HOMOLOGY OF $GL_n$ OVER ALGEBRAICALLY CLOSED FIELDS

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ABSTRACT. In this paper we define higher pre-Bloch groups $p_n(F)$ of a field $F$. When our base field is algebraically closed we study its connection to the homology of the general linear groups with finite coefficient $\mathbb{Z}/l\mathbb{Z}$ where $l$ is a positive integer. As a result of our investigation we give a necessary and sufficient condition for the map $H_n(GL_{n-1}(F), \mathbb{Z}/l\mathbb{Z}) \to H_n(GL_n(F), \mathbb{Z}/l\mathbb{Z})$ to be bijective. We prove that this map is bijective for $n \leq 4$. We also demonstrate that the divisibility of $p_n(\mathbb{C})$ is equivalent to the validity of the Friedlander-Milnor Isomorphism Conjecture for $(n + 1)$-th homology of $GL_n(\mathbb{C})$.

INTRODUCTION

A theorem of Bloch and Wigner, unpublished and in a somewhat different form, asserts the existence of the following exact sequence

$$0 \to \mathbb{Q}/\mathbb{Z} \to H_3(SL_2(\mathbb{C}), \mathbb{Z}) \to p(\mathbb{C}) \to \bigwedge^2 \mathbb{C}^\ast \to K_2(\mathbb{C}) \to 0.$$ 

A similar exact sequence can be obtained for any algebraically closed field. We refer the reader to [3, Appendix A] for a proof of the above exact sequence and for a precise description of the groups and the maps involved (see also [12, 2.12, 2.14]).

The group $p(\mathbb{C})$ is called pre-Bloch group and it has been the source of many interesting ideas and connections. The pre-Bloch group $p(F)$ plays a very important role in the study of scissors congruences of polyhedra in connection with Hilbert’s third problem [2,12], it is related very closely to the third $K$-group $K_3(F)$ which was the main driving force behind Suslin’s solution of the Quillen-Lichtenbaum Conjecture [17], and it is used in establishing certain cases of the Friedlander-Milnor Isomorphism Conjecture [7] for certain lower homology groups [2], and so on.

Thus it is natural to ask whether there is a general notion of higher pre-Bloch groups, and if so, if it carries useful information. In [5, Section 4.4] Loday defines a higher version of the pre-Bloch group, which we denote by $p_n(F)$, such that $p(F) = p_2(F)$, and predicts that it should have a close relation with the homology of general linear groups. We call $p_n(F)$ the $n$-th pre-Bloch group of the field $F$.

Although the definition of $p_n(F)$ is easy, which is in terms of generators and relations, it is difficult to study it directly. In this article we explore its
connection with the homology of the general linear groups. As we will see this connection is very close. Here is our main result.

**Theorem 3.4.** Let $F$ be an algebraically closed field and let $l$ be a positive integer. The following conditions are equivalent

(i) $H_m(\text{GL}_{n-1}, \mathbb{Z}/l\mathbb{Z}) \to H_m(\text{GL}_n, \mathbb{Z}/l\mathbb{Z})$ is injective for $m = n$ and is surjective for $m = n + 1$,

(ii) $p_n(F) \otimes \mathbb{Z}/l\mathbb{Z} = \begin{cases} \mathbb{Z}/l\mathbb{Z} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even.} \end{cases}$

It is known by a theorem of Dupont and Sah that for an algebraically closed field $F$, $p_2(F)$ is divisible [3]. The above theorem suggests that a general version of this fact might be true.

**Conjecture 3.5.** Let $F$ be an algebraically closed field and let $l$ be a positive integer. Then

$$p_n(F) \otimes \mathbb{Z}/l\mathbb{Z} = \begin{cases} \mathbb{Z}/l\mathbb{Z} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even.} \end{cases}$$

From the result of Dupont and Sah the conjecture is true for $n=2$ [3]. In this article we prove it for $n = 3, 4$ and also for all $n$ over the algebraic closure of finite fields. For the latter we use a result of Friedlander concerning the homology of general linear groups over algebraic closure of a finite field.

Here is a strong support for our conjecture.

**Proposition 3.11.** Let $l$ be a positive integer. The following conditions are equivalent

(i) $p_n(\mathbb{C}) \otimes \mathbb{Z}/l\mathbb{Z} = \begin{cases} \mathbb{Z}/l\mathbb{Z} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$ for all $n$,

(ii) $H_n(\text{GL}_{n-1}(\mathbb{C}), \mathbb{Z}/l\mathbb{Z}) \sim \to H_n(\text{GL}_n(\mathbb{C}), \mathbb{Z}/l\mathbb{Z})$ for all $n$,

(iii) $H_n(B\text{GL}_{n-1}(\mathbb{C}), \mathbb{Z}/l\mathbb{Z}) \sim \to H_n(B\text{GL}_n(\mathbb{C})^{\text{top}}, \mathbb{Z}/l\mathbb{Z})$ for all $n$.

Here $B\text{GL}_{n-1}(\mathbb{C})^{\text{top}}$ is the classifying space of $\text{GL}_{n-1}(\mathbb{C})$ with its usual topology and $B\text{GL}_n(\mathbb{C})$ is the classifying space of $\text{GL}_n(\mathbb{C})$ with $\text{GL}_n(\mathbb{C})$ as a discrete group. The condition (iii) is a special case of the Friedlander-Milnor Conjecture on the homology of Lie groups with finite coefficients (see [3,10]).

We briefly outline the organization of the present paper.

In Section 1 we introduce a spectral sequence which will be our main tool in handling the homology of general linear groups. In this section we will prove an important lemma, which is used in the proof of Theorem 3.4.

In Section 2 we define the higher pre-Bloch groups $p_n(F)$ and give some of its properties. In defining these groups we follow Suslin’s approach for the definition of $p(F)$ in [7]. Here we also give some satisfactory description of $p_3(F)$ and $p_4(F)$.
In Section 3 we prove Theorem 3.4 and Proposition 3.11. Here we also prove that Conjecture 3.5 is true for algebraic closure of a finite field.

In Section 4 we show that condition (i) or (ii) of Theorem 3.4 is satisfied for \( n \leq 4 \). Here we also establish a new case of the Friedlander-Milnor conjecture for the fourth homology of \( GL_3(C) \) and \( SL_3(C) \).

In Section 5 some of these ideas are generalized.

**Notation.** Here we establish some notations that is used throughout the paper. In this paper by \( H_i(G) \) of a group \( G \) we mean the integral homology group \( H_i(G, \mathbb{Z}) \). By \( GL_n \) we mean the general linear group \( GL_n(F) \), where \( F \) is an infinite field. By \( k \) we mean \( \mathbb{Z} \) or a prime field and by \( \mathbb{Z}/l \) we mean \( \mathbb{Z}/l\mathbb{Z} \), where \( l \) is a positive integer. If \( A \to A' \) is a homomorphism of abelian groups, by \( A'/A \) we mean \( \text{coker}(A \to A') \).

### 1. The spectral sequences

Let \( C_h(F^n) \) be the free \( k \)-module with a basis consisting of \( \langle (v_0), \ldots, (v_h) \rangle \), where the vectors \( v_0, \ldots, v_h \in F^n \) are in general positions, that is every \( \min\{h + 1, n\} \) of them is linear independent. By \( \langle v_i \rangle \) we mean the line passing through vectors \( v_i \) and 0. Let \( \partial_0 : C_0(F^n) \to C_{-1}(F^n) := k, \sum_i n_i((v_i)) \to \sum_i n_i \) and \( \partial_h = \sum_{i=0}^h (-1)^i d_i : C_h(F^n) \to C_{h-1}(F^n), h \geq 1, \)

where \( d_i((\langle v_0 \rangle, \ldots, \langle v_h \rangle)) = (\langle v_0 \rangle, \ldots, \langle v_{i-1} \rangle, \langle v_i \rangle, \ldots, \langle v_h \rangle) \). It is easy to see that the complex

\[
\begin{align*}
0 &\leftarrow k \leftarrow C_0(F^n) \xleftarrow{\partial_1} \cdots \xleftarrow{\partial_{n-1}} C_{n-1}(F^n) \xleftarrow{\partial_n} C_n(F^n) \leftarrow \cdots
\end{align*}
\]

is exact. Consider the following exact sequence

\[
\begin{align*}
0 &\leftarrow k \leftarrow C_0(F^n) \xleftarrow{\partial_1} \cdots \xleftarrow{\partial_{n-1}} C_{n-1}(F^n) \leftarrow H_{n-1}(X_n, k) \leftarrow 0,
\end{align*}
\]

where \( H_{n-1}(X_n, k) := \ker(\partial_{n-1}) \). We consider \( C_i(F^n) \) as left \( GL_n \)-module in a natural way. If it is necessary we convert this action to the right action by the definition \( m.g := g^{-1}m \) for \( g \in GL_n \) and \( m \in C_i(F^n) \).

