Risk measures and progressive enlargement of filtration: a BSDE approach

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Joint work with Emanuela Rosazza Gianin
Question:

How can we make a risk measure react to shocks in financial markets?
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Example:
Reference risk measure $\rightarrow$ Default event $\rightarrow$ Updated risk measure.
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Solution:
Progressive enlargement of filtration and BSDEs with jumps (BSDEJ).
Why BSDEs and progressive enlargement of filtrations?

**BSDEs**

- BSDEs allow to induce and/or represent dynamic risk measures specifying:
  1. A filtered probability space (i.e., a *probabilistic model*).
  2. A measurable map $g$, called *driver*.
  - Properties of the driver $\leftrightarrow$ properties of the risk measure.
  - The driver can be determined based on investor’s preferences, regulatory requirements, etc... 
- BSDEs flow property $\Rightarrow$ *time-consistency* (i.e., evaluation of risk is recursive).
- Numerical simulation of BSDEs.

**Progressive enlargement of filtration**

- *Reference* filtration $\mathbb{F}$: information available prior to a shock (e.g., default).
- *Progressively enlarged* filtration $\mathbb{G}$: information updated after shock (can be generalized to multiple events).
Why BSDEs? Comparison with literature

Nonlinear expectations and \( g \)-expectations:

- S. Peng. Backward SDE and related \( g \)-expectation. In Backward stochastic differential equations (Paris, 1995–1996), volume 364 of Pitman Res. Notes Math. Ser., pages 141–159. Longman, Harlow, 1997.

- F. Coquet, Y. Hu, J. Mémin, and S. Peng. Filtration-consistent nonlinear expectations and related \( g \)-expectations. Probab. Theory Related Fields, 123(1):1–27, 2002.

Representation of risk measures via BSDEs driven by a Wiener process:

- E. Rosazza Gianin. Risk measures via \( g \)-expectations. Insurance Math. Econom., 39 (1):19–34, 2006.

- P. Barrieu and N. El Karoui. Pricing, hedging, and designing derivatives with risk measures. In Carmona, R. (ed.) Indifference pricing: theory and applications, pages 77–144. Princeton University Press, Princeton, 2009.

Representation of risk measures via BSDEs driven by a Wiener process and a Poisson random measure:

- M. C. Quenez and A. Sulem. BSDEs with jumps, optimization and applications to dynamic risk measures. Stochastic Process. Appl., 123(8):3328–3357, 2013.
1. Mathematical setting
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1. Mathematical setting

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Probabilistic setting

We are given the following objects:

- Finite time horizon $T > 0$.
- A complete probability space $(\Omega, \mathcal{A}, \mathbb{P})$.
- A Borel-measurable set $E \subset \mathbb{R}^m$.
- A Wiener process $W = (W_t)_{t \in [0,T]}$.
- A pair of random variables $(\tau, \zeta) \in \mathbb{R}^+ \times E$. Can be generalized to multiple jumps.
- A random counting measure $\mu(dt \, de) := \delta_{(\tau,\zeta)}(dt \, de)$. 

Information flow:

Reference filtration $F = (F_t)_{t \in [0,T]}$: completed natural filtration generated by $W$.

Progressively enlarged filtration $G = (G_t)_{t \in [0,T]}$: completed natural filtration generated by $W$ and $\mu$.

Initially enlarged filtration $H = (H_t)_{t \in [0,T]}$: completed natural filtration generated by $W$ and $(\tau, \zeta)$. Notice that $F \subset G \subset H$. 

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Probabilistic setting

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Information flow:

- Reference filtration $\mathbb{F} = (\mathcal{F}_t)_{t \in [0,T]}$: completed natural filtration generated by $W$.
- Progressively enlarged filtration $\mathbb{G} = (\mathcal{G}_t)_{t \in [0,T]}$: completed natural filtration generated by $W$ and $\mu$.
- Initially enlarged filtration $\mathbb{H} = (\mathcal{H}_t)_{t \in [0,T]}$: completed natural filtration generated by $W$ and $(\tau, \zeta)$.

