ON THE AVERAGE NUMBER OF SHARP CROSSINGS OF CERTAIN GAUSSIAN RANDOM POLYNOMIALS

S. Shemehsavar, S. Rezakhah*

Abstract

Let \( Q_n(x) = \sum_{i=0}^{n} A_i x^i \) be a random algebraic polynomial where the coefficients \( A_0, A_1, \ldots \) form a sequence of centered Gaussian random variables. Moreover, assume that the increments \( \Delta_j = A_j - A_{j-1}, \) \( j = 0, 1, 2, \ldots \) are independent, assuming \( A_{-1} = 0. \) The coefficients can be considered as \( n \) consecutive observations of a Brownian motion. We obtain the asymptotic behaviour of the expected number of u-sharp crossings of polynomial \( Q_n(x) \). We refer to u-sharp crossings as those zero up-crossings with slope greater than \( u \), or those down-crossings with slope smaller than \( -u \). We consider the cases where \( u \) is unbounded and is increasing with \( n \), where \( u = o(n^{5/4}) \), and \( u = o(n^{3/2}) \) separately.

Keywords and Phrases: random algebraic polynomial, number of real zeros, sharp crossings, expected density, Brownian motion.

AMS(2000) subject classifications. Primary 60H42, Secondary 60G99.

*Faculty of Mathematics and Computer Science, Amirkabir University of Technology, 424 Hafez Avenue, Tehran Iran. Email: Rezakhah@aut.ac.ir, Shemehsavar@aut.ac.ir.
1 Preliminaries

The theory of the expected number of real zeros of random algebraic polynomials was addressed in the fundamental work of M. Kac\[6\]. The works of Logan and Shepp \[7, 8\], Ibragimov and Maslova \[5\], Wilkins \[14\], and Farahmand \[3\] and Sambandham \[12, 13\] are other fundamental contributions to the subject. For various aspects on random polynomials see Bharucha-Reid and Sambandham \[1\], and Farahmand \[4\]. There has been recent interest in cases where the coefficients form certain random processes, Rezakhah and Soltani \[10, 11\], Rezakhah and Shemehsavar \[9\].

Let
\[ Q_n(x) = \sum_{i=0}^{n} A_i x^i, \quad -\infty < x < \infty, \]  
(1.1)

where \( A_0, A_1, \cdots \) are mean zero Gaussian random variables for which the increments \( \Delta_i = A_i - A_{i-1}, \; i = 1, 2, \cdots \) are independent, \( A_{-1} = 0 \). The sequence \( A_0, A_1 \cdots \) may be considered as successive Brownian points, i.e., \( A_j = W(t_j), \; j = 0, 1, \cdots, \) where \( t_0 < t_1 < \cdots \) and \( \{W(t), \; t \geq 0\} \) is the standard Brownian motion. In this physical interpretation, \( \text{Var}(\Delta_j) \) is the distance between successive times \( t_{j-1}, t_j \). Thus for \( j = 1, 2, \cdots \), we have that \( A_j = \Delta_0 + \Delta_1 + \cdots + \Delta_j \), where \( \Delta_i \sim N(0, \sigma_i^2) \) and are independent. Thus \( Q_n(x) = \sum_{k=0}^{n}(\sum_{j=k}^{n} x^j)\Delta_k = \sum_{k=0}^{n} a_k(x)\Delta_k \), and \( Q'_n(x) = \sum_{k=0}^{n}(\sum_{j=k}^{n} jx^{j-1})\Delta_k = \sum_{k=0}^{n} b_k(x)\Delta_k \), where
\[ a_k(x) = \sum_{j=k}^{n} x^j, \quad b_k(x) = \sum_{j=k}^{n} jx^{j-1}, \quad k = 0, \cdots, n. \]  
(1.2)

We say that \( Q_n(x) \) has a zero up-crossing at \( t_0 \) if there exists \( \varepsilon > 0 \) such that \( Q_n(x) \leq 0 \) in \((t_0 - \varepsilon, t_0)\) and \( Q_n(x) > 0 \) in \((t_0, t_0 + \varepsilon)\). Similarly \( Q_n(x) \) is said to have a zero down crossing at \( t_0 \), if \( Q_n(x) > 0 \) in \((t_0 - \varepsilon, t_0)\) and \( Q_n(x) \leq 0 \) in \((t_0, t_0 + \varepsilon)\). We study the asymptotic behavior of the expected
number of, $u$-sharp zero crossings, those zero up-crossings with slope greater than $u > 0$, or those down-crossings with slope smaller than $-u$.

