Multiplet containing components with different masses

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Abstract

A principle possibility for the existence of a multiplet including the components with the different masses is indicated. This paper is dedicated to the memory of Anna Yakovlevna Gelyukh (Kalaida).

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1 Introduction

We start from the citation a very surprising (for us) appraisal of supersymmetry \cite{1,2,3,4} given by Yury Abramovich Golfand during the Conference ”Supersymmetry-85” at the Kharkov State University in 1985. He said \cite{5} that supersymmetry did not justify his hopes to find a generalization of the Poincaré group such that every its representation include the particles of different masses. Golfand and Likhtman had missed their aim, but had instead found supersymmetry, every representation of which contains the fields of different spins.

So, the problem was raised and requires its solution. In the present paper we give a possible solution of the problem of the multiplet which components have the different masses\footnote{Concerning another approach to this problem see the paper \cite{6}.}. We illustrate the solution on the example of the centrally extended \((1 + 1)\)-dimensional Poincaré algebra \cite{7,8,9,10,11,12}.

2 Tensor extension of the Poincaré algebra

In the paper \cite{10} the tensor extension of the Poincaré algebra in \(D\) dimensions

\[
[M_{ab}, M_{cd}] = (g_{ad}M_{bc} + g_{bc}M_{ad}) - (c \leftrightarrow d),
\]

\[
[M_{ab}, P_c] = g_{bc}P_a - g_{ac}P_b,
\]

\[
[P_a, P_b] = Z_{ab},
\]

\[
[M_{ab}, Z_{cd}] = (g_{ad}Z_{bc} + g_{bc}Z_{ad}) - (c \leftrightarrow d),
\]

\[
[P_a, Z_{bc}] = 0,
\]

\[
[Z_{ab}, Z_{cd}] = 0
\]

was introduced and its Casimir operators

\[
Z_{a_1a_2}Z_{a_2a_3} \cdots Z_{a_{2k-1}a_{2k}}Z_{a_{2k}a_1}, \quad (k = 1, 2, \ldots);
\]

\[
P^{a_1}Z_{a_1a_2}Z_{a_2a_3} \cdots Z_{a_{2k-1}a_{2k}}Z_{a_{2k}a_{2k+1}}
\]

\[
+Z^{a_1}Z_{a_1a_2} \cdots Z_{a_{2k-1}a_{2k}}Z_{a_{2k}a_{2k+1}}M_{a_{2k+1}a_1}, \quad (k = 0, 1, 2, \ldots);
\]

\[
\epsilon^{a_1a_2 \cdots a_{2k-1}a_{2k}}Z_{a_1a_2} \cdots Z_{a_{2k-1}a_{2k}}, \quad 2k = D
\]

were constructed. Here \(M_{ab}\) are generators of rotations, \(P_a\) are generators of translations, \(Z_{ab}\) is a tensor generator and \(\epsilon^{a_1 \cdots a_{2k}}, \epsilon^{01 \cdots 2k-1} = 1\) is the totally antisymmetric Levi-Civita tensor in the even dimensions \(D = 2k\).
Generators of the left shifts with a group element \( G \), acting on the function \( f(u) \)

\[
[T(G)f](u) = f(G^{-1}u), \quad u = (x^a, z^{ab}),
\]

have the form

\[
P_a = -\left( \partial_x^a + \frac{1}{2} x^b \partial_{z^{ab}} \right),
\]

\[
Z_{ab} = -\partial_{z^{ab}},
\]

\[
M_{ab} = x_a \partial_{x^b} - x_b \partial_{x^a} + z^c_a \partial_{z^{bc}} - z^c_b \partial_{z^{ac}} + S_{ab}, \tag{2.5}
\]

where coordinates \( x^a \) correspond to the translation generators \( P_a \), coordinates \( z^{ab} \) correspond to the generators \( Z_{ab} \) and \( S_{ab} \) is a spin operator. In the expressions (2.5) \( \partial_u \equiv \frac{\partial}{\partial u} \).

### 3 Two-dimensional case

In the case of the extended two-dimensional Poincaré algebra the Casimir operators (2.2), (2.3) and (2.4) can be expressed as degrees of the following generating Casimir operators:

\[
Z = -\frac{1}{2} \epsilon^{ab} Z_{ab}, \tag{3.1}
\]

\[
C = P^a P_a + Z^{ab} M_{ba}, \tag{3.2}
\]

where \( \epsilon^{ab} = -\epsilon^{ba} \), \( \epsilon^{01} = 1 \) is the completely antisymmetric two-dimensional Levi-Civita tensor. The relations (2.5) can be represented as

\[
P_t \equiv P_0 = -\partial_t - \frac{x}{2} \partial_y, \tag{3.3}
\]

\[
P_x \equiv P_1 = -\partial_x + \frac{t}{2} \partial_y, \tag{3.4}
\]

\[
J = \frac{1}{2} \epsilon^{ab} M_{ab} = -t \partial_x - x \partial_t + S_{01}, \tag{3.5}
\]

\[
Z = \partial_y, \tag{3.6}
\]

where \( t \equiv x^0 \) is a time, \( x \equiv x^1 \) is a space coordinate, \( y \equiv z^{01} \) is a coordinate corresponding to the central element \( Z \) and the space-time metric tensor has the following nonzero components: \( g_{11} = -g_{00} = 1 \).