Notice that: $\mathbb{F} \subset \mathbb{G} \subset \mathbb{H}$. 
The fundamental assumption

The following assumption is essential for most of the following results.

Assumption (Density hypothesis\(^1\))

For any \(t \geq 0\), the conditional distribution of the pair \((\tau, \zeta)\) given \(\mathcal{F}_t\) is absolutely continuous with respect to the Lebesgue measure on \(\mathbb{R}^+ \times E\). In particular, there exists a strictly positive \((\mathbb{R}^+ \times E)\)-indexed \(\mathbb{F}\)-predictable random field \(\gamma\) such that

\[
P((\tau, \zeta) \in C \mid \mathcal{F}_t) = \int_C \gamma_t(\vartheta, e) \, d\vartheta \, de, \quad C \in \mathcal{B}(\mathbb{R}^+) \otimes \mathcal{B}(E), \ t \geq 0.
\]

\(^1\)It is related to Condition (A’) in: J. Jacod. Grossissement initial, hypothèse (H’) et théorème de Girsanov. In Lect. Notes. Math., volume 1118, pages 15–35. Springer-Verlag, 1985.
The fundamental decompositions

Lemma (Callegaro et al., 2013, ESAIM PS; Pham, 2010, SPA)

1. For any \( t \geq 0 \), a random variable \( \xi \) is \( \mathcal{G}_t \)-measurable if and only if it is of the form

\[
\xi(\omega) = \xi^0(\omega) \mathbb{1}_{t < \tau(\omega)} + \xi^1(\omega, \tau(\omega), \zeta(\omega)) \mathbb{1}_{t \geq \tau(\omega)},
\]

for some \( \mathcal{F}_t \)-measurable random variable \( \xi^0 \) and a \( \mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}^+) \otimes \mathcal{B}(E) \)-measurable function \( \xi^1 \).

2. A process \( Y = (Y_t)_{t \geq 0} \) is \( \mathcal{G} \)-predictable if and only if it is of the form

\[
Y_t = Y^0_t \mathbb{1}_{t \leq \tau} + Y^1_t(\tau, \zeta) \mathbb{1}_{t > \tau}, \quad t \geq 0,
\]

where \( Y^0 \) is an \( \mathcal{F} \)-predictable process and \( Y^1 \) is a \( (\mathbb{R}^+ \times E) \)-indexed \( \mathcal{F} \)-predictable random field.

Lemma (Pham, 2010, SPA; Song, 2014, ESAIM PS)

Under the Density hypothesis, any \( \mathcal{G} \)-optional process \( Y = (Y_t)_{t \geq 0} \) can be decomposed as

\[
Y_t = Y^0_t \mathbb{1}_{t < \tau} + Y^1_t(\tau, \zeta) \mathbb{1}_{t \geq \tau}, \quad t \geq 0,
\]

where \( Y^0 \) is an \( \mathcal{F} \)-optional process and \( Y^1 \) is a \( (\mathbb{R}^+ \times E) \)-indexed \( \mathcal{F} \)-optional random field.
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Let us define the following sets:

- \( S_\infty^G[a, b] \), real-valued \( G \)-progressive processes \( Y \) such that:
  \[
  \|Y\|_{S_\infty^G[a, b]} := \text{ess sup}_{t \in [a, b]} |Y_t| < \infty.
  \]

- \( L^2_G[a, b] \), \( \mathbb{R}^d \)-valued \( G \)-predictable processes \( Z \) such that:
  \[
  \|Z\|_{L^2_G[a, b]} := \left( \mathbb{E} \left[ \int_a^b |Z_t|^2 \, dt \right] \right)^{\frac{1}{2}} < \infty.
  \]

- \( L^2(\mu) \), real-valued \( E \)-indexed \( G \)-predictable processes \( U \) such that:
  \[
  \|U\|_{L^2(\mu)} := \left( \mathbb{E} \left[ \int_0^T \int_E |U_s(e)|^2 \mu(ds \, de) \right] \right)^{\frac{1}{2}} < \infty.
  \]

