De Rham model for string topology

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0. Introduction. Let $M$ be a simply connected closed oriented $n$-dimensional manifold, and $LM := \text{Map}_{C^{\infty}(S^1,M)}$ the associated free loop space. String topology of Chas and Sullivan [CS] deals with an ample family of algebraic operations on the ordinary and equivariant homologies of $LM$, the most important being a graded commutative associative product,

$$\boxdot : H_p(LM) \otimes H_q(LM) \longrightarrow H_{p+q}(LM),$$
onumber

on the shifted ordinary homology$^1$, $H_\bullet(LM) := H_{\bullet+n}(LM)$. The product $\boxdot$ is compatible with (in general, non-commutative) Pontrjagin product,

$$\star : H_p(\Omega M) \otimes H_q(\Omega M) \longrightarrow H_{p+q}(\Omega M),$$

in the homology of the subspace, $\Omega M$, of $LM$ consisting of loops based at some fixed point in $M$, and the same product $\boxdot$ is also an extension of the classical intersection product in $H_\bullet(M)$: the natural chain of linear maps

$$(H_\bullet(M), \cap) \xrightarrow{\text{constant loops}} (H_\bullet(LM), \boxdot) \xrightarrow{\text{intersection with the subspace } \Omega M} (H_\bullet(\Omega M), \star)$$

is a chain of homomorphisms of algebras.

Let $(\Lambda^i_M = \bigoplus_{i=0}^n \Lambda^i_M, d)$ be the De Rham differential graded (dg, for short) algebra of $M$. With the help of simplicial techniques it was established in [J] that

$$H_\bullet(LM) = \text{Ext}_{\Lambda^\bullet_M \otimes \Lambda^\bullet_M}^\bullet (\Lambda_M, \Lambda^*_M),$$

i.e. that morphisms in the derived category, $\mathcal{D}(\Lambda_M)$, of $\Lambda_M$-bimodules between the two objects, $\Lambda_M$ and its dual $\Lambda^*_M := \text{Hom}_R(\Lambda_M, R)$, have a nice free loop space interpretation. As $M$ is closed oriented, the Poincare duality asserts that $\Lambda^*_M$ and $\Lambda_M[n]$ are isomorphic in $\mathcal{D}(\Lambda_M)$ so that (*) can be rewritten equivalently as

$$H_\bullet(LM) = \text{Ext}_{\Lambda^\bullet_M \otimes \Lambda^\bullet_M}^\bullet (\Lambda_M, \Lambda_M).$$

Moreover, as it was shown in [CJ, C] using ring spectra and simplicial methods, the latter isomorphism is actually an isomorphism of algebras,

$$(H_\bullet(LM), \boxdot) = \left( \text{Ext}_{\Lambda^\bullet_M \otimes \Lambda^\bullet_M}^\bullet (\Lambda_M, \Lambda_M), \text{Yoneda product} \right),$$

$^1$In this paper all homologies are assumed to be over $\mathbb{R}$. 
i.e. the Chas-Sullivan product is the same thing as the composition in the endomorphism ring of the object $\Lambda_M$ in the derived category of $\Lambda_M$-bimodules. The isomorphism $(**)$ was also studied in the thesis of Tradler [Tr].

This paper offers new geometrically transparent (and down-to-earth) proofs of the string topology main theorems which are based on the theory of iterated integrals [Ch]. In particular, the isomorphism $(\ast)$ gets incarnated as the holonomy map. Which combined with the Thom class of $M$ explains why $(**)$ is an isomorphism of algebras. As a by-product, the resulting De Rham model for string topology provides us with new algorithms for computing both the free loop space homology and the Chas-Sullivan product; see also [CJY, FTV, R] for other approaches.

In fact, our approach to string topology easily generalizes to its “brane” version. Let $f : Z \to M$ be a smooth map from a compact oriented $p$-dimensional manifold to a compact simply connected manifold $M$, and let $L_f$ be defined by the associated pullback diagram,

\[
\begin{array}{c}
L_f \\
\downarrow \\
\downarrow \\
Z \\
\downarrow \quad f \\
M.
\end{array}
\]

The shifted homology, $H_\bullet(L_f) := H_{\bullet+p}(L_f)$, is naturally an associative (but, in general, non-commutative) algebra with respect to the obvious analogue, $\circ$, of the Chas-Sullivan product.

If $Z$ is a point, then $L_f \simeq \Omega M$ and $(H_\bullet(L_f), \circ) = (H_\bullet(\Omega M), \ast)$. Chen [Ch] has found a nice model for the algebra $(H_\bullet(\Omega M), \ast)$ as the homology of the free differential algebra $(R[X], \bar{\partial})$, where $R[X] := \otimes \tilde{H}_\bullet(M)[1]$ is the tensor algebra generated by the reduced homology (with shifted degree) of $M$ and $\bar{\partial}$ is a differential on $R[X]$ which can be computed by a certain iterative procedure (see Lemma 2.1 below). In this paper we generalize Chen’s model of $(H_\bullet(\Omega M), \ast)$ to the case of the Chas-Sullivan algebra, $(H_\bullet(L_f), \circ)$, for an arbitrary smooth map $f : Z \to M$. More precisely, we establish two isomorphisms of algebras,

\[
(H_\bullet(L_f), \circ) \cong (\Lambda_Z \otimes \mathbb{R}[X], d_f),
\]

and

\[
(\Lambda_Z \otimes \mathbb{R}[X], d_f) \cong (\text{Hoch} \ast (\Lambda_M, \Lambda_Z), \text{Hochschild product}),
\]

where $\Lambda_Z$ is the De Rham algebra of $Z$ and $d_f$ is a certain twisting of the natural differential $d + \bar{\partial}$ on $\Lambda_Z \otimes \mathbb{R}[X]$. In the case $f = \text{Id} : M \to M$ the above two results imply $(**)$.

The paper is organized as follows: In Sections 1 and 2 we review basic facts of Chen’s theory of iterated integrals and of a formal power series connection. In Section 3 we use the latter to give a suitable model for the Hochschild cohomology of a dg algebra with finite dimensional cohomology and then illustrate its work in two standard examples. In Section 4 we use the theory of iterated integrals to give a new proof of the isomorphism theorem $(\ast)$. The sections 6-10 are devoted to a new proof of the isomorphism of algebras theorem $(**)$.

In the final section 11 we discuss the de Rham model for the string topology on a “brane” $f : Z \to M$. 

2
1. Iterated integrals. Let $M$ be a smooth $n$-dimensional manifold and $PM$ the space of piecewise smooth paths $\gamma : [0, 1] \to M$. For each simplex, $k = 0, 1, 2, \ldots$,

$$\Delta^k := \{(t_1, \ldots, t_k) \in \mathbb{R}^k \mid 0 \leq t_1 \leq t_2 \leq \ldots \leq t_k \leq 1\},$$

consider a diagram,

$$\xymatrix{PM \times \Delta^k \ar[r]^{ev_k} \ar[d]_{p} & M^{k+2} \\
PM}$$

where $p$ is just the projection on the first factor and $ev_k$ is the evaluation map,

$$ev_k : PM \times \Delta^k \longrightarrow M^{k+2}$$

$$\gamma \times (t_1, \ldots, t_k) \longrightarrow (\gamma(0), \gamma(t_1), \ldots, \gamma(t_k), \gamma(1)).$$

According to Chen [Ch], the space of iterated integrals, $Ch(PM)$, on $PM$ is a subspace of the de Rham space, $\Lambda_{PM}$, of smooth differential forms on $PM$. By definition, $Ch(PM)$ is the image of the following composition of pull-back and push-forward maps,

$$\bigoplus_{k=0}^{\infty} p_* \circ ev^+_k : \bigoplus_{k=0}^{\infty} \Lambda_M \otimes \Lambda^{\otimes k}_M \otimes \Lambda_M \longrightarrow \Lambda_{PM}.$$ 

Chen reserved a special symbol,

$$\int w_1 w_2 \ldots w_k := p_* \circ ev^+_k (1 \otimes w_1 \otimes w_2 \otimes \ldots \otimes w_k \otimes 1),$$

for the restriction of $p_* \circ ev^+_k$ to the subspace $1 \otimes \Lambda^{\otimes k}_M \otimes 1 \subset \Lambda_M \otimes \Lambda^{\otimes k}_M \otimes \Lambda_M$. The latter is a differential form on $PM$ of degree $|w_1| + \ldots + |w_k| - k$ where $|w_i|$ stands for the degree of the form $w_i \in \Lambda_M$, $i = 1, \ldots, k$. If we denote the composition of $ev_k$ with the projection, $M^{k+2} \to M$, to the first (respectively, last) factor by $\pi_0$ (respectively, $\pi_1$), then we can write,

$$Ch(PM) = \text{span} \left\{ \pi_0^* w_0 \wedge \int w_1 w_2 \ldots w_k \wedge \pi_1^* w_{k+1} \right\},$$

where $\{w_i\}_{0 \leq i \leq k+1}$ are arbitrary differential forms on $M$.

