Tail Measures and Regular Variation

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Abstract

A general framework for the study of regular variation (RV) is that of Polish star-shaped metric spaces, while recent developments in [1] have discussed RV with respect to a properly localised boundedness $B$. Along the lines of the latter approach, we discuss the RV of Borel measures and random processes on a general Polish metric spaces $(D, d_D)$. Tail measures introduced in [2] appear naturally as limiting measures of regularly varying time series. We define tail measures on the measurable space $(D, \mathcal{D})$ indexed by $\mathcal{H}(D)$, a countable family of 1-homogeneous coordinate maps, and show some tractable instances for the investigation of RV when $B$ is determined by $\mathcal{H}(D)$. This allows us to study the regular variation of càdlàg processes on $D(\mathbb{R}^l, \mathbb{R}^d)$ retrieving in particular results obtained in [3] for RV of stationary càdlàg processes on the real line removing $l = 1$ therein. Further, we discuss potential applications and open questions.

Key Words: tail measures, regular variation; hidden regular variation; càdlàg processes; max-stable processes; tail processes; spectral tail processes; weak convergence;

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1 Introduction

Let $X(t), t \in T$ be an $\mathbb{R}^d$-valued random process with $T$ a non-empty set (here $d, l, k$ are reserved for positive integers). For given $t_1, \ldots, t_k \in T$ and $A \in \mathcal{B}(\mathbb{R}^d)$ it is of interest for many investigations to determine the asymptotic behaviour as $n \to \infty$ of

$$p_{t_1, \ldots, t_k}(a_n \cdot A) = \mathbb{P}\{(X(t_1), \ldots, X(t_k)) \in a_n \cdot A\}$$

for some positive scaling constants $a_n, n \geq 1$. Such an investigation is reasonable if $a_n \cdot A, n \geq 1$ are Borel absorbing events, i.e., the outer multiplication $\cdot$ satisfies $a_n \cdot A \in \mathcal{B}(\mathbb{R}^{dk})$ and $\lim_{n \to \infty} p_{t_1, \ldots, t_k}(a_n \cdot A) = 0$. Throughout this paper $\mathcal{B}(D)$ stands for the Borel $\sigma$-field on the topological space $D$.

Considering for simplicity the canonical scaling, i.e., $c \cdot A := \{c \cdot a, a \in A\}$ for all $c \in (0, \infty)$, where $\cdot$ is the usual product on $\mathbb{R}$, it is natural to require that the $A$ is separated from the origin (denoted by 0) of $\mathbb{R}^d$, i.e., $A$ is included in the complement of a neighbourhood of 0 in the usual topology. For such $A$, the rate of convergence to 0 of $p_{t_1, \ldots, t_k}(a_n \cdot A)$ is the main topic in the theory of RV of random vectors. Indeed the RV of functions, random processes and Borel measures is important in various research fields and is not confined to probabilistic applications, see e.g., [1] and the references therein.

The problem at hand can be regarded as a scaling approximation discussed for instance in [5] in terms of Kendall’s theorem and is investigated in the framework of RV of measures, in finite or infinite dimensional spaces, see e.g., [1, 3, 6–14].

As, for instance, in [1, 2, 15, 16], we say that $X$ is finite dimensional regularly varying if there exist positive $a_n$’s such that for all $t_1, \ldots, t_k \in T, k \geq 1$, there exists a non-null measure $\nu_{t_1, \ldots, t_k}$ on $\mathcal{B}(\mathbb{R}^{dk})$ satisfying

$$\lim_{n \to \infty} n \mathbb{P}\{(X(t_1), \ldots, X(t_k)) \in a_n \cdot A\} = \nu_{t_1, \ldots, t_k}(A) < \infty$$

for all $\nu_{t_1, \ldots, t_k}$-continuity $A \in \mathcal{B}(\mathbb{R}^{dk})$ separated from 0. The measure $\nu_{t_1, \ldots, t_k}$ is called the exponent measure of $(X(t_1), \ldots, X(t_k))$. If the outer multiplication $\cdot$ is the usual product, it is well-known that the exponent measures are $-\alpha$-homogeneous, i.e., there exists $\alpha > 0$ (not depending on $t_i$’s) such that

$$\nu_{t_1, \ldots, t_k}(z \cdot A) = z^{-\alpha} \nu_{t_1, \ldots, t_k}(A), \quad \forall z \in (0, \infty), \forall (t_1, \ldots, t_k) \in T^k, \forall k \geq 1.$$  (1.1)

RV of Borel measures on some Polish metric space $(D, d_D)$ is investigated in [1, 8, 9, 12, 17]. The recent manuscripts [1, 3] treat RV of measures and processes in terms of a given properly localised boundedness $B$ on $D$ following the ideas in [18]. In [1] several weak conditions are formulated with respect to the scaling and the topology of $D$, see [1][Appendix B: (M1)-(M3), (B1-B3)]. We highlight next some key developments and findings:

**F1** All investigations in the literature, e.g., [8, 10, 19–21] consider RV of random processes with compact parameter space $T \subset \mathbb{R}$, $l \in \mathbb{N}$. Moreover, RV of Borel measures on star-shaped Polish metric spaces are considered. Surprisingly, the non-compact case $T = \mathbb{R}_l$, which is of great interest for the investigation of time series, has been investigated only recently in [3] for stationary stochastically continuous càdlàg random processes when $l = 1$;
The recent manuscripts [1, 3] develop the theory of RV with respect to a properly localised boundedness $\mathcal{B}$. This new approach has several advantages including the unification of RV and hidden RV;

RV of stationary time series can be characterised by tail and spectral tail processes, see [1, 16, 22–24]. See also [12, 25, 26] for non-stationary time series where also local tail processes play a crucial role for the characterisation of RV;

Characterisation of RV of stationary time series in terms of tail measures is first investigated in [2] and further discussed in [3, 22, 25];

There are different definitions of RV useful in various applications, which in view of [9][Thm 3.1] are equivalent for star-shaped Polish metric spaces;

F1–F5 and recent applications developed in [3] motivate the following two topics, which constitute the backbone of the present contribution:

- RV of processes (not necessarily stationary) with non-compact parameter space $T$, or in general RV of Borel measures in non-star-shaped Polish metric spaces with respect to some properly localised boundedness $\mathcal{B}$;
- Basic properties of tail measures in general measure spaces and their relationship with RV;
- Potential applications of RV to càdlàg processes (random fields) with non-compact $T$;
- Discussion on possible different definitions of RV relevant for applications.

RV of stochastically continuous stationary càdlàg processes defined on the real line is recently investigated in [3]. For the case of locally compact $T = \mathbb{R}^l$ the corresponding functional metric spaces (we denote them by $(D, d_{D})$ below) are not radially monotone (or star-shaped, see [12]), which is the case when $T$ is compact. Specifically, for a hypercube $T \subset \mathbb{R}^l, l \geq 1$ and $D(T, \mathbb{R}^d)$ the space of generalized càdlàg functions $T \mapsto \mathbb{R}^d$ (see e.g., [27, 28] for definitions), a metric $d_{D}$ can be chosen so that $D(T, \mathbb{R}^d)$ is Polish and

$$d_{D}(c \cdot f, 0) = cd_{D}(f, 0), \quad \forall c > 0, \forall f \in D(T, \mathbb{R}^d),$$

where $0$ is the zero function. Consequently, $d_{D}(cf, 0)$ is strictly monotone for all $c > 0$ and fixed $f \neq 0$; this is referred to as the radial monotonicity property and has been a key assumption in the treatment of RV of measures in Polish metric spaces, e.g., [8, 29].

When $T = \mathbb{R}^l$, in view of Theorem A.1.(vi) in Appendix, radial monotonicity does not hold. That property is crucial for the proof of [9][Thm 3.1]. Therefore when dealing with $D(\mathbb{R}^l, \mathbb{R}^d)$ the equivalence of different definitions of RV of Borel measures does not follow from the aforementioned theorem, but can be nonetheless confirmed as shown in Lemma 5.2.

Following [1], where a boundedness along with the chosen group action plays a crucial role, we discuss first RV of Borel measures on general Polish metric spaces. From [2, 3, 22, 25] it is known that for particular Polish spaces the limit measure in the definition of RV is a tail measure, which is essentially characterised by the following properties:

P1) $-\alpha$-homogeneity as described by (1.1);

P2) countable indexing by 1-homogeneous maps.

In abstract settings, P1) is introduced under the assumption that $(D, \cdot, \mathcal{D}, \mathbb{R}_\geq, \cdot)$ is a measurable cone with $\mathcal{D}$ being a σ-field on $D$, i.e., the outer multiplication (we prefer here the formulation as a pairing) $(z, f) \mapsto z \cdot f \in D$ for $z \in \mathbb{R}_\geq$ and $f \in D$ is a group action from the multiplicative group $(\mathbb{R}_\geq, \cdot)$ on $D$ and is jointly measurable.

Hereafter, $Z$ is a $D$-valued random element defined on a complete probability space $(\Omega, \mathcal{F}, P)$. The cone measurability assumption and the Fubini-Tonelli theorem yield that

$$\nu_Z(A) = \mathbb{E} \left\{ \int_0^\infty \mathbb{I}(z \cdot Z \in A) \alpha^{-\alpha-1} dz \right\}, \quad A \in \mathcal{D}$$

is a non-negative measure on $\mathcal{D}$ for all $\alpha > 0$. It follows that $\nu = \nu_Z$ satisfies
\( M0 \) \( \nu(t \cdot A) = t^{-\alpha} \nu(A), \ \forall t \in (0, \infty), \forall A \in \mathcal{D} \)

and therefore \( \nu \) is called \(-\alpha\)-homogeneous, with \( \alpha \) referred to as its index.

If the space \( D \) is a countable product of measurable spaces, then \( P2) \) can be introduced with respect to a given 1-homogeneous positive definite (point-separating) measurable map as in [25]. The essential feature of such a map is that it defines a cone on \( D \) and moreover its support can be countably generated by Borel sets of \( \mathbb{R} \). In this paper we do not restrict to such measurable spaces and therefore the countable indexing is introduced below in \( M1) \) with respect to a countable family \( \mathcal{H}(D) \) of \( 1\)-homogeneous measurable maps, see Definition 2.2.

A crucial consequence of both \( P1\)-\( P2) \) is that the introduced tail measures \( \nu \) are \( \sigma \)-finite. Moreover, the family \( \mathcal{H}(D) \) allows us to introduce the families of local tail/ spectral tail processes. The latter are utilised to show that tail measures \( \nu \) possess a stochastic representer \( Z \) such that \( \nu = \nu_Z \) as defined in (1.2).

As in [1], \( \text{RV of general measures } \nu \text{ on } (D, \mathcal{B}(D)) \) is discussed in Section 4.1 with respect to some properly localised boundedness \( \mathcal{B} \) on \( D \). We show that tractable instances for the investigation of \( \text{RV of } \nu \) arise if \( \mathcal{B} \) can be characterised by \( \mathcal{H}(D) \) as in (B5) below, which is in particular the case for some common boundedness on the space of general càdlàg processes, or on \( l \setminus 0 \) defined in [30][p. 877].

In Theorem 4.11 we relate the \( \text{RV of } \text{càdlàg processes on } D(\mathbb{R}^l, \mathbb{R}^d) \) with the \( \text{RV of their restrictions on } D(K, \mathbb{R}^d) \text{ for } K \) a given hypercube on \( \mathbb{R}^l \). Moreover, we present necessary and sufficient conditions for \( \text{RV of } \text{càdlàg processes in Theorem 4.15}. \) Our findings show that \( \text{RV of } \text{càdlàg processes can be investigated without imposing the stationarity assumption.} \)

Besides being more complicated, the non-stationary case is also inevitably less tractable than the stationary one. Despite those limitations, numerous interesting results still continue to hold for non-stationary càdlàg processes, including the equivalence of different definitions of \( \text{RV and Breiman’s lemma (see e.g., [31] for some extensions)} \) with its ramifications, see Section 5.

Another conclusion of this paper is that tractable cases arise for general Polish metric spaces if the boundedness can be intrinsically related to \( \mathcal{H}(D) \). The importance of our results is illustrated also by the wide range of potential applications and open problems discussed in Section 6. Besides, our results for local spectral tail processes, their relationship with tail measures and \( \text{RV are of certain theoretical importance.} \)

Below is a short summary of some new aspects of this contribution:

i) We introduce tail measures, families of local tail/ spectral tail processes for general measure spaces indexed by \( \mathcal{H}(D) \);

ii) All the results on the families of local tail and local spectral tail processes are new for the settings of this paper. Proposition 3.6 is new also for the simpler cases \( D = D(\mathbb{R}^l, \mathbb{R}^d) \) or \( D = D(\mathbb{Z}^l, \mathbb{R}^l) \) and all \( l \geq 1 \);

iii) Theorem 4.7, Theorem 4.11 and the characterisation of the limit measure \( \nu \) in Lemma 4.9 are new also for stationary \( X \) taking values in \( D \) as in ii) above, whereas Lemma 5.2 is new if \( D = D(\mathbb{R}^l, \mathbb{R}^d), l \geq 1 \). Further Theorem 4.15 presents new results for \( X \) with càdlàg sample paths also when \( X \) is stationary and \( l > 1 \);

iv) Our applications in Section 6 include novel results for the tail behaviour of supremum of regularly varying càdlàg processes.

The paper is organized as follows: Section 2 introduces notation and exhibits some preliminary results concluding with our main assumptions. Tail measures, local tail/ spectral tail processes and stochastic representer are discussed in Section 3, whereas the \( \text{RV of Borel measures and random processes} \) is treated in Section 4. Section 5 is dedicated to discussions and some extensions. Potential applications, results for max-stable and \( \alpha \)-stable processes as well as open problems are presented in Section 6. All proofs are relegated to Section 7. In Appendix we review some properties of general càdlàg functions and then display the mapping theorem.

2 Preliminaries

We present first several definitions and notation related to a given metric space. Then we continue with properties of a properly localised boundedness \( \mathcal{B} \) and conclude this section with the formulation of the main assumptions.
2.1 Measurable cones and the family of maps $\mathcal{H}(D)$

Let $(D, d_D)$ be a metric space with corresponding Borel $\sigma$-field $B(D)$ and let $\mathcal{D}$ be another generic $\sigma$-field on $D$. In order to define a homogeneous measure on $\mathcal{D}$ that satisfies M0) we shall assume that a pairing 

$$(z, f) \mapsto z \cdot f \in D, \quad f \in D, z \in \mathbb{R}_> = (0, \infty)$$

(thus $D$ is a cone for the outer multiplication $\cdot$) is a group action of the product group $(\mathbb{R}_>, \cdot)$ on $D$. This simply means

$$z \cdot f = f, \quad (z_1 z_2) \cdot f = z_1 \cdot (z_2 \cdot f), \quad \forall f \in D, \forall z_1, z_2 \in \mathbb{R}_>.$$

**Definition 2.1.** We shall call $(D, \cdot, (\mathbb{R}_>, \cdot))$ a measurable cone, if $D$ is non-empty and the corresponding group action $(z, f) \mapsto z \cdot f, z \in \mathbb{R}_>, f \in D$ of $(\mathbb{R}_>, \cdot)$ on $D$ is $B(\mathbb{R}_>) \times \mathcal{D}/\mathcal{D}$ measurable.

In some cases $D$ possesses a zero element $0_D$, i.e.,

$$z \cdot 0_D = 0_D, \quad \forall z \in \mathbb{R}_> = [0, \infty).$$

In the following we shall write $0$ instead of $0_D$; abusing slightly the notation $0$ will also denote the origin of $\mathbb{R}^m$, $m \in \mathbb{N}$.

Hereafter $Q = \{t_i, i \in \mathbb{N}\}$ is a non-empty subset of a given parameter space $T$.

**Definition 2.2.** We introduce the maps $\| \cdot \|_t : D \mapsto [0, \infty], t \in Q$, which satisfy

$$\| z \cdot f \|_t = z \| f \|_t, \quad \forall f \in D, \forall z \in \mathbb{R}_>$$

and are $\mathcal{D}/B([0, \infty])$-measurable. Suppose further that for all $t \in Q$ there exists $f \in D$ such that $\| f \|_t \in (0, \infty)$ and denote by $\mathcal{H}(D)$ the family of the maps $\| \cdot \|_t, t \in Q$.

In the sequel we shall assume that $\mathcal{H}(D)$ is non-empty. Next, given $f \in D$ and $K \subset T$, we define

$$f^K_\ast = \max_{t \in K \cap Q} \| f \|_t.$$

If $K \cap Q = \emptyset$, interpret $f^K_\ast$ as $0$ and write simply $f_\ast$ if $K = Q$.

Hereafter $\mathcal{H}$ shall denote the class of all maps $\Gamma : D \mapsto \mathbb{R}$ and all maps $\Gamma : D \mapsto [0, \infty]$ which are $\mathcal{D}/B(\mathbb{R})$ and $\mathcal{D}/B([0, \infty])$ measurable, respectively. Write $\mathcal{H}_\lambda, \lambda \geq 0$ for the class of maps $\Gamma \in \mathcal{H}$ such that for all $f \in D$ and some $c > 0$, $\Gamma(c \cdot f) = c^\lambda \Gamma(f)$.

2.2 Boundedness on Polish spaces and $\mathcal{B}$-boundedly finite measures

Consider a non-empty set $D$ equipped with a $\sigma$-field $\mathcal{D}$.

**Definition 2.3.** A measure $\nu$ on $\mathcal{D}$ is a countably additive set-function $\mathcal{D} \mapsto [0, \infty]$ with $\nu(\emptyset) = 0$. We call $\nu$ non-trivial if $\nu(A) \in (0, \infty)$ for some $A \in \mathcal{D}$ and denote the set of non-trivial measures on $\mathcal{D}$ by $\mathcal{M}^+(\mathcal{D})$. If $\mathcal{D} = B(D)$, then $\nu$ is called Borel.

Suppose next that $(D, d_D)$ is a Polish metric space and set $\mathcal{D} = B(D)$. Write $\overline{A}$ and $\partial A$ for the closure and the topological frontier (boundary) of a non-empty set $A \subset D$, respectively. Note that $\partial A = \overline{A} \setminus int(A)$, where $int(A)$ is the interior of $A$. If $\nu \in \mathcal{M}^+(\mathcal{D})$, then the events (i.e., the elements of $\mathcal{D}$) of interest are $A \subset \mathcal{D}$, where $\mathcal{D}$ consists of all events such that $\nu(A) < \infty$. Since $\mathcal{D}$ is in general too large, reducing it to a countably generated set is of great advantage for dealing with properties of $\nu$. This motivates the concept of the properly localised boundedness which is quite general and not restricted to Polish spaces; our definitions below are essentially taken from [1][Appendix B], see also [18, 32].

**Definition 2.4.** A non-empty class $\mathcal{B} = \{A : A \subset D\}$ is called a properly localised boundedness on $D$ if

- **B1)** $\mathcal{B}$ is closed with respect to finite unions and the subsets of elements of $\mathcal{B}$ belong to $\mathcal{B}$;
- **B2)** There exist open sets $O_n \in \mathcal{B}, n \in \mathbb{N}$ such that $\overline{O}_n \subset O_{n+1}, n \in \mathbb{N}$ and $\bigcup_{n=1}^\infty O_n = D$. Moreover for all $A \in \mathcal{B}$ we have $A \subset O_n$ for some $n \in \mathbb{N}$. 5
Remark 2.5. A properly localised boundedness $B$ contains the compact sets of $D$, see [1][Rem B.1.2]. Moreover, all metrically bounded sets of $D$ form a localised boundedness and also the converse holds, namely if $B$ is a properly localised boundedness, then there exists a metric $d'$ on $D$ for which $(D,d')$ is complete and $A ∈ B$ $⟺ A$ is metrically bounded for $d'$, see [1][B.1.3], [18][Rem 2.7].

Throughout the following $B$ denotes a properly localised boundedness on $D$.

Definition 2.6. A Borel measure $ν$ on $D$ that satisfies $ν(A) < ∞$ for all $A ∈ B ∩ D$ is called $B$-boundedly finite. If further $ν$ is non-trivial, then we write $ν ∈ M^+(B)$.

If $F$ is a closed subset of $D$ we set $D_F = D \setminus F$ (assumed to be non-empty), which is again a Polish space. Write $B_F$ for the collections of subsets of $D_F$ with elements $B$ such that

$$d_D(x,F) = \inf_{f ∈ F} d_D(x,f) > ε, \ \ ∀x ∈ B$$

for some $ε > 0$, which may depend on $B$. We can equip $D_F$ with a metric $d_{DF}$, which induces the trace topology on $D_F$ and the elements of $B_F$ are metrically bounded. One instance is the metric given in [1] [Eq. (B.1.4)]. In view of [1][Example B.1.6] $B_F$ is a properly localised boundedness on $D_F$. In the particular case $F = \{a\}$ we write simply $B_α$ and $D_α$, respectively.

The boundedness $B_F$, for $F$ being further a cone, appears in connection with hidden regular variation, see e.g., [18], whereas $B_0$ is the common boundedness used in the definition of RV, see e.g., [8, 12] and references therein.

Hereafter the support of $H ∈ S$ is denoted by $supp(H)$ defined by $supp(H) = \overline{H^{-1}((-∞,0) ∩ (0,∞))}$. Suppose next that $(D, \cdot, S, R_+, ·)$ is a measurable cone and consider the following restrictions for a given properly localised boundedness $B$ on $D$:

B3) For all $A ∈ B$ and all $z ∈ R_+$, we have $z·A ∈ B$;

B4) There exists an open set $A ∈ B$ such that $z·A ⊂ A$ for all $z > 1$. Suppose further that $t·A ⊂ s·A, ∀t > s > 0$ and $\bigcap_{i ≥ 1}(s·A)$ equals the empty set $\emptyset$;

B5) If $H(D)$ is a family of maps as defined in Definition 2.2, then $A ∈ B$ if and only if there exists some index set $K_A ⊂ T$ and $ε_A > 0$ such that $∀f ∈ A$ we have

$$f_{K_A}^* = \sup_{t ∈ K_A ∩ Q} ||f||_t > ε_A.$$ 

Given $Γ ∈ S$ and a measure $ν$ on $S$, write

$$ν[Γ] = \int_D Γ(f)ν(df).$$

Remark 2.7. If $ν ∈ M^+(B)$, then $ν$ is uniquely defined by $ν[Γ]$ for all $Γ : D ↦ R$ bounded continuous supported on $B$. Moreover, $ν$ is $-α$-homogeneous, provided that $ν[Γ_z] = z^{-α}ν[Γ]$ for all bounded continuous $Γ ∈ S$ with support in $B$, with $Γ_z(v) = Γ(z·v), v ∈ D$ and B3) holds. For more details we refer to [1][Appendix B].

