Abstract. We explain how the claims of the KPZ scaling theory are confirmed by a recent proof of Borodin and Corwin on the asymptotics of the semi-discrete directed polymer.
1 Introduction

The 1986 Kardar-Parisi-Zhang (KPZ) equation \[15\] is a stochastic partial differential equation modeling surface growth and, more generally, the motion of an interface bordering a stable against a metastable phase. The scaling theory is an educated guess on the non-universal coefficients in the asymptotics for models in the KPZ universality class. The scaling theory has been developed in a landmark contribution by Krug, Meakin, and Halpin-Healy \[16\]. The purpose of our note is to explain how to apply the scaling theory to the semi-discrete directed polymer. This model has been discussed in depth at the 2010 random matrix workshop at the MSRI and, so-to-speak as a spin-off, Borodin and Corwin \[5\] developed the beautiful theory of Macdonald processes, which provides the tools for an asymptotic analysis of the semi-discrete directed polymer. As we will establish, the scaling theory is consistent with the results in \[5\], thereby providing a highly non-obvious control check.

To place the issue in focus, let me start with a simple example. Assume as given the stationary sequence \(X_j, j \in \mathbb{Z}\), of mean zero random variables and let us consider the partial sums

\[
S_n = \sum_{j=1}^{n} X_j .
\]  

(1.1)

As well studied, it is fairly common that \(S_n/\sqrt{n}\) converges to a Gaussian as \(n \to \infty\), i.e.

\[
\lim_{n \to \infty} \mathbb{P}(S_n \leq \sqrt{D\sqrt{n}s}) = F_G(s) ,
\]  

(1.2)

where \(F_G\) is the distribution function of a unit Gaussian random variable. Here \(F_G\) is the universal object, while the coefficient \(D > 0\) depends on the law \(\mathbb{P}\) and is in this sense model dependent, resp. non-universal. However, using stationarity, \(D\) is readily guessed as

\[
D = \sum_{j=-\infty}^{\infty} \mathbb{E}(X_0X_j) .
\]  

(1.3)

The KPZ class deals with strongly dependent random variables, for which partial sums are of size \(n^{1/3}\) rather than \(n^{1/2}\). \(F_G\) is to be substituted by the GUE Tracy-Widom distribution function, \(F_{GUE}\), which first appeared in the
context of the largest eigenvalue of a GUE random matrix \[ \text{[1] [29]} \]. \( F_{\text{GUE}} \) is defined through a Fredholm determinant as

\[
F_{\text{GUE}}(s) = \det(1 - P_s K_{\text{Ai}} P_s). \tag{1.4}
\]

Here \( K_{\text{Ai}} \) is the Airy kernel,

\[
K_{\text{Ai}}(x, y) = \int_0^\infty d\lambda \text{Ai}(x + \lambda)\text{Ai}(y + \lambda), \tag{1.5}
\]

with \( \text{Ai} \) the Airy function, and \( P_x \) projects onto the interval \( [x, \infty) \). For each \( x \), \( P_x K_{\text{Ai}} P_x \) is a trace class operator in \( L^2(\mathbb{R}) \), hence \( \text{(1.4)} \) is well-defined. To determine the scale coefficient is less obvious than in the example above, but will be explained in due course. Let me stress that the scaling theory is crucial for the proper statistical analysis of either physics \[ \text{[27, 28]} \] or computer experiments \[ \text{[11]} \]. Without this input, the comparison with theoretical results would be considerably less reliable.

Our paper is divided into two parts. We first explain the scaling theory in the context of a specific class of growth models. In the second part the theory is applied to the semi-discrete directed polymer model. The convergence to the GUE Tracy-Widom distribution is established in \[ \text{[5]} \], of course including an expression for the non-universal scale coefficient. Our goal is to explain, how this coefficient can be determined independently, not using the import from the proof in \[ \text{[5]} \].

2 Scaling theory for the single-step model

In the single step growth model the moving surface is described by the graph of the height function \( h(t) : \mathbb{Z} \to \mathbb{Z} \) with the constraint

\[
|h(j + 1, t) - h(j, t)| = 1, \tag{2.1}
\]

hence the name. The random deposition/evaporation events are modeled by a Markov jump process constrained to satisfy \( \text{(2.1)} \). The allowed local moves are then transitions from \( h(j, t) \) to \( h(j, t) \pm 2 \). The dynamics should be invariant under a shift in the \( h \)-direction. Hence the rates for the deposition/evaporation events are allowed to depend only on the local slopes. It is then convenient to switch to height differences