The Stoke’s theorem,

$$dp_* = p_* d + p_* |_{\partial \Delta^k},$$

implies [Ch]

$$d \int w_1 w_2 \ldots w_k = \sum_{i=1}^{k} (-1)^{|w_1| + \ldots + |w_{i-1}| - i} \int w_1 \ldots dw_i \ldots w_k$$

$$- \sum_{i=1}^{k-1} (-1)^{|w_1| + \ldots + |w_i| - i} \int w_1 \ldots (w_i \wedge w_{i+1}) \ldots w_k$$

$$- \pi_0^* w_1 \wedge \int w_2 \ldots w_k$$

$$+ (-1)^{|w_1| + \ldots + |w_{k-1}| - k + 1} \int w_1 \ldots w_{k-1} \wedge \pi_1^* w_k.$$
The wedge product of iterated integrals is again an iterated integral [Ch],
\[
\int w_1 \ldots w_k \wedge \int w_{k+1} \ldots w_{k+l} = \sum_{\sigma \in Sh(k,l)} (-1)^\sigma \int w_{\sigma(1)} w_{\sigma(2)} \ldots w_{\sigma(k+l)},
\]
where the summation goes over the set, $Sh(k,l)$, of all shuffle permutations of type $(k,l)$, and $(-1)^\sigma$ is the Koszul sign computed as usually but with the assumption that the symbols $w_i$ have shifted degree $|w_i| + 1$.

This all means that $Ch(PM)$ is a dg subalgebra of the de Rham algebra $\Lambda_{PM}$. In fact, this subalgebra (together with variants, $Ch(LM) := Ch(PM) |_{LM}$ and $Ch(\Omega M) := Ch(PM) |_{\Omega M}$) encodes much of the essential information about the free path space $PM$ and its more important subspaces, $LM$ and $\Omega M$, of free loops and based loops respectively:

1.1. Fact [Ch]. If $M$ is simply connected, then the cohomology of $Ch(\Omega M)$ (respectively, $Ch(LM)$) equals the de Rham cohomology $H^*(\Omega M)$ (respectively, $H^*(LM)$).

2. Formal power series connection and holonomy map. Let $(A, d)$ be a unital dg algebra over a field $k$ with finite dimensional cohomology, say $\dim H^*(A) = n + 1$. Let $\{[1], [e^i]\}_{1 \leq i \leq n}$ be a homogeneous basis of the graded vector space $H^*(A)$, and $\{1^*, x_i\}_{1 \leq i \leq n}$ the associated dual basis of the shifted dual vector space $Hom_k(H^*(A), k)[1]$. Let $k\langle X \rangle := k\{x_1, \ldots, x_n\}$ be the free graded associative algebra generated by non-commutative indeterminates $x_1, \ldots, x_n$, and $k\langle\langle X\rangle\rangle := k\langle\langle x_1, \ldots, x_n\rangle\rangle$ its formal completion. Finally, let $\{e^i\}$ be arbitrary lifts of the cohomology classes $\{[e^i]\}$ to cycles in $A$.

2.1. Lemma [Ch]. There exists a degree one element $\omega$ in the algebra $A \otimes k\langle\langle X\rangle\rangle$ and a degree one differential, $\partial$, of the algebra $k\langle\langle X\rangle\rangle$ such that

- $\partial I \subset I^2$, where $I$ is the maximal ideal in $k\langle\langle X\rangle\rangle$;
- $\omega \mod A \otimes I^2 = \sum_{i=1}^n e^i \otimes x_i$;
- the Maurer-Cartan equation, $d\omega + \partial \omega + \omega \omega = 0$,

is satisfied.

The proof goes by induction in the tensor powers of $I$ (cf. [Ch, H]).

2.2. Remark. If $(A, d)$ is a non-negatively graded connected and simply connected (i.e. $H^0(A, d) = k$ and $H^1(A, d) = 0$) differential algebra, then the above result holds true with $\omega \in A \otimes k\langle X\rangle \subset A \otimes k\langle\langle X\rangle\rangle$. This case is of major interest to us as it covers the particular example when $A$ is the De Rham algebra $\Lambda_M$ of a connected and simply connected compact manifold. A geometric significance of the differential $\partial$ for $A = \Lambda_M$ was discovered by Chen in the following beautiful

2.3. Fact [Ch]. Let $M$ be a compact simply connected manifold. Then there is a canonical isomorphism of algebras,

\[
(H_*(\Omega M), \text{Pontrjagin product } \ast) = H^*(\mathbb{R}\langle X\rangle, \partial).
\]

As opposite to $k\langle x_1, \ldots, x_n\rangle$, the free graded commutative associative algebra generated by homogeneous indeterminants $x_1, \ldots, x_n$ will be denoted by $k[x_1, \ldots, x_n]$. 

4
Chen’s proof of this statement employs the notion of the holonomy map,

\[ Hol : (C_\bullet(\Omega M), \partial) \rightarrow (\mathbb{R}\langle X \rangle, \partial) \]

\[ \text{simplex } \alpha : \Delta^k \rightarrow \Omega M \rightarrow (-1)^{\lvert \alpha \rvert} \int_{\Delta^k} \alpha^*(T), \]

from the complex of (negatively graded) smooth singular chains in the based loop space \( \Omega M \) to the complex \( (\mathbb{R}\langle X \rangle, \partial) \). The centerpiece is the restriction to \( \Omega M \) of the degree zero element, \( T \in \Lambda_{PM} \otimes k\langle \langle x_1, \ldots, x_n \rangle \rangle \), given by the following iterated integral

\[ T := 1 + \int \omega + \int \omega \omega + \int \omega \omega \omega + \ldots. \]

This element is called the transport of a formal power series connection \( \omega \) from Lemma 2.1, and has two useful properties [Ch]:

- \( dT + \partial T + \pi_0^*(\omega) \wedge T - T \wedge \pi_1^*(\omega) = 0 \), and
- given any two smooth simplices, \( \alpha_1 : \Delta^k \rightarrow PM \) and \( \alpha_2 : \Delta^l \rightarrow PM \), such that for any \( v \in \Delta^k, w \in \Delta^l \) the path \( \alpha_2(w) \) begins where the path \( \alpha_1(v) \) ends (so that the Pontrjagin product \( \alpha_1 \ast \alpha_2 : \Delta^k \times \Delta^l \rightarrow PM \) makes sense), then

\[ \int_{\Delta^k \times \Delta^l} (\alpha_1 \ast \alpha_2)^*(T) = \int_{\Delta^k} \alpha_1^*(T) \int_{\Delta^l} \alpha_2^*(T). \]

As \( T \) restricted to \( \Omega M \) satisfies \( dT = -\partial T \), it is now obvious that \( Hol : (C_\bullet(\Omega M), \partial) \rightarrow (\mathbb{R}\langle X \rangle, \partial) \) is a morphism of dg algebras. Some extra work involving Adam’s cobar construction shows that \( Hol \) is a quasi-isomorphism; see [Ch] for details.

2.4. Remark. The dual of the finitely generated dg algebra \( (k\langle \langle X \rangle \rangle, \partial) \) is a dg coalgebra, i.e. an \( A_\infty \)-structure on the vector space \( H^\bullet(A) \). The latter is precisely a cominimal\(^3\) model for the original dg algebra \((A, d)\) and Chen’s formal power series connection \( \omega \in A \otimes k\langle \langle X \rangle \rangle \) is nothing but an \( A_\infty \) morphism

\[ (H^\bullet(A), \text{cominimal } A_\infty \text{ structure}) \rightarrow (A, d). \]

Thus Chen’s Lemma 2.1 is tantamount to saying that the cohomology, if finite dimensional, of a dg \( k \)-algebra has an induced structure of an \( A_\infty \)-algebra. A fact proved later in greater generality and independently by Kadeishvili [K].

2.5. Remark. A formal power series connection \( \omega \) as described in Lemma 2.1 is not canonical but depends on a number of choices. There is, however, a canonical object underlying this power series which is best described using the language of non-commutative differential geometry.

First we replace the pair \( (k\langle \langle x_1, \ldots, x_n \rangle \rangle, \partial) \) by a germ of the \( n \)-dimensional formal non-commutative smooth graded manifold \( X \) equipped with a degree one smooth vector field \( \partial \).

\(^3\) An \( A_\infty \) structure on a vector space \( V \), that is a codifferential \( Q \) of the free coalgebra \( \otimes^*V[1] \), is called cominimal if the restriction of \( Q \) to the linear bit \( V[1] \subset \otimes^*V[1] \) vanishes. Any \( A_\infty \) algebra over a field of characteristic zero can be represented as a direct sum of a cominimal and a contractible ones, the former being determined uniquely up to an isomorphism.
This means essentially that we enlarge the automorphism group of \((k\langle x_1, \ldots, x_n \rangle, \partial)\) from \(GL(n, k)\) (a change of basis \(x_i \mapsto \sum a_{ij} x_j \) in \(H(A)[1]^{*}\)) to arbitrary formal diffeomorphisms \((x_i \mapsto a_{ij} x_j + a_{ijk} x_j x_k + \ldots)\), i.e. we forget the flat structure in \(H(A)[1]^{*}\).

Next we observe that \(\omega\) defines a flat connection, \(\nabla_{\partial} \simeq \partial + d + \omega\), along the vector field \(\partial\) in the trivial bundle over \(X\) with typical fibre \(A\). It is this flat \(\partial\)-connection \(\nabla_{\omega}\) which is defined canonically and invariantly. Chen’s formal power series \(\omega\) is nothing but a representative of \(\nabla_{\omega}\) in a particular coordinate chart \((x_1, \ldots, x_n)\) on \(X\).