2.3 Main assumptions

Below we write $D(K, R^d)$ for the space of functions $f : K ↦ R^d$. If $K = R^d, K = (0,∞)^l$ or $K$ is a hypercube of $R^l$, then $f ∈ D(K, R^d)$ is assumed to be a càdlàg function, see e.g., [28, 33] for the definition in the less common case $l > 1$. Next, we formulate the following set of assumptions:

A1) $(D, ·, S, R_+, ·)$ is a measurable cone, $Q = \{t_i, i ∈ N\}$ is a subset of some parameter space $T$ and the family of maps $H(D)$ exists;

A2) $(D, d_D)$ is a Polish space with a properly localised boundedness $B$. Further, $| · |$‘s are finite and A1), B3) hold;
A3) Let $D = D(T, R^d)$ with $T = R^l$ or $T = Z_l$ equipped with the Skorohod $J_1$ topology and the corresponding metric $d_2$ which turns it into a Polish space. Set $|f|_l = |f(t)|$, $f \in D$, $t \in T$, where $| \cdot | : R^d \mapsto [0, \infty)$ is a norm on $R^d$. Here $Q$ is a countable dense subset of $T$, the pairing $(z, f) \mapsto z \cdot f = z f$ with $(z f)(t) = z f(t), t \in T$ is the canonical pairing.

Under A3) the assumption A2) holds for $D = D(R^l, R^d)$, which follows from Theorem A1.(i)-(iv). Moreover by Theorem A1.(i), the Borel $\sigma$-field $\mathcal{B}(D)$ agrees with $\mathcal{B}_Q = \sigma(p_x, t \in Q)$. Consequently, $| \cdot |, t \in Q$ are $\mathcal{B}(D)/\mathcal{B}(R)$ measurable and 1-homogeneous and thus $\mathcal{H}(D)$ exists.

Consider next the boundedness $B_0$ defined on $D_0 = D \setminus \{0\}$ with $D = D(R^l, R^d)$ and $0 \in D$ the zero function. In view of Theorem A1.(v) $A \in B_0$ if and only if there exists a hypercube $K_A \subset R^d$ and some $\varepsilon_A > 0$ such that

$$\sup_{x \in K_A} |f|_l = f^*_l > \varepsilon_A, \forall f \in A. \quad (2.1)$$

Remark 2.8. Eq. (2.1) shows that $B_0$ satisfies B5). This is also the case if $D = D(Z^l, R^d)$, see [1]p. 105 for $l = 1$. Note that other properly localised boundedness satisfying B5) exist, for instance $B_0$ on the space $D_0 = l \setminus \{0\}$ defined in [30][p. 877] by metrically bounded sets therein.

3 Tail measures

Tail measures introduced in [2] play a crucial role in the study of RV, see e.g., [1–3, 22, 25]. In the literature so far the main emphasis has been on shift-invariant tail measures and tail measures defined on product spaces. In this section we shall assume that $A1)$ holds and fix some $\alpha > 0$.

3.1 Definition and basic properties

If $\nu$ is a $-\alpha$-homogeneous measure on $\mathcal{D}$ and $A \in \mathcal{D}$ satisfies $z \cdot A = A$ for some positive $z \neq 1$, then

$$\nu(A) \in [0, \infty). \quad (3.1)$$

By the 1-homogeneity of the maps $| \cdot |$, (3.1) implies that $F_\alpha$ defined by

$$F_\alpha = \{ f \in D : f^*_\alpha = 0 \}, \quad f^*_\alpha = \sup_{t \in \mathbb{Q}} |f|_l,$$

satisfies $\nu(F_\alpha) \in [0, \infty)$. Of particular interest are measures $\nu$ such that

M1) $\nu(F_\alpha) = 0$,

since this property is crucial for establishing their $\sigma$-finiteness. Next, we define tail measures on $\mathcal{D}$, supported by the findings of [2], in which tail measures on the product $\sigma$-field of $D = (R^d)^T$ are introduced. See also [1, 22, 25] for special product spaces containing a zero element 0 (we do not assume existence of 0 here) and [3] for $D = D(R, R^d)$.

Definition 3.1. A measure $\nu$ on $\mathcal{D}$ that satisfies M0), M1) is called a tail measure (write $\nu \in \mathcal{M}_\alpha(\mathcal{D})$) if

M2) $p_h := \nu(\{ f \in D : \| f \|_h > 1 \}) \in [0, \infty), \forall h \in \mathbb{Q}$, with $p_{h_0} \in (0, \infty)$ for some $h_0 \in \mathbb{Q}$.

Remark 3.2. The measurability of $\| \cdot \|_h, h \in \mathbb{Q}$ implies $A_h = \{ f \in D : \| f \|_h = 1 \} \in \mathcal{D}, h \in \mathbb{Q}$. If $\nu \in \mathcal{M}_\alpha(\mathcal{D})$, then by M0) and M2) $\nu(A_h) = 0$ for all $h \in \mathbb{Q}$. Consequently, for all $x > 0$

$$p_h(x) = \nu(\{ f \in D : \| f \|_h \geq x \}) = x^{-\alpha} \nu(\{ f \in D : \| f \|_h > 1 \}) = x^{-\alpha} p_h, \ h \in \mathbb{Q} \quad (3.2)$$

and thus if $p_h = 0$, then $p_h(0) = 0$ follows by the countable additivity of $\nu$. Since M2) and (3.1) imply

$$\nu(\{ f \in D : \| f \|_h = \infty \}) = 0, \ \forall h \in \mathbb{Q}, \quad (3.3)$$

then M1) is equivalent to

$$\nu(\{ f \in D : \sup_{t \in \mathbb{Q}} |f|_l \in (0, \infty) \}) = \nu(\{ f \in D : \sup_{t \in \mathbb{Q}, p_t > 0} |f|_l \in (0, \infty) \}) = 0. \quad (3.4)$$
We introduce next the local tail and local spectral tail processes as in \([3.2]\). Our definition of tail measures implies their \(\sigma\)-finitness and as in \([25]\) we have the following result (its proof is omitted).

**Lemma 3.3.** If \(\nu \in \mathcal{M}_\sigma(\mathcal{D})\), then it is \(\sigma\)-finite and \(\nu\) is uniquely determined by its restrictions to \(\{f \in D : |f|_h > 1\}\) for all \(h \in Q\).

Recall that \(Z\) denotes throughout this paper a \(D\)-valued random element defined on a complete probability space \((\Omega, \mathcal{F}, P)\). Suppose next that \(|Z|_h\) are random variables \((rv's)\) for all \(h \in Q\) and further

\[
E(|Z|_{h_0}^2) \in (0, \infty), \quad E(|Z|_h^2) \in [0, \infty), \quad \forall h \in Q, \quad P\{Z_0^2 \neq 0\} = 1
\]

for some \(h_0 \in Q\) and consider \(\nu_Z\) defined in \((1.2)\). Since \(D\) is a measurable cone, then \(\nu_Z\) is the image measure of \((z, f) \mapsto z \cdot f\) with respect to the product measure \(\mu(df) \times \nu_\alpha(dr)\), with

\[
\nu = P \circ Z^{-1}, \quad \nu_\alpha(dr) = \alpha^{-\alpha-1} dr.
\]

Clearly, \(\nu_Z\) satisfies \((M0)-(M1)\) with \(p_h = E(|Z|_{h_0}^2) < \infty\) for all \(h \in Q\) and hence \(\nu_Z \in \mathcal{M}_\sigma(\mathcal{D})\). Hereafter \(R\) is an \(\alpha\)-Pareto \(rv\) with \(P[R > t] = t^{-\alpha}, t \geq 1\), independent of all other random elements. It can be utilised to link \(Z\) and \(\nu_Z\) as in \((3.6)\) below.

If a measure \(\nu\) on \(\mathcal{D}\) has representation \((1.2)\) with \(Z\) satisfying the first two conditions in \((3.5)\), then it follows that for all \(h \in Q, \Gamma \in \mathcal{F}, \varepsilon \in (0, \infty)\) (here \(\mathcal{F}\) is the class of maps defined in Section 2.1)

\[
\int_D \Gamma(f) \mathbb{1}\{|f|_h > \varepsilon\} \nu(df) = \frac{1}{\varepsilon^\alpha} E\{ |Z|_h^\Gamma(\varepsilon R/|Z|_h) \cdot Z)\}
\]

and hence

\[
p_h = \nu\{\{f \in D : |f|_h > 1\}\} = E\{ |Z|_h^2\} \in [0, \infty), \quad \forall h \in Q.
\]

### 3.2 Local tail and local spectral tail processes

We introduce next the local tail and local spectral tail processes as in \([25]\); our setup here is less restrictive compared to that of product spaces dealt with in the aforementioned paper. Recall that \(\nu \in \mathcal{M}_\sigma(\mathcal{D})\) stands for \(\nu\) is a tail measure on \((D, \mathcal{D})\).

**Definition 3.4.** Given \(\nu \in \mathcal{M}_\sigma(\mathcal{D})\) and \(h \in Q\) such that \(p_h > 0\), the local process \(Y^{[h]}\) of \(\nu\) at \(h\) has law \(\nu_h(A) = \nu\{\{f \in A : |f|_h > 1\}\} / p_h, A \in \mathcal{D}\). We call \(\Theta^{[h]} = (|Y^{[h]}|_h)^{-1} \cdot Y^{[h]}\) the local spectral tail process of \(\nu\) at \(h\). If \(p_h = 0\), then set \(Y^{[h]} = R \cdot g, \Theta^{[h]} = g\) with \(g \in D\) satisfying \(|g|_h = 1\).

We shall drop the subscript \(\nu\) for local tail/ spectral tail processes, when there is no ambiguity.

**Remark 3.5.** \(Y^{[h]}\) is a random element from a probability space \((\Omega, \mathcal{F}, P)\) to \((D, \mathcal{D})\) and similarly for \(\Theta^{[h]}\). Take for instance \(\Omega = D, \mathcal{F} = \mathcal{D}, P = \nu_h\) and define \(Y^{[h]} : f \mapsto \mathbb{1}\{|f|_h > 1\} f, f \in D\), which in view of \((A1)\) is a \(\mathcal{F} / \mathcal{D}\) measurable map for all \(h \in Q\). The assumption on \(\nu\) and \((3.3)\) imply that \(|Y^{[h]}|_h, t \in Q\) is a non-negative \(rv\) for all \(h, t \in Q\).

**Proposition 3.6.** If \(\nu \in \mathcal{M}_\sigma(\mathcal{D})\), then for all \(h, t \in Q\)

\[
P\{|Y^{[h]}|_h > 1\} = P\{|\Theta^{[h]}|_h = 1\} = 1
\]

and if \(p_h = 0, p_t > 0\), then \(|Y^{[t]}|_h = |\Theta^{[t]}|_h = 0\) almost surely. Further, for all \(h, t \in Q\) such that \(p_hp_t > 0\)

\[
p_h E\{|\Theta^{[h]}|_t^\alpha \Gamma(\Theta^{[h]}))\} = p_t E\{\mathbb{1}\{|\Theta^{[t]}|_h \neq 0\} \Gamma(\Theta^{[h]}))\}, \quad \forall \Gamma \in \mathcal{F}_0,
\]

and for all \(x > 0\)

\[
p_h E\{\Gamma(x \cdot Y^{[h]})) \mathbb{1}\{x|Y^{[h]}|_t > 1\}) = p_t x^\alpha E\{\Gamma(Y^{[t]})|x|Y^{[t]}|_h > x\})\}, \quad \forall \Gamma \in \mathcal{F}.
\]

Moreover, the law of \(Y^{[h]}\) agrees with that of \(R \cdot \Theta^{[h]}, h \in Q\) and \(Y^{[h]}, h \in Q : p_h > 0\) uniquely determine \(\nu\).
Remark 3.7. For all \( h, t \in Q \) such that \( p_h p_t > 0 \)
\[
p_h \mathbb{E} \left\{ \mathbb{1} \left( |\Theta|^h |_t \neq 0 \right) \Gamma(\Theta^h) \right\} = p_t \mathbb{E} \left\{ \mathbb{1} \left( |\Theta|^h |_h \neq 0 \right) \Gamma(\Theta^h) \right\}, \quad \forall \Gamma \in S_\alpha, \tag{3.10}
\]
see also \([26]\).

If \( \nu = \nu_Z \) is given by (1.2) with \( Z \) satisfying (3.5), then by definition the claim in (3.6) implies
\[
\int_D \Gamma(f) \mathbb{1} \left( \|f\|_h > 1 \right) \nu(df) = p_h \mathbb{E} \left\{ \Gamma(Y^h) \right\} = \mathbb{E} \left\{ \|Z^h\|_h \Gamma((R/\|Z^h\|) \cdot Z) \right\}, \quad \forall h \in Q : p_h > 0, \forall \Gamma \in \mathcal{F}.
\tag{3.11}
\]
If \( p_h = 0 \), then (3.11) still holds taking \( \Gamma \) to be bounded. Consequently, since \( p_h = \mathbb{E} \left\{ \|Z^h\|_h \right\} < \infty \), then \( Z \) determines the laws of \( Y^h \) and \( \Theta^h \) denoted by \( \mathbb{P}_{Y^h} \) and \( \mathbb{P}_{\Theta^h} \), respectively, i.e.,
\[
\mathbb{P}_{Y^h}(\cdot) = p_h^{-1} \mathbb{E} \left\{ \|Z^h\|_h^2 \delta_{(R/\|Z^h\|) \cdot Z}(\cdot) \right\}, \quad \mathbb{P}_{\Theta^h}(\cdot) = p_h^{-1} \mathbb{E} \left\{ \|Z^h\|_h^2 \delta_{(R/\|Z^h\|) \cdot Z}(\cdot) \right\}
\tag{3.12}
\]
for all \( h \in Q \) such that \( p_h > 0 \), with \( \delta_x(\cdot) \) the Dirac point measure of \( x \in \mathbb{R} \).

3.3 Stochastic representers

A measure \( \nu \) on \( \mathcal{F} \) has a stochastic representer \( Z \) satisfying (3.5) if \( \nu \) equals \( \nu_Z \) defined in (1.2). Hereafter \( q_t \geq 0, t \in T \) satisfy \( q_t > 0 \) for all \( t \in Q \) such that \( p_t > 0 \) and we set
\[
S^q(Y^h) = \int_Q \|Y^h\|_t^q q_t \lambda(dt), \quad h \in Q,
\]
where \( \lambda(dt) = \lambda_Q(dt) \) is the counting measure on \( Q \).

If \( q_t \)’s are such that \( \sum_{t \in Q} q_t = 1 \), then we shall consider a \( Q \)-valued rv \( N \) with probability mass function \( q_t, t \in Q \) being independent of all other elements.

Remark 3.8. In view of Remark 3.5 and [34][Cor. 5.8] it is possible to choose \( Y^h, h \in Q \) and \( N \) to be defined in the same probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \) such that all these are independent, which we shall assume below. Moreover, \( q_t \)’s and thus \( N \) can be chosen such that \( \mathbb{E} \{p_N\} < \infty \), with \( p_h = \mathbb{E} \{\|Z^h\|_h^2\} < \infty \). Recall \( Y^h = Y^h |_Q \) is defined with respect to some \( \nu \in \mathcal{M}_\alpha(\mathcal{F}) \).

Below, let \( K(Q) = \{K_n \subset Q, n \in \mathbb{N}\} \) such that \( \cup_{n \geq 1} K_n = Q \). Under assumption A3) \( Q \) is a dense subset of \( T \) and we shall choose \( K_n = [-n, n]^1 \cap Q, n \in \mathbb{N} \).

Definition 3.9. A measure \( \nu \) on \( \mathcal{F} \) is \( K(Q) \)-bounded (compactly bounded when A3) holds) if

M3) \( \nu \{\{f \in D : f_K^\ast > 1\} \} < \infty, \quad \forall K \in K(Q) \).

The next result shows that tail measures have a family of representers \( Z \), which can be utilised to define \( Y^h \) and \( \Theta^h \) via (3.12) and to give an equivalent condition for M3).

Lemma 3.10. If \( \nu \in \mathcal{M}_\alpha(\mathcal{F}) \), then \( \nu \) has stochastic representer \( Z = Z_N \) given by
\[
Z_N = \frac{p_N^{1/\alpha} Y^N}{(S^q(Y^N))^{1/\alpha}}, \tag{3.13}
\]
where the local tail processes \( Y^h, h \in Q \) and \( N, q_t, t \in Q \) are as in Remark 3.8. Further, \( \|Z\|_h = 0 \) if \( p_h = 0 \) and \( \nu \) satisfies M3) if and only if
\[
\mathbb{E} \left\{ \sup_{t \in K \cap Q} |Z|^q_t \right\} < \infty, \quad \forall K \in K(Q). \tag{3.14}
\]

Remark 3.11. (i) The claim that \( \nu \in \mathcal{M}_\alpha(\mathcal{F}) \) is specified by (1.2) for some representer \( Z \) follows also by [35][Thm 1].
(ii) If \( \nu = \nu_Z = \nu_{Z} \in \mathcal{M}_\alpha(\mathcal{D}) \), applying (3.11) \( \forall h \in \mathcal{Q} : p_h > 0 \) we obtain

\[
p_h \mathbb{E}\{\Gamma(\Theta^h)\} = \mathbb{E}\{|Z|^{\alpha} \Gamma((R/|Z|)\cdot Z)\} = \mathbb{E}\{|Z|^{\alpha} \Gamma(Z)\}, \quad \forall \Gamma \in \mathcal{S}_0.
\]

Since by (3.5) we can choose \( q_h > 0, h \in \mathcal{Q} \) such that \( S(Z) = \sum_{h \in \mathcal{Q}} q_h |Z|^{\alpha}_h \in (0, \infty) \) almost surely, then (3.15) implies

\[
\mathbb{E}\{|\Gamma_\alpha(Z)|\} = \sum_{h \in \mathcal{Q}} q_h \mathbb{E}\{Z|^{\alpha}_h \Gamma_\alpha(Z)/S(Z)\} = \sum_{h \in \mathcal{Q}} q_h \mathbb{E}\{|\tilde{Z}|^{\alpha}_h \Gamma_\alpha(\tilde{Z})/S(\tilde{Z})\} = \mathbb{E}\{|\Gamma_\alpha(Z)|\}, \quad \forall \Gamma_\alpha \in \mathcal{S}_\alpha.
\]

Hence, if in Lemma 3.10 \( \mathcal{D} \) is a countable product space or it is equal to \( D(\mathbb{R}, \mathbb{R}^d) \) we retrieve [25][Thm 2.4] and [3][Thm 2.3], respectively.

(iii) Lemma 3.10 together with Lemma 3.6 and (3.12) implies [25][Prop 2.7].

Example 3.12. Assume that A3 holds. Denoting by 0 the zero function we have \( \{0\} \in \mathcal{B}(\mathcal{D}) \). Since \( \| f \|_t = \| f(t) \|, t \in \mathcal{Q} \) with \( \cdot \) a norm on \( T \) and further \( \mathcal{Q} \) is a dense subset of \( T \), then M1) is equivalent to

\[\nu(\{0\}) = 0.\]

If \( \nu \in \mathcal{M}_\alpha(\mathcal{D}) \), then its representer \( Z \) is a random process with almost surely càdlàg sample paths and so are both \( Y|^{h}_\alpha \) and \( \Theta^h \) for all \( h \in \mathcal{T} \).

A direct implication of (3.12) is that if \( \nu \) is shift-invariant (see [25] for definition), then by (3.11) we have the equality in law

\[
Y_h|^{\alpha} \overset{\text{d}}{=} B^h Y^{[0]}, \quad p_h = p_{h_0} \in (0, \infty), \quad \forall h \in \mathcal{T},
\]

where \( B^h f(\cdot) = f(\cdot - h), h \in \mathcal{T} \). Note in passing that in this case (3.9) reads (set below \( Y = Y^{[0]} \))

\[
\mathbb{E}\{\Gamma(x B^h Y) \mathbb{I}\{x|Y|(-h)| > 1\}\} = x^\alpha \mathbb{E}\{\Gamma(Y) \mathbb{I}\{|Y|(-h)| > x\}\}, \quad \forall \Gamma \in \mathcal{S}_0, \forall h \in \mathcal{T}, \forall x > 0,
\]

which for \( \mathcal{T} = \mathbb{Z} \) is stated initially in [22], see [3] for the case \( \mathcal{T} = \mathbb{R}^d \) and [14] for other interesting properties of \( Y \). Also the converse holds, i.e., (3.17) implies that \( \nu \) is shift-invariant.

### 3.4 Constructing \( \nu \) from local tail processes

A given tail measure defines the family of local tail processes \( Y|^{h}_\alpha, h \in \mathcal{Q} \). We discuss in this section the inverse procedure, namely how to calculate \( \nu \in \mathcal{M}_\alpha(\mathcal{D}) \) from the \( Y|^{h}_\alpha \)'s. Hereafter \( q_h \)'s are positive constants and we write \( \lambda \) for counting measure \( \lambda_\mathcal{Q} \) on \( \mathcal{Q} \) or for Lebesgue measure on \( T = \mathbb{R}^d \) and define

\[
\mathcal{E}_K(f) = \int_K \mathbb{I}\{|f|_t > 1\} q_t \lambda(dt), \quad f \in \mathcal{D},
\]

with \( K \) a non-empty subset of \( \mathcal{Q} \) if \( \lambda = \lambda_\mathcal{Q} \) and \( K \) is a non-empty hypercube of \( \mathbb{R}^d \), otherwise.

The next result extends [1][Thm 5.4.2].

