda lattice to the sinh-Gordon and Tcsitcsikica equations. It is shown that the following boundary values problems are integrable:

for the sinh-Gordon equation

\[ p_{rr} + \frac{1}{r} p_r + \frac{1}{r^2} p_{\theta \theta} = 4 \sinh p, \]

\[ (p_r + \frac{i}{a} p_\theta)|_{r=a} = 0; \quad (5) \]

for the Tcsitcsikica equation

\[ q_{rr} + \frac{1}{r} q_r + \frac{1}{r^2} q_{\theta \theta} = e^{2q} - e^{-q} \]

\[ (q_r + \frac{i}{a} q_\theta)|_{r=a} = 0. \quad (6) \]

In the fifth section we give a class of exact solutions of the Toda lattice on a circle with nonhomogeneous Neumann type boundary condition on a circle.

## 2 Liouville equation

In this section we concentrate on boundary value problems for elliptic equations. Suppose that the boundary \( \Gamma \) of a domain \( D \) is parameterized by the equation \( x' = f(t') \) that introduces a local system of coordinates by taking \( t \)-axis along the tangent direction and \( x \)-axis along normal to the curve \( \Gamma \).

Suppose that the differential equation under consideration

\[ E(u) = 0 \]

admits two different Lax representations. For the sake of simplicity we take them rewritten in terms of the new coordinates

\[ Y_x = U(\lambda, u, u_x, \ldots) Y(\lambda) \]
\[ Y_t = V(\lambda, u, u_x, \ldots) Y(\lambda) \]

and

\[ \tilde{Y}_x = \tilde{U}(\tilde{\lambda}, u, u_x, \ldots) \tilde{Y}(\tilde{\lambda}) \]
\[ \tilde{Y}_t = \tilde{V}(\tilde{\lambda}, u, u_x, \ldots) \tilde{Y}(\tilde{\lambda}). \]
where $\lambda, \tilde{\lambda}$ are spectral parameters. Now the equation of the boundary is of the form $x = 0$. We are looking for conditions that allow to relate the equations for $t$ evolution along the boundary, since $x$ is fixed. More precisely, we have the following definition

**Definition 1** A boundary condition

$$\Omega(t, u, u_t, u_x, \ldots) = 0$$  \hspace{1cm} (10)

is integrable if there exists a matrix $F(\lambda, t, u, \ldots)$ and function $h(\lambda)$ such that on the boundary $x = 0$ the function $Y = F(\lambda, t, u, \ldots) \tilde{Y}(\tilde{\lambda})$ is a solution of the equation $Y_t = VY$ for any solution $\tilde{Y}$ of the equation $\tilde{Y}_t = \tilde{V}\tilde{Y}$ with $\tilde{\lambda} = h(\lambda)$, provided the boundary condition holds.

If a boundary condition is integrable in the sense of the definition above this means that the corresponding boundary value problem admits the Lax representation consisting of the two Lax pairs (8) and (9) defined on the domain $D$ such that the eigenfunctions $Y$ and $\tilde{Y}$ satisfy along the boundary an additional boundary condition $(Y - F\tilde{Y})|_\Gamma = 0$.

To consider a circle as a boundary we use polar coordinates $(r, \theta)$. So, the boundary is $r = a$. In polar coordinates the Liouville equation is

$$u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 8e^u.$$  \hspace{1cm} (11)

It admits the Lax pair

$$Y_r = LY, \quad Y_\theta = AY,$$  \hspace{1cm} (12)

where $x = r, \ t = \theta, \ U = L, \ V = A$ and

$$L = \begin{pmatrix}
\frac{e^{u+i\theta}}{2\lambda} + \lambda e^{-i\theta} & -\frac{e^{u+i\theta}}{2\lambda} + \frac{1}{4}u_r + \frac{i}{4r}u_\theta \\
\frac{e^{u+i\theta}}{2\lambda} + \frac{1}{4}u_r + \frac{i}{4r}u_\theta & -\frac{e^{u+i\theta}}{2\lambda} - \lambda e^{-i\theta}
\end{pmatrix}$$  \hspace{1cm} (13)

$$A = ir \begin{pmatrix}
\frac{e^{u+i\theta}}{2\lambda} - \lambda e^{-i\theta} & -\frac{e^{u+i\theta}}{2\lambda} - \frac{1}{4}u_r - \frac{i}{4r}u_\theta \\
\frac{e^{u+i\theta}}{2\lambda} - \frac{1}{4}u_r - \frac{i}{4r}u_\theta & -\frac{e^{u+i\theta}}{2\lambda} + \lambda e^{-i\theta}
\end{pmatrix}$$  \hspace{1cm} (14)
To obtain a second Lax representation we use the Kelvin transformation. The equation (11) is invariant under the Kelvin transformation

$$\bar{r} = \frac{a^2}{r}, \quad \bar{u} = u + 4 \ln \frac{a}{r}. \quad (15)$$

Under such transformation the Lax pair (12) takes form

$$\bar{Y}_r = \bar{L} \bar{Y}, \quad \bar{Y}_\theta = \bar{A} \bar{Y}, \quad (16)$$

where

$$\bar{L} = \begin{pmatrix}
\frac{r^4 e^{u+i\theta}}{2a^4 \lambda} + \bar{\lambda} e^{-i\theta} & -\frac{r^4 e^{u+i\theta}}{2a^4 \lambda} - \frac{r^2 u_r}{4a^2} - \frac{r}{a^2} + \frac{i r}{4a^2} u_\theta \\
\frac{r^4 e^{u+i\theta}}{2a^2 \lambda} - \frac{r^2 u_r}{4a^2} - \frac{r}{a^2} + \frac{i r}{4a^2} u_\theta & -\frac{r^4 e^{u+i\theta}}{2a^4 \lambda} - \bar{\lambda} e^{-i\theta}
\end{pmatrix} \quad (17)$$

