A classification of injective $\text{FI}^m$-modules

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ABSTRACT
In this paper we generalize a shift theorem, which plays a key role in studying representations of $\text{FI}^m$, the product category of the category of finite sets and injections, and classify finitely generated injective $\text{FI}^m$-modules over a field of characteristic 0.

KEYWORDS
$\text{FI}^m$-modules; injective modules; shift theorem

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1. Introduction

1.1. Motivation

The representation theory of infinite combinatorial categories has attracted much attention. It is mainly concerned with how an infinite category acts on a category of modules as they have close relations to (co)homological groups of topological spaces, geometric groups, and algebraic varieties. Among quite a few frequently concerned examples, there is the infinite combinatorial category $\text{FI}$ of finite sets and injections whose representation theoretic and homological properties are extensively studied; see [2, 4].

Due to the importance of the category $\text{FI}$, the structure of finitely generated injective $\text{FI}$-modules is of interest to many mathematicians. In [11], Sam and Snowden firstly classified all injective $\text{FI}$-modules over a field of characteristic 0, and proved that every finitely generated $\text{FI}$-module has finite injective dimension. In [6], Gan and Li give another proof of this fact by introducing the coinduction functor for FI-modules. These results give a deep homological explanation for the following crucial result established by Church, Ellenberg and Farb in [3]: a sequence of representations of symmetric groups over a field of characteristic 0 encoded by an FI-module is representation stable if and only if it is a finitely generated FI-module.

One natural generalization of the category $\text{FI}$ is the product category $\text{FI}^m$ whose representation theory has also been studied; see for instance [5, 8, 9]. However, the classification of the injective $\text{FI}^m$-modules is not covered in [9] and remains as an open problem at that time. In a recent work [13], by extending the method used in [6], the author showed the locally self-injective property of $\text{FI}^m$ over fields of characteristic 0 and further found the external product of finitely generated injective FI-modules being necessarily injective as $\text{FI}^m$-module. In this paper, by generalizing certain concepts in [9] and utilizing an inductive method, the author successfully classifies all finitely generated injective $\text{FI}^m$-modules. Surprisingly, it turns out that the finitely generated indecomposable injective $\text{FI}^m$-modules are already found by the previous work [13], i.e. they are exactly the external tensor product of $m$ many indecomposable injective FI-modules.
1.2. Main results

Before describing the main results of this paper, let us introduce a few notations. Throughout this paper let \( m \) be a positive integer and denote by \([m]\) the set \( \{1, \ldots, m\} \). Let \( \text{FI} \) be the category of finite sets and injections. For brevity, we denote by \( \mathcal{F} \) the full subcategory of \( \text{FI} \) consisting of objects \([n]\), \( n \in \mathbb{N} \) and by \( \mathcal{F}^m \) the product category of \( m \) copies of \( \mathcal{F} \). For any subset \( S \) of \([m]\), we may form a product category \( \mathcal{F}^S \) of \( \mathcal{F} \) indexed by \( S \).

Let \( k \) be a field of characteristic zero. For a locally small category \( C \), a representation of \( C \) or a \( C \)-module over \( k \) is a covariant functor from \( C \) to \( k \)-Mod, the category of vector spaces over \( k \). We denote by \( C \)-Mod the category of all representations of \( C \) over \( k \) and by \( C \)-mod the category of finitely generated representations, which are quotients of direct sums of finitely many representable functors.

Let \( C \) and \( D \) be two locally small categories. There is a way to construct a \( C \times D \)-module from a pair of \( C \)-modules and \( D \)-module. Explicitly, given a \( C \)-module \( V_1 \) and a \( D \)-module \( V_2 \), we define their external tensor product to be the \( C \times D \)-module, denoted by \( V_1 \boxtimes V_2 \), such that
\[(V_1 \boxtimes V_2)(c \times d) = V_1(c) \otimes_k V_2(d)\]
where \( c \in \text{Obj}(C) \) and \( d \in \text{Obj}(D) \). For more details, see [5, Definition 6.4]

Now we are ready to describe the main results of this paper. In [10], Nagpal firstly proved this theorem of \( \text{FI} \)-modules. In this paper, we generalize the shift theorem [9, Proposition 4.10] by taking \( S = [m] \), where \( \Sigma_i \) is the \( i \)-th shift functor. For more details, see Section 2.

**Theorem 1.1.** Let \( V \) be a finitely generated \( \mathcal{F}^m \)-module. Then there exists some positive integer \( N \) such that \( (\prod_{i \in S} \Sigma_i)^N V \) is \( S \)-semi-induced for \( n \geq N \).

The classification of finitely generated injective \( \mathcal{F} \)-module was first accomplished by Sam and Snowden in [11]. Later, Gan and Li gave a new and independent proof for this result in [6], utilizing properties of the coinduction functor which is right adjoint to the shift functor. Extending Gan and Li’s method, we showed in [13] that finitely generated injective \( \mathcal{F}^m \)-module is injective, and furthermore, external tensor products of finitely generated injective \( \mathcal{F} \)-modules are injective \( \mathcal{F}^m \)-modules. The second main theorem strengthens this result.

**Theorem 1.2.** Any finitely generated indecomposable injective \( \mathcal{F}^m \)-module is isomorphic to \( I_1 \boxtimes \cdots \boxtimes I_m \) where each \( I_i \) is a finitely generated indecomposable injective \( \mathcal{F} \)-module for \( i = 1, \ldots, m \).

Since we already know from [11] that an indecomposable injective \( \mathcal{F} \)-module is either an indecomposable \( \mathcal{F} \)-module or a finite dimensional indecomposable injective \( \mathcal{F} \)-module, which can be explicitly constructed, the above theorem actually gives a complete classification of finitely generated injective \( \mathcal{F}^m \)-modules.

2. Preliminaries

In this section we give necessary notations, definitions, and some elementary results used throughout this paper. Since some results are generalizations of corresponded results described in [9] and can be established with the essentially same ideas or arguments, occasionally we omit detailed proofs and suggest the reader to see [9] for details.

2.1. Some notations

Recall that objects in \( \mathcal{F}^m \) are of the form \( n = ([n_1], \ldots, [n_m]) \). We denote by \( n + n' \) the object in \( \mathcal{F}^m \) whose \( i \)-th component is \([n_i + n'_i]\), and by \( 1 \), the object whose \( i \)-th component is the singleton set \([1]\) and all other components are empty sets. The degree of an object \( n \), denoted by \( \deg(n) \), is defined to be the integer \( \sum_i n_i \). For a morphism \( \alpha : n \to n' \), we define \( \deg(\alpha) \) to be the integer \( \deg(n') - \deg(n) \).
deg(n). We also mention that there is a partial order \( \leq \) defined on Obj(\( F^m \)) by specifying \( n \preceq n' \) if \( F^m(n, n') \neq \emptyset \).

Let \( V \) be an \( F^m \)-module. We denote the value of \( V \) on an object \( n \) by \( V(n) \). For a morphism \( \alpha : n \to n' \) in \( F^m \) and an element \( v \in V(n) \), we denote by \( \alpha \cdot v \) the element \( V(\alpha)(v) \in V(n') \). Let \( F^m \)-Mod be the category of all \( F^m \)-modules. It is well known that this category is abelian and has enough projective objects. In particular, for an object \( n \) in \( F^m \), the \( k \)-linearization of the representable functor \( F^m(n, -) \) is a projective \( F^m \)-module. We denote it by \( M(n) \), and we say that an \( F^m \)-module is a free module if it is isomorphic to a direct sum of \( k \)-linearizations of representable functors.

A key technical tool for studying representations of \( F \) is an endofunctor on \( F \)-Mod, which is introduced in [3] and called shift functor. For convenience of the readers, we present here its definition and generalization. There is a self-embedding functor \( \iota \) on \( F \) such that \( \iota([n]) = [n + 1] \), and for a morphism \( f : [n] \to [t] \) in \( F \), \( \iota(f) \) is a morphism from \([n + 1]\) to \([t + 1]\) with

\[
\iota(f)(x) = \begin{cases} 1, & x = 1 \\ f(x - 1) + 1, & x \neq 1 \end{cases}
\]

for element \( x \in [n + 1] \). The shift functor \( \Sigma \) is defined to be the endofunctor on \( F \)-Mod sending an \( F \)-module \( V \) to the \( F \)-module \( V \circ \iota \). Furthermore, there is a natural transformation between the identity functor and the shift functor, so we get a natural homomorphism \( V \to \Sigma V \), and hence obtain the kernel functor \( K \) and the cokernel functor \( D \), which is called the derivative functor in [3].