**Remark 1.1.** Let \( \mathcal{U}(\mathbb{P}^{n-1}) \) be the simplicial set whose for \( 0 \leq h \leq n-1 \) non-degenerate \( h \)-simplices are of the form \( (\langle v_0 \rangle, \ldots, \langle v_h \rangle) \) as in the above and whose face operators are given by \( d_i \). Let \( X_n \) be the geometric realization of \( \mathcal{U}(\mathbb{P}^{n-1}) \). It is well-known that the complex

\[
\begin{align*}
0 &\leftarrow C_0(F^n) \xleftarrow{\partial_1} \cdots \xleftarrow{\partial_{n-1}} C_{n-1}(F^n) \leftarrow 0
\end{align*}
\]

computes the homology of \( X_n \) with coefficient in \( k \). Hence \( H_0(X_n, k) = k, H_i(X_n, k) = 0 \) if \( i \neq 0, n-1 \) and \( H_{n-1}(X_n, k) = \ker(\partial_{n-1}) \).

The exact sequence \( 2 \) induces a first quadrant spectral sequence converging to zero with

\[
E^1_{p,q}(n) = \begin{cases} 
H_q(F^p \times GL_{n-p}, k) & \text{if } 0 \leq p \leq n \\
H_q(GL_n, H_{n-1}(X_n, k)) & \text{if } p = n + 1 \\
0 & \text{if } p \geq n + 2.
\end{cases}
\]
For $1 \leq p \leq n$, and $q \geq 0$ the differential $d^1_{p,q}(n)$ equals $\sum_{i=1}^{p} (-1)^{i+1} H_i(\alpha_{i,p})$
where $\alpha_{i,p} : F^{*p} \times GL_{n-p} \to F^{*p-1} \times GL_{n-p+1}$,
$$(a_1, \ldots, a_p, A) \mapsto (a_1, \ldots, \hat{a_i}, \ldots, a_p, \left( \begin{array}{cc} a_i & 0 \\ 0 & A \end{array} \right))$$
(see the proof of [9, Thm. 3.5] for details). In particular for $0 \leq p \leq n$,
$$d^1_{p,0}(n) = \begin{cases} \text{id}_k & \text{if } p \text{ is odd} \\ 0 & \text{if } p \text{ is even,} \end{cases}$$
so $E^2_{p,0}(n) = 0$ for $p \neq n, n + 1$. In fact this is also true for $p = n, n + 1$.
Applying the right exact functor $H_0$ to the exact sequence
$$C_{n+1}(F^n) \to C_n(F^n) \to H_{n-1}(X_n, k) \to 0$$
we get the exact sequence
$$H_0(GL_n, C_{n+1}(F^n)) \to H_0(GL_n, C_n(F^n)) \to H_0(GL_n, H_{n-1}(X_n, k)) \to 0.$$
The group $GL_n$ acts transitively on the basis $\langle \langle v_0, \ldots, v_n \rangle \rangle$ of $C_n(F^n)$ so $H_0(GL_n, C_n(F^n)) = k$. From this we obtain
$$H_0(GL_n, H_{n-1}(X_n, k)) = \begin{cases} 0 & \text{if } n \text{ is odd} \\ k & \text{if } n \text{ is even.} \end{cases}$$
Consider $F^{n-2}$ as a vector subspace of $F^n$ generated by $e_3, e_4, \ldots, e_n$ (so $GL_{n-2}$ embeds in $GL_n$ as $\text{diag}(1, 1, GL_{n-2})$). Let $L'_*$ and $L_*$ be the complexes
$$0 \leftarrow 0 \leftarrow 0 \leftarrow k \leftarrow C_0(F^{n-2}) \leftarrow \cdots \leftarrow H_{n-3}(X_{n-2}, k) \leftarrow 0$$
$$0 \leftarrow k \leftarrow C_0(F^n) \leftarrow C_1(F^n) \leftarrow C_2(F^n) \leftarrow \cdots \leftarrow H_{n-1}(X_n, k) \leftarrow 0$$
respectively, that is $L'_0 = 0$, $L'_1 = 0$, $L'_2 = k$, $L'_{i+3} = C_i(F^{n-2})$ for $i = 0, \ldots, n - 3$, $L'_{n+1} = H_{n-3}(X_{n-2}, k)$ and $L'_i = 0$ for $i \geq n + 2$, $L_0 = k$, $L_{i+1} = C_i(F^n)$ for $i = 0, \ldots, n - 1$, $L_{n+1} = H_{n-1}(X_n, k)$ and $L_i = 0$ for $i \geq n + 2$. Define the map of complexes $L'_* \xrightarrow{\theta} L_*$, given by
$$(\langle v_1, \ldots, v_j \rangle) \mapsto \theta_k (\langle e_1, \langle v_1, \ldots, v_j \rangle \rangle - (\langle e_1, e_1 + e_2, \langle v_1, \ldots, v_j \rangle \rangle)
+ \langle e_2, e_1 + e_2, \langle v_1, \ldots, v_j \rangle \rangle)$$
This induces a map of bicomplexes
$$L'_* \otimes_{GL_{n-2}} F'_* \to L_* \otimes_{GL_n} F_* \to L_0 \otimes_{GL_n} F_*/L'_* \otimes_{GL_{n-2}} F'_*,$$
where $F_* : \cdots \to F_2 \xrightarrow{\delta_2} F_1 \xrightarrow{\delta_1} F_0 \to 0$ is a free left $GL_n$-resolution of $k$ and $F'_*$ is $F_*$ as $GL_{n-2}$-resolution. Thus one gets the map of spectral sequences
$$E'^r_{p,q}(n) \to E^r_{p,q}(n) \to E'^r_{p,q}(n),$$
where all the three spectral sequences converge to zero. By a similar approach as in the proof of [9, Thm. 3.5], one sees that the spectral sequence $E^r_{p,q}(n)$ is of the form

$$E^r_{p,q}(n) = \begin{cases} E_{p-2,q}(n-2) & \text{if } p \geq 2 \\ 0 & \text{if } p = 0, 1. \end{cases}$$

It is not difficult to see that for $2 \leq p \leq n$ and $q \geq 0$ the map $E^r_{p,q}(n) \rightarrow E^r_{p,q}(n)$ is the map $H_q(\text{inc})$ induced by $\text{inc} : F^{s_p-2} \times \text{GL}_{n-p} \rightarrow F^{s_p} \times \text{GL}_{n-p}$ with $A \mapsto (1, 1, A)$ and by a little work one sees that

$$E^r_{p,q}(n) = E^1_{p,q}(n)/E^r_{p,q}(n)$$

for $0 \leq p \leq n$ (see [9, Section 4]).

From the exact sequence of complexes

$$0 \rightarrow L'_p \otimes_{\text{GL}_{n-2}} F'_* \rightarrow L_p \otimes_{\text{GL}_n} F_* \rightarrow \ldots \rightarrow E^r_{p,q}(n) \rightarrow E^r_{p,q}(n-1) \rightarrow \ldots \rightarrow E^1_{p,0}(n) \rightarrow \ldots \rightarrow E^1_{p,0}(n) \rightarrow 0,$$

we obtain the long exact sequence

$$(4) \quad \ldots \rightarrow E^r_{p,q}(n) \rightarrow E^r_{p,q}(n) \rightarrow E^r_{p,q}(n) \rightarrow E^r_{p,q-1}(n) \rightarrow \ldots \rightarrow E^1_{p,0}(n) \rightarrow E^1_{p,0}(n) \rightarrow E^1_{p,0}(n) \rightarrow 0.$$ This exact sequence is studied in the above if $0 \leq p \leq n, q \geq 0$ and $p = n + 1, q = 0$. We will come back to it later.

Here is an important lemma which is used in the proof of Theorem 3.4.

