Time-Dependent Random Walks and the Theory of Complex Adaptive Systems

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Motivated by novel results in the theory of complex adaptive systems, we analyze the dynamics of random walks in which the jumping probabilities are time-dependent. We determine the survival probability in the presence of an absorbing boundary. For an unbiased walk the survival probability is maximized in the case of large temporal oscillations in the jumping probabilities. On the other hand, a random walker who is drifted towards the absorbing boundary performs best with a constant jumping probability. We use the results to reveal the underlying dynamics responsible for the phenomenon of self-segregation and clustering observed in the evolutionary minority game.

Random walk is one of the most ubiquitous concepts of statistical physics. In fact, it finds applications in virtually every area of physics (see e.g., [1–6] and references therein). Random walks in the presence of absorbing traps are much studied in recent years as models for absorbing-state phase transitions [7,8], polymer adsorption [9], granular segregation [10], and the spreading of an epidemic [11].

In this Letter we analyze the problem of a random walk with an absorbing boundary, in which the jumping probabilities are time-dependent. In addition to the intrinsic interest of such time-dependent random walks, our study is motivated by the flurry of activity in the field of complex adaptive systems.

Agent-based models of complex adaptive systems are attracting significant interest across many disciplines [12]. They find applications in social, biological, physical, and economic sciences. Of particular interest are situations in which members compete for a limited resource, or to be in a minority [13]. In financial markets for instance, more buyers than sellers implies higher prices, and it is therefore better for a trader to be in a minority group of sellers. Predators foraging for food will do better if they hunt in areas with fewer competitors. Rush-hour drivers, facing the choice between two alternative routes, wish to choose the route containing the minority of traffic [14].

One of the most studied models in the field of complex adaptive systems is the minority game (MG) [15], and its evolutionary version (EMG) [13] (see also [16–28] and references therein). The game describes agents who each make a binary decision (e.g., “to buy”/“to sell”), or “taking route A”/“taking route B”) at every point in the game. Profit is made by those who find themselves in the minority group, i.e., who end up selling when most wish to buy, or vice versa. Each winner gains R points (the “prize”), while agents belonging to the majority group lose 1 point (the “fine”). The agents have a common “memory” look-up table, containing the outcomes of recent occurrences. Faced with a given bit string of recent occurrences, each agent chooses the outcome in the memory with probability g, known as the agent’s “gene” value. If an agent score falls below some value d, then its strategy (i.e., its gene value) is modified. In other words, each agent tries to learn from his past mistakes, and to adjust his strategy in order to survive.

Early studies of the EMG were restricted to situations in which the prize-to-fine ratio R was assumed to be equal unity. A remarkable conclusion deduced from the EMG [13] is that a population of competing agents tends to self-segregate into opposing groups characterized by extreme behavior. It was realized that in order to flourish in such situations, an agent should behave in an extreme way (g = 0 or g = 1) [13].

On the other hand, in many real life situations the prize-to-fine ratio may take a variety of different values [25]. A different kind of strategy may be more favorite in such situations. In fact, we know from real life situations that extreme agents not always perform better than cautious ones. In particular, our daily experience indicates that in difficult situations (e.g., when the prize-to-fine ratio is low) human people tend to be confused and indecisive. In such circumstances they usually seek to do the same (rather than the opposite) as the majority.

Based on this qualitative expectation, we have recently extended the exploration of the EMG to include situations in which the prize-to-fine ratio R differs from unity. It was found [25] that an intriguing phase transition exist in the model: “confusion” and “indecisiveness” take over when the prize-to-fine ratio falls below some critical value, in which case “cautious” agents (characterized by $g = \frac{1}{2}$) perform better than extreme ones. In such circumstances agents tend to cluster around $g = \frac{1}{2}$ (see Fig. 1 of Ref. [25]) rather than self-segregate into two opposing groups.

Moreover, it has been demonstrated [25,27] that the population of evolving agents never establishes a genuine stationary distribution. In fact, the probability of a particular agent to win oscillates in time. This fact has been overlooked in former studies of the EMG. The score of each agent may therefore be described by a random walk with time-dependent jumping (winning) probabilities. Thus, the problem of random walkers whose jumping probabilities are time-dependent is of great interest for the understanding of the dynamics of complex adaptive
systems. The aim of the present Letter is to analyze this problem, and to provide an analytical explanation for the phase-transition observed in such systems.