The extended Poincaré algebra (2.1) in this case can be rewritten in the following form (see also [9]):

\[
[P_a, J] = \epsilon^a_b P_b,
\]
\[ [P_a, P_b] = \epsilon_{ab}Z, \]
\[ [P_a, Z] = 0, \quad [J, Z] = 0 \quad (3.7) \]

and for the Casimir operator (3.2) we have the expression
\[ C = P^a P_a - 2ZJ. \quad (3.8) \]

For simplicity let us consider the spin-less case \( S_{01} = 0 \). We define a mass square operator as follows
\[ M^2 \overset{\text{def}}{=} P^a P_a = P_x^2 - P_t^2. \quad (3.9) \]

4 New coordinates

By a transition from \( t, x \) and \( y \) to the new coordinates
\[
x_{\pm} = \frac{t \pm x}{2}, \\
y_\pm = y - \frac{t^2 - x^2}{4}, \quad (4.1)\]

we obtain the following expressions for the generators:
\[
P_- = \partial_{x_-}, \\
P_+ = 2x_- \partial_{y_-} - \partial_{x_+}, \\
J = x_- \partial_{x_-} - x_+ \partial_{x_+}, \\
Z = \partial_{y_-}, \quad (4.5)\]

where
\[ P_{\pm} = P_x \pm P_t. \]

These generators satisfy the following commutation relations:
\[
[P_+, P_-] = -2Z, \\
[J, P_{\pm}] = \pm P_{\pm}, \\
[J, Z] = 0, \\
[P_{\pm}, Z] = 0. \quad (4.6)\]

We see that \( P_{\pm} \) are step-type operators.

The Casimir operator (3.8) in the new coordinates takes the form
\[ C = P_+ P_- + Z - 2ZJ \quad (4.7) \]

and the mass square operator is
\[ M^2 = P_+ P_- + Z, \quad (4.8) \]

in terms of which the Casimir operator (4.7) is expressed as
\[ C = M^2 - 2ZJ. \quad (4.9)\]
5 Multiplet

As a complete set of the commuting operators we choose the Casimir operators $Z$, $C$ and rotation operator $J$. Let us assume that there exist such a state $\Psi_{z,j}(x_+, x_-, y_-)$ that

$$ P_- \Psi_{z,j}(x_+, x_-, y_-) = 0, \quad (5.1) $$

$$ Z \Psi_{z,j}(x_+, x_-, y_-) = z \Psi_{z,j}(x_+, x_-, y_-), \quad (5.2) $$

$$ J \Psi_{z,j}(x_+, x_-, y_-) = j \Psi_{z,j}(x_+, x_-, y_-). \quad (5.3) $$

The equations (4.2) and (5.1) mean that $\Psi_{z,j}(x_+, x_-, y_-)$ independent on the coordinate $x_-$. Then, as a consequence of the relations (4.4), (4.5), (5.2) and (5.3), we come to the following expression for the state $\Psi_{z,j}(x_+, y_-)$:

$$ \Psi_{z,j}(x_+, y_-) = ax_+^{-j} e^{zy_-}, \quad (5.4) $$

where $a$ is some constant.

For the states

$$ P_+^k \Psi_{z,j}(x_+, y_-), \quad (k = 0, 1, 2, \ldots) \quad (5.5) $$

we obtain

$$ JP_+^k \Psi_{z,j}(x_+, y_-) = J P_+^k \Psi_{z,j}(x_+, y_-) = (j + k) P_+^k \Psi_{z,j}(x_+, y_-), \quad (5.6) $$

$$ M^2 P_+^k \Psi_{z,j}(x_+, y_-) = M^2 P_+^k \Psi_{z,j}(x_+, y_-) = (2k + 1) z P_+^k \Psi_{z,j}(x_+, y_-), \quad (5.7) $$

$$ CP_+^k \Psi_{z,j}(x_+, y_-) = CP_+^k \Psi_{z,j}(x_+, y_-) = (1 - 2j) z P_+^k \Psi_{z,j}(x_+, y_-), \quad (5.8) $$

from which we have

$$ J = j + k, \quad (5.9) $$

$$ M^2 = (2k + 1) z, \quad (5.10) $$

$$ C = M^2 - 2z J = (1 - 2j) z. \quad (5.11) $$

The states (5.5) are the components of the multiplet.

By excluding $k$ from the relations (5.9) and (5.10), we come to the Regge type trajectory

$$ J = \alpha(0) + \alpha' M^2 \quad (5.12) $$

with parameters

$$ \alpha(0) = j - \frac{1}{2}, \quad (5.13) $$

$$ \alpha' = \frac{1}{2z}. \quad (5.14) $$
6 Conclusion

Thus, on the example of the centrally extended $(1 + 1)$-dimensional Poincaré algebra we solved the problem of the multiplet which contains the components with the different masses.

It would be interesting to construct the models based on such a multiplet.

Note that, as can be easily seen from the commutation relation

\[
[\partial_{x^a} + A_a, \partial_{x^b} + A_b] = F_{ab},
\]

where $A_a$ is an electromagnetic field and $F_{ab}$ is its strength tensor, the above mentioned extended $D = 2$ Poincaré algebra (3.7) is arisen in fact when an “electron” in the two-dimensional space-time is moving in the constant homogeneous electric field.

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