Pattern Selection of Cracks in Directionally Drying Fracture

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(Received July 9, 1996; accepted for publication October 22, 1996)

We study pattern selection of cracks in directionally drying fractures by analyzing the experimental systems recently devised by C. Allain and L. Limat [Phys. Rev. Lett. 74 (1995) 2981]. Proposing a simple picture of crack formation, we clarify the mechanism of how cracks become regularly arrayed and find that the interval between neighboring cracks is proportional to the 2/3 power of the cell thickness. This result explains well the experimental data of Allain and Limat.

KEYWORDS: fracture, statistical physics, pattern selection, power law, drying

1. Introduction

The study of fracture has developed greatly since Griffith wrote a breakthrough paper in 1920. In particular, recent progress is due largely to the development of well-controlled experimental systems. Recently, Allain and Limat have studied periodically aligned crack patterns using an experimental system consisting of directionally drying fractures. In their experiment, a colloidal suspension was put into a rectangular cell of which one surface was left open in order to allow evaporation, and after a short time, periodically aligned cracks were observed. Similar phenomena have been observed in experiments on drying fractures with different geometries. Since a crack cannot adjust its position after it appears, these experimental results cannot be explained in the same way as periodic pattern formation in convective systems and reaction diffusion systems. We are thus led to consider the mechanism of crack formation in drying fractures.

The question we address is how to determine the interval between neighboring cracks, ٪. Although Allain and Limat gave a theoretical estimate of ٪ and confirmed that its value is of the same order as the experimental result, their theory leads to a relation between ٪ and the system thickness ٪ as ٪/٪ ٪ А − ٪ ln ٪, which does not fit their experimental data well. In this Letter, we propose a simple picture of crack formation in a directionally drying fracture. Based on this picture, we clarify why cracks are formed periodically and derive the scaling relation ٪ ٪ ٪/٪. Further, by comparison with the experimental data, we confirm the validity of the scaling relation.

2. Picture of Crack Formation

The experimental configuration we analyze is illustrated in Fig. 1. We assume that the cell under consideration extends semi-ininitely in the positive y direction and has boundaries at ٪ 0 (front surface), ٪ х ٪ и ٪ х ٪. Further, we are interested in the limiting case ٪ ٪. We first consider the water distribution in the material without cracks. Let ٪ be the water volume fraction. We assume that water evaporates from the front surface at a rate (٪−٪) ٪, where ٪ is the equilibrium value of the water volume fraction, and that there is no flux at the other boundaries. In a bulk region, a diffusion current proportional to the gradient of ٪ is assumed to arise. Then, the time evolution of ٪ is given by the diffusion equation

\[ \frac{\partial \phi}{\partial t} = D \Delta \phi, \]

where ٪ is a diffusion constant. As an initial condition, we assume that ٪ takes a constant value ٪ ٪. Then, ٪ is independent of ٪, and ٪ is expressed by using a Green function of the diffusion equation. Without knowing the explicit form of ٪, we do know the following general aspects of its behavior which are important for the argument given below. First, when ٪ is fixed and ٪ is increased, ٪(٪, ٪) monotonically approaches ٪ with the characteristic variation scale ٪ ٪. Second, ٪(0, ٪) approaches ٪ with the time scale ٪ ٪. Then, the length scale reached by the diffusion due to the evaporation at the front surface, which is denoted by ٪, is estimated as ٪ = ٪ ٪ ٪. In the argument below, ٪ ٪ will be assumed so that we can concentrate on the idealized case that the evaporation at crack surfaces does not cause inhomogeneity in the x direction.

We next discuss elastic properties of the material in which a colloidal gel forms. From a macroscopic viewpoint, the material can be regarded as a homogeneous elastic medium. We thus apply the linear elastic theory to the calculation of the macroscopic stresses. Here, the stresses ٪, ٪ and ٪ are proportional to the sum of the corresponding strains and the volume shrinkage rate.

Fig. 1. Schematic view of experimental setup. The axes ٪, ٪, ٪ are defined as seen. The drying process takes place only at the front surface, whose dimensions are defined as ٪ ٪. Cracks originate at the surface and extend in the y direction.
for simplicity, we assume that separations at the vertical boundaries \((x = \pm L/2)\) occur before crack formation in the bulk. Note that the system after such separation is equivalent to one under stress-free conditions at the vertical boundaries. Whether this assumption is realistic or not depends on the conditions at the interfaces between the elastic material and the cell. However, as we will see later, even in the case that the separation from the boundaries never occurs, the result of pattern selection of cracks is unchanged. In addition, we assume that no-slip boundary conditions are maintained at the horizontal boundaries. Similar to the case of the vertical boundaries, this assumption may not be realistic under some conditions. However, even if the separation from the top plate occurs, we can obtain the same scaling relation on the assumption that uniform evaporation from the front surface dominates.\(^9\) We do not need to consider the situation that separations from both plates occur, because in this case the fracture process stops.