The self-embedding functor and associated shift functor on \( F \)-modules induce \( m \) distinct self-embedding functors and shift functors on \( F^m \)-modules. Explicitly, for \( i \in [m] \), the product category \( F^m \) can be viewed as product \( \mathcal{F}^m \times F \). The \( i \)-th self-embedding functor is defined to be the endofunctor \( \iota_i := \text{Id} \times \iota \) on \( F^m \) where \( \text{Id} \) is the identity functor on the category \( \mathcal{F}^m \). The \( i \)-th shift functor \( \Sigma_i \) is defined to be the endofunctor on \( F^m \)-Mod sending an \( F^m \)-module \( V \) to the \( F^m \)-module \( V \circ \iota_i \). There are also the \( i \)-th kernel functor \( K_i \) and the \( i \)-th derivative functor \( D_i \) defined on the category \( F^m \)-Mod; for their definitions and elementary properties, see [9, Section 2.2]. Remark that the kernel functor commutes with the shift functor, i.e. \( K_i \Sigma_j \cong \Sigma_j K_i \) for all \( i, j \in [m] \).

A main goal of this paper is to extend quite a few results in [9] from the full set \([m]\) to an arbitrary nonempty subset \( S \) of \([m]\), and denote \( -S \) the complement subset \([m] \setminus S \). Then \( F^m \) is the product category of \( F^S \) and \( F^{-S} \). Accordingly, an object \( n = ([n_1], \ldots, [n_m]) \in \text{Obj}(F^m) \) can be written as a product \( s \times t \) for some object \( s \in \text{Obj}(F^S) \) and object \( t \in \text{Obj}(F^{-S}) \). We define the \( S \)-degree of \( n \) to be the degree of \( s \) in the category \( F^S \) and denote it by \( \text{deg}_S(n) \), i.e. \( \text{deg}_S(n) = \sum_{i \in S} n_i \). For a morphism \( \alpha : n \to n' \) in \( F^m \), we define the \( S \)-degree of \( \alpha \), denoted by \( \text{deg}_S(\alpha) \), to be the integer \( \text{deg}_S(n') - \text{deg}_S(n) \). For brevity, we write \( \text{deg}_i \) for \( \text{deg}_{[i]} \) where \( i \in [m] \). We denote by \( \Sigma_S \) the endofunctor on \( F^m \)-Mod which is the direct sum of \( \Sigma_i \) for all \( i \in S \), i.e. \( \Sigma_S = \bigoplus_{i \in S} \Sigma_i \). The endofunctors \( K_S \) and \( D_S \) on the category \( F^m \)-Mod are defined similarly.

In the proofs of main results, we have to deal with the following categories which are generalizations of \( F^m \). Let \( G \) be a finite group, and view it as a category with a single object. We define \( F^m_G \) to be the category as follows: objects of \( F^m_G \) coincide with objects of \( F^m \), and morphisms of \( F^m_G \) are ordered pairs \( (\alpha, g) \) where \( \alpha \in \text{Mor}(F^m) \) and \( g \in G \). Clearly, it is isomorphic to the product category of \( F^m \) and \( G \). The \( S \)-degree of morphism \( (\alpha, g) \) is defined to be the integer \( \text{deg}_S(\alpha) \). Clearly, when \( G \) is the trivial group, then \( F^m_G \) is precisely \( F^m \). Furthermore, we remark that many results from [9] are still valid for \( F^m_G \), so we will restate some of them in this paper without providing detailed proofs. In particular, functors \( \Sigma_i, K_i, \) and \( D_i \) can be extended to the category \( F^m_G \)-Mod in a natural way and we keep the same notations. It is worthy to remark that the category \( F^m_G \) is also locally Noetherian over \( k \); that is, submodules of finitely generated \( F^m_G \)-modules are still finitely generated; see Lemma 4.4.
2.2. S-Torsion theory

In this subsection we introduce a torsion theory with respect to the nonempty subset \( S \subseteq [m] \), and give a few elementary results.

**Definition 2.1.** Let \( V \) be an \( \mathcal{F}_G^m \)-module and \( n \) an object in \( \mathcal{F}_G^m \). An element \( v \in V_n \) is called \( S \)-torsion if there exists a morphism \( \alpha : n \to V_n \) such that \( \deg_S(\alpha) > 0 \), \( \deg_{-S}(\alpha) = 0 \), and \( \alpha \cdot v = 0 \).

Suppose that \( \alpha : n \to n' \) is a morphism in the above definition. Then \( \deg_S(\alpha) > 0 \) means that there exists a certain \( i \in S \) such that \( n'_i > n_i \), and \( \deg_{-S}(\alpha) = 0 \) means that \( n'_i = n_j \) for all \( j \in -S \). Loosely speaking, this means that \( \alpha \) is a morphism along the \( -S \)-direction.

Let \( V \) be a non-zero \( \mathcal{F}_G^m \)-module. It is clear that all \( S \)-torsion elements in \( V \) form a submodule which is called the \( S \)-torsion part of \( V \) and denoted by \( V^S_T \). The \( S \)-torsion free part of \( V \) is defined to be the quotient module \( V/\mathcal{V}^S_T \) and denoted by \( V^S_F \). Then we have a short exact sequence \( 0 \to V^S_T \to V \to V^S_F \to 0 \). We say that an \( \mathcal{F}_G^m \)-module is \( S \)-torsion (resp., \( S \)-torsion free) if and only if its torsion free part (resp., torsion part) is zero. We remark that when taking \( S = [m] \), these definitions coincide with the ones introduced in the paper [9].

There is another type of torsion submodule which is, in some sense, “stronger” than the above one. Let \( S \) be a subset of \([m]\), we denote by \( V^S_{Tor} \) the submodule

\[
V^S_{Tor} := \bigcap_{i \in S} V^{[i]}_T
\]

of \( V \). More transparently, one has

\[
V^S_{Tor} = \bigoplus_{n \in \text{Obj}(\mathcal{F}_G^m)} \{ v \in V(n) \mid \forall i \in S, \exists \alpha_i \in \text{Mor}(\mathcal{F}_G^m), \ deg_i(\alpha_i) > 0, \ deg_{-[m] \setminus [i]}(\alpha_i) = 0, \text{ and } \alpha_i \cdot v = 0 \}.
\]

**Remark 2.2.** Let \( v \) be an element in \( V(n) \) for a certain object \( n \) in \( \mathcal{F}_G^m \). Then \( v \) is contained in \( V^S_T \) if it eventually vanishes along the \( -i \)-direction for a certain \( i \in S \), and \( v \) is contained in \( V^S_{Tor} \) if it eventually vanishes along the \( i \)-th direction for all \( i \in S \). Keeping in mind this intuition, it is easy to see that the quotient module \( V^{[i-1]}_{Tor}/V^{[i]}_{Tor} \) is \( [i] \)-torsion free. Furthermore, if \( V \) is finitely generated, then the submodule \( V^S_{Tor} \) is finite dimensional, a consequence of the locally Noetherian property of \( \mathcal{F}_G^m \) over \( k \).

In the rest of this subsection we state a few elementary results on \( S \)-torsion theory, which have been established in [9] for the special case that \( S = [m] \).

**Lemma 2.3.** Let \( V \) be an \( \mathcal{F}_G^m \)-module. Then:

1. \( V \) is \( S \)-torsion free if and only if \( K_S V = 0 \), or equivalently, \( K_i V = 0 \) for all \( i \in S \).

2. If \( V \) is \( S \)-torsion free, then so is \( \Sigma_i V \) for all \( i \in S \).

3. In a short exact sequence \( 0 \to U \to V \to W \to 0 \), if both \( U \) and \( W \) are \( S \)-torsion free, so is \( V \).

4. In a short exact sequence \( 0 \to U \to V \to W \to 0 \), if \( W \) is \( S \)-torsion free, then the sequence \( 0 \to D_S U \to D_S V \to D_S W \to 0 \) is exact as well. In particular, \( 0 \to D_i U \to D_i V \to D_i W \to 0 \) is exact for all \( i \in S \).

