Representations of Cohomological Hall Algebras and Donaldson–Thomas Theory with Classical Structure Groups

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Abstract: We introduce a new class of representations of the cohomological Hall algebras of Kontsevich and Soibelman, which we call cohomological Hall modules, or CoHM for short. These representations are constructed from self-dual representations of a quiver with contravariant involution $\sigma$ and provide a mathematical model for the space of BPS states in orientifold string theory. We use the CoHM to define a generalization of the cohomological Donaldson–Thomas theory of quivers in which the quiver representations have orthogonal or symplectic structure groups. The associated invariants are called orientifold Donaldson–Thomas invariants. We prove the orientifold analogue of the integrality conjecture for $\sigma$-symmetric quivers. We also formulate precise conjectures regarding the geometric meaning of orientifold Donaldson–Thomas invariants and the freeness of the CoHM of a $\sigma$-symmetric quiver. We prove the freeness conjecture for disjoint union quivers, loop quivers and the affine Dynkin quiver of type $\tilde{A}_1$. We also verify the geometric conjecture in a number of examples. Finally, we construct explicit Poincaré–Birkhoff–Witt-type bases of the CoHM of finite type quivers.

Introduction

Motivation. Motivated by the Donaldson–Thomas (DT) theory of three dimensional Calabi–Yau categories, Kontsevich and Soibelman introduced in [27] the cohomological Hall algebra (CoHA) of a quiver with potential. We briefly recall the connection between DT theory and the CoHA, assuming for simplicity that the potential is zero and the quiver $Q$ is symmetric. Let $\Lambda_Q^+$ be the monoid of dimension vectors of $Q$ and let $\chi : \Lambda_Q^+ \times \Lambda_Q^+ \to \mathbb{Z}$ be the Euler form of $Q$. Denote by $\text{Vect}$ the category of finite dimensional rational vector spaces and by $D^{ib}(\text{Vect})_{\Lambda_Q^+}$ the category of $\Lambda_Q^+$-graded objects of $D^{ib}(\text{Vect})$, the bounded derived category of $\text{Vect}$. The monoid structure of $\Lambda_Q^+$ induces a symmetric monoidal structure on $D^{ib}(\text{Vect})_{\Lambda_Q^+}$. Let $M_D$ be the moduli
stack of representations of $Q$ of dimension vector $d \in \Lambda_Q^+$. The CoHA of $Q$ is then

$$\mathcal{H}_Q = \bigoplus_{d \in \Lambda_Q^+} H^\bullet(M_d)[-\chi(d, d)] \in D^{ib}(\text{Vect})_{\Lambda_Q^+}.$$ 

Here $H^\bullet(M_d)[-\chi(d, d)]$ denotes the singular cohomology of $M_d$ with rational coefficients considered as a complex with trivial differential, with its cohomological degree shifted by $-\chi(d, d)$. The stack of flags of representations of $Q$ defines correspondences between the stacks $M_d, d \in \Lambda_Q^+$, which can be used to give $\mathcal{H}_Q$ the structure of an associative algebra in $D^{ib}(\text{Vect})_{\Lambda_Q^+}$. See Sect. 2.1 for details. There exists an object $V^\text{prim}_Q \in D^{ob}(\text{Vect})_{\Lambda_Q^+}$ such that

$$[\text{Sym}(V^\text{prim}_Q \otimes \mathbb{Q}[u])] = [\mathcal{H}_Q] \in K_0(D^{ib}(\text{Vect})_{\Lambda_Q^+}),$$

where $u$ is an indeterminate of degree $(0, 2) \in \Lambda_Q^+ \times \mathbb{Z}$ and $\text{Sym}(V)$ is the free super-commutative algebra on $V$, the $\mathbb{Z}_2$-grading induced by the cohomological $\mathbb{Z}$-grading. The degree $d \in \Lambda^+_Q$ motivic DT invariant of $Q$ is then defined by

$$\Omega_{Q,d} = [V^\text{prim}_{Q,d}] \in K_0(D^{ib}(\text{Vect})).$$

The Kontsevich–Soibelman integrality conjecture [26] states that

$$\Omega_{Q,d} \in \text{image} \left( K_0(D^b(\text{Vect})) \to K_0(D^{ib}(\text{Vect})) \right)$$

for each $d \in \Lambda_Q^+$. A proof of this conjecture was given in [27]. However, the positivity of $\Omega_Q$ was not proved.

While the definition of $\Omega_Q$ involves only the class of $\mathcal{H}_Q$ in the Grothendieck group $K_0(D^{ib}(\text{Vect})_{\Lambda_Q^+})$, it is natural to expect that understanding the algebra structure of $\mathcal{H}_Q$ will provide additional insights into DT theory. Not unrelated, $\mathcal{H}_Q$ is a model for the algebra of closed oriented BPS states of a quantum field theory or string theory with extended supersymmetry [20,27]. In this direction, Efimov [11] constructed a subobject $V^\text{prim}_Q \otimes \mathbb{Q}[u] \subset \mathcal{H}_Q$, with $V^\text{prim}_Q$ having finite dimensional $\Lambda_Q^+$-homogeneous summands, for which the canonical map

$$\text{Sym}(V^\text{prim}_Q \otimes \mathbb{Q}[u]) \to \mathcal{H}_Q$$

is an algebra isomorphism. This is reviewed in Sect. 2.2. Upon passing to Grothendieck rings, this confirms the integrality and positivity conjectures. Moreover, it follows that there is an isomorphism

$$V^\text{prim}_Q \otimes \mathbb{Q}[u] \simeq \mathcal{H}_Q/(\mathcal{H}_Q, + \cdot \mathcal{H}_Q, +),$$

where $\mathcal{H}_Q, + \subset \mathcal{H}_Q$ is the augmentation ideal. The subobject $V^\text{prim}_Q$ is a cohomologically refined DT invariant [39]. More generally, for an arbitrary quiver with potential $W$ and generic stability $\theta$, it was proved in [6] that the slope $\mu$ cohomological DT invariant $V^\text{prim,}\mu_{Q,W,\mu}$ can be constructed as a subobject of the semistable critical CoHA $\mathcal{H}_{Q,W,\mu}^{\theta-ss}$, that the critical analogue of the map (2) is an isomorphism in $D^{ib}(\text{Vect})_{\Lambda_Q^+}$ and that $V^\text{prim,}\theta_{Q,W,\mu}$
satisfies the integrality conjecture. In this way, $\mathcal{H}_{Q,W,\mu}^{\theta-ss}$ acquires a Poincaré–Birkhoff–Witt-type basis. As an application of these results, the structure of $\mathcal{H}_{Q,W,\mu}^{\theta-ss}$ has been used to give a new proof of the Kać conjecture [5].

While less studied, the representation theory of the CoHA is also relevant to DT theory. Physical arguments suggest that the space of open BPS states in a theory with defects is a representation of the BPS algebra [18]. By the work of [3], such representations are expected to be related to CoHA representations constructed from stable framed objects [38,39]. Framed CoHA representations of quiver categories have been studied in [6,14,41,42].

In this paper we introduce a class of CoHA representations which are defined using orthogonal and symplectic analogues of quiver representations. In the setting of quiver representations over finite fields, the corresponding class of Hall algebra representations was introduced and studied in [43,44]. While the framing construction models open BPS states, the constructions in this paper model (unoriented) BPS states in orientifold string theory. Our formalism provides an extension of DT theory from structure group $GL_n(\mathbb{C})$ to the classical groups $O_n(\mathbb{C})$ and $Sp_{2n}(\mathbb{C})$, in the following sense. If $G$ is a reductive group, then the derived moduli stack of $G$-bundles on a Calabi–Yau threefold $X$ has a canonical $(-1)$-shifted symplectic structure and its truncation has a symmetric perfect obstruction theory [32] which could be used to define the $G$-DT invariants of $X$. The usual DT theory is related to the case $G = GL_n(\mathbb{C})$. For $G$ an orthogonal or symplectic group, $G$-bundles on $X$ are precisely the (frame bundles of) self-dual objects of the category of vector bundles on $X$. We expect that the general setting of orientifold DT theory is that of a three dimensional triangulated Calabi–Yau category together with a duality functor which preserves the Calabi–Yau pairing. The CoHA representations introduced below, and the resulting orientifold DT invariants, provide a concrete realization of this theory in the quiver setting.

Results. Let $Q$ be a quiver with contravariant involution $\sigma$. Let $\Lambda^\sigma_+ \subset \Lambda_+$ be the submonoid of $\sigma$-invariant dimension vectors. The symmetrization map $\Lambda_Q^+ \to \Lambda^\sigma_Q$, $d \mapsto d + \sigma(d)$, gives $D^{ib}(\text{Vect})_{\Lambda_Q^+}$ the structure of a left module category over the monoidal category $D^{ib}(\text{Vect})_{\Lambda^\sigma_Q}$. After fixing some combinatorial data described in Sect. 1.4, $\sigma$ induces a duality functor on the category $\text{Rep}_C(Q)$. Denote by $M^\sigma_e$ the stack of representations of dimension vector $e \in \Lambda^\sigma_+$ which are symmetrically isomorphic to their duals (henceforth, self-dual representations). Set

$$\mathcal{I}_Q = \bigoplus_{e \in \Lambda^\sigma_+} H^\bullet(M^\sigma_e)[-E(e)] \in D^{ib}(\text{Vect})_{\Lambda^\sigma_Q}.$$

The function $E : \Lambda_Q \to \mathbb{Z}$ is the analogue of the Euler form for self-dual representations. Write $M^\sigma_{d,e}$ for the stack of flags of representations $U \subset Z$, with $Z$ self-dual, $U$ isotropic in $Z$ and $\dim U = d$, $\dim Z = d + \sigma(d) + e$. In Theorem 3.1 we prove that the correspondences

$$M^\sigma_d \times M^\sigma_e \leftrightarrow M^\sigma_{d,e} \to M^\sigma_{d+\sigma(d)+e}$$

$$(U, Z // U) \leftrightarrow U \subset Z \leftrightarrow Z,$$

where $//$ is a categorical version of symplectic reduction, give $\mathcal{I}_Q$ the structure of a left $\mathcal{H}_Q$-module object in $D^{ib}(\text{Vect})_{\Lambda^\sigma_Q}$, which we call the cohomological Hall module.
(CoHM). In Theorem 3.3 we use localization in equivariant cohomology to present $\mathcal{I}_Q$ as a signed shuffle module, analogous to the Feigin–Odesskiĭ shuffle algebra structure of $\mathcal{H}_Q$ [27]. The passage from shuffle algebras to signed shuffle modules reflects the passage from Weyl groups of general linear groups to Weyl groups of classical groups.

Suppose that $Q$ is $\sigma$-symmetric, that is, $Q$ is symmetric and the function $E$ satisfies $\sigma^*E = E$. We define the cohomological orientifold DT invariant to be

$$W_{Q, \text{prim}} = \mathcal{I}_Q / (\mathcal{H}_Q, \mathcal{I}_Q) \in D^{lb}(\text{Vect})_{\Lambda^\sigma_+}.$$ 

By picking a splitting, we will often view $W_{Q, \text{prim}}$ as a subobject of $\mathcal{I}_Q$. The motivic orientifold DT invariants of $Q$ are then defined by

$$\Omega_{Q, e}^\sigma = [W_{Q, e, \text{prim}}] \in K_0(D^{lb}(\text{Vect})), \quad e \in \Lambda^\sigma_+.$$ 

The definition of $W_{Q, \text{prim}}$ should be compared with the isomorphism (3). In the geometric interpretation of DT invariants, the infinite factor $\mathbb{Q}[u]$ in the isomorphism (3) can be identified with $H^*(B\mathcal{C}_x)$, where $\mathcal{C}_x$ is the automorphism group of an arbitrary stable representation of $Q$. Since the automorphism group of a stable self-dual representation is necessarily finite, we do not expect to have a similar infinite factor in the definition of $W_{Q, \text{prim}}$.

Our first main result is the following.

**Theorem A** (Theorem 3.4). The orientifold integrality conjecture holds for $\sigma$-symmetric quivers. More precisely, for each $e \in \Lambda^\sigma_+$, we have

$$\Omega_{Q, e}^\sigma \in \text{image} (K_0(D^b(\text{Vect})) \hookrightarrow K_0(D^{lb}(\text{Vect}))).$$

The proof is a modification of Efimov’s proof [11] of the integrality conjecture for $\mathcal{H}_Q$ and relies on the explicit signed shuffle description of $\mathcal{I}_Q$.

To better understand $\Omega_{Q, e}^\sigma$, we study the module theoretic analogue of the map (2). The situation is more complicated than that of the CoHA since $\mathcal{I}_Q$ is not a free $\mathcal{H}_Q$-module. This explains why there is no naive generalization of equation (1) which can be used to define $W_{Q, \text{prim}}$.

**Conjecture A** (Conjecture 3.7). Let $Q$ be $\sigma$-symmetric and assume that $\mathcal{H}_Q$ is super-commutative. For each $e \in \Lambda^\sigma_+$, there is a $\Lambda^\sigma_+ \times \mathbb{Z}$-graded subalgebra $\mathcal{H}_Q(e) \subset \mathcal{H}_Q$ such that the CoHA action map

$$\bigoplus_{e \in \Lambda^\sigma_+} \mathcal{H}_Q(e) \otimes_{\mathbb{Q}} W_{Q, e, \text{prim}} \twoheadrightarrow \mathcal{I}_Q$$

is an isomorphism in $D^{lb}(\text{Vect})_{\Lambda^\sigma_+}$.

Passing to Grothendieck groups, Conjecture A would imply the following orientifold analogue of the factorization (1):

$$\sum_{e \in \Lambda^\sigma_+} [\mathcal{H}_Q(e)] \cdot \Omega_{Q, e}^\sigma \quad \text{(Conj. A)} = [\mathcal{I}_Q].$$
In general, $\Omega_Q$ is insufficient to determine $[\mathcal{H}_Q(e)]$, so that $\Omega^\sigma_Q$ cannot be computed directly from $\Omega_Q$. Instead, a $\mathbb{Z}_2$-equivariant refinement of $\Omega_Q$ is needed.

Turning to geometry, let $PH^\bullet(\mathfrak{M}_{\sigma,e}) = \bigoplus_{n \geq 0} Gr^n_W H_n(\mathfrak{M}_{\sigma,e}^{st})$ be the pure part of the cohomology of the moduli scheme of stable self-dual representations of dimension vector $e$. In Sect. 3.5 we construct a canonical surjection $W_{Q,e}^{prim} \twoheadrightarrow PH^{\bullet - E}(e)(\mathfrak{M}_{\sigma,e}^{st})$ of $\mathbb{Z}$-graded vector spaces.

**Conjecture B** (Conjecture 3.10). If $Q$ is $\sigma$-symmetric, then the surjection $W_{Q,e}^{prim} \twoheadrightarrow PH^{\bullet - E}(e)(\mathfrak{M}_{\sigma,e}^{st})$ is an isomorphism.

Sections 4 and 5 confirm Conjectures A and B in a number of examples. The results of Sect. 4, which focuses on $\sigma$-symmetric quivers, can be summarized as follows.

**Theorem B** (Theorem 4.7). Conjecture A holds if $Q$ is the $m$-loop quiver.

Loop quivers have the special property that $\Omega_Q$ determines the $\mathbb{Z}_2$-equivariant DT invariants. In particular, Theorem B can be used to explicitly compute $\Omega^\sigma_Q$.

**Theorem C** (Theorems 4.2 and 4.10). Conjectures A and B hold for disjoint union quivers and for the symmetric orientation of the affine Dynkin quiver of type $\tilde{A}_1$.

In Sect. 5 we study finite type quivers with involution. The non-trivial task is to describe the CoHM of Dynkin quivers of type $A$.

**Theorem D** (Theorem 5.8). The CoHM of a Dynkin quiver of type $A$ admits two PBW-type bases, each of which is determined by a simple/indecomposable PBW-type basis of $\mathcal{H}_Q$ and the set of simple/indecomposable self-dual representations.

This result categorifies the orientifold quantum dilogarithm identities of [43]. To prove Theorem D, we modify Rimányi’s approach to the study of the CoHA of a finite type quiver [36]. Along the way we prove a number of results of independent interest. For example, Corollary 5.6 states that Thom polynomials of orbit closures of self-dual quiver representations appear as structure constants of the CoHM.

**Remark.** After this paper was written, a revised version of [6] appeared, a key new result of which is the construction of a BPS Lie algebra from $\mathcal{H}_{Q,W}$ [6, §1.7]. Crucial to this construction is the bialgebra property of $\mathcal{H}_{Q,W}$, which was established earlier by Davison in [4, Theorem 1.1]. The bialgebra property can be seen as a cohomological analogue of Green’s theorem [17, Theorem 1].

It would be very interesting to use the CoHM $\mathcal{I}_{Q,W}$ to construct representations of the the BPS Lie algebra. We expect an analogue of Green’s theorem for $\mathcal{I}_{Q,W}$, describing the compatibility of the $\mathcal{H}_{Q,W}$-module and $\mathcal{H}_{Q,W}$-comodule structures of $\mathcal{I}_{Q,W}$, to play a key role in such a construction. However, at present it is not known, even conjecturally, what the precise analogue of the Green’s theorem should be. For recent work in this direction, the reader is referred to [45].

1. Background Material

1.1. Classical groups. We fix notation regarding the classical groups. Each such group $G_n$ is the automorphism group of a pair $(V_n, \langle \cdot, \cdot \rangle)$ consisting of a complex vector space with a nondegenerate bilinear form.
Dynkin types $B_n$ and $D_n$: Let $V_n = \mathbb{C}^{2n+1}$ with basis $x_1, \ldots, x_n, w, y_1, \ldots, y_n$ in type $B_n$ and $V_n = \mathbb{C}^{2n}$ with basis $x_1, \ldots, x_n, y_1, \ldots, y_n$ in type $D_n$. The symmetric form satisfies $(x_i, y_j) = \delta_{i,j}$ and, in type $B_n$, $(w, w) = 1$, all other pairings between basis vectors being zero. Then $G_n$ is the orthogonal group $O_{2n+1}(\mathbb{C})$ or $O_{2n}(\mathbb{C})$.

(i) Dynkin type $C_n$: Let $V_n = \mathbb{C}^{2n}$ with basis $x_1, \ldots, x_n, y_1, \ldots, y_n$. The skew-symmetric form satisfies $(x_i, y_j) = \delta_{i,j}$, all other pairings between basis vectors being zero. Then $G_n$ is the symplectic group $Sp_{2n}(\mathbb{C})$.

Define a (connected) maximal torus

$$T_n = \{ \text{diag}(t_1, \ldots, t_n, (1), t_1^{-1}, \ldots, t_n^{-1}) \mid t_i \in \mathbb{C}^\times \} \leq G_n,$$

omitting the middle 1 except in type $B_n$. For each $1 \leq i \leq n$, there is an associated character $e_i : T_n \to \mathbb{C}^\times$, $t \mapsto t_i$. The positive roots in each type are as follows:

- **Type $B_n$:** $\Delta = \{ e_i \pm e_j \mid 1 \leq i < j \leq n \} \sqcup \{ e_i \mid 1 \leq i \leq n \}$
- **Type $C_n$:** $\Delta = \{ e_i \pm e_j \mid 1 \leq i < j \leq n \} \sqcup \{ 2e_i \mid 1 \leq i \leq n \}$
- **Type $D_n$:** $\Delta = \{ e_i \pm e_j \mid 1 \leq i < j \leq n \}$

The Weyl groups $W = W_n$ with $\deg x_i = 2$. Similarly, if $N > n$, then the variety Mat$^*_{N \times \infty}$ of complex $N \times \infty$ matrices of rank $n$ is $2(N - n)$-connected and carries a free action of $GL_n = GL_n(\mathbb{C})$. The quotients Mat$^*_{N \times \infty} \to \text{Mat}^*_N / GL_n$ form an injective system $\{ E_N \to B_N \}_{N > n}$ of finite dimensional approximations by varieties of the universal bundle $EGL_n \to BGL_n$. More generally, if $G \leq GL_n$ is a closed subgroup, then $\{ E_N \to E_N / G \}_{N > n}$ approximates $EG \to BG$.

Let $G$ act on a variety $X$. The $G$-equivariant cohomology of $X$ is

$$H_G^*(X) = \lim_{\leftarrow} H_*(X \times G, E_N; \mathbb{Q}),$$

where $H^*(-; \mathbb{Q})$ denotes singular cohomology with rational coefficients. Write $H_G^*$ for $H_G^*(\text{Spec}(\mathbb{C}))$. Let $T_{GL_n} \leq GL_n$ be a maximal torus. There are ring isomorphisms

$$H_{GL_n}^* \simeq H^*(BT_{GL_n})^{\mathcal{O}_{GL_n}} \simeq \mathbb{Q}[x_1, \ldots, x_n]^{\mathcal{O}_{GL_n}}$$

with $\deg x_i = 2$. Similarly, if $G_n$ is a classical group of type $B_n$, $C_n$ or $D_n$, then the inclusion $T_n \hookrightarrow G_n$ induces ring isomorphisms

$$H_{G_n}^* \simeq H^*(BT_n)^{\mathcal{O}_{G_n}} \simeq \mathbb{Q}[z_1^2, \ldots, z_2^2]^{\mathcal{O}_{G_n}}$$

with $\deg z_i = 2$. Here it is essential that $G_n$ is the full orthogonal group in type $D_n$.

**Lemma 1.1.** (i) Let $\phi : GL_n \to GL_n$, $g \mapsto (g^{-1})^t$. The induced map $(B\phi)^* : H_{GL_n}^* \to H_{GL_n}^*$ is given by $(B\phi)^* f(x_1, \ldots, x_n) = f(-x_1, \ldots, -x_n)$.

(ii) Let $h : GL_n \hookrightarrow G_n$ be the hyperbolic embedding. The induced map $(Bh)^* : H_{G_n}^* \to H_{GL_n}^*$ is given by $(Bh)^* z_i = x_i$. 

(iii) Let \( \iota : G_n \hookrightarrow GL_{2n+\epsilon} \) be the embedding arising from the description of \( G_n \) given in Sect. 1.1, where \( \epsilon = 1 \) in type \( B_n \) and \( \epsilon = 0 \) otherwise. Under the identification \( H^*_\epsilon(\Lambda_{2n+\epsilon}) \simeq \mathbb{Q}[x_1, \ldots, x_n, (w), y_1, \ldots, y_n]^{\mathbb{S}_{2n+\epsilon}} \), the induced map 
\((Bt)^* : H^*_\epsilon(GL_{2n+\epsilon}) \rightarrow H^*_\epsilon(G_n) \) is determined by

\[(Bt)^*x_i = z_i, \quad (Bt)^*w = 0, \quad (Bt)^*y_i = -z_i.\]

**Proof.** The statements can be proved by picking compatible maximal tori for the domain and codomain of the group homomorphisms. \( \square \)

Finally, recall that \( H^*_\epsilon(X) \) and its compactly supported variant \( H^*_{\epsilon, c}(X) \) have canonical mixed Hodge structures \( [7] \). The pure part of, say, \( H^*_\epsilon(X) \) is by definition

\[PH^*_\epsilon(X) = \bigoplus_{k \geq 0} PH^k_\epsilon(X), \quad PH^k_\epsilon(X) = Gr^W_k H^k_\epsilon(X).\]

Here \( 0 = W_{-1}H^k_\epsilon(X) \subset W_0H^k_\epsilon(X) \subset \cdots \subset W_2H^k_\epsilon(X) = H^k_\epsilon(X) \) is the weight filtration with associated graded

\[Gr^W_k H^k_\epsilon(X) = W_kH^k_\epsilon(X)/W_{k-1}H^k_\epsilon(X).\]

### 1.3. Quiver representations.

Let \( Q \) be a quiver with finite sets of nodes \( Q_0 \) and arrows \( Q_1 \). Let \( \text{Rep}_C(Q) \) be the hereditary abelian category of finite dimensional complex representations of \( Q \). Objects of \( \text{Rep}_C(Q) \) are pairs \( U = (U, u) \), where \( U = \bigoplus_{i \in Q_0} U_i \) is a finite dimensional \( Q_0 \)-graded complex vector space and \( u = \{ U_i \xrightarrow{u_{ij}} U_j \}_{(i \xrightarrow{\alpha} j) \in Q_1} \) is a collection of linear maps. Set \( \Lambda^+_Q = \mathbb{Z}_{\geq 0}Q_0 \) and \( \Lambda_Q = \mathbb{Z}Q_0 \). The dimension vector of \( U \) is \( \text{dim} U \in \Lambda^+_Q \).

The Euler form of \( \text{Rep}_C(Q) \) is

\[\chi(U, V) = \text{dim}_C \text{Hom}(U, V) - \text{dim}_C \text{Ext}^1(U, V).\]

It descends to the following bilinear form on \( \Lambda_Q \):

\[\chi(d, d') = \sum_{i \in Q_0} d_i d'_i - \sum_{(i \xrightarrow{\alpha} j) \in Q_1} d_i d'_j.\]

For each \( d \in \Lambda^+_Q \), let \( R_d = \bigoplus_{i \xrightarrow{\alpha} j} \text{Hom}_C(C^d_i, C^d_j) \). The group \( GL_d = \prod_{i \in Q_0} GL_d_i(\mathbb{C}) \) acts on \( R_d \) by change of basis. Its orbits are in bijection with the isomorphism classes of representations of dimension vector \( d \).

### 1.4. Self-dual quiver representations.

For a detailed discussion of self-dual quiver representations, the reader is referred to \([8]\).

An involution of \( Q \) is a pair of involutions \( \sigma : Q_0 \rightarrow Q_0 \) and \( \sigma : Q_1 \rightarrow Q_1 \) such that

(i) if \( (i \xrightarrow{\alpha} j) \in Q_1 \), then \( (\sigma(j) \xrightarrow{\sigma(\alpha)} \sigma(i)) \in Q_1 \), and
(ii) if \( (i \xrightarrow{\alpha} \sigma(i)) \in Q_1 \), then \( \alpha = \sigma(\alpha) \).
Let $\Lambda^\sigma_{\sigma^+}$ be the subgroup of fixed points of the induced involution $\sigma : \Lambda_Q \to \Lambda_Q$. Set also $\Lambda_{\sigma^+} = \Lambda^+ \cap \Lambda_{\sigma^+}$. The group homomorphism

$$H : \Lambda_Q \to \Lambda^\sigma_{\sigma^+}, \quad d \mapsto d + \sigma(d),$$

(5)

makes $\Lambda^\sigma_{\sigma^+}$ into a $\Lambda_Q$-module.

