ON THE UPPER BOUNDS FOR THE CONSTANTS OF THE HARDY–LITTLEWOOD INEQUALITY

GUSTAVO ARAÚJO, DANIEL PELLEGRINO, AND DIOGO DINIZ P. DA SILVA E SILVA

Abstract. The best known upper estimates for the constants of the Hardy–Littlewood inequality for $m$–linear forms on $\ell_p$ spaces are of the form $(\sqrt{2})^{m-1}$. We present better estimates which depend on $p$ and $m$. An interesting consequence is that if $p \geq m^2$ then the constants have a subpolynomial growth as $m$ tends to infinity.

1. Introduction

Let $K$ be $\mathbb{R}$ or $\mathbb{C}$. Given an integer $m \geq 2$, the Hardy–Littlewood inequality (see [1, 8, 12]) asserts that for $2m \leq p \leq \infty$ there exists a constant $C^K_{m,p} \geq 1$ such that, for all continuous $m$–linear forms $T : \ell_p^n \to K$ and all positive integers $n$,

$$
\left( \sum_{j_1, \ldots, j_m=1}^n |T(e_{j_1}, \ldots, e_{j_m})|^{\frac{2mp}{mp+p-2m}} \right)^{\frac{mp+p-2m}{2mp}} \leq C^K_{m,p} \|T\|.
$$

Using the generalized Kahane-Salem-Zygmund inequality (see [1]) one can easily verify that the exponents $\frac{2mp}{mp+p-2m}$ are optimal. The case $p = \infty$ recovers the classical Bohnenblust–Hille inequality (see [4]). More precisely, it asserts that there exists a constant $B^\text{mult}_{K,m}$ such that for all continuous $m$–linear forms $T : \ell_\infty^n \to K$ and all positive integers $n$,

$$
\left( \sum_{j_1, \ldots, j_m=1}^n |T(e_{j_1}, \ldots, e_{j_m})|^{\frac{2m}{mp+p-2m}} \right)^{\frac{mp+p-2m}{2mp}} \leq B^\text{mult}_{K,m} \|T\|.
$$

From [3, 11] we know that $B^\text{mult}_{K,m}$ has a subpolynomial growth. On the other hand, the best known upper bounds for the constants in [1] are $(\sqrt{2})^{m-1}$ (see [1, 2, 6]). In this paper we show that $(\sqrt{2})^{m-1}$ can be improved to $C^R_{m,p} \leq (\sqrt{2})^{\frac{2m(m-1)}{p}} \left( B^\text{mult}_{R,m} \right)^{\frac{m-2m}{p}}$ for real scalars and to $C^C_{m,p} \leq \left( \frac{2}{\sqrt{p}} \right)^{\frac{2m(m-1)}{p}} \left( B^\text{mult}_{C,m} \right)^{\frac{m-2m}{p}}$ for complex scalars. These estimates are quite better than $(\sqrt{2})^{m-1}$ because $B^\text{mult}_{K,m}$ has a subpolynomial growth. Moreover, our estimates depend on $p$ and $m$ and catch more subtle information. For instance, if $p \geq m^2$ we conclude that $(C^K_{m,p})_{m=1}^\infty$ has a subpolynomial growth. Our main result is the following:

Theorem 1.1. Let $m \geq 2$ be a positive integer and $2m \leq p \leq \infty$. Then, for all continuous $m$–linear forms $T : \ell_p^n \to \mathbb{K}$ and all positive integers $n$, we have

$$
\left( \sum_{j_1, \ldots, j_m=1}^n |T(e_{j_1}, \ldots, e_{j_m})|^{\frac{2mp}{mp+p-2m}} \right)^{\frac{mp+p-2m}{2mp}} \leq C^K_{m,p} \|T\|
$$

with

$$
C^R_{m,p} \leq \left( \sqrt{2} \right)^{\frac{2m(m-1)}{p}} \left( B^\text{mult}_{R,m} \right)^{\frac{m-2m}{p}}
$$

The authors are supported by CNPq Grant 313797/2013-7 - PVE - Linha 2.
and
\[ C_{m,p}^C \leq \left( \frac{2}{\sqrt{p}} \right)^{\frac{2m(m-1)}{p}} (B_{\text{mult}}^{C,m})^{\frac{2m}{p}}. \]

