THE GLOBAL EXISTENCE OF MARTINGALE SOLUTIONS TO
STOCHASTIC COMPRESSIBLE NAVIER-STOKES EQUATIONS WITH
DENSITY-DEPENDENT VISCOSITY

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Abstract. In this paper, we establish the global existence of martingale solutions to the
compressible Navier-Stokes equations with density-dependent viscosity and vacuum driven by
the stochastic external forces. This can be regarded as a stochastic version of Vasseur-Yu’s work
for the corresponding deterministic Navier-Stokes equations [35], in which the global existence
of weak solutions holds for adiabatic exponent $\gamma > 1$. We use vanishing viscosity method and
Jakubowski-Skorokhod’s representation theorem. For the stochastic case, we need to add an
artificial Rayleigh damping term in addition to the artificial terms in $[35, 36]$, to construct
regularized approximated solutions. Moreover, we have to send the artificial terms to 0 in
a different order. It is worth mentioning that, $\gamma > \frac{6}{5}$ is necessary in this paper due to the
multiplicative stochastic noise.

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1. Introduction

Compressible Navier-Stokes equations describe the motion of compressible viscous Newtonian fluid. Practically, besides the structural vibration of the fluid, the fluid will also be affected by random external forces such as humidity, wind, solar radiation, industrial pollution etc. Therefore it is reasonable to add stochastic forces to Navier-Stokes equations. On the other hand, if we consider the random factor in the process of deriving Boltzmann equation, from which we can derive the Navier-Stokes equations formally, then we find that the stochastic factors merely appear as the stochastic external forces in Navier-Stokes equations.

Boltzmann equation describes the dynamic behaviors of molecules in microscopic scale. When the molecules are affected by stochastic factors, we can also get a model in microscopic scale, and then obtain the stochastic macroscopic model. Let \( f = f(t,x,v) \) denotes the distribution function of molecules, which is a stochastic process of \( t, x, \) and \( v \), where \( x = (x_1, x_2, x_3) \) denotes the position, \( v = (v_1, v_2, v_3) \) is the velocity of molecule. The amount of molecules in an infinitesimal volume \( d x \) around \( x, d v \) around \( v \) is \( d N = f(t,x,v) \, d x \, d v \), where \( d x = d x_1 \, d x_2 \, d x_3, \) \( d v = d v_1 \, d v_2 \, d v_3 \). The total number \( N \) of molecules in a given volume \( V \) is

\[
N = \int_V d x \int_{\mathbb{R}^3} f(t,x,v) \, d v. \tag{1.1}
\]

The total number in a unit volume centered at \((t,x)\), is

\[
A \triangleq A(t,x) = \int_{\mathbb{R}^3} f(t,x,v) \, d v, \tag{1.2}
\]

so the density at \((t,x)\) is

\[
\rho(t,x) = AM = M \int_{\mathbb{R}^3} f(t,x,v) \, d v, \tag{1.3}
\]

where \( M \) is the molar mass. The average velocity, a random variable, \( u(t,x) = (u_1,u_2,u_3) \) can be written as

\[
u(t,x) = \frac{1}{A} \int_{\mathbb{R}^3} v f(t,x,v) \, d v = \frac{\int_{\mathbb{R}^3} v f(t,x,v) \, d v}{\int_{\mathbb{R}^3} f(t,x,v) \, d v}, \tag{1.4}
u
\]
i.e., \( \nu \) is the velocity of the fluid. Assume \( x' = x + v \, d t, \) \( v' = v + g \, d t \) at \( t \) and \( d t \), \( g \) is the external force acting on a molecule and every unit mass, the former infinitesimal volume \( d x \, d v \) becomes \( d x' \, d v' \), and we denote \( d N' = f(t + d t, x', v') \), if there are collisions, some molecules outside \( d x \, d v \) will be scattered into \( d x' \, d v' \), so there holds

\[
d N' - d N = J(t,x,v) \, d t \, d x \, d v, \tag{1.5}\]

\( J \) can be determined from the analysis about collision:

\[
J(t,x,v) \, d t = \left( \int_{\mathbb{R}^3} \left( \int \sigma(\Gamma) |w - v| (f(t,x,v') f(t,x,w') - f(t,x,v) f(t,x,w)) \, d \Gamma \right) \, d w \right) \, d t, \tag{1.6}
\]

where \( \sigma(\Gamma) = \sigma((v,w) \rightarrow (v',w')) \) is differential scattering cross section, \( \Gamma \) is the solid angle

So we have

\[
d f + (v \cdot \nabla_x f + g \nabla_v f) \, d t = \left( \int_{\mathbb{R}^3} \left( \int \sigma(\Gamma) |w - v| (f(t,x,v') f(t,x,w') - f(t,x,v) f(t,x,w)) \, d \Gamma \right) \, d w \right) \, d t. \tag{1.7}
\]
We still have the conservation theorem from [17], as in [28], we use the notation \( \langle \cdot \rangle = \frac{\int_\mathbb{R}^3 \cdot f \, d x \, d v}{\int_\mathbb{R}^3 f \, d x \, d v} \). For any conserved quantity \( \chi \), i.e., \( \chi(x, v) + \chi(x, w) = \chi(x, v') + \chi(x, w') \), then \( \chi \) satisfies

\[
\frac{d}{dt} \langle A \chi \rangle + \left( \sum_{i=1}^3 \frac{\partial}{\partial x_i} \langle A v_i \chi \rangle - n \sum_{i=1}^3 \langle A v_i \chi \rangle \right) \, dt
\]

\[
+ \left( -A \sum_{i=1}^3 \langle v_i \frac{\partial \chi}{\partial x_i} \rangle - A \sum_{i=1}^3 \langle g_i \frac{\partial \chi}{\partial v_i} \rangle - A \sum_{i=1}^3 \langle \frac{\partial g_i}{\partial v_i} \chi \rangle \right) \, dt = 0.
\]

Since the molar mass \( M \) is conserved after collision, we take \( \chi = M \), then

\[
\frac{d}{dt} \langle AM v_i \rangle + \sum_{j=1}^3 \frac{\partial}{\partial x_j} \langle AM v_j v_i \rangle \, dt - AM \sum_{j=1}^3 \langle g_i \rangle \, dt - AM \sum_{j=1}^3 \langle \frac{\partial g_i}{\partial v_j} v_j \rangle \, dt = 0.
\]

Write it in macroscopic scale, (see [28]),

\[
\frac{d}{dt} \langle \rho u \rangle + \text{div}(\rho u) \, dt = AM \left( \frac{\partial g_i}{\partial v_i} \right) \, dt.
\]

Take \( \chi = M v_i \), similarly we have

\[
\frac{d}{dt} \langle AM v_i \rangle + \sum_{j=1}^3 \frac{\partial}{\partial x_j} (AM v_j v_i) \, dt - AM \sum_{j=1}^3 \langle g_i \rangle \, dt - AM \sum_{j=1}^3 \langle \frac{\partial g_i}{\partial v_j} v_j \rangle \, dt = 0.
\]

where \( P \triangleq (p_{ij}) \) is the stress tensor,

\[
p_{ij} = \int_\mathbb{R}^3 M (v_i - u_i) (v_j - u_j) f (t, x, v) \, d v = \rho ((v_i - u_i) (v_j - u_j)).
\]

In fact, \( P = \rho \mathbb{I}_3 + 2 \mu \mathbb{D}(u) + \lambda \text{div} u \mathbb{I}_3 \), see [28], \( \lambda \) and \( \mu \) are viscosities with the bulk viscosity \( \lambda + \frac{2}{3} \mu > 0 \), \( \mathbb{I}_3 \) is the identity matrix,

\[
\mathbb{D} u = \frac{\nabla u + \nabla u^\top}{2}
\]

is the deformation tensor.

Consider the right hand side of (1.13), the term \( \rho \sum_{i=1}^3 \langle g_i \rangle \, dt \) should be the dominant term when we discuss the momentum, then \( \rho \sum_{i=1}^3 \langle \frac{\partial g_i}{\partial v_i} v_i \rangle \) should not be very large. But the velocity of molecules \( v_i \) is very large, so the term \( \frac{\partial g_i}{\partial v_i} \) will be small. Air pollution, the greenhouse gases, solar radiation and other factors like flying creatures in the sky may affect the system either. If all these factors are slight and various, they can form Gaussian noise. So it is reasonable to consider the following kinds of stochastic forced equations with many kinds of stochastic factors viewed as Gaussian noise [30], [31]:

\[
\begin{align*}
\rho u + \text{div}(\rho u) &= 0, \\
\frac{d}{dt} (\rho u) + (\text{div}(\rho u \otimes u) + \nabla p - \text{div}(2 \mu \mathbb{D}(u) + \lambda \text{div} u \mathbb{I}_3)) \, dt = \rho F(\rho, u) \, d W.
\end{align*}
\]
\[ \rho \mathbb{F} (\rho, \mathbf{u}) \, dW \text{ is the external multiplicative noise.} \]
\[ W = \sum_{k=1}^{+\infty} e_k \beta_k, \quad dW = \sum_{k=1}^{+\infty} e_k \, d\beta_k, \]  
where \( \beta_k \) is the standard real Brownian motion, \( \{e_k\}_{k=1}^{+\infty} \) is an orthonormal basis in an auxiliary separable Hilbert space \( \mathcal{H} \), which is isometrically isomorphic to \( l^2 \), the space of square-summable sequences. \( \mathcal{H} \) is independent with domain \( \mathbb{T}^3 \). Let \( H \) be a Bochner space, \( \mathbb{F} (\rho, \mathbf{u}) \) is \( H \)-valued and in square summable space \( \mathcal{H} \), denoting the inner product in \( \mathcal{H} \) as \( \langle \cdot, \cdot \rangle \), the inner product
\[ \langle \mathbb{F} (\rho, \mathbf{u}), e_k \rangle = F_k (t, x, \rho, \mathbf{u}) \]  
is a three-dimensional (3D) \( H \)-valued vector function, which shows the strength of external stochastic forces,
\[ \mathbb{F} (\rho, \mathbf{u}) \, dW = \sum_{k=1}^{+\infty} F_k (t, x, \rho, \mathbf{u}) \, d\beta_k, \quad \mathbb{F} (\rho, \mathbf{u}) = \sum_{k=1}^{+\infty} F_k (t, x, \rho, \mathbf{u}) \, e_k, \]
we will write \( F_k (\rho, \mathbf{u}) \) for short in the following statement. We mention that \( \mathbf{u} = \mathbf{u} (w, t, x) \) and \( \rho = \rho (w, t, x) \), \( w \) is a sample in \( \Omega \), \( x \in \mathbb{T}^3 \). To make the notations consistent with the usual deterministic studies in fluid dynamic equations, we write \( \rho (t, x) \) and \( \mathbf{u} (t, x) \) for short.

1.1. The progress in the stochastic forced Navier-Stokes equation. When \( \rho \mathbb{F} (\rho, \mathbf{u}) \equiv 0 \), system \[ \text{(1.19)} \]  
reduces to deterministic Navier-Stokes system. There are rich literatures concerning the well-posedness of weak solutions. Kazhikhov-Shelukhin \[ 26 \]  
obtained the existence of global weak solutions to 1D compressible case in 1977. For an overview of the known results and earlier works in 1D or 2D cases, we refer to \[ 16, 22, 25 \]. For 3D compressible Navier-Stokes equations, Lions \[ 29 \]  
proved the global existence of weak solutions to Navier-Stokes equations for \( \gamma > \frac{5}{3} \) with large initial data in 1998. Then Feireisl-Novotný-Petzeltová in \[ 13 \]  
extended the existence of weak solutions to \( \gamma > \frac{5}{3} \) by using a new test function in 2001. In 2021, Hu \[ 18 \]  
obtained the renormalized global weak solution for \( \frac{5}{3} < \gamma < \frac{3}{2} \) up to a closed set with zero parabolic Hausdorff measure. Now we turn to the stochastic case when \( \rho \mathbb{F} (\rho, \mathbf{u}) \neq 0 \), we can consider the so-called martingale solution of the system, which can be interpreted as weak solution in stochastic version \[ 24, 31 \]. Wang-Wang \[ 37 \]  
obtained global martingale solutions to stochastic Navier-Stokes equations in a bounded domain with non-slip boundary condition in 2015. In 2016, Breit-Hofmanova \[ 8 \]  
showed the existence of global martingale solution on a torus with periodic boundary condition. Smith got global martingale solutions \[ 33 \]  
in which the stochastic forces are Lipschitz continuous in 2017.

The above stochastic result are under constant viscosities. However, in physics the viscosities \( \mu \) and \( \lambda \) usually depend on temperature or density, or both. In isentropic case, the viscosities depend on density, denoted as \( \mu (\rho) \) and \( \lambda (\rho) \). This fact can be derived from the Chapman-Enskorg expansion in Boltzman equation \[ 28 \].

The system of stochastic forced isentropic compressible Navier-Stokes equations with density-dependent viscosity in torus \( \mathbb{T}^3 \) is
\[ \left\{ \begin{array}{l}
\rho_t + \text{div} (\rho \mathbf{u}) = 0, \\
d \rho \mathbf{u} + (\text{div} (\rho \mathbf{u} \otimes \mathbf{u}) + \nabla p - \text{div} \mathcal{T}) \, dt = \rho \mathbb{F} (\rho, \mathbf{u}) \, dW, 
\end{array} \right. \]  
(1.20)
\( \rho \) is the density, \( \mathbf{u} = (u_1, u_2, u_3) \in \mathbb{R}^3 \) denotes the velocity. For isentropic gas, the pressure can be expressed as \( p = a \rho^\gamma \).

\[
T = 2\mu(\rho) \mathbb{D}(\mathbf{u}) + \lambda(\rho) \text{div} \mathbf{u} I_3
\]  

(1.21)

is viscosity stress tensor.

For the case with general density-dependent viscosities, when \( \rho \bar{E}(\rho, \mathbf{u}) \equiv 0 \), Li-Xin \cite{27} got the global existence of Navier-Stokes equations in 2015. Vasseur-Yu \cite{35} considered the density-degenerate case when viscosities satisfies \( \mu(\rho) = \frac{\rho}{2}, \lambda(\rho) = 0 \). They obtained the existence of global weak solutions without added constraint on adiabatic exponent \( \gamma > 1 \) by means of vanishing viscosity method, Bresch-Desjardins relation (B-D entropy) and Mellet-Vasseur type inequality in 2016. Bresch-Desjardins relation (B-D relation) \( \lambda(\rho) = 2(\rho \mu'(\rho) - \mu(\rho)) \) is firstly introduced by Bresch and Desjardins \cite{7} to solve the shallow water problem in 2006, and the relation is necessary to balance the term in deducing B-D entropy so as to get further more regularity of density. Mellet-Vasseur \cite{30} established the compactness of \( \sqrt{\rho} \mathbf{u} \) in \( L^2([0, T] \times \mathbb{T}^3) \) for the Navier-Stokes equations in 2007, they deduced the Mellet-Vasseur type inequality which implies that \( \rho \mathbf{u}^2 \) bounded in \( L^\infty(0, T; L\log L(L(\mathbb{T}^3))) \) rather than \( L^\infty(0, T; L^1(\mathbb{T}^3)) \).

We turn to the case when \( \rho \bar{E}(\rho, \mathbf{u}) \not\equiv 0 \). In 2000, Tornatore \cite{34} considered the global existence and uniqueness of solutions where \( \lambda(\rho) = 1 \) and \( \lambda(\rho) = 1 + \rho^\beta \). In 2020, Breit-Feireisl \cite{1} considered the existence of weak solution of Navier-Stokes-Fourier equations when the viscosities \( \mu(\theta) \) satisfy \( \mu \cdot (1 + \theta) < \mu(\theta) < \bar{\mu} \cdot (1 + \theta) \), where \( \mu \) and \( \bar{\mu} \) are specific positive constants, \( \theta \) is the temperature. In \cite{1}, there is a heat source. And the density and temperature is strictly positive because the Helmholtz function \( H_\theta = \rho e(\rho(\rho, \theta)) - \bar{\theta} s(\rho, \theta) \) should satisfy the coercivity property, where \( e \) is the energy in energy equation, \( \bar{\theta} \) is a non-negative constant, \( s \) is the entropy, see also \cite{12}. There are also results on singular limit \cite{2}, the stationary solutions \cite{4}, and the relative energy \cite{3}. Very recently in 2022, Brzeziński-Dharwal-Zatorska \cite{3} got the sequential stability of the martingale solutions to the stochastic forced compressible Navier-Stokes equations based on the assumption that the existence of global martingale solutions holds. In this paper, we give the global existence of martingale solution of stochastic isentropic compressible Navier-Stokes equations under viscosity-degenerate case.

1.2. Our main result. We consider the stochastic isentropic compressible Navier-Stokes equations with density-dependent viscosities. Once the viscosities depend on density, \( d(\rho \mathbf{u}) \) and \( \text{div} T \) will be degenerate at vacuum \( \rho(\tau, x) = 0 \), generally which will bring the difficulties to deal with the nonlinear term \( \text{div}(\rho \mathbf{u} \otimes \mathbf{u}) \). So here for simplicity, we consider a specific case \cite{35}

\[
\mu(\rho) = \frac{\rho}{2}, \quad \lambda(\rho) = 0.
\]  

(1.22)

Under (1.22), (1.20) turns into

\[
\begin{aligned}
\rho_t + \text{div} (\rho \mathbf{u}) & = 0, \\
d(\rho \mathbf{u}) + (\text{div}(\rho \mathbf{u} \otimes \mathbf{u}) + \nabla p - \text{div}(\rho \mathbf{D} \mathbf{u})) \, d\tau & = \rho \bar{E}(\rho, \mathbf{u}) \, dW.
\end{aligned}
\]  

(1.23)

We consider the global existence of martingale solutions to (1.23) in torus \( \mathbb{T}^3 \), given the initial conditions

\[
\rho|_{\tau=0} = \rho_0, \quad \rho \mathbf{u}|_{\tau=0} = \mathbf{q}_0.
\]  

(1.24)
We assume that there exists \( f = (f_1, f_2, \cdots, f_k, \cdots) \) such that \( F_k(\rho, u) \) satisfies
\[
\|F_k(\rho, u)\|_{L^\infty} + \|\nabla F_k(\rho, u)\|_{L^\infty} + \|\nabla u F_k(\rho, u)\|_{L^\infty} \leq f_k, \quad \sum_{k=1}^{+\infty} f_k^2 < +\infty, \tag{1.25}
\]
here \( L^\infty \)-norm is the maximum of every component of vector or matrix. This condition implies that
\[
\|\rho F_k(\rho, u)\| \leq f_k \cdot (|\rho| + |\rho u|), \tag{1.26}
\]
i.e. \( \rho F_k(\rho, u) \triangleq G_k(\rho, \rho u) \) is Lipschitz continuous with respect to \( \rho \) and \( q = \rho u \). When \( \rho = 0 \), it is reasonable to define \( F_k(\rho, u) = 0 \), when \( \rho > 0 \), we can write \( F_k(\rho, u) = F_k \left( \rho, \frac{\rho \rho}{\rho} \right) \).

For our problem, we firstly give a definition of martingale solution.

**Definition 1.1.** \((\rho, u)\) is a martingale solution to (1.23) if

1. the density \( \rho \) and the velocity \( u \) are random distributions adapted to \((\mathcal{F}_t)_{t \geq 0}\);
2. the equation of continuity holds \( \mathbb{P} \) almost surely: for \( \forall \varphi \in C_c^{\infty}([0, T); C^\infty (T^3)) \),
   \[
   -\int_0^T \int_{T^3} \partial_t \varphi \rho \, dx \, dt = \int_{T^3} \varphi(0, x) \rho_0 \, dx + \int_0^T \int_{T^3} \rho \varphi \, \nabla \varphi \, dx \, dt; \tag{1.26}
   \]
3. the momentum equation holds \( \mathbb{P} \) almost surely: for \( \forall \psi \in C_c^{\infty}([0, T); C^\infty (T^3)) \),
   \[
   -\int_0^T \int_{T^3} \rho \psi \, dx \, dt \quad - \int_0^T \int_{T^3} \rho u \otimes u : \nabla \psi \, dx \, dt \\
   -\int_0^T \int_{T^3} \rho^2 \, \text{div} \psi \, dx \, dt \quad - \int_0^T \int_{T^3} \rho \mathbb{D} u : \nabla \psi \, dx \, dt \\
   = \int_0^T \int_{T^3} \rho \rho \rho(\rho, u) \psi \, dx \, dW + \int_{T^3} \rho_0 u_0 \cdot \psi_0 \, dx. \tag{1.27}
   \]

For the convenience of stating our main theorem, we introduce the notation of the energy

\[
E(t) = \int_{T^3} \left( \frac{1}{2} \rho |u|^2 + a \int_1^\rho \frac{p(z)}{z} \, dz \right) \, dx \tag{1.28}
\]
and the stochastic B-D entropy

\[
\tilde{E}(t) = \int_{T^3} \left( \frac{1}{2} \rho |u + \nabla \log \rho|^2 + a \int_1^\rho \frac{p(z)}{z} \, dz \right) \, dx. \tag{1.29}
\]

Let \( \Lambda \) be the law (see Appendix) of \( \rho_0 \) and \( q_0 \). Our main theorem is as follows.

**Theorem 1.1.** If

\[
\Lambda \{ \rho(0, x) \geq 0 \} = 1, \quad \Lambda \left\{ 0 < \rho \leq \int_{T^3} \rho(0, x) \, dx \leq \bar{\rho} < \infty \right\} = 1. \tag{1.30}
\]

and

\[
\mathbb{E} [E(0)^r] \leq C, \quad \mathbb{E} [\tilde{E}(0)^r] \leq C, \tag{1.31}
\]
\[
\mathbb{E} \left[ \left( \int_{T^3} \rho_0 \left( 1 + |u_0|^2 \right) \ln \left( 1 + |u_0|^2 \right) \, dx \right)^r \right] \leq C, \tag{1.32}
\]
for any \( r > 2 \), then for \( \gamma > \frac{5}{8} \), there exists a martingale solution to (1.23). Moreover, there hold
For our problem, the difficulties and strategies for our problem are analysed as follows.

1. **Loss of regularity of** \( \rho u \), **caused by the stochastic external forces.** More specifically, the Gaussian process in the right hand side of the (1.23) is at most Hölder \(-\frac{1}{2}\) continuous. Since we do not have the estimate of \( \rho u \), the Aubin-Lions’ lemma does not apply to \( \rho u \) in this case either, we should consider the time continuity of \( \rho u \) so as to obtain the compactness of \( \rho u \) in \( C\left([0, T]; L^\frac{3}{2}(T^3)\right) \). Compared with the embedding theorem in [6], poorer regularity holds in spacial space in this paper.

2. **Low regularity of** \( u \) **on account of degenerate viscosities and the trouble in tightness while passing to the limit of the nonlinear term** \( \rho u \otimes u \). Compared with the case of constant viscosities in [6], the equation (1.23) becomes degenerate because the second order derivative of velocity will vanish at vacuum. We do not have the information of \( \|Du\|_{L^2([0, T] \times \mathbb{T}^3)} \) from the term \( \text{div}(\rho Du) \) because \( \rho \) do not have positive pointwise lower bound no matter in deterministic case or stochastic case. We do not have the estimate of \( d u \) either. The lack of regularity of \( u \) is an obstacle to the convergence of approximations for the nonlinear term \( \rho u \otimes u \). To make up the lack of regularity of \( u \), motivated by Vasseur-Yu’s work [36], we construct an approximated scheme by adding artificial damping terms \(-r_1 \rho u^2 - r_2 u \) to the momentum equation, see also [2]. Originally, in shallow water model, the drag terms \( r_0 u \) is in the laminar case \( (r_0 \geq 0) \), and \( r_1 \rho u \otimes u \) is in the turbulent regime. We add the term \( \varepsilon \Delta u \) to control the estimate of \( \Delta u \), see also Zatorska’s work [33] in 2012. Inspired by [36], we added quantum term \( \kappa \rho \left( \nabla \left( \frac{\Delta u}{\sqrt{\rho}} \right) \right) \), artificial pressure term \( \delta \rho \nabla \Delta \rho + \eta \rho \Delta \rho \) and damping terms to provide more regularity for \( \rho \) and \( u \) to obtain the existence of global approximated weak solutions. For the theory about quantum Navier-Stokes equations, please refer to the works of Jüngle [23], Dong [11], Jiang [21] and Giscon-Violet [14].

\[ (1) \text{ the energy inequality} \]

\[
\mathbb{E} \left[ \sup_{t \in [0, T]} E(t)^r \right] \leq C \left( \mathbb{E} \left[ E(0)^r \right] + 1 \right),
\]

(1.33)

\[
\mathbb{E} \left[ \left( \int_0^T \int_{T^3} \rho |Du|^2 \, dx \, ds \right)^r \right] \leq C \left( \mathbb{E} \left[ E(0)^r \right] + 1 \right),
\]

(1.34)

here \( C \) depends on \( r, T \), and \( \sum_{k=1}^{+\infty} f_k^2 \);

\[ (2) \text{ the stochastic Mellet-Vasseur inequality:} \]

\[
\mathbb{E} \left[ \sup_{t \in [0, T]} \hat{E}(t)^r \right] \leq C,
\]

(1.35)

where \( C \) depends on \( r, T, \mathbb{E} \left[ E(0)^r \right], \mathbb{E} \left[ E(0)^{k_i} \right] \) and \( \mathbb{E} \left[ E(0)^{k_i} \right], k_i \) are specific constants, \( i = 1, 2, 3 \);

\[ (3) \text{ the stochastic Mellet-Vasseur inequality:} \]

\[
\mathbb{E} \left[ \left( \int_{T^3} \rho_F \left( 1 + |u_F|^2 \right) \ln \left( 1 + |u_F|^2 \right) \, dx \right)^r \right] \leq C + C_{r,T},
\]

(1.36)

\[
\mathbb{E} \left[ \left( \int_{T^3} \rho_0 \left( 1 + |u_0|^2 \right) \ln \left( 1 + |u_0|^2 \right) \, dx \right)^r \right],
\]

the constant \( C_r \) depends on \( T, r, \mathbb{E} \left[ E(0)^r \right], \mathbb{E} \left[ \hat{E}(0)^r \right] \) and \( C_r \) merely depends on \( r \).
However, since $\rho \mathbf{u}$ in $C \left( [0, T]; L^2 (T^3) \right)$, the existing regularity brought by $-r_1 \rho \mathbf{u}^3 - r_2 \mathbf{u}$ is not enough to get through the convergences of approximations of the nonlinear term $\rho \mathbf{u} \otimes \mathbf{u}$. Therefore we make up the high regularity of $\mathbf{u}$ by adding other artificial damping terms. In physics, Rayleigh damping is linear to mass and stiffness, and later Rayleigh studied the case when the damping term is a higher order of velocity, we adopt the latter form to add artificial Rayleigh damping term $-r_0 \mathbf{u}^3$. In the process of vanishing artificial terms, the order of passing to the limit is different with [35]. In order to get the uniform bound of the stochastic B-D entropy, the term $-r_0 \mathbf{u}^3$ should vanish at the same time with our artificial pressure terms $\delta \rho \nabla \Delta^3 \rho$ and $\eta \rho^{-10}$ and these three terms should be remained after quantum term $\kappa \rho \left( \nabla \left( \frac{\Delta \sqrt{\rho}}{\sqrt{\rho}} \right) \right)$ vanishes.

3. Renormalized solution does not apply in the stochastic case. We use Itô’s formula to deduce the stochastic Mellet-Vasseur type inequality instead of "renormalized" solution. Originally, the renormalized solution are associated with the good property of effective viscous flux. More specifically, the strong convergence of $\nabla p + (\eta + \lambda) \text{div} \, \mathbf{u}$, combined with the convexity of $b(\rho) = \rho^2$, shows the strong convergence of $\rho$ when the conditions of Aubin-Lions’ lemma could not be satisfied. For the concrete procedure of renormalized solution, one can refer to Feireisl’s book [4] or Hoff’s work in 1995 [17]. Compared with [35], they realized that quantum term is an obstacle for Mellet-Vasseur type inequality to provide compactness of $\rho \mathbf{u}^2$, so they let quantum term goes to zero before deriving the Mellet-Vasseur type inequality and reached the finish line [35]. However, in this paper, $\eta$ and $\delta$ are left after $\kappa \to 0$, this means that the good regularity of $\rho$ is still kept while $\kappa \to 0$, hence the strong convergence of $\rho \mathbf{u}$ is kept valid, so we do not "renormalize" the momentum equation as in [35]. Furthermore, when we derive the stochastic B-D entropy and Mellet-Vasseur type inequality, the test function $\mathbf{u}$ should not be applied to the momentum equation so as to avoid dealing trouble brought by $d \rho \mathbf{u}$ and stochastic force.

4. The estimate of multiplicative noise in the layer $\kappa \to 0$. The stochastic term plays a bad role in passing to the limit $\kappa \to 0$. The estimate of stochastic integral for multiplicative noise can only get through for $\gamma > \frac{6}{5}$, and the estimate is necessary for us to get the almost everywhere convergence of $\rho \mathbf{u}$ by the Arzelà-Ascoli’s theorem. Multiplicative noise will change as time goes by and it will influence the system in return. Usually the multiplicative noise is more complicated than addictive noise that will not change with $\mathbf{u}$. If there was an addictive noise here, the condition $\gamma > \frac{6}{5}$ will be not necessary when we do the energy estimate. Notice that the quantum term could not be removed from the regularized system, because it is necessary for the stochastic B-D entropy estimate of the term $I_4$, which arises from the artificial viscosity $\varepsilon \Delta \rho$. The condition in this paper is slightly wider than the result $\gamma > \frac{3}{2}$ in constant viscosity case [6].

Our proof is based on the vanishing viscosity method. We approximate the solutions $\rho$ and $\rho \mathbf{u}$ with solutions which has better regularity. The regularized system is

$$
\left\{ \begin{array}{l}
\rho_t + \text{div} (\rho \mathbf{u}) = \varepsilon \Delta \rho,

\rho \left( \rho \mathbf{u} \right) + (\rho \mathbf{u} \cdot \mathbf{u}) + \nabla p - \text{div} (\rho \mathbf{D} \mathbf{u}) \right) d t + \left( r_0 |\mathbf{u}|^2 \mathbf{u} + r_1 \rho |\mathbf{u}|^2 \mathbf{u} + r_2 \mathbf{u} \right) d t

\kappa \rho \left( \nabla \left( \frac{\Delta \sqrt{\rho}}{\sqrt{\rho}} \right) \right) d t + \rho \mathbf{F} (\rho, \mathbf{u}) d W.
\end{array} \right. \tag{1.37}$$
The initial conditions is $\rho(t, x, \omega)|_{t=0} = \rho_0(x, \omega)$, $(\rho u)(t, x, \omega)|_{t=0} = q_0(x, \omega)$ in $\mathbb{T}^3$. Let $\Lambda$ be the initial law of random variable such that (1.37) holds. By Galerkin approximation, we will firstly project (1.37) in $mD$ approximated space $H_m$ and get the global existence of approximated solutions in $H_m$. Employing Burkholder-Davis-Gundy’s inequality, we can estimate the $r$-th moment of stochastic term. Then we do the energy estimate so that we can figure out the tightness of laws of a sequence of approximated solutions, say $\Pi_m[\rho]$, $\Pi_m[pu]$. By Jakubowski’s extension of Skorokhod’s representation theorem and the energy estimates, we will pass to the limit $m \to +\infty$ to obtain the global existence of martingale solutions of (1.37). Notice that some estimates of higher order derivatives of density are not uniform in $\varepsilon$. So we would establish the stochastic B-D entropy estimate uniformly in $\varepsilon$. Moreover, the stochastic B-D entropy estimate gives the $H^1$ regularity of $\rho^\frac{1}{2}$. Then we vanish the artificial viscosity by sending $\varepsilon \to 0$. By Itô’s formula, passing to the limit $\kappa$ go to zero, we can derive the Mellet-Vasseur type inequality in stochastic version, which gives us the strong convergence of $\rho^\frac{1}{2}u$, $\mathbb{P}$ almost surely, so we can go through the convergence of nonlinear term $\rho u \otimes u$, $\mathbb{P}$ almost surely. Let $n \to \infty$, $\delta \to 0$, $\eta \to 0$, $r_0 \to 0$ at the same time, where $n$ satisfies $|u| \leq n$. Finally take the limit $r_1 \to 0$ and $r_2 \to 0$, then we obtain the global existence of martingale solutions of (1.28) for $\gamma > \frac{9}{4}$, i.e., our theorem 1.1.

In conclusion, the order of taking these limits is:

1. The artificial viscosity $\varepsilon \Delta \rho$ vanishes, i.e., $\varepsilon \to 0$.
2. The quantum term $\kappa \rho \left( \nabla \left( \frac{\Delta \rho}{\sqrt{\rho}} \right) \right)$ vanishes, i.e., $\kappa \to 0$.
3. The artificial pressure terms $\delta \rho \nabla \Delta \rho$ and $\eta \rho^{-10}$ vanish at the same time with the Rayleigh damping term $r_0 u^3$, i.e., $\delta \to 0$, $\eta \to 0$, $r_0 \to 0$, $n \to \infty$ at the same time, where $n$ is the truncation of $u$, $|u| \leq n$.
4. The artificial damping terms $r_1 \rho u^3$ and $r_2 u$ vanish, i.e., $r_1 \to 0$ and $r_2 \to 0$.

The paper is organised as follows. In section 2, we firstly give the global-in-time existence of solutions to stochastic quantum Navier-Stokes system with damping terms and artificial pressures. In section 3, we established the B-D entropy estimate uniformly in $\varepsilon$ under stochastic case and take the limit $\varepsilon \to 0$. In section 4, we have the Mellet-Vasseur inequality in stochastic version, meanwhile let $\delta \to 0$, $\eta \to 0$, $r_0 \to 0$, $n \to \infty$. Also we take the limit $r_1 \to 0$ and $r_2 \to 0$, then it comes our theorem 1.1. Section 5 is the Appendix, we list the basic theories in Stochastic analysis for reader’s convenience and give the process of the derivation from Boltzman equation to Navier-Stokes equation.

2. Global existence of martingale solutions to the regularized system

In this section, we firstly construct parabolic approximation structure to (1.37), apply Galerkin approximation to our system (1.37). Then inspired by [1], we truncate $u$ by $R$, divide the time interval $[0, T]$ into $\frac{T}{h}$ parts, construct an iteration scheme (2.4) and (2.5). Then we take the limit $h \to 0$ and $R \to +\infty$. For any time $t$, the limit of solution of (2.4) and (2.5) satisfies the original system (2.1) if we can pass to the limit $h \to 0$ uniformly in $h$. At the last of this section, we aim to obtain the global existence of martingale solutions in infinite dimensional space by sending $m \to +\infty$.

2.1. Galerkin approximation of the regularized system (1.37) with artificial viscosities, pressure and damping terms in finite dimensional space $H_m$. Let $\{w_k\}_{k=1}^{+\infty}$ be an orthogonal basis of $L^2(\mathbb{T}^3)$, $k = (k_1, k_2, k_3)$, $|k| = \max_{j=1,2,3} k_j$, more precisely, $w_k(x) = e^{2\pi i k \cdot x}$.
Define

$$H_m = \left\{ w = \sum_{m_{\max} \leq |m| \leq m} a_m \cos 2\pi m \cdot x + b_m \sin 2\pi m \cdot x \right\}^3.$$ 

Generally, for any $f \in L^p(T^3)$, $\{w_k\}_{k=1}^{+\infty}$ is an orthogonal basis of $L^p(T^3)$, $\Pi_m[f] = \sum_{|k|=1}^{m} \langle f, w_k \rangle w_k$.

