OBSERVABILITY FOR THE SCHRÖDINGER EQUATION: AN OPTIMAL TRANSPORTATION APPROACH

FRANÇOIS GOLSE AND THIERRY PAUL

ABSTRACT. We establish an observation inequality for the Schrödinger equation on $\mathbb{R}^d$, uniform in the Planck constant $\hbar \in [0,1]$. The proof is based on the pseudometric introduced in [F. Golse, T. Paul, Arch. Rational Mech. Anal. 223 (2017), 57–94]. This inequality involves only effective constants which are computed explicitly in their dependence in $\hbar$ and all parameters involved.

1. Observation inequality for the Schrödinger equation

Consider the Schrödinger equation where the (real-valued) potential $V$ belongs to $C^{1,1}(\mathbb{R}^d)$ is such that the quantum Hamiltonian

$$-\frac{1}{2}\hbar^2 \Delta_y + V(y)$$

has a self-adjoint extension on $\mathcal{H} := L^2(\mathbb{R}^d)$:

$$i\hbar \partial_t \psi(t, y) = \left(-\frac{1}{2}\hbar^2 \Delta_y + V(y)\right) \psi(t, y), \quad \psi|_{t=0} = \psi^{in}.$$ (1)

In the equation above, $\hbar > 0$ the reduced Planck constant, and the particle mass is set to 1.

An observation inequality for the Schrödinger equation (1) is an inequality of the form

$$\|\psi^{in}\|_{\mathcal{H}}^2 \leq C \int_0^T \int_\Omega |\psi(t, x)|^2 dx dt,$$ (2)

for some $T > 0$, where $\Omega$ is an open subset of $\mathbb{R}^d$, and $C \equiv C[T, \Omega]$ is a positive constant, which holds for some appropriate class of initial data $\psi^{in}$ (see equation (2) in [9]).

Note that the r.h.s. of (2) is smaller than $CT$, so that (2) can be satisfied only when $CT \geq 1$. Moreover, it is easy to check that the case $CT = 1$ is possible only when $\Omega = \mathbb{R}^d$, and reduces that way to a tautology.

Therefore we will suppose in the sequel

$$CT > 1.$$

We will say that a compact subset $K$ of $\mathbb{R}^d \times \mathbb{R}^d$, an open set $\Omega$ of $\mathbb{R}^d$ and $T > 0$ satisfy the “(à la) Bardos-Lebeau-Rauch geometric condition” [2] if:

$$(GC) \quad \text{for each } (x, \xi) \in K, \text{ there exists } t \in (0,T) \text{ s.t. } X(t; x, \xi) \in \Omega.$$ (GC)

Let us recall the definition of the Schrödinger coherent state:

$$|q, p\rangle(x) := (\pi \hbar)^{-d/4} e^{-|x-q|^2/2\hbar} e^{ip(x-q^2)/\hbar}$$
providing a decomposition of the identity on $H$ (in a weak sense)

\[
\int_{\mathbb{R}^{2d}} \langle q, p \rangle \langle q, p \rangle \frac{dxdp}{(2\pi\hbar)^d} = I_H.
\]

Let us recall also, for any self-adjoint operator $A$ on $L^2(\mathbb{R}^d)$ and any $\psi \in L^2(\mathbb{R}^d)$, the definition of the standard deviation of $A$ in the state $\psi$, $\Delta_A(\psi) \in [0, +\infty]$:

\[
\Delta_A(\psi) = \sqrt{\left(\langle \psi, A^2 \psi \rangle_{L^2(\mathbb{R}^d)} - \langle \psi, A\psi \rangle_{L^2(\mathbb{R}^d)}^2\right)}.
\]

We define

\[
\Delta(\psi) := \sum_{j=1}^d \left(\Delta_{x_j}^2(\psi) + \Delta_{-i\hbar \partial x_j}^2(\psi)\right).
\]

Let us remark that, by the Heisenberg inequalities, for any $\psi \in H$,

\[
\Delta(\psi)^2 \geq \hbar.
\]

and, for any $(p, q) \in \mathbb{R}^{2d}$,

\[
\Delta([p, q]) = \sqrt{\hbar}.
\]

**Theorem 1.1.** Assume that $V$ belongs to $C^{1,1}(\mathbb{R}^d)$ and that $V^- \in L^{d/2}(\mathbb{R}^d)$.

Let $T > 0$, $\Omega$ be an open subset of $\mathbb{R}^d$, and $K$ be a compact set in $\mathbb{R}^{2d}$ satisfying the Bardos-Lebeau-Rauch condition (GC).

Moreover, let $\delta > 0$ and

\[
\Omega_{\delta} := \{x \in \mathbb{R}^d : \text{dist}(x, \Omega) < \delta\}.
\]

Then the Schrödinger equation (1) satisfies an observability property on $[0, T] \times \Omega_{\delta}$ of the form (2) with constant $C$ for all vectors $\psi \in H$ satisfying

\[
C[T, K, \Omega] \left(\int_K \langle |\psi| \rangle^2 \frac{dxdp}{(2\pi\hbar)^d}\right) - D[T, \text{Lip}(\nabla V)] \frac{\Delta(\psi)}{\delta} \geq \frac{1}{C}
\]

where

\[
C[T, K, \Omega] = \inf_{(x, \xi) \in K} \int_0^T 1_\Omega(X(t; x, \xi)) dt
\]

\[
D[T, \text{Lip}(\nabla V)] = \frac{e^{(1+\text{Lip}(\nabla V)^2)T/2} - 1}{1 + \text{Lip}(\nabla V)^2}.
\]

Moreover, the observation inequality will be satisfied for a non empty set of vectors as soon as $\delta$ satisfies the following, non sharp, bound:

\[
\delta \geq \frac{\sqrt{\hbar}}{D[T, \text{Lip}(\nabla V)] \sqrt{\delta}}
\]

\[
\frac{C[T, K, \Omega](1 - e^{-\frac{\delta^2}{4\hbar}}/(4\pi)^d) + C^{-1}}{d_K}
\]

where $d_K$ is the diameter of $K$.

The first part of Theorem 1.1 is exactly the second part (pure state case) of Corollary 4.2 of Theorem 4.1 in Section 4 below.
Controllability of the quantum dynamics has a long history in mathematics and mathematical physics. Giving an exhaustive bibliography on the subject is by far beyond the scope of the present paper, but the reader can consult the survey article [9] and the literature cited there, together with the important earlier references [3, 4], [10] far beyond the scope of the present paper, but the reader can consult the references [3, 4], [10].

For the bound on \( \delta \), we first remark that the quantity

\[
E[\psi, \delta] := C[T, K, \Omega^\delta]\left(\int_K |\langle \psi|p, q\rangle|^2 \frac{dp dq}{(2\pi \hbar)^2}\right) - D[T, \text{Lip}(\nabla V)] \frac{\Delta(\psi)}{\delta},
\]

needed to be strictly positive for the observability condition to hold true, is a difference between (a quantity proportional to) \( \int_K |\langle \psi|p, q\rangle|^2 \frac{dp dq}{(2\pi \hbar)^2} \leq 1 \) by (3), which evaluates the microlocalization of \( \psi \) on \( K \), and (a quantity proportional to) \( \Delta(\psi) \geq \sqrt{\frac{\delta}{\hbar}} \) by (5), which measures the spreading of \( \psi \) near its average position in phase-space.