**Lemma 1.2.** $E^2_{n,i}(n) = 0$ for all $i$. In particular $E^2_{n,i}(n) = E^2_{n,i}(n) = 0$.

**Proof.** Consider the following commutative diagram with exact columns

$$
\begin{array}{ccc}
0 & \rightarrow & H_{n-1}(X_n, k) \otimes_{\text{GL}_n} F_{i+1} \\
\downarrow & & \downarrow \\
H_{n-1}(X_n, k) \otimes_{\text{GL}_n} F_i & \rightarrow & H_{n-1}(X_n, k) \otimes_{\text{GL}_n} F_{i-1} \\
\downarrow & & \downarrow \\
C_{n-1}(F^n) \otimes_{\text{GL}_n} F_{i+1} & \rightarrow & C_{n-1}(F^n) \otimes_{\text{GL}_n} F_i \\
\downarrow & & \downarrow \\
C_{n-2}(F^n) \otimes_{\text{GL}_n} F_{i+1} & \rightarrow & C_{n-2}(F^n) \otimes_{\text{GL}_n} F_i \\
\end{array}
$$

Set $\sigma_j = (e_1, \ldots, e_{j+1})$ for $0 \leq j \leq n-1$. Let $\sigma_{n-1} \otimes x \in C_{n-1}(F^n) \otimes_{\text{GL}_n} F_i$ represent an element of the group $H_i(\text{GL}_n, C_{n-1}(F^n)) \simeq H_i(F^{x,n}, k)$ such that $d^1_{n,j}(\sigma_{n-1} \otimes x) = 0$. Then

$$(\partial_{n-1}(\sigma_{n-1} \otimes x) = \partial_{n-1}(\sigma_{n-1}) \otimes x \in \text{im}(\text{id}_{C_{n-2}} \otimes \delta_{i+1}).$$

Let $\sigma_{n-2} \otimes \delta_{i+1}(y) = \partial_{n-1}(\sigma_{n-1}) \otimes x$. It is easy to see that

$$\sigma_{n-2} \otimes \delta_{i+1}(y) = \sigma_{n-2} \otimes \left( \sum_{i=0}^{n-1} (-1)^i y_i x \right),$$
where $g_i \in \text{GL}_n$ is the permutation matrix such that
\[ g_i^{-1}(e_1, \ldots, e_{i+1}, \ldots, e_n, e_{i+1}) = (e_1, \ldots, e_n). \]

The inclusions $F^{*n} \subseteq \text{Stab}_{\text{GL}_n}(\sigma_{n-2}) \subseteq \text{GL}_n$ induce the commutative diagram
\[
\begin{array}{c}
\kron{F^{*n}} \kron{F_{i+1}} \\ \downarrow \\
\kron{F_i} \\ \downarrow \\
\kron{F_{i-1}} \\
\end{array} \xrightarrow{\sim} 
\begin{array}{c}
\kron{\text{Stab}_{\text{GL}_n}(\sigma_{n-2})} \kron{F_{i+1}} \\ \downarrow \\
\kron{\text{Stab}_{\text{GL}_n}(\sigma_{n-2})} \kron{F_i} \\ \downarrow \\
\kron{\text{Stab}_{\text{GL}_n}(\sigma_{n-2})} \kron{F_{i-1}} \\
\end{array} \xrightarrow{\sim} 
\begin{array}{c}
\kron{C_{n-2}(\text{GL}_n)} \kron{F_{i+1}} \\ \downarrow \\
\kron{C_{n-2}(\text{GL}_n)} \kron{F_i} \\ \downarrow \\
\kron{C_{n-2}(\text{GL}_n)} \kron{F_{i-1}} \\
\end{array}
\]

Since $H_i(F^{*n}, k) \simeq H_i(\text{Stab}_{\text{GL}_n}(\sigma_{n-2}), k)$, there is a $y' \in F_{i+1}$ such that
\[ 1 \otimes \delta_{i+1}(y') = 1 \otimes \left( \sum_{i=0}^{n-1} (-1)^i g_i x \right) \in k \otimes F^{*n} F_i. \]

Let $\sigma_n = (\langle e_1 \rangle, \ldots, \langle e_n \rangle, \langle e_1 + \cdots + e_n \rangle)$. Clearly $\partial_n(\sigma_n) \in H_n-1(X_n, k)$. If $z = \partial_n(\sigma_n) \otimes x$, then
\[
(j \otimes \text{id}_{F_i})(z) = (-1)^n \sigma_n \otimes x + \sigma_n \otimes \left( \sum_{i=0}^{n-1} (-1)^i g_i x \right)
\]
\[ = (-1)^n \sigma_n \otimes x + \sigma_n \otimes \left( \sum_{i=0}^{n-1} (-1)^i g_i x \right), \]

where $j : H_{n-1}(X_n, k) \hookrightarrow C_{n-1}(\text{GL}_n)$, $\tilde{\sigma}_{n-1} = (\langle e_1 \rangle, \ldots, \langle e_{n-1} \rangle, \langle e_1 + \cdots + e_n \rangle)$ and $g \in \text{Stab}_{\text{GL}_n}(\sigma_{n-2})$ with $g^{-1} \sigma_{n-1} = \sigma_{n-1}$. Since $1 \otimes g_i x = 1 \otimes g_i x$ in $k \otimes \text{Stab}_{\text{GL}_n}(\sigma_{n-2}) F_i$, there exist $y'' \in F_{i+1}$ such that
\[ 1 \otimes \sum_{i=0}^{n-1} (-1)^i g_i x = 1 \otimes \sum_{i=0}^{n-1} (-1)^i g_i x + 1 \otimes \delta_{i+1}(y'') \in k \otimes F^{*n} F_i. \]

Now in $C_{n-1}(\text{GL}_n) \otimes_{\text{GL}_n} F_i$ we have
\[
(j \otimes \text{id}_{F_i})(z) = (-1)^n \sigma_n \otimes x + \sigma_n \otimes \sum_{i=0}^{n-1} (-1)^i g_i x
\]
\[ = (-1)^n \sigma_n \otimes x + \sigma_n \otimes \sum_{i=0}^{n-1} (-1)^i g_i x + \sigma_n \otimes \delta_{i+1}(y'')
\]
\[ = (-1)^n \sigma_{n-1} \otimes x + \sigma_{n-1} \otimes \delta_{i+1}(y' + y''). \]

This completes the proof of the triviality of $E^2_{n,i}(n)$. The triviality of $E'^2_{n,i}(n)$ follows immediately from this, because $E''_{n,i}(n) = E^2_{n-2,i}(n-2)$. The triviality of $E''_{n,i}(n)$ follows from these and applying the Snake lemma to the
following commutative diagram with exact rows
\[
\begin{array}{ccc}
E^1_{n+1,i}(n) & \rightarrow & E^1_{n+1,i}(n) \\
\downarrow & & \downarrow \\
0 & \rightarrow & \ker(d^1_{n,i}(n)) \\
\end{array}
\]

Thus we get the exact sequence
\[
0 \rightarrow \ker(d^1_{n,i}(n)) \rightarrow \ker(d^1_{n,i}(n)) \rightarrow \ker(d^{n+1}_{n,i}(n)) \rightarrow 0.
\]

2. Higher pre-Bloch groups

In this section we define the higher pre-Bloch groups \( p_n(F) \) and investigate some of its properties.

**Definition 2.1.** Set \( t_n^{(k)}(F) := H_1(\text{GL}_n, H_{n-1}(X_n, k)) \). We denote \( t_n^{(Z)}(F) \) by \( t_n(F) \). By convention \( t_n^{(k)}(F) = 0 \) for \( n = 0, 1 \).

From the short exact sequence
\[
0 \rightarrow \partial_{n+1}(C_{n+1}(F^n)) \xrightarrow{\alpha} C_n(F^n) \xrightarrow{\beta} H_{n-1}(X_n, \mathbb{Z}) \rightarrow 0
\]
on one obtains the long exact sequence
\[
\cdots \rightarrow H_1(\text{GL}_n, \partial_{n+1}(C_{n+1}(F^n))) \xrightarrow{H_1(\alpha)} H_1(\text{GL}_n, C_n(F^n)) \xrightarrow{H_1(\beta)} t_n(F) \rightarrow H_0(\text{GL}_n, \partial_{n+1}(C_{n+1}(F^n))) \rightarrow \cdots.
\]

If \( n \) is even, the composition
\[
F^* = H_1(\text{GL}_n, C_n(F^n)) \xrightarrow{H_1(\beta)} t_n(F) \xrightarrow{H_1(\beta)} H_1(\text{GL}_n, C_{n-1}(F^n)) = F^{*n}
\]
is given by \( H_1(j) \circ H_1(\beta)(a) = (a, \ldots, a) \in F^{*n} \), where \( j : H_{n-1}(X_n, k) \rightarrow C_{n-1}(F^n) \). Thus \( H_1(\beta) \) is injective. If \( n \) is odd, the composition
\[
H_1(\text{GL}_n, C_{n+1}(F^n)) \rightarrow H_1(\text{GL}_n, \partial_{n+1}(C_{n+1}(F^n))) \xrightarrow{H_1(\alpha)} H_1(\text{GL}_n, C_n(F^n))
\]
is surjective and so \( H_1(\alpha) \) is surjective. Now from the above long exact sequence one gets the following exact sequences
\[
0 \rightarrow F^* \rightarrow t_n(F) \rightarrow H_0(\text{GL}_n, \partial_{n+1}(C_{n+1}(F^n))) \rightarrow 0, \quad \text{if } n \text{ is even,}
\]
\[
0 \rightarrow t_n(F) \rightarrow H_0(\text{GL}_n, \partial_{n+1}(C_{n+1}(F^n))) \rightarrow \mathbb{Z} \rightarrow 0, \quad \text{if } n \text{ is odd.}
\]

To study \( H_0(\text{GL}_n, \partial_{n+1}(C_{n+1}(F^n))) \) apply the functor \( H_0 \) to
\[
C_{n+2}(F^n) \rightarrow C_{n+1}(F^n) \rightarrow \partial_{n+1}(C_{n+1}(F^n)) \rightarrow 0.
\]

