BOUNDEDNESS OF BILINEAR PSEUDO-DIFFERENTIAL OPERATORS OF $S_{0,0}$-TYPE ON $L^2 \times L^2$

TOMOYA KATO, AKIHIKO MIYACHI, AND NAOHITO TOMITA

Abstract. We extend the known result that the bilinear pseudo-differential operators with symbols in the bilinear Hörmander class $BS^{-n/2}_{0,0}(\mathbb{R}^n)$ are bounded from $L^2 \times L^2$ to $h^1$. We show that those operators are also bounded from $L^2 \times L^2$ to $L^r$ for every $1 < r < 2$. Moreover we give similar results for symbol classes wider than $BS^{-n/2}_{0,0}(\mathbb{R}^n)$. We also give results for symbols of limited smoothness.

1. Introduction

For a bounded measurable function $\sigma(x, \xi_1, \xi_2)$ on $(\mathbb{R}^n)^3$, the bilinear pseudo-differential operator $T_\sigma$ is defined by

$$T_\sigma(f_1, f_2)(x) = \frac{1}{(2\pi)^{2n}} \int_{(\mathbb{R}^n)^2} e^{ix\cdot(\xi_1+\xi_2)} \sigma(x, \xi_1, \xi_2) \hat{f}_1(\xi_1) \hat{f}_2(\xi_2) \, d\xi_1 d\xi_2$$

for $f_1, f_2 \in \mathcal{S}(\mathbb{R}^n)$.

For the boundedness of the bilinear operators $T_\sigma$, we shall use the following terminology. Let $X_1, X_2$, and $Y$ be function spaces on $\mathbb{R}^n$ equipped with quasi-norms $\| \cdot \|_{X_1}, \| \cdot \|_{X_2}$, and $\| \cdot \|_Y$, respectively. If there exists a constant $A$ such that

$$\|T_\sigma(f_1, f_2)\|_Y \leq A \|f_1\|_{X_1} \|f_2\|_{X_2} \text{ for all } f_1 \in \mathcal{S} \cap X_1 \text{ and } f_2 \in \mathcal{S} \cap X_2,$$

then, with a slight abuse of terminology, we say that $T_\sigma$ is bounded from $X_1 \times X_2$ to $Y$ and write $T_\sigma : X_1 \times X_2 \to Y$. The smallest constant $A$ of (1.1) is denoted by $\|T_\sigma\|_{X_1 \times X_2 \to Y}$. If $\mathcal{A}$ is a class of symbols, we denote by $\text{Op}(\mathcal{A})$ the class of all bilinear operators $T_\sigma$ corresponding to $\sigma \in \mathcal{A}$. If $T_\sigma : X_1 \times X_2 \to Y$ for all $\sigma \in \mathcal{A}$, then we write $\text{Op}(\mathcal{A}) \subset B(X_1 \times X_2 \to Y)$.

The bilinear Hörmander symbol class $BS^{m}_{\rho, \delta}(\mathbb{R}^n)$, $m \in \mathbb{R}$, $0 \leq \rho, \delta \leq 1$, consists of all $\sigma(x, \xi_1, \xi_2) \in C((\mathbb{R}^n)^3)$ such that

$$|\partial_x^\alpha \partial_{\xi_1}^{\beta_1} \partial_{\xi_2}^{\beta_2} \sigma(x, \xi_1, \xi_2)| \leq C_{\alpha, \beta_1, \beta_2} (1 + |\xi_1| + |\xi_2|)^{m+\delta|\alpha|-\rho(|\beta_1|+|\beta_2|)}$$

for all multi-indices $\alpha, \beta_1, \beta_2 \in \mathbb{N}_0^n = \{0, 1, 2, \ldots \}^n$.

In the case $\rho = 1$ and $\delta < 1$, the bilinear pseudo-differential operators with symbols in $BS^0_{1, \delta}$ are bilinear Calderón–Zygmund operators in the sense of Grafakos-Torres [14] and they are bounded from $L^p \times L^q$ to $L^r$ with $1 < p, q < \infty$ and $1/r = 1/p + 1/q$ (see Coifman-Meyer [7], Bényi-Torres [3], and Bényi-Maldonado-Naibo-Torres [2]). Here the condition $1/r = 1/p + 1/q$ is necessary since the constant

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function belongs to $BS_{1,0}^0$ and the operator $T_\sigma$ corresponding to $\sigma = 1$ is simply the pointwise product of functions.

In this paper, we shall be interested in the case $\rho = \delta = 0$ and consider only the boundedness of $T_\sigma$ on $L^2 \times L^2$. Recall that $BS_{0,0}^m(\mathbb{R}^n)$ consists of all $\sigma$ satisfying the estimate

$$
|\partial_x^\alpha \partial_\xi^\beta \sigma(x, \xi)| \leq C_{\alpha, \beta_1, \beta_2} (1 + |\xi_1| + |\xi_2|)^m.
$$

(1.2)

Bilinear pseudo-differential operators with symbols in $BS_{0,0}^m(\mathbb{R}^n)$ have some features different from the corresponding linear operators. For the case of linear pseudo-differential operator, which is defined by

$$
\sigma(X, D)f(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix\cdot \xi} \sigma(x, \xi) \hat{f}(\xi) \, d\xi, \quad f \in \mathcal{S}(\mathbb{R}^n),
$$

the celebrated Calderón-Vaillancourt theorem states that the operator $\sigma(X, D)$ is bounded on $L^2(\mathbb{R}^n)$ if the symbol $\sigma(x, \xi)$ satisfies the estimate

$$
|\partial_x^\alpha \partial_\xi^\beta \sigma(x, \xi)| \leq C_{\alpha, \beta}
$$

for all multi-indices $\alpha, \beta \in \mathbb{N}_0^n$ (see [6]). For bilinear operators, innocent generalization of this theorem does not hold. In fact, Bényi-Torres [4] proved that there exists a symbol in $BS_{0,0}^0$ for which the corresponding bilinear pseudo-differential operator is not bounded from $L^2 \times L^2$ to $L^1$. Thus in order to have the inclusion $\text{Op}(BS_{0,0}^0) \subset B(L^2 \times L^2 \rightarrow L^1)$, the order $m$ must be negative. Miyachi–Tomita [20] proved that the inclusion $\text{Op}(BS_{0,0}^m(\mathbb{R}^n)) \subset B(L^2 \times L^2 \rightarrow L^1)$ holds if and only if $m \leq -n/2$. For the critical case $m = -n/2$, it is also proved in [20] that

$$
\text{Op}(BS_{0,0}^{-n/2}(\mathbb{R}^n)) \subset B(L^2 \times L^2 \rightarrow h^1),
$$

(1.3)

where $h^1$ is the local Hardy space of Goldberg [10] (the definition of $h^1$ will be given in the next section).

The purpose of the present paper is to improve (1.3) in three ways. Firstly, we show that the target space $h^1$ in (1.3) can be replaced by $L^r$ with $1 < r \leq 2$ or even by the amalgam space $(L^2, \ell^1)$. (The definition of the amalgam space is given in the next section.) Since $(L^2, \ell^1) \hookrightarrow h^1 \cap L^2$, this is an improvement of (1.3). Secondly, we show that the class $BS_{0,0}^{-n/2}(\mathbb{R}^n)$ can be replaced by a general class. We show that the weight function $(1 + |\xi_1| + |\xi_2|)^{n/2}$ appearing in the definition of $BS_{0,0}^{-n/2}(\mathbb{R}^n)$ (see (1.2)) can be replaced by other functions and, among functions that have certain moderate behavior, we shall characterize all the possible weight functions. Thirdly, we give some refined results concerning operators with symbols of limited smoothness.

To explain our results in more detail, we introduce the following.

**Definition 1.1.** For a nonnegative bounded function $W$ on $\mathbb{R}^n \times \mathbb{R}^n$, we denote by $BS_{0,0}^W(\mathbb{R}^n)$ the set of all those smooth functions $\sigma = \sigma(x, \xi_1, \xi_2)$ on $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n$ such that the estimate

$$
|\partial_x^\alpha \partial_\xi_1^\beta \partial_\xi_2^\gamma \sigma(x, \xi_1, \xi_2)| \leq C_{\alpha, \beta_1, \beta_2} W(\xi_1, \xi_2)
$$

holds for all multi-indices $\alpha, \beta_1, \beta_2 \in \mathbb{N}_0^n$. We shall call $W$ the *weight function* of the class $BS_{0,0}^W(\mathbb{R}^n)$. 


Definition 1.2. We denote by $\mathcal{B}(\mathbb{Z}^n \times \mathbb{Z}^n)$ the set of all those nonnegative functions $V$ on $\mathbb{Z}^n \times \mathbb{Z}^n$ for which there exists a constant $c \in (0, \infty)$ such that the inequality

$$\sum_{\nu_1, \nu_2 \in \mathbb{Z}^n} V(\nu_1, \nu_2) A(\nu_1 + \nu_2) B(\nu_1) C(\nu_2) \leq c \|A\|_{\mathbb{Z}^n} \|B\|_{\mathbb{Z}^n} \|C\|_{\mathbb{Z}^n}$$

holds for all nonnegative functions $A, B, C$ on $\mathbb{Z}^n$.

Now the following is one of the main theorems of this paper.

Theorem 1.3. Let $V$ be a nonnegative bounded function on $\mathbb{Z}^n \times \mathbb{Z}^n$ and let

$$\tilde{V}(\xi, \eta) = \sum_{\nu_1, \nu_2 \in \mathbb{Z}^n} V(\nu_1, \nu_2) 1_q(\xi - \nu_1) 1_q(\eta - \nu_2), \quad (\xi, \eta) \in \mathbb{R}^n \times \mathbb{R}^n,$$

where $Q = [-1/2, 1/2]^n$. Then the following hold.

(1) If there exists an $r \in (0, \infty)$ such that all $T_\sigma \in \text{Op}(BSV_{\mathcal{B}}^{0,0}(\mathbb{R}^n))$ are bounded from $L^2 \times L^2$ to $L^r$, then $V \in \mathcal{B}(\mathbb{Z}^n \times \mathbb{Z}^n)$.

(2) Conversely, if $V \in \mathcal{B}(\mathbb{Z}^n \times \mathbb{Z}^n)$, then all $T_\sigma \in \text{Op}(BSV_{\mathcal{B}}^{0,0}(\mathbb{R}^n))$ are bounded from $L^2 \times L^2$ to the amalgam space $(L^2, \ell^1)$. In particular, all those $T_\sigma$ are bounded from $L^2 \times L^2$ to $L^r$ for all $r \in [1, 2]$ and to $h^1$.

Some typical examples of functions in $\mathcal{B}(\mathbb{Z}^n \times \mathbb{Z}^n)$ are the following.

Example 1.4. The following functions $V$ on $\mathbb{Z}^n \times \mathbb{Z}^n$ belong to the class $\mathcal{B}(\mathbb{Z}^n \times \mathbb{Z}^n)$:

$$V(\nu_1, \nu_2) = (1 + |\nu_1| + |\nu_2|)^{-n/2};$$

$$V(\nu_1, \nu_2) = (1 + |\nu_1|)^{-a_1}(1 + |\nu_2|)^{-a_2}, \quad a_1, a_2 > 0, \quad a_1 + a_2 = n/2;$$

$$V(\nu_1, \nu_2) = \prod_{j=1}^{n} \prod_{i=1}^{2} (1 + |\nu_{i,j}|)^{-a_{i,j}} \quad a_{i,j} > 0, \quad a_{i,j} + a_{j,i} = 1/2;$$

where $\nu_i = (\nu_{i,1}, \ldots, \nu_{i,n}) \in \mathbb{Z}^n$, $i = 1, 2$.

Notice that the bilinear Hörmander class $BS^{-n/2}_{0,0}(\mathbb{R}^n)$ is equal to the class $BSV_{\mathcal{B}}^{0,0}(\mathbb{R}^n)$ of Theorem 1.3 with $V$ of (1.5). Observe that the function (1.6) is bigger than (1.5) and (1.7) is much bigger, and hence the corresponding classes $BSV_{\mathcal{B}}^{0,0}(\mathbb{R}^n)$ are wider than $BS^{-n/2}_{0,0}(\mathbb{R}^n)$. We shall prove that not only (1.5) but also any $V$ in the Lorentz class $\ell^1(\mathbb{Z}^n)$ belongs to $\mathcal{B}(\mathbb{Z}^n \times \mathbb{Z}^n)$. We also prove $\mathcal{B}(\mathbb{Z}^n \times \mathbb{Z}^n)$ contains functions that are generalizations of (1.6) and (1.7).

It will be worthwhile to observe that the claim of Theorem 1.3 (2) for $V$ of (1.6) is equivalent to the following: the bilinear pseudo-differential operators $T_\sigma$ with $\sigma \in BSV_{0,0}(\mathbb{R}^n)$ are bounded from $W^{a_1} \times W^{a_2}$ to $(L^2, \ell^1) \hookrightarrow h^1 \cap L^2$ for all $a_1, a_2$ satisfying the conditions of (1.6), where $W^s = W^s(\mathbb{R}^n)$ denotes the $L^2$-based Sobolev space.

Recently Grafakos–He–Slavíková proved that if the symbol $\sigma(x, \xi, \eta) = \sigma(\xi, \eta)$ does not depend on $x$, and if $\sigma \in BS_{0,0}(\mathbb{R}^n) \cap L^q(\mathbb{R}^{2n})$ with $q < 4$, then $T_\sigma$ is bounded from $L^2 \times L^2$ to $L^1$. In the present paper, we shall show that this result, even in a generalized form, can be deduced from Theorem 1.3.

Not only Theorem 1.3 we also give refined theorems which treat symbols of limited smoothness. For linear pseudo-differential operators, there are several results concerning symbols with limited smoothness. Authors such as Cordes, Coifman-Meyer, Muramatu, Miyachi, Sugimoto, and Bouikhemair investigated minimal smoothness assumptions on the symbols to assure the boundedness.
of linear pseudo-differential operators. As for the $L^2$ boundedness, they proved that, 
roughly speaking, smoothness of symbols up to $n/2$ for each variable $x$ and $\xi$ assures 
the boundedness in $L^2$. For bilinear operators, to the best of the authors’ knowledge, 
there is only one result concerning symbols of limited smoothness, which was 
given by Herbert–Naibo [15]. In [15], the authors proved that symbols of the class 
$BS_{0,0}^m(\mathbb{R}^n)$ with $m < -n/2$ provide bounded bilinear pseudo-differential operators in 
$L^2 \times L^2 \to L^1$ if the smoothness up to $n/2$ for the $x$ variable and up to $n$ for the $\xi_1$ 
and $\xi_2$ variables are assumed. In the present paper, we shall relax the smoothness 
condition of [15] and also give results for general classes which include $BS_{0,0}^m(\mathbb{R}^n)$ of 
critical order $m = -n/2$.

Our method to prove the boundedness of pseudo-differential operators relies on 
the idea of Boulkhemair [5], who treated linear pseudo-differential operators.

We end this section by mentioning the plan of this paper. In Section 2 we 
will give the basic notations used throughout this paper and recall the definitions 
and properties of some function spaces. In Section 3 we give several properties 
of the class $B(\mathbb{Z}^n \times \mathbb{Z}^n)$ and prove that it contains the functions $V$ of Example 
1.4. In Section 4 we prove Theorem 1.3 and also give two other main theorems of 
this paper, Theorems 4.3 and 4.5. The latter theorems treat symbols with limited 
smoothness. In the same section, we also give a proof to the theorem of Grafakos– 
He–Slavíková [13] by using Theorem 1.3. In Section 5 we show the sharpness of our 
main theorems.

2. Preliminaries

2.1. Basic notations. We collect notations which will be used throughout this 
paper. We denote by $\mathbb{R}$, $\mathbb{Z}$, $\mathbb{N}$, and $\mathbb{N}_0$ the sets of real numbers, integers, positive 
integers, and nonnegative integers, respectively. We denote by $Q$ the $n$-dimensional 
unit cube $[-1/2, 1/2]^n$. For $1 \leq p \leq \infty$, $p'$ is the conjugate number of $p$ defined by 
$1/p + 1/p' = 1$. We write $[s] = \max\{n \in \mathbb{Z} : n \leq s\}$ for $s \in \mathbb{R}$. For $x \in \mathbb{R}^d$, we write 
$\langle x \rangle = (1 + |x|^2)^{1/2}$. Thus $\langle (x, y) \rangle = (1 + |x|^2 + |y|^2)^{1/2}$ for $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$.

For two nonnegative functions $A(x)$ and $B(x)$ defined on a set $X$, we write $A(x) \leq B(x)$ for $x \in X$ to mean that there exists a positive constant $C$ such that $A(x) \leq CB(x)$ for all $x \in X$. We often omit to mention the set $X$ when it is obviously 
recognized. Also $A(x) \approx B(x)$ means that $A(x) \leq B(x)$ and $B(x) \leq A(x)$.

We denote the Schwartz space of rapidly decreasing smooth functions on $\mathbb{R}^d$ by 
$S(\mathbb{R}^d)$ and its dual, the space of tempered distributions, by $S'(\mathbb{R}^d)$. The Fourier 
transform and the inverse Fourier transform of $f \in S(\mathbb{R}^d)$ are given by

$$
\mathcal{F}f(\xi) = \hat{f}(\xi) = \int_{\mathbb{R}^d} e^{-ix \cdot \xi} f(x) \, dx, \\
\mathcal{F}^{-1}f(x) = \check{f}(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{ix \cdot \xi} f(\xi) \, d\xi,
$$

respectively. For $m \in S'(\mathbb{R}^d)$, the Fourier multiplier operator is defined by

$$
m(D)f = \mathcal{F}^{-1}[m \cdot \mathcal{F}f].
$$

We also use the notation $(m(D)f)(x) = m(D_x)f(x)$ when we indicate which variable 
is considered.

For a measurable subset $E \subset \mathbb{R}^d$, the Lebesgue space $L^p(E)$, $0 < p \leq \infty$, is the set of 
all those measurable functions $f$ on $E$ such that $
\|f\|_{L^p(E)} = (\int_E |f(x)|^p \, dx)^{1/p} < \infty$.
\[ \|f\|_{L^\infty(E)} = \text{ess sup}_{x \in E} |f(x)| < \infty \] if \(0 < p < \infty\) or \(\|f\|_{L^\infty(E)} = \text{ess sup}_{x \in E} |f(x)| < \infty\) if \(p = \infty\). We also use the notation \(\|f\|_{L^p(E)} = \|f(x)\|_{L^p(E)}\) when we want to indicate the variable explicitly.

