Abstract: We discuss the dynamics of a particular two-dimensional (2D) physical system in the four dimensional (4D) (non-)commutative phase space by exploiting the consistent Hamiltonian and Lagrangian formalisms based on the symplectic structures defined on the 4D (non-)commutative cotangent manifolds. The noncommutativity exists equivalently in the coordinate or the momentum planes embedded in the 4D cotangent manifolds. The signature of this noncommutativity is reflected in the derivation of the first-order Lagrangians where we exploit the most general form of the Legendre transformation defined on the (non-)commutative (co-)tangent manifolds. The second-order Lagrangian, defined on the 4D tangent manifold, turns out to be the same irrespective of the noncommutativity present in the 4D cotangent manifolds for the discussion of the Hamiltonian formulation. A connection with the noncommutativity of the dynamics, associated with the quantum groups on the $q$-deformed 4D cotangent manifolds, is also pointed out.

Keywords: Lagrangian and Hamiltonian formalisms; (non-)commutative (co-)tangent manifolds; symplectic structures; quantum groups

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1 Introduction

In recent years, there has been a great deal of interest in the study of noncommutative geometry because of its very neat and natural appearance in the context of brane configurations related to the dynamics of string theory and matrix model of the M-theory [1-3]. It has been demonstrated that, in a certain specific limit, the whole string dynamics can be described by the minimally coupled gauge theory on a noncommutative space [2]. Even the experimental test for the existence of such kind of noncommutativity has been proposed where the synchrotron radiation has been incorporated in the experimental set up in a very specific way [4]. In such kinds of experimental proposals, it has been argued that only the quantum mechanical approximations are good enough to shed some light on the existence of the noncommutativity [5,6]. This is why a whole lot of studies have gone into the understanding of quantum mechanics of some physical systems on the noncommutative spaces (see, e.g., [7-10] and references therein for details). In particular, the motion of the charged particles in the 2D plane, under the influence of a constant perpendicular magnetic field, has been studied extensively because of the fact that the noncommutativity appears in this problem (i.e. Landau problem) quite naturally [7-10].

In the present scenario of the frontier research in the realm of high energy physics (in particular, connected with the string theory), the noncommutative (NC) effects are presumed to show up only at very high energy scale (perhaps, very near to the Planck scale). One can observe, however, the physical consequences of these NC effects in the low energy effective actions for the physically interesting systems. Equivalently, on the other hand, one can construct the low energy effective actions for the above physical systems by exploiting the basic ideas behind the NC geometry that are supposed to be the benchmarks of physics at very high energy scale. In the latter category, mention can be made of a couple of interesting examples such as the NC Chern-Simons theory [11] and the NC extension of the standard model [12]. One of the most interesting physical system that encompasses the NC structure in its folds at low energy scale is the 2D Landau problem. The purpose of our present paper is to study, systematically, the Hamiltonian and Lagrangian formulation of the 2D Landau problem based on the symplectic structures defined on the 4D cotangent manifolds. We lay emphasis on the classical equations of motion for the charged particle moving on a 2D plane under the influence of a harmonic oscillator potential and a constant magnetic field perpendicular to this 2D plane. We demonstrate that there exist three consistent first-order Lagrangians (FOLs) and three Hamiltonians that lead to the same classical equations of motion when we exploit the Euler-Lagrange equations of motion and the Hamilton’s equations of motion using (i) the canonical Poisson brackets (PBs), and (ii) the nontrivial (noncommutative) Poisson brackets. The nontrivial noncommutativity is found to be equivalently present either in the coordinate plane or in the momentum plane, embedded in the 4D cotangent manifold. The FOLs are defined and derived by the most general form of the Legendre transformations which exploit the above symplectic
structures (see, e.g., [13] for details). It is interesting to point out the fact that the second-order Lagrangian \( L^{(s)}(x, \dot{x}) \), defined in the 4D tangent manifold, turns out to be the same for all the three consistent FOLs and the Hamiltonians. In fact, the signature of the NC structure, present in the PBs, disappears in the derivation of the second-order Lagrangian (SOL) for the system under consideration. This happens, perhaps, because of the fact that the Hamiltonians (and the corresponding PBs) are connected one-another due to a special class of transformations (see, e.g., (3.5) and (3.6) below) on the phase variables under which the sum of the areas of the phase space remains unchanged. Furthermore, we have shown that these special class of transformations are not exactly the canonical transformations because the form of the basic canonical PBs does not remain form invariant.

