Research Article

Existence and Uniqueness of Weak Solutions for a New Class of Convex Optimization Problems Related to Image Analysis

Anas Tiarimti Alaoui and Mostafa Jourhmane

TIAD Laboratory, Department of Mathematics, Faculty of Sciences and Technics, Sultan Moulay Slimane University, Beni Mellal, Morocco

Correspondence should be addressed to Anas Tiarimti Alaoui; a.tiarimti@usms.ma

Received 24 December 2020; Revised 20 March 2021; Accepted 25 March 2021; Published 21 April 2021

Academic Editor: Jen-Chih Yao

Copyright © 2021 Anas Tiarimti Alaoui and Mostafa Jourhmane. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

This paper proposes a new anisotropic diffusion model in image restoration that is understood from a variational optimization of an energy functional. Initially, a family of new diffusion functions based on cubic Hermite spline is provided for optimal image denoising. After that, the existence and uniqueness of weak solutions for the corresponding Euler–Lagrange equation are proven in an appropriate functional space, and a consistent and stable numerical model is also shown. We complement this work by illustrating some experiments on different actual brain Magnetic Resonance Imaging (MRI) scans, showing the proposed model’s impressive results.

1. Introduction

During the last few decades, the digital image has been significantly utilized as a noninvasive medical diagnosis or treatment tool. For instance, a different method has been developed in the electroencephalogram (EEG) (see [1]). Since a digital image contains relevant information that should be carefully extracted from it, many engineers and mathematicians have established numerous techniques and theories [2–4] to collect quantitative or qualitative data. At that point, by analyzing these reliable data and understanding its contents, one can make the right decisions at the diagnosis time.

Along with the meaningful details, digital images carry false information as noise and other undesirable artifacts due to various reasons. This erroneous information can occur during image formation, transmission, or recording processes. Therefore, one needs good models to identify and eliminate these noises while preserving relevant information and structures.

Image denoising is a necessary preprocessing step for many applications that rely on image quality, such as image segmentation and pattern recognition. Hence, how to restore a degraded image to its original form? Many approaches have been developed to process images and showed great interest in using the variational approach [5–15], which attempts to establish mathematical models strictly related to the physical world by defining a diffusion equation from a particular optimization problem, which generally has the following form:

$$\min_u \left\{ F(u) = \int_\Omega \left( \phi(|\nabla u|) + \frac{\rho}{2} (u - u_0)^2 \right) \, d\Omega \right\},$$

(1)

where $\Omega \subset \mathbb{R}^2$ is an open bounded domain, $u_0$ and $u$ are the observed and the reconstructed images defined as functions of $\Omega \subset \mathbb{R}^2 \rightarrow \mathbb{R}$ that associate each pixel $x \in \Omega$ to the gray level $u(x)$ or $u_0(x)$, $\phi: \mathbb{R}^+ \rightarrow \mathbb{R}$ is a nonnegative increasing function with $\phi(0) = 0$, and $\rho \in \mathbb{R}^+$ is a weighting parameter that enables to adjust the influence of the data term in the regularizing term.

One common approach to solve the problem (1) is by seeking the steady-state solution of the following PDE, which corresponds to the Euler–Lagrange equation of the energy functional $F(u)$:
\[
\frac{\partial u}{\partial t} = \nabla \cdot \left[ \frac{\varphi^\prime (|\nabla u|)}{|\nabla u|} \nabla u \right] - \rho (u - u_0). 
\] (2)

Perona and Malik (Perona–Malik) [16] were the first to propose such a model in image processing, which corresponds to nonconvex energy processing. Unfortunately, this property leads to multiple solutions with staircase effects observed in practice [6, 17]. In [8], Charbonnier et al. suggested a strictly convex energy functional to circumvent the Perona–Malik model’s ill-posedness. Besides, many variational approaches have also been developed during the last thirty years. The most famous one was the total variation denoising derived from Perona–Malik or TV models have been proposed in the last twenty years; for more detailed information, we refer to [2, 10–15, 18] and the references therein.

Nevertheless, because of the above model’s isotropy, the diffusion equation (2) is controlled by the function \( s \rightarrow (\varphi^\prime (s)/s) \) and produces the same amount of blurring in all its directions. This means that the process cannot successfully eliminate noises at the edges. Thus, it would be important to consider smoothing along edges by adopting anisotropic diffusion [19].

Furthermore, the type (1) model uses the gradient magnitude as a local descriptor operator for the image edge detector. However, digital images present some difficulties in their discrete structure; they are discrete in space and discrete in intensity value. Consequently, one may need to adapt to the digital image structure by considering differential operators that respond to vertical, horizontal, and diagonal edges and using a consistent approximation. Therefore, motivated by the above reasoning and inspired by Weickert’s anisotropic idea [19], we conceived new convex optimization problems that lead to anisotropic diffusion equations with a novel matrix diffusion tensor.

Therefore, this paper presents a new variational PDE model based on directional edge detectors in Section 2. The construction of a new diffusion function using cubic Hermite spline and the existence and uniqueness of weak solutions are illustrated in Section 3. Next, Section 4 implements a consistent explicit symmetric finite difference approximation that involves the \( 3 \times 3 \) neighborhoods at each point and shows the experiments on different real images. We terminate this work with a conclusion in Section 5.

2. Novel Anisotropic Diffusion Model for Image Restoration

Let us now consider the rectangular domain \( \Omega = (0,a) \times (0,b) \subset \mathbb{R}^2 \) with \( \Gamma = \partial \Omega \) being its boundary. This approach’s general idea is to search for a solution among the minima of some functional energy. The proposed functional energy has a regularizing term that depends on the combination of a function \( \varphi \) and local structures of the image expressed by the four directional derivatives:

\[
\begin{align*}
\partial_{x,1} u &= u_{x,1} = \nabla u \cdot e_1, \\
\partial_{x,2} u &= u_{x,2} = \nabla u \cdot e_2, \\
\partial_{x,12} u &= u_{x,12} = \nabla u \cdot \frac{e_1 + e_2}{|e_1 + e_2|}, \\
\partial_{x,-12} u &= u_{x,-12} = \nabla u \cdot \frac{-e_1 + e_2}{|e_1 + e_2|},
\end{align*}
\] (3)

where \((e_1, e_2)\) is the canonical basis of \( \mathbb{R}^2 \). Hence, our problem consists of solving the following energy minimization problem:

\[
\min_{u \in U} E(u),
\] (4)

such that

\[
E(u) = \int_{\Omega} \left( \frac{\varphi(\|u_{x,1}\|) + \varphi(\|u_{x,2}\|) + \varphi(\|u_{x,12}\|) + \varphi(\|u_{x,-12}\|)}{4} + \frac{\rho}{2} (u - u_0)^2 \right) dx
\]

\[
= \int_{\Omega} L(x, u, u_{x,1}, u_{x,2}) dx,
\] (5)

under the following assumptions:

\[
\begin{aligned}
&\varphi : \mathbb{R}_+ \rightarrow \mathbb{R} \text{ is a } C^2 - \text{function/} \varphi(0) = \varphi^\prime(0) = 0, \\
&\varphi(s) > 0, \varphi^\prime(s) > 0, \quad s > 0, \\
&\varphi^\prime\prime(s) > 0, \quad s \geq 0, \\
&\lim_{s \to +\infty} \frac{\varphi(s)}{s} = +\infty, \quad \rho \in \mathbb{R}_+. 
\end{aligned}
\] (6)