However, this competition is balanced by the smallness of \( D[T, \text{Lip}(\nabla V)] \frac{\Delta(\psi)}{\delta} \) for large values of \( \delta \), namely \( E[\delta, \psi] \geq \frac{1}{\delta} \) when

\[
\delta \geq \frac{D[T, \text{Lip}(\nabla V)] \Delta(\psi)}{C[T, K, \Omega]\left(1 - \int_K |\langle \psi|p, q\rangle|^2 \frac{dp dq}{(2\pi \hbar)^2}\right)} + C^{-1}.
\]

Finally, we remark that, taking \( \psi = \langle p_0, q_0 \rangle \) for some \( (p_0, q_0) \in \mathbb{R}^{2d} \) we have, by (6),

\[
\Delta(\psi) = \sqrt{\delta \hbar},
\]

and, when \( (p_0, q_0) \) belongs to the interior of \( K \),

\[
\int_K |\langle p_0, q_0 |p, q\rangle|^2 \frac{dp dq}{(2\pi \hbar)^2} = 1 - \int_{\mathbb{R}^{2d} \setminus K} e^{-\frac{\|p_0 - p\|^2 + \|q_0 - q\|^2}{\hbar}} \frac{dp dq}{(2\pi \hbar)^d} \geq 1 - \frac{\text{dist}(\langle p_0, q_0 \rangle, \mathbb{R}^{2d}\setminus K)^2}{(4\pi)^d}.
\]

We conclude by picking \( (p_0, q_0) \) such that, for example, \( \text{dist}(\langle p_0, q_0 \rangle, \mathbb{R}^{2d}\setminus K) \geq \frac{\delta}{4\pi} \).

In the present paper, we will be working with the slightly more general Heisenberg equation

\[
(7) \quad i\hbar \partial_t R(t) = \left[-\frac{\hbar^2}{2} \Delta_0 + V(y), R(t)\right], \quad R|_{t=0} = R^{in} \geq 0, \quad \text{trace} R = 1,
\]

equivalent to the Schrödinger equation, modulo a global phase of the wave function, through the passage

\[
\psi \in \mathcal{F} \longrightarrow |\psi\rangle \langle \psi|,
\]

and whose underlying classical dynamics solves the Liouville equation

\[
\partial_t f(t, x, \xi) + \left\{ \frac{1}{2} |\xi|^2 + V(x), f(t, x, \xi) \right\} = 0, \quad \int_{t=0} = f^{in},
\]

where \( f^{in} \) is a probability density on \( \mathbb{R}^d \times \mathbb{R}^d \) having finite second moments.

Corollary 4.2 contains also an equivalent statement for initial conditions which are Töplitz operators. The general case of mixed states can be recovered by the inequality (12) inside the proof of Theorem 4.1.

The core of the paper is Theorem 4.1 in Section 4, whose proof needs the introduction in Section pseudomet of a class of pseudometrics adapted to the Heisenberg

\[
1\text{Note that } \int_{\mathbb{R}^{2d}} K|\langle p|p, q\rangle|^2 \frac{dp dq}{(2\pi \hbar)^2} \text{ is the integral over } K \text{ of the Husimi function of } \psi.
\]
equation (7), introduced in [6] after [5], and whose evolution under (7) is presented in Section 3.

2. A Pseudometric for Comparing Classical and Quantum Densities

This section elaborates on [6], with some marginal improvements.

A density operator on $\mathcal{H}$ is an operator $R \in \mathcal{L}(\mathcal{H})$ such that

$$R = R^* \geq 0, \quad \text{trace}(R) = 1.$$  

The set of all density operators on $\mathcal{H}$ will be denoted by $\mathcal{D}(\mathcal{H})$. We denote by $\mathcal{D}^2(\mathcal{H})$ the set of density operators on $\mathcal{H}$ such that

$$\text{trace}(R^{1/2}(-\hbar^2 \Delta_y + |y|^2) R^{1/2}) < \infty.$$  

If $R \in \mathcal{D}^2(\mathcal{H})$, one has

$$\text{trace}((-\hbar^2 \Delta_y + |y|^2)^{1/2} R (-\hbar^2 \Delta_y + |y|^2)^{1/2}) = \text{trace}(R^{1/2}(-\hbar^2 \Delta_y + |y|^2) R^{1/2}) < \infty$$

as can be seen from the lemma below (applied to $A = \lambda^2 |y|^2 - \hbar^2 \Delta_y$ and $T = R$).

**Lemma 2.1.** Let $T \in \mathcal{L}(\mathcal{H})$ satisfy $T = T^* \geq 0$, and let $A$ be an unbounded operator on $\mathcal{H}$ such that $A = A^* \geq 0$. Then

$$\text{trace}(T^{1/2} A^{1/2}) = \text{trace}(A^{1/2} R A^{1/2}) \in [0, +\infty].$$

**Proof.** The definition of $T^{1/2}$ and $A^{1/2}$ can be found in Theorem 3.35 in chapter V, §3 of [8], together with the fact that $A^{1/2}$ and $T^{1/2}$ are self-adjoint.

If $\text{trace}(T^{1/2} A^{1/2}) < \infty$, then $A^{1/2} T^{1/2} \in \mathcal{L}^2(\mathcal{H})$ and the equality holds by formula (1.26) in chapter X, §1 of [8]. If $\text{trace}(T^{1/2} A^{1/2}) = \infty$, then $\text{trace}(A^{1/2} T^{1/2}) = +\infty$, for otherwise $T^{1/2} A^{1/2}$ and its adjoint $A^{1/2} T^{1/2}$ would belong to $\mathcal{L}^2(\mathcal{H})$, so that $T^{1/2} A^{1/2} \in \mathcal{L}^1(\mathcal{H})$, which would be in contradiction with the assumption that $\text{trace}(T^{1/2} A^{1/2}) = \infty$. \hfill \Box

Let $f \equiv f(x, \xi)$ be a probability density on $\mathbb{R}^d \times \mathbb{R}^d$ such that

$$\int_{\mathbb{R}^d \times \mathbb{R}^d} (|x|^2 + |\xi|^2) f(x, \xi) \, dx \, d\xi < \infty.$$  

A coupling of $f$ and $R$ is a measurable operator-valued function $(x, \xi) \mapsto Q(x, \xi)$ such that, for a.e. $(x, \xi) \in \mathbb{R}^d \times \mathbb{R}^d$,

$$Q(x, \xi) = Q(x, \xi)^* \geq 0, \quad \text{trace}(Q(x, \xi)) = f(x, \xi), \quad \int_{\mathbb{R}^d \times \mathbb{R}^d} Q(x, \xi) \, dx \, d\xi = R.$$  

The second condition above implies that $Q(x, \xi) \in \mathcal{L}^1(\mathcal{H})$ for a.e. $(x, \xi) \in \mathbb{R}^d \times \mathbb{R}^d$. Since $\mathcal{L}^1(\mathcal{H})$ is separable, the notion of strong and weak measurability are equivalent for $Q$. The set of couplings of $f$ and $R$ is denoted by $\mathcal{C}(f, R)$. Notice that the function $(x, \xi) \mapsto f(x, \xi) R$ belongs to $\mathcal{C}(f, R)$.