A triple \( (Y, Z, U) \in S_\infty^G[0, T] \times L^2_G[0, T] \times L^2(\mu) \) is a solution to the BSDEJ if it satisfies:

\[
Y_t = \xi + \int_t^T g(s, Y_s, Z_s, U_s(\cdot)) \, ds - \int_t^T Z_s \, dW_s - \int_t^T \int_E U_s(e) \mu(ds \, de), \quad t \in [0, T],
\]

where:

- \( \xi \) is a \( G_T \)-measurable r.v., the terminal condition.
- \( g : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \times B(E) \to \mathbb{R} \) is a measurable map, the driver.
The decomposition of the BSDEJ

We assume that for any \((y, z, u) \in \mathbb{R} \times \mathbb{R}^d \times B(E)^2\) the map \((\omega, t) \mapsto g(\omega, t, y, z, u)\) is \(\mathcal{G}\)-predictable. Hence the driver can be written (fundamental decomposition):

\[
g(t, y, z, u) = g^0(t, y, z, u)1_{t \leq \tau} + g^1(t, y, z, u, \tau, \zeta)1_{t > \tau}.
\]

The terminal condition can be always decomposed as:

\[
\xi = \xi^01_{T < \tau} + \xi^1(\tau, \zeta)1_{T \geq \tau}.
\]

Idea: solve the BSDEJ through a system of indexed Brownian BSDEs before and after the jump time \(\tau\).

We define \(B(E) := \{f : E \to \mathbb{R}, \text{ Borel-measurable}\}\) equipped with the pointwise convergence topology.
Existence and uniqueness

Immersion hypothesis: Any $\mathbb{F}$-martingale remains a $\mathbb{G}$-martingale.

Theorem (Kharroubi, Lim, 2014, J. Theoret. Prob.)

Under the Density hypothesis and the immersion hypothesis (plus other technical hypotheses), the BSDEJ admits a unique solution $(Y, Z, U)$ on $[0, T]$, where

\[
\begin{aligned}
Y_t &= Y_t^0 \mathbb{1}_{t<\tau} + Y_t^1(\tau, \zeta) \mathbb{1}_{t\geq\tau}, \\
Z_t &= Z_t^0 \mathbb{1}_{t\leq\tau} + Z_t^1(\tau, \zeta) \mathbb{1}_{t>\tau}, \\
U_t(\cdot) &= U_t^0(\cdot) \mathbb{1}_{t\leq\tau} = [Y_t^1(t, \cdot) - Y_t^0] \mathbb{1}_{t\leq\tau}.
\end{aligned}
\]

and $(Y^0, Z^0), (Y^1, Z^1)$ are the unique solutions to the BSDEs

\[
Y_t^1(\vartheta, e) = \xi^1(\vartheta, e) + \int_t^T g^1(s, Y_s^1(\vartheta, e), Z_s^1(\vartheta, e), 0, \vartheta, e) \, ds
- \int_t^T Z_s^1(\vartheta, e) \, dW_s, \quad \vartheta \wedge T \leq t \leq T, \quad (\vartheta, e) \in \mathbb{R}^+ \times E,
\]

\[
Y_t^0 = \xi^0 + \int_t^T g^0(s, Y_s^0, Z_s^0, Y_s^1(s, \cdot) - Y_s^0) \, ds - \int_t^T Z_s^0 \, dW_s, \quad 0 \leq t \leq T.
\]
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Dynamic risk measures

Let $\mathcal{D} \in \{F, G, H\}$.

**Definition (Dynamic risk measure)**

A $\mathcal{D}$-dynamic risk measure is a family $\rho := (\rho_t)_{t \in [0, T]}$ of $\mathcal{D}$-conditional risk measures $\rho_t$.