$$\bar{A} = \frac{ia^2}{r} \begin{pmatrix}
\frac{r^4 e^{u+i\theta}}{2a^4 \lambda} - \bar{\lambda} e^{-i\theta} & -\frac{r^4 e^{u+i\theta}}{2a^4 \lambda} + \frac{r^2 u_r}{4a^2} + \frac{r}{a^2} - \frac{i r u_\theta}{4a^2} \\
\frac{r^4 e^{u+i\theta}}{2a^4 \lambda} + \frac{r^2 u_r}{4a^2} + \frac{r}{a^2} - \frac{i r u_\theta}{4a^2} & -\frac{r^4 e^{u+i\theta}}{2a^4 \lambda} + \bar{\lambda} e^{-i\theta}
\end{pmatrix} \quad (18)$$

We will look for a boundary condition under which there exist a transformation $\bar{\lambda} = h(\lambda)$ and non degenerate matrix $F(\lambda, \theta, u)$ such that $Y(\lambda) = F\bar{Y}(h(\lambda))$ will solve the $Y_\theta(\lambda) = A|_{r=a} Y(\lambda)$ for every solution $\bar{Y}(\lambda)$ of the equation $\bar{Y}_\theta(\lambda) = \bar{A}|_{r=a} \bar{Y}(\lambda)$.

**Lemma 1** The integrable boundary condition is given by

$$u_r|_{r=a} = \frac{-2}{a} \quad (19)$$

and there are two choices for the matrix $F$ and the function $h$

(i) $h(\lambda) = \lambda, \quad F = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad (20)$

(ii) $h(\lambda) = -\lambda, \quad F = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \quad (21)$
Proof. Let $\bar{Y}(\lambda)$ satisfies the equation $\bar{Y}_\theta = \bar{A}\bar{Y}$. On the boundary $r = a$, $Y = F\bar{Y}(h(\lambda))$ has to satisfy $Y_\theta = AY$. Substituting $Y = F\bar{Y}(h(\lambda))$ into $Y_\theta = AY$ and using $\bar{Y}_\theta = \bar{A}\bar{Y}$ for $\bar{Y}(h(\lambda))$ we obtain
\[
\left( \frac{d}{d\theta} F - A(\lambda)F + F\bar{A}(h(\lambda)) \right) \bar{Y}(h(\lambda)) = 0
\] (22)

The above equality holds if
\[
\frac{d}{d\theta} F = A(\lambda)F - F\bar{A}(h(\lambda))
\] (23)

We have an equation for the unknown matrix $F$ and function $h(\lambda)$. To solve the boundary condition (10) with respect to $u_r$ we let $u_r = G(\theta, u, u_\theta)$. Assuming that $F$ does not depend on $u_\theta$ and differentiating (23) twice with respect to $u_\theta$ we obtain $\frac{\partial^2 u_r}{\partial u_\theta^2} = 0$. That is, $u_r = g_1(u, \theta)u_\theta + g_2(u, \theta)$. We substitute the above expression for $u_r$ into (23) and let
\[
F = \begin{pmatrix} f_{11}(u, \lambda, \theta) & f_{12}(u, \lambda, \theta) \\ f_{21}(u, \lambda, \theta) & f_{22}(u, \lambda, \theta) \end{pmatrix}.
\] (24)

Separating terms with $u_\theta$ and without $u_\theta$ in (23) we obtain two sets of equations. We write the first set of equations, terms with $u_\theta$, as
\[
\frac{\partial}{\partial u} f = Pf,
\] (25)

where $f = (f_{11}, f_{12}, f_{21}, f_{22})^T$ and $P$ is a matrix
\[
\begin{pmatrix}
0 & -\left( \frac{a_1}{4} - \frac{i}{4a} \right) & \left( -\frac{a_1}{4} - \frac{i}{4a} \right) & 0 \\
-\left( \frac{a_1}{4} - \frac{i}{4a} \right) & 0 & 0 & \left( -\frac{a_1}{4} - \frac{i}{4a} \right) \\
\left( -\frac{a_1}{4} - \frac{i}{4a} \right) & 0 & 0 & -\left( \frac{a_1}{4} - \frac{i}{4a} \right) \\
0 & \left( \frac{a_1}{4} - \frac{i}{4a} \right) & \left( -\frac{a_1}{4} - \frac{i}{4a} \right) & 0
\end{pmatrix}
\] (26)

We write the second set of equations, terms without $u_\theta$, as
\[
\frac{\partial}{\partial \theta} f = Qf,
\] (27)
where \( Q \) is a matrix

\[
\begin{pmatrix}
\mu - \nu & -\left( \delta + \frac{e^{u+i\theta}}{2h(\lambda)} \right) & \left( -\frac{g_2}{4} - \frac{e^{u+i\theta}}{2\lambda} \right) & 0 \\
-\left( \delta - \frac{e^{u+i\theta}}{2h(\lambda)} \right) & \mu + \nu & 0 & \left( -\frac{g_2}{4} - \frac{e^{u+i\theta}}{2\lambda} \right) \\
\left( \frac{e^{u+i\theta}}{2\lambda} - \frac{g_2}{4} \right) & 0 & -\mu - \nu & \left( \delta + \frac{e^{u+i\theta}}{2h(\lambda)} \right) \\
0 & \left( \frac{e^{u+i\theta}}{2\lambda} - \frac{g_2}{4} \right) & -\left( \delta - \frac{e^{u+i\theta}}{2h(\lambda)} \right) & -\mu + \nu
\end{pmatrix}
\] (28)

with \( \mu = \frac{e^{u+i\theta}}{2\lambda} - \lambda e^{-i\theta}, \nu = \frac{e^{u+i\theta}}{2h(\lambda)} - h(\lambda)e^{-i\theta} \) and \( \delta = \frac{g_2}{4} + \frac{1}{a} \).