The Euler–Lagrange equation of \( E(u) \) (5) is obtained from

\[
\frac{\partial L}{\partial u} + \partial_{x,1} \left( \frac{\partial L}{\partial u_{x,1}} \right) - \partial_{x,2} \left( \frac{\partial L}{\partial u_{x,2}} \right) = 0,
\] (7)

which leads to
\[
\rho ( u - u_0 ) - \left[ \partial_{x_1} \left( \frac{\varphi' \left( |u_{x_1}| \right)}{|u_{x_1}|} u_{x_1} + \frac{\varphi' \left( |u_{x_1}| \right)}{\sqrt{2} |u_{x_{12}}|} u_{x_{12}} - \frac{\varphi' \left( |u_{x_{12}}| \right)}{\sqrt{2} |u_{x_{12}}|} u_{x_{12}} \right) \right. \\
\left. + \partial_{x_2} \left( \frac{\varphi' \left( |u_{x_2}| \right)}{|u_{x_2}|} u_{x_2} + \frac{\varphi' \left( |u_{x_2}| \right)}{\sqrt{2} |u_{x_{12}}|} u_{x_{12}} + \frac{\varphi' \left( |u_{x_{12}}| \right)}{\sqrt{2} |u_{x_{12}}|} u_{x_{12}} \right) \right] = 0.
\]

This is equivalent to

\[
\rho ( u - u_0 ) - \left[ \partial_{x_1} \left( \frac{\varphi' \left( |u_{x_1}| \right)}{|u_{x_1}|} u_{x_1} \right) + \partial_{x_2} \left( \frac{\varphi' \left( |u_{x_2}| \right)}{|u_{x_2}|} u_{x_2} \right) \\
+ \partial_{x_{12}} \left( \frac{\varphi' \left( |u_{x_{12}}| \right)}{|u_{x_{12}}|} u_{x_{12}} \right) \right] = 0.
\]

Let \( g : \mathbb{R} \rightarrow \mathbb{R} \) be a positive function such that \( g(s) = (\varphi'(s)/s) \). We may reformulate equation (9) and present it as follows.

**Proposition 1.** The Euler–Lagrange equation (9) is equivalent to

\[
\rho ( u - u_0 ) - \nabla \cdot \left[ \mathbf{D}_{\nabla u} \nabla u \right] = 0,
\]

where \( \mathbf{D}_{\nabla u} \) is a real symmetric positive definite matrix of \( \mathbb{R}^{2 \times 2} \) defined as follows:

\[
\mathbf{D}_{\nabla u} = \begin{pmatrix}
\frac{g(|u_{x_1}|) + g(|u_{x_{12}}|) + g(|u_{x_{21}}|)}{2} & \frac{g(|u_{x_{11}}|) - g(|u_{x_{12}}|)}{2} \\
\frac{g(|u_{x_{11}}|) - g(|u_{x_{12}}|)}{2} & \frac{g(|u_{x_{21}}|) + g(|u_{x_{22}}|) + g(|u_{x_{21}}|)}{2}
\end{pmatrix}.
\]

**Proof.** First, it is clear that

\[
\partial_{x_1} \left( g \left( \frac{|u_{x_1}|}{u_{x_1}} \right) u_{x_1} \right) + \partial_{x_1} \left( g \left( \frac{|u_{x_2}|}{u_{x_2}} \right) u_{x_2} \right) = \nabla \cdot \begin{pmatrix}
g \left( \frac{|u_{x_1}|}{u_{x_1}} \right) & 0 \\
0 & g \left( \frac{|u_{x_2}|}{u_{x_2}} \right)
\end{pmatrix} u_{x_1} u_{x_2}.
\]

On the other hand, we have
Euler–Lagrange equation (10) can be solved considering the invariable definite, which completes the proof.

Besides, the matrix $D_{\psi u}$ has two eigenvalues $\lambda_{\pm}$ as follows:

\[
\lambda_{\pm} = \frac{1}{2} \left( g(\mid u_{x_1} \mid) + g(\mid u_{x_3} \mid) + g(\mid u_{x_1} \mid) + g(\mid u_{x_3} \mid) \pm \sqrt{\left( g(\mid u_{x_1} \mid) - g(\mid u_{x_3} \mid) \right)^2 + \left( g(\mid u_{x_1} \mid) - g(\mid u_{x_3} \mid) \right)^2} \right). 
\]

Since $g > 0$ and

\[
\lambda_{+} = \frac{2 \left( g(\mid u_{x_1} \mid) g(\mid u_{x_1} \mid) + g(\mid u_{x_1} \mid) g(\mid u_{x_3} \mid) \right)}{2 \lambda_{+}}, 
\]

then we deduce that the matrix $D_{\psi u}$ is symmetric positive definite, which completes the proof. □

By applying the steepest descent method, the Euler–Lagrange equation (10) can be solved considering the steady state of the following evolution equation:

\[
\begin{align*}
\frac{\partial u}{\partial t} &= \nabla \cdot [D_{\psi u} \nabla u] - \rho (u - u_0), & \text{in } \Omega \times (0, T], \\
\langle D_{\psi u} \nabla, n \rangle &= 0, & \text{on } \partial \Omega \times (0, T], \\
u(x; 0) &= u_0(x), & \text{in } \Omega.
\end{align*}
\]
In [20], the existence and uniqueness of weak solutions for problem (16) were proven if the weighting parameter $\rho = 0$.

3. Mathematical Investigation of the Proposed Model

3.1. Description of the Proposed Model. Our model’s main objective is to allow strong directional smoothing within the

\[ g: \mathbb{R}_+ \rightarrow \mathbb{R}^1 \] - function such that \( g(s) > 0, \quad g'(s) < 0, \)
\[ \lim_{s \to \infty} g(s) = 0. \]

Besides, due to the decreasing positivity of the function \( g \), it is evident that our model’s behavior (16) encourages smoothing along edges in the \( \mathbf{e}_1, \mathbf{e}_2, (\mathbf{e}_1 + \mathbf{e}_2) \), or \( (-\mathbf{e}_1 + \mathbf{e}_2) \) directions and ceases across them.

Moreover, we may provide another investigation analysis of our model by examining the eigenvalues-eigenvectors

\[ \eta_k = \left( \frac{g\left| u_{x_i} \right|}{g\left( \left| u_{x_i} \right| \right)} + \sqrt{\left( g\left( \left| u_{x_i} \right| \right) - g\left( \left| u_{x_{11}} \right| \right) \right)^2} \right), \]

provided that \( \left| u_{x_{11}} \right| \neq \left| u_{x_{i\cdot}} \right| \). We can then expand the first equation of (16) into

\[ \frac{\partial u}{\partial t} = \nabla \cdot \left[ \lambda_+ \theta_+ \theta_+^T \nabla u \right] + \nabla \cdot \left[ \lambda_- \theta_- \theta_-^T \nabla u \right] - \rho (u - u_0). \]

Accordingly, the diffusion caused by (20) is measured by the \( \lambda_\pm \) values and oriented toward \( \theta_\pm \). Specifically, it is clear from the expression of \( \lambda_\pm \) that \( \lambda_\pm \geq \lambda_\pm > 0 \), which means that the diffusion toward \( \theta_+ \) is privileged over \( \theta_- \). Furthermore, we can deduce the following results:

(i) In flat areas, we have
\[ \left\{ \begin{array}{l} \left| u_{x_i} \right| = \left| u_{x_{11}} \right|, \\ \left| u_{x_{i\cdot}} \right| = \left| u_{x_{1\cdot}} \right| \end{array} \right\} \Longleftrightarrow \left\{ \begin{array}{l} \left| u_{x_i} \right| = \left| u_{x_{11}} \right| = \left| u_{x_{i\cdot}} \right| = \left| u_{x_{1\cdot}} \right| = 0 \end{array} \right\}, \]

\[ \lambda_\pm = \left( \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} \right)^T, \]

\[ \lambda_\pm = 2 g(0). \]

(ii) At straight edges: \( \lambda_\pm > \lambda_\pm > 0 \).

