ℵ₀-CATEGORICITY OF SEMIGROUPS II

THOMAS QUINN-GREGSON

Abstract. A countable semigroup is ℵ₀-categorical if it can be characterised, up to isomorphism, by its first order properties. In this paper we continue our investigation into the ℵ₀-categoricity of semigroups. Our main results are a complete classification of ℵ₀-categorical orthodox completely 0-simple semigroups, and descriptions of the ℵ₀-categorical members of certain classes of strong semilattices of semigroups.

1. Introduction

A countable structure is ℵ₀-categorical if it is uniquely determined by its first order properties, up to isomorphism. The concept of ℵ₀-categoricity arises naturally from model theory, however it has a purely algebraic formulation, as we explain in Section 2. Significant results exist for both relational and algebraic structures from the point of view of ℵ₀-categoricity, but, until recently, little was known in the context of semigroups. This article is the second of a pair initiating and developing the study of ℵ₀-categorical semigroups. For background and motivation we refer the reader to [10] and our first article [5].

We explore in [5] the behaviour of ℵ₀-categoricity with respect to standard constructions, such as quotients and subsemigroups. For example, ℵ₀-categoricity of a semigroup is inherited by both its maximal subgroups and its principal factors. Differences with the known theory for groups and rings emerged, for example, any ℵ₀-categorical nil ring is nilpotent, but the same is not true for semigroups. While keeping the machinery at a low level, we were able to give, amongst other results, complete classifications of ℵ₀-categorical primitive inverse semigroups and of $E$-unitary inverse semigroups with finite semilattices of idempotents.

For the work in this current article, it is helpful to develop some general strategies and then apply them in various contexts. In view of this, in Section 2, we introduce ℵ₀-categoricity in the setting of (first order) structures. Although we will mostly be working in the context of semigroups, this broader view will be useful for studying certain structures, such as graphs and semilattices, which naturally arise in our considerations of semigroups. Key results from [5] are given in this setting. In particular, we formalise the previously defined concept of ℵ₀-categoricity over a set of subsets; the ℵ₀-categoricity of rectangular bands over any set of subrectangular bands acts as a useful example.

2010 Mathematics Subject Classification. Primary 20M10; Secondary 03C35.

Key words and phrases. ℵ₀-categorical, semigroups, Rees matrix semigroups.

This work forms part of the PhD at the University of York, supervised by Prof. Victoria Gould and funded by EPSRC.
In Section 3 we construct a handy method for dealing with the $\aleph_0$-categoricity of semigroups in which their automorphisms can be built from certain ingredients. This is then used in Section 4 to study the $\aleph_0$-categoricity of strong semilattices of semigroups. The main results of this article are in Section 5 where we continue from [5] our study into the $\aleph_0$-categoricity of completely 0-simple semigroups. We follow a method of Graham and Houghton by considering graphs arising from Rees matrix semigroups, which necessitated our study of $\aleph_0$-categoricity in the general setting of structures.

We shall assume that all structures considered will be of countable cardinality.

2. The $\aleph_0$-categoricity of a structure

We begin by translating a number of results in [5] to the general setting of (first order) structures. Their proofs easily generalize, and as such we shall omit them, referencing only the corresponding result in [5].

A (first order) structure is a set $M$ together with a collection of constants $C$, finitary relations $R$, and finitary functions $F$ defined on $M$. We denote the structure as $(M; R, F, C)$, or simply $M$ where no confusion may arise. Each constant element is associated with a constant symbol, each $n$-ary relation of $M$ is associated with an $n$-ary relational symbol, and each $n$-ary function is associated with an $n$-ary function symbol. The collection $L$ of these symbols is called the signature of $M$. We follow the usual convention of not distinguishing between the constants/relations/functions of $M$, and their corresponding abstract symbols in $L$.

Our main example is that of a semigroup $(S, \cdot)$, where $S$ is a set together with a single (associative) binary operation $\cdot$, and so the associated signature consists of a single binary function symbol.

A property of a structure is first order if it can be formulated within first order predicate calculus. A (countable) structure is $\aleph_0$-categorical if it can be uniquely classified by its first order properties, up to isomorphism.

Given a structure $M$, we say that a pair of $n$-tuples $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_n)$ of $M$ are automorphically equivalent or belong to the same $n$-automorphism type if there exists an automorphism $\phi$ of $M$ such that $a\phi = b$, that is, $a_i\phi = b_i$ for each $i \in \{1, \ldots, n\}$. We denote this equivalence relation as $a \sim_{M,n} b$. We call $\text{Aut}(M)$ oligomorphic if $\text{Aut}(M)$ has only finitely many orbits in its action on $M^n$ for each $n \geq 1$, that is, if each $|M^n/\sim_{M,n}|$ is finite. The central result in the study of $\aleph_0$-categorical structures is the Ryll-Nardzewski Theorem, which translates the concept to the study of oligomorphic automorphism groups (see [10]):

**Theorem 2.1** (The Ryll-Nardzewski Theorem (RNT)). A structure $M$ is $\aleph_0$-categorical if and only if $\text{Aut}(M)$ is oligomorphic.

An immediate consequence of the RNT is that any characteristic substructure inherits $\aleph_0$-categoricity, where a subset is called characteristic if it is invariant under automorphisms of the structure. However, key subsemigroups of a semigroup such as maximal subgroups and principal ideals are not necessarily characteristic, and a more general definition is required:
Definition 2.2. [5] Definition 3.1] Let $M$ be a structure and, for some fixed $t \in \mathbb{N}$, let \{$(X_i : i \in I)$\} be a collection of $t$-tuples of $M$. Let \{$(A_i : i \in I)$\} be a collection of subsets of $M$ with the property that for any automorphism $\phi$ of $M$ such that there exists $i, j \in I$ with $X_i \phi = X_j$, then $\phi|_{A_i}$ is a bijection from $A_i$ onto $A_j$. Then we call \(A = \{(A_i, X_i) : i \in I\}\) a system of $t$-pivoted pairwise relatively characteristic (t-pivoted p.r.c.) subsets (or, substructure, if each $A_i$ is a substructure) of $M$. The $t$-tuple $X_i$ is called the pivot of $A_i$ ($i \in I$). If $|I| = 1$ then, letting $A_1 = A$ and $X_1 = X$, we write \{(A, X)\} simply as $(A, X)$, and call $A$ an $X$-pivoted relatively characteristic $(X$-pivoted r.c.) subset/substructure of $M$.

Definition 2.2 was shown to be of use in regard to, for example, Green's relations. In particular \{(H, e) : e \in E(S)\} was shown to form a system of 1-pivoted p.r.c. subgroups of a semigroup $S$. It then follows from Proposition 2.3 that maximal subgroups inherit $\aleph_0$-categoricity, and moreover there exists only finitely many non-isomorphic maximal subgroups in an $\aleph_0$-categorical semigroup.

Proposition 2.3. [5] Proposition 3.3] Let $M$ be an $\aleph_0$-categorical structure and \{(A_i, X_i) : i \in I\} a system of $t$-pivoted p.r.c. subsets of $M$. Then \{|A_i| : i \in I\}$ is finite. If, further, each $A_i$ forms a substructure of $M$, then \{|A_i| : i \in I\}$ is finite, up to isomorphism, with each $A_i$ $\aleph_0$-categorical.

We use the RNT in conjunction with [5 Lemma 2.8] to prove that a structure $M$ is $\aleph_0$-categorical in the following way. For each $n \in \mathbb{N}$, let $\gamma_1, \ldots, \gamma_r$ be a finite list of equivalence relations on $M^n$ such that $M^n/\gamma_i$ is finite for each $1 \leq i \leq r$ and

$$\gamma_1 \cap \gamma_2 \cap \cdots \cap \gamma_r \not\subseteq \sim_{M,n}.$$

A consequence of the two aforementioned results is that $M$ is $\aleph_0$-categorical. A condition imposed on $n$-tuples of $M$ will naturally translate to an equivalence relation, and we will say that a condition has finitely many choices if its corresponding equivalence relation has finitely many equivalence classes.

Example 2.4. Recalling [5 Example 2.10], consider the equivalence $\sim_{X,n}$ on $n$-tuples of a set $X$ given by

$$(2.1) \quad (a_1, \ldots, a_n) \sim_{X,n} (b_1, \ldots, b_n) \text{ if and only if } [a_i = a_j \iff b_i = b_j, \text{ for each } i, j].$$

A pair of $n$-tuples $a$ and $b$ are $\sim_{X,n}$-equivalent if and only if there exists a bijection $\phi : \{a_1, \ldots, a_n\} \to \{b_1, \ldots, b_n\}$ such that $a_i \phi = b_i$, and the number of $\sim_{X,n}$-classes of $X^n$ is finite, for each $n \in \mathbb{N}$. Note also that if $M$ is a structure then any pair of $n$-automorphically equivalent tuples are clearly $\sim_{M,n}$-equivalent.

Let $M$ be a structure and $\mathcal{A} = \{A_i : i \in I\}$ a collection of subsets of $M$. We may extend the signature of $M$ to include the unary relations $A_i$ ($i \in I$). We denote the resulting structure as $\bar{M} = (M; \mathcal{A})$, which we call a set extension of $M$. If $\mathcal{A} = \{A_1, \ldots, A_n\}$ is finite, then we may simply write $\bar{M}$ as $(M; A_1, \ldots, A_n)$.

Notice that automorphisms of $\bar{M}$ are simply those automorphisms of $M$ which fix each $M_i$ setwise, that is automorphisms $\phi$ such that $A_i \phi = A_i$ ($i \in I$). The set of all such
Lemma 2.5. \cite{5} Lemma 6.2] Let $M$ be a structure with a system of $t$-pivoted p.r.c. subsets \{(A_i, X_i) : i \in I\}. Then $(M; \{A_i : i \in I\})$ is $\aleph_0$-categorical if and only if $M$ is $\aleph_0$-categorical and $I$ is finite.

Lemma 2.6. \cite{5} Lemma 6.3] Let $M$ be a structure, let $t, r \in \mathbb{N}$, and for each $1 \leq k \leq r$ let $X_k \in M^t$. Suppose also that $A_k$ is an $X_k$-pivoted relatively characteristic subset of $M$ for $1 \leq k \leq r$. Then $(M; A_1, \ldots, A_r)$ is $\aleph_0$-categorical if and only if $M$ is $\aleph_0$-categorical.

Consequently, if $S$ is an $\aleph_0$-categorical semigroup and $G_1, \ldots, G_n$ is a collection of maximal subgroups of $S$ then $(S; G_1, \ldots, G_n)$ is $\aleph_0$-categorical.

However, note that not every $\aleph_0$-categorical set extension of a semigroup requires the subsets to be relatively characteristic. We claim that any set extension of a rectangular band by a finite set of subrectangular bands is $\aleph_0$-categorical. This result is of particular use in the next section when considering the $\aleph_0$-categoricity of normal bands.

Recall that every rectangular band can be written as a direct product of a left zero and right zero semigroup. The following isomorphism theorem for rectangular bands will be vital for proving our claim, and follows immediately from \cite{13} Corollary 4.4.3: \cite{13} Corollary 4.4.3]

Lemma 2.7. Let $B_1 = L_1 \times R_1$ and $B_2 = L_2 \times R_2$ be a pair of rectangular bands. If $\phi_L : L_1 \to L_2$ and $\phi_R : R_1 \to R_2$ are a pair of bijections, then the map $\phi : B_1 \to B_2$ given by $(l, r) \phi = (l \phi_L, r \phi_R)$ is an isomorphism, denoted $\phi = \phi_L \times \phi_R$. Conversely, every isomorphism can be constructed this way.

Theorem 2.8. If $B$ is a rectangular band and $B_1, \ldots, B_r$ is a finite list of subrectangular bands of $B$, then $\tilde{B} = (B; B_1, \ldots, B_r)$ is $\aleph_0$-categorical. In particular, a rectangular band is $\aleph_0$-categorical.

Proof. Let $B = L \times R$, where $L$ is a left zero semigroup and $R$ is a right zero semigroup. For each $1 \leq k \leq r$, let

\[
B_k^L = \{ i \in L : (i, j) \in B_k \text{ for some } j \in R \},
\]

\[
B_k^R = \{ j \in R : (i, j) \in B_k \text{ for some } i \in L \}.
\]

Define a pair of equivalence relations $\sigma_L$ and $\sigma_R$ on $L$ and $R$, respectively, by

\[
i \sigma_L j \iff [i \in B_k^L \iff j \in B_k^L], \text{ for each } k,
\]

\[
i \sigma_R j \iff [i \in B_k^R \iff j \in B_k^R], \text{ for each } k.
\]

The equivalence classes of $\sigma_L$ are simply the set $L \setminus \bigcup_{1 \leq k \leq r} B_k^L$ together with certain intersections of the sets $B_k^L$. Since $r$ is finite, it follows that $L/\sigma_L$ is finite, and similarly $R/\sigma_R$ is finite. Let $a = ((i_1, j_1), \ldots, (i_n, j_n))$ and $b = ((k_1, \ell_1), \ldots, (k_n, \ell_n))$ be a pair of $n$-tuples of $B$ under the four conditions that

1. $i_s \sigma_L k_s$ for each $1 \leq s \leq n$.  

\[ \text{ (1) } i_s \sigma_L k_s \text{ for each } 1 \leq s \leq n, \]
and $S = \{\text{morphisms between their components}\}$. For example, for a strong semilattice of semigroups $L$, $N$, we let $\text{Iso}(L; N)$ denote the set of all isomorphisms from $L$ onto $N$. 

