DISTRIBUTIONAL CHAOS AND FACTORS

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Abstract. We show the existence of a dynamical system without any distributionally scrambled pair which is semiconjugated to a distributionally chaotic factor.

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1. Introduction

Semiconjugacy is used as a common tool for proving topological chaos or positive topological entropy. The usual technique is to find a semiconjugacy \( \pi \) with a chaotic system and transfer the chaos to the extension. By continuity of \( \pi \), the topological entropy of the extension is not smaller than the entropy of factor system. Unfortunately, semiconjugacy may not automatically guarantee the distributional chaos, which was introduced in [1]. Authors in [2],[3],[4] developed several techniques for proving distributional chaos via semiconjugacy, usually using a symbolic space as the factor space. Example in [2] shows the existence of distributionally chaotic factor which is semiconjugated to the system with no three-points distributionally scrambled sets. The aim of the paper is to improve this result and find a distributionally chaotic factor which has an extension without any distributionally scrambled pair.

2. Terminology

Let \((X, d)\) be a non-empty compact metric space. Let us denote by \((X, f)\) the topological dynamical system, where \(f\) is a continuous self-map acting on \(X\). We define the forward orbit of \(x\), denoted by \(\text{Orb}^+_f(x)\) as the set \(\{f^n(x) : n \geq 0\}\). Let \((X, f)\) and \((Y, g)\) be dynamical systems on compact metric spaces. A continuous map \(\pi: X \to Y\) is called a semiconjugacy between \(f\) and \(g\) if \(\pi\) is surjective and \(\pi \circ f = g \circ \pi\). In this case we can say that \((Y, g)\) is a factor of the system \((X, f)\) or equivalently \((X, f)\) is an extension of the system \((Y, g)\).

Definition 1. A pair of two different points \((x_1, x_2)\) \(\in X^2\) is called scrambled if
\[
\lim \inf_{k \to \infty} d(f^k(x_1), f^k(x_2)) = 0 \tag{1}
\]
and
\[
\lim \sup_{k \to \infty} d(f^k(x_1), f^k(x_2)) > 0. \tag{2}
\]
A subset \(S\) of \(X\) is called scrambled if every pair of distinct points in \(S\) scrambled. The system \((X, f)\) is called chaotic if there exists an uncountable scrambled set.

Definition 2. For a pair \((x_1, x_2)\) of points in \(X\), define the lower distribution function generated by \(f\) as
\[
\Phi(x_1, x_2)(\delta) = \lim \inf_{m \to \infty} \frac{1}{m} \# \{0 < k < m; d(f^k(x_1), f^k(x_2)) < \delta\},
\]
and the upper distributional function as
\[
\Phi^*(x_1, x_2)(\delta) = \lim \sup_{m \to \infty} \frac{1}{m} \# \{0 < k < m; d(f^k(x_1), f^k(x_2)) < \delta\},
\]
where \(\#A\) denotes the cardinality of the set \(A\).

A pair \((x_1, x_2)\) \(\in X^2\) is called distributionally scrambled of type 1 if
\[
\Phi^*(x_1, x_2) = 1 \text{ and } \Phi(x_1, x_2)(\delta) = 0, \text{ for some } 0 < \delta \leq \text{diam } X,
\]

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distributionally scrambled of type 2 if
\[ \Phi^*_{(x_1, x_2)} \equiv 1 \quad \text{and} \quad \Phi_{(x_1, x_2)} < \Phi^*_{(x_1, x_2)}, \]
distributionally scrambled of type 3 if
\[ \Phi_{(x_1, x_2)} < \Phi^*_{(x_1, x_2)}. \]
The dynamical system \( (X, f) \) is distributionally chaotic of type \( i \) (DC\( i \) for short), where \( i = 1, 2, 3 \), if there is an uncountable set \( S \subset X \) such that any pair of distinct points from \( S \) is distributionally scrambled of type \( i \).