A duality structure on $(Q, \sigma)$ is a pair of functions,

$$s : Q_0 \to \{\pm 1\}, \quad \tau : Q_1 \to \{\pm 1\}$$

such that $s$ is $\sigma$-invariant and $\tau_\alpha \tau_{\sigma(\alpha)} = s_is_j$ for every arrow $i \to j$. A duality structure defines an exact functor $S : \text{Rep}_C^\sigma(Q) \to \text{Rep}_C^\sigma(Q)$. On objects we have

$$S(U)_i = U_{\sigma(i)}^\vee, \quad S(u)_{\alpha} = \tau_\alpha u_{\sigma(\alpha)}^\vee,$$

where $(-)^\vee = \text{Hom}_C(-, \mathbb{C})$. If $\phi : U \to U'$ is a morphism with components $\phi_i : U_i \to U'_{i'}$, then $S(\phi) : S(U') \to S(U)$ has components $S(\phi)_i = \phi_{\sigma(\alpha)}(1)$. Let $\text{ev}_{U_i} : U_i \to U_i^{\vee\vee}$ be the evaluation isomorphism. Then $\Theta_U = \Theta_{i\in Q_0} \cdot \text{ev}_{U_i}$ defines the components of a natural isomorphism $\Theta : 1_{\text{Rep}(Q)} \to S \circ S^\sigma$ which satisfies $\Theta(SU) \circ \Theta_S(U) = \text{id}_{S(U)}$. The triple $(\text{Rep}_C^\sigma(Q), S, \Theta)$ is an abelian category with duality.

A self-dual representation of $Q$ is a pair $(Z, \psi)$ consisting of a representation $Z$ and an isomorphism $\psi_Z : Z \to S(Z)$ which satisfies $\psi_Z \circ \Theta_Z = \psi_Z$. Geometrically, a self-dual representation is a representation $Z$ together with a nondegenerate bilinear form $\langle \cdot, \cdot \rangle$ with the following properties:

(i) $Z_i$ and $Z_j$ are orthogonal unless $i = \sigma(j)$,

(ii) the restriction of $\langle \cdot, \cdot \rangle$ to $Z_i + Z_{\sigma(i)}$ satisfies $\langle x, x' \rangle = s_i(x', x)$, and

(iii) for each arrow $\alpha : i \to j$, the structure maps of $Z$ satisfy

$$\langle z_{\alpha}x, x' \rangle = \tau_\alpha \langle x, z_{\sigma(\alpha)}x' \rangle = 0, \quad x \in Z_i, \quad x' \in Z_{\sigma(j)}.$$

(6)

As a basic example, let $U \in \text{Rep}_C^\sigma(Q)$. Then the hyperbolic representation $H(U)$ is the self-dual representation $$(U \oplus S(U), \begin{pmatrix} 0 & \text{id}(U) \\ \Theta_U & 0 \end{pmatrix}).$$

Fix once and for all a partition $Q_0 = Q_0^0 \sqcup Q_0^\sigma \sqcup Q_0^+$ such that $Q_0^\sigma$ consists of the nodes fixed by $\sigma$ and $\sigma(Q_0^+) = Q_0^\sigma$. Similarly, fix a partition $Q_1 = Q_1^0 \sqcup Q_1^\sigma \sqcup Q_1^+$.

Let $e \in \Lambda_{\sigma^+}$. We will always assume that $e_i$ is even if $i \in Q_0^\sigma$ and $s_i = -1$. The trivial representation $\mathbb{C}^e$ then admits a self-dual structure $\langle \cdot, \cdot \rangle$ which is unique up to $Q_0$-graded isometry. Denote by $R^e_\sigma \subset R_e$ the linear subspace of representations whose structure maps satisfy equation (6) with respect to $\langle \cdot, \cdot \rangle$. There is an isomorphism

$$R^e_\sigma \simeq \bigoplus_{(i \to j) \in Q_1^+} \text{Hom}_C(\mathbb{C}^{e_i}, \mathbb{C}^{e_j}) \oplus \bigoplus_{(i \to \sigma(\alpha)) \in Q_1^+} \text{Bil}_C^{e_{\alpha}}(\mathbb{C}^{e_i}),$$

where $\text{Bil}_C^{e}(\mathbb{C}^{e_i})$ is the vector space of symmetric ($\epsilon = 1$) or skew-symmetric ($\epsilon = -1$) bilinear forms on $\mathbb{C}^{e_i}$. The subgroup $G^e_\sigma \leq \text{GL}_e$ which preserves $\langle \cdot, \cdot \rangle$ is

$$G^e_\sigma \simeq \prod_{i \in Q_0^\sigma} \text{GL}_{e_i}(\mathbb{C}) \times \prod_{i \in Q_0^+} G^e_{e_i},$$
where
\[
\mathbb{G}^e_{s_i} = \begin{cases} 
\text{Sp}_{e_i}(\mathbb{C}) & \text{if } s_i = -1, \\
\text{O}_{e_i}(\mathbb{C}) & \text{if } s_i = 1.
\end{cases}
\]

The group \( \mathbb{G}^e \) acts linearly on \( R^e \). Its orbits are in bijection with the set of isometry classes of self-dual representations of dimension vector \( e \).

Let \( U \in \text{Rep}_C(Q) \). The pair \( (S, \Theta_U) \) gives \( \text{Ext}^i(S(U), U) \) the structure of a representation of \( \mathbb{Z}_2 \). Write \( \text{Ext}^i(S(U), U)^\pm \) for the subspaces of invariant (\( + \)) and anti-invariant (\( - \)) elements and define
\[
\mathcal{E}(U) = \dim_{\mathbb{C}} \text{Hom}(S(U), U)^- - \dim_{\mathbb{C}} \text{Ext}^1(S(U), U)^+.
\]

It was proved in [44, Proposition 3.3] that \( \mathcal{E}(U) \) depends only on the dimension vector of \( U \) and that the resulting function \( \mathcal{E} : \Lambda Q \to \mathbb{Z} \) is
\[
\mathcal{E}(d) = \sum_{i \in Q^e_0} \frac{d_i(d_i - s_i)}{2} + \sum_{i \in Q^e_0} d_{\sigma(i)}d_i - \sum_{(\sigma(i) \overset{\alpha}{\rightarrow} j) \in Q^e_1} \frac{d_i(d_i + \tau_\alpha s_j)}{2} - \sum_{(i \overset{\alpha}{\rightarrow} j) \in Q^e_1} d_{\sigma(i)}d_j.
\] (7)

It is straightforward to verify that the following identity holds:
\[
\mathcal{E}(d + d') = \mathcal{E}(d) + \mathcal{E}(d') + \chi(\sigma(d), d'), \quad d, d' \in \Lambda Q.
\] (8)

Self-dual representations admit reductions along isotropic subrepresentations. More precisely, let \( Z \) be a self-dual representation and let \( U \subset Z \) be an isotropic subrepresentation, that is, a subrepresentation whose underlying complex vector space is isotropic. Then the orthogonal complement \( U^\perp \subset Z \) is a subrepresentation which contains \( U \). The quotient representation \( U^\perp / U \) then inherits a canonical self-dual structure, henceforth denoted by \( Z / U \).

**Remark.** The bounded derived category \( D^b(\Gamma_Q \text{-mod}) \) of the Ginzburg dg algebra \( \Gamma_Q \) is a three dimensional Calabi–Yau category for which \( \text{Rep}_C(Q) \) is the heart of a bounded \( t \)-structure [16]. The data \( (\sigma, s, \tau) \) induces a triangulated duality structure on \( D^b(\Gamma_Q \text{-mod}) \) which, up to a sign, preserves the Calabi–Yau pairings. This gives an abstract version of the three dimensional Calabi–Yau orientifolds considered in the string theory literature [9,22], and explains how this paper fits into the general setting of DT theory of three dimensional Calabi–Yau categories.

### 1.5. Moduli spaces

We recall how the standard theory of stability of quiver representations can be adapted to the self-dual setting. For details, see [25] and [43, §3] in the cases of ordinary and self-dual representations, respectively.

A stability \( \theta \in \text{Hom}_Z(\Lambda Q, \mathbb{Z}) \) is called \( \sigma \)-compatible if it satisfies \( \sigma^* \theta = -\theta \). Fix a \( \sigma \)-compatible stability \( \theta \). The slope of a non-zero representation \( U \) is
\[
\mu(U) = \frac{\theta(\text{dim } U)}{\text{dim } U} \in \mathbb{Q}.
\]
The slope of a self-dual representation is necessarily zero. A self-dual representation \( Z \) is called \( \sigma \)-semistable if \( \mu(U) \leq \mu(Z) \) for all non-zero isotropic subrepresentations \( U \subset Z \). If this inequality is strict, then \( Z \) is called \( \sigma \)-stable. The moduli scheme of \( \sigma \)-semistable self-dual representations of dimension vector \( e \) is the \( \theta \)-linearized geometric invariant theory quotient \( \mathcal{M}_e^{\sigma, \text{st}} = R_e^{\sigma} / \theta G_e^{\sigma} \). The open subscheme \( \mathcal{M}_e^{\sigma, \text{st}} \subset \mathcal{M}_e^{\sigma, \text{ss}} \) parameterizes isometry classes of \( \sigma \)-stable self-dual representations and has at worst orbifold singularities. Up to isometry, a \( \sigma \)-stable self-dual representation admits a unique orthogonal decomposition \( Z = \bigoplus_{i=1}^{k} Z_i \), where \( Z_1, \ldots, Z_k \) are pairwise non-isometric self-dual representations, each of which is stable as an ordinary representation [43, Proposition 3.5]. If \( k = 1 \), then \( Z \) is called regularly \( \sigma \)-stable. By convention \( \mathcal{M}_{0}^{\sigma, \text{st}} = \text{Spec}(\mathbb{C}) \).

### 1.6. A module over the quantum torus

Let \( q^{\frac{1}{\mathbb{Z}}} \) be a formal variable. Following [26, §6.2], the quantum torus \( \hat{T}_Q = \mathbb{Q}(q^{\frac{1}{\mathbb{Z}}})[[\Lambda^+_Q]] \) is the \( \mathbb{Q}(q^{\frac{1}{\mathbb{Z}}}) \)-vector space with topological basis \( \{ t^d \mid d \in \Lambda^+_Q \} \) and multiplication

\[
t^d \cdot t^{d'} = q^{\frac{1}{2}(\chi(d,d') - \chi(d',d))} t^{d + d'}.
\]

If \( Q \) has an involution and duality structure, then, as in [43, §4.1], we can also consider the vector space \( \hat{S}_Q = \mathbb{Q}(q^{\frac{1}{\mathbb{Z}}})[[\Lambda^{\sigma,+}_Q]] \) with topological basis \( \{ e \mid e \in \Lambda^{\sigma,+}_Q \} \). The formula

\[
t^d \ast e = q^{\frac{1}{2}(\chi(d,e) - \chi(e,d) + \mathcal{E}^\sigma(e,d) - \mathcal{E}^\sigma(d,e))} t^{\mathcal{H}(d) + e}.
\]

gives \( \hat{S}_Q \) the structure of a left \( \hat{T}_Q \)-module, as follows easily from equation (8).

### 2. Cohomological Hall Algebras

#### 2.1. Definition of the CoHA

Fix a quiver \( Q \). Let \( \text{Vect} \) be the category of finite dimensional rational vector spaces and let \( D^{ib}(\text{Vect}) \subset D(\text{Vect}) \) be the full subcategory of the unbounded derived category consisting of objects whose cohomological degree is bounded from below. Let \( D^{ib}(\text{Vect})_{\Lambda^+_Q} \) be the category of \( \Lambda^+_Q \)-graded objects of \( D^{ib}(\text{Vect}) \). Denote by [1] : \( D^{ib}(\text{Vect})_{\Lambda^+_Q} \rightarrow D^{ib}(\text{Vect})_{\Lambda^+_Q} \) the cohomological shift functor and by \([-1]\) its inverse. Given \( n \in \mathbb{Z} \), write \([n]\) for the iterated composition \([1]^n\).

Define a monoidal structure \( \boxdot \) on \( D^{ib}(\text{Vect})_{\Lambda^+_Q} \) by

\[
\bigoplus_{d \in \Lambda^+_Q} \mathcal{U}_d \boxdot \bigoplus_{d \in \Lambda^+_Q} \mathcal{V}_d = \bigoplus_{d \in \Lambda^+_Q} \biggl( \bigoplus_{d' + d'' = d} \mathcal{U}_{d'} \boxdot \mathcal{V}_{d''} \biggr) \chi(d', d'') - \chi(d', d'').
\]

Fix \( d', d'' \in \Lambda^+_Q \) and put \( d = d' + d'' \). Let \( \mathbb{C}^{d'} \subset \mathbb{C}^{d} \) be the \( Q_0 \)-graded subspace spanned by the first \( d' \) coordinate directions. Let \( R_{d', d''} \subset R_d \) be the subspace of representations which preserve \( \mathbb{C}^{d'} \) and let \( \text{GL}_{d', d''} \leq \text{GL}_{d} \) be the subgroup which preserves \( \mathbb{C}^{d'} \). The cohomological Hall algebra (henceforth CoHA) of \( Q \) is

\[
\mathcal{H}_Q = \bigoplus_{d \in \Lambda^+_Q} H^*_\text{GL}_d(R_d)[-\chi(d, d)] \in D^{ib}(\text{Vect})_{\Lambda^+_Q}.
\]
For each $d \in \Lambda^+_Q$, write $\mathcal{H}_{Q,d}$ for the degree $d$ summand of $\mathcal{H}_Q$. Then $\mathcal{H}_{Q,d} \in D^{ib}(\text{Vect})$, which we regard as a complex with trivial differential. The $k$th cohomology of $\mathcal{H}_{Q,d}$ is

$$\mathcal{H}_{Q,(d,k)} := H^k(\mathcal{H}_{Q,d}) = H^k_{\text{GL}_d}(R_d).$$

(9)

The CoHA multiplication $\mathcal{H}_Q \boxtimes \mathcal{H}_Q \to \mathcal{H}_Q$ is defined so that its restriction $\mathcal{H}_{Q,d'} \boxtimes \mathcal{H}_{Q,d''}$ is induced by the composition

$$H^k_{\text{GL}_{d'}}(R_{d'}) \otimes H^k_{\text{GL}_{d''}}(R_{d''}) \xrightarrow{\sim} H^{k+k''}_{\text{GL}_{d'} \times \text{GL}_{d''}}(R_{d'} \times R_{d''}) \xrightarrow{\sim} H^{k+k'' + 2\Delta_1}_{\text{GL}_{d'} \times \text{GL}_{d''}}(R_{d'} \times R_{d''}) \xrightarrow{\sim} H^{k+k'' + 2\Delta_1+2\Delta_2}_{\text{GL}_{d}}(R_d),$$

where we have set

$$\Delta_1 = \dim \mathbb{C} R_d - \dim \mathbb{C} R_{d'}, \quad \Delta_2 = \dim \mathbb{C} \text{GL}_{d'}, \dim \text{GL}_{d''} - \dim \mathbb{C} \text{GL}_d.$$

The maps in the composition are constructed from the morphisms

$$R_{d'} \times R_{d''} \xrightarrow{\pi} R_{d'}, \quad \text{GL}_{d'} \times \text{GL}_{d''} \xrightarrow{p} \text{GL}_{d'} \times \text{GL}_{d''} \xrightarrow{\iota} \text{GL}_{d},$$

(10)

The first map in the above composition is the Künneth map, the second is induced by the homotopy equivalences $\pi$ and $p$, the third is pushforward along the $\text{GL}_{d'}, \text{GL}_{d''}$-equivariant closed inclusion $\iota$ and the final is pushforward along $\text{GL}_{d}/\text{GL}_{d'}, \text{GL}_{d''}$, the fibre of $\text{BGL}_{d'}, \text{GL}_{d''} \to \text{BGL}_d$. The degree shift is $\Delta_1 + \Delta_2 = -\chi(d', d'')$. It is shown in [27, Theorem 1] that this multiplication gives $\mathcal{H}_Q$ the structure of an associative algebra object of $D^{ib}(\text{Vect})_{\Lambda^+_Q}$.

The CoHA multiplication can be written explicitly using localization in equivariant cohomology. To do so, identify $\mathcal{H}_{Q,d}$ with the subspace of polynomials in $\{x_{i,1}, \ldots, x_{i,d_i}\}_{i \in Q_0}$ which are invariant under the Weyl group $\mathcal{S}_d = \prod_{i \in Q_0} \mathcal{S}_{d_i}$ of $\text{GL}_d$. We view the product of $f_1 \in \mathcal{H}_{Q,d'}$ and $f_2 \in \mathcal{H}_{Q,d''}$ as a polynomial in $\{x_{i,1}, \ldots, x_{i,d_i}\}_{i \in Q_0}$ by identifying $x'_{i,k}$ and $x''_{j,k}$ with $x_{i,k}$ and $x_{i,d_i+k}$, respectively.

Let $\mathcal{S}_{d', d''} \subset \mathcal{S}_d$ be the set of 2-shuffles of type $(d', d'')$, that is, elements $\{\pi_i\}_{i \in Q_0} \in \mathcal{S}_d$ which satisfy

$$\pi_i(1) < \cdots < \pi_i(d_i'), \quad \pi_i(d_i' + 1) < \cdots < \pi_i(d_i), \quad i \in Q_0.$$

Then $\mathcal{S}_{d', d''}$ acts on polynomials in $\{x_{i,1}, \ldots, x_{i,d_i}\}_{i \in Q_0}$ via the action of $\mathcal{S}_d$.

**Theorem 2.1** [27, Theorem 2]. The product of $f_1 \in \mathcal{H}_{Q,d'}$ and $f_2 \in \mathcal{H}_{Q,d''}$ is

$$f_1 \cdot f_2 = \sum_{\pi \in \mathcal{S}_{d', d''}} \pi \left( f_1(x') f_2(x'') \prod_{(i \to j) \in Q_1} \prod_{b=1}^{d'_j} \prod_{a=1}^{d'_i} (x''_{j,b} - x'_{i,a}) \right).$$

Given $V \in D^{ib}(\text{Vect})$, set

$$\chi_q(V) = \sum_{i \in \mathbb{Z}} (-1)^i \dim \mathcal{Q} H^i(V) q^i \in \mathbb{Z}(q^\frac{1}{2}).$$
The class of \( W \in \text{D}^b(\text{Vect})_{\Lambda_+^Q} \) in \( K_0(\text{D}^b(\text{Vect})_{\Lambda_+^Q}) \) is then

\[
\chi_{q,t}(W) = \sum_{d \in \Lambda_+^Q} \chi_q(W_d) t^d \in \mathbb{Z}(q^{1/2})[\Lambda_+^Q].
\]

With this notation, the motivic DT series of \( Q \) is

\[
A_Q(q^{1/2}, t) := \chi_{q,t}(\mathcal{H}_Q) = \sum_{(d,k) \in \Lambda_+^Q \times \mathbb{Z}} \text{dim}_Q \mathcal{H}_{Q,(d,k)}(-q^{1/2})^k t^d \in \mathbb{Z}(q^{1/2})[\Lambda_+^Q].
\]

It can be written explicitly as

\[
A_Q(q^{1/2}, t) = \sum_{d \in \Lambda_+^Q} \frac{(-q^{1/2})^{\chi(d,d)}}{\prod_{i \in Q_0} \prod_{j=1}^{d_i} (1 - q^j)} t^d.
\]

Passing from motivic DT series to motivic DT invariants is most easily explained for symmetric quivers. We do this in the next section.

2.2. The CoHA of a symmetric quiver. In this section we consider symmetric quivers, that is, quivers whose Euler form \( \chi \) is symmetric. In this case, \( \boxtimes^\text{tw} \) reduces to the symmetric monoidal structure \( \boxtimes \) induced by the monoid structure of \( \Lambda_+^Q \), with no additional cohomological degree shift. In particular, we can regard \( \mathcal{H}_Q \) as a \( \Lambda_+^Q \times \mathbb{Z} \)-graded algebra, the grading of \( \mathcal{H}_Q \) being determined by equation (9).

Define a \( \mathbb{Z}_2 \)-grading on \( \mathcal{H}_Q \) as the reduction modulo two of its \( \mathbb{Z} \)-grading. If the Euler form satisfies

\[
\chi(d, d') \equiv \chi(d, d') \chi(d', d') \mod 2, \quad d, d' \in \Lambda_+^Q
\]

then \( \mathcal{H}_Q \) is a supercommutative algebra. Writing \( a_{ij} \) for the number of arrows from \( i \) to \( j \), equation (11) holds if and only if

\[
a_{ij} \equiv (1 + a_{ii})(1 + a_{jj}) \mod 2
\]

for all distinct \( i, j \in Q_0 \). If \( \chi \) does not satisfy equation (11), then the CoHA multiplication can be twisted by a sign so as to make it supercommutative [27, §2.6]. As we will not use the explicit form of the twist, we do not recall it here.

Write \( \text{Sym}(V) \) for the free supercommutative algebra generated by a \( \Lambda_+^Q \times \mathbb{Z} \)-graded vector space \( V \). The following result was conjectured by Kontsevich and Soibelman [27, Conjecture 1]. We consider \( \mathcal{H}_Q \) as a supercommutative algebra.

**Theorem 2.2** [11, Theorem 1.1]. Let \( Q \) be a symmetric quiver and let \( u \) be a formal variable of degree \((0,2)\). There exists a \( \Lambda_+^Q \times \mathbb{Z} \)-graded rational vector space of the form \( V_Q = V_Q^{\text{prim}} \otimes \mathbb{Q}[u] \) such that \( \text{Sym}(V_Q) \simeq \mathcal{H}_Q \) as algebras. Moreover, each \( \Lambda_+^Q \)-homogeneous summand \( V_{Q,d}^{\text{prim}} \subset V_Q^{\text{prim}} \) is finite dimensional.
Without the supercommutative twist, the isomorphism \( \text{Sym}(V_Q) \cong \mathcal{H}_Q \) is to be understood as objects of \( D^b(\text{Vect})_{\Lambda_Q^+} \). The second part of Theorem 2.2, which is known as the integrality conjecture [26], asserts that \( V_Q^\text{prim} \) is an element of \( D^b(\text{Vect})_{\Lambda_Q^+} \subset D^b(\text{Vect})_{\Lambda_Q^+}' \), the full subcategory of objects whose \( \Lambda_Q^+ \)-homogeneous components lie in \( D^b(\text{Vect}) \).

**Definition.** The motivic DT invariant of a symmetric quiver is the class of \( V_Q^\text{prim} \) in \( K_0(D^b(\text{Vect})_{\Lambda_Q^+}) \):

\[
\Omega_Q(q^{\frac{1}{2}}, t) = \sum_{(d, k) \in \Lambda_Q^+ \times \mathbb{Z}} \dim_{\mathbb{Q}} V_{Q,(d,k)}^{\text{prim}} (-q^{\frac{1}{2}})^k t^d \in \mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}] [\Lambda_Q^+] \]

Since \( Q \) is symmetric, the parity-twisted Hilbert–Poincaré series of \( \mathcal{H}_Q \) coincides with \( A_Q \). Using this observation, Theorem 2.2 implies that \( A_Q \) can be written as a product of \( q \)-Pochhammer symbols \( (t; q)_{\infty} = \prod_{i \geq 0} (1 - q^i t) \).

**Corollary 2.3** [11, Corollary 4.1]. Let \( Q \) be a symmetric quiver. Then

\[
A_Q(q^{\frac{1}{2}}, t) = \prod_{(d, k) \in \Lambda_Q^+ \times \mathbb{Z}} (q^{\frac{k}{2}} t^d; q)_{\infty}^{-\Omega_{Q,(d,k)}}
\]

where \( \Omega_{Q,(d,k)} \) is the coefficient of \( q^{\frac{k}{2}} t^d \) in \( \Omega_Q \).

To end this section, we recall a geometric interpretation of \( V_Q^\text{prim} \). Let \( M^s_d \) be the stack of stable representations of dimension vector \( d \) with respect to the trivial stability, \( \theta = 0 \). The canonical map \( M^s_d \to M^s_d \) is a \( \mathbb{C}^\times \)-gerbe and induces an isomorphism of mixed Hodge structures \( H^\bullet(M^s_d) \cong H^\bullet(M^s_d) \otimes \mathbb{Q}[u] \).

**Theorem 2.4** ([2, Theorem 2.2]). Let \( Q \) be the double of a quiver. For each \( d \in \Lambda_Q^+ \), the restriction map \( H^\bullet_{\text{GL}_d}(R_d) \to H^\bullet(M^s_d) \) induces an isomorphism of \( \mathbb{Z} \)-graded vector spaces \( V_{Q,d}^{\text{prim}} \cong \overline{P H^\bullet - \chi(d,d)}(M^s_d) \).

Other geometric interpretations of \( V_Q^\text{prim} \) (or \( \Omega_Q \)) can be found in [6,21,31].

### 3. Cohomological Hall Modules

We introduce the cohomological Hall module of a quiver, establish its basic properties and formulate the main conjectures regarding its structure.

#### 3.1. Definition of the CoHM

Fix a quiver with involution \((Q, \sigma)\) and duality structure \((s, \tau)\). Using the group homomorphism (5) and equation (8), we verify that \( D^b(\text{Vect})_{\Lambda_Q^+} \) becomes a left module category over the monoidal category \( (D^b(\text{Vect})_{\Lambda_Q^+}, \boxtimes) \) via the formula

\[
\bigoplus_{d \in \Lambda_Q^+} \mathcal{U}_d \boxtimes S^{\text{tw}} \bigoplus_{e \in \Lambda_Q^+} \mathcal{X}_e = \bigoplus_{e \in \Lambda_Q^+} \left( \bigoplus_{(d', e'') \in \Lambda_Q^+ \times \Lambda_Q^+} \mathcal{U}_{d'} \otimes \mathcal{X}_{e''}[\gamma(d', e'')] \right),
\]
where we have set
\[ \gamma(d, e) = \chi(d, e) - \chi(e, d) + \mathcal{E}(\sigma(d)) - \mathcal{E}(d). \]

Fix \( d \in \Lambda^+_Q \) and \( e \in \Lambda^\sigma_+ Q \). Let \( R^\sigma_{d,e} \subset R^\sigma_{H(d)+e} \) be the subspace consisting of structure maps on the orthogonal direct sum \( H(\mathbb{C}^d) \oplus \mathbb{C}^e \) which preserve the canonical \( Q_0 \)-graded isotropic subspace \( \mathbb{C}^d \). Then \( R^\sigma_{d,e} \) can be identified with the subspace of

\[ R_d \oplus R^\sigma_e \oplus \bigoplus_{(i \xrightarrow{\alpha} j) \in Q} \text{Hom}_C(\mathbb{C}^{e_i}, \mathbb{C}^{d_j}) \oplus \bigoplus_{(i \xrightarrow{\alpha} j) \in Q} \text{Hom}_C((\mathbb{C}^{d_i(\sigma)}), (\mathbb{C}^{d_j})) \]
defined by the condition that its component \( \{ u_{\alpha} \} \) in the final summand satisfies \( \Theta_{\mathcal{E}^d} u_{\alpha} = -\tau_{\alpha} u_{\sigma(\alpha)} \). Let \( G^\sigma_{d,e} \leq G^\sigma_{H(d)+e} \) be the subgroup which preserves \( \mathbb{C}^d \).