This whole paper (more precisely, everything starting with Theorem 3.1 below) should have been written using the geometric language of flat \(\partial\)-connections\(^4\) \(\nabla_{\partial}\) on the non-commutative dg manifold \((X, \partial)\). After some hesitation we decided not to do it and work throughout in a particular coordinate patch \((x_1, \ldots, x_n)\); this makes our formulae more transparent but at a price of using a non-canonical, a choice of coordinates dependent object \(\omega\). Of course, ultimately nothing depends on that choice.

### 3. A model for Hochschild cohomology

Let \((A, d)\) be a unital dg \(k\)-algebra with finite dimensional cohomology. Lemma 2.1 says that there is a canonical (see the remark just above) deformation, \(d + \partial + \omega\), of the differential \(d\) in \(A \otimes k\langle \langle X \rangle \rangle\). We shall need below its Lie bracket version,

\[
d_{\omega}a := da + \partial a + [\omega, a], \quad \forall a \in A \otimes k\langle \langle X \rangle \rangle.
\]

Clearly, the Maurer-Cartan equation implies \(d_{\omega}^2 = 0\). If \((M, d)\) is a dg bi-module over \((A, d)\), then formally the same expression as above defines a differential \(d_{\omega}^M\) in \(M \otimes k\langle \langle X \rangle \rangle\).

The following result should be well-known though the author was not able to trace a reference.

#### 3.1. Theorem

(i) The Hochschild cohomology \(Hoch^\bullet(A, A)\) is canonically isomorphic as an associative algebra to the cohomology \(H^\bullet(A \otimes k\langle \langle X \rangle \rangle, d_{\omega})\).

(ii) The Hochschild cohomology \(Hoch^\bullet(A, M)\) is canonically isomorphic as a vector space to the cohomology \(H^\bullet(M \otimes k\langle \langle X \rangle \rangle, d_{\omega}^M)\).

**Proof.** We show the proof of (i) only (the proof of statement (ii) is analogous). The idea is simple: we shall construct a continuous morphism of topological dg algebras,

\[
(C^\bullet(A, A) := \bigoplus_{n \geq 1} Hom_k(A^\otimes n, A)[-n], \cup_{Hoch}, d_{Hoch}) \to (A \otimes k\langle \langle X \rangle \rangle, product \otimes product, d_{\omega}),
\]

which induces an isomorphism in cohomology.

Usually \(C^\bullet(A, A)\) gets identified with the vector space of coderivations of the bar construction on \(A\) as it nicely exposes the dg Lie algebra structure. We are interested, however, in the dg associative algebra structure on \(C^\bullet(A, A)\) and thus have to look for something else.

Let \(A_1\) and \(A_2\) be \(A_\infty\)-algebras, that is, a pair of co-free codifferential coalgebras \((B(A_1), \Delta, Q_1)\) and \((B(A_2), \Delta, Q_2)\), where \(B\) stands for the bar construction,

\[
B(A_i) := \bigoplus_{n \geq 1} (A_i[1])^\otimes n,
\]

\(^4\)Let \((X, \partial)\) be a smooth dg manifold, that is, a graded manifold \(X\) equipped with a degree one smooth vector field \(\partial\) such that \([\partial, \partial] = 0\). Let \(E\) be a vector bundle over \(X\) which we understand as a locally free sheaf of \(\mathcal{O}_X\)-modules, \(\mathcal{O}_X\) being the structure sheaf on \(X\). An \(\partial\)-connection on \(E\) is a linear map \(\nabla_{\partial} : E \to E\) such that \(\nabla_{\partial}(fe) = (-1)^{|f|} \nabla_{\partial}(e) + (\partial f)e\) for any \(f \in \mathcal{O}_X, e \in E\). It is called flat if \(\nabla^2_{\partial} = 0\).
\(\Delta\) for the coproduct,

\[
\Delta : \quad B(A_i) \quad \rightarrow \quad B(A_i) \otimes B(A_i)
\]

\[
[a_1|a_2|\ldots|a_n] \quad \rightarrow \quad \sum_{k=1}^{n-1} [a_1, \ldots, a_k] \otimes [a_{k+1} \ldots|a_n],
\]

and \(Q_i : B(A_i) \rightarrow B(A_i)\) for a degree 1 coderivation of \((B(A_i), \Delta)\) satisfying the equation \(Q_i^2 = 0\). For example, if, say, \(A_2\) is just a dg algebra (which we now assume from now on) with the differential \(d : A_2 \rightarrow A_2\) and the product \(\mu_2 : A_2 \otimes A_2 \rightarrow A_2\), then the associated codifferential is given by

\[
Q_2[a_1|a_2|\ldots|a_n] := \sum_{k=1}^{n} (-1)^{|a_1|+\ldots+|a_{k-1}|+1} [a_1|\ldots|da_k|\ldots|a_n]
\]

\[
- \sum_{k=1}^{n-1} (-1)^{|a_1|+\ldots+|a_k|} [a_1|\ldots|\mu_2(a_k, a_{k+1})|\ldots|a_n].
\]

The vector space of all coalgebra maps,

\[
\text{Comap}(A_1, A_2) := \text{Hom}_{\text{Coalg}}(B(A_1), B(A_2)),
\]

is naturally isomorphic to \(\text{Hom}_k(B(A_1), A_2[1]) = \Pi_{n \geq 1} Hom_k(A_1 \wedge^n, A_2)[1-n]\). Then one can turn \(\text{Comap}(A_1, A_2)[-1]\) into an associative algebra with the product given by (see e.g. [HMS])

\[
f \cup g := \mu_2 \circ (f \otimes g) \circ \Delta, \quad \forall f, g \in \text{Hom}_k(B(A_1), A_2).
\]

A remarkable fact is that every \(A_\infty\) morphism, i.e. every element of the set

\[
\text{Hom}_{A_\infty}(A_1, A_2) := \{\omega \in \text{Hom}_{\text{Coalg}}(B(A_1), B(A_2)) : |\omega| = 0, \omega \circ Q_1 = Q_2 \circ \omega\},
\]

makes \((\text{Comap}(A_1, A_2)[-1], \cup)\) into a dg algebra. Indeed, the defining equation \(\omega \circ Q_1 = Q_2 \circ \omega\) for \(\omega\) (understood as a degree 1 element of \(\text{Hom}_k(B(A_1), A_2)\)) takes the form of a twisting cochain equation,

\[
d \circ \omega + \omega \circ Q_1 + \omega \cup \omega \equiv D\omega + \omega \cup \omega = 0,
\]

where \(D\) is the natural differential in \(\text{Hom}_{\text{complexes}}((B(A_1), Q_1), (A_2, d))\).

Then \((\text{Comap}(A_1, A_2)[-1], \cup, \delta_\omega)\) with the differential defined by

\[
\delta_\omega(f) := Df + \omega \cup f - (-1)^{|f|}f \cup \omega
\]

is obviously a topological dg algebra.

For example, for any dg algebra \(A\), \((\text{Comap}(A, A)[-1], \cup, \delta_{Id})\) is nothing but the Hochschild dg algebra of \(A\).

Now we assume, in the above notations, that \(A_2\) is the dg algebra of Theorem 3.1 and \(A_1\) is its cohomology \(H(A, d)\) equipped with an induced \(A_\infty\)-structure \(Q\). As \(H(A, d)\) is finite-dimensional, the dualization of the standard completion of \((B(H(A, d)), Q)\) is a free topological dg algebra which is nothing but \((k(\langle X \rangle), \partial)\). Thus the dg algebra,

\[
(A \otimes k(\langle X \rangle), \text{product} \otimes \text{product}, d_\omega),
\]

7
is canonically isomorphic to \((\text{Comap}(H(A, d), A)[-1], \cup, \delta_\omega)\). On the other hand, the composition with the \(A_\infty\)-morphism \(\omega \in \text{Comap}(H(A, d), A)\) induces a natural continuous map

\[ \omega^*: \text{Comap}(A, A)[-1] \to \text{Comap}(H(A, d), A)[-1]. \]

A straightforward calculation shows that \(\omega^*\) is actually a continuous morphism of topological dg algebras,

\[ (\text{Comap}(A, A)[-1], \cup, \delta_{Id}) \to (\text{Comap}(H(A, d), A)[-1], \cup, \delta_\omega), \]

that is, a continuous morphism of the topological dg algebras

\[ (C^\bullet(A, A), \cup_{\text{Hoch}}, d_{\text{Hoch}}) \xrightarrow{\omega^*} (A \otimes k\langle\langle X\rangle\rangle, \text{product} \otimes \text{product}, d_\omega). \]

Being continuous, this morphism preserves the decreasing filtrations, \(C_{\geq r}(A, A)\) and \(A \otimes I^r, I\) being the maximal ideal in \(k\langle\langle X\rangle\rangle\), and, as it is easy to see, induces an algebra isomorphism of the \(E_1\) terms of the associated spectral sequences. As filtrations are complete and exhaustive, the spectral sequences converge to \(Hoch^\bullet(A, A)\) and \(H^\bullet(A \otimes k\langle\langle X\rangle\rangle, d_\omega)\) respectively. Then the Comparison Theorem establishes the required isomorphism. \(\square\)

3.2. Remark. If \(A\) is connected and simply connected, then \(k\langle\langle X\rangle\rangle\) can be replaced in the above theorem by \(k\langle X\rangle\).