Cramer and Leadbetter [1967 p 287] have shown that the expected number of total zeros of any Gaussian non stationary process, say $Q_n(x)$, is calculated via the following formula

$$EN(a, b) = \int_a^b dt \int_{-\infty}^{\infty} |y| p_t(0, y) dy$$

where $p_t(z, y)$ denotes the joint density of $Q_n(x) = z$ and its derivative $Q'_n(x) = y$, and

$$p_t(0, y) = \left[2\pi\gamma\sigma(1 - \mu^2)^{1/2}\right]^{-1} \times \exp \left\{ -\frac{\gamma^2 m^2 + 2\mu\gamma\sigma m(y - m') + \sigma^2(y - m')^2}{2\gamma^2\sigma^2(1 - \mu^2)} \right\}$$

in which $m = E(Q_n(x))$, $m' = E(Q'_n(x))$, $\sigma^2 = \text{Var}(Q_n(x))$, $\gamma^2 = \text{Var}(Q'_n(x))$, and $\mu = \text{Cov}(Q_n(x), Q'_n(x))/\gamma\sigma$.

By a similar method as in Farahmand [4] we find that $ES_u(a, b)$, the expected number of $u$-sharp zero crossings of $Q_n(x)$ in any interval $(a, b)$, satisfies

$$ES_u(a, b) = \int_a^b dt \left\{ \int_{-\infty}^{-u} + \int_u^{\infty} \right\} |y| p_t(0, y) dy = \int_a^b f_n(x) dx$$  \hspace{1cm} (1.3)$$

where

$$f_n(x) = \frac{1}{\pi} g_{1,n}(x) \exp(g_{2,n}(x))$$  \hspace{1cm} (1.4)$$

and

$$g_{1,n}(x) = EA^{-2}, \hspace{1cm} g_{2,n}(x) = -\frac{A^2u^2}{2E^2},$$  \hspace{1cm} (1.5)$$

in which

$$A^2 = \text{Var}(Q_n(x)) = \sum_{k=0}^{n} a_k^2(x)\sigma_k^2, \hspace{1cm} B^2 = \text{Var}(Q'_n(x)) = \sum_{k=0}^{n} b_k^2(x)\sigma_k^2,$$

$$D = \text{Cov}(Q_n(x), Q'_n(x)) = \sum_{k=0}^{n} a_k(x)b_k(x)\sigma_k^2, \hspace{1cm} \text{and} \hspace{1cm} E^2 = A^2B^2 - D^2,$$

and $a_k(x)$, $b_k(x)$ is defined by (1.2).
2 Asymptotic behaviour of $ES_u$

In this section we obtain the asymptotic behaviour of the expected number of \( u \)-sharp crossings of \( Q_n(x) = 0 \) given by (1.1). We prove the following theorem for the case that the increments \( \Delta_1 \cdots \Delta_n \) are independent and have the same distribution. Also we assume that \( \sigma_k^2 = 1, \) for \( k = 1 \cdots n. \)

**Theorem (2.1):** Let \( Q_n(x) \) be the random algebraic polynomial given by (1.1) for which \( A_j = \Delta_1 + \cdots + \Delta_j \) where \( \Delta_k, \, k = 1, \ldots, n \) are independent and \( \Delta_j \sim N(0, \sigma_j^2) \) then the expected number of \( u \)-sharp crossings of \( Q_n(x) = 0 \) satisfies