In particular, the brackets equivalently defined in the 2D coordinate or momentum plane do change their form (under the above transformations) which is the root cause of the presence of NC structures in the theory. In the framework of quantum groups, we demonstrate and establish that the above (phase space) area preserving transformations correspond to the \( SL_{q,q^{-1}}(2) \) invariant transformations on the phase variables defined on the 4D \( q \)-deformed cotangent manifolds. The consistency of the above transformations with the structure of the \( q \)-deformed PBs, however, puts a restriction \( q^2 = 1 \) on the deformation parameter. This requirement, in turn, allows only the presence of exact canonical transformations for the phase variables in the framework of transformations generated by the quantum groups.

The contents of our present paper are organized as follows. In section 2, we discuss the canonical brackets and show that the classical dynamical equations of motion for the charged particles on the 2D plane (moving under the influence of a constant perpendicular magnetic field) can be derived from a canonical Hamiltonian. Section 3 is devoted to the discussion of the nontrivial NC brackets which are exploited in the derivation of the same classical dynamical equations from another set of a couple of Hamiltonians. In this section, we also discuss about the special kind of transformations on the phase space variables that connect the (non-)trivial PBs as well as the Hamiltonian functions. We deal with the non-commutativity associated with the quantum groups in section 4 and show its connection with the NC structures associated with the PBs, discussed in the previous sections. Finally, in section 5, we concisely summarize our results and make some concluding remarks.

2 Canonical brackets: Hamiltonian and Lagrangian dynamics

We start off with the following Hamiltonian, describing the motion of the charged particles in a 2D plane under the influence of a two-dimensional harmonic oscillator potential, as \( ^\dagger \)

\[
\hat{H}_1 = \frac{1}{2} p_i p_i + \frac{1}{2} \left( \omega^2 + \frac{k^2}{4} \right) x_i x_i - \frac{k}{2} \varepsilon_{ij} x_i p_j,
\]

\( ^\dagger \)We adopt here the summation convention where all the repeated Latin indices \( i,j,k \ldots = 1,2 \) in the coordinate and/or momentum space and \( A,B,C \ldots = 1,2,3,4 \) in the momentum phase (i.e. cotangent) space are summed over. The antisymmetric Levi-Civita tensor \( \varepsilon_{ij} \) is chosen such that \( \varepsilon_{12} = +1 = -\varepsilon_{12}^\dagger \).
where \((x_i, p_i)\) (with \(i = 1, 2\)) are the canonically conjugate position and momenta variables for the charged particles with individual charge \(e\) and \(\omega\) is the frequency of the harmonic oscillations. The constant \(k = \frac{e B}{c}\) is connected with the charge \(e\), the speed of light \(c\) and the magnetic field \(B\) perpendicular to the 2D plane of oscillation. Here the mass of the particle has been set equal to one for the sake of simplicity of all the later calculations. The dynamical equation of motion for this physical system

\[
\ddot{x}_i = -\omega^2 x_i + k \varepsilon_{ij} \dot{x}_j, \tag{2.2}
\]

can be derived from the above Hamiltonian by exploiting the following canonical Poisson brackets (PBs) among the conjugate variables

\[
\{x_i, p_j\}_{(PB)} = \delta_{ij}, \quad \{x_i, x_j\}_{(PB)} = 0, \quad \{p_i, p_j\}_{(PB)} = 0. \tag{2.3}
\]

With the help of the above canonical brackets (2.3), the canonical symplectic structures on the 4D cotangent manifold, for the Hamiltonian (2.1), can be defined as

\[
\Omega^{AB}(1) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad \Omega_{AB}(1) = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \tag{2.4}
\]

where the entry (1) in the parenthesis of the symplectic structures \(\Omega^{AB}\) and \(\Omega_{AB}\) stand for the subscript of the Hamiltonian in (2.1) and the notation \(z^A = (z^1, z^2, z^3, z^4) \equiv (x_1, x_2, p_1, p_2)\) has been introduced in the definition of the matrix form of the symplectic structure as given below

\[
\Omega^{AB}(1) = \text{Matrix} \left( \{z^A, z^B\}_{(PB)} \right), \quad \Omega^{AB} \Omega_{BC} = \delta^A_C = \Omega_{CM} \Omega^{MA}. \tag{2.5}
\]