By choosing appropriate $a_m$ and $b_m$, $\Pi_m(f) \to f$ in $L^p(T^3)$ as $m \to \infty$, $1 \leq p < +\infty$, refer to Grafakos’s book [15] for adequate setting.

We use the following Galerkin approximation scheme:

$$\begin{aligned}
\rho_t + \text{div}(\rho u) &= \varepsilon \Delta \rho, \\
\frac{d}{dt} \Pi_m[\rho u] + \Pi_m[\text{div}(\rho u \otimes u) + \nabla(a \rho^\gamma) - \text{div}(\rho D u) - \frac{11}{10} \eta \nabla \rho^{-10}] &= t - \Pi_m[\varepsilon \nabla \rho \nabla u] dt - \Pi_m[|\varepsilon \Delta^2 u|] dt \\
&+ \Pi_m[\delta \rho \nabla \Delta^3 \rho] dt + \kappa \Pi_m \left[ \rho \left( \nabla \left( \frac{\Delta \rho}{\sqrt{\rho}} \right) \right) \right] dt + \Pi_m[\rho \varepsilon (\rho, u)] dW.
\end{aligned}$$

(2.1)

By the definition of $\Pi_m$, $\rho_0$ can be approximated by $\Pi_m[\rho_0]$, we can choose suitable coefficients before $w_k$ in the expression such that there exists a positive constant $\hat{\rho}$ and $\Pi_{m_k}[\rho_0]$, a subsequence of $\Pi_m[\rho_0]$, s.t. $\Pi_{m_k}[\rho_0] \geq \hat{\rho}$. By choosing $m_k$, this inequality may not hold when $m_k \to +\infty$, the subsequence is still denoted by $\Pi_m[\rho_0]$ for convenience.

By the maximum principle of the parabolic structure, we have

$$\inf_{x \in T^3} \Pi_m[\rho(0)] e^{-\int_0^t \text{div} u} \leq \rho \leq \sup_{x \in T^3} \Pi_m[\rho(0)] e^{-\int_0^t \text{div} u}.$$  

(2.2)

The space $H_m$ is a space of high regularity, so it naturally holds $\Lambda \{\Pi_m[\rho_0] \leq \hat{\rho}\} = 1$, $\Lambda \{||\Pi_m[u_0]|H_m \leq \alpha\} = 1$, $\rho, \bar{u} \in H_m$ and $\bar{\rho}$ can change with $m$. For the sake of (2.2), We define a cut-off function

$$\chi_R(z) = \begin{cases} 1, & \text{for } \|z\|_{H_m} \leq R, \\
0 \leq \chi_R(z) \leq 1 \in C^\infty((R, R+1)), & \text{for } R \leq \|z\|_{H_{m}} \leq R+1, \\
0, & \text{for } \|z\|_{H_{m}} \geq R+1.
\end{cases}$$

(2.3)

$[u]_R = \chi_R(u)u$ represents the truncation of $u$.

2.2. The global existence of martingale solutions of the regularized approximated system in $H_m$. We get the solution of (2.1) by iteration. Splitting the time interval $[0, T]$ into grid, we set $h$ as step length. We define recursively,

$$\begin{aligned}
\rho_t + \text{div}(\rho(t)[u(nh)]_R) &= \varepsilon \Delta \rho(t), \\
\rho(nh) &= \rho(nh-1) + \lim_{s \to nh} \rho(s), \\
\rho(0) &= \Pi_m[\rho_0].
\end{aligned}$$

(2.4)

The iteration scheme for the momentum equation is

$$\begin{aligned}
\frac{d}{dt} \Pi_m[\rho u] + \Pi_m[\text{div}(\rho(t)[u(nh)]_R \otimes u(nh))] dt + \Pi_m[\chi_R(u(nh))\nabla p(\rho(t))] dt \\
- \Pi_m[\chi_R(u(nh))\text{div}(\rho D u(nh))] dt - \Pi_m\left[\frac{11}{10} \eta \nabla \rho^{-10} \chi_R(u(nh))\right] dt \\
= \Pi_m\left[(-r_0|[u(nh)]^2 [u(nh)]_R - r_1 \rho [u(nh)]^2 [u(nh)]_R - r_2 [u(nh)]_R)|\right] dt \\
- \varepsilon \Pi_m[\chi_R(u(nh))\nabla \rho \nabla u(nh)] dt.
\end{aligned}$$

(2.5)
\[-\varepsilon \Pi_m [\chi_R (u(nh)) \nabla^2 u(nh)] \, dt + \Pi_m [\chi_R (u(nh)) \delta \rho \nabla \triangle \rho] \, dt
\]
\[+ \kappa \Pi_m [\chi_R (u(nh)) \rho \left( \nabla \left( \frac{\triangle \rho}{\sqrt{\rho}} \right) \right)] \, dt + \Pi_m [\chi_R (u(nh)) \rho] \Pi_m [\mathcal{F}(\rho, u(nh))] \, dW,
\]
with initial conditions \( u(nh) = u(nh) = \lim_{s \to nh} u(s), \) \( m_0 = \rho u(0). \) We write it in a new form by introducing a linear mapping \( \mathcal{M} \rho \) as we usually see in the Galerkin approximation,
\[
\mathcal{M} \rho : H_m \to H_m^*, \quad \mathcal{M} \rho (z) = \Pi_m (\rho z),
\]
(2.6)
or
\[
\int_{\mathbb{T}^3} \mathcal{M} \rho (z) \cdot \varphi \, dx \equiv \int_{\mathbb{T}^3} \rho z \cdot \varphi \, dx \quad \text{for all} \ \varphi \in H_m.
\]
(2.7)
From (2.3), \( \mathcal{M} \rho \) is invertible for \( \rho \neq 0, \) we can write \( \rho = S(u(nh), \Pi_m [\rho_0]), \) \( S \) is a Lipschitz-continuous operator. This gives a representation of \( u \) explicitly. Here the stochastic equation admits a unique solution in a time interval \([0, t], \) by introducing a linear mapping \( \mathcal{M} \rho \) as we usually see in the Galerkin approximation,
\[
\mathcal{M} \rho : H_m \to H_m^*, \quad \mathcal{M} \rho (z) = \Pi_m (\rho z),
\]
(2.6)
From (2.3), \( \mathcal{M} \rho \) is invertible for \( \rho \neq 0, \) we can write \( \rho = S(u(nh), \Pi_m [\rho_0]), \) \( S \) is a Lipschitz-continuous operator. This gives a representation of \( u \) explicitly. Here the stochastic equation admits a unique solution in a time interval \([0, t], \) for any \( t, \) that is to say, for any \( h, \) the stochastic equation admits a unique solution in \([nh, (n + 1)h], \) all we need to do is to see whether \( \|u\|_{H_m} \leq C, \) \( C \) is independent of \( R, h, \) time \( \tau, \) \( \tau \in [0, T] \) for \( R \to + \infty. \)

From (2.3), by Burkholder-Davis-Gundy’s inequality (see Appendix) and the norms are equivalent in \( H_m, \) we have the following estimates uniformly in \( h \) and \( \tau: \)
\[
E[\|\rho u(\tau)\|_{H_m}^r] + E[\|u(\tau)\|_{H_m}^r] \leq C \left( E[\|u_0\|_{H_m}^r] + 1 \right),
\]
(2.8)
here \( C \) depends on \( m, R, T, \rho(m), \bar{\rho} \) and \( r. \)

2.2.1. Linearization. The solutions of (2.4) and (2.5) are concerned with \( h, m, R, \varepsilon, \delta, \eta, r_0, r_1, r_2 \) and \( \kappa. \) Since we mainly focus on the limit \( h \to 0 \) in this subsection, for simplicity, we denote the solutions as \( \rho_{m,h} \) and \( u_{m,h}. \) In order to apply the stochastic compactness method to get the convergence of \( \rho_{m,h} \) and \( u_{m,h}, \) we firstly determine the space of \( \rho_{m,h} \) and \( u_{m,h}. \) For (2.1), estimate
\[
E \left[ \Pi_m \|\rho_{m,h} u_{m,h}(t_1, \cdot) - \rho_{m,h} u_{m,h}(t_2, \cdot)\|_{H_m}^r \right] \leq C(t_1 - t_2)^2 \left( E \left[ \|u_0\|_{H_m}^r \right] + 1 \right),
\]
(2.9)
still holds for \( r > 1, 0 \leq t_1, t_2 \leq T, \) because the deterministic nonlinearity terms have been linearized and \( u_{m,h} \) has been truncated by \( R. \) By Kolmogorov-Centov’s continuity theorem,
\[
E \left[ \|u_{m,h}\|_{H_m}^r \right] \leq C \left( E \left[ \|u_0\|_{H_m}^r \right] + 1 \right),
\]
(2.10)
where \( r > 2, \beta \in (0, \frac{1}{2} - \frac{1}{r}), \) here \( C \) only depends on \( m, R, \rho(m), \bar{\rho}, T \) and \( r. \) For \( \rho_{m,h}, \)
\[
\left\| \rho_{m,h}(\tau_1) - \rho_{m,h}(\tau_2) \right\|_{H_m} \leq C(t_1 - t_2) \left( E \left[ \|u_0\|_{H_m}^r \right] + 1 \right),
\]
(2.11)
here \( C \) depends on \( R, m, \) so \( \rho_{m,h} \in C^{0,1}(0, T; H_m), \) by Arzelà-Ascoli’s continuity theorem, \( \{\|\rho_{m,h}\|_{H_m}\} \) is equiv-continuous in \( C([0, T]), \) hence precompact in subset of \( C([0, T]), \) i.e., \( C^{0,1}(0, T; H_m) \to C([0, T]; H_m). \)

Based on the tightness of the \( L[\rho_{m,h}, u_{m,h}, W_{m,h}] \) and the Jakubowski-Skorokhod’s representation theorem (see Appendix), we have the following convergence theorem.

**Lemma 2.1.** The law of \( L[\rho_{m,h}, u_{m,h}, W_{m,h}] \) is tight on \( X_1 = X_\rho \times X_u \times X_W = C([0, T]; H_m) \times C^\nu([0, T]; H_m) \times C([0, T]; \xi), \) \( 0 < \nu < \beta, \beta \in (0, \frac{1}{2} - \frac{1}{r}), \) by the Jakubowski-Skorokhod’s
representation theorem, there exists a family of $\mathcal{X}$-valued Borel measurable random variables \( \{ \rho_{m,h}, u_{m,h}, W_{m,h} \} \) defined on a complete probability space \((\Omega, \mathcal{F}, \mathbb{P})\) such that

(1) \( \mathcal{L} [\rho_{m,h}, u_{m,h}, W_{m,h}] \) on \( \mathcal{X}_1 \) is given by \( \mathcal{L} [\rho_{m,h}, u_{m,h}, W_m] \);
(2) \( \mathcal{L} [\rho_{m,h}, u_{m,h}, W_{m,h}] \) on \( \mathcal{X}_1 \) is a Radon measure;
(3) \( \{ \rho_{m,h}, u_{m,h}, W_{m,h} \} \) converges \( \mathbb{P} \)-almost surely to \( \{ \rho_m, u_m, W_m \} \) in the topology of \( \mathcal{X}_1 \), i.e.,

\[
\rho_{m,h} \rightarrow \rho_m \quad \text{in} \quad C ([0, T]; H_m) \quad \mathbb{P} \text{ almost surely},
\]
\[
u_{m,h} \rightarrow u_m \quad \text{in} \quad C^\nu ([0, T]; H_m) \quad \mathbb{P} \text{ almost surely},
\]
\[
W_{m,h} \rightarrow W_m \quad \text{in} \quad C ([0, T]; \mathcal{Y}) \quad \mathbb{P} \text{ almost surely}.
\]

We skip the proof here, refer to [4] for the adequate proof. The limits \( \rho_m \) and \( u_m \) satisfy the approximated parabolic equation.

**Proposition 2.1.** The process \( \rho_m \) and \( u_m \) satisfy

\[
\Pi_m (\{ \rho_m \}) + \Pi_m [\text{div} (\rho_m [u_m] \rho)] = \Pi_m [\varepsilon \Delta \rho_m]
\]

in distribution and \( \mathbb{P} \) almost surely.

**Proof.** Firstly, the new processes \( \mathcal{L} [u_{m,h}] = \mathcal{L} [u_{m,h}], \mathcal{L} [\rho_{m,h}] = \mathcal{L} [\rho_{m,h}] \), hence \( \rho_{m,h}, u_{m,h} \) still satisfies the continuity equation \( \mathbb{P} \) almost surely. That is to say,

\[
\int_0^t \int_{\mathbb{T}^3} \Pi_m \left( (\rho_{m,h})_t + \text{div} (\rho_{m,h} [u_{m,h}] \rho) \right) dx \, ds = \int_0^t \int_{\mathbb{T}^3} \Pi_m [\varepsilon \Delta \rho_{m,h}] dx \, ds
\]

for all \( \phi \in C^\infty_c ([0, T] \times \mathbb{T}^3) \), and \( t \in [0, T], \mathbb{P} \) almost surely. Taking \( \int_{\mathbb{T}^3} \Pi_m [\rho_{m,h}(t)] \phi(t, x) dx \) for example, \( \rho_{m,h} \rightarrow \rho_m \) in \( C ([0, T]; H_m) \), \( \mathbb{P} \) almost surely, therefore \( \Pi_m [\rho_{m,h}(t, x)] \phi(t, x) \) is uniformly integrable in \( L^1 (\Omega \times [0, T] \times \mathbb{T}^3) \), by Vitali’s convergence theorem, we have

\[
\int_{\mathbb{T}^3} \Pi_m [\rho_{m,h}(t, x)] \phi(t, x) dx \rightarrow \int_{\mathbb{T}^3} \Pi_m [\rho_m(t, x)] \phi(t, x) dx \quad \mathbb{P} \text{ almost surely}.
\]

So we have

\[
\int_{\mathbb{T}^3} \Pi_m [\rho_m(t, x)] \phi(t, x) dx - \int_{\mathbb{T}^3} \Pi_m [\rho_0(0, x)] \phi(0, x) dx
\]

\[
- \int_0^t \int_{\mathbb{T}^3} \Pi_m [\rho_m] (x, s) \phi_x(s, x) dx \, ds + \int_0^t \int_{\mathbb{T}^3} \Pi_m [\text{div} (\rho_m [u_m] \rho)] \phi(s, x) dx \, ds
\]

\[
= \int_0^t \int_{\mathbb{T}^3} \Pi_m [\varepsilon \Delta \rho_m] \phi(s, x) dx \, ds,
\]

for all \( \phi \in C^\infty_c ([0, T] \times \mathbb{T}^3) \), and \( t \in [0, T], \mathbb{P} \) almost surely.

We give new notation here:

\[
\| \cdot \|_{L^p (0, T; L^q (\mathbb{T}^3))} = \| \cdot \|_{L^p L^q} \quad \text{for} \quad 1 \leq p \leq \infty, 1 \leq q \leq \infty,
\]

\[
\| \cdot \|_{L^p (0, T; H^q (\mathbb{T}^3))} = \| \cdot \|_{L^p H^q} \quad \text{for} \quad 1 \leq p \leq \infty, 1 \leq q \leq \infty.
\]

To show that the limits \( \rho_m \) and \( u_m \) satisfy the approximated momentum equation, we firstly give the following convergence theorem of stochastic integral.

**Lemma 2.2.** If \( G_k \rightarrow G \) in \( L^r (\Omega; L^2 ([0, T] \times \mathbb{T}^3)) \) \( \mathbb{P} \) almost surely, \( W_k \rightarrow W \) in \( C ([0, T] \times \mathcal{Y}) \) \( \mathbb{P} \) almost surely. Then

\[
\int_0^t \int_{\mathbb{T}^3} G_k \, dx \, dW_k \rightarrow \int_0^t \int_{\mathbb{T}^3} G \, dx \, dW \quad \mathbb{P} \text{ almost surely}.
\]
Proof. Since $W_k \to W$ as $k \to \infty$, so there exists a constant $k_1 = \max \{1, K_1 \} > 0$, such that for $k > k_1$, $|W_k - W| \leq 1$. And there exists a constant $k_2 = \max \{2, K_2 \} > k_1$, such that for $k > k_2$, $|W_k - W| \leq \frac{1}{k_2}$. All the time all the way, there exists $k_{n+1} = \max \{n+1, K_{n+1} \} > k_n$, when $k > k_{n+1}$, it holds $|W_k - W| \leq \frac{1}{k_n}$.

\[
\mathbb{E} \left[ \left| \int_0^t \int_{T^3} \mathbf{G}_k \, dW_k - \int_0^t \int_{T^3} \mathbf{G} \, dW \right|^r \right] \geq \mathbb{E} \left[ \int_0^t \int_{T^3} (G_k - G) \, dW_k \right] + 2^{r-1} \mathbb{E} \left[ \left| \int_0^t \int_{T^3} \mathbf{G} \, dW_k - \int_0^t \int_{T^3} \mathbf{G} \, dW \right|^r \right]
\]

The first term in right hand side of the above formula goes to zero as $k \to +\infty$. Now we focus on the second term. We define a mollification $\tilde{G} = \frac{1}{t} \int_0^t \mathbb{E} \left[ \mathbf{G} \right] \, ds$, here $t \in \left\{ \frac{1}{k_1}, \frac{1}{k_2}, \ldots, 1, n \right\}$. By Heine's theorem, $\mathbb{G} \to \tilde{G}$ still holds as $t \to 0$.

\[
\mathbb{E} \left[ \int_0^t \int_{T^3} \tilde{G} \, dW_k - \int_0^t \int_{T^3} \tilde{G} \, dW \right] \geq \mathbb{E} \left[ \int_0^t \int_{T^3} (G_k - G) \, dW_k \right] + 2^{r-1} \mathbb{E} \left[ \left| \int_0^t \int_{T^3} \tilde{G} \, dW_k - \int_0^t \int_{T^3} \tilde{G} \, dW \right|^r \right]
\]

The first term will go to 0 as $t \to 0$. For $\mathbb{E} \left[ \left| \int_0^t \int_{T^3} \tilde{G} \, dW_k - \int_0^t \int_{T^3} \tilde{G} \, dW \right|^r \right]$, by Itô’s formula, we can do integration by part for the smooth integrand, and

\[
\frac{d\tilde{G}}{dt} = \frac{1}{t} G(t) + \frac{1}{t^2} \int_0^t \exp \left( \frac{-t-s}{t} \right) G(s) \, ds = \frac{1}{t} G(t) - \frac{1}{t^2} G.
\]

So it holds

\[
\mathbb{E} \left[ \int_0^t \int_{T^3} \tilde{G} \, dW_k - \int_0^t \int_{T^3} \tilde{G} \, dW \right] \geq \frac{1}{t} \mathbb{E} \left[ \int_0^t \int_{T^3} \tilde{G} \, dW_k - \int_0^t \int_{T^3} \tilde{G} \, dW \right]
\]
Since the arbitrariness of \( \phi \) in distribution sense and \( \bar{k} \) almost surely in \( P \), it holds

We should mainly show the convergence of stochastic term in this part.

Now we have a proposition for the approximated momentum equation.

**Proposition 2.2.** The process \((\rho_m, u_m, W_m)\) satisfies

\[
\frac{d}{dt} \Pi_m [(\rho_m u_m)] + \Pi_m \left[ \left( \text{div}(\rho_m u_m) \right) \chi_R(\|u_m\|_{H_m}) \right] dt \\
- \Pi_m \left[ \left( \text{div}(\rho_m \Delta u_m) \right) \chi_R(\|u_m\|_{H_m}) \right] dt \\
= \Pi_m \left[ (-r_0 |u_m|^2 |u_m|_R - r_1 \rho_m |u_m|^2 |u_m|_R - r_2 |u_m|_R) \right] dt \\
+ \Pi_m \left[ \varepsilon \nabla \rho_m \nabla u_m \chi_R(\|u_m\|_{H_m}) \right] dt \\
- \Pi_m \left[ \varepsilon \Delta^2 u_m \chi_R(\|u_m\|_{H_m}) \right] dt + \Pi_m \left[ \frac{11}{10} \rho_m^{-10} \chi_R(\|u_m\|_{H_m}) \right] dt \\
+ \Pi_m \left[ \delta \rho_m \nabla \Delta^3 \rho_m \chi_R(\|u_m\|_{H_m}) \right] dt + \Pi_m \left[ \kappa \rho_m \left( \nabla \left( \frac{\Delta \rho_m}{\sqrt{\rho_m}} \right) \right) \chi_R(\|u_m\|_{H_m}) \right] dt \\
+ \Pi_m \left[ \rho_m \mathbb{F}(\rho_m, u_m) \chi_R(\|u_m\|_{H_m}) \right] dt \text{W},
\]

in distribution sense and \( \bar{P} \) almost surely, \( t \in [0, T] \).

**Proof.** We should mainly show the convergence of stochastic term in this part.

Similar as in [2], by the strong convergences of \( \bar{\rho}_{m,h} \) and \( \bar{u}_{m,h} \), as well the continuity of \( F_k \), it holds

\[
\Pi_m \left[ \rho_{m,h} \Pi_m \left[ F(\bar{\rho}_{m,h}, \bar{u}_{m,h}) \right] \right] \to \Pi_m \left[ \rho_m \Pi_m \left[ F(\rho_m, u_m) \right] \right], \tag{2.24}
\]

\( \bar{P} \) almost surely in \( L^2 \left( \Omega; L^2([0, T]; L^2(\mathbb{T}^3)) \right) \). Combining this with the convergence of \( W_{m,h} \), by lemma 2.2 we have, for all \( \phi \in C_{c}^{\infty}([0, T] \times \mathbb{T}^3) \),

\[
\int_0^t \int_{\mathbb{T}^3} \Pi_m \left[ \rho_{m,h} \Pi_m \left[ F(\bar{\rho}_{m,h}, \bar{u}_{m,h}) \right] \right] \phi dW_{m,h} d x \\
\to \int_0^t \int_{\mathbb{T}^3} \Pi_m \left[ \rho_m \Pi_m \left[ F(\rho_m, u_m) \right] \right] \phi dW_m d x, \quad t \in [0, T], \tag{2.25}
\]
2.2.2. Energy balance and energy estimate for the regularied system. In order to show the norms of random variables are independent on $R$, we firstly derive the energy estimates. In this part, we use the general notation $\rho$ and $u$.

We apply Itô’s formula to the functional

$$g : L^q (T^3) \times H^*_m \to \mathbb{R},$$

$$(\rho, q) \mapsto \frac{1}{2} q M^{-1} [\rho] q,$$  \hspace{1cm} (2.26)

for $q \in H^*_m$, $M^{-1} [\rho]$ maps $H^*_m$ to $H_m$. 

$$\partial_q g (\rho, q) = M^{-1} [\rho] q \in H_m, \quad \partial^2_{qq} g (\rho, q) = M^{-1} [\rho],$$

$$\partial_{\rho} g (\rho, q) = -\frac{1}{2} M^{-1} [\rho] (q \cdot M^{-1} [\rho] q) \in L^q (T^3) \times H^*_m \to \mathbb{R}.$$  \hspace{1cm} (2.27)

Therefore, for $g = \frac{1}{2} \rho |u|^2 = \frac{1}{2} \frac{\rho^2}{\rho}$, according to Itô’s formula (see Appendix), we deduce

$$d \left( \int_{T^3} \frac{1}{2} \rho u^2 \, dx \right)$$

$$= -\frac{1}{2} \int_{T^3} u^2 \rho \, dx + \int_{T^3} u \cdot d (\rho u) \, dx + \left( \frac{1}{2} \int_{T^3} \rho |F(\rho, u)|^2 \, dx \right) \, dt$$

$$= -\frac{1}{2} \int_{T^3} u^2 \rho \, dx + \left( \int_{T^3} \rho F(\rho, u) \cdot u \, dx \right) \, dW$$

$$- \left( \int_{T^3} u \cdot (\text{div} (\rho u \otimes u) + \nabla p (\rho)) \, dx - \int_{T^3} \varepsilon \nabla \rho \nabla u \cdot u \, dx - \varepsilon \int_{T^3} |\nabla u|^2 \, dx \right. \quad (2.28)$$

$$+ \int_{T^3} \text{div} (\rho D u) \, u \, dx + \int_{T^3} \frac{11}{10} \eta \nabla \rho^{-10} \cdot u \, dx + \delta \int_{T^3} \rho \nabla \Delta \rho \cdot u \, dx$$

$$- \int_{T^3} \rho_1 |u|^2 \cdot u \, dx - \int_{T^3} \rho_2 |u|^3 \cdot u \, dx$$

$$- \int_{T^3} \rho_3 \nabla \left( \frac{\nabla \sqrt{\rho}}{\sqrt{\rho}} \right) \cdot u \, dx + \left( \frac{1}{2} \int_{T^3} \rho |F(\rho, u)|^2 \, dx \right) \, dt.$$

Since

$$\left( \int_{T^3} u \cdot \text{div} (\rho u \otimes u) \, dx \right) \, dt = \left( \int_{T^3} \text{div} (\rho u) \frac{1}{2} u^2 \, dx \right) \, dt,$$

so we can deduce that

$$-\frac{1}{2} \int_{T^3} u^2 \rho \, dx \left( \int_{T^3} \text{div} (\rho u \otimes u) \cdot u \, dx \right) \, dt - \left( \int_{T^3} \varepsilon \nabla \rho \nabla u \cdot u \, dx \right) \, dt = 0.$$  \hspace{1cm} (2.29)

Multiply $-\eta \rho^{-11}$ to the mass conservation equation, and integrate on $T^3$, we have

$$\left( \int_{T^3} \frac{11}{10} \eta \nabla \rho^{-10} \, dx \right) \, dt = -d \left( \int_{T^3} \rho_1 \rho^{-10} \, dx \right) + \varepsilon \eta \frac{1}{2} \int_{T^3} |\nabla \rho^{-5}|^2 \, dx \, dt.$$  \hspace{1cm} (2.30)

Similarly,

$$\left( \int_{T^3} \frac{p}{\gamma - 1} \, dx \right) \, dt = -d \left( \int_{T^3} \frac{p}{\gamma - 1} \, dx \right) + \varepsilon \frac{4a}{\gamma} \left( \int_{T^3} |\nabla (\rho^\gamma)|^2 \, dx \right) \, dt,$$  \hspace{1cm} (2.31)

$$\left( \int_{T^3} \delta \rho \nabla \rho \Delta \rho \cdot u \, dx \right) \, dt = -d \left( \int_{T^3} \delta |\nabla \rho|^2 \, dx \right) + \varepsilon \delta \left( \int_{T^3} |\Delta \rho|^2 \, dx \right) \, dt.$$  \hspace{1cm} (2.32)

For term $\kappa \rho \nabla \left( \frac{\nabla \sqrt{\rho}}{\sqrt{\rho}} \right)$, we can use the equality relation $2 \rho \nabla \left( \frac{\nabla \sqrt{\rho}}{\sqrt{\rho}} \right) = \text{div} (\rho \nabla \log \rho)$. Therefore,

$$\int_{T^3} \kappa \rho \nabla \left( \frac{\nabla \sqrt{\rho}}{\sqrt{\rho}} \right) \cdot u \, dx = \kappa \frac{d}{dt} \int_{T^3} |\nabla \sqrt{\rho}|^2 \, dx - \kappa \varepsilon \int_{T^3} \rho |\nabla \log \rho|^2 \, dx.$$  \hspace{1cm} (2.33)
To conclude, we have the energy balance
\[
\int_{T^3} \left( \frac{1}{2} |\mathbf{u}|^2 + \frac{a}{\gamma} \rho^\gamma + \frac{\eta}{10} \rho^{-10} + \kappa |\nabla \sqrt{\rho}|^2 + \frac{\delta}{2} |\nabla \Delta \rho|^2 \right) \, dx \\
+ \varepsilon \frac{4a}{\gamma} \int_0^t \int_{T^3} |\nabla \rho|^2 \, dx \, dt + \varepsilon \frac{11}{25} \int_0^t \int_{T^3} |\nabla \rho^{-5}|^2 \, dx \, dt \\
+ \varepsilon \delta \int_0^t \int_{T^3} |\Delta \rho|^4 \, dx \, dt + \varepsilon \frac{\kappa}{2} \int_0^t \int_{T^3} \rho |\nabla \log \rho|^2 \, dx \, dt + \int_0^t \rho |D \mathbf{u}|^2 \, dx \, dt \\
+ \varepsilon \int_0^t \int_{T^3} |\Delta \mathbf{u}|^2 \, dx \, dt + r_0 \int_0^t \int_{T^3} |\mathbf{u}|^4 \, dx \, dt + r_1 \int_0^t \int_{T^3} \rho |\mathbf{u}|^4 \, dx \, dt + r_2 \int_0^t \int_{T^3} |\mathbf{u}|^2 \, dx \, dt \\
= \int_{T^3} \left( \frac{1}{2} \rho_0 |\mathbf{u}_0|^2 + \frac{a}{2} \rho_0^\gamma + \frac{\eta}{10} \rho_0^{-10} + \kappa |\nabla \sqrt{\rho_0}|^2 + \frac{\delta}{2} |\nabla \Delta \rho_0|^2 \right) \, dx \\
+ \frac{1}{2} \int_0^t \int_{T^3} \rho |\mathbf{F}(\rho, \mathbf{u})|^2 \, dx \, dt + \frac{1}{2} \int_0^t \int_{T^3} \rho \mathbf{F}(\rho, \mathbf{u}) \cdot \mathbf{u} \, dx \, dW.
\] (2.34)

The $C_r$ inequality says for real-valued $r > 1$, $a$ and $b$, by the property of convex functions, it holds $|a + b|^r \leq \max(1, 2^{r-1}) (|a|^r + |b|^r)$. Hence to estimate
\[
\mathbb{E} \left[ \left| \int_0^t \int_{T^3} \rho |\mathbf{F}(\rho, \mathbf{u})|^2 \, dx \, dt \right|^r \right] \\
= \mathbb{E} \left[ \left| \int_0^t \int_{T^3} \rho \left| \sum_{k=1}^{+\infty} \mathbf{F}_k(\rho, \mathbf{u}) \epsilon_k \right|^2 \, dx \, dt \right|^r \right] \\
= \mathbb{E} \left[ \left| \sum_{k=1}^{+\infty} \int_0^t \int_{T^3} \rho |\mathbf{F}_k(\rho, \mathbf{u})|^2 \, dx \, dt \right|^r \right] \\
\] (2.35)

in the right hand side, we only need to consider
\[
\mathbb{E} \left[ \left| \int_0^t \int_{T^3} \rho \mathbf{F}(\rho, \mathbf{u}) \cdot \mathbf{u} \, dx \, dW \right|^r \right] \\
\leq C \mathbb{E} \left[ \left| \int_0^t \int_{T^3} \rho \left| \sum_{k=1}^{+\infty} \mathbf{F}_k(\rho, \mathbf{u}) \epsilon_k \cdot \mathbf{u} \right|^2 \, dx \, ds \right|^\frac{r}{2} \right] \\
= C \mathbb{E} \left[ \left| \sum_{k=1}^{+\infty} \int_0^t \int_{T^3} \rho \mathbf{F}_k(\rho, \mathbf{u}) \cdot \mathbf{u} \, dx \, ds \right|^\frac{r}{2} \right] \\
\] (2.37)

separately, here $C$ depends on $r$.

For any function $\xi, 1 < \gamma < 3, \xi^{\frac{1}{\gamma}} < (1 + \xi)^{\frac{1}{\gamma}} < (1 + \xi)$. By the property of $\mathbf{F}_k$, we do the following energy estimates,
\[
\sum_{k=1}^{+\infty} \int_0^t \int_{T^3} \rho_m |\Pi_m [\mathbf{F}_k(\rho_m, \mathbf{u}_m)]|^2 \, dx \, dt
\]
\[
\|\rho_m\|_{L^1_t L^1_x} \sum_{k=1}^{+\infty} \|F_k (\rho_m, u_m)\|_{L^\infty_t L^1_x}^{2} \\
 \leq \|\rho_m\|_{L^1_t L^1_x} \sum_{k=1}^{+\infty} f_k^2 \leq C \int_0^t \left( \int_{T^3} \rho_m^2 \, d\mathbf{x} \right)^{\frac{1}{2}} \, d\mathbf{s}
\]
uniformly in \(R\) and \(m\), here \(C\) depends on \(\sum_{k=1}^{+\infty} f_k^2\). So by Jensen’s inequality for \(r \geq 1\),

\[
\mathbb{E} \left[ \left( \int_0^t \left( \int_{T^3} \rho_m^2 (s, \mathbf{x}) \, d\mathbf{x} \right)^r \, d\mathbf{s} \right)^{\frac{1}{r}} \right] \leq C \mathbb{E} \left[ \int_0^t \left( \int_{T^3} \rho_m^2 (s, \mathbf{x}) \, d\mathbf{x} \right)^r \, d\mathbf{s} \right]^{\frac{1}{r}} \leq C \left( t^{r-1} \mathbb{E} \left[ \int_0^t \left( \int_{T^3} \rho_m^2 (s, \mathbf{x}) \, d\mathbf{x} \right)^r \, d\mathbf{s} \right] + t^r \right), \quad t \in [0, T],
\]
here \(C\) depends on \(r\) and \(\sum_{k=1}^{+\infty} f_k^2\). For \(2.37\), \(\frac{1}{r} + \frac{1}{2} + \frac{1}{q} = 1\),

\[
\sum_{k=1}^{+\infty} \int_{T^3} \rho_m \Pi_m \left[ F_k (\rho_m, u_m) \right] \cdot u_m \, d\mathbf{x} \\
\leq C \sum_{k=1}^{+\infty} \|\sqrt{\rho_m}\|_{L^\infty_t L^2_x} \|\sqrt{\rho_m} u_m\|_{L^2_x} \|\Pi_m \left[ F_k (\rho_m, u_m) \right]\|_{L^2_x} \\
\leq C \sum_{k=1}^{+\infty} \int_{T^3} \rho_m \Pi_m \left[ F_k (\rho_m, u_m) \right] \cdot u_m \, d\mathbf{x} \\
\leq C \sum_{k=1}^{+\infty} f_k^2 \|\sqrt{\rho_m}\|_{L^\infty_t L^2_x} \|\sqrt{\rho_m} u_m\|_{L^2_x}^2 \\
\leq C \sum_{k=1}^{+\infty} f_k^2 \left( \int_{T^3} \rho_m^2 \, d\mathbf{x} \right)^{\frac{1}{2}} \int_{T^3} \rho_m u_m^2 \, d\mathbf{x} \\
\leq C \sum_{k=1}^{+\infty} f_k^2 \left( \int_{T^3} \rho_m^2 \, d\mathbf{x} + 1 \right) \left( \int_{T^3} \rho_m u_m^2 \, d\mathbf{x} \right) \leq C \left( \int_{T^3} \rho_m^2 \, d\mathbf{x} + 1 \right) \left( \int_{T^3} \rho_m u_m^2 \, d\mathbf{x} \right),
\]
here \(C\) depends on \(r\) and \(\sum_{k=1}^{+\infty} f_k^2\). Due to the property of convex functions, by Jensen’s inequality as used in \(2.39\), we have

\[
\mathbb{E} \left[ \int_0^t \left( \int_{T^3} \rho_m \Pi_m \left[ F (\rho_m u_m) \right] \cdot u_m \, d\mathbf{x} \, d\mathbf{W} \right) \right] \\
\leq C \mathbb{E} \left[ \left( \int_0^t \left( \int_{T^3} \rho_m^2 \, d\mathbf{x} + \int_{T^3} \rho_m (s) u_m^2 (s) \, d\mathbf{x} \right)^2 + 1 \right) \, d\mathbf{s} \right]^{\frac{1}{2}} \\
= C \mathbb{E} \left[ \left( \int_0^t \left( \int_{T^3} \rho_m^2 \, d\mathbf{x} + \int_{T^3} \rho_m (ts) u_m^2 (ts) \, d\mathbf{x} \right)^2 + 1 \right) \, d\mathbf{s} + t \right]^{\frac{1}{2}}
\]
\(2.41\).
The global existence of the classical solutions to the nonlinear regularized Garlerkin scheme $H_r$ for $YACHUN LI AND LICHEN ZHANG$

\[
E(t) = \int_{T_3} \left( \frac{1}{2} \rho_m |u_m|^2 + \frac{a}{\gamma} \rho_m^\gamma + \frac{\eta}{10} \rho_m^{-10} + \frac{\kappa}{2} |\nabla \rho_m|^2 + \frac{\delta}{2} \nabla \Delta \rho_m|^2 \right) \, dx, \quad (2.42)
\]

we have

\[
E \left[ E(t) + \varepsilon \frac{4a}{\gamma} \int_0^t \int_{T_3} |\nabla \rho_m|^2 \, dx \, ds + \varepsilon \eta \frac{11}{25} \int_0^t \int_{T_3} |\nabla \rho_m^{-5}|^2 \, dx \, ds + \varepsilon \delta \int_0^t \int_{T_3} \rho_m |\nabla \rho_m|^2 \, dx \, ds \right. \\
+ \int_0^t \int_{T_3} \rho_m |\mathbb{D} u_m|^2 \, dx \, ds + \varepsilon \int_0^t \int_{T_3} |\Delta u_m|^2 \, dx \, ds \\
+ r_0 \int_0^t \int_{T_3} |u_m|^4 \, dx \, ds + r_1 \int_0^t \int_{T_3} \rho_m |u_m|^4 \, dx \, ds + r_2 \int_0^t \int_{T_3} |u_m|^2 \, dx \, ds \right] \\
\leq C \left( t^{\tilde{r}-1} + t^{-1} \right) \int_0^t E \left[ \varepsilon(s)^r \right] \, ds + t^{\tilde{r} + t} + E \left[ E(0)^r \right].
\]

Therefore,

\[
E[(E(t))^r] \leq C \left( (t^{\tilde{r}-1} + t^{-1}) \int_0^t E \left[ \varepsilon(s)^r \right] \, ds + t^{\tilde{r} + t} + E \left[ E(0)^r \right] \right),
\]

here $C$ depends on $r$ and $\sum_{k=1}^{+\infty} f_k^2$. By Grönwall’s inequality, $t^{\tilde{r} + t} + E \left[ E(0)^r \right]$ is nondecreasing with respect to $t$, it holds

\[
E[\sup_{t \in [0,T]} (E(t))^r] \leq C \left( t^{\tilde{r} + t} + E \left[ E(0)^r \right] \right) e^{C(t^{\tilde{r} + t})} \leq C(E \left[ E(0)^r \right] + 1),
\]

here $C$ depends on $r$, $T$, and $\sum_{k=1}^{+\infty} f_k^2$. Thereupon for any $r \geq 2$,

\[
E \left[ \left( \frac{4a}{\gamma} \int_0^t \int_{T_3} |\nabla \rho_m|^2 \, dx \, ds + \varepsilon \eta \frac{11}{25} \int_0^t \int_{T_3} |\nabla \rho_m^{-5}|^2 \, dx \, ds + \varepsilon \delta \int_0^t \int_{T_3} \rho_m |\nabla \rho_m|^2 \, dx \, ds \right. \\
+ \int_0^t \int_{T_3} \rho_m |\mathbb{D} u_m|^2 \, dx \, ds + \varepsilon \int_0^t \int_{T_3} |\Delta u_m|^2 \, dx \, ds + r_0 \int_0^t \int_{T_3} |u_m|^4 \, dx \, ds + r_1 \int_0^t \int_{T_3} \rho_m |u_m|^4 \, dx \, ds + r_2 \int_0^t \int_{T_3} |u_m|^2 \, dx \, ds \right] \\
\leq C(r,T)(E \left[ E(0)^r \right] + 1),
\]

here $C$ depends on $r$, $T$, and $\sum_{k=1}^{+\infty} f_k^2$.