**Proof.** (1): If \( K_S V \neq 0 \), then there exists a certain \( i \in S \) such that \( K_i V \neq 0 \). By the definition of \( K_i \) and Definition 2.1, the \( \mathcal{F}_G^m \)-module \( V \) contains nonzero \( S \)-torsion elements, and hence is not \( S \)-torsion free. Conversely, if \( V \) is not \( S \)-torsion free, we can find a morphism \( f \in \mathcal{F}_G^m(n, t) \) satisfying the condition in Definition 2.1, and a nonzero element \( v \in V(n) \) such that \( f \cdot v = 0 \). By a simple induction on the degree
of morphisms one can assume that \( \text{deg}_S(f) = 1 \); that is, there is a certain \( i \in S \) such that \( \text{deg}_i(f) = 1 \). Clearly, \( v \in K_i V \), so \( K_S V \neq 0 \).

(2) Assume that \( \Sigma_i V \) is not \( S \)-torsion free; that is, there is some nonzero \( v \in \Sigma_i V(n) = V(n + 1_i) \) for a certain object \( n \) such that \( v \) is sent to 0 by some morphism \( \alpha \) satisfying the condition in Definition 2.1. By the definition of the \( i \)-th shift functor \( \Sigma_i \), \( v \) is sent to 0 by the morphism \( i_i(\alpha) \), which also satisfies the condition specified in Definition 2.1. Thus \( V \) is not \( S \)-torsion free, which is a contradiction.

(3): Applying the exact functor \( \Sigma_S \) one gets a commutative diagram where all vertical rows represent natural maps:

\[
\begin{array}{cccccc}
0 & \to & U & \to & V & \to & W & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \Sigma_S U & \to & \Sigma_S V & \to & \Sigma_S W & \to & 0.
\end{array}
\]

According to statement (1), the maps \( U \to \Sigma_S U \) and \( W \to \Sigma_S W \) are injective, so is the map \( V \to \Sigma_S V \) by the snake Lemma. Therefore, by the first statement, \( V \) is \( S \)-torsion free as well.

(4): Follows from the above commutative diagram and the snake Lemma.

We present some useful properties of the functors \( \Sigma_i \) and \( D_i \).

**Lemma 2.4.** [9, Lemma 2.3] For \( i, j \in [m] \), one has:

1. \( \Sigma_i M(n) \cong M(n) \oplus M(n - 1_i)^{\oplus n_i} \).
2. \( D_i M(n) \cong M(n - 1_i)^{\oplus n_i} \).
3. \( \Sigma_i \circ \Sigma_j = \Sigma_j \circ \Sigma_i \).
4. \( \Sigma_i \circ D_j = D_j \circ \Sigma_i \).

In the situation that \( V \) is finitely generated, one can apply the shift functors to eliminate its torsion part.

**Lemma 2.5.** [9, Lemma 4.8] Let \( V \) be a finitely generated \( \mathcal{T}_G^m \)-module over a field of characteristic 0 and \( i \in [m] \). Then \( K_i \Sigma_i^n V = 0 \) for \( n \) sufficiently large.

### 2.3. Slices and \( S \)-homology groups

In this subsection we introduce \( S \)-homology groups and \( S \)-homological degrees. As before, let \( S \) be a nonempty subset of \([m]\). The category \( \mathcal{T}_G^m \) is a product of the category \( \mathcal{T}_G^S \) and the category \( \mathcal{T}_G^{-S} \). We will see later that the study about the injectivity of any finitely generated \( \mathcal{T}_G^m \)-module can be reduced to that of modules over the subcategory \( \mathcal{R}_s = \text{Aut}(s) \times \mathcal{T}_G^{-S} \) of \( \mathcal{T}_G^m \) for certain object \( s \in \text{Obj}(\mathcal{T}_S) \). In view of this observation, it is necessary to introduce the functor \( \overline{M}(s) \otimes \mathcal{R}_s \) — that produces an \( \mathcal{T}_G^m \)-module from an \( \mathcal{R}_s \)-module.

**Definition 2.6.** For object \( s \in \text{Obj}(\mathcal{T}_S) \), let \( \mathcal{R}_s := \text{Aut}(s) \times \mathcal{T}_G^{-S} \) be the subcategory of \( \mathcal{T}_G^m \). There is a functor

\[
\overline{M}(s) \otimes \mathcal{R}_s : \mathcal{R}_s \cdot \text{Mod} \to \mathcal{T}_G^m \cdot \text{Mod}
\]

where \( \overline{M}(s) := M(s) \otimes k \mathcal{T}_G^{-S} \) is a \((\mathcal{T}_G^m, \mathcal{R}_s)\)-bimodule, \( M(s) \) is free as an \( \mathcal{T}_G^S \)-module, and the category algebra \( k \mathcal{T}_G^{-S} \) is a \((\mathcal{T}_G^m, \mathcal{T}_G^{-S})\)-bimodule. We will denote this functor by \( F_s \) throughout this paper.

In the above definition, the right \( \mathcal{R}_s \)-module structure of \( \overline{M}(s) \) follows from the right \( k \text{Aut}(s) \)-module structure of \( M(s) \) together with the right \( \mathcal{T}_G^{-S} \)-module structure of \( k \mathcal{T}_G^{-S} \). One has that \( \overline{M}(s) \) is free as
right $\mathcal{R}_s$-module since the $\mathcal{S}^S$-module $M(s)$ is free as right $k \text{Aut}(s)$-module. Therefore the functor $F_s$ is exact and preserves projectives.

**Definition 2.7.** For an object $s \in \text{Obj}(\mathcal{S}^S)$ and an $\mathcal{S}_G^m$-module $V$, we define the slice of $V$ on $s$ to be the $\mathcal{S}_G^m$-module (which is also an $\mathcal{R}_s$-module)

$$V[[s]] = \bigoplus_{t \in \text{Obj}(\mathcal{S}^S)} V(s \times t).$$

Explicitly, the $k$-space $V[[s]](n)$ equals $V(n)$ if $n = s \times t$ for certain $t \in \text{Obj}(\mathcal{S}^S)$ and otherwise equals zero; the homomorphism $V[[s]](f)$ equals $V(f)$ if the domain and codomain of $f$ are both of the form $s \times t$, and otherwise is zero homomorphism.

We present here an observation. For a finitely generated $\mathcal{S}_G^m$-module $V$, the slice $V[[s]]$ may not be a submodule of $V$. However, for the homology module $H_0^S(V)$ that is defined in the next paragraph, the slice $H_0^S(V)[[s]]$ is a direct summand of it and it is a direct sum of finitely many such summands. Also, for $i \in s$, the $\mathcal{S}_G^m$-module $K_i V$ is a direct sum of its slices on objects in the category $\mathcal{S}^i$.

Now we are ready to define $S$-homology groups for $\mathcal{S}_G^m$-modules. Let $\mathcal{I}_S$ be the free $k$-module spanned by all morphisms $\alpha$ with $\deg_S(\alpha) > 0$. It is easy to see that $\mathcal{I}_S$ is a two-sided ideal of the category algebra $k\mathcal{S}_G^m$. For $V \in \mathcal{S}_G^m$-Mod, define

$$H_0^S(V) = k\mathcal{S}_G^m/\mathcal{I}_S \otimes_{k\mathcal{S}_G^m} V.$$

Note that the module $H_0^S(V)$ is isomorphic to the quotient $V/\mathcal{I}_S V$. Further, for $i > 0$, we define the $i$-th $S$-homology functor to be the $i$-th left derived functor of $H_0^S$ and denote it by $H_i^S$. The $\mathcal{S}_G^m$-module $H_i^S(V)$ is called the $i$-th $S$-homology group of $V$.

For $i \geq 0$, we define the $i$-th $S$-homological degree to be the integer

$$t_i^S(V) = \sup \{\deg(s) \mid H_i^S(V)[[s]] \neq 0, \ s \in \text{Obj}(\mathcal{S}^S)\}.$$

If the set on the right hand side is empty, we set $t_i^S(V) = -1$.

We collect some elementary results about $S$-homology groups in the following lemmas.

**Lemma 2.8.** For a short exact sequence of $\mathcal{S}_G^m$-module

$$0 \to V' \to V \to V'' \to 0,$$

we have

(1) $t_{i+1}^S(V'') \leq \max\{t_{i+1}^S(V'), t_i^S(V')\}$

(2) $t_i^S(V) \leq \max\{t_i^S(V'), t_i^S(V'')\}$

(3) $t_i^S(V') \leq \max\{t_i^S(V), t_{i+1}^S(V'')\}$

for $i \geq 0$.