**Definition (Conditional risk measure)**

A $\mathcal{D}$-conditional risk measure is a map $\rho_t$ such that:

1. $\rho_t : L^\infty(\mathcal{D}_T) \to L^\infty(\mathcal{D}_t)$, for all $t \in [0, T]$;
2. $\rho_0$ is a static risk measure, i.e., a functional $\rho_0 : L^\infty(\mathcal{D}_T) \to \mathbb{R}$;
3. $\rho_T(\xi) = -\xi$, for all $\xi \in L^\infty(\mathcal{D}_T)$.
Dynamic risk measures

Let $\mathbb{D} \in \{\mathcal{F}, \mathcal{G}, \mathcal{H}\}$.

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3. $\rho_T(\xi) = -\xi$, for all $\xi \in L^\infty(\mathcal{D}_T)$.

**Example**

$\rho_t := \mathbb{E}[-\xi | \mathcal{F}_t]$, $\xi \in L^\infty(\mathcal{F}_T)$, $t \in [0,T]$, is a $\mathcal{F}$-conditional risk measure.
The induced dynamic risk measure

$L^\infty(G_T) \ni \xi \leadsto (Y^\xi, Z^\xi, U^\xi)$, unique solution of the BSDEJ. Define

$$\rho_t(\xi) := Y_t^{-\xi}, \quad t \in [0, T].$$

It is easy to show that $\rho = (\rho_t)_{t \in [0, T]}$ is a $\mathbb{G}$-dynamic risk measure.

**A note**

If $g(t, y, 0, 0) = 0$ for any $t \in [0, T]$ and any $y \in \mathbb{R}$, then

$$\rho_t(\xi) = \mathcal{E}_g(-\xi \mid G_t), \quad t \in [0, T],$$

where $\mathcal{E}_g(\cdot)$ denotes the $\mathbb{G}$-conditional $g$-expectation associated to the BSDEJ.
The \( \mathcal{G} \)-dynamic risk measure induced by the BSDEJ can be decomposed as follows.

**Proposition (C., Rosazza Gianin, 2020)**

Under the assumptions of the existence and uniqueness theorem, there exist a \( \mathcal{F} \)-dynamic risk measure \( \rho^0 := (\rho_t^0)_{t \in [0,T]} \) and a \( \mathcal{H} \)-dynamic risk measure \( \rho^1 := (\rho_t^1)_{t \in [0,T]} \) such that:

\[
\rho_t(\xi) = \rho^0_t(\xi^0) \mathbb{1}_{t < \tau} + \rho^1_t(\xi^0(\tau, \zeta)) \mathbb{1}_{t \geq \tau}, \quad t \in [0, T], \xi \in L^\infty(\mathcal{G}_T).
\]
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Dynamic risk measures, as previously defined, may not be sufficient to assess riskiness of financial positions in a meaningful way.

We can impose on dynamic risk measures some mathematical requirement to reflect financial motivations.
Properties of dynamic risk measures

1. **Zero-one law:** For all \( t \in [0, T] \) and all \( A \in \mathcal{G}_t \):

   \[
   \rho_t(\xi 1_A) = 1_A \rho_t(\xi), \quad \xi \in L^\infty(\mathcal{G}_T).
   \]

2. **Translation invariance:** For all \( t \in [0, T] \) and all \( \eta \in L^\infty(\mathcal{G}_t) \):

   \[
   \rho_t(\xi + \eta) = \rho_t(\xi) - \eta, \quad \xi \in L^\infty(\mathcal{G}_T).
   \]

3. **Positive homogeneity:** For all \( t \in [0, T] \) and all \( \eta \in L^\infty(\mathcal{G}_t), \eta \geq 0 \):

   \[
   \rho_t(\xi \eta) = \eta \rho_t(\xi), \quad \xi \in L^\infty(\mathcal{G}_T).
   \]

4. **Monotonicity:** For all \( \xi, \eta \in L^\infty(\mathcal{G}_T) \), with \( \xi \leq \eta \):

   \[
   \rho_t(\xi) \geq \rho_t(\eta), \quad t \in [0, T].
   \]

5. **Convexity:** For all \( \xi, \eta \in L^\infty(\mathcal{G}_T) \) and all \( \alpha \in [0, 1] \):

   \[
   \rho_t(\alpha \xi + (1 - \alpha)\eta) \leq \alpha \rho_t(\xi) + (1 - \alpha)\rho_t(\eta), \quad t \in [0, T].
   \]