Notation 3.1. Given a pair of structures $M$ and $M'$, we let $\text{Iso}(M; M')$ denote the set of all isomorphisms from $M$ onto $M'$.
Definition 3.2. Let $M$ be an $L$-structure with fixed substructure $M'$. Let $\mathcal{A} = \{M_i : i \in N\}$ be a set of substructures of $M'$ indexed by some $K$-structure $N$ such that $M' = \bigcup_{i \in N} M_i$. Let $N_1, \ldots, N_r$ be a finite partition of $N$, and set $\bar{N} = (N; N_1, \ldots, N_r)$. For each $i, j \in N$, let $\Psi_{i,j}$ be a subset of $\text{Iso}(M_i; M_j)$ under the conditions that

\begin{enumerate}[(1)]
\item if $i, j \in N_k$ for some $1 \leq k \leq r$ then $\Psi_{i,j} \neq \emptyset$,
\item if $\phi \in \Psi_{i,j}$ and $\phi' \in \Psi_{j,l}$ then $\phi \phi' \in \Psi_{i,l}$,
\item if $\phi \in \Psi_{i,j}$ then $\phi^{-1} \in \Psi_{j,i}$,
\item if $\pi \in \text{Aut}(\bar{N})$ and $\phi_i \in \Psi_{i,i}$ for each $i \in N$, then there exists an automorphism of $M$ extending the $\phi_i$.
\end{enumerate}

Letting $\Psi = \bigcup_{i,j \in N} \Psi_{i,j}$, then, under the conditions above, we call $\mathcal{A} = \{M_i : i \in N\}$ an $(M, M'; \bar{N}; \Psi)$-system (in $M$). If $M' = M$ then we may simply refer to this as an $(M; \bar{N}; \Psi)$-system.

By Condition (3.1) if $i, j \in N_k$ for some $k$, then $M_i \cong M_j$. Hence the number of isomorphism types in $\mathcal{A}$ is bounded by $r$. Moreover, it follows from Conditions (3.1), (3.2), and (3.3) that $\Psi_{i,i}$ is a subgroup of $\text{Aut}(M_i)$, for each $i \in N$. If the sets $M_i$ are not pairwise disjoint, then Condition (3.4) should be met with caution. Indeed, if $x \in M_i \cap M_j$ then by taking $\pi$ to be the identity map of $\bar{N}$, we have that $x\phi_i, x\phi_j \in M_i \cap M_j$ for all $\phi_i \in \text{Aut}(M_i)$ and $\phi_j \in \text{Aut}(M_j)$ by Condition (3.4). However, for our work the sets $M_i$ will mostly be pairwise disjoint, or will all intersect at an element which is fixed by every isomorphism between the $M_i$. For example, $M$ could be a semigroup containing a zero, and 0 is the intersection of each of the sets $M_i$.

Note also that no link needs to exist between the signatures $L$ and $K$. For most of our examples they will be the signature of semigroups and the signature of sets (the empty signature), respectively.

Given an $(M; M'; \bar{N}; \Psi)$-system $\mathcal{A} = \{M_i : i \in N\}$ in $M$, we aim to show that, if $N$ is $\aleph_0$-categorical and each $M_i$ possess a stronger notion of $\aleph_0$-categoricity, then $M$ is $\aleph_0$-categorical. The stronger notion that we require comes from the following definition, which generalises the notion of $\aleph_0$-categoricity of set extensions.

Definition 3.3. Let $M$ be a structure and $\Psi$ a subgroup of $\text{Aut}(M)$. Then we say that $M$ is $\aleph_0$-\textit{categorical over $\Psi$} if $\Psi$ has only finitely many orbits in its action on $M^n$ for each $n \geq 1$. We denote the resulting equivalence relation on $M^n$ as $\sim_{M,\Psi,n}$.

By taking $\Psi$ to be those automorphisms which fix certain subsets of $M$ we recover our original definition of $\aleph_0$-categoricity of a set extension. Similarly, by taking $\Psi$ to be those automorphisms which preserve a fixed equivalence relation, or those which fix certain equivalence classes, we obtain a pair of notions defined in [5].

Lemma 3.4. Let $M$ be a structure, and $\mathcal{A} = \{M_i : i \in N\}$ be an $(M, M'; \bar{N}; \Psi)$-system. If $\bar{N}$ is $\aleph_0$-categorical and each $M_i$ is $\aleph_0$-categorical over $\Psi_{i,i}$ then

$$|(M')^n/\sim_{M,n}| < \aleph_0$$

for each $n \geq 1$. 

Proof. Let $\bar{N} = (N; N_1, \ldots, N_r)$ and, for each $1 \leq k \leq r$, fix some $m_k \in N_k$. For each $i \in N_k$, let $\theta_i \in \Psi_{i,m_k}$, noting that such an element exists by Condition (3.1) on $\Psi$. Let $\bar{a} = (a_1, \ldots, a_n)$ and $\bar{b} = (b_1, \ldots, b_n)$ be a pair of $n$-tuples of $M'$, with $a_t \in M_{i_t}$ and $b_t \in M_{j_t}$, and such that $(i_1, \ldots, i_n) \sim_{N,n} (j_1, \ldots, j_n)$ via $\pi \in \text{Aut}(\bar{N})$, say. For each $1 \leq k \leq r$, let $i_{k1}, i_{k2}, \ldots, i_{kn_k}$ be the entries of $(i_1, \ldots, i_n)$ belonging to $N_k$, where $1 < k2 < \cdots < kn_k$, and set

$$a_k = (a_{k1}, \ldots, a_{kn_k}) \in (M')^{n_k}.$$

We similarly form each $b_k$, observing that as $i_{t}\pi = j_t$ for each $1 \leq t \leq n$ and $\pi$ fixes the sets $N_j$ setwise ($1 \leq j \leq r$) the elements $j_{k1}, j_{k2}, \ldots, j_{kn_k}$ are precisely the entries of $(j_1, \ldots, j_n)$ belonging to $N_k$, so that $b_k = (b_{k1}, \ldots, b_{kn_k})$ for some $b_{kt} \in M'$. Notice that as $N_1, \ldots, N_r$ partition $N$ we have $n = n_1 + n_2 + \cdots + n_r$. Since $i_{kt}, j_{kt} \in N_k$ for each $1 \leq t \leq n_k$, we have that $a_{kt}\theta_i, b_{kt}\theta_j$ are elements of $M_{m_k}$. We may thus suppose further that for each $1 \leq k \leq r$,

$$(a_{kt}\theta_{i_{k1}}, \ldots, a_{kn_k}\theta_{i_{kn_k}}) \sim_{M_{m_k}, \Psi_{m_k,m_k}} (b_{k1}\theta_{j_{k1}}, \ldots, b_{kn_k}\theta_{j_{kn_k}})$$

via $a_{k} \in \Psi_{m_k,m_k}$, say (where if $a_k$ is a 0-tuple, then we take $\sigma_k$ to be the identity of $N_{m_k}$). For each $1 \leq k \leq r$ and each $i \in N_k$, let

$$\phi_i = \theta_i\sigma_k\theta^{-1}_{i\pi} : M_i \rightarrow M_{i\pi},$$

noting that $\phi_i \in \Psi_{i,i\pi}$ by Conditions (3.2) and (3.3) on $\Psi$, since $\theta_i, \sigma_k$ and $\theta^{-1}_{i\pi}$ are elements of $\Psi$. Hence, by Condition (3.4) on $\Psi$, there exists an automorphism $\phi$ of $M$ extending each $\phi_i$. For any $1 \leq k \leq r$ and any $1 \leq t \leq n_k$ we have

$$a_{kt}\phi = a_{kt}\phi_{i_{kt}} = a_{kt}\theta_{i_{kt}}\sigma_k\theta^{-1}_{i\pi} = b_{kt}\theta_{j_{kt}}\theta^{-1}_{i\pi} = b_{kt},$$

and so $a \sim_{M,n} b$ via $\phi$. Since $\bar{N}$ is $\aleph_0$-categorical and each $M_i$ are $\aleph_0$-categorical over $\Psi_{i,i}$, the conditions imposed on the tuples $a$ and $b$ have finitely many choices, and so $|(M')^n/ \sim_{M,n}|$ is finite.  

Notice that, by Corollary 2.9, the structure $N$ in the lemma above can simply be a set. In most cases we take $M' = M$, and the result simplifies accordingly by the RNT as follows.

**Corollary 3.5.** Let $M$ be a structure, and $\mathcal{A} = \{M_i : i \in I\}$ be an $(M; \bar{N}; \Psi)$-system. If $\bar{N}$ is $\aleph_0$-categorical and each $M_i$ is $\aleph_0$-categorical over $\Psi_{i,i}$, then $M$ is $\aleph_0$-categorical.

**Example 3.6.** Corollary 3.5 could be used to prove more efficiently how $\aleph_0$-categoricity interplays with the greatest 0-direct decomposition of a semigroup with zero [5, Proposition 5.6]. Indeed, if $S = \bigsqcup^n_{i \in I} S_i$ is the greatest 0-direct decomposition of $S$, and $I_1, \ldots, I_n$ is a finite partition of $I$ corresponding to the isomorphism types of the summands of $S$, then it is a simple exercise to show that $S = \{S_i : i \in I\}$ is an $(S; (I; I_1, \ldots, I_n); \Psi)$-system, where $\Psi$ is the collection of all isomorphisms between summands. Since $(I; I_1, \ldots, I_n)$ is $\aleph_0$-categorical, it follows by Corollary 3.5 that $S$ is $\aleph_0$-categorical if each $S_i$ is $\aleph_0$-categorical (over $\Psi_i = \text{Aut}(S_i)$).
In this section we study the $\aleph_0$-categoricity of strong semilattices of semigroups by making use of our most recent methodology. We are motivated by the work of the author in [18] and [19], where the homogeneity of bands and inverse semigroups are shown to depend heavily on the homogeneity of strong semilattices of rectangular bands and groups, respectively. Recall that a structure is homogeneous if every isomorphism between finitely generated substructures extend to an automorphism. A uniformly locally finite homogeneous structure is $\aleph_0$-categorical [15, Corollary 3.1.3]. Consequently, each homogeneous band is $\aleph_0$-categorical, although the same is not true for homogeneous inverse semigroups.

Let $Y$ be a semilattice. To each $\alpha \in Y$ associate a semigroup $S_\alpha$, and assume that $S_\alpha \cap S_\beta = \emptyset$ if $\alpha \neq \beta$. For each pair $\alpha, \beta \in Y$ with $\alpha \geq \beta$, let $\psi_{\alpha,\beta} : S_\alpha \to S_\beta$ be a morphism such that $\psi_{\alpha,\alpha}$ is the identity mapping and if $\alpha \geq \beta \geq \gamma$ then $\psi_{\alpha,\beta} \psi_{\beta,\gamma} = \psi_{\alpha,\gamma}$. On the set $S = \bigcup_{\alpha \in Y} S_\alpha$ define a multiplication by $a * b = (a \psi_{\alpha,\alpha\beta})(b \psi_{\beta,\alpha\beta})$ for $a \in S_\alpha, b \in S_\beta$, and denote the resulting structure by $S = [Y; S_\alpha; \psi_{\alpha,\beta}]$. Then $S$ is a semigroup, and is called a strong semilattice $Y$ of the semigroups $S_\alpha$ ($\alpha \in Y$). The semigroups $S_\alpha$ are called the components of $S$. We follow the convention of denoting an element $a$ of $S_\alpha$ as $a_\alpha$.

The idempotents of $S = [Y; S_\alpha; \psi_{\alpha,\beta}]$ are given by $E(S) = \bigcup_{\alpha \in Y} E(S_\alpha)$, and if $E(S)$ forms a subsemigroup of $S$ then $E(S) = [Y; E(S_\alpha); \psi_{\alpha,\beta}|_{E(S_\alpha)}]$.

We build automorphisms of strong semilattices of semigroups in a natural way using the following well known result. A proof can be found in [17].

**Theorem 4.1.** Let $S = [Y; S_\alpha; \psi_{\alpha,\beta}]$ be a strong semilattices of semigroups. Let $\pi \in \text{Aut}(Y)$ and, for each $\alpha \in Y$, let $\theta_\alpha : S_\alpha \to S_{\alpha\pi}$ be an isomorphism. Assume further that for any $\alpha \geq \beta$, the diagram

\[
\begin{array}{ccc}
S_\alpha & \xrightarrow{\theta_\alpha} & S_{\alpha\pi} \\
\downarrow{\psi_{\alpha,\beta}} & & \downarrow{\psi_{\alpha\pi,\beta\pi}} \\
S_\beta & \xrightarrow{\theta_\beta} & S_{\beta\pi}
\end{array}
\]

(4.1)

commutes. Then the map $\theta = \bigcup_{\alpha \in Y} \theta_\alpha$ is an automorphism of $S$, denoted $\theta = [\theta_\alpha, \pi]_{\alpha \in Y}$.

We denote the diagram (4.1) by $[\alpha; \beta; \alpha\pi, \beta\pi]$. The map $\pi$ is called the induced (semilattice) automorphism of $Y$, denoted $\theta^\pi$.

Unfortunately, not all automorphisms of strong semilattices of semigroups can be constructed as in Theorem 4.1. We shall call a strong semilattice of semigroups $S$ automorphism-pure if every automorphism of $S$ can be constructed as in Theorem 4.1. For example, every strong semilattice of completely simple semigroups is automorphism-pure [16, Lemma IV.1.8], and so both strong semilattices of groups (Clifford semigroups) and strong semilattices of rectangular bands (normal bands) are automorphism-pure.
Let $S = [Y; S_{\alpha}; \psi_{\alpha,\beta}]$ be a strong semilattice of semigroups. We denote the equivalence relation on $Y$ corresponding to isomorphism types of the semigroups $S_{\alpha}$ by $\eta_S$, so that $\alpha \eta_S \beta \iff S_\alpha \cong S_\beta$. We let $Y^S$ denote the set extension of $Y$ given by $Y^S = (Y; Y/\eta_S)$.