The most general form of the PB between two dynamical variables \(f(z), g(z)\) in the momentum phase (i.e. cotangent) space can be defined as

\[
\{f(z), g(z)\}_{(PB)} = \Omega^{AB} \partial_A f(z) \partial_B g(z), \quad \partial_A = \frac{\partial}{\partial z^A}. \tag{2.6}
\]

In general, the symplectic structures in (2.4) and/or (2.5) can be functions of the phase variables \(z^A\). In such a situation, the most general form of the Legendre transformation (which leads to the derivation of the first-order Lagrangian \(L^{(f)}\)) is given by [13]

\[
L^{(f)}(z, \dot{z}) = z^A \Omega_{AB}(z) \dot{z}^B - H(z), \tag{2.7}
\]

where the general form of the covariant symplectic structure \(\Omega(z)\) is [13]

\[
\Omega(z) = \int_0^1 d\alpha \alpha \Omega(\alpha z). \tag{2.8}
\]
For our case of canonical symplectic structures (listed in (2.4) and satisfying (2.5)), the above formula yields \( \bar{\Omega}_{AB}(1) = \frac{1}{2} \Omega_{AB}(1) \). The first-order Lagrangian (modulo some total derivatives w.r.t. time), for our discussion, is

\[
L^f(z, \dot{z}) = \frac{1}{2} z^A \Omega^{AB}(1) \dot{z}^B - H_1(z),
\]

\[
\equiv p_i \dot{x}_i - \frac{1}{2} p_i p_i - \frac{1}{2} \left( \omega^2 + \frac{1}{4} k^2 \right) x_i x_i + \frac{1}{2} k \varepsilon_{ij} x_i p_j.
\]

The equations of motion, emerging from the above FOL (2.9), are

\[
\dot{x}_i = p_i + \frac{k}{2} \varepsilon_{ij} x_j, \quad \dot{p}_i = -\left( \omega^2 + \frac{1}{4} k^2 \right) x_i + \frac{1}{2} k \varepsilon_{ij} p_j.
\]

Combined together, the equations of motion (2.10) imply our starting equation of motion (2.2) that was derived from the Hamiltonian (2.1) by exploiting the canonical PBs of (2.3).

Now, inserting the equations of motions derived from FOL (2.9) into itself, we obtain the following second-order Lagrangian

\[
L^{(s)}(x, \dot{x}) = \frac{1}{2} \dot{x}_i \dot{x}_i + \frac{k}{2} \varepsilon_{ij} \dot{x}_i \dot{x}_j - \frac{1}{2} \omega^2 x_i x_i.
\]

A couple of comments are in order which would turn out to be quite handy for our further discussions in the later sections. First and foremost, the Euler-Lagrange equation of motion, emerging from (2.11), is nothing but our starting equation of motion (2.2). Second, the canonical momentum \( p_i \) defined from the above SOL in (2.11)

\[
p_i = \frac{\partial L^{(s)}(x, \dot{x})}{\partial \dot{x}_i} = \dot{x}_i - \frac{k}{2} \varepsilon_{ij} x_j,
\]

turns out to be the same as the one derived from the Hamilton’s equations of motion by exploiting the Hamiltonian function (2.1).

3 Nontrivial brackets: Hamiltonian and Lagrangian dynamics

Keeping the dynamical evolution (\( \ddot{x}_i = -\omega^2 x_i + k \varepsilon_{ij} \dot{x}_j \)) of the system intact, it is straightforward to check that the following new Hamiltonians

\[
H_2 = \frac{1}{2} P_i P_i + \frac{1}{2} \omega^2 x_i x_i,
\]

\[
H_3 = \frac{1}{2} P_i P_i + \frac{1}{2} \omega^2 X_i X_i,
\]

expressed in terms of the phase variables \((x_i, P_i)\) and \((X_i, p_i)\) lead to our starting equation of motion (2.2) if we exploit the following nontrivial PBs in the phase space

\[
\{x_i, x_j\}_{(PB)}^{(x,p)} = 0, \quad \{x_i, P_j\}_{(PB)}^{(x,p)} = \delta_{ij}, \quad \{P_i, P_j\}_{(PB)}^{(x,p)} = k \varepsilon_{ij},
\]

