Equivalence of coefficients extraction of one-loop master integrals

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Abstract
Now, there have been many different methods to calculate one-loop amplitudes. Two of them are the unitarity cut method and the generalized unitarity cut method. In this short paper, we present an explicit connection between these two methods, especially how the extractions of triangle and bubble coefficients are equivalent to each other.

Keywords: scattering amplitude, one loop correction

1. Introduction

Experiments of high-energy physics such as the LHC requires calculations of cross sections of processes involving multiple particles. Precise theoretical predictions of these processes need calculations of Feynman diagrams at the one-loop level and beyond. However, loop calculation is tedious and very inefficient using the standard method. In recent years new methods have been developed to prompt the calculation and avoid laborious work. Currently, one-loop calculation has been considered as a solved problem and the focus is the higher loop calculations.

For loop calculation, the main approach is the Passarino–Veltman (PV) reduction method [1]. The reduction can be divided into two categories: the reduction at the integrand level, and the reduction at the integral level. For a one-loop case, the efficient integrand level reduction is introduced by Ossola, Papadopoulos and Pittau in [2]. But for the integral level reduction, the PV reduction method is inefficient.

Another two different methods, the unitarity cut method and the generalized unitarity cut method, are two mainly used methods now.

If we take the imaginary part of a given branch at both sides, we will have

\[ \text{Im} A^{1-\text{loop}} = \sum_i C_i \text{Im} I_i. \tag{1.3} \]

Thus, if we can calculate the imaginary part easily, we can extract the master coefficients by comparing both sides. By Cutkosky rules [5], the calculation of the imaginary part is carried out following phase space integration

\[ \Delta A^{1-\text{loop}} = \int d\mu \ A_{\text{Left}}^{\text{tree}} \times A_{\text{Right}}^{\text{tree}}. \tag{1.4} \]

\[ A^{1-\text{loop}} = \sum C_i I_i. \tag{1.1} \]

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where the Lorentz-invariant phase space (LIPS) measure is defined by
\[ d\mu = d^4\ell_1 \ d^4\ell_2 \ \delta^4(\ell_1 + \ell_2 - K) \ \delta^{(+)}(\ell_1^2) \ \delta^{(+)}(\ell_2^2). \tag{1.5} \]

Here, the superscript \((+)\) on the delta functions for the cut propagators denotes the choice of a positive-energy solution. Although the integration has been simplified to two dimensions, carrying it out is still a difficult task. The breakthrough comes after realizing that, by holomorphic anomaly, such a two-dimensional phase space integration can be translated to read out the residue of corresponding poles \([6–9]\). Using this technique, analytic expressions of coefficients of one-loop master integrals have been given in the series papers \([10–13]\).

Inspired by the double cut for the imaginary part, multiple cuts have also been proposed in \([14, 15]\). In particular, in \([16]\) it has been shown that putting four propagators on shell, one can read out the coefficient of boxes as the multiplication of four on-shell tree-level amplitudes at the four corners. This generalized unitarity cut method has been further developed in \([17, 18]\).

Both methods, i.e. the unitarity cut method and the generalized unitarity cut method, have solved the one-loop integral level reduction completely. However, the connection between these two methods has still not been clearly demonstrated. It is our purpose in this short paper to reveal the equivalence of these two methods.

The plan of the paper is as follows. In section 2 we have reviewed the two methods, and in section 3 we will present our proof of the equivalence of these two methods.

2. Review of the unitarity cut method and generalized unitarity cut method

In this section, we will review both methods to establish the basis for our investigation. In the first subsection, we will briefly review the unitarity cut method and write down the major formula. We will review the generalized unitarity cut method in the second subsection.

2.1. Review of the unitarity cut method

The unitary property of the S-matrix means \(S^\dagger S = 1\). Writing \(S = 1 + iT\), we have \(2\text{Im } T = T^\dagger T\), which is the familiar optical theorem. Expanding this equation by the order of the coupling constant, we see that the imaginary part of the one-loop amplitude is related to a product of two on-shell tree-level amplitudes. This imaginary part should be viewed more generally as a discontinuity across the branch cut singularity of the amplitude—in a kinematic configuration where one kinematic invariant momentum, say \(K^2\), is positive, while all others are negative. This condition isolates the momentum channel \(K\) of our interest; \(K\) is the sum of some of the external momenta.

As we have mentioned in the introduction, for the one-loop amplitude the Cutkosky rule gives the integral \((1.4)\). Now, we discuss how to carry out the phase space integration.

Since we are trying to compare with the method given in \([18]\), we will focus on massless theory in pure 4D, thus we rewrite it as
\[ A^{1\text{-loop}} = -it(4\pi)^2 \int \frac{d^4\ell}{(2\pi)^2} \ \delta^{(+)}(\ell^2) \ \delta^{(+)}((\ell - K)^2) \ \mathcal{T}^{(N)}(\ell), \tag{2.1} \]

where the integrand can be generally represented as
\[ \mathcal{T}^{(N)}(\ell) = \frac{\prod_{j=1}^{n+k} (-2\ell \cdot P_j)}{\prod_{i=1}^{k} D_i(\ell)}, \quad D_i(\ell) = (\ell - K_i)^2. \tag{2.2} \]