**Proposition 4.2.** Let $S = [Y; S_{\alpha}; \psi_{\alpha,\beta}]$ be automorphism-pure and $\aleph_0$-categorical. Then each $S_{\alpha}$ is $\aleph_0$-categorical and $Y^S$ is $\aleph_0$-categorical, with $Y/\eta_S$ finite.

**Proof.** For each $\alpha \in Y$ fix some $x_\alpha \in S_{\alpha}$. We claim that $\{(S_{\alpha}, x_\alpha) : \alpha \in Y\}$ forms a system of 1-pivoted p.r.c. subsemigroups of $S$. Indeed, let $\theta$ be an automorphism of $S$ such that $x_\alpha \theta = x_\beta$ for some $\alpha, \beta \in Y$. Since $S$ is automorphism-pure, there exists $\pi \in \text{Aut}(Y)$ and isomorphisms $\theta_\alpha : S_{\alpha} \to S_{\alpha\pi} (\alpha \in Y)$ such that $\theta = [\theta_\alpha, \pi]_{\alpha \in Y}$. Hence $S_\alpha \theta = S_\beta$, and the claim follows. Consequently, by the $\aleph_0$-categoricity of $S$ and Proposition 2.3, each $S_{\alpha}$ is $\aleph_0$-categorical and $Y/\eta_S$ is finite.

Let $a = (\alpha_1, \ldots, \alpha_n)$ and $b = (\beta_1, \ldots, \beta_n)$ be a pair of $n$-tuples of $Y$ such that there exists $a_\alpha a_{\alpha_k} \in S_{\alpha\alpha_k}$ and $b_{\beta_k} \in S_{\beta\beta_k}$ with $(a_{\alpha_1}, \ldots, a_{\alpha_n}) \sim_{S,n} (b_{\beta_1}, \ldots, b_{\beta_n})$ via $[\theta_\alpha, \pi]_{\alpha \in Y} \in \text{Aut}(S)$, say. Since $\pi' \in \text{Aut}(Y)$ and $S_{\alpha} \cong S_{\alpha},$ for each $\alpha \in Y$, it follows that $\pi' \in \text{Aut}(Y^S)$. Moreover, $a_k \pi' = b_k$ for each $k$, so that $a \sim_{Y^S,n} b$ via $\pi'$. We have thus shown that

$$|Y^n/\sim_{Y^S,n}| \leq |S^n/\sim_{S,n}| < \aleph_0,$$

as $S$ is $\aleph_0$-categorical. Hence $Y^S$ is $\aleph_0$-categorical. \qed

A natural question arises: how can we built an $\aleph_0$-categorical strong semilattice of semigroups from an $\aleph_0$-categorical semilattice and a collection of $\aleph_0$-categorical semigroups? In this paper we will only be concerned with the $\aleph_0$-categoricity of strong semilattices of semigroups in which all connecting morphisms are injective or all are constant. For arbitrary connecting morphisms, the problem of assessing $\aleph_0$-categoricity appears to be difficult to capture in a reasonable way. Examples of more complex $\aleph_0$-categorical strong semilattices of semigroups arise from [13], where the *universal* normal band is shown to have surjective but not injective connecting morphisms. We first study the case where each connecting morphism is a constant map.

Suppose that $Y$ is a semilattice and, for each $\alpha \in Y$, $S_\alpha$ is a semigroup containing an idempotent $e_\alpha$. For each $\alpha \in Y$ let $\psi_{\alpha,\alpha}$ be the identity automorphism of $S_\alpha$, and for $\alpha > \beta$ let $\psi_{\alpha,\beta}$ be the constant map with image $\{e_\beta\}$. We follow the notation of [23] and let $\psi_{\alpha,\beta} := C_{\alpha,e_\beta}$ for each $\alpha > \beta$ in $Y$. It is easy to check that $\psi_{\alpha,\beta} \psi_{\beta,\gamma} = \psi_{\alpha,\gamma}$ for all $\alpha \geq \beta \geq \gamma$ in $Y$, so that $S = [Y; S_{\alpha}; C_{\alpha,e_\beta}]$ forms a strong semilattice of semigroups. We call $S$ a *constant strong semilattice of semigroups*.

**Definition 4.3.** If $S = [Y; S_{\alpha}; C_{\alpha,e_\beta}]$ is a constant strong semilattice of semigroups, then we denote the subset of $\text{Iso}(S_{\alpha}; S_{\beta})$ consisting of those isomorphisms which map $e_\alpha$ to $e_\beta$ as $\text{Iso}(S_{\alpha}; S_{\beta})^{[e_\alpha,e_\beta]}$. Notice that the set $\text{Iso}(S_{\alpha}; S_{\beta})^{[e_\alpha,e_\beta]}$ is simply the subgroup $\text{Aut}(S_{\alpha}; \{e_\alpha\})$ of $\text{Aut}(S_{\alpha})$. We may then define a relation $v_S$ on $Y$ by

$$\alpha v_S \beta \iff \text{Iso}(S_{\alpha}; S_{\beta})^{[e_\alpha,e_\beta]} \neq \emptyset,$$

so that $v_S \subseteq \eta_S$. 
The relation \( \nu_S \) is reflexive since \( 1_{S_\alpha} \in \text{Aut}(S_\alpha; \{ e_\alpha \}) \) for each \( \alpha \in Y \), and it easily follows that \( \nu_S \) forms an equivalence relation on \( Y \).

**Proposition 4.4.** Let \( S = [Y; S_\alpha; C_{\alpha,e_\beta}] \) be such that \( Y/\nu_S = \{ Y_1, \ldots, Y_r \} \) is finite, \( Y = (Y; Y_1, \ldots, Y_r) \) is \( \aleph_0 \)-categorical and each \( S_\alpha \) is \( \aleph_0 \)-categorical. Then \( S \) is \( \aleph_0 \)-categorical.

**Proof.** We prove that \( \{ S_\alpha : \alpha \in Y \} \) forms an \(( S; Y; \Psi)\)-system for some \( \Psi \). For each \( \alpha, \beta \in Y \), let \( \Psi_{\alpha,\beta} = \text{Iso}(S_\alpha; S_\beta)^{[e_\alpha; e_\beta]} \) and fix \( \Psi = \bigcup_{\alpha, \beta \in Y} \Psi_{\alpha,\beta} \). Then Conditions (3.1), (3.2) and (3.3) are seen to be satisfied since \( \nu_S \) forms an equivalence relation on \( Y \). Let \( \pi \in \text{Aut}(Y) \) and, for each \( \alpha \in Y \), let \( \theta_\alpha \in \Psi_{\alpha,\pi} \). We claim that \( \theta = [\theta_\alpha; \pi]_{\alpha \in Y} \) is an automorphism of \( S \). Indeed, for any \( s_\alpha \in S_\alpha \) and any \( \beta < \alpha \) we have

\[
s_\alpha C_{\alpha,e_\beta} \theta_\beta = e_\beta \theta_\beta = e_\beta \pi = s_\alpha \theta_\alpha C_{\alpha,e_\beta} e_\pi
\]

so that the diagram \([\alpha, \beta; \alpha \pi, \beta \pi]\) commutes. Moreover \([\alpha, \alpha; \alpha \pi, \alpha \pi]\) commutes as

\[
s_\alpha 1_{S_\alpha} \theta_\alpha = s_\alpha \theta_\alpha = s_\alpha \theta_\alpha 1_{S_\alpha},
\]

and the claim follows by Theorem 4.1. Since \( \theta \) extends each \( \theta_\alpha \), we have that \( \{ S_\alpha : \alpha \in Y \} \) is an \(( S; Y; \Psi)\)-system. Moreover, as \( S_\alpha \) is \( \aleph_0 \)-categorical, it is \( \aleph_0 \)-categorical over \( \Psi_{\alpha,\alpha} = \text{Aut}(S_\alpha; \{ e_\alpha \}) \) by [5, Lemma 2.6]. Hence \( S \) is \( \aleph_0 \)-categorical by Corollary 3.5. \( \square \)

Examining our two main classes of automorphism-pure strong semilattices of semigroups: Clifford semigroups and normal bands, the result above reduces accordingly. If \( S = [Y; G_\alpha; C_{\alpha,e_\beta}] \) is a constant strong semilattice of groups, then \( e_\alpha \) is the identity of \( G_\alpha \), and so \( \text{Iso}(G_\alpha; G_\beta) = \text{Iso}(G_\alpha; G_\beta)^{[e_\alpha; e_\beta]} \) for each \( \alpha, \beta \in Y \). On the other hand, if \( S = [Y; B_\alpha; C_{\alpha,e_\beta}] \) is a constant strong semilattice of rectangular bands, then it follows from Lemma 2.7 that \( \text{Iso}(B_\alpha; B_\beta) \neq 0 \) if and only if \( \text{Iso}(B_\alpha; B_\beta)^{[e_\alpha; e_\beta]} \neq 0 \), for any \( e_\alpha \in B_\alpha, e_\beta \in B_\beta \). In both cases we therefore have \( \nu_S = \eta_S \). Moreover, each rectangular band \( B_\alpha \) is \( \aleph_0 \)-categorical by Theorem 2.8 and the following result is then immediate by Propositions 4.2 and 4.4.

**Corollary 4.5.** Let \( S = [Y; S_\alpha; C_{\alpha,e_\beta}] \) be a constant strong semilattice of rectangular bands (groups). Then \( S \) is \( \aleph_0 \)-categorical if and only if \( Y^S \) is \( \aleph_0 \)-categorical, with \( Y/\eta_S \) finite (and each group \( S_\alpha \) is \( \aleph_0 \)-categorical).

We now consider the \( \aleph_0 \)-categoricity of a strong semilattice of semigroups \( S = [Y; S_\alpha; \psi_{\alpha,\beta}] \) such that each connecting morphism is injective. For each \( \alpha > \beta \) in \( Y \), we abuse notation somewhat by denoting the isomorphism \( \psi_{\alpha,\beta}^{-1}|_{\text{Im} \psi_{\alpha,\beta}} \) simply by \( \psi_{\alpha,\beta}^{-1} \). We observe that if \( \alpha > \beta > \gamma \) and \( x_\gamma \in \text{Im} \psi_{\alpha,\gamma} \), say \( x_\gamma = x_\alpha \psi_{\alpha,\gamma} \), then

\[
x_\gamma \psi_{\alpha,\gamma}^{-1} \psi_{\alpha,\beta} = x_\alpha \psi_{\alpha,\gamma} \psi_{\alpha,\gamma}^{-1} \psi_{\alpha,\beta} = x_\alpha \psi_{\alpha,\beta} = x_\gamma \psi_{\beta,\gamma}^{-1}.
\]

Hence, on the restricted domain \( \text{Im} \psi_{\alpha,\gamma} \), we have

\[
\psi_{\alpha,\gamma}^{-1} \psi_{\alpha,\beta} = \psi_{\beta,\gamma}^{-1}.
\]

(4.2)

If \( Y \) has a zero (i.e. a minimum element under the natural order) we may define an equivalence relation \( \xi_S \) on \( Y \) by \( \alpha \xi_S \beta \) if and only if \( S_\alpha \psi_{\alpha,0} = S_\beta \psi_{\beta,0} \). If \( \alpha \xi_S \beta \) then \( \psi_{\alpha,0} \psi_{\beta,0}^{-1} \) is an isomorphism from \( S_\alpha \) onto \( S_\beta \), and so \( \xi_S \subseteq \eta_S \).
Proposition 4.6. Let $S = [Y; S_\alpha; \psi_{\alpha, \beta}]$ be such that each $\psi_{\alpha, \beta}$ is injective. Let $Y$ be a semilattice with zero and $Y/\mathcal{F}_S = \{Y_1, \ldots, Y_r\}$ be finite, with

$$\{S_\alpha \psi_{\alpha, 0} : \alpha \in Y\} = \{T_1, \ldots, T_r\}.$$ 

Then $S$ is $\aleph_0$-categorical if both $Y = (Y_1, \ldots, Y_r)$ and $S_0 = (S_0; T_1, \ldots, T_r)$ are $\aleph_0$-categorical. Moreover, if $S$ is automorphism-pure and $\aleph_0$-categorical, then conversely both $Y$ and $S_0$ are $\aleph_0$-categorical.

Proof. Suppose first that both $Y$ and $S_0$ are $\aleph_0$-categorical. Let $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_n)$ be $n$-tuples of $S$ with $(\alpha_1, \ldots, \alpha_n) \sim_Y (\beta_1, \ldots, \beta_n)$ via $\pi \in \text{Aut}(Y)$, say. Suppose further that

$$(a_1 \psi_{\alpha_1}, \ldots, a_n \psi_{\alpha_n}) \sim_{S_0, n} (b_1 \psi_{\beta_1}, \ldots, b_n \psi_{\beta_n})$$

via $\theta_0 \in \text{Aut}(S_0)$, say. Then for each $\alpha \in Y$ we have $S_\alpha \psi_{\alpha, 0} = S_{\alpha \pi} \psi_{\alpha \pi, 0}$, and so we can take an isomorphism $\theta_\alpha : S_\alpha \to S_{\alpha \pi}$ given by

$$\theta_\alpha = \psi_{\alpha, 0} \theta_0 \psi_{\alpha, 0}^{-1}.$$ 

For each $\alpha \geq \beta$ in $Y$, the diagram $[\alpha, \beta; \alpha \pi, \beta \pi]$ commutes as

$$\psi_{\alpha, \beta} \theta_\beta = \psi_{\alpha, \beta} \psi_{\beta, 0} \theta_0 \psi_{\beta, 0}^{-1} = \psi_{\alpha, 0} \theta_0 \psi_{\beta, 0}^{-1},$$

where the penultimate equality is due to (4.2) as $\text{Im} \psi_{\alpha, 0} = \text{Im} \psi_{\beta, 0} = (\text{Im} \psi_{\alpha, 0}) \theta_0$. Hence $\theta = [\theta, \pi]_{\alpha \in Y}$ is an automorphism of $S$ by Theorem 4.1. Furthermore,

$$a_{\alpha k} \theta = a_{\alpha k} \theta_\alpha = a_{\alpha k} \psi_{\alpha k, 0} \theta_0 \psi_{\alpha k, 0}^{-1} = b_{\beta k} \psi_{\beta, 0} \psi_{\beta, 0}^{-1} = b_{\beta k}$$

for each $1 \leq k \leq n$, so that $a \sim_{S, n} b$ via $\theta$. We thus have that

$$|S^n / \sim_{S, n}| \leq |Y^n / \sim_Y, n| \cdot |S_0^n / \sim_{S_0, n}| < \aleph_0$$

and so $S$ is $\aleph_0$-categorical.

