On Hecke theory for Hermitian modular forms

by

Adrian Hauffe-Waschbüsch and Aloys Krieg

Dedicated to Murugesan Manickam on the occasion of his 60th birthday.

Abstract. In this paper we outline the Hecke theory for Hermitian modular forms in the sense of Hel Braun for arbitrary class number of the attached imaginary-quadratic number field. The Hecke algebra turns out to be commutative. Its inert part has a structure analogous to the case of the Siegel modular group and coincides with the tensor product of its $p$-components for inert primes $p$. This leads to a characterization of the associated Siegel-Eisenstein series. The proof also involves Hecke theory for particular congruence subgroups.

Keywords. Hecke algebra, Hermitian modular group, cusp forms, Siegel-Eisenstein series

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1 Introduction

The Hermitian modular group associated with an imaginary-quadratic number field $\mathbb{K}$ was introduced by H. Braun [1], [2] as an analogue of the Siegel modular group. The case of class number $> 1$ leads to number theoretical complications. If one wants to consider the Hecke theory as for instance by Freitag [8], there are only a few concrete results (cf. [5], [11]). Most authors consider the situation over local fields (cf. [16]).

In this paper we show that each double coset contains a matrix in block diagonal form. Hence the Hecke algebra is commutative. Moreover we characterize a particular subalgebra of the Hecke algebra, which is related to inert primes. As a consequence we obtain a characterization of the Siegel-Eisenstein series, which was available up to now only in the case of class number 1 (cf. [13]). Many of our results are similar to the investigations by M. Manickam [14] on Jacobi forms.

2 The Hecke algebra for the Hermitian modular group

Throughout the paper let

$$\mathbb{K} = \mathbb{Q}(\sqrt{-m}) \subset \mathbb{C}, \ m \in \mathbb{N} \text{ squarefree},$$

be an imaginary-quadratic number field. Its discriminant and ring of integers are

$$d_\mathbb{K} = \begin{cases} -m & \text{and } \mathcal{O}_\mathbb{K} = \mathbb{Z} + \mathbb{Z} \omega_\mathbb{K} = \left\{ \mathbb{Z} + \mathbb{Z}(1 + \sqrt{-m})/2 \right\} \text{ if } m \equiv 3 \text{ (mod 4)}, \\ -4m & \mathbb{Z} + \mathbb{Z} \sqrt{-m} \text{ if } m \equiv 1, 2 \text{ (mod 4)}. \end{cases}$$

Denote its class number by $h_\mathbb{K}$ and the associated primitive real Dirichlet character mod $|d_\mathbb{K}|$ by $\chi_\mathbb{K}$.

Define the set of integral unitary similitudes of factor $q \in \mathbb{N}$ by

$$\Delta_n(q) := \{ M \in \mathcal{O}_{\mathbb{K}}^{2n \times 2n}; \ J[M] := \mathbb{M}^t J M = q J \}, \ J = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}, \ I = \begin{pmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{pmatrix}.$$  

Moreover let

$$\Delta_n(q) := \bigcup_{\ell=0}^{\infty} \Delta_n(q^\ell), \ \Delta_n = \bigcup_{q \in \mathbb{N}} \Delta_n(q).$$

Then

$$\Gamma_n := \Delta_n(1) \subseteq U(n, n; \mathbb{C}) := \{ M \in \mathbb{C}^{2n \times 2n}; \ J[M] = J \}$$

is the Hermitian modular group of degree $n$. Given $q \in \mathbb{N}$ let

$$\Gamma_n[q] = \{ M \in \Gamma_n; \ M \equiv I \text{ (mod q)} \}$$

stand for the principal congruence subgroup of level $q$. We will always assume a block
decomposition

\[ M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Delta_n, \ A, B, C, D \in \mathcal{O}_K^{n \times n}. \]

**Lemma 1.** Given \( M \in \Delta_n(q) \) then

\[ \sharp(\Gamma_n \backslash \Gamma_n M \Gamma_n) < \infty. \]

**Proof.** Use \( M^{-1} \Gamma_n M \cap \Gamma_n \supseteq \Gamma_n[q], \) hence

\[ \sharp(\Gamma_n \backslash \Gamma_n M \Gamma_n) = [\Gamma_n : \Gamma_n \cap M^{-1} \Gamma_n M] \leq [\Gamma_n : \Gamma_n[q]] \leq q^{8n^2} < \infty. \]

Hence \((\Gamma_n, \Delta_n)\) fulfills the Hecke-condition (cf. [8], [12]).