Here, \(N\) is defined as the degree of amplitude, which is just equal to \(n\) and is the half power of momentum in the fraction. To carry out the integration, we use the spinor technique to write the loop momentum as \(\ell = i\lambda\bar{\lambda}\), with \(\lambda, \bar{\lambda} \in \mathbb{C}P^1\), and the measure can be expressed as follows
\[
\int d\mu (\bullet) = \int d^4\delta(\ell^2)\delta((\ell - K)^2) (\bullet) = \\
\int_0^\infty dt \int d\lambda \lambda \ d\bar{\lambda} \delta(K^2 - t\langle|\lambda|\bar{\lambda}\rangle) (\bullet) = \\
\int_{\mathbb{R}^n} d\lambda \lambda \ d\bar{\lambda} \frac{K^2}{\langle|\lambda|\bar{\lambda}\rangle^2} (\bullet),
\]

where the \(t\) integration has been carried out. After the \(t\) integration, the \(\mathcal{T}^{(N)}(\ell)\) becomes
\[ \mathcal{T}^{(N)}(\lambda, \bar{\lambda}) = \frac{(K^2)^n \prod_{j=1}^{n+k} \langle|\lambda| R_j |\bar{\lambda}\rangle}{\prod_{i=1}^{k} \langle|\lambda| Q_i |\bar{\lambda}\rangle}, \tag{2.4} \]

where \(Q_i = \frac{k_i^2}{K_i} K - K_i\) and \(R_i = -P_j\). We also define the integrand
\[ I_{\text{term}} = \frac{K^2}{\langle|\lambda|\bar{\lambda}\rangle^2} \mathcal{T}^{(N)}(\lambda, \bar{\lambda}) = \frac{\langle G(\lambda) \prod_{j=1}^{n+k} |a_j| |\bar{\lambda}\rangle}{\langle|\lambda|\bar{\lambda}\rangle^2 \prod_{i=1}^{k} \langle|\lambda| Q_i |\bar{\lambda}\rangle}, \tag{2.5} \]

where \(G(\lambda)\) is constant, and \(|a_j| = \langle|\lambda| R_j\rangle\).

The expression \(\mathcal{T}^{(N)}(\lambda, \bar{\lambda})\) contains all the information of coefficients of boxes, triangles and bubbles. To disentangle the information, canonical splitting has been given in \([12, 13, 19–21]\)
\[ \frac{|a| |\bar{\lambda}\rangle}{\langle|\lambda| Q_1 |\bar{\lambda}\rangle\langle|\lambda| Q_2 |\bar{\lambda}\rangle} = \frac{|a| Q_1 |\bar{\lambda}\rangle}{\langle|\lambda| Q_2 |\bar{\lambda}\rangle\langle|\lambda| Q_2 |\bar{\lambda}\rangle} + \frac{|a| Q_2 |\bar{\lambda}\rangle}{\langle|\lambda| Q_2 |\bar{\lambda}\rangle\langle|\lambda| Q_2 |\bar{\lambda}\rangle}. \tag{2.6} \]
After making the splitting, we get the canonical splitting [12]

The second line contains all the information of the coefficients of triangles and boxes, while the first line contains purely the information of the coefficients of bubbles. More explicitly, using holomorphic anomaly integration of equation (2.3) and taking residues of various poles at the first line, we get the bubble coefficients. The detailed process is in [19]. Based on the above canonical splitting, we can extract coefficients of various master integrals. The algebraic expressions are summarized as follows [22]:

- **Box coefficients**

The coefficient of the box, identified by the two cut propagators along with \( D_r \) and \( D_s \), is given by

\[
C[K_r, K_s, K] = \frac{1}{2} \left( T^{(N)}(t) D_r(t) D_s(t) \big|_{h \rightarrow t, r, s \rightarrow r, s} + [P_{r,1} \leftrightarrow P_{s,2}] \right).
\]

(2.8)

We define \( Q_i \) and \( Q_r \) as follows

\[
Q_i = \frac{K_r^2}{K^2} K_i - K_r,
\]

\[
Q_r = \frac{K_r^2}{K^2} K_s - K_r.
\]

(2.9)

And define the auxiliary vectors \( P_{r,1} \) and \( P_{s,2} \) as the null linear combinations of \( Q_i \) and \( Q_r \)

\[
P_{r,1} = Q_i + \left( -\frac{Q_i \cdot Q_r + \sqrt{\Delta_{rs}}}{Q_i^2} \right) Q_r,
\]

\[
P_{s,2} = Q_i + \left( -\frac{Q_i \cdot Q_r - \sqrt{\Delta_{rs}}}{Q_i^2} \right) Q_r,
\]

\[
\Delta_{rs} = (Q_i \cdot Q_r)^2 - Q_i^2 Q_r^2.
\]

(2.10)

### Triangle coefficients

If \( N < -1 \), the triangle coefficients are zero. If \( N \geq -1 \), the coefficient of the triangle, identified by the two cut propagators along with \( D_s \), is given by

\[
C[K_s, K] = \frac{1}{2(N+1)!} \sqrt{\Delta_{ss}^{N+1}} \left( P_{r,1} P_{s,2} \right)^{N+1}
\]

\[
\times \left[ \text{d}^{N+1} T^{(N)}(t) D_s(t) \left\{ \left( \lambda K \right)^{N+1} \right\}_{h \rightarrow \tau_q, \lambda \rightarrow \tau, r, s} + [P_{s,1} \leftrightarrow P_{s,2}] \right]_{\tau \rightarrow 0}.
\]

(2.11)

Here, we use the following definitions. The vectors \( P_{r,1} \) and \( P_{s,2} \) are null linear combinations of \( Q_i \) and \( K \)

\[
P_{r,1} = Q_i + \left( -\frac{Q_i \cdot K + \sqrt{\Delta_{rs}}}{K^2} \right) K_s,
\]

\[
P_{s,2} = Q_i + \left( -\frac{Q_i \cdot K - \sqrt{\Delta_{rs}}}{K^2} \right) K_s.
\]

(2.12)

The effect of the multiple derivative of the parameter \( \tau \), evaluated at \( \tau = 0 \), is simply to pick out a term in the series expansion.