**Proof.** The conclusion follows from the long exact sequence

$$\cdots \to H_i^S(V') \to H_i^S(V) \to H_i^S(V'') \to H_{i-1}^S(V') \to \cdots \to H_0^S(V') \to H_0^S(V) \to H_0^S(V'') \to 0.$$

□

**Lemma 2.9.** If $V \in \mathcal{S}_G^m$-Mod is nonzero, then $t_0^S(D_S V) = t_0^S(V) - 1$. 

Proof. As explained in [7, Lemma 1.5], we only need to deal with the case that \( t_0^S(V) \) is finite. If \( t_0^S(V) = 0 \), then an argument similar to the following one will yield \( D_5P = 0 \) hence \( D_5V = 0 \), as required. Now we assume that \( t_0^S(V) > 0 \). Since \( V \) is finitely generated, we may find a surjection \( P \to V \) where \( P \) is a finitely generated projective \( \mathcal{F}_G^m \)-modules with \( t_0^S(P) = t_0^S(V) \). Since the functor \( D_5 \) is right exact, the map \( D_5P \to D_5V \) is surjective. By Lemma 2.4, we have that \( t_0^S(D_5P) = t_0^S(P) - 1 \). Then one can deduce that \( t_0^S(D_5V) \leq t_0^S(D_5P) = t_0^S(P) - 1 = t_0^S(V) - 1 \).

On the other hand, there is an object \( s \in \text{Obj}(\mathcal{F}_G) \) such that \( \deg(s) = t_0^S(V) \geq 1 \) and \( H_0^S(V)([s]) \neq 0 \). Now let \( V' \) be the submodule of \( V \) generated by \( V[[t]] \) such that \( s \nleq t \) and \( t \in \text{Obj}(\mathcal{F}_G) \). Then \( V([s]) \nsubseteq V' \) since otherwise one should have \( H_0^S(V)([s]) = 0 \). The surjection \( V \to V/V' \to 0 \) induces a surjection \( D_5V \to D_5V/V' \to 0 \), and hence \( t_0^S(D_5V) = t_0^S(D_5(V/V')) \). Note that \( (V/V')[[t]] \neq 0 \) implies \( s \nleq t \) for \( t \in \text{Obj}(\mathcal{F}_G) \). Thus, for \( i \in S \), we have that \( \Sigma^i(V/V')[[t]] \neq 0 \) implies \( t \nleq s - 1 \) and that \( D_i(V/V')[[t]] \neq 0 \) implies \( t \nleq s - 1 \). As a result, we have \( t_0^S(D_i(V/V')) \geq \deg(s) - 1 = t_0^S(V) - 1 \) and so is \( t_0^S(D_5(V/V')) \). Combining the two inequalities, one obtains that \( t_0^S(D_5(V)) \geq t_0^S(D_5(V/V')) \geq t_0^S(V) - 1 \), as required. \( \square \)

3. S-induced modules and S-semi-induced modules

In this section we consider \( S \)-induced modules and \( S \)-semi-induced modules, which are generalizations of induced modules and semi-induced modules considered in [9]. All modules considered in the rest of this paper are finitely generated unless otherwise specified.

Definition 3.1. A finitely generated \( \mathcal{F}_G^m \)-module \( V \) is said to be \( S \)-induced if \( V \cong F_s(W) \) for certain object \( s \) in \( \mathcal{F}_G \) and some \( R_k \)-module \( W \).

The \( S \)-induced module \( V \) has the following universal property. Let \( W \) be an \( R_k \)-module (also viewed as an \( \mathcal{F}_G^m \)-module) and \( N \) an \( \mathcal{F}_G^m \)-module. Then a homomorphism \( W \to N \) as \( \mathcal{F}_G^m \)-module uniquely extends to a homomorphism \( F_s(W) \to N \) as \( \mathcal{F}_G^m \)-module.

Definition 3.2. An \( \mathcal{F}_G^m \)-module \( V \) is said to be \( S \)-semi-induced if it has a finite filtration

\[
0 = V^0 \subseteq V^1 \subseteq \cdots \subseteq V^n = V
\]

such that for each \( i \), the quotient module \( V^{i+1}/V^i \) is \( S \)-induced.

We remark that when \( S = \{m\} \), \( S \)-semi-induced modules coincide with relative projective modules defined in [9]. Further, over a field of characteristic zero, relative projective modules coincide with projective modules.

Lemma 3.3. Let \( V \) be an \( \mathcal{F}_G^m \)-module generated by its slice \( V[[s]] \) for a certain object \( s \in \text{Obj}(\mathcal{F}_G) \). One has:

(1) The following are equivalent:

- \( V \) is an \( S \)-induced module;
- \( H_i^S(V) = 0 \) for all \( i \geq 1 \);
- \( H_1^S(V) = 0 \).

(2) If \( V \) is \( S \)-induced, then it is \( S \)-torsion free.

(3) If \( t_0^S(V) \leq t_0^S(P) \), then \( V \) is \( S \)-induced.

(4) If \( V \) is \( S \)-induced, then \( \Sigma_i V \) is \( S \)-semi-induced and \( D_i V \) is \( S \)-induced for all \( i \in S \).
Proof. (1): Suppose that \( V \) is an \( S \)-induced module. Then one has \( V \cong M(s) \otimes_{R_s} V[[s]] \) where \( R_s \) is defined as in Definition 2.6. Take a projective presentation \( 0 \to W \to P \to V[[s]] \to 0 \) of the \( R_s \)-module \( V[[s]] \). Applying the functor \( M(s) \otimes_{R_s} - \), we get a projective presentation of the \( \mathcal{F}^m_G \)-module \( V \) as follows

\[
0 \to M(s) \otimes_{R_s} W \to M(s) \otimes_{R_s} P \to (M(s) \otimes_{R_s} V[[s]]) \cong V \to 0.
\]

Applying the functor \( k^{\mathcal{F}^m/m} / S \otimes k^{\mathcal{F}^m/m} \) we recover the original short exact sequence. That is, \( H_S^0(V) = 0 \). Replacing \( V \) by \( V' = M(s) \otimes_{R_s} W \) we deduce that \( H_S^0(V) = 0 \). Recursively, for every \( i \geq 1 \), one gets \( H_S^i(V) = 0 \).

Conversely, suppose that \( H_S^0(V) = 0 \). Since \( V \) is generated by \( V[[s]] \), there is a short exact sequence of \( \mathcal{F}^m_G \)-module

\[
0 \to K \to N \to V \to 0.
\]

where \( N = M(s) \otimes_{R_s} V[[s]] \). The long exact sequence of \( \mathcal{F}^m \)-modules

\[
\cdots \to H_S^0(V) \to H_S^0(K) \to H_S^0(N) = V[[s]] \to H_S^0(V) = V[[s]] \to 0
\]

implies that \( H_S^0(K) = 0 \). That is, \( K = 0 \), and hence \( V \cong N \) is \( S \)-induced.

(2): By definition, we have that \( V \cong M(s) \otimes_{R_s} V[[s]] \). Let \( s' \) and \( s'' \) be objects in \( \mathcal{F}^S \) with \( s < s' \leq s'' \), \( t \) an object in \( \mathcal{F}^{S'}_G \), and \( f : s' \times t \to s'' \times t \) a morphism in \( \mathcal{F}^m_G \). By an argument similar to the proof of [9, Lemma 4.2(2)], we have that \( f \cdot v \neq 0 \) for non-zero element \( v \in V(s' \times t) \). Therefore, \( V \) is \( S \)-torsion free.

(3): Again, consider exact sequences (3.1) and (3.2). Since \( t_{S}^0(V) \leq t_{S}^0(V) \), by Lemma 2.8 we know that \( t_{S}^0(K) \leq \max\{t_{S}^0(V), t_{S}^0(N) = t_{S}^0(V)\} = t_{S}^0(V) = \deg(s) \), which means that \( H_{S}^0(K)[[t]] \neq 0 \) only if \( \deg(t) \leq \deg(s) \) for \( t \in \text{Obj}(\mathcal{F}^S) \). The sequence (3.1) yields a short exact sequence of \( R_s \)-modules

\[
0 \to K[[s]] \to N[[s]] \to V[[s]] \to 0.
\]

So \( K[[s]] = 0 \) since \( N[[s]] = V[[s]] \). We have that \( N[[s']] \neq 0 \) only if \( s' \succ s \) by our construction of \( N \), so \( N \) is the submodule \( K \) of \( N \). Putting the established results together we obtain that \( H_{S}^0(K) = 0 \), so \( K = 0 \). Therefore, \( V \cong N \) is \( S \)-induced.