Lemma 3.13. Let \( \nu \in \mathcal{M}_\alpha(\mathcal{D}) \) be given. Suppose that for some \( H \in \mathcal{S}_0 \), there exists \( \varepsilon_\mathcal{H} > 0 \) and some non-empty \( K_\mathcal{H} \subset \mathcal{Q} \) such that for all \( f \in \mathcal{D} \) satisfying \( f|^{\mathcal{H}}_K \leq \varepsilon_\mathcal{H} \) we have \( H(f) = 0 \). If \( \int_K p_t q_t \lambda(dt) \in (0, \infty) \) for some \( \mathcal{K} \subset \mathcal{Q} \) such that \( K_\mathcal{H} \subset \mathcal{K} \), then

\[
\nu[H] = e^{-\alpha} \int_K \mathbb{E}\left\{\frac{H(x \cdot Y|^{\mathcal{H}}_K)}{\mathcal{E}_K(x \cdot Y|^{\mathcal{H}}_K)}\right\} p_t q_t \lambda(dt), \quad \forall \varepsilon \in (0, \varepsilon_\mathcal{H}).
\]

Remark 3.14. (i) Taking \( H(f) = \mathbb{I}\{(\sup_{s \in K} |f|_s)_t > 1\} \) for some non-empty \( \mathcal{K} \subset \mathcal{Q} \), for \( q_t \)'s as chosen in Lemma 3.13 we obtain from (3.20) that M3) is equivalent to

\[
\int_K \mathbb{E}\left\{\frac{H(x \cdot Y|^{\mathcal{H}}_K)}{\mathcal{E}_K(x \cdot Y|^{\mathcal{H}}_K)}\right\} p_t q_t \lambda(dt) = \int_K \mathbb{E}\left\{\frac{1}{\mathcal{E}_K(x \cdot Y|^{\mathcal{H}}_K)}\right\} p_t q_t \lambda(dt) < \infty, \quad \forall K \in \mathcal{Q}.
\]

since \( P\{|Y|_t > 1\} = 1, \forall t \in \mathcal{Q} \) implies that \( H(Y|^{\mathcal{H}}_t) = 1 \) almost surely for all \( t \in K \);
(ii) Under A3), by (2.1) an \( -\alpha \)-homogeneous Borel measure \( \nu \) on \( \mathcal{B}(D) \) is \( \mathcal{B}_0 \)-boundedly finite if and only if M3) holds, or equivalently (3.21) is satisfied.

We have seen that local tail/spectral tail processes can be defined directly through the representer \( Z \) of a tail measure \( \nu = \nu_Z \). We may also define such families without referring to \( Z \) as follows.

**Definition 3.15.** The family of \( D \)-valued random elements \( Y^{[h]}, h \in \mathcal{Q} \) is called a family of tail processes with index \( \alpha > 0 \), if for given weights \( p_h \in [0,\infty) \), \( h \in \mathcal{Q} \), with \( p_h > 0 \) for some \( h_0 \in \mathcal{Q} \) both (3.7) and (3.9) are satisfied and further \( \mathbb{P}\{ |Y^{[t]}|_h = 0 \} = 1 \) if \( p_h = 0, p_t > 0 \).

Similarly, \( D \)-valued spectral tail processes \( \Theta^{[h]}, h \in \mathcal{Q} \) can be defined without reference to some measure \( \nu \). As shown next a family of tail processes defines uniquely a tail measure on \( D \).

**Lemma 3.16.** Let \( Y^{[h]}, h \in \mathcal{Q} \) and \( \Theta^{[h]}, h \in \mathcal{Q} \) be a family of \( D \)-valued tail and spectral tail processes, respectively with index \( \alpha > 0 \) and let \( q_h \)'s be positive constants such that \( \sum_{h \in \mathcal{Q}} p_h q_h < (0, \infty) \).

(i) If \( Y^{[h]}, h \in \mathcal{Q}, N \) are defined as in Remark 3.8, then there exists a unique \( \nu \in \mathcal{M}_\alpha(\mathcal{D}) \) such that its local tail processes are \( Y^{[h]}, h \in \mathcal{Q} \);

(ii) \( \nu \) defined in (i) above satisfies M3) if and only if

\[
\int_{K} \mathbb{E}\left\{ \frac{1}{\mathbb{K}} \right\} p_h q_h \lambda(dt) < \infty, \quad \forall K \in K(\mathcal{Q}),
\]

which under A3) with \( T = \mathbb{R}^d \) is true also for \( \lambda(dt) \) the Lebesgue measure on \( T \) and all compact \( K \subset T \) with \( q_h \) as in Lemma 3.13;

(iii) \( \mathcal{R} \Theta^{[h]}, h \in \mathcal{Q} \) is a family of tail processes with index \( \alpha > 0 \).

**Example 3.17.** Suppose that A3) holds and let \( Z \) satisfy (3.5) for all \( h \in T \). Assume that \( \tilde{Z} = B^h Z \) and \( Z \) satisfy (3.15) for all \( h \in T \), which implies \( \mathbb{E}\{ |Z|_h^2 \} = C \in (0, \infty), \forall h \in T \) and \( \nu_Z \) has local processes at \( h \in T \) given by

\[
Y^{[h]} = B^h Y, \quad Y = Y^{[0]},
\]

with \( Y^{[0]} \) having law \( \mathbb{E}\{ |Z|_0^2 \delta_{\mathbb{R}^d/\mathbb{R}^d}() \}/C \). For \( \nu_{\sigma Z} \) with representer \( \sigma Z, \sigma \in \mathcal{D}, \sigma \neq 0 \) its local tail processes are given by

\[
Y^{[h]}(t) = \frac{\sigma(t)}{\sigma(h)} B^h Y(t), \quad \forall t \in T, \forall h : \sigma(h) \neq 0.
\]

## 4 RV of measures and random elements

We first discuss the RV of Borel measures and \( D \)-valued random elements assuming A2). Subsequently, we study in detail the RV of processes with cadlag paths.

### 4.1 \( \mathcal{B} \)-boundedly finite Borel measures

RV of Borel measures on Polish metric spaces is discussed in [8, 9, 12, 13, 17]. We follow the treatment of RV in [1], where some properly localised boundedness \( \mathcal{B} \) plays a crucial role.

Throughout this section we suppose that A2) holds.

Let next \( \nu_z, z > 0 \) be \( \mathcal{B} \)-boundedly finite measures on \( \mathcal{B}(D) \) and recall our notation \( \nu[H] = \int_D H(f)\nu(df) \).

**Definition 4.1.** \( \nu_z \) converges \( \mathcal{B} \)-vaguely to some Borel measure \( \nu \) as \( z \to \infty \) (denote this by \( \nu_z \xrightarrow{\mathcal{B}} \nu \)) if

\[
\lim_{z \to \infty} \nu_z[H] = \nu[H]
\]

is valid for all continuous and bounded maps \( H : D \to \mathbb{R} \) with \( \text{supp}(H) \in \mathcal{B} \).
In the sequel \( g, g' \) are two maps \( \mathbb{R}_+ \mapsto \mathbb{R}_+ \) and for some \( \mu \in \mathcal{M}^+(\mathcal{B}) \) we set
\[
\mu_z(A) = g(z)\mu(z\cdot A), \quad \mu'_z(A) = g'(z)\mu(z\cdot A), \quad A \in \mathcal{B}(\mathcal{D}), \ z \in \mathbb{R}_+.
\]

**Lemma 4.2.** If \( \mu_z, \mu'_z \) converge \( \mathcal{B} \)-vaguely to \( \nu \in \mathcal{M}^+(\mathcal{B}) \) and \( \nu' \in \mathcal{M}^+(\mathcal{B}) \), respectively, as \( z \to \infty \), then \( \lim_{z \to \infty} g'(z)/g(z) = c \) and \( \nu' = c\nu \) for some \( c \in (0, \infty) \).

**Definition 4.3.** \( \mu \in \mathcal{M}^+(\mathcal{B}) \) is regularly varying with scaling function \( g \), if \( \mu_z \xrightarrow{v,R} \nu \in \mathcal{M}^+(\mathcal{B}) \), abbreviate this as \( \mu \in \mathcal{R}(g, \mathcal{B}, \nu) \).

**Remark 4.4.**
(i) In view of \([1][\text{Cor B.1.19}]) \( \mu \in \mathcal{R}(g, \mathcal{B}, \nu) \) if and only if there exist open sets \( O_k \in \mathcal{B}, k \in \mathbb{N} \) satisfying \( B2 \) such that for all positive integers \( k \)
\[
\nu(\partial O_k) = 0, \ g(z)\mu(z\cdot (O_k \cap \cdot)) \xrightarrow{w} \nu(O_k \cap \cdot), \ z \to \infty.
\]

(ii) Let \( \nu_Z \in \mathcal{M}_\alpha(\mathcal{D}) \) and write \( \mathbb{P}_Z \) for the law of \( Z \). The \( -\alpha \)-homogeneity implies that \( \nu \in \mathcal{R}(g, \mathcal{B}, \nu_Z) \) with \( g(x) = x^\alpha \).

Note that \( \nu_Z \) is the mean measure of the Poisson Point Process (PPP) \( \mathcal{N}_Z \) on \( \mathcal{D} \), which is defined by
\[
\mathcal{N}_Z(\cdot) = \sum_{i=1}^{\infty} \delta_{P_i, Z(\cdot)}(\cdot),
\]
with \( \sum_{i=1}^{\infty} \delta_{P_i, Z(\cdot)} \) being a PPP on \((0, \infty) \times \mathcal{D} \) with mean measure \( \nu_Z(\cdot) \circ \mathbb{P}_Z(\cdot) \) and \( Z(\cdot) \)'s being independent copies of \( Z \).

Write next \( g \in \mathcal{R}_\alpha \), if
\[
\lim_{z \to \infty} g(zt)/g(z) = t^\alpha, \ \forall t > 0
\]
for a non-negative rv \( W \) we write \( W \in \mathcal{R}_\alpha \) if \( 1/\mathbb{P}\{W > t\} \in \mathcal{R}_\alpha \).

Set \( H_t(f) := |f|/t, f \in \mathcal{D} \) and recall the definition of \( p_h(x) \) in \((3.2)\).

**Lemma 4.5.** Let \( \mu \in \mathcal{R}(g, \mathcal{B}, \nu) \), where \( g \) is Lebesgue measurable.
(i) If \( g \in \mathcal{R}_\alpha \) for some \( \alpha > 0 \), then \( \nu \) is \(-\alpha\)-homogeneous;
(ii) If \( p_{h_0}(x) \in (0, \infty) \) for some \( h_0 \in \mathcal{Q}, x > 0 \) and further \( H_t^{-1}(B) \in \mathcal{B} \) for all \( B \in \mathcal{B}(\mathcal{D}(0, \infty)) \) separated from 0 satisfying also \( \nu(Disc(H_{h_0})) = 0 \), then \( g \in \mathcal{R}_\alpha \) for some \( \alpha > 0 \) and \( \nu \) is \(-\alpha\)-homogeneous;
(iii) Suppose that \( B4 \) holds. If \( \mu(k\cdot A) > 0 \) for almost all \( k > M > 0 \) and the group action is continuous, then \( g \in \mathcal{R}_\alpha \) for some \( \alpha > 0 \) and \( \nu \) is \(-\alpha\)-homogeneous.

### 4.2 D-valued random elements

Consider next a D-valued random element \( X \) defined on a complete probability space \((\Omega, \mathcal{F}, \mathbb{P})\). Define for some \( g : \mathbb{R}_+ \mapsto \mathbb{R}_+ \)
\[
\mu_z(A) = g(z)\mathbb{P}\{X \in z\cdot A\}, \ A \in \mathcal{F}, z > 0.
\]

**Definition 4.6.** The random element \( X \) is called RV with respect to \( g \) and \( \nu \in \mathcal{M}^+(\mathcal{B}) \), if \( \mu_z \xrightarrow{v,R} \nu \) as \( z \to \infty \). Abbreviate this as \( X \in \mathcal{R}(g, \mathcal{B}, \nu) \) and when \( g \in \mathcal{R}_\alpha \) write \( X \in \mathcal{R}_\alpha(g, \mathcal{B}, \nu) \).

If \( X \in \mathcal{R}(g, \mathcal{B}, \nu) \), with Lebesgue measurable \( g \), under the assumptions of Lemma 4.5.(ii) we have that \( |X|_{h_0} \in \mathcal{R}_\alpha \) implies \( \nu \) is \(-\alpha\)-homogeneous. If further the conditions of Lemma 4.5.(ii) hold for all \( h \in \mathcal{Q} \), then Theorem A.2 yields
\[
\lim_{z \to \infty} \frac{\mathbb{P}\{|X|_h > z\}}{\mathbb{P}\{|X|_{h_0} > z\}} = \frac{p_h}{p_{h_0}} \in [0, \infty), \ \forall h \in \mathcal{Q}, \ p_{h_0} \in (0, \infty). \quad (4.2)
\]

Under \( A2 \), we present in the next result a sufficient condition for the RV of \( X \) when \( \mathcal{B} \) is determined by the countable family of maps \( \mathcal{H}(\mathcal{D}) \) as in \( B5 \).
Theorem 4.7. Let $X$ be such that $|X|_{t_0} \in \mathcal{R}_\alpha$ for some $t_0 \in \mathcal{Q}$ and (4.2) holds. Assume that $\forall h \in \mathcal{Q}: p_h > 0$ conditionally on $|X|_h > z$, $z^{-1} \cdot X$ converges weakly on $(D, d_B)$ to $Y^{[h]}$ as $z \to \infty$. Suppose further that
\[ \lim_{z \to \infty} \frac{\mathbb{P}\{X_K > \varepsilon z\}}{\mathbb{P}\{|X|_{t_0} > z\}} < \infty, \quad \forall \varepsilon > 0, \forall K \in \mathcal{K} (\mathcal{Q}) \] (4.3)
and for some $c > 1$ and positive $q_t$'s such that $\sum_{t \in \mathcal{Q}} \max(1, p_t) q_t < \infty$
\[ \lim_{z \to \infty} \sup_{\eta > 0} \mathbb{P}\{X_K > \varepsilon z, \mathcal{E}_K (\varepsilon z^{-1} \cdot X) \leq \eta\} = 0, \quad \forall \varepsilon > 0, \forall K \in \mathcal{K} (\mathcal{Q}). \] (4.4)

If $Y^{[h]}$, $h \in \mathcal{Q}$, is a family of tail processes, $\mathcal{E}_K (\cdot)$ defined in (3.19) is almost surely continuous with respect to the law of $Y^{[h]}$, for all $h \in \mathcal{Q}$ and further $\mathcal{B}$ satisfies B5, then there exists a $\mathcal{B}$-boundedly finite Borel tail measure $\nu \in \mathcal{M}_\alpha (\mathcal{D})$ such that $\nu$ holds and $X \in \mathcal{R}_\alpha (g, \mathcal{B}, \nu)$, $g(t) = p_{t_0}/\mathbb{P}\{|X|_{t_0} > t\}$.

4.3 Càdlàg processes

In this section we assume $A3$ and consider a $D$-valued random process $X$ defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and not identical to $0$. Further below $g : \mathbb{R}_+ \mapsto \mathbb{R}_+$ is a Lebesgue measurable function. Alternatively to the definition in the Introduction, as in [3], $X$ is called finite dimensional regularly varying if for all $t_1, \ldots, t_k \in T, k \geq 1$ there exists a non-trivial Borel measure $\nu_{t_1, \ldots, t_k}$ on $\mathcal{B} (\mathbb{R}^d)$ satisfying
\[ \lim_{z \to \infty} \inf_{\alpha \in \mathbb{R}} g(z) \mathbb{P}\{(z^{-1} \cdot X(t_1), \ldots, z^{-1} \cdot X(t_k)) \in A\} = \nu_{t_1, \ldots, t_k} (A) < \infty \] (4.5)
for all $A \in \mathcal{B} (\mathbb{R}^d)$ separated from $0$ in $\mathbb{R}^d$ such that $\nu_{t_1, \ldots, t_k} (\partial A) = 0$. Moreover the measures $\nu_{t_1, \ldots, t_k}$ are $\alpha$-homogeneous for some $\alpha > 0$ and $g \in \mathcal{R}_\alpha$. If we set $\nu_{t_1, \ldots, t_k} (\{0\}) = 0$, then $\nu_{t_1, \ldots, t_k}$ is a tail measure on $(\mathbb{R}^d)^k$ with index $\alpha$. Since any norm $\| \cdot \|$ on $\mathbb{R}^d$ is continuous, 1-homogeneous and $[0] = 0$, Remark 6.2 implies that
\[ \|X\|_{t_0} = |X (t_0)| \in \mathcal{R}_\alpha. \]

In view of [2][Thm 2.1] there exists $\nu'$ on $(\mathbb{R}^d)^T$ equipped with the cylindrical $\sigma$-field such that $\nu_{t_1, \ldots, t_k}$ is its projection on the corresponding subspace. From the aforementioned reference $\nu'$ is $-\alpha$-homogeneous and moreover M1 holds for all $h \in T$. Denote by $Y^{[h]}$, $h \in T$ and $\Theta^{[h]}$, $h \in T$ the local tail and local spectral tail processes of $\nu'$, respectively.

Utilising [25][Lem 3.5], [12][Prop 3.1, Thm 4.1.5.1] and [2][Thm 12.1] the finite RV of $X$ implies that (4.2) holds and further (we use the notation of [25] below for convergence in distribution):

(i) for all $h$ such that $p_h = \nu' (\{f \in D : \|f\|_h > 1\}) > 0$ and all $t_j \in T, 1 \leq j \leq k$
\[ \lim_{z \to \infty} \mathcal{L} \left( z^{-1} \cdot X(t_1), \ldots, z^{-1} \cdot X(t_k) \mid \|X\|_h > x \right) = \mathcal{L} \left( Y^{[h]} (t_1), \ldots, Y^{[h]} (t_k) \right); \] (4.6)

(ii) for all $h \in T$ such that $p_h > 0$ and all $t_j \in T, 1 \leq j \leq k$ we have
\[ \lim_{z \to \infty} \mathcal{L} \left( \frac{1}{\|X\|_h} \cdot X(t_1), \ldots, \frac{1}{\|X\|_h} \cdot X(t_k) \mid \|X\|_h > u \right) = \mathcal{L} \left( \Theta^{[h]} (t_1), \ldots, \Theta^{[h]} (t_k) \right). \] (4.7)

We focus next on $D = D(\mathbb{R}^1, \mathbb{R}^d)$ and discuss RV on $D_0 = D \setminus \{0\}$ equipped with the boundedness $\mathcal{B}_0$ defined in Section 2 via (2.1). The case $D = D(\mathbb{Z}^1, \mathbb{R}^d)$ and some more general product spaces are already investigated in [25].

Definition 4.8. $X$ is called RV with limit measure $\nu \in \mathcal{M}^+ (\mathcal{B}_0)$ if $g(z) \mathbb{P}\{z^{-1} X \in \cdot\} \overset{\nu_0}{\to} \nu$ as $z \to \infty$ for some Lebesgue measurable $g : \mathbb{R}_+ \mapsto \mathbb{R}_+$.

A $\mathcal{B}_0$-boundedly finite measure $\nu$ on $\mathcal{B}(D_0)$, i.e., $\nu \in \mathcal{M}^+ (\mathcal{B}_0)$ can be uniquely extended to a measure $\nu^*$ on $\mathcal{D} = \mathcal{B}(D)$ by
\[ \nu^* (\{0\}) = 0, \quad \nu^*(A) = \nu(A \cap \{f \in D : f \neq 0\}), \quad A \in \mathcal{D}. \] (4.8)
If $\nu \in M^+(B_0)$ is $-\alpha$-homogeneous, then $\nu^*$ is also $-\alpha$-homogeneous and since $\nu^*(\{0\}) = 0$ is equivalent to (M1) we have that $\nu^*$ is a tail measure on $\mathcal{D}$ with index $\alpha > 0$. Given such $\nu$ we shall write for notational simplicity $\nu$ instead of $\nu^*$ and hence $\nu \in M^+(B_0) \cap M_\alpha(\mathcal{D})$ means $\nu \in M^+(B_0)$ and $\nu^* \in M_\alpha(\mathcal{D})$.

Since $D$ is not star-shaped, for a RV $X$ with limit measure $\nu$ we cannot apply at this point [9][Thm 3.1] to conclude that $\nu$ is $-\alpha$-homogeneous. The next result shows that $\nu^*$ is even a compactly bounded tail measure. As in the previous section we set below $p_h = \nu(\{f \in D : \|f\|_h > 1\})$. In the rest of this section $l_0 \in T$ is such that $p_{l_0} > 0$.

**Lemma 4.9.** If $X$ defined on the complete non-atomic probability space $(\Omega, \mathcal{F}, P)$ is RV with limit measure $\nu \in M^+(B_0)$, then $g \in R_\alpha$ for some $\alpha > 0$ and $\nu$ extends uniquely to a tail measure on $\mathcal{D}$ with index $\alpha$. Moreover, we can take $g(t) = p_{l_0}/P[\|X\|_{l_0} > t]$ and $\nu = \nu_2$ with representor $Z$ satisfying $P[Z \neq 0] = 1$. Further the local tail processes $Y^{[h]}$, $h \in Q$, of $\nu$ are all defined on $(\Omega, \mathcal{F}, P)$ and both (3.14), (3.21) hold for all compact $K \subset T = \mathbb{R}^I$.

In view of the Lemma 4.9 we can adopt the following equivalent definition.

**Definition 4.10.** $X$ is regularly varying with $\nu \in M^+(B_0) \cap M_\alpha(\mathcal{D})$ (abbreviated $X \in R_\alpha(B_0, \nu)$), if $\|X\|_{l_0} \in R_\alpha$ and

$$p_{l_0}P[\|X\|_{l_0} > 1]/P[\|X\|_{l_0} > z] \xrightarrow{\nu_2} \nu(\cdot).$$

Next, we shall utilise Remark 4.4.(i) and the explicit structure of $B_0$ described in (2.1).

In the rest of this section assume without loss of generality that $t_0 = 0 \in \mathbb{R}^I$ and this will be the assumption also for RV of $X_U$, the restriction of $X$ on $U = [a, b]$ a hypercube of $\mathbb{R}^I$ that contains $[-1, 1]^I$.

Denote by $D_U = D(U, \mathbb{R}^I)$ the space of càdlàg functions $f : U \rightarrow \mathbb{R}^I$ with $U$ some hypercube in $\mathbb{R}^I$ that contains $[-1, 1]^I$, which is also a Polish space, see e.g., [36][Lem 2.4]. Define the boundedness $B_0(D_U)$ with respect to the zero function of $D_U$ denoted by $B_0$, $B_0(D_U)$ can be characterised by (2.1) with obvious modifications. An analogous result to Lemma 4.9 can be formulated for $\mu \in M^+(B_0(D_U))$ with $T = U$ and hence we can define RV of a $D_U$-valued random element similarly to that of $D$-valued random elements. Further, we extend $\mu$ uniquely to a tail measure on $\mathcal{B}(D_U)$ as above.