### Bubble coefficients

There is just one bubble in the cut channel \( K \). If \( N < 0 \), the coefficient is zero. If \( N \geq 0 \), the coefficient is

\[
C[K] = K^2 \sum_{q=0}^{N} \left( \frac{(-1)^q}{q!} \frac{d^q}{ds^q} \left( B_{r,1}^{(0)}(s) \right) \right) + \sum_{r=1}^{N} \left( B_{r,1}^{(0)}(s) - B_{r,1}^{(0)}(s) \right) \bigg|_{s=0}.
\]

(2.13)

where

\[
B_{r,1}^{(0)}(s) = \frac{d^N}{ds^N} \left( \frac{2\eta \cdot K)^{m+1}(\lambda K)\lambda^N}{N!\eta^N K! \eta^N (m+1)(K^2)^{m+1}(\lambda K)^{N+1}} T^{(N)}(t) \bigg|_{h \rightarrow \tau_q, \lambda \rightarrow \tau, r, s} \right)
\]

\[
B_{r,1}^{(0)}(s) = \frac{d^N}{ds^N} \left( \frac{(-1)^{b+1}}{b!(m+1)\sqrt{\Delta_{rs}^{b+1}}(P_{r,1} P_{s,2})} \right) \frac{d^b}{d\tau^b} \left( \frac{\left( \lambda K P_{r,1} \right)^{m+1}(\lambda Q_i \eta \lambda)^b (\lambda K \lambda)^{N+1}}{\left( \lambda K P_{r,1} \right)^{m+1}(\lambda \eta K \lambda)^{N+1}} T^{(N)}(t) D_s(t) \bigg|_{h \rightarrow \tau_q, \lambda \rightarrow \tau, r, s} \right)
\]
Here, \( \eta, \eta' \) are arbitrary spinors, which should be generic in the sense that they do not coincide with any spinors from massless external legs. The above expressions for bubble coefficients look complicated. However, it is just the calculations of residues of various poles in the first line of canonical splitting (2.7). As will become clear in section 3, our comparison will be carried out at the level of (2.7) only.

2.2. Review of the generalized unitarity cut method

As we have mentioned, our purpose in this paper is to establish the explicit relation between the unitarity cut method and the generalized unitarity cut method proposed in [18]. In this subsection, we will briefly review their results.

The key idea of their method is the generalization of formula (1.3) with multiple cut

\[
\text{Cut}^{(n)}_{[\ell_1, \ldots, \ell_n]} = \sum_c C^c_\ell \text{Cut}^{(n)}_{[\ell_1, \ldots, \ell_n]} I^c_\ell + \ldots + \sum_c C^c_\ell \text{Cut}^{(n)}_{[\ell_1, \ldots, \ell_n]} I^c_\ell,
\]

where the sum of \( c \) is over all different channels, and \( \text{Cut}^{(n)}_{[\ell_1, \ldots, \ell_n]} \) means to cut \( n \) propagators \( D_{\ell_1, \ldots, \ell_n} \). Based on this formula, when applying to \( D = 4 \), coefficients of master integrals can be read out as follows:

- **Box coefficients**
  For the quadruple cut, \( \text{Cut}^{(4)} \) is simply proportional to \( C_4 \). That is because the quadruple cut renders \( I_0(n < 4) \) zero [16, 23]. To be concrete,

\[
C_4 = \frac{1}{2} \sum_{\ell \in S} A_{4,1}(\ell) A_{4,2}(\ell) A_{R,1}(\ell) A_{R,2}(\ell),
\]

where \( S \) is the solution set for the four delta functions of the cut propagators

\[
S = \{ \ell \mid |\ell|^2 = 0, (\ell - K)^2 = 0, (\ell - K)^2 = 0 \}.
\]

- **Triangle coefficients**
  For triangle coefficients in 4D, it is not so lucky because triple cuts cannot fix the internal momentum completely and there is a free parameter left. A consequence of this freedom is that some box integrals will contribute to triple cuts. Thus, we need to have a cleverer way to disentangle their information. The way to do so is by following [18]. Suppose we cut propagators \( \ell_1^2 = \ell_2^2 = (\ell - K)^2, \ell_1^2 = (\ell - K)^2 \). Without loss of generality, we can choose the external conditions with a Lorentz boost to be

\[
K_{ni} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},
\]

\[
(K_i)_{ni} = \begin{pmatrix} E_+ & 0 & 0 & 0 \\ 0 & E_+ & 0 & 0 \\ 0 & 0 & E_+ & 0 \\ 0 & 0 & 0 & E_+ \end{pmatrix},
\]