\end{enumerate}
\[
\{X_i, X_j\}_{(PB)}^{(x,p)} = \frac{k}{\omega^2} \varepsilon_{ij}, \quad \{X_i, p_j\}_{(PB)}^{(x,p)} = \delta_{ij}, \quad \{p_i, p_j\}_{(PB)}^{(x,p)} = 0. \quad (3.4)
\]

Here the new phase variables \((x_i, P_i)\) and \((X_i, p_i)\) parametrize the transformed versions of the cotangent manifolds that are obtained by exploiting a certain specific types of transformation on the original phase variables \((x_i, p_i)\) (which earlier described the original cotangent manifold). Furthermore, it is evident that the brackets in (3.3) and (3.4) are defined for the Hamiltonians in (3.1) and (3.2), respectively.

As far as the the nontrivial PBs in (3.3) and (3.4) are concerned, the noteworthy points, at this stage, are as follows. First, it is interesting to pin-point that the PB between the co-ordinates and momenta variables remain the same in the canonical brackets (2.3) as well as the nontrivial brackets (3.3) and (3.4) even though, as is clear, a certain specific types of transformation have been performed on the original phase space variables \((x_i, p_i)\) to make them the new pairs of variables \((x_i, P_i)\) and \((X_i, p_i)\), respectively. Second, for the Hamiltonian (3.1), it is the momenta variables \(P_i\) (describing a momentum plane in the transformed cotangent manifold) that are required to be noncommutative (vis-à-vis the brackets in (2.3)) for the dynamical equations of motion (2.2) to remain intact. Third, it is the coordinates \(X_i\) (defining a coordinate plane in the transformed cotangent manifold) that are required to possess noncommutative nature (vis-à-vis the brackets in (2.3)) for the starting equation of motion (2.2) to remain unchanged. Fourth, it can be checked that the following specific transformations on the phase variables

\[
p_i \rightarrow P_i = p_i + \frac{k}{\omega^2} \varepsilon_{ij} x_j, \quad x_i \rightarrow X_i = x_i, \quad H_1(z) \rightarrow H'_1(z) = H_2(z), \quad (3.5)
\]

\[
p_i \rightarrow P'_i = p_i, \quad x_i \rightarrow X'_i = x_i - \frac{k}{2\omega^2} \varepsilon_{ij} p_j, \quad \tilde{H}_1(z) \rightarrow \tilde{H}'_1(z) = H_3(z), \quad (3.6)
\]

relate \(H_1(z)\) to \(H_2(z)\) and \(\tilde{H}_1(z)\) to \(H_3(z)\) where (i) the phase variables \(z\)’s come in different guises for the different types of Hamiltonian, and (ii) the Hamiltonian \(\tilde{H}_1(z)\)

\[
\tilde{H}_1(z) = \frac{1}{2} \left(1 + \frac{k^2}{4 \omega^2}\right) p_i p_i + \frac{1}{2} \omega^2 x_i x_i - \frac{k}{2} \varepsilon_{ij} x_i p_j, \quad (3.7)
\]

is obtained from \(H_1(z)\) by the following canonical transformations

\[
\begin{align*}
x_i &\rightarrow X_i = \omega^{-1} p_i, \quad p_i \rightarrow P_i = -\omega x_i, \\
\{X_i, P_j\}_{(PB)}^{(x,p)} &= \delta_{ij}, \quad \{X_i, X_j\}_{(PB)}^{(x,p)} = \{P_i, P_j\}_{(PB)}^{(x,p)} = 0, \quad (3.8)
\end{align*}
\]

which show that \(H_1(z)\) and \(\tilde{H}_1(z)\) are canonically equivalent. Fifth, the transformations in (3.5) and (3.6) imply the noncommutativity present in (3.3) and (3.4). Sixth, the canonical equivalence of \(H_1(z)\) and \(\tilde{H}_1(z)\) demonstrates that the noncommutativity of (3.3) and (3.4) are equivalent too. Seventh, it is obvious that, under the above transformations, the
sum of the area elements in the momentum phase (cotangent) space remain invariant (i.e. 
\( dX_i \, dP_i = dx_i \, dp_i \)) for the transformations in (3.5) and (3.6).