The uniformly local \(L^2\) space, denoted by \(L^2(u)(\mathbb{R}^n)\), consists of all those measurable functions \(f\) on \(\mathbb{R}^n\) such that
\[
\|f\|_{L^2(u)(\mathbb{R}^n)} = \sup_{\nu \in \mathbb{Z}^n} \left( \int_Q |f(x + \nu)|^2 \, dx \right)^{1/2} < \infty
\]
(this notion can be found in [18, Definition 2.3]).

Let \(K\) be a countable set. We define the sequence spaces \(\ell^q(K)\) and \(\ell^{q,\infty}(K)\) as follows. The space \(\ell^q(K)\), \(1 \le q \le \infty\), consists of all those complex sequences \(a = \{a_k\}_{k \in K}\) such that \(\|a\|_{\ell^q(K)} = \left( \sum_{k \in K} |a_k|^q \right)^{1/q} < \infty\) if \(1 \le q < \infty\) or \(\|a\|_{\ell^\infty(K)} = \sup_{k \in K} |a_k| < \infty\) if \(q = \infty\). For \(1 \le q < \infty\), the space \(\ell^{q,\infty}(K)\) is the set of all those complex sequences \(a = \{a_k\}_{k \in K}\) such that
\[
\|a\|_{\ell^{q,\infty}(K)} = \sup_{t > 0} \left\{ t \# \{k \in K : |a_k| > t\} \right\}^{1/q} < \infty,
\]
where \(\#\) denotes the cardinality of a set. Sometimes we write \(\|a\|_{\ell^q} = \|a_k\|_{\ell^q(K)}\) or \(\|a\|_{\ell^{q,\infty}} = \|a_k\|_{\ell^{q,\infty}(K)}\). If \(K = \mathbb{Z}^n\), we usually write \(\ell^q\) or \(\ell^{q,\infty}\) for \(\ell^q(\mathbb{Z}^n)\) or \(\ell^{q,\infty}(\mathbb{Z}^n)\).

Let \(X, Y, Z\) be function spaces. We denote the mixed norm by
\[
\|f(x, y, z)\|_{X \times Y \times Z} = \left\| f(x, y, z) \right\|_{X \times Y \times Z}.
\]
(Here pay special attention to the order of taking norms.) We shall use these mixed norms for \(X, Y, Z\) being \(L^p\) or \(\ell^q\). Recall that the Minkowski inequality implies
\[
\|f(x, y)\|_{L^p_1 L^q_2} \le \|f(x, y)\|_{L^p_1 L^q_2}, \quad \text{if } p \le q.
\]

2.2. Local Hardy space \(h^1\) and the space \(bmo\). We recall the definition of the local Hardy space \(h^1(\mathbb{R}^n)\) and the space \(bmo(\mathbb{R}^n)\).

Let \(\phi \in S(\mathbb{R}^n)\) be such that \(\int_{\mathbb{R}^n} \phi(x) \, dx \neq 0\). Then, the local Hardy space \(h^1(\mathbb{R}^n)\) consists of all \(f \in S'(\mathbb{R}^n)\) such that \(\|f\|_{h^1} = \|\sup_{0 < t < 1} |\phi_t \ast f|\|_{L^1} < \infty\), where \(\phi_t(x) = t^{-n} \phi(x/t)\). It is known that \(h^1(\mathbb{R}^n)\) does not depend on the choice of the function \(\phi\), and that \(h^1(\mathbb{R}^n) \hookrightarrow L^1(\mathbb{R}^n)\).

The space \(bmo(\mathbb{R}^n)\) consists of all locally integrable functions \(f\) on \(\mathbb{R}^n\) such that
\[
\|f\|_{bmo} = \sup_{|R| \le 1} \frac{1}{|R|} \int_R |f(x) - f_R| \, dx + \sup_{|R| \ge 1} \frac{1}{|R|} \int_R |f(x)| \, dx < \infty,
\]
where \(f_R = \frac{1}{|R|} \int_R f\), and \(R\) ranges over the cubes in \(\mathbb{R}^n\).

It is known that the dual space of \(h^1(\mathbb{R}^n)\) is \(bmo(\mathbb{R}^n)\). See Goldberg [10] for more details about \(h^1\) and \(bmo\).

2.3. Amalgam spaces. For \(1 \le p, q \le \infty\), the amalgam space \((L^p, \ell^q)(\mathbb{R}^n)\) is defined to be the set of all those measurable functions \(f\) on \(\mathbb{R}^n\) such that
\[
\|f\|_{(L^p, \ell^q)(\mathbb{R}^n)} = \|f(x + \nu)\|_{L^p(Q) \ell^q(\mathbb{Z}^n)} = \left\{ \sum_{\nu \in \mathbb{Z}^n} \left( \int_Q |f(x + \nu)|^p \, dx \right)^{q/p} \right\}^{1/q} < \infty
\]
with usual modification when \(p\) or \(q\) is infinity. Obviously, \((L^p, \ell^p) = L^p\) and \((L^2, \ell^\infty) = L^2_{ul}\). For \(1 \le p, q < \infty\), the duality \((L^p, \ell^q)^\ast = (L^q, \ell^p)^\ast\) holds. If \(p_1 \ge p_2\) and \(q_1 \le q_2\), then \((L^{p_1}, \ell^{q_1}) \hookrightarrow (L^{p_2}, \ell^{q_2})\). In particular, \((L^2, \ell^1) \hookrightarrow L^r\) for \(1 \le r \le 2\). In the case \(r = 1\), the stronger embedding \((L^2, \ell^1) \hookrightarrow h^1\) holds. This
last fact follows from the embedding $bmo \hookrightarrow (L^2, \ell^\infty)$ and the duality $(h^1)' = bmo$. For $1 \leq p, q_1, \ldots, q_n \leq \infty$, we also define the space $(L^p, \ell^{q_1} \ldots \ell^{q_n})(\mathbb{R}^n)$ by the mixed norm

$$
\|f\|_{(L^p, \ell^{q_1} \ldots \ell^{q_n})(\mathbb{R}^n)} = \|f(x + \nu)\|_{L^p_\nu(Q)\ell^{q_1}_\nu(z) \ldots \ell^{q_n}_\nu(z)},
$$

where $\nu = (\nu_1, \ldots, \nu_n) \in \mathbb{Z}^n$. See Fournier–Stewart [9] and Holland [10] for more properties of amalgam spaces.

3. Class $\mathcal{B}$

In this section, we give several properties of the class $\mathcal{B}(\mathbb{Z}^n \times \mathbb{Z}^n)$ introduced in Definition [12]. We also introduce the class $\mathcal{M}(\mathbb{R}^d)$, which will be used in the next section.

Proposition 3.1.  
(1) Every function in the class $\mathcal{B}(\mathbb{Z}^n \times \mathbb{Z}^n)$ is bounded.

(2) A nonnegative function $V = V(\nu_1, \nu_2)$ on $\mathbb{Z}^n \times \mathbb{Z}^n$ belongs to $\mathcal{B}(\mathbb{Z}^n \times \mathbb{Z}^n)$ if and only if $V(\nu_1 + \nu_2, -\nu_2)$ or $V(-\nu_1, \nu_1 + \nu_2)$ belongs to $\mathcal{B}(\mathbb{Z}^n \times \mathbb{Z}^n)$.

(3) The class $\mathcal{B}(\mathbb{Z}^n \times \mathbb{Z}^n)$ is not rearrangement invariant, i.e., there exists a function $V$ on $\mathbb{Z}^n \times \mathbb{Z}^n$ and a bijection $\Phi : \mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{Z}^n \times \mathbb{Z}^n$ such that $V \in \mathcal{B}(\mathbb{Z}^n \times \mathbb{Z}^n)$ but $V \circ \Phi \notin \mathcal{B}(\mathbb{Z}^n \times \mathbb{Z}^n)$.

(4) Let $d, d' \in \mathbb{Z}$, $V \in \mathcal{B}(\mathbb{Z}^d \times \mathbb{Z}^{d'})$, and $V' \in \mathcal{B}(\mathbb{Z}^{d'} \times \mathbb{Z}^d)$. Then the function

$$
W((\mu_1, \mu'_1), (\mu_2, \mu'_2)) = V(\mu_1, \mu_2)V'(\mu'_1, \mu'_2), \quad \mu_1, \mu_2 \in \mathbb{Z}^d, \quad \mu'_1, \mu'_2 \in \mathbb{Z}^{d'},
$$

belongs to $\mathcal{B}(\mathbb{Z}^{d+d'} \times \mathbb{Z}^{d+d'})$.

Proof. (1) If $V$ satisfies (1.4), then applying it to the case where each of $A, B, C$ is a defining function of one point we easily find $V(\nu_1, \nu_2) \leq c$.

(2) This can be easily proved by a simple change of variables.

(3) First observe that the function $V(\nu_1, \nu_2) = \langle \nu_1 \rangle^{-n/2-\epsilon}$ with $\epsilon > 0$ belongs to $\mathcal{B}(\mathbb{Z}^n \times \mathbb{Z}^n)$. In fact for this $V$ and for $B(\nu_1) \in \ell^2_{\nu_1}(\mathbb{Z}^n)$, the function $VB$ belongs to $\ell^1(\mathbb{Z}^n)$ and the inequality (1.4) can be easily checked by the use of Hölder’s inequality. (See also Proposition 3.2 below.) On the other hand, for $\alpha > 0$, the function

$$
W(\nu_1, \nu_2) = ((\nu_1, \nu_2))^{-n/2+\alpha}, \quad (\nu_1, \nu_2) \in \mathbb{Z}^n \times \mathbb{Z}^n
$$

does not belong to $\mathcal{B}(\mathbb{Z}^n \times \mathbb{Z}^n)$. In fact, for $A(\mu) = B(\mu) = C(\mu) = \langle \mu \rangle^{-n/2-\alpha/4} \in \ell^2_{\mu}(\mathbb{Z}^n)$, it is easy to see that

$$
\sum W(\nu_1, \nu_2)A(\nu_1 + \nu_2)B(\nu_1)C(\nu_2) = 0.
$$

For $j \in \mathbb{N}_0$, set

$$
E_j(V) = \{(\nu_1, \nu_2) \in \mathbb{Z}^n \times \mathbb{Z}^n | 2^{-j-1} < V(\nu_1, \nu_2) \leq 2^{-j}\},
$$

$$
E_j(W) = \{(\nu_1, \nu_2) \in \mathbb{Z}^n \times \mathbb{Z}^n | 2^{-j-1} < W(\nu_1, \nu_2) \leq 2^{-j}\}.
$$

Then both $\{E_j(V)\}_{j \in \mathbb{N}_0}$ and $\{E_j(W)\}_{j \in \mathbb{N}_0}$ are partitions of $\mathbb{Z}^n \times \mathbb{Z}^n$, each $E_j(V)$ is an infinite set, and $E_j(W)$ is a finite set. It is easy to construct a bijection $\Phi$ of $\mathbb{Z}^n \times \mathbb{Z}^n$ onto itself such that

$$
\Phi(E_j(W)) \subset E_0(V) \cup \cdots \cup E_j(V) \quad \text{for all } j \in \mathbb{N}_0.
$$

Then $W \leq 2^{-j}$ and $V \circ \Phi > 2^{-j-1}$ on each $E_j(W)$, we have $W < 2V \circ \Phi$ on the whole $\mathbb{Z}^n \times \mathbb{Z}^n$. Since $W \notin \mathcal{B}(\mathbb{Z}^n \times \mathbb{Z}^n)$, we have $V \circ \Phi \notin \mathcal{B}(\mathbb{Z}^n \times \mathbb{Z}^n)$.

(4) Let $A, B, C$ be nonnegative functions on $\mathbb{Z}^{d+d'}$ and consider the sum

$$
\sum_{(\mu_1, \mu'_1), (\mu_2, \mu'_2) \in \mathbb{Z}^{d+d'}} V(\mu_1, \mu_2)V'(\mu'_1, \mu'_2)A(\mu_1, \mu'_1) + (\mu_2, \mu'_2))B(\mu_1, \mu'_1)C(\mu_2, \mu'_2).
$$
If we first take the sum over $\mu_1, \mu_2 \in \mathbb{Z}^d$, then the assumption $V \in \mathcal{B}(\mathbb{Z}^d \times \mathbb{Z}^d)$ implies that the above sum is bounded by a constant times

$$\sum_{\mu_1', \mu_2' \in \mathbb{Z}^d} V'(\mu_1', \mu_2') \|A(\mu_1, \mu_1' + \mu_2')\|_{\ell^2_\nu(\mathbb{Z}^d)} \|B(\mu_1, \mu_1')\|_{\ell^2_\mu(\mathbb{Z}^d)} \|C(\mu_2, \mu_2')\|_{\ell^2_\mu(\mathbb{Z}^d)}.$$

Now $V' \in \mathcal{B}(\mathbb{Z}^d \times \mathbb{Z}^d)$ implies that the last sum is bounded by a constant times

$$\|A(\mu_1, \mu_1')\|_{\ell^2_\nu(\mathbb{Z}^d)} \|B(\mu_1, \mu_1')\|_{\ell^2_\mu(\mathbb{Z}^d)} \|C(\mu_2, \mu_2')\|_{\ell^2_\mu(\mathbb{Z}^d)} \|B\|_{\ell^2(\mathbb{Z}^d \times \mathbb{Z}^d)} \|C\|_{\ell^2(\mathbb{Z}^d \times \mathbb{Z}^d)}.$$

Thus the function $W$ of (4) belongs to $\mathcal{B}(\mathbb{Z}^{d+d'} \times \mathbb{Z}^{d+d'})$. 

\begin{proposition}
Suppose a nonnegative function $V$ on $\mathbb{Z}^n \times \mathbb{Z}^n$ is one of the following forms:

$$V(\nu_1, \nu_2) = V_0(\nu_1), \quad V_0(\nu_1 + \nu_2).$$

Then $V \in \mathcal{B}(\mathbb{Z}^n \times \mathbb{Z}^n)$ if and only if $V_0 \in \ell^2(\mathbb{Z}^n)$. In particular, a nonzero constant function does not belong to $\mathcal{B}(\mathbb{Z}^n \times \mathbb{Z}^n)$.

\end{proposition}

\begin{proof}
We use the following fact: if $K$ is a nonnegative function on $\mathbb{Z}^n$, then the inequality

$$\left\| \sum_{\nu \in \mathbb{Z}^n} K(\nu_1 - \nu_2)X(\nu_2) \right\|_{\ell^2_\nu(\mathbb{Z}^n)} \leq C\|X\|_{\ell^2(\mathbb{Z}^n)}$$

holds for all nonnegative functions $X$ on $\mathbb{Z}^n$ if and only if $\|K\|_{\ell^2(\mathbb{Z}^n)} \leq c$. Here is a proof. Consider the case where $K(\nu) = 0$ except for finitely many $\nu$’s. Then, by the $L^2$ theory of Fourier analysis for periodic functions, it is easy to see that the inequality \((\mathbf{x})\) holds for all nonnegative $X$ if and only if the function $k(x) = \sum_{\nu \in \mathbb{Z}^n} K(\nu)e^{2\pi i \nu \cdot x}$ satisfies $\|k\|_{L^\infty(Q)} \leq c$. But since $K$ is nonnegative, we have $\|k\|_{L^\infty(Q)} = \|K\|_{\ell^2(\mathbb{Z}^n)}$ and thus $\|K\|_{\ell^2(\mathbb{Z}^n)} \leq c$. The general case follows by a limiting argument.

Now suppose $V(\nu_1, \nu_2) = V_0(\nu_1)$ and $V \in \mathcal{B}(\mathbb{Z}^n \times \mathbb{Z}^n)$. Then, by a change of variables, the inequality \((\mathbf{y})\) is written as

$$\sum_{\nu_1, \nu_2 \in \mathbb{Z}^n} V_0(\nu_1 - \nu_2)A(\nu_1)B(\nu_1 - \nu_2)C(\nu_2) \leq c\|A\|_{\ell^2} \|B\|_{\ell^2} \|C\|_{\ell^2}.$$

By the fact mentioned above, this inequality holds if and only if $\|V_0B\|_{\ell^1} \leq c\|B\|_{\ell^2}$, which is equivalent to $\|V_0\|_{\ell^1} \leq c$.

The cases $V(\nu_1, \nu_2) = V_0(\nu_2)$ and $V(\nu_1, \nu_2) = V_0(\nu_1 + \nu_2)$ are proved in a similar way or by the use of Proposition \(\mathbf{3.1}\) (2).

\end{proof}

\begin{proposition}
Let $2 < p_1, p_2 < \infty$, $1/p_1 + 1/p_2 = 1/2$, and let $f_1 \in \ell^{p_1, \infty}(\mathbb{Z}^d)$ and $f_2 \in \ell^{p_2, \infty}(\mathbb{Z}^d)$ be nonnegative sequences. Then the functions $f_1(\nu_1)f_2(\nu_2)$, $f_1(\nu_1 + \nu_2)f_2(\nu_2)$, and $f_1(\nu_1)f_2(\nu_1 + \nu_2)$ belong to $\mathcal{B}(\mathbb{Z}^d \times \mathbb{Z}^d)$.

\end{proposition}

\begin{proof}
By Proposition \(\mathbf{3.1}\) (2), it is sufficient to prove that $f_1(\nu_1)f_2(\nu_2)$ belongs to $\mathcal{B}(\mathbb{Z}^d \times \mathbb{Z}^d)$. Let $A, B, C$ be nonnegative functions on $\mathbb{Z}^d$.