Let \( \partial_k(G) \subseteq \mathcal{O}_K \) stand for the ideal generated by all \( k \times k \) subdeterminants of an integral matrix \( G \), which is invariant under multiplication with unimodular matrices. Then [1], Theorem 1, resp. [2], Lemma 1, implies

**Lemma 2.** If \( M \in \Delta_n \) there exist \( L^*, L' \in \Gamma_n \) such that

\[ L^* M = \begin{pmatrix} A^* & B^* \\ 0 & D^* \end{pmatrix}, \ \mathcal{O}_K \det A^* = \partial_n \begin{pmatrix} A \\ C \end{pmatrix}, \]

\[ M L' = \begin{pmatrix} A' & 0 \\ C' & D' \end{pmatrix}, \ \mathcal{O}_K \det A' = \partial_n (A, B). \]

The next step is a block diagonal decomposition in double cosets.

**Lemma 3.** Given \( M \in \Delta_n \) there exist \( L_1, L_2 \in \Gamma_n \) such that

\[ L_1 M L_2 = \begin{pmatrix} A^* & 0 \\ 0 & A^* H \end{pmatrix} \text{ for some } H = T^r \in \mathcal{O}_K^{n \times n}. \]

**Proof.** Choose \( A^* \) such that \(|\det A^*|\) is minimal among all the matrices

\[ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_n M \Gamma_n \text{ with } \det A \neq 0. \]

Let \( M^* = \begin{pmatrix} A^* & B^* \\ C^* & D^* \end{pmatrix} \in \Gamma_n M \Gamma_n \). Then

\[ \partial_n \begin{pmatrix} A^* \\ C^* \end{pmatrix} = \partial_n (A^*, B^*) = (\det A^*) \mathcal{O}_K \]

follows from Lemma [2] hence

\( A^{*^{-1}} B^* \) and \( C^* A^{*^{-1}} \)
are integral and Hermitian. Therefore we get a matrix
\[
\begin{pmatrix}
A^* & 0 \\
0 & D'
\end{pmatrix} \in \Gamma_n M \Gamma_n.
\]
As \((A\ast D') \in \Gamma_n M \Gamma_n\) we conclude
\[
A^{*-1} D' = H \in \mathcal{O}_K^{n \times n} \quad \text{and} \quad H = \overline{H}^{tr}.
\]

A simple consequence is

**Corollary 1.** Given \(M \in \Delta_n\) then
\[
\Gamma_n M \Gamma_n = \Gamma_n M^{tr} \Gamma_n.
\]

**Proof.** We assume \((A\ast D') \in \Gamma_n M \Gamma_n\) due to Lemma 3. By means of \([6]\), Theorem 2.2, there are \(U, V \in GL_n(\mathcal{O}_K)\) such that
\[
UAV = A^{tr}.
\]
Hence
\[
\begin{pmatrix}
U & 0 \\
0 & U^{tr-1}
\end{pmatrix}
\begin{pmatrix}
A & 0 \\
0 & D
\end{pmatrix}
\begin{pmatrix}
V & 0 \\
0 & V^{tr-1}
\end{pmatrix} = \begin{pmatrix}
A^{tr} & 0 \\
0 & D^*
\end{pmatrix},
\]
\(J[M] = qJ\) then implies \(D^* = D^{tr}\). \(\square\)

As \(M \mapsto M^{tr}\) is an involution which keeps the double cosets invariant, we conclude from \([8]\) or \([12]\).

**Theorem 1.** \((\Gamma_n, \Delta_n)\) is a Hecke pair. The Hecke algebra \(\mathcal{H}(\Gamma_n, \Delta_n)\) is commutative.

Our next aim is to describe particular products in this Hecke algebra. Therefore we need

**Lemma 4.** Let \(q, r \in \mathbb{N}\) be coprime and \(d_K \neq -3, -4\). Then
\[
\Gamma_n[q] \cdot \Gamma_n[r] = \Gamma_n.
\]

**Proof.** As the principal congruence subgroups are normal, we may restrict to generators of \(\Gamma_n\). We use the generators from \([4]\), Theorem 2.1, for which the claim follows by a simple calculation of the form
\[
\begin{pmatrix}
1 & \ell H \\
0 & I
\end{pmatrix}
\begin{pmatrix}
1 & H \\
0 & I
\end{pmatrix} \in \Gamma_n[r]
\]
for \(H = \overline{H}^{tr} \in \mathcal{O}_K^{n \times n}\) and some \(\ell \in \mathbb{N}, q \mid \ell\). \(\square\)

An application is described in
Corollary 2. Given $M \in \Delta_n(q), r \in \mathbb{N}$, $\gcd(q, r) = 1$ then
\[ \Gamma_n M \Gamma_n = \Gamma_n M \Gamma_n [r]. \]

Proof. Clearly $M^{-1} \Gamma_n M \cap \Gamma_n \supseteq \Gamma_n [q]$ holds. Now apply Lemma 4.