With the nontrivial brackets defined in (3.3), it can be seen that the analogues of the
symplectic structures in (2.4) (and satisfying (2.5)) for the transformed cotangent manifold
are
\[
\Omega^{AB}(2) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & k \\ 0 & -1 & -k & 0 \end{pmatrix}, \quad \Omega_{AB}(2) = \begin{pmatrix} 0 & k & -1 & 0 \\ -k & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},
\]
(3.9)

where the notation \( z^A = (z^1, z^2, z^3, z^4) = (x_1, x_2, P_1, P_2) \) and the definitions of (2.5) have
been taken into account. In terms of the covariant symplectic structure \( \Omega_{AB}(2) \), defined
on the transformed cotangent manifold, we obtain the following FOL (modulo some total
derivatives w.r.t. time) for the case under consideration; namely,
\[
L_2^{(f)}(z, \dot{z}) = \frac{1}{2} z^A \Omega_{AB}(2) \dot{z}^B - H_2(z),
\]
\[
\equiv p_i \dot{x}_i + \frac{1}{2} k \varepsilon_{ij} x_i \dot{x}_j - \frac{1}{2} p_i p_i - \frac{1}{2} \omega^2 x_i x_i.
\]
(3.10)

The following equations of motion emerge from the above FOL
\[
P_i = \dot{x}_i, \quad \dot{P}_i = -\omega^2 x_i + k \varepsilon_{ij} \dot{x}_j,
\]
(3.11)

which, ultimately, lead to the derivation of the dynamical equation of motion (2.2), we
started with. It will be noticed that the above equations of motion (3.11) are quite different
in appearance from their counterparts in (2.10) which have been derived from the FOL \( L_2^{(f)} \).

Now let us concentrate on the nontrivial brackets defined in (3.4). Here the analogues
of the symplectic structures in (2.4) (that satisfy (2.5)) are
\[
\Omega^{AB}(3) = \begin{pmatrix} 0 & (k/\omega^2) & 1 & 0 \\ -(k/\omega^2) & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad \Omega_{AB}(3) = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & (k/\omega^2) \\ 0 & 1 & -(k/\omega^2) & 0 \end{pmatrix},
\]
(3.12)

where the notation \( z^A = (z^1, z^2, z^3, z^4) = (X_1, X_2, p_1, p_2) \) has been chosen in the analogues
of the definitions (2.5) for the case under consideration. With the help of the covariant
symplectic structure in (3.12), one can obtain the FOL \( L_3^{(f)} \) (modulo some total derivatives
w.r.t. time) as
\[
L_3^{(f)}(z, \dot{z}) = \frac{1}{2} z^A \Omega_{AB}(3) \dot{z}^B - H_3(z),
\]
\[
\equiv p_i \dot{X}_i + \frac{k}{2\omega^2} \varepsilon_{ij} p_i \dot{p}_j - \frac{1}{2} p_i p_i - \frac{1}{2} \omega^2 X_i X_i.
\]
(3.13)
The above Lagrangian leads to the derivation of the Euler-Lagrange equations of motion that are first-order in time derivative on $X_i$ and $p_i$. These $\dot{X}_i$ and $\dot{p}_i$ are intertwined together in one equation as

$$\dot{X}_i = p_i - \frac{k}{\omega^2} \varepsilon_{ij} \dot{p}_j, \quad \dot{p}_i = -\omega^2 X_i.$$  (3.14)

Combined together, the above equations of motion lead to the derivation of the same dynamical equations of motion (i.e. $\ddot{X}_i = -\omega^2 X_i + k \varepsilon_{ij} \dot{X}_j$) as we started with in (2.2). It will be noticed that even though the equations of motion are expressed, ultimately, in $X_i, \dot{X}_i, \ddot{X}_i$ (which are different from $x_i, \dot{x}_i, \ddot{x}_i$ in (2.2)), the evolution of the dynamics of the physical system w.r.t. time is the same as in (2.2). It is very interesting to check that the SOL (that emerges from the FOL (3.13)) has exactly the same appearance as the one in (2.11), albeit expressed in terms of $X_i$ and $\dot{X}_i$.