The equations (25) and (27) must be compatible. This leads to the following compatibility condition

\[
(P_\theta - Q_u + [P, Q]) f = 0,
\] (29)

where \([P, Q]\) is a commutator of \( P \) and \( Q \). The matrix \((P_\theta - Q_u + [P, Q])\) is nonzero. To have nonzero solution \( f \) the determinant of \((P_\theta - Q_u + [P, Q])\) must be zero. It gives the following equality

\[
\frac{a^6 e^{-4i\theta}}{16} \left( h^2(\lambda)(ag_1(u, \theta) - i)^2 - (ag_1(u, \theta) + i)^2 \lambda^2 \right)^2 = 0.
\] (30)

The above equality holds if either

(1) \( h(\lambda) = \beta \lambda \) and \( g_1 = \frac{\beta(1+\beta)}{a(1-\beta)} \) where \( \beta \in \mathbb{R}\{\{-1, 1\}\} \) or

(2) \( h(\lambda) = \lambda \) and \( g_1 = 0 \) or

(3) \( h(\lambda) = -\lambda \) and \( g_1 = 0 \).

One can show that in the case (1) there is no vector \( f \) to satisfy equations (25) and (27). In the case (2) one has the only solution \( f = q(\lambda)(1, 0, 0, 1)^T \) if \( g_2 = \frac{-a}{2} \). That gives the boundary condition (19), function \( h \) and matrix \( F \) given by (20).

The case (3) is similar to the case (2) and gives same boundary condition (19), function \( h \) and matrix \( F \) given by (21).

□

From the above lemma we have the following integrable boundary value problem with corresponding Lax pairs (we have two Lax pairs for the problem).
\[ \begin{align*}
\bullet \quad r < a & \quad u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 8e^u, \\
& \quad Y_r(\lambda) = L(\lambda)Y(\lambda), \quad \text{and} \quad \bar{Y}_r(\lambda) = \bar{L}(\lambda)\bar{Y}(\lambda), \\
& \quad Y_\theta(\lambda) = A(\lambda)Y(\lambda), \quad \text{and} \quad \bar{Y}_\theta(\lambda) = \bar{A}(\lambda)\bar{Y}(\lambda), \\
nr = a & \quad u_r|_{r=a} = -\frac{2}{a}, \\
& \quad Y(\lambda) = F\bar{Y}(-\lambda) \\
\end{align*} \]

where \( F \) is given by (20), \( L \) is given by (13), \( A \) is given by (14) and \( \bar{L} \) is given by (17), \( \bar{A} \) is given by (18).

\[ \begin{align*}
\bullet \quad r < a & \quad u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 8e^u, \\
& \quad Y_r(\lambda) = L(\lambda)Y(\lambda), \quad \text{and} \quad \bar{Y}_r(\lambda) = \bar{L}(\lambda)\bar{Y}(\lambda), \\
& \quad Y_\theta(\lambda) = A(\lambda)Y(\lambda), \quad \text{and} \quad \bar{Y}_\theta(\lambda) = \bar{A}(\lambda)\bar{Y}(\lambda), \\
nr = a & \quad u_r|_{r=a} = -\frac{2}{a}, \\
& \quad Y(\lambda) = F\bar{Y}(-\lambda) \\
\end{align*} \]

where \( F \) is given by (21), \( L \) is given by (13), \( A \) is given by (14) and \( \bar{L} \) is given by (17), \( \bar{A} \) is given by (18).

3 Two-dimensional Toda Lattice

We make the same assumption, as in the case of the Liouville equation, for the coordinates. Hence, boundary is given by \( x = 0 \). Again we suppose that the differential equation under consideration admits two different Lax representations

\[ \begin{align*}
Y_x &= UY \\
Y_t &= VY \\
\bar{Y}_x &= \bar{U}\bar{Y} \\
\bar{Y}_t &= \bar{V}\bar{Y}
\end{align*} \]  

For the two dimensional Toda lattice equation \( U, V, \bar{U} \) and \( \bar{V} \) in (31) are linear operators.

**Definition 2** A boundary condition

\[ \Omega(u) = 0 \]  

is integrable if there exists a linear differential operator \( A \) such that on the boundary \( x = 0 \) we have \( \bar{Y} = AY \) is a solution of \( \bar{Y}_t = \bar{V}\bar{Y} \) for any solution \( Y \) of \( Y_t = VY \), provided the boundary condition holds.
To consider a circle as a boundary we use polar coordinates \((r, \theta)\). So, the boundary is \(r = a\). The two dimensional Toda lattice equation in polar coordinates becomes

\[
    u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = \omega(n - 1) - \omega(n),
\]

where \(\omega(n) = \exp(u(n) - u(n + 1))\). The above equation admits a Lax pair

\[
    \varphi_{1,r}(n) = \frac{e^{i\theta}}{2} \varphi_1(n + 1) - \frac{1}{2} \left( u_r(n) - \frac{i}{r} u_\theta(n) \right) \varphi_1(n) - \\
    \frac{e^{-i\theta}}{2} \omega(n - 1) \varphi_1(n - 1),
\]

\[
    \varphi_{1,\theta}(n) = \frac{ire^{i\theta}}{2} \varphi_1(n + 1) - \frac{ire}{2} \left( u_r(n) - \frac{i}{r} u_\theta(n) \right) \varphi_1(n) + \\
    + \frac{ire^{-i\theta}}{2} \omega(n - 1) \varphi_1(n - 1)
\]

To obtain other Lax representations we use symmetries of the equation (33).

