Approximate quantum encryption with faster key expansion

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Abstract
Perfect encryption of a qubit state using the Quantum One-Time Pad (QOTP) requires 2 classical key bits. More generally, perfect encryption of a $2^n$-dimensional state requires $2^n$ classical bits. However, almost-perfect encryption, with information-theoretic security, can be achieved with only little more than 1 key bit per qubit. It has been shown that key length $n + 2 \log \frac{1}{\varepsilon}$ suffices to encrypt $n$ qubits in such a way that the cipherstate has trace distance $\leq \varepsilon$ from the fully mixed state. In this paper, we present a fast key expansion method to create a $2^n$-bit pseudorandom string which is then used as a QOTP key. In this expansion we make use of $2^n$ bits of public randomness which are included as a classical part of the cipherstate. Our key expansion is a factor 2 faster than the previous fastest scheme, while achieving the shortest known key length $n + 2 \log \frac{1}{\varepsilon}$.

1 Introduction

1.1 Encryption of quantum states
An encryption scheme is called perfect if the ciphertext reveals no information whatsoever about the message that was encrypted. In the case of classical messages, the length of the key required to achieve perfect encryption is at least the Shannon entropy of the message. The Vernam cipher [Ver18] (also known as One-Time Pad, OTP) performs a bitwise xor of an $n$-bit message and an $n$-bit key; it achieves perfect encryption for any probability distribution of the message.

An equivalent of the Vernam cipher exists for the encryption of quantum states [AMTdW00, BR03, Leu02]. This is known as the quantum Vernam cipher, Quantum One-Time Pad (QOTP) or private quantum channel. In order to perfectly encrypt any $n$-qubit state, the necessary and sufficient key length is $2n$ bits. In its simplest form, QOTP encryption and decryption work by applying to each individual qubit a Pauli operation from the set $\{1, \sigma_x, \sigma_y, \sigma_z\}$. The choice of Pauli operations constitutes the key. For someone who does not know this key, the state after encryption equals the fully mixed state regardless of the original state.

It is possible to get $\varepsilon$-close to the fully mixed state by a randomization process that takes fewer than $2n$ key bits [HLSW04, AS04, DN06, Aub09, ŠdV17]. This is called approximate randomization or almost-perfect encryption. The $\varepsilon$-close' property can be expressed as a distance with respect to different norms, e.g. the 1-norm (trace norm) or the $\infty$-norm (maximum absolute eigenvalue). In this paper, we consider only the 1-norm, since it expresses the indistinguishability of states and it is a universally composable measure of security [Can01, BOHL+05, FS09]. Table 1 summarizes known results on approximate randomization, including this work, focusing on the 1-norm.

Approximate randomization is related to quantum encryption with entropic security [Des09, DD10]. Entropically secure encryption needs assumptions on the adversary’s prior knowledge about the plaintext. The approximate randomization scenario is the special case where it is known that Eve is not entangled with the plaintext state, but no other assumption is made. In the current paper, we will not consider entropic security in general.

Hayden et al. [HLSW04] showed that approximate randomization is possible with a key length of $n + \log n + 2 \log \frac{1}{\varepsilon}$ by using sets of random unitaries. Random selection and storage of unitary matrices are very inefficient. Ambainis and Smith [AS04] introduced far more efficient schemes that work with a pseudorandom sequence that selects Pauli operators as in the QOTP. In one of them, they expand the

\[ n + 2 \log \frac{1}{\varepsilon} + \log 134 \]
|                       | Key length $\ell$ | Ciphertext length | Randomization process |
|-----------------------|-------------------|-------------------|-----------------------|
| [HLSW04]              | $n + \log n + 2 \log \frac{1}{\varepsilon}$ | $n$ qubits | Random unitaries (non-Haar, e.g. Pauli) |
| [AS04]                | $n + 2 \log n + 2 \log \frac{1}{\varepsilon}$ | $n$ qubits | Pseudorandom QOTP based on small-bias sets. Key expansion takes $O(n^2)$ operations. |
| [AS04]                | $n + 2 \log \frac{1}{\varepsilon}$ | $n$ qubits + $2n$ bits | Pseudorandom QOTP based on small-bias sets. Key expansion takes $\approx 6n\log_3 n$ operations. |
| [DN06]                | $n + 2 \log \frac{1}{\varepsilon} + 4$ | $n$ qubits | Pseudorandom QOTP based on small-bias sets. Key expansion takes $O(n^2\log n)$ operations. |
| [ŠdV17]               | $n + 2 \log \frac{1}{\varepsilon}$ | $n$ qubits | Pseudorandom QOTP based on huge Common Reference String. |
| This paper            | $n + 2 \log \frac{1}{\varepsilon}$ | $n$ qubits + $2n$ bits | Pseudorandom QOTP based on affine function in $\text{GF}(2^\ell)$. Key expansion takes $\approx 3n\log_3 n$ operations. |