(4): As shown in the proof of statement (1), there is a short exact sequence \( 0 \to K \to P \to V \to 0 \) such that \( P \) is a projective \( \mathcal{F}^m_G \)-module generated by \( P[[s]] \). By the previous arguments we know that all terms in this sequence are \( S \)-induced modules generated by their slice on the object \( s \), and hence are \( S \)-torsion free. By Statement (4) of Lemma 2.3, for each \( i \in S \), we get a short exact sequence \( 0 \to D_iK \to D_iP \to D_iV \to 0 \). Since \( D_iV = 0 \) whenever \( s_i \), the \( i \)-th component of \( s \), is zero, without loss of generality we assume that \( s_i > 0 \). Then \( D_iV \) is generated by its slice on the object \( s - 1_i \) since so is \( D_iP \). Replacing \( V \) by \( K \) one knows that \( D_iK \) is also generated by its slice on \( s - 1_i \). The module \( D_iP \) being projective implies that \( H_{S}^0(D_iP) = 0 \), the long exact sequence of homology groups induced by \( 0 \to D_iK \to D_iP \to D_iV \to 0 \) tells us that \( t_{S}^0(D_iV) = t_{S}^0(D_iP) \geq t_{S}^0(D_iK) \geq t_{S}^0(D_iV) \). By statement (3), \( D_iV \) is \( S \)-induced. But \( D_iV \cong \Sigma_i V / V \), so \( \Sigma_i V \) is \( S \)-semi-induced since \( V \) is \( S \)-induced.

In the following proposition we describe two homological characterizations of \( S \)-semi-induced modules.

**Proposition 3.4.** For \( V \in \mathcal{F}^m_G \)-mod, the following are equivalent:

1. \( V \) is an \( S \)-semi-induced module;
2. \( H_S^i(V) = 0 \) for all \( i \geq 1 \);
3. \( H_S^1(V) = 0 \).

**Proof.** (1) \( \Rightarrow \) (2): Let

\[
0 = V^0 \subseteq V^1 \subseteq \cdots \subseteq V^m = V
\]
be the filtration described in Definition 3.2. We prove by making induction on the superscripts of modules in this filtration. By statement (1) of Lemma 3.3 we have that \( H_i^S(V^k) = 0 \). Assume that \( H_i^S(V^k) = 0 \). We have a short exact sequence

\[
0 \to V^k \to V^{k+1} \to V^{k+1}/V^k \to 0
\]

where \( V^{k+1}/V^k \) is S-induced. Again, by statement (1) of Lemma 3.3, we have that \( H_i^S(V^{k+1}/V^k) = 0 \) for \( i \geq 1 \). The long exact sequence

\[
\cdots \to H^S_i(V^k) = 0 \to H^S_i(V^{k+1}) \to H^S_i(V^{k+1}/V^k) = 0 \to \cdots
\]

tells us that \( H^S_i(V^{k+1}) = 0 \). The conclusion then follows by induction.

(2) \( \Rightarrow \) (3): trivial.

(3) \( \Rightarrow \) (1): On the one hand, the comment following Definition 2.7 says that the \( F^m_G \)-module \( H_0^S(V) \) is a direct sum of \( F^m_G \)-modules \( W_j \), where \( W_j = H_0^S(V)[[s_j]] \) is an \( R_s \)-module (also an \( F^m_G \)-module) for a set of objects \( s_j \) in \( F^S \) and \( 1 \leq j \leq l \) for some positive integer \( l \). In other words, \( H_0^S(V) \cong \bigoplus_{j=1}^l W_j \).

On the other hand, we obtain a short exact sequence of \( F^m_G \)-modules

\[
0 \to K \to N \to V \to 0.
\]

where \( N := \bigoplus_{j=1}^l F_s(W_j) \) and \( K \) is the kernel of the natural surjection \( N \to V \). Since \( H_0^S(V) = 0 \) and \( H_0^S(N) = \bigoplus_{j=1}^l W_j \), we have the long exact sequence

\[
\cdots \to H^S_0(V) = 0 \to H^S_0(K) \to \bigoplus_{j=1}^l W_j \to H^S_0(V) \to 0.
\]

Since \( H^S_0(V) \cong \bigoplus_{j=1}^l W_j \), the kernel \( H^S_0(K) \) is zero hence \( K = 0 \). Therefore \( V \cong N \) is S-semi-induced.

\[\square\]

**Corollary 3.5.** If \( V \) is S-semi-induced, then it is S-torsion free, and \( \Sigma_i V \) and \( D_i V \) are S-semi-induced as well for all \( i \in S \).

**Proof.** Suppose that \( V \) admits a filtration

\[
0 = V^0 \subseteq V^1 \subseteq \cdots \subseteq V^m = V
\]

with each factor S-induced. The first part of the statement follows from statement (2) of Lemma 3.3 and the fact that S-torsion free modules are closed under extension. Now we prove the second part of the statement. For \( i \in S \), applying the exact functor \( \Sigma_i \) to this filtration we get a filtration of \( \Sigma_i V \) with factor

\[
\Sigma_i V^k / \Sigma_i V^{k-1} \cong \Sigma_i (V^k / V^{k-1})
\]

which is S-semi-induced by statement (4) of Lemma 3.3. Therefore, \( \Sigma_i V \) is S-semi-induced. A similar argument together with statement (4) of Lemma 2.3 shows that \( D_i V \) is S-semi-induced.

\[\square\]

The following lemma is crucial for us to prove the first main result of this paper.

**Lemma 3.6.** Let \( V \) be an S-torsion free \( F^m_G \)-module. If \( D_S V \) is S-semi-induced, so is \( V \).

**Proof.** The conclusion holds for \( V = 0 \) trivially, so we may assume that \( V \neq 0 \). Since \( V \) is finitely generated, the set

\[
O = \{ s \in \text{Obj}(F^S) \mid H_0^S(V)[[s]] \neq 0 \}
\]

is finite. We prove by an induction on the cardinality of \( O \). The conclusion holds clearly if \( |O| = 0 \). For \( |O| \geq 1 \), we choose an object \( s \in O \) such that \( \deg(s) \) is maximal; that is, \( \deg(s) = t_0^S(V) \) (of course, this
Let $V$ be a finitely generated module. Consider the short exact sequence $0 \rightarrow V' \rightarrow V \rightarrow V'' \rightarrow 0$. We claim that $V''$ is an $S$-induced $\mathcal{F}_G^m$-module. To see this, from the long exact sequence of homology groups one has

$$t_0^S(V') \leq \max\{t_0^S(V'), t_0^S(V)\} \leq \max\{t_0^S(V), t_0^S(V')\},$$

where the second inequality follows from $t_0^S(V') \leq t_0^S(V)$ by our construction of $V'$. Furthermore, let $0 \rightarrow W \rightarrow P \rightarrow V \rightarrow 0$ be a short exact sequence of $\mathcal{F}_G^m$-modules such that $P$ is a free $\mathcal{F}_G^m$-module satisfying $t_0^S(V) = t_0^S(P)$. Applying $D_S$ we get another short exact sequence $0 \rightarrow D_SW \rightarrow D_SP \rightarrow D_SV \rightarrow 0$ such that $D_SP$ is also free and satisfies $t_0^S(D_SP) = t_0^S(D_SV)$ by Lemma 2.9. Then one has

$$t_0^S(V) \leq t_0^S(W) = t_0^S(D_SW) + 1 \leq \max\{t_0^S(D_SV), t_0^S(D_SW)\} + 1 = t_0^S(D_SV) + 1 = t_0^S(D_SV),$$

where the third inequality follows from $H^0_1(D_SV) = 0$ by Proposition 3.4. Putting the above two inequalities together we conclude that $t_0^S(V'') \leq t_0^S(V) = t_0^S(V'')$. By statement (3) of Lemma 3.3, $V''$ is $S$-induced as claimed.

The conclusion follows after we show that $V'$ is $S$-semi-induced. By the induction hypothesis, it suffices to show that $D_SV'$ is $S$-semi-induced. Since $V''$ is $S$-induced, so is $D_SV''$ by statement (4) of Lemma 3.3 and it is $S$-torsion free by statement (2) of Lemma 3.3. By statement (4) of Lemma 2.3, we get a short exact sequence $0 \rightarrow D_SV' \rightarrow D_SV \rightarrow D_SV'' \rightarrow 0$. The long exact sequence of homology groups tells us that $H^0_1(D_SV') = 0$, so $D_SV'$ is $S$-semi-induced by Proposition 3.4.