Thus we get the exact sequence
\[
(C_{n+2}(F^n))_{\text{GL}_n} \rightarrow (C_{n+1}(F^n))_{\text{GL}_n} \rightarrow H_0(\text{GL}_n, \partial_{n+1}(C_{n+1}(F^n))) \rightarrow 0.
\]

Let \( E_n = \sum_{i=1}^{n} e_i \in F^{*n}, a = \sum_{i=1}^{n} a_i e_i \in F^{*n} \), where \( a_i \in F^* \setminus \{1\} \) and \( a_i \neq a_j \) if \( i \neq j \). Denote the orbit of the frame \( \langle e_1, \ldots, e_n, (E_n), (a) \rangle \in C_{n+1}(F^n) \) by \( p(a) \) and orbit of the frame \( \langle e_1, \ldots, e_n, (E_n), (a), (b) \rangle \in C_{n+2}(F^n) \) by \( p(a, b) \).
$C_{n+2}(F^n)$ by $p(a, b)$, where $b = \sum_{i=1}^n b_i e_i \in F^{*n}$, $b_i \in F^* - \{1\}$, $b_i \neq b_j$ if $i \neq j$ and $a_i \neq b_j$ for all $i, j$. We see that

$$(C_{n+1}(F^n))_{\text{GL}_n} = \prod_a \mathbb{Z}_p(a), \quad (C_{n+2}(F^n))_{\text{GL}_n} = \prod_{a, b} \mathbb{Z}_p(a, b).$$

A direct computation shows that

$$\overline{\partial_{n+2}(p(a, b))} = \sum_{i=1}^n (-1)^i + 1 p\left(\frac{b_1 - b_i}{a_1 - a_i}, \ldots, \frac{b_i - b_i}{a_i - a_i}, \ldots, \frac{b_n - b_i}{a_n - a_i}, \frac{b_i}{a_i}\right) + (-1)^n p\left(\frac{b_1}{a_1}, \ldots, \frac{b_n}{a_n}\right) - (-1)^n p(a) + (-1)^n p(b).$$

Therefore $H_0(\text{GL}_n, \partial_{n+1}(C_{n+1}(F^n)))$ is generated by $[a_1; \ldots; a_n] \in \mathbb{P}^{n-1}$, $a_i \in F^* - \{1\}$, $a_i \neq a_j$ if $i \neq j$, and relations

$$[b_1; \ldots; b_n] - [a_1; \ldots; a_n] + \left[\frac{b_1}{a_1}; \ldots; \frac{b_n}{a_n}\right] - \sum_{i=1}^n (-1)^i + 1 \left[\frac{b_1 - b_i}{a_1 - a_i}; \ldots; \frac{b_i - b_i}{a_i - a_i}; \ldots; \frac{b_n - b_i}{a_n - a_i}; \frac{b_i}{a_i}\right] = 0,$$

where $b_i$ are as above. If in the above we replace $a_i/a_n$ and $b_i/b_n$ with $a_i$ and $b_i$ respectively, one sees that the group $H_0(\text{GL}_n, \partial_{n+1}(C_{n+1}(F^n)))$ is generated by the symbols $[a_1, \ldots, a_{n-1}]$, $a_i \in F^* - \{1\}$, $a_i \neq a_j$ if $i \neq j$ and relations

$$[b_1, \ldots, b_{n-1}] - [a_1, \ldots, a_{n-1}] + \left[\frac{b_1}{a_1}; \ldots; \frac{b_{n-1}}{a_{n-1}}\right] - \left[\frac{b_1 - 1}{a_1 - 1}; \ldots; \frac{b_{n-1} - 1}{a_{n-1} - 1}\right] - \sum_{i=1}^{n-1} (-1)^{i + 1} \left[\frac{b_i b_i - 1}{a_i a_i - 1}; \ldots; \frac{b_i b_i - 1}{a_i a_i - 1}; \ldots; \frac{b_i b_i - 1}{a_i a_i - 1}; \frac{b_i - 1}{a_i - 1}\right] = 0,$$

where $b_i \in F^* - \{1\}$, $b_i \neq b_j$ if $i \neq j$ and $a_i \neq b_j$ for all $i, j$. When $n = 2$ this is the definition of $p(F)$.

Thus one can think of $H_0(\text{GL}_n, \partial_{n+1}(C_{n+1}(F^n)))$ as a natural generalization of $p(F)$ for $n \geq 3$. So we allow ourself to make the following definition.

**Definition 2.2.** The group $H_0(\text{GL}_n, \partial_{n+1}(C_{n+1}(F^n)))$ is called the $n$-th pre-Bloch group of $F$ and we denote it by $p_n(F)$.

From the above we have the following exact sequences

$$0 \rightarrow F^* \rightarrow t_n(F) \rightarrow p_n(F) \rightarrow 0, \quad \text{if } n \text{ is even},$$

$$0 \rightarrow t_n(F) \rightarrow p_n(F) \rightarrow \mathbb{Z} \rightarrow 0, \quad \text{if } n \text{ is odd}.$$

**Remark 2.3.** In [18, 2.7] Yagunov defines another version of higher pre-Bloch groups, denoted by $\varphi^n(F)$. He also defines the classical pre-Bloch
group $\varphi^n(F)_{cl}$. Our definition of the pre-Bloch group is very close to his definition of the classical pre-Bloch group. In fact

$$\varphi^n(F)_{cl} = \begin{cases} \ker(p_n(F) \to \mathbb{Z}) & \text{if } n \text{ is odd} \\ p_n(F) & \text{if } n \text{ is even.} \end{cases}$$

See [18, 3.11] for the relation between $\varphi^n(F)_{cl}$ and $\varphi^n(F)$.

Since $E_{n+1,1}^1(n) = t_n(F)$ for $k = \mathbb{Z}$, from the exact sequence (1) we have the following exact sequence

$$(7) \quad t_{n-2}(F) \to t_n(F) \to E_{n+1,1}^1(n) \to 0.$$ 

An easy calculation shows that $\ker(d_{n,1}^1(n)) \subseteq F^* \oplus p_n(F)$ is generated by elements of the form

$$A = \begin{cases} (a_1, a_2, \ldots, a_j) & \text{if } n = 2j \\ (a_1, 1, \ldots, a_j, 1, \prod_{i=1}^j a_i^{-1}) & \text{if } n = 2j + 1. \end{cases}$$

This proves that $\ker(d_{n,1}^1(n)) \simeq F^{* \lfloor n/2 \rfloor}$. So we have a surjective map $t_n(F) \to F^{* \lfloor n/2 \rfloor}$. Using (7) we obtain a surjective map $E_{n+1,1}^1(n) \to F^*$. It is not difficult to see that the composition $t_n(F) \to E_{n+1,1}^1(n) \to F^*$ splits the exact sequence (5) for $n$ even. So we have proved the following lemma.

**Lemma 2.4.** Let $n \geq 2$. $t_n(F) \simeq F^* \oplus p_n(F)$ if $n$ is even and $p_n(F) \simeq \mathbb{Z} \oplus t_n(F)$ if $n$ is odd.

In the following lemma we give some satisfactory description of the group $t_n(F), \ n = 3, 4,$ for arbitrary infinite field $F$. This also gives a better description of $p_n(F)$ for $n = 3, 4$.

**Lemma 2.5.** Let $F$ be an infinite field. Then

(i) $t_2(F) \simeq F^* \oplus p_2(F)$,

(ii) $t_3(F) \simeq F^*$, therefore $p_3(F) \simeq F^* \oplus \mathbb{Z}$,

(iii) $t_4(F) \simeq F^* \oplus p_4(F)$ and there is an exact sequence

$$t_2(F) \to t_4(F) \to F^* \to 1.$$ 

**Proof.** (i) This part has already been proven in Lemma 2.4.

(ii) By Lemma 1.2 and [10, Cor. 3.5] the $E_{p,q}^2(3)$-terms are of the following form

* * 
0 0 * * * 0 
0 0 0 0 * 0 
0 0 0 0 $E_{4,1}^2(3)$ 0 
0 0 0 0 0 0 0 0 0 0 0 0 0 0 ...

By a similar arguments as in the proof of [10, Lemma 3.6] we have $E_{0,4}^3(3) = 0$. Since the spectral sequence converges to zero, $E_{4,1}^2(3) = E_{4,1}^\infty(3) = 0.$
Therefore $t_3(F) \simeq F^*$.