Now, if $\nu \in M^+(B_0)$ and thus $\nu^* \in M_\alpha(\mathcal{D})$, we can define its projection with respect to $U$ denoted by $\nu_U^*$ as the tail measure on $\mathcal{B}(D_U)$ determined uniquely by $Y^{[h]}_U$, $h \in U$, where $Y^{[h]}$'s are the local tail processes of $\nu^*$, since their restriction on $U$ denoted by $Y^{[h]}_U$ yields a family of tail processes on $D_U$. Write then $\nu_U$ for the restriction of $\nu$ on $\mathcal{B}(D_U \setminus \{0_U\})$.

Let $p_U : D \rightarrow D_U$, with $p_U(f) = f_U$ be the restriction of $f \in D$ on $U$. In view of Theorem A.1, (ix) we can find $a_n, b_n$ such that $\nu^*(\text{Disc}(p_U)) = 0$ and $[-n, n]^I \subset [a_n, b_n] =: U_n$ for each given positive integer $n$.

**Theorem 4.11.** If $U_n, n \in \mathbb{N}$ is as above and $X \in R_\alpha(B_0, \nu)$, then $$X_{U_n} \in R_\alpha(B_0(D_{U_n}), \nu^{(n)}), \quad \nu^{(n)} = \nu|_{U_n}, \quad \forall n \in \mathbb{N}.$$ Conversely, if $X_{U_n} \in R_\alpha(B_0(D_{U_n}), \nu^{(n)})$ for all $n \in \mathbb{N}$, then

$$X \in R_\alpha(B_0, \nu), \quad \nu|_{U_n} = \nu^{(n)}, \quad \forall n \in \mathbb{N}, \quad \nu \in M^+(B_0) \cap M_\alpha(\mathcal{D}).$$  

(4.9)

**Remark 4.12.** (i) Both Lemma 4.9 and Theorem 4.11 hold also for $D = D(Z^I, \mathbb{R}^I)$;

(ii) If there is a $D$-valued random element $Z$ such that

$$P[Z \neq 0] = 1, \quad \mathbb{E}[\|Z\|_o^\alpha] \in [0, \infty), \quad \forall t \in T, \mathbb{E}[\|Z\|_t^\alpha] > 0, \quad \mathbb{E}\left(\sup_{t \in K \cap Q} \|Z\|_t^\alpha\right) < \infty, \quad \forall K \subset \mathbb{R}^I,$$

with $K \subset T$ compact and $\nu^{(n)} = \nu_{Z_n}, \forall n \in \mathbb{N}$, where

$$Z_n(t) = c_n^{1/\alpha} Z(t) \sup_{t \in U_n} |Z|_t > 0, \quad t \in U_n, \quad c_n = \mathbb{P}\left(\sup_{t \in U_n} |Z|_t > 0\right) > 0.$$  

(4.11)
then it follows from the proof of Theorem 4.11 that (4.9) holds with \( \nu = \nu_Z \). Conversely, if \( \nu \) has representer \( Z \), then \( \nu^{(n)} = \nu_{Z^n}, \forall n \in \mathbb{N} \) holds.

Consider next an \( \mathbb{R}^d \)-valued max-stable random process \( X(t), t \in T \) given via its de Haan representation (e.g., [10, 37])

\[
X(t) = \max_{i \geq 1} \Gamma_i^{-1/\alpha} Z^{(i)}(t), \quad t \in T.
\]

(4.12)

Here \( \Gamma_i = \sum_{k=1}^i \varepsilon_k \), where \( \varepsilon_k, k \geq 1 \) are independent unit exponential r.v.'s being independent of \( Z^{(i)} \)'s which are independent copies of \( Z(t), t \in \mathbb{R}^d \) with almost surely sample paths in \( D \) satisfying (4.10). In view of [38] \( X \) is max-stable. Commonly, \( Z \) is referred to as a spectral process of \( X \). Let \( \nu_Z \) be the tail measure corresponding to \( Z \), which is compactly-bounded by (4.10). The law of \( X \) is uniquely determined by \( \nu_Z \) or the local tail processes of \( \nu_Z \), see [25]. Moreover, as shown in [39, 40] \( X \) is stationary if and only if (see also [3][Thm 2.3] for the case \( l = 1 \))

\[
\mathbb{E}\{\|Z(h)\|^n F(Z)\} = \mathbb{E}\{\|Z(0)\|^n F(B^h Z)\}, \quad \forall F \in \mathcal{S}_0, \forall h \in T
\]

holds. It follows that (4.13) is also equivalent with \( \nu_Z \) is shift-invariant, see also [3][Thm 2.3] discussing \( l = 1 \).

**Corollary 4.13.** If \( X \) is given by (4.12) with \( Z \) satisfying (4.10), then \( X \in \mathcal{R}_\alpha(B_0, \nu_Z) \).

**Example 4.14 (Brown-Resnick max-stable processes).** Let

\[
Z(t) = (e^{W_1(t)}, \ldots, e^{W_d(t)}), \quad W_i(t) = V_i(t) - \alpha Var(V_i(t))/2, \quad 1 \leq i \leq d, t \in T = \mathbb{R}^d,
\]

with \( \alpha > 0 \), \( (V_1(t), \ldots, V_d(t)), t \in T \) a centered \( \mathbb{R}^d \)-valued Gaussian process with almost surely continuous sample paths such that \( V_i(0) = 0, i \leq d \) almost surely. In the light of [41][Cor. 6.1], Eq. (4.10) holds, and thus by Remark 3.14, \( \nu_Z \) is \( B_0 \)-boundedly finite on \( D = C(\mathbb{R}^l, \mathbb{R}^d) \), the space of continuous functions \( f : \mathbb{R}^l \rightarrow \mathbb{R}^d \) equipped with a metric that turns it into a Polish space. Consider the max-stable process \( X \) with spectral process \( Z \). Corollary 4.13 implies \( X \in \mathcal{R}_\alpha(B_0, \nu_Z) \).

We focus next on \( D = D(\mathbb{R}^l, \mathbb{R}^d) \) and utilise Theorem 4.7 since \( B_0 \) is determined by the family of maps \( \mathcal{H}(D) \). See Remark 4.16 and Section A for the definition of \( w, w' \) and \( w'' \) that appear below.

**Theorem 4.15.** Let \( X \) be defined on a complete non-atomic probability space \((\Omega, \mathcal{F}, \mathbb{P})\) and let \( t_0 = 0 \). The following statements are equivalent:

(i) \( X \in \mathcal{R}_\alpha(B_0, \nu) \) with \( \|X\|_{t_0} \in \mathcal{R}_\alpha \);

(ii) Eq. (4.5) holds for all \( t_1, \ldots, t_k \in T_0, k \geq 1 \) for some \( T_0 \) such that \( T \setminus T_0 \) is countable and

\[
\lim_{\eta \downarrow 0} \sup_{z \to \infty} \frac{\mathbb{P}\{|w'(X, K, \eta) > \varepsilon z\}}{\mathbb{P}\{|X|_{t_0} > z\}} = 0, \quad \forall \varepsilon > 0, \forall K \in K(\mathbb{Q}).
\]

(4.14)

(iii) Eq. (4.2) holds and for all \( h \in T_0 \) for some \( T_0 \) such that \( T \setminus T_0 \) is countable

\[
\lim_{\eta \downarrow 0} \sup_{z \to \infty} \frac{\mathbb{P}\{|w'(X, K, \eta) > \varepsilon z, |X|_h \leq z\}}{\mathbb{P}\{|X|_{t_0} > z\}} = 0, \quad \forall \varepsilon > 0, \forall K \in K(\mathbb{Q}).
\]

(4.15)

Further if \( p_h > 0 \), then conditionally on \( |X|_h > z, z^{-1}X \) converges weakly on \( (D, d_D) \) to \( Y[h] \) as \( z \to \infty \), where \( Y[h] \)’s are \( D \)-valued random processes defined on \((\Omega, \mathcal{F}, \mathbb{P})\) being further the local tail processes of a tail measure \( \nu \) on \( D \) with index \( \alpha > 0 \);

(iv) Let \( s_k < t_k, k \in \mathbb{N} \) be given constants satisfying \( -\lim_{k \to \infty} s_k = \lim_{k \to \infty} t_k = \infty \) and set \( K_k = [s_k, t_k] \). There exists \( B_0(D_{K_k}) \)-boundedly finite Borel measures \( \nu_k \) with \( \nu_k^{(n)} \) its corresponding tail measure on \( \mathcal{P}(D_{K_k}) \) with index \( \alpha > 0 \). Suppose that \( \nu_k(\{f \in D : f(t) \neq f(t-\cdot)\}) = 0 \) for \( t \in [s_k, t_k] \) and \( X_{K_k} \in \mathcal{R}_\alpha(B_0(D_{K_k}), \nu_k) \) for all \( k \in \mathbb{N} \) with \( \|X\|_{t_0} \in \mathcal{R}_\alpha \).

**Remark 4.16.** (i) For \( l > 1 \) and \( D = D(\mathbb{R}^l, \mathbb{R}^d) \), if (i) holds, then the weak convergence in (7.10) below and Theorem A.1, (ii) imply Theorem 4.15, (ii) where (4.14) is substituted by

\[
\lim_{\eta \downarrow 0} \sup_{z \to \infty} \frac{\mathbb{P}\{|w'(X, K, \eta) > \varepsilon z, \sup_{t \in [-k,k]} |X|_t > z/k\}}{\mathbb{P}\{|X|_{t_0} > z\}} = 0, \quad \forall K \in K(\mathbb{Q}),
\]

(4.16)
\[
\lim_{m \to \infty} \limsup_{z \to \infty} \frac{\mathbb{P}\{\sup_{t \in [-k,k]}|X_t| > mz\}}{\mathbb{P}\{|X_{t_0}| > z\}} = 0, \tag{4.17}
\]

with \(k \in \mathbb{N}, \varepsilon > 0\) arbitrary. Conversely, Theorem 4.15(ii) with the above modification implies Theorem 4.15(i) and similarly Theorem 4.15(iii) therein can be modified to yield the equivalence with Theorem 4.15(i).

(ii) If instead of \(D(\mathbb{R}^1, \mathbb{R}^d)\) we consider \(C(\mathbb{R}^1, \mathbb{R}^d)\), then Theorem 4.15 holds with \(w\) instead of \(w'\). This follows since in this case we can substitute \(w'\) by \(w\) in (4.16).

(iii) By [43][Eq. (12.28)] and (4.14) when \(l = 1\)
\[
\lim_{n \to 0} \limsup_{z \to \infty} \frac{\mathbb{P}\{w'(X,K,\eta) > \varepsilon z\}}{\mathbb{P}\{|X_{t_0}| > z\}} = 0, \quad \forall \varepsilon > 0, \forall K \in K(Q). \tag{4.18}
\]

It follows using [43][Eq. (12.32)] and [8][Thm 10] that (4.14) can be substituted by (4.18).

Example 4.17 (Random scaling). Under A3) let \(\nu_2\) be a compactly-bounded tail measure on \(D\). Let \(R\) be an \(\alpha\)-Pareto rv independent of \(Z\) and set \(X(t) = RZ(t), t \in T\). Utilising [10][Lem 2.3 (2)], since \(\mathbb{E}\{\sup_{t \in K} |Z|^\alpha\} < \infty\) for all compact \(K \in \mathbb{R}^1\), it follows from Remark 4.12(ii) and Remark 4.16(i) that \(X \in \mathcal{R}_0(\mathcal{B}_0, \nu)\). We note that this example is discussed in [10] for compact \(T\).

Example 4.18 (Scaled & shifted processes). Suppose that \(X \in \mathcal{R}_0(\mathcal{B}_0, \nu)\) is a \(D\)-valued random element, \(Y^{[h]}, h \in T\) are the local processes of a \(K(Q)\)-bounded Borel tail measure \(\nu\) on \(D\). Let \(X^{[\bar{\sigma}, \sigma]}(t) = \sigma(t)X(t) + f(t), t \in T\), with \(f, \sigma \in D\) such that \(\sigma \in D\) is continuous and \(\sigma(t) \neq 0\) for all \(t \in T\). Note that if \(T = \mathbb{R}^1\), then \(\lim_{m \to 0} w'(f,K,\delta) = 0, \forall K \in K(Q)\), see Theorem A1(i). Using Remark 4.16(i) we have \(X \in \mathcal{R}_0(\mathcal{B}_0, \nu_0)\), where the tail measure
\[
\nu_0(A) = \nu\{f \in D : \sigma f \in A\}, \quad A \in \mathcal{D} \tag{4.19}
\]
has local tails given by \(Y^{[h]}(t) = Y^{[h]}[\sigma(t)/\sigma(h)]\) for all \(h, t \in T\).

5 Discussions

We shall consider first another common definition of RV, in terms of sequences, see [5] and also [44] for a recent full account. The second part of our discussions is dedicated to RV under transformations and then we conclude with a short section on stationary càdlàg processes.

5.1 An alternative definition of RV

Suppose that (A2) holds and let in the following \(a_n > 0, n \in \mathbb{N}\) be a non-decreasing sequence of constants such that
\[
\lim_{n \to \infty} a_n = t^\alpha, \quad \forall t > 0,
\]
where \([x]\) denotes the integer part of \(x\). For such constants we write \(a_n \in \mathcal{R}_\alpha\). Another common and less restrictive definition of RV (see e.g., [1][Thm B.2.1]) is the following:

Definition 5.1. \(\mu \in \mathcal{M}^+(B)\) is regularly varying if for \(a_n \in \mathcal{R}_{1/\alpha}\)
\[
\mu_n(A) = n\mu(a_n \cdot A), \quad A \in \mathcal{B}(D)
\]
converges \(\mathcal{B}\)-vaguely to some \(\nu \in \mathcal{M}^+(B)\) as \(n \to \infty\), abbreviate this as \(\mu \in \mathcal{R}_\alpha(a_n, B, \nu)\).

If \(\mu \in \mathcal{R}_\alpha(g, B, \nu)\) and \(g \in \mathcal{R}_\alpha\), Lemma 4.2 yields \(\mu \in \mathcal{R}_\alpha(g_* B, \nu)\) for any Lebesgue measurable \(g_* : \mathbb{R}_+ \to \mathbb{R}_+\) such that \(\lim_{z \to \infty} g(z)/g_*(z) = 1\). Since \(g \in \mathcal{R}_\alpha\), we can choose \(g_* \in \mathcal{R}_\alpha\) asymptotically non-decreasing. Taking then \(a_n = g_*^{-1}(n), n \geq 1\) with \(g^{-1}\) an asymptotic inverse of \(g\), it follows that
\[
\mu \in \mathcal{R}_\alpha(g, B, \nu) \implies \mu \in \mathcal{R}_\alpha(a_n, B, \nu).
\]
The inverse implication above (and thus the equivalence of both definitions of RV) can be proven under [1][M1)-(M3),(B1)-(B3), p. 521/522, see [1][Thm B.2.2].
The Definition 5.1 can be naturally extended to D-valued random processes \( X \), which is abbreviate as

\[
X \in \mathcal{R}_\alpha(a_n, B, \nu).
\]

Both definitions of RV for càdlàg processes are equivalent as we show next.

**Lemma 5.2.** If \( \nu, X \) are as in Theorem 4.11, then \( X \in \mathcal{R}_\alpha(B_0, \nu) \) is equivalent to \( X \in \mathcal{R}_\alpha(a_n, B, \nu) \), where \( a_n \) is such that \( n \mathbb{P}\{\|X\|_{t_0} > a_n\} = p_{t_0} \) for all large \( n \in \mathbb{N} \).

**Example 5.3.** We consider the setup of \([1][Prop 2.1.13] assuming A3\). Let \( X \in \mathcal{R}_\alpha(a_n, B, \nu) \) with \( \|X\|_{t_0} \in \mathcal{R}_\alpha \) for some \( t_0 \in T \) and let \( \Gamma : T \to \mathbb{R}^K \) be a random map independent of \( X \) defined on \( (\Omega, \mathcal{F}, \mathbb{P}) \) and let \( \| \cdot \| \) be some norm on \( \mathbb{R}^d \). Suppose that almost surely \( \Gamma(u) = \epsilon \Gamma(u), \forall \epsilon > 0, u \in \mathbb{R}^d \) for some \( \gamma > 0 \). Assume that \( \Gamma \) is almost surely continuous satisfying (4.16). If further (4.17) holds with \( X \) substituted by \( \Gamma \circ X \) and for some \( \epsilon > 0 \)

\[
\mathbb{E}\left\{ \sup_{\|u\| \leq 1} |\Gamma(u)|^{\alpha/\gamma + \epsilon} \right\} < \infty, \tag{5.1}
\]

where \( \alpha > 0 \) is the index of \( \nu_0 \), then \( \Gamma(X) \in \mathcal{R}_\alpha(a_n, B_0, \mathbb{E}\{\nu \circ \Gamma^{-1}\}) \), provided that \( \mathbb{E}\{\nu \circ \Gamma^{-1}\} \) is non-trivial. A particular instance of interest is \( \Gamma(t) = AX(t), t \in T \) with \( A \) a \( k \times l \) real matrix satisfying \([1][Eq. (2.1.14)]\).

### 5.2 Transformations

We shall focus in this section on \( D = D(\mathbb{R}, \mathbb{R}^d) \). The next lemma is a restatement of \([21][Lem 3.2] \) for our setup.

**Lemma 5.4.** If \( X \in \mathcal{R}_\alpha(a_n, B_0, \nu) \) and \( \sigma \) is a \( D \)-valued random process independent of \( X \), then \( \sigma X \in \mathcal{R}_\alpha(a_n, B_0, \mathbb{E}\{\nu_{\sigma}\}) \), with \( \nu_{\sigma} \) defined in (4.19), provided that

\[
\mathbb{E}\left\{ (\sup_{t \in K} |\sigma(t)|)^{\alpha + \epsilon} \right\} < \infty, \quad \forall K \in \mathcal{K}(Q) \tag{5.2}
\]

for some \( T_0 \) such that \( T \setminus T_0 \) is countable and

\[
\mathbb{P}\{\sigma(t) \neq 0\} > 0, \quad \forall t \in T_0. \tag{5.3}
\]

In view of Theorem 4.15.(iv) Lemma 5.4 can be extended considering \( X_i, i \leq m \) independent copies of \( X \) and \( \sigma_i, i \leq m \) \( D \)-valued random processes. Then \([21][Lem 3.3] \) can be restated by imposing (5.2) and (5.3) on all \( \sigma_i \)’s.

**Lemma 5.5.** If \( X \in \mathcal{R}_\alpha(a_n, B_0, \nu) \), then \([21][Thm 3.3] \) holds also if in the assumptions therein \( |\sigma_j|_{\infty} \) is substituted by \( \sup_{t \in K} |\sigma_j(t)| \), for all compacts \( K \in \mathcal{K}(Q) \).

### 5.3 Stationary càdlàg processes

Under the settings of Section 4.3 assume further that \( X \) is stationary. Hence \( \|X\|_{t_0} \in \mathcal{R}_\alpha \) for some \( t_0 \in T \) implies \( \|X\|_t \in \mathcal{R}_\alpha \) at all \( t \in T \) and thus

\[
p_t = p_0 \in (0, \infty), \quad \forall h \in T.
\]

It follows easily using (4.6) or directly by Theorem 4.15 and \([23][Thm 3.2] \) that if \( X \in \mathcal{R}_\alpha(B_0, \nu) \) holds, then the local tail processes \( Y^{[h]}, h \in T \) are given by (3.23) which implies that the corresponding tail measure \( \nu \) is shift-invariant. Moreover, the converse holds i.e., if \( \nu \) is shift-invariant, then (3.23) holds, see also \([25] \). With this additional knowledge on \( \nu \), the counterpart of Theorem 4.15 for stationary \( X \) can be easily reformulated.

**Example 5.6 (Stationary Brown-Resnick max-stable processes).** Let \( Z, X, W \) be as in Example 4.14, \( d = 1 \) and suppose that \( X \) is stationary, which follows if \( \text{Var}(W_1(t) - W_1(s)), s, t \in T \) depends only on \( t - s \) for all \( s, t \in T \). We have that \( X \in \mathcal{R}_\alpha(B_0, \nu_Z) \) with \( \nu_Z \) having representer \( Z \) and being shift-invariant. Note that since \( |Z(0)| = 1 \) almost surely, then \( \Theta = Z \) is the local spectral process of \( \nu_Z \) at 0 and thus

\[
Y^{[h]}(t) = e^{\alpha \eta + W_1(t)}, \quad t \in T,
\]

with \( \eta \) a unit exponential rv independent of \( W_1 \). Hence (3.18) reads for \( \alpha = 1, x > 0 \)

\[
\mathbb{E}\{\Gamma(xe^{\eta + B W_1})1(W_1(-h) + \eta < -ln x)\} = x \mathbb{E}\{\Gamma(e^{\eta + W_1})1(W_1(h) + \eta > ln x)\}, \quad \forall h \in T, \forall \Gamma \in \mathcal{F}. \tag{5.4}
\]
6 Applications & open questions

We first mention four applications considering $X \in \mathcal{R}(g, \mathcal{B}, \nu)$ as in Section 4.2 assuming A2).