For instance, we can assume that \( u_{x_i} = 0 \) and \( \left| u_{x_{1\cdot}} \right| \gg 0 \). Then, we obtain \( \left| u_{x_{1\cdot}} \right| = 0 \) and \( \left| u_{x_{11}} \right| \gg 0 \), which implies that

\[ \lambda_\pm = \lambda_\pm = 2 g(0) > 0. \]

The diffusion is anisotropic and oriented along the \( (-\mathbf{e}_1 + \mathbf{e}_2) \) direction.

Consequently, the diffusion process is anisotropic and oriented along the \( (-\mathbf{e}_1 + \mathbf{e}_2) \) direction.

(iii) Corners: \( \lambda_\pm > \lambda_\pm > 0 \).

According to the characteristics of the function \( g \), diffusion is anisotropic and oriented along \( \theta_+ \) and \( \theta_- \) directions.

In fact, the difference \( \lambda_\pm - \lambda_\pm = (g(|u_{x_{11}}|) - g(|u_{x_{1\cdot}}|))^2 + (g(|u_{x_{1\cdot}}|) - g(|u_{x_{11}}|))^2 \) gives insights into our diffusion model’s anisotropic property. In other words, it indicates the isotropic diffusion for the zero value and the anisotropic diffusion for larger values.

3.2. New Adaptive Diffusion Function Using Hermite Spline.

For effective image denoising and better control of the diffusion via the process (16), we will approximate numerically the unknown diffusion function \( g \) that meets the requirements \( (H_1 + H_2) \) by using Hermite spline [21]. Therefore, we can use cubic Hermite spline to interpolate...
numeric data specified at 0, k, and \( h \geq 1 \) \((0 < k < h)\) to obtain a function that meets the requirements \((H_1 + H_2)\). To this end, we propose a function \( g \) as follows:

\[
g(s) = \begin{cases} 
  p_0 P_{1,0k}(s) + v_0 P_{2,0k}(s) + \sum_{i=0}^{k} p_k P_{1,ik}(s) + v_k P_{2,ik}(s), & s \in [0,k], \\
  p_k P_{1,hk}(s) + v_k P_{2,hk}(s) + \sum_{i=0}^{h} p_{h,i} P_{1,hk}(s) + v_{h,i} P_{2,hk}(s), & s \in [k,h], \\
  p_{h,i} g_{h,i}(s) + v_{h,i} g_{h,i}(s), & s \in [h,\infty], 
\end{cases}
\]  

(23)

where \( p \) and \( v \) are the coefficients used to define the position and the velocity vector at a specific point, \( k \) and \( h \) are two threshold parameters, and \( \{P_{i,j}\} \) is a family of the basis functions composed of polynomials of degree 3 used on the interval \([c,d]\) such that

\[
Pg_{1,cd}(s) = \frac{(s - d)^2 (2s + d - 3c)}{(d - c)^2}, \\
Pg_{2,cd}(s) = \frac{(s - d)^2 (s - c)}{(d - c)^2}.
\]

(24)

And, we may consider

\[
\begin{align*}
  g_{h,1}(s) &= \frac{h}{\log(h) + 2} \frac{2s \log(s) + 1)}{s^2} - h \log(h), \\
  g_{h,2}(s) &= \frac{h^2}{\log(h) + 2} \frac{s \log(s) + 1)}{s^2} - h \log(h) + 1.
\end{align*}
\]

(25)

Hence, we can reformulate the expression of \( g \) as follows:

\[
g(s) = \sum_{i=0}^{3} A_{0k,i} s^i, s \in [0,k], \sum_{i=0}^{3} A_{kh,i} s^i, s \in [k,h], A_{h,i} \frac{\log(s) + 1}{s}, s \in [h,\infty], \\
+ A_{h,1} \frac{1}{s}, s \in [h,\infty],
\]

(26)

By considering the continuity of \( \phi \) at \( k \) and \( h \), and using \( \phi(0) = 0 \), it follows then

\[
\phi(s) = \sum_{i=0}^{3} A_{0k,i} s^i + \sum_{i=0}^{3} A_{kh,i} s^i + \left( A_{0k,i} - A_{kh,i} \right) s^i i^2, \\
  s \in [k,h], A_{h,i} \log(s) + A_{h,i} \log(s) + A_{h,i} \log(s), s \in [h,\infty],
\]

(27)

where

\[
A_{h,0} = \sum_{i=0}^{3} \left( \frac{A_{0k,i} - A_{kh,i}}{i + 2} \right) s + A_{h,i} \log(h) - A_{h,i} \log(h).
\]

(28)

The values of the coefficients \( A_{0k,i} \) and \( A_{kh,i} \) are determined experimentally under the conditions of \((H_1 + H_2)\) on the intervals \([0,k]\) and \([k,h]\). Besides, we may introduce some sufficient conditions on the coefficients \( A_{h,i} \) to ensure that the functions \( \phi \) and \( g \) satisfy \((H_1 + H_2)\) on \([h,\infty]::

\[
\begin{align*}
  A_{h,2} &> 0, \\
  A_{h,1} &< h A_{h,2}, \\
  A_{h,1} &\geq - \frac{h \log(h)}{2} A_{h,2}.
\end{align*}
\]

(29)

Now, we establish a useful growth condition for the function \( \phi \).

**Proposition 2.** For all \( s \in \mathbb{R}_+ \), we can prove that

\[
\phi(s) \geq As + B,
\]

(30)

where \( A > 0 \) and \( B \in \mathbb{R} \).

**Proof.** We know that \( g'(s) \leq 0 \). Then,

\[
s \in [0,k]: \phi'(s) = sg(s) + g(k)s = g(k)(s^2/2) + C \geq g(k)s - (g(k)/2) + C \text{ with } C \in \mathbb{R}.
\]
3.3. Existence and Uniqueness. In this section, we investigate the existence and uniqueness of weak solutions of the Euler–Lagrange equation associated with the energy functional $E(u)$ defined in (5). For $\rho > 0$ and $u_0 \in L^2(\Omega)$, we consider

$$
\begin{align*}
\rho (u - u_0) - \text{div} (D_v u \nabla u) &= 0, \quad \text{in } \Omega, \\
\langle D_v u \nabla u, \mathbf{n} \rangle &= 0, \quad \text{on } \partial \Omega.
\end{align*}
$$

(31)

First, we introduce a new functional space $L \log L^h(\Omega)$:

$$
L \log L^h(\Omega) = \left\{ u : \Omega \rightarrow \mathbb{R} | \int_{\Omega \cap |x| \leq h} |u| \log(|u|) \, dx < \infty \right\}.
$$

(32)

Next, we define a weak solution for problem (31).

**Definition 1** A function $u \in L^2(\Omega) \cap W^{1,1}(\Omega)$ with $\partial_x u \in L \log L^h(\Omega)$ for $i = 1, 2$ is called a weak solution for problem (31) if, for any $v \in C^1(\bar{\Omega})$ we have

$$
\int_{\Omega} \rho (u - u_0) \, dx + \int_{\Omega} (D_v u \nabla u - \nabla u) \cdot \nabla v \, dx = 0.
$$

(33)

And when $v$ is a constant function, we obtain

$$
U = \left\{ u \in L^2(\Omega) \cap W^{1,1}(\Omega) | \partial_x u \in L \log L^h(\Omega) \right\}
$$

and the functional $E$ as defined in (5). It is obvious that $u_{0,\Omega} \in U$ knowing that $u_{0,\Omega} = (1/|\Omega|) \int_{\Omega} u_0 \, dx$.