where \( P_{ni} = p_{(n')}_{ni} = \begin{pmatrix} p^0 + p^3 & p^1 - ip^2 \\ p^1 + ip^2 & p^0 - p^3 \end{pmatrix} \). Now, the on-shell condition gives

\[
\ell_i = \begin{pmatrix} \alpha_i^+ \\ \ell_i \\ \alpha_i^- \end{pmatrix} = \begin{pmatrix} \alpha_i^+ \cos\theta - i\alpha_i^- \sin\theta \\ \ell_i \\ \alpha_i^- \cos\theta + i\alpha_i^+ \sin\theta \end{pmatrix},
\]

with

\[
\vec{\ell} = \alpha_i^+ \alpha_i^- \equiv r^2 = -E_+ E_+ (1 + E_+)(1 + E_+) \left( E_+ - E_-^2 \right),
\]

and \( \alpha_i^+ \) are some definite functions of \( E_+ \) whose explicit expressions are not important in the derivation. Here, we only discuss under the condition \( r^2 > 0 \), while the results of other regions of \( r^2 \) can be obtained by analytic continuation. Integrate out all delta functions, we have

\[
\text{Cut}^{(3)} \propto \int_{0}^{2\pi} d\theta F(r \cos\theta, r \sin\theta),
\]

where \( F = A_1 A_2 A_3 \) is the factorized tree amplitude after the triple cut. To find the proportionality constant, we consider the simplest case when the loop amplitude

\[
A^{\text{one-loop}} = \int \frac{d^4\ell}{(2\pi)^4} \frac{1}{(\ell^2 + P_1^2)(\ell^2 - P_3^2)}. \]

We expect \( \text{Cut}^{(3)} = C_3 = 1 \), because no box integrals exist. Now, \( A_1 A_2 A_3 = 1 \) and we find the proportionality constant is equal to \( 1/2\pi \).

For further derivation, we change the variable to \( z = r \cos\theta \). The integral becomes

\[
\text{Cut}^{(3)} = \frac{1}{2\pi} \int_{-r}^{r} \frac{dz}{\sqrt{r^2 - z^2}} \left[ F(z, \sqrt{r^2 - z^2}) + F(z, -\sqrt{r^2 - z^2}) \right].
\]

Then, we consider \( z \) to be a complex variable. That is because we want to identify the box integral contribution from \( \text{Cut}^{(3)} \), and the momenta that satisfy the quadruple cut are always complex. The integrand has a branch cut which can be taken to be \((−r, r)\), and the integral itself can be rewritten to be a contour integral encircling the branch cut in a clockwise direction

\[
\text{Cut}^{(3)} = \frac{1}{4\pi} \int_{C_0} \frac{dz}{\sqrt{r^2 - z^2}} \left[ F(z, \sqrt{r^2 - z^2}) + F(z, -\sqrt{r^2 - z^2}) \right].
\]
The integrand of the above integral has simple poles on the complex $z$ plane. Simple poles at finite $z$ come from the remaining propagators in $A_1(z)A_2(z)A_3(z)$. When $z$ approaches these poles, an additional propagator goes on shell, and the integrand is further factorized to be a product of four pieces of tree amplitudes $A'_1(z)A'_2(z)A'_3(z)$, which is proportional to $C_2$ for some channel $c$. It is now clear that to eliminate the contributions from box integrals, we can simply drop the residues at finite poles, and take the residue at infinity only. So, we deform the contour to obtain

$$C_2 = \frac{1}{4\pi}\int_C \frac{dz}{\sqrt{r^2 - z^2}} [F(z, \sqrt{r^2 - z^2}) + F(z, -\sqrt{r^2 - z^2})],$$

where the contour $C$ encircles the pole at infinity.

- **Bubble coefficients**

For bubble coefficients, as a triple cut case, a double cut will contain contributions from boxes and triangles. The separation of the bubble part from the others is carried out in [18] as follows

$$C_2 = \int_C \frac{dL}{N_L} \mathcal{M}_L(\ell_1, \ell_2) \mathcal{M}_R(\ell_1, \ell_2),$$

where we deformed the loop momentum using the BCFW (Britto–Cachazo–Feng–Witten recursion relation) shift [24, 25] with $\ell_1 = \ell_1 - q$ and $\ell_2 = \ell_2 + q$ with $q$ a reference momentum, which keeps $\ell_1, \ell_2$ on shell as well as momentum conservation. The contour $C$ encircles the pole at infinity.

### 3. Connection between these two methods

Having roughly reviewed the two methods in the previous section, in this section we will show their connection explicitly.

#### 3.1. Connection between input amplitude

The main difference between these two methods is the number of cuts implemented on the loop propagators. For the unitarity cut method with double cuts, the input is always the multiplication of two on-shell tree-level amplitudes, i.e.

$$\text{Input}_2 = T^{(N)}(\ell) = A_L A_R,$$

where we have assumed the propagators $(\ell - K)^2$ have been cut. However, for the generalized unitarity cut method, depending on which coefficient we are looking for, the input is different. For the triangle coefficient, the triple cut is needed and the input is

$$\text{Input}_3 = A_{L,s} A_{L,s} A_{R,R},$$

where an extra propagator $D_0 = (\ell - K)^2$ has been cut. For the box coefficient, the quadruple cut is needed and the input is

$$\text{Input}_4 = A_{L,s} A_{L,s} A_{R,R},$$

where two extra propagators $D_0 = (\ell - K)^2$ and $D_0 = (\ell - K)^2$ have been cut.