3.3. Example. Let \(M\) be an \(n\)-dimensional sphere \(S^n\). Let \(A = \mathbb{R}[\nu]\) be a dg subalgebra of the de Rham algebra \(\Lambda_M\) generated over \(\mathbb{R}\) by a volume form \(\nu\). As \(A\) is quasi-isomorphic to \(\Lambda_M\), \(Hoch^\bullet(A, A) = Hoch^\bullet(\Lambda_M, \Lambda_M)\).

Clearly, the data \(\partial = 0, \omega = \nu \otimes x, |x| = 1 - n\), is a solution of the Maurer-Cartan equation for \(A\),

\[ d\omega + \partial\omega + \omega\omega = 0. \]

Hence, by Theorem 3.1, one gets immediately

\[ Hoch^\bullet(\Lambda_M, \Lambda_M) = \ker : [\nu \otimes x, ] : \mathbb{R}[\nu] \otimes \mathbb{R}[x] \to \mathbb{R}[\nu] \otimes \mathbb{R}[x] \]

\[ \text{im} : [\nu \otimes x, ] : \mathbb{R}[\nu] \otimes \mathbb{R}[x] \to \mathbb{R}[\nu] \otimes \mathbb{R}[x] \]

\[ = \begin{cases} \mathbb{R}[\nu, x]/(\nu^2), & |\nu| = n, |x| = 1 - n, \\ \mathbb{R}[\nu, \mu, \tau]/(\nu^2, \mu^2, \nu \mu, \nu \tau), & |\nu| = n, |\mu| = 1, |\tau| = 2 - 2n \text{ if } n \text{ is even.} \end{cases} \]

Which is in accordance with the earlier calculations [CJY, FTV]. The key to the \(n\) even case is

\[ \mu \simeq \nu \otimes x, \quad \tau \simeq 1 \otimes x^2. \]

3.4. Example. Let \(M\) be an \(n\)-dimensional complex projective space \(\mathbb{C}P^n\). Let \(A = \mathbb{R}[h]/h^{n+1}\) be a dg subalgebra of the de Rham algebra \(\Lambda_M\) generated over \(\mathbb{R}\) by the standard Kähler form \(h\). As \(A\) is quasi-isomorphic to \(\Lambda_M\), \(Hoch^\bullet(A, A) = Hoch^\bullet(\Lambda_M, \Lambda_M)\).

The data,

\[ \omega = \sum_{i=1}^{n} h^N \otimes x_i, \quad |x_i| = 1 - 2i, \]
and
\[ \delta = - \sum_{i=2}^{n} \sum_{j=1}^{i-1} x_j x_{i-j} \frac{\partial}{\partial x_i}, \]
give a solution of the Maurer-Cartan equations for \( A \). Plugging these into Theorem 3.1, one gets after a straightforward inspection,
\[ \text{Hoch}^* (\Lambda_M, \Lambda_M) = \mathbb{R}[h, \mu, \nu]/(h^{n+1}, h^n \mu, h^n \nu), |h| = 2, |\mu| = 1, |\nu| = -2n. \]
This is in agreement with a spectral sequence calculation made earlier in [CJY]. The key to the answer is
\[ h \simeq h \otimes 1, \quad \mu \simeq \sum_{i=1}^{n} i h^{\wedge i} \otimes x_i, \quad \nu \simeq \sum_{i+j=n+1} 1 \otimes x_i x_j. \]

### 4. Holonomy map for the free loop space.

Let \( M \) be a compact smooth manifold, and \( \omega \in \Lambda_M \otimes \mathbb{R} \langle \langle X \rangle \rangle \) a formal power series connection of the de Rham algebra \( (\Lambda_M, d) \) (see Lemma 2.1 and Remark 2.4). As \( \Lambda_M^* \) is naturally a dg bi-module over \( (\Lambda_M, d) \), the connection \( \omega \) gives rise to a differential, \( d^* \omega = d^* + \delta + [\omega, \ldots] \), on \( \text{Hom}_{\mathbb{R}}(\Lambda_M, \mathbb{R} \langle \langle X \rangle \rangle) \) (cf. section 3). Explicitly, for any \( f \in \text{Hom}_{\mathbb{R}}(\Lambda_M, \mathbb{R} \langle \langle X \rangle \rangle) \) and any \( \ldots \in \Lambda_M \),
\[ (d^*_\omega f) (\ldots) = -(-1)^{|f|} f(d \ldots) + \delta f(\ldots) + \sum_l (-1)^{|l|} [p_l(X), f(w_l \wedge \ldots)], \]
where we used a decomposition of a formal power series connection \( \omega \in \mathbb{R} \langle \langle X \rangle \rangle \otimes \Lambda_M \) into a sum of tensor products, \( \omega = \sum_l p_l(X) \otimes w_l \).

The transport \( T \) of \( \omega \) when restricted to the free loop subspace, \( LM \), of the path space \( PM \) satisfies the equation,
\[ dT + \delta T + \pi^*(\omega) T - T \pi^*(\omega) = 0, \]
where \( \pi : LM \to M \) is the evaluation map \( \pi_0 |_{LM} = \pi_1 |_{LM} \).

Let \((C^* (LM), \partial)\) be the complex of (negatively graded) smooth singular chains in the free loop space.

#### 4.1. Theorem. The map
\[ \text{Hol} : \quad (C^* (LM), \partial) \quad \rightarrow \quad (\text{Hom}_{\mathbb{R}}(\Lambda_M, \mathbb{R} \langle \langle X \rangle \rangle), d^*_\omega) \]
\[ \text{simplex } \alpha : \Delta^* \rightarrow LM \quad \rightarrow \quad \text{Hol}(\alpha) : \Lambda_M \rightarrow \mathbb{R} \langle \langle X \rangle \rangle \]
\[ (\ldots) \rightarrow (-1)^{|\alpha|} \int_{\Delta^*} \alpha^* (\pi^*(\ldots) \wedge T), \]
is a morphism of complexes. Moreover, it induces an isomorphism in cohomology if \( M \) is simply connected.

**Proof.** That \( \text{Hol} \) commutes with the differentials is nearly obvious,
\[ \text{Hol}(\partial \alpha) = (-1)^{|\alpha|-1} \int_{\partial \Delta^*} \alpha^* (\pi^*(\ldots) \wedge T) \]
\[ = (-1)^{|\alpha|-1} \int_{\Delta^*} d\alpha^* (\pi^*(\ldots) \wedge T) \]
\[ = (-1)^{|\alpha|} \int_{\Delta^*} \alpha^* (\pi^*(-d \ldots) \wedge T + \pi^*(\ldots) \wedge (\delta T + \pi^*(\omega) T - T \pi^*(\omega)) \]
\[ = d^*_\omega \text{Hol}(\alpha). \]
For the second part, consider two increasing filtrations,
\[ F_p \text{Hom}_R(\Lambda M, R\langle\langle X\rangle\rangle) := \{ f \in \text{Hom}_R(\Lambda M, R\langle\langle X\rangle\rangle) \mid f(w) = 0 \forall w \in \Lambda^{\geq p} M \} \]
and
\[ F_p C_*(LM) := \begin{cases} 
\text{simplexes } \Delta^\bullet \to LM \text{ whose composition with } \pi \\
\text{land in the skeleton } M^p \\
\text{for an appropriate smooth triangulation of } M 
\end{cases} \]
Clearly, the map Hol preserves the filtrations. The Leray-Serre spectral sequence associated with the filtration \( F_p C_*(LM) \) has
\[ E^2_{\bullet,\bullet} = H_\bullet(M) \otimes H_\bullet(\Omega M). \]
As \( M \) is simply connected, the formal power series connection \( w = \sum_l p_l(x_1, \ldots, x_n) \otimes w_l \) has all \( w_l \in \Lambda^{\geq 2} M \). Then using Fact 2.3, it is easy to compute the \( E^2 \) term of the spectral sequence associated to \( F_p \text{Hom}_R(\Lambda M, R\langle\langle X\rangle\rangle) \),
\[ \hat{E}^2_{\bullet,\bullet} = (H^*(M))^* \otimes H_\bullet(\Omega M). \]
Thus Hol provides an isomorphism of the \( E^2 \) terms. Then the standard spectral sequences comparison argument finishes the proof. \( \square \)

The above Theorem in conjunction with Theorem 3.1(ii) immediately implies the Jones isomorphism [J]:

4.2. Corollary. If \( M \) is simply connected, the holonomy map induces the isomorphism,
\[ H_\bullet(LM, R) = \text{Ext}^\bullet_{\Lambda M \otimes \Lambda M}(\Lambda M, \Lambda M^*). \]

5. BV operator and an odd holonomy map. The holonomy map Hol can also be understood as a degree 0 morphism of complexes,
\[ \widetilde{Hol} : (\Lambda M, d) \to (\Lambda_{LM} \otimes R\langle\langle X\rangle\rangle, d_{\pi^*(\omega)} := d + \partial + [\pi^*(\omega), \ldots]) \]
\[ \Xi \to \pi^*(\Xi) \wedge T \]
Consider now a degree -1 linear map
\[ \Psi : \Lambda M \otimes R\langle\langle X\rangle\rangle \to \Lambda_{LM} \otimes R\langle\langle X\rangle\rangle \]
\[ \Xi \to \sum_{n=0}^{\infty} \sum_{i=0}^{n} \int_0^{\omega_1} \cdots \int_0^{\omega_n} \Xi \omega \cdots \omega_n \]
5.1. Lemma. For any \( \Xi \in \Lambda M \otimes R\langle\langle X\rangle\rangle \),
\[ [T, \pi^*(\Xi)] = d_{\pi^*(\omega)} \Psi(\Xi) + \Psi(d_\omega \Xi). \]
Proof is a straightforward calculation of the r.h.s.