The cohomological Hall module (henceforth CoHM) is

\[ \mathcal{I}_Q = \bigoplus_{e \in \Lambda^\sigma_+} \mathcal{H}^\bullet_{G^\sigma_e}(R^\sigma_e)[−\mathcal{E}(e)] \in D^{ib}(\text{Vect})_{\Lambda^\sigma_+}. \]

Write \( \mathcal{I}_{Q,e} \in D^{ib}(\text{Vect}) \) for the degree \( e \in \Lambda^\sigma_+ \) summand of \( \mathcal{I}_Q \) and set

\[ \mathcal{I}_{Q,(e,l)} := H^l(\mathcal{I}_{Q,e}) = H^l_{G^\sigma_e}(R^\sigma_e). \quad (12) \]

The CoHA action map \( \ast : \mathcal{H}_Q \boxtimes \mathcal{S}^{\text{tw}} \mathcal{I}_Q \to \mathcal{I}_Q \) is defined so that its restriction to \( \mathcal{H}_Q \boxtimes \mathcal{S}^{\text{tw}} \mathcal{I}_{Q,e} \) is induced by the composition

\[ H^k_{G^\sigma_e}(R_d) \otimes H^l_{G^\sigma_e}(R^\sigma_e) \xrightarrow{\sim} H^{k+l}_{G^\sigma_e \times G^\sigma_e}(R_d \times R^\sigma_e) \xrightarrow{\sim} \]

\[ H^{k+l}_{G^\sigma_{d,e}}(R^\sigma_{d,e}) \to H^{k+l+\delta_1}_{G^\sigma_{d,e}}(R^\sigma_{H(d)+e}) \to H^{k+l+2\delta_1+2\delta_2}_{G^\sigma_{H(d)+e}}(R^\sigma_{H(d)+e}), \]

where we have set

\[ \delta_1 = \dim_{\mathbb{C}} R^\sigma_{H(d)+e} - \dim_{\mathbb{C}} R^\sigma_{d,e}, \quad \delta_2 = \dim_{\mathbb{C}} G^\sigma_{d,e} - \dim_{\mathbb{C}} G^\sigma_{H(d)+e}. \]

The maps in this composition are defined analogously to those appearing in the CoHA multiplication, although with the maps (10) having been replaced by

\[ R_d \times R^\sigma_e \xleftarrow{\pi} R^\sigma_{d,e} \xrightarrow{i} R^\sigma_{H(d)+e}, \quad \text{GL}_d \times G^\sigma_e \xleftarrow{p} G^\sigma_{d,e} \xrightarrow{j} G^\sigma_{H(d)+e}. \]

A short calculation shows that \( \delta_1 + \delta_2 = -\chi(d, e) - \mathcal{E}(\sigma(d)) \).

**Theorem 3.1.** The map \( \ast \) gives \( \mathcal{I}_Q \) the structure of a left \( \mathcal{H}_Q \)-module object of \( D^{ib}(\text{Vect})_{\Lambda^\sigma_+} \).

**Proof.** The commutative diagram used to prove associativity of the CoHA in [27, §2.3] has a natural modification in the self-dual setting, obtained by requiring that the structure maps and isometry groups preserve multi-step isotropic flags. This modified commutative diagram establishes the \( \mathcal{H}_Q \)-module structure of \( \mathcal{I}_Q \). \( \square \)
The numerical Witt group \( W(Q) \) is the abelian group defined by the exact sequence

\[
\Lambda_Q \xrightarrow{H} \Lambda_Q^\sigma \xrightarrow{\nu} W(Q) \to 0.
\]

Explicitly, \( W(Q) \cong \prod_{i \in Q_0^\sigma} \mathbb{Z}_2 \) with \( \nu \) sending a dimension vector to its parities at the nodes \( Q_0^\sigma \). The following result is immediate.

**Proposition 3.2.** For each \( w \in W(Q) \), the direct sum

\[
\mathcal{I}_Q^w = \bigoplus_{\{v \in \Lambda_Q^\sigma | \nu(v) = w\}} \mathcal{I}_{Q,v}
\]

is a \( \mathcal{H}_Q \)-submodule of \( \mathcal{I}_Q \) and \( \mathcal{I}_Q = \bigoplus_{w \in W(Q)} \mathcal{I}_Q^w \) as \( \mathcal{H}_Q \)-modules. Moreover, \( \mathcal{I}_Q^w \) is trivial unless \( s_i = 1 \) for all \( i \in Q_0^\sigma \) with \( w_i = 1 \).

The motivic orientifold DT series of \( Q \) is the class of \( \mathcal{I}_Q \) in \( K_0(D^b(Vect)_{\Lambda_Q^\sigma}) \):

\[
A_Q^\sigma(q^{\frac{1}{2}}, \xi) := \chi_{q,I}(\mathcal{I}_Q) = \sum_{(e,I) \in \Lambda_Q^\sigma \times \mathbb{Z}} \dim_{\mathcal{H}_Q,I,e}(-q^{\frac{1}{2}})^{\chi_e} \xi \in \mathbb{Z}(q^{\frac{1}{2}})[\Lambda_Q^\sigma].
\]

Using the equivariant contractibility of \( R^\sigma_e \) and the isomorphisms (4), we find

\[
A_Q^\sigma = \sum_{e \in \Lambda_Q^\sigma} \frac{(-q^{\frac{1}{2}})^{\mathcal{E}(e)}}{\prod_{i \in Q_0^\sigma} \prod_{j=1}^{\mathcal{E}(e)} (1 - q^{e_j}) \prod_{i \in Q_0^\sigma} \prod_{j=1}^{\mathcal{E}(e)} (1 - q^{2j})} \xi^e.
\]  

In what follows we will view \( A_Q^\sigma \) as an element of the \( \mathbb{T}_Q \)-module \( \hat{S}_Q \).

**Remark.** Also inspired by DT theory, in [43] a different series was associated to a quiver with duality structure. Given a finite field \( \mathbb{F}_q \) with \( q \) odd, the \( \mathcal{E} \)-weighted number of \( \mathbb{F}_q \)-rational points of the stack of self-dual representations of \( Q \) is

\[
\mathcal{A}_{Q,\mathbb{F}_q}(\xi) = \sum_{Z} \frac{(-q^{\frac{1}{2}})^{\mathcal{E}(dim Z)}}{\#Aut_Z(Z)} \xi^{\dim Z},
\]

the sum being over isometry classes of self-dual representations. A comparison of equation (13) and a renormalized version of [43, Proposition 4.2] shows that \( A_Q^\sigma(q^{-\frac{1}{2}}, \xi) = \mathcal{A}_{Q,\mathbb{F}_q}(\xi) \). It follows that the cohomological and finite field approaches to orientifold DT theory are compatible.

### 3.2. The CoHM as a signed shuffle module

Let \( d \in \Lambda_Q^+ \) and \( e \in \Lambda_Q^{\sigma, +} \). Using the isomorphism (4), we identify \( \mathcal{I}_{Q,e} \) with the vector space of \( \prod_{i \in Q_0^\sigma} \mathbb{S}_{e_i} \times \prod_{i \in Q_0^\sigma} \mathbb{S}_{\mathcal{Q}_i} \) invariant polynomials in the variables

\[
\{z_{i,1}, \ldots, z_{i,e_i}\}_{i \in Q_0^\sigma}, \quad \{z_{i,1}^2, \ldots, z_{i,e_i}^2\}_{i \in Q_0^\sigma}.
\]

We will also identify polynomials in

\[
\{x_{i,1}, \ldots, x_{i,d_i}\}_{i \in Q_0^\sigma}, \quad \{z_{i,1}^\prime, \ldots, z_{i,e_i}^\prime\}_{i \in Q_0^\sigma}, \quad \{z_{i,1}^\prime, \ldots, z_{i,e_i}^\prime\}_{i \in Q_0^\sigma}.
\]
with polynomials in
\[ \{z_{i,1}, \ldots, z_{i,d_i+e_i+d_{\sigma(i)}} \}_{i \in Q_0^+}, \quad \{z_{i,1}, \ldots, z_{i,d_i+\left\lceil \frac{e_i}{2} \right\rceil} \}_{i \in Q_0^\sigma} \] (14)
via the assignments
\[ x'_{i,j} \mapsto z_{i,j}, \quad z''_{i,j} \mapsto z_{i,d_i+j}, \quad x'_{\sigma(i),j} \mapsto -z_{i,d_i+e_i+j}, \quad i \in Q_0^+ \]
and
\[ x'_{i,j} \mapsto z_{i,j}, \quad z''_{i,j} \mapsto z_{i,d_i+j}, \quad i \in Q_0^\sigma. \]
The signs in this identification arise from the first part of Lemma 1.1.

Given \( m, n, p \in \mathbb{Z}_{\geq 0} \), let \( \mathfrak{sh}_{m,n,p} \subset \mathfrak{S}_{m+n+p} \) be the set of 3-shuffles of type \((m, n, p)\). The set of \( \sigma \)-shuffles of type \((d, e) \in \Lambda_Q^+ \times \Lambda_Q^{\sigma,+}\) is defined to be
\[ \mathfrak{sh}_{d,e}^\sigma = \prod_{i \in Q_0^+} \mathfrak{sh}_{d_i,e_i,d_{\sigma(i)}} \times \prod_{i \in Q_0^\sigma} \left( \mathbb{Z}_2 \times \mathfrak{sh}_{d_i,\left\lceil \frac{e_i}{2} \right\rceil} \right). \]
There is a natural action of \( \mathfrak{sh}_{d,e}^\sigma \) on the space of polynomials in the variables (14), the shuffle factors acting through the symmetric group and the factors of \( \mathbb{Z}_2 \) acting by signs on the first \( d_i \) elements of \( \{z_{i,1}, \ldots, z_{i,d_i+\left\lceil \frac{e_i}{2} \right\rceil} \}_{i \in Q_0^\sigma} \).

For each \( i \in Q_0 \), define \( \varepsilon_i : \Lambda_Q \to \{0, 1\}, \ e \mapsto e_i \mod 2 \). Introduce the notation \( \leq_+ 1 \) for \( \leq \) and \( \leq_{-1} \) for \( < \).

**Theorem 3.3.** Let \( f \in \mathcal{H}_{Q,d} \) and \( g \in \mathcal{I}_{Q,e} \). The CoHA action map is given by
\[ f \star g = \sum_{\pi \in \mathfrak{sh}_{d,e}^\sigma} \prod_{a \in Q_0^+ \cup Q_0^\sigma} V_a(x', z'') \prod_{i \in Q_0^+ \cup Q_0^\sigma} D_i(x', z''), \]
where the numerators and denominators are defined as follows:
If \( i \in Q_0^+ \), then
\[ D_i = \prod_{k=1}^{e_i} \prod_{l=1}^{d_i} (z''_{i,k} - x'_{i,l}) \prod_{m=1}^{d_{\sigma(i)}} \prod_{l=1}^{d_i} (-x'_{\sigma(i),m} - x'_{i,l}) \prod_{m=1}^{d_{\sigma(i)}} (-x'_{\sigma(i),m} - z''_{i,k}). \]
If \( i \in Q_0^\sigma \), then
\[ D_i = g_i(x'_{i,1}, \ldots, x'_{i,d_i}) \prod_{1 \leq k < l \leq d_i} (-x'_{i,k} - x'_{i,l}) \prod_{l=1}^{d_i} \prod_{1 \leq k \leq l} (x''_{i,k} - z''_{i,k}). \]
with
\[ g_i(x_{i,1}, \ldots, x_{i,d_i}) = \begin{cases} \prod_{l=1}^{d_i} (-x_{i,l}) & \text{if } G_{2d_i+e_i}^i \text{ is of type } B, \\ \prod_{l=1}^{d_i} (-2x_{i,l}) & \text{if } G_{2d_i+e_i}^i \text{ is of type } C, \\ 1 & \text{if } G_{2d_i+e_i}^i \text{ is of type } D. \end{cases} \]
If \( i \rightarrow j \in Q^\sigma_1 \), then \( V_\alpha = \tilde{V}_\alpha^{(i)} \cdot \tilde{V}_\alpha^{(j)} \prod_{m=1}^{d_{\sigma(j)}} \prod_{l=1}^{d_i} (-x'_{\sigma(i),m} - x'_{i,l,j}) \) where

\[
\tilde{V}_\alpha^{(i)} = \begin{cases}
\prod_{m=1}^{d_{\sigma(j)}} \prod_{k=1}^{d_i} (-x'_{\sigma(i),m} - z''_{i,k}) & \text{if } i \not\in Q^\sigma_0, \\
\prod_{m=1}^{d_{\sigma(j)}} \prod_{l=1}^{d_i} (x''_{i,l,j} - z''_{i,k}) \prod_{l=1}^{d_i} (-x'_{\sigma(i),m}) & \text{if } i \in Q^\sigma_0.
\end{cases}
\]

and

\[
\tilde{V}_\alpha^{(j)} = \begin{cases}
\prod_{k=1}^{d_i} (z''_{i,k} - x'_{i,l,j}) & \text{if } j \not\in Q^\sigma_0, \\
\prod_{l=1}^{d_i} (x''_{i,l,j} - z''_{i,k}) \prod_{l=1}^{d_i} (-x'_{\sigma(i),m}) & \text{if } j \in Q^\sigma_0.
\end{cases}
\]

If \( \sigma(i) \rightarrow i \in Q^\sigma_1 \), then \( V_\alpha = \tilde{V}_\alpha \prod_{1 \leq j \leq \gamma_\alpha} (x'_{\sigma(i),j} - x'_{\sigma(i),k}) \) where

\[
\tilde{V}_\alpha = \begin{cases}
\prod_{k=1}^{d_i} (z''_{i,k} - x'_{\sigma(i),l}) & \text{if } i \not\in Q^\sigma_0, \\
\prod_{l=1}^{d_i} (x''_{\sigma(i),l} - z''_{i,k}) \prod_{l=1}^{d_i} (-x'_{\sigma(i),m}) & \text{if } i \in Q^\sigma_0.
\end{cases}
\]

Proof. Interpret \( f \) and \( g \) as classes in \( H^* (BGL_d \times BG^\sigma_\cdot) \). Let \( \text{eu}_{G^\sigma_{d,e}} (N_{R^\sigma_{H(d)+e}/R^\sigma_{d,e}}) \) be the \( G^\sigma_{d,e} \)-equivariant Euler class of the fibre of the normal bundle to \( R^\sigma_{d,e} \subset R^\sigma_{H(d)+e} \) at the origin. Then

\[
f \ast g = \int_{[G^\sigma_{H(d)+e}/G^\sigma_{d,e}]} f \cdot g \cdot \text{eu}_{G^\sigma_{d,e}} (N_{R^\sigma_{H(d)+e}/R^\sigma_{d,e}}),
\]

where \([G^\sigma_{H(d)+e}/G^\sigma_{d,e}]\) is the \( G^\sigma_{H(d)+e} \)-equivariant fundamental class of \( G^\sigma_{H(d)+e}/G^\sigma_{d,e} \). As in [27, §2.4], this integral can be computed by equivariant localization with respect to the maximal torus \( T = T_{H(d)+e} \leq G^\sigma_{H(d)+e} \).

Let \( U \in R_d \) and \( Z \in R^\sigma_{H(d)+e} \). Observe that an inclusion \( U \hookrightarrow Z \) is isotropic if and only if, for each arrow \( \alpha : i \rightarrow j \), we have a commutative diagram

\[
\begin{array}{ccc}
U_i & \hookrightarrow (U^\perp)_i & \hookrightarrow Z_i \\
\downarrow_{u_\alpha} & & \downarrow_{z_\alpha} \\
U_j & \hookrightarrow (U^\perp)_j & \hookrightarrow Z_j.
\end{array}
\]
We start by computing the equivariant Euler class of the tangent space at a T-fixed point of $G^\sigma_{H(d)+e}/G^\sigma_{d,e}$. The horizontal arrows in (15) lead to an isomorphism

$$G^\sigma_{H(d)+e}/G^\sigma_{d,e} \simeq \prod_{i \in Q_0^+} \text{Fl}(d_i, e_i, d_{\sigma(i)}) \times \prod_{i \in Q_0^-} \text{IGr}^\sigma_{i} (d_i, 2d_i + e_i)$$

where $\text{Fl}(a_1, a_2, a_3)$ is the variety of flags $0 = V_0 \subset V_1 \subset V_2 \subset \mathbb{C}^{a_1+a_2+a_3}$ with $\dim V_i / V_{i-1} = a_i$ and $\text{IGr}^i(a, b)$ is the variety of $a$-dimensional isotropic subspaces of a $b$-dimensional orthogonal ($s = 1$) or symplectic ($s = -1$) vector space. The T-fixed points of $\text{Fl}(d_i, e_i, d_{\sigma(i)})$ are two-step coordinate flags and so can be labelled by disjoint pairs of increasing sequences in $(1, \ldots, d_i + e_i + d_{\sigma(i)})$ of the form $\pi = \{a_1 < \cdots < a_{d_i}; \ b_1 < \cdots < b_{e_i}\}$. Such pairs are in bijection with $\mathfrak{h}_{d_i, e_i, d_{\sigma(i)}}$. The T-character of the tangent space to $\text{Fl}(d_i, e_i, d_{\sigma(i)})$ at the flag $U_i \subset (U^\perp)_i \subset \mathbb{Z}_i$ corresponding to the trivial shuffle is the product of the following weights:

$$\text{Hom}_C(U_i, (Z // U)_i) \sim \prod_{k=1}^{d_i} \prod_{l=1}^{d_i} (\zeta_{i,k}' - x_{i,l}')$$

$$\text{Hom}_C(U_i, U_{\sigma(i)}^\vee) \sim \prod_{m=1}^{d_{\sigma(i)}} \prod_{l=1}^{d_{\sigma(i)}} (-x_{\sigma(i),m}' - x_{i,l}')$$

$$\text{Hom}_C((Z // U)_i, U_{\sigma(i)}^\vee) \sim \prod_{m=1}^{d_{\sigma(i)}} \prod_{k=1}^{d_{\sigma(i)}} (-x_{\sigma(i),m}' - z_{i,k}'').$$

Here we have indicated the summands of the tangent space and their corresponding weights. Similarly, T-fixed points of $\text{IGr}^\sigma_{i}(d_i, 2d_i + e_i)$ are isotropic coordinate planes and so are in bijection with $\mathbb{Z}_2^{d_i} \times \mathfrak{h}_{d_i, d_i + [\frac{e_i}{2}]}$ via the map

$$\mathbb{Z}_2^{d_i} \times \mathfrak{h}_{d_i, d_i + [\frac{e_i}{2}]} \ni (p, \pi) \mapsto \text{span}_C \{v_{\pi(1), p_1}, \ldots, v_{\pi(d_i), p_{d_i}}\},$$

where, in the notation of Sect. 1.1, $v_{i,p}$ is $x_i$ or $y_i$ if $p = 1$ or $p = -1$, respectively. The T-character of the tangent space to $\text{IGr}^\sigma_{i}(d_i, 2d_i + e_i)$ at such a fixed point is the product of the positive roots of $G^\sigma_{2d_i+e_i}$ which are not in the corresponding parabolic Lie subalgebra. Using the conventions of Sect. 1.1, we arrive at the stated form of the denominators $D_i$.

Consider now the restriction of $\text{eu}_{G^\sigma_{d,e}}(N_{R^\sigma_{H(d)+e}/R^\sigma_{d,e}})$ to a T-fixed point. From the vertical arrows of (15), the contribution $V_\alpha$ of $\alpha \in Q_1^+$ to $\text{eu}_{G^\sigma_{d,e}}(N_{R^\sigma_{H(d)+e}/R^\sigma_{d,e}})$ is the product of the following weights:

$$\text{Hom}_C(U_i, (Z // U)_j) \sim \begin{cases} \prod_{k=1}^{d_i} \prod_{l=1}^{d_i} (\zeta_{i,k}' - x_{i,l}') & \text{if } j \notin Q_0^\sigma, \\ \prod_{k=1}^{d_i} \prod_{l=1}^{d_i} (\zeta_{i,k}' - x_{i,l}') & \text{if } j \in Q_0^\sigma \end{cases}$$

$$\text{Hom}_C(U_i, U_{\sigma(j)}^\vee) \sim \prod_{m=1}^{d_{\sigma(j)}} \prod_{l=1}^{d_{\sigma(j)}} (-x_{\sigma(j),m}' - x_{i,l}').$$
\[
\text{Hom}_\mathbb{C}((\mathbb{Z} / \mathbb{U})_i, U_{\sigma(j)}) \sim \begin{cases} 
\prod_{m=1}^{d_{\sigma(j)}} \prod_{k=1}^{e_i} (-x'_{\sigma(j),m} - z''_{i,k}) & \text{if } i \not\in Q_0^\sigma, \\
\prod_{m=1}^{d_{\sigma(j)}} \prod_{k=1}^{e_i} (\sigma_2'_{\sigma(j),m} - z''_{i,k}) \prod_{m=1}^{d_{\sigma(j)}} (-x'_{\sigma(j),m})^{e_i(e)} & \text{if } i \in Q_0^\sigma.
\end{cases}
\]

The contribution of an arrow \((\alpha : \sigma(i) \to i) \in Q_1^\sigma\) is computed similarly. Putting together the above calculations completes the proof. \qed

3.3. The orientifold integrality conjecture for \(\sigma\)-symmetric quivers. In the self-dual setting it is natural to impose the following stronger notion of symmetry.

**Definition.** A quiver with involution and duality structure is called \(\sigma\)-symmetric if it is symmetric and the equality \(E(d) = E(\sigma(d))\) holds for all \(d \in \Lambda_Q\).

Concretely, by using equation (7), we see that a symmetric quiver is \(\sigma\)-symmetric if and only if, for each \(i \in Q_0\), the following equality holds:

\[
\sum_{(\sigma(i) \to i) \in Q_1^\sigma} \tau_\alpha = \sum_{(i \to \sigma(i)) \in Q_1^\sigma} \tau_\alpha.
\]

Unlike all other places in the paper, the sums in equation (16) run over arrows with fixed head and tail. If \(Q\) is \(\sigma\)-symmetric, then \(\mathcal{X}^{S}\text{-tw}\) reduces to the untwisted \(D^{lb}(\text{Vect})_{\Lambda_Q^+}\) module structure \(\mathcal{X}^{S}\) determined by the \(\Lambda_Q^+\)-module \(\Lambda_Q^\sigma\), with no additional cohomological degree shift. In particular, the CoHM of a \(\sigma\)-symmetric quiver is a \(\Lambda_Q^\sigma \times \mathbb{Z}\)-graded \(\mathcal{H}_Q\)-module, the homogeneous components of \(\mathcal{I}_Q\) being given by equation (12).

**Remark.** In general, we do not know if \(\mathcal{I}_Q\) can be twisted so as to become a module over the supercommutative twist of \(\mathcal{H}\). We therefore consider \(\mathcal{H}_Q\) with its standard multiplication.

Let \(\mathcal{H}_{Q,+} = \oplus_{d \neq 0} \mathcal{H}_{Q,d}\) be the augmentation ideal of \(\mathcal{H}_Q\).

**Definition.** The cohomological orientifold DT invariant of a \(\sigma\)-symmetric quiver \(Q\) is

\[
W_Q^{\text{prim}} = \mathcal{I}_Q / (\mathcal{H}_{Q,+} \ast \mathcal{I}_Q) \in D^{lb}(\text{Vect})_{\Lambda_Q^\sigma+}.
\]

By picking a vector space splitting, we can regard \(W_Q^{\text{prim}}\) as a subobject of \(\mathcal{I}_Q\).

The following result confirms the orientifold analogue of the integrality conjecture.

**Theorem 3.4.** The \(\Lambda_Q^\sigma+\)-homogeneous summands of \(W_Q^{\text{prim}}\) are finite dimensional.

**Proof.** We generalize the proof of the integrality conjecture from [11, §3]. Define

\[
X_{Q,d} = \mathbb{Q}[x_{i,j} \mid i \in Q_0, 1 \leq j \leq d_i], \quad d \in \Lambda_Q^+
\]

and

\[
Z_{Q,e} = \mathbb{Q}[z_{i,j} \mid i \in Q_0^+, 1 \leq j \leq e_i] \otimes \mathbb{Q}[z_{i,j} \mid i \in Q_0^\sigma, 1 \leq j \leq \lfloor \frac{e_i}{2} \rfloor], \quad e \in \Lambda_Q^\sigma+.
\]

considered as \( \mathbb{Z} \)-graded algebras with generators in degree two. The Weyl groups \( \mathfrak{S}_d \) and \( \mathfrak{W}_e \) of \( \text{GL}_d \) and \( \mathfrak{G}_e \) act on \( X_{Q,d} \) and \( Z_{Q,e} \), respectively. Up to constant degree shifts, we have \( \mathcal{H}_{Q,d} = X_{\hat{\mathfrak{S}}_d,d} \) and \( \mathcal{I}_{Q,e} = Z_{\mathfrak{W}_e}^{\mathfrak{M}_e} \) as \( \mathbb{Z} \)-graded vector spaces.

Keeping the notation of Theorem 3.3, define
\[
\mathcal{K}_{d',e''}^{\mathfrak{M}_e}(x', z'') = \prod_{\alpha \in Q_1^{\subset} \cup Q_0^{\subset}} V_\alpha(x', z'') \prod_{\mu \in Q_0^+ \cup Q_0^\cdot} D_\mu(x', z'').
\]

Let \( Z_{Q,e}^{\mathfrak{loc}} \) be the localization of \( Z_{Q,e} \) at all factors of the denominators \( D_i \) of \( \mathcal{K}_{d',e''}^{\mathfrak{M}_e} \), as \((d', e'')\) ranges over the subset of elements of \( \Lambda^+ \times \Lambda^+ \) which satisfy \( H(d') + e'' = e \) and \( d' \neq 0 \). Let \( L_{Q,e} \subset Z_{Q,e}^{\mathfrak{loc}} \) be the smallest \( \mathfrak{W}_e \)-stable \( Z_{Q,e} \)-submodule which contains \( \mathcal{K}_{d',e''}^{\mathfrak{M}_e} \) for all \((d', e'')\) as above. We claim that \( L_{Q,e}^{\mathfrak{M}_e} \) is the image of the action map
\[
\bigoplus_{(d', e'') \in \Lambda_0^+ \times \Lambda_0^+, H(d') + e'' = e, d' \neq 0} \mathcal{H}_{Q,d'} \otimes^S \mathcal{I}_{Q,e''} \rightarrow \mathcal{I}_{Q,e}.
\]

That the image of \( \star \) is contained in \( L_{Q,e}^{\mathfrak{M}_e} \) follows from the observation that \( L_{Q,e}^{\mathfrak{M}_e} \) is \( \mathbb{Q} \)-linearly spanned by \( \mathfrak{M}_e \)-symmetrizations of elements of the form
\[
f \circ g \mathcal{K}_{d',e''}, \quad f \in X_{Q,d'}, \quad g \in Z_{Q,e''},
\]
with \((d', e'')\) as above. For the reverse inclusion, suppose that we are given an element of the form (17). By first symmetrizing with respect to \( \mathfrak{S}_d \) and \( \mathfrak{W}_e \), both of which are subgroups of \( \mathfrak{W}_e \), we may assume that \( f \in \mathcal{H}_{Q,d'} \) and \( g \in \mathcal{I}_{Q,e''} \). Then, up to a non-zero constant, the \( \mathfrak{M}_e \)-symmetrization of \( f \circ g \mathcal{K}_{d',e''} \) is \( f \star g \).