Conversely, suppose $S$ is automorphism-pure and $\aleph_0$-categorical. For each $1 \leq k \leq r$, fix some $\gamma_k \in Y_k$, where we assume w.l.o.g. that $S_{\gamma_k} \psi_{\gamma_k, 0} = T_k$. For each $\alpha \in Y$, fix some $x_\alpha \in S_\alpha$. Let $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_n)$ be $n$-tuples of $S$ such that

$$(x_{\alpha_1}, \ldots, x_{\alpha_n}, x_{\gamma_1}, \ldots, x_{\gamma_r}) \sim_{S^n, n+r} \psi_{\alpha_1, 0} (x_{\beta_1}, \ldots, x_{\beta_n}, x_{\gamma_1}, \ldots, x_{\gamma_r}),$$

via $\theta \in \text{Aut}(S)$, say. Since $S$ is automorphism-pure there exists $\pi \in \text{Aut}(Y)$ and $\theta_\alpha \in \text{Iso}(S_\alpha; S_{\alpha \pi})$ such that $\theta = [\theta, \pi]_{\alpha \in Y}$. The automorphism $\pi$ fixes each $\gamma_k$, so that $S_{\gamma_k} \theta = S_{\gamma_k}$. Hence, as the diagram $[\gamma_k, 0; \gamma_k, 0]$ commutes for each $k$, we have

$$T_k = S_{\gamma_k} \psi_{\gamma_k, 0} = (S_{\gamma_k} \theta_{\gamma_k}) \psi_{\gamma_k, 0} = S_{\gamma_k} \psi_{\gamma_k, 0} \theta_0 = T_k \theta_0 = T_k \theta.$$ 

If $\alpha \in Y_k$ then, by the commutativity of the diagram $[\alpha, 0; \alpha \pi, 0]$, we therefore have

$$S_\alpha \psi_{\alpha, 0} = T_k = T_k \theta_0 = S_\alpha \psi_{\alpha, 0} \theta_0 = S_\alpha \theta_\alpha \psi_{\alpha, 0} = S_{\alpha \pi} \psi_{\alpha, 0},$$

and so $\pi \in \text{Aut}(Y)$. We have shown that

$$|Y^n / \sim_Y, n| \leq |S_0^n / \sim_{S_0, n+r}| < \aleph_0.$$
and so $\mathcal{Y}$ is $\aleph_0$-categorical. Now suppose $\xi$ and $d$ are $n$-tuples of $\mathcal{S}_0$ such that

$$(\xi, x_{\gamma_1}, \ldots, x_{\gamma_r}) \sim_{s,n+r} (d, x_{\gamma_1}, \ldots, x_{\gamma_r}),$$

via $\theta' = [\theta'_\alpha, \pi']_{\alpha \in \mathcal{Y}} \in \text{Aut}(S)$, say. Then arguing as before we have that $T_k \theta' = T_k$ for each $k$, and it follows that $\theta'_0 \in \text{Aut}(\mathcal{S}_0)$, with $\xi \theta'_0 = d$. Hence

$$|S_0^n/\sim_{S_0,n}| \leq |S_0^{n+r}/\sim_{S_0,n+r}| < \aleph_0$$

and so $\mathcal{S}_0$ is $\aleph_0$-categorical. \hfill $\Box$

Note that if $Y$ is finite, then the meet of all the elements of $Y$ is a zero. Moreover, as $Y$ is finite, it is $\aleph_0$-categorical over any set of subsets by the RNT, and so the result above simplifies accordingly in this case:

**Corollary 4.7.** Let $S = [Y; \mathcal{S}_0; \psi_{\alpha,\beta}]$ be such that $Y$ is finite and each $\psi_{\alpha,\beta}$ is injective. If $\mathcal{S}_0 = (\mathcal{S}_0; \{S_\alpha \psi_{\alpha,0} : \alpha \in Y\})$ is $\aleph_0$-categorical then $S$ is $\aleph_0$-categorical. Conversely, if $S$ is automorphism-pure and $\aleph_0$-categorical then $\mathcal{S}_0$ is $\aleph_0$-categorical.

For a Clifford semigroup $S$, the property that the connecting morphisms are injective is equivalent to $S$ being is $E$-unitary, that is, such that for all $e \in E(S)$ and all $s \in S$, if $es \in E$ then $s \in E(S)$ [13, Exercise 5.20]. Since Clifford semigroups are automorphism-pure, we therefore have the following simplification of Proposition 4.6.

**Corollary 4.8.** Let $S = [Y; G_\alpha; \psi_{\alpha,\beta}]$ be an $E$-unitary Clifford semigroup. Let $Y$ be a semilattice with zero and $Y/\xi_S$ be finite. Then $S$ is $\aleph_0$-categorical if and only if $(Y; Y/\xi_S)$ and $(\mathcal{S}_0; \{S_\alpha \psi_{\alpha,0} : \alpha \in Y\})$ are $\aleph_0$-categorical. In particular, if $Y$ is finite then $S$ is $\aleph_0$-categorical if and only if $(\mathcal{S}_0; \{S_\alpha \psi_{\alpha,0} : \alpha \in Y\})$ is $\aleph_0$-categorical.

**Example 4.9.** We use the work of Apps [1] to construct examples of $\aleph_0$-categorical $E$-unitary Clifford semigroups as follows. Let $G$ be an $\aleph_0$-categorical group and $H_1 < H_2 < \cdots$ a characteristic series in $G$, so that each $H_i$ is a characteristic subgroup of $G$ and $H_i$ is a subgroup of $H_{i+1}$. Apps proved that such a series must be finite, and there exists a characteristic series $\{1\} = G_0 < G_1 < G_2 < \cdots < G_n = G$ with each $G_i/G_{i-1}$ a characteristically simple $\aleph_0$-categorical group. For each $0 \leq i \leq n$, let $K_i = G_i \times \{i\}$ be an isomorphic copy of $G_i$. For each $0 \leq i \leq j \leq n$, let $\psi_{i,j} : K_i \to K_j$ be the map given by $(x,i)\psi_{i,j} = (x,j)$. Then we may form a strong semilattice of the groups $K_i$ by taking $S = [Y; K_i; \psi_{i,j}]$, where $Y$ is the set $\{0,1,\ldots,n\}$ with the reverse ordering $0 > 1 > 2 > \cdots > n$. Notice that $S$ is $E$-unitary as each connecting morphism is injective. Moreover, each $K_i \psi_{i,n} = G_i \times \{n\}$ is a characteristic subgroup of $K_n = G_n \times \{n\}$. Hence, by Lemma 2.6, $(K_n; \{K_i \psi_{i,n} : 1 \leq i \leq n\})$ is $\aleph_0$-categorical. Since $Y$ is finite, we have that $(Y; Y/\xi_S)$ is $\aleph_0$-categorical, and so $S$ is $\aleph_0$-categorical by Corollary 4.8.

If $S = [Y; S_\alpha; \psi_{\alpha,\beta}]$ is such that each connecting morphism is an isomorphism, then $Y/\xi_S = \{Y\}$, and so the result above simplifies accordingly. However we can prove a more general result directly (without the condition that $Y$ has a zero) with aid of the following proposition. The result is folklore, but a proof can be found in [17].
Proposition 4.10. Let $S = [Y; S_\alpha; \psi_{\alpha,\beta}]$ be such that each $\psi_{\alpha,\beta}$ is an isomorphism. Then $S \cong S_\alpha \times Y$ for any $\alpha \in Y$. Conversely, if $T$ is a semigroup and $Z$ is a semilattice then $T \times Z$ is isomorphic to a strong semilattice of semigroups such that each connecting morphism is an isomorphism.

Corollary 4.11. Let $S = [Y; S_\alpha; \psi_{\alpha,\beta}]$ be such that each $\psi_{\alpha,\beta}$ is an isomorphism. If $S_\alpha$ and $Y$ are $\aleph_0$-categorical, then $S$ is $\aleph_0$-categorical. Moreover, if $S$ is automorphism-pure then the converse holds.

Proof. By Proposition 4.10, $S$ is isomorphic to $S_\alpha \times Y$ for any $\alpha \in Y$. The first half of the result then follows as $\aleph_0$-categoricity is preserved by finite direct product [8].

If $S$ is automorphism-pure then the converse holds by Proposition 4.2, as $(Y; Y/\eta_S)$ being $\aleph_0$-categorical clearly implies $Y$ is $\aleph_0$-categorical. □

5. $\aleph_0$-CATeROICAL REES MATRIX SEMIGROUPS

A semigroup $S$ is called simple (0-simple) if it has no proper ideals (if its only proper ideal is $\{0\}$ and $S^2 \neq \{0\}$). A simple (0-simple) semigroup is called completely simple (completely 0-simple) if contains a primitive idempotent, i.e. a non-zero idempotent $e$ such that for any non-zero idempotent $f$ of $S$,

$$ef = fe = f \Rightarrow e = f.$$ 

By Rees Theorem [20], to study the $\aleph_0$-categoricity of a completely 0-simple semigroup, it is sufficient to consider Rees matrix semigroups:

Theorem 5.1 (The Rees Theorem). Let $G$ be a group, let $I$ and $\Lambda$ be non-empty index sets and let $P = (p_{\lambda,i})$ be an $\Lambda \times I$ matrix with entries in $G \cup \{0\}$. Suppose no row or column of $P$ consists entirely of zeros (that is, $P$ is regular). Let $S = (I \times G \times \Lambda) \cup \{0\}$, and define multiplication $\ast$ on $S$ by

$$(i, g, \lambda) \ast (j, h, \mu) = \begin{cases} (i, gp_{\lambda,j}h, \mu) & \text{if } p_{\lambda,j} \neq 0 \\ 0 & \text{else} \end{cases}$$

$$0 \ast (i, g, \lambda) = (i, g, \lambda) \ast 0 = 0 \ast 0 = 0.$$ 

Then $S$ is a completely 0-simple semigroup, denoted $M^0[G; I, \Lambda; P]$, and is called a (regular) Rees matrix semigroup (over $G$). Conversely, every completely 0-simple semigroup is isomorphic to a Rees matrix semigroup.

The matrix $P$ is called the sandwich matrix of $S$. Note that if $S$ is a completely simple semigroup, then $S^0$ is isomorphic to a Rees matrix semigroup with sandwich matrix without zero entries [13, Section 3.3]. Consequently, by the Rees Theorem and [5] Corollary 2.12, to examine the $\aleph_0$-categoricity of both completely simple and completely 0-simple semigroups, it suffices to study Rees matrix semigroups (having 0, as in our convention above).

A fundamental discovery in [5] was that to understand the $\aleph_0$-categoricity of an arbitrary semigroup, it is necessary to study $\aleph_0$-categorical completely (0-)simple semigroups. Indeed, they arise as principal factors of an $\aleph_0$-categorical semigroup, as well as giving examples of 0-direct indecomposable summands in a semigroup with zero.
In [5] the \( \aleph_0 \)-categoricity of Rees matrix semigroups over identity matrices (known as Brandt semigroups) were determined, although we deferred the general case to this current article.

Given a Rees matrix semigroup \( S = \mathcal{M}^0[G; I, \Lambda; P] \) with \( P = (p_{\lambda,i}) \), we let \( G(P) \) denote the subset of \( G \) of all non-zero entries of \( P \), that is, \( G(P) := \{ p_{\lambda,i} : p_{\lambda,i} \neq 0 \} \). The idempotents of \( S \) are easily described [13 Page 71]:

\[
E(S) = \{(i, p_{\lambda,i}^{-1}, \lambda) : p_{\lambda,i} \neq 0 \}.
\]

Since there exists a simple isomorphism theorem for Rees matrix semigroups [13 Theorem 3.4.1] (see Theorem 5.9), we should be hopeful of achieving a thorough understanding of \( \aleph_0 \)-categorical Rees matrix semigroups via the RNT. However, from the isomorphism theorem it is not clear how the \( \aleph_0 \)-categoricity of the semigroup \( \mathcal{M}^0[G; I, \Lambda; P] \) effects the sets \( I \) and \( \Lambda \). We instead follow a technique of Graham [6] and Houghton [11] of constructing a bipartite graph from the sets \( I \) and \( \Lambda \).

A bipartite graph is a (simple) graph whose vertices can be split into two disjoint non-empty sets \( L \) and \( R \) such that every edge connects a vertex in \( L \) to a vertex in \( R \). The sets \( L \) and \( R \) are called the left set and the right set, respectively. Formally, a bipartite graph is a triple \( \Gamma = \langle L, R, E \rangle \) such that \( L \) and \( R \) are non-empty trivially intersecting sets and

\[
E \subseteq \{ \{x, y\} : x \in L, y \in R \}.
\]

We call \( L \cup R \) the set of vertices of \( \Gamma \) and \( E \) the set of edges. An isomorphism between a pair of bipartite graphs \( \Gamma = \langle L, R, E \rangle \) and \( \Gamma' = \langle L', R', E' \rangle \) is a bijection \( \psi : L \cup R \to L' \cup R' \) such that \( L\psi = L' \), \( R\psi = R' \), and \( \{l, r\} \in E \) if and only if \( \{l\psi, r\psi\} \in E' \). We are therefore regarding bipartite graphs in the signature \( L_{BG} = \{Q_L, Q_R, E\} \), where \( Q_L \) and \( Q_R \) are unary relations, which correspond to the sets \( L \) and \( R \), respectively, and \( E \) is a binary relation corresponding to the edge relation (here we abuse the notation somewhat by letting \( E \) denote the edge relation and the set of edges).