Since

$$
0 \leq \inf_{u \in U} E(u) \leq E(u_{0,\Omega}) = \frac{\rho}{2} \int_{\Omega} (u_{0,\Omega} - u_0)^2 \, dx,
$$

(40)

then we can construct a minimizing sequence $\{u_m\}_{m=1}^{\infty}$ in $U$ such that $E(u_m) \leq E(u_{0,\Omega}) + 1$ and

$$
\lim_{m \to \infty} E(u_m) = \inf_{u \in U} E(u).
$$

(41)

Besides,

$$
\int_{\Omega} u_m^2 \, dx = \int_{\Omega} (u_m - u_0 + u_0 - u_{0,\Omega} + u_{0,\Omega})^2 \, dx
\leq 4 \int_{\Omega} (u_m - u_0)^2 \, dx + 4 \int_{\Omega} (u_0 - u_{0,\Omega})^2 \, dx + 2 \int_{\Omega} u_{0,\Omega}^2 \, dx
\leq \frac{8}{\rho} \left( E(u_0) + E(u_{0,\Omega}) \right) + 2u_{0,\Omega}^2 |\Omega|
\leq \frac{8}{\rho} \left( 2E(u_{0,\Omega}) + 1 \right) + 2u_{0,\Omega}^2 |\Omega|.
$$

(42)

Before stating our main theorem, we need first to introduce a useful lemma.

**Lemma 1** (uniform integrability and weak convergence [22]). Assume that $\Omega$ is bounded and let $\{u_m\}_{m=1}^{\infty}$ be a sequence of functions in $L^1(\Omega)$ satisfying

$$
\sup_m \|u_m\|_{L^1(\Omega)} < \infty.
$$

(35)

Suppose also

$$
\lim_{l \to \infty} \sup_m \int_{\Omega \cap |x| \geq l} |u_m| \, dx = 0.
$$

(36)

Then, there exist a subsequence $\{u_{m_j}\}_{j=1}^{\infty}$ and $\bar{u} \in L^1(\Omega)$ such that

$$
u_{m_j} \rightharpoonup \bar{u} \text{ weakly in } L^1(\Omega).
$$

(37)

Now, we state our main theorem.

**Theorem 1.** There exists a unique weak solution for problem (31).

Proof. We consider the variational problem

$$
\min_{u \in U} \{ E(u) | u \in U \},
$$

(38)

where

$$
\int_{\Omega} u \, dx = \int_{\Omega} u_0 \, dx.
$$

(39)

It follows then

$$
\sup_m \|u_m\|_{L^2(\Omega)} < \infty.
$$

(43)

On the other hand, from Proposition 2, we may find $\alpha > 0$ and $\beta \in \mathbb{R}$ such that

$$
E(u_{0,\Omega}) + 1 > E(u_m) \geq \int_{\Omega} \left[ \varphi \left( |\partial_x u_m| \right) + \varphi \left( |\partial_x u_m| \right) \right] \, dx
\geq a \|\nabla u_m\|_{L^1(\Omega)} + \beta.
$$

(44)

Then, we get

$$
\sup_m \|\nabla u_m\|_{L^1(\Omega)} < \infty.
$$

(45)

Moreover, we have

$$
\lim_{x \to \infty} \frac{\varphi(s)}{s \log(s)} = A_{h,2} > 0.
$$

(46)

Then, for $i = 1, 2$ there exist $l \geq h \geq 1$ and positive constants $C$ and $C_1$ (independent of $l$) such that
\begin{equation}
\int_{\Omega} \int_{[\partial_{\nu} u_m \geq 2]} \phi\left(\left|\partial_{\nu} u_m\right|\right) dx \\
\leq C \int_{\Omega} \int_{[\partial_{\nu} u_m \geq h]} \left|\partial_{\nu} u_m\right| \log \left(\left|\partial_{\nu} u_m\right|\right) dx \leq C_1.
\end{equation}

On the other hand, we know that
\begin{equation}
\lim_{j \to +\infty} \frac{\varphi(s)}{s} = +\infty.
\end{equation}

Then, given \(\varepsilon > 0\), let \(\eta_\varepsilon = C_1/\varepsilon\) and choose \(l \geq h \geq 1\) such that for all \(s \geq l\), we have \(\varphi(s) > \eta_\varepsilon s\). Hence, for \(i = 1, 2\), we obtain
\begin{equation}
\int_{\Omega} \int_{[\partial_{\nu} u_m \geq 2]} \left|\partial_{\nu} u_m\right| dx \leq \frac{1}{\eta_\varepsilon} \int_{\Omega} \int_{[\partial_{\nu} u_m \geq l]} \varphi\left(\left|\partial_{\nu} u_m\right|\right) dx \\
\leq \frac{C_1}{\eta_\varepsilon} = \varepsilon,
\end{equation}
and this is true for all \(m\) and arbitrary \(\varepsilon > 0\). It follows then that, for \(i = 1, 2\), we have
\begin{equation}
\lim_{j \to +\infty} \sup_m \int_{\Omega} \int_{[\partial_{\nu} u_m \geq 2]} \left|\partial_{\nu} u_m\right| dx = 0.
\end{equation}

From (43), (45)–(50) and by Lemma 1 and the weak compactness of \(L^2(\Omega)\), we can find a subsequence \(\left\{u_{m_j}\right\}_{j=1}^{\infty}\) of \(\left\{u_m\right\}_{m=1}^{\infty}\) and a function \(\bar{u} \in L^2(\Omega) \cap W^{1,1}(\Omega)\) such that
\begin{equation}
u u_{m_j} \rightharpoonup \nu \bar{u} \ \text{weakly in} \ L^2(\Omega),
\end{equation}
\begin{equation}\nabla u_{m_j} \rightharpoonup \nabla \bar{u} \ \text{weakly in} \ L^1(\Omega).
\end{equation}

Therefore, we have
\begin{equation}
\int_{\Omega} \bar{u} dx = \lim_{j \to +\infty} \int_{\Omega} u_{m_j} dx = \int_{\Omega} u_0 dx.
\end{equation}

Additionally, knowing that the function \(f(s) = s \log(s)\) for \(s \geq 1\) is increasing and convex, the function \(f(|\nu|)\) is also convex for all \(s \geq 1\). Therefore, we obtain for \(i = 1, 2\)
\begin{equation}
f\left(\left|\partial_{\nu} \bar{u}\right|\right) \leq f\left(\left|\partial_{\nu} u_{m_j}\right|\right) + f'\left(\left|\partial_{\nu} \bar{u}\right|\right) \left(\partial_{\nu} \bar{u} - \partial_{\nu} u_{m_j}\right).
\end{equation}

Integrating the above inequality over \(\Omega_M \cap \{h \leq |\partial_{\nu} u_{m_j}|\}\) with \(\Omega_M = \Omega \cap \{h \leq |\partial_{\nu} \bar{u}| \leq M\}\) with \(M > 0\), we have
\begin{equation}
\int_{\Omega_M} f\left(\left|\partial_{\nu} \bar{u}\right|\right) dx \leq \int_{\Omega_M} \int_{[\partial_{\nu} \bar{u} \geq h]} f\left(\left|\partial_{\nu} u_{m_j}\right|\right) dx \\
+ \int_{\Omega \cap \{h \leq |\partial_{\nu} \bar{u}| \leq M\}} f'\left(\left|\partial_{\nu} \bar{u}\right|\right) \left|\partial_{\nu} \bar{u} - \partial_{\nu} u_{m_j}\right| dx \\
\leq \int_{\Omega_M} \int_{[\partial_{\nu} \bar{u} \geq h]} f\left(\left|\partial_{\nu} u_{m_j}\right|\right) dx \\
+ \int_{\Omega \cap \{h \leq |\partial_{\nu} \bar{u}| \leq M\}} f'\left(\left|\partial_{\nu} \bar{u}\right|\right) \chi_{\{h \leq |\partial_{\nu} \bar{u}| \leq M\}} \left|\partial_{\nu} \bar{u} - \partial_{\nu} u_{m_j}\right| dx.
\end{equation}