We set

$$E_i(j) = \{\nu \in \mathbb{Z}^d \mid 2^{-(j+1)/p_i} < f_i(\nu) \leq 2^{-j/p_i}\}, \quad i = 1, 2, \quad j \in \mathbb{Z}.$$

Our assumption $f_i \in \ell^{p_i, \infty}(\mathbb{Z}^d)$ implies the estimate

$$\#(E_i(j)) \lesssim 2^j.$$
Since \( \{E_i(j)\}_{j \in \mathbb{Z}} \) gives a decomposition of the set \( \{ \nu \in \mathbb{Z}^d : f_i(\nu) > 0 \} \), the sum on the left hand side of (1.2) for \( V(\nu_1, \nu_2) = f_1(\nu_1)f_2(\nu_2) \) is written as

\[
\sum_{\nu_1, \nu_2 \in \mathbb{Z}^d} f_1(\nu_1)f_2(\nu_2)A(\nu_1 + \nu_2)B(\nu_1)C(\nu_2)
\]

\approx \sum_{j_1, j_2 \in \mathbb{Z}} \sum_{\nu_1 \in E_1(j_1)} \sum_{\nu_2 \in E_2(j_2)} 2^{-j_1/p_1}2^{-j_2/p_2} A(\nu_1 + \nu_2)B(\nu_1)C(\nu_2).

Fix \( j_1, j_2 \) and consider the sum over \( \nu_1 \in E_1(j_1) \) and \( \nu_2 \in E_2(j_2) \). If \( j_1 \leq j_2 \), then we apply the Cauchy–Schwarz inequality first to the sum over \( \nu_2 \) and then to the sum over \( \nu_1 \) to obtain

\[
\sum_{\nu_1 \in E_1(j_1)} \sum_{\nu_2 \in E_2(j_2)} 2^{-j_1/p_1}2^{-j_2/p_2} A(\nu_1 + \nu_2)B(\nu_1)C(\nu_2)
\]

\leq \sum_{\nu_1 \in E_1(j_1)} 2^{-j_1/p_1}2^{-j_2/p_2} \| A \|_{L^2(\mathbb{Z}^d)} B(\nu_1) \| C \|_{L^2(\mathbb{Z}^d)}

\leq 2^{-j_1/p_1}2^{-j_2/p_2} (\| E_1(j_1) \|)^{1/2} \| A \|_{L^2(\mathbb{Z}^d)} \| B \|_{L^2(\mathbb{Z}^d)} \| C \|_{L^2(\mathbb{Z}^d)}

\leq 2^{-(j_2-j_1)/p_2} \| A \|_{L^2(\mathbb{Z}^d)} \| B \|_{L^2(\mathbb{Z}^d)} \| C \|_{L^2(\mathbb{Z}^d)}.

where the last \( \lesssim \) follows from the estimate \( \| E_1(j_1) \| \lesssim 2^{j_1} \) (see (3.2)) and the equality \( 1/p_1 + 1/p_2 = 1/2 \). Similarly, if \( j_1 > j_2 \), then we apply the Cauchy–Schwarz inequality first to the sum over \( \nu_1 \) and then to the sum over \( \nu_2 \) to obtain the same estimate as above but with the factor \( 2^{-(j_1-j_2)/p_1} \) replaced by \( 2^{-(j_1-j_2)/p_2} \).

Thus in either case we have

\[
\sum_{\nu_1 \in E_1(j_1)} \sum_{\nu_2 \in E_2(j_2)} 2^{-j_1/p_1}2^{-j_2/p_2} A(\nu_1 + \nu_2)B(\nu_1)C(\nu_2)
\]

\lesssim (2^{-|j_1-j_2|/p_2} + 2^{-|j_1-j_2|/p_1}) \| A \|_{L^2(\mathbb{Z}^d)} \| B \|_{L^2(\mathbb{Z}^d)} \| C \|_{L^2(\mathbb{Z}^d)}.

By the Schur lemma, the sum of the above over \( j_1, j_2 \in \mathbb{Z} \) is bounded by

\[
\| A \|_{L^2(\mathbb{Z}^d)} \| B \|_{L^2(\mathbb{Z}^d)} \| C \|_{L^2(\mathbb{Z}^d)} \leq \| A \|_{L^2(\mathbb{Z}^d)} \| B \|_{L^2(\mathbb{Z}^d)} \| C \|_{L^2(\mathbb{Z}^d)}.

(For the Schur lemma, see, e.g., [12, Appendix A].) \( \Box \)

**Proposition 3.4.** All nonnegative functions in the class \( \ell^4,\infty(\mathbb{Z}^d \times \mathbb{Z}^d) \) belong to \( B(\mathbb{Z}^d \times \mathbb{Z}^d) \).

**Proof.** By appropriately extending functions on \( \mathbb{Z}^d \) and \( \mathbb{Z}^d \times \mathbb{Z}^d \) to functions on \( \mathbb{R}^d \) and \( \mathbb{R}^d \times \mathbb{R}^d \), it is sufficient to prove the inequality

\[
\int_{\mathbb{R}^d \times \mathbb{R}^d} V(x_1, x_2)A(x_1 + x_2)B(x_1)C(x_2) \, dx_1 dx_2
\]

\[\lesssim \| V \|_{L^4,\infty(\mathbb{R}^d \times \mathbb{R}^d)} \| A \|_{L^2(\mathbb{R}^d)} \| B \|_{L^2(\mathbb{R}^d)} \| C \|_{L^2(\mathbb{R}^d)}
\]

for nonnegative measurable functions \( V, A, B, C \) on the corresponding Euclidean spaces. We shall derive this inequality from the inequality

\[
\int_{\mathbb{R}^d \times \mathbb{R}^d} V(x_1, x_2)A(x_1 + x_2)B(x_1)C(x_2) \, dx_1 dx_2
\]

\[\lesssim \| V \|_{L^4(\mathbb{R}^d \times \mathbb{R}^d)} \| A \|_{L^2(\mathbb{R}^d)} \| B \|_{L^2(\mathbb{R}^d)} \| C \|_{L^2(\mathbb{R}^d)}
\]
by using real interpolation. It is known that \((3.4)\) holds if and only if the following two conditions are satisfied:

\[
\begin{align*}
2 \frac{1}{q_0} + \frac{1}{q_1} + \frac{1}{q_2} + \frac{1}{q_3} &= 2, \\
0 \leq \frac{1}{q_i} &\leq 1 - \frac{1}{q_0}, \quad i = 1, 2, 3.
\end{align*}
\]

For the reader’s convenience, here we give a proof of the fact that \((3.4)\) holds under the assumptions \((3.5)\) and \((3.6)\). It is sufficient to show

\[
\|(A(x_1 + x_2)B(x_1)C(x_2))\|_{L^{q_0}_{x_1,x_2}} \lesssim \|A\|_{L^{q_1}} \|B\|_{L^{q_2}} \|C\|_{L^{q_3}}.
\]

In the case \(q_0' = \infty\), \((3.6)\) implies \(q_1 = q_2 = q_3 = \infty\) and \((3.7)\) is obvious. We assume \(q_0' < \infty\). Take \(\alpha, \beta, \gamma, \delta \in [1, \infty]\) that satisfy \(1/\delta + 1/\gamma = 1\) and \(1 + 1/\delta = 1/\alpha + 1/\beta\). Then writing \(B(\mu) = B(-\mu)\) and using Hölder’s inequality and Young’s inequality for convolution, we have

\[
\begin{align*}
\|A(x_1 + x_2)B(x_1)C(x_2)\|_{L^{q_0}_{x_1,x_2}} &= \int (A^{q_0'} * \tilde{B}^{q_0'})(x_2)C(x_2)^{q_0'} dx_2 \\
&\leq \|A^{q_0'} * \tilde{B}^{q_0'}\|_{L^\delta} \|C^{q_0'}\|_{L^\gamma} \leq \|A^{q_0'}\|_{L^{q_0}} \|\tilde{B}^{q_0'}\|_{L^\delta} \|C^{q_0'}\|_{L^\gamma} \\
&= (\|A\|_{L^{q_0'}} \|B\|_{L^{q_0'}} \|C\|_{L^{q_0'}})^{q_0'}.
\end{align*}
\]

By choosing \(\alpha, \beta, \gamma\) such that \(\alpha q_0' = q_1, \beta q_0' = q_2, \) and \(\gamma q_0' = q_3\), we obtain \((3.7)\) with the constant in \(\lesssim\) equal to 1.

From \((3.4)\), it follows by duality that the trilinear map

\[
T(A, B, C)(x_1, x_2) = A(x_1 + x_2)B(x_1)C(x_2)
\]

satisfies the estimate

\[
\|T(A, B, C)\|_{L^{q_0}_{x_1,x_2}} \lesssim \|A\|_{L^{q_0}_{x_1,x_2}} \|B\|_{L^{q_0}_{x_1,x_2}} \|C\|_{L^{q_0}_{x_1,x_2}}
\]

for all \((q_i)\) satisfying \((3.5)\) and \((3.6)\). Hence, by the real interpolation for multilinear operators (see Janson [17]), it follows that if \((q_i)\) satisfy \((3.5)\) and also satisfy the strict inequalities

\[
0 < \frac{1}{q_i} < 1 - \frac{1}{q_0}, \quad i = 1, 2, 3,
\]

then the Lorentz norm estimate

\[
\|T(A, B, C)\|_{L^{q_0', r_0'}_{x_1,x_2}} \lesssim \|A\|_{L^{q_1', r_1'}_{x_1,x_2}} \|B\|_{L^{q_2', r_2'}_{x_1,x_2}} \|C\|_{L^{q_3', r_3'}_{x_1,x_2}}
\]

holds for all \((r_i)\) such that

\[
r_i \in [1, \infty], \quad i = 0, 1, 2, 3, \quad \text{and} \quad \frac{1}{r_0} + \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} = 1.
\]

By duality again, this implies that the inequality

\[
\int_{\mathbb{R}^d \times \mathbb{R}^d} V(x_1, x_2)A(x_1 + x_2)B(x_2)C(x_2) \, dx_1 dx_2 \lesssim \|V\|_{L^{q_0', r_0'}_{x_1,x_2}} \|A\|_{L^{q_1', r_1'}_{x_1,x_2}} \|B\|_{L^{q_2', r_2'}_{x_1,x_2}} \|C\|_{L^{q_3', r_3'}_{x_1,x_2}}
\]
holds for all \((q_i)\) and \((r_i)\) satisfying \((3.5), (3.8),\) and \((3.9).\) In particular, by taking \(q_0 = 4, q_1 = q_2 = q_3 = 2, r_0 = r_1 = \infty,\) and \(r_2 = r_3 = 2,\) we obtain

\[
\int_{\mathbb{R}^d \times \mathbb{R}^d} V(x_1, x_2) A(x_1 + x_2) B(x_1) C(x_2) \, dx_1 dx_2 \\
\lesssim \|V\|_{L^{4,\infty}(\mathbb{R}^d \times \mathbb{R}^d)} \|A\|_{L^{2,\infty}(\mathbb{R}^d)} \|B\|_{L^2(\mathbb{R}^d)} \|C\|_{L^2(\mathbb{R}^d)},
\]

which a fortiori implies \((3.3).\)

**Remark 3.5.** The basic idea of using real interpolation to derive \((3.10)-(3.9)\) from \((3.4)\) is given in the paper of Perry [22, Appendix A]. Theorem A.3 in this Appendix A, written by M. Christ, gives a sufficient condition to derive inequality of the form \((3.10)-(3.9)\) from the inequality of the form \((3.4).\) In this general theorem, the sufficient condition is expressed in terms of \((q_j)\) and subspaces of \(\mathbb{R}^d.\) If \(d = 1,\) then by applying this theorem we can conclude that \((3.10)-(3.9)\) holds for all \((q_j)\) satisfying \((3.8).\) However, if \(d \geq 2,\) the case \((3.8)\) does not satisfy the very condition of the theorem.

**Remark 3.6.** It is also possible to prove Proposition 3.3 by the same method as in Proof of Proposition 3.4. In fact, by using Hölder’s inequality and Young’s inequality, we see that the inequality

\[
(3.11) \quad \int_{\mathbb{R}^d \times \mathbb{R}^d} f_1(x_1) f_2(x_2) A(x_1 + x_2) B(x_1) C(x_2) \, dx_1 dx_2 \\
\lesssim \|f_1\|_{L^{p_1}} \|f_2\|_{L^{p_2}} \|A\|_{L^{q_1}} \|B\|_{L^{q_2}} \|C\|_{L^{q_3}}
\]

holds for

\[
(3.12) \quad 1/p_1 + 1/p_2 + 1/q_1 + 1/q_2 + 1/q_3 = 2, \\
(3.13) \quad 0 \leq 1/p_1, 1/p_2, 1/q_1, 1/q_2, 1/q_3 \leq 1, \\
(3.14) \quad 0 \leq 1/q_2 + 1/p_1 \leq 1, \\
(3.15) \quad 0 \leq 1/q_3 + 1/p_2 \leq 1.
\]

Hence, by the same argument of interpolation as in Proof of Proposition 3.4, we see that \((3.11)\) holds with the Lebesgue norms replaced by appropriate Lorentz norms if the equality \((3.12)\) holds and if all the inequalities \((3.13), (3.14),\) and \((3.15)\) hold with strict inequalities. Thus, in particular, for \(q_1 = q_2 = q_3 = 2\) and for \(p_1, p_2\) satisfying \(0 < 1/p_1, 1/p_2 < 1/2\) and \(1/p_1 + 1/p_2 = 1/2,\) we have

\[
\int_{\mathbb{R}^d \times \mathbb{R}^d} f_1(x_1) f_2(x_2) A(x_1 + x_2) B(x_1) C(x_2) \, dx_1 dx_2 \\
\lesssim \|f_1\|_{L^{p_1,\infty}} \|f_2\|_{L^{p_2,\infty}} \|A\|_{L^{q_1,\infty}} \|B\|_{L^{q_2,\infty}} \|C\|_{L^{q_3}},
\]

which a fortiori implies the conclusion of Proposition 3.3.

Here we give a proof of the assertion of Example 1.4.

**Proof of Example 1.4.** The function \((1.5)\) is in \(\ell^{4,\infty}(\mathbb{Z}^{2n})\) and hence it belongs to \(B(\mathbb{Z}^n \times \mathbb{Z}^n)\) by Proposition 3.3. The fact that the functions \((1.6)\) and \((1.7)\) belong to \(B(\mathbb{Z}^n \times \mathbb{Z}^n)\) can be seen by the use of Propositions 3.3 and 3.1 (4).

We introduce the following.
**Definition 3.7.** Let $d \in \mathbb{N}$. We say that a continuous function $F: \mathbb{R}^d \to (0, \infty)$ is of moderate class if there exists an $N = N_F > 0$ such that

\begin{equation}
F(\xi)^2 \ast \langle \xi \rangle^{-N} = \int_{\mathbb{R}^d} F(\eta)^2 \langle \xi - \eta \rangle^{-N} \, d\eta \approx F(\xi)^2 \quad \text{for all } \xi \in \mathbb{R}^d,
\end{equation}

where the implicit constants in $\approx$ may depend on $F$. We denote by $\mathcal{M}(\mathbb{R}^d)$ the set of all functions on $\mathbb{R}^d$ of moderate class.

Here are some simple properties of the class $\mathcal{M}(\mathbb{R}^d)$.

**Proposition 3.8.**  
(1) If the relation (3.16) holds for an $N > 0$, then the same relation, possibly with different constants in $\approx$, holds if $N$ is replaced by $N' > \max\{N, d\}$.

(2) If $F \in \mathcal{M}(\mathbb{R}^d)$ and $N > 0$ satisfy (3.16), then

\[ F(\xi) \langle \xi \rangle^{-N/2} \lesssim F(\xi + \zeta) \lesssim F(\xi) \langle \zeta \rangle^{N/2} \quad \text{for all } \xi, \zeta \in \mathbb{R}^d. \]

(3) Let $d = d_1 + d_2$ with $d_1, d_2 \in \mathbb{N}$. Then a continuous function $F: \mathbb{R}^d \to (0, \infty)$ belongs to the class $\mathcal{M}(\mathbb{R}^d)$ if and only if the relation

\begin{equation}
\int_{\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}} F(\eta_1, \eta_2)^2 \langle \xi - \eta_1 \rangle^{-N_1} \langle \xi - \eta_2 \rangle^{-N_2} \, d\eta_1 d\eta_2 \approx F(\xi_1, \xi_2)^2
\end{equation}

holds for any sufficiently large $N_1 > 0$ and $N_2 > 0$.

(4) If $d_1, d_2 \in \mathbb{N}$, $F_1 \in \mathcal{M}(\mathbb{R}^{d_1})$, and $F_2 \in \mathcal{M}(\mathbb{R}^{d_2})$, then the function $F(\xi_1, \xi_2) = F_1(\xi_1)F_2(\xi_2)$ belongs to $\mathcal{M}(\mathbb{R}^{d_1 + d_2})$.

**Proof.** The assertion (1) follows once we make the convolution of the functions in (3.16) with the function $\langle \xi \rangle^{-N'}$ and use the fact that $\langle \xi \rangle^{-N} \ast \langle \xi \rangle^{-N'} \approx \langle \xi \rangle^{-N}$ if $N' > \max\{N, d\}$. The assertion (2) follows from the inequalities

\[ \langle \xi \rangle^{-N} \langle \zeta \rangle^{-N} \lesssim \langle \xi + \zeta \rangle^{-N} \lesssim \langle \xi \rangle^{-N} \langle \zeta \rangle^{-N}. \]

To prove the assertion (3), first observe that if the relation (3.17) holds then the same relation holds if $N_i$ are replaced by $N'_i > \max\{N_i, d_i\}$, $i = 1, 2$. This is proved by the same reasoning as in the proof of (1). Using this fact, the fact of (1), and the obvious inequalities

\[ \langle \xi_1, \xi_2 \rangle^{-2N} \leq \langle \xi_1 \rangle^{-N} \langle \xi_2 \rangle^{-N} \leq \langle \xi_1, \xi_2 \rangle^{-N} \leq \langle \xi_1 \rangle^{-N/2} \langle \xi_2 \rangle^{-N/2}, \]

we can easily prove (3). Finally the assertion (4) easily follows from (3). \hfill \square

Finally we give a general result concerning the classes $\mathcal{B}$ and $\mathcal{M}$.

**Proposition 3.9.** For any $V \in \mathcal{B}(\mathbb{Z}^d \times \mathbb{Z}^d)$, there exists a function $V^* \in \mathcal{M}(\mathbb{R}^{2d})$ such that $V(\nu_1, \nu_2) \leq V^*(\nu_1, \nu_2)$ for all $(\nu_1, \nu_2) \in \mathbb{Z}^d \times \mathbb{Z}^d$ and the restriction of $V^*$ to $\mathbb{Z}^d \times \mathbb{Z}^d$ belongs to $\mathcal{B}(\mathbb{Z}^d \times \mathbb{Z}^d)$.