(i) for \( u = o(n^{5/4}) \)

\[
ES_u(-\infty, \infty) = \frac{1}{\pi} \log(2n + 1) + \frac{1}{\pi} (1.920134502) \\
+ \frac{1}{\pi \sqrt{2n}} \left( -\pi + 2 \arctan \left( \frac{1}{2 \sqrt{2n}} \right) \right) + C_1 \frac{1}{n \pi} \\
+ \frac{u^2}{n^3 \pi} \left( 19.05803659 - \frac{8}{3} \ln(n^3 + 1) \right) \\
+ \frac{u^4}{n^6 \pi} \left( -34989.96324 + \frac{32}{3} \ln(n^6 + 1) \right) + o(n^{-1})
\]

where \( C_1 = -0.7190843756 \) for \( n \) even and \( C_1 = 1.716159410 \) for \( n \) odd.

(ii) for \( u = o(n^{3/2}) \)

\[
ES_u(-\infty, \infty) = \frac{1}{\pi} \log(2n + 1) + \frac{1}{\pi} (1.920134502) + o(1)
\]

**Proof:** Due to the behaviour of \( Q_n(x) \), the asymptotic behaviour is treated separately on the intervals \( 1 < x < \infty, -\infty < x < -1, \, 0 < x < 1 \) and \( -1 < x < 0. \) For \( 1 < x < \infty, \) using (1.3), the change of variable \( x = 1 + \frac{t}{n} \) and the equality \( \left( 1 + \frac{t}{n} \right)^n = e^t \left( 1 - \frac{t^2}{n} \right) + O \left( \frac{1}{n} \right) \), we find that

\[
ES_u(1, \infty) = \frac{1}{n} \int_{0}^{\infty} f_n(1 + \frac{t}{n})dt,
\]
where \( f_n(\cdot) \) is defined by (1.4). Using (1.5), and by tedious manipulation we have that
\[
g_{2,n}(1 + \frac{t}{n}) = o(n^{-2})
\]
and
\[
n^{-1}g_{1,n}(1 + \frac{t}{n}) = \left( R_1(t) + \frac{S_1(t)}{n} + O\left( \frac{1}{n^2} \right) \right), \quad (2.1)
\]
where
\[
R_1(t) = \frac{\sqrt{(4t-15)e^{4t}+(24t+32)e^{3t} - e^{2t}(8t^3+12t^2+36t+18) + 8e^t + 1}}{2t (-1 - 3e^t + 4e^t + 2te^t)}
\]
and \( S_1(t) = S_{11}(t)/S_{12}(t) \) in which
\[
S_{11}(t) = -0.25 \left( (4t^2 - 6t - 27) e^{6t} + (156 - 84t + 116t^2 - 24t^3) e^{5t} + (16t^5 - 72t^4 + 96t^3 - 212t^2 + 220t - 331)e^{4t} + (328 - 168t + 128t^2 - 104t^3)e^{3t} + (8t^4 + 8t^3 - 32t^2 + 42t - 153)e^{2t} + (28 - 4t - 4t^2)e^t - 1 \right)
\]
\[
S_{12}(t) = \left( 2e^{2t} - 1 + 4e^t - 3e^{2t} \right)^2 \\
\times (1 - 8t^3 e^{2t} - 12e^{2t} t^2 + 8e^t t - 18e^{2t} t - 36e^{2t} t^2 - 15e^t + 32e^t + 24e^3 t + 4t e^4) \right)^{1/2}
\]
One can easily verify that as \( t \to \infty \),
\[
R_1(t) = \frac{1}{2t^{3/2}} + O(t^{-2}), \quad S_1(t) = -\frac{1}{8t^{1/2}} + O(t^{-3/2}).
\]
As (2.1) can not be integrated term by term, we use the equality
\[
\frac{I_{[t>1]}}{8n\sqrt{t}} = \frac{I_{[t>1]}}{8n\sqrt{t} + t\sqrt{t}} + O\left( \frac{1}{n^2} \right), \quad (2.2)
\]
where
\[
I_{[t>1]} = \begin{cases} 
1 & \text{if } t \geq 1 \\
0 & \text{if } t < 1 
\end{cases}
\]
Thus by (2.2) we have that
\[
\frac{1}{n} f_n\left( 1 + \frac{t}{n} \right) = -\frac{I_{[t>1]}}{\pi(8n\sqrt{t} + t\sqrt{t})} + \frac{R_1(t)}{\pi} + \frac{1}{\pi} \left( \frac{S_1(t)}{n} + \frac{I_{[t>1]}}{8n\sqrt{t}} \right) + O\left( \frac{1}{n^2} \right). 
\]
This expression is term by term integrable, and provides that
\[
ES_u(1, \infty) = \frac{1}{n} \int_0^{\infty} f_n \left( 1 + \frac{t}{n} \right) dt = \frac{1}{2\pi\sqrt{2n}} \left( -\pi + 2 \arctan \left( \frac{1}{2\sqrt{2n}} \right) \right) + \frac{1}{\pi} \int_0^{\infty} R_1(t) dt + \frac{1}{pn} \int_0^{\infty} \left( S_1(t) + \frac{I_{[t>1]}}{8\sqrt{t}} \right) dt + O \left( \frac{1}{n^2} \right),
\]
where \( \int_0^{\infty} R_1(t) dt = 0.7348742023 \), and \( \int_0^{\infty} \left( S_1(t) + I_{[t>1]}(8\sqrt{t})^{-1} \right) dt = -0.2496371198. \)