Let \( \Gamma = \langle L, R, E \rangle \) be a bipartite graph. Then \( \Gamma \) is called complete if, for all \( x \in L, y \in R \), we have \( \{x, y\} \in E \). If \( E = \emptyset \) then \( \Gamma \) is called empty. If each vertex of \( \Gamma \) is incident to exactly one edge, then \( \Gamma \) is called a perfect matching. The complement of \( \Gamma \) is the bipartite graph \( \langle L, R, E' \rangle \) with

\[
E' = \{ \{x, y\} : x \in L, y \in R, \{x, y\} \not\in E \}.
\]

Hence an empty bipartite graph is the complement of a complete bipartite graph, and vice-versa. We call \( \Gamma \) random if, for each \( k, \ell \in \mathbb{N} \), and for every distinct \( x_1, \ldots, x_k, y_1, \ldots, y_\ell \) in \( L \) (in \( R \)) there exists infinitely many \( u \in R \) (\( u \in L \)) such that \( \{u, x_i\} \in E \) but \( \{u, y_j\} \not\in E \) for each \( 1 \leq i \leq k \) and \( 1 \leq j \leq \ell \).

Clearly, for each pair \( n, m \in \mathbb{N}^* = \mathbb{N} \cup \{0\} \), there exists a unique (up to isomorphism) complete bipartite graph with left set of size \( n \) and right set of size \( m \), which we denote as \( K_{n,m} \). There also exists a unique, up to isomorphism, perfect matching with left and right sets of size \( n \), denoted \( P_n \). Similar uniqueness holds for the empty bipartite graph \( E_{n,m} \) with left set of size \( n \) and right set of size \( m \), and the complement of the perfect matching.
Proposition 5.3. Let \( \Gamma = \langle L, R, E \rangle \) be a bipartite graph with \( \mathcal{C}(\Gamma) = \{ \Gamma_i : i \in A \} \). Let \( \pi \) be a bijection of \( A \) and \( \phi_i : \Gamma_i \to \Gamma_{i\pi} \) an isomorphism for each \( i \in A \). Then \( \bigcup_{i \in I} \phi_i \) is an automorphism of \( \Gamma \). Conversely, every automorphism of \( \Gamma \) can be constructed in this way.

Proposition 5.4. Let \( \Gamma = \langle L, R, E \rangle \) be a bipartite graph with \( \mathcal{C}(\Gamma) = \{ \Gamma_i : i \in A \} \). Then \( \Gamma \) is \( \aleph_0 \)-categorical if and only if each connected component is \( \aleph_0 \)-categorical and \( \mathcal{C}(\Gamma) \) is finite, up to isomorphism.

Proof. \((\Rightarrow)\) By Proposition 5.3 we have that, for any choice of \( x_i \in \Gamma_i \) (\( i \in A \)), the set \( \{ (\Gamma_i, x_i) : i \in A \} \) forms a system of 1-pivoted p.r.c. sub-bipartite graphs of \( \Gamma \). The result then follows from Proposition 2.3.

\((\Leftarrow)\) First we show that \( \mathcal{C}(\Gamma) \) forms a \((\Gamma; \bar{A}; \Psi)\)-system in \( \Gamma \) for some \( \bar{A} \) and \( \Psi \). Let \( A_1, \ldots, A_r \) be the finite partition of \( A \) corresponding to the isomorphism types of the connected components of \( \Gamma \), that is, \( \Gamma_i \cong \Gamma_j \) if and only if \( i, j \in A_k \) for some \( k \). Fix \( \bar{A} = (A; A_1, \ldots, A_r) \). For each \( i, j \in A \), let \( \Psi_{i,j} = \text{Iso}(\Gamma_i; \Gamma_j) \) and fix \( \Psi = \bigcup_{i,j \in A} \Psi_{i,j} \). Then \( \Psi \) clearly satisfy Conditions (A), (B) and (C). Let \( \pi \in \text{Aut}(\bar{A}) \) and, for each \( i \in A \), let \( \phi_i \in \Psi_{i,i\pi} \). Then by Proposition 5.3 \( \phi = \bigcup_{i \in A} \phi_i \) is an automorphism of \( \Gamma \), and so \( \Psi \) satisfies Condition (D). Hence \( \mathcal{C}(\Gamma) \) forms an \((\Gamma; \bar{A}; \Psi)\)-system. Each \( \Gamma_i \) is \( \aleph_0 \)-categorical (over \( \Psi_{i,i} = \text{Aut}(\Gamma_i) \)) and \( \bar{A} \) is \( \aleph_0 \)-categorical by Corollary 2.9 and so \( \Gamma \) is \( \aleph_0 \)-categorical by Corollary 3.5. \( \square \)
Definition 5.5. Let $S = \mathcal{M}^0[G; I, \Lambda; P]$ be a Rees matrix semigroup with $P = (p_{\lambda,i})$. Then we form a bipartite graph $\Gamma(P) = \langle I, \Lambda, E \rangle$ with edge set

$$E = \{\{i, \lambda\} : p_{\lambda,i} \neq 0\},$$

which we call the induced bipartite graph of $S$.

The above construct has long been fundamental to the study of Rees matrix semigroups, and has its roots in a paper by Graham in [6]. Here, it is used to describe the maximal nilpotent subsemigroups of a Rees matrix semigroup, where a semigroup is nilpotent if some power is equal to $\{0\}$. All maximal subsemigroups of a finite Rees matrix semigroup were described in the same paper, a result which was later extended in [7] to arbitrary finite semigroups. In [12], Howie used the induced bipartite graph to describe the subsemigroup of a Rees matrix semigroup generated by its idempotents. Finally, in [11], Houghton described the homology of the induced bipartite graph, and a detailed overview of his work is given in [21].

Example 5.6. Let $S = \mathcal{M}^0[G; \{1, 2, 3\}, \{\lambda, \mu\}; P]$ where

$$P = \begin{bmatrix} 1 & 2 & 3 \\ a & b & 0 \\ 0 & c & d \end{bmatrix} \begin{array}{c} \lambda \\ \mu \end{array}$$

Then the induced bipartite graph of $S$ is given in Figure 1.

![Figure 1. Induced bipartite graph](image)

Example 5.7. Let $S = \mathcal{M}^0[G; I, \Lambda; P]$ be such that $P$ has no zero entries, so that $S$ is isomorphic to a completely simple semigroup with zero adjoined. Then $\Gamma(P)$ is a complete bipartite graph.

Notation 5.8. Let $S = \mathcal{M}^0[G; I, \Lambda; P]$ be a Rees matrix semigroup. Given an $n$-tuple $\alpha = ((i_1, g_1, \lambda_1), \ldots, (i_n, g_n, \lambda_n))$ of $S^\ast$, we denote $\Gamma(\alpha)$ as the $2n$-tuple $(i_1, \lambda_1, \ldots, i_n, \lambda_n)$ of $\Gamma(P)$.

Following [2], we adapt the isomorphism theorem for Rees matrix semigroups to explicitly highlight the role of the induced bipartite graph:
Theorem 5.9. Let $S_1 = \mathcal{M}^0[G_1; I_1, \Lambda_1; P_1]$ and $S_2 = \mathcal{M}^0[G_2; I_2, \Lambda_2; P_2]$ be a pair of Rees matrix semigroups with sandwich matrices $P_1 = (p_{\lambda,i})$ and $P_2 = (q_{\mu,j})$, respectively. Let $\psi \in \text{Iso}(\Gamma(P_1); \Gamma(P_2))$, $\theta \in \text{Iso}(G_1; G_2)$, and $u_i, v_{\lambda} \in G_2$ for each $i \in I_1, \lambda \in \Lambda_1$. Then the mapping $\phi : S_1 \to S_2$ given by

$$(i, g, \lambda)\phi = (i\psi, u_i(g\theta)v_{\lambda}, \lambda\psi)$$

is an isomorphism if and only if $p_{\lambda,i}\theta = v_{\lambda} \cdot q_{\lambda\psi, i\psi} \cdot u_i$ whenever $p_{\lambda,i} \neq 0$. Moreover, every isomorphism from $S_1$ to $S_2$ can be described in this way.

The isomorphism $\phi$ will be denoted as $(\theta, \psi, (u_i)i \in I, (v_{\lambda})\lambda \in \Lambda)$. We also denote the induced group isomorphism $\theta$ as $\phi_{G_1}$, and the induced bipartite graph isomorphism $\psi$ as $\phi_{\Gamma(P_1)}$, so that $\phi = (\phi_{G_1}, \psi_{\Gamma(P_1)}, (u_i)i \in I_1, (v_{\lambda})\lambda \in \Lambda_1)$. Note that the induced group isomorphism is not uniquely defined by $\phi$. That is, there may exist $\theta' \in \text{Iso}(G_1; G_2)$ and $u'_i, v'_{\lambda} \in G_2$, such that $\theta' \neq \theta$ but $\phi = (\theta', \psi, (u'_i)i \in I_1, (v'_{\lambda})\lambda \in \Lambda_1)$. Examples of this phenomenon will occur throughout this work.

The composition and inverses of isomorphisms between Rees matrix semigroups behave in a natural way as follows, and a proof can be found in [17].

Corollary 5.10. Let $S_k = \mathcal{M}^0[G_k; I_k, \Lambda_k; P_k]$ $(k = 1, 2, 3)$ be Rees matrix semigroups. Then for any pair of isomorphisms $\phi = (\theta, \psi, (u_i)i \in I_1, (v_{\lambda})\lambda \in \Lambda_1) \in \text{Iso}(S_1; S_2)$ and $\phi' = (\theta', \psi', (u'_i)i \in I_2, (v'_{\lambda})\lambda \in \Lambda_2) \in \text{Iso}(S_2; S_3)$ we have:

(i) $\phi\phi' = (\theta\theta', \psi\psi', (u'_{i\psi}(u_i\theta'))i \in I_1, ((v_{\lambda\theta'})v_{\lambda\psi'})\lambda \in \Lambda_1)$;

(ii) $\phi^{-1} = (\theta^{-1}, \psi^{-1}, ((u_i\theta^{-1})^{-1}\theta^{-1})i \in I_2, ((v_{\lambda\psi^{-1}})^{-1}\theta^{-1})\lambda \in \Lambda_2)$.

Let $\Gamma = \langle L, R, E \rangle$ be a bipartite graph. For each $n \in \mathbb{N}$, we let $\sigma_{\Gamma,n}$ be the equivalence relation on $\Gamma^n$ given by

$$(x_1, \ldots, x_n) \sigma_{\Gamma,n} (y_1, \ldots, y_n) \Leftrightarrow [x_i \in L \Leftrightarrow y_i \in L, \text{ for each } 1 \leq i \leq n].$$

Since each entry of an $n$-tuple of $\Gamma$ lies in either $L$ or $R$ we have that

$$|\Gamma^n/\sigma_{\Gamma,n}| = 2^n,$$

for each $n$. Moreover, as the automorphisms of $\Gamma$ fixes the sets $L$ and $R$, it easily follows that $\sim_{\Gamma,n} \subseteq \sigma_{\Gamma,n}$.

Proposition 5.11. If $S = \mathcal{M}^0[G; I, \Lambda; P]$ is $\aleph_0$-categorical, then $G$ and $\Gamma(P)$ are $\aleph_0$-categorical.

Proof. Since $G$ is isomorphic to the non-zero maximal subgroups of $S$, it is $\aleph_0$-categorical by [3] Corollary 3.7. Now let $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_n)$ be a pair of $\sigma_{\Gamma,n}$-related $n$-tuples of $\Gamma(P)$. Let $i_1 < i_2 < \cdots < i_s$ and $j_1 < j_2 < \cdots < j_t$ be the indexes of entries of $a$ lying in $L$ and $R$, respectively (noting that the same is true for $b$ as $a \sigma_{\Gamma,n} b$). Suppose further that there exists $i \in I, \lambda \in \Lambda$ such that the $n$-tuples

$$((a_{i_1}, 1, \lambda), \ldots, (a_{i_s}, 1, \lambda), (i, 1, a_{j_1}), \ldots, (i, 1, a_{j_t})) \quad \text{and}$$

$$((b_{i_1}, 1, \lambda), \ldots, (b_{i_s}, 1, \lambda), (i, 1, b_{j_1}), \ldots, (i, 1, b_{j_t})).$$
are automorphically equivalent via \( \phi \in \text{Aut}(S) \), say. Then \( a_i, \phi_{\Gamma(P)} = b_i \) and \( a_{j,r}, \phi_{\Gamma(P)} = b_{j,r} \), for each \( 1 \leq r \leq s \) and \( 1 \leq r' \leq t \) by Theorem 5.9. Hence \( a \sim_{\Gamma(P)} b \) via \( \phi_{\Gamma(P)} \), and we have thus shown that

\[
|\Gamma(P) / \sim_{\Gamma(P)}| \leq 2^n \cdot |S^n / \sim_S|.
\]

Hence \( \Gamma(P) \) is \( \aleph_0 \)-categorical by the \( \aleph_0 \)-categoricity of \( S \). \( \square \)

In the next subsection we construct a counterexample to the converse of Proposition 5.11. Our method will be to transfer the concept of the connected components of bipartite graphs to corresponding subsemigroups of Rees matrix semigroups.