**Proof.** Suppose $V \in \mathcal{B}(\mathbb{Z}^d \times \mathbb{Z}^d)$ and suppose the inequality (1.4) holds. We may assume $V$ is not identically equal to 0. By translation of variables, we see that the inequality

\begin{equation}
\sum_{\nu_1, \nu_2 \in \mathbb{Z}^d} V(\nu_1 - \mu_1, \nu_2 - \mu_2)A(\nu_1 + \nu_2)B(\nu_1)C(\nu_2) \leq c\|A\|_{\ell^2(\mathbb{Z}^d)}\|B\|_{\ell^2(\mathbb{Z}^d)}\|C\|_{\ell^2(\mathbb{Z}^d)}
\end{equation}

holds for all $(\mu_1, \mu_2)$.
holds for all $(\mu_1, \mu_2) \in \mathbb{Z}^d \times \mathbb{Z}^d$ with the same constant $c$ as in (1.4). Take a number $N > 2d$. Multiplying (3.18) by $\langle (\mu_1, \mu_2) \rangle^{-N}$ and taking sum over $(\mu_1, \mu_2) \in \mathbb{Z}^d \times \mathbb{Z}^d$, we see that the function

$$G(\nu_1, \nu_2) = \sum_{\mu_1, \mu_2 \in \mathbb{Z}^d} V(\nu_1 - \mu_1, \nu_2 - \mu_2) \langle (\mu_1, \mu_2) \rangle^{-N}$$

also belongs to the class $\mathcal{B}(\mathbb{Z}^d \times \mathbb{Z}^d)$. We shall show that the function

$$V^*(\xi_1, \xi_2) = \left( \sum_{\mu_1, \mu_2 \in \mathbb{Z}^d} V(\mu_1, \mu_2)^2 \langle (\xi_1 - \mu_1, \xi_2 - \mu_2) \rangle^{-2N} \right)^{1/2}, \quad (\xi_1, \xi_2) \in \mathbb{R}^d \times \mathbb{R}^d,$

has the desired properties. First, $V^*$ is a positive continuous function on $\mathbb{R}^{2d}$. For $N' > 2N$, we have

$$\int_{\mathbb{R}^{2d}} V^*(\xi_1, \xi_2)^2 \langle (\eta_1 - \xi_1, \eta_2 - \xi_2) \rangle^{-N'} d\xi_1 d\xi_2$$

$$= \int_{\mathbb{R}^{2d}} \sum_{\mu_1, \mu_2 \in \mathbb{Z}^d} V(\mu_1, \mu_2)^2 \langle (\xi_1 - \mu_1, \xi_2 - \mu_2) \rangle^{-2N} \langle (\eta_1 - \xi_1, \eta_2 - \xi_2) \rangle^{-N'} d\xi_1 d\xi_2$$

$$\leq \sum_{\mu_1, \mu_2 \in \mathbb{Z}^d} V(\mu_1, \mu_2)^2 \langle (\xi_1 - \mu_1, \eta_2 - \xi_2) \rangle^{-2N} = V^*(\eta_1, \eta_2)^2.$$

Hence $V^* \in \mathcal{M}(\mathbb{R}^{2d})$. Obviously $V^*(\nu_1, \nu_2) \geq V(\nu_1, \nu_2)$. Finally, since $V^*(\nu_1, \nu_2) \leq G(\nu_1, \nu_2)$ (because $\| \cdot \|_{\mathcal{E}} \leq \| \cdot \|_{\ell_1}$) and since $G \in \mathcal{B}(\mathbb{Z}^d \times \mathbb{Z}^d)$, the restriction of $V^*$ to $\mathbb{Z}^d \times \mathbb{Z}^d$ also belongs to $\mathcal{B}(\mathbb{Z}^d \times \mathbb{Z}^d)$.

4. Main results

4.1. Key proposition. Proposition 4.1 to be given below plays a crucial role in our argument. In fact, it already contains the essential part of Theorem 1.3 (2) and Theorems 4.3 and 4.3 that will be given in Subsections 4.2 and 4.3. The basic idea of the arguments of Subsections 4.1, 4.3 goes back to Boulkhemair [5, Theorem 5].

**Proposition 4.1.** Let $W \in \mathcal{M}(\mathbb{R}^{2n})$ and suppose the restriction of $W$ to $\mathbb{Z}^n \times \mathbb{Z}^n$ belongs to the class $\mathcal{B}(\mathbb{Z}^n \times \mathbb{Z}^n)$. For $j = 1, \ldots, n$, let $R_{0,j}, R_{1,j}, R_{2,j} \in [1, \infty)$, $1 \leq r_j \leq 2$, $2 \leq p_{1,j}, p_{2,j} \leq \infty$, and $1/r_j = 1/p_{1,j} + 1/p_{2,j}$. Suppose $\sigma$ is a bounded continuous function on $(\mathbb{R}^n)^3$ such that supp $\mathcal{F}\sigma \subset \prod_{i=0}^n(\prod_{j=1}^n[-R_{0,j}, R_{1,j}])$. Then

$$\left| \int_{\mathbb{R}^n} T_\sigma(f_1, f_2)(x)g(x) \, dx \right|$$

$$\leq \left( \prod_{j=1}^n R_{0,j}^{1/2} R_{1,j}^{1/p_{1,j}} R_{2,j}^{1/p_{2,j}} \right) \|W(\xi_1, \xi_2)^{-1}\sigma(x, \xi_1, \xi_2)\|_{L^2((\mathbb{R}^n)^3)}$$

$$\times \|f_1\|_{L^2} \|f_2\|_{L^2} \|g\|_{L^2(\mathbb{R}^n)^n}.$$

**Proof.** We rewrite the integral on the left hand side of (4.1). Take a function $\kappa \in \mathcal{S}(\mathbb{R})$ such that $\widehat{\kappa} = 1$ on $[-1, 1]$ and define the functions $\theta_i$, $i = 0, 1, 2$, by

$$\theta_i(\xi_1, \ldots, \zeta_n) = R_{i,1} \cdots R_{i,n} \kappa(R_{i,1} \xi_1) \cdots \kappa(R_{i,n} \zeta_n), \quad (\xi_1, \ldots, \zeta_n) \in \mathbb{R}^n,$$
Then \( \hat{\theta}_0 \otimes \hat{\theta}_1 \otimes \hat{\theta}_2 = 1 \) on \( \text{supp} \mathcal{F} \sigma \) and hence \( \sigma \) can be written as
\[
\sigma(x, \xi_1, \xi_2) = \int_{\mathbb{R}^3} \sigma(y, \eta_1, \eta_2) \hat{\theta}_0(x - y) \hat{\theta}_1(\xi_1 - \eta_1) \hat{\theta}_2(\xi_2 - \eta_2) \, dy \, d\eta_1 \, d\eta_2.
\]
Thus the integral on the left hand side of (4.1) is written as
\[
I := (2\pi)^2 \int_{\mathbb{R}^n} T_\sigma(f_1, f_2)(x) g(x) \, dx
\]
(4.2)
\[
= \int_{(\mathbb{R}^n)^6} e^{ix(\xi_1 + \xi_2)} \sigma(y, \eta_1, \eta_2) \times \theta_0(x - y)g(x)\theta_1(\xi_1 - \eta_1)\hat{f}_1(\xi_1)\theta_2(\xi_2 - \eta_2)\hat{f}_2(\xi_2) \, dX,
\]
where \( dX = dx \, d\xi_1 \, d\xi_2 \, dy \, d\eta_1 \, d\eta_2 \).

Recall that \( Q = \{-1/2, 1/2\}^n \) is the \( n \)-dimensional unit cube. Since \( \mathbb{R}^n \) is a disjoint union of the cubes \( \tau + Q, \tau \in \mathbb{Z}^n \), integral of a function on \( \mathbb{R}^n \) can be written as
\[
\int_{\mathbb{R}^n} F(x) \, dx = \sum_{\tau \in \mathbb{Z}^n} \int_Q F(x + \tau) \, dx.
\]

By using this formula, we rewrite the integral in (4.2) as
\[
I = \sum_{\nu, \mu \in (\mathbb{Z}^n)^3} \int_{Q^6} e^{i(x + \nu_0)(\xi_1 + \nu_1 + \xi_2 + \nu_2)} \sigma(y + \mu_0, \eta_1 + \mu_1, \eta_2 + \mu_2) \times \theta_0(x + \nu_0 - y - \mu_0)g(x + \nu_0) \times \theta_1(\xi_1 + \nu_1 - \eta_1 - \mu_1)\hat{f}_1(\xi_1 + \nu_1) \times \theta_2(\xi_2 + \nu_2 - \eta_2 - \mu_2)\hat{f}_2(\xi_2 + \nu_2) \, dX,
\]
where \( \nu = (\nu_0, \nu_1, \nu_2), \mu = (\mu_0, \mu_1, \mu_2) \in (\mathbb{Z}^n)^3 \).

We rewrite the exponential term as
\[
e^{i(x + \nu_0)(\xi_1 + \nu_1 + \xi_2 + \nu_2)} = e^{i(\nu_1 + \nu_2)x} e^{i\nu_0 \xi_1} e^{i\mu_0 \xi_2} e^{i\nu_0(\nu_1 + \nu_2)} \sum_{\alpha = \beta + \gamma} \frac{i^{\lvert \alpha \rvert}}{\beta! \gamma!} x^\alpha \xi_1^\beta \xi_2^\gamma.
\]

Now the variables \( x, \xi_1, \xi_2 \) are separated and \( I \) is written as
\[
I = \sum_{\nu, \mu \in (\mathbb{Z}^n)^3} \sum_{\alpha = \beta + \gamma} \frac{i^{\lvert \alpha \rvert}}{\beta! \gamma!} e^{i\nu_0(\nu_1 + \nu_2)} \int_{Q^3} \sigma(y + \mu_0, \eta_1 + \mu_1, \eta_2 + \mu_2) \times \left( \int_Q e^{i(\nu_1 + \nu_2)x} \theta_0(x + \nu_0 - y - \mu_0)g(x + \nu_0) \, dx \right) \times \left( \int_Q e^{i\nu_0 \xi_1} \theta_1(\xi_1 + \nu_1 - \eta_1 - \mu_1)\hat{f}_1(\xi_1 + \nu_1) \, d\xi_1 \right) \times \left( \int_Q e^{i\nu_0 \xi_2} \theta_2(\xi_2 + \nu_2 - \eta_2 - \mu_2)\hat{f}_2(\xi_2 + \nu_2) \, d\xi_2 \right) \, dy \, d\eta_1 \, d\eta_2.
\]

We take a sufficiently large even positive integer \( N \). Then, since \( \langle z \rangle^N \) is a polynomial of \( z \) of order \( N \), we can write
\[
(4.3) \quad \langle \nu_0 - \mu_0 \rangle^N = \sum_{|\alpha_1 + \alpha_2 + \alpha_3| \leq N} C_{\alpha_1, \alpha_2, \alpha_3} (x + \nu_0 - y - \mu_0)^{\alpha_1} x^{\alpha_2} y^{\alpha_3}
\]
and hence

\[
\theta_0(x + \nu_0 - y - \mu_0) \\
= (\nu_0 - \mu_0)^{-N} \theta_0(x + \nu_0 - y - \mu_0) \sum_{|\alpha_1 + \alpha_2 + \alpha_3| \leq N} C_{\alpha_1, \alpha_2, \alpha_3}(x + \nu_0 - y - \mu_0)^{\alpha_1} x^{\alpha_2} y^{\alpha_3}
\]

\[
= (\nu_0 - \mu_0)^{-N} \sum_{|\alpha_1 + \alpha_2 + \alpha_3| \leq N} C_{\alpha_1, \alpha_2, \alpha_3} \tilde{\theta}_0^{\alpha_1}(x + \nu_0 - y - \mu_0)^{\alpha_1} x^{\alpha_2} y^{\alpha_3},
\]

where \(\tilde{\theta}_0^{\alpha_1}(z) = \theta_0(z)z^{\alpha_1}\). We also rewrite the \(\theta_1(\ldots)\) and \(\theta_2(\ldots)\) in the same way. Thus we obtain

\[
I = \sum_{\alpha = \beta + \gamma} \sum_{|\alpha_1 + \alpha_2 + \alpha_3| \leq N} \sum_{|\beta_1 + \beta_2 + \beta_3| \leq N} \sum_{|\gamma_1 + \gamma_2 + \gamma_3| \leq N} \frac{i^{[\alpha]} C_{\alpha, \beta, \gamma}}{[\beta!]^2} e^{i\nu_0 (\nu_1 + \nu_2)}
\]

\[
\times \sum_{\nu, \mu} \int_{Q^3} \sigma(y + \mu_0, \eta_1 + \mu_1, \eta_2 + \mu_2)^y^{\alpha_3} \eta_1^{\beta_1} \eta_2^{\gamma_3}
\]

\[
\times \langle \nu_0 - \mu_0 \rangle^{-N} \langle \nu_1 - \mu_1 \rangle^{-N} \langle \nu_2 - \mu_2 \rangle^{-N}
\]

\[
\times \left( \int_Q e^{i(\nu_1 + \nu_2)x} \theta_0^{\alpha_1}(x + \nu_0 - y - \mu_0)g(x + \nu_0)x^{\alpha_2} dx \right)
\]

\[
\times \left( \int_Q e^{i\nu_0 \xi_1} \theta_1^{\beta_1}(\xi_1 + \nu_1 - \eta_1 - \mu_1)h_1(\xi_1 + \nu_1)\xi_1^{\beta_2} d\xi_1 \right)
\]

\[
\times \left( \int_Q e^{i\nu_0 \xi_2} \theta_2^{\gamma_1}(\xi_2 + \nu_2 - \eta_2 - \mu_2)f_2(\xi_2 + \nu_2)\xi_2^{\gamma_2} d\xi_2 \right) dyd\eta_1d\eta_2
\]

\[
= \sum_{\alpha, \beta, \gamma} \frac{i^{[\alpha]} C_{\alpha, \beta, \gamma}}{[\beta!]^2} e^{i\nu_0 (\nu_1 + \nu_2)} I_{\alpha, \beta, \gamma},
\]

where \(C_{\alpha, \beta, \gamma} = C_{\alpha_1, \alpha_2, \alpha_3} C_{\beta_1, \beta_2, \beta_3} C_{\gamma_1, \gamma_2, \gamma_3}\) is the product of the constants in (4.3), \(I_{\alpha, \beta, \gamma}\) denotes the part \(\sum_{\nu, \mu} \int_{Q^3} \ldots dyd\eta_1d\eta_2\) of the formula, and \(\alpha = (\alpha_1, \alpha_2, \alpha_3), \beta = (\beta_1, \beta_2, \beta_3), \gamma = (\gamma_1, \gamma_2, \gamma_3)\).

Now we shall estimate \(I\). Notice that in the last expression of \(I\), the sums over \(\alpha_i, \beta_i, \gamma_i\) are taken over finite sets and the sum over \(\alpha, \beta, \gamma \in (\mathbb{N}_0)^n\), \(\alpha = \beta + \gamma\), has the factor \(1/([\beta!]^2)\). Hence, in order to prove the estimate for \(I\), it is sufficient to show that \(I_{\alpha, \beta, \gamma}\) is bounded by the right hand side of (4.1) uniformly in \(\alpha, \beta, \gamma \in (\mathbb{N}_0)^n\).
Using the obvious estimate $|y^{\alpha_1} \eta_1^{\beta_1} \eta_2^{\gamma_1}| \leq 1$ for $y, \eta_1, \eta_2 \in Q$ and using the Cauchy–Schwarz inequality with respect to the integral over $y, \eta_1, \eta_2$, we obtain

\[
|I_{\alpha,\beta,\gamma}| \leq \sum_{\nu,\mu \in \mathbb{N}^3} W(\mu_1, \mu_2)^{-1} \|\sigma(y + \mu_0, \eta_1 + \mu_1, \eta_2 + \mu_2)\|_{L^2_{\nu_1, \nu_2}(Q)}^3
\times W(\mu_1, \mu_2)(\nu_0 - \mu_0)^{-N}(\nu_1 - \mu_1)^{-N}(\nu_2 - \mu_2)^{-N}
\times \left\| \int_Q e^{i(\nu_1 + \nu_2) \cdot x} \tilde{\theta}_0^{\alpha_1}(x + \nu_0 - y - \mu_0) g(x + \nu_0) x^{\alpha_1 + \alpha_2} \, dx \right\|_{L^2_{\nu_1}(Q)}
\times \left\| \int_Q e^{i\nu_0 \xi_1 \tilde{\theta}_1^{\beta_1}(\xi_1 + \nu_1 - \eta_1 - \mu_1) \hat{f}_1(\xi_1 + \nu_1) \xi_1^{\beta_1 + \beta_2} \, d\xi_1 \right\|_{L^2_{\eta_1}(Q)}
\times \left\| \int_Q e^{i\nu_0 \xi_2 \tilde{\theta}_2^{\gamma_1}(\xi_2 + \nu_2 - \eta_2 - \mu_2) \hat{f}_2(\xi_2 + \nu_2) \xi_2^{\gamma_1 + \gamma_2} \, d\xi_2 \right\|_{L^2_{\eta_2}(Q)}.
\]