AP1) A well-known application of RV is the derivation of the tail behaviour of $H(X)$, for a given functional of interest $H$. When $X$ has càdlàg sample paths, a canonical choice is $H(f) = H_X(K) = \sup_{t \in K} |f|$, $f \in \mathcal{D}$, with $K$ a compact set in $\mathbb{R}^l$, or $H(f) := H_A^\ast(f) = \int_A f(t) \lambda(dt)$, $f \in \mathcal{D}$, with $A$ a bounded Borel set in $\mathbb{R}^l$ of positive Lebesgue measure. Conditions on $H$ for tractable tail behaviour of $H_X$ are presented in Remark 6.2 below;

AP2) As already discussed in several contributions, see e.g., [45][Prop 2.3], RV implies the convergence of the collection of probability measures $\mu_z(A) = \mathbb{P}\{z^{-1} \cdot X \in A \mid z^{-1} \cdot \mathcal{B} \}$ for all $A \in \mathcal{A}$, as $z \to \infty$. Assuming, additionally, that the Borel set $\mathcal{B}$ belongs to $\mathcal{B}$ and is $\nu$-continuous (i.e., $\nu(B) > 0$) with $\nu(B) > 0$, we obtain

$$\mu_z \xrightarrow{\nu, B} \mu, \ z \to \infty, \quad (6.1)$$

where $\mu_z = \nu(\cdot \cap B)/\nu(B)$. This application is useful for the formulation of conditional limit results, as already shown in the aforementioned contribution;

AP3) An interesting application considered for the discrete setup is developed recently in [31] for the product of RV random matrices. The results therein can be extended to the product of random matrix functions, making use of Theorem 4.11, Theorem 4.15, and ideas given in the aforementioned contribution. Moreover, extensions to more general homogeneous functionals can be also obtained using further Remark 6.2;

AP4) One advantage of introducing the RV with respect to some boundedness is that it includes also the concept of hidden RV discussed for instance in [7, 17, 45]. Indeed, in our settings hidden RV corresponds to the choice of $\mathcal{B}_F$ with $F \subset \mathcal{D}$ closed being the boundedness on $\mathcal{D}_F = \mathcal{D} \setminus \{F\}$ defined in Section 2.2. Since by definition we need $\mathcal{D}_F = \mathcal{D} \setminus F$ to be a measurable cone, we shall further assume that $F$ is a cone, i.e.,

$$z \cdot F \subset F, \ \forall z \in \mathbb{R}_>. \quad \text{(6.2)}$$

Define similarly $\mathcal{B}_F'$ with respect to $\mathcal{D}_F'$, with $F' \subset \mathcal{D}'$ a closed cone. The next result is useful when considering maps of hidden regularly varying processes.

**Lemma 6.1.** Let $H : \mathcal{D} \mapsto \mathcal{D}'$ be $\mathcal{B}(\mathcal{D})/\mathcal{B}(\mathcal{D}')$ measurable with $H(F) = F'$. Let further $\nu_z, z > 0$ be $\mathcal{B}_F$-boundedly finite measures on $\mathcal{B}(\mathcal{D}_F)$ and let $\nu$ be a $\mathcal{B}_F'$, boundedly-finite measure on $\mathcal{B}(\mathcal{D}_F')$. If one of the following conditions

(i) $H$ is uniformly continuous;

(ii) $\mathcal{D}$ or $\mathcal{D}'$ are compact and $H$ is continuous;

(iii) $\nu(\text{Disc}(H)) = 0$, $H$ is a continuous and one-to-one if restricted on $F$, which has finite number of elements

is satisfied and $\nu_z \xrightarrow{\nu, \mathcal{B}_F} \nu$, then $\nu_z \circ H^{-1} \xrightarrow{\nu, \mathcal{B}_F'} \nu \circ H^{-1}$.

**Remark 6.2.** Under the conditions of Lemma 6.1, if $H(c \cdot f) = c \cdot H(f)$, $\forall c > 0$, $f \in \mathcal{D}$ and $\nu \circ H^{-1}$ is non-trivial, then $\mathcal{B}(\mathcal{D}_F')$ implies $\mu \circ H^{-1} \in \mathcal{B}(\mathcal{D}_F')$ and thus under the settings of [25] we retrieve the claim of Lemma 3.2 therein.

The applications and findings of [3] for $T = \mathbb{R}^l, l = 1$ can be extended to our general case of stationary $X$ for all integer $l > 1$, by alluding to the methodology developed therein, together with Theorem 4.11. We do not presently repeat all calculations, but rather mention a few details and some new results on the tail behaviour of supremum of càdlàg processes. The rest of this section considers random processes $X(t), t \in T$ with $T = \mathbb{Z}^l$ or $T = \mathbb{R}^l$. In the latter case we assume that $X$ has càdlàg sample paths. Further, $\nu_Z$ is a tail measure with representer $Z$, which has almost surely càdlàg sample paths if $T = \mathbb{R}^l$. Suppose next that $\mathbb{E}(||Z(0)||^\alpha) = 1$, where $|| \cdot ||$ is a norm on $\mathbb{R}^d$.

6.1 Stationary case

We now consider the case when $X$ is stationary and $X \in \mathcal{R}_\alpha(a_n, \mathcal{B}_0, \nu_Z)$. Hence $\nu_Z$ is shift-invariant and therefore uniquely determined by $Y = Y[0]$. Note in passing that $\mathbb{E}(||Z(0)||^\alpha) = 1$ implies that $p_h = \mathbb{E}(||Z(h)||^\alpha) = 1$, for all $h \in T$. The determination of the tail behaviour of $H_K(X)$, with $H_K$ the supremum functional in AP1), is a classical interesting
problem of probability theory. As already demonstrated in [3], our results can be applied to consider both the case \( K \) does not depend on \( n \) and \( K = K_n = [0, n]^l \), when \( n \) tends to infinity. We first state a general upper bound for the growth of the supremum

\[
M_n = \sup_{t \in [0, n] \cap T} \|X(t)\|
\]

assuming for simplicity that \( \|X(0)\| \) is a unit \( \alpha \)-Fréchet rv with df \( e^{-x^\alpha}, x > 0 \).

**Proposition 6.3.** If \( X \in \mathcal{R}_\alpha(a_n, B_0, \nu_Z) \) where \( a_n = n^{1/\alpha} \) and \( \|X(0)\| \) is a unit \( \alpha \)-Fréchet rv, then for all \( x > 0 \)

\[
\limsup_{n \to \infty} \mathbb{P}\{M_n > a_n^\alpha x\} \leq \theta_Y x^{-\alpha}, \quad \theta_Y = \mathbb{E}\left\{\frac{1}{\int_\mathbb{R}_1 \mathbb{I}(\|Y(t)\| > 1) \lambda(dt)}\right\} \in (0, \infty),
\]

provided that for \( T = \mathbb{R}^l \) we have \( I_{k,n} = \mathbb{E}\left\{1/ \int_{s \in [-k,k]^l} \mathbb{I}(\|X(s)\| > a_n \|X(0)\| > a_n) \lambda(ds)\right\} \) is bounded for all \( n \) large and some \( k > 0 \).

Note that (6.2) is shown in [1][Lem 7.5.4] for \( T = \mathbb{Z}^l, l = 1 \) and \( I_{k,n} \leq 1 \) for all positive integers \( l, k, n \) if \( T = \mathbb{Z}^l \).

In the particular case that

\[
\varepsilon(Y) = \int_T \mathbb{I}(\|Y(t)\| > 1) \lambda(dt) = \infty
\]

almost surely, then \( \theta_Y = 0 \). Hence Proposition 6.3 implies the following convergence in probability

\[
a_n^{-1} M_n \overset{P}{\to} 0, \quad n \to \infty.
\]

(6.3)

In order to establish weak convergence of \( a_n^{-1} M_n \) to some Fréchet rv as \( n \to \infty \), we have to guarantee the positivity of \( \theta_Y \). In both the discrete setup (cf., [16, 24, 46]) and the continuous case with \( l = 1 \) dealt with in [3], it is known that \( \theta_Y > 0 \) follows from the anticlustering condition of [15], which we now present for both cases \( A = \mathbb{Z}^l \) and \( A = \mathbb{R}^l \).

We say that \( f \) is a scaling function, if \( f : (0, \infty) \to (0, \infty) \) is non-decreasing and unbounded, and set \( |x|_\infty = \max_{1 \leq i \leq l} |x_i|, x \in \mathbb{R}^l \).

**Condition 6.4 (C(A)).** There exist scaling functions \( a \) and \( r \) such that

\[
\lim_{t \to \infty} \limsup_{y \to \infty} \mathbb{P}\left\{\sup_{t \leq |s| \leq r(y), s \in A} \mathbb{I}(|X(s)| > a(y) x \mathbb{I}(|X(0)| > a(y)) = 0, \quad \forall x > 0.\right\}
\]

(6.4)

From the anticlustering condition we may derive important properties of \( Y \) and \( Z \), in particular that \( \theta_Y > 0 \).

**Lemma 6.5.** Under the assumptions of Proposition 6.3, if \( T = \mathbb{R}^l \) and Condition 6.1 (C(\mathbb{Z}^l)) holds, then

\[
\mathbb{P}\left\{\int_T \|Y(t)\|^\alpha \lambda(dt) \in (0, \infty)\right\} = 1.
\]

(6.5)

As in [3], we say that the shift-invariant tail measure \( \nu_Z \) is dissipative if (6.5) holds almost surely. Along the same lines of the aforementioned paper, it follows that \( \nu_Z \) is dissipative if and only if \( \varepsilon(Y) \) and \( \int_{\mathbb{R}^l} \|Z(t)\|^\alpha \lambda(dt) \) are almost surely positive and finite, implying in particular that \( \theta_Y > 0 \).

Moreover, if \( \nu_Z \) is dissipative, the PPP \( N \) defined in Remark (4.4).(ii), has the following representation

\[
N() = \sum_{i=1}^{\infty} \delta_{P_i, \tau_i, Q^{(i)}()}(\cdot),
\]

where \( \sum_{i=1}^{\infty} \delta_{P_i, \tau_i, Q^{(i)}()}(\cdot) \) is a PPP on \( (0, \infty) \times \mathbb{R}^l \times D \), with mean measure \( \theta_Y \lambda(\cdot) \otimes \nu_a(\cdot) \otimes \mathbb{P}_Q(\cdot) \) and \( Q^{(i)} \)'s are independent copies of \( Q \) with law

\[
\mathbb{P}_Q(\cdot) = \theta_Y^{-1} \mathbb{E}\{\delta_{Y/\sup_{t \in \mathbb{R}^l} \|Y(t)|/\varepsilon(Y)}\}\}
\]

The dissipative representation of \( N \) is key to the so-called \( m \)-dependent approximation (see [25] for the definition). Specifically, if \( \nu_Z \) is dissipative, the max-stable stationary process \( X \) defined in (4.12) (recall \( Z \) has non-negative components) has the dissipative representation

\[
X^{(m)}(t) = \max_{i \geq 1} P_i Q^{(i)}(t - \tau_i), t \in \mathbb{T}, \text{ which has an } m \text{-approximation given by}
\]

\[
X^{(m)}(t) = \max_{i \geq 1} P_i Q^{(i)}(t - \tau_i) \mathbb{I}(|t - \tau_i| \leq m), \quad t \in \mathbb{T}, m > 0.
\]
Similarly, an $m$-approximation can be derived for the $\alpha$-stable stationary $X$ with $\alpha \in (0, 2)$ defined by substituting max with $\sum$ in (4.12) and in the above dissipative representation. In order to avoid centering, when $\alpha \in [1, 2)$, as in [3], $Z$ is further assumed to be symmetric. In both cases, $(X, X^{(m)})$ is stationary. The next results extends [3][Thm 4.1, Cor 4.3, Thm 4.5, Cor 4.6] (note that $TP\{\cdot\}$ should be $P\{\cdot\}$ therein) to $l \geq 1$. Related results are derived also in [47-50].

**Theorem 6.6.** If $X$ as above is max-stable or $\alpha$-stable, then $X \in \widetilde{R}_\alpha(\alpha_n, B_0, \nu_Z)$ with $\alpha_n = n^{1/\alpha}$. Moreover, we have

$$a_n^{-1}M_n \xrightarrow{d} \eta_X^{1/\alpha}V, \quad n \to \infty,$$

with $V$ an $\alpha$-Fréchet rv and $\eta_X = \theta_Y < \infty$.

**Remark 6.7.** In the literature $\theta_Y$ is commonly referred to as the candidate extremal index, which under the assumptions of Theorem 6.6 is equal to the extremal index $\eta_X$ of $X$. As shown in [51], it is possible to have $\eta_X < \theta_Y$. Note in passing that when (6.6) holds and $X$ as in Proposition 6.3 is stationary and regularly varying with tail process $Y$, then (6.2) implies

$$\eta_X \leq \theta_Y,$$

which for $T = Z$ is shown in [1][Lem 7.5.4].

### 6.2 Non-stationary case

The non-stationary case is significantly less tractable, when compared to the stationary one. Yet, there are a few exceptions, for instance $X_{f, \sigma}$ defined in Example 4.18, with $X$ stationary and regularly varying. Under some growth restrictions on $f$ and $\sigma$, Proposition 6.3 and Theorem 6.6 can be extended for $X_{f, \sigma}$.

We shift our focus below to an interesting special case, namely $X(t) = RZ(t)$, $t \in T$, with $R$ a non-negative rv independent of $Z$, which is the representer of some shift-invariant tail measure $\nu_Z$ on $\mathcal{D}$.

Assume next that

$$\lim_{x \to \infty} x^\alpha \mathbb{P}\{R > x\} = 1,$$

for some $\alpha > 0$. Applying Corollary 4.13 we have that $X \in \widetilde{R}_\alpha(\alpha_n, B_0, \nu_Z)$ with $\alpha_n = n^{1/\alpha}$. Moreover, in view of AP1) or directly by [31][Lem 1.1] for all $n > 0$

$$\lim_{x \to \infty} x^{-\alpha} \mathbb{P}\left\{ \sup_{t \in [0, n]^{l \cap T}} \|X(t)\| > x \right\} = \mathbb{E}\left\{ \sup_{t \in [0, n]^{l \cap T}} \|Z(t)\|^\alpha \right\} \in (0, \infty).$$

We consider next what happens when $n$ tends to infinity.

**Lemma 6.8.** If $R$ possesses a probability density function $f$ such that $f(s) \leq cs^{-\alpha-1}$ for some $c > 0$ and all $s$ large, then

$$\limsup_{n \to \infty} \mathbb{P}\left\{ \sup_{t \in [0, n]^{l \cap T}} \|X(t)\| > a_n^l x \right\} \leq C\theta_Y x^{-\alpha}$$

is valid for all $x > 0$ and some fix $C > 0$.

**Example 6.9** (Stationary $Z$). Consider $R$ as in Lemma 6.8 assuming further that $Z$ is stationary. Clearly, $\nu_Z$ is shift-invariant and moreover by [32][Cor 1, Rem 1,ii)] we have that $\theta_Y = 0$. Consequently, (6.8) implies (6.3).

### 6.3 Open questions

RV in the discrete setup $T = Z^l$ has played an important role in the derivation of large deviation type results, as considered in e.g., [53, 54]. Also for the discrete setup, [1, 55, 56] have shown that RV and shift-invariant tail measures are crucial for the estimation of various functionals of time series.

The applications of RV to stationary processes are abundant, while the non-stationary case is rather intractable. Even for the simple case $X(t) = RZ(t)$ as in the previous section the asymptotic approximation of $a_n^{-l}M_n$ could not be derived for general $\nu_Z$. In the recent contribution [57], periodic sequences have been considered for the discrete setup.

With motivation from the aforementioned results and developments we formulate below four open questions.
In the proof below we use several times the Fubini-Tonelli theorem which is applicable to the Fubini-Tonelli theorem w hich is applicable

$$\gamma_n \mathbb{P} \{ X_n \in \cdot \} \overset{\text{as } \mu}{\rightarrow} \mu(\cdot), \ n \to \infty,$$

with \( \mu \) a non-trivial Borel measure. Since \( \gamma_n \) is general, \( \mu \) does not need to be a tail measure. It is of interest to consider non-compact \( T \), for instance \( T = \mathbb{R}^d \) and the special case where \( \mu \) is obtained as a transform of the product of two tail measures as in the results derived in [13]. It remains to be investigated if such extensions yield significant applications;

**OP2** The applications discussed in [1, 54–56, 58] for \( T = \mathbb{Z}^d \) can be considered also in the non-discrete setup \( T = \mathbb{R}^d \) using additionally our findings related to RV and tail measures. Still several technical conditions in the aforementioned papers need to be translated to the non-discrete settings, which is not an easy task;

**OP3** Does RV of periodic processes on \( T = \mathbb{R}^d \) offer some technical advantages in the analysis of related questions posed in [57]? In particular it is of some interest to relate (and estimate) the extremal index of periodic processes in terms of the corresponding \( Y^{[b]}; \)

**OP4** Several findings and applications in [3] are derived based on Condition 6.4 (C(\( \mathbb{R}^d \))). As shown in Lemma 6.5 the weaker Condition 6.4 (C(\( \mathbb{Z}^d \))) can instead be imposed even when \( T = \mathbb{R}^d \). It is of interest to investigate if the weaker condition Condition 6.4 (C(\( \mathbb{Z}^d \))) can be imposed in the applications discussed in [3] and also to characterise stationary processes \( X \) for which both conditions are equivalent.

7 Proofs

**Proof of Proposition 3.6** In the proof below we use several times the Fubini-Tonelli theorem which is applicable since \( \nu \) is \( \sigma \)-finite. Since \( \| \cdot \|_t, t \in \mathcal{Q} \) is measurable, 1-homogeneous and the outer multiplication \((z, f) \mapsto z \cdot f \) is jointly measurable, then for all maps \( \Gamma \in \mathcal{F} \) and all \( h, t \in \mathcal{Q} \) such that \( p_h p_t > 0 \) and all \( x > 0 \), by the definition of \( Y^{[b]}, Y^{[t]} \) and \(-\alpha\)-homogeneity of \( \nu \)

\[
\begin{align*}
p_h \mathbb{E} \left\{ \Gamma(x \cdot Y^{[b]}) \mathbb{I} \left( \| Y^{[b]} \|_t > x \right) \right\} &= \int_D \Gamma(x \cdot f) \mathbb{I}(\| f \|_t > 1, \| f \|_h > 1) \nu(df) \\
&= x^\alpha \int_D \Gamma(f) \mathbb{I}(\| f \|_t > 1, \| f \|_h > x) \nu(df) \\
&= x^\alpha \int_D \mathbb{I}(\| f \|_h > x, \| f \|_t > 1) \nu(df) \\
&= x^\alpha p_t \mathbb{E} \left\{ \Gamma(Y^{[t]}) \mathbb{I} \left( \| Y^{[t]} \|_h > x \right) \right\}.
\end{align*}
\]

If \( p_h = 0 \) and \( p_t > 0 \), as above taking \( \Gamma \) bounded by some constant \( C > 0 \) we obtain

\[
\begin{align*}
x^\alpha p_t \mathbb{E} \left\{ \Gamma(Y^{[t]}) \mathbb{I} \left( \| Y^{[t]} \|_h > x \right) \right\} &= \int_D \Gamma(x \cdot f) \mathbb{I}(\| f \|_t > 1, \| f \|_h \geq 1) \nu(df) \\
&\leq C \int_D \mathbb{I}(\| f \|_h \geq 1) \nu(df) \\
&= C \int_D \mathbb{I}(\| f \|_h > 1) \nu(df) \\
&= C p_h \\
&= 0
\end{align*}
\]

for all \( x \in (0, \infty) \), where the third last equality follows from (3.2). Since \( \| \cdot \|_h \) is non-negative we have thus \( \mathbb{P}\{\| Y^{[t]} \|_h = 0\} = 1 \) and further (3.9) holds.

A direct implication of (3.9) is that \( R = \| Y^{[b]} \|_h \) is an \( \alpha \)-Pareto rv. In particular, for all \( x \in (1, \infty), h \in \mathcal{Q} \) using (3.9) and that \( \mathbb{P}\{\| Y^{[b]} \|_h > 1 / x\} = 1 \) we obtain

\[
\mathbb{P}\{\| Y^{[b]} \|_h^{-1} \cdot Y^{[b]} \in A, \| Y^{[b]} \|_h > x\}
\]

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implying that \( R \) is independent of \( \Theta^{[h]} = |Y^{[h]}|^{-1}Y^{[h]} \). Hence \( P\{|\Theta^{[h]}| = 0\} = 1 \) for \( h, t \) such that \( p_h = 0, p_t > 0 \) follows from \( P\{|Y^{[t]}| = 0\} = 1 \) shown above and the 1-homoogeneity of \( |.| \). By the definition for all \( h \in \mathbb{Q} \) such that \( p_h > 0 \) we have that \( P\{|\Theta^{[h]}| = 1\} = 1 \) and thus (3.7) follows. Further for all \( h, t \in \mathbb{Q} \) such that \( p_h p_t > 0 \) and all \( \Gamma \in \mathcal{S}_h \) by (3.9) (recall \( v_\alpha(dr) = \alpha^{-\alpha-1} dr \))

\[
\begin{align*}
\nu &\{\{ f \in A : |f_t| > 1 \} = \nu(A \cap A_{1h}) , \ A \in \mathcal{A} , h \in \mathbb{Q} : p_h > 0 \}.
\end{align*}
\]
Since $\nu$ and $\nu_Z$ have local spectral tail process $\Theta^{[h]}$ and $\Theta^{[h]}_Z$, $h \in \mathcal{Q}$, respectively, by Proposition 3.6 $\nu = \nu_Z$. The latter and (M1) imply
\[
0 = \nu \left( \left\{ f \in D : \sup_{t \in \mathcal{Q}} |f|_t = 0 \right\} \right) = \int_0^\infty \mathbb{P} \left( r \sup_{t \in \mathcal{Q}} |Z|_t = 0 \right) v_\alpha(dr)
\]
and thus $\mathbb{P} \left( \sup_{t \in \mathcal{Q}} |Z|_t = 0 \right) = 0$. Hence using further (3.15) we establish (3.5). Since $\nu = \nu_Z$ yields
\[
\nu \left( \left\{ f \in D : \sup_{t \in K} \|f\|_t > 1 \right\} \right) = \int_0^\infty \mathbb{P} \left( r \sup_{t \in K} \|Z\|_t > 1 \right) v_\alpha(dr) = \mathbb{E} \left( \sup_{t \in K} |Z|_t^\alpha \right)
\]
for all $K \subset \mathcal{Q}$, then (3.14) is equivalent to (M3) establishing the proof.

\[\] Proof of Lemma 3.13 Let the $\mathcal{G}/\mathcal{B}(\mathbb{R})$ measurable map $H : D \to \mathbb{R}$ be such that for some $\varepsilon_H > 0$ and all $f \in D$ we have $H(f) = 0$ if $f_{K_0} \leq \varepsilon_H$. Hence for all sets $K$ such that $K_0 \subset K \subset T$, since $q_i$'s are positive and the maps $\| \cdot \|_t : D \to [0, \infty]$, $\forall t \in \mathcal{Q}$ are 1-homogeneous
\[
H(f) = H(f)1(\|f\|_K > \varepsilon) = H(f)1(\mathcal{E}_K^q(\varepsilon^{-1}f) > 0), \quad \forall \varepsilon \in (0, \varepsilon_H), \forall f \in D.
\]
If $\lambda = \lambda_\mathcal{Q}$ is the counting measure on $\mathcal{Q}$, then $\mathcal{E}_K^q(\cdot) \in \mathcal{F}_\mathcal{Q}$. Since further by assumption $I = \int_K p_i q_i \lambda(dt) < \infty$, by (M0) and the $\sigma$-finiteness of $\nu$, applying the Fubini-Tonelli theorem we obtain
\[
\int_D \mathcal{E}_K^q(\varepsilon^{-1}f) \nu(df) = \varepsilon^{-\alpha} \int_K \int_D 1(\|f\|_t > 1) \nu(df) p_i q_i \lambda(dt)
\]
and hence $\nu(\{ f \in D : \mathcal{E}_K^q(\varepsilon^{-1}f) = \infty \}) = 0$. By (3.11),(7.2) and the Fubini-Tonelli theorem
\[
\nu[H] = \int_D H(f) \nu(df) = \int_D H(f)1(\mathcal{E}_K^q(\varepsilon^{-1}f) > 0) \nu(df)
\]
\[
= \int_K \int_D H(f)1(\mathcal{E}_K^q(\varepsilon^{-1}f) > 0) \frac{\mathcal{E}_K^q(\varepsilon^{-1}f)}{\mathcal{E}_K^q(f)}(f) \nu(df)
\]
\[
= \varepsilon^{-\alpha} \int_K \int_D H(f)1(\|f\|_t > 1) \frac{1(\|f\|_t > \varepsilon)}{\mathcal{E}_K^q(f)}(f) \nu(df) p_i q_i \lambda(dt)
\]
\[
= \varepsilon^{-\alpha} \int_K \mathbb{E} \left\{ \frac{H(\varepsilon^{-1}Y)}{\mathcal{E}_K^q(\varepsilon^{-1}Y)} \right\} p_i q_i \lambda(dt)
\]
establishing the claim. \[\]

Proof of Lemma 3.16 (i): Let $\nu$ be defined through a stochastic representer $Z = Z_N$ as in Lemma 3.10. Since $\nu$ satisfies (M0)-(M1), then by Lemma 3.3 $\nu$ is unique, hence the claim follows.