3. Distributional chaos and factors

We will show the existence of a system without any distributionally scrambled pair which is semi-conjugated to a distributionally chaotic factor. This system is three-dimensional union of countably many homocentric cylinders with unit height and converging radius. First we state the following technical lemma about rotation on circle. Let \( u \in S \) and \( v \in S \) be determined by normed angles \( \phi_u \in I \) and \( \phi_v \in I \). These points rotate along the circle by different angles \( r_u \in I \), respectively \( r_v \in I \), i.e.
\[
\phi_u \mapsto (\phi_u + r_u) \mod 1 \\
\phi_v \mapsto (\phi_v + r_v) \mod 1. \tag{3}
\]
We denote the relative angle of rotation by \( \Delta r = |r_u - r_v| \) and assume that the metric on \( S \) is \( \rho(\alpha, \beta) = \min\{|\alpha - \beta|, 1 - |\alpha - \beta|\} \).

**Lemma 1.** For every number \( \delta > 0 \) and every integer \( p > \frac{2}{\Delta r} \), the following estimation holds:
\[
\frac{1}{p} \# \{0 < i < p; \rho((\phi_u + ir_u) \mod 1, (\phi_v + ir_v) \mod 1) < \delta \} < 3\delta.
\]

**Proof.** Because \( \rho((\phi_u + ir_u) \mod 1, (\phi_v + ir_v) \mod 1) = \rho(\phi_u, (\phi_v + i\Delta r) \mod 1) \), it is sufficient to show
\[
\frac{1}{p} \# \{0 < i < p; \rho(\phi_u, (\phi_v + i\Delta r) \mod 1) < \delta \} < 3\delta.
\]
The expression \( |p \cdot \Delta r| \) determines the number of turns that the point \( v \) makes along the circle by rotation through the angle \( \Delta r \) after \( p \) iterations, where \( [x] \) denotes the integer part of \( x \). The maximal number of iterations \( i \) during one turn, for which \( \rho(\phi_u, (\phi_v + i\Delta r) \mod 1) < \delta \), is \( \frac{2\delta}{\Delta r} \). It follows
\[
\frac{1}{p} \# \{0 < i < p; \rho(\phi_u, (\phi_v + i\Delta r) \mod 1) < \delta \} < \frac{1}{p} \left[ \left| p \cdot \Delta r \right| \frac{2\delta}{\Delta r} + \frac{2\delta}{\Delta r} \right] < 2\delta + \frac{2\delta}{p\Delta r}.
\]
Because \( p > \frac{2}{\Delta r} \), we can estimate the second term by \( \delta \), i.e. \( \frac{2\delta}{p\Delta r} < \delta \).
\( \square \)

**Theorem 1.** There exists a DC1 dynamical system \( (Y, f) \) which is semi-conjugated to an extension \( (X, F) \) which possess no distributionally scrambled pair (of type 1 or 2).

**Proof.** The space \( X \) is defined
\[
X = \left( \left\{(2 - \frac{1}{k}) \cos 2\pi \phi, (2 - \frac{1}{k}) \sin 2\pi \phi : k \in \mathbb{N}, \phi \in I \right\} \cup \left\{2 \cos 2\pi \phi, 2 \sin 2\pi \phi : \phi \in I \right\} \right) \times I,
\]
where \( I \) is the unit interval. Each point \( u = [r_u \cos 2\pi \phi_u, r_u \sin 2\pi \phi_u, z_u] \) in \( X \) is determined by its angle \( \phi_u \in I \), radius \( r_u \in \{2 - \frac{1}{k} : k \in \mathbb{N}\} \cup \{2\} \) and height \( z_u \in I \).
The space is endowed with max-metric
\[
d(u, v) = \max\{|r_u - r_v|, |z_u - z_v|, \rho(\phi_u, \phi_v)|,
\]
where \( \rho(\phi_u, \phi_v) = \min\{|\phi_u - \phi_v|, 1 - |\phi_u - \phi_v|\} \). We define the mapping \( F : X \to X \) as identity on the limit cylinder,
\[ [2 \cos 2\pi \phi, 2 \sin 2\pi \phi, z] \mapsto [2 \cos 2\pi \phi, 2 \sin 2\pi \phi, z], \]
and as a composition of rotation and continuous mapping \( g \) on inner cylinders,
\[ ((2 - \frac{1}{k}) \cos 2\pi \phi, (2 - \frac{1}{k}) \sin 2\pi \phi, z) \mapsto [(2 - \frac{1}{k + 1}) \cos 2\pi (\phi + \Psi(k, z)), (2 - \frac{1}{k + 1}) \sin 2\pi (\phi + \Psi(k, z)), g_k(z)]. \]
To define \( g_k : I \rightarrow I \) and \( \Psi : \mathbb{N} \times I \rightarrow I \), let \( \{ r_i \}_{i=1}^\infty = \mathbb{Q} \cap (0,1) \) be a sequence of all rationals in \((0,1)\), and \( m_1 < m_2 < m_3 < \ldots \) an increasing sequence of integers which we specify later. Then