We consider a particular case. Let $M \in \Delta_n(q), \gcd(q, r) = 1$ and $M \equiv I \pmod{r}$ as well as
\[ \Gamma_n M \Gamma_n = \bigcup_{1 \leq j \leq \ell} \Gamma_n L_j, \quad L_j \equiv I \pmod{r} \]
due to Corollary 2. Then we immediately obtain

(1) \[ \Gamma_n [r] M \Gamma_n [r] = \bigcup_{1 \leq j \leq \ell} \Gamma_n [r] L_j \]
as well as

(2) \[ \Gamma_n M \Gamma_n = \bigcup_{1 \leq j \leq \ell} \Gamma_n R L_j R^{-1} \text{ for } R \in \Delta_n(r). \]

An immediate consequence is

Corollary 3. Given $M \in \Delta_n(q), L \in \Delta_n(r)$ with coprime $q, r \in \mathbb{N}$, then
\[ \Gamma_n M \Gamma_n \cdot \Gamma_n L \Gamma_n = \Gamma_n M L \Gamma_n. \]

Proof. We choose decompositions
\[ \Gamma M \Gamma_n = \bigcup_i \Gamma_n M K_i, \quad K_i \in \Gamma_n [r], \quad \Gamma_n L \Gamma_n = \bigcup_j \Gamma_n L R_j \]
due to Corollary 2. Clearly the right cosets
\[ \Gamma_n M K_i L R_j \]
are mutually disjoint and contained in $\Gamma_n M L \Gamma_n$. Thus the claim follows.

In the case of $h_K = 1$ the Hecke algebra coincides with the tensor product of its primary components
\[ \mathcal{H}_{n,p} = \mathcal{H}(\Gamma_n, \Delta_{n,p}), \quad p \text{ prime}. \]
In this situation the structure is described in [11]. If $h_K > 1$ this result is no longer true (cf. [5], 3.3.6), e.g. $K = \mathbb{Q}(\sqrt{-5})$
\[ \Gamma_2 \text{ diag } (1, 1 + \sqrt{-5}, 6, 1 + \sqrt{-5}) \Gamma_2 \notin \bigotimes_p \mathcal{H}_{n,p}. \]
Many authors define the Hecke algebra as the tensor product of its $p$-components (cf. [16]). But the tensor product is a proper subalgebra of $\mathcal{H}(\Gamma_n, \Delta_n)$ in general.

The example shows that it is much more difficult to look at the decomposition of double cosets.

Lemma 5. Let $M \in \Delta_n(q)$, $q = r_1 r_2 \in \mathbb{N}$, where $r_1$ is a product of split or ramified primes and $r_2$ a product of inert primes. Then there exist $M_j \in \Delta_n(r_j)$, $j = 1, 2$, such that

$$\Gamma_n M \Gamma_n = \Gamma_n M_1 \Gamma_n \cdot \Gamma_n M_2 \Gamma_n.$$ 

Proof. We may assume $M = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ due to Lemma \[\] and consider the determinantal divisors. Let

$$\partial_k(A) = \mathcal{I}_k \cdot \mathcal{O}_K a_k, \quad k = 1, \ldots, n,$$

where $a_k \in \mathbb{N}$ divides $r_2^n$ and $\mathcal{I}_k$ is not divisible by $p \mathcal{O}_K$ for any inert prime $p$. In view of [6], Theorem 2.1, there exist

$$A_j \in \mathcal{O}_K^{n \times n}, \quad j = 1, 2, \quad \partial_k(A_1) = \mathcal{I}_k, \quad \partial_k(A_2) = \mathcal{O}_K a_k, \quad k = 1, \ldots, n.$$