A few comments are in order as far as the transformations (3.5) and (3.6) of the phase variables (defining the transformed cotangent space) are concerned. It should be noticed that, unlike transformations (3.8), these transformations are not exactly canonical in nature even though they preserve the invariance of the sum of the area elements in the cotangent manifold. To be precise, it should be noted that, for a transformation $(x_i, p_i) \rightarrow (X_i, P_i)$, to be canonical [14], the following basic PBs defined in the transformed cotangent manifold

$$\{X_i, P_j\}^{(x,p)}_{(PB)} = \delta_{ij}, \quad \{X_i, X_j\}^{(x,p)}_{(PB)} = 0, \quad \{P_i, P_j\}^{(x,p)}_{(PB)} = 0,$$  (3.15)

should retain their form. Here the superscripts on the brackets denote the definition of the PBs w.r.t. the original phase variables $(x_i, p_i)$. It is straightforward to check that the above brackets are not preserved under the transformations (3.5) and (3.6) if we exploit the basic PBs defined in (2.3). Thus, the latter transformations are a very special class of transformations in the phase space which are not canonical in nature per se but they preserve the sum of the areas in the phase space.

4 Deformed $q$-brackets: connection with quantum groups

In this section, we briefly recapitulate some of the key and pertinent points of our earlier works [15,16] on the consistent construction of the dynamics in the noncommutative phase space where the noncommutativity (i.e. $q$-deformation) was introduced in the $q$-deformed $2N$ dimensional cotangent manifold corresponding to a given $N$-dimensional configuration manifold. In our present context (where we are discussing a 2D physical system), the following relationships among the phase variables $(x_i, p_i)$ (with $i = 1, 2$) on the 4D cotangent manifold

$$x_i x_j = x_j x_i, \quad p_i p_j = p_j p_i, \quad x_i p_j = q p_j x_i,$$  (4.1)

play a very crucial and decisive role in our whole discussion of the dynamics. In fact, the above $q$-deformation is chosen in the phase space so that the quantum group invariance
and the ordinary rotational invariance can be maintained together \footnote{This deformation can be easily generalized to the 2N-dimensional Minkowski cotangent manifolds for the dynamical discussion of a physical relativistic system where the quantum group invariance and Lorentz invariance are respected together for any arbitrary ordering of the spacetime indices \cite{15,16}.} for any arbitrary ordering of the indices \(i\) and \(j\) (see, e.g., \cite{15} for details). We shall concentrate here on the following general transformations on the phase variables \((x_i, p_i)\) defined on the 4D \(q\)-deformed cotangent manifold

\[
x_i \rightarrow X_i = A x_i + B p_i, \quad p_i \rightarrow P_i = C x_i + D p_i. \tag{4.2}
\]

In the above, the original phase variables \((x_i, p_i)\) and the transformed sets of the phase variables \((X_i, P_i)\) are assumed to commute with the elements \(A, B, C, D\) of a \(2 \times 2\) matrix corresponding to the quantum group \(GL_{q,p}(2)\). Here the deformation parameters \(q, p\) are the non-zero complex numbers (i.e. \(q, p \in \mathbb{C}/\{0\}\)) corresponding to the most general quantum group deformation of the ordinary general linear group \(GL(2)\) of \(2 \times 2\) nonsingular matrices.

In fact, the elements of the quantum group \(GL_{q,p}(2)\) obey (see, e.g., \cite{17} for details)

\[
AB = pBA, \quad AC = qCA, \quad BC = (q/p)CB, \quad BD = qDB, \\
CD = pDC, \quad AD - DA = (p - q^{-1})BC = (q - p^{-1})CB. \tag{4.3}
\]

It can be checked that the relationships in (4.1) remain form-invariant under the general transformations (4.2) \emph{only for the choice} \(pq = 1\). In other words, the \(q\)-deformation (4.1) in the 4D cotangent manifold remains form-invariant under the transformations generated by the quantum group \(GL_{q,q^{-1}}(2)\). It should be noted that these transformations are different from the transformations generated by the single parameter \(q\)-deformed quantum group \(GL_q(2)\). In fact, the latter is the limiting case of \(GL_{q,p}(2)\) when \(q = p\). Under the restriction \(pq = 1\), the relations (4.3) yield the following braiding relationships among the rows and columns

\[
AB = q^{-1}BA, \quad AC = qCA, \quad BC = q^2 CB, \quad BD = qDB, \\
CD = q^{-1}DC, \quad AD = DA. \tag{4.4}
\]