For \(-\infty < x < -1\), using \( x = -1 - \frac{t}{n} \), we have \( ES_u(-\infty, -1) = \frac{1}{n} \int_0^{\infty} f_n \left( -1 - \frac{t}{n} \right) dt, \) where \( f_n(\cdot) \) is defined by (1.4), and by (1.5)
\[
n^{-1} g_{1,n} \left( -1 - \frac{t}{n} \right) = \left( R_2(t) + \frac{S_2(t)}{n} + O \left( \frac{1}{n^2} \right) \right) \tag{2.3}
\]
where
\[
R_2(t) = 1/2 \sqrt{-2e^{2t} + e^{4t} + 1 - 12e^{2t}t^2 - 8t^3e^{2t} + 4te^{4t} - 4te^{2t}}/t^2(e^{2t} - 1 + 2te^{2t})^2;
\]
for \( n \) even \( S_2(t) = \frac{S_{21}(t) + S_{22}(t)}{4S_{23}(t)} \), and for \( n \) odd \( S_2(t) = \frac{S_{21}(t) - S_{22}(t)}{4S_{23}(t)} \)
in which
\[
S_{21}(t) = 1 + \left( -8t^4 + 30t - 8t^3 + 48t^2 - 3 \right)e^{2t} + \left( 3 - 12t + 52t^2 + 96t^3 + 40t^4 - 16t^5 \right)e^{4t} - \left( 18t + 4t^2 + 1 \right)e^{6t},
\]
\[
S_{22}(t) = \left( 8t + 32t^3 + 40t^2 \right)e^{3t} + \left( -8t^2 - 12t \right)e^{5t} + 4e^{4t},
\]
\[
S_{23}(t) = \left( e^{4t}(4t + 1) - 2e^{2t}(1 + 2t + 6t^2 + 4t^3) + 1 \right)^{1/2} \left( e^{2t}(2t + 1) - 1 \right)^2.
\]
and
\[
g_{2,n} \left( -1 - \frac{t}{n} \right) = \left( \frac{u^2}{n^3} \right) G_{2,1}(t) + o \left( \frac{1}{n} \right) \tag{2.4}
\]
where
\[
G_{2,1}(t) = \frac{16 \left( 2e^{2t} + e^{4t} - 1 \right) t^3}{2e^{2t} - e^{4t} + 12e^{2t}t^2 - 1 + 8e^{2t}t^3 - 4e^{4t}t + 4e^{2t}}.
\]
It can be seen that as \( t \to \infty, \)
\[
R_2(t) = \frac{1}{2t^{5/2}} + O(t^{-2}), \quad S_2(t) = -\frac{1}{8t^{1/2}} + O(t^{-3/2}), \quad G_{2,1}(t) = o(e^{-t}).
\]
where we observe that