In [6], one considers the following “pseudometric”: for each probability density $f$ on $\mathbb{R}^d \times \mathbb{R}^d$ and each $R \in \mathcal{D}^2(\mathcal{H})$,

$$E_{h, \lambda}(f, R) := \inf_{Q \in \mathcal{C}(f, R)} \left( \int_{\mathbb{R}^d \times \mathbb{R}^d} \text{trace}_\mathcal{H}(Q(x, \xi)^{1/2} c_\lambda(x, \xi, y, h D_y) Q(x, \xi)^{1/2}) \, dx \, d\xi \right)^{1/2}$$

where the quantum transportation cost is the quadratic differential operator in $y$, parametrized by $(x, \xi) \in \mathbb{R}^d \times \mathbb{R}^d$,

$$c_\lambda(x, \xi, y, h D_y) := \lambda^2 |x - y|^2 + |\xi - h D_y|^2, \quad D_y := -i \nabla_y.$$
Lemma 2.2. If \( R \in \mathcal{D}^2(S) \) while \( f \) is a probability density on \( \mathbb{R}^d \times \mathbb{R}^d \) with finite second moment (10), one has

\[
\int_{\mathbb{R}^d \times \mathbb{R}^d} \text{trace}_\delta(Q(x, \xi)\frac{1}{2}c(x, \xi, y, hD_y)Q(x, \xi)\frac{1}{2})dxd\xi
= \int_{\mathbb{R}^d \times \mathbb{R}^d} \text{trace}_\delta(c(x, \xi, y, hD_y)\frac{1}{2}Q(x, \xi)c(x, \xi, y, hD_y)\frac{1}{2})dxd\xi
\leq 2 \int_{\mathbb{R}^d \times \mathbb{R}^d} (\lambda^2|x|^2 + |\xi|^2) f(x, \xi)dxd\xi + 2 \text{trace}(R^{1/2}(-h^2\Delta_y + \lambda^2|y|^2)R^{1/2}) < \infty
\]
for each \( Q \in \mathcal{C}(f, R) \).

Proof. Notice that \( c_\lambda(x, \xi, y, hD_y) \leq 2\lambda^2(|x|^2 + |y|^2) + 2(|\xi|^2 - h^2\Delta_y) = 2(\lambda^2|x|^2 + |\xi|^2) + 2(\lambda^2|y|^2 - h^2\Delta_y) \)
so that

\[
\int_{\mathbb{R}^d \times \mathbb{R}^d} \text{trace}_\delta(Q(x, \xi)\frac{1}{2}c(x, \xi, y, hD_y)Q(x, \xi)\frac{1}{2})dxd\xi
\leq 2 \int_{\mathbb{R}^d \times \mathbb{R}^d} \text{trace}_\delta(Q(x, \xi)\frac{1}{2}(\lambda^2|x|^2 + |\xi|^2)Q(x, \xi)\frac{1}{2})dxd\xi
+ 2 \int_{\mathbb{R}^d \times \mathbb{R}^d} \text{trace}_\delta(Q(x, \xi)\frac{1}{2}(\lambda^2|y|^2 - h^2\Delta_y)Q(x, \xi)\frac{1}{2})dxd\xi.
\]

First

\[
\int_{\mathbb{R}^d \times \mathbb{R}^d} \text{trace}_\delta(Q(x, \xi)\frac{1}{2}(\lambda^2|x|^2 + |\xi|^2)Q(x, \xi)\frac{1}{2})dxd\xi
\]

\[
= \int_{\mathbb{R}^d \times \mathbb{R}^d} (\lambda^2|x|^2 + |\xi|^2) \text{trace}_\delta(Q(x, \xi))dxd\xi
\]

\[
= \int_{\mathbb{R}^d \times \mathbb{R}^d} (\lambda^2|x|^2 + |\xi|^2)f(x, \xi)dxd\xi.
\]

Since \( R \in \mathcal{D}^2(S) \), one has

\[
\text{trace}_\delta(R^{1/2}(\lambda^2|y|^2 - h^2\Delta_y)R^{1/2}) = \text{trace}_\delta(\lambda^2|y|^2 - h^2\Delta_y)^{1/2}R(\lambda^2|y|^2 - h^2\Delta_y)^{1/2})
\]

\[
= \int_{\mathbb{R}^d \times \mathbb{R}^d} \text{trace}_\delta((\lambda^2|y|^2 - h^2\Delta_y)^{1/2}Q(x, \xi)dxd\xi(\lambda^2|y|^2 - h^2\Delta_y)^{1/2}) < \infty,
\]

where the first equality is (9), while the second follows from the monotone convergence theorem (Theorem 1.27 in [11]) applied to a spectral decomposition of the harmonic oscillator \( \lambda^2|y|^2 - h^2\Delta_y \).

In particular

\[
\text{trace}_\delta((\lambda^2|y|^2 - h^2\Delta_y)^{1/2}Q(x, \xi)(\lambda^2|y|^2 - h^2\Delta_y)^{1/2}) < \infty
\]

for a.e. \((x, \xi) \in \mathbb{R}^d \times \mathbb{R}^d \). Applying Lemma 2.1 to \( A = \lambda^2|y|^2 - h^2\Delta_y \) and \( T = Q(x, \xi) \)
for a.e. \((x, \xi) \in \mathbb{R}^d \times \mathbb{R}^d \), one has

\[
\text{trace}_\delta((\lambda^2|y|^2 - h^2\Delta_y)^{1/2}Q(x, \xi)(\lambda^2|y|^2 - h^2\Delta_y)^{1/2})
= \text{trace}_\delta(Q(x, \xi)^{1/2}(\lambda^2|y|^2 - h^2\Delta_y)Q(x, \xi)^{1/2})
\]

for a.e. \((x, \xi) \in \mathbb{R}^d \times \mathbb{R}^d \). Integrating both sides of this equality over \( \mathbb{R}^d \times \mathbb{R}^d \), one finds that

\[
\int_{\mathbb{R}^d \times \mathbb{R}^d} \text{trace}_\delta(Q(x, \xi)^{1/2}(\lambda^2|y|^2 - h^2\Delta_y)Q(x, \xi)^{1/2})dxd\xi
= \text{trace}_\delta(((\lambda^2|y|^2 - h^2\Delta_y)^{1/2}R(\lambda^2|y|^2 - h^2\Delta_y)^{1/2}) < \infty.
\]
In particular
\[
\text{trace}_\mu(Q(x, \xi)^{1/2}c(x, \xi, y, hD_y)Q(x, \xi)^{1/2}) < \infty
\]
for a.e. \((x, \xi) \in \mathbb{R}^d \times \mathbb{R}^d\). Applying again Lemma 2.1 with \(A = c(x, \xi, y, hD_y)\) and \(T = Q(x, \xi)\) for all such \((x, \xi)\) shows that
\[
\text{trace}_\mu(Q(x, \xi)^{1/2}c(x, \xi, y, hD_y)Q(x, \xi)^{1/2}) = \text{trace}_\mu(c(x, \xi, y, hD_y)^{1/2}Q(x, \xi)c(x, \xi, y, hD_y)^{1/2})
\]
for a.e. \((x, \xi) \in \mathbb{R}^d\), and the equality in the lemma follows from integrating both sides of this last identity over \(\mathbb{R}^d \times \mathbb{R}^d\).

The main properties of this pseudo-metric are recalled in the following theorem. Before stating it, we recall some fundamental notions and introduce some notations.

The Wigner transform of \(R \in \mathcal{D}(\mathcal{F})\) is
\[
W_h[R](x, \xi) = \frac{1}{(2\pi)^{d+1}} \int_{\mathbb{R}^{2d}} r(x + \frac{1}{2}hy, x - \frac{1}{2}hy)e^{-iy\cdot \xi}dy
\]
where \(r\) is the integral kernel of \(R\). Obviously \(W_h[R]\) is real-valued, but in general \(W_h[R]\) is not a.e. nonnegative in general.