Table 1: Results on almost-perfect encryption of $n$ qubits, with security definition in terms of the trace distance: $\|\text{cipherstate} - \text{fully mixed state}\|_1 \leq \varepsilon$. The listed complexity for the finite field multiplications is based on the fastest known implementation and shows only the number of AND operations. (See Section 3.3).

key using small-bias sets and achieve key length $n + 2 \log n + 2 \log \frac{1}{\varepsilon}$. This scheme is length-preserving, i.e. the cipherstate consists of $n$ qubits. In another construction, they expand the key by multiplying it with a random binary string of length $2n$; this string becomes part of the cipherstate. The key length is reduced to $n + 2 \log \frac{1}{\varepsilon}$. Dickinson and Nayak [DN06] improved the small-bias based scheme of [AS04] and achieved key length $n + 2 \log \frac{1}{\varepsilon} + 4$. Škorić and de Vries [ŠdV17] described a pseudorandom QOTP scheme that has key length $n + 2 \log \frac{1}{\varepsilon}$, but they need an exponentially large common random string to be stored somewhere.

In all the schemes that expand the key, the expansion can be done efficiently, with time complexity $O(n^2\log n)$ or better, because these schemes are based on Galois field multiplication which takes $O(n\log n)$ time (see e.g. Theorem 8.7 in [AHU74] and its corollary).

1.2 Contribution and outline

We modify the second scheme of Ambainis and Smith [AS04]. Instead of expanding the key by multiplying in $\text{GF}(2^{2n})$, we append to the key an affine function of the key. The two parameters of the affine function are drawn at random and become part of the cipherstate. The resulting encryption scheme still has key length $\ell = n + 2 \log \frac{1}{\varepsilon}$, which is the shortest length presented in the literature. The cipherstate consists of $n$ qubits and $2n$ classical bits.

The key expansion is roughly twice as fast as [AS04] because we multiply in $\text{GF}(2^\ell)$ instead of $\text{GF}(2^{2n})$.

The outline of the paper is as follows. In Section 2.1 we introduce notation, and in Section 2.2 the desired ’$\varepsilon$-randomizing’ security property is specified. The QOTP is briefly recalled in Section 2.3. In Section 3 we give the details of our construction, the security proof and the complexity of the key expansion. We conclude with a brief discussion in Section 4.

2 Preliminaries

2.1 Notation

Expectation over a random variable $X$ is written as $E_X$. We denote the space of density matrices on Hilbert space $\mathcal{H}$ as $\mathcal{D}(\mathcal{H})$. The single-qubit Hilbert space is $\mathcal{H}_2$. A bipartite state comprising $2n$ bits to be costly. Classical storage and transmission are ‘for free’ compared to quantum resources.

\[\text{We do not consider the additional 2n bits to be costly. Classical storage and transmission are 'for free' compared to quantum resources.}\]
subsystems ‘A’ and ‘B’ is written as $\rho^{AB} \in \mathcal{D}(\mathcal{H}_A \otimes \mathcal{H}_B)$. The state of a subsystem is obtained by taking the partial trace over the other subsystem, e.g. $\rho^A = \text{tr}_B \rho^{AB}$. The identity operator on $\mathcal{H}$ is denoted by $1_\mathcal{H}$; we will simply write $1$ when the Hilbert space is clear from the context. Similarly we write $\tau^\mathcal{H}$ for the fully mixed state $1_\mathcal{H}/\text{dim}(\mathcal{H})$, often omitting the superscript. Let $M$ be an operator with eigenvalues $\lambda_i$. The Schatten 1-norm of $M$ is given by $\|M\|_1 = \text{tr} \sqrt{M^\dagger M} = \sum_j |\lambda_i|$. The induced ‘trace’ distance between states $\rho, \sigma$ is $\|\rho - \sigma\|_1$.