Since \(f'\left(|\partial_{\nu} \bar{u}|\right) \chi_{\{k \leq |\partial_{\nu} \bar{u}| \leq M\}} \in L^\infty(\Omega)\) and by letting \(j \to +\infty\), we get
\begin{equation}
\int_{\Omega} \int_{\Omega_M} f\left(\left|\partial_{\nu} \bar{u}\right|\right) dx \leq \liminf_{j \to +\infty} \int_{\Omega} \int_{\Omega_M} f\left(\left|\partial_{\nu} u_{m_j}\right|\right) dx < \infty.
\end{equation}

Then, by letting \(M \to +\infty\), we deduce
\begin{equation}
\int_{\Omega} \int_{\Omega_M} f\left(\left|\partial_{\nu} \bar{u}\right|\right) dx < \infty.
\end{equation}

It follows then \(\partial_{\nu} \bar{u} \in L\log L^b(\Omega)\). Besides, we havet
\begin{equation}
\int_{\Omega} (\bar{u} - u_0)^2 dx \leq \liminf_{j \to +\infty} \int_{\Omega} (u_{m_j} - u_0)^2 dx.
\end{equation}

And by following the same reasoning as set out above, we know that the function \(\varphi(s)\) for \(s \geq 0\) is increasing and convex. Then, we can easily deduce that
\begin{equation}
\int_{\Omega} \varphi\left(\left|\partial_{\nu} \bar{u}\right|\right) dx \leq \liminf_{j \to +\infty} \int_{\Omega} \varphi\left(\left|\partial_{\nu} u_{m_j}\right|\right) dx.
\end{equation}

Therefore, by combining (57) with (58), we conclude
\begin{equation}
E(\bar{u}) \leq \liminf_{j \to +\infty} E(u_{m_j}) = \inf_{u \in U} E(u),
\end{equation}
which signifies that \(\bar{u} \in U\) is a minimizer of the energy functional \(E(u)\), i.e.,
\begin{equation}
E(\bar{u}) = \min_{u \in U} E(u).
\end{equation}

Furthermore, for all \(v \in C^1(\bar{\Omega})\) and for all \(t \in \mathbb{R}\), we have \(\bar{u} + t(v - \nu_\Omega) \in U\) with \(\nu_\Omega = (1/|\Omega|) \int_\Omega v dx\). Then, \(r(0) \leq r(t)\), where
\begin{equation}
r(t) = E(\bar{u} + t(v - \nu_\Omega)).
\end{equation}

Hence, we have \(r'(0) = 0\), which means
\begin{equation}
\int_{\Omega} \rho(\bar{u} - u_0)(v - \nu_\Omega) dx + \int_{\Omega} \mathbf{D}_{\nu} \nabla \bar{u} \cdot \nabla v dx = 0.
\end{equation}

Because of (52), we have
\begin{equation}
\int_{\Omega} \rho(\bar{u} - u_0) v dx + \int_{\Omega} \mathbf{D}_{\nu} \nabla \bar{u} \cdot \nabla v dx = 0.
\end{equation}

We conclude then that \(\bar{u}\) is a weak solution for problem (31).

Now, assuming that there is another minimizer \(\bar{u}\) of \(E\) and using the fact that \(E\) is strictly convex (5) and (6), we have
\begin{equation}
\min_{u \in U} E(u) = \frac{1}{2} E(\bar{u}) + \frac{1}{2} E(\bar{u}) > E\left(\frac{\bar{u} + \bar{u}}{2}\right) \geq \min_{u \in U} E(u),
\end{equation}
a contradiction. Thus, there is only one minimizer, which completes the proof.
4. Numerical Implementation and Experimental Results

4.1. Consistency and Stability of Finite Difference Approximation. For a consistent and stable discretization scheme, one can use the following accurate finite difference scheme:

\[
\begin{align*}
\frac{u_t(x_i, y_j; t_n)}{\delta_t} &= \frac{u(x_{i+1/2}, y_j; t_n) - u(x_{i-1/2}, y_j; t_n)}{\delta_t} + O(\delta_t^2), \\
\frac{u_x(x_i, y_j; t_n)}{\delta_x} &= \frac{u(x_{i+1/2}, y_j; t_n) - u(x_{i-1/2}, y_j; t_n)}{\delta_x} + O(\delta_x^2), \\
\frac{u_y(x_i, y_j; t_n)}{\delta_y} &= \frac{u(x_i, y_{j+1/2}; t_n) - u(x_i, y_{j-1/2}; t_n)}{\delta_y} + O(\delta_y^2), \\
\frac{u_{xx}(x_i, y_j; t_n)}{\delta_{xx}} &= \frac{u(x_{i+1/2}, y_{j+1/2}; t_n) - u(x_{i-1/2}, y_{j-1/2}; t_n)}{\delta_{xx}} + O(\delta_{xx}^2), \\
\frac{u_{yy}(x_i, y_j; t_n)}{\delta_{yy}} &= \frac{u(x_{i+1/2}, y_{j+1/2}; t_n) - u(x_{i-1/2}, y_{j-1/2}; t_n)}{\delta_{yy}} + O(\delta_{yy}^2), \\
\end{align*}
\]

(65)

By assuming \( \delta = 1 \) and denoting

\[
\begin{align*}
g^n_{N,i} &= g\left(\frac{1}{2}\Delta_N u^n_{i,j}\right), \\
g^n_{E,i} &= g\left(\Delta_E u^n_{i,j}\right), \\
g^n_{S,i} &= g\left(\Delta_S u^n_{i,j}\right), \\
g^n_{W,i} &= g\left(\Delta_W u^n_{i,j}\right), \\
g^n_{E,N,i} &= g\left(\frac{\Delta_{EN} u^n_{i,j}}{\sqrt{2}}\right), \\
g^n_{W,E,i} &= g\left(\frac{\Delta_{WE} u^n_{i,j}}{\sqrt{2}}\right), \\
g^n_{S,E,i} &= g\left(\frac{\Delta_{SE} u^n_{i,j}}{\sqrt{2}}\right), \\
g^n_{W,S,i} &= g\left(\frac{\Delta_{WS} u^n_{i,j}}{\sqrt{2}}\right), \\
g^n_{N,W,i} &= g\left(\frac{\Delta_{NW} u^n_{i,j}}{\sqrt{2}}\right), \\
\end{align*}
\]

(66)

then we may approximate the solution for problem (16) by the above scheme to obtain the following discrete diffusion filter:
\[ u_{i,j}^{n+1} = u_{i,j}^n + \delta_i \left[ g_N \Delta_N u + g_E \Delta_E u + g_s \Delta_s u + g_W \Delta_W u + \frac{g_{NE} \Delta_{NE} u + g_{SE} \Delta_{SE} u + g_{SW} \Delta_{SW} u + g_{NW} \Delta_{NW} u}{2} \right]_{i,j} \]

where \( u_{i,j}^0 \) is the degraded image, \( 1 \leq i \leq N, 1 \leq j \leq M, \) and \( n \geq 0 \). Furthermore, we use the discrete Neumann boundary condition:

\[
\begin{aligned}
    u_{0,j}^n &= u_{1,j}^n, & u_{N+1,j}^n &= u_{N,j}^n, & \text{for } 1 \leq j \leq M, \\
    u_{i,0}^n &= u_{i,1}^n, & u_{i,M+1}^n &= u_{i,M}^n, & \text{for } 1 \leq i \leq N, \\
    u_{0,0}^n &= u_{1,1}^n, & u_{N+1,0}^n &= u_{N,1}^n, \\
    u_{0,M+1}^n &= u_{1,M}^n, & u_{N+1,M+1}^n &= u_{N,M}^n.
\end{aligned}
\]

Then, using the filter (67) on every initial image \( u^0 \) yields a unique sequence \( (u^n)_{n \in \mathbb{N}} \). Besides, due to the continuity of the function \( g \), for every finite \( n, u^n \) depends continuously on \( u^0 \). Thus, under a specific condition, equation (67) satisfies the following maximum-minimum principle, which describes a stability property for the discrete scheme.