Note that \((\Lambda M \otimes 1, d)\) is naturally a subcomplex of \((\Lambda M \otimes R\langle\langle X\rangle\rangle, d_\omega)\).
5.2. Corollary. The map $\bar{\Psi}_0 := \bar{\Psi}|_{\Lambda_M \otimes 1}$ induces a degree $-1$ morphism of complexes,

$$(\Lambda_M, d) \longrightarrow (\Lambda_M \otimes \mathbb{R}(\langle X \rangle), d_{\pi^*(\omega)})$$

and, through pairing with chains, a degree $-1$ morphism of complexes,

$$\Psi : \left( C_\bullet(\Lambda M), \partial \right) \longrightarrow \left( \text{Hom}_{\mathbb{R}}(\Lambda M, \mathbb{R}(\langle X \rangle)), d_\omega^* \right)$$

simplex $\alpha : \Delta^\bullet \rightarrow \Lambda M$ $\longrightarrow$ $\Psi(\alpha) : \Lambda M \rightarrow \mathbb{R}(\langle X \rangle)$

$$(\ldots) \rightarrow \int_{\Delta^\bullet} \alpha^* \left( \bar{\Psi}_0(\ldots) \right).$$

There is a natural action of the circle group $S^1$ on the free loop space,

$$\text{Rot} : \Lambda M \times S^1 \rightarrow \Lambda M,$$

by rotating the loops, $\text{Rot} : (\gamma(t), s) \rightarrow \gamma(t + s)$. This action induces a degree $-1$ operator on (negatively graded) chains,

$$\Delta_{BV} : C_\bullet(\Lambda M) \longrightarrow C_{\bullet+1}(\Lambda M)$$

$$\alpha : \Delta^\bullet \rightarrow \Lambda M \longrightarrow \Delta_{BV}(\alpha) : \Delta^\bullet \times S^1 \rightarrow \Lambda M$$

$$(z, s) \rightarrow \text{Rot}(\alpha(z), s).$$

This operator commutes with the differential $\partial$ and hence induces an operator on singular homology, $\Delta_{BV} : H_\bullet(\Lambda M) \rightarrow H_{\bullet+1}(\Lambda M)$, which satisfies the condition $\Delta^2 = 0$ as its iteration $\Delta_{BV}^k$ increases the geometric dimension of the input simplex only by one [CS].

5.3. Proposition. $\text{Hol} \circ \Delta_{BV} = \Psi$.

Proof. Let $q : \Lambda M \times S^1 \rightarrow \Lambda M$ be the natural projection. It was shown in [GJP] that the $S^1$ rotation affects an arbitrary iterated integral as follows,

$$q_* \circ \text{Rot}^* \left( \pi^*(w_0) \wedge \int w_1 w_2 \ldots w_k \right) = \sum_{i=0}^r (-1)^r \int w_i \ldots w_k w_0 w_1 \ldots w_{i-1},$$

where $r = (|w_0| + |w_1| + \ldots + |w_{i-1}| - i)(|w_i| + \ldots + |w_k| - k + i)$. Applying this formula to $\pi^*(\ldots) \wedge T$, $\forall(\ldots) \in \Lambda_M$, one immediately obtains the required result. \(\Box\)

6. Poincare duality. Here is a deformed version of the classical duality:

6.1. Theorem. For any smooth compact oriented manifold $M$ the linear map

$$\mathcal{P} : (\Lambda_M \otimes \mathbb{R}(\langle X \rangle), d_{\omega}) \longrightarrow (\text{Hom}_{\mathbb{R}}(\Lambda_M, \mathbb{R}(\langle X \rangle)), d_{\omega}^*)$$

$$\Xi \longrightarrow \mathcal{P}(\Xi) : \Lambda M \rightarrow \langle X \rangle,$$

$$(\ldots) \rightarrow \int_M \Xi \wedge (\ldots),$$

is a degree $-n$ quasi-isomorphism of complexes.

Proof. Checking commutativity with differentials is just an elementary application of the Stokes theorem. The map $\mathcal{P}$ obviously preserves natural filtrations of both sides by powers of
the maximal ideal in \( \mathbb{R}\langle X \rangle \). At the \( E_1^{1,1} \) level of the associated spectral sequences this map induces, by the classical Poincare duality, a degree \(-n\) isomorphism,

\[
P : H^{\bullet}(M) \otimes \mathbb{R}\langle X \rangle^\bullet \to \text{Hom}_\mathbb{R}(H^{\bullet}(M), \mathbb{R}) \otimes \mathbb{R}\langle X \rangle^\bullet.
\]

Hence the spectral sequences comparison theorem delivers the required result. \( \square \)

6.2. Corollary. If \( M \) is simply connected, then, for any cycle \( \alpha : \Delta^\bullet \to LM \) in \( (C_\bullet(LM), \partial) \), there exists a cycle \( \Xi_\alpha \in (\Lambda_M \otimes \mathbb{R}\langle X \rangle, d_\omega) \) such that,

\[
(-1)^{[\alpha]} \int_{\Delta^\bullet} \alpha^* (T \wedge \pi^* (\ldots)) = \int_M \Xi_\alpha \wedge (\ldots) + d_\omega H(\ldots),
\]

for all \((\ldots) \in \Lambda_M\) and some \( H \in \text{Hom}_\mathbb{R}(\Lambda_M, \mathbb{R}\langle X \rangle)\).

Our next task is to find an explicit expression of such a cycle \( \Xi_\alpha \) in terms of iterated integrals and a formal power series connection.

7. Thom class and a suitable homotopy. Let \( i : M \hookrightarrow M \times M \) be the diagonal embedding, and \([U] \in H^n(M \times M, M \times M \setminus i(M))\) the associated Thom class. For any tubular neighbourhood, \( \text{Tub} \), of \( i(M) \) in \( M \times M \) one can represent \([U]\) by a closed \( n\)-form \( U \in \Lambda_{M \times M} \) such that

- support \( U \subset \text{Tub} \), and
- \( \text{pr}_1(U) = 1 \), where \( \text{pr} : M \times M \to M \) is the projection to the first factor and 1 is the constant function on \( M \).

Let us consider a commutative diagram of maps,

\[
\begin{array}{ccc}
M \times LM & \xrightarrow{p_y} & LM \\
\downarrow{p_x} & & \downarrow{\pi} \\
M & & M
\end{array}
\]

where \( p_x \) and \( \hat{p}_y \) are natural projections to, respectively, the first and second factors, and let us introduce a list of notations,

\[
\begin{align*}
\widehat{Tub} & := (p_x \times p_y)^{-1}(\text{Tub}) \subset M \times LM, \\
\hat{T}_y & := (\hat{p}_y)^* T \quad \in \Lambda_{LM \times M} \otimes R\langle X \rangle, \\
\hat{U}_{x,y} & := (p_x \times p_y)^* (U) \quad \in \Lambda_{LM \times M} \otimes R\langle X \rangle, \\
\hat{\omega}_x & := (p_x)^* \omega \quad \in \Lambda_{LM \times M} \otimes R\langle X \rangle, \\
\hat{\omega}_y & := (p_y)^* \omega \quad \in \Lambda_{LM \times M} \otimes R\langle X \rangle,
\end{align*}
\]

(so that a symbol with the hat always stands for the object living on \( M \times LM \)). A point in \( \widehat{Tub} \) is a pair \((x, l_y)\) consisting of a point \( x \in M \) and a loop \( l_y \) in \( LM \) based in \( y := \pi(l) \) such that \((x, y) \in \text{Tub}\).
Let $\nabla$ be an arbitrary smooth connection on $M$. For any two sufficiently close to each other points $x, y \in M$ there is a unique vector $V \in T_x M$ such that $y = \exp_x V$. For any $s \in [0, 1]$ define the path

$$P_{s \cdot x, y} : [0, 1] \rightarrow M$$

$$t \rightarrow \exp_x((1 - s)tV + sV)$$

which is just a $[0, 1]$-parameterized geodesic from $s \cdot x := \exp_x (sV)$ to $y = \exp_x V$. Such a path allows us in turn to introduce a smooth map

$$F : \hat{T}ub \times [0, 1] \rightarrow \hat{T}ub$$

$$(x, l_y) \times s \rightarrow (x, (P_{s \cdot x, y} \ast l) \ast P_{s \cdot x, y}^{-1})$$

where $\ast$ is the Pontrjagin products of paths.