1. Reflection on \(\theta\)

\[
    \tilde{\theta} = -\theta;
\]

2. The Kelvin transformation

\[
    \tilde{r} = \frac{a}{r}, \quad \tilde{u} = u + 4n \ln \frac{a}{r};
\]

3. Reflection on \(n\)

\[
    \tilde{u} = -u(-n).
\]

Using the transformation (36) we obtain the following Lax representation

\[
    \varphi_{2,r}(n) = \frac{e^{-i\theta}}{2} \varphi_2(n + 1) - \frac{1}{2} \left( u_r(n) + \frac{i}{r} u_\theta(n) \right) \varphi_2(n) - \\
    - \frac{e^{i\theta}}{2} \omega(n - 1) \varphi_2(n - 1),
\]
\( \varphi_{2, \theta}(n) = \frac{ire^{-i\theta}}{2} \varphi_2(n + 1) - \frac{ir}{2} \left( u_r(n) + \frac{i}{r} u_\theta(n) \right) \varphi_2(n) + \frac{ire^{i\theta}}{2} \omega(n - 1) \varphi_2(n - 1) \) \quad (40)

Using the Kelvin transformation (37) we obtain the following Lax representation

\[
\varphi_{3, r}(n) = \frac{e^{i\theta}}{2} \varphi_3(n + 1) - \frac{1}{2} \left( \frac{-r^2}{a^2} u_r(n) + 4n \frac{r}{a^2} - \frac{ir}{a^2} u_\theta \right) \varphi_4(n) - \frac{-r^4 e^{-i\theta}}{2a^4} \omega(n - 1) \varphi_3(n - 1), \quad (41)
\]

\[
\varphi_{3, \theta}(n) = \frac{ia^2 e^{i\theta}}{2r} \varphi_3(n + 1) - \frac{ia^2}{2r} \left( \frac{-r^2}{a^2} u_r(n) + 4n \frac{r}{a^2} - \frac{ir}{a^2} u_\theta \right) \varphi_3(n) + \frac{ir^3 e^{-i\theta}}{2a^4} \omega(n - 1) \varphi_3(n - 1). \quad (42)
\]

Using the transformations (38) we obtain the following Lax representation

\[
\varphi_{4, r}(n) = \frac{e^{i\theta}}{2} \varphi_4(n - 1) - \frac{1}{2} \left( -u_r(n) + \frac{i}{r} u_\theta(n) \right) \varphi_4(n) - \frac{e^{-i\theta}}{2} \omega(n) \varphi_4(n + 1), \quad (43)
\]

\[
\varphi_{4, \theta}(n) = \frac{ire^{i\theta}}{2} \varphi_4(n - 1) - \frac{ir}{2} \left( -u_r(n) + \frac{i}{r} u_\theta(n) \right) \varphi_4(n) + \frac{ire^{-i\theta}}{2} \omega(n) \varphi_4(n + 1). \quad (44)
\]

According to Definition 2, to obtain the integrable boundary conditions we relate the equations for \( \theta \) evolution of the above Lax representations, on the boundary \( r = a \). We consider the case when the eigenfunctions are related by multiplication operator \( \varphi_i = A(\theta, n, u, \ldots) \cdot \varphi_j \).

It turns out (see the lemma 7) that that Lax pairs corresponding to the Kelvin transformation (37) and the symmetry (38) are gauge equivalent. A solution of (42) transforms to a solution of (44) without any boundary.
conditions. So, for boundary value problems (3) and (4) we have two possible Lax pairs.

In the lemma 2 we derive the Lax pair for the boundary value problem (2).

Lemma 2 Let \( \varphi_1(n) \) be a solution of the equation (35) then on the boundary \( r = a \) a function \( \varphi_2(n) = A \cdot \varphi_1(n) \), where
\[
A = e^{2in + g(\theta)}, \quad g(\theta) \text{ is an arbitrary function of } \theta,
\] is a solution of the equation (40) provided the following boundary conditions
\[
u(n) = 2in + g(\theta) + k(n), \quad k(n) \text{ is an arbitrary function of } n,
\] holds (for all \( n \)).

Proof. On the boundary \( r = a \) we substitute \( \varphi_2(n) = A(n, \theta, u, \ldots) \cdot \varphi_1(n) \) into the equation (40) and use (35) for \( \varphi_1, \theta(n) \). The resulting equation holds if the coefficients of \( \varphi_1(n + 1) \), \( \varphi_1(n) \) and \( \varphi_1(n - 1) \) are zero. Thus we obtain the equations
\[
\frac{iae^{i\theta}}{2} A(n) = \frac{iae^{-i\theta}}{2} A(n + 1)
\]
\[
A_\theta(n) - \frac{ia}{2} \left( u_r(n) - \frac{iu_\theta(n)}{a} \right) A(n) = -\frac{ia}{2} \left( u_r(n) + \frac{iu_\theta(n)}{a} \right) A(n)
\]
\[
\frac{iae^{-i\theta}}{2} \omega(n - 1) A(n) = \frac{iae^{i\theta}}{2} \omega(n - 1) A(n - 1)
\]
From the equations (47), (49) we have that \( A(n) = e^{2in} A(n - 1) \). Hence, \( A(n) = e^{2in} b(\theta) \) where \( b(\theta) \) is a function of \( \theta \) only. Substituting \( A(n) = e^{2in} b(\theta) \) into the equation (48) we obtain
\[
b_\theta + (2in - u_\theta(n))b = 0.
\]
Since the function \( b \) does not depend on \( n \) we have that the coefficient of \( b \) in the above equation does not depend on \( n \), so \( u_\theta(n) = 2in + h(\theta) \). Integrating with respect to \( \theta \) we obtain the boundary condition (46), where \( k \) is an arbitrary function of \( n \) and \( g(\theta) = \int h(\theta) d\theta \). Then solving the equation (48), assuming that the found boundary condition holds, we obtain \( A = e^{2in + \int g(\theta) d\theta} \), the expression (45) for \( A \). □
In a similar way, from the next lemmas we have Lax pairs for the boundary value problems (3) and (4).