By virtue of the properties of the moderate function $W$ as given in Proposition 3.8 (2) and (3), we have

\[
\sup_{\mu_0, \mu_1, \mu_2} \left\{ W(\mu_1, \mu_2)^{-1} \|\sigma(y + \mu_0, \eta_1 + \mu_1, \eta_2 + \mu_2)\|_{L^2_{\nu_1, \nu_2}(Q)} \right\} \approx \|W(\xi_1, \xi_2)^{-1} \sigma(x, \xi_1, \xi_2)\|_{L^2_{\nu_1}(\mathbb{R}^n)}
\]

and

\[
\|W(\mu_1, \mu_2)(\nu_0 - \mu_0)^{-N}(\nu_1 - \mu_1)^{-N}(\nu_2 - \mu_2)^{-N}\|_{L^2_{\nu_0, \nu_1, \nu_2}} \approx W(\nu_1, \nu_2)
\]

if $N$ is chosen sufficiently large. Hence, applying the Cauchy–Schwarz inequality to the sum over $\mu = (\mu_0, \mu_1, \mu_2)$ in (4.4), and using (4.5) and (4.6), we obtain

\[
|I_{\alpha,\beta,\gamma}| \leq \|W(\xi_1, \xi_2)^{-1} \sigma(x, \xi_1, \xi_2)\|_{L^2_{\nu_1}(\mathbb{R}^n)} \sum_{\nu_0, \nu_1, \nu_2 \in \mathbb{Z}^n} W(\nu_1, \nu_2)
\times \left\| \int_Q e^{i(\nu_1 + \nu_2) \cdot x} \tilde{\theta}_0^{\alpha_1}(x + \nu_0 - y - \mu_0) g(x + \nu_0) x^{\alpha_1 + \alpha_2} \, dx \right\|_{L^2_{\nu_1}(Q)}
\times \left\| \int_Q e^{i\nu_0 \xi_1 \tilde{\theta}_1^{\beta_1}(\xi_1 + \nu_1 - \eta_1 - \mu_1) \hat{f}_1(\xi_1 + \nu_1) \xi_1^{\beta_1 + \beta_2} \, d\xi_1 \right\|_{L^2_{\eta_1}(Q)}
\times \left\| \int_Q e^{i\nu_0 \xi_2 \tilde{\theta}_2^{\gamma_1}(\xi_2 + \nu_2 - \eta_2 - \mu_2) \hat{f}_2(\xi_2 + \nu_2) \xi_2^{\gamma_1 + \gamma_2} \, d\xi_2 \right\|_{L^2_{\eta_2}(Q)}.
\]

In what follows, we will simply write

\[
A_\alpha(\nu_0, \nu_0) = \left\| \int_Q e^{i\nu_0 \cdot x} \tilde{\theta}_0^{\alpha_1}(x + \nu_0 - y - \mu_0) g(x + \nu_0) x^{\alpha_1 + \alpha_2} \, dx \right\|_{L^2_{\nu_1}(Q)}
\]

\[
B_\beta(\nu_0, \nu_1) = \left\| \int_Q e^{i\nu_0 \xi_1 \tilde{\theta}_1^{\beta_1}(\xi_1 + \nu_1 - \eta_1 - \mu_1) \hat{f}_1(\xi_1 + \nu_1) \xi_1^{\beta_1 + \beta_2} \, d\xi_1 \right\|_{L^2_{\eta_1}(Q)}
\]

\[
C_\gamma(\nu_0, \nu_2) = \left\| \int_Q e^{i\nu_0 \xi_2 \tilde{\theta}_2^{\gamma_1}(\xi_2 + \nu_2 - \eta_2 - \mu_2) \hat{f}_2(\xi_2 + \nu_2) \xi_2^{\gamma_1 + \gamma_2} \, d\xi_2 \right\|_{L^2_{\eta_2}(Q)}.
\]
Thus the inequality (4.7) is written as

\begin{equation}
|I_{\alpha,\beta,\gamma}| \lesssim \|W(\xi_1, \xi_2)^{-1}\sigma(x, \xi_1, \xi_2)\|_{L^2_\nu((\mathbb{R}^n)^2)} \|I_{\alpha,\beta,\gamma}|
\end{equation}

with

\begin{equation}
I_{\alpha,\beta,\gamma} = \sum_{\nu_0, \nu_1, \nu_2 \in \mathbb{Z}^n} W(\nu_1, \nu_2) A_\alpha(\nu_1 + \nu_2, \nu_0) B_\beta(\nu_0, \nu_1) C_\gamma(\nu_0, \nu_2).
\end{equation}

We shall estimate \(I_{\alpha,\beta,\gamma}\).

To the sum over \(\nu_1, \nu_2\) in (4.9), we apply the \(\ell^2\) estimate assured by our assumption that \(W\) restricted to \(\mathbb{Z}^n \times \mathbb{Z}^n\) belongs to the class \(B(\mathbb{Z}^n \times \mathbb{Z}^n)\) to obtain

\[I_{\alpha,\beta,\gamma} \lesssim \sum_{\nu_0 \in \mathbb{Z}^n} \|A_\alpha(\nu_3, \nu_0)\|_{\ell^2_{\nu_3}} \|B_\beta(\nu_0, \nu_1)\|_{\ell^2_{\nu_0}} \|C_\gamma(\nu_0, \nu_2)\|_{\ell^2_{\nu_2}}.
\]

To estimate the sum over \(\nu_0 = (\nu_{0,1}, \ldots, \nu_{0,n}) \in \mathbb{Z}^n\), we use the Hölder inequality with the exponents \(1 = 1/r_j + 1/p_{1,j} + 1/p_{2,j}\). Thus we have

\begin{equation}
I_{\alpha,\beta,\gamma} \lesssim \|A_\alpha(\nu_3, \nu_0)\|_{\ell^2_{\nu_3}} \|B_\beta(\nu_0, \nu_1)\|_{\ell^2_{\nu_0}} \|C_\gamma(\nu_0, \nu_2)\|_{\ell^2_{\nu_0}}.
\end{equation}

The norm of \(A_\alpha\) in (4.10) is estimated by the use of the Parseval identity in \(\ell^2_{\nu_3}\) as follows:

\[\|A_\alpha(\nu_3, \nu_0)\|_{\ell^2_{\nu_3}} \|\tilde{\theta}_0^\alpha(\nu_0)\|_{L^2_\nu((\mathbb{R}^n)^n)} = \|g\|_{L^2_\nu((\mathbb{R}^n)^n)},\]

where the inequality \(\leq\) on the fourth line holds because \(|x^{\alpha_1 + \alpha_2}| \leq 1\) for \(x \in Q\) and \(\|F(\mu_0)\|_{L^2_\nu((\mathbb{R}^n)^n)} \leq \|F\|_{L^2_\nu((\mathbb{R}^n)^n)}\). Recall that \(\tilde{\theta}_0^{\alpha_1}(y)\) is defined by

\begin{equation}
\tilde{\theta}_0^{\alpha_1}(y) = R_0^{1-\alpha_1_1} \cdots R_0^{1-\alpha_1_n} \kappa(R_0, y_1) \cdots \kappa(R_0, y_n) R_0^{\alpha_1_1} \cdots R_0^{\alpha_1_n}.
\end{equation}

Thus, since a function of the form \(\kappa(z)z^n\) belongs to the Schwartz class \(S(\mathbb{R})\) and since \(R_{0,j} \geq 1\), we have

\begin{equation}
\|\tilde{\theta}_0^{\alpha_1}\|_{L^2_\nu((\mathbb{R}^n)^n)} \approx \prod_{j=1}^n R_0^{1/2-\alpha_1,j} \leq \prod_{j=1}^n R_0^{1/2}.
\end{equation}

Therefore

\begin{equation}
\|A_\alpha(\nu_3, \nu_0)\|_{\ell^2_{\nu_3}} \|\tilde{\theta}_0^{\alpha_1}\|_{L^2_\nu((\mathbb{R}^n)^n)} \lesssim \left(\prod_{j=1}^n R_0^{1/2}\right) \|g\|_{L^2_\nu((\mathbb{R}^n)^n)}.
\end{equation}
For the norm of $B_\beta$ in (4.10), we use (2.1), the Hausdorff–Young inequality for $\ell^{p_1,1}_{\nu_0,1}$, and the inequality $\| \cdot \|_{p_1} \leq \| \cdot \|_{p_1,1}$ to obtain

$$
\|B_\beta(v_0, \nu_1)\|_{\ell^{p_1,1}_{\nu_0,1} \cdots \ell^{p_1,n}_{\nu_0,n}} \\
= \left\| \int_{Q} e^{ip_1,1\xi_1 \theta_1^\beta (\xi_1 + \nu_1 - \eta_1 - \mu_1)} \hat{f}_1(\xi_1 + \nu_1) \xi_1^{\beta + \beta_2} d\xi_1 \right\|_{L^{p_1,1}_{\beta_1}(Q)\ell^{p_1,2}_{\nu_1} \ell^{p_1,3}_{\nu_2} \cdots \ell^{p_1,n}_{\nu_0,n}} \\
\leq \| \cdots \|_{\ell^{p_1,1}_{\nu_0,1} \ell^{p_1,2}_{\nu_1} \ell^{p_1,3}_{\nu_2} \cdots \ell^{p_1,n}_{\nu_0,n}} \\
\leq \left\| \int_{Q^{n-1}} e^{i(v_0,2, \ldots ,v_0,n) \cdot (\xi_1, \ldots ,\xi_1,n)} \theta_1^\beta (\xi_1 + \nu_1 - \eta_1 - \mu_1) \xi_1^{\beta + \beta_2} d\xi_1,2 \cdots d\xi_1,n \right\|_{L^{p_1,1}(I)L^{p_1,2}(Q)\ell^{p_1,2}_{\nu_1} \ell^{p_1,3}_{\nu_2} \cdots \ell^{p_1,n}_{\nu_0,n}} \\
\leq \| \cdots \|_{L^{p_1,1}(I)\ell^{p_1,2}_{\nu_1} \ell^{p_1,3}_{\nu_2} \cdots \ell^{p_1,n}_{\nu_0,n}}
$$

where $I = [-1/2, 1/2]$. We then repeat the same arguments for $\ell^{p_1,2}_{\nu_0,2}, \ldots, \ell^{p_1,n}_{\nu_0,n}$ in this order to obtain

$$
\|B_\beta(v_0, \nu_1)\|_{\ell^{p_1,1}_{\nu_0,1} \cdots \ell^{p_1,n}_{\nu_0,n}} \\
\leq \left\| \int_{Q^{n-1}} e^{i(v_0,2, \ldots ,v_0,n) \cdot (\xi_1, \ldots ,\xi_1,n)} \theta_1^\beta (\xi_1 + \nu_1 - \eta_1 - \mu_1) \xi_1^{\beta + \beta_2} d\xi_1,2 \cdots d\xi_1,n \right\|_{L^{p_1,1}(I)L^{p_1,2}(Q)\ell^{p_1,2}_{\nu_1} \ell^{p_1,3}_{\nu_2} \cdots \ell^{p_1,n}_{\nu_0,n}} \\
\leq \| \cdots \|_{L^{p_1,1}(I)\ell^{p_1,2}_{\nu_1} \ell^{p_1,3}_{\nu_2} \cdots \ell^{p_1,n}_{\nu_0,n}}.
$$

Changing variables $\xi_1,j \rightarrow \xi_1,j + \eta_1,j$ for $j = 1, \ldots, n$, and using (2.1), we have

$$
\left\| \theta_1^\beta (\xi_1 - \eta_1) \hat{f}_1(\xi_1) \right\|_{L^{p_1,n}_{\xi_1,n}(\mathbb{R}) \cdots L^{p_1,1}_{\xi_1,1}(\mathbb{R}) L^{p_1,n}_{\xi_1,n}(\mathbb{R})} \\
= \left\| \theta_1^\beta (\xi_1 + \eta_1) \hat{f}_1(\xi_1 + \eta_1) \right\|_{L^{p_1,n}_{\xi_1,n}(\mathbb{R}) \cdots L^{p_1,1}_{\xi_1,1}(\mathbb{R}) L^{p_1,n}_{\xi_1,n}(\mathbb{R})} \\
\leq \left\| \theta_1^\beta (\xi_1) \hat{f}_1(\xi_1 + \eta_1) \right\|_{L^{p_1,n}_{\xi_1,n}(\mathbb{R}) \cdots L^{p_1,1}_{\xi_1,1}(\mathbb{R}) L^{p_1,n}_{\xi_1,n}(\mathbb{R})} \\
= \left\| \theta_1^\beta (\xi_1) \right\|_{L^{p_1,n}_{\xi_1,n}(\mathbb{R}) \cdots L^{p_1,1}_{\xi_1,1}(\mathbb{R})} \left\| \hat{f}_1 \right\|_{L^{2}}.
$$

For the mixed norm of $\theta_1^\beta$ in the last expression, by the same reason as we deduced (4.12) from (4.11), we have

$$
\left\| \theta_1^\beta (\xi_1) \right\|_{L^{p_1,n}_{\xi_1,n}(\mathbb{R}) \cdots L^{p_1,1}_{\xi_1,1}(\mathbb{R})} \leq \prod_{j=1}^{n} R_{1,j}^{1/p_1,j} \leq \prod_{j=1}^{n} R_{1,j}^{1/p_1,j}.
$$

Also $\| \hat{f}_1 \|_{L^2} \approx \| f_1 \|_{L^2}$ by Plancherel’s theorem. Now combining the inequalities obtained above, we get

$$
(4.14) \quad \|B_\beta(v_0, \nu_1)\|_{\ell^{p_1,1}_{\nu_0,1} \cdots \ell^{p_1,n}_{\nu_0,n}} \leq \left( \prod_{j=1}^{n} R_{1,j}^{1/p_1,j} \right) \| f_1 \|_{L^2}.
$$
Similarly, we have

\[\|C_\gamma(v_0, v_2)\|_{L^2(\mathbb{R}^n)^{p_0^1, \ldots, p_0^m}} \lesssim \left( \prod_{j=1}^n P_{2,j}^{1/p_2,j} \right) \|f_2\|_{L^2}.\]

The desired inequality (4.1) now follows from (4.8), (4.10), (4.13), (4.14), and (4.15). This completes the proof of Proposition 4.1.

4.2. A theorem for symbols with limited smoothness. From Proposition 4.1, we shall deduce a theorem concerning bilinear pseudo-differential operators \(T_\sigma\) with symbols of limited smoothness. To measure the smoothness of such symbols, we shall use Besov type norms. To define the Besov type norms, we use the partition of unity given as follows. Let \(d \in \mathbb{N}\). Take a \(\phi \in \mathcal{S}(\mathbb{R}^d)\) such that \(\phi(\xi) = 1\) for \(|\xi| \leq 1\) and \(\text{supp} \phi \subset \{\xi \in \mathbb{R}^d : |\xi| \leq 2\}\). We put \(\psi(\xi) = \phi(\xi) - \phi(2\xi)\). Then \(\text{supp} \psi \subset \{\xi \in \mathbb{R}^d : 1/2 \leq |\xi| \leq 2\}\). We set \(v_0 = \phi\) and \(v_k = \psi(\cdot/2^k)\) for \(k \in \mathbb{N}\). Then \(\sum_{k=0}^\infty \psi_k(\xi) = 1\) for all \(\xi \in \mathbb{R}^d\). We shall call \(\{\psi_k\}_{k \in \mathbb{N}_0}\) a Littlewood–Paley partition of unity on \(\mathbb{R}^d\). It is easy to see that the Besov type norms given in the following definition do not depend, up to the equivalence of norms, on the choice of Littlewood–Paley partition of unity.

**Definition 4.2.** Let \(W \in \mathcal{M}(\mathbb{R}^{2n})\). Let \(\{\psi_k\}_{k \in \mathbb{N}_0}\) be a Littlewood–Paley partition of unity on \(\mathbb{R}\). For

\[s = (s_0, 1, \ldots, s_{0,n}, s_{1,1}, \ldots, s_{n,1}, s_{2,1}, \ldots, s_{2,n}) \in [0, \infty)^{3n},\]

\[k = (k_0, 1, \ldots, k_{0,n}, k_{1,1}, \ldots, k_{1,n}, k_{2,1}, \ldots, k_{2,n}) \in (\mathbb{N}_0)^{3n},\]

and \(\sigma = \sigma(x, \xi_1, \xi_2) \in L^\infty(\mathbb{R}_x^n \times \mathbb{R}_{\xi_1}^n \times \mathbb{R}_{\xi_2}^n)\), we write \(s \cdot k = \sum_{i=0}^2 \sum_{j=1}^n s_i k_i j\) and

\[\Delta_k \sigma(x, \xi_1, \xi_2) = \left( \prod_{j=1}^n \psi_{k_0,j}(D_{x_j}) \psi_{k_{1,j}}(D_{\xi_1,j}) \psi_{k_{2,j}}(D_{\xi_2,j}) \right) \sigma(x, \xi_1, \xi_2).
\]

We denote by \(BS_{0,0}^W(s; \mathbb{R}^n)\) the set of all \(\sigma \in L^\infty((\mathbb{R}^n)^3)\) for which the following norm is finite:

\[\|\sigma\|_{BS_{0,0}^W(s; \mathbb{R}^n)} = \sum_{k \in (\mathbb{N}_0)^{3n}} 2^{sk} \|W(\xi_1, \xi_2)^{-1} \Delta_k \sigma(x, \xi_1, \xi_2)\|_{L^2(\mathbb{R}^n, x, \xi_1, \xi_2)}.
\]

In terms of these notations, the theorem reads as follows.

**Theorem 4.3.** Let \(W \in \mathcal{M}(\mathbb{R}^{2n})\) and suppose the restriction of \(W\) to \(\mathbb{Z}^n \times \mathbb{Z}^n\) belongs to the class \(\mathcal{B}(\mathbb{Z}^n \times \mathbb{Z}^n)\). Let \(r_j \in [1, 2], j = 1, \ldots, n\). Then the bilinear pseudo-differential operator \(T_\sigma\) is bounded from \(L^2(\mathbb{R}_x^n) \times L^2(\mathbb{R}_y^n)\) to the amalgam space \((L^2, l^r, \ldots, l^r)(\mathbb{R}^n)\) if \(\sigma \in BS_{0,0}^W(s; \mathbb{R}^n)\) with \(s = (s_{ij})_{i,j}\) satisfying

\[s_{0,j} = 1/2, \quad s_{1,j} + s_{2,j} \geq 1/r_j - 1/2, \quad s_{1,j} + s_{2,j} = 1/r_j\]

for \(j = 1, \ldots, n\). If in addition \(r_1 = \cdots = r_n = r \in [1, 2]\), then \(T_\sigma\) is bounded from \(L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)\) to \(L^r(\mathbb{R}^n)\) when \(1 < r \leq 2\) or to \(h^1(\mathbb{R}^n)\) when \(r = 1\).