The theorem is therefore equivalent to the statement that, for each \( e \in \Lambda_0^+ \), the subspace \( L_{Q,e}^{\mathfrak{M}_e} \subset \mathcal{I}_{Q,e} \) has finite codimension. Let \( \hat{Q} \) be the quiver with involution obtained from \( Q \) by adding a loop at each node, with duality structure \( \tau = -1 \) for nodes in \( Q_0^\cdot \). Then \( L_{Q,e}^{\mathfrak{M}_e} \subset L_{\hat{Q},e} \), and it suffices to prove that \( L_{\hat{Q},e}^{\mathfrak{M}_e} \subset \mathcal{I}_{\hat{Q},e} = \mathcal{I}_{Q,e} \) has finite codimension. We therefore focus on \( \hat{Q} \) which, for ease of notation, we henceforth denote by \( Q \). In this case \( L_{Q,e} \subset \mathcal{I}_{Q,e} \) and we need not localize. Then we have an injection
\[
\mathcal{I}_{Q,e} / L_{Q,e}^{\mathfrak{M}_e} \hookrightarrow Z_{Q,e} / L_{Q,e}
\]
and it suffices to prove that \( L_{Q,e} \subset Z_{Q,e} \) has finite codimension. Interpret \( Z_{Q,e} \) as the algebra of functions on the affine space \( \mathbb{Q}^D \) of dimension
\[
D = \sum_{i \in Q_0^+} e_i + \sum_{i \in Q_0^\cdot} \left\lfloor \frac{e_i}{2} \right\rfloor
\]
and suppose that all elements of \( L_{Q,e} \) vanish at some point \( z \in \mathbb{Q}^D \). It suffices to prove that \( z = 0 \). Suppose then that \( z \neq 0 \). Replacing \( z \) with a \( \mathfrak{W}_e \)-translate if necessary, we will show that we can write \( z = \{z_i\}_{i \in Q_0^+ \cup Q_0^\cdot} \) as
\[
z_i = (x_{i,1}', \ldots, x_{i,d}', z_{i,1}', \ldots, z_{i,d}', -x_{\sigma(i),1}, \ldots, -x_{\sigma(i),d'}), \quad i \in Q_0^+
\]
and

\[ z_i = (x'_{i,1}, \ldots, x'_{i,d'_i}, z''_{i,1}, \ldots, z''_{i,1/2}), \quad i \in Q_0^\sigma \]

for some \( d' \neq 0 \), with the property that \( K^\sigma_{d',e''}(x', z'') \neq 0 \), yielding a contradiction. To do so, let \( z'' \) be the collection of vanishing coordinates of \( z \) and let \( x \) be what remains. By assumption \( x \neq 0 \). A pair of the form \( \{a, -a\} \), for some \( a \in \mathbb{Q} \), is called a ±-pair.

We can use the \( \mathfrak{W}_e \)-action to ensure that the following three conditions hold:

(i) There are no ±-pairs among all \( Q^+_0 \)-variables, and there are no ±-pairs among all \( Q^-_0 \)-variables. Indeed, we can use the symmetric subgroup of \( \mathfrak{W}_e \) on the \( Q^\pm_0 \)-variables to ensure that if \( a \) appears in the \( Q^+_0 \) variables, then \(-a\), if it occurs at all, occurs only in the \( Q^-_0 \) variables.

(ii) There are no ±-pairs among all \( Q^0_0 \)-variables. This can be ensured by using the action of the sign-change subgroup of \( \mathfrak{W}_e \) on the \( Q^0_0 \)-variables.

(iii) There are no ±-pairs among all \( Q^+_1 \)- and \( Q^-_0 \)-variables. This can be ensured (while preserving the previous condition) by using the action of the sign-change subgroup of \( \mathfrak{W}_e \).

By Theorem 3.3, the non-vanishing \( K^\sigma_{d',e''}(x', 0) \neq 0 \) is equivalent to the following conditions:1

(a) \( \prod_{m=1}^{d_n(i)} \prod_{l=1}^{d_l(i)} (-x'_{\sigma(i),m} - x'_{i,l}) \neq 0 \) if \( i \in Q^+_0 \).

(b) \( \prod_{1 \leq k < l \leq d_i} (x'_{i,k} + x'_{i,l}) \neq 0 \) if \( i \in Q^-_0 \).

(c) \( \prod_{1 \leq j \leq k \leq d_{\sigma(i)}} (-x'_{\sigma(i),j} - x'_{\sigma(i),k}) \neq 0 \) if \( (\sigma(i) \xrightarrow{\sigma} i) \in Q^+_1 \).

(d) \( \prod_{m=1}^{d_n(i)} \prod_{l=1}^{d_l(i)} (-x'_{\sigma(j),m} - x'_{i,l}) \neq 0 \) if \( (i \xrightarrow{\sigma} j) \in Q^-_1 \).

It is straightforward to verify that conditions (i)–(iii) imply conditions (a)–(d). This completes the proof. \( \square \)

**Definition.** The motivic orientifold DT invariant of a \( \sigma \)-symmetric quiver \( Q \) is the class of \( W^\text{prim}_Q \) in \( K_0(D^b(\mathbf{Vect})_{\Lambda^\sigma,+}) \):

\[
\Omega^\sigma_Q(q^{1/2}, \xi) = \sum_{(e,l) \in \Lambda^\sigma,+ \times \mathbb{Z}} \text{dim}_Q W^\text{prim}_Q(e,l)(-q^{1/2})^l \xi^e \in \mathbb{Z}[q^{1/2}, q^{-1/2}]\llbracket \Lambda^\sigma,+ \rrbracket.
\]

By Theorem 3.4, the numerical orientifold DT invariant can be defined as the \( q^{1/2} \mapsto 1 \) specialization of \( \Omega^\sigma_Q(q^{1/2}, \xi) \). Note that we do not remove from \( W^\text{prim}_Q \) a factor of the form \( \mathbb{Q}[u] \) (or any other factor). In the ordinary setting this factor compensates for the difference between the cohomologies of the moduli stack and moduli scheme of stable representations. In the self-dual setting the analogous cohomology groups are isomorphic; see Lemma 3.8 below.

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1 Because signs are included in the definition of \( x' \), additional sign substitutions (as occur in Theorem 3.3) are not needed in these equations.
3.4. Freeness for $\sigma$-symmetric quivers. In this section we formulate for $\mathcal{I}_Q$ an analogue of the freeness of the CoHA (Theorem 2.2). We begin with the following simple result.

**Lemma 3.5.** Let $Q$ be $\sigma$-symmetric. Define a $\mathbb{Z}_2$-grading on $\mathcal{I}_Q$ by the reduction modulo two of its $\mathbb{Z}$-grading. Then $\mathcal{I}_Q$ is a super $\mathcal{H}_Q$-module.

**Proof.** For an arbitrary quiver with involution, the following equality holds:

$$\chi(d, d') = \chi(\sigma(d'), \sigma(d)), \quad d, d' \in \Lambda_Q. \quad (18)$$

In the $\sigma$-symmetric case, the parity of elements of $\mathcal{H}_{Q,(d,k)} \ast \mathcal{I}_{Q,(e,l)}$ is $\mathcal{E}(H(d) + e)$. Working modulo two, we compute

$$\mathcal{E}(H(d) + e) \equiv \mathcal{E}(d) + \mathcal{E}(\sigma(d)) + \chi(d, d) + \mathcal{E}(e) + \chi(d, e) + \mathcal{E}(\sigma(d), e)$$

$$\equiv \mathcal{E}(d) + \mathcal{E}(\sigma(d)) + \chi(d, d) + \mathcal{E}(e) + \chi(e, d)$$

$$\equiv \mathcal{E}(d) + \mathcal{E}(\sigma(d)) + \chi(d, d) + \mathcal{E}(e)$$

$$\equiv \chi(d, d) + \mathcal{E}(e).$$

The first equality follows from equation (8), the second from equation (18), the third from symmetry of $Q$ and the last from $\sigma$-symmetry of $Q$. Since $\chi(d, d) + \mathcal{E}(e)$ is the sum of the parities of elements of $\mathcal{H}_{Q,(d,k)}$ and $\mathcal{I}_{Q,(e,l)}$, the lemma follows. \(\square\)

Note that a duality structure induces linear isomorphisms $R_d \rightarrow R_{\sigma(d)}$ which are equivariant with respect to the group isomorphisms $GL_d \rightarrow GL_{\sigma(d)}$, $\{g_i\}_{i \in Q_0} \mapsto \{(g_{\sigma(i)})^{-1}\}_{i \in Q_0}$. This induces a morphism of stacks

$$M_d \rightarrow M_{\sigma(d)}. \quad (19)$$

These morphisms in turn define an algebra anti-involution $S_{\mathcal{H}} : \mathcal{H}_Q \rightarrow \mathcal{H}_Q$. Explicitly, using the first part of Lemma 1.1, we find that

$$S_{\mathcal{H}}(f)(\{x_i,j\}_{i \in Q_0, 1 \leq j \leq d_{\sigma(i)}} = f(\{\tilde{x}_i,j\}_{i \in Q_0, 1 \leq j \leq d_i})|_{\tilde{x}_i,j = -x_{\sigma(i)}, j}, \quad f \in \mathcal{H}_{Q,d}. \quad (20)$$

The following result is a module theoretic analogue of the (almost) supercommutativity of $\mathcal{H}_Q$.

**Proposition 3.6.** Let $Q$ be a $\sigma$-symmetric quiver. The equality

$$S_{\mathcal{H}}(f) \ast g = (-1)^{\chi(e,d) + \mathcal{E}(d)} f \ast g$$

holds for all $f \in \mathcal{H}_{Q,d}$ and $g \in \mathcal{I}_{Q,e}$.

**Proof.** Let $\sigma \in \mathfrak{sh}_{Q,e}^\sigma$ be the $\sigma$-shuffle defined by the following maps of ordered sets:

$$[d_i] \sqcup [e_i] \sqcup [d_{\sigma(i)}] \mapsto [d_{\sigma(i)}] \sqcup [e_i] \sqcup [d_i], \quad i \in Q_0^+$$

$$[d_i] \sqcup [\lfloor \frac{e_i}{2} \rfloor] \mapsto [-d_i] \sqcup [\lfloor \frac{e_i}{2} \rfloor], \quad i \in Q_0^-.$$

Here $[n] = \{z_1 < \cdots < z_n\}$. Precomposition with $\sigma$ defines a bijection $\mathfrak{sh}_{\sigma(d),e}^\sigma \rightarrow \mathfrak{sh}_{\sigma(e),d}^\sigma$. Equation (20) shows that, after identifying variables as in Sect. 3.2, the polynomials $f$ and $S_{\mathcal{H}}(f)$ differ exactly by the application of $\sigma$. Note that $\sigma$ fixes the element $g$. 

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Using the explicit form of $K_{d,e}^\sigma$ given by Theorem 3.3, we will show that

$$\varpi(K_{\sigma(d),e}^\sigma) = (-1)^{\chi(e,d)+\mathcal{E}(d)}K_{d,e}^\sigma.$$  \hfill (21)

Applying $\varpi$ to a factor $D_i$ of $K_{\sigma(d),e}^\sigma$, $i \in Q_0^\sigma$, results in multiplication by $(-1)^{\frac{d_i(d_i+1)}{2}}$ in types $B$ and $C$ and $(-1)^{\frac{d_i}{2}}$ in type $D$. If $i \in Q_0^\sigma$, then the result is multiplication by $(-1)^{\epsilon_i d_i + d_i \sigma(i) + \epsilon_i d_i(i)}$. The sign change of the denominator of $K_{\sigma(d),e}^\sigma$ is thus $(-1)^{\chi Q_0^\sigma(\sigma(d), e) + \mathcal{E} Q_0^\sigma(\sigma(d))}$, the subscripts indicating that only summands of $\chi$ and $\mathcal{E}$ associated to $Q_0$ are included. Similarly, $\varpi$ acts on $V$ by multiplication by $(-1)^{d_i d_{\sigma(i)} + d_i d_{\sigma(i)} + d_i d_{e_j}}$ for $(i \rightarrow j) \in Q_1^\sigma$ and by $(-1)^{\epsilon_i d_{\sigma(i)} + d_{\sigma(i)} d_{\sigma(i)} + d_i d_{e_j}}$ for $(\sigma(i) \rightarrow i) \in Q_1^\sigma$. The sign change of the numerator is thus $(-1)^{\chi Q_1^\sigma(\sigma(d), e) + \mathcal{E} Q_1^\sigma(\sigma(d))}$. Equation (21) now follows from $\sigma$-symmetry.

Combining the above observations, we compute

$$S_H(f) \ast g = \sum_{\pi \in \mathfrak{h}_{\sigma(d),e}^\sigma} \pi(S(f)g K_{\sigma(d),e}^\sigma)$$

$$= \sum_{\pi \in \mathfrak{h}_{\sigma(d),e}^\sigma} \pi(\varpi(f)g K_{\sigma(d),e}^\sigma)$$

$$= (-1)^{\chi(e,d)+\mathcal{E}(d)} \sum_{\pi \in \mathfrak{h}_{\sigma(d),e}^\sigma} \pi \circ \varpi(fg K_{d,e}^\sigma)$$

$$= (-1)^{\chi(e,d)+\mathcal{E}(d)} \sum_{\pi' \in \mathfrak{h}_{d,e}^\sigma} \pi'(fg K_{d,e}^\sigma),$$

which is equal to $(-1)^{\chi(e,d)+\mathcal{E}(d)}f \ast g$. \hfill $\Box$

Since $S_H$ is an algebra anti-involution, the image of the multiplication map $H_{Q,+} \boxtimes H_{Q,+} \rightarrow H_Q$, and hence $V_Q$, inherits the structure of a $\mathbb{Z}_2$-representation. Moreover, $S_H$ acts on $\mathbb{Q}[u]$ by sending $u^n$ to $(-u)^n$ and we have $V_Q = V_Q^{\text{prim}} \otimes \mathbb{Q}[u]$ as $\mathbb{Z}_2$-representations. Indeed, setting $\sigma_d = \sum_{i \in Q_0} \sum_{j=1}^{d_i} x_{i,j}$ in degree $(0,2)$, there is an isomorphism (see [11, §3])

$$V_Q \simeq \bigoplus_{d \in \Lambda_{Q,+}^d} \left( V_{Q,d}^{\text{prim}} \otimes \mathbb{Q}[\sigma_d] \right).$$

If $V_Q^{\text{prim}}$ is interpreted geometrically as in Theorem 2.4 or [31], then its $\mathbb{Z}_2$-representation structure coincides with that induced by the geometric $\mathbb{Z}_2$-action on $\bigsqcup_{d \in \Lambda_{Q,+}^d} M^{\text{st}}_{Q,d}$.

Motivated by Proposition 3.6, define, for each $e' \in \Lambda_{Q,+}^\sigma$, an $e'$-twisted $\mathbb{Z}_2$-representation on $H_Q$ by requiring that the generator of $\mathbb{Z}_2$ acts by

$$f \mapsto (-1)^{\chi(e',d)+\mathcal{E}(d)} S_H(f), \quad f \in H_{Q,d}. \hfill (22)$$

Define a $\Lambda_{Q,+}^{\sigma} \times \mathbb{Z}$-graded $e'$-twisted $\mathbb{Z}_2$-representation $\tilde{V}_Q$ by

$$\tilde{V}_{Q,e} = \bigoplus_{d \in \Lambda_{Q,+}^{\sigma}, H(d)=e} V_{Q,d}, \quad e \in \Lambda_{Q,+}^{\sigma}.$$
Let \((\tilde{V}_Q)_{(Z_2,e')}\) be the space of \(Z_2\)-coinvariants of \(\tilde{V}_Q\). Identifying invariants and coinvariants, we obtain a \(\Lambda_Q^{\sigma,+} \times Z\)-graded subalgebra
\[
\text{Sym}((\tilde{V}_Q)_{(Z_2,e')}) \subset \text{Sym}(V_Q).
\]
When \(\mathcal{H}_Q\) is supercommutative, we denote by \(\mathcal{H}_Q(e')\) the subalgebra \(\text{Sym}((\tilde{V}_Q)_{(Z_2,e')})\) of \(\text{Sym}(V_Q) \simeq \mathcal{H}_Q\). The cyclic \(\mathcal{H}_Q\)-module \(\mathcal{H}_Q \ast g \subset I_Q\) generated by \(g \in I_Q,e'\) is actually cyclic for \(\mathcal{H}_Q(e')\). Indeed, Proposition 3.6 implies that the \((\mathbb{Z}_2, e')\)-anti-invariants of \(\tilde{V}_Q\), and hence the ideal of \(\text{Sym}(V_Q) \simeq \mathcal{H}_Q\) that they generate, annihilates \(g\).

We now state the main conjecture regarding the structure of \(I_Q\).

**Conjecture 3.7.** Let \(Q\) be a \(\sigma\)-symmetric quiver. Then the CoHA action map
\[
\bigoplus_{e \in \Lambda_Q^{\sigma,+}} \text{Sym}((\tilde{V}_Q)_{(Z_2,e)}) \boxtimes S W_{Q,e}^{\text{prim}} \to I_Q
\]
is an isomorphism in \(D^{\text{lb}}(\text{Vect})_{\Lambda_Q^{\sigma,+}}\). Moreover, if \(\mathcal{H}_Q\) is supercommutative, then for each \(e \in \Lambda_Q^{\sigma,+}\), the restriction to the summand \(\mathcal{H}_Q(e) \boxtimes S W_{Q,e}^{\text{prim}}\) is a \(\mathcal{H}_Q(e)\)-module isomorphism onto its image.

When \(\mathcal{H}_Q\) is not supercommutative, the action map in Conjecture 3.7 is defined via the \(D^{\text{lb}}(\text{Vect})_{\Lambda_Q^{\sigma,+}}\)-isomorphism \(\text{Sym}(V_Q) \simeq \mathcal{H}_Q\); see the comments after Theorem 2.2. Some instances of Conjecture 3.7 will be proved in Sect. 4.

**Remark.** The morphism of stacks (19) induces an involution of the stack \(M^s\) of stable representations and \(H^\bullet(M^s/Z_2) \simeq H^\bullet(M^s)_{Z_2}\) as mixed Hodge structures. The algebra \(\text{Sym}((\tilde{V}_Q)_{(Z_2,e)})\) is not \(\text{Sym}(PH^\bullet(M^s/Z_2))\), but instead isomorphic to \(\text{Sym}(PH^\bullet(M^s)_{Z_2,e})\), where the non-geometric \(e\)-twisted \(Z_2\)-action (22) is used.

Conjecture 3.7 implies a factorization of the orientifold DT series in terms of orientifold DT invariants and what we will call \(Z_2\)-equivariant DT invariants; cf. Corollary 2.3. To introduce the latter invariants, denote by \(\text{Rep}_{Z}(Z_2)\) the category of finite dimensional \(Z\)-graded rational \(Z_2\)-representations.

**Definition.** Let \(e' \in \Lambda_Q^{\sigma,+}\). The \(Z_2\)-equivariant motivic DT invariant of \(Q\) is the class of the \(e'\)-twisted \(Z_2\)-representation \(\tilde{V}_Q^{\text{prim}}\) in \(K_0(D^b(\text{Rep}_{Z}(Z_2)))_{\Lambda_Q^{\sigma,+}}\):
\[
\tilde{\Omega}_{e'} = \sum_{(e,k) \in \Lambda_Q^{(\sigma,+)} \times Z} \left( \dim_Q \left( \tilde{V}_Q^{\text{prim}}(e,k)^+ \right) + \dim_Q \left( \tilde{V}_Q^{\text{prim}}(e,k)^- \right) \right) (-q^{-\frac{1}{2}})^k \xi^e,
\]
\[
\in \mathbb{Z}(\frac{1}{2})[\Lambda_Q^{(\sigma,+)}][\eta]/(\eta^2 - 1).
\]
Here \((-)^\pm\) denotes the subspace of \(e'\)-twisted (anti-)invariants and \(\eta\) is the class of the degree zero sign representation.

The graded character of the \(Z_2\)-representation \(Q[u]\) is \(\frac{1+\eta}{1-q^2}\). Using this, we find
\[
[(\tilde{V}_Q)_{(Z_2,e')}\] = \frac{1}{1-q^2} \sum_{(e,k) \in \Lambda_Q^{(\sigma,+)} \times Z} (\tilde{\Omega}_{Q,e,k}^{e,+} + \tilde{\Omega}_{Q,e,k}^{e,-}) q^{-\frac{1}{2}} \xi^e.
\]
It follows that the parity-twisted Hilbert–Poincaré series of \( \text{Sym}(\tilde{\mathcal{V}}_Q((\mathbb{Z},e'))) \) is

\[
A_Q(e') = \prod_{(e,k) \in \Lambda_0^{\sigma^+,+} \times \mathbb{Z}} (q^{\frac{1}{2} + \delta_{-1, \lambda}} \xi_e ; q^2)_{\infty}^{\omega_Q(e',k)} \in \mathbb{Z}(q^{\frac{1}{2}})[\Lambda_0^{\sigma^+}].
\]

Passing to Grothendieck groups, Conjecture 3.7 would imply the factorization

\[
A_Q^{\sigma} \overset{\text{(Conj. 3.7)}}{=} \sum_{e \in \Lambda_0^{\sigma^+,+}} A_Q(e) \cdot \Omega_Q^\sigma(e, \xi_e), \quad (23)
\]

interpreted as an equality in \( \hat{\mathbb{H}}_Q \) with its commutative multiplication. Equation (23) uniquely determines \( \Omega_Q^\sigma \) from \( A_Q^{\sigma} \) and \( \omega_Q^\sigma \).

3.5. Orientifold DT invariants and Hodge theory. We continue to assume that \( Q \) is \( \sigma \)-symmetric. In this section we describe a connection between \( W^\text{prim}_{Q,e} \) and the Hodge theory of \( \mathcal{M}_e^{\sigma,\text{st}} \). We use the trivial stability, \( \theta = 0 \), throughout. We begin with the following basic lemma.

**Lemma 3.8.** Let \( e \in \Lambda_0^{\sigma^+,+} \).

(i) The canonical map

\[
H^\bullet(\mathcal{M}_e^{\sigma,\text{st}}) \to H^\bullet_{G_e}(R^\sigma_{e,\text{st}})(24)
\]

is an isomorphism of mixed Hodge structures.

(ii) For each \( k \geq 0 \), the subspace \( W_{k+1} H^k(\mathcal{M}_e^{\sigma,\text{st}}) \subset H^k(\mathcal{M}_e^{\sigma,\text{st}}) \) is trivial and, dually, \( W_k H^c(\mathcal{M}_e^{\sigma,\text{st}}) = H^c(\mathcal{M}_e^{\sigma,\text{st}}) \).

**Proof.** Since \( H^\bullet_{G_e}(R^\sigma_{e,\text{st}}) \simeq H^\bullet(\mathcal{M}_e^{\sigma,\text{st}}) \) and \( M_e^{\sigma,\text{st}} \to \mathcal{M}_e^{\sigma,\text{st}} \) is a coarse moduli scheme for the smooth Deligne–Mumford stack \( M_e^{\sigma,\text{st}} \), Edidin [10, Theorem 4.40] implies that (24) is a graded vector space isomorphism. In the notation of Sect. 1.2, the morphisms \( R^\sigma_{e,\text{st}} \times G_e E_N \to M_e^{\sigma,\text{st}} \) approximate the morphism \( R^\sigma_{e,\text{st}} \times G_e E G_e \to \mathcal{M}_e^{\sigma,\text{st}} \) and respect mixed Hodge structures. Passing to the limit then shows that (24) is a morphism of mixed Hodge structures.

The second statement follows from [7, Théorème 8.2.4 (iv)] and Poincaré duality. \( \square \)

The next result gives a partial analogue of Theorem 2.4.

**Proposition 3.9.** For each \( e \in \Lambda_0^{\sigma^+,+} \), the restriction \( H^\bullet_{G_e}(R^\sigma_{e,\text{st}}) \to H^\bullet_{G_e}(R^\sigma_{e,\text{st}}) \) factors through a surjective map \( W^\text{prim}_{Q,e} \to \mathbb{P} H^\bullet(-e)(\mathcal{M}_e^{\sigma,\text{st}}) \) of \( \mathbb{Z} \)-graded vector spaces.

**Proof.** Noting that \( \dim_c M_e^\sigma = -e(e) \), Poincaré duality for smooth Artin stacks gives a perfect pairing

\[
H^\bullet_{G_e}(R^\sigma_{e}) \otimes_{\mathbb{Q}} H^2_{e,G_e}(-2e) \to \mathbb{Q}(-e(e)).
\]
By Deligne [7, Théorème 9.1.1], the mixed Hodge structure on \( H^i_{c,G_e} (R_e^\sigma) \) is pure of weight \( i \). Hence \( H^i_{c,G_e} (R_e^\sigma) \) is also pure of weight \( i \). Consider the long exact sequence of mixed Hodge structures associated to the pair \( (R_e^\sigma, R_e^\sigma) \):

\[
\cdots \to H^{i-1}_{c,G_e} (R_e \setminus R_e^\sigma) \to H^i_{c,G_e} (R_e^\sigma) \to H^i_{c,G_e} (R_e^\sigma, R_e^\sigma, \ldots) \to \cdots.
\]

Since the weights of \( H^{i-1}_{c,G_e} (R_e \setminus R_e^\sigma) \) are bounded above by \( i - 1 \), the restriction \( P H^i_{c,G_e} (R_e^\sigma) \to H^i_{c,G_e} (R_e^\sigma) \) is an injection and, dually, \( H^i_{c,G_e} (R_e^\sigma) \to P H^i_{c,G_e} (R_e^\sigma) \) is a surjection. Here we have used the second part of Lemma 3.8.