**Theorem 2** (discrete extremum principle). For an iteration step satisfying

\[ 0 < \delta_i < \frac{1}{\rho + 6\max_{s \in \mathbb{R}} g(s)} \]

scheme (67) satisfies

\[ \min_{i,j} u_{i,j}^n \leq u_{i,j}^n \leq \max_{i,j} u_{i,j}^n, \]

for all \( 1 \leq i \leq N, 1 \leq j \leq M, \) and \( n \in \mathbb{N} \).

**Proof.** For all \( 1 \leq i \leq N, 1 \leq j \leq M, \) and \( n \in \mathbb{N} \), we have

\[ u_{i,j}^{n+1} = \delta_i \left( g_N u_{i,j+1}^n + g_E u_{i+1,j}^n + g_s u_{i,j-1}^n + g_W u_{i-1,j}^n \right) + \frac{\delta_i}{2} \left( g_{NE} u_{i+1,j+1}^n + g_{SE} u_{i+1,j-1}^n + g_{SW} u_{i-1,j+1}^n + g_{NW} u_{i-1,j-1}^n \right) + \frac{g_{NE} u_{i+1,j}^n + g_{SE} u_{i,j-1}^n + g_{SW} u_{i,j+1}^n + g_{NW} u_{i,j-1}^n}{2} + \rho \left( u_{i,j}^n + \rho \delta_i u_{i,j}^0 \right) \]

Since \( 0 < \delta_i < (1/(\rho + 6\max_{s \in \mathbb{R}} g(s))), \rho \in \mathbb{R}_+, \) and \( g > 0 \), the following inequality holds:

\[ 1 - \delta_i \left( g_N u_{i,j}^n + g_E u_{i,j}^n + g_s u_{i,j}^n + g_W u_{i,j}^n + \frac{g_{NE} u_{i,j}^n + g_{SE} u_{i,j}^n + g_{SW} u_{i,j}^n + g_{NW} u_{i,j}^n}{2} + \rho \right) > 0. \]

It follows then

\[ u_{i,j}^{n+1} < \delta_i \left( g_{NE} u_{i,j}^n + g_{SE} u_{i,j}^n + g_{SW} u_{i,j}^n \right) \max_{i,j} u_{i,j}^n \]

\[ + \frac{\delta_i}{2} \left( g_{NE} u_{i,j}^n + g_{SE} u_{i,j}^n + g_{SW} u_{i,j}^n \right) \max_{i,j} u_{i,j}^n \]

\[ + \left[ 1 - \delta_i \left( g_N u_{i,j}^n + g_E u_{i,j}^n + g_s u_{i,j}^n + g_W u_{i,j}^n + \frac{g_{NE} u_{i,j}^n + g_{SE} u_{i,j}^n + g_{SW} u_{i,j}^n + g_{NW} u_{i,j}^n}{2} + \rho \right) \right] \max_{i,j} u_{i,j}^n + \rho \delta_i \max_{i,j} u_{i,j}^0 \]

\[ = (1 - \rho \delta_i) \max_{i,j} u_{i,j}^n + \rho \delta_i \max_{i,j} u_{i,j}^0. \]
Similarly, we have

\[ u_{n+1}^{ij} > \delta_1 \left( g_{N_{n+1}}^{ij} + g_{E_{n+1}}^{ij} + g_{S_{n+1}}^{ij} + g_{W_{n+1}}^{ij} \right) \min_{i,j} u_{n}^{ij} \]

\[ + \frac{\delta_2}{2} \left( g_{NE_{n+1}}^{ij} + g_{SE_{n+1}}^{ij} + g_{SW_{n+1}}^{ij} + g_{NW_{n+1}}^{ij} \right) \min_{i,j} u_{n}^{ij} \]

\[ + \left[ 1 - \delta_1 \left( g_{N_{n+1}}^{ij} + g_{E_{n+1}}^{ij} + g_{S_{n+1}}^{ij} + g_{W_{n+1}}^{ij} \right) + \frac{g_{NE_{n+1}}^{ij} + g_{SE_{n+1}}^{ij} + g_{SW_{n+1}}^{ij} + g_{NW_{n+1}}^{ij}}{2} + \rho \right] \min_{i,j} u_{n}^{ij} + \rho \delta_1 \min_{i,j} u_{n}^{0} \]

\[ = (1 - \rho \delta_1) \min_{i,j} u_{n}^{ij} + \rho \delta_1 \min_{i,j} u_{n}^{0} \]

Consequently, the result (70) is true for \( n = 1 \) because

\[ \min_{i,j} u_{n}^{ij} = (1 - \rho \delta_1) \min_{i,j} u_{n}^{0} + \rho \delta_1 \min_{i,j} u_{n}^{0} < u_{n}^{ij} \]

\[ < (1 - \rho \delta_1) \max_{i,j} u_{n}^{ij} + \rho \delta_1 \max_{i,j} u_{n}^{0} = \max_{i,j} u_{n}^{ij}. \]

(75)

Therefore, by induction, it is easy to prove the result (70), which completes the proof.

### 4.2. Experimental Procedures and Results

#### 4.2.1. Experimental Procedures

This section is devoted to comparing our model (16), as an image denoising algorithm, with the ones proposed by Wang and Zhou (Wang–Zhou) [10] and Maiseli [15]. All the experiments are conducted under Windows 10 and GNU Octave version 6.2.0, running on a laptop with an Intel® Core™ i7-10510U (8 MB cache, 4 Core, up to 4.9 GHz), 16 GB memory (LPDDR3, Dual-channel, 2133 MHz), and 512 GB storage (PCIe, SSD, 3 × 4). The experiments were done on three actual MRI scans (Figure 1) affected by different \( \sigma^2 \)-values of zero-mean white Gaussian noise and restored using our filter (67) with the boundary-initial conditions (68), provided that the diffusion function (23) satisfies the assumptions \((H_1 + H_2)\).

We also used the discrete diffusion filter as described in [15] for Wang-Zhou and Maiseli models with the following diffusion functions:

**Wang–Zhou** [10] diffusion function:

\[ g(s) = \frac{1}{s + 1} + \frac{\log(s + 1)}{s}, \quad s \geq 0. \]

(76)

**Maiseli** [15] diffusion function is a combination between the Perona–Malik diffusion function [16] and the Charbonnier diffusion function [8]:

\[ g(s) = \begin{cases} 
\frac{1}{1 + (s/k_1)^2} & s < k_1, \\
1 & \sqrt{1 + (s/k_2)^2} & s \geq k_1,
\end{cases} \]

(77)

where \( k_1 \) and \( k_2 \) are positive constants.

Besides, in order to evaluate the quality of the restored images from different image denoising methods, we used two image quality metrics:

(i) **Peak signal-to-noise ratio (PSNR)** [26]:

\[ \text{PSNR} = 10 \log_{10} \left( \frac{255^2 MN}{\| u - I \|_2^2} \right). \]

(78)

where \( f \) is the original or the uncorrupted image and \( u \) is the distorted or the restored image. PSNR is one of the oldest image quality metrics evaluating an image's signal strength relative to noise, and it is always positive. We evaluate the PSNR metric by using the Octave built-in function “psnr.” However, due to its limitations and its failure in some circumstances as an adequate measure of visual quality [27], we used another metric that significantly correlates with the human visual perception.