(iii) For $n = 4$ we look at the spectral sequence $E_{p,q}^2(4)$. By Lemma 1.2 and [11] Thm. 5.5 the $E_{p,q}^2(4)$-terms are of the following form

\[
\begin{array}{cccccc}
* & * & 0 & 0 & * & 0 \\
0 & 0 & * & * & 0 & * \\
0 & 0 & 0 & 0 & * & 0 \\
0 & 0 & 0 & 0 & 0 & E_{5,1}^2(4) \\
0 & 0 & 0 & 0 & 0 & 0 & \ldots
\end{array}
\]

With a similar argument to the case $n = 3$, using the results of [11] one can show that $E_{0,5}^3(4) = 0$ (see the proof of [10] Lemma 3.6)). With a little bit work one can prove that $E_{2,3}^2 = 0$ (see [10] or [11] to get an idea how one can do that). An easy analysis of the spectral sequence shows that $E_{5,1}^2(4) = E_{5,1}^\infty(4) = 0$. The exact sequence follows from this and the exact sequence (7). \qed

### 3. Homology of $GL_n$ with finite coefficient

In this section we show that over an algebraically closed field $F$ the pre-Bloch group $p_n(F)$ is closely related to the homology of $GL_n$ with finite coefficient.

**Lemma 3.1.** Let $F$ be an infinite field, $k$ a prime field and assume that $n \geq 3$, $j \geq 0$ be integers such that $n + 1 \geq j$. Let $H_q(\text{inc}) : H_q(GL_{n-2}, k) \to H_q(GL_{n-1}, k)$ be surjective for $0 \leq q \leq j - 1$. Then the following conditions are equivalent;

(i) $H_j(\text{inc}) : H_j(GL_{n-1}, k) \to H_j(GL_n, k)$ is surjective,

(ii) $H_j(\text{inc}) : H_j(F^* \times GL_{n-1}, k) \to H_j(GL_n, k)$ is surjective.

**Proof.** See [9] Lem. 4.1]. \qed

**Lemma 3.2.** Let $F$ be an infinite field, $k$ a prime field and assume that $n \geq 3$, $j \geq 0$ be integers such that $n \geq j$. Let $H_q(\text{inc}) : H_q(GL_{m-1}, k) \to H_q(GL_m, k)$ be isomorphism for $m = n, n-1$ and $0 \leq q \leq \min\{j-1, m-2\}$. Then the following conditions are equivalent;

(i) $H_j(\text{inc}) : H_j(GL_{m-1}, k) \to H_j(GL_m, k)$ is bijective,

(ii) $H_j(F^* \times GL_{m-2}, k) \xrightarrow{\tau_2} H_j(F^* \times GL_{m-1}, k) \xrightarrow{\tau_1} H_j(GL_m, k) \to 0$ is exact, where $\tau_1 = H_j(\text{inc})$ and $\tau_2 = H_j(\alpha) - H_j(\text{inc})$, $\alpha : (a,b,A) \mapsto (b, \text{diag}(a,A))$.

**Proof.** See [9] Lem. 4.2]. \qed

**Proposition 3.3** (Stability). Let $F$ be an algebraically closed field. Then $H_m(GL_{n-1}, \mathbb{Z}/l) \to H_m(GL_n, \mathbb{Z}/l)$ is surjective if $m \leq n$ and is injective if $m \leq n - 1$.

**Proof.** These results are already known and immediately follow from Suslin’s homological stability theorem [16] Thm. 3.4]. But for this special case we
give a proof that is much easier than Suslin’s proof. Here we may assume that \( l \) is a prime. The proof is by induction on \( n \). If \( n = 1 \) then everything is obvious. Assume the induction hypothesis, that is \( H_m(\text{GL}_{j-1}, \mathbb{Z}/l) \to H_m(\text{GL}_j, \mathbb{Z}/l) \) is surjective if \( m \leq j \) and is bijective if \( m \leq j - 1 \), where \( 1 \leq j \leq n - 1 \). Consider the spectral sequence \( E^{pq}_{2,2}(n) \) with \( k = \mathbb{Z}/l \). It is sufficient to prove that \( E^{pq}_{2,2}(n) = 0 \) if \( p + q \leq n + 1 \), \( 0 \leq q \leq n - 2 \) and \( E^{pq}_{2,2,n-1}(n) = 0 \). Because then we obtain \( E^{pq}_{2,0,m}(n) = 0 \) for \( 0 \leq m \leq n \) and \( E^{pq}_{1,1}(n) = 0 \) for \( 0 \leq m \leq n - 1 \) and by applying Lemmas \( \text{[4.1]} \) and \( \text{[3.2]} \) we get the desired results. The proof is analogue (and even easier) than the proof of \([9, \text{Thm. 4.3}]\). So we refer the reader to the proof of that theorem. Note that here one must use the fact that \( H_{2i+1}(F^*, \mathbb{Z}/l) = 0 \) for \( i \geq 0 \), since \( F^* \) is a divisible group \([2, \text{Prop. 4.7}]\). 

**Theorem 3.4.** Let \( F \) be algebraically closed. The following conditions are equivalent

(i) \( H_m(\text{GL}_{n-1}, \mathbb{Z}/l) \to H_m(\text{GL}_n, \mathbb{Z}/l) \) is injective for \( m = n \) and is surjective for \( m = n + 1 \).

(ii) \( p_n(F) \otimes \mathbb{Z}/l = \begin{cases} \mathbb{Z}/l & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even.} \end{cases} \)

**Proof.** We may assume that \( l \) is a prime. Again we look at the spectral sequence \( E^{pq}_{2,2}(n) \) with \( k = \mathbb{Z}/l \). By Lemma \( \text{[4.2]} \) \( E^{pq}_{2,2}(n) = 0 \). By Proposition \( \text{[3.3]} \) and a similar argument as the proof of \([9, \text{Thm. 4.3}]\) one can show that \( E^{pq}_{2,2}(n) = 0 \) for \( p + q = n + 2 \), where \( 3 \leq q \leq n \). Since the spectral sequence converges to zero one sees that \( E^{pq}_{2,2,n+1}(n) = 0 \) if and only if \( E^{pq}_{2,1,n}(n) = 0 \) and \( E^{pq}_{1,2,n}(n) = 0 \). Note that by Lemma \( \text{[2.4]} \) \( p_n(F) \) has the desired property if and only if \( t_n(F) \otimes \mathbb{Z}/l = 0 \) if and only if \( t_n(\mathbb{Z}/l)(F) = 0 \) (use Remark \( \text{[1.1]} \)). By Proposition \( \text{[1.1]} \) this theorem is true for \( n = 2, 3 \). Thus by induction we may assume that the theorem is true for lower cases.

By \([7]\) and the induction step \( E^{pq}_{2,2,n+1,1}(n) = 0 \) if and only if \( t_n(\mathbb{Z}/l)(F) = 0 \). By Lemmas \( \text{[3.1]} \) and \( \text{[3.2]} \) we have \( E^{pq}_{2,1,n}(n) = E^{pq}_{0,2,1,n}(n) = 0 \) if and only if the map \( H_m(\text{GL}_{n-1}, \mathbb{Z}/l) \to H_m(\text{GL}_n, \mathbb{Z}/l) \) is injective for \( m = n \) and is surjective for \( m = n + 1 \). This completes the proof of the theorem. 

So it is convenient to make the following conjecture (which easily follows from Conjecture \( \text{[3.3]} \) using Lemma \( \text{[2.4]} \)).

**Conjecture 3.5.** Let \( F \) be algebraically closed. Then

\[ p_n(F) \otimes \mathbb{Z}/l = \begin{cases} \mathbb{Z}/l & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even.} \end{cases} \]

**Remark 3.6.** (i) By a theorem of Dupont and Sah Conjecture \( \text{[3.5]} \) is true for \( n = 2 \) \([3, \text{App. A}]\) and by lemma \( \text{[2.5]} \) it is also true for \( n = 3, 4 \).

(ii) Conjecture \( \text{[3.5]} \) is related very closely to a conjecture of Yagunov \([18, \text{Conj. 0.2}]\). In \([18]\) he defines certain pre-Bloch groups \( \varphi^n(F) \) and conjectures...
that they are divisible. By Remark 2.3 and [18, Prop. 3.11], up to 2-torsion, Conjecture 3.5 implies Conjecture 0.2 from [18].

In the rest of this section we prove certain results that support Conjecture 3.5. First a theorem due to Friedlander.

**Theorem 3.7.** Let $\overline{F}_q$ be the algebraic closure of the finite field $F_q$. Then

(i) $H_i(\text{GL}_n(\overline{F}_q)) \rightarrow H_i(\text{GL}_{n+1}(\overline{F}_q))$ is isomorphism for $i \leq 2n - 1$,

(ii) $H_i(\text{SL}_n(\overline{F}_q)) \rightarrow H_i(\text{SL}_{n+1}(\overline{F}_q))$ is isomorphism for $i \leq 2n - 1$.