An unbiased random walk. We study a discrete-time random walk on the nonnegative integers, \( x_t = 0, 1, 2, \ldots \), with the origin absorbing. Initially the walker is at \( x_0 = 1 \) (We shall later generalize our results to include situations in which \( x_0 > 1 \)). The probability \( p(t) \) to step to the right is time-dependent and is given by \( p(t) = \frac{1}{2} - (-1)^t \varepsilon \), where \( |A| \) is the amplitude of the temporal oscillations in the jumping probabilities \((-\frac{1}{2} < A < \frac{1}{2})\).

We denote by \( y \) the location of the rightmost site yet visited. The probability \( P(x, y, t) \) for a random walker to be at a position \( x \) at time \( t \) follows the evolution equation

\[
P(x, y, t + 2) = [(\frac{1}{2} + A)^2 + (\frac{1}{2} - A)^2]P(x, y, t) + (\frac{1}{2} + A)(\frac{1}{2} - A)[P(x - 2, y, t) + P(x + 2, y, t)] ,
\]

(1) for \( x \leq y - 2 \). We also have \( P(0, y, t) = 0 \), representing the absorbing boundary at the origin. Denoting \( D(y, t) \equiv P(y, y, t) \), the additional boundary conditions are

\[
D(y, t + 2) = (\frac{1}{4} - A^2)[P(y - 2, y, t) + P(y - 2, y - 1, t) + D(y - 2, t)] + [\frac{1}{2} + (-1)^y A^2]D(y, t) ,
\]

(2)

\[
P(y - 2, y, t + 2) = (\frac{1}{4} - A^2)[P(y - 4, y, t) + D(y, t)] + 2(\frac{1}{4} + A^2)P(y - 2, y, t) ,
\]

(3)

\[
P(y - 2, y - 1, t + 2) = (\frac{1}{4} - A^2)P(y - 4, y - 1, t) + 2(\frac{1}{4} + A^2)P(y - 2, y - 1, t) + \frac{1}{2} - (-1)^y A^2 D(y - 2, t) .
\]

(4)

It proofs useful to define the generating function \( \hat{P}(x, y, z) = \sum_{t=0}^{\infty} z^t P(x, y, t) \) (and similarly for \( D \)). Multiplying Eqs. (1)-(4) by \( z^t \) and summing over \( t \), one finds that \( \hat{P}(x, y, z) \) satisfies the recursion relation

\[
z^{-2} \hat{P}(x, y) = (\frac{1}{4} - A^2)[\hat{P}(x - 2, y) + \hat{P}(x + 2, y)] + 2(\frac{1}{4} + A^2)\hat{P}(x, y) ,
\]

(5)

(we drop the argument \( z \) for brevity), subject to the boundary conditions

\[
z^{-2} \hat{D}(y) = (\frac{1}{4} - A^2)[\hat{P}(y - 2, y) + \hat{P}(y - 2, y - 1) + \hat{D}(y - 2)] + [\frac{1}{2} + (-1)^y A^2] \hat{D}(y) ,
\]

(6)

\[
z^{-2} \hat{P}(y - 2, y) = (\frac{1}{4} - A^2)[\hat{P}(y - 4, y) + \hat{D}(y)] + 2(\frac{1}{4} + A^2)\hat{P}(y - 2, y) ,
\]

(7)

and

\[
z^{-2} \hat{P}(y - 2, y - 1) = (\frac{1}{4} - A^2)\hat{P}(y - 4, y - 1) + 2(\frac{1}{4} + A^2)\hat{P}(y - 2, y - 1) + [\frac{1}{2} - (-1)^y A^2] \hat{D}(y - 2) .
\]

(8)

The solution of Eq. (5) subject to the boundary condition at the origin, \( \hat{P}(0, y) = 0 \), is \( \hat{P}(x, y) = (\lambda^x - \lambda^{-x})B(y) \), where \( \lambda \) is a function of \( z \) and \( A \). To determine the survival probability as \( t \to \infty \), we shall analyze the behavior of \( \hat{P}(x, y, z) \) as \( z \to 1 \). In this limit

\[
C(x) \equiv \lambda^x - \lambda^{-x} = 2 \sinh \left( \frac{\epsilon}{2(\frac{1}{4} - A^2)} \right) ,
\]

(9)

where \( \epsilon = 1 - z \).