Although the input amplitudes seem to be different, they can be easily related by the factorization property of the tree amplitude. For example,

$$A_L \times A_R \times D_0 |_{D_0 = 0} = \sum_c A_{L,c} A_{L,c} A_{R,R} D_0 |_{D_0 = 0} = A_{L,s} A_{L,s} A_{R,R}.$$

After calculating each term in the formula, it is easy to show the relationship of the left and right side of the equation $T^{(N)}(\ell) = A_L \times A_R$

$$T^{(N)}(\ell) \cdot D_0(\ell) |_{D_0 = 0} = A_{L,s} A_{L,s} A_{R,R}.$$

These relations will be used when we prove the equivalence of the two methods.

#### 3.2. The equivalence of the box coefficient between two methods

Let us recall the box coefficients of the unitarity cut method in (2.8)

$$C_{(K_s, K_s, K)} = \frac{1}{2} \left\{ T^{(N)}(\ell) \cdot D_0(\ell) |_{D_0 = 0} + [P_{s,1} \leftrightarrow P_{s,2}] \right\}$$

The role of the replacement of $|\lambda\rangle$ and $|\bar{\lambda}\rangle$ is to put the propagators $D_0(\ell)$ and $D_0(\ell)$ on shell. To see it, one can see that using the definition in (2.10), $Q_s$ and $Q_{s'}$ can be expressed in terms of $P_{s,1}$ and $P_{s,2}$ as

$$Q_s = \frac{\sqrt{\Delta_{s,1}} + Q_s \cdot Q_{s,1}}{2 \sqrt{\Delta_{s,1}}},$$

$$Q_{s'} = \frac{\Delta_{s,1}}{2 \sqrt{\Delta_{s,1}}} (P_{s,1} - P_{s,2}).$$

Thus, we have

$$\langle \lambda | Q_s | \bar{\lambda} \rangle = \frac{\Delta_{s,1}}{2 \sqrt{\Delta_{s,1}}} (\langle \lambda | P_{s,1} | \bar{\lambda} \rangle + \langle \lambda | P_{s,2} | \bar{\lambda} \rangle),$$

$$\langle \lambda | Q_{s'} | \bar{\lambda} \rangle = \frac{\Delta_{s,1}}{2 \sqrt{\Delta_{s,1}}} (\langle \lambda | P_{s,1} | \bar{\lambda} \rangle - \langle \lambda | P_{s,2} | \bar{\lambda} \rangle).$$
which are zero after the substitutions $|\tilde{\lambda}| \rightarrow |P_{a1,1}|$, $|\lambda| \rightarrow |P_{a1}|$, or $|\tilde{\lambda}| \rightarrow |P_{a2,1}|$, $|\lambda| \rightarrow |P_{a2,2}|$. Namely, propagators
\[
D_\ell = (\ell - K_\ell)^2 = \frac{K^2(\lambda Q_\ell |\tilde{\lambda}|)}{\lambda |\tilde{\lambda}|} = 0,
\]
\[
D_z = (\ell - K_z)^2 = \frac{K^2(\lambda Q_z |\tilde{\lambda}|)}{\lambda |\tilde{\lambda}|} = 0
\] (3.9)
are indeed on shell after the above substitutions. With this observation and (3.5), the box coefficient (3.6) is simply
\[
C[K_\tau, K_\tau, K] = \frac{1}{2} \sum_{\ell \in S} A_{L_\tau,1}(\ell)A_{L_\tau,2}(\ell)A_{R_\tau,1}(\ell)A_{R_\tau,2}(\ell),
\] (3.10)
which is the same as result (2.15) coming from the generalized unitarity cut method. Thus, we have proved that the expressions of box coefficients for the unitarity cut method and generalized unitarity cut method are indeed the same.

3.3. Triangle coefficients

For the triangle coefficient, the connection between the two methods is not so obvious. The main difference is the choice of parameter for cut on-shell complexified momentum. In the unitarity cut method, we parameterize the momentum as $|\ell| \rightarrow |P_{a1} - \tau P_{a2}|$, where a complex parameter $\tau$ is introduced. In the generalized unitarity cut method, after putting three propagators on shell, we introduce a parameter $z$ corresponding to the remaining degrees of freedom of the loop momentum. Although $\tau$ and $z$ seem to be very different, there is a simple relation between these two parameters, which we will derive by direct calculation. Using this relation, we will prove the equivalence of the unitarity cut method and generalized unitarity cut method in computing the triangle coefficient.