**Proof.** The assertion concerning the boundedness to \(L^r\) or to \(h^1\) directly follows from the assertion for the amalgam space with the aid of the embeddings \((L^2, l^r) \hookrightarrow L^r\) for \(1 < r \leq 2\) and \((L^2, l^1) \hookrightarrow h^1\).
The boundedness to the amalgam space follows from Proposition 4.1. We decompose the symbol $\sigma$ by using the Littlewood–Paley partition:

$$\sigma(x, \xi_1, \xi_2) = \sum_{k \in (\mathbb{N}_0)^3n} \Delta_k \sigma(x, \xi_1, \xi_2)$$

Then the support of $\mathcal{F}(\Delta_k \sigma)$ is included in $\prod_{i=0}^{2} \prod_{j=1}^{n} [-R_{i,j}, R_{i,j}]$ with $R_{i,j} = 2^{k_{i,j}+1}$. Take $p_{1,j}, p_{2,j}$ such that $1/r_j - 1/2 \leq 1/p_{1,j}, 1/p_{2,j} \leq 1/2$ and $1/p_{1,j} = 1/p_{2,j} = 1/r_j$ for $j = 1, \ldots, n$. Then Proposition 4.1 and the duality between amalgam spaces yield

$$\|T_{\Delta_k \sigma}\|_{L^2 \times L^2 \to (L^2, \ell'^1 \ldots \ell'^n)} \lesssim \left( \prod_{j=1}^{n} (2^{k_{0,j}})^{1/2} (2^{k_{1,j}})^{1/p_{1,j}} (2^{k_{2,j}})^{1/p_{2,j}} \right) \|W(\xi_1, \xi_2)^{-1} \Delta_k \sigma(x, \xi_1, \xi_2)\|_{L^2_{ul}}.$$

Taking sum over $k \in (\mathbb{N}_0)^3n$, we obtain

$$\|T\|_{L^2 \times L^2 \to (L^2, \ell'^1 \ldots \ell'^n)} \leq \sum_{k \in (\mathbb{N}_0)^3n} \|T_{\Delta_k \sigma}\|_{L^2 \times L^2 \to (L^2, \ell'^1 \ldots \ell'^n)} \lesssim \sum_{k \in (\mathbb{N}_0)^3n} \left( \prod_{j=1}^{n} (2^{k_{0,j}})^{1/2} (2^{k_{1,j}})^{1/p_{1,j}} (2^{k_{2,j}})^{1/p_{2,j}} \right) \|W(\xi_1, \xi_2)^{-1} \Delta_k \sigma(x, \xi_1, \xi_2)\|_{L^2_{ul}} = \|\sigma\|_{BS^{W_0^*, (s_0, s_1, s_2; \mathbb{R}^n)}}$$

with $s_{0,j} = 1/2, s_{1,j} = 1/p_{1,j}$, and $s_{2,j} = 1/p_{2,j}$, which is the desired result. \hfill \blacksquare

4.3. Another theorem for symbols with limited smoothness. In this subsection, we give a variant of Theorem 4.3. Here to measure the smoothness of symbols, we use different Besov type norms which are defined below. It is easy to see that these Besov type norms also do not depend, up to the equivalence of norms, on the choice of the Littlewood–Paley partition of unity involved in the definition.

**Definition 4.4.** Let $W \in \mathcal{M}(\mathbb{R}^{2n})$. Let $\{\psi^{(n)}_k\}_{k \in \mathbb{N}_0}$ be a Littlewood–Paley partition of unity on $\mathbb{R}^n$ and write

$$\Delta^*_k \sigma(x, \xi_1, \xi_2) = \psi^{(n)}_{k_0}(D_x) \psi^{(n)}_{k_1}(D_{\xi_1}) \psi^{(n)}_{k_2}(D_{\xi_2}) \sigma(x, \xi_1, \xi_2)$$

for $k = (k_0, k_1, k_2) \in (\mathbb{N}_0)^3$. For $s_0, s_1, s_2 \in [0, \infty)$, we denote by $BS^{W_0^*, (s_0, s_1, s_2; \mathbb{R}^n)}$ the set of all $\sigma \in L^\infty((\mathbb{R}^n)^3)$ for which the following norm is finite:

$$\|\sigma\|_{BS^{W_0^*, (s_0, s_1, s_2; \mathbb{R}^n)}} = \sum_{k_0, k_1, k_2 \in \mathbb{N}_0} 2^{s_0 k_0 + s_1 k_1 + s_2 k_2} \|W(\xi_1, \xi_2)^{-1} \Delta^*_k \sigma(x, \xi_1, \xi_2)\|_{L^2_{ul}(\mathbb{R}^{3n})}.$$

The following theorem can be deduced from Proposition 4.1 just in the same way as in Proof of Theorem 4.3. We omit the proof.

**Theorem 4.5.** Let $W \in \mathcal{M}(\mathbb{R}^{2n})$ and suppose the restriction of $W$ to $\mathbb{Z}^n \times \mathbb{Z}^n$ belongs to the class $\mathcal{B}(\mathbb{Z}^n \times \mathbb{Z}^n)$. Let $r \in [1, 2]$. Then the bilinear pseudo-differential operator $T_\sigma$ is bounded from $L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ to the amalgam space $(L^2, \ell^r)(\mathbb{R}^n)$ if $\sigma \in BS^{W_0^*, (s_0, s_1, s_2; \mathbb{R}^n)}$ with $s_0 = n/2, s_1, s_2 \geq n/r - n/2$, and $s_1 + s_2 = n/r$. In particular, under the same assumptions, $T_\sigma$ is bounded from $L^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ to $L^r(\mathbb{R}^n)$ in the case $1 < r \leq 2$ or to $h^1(\mathbb{R}^n)$ in the case $r = 1$. 
Remark 4.6. We should compare Theorems 4.3 and 4.5. In fact, the assertion of
Theorem 4.3 for the case 1 \( \leq r < 2 \) is covered by Theorem 4.5. To see this, we
denote by \( BS^{W}_{0,0}(t_{0})^{n}, (t_{1})^{n}, (t_{2})^{n}; \mathbb{R}^{n} \) the class \( BS^{W}_{0,0}(s; \mathbb{R}^{n}) \) with \( s = (s_{i,j}) \) given by
\[
    s_{0,j} = t_{0}, \quad s_{1,j} = t_{1}, \quad s_{2,j} = t_{2}, \quad j = 1, \ldots, n.
\]

With this special class of symbols, Theorem 4.3 for the case \( r_{1} = \cdots = r_{n} = r \) asserts
that \( T_{\sigma} \) is bounded from \( L^{2}(\mathbb{R}^{n}) \times L^{2}(\mathbb{R}^{n}) \) to the amalgam space \( (L^{2}, \ell^{r})(\mathbb{R}^{n}) \) if \( \sigma \in BS^{W}_{0,0}(t_{0})^{n}, (t_{1})^{n}, (t_{2})^{n}; \mathbb{R}^{n} \) with \( t_{0} = 1/2, t_{1}, t_{2} \geq 1/\lambda - 1/2, \) and \( 1/t_{1} + 1/t_{2} = 1/r \).
This assertion is stronger than Theorem 4.5 in the case \( 1 \leq r < 2 \). This follows
from the fact that the inclusion
\[
    BS^{W}_{0,0}(nt_{0}, nt_{1}, nt_{2}; \mathbb{R}^{n}) \hookrightarrow BS^{W}_{0,0}(t_{0})^{n}, (t_{1})^{n}, (t_{2})^{n}; \mathbb{R}^{n}
\]
holds for \( t_{0}, t_{1}, t_{2} > 0 \). This inclusion, in a slightly different form, is already proved
in [5] Appendix A2 (i). Here we give a brief proof for the reader’s convenience. To prove
(4.16), notice that
\[
    \Delta_{k} \sigma(x, \xi_{1}, \xi_{2}) = \sum_{m} \Delta_{k} \Delta_{m}^{*} \sigma(x, \xi_{1}, \xi_{2})
\]
and that \( \Delta_{k} \Delta_{m}^{*} \neq 0 \) only if
\[
    \max\{k_{1}, \ldots, k_{n}\} - c < m_{i} < \max\{k_{1}, \ldots, k_{n}\} + c, \quad i = 0, 1, 2,
\]
where \( c \) is a constant depending only on \( n \). Using the property of \( W \in \mathcal{M}(\mathbb{R}^{2n}) \)
given in Proposition 5.3 (2), we see that the estimate
\[
    ||W(\xi_{1}, \xi_{2})^{-1} \Delta_{k} \tau(x, \xi_{1}, \xi_{2})||_{L_{ul}^{2}} \lesssim ||W(\xi_{1}, \xi_{2})^{-1} \tau(x, \xi_{1}, \xi_{2})||_{L_{ul}^{2}}
\]
with an implicit constant independent of \( k \) holds for all bounded functions \( \tau \) on
\( (\mathbb{R}^{n})^{3} \). Thus from (4.17) we have
\[
    ||W(\xi_{1}, \xi_{2})^{-1} \Delta_{k} \sigma(x, \xi_{1}, \xi_{2})||_{L_{ul}^{2}} \lesssim \sum_{m} ||W(\xi_{1}, \xi_{2})^{-1} \Delta_{m}^{*} \sigma(x, \xi_{1}, \xi_{2})||_{L_{ul}^{2}}.
\]
If \( t_{0}, t_{1}, t_{2} > 0 \), then we have
\[
    \sum_{k_{0}+k_{1}+k_{2}=k} 2^{(k_{0}+k_{1}+k_{2})+t_{0}(k_{1}+k_{2})} \approx 2^{nt_{0}+nt_{1}+nt_{2}}.
\]
Hence, from (4.19), we obtain
\[
    ||\sigma||_{BS^{W}_{0,0}(nt_{0}, nt_{1}, nt_{2}; \mathbb{R}^{n})} \lesssim ||\sigma||_{BS^{W}_{0,0}(t_{0})^{n}, nt_{1}, nt_{2}; \mathbb{R}^{n}}
\]
as desired.

4.4. Symbols with classical derivatives. In this subsection, we show that symbols
that have classical derivatives up to certain order satisfy the conditions of
Theorems 4.3 and 4.5.

Proposition 4.7. Let \( \sigma = \sigma(x, \xi_{1}, \xi_{2}) \) be a bounded measurable function on \( (\mathbb{R}^{n})^{3} \)
and \( W \in \mathcal{M}(\mathbb{R}^{2n}) \).

(1) Let \( s = (s_{i,j}) \in [0, \infty)^{3n} \). Suppose
\[
    |\partial_{x}^{\alpha_{1}} \cdots \partial_{x}^{\alpha_{n}} \partial_{\xi_{1}}^{\alpha_{1},0} \cdots \partial_{\xi_{1}}^{0,\alpha_{1}} \partial_{\xi_{2}}^{\alpha_{2},0} \cdots \partial_{\xi_{2}}^{0,\alpha_{2}} \sigma(x, \xi_{1}, \xi_{2})| \leq W(\xi_{1}, \xi_{2})
\]
for \( \alpha_{i,j} \leq s_{i,j} + 1 \). Then \( \sigma \in BS^{W}_{0,0}(s; \mathbb{R}^{n}) \).
(2) Let $s_0, s_1, s_2 \in [0, \infty)$. Suppose

$$|\partial_x^{\alpha_0} \partial_{\xi_1}^{\alpha_1} \partial_{\xi_2}^{\alpha_2} \sigma(x, \xi_1, \xi_2)| \leq W(\xi_1, \xi_2)$$

for $\alpha_i \in (N_0) \cap \mathbb{N}$ with $|\alpha_i| \leq |s_i| + 1$. Then $\sigma \in B_{s_0}^{W*}(s_0, s_1, s_2; \mathbb{R}^n)$.

To be precise, the above assumptions should be understood that the derivatives of $\sigma$ taken in the sense of distribution are functions in $L^\infty(\mathbb{R}^{3n})$ and they are bounded by $W(\xi_1, \xi_2)$ almost everywhere.

**Proof.** It is sufficient to treat $\sigma$ of class $C^\infty$. In fact, by using appropriate mollifier we can derive the result for general $\sigma$ from the result for $\sigma$ of class $C^\infty$. Since the claims (1) and (2) can be proved in almost the same way, here we shall give a proof of (2) and leave the proof of (1) to the reader.

Suppose $\sigma$ is $C^\infty$ and satisfies the assumption of (2). We write $N_i = |s_i| + 1$ and $\psi = \psi^{(n)}$.

First consider $\Delta_k \sigma$ for $k_0, k_1, k_2 \geq 1$. Recall that $\psi_k(\xi) = \psi(\xi/2^k)$ for $k \geq 1$ and $\psi \in C^\infty_c(\mathbb{R}^n)$ satisfies $\text{supp } \psi \subset \{1/2 \leq |\xi| \leq 2\}$. The inverse Fourier transform $\hat{\psi}$ satisfies the moment condition $\int x^n \hat{\psi}(x) \, dx = \partial^n_i \partial^n_j \hat{\psi}(0) = 0$. Thus, using the Taylor expansion with respect to the third variable of the symbol, we have

$$\Delta_k \sigma(x, \xi_1, \xi_2) = 2^n(k_0 + k_1 + k_2) \int \frac{\partial^{\alpha_0} \partial^{\alpha_1} \partial^{\alpha_2}}{\alpha_0! \alpha_1! \alpha_2!} \sigma(2^k y, 2^k \eta_1, 2^k \eta_2)$$

$$\times \sigma(x - y, \xi_1 - \eta_1, \xi_2 - \eta_2) - \sum_{|\alpha_2| < N_2} \frac{(-\eta_2)^{\alpha_2}}{\alpha_2!} \sigma(2^k y, 2^k \eta_1, 2^k \eta_2)$$

$$\times \sum_{|\alpha_2| = N_2} \frac{(-\eta_2)^{\alpha_2}}{\alpha_2!} \int_0^1 N_2(1 - t_2)^{N_2-1}(\partial^{\alpha_2}_{\xi_2} \sigma)(x - y, \xi_1 - \eta_1, \xi_2 - t_2 \eta_2) \, dt_2 \, dY,$$

where $dY = dy dt_1 dt_2$. Repeating the same argument to the variables $\eta_1$ and $y$, we obtain

$$\Delta_i \sigma(x, \xi_1, \xi_2) = 2^n(k_0 + k_1 + k_2) \sum_{|\alpha_0| = N_0} \frac{1}{\alpha_0!} \sum_{|\alpha_1| = N_1} \frac{1}{\alpha_1!} \sum_{|\alpha_2| = N_2} \frac{1}{\alpha_2!}$$

$$\times \int \frac{\partial^{\alpha_0} \partial^{\alpha_1} \partial^{\alpha_2}}{\alpha_0! \alpha_1! \alpha_2!} \sigma(2^k y, 2^k \eta_1, 2^k \eta_2)$$

$$\times \int \frac{\partial^{\alpha_0} \partial^{\alpha_1} \partial^{\alpha_2}}{\alpha_0! \alpha_1! \alpha_2!} \sigma(2^k y, 2^k \eta_1, 2^k \eta_2)$$

$$\times \int [\prod_{i=0}^{N_1} (1 - t_i)^{N_1-1}] \sigma(x - t_0 y, \xi_1 - t_1 \eta_1, \xi_2 - t_2 \eta_2) \, dT \, dY,$$

where $dT = dt_0 dt_1 dt_2$. If $\sigma$ satisfies the assumption of (2), then for $\alpha_0, \alpha_1, \alpha_2$ with $|\alpha_i| = N_i$ we have

$$|\partial^{\alpha_0} \partial^{\alpha_1} \partial^{\alpha_2} \sigma(x - t_0 y, \xi_1 - t_1 \eta_1, \xi_2 - t_2 \eta_2)| \leq W(\xi_1 - t_1 \eta_1, \xi_2 - t_2 \eta_2)$$

$$\lesssim W(\xi_1, \xi_2) \langle \eta_1 \rangle^L \langle \eta_2 \rangle^L,$$
where the latter inequality follows from the assumption $W \in \mathcal{M}(\mathbb{R}^{2n})$ and $L$ is a constant depending on $W$ (see Proposition 3.8 (2)). Hence

$$
|\Delta_k^s \sigma(x,\xi_1,\xi_2)| \lesssim 2^{n(k_0+k_1+k_2)}W(\xi_1,\xi_2)
$$

$$
\times \int_{(\mathbb{R}^n)^3} |\tilde{\psi}(2^{k_0}y)| |y| |\tilde{\psi}(2^{k_1}\eta_1)| |\eta_1|^N_1(\eta_1)^L |\tilde{\psi}(2^{k_2}\eta_2)| |\eta_2|^N_2(\eta_2)^L \, dY
\lesssim 2^{-k_0N_0} 2^{-k_1N_1} 2^{-k_2N_2} W(\xi_1,\xi_2)
$$

for all $x,\xi_1,\xi_2 \in \mathbb{R}^n$ and all $k_0, k_1, k_2 \geq 1$.

If one of $k_i$ is zero, then by avoiding usage of the moment condition and the Taylor expansion for the corresponding variables, we also obtain the same conclusion as above.

Thus we have

$$
\|W(\xi_1,\xi_2)^{-1} \Delta_k^s \sigma(x,\xi_1,\xi_2)\|_{L^2_{\mu}} \lesssim 2^{-k_0N_0} 2^{-k_1N_1} 2^{-k_2N_2}
$$

for all $k_0, k_1, k_2 \in \mathbb{N}_0$. Since $N_i = \lceil s_i \rceil + 1 > s_i$, the above inequalities imply

$$
\sum_{k_0,k_1,k_2 \in \mathbb{N}_0} 2^{s_0k_0+s_1k_1+s_2k_2} \|W(\xi_1,\xi_2)^{-1} \Delta_k^s \sigma(x,\xi_1,\xi_2)\|_{L^2_{\mu}} \lesssim 1.
$$

This completes the proof. 

\[\square\]

4.5. Proof of Theorem 1.3. Here we give a proof of Theorem 1.3.

**Proof.** We prove the assertion (2) first. Suppose $V \in \mathcal{B}(\mathbb{Z}^n \times \mathbb{Z}^n)$ and $\sigma \in \mathcal{B}^V_{S_{0,0}}(\mathbb{R}^n)$. We take a function $V^*$ as mentioned in Proposition 3.9. By Proposition 3.8 (2), it follows that $\tilde{V} \lesssim V^*$ and hence $\sigma \in \mathcal{B}^V_{S_{0,0}}(\mathbb{R}^n)$. Proposition 4.4 implies that $\sigma$ also satisfies the assumptions of Theorems 4.3 and 4.5 with $W = V^*$ and $r_1 = \cdots = r_n = r = 1$, and the boundedness of $T_\sigma$ follows.