2.2.3. The global existence of the classical solutions to the nonlinear regularized Garlerkin scheme in $H_m$. Define $\tau_R$, a stopping time (see Appendix), until which $(\rho_m, u_m)$ is the well-defined weak solution in $C([0,T]; H_m) \times C_t^\nu([0,T]; H_m)$.

\[
\tau_R = \inf \left\{ t \in [0,T] \mid \|u_m\|_{H_m} > R \right\}.
\]
Denoting the solutions as \((\rho_R, u_R)\) on \([0, \tau_R]\), for this layer we focus on \(R \to +\infty\), we need show that
\[
\mathbb{P} \left( \sup_{R \in \mathbb{N}} \tau_R = T \right) = 1. \tag{2.48}
\]
In other words, till time \(T\) the solutions will not blow up as \(R \to \infty\). Since (2.2), we only need consider \(\|u_R\|_{H_m}\) and well-posedness of \(\rho_m\) follows. From (2.46) we know that \(\|\nabla u_R\|_{L^2_{t}L^6_x} \leq C\), \(\mathbb{P}\) almost surely, this implies that \(\|\nabla u_R\|_{L^2_{t}L^6_x} \leq C\), \(\mathbb{P}\) almost surely, here \(C\) depends on \(r, T\), and \(\sum_{k=1}^{+\infty} f_k^2\). Since \(E \left[ \left( 0 \int_0^T |u_R|^2 d s \right)^r \right] \leq C(E(0)^r + 1)\), it holds
\[
\|u_R\|_{H_m} \leq C, \quad \|\text{div} u_R\|_{H_m} \leq C, \quad \mathbb{P} \text{ almost surely}. \tag{2.49}
\]
This shows that \(\tau_R = T\) for \(R\) large enough.

2.3. \(m\) goes to infinity: the global existence of martingale solutions to regularized system \([12,37]\) in infinite dimensional space. The solution in this layer is concerned with \(m, \varepsilon, \kappa, \delta, \eta, r_0, r_1, r_2\), but we denote \(\rho_m\) as the solution of the Galerkin approximation scheme after taking \(R \to +\infty\) for convenience. For the space of \(u_m\), we try not use the regular property from \(\nabla^2 u_m\) because we will let \(\varepsilon\) goes to zero finally. In conclusion, from the estimate (2.42), (2.43) and (2.46), we get the following bounds
\[
E \left[ \left( \sup_{t \in (0, T]} \|\rho_m\|_{L^2_x}^r \right) \right] \leq C, \quad E \left[ \sup_{t \in (0, T]} \left( \eta \|\rho_m^{-1}\|_{L^5_x}^{10} \right)^r \right] \leq C, \quad E \left[ \sup_{t \in (0, T]} \left( \frac{\kappa}{2} \left\| \nabla \rho_m^{\frac{1}{2}} \right\|_{L^2_x}^2 \right)^r \right] \leq C, \quad E \left[ \sup_{t \in (0, T]} \left( \frac{\delta}{2} \left\| \nabla^{4} \rho_m \right\|_{L^2_x}^2 \right)^r \right] \leq C, \quad E \left[ \left( \varepsilon \eta \left\| \nabla \rho_m^{-5} \right\|_{L^2_x}^2 \right)^r \right] \leq C, \quad E \left[ \left( \frac{4\varepsilon}{\gamma} \left\| \nabla \rho_m \right\|_{L^2_x}^2 \right)^r \right] \leq C, \quad E \left[ \left( \varepsilon \delta \left\| \nabla^{5} \rho_m \right\|_{L^2_x}^2 \right)^r \right] \leq C, \quad E \left[ \left( \varepsilon \kappa \left\| \nabla \rho_m \right\|_{L^2_x}^2 \right)^r \right] \leq C, \tag{2.50}
\]
we also have the bound concerning \(u_m\):
\[
E \left[ \left( \varepsilon \|u_m\|_{L^2_{t}W^2_{2,2}}^r \right) \right] \leq C, \quad E \left[ \left( \sup_{t \in (0, T]} \|\rho_m u_m\|_{L^1_t}^2 \right)^r \right] \leq C, \quad E \left[ \left( \sup_{t \in (0, T]} \left\| \rho_m \nabla u_m \right\|_{L^2_x}^{2+ \frac{2}{\gamma}} \right)^r \right] \leq C, \quad E \left[ \left( \left\| \rho_m \nabla u_m \right\|_{L^2_x}^{2+ \frac{2}{\gamma}} \right)^r \right] \leq C, \quad E \left[ \left( \frac{1}{\gamma} \left\| \rho_m \Delta u_m \right\|_{L^2_x}^2 \right)^r \right] \leq C, \quad E \left[ \left( \frac{1}{\gamma} \left\| \rho_m \Delta u_m \right\|_{L^2_x}^2 \right)^r \right] \leq C, \quad E \left[ \left( \|r_0 \|_{L^2_{t}L^4_x}^4 \right)^r \right] \leq C, \quad E \left[ \left( r_1 \left\| \rho_m \nabla u_m \right\|_{L^2_x}^4 \right)^r \right] \leq C, \quad E \left[ \left( r_2 \left\| \rho_m \nabla u_m \right\|_{L^2_x}^4 \right)^r \right] \leq C, \tag{2.51}
\]
where \(C\) depends on \(T, r, \sum_{k=1}^{+\infty} f_k^2\), and \(E[E(0)^r]\).

The following lemma is Jüngle’s original idea \([23]\).

**Lemma 2.3.** For any smooth positive function \(f(x)\), it holds
\[
\int_{\mathbb{T}^3} f \mathbb{E} \left[ \left| \nabla^2 \log f \right|^2 d x \right] \geq \frac{1}{7} \int_{\mathbb{T}^3} \left| \nabla^2 f \right|^2 d x + \frac{1}{8} \int_{\mathbb{T}^3} \left| \nabla f \right|^4 d x. \tag{2.52}
\]
Remark 2.1. For $f(x)$ satisfies $\int_{\mathbb{T}^3} f \left| \nabla^2 \log f \right|^2 \, dx \leq C$, though there might be vacuum, it appears on an zero-measure area. On the zero-measure area, $f(x)$ can be redefined as a positive constant $f_1$, which will not cause any change to the integral, therefore the inequality stays valid as long as $\int_{\mathbb{T}^3} f \left| \nabla^2 \log f \right|^2 \, dx \leq C$.

According to the energy estimates $\text{(2.50)}$, we know that
\begin{align}
\mathbb{E} \left[ \left( \varepsilon \kappa \left\| \frac{1}{\rho_m^3} \right\|_{L^2 L^2_3} \right)^r \right] &\leq C, \quad (2.53) \\
\mathbb{E} \left[ \left( \varepsilon \kappa \left\| \nabla \left( \frac{1}{\rho_m} \right) \right\|_{L^4 L^4_3} \right)^2 \right] &\leq C, \quad (2.54)
\end{align}
where $C$ depends on $T, r, \sum_{k=1}^{+\infty} f_k^2$ and $\mathbb{E} [E(0)^r]$.

To show the convergence, we will proceed in four steps.

**Step1:** Choose the path space.

Now we roughly give the estimate for $\rho_m, \rho_m u_m$, and $\rho_m^{\frac{1}{n}}$.

**Lemma 2.4.**
\begin{align}
\mathbb{E} \left[ \left( \left\| (\rho_m)_t \right\|_{L^2 L^2_3} \right)^r \right] &\leq C, \quad \mathbb{E} \left[ \left( \left\| \left( \frac{1}{\rho_m} \right)_t \right\|_{L^2 L^2_3} \right)^r \right] \leq C, \quad (2.55) \\
\mathbb{E} \left[ \left( \left\| \frac{1}{\rho_m} \right\|_{L^2 H^2_3} \right)^r \right] &\leq C, \quad \mathbb{E} \left[ \left( \left\| \nabla \left( \frac{1}{\rho_m} \right) \right\|_{L^4 L^4_3} \right)^r \right] \leq C, \quad (2.56) \\
\mathbb{E} \left[ \left( \left\| \rho_m \right\|_{L^2 L^2_3} \right)^r \right] &\leq C, \quad \mathbb{E} \left[ \left( \left\| \rho_m u_m \right\|_{L^2 L^2_3} \right)^r \right] \leq C, \quad (2.57)
\end{align}
$C$ is merely dependent on $T, r, r_1^{-1}, \delta^{-\frac{1}{\kappa}}, \varepsilon^{-1} \kappa^{-1}$, and $\text{(2.55)}$ will be independent on $\delta$ after $\varepsilon \to 0$.

**Proof.**
\begin{align}
(\rho_m)_t &= -\rho_m \text{div } u_m - \nabla \rho_m \cdot u_m + \varepsilon \Delta \rho_m \\
&= -\sqrt{\rho_m} \sqrt{\rho_m} \text{div } u_m - \left( \frac{1}{\rho_m} \right) \left( \frac{1}{\rho_m} \right) \left( 4 \nabla \rho_m \right) + \varepsilon \Delta \rho_m. \tag{2.58}
\end{align}

Now we give an estimate independent on $\delta^{-1}$. On the one hand,
\begin{align}
\mathbb{E} \left[ \left\| \frac{1}{\rho_m} \frac{1}{\rho_m} \text{div } u_m \right\|_{L^2 L^2_3} \right]^r &\leq \mathbb{E} \left[ \left( \left\| \frac{1}{\rho_m} \frac{1}{\rho_m} \text{div } u_m \right\|_{L^2 L^2_3} \right)^r \right] \leq C. \tag{2.59}
\end{align}
On the other hand,
\begin{align}
\mathbb{E} \left[ \left\| \nabla \rho_m \left( \frac{1}{\rho_m} u_m \right) \right\|_{L^2 L^2_3} \right]^r &\leq \mathbb{E} \left[ \left( \left\| \nabla \rho_m \left( \frac{1}{\rho_m} u_m \right) \right\|_{L^2 L^2_3} \right)^r \right] \leq C, \tag{2.60}
\end{align}
so
\begin{align}
\mathbb{E} \left[ \left\| \rho_m \right\|_{L^2 L^2_3} \right] \leq C. \tag{2.61}
\end{align}
$C$ depends on $T, r, \varepsilon^{-1}$ and $\kappa^{-1}$. Meanwhile,
\begin{align}
2 \left( \frac{1}{\rho_m} \right)_t &= -\rho_m \text{div } u_m - 2 \nabla \rho_m \cdot u_m + \varepsilon \frac{\Delta \rho_m}{\rho_m} \\
&= -\rho_m \text{div } u_m - 8 \rho_m \nabla \rho_m \cdot u_m + \varepsilon \frac{\Delta \rho_m}{\rho_m}, \tag{2.62}
\end{align}
Since (2.50) and $H^8 (T^3) \rightarrow C^{6,3} (T^3)$, so
\[
E \left[ \left( \left\| \frac{\Delta \rho_m}{\rho_m^{1/3}} \right\|_{L^2_1 L^2_3} \right)^2 \right] \leq C, \tag{2.63}
\]
$C$ depends on $T, r, \varepsilon^{-1} \eta^{-1}$ and $\varepsilon^{-1} \delta^{-1}$. Hence
\[
E \left[ \left( \left\| \nabla \rho_m^{1/3} \right\|_{L^2_1 L^2_3} \right)^2 \right] \leq C, \tag{2.64}
\]
for $C$ depends on $T, r, r_1^{-1}, \varepsilon \eta^{-1}, \varepsilon \delta^{-1}$ and $\varepsilon^{-1} \kappa^{-1}$. Notice that this estimate is independent on $\delta^{-1}$ as $\varepsilon$ goes to zero before $\delta$ goes to zero.
\[
E \left[ \left( \left\| \nabla \rho_m^{1/3} \right\|_{L^2_1 L^2_3} \right)^2 \right] \leq C,
\]
gives
\[
E \left[ \left( \left\| \rho_m \right\|_{L^1_2} \right)^2 \right] \leq C, \tag{2.65}
\]
also $\rho_m \in L^{\infty}(0, T; L^1 (T^3))$, by Hölder’s inequality we have
\[
E \left[ \left( \left\| \rho_m \right\|_{L^1_2} \right)^2 \right] \leq E \left[ \left\| \rho_m \right\|_{L^1_{2r}} \right]^2 \left\| \rho_m \right\|_{L^1_2} \leq C, \tag{2.66}
\]
here $C$ depends on $T, r$ and $\varepsilon^{-1}$.
\[
E \left[ \left( \left\| \rho_m u_m \right\|_{L^1_2 L^2_3} \right)^2 \right] \leq E \left[ \left\| \rho_m \right\|_{L^1_{2r}} \right]^2 \left\| u_m \right\|_{L^1_{1r} L^2_3} \leq C, \tag{2.67}
\]
due to $H^2 (T^3) \rightarrow C^{0, \frac{1}{2}} (T^3)$, $\rho_m u_m \in L^{\infty}([0, T]; L^2 (T^3))$, here $C$ depends on $T, r$ and $\varepsilon^{-1} \kappa^{-1}$. We calculate
\[
\nabla (\rho_m u_m) = 2 \nabla \rho_m^{1/3} \otimes \rho_m^{1/3} u_m + \rho_m^{1/3} \cdot \rho_m^{1/3} \nabla u_m, \tag{2.68}
\]
so we have $\nabla (\rho_m u_m) \in L^2 \left(0, T; L^{\frac{3}{2}} (T^3)\right)$, $W^{1, \frac{3}{2}} (T^3) \hookrightarrow L^3 (T^3)$. $\rho_m u_m$ are bounded in $L^2 \left([0, T]; L^2 (T^3)\right)$. This is the end of the proof.

The path space is
\[
X_2 = \mathcal{X}_{\rho_0} \times \mathcal{X}_{\rho_0 u_0} \times \mathcal{X}_{\frac{1}{\rho_0^{1/3}} u_0} \times \mathcal{X}_\rho \times \mathcal{X}_u \times \mathcal{X}_{\rho u} \times \mathcal{X}_W, \tag{2.69}
\]
where
\[
\begin{align*}
\mathcal{X}_{\rho_0} &= L^7 (T^3) \cap L^1 (T^3) \cap L^{-10} (T^3) \cap H^9 (T^3), \\
\mathcal{X}_{\rho_0 u_0} &= L^1 (T^3), \mathcal{X}_{\frac{1}{\rho_0^{1/3}} u_0} = L^2 (T^3), \\
\mathcal{X}_\rho &= L^2 \left([0, T]; H^{10} (T^3)\right) \cap L^2 \left([0, T]; W^{1,3} (T^3)\right) \cap L^{\frac{11}{6}} \left([0, T]; T^3\right), \\
\mathcal{X}_u &= L^2 \left(0, T; T^3\right) \cap L^4 \left(0, T; T^3\right), \\
\mathcal{X}_{\rho u} &= L^2 \left(0, T; W^{1, \frac{7}{2}} (T^3)\right) \cap C \left(0, T; L^{\frac{3}{2}} (T^3)\right), \\
\mathcal{X}_W &= C \left(0, T; \mathcal{S}\right).
\end{align*}
\]

**Step 2:** Show the tightness of the laws.

**Proposition 2.3.** \(\mathcal{L} \left[ \rho_{0,m}, \rho_{0,m} u_{0,m}, \rho_{0,m} u_{0,m}^*, \rho_{m}, u_{m}, \rho_{m} u_{m}, W_m \right]; m \in \mathbb{N} \) is tight on \(\mathcal{X}_m\).

To prove this proposition, we only need prove proposition 2.4 to proposition 2.9 singly.

**Proposition 2.4.** The set \(\mathcal{L} [\rho_{0,m}]; m \in \mathbb{N} \) is tight on \(\mathcal{X}_{\rho_0,m}\).
Probability measures on the σ-algebra of Borel sets of any Polish space are Radon measures, therefore probability measures are inner regular or tight, see Appendix. We recall that $L^p(T^3)$ ($1 \leq p < +\infty$) is separable, $L^\infty(T^3)$ is inseparable, and $C([a,b])$, space of continuous functions in $[a,b]$ is separable. $L^p(1 \leq p < +\infty)$ is complete so that it is a Polish space. Consequently, $\mathcal{L} [\rho_0,m]$ is tight on $X_{\rho_0,m} = L^3(T^3)$.

Similarly, we get the tightness of measures generated by $\rho_0,m \mathbb{u}_0,m, \rho_0,m \mathbb{u}_0,m$.

**Proposition 2.5.** The set $\{\mathcal{L} [\rho_0,m \mathbb{u}_0,m] : m \in \mathbb{N}\}$ is tight on $X_{\rho_0,u}$, $\{\mathcal{L} [\frac{1}{2} \rho_0,m \mathbb{u}_0,m] : m \in \mathbb{N}\}$ is tight on $X_{\frac{1}{2} \rho_0,u}$.

**Proposition 2.6.** The set $\{\mathcal{L} [\mathbb{u}_m] : m \in \mathbb{N}\}$ is tight on $X_u$.

**Proof.:** For any given $t > 0$, $E [\|\mathbb{u}_m\|_{L_t^1 L_t^2}^{2r}] \leq C (r_0^{-1})^{-1}$, let $L \geq \frac{1}{t}$, then the set

\[ B_L = \{ \mathbb{u}_m \in L^4([0,T] \times T^3) | \|\mathbb{u}_m\|_{L_t^1 L_t^2} \leq L \} \]

is relatively compact in $X_u$, hence

\[ \mathcal{L} [\mathbb{u}_m](B^c_L) = \mathbb{P}(\|\mathbb{u}_m\|_{L_t^1 L_t^2} \geq L) \leq \frac{E [\|\mathbb{u}_m\|_{L_t^1 L_t^2}^{2r}]}{L^{2r}} \leq \frac{C}{L^{2r}} \leq t. \]

**Proposition 2.7.** The set $\{\mathcal{L} [\rho_0,m] : m \in \mathbb{N}\}$ is tight on $X_{\rho_0}$.

**Proof.** For the more information about $\rho_0$. We calculate that

\[ \triangle \rho_0 = 2 |\nabla \rho_0|^2 + 2 \rho_0 \rho_0 \triangle \rho_0. \]

Since $\rho_0,m \in L^r(\Omega; L^\infty(0,T; H^1(T^3)))$ (for $\kappa > 0$ in energy $E(t)$), $H^1(T^3) \hookrightarrow L^6(T^3)$, $\rho_0,m \in L^r(\Omega; L^2(0,T; H^2(T^3)))$, hence $\rho_0,m \triangle \rho_0,m \in L^r(\Omega; L^2(0,T; L^2(T^3)))$. Similarly, $|\nabla \rho_0,m|^2 \in L^r(\Omega; L^2(0,T; L^2(T^3)))$. This gives the proposition.

**Remark 2.2.** $L^2(0,T; W^{2,\frac{2}{3}}(T^3))$ is continuously embedded in $L^2([0,T] \times T^3)$ and $L^2(0,T; L^2(T^3))$. $W^{2,\frac{2}{3}}(T^3)$ and $L^2(T^3)$ are reflexive spaces, applying the Aubin-Lion’s lemma, we have the strong convergence of $\rho_0$:

\[ \rho_0 \rightarrow \rho \quad \text{in} \quad L^r(\Omega; L^2([0,T]; W^{1,3}(T^3))) . \]

**Remark 2.3.** In addition, by \textbf{(2.35)}, $\rho_0,m \in L^r(\Omega; L^2([0,T]; H^2(T^3)))$, $H^2(T^3) \hookrightarrow W^{1,6}(T^3)$, $W^{1,6}(T^3)$ is continuously embedded in $L^2(T^3)$, therefore, the strong convergence of $\rho_0,m$ holds:

\[ \rho_0,m \rightarrow \rho \quad \text{in} \quad L^r(\Omega; L^2([0,T]; W^{1,6}(T^3))) . \]

**Proposition 2.8.** The set $\{\mathcal{L} [\rho_0,m \mathbb{u}_m] : m \in \mathbb{N}\}$ is tight on $X_{\rho_0,u}$.

**Proof.** We firstly consider the deterministic part, and we denote

\[ Y_m(t) = \Pi_m[\rho_0,m \mathbb{u}_m(0)] - \int_0^t \Pi_m[\text{div}(\rho_0,m \mathbb{u}_m \otimes \mathbb{u}_m)] \, ds + \int_0^t \Pi_m[\text{div}(\rho_0,m \mathbb{u}_m)] \, ds \]

\[ - \int_0^t \Pi_m[\nabla \rho_0,m] \, ds + \int_0^t \frac{11}{10} \Pi_m[\nabla \rho_0,m^{10}] \, ds + \int_0^t \delta \Pi_m[\rho_0 \nabla \Delta^9 \rho_0,m] \, ds \]
that is bounded in formly in together with the Kolmogorov-Centov’s theorem, it holds
\[ \frac{\partial}{\partial t} \int_{\mathbb{R}^3} \left[ G_{\mu} \right] (\psi) \ d\mu \]
\[ = \int_{\mathbb{R}^3} \left[ G_{\mu} \right] (\psi) \ d\mu - \int_{\mathbb{R}^3} \left[ G_{\mu} \right] (\psi) \ d\mu = 0. \]

In fact, \( \frac{\partial}{\partial t} \int_{\mathbb{R}^3} \left[ G_{\mu} \right] (\psi) \ d\mu \)
\[ \leq C(T, r), \quad \forall \ v \leq \frac{1}{2}, \ l > \frac{5}{2}. \]

We claims that \( \frac{\partial}{\partial t} \left[ G_{\mu} \right] (\psi) \)
\[ \leq C(T, r), \quad \forall \ v \leq \frac{1}{2}, \ l > \frac{5}{2}. \]

In fact, \( \frac{\partial}{\partial t} \left[ G_{\mu} \right] (\psi) \)
\[ \leq C(T, r), \quad \forall \ v \leq \frac{1}{2}, \ l > \frac{5}{2}. \]

We claims that \( \frac{\partial}{\partial t} \left[ G_{\mu} \right] (\psi) \)
\[ \leq C(T, r), \quad \forall \ v \leq \frac{1}{2}, \ l > \frac{5}{2}. \]
By Kolmogorov-Centov's continuity theorem (see Appendix),

\[ \| \rho_m \|_{L^2(\mathbb{T}^3)} \| \Pi_m[F_k(\rho_m, u_m)] \|_{L^\infty(\mathbb{T}^3)} \]

\[ \leq f_k \| \rho_m \|_{L^\infty L^2} \leq f_k \| \rho_m \|_{L^\infty H^1_1}. \]

(2.79)

uniformly in \( m, r > 2, \zeta \in (0, \frac{1}{2} - \frac{1}{r}) \). Combined with the result of deterministic part (2.77), with \( C \) dependent on \( T \) and \( r \), it holds

\[ \| \Pi_m[\rho_m u_m] \|_{C^t([0,T];L^2(\mathbb{T}^3))} \leq C, \quad \mathbb{P} \text{ almost surely,} \quad r > 4. \]

(2.81)

The embedding \( C^\kappa ([0,T];W^{-l,2}(\mathbb{T}^3)) \hookrightarrow C^{\kappa - \varepsilon} ([0,T];W^{-k,2}(\mathbb{T}^3)), k > l \), implies the tightness of \( L[\rho_m u_m] \in C^\kappa ([0,T];W^{-k,2}(\mathbb{T}^3)) \) for \( k > l > \frac{3}{2} \).

Since the quantum term provides a better regularity in the energy estimates, we have \( \Pi_m[\rho_m u_m] \) belongs to \( L^\infty ([0,T];L^{3/2}(\mathbb{T}^3)) \cap C^\kappa ([0,T];W^{-k,2}(\mathbb{T}^3)) \), which can be compactly embedded in \( C([0,T];L^2(\mathbb{T}^3)) \). In fact,

\[ E \left[ \left( \sup_{t \in [0,T]} \| \Pi_m[\rho_m u_m(t_2) - \rho_m u_m(t_1)] \|_{L^{3/2}(\mathbb{T}^3)} \right)^p \right] \]

\[ \leq E \left[ \left( \sup_{\phi \in L^1(\mathbb{T}^3), \| \phi \|_{L^1} = 1} \int_{\mathbb{T}^3} \Pi_m[\rho_m u_m(t_2) - \rho_m u_m(t_1)] \phi \, d x \right)^p \right] \]

\[ = E \left[ \left( \sup_{\phi \in L^1(\mathbb{T}^3), \| \phi \|_{L^1} = 1} \left( \int_{\mathbb{T}^3} \Pi_m[\rho_m u_m(t_2) - \rho_m u_m(t_1)] (\phi - \phi \star \eta) \, d x + \int_{\mathbb{T}^3} \Pi_m[\rho_m u_m(t_2) - \rho_m u_m(t_1)] \phi \star \eta \, d x \right) \right)^p \right], \]

where \( \eta \) is a mollifier and \( \| \phi \star \eta \|_{L^1(\mathbb{T}^3)} \rightarrow \| \phi \|_{L^1(\mathbb{T}^3)} \) as \( \varepsilon \) goes to 0. Since \( L^p(\mathbb{T}^3) \), \( 1 < p < \infty \) is uniformly convex space, due to Hanner’s inequality, weak convergence and convergence in norm implies \( \| \phi \star \eta - \phi \|_{L^1(\mathbb{T}^3)} \rightarrow 0 \). By Hölder’s inequality, the first term in the above formula will go to 0. By stochastic Fubini’s theorem,

\[ \int_{\mathbb{T}^3} \Pi_m[\rho_m u_m(t_2) - \rho_m u_m(t_1)] \phi \star \eta \, d x \]

\[ = \int_{\mathbb{T}^3} \left( \int_{t_1}^{t_2} \frac{d Y_m(t)}{d t} + \int_{t_1}^{t_2} \rho_m E(\rho_m, u_m) \, d W \right) \phi \star \eta \, d x \]

\[ = \int_{t_1}^{t_2} \left( \int_{\mathbb{T}^3} \frac{d Y_m(t)}{d t} \phi \star \eta \, d x \, d t + \int_{t_1}^{t_2} \rho_m E(\rho_m, u_m) \phi \star \eta \, d x \, d W \right) \]

\[ \leq \left\| \frac{d Y_m(t)}{d t} \right\|_{L^{3/2}([0,T];W^{-1,2}(\mathbb{T}^3))} \| \phi \star \eta \|_{W^{1,2}(\mathbb{T}^3)} (t_2 - t_1)^{\frac{1}{4}} \]

\[ + \int_{t_3}^{t_2} \int_{t_1}^{t_2} \rho_m E(\rho_m, u_m) \, d W \phi \star \eta \, d x \]

(2.83)

It is clear that, with \( C \) dependent on \( T \) and \( r \),

\[ E \left[ \left\| \int_{t_1}^{t_2} \rho_m E(\rho_m, u_m) \, d W \right\|_{L^{3/2}(\mathbb{T}^3)}^r \right] \]

\[ \leq \left\| \int_{t_1}^{t_2} \rho_m E(\rho_m, u_m) \, d W \right\|_{L^{3/2}(\mathbb{T}^3)}^r \]

(2.84)
Therefore we have
\[
\mathbb{E} \left[ \left( \| \Pi_m [\rho_m u_m(t_2) - \rho_m u_m(t_1)] \|_{L^2(T^3)} \right)^r \right] \leq C(t_2 - t_1)^\frac{r}{4}.
\]
(2.85)
\[
\| \Pi_m [\rho_m u_m] \|_{L^2(T^3)} \text{ is continuous in } [0, T] \text{ uniformly in } m \text{ in } \mathbb{P}-\text{almost surely. Arzela-Ascoli's theorem yields the compactness of } \{ \Pi_m [\rho_m u_m] \}, \text{ i.e.}
\]
\[
\| \Pi_m [\rho_m u_m] \|_{L^2(T^3)} \rightarrow \| q \|_{L^2(T^3)} \text{ in } C([0, T]) \text{ in } \mathbb{P} \text{ almost surely.}
\]
(2.86)
Since \( L^2(T^3) \) is an uniformly convex space, therefore we have
\[
\Pi_m [\rho_m u_m] \rightarrow q \text{ in } C \left( [0, T]; L^2(T^3) \right) \text{ in } \mathbb{P} \text{ almost surely.}
\]
(2.87)
Hence \( \mathcal{L} [\rho_m u_m] \) is tight in \( X_{\rho u} \). And note that we do not need \( r > 4 \) or \( r > 2 \) here.
Since \( \left\{ \rho_m \bar{u}_m \right\} \) converges weakly in \( L^r(\Omega; L^\infty([0, T]; L^2(T^3))) \), \( \left\{ \bar{\rho}_m \right\} \) converges weakly to \( \rho_{\bar{q}} \) in \( L^r(\Omega; L^\infty([0, T]; L^6(T^3))) \), \( \bar{\rho}_m \rightarrow \rho_{\bar{q}} \) in \( L^r(\Omega; L^2([0, T]; W^{1,6}(T^3))) \),
\[
\mathbb{E} \left[ \| \bar{u}_m \|_{L^r(T^3)}^r \right] \leq C, \text{ } C \text{ depends on } T, r \text{ and } r_0^{-1}, \text{ so } \bar{\rho}_m u_m \rightarrow \rho u \text{ in } L^r(\Omega; C([0, T]; L^2(T^3))).
\]
Therefore \( q = \rho u \) in \( C \left( [0, T]; L^2(T^3) \right) \text{ in } \mathbb{P} \text{ almost surely. This complete the proof.}

**Proposition 2.9.** \( \mathcal{L}[W_m] \) is a tight Radon measure on a Polish space \( C \left( [0, T]; \mathcal{F} \right) \).

**Step3:** Pass to the limit \( m \rightarrow \infty \).
\( X_2 \) is a sub-Polish space instead of Polish space because weak topologies are generally not measurable. So our stochastic compactness argument is based on the Jakubowski’s extension of Skorokhod’s representation theorem.

**Proposition 2.10.** There exists a family of \( X_2 \)-valued Borel measurable random variables
\[
\left\{ \bar{\rho}_{0,m}, \bar{\rho}_{0,m} u_{0,m}, \bar{\rho}_{0,m}^2 \bar{u}_{0,m}, \bar{\rho}_m, u_m, \bar{\rho}_m u_m, W_m \right\}, \text{ } m \in \mathbb{N},
\]
defined on a complete probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \) and \( \left\{ \rho_0, \rho_0 u_0, \rho_0^2 u_0, \rho, \rho u, W \right\} \), such that (up to a sequence):

1. For all \( m \in \mathbb{N} \), \( \mathcal{L} \left[ \bar{\rho}_{0,m}, \bar{\rho}_{0,m} u_{0,m}, \bar{\rho}_{0,m}^2 \bar{u}_{0,m}, \bar{\rho}_m, u_m, \bar{\rho}_m u_m, W_m \right], \text{ } m \in \mathbb{N} \text{ coincides with} \mathcal{L} \left[ \rho_0, \rho_0 u_0, \rho_0^2 u_0, \rho, \rho u, W \right] \text{ on } X_2.

2. \( \mathcal{L} \left[ \rho_0, \rho_0 u_0, \rho_0^2 u_0, \rho, \rho u, W \right] \) on \( X_2 \) is a Radon measure.

3. \( \left\{ \bar{\rho}_{0,m}, \rho_{0,m} u_m, \rho_{0,m}^2 \bar{u}_m, \bar{\rho}_m, u_m, \bar{\rho}_m u_m, W_m, \text{ } m \in \mathbb{N} \right\} \text{ converges to} \left\{ \rho_0, \rho_0 u_0, \rho_0^2 u_0, \rho, \rho u, W \right\} \text{ in the topology of } X_2, \text{ } \mathbb{P} \text{ almost surely as } m \rightarrow +\infty.