\[ \frac{1}{n} f_n(-1 - \frac{t}{n}) = -\frac{I_{[t>1]}(8n\sqrt{t} + t\sqrt{t})}{\pi(8n\sqrt{t} + t\sqrt{t})} + \frac{1}{\pi} \left( R_2(t) + \frac{1}{n} (S_2(t) + \frac{I_{[t>1]}}{8\sqrt{t}}) \right) + \frac{u^2}{n^3} R_2(t)G_{2,1}(t) + \frac{u^4}{2n^6} R_2(t)G_{2,1}^2(t) \] + o\left( n^{-1} \right). \]

Thus

\[ ES_u(-\infty, -1) = \frac{1}{n} \int_0^\infty f_n(-1 - \frac{t}{n}) = \frac{1}{2\pi \sqrt{2n}} \left( -\pi + 2 \arctan\left( \frac{1}{2\sqrt{2n}} \right) \right) \]
\[ + \frac{1}{\pi} \int_0^\infty R_2(t) dt + \frac{1}{\pi} \int_0^\infty \left( S_2(t) + \frac{I_{[t>1]}}{8\sqrt{t}} \right) dt \]
\[ + \frac{1}{n^3} \int_0^\infty R_2(t)G_{2,1}(t) dt + \frac{1}{n^6} \int_0^\infty \frac{1}{2} R_2(t)G_{2,1}^2(t) dt + o\left( n^{-1} \right) \]

where \( \int_0^\infty R_2(t) dt = 1.095640061 \), and

\[ \int_0^\infty R_2(t)G_{2,1}(t) dt = -2.418589510, \quad \int_0^\infty \frac{1}{2} R_2(t)G_{2,1}^2(t) dt = 7.057233216, \]

and for \( n \) odd \( \int_0^\infty \left( S_2(t) + \frac{I_{[t>1]}}{8\sqrt{t}} \right) dt = \int_0^\infty \frac{1}{2} R_2(t)G_{2,1}^2(t) dt = -0.677136959. \)

For \( 0 < x < 1, \) let \( x = 1 - \frac{t}{n+1}, \) then \( ES_u(0, 1) = \left( \frac{n}{n+1} \right) \int_0^\infty f_n \left( 1 - \frac{t}{n+1} \right) dt, \)
where \( f_n(\cdot) \) is defined by (1.4), and by (1.5)

\[ g_{2,n} \left( 1 - \frac{t}{n+1} \right) = o(n^{-2}), \quad (2.5) \]

and

\[ \left( \frac{n}{(n+t)^2} \right) g_{1,n} \left( 1 - \frac{t}{n+t} \right) = \left( 1 - \frac{2t}{n} + O \left( \frac{1}{n^2} \right) \right) \left( R_3(t) + \frac{S_3(t)}{n} + O \left( \frac{1}{n^2} \right) \right) \]
\[ = \left( R_3(t) + \frac{S_3(t) - 2tR_3(t)}{n} \right) + O \left( \frac{1}{n^2} \right), \quad (2.6) \]

where we observe that \( R_3(t) \equiv R_1(-t) \) and \( S_3(t) = S_{31}(t)/S_{32}(t), \) in which

\[ S_{31}(t) = \left( \left( 7t^2 - \frac{69}{2} t - \frac{63}{4} \right) e^{-6t} + (6t^3 + 35t - 55t^2 + 39) e^{-5t} \right) + \left( 49t - 4t^5 + 22t^4 + 91t^2 - \frac{63}{4} - 12t^3 \right) e^{-4t} + \left( -6t^3 - 30 - 44t^2 - 66t \right) e^{-3t} + \left( \frac{35}{2} + 2t^4 - 6t^2 + 16t^2 + \frac{123}{4} \right) e^{-2t} + \left( -9 - t - t^2 \right) e^{-t} + 3/4, \]
and \( S_{32}(t) \equiv S_{12}(-t) \). Now as \( t \to \infty \), \( R_3(t) = \frac{1}{2t} + O(t^{-1/2}e^{-t/2}) \), and \( S_3(t) = \frac{3}{4} + O(t^2e^{-t}). \)