Now we are ready to prove the first theorem in the Introduction of this paper.

**A proof of Theorem 1.1.** By Lemma 2.5, there exists a positive integer $c$ such that $K_S(\prod_{i \in S} \Sigma_i)^cV = 0$. Therefore $(\prod_{i \in S} \Sigma_i)^cV$ is $S$-torsion free by statement (1) of Lemma 2.3. Now we make induction on $t_0^S(V)$. Assume that the conclusion holds for any $W \in \mathcal{F}_G^m$-mod with $t_0^S(W) < t_0^S(V)$. By Lemma 2.9, we have that $t_0^S(D_SW) < t_0^S(V)$. By the induction hypothesis, there exists some integer $l > 0$ such that $(\prod_{i \in S} \Sigma_i)^lD_SW$ is $S$-semi-induced. Set $n := \max\{c, l\}$. By Corollary 3.5 and the statement (4) of Lemma 2.4, we have that

$$D_S\left(\prod_{i \in S} \Sigma_i\right)^nV \cong \left(\prod_{i \in S} \Sigma_i\right)^nD_SV = \left(\prod_{i \in S} \Sigma_i\right)^{n-l}\left(\prod_{i \in S} \Sigma_i\right)^lD_SV,$$

is $S$-semi-induced. By statement (2) of Lemma 2.3, we have that

$$\left(\prod_{i \in S} \Sigma_i\right)^nV = \left(\prod_{i \in S} \Sigma_i\right)^{n-c}\left(\prod_{i \in S} \Sigma_i\right)^cV$$

is $S$-torsion free. By Lemma 3.6, we have that $(\prod_{i \in S} \Sigma_i)^nV$ is $S$-semi-induced.

An immediate corollary is:

**Corollary 3.7.** Let $V$ be a finitely generated $\mathcal{F}_G^m$-module. If $V$ is $S$-torsion free then $V$ can be embedded into some finitely generated $S$-semi-induced $\mathcal{F}_G^m$-module.

**Proof.** This follows from Theorem 1.1, statements (1) and (2) of Lemma 2.3 and the fact that the functor $\Sigma_i$ preserves finitely generated modules. Explicitly, there is an injective homomorphism $V \rightarrow (\prod_{i \in S} \Sigma_i)^nV$. 

---

**4. A classification of indecomposable injective $\mathcal{F}_G^m$-modules**

In this section we classify all indecomposable injective $\mathcal{F}_G^m$-modules. For this purpose, we firstly construct a class of modules that finitely cogenerates the category $\mathcal{F}_G^m$-mod and show that they are
injective. Consequently, any finitely generated injective $\mathcal{F}_G^m$-module is isomorphic to a direct summand of a finite direct sum of modules in this class.

Let us introduce a few necessary notions. Let $\mathcal{C}$ be a small category. We denote by $\mathcal{C} \text{-} \text{inj}$ the category of all finitely generated injective $\mathcal{C}$-modules. For a $\mathcal{C}$-module $V$ and a class $\mathcal{U}$ of $\mathcal{C}$-modules, we say that $V$ is finitely cogenerated by $\mathcal{U}$ if there exists a finite set $X$ and a map $f : X \to \mathcal{U}$ such that the $\mathcal{C}$-module $V$ can be embedded into the $\mathcal{C}$-module $\bigoplus_{x \in X} f(x)$.

As before, let $S$ be a nonempty subset of $[m]$. We denote by $\mathcal{U}_G^S$ the class of $\mathcal{F}_G^m$-modules

\[ U_G^S := \{(\bigotimes_{I \in S} I) \otimes kG \mid I \in \mathcal{F}_G^{m}\} \]

We usually omit the subscript $G$ of $U_G^S$ when there is no ambiguity. For brevity, we denote by $\mathcal{U}^n$ the class $\mathcal{U}^{[m]}$ for $1 \leq n \leq m$.

**Definition 4.1.** Let $j : \mathcal{F}_G^m \rightarrow \mathcal{F}_G^m, \alpha \mapsto (\alpha, 1)$ be the embedding functor where $\alpha \in \text{Mor}(\mathcal{F}_G^m)$. It induces a pair $(\text{Ind}, \text{Res})$ of functors

\[
\text{Res} : \mathcal{F}_G^m \text{-Mod} \rightarrow \mathcal{F}_G^m \text{-Mod}; \quad \text{Ind} : \mathcal{F}_G^m \text{-Mod} \rightarrow \mathcal{F}_G^m \text{-Mod}
\]

as follows: The functor $\text{Res}$ sends an $\mathcal{F}_G^m$-module $W$ to the $\mathcal{F}_G^m$-module $W \circ j$, and the functor $\text{Ind}$ sends an $\mathcal{F}_G^m$-module $V$ to the $\mathcal{F}_G^m$-module $W \otimes kG = k \mathcal{F}_G^m \otimes_{\mathcal{F}_G^m} V$ and sends an $\mathcal{F}_G^m$-morphism $\varphi : V \rightarrow V'$ to $\mathcal{F}_G^m$-morphism $\varphi \otimes \text{Id} : V \otimes kG \rightarrow V' \otimes kG$.

We will show in Lemma 4.2 that $(\text{Ind}, \text{Res})$ is an adjoint pair. Moreover, the functors $\text{Res}$ and $\text{Ind}$ are both exact functors, so $\text{Ind}$ preserves projectives and $\text{Res}$ preserves injectives.

**Lemma 4.2.** The functor $\text{Res}$ is right adjoint to $\text{Ind}$.

**Proof.** We prove by constructing the adjunction directly. For an $\mathcal{F}_G^m$-module $V$ and an $\mathcal{F}_G^m$-module $W$, let

\[
\theta_{VW} : \text{Hom}_{\mathcal{F}_G^m}(\text{Ind}(V), W) \rightarrow \text{Hom}_{\mathcal{F}_G^m}(V, \text{Res}(W))
\]

be the map such that for an $\mathcal{F}_G^m$-module homomorphism $\varphi : V \otimes kG \rightarrow W$ we have that

\[
\theta_{VW}(\varphi)(v) = \varphi(v \otimes 1_G)
\]

for $v \in V$. Conversely, one can define a map

\[
\theta_{VW}^{-1} : \text{Hom}_{\mathcal{F}_G^m}(V, \text{Res}(W)) \rightarrow \text{Hom}_{\mathcal{F}_G^m}(\text{Ind}(V), W)
\]

such that for an $\mathcal{F}_G^m$-module homomorphism $f : V \rightarrow W$,

\[
\theta_{VW}^{-1}(f)(v \otimes k g) = (e_n, g) \cdot f(v)
\]

for $v \in V(n), g \in G$, and $e_n$ is the identity morphism on the object $n \in \text{Obj}(\mathcal{F}_G^m)$. It is routine to check that the homomorphisms $\theta_{VW}(\varphi)$ and $\theta_{VW}^{-1}(f)$ are well-defined, that the map $\theta_{VW}$ is a bijection with inverse $\theta_{VW}^{-1}$, and that $\theta : \text{Hom}_{\mathcal{F}_G^m}(\text{Ind}(-), -) \rightarrow \text{Hom}_{\mathcal{F}_G^m}(-, \text{Res}(-))$ is a natural equivalence. \qed

The following lemma says that the functor $\text{Res}$ preserves finitely generated modules.

**Lemma 4.3.** An $\mathcal{F}_G^m$-module $V$ is finitely generated if and only if $\text{Res}(V)$ is finitely generated as $\mathcal{F}_G^m$-module.

**Proof.** The if part is trivial. For the only if part, suppose that the $\mathcal{F}_G^m$-module $V$ has a finite set $X$ of generators. Then the set

\[
X' := \{(1, g) \cdot x \mid g \in G, x \in X\}
\]

is a finite set since $G$ is finite. Moreover, $X'$ is a set of generators of the $\mathcal{F}_G^m$-module $\text{Res}(V)$ since any morphism $(\alpha, g)$ in the category $\mathcal{F}_G^m$ can be decomposed as $(\alpha, 1) \circ (1, g)$. Therefore, the $\mathcal{F}_G^m$-module $\text{Res}(V)$ is finitely generated. \qed
As we mentioned before, the category $\mathcal{F}_G^m$ is locally Noetherian over $k$. Here we give a new proof using the functors $\text{Ind}$ and $\text{Res}$.