A straightforward modification of [2, Lemma 2.1] shows that the composition of the action map

\[
\bigoplus_{(d',e') \in \Lambda^+ \times \Lambda^+} \mathcal{H}_{Q,d'} \boxtimes \mathcal{I}_{Q,e'} \to \mathcal{I}_{Q,e} = H^{* - \mathcal{E}(e)}_{G_e^e} (R_e^\sigma)
\]

with the restriction \( H^{* *}_{G_e^e} (R_e^\sigma) \to H^{* *}_{G_e^e} (R_e^\sigma, \ldots) \simeq H^{* *}(\mathcal{M}_e^\sigma, \ldots) \) is zero. Combined with the previous paragraph, this shows that \( W_{Q,e}^{prim} \to P H^{* * - \mathcal{E}(e)}(\mathcal{M}_e^\sigma, \ldots) \) is surjective. \( \square \)

The proof of injectivity in Theorem 2.4 relies on the interpretation of \( \Omega_Q \) in terms of the cohomology of smooth Nakajima quiver varieties [21, Corollary 1.5]. Since Nakajima varieties are singular in the self-dual setting, it is unclear if the proof from [2] can be adapted. In any case, it is natural to make the following conjecture.

**Conjecture 3.10.** The surjection \( W_{Q,e}^{prim} \to P H^{* * - \mathcal{E}(e)}(\mathcal{M}_e^\sigma, \ldots) \) is an isomorphism.

We verify Conjecture 3.10 in some examples in Sect. 4. In view of [31, Theorem 4.6], it is also natural to conjecture that \( W_{Q,e}^{prim} \) is isomorphic to the intersection cohomology \( IC^{* * - \mathcal{E}(e)}(\mathcal{M}_e^\sigma, \ldots) \) of the closure of \( \mathcal{M}_e^{\sigma, \ldots} \subset \mathcal{M}_e^{\sigma, ss} \). This can be verified in all examples in which Conjecture 3.10 is verified below.

### 3.6. The critical semistable CoHM

We define the CoHM of a quiver with potential and stability, generalizing the construction of Sect. 3.1. We will use the semistable generalization in later sections of the paper. The critical generalization, however, appears only in the present section.

Fix a stability \( \theta \) and potential \( W \in \mathbb{C} Q / [\mathbb{C} Q, \mathbb{C} Q] \). Let \( d', d'' \in \Lambda^+ \) and set \( d = d' + d'' \). Let \( R^{\theta, ss}_d \subset R_d \) be the open subvariety of semistable representations and define \( R^{\theta, ss}_{d', d''} = R_{d', d''} \cap R^{\theta, ss}_d \). The trace maps

\[
\text{tr}(W)_d : R^{\theta, ss}_d \to \mathbb{C}, \quad \text{tr}(W)_{d', d''} : R^{\theta, ss}_{d', d''} \to \mathbb{C}
\]

are invariant under \( \text{GL}_{d'} \) and \( \text{GL}_{d', d''} \), respectively. For ease of exposition, we assume that, for each \( d \in \Lambda^+ \), the only critical value of \( \text{tr}(W)_d \) is zero. Recall that the full subcategory of \( \text{Rep}_\mathbb{C}(Q) \) consisting of the zero object and all semistable representations
of a fixed slope is abelian. If $\mu(d') = \mu(d'')$, then by restriction of the maps (10) we get

$$R^\theta_{d'} \times R^\theta_{d''} \xleftarrow{\pi} R^\theta_{d'} \xrightarrow{i} R^\theta_d,$$

along which the trace maps pull back according to

$$i^* \operatorname{tr}(W)_d = \operatorname{tr}(W)_{d',d''} = \pi^* (\operatorname{tr}(W)_{d'} \oplus \operatorname{tr}(W)_{d''}).$$

Let $\varphi_{\operatorname{tr}(W)_d} \in D^b_{\text{const}}(R^\theta_{d'})$ be the sheaf of vanishing cycles of $\operatorname{tr}(W)_d$, henceforth denoted by $\varphi_{\operatorname{tr}(W)_d}$. See [24] for background on vanishing cycles. By assumption, $\varphi_{\operatorname{tr}(W)_d}$ is supported on $(\operatorname{tr}(W)_d)^{-1}(0)$. The slope $\mu$ semistable critical CoHA $[27, \S 7]$ has underlying object

$$\mathcal{H}^\theta_{Q,W,\mu} = \bigoplus_{d \in \Lambda^+_Q,\mu} H^\bullet_{c,\GL_d}(R^\theta_{d'}, \varphi_{\operatorname{tr}(W)_d})^\vee[\chi(d,d)] \in D^b(\operatorname{Vect})_{\Lambda^+_Q,\mu},$$

where $\Lambda^+_Q,\mu = \{d \in \Lambda^+_Q \mid \mu(d) = \mu\} \cup \{0\}$. An associative product is defined on $\mathcal{H}^\theta_{Q,W,\mu}$ analogously to Sect. 2.1; see [27, \S 7], [4, \S 3.2] for details. When $\theta$ and $W$ are trivial, the CoHA $\mathcal{H}^\theta_{Q,W,\mu}$ reduces to $\mathcal{H}_Q$. The inclusions $R^\theta_d \hookrightarrow R_d$ induce an algebra homomorphism $\mathcal{H}^\theta_{Q,W,\mu} \to \mathcal{H}^\theta_{Q,W,\mu}$, where $\mathcal{H}^\theta_{Q,W,\mu} \subset \mathcal{H}_Q$ is the subalgebra associated to the submonoid $\Lambda^+_Q,\mu \subset \Lambda^+_Q$.

Suppose now that $Q$ has an involution and duality structure and that $\theta$ is $\sigma$-compatible. A potential is called $S$-compatible if the trace maps are invariant under the isomorphisms $R_d \sim R_{\sigma(d)}$ induced by $S$. In this case the maps

$$\operatorname{tr}(W)_c : R^\sigma_{c,\theta} \to \mathbb{C}, \quad \operatorname{tr}(W)_{d',e'} : R^\sigma_{d',e'} \to \mathbb{C}$$

are invariant under $G^\sigma_c$ and $G^\sigma_{d',e'}$, respectively.

**Lemma 3.11.** Let $X$ be a complex manifold and $f : X \to \mathbb{C}$ a holomorphic function whose only critical value is zero. For any $c \in \mathbb{R}_{>0}$, there is a canonical isomorphism of vanishing cycle functors $\varphi_f \simeq \varphi_{c,f}$. In particular, $\varphi_f \otimes \varphi_{c,f} \simeq \varphi_{c,f} \otimes \varphi_f$.

**Proposition 3.12.** Let $\theta$ be a $\sigma$-compatible stability and $W$ an $S$-compatible potential. Then

$$\mathcal{T}^\theta_{Q,W} = \bigoplus_{e \in \Lambda^+_Q,\mu} H^\bullet_{c,\GL_d}(R^\sigma_{c,\theta} \varphi_{\operatorname{tr}(W)_d}^\vee[\mathcal{E}(e)]) \in D^b(\operatorname{Vect})_{\Lambda^+_Q,\mu}$$

has a cohomological Hall module structure over $\mathcal{H}^\theta_{Q,W,\mu=0}$. Moreover, the map $\mathcal{T}_{Q,W} \to \mathcal{T}^\theta_{Q,W}$ induced by the $G^\sigma_c$-equivariant open inclusions $R^\sigma_{c,\theta} \hookrightarrow R^\sigma_c$ is a module homomorphism over $\mathcal{H}^\theta_{Q,W,\mu=0} \to \mathcal{H}^\theta_{Q,W,\mu=0}$.

**Proof.** Let $U \subset Z$ be an isotropic subrepresentation. If $U$ is semistable of slope zero and $Z \parallel U$ is $\sigma$-semistable, then $Z$ is $\sigma$-semistable. Indeed, in this situation we obtain a pair of short exact sequences of representations,

$$0 \to U \to U \parallel \to Z \parallel U \to 0, \quad 0 \to U \parallel \to Z \to S(U) \to 0. \quad (25)$$

Since $Z \parallel U$ is $\sigma$-semistable, it is semistable [43, Proposition 3.2]. The sequences (25) then imply that $U \parallel$ and $Z$ are semistable of slope zero. Hence $Z$ is $\sigma$-semistable.
Using the previous paragraph, for each pair \((d,e) \in \Lambda_{Q,\mu=0}^+ \times \Lambda_{Q}^+\), we obtain morphisms \(R_d^{\theta,ss} \times R_e^{\sigma,\theta,ss} \xrightarrow{\pi} R_{d,e}^{\theta,ss} \xhookrightarrow{i} R_{H(d)+e}^{\sigma,\theta,ss}\), which satisfy

\[ i^* \text{tr}(W)^{\sigma}_{H(d)+e} = \text{tr}(W)^{\sigma}_{d,e} = \pi^* \left( 2 \text{tr}(W)^{\sigma}_d \boxplus \text{tr}(W)^{\sigma}_e \right). \]

Combining Lemma 3.11 with the Thom–Sebastiani isomorphism \([30]\) gives

\[ H^\bullet \mathcal{C}, \mathbb{G}_{d}(R^{\theta,ss}_d, \phi_{\text{tr}(W)^{\sigma}}^e) \xrightarrow{\sim} H^\bullet \mathcal{C}, \mathbb{G}_{e}(R^{\theta,ss}_e, \phi_{\text{tr}(W)^{\sigma}}^d) \]

Here we have used that zero is the only critical value of \(\text{tr}(W)^{\sigma}_e\), as follows from the corresponding (assumed) property of \(\text{tr}(W)^{\sigma}_d\); see \([4, \text{Remark 2.10}]\). From this point on the construction of the \(H^{\theta,ss}_{Q,\mu=0}\)-module structure of \(I_{\theta,ss}^{\sigma,\theta,ss}\) is the obvious common generalization of \([27, \S 7]\) and Sect. 3.1. We omit the details.

The second statement follows from the fact that the diagram

\[
\begin{array}{ccc}
R_{d,e}^{\theta,ss} & \xlongleftarrow{\pi} & R_d^{\sigma} \\
\downarrow & & \downarrow \\
R_d^{\theta,ss} \times R_e^{\sigma,\theta,ss} & \xlonghookrightarrow{} & R_d \times R_e^{\sigma}
\end{array}
\]

is Cartesian, which in turn follows from the first paragraph of the proof. \(\blacksquare\)

Consider the case in which \(Q\) is \(\sigma\)-symmetric and the potential is zero. Define \(W_{Q,\theta}^{\text{prim}} = T^{\theta,ss}_{Q} / (H^{\theta,ss}_{Q,\mu=0,+} \ast T^{\theta,ss}_{Q})\) with associated motivic invariant \(\Omega_{Q}^{\sigma,\theta}\). When \(\theta\) is zero, this reduces to the definition of Sect. 3.3. As in Sect. 3.4, we expect \(T^{\theta,ss}_{Q}\) to be a direct sum of free modules over subalgebras of \(H^{\theta,ss}_{Q,\mu=0}\), leading to a factorization

\[ A_{Q,\theta,ss}^{\sigma,\theta} (\text{Conj.}) = \sum_{e \in \Lambda_{Q}^+} A_{Q,\mu=0}^{\theta,ss} \cdot \Omega_{Q,e}^{\sigma,\theta}^e. \]

If such a factorization indeed exists, then \(\Omega_{Q}^{\sigma,\theta}\) is independent of \(\theta\). This follows from a short argument using the wall-crossing formula

\[ A_Q^\sigma = \prod_{\mu \in \mathbb{Q}_{>0}} A_{Q,\mu}^{\theta,ss} \ast A_{Q}^{\sigma,\theta,ss} \tag{26} \]

proved in \([43, \text{Theorem 4.5}]\).

To end this section, we describe the expected structure of \(I_{Q,W}\). Sections 4 and 5 give evidence for these expectations when \(W\) vanishes. Let \(Q, W\) and \(\theta\) be arbitrary. Motivated by the existence and uniqueness of Harder–Narasimhan (HN) filtrations, in \([27, \S 5.2]\) (see also \([3, \S 8.1]\)) it was asked if there exist algebra embeddings \(H_{Q,W,\mu}^{\theta,ss} \hookrightarrow H_{Q,W}\) such that the slope-ordered CoHA multiplication

\[ \boxtimes_{\mu \in \mathbb{Q}} H_{Q,W,\mu}^{\theta,ss} \rightarrow H_{Q,W} \]
is an isomorphism in $D^{lb}(\text{Vect})_{\Lambda^+_Q}$. When $\theta$ is generic, $\mathcal{H}^{\theta-ss}_{Q,W,\mu}$ is expected to be the quantized enveloping algebra of a Lie superalgebra structure on $V^{\text{prim},\theta}_{Q,W,\mu} \otimes \mathbb{Q}[u]$. In this way, $\mathcal{H}_{Q,W}$ obtains a Poincaré–Birkhoff–Witt-type basis [6]. Conjecturally, $V^{\text{prim},\theta}_{Q,W}$ can be interpreted as a space of single-particle BPS states in an oriented string theory.

Similarly, each self-dual representation $Z$ has a unique $\sigma$-HN filtration [43, Proposition 3.3], that is, an isotropic filtration

$$0 = U_0 \subset U_1 \subset \cdots \subset U_r \subset Z$$

such that $U_1/U_0, \ldots, U_r/U_{r-1}$ are semistable with strictly decreasing positive slopes and $Z/\sum U_r$ is zero or $\sigma$-semistable. It is then natural to ask for a $\mathcal{H}^{\theta-ss}_{Q,W,\mu=0}$-module embedding $\mathcal{T}^{\theta-ss}_{Q,W} \hookrightarrow \mathcal{I}_{Q,W}$ such that the slope-ordered CoHA action

$$\star : \mathcal{T}^{\theta-ss}_{Q,W,\mu} \mathcal{H}^{\theta-ss}_{Q,W,\mu} \mathcal{S}^{\text{tw}} \mathcal{T}^{\theta-ss}_{Q,W} \to \mathcal{I}_{Q,W}$$

(27)
is an isomorphism in $D^{lb}(\text{Vect})_{\Lambda^+_Q}$. Together with the natural extension of Conjecture 3.7 to $\mathcal{I}^{\theta-ss}_{Q,W}$, an isomorphism of the form (27) would determine a PBW-type basis of $\mathcal{I}_{Q,W}$ in terms of $W^{\text{prim},\theta}_{Q,W}$ and the PBW-type bases of $\mathcal{H}^{\theta-ss}_{Q,W,\mu=0}$. Conjecturally, $W^{\text{prim},\theta}_{Q,W}$ can be interpreted as a space of single-particle BPS states of the orientifold of the theory which $V^{\text{prim},\theta}_{Q,W,\mu}$ describes. Decompositions similar to (27) occur in physical definitions of unoriented BPS invariants [37,40].

Remark. The natural generality of this section, which we do not pursue in this paper, is to view $\mathcal{T}^{\theta-ss}_{Q,W}$ as an object of $D^{lb}(\text{MMHS})_{\Lambda^+_Q}$ instead of $D^{lb}(\text{Vect})_{\Lambda^+_Q}$, where MMHS is a category of monodromic mixed Hodge structures. The formulation of this section would then be obtained by forgetting all Hodge theoretic information. The critical CoHA $\mathcal{H}^{\theta-ss}_{Q,W,\mu}$ is constructed as an algebra object of $D^{lb}(\text{MMHS})_{\Lambda^+_Q}$ in [4,6,27].

4. $\sigma$-Symmetric Examples

4.1. Disjoint union quivers. Let $Q$ and $Q'$ be quivers. The disjoint union quiver $Q \sqcup Q'$ has nodes $Q_0 \sqcup Q'_0$ and arrows $Q_1 \sqcup Q'_1$. The opposite quiver $Q^{\text{op}}$ has nodes $Q_0$ and an arrow $j \to i$ for each arrow $i \to j$ of $Q$.

Lemma 4.1. There are canonical algebra isomorphisms

$$\mathcal{H}_{Q \sqcup Q'} \simeq \mathcal{H}_Q \otimes \mathcal{H}_{Q'}, \quad \mathcal{H}_{Q^{\text{op}}} \simeq \mathcal{H}_Q^{\text{op}}$$

where $\mathcal{H}_Q^{\text{op}}$ is the opposite algebra of $\mathcal{H}_Q$.

Proof. The isomorphism $\mathcal{H}_{Q \sqcup Q'} \simeq \mathcal{H}_Q \otimes \mathcal{H}_{Q'}$ is the pullback along the isomorphisms $R_d(Q) \times R_{d'}(Q') \to R_{(d,d')}((Q \sqcup Q'))$. The isomorphism $\mathcal{H}_{Q^{op}} \simeq \mathcal{H}_Q^{op}$ is induced by the linear isomorphisms $R_d(Q) \to R_d(Q^{op})$ which send a representation to its transpose and are equivariant with respect to the group isomorphisms $\text{GL}_d \to \text{GL}_d$, $\{g_i\}_{i \in Q_0} \mapsto ((g_i^{-1})^t)_{i \in Q_0}$. \quad \Box

\footnote{2 Analogously to Conjecture 3.7, we should really restrict to a subalgebra of $\mathcal{H}^{\theta-ss}_{Q,W,\mu=0}$.}
The quiver $Q^\sqcup = Q \sqcup Q^{\text{op}}$ has a canonical involution which swaps the nodes and arrows of $Q$ and $Q^{\text{op}}$. Representations of $Q^\sqcup$ are of the form $U_1 \oplus S(U_2)$ for $U_1, U_2 \in \text{Rep}_\mathbb{C}(Q)$. Self-dual representations are hyperbolics on representations of $Q$. It follows that there is a vector space isomorphism $I_{Q^\sqcup} \to \mathcal{H}_Q$. Lemma $4.1$ implies that $I_{Q^\sqcup}$ is a $\mathcal{H}_Q \otimes \mathcal{H}_Q^{\text{op}}$-module. Similarly, $\mathcal{H}_Q$ is the regular left $\mathcal{H}_Q$-bimodule.

**Theorem 4.2.** The map $I_{Q^\sqcup} \to \mathcal{H}_Q$ is an isomorphism of left $\mathcal{H}_Q \otimes \mathcal{H}_Q^{\text{op}}$-modules.

**Proof.** The action of $f_1 \otimes f_3 \in \mathcal{H}_Q \otimes \mathcal{H}_Q^{\text{op}}$ on $f_2 \in \mathcal{H}_Q$ is $f_1 \cdot f_2 \cdot f_3 \in \mathcal{H}_Q$, which is in turn the image of $f_1 \otimes f_2 \otimes f_3$ under the composition (omitting degree shifts)

$$3 \bigotimes_{k=1}^3 H_{\text{GL}_d}(R_{d_k}) \xrightarrow{\sim} H_{\text{GL}_{d_1, d_2, d_3}}(R_{d_1, d_2, d_3}) \to H_{\text{GL}_{d_1, d_2, d_3}}(R_{d_1, d_2, d_3}).$$

The isomorphism $R_d \simeq R_{\sigma(H)}^\sigma$ identifies $R_{\sigma(H)}^\sigma \subset R_{1+\sigma(H)}^\sigma$, which is $d_1, d_2, d_3 \subset R_{d_1, d_2, d_3}$ which stabilizes the $\mathcal{H}_Q$-graded flag

$$\mathbb{C}^{d_1} \subset (\mathbb{C}^{d_2})^\perp \cap \mathbb{C}^{d_1+d_2+d_3} \subset \mathbb{C}^{d_1+d_2+d_3}$$

and identifies $\mathcal{G}_{d_1, d_2, d_3} \subset \mathcal{G}_{d_1+d_2+d_3}$ with $\text{GL}_{d_1, d_2, d_3} \subset \text{GL}_{d_1+d_2+d_3}$. Using these observations, we find that $(f_1 \otimes f_3) * f_2 = f_1 \cdot f_2 \cdot f_3$. That the isomorphism $I_{Q^\sqcup} \simeq \mathcal{H}_Q$ respects the gradings follows from the equality

$$\mathcal{E}_{Q^\sqcup}(U_1 \oplus S(U_2)) = \chi_Q(U_2, U_1), \quad (28)$$

which holds for all $U_1, U_2 \in \text{Rep}_\mathbb{C}(Q)$. □

**Corollary 4.3.** Conjectures 3.7 and 3.10 hold for $Q^\sqcup$.

**Proof.** By equation (28), $Q^\sqcup$ is $\sigma$-symmetric if and only if $Q$ is symmetric. Assume then that $Q$ is symmetric and consider $\mathcal{H}_Q$ with its supercommutative multiplication. Theorems 2.2 and 4.2 give algebra isomorphisms

$$\mathcal{H}_{Q^\sqcup} \simeq \mathcal{H}_Q \otimes \mathcal{H}_Q^{\text{op}} \simeq \text{Sym}(\mathcal{V}_Q^{\text{prim}} \oplus \mathcal{S}_\mathcal{H}(\mathcal{V}_Q^{\text{prim}})) \otimes \mathbb{Q}[u]).$$

Lift the supercommutative twist of $\mathcal{H}_Q$ to $I_{Q^\sqcup}$ by taking $I_{Q^\sqcup}$ to be the regular super $\mathcal{H}_Q$-bimodule. Then $I_{Q^\sqcup}$ is a free module with basis $1_0 \in I_{Q^\sqcup, 0}$ over the subalgebra of $\mathcal{H}_{Q^\sqcup}$ generated by the image of

$$V_Q \leftrightarrow V_Q \oplus \mathcal{S}_\mathcal{H}(V_Q) \simeq (V_Q^{\text{prim}} \oplus \mathcal{S}_\mathcal{H}(V_Q^{\text{prim}})) \otimes \mathbb{Q}[u], \quad v \mapsto v + \mathcal{S}_\mathcal{H}(v).$$

Conjecture 3.7 follows.

Conjecture 3.10 holds since $\mathcal{M}^{e, \text{st}}(Q^\sqcup) = \emptyset$ if $e \neq 0$. Indeed, any self-dual representation of $Q^\sqcup$ is destabilized by the subrepresentation supported on $Q$. □

Similarly, $I_{Q^\sqcup}$ is a rank one free $\mathcal{H}_Q$-module. This statement can be interpreted as the PBW factorization (27) corresponding to a $\sigma$-compatible stability $\theta$ whose restriction to $\Lambda_Q^\sqcup \subset \Lambda_Q^{\sigma, \text{st}}$ is positive. Again, we have $\mathcal{M}^{e, \theta, \text{st}}(Q^\sqcup) = \emptyset$ if $e \neq 0$. 


4.2. Loop quivers. Let \( L_m \) be the quiver with one node and \( m \geq 0 \) loops. It is symmetric and \( \mathcal{H}_{L_m} \) is supercommutative. If \( f_1 \in \mathcal{H}_{L_m,d'} \) and \( f_2 \in \mathcal{H}_{L_m,d''} \), then

\[
\begin{align*}
\pi(f_1(x'_1, \ldots, x_{d'}) f_2(x''_1, \ldots, x_{d''})) = \sum_{\pi \in \mathfrak{A}_{d',d''}} \pi\left( \prod_{l=1}^{d'} \prod_{k=1}^{d''} (x''_k - x'_k)^{m-1} \right).
\end{align*}
\]

The quiver \( L_m \) has a unique involution. A duality structure is determined by signs \( s \) and \( \tau_1, \ldots, \tau_m \). Suppose that \( \tau_+ \) of the latter are positive and \( \tau_- = m - \tau_+ \) are negative. The quiver \( L_m \) is \( \sigma \)-symmetric. When \( s = 1 \), Proposition 3.2 gives \( \mathcal{I}_{L_m} = I_{L_m}^D \oplus I_{L_m}^B \), the summands corresponding to even and odd dimensional self-dual representations, respectively. When \( s = -1 \), we write \( \mathcal{I}_{L_m}^C \) for \( \mathcal{I}_{L_m} \). Given \( f \in \mathcal{H}_{L_m,d} \) and \( g \in \mathcal{I}_{L_m,e} \), we have

\[
\begin{align*}
 f \star g = 2^{(s-\frac{1}{2})d} \sum_{\pi \in \mathfrak{A}_{d',d''}} \pi\left( f(x_1, \ldots, x_d) g(z_1, \ldots, z_{\frac{d}{2}}) \right) \\
\times \prod_{i=1}^{d} (-x_i)^{N(s, \tau)} \left( \prod_{1 \leq i < j < d} (-x_i - x_j) \prod_{i=1}^{d} \prod_{j=1}^{\frac{d}{2}} (x_i^2 - z_j^2) \right)^{m-1},
\end{align*}
\]

where

\[
N(s, \tau) = \begin{cases} 
 m + \tau_+ - 1 & \text{in type } B, \\
 \tau_- - 1 & \text{in type } C, \\
 \tau_+ & \text{in type } D.
\end{cases}
\]

Since the cases \( m \in \{0, 1\} \) serve as building blocks for more complicated examples, we will describe them in detail.

4.2.1. Zero loops The algebra \( \mathcal{H}_{L_0} \) is free supercommutative on the odd variables \( x^i \in \mathcal{H}_{L_{0,1}}, i \geq 0 \), of degree \((1, 2i + 1) \) [27, §2.5]. Explicitly, if \( i = (i_1, \ldots, i_1) \in \mathbb{Z}_{\geq 0}^d \) is a strictly decreasing sequence, then \( x^{i_1} \ldots x^{i_d} = s_{i_1} \cdots s_{i_d} \). Here \( s_\lambda \) is the Schur polynomial of a partition \( \lambda \) and \( \delta_r = (r - 1, \ldots, 1, 0) \). Hence \( \mathcal{V}_{L_0}^\text{prim} = \mathbb{Q} \cdot 1 = \mathbb{Q}(1,1) \).

Let \( \phi : \mathcal{H}_{L_0} \to \mathcal{H}_{L_0} \) be the algebra automorphism determined by \( \phi(x^i) = 2x^i \) and let \( (\mathcal{I}_{L_0}^D)_\phi \) be the corresponding \( \phi \)-twisted \( \mathcal{H}_{L_0} \)-module. Using the explicit form of \( \star \), we see that \( (\mathcal{I}_{L_0}^D)_\phi \simeq \mathcal{I}_{L_0}^C[1] \) as graded \( \mathcal{H}_{L_0} \)-modules, where [1] denotes \( \Lambda^\sigma_{Q} \)-degree shift by one. We therefore restrict attention to \( \mathcal{I}_{L_0}^D \) in what follows.

Given \( f \in \mathbb{Q}[x_1, \ldots, x_d] \), set \( \tilde{f}(x_1, \ldots, x_d) = f(x^2_1, \ldots, x^2_d) \). Let \( i \) be a strictly decreasing partition of length \( d \). Short induction arguments show the following:

(i) Type \( B \): If all \( i \) are odd, then \( s_{i-\delta_d} \star 1_{L_0}^D = (-2)^d \tilde{s}_{i-\delta_d} \).