Next, we shall prove the assertion (1). The basic idea of this part of proof goes back to [20] Proof of Lemma 6.3].

Let $V$ be a nonnegative bounded function on $\mathbb{Z}^n \times \mathbb{Z}^n$ and $0 < r < \infty$. We assume $\text{Op}(BS_{0,0}^V) \subset B(L^2 \times L^2 \to L^r)$ with $V$ defined as in Theorem 1.3. By the closed graph theorem, it follows that there exist a positive integer $M$ and a positive constant $C$ such that

$$
\|T_{\varphi,\tilde{\varphi}}\|_{L^2 \times L^2 \to L^r} \leq C \max_{|\alpha|,|\beta|,|\gamma| \leq M} \|\tilde{V}(\xi_1,\xi_2)^{-1} \Delta_k^s \sigma(x,\xi_1,\xi_2)\|_{L^\infty}
$$

for all bounded smooth functions $\sigma$ on $(\mathbb{R}^n)^3$ (see [11] Lemma 2.6]). Our purpose is to prove the inequality (1.3). For this, it is sufficient to consider $A, B, C \in \ell^2(\mathbb{Z}^n)$ such that $A(\mu) = B(\mu) = C(\mu) = 0$ except for a finite number of $\mu \in \mathbb{Z}^n$.

Take $\varphi, \tilde{\varphi} \in S(\mathbb{R}^n)$ such that

$$
\text{supp } \tilde{\varphi} \subset [-1/2, 1/2]^n, \quad \tilde{\varphi} = 1 \text{ on } [-1/4, 1/4]^n, \quad \text{supp } \varphi \subset [-1/4, 1/4]^n,
$$

$$
|F^{-1}\varphi| \geq 1 \text{ on } [-\pi, \pi]^n.
$$

(4.22)

Take a sequence of real numbers $\{\epsilon_k\}_{k \in \mathbb{Z}^n}$ such that $\sup_{k \in \mathbb{Z}^n} |\epsilon_k| \leq 1$, and set

$$
\sigma(\xi_1,\xi_2) = \sum_{k_1,k_2 \in \mathbb{Z}^n} \epsilon_{k_1+k_2} V(k_1,k_2) \tilde{\varphi}(\xi_1-k_1) \varphi(\xi_2-k_2).
$$
Then we have
\begin{equation}
|\partial_{\xi_1}^{\beta_1} \partial_{\xi_2}^{\beta_2} \sigma(\xi_1, \xi_2)| \leq C_{\beta_1, \beta_2} \bar{V}(\xi_1, \xi_2)
\end{equation}
with $C_{\beta_1, \beta_2}$ independent of the sequence $\{\epsilon_k\}$. We define $f_1, f_2 \in S(\mathbb{R}^n)$ by
\begin{align*}
\hat{f}_1(\xi_1) &= \sum_{\nu_1 \in \mathbb{Z}^n} B(\nu_1) \varphi(\xi_1 - \nu_1), \\
\hat{f}_2(\xi_2) &= \sum_{\nu_2 \in \mathbb{Z}^n} C(\nu_2) \varphi(\xi_2 - \nu_2).
\end{align*}
Then $f_1(x) = \sum_{\nu_1 \in \mathbb{Z}^n} B(\nu_1) e^{i\nu_1 \cdot x} \mathcal{F}^{-1} \varphi(x)$ and hence, using Parseval’s identity and (4.22), we have $\|f_1\|_{L^2} \approx \|B\|_{\ell^2}$. Similarly $\|f_2\|_{L^2} \approx \|C\|_{\ell^2}$. From the situation of the supports of $\varphi$ and $\hat{\varphi}$, we have
\begin{align*}
T_\sigma(f_1, f_2)(x) &= \sum_{\nu_1, \nu_2 \in \mathbb{Z}^n} \epsilon_{\nu_1+\nu_2} V(\nu_1, \nu_2) B(\nu_1) C(\nu_2) e^{i(\nu_1+\nu_2) \cdot x} \mathcal{F}^{-1} \varphi(x)^2 \\
&= \sum_k \epsilon_k d_k e^{ik \cdot x} \mathcal{F}^{-1} \varphi(x)^2,
\end{align*}
where
\begin{equation}
d_k = \sum_{\nu_1+\nu_2 = k} V(\nu_1, \nu_2) B(\nu_1) C(\nu_2).
\end{equation}
Notice that $d_k \neq 0$ only for a finite number of $k$’s by virtue of our assumptions on $B$ and $C$.

Now from (4.21), (4.23), and from the estimates of the $L^2$ norms of $f_1$ and $f_2$ mentioned above, we have
\[\|T_\sigma(f_1, f_2)\|_L \lesssim \|f_1\|_{L^2} \|f_2\|_{L^2} \approx \|B\|_{\ell^2} \|C\|_{\ell^2}.\]

By (4.22), we have
\[\|T_\sigma(f_1, f_2)\|_{L^r} ^r \geq \int_{[-\pi, \pi]^n} \left| \sum_k \epsilon_k d_k e^{ik \cdot x} \right| ^r dx.
\]
Hence
\begin{equation}
\int_{[-\pi, \pi]^n} \left| \sum_k \epsilon_k d_k e^{ik \cdot x} \right| ^r dx \lesssim (\|B\|_{\ell^2} \|C\|_{\ell^2}) ^r.
\end{equation}
It should be noticed that the implicit constant in (4.25) does not depend on $\{\epsilon_k\}$.

We choose $\epsilon_k = \epsilon_k(\omega)$ to be identically distributed independent random variables on a probability space, each of which takes $+1$ and $-1$ with probability $1/2$. Then integrating over $\omega$ and using Khintchine’s inequality, we have
\begin{equation}
\int \left( \text{the left hand side of (4.25)} \right) dP(\omega) \approx \left( \sum_k |d_k|^2 \right) ^{r/2}
\end{equation}
(for Khintchine’s inequality, see, e.g., [11, Appendix C]).

Combining (4.24), (4.25), and (4.26), we obtain
\[\left\| \sum_{\nu_1+\nu_2 = k} V(\nu_1, \nu_2) B(\nu_1) C(\nu_2) \right\|_{\ell^2_k} \lesssim \|B\|_{\ell^2} \|C\|_{\ell^2},\]
which is equivalent to (1.14). This completes the proof of Theorem 1.3. □
4.6. A theorem of Grafakos–He–Slavíková with some generalization. The theorem given below is a generalization of the theorem of Grafakos–He–Slavíková \[15\]. We shall prove this theorem by using Theorem \[1.3\].

**Theorem 4.8.** Suppose \( \sigma \in BS_{0,0}^{0}(\mathbb{R}^{n}) \) with the notation of \[1.2\] and suppose the function \( V(\xi_{1}, \xi_{2}) = \sup_{x \in \mathbb{R}^{n}} |\sigma(x, \xi_{1}, \xi_{2})| \) belongs to \( L_{\xi_{1}, \xi_{2}}^{q}(\mathbb{R}^{2n}) \) for some \( 0 < q < 4 \). Then the bilinear pseudo-differential operator \( T_{\sigma} \) is bounded from \( L^{2} \times L^{2} \) to the amalgam space \( (L^{2}, \ell^{1}) \). In particular, \( T_{\sigma} \) is bounded from \( L^{2} \times L^{2} \) to \( h^{1} \cap L^{2} \).

**Proof.** We assume \( V \in L^{q}(\mathbb{R}^{2n}) \) with \( 1 \leq q < 4 \). The assumption \( q \geq 1 \) gives no additional restriction since \( \sigma \) already belongs to \( L^{\infty} \) by the assumption \( \sigma \in BS_{0,0}^{0}(\mathbb{R}^{n}) \). In the following argument, \( N \) denotes a fixed sufficiently large positive number that depends only on the dimension \( n \).

We take a Littlewood-Paley partition of unity \( \{ \psi_{k} \} \) on \( \mathbb{R}^{3n} \) and decompose \( \sigma \) as

\[
\sigma(x, \xi_{1}, \xi_{2}) = \sum_{k=0}^{\infty} \psi_{k}(D_{x, \xi_{1}, \xi_{2}})\sigma(x, \xi_{1}, \xi_{2}) = \sum_{k=0}^{\infty} \sigma_{k}(x, \xi_{1}, \xi_{2}).
\]

In order to show \( T_{\sigma} : L^{2} \times L^{2} \to (L^{2}, \ell^{1}) \), we shall prove

\[
\sum_{k=0}^{\infty} \| T_{\sigma_{k}} \|_{L^{2} \times L^{2} \to (L^{2}, \ell^{1})} < \infty.
\]

We define \( V_{k} \) by

\[
V_{k}(\xi_{1}, \xi_{2}) = \int_{\mathbb{R}^{2n}} V(\eta_{1}, \eta_{2})2^{kn}(1 + 2^{k}|\xi_{1} - \eta_{1}| + 2^{k}|\xi_{2} - \eta_{2}|)^{-N} d\eta_{1}d\eta_{2}.
\]

We shall derive estimates of \( \sigma_{k} \) in terms of \( V_{k} \).

Firstly,

\[
|\partial_{x, \xi_{1}, \xi_{2}}^{\alpha} \sigma_{k}(x, \xi_{1}, \xi_{2})| \leq C_{\alpha}2^{k|\alpha|}V_{k}(\xi_{1}, \xi_{2}).
\]

To see this, consider first the case \( k \geq 1 \). Then recall that the function \( \psi_{k} \) is of the form \( \psi_{k} = \psi(2^{-k} \cdot) \) with \( \psi \in \mathcal{S}(\mathbb{R}^{3n}) \). Hence the derivative on the left hand side can be written as

\[
\partial_{x, \xi_{1}, \xi_{2}}^{\alpha} \sigma_{k}(x, \xi_{1}, \xi_{2}) = \left( (\partial_{x}^{\alpha} \mathcal{F}^{-1}\psi_{k}) \ast \sigma \right)(x, \xi_{1}, \xi_{2})
\]

\[
= \int_{\mathbb{R}^{3n}} 2^{3kn}2^{k|\alpha|}(\partial_{x}^{\alpha} \mathcal{F}^{-1}\psi)(2^{k}(x - y, \xi_{1} - \eta_{1}, \xi_{2} - \eta_{2}))\sigma(y, \eta_{1}, \eta_{2}) dyd\eta_{1}d\eta_{2}.
\]

Since \( \psi \in \mathcal{S} \) and since \( \sigma \) is bounded by \( V \), the integrand on the right hand side is bounded by

\[
C_{\alpha}2^{3kn}2^{k|\alpha|}(1 + 2^{k}|x - y|)^{-N}(1 + 2^{k}|\xi_{1} - \eta_{1}| + 2^{k}|\xi_{2} - \eta_{2}|)^{-N}V(\eta_{1}, \eta_{2})
\]

and thus the estimate \( (4.28) \) follows. Proof for \( k = 0 \) is similar.

Secondly,

\[
|\partial_{x}^{\alpha} \sigma_{k}(x, \xi_{1}, \xi_{2})| \leq C_{\alpha,l}2^{-kL},
\]

where \( L \in \mathbb{N} \) can be taken arbitrarily large. For \( k = 0 \), this estimate is obvious from the assumption \( \sigma \in BS_{0,0}^{0} \). Suppose \( k \geq 1 \). We write \( X = (x, \xi_{1}, \xi_{2}) \) and \( Y = (y, \eta_{1}, \eta_{2}) \). Then, since \( \psi_{k}(X) = \psi(2^{-k}X) \) and \( \mathcal{F}^{-1}\psi \) satisfies the moment condition \( \int X^{\alpha}\mathcal{F}^{-1}\psi(X) \, dX = 0 \), we have

\[
\partial_{x}^{\alpha} \sigma_{k}(X) = (\mathcal{F}^{-1}\psi_{k} \ast (\partial^{\alpha}\sigma))(X)
\]
Since $\psi \in S$ and since the derivatives of $\sigma$ are bounded, the integrand on the right hand side is bounded by

$$C_{\alpha, L} 2^{3kn} (1 + 2^k |Y|)^{-N-L} |Y|^L = C_{\alpha, L} 2^{-kL} 2^{3kn} (1 + 2^k |Y|)^{-N-L} 2^k |Y|^L$$

and thus the estimate (4.29) follows.

For bilinear pseudo-differential operators, a simple change of variables yields the formula

$$T_{\sigma_k} (f_1, f_2)(2^k x) = T_{\tilde{\sigma}_k} (f_1(2^k \cdot), f_2(2^k \cdot))(x).$$

For the norm of $(L^2, \ell^1)(\mathbb{R}^n)$, there exists a real number $a$ such that

$$\|g(\lambda)\|_{(L^2, \ell^1)(\mathbb{R}^n)} \lesssim \lambda^a \|g\|_{(L^2, \ell^1)(\mathbb{R}^n)}$$

for $0 < \lambda \leq 1$. (In fact, we can take $a = -n$ and this is the optimal number; however, the exact value of $a$ is not necessary for our argument.) For the $L^2$ norm, we have

$$\|g(\lambda)\|_{L^2(\mathbb{R}^n)} = \lambda^{-n/2} \|g\|_{L^2(\mathbb{R}^n)}, \quad \lambda > 0.$$
5. Sharpness of the theorems

In this section, we shall prove that our main theorems, Theorems 1.3, 4.3 and 4.5, are sharp in several senses. Here we consider the cases of the following special weights:

\[ W_m(\xi_1, \xi_2) = (\langle \xi_1, \xi_2 \rangle)^m, \quad m \in (-\infty, 0], \]
\[ W_{m_1, m_2}(\xi_1, \xi_2) = \langle \xi_1 \rangle^{m_1} \langle \xi_2 \rangle^{m_2}, \quad m_1, m_2 \in (-\infty, 0]. \]

We denote the class \( BS_{0,0}^W(\mathbb{R}^n) \) of Definition 4.1 for \( W = W_m \) and \( W = W_{m_1, m_2} \) simply by \( BS_{0,0}^m(\mathbb{R}^n) \) and \( BS_{0,0}^{(m_1, m_2)}(\mathbb{R}^n) \), respectively. Thus the class \( BS_{0,0}^m(\mathbb{R}^n) \) is the same as the one defined by (1.2).

5.1. Sharpness of the order \(-n/2\). We have already observed that \( W_m \) with \( m = -n/2 \) and \( W_{m_1, m_2} \) with \( m_1, m_2 < 0 \) and \( m_1 + m_2 = -n/2 \) belong to \( B(\mathbb{Z}^n \times \mathbb{Z}^n) \) (see Example 1.4 and the proof given in Section 3). Here we shall see that these are critical weights among the weights \( W_m \) and \( W_{m_1, m_2} \). Firstly, the weight \( W_m \) with \(-n/2 < m \leq 0\) does not belong to \( B(\mathbb{Z}^n \times \mathbb{Z}^n) \) as we have already observed in Proof of Proposition 3.1 (3). Next, the weight \( W_{m_1, m_2} \) with \( m_1, m_2 \in (-\infty, 0] \) does not belong to \( B(\mathbb{Z}^n \times \mathbb{Z}^n) \) if \( m_1 + m_2 > -n/2 \) or if \( m_1 + m_2 = -n/2 \) and \( m_1 m_2 = 0 \). To show this, observe that \( W_{m_1, m_2}(\xi_1, \xi_2) \leq W_{m_1, m_2}(\xi_1, \xi_2) \). Thus if \( W_{m_1, m_2} \in B(\mathbb{Z}^n \times \mathbb{Z}^n) \) then \( W_{m_1, m_2} \in B(\mathbb{Z}^n \times \mathbb{Z}^n) \), which is possible only when \( m_1 + m_2 \leq -n/2 \). Also Proposition 3.2 implies that the functions \( W_{0, -n/2} \) and \( W_{-n/2, 0} \) do not belong to \( B(\mathbb{Z}^n \times \mathbb{Z}^n) \).

5.2. Sharpness of \( r \in [1, 2] \). The next proposition shows that the range \( 1 \leq r \leq 2 \) in Theorems 1.3, 4.3 and 4.5 is in a sense optimal.

**Proposition 5.1.** Let \( 0 < r < \infty, m \in (-\infty, 0], \) and assume \( \text{Op}(BS_{0,0}^m(\mathbb{R}^n)) \subset B(L^2 \times L^2 \to L^r) \). Then \( r \geq 1 \). Moreover, \( r \leq 2 \) in the case \( m = -n/2 \).

**Proof.** If the symbol \( \sigma(x, \xi_1, \xi_2) \) is independent of \( x \), then \( \sigma \) is called a Fourier multiplier and \( T_\sigma \) is called a bilinear Fourier multiplier operator. For bilinear Fourier multiplier operators, the following is known: if a nonzero Fourier multiplier operator \( T_\sigma \) is bounded from \( L^p \times L^q \) to \( L^r \), \( 1 \leq p, q < \infty \), and \( 0 < r < \infty \), then \( 1/p + 1/q \geq 1/r \) (see [14, Proposition 5] and [12, Proposition 7.3.7]). Let \( \sigma(\xi_1, \xi_2) \) be a nonzero function in \( S((\mathbb{R}^n)^2) \). Then, since \( \sigma(\xi_1, \xi_2) \) belongs to \( BS_{0,0}^m(\mathbb{R}^n) \) for all \( m \leq 0 \), the assumption of the proposition implies \( T_\sigma : L^2 \times L^2 \to L^r \). Hence, by the fact mentioned above, we must have \( 1/2 + 1/2 \geq 1/r \), that is, \( r \geq 1 \).