As the evaporation proceeds, crack formation occurs first at a time \(t_1\) satisfying the equation
\[
\max \left\{ E(L, C(t_1)) - E(l, C(t_1)) - E(L - l, C(t_1)) \right\} = \Gamma H,
\]
where \(\Gamma\) is the surface energy per unit length, and \(E(L, C(t))\) is the elastic energy of the material with the horizontal length \(L\) and the volume shrinkage rate \(C(t)\) at time \(t\). The value of \(l\) maximizing the quantity\( E(L, C(t_1)) - E(l, C(t_1)) - E(L - l, C(t_1))\), denoted by \(L_1\), specifies the position where the crack is formed. From the geometrical symmetry of the problem, we expect \(L_1 = L/2\). This implies that a crack is formed first at the center of the strip. (This will be confirmed later.)

In this way, at a time \(t_1\), there are two strips with a horizontal length \(L/2\). We should note here that each strip has the same boundary conditions as the original one. Thus, by replacing variables \(t_1, L\) in eq. (3) with \(t_2, L_1\), we know the time \(t_2\) at which the next crack formation occurs at the center of each strip with the length \(L_1/2\). (Note that these crack formations occur simultaneously.) As bisected strips are further bisected in succession, this process repeats, producing strips with equal horizontal length until the evaporation ceases. This is the reason the cracks show spatial periodicity. The interval between cracks \(\lambda\), which is identical to the horizontal size of strips at \(t = \infty\), is determined by the maximum length \(2^{-n}L\) \((n: \text{integer})\) shorter than the length 2\(\lambda\), satisfying
\[
E(2\lambda, C_\infty) - 2E(\lambda, C_\infty) = \Gamma H.
\]
Also, as easily checked, \(\lambda\) satisfies the inequality
\[
\lambda_1 < \lambda < 2\lambda_1.
\]
Therefore, we can estimate the value of \(\lambda\) if we can derive an expression for the elastic energy of the strip. In the following paragraphs, we will derive an expression for \(E(l, C)\).

3. Power Law of Crack Intervals

In order to evaluate the expression of the elastic energy of a strip with a horizontal size \(l\), we consider a quasi-one
dimensional spring network which is composed of a chain of \(N\) springs along the centerline and vertical springs connecting each node to a fixed position at the boundary (see Fig. 2). The ends of the horizontal springs have no constraints because free boundary conditions are imposed at the vertical boundaries of the strip. Here, the natural lengths of the horizontal and the vertical springs at \(t = 0\) are given by \(a = 1/N\) and \(H/2\), respectively, and the attachment points are assumed to be positioned regularly, with a period \(a\). Further, the spring constants of the horizontal and the vertical springs are denoted by \(k_1\) and \(k_2\), respectively. This effective spring network resembles Meakin’s model\(^{10}\) when the model is supplemented with a breaking rule. We note, however, that according to our picture of crack formation discussed above, the horizontal springs are broken deterministically, as in Hayakawa’s model,\(^{11}\) not probabilistically, as in Meakin’s model.

The elastic energy of the spring network is expressed by

\[
\frac{N-1}{2} k_1 \sum_{i=0}^{N-1} \left[ u_{i+1} - u_i + a \frac{C}{3} \right]^2 + 2 \frac{N}{2} k_2 \left[ \sqrt{u_i^2 + \left( \frac{H}{2} \right)^2} - \frac{H}{2} \left( 1 - \frac{C}{3} \right) \right]^2.
\]

Here, \(u_i\) is the displacement of the \(i\)-th node from the reference point \(x_i = ia - l/2\). Note that the linear shrinkage rate of springs is given by \(C/3\). Under the assumption \((u_i/H)^2 \ll C/3 \ll 1\), expanding eq. (6) in \(u_i/H\) and ignoring terms of higher order than \((u_i/H)^2\), eq. (6) reduces to

\[
\frac{N-1}{2} k_1 \sum_{i=0}^{N-1} \left[ u_{i+1} - u_i + a \frac{C}{3} \right]^2 + \sum_{i=0}^{N-1} k_2 \left( \frac{C}{6} \right)^2 + \frac{C}{3} u_i^2.
\]

Further, in order to make it possible to develop an analytical argument, we take the continuum limit \((a \rightarrow 0\) with fixing \(L\)) of eq. (7). First, we introduce a variable \(\bar{u}\) well-defined in this limit by

\[
u_i = H \bar{u} \left( \frac{x_i}{H} \right).
\]