5.1. **Connected Rees components.** Let \( S_k = M^0[G; I_k, \Lambda_k; P_k] \) \( (k \in A) \) be a collection of Rees matrix semigroups with \( P_k = (p^{(k)}_{\lambda,i}) \) and \( S_k \cap S_\ell = \{0\} \) for each \( k, \ell \in A \). Then we may form a single Rees matrix semigroup \( S = M^0[G; I, \Lambda; P] \), where \( I = \bigcup_{k \in A} I_k \), \( \Lambda = \bigcup_{k \in A} \Lambda_k \), and \( P = (p_{\lambda,i}) \) is the \( \Lambda \) by \( I \) matrix defined by

\[
p_{\lambda,i} = \begin{cases} 
p^{(k)}_{\lambda,i} & \text{if } \lambda, i \in \Gamma(P_k), \text{ for some } k \\
0 & \text{else.}
\end{cases}
\]

That is, \( P \) is the block matrix

\[
P = \begin{bmatrix}
P_1 & 0 & 0 & \cdots \\
0 & P_2 & 0 & \cdots \\
0 & 0 & P_3 & \cdots \\
\vdots & \vdots & \ddots & \ddots
\end{bmatrix}.
\]

We denote \( S \) by \( \otimes_{k \in A} S_k \). The subsemigroups \( S_k \) of \( S \) are called **Rees components** of \( S \). Notice that each \( \Gamma(P_k) \) is a union of connected components of \( \Gamma(P) \). The subsemigroup \( S_k \) will be called a **connected Rees component** of \( S \) if \( \Gamma(P_k) \) is connected (and is therefore a connected component of \( \Gamma(P) \)).

Conversely, for any Rees matrix semigroup \( S = M^0[G; I, \Lambda; P] \) there exists partitions \( \{I_k : k \in A\} \) and \( \{\Lambda_k : k \in A\} \) of \( I \) and \( \Lambda \), respectively, such that \( \mathcal{C}(\Gamma(P)) = \{\Lambda_k \cup I_k : k \in A\} \). Consequently, for each \( k \in A \), the subsemigroup \( S_k = M^0[G; I_k, \Lambda_k; P_k] \) of \( S \) is a connected Rees component, where \( P_k \) is the \( \Lambda_k \times I_k \) submatrix of \( P \), and are such that \( S_kS_\ell = 0 \) for all \( k \neq \ell \). Following the work of Graham [6], we may then permute the rows and columns of \( P \) if necessary to assume w.l.o.g. that \( P \) is a block matrix of the form (5.1).

Note that if \( S \) is a Rees matrix semigroup with connected Rees components \( \{S_k : k \in A\} \) then clearly

\[
E(S) = \bigcup_{k \in A} E(S_k).
\]

Using the fact that automorphisms of \( \Gamma(P) \) arise as collections of isomorphisms between its connected components, we obtain an alternative description of automorphisms of a Rees matrix semigroups. The proof is a simple exercise, and can be found in [17].
Corollary 5.12. Let $S = \bigoplus_{k \in A} S_k = M^0[\Gamma(P); \Lambda; P]$ be a Rees matrix semigroup such that each $S_k = M^0[I; \Lambda; P]$ is a connected Rees component of $S$. Let $\pi$ be a bijection of $A$ and, for each $k \in A$, let $\phi_k = (\theta, \psi_k, (u_i^{(k)})_{i \in I}, (v_{\lambda}^{(k)})_{\lambda \in \Lambda})$ be an isomorphism from $S_k$ to $S_{k\pi}$. Then $\phi = (\theta, \psi, (u_i)_{i \in I}, (v_{\lambda})_{\lambda \in \Lambda})$ is an automorphism of $S$, where $\psi = \bigcup_{k \in A} \psi_k$, and if $i, \lambda \in \Gamma(P)$ then $u_i = u_i^{(k)}$ and $v_{\lambda} = v_{\lambda}^{(k)}$. Moreover, every automorphism of $S$ can be described in this way.

We observe that the induced group automorphisms of the isomorphisms $\phi_k$ above must all be equal.

Recall that if $S = M^0[\Gamma(P); \Lambda; P]$ is $\aleph_0$-categorical, then $\Gamma(P)$ is $\aleph_0$-categorical by Proposition 5.11 and thus $\mathcal{C}(\Gamma(P))$ is finite, up to isomorphism, with each connected component being $\aleph_0$-categorical by Proposition 3.4. We extend this result to the set of all connected Rees components of $S$ as follows:

Proposition 5.13. Let $S = \bigoplus_{k \in A} S_k$ be an $\aleph_0$-categorical Rees matrix semigroup such that each $S_k$ is a connected Rees component of $S$. Then each $S_k$ is $\aleph_0$-categorical and $S$ has finitely many connected Rees components, up to isomorphism.

Proof. We claim that $\{(S_k, a_k) : k \in A\}$ is a system of 1-pivoted p.r.c. subsemigroups of $S$ for any $a_k \in S_k$, to which the result follows by Proposition 2.3. Indeed, let $\phi$ be an automorphism of $S$ such that $a_k\phi = a_l$ for some $k, l$. Then, by Corollary 5.12 there exists a bijection $\pi$ of $A$ with $S_k\phi = S_{k\pi} = S_l$ as required.\hfill\square

Our interest is now in attaining a converse to the proposition above, since it would provide us with a method for building ‘new’ $\aleph_0$-categorical Rees matrix semigroups from ‘old’. With the aid of Lemma 3.4 we shall prove that a converse exists in the class of Rees matrix semigroups over finite groups. The case where the maximal subgroups are infinite is an open problem.

Given a pair $S = M^0[I; \Lambda; P]$ and $S' = M^0[I'; \Lambda'; Q]$ of Rees matrix semigroups over a group $G$, we denote $\text{Iso}(S; S')(1_G)$ as the set of isomorphisms between $S$ and $S'$ with trivial induced group isomorphism. That is, $\text{Iso}(S; S')(1_G)$ is the set
\[
\{\phi \in \text{Iso}(S; S') : \exists \psi \in \text{Iso}(\Gamma(P); \Gamma(Q)) \text{ and } u_i, v_{\lambda} \in G \text{ such that } \phi = (1_G, \psi, (u_i)_{i \in I}, (v_{\lambda})_{\lambda \in \Lambda})\}.
\]
If $S = S'$ we denote this simply as $\text{Aut}(S)(1_G)$, and notice that $\text{Aut}(S)(1_G)$ is a subgroup of $\text{Aut}(S)$ by Corollary 5.10.

Lemma 5.14. Let $S = M^0[I; \Lambda; P]$ be a Rees matrix semigroup over a finite group $G$. Then $S$ is $\aleph_0$-categorical if and only if $S$ is $\aleph_0$-categorical over $\text{Aut}(S)(1_G)$.

Proof. Let $S$ be $\aleph_0$-categorical with $G = \{g_1, \ldots, g_r\}$. Let $\underline{a}$ and $\underline{b}$ be a pair of $n$-tuples of $S$. For some fixed $p_{\mu, j} \neq 0$, let $\underline{g}$ be the $r$-tuple of $S$ given by $\underline{g} = ((j, g_1, \mu), \ldots, (j, g_r, \mu))$, and suppose $(\underline{a}, \underline{g}) \sim_{S, n+r} (\underline{b}, \underline{g})$ via $\phi = (\theta, \psi, (u_i)_{i \in I}, (v_{\lambda})_{\lambda \in \Lambda})$, say. Then, for each $1 \leq k \leq r$, we have
\[
(j, g_k, \mu)\phi = (j\psi, u_j(g_k\theta)v_{\mu}, \mu\psi) = (j, g_k, \mu),
\]
so that \( g_k \theta = u_j^{-1} g_k v^{-1}_\mu \). For each \( i \in I, \lambda \in \Lambda \), let \( \bar{v}_i = u_i u_j^{-1} \) and \( \bar{v}_\lambda = v^{-1}_\mu v_\lambda \). Then
\[
(i \psi, \bar{u}_i g_k \bar{v}_\lambda, \lambda \psi) = (i \psi, (u_i u_j^{-1}) g_k (v^{-1}_\mu v_\lambda), \lambda \psi)
= (i \psi, u_i (g_k \theta) v_\lambda, \lambda \psi)
= (i, g_k, \lambda) \phi,
\]
for any \((i, g_k, \lambda) \in S\), so that \( \phi = (1_G, \psi, (\bar{u}_i)_{i \in I}, (\bar{v}_\lambda)_{\lambda \in \Lambda}) \in \text{Aut}(S)(1_G) \). Consequently, \((a, g) \sim S, \text{Aut}(S)(1_G), n+r \) \((b, g)\) and in particular \((a, g) \sim S, \text{Aut}(S)(1_G), n \sim b\). We have thus shown that
\[
|S^n / \sim S, \text{Aut}(S)(1_G), n| \leq |S^{n+r} / \sim S, \text{Aut}(S)(1_G)| < \aleph_0,
\]
as \(S\) is \(\aleph_0\)-categorical. Hence \(S\) is \(\aleph_0\)-categorical over \(\text{Aut}(S)(1_G)\).

The converse is immediate. \( \square \)

We are now able to prove our desired converse to Proposition \(5.13\) in the case where the maximal subgroups are finite.

**Theorem 5.15.** Let \( S = \mathcal{M}^0[G; I, \Lambda; P] \) be a Rees matrix semigroup such that \( G \) is finite. Then \( S\) is \(\aleph_0\)-categorical and \(S\) has only finitely many connected Rees components, up to isomorphism.

**Proof.** \((\Rightarrow)\) Immediate from Proposition \(5.13\)

\((\Leftarrow)\) Since \( S \) is regular with finite maximal subgroups, to prove \( S \) is \(\aleph_0\)-categorical, it suffices by \([5, \text{Corollary 3.13}]\) to show that \(|E(S)^n / \sim_{S,n}|\) is finite, for each \( n \in \mathbb{N}\). Let \( \{S_k : k \in A\} \) be the set of connected Rees components of \( S \), which is finite up to isomorphism and with each \( S_k \) being \(\aleph_0\)-categorical. Define a relation \( \eta \) on \( A \) by \( i \sim j \) if and only if \( \text{Iso}(S_i; S_j)(1_G) \neq \emptyset \). By Corollary \(5.10\), we have that \( \eta \) is an equivalence relation.

We first prove that \( A/\eta \) is finite. Suppose for contradiction that there exists an infinite set \( X \) of pairwise \( \eta \)-inequivalent elements of \( A \). Since \( S \) has finitely many connected components up to isomorphism, there exists an infinite subset \( \{i_r : r \in \mathbb{N}\} \) of \( X \) such that \( S_{i_n} \cong S_{i_m} \) for each \( n, m \). Fix an isomorphism \( \phi_{i_n} : S_{i_n} \to S_{i_1} \) for each \( n \in \mathbb{N}\). Then as \( \text{Aut}(G) \) is finite, there exists distinct \( n, m \) such that \( \phi_{i_n}^G = \phi_{i_m}^G \), and so \( \phi_{i_n} \phi_{i_m}^{-1} \in \text{Iso}(S_{i_n}; S_{j_m})(1_G) \) by Corollary \(5.10\). Hence \( i_n \eta i_m \), a contradiction, and so \( A/\eta \) is finite.

Let \( S' = \bigcup_{k \in A} S_k \), noting that \( S' \) is the 0-direct union of the \( S_k \), and in particular is a subsemigroup of \( S \). Let \( A/\eta = \{A_1, \ldots, A_r\} \) and set \( \bar{A} = (A; A_1, \ldots, A_r) \). For each \( i, j, k \in A \), let \( \Psi_{i,j} = \text{Iso}(S_i; S_j)(1_G) \) and fix \( \Psi = \bigcup_{i,j \in A} \Psi_{i,j} \). We prove that \( \{S_k : k \in A\} \) forms an \((S; S'; \bar{A}; \Psi)-system\) in \( S \). First, by our construction, if \( i \in A_m \) for some \( m \) then \( \Psi_{i,j} \neq \emptyset \), and so \( \Psi \) satisfies Condition (3.1). Furthermore, it follows immediately from Corollary \(5.10\) that \( \Psi \) satisfies Conditions (3.2) and (3.3). Finally, take any \( \pi \in \text{Aut}(\bar{A}) \) and, for each \( k \in A \), let \( \phi_k \in \Psi_{k,k'} \). Then as \( \phi_k^G = 1_G \) for each \( k \in A \), we may construct an automorphism \( \phi \) of \( S \) from the set of isomorphisms \( \{\phi_k : k \in A\} \) by Corollary \(5.12\). Hence, as \( \phi \) extends each \( \phi_k \) by construction, we have that \( \{S_k : k \in A\} \) forms an \((S; S'; \bar{A}; \Psi)-system\) as required. Since \( S_k \) is \(\aleph_0\)-categorical, it is \(\aleph_0\)-categorical over \( \Psi_{k,k} = \text{Aut}(S_k)(1_G) \) by Lemma \(5.14\). By Corollary \(2.9\), \( \bar{A} \) is \(\aleph_0\)-categorical, and so
\[
|(S')^n / \sim_{S,n}| < \aleph_0
\]
by Lemma 3.4. Given that \(E(S) \subseteq S'\) by (5.2), we therefore have that
\[
|E(S)|/ \sim_{S,n} \leq |(S')/ \sim_{S,n}| < \aleph_0.
\]
Hence \(S\) is \(\aleph_0\)-categorical. \(\Box\)

**Open Problem 5.16.** Does Theorem 5.15 hold if \(G\) is allowed to be any \(\aleph_0\)-categorical group?

We now have a simple tool for constructing a counterexample to the converse of Proposition 5.11. Indeed, by Proposition 5.13 it suffices to find a Rees matrix semigroup over an \(\aleph_0\)-categorical group with \(\aleph_0\)-categorical induced bipartite graph, but with infinitely many non-isomorphic connected Rees components.