Define $D_j = r_j A_j^{p-1}$. Then we have

$$M_j = \begin{pmatrix} A_j & 0 \\ 0 & D_j \end{pmatrix} \in \Delta_n(r_j), \quad j = 1, 2,$$

$$\Gamma_n M_1 \Gamma_n \cdot \Gamma_n M_2 \Gamma_n = \Gamma_n M_1 M_2 \Gamma_2$$

by means of Corollary \[\] As $\mathcal{I}_k$ and $\mathcal{O}_K a_k$ are coprime, we conclude

$$\partial_k(A_1 A_2) = \partial_k(A_1) \cdot \partial_k(A_2) = \partial_k(A), \quad k = 1, \ldots, n,$$

from [6], Theorem 4.2, or [5], Satz 2.6.8. Then

$$\Gamma_n M_1 M_2 \Gamma_2 = \Gamma_n M \Gamma_n$$

follows from [6], Theorem 2.2. \[\]

3 The inert part of the Hecke algebra

Lemma \[\] shows that it is interesting to have a closer look at the inert part defined by

$$\Delta_n^{\text{inert}} = \bigcup_{q \in \mathbb{N}} \Delta_n(q)$$

and call

$$\mathcal{H}_n^{\text{inert}} = \mathcal{H}(\Gamma_n, \Delta_n^{\text{inert}})$$

the inert part of the Hecke algebra.
Given \( M \in \Delta_n(q) \), where \( q \) is only divided by inert primes, we conclude that \( \partial_k(M) = \phi \Delta r \), where \( r | q^n \). Thus we can apply Theorem 1 as well as \([6]\), Theorem 2.2, in order to obtain the elementary divisor theorem similar to the case of the Siegel modular group (cf. \([8\), \([12]\]).

**Theorem 2.** Given \( M \in \Delta_n(q) \subseteq \Delta_n^{\text{inert}} \) the double coset \( \Gamma_n M \Gamma_n \) contains a unique representative

\[
\text{diag}(a_1, \ldots, a_n, d_1, \ldots, d_n), \quad a_j, d_j \in \mathbb{N}, \quad a_j d_j = q, \quad a_1 | a_2 | \ldots | a_n | d_{n-1} | \ldots | d_1.
\]

In this case the elementary divisor theorem holds. Next we have a look at right coset representatives.

**Corollary 4.** Given \( M \in \Delta_n(q) \subseteq \Delta_n^{\text{inert}} \) the right coset \( \Gamma_n M \) possesses a representative of the form

\[
\left( \begin{array}{cc} A & B \\ 0 & D \end{array} \right), \quad A = \overline{D}^{tr-1},
\]

where \( D \) is an upper triangular matrix with diagonal entries \( d_j \in \mathbb{N}, \quad d_j | q, \quad j = 1, \ldots, n \).

Now we use Corollary 3 in order to get

**Corollary 5.** \( \mathcal{H}_n^{\text{inert}} = \bigotimes_{p \text{ inert}} \mathcal{H}(\Gamma_n, \Delta_n,p) \).

In this case one can directly adopt the proofs, which are given for the Siegel modular group in \([8\) or \([12\].

Next we consider generators.

**Corollary 6.** Let \( p \) be an inert prime. Then \( \mathcal{H}(\Gamma_n, \Delta_n,p) \) is generated by the double cosets

\[
\mathcal{T}_n(p) = \Gamma_n \left( \begin{array}{cc} I & 0 \\ 0 & pI \end{array} \right) \Gamma_n,
\]

\[
\mathcal{T}_{n,j}(p^2) = \Gamma_n \text{diag} \left( 1, \ldots, 1, p, \ldots, p, p^2, \ldots, p^2, p, \ldots, p \right) \Gamma_n, \quad j = 0, \ldots, n-1
\]

which are algebraically independent.

Given \( M \in \Delta_n(q) \subseteq \Delta_n^{\text{inert}} \) we choose a representative

\[
M^* = \left( \begin{array}{cc} A & B \\ 0 & D \end{array} \right), \quad A = \left( \begin{array}{cc} A_1 & 0 \\ \alpha & \alpha \end{array} \right), \quad B = \left( \begin{array}{cc} B_1 & * \\ * & * \end{array} \right), \quad D = \left( \begin{array}{cc} D_1 & d \\ 0 & \delta \end{array} \right)
\]

in \( \Gamma_n M \) and define for \( k \in \mathbb{Z}, n \geq 2 \)

\[
\phi_k(\Gamma_n M) = \delta^{-k} \Gamma_{n-1} M_1, \quad M_1 = \left( \begin{array}{cc} A_1 & B_1 \\ 0 & D_1 \end{array} \right) \in \Delta_{n-1}(q).
\]
This map can be extended to a homomorphism of Hecke algebras (cf. [8], [10], [11], [12]). The main result is

**Corollary 7.** If $p$ is an inert prime and $n \geq 2$ one has

$$\phi_k(T_n(p)) = (p^{2n-1-k} + 1)T_{n-1}(p).$$

Note that we also need the Hecke algebra for $\Gamma_n[r]$, i.e.

$$T_n^r(p) = \Gamma_n[r] \begin{pmatrix} I & 0 \\ 0 & pI \end{pmatrix} \Gamma_n[r].$$

If $p \equiv 1 \pmod{r}$ we have the same result as above due to (1).