6. **Fatou property:** For any sequence \( (\xi_n)_{n \in \mathbb{N}} \subset L^\infty(\mathcal{G}_T) \) and \( \xi \in L^\infty(\mathcal{G}_T) \) such that \( \xi_n \to \xi \):

   \[
   \rho_t(\xi) \leq \liminf_{n \to \infty} \rho_t(\xi_n), \quad t \in [0, T].
   \]

7. **Time-consistency:** For any \( \mathcal{G}_t \)-stopping time \( \sigma \leq T \), and \( \xi \in L^\infty(\mathcal{G}_T) \):

   \[
   \rho_t(\xi) = \rho_t(-\rho_{\sigma}(\xi)), \quad t \leq \sigma.
   \]
Proposition (C., Rosazza Gianin, 2020)

Under the assumptions of the existence and uniqueness theorem, the dynamic risk measure $\rho$ satisfies the following properties:

1. **Zero-one law** if either $g(t, 0, 0, 0) = 0$, $\mathbb{P}$-a.s., for all $t \in [0, T]$, or both $\rho^0$ and $\rho^1$ satisfy this property.

2. **Translation invariance** if either $g$ does not depend on $y$ or both $\rho^0$ and $\rho^1$ satisfy this property.

3. **Positive homogeneity** if either $g$ is positively homogeneous with respect to $(y, z, u)$, $\mathbb{P}$-a.s., for all $t \in [0, T]$, or both $\rho^0$ and $\rho^1$ satisfy this property.

4. **Monotonicity**.

5. **Convexity** if either $g$ is convex with respect to $(y, z, u)$, $\mathbb{P}$-a.s., for all $t \in [0, T]$, or both $\rho^0$ and $\rho^1$ satisfy this property.

6. **Strong time-consistency**.
Proposition (C., Rosazza Gianin, 2020)

Let the assumptions of the existence and uniqueness theorem hold. Let \( \bar{\xi}, \hat{\xi} \in L^\infty(\mathcal{G}_T) \) and denote by \((\bar{Y}, \bar{Z}, \bar{U})\) (resp. \((\hat{Y}, \hat{Z}, \hat{U})\)) the solution to the BSDEJ with driver \( g \) and terminal condition \( \bar{\xi} \) (resp. \( \hat{\xi} \)).

Suppose, moreover, that for each \((\vartheta, e) \in \mathbb{R}^+ \times E:\)
\[
\|\bar{Y}^0 - \hat{Y}^0\|_{S\infty[0,T]} \leq K^0 \|\bar{\xi}^0 - \hat{\xi}^0\|_{L^\infty},
\]
\[
\|\bar{Y}^1(\vartheta, e) - \hat{Y}^1(\vartheta, e)\|_{S\infty[\vartheta,T]} \leq K^1(\vartheta, e) \|\bar{\xi}^1(\vartheta, e) - \hat{\xi}^1(\vartheta, e)\|_{L^\infty},
\]

for some finite constants \( K^0, K^1(\vartheta, e) > 0 \), and that
\[
\sup_{(\vartheta, e) \in \mathbb{R}^+ \times E} K^1(\vartheta, e) \|\bar{\xi}^1(\vartheta, e) - \hat{\xi}^1(\vartheta, e)\|_{L^\infty} < +\infty.
\]

Then there exists a finite constant \( M > 0 \) such that
\[
\|\bar{Y} - \hat{Y}\|_{S\infty[0,T]} \leq 2M.
\]

Proposition (C., Rosazza Gianin, 2020)

Under the assumptions of the existence and uniqueness theorem and the above Proposition, the dynamic risk measure \( \rho \) satisfies the Fatou property.
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Example: dynamic entropic risk measure

Suppose that an agent wants to assess the riskiness of a financial position $\xi$ (that we assume bounded) and that she/he evaluates her/his preferences based on the utility function $u(x) := -\gamma e^{-\frac{x}{\gamma}}$, where $\gamma > 0$ is the risk tolerance parameter.