**Remark 2.4.** By construction, we know that \( \rho_{0,m} \rightarrow \rho_0 \). Since the law of \( \bar{\rho}_{0,m} \) coincides with the law of \( \rho_{0,m} \) on \( X_2 \), it also holds \( \bar{\rho}_{0,m} \rightarrow \rho_0 \) in \( L^2(T^3) \) \( \mathbb{P} \text{ almost surely as } m \rightarrow +\infty \).

Next we show the convergence of the initial energy when \( m \) goes to infinity. For \( \rho u^2 \) in the space of \( L^2([0, T]; L^1(T^3)) \), we know that
can be seen from convergences of the continuity equation in weak sense by Vitali’s convergence theorem. So well as the weak spaces. We mainly see whether the convergence of term associated with stochastic forces in the equation (2.88)

We should pay attention to whether the stochastic integral makes sense or not after taking limits. And usually the condition is the progressively measurability of the integrand and $$\mathbb{E} \left[ \int_0^T \left( \int_{T_3} \rho F \, d\tau \right)^2 \, dt \right] < +\infty$$. For a right-continuous or left-continuous process, if it is $$\tilde{F} = \sigma_t (\sigma_t[\rho] \cup \sigma_t[\rho u]) \cup \sigma_t[W]$$-adapted, then it is progressively measurable. Obviously, $$\int_{T_3} \rho F_k (\rho, u) \, d\tau$$ is continuous in time on account of the property of $$\rho \in C \left( [0, T] ; L^2 (\mathbb{T}^3) \right)$$ and our assumption on $$\mathbf{F}_k$$. Another necessary condition is that $$\int_{T_3} \rho F_k (\rho, u) \, d\tau$$ is $$\mathcal{F}$$-adapted, this can be seen from $$|\rho F_k (\rho, u)| \leq f_k (\rho + |\rho u|)$$.

For this $$m \to \infty$$ layer, damping terms provide $$\bar{u}_m \to \mathbf{u}$$ weakly in $$L^4 ([0, T] \times \mathbb{T}^3)$$, $$\bar{\rho}_m \bar{u}_m \to \rho \mathbf{u}$$ strongly in $$C \left( [0, T] ; L^\frac{7}{2} (\mathbb{T}^3) \right)$$, $$\bar{\rho}_m \to \rho$$ weakly in $$L^2 ([0, T] ; H^9 (\mathbb{T}^3))$$, so $$\bar{\rho}_m, \bar{u}_m$$ satisfy the continuity equation in weak sense by Vitali’s convergence theorem. So well as the weak convergences of $$\rho (\bar{\rho}_m), \bar{\rho}_m^m \to \bar{\rho}, \bar{\rho}_m \to \bar{\rho}$$ in the corresponding spaces. We mainly see whether the convergence of term associated with stochastic forces in the momentum conservation equation holds, more precisely, the convergence of integral

$$\int_0^T \int_{T_3} \Pi_m \left[ \rho_m \Pi_m \left[ \mathbf{F} (\rho_m, \mathbf{u}_m) \right] \right] \, d\bar{W}_m \, d\tau$$.

**Lemma 2.5.** $$\int_0^T \int_{T_3} \Pi_m \left[ \rho_m \mathbf{F} (\rho_m, \mathbf{u}_m) \right] \, d\bar{W}_m \, d\tau \to \int_0^T \int_{T_3} \rho F (\rho, u) \, dW \, d\tau$$, almost surely.

To prove this lemma, we firstly give the following proposition.

**Proposition 2.11.** We claim that $$|\{ (t, x) | \rho = 0 \}| = 0$$ holds $$\mathbb{P}$$ almost surely.

**Proof.** Since $$\mathbb{E} \left[ \left\| \rho \right\|_{L^\infty_t L^\infty_x} \right] \leq \mathbb{E} \left[ \left\| D^2 \rho^{-1} \right\|_{L^\infty_t L^2_x} \right]$$, compute

$$D^2 \rho^{-1} = D \left( D (\rho^{-1}) \right) = D (\rho^{-2} D \rho) = 2 \rho^{-3} D \rho \otimes D \rho - \rho^{-2} D^2 \rho, \quad \text{ (2.89)}$$

so by Hölder’s inequalities,

$$\mathbb{E} \left[ \left\| \rho \right\|_{L^\infty_t L^\infty_x} \right] \leq C_r \mathbb{E} \left[ \left\| \rho^3 \right\|_{L^\infty_t L^2_x} \right] + \mathbb{E} \left[ \left\| D^2 \rho \right\|_{L^\infty_t L^2_x} \right]$$

$$\leq C_r \mathbb{E} \left[ \left\| \rho \right\|_{L^\infty_t L^1_0} \left\| \rho \right\|_{L^\infty_t L^2_0} \right] + C_r \mathbb{E} \left[ \left\| \rho^{-1} \right\|_{L^\infty_t L^{1/2}} \right] \quad \text{ (2.90)}$$

$$\leq C_r \mathbb{E} \left[ \left| \mathbb{E} (t) \right|^{1/2} \right] + C_r \mathbb{E} \left[ \left| \mathbb{E} (0) \right|^{1/2} \right] \leq C_r \left( \mathbb{E} \left[ \left| \mathbb{E} (0) \right|^{1/2} \right] + \left( \mathbb{E} \left[ \left| \mathbb{E} (0) \right|^{1/2} \right] \right)^{1/2} \right)$$,

for any $$r > 1$$, where $$C_r = \max \{ 1, 2^{r-1} \} = 2^{r-1}$$ may be different in different inequality. The last inequality holds up to a constant $$K^r$$ which comes from the Burkholder-Davis-Gundy’s inequality, that is equivalent to $$\epsilon^r$$, see Appendix. For any $$M > 3e \max \left\{ 1, \mathbb{E} (0)^{1/2} \right\}$$ large enough, if we could take $$\left\| \rho \right\|_{L^\infty_t L^\infty_x} \geq M$$, denote $$\Omega_M = \{ \omega \in \Omega | \left\| \rho^{-1} (\omega) \right\|_{L^\infty_t L^\infty_x} \geq M \}$$, since

$$C_r \left( \mathbb{E} \left[ \left| \mathbb{E} (0) \right|^{1/2} \right] + \left( \mathbb{E} \left[ \left| \mathbb{E} (0) \right|^{1/2} \right] \right)^{1/2} \right)$$
\[\geq \mathbb{E} \left[ \left\| \rho^{-1} \right\|^r_{L^1_t L^\infty_x} \right] = \int_\Omega \left\| \rho^{-1} \right\|^r_{L^1_t L^\infty_x} \, d\mathbb{P}(\omega) \] (2.91)

\[= \int_{\Omega_M} \left\| \rho^{-1} \right\|^r_{L^1_t L^\infty_x} \, d\mathbb{P}(\omega) + \int_{\Omega_M^c} \left\| \rho^{-1} \right\|^r_{L^1_t L^\infty_x} \, d\mathbb{P}(\omega) \]

\[\geq \mathbb{P}[\Omega_M] M^r + \int_{\Omega_M^c} \left\| \rho^{-1} \right\|^r_{L^1_t L^\infty_x} \, d\mathbb{P}(\omega),\]

so

\[\mathbb{P}\{\omega \in \Omega \mid \left\| \rho^{-1}(\omega) \right\|^r_{L^1_t L^\infty_x} > M\} \]

\[\leq \frac{2^{r-1}}{3^r} \left( \mathbb{E} \left[ \left( \frac{E(0)_{1/\rho}}{\left( \max\{1, E(0)_{1/\rho}\} \right)^r} \right) \right] + \mathbb{E} \left[ \left( \frac{E(0)_{1/\rho}}{\left( \max\{1, E(0)_{1/\rho}\} \right)^r} \right) \right] \right)^{\frac{1}{r}},\] (2.92)

where \(M\) is definite number, which can be put in the expectation. The arbitrariness of \(r\) and dominated convergence theorem deduce \[\mathbb{P}\{\omega \in \Omega \mid \left\| \rho^{-1}(\omega) \right\|^r_{L^1_t L^\infty_x} \geq 0\} = 0.\] This shows that \(\rho > 0\) for almost everywhere in \(\Omega \times (0, T) \times \mathbb{T}^3\). The set of \(\rho = 0\) measures zero. To see more general deterministic inequality please refer to [1].

Now we turn to the proof of lemma [2.5].

**Proof.** Recalling the definition of \(F_k(\rho, u)\), when \(\rho = 0\), it is reasonable to define \(F(\rho, \rho u) = 0\), when \(\rho \neq 0\), we can write \(F_k(\rho, u) = F_k \left( \rho, \frac{\rho u}{\rho} \right)\). \(F_k(\rho, u) = \sum_{k=1}^{+\infty} F_k(\rho, u)e_k\), \(|\rho F_k| \leq f_k(\rho + |\rho u|)\), \(\sum_k f_k^2 < +\infty\). Define \(O' = \{(t, x) \mid \rho < \frac{1}{2}\}\), for \(\rho \in (O')^c\), without loss of generality, when \(m\) large enough, such that \(\rho_m > \frac{1}{2}\). As we have derived before, \(\bar{\rho}_m\bar{u}_m\) converges strongly to \(\rho u\) in \(C\left( [0, T]; L^2_x (\mathbb{T}^3) \right)\), therefore converges almost everywhere up to a subsequence. \(\bar{\rho}_m \to \rho\) strongly in \(C\left( [0, T]; W^{1,3}_x (\mathbb{T}^3) \right)\), so converges almost everywhere up to a subsequence. We claim that \(F_k \left( \bar{\rho}_m, \frac{\bar{\rho}_m \bar{u}_m}{\bar{\rho}_m} \right) \to F_k (\rho, \frac{\rho u}{\rho})\) for the sake of the strong convergence of \(\bar{\rho}_m F_k (\bar{\rho}_m, \frac{\bar{\rho}_m \bar{u}_m}{\bar{\rho}_m}) \to \rho F_k (\rho, \frac{\rho u}{\rho})\). Actually, for the situation in \(O'\), \(\rho \neq 0\), we know that

\[= \hat{\rho}_m F_k \left( \frac{\bar{\rho}_m \bar{u}_m}{\bar{\rho}_m}, \frac{\hat{\rho}_m \hat{u}_m}{\hat{\rho}_m} \right) - \rho F_k \left( \rho, \frac{\rho u}{\rho} \right),\] (2.93)

\[\rho F_k \left( \rho, \frac{\hat{\rho}_m \hat{u}_m}{\hat{\rho}_m} \right) - \rho F_k \left( \rho, \frac{\rho u}{\rho} \right) \]

\[\leq f_k (|\bar{\rho}_m - \rho| + |\bar{\rho}_m \bar{u}_m - \rho u| + |\bar{\rho}_m - \rho| F_k \left( \frac{\bar{\rho}_m \bar{u}_m}{\bar{\rho}_m} \right)\).

Therefore we have

\[F_k \left( \frac{\bar{\rho}_m \bar{u}_m}{\bar{\rho}_m}, \frac{\rho u}{\rho} \right) \to 0\] almost everywhere as \(m \to +\infty\) for \(\rho > 0\). (2.95)
When $\rho = 0$, \(|\bar{\rho}_m F_k(\bar{\rho}_m, \bar{\rho}_m^m)\) \leq f_k (|\bar{\rho}_m| + |\bar{\rho}_m^m|)$, this implies \(\left| F_k(\rho, \frac{\bar{\rho}_m^m}{\bar{\rho}_m}) - 0 \right| \leq f_k \).

That is to say, the integrand is bounded by \(f_k\), this gives that

\[
\int_0^T \int_{\mathcal{O}} |F_k(\bar{\rho}_m, \bar{\bar{\rho}}_m) - F_k(\rho, u)|^2 \, dx \, dt \to 0 \quad \text{as} \quad m \to +\infty. \tag{2.96}
\]

So (2.96) along with (2.95) gives

\[
\int_0^T \int_{\mathcal{O}} |F_k(\bar{\rho}_m, \bar{\bar{\rho}}_m) - F_k(\rho, u)|^2 \, dx \, dt \\
= \int_0^T \int_{\mathcal{O} \cap \{\rho = 0\}} |F_k(\bar{\rho}_m, \bar{\bar{\rho}}_m) - F_k(\rho, u)|^2 \, dx \, dt \\
+ \int_0^T \int_{\mathcal{O} \cap \{\rho \neq 0\}} |F_k(\bar{\rho}_m, \bar{\bar{\rho}}_m) - F_k(\rho, u)|^2 \, dx \, dt \\
\to 0 \quad \text{as} \quad m \to +\infty. \tag{2.97}
\]

Therefore,

\[
\int_0^T \int_{\mathcal{O}^3} |F_k(\bar{\rho}_m, \bar{\bar{\rho}}_m) - F_k(\rho, u)|^2 \, dx \, dt \\
= \int_{\mathcal{O}} |F_k(\bar{\rho}_m, \bar{\bar{\rho}}_m) - F_k(\rho, u)|^2 \, dx \, dt + \int_{(\mathcal{O}^c)} |F_k(\bar{\rho}_m, \bar{\bar{\rho}}_m) - F_k(\rho, u)|^2 \, dx \, dt \\
\to 0 \quad \text{as} \quad m \to +\infty. \tag{2.98}
\]

Consequently, we have

\[
F_k(\bar{\rho}_m, \bar{\bar{\rho}}_m) \to F_k(\rho, u) \quad \text{in} \quad L^2([0, T] \times \mathbb{T}^3). \tag{2.99}
\]

Therefore, \(\Pi_m [F_k(\bar{\rho}_m, \bar{\bar{\rho}}_m)] \to F_k(\rho, u) \quad \text{in} \quad L^2([0, T] \times \mathbb{T}^3)\) thanks to the continuity of projection operator \(\Pi_m\). On the other hand, we have the strong convergence of \(\bar{\rho}_m\) in \(L^2([0, T] \times \mathbb{T}^3)\), so

\[
\Pi_m [\bar{\rho}_m \Pi_m [F_k(\bar{\rho}_m, \bar{\bar{\rho}}_m)]] \to \rho F_k(\rho, u) \quad \text{in} \quad L^1([0, T] \times \mathbb{T}^3). \tag{2.100}
\]

To get the convergence of the stochastic integral, we need the strong convergence of

\[
\Pi_m [\bar{\rho}_m \Pi_m [F_k(\bar{\rho}_m, \bar{\bar{\rho}}_m)]]
\]

in \(L^2([0, T] \times \mathbb{T}^3)\). Since

\[
\int_0^T \int_{\mathbb{T}^3} \Pi_m \bar{\rho}_m F_k(\bar{\rho}_m, \bar{\bar{\rho}}_m)^2 \, dx \, dt \leq \sum_{k=1}^{+\infty} f_k^2 \int_0^T \int_{\mathbb{T}^3} \bar{\rho}_m^2 \, dx \, dt, \tag{2.101}
\]

so \(\bar{\rho}_m \Pi_m [F_k(\bar{\rho}_m, \bar{\bar{\rho}}_m)] \to \rho F_k(\rho, u) \quad \text{in} \quad L^r(\Omega; L^2([0, T] \times \mathbb{T}^3))\). By dominated convergence theorem and the strong convergence of \(\bar{\rho}_m\), we have

\[
\int_0^T \int_{\mathbb{T}^3} |\bar{\rho}_m \Pi_m [F_k(\bar{\rho}_m, \bar{\bar{\rho}}_m)] - \rho F_k(\rho, u)|^2 \, dx \, dt \\
= \int_0^T \int_{\mathbb{T}^3} |\bar{\rho}_m \Pi_m [F_k(\bar{\rho}_m, \bar{\bar{\rho}}_m)] - \rho \Pi_m [F_k(\bar{\rho}_m, \bar{\bar{\rho}}_m)]| \\
+ \rho \Pi_m [F_k(\bar{\rho}_m, \bar{\bar{\rho}}_m)] - \rho F_k(\rho, u)|^2 \, dx \, dt \\
\leq 2 \int_0^T \int_{\mathbb{T}^3} |\bar{\rho}_m \Pi_m [F_k(\bar{\rho}_m, \bar{\bar{\rho}}_m)] - \rho \Pi_m [F_k(\bar{\rho}_m, \bar{\bar{\rho}}_m)]|^2 \, dx \, dt \tag{2.102}
\]
\[ + 2 \int_0^T \int_{\mathbb{T}^3} |\rho \Pi_m[\mathbf{F}_k(\bar{\rho}_m, \mathbf{u}_m)] - \rho \mathbf{F}_k(\rho, \mathbf{u})|^2 \, dx \, dt \]
\[ \leq 2f_k^2 \int_0^T \int_{\mathbb{T}^3} |\bar{\rho}_m - \rho|^2 \, dx \, dt + \|\rho\|_{L^\infty_t L^2_x} \int_0^T \int_{\mathbb{T}^3} |\Pi_m[\mathbf{F}_k(\bar{\rho}_m, \mathbf{u}_m)] - \mathbf{F}_k(\rho, \mathbf{u})|^2 \, dx \, dt \]
\[ \to 0 \text{ as } m \to +\infty, \text{ by } \|\rho\|_{L^\infty_t L^\infty_x} \leq \|\rho\|_{L^\infty_t H^0_x}. \]

This shows that
\[ \Pi_m[\bar{\rho}_m \Pi_m[\mathbf{F}_k(\bar{\rho}_m, \mathbf{u}_m)]] \to \rho \mathbf{F}_k(\rho, \mathbf{u}) \text{ in } L^2([0, T] \times \mathbb{T}^3). \quad (2.103) \]

Now combined with (2.103) and \(\bar{W}_m \to W\), by lemma 2.2 it holds
\[ \int_0^T \int_{\mathbb{T}^3} \Pi_m[\bar{\rho}_m \mathbf{F}_k(\bar{\rho}_m, \mathbf{u}_m)] \, dW_m \, dx \rightarrow \int_0^T \int_{\mathbb{T}^3} \rho \mathbf{F}_k(\rho, \mathbf{u}) \, dW \, dx \text{ as } m \to +\infty \bar{\mathbb{P}} \text{ almost surely.} \]

This completes the proof of the lemma.

**Remark 2.5.** Notice that even \(\eta\) and \(\delta\) will vanish finally, this method still work through without good regularity of \(\rho\), because later on we will consider the convergence of \(\rho \mathbf{F}_k(\rho, \mathbf{u})\) instead of \(\mathbf{F}_k(\rho, \mathbf{u})\).

**Step 4:** What will hold after taking the limit?

By Vitali’s convergence theorem and lemma 2.5 we have the following proposition.

**Proposition 2.12.** \((\rho, \mathbf{u})\) is a martingale solution to the stochastic B-D entropy balance of system

\[ \begin{cases} 
\rho_t + \text{div}(\rho \mathbf{u}) = \varepsilon \Delta \rho, \\
\rho \mathbf{u} + (\text{div}(\rho \mathbf{u} \otimes \mathbf{u}) + \nabla (a \rho^\gamma)) - \text{div}(\rho \mathbf{D} \mathbf{u}) - \frac{1}{10} \mathbf{H} \rho \rho^{-10} \nabla \rho^\gamma \mathbf{u} \mathbf{D} \mathbf{u} \right)d t \\
= \left( -r_0 |\mathbf{u}|^2 \mathbf{u} - r_1 \rho |\mathbf{u}|^2 \mathbf{u} \right) d t - \left( \varepsilon \nabla \rho \mathbf{u} \right) d t - \left( \varepsilon \Delta \rho \mathbf{u} \right) d t \\
+ \left( \delta \rho \nabla \Delta \rho \right) d t + \kappa \left( \rho \left( \mathbf{D} \mathbf{u} \right) \right) d t + (\rho \mathbb{W}(\rho, \mathbf{u})) \right)d W. 
\end{cases} \quad (2.104) \]

Finally, for the converged weak solution \(\rho, \rho \mathbf{u}\), we still compute the energy balance:

\[ \begin{align*} 
\int_{\mathbb{T}^3} \left( \frac{1}{2} \rho |\mathbf{u}|^2 + \frac{a}{\gamma} \rho^\gamma + \frac{1}{10} \rho^{-10} + \frac{\kappa}{2} |\nabla \sqrt{\rho}|^2 + \frac{\delta}{2} |\nabla \Delta \rho|^2 \right) d x \\
+ \left( \frac{\varepsilon}{\gamma} \right) \int_{\mathbb{T}^3} |\nabla \rho^\gamma|^2 d x + \varepsilon \eta \frac{11}{25} \int_{\mathbb{T}^3} |\nabla \rho^{-5}|^2 d x + \varepsilon \int_{\mathbb{T}^3} |\Delta \mathbf{u}|^2 d x \\
+ \varepsilon \delta \int_{\mathbb{T}^3} |\Delta \rho|^2 d x + \varepsilon \kappa \int_{\mathbb{T}^3} \rho |\nabla \log \rho|^2 d x + \int_{\mathbb{T}^3} \rho |\mathbf{D} \mathbf{u}|^2 d x \\
+ r_0 \int_{\mathbb{T}^3} |\mathbf{u}|^4 d x d t + r_1 \int_{\mathbb{T}^3} \rho |\mathbf{u}|^4 d x d t + r_2 \int_{\mathbb{T}^3} |\mathbf{u}|^2 d x d t \\
= \left( \frac{1}{2} \int_{\mathbb{T}^3} \rho \left( \mathbb{W}(\rho, \mathbf{u}) \right) \right)^2 d x d t + \left( \frac{1}{2} \int_{\mathbb{T}^3} \rho \mathbf{W}(\rho, \mathbf{u}) \cdot \mathbf{u} d x \right) d W. 
\end{align*} \quad (2.105) \]

We denote the energy \(E(t) = \int_{\mathbb{T}^3} \left( \frac{1}{2} \rho |\mathbf{u}|^2 + \frac{a}{\gamma} \rho^\gamma + \frac{1}{10} \rho^{-10} + \frac{\delta}{2} |\nabla \sqrt{\rho}|^2 + \frac{\delta}{2} |\nabla \Delta \rho|^2 \right) d x \). Then the estimates

\[ E[E(t)^r] \leq E[E(0)^r], \quad r \geq 2, \quad (2.106) \]

and

\[ \begin{align*} 
E \left( \int_0^t \int_{\mathbb{T}^3} |\nabla \rho^\gamma|^2 d x d s + \varepsilon \eta \frac{11}{25} \int_0^t \int_{\mathbb{T}^3} |\nabla \rho^{-5}|^2 d x d s \\
+ \varepsilon \delta \int_0^t \int_{\mathbb{T}^3} |\Delta \rho|^2 d x d s + \varepsilon \kappa \int_0^t \int_{\mathbb{T}^3} \rho |\nabla \log \rho|^2 d x d s \\
\end{align*} \]
where $$\kappa$$ estimate independent of Lemma 3.1. The stochastic B-D entropy balance of system $$\varepsilon$$ construct the energy estimate uniformly in $$C$$.

Denote $$\int_{\Omega} \frac{1}{2} \rho |u + \nabla \log \rho|^2 + \eta \frac{1}{10} \rho^{-10} + \frac{\delta}{2} |\nabla \triangle \rho|^2 + \frac{\kappa}{2} |\nabla \sqrt{\rho}|^2 + a \int_{t}^{\rho} \frac{p(z)}{z} d z \ d x$$

$$+ (\frac{4a}{T} \int_{\Omega} |\nabla \rho|^2 |^2 d x + \frac{11}{25} \varepsilon \int_{\Omega} |\nabla \rho^{-5}|^2 d x + \varepsilon \int_{\Omega} |\triangle u|^2 d x$$

$$= \varepsilon \int_{\Omega} \frac{1}{2} \rho |u + \nabla \log \rho|^2 + \frac{\kappa}{2} |\nabla \sqrt{\rho}|^2 + a \int_{t}^{\rho} \frac{p(z)}{z} d z \ d x \ d W + \int_{\Omega} \frac{1}{2} \rho |F(\rho, u)|^2 \ d x \ d t$$

$$+ \int_{\Omega} \rho F(\rho, u) \cdot \nabla \log \rho \ d x \ d W$$

$$+ \varepsilon \int_{\Omega} |\nabla \rho|^2 |\nabla \rho|^2 d x + \varepsilon \int_{\Omega} |\nabla \rho \nabla \rho \log \rho \ d x \ d t - \varepsilon \int_{\Omega} \nabla (\rho u) \cdot \nabla \log \rho \ d x$$

$$- \varepsilon \int_{\Omega} \rho \nabla \triangle \log \rho \ d x - \rho \int_{\Omega} |u|^2 u \cdot \nabla \log \rho \ d x$$

$$- \int_{\Omega} \rho |u|^2 u \cdot \nabla \log \rho \ d x - \int_{\Omega} u \cdot \nabla \log \rho \ d x \ d t$$

$$\triangle I_1 + I_2 + I_3 + I_4 + I_5 + I_6 + I_7 + I_8 + I_9 + I_{10},$$

where $$\int_{\Omega} \frac{1}{2} \rho |u + \nabla \log \rho|^2 \ d x \ d t = \int_{\Omega} |\nabla \rho|^2 |^2 d x \ d t = \int_{\Omega} |\nabla \rho|^2 \cdot \nabla \log \rho \ d x \ d t.$$

Proof. Denote $$f = f(\varphi, q) = \rho u \cdot \nabla \log \rho = q \varphi, q = \rho u, f_\varphi = q = \rho u, f_q = \varphi, f_{qq} = 0,$$ by Itô’s formula, it holds

$$d \int_{\Omega} \rho u \cdot \nabla \log \rho \ d x = \int_{\Omega} \rho u \cdot d \nabla \log \rho \ d x + \int_{\Omega} d (\rho u) \cdot \nabla \log \rho \ d x. \quad (3.2)$$
Here "d" outside the integral in the above equality means differentiating with respect to time $t$. The first term

$$\int_{T^3} d(\rho u) \cdot \nabla \log \rho \, d x$$

$$= \left( \int_{T^3} - \text{div} \, (\rho u \otimes u) \cdot \nabla \log \rho \, d x - \int_{T^3} \nabla \rho^2 \cdot \nabla \log \rho \, d x \right.$$  

$$+ \frac{11}{10} \eta \int_{T^3} \nabla \rho^{-10} \cdot \nabla \log \rho \, d x + \int_{T^3} \text{div} \, (\rho D u) \cdot \nabla \log \rho \, d x$$  

$$+ \varepsilon \int_{T^3} (\nabla \rho \nabla u) \cdot \nabla \log \rho \, d x - r_0 \int_{T^3} |u|^2 u \cdot \nabla \log \rho \, d x$$  

$$- r_1 \int_{T^3} \rho |u|^2 u \cdot \nabla \log \rho \, d x - r_2 \int_{T^3} u \cdot \nabla \log \rho \, d x$$  

$$- \varepsilon \int_{T^3} \Delta^2 u \cdot \nabla \log \rho \, d x + \kappa \int_{T^3} \text{div} \, (\rho \nabla^2 \log \rho) \cdot \nabla \log \rho \, d x$$  

$$+ \delta \int_{T^3} \rho \nabla \Delta^9 \cdot \nabla \log \rho \, d x \right) \, d t + \int_{T^3} \rho \mathcal{F}(\rho, \rho u) \, d W \cdot \nabla \log \rho \, d x,$$

the second term

$$\int_{T^3} \rho u \cdot d(\nabla \log \rho) \, d x = \int_{T^3} \text{div} \, (\rho u) \left( \frac{\text{div} \, (\rho u) - \varepsilon \Delta \rho}{\rho} \right) \, d x \, d t.$$

After the simplification of the deterministic part \[3.3\], we have

$$\int_{T^3} d \left( \frac{1}{2} \rho |\nabla \log \rho|^2 \, d x \right)$$

$$= \left( - \int_{T^3} \rho \nabla u : \nabla \log \rho \otimes \nabla \log \rho \, d x - \int_{T^3} \Delta \rho \text{div} \, u \, d x + \varepsilon \int_{T^3} \frac{\nabla \rho \cdot \nabla \Delta \rho}{\rho} \, d x \right) \, d t,$$

Together with the energy balance \[2.34\] with $d \left( \int_{T^3} \rho u^2 \, d x \right)$, \[3.1\] follows.

### 3.1.2. The energy estimates for stochastic B-D entropy balance \[3.1\].

For

$$I_1 = \left( \int_{T^3} \rho \mathcal{F}(\rho, u) \cdot u \, d x \right) \, d W,$$
\( \int_{T^3} \rho F_k(\rho, u) \cdot u \, d x \) is continuous with respect to \( t \), by Burkholder-Davis-Gundy's inequality,

\[
\begin{align*}
\mathbb{E} \left[ \left\| \int_0^t \int_{T^3} \rho F(\rho, u) \cdot u \, d x \, d W \right\|^r \right] \\
\leq C \mathbb{E} \left[ \left( \int_0^t \left\| \int_{T^3} \rho F(\rho, u) \cdot u \, d x \right\|^2 \, d s \right)^{\frac{r}{2}} \right] \\
\leq C \mathbb{E} \left[ \left( \int_0^t \rho \sum_{k=1}^{+\infty} F_k(\rho, u) e_k \cdot u \, d x \right)^2 \, d s \right]^{\frac{r}{2}} \hspace{1cm} (3.7)
\end{align*}
\]

for \( r > 2 \), here \( C \) depends on \( r \).

\[
\left\| \int_{T^3} \rho u \cdot F_k(\rho, u) \, d x \right\|^2 \leq \left\| \sqrt{\rho} L_{2^2(T^3)} \left( \sqrt{\rho} u \right) \right\| L_{2^2(T^3)} \left( \left\| F_k(\rho, u) \right\| L_{1^2(T^3)} \right)^2
\leq C \left\| \sqrt{\rho} L_{2^2(T^3)} \left( \sqrt{\rho} u \right) \right\| L_{2^2(T^3)} \left( \left\| F_k(\rho, u) \right\| L_{\infty(T^3)} \right)^2
= C \left\| \sqrt{\rho} L_{2^2(T^3)} \left( \sqrt{\rho} u \right) \right\| L_{2^2(T^3)} \right|^2,
\]

where \( \frac{1}{2} + \frac{1}{2} + \frac{1}{q} = 1 \), \( C \) depends on the domain and \( \sum_{k=1}^{+\infty} f_k^2 \).

\[
\mathbb{E} \left[ \left| \left\| \sqrt{\rho} \right\|_{L^2_2(T^3)} \right|^2 \right] \leq C \left( \mathbb{E} \left[ E(0)^r \right] + 1 \right) \text{ and } \mathbb{E} \left[ \left| \left\| \sqrt{\rho} u \right\|_{L^2_2(T^3)} \right|^2 \right] \leq C \left( \mathbb{E} \left[ E(0)^r \right] + 1 \right) \text{ imply}
\]

\[
\mathbb{E} \left[ \left| \int_0^t \int_{T^3} \rho F(\rho, u) \cdot u \, d x \, d W \right| \right] \leq C \mathbb{E} \left[ \int_0^t \left\| \sqrt{\rho} L_{2^2(T^3)} \left( \sqrt{\rho} u \right) \right\| L_{2^2(T^3)} \left( \left\| F_k(\rho, u) \right\| L_{1^2(T^3)} \right)^2 \, d s \right]^{\frac{r}{2}} \hspace{1cm} (3.9)
\]

with \( C \) depends on \( r, t \) and \( \sum_{k=1}^{+\infty} f_k^2 \).

For term \( I_2 = \left( \int_{T^3} \frac{1}{2} \rho F(\rho, u) \cdot u \, d x \right) \) d \( s \),

\[
\int_{T^3} \frac{1}{2} \rho \left| F(\rho, u) \right|^2 \, d x = \int_{T^3} \frac{1}{2} \rho \sum_{k=1}^{+\infty} F_k(\rho, u) e_k \, d x \leq \| \rho \|_{L_1(T^3)} \frac{1}{2} \sum_{k=1}^{+\infty} \| F_k(\rho, u) \|_{L_{\infty(T^3)}} \leq \| \rho \|_{L_1(T^3)} \frac{1}{2} \sum_{k=1}^{+\infty} \| F_k(\rho, u) \|_{L_{\infty(T^3)}} \hspace{1cm} (3.10)
\]

\[
\leq C \left( \int_{T^3} \rho^\gamma \, d x + 1 \right),
\]

therefore for \( r > 1 \) by Jensen's inequality,

\[
\mathbb{E} \left[ \left| \int_0^t \int_{T^3} \frac{1}{2} \rho F(\rho, u) \, d x \, d s \right| \right] \leq C \mathbb{E} \left[ \int_0^t \left( \int_{T^3} \rho^\gamma \, d x + 1 \right) \, d s \right] \hspace{1cm} (3.11)
\]

\[
\leq C \left( \mathbb{E} \left[ e(s)^r \right] + 1 \right) \leq C \left( \mathbb{E} \left[ E(0)^r \right] + 1 \right),
\]
where $C$ depends on $r, t$ and $\sum_{k=1}^{+\infty} f_k^2$.

For $I_3 = \left( \int_{T^3} \rho F(\rho, u) \cdot \nabla \log \rho \, dx \right) dW$,

$$
\mathbb{E} \left[ \left( \int_0^t \int_{T^3} \rho F(\rho, u) \cdot \nabla \log \rho \, dx \, dW \right)^r \right] 
\leq C \mathbb{E} \left[ \left( \int_0^t \int_{T^3} \rho F(\rho, u) \cdot \nabla \log \rho \, dx \, dW \right)^{\frac{r}{2}} \right] 
\leq C \mathbb{E} \left[ \left( \int_0^t \int_{T^3} \rho F(\rho, u) \cdot \nabla \log \rho \, dx \, dW \right)^{\frac{r}{2}} \right] 
$$

(3.12)

where $C$ depends on $r$ and $\sum_{k=1}^{+\infty} f_k^2$. Based on \textcolor{red}{(2.106)}\textcolor{red}, by Hölder’s inequality and Cauchy’s inequality,

$$
\mathbb{E} \left[ \left( \int_0^t \int_{T^3} \rho F(\rho, u) \cdot \nabla \log \rho \, dx \, dW \right)^r \right] 
\leq C \mathbb{E} \left[ \left( \int_{0}^{+\infty} \sum_{k=1}^{+\infty} \left\| F_k(\rho, u) \right\|_{L^\infty(T^3)} \left\| \rho \nabla \log \rho \right\|_{L^1(T^3)} \, ds \right)^{\frac{r}{2}} \right] 
\leq C \mathbb{E} \left[ \left( \int_{0}^{+\infty} \sum_{k=1}^{+\infty} \left\| F_k(\rho, u) \right\|_{L^\infty(T^3)} \left\| \rho \nabla \log \rho \right\|_{L^1(T^3)} \, ds \right)^{\frac{r}{2}} \right] 
$$

(3.13)

For the term $I_4 = \varepsilon \int_{T^3} \frac{\nabla \rho \cdot \nabla \Delta \rho}{\rho} \, dx \, ds$, for $\kappa > 0$, $\sqrt{\varepsilon} \Delta y^2 \rho \in L^2([0, t] \times T^3), H^{10}(T^3) \rightarrow C^{8,7}(T^3)$. We do the following estimate following estimates

$$
\mathbb{E} \left[ \left( \int_0^t \varepsilon \int_{T^3} \frac{\nabla \rho \cdot \nabla \Delta \rho}{\rho} \, dx \, ds \right)^r \right] 
\leq \mathbb{E} \left[ \left( \int_0^t \varepsilon \int_{T^3} \left\| \frac{\nabla \rho \cdot \nabla \Delta \rho}{\rho} \right\|_{L^2} \, dx \, ds \right)^{\frac{r}{2}} \right] 
\leq \mathbb{E} \left[ \left( \int_0^t \varepsilon \int_{T^3} \left\| \nabla \rho \cdot \nabla \Delta \rho \right\|_{L^2} \, dx \, ds \right)^{\frac{r}{2}} \right] 
$$
\[ \leq \mathbb{E} \left[ \left( \| \rho^{-1} \|_{L^\infty_t L^1} \| \nabla \sqrt{\rho} \|_{L^\infty_t L^1} \right)^r \left( \varepsilon \int_0^t \left( \int_{T^3} |\nabla \Delta \rho|^5 \, dx \right)^{\frac{1}{5}} \, ds \right) \right] \]  

(3.14)

\[ \leq \mathbb{E} \left[ \left( \| \rho^{-1} \|_{L^\infty_t L^1} \| \nabla \sqrt{\rho} \|_{L^\infty_t L^1} \right)^r \left( \varepsilon^{\frac{7}{5}} \delta^{-\frac{7}{10}} \int_0^t \left( \int_{T^3} |(\varepsilon \delta)^{\frac{7}{10}} \Delta^5 \rho|^2 \, dx \right)^{\frac{1}{5}} \, ds \right) \right] \]

\[ \leq \mathbb{E} \left[ \left| \frac{\varepsilon^{\frac{7}{5}} \delta^{-\frac{7}{10}}}{\varepsilon^{\frac{7}{5}} \delta^{-\frac{7}{10}}} \mathbb{E} \left[ \left( \int_0^t \left( \int_{T^3} |(\varepsilon \delta)^{\frac{7}{10}} \Delta^5 \rho|^2 \, dx \right)^{\frac{1}{5}} \, ds \right) \right] \right] \]

\[ \leq C \varepsilon^{\frac{7}{5}}, \]

here \( C \) depends on \( \delta^{-\frac{7}{5}}, \eta^{-\frac{7}{10}}, \kappa \), \( r, t \) and \( \mathbb{E} \left[ E(0) \right] \), note that we have the upper bound of \( \left( \mathbb{E} \left[ e(s) \right] \right)^{\frac{1}{r}} \leq C \left( \left( \mathbb{E} \left[ E(0) \right] \right)^{\frac{1}{r}} + 1 \right) \). This estimate is fine because we will send \( \varepsilon \) to 0 firstly in the approximation scheme.