Since the relation (2.6) is not term by term integrable we use the equality
\[
\frac{I_{[t>1]}(t)}{2t} - \frac{I_{[t>1]}(t)}{4n+2t} = \frac{I_{[t>1]}(t)}{2t} - \frac{I_{[t>1]}(t)}{4n+2t} + O\left(\frac{1}{n^2}\right),
\]
(2.7)
can be written as
\[
\frac{n}{(n+t)^2}f_n(1 - \frac{t}{n+t}) = \frac{1}{\pi} \left( R_3(t) - \frac{I_{[t>1]}(t)}{2t} \right) + \frac{1}{\pi} \left( \frac{I_{[t>1]}(t)}{2t} - \frac{I_{[t>1]}(t)}{4n+2t} \right) + \frac{1}{n\pi} \left( S_3(t) - 2tR_3(t) + \frac{I_{[t>1]}(t)}{4} \right) + O\left(\frac{1}{n^2}\right)
\]

Thus we have that
\[
ES_u(0,1) = \frac{n}{(n+t)^2} \int_0^\infty f_n(1 - \frac{t}{n+t})dt = \frac{1}{\pi} \int_0^\infty \left( R_3(t) - \frac{I_{[t>1]}(t)}{2t} \right) dt + \frac{1}{2\pi}(\log(4n+2) - \log(2)) + \frac{1}{n\pi} \int_0^\infty \left( S_3(t) - 2tR_3(t) + \frac{I_{[t>1]}(t)}{4} \right) dt + O\left(\frac{1}{n^2}\right)
\]
(2.8)
where \( \int_0^\infty \left( R_3(t) - \frac{I_{[t>1]}(t)}{2t} \right) dt = -0.2897712456 \) and \( \int_0^\infty \left( S_3(t) - 2tR_3(t) + \frac{I_{[t>1]}(t)}{4} \right) dt = 0.498174649. \)

For \(-1 < x < 0\), let \( x = -1 + \frac{t}{n+t} \), then \( ES_u(-1,0) = \left( \frac{n}{(n+t)^2} \right) \int_0^\infty f_n \left( -1 + \frac{t}{n+t} \right) dt \),

again by (1.4) and (1.5) we have
\[
\left( \frac{n}{(n+t)^2} \right) g_{1,n}(t) = \left( \frac{n^2}{(n+t)^2} \right) \left( R_4(t) + \frac{S_4(t)}{n} + O\left(\frac{1}{n^2}\right) \right),
\]
(2.9)
in which \( R_4(t) \equiv R_2(-t) \).

for \( n \) even \( S_4(t) = \frac{S_{41}(t) + S_{42}(t)}{4S_{43}(t)} \), and for \( n \) odd \( S_4(t) = \frac{S_{41}(t) - S_{42}(t)}{4S_{43}(t)} \)

In which
\[
S_{41}(t) = 8 \left( -9/4t + 6t^2 - 3t^3 - \frac{9}{8} + t^4 \right) e^{-2t} + 8 \left( 15t^4/4 - 3/2t - 22t^3 + 9/8 + 19/2t^2 - 2t^5 \right) e^{-4t} + 8 \left( \frac{15}{4}t - 7/2t^2 - 3/8 \right) e^{-6t} + 3,
\]
}\( n \) even
\[
\frac{n^2}{(n+t)^2} \left( R_4(t) + \frac{S_4(t)}{n} + O\left(\frac{1}{n^2}\right) \right),
\]
(2.9)
in which \( R_4(t) \equiv R_2(-t) \).