At this stage, it is important to note that it is the \emph{individual pairs} of the original phase variables \((x_i, p_i)\) (with \(i = 1, 2\)) that change to the transformed phase variables \((X_i, P_i)\) (with \(i = 1, 2\)) under the quantum group \(GL_{q,q^{-1}}(2)\). Thus, for our present discussion, there are two sets of quantum groups \(GL_{q,q^{-1}}(2)\) which are responsible for the transformations of the original two pairs of phase variables \((x_1, p_1)\) and \((x_2, p_2)\) in (4.2). This statement can be mathematically expressed, in the matrix form, as given below

\[
\begin{pmatrix} x_1 \\ p_1 \end{pmatrix} \rightarrow \begin{pmatrix} X_1 \\ P_1 \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} x_1 \\ p_1 \end{pmatrix}, \\
\begin{pmatrix} x_2 \\ p_2 \end{pmatrix} \rightarrow \begin{pmatrix} X_2 \\ P_2 \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} x_2 \\ p_2 \end{pmatrix}, \tag{4.5}
\]

where \(A, B, C, D\) are the elements of the quantum group \(GL_{q,q^{-1}}(2)\) that satisfy (4.4).
Now the stage is set for establishing the connection between the noncommutativity discussed in the previous sections and the noncommutativity due to the quantum groups in the present section. As argued in the paragraph after (3.4), one of the key as well as crucial features of the transformations in (3.5) and (3.6) is the invariance of the sum of the phase space areas. In the language of the $GL_{q,q^{-1}}(2)$ invariant differential geometry defined on the 4D cotangent manifold (see, e.g., [15] for details), this statement can be mathematically captured in the following wedge products and their transformations

\[
\begin{align*}
(dx_i \wedge dx_i) & \rightarrow (dX_i \wedge dX_i) = 0, \\
(dx_i \wedge dp_i) & \rightarrow (dX_i \wedge dP_i) = 0,
\end{align*}
\]

where, in addition to (4.4), we have used the following relationships [15]

\[
\begin{align*}
(dx_i \wedge dx_i) = 0, & \quad (dp_i \wedge dp_i) = 0, & \quad (dx_i \wedge dp_i) = -q (dp_i \wedge dx_i). 
\end{align*}
\]

This shows that, for the area preserving phase space transformations in (4.6), we have

\[
AD - q^{-1}BC = AD - qCB = DA - q^{-1}BC = DA - qCB = 1,
\]

which is nothing but setting the $q$-determinant (see, e.g., [17] for details) of the $2 \times 2$ $GL_{q,q^{-1}}(2)$ matrix equal to one. In more sophisticated language, the requirement of the area preserving phase transformations in (4.5) entails upon the quantum group $GL_{q,q^{-1}}(2)$ to become $SL_{q,q^{-1}}(2)$ as the $q$-determinant $AD - q^{-1}BC = 1$. From the point of view of the structure of the PBs defined in the canonical form (cf. (2.3)) and the nontrivial forms (3.3) and (3.4), it can be seen that the crucial common feature is

\[
\{x_i, p_j\}_{(PB)}^{(x,p)} = \delta_{ij}, \quad \{x_i, P_j\}_{(PB)}^{(x,p)} = \delta_{ij}, \quad \{X_i, P_j\}_{(PB)}^{(x,p)} = \delta_{ij}.
\]

In other words, the PBs between the original canonically conjugate variables $(x_i, p_i)$ and the transformed phase variables $(X_i, P_j)$ and $(x_i, P_j)$ remain unchanged. For the same to remain sacrosanct even in the case of quantum group transformations in (4.5) (or (4.2)), it can be seen that the following condition

\[
\{X_i, P_j\}_{(PB)}^{(q)} = (AD - qBC) \delta_{ij} \rightarrow \left\{X_i, P_j\right\}_{(PB)}^{(q)} = \delta_{ij},
\]

has to be satisfied. In the derivation of the above, we have exploited the following $q$-deformed basic Poisson brackets for the phase variables $(x_i, p_i)$ [15]