Substituting Eqs. (7) and (8) in Eq. (6) one obtains the recursion relation

\[
\left[(2 + q) \frac{\hat{C}(y - 2)}{C(y - 4)} - (1 + q)\right] \hat{B}(y) = \frac{1}{2} q \hat{B}(y - 1) + \frac{1}{2} (1 + q) \frac{\hat{C}(y - 2)}{C(y - 4)} \hat{B}(y - 2) ,
\]

(10)

whose asymptotic solution is

\[
\hat{B}(y) = \frac{1}{\sinh^2(\sqrt{s\epsilon})} \frac{1}{\sqrt{1 + (-1)^y A^2}} ,
\]

(11)

where \( q \equiv \frac{1}{2} + (-1)^y A \) and \( s \equiv \frac{1}{2(\frac{1}{4} - A^2)} \).

The survival probability is determined by \( S(t) = \sum_{y=0}^{\infty} \sum_{x=0}^{y} P(x, y, t) \). Taking cognizance of Eqs. (9) and (11) one finds that the singular behavior of its generating function, \( \hat{S}(z) \), is dominated by

\[
\hat{S}(z) = \sum_{y=0}^{\infty} \hat{B}(y) \sum_{x=0}^{y} \hat{C}(x) = \sqrt{\frac{2(\frac{1}{2} + A)}{2 - A}} e^{-\frac{\epsilon}{2}} .
\]

(12)

Nothing that for large \( t \), the coefficient of \( z^t \) in \( (1 - z)^{-\frac{1}{2}} \) is \( 1/\sqrt{\pi t} \), we have that the asymptotic behavior of the survival probability is

\[
S(t; x_0 = 1) = \sqrt{\frac{\frac{1}{2} + A}{2 - A}} \sqrt{\frac{2}{\pi}} e^{-\frac{\epsilon}{2}} .
\]

(13)

Finally, it is straightforward to prove the recursion relations \( S(t; x_0 = 2n + 1, A) = (1 + \frac{\epsilon}{2 - A})S(t; x_0 = 1, A) \) and \( S(t; x_0 = 2n, A) = (\frac{1}{2 - A})S(t; x_0 = 1, -A) \). Thus,
the survival probability for an arbitrary value of the initial position \(x_0\) is given by

\[
S(t) = \frac{x_0 + [1 - (-1)^x_0]A}{\sqrt{1 - 4A^2}} \sqrt{\frac{2}{\pi t}}. \tag{14}
\]

We therefore conclude that an unbiased random walker whose jumping probabilities display large temporal oscillations (i.e., \(|A| \simeq \frac{1}{2}\)) has the largest survival probability. On the other hand, the survival probability \(S(t)\) has a minimum at \(A = 0\) for even values of \(x_0\) (corresponding to a fixed, time-independent jumping probability), and at \(A = \frac{1}{2x_0}\) for odd values of \(x_0\).

Figure 1 displays the survival probability as a function of \(A\), the amplitude of the temporal oscillations in the jumping probabilities. We find an excellent agreement between the analytically predicted results [see Eq. (14)] and the numerical ones.

![Figure 1](image)

**FIG. 1.** The survival probability \(S(t;x_0)\sqrt{\frac{2}{\pi t}}\) as a function of the amplitude of the temporal oscillations in the jumping probabilities. We display results for different values of the initial location: \(x_0 = 1, 2, 3, 4\). Theoretical results are represented by solid curves.

A biased random walk. We next consider situations in which the step length to the left is larger than the step length \(R\) to the right (where \(R \leq 1\)). Thus, the walker is drifted towards the absorbing boundary. We shall also generalize the analysis to include situations in which the jumping probability to the left is different from the corresponding jumping probability to the right. The (time-dependent) jumping probability to the right is now given by \(p(t) = \frac{1}{2} - \varepsilon - (-1)^t A\), where \(-\frac{1}{2} \leq \varepsilon < A < \frac{1}{2} - \varepsilon\).