2.2 Security definitions

We use standard definitions of encryption and $\varepsilon$-randomization.

**Definition 2.1 (Encryption scheme)** An encryption scheme with classical key space $\mathcal{K}$, quantum message space $\mathcal{H}$ and quantum ciphertext space $\mathcal{H}'$ consists of a pair $(\text{Enc}, \text{Dec})$. Here $\text{Enc} : \mathcal{K} \times \mathcal{D}(\mathcal{H}) \to \mathcal{D}(\mathcal{H}')$ is a (possibly randomized) algorithm that takes as input a classical key $k \in \mathcal{K}$ and a quantum state $\varphi \in \mathcal{D}(\mathcal{H})$, and outputs a quantum state $\omega = \text{Enc}(k, \varphi) \in \mathcal{D}(\mathcal{H}')$ called the cipherstate. $\text{Dec} : \mathcal{K} \times \mathcal{D}(\mathcal{H}') \to \mathcal{D}(\mathcal{H})$ is an algorithm that takes as input a key $k \in \mathcal{K}$ and a state $\omega \in \mathcal{D}(\mathcal{H}')$, and outputs a state $\text{Dec}(k, \omega) \in \mathcal{D}(\mathcal{H})$. It must hold that $\forall_{k \in \mathcal{K}, \varphi \in \mathcal{D}(\mathcal{H})} \text{Dec}(k, \text{Enc}(k, \varphi)) = \varphi$.

Note that Def. 2.1 allows the cipherstate space to be larger than the plaintext space, $\text{dim } \mathcal{H}' > \text{dim } \mathcal{H}$. We will be working with the special case where the cipherstate consists of a quantum state of the same dimension as the input ($n$ qubits), accompanied by classical information.

The effect of the encryption, with the key unknown to the adversary, can be described as a completely positive trace preserving (CPTP) map $R : \mathcal{D}(\mathcal{H}) \to \mathcal{D}(\mathcal{H}')$ as follows,

$$R(\varphi) = \sum_{k \in \mathcal{K}} \frac{1}{|\mathcal{K}|} \text{Enc}(k, \varphi).$$

**Definition 2.2 ($\varepsilon$-Randomizing)** Let $\varepsilon \geq 0$. A CPTP linear operator $R : \mathcal{D}(\mathcal{H}) \to \mathcal{D}(\mathcal{H}')$ is said to be $\varepsilon$-randomizing with respect to a norm $\| \cdot \|$ if

$$\forall_{\varphi \in \mathcal{D}(\mathcal{H})} \| R(\varphi) - \tau^\mathcal{H'} \| \leq \varepsilon.$$  

We say that $R$ is completely randomizing if $\varepsilon = 0$.

Def. 2.2 is a slight modification of Def. 1.1 in [DN06]; the difference is that we allow $\text{dim } \mathcal{H}' \neq \text{dim } \mathcal{H}$.

It is important to note that randomization as specified in Def. 2.2 implies that $R(\varphi)$ is practically independent of any Eve who is correlated only classically with $\varphi$. Let $\rho^{E \Phi}$ denote the bipartite state of Eve and the quantum ‘plaintext’ state. We write $\varphi = \rho^{E \Phi} = \text{tr}_E \rho^{E \Phi}$. Without entanglement between the $E$ and $\Phi$ subsystems, the state is separable, $\rho^{E \Phi} = \sum_k \rho_k^{E} \otimes \varphi_k$. We write $\tau = \tau^\mathcal{H'}$. Furthermore $\Phi'$ denotes the result of the operation $R$ on the $\Phi$ subsystem, and we write $\varphi' = R(\varphi)$. Repeated use of the triangle inequality, followed by (2) yields $\| \rho^{E \Phi'} - \rho^{E \Phi} \| = \| \sum_k \rho_k^{E} \otimes (\varphi'_k - \varphi_k) \| \leq \sum_k \| \rho_k^{E} \| \| \varphi'_k - \varphi_k \| \leq \sum_k \| \rho_k^{E} \| \| \varphi'_k - \varphi_k \| \leq 2 \varepsilon$.