(ii) Type \( D \): If all \( i \) are even, then \( s_{i-\delta_d} \star 1_{L_0}^D = 2^d \tilde{s}_{i-\delta_d} \).

Let \( \mathcal{H}_{L_0}^{\text{even}} \) and \( \mathcal{H}_{L_0}^{\text{odd}} \) be the algebras generated by \( \{x^{2i}\}_{i \geq 0} \) and \( \{x^{2i+1}\}_{i \geq 0} \), respectively. Equivalently,

\[
\mathcal{H}_{L_0}^{\text{even}} = \text{Sym}(\mathcal{V}_{L_0}^{\text{prim}} \otimes \mathbb{Q}[u^2]), \quad \mathcal{H}_{L_0}^{\text{odd}} = \text{Sym}(\mathcal{V}_{L_0}^{\text{prim}} \otimes u\mathbb{Q}[u^2]).
\]

These are the subalgebras of the CoHA introduced above Lemma 3.5.
Proposition 4.4. (i) The $\mathcal{H}_{L_0}^{\text{odd}}$-module $T_{L_0}^B$ is free with basis $1_1^\sigma \in T_{L_0,1}^B$.
(ii) The $\mathcal{H}_{L_0}^{\text{even}}$-module $T_{L_0}^D$ is free with basis $1_0^\sigma \in T_{L_0,0}^D$.

In particular, $\Omega_{L_0}^B = \xi$, $\Omega_{L_0}^C = 1$ and $\Omega_{L_0}^D = 1$. Conjectures 3.7 and 3.10 hold for $L_0$.

Proof. The map $i \mapsto i-1$ is a bijection between the set of strictly decreasing purely odd partitions of length $d$ and the set of strictly decreasing partitions of length $d$. Since the Schur functions $\tilde{S}_i$ parameterized by the former set are a basis of $T_{L_0,2d+1}^B$, the first statement follows. In type $D$ we use instead the bijection $i \mapsto i/2$ between the sets of strictly decreasing purely even and strictly decreasing partitions.

That Conjecture 3.10 holds follows from the observations

$$\mathcal{M}_{2e}^{\text{sp},st} = \emptyset, \quad e \geq 1, \quad \mathcal{M}_{e}^{\alpha, st} = \begin{cases} \text{Spec}(\mathbb{C}) & \text{if } e = 1, \\ \emptyset & \text{if } e \geq 2, \end{cases}$$

with $\text{sp}$ and $\alpha$ indicating type $C$ or types $B$ or $D$, respectively. \hfill $\square$

4.2.2. One loop The algebra $\mathcal{H}_{L_1}$ is free supercommutative on the even variables $x^i \in \mathcal{H}_{L_1,1}, i \geq 0$, of degree $(1, 2i)$ [27, §2.5]. Explicitly, $x^{i_1} \cdots x^{i_d} = N(i)m_i$, where

$$N(i) = \prod_{k \geq 0} \# \{ j \geq 1 \mid i_j = k \}! \in \mathbb{Z}$$

and $m_i$ is the monomial symmetric polynomial. Hence $V_{L_1}^{\text{prim}} = \mathbb{Q} \cdot 1_1 = \mathbb{Q}_{(1,0)}$.

Similar to the case $m = 0$, we have module isomorphisms $T_{L_1}^B \cong T_{L_1}^C(1)$ if $\tau = 1$ and $(T_{L_1}^B)^\circ \cong T_{L_1}^C(1) \cong (T_{L_1}^D)^\circ(1)$ if $\tau = -1$. So we consider only $T_{L_1}^C$ if $\tau = 1$ and $T_{L_1}^B$ if $\tau = -1$. Given a partition $i$ of length $d$, we find:

(i) Type $B$, $\tau = -1$: If $i$ is purely even, then $m_i \star 1_0^\sigma = 2^d \tilde{m}_{i/2}$.
(ii) Type $C$, $\tau = 1$: If $i$ is purely odd, then $m_i \star 1_0^\sigma = (-2)^d \tilde{m}_{i/2}$.
(iii) Type $D$, $\tau = 1$: If $i$ is purely odd, then $m_i \star 1_2^\sigma = (-2)^d \tilde{m}_{i/2}(\omega_{\sigma}, \phi_i)$, where $(i, \phi_i)$ is the length $d + e$ partition obtained by appending $e$ zeros to $i$.

Let $\mathcal{H}_{L_1}^{\text{even}} = \text{Sym}(V_{L_1}^{\text{prim}} \otimes \mathbb{Q}[u^2])$ and $\mathcal{H}_{L_1}^{\text{odd}} = \text{Sym}(V_{L_1}^{\text{prim}} \otimes u\mathbb{Q}[u^2])$.

Proposition 4.5. (i) If $\tau = -1$, then $T_{L_1}^B$ is a free $\mathcal{H}_{L_1}^{\text{even}}$-module with basis $1_0^\sigma \in T_{L_1,1}^B$.
(ii) If $\tau = 1$, then $T_{L_1}^C$ is a free $\mathcal{H}_{L_1}^{\text{odd}}$-module with basis $1_0^\sigma \in T_{L_1,1}^C$.
(iii) If $\tau = 1$, then $T_{L_1}^D$ is a free $\mathcal{H}_{L_1}^{\text{odd}}$-module with basis $1_2^\sigma \in T_{L_1,2e}^D$, $e \geq 0$.

In particular, if $\tau = -1$, then $\Omega_{L_1}^B = \xi$, $\Omega_{L_1}^C = 1$ and $\Omega_{L_1}^D = 1$ while if $\tau = 1$, then

$$\Omega_{L_1}^B = \frac{q^{-1}\xi}{1 - q^{-1}\xi^2}, \quad \Omega_{L_1}^C = 1, \quad \Omega_{L_1}^D = \frac{1}{1 - q^{-1}\xi^2}.$$

Conjectures 3.7 and 3.10 hold for $L_1$. 

Proof. Freeness is proved similarly to Proposition 4.4. When \( \tau = 1 \) we have
\[
\mathcal{M}_{2e}^{\text{sp, st}} = \emptyset, \quad e \geq 1, \quad \mathcal{M}_{e}^{\text{sp, st}} = \begin{cases} \text{Spec}(\mathbb{C}) & \text{if } e = 1, \\ \emptyset & \text{if } e \geq 2 \end{cases}
\]
while for \( \tau = -1 \) we have \( \mathcal{M}_{2e}^{\text{sp, st}} = \emptyset \) and
\[
\mathcal{M}_{e}^{\text{o}} = \text{Symm}_{e \times e} \mathbb{O}_e \simeq \text{Sym}^{e} \mathbb{C}, \quad \mathcal{M}_{e}^{\text{sp, st}} \simeq (\text{Sym}^{e} \mathbb{C}) \setminus \Delta, \quad e \geq 1
\]
where \( \text{Symm}_{e \times e} \) is the variety of symmetric \( e \times e \) matrices and \( \Delta \) is the big diagonal.

Conjecture 3.10 is now immediate except in type \( D \) with \( \tau = 1 \), where it reads
\[
PH^{0}(\mathcal{M}_{e}^{\text{sp, st}}) \simeq \mathbb{Q}(0), \quad PH^{k}(\mathcal{M}_{e}^{\text{sp, st}}) = 0, \quad e, k \geq 1
\]
which in turn follows from the isomorphism \( H^{*}(\text{Sym}^{e} \mathbb{C}) \setminus \Delta) \simeq H^{*}(\mathbb{C} \setminus \{0\}). \square

4.2.3. Higher loops The situation is more complicated when \( m \geq 2 \) since \( \mathcal{H}_{L_{m}} \) is not finitely generated and \( \mathcal{M}_{L_{m}} \) is not finitely generated over \( \mathcal{H}_{L_{m}} \).

Note that once we fix a type \( B, C \) or \( D \), the sign \( (-1)^{x(e,d)+e(d)} \) which appears in the definition of equivariant DT invariants is independent of \( e \). We therefore write \( \mathcal{H}_{Q} \) for the subalgebra \( \mathcal{H}_{Q}(e) \subset \mathcal{H}_{Q} \). Each homogeneous summand \( \mathcal{H}_{L_{m},(d,k)} \subset \mathcal{H}_{L_{m}} \) is isotypical as a twisted \( \mathbb{Z}_{2} \)-representation. It follows that the equivariant DT invariants are
\[
\tilde{\Omega}^{+}_{L_{m},(2d,k)} = \begin{cases} \Omega_{L_{m},(d,k)} & \text{if } \chi(e,d) + E(d) + \frac{k-x(d)}{2} = 0 \mod 2, \\ 0 & \text{if } \chi(e,d) + E(d) + \frac{k-x(d)}{2} = 1 \mod 2 \end{cases}
\]
and
\[
\tilde{\Omega}^{-}_{L_{m},(2d,k)} = \begin{cases} 0 & \text{if } \chi(e,d) + E(d) + \frac{k-x(d)}{2} = 0 \mod 2, \\ \Omega_{L_{m},(d,k)} & \text{if } \chi(e,d) + E(d) + \frac{k-x(d)}{2} = 1 \mod 2 \end{cases}
\]
Let \( (d^{1}, \ldots , d^{n}; e^{\infty}) \in (\Lambda_{L_{m}}^{+})^{n} \times \Lambda_{L_{m}}^{\sigma^{+}} \). Put \( e = \sum_{p=1}^{n} H(d^{p}) + e^{\infty} \). Keeping the notation from the proof of Theorem 3.4 (but dropping the subscript \( L_{m} \)), we have an algebra isomorphism
\[
Z_{e} \simeq X_{d^{1}} \otimes \cdots \otimes X_{d^{n}} \otimes Z_{e^{\infty}} = Z_{d^{*}, e^{\infty}}
\]
which induces an algebra embedding
\[
i : \mathcal{I}_{e} \hookrightarrow \mathcal{H}_{d^{1}} \otimes \cdots \otimes \mathcal{H}_{d^{n}} \otimes \mathcal{I}_{e^{\infty}} = \mathcal{I}_{d^{*}, e^{\infty}}.
\]
Setting \( \mathcal{M}_{d^{*}, e^{\infty}} = \prod_{p=1}^{n} \mathcal{G}_{d^{p}} \times \mathcal{M}_{\mathcal{F}} \), we have \( \mathcal{I}_{d^{*}, e^{\infty}} \simeq Z_{d^{*}, e^{\infty}}^{\mathcal{M}_{d^{*}, e^{\infty}}} \). As in [11], for each \( d \in \Lambda_{L_{m}}^{+} \), define a subalgebra of \( X_{d}^{\text{prim}} = \mathbb{Q}[x_{k} - x_{l} \mid 1 \leq k < l \leq d] \subset X_{d} \) and let \( J_{d} \) be the minimal \( \mathcal{G}_{d} \)-stable \( X_{d}^{\text{prim}} \)-submodule of \( X_{d} \) which contains, for each non-trivial decomposition \( d = d^{1} + d^{\prime} \), the kernel \( K_{d^{1}, d^{\prime}}(x, x^{\prime}) \). Define a \( \mathcal{M}_{d^{*}, e^{\infty}} \)-stable ideal
\[
L_{d^{*}, e^{\infty}} = J_{d^{1}} Z_{d^{*}, e^{\infty}} + \cdots + J_{d^{n}} Z_{d^{*}, e^{\infty}} + (L_{e^{\infty}} \cap Z_{e^{\infty}}) Z_{d^{*}, e^{\infty}} \subset Z_{d^{*}, e^{\infty}}.
\]
Note that \( L_{e^{\infty}} \subset Z_{e^{\infty}} \) except in type \( C \) with \( \tau_{-} = 0 \).

Write \( \zeta_{e}^{(p)} \in X_{d^{p}}, 1 \leq p \leq n \) and \( \zeta_{e}^{(\infty)} \in Z_{e^{\infty}} \) for the standard algebra generators, considered as elements of \( Z_{d^{*}, e^{\infty}} \).
Lemma 4.6. The following elements are not zero-divisors in $Z_{d^*} \cdot e^\infty / L_{d^*} \cdot e^\infty$:

(i) $z_k^{(p)} + z_l^{(p)}$ for $1 \leq k \leq l \leq d^p$ and $1 \leq p \leq n$, and

(ii) $z_k^{(q)} \pm z_l^{(p)}$ for $1 \leq k \leq d^q$, $1 \leq l \leq d^p$ and $1 \leq p < q \leq \infty$.

Proof. The statement and its proof are variations of [11, Claim, pg. 1141]; we include it for completeness. Any element $g \in Z_{d^*} \cdot e^\infty$ can be written uniquely as $g = \sum_{v=0}^{N} g_v \sigma_{d^p}^v$ for some elements $g_v \in X_{d^1} \otimes \cdots \otimes X_{d^{p-1}} \otimes X_{d^p}^{\text{prim}} \otimes X_{d^{p+1}} \otimes \cdots \otimes X_{d^n} \otimes Z_{e^\infty}$. Then $g \notin L_{d^*} \cdot e^\infty$ if and only if $g_v \notin L_{d^*} \cdot e^\infty$ for some $v \in \{0, \ldots, N\}$.

(i) Observe that we can write

$$z_k^{(p)} = \frac{1}{d^p} \sigma_{d^p} - \frac{1}{d^p} \sum_{r \neq k} (z_r^{(p)} - z_k^{(p)}).$$

Since $\sum_{r \neq k} (z_r^{(p)} - z_k^{(p)}) \in A_{d^p}^{\text{prim}}$, the sum $z_k^{(p)} + z_l^{(p)}$ is of the form $c \sigma_{d^p} + a$ for $c \in \mathbb{Q}^\infty$ and $a \in A_{d^p}^{\text{prim}}$. Then

$$(z_k^{(p)} + z_l^{(p)})g = cg_N \sigma_{d^p}^{N+1} + \sum_{v=0}^{N} g_v' \sigma_{d^p}^v.$$ 

It follows that $(z_k^{(p)} + z_l^{(p)})g \notin L_{d^*} \cdot e^\infty$.

(ii) We have

$$z_k^{(q)} \pm z_l^{(p)} = (z_k^{(q)} \pm z_l^{(p)}) \mp \frac{1}{d^p} \sigma_{d^p} \pm \frac{1}{d^p} \sigma_{d^p}.$$ 

Since $z_l^{(p)} - \frac{1}{d^p} \sigma_{d^p} \in A_{d^p}^{\text{prim}}$, the remainder of the proof is as in part (i). \qed

Let $\{v_{d, \beta}\}_{1 \leq \beta \leq \dim V_{d}^{\text{prim}}}$ be a homogeneous basis of $V_{d}^{\text{prim}}$. Then $\{v_{d, \beta} \sigma_{d}^m\}_{d, \beta, m}$ is a basis of $V_{L_m}$ with a lexicographic order $\geq$. Let also $\{w_{e, \beta}\}_{1 \leq \beta \leq \dim W_{e}^{\text{prim}}}$ be a homogeneous basis of $W_{e}^{\text{prim}}$. For each $e \in \Lambda_{L_m}^{\sigma^+, \sigma^-}$, let Seq$_e^\sigma$ be the set of all sequences

$$(v_{d^1, \beta_1} \sigma_{d^1}^{h_1}, \ldots, v_{d^n, \beta_n} \sigma_{d^n}^{h_n}, w_{e}^\infty, \beta_\infty)$$

which have the following properties:

(i) Each $d^p$, $1 \leq p \leq n$, is non-zero. We allow $e^\infty$ to be zero.

(ii) The equality $e = \sum_{p=1}^{n} H(d^p) + e^\infty$ holds.

(iii) The inequalities $v_{d^1, \beta_1} \sigma_{d^1}^{h_1} \geq \cdots \geq v_{d^n, \beta_n} \sigma_{d^n}^{h_n}$ hold.

(iv) If $(d^p, \beta_p, h_p) = (d^{p+1}, \beta_{p+1}, h_{p+1})$, then $\chi(d^p, d^p) \equiv 0 \mod 2$.

(v) If $v_{d, \beta, h} \in H(d^p, k)$, then $\chi(e^\infty, d^p) + E(d^p) \equiv \frac{k + 2h_p - \chi(d^p, d^p)}{2} \mod 2$.

A lexicographic order $\geq$ is defined on Seq$_e^\sigma$ by first comparing the CoHM components and then comparing the CoHA sequences.

Theorem 4.7. Conjecture 3.7 holds for m-loop quivers.
Proof. As we have seen that the theorem is true if \( m \leq 1 \), we will assume that \( m \geq 2 \) in what follows. The general structure of the proof is similar to [11, §3].

Given \( t \in \text{Seq}_e^\sigma \), let \( M_t \in \mathcal{I}_e \) be the corresponding ordered product. In view of condition (iv) of \( \text{Seq}_e^\sigma \), to prove the theorem it suffices to show that, for each strictly decreasing sequence \( t_1 > \cdots > t_r \) in \( \text{Seq}_e^\sigma \) and each tuple \( (\lambda_1, \ldots, \lambda_r) \in (\mathbb{Q}^\times)^r \), we have \( \sum_{i=1}^r \lambda_i M_{t_i} \neq 0 \).

Let \( \text{dim} \, t_1 = (d^1, \ldots, d^n; e^\infty) \) be the underlying sequence of dimension vectors of \( t_1 \). Suppose that \( a \in \{1, \ldots, r\} \) is minimal with the property that \( \text{dim} \, t_{a+1} < \text{dim} \, t_1 \). Assume that we are not in type \( C \) with \( \tau_- = 0 \). We claim that

\[
\iota(M_{t_i}) \in (L_{d^i, e^\infty})^{\text{dim} \, t_i}, \quad a + 1 \leq i \leq r.
\]

Indeed, suppose, for example, that \( \widetilde{e}^\infty \prec e^\infty \), where \( \widetilde{e}^\infty \) is the \( \Lambda_{L_m}^+ \)-component of \( \text{dim} \, t_i \). We claim that

\[
\iota(M_{t_i}) \in (L_{e^\infty} Z_{d^i, e^\infty})^{\text{dim} \, t_i} \subset (L_{d^i, e^\infty})^{\text{dim} \, t_i, e^\infty}. \tag{29}
\]

To see this, note that the underlying shuffle of a \( \sigma \)-shuffle \( \tilde{\pi} \in \text{sh}_{\text{dim} \, t_i}^\sigma \) defines a bijection

\[
\{z_1^{(\infty)}, \ldots, z_i^{(\infty)} \} \leftrightarrow \{z_i^{(q)} \mid (i, q) \in S(\tilde{\pi}) \}.
\]

Here \( z_k^{(q)} \) are canonical algebra generators of \( Z_e \), but partitioned according to \( t_i \), as opposed to the variables \( z_k^{(q)} \), which are partitioned according to \( t_1 \). Let \( e' \in \Lambda_{L_m}^+ \) be such that \( \lfloor \frac{e'}{2} \rfloor \) is the number of points in \( S(\tilde{\pi}) \) whose second component is \( \infty \), and let \( d' \in \Lambda_{L_m}^+ \) be the difference \( \#S(\tilde{\pi}) - \lfloor \frac{e'}{2} \rfloor \). Then, up to the action of sign changes, the \( \tilde{\pi} \)-th term in \( \iota(M_{t_i}) \) has a factor of \( K_{d', e'} \), as follows from the explicit expressions for the kernels \( K \) and \( K^\sigma \), and hence lies in \( L_{e^\infty} Z_{d^i, e^\infty} \). This gives the inclusion (29). The general case can be argued in the same way.

To finish the proof, it suffices to show that

\[
\sum_{i=1}^r \lambda_i M_{t_i} \notin L_{d, e^\infty}^{\text{dim} \, t_i, e^\infty}.
\]

We focus on a single summand \( M_{t_i} \) and, for ease of notation, write \( f_p \) and \( g_\infty \) for \( v_{d^p, \beta_p} \sigma_{d^p} \) and for \( w_{e^\infty, \beta^\infty} \), respectively. We claim that

\[
\iota(f_1 \cdots f_n \star g_\infty) \equiv 2^n F_{d^\bullet, e^\infty} \mod L_{d^\bullet, e^\infty}^{\text{dim} \, t_i, e^\infty}, \tag{30}
\]

where \( F_{d^\bullet, e^\infty} \) is a product of terms of the form \((z_k^{(q)})^2 - (z_k^{(p)})^2, 1 \leq p < q \leq \infty, the sign \( s(\pi) \in \{\pm\} \) is a Koszul sign, \( K_{d^p} = K_{d^p, 0}(x_1^{(p)}, \ldots, x_{d^p}^{(p)}) \) and \( \mathcal{G}_n^d \) is the set

\[
\{\pi \in \mathcal{G}_n \mid d^p = \sigma^p(\pi), 1 \leq p \leq n \}.
\]

To see this, first observe that the \( \sigma \)-shuffles \( \pi \in \text{sh}_{\text{dim} \, t_i}^\sigma \) whose underlying shuffles do not lie in \( \mathcal{G}_n^d \) produce a term which lies in \( L_{d^\bullet, e^\infty}^{\text{dim} \, t_i, e^\infty} \). Consider then the action of the
sign-change group $\mathbb{Z}_2^P \subset \mathfrak{sh}_{d*,e^\infty}$. If $\pm 1 \neq \tilde{\pi} \in \mathbb{Z}_2^P$, then, up to a permutation, there is a non-trivial decomposition $d^P = d^{P'} + d^{P''}$ such that

$$\tilde{\pi}(f_p K^A_{d^P}) = f_p(x'_1', \ldots, x'_{d^{P'}}', -x''_i, \ldots, -x''_{d^{P''}},) K^A_{d^{P'}}(x') K^A_{d^{P''}}(-x'') K_{d^{P'}} K_{d^{P''}}(x', x''),$$

as follows from the explicit expression for $K^A_{d^P}$. Because of the last factor, the right hand side of this equation lies in $J_{d^P}$. On the other hand, if $\tilde{\pi} = -1$, then the fifth defining condition of $\text{Seq}_e$ implies that $\tilde{\pi}(f_p K^A_{d^P}) = f_p K^A_{d^P}$. Summing over the sign-change groups therefore gives the factor $2^n$ in equation (30).

By the second part of Lemma 4.6, the prefactor $F_{d^P*}^{p^*} e^\infty$ is not a zero-divisor in $I_{d^P*} e^\infty / L_{d^P*} e^\infty$. Similarly, by Lemma 4.6, the element $K_{d^P}^A$ is not a zero-divisor in $\mathcal{H}_{d_P^P} / J_{d^P}^{\sigma_{d^P}}$. Since $f_p \not\in J_{d^P}$, $1 \leq p \leq n$, and $g_{e^\infty} \not\in L_{e^\infty}$ by definition, it follows that $f_1 \cdots f_n \not\in L_{d^P*} e^\infty$. This completes the proof outside of type $C$ with $\tau_\infty = 0$.

A minor modification is required in type $C$ with $\tau_\infty = 0$, as in this case $K_{d^P}^A$ has a non-trivial denominator, namely $x_1 \cdots x_d$. By the first part of Lemma 4.6, the elements $x_k^{(p)}$, $1 \leq p \leq n$, are not zero-divisors in $Z_{d^P*} e^\infty / L_{d^P*} e^\infty$. Define the localization

$$Z_{d^P*} e^\infty = Z_{d^P*} e^\infty[(z_i^{(p)})^{-1} | 1 \leq i \leq d^P, 1 \leq p \leq n]$$

and put $I_{d^P*} e^\infty = (Z_{d^P*} e^\infty)^{\mathfrak{sl}_{d^P*} e^\infty}$. Let $\eta : Z_{d^P*} e^\infty \rightarrow Z_{d^P*} e^\infty$ be the canonical map and define $L_{d^P*} e^\infty = \eta(L_{d^P*} e^\infty) Z_{d^P*} e^\infty$. We obtain an inclusion

$$\eta : I_{d^P*} e^\infty / L_{d^P*} e^\infty \hookrightarrow I_{d^P*} e^\infty / (L_{d^P*} e^\infty)^{\mathfrak{sl}_{d^P*} e^\infty}.$$

The expression (30) now computes $\eta(M_i)$ and the remainder of the above proof can be applied without change. □

**Corollary 4.8.** The orientifold DT invariants of $L_m$ are uniquely determined by the equation $A_{L_m}^\sigma = \Lambda_{L_m}^\sigma \Omega_{L_m}^\sigma$.

**Proof.** This follows immediately from Theorem 4.7 and equation (23). □

Since $\Omega_{L_m}$ has been computed by Reineke [35, Theorem 6.8] and $A_{L_m}^\sigma$ is given explicitly by equation (13), Corollary 4.8 can be used to compute $\Omega_{L_m}^\sigma$.

**Example.** Let $m = 2$ with $(s, \tau) = (1, -1)$. We have

$$\Omega_{L_2} = -q^{-\frac{1}{2}} t + q^{-2} r^2 - q^{-\frac{9}{2}} r^3 + q^{-8}(1 + q^2)t^4 + O(t^5)$$

and

$$\tilde{\Omega}_{L_2} = -q^{-\frac{9}{2}} \xi^6 + q^{-8}(1 + q^2)\xi^8 + O(\xi^{10}), \quad \tilde{\Omega}_{L_2} = -q^{-\frac{3}{2}} \xi^2 + q^{-2} \xi^4 + O(\xi^{10}).$$

Using Corollary 4.8, we compute

$$\Omega_{L_2} = \xi - q^{-\frac{3}{2}} \xi^3 + q^{-5}(1 + q^2)\xi^5 - q^{-\frac{21}{4}}(1 + q^2 + 2q^4 + q^6)\xi^7$$

$$+ q^{-18}(1 + q^2 + 2q^4 + 3q^6 + 4q^8 + 3q^{10} + q^{12})\xi^9 + O(\xi^{11}).$$

Up to $\Lambda_{Q}^{\sigma^+}$-degree five, minimal generators of $I_{L_2}^B$ are $1^\sigma_1$, $1^\sigma_2$, $1^\sigma_3$ and $z_1^2 + z_2^2$. 