**Proof.** See [4, Thm. 3]. □

**Corollary 3.8.** (i) Conjecture 3.5 is true for $F = \overline{F}_q$.

(ii) Let $n \geq 3$. Then $t_n(\overline{F}_q)$ is a torsion divisible group. In particular

\[ p_n(\overline{F}_q) \otimes \mathbb{Q} = \begin{cases} \mathbb{Q} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even}. \end{cases} \]

**Proof.** (i) By 2.5 we may assume $n \geq 4$. The conjecture follows from Theorems 3.7 and 3.4.

(ii) Again we may assume $n \geq 4$. Look at the spectral sequence $E_{r,s}^1(n)$ with $k = \mathbb{Q}$. Since $\overline{F}_q$ is torsion,

\[ E_{r,s}^1(n) = \begin{cases} H_s(\text{GL}_{n-r}(\overline{F}_q), \mathbb{Q}) & \text{if } 0 \leq r \leq n \\ H_s(\text{GL}_n(\overline{F}_q), H_{n-1}(X_n, \mathbb{Q})) & \text{if } r = n + 1 \\ 0 & \text{if } r \geq n + 2. \end{cases} \]

Now by an easy analysis of this spectral sequence, using 3.7, we see that $p_n(\overline{F}_q)$ is torsion. The rest follows from Lemma 2.4. □

**Corollary 3.9.** Let $\text{char}(F) \neq 0$. Then the following conditions are equivalent

(i) $p_n(F) \otimes \mathbb{Z}/l = \begin{cases} \mathbb{Z}/l & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$ for all $n$,

(ii) $H_n(\text{GL}_{n-1}, \mathbb{Z}/l) \sim H_n(\text{GL}_n, \mathbb{Z}/l)$ for all $n$.

**Proof.** It is sufficient to prove that in part (i) of Theorem 3.4 the surjectivity follows from the injectivity. If $\text{char}(F) = p \neq 0$, then it contains a copy of $\mathbb{F}_p$. Consider the commutative diagram

\[
\begin{array}{ccc}
H_{n+1}(\text{GL}_{n-1}(\overline{F}_p), \mathbb{Z}/l) & \rightarrow & H_{n+1}(\text{GL}_{n-1}(F), \mathbb{Z}/l) \\
\downarrow & & \downarrow \\
H_{n+1}(\text{GL}_n(\overline{F}_p), \mathbb{Z}/l) & \rightarrow & H_{n+1}(\text{GL}_n(F), \mathbb{Z}/l) \\
\downarrow & & \downarrow \\
H_{n+1}(\text{GL}(\overline{F}_p), \mathbb{Z}/l) & \sim & H_{n+1}(\text{GL}(F), \mathbb{Z}/l).
\end{array}
\]

By 3.7 the left column maps are bijective. By a theorem of Suslin [14, Cor. 1] the bottom row map is bijective. Now the claim follows easily. □
For a topological group $G$ let $BG^{\text{top}}$ be its classifying space with its underlying topology and $BG$ be its classifying space as a topological group with discrete topology. By the functorial property of $B$ we have a natural map $\psi : BG \to BG^{\text{top}}$.

**Conjecture 3.10** (Friedlander-Milnor Conjecture). *Let $G$ be a Lie group. The canonical map $\psi : BG \to BG^{\text{top}}$ induces isomorphism of homology and cohomology with any finite abelian coefficient group.*

See [7] and [12] for more information in this direction. Here is a strong support for Conjecture 3.5.

**Proposition 3.11.** The following conditions are equivalent

(i) $p_n(\mathbb{C}) \otimes \mathbb{Z}/l = \begin{cases} \mathbb{Z}/l & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$ for all $n$,

(ii) $H_n(\text{GL}_{n-1}(\mathbb{C}), \mathbb{Z}/l) \xrightarrow{\sim} H_n(\text{GL}_n(\mathbb{C}), \mathbb{Z}/l)$ for all $n$,

(iii) $H_n(B\text{GL}_{n-1}(\mathbb{C}), \mathbb{Z}/l) \xrightarrow{\sim} H_n(B\text{GL}_{n-1}(\mathbb{C})^{\text{top}}, \mathbb{Z}/l)$ for all $n$.

**Proof.** It is well-known that

\[ \text{GL}_{n-1}(\mathbb{C}) \to \text{GL}_n(\mathbb{C}) \to \text{GL}_n(\mathbb{C})/\text{GL}_{n-1}(\mathbb{C}) \]

is a fibration and $\text{GL}_n(\mathbb{C})/\text{GL}_{n-1}(\mathbb{C})$ is $(2n-2)$-connected [8 Thm. 3.15]. Hence $\pi_i(\text{GL}_{n-1}(\mathbb{C})) \to \pi_i(\text{GL}_n(\mathbb{C}))$ is injective if $i \leq 2n-3$ and surjective if $i \leq 2n-2$ which imply that

\[ \pi_j(B\text{GL}_{n-1}(\mathbb{C})^{\text{top}}) \to \pi_j(B\text{GL}_n(\mathbb{C})^{\text{top}}) \]

is injective if $j \leq 2n-2$ and surjective if $j \leq 2n-1$. Therefore

\[ H_j(B\text{GL}_{n-1}(\mathbb{C})^{\text{top}}, \mathbb{Z}) \to H_j(B\text{GL}_n(\mathbb{C})^{\text{top}}, \mathbb{Z}) \]

is injective for $j \leq 2n-2$ and surjective for $j \leq 2n-1$ [13 Chap. 7, Sec. 5. Thm. 9]. We call this topological stability.

Here we prove (i) $\iff$ (ii). The proof of (ii) $\iff$ (iii) is similar and easier. (i) $\Rightarrow$ (ii) This immediately follows from [3,4 (ii) $\Rightarrow$ (i) By [3,4 it is sufficient to prove that

\[ H_{n+1}(\text{GL}_{n-1}(\mathbb{C}), \mathbb{Z}/l) \to H_{n+1}(\text{GL}_n(\mathbb{C}), \mathbb{Z}/l) \]

is surjective. For this we look at the following commutative diagram

\[ \begin{array}{ccc} H_{n+1}(B\text{GL}_{n-1}(\mathbb{C}), \mathbb{Z}/l) & \to & H_{n+1}(B\text{GL}_{n-1}(\mathbb{C})^{\text{top}}, \mathbb{Z}/l) \\ \downarrow & & \downarrow \\ H_{n+1}(B\text{GL}_n(\mathbb{C}), \mathbb{Z}/l) & \to & H_{n+1}(B\text{GL}_n(\mathbb{C})^{\text{top}}, \mathbb{Z}/l) \\ \downarrow & & \downarrow \\ H_{n+1}(B\text{GL}(\mathbb{C}), \mathbb{Z}/l) & \xrightarrow{\sim} & H_{n+1}(B\text{GL}(\mathbb{C})^{\text{top}}, \mathbb{Z}/l). \end{array} \]

By a theorem of Suslin the last row is isomorphism [13 Cor. 4.8] and by topological stability the column maps in the right are isomorphism for $n \geq 3$. 
By a result of Milnor, for any Lie group $G$ with a finite number of connected components the map

$$H_i(BG, \mathbb{Z}/l) \rightarrow H_i(BG^{\text{top}}, \mathbb{Z}/l)$$

is always surjective \cite[Thm. 1]{7}. So the row maps of the diagram are surjective. By (ii) and \cite[3.3]{3.3} the bottom column map in the left is isomorphism, so the middle row map is isomorphism. All these imply that the first column map in the left of the diagram is surjective. \qed

Remark 3.12. The original goal of Loday to introduce the higher pre-Bloch groups in \cite{5} was that it might help one to study $H_{n+1}^{\text{GL}_n}/H_{n+1}(\text{GL}_{n-1})$, which is motivated by the Bloch-Wigner exact sequence and also by a result of Suslin which describes the quotient group $H_n(\text{GL}_n)/H_n(\text{GL}_{n-1})$ explicitly \cite[Thm. 3.4]{16}.

It is easy to define a natural map

$$\eta_n : H_{n+1}(\text{GL}_n)/H_{n+1}(\text{GL}_{n-1}) \rightarrow \mathfrak{p}_n(F).$$

This map can be constructed using exact sequence \cite{2}. From the short exact sequence $0 \rightarrow \partial_1(C_1(F^n)) \rightarrow C_0(F^n) \rightarrow \mathbb{Z} \rightarrow 0$ we get the connecting homomorphism $H_{n+1}(\text{GL}_n) \rightarrow H_n(\text{GL}_n, \partial_1(C_1(F^n)))$. Iterating this process we get a homomorphism $\eta_n : H_{n+1}(\text{GL}_n) \rightarrow t_n(F)$. Since the epimorphism $C_0(F^n) \rightarrow \mathbb{Z} \rightarrow 0$ has a $\text{GL}_{n-1}$-equivariant section $m \mapsto m(\langle e_n \rangle)$, the restriction of $\eta_n$ to $H_{n+1}(\text{GL}_{n-1})$ is zero. Thus we obtain a homomorphism

$$H_{n+1}(\text{GL}_n)/H_{n+1}(\text{GL}_{n-1}) \rightarrow t_n(F).$$

The Composition of $\eta_n$ with the map $t_n(F) \rightarrow \mathfrak{p}_n(F)$, constructed in the previous section, gives us the map that we are looking for. This map also can be constructed on the level of complexes. For details of this approach see \cite{18}.