$$\Downarrow$$

Dynamic entropic risk measure:

$$\rho_t(\xi) := \gamma \log \left( \mathbb{E}_t \left[ e^{-\frac{\xi}{\gamma}} \mid \mathcal{G}_t \right] \right), \quad t \in [0, T].$$

How can we make this risk measure react to shocks in the financial market? Is the agent allowed to change her/his preferences based on this event? Can we make $\gamma$ depend on it in a time-consistent way?
Example: dynamic entropic risk measure

We use the parameter $\gamma$ to introduce a dependence of the risk aversion of the agent (the inverse of $\gamma$) on the possible default times and values.

After default the agent becomes more risk averse.
Example: dynamic entropic risk measure

We use the parameter $\gamma$ to introduce a dependence of the risk aversion of the agent (the inverse of $\gamma$) on the possible default times and values.

After default the agent becomes more risk averse.

Suppose that the driver $g$ of the BSDEJ is $g(\omega, t, z) := \frac{1}{2}\|z\|^2 f(t, \tau(\omega), \zeta(\omega))$, where

$$f(t, \vartheta, e) = \begin{cases} 1, & \text{if } t \leq \vartheta, \\ \frac{1}{\gamma(\vartheta, e)}, & \text{if } t > \vartheta, \end{cases}$$

and $\gamma: \mathbb{R}^+ \times E \to (0, 1)$ is a measurable function. The driver can be decomposed as

$$g(t, z) = g^0(z) 1_{t \leq \tau} + g^1(z, \tau, \zeta) 1_{t > \tau},$$

$$g^0(z) = \frac{1}{2}\|z\|^2, \quad g^1(z, \vartheta, e) = \frac{1}{2\gamma(\vartheta, e)}\|z\|^2.$$
Example: dynamic entropic risk measure

Define the $\mathbb{G}$-dynamic risk measure $\rho_t(\xi) := Y_t^{-\xi}$, $t \in [0, T]$, $\xi \in L^\infty(\mathbb{G}_T)$. Then

$$\rho_t(\xi) = \rho^0_t(\xi^0) \mathbb{1}_{t<\tau} + \rho^1_t(\xi^1(\tau, \zeta)) \mathbb{1}_{t\geq\tau},$$

where $\rho^0_t(\xi^0) = Y^0$, with

$$Y^0_t = -\xi^0 + \int_t^T g^0(Z^0_s) \, ds - \int_t^T Z^0_s \, dW_s, \quad 0 \leq t \leq T$$

and, on the event $\{t \geq \tau\}$, $\rho^1_t(\xi^1(\tau, \zeta)) = Y^1(\tau, \zeta)$, with

$$Y^1_t(\vartheta, e) = -\xi^1(\vartheta, e) + \int_t^T g^1(Z^1_s(\vartheta, e), \vartheta, e) \, ds - \int_t^T Z^1_s(\vartheta, e) \, dW_s, \quad \vartheta \wedge T \leq t \leq T.$$

More explicitly

$$\rho^0_t(\xi^0) = \log \mathbb{E}[e^{-\xi^0} \ | \ F_t], \quad t \in [0, T], \quad \text{Reference risk measure},$$

$$\rho^1_t(\xi^1(\tau, \zeta)) = \gamma(\tau, \zeta) \log \mathbb{E}[e^{-\frac{\xi^1(\tau, \zeta)}{\gamma(\tau, \zeta)}} \ | \ \mathcal{H}_t], \quad \text{on} \ \{t \geq \tau\}, \quad \text{Updated risk measure}.$$
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The dual representation

Suppose that the $\mathcal{G}$-dynamic risk measure $\rho$ induced by the BSDEJ satisfies the zero-one law, translation invariance, convexity and Fatou properties (monotonicity is granted by the comparison theorem).