Instead of the Wigner transform, one can consider a mollified variant thereof, the Husimi transform of \(R\), that is
\[
\tilde{W}_h[R](x, \xi) = (e^{h\Delta_x/4}W_h[R])(x, \xi) \geq 0 \text{ for a.e. } (x, \xi) \in \mathbb{R}^d \times \mathbb{R}^d.
\]
The Schrödinger coherent state is
\[
|q, p\rangle(x) := (\pi \hbar)^{-d/4}e^{-|x-q|^2/2\hbar}e^{ip(x-q)/\hbar}.
\]
For each Borel probability measure \(\mu\) on \(\mathbb{R}^d \times \mathbb{R}^d\), one defines the Töplitz operator with symbol \((2\pi \hbar)^d\mu:\)
\[
\text{OP}_h^T[(2\pi \hbar)^d\mu] := \int_{\mathbb{R}^d \times \mathbb{R}^d} |q, p\rangle(q, p)\mu(dqdp) \in \mathcal{D}(\mathcal{F})
\]

**Proposition 2.3.** For each probability density \(f\) and each Borel probability measure \(\mu\) on \(\mathbb{R}^d \times \mathbb{R}^d\) with finite second order moment (10). Then
\[
\text{OP}_h^T[(2\pi \hbar)^d\mu] \in \mathcal{D}^2(\mathcal{F}),
\]
and one has
\[
E_{h, \lambda}(f, \text{OP}_h^T[(2\pi \hbar)^d\mu])^2 \leq \max(1, \lambda^2) \text{dist}_{\text{MK}, 2}(f, \mu)^2 + \frac{1}{2}(\lambda^2 + 1)\hbar.
\]

**Proof.** Let \(P(x, \xi, dqdp)\) be an optimal coupling of \(f(x, \xi)\) and \(\mu(dqdp)\) for \(\text{dist}_{\text{MK}, 2}\).

Set \(Q(x, \xi) := \text{OP}_h^T[(2\pi \hbar)^dF(x, \xi, \cdot)]\). Then \(Q \in \mathcal{C}(f, \text{OP}_h^T[(2\pi \hbar)^d\mu])\) according to Lemma 3.1 in [6], so that
\[
E_{h, \lambda}(f, \text{OP}_h^T[(2\pi \hbar)^d\mu])^2 \leq \int_{\mathbb{R}^d \times \mathbb{R}^d} \text{trace}_\mu(Q(x, \xi)^{1/2}c(x, \xi, y, hD_y)Q(x, \xi)^{1/2})dxd\xi.
\]
For each \(p, q \in \mathbb{R}^d\), one has
\[
\text{trace}_\mu(c(x, \xi, y, hD_y)^{1/2}|q, p\rangle(q, p)c(x, \xi, y, hD_y)^{1/2}) = \langle q, p|c(x, \xi, y, hD_y)|q, p\rangle = \lambda^2|x-q|^2 + |\xi - p|^2 + \frac{1}{2}(\lambda^2 + 1)\hbar
\]
according to fla. (55) in [5]. For each finite positive Borel measure \( m \) on \( \mathbb{R}^d \times \mathbb{R}^d \), one has
\[
\text{trace}_B (c_\lambda(x, \xi, y, hD_y)^{1/2} \mathcal{OP}_h^T[(2\pi \hbar)^d m] c_\lambda(x, \xi, y, hD_y)^{1/2})
= \int_{\mathbb{R}^d \times \mathbb{R}^d} (\lambda^2 |x - q|^2 + |\xi - p|^2 + \frac{1}{2}(\lambda^2 + 1)\hbar) m(dqdp) .
\]
by the monotone convergence theorem (Theorem 1.27 in [11]) applied to a spectral decomposition of the transportation cost operator \( c_\lambda(x, \xi, y, hD_y) \), which is a shifted harmonic oscillator.

Specializing this formula to the case \( x = \xi = 0 \) and \( m = \mu \) shows that the operator \( \mathcal{OP}_h^T[(2\pi \hbar)^d \mu] \in \mathcal{D}^2(\mathcal{F}) \).

Specializing this formula to the case \( m = P(x, \xi, dqdp) \) and integrating in \((x, \xi)\) shows that
\[
\int_{\mathbb{R}^d \times \mathbb{R}^d} \text{trace}_B (c_\lambda(x, \xi, y, hD_y)^{1/2} \mathcal{OP}_h^T[(2\pi \hbar)^d P(x, \xi, \cdot)] c_\lambda(x, \xi, y, hD_y)^{1/2}) dx d\xi
= \int_{\mathbb{R}^d \times \mathbb{R}^d} \int_{\mathbb{R}^d \times \mathbb{R}^d} (\lambda^2 |x - q|^2 + |\xi - p|^2) P(x, \xi, dqdp) + \frac{1}{2}(\lambda^2 + 1)\hbar
= \text{dist}_{\text{MK}}(f, \mu)^2 + \frac{1}{2}(\lambda^2 + 1)\hbar
\]
and since \( Q : (x, \xi) \mapsto \mathcal{OP}_h^T[(2\pi \hbar)^d P(x, \xi, \cdot)] \) belongs to \( C(f, \mathcal{OP}_h^T[(2\pi \hbar)^d \mu]) \),
\[
\int_{\mathbb{R}^d \times \mathbb{R}^d} \text{trace}_B (Q(x, \xi)^{1/2} c_\lambda(x, \xi, y, hD_y) Q(x, \xi)^{1/2}) dx d\xi
= \int_{\mathbb{R}^d \times \mathbb{R}^d} \text{trace}_B (c_\lambda(x, \xi, y, hD_y)^{1/2} Q(x, \xi) c_\lambda(x, \xi, y, hD_y)^{1/2}) dx d\xi .
\]
With the previous equality and the inequality above, the proof is complete. \( \square \)

3. Evolution of the pseudo-metric under the Schrödinger dynamics

Denote by \( t \mapsto (X(t; x, \xi), \Xi(t; x, \xi)) \) the solution of the Cauchy problem for the Hamiltonian system
\[
\dot{X} = \Xi, \quad \dot{\Xi} = -\nabla V(X), \quad (X(0; x, \xi), \Xi(0; x, \xi)) = (x, \xi) .
\]
Since \( V \in C^{1,1}(\mathbb{R}^d) \), this solution is defined for all \( t \in \mathbb{R} \), for all \( x, \xi \in \mathbb{R}^d \). Henceforth, we denote by \( \Phi_t \), the map \((x, \xi) \mapsto \Phi_t(x, \xi) := (X(t; x, \xi), \Xi(t; x, \xi)) \), and by \( H = H(x, \xi) := \frac{1}{2}(|\xi|^2 + V(x)) \) the Hamiltonian.

On the other hand, assume that \( V^- \in L^{1/2}(\mathbb{R}^d) \), so that \( H = -\frac{1}{2}h^2 \Delta + V \) is self-adjoint on \( \mathcal{F} \) by Lemma 4.8b in chapter VI, §4 of [8]. Then \( U(t) := \exp(itH/h) \) is a unitary group on \( \mathcal{F} \).

**Theorem 3.1.** Let \( f^{in} \) be a probability density on \( \mathbb{R}^d \times \mathbb{R}^d \) which satisfies (10), and let \( R^{in} \in \mathcal{D}^2(\mathcal{F}) \). For each \( t \geq 0 \), set
\[
R(t) := U(t)^* R^{in} U(t) , \quad f(t, X, \Xi) := f^{in}(\Phi_{-t}(X, \Xi)) \quad \text{for a.e.} \quad (X, \Xi) \in \mathbb{R}^d \times \mathbb{R}^d .
\]
Then, for each \( \lambda > 0 \) and each \( t \geq 0 \), one has
\[
E_{h, \lambda}(f(t, \cdot, \cdot), R(t)) \leq E_{h, \lambda}(f^{in}, R^{in}) \exp \left( \frac{\lambda}{2t} \left( \text{Lip}(\nabla V)^2 \right) \right) .
\]
This theorem is a slight improvement of Theorem 2.7 in [6] in the special case \( N = 1 \). For the sake of being complete, we recall the argument in [6], with the appropriate modifications.
Proof. Let $Q^{in} \in \mathcal{C}(f^{in}, R^{in})$. Set

$$Q(t, X, \Xi) := U(t)^* Q^{in} \circ \Phi_{-t}(X, \Xi) U(t)$$

for all $t \in \mathbb{R}$ and a.e. $(x, \xi) \in \mathbb{R}^d \times \mathbb{R}^d$, and

$$\mathcal{E}(t) := \int_{\mathbb{R}^{2d}} \text{trace}_3 (Q(t, X, \Xi)^{1/2} c_\lambda(X, \Xi, y, hD_y) Q(t, X, \Xi)^{1/2}) dxd\Xi.$$