In the lemma 3 and the lemma 4 we derive the Lax pair for the boundary value problem (3).

**Lemma 3** Let \( \varphi_1(n) \) be a solution of the equation (53) then on the boundary \( r = a \) a function \( \varphi_3(n) = A \cdot \varphi_1(n) \), where

\[
A = e^{ia \int g(\theta) d\theta}, \quad g(\theta) \text{ is an arbitrary function of } \theta,
\]

is a solution of the equation (42) provided the following boundary conditions

\[
u_r(n) = \frac{2n}{a} + g(\theta)
\]

holds.

**Proof** On the boundary \( r = a \) we substitute \( \varphi_3(n) = A(n) \cdot \varphi_1(n) \) into the equation (42) and use (53) for \( \varphi_{1,\theta}(n) \). The resulting equation holds if the coefficients of \( \varphi_1(n + 1) \), \( \varphi_1(n) \) and \( \varphi_1(n - 1) \) are zero. Thus we obtain the equations

\[
\frac{iae^{i \theta}}{2} A(n) = \frac{iae^{i \theta}}{2} A(n + 1)
\]

\[
A_{\theta}(n) - \frac{ia}{2} \left( u_r(n) - \frac{iu_{\theta}(n)}{a} \right) A(n) = \frac{ia}{2} \left( u_r(n) - \frac{4n}{a} + \frac{iu_{\theta}(n)}{a} \right) A(n)
\]

\[
\frac{iae^{-i \theta}}{2} \omega(n - 1) A(n) = \frac{iae^{-i \theta}}{2} \omega(n - 1) A(n - 1)
\]

From the equations (53), (55) we have that \( A \) does not depend on \( n \). Hence, the coefficient of \( A \) in (54) must be a function of \( \theta \) only. This gives the boundary condition (52). Then solving the equation (54), assuming that (52) holds, we obtain the expression (51) for \( A \). □

**Lemma 4** Let \( \varphi_1(n) \) be a solution of the equation (53) then on the boundary \( r = a \) a function \( \varphi_4(n) = A \cdot \varphi_1(n) \), where

\[
A = e^{2i \theta + u(n) + ia \int g(\theta) d\theta}, \quad g(\theta) \text{ is an arbitrary function of } \theta,
\]

is a solution of the equation (42) provided the following boundary conditions

\[
u_r(n) = \frac{2n}{a} + g(\theta)
\]

holds.
Proof. On the boundary \( r = a \) we substitute \( \varphi_1(n) = A(n, \theta, u, \ldots) \cdot \varphi_1(n) \) into the equation (44) and use (42) for \( \varphi_2(n) \). The resulting equation holds if the coefficients of \( \varphi_2(n+1) \), \( \varphi_2(n) \) and \( \varphi_2(n-1) \) are zero. Thus we obtain the equations

\[
\frac{iae^{i\theta}}{2}A(n) = \frac{iae^{-i\theta}}{2}\omega(n)A(n+1)
\]

(58)

\[
A_{\theta}(n) - \frac{ia}{2}\left( u_r(n) - \frac{iu_{\theta}(n)}{a} \right)A(n) = -\frac{ia}{2}\left( -u_r(n) + \frac{iu_{\theta}(n)}{a} \right)A(n)
\]

(59)

\[
\frac{iae^{-i\theta}}{2}\omega(n-1)A(n) = \frac{iae^{i\theta}}{2}A(n-1)
\]

(60)

From the equations (58), (60) we have that \( A(n) = e^{-2i\theta}\omega(n)A(n+1) \). Hence, \( A(n) = e^{2i\theta n + u(n)}b(\theta) \) where \( F(\theta) \) is a function of \( \theta \) only. Substituting \( A(n) = e^{-2i\theta n}b(\theta) \) into the equation (59) we obtain

\[
b_{\theta} - (2in - iau_r(n))b = 0.
\]

(61)

Since the function \( b \) does not depend on \( n \) we have that the coefficient of \( b \) in the above equation does not depend on \( n \). This gives the boundary condition (57). Then solving the equation (59), assuming that the found boundary condition holds, we obtain the expression (56) for \( A \). □

In the lemma 5 and the lemma 6 we derive the Lax pair for the boundary value problem 4.

**Lemma 5** Let \( \varphi_2(n) \) be a solution of the equation (40) then on the boundary \( r = a \) a function \( \varphi_3(n) = A \cdot \varphi_2(n) \), where

\[
e^{-2i\theta n + ia \int g(\theta) d\theta}, \quad g(\theta) \text{ is an arbitrary function of } \theta,
\]

(62)

is a solution of the equation (43) provided the following boundary conditions

\[
u_r(n) = -\frac{i}{a}u_{\theta}(n) + g(\theta)
\]

(63)

holds.