For \( I_5 = \varepsilon \left( \int_{T^3} \nabla \rho \nabla u \nabla \log \rho \, dx \right) \, ds \), we treat it similarly as in Zatorska’s work [5], the process is still given here. By Sobolev-Poincaré’s inequalities on torus, \( \int_{T^3} \nabla u \, dx = 0 \), we have \( \| \nabla u \|_{L^6(T^3)} \leq C \| \Delta u \|_{L^2(T^3)} \),

\[ \left| \int_{T^3} \varepsilon \nabla \rho \nabla u \nabla \log \rho \, dx \right| = \left| \int_{T^3} \varepsilon \nabla \rho \nabla u \nabla \log \rho \, dx \right| \leq C \| \nabla u \|_{L^6(T^3)} \| \rho^{-1} \|_{L^1(T^3)} \| \rho \|_{W^{1, \frac{30}{17}}(T^3)} \leq C \| \Delta u \|_{L^2(T^3)} \| \rho^{-1} \|_{L^{10}(T^3)} \| \rho \|_{W^{1, \frac{30}{17}}(T^3)} \],

by \( C_r \)-inequality. Hence by Hölder’s inequality and Cauchy’s inequality,

\[ \mathbb{E} \left[ \left( \int_0^t \int_{T^3} \varepsilon \nabla \rho \nabla u \nabla \log \rho \, dx \, ds \right)^{\frac{1}{r}} \right] \]

\[ \leq \mathbb{E} \left[ \left( \int_0^t \frac{C}{r} \| \Delta u \|_{L^2(T^3)} \| \rho^{-1} \|_{L^{10}(T^3)} \| \rho \|_{W^{1, \frac{30}{17}}(T^3)} \, ds \right)^{\frac{1}{r}} \right] \]

(3.16)

\[ \leq \mathbb{E} \left[ \left( \int_0^t \left( \frac{C}{r} \| \Delta u \|_{L^2(T^3)} \| \rho^{-1} \|_{L^{10}(T^3)} \| \rho \|_{L^{10}(T^3)} \| \rho \|_{W^{1, \frac{30}{17}}(T^3)} \right)^{\frac{1}{r}} \, ds \right)^{\frac{1}{r}} \right] \]

\[ \leq \varepsilon^{\frac{7}{5}} \left( \mathbb{E} \left[ \left( \int_0^t \| \Delta u \|_{L^2(T^3)} \| \rho^{-1} \|_{L^{10}(T^3)} \| \rho \|_{L^{10}(T^3)} \| \rho \|_{W^{1, \frac{30}{17}}(T^3)} \, ds \right)^{\frac{1}{r}} \right] \right)^{\frac{1}{r}} \]

\[ \leq \frac{\varepsilon^{\frac{7}{5}}}{3} \mathbb{E} \left[ \left( \int_0^t \| \Delta u \|_{L^2(T^3)} \| \rho^{-1} \|_{L^{10}(T^3)} \| \rho \|_{L^{10}(T^3)} \| \rho \|_{W^{1, \frac{30}{17}}(T^3)} \, ds \right)^{\frac{1}{r}} \right] + \varepsilon^{\frac{7}{5}} 3 \mathbb{E} \left[ \left( \int_0^t \| \rho^{-1} \|_{L^{10}(T^3)} \| \rho \|_{L^{10}(T^3)} \| \rho \|_{L^{10}(T^3)} \| \rho \|_{W^{1, \frac{30}{17}}(T^3)} \, ds \right)^{\frac{1}{r}} \right] \]

\[ \leq \frac{\varepsilon^{\frac{7}{5}}}{3} \mathbb{E} \left[ \left( \int_0^t \| \Delta u \|_{L^2(T^3)} \| \rho^{-1} \|_{L^{10}(T^3)} \| \rho \|_{L^{10}(T^3)} \| \rho \|_{W^{1, \frac{30}{17}}(T^3)} \, ds \right)^{\frac{1}{r}} \right] + C \varepsilon^{\frac{7}{5}}, \]

here \( C \) depends on \( \delta^{-\frac{7}{5}}, \eta^{-\frac{7}{10}}, r, t \) and \( \mathbb{E} \left[ E(0)^{k_1} \right], k_1 \) is a specific positive constant. The last inequality can be obtained similarly as estimate for \( I_4 \).
For \( I_6 = -\varepsilon \left( \int_{T^3} \text{div}(\rho \mathbf{u}) \frac{\Delta \rho}{\rho} \, d\mathbf{x} \right) \, d\mathbf{s} \) and \( I_7 = -\varepsilon \left( \int_{T^3} \mathbf{u} \cdot \nabla (\Delta \log \rho \, d\mathbf{x}) \right) \, d\mathbf{s} \), similarly as \( I_5 \), it holds

\[
\mathbb{E} \left[ \left( \int_0^t \int_{T^3} -\varepsilon \frac{\text{div}(\rho \mathbf{u}) \Delta \rho}{\rho} \, d\mathbf{x} \, d\mathbf{s} \right)^r \right] \leq \frac{\varepsilon^r}{3} \mathbb{E} \left[ \left( \int_0^t \| \mathbf{u} \|^2_{L^2(T^3)} \, d\mathbf{s} \right)^r \right] + C \varepsilon^\frac{r}{2}, \tag{3.17}
\]

here \( C \) depends on \( \delta^{-\frac{1}{2}}, \eta^{-\frac{1}{2}}, \kappa^{-\frac{1}{2}}, r, t \) and \( \mathbb{E} \left[ E(0)^{k_2 r} \right] \), \( k_2 \) is a specific positive constant.

\[
\mathbb{E} \left[ \left( -\varepsilon \int_0^t \int_{T^3} \Delta \mathbf{u} \cdot \nabla \log \rho \, d\mathbf{x} \, d\mathbf{s} \right)^r \right] \leq \frac{\varepsilon^r}{3} \mathbb{E} \left[ \left( \int_0^t \| \Delta \mathbf{u} \|^2_{L^2(T^3)} \, d\mathbf{s} \right)^r \right] + C \varepsilon^\frac{r}{2}, \tag{3.18}
\]

here \( C \) depends on \( \delta^{-\frac{1}{2}}, \eta^{-\frac{1}{2}}, \kappa^{-\frac{1}{2}}, r, t \) and \( \mathbb{E} \left[ E(0)^{k_3 r} \right] \), \( k_3 \) is a specific positive constant.

About \( I_8 = -r_0 \left( \int_{T^3} |\mathbf{u}|^2 \mathbf{u} \cdot \nabla \log \rho \, d\mathbf{x} \right) \, d\mathbf{s} \),

\[
\left| -r_0 \int_0^t \int_{T^3} |\mathbf{u}|^2 \mathbf{u} \cdot \nabla \log \rho \, d\mathbf{x} \, d\mathbf{s} \right| \leq r_0 \int_0^t \int_{T^3} \rho^2 \left| \mathbf{u} \right|^3 \left| \nabla \rho \right| \frac{1}{\rho^{\frac{4}{3}}} \, d\mathbf{x} \, d\mathbf{s} \leq r_0 \| \rho^{-\frac{1}{4}} \|_{L_t^\infty L_x^\infty} \| \rho^2 \mathbf{u}^3 \|_{L_t^\infty L_x^4} \| \nabla \rho \|_{L_t^\infty L_x^4} \leq r_0 \| \rho^{-\frac{1}{4}} \|_{L_t^\infty L_x^\infty} \| \rho^2 \mathbf{u}^3 \|_{L_t^\infty L_x^4} \| \nabla \Delta^\frac{1}{4} \rho \|_{L_t^\infty L_x^4}. \tag{3.19}
\]

Therefore

\[
\mathbb{E} \left[ \left| -r_0 \int_0^t \int_{T^3} |\mathbf{u}|^2 \mathbf{u} \cdot \nabla \log \rho \, d\mathbf{x} \, d\mathbf{s} \right|^r \right] \leq C, \tag{3.20}
\]

where \( C \) depends on \( r, t, r_0, \delta^{-1}, \delta^{-1} \) and \( \mathbb{E} \left[ E(0)^r \right] \). This requires the terms \( \eta \nabla \rho^{-10} \) and \( \delta \nabla \Delta^9 \rho \) to be left after \( r_0 |\mathbf{u}|^2 \mathbf{u} \) vanishes.

As for \( I_9 = -r_1 \left( \int_{T^3} \rho |\mathbf{u}|^2 \mathbf{u} \cdot \nabla \log \rho \, d\mathbf{x} \right) \, d\mathbf{s} \),

\[
\left| -r_1 \int_{T^3} \rho |\mathbf{u}|^2 \mathbf{u} \cdot \nabla \log \rho \, d\mathbf{x} \right| = \left| -r_1 \int_{T^3} |\mathbf{u}|^2 \mathbf{u} \cdot \nabla \rho \, d\mathbf{x} \right| \leq C \int_{T^3} r_1 \rho |\mathbf{u}|^2 |\nabla \mathbf{u}| \, d\mathbf{x} \leq C \int_{T^3} r_1 \rho |\mathbf{u}|^4 \, d\mathbf{x} + \frac{1}{8} \int_{T^3} \rho |\nabla |^2 \mathbf{u} |^2 \, d\mathbf{x},
\]

\( \frac{1}{8} \int_{T^3} \rho |\nabla |^2 \mathbf{u} |^2 \, d\mathbf{x} \) can be controlled by \( \int_{T^3} \rho |\nabla |^2 \mathbf{u} |^2 \, d\mathbf{x} \), by energy estimate \( \Box \), we have

\[
\mathbb{E} \left[ \left| \int_0^t \left| -r_1 \int_{T^3} \rho |\mathbf{u}|^2 \mathbf{u} \cdot \nabla \log \rho \, d\mathbf{x} \right|^r \right| \right] \leq C, \tag{3.22}
\]

where \( C \) depends on \( r, t \) and \( \mathbb{E} \left[ E(0)^r \right] \).

About \( I_{10} = -r_2 \left( \int_{T^3} \mathbf{u} \cdot \nabla \log \rho \, d\mathbf{x} \right) \, d\mathbf{s} \),

\[
- r_2 \int_{T^3} \mathbf{u} \cdot \nabla \log \rho \, d\mathbf{x} = - r_2 \int_{T^3} \mathbf{u} \cdot \nabla \rho \rho \, d\mathbf{x} = r_2 \int_{T^3} (\log \rho)_{t} \, d\mathbf{x} - \varepsilon r_2 \int_{T^3} \frac{\Delta \rho}{\rho} \, d\mathbf{x}. \tag{3.23}
\]

We denote \( \log_+ \rho = \log \max \{ \rho, 1 \}, \log_\rho = -\log \min \{ \rho, 1 \} \), therefore

\[
\int_{T^3} \log_+ \rho \, d\mathbf{x} \leq \int_{T^3} \rho \mathbf{I}_{(\rho>1)} \, d\mathbf{x} \leq \int_{T^3} \rho^\gamma \mathbf{I}_{(\rho>1)} \, d\mathbf{x} \leq \int_{T^3} \rho^\gamma \, d\mathbf{x}.
\]

So

\[
- r_2 \int_0^t \int_{T^3} \mathbf{u} \cdot \nabla \log \rho \, d\mathbf{x} \, d\mathbf{s} \tag{3.24}
\]
\[
\leq r_2 \int_{T^3} \left( \log_+ \rho - \log_- \rho \right)(t) \, d x + r_2 \int_{T^3} \left( \log_- \rho_0 - \log_+ \rho_0 \right) \, d x + \varepsilon r_2 \int_0^t \int_{T^3} \frac{\Delta \rho}{\rho} \, d x \, d s \\
\leq r_2 \int_{T^3} \log_+ \rho(t) \, d x - r_2 \int_{T^3} \log_- \rho(t) \, d x + r_2 \int_{T^3} \log_+ \rho_0 \, d x + \varepsilon r_2 \int_0^t \int_{T^3} \frac{\Delta \rho}{\rho} \, d x \, d s \\
\leq r_2 \int_{T^3} \rho^7(t) \, d x - r_2 \int_{T^3} \log_- \rho(t) \, d x + r_2 \int_{T^3} \log_- \rho_0 \, d x + \varepsilon r_2 \int_0^t \int_{T^3} \frac{\Delta \rho}{\rho} \, d x \, d s.
\]

We will move the term \(-r_2 \int_{T^3} \log_- \rho \, d x\) to the left hand side firstly, next take the expectation of its \(r\)-th power

\[
E \left[ r_2 \int_{T^3} \rho^7 \, d x + r_2 \int_{T^3} \log_- \rho \, d x + \varepsilon r_2 \int_0^t \int_{T^3} \frac{\Delta \rho}{\rho} \, d x \, d s \right]^r \\
\leq r_2 C_r E \left[ \int_{T^3} \rho^7 \, d x \right]^r + r_2 C_r E \left[ \int_{T^3} \log_- \rho_0 \, d x \right]^r \\
+ C_r E \left[ \varepsilon r_2 \int_0^t \int_{T^3} \frac{\Delta \rho}{\rho} \, d x \, d s \right]^r \\
\leq C_1 + C_r E \left[ r_2 \int_{T^3} \log_- \rho_0 \, d x \right]^r + C_2 \varepsilon^\frac{r}{2},
\]

where \(C_1\) depends on \(r, r_2, t\) and \(E[E(0)^r]\), \(C_2\) depends on \(r, r_2, t, \delta^{-1}, \eta^{-1}\) and \(E[E(0)^r]\). Finally, we denote

\[
\tilde{E} = \int_{T^3} \left( \frac{1}{2} \rho |u + \nabla \log \rho|^2 + \frac{\eta}{10} \rho^{-10} + \frac{\delta}{2} |\nabla \Delta^4 \rho|^2 + \frac{\kappa}{2} |\nabla \sqrt{\rho}|^2 \\
+ a \int_1^x \frac{p(z)}{z} \, d z + r_2 \int_{T^3} \log_- \rho \, d x, \right)
\]

and obtain the stochastic B-D entropy:

\[
E \left[ \left( \tilde{E}(t) + \int_0^t \int_{T^3} \left( \varepsilon \frac{4a}{\gamma} |\nabla \rho|^2 + \varepsilon \eta \frac{11}{25} |\nabla \rho^{-5}|^2 + \varepsilon \delta |\Delta^5 \rho|^2 + \varepsilon \kappa \rho |\nabla \log \rho|^2 \right)^r \right) \right] \\
\leq C_3 \varepsilon^\frac{r}{2} + C_4 + E \left[ \left( \int_0^t \int_{T^3} \rho |\nabla \log \rho|^2 \, d x \, d s \right)^r \right] + \varepsilon E \left[ \left( \int_0^t \|u\|_{H^2(T^3)}^2 \, d s \right)^r \right],
\]

where \(C_3\) depends on \(\delta^{-\frac{1}{2}}, \eta^{-\frac{1}{2}}, \kappa^{-\frac{1}{2}}, r, t, E[E(0)^r], E[E(0)^{\frac{1}{2}}], E[E(0)^{k_1 r}], E[E(0)^{k_2 r}], \text{and} E[E(0)^{k_3 r}]\), \(C_4\) depends on \(r, t\) and \(E[E(0)^r]\). This shows that

\[
E \left[ \left( \tilde{E}(t) + \int_0^t \int_{T^3} \left( \varepsilon \frac{4a}{\gamma} |\nabla \rho|^2 + \varepsilon \eta \frac{11}{25} |\nabla \rho^{-5}|^2 + \varepsilon \delta |\Delta^5 \rho|^2 + \varepsilon \kappa \rho |\nabla \log \rho|^2 \right)^r \right) \right] \\
\leq C_3 \varepsilon^\frac{r}{2} + C_4 + E \left[ \left( \int_0^t \int_{T^3} \rho |\nabla \log \rho|^2 \, d x \, d s \right)^r \right] + C_r E \left[ \tilde{E}(0)^r \right].
\]
Hence
\[ \mathbb{E} \left[ \left( \hat{E}(t) \right)^r \right] \leq C_3 \varepsilon^2 + C_4 + \mathbb{E} \left[ \left( \int_0^t \int_{T^3} \rho |\nabla \log \rho|^2 \, dx \, ds \right)^r \right] + C_r \mathbb{E} \left[ \hat{E}(0)^r \right]. \] (3.29)
Therefore,
\[ \mathbb{E} \left[ \hat{E}(t)^r \right] \leq C_3 \varepsilon^2 + C_4 + \int_0^t \mathbb{E} \left[ \hat{E}(s)^r \right] \, ds + C_r \mathbb{E} \left[ \hat{E}(0)^r \right]. \] (3.30)
By Grönwall’s inequality, \( C_3 \varepsilon^2 + C_4 + C_r \mathbb{E} \left[ \hat{E}(0)^r \right] \) is nondecreasing with respect to \( t \), it holds
\[ \mathbb{E} \left[ \hat{E}(t)^r \right] \leq C, \] (3.31)
where \( C \) depends on \( \delta^{-\frac{1}{2}}, \eta^{-\frac{1}{2}}, \kappa^{-\frac{1}{2}}, r, t, \mathbb{E} [E(0)^*], \mathbb{E} [E(0)^{\frac{r}{2}}] \) and \( \mathbb{E} [E(0)^{kr}], \mathbb{E} [\hat{E}(0)^r], i = 1, 2, 3 \). Thereupon
\[ \mathbb{E} \left[ \sup_{t \in [0,T]} \hat{E}(t)^r \right] \leq C. \] (3.32)

where \( C \) depends on \( \delta^{-\frac{1}{2}}, \eta^{-\frac{1}{2}}, \kappa^{-\frac{1}{2}}, r, T, \mathbb{E} [E(0)^*], \mathbb{E} [E(0)^{\frac{r}{2}}] \) and \( \mathbb{E} [E(0)^{kr}], \mathbb{E} [\hat{E}(0)^r], i = 1, 2, 3 \).

### 3.2. Passing to the limit \( \varepsilon \to 0 \)

In this section, the solutions of (3.104) may concern with \( \varepsilon, \delta, \eta, \kappa, r_1, r_2 \), for simplicity, we use the notation \( \rho_\varepsilon \) and \( u_\varepsilon \) because we concentrate on the limit \( \varepsilon \to 0 \) in this layer.

**Step 1:** Choose the path space of density and velocity.

From (3.33), if we assume that \( \mathbb{E} [E(0)^r] \) and \( \mathbb{E} [\hat{E}(0)^r] \) are bounded, Recall lemma (2.3) for any smooth function \( \rho_{\varepsilon}(x) \), it holds
\[ \mathbb{E} \left[ \kappa^r \left\| \rho_{\varepsilon}^{\frac{1}{2}} \right\|_{L^2_H}^{2r} \right] \leq C, \quad \mathbb{E} \left[ \kappa^r \left\| \nabla \left( \rho_{\varepsilon}^{\frac{1}{2}} \right) \right\|_{L^1_L}^{4r} \right] \leq C, \] (3.34)
uniformly in \( \varepsilon, \delta, \eta, \kappa \). The following estimates hold independent of \( \varepsilon \):
\[ \mathbb{E} \left[ \sup_{t \in [0,T]} \eta^r \left\| \rho_{\varepsilon}^{-1} \right\|_{L^1_L}^{10r} \right] \leq C, \quad \mathbb{E} \left[ \sup_{t \in [0,T]} \left( \frac{1}{2} \right)^r \left\| \nabla \sqrt{\rho_{\varepsilon}} \right\|_{L^2_L}^{2r} \right] \leq C, \]
\[ \mathbb{E} \left[ \sup_{t \in [0,T]} \delta^r \left\| \nabla \Delta \rho_{\varepsilon} \right\|_{L^2_L}^{2r} \right] \leq C, \quad \mathbb{E} \left[ \eta^r \left\| \nabla \rho_{\varepsilon}^{-5} \right\|_{L^1_L}^{2r} \right] \leq C, \] (3.35)
\[ \mathbb{E} \left[ \gamma^r \left\| \nabla \rho^\varepsilon \right\|_{L^2_t L^2_x}^{2r} \right] \leq C, \quad \mathbb{E} \left[ \delta^r \left\| \Delta^5 \rho^\varepsilon \right\|_{L^2_t L^2_x}^{2r} \right] \leq C, \]

there also holds the bound concerning \( u \):
\[ \begin{align*}
\mathbb{E} \left[ \sup_{t \in [0,T]} \left\| \rho^\varepsilon u \right\|_{L^2_x}^2 \right] \leq C, & \quad \mathbb{E} \left[ \sup_{t \in [0,T]} \left\| \partial_\tau \rho^\varepsilon u \right\|_{L^2_x}^{2r} \right] \leq C, \\
\mathbb{E} \left[ \left\| \partial_\tau \rho^\varepsilon u \right\|_{L^2_t L^2_x}^{2r} \right] \leq C, & \quad \mathbb{E} \left[ \| r^\varepsilon_0 \|_{L^2_t L^2_x}^{4r} \right] \leq C, \\
\mathbb{E} \left[ \| r^\varepsilon_0 \|_{L^2_t L^2_x}^{4r} \right] \leq C, & \quad \mathbb{E} \left[ \| r^\varepsilon_0 \|_{L^2_t L^2_x}^{4r} \right] \leq C,
\end{align*} \tag{3.36} \]

where \( C \) depends on \( \delta^{-\frac{1}{2}}, \eta^{-\frac{1}{2}}, \kappa^{-\frac{1}{2}}, r, T, \mathbb{E} \left[ E(0)^r \right], \mathbb{E} \left[ E(0)^{\frac{m}{2}} \right] \) and \( \mathbb{E} \left[ E(0)^{k_1 r} \right], i = 1, 2, 3. \) Similar as in the last section, we give the following estimate from the mass conservation equation without a proof here
\[ \begin{align*}
\mathbb{E} \left[ \| \rho^\varepsilon_0 \|_{L^2_t L^2_x} \right] \leq C, & \quad \mathbb{E} \left[ \| \rho^\varepsilon u \|_{L^2_t L^2_x} \right] \leq C, \\
\mathbb{E} \left[ \| \rho^\varepsilon \|_{L^2_t L^2_x} \right] \leq C, & \quad \mathbb{E} \left[ \| \rho^\varepsilon \|_{L^2_t H^1_x} \right] \leq C,
\end{align*} \tag{3.37} \]

where \( C \) depends on \( \delta^{-\frac{1}{2}}, \eta^{-\frac{1}{2}}, \kappa^{-\frac{1}{2}}, r, t, \mathbb{E} \left[ E(0)^r \right], \mathbb{E} \left[ E(0)^{\frac{m}{2}} \right] \) and \( \mathbb{E} \left[ E(0)^{k_1 r} \right]. \) \tag{3.38} \]

will be independent on \( \delta \) after \( \varepsilon \to 0. \) Similar as the estimates we have done in \( m \) layer, we have the strong convergence of \( \rho^\varepsilon \)
\[ \rho^\varepsilon \to \rho \quad \text{in} \quad L^2 \left( [0,T]; W^{1,3} \left( \mathbb{T}^3 \right) \right) \quad \widehat{\mathbb{P}} \text{ almost surely}, \tag{3.39} \]

and the strong convergence of \( \rho^\varepsilon \)
\[ \rho^\varepsilon \to \rho \quad \text{in} \quad L^2 \left( [0,T]; W^{1,6} \left( \mathbb{T}^3 \right) \right) \quad \widehat{\mathbb{P}} \text{ almost surely}, \tag{3.40} \]

uniformly in \( \varepsilon, \eta \) and \( \delta. \) \( W^{1,6} \left( \mathbb{T}^3 \right) \hookrightarrow C^{0,\frac{3}{2}} \left( \mathbb{T}^3 \right), \rho^\varepsilon u \in L^\infty \left( [0,T]; L^2 \left( \mathbb{T}^3 \right) \right), \) so \( \{ \rho^\varepsilon u \} \) converges weakly to \( \rho u \) in \( L^2 \left( [0,T] \times \mathbb{T}^3 \right). \)

Next, we choose the path space
\[ \mathcal{X}_3 = \mathcal{X}_{\rho^0} \times \mathcal{X}_{\rho^0 u^0} \times \mathcal{X}_\rho \rho^\varepsilon u \times \mathcal{X}_u \times \mathcal{X}_{\rho^0 u} \times \mathcal{X}_W, \tag{3.41} \]

where
\[ \begin{align*}
\mathcal{X}_{\rho^0} &= L^7 \left( \mathbb{T}^3 \right) \cap L^1 \left( \mathbb{T}^3 \right) \cap L^{-10} \left( \mathbb{T}^3 \right) \cap H^9 \left( \mathbb{T}^3 \right), \\
\mathcal{X}_{\rho^0 u^0} &= L^1 \left( \mathbb{T}^3 \right), \mathcal{X}_{\rho^\varepsilon u} = L^2 \left( \mathbb{T}^3 \right), \\
\mathcal{X}_\rho &= L^2 \left( [0,T]; H^{10} \left( \mathbb{T}^3 \right) \right) \cap L^2 \left( [0,T]; W^{1,3} \left( \mathbb{T}^3 \right) \right) \\
&\cap L^{\frac{8}{3}} \left( [0,T] \times \mathbb{T}^3 \right) \cap C \left( [0,T]; L^{1,3} \left( \mathbb{T}^3 \right) \right), \\
\mathcal{X}_u &= L^2 \left( [0,T] \times \mathbb{T}^3 \right) \cap L^4 \left( [0,T] \times \mathbb{T}^3 \right), \\
\mathcal{X}_{\rho^0 u} &= L^2 \left( [0,T]; W^{1,3} \left( \mathbb{T}^3 \right) \right) \cap C \left( [0,T]; L^\frac{8}{3} \left( \mathbb{T}^3 \right) \right), \\
\mathcal{X}_W &= C \left( [0,T]; \delta \right). \tag{3.42} \end{align*} \]

**Step 2:** The tightness of the laws and the limit \( \varepsilon \to 0. \)

Similar as in last section, we have the following proposition without a proof here.
Proposition 3.1. \( \{ \mathcal{L} \left[ \rho_{0,\varepsilon}, \rho_{0,\varepsilon}^\frac{1}{2} u_{0,\varepsilon}, \bar{\rho}_{0,\varepsilon}, \bar{u}_{0,\varepsilon}, \rho_{\varepsilon}, \bar{\rho}_{\varepsilon}, \bar{u}_{\varepsilon}, W_\varepsilon \right] \} \) is tight on \( X_3 \).

Proposition 3.2. There exists a family of \( X_3 \)-valued Borel measurable random variables
\[
\left\{ \rho_{0,\varepsilon}, \rho_{0,\varepsilon}^\frac{1}{2} u_{0,\varepsilon}, \bar{\rho}_{0,\varepsilon}, \bar{u}_{0,\varepsilon}, \rho_{\varepsilon}, \bar{\rho}_{\varepsilon}, \bar{u}_{\varepsilon}, W_\varepsilon \right\},
\]
and \( \left\{ \rho_0, \rho_0 u_0, \rho_0^\frac{1}{2} u_0, \rho, \bar{\rho} u, W \right\} \) defined on a new complete probability space, we still denote it as \((\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})\), for convenience we still use the same notation in the following sections, such that (up to a subsequence):

1. For all \( \varepsilon \in \mathbb{N} \), \( \mathcal{L} \left[ \rho_{0,\varepsilon}, \rho_{0,\varepsilon}^\frac{1}{2} u_{0,\varepsilon}, \bar{\rho}_{0,\varepsilon}, \bar{u}_{0,\varepsilon}, \rho_{\varepsilon}, \bar{\rho}_{\varepsilon}, \bar{u}_{\varepsilon}, W_\varepsilon \right] \) coincides with \( \mathcal{L} \left[ \rho_0, \rho_0 u_0, \rho_0^\frac{1}{2} u_0, \rho, \bar{\rho} u, W \right] \) on \( X_3 \).

2. \( \mathcal{L} \left[ \rho_0, \rho_0 u_0, \rho_0^\frac{1}{2} u_0, \rho, \bar{\rho} u, W \right] \) on \( X_3 \) is a Radon measure.

3. Random variables \( \left\{ \rho_{0,\varepsilon}, \rho_{0,\varepsilon}^\frac{1}{2} u_{0,\varepsilon}, \bar{\rho}_{0,\varepsilon}, \bar{u}_{0,\varepsilon}, \rho_{\varepsilon}, \bar{\rho}_{\varepsilon}, \bar{u}_{\varepsilon}, W_\varepsilon \right\} \) converges to \( \left\{ \rho_0, \rho_0 u_0, \rho_0^\frac{1}{2} u_0, \rho, \bar{\rho} u, W \right\} \) in the topology of \( X_3 \), \( \tilde{\mathbb{P}} \) almost surely as \( \varepsilon \to 0 \).

**Step3:** The system after taking the limit.

Obviously, the stochastic integral makes sense.

For this \( \varepsilon \to 0 \) layer, thanks to the strong convergences of \( \bar{\rho}_{\varepsilon} \) and \( \bar{\rho}_{\varepsilon} \bar{u}_{\varepsilon} \), we have the almost everywhere convergences of \( \bar{\rho}_{\varepsilon} \) and \( \bar{\rho}_{\varepsilon} \bar{u}_{\varepsilon} \), according to Vitali’s convergence theorem, \( \rho \) satisfies the new equation
\[
\rho_t + \text{div}(\rho u) = 0,
\]
in distribution \( \mathbb{P} \) almost surely.

**Proposition 3.3.** \((\rho, u)\) satisfies the new system involving \( \eta, \delta, \kappa \):
\[
\begin{align*}
\rho_t + \text{div}(\rho u) &= 0, \\
\frac{d}{dt}(\rho u) + (\text{div}(\rho u \otimes u) + \nabla (\rho \eta^2) - \text{div}(\rho \nabla u)) dt &= \left( -r_0 u, u - r_3 u^2 - r_2 u \right) dt \\
&\quad + \kappa \rho \left( \nabla \left( \frac{\frac{\delta}{\sqrt{\Gamma}}}{\frac{\delta}{\sqrt{\Gamma}}} \right) \right) dt + \rho \mathbb{E}(\rho, u) dW
\end{align*}
\]
in distribution \( \mathbb{P} \) almost surely.

**Proof.** For \( \varphi \) in \( L^2 ([0,T]; H^9 (\mathbb{T}^3)) \),
\[
E \left[ \int_0^T \int_{\mathbb{T}^3} \varepsilon \Delta \bar{\rho}_{\varepsilon} \varphi \, dx \, dt \right] \leq \varepsilon \left( \delta^{-\frac{3}{2}} \right)^2 E \left[ \left\| \nabla \bar{\rho}_{\varepsilon} \right\|_{L_t^2 H_x^{10}}^r \left\| \varphi \right\|_{L_t^2 H_x^2}^r \right] \to 0, \quad \text{as } \varepsilon \to 0. \tag{3.45}
\]
Correspondingly,
\[
E \left[ \int_0^T \int_{\mathbb{T}^3} \varepsilon \nabla \bar{\rho}_{\varepsilon} \nabla \bar{u}_{\varepsilon} \varphi \, dx \, dt \right] \leq E \left[ \left( \varepsilon \left\| \nabla \sqrt{\rho_{\varepsilon}} \right\|_{L_t^\infty L_x^2} \left\| \sqrt{\rho_{\varepsilon}} \nabla \bar{u}_{\varepsilon} \right\|_{L_t^2 L_x^2} \left\| \varphi \right\|_{L_t^2 H_x^2} \right)^r \right] \to 0 \quad \text{as } \varepsilon \to 0. \tag{3.46}
\]
The weak convergences of \( \nabla \tilde{\rho}_x^{-5}, \nabla^3 \tilde{\rho}_x, \nabla \tilde{\rho}_x^2 \log \tilde{\rho}_x \) still hold in the corresponding spaces. Next we see whether the convergence of term associated with stochastic forces in the momentum conservation equation. Since \( \tilde{\rho}_x, \tilde{u}_x \rightarrow \rho u \) strongly in \( L^2([0, T] \times \mathbb{T}^3) \) uniformly in \( \varepsilon, \eta, \delta \) and \( \tilde{\mathbb{P}} \) almost surely, \( \tilde{\rho}_x \rightarrow \rho \) strongly in \( L^2([0, T]; W^{1,3}(\mathbb{T}^3)) \) uniformly in \( \varepsilon, \eta, \delta \) and \( \tilde{\mathbb{P}} \) almost surely, together with \( \tilde{\rho}_x \tilde{F}_k(\tilde{\rho}_x, \tilde{u}_x) \leq (\tilde{\rho}_x + \tilde{\rho}_x \tilde{u}_x) \) implies the almost everywhere convergence of \( \tilde{\rho}_x \tilde{F}_k(\tilde{\rho}_x, \tilde{u}_x) \). By repeating the procedure of the \( m \) layer, we can still have the weak convergence of stochastic term. More precisely, we have

\[
\int_0^T \int_{\mathbb{T}^3} \tilde{\rho}_x \tilde{F}_k(\tilde{\rho}_x, \tilde{u}_x) \, dW \, dx \rightarrow \int_0^T \int_{\mathbb{T}^3} \rho \tilde{F}_k(\rho, u) \, dW \, dx \quad (3.48)
\]

in distribution and \( \tilde{\mathbb{P}} \) almost surely.