Note that $F_1 = F|_{\hat{T}ub \times 1}$ is the identity map $\hat{T}ub \rightarrow \hat{T}ub$, while $F_0 = F|_{\hat{T}ub \times 0}$ factors through the “based diagonal”,

$$\hat{T}ub \hookrightarrow M \times LM \twoheadrightarrow LM \quad \pi \times \text{Id} \hookrightarrow M \times LM$$

$$(x, l_y) \rightarrow (P_{x, y} \ast l) \ast P_{x, y}^{-1}$$

In particular, the pullback map $F_0^* : \Lambda_{\hat{T}ub} \rightarrow \Lambda_{\hat{T}ub}$ vanishes on any differential form which has a factor of the type $p^*_y(\ldots) - p^*_x(\ldots), \forall(\ldots) \in \Lambda_M$.

Next we decompose, as usual, the pullback map,

$$F^* = 'F^* \oplus "F^*,$$

into a bit, $'F^*$, containing $ds$ and a bit, $"F^*$, not containing $ds$ and introduce a homotopy operator,

$$h : \Lambda_{\hat{T}ub}^\bullet \rightarrow \Lambda_{\hat{T}ub}^{\bullet - 1}$$

$$(\ldots) \rightarrow \int_0^1 'F^*(\ldots),$$

such that

$$\text{Id} = F_0^* + dh + hd.$$
Extending by zero differential forms with compact support in $\overline{Tub}$ to the whole $M \times LM$ we can summarize our achievements in this section as follows.

**7.2. Fact.** There exists a morphism of $p^*_x(\Lambda_M)$-modules,

$$H = (-1)^n \hat{U}_{x,y} \wedge h : \Lambda^{\bullet}_{M \times LM} \to \Lambda^{\bullet+n-1}_{M \times LM},$$

such that, for any $(\ldots) \in \Lambda_M$ and any $\Xi \in \Lambda_{M \times LM},$

$$\hat{U}_{x,y} \wedge \Xi \wedge (p^*_y(\ldots) - p^*_x(\ldots)) = dH \left( \Xi \wedge (p^*_y(\ldots) - p^*_x(\ldots)) \right) + (-1)^n H \left( d\Xi \wedge (p^*_y(\ldots) - p^*_x(\ldots)) \right) + (-1)^n + |\Xi| H \left( \Xi \wedge (p^*_y(\ldots) - p^*_x(\ldots)) \right)$$

In particular, extending $H$ by $\mathbb{R}\langle \langle X \rangle \rangle$-linearity to $\Lambda_{M \times LM} \otimes \mathbb{R}\langle \langle X \rangle \rangle$, taking $\Xi = \hat{T}_y$ and using the equation, $d\hat{T}_y + \partial \hat{T}_y + [\pi^*_y(\omega), \hat{T}_y] = 0,$ we get

$$\hat{U}_{x,y} \wedge \hat{T}_y \wedge (p^*_y(\ldots) - p^*_x(\ldots)) = (d + \partial)H \left( \hat{T}_y \wedge (p^*_y(\ldots) - p^*_x(\ldots)) \right) + (-1)^n H \left( [\hat{\omega}_y, \hat{T}_y] \wedge (p^*_y(\ldots) - p^*_x(\ldots)) \right) + (-1)^n H \left( \hat{T}_y \wedge (p^*_y(\ldots) - p^*_x(\ldots)) \right) = (-1)^n H \left( [\hat{\omega}_y - \hat{\omega}_x, \hat{T}_y] \right) \wedge p^*_x(\ldots) + (d + \partial)H \left( \hat{T}_y \wedge (p^*_y(\ldots) - p^*_x(\ldots)) \right) + (-1)^n H \left( [p^*_y(\omega \wedge \ldots) - p^*_y(\omega \wedge \ldots), \hat{T}_y] \right) + (-1)^n H \left( \hat{T}_y \wedge (p^*_y(\ldots) - p^*_x(\ldots)) \right)$$

**8. An inversion of the deformed Poincare duality map.** We assume from now on that $M$ is a compact simply connected manifold. For any cycle $\alpha : \Delta^\bullet \to LM$ in $(C^\bullet(LM), \partial)$ there is associated a diagram of maps,

$$\begin{array}{ccc}
M \times \Delta^\bullet & \xrightarrow{p} & M \\
\downarrow{\text{Id} \times \alpha} & & \downarrow{\text{Id} \times \alpha} \\
M & \xrightarrow{\text{Id} \times \alpha} & M \times LM
\end{array}$$

Then, for any $(\ldots) \in \Lambda_M,$ we have

$$(-1)^{|\alpha|} Hol(\alpha)(\ldots) = \int_{\Delta^\bullet} \alpha^* (T \wedge \pi^*(\ldots)) = \int_{M \times \Delta^\bullet} (\text{Id} \times \alpha)^* \left( \hat{U}_{x,y} \wedge \hat{T}_y \wedge p^*_y(\ldots) \right) = \int_{M \times \Delta^\bullet} (\text{Id} \times \alpha)^* \left( \hat{U}_{x,y} \wedge \hat{T}_y \wedge p^*_x(\ldots) \right) + \int_{M \times \Delta^\bullet} (\text{Id} \times \alpha)^* \left( \hat{U}_{x,y} \wedge \hat{T}_y \wedge (p^*_y(\ldots) - p^*_x(\ldots)) \right)$$
Hence, setting
\[
\Xi_\alpha = p_* \circ (\text{Id} \times \alpha)^* \left( \tilde{U}_{x,y} \wedge \tilde{T}_y + (-1)^n H \left( [\tilde{\omega}_y - \tilde{\omega}_x, \tilde{T}_y] \right) \right) \in \Lambda_M \otimes \mathbb{R}(X)
\]
and
\[
H_\alpha : \Lambda_M \to \mathbb{R}(X)
\]
\[
(\ldots) \to \int_{M \times \Delta} H \left( \tilde{T}_y \wedge (p_y^*(\ldots) - p_x^*(\ldots)) \right)
\]
we get,
\[
(-1)^{\alpha} Hol(\alpha)(\ldots) = \int_M \Xi_\alpha \wedge (\ldots) + d_\omega H_\alpha(\ldots),
\]
where we have used a calculation at the end of 7.2.

We almost proved the following

8.1. Theorem. The degree \(n\) linear map
\[
(C_\bullet(LM), \partial) \to (\Lambda_M \otimes \mathbb{R}(X), d_\omega)
\]
\[
\alpha \to (-1)^{\alpha} \Xi_\alpha
\]
is a quasi-isomorphism of complexes whose composition with \(\mathcal{P}\) induces on cohomology the map \(\text{Hol}\):

\[
\xymatrix{ H_\bullet(LM) \ar[r]^{H^\bullet+n} & H^\bullet(\text{Hom}_\mathbb{R}(\Lambda_M \otimes \mathbb{R}(X), d_\omega)) \\
\text{Hol} \ar[u] & \text{\tau} \ar[l] \ar[u]}
\]

Proof. We need only to check that the map is well defined, i.e. that \(\Xi_\alpha\) is a cycle. The latter can be suitably represented as
\[
\Xi_\alpha = p_* \circ (\text{Id} \times \alpha)^* \left( \tilde{U}_{x,y} \wedge \tilde{T}_y + h([\tilde{\omega}_y - \tilde{\omega}_x, \tilde{T}_y]) \right)
\]
where we understand \(\tilde{T}_y + h([\tilde{\omega}_y - \tilde{\omega}_x, \tilde{T}_y])\) as a differential form on \(\tilde{T}\) and \(\tilde{U}_{x,y} \wedge (\tilde{T}_y + h([\tilde{\omega}_y - \tilde{\omega}_x, \tilde{T}_y]))\) as a differential form on \(M \times LM\) (extended by zero from \(\tilde{T}\)).

As opposite to \(d_\omega \tilde{T}_y = 0\), we have,
\[
d_{\tilde{\omega}} \left( \tilde{T}_y + h([\tilde{\omega}_y - \tilde{\omega}_x, \tilde{T}_y]) \right) = - \left[ \tilde{\omega}_y - \tilde{\omega}_x, \tilde{T}_y \right] + (d + \partial) h([\tilde{\omega}_y - \tilde{\omega}_x, \tilde{T}_y]) - \left[ \tilde{\omega}_x, h([\tilde{\omega}_y - \tilde{\omega}_x, \tilde{T}_y]) \right]
\]
\[
= - h(d[\tilde{\omega}_y - \tilde{\omega}_x, \tilde{T}_y]) + \partial h([\tilde{\omega}_y - \tilde{\omega}_x, \tilde{T}_y]) + h \left( \tilde{\omega}_x, [\tilde{\omega}_y - \tilde{\omega}_x, \tilde{T}_y] \right)
\]
\[
= h([\tilde{\omega}_y \tilde{\omega}_y - \tilde{\omega}_x \tilde{\omega}_x, \tilde{T}_y]) - h(\tilde{\omega}_y - \tilde{\omega}_x, [\tilde{\omega}_y, \tilde{T}_y]) - h(\tilde{\omega}_x, [\tilde{\omega}_y - \tilde{\omega}_x, \tilde{T}_y]) = 0.
\]

Pushing down this equation along \(\alpha : \Delta^\bullet \to LM\) finally gives \(d_\omega \Xi_\alpha = 0\). \(\square\)

9. Inverse Poincare map revised. As the calculation in beginning of Section 8 shows, the class \(\Xi_\alpha = p_* \circ (\text{Id} \times \alpha)^* \left( \tilde{U}_{x,y} \wedge (\tilde{T}_y + h([\tilde{\omega}_y - \tilde{\omega}_x, \tilde{T}_y])) \right)\) provides us with a natural lift of
the holonomy map $[\text{Hol}] : H_\bullet (LM) \to H^\bullet (\text{Hom}_R(\Lambda_M \otimes \mathbb{R}(X), d_\omega^*)$ along the deformed Poincare duality map, $[\mathcal{P}] : H^\bullet (\Lambda_M \otimes \mathbb{R}(X), d_\omega) \to H^\bullet (\text{Hom}_R(\Lambda_M \otimes \mathbb{R}(X), d_\omega^*)$. However, it is not unique.