2. The proof

We recall that the Khinchin inequality (see [5]) asserts that for any \( 0 < q < \infty \), there are positive constants \( A_q \), \( B_q \) such that regardless of the scalar sequence \( (a_j)_{j=1}^\infty \) in \( \ell_2 \) we have
\[ A_q \left( \sum_{j=1}^\infty |a_j|^2 \right)^{\frac{1}{2}} \leq \left( \int_0^1 \sum_{j=1}^\infty a_j r_j(t) \, dt \right)^{\frac{1}{q}} \leq B_q \left( \sum_{j=1}^\infty |a_j|^2 \right)^{\frac{1}{2}}, \]
where \( r_j \) are the Rademacher functions. More generally, from the above inequality together with the Minkowski inequality we know that
\[ A_q^m \left( \sum_{j_1,...,j_m=1}^\infty |a_{j_1,...,j_m}|^2 \right)^{\frac{1}{2}} \leq \left( \int_I \sum_{j_1,...,j_m=1}^\infty a_{j_1,...,j_m} r_{j_1}(t_1)...r_{j_m}(t_m) \, dt_1...dt_m \right)^{\frac{1}{q}} \leq B_q^m \left( \sum_{j_1,...,j_m=1}^\infty |a_{j_1,...,j_m}|^2 \right)^{\frac{1}{2}} \]
for \( I = [0,1]^m \) and all \( (a_{j_1,...,j_m})_{j_1,...,j_m=1}^\infty \) in \( \ell_2 \). The notation of the constant \( A_q \) above will be used in all this paper.

Let \( 1 \leq s \leq 2 \) and
\[ \lambda_0 = \frac{2s}{ms + s - 2m + 2}. \]
Since
\[ \frac{m - 1}{s} + \frac{1}{\lambda_0} = \frac{m + 1}{2}, \]
from the generalized Bohnenblust–Hille inequality (see [1]) we know that there is a constant \( C_m \geq 1 \) such that for all \( m \)-linear forms \( T : \ell_\infty^n \times \cdots \times \ell_\infty^n \to \mathbb{K} \) we have
\[ \left( \sum_{j_1=1}^n \left( \sum_{j_i=1}^n |T(e_{j_1},...,e_{j_m})|^s \right)^{\frac{1}{s}} \right)^{\frac{s}{\lambda_0}} \leq C_m \|T\|. \]
Above, \( \sum_{j_k=1}^n \) means the sum over all \( j_k \) for all \( k \neq i \). If we choose
\[ s = \frac{2mp}{mp + p - 2m} \quad \text{if } p < \infty, \]
\[ s = \frac{2m}{m + 1} \quad \text{if } p = \infty, \]
we have
\[ \lambda_0 \leq s \leq 2, \]
and \( \lambda_0 = s \) when \( p = \infty \).

The multiple exponent
\[ (\lambda_0, s, s, ..., s) \]
can be obtained by interpolating the multiple exponents \((1, 2, ..., 2)\) and \(\left( \frac{2m}{m+1}, ..., \frac{2m}{m+1} \right)\) with, respectively,
\[ \theta_1 = 2 \left( \frac{1}{\lambda_0} - \frac{1}{s} \right) \]
\[ \theta_2 = m \left( \frac{2}{s} - 1 \right), \]
in the sense of [1].

It is thus important to control the constants associated to the multiple exponents \((1, 2, ..., 2)\) and \(\left( \frac{2m}{m+1}, ..., \frac{2m}{m+1} \right)\).
The exponent $\left(\frac{2m}{m+1}, \ldots, \frac{2m}{m+1}\right)$ is the classical exponent of the Bohnenblust–Hille inequality and the estimate of the constant associated to $(1, 2, \ldots, 2)$ is well-known (we present the details for the sake of completeness). In fact, in general, for the exponent $\left(\frac{2k}{k+1}, \ldots, \frac{2k}{k+1}, 2, \ldots, 2\right)$ (with $\frac{2k}{k+1}$ repeated $k$ times and $2$ repeated $m-k$ times), using the multiple Khinchin inequality \(^4\), we have, for all $m$-linear forms $T : l_\infty^m \times \cdots \times l_\infty^m \to K$,