Just as the earlier works [HLSW04, AS04, DN06, ŠdV17] we will use Def. 2.2 (with the 1-norm) as our security definition.

2.3 The Quantum One Time Pad (QOTP)

Let $\mathcal{H}_2$ denote the Hilbert space of a qubit. Let $Z$ and $X$ be single-qubit Pauli operators, in the standard basis given by $(\frac{1}{\sqrt{2}} |0\rangle + \frac{1}{\sqrt{2}} |1\rangle)$ and $(\frac{1}{\sqrt{2}} |0\rangle - \frac{1}{\sqrt{2}} |1\rangle)$. For QOTP encryption of one qubit, the key consists of two bits $s, t \in \{0, 1\}$. The encryption of a state $\varphi \in \mathcal{D}(\mathcal{H}_2)$ is given by $X^sZ^t\varphi Z^tX^s$. Decryption is the same operation as encryption. For someone who does not know the key, the state after encryption is $\frac{1}{2} \sum_{s, t \in \{0, 1\}} X^sZ^t\varphi Z^tX^s = \frac{1}{2}$ for any $\varphi$. Hence the QOTP is completely randomizing.

The simplest way to encrypt an $n$-qubit state $\varphi \in \mathcal{D}(\mathcal{H}_2^\otimes n)$ is to encrypt each qubit independently. The key is $\beta = (\beta_1, \ldots, \beta_n) \in \{0, 1\}^{2n}$, with $\beta_i = (s_i, t_i)$. In the rest of the paper we will use the following shorthand notation for the QOTP cipherstate,

$$F_{\beta}(\varphi) = U_{\beta} \varphi U_{\beta}^\dagger$$

where $U_{\beta} = \bigotimes_{i=1}^n X^{s_i}Z^{t_i}$.  

(3)

It holds that $2^{-2n} \sum_{\beta \in \{0, 1\}^{2n}} F_{\beta}(\varphi) = 1/2^n$ for any $\varphi \in \mathcal{D}(\mathcal{H}_2^\otimes n)$.
3 Our result on approximate randomization

3.1 The construction

We encrypt an $n$-qubit state $\varphi \in \mathcal{D}(\mathcal{H}_2^\otimes n)$ using a key $k \in \{0, 1\}^\ell$ where $\ell > n$, and $\ell$ is an even integer. We construct a pseudorandom sequence $b \in \{0, 1\}^{2n}$ by expanding $k \in \{0, 1\}^\ell$ as follows. Two strings $u \in \{0, 1\}^\ell$, $v \in \{0, 1\}^{2n-\ell}$ are drawn at random. They are interpreted as elements of $\text{GF}(2^\ell)$. Note that $2n - \ell < n$. The string $b$ is constructed by concatenating $k$ with an affine function of $k$.

$$b(k, u, v) = k \|(uk + v)_{\text{lsb}}.$$ \hfill (4)

The subscript ‘lsb’ (Least Significant Bits) stands for taking the last $2n - \ell$ bits of the string; in the finite field representation, this corresponds to taking a polynomial in $x$ modulo $x^{2n-\ell}$. In (4) the multiplication and addition are operations in $\text{GF}(2^\ell)$. Instead of $(uk + v)_{\text{lsb}}$ we can also write $(uk)_{\text{lsb}} + v$. The cipherstate is given by

$$\text{Enc}(k, \varphi) = (u, v, F_{b(k, u, v)}(\varphi)) \hfill (5)$$

with $F$ the QOTP encryption as defined in (3) and $b(\cdot, \cdot, \cdot)$ as defined by (4). The parameters $u, v$ are a classical part of the cipherstate.