\[
g_k = \begin{cases} 
    h_l & \text{if } m_{3l+1} \leq k < m_{3l+2} \\
    \text{Id} & \text{if } m_{3l+2} \leq k < m_{3l+3} \\
    h_l^{-1} & \text{if } m_{3l+3} \leq k < m_{3l+4} 
\end{cases} \quad \text{for } k, l \in \mathbb{N}_0 \tag{5}
\]

where \( h_l : I \rightarrow I \) is a continuous strictly increasing mapping with three fixed points \(0, 1, r_l\) and \( \lim_{l \to \infty} ||h_l - \text{Id}|| = 0 \); \( h_l(x) < x \) for \( x \in (0, r_l) \); \( h_l(x) > x \) for \( x \in (r_l, 1) \).

The sequence \( \{ m_i \}_{i=1}^\infty \) is defined in the following way:

\[
m_{3l+2} - m_{3l+1} = m_{3l+4} - m_{3l+3} = n_l,
\]

where \( n_l \) is integer satisfying

\[
h_l^{n_l}([0, r_l - \frac{1}{l}]) \subset [0, \frac{1}{l}] \quad \land \quad h_l^{n_l}([r_l + \frac{1}{l}, 1]) \subset (1 - \frac{1}{l}, 1],
\]

and simultaneously \( \{ m_i \}_{i=1}^\infty \) can be chosen such that

\[
m_{3l+3} - m_{3l+2} > \frac{2l}{\varepsilon_l}, \quad \text{where } \varepsilon_l = \min\{h_l^{n_l}(\frac{1}{l}), 1 - h_l^{n_l}(1 - \frac{1}{l})\},
\]

\[
\lim_{l \to \infty} \frac{m_{3l+1}}{m_{3l+2}} = \lim_{l \to \infty} \frac{m_{3l+3}}{m_{3l+4}} = 1, \\
\lim_{l \to \infty} \frac{m_{3l+2}}{m_{3l+3}} = 0. \tag{8}
\]

The angle of rotation \( \Psi : \mathbb{N} \times I \rightarrow I \) is defined as

\[
\Psi(k, z) = \begin{cases} 
    z_l & \text{if } 1 \leq k < m_4 \\
    z/l & \text{if } m_{3l+1} \leq k < m_{3l+4} 
\end{cases} \quad l \in \mathbb{N}.
\]

The factor space \( Y \) is simply \( X \) with fixed \( \phi = 0 \), i.e. for each point \( y \in Y \),

\[
y = [2 - \frac{1}{k}, 0, z] \quad \text{or} \quad [2, 0, z], \quad k \in \mathbb{N}, z \in I.
\]

To simplify the notation, we skip the second zero coordinate and treat \( Y \) as a two-dimensional space. The space \( Y \) is union of converging sequence of unit fibers and the limit fiber,

\[
Y = \{ 2 - \frac{1}{k} : k \in \mathbb{N} \} \times I \cup \{ 2 \} \times I.
\]