$$
\left( \sum_{j_1, \ldots, j_k=1}^{n} \left( \sum_{j_{k+1}, \ldots, j_m=1}^{n} |T(e_{j_1}, \ldots, e_{j_m})|^2 \right)^{\frac{1}{2} \frac{2k}{k+1}} \right)^{\frac{k+1}{2k}} \\
\leq \left( \sum_{j_1, \ldots, j_k=1}^{n} \left( A^{-\frac{(m-k)}{k+1}} \int_{[0,1]^{m-k}} \left| T(e_{j_1}, \ldots, e_{j_{k-1}}, \sum_{j_{k+1}=1}^{n} r_{j_{k+1}}(t_{k+1}) \ldots r_{j_m}(t_m) T(e_{j_1}, \ldots, e_{j_m}) dt_{k+1} \ldots dt_m \right) \right)^{\frac{2k}{k+1}} \right)^{\frac{k+1}{2k}} \\
= A^{-\frac{(m-k)}{k+1}} \int_{[0,1]^{m-k}} \left( \sum_{j_1, \ldots, j_k=1}^{n} r_{j_{k+1}}(t_{k+1}) \ldots r_{j_m}(t_m) e_{j_1}, \ldots, e_{j_m} \right)^{\frac{2k}{k+1}} \right)^{\frac{k+1}{2k}} \\
\leq A^{-\frac{(m-k)}{k+1}} \sup_{t_{k+1}, \ldots, t_m \in [0,1]} B_{R, k}^{\text{mult}} \left\| T(e_{j_1}, \ldots, e_{j_{k-1}}, \sum_{j_{k+1}=1}^{n} r_{j_{k+1}}(t_{k+1}) \ldots r_{j_m}(t_m) e_{j_1}, \ldots, e_{j_m}) \right\| \right)^{\frac{k+1}{2k}} \\
= A^{-\frac{(m-k)}{k+1}} B_{R, k}^{\text{mult}} \| T \|^{\frac{k+1}{2k}}.
$$

So, choosing $k = 1$, since $A_1 = \left(\sqrt{2}\right)^{-1}$ and $B_{R, 1}^{\text{mult}} = 1$, we conclude that the constant associated to the multiple exponent $(1, 2, \ldots, 2)$ is $\left(\sqrt{2}\right)^{-m-1}$.

Therefore, the optimal constant associated to the multiple exponent

$$(\lambda_0, s, s, \ldots, s)$$

is less or equal (for real scalars) than

$$
\left(\frac{m-1}{\sqrt{2}}\right)^{2(\frac{\lambda_0}{\lambda_0} - \frac{1}{\lambda_0})} \left( B_{R, m}^{\text{mult}} \right)^{m(\frac{1}{2} - 1)}
$$

i.e.,

$$
C_m \leq \left(\frac{m-1}{\sqrt{2}}\right)^{\frac{2m(m-1)}{p}} \left( B_{R, m}^{\text{mult}} \right)^{\frac{p-2m}{p}}.
$$

More precisely, \(^5\) is valid with $C_m$ as above. For complex scalars we can use the Khinchin inequality for Steinhaus variables and replace $\sqrt{2}$ by $\frac{2}{\sqrt{p}}$ as in \(^\text{[10]}\).

Let

$$
\lambda_j = \frac{\lambda_0 p}{p - \lambda_0 j}
$$

for all $j = 0, \ldots, m$. Note that

$$
\lambda_m = s
$$

and that

$$
\left( \frac{p}{\lambda_j} \right)^* = \frac{\lambda_{j+1}}{\lambda_j}
$$

for all $j = 0, \ldots, m - 1$. 

Let us suppose that $1 \leq k \leq m$ and that
\[
\left( \sum_{j_i=1}^{n} \left( \sum_{j_i=1}^{n} \left| T(e_{j_1}, \ldots, e_{j_m}) \right|^s \right)^{\frac{1}{s}} \right) \cdot \left( \sum_{j_{k+1}=1}^{m-k} \left( \sum_{j_{k+1}=1}^{m-k} \left| T(e_{j_{k+1}}, \ldots, e_{j_m}) \right|^s \right)^{\frac{1}{s}} \right) \leq C_m \|T\|
\]
is true for all continuous $m$–linear forms $T : \ell_p^n \times \cdots \times \ell_p^n \times \ell_\infty^n \times \cdots \times \ell_\infty^n \to \mathbb{K}$ and for all $i = 1, \ldots, m$. Let us prove that
\[
\left( \sum_{j_i=1}^{n} \left| T(e_{j_1}, \ldots, e_{j_m}) \right|^s \right)^{\frac{1}{s}} \leq C_m \|T\|
\]
for all continuous $m$–linear forms $T : \ell_p^n \times \cdots \times \ell_p^n \times \ell_\infty^n \times \cdots \times \ell_\infty^n \to \mathbb{K}$ and for all $i = 1, \ldots, m$.