**Lemma 4.4.** The category $\mathcal{F}_G^m$ is locally Noetherian over a commutative Noetherian ring. That is, any $\mathcal{F}_G^m$-submodule of finitely generated $\mathcal{F}_G^m$-module is still finitely generated.

**Proof.** Suppose that $U$ is an $\mathcal{F}_G^m$-submodule of some finitely generated $\mathcal{F}_G^m$-module $V$. Since the functor $\text{Res}$ is exact, $\text{Res}(U)$ is a submodule of $\text{Res}(V)$ as $\mathcal{F}_G^m$-module. By Lemma 4.3, $\text{Res}(V)$ is finitely generated. By the Noetherian property of $\mathcal{F}_G^m$, see [9, Theorem 1.1], $\text{Res}(U)$ is finitely generated. Again, by Lemma 4.3, the $\mathcal{F}_G^m$-module $U$ is finitely generated as desired. □

In classical group representation theory, it is well known that for a finite group inclusion $H \leq G$, any $H$-projective $kG$-module $V$ is a direct summand of the $kG$-module $V \downarrow^G_H \oplus \uparrow^G_H$ where $\downarrow^G_H$ is the restriction functor and $\uparrow^G_H$ is the induction functor; see [12, Proposition 11.3.4]. We have the following similar result.

**Lemma 4.5.** Let $V$ be an $\mathcal{F}_G^m$-module. Then $V$ is isomorphic to a direct summand of the $\mathcal{F}_G^m$-module $W = \text{Ind}(\text{Res}(V))$.

**Proof.** For brevity, we write $g \cdot v$ for $(id_n, g) \cdot v$ where $n \in \text{Obj}(\mathcal{F}_G^m)$, $v \in V(n)$, and $g \in G$. We prove by constructing a pair of $\mathcal{F}_G^m$-module homomorphisms $\varphi : V \to W$ and $\epsilon : W \to V$ such that $\epsilon \varphi = \text{Id}_V$. For an object $n \in \text{Obj}(\mathcal{F}_G^m)$, the components of $\varphi$ and $\epsilon$ on $n$ are given by

$$
\varphi_n : V(n) \to \text{Res}(V)(n) \otimes_k kG, \quad v \mapsto \frac{1}{|G|} \sum_{g \in G} g^{-1} \cdot v \otimes_k g
$$

for $v \in V(n)$ and

$$
\epsilon_n : \text{Res}(V)(n) \otimes_k kG \to V(n), \quad u \otimes_k g \mapsto g \cdot u
$$

for $u \in \text{Res}(V)(n)$ and $g \in G$. It remains to check that $\epsilon_n \varphi_n = 1$ and that both $\varphi$ and $\epsilon$ are $\mathcal{F}_G^m$-module homomorphisms, which are routine. □

The functor $\text{Ind}$ preserves injective modules. That is:

**Lemma 4.6.** Suppose that $I$ is an injective $\mathcal{F}_G^m$-module. Then $\text{Ind}(I) = I \boxtimes kG$ is an injective $\mathcal{F}_G^m$-module.

**Proof.** Let $V$ be any $\mathcal{F}_G^m$-module. We have that

$$
\text{Ext}_{\mathcal{F}_G^m}(V, \text{Ind}(I)) \subseteq \text{Ext}_{\mathcal{F}_G^m}(\text{Ind} \circ \text{Res}(V), \text{Ind}(I))
$$

$$
= \text{Ext}_{\mathcal{F}_G^m}(\text{Res}(V), \text{Ind}(kG))
$$

$$
\cong \text{Ext}_{\mathcal{F}_G^m}(\text{Res}(V), \bigoplus_{g \in G} I)
$$

$$
\cong \bigoplus_{g \in G} \text{Ext}_{\mathcal{F}_G^m}(\text{Res}(V), I)
$$

$$
= 0.
$$

where the first inclusion follows from Lemma 4.5 together with the fact that the functor Ext is additive and the first identity follows from the Eckmann Shapiro’s Lemma. Therefore, the $\mathcal{F}_G^m$-module $\text{Ind}(I)$ is injective. □

An immediate corollary is:
Corollary 4.7. Every module in the class \( \mathcal{U}^S \) is a finitely generated injective \( \mathcal{F}_G^S \)-module.

Proof. This follows immediately by Lemma 4.6 and [13, Theorem 1.1].

In the following lemma we prove the main result of this section for the special case that \( m = 1 \).

Lemma 4.8. Every finitely generated \( \mathcal{F}_G \)-module \( V \) is finitely cogenerated by \( \mathcal{U}^1 \).

Proof. There is a short exact sequence

\[
0 \to V_T \to V \to V_F \to 0
\]

where \( V_T \) is the torsion part and \( V_F \) is the torsion free part of \( V \). By Lemma 4.4, the torsion part \( V_T \) is finitely generated, and hence finite dimensional. Therefore \( V_T \) can be embedded into a finite dimensional injective \( \mathcal{F}_G \)-modules which lies in \( \mathcal{U}^1 \). The module \( V_F \) is \([1]\)-torsion free, so by Proposition 3.7 it can be embedded into some finitely generated \([1]\)-semi-induced \( \mathcal{F}_G \)-module which is projective by the comment following Definition 3.2. But any finitely generated projective \( \mathcal{F}_G \)-module is a direct summand of a finite direct sum of \( \mathcal{F}_G \)-modules of the form \( M(n) \otimes kG \) which lies in \( \mathcal{U}^1 \). The conclusion then follows.

Lemma 4.9. Notation as before. Let \( s \) be an object in \( \mathcal{F}^S \) and \( W \) an \( \mathcal{R}_s \)-module. If \( W \cong W' \otimes k \text{Aut}(s) \) for some \( \mathcal{F}_G^S \)-module \( W' \), then \( F_s(W) \cong M(s) \otimes W' \) as \( \mathcal{F}_G^m \)-modules. In other words, there is an isomorphism of \( \mathcal{F}_G^m \)-modules

\[
\theta : \overline{M}(s) \otimes_{k(\mathcal{F}^{-S} \times G \times \text{Aut}(s))} W \cong M(s) \otimes W'
\]

where \( M(s) \) is a free \( \mathcal{F}^S \)-module and \( \overline{M}(s) = k\mathcal{F}^{-S} \otimes kG \otimes M(s) \) is an \((\mathcal{F}_G^m, \mathcal{R}_s)\)-bimodule.

Proof. For an object \( n = n_S \times n_T \in \text{Obj}(\mathcal{F}_G^m) \) where \( n_S \in \text{Obj}(\mathcal{F}^S) \) with \( n_S \simeq s \) and \( n_T \in \text{Obj}(\mathcal{F}^{-S}) \), the map \( \theta_n \), which is the component of \( \theta \) on \( n \), is given by

\[
\theta_n : (\alpha \otimes_k g \otimes_k \beta) \otimes_{k(\mathcal{F}^{-S} \times G \times \text{Aut}(s))} (w' \otimes_k \sigma) \mapsto (\alpha, g) \cdot w' \otimes_k \beta \sigma
\]

where \( \alpha \) is a morphism in \( \mathcal{F}^{-S} \) with codomain \( n_T, g \in G, \beta \in \mathcal{F}^S(s, n_S), w' \in W' \), and \( \sigma \in \text{Aut}(s) \). Since

\[
(\alpha \otimes_k g \otimes_k \beta) \otimes_{k(\mathcal{F}^{-S} \times G \times \text{Aut}(s))} (w' \otimes_k \sigma) = (id_{n_T} \otimes 1_G \otimes_k \beta \sigma) \otimes_{k(\mathcal{F}^{-S} \times G \times \text{Aut}(s))} ((\alpha, g) \cdot w' \otimes_k id_s)
\]

with \( \beta \sigma \in \mathcal{F}^S(s, n_S) \) and \( (\alpha, g) \cdot w' \in W'(n_T) \), the map \( \theta_n \) can be simplified as

\[
\theta_n : (id_{n_T} \otimes 1_G \otimes_k \beta) \otimes_{k(\mathcal{F}^{-S} \times G \times \text{Aut}(s))} (w' \otimes_k id_s) \mapsto \beta \otimes_k w'
\]

for \( \beta \in \mathcal{F}^S(s, n_S) \) and \( w' \in W'(n_T) \). The map \( \theta_n \) is easily seen to be bijective. It is routine to check that \( \theta \) is an \( \mathcal{F}_G^m \)-module homomorphism.