3.2 Security proof

Eve sees the parameters $u, v$ but she does not know $k$. From her point of view the state of the qubits is

$$R_{uv}(\varphi) \overset{\text{def}}{=} \frac{1}{2^\ell} \sum_{k \in \{0, 1\}^\ell} F_{b(k, u, v)}(\varphi). \hfill (6)$$

Lemma 3.1 It holds that

$$\mathbb{E}_{uv} R_{uv}(\varphi) = \tau. \hfill (7)$$

Proof: We write $\mathbb{E}_{uv} R_{uv}(\varphi) = \mathbb{E}_u [\mathbb{E}_k F_{b(k, u, v)}(\varphi)]$. Next, $\mathbb{E}_k F_{b(k, u, v)}(\varphi) = \mathbb{E}_{\beta \in \{0, 1\}^{2n}} F_\beta(\varphi) = \tau$. The first equality follows from the fact that for any fixed $u$, the $k$ and $v$ together can create any string in $\{0, 1\}^{2n}$ in precisely one way. The second equality is due to the fact that the QOTP is completely randomizing. \square

Lemma 3.2 Let $f$ be any (possibly operator valued) function acting on $\{0, 1\}^{2n}$. It holds that

$$\mathbb{E}_{kk'uv} f(b(k, u, v)) f(b(k', u, v)) = 2^{-\ell} \mathbb{E}_\beta f(\beta) f(\beta) + \mathbb{E}_{\beta \beta'} f(\beta) f(\beta') - 2^{-\ell} \mathbb{E}_{gg'} f(k||g) f(k||g'). \hfill (8)$$

Here $\beta, \beta' \in \{0, 1\}^{2n}$, and $\mathbb{E}_\beta$ stands for $2^{-2n} \sum_\beta$. Similarly, $g, g' \in \{0, 1\}^{2n-\ell}$ and $\mathbb{E}_g$ stands for $2^{\ell-2n} \sum_g$.

Proof: We first look at the summation terms with $k' \neq k$. We write $b(k, u, v) = k||g$ and $b(k', u, v) = k'||g'$. Consider $k, k'$ fixed. For every combination $(g, g')$ there are exactly $2^{2\ell-2n}$ values of $(u, v)$ that yield $(uk)_{\text{lsb}} + v = g$, $(uk')_{\text{lsb}} + v = g'$.\footnote{When the two equations are added the $v$ disappears and we get $[u(k+k')]_{\text{lsb}} = g + g'$, which has $2^\ell / 2^{2n-\ell}$ solutions $u$. Then, at fixed $k, k', g, g'$, $u$ the solution for $v$ is unique.} This allows us to rewrite the summations as

$$\sum_{kk':k' \neq k} \sum_{uv} f(b(k, u, v)) f(b(k', u, v)) = 2^{2\ell-2n} \sum_{kk' \neq k} \sum_{gg'} f(k||g) f(k'||g') \hfill (9)$$

$$= 2^{2\ell-2n} \left( \sum_{kgk'g'} f(k||g) f(k'||g') - \sum_{kgg'} f(k||g) f(k||g') \right). \hfill (10)$$

Next we look at the $k' = k$ terms. The summation over $k$ and $v$ spans all the possible $\beta$’s. Hence we can write

$$\sum_k \sum_{uv} f(b(k, u, v)) f(b(k, u, v)) = \sum_u \sum_\beta f(\beta) f(\beta). \hfill (11)$$
Combining the $k' = k$ and $k' \neq k$ parts we get
\[
\sum_{k\neq k'} f(b(k, u, v)) f(b(k', u, v)) = 2^f \sum_{\beta} f(\beta) f(\beta') + 2^{2^f-2n} \sum_{\beta' \neq \beta} f(\beta) f(\beta') - 2^{2^f-2n} \sum_{k \neq k'} f(k\|g) f(k\|g').
\] (12)

In order to go from summations to expectations we divide (12) by a factor $(2^f)^32^{2n-\ell} = 2^{2\ell+2n}$. □

**Theorem 3.3** The randomizing map $R_{uv} : \mathcal{D}(\mathcal{H}_2^\otimes n) \to \mathcal{D}(\mathcal{H}_2^\otimes n)$ as described in (6) satisfies
\[
\forall \varphi \in \mathcal{D}(\mathcal{H}_2^\otimes n), \quad \mathbb{E}_{uv}\|R_{uv}(\varphi) - \tau\|_1 < \sqrt{2^{n-\ell}}.
\] (13)