The initial case (the case $k = 0$) is precisely \(4\) with $C_m$ as in \(5\).

Consider
\[
T \in \mathcal{L}(\ell_p^n \times \cdots \times \ell_p^n; \ell_\infty^n; \mathbb{K})
\]
and for each $x \in B_{\ell_p^n}$ define
\[
T^{(x)} : \ell_p^n \times \cdots \times \ell_p^n \times \ell_\infty^n \times \cdots \times \ell_\infty^n \to \mathbb{K}
\]
with $xz(k) = (x_j z_j^{(k)})_{j=1}^{n}$. Observe that
\[
\|T\| = \sup \{ \|T^{(x)}\| : x \in B_{\ell_p^n} \}.
\]

By applying the induction hypothesis to $T^{(x)}$, we obtain
\[
\left( \sum_{j_i=1}^{n} \left( \sum_{j_i=1}^{n} \left| T(e_{j_1}, \ldots, e_{j_m}) \right|^s \right)^{\frac{1}{s}} \right) \cdot \left( \sum_{j_{k+1}=1}^{m-k} \left( \sum_{j_{k+1}=1}^{m-k} \left| T(e_{j_{k+1}}, \ldots, e_{j_m}) \right|^s \right)^{\frac{1}{s}} \right) \leq C_m \|T^{(x)}\|
\]
for all $i = 1, \ldots, m$.

We will analyze two cases:

1) $i = k$

Since
\[
\left( \frac{p}{\lambda_{j-1}} \right)^s = \frac{\lambda_j}{\lambda_{j-1}}
\]
for all \( j = 1, \ldots, m \), we conclude that

\[
\left( \sum_{j_k=1}^{n} \left( \sum_{j_k=1}^{n} | T(e_{j_1}, \ldots, e_{j_m}) |^s \right)^{\frac{1}{s}} \right)^{\frac{1}{k}} \leq C_m \| T \|.
\]

where the last inequality holds by (6).

2) \( i \neq k \)

It is important to note that \( \lambda_{k-1} < \lambda_k \leq s \). Denoting, for \( i = 1, \ldots, m \),

\[
S_i = \left( \sum_{j_i=1}^{n} | T(e_{j_1}, \ldots, e_{j_m}) |^s \right)^{\frac{1}{s}}
\]

we get

\[
\sum_{j_k=1}^{n} \left( \frac{\sum_{j_k=1}^{n} | T(e_{j_1}, \ldots, e_{j_m}) |^s}{S_i^{s-\lambda_k}} \right)^{\frac{1}{s}} = \sum_{j_k=1}^{n} S_i^{\lambda_k} = \sum_{j_k=1}^{n} S_i^{\lambda_{k-1}s - \lambda_k} S_i^s \]

\[
= \sum_{j_k=1}^{n} \sum_{j_k=1}^{n} \frac{\sum_{j_k=1}^{n} | T(e_{j_1}, \ldots, e_{j_m}) |^s}{S_i^{s-\lambda_k}} = \sum_{j_k=1}^{n} \sum_{j_k=1}^{n} \frac{| T(e_{j_1}, \ldots, e_{j_m}) |^s}{S_i^{s-\lambda_k}} \frac{S_i^{s-\lambda_{k-1}}}{S_i^{s-\lambda_k}} \frac{S_i^{s-\lambda_{k-1}}}{S_i^{s-\lambda_k-1}}.
\]
Therefore, using Hölder’s inequality twice we obtain

\[
\sum_{j_k=1}^n \left( \sum_{j_1=1}^n |T(e_{j_1}, \ldots, e_{j_m})|^s \right)^{\frac{1}{s}} \leq \sum_{j_k=1}^n \left( \sum_{j_1=1}^n \frac{|T(e_{j_1}, \ldots, e_{j_m})|^s}{S_i^{\lambda_k-\lambda_k-1}} \right)^{\frac{\lambda_k}{s}} \left( \sum_{j_k=1}^n \left( \sum_{j_1=1}^n |T(e_{j_1}, \ldots, e_{j_m})|^s \right)^{\frac{1}{s}} \right)^{\frac{\lambda_k-\lambda_k-1}{s}}.
\]