for \( n \) even \( S_4(t) = \frac{S_{41}(t) + S_{42}(t)}{4S_{43}(t)} \), and for \( n \) odd \( S_4(t) = \frac{S_{41}(t) - S_{42}(t)}{4S_{43}(t)} \)

In which
\[
S_{41}(t) = 8 \left( -9/4t + 6t^2 - 3t^3 - \frac{9}{8} + t^4 \right) e^{-2t} + 8 \left( 15t^4/4 - 3/2t - 22t^3 + 9/8 + 19/2t^2 - 2t^5 \right) e^{-4t} + 8 \left( \frac{15}{4}t - 7/2t^2 - 3/8 \right) e^{-6t} + 3,
\]
}\( n \) even
Thus we have that
\[ g_{2,n} \left( -1 + \frac{t}{n + t} \right) = \left( \frac{u^2}{n^3} \right) G_{2,1}(t) + o(n^{-1}) \]  
(2.10)

where
\[ G_{2,1}(t) = -\frac{16 \left( -e^{-2t} + 1 + 2e^{-2t}t^3 \right)}{\left( -4e^{-4t} + e^{-4t} - 12e^{-2t}t^2 + 4e^{-2t} + 1 + 8e^{-2t}t^3 - 2e^{-2t} \right)} \]

As \( t \to \infty \) we have
\[ R_4(t) = \frac{1}{2t} + O(t^{1/2}e^{-t}), \quad S_4(t) = \frac{3}{4} + O(te^{-t}), \quad G_{2,1}(t) = -16t^3 + O(t^4e^{-2t}). \]

We have that
\[ \frac{n^2}{(n + t)^2} f_n(-1 + \frac{t}{n + t}) = \frac{1}{\pi} \left( 1 - \frac{2t}{n} + O\left( \frac{1}{n^2} \right) \right) \left( R_4(t) + \frac{S_4(t)}{n} + O\left( \frac{1}{n^2} \right) \right) \times \left( 1 + \frac{u^2}{n^3} G_{2,1}(t) + \frac{u^4}{2n^6} G_{2,1}^2(t) + o(n^{-1}) \right) \]
\[ = \frac{1}{\pi} \left\{ R_4(t) + \frac{S_4(t) - 2t R_4(t)}{n} + \frac{u^2}{n^3} R_4(t) G_{2,1}(t) \right. \]
\[ + \frac{u^4}{2n^6} R_4(t) G_{2,1}^2(t) \left\} + o(n^{-1}) \]

Since this is not term by term integrable we use (2.7) and the following equalities to solve this
\[ \frac{8u^2 t^2}{n^3} = \frac{8u^2 t^2}{n^3 + \exp(t^3)} + o(n^{-2}), \quad \frac{64u^4 t^5}{n^6} = \frac{64u^4 t^5}{n^6 + \exp(t^6)} + o(n^{-2}) \]

Thus we have that
\[ E S_u(-1, 0) = \frac{n^2}{(n + t)^2} \int_0^\infty f_n(-1 + \frac{t}{n + t}) dt \]
\[ = \frac{1}{2\pi} (\log(2n + 1)) + \frac{1}{\pi} \left\{ \int_0^\infty (R_4(t) - \frac{I_{[t>1]}}{2t}) dt \right\} + \frac{1}{n} \int_0^\infty \left( S_4(t) - 2t R_4(t) + \frac{I_{[t>1]}}{4t} \right) dt \]
\[ + \frac{u^2}{n^3} \int_0^\infty (R_4(t) G_{2,1}(t) + 8t^2) dt - u^2 \int_0^\infty \frac{8t^2}{n^3 + \exp(t^3)} dt \]
\[ + \frac{u^4}{n^6} \int_0^\infty \left( \frac{1}{2} R_4(t) G_{2,1}^2(t) - 64t^5 \right) dt + u^4 \int_0^\infty \frac{64t^5}{n^6 + \exp(t^6)} dt \]
\[ + o(n^{-1}) \]  
(2.11)
where \( f_0^\infty (R_4(t) - \frac{I[t > 1]}{2t})dt = 0.3793914851 \), and
\[
\int_0^\infty (R_4(t)G_{2,1}(t) + 8t^2)dt = 21.47662610, \quad \int_0^\infty \left( \frac{1}{2} R_4(t)G_{2,1}^2(t) - 64t^5 \right)dt = -34997.02047
\]
and we have for \( n \) even \( \int_0^\infty (S_4(t) - 2t R_4(t) + \frac{I[t > 1]}{4})dt = -0.4999081999 \), and for \( n \) odd \( \int_0^\infty (S_4(t) - 2t R_4(t) + \frac{I[t > 1]}{4})dt = 1.499908200 \). Also we have that
\[
\int_0^\infty \frac{8t^2}{n^3 + \exp(t^3)} dt = \frac{8\ln(n^3 + 1)}{3n^3}, \quad \int_0^\infty \frac{64t^5}{n^6 + \exp(t^6)} dt = \frac{32\ln(n^6 + 1)}{3n^6}.
\]
So we arrive at the first assertion of the theorem.