**Example 5.17.** Let \(G\) be an \(\aleph_0\)-categorical infinite abelian group with identity element 1, and \(\{g_i : i \in \mathbb{N}\}\) be an enumeration of its non-identity elements (such a group exists by the work of Rosenstein [22], taking \(G = \bigoplus_\mathbb{N} \mathbb{Z}_2\), for example). Let \(I_k = \{i_k^s : s \in \mathbb{N}\}\) and \(\Lambda_k = \{\lambda_k^t : t \in \mathbb{N}\}\) be finite sets for each \(k \in \mathbb{N}\). Let \(P_k = (p_{\lambda_k^t, i_k^s})\) be the \(\Lambda_k \times I_k\) matrix such that \(p_{\lambda_k^t, i_k^s} = g_m\) for each \(1 \leq m \leq k\), and all other entries being 1, that is,
\[
P_k = \begin{bmatrix}
g_1 & 1 & 1 & \cdots & 1 & 1 & \cdots \\
1 & g_2 & 1 & \cdots & 1 & 1 & \cdots \\
1 & 1 & \ddots & \ddots & \vdots & \vdots & \\
\vdots & \vdots & \ddots & 1 & 1 & \cdots & \\
1 & 1 & \cdots & 1 & g_k & 1 & \cdots \\
1 & 1 & \cdots & 1 & 1 & \cdots & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & 
\end{bmatrix}
\]
Then each \(\Gamma(P_k)\) is a complete bipartite graph, isomorphic to \(K_{\aleph_0, \aleph_0}\), and is thus \(\aleph_0\)-categorical by Theorem 5.2. For each \(k \in \mathbb{N}\), let \(S_k\) be the connected Rees matrix semigroup \([G; I_k, \Lambda_k; P_k]\), and set
\[
\bigotimes_{k \in \mathbb{N}} S_k = \mathcal{M}^{0}[G; I, \Lambda; P].
\]
Then \(\Gamma(P)\), being the disjoint union of the pairwise isomorphic \(\aleph_0\)-categorical bipartite graphs \(\Gamma(P_k)\), is \(\aleph_0\)-categorical by Theorem 3.4.

We claim that \(S_k \cong S_\ell\) if and only if \(k = \ell\). Let \((\theta, \psi, (u_i)_{i \in I_k}, (v_\lambda)_{\lambda \in \Lambda_k})\) be an isomorphism between \(S_k\) and \(S_\ell\), and assume w.l.o.g. that \(k \geq \ell\). Since there exists only finitely many rows of \(P_k\) and \(P_\ell\) which have non-identity entries, there exists \(\lambda_k^s \in \Lambda_k\) such that both row \(\lambda_k^s\) of \(P_k\) and row \(\lambda_\ell^s\psi\) of \(P_\ell\) consist entirely of identity entries. Hence, for each \(i_k^t \in I_k\),
\[
p_{\lambda_k^s, i_k^t}^{(k)} \theta = 1 = 1 = v_{\lambda_k^s} \cdot p_{\lambda_\ell^s \psi, i_\ell^t}^{(\ell)} \cdot u_{i_k^t} = v_{\lambda_k^s} u_{i_k^t}
\]
by Theorem 5.9 so that
\[
v_{\lambda_k^s}^{-1} = u_{i_k^1} = u_{i_k^2} = \cdots = u,
\]
say. Dually, by considering the columns of \(P_k\) and \(P_\ell\), we have
\[
v_{\lambda_\ell^s} = v_{\lambda_\ell^s} = \cdots = u^{-1},
\]
since \( v_{\lambda s}^{-1} = u \). We therefore have, for each \( 1 \leq m \leq k \),
\[
g_m \theta = p_{(k)}^{(k)} \lambda_m \theta = u^{-1} \cdot p_{(k)}^{(\ell)} \lambda_m \psi \cdot u = p_{(k)}^{(\ell)} \lambda_m \psi \cdot u \in \{g_1, \ldots, g_\ell\}
\]
as \( G \) is abelian. It follows that the automorphism \( \theta \) maps \( \{g_1, \ldots, g_k\} \) to \( \{g_1, \ldots, g_\ell\} \). Since \( k \geq \ell \), this forces \( k = \ell \), thus proving the claim. We have shown that \( \mathcal{M}^0[G; I, \Lambda; P] \) has infinitely many non-isomorphic connected Rees components, and is therefore not \( \aleph_0 \)-categorical by Proposition 5.13.

### 5.2. Labelled bipartite graphs.

In Example 5.17, the problem which arose was that by shifting from the sandwich matrix \( P = (p_{\lambda i}) \) to the induced bipartite graph \( \Gamma(P) \) we have “forgotten” the value of the entries \( p_{\lambda i} \). In this subsection we extend the construction of the induced bipartite graph of a Rees matrix semigroup to attempt to rectifying this problem, as well as to build classes of \( \aleph_0 \)-categorical Rees matrix semigroups. Further examples of \( \aleph_0 \)-categorical Rees matrix semigroups can then be built using Theorem 5.15.

**Definition 5.18.** Let \( \Gamma = \langle L, R, E \rangle \) be a bipartite graph, \( \Sigma \) a set, and \( f : E \rightarrow \Sigma \) a surjective map. Then the triple \((\Gamma, \Sigma, f)\) is called a \( \Sigma \)-labelled \( (by \ f) \) bipartite graph, which we denote as \( \Gamma^f \).

A pair of \( \Sigma \)-labelled bipartite graphs \( \Gamma^f = (\Gamma, \Sigma, f) \) and \( \Gamma^{f'} = (\Gamma', \Sigma, f') \) are isomorphic if there exists an isomorphism \( \psi : \Gamma \rightarrow \Gamma' \) which preserves labels, that is, such that
\[
\{x, y\} f = \sigma \iff \{x\psi, y\psi\} f' = \sigma.
\]
This gives rise to a natural signature in which to consider \( \Sigma \)-labelled bipartite graphs as follows. For each \( \sigma \in \Sigma \), take a binary relation symbol \( E_\sigma \) and let
\[
L_{BG\Sigma} = L_{BG} \cup \{E_\sigma : \sigma \in \Sigma\}.
\]
Then we call \( L_{BG\Sigma} \) the signature of \( \Sigma \)-labelled bipartite graphs, where \( \{x, y\} \in E_\sigma \) if and only if \( \{x, y\} \in E \) and \( \{x, y\} f = \sigma \).

Let \( \Gamma^f \) be a \( \Sigma \)-labelled bipartite graph. Then for any set \( \Sigma' \) and bijection \( g : \Sigma \rightarrow \Sigma' \), we can form a \( \Sigma' \)-labelling of \( \Gamma \) simply by taking \( \Gamma^{fg} \), which we call a relabelling of \( \Gamma^f \). Notice that if \( \psi \) is an automorphism of \( \Gamma \), then \( \psi \in \text{Aut}(\Gamma^f) \) if and only if \( \psi \in \text{Aut}(\Gamma^{fg}) \). Indeed, if \( \psi \in \text{Aut}(\Gamma^f) \) then for any edge \( \{x, y\} \) of \( \Gamma \) we have
\[
\{x, y\} fg = \sigma' \iff \{x, y\} f = \sigma' g^{-1} \iff \{x\psi, y\psi\} f = \sigma' g^{-1} \iff \{x\psi, y\psi\} f g = \sigma',
\]
since \( g \) is a bijection. The converse is proven similarly, and the following result is then immediate from the RNT.

**Lemma 5.19.** Let \( \Gamma^f \) be a \( \Sigma \)-labelling of a bipartite graph \( \Gamma \). Then \( \Gamma^f \) is \( \aleph_0 \)-categorical if and only if any relabelling of \( \Gamma^f \) is \( \aleph_0 \)-categorical.

**Lemma 5.20.** If \( \Gamma^f = (\Gamma, \Sigma, f) \) is an \( \aleph_0 \)-categorical labelled bipartite graph then \( \Sigma \) is finite and \( \Gamma \) is \( \aleph_0 \)-categorical.

**Proof.** For each \( \sigma \in \Sigma \), let \( \{x_\sigma, y_\sigma\} \) be an edge in \( \Gamma \) such that \( \{x_\sigma, y_\sigma\} f = \sigma \). Then \( \{(x_\sigma, y_\sigma) : \sigma \in \Sigma\} \) is a set of distinct 2-automorphism types of \( \Gamma^f \), and so \( \Sigma \) is finite by the
RNT. Since automorphisms of $\Gamma^f$ induce automorphisms of $\Gamma$, the final result is immediate from the RNT.

A consequence of the previous pair of lemmas is that, in the context of $\aleph_0$-categoricity, it suffices to consider finitely labelled bipartite graphs, with labelling set $m = \{1, 2, \ldots, m\}$ for some $m \in \mathbb{N}$.

**Lemma 5.21.** Let $\Gamma^f = ((L, R, E), m, f)$ be an $m$-labelled bipartite graph such that either $L$ or $R$ are finite. Then $\Gamma^f$ is $\aleph_0$-categorical.

**Proof.** Without loss of generality assume that $L = \{l_1, l_2, \ldots, l_t\}$ is finite. Define a relation $\tau$ on $R$ by $y \tau y'$ if and only if $y$ and $y'$ are adjacent to the same elements in $L$ and $\{l_i, y\}f = \{l_i, y'\}f$ for each such $l_i \in L$. Note that since both $L$ and $m$ are finite, $R$ has finitely many $\tau$-classes, say $R_1, \ldots, R_t$. Considering $R$ simply as a set, fix $A = (R; R_1, \ldots, R_t)$.

Since $L$ is finite, to prove that $\Gamma^f$ is $\aleph_0$-categorical it suffices to show that $(\Gamma^f \setminus L)^n = R^n$ has finitely many $\sim_{\Gamma^f, n}$-classes for each $n \in \mathbb{N}$ by a simple generalization of [5, Proposition 2.11]. Let $a = (r_1, \ldots, r_n)$ and $b = (r'_1, \ldots, r'_n)$ be $n$-tuples of $R$ such that $a \sim_{A, n} b$ via $\psi \in \text{Aut}(A)$, say. We claim that the map $\hat{\psi} : \Gamma^f \rightarrow \Gamma^f$ which fixes $L$ and is such that $\hat{\psi}|_R = \psi$ is an automorphism of $\Gamma^f$. Indeed, as $\psi$ setwise fixes the $\tau$-classes, we have $(\lambda, \lambda\psi) \in \tau$ for each $\lambda \in R$. Hence $\lambda$ and $\lambda\psi$ are adjacent to the same elements in $L$, and so

\[
\{l_i, \lambda\} \in E \iff \{l_i, \lambda\psi\} \in E \iff \{l_i\hat{\psi}, \lambda\hat{\psi}\} \in E,
\]

so that $\hat{\psi}$ is an automorphism of $\Gamma$. Similarly $\{l_i, \lambda\}f = \{l_i, \lambda\psi\}f = \{l_i\hat{\psi}, \lambda\hat{\psi}\}f$, so that $\hat{\psi}$ preserves labels. This proves the claim.

For each $1 \leq k \leq n$ we have $r_k\hat{\psi} = r_k\psi = r'_k$, so that $a \sim_{\Gamma^f, n} b$. Consequently,

\[
|(\Gamma^f \setminus L)^n / \sim_{\Gamma^f, n}| \leq |A^n / \sim_{A, n}|.
\]

The set extension $A$ is $\aleph_0$-categorical by Corollary 2.9 and so $|A^n / \sim_{A, n}|$ is finite for each $n \geq 1$. Hence $\Gamma^f$ is $\aleph_0$-categorical. \hfill $\Box$

**Lemma 5.22.** Let $\Gamma^f = ((L, R, E), m, f)$ be such that there exists $p \in m$ with $\{x, y\}f = p$ for all but finitely many edges in $\Gamma$. Then $\Gamma^f$ is $\aleph_0$-categorical if and only if $\Gamma$ is $\aleph_0$-categorical.

**Proof.** Suppose $\Gamma$ is $\aleph_0$-categorical, and that $\{l_1, r_1\}, \ldots, \{l_t, r_t\}$ are precisely the edges of $\Gamma$ such that $\{l_k, r_k\}f \neq p$, where $l_k \in L$ and $r_k \in R$. Let $a$ and $b$ be $n$-tuples of $\Gamma^f$ such that

\[
(a, l_1, r_1, \ldots, l_t, r_t) \sim_{\Gamma^f, n+2t} (b, l_1, r_1, \ldots, l_t, r_t)
\]

via $\psi \in \text{Aut}(\Gamma)$, say. We claim that $\psi$ is an automorphism of $\Gamma^f$. For each $1 \leq k \leq t$ we have $l_k\psi = l_k$ and $r_k\psi = r_k$ so that

\[
\{l_k, r_k\}f = \{l_k\psi, r_k\psi\}f.
\]

It follows that $\{l, r\}f = p$ if and only if $\{l\psi, r\psi\}f = p$, and so $\psi$ preserves all labels, thus proving the claim. Consequently, $a \sim_{\Gamma^f, n} b$ via $\psi$, so that

\[
|(\Gamma^f)^n / \sim_{\Gamma^f, n}| \leq |\Gamma^{n+2t} / \sim_{\Gamma^f, n+2t}| < \aleph_0
\]
by the $\aleph_0$-categoricity of $\Gamma$. Hence $\Gamma^f$ is $\aleph_0$-categorical.

The converse is immediate from Lemma 5.20.

**Definition 5.23.** Given a Rees matrix semigroup $S = \mathcal{M}^0[G; I, \Lambda; P]$, we form a $G(P)$-labelling of the induced bipartite graph $\Gamma(P) = \langle I, \Lambda, E \rangle$ of $S$ in the natural way by taking the labelling $f : E \to G(P)$ given by

$$\{i, \lambda\}f = p_{\lambda,i}.$$

We denote the labelled bipartite graph by $\Gamma(P)^l$, which we call the *induced labelled bipartite graph* of $S$.