(7)

Now we investigate the first factor in (7). From Hölder’s inequality and (6) it follows that

\[
\left( \sum_{j_k=1}^n \left( \sum_{j_1=1}^n |T(e_{j_1}, \ldots, e_{j_m})|^s \right)^{\frac{1}{s}} \right)^{\frac{\lambda_k}{s}} \leq \left( C_m \|T\| \right)^{\frac{\lambda_k}{s}}.
\]

(8)

We know from the case \( i = k \) that

\[
\sum_{j_k=1}^n \left( \sum_{j_1=1}^n |T(e_{j_1}, \ldots, e_{j_m})|^s \right)^{\frac{1}{s}} \leq \left( C_m \|T\| \right)^{\frac{(\lambda_k-\lambda_k-1)s}{s}}.
\]

(9)

Now we investigate the first factor in (7). From Hölder’s inequality and (8) it follows that

\[
\left( \sum_{j_k=1}^n \left( \sum_{j_1=1}^n |T(e_{j_1}, \ldots, e_{j_m})|^s \right)^{\frac{1}{s}} \right)^{\frac{\lambda_k}{s}} = \left\| \sum_{j_k=1}^n \left( \sum_{j_1=1}^n |T(e_{j_1}, \ldots, e_{j_m})|^s \right)^{\frac{1}{s}} \right\|^\lambda_k \leq \left( C_m \|T\| \right)^\lambda_k.
\]

Replacing (8) and (9) in (7) we finally conclude that

\[
\sum_{j_k=1}^n \left( \sum_{j_1=1}^n |T(e_{j_1}, \ldots, e_{j_m})|^s \right)^{\frac{1}{s}} \leq \left( C_m \|T\| \right)^{\frac{\lambda_k}{s}} \left( C_m \|T\| \right)^{\frac{\lambda_k-\lambda_k-1}{s}} = \left( C_m \|T\| \right)^\lambda_k.
\]

Since \( \lambda_m = s \) the proof is done.

3. Constants with subpolynomial growth

The optimal constants of the Khinchin’s inequality (these constants are due to U. Haagerup [7]) are

\[
A_q = \sqrt{2} \left( \frac{\Gamma \left( \frac{q+1}{2} \right)}{\sqrt{\pi^q}} \right)^{\frac{1}{q}}
\]

for \( q > q_0 \approx 1.847 \) and

\[
A_q = 2^{\frac{q-1}{2}}
\]
for \( q \leq q_0 \), where \( q_0 \in (0, 2) \) is the unique real number satisfying
\[
\Gamma\left(\frac{q_0 + 1}{2}\right) = \frac{\sqrt{\pi}}{2}.
\]
For complex scalars if we use the Khinchin inequality for Steinhaus variables we have
\[
A_q = \left( \Gamma\left(\frac{q + 2}{2}\right) \right)^{\frac{1}{q}}
\]
for all \( 1 \leq q < 2 \) (see [9]).

The best known upper estimates for \( B^{\text{mult}}_{R,m} \) and \( B^{\text{mult}}_{C,m} \) (from [3]) are
\[
B^{\text{mult}}_{R,m} \leq \prod_{j=2}^{m} A_{\frac{j-1}{2}}^{2j}.
\]
Combining these results we have
\[
C^{\text{R}}_{m,p} \leq \left( \frac{2^{m^2 - m - 2m} + 446381}{55440} \prod_{j=14}^{m} \left( \frac{\frac{3}{2} - \frac{1}{\sqrt{\pi}}}{j} \right)^{\frac{1}{2j}} \right)^{\frac{p-2m}{p}}
\]
for \( m \geq 14 \).
\[
C^{\text{R}}_{m,p} \leq \left( \frac{2}{\sqrt{\pi}} \right)^{\frac{2m(m-1)}{p}} \left( \prod_{j=2}^{m} \frac{2^{-j}}{j} \right)^{\frac{p-2m}{p}}
\]
for \( 2 \leq m \leq 13 \).