We shall use below another geometrically transparent lift of $[\mathcal{P}]^{-1} \circ [\text{Hol}]$ from the cohomology to the chain level. Note in this connection that $\Xi^\alpha$ has to be specified only up to a $d_\omega$-exact term implying that the form $\hat{T}_y + h(\hat{\omega}_y - \hat{\omega}_x, \hat{T}_y)$ on $\hat{\text{Tub}}$ has to be specified only up to a $d_\omega$-exact term.

Recall that for a tubular neighbourhood, $\text{Tub}$, of the diagonal in $M \times M$, the subspace $\hat{\text{Tub}} \subset M \times LM$ consists of pairs $(x, l_y)$, $x$ being a point $x \in M$ and $l_y$ a loop in $LM$ based at $y := \pi(l_y)$, such that $(x, y) \in \text{Tub}$. Let $P_{x,y}$ be the geodesic from $x$ to $y$ for some fixed affine connection on $M$. Consider a map, $i_{x,y} : \hat{\text{Tub}} \to LM$

$$(x, l_y) \mapsto (P_{x,y} \star l_y) \star P_{x,y}^{-1}.$$  

9.1. Proposition. $\hat{T}_y + h(\hat{\omega}_y - \hat{\omega}_x, \hat{T}_y) = i_{x,y}^* T \mod \text{Im} d_\omega$.

Proof. Comparing the map $i_{x,y}$ with the homotopy $F$ in Section 7, one notices that $F_0$ is precisely the composition of $i_{x,y}$ with the natural embedding,

$$\text{emb} : LM \longrightarrow M \times LM \quad l \mapsto (\pi(l), l).$$

Then applying the equality of maps, $\text{Id} = i_{x,y} \circ \text{emb} + dh + hd$, to $\hat{T}_y$ one obtains,

$$i_{x,y}^* T = \hat{T}_y - dh(\hat{T}_y)) - h(d\hat{T}_y))$$

$$= \hat{T}_y + h(\hat{\omega}_y - \hat{\omega}_x, \hat{T}_y) - d_\omega h(\hat{T}_y).$$

9.2. Corollary. The degree $n$ linear map

$$\text{Hol} : (G\bullet (LM), \partial) \longrightarrow (\Lambda_M \otimes \mathbb{R}(X), d_\omega)$$

$$(-1)\alpha \longrightarrow \Gamma_\alpha := p_\ast \circ (\text{Id} \times \alpha)\ast (\hat{U}_{x,y} \wedge i_{x,y}^* T)$$

is a quasi-isomorphism of complexes making the following diagram

$$\begin{array}{c}
H^{\bullet+n}(\Lambda_M \otimes \mathbb{R}(X), d_\omega)) \\
\text{Hol} \downarrow \text{Id} \downarrow
\end{array}$$

$$\begin{array}{c}
H_\bullet (LM) \text{Hol} \longrightarrow H^\bullet (\text{Hom}_R(\Lambda_M \otimes \mathbb{R}(X), d_\omega^*)
\end{array}$$

commutative.

10. Compatibility with products. Now we have all the threads to show a new proof of a remarkable result of [CJ, C, Tr] which establishes an isomorphism of algebras,

$$(H_\bullet (LM), \text{Chas-Sullivan product } \otimes) = (\text{Ext}_{\Lambda_M \otimes \Lambda_M}^\bullet (\Lambda_M, \Lambda_M) \text{, Yoneda product}).$$
It follows immediately from Theorem 3.1(i) and the following

**10.1. Theorem.** The degree 0 linear map

\[
\text{Hol} : (C_{•+n}(LM), ∂) \longrightarrow (A_M \otimes \mathbb{R}\langle X \rangle, d_\omega)
\]

\[
\alpha \quad \longrightarrow \quad (-1)^{|\alpha|} \Gamma_\alpha
\]

induces on cohomology a quasi-isomorphism of algebras,

\[
(H_{•}(LM), \otimes) \longrightarrow H^•(A_M \otimes \mathbb{R}\langle X \rangle, d_\omega).
\]

**Proof.** We shall use small Latin letters, “coordinate variables”, to distinguish various copies of \(M\) and \(LM\), e.g. \(M_x, M_y, LM_y\), etc.

Let \(α_y : \Delta^• → LM_y\) and \(α_z : \Delta^• → LM_z\) be any two cycles in \((C_{•+n}(LM), ∂)\) such that their projections, \(π_y \circ α_y : \Delta^• → M_y\) and \(π_z \circ α_z : \Delta^• → M_z\) intersect transversally at a cycle \(π_y \circ α_y \cap π_z \circ α_z : \Delta^•_{yz} → M\), where

\[
\Delta^•_{yz} := (π_y \circ α_y \cap π_z \circ α_z)^{-1} (\{π_y \circ α_y(\Delta^•_{y}) \times π_z \circ α_z(\Delta^•_{z})\} \cap \{\text{diagonal in } M_y \times M_z\}).
\]

Chas and Sullivan [CS] define the product chain, \(α_y \otimes α_z\), as

\[
α_1 \otimes α_2 : \Delta^•_{yz} → LM
\]

\[
z \quad → \quad α_1(z) \ast α_2(z).
\]

The theorem will follow if we show that

\[
Γ_α_y \land Γ_α_z = Γ_{α_y \otimes α_z} \mod \text{Im } d_ω.
\]

(⋆)

In the obvious notations associated with the commutative diagram,

\[
\begin{array}{ccc}
M_x \times \Delta^•_y \times \Delta^•_z & \xrightarrow{p} & M_x \\
\downarrow \text{Id} \times α_y \times α_z & & \text{Id} \times α_y \times α_z \\
M_x \times LM_y \times LM_z & \xleftarrow{p} & M_x \times \Delta^•_y \times \Delta^•_z
\end{array}
\]

we can represent the l.h.s. of (⋆) as follows

\[
Γ_α_y \land Γ_α_z = p_+ (\text{Id}_x \times α_y \times α_z)^*(U_{x,y} \land i_{x,y}^* T \land i_{x,z}^* T).
\]

Using homotopy 7.2 we can transform \(\ldots \land U_{x,z} \land \ldots\) into \(\ldots \land \hat{U}_{y,z} \land \ldots\) \(mod d_ω\)-exact terms:

\[
\hat{U}_{x,y} \land \hat{U}_{x,z} \land i_{x,y}^* T \land i_{x,z}^* T = \hat{U}_{x,y} \land \hat{U}_{y,z} \land i_{x,y}^* T \land i_{x,z}^* T + dH_{x,y} (U_{x,z} - \hat{U}_{y,z}) \land i_{x,y}^* T \land i_{x,z}^* T
\]

\[
+ H (U_{x,z} - \hat{U}_{y,z}) \land d(i_{x,y}^* T \land i_{x,z}^* T)
\]

\[
= \hat{U}_{x,y} \land \hat{U}_{y,z} \land T_{x,y} \land T_{x,z} + d_{ω} H_{x,y} (U_{x,z} - \hat{U}_{y,z}) \land i_{x,y}^* T \land i_{x,z}^* T.
\]

Hence

\[
Γ_α_y \land Γ_α_z = p_+ (\text{Id}_x \times α_y \times α_z)^*(U_{x,y} \land \hat{U}_{y,z} \land i_{x,y}^* T \land i_{x,z}^* T) \mod \text{Im } d_ω.
\]

17
Let $\text{Tub}_{y,z}$ be a tubular neighbourhood of the diagonal in $M_y \times M_z$ and let $\overline{\text{Tub}}_{y,z}$ be its pre-image under the natural evaluation projection,

$$\pi_y \times \pi_z : LM_y \times LM_z \to M_y \times M_z.$$ 

Similarly, let $\text{Tub}_{x,y,z}$ be a tubular neighbourhood (supporting the form $U_{x,y} \wedge U_{y,z}$) of the small diagonal in $M_x \times M_y \times M_z$ and let $\overline{\text{Tub}}_{x,y,z}$ be its pre-image under the natural map,

$$\text{Id}_x \times \pi_y \times \pi_z : M_x \times LM_y \times LM_z \to M_x \times M_y \times M_z.$$ 