(ii): The claim follows by showing that (7.1) holds for all compact $K \subset T$, which is implied by Remark 3.14. When (A3) holds we can use additionally Remark 3.14(iii).

(iii): Let $\Theta^{[h]}$, $h \in \mathcal{Q}$ be a family of spectral tail processes satisfying (3.7) and (3.8) and set $Y^{[h]} = R \cdot \Theta^{[h]}$, $h \in \mathcal{Q}$. For all $h,t \in \mathcal{Q}$ such that $p_h p_t > 0$ we have by (3.8) that $\| \Theta^{[h]} \|_t$ is positive with non-zero probability and almost surely finite. Given $x > 0$ and $\Gamma \in \mathcal{F}_\mathcal{Q}$ by the 1-homogeneity of $\| \cdot \|_t$, $t \in \mathcal{Q}$ and the independence of the $\alpha$-Pareto r.v $R$ with $\Theta^{[h]}$'s (set $B_h = \| \Theta^{[h]} \|_h$ and recall that $P\{B_t = 1\} = 1$ and $v_\alpha(dr) = \alpha^{r-1}dr$)
\[
x^\alpha p_t \mathbb{E} \left\{ \mathbb{E} \left[ \Gamma(\varepsilon^{[h]})(\|Y^{[h]}\|_h > x) \right] \right\}
\]
\[
= x^\alpha p_t \int_0^\infty \mathbb{E}(\Gamma(r \cdot \Theta^{[h]}))(\|r B_h > x, r > 1, 0 < B_h < \infty) v_\alpha(dr)
\]
\[
= p_t \int_0^\infty \mathbb{E} \left\{ B_h \mathbb{E} \left[ \Gamma(\varepsilon^{[h]})(r \cdot \Theta^{[h]})(\|r B_h > x, r > 1, 0 < B_h < \infty) \right] v_\alpha(dr)
\]
\[
= p_t \int_0^\infty \mathbb{E} \left\{ \Gamma((r x) \cdot \Theta^{[h]})(\|r > 1, r \|Y^{[h]}\|_h > 1, 0 < \|Y^{[h]}\|_h < \infty) \right\} v_\alpha(dr)
\]
\[
= p_h \mathbb{E} \left\{ \Gamma(x \cdot Y^{[h]}) I \left( \left| Y^{[h]} \right|_t > 1/x \right) \right\},
\]

where we used (3.8) in the second last line above and \( \mathbb{P}\{B_h \in [0, \infty)\} \) which follows by Remark 3.5, hence the proof is complete.

**Proof of Lemma 4.2** The claim follows with the same arguments as given in the proof of [1][Thm B.2.2 (b)].

**Proof of Lemma 4.5** For all \( z \in \mathbb{R}_+ \), if \( \Gamma : D \to \mathbb{R}_+ \) is a bounded continuous map and \( \text{supp}(H) \in \mathcal{B} \), then by assumption \( B3) \) and the continuity of the pairing \( (z, f) \mapsto z \cdot f \), also \( \Gamma(z) = \Gamma(z \cdot f), f \in D \) is a bounded continuous map supported on \( \mathcal{B} \) for all \( z \in \mathbb{R}_+ \). Consequently, the assumption \( g \in \mathcal{R}_\alpha \) implies

\[
\mu[\Gamma] = \lim_{x \to \infty} \mu_{xz}[\Gamma(xz \cdot \cdot)] = \lim_{x \to \infty} \frac{g(xz)}{g(x)} g(x) \mu[\Gamma(xz \cdot \cdot)] = z^\alpha \mu[\Gamma_z] = \nu_s[\Gamma], \quad \forall z \in \mathbb{R}_+.
\]

Since \( z \) can be chosen arbitrary

\[
\mu[\Gamma_s] = \nu_{z/s}[\Gamma_s] = s^{-\alpha} z^\alpha \nu[\Gamma_z] = s^{-\alpha} \nu[\Gamma], \quad \forall s \in \mathbb{R}_+,
\]

hence the claim (i) follows from Remark 2.7.

By assumption \( H_{\text{top}}(f) = \|f\|_{\text{top}} \) is a \( \mathcal{B}(D)/\mathcal{B}(\mathbb{R}) \) measurable function. The assumption that \( H_{\text{top}}^{-1}(B) \in \mathcal{B} \) for all \( B \in \mathcal{B}(\mathbb{R}) \) with \( B \in \mathcal{B}_0(\mathbb{R}) \) and \( \nu(\text{Disc}(H_{\text{top}})) = 0 \) imply in view of Theorem A.2 that

\[
\lim_{z \to \infty} \mathbb{P}\{|X|_{t_0} \in z \cdot A\} = \nu\{|f \in D : \|f\|_{t_0} \in A\} =: \nu_{z}(A), \quad \forall A \in \mathcal{B}_0(\mathbb{R}) \cap \mathcal{B}(\mathbb{R}).
\]

Since for \( A = (x, \infty) \) we have \( \nu_{z}(A) = p_h(x) \in (0, \infty) \), the measure \( \nu \) is non-zero and hence the assumption that \( g \) is Lebesgue measurable implies that \( g \in \mathcal{R}_\alpha \) for some \( \alpha > 0 \) using for instance [1][Thm 1.1]. Hence by statement (i) we have that \( \nu \) is \(-\alpha\)-homogeneous and thus statement (ii) holds.

We show next (iii) along the lines of [1][Thm B.2.2]. Since \( \nu \) is non-trivial and \( \mathcal{B}\)-boundedly finite, we can find an open set \( A \) such that \( A \in \mathcal{B} \) and \( \nu(A) \in (0, \infty) \). Further, by our assumption \( z \cdot A \subset A \) for all \( z \geq 1 \) (thus \( A \) is a semi-cone). As shown in [1][p. 521]

\[
t \cdot A \subset z \cdot A, \quad \forall t \geq s > 0.
\]

Consequently, \( z \mapsto \nu(z \cdot A) \) is decreasing and by \( B3) \) also finite for all \( z \in \mathbb{R}_+ \). Further by \( B4) \)

\[
t \cdot A \subset z \cdot A, \quad \forall t \geq s > 0
\]

implies that \( \nu(\partial(z \cdot A)) = 0 \) for almost all \( z \in \mathbb{R}_+ \). Hence by assumption we can find some \( k > 0 \) such that \( \nu(\partial(A_k)) = 0, A_k = k \cdot A \) and further \( \nu(A_k) \in (0, \infty) \). By the continuity of the pairing we have that \( z \cdot A \) is open for almost all \( z \in \mathbb{R}_+ \), and then \( \mu \in \mathcal{R}\{g, \mathcal{B}, \nu\} \) implies for almost all \( s \in \mathbb{R}_+ \)

\[
\lim_{z \to \infty} \frac{g(z/s)}{g(z)} = \lim_{z \to \infty} \frac{g(z/s) \mu((z/s) \cdot (kz \cdot A))}{\mu(kz \cdot A)} = \frac{\nu(s \cdot A_k)}{\nu(A_k)} < \infty,
\]

where the last inequality follows since \( s \cdot A_k = (ks) \cdot A \in \mathcal{B} \) and \( \nu \) is \( \mathcal{B}\)-boundedly finite. Note that since \( g \) is non-negative and

\[
\lim_{z \to \infty} g(z) \mu((kz \cdot A) = \nu(A_k) \in (0, \infty)
\]

we have that \( \mu((kz \cdot A) \) is positive and finite for all \( z \) large, hence \( \mu((kz \cdot A)/\mu((kz \cdot A) = 1 \) for all \( z \) large justifying the second expression in (7.4).

Next, by the countable additivity of \( \nu \), (7.3) and the assumption \( \cap_{z \geq 1}(z \cdot A) \) is empty we have

\[
\lim_{z \to \infty} \nu(z \cdot A_k) = \nu(\emptyset) = 0
\]

and by (7.3) we have that the limit in (7.4) cannot be constant. Then, since \( g \) is Lebesgue measurable, by [1][Thm 1.1.2] \( g \in \mathcal{R}_\alpha \) and moreover necessarily \( \alpha > 0 \), hence statement (iii) follows.

**Proof of Theorem 4.7** Let \( H : D \to \mathbb{R}, \text{supp}(H) \in \mathcal{B} \) be a bounded continuous map. The assumption on \( \mathcal{B} \) implies that there exists \( \varepsilon > 0 \) and some \( K \subset Q \) such that \( H(f) = 0 \), if \( f_k = \sup_{t \in K \cap \mathbb{Q}} |f|_t \leq c \varepsilon \) for some fixed given \( c > 1 \). Hence we have

\[
H(f) = H(f) \mathbb{I}(\mathcal{E}_K^c((c\varepsilon)^{-1} \cdot f > 0) = H(f) \mathbb{I} \left( \sup_{t \in K \cap \mathbb{Q}} |f|_t > c \varepsilon \right), \quad \forall f \in D.
\]
Recall that $\mathcal{E}_K^q$ is defined by

$$\mathcal{E}_K^q(f) = \int_K \|f|_t > 1\| q_t \lambda(dt), \quad f : D \mapsto \mathbb{R}^d,$$

with $q_t$'s positive constants. Next, for all $\eta, z$ positive, by the Fubini-Tonelli theorem,

$$\mathbb{E}\{H(z^{-1} \cdot X)\} \geq \mathbb{E}\{H(z^{-1} \cdot X)\|((\varepsilon z)^{-1} \cdot X) > \eta\}\]

$$= \mathbb{E}\{H(z^{-1} \cdot X) \mathbb{E}_K^q((\varepsilon z)^{-1} \cdot X) > \eta\} \mathbb{E}_K^q((\varepsilon z)^{-1} \cdot X)\}

$$= \int_K \mathbb{E}\{H(z^{-1} \cdot X) \mathbb{E}_K^q((\varepsilon z)^{-1} \cdot X) > \eta\} \|\|X\|_t > \varepsilon\| \} q_t \lambda(dt).$$

Note that for the derivation of the second equality above we have used that $\mathcal{E}_K^q((\varepsilon z)^{-1} \cdot X)$ is finite almost surely, which is consequence of the choice of $q_t$'s since

$$\mathbb{E}\{\mathcal{E}_K^q((\varepsilon z)^{-1} \cdot X)\} = \mathbb{E}\{\int_K \|((\varepsilon z)^{-1} \cdot X) > \eta\} \} \leq \int_Q \max(1, p_t) q_t \lambda(dt) < \infty.$$

The assumption of the continuity of the pairing $(z, f) \mapsto z \cdot f$ implies that $H_z(f) = H(\varepsilon \cdot f) : D \mapsto \mathbb{R}_+$ is also a bounded continuous map. Moreover, by $B3$ $H_z$ satisfies supp$(H_z) \in \mathcal{B}$. Hence, by the RV of $|X|_{t_0}$ condition $(4.2)$, the continuity of $H_z$ and the fact that $\mathcal{E}_K^q(f), f \in D$ is almost surely continuous with respect to the law of $Y^{[i]}$ (hence the continuous mapping theorem can be applied) and the dominated convergence theorem, for almost all $\eta > 0$ we obtain

$$\lim_{z \rightarrow \infty} \mathbb{E}\{H(z^{-1} \cdot X) \mathbb{E}_K^q((\varepsilon z)^{-1} \cdot X) > \eta\} = \lim_{z \rightarrow \infty} \mathbb{E}\{H(z^{-1} \cdot X) \mathbb{E}_K^q((\varepsilon z)^{-1} \cdot X) > \eta\} \mathbb{P}\{|X|_{t_0} > z\} \|\|X\|_t > \varepsilon\| \} q_t \lambda(dt)

$$= \frac{1}{\varepsilon^\alpha p_t} \int_K \mathbb{E}\{H(\varepsilon \cdot Y^{[i]}) \mathbb{E}_K^q(Y^{[i]}) > \eta\} \mathbb{E}_K^q(Y^{[i]}) p_t \lambda(dt).$$

The monotone convergence theorem leads to (recall $(7.5)$)

$$\lim_{\eta \rightarrow 0} \int_\varepsilon^{-\alpha} \int_K \mathbb{E}\{H(\varepsilon \cdot Y^{[i]}) \mathbb{E}_K^q(Y^{[i]}) > \eta\} \mathbb{E}_K^q(Y^{[i]}) p_t \lambda(dt) = \varepsilon^{-\alpha} \int_K \mathbb{E}\{H(\varepsilon \cdot Y^{[i]}) \mathbb{E}_K^q(Y^{[i]}) \mathbb{P}\{|X|_{t_0} > z\} \|\|X\|_t > \varepsilon\| \} q_t \lambda(dt) = \nu[H].$$

By the above and $(4.4)$ for some $C^* > 0$

$$\lim_{\eta \rightarrow 0} \sup_{\varepsilon > 0} \sup_{z \rightarrow \infty} \mathbb{E}\{H(z^{-1} \cdot X) \mathbb{E}_K^q((\varepsilon z)^{-1} \cdot X) > \eta, \mathcal{E}_K^q((\varepsilon z)^{-1} \cdot X) \leq \eta\} \mathbb{P}\{|X|_{t_0} > z\} \|\|X\|_t > \varepsilon\| \} q_t \lambda(dt)$$

$$\leq C^* \lim_{\eta \rightarrow 0} \sup_{z \rightarrow \infty} \mathbb{P}\{|X|_{t_0} > \varepsilon z, \mathcal{E}_K^q((\varepsilon z)^{-1} \cdot X) \leq \eta\} = 0.$$

Consequently, $(7.6)$ yields

$$\lim_{z \rightarrow \infty} \mathbb{P}\{|X|_{t_0} > z\} \mathbb{E}\{H(z^{-1} \cdot X) \mathbb{E}_K^q(Y^{[i]}) \} = \nu[H].$$

Since $H$ is bounded, then for some constant $\tilde{C} > 0$ by $(4.3)$ and $(7.5)$

$$\lim_{z \rightarrow \infty} \mathbb{P}\{|X|_{t_0} > z\} \mathbb{E}\{H(z^{-1} \cdot X) \mathbb{E}_K^q(Y^{[i]}) \} \leq \tilde{C} \lim_{z \rightarrow \infty} \mathbb{P}\{|X|_{t_0} > \varepsilon z\} \mathbb{P}\{|X|_t > \varepsilon\} < \infty$$

implying $\nu[H] < \infty$. By the assumption $Y^{[i]}, h \in Q$ is a family of tail processes, hence in view of Lemma 3.16(i) it defines a unique tail measure $\nu_*$. From Lemma 3.13 we have that $\nu_*[H] = \nu[H]$. In view of Remark 2.7, $\nu_*[H]$ uniquely

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defines \( \nu_* \) for \( H \) as chosen above. Consequently, \( \nu = \nu_* \) follows. In the above calculations we choose \( q_i \)'s positive such that \( \sum_{i \in \mathbb{G}} \max(1, p_i) q_i < \infty \) and take \( K \) such that \( t_0 \in K \), which is possible. Since \( Y_{[h]}^t, h \in \mathcal{Q} \) is a family of tail processes and \( \nu[H] < \infty \), then Remark 3.14(ii) implies that \( \nu \) is a \( \mathcal{B} \)-boundedly finite non-trivial Borel tail measure and thus the claim follows by the definition of RV.

**Proof of Lemma 4.9** We can extend \( \nu \) to \( \mathcal{D} \) as in (4.8). Note that (set \( f_0^\mathcal{Q} = \max_{t_i \in \mathcal{Q}} \| f \|_{t_i} \))

\[
D = \bigcup_{t_i \in \mathcal{Q}} \bigcup_{k \in \mathbb{N}} \left\{ \{ f \in \mathcal{D} : f_0^\mathcal{Q} = 0 \} \cup \{ f \in \mathcal{D} : \| f \|_{t_i} > 1/k \} \right\} = \bigcup_{t_i \in \mathcal{Q}} \bigcup_{k \in \mathbb{N}} (A_0 \cup A_{k_i}).
\]

The joint measurability of the the outer multiplication and the measurability of \( \cdot \cdot \cdot \)'s yield

\[
A_0 \in \mathcal{D}, \quad A_{k_i} \in \mathcal{D}, \quad \forall k \in \mathbb{N}, \forall t_i \in \mathcal{Q}.
\]

Since \( \nu \) is non-zero it follows that it is impossible that \( p_t(x) = \nu(\{ f \in \mathcal{D} : \| f \|_t > x \}) = 0 \) for all \( t \in \mathcal{Q} \) and all \( x \) in a dense set of \( \mathbb{R}_+ \). Hence the assumption that \( \nu \) is non-trivial implies that for some \( t_0 \in \mathcal{T}, x > 0 \)

\[
p_{t_0}(x) = \nu(\{ f \in \mathcal{D} : \| f \|_{t_0} > x \}) \in (0, \infty).
\]

(7.7)

Since \( \nu \) is \( \mathcal{B}_0 \)-boundedly finite and we have set \( \nu(\{ 0 \}) = 0 \), it follows utilising further (2.1) that \( \nu \) is \( \sigma \)-finite. In view of Theorem A.1,(viii) for a countable dense set \( \mathcal{Q} \)

\[
\nu(\text{Disc}(p_t(\mathcal{D}))) = 0, \quad \forall t \in \mathcal{Q}.
\]

Set \( \nu_\alpha(A) = g(z)\mathbb{P}\{ x^{-1} \cdot X \in A \}, A \in \mathcal{D} \). By Theorem A.2 \( \nu_\alpha \circ p_n^{-1} = \nu \circ (p_0^{-1} \circ p_n^{-1}) \), whenever \( p_0 > 0 \) and in particular by (7.7) this holds for \( t = t_0 \). This then implies (use for instance [1][Thm B.2.2]) that \( g \in \mathcal{R}_\alpha \) for some \( \alpha > 0 \) and hence \( \nu \) is \( -\alpha \)-homogeneous follows from Lemma 4.5, statement (i). By the above we can take

\[
g(t) = p_{t_0}/\mathbb{P}\{ \| X \|_{t_0} > t \}.
\]

Since \( \nu(\text{Disc}(p_t(\mathcal{D}))) = 0, \forall t \in \mathcal{Q} \) we have (4.2) holds and thus \( \nu \) satisfies M2). We conclude that \( \nu \) is a tail measure on \( \mathcal{D} \) with index \( \alpha \). In view of Lemma 3.10 we have that \( \nu = \nu_Z \) and \( \mathbb{P}\{ \sup_{t \in \mathcal{Q}} \| Z \|_t > 0 \} = 1 \). Since \( \| x \| = 0 \) if and only if \( x = 0 \) and \( Z \) has almost surely càdlàg sample paths, then \( \mathbb{P}\{ \| Z \| = 0 \} = 0 \). The last two claims follow from Remark 3.14(ii).

**Proof of Theorem 4.11** Since by the choice of \( U_n \) we have \( \nu(\text{disc}(p_{U_n})) = 0 \), the first implication is direct consequence of the continuous mapping theorem (utilising Theorem A.1,(viii)-(x)) and the characterisation of \( \mathcal{B}_0 \) and \( \mathcal{B}_0(\mathcal{D}_{U_n}) \). In particular \( \nu^{(n)} = \nu \circ p_{U_n}^{-1} \) and it follows easily that the local tail processes of \( \nu^{(n)} \) denoted by \( Y_{n}^{[h]} \), \( h \in \mathcal{Q}_n = \mathcal{Q} \cap U_n \) satisfy

\[
Y_n^{[h]} = Y_{U_n}^{[h]}
\]

almost surely, where \( Y_{[h]} \)'s are the local tail processes of \( \nu \) and \( Y_{U_n}^{[h]} \) is their restriction on \( U_n \).

We show next the converse assuming \( X_{U_n} \in \mathcal{R}_\alpha(\mathcal{B}_0(\mathcal{D}_{U_n}), \nu^{(n)}) \) for all \( n \in \mathbb{N} \).