**Proof:** For any $\varphi$ we have
\[
\mathbb{E}_{uv}\|R_{uv}(\varphi) - \tau\|_1 = \mathbb{E}_{uv}\text{tr} \sqrt{(R_{uv}(\varphi) - \tau)^2}
\]
\[
\overset{\text{Jensen}}{\leq} \text{tr} \sqrt{\mathbb{E}_{uv}(R_{uv}(\varphi) - \tau)^2}
\]
\[
\overset{\text{Lemma 3.1}}{=} \text{tr} \sqrt{\mathbb{E}_{uv}[R_{uv}(\varphi)]^2 - \tau^2}
\]
\[
\overset{\text{Lemma 3.2}}{=} \text{tr} \sqrt{2^{-\ell}\mathbb{E}_{\beta} F_{\beta}(\varphi) F_{\beta}(\varphi) - 2^{-\ell}\mathbb{E}_{k}[\mathbb{E}_{g} F_{k\|g}(\varphi)]\mathbb{E}_{g'} F_{k\|g'}(\varphi)}.
\] (18)

In the last step we used $\mathbb{E}_{\beta\beta'} F_{\beta}(\varphi) F_{\beta'}(\varphi) = \tau^2$. Next we use $F_{\beta}(\varphi) F_{\beta}(\varphi) = F_{\beta}(\varphi^2)$ and $\mathbb{E}_{\beta\beta'} F_{\beta}(\varphi^2) = \tau \text{tr} (\varphi^2)$, yielding
\[
\mathbb{E}_{\beta\beta'} F_{\beta}(\varphi) F_{\beta'}(\varphi) = \tau \text{tr} (\varphi^2).
\] (19)

Furthermore we write $\varphi = \sum_{abcd} \varphi_{abcd}|e_a^L e_b^R)(e_c^L e_d^R|$ where the index ‘L’ stands for the first $\ell/2$ qubits and ‘R’ stands for the final $n - \ell/2$ qubits; $e^L$ is a basis for the L subsystem and likewise $e^R$ for R. The marginal state of the L subsystem is given by $\varphi^L = \text{tr}_R \varphi = \sum_{uv} (\sum_{bd} \varphi_{abcd}|e_b^L)(e_d^L|$. We note that $\mathbb{E}_{g} F_{k\|g}(\varphi) = F_k(\varphi^L) \otimes \varphi^R$, which gives
\[
\mathbb{E}_{k}[\mathbb{E}_{g} F_{k\|g}(\varphi)]\mathbb{E}_{g'} F_{k\|g'}(\varphi)] = \mathbb{E}_{k} F_k(\varphi^L) F_k(\varphi^L) \otimes (\varphi^R)^2 = \varphi^L \otimes (\varphi^R)^2 \text{tr}_L((\varphi^L)^2).
\] (20)

We substitute (19) and (20) into (18). Since the operator under the square root is diagonal, the $\text{tr} \sqrt{\cdot}$ is readily computed and gives
\[
\mathbb{E}_{uv}\|R_{uv}(\varphi) - \tau\|_1 \leq \sqrt{2^{n-\ell} \text{tr} \varphi^2 - 2^{-\ell} \text{tr} (\varphi^L)^2}
\]
\[
\overset{\text{Lemma 3.1}}{\leq} \sqrt{2^{n-\ell} - 2^{-\ell}}
\]
\[
\overset{\text{Lemma 3.2}}{<} \sqrt{2^{n-\ell}}.
\] (21)

In (22) we used $\text{tr} \varphi^2 \leq 1$ and $\text{tr} (\varphi^L)^2 \geq 2^{-\ell/2}$. □

**Theorem 3.4** Our scheme is $\varepsilon$-randomizing (Def. 2.2) with respect to the 1-norm when the key length is set to $\ell = n + 2 \log \frac{1}{\varepsilon}$.