**Proof.** On the boundary \( r = a \) we substitute \( \varphi_3(n) = A(n, \theta, u, \ldots) \cdot \varphi_2(n) \) into the equation (42) and use (44) for \( \varphi_2(n) \). The resulting equation holds
if the coefficients of $\varphi_2(n+1)$, $\varphi_2(n)$ and $\varphi_2(n-1)$ are zero. Thus we obtain the equations

\[
\frac{iae^{-i\theta}}{2} A(n) = \frac{iae^{i\theta}}{2} A(n+1)
\]  
(64)

\[
A_\theta(n) - \frac{ia}{2} \left( u_r(n) + \frac{iu_\theta(n)}{a} \right) A(n) = -\frac{ia}{2} \left( -u_r(n) + \frac{4n}{r} - \frac{iu_\theta(n)}{a} \right) A(n)
\]  
(65)

\[
\frac{iae^{i\theta}}{2} A(n) = \frac{iae^{-i\theta}}{2} A(n-1)
\]  
(66)

From the equations (64), (66) we have that $A(n) = e^{-2i\theta} A(n-1)$. Hence, $A(n) = e^{-2i\theta n} b(\theta)$ where $b(\theta)$ is a function of $\theta$ only. Substituting $A(n) = e^{-2i\theta n} b(\theta)$ into the equation (65) we obtain

\[
b_\theta - ia(u_r(n) + \frac{i}{a} u_\theta(n)) b = 0.
\]  
(67)

Since the function $b$ does not depend on $n$ we have that the coefficient of $b$ in the above equation does not depend on $n$. This give the boundary condition (63). Then solving the equation (65), assuming that the found boundary condition holds, we obtain the expression (62) for $A$. □

**Lemma 6** Let $\varphi_2(n)$ be a solution of the equation (40) then on the boundary $r = a$ a function $\varphi_4(n) = A \cdot \varphi_2(n)$, where

\[
A = e^{u(n) + \int g(\theta) d\theta}, \quad g(\theta) \text{ is an arbitrary function of } \theta,
\]  
(68)

is a solution of the equation (44) provided the following boundary conditions

\[
u_r(n) = -\frac{i}{a} u_\theta(n) + g(\theta)
\]  
(69)

holds.

**Proof.** On the boundary $r = a$ we substitute $\varphi_4(n) = A(n, \theta, u, \ldots) \cdot \varphi_2(n)$ into equation (44) and use (10) for $\varphi_{2,\theta}(n)$. The resulting equation holds if the coefficients of $\varphi_2(n+1)$, $\varphi_2(n)$ and $\varphi_2(n-1)$ are zero. Thus we obtain the equations

\[
\frac{iae^{-i\theta}}{2} A(n) = \frac{iae^{-i\theta}}{2} \omega(n) A(n+1)
\]  
(70)
\[ A_\theta(n) - \frac{ia}{2} \left( u_r(n) + \frac{iu_\theta(n)}{a} \right) A(n) = -\frac{ia}{2} \left( -u_r(n) + \frac{iu_\theta(n)}{a} \right) A(n) \quad \text{(71)} \]

\[ \frac{iace^{i\theta}}{2} \omega(n-1) A(n) = \frac{iace^{i\theta}}{2} A(n-1) \quad \text{(72)} \]

From the equations (70), (72) we have that \( A(n) = \omega(n) A(n+1) \). Hence, \( A(n) = e^{\omega(n)} F(\theta) \) where \( F(\theta) \) is a function of \( \theta \) only. Substituting \( A(n) = e^{\omega(n)} b(\theta) \) into the equation (71) we obtain

\[ b_\theta + (-iau_r(n) + u_\theta(n)) b = 0. \quad \text{(73)} \]

Since the function \( b \) does not depend on \( n \) we have that the coefficient of \( b \) in the above equation does not depend on \( n \). We obtain the boundary condition (69). Solving the equation (71) and assuming that the found boundary condition holds, we obtain the expression (68) for \( A \). □

In Lemma 7 we show that the Lax representations corresponding to the Kelvin transformation (37) and the symmetry (38) are equivalent.

**Lemma 7.** Let \( \varphi_3(n) \) be a solution of the equation (42) then on the boundary \( r = a \) a function \( \varphi_4(n) = A \cdot \varphi_2(n) \), where

\[ A = e^{2i\theta n + \omega(n)} \quad \text{(74)} \]

is a solution of the equation (44).

**Proof.** On the boundary \( r = a \) we substitute \( \varphi_4(n) = A(n, \theta, u, \ldots) \cdot \varphi_3(n) \) into the equation (44) and use (40) for \( \varphi_{3, \theta}(n) \). The resulting equation holds if the coefficients of \( \varphi_3(n+1), \varphi_3(n) \) and \( \varphi_3(n-1) \) are zero. Thus we obtain the equations

\[ \frac{iace^{i\theta}}{2} A(n) = \frac{iace^{-i\theta}}{2} \omega(n) A(n+1) \quad \text{(75)} \]

\[ A_\theta(n) - \frac{ia}{2} \left( -u_r(n) + 4n/a - \frac{iu_\theta(n)}{a} \right) A(n) = -\frac{ia}{2} \left( -u_r(n) + \frac{iu_\theta(n)}{a} \right) A(n) \quad \text{(76)} \]

\[ \frac{iace^{-i\theta}}{2} \omega(n-1) A(n) = \frac{iace^{i\theta}}{2} A(n-1) \quad \text{(77)} \]

From the equations (70), (72) we have that \( A(n) = e^{-2i\theta} \omega(n) A(n+1) \). Hence, \( A(n) = e^{2i\theta n + \omega(n)} b(\theta) \) where \( b(\theta) \) is a function of \( \theta \) only. Substituting \( A(n) = e^{2i\theta n + \omega(n)} b(\theta) \) into the equation (71) we obtain

\[ b_\theta = 0. \quad \text{(78)} \]
Hence the function $b$ is a constant. This gives us the expression (68) for $A$. □

From the above lemmas we have the following list of integrable boundary value problems with corresponding Lax pairs. Some of the integrable boundary value problems admit two different Lax pairs. We give both Lax pairs in the list.