Step I (existence of \( \nu \)):

The sets \( U_n \) are increasing and \( \bigcup_{n=1}^{\infty} U_n = \mathbb{R}^1 \). Each measure \( \nu^{(n)}, n \in \mathbb{N} \) is \( K(\mathcal{Q}_n) \)-bounded (or compactly-bounded) with \( \mathcal{Q}_n = \mathcal{Q} \cap U_n \) and has a unique family of corresponding tail processes \( Y_{n}^{[h]} \), \( h \in \mathcal{Q} \). Since all spaces \( \mathcal{D}_{U_n}, \mathcal{D} \) are Polish we can consider all local tail processes to be defined on the same non-atomic complete probability space, [50][Lem p. 1276].

Applying the continuous mapping theorem (we utilise Theorem A.1,(viii)-(x)) to the projection of \( U_{n+1} \) to \( U_n \) denoted by \( p_{U_{n+1},U_n} \) shows that \( \nu^{(n)} = \nu^{(n+1)} \circ p_{U_{n+1},U_n}^{-1} \). It follows that the restriction of tail processes \( Y_{n+1}^{[h]} \) of \( \nu^{(n+1)} \) on \( U_n \) denoted by \( Y_{U_n}^{[h]} \), \( h \in U_n \) are also tail processes. By the uniqueness of the family of the tail processes it follows that \( \nu^{(n)} \) has local tail processes \( Y_{U_n}^{[h]} \), \( h \in \mathcal{Q} \), i.e., \( Y_{U_n}^{[h]} = Y_{U_n}^{[h]} \) almost surely. We can extend all \( Y_{U_n}^{[h]} \)s to be càdlàg processes on \( \mathcal{D} \). Applying Theorem A.1,(vii) or Theorem A.1,(xi) we obtain that \( Y_{U_n}^{[h]} \) converges weakly on \( \mathcal{D} \) as \( n \to \infty \) to a D-valued random element \( Y^{[h]} \). It follows easily that \( Y_{U_n}^{[h]} \) restricted on \( U_n \) coincides almost surely with \( Y_{U_n}^{[h]} \) and moreover \( Y_{U_n}^{[h]} \), \( h \in \mathcal{Q} \) is a family of tail processes. Let \( \nu \) denote the corresponding tail measure defined by (3.20). By the definition of local tail processes and the above we have that

\[
\nu_{U_n} = \nu^{(n)} = \nu_{Z_n}, \quad n \in \mathbb{N},
\]

(7.8)

where \( Z \) is a representor of \( \nu \) constructed from \( Y_{[h]} \)'s and \( Z_n \) is given by (4.11). Hence we have

\[
\mathbb{E}(\{ |Z_{[0]}| \}) = \nu^{(n)}(\{ f \in \mathcal{D} : \| f \| > 1 \}) \to p_0 = \mathbb{E}(\{ |Z_{[0]}| \}) < \infty, \quad n \to \infty.
\]

(7.9)
It follows that \( Y^{[h]} \) satisfies (3.21) (since that holds for \( Y^{[h]}_n \)-s) and hence \( \nu \) is \( \mathcal{B}_0 \)-boundedly finite.

Step II (RV of \( X \)):

By (2.1) and the definition of \( \| \cdot \|_1 \), see A3 and (2.1) the boundedness \( \mathcal{B}_0 \) on \( D_0 \) can be generated (see also [1][Example B.1.7]) by the open sets

\[
O^\infty_k = \left\{ f \in D : \sup_{t \in [-k,k] \cap \mathbb{Q}} |f|_{L} > 1/k \right\}, \quad k \in \mathbb{N}.
\]

Since \( \nu \) is \(-\alpha\)-homogeneous by Remark 3.2 \( \nu(\partial O^\infty_k) = 0 \) for all \( k \in \mathbb{N} \).

By (7.9) we can assume without loss of generality that \( p_0 = 1 \). In view of Remark 4.4(i) and recalling that \( \nu \) is \( \mathcal{B}_0 \)-boundedly finite, the claim follows if we show the following weak convergence:

\[
\mu_{k,z}(\cdot) = \mathbb{P}\{z^{-1} \cdot X \in \cdot \cap O^\infty_k\}/\mathbb{P}\{|X|_0 > z\} \xrightarrow{w} \nu(\cdot \cap O^\infty_k) =: \nu_k(\cdot), \quad z \to \infty
\]

(7.10)

for all positive integers \( k \). Note that \( \nu_k \) is a finite measure and set

\[
B^n_k = \left\{ f \in D_n : \sup_{t \in [-k,k] \cap \mathbb{Q}} |f|_{L} > 1/k \right\}.
\]

Next, fix an integer \( k > 0 \). In the light of Theorem A.1.(xi) the stated weak convergence is equivalent to

\[
\mu_{k,z}(\cdot) = \mathbb{P}\{z^{-1} \cdot X_{U_n} \in \cdot \cap B^n_k\}/\mathbb{P}\{|X|_0 > z\} \xrightarrow{w} \nu^*_k(\cdot), \quad z \to \infty,
\]

where

\[
\nu^*_k(A) = \nu_k(\{f \in D : f_{U_n} \in A\}), \quad A \in \mathcal{D}(D_{U_n})
\]

for all \( n \) large. The properly localised boundedness \( \mathcal{B}_0(U_n) \) can be generated by the open sets on \( D_n \) given by

\[
O^n_k = \left\{ f \in D_n : \sup_{t \in [-k,k] \cap \mathbb{Q}} |f|_{L} > 1/k \right\}, \quad n \in \mathbb{N}.
\]

In particular, for all \( n \) large, \( B^n_k \subset O^n_k \), implying \( B^n_k \subset \mathcal{B}_0(U_n) \). Moreover, \( \nu^{(n)}(\partial B^n_k) = 0 \), since \( \nu^{(n)} \) is \(-\alpha\)-homogeneous and so we can apply Remark 3.2. By the assumption \( X_{U_n} \in \mathcal{D}_a(\mathcal{B}_0(U_n), \nu^{(n)}) \) Remark 4.4(i) implies the weak convergence

\[
\mu_{k,z}(\cdot) \xrightarrow{w} \nu^{(n)}(\cdot \cap B^n_k) = \nu^*_k(\cdot), \quad z \to \infty,
\]

where the last equality above follows from (7.8) establishing the proof.

\[\square\]

**Proof of Corollary 4.13** It follows as in the proof of [10][Thm 3.3] that \( X \) has almost surely sample paths in \( D(\mathbb{R}^d, \mathbb{R}^d) \).

By [10][Lem 3.1, Thm 3.3] we have that \( X_{U_n} \) is regularly varying with \( U_n \) as in Theorem 4.11. Moreover, \( X_{U_n} \) has de Haan representation (4.12) with \( Z \)'s independent copies of \( Z \) determined in (4.11). The claim follows from the converse in the aforementioned theorem and Remark 4.12.(ii). Note that the case \( l = 1 \) is already shown in [3][Thm 4.1] using a direct proof.

\[\square\]

**Proof of Theorem 4.15 (i) \implies (ii):** Since \( \nu = K(Q) \)-bounded (and also \( \mathcal{B}_0 \)-boundedly finite) it has a D-valued representer \( Z \) that satisfies (3.14), which in view of Lemma 3.10 is equivalent with M3. As mentioned for instance in [27][p. 205] the set of stochastic continuity points of \( Z \) denoted by \( Z_F \), i.e., \( t \in Z_F \) such that \( \mathbb{P}\{Z(t) \neq Z(t-)\} = 0 \) is the same as the set of points \( \{t \in T : p \circ Z^{-1}(\{f \in D : p_t \text{ is continuous at } f\}) = 0 \} \), i.e., \( p_t : D \mapsto \mathbb{R}^d \) is continuous almost everywhere \( p \circ Z^{-1} \). Hence for all \( t \in Z_F \) we have

\[
\nu(\{f \in D : f(t) \neq f(t-)\}) = \int_0^\infty \mathbb{P}\{Z(t) \neq Z(t-)\}v_{(t)}(dr) = 0
\]

and thus \( p_t, t \in Z_F \) is \( \nu \)-continuous almost everywhere.

Let \( Q \subset Z_F \) be a dense set of \( T \) and let \( a < b, a, b \in Z_F \) be given and set \( K = [a,b] \). The existence of \( T_{a,b} \subset K \) which is up to a countable set equal \( K \) such that (4.5) holds for all \( t_1, \ldots, t_k \in T_{a,b}, k \geq 1 \) follows by arguments mentioned in [3] where the stationarity has not been used and the proof relies on [8][Thm 10, (ii) \implies (i)]. In the rest of the proof, by the equivalence of the norms on \( \mathbb{R}^d \) we shall suppose without loss of generality that \( \| \cdot \| \) equals the norm \( \| \cdot \|_a \) on \( \mathbb{R}^d \) used also for the definitions of \( w' \) and \( w'' \) below.

Taking \( T_0 \) to be the union of \( T_{a,b} \)'s, then (4.5) holds for all \( t_1, \ldots, t_k \in T_0, k \geq 1 \), with \( T_0 \subset T \) such that \( T \setminus T_0 \) is countable. Moreover, from [8][Eq. (7),(8),(9)] and [43][Eq. (12.32)] we obtain (4.14).

(ii) \implies (iii): Condition (4.15) follows immediately from (4.14). For all \( h \in T_0, \varepsilon > 0, z > 0 \)

\[
\mathbb{P}\{w'(X, K, \delta) > \varepsilon z, |X|h > z\} \leq \mathbb{P}\{w'(X, K, \delta) > \varepsilon z\}
\]

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and thus if \( p_h > 0 \), in view of (4.14) and (4.2) we obtain
\[
\lim_{d \to 0} \lim_{z \to \infty} \mathbb{P}\{w'(X, K, \delta) > \varepsilon z \mid |X|_h > z\} = 0. \tag{7.11}
\]

Using [3][Thm B.1], (4.6) and (7.11) we obtain the weak convergence in \((D, d_{D})\) of \( z^{-1} \cdot X \) conditionally on \(|X|_h > z\) to \( Y^{[h]} \) and further the limiting process \( Y^{[h]} \) has almost surely paths in \( D \). Since \( D \) is Polish, then by [59][Lem p. 1276] \( Y^{[h]} \) can be realised as a random element on the non-atomic probability space \((\Omega, \mathcal{F}, \mathbb{P})\).

In order to establish the claim we need to show that \( Y^{[h]}(t), h \in Q \) are local tail processes of a tail measure \( \nu \) on \( \mathcal{F} \). By Proposition 3.6 we have that \( Y^{[h]}(t), t \in Q \) are local tail processes when we take \( T = Q \), i.e., (3.7) and (3.9) hold. Since by Theorem A.1, (iii) \( \sigma(p, t \in Q) = \sigma(D) \) it follows that for all \( h, t \in Q \) such that \( p_h p_t > 0 \) (3.9) holds also for \( Y^{[h]}(t), t \in T \) and thus \( Y^{[h]}(t), h \in Q \) are local tail processes as \( D \)-valued random elements. Hence by Lemma 3.16, (i), there exists a unique tail measure \( \nu \) corresponding to these tail processes.

(iii) \( \Rightarrow \) (i): Let \( Q \subset T_0 \) be a dense subset of \( T \) and let \( w', w'' \) be the two moduli on \( D \) defined in (A.1), (A.3) below. For all \( z \) sufficiently large and \( \delta, \varepsilon \) positive
\[
\mathbb{P}\{X^*_K > \varepsilon z\} \leq \mathbb{P}\{w'(X, K, \delta) > \varepsilon z / 2\} + \mathbb{P}\{w'(X, K, \delta) < \varepsilon z / 2, X^*_K > \varepsilon z\} \\
\leq \mathbb{P}\{w'(X, K, \delta) > \varepsilon z / 2\} + \mathbb{P}\{\max_{0 \leq t \leq m} |X(t_i)| > \varepsilon z / 2\}
\]
for all \( K = [-n, n], n \in \mathbb{N} \) and every sequence \((t_0, \ldots, t_m) \in Q^m \) such that \(-n = t_0 < \cdots < t_i = n \) and \( t_i - t_{i-1} \leq \delta \) for \( i \leq m \) since \( w'' \) is dominated by \( w' \), which is shown in [13, Eq. (12.28)]. By the assumptions (7.11) holds, which together with (4.15) implies (4.14). Consequently, the arbitrary choice of \( \delta > 0 \) yields (4.3). In particular, (4.3) implies that
\[
p_i \leq C p_{i_0} < \infty, \quad \forall t \in K \tag{7.12}
\]
for some constant \( C > 0 \).

A crucial implication of Proposition 3.6 is that \( Y^{[h]} \) has the same law as \( R \cdot \Theta^{[h]} \) with \( R \) an \(-\alpha\)-Pareto rv independent of the local spectral tail process \( \Theta^{[h]} \). This implies that \( E^\mathcal{F}_K(f), f \in D \) is almost surely continuous with respect to the law of \( Y^{[h]} \) (see also [1][Rem 6.1.6]). Recall that
\[
E^\mathcal{F}_K(f) = \int_K \mathbb{1}\{|f|_t > 1\} q_t \lambda(dt), \quad f : T \mapsto \mathbb{R}^d,
\]
with \( \lambda = \lambda_Q \) counting measure on \( Q \) and we take \( q_t > 0, t \in Q \) satisfying \( \sum_{t \in K} \max(1, p_t) q_t < \infty \). In view of (7.12) the last condition is satisfied if we show that positive \( q_t \)'s can be chosen such that \( \sum_{t \in Q} q_t < \infty \).

The proof of the claim follows by showing that condition (4.4) in Lemma 4.7 holds for \( c = 2 \) and appropriate \( q_t \)'s constructed below.

Consider therefore the following construction of a density \( q \), which is needed for the proof below. Consider a compact set \( K \subset [0, 1] \). For fixed \( k \in \mathbb{N} \), we pick a single arbitrary and distinct point from each of the sets
\[
s^{(k)}_j \in B(m2^{-k}, 2^{-k}) \cap Q, \quad m = 0, \ldots, 2^k,
\]
where \( B(a, r) \) denotes the closed ball with center in \( a \) and radius \( r \). Notice that for any \( k \), the above system of balls covers \( Q \cap [0, 1] \). Assign the same point density
\[
q_k = 2^{-2k - 2}
\]
to each of the \( 2^k \) distinct points \( s^{(k)}_j \). The sum of all these masses is equal to
\[
\sum_{k=0}^{\infty} \sum_{j=0}^{2^k} 2^{-2k - 2} = \sum_{k=0}^{\infty} 2^{-k - 2} = \frac{1}{2}.
\]

Spreading out the remaining mass \( 1/2 \) among the non-chosen points in \( K \cap Q \), we obtain
\[
\int_K q_t \lambda(dt) = 1.
\]
Now consider an interval of length $v-u$ with $u, v \in K \cap \mathbb{Q}$. Let $n_1$ be the smallest natural number such that $2^{-n_1} < v-u$. In particular it follows that $(v-u)/2 \leq 2^{-n_1}$. Hence, there exists a ball $B_1 = B(m_12^{-n_1-2}, 2^{-n_1-2}) \subset (u, v)$. In particular, there is $s_1 \in B_1 \cap \mathbb{Q}$ having mass

$$q_{s_1} = 2^{-2(n_1+2)-2} = 2^{-6}(2^{-n_1})^2 \geq 2^{-6} \left( \frac{v-u}{2} \right)^2.$$

For a general $K$ not included in $[0,1]$, a constant $C(K)$ can similarly be found such that $s_1 \in [u,v)$ and

$$q_{s_1} \geq C(K)(v-u)^2.$$

If $f \in D$ is such that $w'(f, K, \eta) \leq \varepsilon/2$, and $\mathcal{E}_K^q((2\varepsilon)^{-1} \cdot f) > 0$, then there exists $t \in [a,b]$ such that $f(t) > 2\varepsilon$ and an interval $[u,v)$ such that

$$t \in [u,v), \quad v-u \geq \eta, \quad \sup_{|u\leq a', b'<v} \|f(s) - f(s')||_* \leq \varepsilon.$$

Consequently, $f(s) > \varepsilon$ for all $s \in [u,v)$ and

$$\mathcal{E}_K^q(\varepsilon^{-1} f) = \int_K \mathbb{I}(|f| > \varepsilon) q_t \lambda(dt) = \int_K \mathbb{I}(|f(t)||_* > \varepsilon) q_t \lambda(dt) \geq \int_{[u,v)} q_t \lambda(dt) \geq C(K)(v-u)^2 \geq C(K)\eta^2.$$

Since $X$ has almost surely locally bounded sample paths, the above yields for some constant $\tilde{C} > 0$

$$\mathbb{P}\{X_K^* > 2\varepsilon \varepsilon, \mathcal{E}_K^q((\varepsilon \varepsilon)^{-1} \cdot X) \leq \eta\} \leq \tilde{C}\mathbb{P}\{w'(X, K, \sqrt[\eta/C(K)](\eta/C(K)) > \varepsilon\},$$

hence (4.14) implies condition (4.4) for $c = 2$ and thus the claim follows from Lemma 4.7.

(iii) $\implies$ (iv): As shown above when (iii) holds, the tail measure $\nu$ with index $\alpha$ is $B_0$-boundedly finite. By Lemma 3.10 and Remark 3.14.(ii) $\nu = \nu_\alpha$ satisfying further M3 which is equivalent with (3.14) as mentioned above. In particular with representer $Z_{K_\alpha}$ is a $B_0(\mathcal{D}_{K_\alpha})$-boundedly finite tail measure with index $\alpha > 0$ on $\mathcal{B}(\mathcal{D}_{K_\alpha})$ for all compact intervals $K_\alpha \subset \mathbb{R}$ containing some open interval that includes 0.

Let $s_k < t_k, k \in \mathbb{N}$ in $T_0$ satisfying

$$\lim_{k \to \infty} s_k = \lim_{k \to \infty} t_k = \infty.$$

Suppose for simplicity that $-s_k, t_k, k \geq 1$ are strictly increasing positive sequences and chose them to belong to $Z_\alpha$. This is possible since $T_0$ and $Z_\alpha$ are equal up to a countable set. In view of (4.14) we have that for all $\varepsilon > 0$

$$\lim_{k \to \infty} \sup_{\eta > 0} \mathbb{P}\{w'(X, K_\alpha, \eta) > \varepsilon\} = 0$$

and thus using further [43][Eq. (12.31)] and the definition of $\| \cdot \|_{\alpha}$ as well as the equivalence of the norms on $\mathbb{R}^d$

$$\lim_{k \to \infty} \sup_{\eta > 0} \mathbb{P}\{\|X(s_k + \eta) - X(s_k)\| > \varepsilon\} = \lim_{k \to \infty} \sup_{\eta > 0} \mathbb{P}\{\|X(t_k - \eta) - X(t_k - \eta)\| > \varepsilon\} = 0.$$

Since by [43][p. 132] almost surely

$$w(X, [s_k, s_k + \eta], \eta) \leq 2[w''(X, K_\alpha, \eta) + \|X(s_k + \eta) - X(s_k)||_*],$$

$$w(X, [t_k - \eta, t_k], \eta) \leq 2[w''(X, K_\alpha, \eta) + \|X(t_k - \eta) - X(t_k - \eta)||_*],$$

then it follows as in the proof (iii) $\implies$ (i) (along the lines of [8][Thm 10,(i)]) that $X_K \in \mathcal{R}_\alpha(B_0(\mathcal{D}_{K_\alpha}), \nu_\alpha)$.

(iv) $\implies$ (ii): The proof follows from [8][Thm 10,(ii)] and [43][Eq. (12.32)]. \hfill\Box

**Proof of Lemma 5.2** Let $U_n, n \in \mathbb{N}$, $\nu^{(n)}$ be as in Theorem 4.11. By Theorem A.2 $X \in \mathcal{R}_\alpha(a_n, B_0, \nu)$ implies $X_{U_n} \in \mathcal{R}_\alpha(a_n, B_0, \nu^{(n)})$ for all $n \in \mathbb{N}$. The Polish space $\mathcal{D}_{\alpha}$ is a star-shaped metric space and thus [9][Thm 3.1] implies $X_{U_n} \in \mathcal{R}_\alpha(a_n, B_0, \nu^{(n)})$, hence the claim follows from Theorem 4.11. \hfill\Box

**Proof of Proposition 6.3** Note first that by the assumption on $|X(0)|$ for any $c > 0$ we have

$$\lim_{n \to \infty} n^c \mathbb{P}\{|X(0)| > (a_n x)^c\} = x^{-c/\alpha}$$
for all $x > 0$. We consider for simplicity only the case $T = \mathbb{R}^l$. By the stationarity of $X$, using [60][Thm 2.1] we obtain

$$\lim_{m \to \infty} \lim_{n \to \infty} nm^{-l} \mathbb{P}\left\{ \sup_{t \in [0,m]^l} |X(t)| > a_n \right\}$$

$$= \lim_{m \to \infty} m^{-l} \lim_{n \to \infty} n \mathbb{P}\left\{ \|X(0)\| > a_n \right\} \mathbb{E}\left\{ \frac{1}{J_{\in [0,m]^l}} \mathbb{I}\{\|X(s)\| > a_n\} \lambda(ds) \right\} \mathbb{P}\left\{ \|X(t)\| > a_n \right\} \lambda(dt)$$

$$= \lim_{m \to \infty} m^{-l} \lim_{n \to \infty} n \mathbb{P}\left\{ \|X(0)\| > a_n \right\} \mathbb{E}\left\{ \frac{1}{J_{\in [0,m]^l}} \mathbb{I}\{\|B^i X(s)\| > a_n\} \lambda(ds) \right\} \mathbb{P}\left\{ \|X(t)\| > a_n \right\} \lambda(dt)$$

$$= \lim_{n \to \infty} m^{-n} \mathbb{P}\left\{ \|X(t)\| > a_n \right\} \mathbb{E}\left\{ \frac{1}{J_{\in [0,m]^l}} \mathbb{I}\{\|Y(s)\| > 1\} \lambda(ds) \right\}$$

$$= \mathbb{E}\left\{ \frac{1}{J_{\in \mathbb{R}^l}} \mathbb{I}\{\|Y(t)\| > 1\} \lambda(dt) \right\}.$$

where the third last line follows from the weak convergence of $X/a_n$ conditioned on $|X(0)| > a_n$ to the tail process $Y$ and continuous mapping theorem and the second last line is consequence of dominated convergence theorem. Using again the stationarity of $X$ and the above bound gives (write $[n/m]$ for the largest integer smaller than $n/m$)

$$\limsup_{n \to \infty} \mathbb{P}\{M_n > a_n^l x\} \leq \limsup_{m \to \infty} \limsup_{n \to \infty} \mathbb{P}\{M_m > a_n^l x\}$$

$$= \lim_{m \to \infty} \frac{1}{m^l} \lim_{n \to \infty} n \mathbb{P}\{M_m > a_n^l x\}$$

$$= \mathbb{E}\left\{ \frac{1}{J_{\in \mathbb{R}^l}} \mathbb{I}\{\|Y(t)\| > 1\} \lambda(dt) \right\} x^{-\alpha}.$$

The finiteness of the expectation above follows from (3.21), hence the proof is complete. □

**Proof of Lemma 6.1** The claim under the first two conditions follows by Theorem A.2 and identical arguments as in[17][Cor 2.1, 2.2]. The last claim is again consequence of Theorem A.2 if we show that

$$H^{-1}(B) \in \mathcal{B}, \quad \forall B \in \mathcal{B}' \cap \mathcal{B}(D') \quad \nu(\text{Disc}(H)) = 0. \quad (7.13)$$

If $B' \in \mathcal{B}'$, then by definition $d_D(f', F') > \varepsilon$ for all $f' \in B'$ and some $\varepsilon > 0$. By continuity of $H$ for all $f \in F$ we can find $\delta_f > 0$ such that for all $x \in D$ satisfying

$$d_D(f, x) \leq \delta_f$$

we have $d_D(H(f), H(x)) \leq \varepsilon$. Since $F$ has finite number of elements, then $\delta = \min_{f \in F} \delta_f > 0$. Since $H(F) = F'$ and $d_D(f', F') > \varepsilon$ for all $f' \in B'$, then $d_D(F, H^{-1}(B')) > \delta$ and thus (7.13) follows establishing the claim. Note that when $F$ has one element this is already proven in [9][(ii), p. 125]. □

**Proof of Lemma 6.5** Since $\mathbb{P}\{Z \neq 0\} = \mathbb{P}\{\|Y(0)\| > 1\} = 1$, then the integrals in (6.5) are almost surely positive. Clearly, RV of $X$ implies the RV of $X(t), t \in \mathbb{Z}^l$ with limit measure which is shift-invariant and has tail process $Y(t), t \in \mathbb{Z}^l$. As shown in [5][Lem 3.5] we obtain

$$\|Y(t)\| \to 0, \quad \|t\| \to \infty, \quad t \in \mathbb{Z}^l$$

almost surely, which in view of [25][Prop 2.18] is equivalent with

$$\mathbb{P}\left\{ \sum_{t \in \mathbb{Z}^l} |Y(t)|^\alpha < \infty \right\} = 1. \quad (7.14)$$
We have that $Y^*(t) = |Y(t)|, t \in \mathbb{Z}$ is a tail process with representation $R \Theta^*(t) = R[\Theta^{[0]}(t)], t \in \mathbb{Z}$, where $R$ is $\alpha$-Pareto independent of $\Theta^*$, which is a spectral tail process. Hence in view of [52][4.6] we have that (7.14) is equivalent with $\mathbb{P}(\int_{\mathbb{R}} |Y(t)|^{\alpha} \lambda(dt) < \infty) = 1$ establishing the claim.