Then the system \((X, F)\) is semiconjugated with skew-product map \( f : Y \rightarrow Y \), which is identity on the limit fiber,

\[
[2, z] \mapsto [2, z],
\]

and which is \( g_k \) on inner fibers,

\[
[2 - \frac{1}{k}, z] \mapsto [2 - \frac{1}{k+1}, g_k(z)], \quad k \in \mathbb{N}.
\]

I. The factor system \((Y, f)\) is DC1.

We show that set \( S = \{ 1 \} \times I \) is a distributionally scrambled set, i.e. for any pair of distinct points \((u, v) \in S^2\),

\[
\Phi^*_{(u,v)}(\epsilon) = 1 \quad \text{and} \quad \Phi^*_{(u,v)}(\epsilon) = 0, \quad \text{where } \epsilon < 1. \tag{9}
\]

Since \( \{ r_i \}_{i=1}^\infty \) is dense in \( I \) and by (9), we can find a sequence \( \{ s_k \}_{k=1}^\infty \) such that \( d(f^i(u), f^i(v)) < \frac{1}{k} \), for \( m_{3s_k+2} \leq i < m_{3s_k+3} \), and therefore, by (9), \( \Phi^*_{(u,v)} \equiv 1 \). Suppose \( u^2 > v^2 \), where \( x^2 \) denotes the second coordinate of a point \( x \). We can find another subsequence \( \{ q_k \}_{k=1}^\infty \) such that \( d(f^i(u), f^i([1, 1])) < \frac{1}{q_k} \) and simultaneously \( d(f^i(v), f^i([0, 1])) < \frac{1}{q_k} \), for \( m_{3q_k+2} \leq i < m_{3q_k+3} \). Since \( f \) preserves the distance between the endpoints of any fiber, \( d(f^i([1, 1]), f^i([0, 1])) = 1 \), for \( i \geq 0 \), we can conclude, by (9), \( \Phi^*_{(u,v)}(\epsilon) = 0 \), for any \( \epsilon < 1 \).
II. \((X, F)\) has no distributionally scrambled pair

We claim \(\Phi^*_{(u, v)} < 1\) for any pair of distinct points in \(X\). Let \(X_0\) be the limit cylinder \(X_0 = \{[2 \cos 2\pi \phi, 2 \sin 2\pi \phi] : \phi \in I\} \times I\) and \(\bar{X} = X \setminus X_0\). Consider 4 possible cases:

\(a)\) \((u, v) \in \bar{X}\) with \(z_u = z_v = z, k_u = k_v = k, \phi_u \neq \phi_v.\)

The angle of rotation is the same for both \(u\) and \(v\), \(\Psi(k_u, z_u) = \Psi(k_v, z_v) = \Psi(k, z)\), hence, by (11),

\[d(F(u), F(v)) = \rho(\phi_u + \Psi(k, z), \phi_v + \Psi(k, z)) = \rho(\phi_u, \phi_v) = d(u, v)\]

\(F\) is isometric in this case and \(\Phi^*_{(u, v)} \neq 1\).

\(b)\) \((u, v) \in \bar{X}\) with \(z_u \neq z_v, k_u = k_v = k, \phi_u \neq \phi_v.\)

Without loss of generality suppose \(k = 1\) (otherwise consider the pre images \((F^{-1}(u), F^{-1}(v))\) and let \(L\) be an integer such that \(|z_u - z_v| > \frac{1}{L}\). It is sufficient to show that there is \(0 < \delta < \frac{1}{3}\), for which

\[
\frac{1}{m_{3L+3} - m_{3L+2}} \# \{m_{3L+2} < i < m_{3L+3}; d(F^i(u), F^i(v)) < \delta\} < 3\delta.
\]

Since \(d\) is max-metric, it is sufficient to prove

\[
\frac{1}{m_{3L+3} - m_{3L+2}} \# \{m_{3L+2} < i < m_{3L+3}; \rho(\phi_F(u), \phi_F(v)) < \delta\} < 3\delta.
\]