Under these assumptions, $\rho$ admits the dual (or robust) representation

$$\rho_t(\xi) = \text{ess sup}_{Q \in \mathcal{Q}} \{ E_Q[-\xi | G_t] - \alpha_t(Q) \}, \quad \xi \in L^\infty(\mathcal{G}_T), \ t \in [0, T],$$

where $\mathcal{Q} := \{ Q, \text{ probability measures on } (\Omega, \mathcal{G}_T), \text{ such that } Q \sim P|_{\mathcal{G}_T} \}$. The map $\alpha_t$ is the $\mathcal{G}$-penalty term:

$$\alpha_t(Q) = \text{ess sup}_{\xi \in L^\infty(\mathcal{G}_T)} \{ E_Q[-\xi | G_t] - \rho_t(\xi) \} = \text{ess sup}_{\xi \in L^\infty(\mathcal{G}_T), \rho_t(\xi) \leq 0} \{ E_Q[-\xi | G_t] \}, \quad Q \in \mathcal{Q}.$$
The dual representation

Suppose that the \( G \)-dynamic risk measure \( \rho \) induced by the BSDEJ satisfies the zero-one law, translation invariance, convexity and Fatou properties (monotonicity is granted by the comparison theorem).

Under these assumptions, \( \rho \) admits the dual (or robust) representation

\[
\rho_t(\xi) = \text{ess sup}_{Q \in \mathcal{Q}} \{ \mathbb{E}_Q[-\xi | G_t] - \alpha_t(Q) \}, \quad \xi \in L^\infty(G_T), \ t \in [0, T],
\]

where \( \mathcal{Q} := \{ \mathbb{Q}, \text{probability measures on } (\Omega, G_T), \text{ such that } \mathbb{Q} \sim \mathbb{P}|_{G_T} \} \). The map \( \alpha_t \) is the \( G \)-penalty term:

\[
\alpha_t(Q) = \text{ess sup}_{\xi \in L^\infty(G_T)} \{ \mathbb{E}_Q[-\xi | G_t] - \rho_t(\xi) \} = \text{ess sup}_{\xi \in L^\infty(G_T), \rho_t(\xi) \leq 0} \{ \mathbb{E}_Q[-\xi | G_t] \}, \quad Q \in \mathcal{Q}.
\]

**Question**

Can we decompose the penalty term as we did with the dynamic risk measure \( \rho \)?
Decomposition of the penalty term

The candidate penalty terms to provide a decomposition of the $\mathbb{G}$-penalty $\alpha$ are those associated to the $\mathbb{F}$-risk measure $\rho^0$ and the $\mathbb{H}$-risk measure $\rho^1$, i.e.:

$$
\alpha^0_t(Q^0) = \text{ess sup}_{\xi^0 \in L^\infty(\mathcal{F}_T)} \left\{ \mathbb{E}_{Q^0} [-\xi^0 | \mathcal{F}_t] - \rho^0_t(\xi^0) \right\}, \quad Q^0 \in Q^0,
$$

$$
\alpha^1_t(Q^1) = \text{ess sup}_{\xi^1 \in L^\infty(\mathcal{H}_T)} \left\{ \mathbb{E}_{Q^1} [-\xi^1 | \mathcal{H}_t] - \rho^1_t(\xi^1) \right\}, \quad Q^1 \in Q^1,
$$

where

$$Q^0 := \{Q^0, \text{ probability measures on } (\Omega, \mathcal{F}_T), \text{ such that } Q^0 \sim \mathbb{P}_{|\mathcal{F}_T} \},$$

$$Q^1 := \{Q^1, \text{ probability measures on } (\Omega, \mathcal{H}_T), \text{ such that } Q^1 \sim \mathbb{P}_{|\mathcal{H}_T} \}.$$
Decomposition of the penalty term

Recall that $\mathcal{Q} := \{\mathcal{Q}, \text{ probability measures on } (\Omega, \mathcal{G}_T), \text{ such that } \mathcal{Q} \sim \mathbb{P}|_{\mathcal{G}_T} \}$. Define:

$$\mathcal{Q}^{\rightarrow} := \{\mathcal{Q} \in \mathcal{Q}: \text{ any } \mathcal{F}-\text{martingale is a } \mathcal{G}-\text{martingale under } \mathcal{Q} \}.$$