### 3.3. Energy estimate for the solutions after \( \varepsilon \rightarrow 0 \)

\( \varepsilon \) satisfies the energy estimate \( (3.31) \) and \( (3.33) \). We should also take a limit \( \varepsilon \rightarrow 0 \) to get a new energy estimate which is satisfied by \( u \), the limit of \( \tilde{u}_x \). By the lower semi-continuity of the integrals of convex functions, we pass to the limit \( \varepsilon \rightarrow 0 \) in \( (3.27) \) thanks to \( (3.33) \). More precisely,

\[
\mathbb{E} \left[ \tilde{E}(t) \right] \leq \mathbb{E} \left[ \tilde{E}(0) \right] + C_r \left[ \left( \int_0^t \tilde{E}(s) \, ds \right) \right] + C_r \mathbb{E} \left[ \left( \int_0^t \tilde{E}(s) \, ds \right) \right]
\]

holds independent with \( \delta^{-\frac{1}{2}}, \eta^{-\frac{1}{2}} \) and \( \kappa^{-\frac{1}{2}} \), here \( \hat{C} \) depends on \( r, T \) and \( \mathbb{E}[E(0)^r] \). Consequently, by Grönwall’s inequality, we have the following estimate,

\[
\tilde{E}(t) = \int_{\mathbb{T}^3} \left( \frac{1}{2} \rho |u + \nabla \log \rho|^2 + \eta \rho^{-10} + \frac{\delta}{2} |\nabla \Delta \rho|^2 + \frac{\kappa}{2} |\nabla \sqrt{\rho}|^2 + a \int_1 \frac{\rho(z)}{z} \, dz \right) \, dx,
\]

\[
\mathbb{E} \left[ \sup_{t \in [0, T]} \tilde{E}(t) \right] \leq C,
\]

(3.50)

where \( C \) depends on \( r, T, \mathbb{E}[E(0)^r], \mathbb{E}[E(0)^r] \). Thereupon,

\[
\mathbb{E} \left[ \left( \int_0^t \int_{\mathbb{T}^3} \frac{\rho}{2} |\nabla u - \nabla^T u|^2 \, dx \, ds \right) \right] \leq C,
\]

\[
\mathbb{E} \left[ \left( \int_0^t \int_{\mathbb{T}^3} \rho |u|^4 \, dx \, ds \right) \right] \leq C,
\]

(3.51)
here $C$ depends on $r, T, \mathbb{E}[E(0)^r]$ and $\mathbb{E}[\tilde{E}(0)^r]$.

4. Global existence of martingale solutions

In this section, when we take $\kappa \to 0$, the high regularity of $\rho$ will be lost, which will cause the lack of regularity of $\rho u$. Therefore our target turns to derive the Mellet-Vasseur type inequality in stochastic version which guarantees the strong convergence of $\rho u$.

4.1. Approximation of Mellet-Vasseur type inequality in stochastic version. As in [35], we define two $C^\infty$ cut-off functions $\phi_K$:

$$\phi_K(\rho) = \begin{cases} 1, & \text{for any } \rho < K, \\ 0, & \text{for any } \rho > 2K, \end{cases}$$ (4.1)

where $K > 0$ is any number real number, $|\phi_K'(\rho)| \leq \frac{1}{K}$. For any $\rho > 0$, there exists $C > 0$ such that

$$|\phi_K'(\rho)\sqrt{\rho}| + \left|\frac{\phi_K(\rho)}{\sqrt{\rho}}\right| \leq C,$$

where $C$ depends on $K$. Set $v = \phi_K(\rho)u$. Define $\varphi_n(u) = \tilde{\varphi}_n(|u|^2) \in C^1(\mathbb{R}^3)$, where $\tilde{\varphi}_n$ is given on $\mathbb{R}^+$ by

$$\tilde{\varphi}_n(y) = \begin{cases} (1 + y) \log(1 + y), & 0 \leq y < n, \\ 2(1 + \log(1 + n))y - (1 + y) \log(1 + y) + 2(\log(1 + n) - n), & n \leq y \leq C_n, \\ e(1 + n)^2 - 2n - 2, & y \geq C_n, \end{cases}$$ (4.2)

with $\tilde{\varphi}_n(0) = 0$, and $C_n = e(1 + n)^2 - 1$.

$$\tilde{\varphi}_n'(y) = \begin{cases} 1 + \log(1 + y), & 0 \leq y \leq n, \\ 1 + 2\log(1 + n) - \log(1 + y) \geq 0, & n < y < C_n, \\ 0, & y \geq C_n. \end{cases}$$ (4.3)

and $\tilde{\varphi}_n'(y) \leq 1 + \log(1 + n)$, if $n < y \leq C_n$.

$$\tilde{\varphi}_n''(y) = \begin{cases} \frac{1}{1+y}, & 0 \leq y \leq n, \\ -\frac{1}{(1+y)^2}, & n < y < C_n, \\ 0, & y \geq C_n, \end{cases}$$ (4.4)

$\tilde{\varphi}_n(y)$ is a nondecreasing function with respect to $y$ for any fixed $n$, and it is a nondecreasing function with respect to $n$ for any fixed $y$.

$$\tilde{\varphi}_n(y) \to (1 + y) \log(1 + y) \text{ almost everywhere as } n \to \infty.$$ (4.5)

For any $u \in \mathbb{R}^3$, we denote $\nabla_u$ as taking derivative with respect to $u$, we have

$$\nabla_u \varphi_n(u) = 2\tilde{\varphi}_n'(|u|^2) u,$$ (4.6)

$$\nabla_u^2 \varphi_n(u) = 2\left(2\tilde{\varphi}_n''(|u|^2) u \otimes u + I_3 \tilde{\varphi}_n'(|u|^2)\right).$$ (4.7)

The solution is concerned with $\kappa, K, n, \eta, \delta, r_0, r_1, r_2$, for convenience, we denotes $\rho_\kappa, u_\kappa$ as the solution to (3.34) because we will concentrate on $\kappa \to 0$ next. In last section, we know that $\rho_\kappa, \rho_\kappa u_\kappa$ satisfies (3.34) weakly $\tilde{P}$ almost surely.

Multiply $\phi_K(\rho_\kappa)$ on the both side of (3.34), we have

$$d(\rho_\kappa v_\kappa) + (\text{div}(\rho_\kappa u_\kappa \otimes v_\kappa) - \text{div} S_\kappa + R_\kappa)\, dt = \phi_K(\rho_\kappa)\varphi(\rho_\kappa, u_\kappa)\, dW_\kappa,$$ (4.8)
Lemma 4.1. For any weak solution to (4.11), by Itô’s formula, there holds
\[
\begin{align*}
\frac{d}{dt} \left[ \int_{\mathcal{T}_3} \rho_\kappa \nabla \varphi_n(\nu_\kappa) \, d x \right] &= \left( \int_{\mathcal{T}_3} \nabla \varphi_n(\nu_\kappa) \mathbf{R}_\kappa \, d x \right) \, dt \\
&\quad + \left( \int_{\mathcal{T}_3} \nabla \varphi_n(\nu_\kappa) \cdot \text{div} \, S_\kappa \, d x \right) \, dt \\
&\quad + \left( \int_{\mathcal{T}_3} \nabla \varphi_n(\nu_\kappa) \, d W_\kappa \right) \\
&\quad + \left( \int_{\mathcal{T}_3} \rho_\kappa \phi_K(\rho_\kappa) \mathbb{F}(\rho_\kappa, \nu_\kappa) \, d x \right) \, dt.
\end{align*}
\]
\[
\begin{align*}
\text{Proof.} \quad &\text{Let } h(\rho_\kappa, \rho_\kappa \nu_\kappa) = \rho_\kappa \left( 1 + |\nu_\kappa|^2 \right) \ln \left( 1 + |\nu_\kappa|^2 \right) - \rho_\kappa + \frac{|\nu_\kappa|^2}{\rho_\kappa} \\
&\quad \times \left( 1 + |\nu_\kappa|^2 \right) \ln \left( 1 + |\nu_\kappa|^2 \right) - 2|\nu_\kappa|^2 \\
&\quad \times \left( 1 + |\nu_\kappa|^2 \right) \ln \left( 1 + |\nu_\kappa|^2 \right) - 2|\nu_\kappa|^2 \\
&\quad \times \left( 1 + |\nu_\kappa|^2 \right) \ln \left( 1 + |\nu_\kappa|^2 \right) - 2|\nu_\kappa|^2
\end{align*}
\]
so for
\[
\begin{align*}
\frac{d}{dt} \left( h(\rho_\kappa, \rho_\kappa \nu_\kappa) \right) &= h_{\rho_\kappa} \, d\rho_\kappa + \nabla_{(\rho_\kappa \nu_\kappa)} h \, d(\rho_\kappa \nu_\kappa) + \frac{\nabla^2 (\rho_\kappa \nu_\kappa) h}{2} \langle d(\rho_\kappa \nu_\kappa), d(\rho_\kappa \nu_\kappa) \rangle \\
&= \left( \left( 1 + |\nu_\kappa|^2 \right) \ln \left( 1 + |\nu_\kappa|^2 \right) - 2|\nu_\kappa|^2 \right) \, d\rho_\kappa \\
&\quad + \left( 2\nu_\kappa \ln \left( 1 + |\nu_\kappa|^2 \right) + 2\nu_\kappa \right) \, d(\rho_\kappa \nu_\kappa) + \frac{\nabla^2 (\rho_\kappa \nu_\kappa) h}{2} \langle d(\rho_\kappa \nu_\kappa), d(\rho_\kappa \nu_\kappa) \rangle \\
&= \left( \left( 1 + |\nu_\kappa|^2 \right) \ln \left( 1 + |\nu_\kappa|^2 \right) - 2|\nu_\kappa|^2 \right) \, d\rho_\kappa \\
&\quad + \left( 2\nu_\kappa \ln \left( 1 + |\nu_\kappa|^2 \right) + 2\nu_\kappa \right) \, d(\rho_\kappa \nu_\kappa) \\
&\quad + \left( 2\nu_\kappa \ln \left( 1 + |\nu_\kappa|^2 \right) + 2\nu_\kappa \right) \, d(\rho_\kappa \nu_\kappa) \\
&\quad + \rho_\kappa \phi_K(\rho_\kappa) \mathbb{F}(\rho_\kappa, \nu_\kappa) \, d W_\kappa \\
&\quad + \rho_\kappa \phi_K(\rho_\kappa) \mathbb{F}(\rho_\kappa, \nu_\kappa) \left( \ln \left( 1 + |\nu_\kappa|^2 \right) + 2 \frac{\rho_\kappa \nu_\kappa \otimes \rho_\kappa \nu_\kappa}{|\rho_\kappa \nu_\kappa|^2} \right) \, d W_\kappa.
\end{align*}
\]
Take integration on torus, we have
\[
\int_{\mathbb{T}^3} h(\rho_\kappa, \rho_\kappa \mathbf{v}_\kappa) \, d\mathbf{x}
\]
\[
= - \int_{\mathbb{T}^3} \int_0^1 \left( -|\mathbf{v}_\kappa|^2 - 1 \right) \ln (1 + |\mathbf{v}_\kappa|^2) \, d(\rho_\kappa \mathbf{u}_\kappa) \, ds \, d\mathbf{x} 
- \int_{\mathbb{T}^3} \int_0^1 2 \mathbf{v}_\kappa \cdot \left( (\rho_\kappa \mathbf{u}_\kappa \cdot \nabla) \mathbf{v}_\kappa \right) \, d\mathbf{x} \, ds 
- \int_{\mathbb{T}^3} \int_0^1 2 \mathbf{v}_\kappa \ln (1 + |\mathbf{v}_\kappa|^2) \cdot \left( (\rho_\kappa \mathbf{u}_\kappa \cdot \nabla) \mathbf{v}_\kappa \right) \, d\mathbf{x} \, ds 
+ \int_{\mathbb{T}^3} \int_0^1 (2 \mathbf{v}_\kappa \ln (1 + |\mathbf{v}_\kappa|^2) + 2 \mathbf{v}_\kappa) (\nabla \mathbf{S}_\kappa - \mathbf{R}_\kappa) \, d\mathbf{x} \, ds 
+ \int_{\mathbb{T}^3} \int_0^1 (2 \mathbf{v}_\kappa \ln (1 + |\mathbf{v}_\kappa|^2) + 2 \mathbf{v}_\kappa) \rho_\kappa \phi_K(\rho_\kappa) F(\rho_\kappa, \mathbf{u}_\kappa) \, d\mathbf{x} \, ds 
+ \int_{\mathbb{T}^3} \int_0^1 \rho_\kappa \phi_K^2(\rho_\kappa) F(\rho_\kappa, \mathbf{u}_\kappa) \left( (1 + \ln (1 + |\mathbf{v}_\kappa|^2)) \mathbb{I}_3 + \frac{2 \rho_\kappa \mathbf{v}_\kappa \otimes \rho_\kappa \mathbf{v}_\kappa}{|\rho_\kappa \mathbf{v}_\kappa|^2 + |\rho_\kappa|^2} \right) F(\rho_\kappa, \mathbf{u}_\kappa) \, d\mathbf{x} \, ds 
+ \int_{\mathbb{T}^3} f (\rho_{\kappa,0}, \rho_{\kappa,0} \mathbf{v}_{\kappa,0}) \, d\mathbf{x}. 
\]

The first three integrals in the right hand side will cancel. So we have
\[
\int_{\mathbb{T}^3} h(\rho_\kappa, \rho_\kappa \mathbf{v}_\kappa) \, d\mathbf{x}
\]
\[
= \int_{\mathbb{T}^3} \int_0^1 (2 \mathbf{v}_\kappa \ln |\mathbf{v}_\kappa|^2 + 2 \mathbf{v}_\kappa) (\nabla \mathbf{S}_\kappa - \mathbf{R}_\kappa) \, d\mathbf{x} \, ds 
+ \int_{\mathbb{T}^3} \int_0^1 (2 \mathbf{v}_\kappa \ln |\mathbf{v}_\kappa|^2 + 2 \mathbf{v}_\kappa) \phi_K(\rho_\kappa) \rho_\kappa F(\rho_\kappa, \mathbf{u}_\kappa) \, d\mathbf{x} \, ds 
+ \int_{\mathbb{T}^3} \int_0^1 \rho_\kappa \phi_K^2(\rho_\kappa) F(\rho_\kappa, \mathbf{u}_\kappa) \left( (1 + \ln (1 + |\mathbf{v}_\kappa|^2)) \mathbb{I}_3 + \frac{2 \rho_\kappa \mathbf{v}_\kappa \otimes \rho_\kappa \mathbf{v}_\kappa}{|\rho_\kappa \mathbf{v}_\kappa|^2 + |\rho_\kappa|^2} \right) F(\rho_\kappa, \mathbf{u}_\kappa) \, d\mathbf{x} \, ds 
+ \int_{\mathbb{T}^3} f (\rho_{\kappa,0}, \rho_{\kappa,0} \mathbf{v}_{\kappa,0}) \, d\mathbf{x}. 
\]

Similarly formula holds for $|\mathbf{v}_\kappa|^2 \geq n^2$, with setting
\[
h(\rho_\kappa, \rho_\kappa \mathbf{v}_\kappa) = \rho_\kappa \left( 2 (1 + \ln (1 + n)) \frac{|\rho_\kappa \mathbf{v}_\kappa|^2}{\rho_\kappa^2} - 1 + \frac{|\rho_\kappa \mathbf{v}_\kappa|^2}{\rho_\kappa^2} \right) \ln \left( 1 + \frac{|\rho_\kappa \mathbf{v}_\kappa|^2}{\rho_\kappa^2} \right) + 2 (\ln (1 + n) - n). 
\]

In conclusion, we have
\[
\int_{\mathbb{T}^3} \rho_\kappa \mathbf{u}_\kappa \, d\mathbf{x} = - \int_0^1 \int_{\mathbb{T}^3} \nabla \mathbf{v}_\kappa \mathbf{u}_\kappa \mathbf{R}_\kappa \, d\mathbf{x} \, ds \int_0^1 \int_{\mathbb{T}^3} \nabla \mathbf{S}_\kappa : \nabla (\nabla \mathbf{v}_\kappa \mathbf{u}_\kappa) \, d\mathbf{x} \, ds 
+ \int_0^1 \int_{\mathbb{T}^3} \nabla \mathbf{v}_\kappa \mathbf{u}_\kappa \mathbf{R}_\kappa \phi_K(\rho_\kappa) F(\rho_\kappa, \mathbf{u}_\kappa) \, d\mathbf{x} \, dW_\kappa 
+ \int_{\mathbb{T}^3} \int_0^1 \rho_\kappa \phi_K^2(\rho_\kappa) F(\rho_\kappa, \mathbf{u}_\kappa) \mathbf{v}_\kappa \mathbf{u}_\kappa \mathbf{F}(\rho_\kappa, \mathbf{u}_\kappa) \, d\mathbf{x} \, ds 
+ \int_{\mathbb{T}^3} \rho_{\kappa,0} \mathbf{u}_\kappa(\mathbf{v}_\kappa,0) \, d\mathbf{x}. 
\]

This implies the lemma.
4.2. Vanishing the quantum term: $\kappa \to 0$.

**Step1:** Choose the path space.

We study the convergence of $\rho_\kappa$.

$$
\mathbb{E} \left[ \left\| \frac{1}{\rho_\kappa} \frac{1}{\rho_\kappa} \text{div} u_\kappa \right\|_{L_t^2 L_x^2} \right] \leq \mathbb{E} \left[ \left\| \frac{1}{\rho_\kappa} \right\|_{L_t^\infty L_x^6} \left\| \rho_\kappa \text{div} u_\kappa \right\|_{L_t^2 L_x^2} \right] \leq C, \tag{4.18}
$$

Together with

$$
\mathbb{E} \left[ \left\| \frac{1}{\rho_\kappa} \left( \frac{1}{\rho_\kappa} u_\kappa \right) \right\|_{L_t^2 L_x^2} \right] \leq \mathbb{E} \left[ \left\| \nabla \frac{1}{\rho_\kappa} \right\|_{L_t^\infty L_x^2} \left\| \frac{1}{\rho_\kappa} \right\|_{L_t^6 L_x^6} \left\| \rho_\kappa u_\kappa \right\|_{L_t^1 L_x^4} \right] \leq C, \tag{4.19}
$$

implies

$$
\mathbb{E} \left[ \left\| (\rho_\kappa) \right\|_{L_t^1 L_x^4} \right] \leq C, \tag{4.20}
$$

uniformly in $\delta, \eta$, here $C$ depends on $r_1^{-1}$.

We calculate that

$$
\left| \nabla \rho_\kappa \right| = 2 \left| \nabla \frac{1}{\rho_\kappa} \right| \frac{1}{\rho_\kappa}, \tag{4.21}
$$

so $\{\rho_\kappa\}$ is bounded in $L^\infty \left(0, T; W^{1,1} (\mathbb{T}^3)\right)$, by Aubin-Lion’s lemma we know that the strong convergence of $\rho_\kappa$ in $L^2 \left(0, T; L^2 (\mathbb{T}^3)\right)$ still holds.

**Step2:** The tightness of the laws.

We then consider the regularity of $\rho_\kappa u_\kappa$, and we have the following tightness.

**Proposition 4.1.** The set $\{\mathcal{L} [\rho_\kappa u_\kappa]\}$ is tight on $\mathcal{X}_{\rho \mu, \kappa} = C \left([0, T]; L^2 (\mathbb{T}^3)\right)$.

Proof: We firstly consider the deterministic part, and we denote

$$
Y_\kappa (t) = \rho_\kappa v_\kappa (0) - \int_0^t \text{div} (\rho_\kappa u_\kappa \otimes v_\kappa) \, ds + \int_0^t \text{div} S_\kappa \, ds - \int_0^t R_\kappa \, ds,
$$

where

$$
S_\kappa = \rho_\kappa \phi_K (\rho_\kappa) \left( \nabla u_\kappa + \kappa \frac{\Delta \sqrt{\rho_\kappa}}{\sqrt{\rho_\kappa}} I_3 \right),
$$

$$
R_\kappa = \rho_\kappa^2 u_\kappa \phi'_K (\rho_\kappa) \text{div} u_\kappa + 2 \rho_\kappa \nabla \rho_\kappa \phi (\rho_\kappa) + \rho_\kappa \nabla \phi (\rho_\kappa) \text{div} u_\kappa
$$

$$
+ r_0 u_\kappa \phi (\rho_\kappa) + r_1 \rho_\kappa |u_\kappa|^2 u_\kappa \phi (\rho_\kappa) + r_2 |u_\kappa|^2 u_\kappa \phi (\rho_\kappa)
$$

$$
+ 2 r_3 \phi (\rho_\kappa) \nabla \sqrt{\rho_\kappa} \Delta \sqrt{\rho_\kappa}
$$

$$
- \frac{11}{10} \rho_\kappa^{-10} \phi (\rho_\kappa) - \delta \rho_\kappa \nabla \Delta^9 \rho_\kappa \phi (\rho_\kappa) + \kappa \sqrt{\rho_\kappa} \phi (\rho_\kappa) \Delta \sqrt{\rho_\kappa}.
$$

In this layer, $\kappa$ may go to 0. The high regularity on $\sqrt{\rho}$ and $\rho^{\frac{2}{3}}$ will not hold uniformly on $\kappa$, but we can use $\rho' \in L^1 \left([0, T]; L^2 (\mathbb{T}^3)\right)$, $\rho \in L^\infty \left([0, T]; L^7 (\mathbb{T}^3)\right)$ uniformly in $\kappa$ to conclude that $\frac{dY_\kappa (t)}{dt}$ is bounded in $L^\beta \left(\Omega; L^\delta \left([0, T]; W^{-1,2} (\mathbb{T}^3)\right)\right)$ for $l > \frac{2}{5}$ uniformly in $\kappa$. Hence by multiplying a test function and then integrating with respect to $x$, taking the expectation, $Y_\kappa (t)$ converges strongly in $C^\varsigma \left([0, T]; W^{-1,2} (\mathbb{T}^3)\right)\), $\varsigma \leq \frac{2}{5}$ uniformly in $m$, $\tilde{\mathbb{F}}$ almost surely.
Next, we study the time regularity of stochastic integral. Applying the Burkholder-Davis-Gundy’s inequality and Hölder’s inequality, for $\gamma \geq \frac{4}{3}$, we obtain

\[
E \left[ \left( \left\| \int_{t_1}^{T_2} \phi_K(\rho_\kappa) \rho_\kappa F(\rho_\kappa, u_\kappa) \, dW_\kappa \right\|_{L^{\frac{3}{4}}(T^3)} \right)^{\frac{q}{2}} \right] \\
\leq E \left[ \left( \int_{T_1}^{T_2} \left( \int_{t_1}^{t_2} |\rho_\kappa F(\rho_\kappa, u_\kappa)|^2 \, d\tau \right)^{\frac{q}{2}} \, d\tau \right) \right] \\
\leq E \left[ \left( \int_{T_1}^{T_2} \left( \int_{t_1}^{t_2} |\rho_\kappa F(\rho_\kappa, u_\kappa)|^\frac{3q}{2} \, d\tau \right) \right) \right] \frac{2}{q} \\
\leq E \left[ \left( \sum_{n=1}^{\infty} \|F(\rho_\kappa, u_\kappa)\|_{L^{\frac{3}{4}}(T^3)} \int_{t_1}^{t_2} (\rho_\kappa)^{\frac{3q}{2}} \, d\tau \right) \right] \frac{2}{q} \\
\leq E \left[ \left( \sum_{n=1}^{\infty} \left( \int_{t_1}^{t_2} \left( \rho_\kappa F(\rho_\kappa, u_\kappa) \right)^{\frac{3q}{2}} \, d\tau \right) \right)^{\frac{2}{q}} \right] \\
\leq \left( \tau_2 - \tau_1 \right)^{\frac{2}{q}} C.
\]

Here we used $\rho_\kappa^\gamma \in L^1(0, T; L^3(T^3))$, $\rho \in L^{\infty}(0, T; L^7(T^3))$ uniformly in $\kappa$ and $\frac{3q}{2} \geq 2$ when $\gamma \geq \frac{4}{3}$ so as to deduce $f_k^{\frac{3q}{2}} \leq f_k^2$, correspondingly, $\sum_{n=1}^{\infty} f_k^{\frac{3q}{2}} \leq C$. The $C^0$ in time continuity can not be derived any more, we can only have $\rho_\kappa F(\rho_\kappa, u_\kappa) \, dW_\kappa \in C \left( [0, T]; L^{\frac{3q}{2}}(T^3) \right)$.

Consequently, $\rho_\kappa F(\rho_\kappa, u_\kappa) \, dW_\kappa \in C \left( [0, T]; L^2(T^3) \right)$ for $\gamma \geq \frac{4}{3}$.

For $\frac{6}{3} < \gamma < 2$, by interpolation inequality,

\[
E \left[ \left( \int_{t_1}^{t_2} \phi_K(\rho_\kappa) \rho_\kappa F(\rho_\kappa, u_\kappa) \, dW_\kappa \right)^{\frac{q}{2}} \right] \\
\leq \left( \int_{t_1}^{t_2} \left( \int_{t_1}^{t_2} \rho_\kappa F(\rho_\kappa, u_\kappa)^2 \, d\tau \right)^{\frac{q}{2}} \, d\tau \right) \\
\leq \left( \sum_{n=1}^{\infty} \|F(\rho_\kappa, u_\kappa)\|^q_{L^{\infty}(T^3)} \int_{t_1}^{t_2} (\rho_\kappa)^2 \, d\tau \right)^{\frac{q}{2}}
\]
\begin{align}
\leq & \mathbb{E} \left[ \left( \sum_{n=1}^{\infty} f_k^2 \int_{\tau_1}^{\tau_2} \|\rho_n\|_{L^2(T^3)}^2 \, dt \right)^{\frac{3}{2}} \right] \tag{4.25} \\
\leq & \mathbb{E} \left[ \left( \sum_{n=1}^{\infty} f_k^2 \int_{\tau_1}^{\tau_2} \left( \|\rho_n\|_{L^3(T^3)}^{\frac{3}{2}} \|\rho_n\|_{L^3,\gamma(T^3)}^{\frac{3}{2}} \right)^2 \, dt \right)^{\frac{3}{2}} \right] \\
\leq & \mathbb{E} \left[ \left( \sum_{n=1}^{\infty} f_k \|\rho_n\|_{L^3(T^3)}^{3-1} \int_{\tau_1}^{\tau_2} \left( \int_{T^3} \rho_n^{3\gamma} \, dx \right)^{\frac{1}{3}} \, dt \right)^{\frac{3}{2}} \right] \\
\leq & C(\|\rho_n\|_{L^3(T^3)}^{3-1}) \mathbb{E} \left[ \left( \int_{\tau_1}^{\tau_2} \left( \int_{T^3} \rho_n^{3\gamma} \, dx \right)^{\frac{1}{3}} \, dt \right)^{\frac{3}{2}} \right] \\
\leq & C(\|\rho_n\|_{L^3(T^3)}^{3-1}) \mathbb{E} \left[ \left( \int_{\tau_1}^{\tau_2} \left( \int_{T^3} \rho_n^{3\gamma} \, dx \right)^{\frac{1}{3}} \, dt \right)^{\frac{3}{2}} \right] \\
\leq & C(\tau_2 - \tau_1)^{\frac{r(\gamma-1)}{4}} \\
\end{align}

holds uniformly in \(\kappa, r_0, r_1\).

Thereupon,
\[
\rho_n \mathbb{F}(\rho_n, u_n) \, dW_\kappa \in C \left( [0, T]; L^2 \left( T^3 \right) \right), \quad \text{for } \frac{6}{5} < \gamma < 3. \tag{4.26}
\]

It is a pity that the adiabatic exponent of isobutane is 1.079, our results can not be used to this kinds of gas. Our results can apply to the usual gases in the air.

\begin{align}
\mathbb{E} \left[ \left( \int_{T^3} \left( \int_{t_1}^{t_2} dY_n(t) \right)^2 \, dx \right) \right] \\
= \mathbb{E} \left[ \left( \int_{T^3} \left( \int_{t_1}^{t_2} dY_n(t) \right)^2 \, dx \right) \right] \\
= \mathbb{E} \left[ \left( \int_{T^3} \left( \int_{T^3} \frac{dY_n(t)}{dt} \, dt \phi \, dx \right) \right)^{\frac{1}{2}} \right] \tag{4.27} \\
= \mathbb{E} \left[ \left( \int_{T^3} \left( \int_{t_1}^{t_2} \frac{dY_n(t)}{dt} \phi \, dx \, dt \right) \right)^{\frac{1}{2}} \right] \\
\leq \|\phi\|_{L^{\infty}(0, T; W^{l,2}(T^3))} \mathbb{E} \left[ \left( \int_{T^3} \left( \int_{t_1}^{t_2} \frac{dY_n(t)}{dt} \phi \, dx \, dt \right) \right)^{\frac{1}{2}} \right] \\
= \left( t_2 - t_1 \right)^{\frac{1}{2}},
\end{align}

together with \[1220, \|\rho_n u_n\|_{L^2(T^3)}^2\) is continuous in \([0, T]\) uniformly in \(\kappa, \mathbb{P}\) almost surely. Arzelà-Ascoli’s theorem yields the compactness of \(\{\rho_n u_n\}, \) i.e.,
\[
\rho_n u_n \to \rho u \text{ in } C \left( [0, T]; L^2 \left( T^3 \right) \right), \quad \mathbb{P} \text{ almost surely.} \tag{4.28}
\]

Take the space \(\mathcal{X}_4 = \mathcal{X}_{\rho_0} \times \mathcal{X}_{\rho_0 u_0} \times \mathcal{X}_{\frac{1}{\rho_0} u_0} \times \mathcal{X}_\rho \times \mathcal{X}_u \times \mathcal{X}_{\rho u} \times \mathcal{X}_W, \) where \(\mathcal{X}_{\rho_0} = L^7 \left( T^3 \right), \mathcal{X}_{\rho_0 u_0} = L^1 \left( T^3 \right), \mathcal{X}_{\frac{1}{\rho_0} u_0} = L^2 \left( T^3 \right), \mathcal{X}_\rho = L^2 \left( 0, T; L^2 \left( T^3 \right) \right) \cap L^{\frac{2}{\gamma}} \left( [0, T] \times T^3 \right), \mathcal{X}_u = L^2 \left( [0, T] \times T^3 \right) \cap L^4 \left( [0, T] \times T^3 \right), \mathcal{X}_{\rho u} = C \left( [0, T]; L^{\frac{4}{\gamma}} \left( T^3 \right) \right), \mathcal{X}_W = C \left( [0, T]; \mathfrak{S} \right).

**Step 3:** The convergence \(\kappa \to 0.\)

We have the corresponding tightness and Skorokhod’s representation theorem.
Proposition 4.2. There exists a family of $\mathcal{X}_4$-valued Borel measurable random variables

$$\left\{ \bar{\rho}_{0, \kappa}, \bar{\rho}_{0, \kappa} \bar{u}_{0, \kappa}, \rho_{0, \kappa}^{1/2} \bar{u}_{0, \kappa}, \rho_{0, \kappa}, \bar{u}_{\kappa}, \bar{u}_{0, \kappa}, W_\kappa \right\},$$

and

$$\left\{ \rho_0, \rho_0 \bar{u}_0, \rho_0^{1/2} \bar{u}_0, \rho, \rho \bar{u}, W \right\}$$

defined on a new complete probability space, we still denote it as $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}})$, for convenience we still use the same notation in the following sections, such that (up to a subsequence):

1. For all $\kappa \in \mathbb{N}$, $\mathcal{L} \left[ \rho_{0, \kappa}, \rho_{0, \kappa} \bar{u}_{0, \kappa}, \rho_{0, \kappa}^{1/2} \bar{u}_{0, \kappa}, \rho_{0, \kappa}, \bar{u}_{\kappa}, \bar{u}_{0, \kappa}, W_\kappa \right]$, coincides with

$$\mathcal{L} \left[ \rho_0, \rho_0 \bar{u}_0, \rho_0^{1/2} \bar{u}_0, \rho, \rho \bar{u}, W \right]$$
on $\mathcal{X}_4$.

2. $\mathcal{L} \left[ \rho_0, \rho_0 \bar{u}_0, \rho_0^{1/2} \bar{u}_0, \rho, \rho \bar{u}, W \right]$ on $\mathcal{X}_4$ is a Radon measure.

3. The family of random variables $\left\{ \bar{\rho}_{0, \kappa}, \bar{\rho}_{0, \kappa} \bar{u}_{0, \kappa}, \rho_{0, \kappa}^{1/2} \bar{u}_{0, \kappa}, \rho_{0, \kappa}, \bar{u}_{\kappa}, \bar{u}_{0, \kappa}, W_\kappa \right\}$ converges to

$$\left\{ \rho_0, \rho_0 \bar{u}_0, \rho_0^{1/2} \bar{u}_0, \rho, \rho \bar{u}, W \right\}$$
in the topology of $\mathcal{X}_4$, $\mathbb{P}$ almost surely as $\kappa \to 0$.

**Step 4:** The estimate after taking the limit.

Let us be consistent with Vasseur-Yu’s work, in which they handle the deterministic terms by taking $K = \kappa^{-\frac{1}{2}}$.

$$\mathbb{E} \left[ \int_{T^3} \bar{\rho}_{\kappa} \varphi_{n}(\bar{v}_{\kappa}) \, dx \right]\leq C_r \mathbb{E} \left[ -\int_0^t \int_{T^3} \nabla \varphi \cdot \nabla (\bar{v}_{\kappa}) \, dx \, ds - \int_0^t \int_{T^3} S_{\kappa} : \nabla (\nabla \varphi \cdot \nabla (\bar{v}_{\kappa})) \, dx \, ds \right]$$

$$+ C_r \mathbb{E} \left[ \int_0^t \int_{T^3} \nabla \varphi \cdot \nabla (\bar{v}_{\kappa}) \, dx \, ds \right]$$

$$+ C_r \mathbb{E} \left[ \int_0^t \int_{T^3} \bar{\rho}_{\kappa} \phi_{K}(\bar{\rho}_{\kappa}) \bar{\rho}_{\kappa} \mathcal{F} (\bar{\rho}_{\kappa}, \bar{u}_{\kappa}) \, dx \, dW \right]$$

$$+ C_r \mathbb{E} \left[ \int_0^t \int_{T^3} \bar{\rho}_{\kappa} \phi_{K}^2(\bar{\rho}_{\kappa}) \mathcal{F} (\bar{\rho}_{\kappa}, \bar{u}_{\kappa}) \nabla \varphi \cdot \nabla (\bar{v}_{\kappa}) \mathcal{F} (\bar{\rho}_{\kappa}, \bar{u}_{\kappa}) \, dx \, ds \right]$$

$$\leq C,$$

where $C$ depends on $\mathbb{E}[\langle E(0) \rangle^r]$, $r$, $t$ and $n$, $C_r$ is a constant merely dependent of $r$.