Since \(|h^n_L(z_u) - h^n_L(z_v)| > \epsilon_L\) (see definition of \(\epsilon_L\) in (10)), and \(|h^n_L(z_u) - h^n_L(z_u)|\) is the minimal distance between trajectories of \(u\) and \(v\) between times \(m_{3L+1}\) and \(m_{3L+4}\), it follows

\[
\min_{\delta L + 1 < k \leq 3L + 4} |g_k \circ g_{k-1} \circ \ldots \circ g_{3L+1}(z_u) - g_k \circ g_{k-1} \circ \ldots \circ g_{3L+1}(z_v)| > \epsilon_L.
\]

Denote the relative angle of rotation of points with height \(z_u\) and \(z_v\) in the \(k\)-th cylinder by \(\Delta \Psi_k(z_u, z_v) = |\Psi(k, z_u) - \Psi(k, z_v)| = \frac{|z_u - z_v|}{L}\). By (11),

\[
\Delta \Psi_k(g_k \circ g_{k-1} \circ \ldots \circ g_{3L+1}(z_u), g_k \circ g_{k-1} \circ \ldots \circ g_{3L+1}(z_v)) > \frac{\epsilon_L}{L},\quad \text{for } m_{3L+1} \leq k < m_{3L+4}.
\]

Since \(m_{3L+3} - m_{3L+2} > 2L\), we can use Lemma 1 and conclude, for any \(\delta > 0\),

\[
\frac{1}{m_{3L+3} - m_{3L+2}} \# \{m_{3L+2} < i < m_{3L+3}; \rho(\phi_F^i(u), \phi_F^i(v)) < \delta\} < 3\delta.
\]

We obtain the result for any \(l > L\) using the same argument, since for every \(l > L\), \(|z_u - z_v| > \frac{1}{L}\).

\(c)\) \((u, v) \in \bar{X}\) with \(z_u \neq z_v, k_u \neq k_v, \phi_u \neq \phi_v.\)

Without loss of generality suppose \(k_u = 1\) and \(k_v = p\). If \(|z_u - z_v| > \frac{1}{L}\), then by case \(b)\)

\[# \{m_{4L+2} + p < i < m_{4L+3} - p; \rho(\phi_F^i(u), \phi_F^i(v)) < \delta\} < 3\delta \cdot (m_{4L+3} - m_{4L+2})\]

and hence

\[
\frac{1}{m_{4L+3} - m_{4L+2}} \# \{m_{4L+2} < i < m_{4L+3}; \rho(\phi_F^i(u), \phi_F^i(v)) < \delta\} < 3\delta + \frac{2p}{m_{4L+3} - m_{4L+2}} < 1,
\]

for sufficiently large \(L\).

\(d)\) \(u \in \bar{X}\) and \(v \in X_0\)

Since \(v \in X_0\) is fixed and \(\phi_v = \phi_{F(v)}\), we can find another point in \(\bar{X}, w = [(2 - \frac{1}{k_v}) \cos 2\pi \phi_v, (2 - \frac{1}{k_v}) \sin 2\pi \phi_v, 0]\), which is also fixed under rotation. Therefore

\[
\rho(\phi_F(u), \phi_F(v)) = \rho(\phi_F(u), \phi_F(w))
\]

and we can apply case \(b)\) or \(c)\) to investigate the pair \((u, w)\) instead of \((u, v)\). \(\square\)

Remark. Notice that the upper distributional function for the extension remains positive, \(\Phi^*_{(u, v)} > 0\), for any pair of distinct points in \(S \times \{1\}\). By (10), \(\Phi_{(u, v)} < \Phi^*_{(u, v)}\), hence the system \((X, F)\) is distributionally chaotic of type 3. This fact implies an open question: \textit{Is there a DC3 system which is semiconjugated to an extension without any distributionally scrambled pairs of type 3?}
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