Now for the proof of the second part of the theorem, that is for the case \( k = o\left(\frac{n}{2}\right) \) we study the asymptotic behavior of \( ES_u(a, b) \) for different intervals \((-\infty, -1), (-1, 0), (0, 1) \) and \((1, \infty) \) separately.

For \( 1 < x < \infty \), using the change of variable \( x = 1 + \frac{t}{n} \), and by \( (1.4), (1.5), (2.1) \) we find that
\[
\frac{1}{n} \int_0^\infty f_n(1 + \frac{t}{n})dt = \frac{1}{\pi} \int_0^\infty R_1(t)dt + o(1)
\]
where \( \int_0^\infty R_1(t)dt = 0.734874192 \).

For \( -\infty < x < -1 \) using the change of variable \( x = -1 - \frac{t}{n} \), and by the fact that \( u^2/n^3 = o(1), \) as \( n \to \infty \), we have that
\[
\exp\{u^2G_{2,1}(-1 + t/n)/n^3\} = 1 + o(1)
\]
Therefore the relations \( (2.4) \) implies that \( \exp\{g_2(-1 - t/n)\} = 1 + o(1) \). Thus by the relations \( (1.4), (1.5), (2.3), (2.4) \) we have that
\[
\frac{1}{n} \int_0^\infty f_n(-1 - \frac{t}{n})dt = \frac{1}{\pi} \int_0^\infty R_2(t)dt + o(1)
\]
where \( \int_0^\infty R_2(t)dt = 1.095640061 \).

For \( 0 < x < 1 \) using the change of variable \( x = 1 - \frac{t}{n+t} \), and relations \( (1.4), (1.5), (2.5), (2.6) \), and by using the equality \( (2.7) \) we have that
\[
\frac{n}{(n+t)^2} \int_0^\infty f_n(-1 - \frac{t}{n+t})dt = \frac{1}{\pi} \int_0^\infty (R_3(t) - \frac{I[t > 1]}{2t})dt + \frac{1}{2\pi} \log(2n + 1) + o(1)
\]
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where \( \int_0^\infty (R_3(t) - \frac{I[t>1]}{2t}) dt = -0.28977126 \).

For \(-1 < x < 0\), using the change of variable \( x = -1 + \frac{t}{n+t} \), (2.9), and by the same reasoning as above, the case \(-\infty < x < -1\), we have that \( \exp\{g_2(-1 + \frac{t}{n+t})\} = 1 + o(1) \). Thus by using the relations (2.9),(2.10), and by using the equality (2.7) we have that

\[
\frac{n}{(n+t)^2} \int_0^\infty f_n(-1 + \frac{t}{n+t}) dt = \frac{1}{\pi} \int_0^\infty (R_4(t) - \frac{I[t>1]}{2t}) dt + \frac{1}{2\pi} \log (2n + 1) + o(1)
\]

where \( \int_0^\infty (R_4(t) - \frac{I[t>1]}{2t}) dt = 0.3793914850 \). This complete the proof of the theorem.

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