From the stochastic B-D entropy we know

$$- r_2 \mathbb{E} \left[ \int_{T^3} \log \bar{\rho}_{\kappa} \, dx \right] \leq C, \text{ for any } r > 2.$$  \hfill (4.30)

Where $C$ depends on $r, T, \mathbb{E}[\langle E(0) \rangle^r], \mathbb{E}[\langle \dot{E}(0) \rangle^r]$ . A.Vassuer and C.Yu argued that

$$\int_{\Omega} \left( \ln \left( \frac{1}{\rho} \right) \right)_+ \, dx \leq \int_{\Omega} \lim \inf_{\kappa \to 0} \left( \ln \left( \frac{1}{\bar{\rho}_{\kappa}} \right) \right)_+ \, dx \leq \lim \inf_{\kappa \to 0} \int_{\Omega} \left( \ln \left( \frac{1}{\bar{\rho}_{\kappa}} \right) \right)_+ \, dx,$$

by Fatou’s lemma, so $\left( \ln \left( \frac{1}{\rho} \right) \right)_+$ is bounded in $L^\infty (0, T; L^1(T^3))$. Therefore (4.30) allows us to conclude that

$$|\{x \mid \rho(t, x) = 0\}| = 0, \text{ for almost every } t, \omega.$$  \hfill (4.32)
For almost every \((t, x)\), \(\rho(t, x) \neq 0\),
\[
\bar{u}_\kappa(t, x) = \frac{\tilde{\rho}_\kappa \bar{u}_\kappa}{\tilde{\rho}_\kappa} \to u(t, x)
\]
and
\[
\bar{v}_\kappa \to u(t, x)
\]
as \(\kappa \to 0\). \(\phi_K(\tilde{\rho}_\kappa) \to 1\) almost everywhere as \(K \to \infty\), \(\varphi'_n \leq 1 + \ln(1 + C_n)\) uniformly in \(K\) for fixed \(n\), therefore \(\nabla \varphi_n \varphi_n(\bar{v}_\kappa) \to \nabla \varphi_n(u)\) almost everywhere, \(\eta \nabla \tilde{\rho}_\kappa^{10} \phi_K(\tilde{\rho}_\kappa) + \delta \nabla \Delta \rho \nabla \tilde{\rho}_\kappa \phi_K(\tilde{\rho}_\kappa) \to \eta \nabla \rho_0^2 u + \eta \nabla \rho_0^{10} + \delta \rho_0 \Delta \rho\) almost everywhere, these three terms are \(L^1\) integrable. By Lebesque’s dominated convergence theorem, taking the limit \(K \to \infty\), i.e. \(\kappa \to 0\), we have
\[
\int_{T_3} \tilde{\rho}_\kappa \varphi_n(\phi_K(\tilde{\rho}_\kappa) u_\kappa) \, d x \to \int_{T_3} \rho \varphi_n(u) \, d x \quad \bar{P} \text{ almost surely.} \tag{4.33}
\]
For the pressure term, integration by parts implies
\[
- \int_{T_3} \int_0^t \nabla \varphi_n(\bar{v}_\kappa) \nabla \tilde{\rho}_\kappa^2 \phi_K(\tilde{\rho}_\kappa) \, d s \, d x \tag{4.34}
\]
\[
= \int_{T_3} \int_0^t \nabla \varphi_n(\bar{v}_\kappa) : \nabla \bar{v}_\kappa \tilde{\rho}_\kappa^2 \, d s \, d x + \int_{T_3} \int_0^t \varphi_n(\bar{v}_\kappa) \nabla \phi_K(\tilde{\rho}_\kappa) \tilde{\rho}_\kappa^4 \, d s \, d x \triangleq P_1 + P_2,
\]
where
\[
P_1 = \int_{T_3} \int_0^t \tilde{\rho}_\kappa^2 \tilde{\rho}_\kappa^2 \phi_K(\tilde{\rho}_\kappa) \nabla \varphi_n(\bar{v}_\kappa) : \nabla u_\kappa \, d s \, d x \tag{4.35}
\]
\[
+ \int_{T_3} \int_0^t \tilde{\rho}_\kappa^2 \phi_K(\tilde{\rho}_\kappa) \nabla \varphi_n(\bar{v}_\kappa) : \nabla \phi_K(\tilde{\rho}_\kappa) \otimes u_\kappa \, d s \, d x \triangleq P_{11} + P_{12}.
\]
Thanks to \(\phi'_K(\tilde{\rho}_\kappa) \tilde{\rho}_\kappa^4 \leq \frac{4}{K^2} \),
\[
E[|P_2|^r] \leq C E \left[ \kappa^{-\frac{1}{4}} \left\| \tilde{\rho}_\kappa^2 \phi'_K(\tilde{\rho}_\kappa) \right\|_{L^\infty_t L^\infty_x} \left\| \kappa^{-\frac{1}{4}} \nabla \tilde{\rho}_\kappa^4 \right\|_{L^4_t L^4_x} \right]^r \tag{4.36}
\]
\[
\leq C \left( \kappa^{-\frac{1}{4}} \frac{2}{K^2} \right)^r = C \kappa^{\frac{r}{2}} \to 0 \text{ as } \kappa \to 0,
\]
here \(C\) depends on \(E[(|E(0)|)^r], r, t\) and \(n\).
\[
E[|P_{12}|^r] = E \left[ \left\| \int_{T_3} \int_0^t \tilde{\rho}_\kappa^2 \phi'_K(\tilde{\rho}_\kappa) \nabla \varphi_n(\bar{v}_\kappa) : \phi'_K(\tilde{\rho}_\kappa) \tilde{\rho}_\kappa^2 \nabla \bar{u}_\kappa \, d s \, d x \right\|^r \right]
\]
\[
\leq C E \left[ \kappa^{-\frac{1}{2}} \left\| \tilde{\rho}_\kappa^2 \phi'_K(\tilde{\rho}_\kappa) \right\|_{L^\infty_t L^\infty_x} \left\| \kappa^{-\frac{1}{4}} \nabla \tilde{\rho}_\kappa^4 \right\|_{L^4_t L^4_x} \right]^r \tag{4.37}
\]
\[
\leq C \left( \kappa^{-\frac{1}{2}} \frac{2}{K^2} \right)^r = C \kappa^{\frac{r}{2}} \to 0 \text{ as } \kappa \to 0,
\]
here we used \(\nabla \varphi_n(\bar{v}_\kappa) \leq C(n)\), \(C\) depends on \(E[(|E(0)|)^r], r, t\) and \(n\). Since
\[
\nabla \varphi_n(\bar{v}_\kappa) : \nabla \bar{u}_\kappa = 4 \varphi''_n (|\bar{v}_\kappa|^2) \varphi'_n \otimes \bar{v}_\kappa : \nabla \bar{u}_\kappa + 2 \varphi'_n (|\bar{v}_\kappa|^2) \text{ div } \bar{u}_\kappa,
\]
we have
\[
P_{11} = 4 \int_{T_3} \int_0^t \phi_K(\tilde{\rho}_\kappa) \tilde{\rho}_\kappa^2 |\nabla \bar{u}_\kappa| \, d s \, d x + 4 \int_{T_3} \int_0^t \phi_K(\tilde{\rho}_\kappa) \tilde{\rho}_\kappa^2 |\text{div } \bar{u}_\kappa| |\varphi_n(\bar{v}_\kappa)| \, d s \, d x.
\]
\[ \begin{align*}
\leq & 4 \int_{T_{\lambda}}^{t} \int_{0}^{T_{\lambda}} \tilde{\rho}_{\kappa} | \nabla \tilde{\mathbf{u}}_{\kappa} |^2 + 4 \int_{T_{\lambda}}^{t} \int_{0}^{T_{\lambda}} \tilde{\rho}_{\kappa}^{2\gamma-1} d s d x + 2 \int_{T_{\lambda}}^{t} \int_{0}^{T_{\lambda}} \phi_{K}^{2}(\tilde{\rho}_{\kappa}) \tilde{\rho}_{\kappa}^{2} | \nabla \tilde{\mathbf{u}}_{\kappa} | \varphi_{\kappa}'(| \tilde{\mathbf{u}}_{\kappa} |^2) d s d x \quad (4.39) \\
\leq & C + 2 \int_{T_{\lambda}}^{t} \int_{0}^{T_{\lambda}} \phi_{K}(\tilde{\rho}_{\kappa}) \tilde{\rho}_{\kappa} | | \nabla \tilde{\mathbf{u}}_{\kappa} |^2 \varphi_{\kappa}'(| \tilde{\mathbf{u}}_{\kappa} |^2) d s d x + 2 \int_{T_{\lambda}}^{t} \int_{0}^{T_{\lambda}} \varphi_{\kappa}'( | \tilde{\mathbf{u}}_{\kappa} |^2 ) \tilde{\rho}_{\kappa}^{2\gamma-1} d s d x, \\
C \text{ depends on } \mathbb{E}(E(0))^r, \quad r, t \text{ and } n, \quad \text{we do not take expectation here because we will estimate it together with } \tilde{\rho}_{\kappa} \phi_{K}(\tilde{\rho}_{\kappa}) | \nabla \tilde{\mathbf{u}}_{\kappa} | \text{ in } S_{\kappa}.
\end{align*} \]

\[ S_{\kappa} = \tilde{\rho}_{\kappa} \phi_{K}(\tilde{\rho}_{\kappa}) \left( \nabla \tilde{\mathbf{u}}_{\kappa} + \kappa \frac{\Delta \sqrt{ \tilde{\rho}_{\kappa} \mathbb{I} } }{ \sqrt{ \tilde{\rho}_{\kappa} } } \right) \triangleq S_{1} + S_{2}, \quad (4.40) \]

\[ \begin{align*}
\int_{T_{\lambda}}^{t} \int_{0}^{T_{\lambda}} \nabla \nabla \varphi_{\kappa}(\mathbf{v}_{\kappa}) : S_{1} d x d s &= \\
= & \int_{T_{\lambda}}^{t} \int_{0}^{T_{\lambda}} \nabla \left( \nabla \varphi_{\kappa}(\mathbf{v}_{\kappa}) \right) : \tilde{\rho}_{\kappa} \phi_{K}(\tilde{\rho}_{\kappa}) | \nabla \tilde{\mathbf{u}}_{\kappa} | d x d s \\
= & \int_{T_{\lambda}}^{t} \int_{0}^{T_{\lambda}} \tilde{\rho}_{\kappa} \phi_{K}(\tilde{\rho}_{\kappa}) \nabla \varphi_{\kappa}(\mathbf{v}_{\kappa}) \nabla \tilde{\mathbf{u}}_{\kappa} : | \nabla \tilde{\mathbf{u}}_{\kappa} | d x d s \\
& + \int_{T_{\lambda}}^{t} \int_{0}^{T_{\lambda}} \tilde{\rho}_{\kappa} \phi_{K}(\tilde{\rho}_{\kappa}) \mathbf{u}_{\kappa}^{T} \nabla \varphi_{\kappa}(\mathbf{v}_{\kappa}) | \nabla \tilde{\mathbf{u}}_{\kappa} | \nabla \left( \phi_{K}(\tilde{\rho}_{\kappa}) \right) d x d s \\
\triangleq & A_{1} + A_{2}. \end{align*} \]

Similarly, \( \mathbb{E} [ | A_{2} |^{r} ] \to 0 \) as \( \kappa \to 0 \).

\[ A_{1} = 2 \int_{T_{\lambda}}^{t} \int_{0}^{T_{\lambda}} \phi_{K}^{2}(\tilde{\rho}_{\kappa}) \varphi_{\kappa}'( | \tilde{\mathbf{u}}_{\kappa} |^2 ) \tilde{\rho}_{\kappa} \nabla \tilde{\mathbf{u}}_{\kappa} : | \nabla \tilde{\mathbf{u}}_{\kappa} | d x d s \]

\[ 
+ 4 \int_{T_{\lambda}}^{t} \int_{0}^{T_{\lambda}} \phi_{K}(\tilde{\rho}_{\kappa}) \varphi_{\kappa}'( | \tilde{\mathbf{u}}_{\kappa} |^2 ) \mathbf{v}_{\kappa} \otimes \mathbf{v}_{\kappa} \nabla \tilde{\mathbf{u}}_{\kappa} : | \nabla \tilde{\mathbf{u}}_{\kappa} | d x d s \quad (4.42) \]

\[ \geq 2 \int_{T_{\lambda}}^{t} \int_{0}^{T_{\lambda}} \phi_{K}^{2}(\tilde{\rho}_{\kappa}) \varphi_{\kappa}'( | \tilde{\mathbf{u}}_{\kappa} |^2 ) \tilde{\rho}_{\kappa} | | \nabla \tilde{\mathbf{u}}_{\kappa} |^2 | d s d x - 4 \int_{T_{\lambda}}^{t} \int_{0}^{T_{\lambda}} \phi_{K}^{2}(\tilde{\rho}_{\kappa}) \tilde{\rho}_{\kappa} | | \nabla \tilde{\mathbf{u}}_{\kappa} |^2 | d s d x, \]

\[ - A_{1} + P_{11} \leq (-2 + 2) \int_{T_{\lambda}}^{t} \int_{0}^{T_{\lambda}} \phi_{K}^{2}(\tilde{\rho}_{\kappa}) \varphi_{\kappa}'( | \tilde{\mathbf{u}}_{\kappa} |^2 ) \tilde{\rho}_{\kappa} | | \nabla \tilde{\mathbf{u}}_{\kappa} |^2 | d s d x \quad (4.43) \]

\[ + 4 \int_{T_{\lambda}}^{t} \int_{0}^{T_{\lambda}} \phi_{K}(\tilde{\rho}_{\kappa}) \tilde{\rho}_{\kappa} | | \nabla \tilde{\mathbf{u}}_{\kappa} |^2 | d s d x + 2 \int_{T_{\lambda}}^{t} \int_{0}^{T_{\lambda}} \varphi_{\kappa}'( | \tilde{\mathbf{u}}_{\kappa} |^2 ) \tilde{\rho}_{\kappa}^{2\gamma-1} | d s d x, \]

\[ \mathbb{E} [ | - A_{1} + P_{11} |^{r} ] \leq C + 2 \int_{T_{\lambda}}^{t} \int_{0}^{T_{\lambda}} \varphi_{\kappa}'( | \tilde{\mathbf{u}}_{\kappa} |^2 ) \tilde{\rho}_{\kappa}^{2\gamma-1} | d s d x, \quad (4.44) \]

\( C \) depends on \( \mathbb{E}(E(0))^r, \quad r, t \) and \( n \).

\[ \mathbb{E} \left[ \left( \int_{T_{\lambda}}^{t} \int_{0}^{T_{\lambda}} 2 \kappa | \nabla \varphi_{\kappa}(\mathbf{v}_{\kappa}) \phi_{K}(\tilde{\rho}_{\kappa}) \nabla \sqrt{ \tilde{\rho}_{\kappa} \Delta } \tilde{\rho}_{\kappa} | d s d x \right)^{r} \right] \]

\[ \leq C_{K \kappa} \prod_{n=1}^{N_{\kappa}} \left( \prod_{n=1}^{N_{\kappa}} \left( \frac{1}{L_{n}\Delta \tilde{\rho}_{\kappa}^{2}} \right)^{r} \right) \left( \frac{1}{L_{n}^{2}} \right)^{r} \]  \( \leq C \kappa^{\frac{1}{2}} \) as \( \kappa \to 0 \), \( \mathbb{E} \left[ \left( \int_{T_{\lambda}}^{t} \int_{0}^{T_{\lambda}} \nabla \varphi_{\kappa}(\mathbf{v}_{\kappa}) S_{2} d s d x \right)^{r} \right] \to 0. \quad (4.46) \]

\[ \int_{T_{\lambda}}^{t} \int_{0}^{T_{\lambda}} \nabla \varphi_{\kappa}(\mathbf{v}_{\kappa}) \left( r_{0} \tilde{\mathbf{u}}_{\kappa} + r_{1} \tilde{\rho}_{\kappa} | \tilde{\mathbf{u}}_{\kappa} |^2 \tilde{\mathbf{u}}_{\kappa} + r_{2} | \tilde{\mathbf{u}}_{\kappa} |^2 \tilde{\mathbf{u}}_{\kappa} \right) \phi_{K}(\tilde{\rho}_{\kappa}) | d s d x \leq 0. \quad (4.47) \]
\[
\int_{T^3} \int_0^t \tilde{\rho}_\kappa \phi_K^2(\tilde{\rho}_\kappa) F(\tilde{\rho}_\kappa, \tilde{u}_\kappa) \nabla_{\tilde{\nu}}^2 \varphi_n(\tilde{\nu}_\kappa) F(\tilde{\rho}_\kappa, \tilde{u}_\kappa) \, d s \, d x
\]
(4.48)
\[
\rightarrow \int_{T^3} \int_0^t \rho F(\rho, u) \nabla_{\nu}^2 \varphi_n(\nu) F(\rho, u) \, d s \, d x,
\]

\[\tilde{P}\] almost surely according to Lebesgue’s dominated convergence theorem. The terms
\[
- \int_{T^3} \int_0^t \nabla_{\tilde{\nu}}^2 \varphi_n(\tilde{\nu}_\kappa) \rho_{\kappa} \phi_K(\tilde{\rho}_\kappa) \, d \tilde{u}_\kappa \, d s \, d x,
\]
(4.49)
and
\[
- \int_{T^3} \int_0^t \nabla_{\tilde{\nu}}^2 \varphi_n(\tilde{\nu}_\kappa) \rho_{\kappa} \nabla \phi_K(\tilde{\rho}_\kappa) \, d \tilde{u}_\kappa \, d s \, d x,
\]
(4.50)
will go to zero due to \(\phi_K'(\tilde{\rho}_\kappa) \rho_{\kappa} \leq \frac{4}{K^2}\). Lemma 2.2 implies that
\[
\int_0^t \int_{T^3} \nabla_{\tilde{\nu}} \varphi_n(\tilde{\nu}_\kappa) \phi_K(\tilde{\rho}_\kappa) \rho_{\kappa} F(\rho, u) \, d x \, d W \rightarrow \int_0^t \int_{T^3} \nabla_{\nu} \varphi_n(\nu) \rho F(\rho, u) \, d x \, d W.
\]
(4.51)

\[\tilde{P}\] almost surely as \(\kappa \to 0\).

Therefore, we have the following lemma.

**Lemma 4.2.** The limit of solutions satisfy
\[
\int_{T^3} \rho \phi_n(u) \, d x = - \int_0^t \int_{T^3} \nabla \phi_n(u) R \, d x \, d s + \int_0^t \int_{T^3} \nabla \phi_n(u) \rho \, d x \, d W
\]
(4.52)
\[
+ \int_{T^3} \int_0^t \rho F(\rho, u) \nabla_{\nu}^2 \phi_n(u) F(\rho, u) \, d s \, d x + \int_{T^3} \rho \phi_n(u) \, d x,
\]
and
\[
E \left[ - \int_\Omega \rho \phi(u) \, d x + \int_0^T \int_\Omega \phi'(u) R \, d x \, d t \right] \leq \bar{C} + C_r E \left[ \int_0^T \int_\Omega (1 + \phi'(|u|^2)) \rho^{2\gamma - 1} \, d x \, d t \right]
\]
(4.53)
\[
+ C_r E \left[ \psi(0) \int_\Omega \rho \phi(u) \, d x \right] + C_r E \left[ \int_{T^3} \int_0^T \nabla \phi_n(u) \rho F(\rho, u) \, d W \, d x \right],
\]
where
\[
R = \eta \nabla \rho^{-10} + \delta \rho \nabla \Delta^9 \rho,
\]
\(\bar{C}\) depends on \(T, r, E[\phi(0)^r]\) and \(E[\tilde{E}(0)^r]\), \(C_r\) merely depends on \(r\).

**Lemma 4.3.** The pair of limits \((\rho, u)\) is a martingale solution to
\[
\begin{cases}
\rho_t + \text{div}(\rho u) = 0, \\
d(\rho u) + (\text{div}(\rho u) \otimes u) + \nabla (\rho u) - \text{div}(\rho D u) - \frac{11}{10} \eta \nabla \rho^{-10} \, d t \\
= (-r_0|u|^2 u - r_1 \rho |u|^2 u - r_2 u) \, d t + \delta \rho \nabla \Delta^9 \rho \, d t + \rho F(\rho, u) \, d W.
\end{cases}
\]
(4.55)

**Proof.** The weak convergences of \(\nabla \rho_{\kappa}, \rho_{\kappa} \nabla^2 \log \rho_{\kappa}\) still hold in the corresponding spaces. By Vitali’s convergence theorem, the limits of \(\tilde{\rho}_\kappa\) and \(\tilde{u}_\kappa\) satisfy the mass conservation equation, we just consider the convergence of stochastic forces. We show what is limit of stochastic integral
\[
\int_0^T \int_{T^3} \tilde{\rho}_\kappa F(\tilde{\rho}_\kappa, \tilde{u}_\kappa) \, d W \, d x.
\]
Since $E \left[ \| \tilde{\rho}_n \|_{L^q_t L^r_x} \right] \leq C$, where $C$ depends on $r, T, E[0]$, and $E[E(0)]$, so we have

$$E \left[ \left( \int_0^T \int_{\mathbb{T}^3} |\tilde{\rho}_n F(\tilde{\rho}_n, \tilde{u}_n)|^2 \, dx \, dt \right)^r \right]$$

$$= E \left[ \left( \int_0^T \int_{\mathbb{T}^3} |\tilde{\rho}_n|^2 \sum_{k=1}^{+\infty} F_k(\tilde{\rho}_n, \tilde{u}_n)^2 \, dx \, dt \right)^r \right]$$

$$\leq \left( \sum_{k=1}^{+\infty} f_k^2 \right)^r E \left[ \int_0^T |\tilde{\rho}_n|^2 \, dx \, dt \right]^r$$

$$\leq C,$$

under our assumption $\gamma > \frac{6}{5}$, $C$ depends on $T, r, E[0]$ and $E[E(0)]$. In addition, we have derived the strong convergence of $\tilde{\rho}_n$ in $L^2 \left( 0, T; L^2_t (\mathbb{T}^3) \right)$ and the strong convergence of $\tilde{\rho}_n \tilde{u}_n$ in $L^2 \left( 0, T; L^2_t (\mathbb{T}^3) \right)$, hence we know $\tilde{\rho}_n \to \rho$ almost everywhere, $\tilde{\rho}_n \tilde{u}_n \to \rho \tilde{u}$ almost everywhere up to a subsequence. Therefore $\tilde{\rho}_n F(\tilde{\rho}_n, \tilde{u}_n) \to \rho \tilde{F}(\rho, \tilde{u})$ almost everywhere. By dominated convergence theorem we have

$$\int_0^t \int_{\mathbb{T}^3} |\tilde{\rho}_n F(\tilde{\rho}_n, \tilde{u}_n)|^2 \, dx \, dt \to \int_0^t \int_{\mathbb{T}^3} |\rho \tilde{F}(\rho, \tilde{u})|^2 \, dx \, dt, \quad \tilde{\mathbb{P}} \text{ almost surely.} \quad (4.57)$$

The convergence of the stochastic integral comes from this and the strong convergence of $\tilde{W}_n \to W$ in $C([0, T]; \tilde{\mathcal{F}})$.

### 4.3. Vanishing the artificial pressure terms and Rayleigh damping term: $\delta, \eta, r_0 \to 0$

Now the solutions are concerned with $n, \delta, \eta, r_0, r_1$ and $r_2$. For simplicity, we use the specific parameter as corresponding foot script when we take the limit.

We take $n \to +\infty$ to derive a new stochastic Mellet-Vasseur type inequality. Recall that the process of stochastic B-D entropy estimates, estimate of the term $\eta_0 u^3$ is dependent on $\delta^{-1}$ and $\eta^{-1}$, which will cause trouble if we firstly take $\delta, \eta \to 0$. So we will take the limit $n \to \infty, \delta, \eta, r_0 \to 0$ at the same time, then we can do the energy estimate for the system, which shows the terms like $E \left[ \left( \int_{\mathbb{T}^3} \delta |\nabla \nabla^4 \rho_\delta |^2 \, dx \right)^r \right]$ in the energy estimate will vanish after we take the limit $\delta \to 0$, so do in the stochastic B-D entropy.

**Step 1:** Choose the path space, show the tightness of the laws and the convergence.

We will take $\delta, \eta$ and $r_0$ as negative powers of $n$, so we can use the notation $\rho_n$ instead of $\rho_\delta$ or $\rho_\eta$. This lemma shows the right hand side of inequality can be bounded by the initial energy and the lemma implies the strong convergence of $\frac{1}{\rho_n^2} \tilde{u}_n \in L^r (\Omega; L^2 ([0, T] \times \mathbb{T}^3))$, refer to Mellet-Vasseur’s job [34].

Take the space $\mathcal{X}_5 = \mathcal{X}_{\rho_0} \times \mathcal{X}_{\tilde{\rho}_0^{\frac{1}{2}} u_0} \times \mathcal{X}_{\rho_0 \rho_0 u_0} \times \mathcal{X}_\rho \times \mathcal{X}_{\tilde{\rho}_0^{\frac{1}{2}} u_0} \times \mathcal{X}_W$, where $\mathcal{X}_{\rho_0} = L^2 (\mathbb{T}^3)$, $\mathcal{X}_{\tilde{\rho}_0^{\frac{1}{2}} u_0} = L^2 (\mathbb{T}^3)$, $\mathcal{X}_{\rho_0 \rho_0 u_0} = L^1 ([0, T]) \cap L^2 (\mathbb{T}^3)$, $\mathcal{X}_\rho = L^2 (0, T; L^2 (\mathbb{T}^3)) \cap L^\infty (0, T] \times \mathbb{T}^3)$, $\mathcal{X}_{\tilde{\rho}_0^{\frac{1}{2}} u_0} = L^2 ([0, T] \times \mathbb{T}^3)$, $\mathcal{X}_W = C ([0, T]; \tilde{\mathcal{F}})$. Therefore, we have the corresponding tightness and Skorokhod representation theorem.
Proposition 4.3. There exists a family of $\mathcal{X}_5$-valued Borel measurable random variables
\[
\left\{ \tilde{\rho}_{0,n}, \tilde{\rho}_{0,n}^2 \tilde{u}_{0,n}, \tilde{p}_n, \tilde{p}_n^2 \tilde{u}_n, \tilde{W}_n \right\}
\]
and \( \left\{ \rho_{0}, \rho_{0}^2 \rho_{0} u_{0}, \rho_{0} u_{0, 0}, \rho_{0}^2 u_{0}, W \right\} \), defined on a new complete probability space \((\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})\), such that (up to a sequence) for \( \gamma > \frac{5}{4} \):

1. \( \mathcal{L} \left[ \tilde{\rho}_{0,n}, \tilde{\rho}_{0,n}^2 \tilde{u}_{0,n}, \tilde{p}_n, \tilde{p}_n^2 \tilde{u}_n, \tilde{W}_n \right] \) and \( \mathcal{L} \left[ \rho_{0,n}, \rho_{0,n}^2 \rho_{0,n} u_{0,n}, \rho_n, \rho_n^2 u_n, W_n \right] \)
coincides with each other on \( \mathcal{X}_5 \).

2. \( \mathcal{L} \left[ \rho_{0}, \rho_{0}^2 \rho_{0} u_{0}, \rho_{0} u_{0}, \rho_{0}^2 u_{0}, W \right] \) on \( \mathcal{X}_5 \) is a Radon measure.

3. Random variables \( \left\{ \tilde{\rho}_{0,n}, \tilde{\rho}_{0,n}^2 \tilde{u}_{0,n}, \tilde{p}_n, \tilde{p}_n^2 \tilde{u}_n, \tilde{W}_n \right\} \) converges to
\( \left\{ \rho_{0}, \rho_{0}^2 \rho_{0} u_{0}, \rho_{0} u_{0}, \rho_{0}^2 u_{0}, W \right\} \) in the topology of \( \mathcal{X}_5 \), \( \tilde{\mathbb{P}} \) almost surely as \( n \to \infty \).

Step 2: The estimate for the limit system.

Recall (5.33) and (5.31), by Gagliardo-Nirenberg interpolation inequality, \( \delta \frac{n}{\Omega} | \nabla \Delta^4 \tilde{\rho}_n | \in L^{\frac{2n}{\Omega}} \left( [0, T]; L^3 (\mathbb{T}^3) \right) \), refer to [36]. Therefore, for any \( \iota > 0 \), Chebyshev’s inequality shows this highest order derivative of \( \rho \) term goes to 0 \( \tilde{\mathbb{P}} \) almost everywhere. For \( \varphi \in C^\infty_c ([0, T); C^\infty (\mathbb{T}^3)) \),

\[
\begin{align*}
\mathbb{P} \left[ \left| \int_0^T \int_{\mathbb{T}^3} \delta \tilde{\rho}_n \Delta^9 \tilde{\rho}_n \varphi \, d x \, d t \right| > \iota \right] & \leq C \frac{E \left[ \int_0^T \int_{\mathbb{T}^3} \delta \Delta^5 \rho_n \nabla \Delta^4 (\rho_n \varphi) \, d x \, d t \right]}{\iota^r} \\
& \leq C \frac{E \left[ | \varphi |_0^T \int_{\Omega} \sqrt{\delta} | \nabla \Delta^5 \tilde{\rho}_n | \, \delta \frac{n}{\Omega} | \nabla \Delta^4 \tilde{\rho}_n | \, \delta \frac{n}{\Omega} \, d x \, d t \right]}{\iota^r} \\
& \leq \frac{C (| \varphi |_0^T \int_{\Omega} \sqrt{\delta} | \nabla \Delta^5 \tilde{\rho}_n | \, | \nabla \Delta^4 \tilde{\rho}_n | \, d x \, d t \right]}{\iota^2} \\
& \to 0 \quad \text{as} \quad \delta \to 0.
\end{align*}
\]
by Aubin-Lion’s lemma, the strong convergence of \( \bar{\rho} \) with \( \eta \nabla \rho \cdot \nabla \phi \rightarrow 0 \) for any \( \eta \nabla \rho \cdot \nabla \phi \rightarrow 0 \).

So we have
\[
\begin{align*}
\mathbb{E} \left[ \left( \int_0^T \int_T^3 \delta \bar{\rho}_n \nabla \Delta^9 \bar{\rho}_n \cdot \nabla \bar{\eta}_n \phi \left( \bar{u}_n \right) d x d t \right)^r \right] \\
\leq 3' C_n \mathbb{E} \left[ \left( \int_0^T \int_T^3 \delta \left| \Delta^5 \bar{\rho}_n \right| \left| \nabla \Delta^4 \bar{\rho}_n \right| d x d t \right)^r \right] \\
= 3' C_n \mathbb{E} \left[ \left( \frac{\delta}{\delta \rho} \int_0^T \int_T^3 \delta \left| \Delta^5 \bar{\rho}_n \right| \frac{\delta}{\delta \rho} \left| \nabla \Delta^4 \bar{\rho}_n \right| d x d t \right)^r \right] \\
\leq 3' C_n \left( \frac{\delta}{\delta \rho} \right)^r \left( \mathbb{E} \left[ \frac{\delta^2}{\delta \rho^2} \left| \Delta^5 \bar{\rho}_n \right| L_2^2 L_2^2 \right] \right)^{2r} \left( \mathbb{E} \left[ \left| \nabla \rho \Delta^4 \bar{\rho}_n \right| L_2^r L_2^r \right] \right)^{\frac{1}{2}} \left( \mathbb{E} \left[ \left| \nabla \rho \Delta^4 \bar{\rho}_n \right| L_2^r L_2^r \right] \right)^{\frac{1}{2}} \\
\leq 3' C \left( \frac{\delta}{\delta \rho} \right)^r,
\end{align*}
\]

for any \( \delta < n^{-\alpha} \), \( \alpha > 76 \), where \( C \) depends on \( r, T, \mathbb{E} \left( E(0)^r \right) \) and \( \mathbb{E} \left( \bar{E}(0)^r \right) \).

Then, we consider the limit \( \eta \rightarrow 0 \). We claim that \( \eta \int_T^3 \bar{\rho}_n^{-10} d x \rightarrow 0 \) as \( \eta \rightarrow 0 \). In fact,
\[
\left( \left\{ x \mid \rho(t, x) = 0 \right\} \right) = 0 \quad \text{for almost every } t, x, \omega, \tag{4.62}
\]

by Aubin-Lion’s lemma, the strong convergence of \( \bar{\rho}_n \) in \( L^2(0, T; L^1(T^3)) \) still holds, so \( \bar{\rho}_n^{-10} \rightarrow \rho^{-10} \) almost everywhere. So the \( \eta \int_T^3 \bar{\rho}_n^{-10} d x \) in energy will vanish as \( \eta \rightarrow 0 \). Next, we deal with \( \mathbb{E} \left[ \left( \int_0^T \eta \left( \int_T^3 \eta \bar{\rho}_n^{-10} \nabla \bar{\eta}_n \phi \left( \bar{u}_n \right) d x d t \right)^r \right] \right. \) and
\[
\begin{align*}
\mathbb{E} \left[ \left( \int_0^T \int_T^3 \eta \left( \frac{11}{20} \left| \nabla \rho \right|^2 d x d s \right)^r \right] \leq C,
\end{align*}
\]

we calculate
\[
\left| \nabla \bar{\rho}_n^{-10} \right| = 2 \left| \nabla \bar{\rho}_n \right| \bar{\rho}_n^{-11} = 2 \left| \nabla \bar{\rho}_n \right| \frac{1}{2} \bar{\rho}_n^{-10} \frac{1}{2}. \tag{4.63}
\]
By Hölder’s inequality,
\[
\eta \frac{11}{20} \left\| \rho_n^{-10^{-\frac{1}{2}}} \right\|_{L_t^1 L_x^{\frac{60}{3}}} \leq \eta^2 \left\| \rho_n^{-10} \right\|_{L_t^1 L_x^{\frac{5}{3}}} \left\| \rho_n^{-\frac{1}{2}} \right\|_{L_t^\infty L_x^{\frac{20}{3}}} \leq C,
\]
where \(C\) depends on \(r, T, \mathbb{E}[E(0)^r]\) and \(\mathbb{E}[\hat{E}(0)^r]\). So by Hölder’s inequality,
\[
\eta \frac{11}{20} \int_0^T \int_{\mathbb{T}^3} \nabla \rho_n^{-10} \, dx \, dt = \eta \frac{11}{20} \int_0^T \int_{\mathbb{T}^3} \rho_n^{-10-\frac{1}{2}} \nabla \rho_n^{-\frac{1}{2}} \, dx \, dt \tag{4.65}
\]
\[
\leq C \sup_{t \in [0,T]} \left( \int_{\mathbb{T}^3} \left\| \nabla \rho_n^{-\frac{1}{2}} \right\|^2 \, dx \right)^{\frac{1}{2}} \int_0^T \eta \frac{11}{20} \left( \int_{\mathbb{T}^3} \rho_n^{-10-\frac{1}{2}} \, dx \right)^{\frac{1}{2}} \, dt \tag{4.66}
\]
here \(C\) depends on the length of the period of the area. So \(\eta^{\frac{23}{200}} \nabla \rho_n^{-10} \in L_{T^3}^{\frac{60}{3}}(0, T; L^1(\mathbb{T}^3))\).

\[
\mathbb{E} \left[ \left( \int_0^T \int_{\mathbb{T}^3} \eta \nabla \rho_n^{-10} \nabla \varphi_n(\tilde{u}_n) \, dx \, dt \right)^r \right] \leq \mathbb{E} \left[ \left( \frac{1}{2} \left(2e(1 + n)^2 - 1\right)^\frac{1}{2} \left(1 + \ln(1 + n)\right) \int_0^T \int_{\mathbb{T}^3} \eta \nabla \rho_n^{-10} \, dx \, dt \right)^r \right] \leq \mathbb{E} \left[ \left(2e(1 + n)^2 - 1\right)^\frac{1}{2} \left(1 + \ln(1 + n)\right) \eta^{\frac{23}{200}} \left( \int_0^T \int_{\mathbb{T}^3} \eta \nabla \rho_n^{-10} \, dx \, dt \right)^r \right] \tag{4.67}
\]

for any \(\eta < n^{-\beta}, \beta > 2\frac{200}{607}, \mathbb{E} \left[ \left( \frac{1}{2} \left(2e(1 + n)^2 - 1\right)^\frac{1}{2} \left(1 + \ln(1 + n)\right) \eta^{\frac{23}{200}} \left( \int_0^T \int_{\mathbb{T}^3} \eta \nabla \rho_n^{-10} \nabla \varphi_n(\tilde{u}_n) \, dx \, dt \right)^r \right) \right] \rightarrow 0 \text{ as } n \rightarrow \infty.