3.3.1. Connection between parameter $\tau$ and $z$. To find the connection between $\tau$ and $z$, we notice that they both show up in the loop momentum $\ell$. We first calculate $\ell$ by the unitarity cut method. For comparison with [18], we put $\ell^2 = (\ell - K_\ell)^2 = 0$ and use the external condition (2.17); thus by the definition (2.12) of $Q_\tau$, $P_{a1}$ and $P_{a2}$, we obtain
\[
Q_\tau = \frac{K^2}{K^2} K - K_\tau = \begin{pmatrix}
-E_\tau E_\tau - E_+ & 0 & 0 & \alpha_+^0 & 0 \\
0 & -E_\tau E_\tau - E_- & 0 & 0 & \alpha_-^0 \\
\end{pmatrix}
\]
\[
P_{a1} = \begin{pmatrix}
\alpha_+^0 - \alpha_\tau & 0 & 0 & 1 & 0 \\
0 & \alpha_\tau - \alpha_-^0 & 0 & 0 & 0 \\
\end{pmatrix}
\]
\[
P_{a2} = \begin{pmatrix}
0 & 0 & 0 & 0 & \alpha_-^0 - \alpha_+^0 \\
0 & -1 & 0 & 0 & \alpha_+^0 - \alpha_-^0 \\
\end{pmatrix}
\] (3.11)
Since we have $\xi_{ab} = i\lambda_0 \tilde{\lambda}_0$ and the substitution $|\tilde{\lambda}| = Q_\tau |\lambda|$ and $|\lambda| = |P_{a1} - \tau P_{a2}|$, we can derive
\[
|\lambda|_{|\ell|} = |P_{a1}||\hat{\lambda}_{|\ell|} = (\frac{1}{\tau})^\frac{1}{2},
\]
\[
|\lambda|_{|\ell|} = a_{ab}(Q_\tau)|\lambda|_{|\ell|} = (\frac{1}{\tau})^\frac{1}{2},
\]
\[
t = \frac{K^2}{\lambda |\tilde{\lambda}|} = \frac{1}{(\alpha_+^0 - \alpha_-^0)} (3.12)
\]
and finally
\[
\ell = t |\lambda|_{|\ell|},
\]
\[
t = \frac{1}{(\alpha_+^0 - \alpha_-^0)} (3.13)
\]
where $t = \frac{\alpha_+^0}{\alpha_+^0 - \alpha_-^0}$ and $\tilde{t} = \frac{-\alpha_-^0}{\alpha_+^0 - \alpha_-^0}$. Notice that $\alpha_+^0$ and $\alpha_-^0$ defined above are the same as formula (140) in [18], which shows the on-shell conditions are satisfied. Similarly, for the substitution $|\lambda| = |P_{a2} - \tau P_{a1}|$, we have $l = \frac{\alpha_-^0}{\alpha_+^0 - \alpha_-^0}$ and $\tilde{l} = \frac{-\alpha_+^0}{\alpha_+^0 - \alpha_-^0}$.

Next, we consider the generalized unitarity cut method. By comparing to the definition of loop momentum $\ell$ in (2.18), we find the relation between parameter $\tau$ and $z$
\[
z = \rho \cos \theta = \frac{\tilde{l} + l}{2} = \frac{1}{2(\alpha_\tau - \alpha_-^0)}(\alpha_+^0 \tau - \alpha_-^0 \tau)
\] (3.14)
for the substitution $|\lambda| = |P_{a2} - \tau P_{a1}|$. For another substitution $|\lambda| = |P_{a1} - \tau P_{a2}|$, it is just to put $\tau \rightarrow \frac{1}{\tau}$.

It is useful to figure out how the complex plane transforms when we change the variable by formula (3.14). Remember that following [18], we suppose $r^2 = -\alpha_-^0 \alpha_\tau / (\alpha_+^0 - \alpha_-^0)^2 > 0$ all along our derivation, while the result of $r^2 \leq 0$ can be obtained by analytic continuation. This transformation maps the whole $\tau$ plane to two sheets of the $z$ plane (because $\tau = z \pm \sqrt{z^2 - 1}$), $|\tau| > 1$ to one and $|\tau| < 1$ to another, as figure 1 shows. The points on the $z$ plane are branch points, and the dashed lines are branch cuts. We choose $\tau = z + \sqrt{z^2 - 1}$, as well as $\arg(z) = 0$ on the $x$ axis of the first sheet. Under this convention, the + and − in the figure means $\sqrt{1 - z^2}$ to be positive or negative near the branch cut.

Notice that the functions $f(z, \sqrt{1 - z^2})$ and $f(z, -\sqrt{1 - z^2})$ can be viewed as the same function $f(z)$
on different sheets. Thus, we can rewrite formulas (2.23) to be

\[
\text{Cut}^{(3)} = \frac{1}{4\pi} \int_C \frac{dz}{\sqrt{1 - z^2}} A_1 A_2 A_3(z)
\]

\[
= \frac{1}{4\pi} \int_{\mathcal{C}} \frac{d\tau}{i\tau} A_1 A_2 A_3(\tau),
\]

where the contour \( C \) on the \( z \) plane and the corresponding contour \( \mathcal{C} \) on the \( \tau \) plane are shown in figure 1. To rescale back we simply put \( \tau \to 1 - \tau^2 \) in the integrand.

### 3.3.2. Connection between residues

Now we come to the explicit formula of the triangle coefficient. In the unitarity cut method, the triangle coefficient comes from

\[
\mathcal{C}[\mathcal{K}, \mathcal{K}] = \frac{1}{2(N + 1)! \sqrt{\Delta}} \langle P_{\mathcal{K}} \rangle^{N+1}
\]

\[
\times \frac{d^{N+1}}{d\tau^{N+1}} (T^{\mathcal{K}}(\tau) D_\tau(\tau) \langle |\mathcal{K}| \rangle N+1 \mid_{\tau = 0})
\]

\[
+ \langle P_{\mathcal{K}} \rangle^{N+1} \mid_{\tau = 0}.
\]

It can be easily shown

\[
\sqrt{\Delta} = \alpha_i - \alpha_i^+,
\]

\[
\langle P_{\mathcal{K}} \rangle = 1,
\]

\[
\langle |\mathcal{K}| \rangle = \tau (\alpha_i - \alpha_i^+).
\]

Note that these equations are valid for two substitutions.