In order to survive under such conditions, the walker must step to the right more times than he steps to the left. More precisely, at least a fraction \(f = \frac{t^+}{t}\) of his steps must be to the right (at any given point in the walk). The chance for the mean number of right-steps to deviate from its average is exponentially decreasing with time. Using an analysis along the same lines as before, one finds that the asymptotic large \(t\) limit of the survival probability is now given by [29]

\[
S(t) = bt^2 e^{-Ft}, \tag{15}
\]

where the associated entropy function (or large-deviation function) reads

\[
F = \frac{1}{2} \left[ f_+ \ln \left( \frac{f_+}{p_+} \right) + (1 - f_+) \ln \left( \frac{1 - f_+}{1 - p_+} \right) \right] + f_- \ln \left( \frac{f_-}{p_-} \right) + (1 - f_-) \ln \left( \frac{1 - f_-}{1 - p_-} \right), \tag{16}
\]

with \(p_\pm = \frac{1}{2} - \varepsilon \pm A\), and

\[
f_\pm = f \pm \left[ 1 - 4\varepsilon^2 + 4A^2 \right] - \sqrt{\left(1 - 4\varepsilon^2 + 4A^2\right)^2 - 64f(1 - f)A^2}/8A. \tag{17}
\]

The explicit expression of the prefactor \(b = b(R)\) is not important for the analysis.

The survival probability at late times is dominated by the exponential factor \(e^{-Ft}\). We note that \(e^{-F}\) [and thus also \(S(t)\)] is a monotonic decreasing function of \(A\). One therefore concludes that a biased random walker (one who is drifted towards the absorbing boundary) with a constant jumping probability (i.e., with \(A = 0\)) has the largest survival probability. On the other hand, a (biased) random walker with large temporal oscillations in his jumping probabilities has the poorest chances of survival.

**Complex adaptive systems.** We now apply the results derived in the present work to provide an explanation for the intriguing phase-transition observed in the EMG (transition from self-segregation to clustering (with an intermediate M-shaped phase), as the value of the prize-to-fine ratio \(R\) falls below some critical value).

As mentioned, it has been demonstrated [25] that the population of evolving agents never establishes a true stationary distribution. In fact, the probability of a particular agent to win (to step to the right in the terminology of this Letter) oscillates in time, the amplitude being dependent on the particular gene-value of the agent [27]. “Extreme” agents (with \(g = 0, 1\)) have an oscillation amplitude which is larger than the corresponding amplitude of “cautious” agents (those with \(g = \frac{1}{2}\)). In fact, the winning probability of \(g \simeq \frac{1}{2}\) agents is practically constant in time. We therefore write \(A^2 = a^2(R)(g - \frac{1}{2})^2\) [33].

One should also take into account the self-interaction (or so-called market impact in financial market terminology) that agents in such systems experience [18]. An agent has a smaller probability of winning when participating in the game as compared to an outsider, someone whose action does not affect the outcome. The self-interaction term has the form \(\varepsilon(g) = \frac{a^2}{\sqrt{N}} g(1 - g)\) [18].
Substituting the expressions for $A(g)$ and $\varepsilon(g)$ in Eq. (16) one finds that $e^{-F}$ [and thus also $S(t)$] has three qualitatively different forms, depending on the precise value of $R$ (the step-lengths ratio, or equivalently the prize-to-fine ratio). Figure 2 displays the function $e^{-F}$, which determines the long-time survival probability of the agents. This figure demonstrates the phase-transition from self-segregation to clustering observed in the evolutionary minority game [13,25] (compare, in particular, with the numerical results presented in Fig. 1 of [25]).

There is also an intermediate phase [for $R_c^{(2)} < R < R_c^{(1)}$], in which the survival probability has an M-shaped distribution. This behavior is a direct consequence of two opposing factors: (a) the fact that a biased random walker with a constant jumping probability ($A = 0$, or equivalently $g = \frac{1}{2}$) has the largest survival probability, and (b) the fact that the market impact (which decreases the winning probability) is the largest for $g = \frac{1}{2}$ agents. Note that the phase-transition from self-segregation to clustering becomes sharp as the number of agents $N$ increases. In fact, the intermediate (M-shaped) phase disappears in the $N \to \infty$ limit, in which case the transition occurs at $R_c^{(1)} = R_c^{(2)} = 1$ [see Eq. (18)].

In summary, in this Letter we have analyzed the dynamics of random walks with time-dependent jumping probabilities. In particular, we have calculated the survival probability of such walkers in the presence of an absorbing boundary. It was shown that the best strategy to be adopted by the walkers depends on the precise value of the step-lengths ratio $R$. In the unbiased case ($R = 1$) the survival probability is maximized by a walker who has large temporal oscillations in his jumping probabilities. On the other hand, a random walker who is drifted towards the absorbing boundary is better off keeping his jumping probabilities constant (i.e., time-independent).

Furthermore, we have shown that the results, when applied to the theory of complex adaptive systems, provide a direct analytical explanation for the phase-transition (from self-segregation to clustering) observed in the evolutionary minority game.

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