**Proof:** From the adversary’s point of view the cipherstate is $\mathbb{E}_{uv}|uv\rangle\langle uv| \otimes R_{uv}(\varphi)$. We have to prove that this is $\varepsilon$-close to the fully mixed state on the whole output space, i.e. to $\mathbb{E}_{uv}|uv\rangle\langle uv| \otimes \tau$. We get
\[
\left\| \mathbb{E}_{uv}|uv\rangle\langle uv| \otimes [R_{uv}(\varphi) - \tau] \right\|_1 = \mathbb{E}_{uv}\left\| R_{uv}(\varphi) - \tau \right\|_1 \overset{\text{Th. 3.3}}{\leq} \sqrt{2^{n-\ell}}.
\] (24)

Substituting $\ell = n + 2 \log \frac{1}{\varepsilon}$ into the final expression yields $\varepsilon$. □

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4Here we use that $\ell$ is even.
3.3 Complexity of the key expansion

We briefly comment on the complexity of our key expansion compared to [AS04]. Multiplication in GF($2^n$) has time complexity $O(n \log n)$ [AHU74, Can89] whereas addition (subtraction) consists of $n$ XOR operations. Mateer [Mat08] introduced an improved version of Schönhage’s multiplication algorithm [Sch77]. If $m$ is of the form $3^\ell$ and $\kappa$ is a power of two, then multiplication of two elements in GF($2^{2m}$) requires $\frac{17}{3}m \log_3 m$ bit-AND operations and at least $\frac{22}{5}m \log m \log(\log m) + \frac{3}{5}m \log m - \frac{3}{5}m + \frac{1}{5}\sqrt{m}$ bit-XORs. If $\kappa$ is not a power of two, then the number of ANDs slightly increases to $6m \log_3 m$ while the bound on the XORs stays the same.

Our key expansion consists of one multiplication in GF($2^\ell$) and one addition in GF($2^{2n-\ell}$) or, since $\ell$ asymptotically almost equals $n$, roughly speaking one multiplication and one addition in GF($2^n$). With Mateer’s multiplication for general $\kappa$, this yields a total cost of $3n \log_3 n - 3n \log_3 2 \text{ ANDs}$ and $\geq n \log n \left( \frac{22}{5} \log \log \frac{2}{3} + \frac{3}{5} \right) - n \left( \frac{22}{5} \log \log \frac{2}{3} + \frac{3}{5} \right) + \mathcal{O}(\sqrt{n})$ XORs for our key expansion.

In [AS04] the $\ell$-bit key $k$ is multiplied by a string $\alpha \in \{0,1\}^{2n}$, and the multiplication is in GF($2^{2n}$). If we write $\alpha = L||R$ and take $\ell \approx n$ then this can be reorganized into the following steps: (i) a polynomial multiplication $k \cdot R$ without modular reduction; (ii) a polynomial multiplication $k \cdot L$ shifted by $n$ positions, resulting in a polynomial of degree at most $3n$, followed by GF($2^n$) modular reduction; (iii) addition of the two above contributions. As we count two XORs per monomial that needs to be reduced\(^5\), we see that the addition in step (iii) precisely compensates the missing reduction in step (i). Furthermore the number of monomials that needs reducing in step (ii) is $n$, which is the same as in GF($2^n$) multiplication. Hence the cost of computing $k \cdot \alpha$ equals the cost of two GF($2^n$) multiplications. Since in GF($2^n$) multiplication is much more expensive than addition, we see that our key expansion is a factor 2 cheaper than [AS04].

4 Discussion

We get the same shortest key length $\ell = n + 2 \log \frac{1}{3}$ reported in other studies, but with a key expansion that is twice as efficient as the fastest one [AS04] in the literature. It is interesting to note that the security proof in [AS04] uses Fourier analysis and $\delta$-biased families, and invokes Cayley graphs for intuition, whereas our proof is more straightforward.

A small improvement to our scheme could be to draw the parameter $u$ from $\{0,1\}^{2n-\ell}$ instead of $\{0,1\}^\ell$.

It would be interesting to see how our scheme behaves regarding entropic security [Des09, DD10]. This is left for future work.

Acknowledgements

We thank Tanja Lange and Dan Bernstein for discussions on multiplication complexity. Part of this work was supported by the Dutch Startimpuls NAQT KAT-2 and NGF Quantum Delta NL KAT-2.

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\(^5\)With $m = 3^n$ it is possible to use the trinomial $x^{2m} + x^m + 1$ as the irreducible polynomial, which allows for efficient reduction. If we depart from the $3^n$ form, irreducible polynomials of degree 5 may become necessary, which leads to more costly modular reduction. It has been shown [BCP19] that irreducible pentanomials can be chosen such that no more than three XORs are required per monomial reduction.
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