We concentrate on the first term on the derivative. Under the substitution \( |\lambda| = Q |\lambda| \) and \( |\lambda| = |P_{\mathcal{K}} \rangle \), we put the propagator \( 1/D_\tau(\tau) \) on shell. Using the fact that \( T^{\mathcal{K}}(\tau) D_\tau(\tau) \) has a Laurent expansion \( \sum_{n=-(N+1)} a_n \tau^n \) around \( \tau = 0 \), we can conclude that

\[
\frac{1}{(N + 1)! \sqrt{\Delta}} \langle P_{\mathcal{K}} \rangle^{N+1}
\]

\[
\times \frac{d^{N+1}}{d\tau^{N+1}} (T^{\mathcal{K}}(\tau) D_\tau(\tau) \langle |\mathcal{K}| \rangle N+1 \mid_{\tau = 0})
\]

\[
= \frac{1}{(N + 1)! \sqrt{\Delta}} \langle P_{\mathcal{K}} \rangle^{N+1} \mid_{\tau = 0}.
\]

where at the last equal sign we use (3.5), and we denote the factorized tree amplitude \( A_{\mathcal{K},1} A_{\mathcal{K},2} A_R \) by \( A_{\mathcal{K},1} A_{\mathcal{K},2} \). For the second term in the derivative, we simply set \( \tau \to 1/\tau \) in \( T^{\mathcal{K}}(\tau) D_\tau(\tau) \), while \( \langle |\mathcal{K}| \rangle \) remains proportional to \( \tau \). We obtain

\[
\frac{1}{(N + 1)! \sqrt{\Delta}} \langle P_{\mathcal{K}} \rangle^{N+1}
\]

\[
\times \frac{d^{N+1}}{d\tau^{N+1}} (T^{\mathcal{K}}(1/\tau) D_{1/\tau}(\tau) \langle |\mathcal{K}| \rangle N+1 \mid_{\tau = 0})
\]

\[
= \frac{1}{(N + 1)! \sqrt{\Delta}} \langle P_{\mathcal{K}} \rangle^{N+1} \mid_{\tau = 0}.
\]

where we use \( \text{Res}_{\tau = 0} (z) = \text{Res}_{\tau = 0} \left( \frac{-f(z)}{z^2} \right) \).

Combining the above results,

\[
\mathcal{C}[\mathcal{K}, \mathcal{K}] = \frac{1}{2} \text{Res}_{\tau = 0} \left( \frac{A_1 A_2 A_3}{\tau} \right) - \frac{1}{2} \text{Res}_{\tau = 0} \left( \frac{A_1 A_2 A_3}{\tau} \right).
\]

\[
\text{(3.21)}
\]
It is now clear that what formula (3.17) really does is to compute the residue of \( \frac{A_iA_j}{z^{2}} \) at \( \tau = 0 \) and \( \tau = \infty \).

After figuring out the meaning of formula (3.17), we turn to the generalized unitarity cut method. We have already shown that the triple cut can be computed by (3.16)

\[
\text{Cut}^{(3)} = \frac{1}{4\pi i} \int_{C} \frac{dz}{z} - \frac{d^2r}{\tau} A_1 A_2 A_3 (z) = \frac{1}{4\pi i} \int_{C} \frac{dr}{\tau} A_1 A_2 A_3 (\tau). \tag{3.22}
\]

To get rid of the influence of the box coefficient, we stretch the contour \( C \) to be two infinitely large loops, which only contain the residue on \( z = \infty \). Now, the contour \( C \) becomes an infinitesimal loop encircling \( \tau = 0 \) and an infinitely large loop encircling \( \tau = \infty \). Thus, we come to the final step

\[
C_3 = i \text{Res}_{\tau=\infty, \text{contour sheets}} \left( \frac{A_1 A_2 A_3}{\sqrt{r^2 - z^2}} \right) = \frac{1}{2} \text{Res}_{\tau=0, \infty} \left( \frac{A_1 A_2 A_3}{\tau} \right). \tag{3.23}
\]

We have proven that the formulas for the triangle coefficient in the two methods are indeed equal.

### 3.4. Bubble coefficients

The core part in evaluating the bubble coefficient of the master integral is to split it from other master integrals. In the unitarity cut method, the procedure is carried out by recognizing it from analytical properties of other master integrals, which behave as pure logarithms. In the generalized unitarity cut method, it is carried out by recognizing it as the infinite pole in the integrand [18].

\[
-\frac{1}{z_i} \text{Res}_{z=z_i} I_{\text{term}} (z) = \frac{1}{\langle \lambda | Q, \bar{\lambda} | \lambda \rangle} \prod_{j=1}^{p_i+1} \left( \left( \langle \lambda | Q | \bar{\lambda} \rangle - |a_j| |K| \lambda \rangle \right) \right) \frac{\prod_{n=1}^{r_{i+1}} \left( \langle \lambda | Q | \bar{\lambda} \rangle - |a_j| |K| \lambda \rangle \right)}{\langle \lambda | Q, \bar{\lambda} | \lambda \rangle \prod_{n=1}^{r_{i+1}} \left( \langle \lambda | Q | \bar{\lambda} \rangle - |a_j| |K| \lambda \rangle \right)}.
\tag{3.29}
\]