Since $\Phi_t$ leaves the phase space volume element $dxd\xi$ invariant

$$\mathcal{E}(t) = \int_{\mathbb{R}^{2d}} \text{trace}_3 (\sqrt{Q^{in}(x, \xi)} U(t) c_\lambda(\Phi_t(x, \xi), y, hD_y) U(t)^* \sqrt{Q^{in}(x, \xi)}) dxd\xi.$$

By construction, $Q(t, \cdot, \cdot) \in \mathcal{C}(f(t, \cdot, \cdot), R(t))$. Indeed, for a.e. $(X, \Xi) \in \mathbb{R}^d$,

$$0 \leq Q^{in}(\Phi_{-t}(X, \Xi)) = Q^{in}(\Phi_{-t}(X, \Xi))^* \in \mathcal{L}(H)$$

so that $Q(t, X, \Xi) \in \mathcal{L}(H)$ satisfies

$$Q(t, X, \Xi) = U(t) Q^{in}(\Phi_{-t}(X, \Xi)) U(t)^*$$

$$= U(t) Q^{in}(\Phi_{-t}(X, \Xi))^* U(t)^* = Q(t, X, \Xi)^* \geq 0.$$

Besides

$$\text{trace}_3 (Q(t, X, \Xi)) = \text{trace}_3 (Q^{in}(\Phi_{-t}(X, \Xi))) = f^{in}(\Phi_{-t}(X, \Xi)) = f(t, X, \Xi)$$

while

$$\int_{\mathbb{R}^d \times \mathbb{R}^d} Q(t, X, \Xi) dXd\Xi = U(t) \left( \int_{\mathbb{R}^d \times \mathbb{R}^d} Q^{in}(\Phi_{-t}(X, \Xi)) dXd\Xi \right) U(t)^*$$

$$= U(t) \left( \int_{\mathbb{R}^d \times \mathbb{R}^d} Q^{in}(x, \xi) dxd\xi \right) U(t)^* = U(t) R^{in} U(t)^* = R(t).$$

In particular

$$\mathcal{E}(t) \geq E_{h, \lambda}(f(t), R(t)), \quad \text{for each } t \geq 0.$$

Let $e_j(x, \xi, \cdot)$ for $j \in \mathbb{N}$ be a $\mathcal{H}$-complete orthonormal system of eigenvectors of $Q^{in}(x, \xi)$ for a.e. $x, \xi \in \mathbb{R}^d$. Hence

$$\text{trace}_3 (\sqrt{Q^{in}(x, \xi)} U(t) c_\lambda(\Phi_t(x, \xi), y, hD_y) U(t)^* \sqrt{Q^{in}(x, \xi)})$$

$$= \sum_{j \in \mathbb{N}} \rho_j(x, \xi) \langle U(t) c_j(x, \xi) | c_\lambda(\Phi_t(x, \xi), y, hD_y) | U(t) c_j(x, \xi) \rangle$$

where $\rho_j(x, \xi)$ is the eigenvalue of $Q^{in}(x, \xi)$ defined by

$$Q^{in}(x, \xi) e_j(x, \xi) = \rho_j(x, \xi) e_j(x, \xi), \quad \text{for a.e. } (x, \xi) \in \mathbb{R}^d \times \mathbb{R}^d.$$

If $\phi \equiv \phi(y) \in C_\infty^\infty(\mathbb{R}^d)$, the map

$$t \mapsto \langle U(t) \phi | c_\lambda(\Phi_t(x, \xi), y, hD_y) | U(t) \phi \rangle$$

is of class $C^1$ on $\mathbb{R}$, and one has

$$\frac{d}{dt} \langle U(t) \phi | c_\lambda(\Phi_t(x, \xi), y, hD_y) | U(t) \phi \rangle$$

$$= \left\{ \frac{i}{h} \mathcal{H} U(t) \phi | c_\lambda(\Phi_t(x, \xi), y, hD_y) | U(t) \phi \right\}$$

$$+ \langle U(t) \phi | c_\lambda(\Phi_t(x, \xi), y, hD_y) \left| \frac{i}{h} \mathcal{H} U(t) \phi \right\rangle$$

$$+ \langle U(t) \phi | [H(\Phi_t(x, \xi)), c_\lambda(\Phi_t(x, \xi), y, hD_y)] | U(t) \phi \rangle.$$
In other words
\[
\frac{d}{dt} (U(t) \phi|c_\lambda(\Phi_t(x, \xi), y, hD_y)) U(t) \phi
\]
\[
= \left\{ U(t) \phi \left[ i \hbar \{ \mathcal{H}, c_\lambda(\Phi_t(x, \xi), y, hD_y) \} \right] U(t) \phi \right\}
\]
\[
+ (U(t) \phi | \{ H(\Phi_t(x, \xi)), c_\lambda(\Phi_t(x, \xi), y, hD_y) \}) | U(t) \phi \right).
\]

A straightforward computation shows that
\[
\{ H(\Phi_t(x, \xi)), c_\lambda(\Phi_t(x, \xi), y, hD_y) \} = \lambda \frac{d}{dt} \sum_{k=1}^d \left( (X_k - y_k)(\Xi_k - hD_{y_k}) + (\Xi_k - hD_{y_k})(X_k - y_k) \right)
\]
\[
- \sum_{k=1}^d ((\partial_k V(X) - \partial_k V(y))(\Xi_k - hD_{y_k}) + (\Xi_k - hD_{y_k})(\partial_k V(X) - \partial_k V(y)))
\]
\[
\leq \lambda \sum_{k=1}^d (\lambda^2 |X_k - y_k|^2 + |\Xi_k - hD_{y_k}|^2) + \frac{1}{\lambda} \sum_{k=1}^d \left( \lambda^2 |\partial_k V(X) - \partial_k V(y)|^2 + |\Xi_k - hD_{y_k}|^2 \right)
\]
\[
\leq \lambda \sum_{k=1}^d (\lambda^2 |X_k - y_k|^2 + |\Xi_k - hD_{y_k}|^2) + \frac{\text{Lip}(\nabla V)^2}{\lambda} \sum_{k=1}^d \left( \lambda^2 |X_k - y_k|^2 + |\Xi_k - hD_{y_k}|^2 \right)
\]
\[
\leq \left( \lambda + \frac{\text{Lip}(\nabla V)^2}{\lambda} \right) c_\lambda(X, \Xi, y, hD_y).
\]