\[
\eta_j(t) = h(j + 1, t) - h(j, t), \quad \eta_j(t) = \pm 1. \tag{2.2}
\]
A single growth step at the bond \((j, j + 1)\) is given by
\[
\eta \rightarrow \eta^{j,j+1},
\]  
(2.3)
where \(\eta^{j,j+1}\) is the configuration with the slopes at \(j\) and \(j + 1\) interchanged. If the corresponding rates are denoted by \(c_{j,j+1}(\eta)\), depending on the local neighborhood of \((j, j + 1)\), the Markov generator reads
\[
Lf(\eta) = \sum_{j \in \mathbb{Z}} c_{j,j+1}(\eta)\left(f(\eta^{j,j+1}) - f(\eta)\right).
\]  
(2.4)

The slope field \(\eta\) is locally conserved, in the sense that the sum \(\sum_{j=a}^{b} \eta(j, t)\) changes only through the fluxes at the two boundaries \(a, b\). To keep things concretely, in the following we consider only the wedge initial condition
\[
h(j, 0) = |j|.
\]  
(2.5)

The scaling theory is based on the assumption. The spatially ergodic and time stationary measures of the slope process \(\eta(t)\) are precisely labeled by the average density
\[
\rho = \lim_{a \to \infty} \frac{1}{2a + 1} \sum_{|j| \leq a} \eta_j
\]  
(2.6)
with \(|\rho| \leq 1\).

Our assumption has been formulated more than 30 years ago. Except for special cases, it remains open even today, see the book by Liggett [17] for more details. The stationary measures from the assumption are denoted by \(\mu_\rho\), as a probability measure on \((-1, 1)^\mathbb{Z}\).

Given \(\mu_\rho\) one defines two natural quantities:

- the average steady state current
  \[
  j(\rho) = \mu_\rho(c_{0,1}(\eta)(\eta_0 - \eta_1))
  \]  
(2.7)

- the integrated covariance of the conserved slope field
  \[
  A(\rho) = \sum_{j \in \mathbb{Z}} \left(\mu_\rho(\eta_j\eta_j) - \mu_\rho(\eta_j)^2\right).
  \]  
(2.8)
We first notice that for long times there is a law of large numbers stating that
\[ h(j,t) \simeq t\phi(j/t) \tag{2.9} \]
for large \( j, t \) with a deterministic profile function \( \phi \). In fact \( \phi \) is the Legendre transform of \( j \) in the sense that
\[ \phi(y) = \sup_{|\rho| \leq 1} \left( y\rho - j(\rho) \right) . \tag{2.10} \]
The argument is based on the hydrodynamic limit for nonreversible lattice gases \[26\], which asserts that on the macroscopic scale the density \( \rho(x,t) \) of the conserved field satisfies
\[ \frac{\partial}{\partial t} \rho(x,t) + \frac{\partial}{\partial x} j(\rho(x,t)) = 0 \tag{2.11} \]
with initial condition
\[ \rho(x,0) = \begin{cases} 1, & x \geq 0, \\ -1, & x < 0. \end{cases} \tag{2.12} \]

The entropy solution to (2.11), (2.12) is indeed given by (2.9), (2.10).

The average current can be fairly arbitrary, except for the linear behavior near \( \rho = \pm 1 \). On the other hand \( \phi \) is convex up with \( \phi(x) = |x| \) for \( |x| \geq x_c \). At points, where \( \phi \) is either linear or cusp-like, the fluctuations may be different from the generic case and have to be discussed separately. We set
\[ \lambda(\rho) = -j''(\rho) . \tag{2.13} \]

**Conjecture** (KPZ class). Let \( y \) be such that \( \phi \) is twice differentiable at \( y \) with \( \phi''(y) \neq 0 \) and set \( \rho = \phi'(y) \), \( |\rho| < 1 \). If \( A(\rho) \) is small and \( \lambda(\rho) \neq 0 \), then
\[ \lim_{t \to \infty} \mathbb{P} \left( h([yt],t) - t\phi(y) \leq -\left( -\frac{1}{2} \lambda A^2 \right)^{1/3} t^{1/3} s \right) = F_{\text{GUE}}(s) . \tag{2.14} \]

\([\cdot]\) denotes integer part. Since \( \phi''(y) > 0 \), because of Legendre transform \( \lambda(\rho) < 0 \).