(ii) **Structural similarity index (SSIM)** [28]:

\[ \text{SSIM} = \frac{(2 \mu_u \mu_f + c_1) (2 \sigma_{uf} + c_2)} {\left( \mu_u^2 + \mu_f^2 + c_1 \right) \left( \sigma_u^2 + \sigma_f^2 + c_2 \right)} \]

(79)

where \( c_1 \) and \( c_2 \) are tuning parameters. \( \mu_u, \sigma_u^2 \) and \( \sigma_{ul} \) stand for the mean, variance, and covariance, respectively. It is a method for measuring the similarity between a degraded image and a perfect one, and it is bounded between zero and one. For a good similarity between the original and the restored images, we need higher values of SSIM-index. In all experiments, we estimate the SSIM-index using “ssim.m” with automatic downsampling, downloaded from the website: https://ece.uwaterloo.ca/z70wang/research/ssim/.

In all experiments and to prioritize SSIM-index over PSNR, we used a combined metric

\[ \text{IQM} = \text{PSNR} + 100 \times \text{SSIM}, \]

(80)

to quantify the quality of the restored images. The iterations are stopped when the IQM value reaches the maximum.
The image denoising process using the proposed model (67) can be implemented as shown in Algorithm 1. First, we consider $I$ as one of the three original brain MRI scans (Figure 1) and generate in it a Gaussian white noise with zero-mean and $\sigma^2$-value, using the Octave built-in function “imnoise.” Next, the initial value $u^0$ is set to be the noisy

\begin{verbatim}
input: I.  
output: u, iter, IQM.  
(1) Initialize: I;  
(2) $u^0 \leftarrow \text{imnoise}(I, "gaussian", 0, \sigma^2);  
(3) IQM_0 \leftarrow \text{PSNR}(I, u^0) + 100 \times \text{SSIM}(I, u^0);  
(4) $u \leftarrow u^0;  
(5) iter \leftarrow 0;  
(6) IQM \leftarrow IQM_0;  
(7) n \leftarrow 0;  
(8) While true do  
(9) $\Delta u = [\text{zeros}(1, M); (u^n(1: N - 1, : ) - u^n(2: N, : ))];  
(10) $\Delta u_{\alpha} = [(\Delta u^n)_{\alpha}; \text{zeros}(1, M)];  
(11) $\Delta u_{\beta} = [(u^n(1: 2: M) - u^n(:, 1: M - 1)] \text{zeros}(N, 1)];  
(12) $\Delta W_{\alpha} = [\text{zeros}(N, 1); (\Delta u^n(1: 1: M - 1)]];  
(13) $\Delta W_{\beta} = [(\Delta u^n)_{\beta}; \text{zeros}(1, M)];  
(14) $\Delta W = [(\Delta W_{\alpha})_{\beta}; \text{zeros}(N - 1, 1)\text{zeros}(N - 1, 1)];  
(15) $\Delta W = [(\Delta W_{\alpha})_{\beta}; \text{zeros}(N - 1, 1); \text{zeros}(1, M)];  
(16) $\Delta W = [(\Delta W_{\alpha})_{\beta}; \text{zeros}(N - 1, 1); \text{zeros}(N - 1, 1)];  
(17) \gamma_{\alpha} \leftarrow \gamma_{\alpha}^2 (|\Delta_{\alpha} u|^2), \alpha \in \Lambda_1 = \{N, E, S, W\};  
(18) \gamma_{\beta} \leftarrow \gamma_{\beta}^2 (|\Delta_{\beta} u|^2/\sqrt{2}), \beta \in \Lambda_2 = \{NE, SE, SW, NW\};  
(19) $u^{n+1} = u^n + \delta \gamma \sum_{\alpha \in \Lambda_1, \beta \in \Lambda_2} \gamma_{\alpha} \beta \gamma_{\beta} \Delta_{\alpha} u_{\alpha} \Delta_{\beta} u_{\beta} - \rho (u^n - u^{n+1});  
(20) IQM_{n+1} \leftarrow \text{PSNR}(I, u^{n+1}) + 100 \times \text{SSIM}(I, u^{n+1});  
(21) if (IQM_{n+1} > IQM) then  
(22) IQM \leftarrow IQM_{n+1};  
(23) iter \leftarrow n + 1;  
(24) $u \leftarrow u^{n+1};  
(25) else  
(26) Break;  
(27) end  
(28) n \leftarrow n + 1;  
(29) end  
\end{verbatim}

Algorithm 1: Proposed image denoising algorithm.
Table 1: PSNR values of the images in Figure 1 are affected by different $\sigma^2$ values of zero-mean white Gaussian noise and restored from different methods with their corresponding iteration number.

| $\sigma^2$ | Noisy | Wang and Zhou [10] | Maiseli [15] | Proposed |
|------------|-------|-------------------|--------------|----------|
|            | PSNR  | PSNR              | Iter         | PSNR     | Iter     |
| Patient 30 |       |                   |              |          |          |
| 0.005      | 23.4404 | 32.8436            | 36           | 33.4928  | 19       |
| 0.010      | 20.6935 | 30.9182            | 52           | 31.5063  | 27       |
| 0.015      | 19.0923 | 29.6302            | 64           | 30.1488  | 33       |
| Patient 50 |       |                   |              |          |          |
| 0.005      | 23.6551 | 31.0313            | 29           | 31.3403  | 15       |
| 0.010      | 20.7956 | 29.0496            | 42           | 29.2157  | 22       |
| 0.015      | 19.1404 | 27.8208            | 52           | 27.8922  | 28       |
| Patient 55 |       |                   |              |          |          |
| 0.005      | 24.0326 | 31.3462            | 30           | 31.4897  | 17       |
| 0.010      | 21.2339 | 29.1926            | 43           | 29.1900  | 24       |
| 0.015      | 19.5726 | 27.8086            | 53           | 27.7439  | 29       |

Table 2: SSIM-index values of the images in Figure 1 are affected by different $\sigma^2$ values of zero-mean white Gaussian noise and restored from different methods with their corresponding iteration number.

| $\sigma^2$ | Noisy | Wang and Zhou [10] | Maiseli [15] | Proposed |
|------------|-------|-------------------|--------------|----------|
|            | SSIM  | SSIM              | Iter         | SSIM     | Iter     |
| Patient 30 |       |                   |              |          |          |
| 0.005      | 0.7022 | 0.9089            | 36           | 0.9230   | 19       |
| 0.010      | 0.5889 | 0.8649            | 52           | 0.8899   | 27       |
| 0.015      | 0.5180 | 0.8282            | 64           | 0.8619   | 33       |
| Patient 50 |       |                   |              |          |          |
| 0.005      | 0.7832 | 0.9139            | 29           | 0.9157   | 15       |
| 0.010      | 0.6767 | 0.8650            | 42           | 0.8702   | 22       |
| 0.015      | 0.6099 | 0.8285            | 52           | 0.8367   | 28       |
| Patient 55 |       |                   |              |          |          |
| 0.005      | 0.7174 | 0.9065            | 30           | 0.9155   | 17       |
| 0.010      | 0.6041 | 0.8559            | 43           | 0.8706   | 24       |
| 0.015      | 0.5358 | 0.8138            | 53           | 0.8327   | 29       |

Figure 2: Continued.
image in the while-loop. After that, a while-loop is executed, and in each iteration, our filter is used to build a new image with a new IQM value. Finally, when the stopping criterion is confirmed, we obtain the restored image with the corresponding iteration.