Hence
\[
\{ U(t) \phi | c_\lambda(\Phi_t(x, \xi), y, hD_y) \} | U(t) \phi \leq \langle \phi | c_\lambda(x, \xi, y, hD_y) | \phi \rangle
\]
\[
+ \left( \lambda + \frac{\text{Lip}(\nabla V)^2}{\lambda} \right) \int_0^t \langle U(s) \phi | c_\lambda(\Phi_s(x, \xi), y, hD_y) | U(s) \phi \rangle ds
\]
so that
\[
\langle U(t) \phi | c_\lambda(\Phi_t(x, \xi), y, hD_y) | U(t) \phi \rangle \leq \langle \phi | c_\lambda(x, \xi, y, hD_y) | \phi \rangle \exp \left( \left( \lambda + \frac{\text{Lip}(\nabla V)^2}{\lambda} \right) t \right)
\]
for each \( \phi \in C_c^\infty(\mathbb{R}^d) \). By density of \( C_c^\infty(\mathbb{R}^d) \) in the form domain of \( c_\lambda(x, \xi, y, hD_y) \)
\[
0 \leq \langle U(t) e_j(x, \xi) | c_\lambda(\Phi_t(x, \xi), y, hD_y) | U(t) e_j(x, \xi) \rangle
\]
\[
\leq \langle e_j(x, \xi) | c_\lambda(x, \xi, y, hD_y) | e_j(x, \xi) \rangle \exp \left( \left( \lambda + \frac{\text{Lip}(\nabla V)^2}{\lambda} \right) t \right)
\]
for a.e. \( (x, \xi) \in \mathbb{R}^d \times \mathbb{R}^d \), so that
\[
\text{trace}_{\delta}(\sqrt{Q^{in}(x, \xi)} U(t) c_\lambda(\Phi_t(x, \xi), y, hD_y) U(t)^* \sqrt{Q^{in}(x, \xi)})
\]
\[
= \sum_{j \in \mathbb{N}} \rho_j(x, \xi) \langle U(t) e_j(x, \xi) | c_\lambda(\Phi_t(x, \xi), y, hD_y) | U(t) e_j(x, \xi) \rangle
\]
\[
\leq \exp \left( \left( \lambda + \frac{\text{Lip}(\nabla V)^2}{\lambda} \right) t \right) \sum_{j \in \mathbb{N}} \rho_j(x, \xi) \langle e_j(x, \xi) | c_\lambda(x, \xi, y, hD_y) | e_j(x, \xi) \rangle
\]
\[
= \exp \left( \left( \lambda + \frac{\text{Lip}(\nabla V)^2}{\lambda} \right) t \right) \text{trace}_{\delta}(\sqrt{Q^{in}(x, \xi)} c_\lambda(x, \xi, y, hD_y) \sqrt{Q^{in}(x, \xi)}).
\]
Integrating both side of this inequality over $\mathbb{R}^d \times \mathbb{R}^d$ shows that

$$\mathcal{E}(t) \leq \mathcal{E}(0) \exp\left(\frac{\lambda + \text{Lip}(\nabla V)^2}{\lambda} t\right).$$

Hence, for each $t \geq 0$ and each $Q^\text{in} \in \mathcal{C}(f, R)$, one has

$$E_{h, \lambda}(f(t), R(t))^2 \leq \mathcal{E}(0) \exp\left(\frac{\lambda + \text{Lip}(\nabla V)^2}{\lambda} t\right).$$

Minimizing the right hand side of this inequality as $Q^\text{in}$ runs through $\mathcal{C}(f^\text{in}, R^\text{in})$, one arrives at the inequality

$$E_{h, \lambda}(f(t), R(t)) \leq E_{h, \lambda}(f^\text{in}, R^\text{in}) \exp\left(\frac{1}{2} \left(\frac{\lambda + \text{Lip}(\nabla V)^2}{\lambda}\right) t\right).$$

\[\square\]

4. The observation inequality

In this section, we state and prove an observation inequality for the Schrödinger equation.

Let $K$ be a compact subset of $\mathbb{R}^d \times \mathbb{R}^d$, let $\Omega$ be an open set of $\mathbb{R}^d$ and let $T > 0$. We recall the “geometric condition” à la Bardos-Lebeau-Rauch [2] for this problem:

$$(\text{GC}) \quad \text{for each } (x, \xi) \in K, \text{ there exists } t \in (0, T) \text{ s.t. } X(t; x, \xi) \in \Omega.$$

**Theorem 4.1.** Assume that $V$ belongs to $C^{1,1}(\mathbb{R}^d)$ and that $V^- \in L^{d/2}(\mathbb{R}^d)$. Let $T > 0$, let $K \subset \mathbb{R}^d \times \mathbb{R}^d$ be compact and let $\Omega \subset \mathbb{R}^d$ be an open set of $\mathbb{R}^d$ satisfying $(\text{GC})$. Let $\chi \in \text{Lip}(\mathbb{R}^d)$ be such that $\chi(x) > 0$ for each $x \in \Omega$.

For each $t \geq 0$, set

$$R(t) := U(t)^* R^\text{in} U(t), \quad f(t, X, \Xi) := f^\text{in}(\Phi^{-1}(X, \Xi)) \quad \text{for a.e. } (X, \Xi) \in \mathbb{R}^d \times \mathbb{R}^d.$$

Then, when $R^\text{in}$ is a pure state $|\psi^\text{in}\rangle \langle \psi^\text{in}|$, one has

$$\int_0^T \int_{\mathbb{R}^d} \chi(x) |\psi(t, x)|^2 dx dt \geq \inf_{(\chi, \xi) \in K} \int_0^T \chi(X(t; x, \xi)) dt \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \varpi_h[\psi^\text{in}](x, \xi) dx d\xi$$

$$- 4\text{Lip}(\chi) \exp\left(\frac{\frac{1}{2}(1 + \text{Lip}(\nabla V)^2)}{2(1 + \text{Lip}(\nabla V)^2)} T\right) - 1 \Delta(\psi^\text{in}).$$

When $R^\text{in} := \text{OP}^T[(2\pi \hbar)^d f^\text{in}]$ is a Töplitz operator of symbol a probability density $f^\text{in}$ on $\mathbb{R}^d \times \mathbb{R}^d$ with support in $K$,

$$\int_0^T \text{trace}(\chi R(t)) dt \geq \inf_{(\chi, \xi) \in K} \int_0^T \chi(X(t; x, \xi)) dt$$

$$- \text{Lip}(\chi) C(T, \text{Lip}(\nabla V)) \sqrt{2dh}$$

where

$$C(T, L) = \inf_{\lambda > 0} \frac{\exp\left(\frac{1}{2} \left(\frac{\lambda + \frac{L^2}{\lambda}}{\lambda} + \frac{1}{2}\right) T\right)}{\left(\lambda + \frac{L^2}{\lambda}\right)} \frac{1}{\sqrt{1 + \frac{1}{\lambda^2}}}.$$ 

In particular, setting $\lambda = L$

$$C(T, L) \leq e^{LT} - 1 \sqrt{1 + \frac{1}{L^2}}.$$
In fact, one can eliminate all mention of the cutoff function $\chi$ in the final statement, as follows.

**Corollary 4.2.** Under the same assumptions as in Theorem 4.1, one has

$$C[T, K, \Omega] := \inf_{(x, \xi) \in K} \int_0^T 1_{\Omega}(X(t; x, \xi)) dt > 0,$$

and for each $\delta > 0$, denoting $\Omega_{\delta} := \{ x \in \mathbb{R}^d \mid \text{dist}(x, \Omega) < \delta \}$,

$$\int_0^T \text{trace}(1_{\Omega_{\delta}} R(t)) dt \geq C[T, K, \Omega] - C(T, \text{Lip}(\nabla V)) \frac{\sqrt{2\hbar}}{\delta}$$

in the Töplitz case, and

$$\int_0^T \int_{\Omega_{\delta}} |\psi(t, x)|^2 dx dt \geq \inf_{(x, \xi) \in K} \int_0^T 1_{\Omega}(X(t; x, \xi)) dt \int_{(x, \xi) \in K} \int_{(x, \xi) \in K} \frac{W_h[\psi^{in}](x, \xi) dx d\xi}{\Delta(\psi^{in})}$$

$$- 4 \exp\left(\frac{1}{2} \left(1 + \text{Lip}(\nabla V)^2\right)^T\right) - 1 \Delta(\psi^{in}) \frac{\delta}{\sqrt{2(1 + \text{Lip}(\nabla V)^2)}}$$

in the pure state case.