On the scale \( (-\frac{1}{2} \lambda A^2 t)^{1/3} \) the height fluctuations are governed by the Tracy-Widom distribution. \( \lambda A^2 \) is the model dependent coefficient which, at least in principle, can be computed once \( \mu_\rho \) is available. \( \lambda \equiv 0 \) for reversible slope dynamics, since \( j = 0 \). In that case the fluctuations are of scale \( t^{1/4} \).
and Gaussian, see [26], Part II, Chapter 3, for a discussion. For nonreversible slope dynamics one can still arrange for \( \lambda \equiv 0 \). If \( j \) is non-zero, there could be isolated points at which \( \lambda \) vanishes. At a cubic inflection point of \( j \) the height fluctuations are expected to be of order \( (t(\log t)^{1/2})^{1/2} [21, 9] \).

In our set-up the conjecture has been proved by Tracy and Widom [30] for the PASEP. In this case the exchange \( +-- \) to \(--+ \) occurs with rate \( p \) and the exchange \(--+ \) to \( +-- \) with rate \( 1-p, 0 \leq p < \frac{1}{2} \). Then \( \mu, \rho \) is a Bernoulli measure, hence
\[
A(\rho) = 1 - \rho^2, \quad j(\rho) = \frac{1}{2}(2p - 1)(1 - \rho^2), \quad -\frac{1}{2} \lambda A^2 = \frac{1}{2}(1 - 2p)(1 - \rho^2)^2. \tag{2.15}
\]
The profile function is
\[
\phi(y) = \frac{1}{2}(1 - 2p)(1 + (y/(1 - 2p))^2) \quad \text{for } |y| \leq 1 - 2p \text{ and } \phi(y) = |y| \text{ for } |y| \geq 1 - 2p.
\]
For the totally asymmetric case, \( p = 0 \), the limit (2.14) has been established before by Johansson [14]. The PushTASEP falls also under our scheme with a proof by Borodin and Ferrari [6]. The scaling theory is further confirmed for growth models different from single-step, to mention the discrete time TASEP [14], the polynuclear growth model [22], and the KPZ equation [2, 25].

The theory of Macdonald process [5] has brought a \( q \)-deformed version of the TASEP in focus. For us here it is a further example for which the non-universal constants can be computed. For the rate in (2.14) we set
\[
c_{j,j+1}(\eta) = \frac{1}{4}(1 - \eta_j)(1 + \eta_{j+1})g\left((n_j^- (\eta_j))\right), \tag{2.16}
\]
where \( n_j^- \) is the number of consecutive \( - \) slopes to the left of site \( j \). \( g(0) = 0, g(j) > 0 \) for \( j > 0 \), and \( g \) increases at most linearly. The \( q \)-TASEP is the special case where \( g(j) = 1 - q^j, 0 \leq q < 1 \), with the TASEP recovered in the limit \( q \to 0 \). The slope system maps onto the totally asymmetric zero range process for length of consecutive gaps between \( + \) slope, denoted by \( Y_j, j \in \mathbb{Z}, Y_j = 0, 1, \ldots \). In the stationary measure the \( Y_j \)'s are i.i.d. and
\[
\mathbb{P}(Y_0 = k) = \begin{cases} Z(\alpha)^{-1}, & k = 0, \\ Z(\alpha)^{-1}\left(\prod_{j=1}^k g(j)\right)^{-1}\alpha^k, & k = 1, 2, \ldots, \end{cases} \tag{2.17}
\]
where
\[
Z(\alpha) = 1 + \sum_{k=1}^{\infty} \left(\prod_{j=1}^k g(j)\right)^{-1}\alpha^k \tag{2.18}
\]
and \( \alpha > 0 \) such that \( Z(\alpha) < \infty \). The translation invariant, time stationary measures for the \( \eta(t) \)-process with rates (2.16) are stationary renewal processes on \( \mathbb{Z} \) with renewal distribution (2.17), (2.18).
The coefficient $A$ can be computed from
\[
\lim_{N \to \infty} \frac{1}{N} \log \langle \exp \left[ \lambda \sum_{j=1}^{N} \eta_j \right] \rangle_\alpha = r(\lambda),
\] (2.19)
where the average is over the stationary renewal process with parameter $\alpha$. Then $\rho = r'(\lambda)$ and $A = r''(\lambda)$ at $\lambda = 0$. The rate function $r$ is implicitly determined by
\[
r(\lambda) = -\lambda - \log z(\lambda), \quad \frac{1}{Z(\alpha)} z(\lambda) Z(\alpha z(\lambda)) e^{2\lambda} = 1.
\] (2.20)
$\rho$, $A$ are computed by successive differentiations. The result is best expressed through $G(\alpha) = \log Z(\alpha)$. Then
\[
\frac{1}{2}(1 + \rho) = (1 + \alpha G')^{-1},
\] (2.21)
\[
A = 4(1 + \alpha G')^{-3} \alpha (\alpha G')' = -\alpha (1 + \rho) \frac{d\rho}{d\alpha}.
\] (2.22)
The average current is given by
\[
j(\rho) = -2\langle c_{0,1} \rangle_\alpha, \quad j = -\alpha (1 + \rho).
\] (2.23)
One can use (2.23) together with (2.21) to work out $\lambda$. But no particularly illuminating formula results for the combination $\lambda A^2$.