4 Hermitian modular forms

Let

$$\mathbb{H}_n := \{ Z \in \mathbb{C}^{n \times n}; \frac{1}{2} (Z - Z^t) > 0 \}$$

denote the Hermitian half-space of degree $n$, where $>$ resp. $\geq 0$ stands for positive definite resp. positive semi-definite. Given $f: \mathbb{H}_n \rightarrow \mathbb{C}, M = (A B) \in \Delta_n$ we define for $k \in \mathbb{Z}$

$$f \mid_k M: \mathbb{H}_n \rightarrow \mathbb{C}, \, Z \mapsto \det(CZ + D)^{-k}f((Az + B)(CZ + D)^{-1}).$$

The vector space $\mathcal{M}(\Gamma_n, k)$ of *Hermitian modular forms* consists of all holomorphic functions $f: \mathbb{H}_n \rightarrow \mathbb{C}$ satisfying

$$f \mid_k M = f \quad \text{for all} \quad M \in \Gamma_n$$

with the usual additional condition of boundedness for $n = 1$, where we deal with classical elliptic modular forms for $SL_2(\mathbb{Z})$. Each $f \in \mathcal{M}(\Gamma_n, k)$ possesses a Fourier expansion of the form

$$f(Z) = \sum_{T \in \Lambda_n, T \geq 0} \alpha_f(T) e^{2\pi i \text{trace}(TZ)},$$

where $T = (t_{ij}) \in \Lambda_n$ means $T = T^t$, $t_{jj} \in \mathbb{Z}$, $t_{ij} \in \frac{1}{\sqrt{d_K}} \mathcal{O}_K$ for $i \neq j$.

The subspace of *cusp forms* $\mathcal{C}(\Gamma_n, k)$ is characterized by

$$\alpha_f(T) \neq 0 \Rightarrow T > 0.$$
Moreover we define the Siegel $\phi$-operator by

$$f|_\phi : \mathbb{H}_{n-1} \to \mathbb{C}, \quad Z_1 \mapsto \lim_{y \to \infty} f \left( \frac{Z_1}{iy} \right) = \sum_{T_1 \in \Lambda_{n-1}, T_1 \geq 0} \alpha_f \left( \frac{T_1}{0 \ 0 \ 0} \right) e^{2\pi i \text{trace} (T_1 Z_1)}.$$

If $h_K = 1$ then $f$ is a cusp form if and only if $f|_\phi \equiv 0$. This is more complicated for $h_K > 1$ (cf. [3], Lemma 1). Therefore let

$$R_U = \begin{pmatrix} U^r & 0 \\ 0 & U^{-1} \end{pmatrix} \in U(n, n; \mathbb{K}) \text{ for } U \in GL_n(\mathbb{K}).$$

**Theorem 3.** Let $n \geq 2$ and let $I_j = \langle u_j, 1 \rangle, \ u_j \in \mathbb{K}, \ j = 1, \ldots, h, \ h = h_K$, be a set of representatives of the ideal classes in $\mathbb{K}$. Then $f \in \mathcal{M}(\Gamma_n, k)$ is a cusp form if and only if

$$f|_{kR^{(n)}_{U_j}}|_\phi \equiv 0, \quad U_j = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ 0 & \cdots & 0 & \overline{u}_j \end{pmatrix}, \ j = 1, \ldots, h.$$

**Proof.** Let $T_0 \in \Lambda_n, \ T_0 \geq 0, \ det T_0 = 0$. Then there exists $0 \neq g \in \mathcal{O}_K^0$ with $T_0g = 0$. Next we determine $U \in GL_n(\mathcal{O}_K)$ and $1 \leq j \leq n$ such that

$$T_0 = U\overline{T}U^r, \quad T_0 = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ u_j \end{pmatrix} \cdot \lambda, \ 0 \neq \lambda \in \mathcal{O}_K,$$

hence

$$g = U^r \overline{T} U_j e_n \cdot \lambda, \quad T_0 [U^r \overline{T} U_j] = \begin{pmatrix} * & 0 \\ 0 & 0 \end{pmatrix}.$$

In view of

$$f|_{kR_{U_j}}|_{kR_U} = f|_{kR_{U_j}}|_{kR_{U_j}}, \quad (\det U)^k \sum_{T \in \Lambda, T \geq 0} \alpha_f (T) e^{2\pi i \text{trace} (U \overline{T} U^r \overline{T}^r Z)}$$

the application of $\phi$ yields $\alpha_f (T_0) = 0$. Hence $f$ is a cusp form. \hfill \Box

Now we have a closer look at the choice of $u_j$ in Theorem 3.