The corollary can be used to obtain an observation inequality for Töplitz operators as “test observables” as follows: let $T > 0$ be an observation time, let $K \subset \mathbb{R}^d \times \mathbb{R}^d$ be a compact subset of the phase-space supporting the initial data, and let $\Omega \subset \mathbb{R}^d$ be the open set where one observes the solution of the Schrödinger equation on the time interval $[0, T]$. Assume that $T, K, \Omega$ satisfies the geometric condition (GC). With these data, one computes $C[T, K, \Omega] > 0$. Choose then $\hbar, \delta > 0$ so that

$$\frac{\hbar}{\delta^2} < \frac{C[T, K, \Omega]^2}{2dC(T, \text{Lip}(\nabla V))^2}.$$

Then the Heisenberg equation (7) satisfies the observability property on $[0, T] \times \Omega_{\delta}$ for all Töplitz initial density operators whose symbol is supported in $K$.

**Proof of the corollary.** Since $\Omega$ is open, the function $1_{\Omega}$ is lower semicontinuous. According to condition (GC), for each $(x, \xi) \in K$, there exists $t_{x, \xi} \in (0, T)$ such that $1_{\Omega}(X(t_{x, \xi}; x, \xi)) = 1$. Since the set

$$\{ t \in (0, T) \mid 1_{\Omega}(X(t; x, \xi)) > 1/2 \}$$

is open, there exists $\eta_{x, \xi} > 0$ such that

$$[t_{x, \xi} - \eta_{x, \xi}, [t_{x, \xi} + \eta_{x, \xi}] \subset (0, T)$$

and then

$$\int_0^T 1_{\Omega}(X(t; x, \xi)) dt \geq 2\eta_{x, \xi} > 0, \quad \text{for each } (x, \xi) \in K.$$

By Fatou’s lemma, the function

$$(x, \xi) \mapsto \int_0^T 1_{\Omega}(X(t; x, \xi)) dt$$

is lower semicontinuous, and positive on $K$. Hence

$$C[T, K, \Omega] := \inf_{(x, \xi) \in K} \int_0^T 1_{\Omega}(X(t; x, \xi)) dt > 0.$$
Apply Theorem 4.1 with $\chi$ defined as follows:

$$
\chi_\delta(x) = \left(1 - \frac{\text{dist}(x, \Omega)}{\delta}\right)_+,
$$
in which case $\text{Lip}(\chi) = \frac{1}{\delta}$.

One concludes by observing that

$$
\int_0^T \text{tr}(\chi_\delta R(t)) dt \int_0^T \text{tr}(1_{\Omega_\delta} R(t)) dt,
$$
whereas

$$
\int_0^T 1_{\Omega}(X(t; x, \xi)) dt \leq \int_0^T \chi_\delta(X(t; x, \xi)) dt.
$$

\[\square\]

**Proof.** Notice that

$$
\text{tr}(\chi(R(t))) - \iint_{\mathbb{R}^d \times \mathbb{R}^d} \chi(x) f(t, x, \xi) dxd\xi
= \iint_{\mathbb{R}^d \times \mathbb{R}^d} \text{tr}_B ((\chi(y) - \chi(x)) Q(t, x, \xi)) dxd\xi
$$

for each $Q \equiv Q(t, x, \xi) \in \mathcal{C}(f(t), R(t))$. Hence

$$
\left| \text{tr}(\chi R(t)) - \iint_{\mathbb{R}^d \times \mathbb{R}^d} \chi(x) f(t, x, \xi) dxd\xi \right|
= \left| \iint_{\mathbb{R}^d \times \mathbb{R}^d} \text{tr}_B ((\chi(y) - \chi(x)) Q(t, x, \xi)) dxd\xi \right|
\leq \iint_{\mathbb{R}^d \times \mathbb{R}^d} |\text{tr}_B ((\chi(y) - \chi(x)) Q(t, x, \xi))| dxd\xi
\leq \text{Lip}(\chi) \iint_{\mathbb{R}^d \times \mathbb{R}^d} \text{tr}_B (Q(t, x, \xi)^{1/2}|x-y| Q(t, x, \xi)^{1/2}) dxd\xi
\leq \text{Lip}(\chi) \iint_{\mathbb{R}^d \times \mathbb{R}^d} \text{tr}_B \left( Q(t, x, \xi)^{1/2} \frac{1}{2} (|x-y|^2 + \frac{1}{\epsilon}) Q(t, x, \xi)^{1/2} \right) dxd\xi.
$$

Minimizing in $\epsilon > 0$ shows that

$$
\left| \text{tr}_B (\chi R(t)) - \iint_{\mathbb{R}^d \times \mathbb{R}^d} \chi(x) f(t, x, \xi) dxd\xi \right|
\leq \text{Lip}(\chi) \left( \iint_{\mathbb{R}^d \times \mathbb{R}^d} \text{tr}_B \left( Q(t, x, \xi)^{1/2}|x-y|^2 Q(t, x, \xi)^{1/2} \right) dxd\xi \right)^{1/2}
\leq \frac{\text{Lip}(\chi)}{\lambda} \left( \iint_{\mathbb{R}^d \times \mathbb{R}^d} \text{tr}_B \left( Q(t, x, \xi)^{1/2} c_\lambda(x, \xi, y, hD_y) Q(t, x, \xi)^{1/2} \right) dxd\xi \right)^{1/2}
$$

This holds for each $Q(t) \in \mathcal{C}(f(t), R(t))$; minimizing in $Q(t) \in \mathcal{C}(f(t), R(t))$ leads to the bound

$$
\left| \text{tr}_B (\chi R(t)) - \iint_{\mathbb{R}^d \times \mathbb{R}^d} \chi(x) f(t, x, \xi) dxd\xi \right| \leq \frac{\text{Lip}(\chi)}{\lambda} E_{h, \lambda}(f(t), R(t)).
$$
By Theorem 3.1
\[ \left| \text{trace}_\mathcal{H}(\chi R(t)) - \int_{\mathbb{R}^d} \chi(x) f(t, x, \xi) \, dx \, d\xi \right| \leq \frac{\text{Lip}(\chi)}{\lambda} E_{h, \lambda}(f^{in}, R^{in}) \exp \left( \frac{1}{2} \left( \lambda + \frac{\text{Lip}(\nabla V)^2}{\lambda} \right) T \right). \]

On the other hand
\[ \int_{\mathbb{R}^d} \chi(x) f(t, x, \xi) \, dx \, d\xi = \int_{\mathbb{R}^d} \chi(x) f^{in}(X(t; x, \xi), \xi(t; x, \xi)) \, dx \, d\xi \]
\[ = \int_{\mathbb{R}^d} \chi(X(t; x, \xi)) f^{in}(x, \xi) \, dx \, d\xi. \]

Hence
\[ \int_0^T \text{trace}(\chi R(t)) \, dt \geq \int_{\mathbb{R}^d} \left( \int_0^T \chi(X_t(x, \xi)) \, dt \right) f^{in}(x, \xi) \, dx \, d\xi \]
\[ - \frac{\text{Lip}(\chi)}{\lambda} E_{h, \lambda}(f^{in}, R^{in}) \int_0^T \exp \left( \frac{1}{2} \left( \lambda + \frac{\text{Lip}(\nabla V)^2}{\lambda} \right) t \right) dt \]
\[ \geq \int_{\mathbb{R}^d} \left( \int_0^T \chi(X_t(x, \xi)) \, dt \right) f^{in}(x, \xi) \, dx \, d\xi \]
\[ - \frac{\text{Lip}(\chi)}{\lambda} \exp \left( \frac{1}{2} \left( \lambda + \frac{\text{Lip}(\nabla V)^2}{\lambda} \right) T \right) - 1 E_{h, \lambda}(f^{in}, R^{in}). \]