Then, when this expression is substituted into eq. (7), \(E\) should not depend on \(a\) in this limit. By noting that \(u_{i+1} - u_i = a \bar{u}' + O(a^2)\) where the prime refers to differentiation with respect to the argument \(x_i/H\), this requirement leads to the conditions \(k_1 \sim 1/a\) and \(k_2 \sim a\). Further, since it seems natural to assume that \(k_1\) and \(k_2\) do not depend on the horizontal length \(l\), we can express \(k_1\) and \(k_2\) in this limit as

\[
k_1 = \frac{H}{a} K, \quad k_2 = \frac{1}{2} \left( \frac{a}{H} \right)^2 \kappa,
\]

where \(K\) is related to the Young modulus of the two dimensional elastic material, and \(\kappa\) is a non-dimensional quantity of order unity. As a result, we obtain the following expression of the elastic energy in the continuum limit:

\[
\frac{KH^2}{2} \int_{-l/2H}^{l/2H} \bar{u}'' - \kappa C \bar{u}' = 0,
\]

under stress-free boundary conditions at both side ends \((\bar{x} = \pm l/2H)\):

\[
\bar{u}' + \frac{C}{3} = 0.
\]

Solving this differential equation, we obtain an analytic expression for the displacement field \(\bar{u}\) as

\[
\bar{u} = \frac{C}{3\kappa} \frac{\sinh \left( \frac{kC}{3} \bar{x} \right)}{\cosh \left( \frac{kC}{3} \frac{l}{2H} \right)},
\]

where \(-l/2H \leq \bar{x} \leq l/2H\). Here, note that eq. (13) is valid only in the case

\[
\frac{\sqrt{kC}}{3} \frac{l}{H} \ll 1,
\]

because we have assumed \(\bar{u}' \ll C/3 \ll 1\) in the derivation of eq. (7). By substituting eq. (13) into eq. (10), an expression for the elastic energy \(E(l,C)\) is derived as

\[
E(l,C) = \frac{K C^2 H l}{18} \left[ 1 + \frac{\kappa}{4} \right] - \frac{3}{4 \kappa \cosh \left( \frac{kC}{3} \frac{l}{2H} \right))}
\]

Further, under the condition of eq. (14), eq. (15) becomes

\[
E(l,C) = \frac{K C^2 H l}{72} \left[ \kappa + \frac{\kappa C}{9} \left( \frac{l}{H} \right)^2 \right].
\]

As expected, it is easily confirmed that \(E(L,C) - E(l,C)\) is maximum when \(l = L/2\), because we find, using eq. (15), that \(\partial E(l,C)/\partial l\) is a monotonically increasing function of \(l\). Thus from eqs. (4) and (16), we obtain

\[
\chi^2 = \gamma H^2,
\]
smaller than $L$, the effect of the evaporation at crack surfaces must be taken into account. In such a case, the above simple picture cannot be applied to the crack formation process because inhomogeneity of the water volume fraction arises. In this case, the crack spacing may have some distribution. Still, we believe that the scaling relation given by eq. (19) will give a first-order approximation for the average value of crack spacing. Second, we assumed that the separation of the strip from the vertical boundary occurs before the appearance of cracks. However, even in the case that the separation from the boundaries never occurs, the vertical boundaries of the slice of material lying between cracks after their formation will satisfy the stress-free boundary conditions. Therefore, by applying the above discussion to such a slice, we can again obtain the result eq. (19).

We note that our proposed mechanism of pattern selection is completely different from one proposed by Allain and Limat. They speculated that a new crack forms near a pre-existing crack based on the assumption that drying takes place mainly at crack surfaces. In order to determine which picture is more appropriate, one has only to observe the time course of crack formation. According to our picture, a crack should form in such a way that it bisects strips. On the other hand, if propagation of crack formation is observed, this may support their idea.

Finally, we briefly discuss the crack patterns which appear subsequent to the crack formation. As soon as a crack is formed, it extends along the $y$ direction and is arrested at the position where the energy release rate for the crack is equal to the surface energy. After an evaporation time $D/J^2$, there is no crack formation at the front surface. Then, one may expect that the individual crack tips will all extend to the same position. However, as shown by Bahr et al., a zig-zag type or more complicated arrangement occurs when a certain condition is satisfied. In fact, we can see such patterns in the experiment performed by Allain and Limat. Further, in their experiment, a more interesting crack pattern appeared due to secondary branching from the elongating cracks. Such patterns will be studied by devising an extended version of the present model.

### Acknowledgements

We thank M. Sano, A. Nakahara and K. Sekimoto for showing us experiments of drying fracture and for stimulating discussions. We also thank F. Takagi and Y. Hayakawa for valuable discussions. This research was supported in part by JSPS Research Fellowships for Young Scientists and by the Grants-in-Aid for Science Research Fund from the Ministry of Education, Science, Sports and Culture Nos. 07238206 and 07640505.

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