The form $\hat{i}_y^* \wedge i_z^* T$ is supported in $M_x \times \overline{\text{Tub}}_{y,z} \subset M_x \times LM_y \times LM_z$. As $\text{Tub}_{y,z}$ is homotopy equivalent to $M_y$, there exists a one parameter family of maps,

$$G : \overline{\text{Tub}}_{y,z} \times [0,1] \to \overline{\text{Tub}}_{y,z}$$

with the property that $G_1 = G|_{\overline{\text{Tub}}_{y,z} \times 1}$ is the identity map while $G_0 = G|_{\overline{\text{Tub}}_{y,z} \times 0}$ factors through the composition

$$\overline{\text{Tub}}_{y,z} \xrightarrow{r} LM_y \times M_y \xrightarrow{LM_y} LM_y \times LM_z$$

for a smooth map $r$. Then one applies the associated to $\text{Id}_x \times G$ homotopy equality of linear maps,

$$\text{Id} = \text{Id}_x \otimes (G_0^* + dhG + h_G d) : \Lambda_{\overline{\text{Tub}}_{x,y,z}} \to \Lambda_{\overline{\text{Tub}}_{x,y,z}};$$

to the form $i_{x,y}^* T \wedge i_{x,z}^* T$ and gets

$$i_{x,y}^* T \wedge i_{x,z}^* T = \hat{i}_{x,y}^* (q_1^* T \wedge q_2^* T) \mod \text{Im} d\tilde{\omega}_z,$$

where $q_1$ and $q_2$ are the natural projections,

$$\begin{array}{ccc}
LM_x & \xrightarrow{q_1} & M_x \\
\downarrow & & \downarrow \\
LM_x & \xleftarrow{q_2} & LM_x
\end{array}$$

and $\hat{i}_{x,y}$ is given by (cf. with $i_{x,y}$ in the beginning of Sect. 9)

$$\hat{i}_{x,y} : \overline{\text{Tub}}_{x,y,z} \to LM_x \times M_x \xrightarrow{LM_x} (x, l_y, l_z) \to (P_{x,y} \ast r(l_y, l_z)) \ast P_{x,y}^{-1}.$$ 

There is a natural Pontrjagin product map

$$LM_x \times M_x \xrightarrow{\ast} LM_x,$$

and the basic property of the transport of a formal power series connection says that

$$\ast^* T = p_1^* (T) \wedge p_2(T).$$

Hence we can eventually write

$$\hat{U}_{x,y} \wedge \hat{U}_{x,z} \wedge i_{x,y}^* T \wedge i_{x,z}^* T = \hat{U}_{x,y} \wedge \hat{U}_{y,z} \wedge \hat{i}_{x,y}^* T \mod \text{Im} d\tilde{\omega}_z,$$
which implies
\[ \Gamma_{\alpha_y} \wedge \Gamma_{\alpha_z} = p_* \circ (\text{Id}_x \times \alpha_y \times \alpha_z)^* \left( \hat{U}_{x,y} \wedge \hat{U}_{y,z} \wedge i_{x,y}^* T \right) \mod \text{Im } d_\omega. \]

Recalling the definition of \( \alpha_y \oplus \alpha_z \) and using transversality of \( p_y \circ \alpha_y \) and \( p_z \circ \alpha_z \) to get rid of the Thom class \( U_{y,z} \) through \( \mathcal{Z} \)-integration one finally obtains
\[ \Gamma_{\alpha_y} \wedge \Gamma_{\alpha_z} = p_* \circ (\text{Id}_x \times \alpha_y \oplus \alpha_z)^* \left( \hat{U}_{x,y} \wedge i_{x,y}^*(T) \right) \mod \text{Im } d_\omega. \]
Which is precisely the desired result \((\star)\). \hfill \Box

11. Loops based at a brane. If \( f : Z \to M \) is a smooth map from a compact oriented \( p \)-dimensional manifold to a compact simply connected manifold \( M \), then one can consider a space, \( L_f \), defined by the pullback diagram,

\[
\begin{array}{ccc}
L_f & \xrightarrow{\nu} & LM \\
\mu \downarrow & & \downarrow \pi \\
Z & \xrightarrow{f} & M \\
\end{array}
\]

The shifted homology, \( H_*(L_f) := H_{*-p}(LM) \), is obviously an associative algebra with respect to the Chas-Sullivan product \( \otimes \) (see [CS, S]). If \( f \) is an embedding, then \( L_f \) is a subspace of \( LM \) consisting of loops in \( M \) which are based at \( Z \subset M \).

The pull-back map on differential forms, \( f^* : \Lambda_M \to \Lambda_Z \), is naturally extended to the tensor product \( \Lambda_M \otimes \mathbb{R}\langle X \rangle \) as \( f^* \otimes \text{Id} \) and is denoted by the same symbol \( f^* \). The map
\[
f^* : (\Lambda_M \otimes \mathbb{R}\langle X \rangle, d_\omega) \longrightarrow (\Lambda_Z \otimes \mathbb{R}\langle X \rangle, d_{f^*}(\omega))
\]
is obviously a morphism of dg algebras.

The proofs of Theorems 4.1, 6.1 and 10.1 apply with trivial changes to their “brane” versions:

11.1. Theorem. The map
\[
\text{Hol} : \quad (C\bullet (L_f), \partial) \quad \longrightarrow \quad \left( \text{Hom}_{\mathbb{R}}(\Lambda_Z, \mathbb{R}\langle X \rangle), d_{f^*}(\omega) \right)
\]

simplex \( \alpha : \Delta^\bullet \to L_f \quad \longrightarrow \quad \text{Hol}(\alpha) : \Lambda_Z \to \mathbb{R}\langle X \rangle \)

\[
\longrightarrow \quad (-1)^{|\alpha|} \int_{\Delta^\bullet} \alpha^* \{ \mu^*(\ldots) \wedge \nu^*(T) \},
\]
is a quasi-isomorphism of complexes.

11.2. Theorem. The linear map
\[
\mathcal{P} : \quad (\Lambda_Z \otimes \mathbb{R}\langle X \rangle, d_{f^*}(\omega)) \quad \longrightarrow \quad \left( \text{Hom}_{\mathbb{R}}(\Lambda_Z, \mathbb{R}\langle X \rangle), d_{f^*}(\omega) \right)
\]

\[
\Xi \quad \longrightarrow \quad \mathcal{P}(\Xi) : \Lambda_Z \to \mathbb{R}\langle X \rangle,
\]

\[
\longrightarrow \quad \int_Z \Xi \wedge (\ldots),
\]
is a degree \(-p\) quasi-isomorphism of complexes.
11.3. **Theorem.** The degree $p$ linear map

$$\text{Hol} : (C_\bullet (Lf), \partial) \longrightarrow (\Lambda Z \otimes \mathbb{R}\langle X \rangle, d_{f^\ast(o)})$$

$(-1)^{|\alpha|} \alpha \longrightarrow p_\ast \circ (f \times \mu \circ \alpha)^\ast \left( \hat{U}_{x,y} \wedge i_{x,y}^\ast T \right)$

associated with the diagram,

$$
\begin{array}{ccc}
Z \times \Delta^\bullet & \xrightarrow{p} & Z \\
\downarrow{f \times \mu \circ \alpha} & & \downarrow{f \times \mu \circ \alpha} \\
M \times LM & \xrightarrow{p} & M \times LM
\end{array}
$$

is a morphism of complexes inducing on cohomology an isomorphism of algebras,

$$(H_\ast(Lf), \cap) \rightarrow H_\ast(A_Z \otimes \mathbb{R}\langle X \rangle, d_{f^\ast(o)}).$$

Moreover, if $f$ is an embedding, then the pull-back map $f^\ast : H_\ast(A_M \otimes \mathbb{R}\langle X \rangle, d_o) \rightarrow H_\ast(A_Z \otimes \mathbb{R}\langle X \rangle, d_{f^\ast(o)})$ corresponds to the intersection map $\cap_{[L_f]} : H_\ast(LM) \rightarrow H_\ast(Lf)$.

11.4. **Example.** Let $Z$ be the projective space $\mathbb{CP}^m$ and let $f : Z \rightarrow M = \mathbb{CP}^n$, $m < n$, be a linear embedding. A straightforward calculation using Theorem 11.3 gives,

$$(H_\ast(L_f, \cap)) = H_\ast\left( \mathbb{R}[h]/h^{m+1} \otimes \mathbb{R}\langle x_1, \ldots, x_n \rangle, d_{f^\ast(o)} = \emptyset + \left[ \sum_{i=1}^m h^i \otimes x_i, \ldots \right] \right)$$

$$= \mathbb{R}[h, \nu, x]/h^{m+1}, \quad |h| = 2, |\nu| = -2n, |x| = -1,$$

where $\emptyset$ is the same as in Example 3.4. (The key to the r.h.s. is $x = \sum_{i=1}^{m+1} ih^{i-1} \otimes x_i, \nu \sim \sum_{i+j=n+1} 1 \otimes x_i x_j$.) Moreover the intersection map

$$
\begin{array}{ccc}
H_\ast(L\mathbb{CP}^n) & \xrightarrow{\cap_{[L_f]}} & H_\ast(Lf) \\
\| & & \|
\end{array}
\xrightarrow{\|}
\begin{array}{ccc}
\mathbb{R}[h, \mu, \nu]/(h^{n+1}, h^n \mu, h^n \nu) & \longrightarrow & \mathbb{R}[h, \nu, x]/h^{m+1}
\end{array}
$$

is given on generators as follows (see Example 3.4 again for the notations used in the l.h.s.),

$$
\begin{array}{c}
h \rightarrow h \\
\nu \rightarrow \nu \\
\mu \rightarrow hx.
\end{array}
$$

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