Note that, unlike the corresponding case for the induced bipartite graph $\Gamma(P)$, there exists isomorphic Rees matrix semigroups with non-isomorphic induced labelled bipartite graphs. For example, let $G$ be a non-trivial group and $P$ and $Q$ be $1 \times 2$ matrices over $G \cup \{0\}$ given by

$$P = \begin{pmatrix} 1 & a \end{pmatrix} \quad Q = \begin{pmatrix} 1 & 1 \end{pmatrix}$$

where $a \notin \{0, 1\}$. Let $S = \mathcal{M}^0[G; 2, 1; P]$ and $T = \mathcal{M}^0[G; 2, 1; Q]$, noting that $\Gamma(P) = \Gamma(Q)$ (and are isomorphic to $K_{2,1}$). Then $(1_G, 1_{\Gamma(P)}, (a_i)_{i \in 2}, (v_{\lambda})_{\lambda \in 1})$ is an isomorphism from $S$ to $T$, where $u_1 = 1 = v_1$, and $u_2 = a$. However, since $\Gamma(P)^l$ and $\Gamma(Q)^l$ have different labelling sets, they are not isomorphic.

**Proposition 5.24.** Let $S = \mathcal{M}^0[G; I, \Lambda; P]$ be such that $G$ and $\Gamma(P)^l$ are $\aleph_0$-categorical. Then $S$ is $\aleph_0$-categorical.

**Proof.** Since $\Gamma(P)^l$ is $\aleph_0$-categorical, the set $G(P)$ is finite by Lemma 5.20, say $G(P) = \{x_1, \ldots, x_r\}$. Let $a = ((i_1, g_1, \lambda_1), \ldots, (i_n, g_n, \lambda_n))$ and $b = ((j_1, h_1, \mu_1), \ldots, (j_n, h_n, \mu_n))$ be a pair of $n$-tuples of $S^*$ under the pair of conditions that

1. $(g_1, \ldots, g_n, x_1, \ldots, x_r) \sim_{G, n+r} (h_1, \ldots, h_n, x_1, \ldots, x_r)$,
2. $\Gamma(a) \sim_{\Gamma(P)^l, 2n} \Gamma(b)$,

via $\theta \in \text{Aut}(G)$ and $\psi \in \text{Aut}(\Gamma(P)^l)$, respectively (noting the use of Notation 5.8 here). We claim that $\phi = (\theta, \psi, (1)_{i \in I}, (1)_{\lambda \in \Lambda})$ is an automorphism of $S$. Indeed, if $p_{\lambda,i} \neq 0$ for some $i \in I$, $\lambda \in \Lambda$, then $p_{\lambda,i} = x_k$ for some $k$, so that $\{i, \lambda\}f = \{i\psi, \lambda\psi\}f = x_k$. Consequently, $p_{\lambda,i} = x_k \theta = x_k = p_{\lambda,\psi,}\psi$, and claim follows by Theorem 5.9. Hence

$$(i_t, g_t, \lambda_t)\phi = (i_t\psi, g_t\theta, \lambda_t\psi) = (j_t, h_t, \mu_t)$$

for each $1 \leq t \leq n$, so that

$$|(S^*)^n / \sim |S, n| \leq |G^{n+r} / \sim_{G,n+r} | \cdot |(\Gamma(P)^l)^{2n} / \sim_{\Gamma(P)^l, 2n}| < \aleph_0,$$

as $G$ and $\Gamma(P)^l$ are $\aleph_0$-categorical. Hence $S$ is $\aleph_0$-categorical by [5, Proposition 2.11].

The converse however fails to hold in general, and a counterexample will be constructed at the end of the section. Despite this, the proposition above enables us to produce concrete examples of $\aleph_0$-categorical Rees matrix semigroups. For example, the result below is immediate from Lemma 5.21.
Corollary 5.25. Let $S$ be a Rees matrix semigroup over an $\aleph_0$-categorical group having sandwich matrix $P$ with finitely many rows or columns, and $G(P)$ being finite. Then $S$ is $\aleph_0$-categorical.

Similarly, Lemma 5.22 may be used in conjunction with Proposition 5.24 to obtain:

Corollary 5.26. Let $S = M^0[G; I, \Lambda; P]$ be a Rees matrix semigroup such that $G$ and $\Gamma(P)$ are $\aleph_0$-categorical, and all but finitely many of the non-zero entries of $P$ are the identity of $G$. Then $S$ is $\aleph_0$-categorical.

Following [14], we call a completely 0-simple semigroup $S$ pure if it is isomorphic to a Rees matrix semigroup with sandwich matrix over $\{0, 1\}$. In [11], Houghton considered trivial cohomology classes of Rees matrix semigroups, a property which is proven in Section 2 of his article to be equivalent to being pure. Hence, by [11] Theorem 5.1, a completely 0-simple semigroup is pure if and only if, for each $a, b \in S$,

$[a, b \in \langle E(S) \rangle \text{ and } a H b] \Rightarrow a = b$.

It follows that all orthodox completely 0-simple semigroups are necessarily pure, but the converse is not true in general. Indeed, a completely 0-simple semigroup is orthodox if and only if it is isomorphic to a Rees matrix semigroup with sandwich matrix over $\{0, 1\}$ and with induced bipartite graph a disjoint union of complete bipartite graphs [9, Theorem 6]. Hence, in this case, it can be easily shown that the isomorphism types of the connected Rees components depends only on the isomorphism types of the induced (complete) bipartite graphs.

We observe that if the sandwich matrix of a Rees matrix semigroup is over $\{0, 1\}$ then $\Gamma(P)^I$ is simply labelled by $\{1\}$. Therefore all automorphisms of $\Gamma(P)$ automatically preserve the labelling, and so $\Gamma(P)^I$ is $\aleph_0$-categorical if and only if $\Gamma(P)$ is $\aleph_0$-categorical. The following result is then immediate from Proposition 5.11 and Corollary 5.26.

Corollary 5.27. A pure Rees matrix semigroup $M^0[G; I, \Lambda; P]$ is $\aleph_0$-categorical if and only if $G$ and $\Gamma(P)$ are $\aleph_0$-categorical.

Furthermore, since complete bipartite graphs are $\aleph_0$-categorical by Theorem 5.2, a disjoint union of complete bipartite graphs is $\aleph_0$-categorical if and only if it has finitely many connected components, up to isomorphism, by Proposition 5.4. The corollary above thus reduces in the orthodox case as follows.

Corollary 5.28. Let $S = M^0[G; I, \Lambda; P]$ be an orthodox Rees matrix semigroup. Then $S$ is $\aleph_0$-categorical if and only if $G$ is $\aleph_0$-categorical and $\Gamma(P)$ has finitely many connected components, up to isomorphism.

In [5] we studied inverse completely 0-simple semigroups, that is, Brandt semigroups. These are necessarily orthodox, and are isomorphic to a Rees matrix semigroup of the form $M^0[G; I, I; P]$ where $P$ is the identity matrix, that is, $p_{ii} = 1$ and $p_{ij} = 0$ for each $i \neq j$ in $I$, and are denoted $B^0[G; I]$. Since the induced bipartite graph of a Brandt semigroup is a perfect matching, it is $\aleph_0$-categorical by Theorem 5.2. Corollary 5.28 then simplifies to obtain our classification of $\aleph_0$-categorical Brandt semigroups [5, Proposition 4.3], which
states that a Brandt semigroup over a group $G$ is $\aleph_0$-categorical if and only if $G$ is $\aleph_0$-categorical.

We are now able to construct a simple counterexample to the converse of Proposition 5.24. Let $G = \{g_i : i \in \mathbb{N}\}$ be an infinite $\aleph_0$-categorical group. Let

$$S = M_0[|G; \mathbb{N}, \mathbb{N}; P|] = B_0[|G; \mathbb{N}|] = T = M_0[|G; \mathbb{N}, \mathbb{N}; Q|],$$

where $Q = (q_{i,j})$ is such that $q_{i,i} = g_i$ and $q_{i,j} = 0$ for each $i \neq j$. Then $\Gamma(P) = \Gamma(Q)$ (and are isomorphic to $P_\mathbb{N}$) and $(1_G, 1_{\Gamma(P)}, (g_i^{-1})_{i \in \mathbb{N}}, (1)_{\lambda \in \mathbb{N}})$ is an isomorphism from $S$ to $T$ by Theorem 5.9 since

$$p_{i,i}^{-1} = 1 = g_i g_i^{-1} = 1 \cdot q_{i,i} \cdot g_i^{-1},$$

for each $i \in \mathbb{N}$. Since $S$ is $\aleph_0$-categorical by the $\aleph_0$-categoricity of $G$, the same is true of $T$. However, $\Gamma(Q)^f$ is a $G$-labelling, and is thus not $\aleph_0$-categorical by Lemma 5.20. Hence $T$ is our desired counterexample.

Open Problem 5.29. Does there exist an $\aleph_0$-categorical connected Rees matrix semigroup over a finite group which is not isomorphic to a Rees matrix semigroup with $\aleph_0$-categorical induced labelled bipartite graph?

To further incorporate the link between the induced bipartite graph of a Rees matrix semigroup and the entries of the sandwich matrix, we could instead introduce the stronger notion of an induced group labelled bipartite graph. A group labelled bipartite graph is a $G$-labelled bipartite graph $\Gamma^f = ((L, R, E), G, f)$, for some group $G$, where an automorphism of $\Gamma^f$ is a pair $(\psi, \theta) \in \text{Aut}(\Gamma) \times \text{Aut}(G)$ such that, for each $\ell \in L$, $r \in R$,

$$(\ell, r)f = g \iff (\ell \psi, r \psi)f = g \theta.$$  

However, group labelled bipartite graphs do not appear to be first order structures.

Let $S = M_0[|G; I, \Lambda; P|]$ be such that $G(P)$ forms a subgroup of $G$. Then we may define the induced group labelled bipartite graph of $S$ as the $(G(P))$-labelled bipartite graph $\Gamma(S)^f$, with automorphisms being pairs $(\psi, \theta) \in \text{Aut}(\Gamma) \times \text{Aut}(G(P))$ such that $p_{\lambda \psi, i \psi} = p_{\lambda i \theta}$ for each $i \in I, \lambda \in \Lambda$. Notice that if $(\psi, \theta)$ is an automorphism of the induced group labelled bipartite graph of $S$ and is such that $\theta$ extends to an automorphism $\theta'$ of $G$, then $(\theta', \psi, (1)_{i \in I}, (1)_{\lambda \in \Lambda})$ is clearly an automorphism of $S$. However, we do not in general obtain all automorphisms of $S$ in this way. Similar problems therefore arise in regard to when $\aleph_0$-categoricity of $S$ passes to its induced group labelled bipartite graph (by which we mean the induced group labelled bipartite graph has an oligomorphic automorphism group).

References

[1] A. B. Apps, ‘On the structure of $\aleph_0$-categorical groups’, J. Algebra 81 (1982) 320–339.
[2] J. Araújo, P.V. Bünau, J.D. Mitchell, M. Neunhöffer, ‘Computing automorphisms of semigroups’, J. Symbolic Comput. 45 (2010) 373–392.
[3] P. Erdős, J. Spencer, ‘Probabilistic Methods in Combinis.’, New York Academic Press (1974).
[4] M. Goldstern, R. Grossberg, M. Kojman, ‘Infinite homogeneous bipartite graphs with unequal sides’, Discrete Math. 149 (1996) 69–82.
[5] V. Gould, T. Quinn-Gregson, ‘$\aleph_0$-categoricity of semigroups’, arXiv:1802.05703.
[6] R. L. Graham, ‘On finite 0-simple semigroups and graph theory’, *Math. Systems Theory*, 2 (1968) 325–339.
[7] N. Graham, R. Graham, and J. Rhodes, ‘Maximal subsemigroups of finite semigroups’, *J. Combin. Theory* 4 (1968) 203–209.
[8] A. Grzegorczyk, ‘Logical uniformity by decomposition and categoricity in $\mathbb{N}_0$’, *Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys.* 16 (1968) 687–692.
[9] T. E. Hall, ‘On regular semigroups whose idempotents form a subsemigroup’, *Bull. Austral. Math. Soc.* 1 (1969) 195–208.
[10] W. Hodges, ‘Model theory’ *Cambridge University Press* (1993).
[11] C. H. Houghton, ‘Completely 0-simple semigroups and their associated graphs and groups’, *Semigroup Forum* 14 (1977) 41–67.
[12] J. M. Howie, ‘Idempotents in completely 0-simple semigroups’, *Glasgow Math. J.* 19 (1978) 109–113.
[13] J. M. Howie, *Fundamentals of semigroup theory*, Oxford University Press (1995).
[14] M. Jackson, M. Volkov, ‘Undecidable problems for completely 0-simple semigroups’, *J. Pure Appl. Algebra* 213 (2009) 1961–1978.
[15] H. D. Macpherson, ‘A survey of homogeneous structures’, *Discrete Math.* 311 (2011) 1599–1634.
[16] M. Petrich, N. R. Reilly, ‘Completely regular semigroups’, *Wiley, New York* (1999).
[17] T. Quinn-Gregson, ‘Homogeneity and $\mathbb{N}_0$-categoricity of semigroups’ PhD thesis, *University of York* (2017).
[18] T. Quinn-Gregson, ‘Homogeneous bands’ *Adv. Math.* 328 (2018) 623-660.
[19] T. Quinn-Gregson, ‘Homogeneity of inverse semigroups’ *Submitted, arXiv:1706.00975* (2017).
[20] D. Rees, ‘On semi-groups’, *Proc. Cambridge Phil. Soc.* 36 (194) 387–400.
[21] J. Rhodes, B. Steinberg, ‘The $q$-theory of finite semigroups’, *Springer Monographs in Maths.* (2009).
[22] J. G. Rosenstein, ‘$\mathbb{N}_0$-categoricity of groups’, *J. Algebra* 25 (1973) 435–467.
[23] S. Worawiset, ‘The structure of endomorphism monoids of strong semilattices of left simple semigroups’, PhD thesis, *University of Oldenburg* (2011).