The following result shows that the category \( \mathcal{F}_G^m \)-mod has enough injectives, and furthermore gives a classification of finitely generated injective \( \mathcal{F}_G^m \)-modules.

Theorem 4.10. Any finitely generated \( \mathcal{F}_G^m \)-module is finitely cogenerated by \( \mathcal{U}_G^m \).

Proof. We prove by an induction on \( m \). By Lemma 4.8, the conclusion holds for \( m = 1 \). Suppose that the conclusion holds for all \( n \) with \( 1 \leq n < m \), and let \( V \) be a finitely generated \( \mathcal{F}_G^m \)-module. Then \( V \) admits a finite filtration

\[
0 \subseteq V_{tor}^m \subseteq \cdots \subseteq V_{tor}^i \subseteq \cdots \subseteq V_{tor}^1 \subseteq V
\]

By Corollary 4.7 and the horseshoe Lemma, it suffices to show the conclusion for each quotient \( V_{tor}^{i+1} / V_{tor}^i \) and \( V_{tor}^m \). But \( V_{tor}^m \) is finite dimensional, so can be embedded into a certain finite dimensional injective module, which lies in \( \mathcal{U}_G^m \). Therefore, we only need to show the conclusion for the
quotient modules $V^{[i-1]}_{tor}/V^{[i]}_{tor}$, which is $[i]$-torsion free. Instead, we prove a stronger result; that is, we prove the conclusion for any nonempty subset $S$ of $[m]$ and any $S$-torsion free $\mathcal{F}^m_G$-modules. By Corollary 3.7, $S$-torsion free modules can be embedded into $S$-semi-induced modules, so it suffices to show that $S$-semi-induced modules is finitely cogenerated by $U^m_G$. By Definition 3.2, it turns out to show the conclusion for $S$-induced $\mathcal{F}^m_G$-modules.

Let $V$ be a finitely generated $S$-induced $\mathcal{F}^m_G$-module. Then it is isomorphic to $F_s(W)$ for some finitely generated $\mathcal{R}_s$-module $W$ and $s \in \text{Obj}(\mathcal{F}^S)$. Put $G' = G \times \text{Aut}(s)$. Note that $\mathcal{R}_s = \mathcal{F}^{-S} \times G'$ and $|S| < m$. Then by the induction hypothesis, the module $W$ is finitely cogenerated by $U^m_G$. Without loss of generality, assume that the $\mathcal{R}_s$-module $W$ can be embedded as below

$$0 \to W \to E \boxtimes k\text{Aut}(s)$$

where $E \in U^S_G$. By the exactness of $F_s$, the $S$-induced module $V \cong F_s(W)$ can be embedded into $F_s(E \boxtimes k\text{Aut}(s))$. By Lemma 4.9, $F_s(E \boxtimes k\text{Aut}(s)) \cong M(s) \boxtimes E$ which lies in $U^m_G$. This finishes the proof.

From now on we focus on the category $\mathcal{F}^m$. By the above theorem, the class $U^m_1$ of injective $\mathcal{F}^m$-modules finitely cogenerates the category $\mathcal{F}^m$-mod, where $1$ is the trivial group. Consequently, any finitely generated injective $\mathcal{F}^m$-module is a direct summand of a finite direct sum of modules in $U^m_1$. In the rest of this paper we give an explicit description of indecomposable injective $\mathcal{F}^m$-modules. It turns out that they coincide with external tensor products of indecomposable injective $\mathcal{F}$-modules, which are either indecomposable projective $\mathcal{F}$-modules or finite dimensional indecomposable injective $\mathcal{F}$-modules by [11] or [6].

**Lemma 4.11.** Let $I_1, \ldots, I_m$ be indecomposable injective $\mathcal{F}$-module. Then $I_1 \boxtimes \cdots \boxtimes I_m$ is an indecomposable injective $\mathcal{F}^m$-module and admits a local endomorphism ring.

**Proof.** We only show the conclusion for $m = 2$ since the general case can be proved similarly. Since $I_1 \boxtimes I_2$ is injective by [13, Theorem 1.1], it remains to show that it is indecomposable. By the classification of indecomposable $\mathcal{F}$-modules in [11] or [6], we obtain three cases:

1. both $I_1$ and $I_2$ are indecomposable projective $\mathcal{F}$-modules;
2. both $I_1$ and $I_2$ are indecomposable finite dimensional injective $\mathcal{F}$-modules;
3. one of $I_1$ and $I_2$ is an indecomposable projective $\mathcal{F}$-module, and the other one is an indecomposable finite dimensional injective $\mathcal{F}$-module.

Furthermore, $I_1$ is an indecomposable finite dimensional injective module if and only if $DI_1$ is an indecomposable projective $\mathcal{F}^{\text{op}}$-module, where $D = \text{Hom}_k(-, k)$ is the usual dual functor and $\mathcal{F}^{\text{op}}$ is the opposite category of $\mathcal{F}$.

In case (1), the $\mathcal{F}^2$-module $I_1 \boxtimes I_2$ is an indecomposable projective $\mathcal{F}^2$-module, so the conclusion holds. In case (2), $D(I_1)$ and $D(I_2)$ are indecomposable projective $\mathcal{F}^{\text{op}}$-modules, so $D(I_1 \boxtimes I_2) \cong D(I_1) \boxtimes D(I_2)$ is an indecomposable projective $(\mathcal{F}^2)^{\text{op}}$-module. Consequently, the $\mathcal{F}^2$-module $I_1 \boxtimes I_2$ is also indecomposable. Furthermore, it is easy to see that the endomorphism ring of $I_1 \boxtimes I_2$ is local in both cases by [1, Lemma 25.4].

Now we focus on case (3). Without loss of generality we can assume that $I_1$ is an indecomposable projective $\mathcal{F}$-module and $I_2$ is an indecomposable finite dimensional injective $\mathcal{F}$-module. We want to show that the endomorphism ring of $I_1 \boxtimes I_2$ is local. Suppose that $I_1$ is induced from an irreducible left $k\text{Aut}([n])$-module $U$ (that is, $I_1 \cong k\mathcal{F} \otimes_{k\text{Aut}([n])} U$) and the $\mathcal{F}^{\text{op}}$-module $D(I_2)$ is induced from an irreducible right $k\text{Aut}([I])$-module $W$. Since $I_1 \boxtimes I_2 \cong F_s(U \boxtimes I_2)$ where $s = [n]$, by the universal property of $S$-induced module, we obtain a ring isomorphism $\text{End}_{\mathcal{F}^2}(I_1 \boxtimes I_2) \cong \text{End}_{\mathcal{R}_s}(U \boxtimes I_2)$. By
Lemma 4.6, the $R_s$-module $U \boxtimes I_2$ is injective. Moreover, it is indecomposable since the dual $R_s^{op}$-module $D(U \boxtimes I_2) \cong D(U) \boxtimes D(I_2)$ is induced from the irreducible right $k(\text{Aut}(\langle n \rangle) \times \text{Aut}(\langle l \rangle))$-module $D(U) \boxtimes W$. Therefore by [1, Lemma 25.4], the endomorphism ring $\text{End}_{R_s}(U \boxtimes I_2)$ is local as desired.

As a corollary, we have:

**Corollary 4.12.** Let $E_i = I_i^1 \boxtimes \cdots \boxtimes I_i^m$ be an $F^m$-module for $i \in [n]$, where each $I_i^j$ is an indecomposable injective $F$-module for $j \in [m]$. Then any indecomposable direct summand of the $F^m$-module $\bigoplus_{i=1}^n E_i$ is isomorphic to certain $E_i$.

**Proof.** This follows from Lemma 4.11 and [1, Theorem 12.6(2)].

Now we are ready to prove the second theorem in the Introduction of this paper.

**A proof of Theorem 1.2.** By Theorem 4.10 and Corollary 4.7, any indecomposable injective $F^m$-module $I$ is isomorphic to an indecomposable direct summand of a finite direct sum of modules in $U^m$. Since the external tensor product $\boxtimes$ commutes with the direct sum $\bigoplus$, such a finite direct sum can be written as $\bigoplus_{i=1}^n E_i$ where $E_i$ is described in Corollary 4.12. Thus, its indecomposable direct summand is isomorphic to a certain $E_i$, as desired.

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