Example. Let \( m = 3 \) with \((s, \tau) = (1, 1)\). We have
\[
\Omega_{L_3} = q^{-1}t + q^{-4}t^2 + q^{-9}(1 + q^2 + q^3)t^3
+ q^{-16}(1 + q^2 + q^3 + 2q^4 + q^5 + 2q^6 + q^7 + q^8)t^4 + O(t^5)
\]
from which we compute
\[
\Omega_{L_3}^D = 1 + q^{-4}(1 + q^2)\xi^2 + q^{-12}(1 + q^2 + 2q^4 + 2q^6 + q^8)\xi^4
+ q^{-24}(1 + q^2 + 2q^4 + 3q^6 + 4q^8 + 5q^{10} + 6q^{12} + 4q^{16} + q^{18})\xi^6
+ q^{-40}(1 + q^2 + 2q^4 + 3q^6 + 5q^8 + 6q^{10} + 9q^{12} + 11q^{14} + 14q^{16} + 16q^{18}
+ 19q^{20} + 20q^{22} + 21q^{24} + 19q^{26} + 14q^{28} + 6q^{30} + q^{32})\xi^8 + O(\xi^{10}).
\]
Finally, we give some results for arbitrary \( m \) and small dimension vector. Define the quantum integers by \([0]_q = 0\) and \([n]_q = q^n - 1\) if \( n \in \mathbb{Z}_{\geq 1} \). Then we have
\[
\Omega_{L_m} = (-q^{\frac{1}{2}})^{1-m}t + q^{2(1-m)}\left[\frac{m}{2}\right]q^2 + O(t^3).
\]
Suppose that \( \tau = -1 \). Using Corollary 4.8, we find
\[
\Omega_{L_m}^C = 1 + (-q^{\frac{1}{2}})^{3(1-m)}\left[\frac{m}{2}\right]q^2 \xi^2
+ q^{5(1-m)}\left\{\begin{array}{c}
\left[\frac{m}{2}\right]q^4 (\frac{3m-2}{2})q^2 + q^m (\frac{2m-2}{2})q^2 + q^{2m} (\frac{m}{2})q^2 \\
\left[\frac{m-1}{2}\right]q^4 (\frac{3m-1}{2})q^2 + q^{m-1} (\frac{m}{2})q^2 + q^{2m-2} (\frac{m-1}{2})q^2 \\
\left[\frac{m}{2}\right]q^2 (\frac{3m-2}{4})q^4 + q^m (\frac{2m}{4})q^4 + q^{2m} (\frac{m-2}{4})q^4 \\
\left[\frac{m-1}{2}\right]q^2 (\frac{3m-1}{4})q^4 + q^{m-1} (\frac{m-2}{4})q^4 + q^{2m-2} (\frac{m+1}{4})q^4
\end{array}\right\} \xi^4
+ O(\xi^6).
\]
The rows of the braces correspond to classes of \( m \) modulo four, with \( m \equiv 0 \) in the top row increasing to \( m \equiv 3 \) in the bottom row. Similarly, we find
\[
\Omega_{L_m}^D = 1 + q^{3(1-m)}\left[\frac{m}{4}\right]q^2 \left[\frac{m + 2}{4}\right]q^4 + O(\xi^6).
\]
Note that \( \mathcal{M}_{L_m,2} = \emptyset \), consistent with the vanishing \( \Omega_{L_m,2}^D = 0 \).

4.3. Symmetric \( \tilde{A}_1 \) quiver. Let \( Q \) be the affine \( \tilde{A}_1 \) quiver
\[
\begin{array}{c}
1 \\
\alpha \\
2
\end{array}
\]
A representation of dimension vector \((d_1, d_2)\) is a pair of complex matrices \((u_\alpha, u_\beta) \in \text{Mat}_{d_2 \times d_1} \times \text{Mat}_{d_1 \times d_2}\). For \( \theta = (1, -1) \) the semistable representations are
(i) the direct sums of simple representations \( S_1^{\oplus k}, k \geq 1, \) having slope 1,
(ii) the direct sums of simple representations \( S_2^{\oplus k}, k \geq 1, \) having slope \(-1, \) and
(iii) the pairs \((u_\alpha, u_\beta) \in \text{GL}_d(\mathbb{C}) \times \text{Mat}_{d \times d}, d \geq 1\), having slope 0.

The algebra \(\mathcal{H}_Q\) is supercommutative. For \(f_1 \in \mathcal{H}_{Q,d'}\) and \(f_2 \in \mathcal{H}_{Q,d''}\), we have

\[
f_1 \cdot f_2 = \sum_{\pi \in \mathfrak{h}_{d',d''}} \pi \left( f_1(x'_1, \ldots, x'_{d'_1}, y'_1, \ldots, y'_{d'_2}) f_2(x''_1, \ldots, x''_{d''_1}, y''_1, \ldots, y''_{d''_2}) \right) \times \prod_{j=1}^{d''_1} \prod_{i=1}^{d'_1} (y''_j - x'_i) \prod_{j=1}^{d''_2} \prod_{i=1}^{d'_2} (x''_j - y'_i),
\]

The semistable algebras \(\mathcal{H}_{Q,\mu=1}^{\theta,ss}\) and \(\mathcal{H}_{Q,\mu=-1}^{\theta,ss}\) are isomorphic to \(\mathcal{H}_{L_0}\) and embed canonically as subalgebras of \(\mathcal{H}_Q\). On the other hand, the inclusion

\[
\text{Mat}_{d \times d} \hookrightarrow \text{GL}_d(\mathbb{C}) \times \text{Mat}_{d \times d}, \quad u_\beta \mapsto (I_{d \times d}, u_\beta)
\]
descends to an isomorphism from the stack of \(d\)-dimensional representations of \(L_0\) to the stack of \((d, d')\)-dimensional semistable representations of \(Q\). This induces an algebra isomorphism \(\mathcal{H}_{Q,\mu=0}^{\theta,ss} \simeq \mathcal{H}_{L_1}\). The map

\[
\Psi_0 : \mathcal{H}_{L_1} \to \mathcal{H}_Q, \quad x^i \mapsto x^i y^0
\]
extends to an algebra embedding. It is proved in [14, Proposition 2.4] that the slope-ordered CoHA multiplication

\[
\mathcal{H}_{Q,\mu=1}^{\theta,ss} \boxtimes \mathcal{H}_{Q,\mu=0}^{\theta,ss} \boxtimes \mathcal{H}_{Q,\mu=-1}^{\theta,ss} \to \mathcal{H}_Q, \quad a \otimes b \otimes c \mapsto a\Psi_0(b)c
\]
is an algebra isomorphism. In particular, there is a decomposition

\[
\mathcal{V}^{\text{prim}}_Q = \bigoplus \mathbb{Q} : 1_{(1,0)} \oplus \mathbb{Q} : 1_{(1,1)} \oplus \mathbb{Q} : 1_{(0,1)}.
\]

Let \(\sigma\) be the involution of \(Q\) that swaps the nodes and fixes the arrows. Then

\[
\mathcal{E}(d_1, d_2) = d_1 d_2 - \frac{d_1(d_1 + s \tau_\alpha)}{2} - \frac{d_2(d_2 + s \tau_\beta)}{2}.
\]

It follows that \(Q\) has two inequivalent \(\sigma\)-symmetric duality structures, say \(s = 1\) and \(\tau = \pm 1\). The structure maps of a self-dual representation are symmetric (resp. skew-symmetric) matrices if \(\tau = 1\) (resp. \(\tau = -1\)). For \(f \in \mathcal{H}_{Q,(d_1,d_2)}\) and \(g \in \mathcal{I}_{Q,(e,e)}\), we have

\[
f \ast g = \sum_{\pi \in \mathfrak{h}_{\mu,\nu}^{d_1,d_2}} \pi \left( f(x_1, \ldots, x_{d_1}, y_1, \ldots, y_{d_2}) g(z_1, \ldots, z_e) \right) \times \prod_{1 \leq j \leq l \leq d_1} (-x_j - x_l) \prod_{l=1}^{d_1} \prod_{k=1}^{e} (-z_k - x_l) \prod_{1 \leq j \leq m \leq d_2} (-y_j - y_m) \prod_{m=1}^{d_2} \prod_{k=1}^{e} (z_k - y_m) \prod_{l=1}^{d_1} \prod_{k=1}^{e} (z_k - x_l) \prod_{k=1}^{e} \prod_{m=1}^{d_2} (-y_m - x_l).
\]

The stability \(\theta\) is \(\sigma\)-compatible. We have

\[
\tau = 1 : \quad R_{(e,e)}^{\sigma,\theta,ss} = (\text{Symm}_{e \times e} \cap \text{GL}_e(\mathbb{C})) \times \text{Symm}_{e \times e}.
\]
and

\[ \tau = -1 : \quad R_{(e,e)}^{\sigma,\theta-ss} = (\text{Skew}_{e \times e} \cap \text{GL}_e(\mathbb{C})) \times \text{Skew}_{e \times e}. \]

The group \( G^\sigma_{(e,e)} = \text{GL}_e(\mathbb{C}) \) acts on \( R_{(e,e)}^{\sigma,\theta-ss} \) by congruences. It follows that the stack of semistable self-dual representations is isomorphic to the stack of self-dual representations of \( L_1 \) with \((s_{L_1}, \tau_{L_1}) = (\tau, +1) \). The induced map \( I_Q^{\theta-ss} \sim I_{L_1} \) is an isomorphism over \( \mathcal{H}^{\theta-ss}_{Q,\mu=0} \sim \mathcal{H}_{L_1} \).

**Lemma 4.9.** The kernel of the restriction map \( I_Q \to I_Q^{\theta-ss} \) in dimension vector \((e, e) \in \Lambda^+_Q \) is the image of the action map

\[
\bigoplus_{d=1}^e \mathcal{H}_{Q,(d,0)} \boxtimes^S I_Q,(e-d,e-d) \to I_Q,(e,e).
\]

**Proof.** Let \( Z \) be a self-dual representation determined by matrices \((u_\alpha, u_\beta)\). Then \( 0 \subset \ker u_\alpha \subset Z \) is the \( \sigma \)-HN filtration. The \( \sigma \)-HN strata of \( R^\sigma_Z \) are therefore the locally closed subsets consisting of self-dual representations with fixed \( \dim_{\mathbb{C}} \ker u_\alpha \) and the closure of a stratum is a union of strata. Using this observation, [14, Lemma 2.1] can be applied with only straightforward modifications to prove the lemma. In slightly more detail, the methods of [14] can be used to prove the lemma for the Chow theoretic Hall module, defined in the same way as \( I_Q \) but using equivariant Chow groups instead of equivariant cohomology. In the case at hand, the (semistable) cohomological and Chow theoretic Hall modules are isomorphic, as can be verified directly. Hence the lemma also holds in the cohomological case. \( \square \)

Let \( \tilde{\mathcal{H}}_Q \subset \mathcal{H}_Q \) be the subalgebra generated by

\[
(\mathbb{Q} \cdot 1_{(1,0)} \otimes \mathbb{Q}[u]) \oplus (\mathbb{Q} \cdot 1_{(1,1)} \otimes u \mathbb{Q}[u^2]) \subset V_Q.
\]

Then \( \tilde{\mathcal{H}}_Q \cong \mathcal{H}_{Q,\mu=1}^{\theta-ss} \otimes \mathcal{H}_{Q,\mu=0}^{\theta-ss,odd} \) as algebras, the second factor being the image of \( \mathcal{H}_{L_1}^{\theta-ss} \). The map sending \( 1_0^\sigma \in I_{L_1,0} \) to \( 1_{(0,0)}^\sigma \in I_{Q,(0,0)} \) extends to a \( \mathcal{H}_{Q,\mu=0}^{\theta-ss,odd} \)-module embedding \( I_Q^{\theta-ss} \hookrightarrow I_Q \).

**Theorem 4.10.** The \( \mathcal{H}_{Q,\mu=0}^{\theta-ss,odd} \)-module \( I_Q^{\theta-ss} \) is free with basis \( 1_0^\sigma \in I_Q^{\theta-ss,(0,0)} \) if \( \tau = -1 \) and basis \( 1_{(e,e)}^\sigma \in I_Q^{\theta-ss,(e,e)}, e \geq 0, \) if \( \tau = 1 \). Moreover, the action map \( \mathcal{H}_{Q,\mu=1}^{\theta-ss} \boxtimes^S I_Q^{\theta-ss} \to I_Q \) is an isomorphism of \( \tilde{\mathcal{H}}_Q \)-modules. In particular, \( I_Q \) is a free \( \tilde{\mathcal{H}}_Q \)-module.

**Proof.** The first statement follows from Proposition 4.5 and the \( \mathcal{H}_{L_1} \)-module isomorphism \( I_Q^{\theta-ss} \cong I_{L_1} \). Direct calculation shows that the restriction map \( I_Q \to I_Q^{\theta-ss} \) is surjective. From this and Lemma 4.9, we conclude that the action map is surjective. The wall-crossing formula (26) reads \( A_{Q,\mu=1}^{\theta-ss} \ast A_{\sigma}^{\theta-ss} = A_{\sigma}^{\theta} \), implying that the Hilbert–Poincaré series of \( \mathcal{H}_{Q,\mu=1}^{\theta-ss} \boxtimes^S I_Q^{\theta-ss} \) and \( I_Q \) are equal. Hence the action map is a \( \Lambda^+_Q \times \mathbb{Z} \)-graded vector space isomorphism. That this map respects \( \tilde{\mathcal{H}}_Q \)-module structures is clear. \( \square \)

**Corollary 4.11.** The motivic orientifold DT invariants are \( \Omega^\sigma_Q = 1 \) if \( \tau = -1 \) and \( \Omega^\sigma_Q = (1 - q^{-\frac{1}{2}} \xi^{(1,1)})^{-1} \) if \( \tau = 1 \). Conjecture 3.10 holds for \( Q \).
Proof. The statement for $\tau = -1$ follows from Theorem 4.10, so we restrict attention to the case $\tau = 1$. Theorem 4.10 implies that $(1 - q^{-\frac{1}{2}})\xi(1,1))^{-1}$ is a coefficientwise upper bound for $\Omega^1_{Q,e}$. To see that it is also a lower bound, observe that since the unshifted cohomological degree of elements of $H_{Q,e} \star \mathcal{I}_{Q,e}$ is at least

$$-2\mathcal{E}(d) = (d_1 - d_2)^2 + d_1 + d_2 > 0,$$

the element $1^\sigma_{(e,e)}$ is nonzero in $W^\text{prim}_{Q,e}$. Hence, $\Omega^\sigma_{Q,e}$ is as stated. To verify Conjecture 3.10, we must prove that $PH^\bullet((M^\sigma_{(e,e)} \star e)^{\text{st}}(1,1)) \simeq Q(0)$. We have $M^\sigma_{(e,e)} \simeq \mathbb{C}^\times$, consisting of regularly $\sigma$-stable representations. Moreover, these are the only regularly $\sigma$-stable self-dual representations, from which it follows that $M^\sigma_{(e,e)} \simeq (\text{Sym}^e \mathbb{C}^\times) \setminus \Delta$. Consider the open inclusions

$$(\text{Sym}^e \mathbb{C}^\times) \Delta \hookrightarrow (\text{Sym}^e \mathbb{C}) \Delta \hookrightarrow \text{Sym}^e \mathbb{P}^1.$$ 

Since $\text{Sym}^e \mathbb{P}^1$ is a smooth compactification of both $\text{Sym}^e \mathbb{C}^\times$ and $\text{Sym}^e \mathbb{C}$, we obtain a commutative diagram

$$\begin{array}{ccc}
H^\bullet(\text{Sym}^e \mathbb{P}^1) & \longrightarrow & H^\bullet((\text{Sym}^e \mathbb{C}) \setminus \Delta) \\
\downarrow & & \downarrow \\
PH^\bullet((\text{Sym}^e \mathbb{C}) \setminus \Delta) & \longrightarrow & PH^\bullet((\text{Sym}^e \mathbb{C}^\times) \setminus \Delta),
\end{array}$$

the surjections following from [33, Proposition 6.29]. Hence $i^*$ is also surjective. Since $PH^\bullet((\text{Sym}^e \mathbb{C}) \setminus \Delta) \simeq Q(0)$, we also have $PH^\bullet((\text{Sym}^e \mathbb{C}^\times) \setminus \Delta) \simeq Q(0)$. □

Remarks. (i) The isomorphism $H^\theta_{Q,e} \simeq \mathbb{C}^S \mathcal{I}_{Q,e}$ of Theorem 4.10 is an instance of the PBW factorization (27).

(ii) Let $\theta = (1, -1)$. If $\tau = 1$, then $M_{(e,e)}^{\sigma, \text{st}} \simeq (\text{Sym}^e \mathbb{C}) \setminus \Delta$ and the proof of Corollary 4.11 shows that $i^*: PH^\bullet(M_{(e,e)}^{\sigma, \text{st}}) \simeq PH^\bullet(M_{(e,e)}^{\sigma, \text{st}})$. This is an example of the lack of wall-crossing for $\sigma$-symmetric quivers.

5. Cohomological Hall Modules of Finite Type Quivers

A quiver is called finite type if it has only finitely many indecomposable representations up to isomorphism. Gabriel proved that a quiver is finite type if and only if it is a finite disjoint union of quivers whose underlying graphs are $ADE$ Dynkin diagrams [15]. Note that a finite type quiver is symmetric if and only if it is a finite collection of points. Indecomposable finite type quivers with involution are either of Dynkin type $A$ or of the form $ADE^U$. By Theorem 4.2, the CoHM of an $ADE^U$ quiver can be described entirely in terms of the CoHA of the $ADE$ quiver, whose structure we will recall in Sect. 5.1. The main task is therefore to describe the CoHM of a quiver of Dynkin type $A$. 

5.1. Finite type CoHA. Let $Q$ be a connected finite type quiver. We assume that $Q$ is not of type $E_8$; for this case see [36, Remark 11.3]. The sets $\Pi$ and $\Delta$ of positive simple and positive roots of $Q$ are in bijection with the sets of isomorphism classes of simple and indecomposable representations of $Q$, respectively [15]. Identify $\Delta$ with a subset of $\Lambda_+^\ast_Q$ and write $I_\beta$ for the indecomposable representation of dimension vector $\beta \in \Delta$. Fix a total order $\beta_1 < \cdots < \beta_N$ on $\Delta$ such that $\text{Hom}(I_{\beta_i}, I_{\beta_j}) = 0 = \text{Ext}^1(I_{\beta_j}, I_{\beta_i})$ if $i < j$. Such an order exists because the Auslander–Reiten quiver $\Gamma(Q)$ of $Q$ is acyclic.

Fix a positive root $\beta \in \Delta$. Consider

$$\mathcal{H}_Q^{(\beta)} = \bigoplus_{n \geq 0} H_{\text{GL}_{n\beta}}^\bullet (R_{n\beta})[-\chi(n\beta, n\beta)]$$

and

$$\mathcal{H}_Q^{(\beta), \simeq} = \bigoplus_{n \geq 0} H_{\text{GL}_{n\beta}}^\bullet (\eta_{\tilde{I}_{\beta}^\oplus n})[-\chi(n\beta, n\beta)],$$

where $\eta_{\tilde{I}_{\beta}^\oplus n} \subset R_{n\beta}$ is the $\text{GL}_{n\beta}$-orbit of representations which are isomorphic to $I_{\beta}^\oplus n$.

Then $\mathcal{H}_Q^{(\beta)}$ is a subalgebra of $\mathcal{H}_Q$ and the natural Hall-type product on $\mathcal{H}_Q^{(\beta), \simeq}$ is such that the restriction map $\rho : \mathcal{H}_Q^{(\beta)} \to \mathcal{H}_Q^{(\beta), \simeq}$ is a surjective algebra homomorphism. Moreover, $\mathcal{H}_Q^{(\beta), \simeq} \simeq \mathcal{H}_Q^{(\beta), \simeq}$ as algebras. Let $\{\tilde{x}^i\}_{i \geq 0}$ be the corresponding algebra generators of $\mathcal{H}_Q^{(\beta), \simeq}$, as defined in Sect. 4.2.1. Choose\(^3\) $i(\beta) \in Q_0$ such that $\dim \mathcal{C}(I_{\beta})_{i(\beta)} = 1$.

Define a section $\psi$ of $\rho$ by $\psi(\tilde{x}^i) = x^i_{i(\beta)}$ and write $\mathcal{H}_Q^{(\beta)} \subset \mathcal{H}_Q$ for the isomorphic image of $\psi$. Elements of $\mathcal{H}_Q^{(\beta)}$ depend only on the variables associated to the node $i(\beta)$.

When $Q$ is of type $A_2$, the following result was stated as [27, Proposition 2.1].

**Theorem 5.1** [36, Theorem 11.2]. The ordered CoHA multiplication maps

$$\boxtimes_{\alpha \in \Pi} \mathcal{H}_Q^{(\alpha)} \to \mathcal{H}_Q, \quad \boxtimes_{\beta \in \Delta} \mathcal{H}_Q^{(\beta)} \to \mathcal{H}_Q$$

are isomorphisms in $D^{lb}(\text{Vect})_{\Lambda_+^\ast_Q}$.

5.2. Resolutions of orbit closures and quantum dilogarithm identities. Let $(Q, \sigma)$ be of Dynkin type $A$. Then $Q$ has two inequivalent duality structures which, for concreteness, we take to be $(\tau, s) = (-1, 1)$ and $(\tau, s) = (-1, -1)$, giving orthogonal or symplectic representations in the language of [8], respectively. In type $A_{2n}$ (resp. $A_{2n+1}$) all orthogonal (symplectic) representations are hyperbolic. In the remaining two cases, henceforth referred to as non-hyperbolic, each $\sigma$-invariant positive root admits a unique self-dual structure.

To describe $\mathcal{I}_Q$ we will modify Rimányi’s approach to the study of $\mathcal{H}_Q$. Fix $d^\ast = (d^1, \ldots, d^r) \in (\Lambda_+^\ast_Q)^r$, $e^\infty \in \Lambda_+^{\infty}$, and put $e = \sum_{i=1}^r H(d^i) + e^\infty$. Let $G_{d^\ast, e^\infty}^\sigma \leq G_e^\sigma$ be the stabilizer of a $Q_0$-graded isotropic flag of $\mathbb{C}^r$ of the form

$$0 = U_0 \subset U_1 \subset \cdots \subset U_r \subset \mathbb{C}^r$$

\(^3\) Such a choice cannot always be made in type $E_8$.
Lemma 5.2. Let \( f_k \dim \mathbb{C}/U_1 = d^k \) and \( \dim \mathbb{C}/U_r = e^\infty \). Let \( \text{Fl}_{d^*,e^\infty}^\sigma \simeq G_e^\sigma / G_{d^*,e^\infty}^\sigma \) be the corresponding isotropic flag variety. Each flag \( U_\bullet \) can be extended to a flag of length \( 2r + 1 \) by setting \( U_{2r-k+1} = U_k^\perp \) for \( k = 0, \ldots, r \).

For each \( k = 1, \ldots, 2r + 1 \), let \( \mathcal{V}_i / k \rightarrow \text{Fl}_{d^*,e^\infty}^\sigma \) be the tautological vector bundle parameterizing the \( k \)th subspace of \( \mathbb{C}/e \) at \( i \in \mathcal{Q}_0 \). The quotient \( \mathcal{F}_i / k = \mathcal{V}_i / \nu_{i, k} / \nu_{i, k-1} \) has rank \( d_i^k \). The self-dual structure on \( \mathbb{C}/e \) induces isomorphisms \( \mathcal{F}_i / k \simeq \mathcal{F}_\sigma(i), 2r+1-k \). This gives vector bundle isomorphisms

\[
\text{Hom}(\mathcal{F}_i / k, \mathcal{F}_j, i) \simeq \text{Hom}(\mathcal{F}_j / i, \mathcal{F}_i / k) \simeq \text{Hom}(\mathcal{F}_\sigma(j), 2r+1-l, \mathcal{F}_\sigma(i), 2r+1-k)
\]

which induce a linear \( \mathbb{Z}_2 \)-action on

\[
\mathcal{G} = \bigoplus_{1 \leq k < l \leq 2r+1} \text{Hom}(\mathcal{F}_i / k, \mathcal{F}_j, i).
\]

Denote by \( \mathcal{G}^\sigma \subset \mathcal{G} \) the weight \(-1\) subbundle.

The following result is motivated by Rimányi [36, Lemmas 8.1 and 8.2].

Lemma 5.3. Let \( f_k \in \mathcal{H}_{\mathcal{Q}, d^k}, k = 1, \ldots, r \), and \( g_\infty \in \mathcal{I}_{\mathcal{Q}, e^\infty} \). Then

\[
(f_1 \cdots f_r) \star g_\infty = \pi^\sigma \left[ \prod_{k=1}^{r} f_k(\mathcal{F}_{i, k}) \right] g_\infty(\mathcal{F}_{i, 0}) \mathcal{E}_G^\sigma(\mathcal{G}^\sigma),
\]

where \( \pi^\sigma : \text{Fl}_{d^*,e^\infty}^\sigma \rightarrow \text{Spec}(\mathbb{C}) \) is the structure map and \( \mathcal{E}_G^\sigma(\mathcal{G}^\sigma) \) is the \( G^\sigma -e \)-equivariant Euler class of \( G^\sigma \rightarrow \text{Fl}_{d^*,e^\infty}^\sigma \).

Proof. Like the left hand side, the right hand side of the claimed equality can be computed by localization with respect to \( T_e \leq G_e^\sigma \). The \( T_e \)-fixed points of \( \text{Fl}_{d^*,e^\infty}^\sigma \) are those appearing in the proof of the \( r \)-fold iteration of Theorem 3.3. Since the \( T_e \)-weights of \( \mathcal{E}_G^\sigma(\mathcal{G}^\sigma) \) and \( \mathcal{E}_{G_d,e^\infty}^\sigma \mathcal{R}_{1,e^\infty}^\sigma \) at fixed points agree, the lemma follows. \( \square \)

Define a \( G_e^\sigma \)-stable closed subvariety of \( \text{Fl}_{d^*,e^\infty}^\sigma \times R_e^\sigma \) by

\[
\Sigma^\sigma = \{(U_\bullet, z) \in \text{Fl}_{d^*,e^\infty}^\sigma \times R_e^\sigma | z_\alpha(U_{i, k}) \subset U_{j, k}, \ (\alpha : i \rightarrow j) \in \mathcal{Q}_1, \ k = 1, \ldots, r \}.
\]

It has a \( G_e^\sigma \)-equivariant fundamental class

\[
[\Sigma^\sigma] \in H_{G_e^\sigma}^\bullet(\text{Fl}_{d^*,e^\infty}^\sigma \times R_e^\sigma) \simeq H_{G_e^\sigma}^\bullet(\text{Fl}_{d^*,e^\infty}^\sigma).
\]

Lemma 5.3. The equality \( \mathcal{E}_G^\sigma(\mathcal{G}^\sigma) = [\Sigma^\sigma] \) holds in \( H_{G_e^\sigma}^\bullet(\text{Fl}_{d^*,e^\infty}^\sigma) \).