In the light of Conjecture \cite{3.5}, it is convenient to ask the following question.

**Question.** Let $F$ be algebraically closed. Is $H_{n+1}(\text{GL}_n)/H_{n+1}(\text{GL}_{n-1})$ divisible?

By \cite{3.7} the answer to this question is positive if $F = \overline{\mathbb{Q}}$ and in the next section we show that the answer also is positive for $n \leq 4$. Using Theorem \cite{3.4} one can show that Conjecture \cite{3.5} gives a positive answer to the above question for $n$ even (see the proof of \cite{4.4}(ii)).

4. Lower degree homology groups

Here we demonstrate that the equivalence conditions in Theorem \cite{3.4} are true for $n \leq 4$. In this section we assume that $F$ is algebraically closed, unless we mention it.

**Proposition 4.1.** We have

(i) $H_2(\text{GL}_2, \mathbb{Z}/l) \cong H_2(\text{GL}_2, \mathbb{Z}/l) = 0 = H_3(\text{GL}_1, \mathbb{Z}/l) \rightarrow H_3(\text{GL}_2, \mathbb{Z}/l)$

(ii) $H_3(\text{GL}_2, \mathbb{Z}/l) \cong H_3(\text{GL}_3, \mathbb{Z}/l)$ and $H_4(\text{GL}_2, \mathbb{Z}/l) \rightarrow H_4(\text{GL}_3, \mathbb{Z}/l)$,
(iii) $H_4(GL_4, \mathbb{Z}/l) \sim H_4(GL_4, \mathbb{Z}/l)$ and $H_5(GL_3, \mathbb{Z}/l) \to H_5(GL_4, \mathbb{Z}/l)$,
(iv) $H_4(GL_3)/H_4(GL_2)$ and $H_5(GL_4)/H_5(GL_3)$ are divisible.

Proof. By Theorem 3.4 to proof (i), (ii) and (iii) it is sufficient to prove that $p_2(F)$ and $p_4(F)$ are divisible and $p_3(F) \otimes \mathbb{Z}/l = \mathbb{Z}/l$. Dupont and Sah [3 Thm. 5.1] proved that $p_2(F)$ is divisible. The rest follows from this and Lemma 2.5. We should mention that to prove Lemma 2.5 we used the main results of [10] and [11], which are difficult since are on an arbitrary infinite field. The proof of those results has great simplification over an algebraically closed field and homology with $\mathbb{Z}/l$ coefficient, for example the proof of Lemmas 5.2, 5.3 and 5.4 in [11] are easy (some even trivial) as $H_{2i+1}(F^*, \mathbb{Z}/l) = 0$ for $i \geq 0$.

(iv) Consider the following commutative diagram with exact rows

$$
\begin{array}{c}
0 \to H_4(GL_2) \otimes \mathbb{Z}/l \to H_4(GL_2, \mathbb{Z}/l) \to \text{Tor}_1^2(H_3(GL_2), \mathbb{Z}/l) \to 0 \\
\downarrow \downarrow \downarrow \\
0 \to H_4(GL_3) \otimes \mathbb{Z}/l \to H_4(GL_3, \mathbb{Z}/l) \to \text{Tor}_1^2(H_3(GL_3), \mathbb{Z}/l) \to 0.
\end{array}
$$

Since $H_3(GL_3) \simeq H_3(GL_2) \oplus K_*^M(F)$ [10 Cor. 5.5], and since $K_*^M(F)$ is uniquely divisible for $i \geq 2$ [11, 5.2], the right column map is isomorphism. The middle column map is surjective by (ii). Thus the left column map is surjective too. This shows that $H_4(GL_3)/H_4(GL_2)$ is $l$-divisible. The proof of divisibility of the group $H_5(GL_4)/H_5(GL_3)$ is analogue and for this one should use (iii) and [11 Cor. 5.7].

Proposition 4.2. (i) $H_3(SL_2, \mathbb{Z}/l) = H_3(SL_3, \mathbb{Z}/l) = 0$, so $H_3(SL_2)$ and $H_3(SL_3)$ are divisible,
(ii) $H_4(SL_2, \mathbb{Z}/l) \to H_4(SL_3, \mathbb{Z}/l) \to H_4(SL_4, \mathbb{Z}/l)$ and $H_5(SL_3, \mathbb{Z}/l) \to H_5(SL_4, \mathbb{Z}/l)$,
(iii) $H_4(SL_3)/H_4(SL_2)$ and $H_5(SL_4)/H_5(SL_3)$ are divisible.

Proof. Since $H_i(SL, \mathbb{Z}/l) \sim H_i(GL, \mathbb{Z}/l)$ for all $i$, by homology stability theorem 5.3 and Prop. 4.1(i) one gets $H_3(SL_3, \mathbb{Z}/l) = 0$. The exact sequence

$$
1 \to SL_n \to GL_n \to F^* \to 1
$$

induces the Lyndon-Hochschild-Serre spectral sequence

$$
nE^2_{p,q} = H_p(F^*, H_q(SL_n, \mathbb{Z}/l)) \Rightarrow H_{p+q}(GL_n, \mathbb{Z}/l).
$$

It is easy to see that $H_q(SL_n, \mathbb{Z}/l) = 0$ for $q = 1, 2$, thus $nE^2_{p,q} = 0$ for $q = 1, 2$. Triviality of $H_i(SL_n, \mathbb{Z}/l)$ for $i$ odd, implies that for $p$ odd

$$
nE^2_{p,q} = H_p(F^*, H_q(SL_n, \mathbb{Z}/l)) = H_p(F^*, \mathbb{Z}/l) \otimes H_q(SL_n, \mathbb{Z}/l) = 0.
$$

Since the above exact sequence splits, $nE_{p,0}^2$ are trivial maps. Thus $2E^2_{0,3} = 2E^2_{0,3} = H_3(SL_2, \mathbb{Z}/l) = H_3(GL_2, \mathbb{Z}/l) = 0$, which imply $2E^2_{2,3} = 0$. So we obtain the exact sequences

$$
0 \to H_4(SL_2, \mathbb{Z}/l) \to H_4(GL_2, \mathbb{Z}/l) \to H_4(F^*, \mathbb{Z}/l) \to 0,
$$

(8)
(9) \(2E_{2,4}^2 \rightarrow H_5(\text{SL}_2, \mathbb{Z}/l) \rightarrow H_5(\text{GL}_2, \mathbb{Z}/l) \rightarrow 0.\)

With an analogue argument for \(nE_{p,q}^2, n = 3, 4,\) we obtain the exact sequences

(10) \(0 \rightarrow H_4(\text{SL}_3, \mathbb{Z}/l) \rightarrow H_4(\text{GL}_3, \mathbb{Z}/l) \rightarrow H_4(F^*, \mathbb{Z}/l) \rightarrow 0,\)

(11) \(3E_{2,4}^2 \rightarrow H_5(\text{SL}_3, \mathbb{Z}/l) \rightarrow H_5(\text{GL}_3, \mathbb{Z}/l) \rightarrow 0,\)

(12) \(0 \rightarrow H_4(\text{SL}_4, \mathbb{Z}/l) \rightarrow H_4(\text{GL}_4, \mathbb{Z}/l) \rightarrow H_4(F^*, \mathbb{Z}/l) \rightarrow 0,\)

(13) \(4E_{2,4}^2 \rightarrow H_5(\text{SL}_4, \mathbb{Z}/l) \rightarrow H_5(\text{GL}_4, \mathbb{Z}/l) \rightarrow 0.\)

If \(m \leq n,\) then there is a natural map of spectral sequences

\[mE_{p,q}^2 \rightarrow nE_{p,q}^2.\]

Now the isomorphism \(H_4(\text{SL}_3, \mathbb{Z}/l) \sim H_4(\text{SL}_4, \mathbb{Z}/l)\) can be deduced from the natural map from exact sequence (10) to exact sequence (12) and the corresponding result for GL in Proposition 4.1. This isomorphism implies that \(3E_{2,4}^2 \sim 4E_{2,4}^2.\) From this, (11(iii)) and exact sequences (11) and (13) we obtain the surjectivity \(H_5(\text{SL}_3, \mathbb{Z}/l) \rightarrow H_5(\text{SL}_4, \mathbb{Z}/l).\) The proof of (iii) is analogue to the case GL in 4.1(iv) using [10, Cor. 6.2] and [11, Prop. 5.8].

Here is a new case of the Friedlander-Milnor Conjecture.