4.2.2. Experimental Results. Several tests have been performed on the images in Figure 1 to obtain the best possible parameters:

The time step:

\[ \delta_t = 0.10917. \]  (81)

The weighting parameter:

\[ \rho = 0.00577. \]  (82)

The threshold parameters for Maiseli diffusion function (77):

\[ k_1 = 2.82261, \quad k_2 = 8.39061. \]  (83)

The different parameters for our diffusion function (23):

\[
\begin{align*}
    p_0 &= 1.46493, & v_0 &= -0.00003, & k &= 16.76863, \\
    p_k &= 0.60493, & v_k &= -0.00735, & h &= 195.5899, \\
    p_h &= 0.15094, & v_h &= -0.00002.
\end{align*}
\]  (84)

Tables 1 and 2 show quantitative results on real images, corrupted with various white Gaussian noises, filtered by different filters. From Table 1, it is clear that the PSNR value of the restored images via Maiseli method has, in most cases, higher values with slight differences over the other methods. However, as shown in Table 2, when it comes to the SSIM-index, the proposed method reveals impressive results against the Wang-Zhou and Maiseli methods. Additionally, our method converges more rapidly to the solution than the others with the less iteration number.

From a visual comparison, Figure 2 shows that the denoising process using our diffusion function removes noises more efficiently and preserves essential image features. Besides, it can be seen from different images that all the restored images via Wang–Zhou and Maiseli methods are more affected by blocky artifacts than the proposed one, despite the higher values in PSNR. This can be attributed to the fact that the Wang–Zhou and Maiseli methods have lower SSIM-indices.

5. Conclusion

This paper presented a new anisotropic diffusion model for image restoration, eliminating the corruption caused by white Gaussian noise. The existence and uniqueness of weak solutions of functions in Orlicz–Sobolev space that minimize the energy functional \( E \) have been proven, provided that the functions \( \varphi \) and \( g \) satisfy some specific conditions \((H_1 + H_2)\). Besides, we have established a consistent and stable numerical model for the denoising process and used the cubic Hermite spline to approximate the best possible diffusion function that revealed its efficiency regarding the
optimal image denoising via (67). We have also proved that our method provides better results compared to the Wang–Zhou and Maiseli methods.

The next stage of the research attempts to prove the existence and uniqueness of weak solutions for the evolution problem (16) and use different numerical methods, such as finite element and mixed finite element methods.

**Data Availability**

No data were used to support this study.

**Conflicts of Interest**

The authors declare that there are no conflicts of interest regarding the publication of this paper.

**References**

[1] M. Jourhmane, “Méthodes numériques de résolution d’un problème d’électro-encéphalographie,” Ph.D. thesis, University of Rennes 1, Rennes, France, 1993.

[2] G. Aubert and P. Kornprobst, *Mathematical Problems in Image Processing: Partial Differential Equations and the Calculus of Variations*, Springer, New York, NY, USA, 2006.

[3] J.-C. Pinoli, *Mathematical Foundations of Image Processing and Analysis, Volume 1*, Wiley-ISTE, Hoboken, NJ, USA, 2014.

[4] J.-C. Pinoli, *Mathematical Foundations of Image Processing and Analysis, Volume 2*, Wiley-ISTE, Hoboken, NJ, USA, 2014.

[5] L. I. Rudin, S. Osher, and E. Fatemi, “Nonlinear total variation based noise removal algorithms,” *Physica D: Nonlinear Phenomena*, vol. 60, no. 1-4, pp. 259–268, 1992.

[6] Y. L. You, W. Xu, A. Tannenbaum, and M. Kaveh, “Behavioral analysis of anisotropic diffusion in image processing,” *IEEE Transactions on Image Processing: A Publication of the IEEE Signal Processing Society*, vol. 5, no. 11, pp. 1539–1553, 1996.

[7] G. Aubert and L. Vese, “A variational method in image recovery,” *SIAM Journal on Numerical Analysis*, vol. 34, no. 5, pp. 1948–1979, 1997.

[8] P. Charbonnier, L. Blanc-Feraud, G. Aubert, and M. Barlaud, “Deterministic edge-preserving regularization in computed imaging,” *IEEE Transactions on Image Processing*, vol. 6, no. 2, pp. 298–311, 1997.

[9] C. A. Z. Barcelos and Y. Chen, “Heat flows and related minimization problem in image restoration,” *Computers & Mathematics with Applications*, vol. 39, no. 5-6, pp. 81-97, 2000.

[10] L. Wang and S. Zhou, “Existence and uniqueness of weak solutions for a nonlinear parabolic equation related to image analysis,” *Journal of Partial Differential Equations*, vol. 19, no. 2, pp. 97–112, 2006.

[11] T. Barbu, V. Barbu, V. Biga, and D. Coca, “A pde variational approach to image denoising and restoration,” *Nonlinear Analysis: Real World Applications*, vol. 10, no. 3, pp. 1351–1361, 2009.

[12] B. Wu, E. A. Ogada, J. Sun, and Z. Guo, “A total variation model based on the strictly convex modification for image denoising,” *Abstract and Applied Analysis*, vol. 2014, Article ID 948392, 16 pages, 2014.

[13] T. Barbu and C. Moroșanu, “Image restoration using a nonlinear second-order parabolic pde-based scheme,” *Analele Universitatii “Ovidius” Constanta - Seria Matematica*, vol. 25, no. 1, pp. 33–48, 2017.

[14] P. Li and S. Li, “Weak solutions for a class of generalised image restoration models,” *International Journal of Dynamical Systems and Differential Equations*, vol. 8, no. 3, pp. 190–203, 2018.

[15] B. J. Maiseli, “On the convexification of the perona-malik diffusion model,” *Signal, Image and Video Processing*, vol. 14, no. 6, pp. 1283–1291, 2020.

[16] P. Perona and J. Malik, “Scale-space and edge detection using anisotropic diffusion,” *IEEE Transactions on Pattern Analysis and Machine Intelligence*, vol. 12, no. 7, pp. 629–639, 1990.

[17] R. T. Whitaker and S. M. Pizer, “A multi-scale approach to nonuniform diffusion,” *CVGIP: Image Understanding*, vol. 57, no. 1, pp. 99–110, 1993.

[18] P. Chen, “Existence and uniqueness of weak solutions for a class of nonlinear parabolic equations,” *Electronic Research Announcements in Mathematical Sciences*, vol. 24, pp. 38–52, 2017.

[19] J. Weickert, *Anisotropic Diffusion in Image Processing*, Teubner Verlag, Stuttgart, Germany, 1998.

[20] A. Tiarimti Alaoui and M. Jourhmane, “Existence and uniqueness of weak solutions for novel anisotropic nonlinear diffusion equations related to image analysis,” *Journal of Mathematics*, vol. 2021, Article ID 5553126, 18 pages, 2021.

[21] J. Stoer and R. Bulirsch, *Interpolation*, Springer, New York, NY, USA, 2002.

[22] L. C. Evans and R. F. Gariepy, *Measure Theory and Fine Properties of Functions*, Taylor & Francis Group, Abingdon, UK, 2015.

[23] F. Gaillard, “Normal brain mr radiopaedia.org, rID: 42777,” 2016.

[24] F. Gaillard, “Normal brain mri (tle protocol) radiopaedia.org, rID: 40748,” 2015.

[25] F. Gaillard, “Normal brain mr radiopaedia.org, rID: 51158,” 2017.

[26] L. C. Evans and R. F. Gariepy, *Measure Theory and Fine Properties of Functions*, Taylor & Francis Group, Abingdon, UK, 2015.

[27] Z. Wang and A. C. Bovik, “Mean squared error: love it or leave it? A new look at signal fidelity measures,” *IEEE Signal Processing Magazine*, vol. 26, no. 1, pp. 98–117, 2009.

[28] Z. Wang, A. C. Bovik, H. R. Sheikh, and E. P. Simoncelli, “Image quality assessment: from error visibility to structural similarity,” *IEEE Transactions on Image Processing*, vol. 13, no. 4, pp. 600–612, 2004.