Combining these results we have
\[
C^{\text{C}}_{m,p} \leq \left( \frac{2}{\sqrt{\pi}} \right)^{\frac{2m(m-1)}{p}} \left( \prod_{j=2}^{m} \left( 2 - \frac{1}{j} \right) \frac{1}{j} \right)^{\frac{p-2m}{p}}.
\]
From [3] we know that there is a constant \( \kappa > 0 \) such that
\[
B^{\text{mult}}_{R,m} \leq \kappa m^{\frac{2m^2 - 2m - 1}{2}} < \kappa m^{0.37},
\]
\[
B^{\text{mult}}_{C,m} \leq \kappa m^{\frac{2m - 2m}{2}} < \kappa m^{0.22},
\]
for all \( m \), where \( \gamma \) is the Euler-Mascheroni constant. We thus conclude that if \( p \geq m^2 \) then \( \left( C^{\text{K}}_{m,p} \right)_{m=1}^{\infty} \) has a subpolynomial growth.

REFERENCES

[1] N. Albuquerque, F. Bayart, D. Pellegrino and J. Seoane-Sepúlveda, Sharp generalizations of the multilinear Bohnenblust–Hille inequality, J. Funct. Anal. 266 (2014), 3726–3740.
[2] N. Albuquerque, F. Bayart, D. Pellegrino and J. Seoane-Sepúlveda, Optimal Hardy–Littlewood type inequalities for polynomials and multilinear operators, arXiv:1311.3177v3 [math.FA].
[3] F. Bayart, D. Pellegrino and J. B. Seoane-Sepúlveda, The Bohr radius of the \( n \)-dimensional polydisc is equivalent to \( \sqrt{\frac{\log n}{n}} \), arXiv:1310.2834v2 [math.FA] 15 Oct 2013.
[4] H. F. Bohnenblust and E. Hille, On the absolute convergence of Dirichlet series, Ann. of Math. 32 (1931), 600–622.
[5] J. Diestel, H. Jarchow and A. Tonge, Absolutely summing operators, Cambridge University Press, Cambridge, 1995.
[6] V. Dimant and P. Sevilla-Peris, Summation of coefficients of polynomials on \( \ell_p \) spaces, arXiv:1309.6063v1 [math.FA].
[7] U. Haagerup, The best constants in the Khinchine inequality, Studia Math. 70 (1982) 231–283.
[8] G. Hardy and J. E. Littlewood, Bilinear forms bounded in space \( \ell_p \), Quart. J. Math. 5 (1934).
[9] H. König, On the best constants in the Khintchine inequality for variables on spheres. Math. Seminar, Universität Kiel, 1998.
[10] D. Nuñez-Alarcón, D. Pellegrino, J.B. Seoane-Sepúlveda, On the Bohnenblust–Hille inequality and a variant of Littlewood’s 4/3 inequality, J. Funct. Anal. 264 (2013), 326–336.
[11] D. Nuñez, D. Pellegrino and J.B. Seoane, D. M. Serrano-Rodriguez, There exist multilinear Bohnenblust-Hille constants \( (C_n)_{n=1}^{\infty} \) with \( \lim_{n \rightarrow \infty} (C_{n+1} - C_n) = 0 \), J. Funct. Anal. 264 (2013), 429–463.
[12] T. Praciano-Pereira, On bounded multilinear forms on a class of \( \ell_p \) spaces. J. Math. Anal. Appl. 81 (1981), 561–568.
Departamento de Matemática, Universidade Federal da Paraíba, 58.051-900 - João Pessoa, Brazil.
E-mail address: gdasarujo@gmail.com

Departamento de Matemática, Universidade Federal da Paraíba, 58.051-900 - João Pessoa, Brazil.
E-mail address: pellegrino@pq.cnpq.br and dmpellegrino@gmail.com

Unidade Academica de Matematica e Estatistica, Universidade Federal de Campina Grande, Caixa Postal 10044, 58.429-270 - Campina Grande- PB, Brazil.
E-mail address: diogo@dme.ufcg.edu.br