The list of Lax pairs for two dimensional Toda lattice

- $r < a$
  
  $u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta \theta} = \omega(n - 1) - \omega(n)$,
  
  $\varphi_1, r(n) = U_1 \varphi_1(n)$, and $\varphi_2, r(n) = U_2 \varphi_2(n)$,
  
  $\varphi_1, \theta(n) = V_1 \varphi_1(n)$, and $\varphi_2, \theta(n) = V_2 \varphi_2(n)$,

  $r = a$
  
  $u(n) = 2 \sin \theta + g(\theta) + k(n)$,
  
  $\varphi_2 = e^{2i\theta n + g(\theta) \theta} \varphi_1$

  where action of operator $U_1$ is given by (34), $V_1$ is given by (35) and $U_2$ is given by (39), $V_2$ is given by (40).

- $r < a$
  
  $u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta \theta} = \omega(n - 1) - \omega(n)$,
  
  $\varphi_1, r(n) = U_1 \varphi_1(n)$, and $\varphi_3, r(n) = U_3 \varphi_3(n)$,
  
  $\varphi_1, \theta(n) = V_1 \varphi_1(n)$, and $\varphi_3, \theta(n) = V_3 \varphi_3(n)$,

  $r = a$
  
  $u_r(n) = \frac{2n}{a^2} + g(\theta)$,
  
  $\varphi_3 = e^{i a \int g(\theta) d\theta} \varphi_1$

  where action of operator $U_1$ is given by (34), $V_1$ is given by (35) and $U_3$ is given by (41), $V_3$ is given by (42).

- $r < a$
  
  $u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta \theta} = \omega(n - 1) - \omega(n)$,
  
  $\varphi_1, r(n) = U_1 \varphi_1(n)$, and $\varphi_4, r(n) = U_4 \varphi_4(n)$,
  
  $\varphi_1, \theta(n) = V_1 \varphi_1(n)$, and $\varphi_4, \theta(n) = V_4 \varphi_4(n)$,

  $r = a$
  
  $u_r(n) = \frac{2n}{a^2} + g(\theta)$,
  
  $\varphi_4 = e^{2i\theta n + u(n) + ia \int g(\theta) d\theta} \varphi_1$

  where action of operator $U_1$ is given by (34), $V_1$ is given by (35) and $U_4$ is given by (43), $V_4$ is given by (44).

- $r < a$
  
  $u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta \theta} = \omega(n - 1) - \omega(n)$
\[ \varphi_{2,r}(n) = U_2 \varphi_2(n), \quad \text{and} \quad \varphi_{3,r}(n) = U_3 \varphi_3(n), \]
\[ \varphi_{2,\theta}(n) = V_2 \varphi_2(n), \quad \text{and} \quad \varphi_{3,\theta}(n) = V_3 \varphi_3(n), \]
\[ r = a \quad u_r(n) = -i a u_\theta(n) + g(\theta), \]
\[ \varphi_3 = e^{-\frac{2i}{a} \theta} \int g(\theta) d\theta \varphi_2 \]
where action of operator \( U_2 \) is given by (39), \( V_2 \) is given by (40) and \( U_3 \) is given by (41), \( V_3 \) is given by (42).

\[ r < a \quad u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = \omega(n - 1) - \omega(n) \]
\[ \varphi_{2,r}(n) = U_2 \varphi_2(n), \quad \text{and} \quad \varphi_{4,r}(n) = U_4 \varphi_4(n), \]
\[ \varphi_{2,\theta}(n) = V_2 \varphi_2(n), \quad \text{and} \quad \varphi_{4,\theta}(n) = V_4 \varphi_4(n), \]
\[ r = a \quad u_r(n) = -i a u_\theta(n) + g(\theta), \]
\[ \varphi_4 = e^{u(n) + \frac{i}{a} \int g(\theta) d\theta} \varphi_2 \]
where action of operator \( U_2 \) is given by (39), \( V_2 \) is given by (40) and \( U_4 \) is given by (43), \( V_4 \) is given by (44).

4 Reductions of two dimensional Toda lattice equation

In this section we obtain integrable boundary conditions for the sinh-Gordon and Tcsitseika equations as reductions of integrable boundary conditions of the two dimensional Toda lattice equation.

To reduce the two dimensional Toda lattice equation to the sinh-Gordon equation we put periodicity condition \( u(n) = u(n + 2) \) for all \( n \), where \( u \) satisfies (33). Then for \( p = u(0) - u(1) \) we have

\[ p_{rr} + \frac{1}{r} p_r + \frac{1}{r^2} p_{\theta\theta} = 4 \sinh p, \quad (79) \]

the sinh-Gordon equation in the polar coordinates. Only the boundary condition of the problem (4) is consistent with periodicity constraint \( u(n + 2) = u(n) \). It gives

\[ p_r + \frac{i}{a} p_\theta = 0 \quad (80) \]
on the boundary \( r = a \). Evidently by changing \( p = i v \) we get \( v_{rr} + \frac{1}{r} v_r + \frac{1}{r^2} v_{\theta\theta} = 4 \sin v \) and \( v_r + \frac{i}{a} v_\theta = 0 \).
To reduce the two dimensional Toda lattice equation to the Tcsitcseika equation we put $u(n) = u(n+3)$ and $u(n) = -u(2-n)$. Then for $q = u(0)$ we have
\[ q_{rr} + \frac{1}{r} q_r + \frac{1}{r^2} q_{\theta\theta} = e^{2q} - e^{-q} \]  
(81)
the Tcsitcseika equation in polar coordinates. Again only the boundary condition of the problem (4) is consistent with periodicity constraint $u(n) = u(n+3)$ and $u(n) = -u(2-n)$. It gives
\[ q_r + \frac{i}{a} q_\theta = 0 \]  
(82)
on the boundary $r = a$.