In addition to (2.14) there is a second scale, which will play no role here, but should be mentioned. Instead of the height statistics at the single point $[yt]$ one could consider, for example, the joint distribution of $h(j_1, t)$, $h(j_2, t)$, referred to as transverse correlations. The transverse scale tells us at which separation $|j_1 - j_2|$ there are nontrivial correlations in the limit $t \to \infty$. The KPZ scaling theory asserts that this scale is
\[
(\frac{1}{2} \lambda^2 At)^{2/3}.
\] (2.24)
The factor $1/2$ comes from the requirement that the limit joint distribution is the two-point distribution of the Airy process. Corresponding predictions hold for the multi-point statistics. Also when considering the two-point function of the stationary $\eta(t)$ process, $E(\eta_0(0)\eta_j(t)) = E(\eta_0(0))^2$, up to a shift linear in $t$, $j$ has to vary on the scale of (2.24) to have a nontrivial limit as $t \to \infty$. The scale (2.24) is confirmed for the PNG [24], TASEP [3], and PushTASEP [6] two-point function in case of step initial conditions, for the stationary TASEP [10] and stationary KPZ equation [13], and for TASEP and PNG [7, 8] in case of flat initial conditions.
3 The semi-discrete directed polymer model

Our starting point is a very particular discretization of the stochastic heat equation as
\[ dZ_j = Z_{j-1} dt + Z_j db_j. \] (3.1)
Here \( j \in \mathbb{Z}, \ t \geq 0, \) and \( \{b_j(t), j \in \mathbb{Z}\} \) is a collection of independent standard Brownian motions. The analogue of the wedge initial condition is
\[ Z_j(0) = \delta_{j,0}. \] (3.2)
Hence \( Z_j(t) = 0 \) for \( j < 0 \) and
\[ dZ_j = Z_{j-1} dt + Z_j db_j, \quad j = 1, 2, \ldots, \]
\[ dZ_0 = Z_0 db_0. \] (3.3)
Let us introduce the totally asymmetric random walk, \( w(t), \) on \( \mathbb{Z} \) moving with rate 1 to the right. Denoting by \( \mathbb{E}_0 \) expectation for \( w(t) \) with \( w(0) = 0, \) one can represent
\[ Z_j(t) = \mathbb{E}_0 \left( \exp \left[ \int_0^t db_w(s) \right] \delta_{w(t),j} \right) e^t. \] (3.4)
\( Z_j(t) \) is the random partition function of the directed polymer \( w(t), \) length \( t, \) endpoints 0 and \( j, \) in the random potential \( db_j(s)/ds. \) This model was first introduced by O’Connell and Yor [20], see also [18, 19]. In the zero temperature limit one would maximize over the term in the exponential at fixed \( \{b_j(s)\} \) and fixed endpoints, see [12] for an early study. The statistics of the maximizer is closely related to GUE and Dyson’s Brownian motion [3].

The height corresponds to the random free energy and we set
\[ h_j(t) = \log Z_j(t), \quad j \geq 0, \ t > 0. \] (3.5)
h_{j} is the solution to
\[ dh_j = e^{h_j-1-h_j} dt + db_j \] (3.6)
and the slope \( u_j = h_{j+1} - h_j \) is governed by
\[ du_j = (e^{-u_j} - e^{-u_{j-1}}) dt + db_{j+1} - db_j, \quad j = 1, 2, \ldots, \]
\[ du_0 = e^{-u_0} + db_1 - db_0. \] (3.8)
The slope $u_j(t)$ is locally conserved.