Next we show that \( r \leq 2 \) in the case \( m = -n/2 \). Assume that \( T_\sigma : L^2 \times L^2 \to L^r \) for all \( \sigma \in BS_{0,0}^{-n/2} \). Let \( \Psi \in S((\mathbb{R}^n)^2) \) and \( \psi \in S(\mathbb{R}^n) \) be such that \( \Psi(\zeta) = 1 \) on \( \{2^{-1/4} \leq |\zeta| \leq 2^{1/4}\} \), \( \supp \Psi \subset \{2^{-1/2} \leq |\zeta| \leq 2^{1/2}\} \), \( \supp \psi \subset \{2^{-3/4} \leq |\eta| \leq 2^{-1/4}\} \), and \( \psi \neq 0 \). We set

\[ \sigma(\xi_1, \xi_2) = \sum_{j \in \mathbb{N}_0} 2^{-j n/2} \Psi(2^{-j}(\xi_1, \xi_2)), \quad (\xi_1, \xi_2) \in \mathbb{R}^{2n}, \]
\[ \hat{f}_{1,k}(\eta) = \hat{f}_{2,k}(\eta) = 2^{-kn/2} \psi(2^{-k}\eta), \quad \eta \in \mathbb{R}^n, \quad k \in \mathbb{N}_0. \]

Then \( \sigma \in BS_{0,0}^{-n/2} \) (in fact, \( \sigma \in BS_{1,0}^{-n/2} \)) and \( \|f_{1,k}\|_{L^2} = \|\psi\|_{L^2} \) does not depend on \( k \). From the support conditions on \( \Psi \) and \( \psi \), we see that \( \Psi(2^{-j}(\xi_1, \xi_2)) \hat{f}_{1,k}(\xi_1) \hat{f}_{2,k}(\xi_2) \)
equals \( \hat{f}_{1,k}(\xi_1)\hat{f}_{2,k}(\xi_2) \) if \( j = k \) and vanishes if \( j \neq k \). Thus

\[
T_\sigma(f_{1,k}, f_{2,k})(x) = 2^{-kn/2} (2^{kn/2}\psi(2^kx))^2 = 2^{kn/2}\psi(2^kx)^2.
\]

Hence our assumption implies that

\[
2^{kn(1/2-1/r)} \approx \| T_\sigma(f_{1,k}, f_{2,k}) \|_{L^r} \lesssim \| f_{1,k} \|_{L^2} \| f_{2,k} \|_{L^2} \approx 1, \quad k \in \mathbb{N}_0,
\]

which is possible only when \( 1/2 - 1/r \leq 0 \), namely \( r \leq 2 \).

\[\Box\]

5.3. **Sharpness of** \( s_0, s_1, s_2 \) **in Theorem** \([4,5]\) **In this subsection, we shall prove that the conditions on** \( s_0, s_1, s_2 \) **in Theorem** \([4,5]\) **are sharp. First we shall prove the following.**

**Proposition 5.2.** Let \( 1 \leq r \leq 2 \) and \( s = (s_0, s_1, s_2) \in [0, \infty)^3 \). If all bilinear pseudo-differential operators \( T_\sigma \) with symbols \( \sigma \) on \( (\mathbb{R}^n)^3 \) satisfying

\[
(5.1) \quad \sup_{k \in (\mathbb{N}_0)^3} 2^{k/s} \| \langle (\xi_1, \xi_2) \rangle^{n/2} \Delta_k^s \sigma(x, \xi_1, \xi_2) \|_{L^\infty_{x,\xi_1,\xi_2}(\mathbb{R}^n)^3} < \infty
\]

are bounded from \( L^2 \times L^2 \) to \( L^r \), then \( s_0 \geq n/2, s_1, s_2 \geq n/r-n/2, \) and \( s_1+s_2 > n/r \).

\[\text{Proof.}\]

In this proof, we use nonnegative functions \( \varphi, \theta \in \mathcal{S}(\mathbb{R}^n) \) such that \( \varphi(x) = 1 \) on \( \{ |x| \leq 1 \} \), \( \text{supp} \varphi \subset \{ |x| \leq 2 \} \), \( \text{supp} \theta \subset \{ 1/2 \leq |\xi| \leq 2 \} \), and \( \theta \neq 0 \). Let \( N_i \) be a nonnegative integer satisfying \( N_i \geq s_i \) for \( i = 0, 1, 2 \).

We first prove the necessity of the condition \( s_0 \geq n/2 \). Set

\[
\sigma(x, \xi_1, \xi_2) = \varphi(x) e^{-ix \cdot (\xi_1+\xi_2)} \langle (\xi_1, \xi_2) \rangle^{-s_0-n/2},
\]

\[
\hat{f}_{1,j}(\eta) = \hat{f}_{2,j}(\eta) = 2^{-jn/2}\theta(2^{-j}\eta), \quad j \in N_0.
\]

Since

\[
(5.2) \quad |\partial_x^{a_0} \partial_{\xi_1}^{a_1} \partial_{\xi_2}^{a_2} \sigma(x, \xi_1, \xi_2)| \leq C_{a_0, a_1, a_2} \langle (\xi_1, \xi_2) \rangle^{-s_0-n/2+|a_0|},
\]

in the same way as in Proof of Proposition \([4,7]\) (see the argument around \([1,20]\)), we have

\[
\Delta_k^s \sigma(x, \xi_1, \xi_2) \lesssim \langle (\xi_1, \xi_2) \rangle^{-s_0-n/2+N_02(k_0+k_1+k_2)n} 
\]

\[
\times \int_{(\mathbb{R}^n)^3} \hat{\psi}(2^k y) \langle y \rangle^{N_0} |\hat{\psi}(2^k \eta_1)| |\hat{\psi}(2^k \eta_2)| \eta_1^{N_1} |\eta_2|^{N_2} dY 
\]

\[
\approx \langle (\xi_1, \xi_2) \rangle^{-s_0-n/2+N_02^{-k_0N_0}2^{-k_1N_1}2^{-k_2N_2}}
\]

for all \( k_0, k_1, k_2 \in \mathbb{N} \), where \( dY = dy \eta_1 \eta_2 \). If we use \((5.2)\) with \( a_0 = 0 \) and the expression

\[
\Delta_k^s \sigma(x, \xi_1, \xi_2) 
\]

\[
= 2^{(k_0+k_1+k_2)n} \sum_{|a_1| = N_1} \frac{1}{a_1!} \sum_{|a_2| = N_2} \frac{1}{a_2!} 
\]

\[
\times \int_{(\mathbb{R}^n)^3} \hat{\psi}(2^k y) \hat{\psi}(2^k \eta_1) (-\eta_1)^{a_1} \hat{\psi}(2^k \eta_2) (-\eta_2)^{a_2} 
\]

\[
\times \int_{[0,1]^2} \left( \prod_{i=1}^2 N_i(1-t_i)^{N_i-1} \right) (\partial_{\xi_1}^{a_1} \partial_{\xi_2}^{a_2} \sigma)(x-y, \xi_1-t_1 \eta_1, \xi_2-t_2 \eta_2) dt_1 dt_2 dY
\]

\[\Box\]
instead of (1.20), we have

\[ |\Delta_k^j \sigma(x, \xi_1, \xi_2)| \lesssim \langle (\xi_1, \xi_2) \rangle^{-s_0-n/2-k_1N_1-2-k_2N_2} \]

for \( k_1, k_2 \in \mathbb{N} \). It is also easy to see that the above estimates actually hold for all \( k_1, k_2, k_3 \in \mathbb{N}_0 \). Hence, taking \( 0 \leq \theta_0 \leq 1 \) satisfying \( s_0 = N_0 \theta_0 \), we have

\[
|\Delta_k^j \sigma(x, \xi_1, \xi_2)| = |\Delta_k^j \sigma(x, \xi_1, \xi_2)|^{1-\theta_0} |\Delta_k^j \sigma(x, \xi_1, \xi_2)|^{\theta_0} 
\lesssim \langle (\xi_1, \xi_2) \rangle^{-s_0-n/2-k_1N_1-2-k_2N_2} \langle (\xi_1, \xi_2) \rangle^{-s_0-n/2+k_1N_0+k_2N_2} 
\times \langle (\xi_1, \xi_2) \rangle^{-s_0-n/2-k_1N_0-k_2N_2} \langle (\xi_1, \xi_2) \rangle^{-s_0-n/2+k_1N_0+k_2N_2},
\]

which implies that \( \sigma \) satisfies (5.1). Then, since

\[ T_\sigma(f_{1,j}, f_{2,j})(x) = \left( \frac{2^{-jn}}{(2\pi)^{2n}} \int_{(\mathbb{R}^n)^2} \langle (\xi_1, \xi_2) \rangle^{-s_0-n/2} \theta(2^{-j} \xi_1) \theta(2^{-j} \xi_2) d\xi_1 d\xi_2 \right) \varphi(x) \]

and since

\[ 2^{-jn} \int_{(\mathbb{R}^n)^2} \langle (\xi_1, \xi_2) \rangle^{-s_0-n/2} \theta(2^{-j} \xi_1) \theta(2^{-j} \xi_2) d\xi_1 d\xi_2 \approx 2^{j(-s_0+n/2)}, \]

it follows from our assumption that

\[ 2^{j(-s_0+n/2)} \approx \|T_\sigma(f_{1,j}, f_{2,j})\|_{L^r} \lesssim \|f_{1,j}\|_{L^2} \|f_{2,j}\|_{L^2} \approx 1, \quad j \in \mathbb{N}_0. \]

This is possible only if \(-s_0 + n/2 \leq 0\), namely \( s_0 \geq n/2 \).

We next prove the necessity of the condition \( s_i \geq r/n - n/2, i = 1, 2 \). Set

\[ \sigma(x, \xi_1, \xi_2) = \langle x \rangle^{-s_2} e^{-ix \cdot \xi_1} \varphi(\xi_1) \varphi(\xi_2), \]

\[ \hat{f}_{1,j}(\xi_1) = \varphi(\xi_1), \quad \hat{f}_{2,j}(\xi_2) = 2^{jn/2} \varphi(2^j \xi_2), \quad j \in \mathbb{N}_0. \]

Since

\[ |\partial_{\xi_1}^{s_1} \partial_{\xi_2}^{s_2} \sigma(x, \xi_1, \xi_2)| \leq C_{\alpha_0, \alpha_1, \alpha_2} \langle x \rangle^{-s_1+|\alpha_1|} \langle (\xi_1, \xi_2) \rangle^{-n/2}, \]

by the same argument as above,

\[ |\Delta_k^j \sigma(x, \xi_1, \xi_2)| \lesssim \begin{cases} 
\langle x \rangle^{-s_1} \langle (\xi_1, \xi_2) \rangle^{-n/2} 2^{-k_1N_0} 2^{-k_2N_2} 
\langle x \rangle^{-s_1 + N_1} \langle (\xi_1, \xi_2) \rangle^{-n/2} 2^{-k_1N_0} 2^{-k_2N_2}.
\end{cases} \]

In the same way as in (5.3), but replacing \( \theta_0 \) by \( 0 \leq \theta_1 \leq 1 \) satisfying \( s_1 = N_1 \theta_1 \), we have

\[ |\Delta_k^j \sigma(x, \xi_1, \xi_2)| \lesssim \langle (\xi_1, \xi_2) \rangle^{-n/2} 2^{-k_1N_0} 2^{-k_1s_1} 2^{-k_2N_2}, \]

which implies that \( \sigma \) satisfies (5.1). On the other hand, since \( \varphi(2^j \xi_2) \varphi(\xi_2) = \varphi(2^j \xi_2) \) for \( j \geq 1 \), we have

\[ T_\sigma(f_1, f_{2,j})(x) = \langle x \rangle^{-s_1} \left( \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \varphi(\xi_1)^2 d\xi_1 \right) 2^{-jn/2} \varphi(2^{-j} x), \quad j \geq 1, \]

and thus

\[ \|T_\sigma(f_1, f_{2,j})\|_{L^r} \approx \|\langle x \rangle^{-s_1} 2^{-jn/2} \varphi(2^{-j} x)\|_{L^r} \geq \|\langle x \rangle^{-s_1} 2^{-jn/2} \varphi(2^{-j} x)\|_{L^r(\{x| \leq 2^j\})} \]

\[ \gtrsim 2^{-j(s_1+n/r-n/2)}. \]

Thus our assumption implies that

\[ 2^{j(-s_1+n/r-n/2)} \lesssim \|T_\sigma(f_1, f_{2,j})\|_{L^r} \lesssim \|f_1\|_{L^2} \|f_{2,j}\|_{L^2} \approx 1, \quad j \geq 1, \]
which is possible only if \(-s_1 + n/r - n/2 \leq 0\), namely \(s_1 \geq n/r - n/2\). By interchanging the roles of \(\xi_1\) and \(\xi_2\), we also have \(s_2 \geq n/r - n/2\).

Finally we prove the necessity of the condition \(s_1 + s_2 > n/r\). Set
\[
\sigma(x, \xi_1, \xi_2) = \langle x \rangle^{-s_1 - s_2} e^{-ix(\xi_1 + \xi_2)} \varphi(\xi_1) \varphi(\xi_2),
\]
\[
\hat{f}_1 = \hat{f}_2 = \varphi.
\]
Since
\[
|\partial^{n_0}_{x_i} \partial^{n_1}_{\xi_1} \partial^{n_2}_{\xi_2} \sigma(x, \xi_1, \xi_2)| \leq C_{\alpha_0, \alpha_1, \alpha_2} \langle x \rangle^{-s_1 - s_2 + |\alpha_1| + |\alpha_2|} \langle (\xi_1, \xi_2) \rangle^{-n/2},
\]
by the same argument as above,
\[
|\Delta_k^s \sigma(x, \xi_1, \xi_2)| \lesssim \begin{cases} 
\langle x \rangle^{-s_1 - s_2} \langle (\xi_1, \xi_2) \rangle^{-n/2} 2^{-k_1 N_0} & 
\langle x \rangle^{-s_1 - s_2 + N_1} \langle (\xi_1, \xi_2) \rangle^{-n/2} 2^{-k_0 N_0} 2^{-k_1 N_1} 
\langle x \rangle^{-s_1 - s_2 + N_2} \langle (\xi_1, \xi_2) \rangle^{-n/2} 2^{-k_0 N_0} 2^{-k_2 N_2} & 
\langle x \rangle^{-s_1 - s_2 + N_1 + N_2} \langle (\xi_1, \xi_2) \rangle^{-n/2} 2^{-k_0 N_0} 2^{-k_1 N_1} 2^{-k_2 N_2}.
\end{cases}
\]
Taking \(0 \leq \theta_i \leq 1\) satisfying \(s_i = N_i \theta_i\) for \(i = 1, 2\), we have
\[
|\Delta_k^s \sigma| = |\Delta_k^s |^{(\theta_1)(1-\theta_2)} |\Delta_k^s |^\theta_1(1-\theta_2) |\Delta_k^s |^{(1-\theta_1)\theta_2} |\Delta_k^s |^{\theta_1 \theta_2} 
\lesssim \langle x \rangle^{-s_1 - s_2} \langle (\xi_1, \xi_2) \rangle^{-n/2} 2^{-k_0 N_0} (1-\theta_1)(1-\theta_2) 
\times \langle x \rangle^{-s_1 - s_2 + N_1} \langle (\xi_1, \xi_2) \rangle^{-n/2} 2^{-k_0 N_0} 2^{-k_1 N_1} \theta_1(1-\theta_2) 
\times \langle x \rangle^{-s_1 - s_2 + N_2} \langle (\xi_1, \xi_2) \rangle^{-n/2} 2^{-k_0 N_0} 2^{-k_2 N_2} (1-\theta_1)\theta_2 
\times \langle x \rangle^{-s_1 - s_2 + N_1 + N_2} \langle (\xi_1, \xi_2) \rangle^{-n/2} 2^{-k_0 N_0} 2^{-k_1 N_1} 2^{-k_2 N_2} \theta_1 \theta_2 

= \langle (\xi_1, \xi_2) \rangle^{-n/2} 2^{-k_0 N_0} 2^{-k_1 N_1} 2^{-k_2 N_2},
\]
which implies that \(\sigma\) satisfies (5.1). Therefore, since
\[
T_\sigma(f_1, f_2)(x) = \langle x \rangle^{-s_1 - s_2} \prod_{i=1}^2 \left( \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \varphi(\xi_i)^2 d\xi_i \right),
\]
it follows from our assumption that \(\langle x \rangle^{-s_1 - s_2}\) belongs to \(L^r\). This is possible only if \(r(-s_1 - s_2) < -n\), namely \(s_1 + s_2 > n/r\).

In the corollary below, \(BS^{-n/2}_{s_0, s_1, s_2}(\mathbb{R}^n)\) denotes the class \(BS^{-n/2}_{s_0, s_1, s_2}(\mathbb{R}^n)\) of Definition 1.4 for \(W(\xi_1, \xi_2) = W_{-n/2}(\xi_1, \xi_2) = \langle (\xi_1, \xi_2) \rangle^{-n/2}\).

**Corollary 5.5.** Let \(1 \leq r \leq 2\) and \(s = (s_0, s_1, s_2) \in [0, \infty)^3\). Assume all bilinear pseudo-differential operators \(T_\sigma\) with \(\sigma \in BS^{-n/2}_{s_0, s_1, s_2}(\mathbb{R}^n)\) are bounded from \(L^2 \times L^2\) to \(L^r\). Then \(s_0 \geq n/2\), \(s_1, s_2 \geq n/r - n/2\), and \(s_1 + s_2 \geq n/r\).

**Proof.** Observe that all \(\sigma\) satisfying (5.1) with \(s_i\) replaced by \(s_i + \epsilon\) with \(\epsilon > 0\) belong to \(BS^{-n/2}_{s_0, s_1, s_2}(\mathbb{R}^n)\). Hence, if the assumption of the corollary holds, then, by Proposition 5.2, we must have \(s_0 + \epsilon \geq n/2\), \(s_1 + \epsilon, s_2 + \epsilon \geq n/r - n/2\), and \(s_1 + \epsilon + s_2 + \epsilon > n/r\) for \(\epsilon > 0\). Since \(\epsilon > 0\) is arbitrary, we obtain the conclusion.

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(T. Kato and N. Tomita) DEPARTMENT OF MATHEMATICS, GRADUATE SCHOOL OF SCIENCE, OSAKA UNIVERSITY, TOYONAKA, OSAKA 560-0043, JAPAN
(A. Miyachi) Department of Mathematics, Tokyo Woman's Christian University, Zempukuji, Suginami-ku, Tokyo 167-8585, Japan

E-mail address, T. Kato: t.katou@cr.math.sci.osaka-u.ac.jp
E-mail address, A. Miyachi: miyachi@lab.twcu.ac.jp
E-mail address, N. Tomita: tomita@math.sci.osaka-u.ac.jp