(11)
\[ \geq \inf_{(x, \xi) \in K} \int_0^T \chi(X(t; x, \xi)) \, dt \int_{(x, \xi) \in K} f^{in}(x, \xi) \, dx \, d\xi \]
\[ - \frac{\text{Lip}(\chi)}{\lambda} \exp \left( \frac{1}{2} \left( \lambda + \frac{\text{Lip}(\nabla V)^2}{\lambda} \right) T \right) - 1 E_{h, \lambda}(f^{in}, R^{in}). \]

In particular, putting \( f^{in} = \tilde{W}_h[R^{in}] \) and \( \lambda = 1 \), one obtains
\[ \int_0^T \text{trace}(\chi R(t)) \, dt \geq \inf_{(x, \xi) \in K} \int_0^T \chi(X(t; x, \xi)) \, dt \int_{(x, \xi) \in K} (\tilde{W}_h[R^{in}](x, \xi)) \, dx \, d\xi \]
\[ - \frac{\text{Lip}(\chi)}{\lambda} \exp \left( \frac{1}{2} \left( \lambda + \frac{\text{Lip}(\nabla V)^2}{\lambda} \right) T \right) - 1 E_{h, \lambda}(\tilde{W}_h[R^{in}], R^{in}). \]

(12)

For \( R^{in} = |\psi^{in}\rangle \langle \psi^{in}| \), we know by Proposition 9.1. in [7] that \( E_{h, 1}(\tilde{W}_h[R^{in}], R^{in}) \leq 2\Delta(R^{in}) \) and we get the conclusion of Theorem 4.1 in the pure state case.

If \( f^{in} \) is any compactly supported probability density, the inequality (11) that
\[ \int_0^T \text{trace}(\chi R(t)) \, dt \geq \inf_{(x, \xi) \in \text{supp}(f^{in})} \int_0^T \chi(X(t; x, \xi)) \, dt \]
\[ - \frac{\text{Lip}(\chi)}{\lambda} \exp \left( \frac{1}{2} \left( \lambda + \frac{\text{Lip}(\nabla V)^2}{\lambda} \right) T \right) - 1 E_{h, \lambda}(f^{in}, R^{in}). \]
Now, if $R_\hbar$ is the Töplitz operator with symbol $(2\pi \hbar)^d \mu^{in}$, where $\mu^{in}$ is a Borel probability measure on $\mathbb{R}^d \times \mathbb{R}^d$, 

$$\int_0^T \text{trace}(\chi R(t)) dt \geq \inf_{(x,\xi) \in \text{supp}(f^{in})} \int_0^T \chi(X(t;x,\xi)) dt - \frac{\text{Lip}(\chi)}{\lambda} \exp \left( \frac{1}{2} \left( \lambda + \frac{\text{Lip}(\nabla V)^2}{\lambda} \right) T \right) - \frac{1}{2} \left( \lambda + \frac{\text{Lip}(\nabla V)^2}{\lambda} \right) \sqrt{\max(1, \lambda^2) \text{dist}_{MK,2}(f^{in}, \mu^{in})^2 + \frac{1}{2}(\lambda^2 + 1) \hbar}.$$ 

In particular, if $R_\hbar = \text{OP}_\hbar^T [(2\pi \hbar)^d f^{in}]$, one has 

$$\int_0^T \text{trace}(\chi R(t)) dt \geq \inf_{(x,\xi) \in \text{supp}(f^{in})} \int_0^T \chi(X(t;x,\xi)) dt - \frac{\text{Lip}(\chi)}{\lambda} \exp \left( \frac{1}{2} \left( \lambda + \frac{\text{Lip}(\nabla V)^2}{\lambda} \right) T \right) - \frac{1}{2} \left( \lambda + \frac{\text{Lip}(\nabla V)^2}{\lambda} \right) \sqrt{\frac{1}{2}(\lambda^2 + 1) \hbar}.$$ 

Maximizing the right hand side as $\lambda$ runs through $(0, +\infty)$, one finds that 

$$\int_0^T \text{trace}(\chi R(t)) dt \geq \inf_{(x,\xi) \in \text{supp}(f^{in})} \int_0^T \chi(X(t;x,\xi)) dt - \text{Lip}(\chi) C(T, \text{Lip}(\nabla V)) \sqrt{2 \hbar},$$

where 

$$C(T, L) := \inf_{\lambda > 0} \frac{\exp \left( \frac{1}{2} \left( \lambda + \frac{L^2}{\lambda} \right) T \right) - 1}{\lambda^2 + L^2} \sqrt{\lambda^2 + 1}.$$ 

If $L > 0$, one can take $\lambda = L$ so that 

$$C(T, L) \leq \frac{e^{LT} - 1}{2L^2} \sqrt{1 + L^2}.$$ 

Notice that, in the case where $L = 0$, one can choose $\lambda = 2r/T$ with 

$$r e^r = 2(e^r - 1), \quad r > 0, \quad \lambda = 2r/T,$$

and find that 

$$C(T, 0) \leq \frac{e^r - 1}{4r^2} T^2 \sqrt{1 + \frac{4r^2}{T^2}}.$$ 

**Acknowledgments.** We would like to thank warmly Claude Bardos for having read the first version of this paper and mentioned several references.

**References**

[1] N. Anantharaman, M. Léautaud, F. Macià: *Wigner measures and observability for the Schrödinger equation on the disk*, Invent. Math. 206 (2016), 485–599.

[2] C. Bardos, G. Lebeau, J. Rauch: *Sharp sufficient conditions for the observation, control and stabilization of waves from the boundary*, SIAM J. Control Opti. 30 (1992), 1024–1065.
[3] C. Fabre: Quelques résultats de contrôlabilité exacte de l’équation de Schrödinger. Application à l’équation des plaques vibrantes. (French) [Exact controllability of the Schrödinger equation. Application to the vibrating-plate equation] C. R. Acad. Sci. Paris Sér. I Math. 312 (1991), 61–66.

[4] C. Fabre: Résultats de contrôlabilité exacte interne pour l’équation de Schrödinger et leurs limites asymptotiques, Asymptotic Analysis 5 (1992), 343–379.

[5] F. Golse, C. Mouhot, T. Paul: On the Mean Field and Classical Limits of Quantum Mechanics, Commun. Math. Phys. 343 (2016), 165–205.

[6] F. Golse, T. Paul: The Schrödinger Equation in the Mean-Field and Semiclassical Regime, Arch. Rational Mech. Anal. 223 (2017), 57–94.

[7] F. Golse, T. Paul: Semiclassical evolution with low regularity, preprint hal-02619489 and arXiv:2011.14884, to appear in J. Math. Pures et Appl..

[8] T. Kato: “Perturbation Theory for Linear Operators”, Springer Verlag, Berlin, Heidelberg, 1966, 1976.

[9] C. Laurent: Internal control of the Schrödinger equation, Math. Control Relat. Fields 4 (2014), 161–186.

[10] G. Lebeau, Contrôle de l’équation de Schrödinger. (French) [Control of the Schrödinger equation] J. Math. Pures Appl. (9) 71 (1992), 267–291.

[11] W. Rudin: “Real and Complex Analysis”, Mc Graw Hill, Singapore, 1986.

(F.G.) CMLS, École polytechnique, CNRS, Université Paris-Saclay, 91128 Palaiseau Cedex, France
Email address: francois.golse@polytechnique.edu

(T.P.) Laboratoire J.-L. Lions, Sorbonne Université & CNRS, boîte courrier 187, 75252 Paris Cedex 05, France
Email address: thierry.paul@upmc.fr