**Lemma 6.** Let $d_K \neq -4, -8$ and $p$ be an odd prime, $p | d_K$. Then representatives of the ideal classes $I_j = \langle u_j, 1 \rangle, \ u_j \in \mathbb{K}$, may be chosen such that we find an $N \in \mathbb{N}$ with the
properties

\[ p \nmid N \text{ and } Nu_j \in \mathcal{O}_K, \ j = 1, \ldots, h_K. \]

**Proof.** According to [7], p. 211, \( u_j \) may be chosen in the form

\[ u_j = \frac{\beta_j + \sqrt{d_K}}{2\alpha_j}, \quad \beta_j^2 - d_K = 4\alpha_j \gamma_j, \quad \alpha_j, \gamma_j \in \mathbb{N}, \beta_j \in \mathbb{Z}. \]

As \( \alpha_j \in \mathbb{N} \) let \( N_j \in \mathbb{N} \) be minimal such that \( N_j u_j \in \mathcal{O}_K \), we may assume \( p \mid \alpha_j \) as we are done otherwise. Then \( p \mid \beta_j \) follows. As \( p^2 \nmid d_K \) we obtain

\[ p^2 \nmid (\beta_j^2 - d_K), \quad p^2 \nmid \alpha_j. \]

Thus we may choose

\[ u_j^* = \frac{2\alpha_j}{p(\beta_j + \sqrt{d_K})}, \quad \langle u_j^*, 1 \rangle K^* = \langle u_j, 1 \rangle K^* \]

and

\[ N_j = \frac{\beta_j^2 - d_K}{p} \in \mathbb{N} \text{ satisfies } Nu_j^* \in \mathcal{O}_K, \quad p \nmid N. \]

Then \( N = N_1 \cdots N_h_K \) is a solution.

Next we need a purely number theoretical assertion on the existence of such primes.

**Lemma 7.** Let \( d_K \neq -4, -8 \) and suppose that there is an odd prime divisor of \( d_K \), which does not divide \( N \in \mathbb{N} \). Then there exist infinitely many inert primes \( p \equiv 1 \mod N \).

**Proof.** At first assume \( m \equiv 3 \mod 4 \). Let \( \ell = \gcd(N, m) \). Then \( m \nmid \ell \) because of \( m \nmid N \). We find \( a \in \mathbb{N} \) with \( \left( \frac{a}{m/\ell} \right) = -1 \). Dirichlet’s prime number theorem asserts the existence of infinitely many primes \( p \) satisfying

\[ p \equiv 1 \mod 4N, \quad p \equiv a \mod m/\ell, \]

since the modules are coprime. Quadratic reciprocity yields

\[ \chi_K(p) = \left( \frac{-m}{p} \right) = \left( \frac{-1}{p} \right) \left( \frac{p}{m/\ell} \right) = -1. \]

The other cases are dealt with in a similar way.

## 5 Hecke operators

Given \( f \in \mathcal{M}(\Gamma_n, k) \) we define the Hecke operator \( \Gamma_n M \Gamma_n, \ M \in \Delta_n, \) acting on \( f \) by

\[ f \mid \Gamma_n M \Gamma_n = \sum_{L \in \mathcal{M}(\Gamma_n, k)} f \mid L \in \mathcal{M}(\Gamma_n, k). \]

(3)
This definition is linearly extended on \( \mathcal{H}(\Gamma_n, \Delta_n) \). Moreover we apply the analogous definition for subgroups of \( \Gamma_n \).

**Lemma 8.** Hecke operators map cusp forms on cusp forms.

**Proof.** We may choose \( L = \left( \begin{array}{cc} A & B \\ 0 & D \end{array} \right) \) in \( \mathfrak{B} \) due to Lemma \( \mathfrak{2} \)

\[
 f_k \mid (A B \ 0 \ D)(Z) = \sum_{T \in \Lambda_n, T > 0} (\det D)^{-k} \alpha_f(T) e^{2\pi i \text{trace}(TBD^{-1} + T|A|Z/q)}
\]

if \( M \in \Delta_n(q) \). Hence only positive definite matrices appear in the Fourier expansion. \( \square \)

Next we consider the eigenvalues of Hecke operators.