**Proposition (C., Rosazza Gianin, 2020)**

For any $t \in [0, T]$ and any $\mathcal{Q} \in \mathcal{Q}^{\rightarrow}$ the following holds for the $\mathcal{G}$-penalty $\alpha$

$$\alpha_t(\mathcal{Q}) \geq k_t(\mathcal{Q})\alpha_t^0(\mathcal{Q}^0), \quad \text{on } \{t < \tau\}, \quad \alpha_t(\mathcal{Q}) = \alpha_t^1(\mathcal{Q}^1), \quad \text{on } \{t \geq \tau\},$$

where $\mathcal{Q}^0$ and $\mathcal{Q}^1$ are probability measures on $(\Omega, \mathcal{F}_T)$ and $(\Omega, \mathcal{H}_T)$, respectively, such that

$$d\mathcal{Q}^0 = \mathbb{E}[L | \mathcal{F}_T] d\mathbb{P}|_{\mathcal{F}_T}, \quad d\mathcal{Q}^1 = L d\mathbb{P}|_{\mathcal{H}_T}, \quad L := \frac{d\mathcal{Q}}{d\mathbb{P}|_{\mathcal{G}_T}},$$

and $k_t(\mathcal{Q})$ is a $\mathcal{F}_t$-measurable random variable satisfying $k_t(\mathcal{Q}) \geq 1 \mathbb{P}$-a.s.
Decomposition of the penalty term

Recall that $Q := \{Q, \text{ probability measures on } (\Omega, \mathcal{G}_T), \text{ such that } Q \sim \mathbb{P}|_{\mathcal{G}_T}\}$. Define:

$$Q^\rightarrow := \{Q \in Q : \text{ any } \mathcal{F}\text{-martingale is a } \mathcal{G}\text{-martingale under } Q\}.$$  

Proposition (C., Rosazza Gianin, 2020)

For any $t \in [0, T]$ and any $Q \in Q^\rightarrow$ the following holds for the $\mathcal{G}$-penalty $\alpha$

$$\alpha_t(Q) \geq k_t(Q)\alpha_t^0(Q^0), \quad \text{on } \{t < \tau\}, \quad \alpha_t(Q) = \alpha_t^1(Q^1), \quad \text{on } \{t \geq \tau\},$$

where $Q^0$ and $Q^1$ are probability measures on $(\Omega, \mathcal{F}_T)$ and $(\Omega, \mathcal{H}_T)$, respectively, such that

$$dQ^0 = \mathbb{E}[L | \mathcal{F}_T] d\mathbb{P}|_{\mathcal{F}_T}, \quad dQ^1 = L d\mathbb{P}|_{\mathcal{H}_T}, \quad L := \frac{dQ}{d\mathbb{P}|_{\mathcal{G}_T}},$$

and $k_t(Q)$ is a $\mathcal{F}_t$-measurable random variable satisfying $k_t(Q) \geq 1$ $\mathbb{P}$-a.s.

The stochastic factor $k_t(Q)$ is linked to the ratio $\frac{Q(\tau > t | \mathcal{F}_t)}{\mathbb{P}(\tau > t | \mathcal{F}_t)}$, $t \in [0, T]$. Financially, it represents an added penalization due to lack of information prior to the default event.
Conclusion and future developments

Summary:

- Definition of dynamic risk measure induced by a BSDEJ in a progressive enlargement of filtration setting.
- Link between properties of the dynamic risk measure and properties of the driver and/or the decomposed dynamic risk measures.
- Partial results for the decomposition of the penalty term appearing in the dual representation of the dynamic risk measure.
- Updating preferences or risk-aversion feature.

To do:

- Find a BSDEJ representation for a given dynamic risk measure (more generally, for a non-linear expectation) in a progressive enlargement of filtration setting.
- Unspecified (possibly infinite) number of jumps.
Thank you for your attention!

Talk based on: A. Calvia, E. Rosazza Gianin, *Risk measures and progressive enlargement of filtration: a BSDE approach*, SIAM J. Financial Math., 11 (2020), pp. 815-848.