5 Some solutions of the boundary value problem

In this section we give an example of solutions for the special case of the boundary value problem
\[ u_{rr} + \frac{1}{r^2} u_{\theta\theta} + \frac{1}{r} u_r = \omega(n-1) - \omega(n), \quad u_r(n)|_{r=a} = \frac{2n}{a} + g(\theta) \]  
(83)
where $\omega(n) = \exp(u(n) - u(n+1))$. We assume that $g(\theta) = 0$ and look for spherically symmetric solution. That is $u$ is a function of $r$ only. The boundary value problem (83) reduces to
\[ u_{rr} + \frac{1}{r} u_r = \omega(n-1) - \omega(n), \quad u_r(n)|_{r=a} = \frac{2n}{a}. \]  
(84)
Let us introduce new variables $t = \ln r$ and $v(n, t) = u(n, r) - 2n \ln r$. Then the boundary value problem (84) becomes
\[ v_{tt} = \ddot{\omega}(n-1) - \ddot{\omega}(n), \quad u_t(n)|_{t=0} = 0, \]  
(85)
where $\ddot{\omega}(n) = \exp\{v(n) - v(n+1)\}$. As solutions of the above boundary value problem we can take even solitons of Toda lattice equation in one dimension. Following [18], (see pp. 494-498), the general $N$-soliton solution is given in terms of the data \{c, z_j, \gamma_j\} such that

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I. The quantities $z_j$ lie in the interval $-1 < z_j < 1$ and are pairwise disjoint.

II. $e^{-c} = \prod_{j=1}^{N} z_j^2$.

III. The quantities $m_j(0) = \frac{\gamma_j}{\dot{a}(z_j)}$, where $a(z) = \prod_{j=1}^{N} \text{sign}_j \frac{z - z_j}{zz_j - 1}$ and dot means derivative with respect to $z$, are positive.

The $N$-soliton solution is given by

$$v(n, t) = c + \ln \frac{\det M(n, t)}{\det M(n-1, t)}$$

(86)

where $M(n, t)$ is a matrix with entries $M_{ij}(n, t) = \delta_{ij} + \sqrt{m_i(t)m_j(t)(z_iz_j)^{n+1}} \frac{1}{1 - z_iz_j}$

and $m_j(t) = \frac{e^{-(z_j-z_j^{-1})t}\gamma_j}{\dot{a}(z_j)}$, $i, j = 1, \ldots N$.

The even solitons are describe by the following lemma.

**Lemma 8** Let $N = 2k$ and the data $z_j, \gamma_j, j = 1 \ldots N$ satisfy $z_i = -z_{N-i+1}$, $\gamma_i = -\gamma_{N-i+1}, i = 1 \ldots k$. Then the $N$ soliton solution (86) is even function of $t$.

**Proof.** With our choice of the initial data the elements of matrix $M(n, t)$, that are symmetric with respect to the "center" of the matrix, are equal. If $t$ is changed to $-t$ then every element of $M(n, t)$ is replaced by the element symmetric to it with respect the "center" of the matrix. Hence determinant of $M(n, t)$ is equal to the determinant of $M(n, -t)$ and $v(n, t) = v(n, -t)$. From $v(n, t) = v(n, -t)$ it follows that $v'(t)|_{t=0} = 0$.

We give an example of the solutions described in the above lemma. For $N = 2$ we put $z_1 = z_0$, $z_2 = -z_0$, $c = -4 \ln z_0$ and $\gamma_1 = -\gamma_0, \gamma_2 = \gamma_0$ where $0 < z_0 < 1$ and $\gamma_0 > 0$. Then the data satisfy the conditions I, II, III and the conditions of the lemma. With such data one has the following solution of (85)

$$v(n, t) = c + \ln \frac{1 + \gamma_0(1 + z_0^2)z_0^{2n+1} \cosh[(z_0 - z_0^{-1})t] + \gamma_0^2 z_0^{4n+4}}{1 + \gamma_0(1 + z_0^2)z_0^{2n-1} \cosh[(z_0 - z_0^{-1})t] + \gamma_0^2 z_0^{4n}}$$

(87)
Hence the boundary value problem \((84)\) has the following solution

\[
\begin{align*}
\quad
u(n, r) &= c + \ln \frac{1 + \frac{1}{2} \gamma_0 (1 + z_0^2) z_0^{2n+1}(r^{(z_0-z_0^{-1})} + r^{-((z_0-z_0^{-1})}) + \gamma_0^2 z_0^{4n+4}}{1 + \frac{1}{2} \gamma_0 (1 + z_0^2) z_0^{2n-1}(r^{(z_0-z_0^{-1})} + r^{-((z_0-z_0^{-1})}) + \gamma_0^2 z_0^{4n}}}
\end{align*}
\]

(88)

6 Conclusion

In the present paper we apply the method of involutions to boundary value problems for soliton equations on bounded regions. As illustrative models we consider Neumann type boundary value problem on a circle for the Liouville equation and initial boundary value problem for the two dimensional Toda lattice equation. The Lax representations for the boundary value problems are represented. We considered some reductions of the integrable boundary value problems in the case of the two dimensional Toda lattice equation. We also constructed a class of solutions satisfying one of the found boundary conditions.

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