**Lemma 9.** Let \( p \) be an inert prime and let \( f \in \mathcal{M}_k(\Gamma_n, k) \) with \( \alpha_f(0) \neq 0 \) as well as \( f \mid \mathcal{T}_n(p) = \lambda f \) for some \( \lambda \in \mathbb{C} \). Then

\[
 \lambda = \prod_{j=1}^n (p^{2j-1-k} + 1).
\]

**Proof.** Use Corollary \( \mathfrak{7} \) as well as

\[
 f_k \mid \mathcal{T}_n(p) \mid \phi = f_k \mid \phi \left| \mathcal{T}_n(p) \right.
\]

\[
 = (p^{2n-1-k} + 1)f_k \mid \phi \left| \mathcal{T}_{n-1}(p) \right.
\]

as well as

\[
 f \mid \phi^n = \alpha_f(0).
\]

After \( n \) steps the result follows. \( \square \)

Next we consider the other extreme case of cusp forms.

**Lemma 10.** Let \( f \in \mathcal{M}(\Gamma_n[q], k) \) be a cusp form. Let \( p \) be an inert prime, \( p \equiv 1(\mod q) \)

and \( f \mid \mathcal{T}_n[q]M\Gamma_n[q] = \lambda f \). Then

\[
 |\lambda| \leq p^{-kn/2} \prod_{j=1}^n (p^{2j-1} + 1).
\]

**Proof.** There exists \( Z_0 \in \mathbb{H}_n \) such that the function

\[
 \mathbb{H}_n \to \mathbb{R}, \quad Z \mapsto (\det Y)^{k/2}|f(Z)|,
\]

attains its maximum at \( Z_0 \) due to \( \mathfrak{3} \). Then the result follows in the same way as in \( \mathfrak{8} \),

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Hilfssatz IV.4.8, because of
\[ \sharp(\Gamma_n[q]\backslash\Gamma_n[q]\{\Gamma_n[p]\}) = \prod_{j=1}^{n}(p^{2j-1} + 1) \]
due to (2) as well as the case \( k = 0 \) in Lemma 9.

Next we need an assertion on iterative \( \phi \)-operators.

**Lemma 11.** Let \( f \in \mathcal{M}(\Gamma_n, k) \), \( R_j \in U(j, j; \mathbb{K}) \), \( j = 1, \ldots, n \). Then
\[ f \mid R_n \mid \phi \mid R_{n-1} \mid \phi \mid R_1 \mid \phi = \lim_{y \to \infty} f(iyI) = c \cdot \alpha f(0) \]
for some \( c \neq 0 \).

**Proof.** As \( f \mid R_n \mid \phi \mid R_{n-1} \mid \phi \mid R_1 \mid \phi = f \mid R \mid \phi^n \),
where
\[ R = R_n \cdot (R_{n-1} \times I) \cdots (R_1 \times I) \in U(n, n; \mathbb{K}). \]
Now use Lemma 2 and (4).

We give an application to the characterization of cusp forms. Therefore we use the special matrices \( R_{U_{i}} \) from Theorem 3.

**Lemma 12.** Let \( f \in \mathcal{M}(\Gamma_n, k) \), \( 1 \leq j \leq n \). Then
\[ f \mid R \mid \phi^j \equiv 0 \quad \text{for all} \quad R \in U(n, n; \mathbb{K}) \]
holds if and only if this is true for
\[ R = R_{U_{i}} \cdot (R_{U_{i-1}} \times I) \cdots (R_{U_{i-j+1}} \times I) \in U(n, n; \mathbb{K}) \]
for all \( V_{\ell} \in GL_{\ell}(\mathcal{O}_{\mathbb{K}}) \) and \( i_{\ell} \in \{1, \ldots, h_{\mathbb{K}}\} \), \( \ell = n, \ldots, n - j + 1 \).