**Proof of Theorem 4.6** The RV of $X$ being max-stable has been shown in Corollary 4.13. If $X$ is $\alpha$-stable RV can be established as in [3]. Alternatively, since for this case Remark 4.12(ii) holds and RV of $X$ for compact $T$ has been established in [10], the RV of $X$ follows from Theorem 4.11. If $\theta_Y = 0$, then (6.6) follows from (6.2) and when $\theta_Y > 0$ the corresponding proofs in [3] can be modified to cover the case $l > 1$.

**Proof of Lemma 6.8** For all $x > 0, n \in \mathbb{N}$ and all $y > 0$ large

$$\mathbb{P}(a_n^{-l}M_n > x) = \mathbb{P}(a_n^{-l}M_n > x, R \leq y) + \mathbb{P}(a_n^{-l}M_n > x, R > y)$$

$$\leq \mathbb{P}\left( \sup_{t \in [0,n]\cap T} |Z(t)| \geq a_n^l y \right) + \mathbb{P}\left( M_n > a_n^l x, R > y \right)$$

$$\leq \frac{y^\alpha}{n^l x^\alpha} \mathbb{E}\left[ \sup_{t \in [0,n]\cap T} |Z(t)|^\alpha \right] + c \int_y^\infty s^{-\alpha-1} \mathbb{P}\left( \sup_{t \in [0,n]\cap T} |Z(t)| > a_n^l x/s \right) ds,$$

where we used the Markov inequality and the assumption $f(s) \leq cs^{-\alpha-1}$ for all $s$ large to derive the last line above. The shift-invariance of $\nu_Z$ implies (4.13). Hence as in [3, 61] it follows that

$$\lim_{n \to \infty} n^{-l} \mathbb{E}\left[ \sup_{t \in [0,n]\cap T} |Z(t)|^\alpha \right] = \mathbb{E}\left( \sup_{t \in T} |Y(t)|^\alpha / \int_T |Y(t)|^\alpha \lambda(dt) \right) = \theta_Y.$$

Given $\varepsilon > 0$ for all large $y$ we have that

$$\int_y^\infty s^{-\alpha-1} \mathbb{P}\left( \sup_{t \in [0,n]\cap T} |Z(t)| > a_n^l x/s \right) ds < (1 + \varepsilon) \int_y^\infty s^{-\alpha-1} e^{-s^{-\alpha}} \mathbb{P}\left( \sup_{t \in [0,n]\cap T} |Z(t)| > a_n^l x/s \right) ds$$

$$= (1 + \varepsilon) \alpha^{-1} \mathbb{P}\left( M_n^* > a_n^l x \right),$$

where $M_n^*$ is defined as $M_n$ taking $R = \Gamma_1^{-1/\alpha}$ with $\Gamma_1$ a unit exponential. If $\tilde{X}$ is a max-stable process with representation (4.12) where $Z^{(i)}$'s are independent copies of $|Z|$, then we have

$$M_n^* \leq \sup_{t \in [0,n]\cap T} \tilde{X}(t), \quad n \in \mathbb{N}$$

almost surely. Since when $\nu_Z$ is shift-invariant (4.13) holds and thus as mentioned before by Corollary 4.13 $\tilde{X}$ is stationary and therefore regularly varying as shown in Theorem 6.6. Hence applying Proposition 6.3 yields (6.8) establishing the claim.

**Appendix A** Space $D(\mathbb{R}^l, \mathbb{R}^d)$ & the mapping theorem

The space of generalised càdlàg functions $f : \mathbb{R}^l \mapsto \mathbb{R}^d$ denoted by $D = D(\mathbb{R}^l, \mathbb{R}^d)$ is the most commonly used when defining random processes. If $U$ is a hypercube of $\mathbb{R}^l$ we define similarly $D_U = D(U, \mathbb{R}^d)$ which is Polish (see e.g., [36][Lem 2.4]) and will be equipped with the $J_1$-topology, the corresponding metric and its Borel $\sigma$-field. The case $l = 1$ is the most extensively studied in numerous contributions as highlighted in Section 2. There are only a few articles dealing with properties of $D$ when $l > 1$ focusing mainly on the space of càdlàg functions $D([0, \infty)^l, \mathbb{R}^d)$, see [33, 62]. The definition of $D$ for $l \geq 1$ needs some extra notation and therefore we directly refer to [62] omitting the details.

Let $\mathcal{Q}$ be a dense set of $\mathbb{R}^l$. Given a hypercube $[a, b] \subset \mathbb{R}^l$ set $K = [a, b]$ and write $P_k(K, \eta), \eta > 0$ for a partition of $K = \bigcup_{i=1}^k K_i$ by disjoint hypercubes $K_i = [a_i, b_i] \subset T, i \leq k$ with smallest length of $[a_i, b_i]$'s exceeding $\eta$ and let $P(K, \eta)$ be the set of all such partitions. We define for a given norm $\| \cdot \|$, on $\mathbb{R}^d$ and $\eta > 0, f \in D$

$$w(f, K, \eta) = \sup_{s, t \in K \cap \mathcal{Q}} |f(t) - f(s)|_*, \quad w^r(f, K, \eta) = \inf_{P_k(K, \eta) \in P(K, \eta)} \max_{1 \leq i \leq k} \max_{s, t \in K \cap \mathcal{Q}} \|f(t) - f(s)|_* ,$$

(A.1)
Let $\tau$ be time changes $\mathbb{R}^l \mapsto \mathbb{R}^l$, i.e., its components denoted by $\tau_i : \mathbb{R} \mapsto \mathbb{R}, i \leq l$ are strictly increasing, continuous, $\tau_i(-\infty) = -\tau_i(\infty) = -\infty$ and such that their slope norm

$$
\|\tau_i\| = \sup_{s \neq t, s, t \in \mathbb{R}} |\ln(\tau_i(t) - \tau_i(s))/(t - s)|
$$

is finite. Write $\Lambda$ for the set of all $\tau$'s. As in [62] we define the metric $d_D$ for all $f, g \in D$ by

$$
d_D(f, g) = \sum_{j=1}^{\infty} 2^{-j} \min(1, d_N(j)(f, g)), \quad f, g \in D,
$$

where $N(j), j \leq \mathbb{N}$ is an enumeration of $\mathbb{N}^l$ and $d_N(j)(f, g)$ is as in [62][Eq. (2.26)], i.e.,

$$
d_N(f, g) = \inf_{\tau \in \Lambda} \left( \sum_{i=1}^{l} \|\tau_i\| + \max_{t \in \mathbb{R}^l \cap Q} \|(k_N(\tau(t)) \cdot f(\tau(t)) - k_N(t) \cdot g(t))_*\right), \quad N = (N_1, \ldots, N_l) \in \mathbb{N}^l,
$$

where $\cdot$ is the Hadamard product (the usual component-wise product) with $k_N : \mathbb{R}^l \mapsto \mathbb{R}^d$ having components

$$
k_Ni(t) = 1, \quad t \in [-N_i, N_i], \quad k_Ni(t) = 0, \quad t \in [-N_i - 1, N_i + 1]^c,$$

and for other $t \in \mathbb{R}^l$ the components $k_Ni(t), i \leq d$ are defined by linear interpolation. Here the hypercube $[-N, N]$ is defined as usual for $N \in \mathbb{N}^l$. Since for all $N(j), j \in \mathbb{N}^l$ such that $[-N(j), N(j)] \subset [-N, N] = [-k, k]^d, k \in \mathbb{N}$ we have $d_N(j)(f, g) \leq d_N(f, g)$ by the definition of $d_N$ it is clear that

$$
d_N(j)(f, g) \leq \sup_{t \in \mathbb{R}^l \cap [-k, k]^d} \|f - g\|_*
$$

we conclude that for all $\eta > 0$ we can find $k > 0$ independent of $f$ and $g$ such that

$$
d_D(f, g) \leq \sup_{t \in \mathbb{R}^l \cap [-k, k]^d} \|f - g\|_* + \eta
$$

holds for all $k$ sufficiently large. Let $J_1$ be the Skorohod topology on $D$, i.e., the smallest topology on $D$ such that $\lim_{n \to \infty} f_n = f$ holds if and only if there exists $\tau_n \in \Lambda$ such that:

1. $J_{1a}) \lim_{n \to \infty} \sup_{s \in \mathbb{R}} |\tau_m(s) - s| = 0, \quad \forall 1 \leq i \leq l$;

2. $J_{1b}) \lim_{n \to \infty} \sup_{t \in [-N, N]} \|f_\delta(\tau_n(t)) - f(t)\|_* = 0, \quad \forall N \in \mathbb{N}^l$.

Let $X_\mathcal{F}$ denote the set of stochastic continuity points $u \in \mathcal{T}$ of the $\mathcal{D}$-valued random element $X$ defined on $(\Omega, \mathcal{F}, \mathbb{P})$, i.e., $X_\mathcal{F}$ consists of all $u \in \mathcal{T}$ such that the image measure $\mathbb{P} \circ X^{-1}$ assigns mass 0 to the event $\{f \in \mathcal{D} : f$ is discontinuous at $u\}$. Write similarly $T_\nu$ for the sets of continuity points of a measure $\nu$ on $\mathcal{F}$. The next result is largely a collection of several known results in the literature.

**Theorem A.1.** (i) $f \in D$ if and only if

$$
\lim_{\delta \to 0} w^\prime(f, K, \delta) = 0, \quad \sup_{t \in \mathbb{R}^l \cap K} \|f(t)\|_* < \infty, \quad \forall K \in K(\mathbb{Q});
$$

(ii) $(D, d_D)$ is a Polish space and $d_D$ generates the $J_1$ topology;

(iii) $\sigma(p_t, t \in \mathbb{Q}) = \mathcal{B}(D)$;

(iv) The pairing $(z, f) \mapsto zf$ which is a group action of $\mathbb{R}^\times$ on $D$ is continuous in the product topology on $\mathbb{R}^\times \times D$;

(v) $A \in \mathcal{B}_0$ if and only if (2.1) holds for some $\varepsilon_A > 0$ and some hypercube $K_A \subset \mathbb{R}^l$;

(vi) For all $f \in D$ such that $\|f(0)\| \geq 1$ we have $d_D(cf, 0) = 1$ for all $c > 1$;
(vii) A sequence of $D$-valued random elements $X_n, n \geq 1$ defined on an non-atomic probability space $(\Omega, \mathcal{F}, \mathbb{P})$ converges weakly as $n \to \infty$ with respect to the $J_1$ topology to a $D$-valued random element $X$ defined on the same probability space, if $(X_n(t_1), \ldots, X_n(t_k))$ converge in distribution as $n \to \infty$ to $(X(t_1), \ldots, X(t_k))$ for all $t_1, \ldots, t_k$ in $X_T$ and further
\[
\lim_{n \to \infty} \sup_{\eta \in \mathcal{D}} \mathbb{P} \{ \nu'(X_n, K, \eta) \geq \varepsilon \} = 0, \quad \forall \varepsilon > 0, \forall K \in \mathcal{K}(\mathbb{Q}),
\]
\[
\lim_{m \to \infty} \sup_{n \to \infty} \mathbb{P} \left\{ \sup_{t \in [-n, n]} |X_n(t)| \geq m \right\} = 0, \quad \forall k \in \mathbb{N};
\]
(viii) The sets $X_T$ and $T_\nu$ for $\nu$ a $\sigma$-finite measure on $\mathcal{B}(D)$ is dense in $\mathbb{T}$ and $\nu(\text{disc}(\mathbb{P})) = 0, \forall t \in T_\nu$.

(ix) If $\nu$ is a $\sigma$-finite measure on $\mathcal{B}(D)$, then for any hyperspace $A \subset \mathbb{R}^t$ with corners in $T_\nu$, we have $\nu(\text{Disc}(\mathbb{P}_A)) = 0$, where $\mathbb{P}_A : D \mapsto D(A, \mathbb{R}^d) = D_A$ with $\mathbb{P}_A(f) = f_A, f \in D$ the restriction of $f$ on $A$. Moreover, we can find increasing hyperspaces $A_k, k \in \mathbb{N}$ with $\nu(\text{Disc}(\mathbb{P}_A)) = 0$ such that $[-k, k]^t \subset A_k$ for all $k \in \mathbb{N}$.

(x) If $A_k$ is as in (ix), the projection map $\mathbb{P}_{A_k} : D_{A_k} \mapsto D_{A_k}$ with $A_k \subset A_\nu$ or $A_{\nu} = \mathbb{R}^t$ is measurable.

(xi) Let $\nu_n, n \in \mathbb{N}, \nu$ be finite measures on $\mathcal{B}(D)$ and let $A_k, k \in \mathbb{N}$ be as specified in (ix) above. If $\nu_n \circ \mathbb{P}^{-1}_{A_k} \mapsto \nu \circ \mathbb{P}^{-1}_{A_k}, \forall k \in \mathbb{N}$, then $\nu_n \mapsto \nu$ as $n \to \infty$.

**Proof of Theorem A.1** (i): This is shown in [62][Thm 2.1] for the case $D([0, \infty]^t, \mathbb{R}^d)$ and can be proved with similar arguments for $D(\mathbb{R}_+, \mathbb{R}^d)$.

(ii): For the case $D([0, \infty]^t, \mathbb{R}^d)$ this is shown in [62][Thm 2.2]. The case $D(\mathbb{R}_+, \mathbb{R}^d)$ follows with the same arguments, see also [33].

(iii): The last two equalities are shown in [33][Thm 3.2] for the case $D([0, \infty]^t, \mathbb{R}^d)$ (see also [62][Thm 2.3]) and can be shown with similar arguments for our setup.

(iv): We need to show that for all positive sequences $a_n \to a > 0$ as $n \to \infty$ and any $f_n, f \in D$ such that $\lim_{n \to \infty} d_D(f_n, f) = 0$. By the characterisation of the Skorohod topology there exists $\tau_n$ such that $J_1(a)$ and $J_1(b)$ hold. Since
\[
|a_n f_n(\tau_n(t)) - a f(t)||_* = |a_n f_n(\tau_n(t)) - f(t)||_* + |a_n - a||f(t)||_*,
\]
and $\sup_{t \in K} |f(t)||_*$ is finite for any compact $K \subset \mathbb{R}^t$, then the claim follows.

(v) By the equivalence of the norms on $\mathbb{R}^d$ and the definition of $\| \cdot \|_t$ in the formulation of $A3)$, we can assume without loss of generality that $\|f\|_1 = \|f(t)\|_*, f \in D, t \in T$.

We have that $A \in B_0$ if and only if there exists $\varepsilon_A > 0$ such that for all $f \in A$ we have $d_D(f, 0) > \varepsilon_A$. Hence for such $A$, by (A.5) there exists $\varepsilon' \in (0, \varepsilon_A)$ and some hypercube $K_A$ such that
\[
f_{K_A}^* = \sup_{t \in K_A \cap \mathbb{Q}} \|f(t)\|_* > \varepsilon'
\]
for all $f \in A$.

Conversely, if for all $f \in A$ we have $f_{K_A}^* > \varepsilon > 0$ and thus $f_{[-k,k]^t}^* > \varepsilon$ for all $k$ sufficiently large, since
\[
d_D(f, 0) \geq \sup_{t \in [-k,k]^t \cap \mathbb{Q}} \|f(t)\|_* = f_{[-k,k]^t}^*, \forall N \in \mathbb{N} \setminus [-k,k]^t,
\]
then by definition of $d_D$ we have that $d_D(f, 0) \geq f_{[-k,k]^t}^* > \varepsilon'$ for some $\varepsilon' > 0$ and all $f \in A$, this means that $A \in B_0$ by the definition of $B_0$ establishing the claim.

(vi): By the definition, for all $c > 0, f \in D$ and $N(j) \in \mathbb{N}^d$ (recall $0$ denotes the zero function in $D$)
\[
d_{N(j)}(cf, 0) = (cf)^{[-N(j),N(j)]} = cf_{[-N(j),N(j)]}^* \geq c|f(0)|.
\]
Hence if $|f(0)| \geq 1$, then
\[
d_D(cf, 0) = \sum_{j=1}^{\infty} 2^{-j} \min(1, d_{N(j)}(cf, 0)) = 1 = d_D(f, 0), \quad c > 1.
\]

(vii): The tightness criteria is given in [62][Thm 2.4]. The claim follows now from [37][Thm 5.5].

(viii): The fact that $X_T$ is dense in $\mathbb{R}^t$ is shown in [33][p. 182] for $D(D([0, \infty]^t, \mathbb{R}^d)$ and hence the claim follows for $D = D(\mathbb{R}_+, \mathbb{R}^d)$. We can use that result and $\sigma$-finiteness of $\nu$ to prove that $T_\nu$ is also dense in $\mathbb{R}^t$. Next, for all $t \in T_\nu$ we
have that \( p_i \) is continuous for almost all \( f \in D \) with respect to \( \nu \) if and only if \( \nu(\{ f \in D : f_i \neq f_{i-} \}) = 0 \), hence the claim follows.

\((ix)\): The proof for \( A \) is along the lines of [33][Lem 4.2] for a probability measure on \( D \) and the argument can be extended to a \( \sigma \)-finite measure \( \nu \). Since \( T_\nu \) is dense in \( \mathbb{R}^l \), then \( A_\nu \) with the stated property exists.

\((x)\): The case \( l = 1 \) is shown for instance in [63][Lem 9.20], where \( A_\nu = \mathbb{R} \). The general case \( l \) is a positive integer and \( A_\nu \) is a hypercube that includes \( A_\nu \) with similar arguments as therein using further (iii) above.

\((xi)\): For probability measures \( \nu_n, \nu \) this is shown in [33][Thm 4.1] and the remark about proper sequences after the proof of [33][Thm 4.1]. However, the proof of the aforementioned theorem as well as the corresponding result [64][Thm 3.3] have a non-fatal gap, namely the projection map denoted by \( r_\nu \) therein has not been shown to be measurable and therefore the mapping theorem cannot be claimed. The measurability of \( r_\nu \) for \( l = 1 \) is shown in [65][Lem 2.3] and the case \( l > 1 \) is claimed in (x) above. The claim for finite non-null measures \( \nu_n, \nu \) follows then, since the weak convergence implies that \( \lim_{n \to \infty} \nu_n(D) = \nu(D) \in (0, \infty) \) and hence \( \nu_n/\nu(D), \nu/\nu(D) \) are probability measures and we have the corresponding weak convergence.

Concluding, we present the mapping theorem under the assumption \( A2 \) for both \( D \) and \( D' \) equipped with properly localised boundednesses \( B \) and \( B' \), respectively.

**Theorem A.2. ([1][Thm B.1.2])** Let \( \nu_z, z > 0 \) be \( B \)-boundedly finite measures on \( \mathcal{B}(D) \) and let \( \nu \) be a measure on \( \mathcal{B}(D') \). If \( H : D \mapsto D' \) is \( \mathcal{B}(D)/\mathcal{B}(D') \) measurable and \( \nu_z \xrightarrow{\nu} \nu \), then \( \nu_z \circ H^{-1} \xrightarrow{\nu} \nu \circ H^{-1} \), provided that

\[
H^{-1}(B) \in B, \quad \forall B \in B' \cap \mathcal{B}(D') \quad \text{and} \quad \nu(\text{Disc}(H)) = 0. \tag{A.6}
\]

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