\[
\begin{align*}
\{x_i, p_j\}_{(PB)}^{(q)} = \delta_{ij}, & \quad \{p_i, x_j\}_{(PB)}^{(q)} = -q \delta_{ij}, & \quad \{p_i, P_j\}_{(PB)}^{(q)} = \{x_i, x_j\}_{(PB)}^{(q)} = 0,
\end{align*}
\]

which crucially depend on the choice of the symplectic structures on the $q$-deformed 4D cotangent manifold. In our earlier work [15], we have chosen the following

\[
\Omega^{AB}(q) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -q \\ -q & 0 & 0 & 0 \\ 0 & -q & 0 & 0 \end{pmatrix}, \quad \Omega_{AB}(q) = \begin{pmatrix} 0 & 0 & -q^{-1} & 0 \\ 0 & 0 & 0 & -q^{-1} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.
\]
It is now unequivocally clear that the requirement of the equality and consistency between (4.9) and (4.10) imposes the following restriction on the elements of the $GL_{q,q^{-1}}(2)$:

$$AD - qBC = AD - q^3CB = DA - qBC = DA - q^3CB = 1. \quad (4.13)$$

This shows that, for a theory in which the canonical brackets (4.9) and the area preserving transformations (i.e., $dX_i \wedge dP_i = dx_i \wedge dp_i$) are respected together, we obtain a restriction $q^2 = 1$ on the deformation parameter. This condition emerges from the equality of (4.8) and (4.13). With the $q$-brackets defined in (4.11), it is straightforward to check (using the $q$-commutation relations (4.4)) that the following brackets, defined in the 2D coordinate as well as the momentum plane of the 4D $q$-deformed cotangent manifold, are true; namely,

$$\{X_i, X_j\}_{(PB)}^{(q)} = (AB - qBA) \delta_{ij} \equiv (1 - q^2) AB \delta_{ij},$$

$$\{P_i, P_j\}_{(PB)}^{(q)} = (CD - qDC) \delta_{ij} \equiv (1 - q^2) CD \delta_{ij}. \quad (4.14)$$

These $q$-brackets turn out to be zero for $q^2 = 1$. Thus, it is clear that the NC structure in the 2D coordinate as well as momentum planes (cf. (3.3) and (3.4)) does not exist for our quantum group considerations if we require the consistency between the $q$-deformed PBs and the area preserving $SL_{q,q^{-1}}(2)$ invariant phase space transformations. Hence, it is obvious that for $q^2 = 1$, the quantum group considerations allow the phase space transformations that are exactly canonical in nature.

5 Conclusions

In the present investigation, we have established (i) a connection between the canonical PBs and the nontrivial NC brackets, and (ii) the relation between the canonical Hamiltonian and the nontrivial Hamiltonians. These connections emerge because of the existence of a special class of symmetry transformations (cf. (3.5) and (3.6)) on the phase variables that preserves the sum of the areas on the (un-)transformed cotangent manifolds. This symmetry transformation is, however, shown not to be exactly canonical in nature unlike the canonical transformations of (3.8) which leave the canonical brackets form-invariant. As far as the dynamics on the 4D tangent manifold is concerned, it remains unaffected by the NC structures present in the PBs equivalently defined on the 2D coordinate or momentum planes that are embedded in the 4D cotangent manifolds. This fact is captured in the appearance of the second-order Lagrangian $L^{(s)}(x, \dot{x})$ which remains unchanged irrespective of the noncommutativity present in the PBs defined on the 4D cotangent manifold. The above cited “sum of the area preserving transformation” is also reflected in the realm of quantum group transformations on the phase variables where the transformations, generated by $GL_{q,q^{-1}}(2)$, are restricted to become the transformations corresponding to the quantum group $SL_{q,q^{-1}}(2)$. The requirement of the consistency of the above transformations with the structure of the $q$-deformed PBs, however, puts a restriction $q^2 = 1$ on the deformation.
parameter. It is interesting to point out that, in a completely different context, exactly the same restriction (i.e. $q = \pm 1$) has been shown to exist for the requirement of equivalence between the gauge- and reparametrization invariances for a $q$-deformed relativistic (super-)particle moving in an undeformed $D$-dimensional target space [16].

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