**Proof.** Apply the same arguments as in the proof of Theorem 3 and Lemma 11.

**Remark 1.** If \( f \in \mathcal{M}(\Gamma_n, k) \) is symmetric, i.e. \( f(Z^{i\ell}) = f(Z) \), and \( M \in \Delta_n \) with \( \det M \in \mathbb{R}_{+} \), we observe
\[ f \mid \Gamma_n M \Gamma_n(Z^{i\ell}) = f \mid \Gamma_n \Gamma_n M \Gamma_n(Z). \]
We conclude \( \Gamma_n \Gamma_n = \Gamma_n M \Gamma_n \) for \( M \in \Delta_n^{\text{inert}} \) from Theorem 2. Thus these Hecke operators map the subspace of symmetric Hermitian modular forms on itself.
6 The Siegel-Eisenstein series

According to [1] we may define the Siegel-Eisenstein series

\[ E_k^{(n)}(Z) = \sum_{M \in \Gamma_n \setminus \Gamma_n, \ 0 \ \mathbf{M}} 1 \ | \ M(Z), \ Z \in \mathbb{H}_n, \]

for even \( k > 2n, d \neq -3, -4, \) where

\[ \Gamma_{n,0} = \left\{ \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} \in \Gamma_n \right\}. \]

We have

\[ E_k^{(n)} | \phi = E_k^{(n-1)}, \ E_k^{(0)} := 1. \]

We can take the same proof as in [8], IV.4.7, in order to get

**Lemma 13.** Let \( k > 2n \) be even, \( d \neq -3, -4, M \in \Delta_n. \) Then

\[ E_k^{(n)} | \Gamma_n M \Gamma_n = \lambda E_k^{(n)}. \]

We obtain our final result and recall the definition of \( N \) from Lemma 6

**Theorem 4.** Let \( k > 2n, d \neq -3, -4. \) Let \( p \) be an inert prime

\[ p \equiv 1 \ \text{mod} \ N^{2n-2} \]

and \( f \in \mathcal{M}_k(\Gamma_n, k) \) satisfying

\[ \alpha_f(0) = 1 \ \text{and} \ f | \mathcal{T}_n(p) = \lambda f \]

for some \( \lambda \in \mathbb{C}. \) Then

\[ f = E_k^{(n)}. \]

**Proof.** The case \( n = 1 \) is clear from the classical theory as \( E_k^{(1)} \) coincides with the normalized elliptic Eisenstein series. Let \( n \geq 2. \) Since the constant term of the Fourier expansion is non-zero, we can apply Lemma 9. If \( f \neq E_k^{(n)} \), there exists a minimal \( j, 1 \leq j \leq n \) such that

\[ (f - E_k^{(n)}) | k \ \mathcal{R} \phi^j \equiv 0 \ \text{for all} \ R \in U(n, n; \mathbb{K}). \]

This means that the non-zero Fourier coefficients have rank > \( n - j \). Now apply Lemma 12 and assume

\[ \tilde{f} := (f - E_k^{(n)}) | k \ \mathcal{R} \phi^{j-1} \neq 0. \]

for an \( R \in U(n, n; \mathbb{K}) \) quoted there. Thus \( \tilde{f} \in \mathcal{M}(\Gamma_n - j + 1[N^{2j-2}], k) \) is a cusp form. We
conclude
\[ \tilde{f} \mid T_{n-j+1}^{N^{2j-2}}(p) = \lambda \tilde{f}, \]
\[ \lambda = \prod_{\ell=1}^{n-j-1} (p^{2\ell-1-k} + 1) > 1. \]

But \( \tilde{f} \) is a cusp form. Therefore we can apply Lemma 10 in order to get
\[ |\lambda| \leq p^{-k(n-j+1)/2} \prod_{\ell=1}^{n-j+1} (p^{2\ell-1} + 1) < p^{-k(n-j+1)/2} \prod_{\ell=1}^{n-j+1} p^{2\ell} = p^{(n-j+1)(n-j+2-k)/2} \leq 1 \]
in view of \( k > 2n \). This contradicts \( \lambda > 1 \) and yields the claim.

Remark 2. a) The cases \( d_K = -3, -4 \) are excluded because of the additional units. As \( h_K = 1 \) in these cases, the results are contained in [13], where the proof is only valid for class number 1. Due to our proof here the results in [13] are also valid for arbitrary \( K \). Moreover these considerations fill the gap in [13] such that the results of section 8 there are true for arbitrary \( h_K \).

b) If \( d_K = -3, -4 \) one has to impose the condition that \( k \) is divisible by the number of units in \( \mathcal{O}_K \). Alternatively for arbitrary even \( k \) one has to restrict the summation to \( \Gamma_n \cap SL_{2n}(\mathcal{O}_K) \) or to insert the factor \((\det M)^{-k/2}\) in the definition of \( E_k^{(n)} \).

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