**Corollary 4.3.** Let \(G = \text{GL}_3(\mathbb{C})\) or \(\text{SL}_3(\mathbb{C}).\) Then for any finite abelian groups \(A,\)

\[H_4(BG, A) \sim H_4(BG^{\text{top}}, A).\]

**Proof.** We may assume \(A = \mathbb{Z}/l,\) where \(l\) is a prime. Now a similar argument as in the proof of Proposition 3.11 using 4.1 and 4.2 will prove this claim. □

**Corollary 4.4.** Let \(n \geq 3.\) Then

(i) \(H_4(\text{SL}_n, \mathbb{Z}/l) \sim \begin{cases} 0 & \text{if } \text{char}(F) \mid l \\ \mathbb{Z}/l & \text{if } \text{char}(F) \nmid l, \end{cases}\)

(ii) \(H_4(\text{SL}_n)\) is uniquely divisible.

**Proof.** We may assume that \(l\) is a prime. By a result of Suslin, the \(K\)-theory of algebraically closed fields with finite coefficient, \(K_i(F, \mathbb{Z}/l),\) does not depend on the field and \(K_i(F, \mathbb{Z}/l)\) is trivial if

(1) \(i\) is odd,

(2) \(i \geq 1\) when \(\text{char}(F) = l \neq 0\)

(see [14] and [15, Cor. 3.13]). This implies that the group \(H_i(\text{SL}_3, \mathbb{Z}/l)\) does not depend on \(F\) and \(H_i(\text{SL}_n, \mathbb{Z}/l)\) is trivial in the above cases (see [14, Cor. 1, Cor. 2]).

(i) To prove this claim it is sufficient to prove it for \(F = \mathbb{C}.\) It is well-known
that $\text{SL}_n(\mathbb{C})$, as a Lie group, is 2-connected and $\pi_3(\text{SL}_n(\mathbb{C})) \simeq \mathbb{Z}$. This implies that $\text{BSL}_n(\mathbb{C})^{\text{top}}$ is 3-connected and $\pi_4(\text{BSL}_n(\mathbb{C})^{\text{top}}) \simeq \mathbb{Z}$. Therefore

$$H_4(\text{BSL}_n(\mathbb{C})^{\text{top}}, \mathbb{Z}) \simeq \mathbb{Z}.$$ 

From Cor. 4.3 we have

$$H_4(\text{SL}_n(\mathbb{C}), \mathbb{Z}/l) \simeq H_4(\text{BSL}_n(\mathbb{C}), \mathbb{Z}/l) \simeq H_4(\text{BSL}_n(\mathbb{C})^{\text{top}}, \mathbb{Z}/l) \simeq \mathbb{Z}/l.$$

(ii) The exact sequence $0 \rightarrow \mathbb{Z} \rightarrow \mathbb{Z} \rightarrow \mathbb{Z}/l \rightarrow 0$ induces the long exact sequence

$$\cdots \rightarrow H_4(\text{SL}) \rightarrow H_4(\mathbb{SL}) \rightarrow H_4(\text{SL}, \mathbb{Z}/l) \rightarrow H_3(\text{SL}) \rightarrow \cdots.$$

For $n \geq 4$ the claim follows from (i), the triviality of $H_5(\text{SL}, \mathbb{Z}/l)$ (and of $H_4(\text{SL}, \mathbb{Z}/l)$ if $\text{char}(F) = l \neq 0$) and the following fact

$$H_3(\text{SL}) \simeq V \oplus \begin{cases} \mathbb{Q}/\mathbb{Z} & \text{if char}(F) = 0 \\ \mathbb{Q}/\mathbb{Z}[\frac{1}{p}] & \text{if char}(F) = p \neq 0, \end{cases}$$

where $V$ is a uniquely divisible group [17]. The case $n = 3$ follows from this and the fact that $H_4(\text{SL}_4) \simeq H_4(\text{SL}_3) \oplus K_4^M(F)$ [11] Prop. 5.8. Note that $K_4^M(F)$ is uniquely divisible.

\textbf{Example 4.5.} (i) Propositions 4.1 and 4.2 are true if $F = \mathbb{R}$ and $2 \nmid l$. Because then $p_2(\mathbb{R})$ is divisible [12 2.14, 4.1(a)] and $H_i(\mathbb{R}, \mathbb{Z}/l) = 0$ for $i \geq 1$. For example, a similar argument as in the above shows that the groups

$$H_4(\text{GL}_3(\mathbb{R}))/H_4(\text{GL}_2(\mathbb{R})) \quad \text{and} \quad H_5(\text{GL}_4(\mathbb{R}))/H_5(\text{GL}_3(\mathbb{R}))$$

are $l$-divisible (see [10 Cor. 5.5], [11 Example 1]). The same is true if one replaces GL with SL.

(ii) Corollary 4.4, homology stability theorem 3.3 and exact sequences [10] and [12] imply that for $n \geq 3$

$$H_4(\text{GL}_n, \mathbb{Z}/l) \simeq \mathbb{Z}/l \oplus H_4(\text{F}^*, \mathbb{Z}/l) \simeq \mathbb{Z}/l \oplus \mathbb{Z}/l.$$

Using [4.4] and [11 Cor. 5.7] it is easy to prove that $H_4(\text{GL}_n)$ is uniquely divisible for $n \geq 3$.

5. SOME GENERALIZATIONS

One can generalize Theorem 3.4 as follows;

\textbf{Proposition 5.1.} Let $F$ be an infinite field and let $k = \mathbb{Q}$ or $k = \mathbb{Z}/l$, $l$ a prime, such that $K_2^M(F) \mathbin{\hat{\otimes}} k = 0$. Then the following are equivalent;

(i) For $3 \leq m \leq n$ the map $H_m(\text{GL}_{m-1}, k) \rightarrow H_m(\text{GL}_m, k)$ is injective and the map $H_{m+1}(\text{GL}_{m-1}, k) \rightarrow H_{m+1}(\text{GL}_m, k)$ is surjective.

(ii) For $3 \leq m \leq n$, the complex $k^{(m)}_{m-2}(F) \rightarrow k^{(m)}_m(F) \rightarrow H_1(\text{F}^*, k) \rightarrow 0$ is exact.

\textbf{Proof.} Clearly $K_i^M(F) \mathbin{\hat{\otimes}} k = 0$ for $i \geq 2$. The proof is similar to the proof of Theorem 3.4. We leave the details to the reader. \qed
**Example 5.2.** Here are some examples of pairs of fields \((F, k)\) such that \(K^M_2(F) \otimes k = 0:\)

1. \(F\) any global field and \(k = \mathbb{Q}\) (see [1]),
2. \(F\) a perfect field of \(\text{char}(F) = l \neq 0\) and \(k = \mathbb{Z}/l\) (see [1]),
3. \(F\) algebraically closed and \(k = \mathbb{Z}/l\) (see Sections 3 and 1),
4. \(F\) a local field and \(k = \mathbb{Z}/l\) (see [6, Example 1.7]),
5. \(F = \mathbb{R}\) and \(k = \mathbb{Z}/l\), \(l \nmid |\mu(F)|\) (see [6, Example 1.6]),
6. \(F = \mathbb{F}_q\) and \(k\) any prime field. (see [6, Example 1.5])
7. \(F = \mathbb{Q}\) and \(k\) any prime field [2, Cor. 10.21].

To give more examples first we state a result of Milnor [6, §2].

**Theorem.** For every field \(F\) we have an exact sequence

\[
0 \to K^M_{n+1}(F) \to K^M_{n+1}(F(t)) \to \bigoplus_{P \in \text{Spec}(F[t])} K^M_n(F[t]/P) \to 0.
\]

The following pairs of fields \((F, k)\) with the desired property follow from this theorem and the above cases;

8. \(F = \mathbb{F}_q(T)\) or \(F = \mathbb{F}_q(T)\) and \(k = \mathbb{Q}\) or \(k = \mathbb{Z}/l\), \(l \neq \text{char}(F)\),
9. \(F = E(t)\), \(E\) algebraically closed and \(k = \mathbb{Z}/l\),
10. \(F = \mathbb{R}(t)\) and \(k = \mathbb{Z}/l, l \neq 2\).

By Lemma 2.4 Conjecture 3.5 immediately follows from the following conjecture.

**Conjecture 5.3.** Let \(n \geq 3\). Then \(t^{(k)}_{n-2}(F) \to t^{(k)}_n(F) \to H_1(F^*, k) \to 0\) is exact.

By Lemma 2.5 this conjecture is true for \(n = 3, 4\). By 3.8 it is also true for the algebraic closure of a finite field when \(k\) is a prime field. We should mention that the surjectivity of \(t^{(k)}_n(F) \to H_1(F^*, k)\) is proven in Section 2.

We have the following results analogue to Prop. 4.1.

**Proposition 5.4.** Let the pair \((F, k)\) be as in Example 5.2. Then

(i) \(H_3(\text{GL}_2, k) \to H_3(\text{GL}_3, k)\) and \(H_4(\text{GL}_2, k) \to H_4(\text{GL}_3, k)\),

(ii) \(H_4(\text{GL}_3, k) \to H_4(\text{GL}_4, k)\) and \(H_5(\text{GL}_3, k) \to H_5(\text{GL}_4, k)\).

**Proof.** The proof is similar to the proof of [1] using [2, 5] \(\Box\)

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