We begin with expression (2.25)

\[
C_2 = \int dLISP \int_{C} \frac{dz}{z} A_l (z) \times A_R (z). \tag{3.24}
\]

However, after two cuts, the contribution coming from triangles and boxes will appear as some remaining propagators in the form of \( \frac{1}{(z - K)_{\tau}} \). When we do the contour integrals at infinity, i.e. when the contour \( C \) is an infinitely large loop, we have

\[
\int_{C} \frac{dz}{z} T^{(N)} (z) = 2\pi i T^{(N)} (0) + 2\pi i \sum_{i} \frac{1}{z_i} \text{Res}_{z=z_i} T^{(N)} (z), \tag{3.25}
\]

with some \( z_i \) that put a remaining propagator \( D_i \) on shell. Rewriting the above formula as

\[
T^{(N)} (0) = -\frac{1}{4\pi i} \int_{C} \frac{dz}{z} T^{(N)} (z) = -\frac{1}{2\pi i} \int_{C} \frac{dz}{z} T^{(N)} (z).
\]

Since we can exchange the order of integration, we integrate \( t \) in \( dLISP \) as in (2.3), then the integrand becomes (2.5). Thus, we have

\[
I_{\text{term}} (0) = -\frac{1}{2\pi i} \int_{C} \frac{dz}{z} I_{\text{term}} (z) = -\frac{1}{4\pi i} \int_{C} \frac{dz}{z} T^{(N)} (z).
\]

We can see that the right-hand side gives a splitting of the input integrand. We want to show such a splitting is nothing but the canonical splitting in the unitarity cut method (2.7).

Now, we calculate the residues of the poles for finite \( z_i \). Since \( G (\lambda) \) is pure holomorphic, we omit it during our derivation and put it back only at the end. The BCFW deformation is given by

\[
\frac{1}{z_i} \text{Res}_{z=z_i} I_{\text{term}} (z) = \frac{1}{\langle \lambda | Q, \bar{\lambda} | \lambda \rangle} \prod_{j=1}^{p_i+1} \left( |a_j| \bar{\lambda} - |a_j| |K| \lambda \right).
\]

For a certain propagator \( D_i \) on shell, we have \( z_i = \frac{\langle \lambda | Q, \bar{\lambda} | \lambda \rangle}{\langle \lambda | Q, \bar{\lambda} | \lambda \rangle} \) and then we have

\[
-\frac{1}{z_i} \text{Res}_{z=z_i} I_{\text{term}} (z) = \frac{1}{\langle \lambda | Q, \bar{\lambda} | \lambda \rangle} \prod_{j=1}^{p_i+1} \left( |a_j| \bar{\lambda} - |a_j| |K| \lambda \right) \frac{\prod_{n=1}^{r_{i+1}} \left( |a_j| \bar{\lambda} - |a_j| |K| \lambda \right)}{\langle \lambda | Q, \bar{\lambda} | \lambda \rangle \prod_{n=1}^{r_{i+1}} \left( |a_j| \bar{\lambda} - |a_j| |K| \lambda \right)}.
\]

Using a generalized version of the Schouten identity

\[
|\bar{\lambda}| \langle \lambda | Q | \bar{\lambda} \rangle - |K| \langle \lambda | Q | \bar{\lambda} \rangle + |Q| \langle \lambda | Q | \bar{\lambda} \rangle = 0,
\]

\[
|\bar{\lambda}| \langle \lambda | Q | \bar{\lambda} \rangle - |K| \langle \lambda | Q | \bar{\lambda} \rangle + |Q| \langle \lambda | Q | \bar{\lambda} \rangle = 0,
\]

we have

\[
C_2 = \int dLISP \int_{C} \frac{dz}{z} A_l (z) \times A_R (z) \tag{3.24}
\]

These two choices are equivalent to each other. Using \( \ell_2 = \ell + K \), we have \( |\ell_2| \sim |K| \).

\footnote{The deformation null momenta \( q \) can have two choices: \(|\ell| |\ell| \) or \(|\ell| |\ell| \). These two choices are equivalent to each other. Using \( \ell_2 = \ell + K \), we have \( |\ell_2| \sim |K| \).}
the residue term will represent as

\[
\frac{1}{\lambda|Q_0\rangle|\lambda|K\rangle^{n+2}} \prod_{j=1}^{n+k} \left( -[a_j|Q_0\rangle|\lambda|K\rangle|\tilde{\lambda}| \right)
\times (-1)^{n+1} \frac{1}{\lambda|K\rangle|\lambda|Q_0\rangle|\lambda|K\rangle^{n+1}} \prod_{\omega_i} \left( -\langle \lambda|Q_0\rangle|\lambda\right) 
\]

Putting it back to (3.27), and comparing with (2.7), we see that the first line of (2.7) is nothing, but the part \( \frac{1}{2\pi i} \int_{\mathcal{C}} dz_i \text{term}(z) \), thus (3.24) is nothing, but taking residues of the first line of (2.7). Thus, we have shown the equivalence of getting bubble coefficients in these two methods.

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