Proof. This can be proved in the same way as [36, Lemma 8.3]. \( \square \)

The duality structure on \( \text{Rep}_G(\mathbb{Q}) \) defines an involution of \( \Gamma(\mathbb{Q}) \), sending an indecomposable representation \( I \) to \( S(I) \). This involution preserves the levels of \( \Gamma(\mathbb{Q}) \) which, being in type \( A \), are the orbits of the Auslander–Reiten translation. Fix a partition \( \Delta = \Delta^- \cup \Delta^0 \cup \Delta^+ \) such that \( \Delta^0 \) is fixed pointwise by \( S \) and \( S(\Delta^-) = \Delta^+ \). Without loss of generality, we can assume that \( \beta_u < S(\beta_u) \) for all \( \beta_u \in \Delta^- \). Write

\[
\Delta^- = \{\beta_{u_1} < \cdots < \beta_{u_r}\}.
\]
Lemma 5.4. Every self-dual representation \( Z \) has a unique isotropic filtration

\[
0 = U_0 \subset U_1 \subset \cdots \subset U_r \subset Z
\]
such that \( U_j / U_{j-1} \cong I_{\beta_{u_j}}^{m_{u_j}}, j = 1, \ldots, r, \) and \( Z / U_r \cong \bigoplus_{\beta \in \Delta^+} I_{\beta}^{m_{\beta}} \).

Proof. Any self-dual representation can be written uniquely as an orthogonal direct sum of indecomposable self-dual representations. Explicitly, we have

\[
Z = \bigoplus_{l=1}^{r} H(I_{\beta_{u_l}}) \otimes m_{u_l} \bigoplus_{\beta \in \Delta^+} I_{\beta}^{m_{\beta}}
\]  

(31)

for some \( m_{u_l} \in \mathbb{Z}_{\geq 0} \). Setting \( U_j = \bigoplus_{l=1}^{j} I_{\beta_{u_l}}^{m_{u_l}} \) gives a filtration with the desired properties.

Suppose that \( U_j' \subset Z \) is another filtration with the stated properties. The ordering \( \beta_{u_1} < \cdots < \beta_{u_r} \) implies that \( U_r' \simeq U_r \). So it suffices to show that there is a unique isotropic embedding \( U_r \hookrightarrow Z \). To do so, first note that \( \text{Hom}(I_{\beta}, I_{\beta'}) = 0 \) for all \( \beta \in \Delta^- \) and \( \beta' \in \Delta^+ \). Indeed, if \( \text{Hom}(I_{\beta}, I_{\beta'}) \neq 0 \), then \( \text{Hom}(I_{\beta'}, S(I_{\beta})) \neq 0 \). Hence \( \beta > \beta' \) and \( \beta' > S(\beta) \) so that \( \beta > S(\beta) \), a contradiction. It follows that the summand \( U_1 \subset U_r \) maps isomorphically onto \( I_{\beta_{u_1}}^{m_{u_1}} \). While \( U_2 \subset U_r \) could potentially map non-trivially to \( S(I_{\beta_{u_1}}) \), this would contradict the condition that \( U_r \) be isotropic. Hence \( U_2 \) maps isomorphically onto \( I_{\beta_{u_1}}^{m_{u_1}} \) and \( I_{\beta_{u_2}}^{m_{u_2}} \). Continuing in this way we see that \( U_r \hookrightarrow M \) is the canonical isotropic embedding. \( \square \)

We derive two results using Lemma 5.4. The first is a self-dual extension of a theorem of Reineke [34, Theorem 2.2] and appears in the unpublished thesis of Lovett [28]. For each \( Z \in R_{\mathcal{G}}^\sigma \), let \( \eta_Z^\sigma \subset R_{\mathcal{G}}^\sigma \) be the \( \mathcal{G}^\sigma \)-orbit of \( Z \) and let \( \eta_Z^\sigma \) be its closure. Elements of \( \eta_Z^\sigma \) are called self-dual degenerations of \( Z \).

Theorem 5.5 [28]. Keeping the notation of Lemma 5.4, set \( d^j = m_j \beta_j \), \( j = 1, \ldots, r \), and \( e^\infty = \dim Z / U_r \). Then the canonical morphism \( \pi_Z^\sigma : \Sigma^\sigma \to R_{\mathcal{G}}^\sigma \) is a \( \mathcal{G}^\sigma \)-equivariant resolution of \( \eta_Z^\sigma \).

Proof. This is proved for \( Q \) of type \( A_3 \) in [29, Proposition 2.3]. We will prove the general case using a self-dual version of Reineke’s argument.

The variety \( \Sigma^\sigma \) is smooth, being the total space of a vector bundle over \( \text{Fl}_{d^j, e^\infty}^{\sigma} \), and the morphism \( \pi_Z^\sigma \) is proper and equivariant. We prove that \( \pi_Z^\sigma(\Sigma^\sigma) = \eta_Z^\sigma \). If \( Y \in \pi_Z^\sigma(\Sigma^\sigma) \), then there exists an isotropic filtration

\[
0 = V_0 \subset V_1 \subset \cdots \subset V_r \subset N, \quad \dim V_j / V_{j-1} = d^j.
\]

Since \( \text{Ext}^1(I_{\beta}, I_{\beta}) = 0 \) for all \( \beta \in \Delta \), Voigt's lemma implies that \( V_j / V_{j-1} \) is a degeneration of \( I_{\beta_{u_j}}^{m_{u_j}} \). Similarly, \( \text{Ext}^1(I_{\beta}, I_{\beta'}) = 0 \) for all \( \beta, \beta' \in \Delta^\sigma \) and \( Y / V_r \) is a degeneration of \( \bigoplus_{\beta \in \Delta^\sigma} I_{\beta}^{m_{\beta}} \). We apply [34, Lemma 2.3] to conclude that \( Y \) is a degeneration of \( Z \). It is proved in [8, Theorem 2.6] that two self-dual representations are isometric if and only if they are isomorphic. Using this, we see that \( Y \) is in fact a self-dual degeneration of \( Z \). Hence \( \eta_Z^\sigma \subset \pi_Z^\sigma(\Sigma^\sigma) \subset \eta_Z^\sigma \), implying \( \pi_Z^\sigma(\Sigma^\sigma) = \eta_Z^\sigma \).
It remains to show that $\pi^\sigma_Z$ restricts to a bijection over $\eta^\sigma_Z$. Consider an arbitrary isotropic filtration

$$0 = U_0 \subset U_1 \subset \cdots \subset U_r \subset Z, \quad \dim U_j / U_{j-1} = d_j.$$ 

Arguing as above, $U_j / U_{j-1}$ and $Z // U_r$ are degenerations of $I_{\beta_{u_j}} \oplus \mu \beta_{u_j}$ and $\oplus_{\beta_u \in \Delta^\sigma} I_{\beta_u} \oplus m_u$, respectively. Since $\text{Hom}(I_{\beta_i}, I_{\beta_j}) = 0$ if $i < j$, we can apply [34, Lemma 2.3] to conclude that $U_j / U_{j-1} \simeq I_{\beta_{u_j}} \oplus \mu \beta_{u_j}$ and that $Z // U_r \simeq \oplus_{\beta_u \in \Delta^\sigma} I_{\beta_u} \oplus m_u$. Lemma 5.4 now implies that $U_\bullet \subset Z$ is the canonical filtration.

We can now prove an analogue of [36, Theorem 10.1].

**Corollary 5.6** Keeping the above notation, the equality

$$[\eta^\sigma_Z] = (1_{m_u \beta_{u_1}, \ldots, 1_{m_u \beta_{u_r}}}) \star 1^\sigma_{\sum_{\beta_u \in \Delta^\sigma} m_u \beta_u}$$

holds in $I_Q$.

**Proof** Theorem 5.5 implies that $\pi^\sigma \left[ \Sigma^\sigma \right] = [\eta^\sigma_Z]$. The desired equality then follows from Lemmas 5.2 and 5.3. $\square$

The class $[\eta^\sigma_Z] \in H^\bullet_{Q^\sigma}(R_c^\sigma)$ is the Thom polynomial of the orbit $\eta^\sigma_Z \subset R_c^\sigma$. These classes play the role of quiver polynomials [1] in the self-dual setting.

**Example** Let $Q = \begin{array}{c} 1 \\ 2 \end{array}$ and set $Q^+_0 = \{1\}$. If $f \in \mathcal{H}_{Q,(d_1,d_2)}$ and $g \in I_{Q,(e,e)}$, then

$$f \star g = \sum_{\pi \in \mathcal{H}_{d_1,d_2}} \pi \cdot \left( f(x_1, \ldots, x_{d_1}, y_1, \ldots, y_{d_2}) g(z_1, \ldots, z_e) \right) \times \prod_{1 \leq i < j \leq d_1} (-x_i - x_j) \prod_{1 \leq i \leq d_2} (-y_i - x_i) \prod_{m=1}^{d_2} \prod_{k=1}^{e} (-y_l - z_k).$$

For orthogonal representations this gives

$$1_{(d,0)} \star 1^\sigma_{(e,e)} = \sum_{\pi \in \mathcal{H}_{d,e}} \pi \cdot \left( \prod_{1 \leq i < j \leq d} (-x_i - x_j) \right) = s(d-1, \ldots, 1,0,e)(-z)$$

while for symplectic representations

$$1_{(d,0)} \star 1^\sigma_{(e,e)} = \sum_{\pi \in \mathcal{H}_{d,e}} \pi \cdot \left( \prod_{1 \leq i < j \leq d} (-x_i - x_j) \right) = 2^d s(d-2,1,0,e)(-z).$$

Corollary 5.6 implies that $1_{(d,0)} \star 1^\sigma_{(e,e)}$ is the Thom polynomial of the orbit of matrices having rank $e$ in the $\text{GL}_{d+e}$ representation $\wedge^2 \mathbb{C}^{d+e}$ or $\text{Sym}^2 \mathbb{C}^{d+e}$, respectively. These polynomials were computed using different methods in [13,19,23].
Turning to the second application of Lemma 5.4, define putative indecomposable orientifold DT invariants $\Omega_{Q,e}^\sigma,\ind$ to be one if $e \in \Lambda_{Q,e}^\sigma,+$ is a sum of pairwise distinct positive roots, each of which is the dimension vector of an indecomposable representation which admits a self-dual structure. Otherwise, set $\Omega_{Q,e}^\sigma,\ind = 0$. Define $\Omega_{Q,e}^\sigma,\simp$ similarly, with simplicity used in place of indecomposability. Put $\Omega_{Q,e}^\sigma,\,... = 1$ in both cases. Set also $\Pi^+ = \Pi \cap \Delta^+$ and $\Pi^\sigma = \Pi \cap \Delta^\sigma$. Let $h = 0$ in the hyperbolic case and $h = 1$ otherwise.

Recall that $A_{L_0}(q^{1/2}, t) = (q^{1/2}t; q)_\infty = E_q(t)$ is the quantum dilogarithm.

**Theorem 5.7** The identity

$$\prod_{\alpha \in \Pi^+} E_q(t^\alpha) \star \sum_{\varnothing \subseteq \pi \subseteq \Pi^\sigma} \prod_{\alpha \in \pi} E_q(z(q^{-1/2} + h t^\alpha) \star \Omega_{Q,\pi}^\sigma,\ind \xi^\pi$$

$$= \prod_{\beta \in \Delta^-} E_q(t^\beta) \star \sum_{\varnothing \subseteq \pi \subseteq \Delta^\sigma} \left( \prod_{\beta \in \pi} E_q(z(q^{-1/2} + h t^\beta) \cdot \prod_{\beta \notin \pi} E_q(z(q^{-1/2} t^\beta)) \right) \star \Omega_{Q,\pi}^\sigma,\ind \xi^\pi$$

holds in $\hat{S}_Q$, where we have written $\xi^\pi$ for $\xi^{\sum_{\beta \in \pi} \beta}$ and similarly for $\Omega_{Q,\pi}^\sigma,\,...$.

**Proof** It is easy to construct a $\sigma$-compatible stability $\theta_{\simp}$ whose stable representations are the simple representations and whose order by increasing slope agrees with $<$. Existence and uniqueness of $\sigma$-HN filtrations implies a factorization of the identity characteristic function in the finite field Hall module of $Q$. Applying the Hall module integration map [43, Theorem 4.1] to this factorization gives the left hand side of the desired equality. Lemma 5.4 leads to a second factorization of the identity characteristic function, the integral of which gives the right hand side. \(\square\)

### 5.3. CoHM of type A quivers.

For each $\beta \in \Delta^\sigma$, let

$$T_{Q}^{(\beta)} = \bigoplus_{n \geq 0} H_{G_{\beta}^{\sigma}}^{\bullet}(R_{\beta}^\sigma)[-\mathcal{E}(n\beta)]$$

and

$$T_{Q}^{(\beta),\simeq} = \bigoplus_{n \geq 0} H_{G_{\beta}^{\sigma}}^{\bullet}(\eta_{I_{\beta}^{\sigma}})^{-}[\mathcal{E}(n\beta)],$$

considered as modules over $H_{Q}^{(\beta)}$ and $H_{Q}^{(\beta),\simeq}$, respectively. We have $T_{Q}^{(\beta),\simeq} \simeq T_{L_0}$ compatibly with $H_{Q}^{(\beta),\simeq} \simeq H_{L_0}$, where the duality structure on $L_0$ is $s_{L_0} = -1$ in the hyperbolic case and $s_{L_0} = 1$ in the non-hyperbolic case. The structure of $T_{Q}^{(\beta),\simeq}$ is therefore determined by Proposition 4.4. The restriction map $\rho^\sigma : T_{Q}^{(\beta)} \to T_{Q}^{(\beta),\simeq}$ is a surjective module homomorphism over $\rho : H_{Q}^{(\beta)} \to H_{Q}^{(\beta),\simeq}$. Using the identification $T_{Q}^{(\beta),\simeq} \simeq T_{L_0}$, define a section of $\rho^\sigma$ by

$$\psi : \begin{cases} x^{2i_1+1} \cdots x^{2i_d+1} \star 1^\sigma_0 & \mapsto \psi(x^{2i_1+1} \cdots x^{2i_d+1}) \star 1^\sigma_0 \quad \text{in type } B, \\ x^{2i_1+1} \cdots x^{2i_d+1} \star 1^\sigma_0 & \mapsto \psi(x^{2i_1+1} \cdots x^{2i_d+1}) \star 1^\sigma_0 \quad \text{in type } C, \\ x^{2i_1} \cdots x^{2i_d} \star 1^\sigma_0 & \mapsto \psi(x^{2i_1} \cdots x^{2i_d}) \star 1^\sigma_0 \quad \text{in type } D. \end{cases}$$
The map $\psi^\sigma$ is a module embedding over the restriction of $\psi$ to the appropriate even/odd subalgebra of $H_Q^{(\beta)}$. Write $I_Q^{(\beta)}$ for the image of $\psi^\sigma$ in types $C$ or $D$ and $I_Q^{(\beta),+}$ for the image of $\psi^\sigma$ in type $B$.

In the non-hyperbolic case, for each subset $\emptyset \subseteq \pi \subseteq \Delta^\sigma$, define

$$I_Q^{(\pi),\text{ind}} = \bigotimes_{\beta \in \Delta^\sigma \setminus \pi} I_Q^{(\beta),\text{even}} \otimes \bigotimes_{\beta \in \pi} I_Q^{(\beta),\text{odd}}.$$  

This is a rank one free $\bigotimes_{\beta \in \Delta^\sigma \setminus \pi} H(\beta),\text{even} \otimes \bigotimes_{\beta \in \pi} H(\beta),\text{odd}$-module. If $\emptyset \subseteq \pi \subseteq \Pi^\sigma$, then $I_Q^{(\pi),\text{simp}}$ is defined similarly. In the hyperbolic case $I_Q^{(\emptyset)}$ is still defined.

We now come to the main result of this section.

**Theorem 5.8** Let $Q$ be of Dynkin type $A$. The ordered CoHA action maps

$$\left( \left\langle \left(\bigotimes_{\alpha \in \Pi^+} H^{(\alpha)}_Q \right) \otimes S^{\text{tw}} \bigoplus_{\emptyset \subseteq \pi \subseteq \Pi^\sigma} \Omega^{\sigma,\text{simp}}_{Q,\pi} \cdot I_Q^{(\pi),\text{simp}} \right) \rightarrow I_Q \right)$$

(32)

and

$$\left( \left\langle \left(\bigotimes_{\beta \in \Delta^-} H^{(\beta)}_Q \right) \otimes S^{\text{tw}} \bigoplus_{\emptyset \subseteq \pi \subseteq \Delta^\sigma} \Omega^{\sigma,\text{ind}}_{Q,\pi} \cdot I_Q^{(\pi),\text{ind}} \right) \rightarrow I_Q \right)$$

(33)

are isomorphisms in $D^\text{ib}(\text{Vect})_{\Lambda^\sigma_Q}$.

**Proof** Let $f_\alpha \in H^{(\alpha)}_Q$ and $g \in \bigoplus_{\emptyset \subseteq \pi \subseteq \Pi^\sigma} \Omega^{\sigma,\text{simp}}_{Q,\pi} \cdot I_Q^{(\pi),\text{simp}}$. Concretely, $g$ is constant if $\Pi^\sigma = \emptyset$ and is a symmetric polynomial in the variables associated to the node $Q^\sigma_0$ otherwise. Taking into account the ordering of the roots, Theorem 3.3 gives

$$\left( \prod_{\alpha \in \Pi^+} f_\alpha \right) \ast g = \left( \prod_{\alpha \in \Pi^+} f_\alpha \right) g,$$

the products on the right hand side being ordinary polynomial multiplication. It follows that the map (32) is an isomorphism.

To show that the map (33) is an isomorphism we use an argument similar to [36, Theorem 11.2]. Fix non-negative integers $\{m_u\}_{\beta_u \in \Delta}$ such that $m_{S(u)} = m_u$. Let $Z$ be the self-dual representation determined by equation (31) and let $e = \dim Z$. The isometry group of $Z$ is homotopy equivalent to

$$\prod_{\beta_u \in \Delta^-} \text{GL}_{m_u} \times \prod_{\beta_u \in \Delta^\sigma} \mathbb{G}^{m_u}_{m_u},$$

where $\mathbb{G}^{m_u}_{m_u}$ is a symplectic group in the hyperbolic case and is an orthogonal group otherwise.

Define sets $T_{i,k,v} \subseteq \{1, \ldots, e_i\}$ for $i \in Q_0$, $k = 1, \ldots, |\Delta|$ and $v = 1, \ldots, m_{u_k}$ by requiring $|T_{i,k,v}| = 1$ if $\dim_{C}(I_{\beta_{u_k}})_i = 1$ and $T_{i,k,v} = \emptyset$ otherwise, and

$$T_{i,1,1} \sqcup \cdots \sqcup T_{i,|\Delta|,m_{|\Delta|}} = \{1, \ldots, e_i\}.$$
as ordered sets. Write \( \{\epsilon_{i,1}, \ldots, \epsilon_{i,e_i}\} \) for the standard basis of \( \mathbb{C}^{e_i} \). Let \( A_{k,v} \) be the indecomposable representation of type \( \beta_{u_k} \) with basis \( \{\epsilon_{i,j}\}_{i \in Q_0, j \in T_{i,k,v}} \) and put

\[
\Phi^\sigma = \bigoplus_{k=1}^{\lfloor \Delta \rfloor} \bigoplus_{v=1}^{m_{u_k}} A_{k,v}.
\]

Define a self-dual structure on \( \Phi^\sigma \) by requiring that

1. \( A_{k,v} \oplus A_{S(k),v} \) be hyperbolic if \( \beta_{u_k} \in \Delta^- \) and \( v = 1, \ldots, m_{u_k} \),
2. \( A_{k,v} \oplus A_{k,\lfloor \frac{m_{u_k}}{2} \rfloor + v} \) be hyperbolic if \( \beta_{u_k} \in \Delta^\sigma \) and \( v = 1, \ldots, \lfloor \frac{m_{u_k}}{2} \rfloor \), and
3. \( A_{k,m_{u_k}} \) have its canonical self-dual structure if \( \beta_{u_k} \in \Delta^\sigma \) and \( m_{u_k} \) is odd.

Then \( \Phi^\sigma \) and \( Z \) are isometric. The restriction map

\[
\rho^\sigma_Z : H^*_Q(\mathbb{R}^e) \rightarrow H^*_Q(\eta^\sigma_Z) \simeq H^*(BAut_S(\Phi^\sigma))
\]

can be computed by identifying \( H^*_Q(\mathbb{R}^e) \) and \( H^*(BAut_S(\Phi^\sigma)) \) with appropriately symmetric polynomials in variables \( \{z_i,j\} \) and \( \{\theta_{k,v}\} \), respectively, and using Lemma 1.1. We find that

(i) if \( i \in Q_0^+ \) and \( j \in T_{i,k,v} \), then

\[
\rho^\sigma_M(z_{i,j}) = \begin{cases} 
\theta_{k,v} & \text{if } \beta_{u_k} \in \Delta^-, \\
-\theta_S(k),v & \text{if } \beta_{u_k} \in \Delta^+, \\
\theta_{k,v} & \text{if } \beta_{u_k} \in \Delta^\sigma \text{ and } 1 \leq j \leq \lfloor \frac{m_{u_k}}{2} \rfloor, \\
-\theta_{k,v} & \text{if } \beta_{u_k} \in \Delta^\sigma \text{ and } \lfloor \frac{m_{u_k}}{2} \rfloor + 1 \leq j \leq 2\lfloor \frac{m_{u_k}}{2} \rfloor, \\
0 & \text{if } \beta_{u_k} \in \Delta^\sigma \text{ and } j = m_{u_k} \text{ is odd},
\end{cases}
\]

and

(ii) if \( i \in Q_0^- \) and \( j \in T_{i,k,v} \), then

\[
\rho^\sigma_M(z_{i,j}) = \begin{cases} 
\theta_{k,v} & \text{if } \beta_{u_k} \in \Delta^-, \\
-\theta_S(k),v & \text{if } \beta_{u_k} \in \Delta^+, \\
\theta_{k,v} & \text{if } \beta_{u_k} \in \Delta^\sigma.
\end{cases}
\]

Let \( f_k \in \mathcal{H}^{(\beta_{u_k})}_{Q,m_{u_k}} \) and let \( g_u \) be an element of \( \mathcal{I}^{(\beta_u)}_{Q,m_{u} \beta_{u}} \) or \( \mathcal{I}^{(\beta_u),+}_{Q,m_{u} \beta_{u}} \), depending on the parity of \( m_u \). We will show that the image of

\[
\left( \bigotimes_{\beta_{u_k} \in \Delta^-} f_k \right) \bigotimes_{\beta_{u_k} \in \Delta^\sigma} g_u
\]

under the map (33) is non-zero by showing that its image under \( \rho^\sigma_Z \) is non-zero. Since \( \pi^\sigma_Z : \Sigma^\sigma \rightarrow R^e_\sigma \) is an equivariant resolution of \( \eta^\sigma_Z \) (Theorem 5.5), there is a single \( T_e \)-fixed point over the \( T_e \)-fixed point \( \Phi^\sigma \in \eta^\sigma_Z \). The image of (34) under \( \rho^\sigma_Z \) therefore consists of the single term

\[
\prod_{\beta_{u_k} \in \Delta^-} f_k(\theta_{u_k,1}, \ldots, \theta_{u_k,m_{u_k}}) \left( \prod_{\beta_{u} \in \Delta^\sigma} g_u(z) \right) K^{(r),\sigma}(z)_{z \mapsto \theta},
\]
where $K^{(r),\sigma}(z)$ is the $r$-fold iteration of the CoHM kernel. By assumption, $g_u$ is of the form $f_u \ast 1_{0/\beta_u}$ for some

$$f_u(x_i(\beta_u), 1, \ldots, x_i(\beta_u), (\frac{m_u}{\Delta})) \in \mathcal{H}_{\{\frac{m_u}{\Delta}\}}^{(\beta_u), \text{even/odd}}.$$ 

Then $g_u(z)_{z \rightarrow \theta} = f_u(\theta_u, 1, \ldots, \theta_u, (\frac{m_u}{\Delta})) \ast 1_{0/1}$, computed in $\mathcal{I}_{L_0}$ with the appropriate duality structure. Each factor $g_u(z)_{z \rightarrow \theta}$ is thus non-zero and the non-vanishing of (35) is equivalent to the non-vanishing of $K^{(r),\sigma}(z)_{z \rightarrow \theta}$. By Corollary 5.6 we have

$$K^{(r),\sigma}(z)_{z \rightarrow \theta} = \rho^*_Z([\eta^*_Z]) = \text{eu}_{\text{Aut}_S}(Z)(N_{R_f^\sigma/\eta^*_Z}),$$

That $\text{eu}_{\text{Aut}_S}(Z)(N_{R_f^\sigma/\eta^*_Z})$ is non-zero can be shown by a straightforward modification of the argument in the ordinary case [12, Corollary 3.15].

We have proved that the restriction of the map (33) to the subspace spanned by elements of the form (34) is injective. To see that (33) itself is injective, let $m^{(t)}_\beta \in \mathbb{Z}_{\geq 0}$, $t = 1, \ldots, N$, satisfy $\sum_{\beta \in \Delta} m^{(t)}_\beta = e$ and consider a linear relation

$$\sum_{t=1}^N a_t \prod_{\beta \in \Delta^-} f^{(t)}_\beta \ast g^{(t)} = 0$$

for some $a_t \in \mathbb{Q}$ and non-zero $f^{(t)}_\beta$ and $g^{(t)}$ as above. Let $Z^{(t)}$ be the self-dual representation associated to $m^{(t)}_\beta$ by equation (31). It suffices to consider the case in which the $Z^{(t)}$ are pairwise non-isometric. Relabelling if necessary, assume that $\eta^*_Z(1) \not\subset \eta^*_Z(0)$, or equivalently $\eta^*_Z(1) \cap \eta^*_Z(0) = \emptyset$, for each $t \geq 2$. Each summand in the localization sum presentation of $[\eta^*_Z(1)]$ (see Corollary 5.6) is then annihilated by $\rho^*_Z(1)$, giving

$$0 = \rho^*_Z(1) \sum_{t=1}^N a_t \prod_{\beta \in \Delta^-} f^{(t)}_\beta \ast g^{(t)} = a_1 \rho^*_Z(1) \left( \prod_{\beta \in \Delta^-} f^{(1)}_\beta \ast g^{(1)} \right).$$

The previous paragraph implies that $a_1 = 0$. Repeating this argument we find that $a_1 = \cdots = a_N = 0$, proving injectivity of (33).

To complete the proof, note that, together with the isomorphism (32), Theorem 5.7 implies that the Hilbert–Poincaré series of the domain and codomain of (33) are equal. Since the map (33) is injective, it is also surjective.

The isomorphism (32) is the PBW factorization (27) associated to the stability $\theta_{\text{simp}}$ described in the proof of Theorem 5.7. We expect an analogous statement to hold for the isomorphism (33), with $\theta_{\text{simp}}$ replaced by a $\sigma$-compatible stability $\theta_{\text{ind}}$ whose stable objects are the indecomposable representations and whose order by increasing slope is opposite to $\prec$. Without the assumption of $\sigma$-compatibility, such a stability is known to exist and in many examples, such as equioriented quivers, we can check directly that it may be chosen $\sigma$-compatibly.

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