Somewhat unexpectedly, one can still find the stationary and translation invariant measures for the interacting diffusions (3.7) on the lattice $\mathbb{Z}$. They are labeled by a parameter $r > 0$ and are of product form. The single site measure is a double exponential of the form

$$
\mu_r(dx) = \Gamma(r)^{-1} e^{-e^{-x}} e^{-rx} dx, \quad r > 0.
$$

Averages with respect to $\mu_r$ are denoted by $\langle \cdot \rangle_r$. The parameters of the scaling theory are now easily computed. We find

$$
\rho = \langle u_0 \rangle_r = -\psi(r)
$$

with $\psi = \Gamma'/\Gamma$ the Digamma function on $\mathbb{R}^+$. Note that $\psi' > 0$, $\psi'' < 0$, and $\rho$ ranges over $\mathbb{R}$. From (3.7) the random current is $-e^{-u_j} dt - db_{j+1}$ and hence the average current

$$
j = -\langle e^{-u_0} \rangle_r = -r.
$$

Finally

$$
A(r) = \langle u_0^2 \rangle_r - \langle u_0 \rangle_r^2 = \psi'(r).
$$

Since the initial conditions force $\phi$ to be convex down, the signs from Section 2 are reversed. In particular, the sup in (2.10) is replaced by the inf and

$$
\phi(y) = \inf_{\rho \in \mathbb{R}} (-y\rho - j(-\rho)), \quad y \geq 0.
$$

$\phi(0) = 0$, $\phi'' < 0$, and $\phi$ has a single strictly positive maximum before dropping to $-\infty$ as $y \to \infty$. Thus $t\phi(y/t)$ reproduces the singular initial conditions for (3.6) as $t \to 0$. Moriarty and O’Connell [18] prove that

$$
\lim_{N \to \infty} \frac{1}{N} h_N(\kappa N) = f(\kappa)
$$

with

$$
f(\kappa) = \inf_{s \geq 0} (\kappa s - \psi(s)).
$$

The scaling theory claims that

$$
\lim_{t \to \infty} \frac{1}{t} h_{[yt]}(t) = \phi(y), \quad y > 0.
$$
Hence, using that $\psi' > 0$ and $y\kappa = 1$, we have

$$
\phi(y) = \frac{1}{\kappa} f(\kappa) = \inf_{s \geq 0} (s - y\psi(s)) = \inf_{s \in \mathbb{R}} (\psi^{-1}(\bar{s}) - y\bar{s}) ,
$$

(3.17)

in agreement with (3.13).

The asymptotic analysis of the height fluctuations is due to Borodin and Corwin with the result

**Theorem** (5.2.12 of [5]). There exists a $\kappa^*$ such for $0 < \kappa^* < \kappa$ it holds

$$
\lim_{n \to \infty} \mathbb{P}(h_n(\kappa n) - nf(\kappa) \leq (-2f''(\kappa))^{-1/3} n^{1/3}s) = F_{\text{GUE}}(s) .
$$

(3.18)

According to (3.11), $\lambda = -j'' > 0$. Hence in (2.14) $-\lambda$ is replaced by $\lambda$ and the sign to the right of $\leq$ is $+$. To see whether with these changes the scaling theory is confirmed, we start from

$$
h_{|yt}(t) = t\phi(y) + \left(\frac{1}{2}\lambda A^2 t\right)^{1/3} \xi_{\text{TW}}
$$

(3.19)

with $\xi_{\text{TW}}$ a GUE Tracy-Widom distributed random variable, hence

$$
h_n(\kappa n) = \kappa n\phi(\kappa^{-1}) + \left(\frac{1}{2}\lambda A^2 \kappa n\right)^{1/3} \xi_{\text{TW}}.
$$

(3.20)

Now $\rho = -\psi(r(\rho))$ is differentiated as

$$
1 = -\psi' r' , \quad 0 = \psi''(r')^2 + \psi' r''.
$$

(3.21)

Since $-\lambda = j''(\rho) = -r''(\rho)$ and $A(r) = \psi'(r)$, one has

$$
\lambda A^2 = \psi'' r' .
$$

(3.22)

Since $y = j'(\rho) = -r'(\rho)$ and $y\kappa = 1$, we conclude

$$
\lambda A^2 \kappa = -\psi''
$$

(3.23)

and, since $f$ is the Legendre transform of $\psi$,

$$
\lambda A^2 \kappa = -\frac{1}{f''},
$$

(3.24)

in agreement with (3.18).
4 Conclusion

The KPZ scaling theory makes a prediction on the non-universal coefficients for models in the KPZ class and has been confirmed for PASEP, discrete TASEP, and PNG. We add to this list the semi-discrete directed polymer. The corresponding stochastic “particle” model is a system of diffusions, \( u_j(t) \), with nearest neighbor interactions such that the sums \( \sum_j u_j(t) \) are locally conserved. This model has a flavor rather distinct from driven lattice gases. Still the long time asymptotics in all models is the Tracy-Widom statistics.

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