With the same \(\varphi \in C^\infty_c([0, T); C^\infty(\mathbb{T}^3)),\) for any \(\iota > 0,
\[
\mathbb{E} \left[ \int_0^T \int_{\mathbb{T}^3} \eta \nabla \rho_n^{-10} \, dx \, dt \right] > \iota \leq \mathbb{E} \left[ \int_0^T \int_{\mathbb{T}^3} \eta \nabla \rho_n^{-10} \varphi \, dx \, dt \right] \leq \mathbb{E} \left[ \int_0^T \int_{\mathbb{T}^3} \eta \nabla \rho_n^{-10} \varphi \, dx \, dt \right] \leq \mathbb{E} \left[ \int_0^T \int_{\mathbb{T}^3} \eta \nabla \rho_n^{-10} \varphi \, dx \, dt \right] \leq C \eta^{\frac{\iota}{r}} \rightarrow 0 \text{ as } \eta \rightarrow 0,
\]
where \(C\) depends on \(r, T, \mathbb{E}[E(0)^r]\) and \(\mathbb{E}[\hat{E}(0)^r]\), the fact that \(\iota\) is arbitrary shows this term in equation goes to 0 \(\mathbb{P}\) almost surely.

From (2.90),
\[
\mathbb{E} \left[ \left\| \rho_n^{-1} \right\|_{L_t^\infty L_x^\infty} \right] \leq C_r \eta^{-\frac{1}{2}r} \left( \mathbb{E} \left[ r^{-1} \left\| \rho_n^{-1} \right\|_{L_t^r L_x^{10r}} \right] \right)^{\frac{1}{20r}} \delta^{-r} \left( \mathbb{E} \left[ \left( \delta \left\| \rho_n^{-1} \right\|_{L_t^r H_x^{20r}} \right) \right] \right)^{\frac{1}{20r}} \]
Lemma 4.4. For $\varpi \in (0,2)$, $\varpi$ is small, 
\[ \mathbb{E} \left[ \left( \int_{T^3} \rho (1 + |u|^2) \ln (1 + |u|^2) \, dx \right)^r \right] \]
Lemma 4.6. For system

\[ E = \int_{\mathbb{T}^3} \left( \frac{1}{2} \rho |u + \nabla \log \rho|^2 + a \int_1^\rho \frac{p(z)}{z} \, dz + r_2 \int_{\mathbb{T}^3} \log - \rho \right) \, dx, \]

so there exists a constant \( C \), such that

\[ \left( \int_{\mathbb{T}^3} \rho \left( 2 + \ln (1 + |u|^2) \right)^{\frac{2}{3}} \, dx \right)^{\frac{3}{2}} \leq C \int_{\mathbb{T}^3} \rho u^2 \, dx, \]

\( \bar{C} \) and \( C_r \) share the same meaning as in lemma 4.2.

\[ \bar{C} = \frac{1}{2} \rho |u|^2 + a \int_1^\rho \frac{p(z)}{z} \, dz + r_2 \int_{\mathbb{T}^3} \log - \rho \]}

Lemma 4.5. The pair \((\rho, u)\) is a martingale solution to

\[ \begin{cases} 
\rho_t + \text{div}(\rho u) = 0, \\
\text{d} (\rho u) + \text{div}(\rho u \otimes u) + \nabla \rho \gamma - \text{div}(\rho \partial_t u) = (-r_1 \rho |u|^2 u - r_2 u) \, dt + \rho \partial_t (\rho, u) \, dW. 
\end{cases} \] (4.73)

Proof. The weak convergence of \( \nabla \tilde{\rho}_n \), still holds in the corresponding spaces. By Vitali's convergence theorem, \( \rho \) and \( u \) satisfy the mass conservation equation, it suffices to see whether the convergence of term associated with stochastic forces holds or not in the momentum conservation equation. Since we still have the strong convergence of \( \tilde{\rho}_n \) in \( L^2 \left( 0, T; L^2 (\Omega) \right) \) and the strong convergence of \( \tilde{\rho}_n \tilde{u}_n \) in \( L^2 \left( [0, T] \times \mathbb{T}^3 \right) \), hence \( \tilde{\rho}_n \to \rho \) almost everywhere, \( \tilde{\rho}_n \tilde{u}_n \to \rho u \) almost everywhere up to a subsequence. Therefore \( \tilde{\rho}_n \partial_t (\tilde{\rho}_n, \tilde{u}_n) \to \rho \partial_t (\rho, u) \) almost everywhere.

By dominated convergence theorem we have

\[ \int_0^t \int_{\mathbb{T}^3} \left| \tilde{\rho}_n \partial_t (\tilde{\rho}_n, \tilde{u}_n) \right|^2 \, dx \, dt \to \int_0^t \int_{\mathbb{T}^3} \left| \rho \partial_t (\rho, u) \right|^2 \, dx \, dt, \quad \bar{P} \text{ almost surely.} \] (4.74)

Lemma 4.6. For system (4.73),

\[ \mathbb{E} \left[ \sup_{t \in [0,T]} E(t)^r \right] \leq C \left( t^2 + T^r + \mathbb{E} \left[ E(0)^r \right] \right) e^{C \left( t^2 + T^r \right)} \leq C \left( \mathbb{E} \left[ E(0)^r \right] + 1 \right), \] (4.75)

for any \( r > 2 \),

\[ \begin{align*}
\mathbb{E} \left[ \left( \int_0^t \int_{\mathbb{T}^3} \rho |\partial_t u|^2 \, dx \, ds + r_1 \int_0^t \int_{\mathbb{T}^3} \rho |u|^4 \, dx \, ds + r_2 \int_0^t \int_{\mathbb{T}^3} |u|^2 \, dx \, ds \right)^r \right] & \\
& \leq C(\mathbb{E} \left[ E(0)^r \right] + 1), \quad (4.76)
\end{align*} \]

here \( C \) depends on \( r, T, \) and \( \sum_{k=1}^{+\infty} f_k^2 \).

Lemma 4.7. The stochastic B-D entropy estimates become

\[ \tilde{E} = \int_{\mathbb{T}^3} \left( \frac{1}{2} \rho |u + \nabla \log \rho|^2 + a \int_1^\rho \frac{p(z)}{z} \, dz + r_2 \int_{\mathbb{T}^3} \log - \rho \right) \, dx, \]

\[ \text{and } \mathbb{E} \left[ \sup_{t \in [0,T]} \tilde{E}(t)^r \right] \leq C, \] (4.78)
\[
\mathbb{E} \left[ \left( r_1 \int_0^t \int_{\mathbb{T}^3} \rho \mathbf{u}^4 \, d\mathbf{x} \, ds \right)^r \right] \leq C, \quad \mathbb{E} \left[ \left( r_2 \int_0^t \int_{\mathbb{T}^3} |\mathbf{u}|^2 \, d\mathbf{x} \, ds \right)^r \right] \leq C, \quad (4.79)
\]

where \( C \) depends on \( \delta^{-\frac{1}{2}}, \eta^{-\frac{1}{2}}, \kappa^{-\frac{1}{2}}, r, T, \mathbb{E}[E(0)^r], \mathbb{E}[E(0)^{\kappa r}] \) and \( \mathbb{E}[E(0)^{k_i r}], i = 1, 2, 3 \) \( k_i \) are specific constants.

### 4.4. Vanishing the drag forces: \( r_1 \to 0, r_2 \to 0. \)

**Step 1:** Choose the path space, show the tightness of the laws and the convergence.

Actually, the martingale solution in this section is still concerned with \( r_0 \) and \( r_1 \), we denote it as \( \tilde{\rho}_\cdot^\frac{1}{2} \tilde{\mathbf{u}}_\cdot^\frac{1}{2} \) for convenience. By the Mellet-Vasseur’s inequality, we have the strong convergence of \( \tilde{\rho}_\cdot^\frac{1}{2} \tilde{\mathbf{u}}_\cdot^\frac{1}{2} \) in \( L^2 \left( [0, T] \times \mathbb{T}^3 \right) \) \( \mathbb{B} \). We study the convergence of \( \tilde{\rho}_\cdot \).

\[
\mathbb{E} \left[ \left\| \tilde{\rho}_\cdot^\frac{1}{2} \tilde{\rho}_\cdot^\frac{1}{2} \text{div} \tilde{\mathbf{u}}_\cdot \right\|_{L_t^2 L_x^2}^r \right] \leq \mathbb{E} \left[ \left\| \tilde{\rho}_\cdot^\frac{1}{2} \text{div} \tilde{\mathbf{u}}_\cdot \right\|_{L_t^\infty L_x^2}^r \right] \leq C, \quad (4.80)
\]

together with

\[
\mathbb{E} \left[ \left\| \nabla \tilde{\rho}_\cdot^\frac{1}{2} \left( \tilde{\rho}_\cdot^\frac{1}{2} \tilde{\mathbf{u}}_\cdot \right) \right\|_{L_t^2 L_x^4}^r \right] \leq \mathbb{E} \left[ \left\| \nabla \tilde{\rho}_\cdot^\frac{1}{2} \right\|_{L_t^\infty L_x^2}^r \right] \leq C, \quad (4.81)
\]

implies

\[
\mathbb{E} \left[ \left\| (\rho_\cdot)_t \right\|_{L_t^2 L_x^1}^r \right] \leq C, \quad (4.82)
\]

uniformly in \( r_1, r_2 \).

We calculate that

\[
|\nabla \tilde{\rho}_\cdot| = 2 \left| \nabla \tilde{\rho}_\cdot^\frac{1}{2} \right| \tilde{\rho}_\cdot^\frac{1}{2}, \quad (4.83)
\]

so \( \tilde{\rho}_\cdot \) is bounded in \( L^\infty \left( 0, T; W^{1,1} \left( \mathbb{T}^3 \right) \right) \), by Aubin-Lion’s lemma we know that the strong convergence of \( \tilde{\rho}_\cdot \) in \( L^2 \left( 0, T; \frac{L^4}{2} \left( \mathbb{T}^3 \right) \right) \) still holds.

Take the space \( \mathcal{X}_6 = \mathcal{X}_{\rho_0} \times \mathcal{X}_{\frac{1}{2} \tilde{\mathbf{u}}_0} \times \mathcal{X}_{\rho} \times \mathcal{X}_{\frac{1}{2} \tilde{\mathbf{u}}} \times \mathcal{X}_W \), where \( \mathcal{X}_{\rho_0} = L^2 \left( \mathbb{T}^3 \right) \), \( \mathcal{X}_{\frac{1}{2} \tilde{\mathbf{u}}_0} = L^2 \left( \mathbb{T}^3 \right) \), \( \mathcal{X}_\rho = L^2 \left( [0, T] \times \mathbb{T}^3 \right), \mathcal{X}_{\frac{1}{2} \tilde{\mathbf{u}}} = L^2 \left( [0, T] \times \mathbb{T}^3 \right), \mathcal{X}_W = C([0, T]; \mathfrak{S}) \). Similarly, we have the following convergence.

**Proposition 4.4.** There exists a family of \( \mathcal{X}_6 \)-valued Borel measurable random variables

\[
\left\{ \tilde{\rho}_0^\frac{1}{2}, \tilde{\rho}_0^\frac{1}{2} \tilde{\mathbf{u}}_0, \tilde{\rho}_0^\frac{1}{2} \tilde{\mathbf{u}}_\cdot, \tilde{\rho}_\cdot^\frac{1}{2} \tilde{\mathbf{u}}_\cdot, \tilde{\rho}_\cdot^\frac{1}{2} \tilde{\mathbf{u}}_\cdot, \tilde{\rho}_\cdot^\frac{1}{2} \tilde{\mathbf{W}} \right\}
\]

and \( \left\{ \rho_0, \rho_0^\frac{1}{2} \mathbf{u}_0, \rho, \rho^\frac{1}{2} \mathbf{u}, \mathbf{W} \right\} \), defined on a new complete probability space \( (\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}) \), such that (up to a sequence) for \( \gamma > \frac{2}{7} \)

1. \( \mathcal{L} \left[ \tilde{\rho}_0^\frac{1}{2} \tilde{\rho}_0^\frac{1}{2} \tilde{\mathbf{u}}_0, \tilde{\rho}_\cdot^\frac{1}{2} \tilde{\mathbf{u}}_\cdot, \tilde{\rho}_\cdot^\frac{1}{2} \tilde{\mathbf{W}} \right] \) and \( \mathcal{L} \left[ \rho_0, \rho_0^\frac{1}{2} \mathbf{u}_0, \rho, \rho^\frac{1}{2} \mathbf{u}, \mathbf{W} \right] \)

2. \( \mathcal{L} \left[ \rho_0, \rho_0^\frac{1}{2} \mathbf{u}_0, \rho, \rho^\frac{1}{2} \mathbf{u}, \mathbf{W} \right] \) on \( \mathcal{X}_6 \) is a Radon measure.

3. Random variables \( \left\{ \tilde{\rho}_0^\frac{1}{2} \tilde{\rho}_0^\frac{1}{2} \tilde{\mathbf{u}}_0, \tilde{\rho}_\cdot^\frac{1}{2} \tilde{\mathbf{u}}_\cdot, \tilde{\rho}_\cdot^\frac{1}{2} \tilde{\mathbf{W}} \right\} \) converges \( \tilde{\mathbb{P}} \) almost surely to \( \left\{ \rho_0, \rho_0^\frac{1}{2} \mathbf{u}_0, \rho, \rho^\frac{1}{2} \mathbf{u}, \mathbf{W} \right\} \) in the topology of \( \mathcal{X}_6 \), i.e.

\[
\tilde{\rho}_0^\frac{1}{2} \tilde{\mathbf{u}}_0^\frac{1}{2} \text{in } L^\gamma \left( \mathbb{T}^3 \right); \quad \tilde{\rho}_\cdot^\frac{1}{2} \tilde{\mathbf{u}}_\cdot^\frac{1}{2} \text{in } L^\gamma \left( \mathbb{T}^3 \right);
\]
\[ \bar{\rho}_\tau \rightarrow \rho \text{ in } L^2 \left( [0, T); L^2 \left( \mathbb{T}^3 \right) \right); \]
\[ \bar{\rho}_\tau \rightarrow \rho \text{ in } L^2 \left( [0, T] \times \mathbb{T}^3 \right); \]
\[ \bar{\rho}^{1/2}_\tau \bar{u}_\tau \rightarrow \rho^{1/2} u \text{ in } L^2 \left( [0, T); L^2 \left( \mathbb{T}^3 \right) \right); \]
\[ \bar{W}_\tau \rightarrow W \text{ in } C \left( [0, T]; \mathcal{F} \right); \]
\[ \bar{P} \text{ almost surely.} \]

**Step 2:** The pair \((\rho, u)\) is a global martingale solution to our system \([1.23]\), which finally proves our main theorem.

**Lemma 4.8.** The pair \((\rho, u)\) is a global-in-time martingale solution to system \([1.23]\).

**Proof.** The proof is similar to the last section. It suffices to prove that terms involving \(r_1 \bar{\rho}_\tau |\bar{u}_\tau|^2 \bar{u}_\tau\) and \(r_2 \bar{u}_\tau\) tend to zero in distribution sense as \(r_2 \rightarrow 0\) and \(r_1 \rightarrow 0 \bar{P} \) almost surely.

Let \(H\) be any test function in \(C^\infty \left( [0, T] \times \mathbb{T}^3 \right)\), actually in \(L^2 \left( [0, T] \times \mathbb{T}^3 \right) \cap L^2 \left( [0, T]; L^\infty \left( \mathbb{T}^3 \right) \right)\) is enough. For \(r_1 \bar{\rho}_\tau |\bar{u}_\tau|^2 \bar{u}_\tau\),

\[ E \left[ \left( \int_0^T \int_{\mathbb{T}^3} r_1 \bar{\rho}_\tau |\bar{u}_\tau|^2 \bar{u}_\tau |H| \, d x \, d t \right)^r \right] \]
\[ \leq E \left[ \left( r_1^\frac{1}{r} \int_0^T \int_{\mathbb{T}^3} \frac{1}{r} \bar{\rho}_\tau |\bar{u}_\tau|^2 \bar{u}_\tau \left\| H \right\|_{L^2 L^\infty} \right)^r \right] \rightarrow 0 \]
as \(r_1 \rightarrow 0\). The term \(r_2 \bar{u}_\tau\)

\[ E \left[ \left( \int_0^T \int_{\mathbb{T}^3} r_2 \bar{u}_\tau |H| \, d x \, d t \right)^r \right] \leq E \left[ \left( \int_0^T \int_{\mathbb{T}^3} \frac{1}{r} r_2 \bar{u}_\tau \left\| H \right\|_{L^2 L^\infty} \right)^r \right] \]
\[ \leq E \left[ \left( r_2^\frac{1}{r} \int_0^T \int_{\mathbb{T}^3} \bar{u}_\tau \left\| H \right\|_{L^2 L^2} \right)^r \right] \rightarrow 0 \]
as \(r_2 \rightarrow 0\). Letting \(r_1 \rightarrow 0\) and \(r_2 \rightarrow 0\), one obtains that

\[ \int_{\mathbb{T}^3} \rho u \cdot H \, d x \bigg|_{t=0}^{t=T} + \int_0^T \int_{\mathbb{T}^3} \rho u |H| \, d x \, d t - \int_0^T \int_{\mathbb{T}^3} \rho u \otimes u : \nabla H \, d x \, d t \]
\[ - \int_0^T \int_{\mathbb{T}^3} \rho' \text{div} H \, d x \, d t - \int_0^T \int_{\mathbb{T}^3} \rho \nabla u : \nabla H \, d x \, d t \]
\[ = \int_0^T \int_{\mathbb{T}^3} \rho \mathbb{F}(\rho, u) H \, d x \, d W, \]
\[ \bar{P} \text{ almost surely.} \]

5. **Appendix: Some preliminaries in stochastic analysis**

In this section, for the convenience of the reader, we list the basic measure theory and theorems in stochastic analysis. Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a stochastic basis. A filtration \((\mathcal{F}_t)_{t \in \mathbb{T}}\) is a family of \(\sigma\)-algebras on \(\Omega\) indexed by \(\mathbb{T}\) such that \(\mathcal{F}_s \subseteq \mathcal{F}_t \subseteq \mathcal{F}, s \leq t, s, t \in \mathbb{T}\). A measurable space together with a filtration is called a filtered space.

1. (Adapted stochastic process.) A stochastic process \(X\) is \(\mathcal{F}\)-adapted if \(X_t\) is \(\mathcal{F}_t\)-measurable for every \(t \in \mathbb{T}\);

2. (Wiener Process.) An \(\mathbb{R}^m\) -valued stochastic process \(W\) is called an \((\mathcal{F}_t)\)-adapted Wiener process, provided:

   a. \(W\) is \((\mathcal{F}_t)\)-adapted;
(b) \( W(0) = 0 \), \( \mathbb{P} \) almost surely;
(c) \( W \) has continuous trajectories: \( t \mapsto W(t) \) is continuous \( \mathbb{P} \) almost surely;
(d) \( W \) has independent increments: \( W(t) - W(s) \) is independent of \( \mathcal{F}_s \) for all \( 0 \leq s \leq t < \infty \).

(3) (Martingale.) A process \( X \) is called a martingale if
- \( X \) is adapted;
- \( X_t \) is integrable for every \( t \in \mathbb{T} \);
- \( X_s = E[X_t | \mathcal{F}_s] \) whenever \( s \leq t, s, t \in \mathbb{T} \).

(4) (Stopping time.) On \( (\Omega, \mathcal{F}, \mathbb{P}) \), a random time is a measurable mapping \( \tau : \Omega \to \mathbb{T} \cup \{\infty\} \). A random time is a stopping time if \( \{\tau \leq t\} \in \mathcal{F}_t \) for every \( t \in \mathbb{T} \). For a process \( X \) and a subset \( V \) of the state space we define the hitting time of \( X \) in \( V \) as
\[
\tau_V(\omega) = \inf \{t \in \mathbb{T} : X_t(\omega) \in V \}
\]
If \( X \) is a continuous adapted process and \( V \) is closed, then \( \tau_V \) is a stopping time.

(5) (Progressive measurability.) Let \( (\Omega, \mathcal{F}, \mathbb{P}) \) be a filtered space. Stochastic process \( X \) is progressively measurable or simply progressively measurable, if for \( \omega \in \Omega \), \( (\omega, s) \mapsto X(s, \omega), s \leq t \) is \( \mathcal{F}_s \otimes \mathcal{B}(\mathbb{T} \cap [0, t]) \)-measurable for every \( t \in \mathbb{T} \). In particular, progressively measurable processes are automatically adapted. The reciprocal is true if the paths of the process are regular enough. Let \( X \) be right-continuous or left-continuous \( \mathcal{F} \)-adapted process. Then \( X \) is progressively measurable. Any measurable and adapted process admits a progressive modification.

(6) (Stochastic integral.) By \( \mathcal{M}_c^2 \) we denote the space of square integrable continuous martingales \( M \) such that \( M_0 = 0 \). For \( M \) in \( \mathcal{M}_c^2 \), we denote by \( \langle M \rangle \) the continuous natural process of bounded variation in the Doob-Meyer decomposition of the sub-martingale \( M^2 \). That is, \( M^2 = \text{martingale} + \langle M \rangle \). We call \( \langle M \rangle \) the quadratic variations of \( M \). Let \( \mathcal{L}^2(M) \) be the space of progressively measurable process \( H \) such that \( E \left[ \int_0^t H^2 \mathbb{d} \langle M \rangle \right] < +\infty \). There exists a unique continuous linear functional \( \int \cdot \mathbb{d} M : \mathcal{L}^2(M) \to \mathcal{M}_c^2 \) coinciding with the elementary stochastic integral on \( \mathcal{S} \) and for which holds
\[
E \left[ \left( \int_0^t H \mathbb{d} M \right)^2 \right] = E \left[ \int_0^t H^2 \mathbb{d} \langle M \rangle \right].
\]
Furthermore, the following properties holds
- Linearity: \( \int (\alpha H + \beta G) \mathbb{d} M = \alpha \int H \mathbb{d} M + \beta \int G \mathbb{d} M \) for \( \alpha \) and \( \beta \) constants;
- Stopping property: \( \int 1_{\{\tau \leq t\}} H \mathbb{d} M = \int H \mathbb{d} M^\tau = \int_0^{\tau^\wedge \tau} H \mathbb{d} M \);
- Itô-Isometry: for every \( t \),
\[
E \left[ \left( \int_0^t H \mathbb{d} M \right)^2 \right] = E \left[ \int_0^t H^2 \mathbb{d} \langle M \rangle \right].
\]

(7) (Local Martingale.) We call an increasing sequence of stopping times \( \{\tau^n\} \) such that \( \tau^n \nearrow \infty \) a localizing sequence of stopping times. Let \( \mathcal{M}_{c}^{\loc} \) be the set of adapted continuous processes such that there exists a localizing sequence of stopping times \( \{\tau^n\} \) with \( M^\tau^n \in \mathcal{M}_c^2 \). Given such a process \( M \in \mathcal{M}_{c}^{\loc} \), with corresponding localizing sequence of stopping times \( \{\tau^n\} \), we can define \( \langle M^\tau^n \rangle \) for every \( n \) and it holds \( \langle M^\tau^n \rangle = \langle M^{\tau^{n+1}} \rangle \) on \( [0, \tau^n] \). Hence we can define \( \langle M \rangle \). Following the same idea, we define \( \mathcal{L}^{\loc}(M) \) for
$M \in L^{loc}$ as the set of progressive measurable processes such that $\int_0^t H^2 \, d\langle M \rangle < \infty$, $\mathbb{P}$-almost surely for every $t \in T$ that allows to define locally the stochastic integral.

(8) (Semi-Martingale.) A semi-martingale is a process $X = X_0 + M + A$, where $A$ is the difference of two increasing continuous processes and $M$ is a continuous local martingale. For two semi-martingales $X = X_0 + M + A$ and $Y = Y_0 + N + B$, we define

- the quadratic variations: $\langle X \rangle \triangleq \langle M \rangle$;
- the co-variations: $\langle X, Y \rangle \triangleq \langle (X + Y) - (X - Y) \rangle / 4 = \langle (M + N) - (M - N) \rangle / 4 = \langle M, N \rangle$.

For every progressive process $H$ such that $\int_0^t |H| \, d\langle A \rangle < \infty$ and $\int_0^t |H|^2 \, d\langle M \rangle < \infty$ almost surely for every $t$, we can therefore define

$$\int H \, dX \triangleq \int H \, dM + \int H \, dA.$$

(9) (Radon measure.) Let $\mathcal{M}$ be a measure on the $\sigma$-algebra with its element belonging to a Hausdorff topological space $S$. The measure $\mathcal{M}$ is called inner regular or tight if for any open set $U$,

$$\mathcal{M}(U) = \sup_{K \subseteq U} \{ \mathcal{M}(K) : K \text{ is compact} \}.$$

The measure $\mathcal{M}$ is called outer regular if for any Borel set $B$,

$$\mathcal{M}(B) = \inf_{B \supseteq U} \{ \mathcal{M}(U) : U \text{ is open} \}.$$

The measure $\mathcal{M}$ is called locally finite if every point of $S$ has a neighborhood $U$ for which $\mathcal{M}(U)$ is finite. The measure $\mathcal{M}$ is called a Radon measure if it is inner regular, outer regular and locally finite. A measure on $\mathbb{R}$ is a Radon measure if and only if it is a locally finite Borel measure. Since a probability measure is globally finite, and hence a locally finite measure, every probability measure on a Radon space is also a Radon measure. In particular a separable complete metric space $(S, \mathcal{M})$ is a Radon space.

(10) (Law of random variable.) Let $(S, \mathcal{S})$ be a measurable space. An $S$-valued random variable is a measurable mapping $U : (\Omega, \mathcal{F}) \to (X, \mathcal{S})$. We denote by $\mathcal{L}[U]$ or also $L_S[U]$ the law of $U$ on $S$, that is, $\mathcal{L}[U]$ is the pushforward probability measure on $(S, \mathcal{S})$ given by

$$\mathcal{L}[U](A) = \mathbb{P}(U \in A), \quad A \in \mathcal{S}. \quad (5.1)$$

In measure theory, a pushforward measure is obtained by using a measurable function transferring a measure from one measurable space to another space.

In the paper, the following theorems are employed, we stated them in this section.

(1) (Jakubowski’s extension of Skorokhod’s representation theorem.) Let $(S, \tau)$ be a sub-Polish space and let $\mathcal{S}$ be the $\sigma$-field generated by $\{f_n : n \in \mathbb{N}\}$. If $(\mu_n)_{n \in \mathbb{N}}$ is a tight sequence of probability measures on $(S, \mathcal{S})$, then there exists a subsequence $(n_k)$ and $S$-valued Borel measurable random variables $(U_k)_{k \in \mathbb{N}}$ and $U$ defined on the standard probability space $([0, 1], \mathcal{B}([0, 1]), \mathbb{P})$, such that $\mu_{n_k}$ is the law of $U_k$ and $U_k(\omega)$ converges to $U(\omega)$ in $S$ for every $\omega \in [0, 1]$. Moreover, the law of $U$ is a Radon measure.
Let $M$ be a continuous local martingale. Let $M^* = \max_{0 \leq s \leq t} |M(s)|$, for any $m > 0$, there exist constants $K^m$ and $K_m$ such that
\[
K_m \mathbb{E} [(\langle M \rangle_T)^m] \leq \mathbb{E} [(M^*_T)^{2m}] \leq K^m \mathbb{E} [(\langle M \rangle_T)^m],
\]
for every stopping time $T$. For $m \geq 1$, $K^m = \left( \frac{2m}{2^m - 1} \right)^{2m(2m - 2) / 2}$, which is equivalent to $e^m$ as $m \to \infty$.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $\bar{X}$ a process on $[0, T]$. Suppose that
\[
\mathbb{E} \left[ |\bar{X}_t - \bar{X}_s|^\alpha \right] \leq C|t - s|^{1 + \beta},
\]
for every $s < t \leq T$ and some strictly positive constants $\alpha, \beta$ and $C$. Then $\bar{X}$ admits a continuous modification $X$, $\mathbb{P} [X_t = \bar{X}_t] = 1$ for every $t$, $X$ is locally Hölder continuous for every exponent $0 < \gamma < \frac{\beta}{\alpha}$, that is
\[
\mathbb{P} \left\{ \omega : \sum_{0 < t - s < h(\omega), t, s \leq T} \frac{|X_t(\omega) - X_s(\omega)|}{|t - s|^{\gamma}} \leq \delta \right\} = 1,
\]
where $h$ is an almost surely strictly positive random variable and $\delta > 0$.

Let $X$ and $Y$ be semi-martingales, then it holds
\[
XY = X_0Y_0 + \int X \, dY + \int Y \, dX + \int d\langle X, Y \rangle
= X_0Y_0 + \int X \, dY + \int Y \, dX + \langle X, Y \rangle,
\]
$\langle X, Y \rangle$ means the co-variation of $X$ and $Y$. In particular, this formula can be used to calculate integration by part. More precisely, if $X$ is continuous, then $\langle dX, dY \rangle = 0$ and it holds $\int X \, dY = -\int Y \, dX + XY - X_0Y_0$.

References

[1] D. Breit and E. Feireisl. Stochastic Navier–Stokes–Fourier equations. Indiana Univ. Math. J., 69(3):911–975, 2020.
[2] D. Breit, E. Feireisl, and M. Hofmanová. Incompressible limit for compressible fluids with stochastic forcing. Arch. Ration. Mech. Anal., 222(2):895–926, 2016.
[3] D. Breit, E. Feireisl, and M. Hofmanová. Compressible fluids driven by stochastic forcing: The relative energy inequality and applications. Comm. Math. Phys., 350(2):443–473, 2017.
[4] D. Breit, E. Feireisl, and M. Hofmanová. Stochastically forced compressible fluid flows. Walter de Gruyter GmbH, Berlin, 2018.
[5] D. Breit, E. Feireisl, M. Hofmanová, and B. Malowski. Stationary solutions to the compressible navier-stokes system driven by stochastic forcing. Probab. Theory Relat. Fields, 174(3–4):981–1032, 2016.
[6] D. Breit and M. Hofmanová. Stochastic Navier–Stokes equations for compressible fluids. Indiana Univ. Math. J., 65:1183–1250, 2016.
[7] D. Bresch and B. Desjardins. On the construction of approximate solutions for the 2d viscous shallow water model and for compressible Navier–Stokes models. J. Math. Pures. Appl., 86(4):362–368, 2006.
[8] Z. Brzeźniak, M. Capiński, and F. Flandoli. Stochastic navier-stokes equations with multiplicative noise. *Stoch. Anal. Appl.*, 10(5):523–532, 1992.

[9] Z. Brzeźniak, G. Dhariwal, and E. Zatorska. Sequential stability of weak martingale solutions to stochastic compressible navier-stokes equations with viscosity vanishing on vacuum, https://arxiv.org/abs/2201.02070. 2022.

[10] D. L. Burkholder, B. J. Davis, and R. F. Gundy. Berkeley symposium on mathematical statistics and probability: Integral inequalities for convex functions of operators on martingales. 2:223–240, 1945-1971.

[11] J. Dong. A note on barotropic compressible quantum Navier–Stokes equations. *Nonlinear Anal.*, 73(4):854–856, 2010.

[12] E. Feireisl and A. Novotný. *Singular Limits in Thermodynamics of Viscous Fluids*. Springer Science and Business Media, Berlin, 2009.

[13] E. Feireisl, A. Novotný, and H. Petzeltová. On the existence of globally defined weak solutions to the Navier–Stokes equations. *J. Math. Fluid Mech.*, 3:358–392, 2001.

[14] M. Gisclon and L. Violet. About the barotropic compressible quantum Navier–Stokes equations. *Nonlinear Anal.*, 128:106–121, 2015.

[15] L. Grafakos. *Classical Fourier analysis*. Springer-Verlag, New York, 2008.

[16] D. Hoff. Spherically symmetric solutions of the Navier–Stokes equations for compressible isothermal flow with large discontinuous initial data. *Indiana U. Math. J.*, 41:1225–1302, 1992.

[17] D. Hoff. Strong convergence to global solutions for multidimensional flows of compressible, viscous fluids with polytropic equations of state and discontinuous initial data. *Arch. Rational Mech. Anal.*, 132:1–14, 1995.

[18] X. Hu. Weak solutions for compressible isentropic Navier–Stokes equations in dimensions three. *Arch. Ration. Mech. Anal.*, 242, 2021.

[19] K. Itô. Stochastic integral. *Proc. Imp. Acad. Tokyo*, 20:519–524, 1944.

[20] A. Jakubowski. Short communication: The almost sure Skorokhod representation for subsequences in nonmetric spaces. *Theor. Probab. Appl.*, 42(1):167–174, 1998.

[21] F. Jiang. A remark on weak solutions to the barotropic compressible quantum Navier–Stokes equations. *Nonlinear Anal.*, 12:1733–1735, 2011.

[22] S. Jiang and P. Zhang. On spherically symmetric solutions of the compressible isentropic Navier–Stokes equations. *Commun. Math. Phys.*, 251:559–581, 2001.

[23] A. Jüngel. Global weak solutions to compressible Navier–Stokes equations for quantum fluids. *SIAM J. Math. Anal.*, 42(3):1025–1045, 2010.

[24] I. Karatzas and S. Shreve. *Brownian motion and stochastic calculus*. Springer-Verlag, New York, 1988.

[25] A. V. Kazhikhov and V. A. Vaigant. On the existence of global solutions to two-dimensional Navier–Stokes equations of a compressible viscous fluid (in Russian). *Doklady Mathematics*, 3:897–900, 1977.

[26] V. Kazhikhov and V. V. Shelukhin. Unique global solution with respect to time of initial–boundary–value problems for one-dimensional equations of a viscous gas. *J. Appl. Math. Mech.*, 41:273–282, 1977.

[27] J. Li and Z. Xin. Global existence of weak solutions to the barotropic compressible Navier–Stokes flows with degenerate viscosities. https://arxiv.org/abs/1504.06826, 2015.
[28] T. Li and T. Qin. *Physics and partial differential equations, Vol.2*. SIAM, and Higher Education Press, Philadelphia, 2012.

[29] P. L. Lions. *Mathematical topics in fluid dynamics: Vol.2*. Oxford Science Publication, Oxford, 1998.

[30] A. Mellet and A. Vasseur. On the barotropic compressible Navier–Stokes equations. *Commun. Partial. Differ. Equ.*, 32(3):431–452, 2007.

[31] Z. Qian and J. Ying. *An introduction to stochastic analysis*. Fudan Press, Shanghai, 2017.

[32] A. V. Skorokhod. Limit theorems for stochastic processes with independent increments. *Theory Probab. its Appl.*, 2:138–171, 1957.

[33] S. Smith. Random perturbations of viscous, compressible fluids: Global existence of weak solutions. *SIAM J. Math. Anal.*, 49(6):4521–4578, 2017.

[34] E. Tornatore. Global solution of bi-dimensional stochastic equation for a viscous gas. *Nonlinear Differ. Equ. Appl.*, 7:343–360, 12, 2000.

[35] A. Vasseur and C. Yu. Existence of global weak solutions for three-dimensional degenerate compressible Navier–Stokes equations. *Invent. Math.*, 206:935–974, 2016.

[36] A. Vasseur and C. Yu. Global weak solutions to the compressible quantum Navier–Stokes with damping. *SIAM J. Math. Anal.*, 48(2):1489–1511, 2016.

[37] D. Wang and H. Wang. Global existence of martingale solutions to the three-dimensional stochastic compressible Navier–Stokes equations. *Differ. Integral Equ.*, 28(11-12):1105–1154, 2015.

[38] E. Zatorska. On the flow of chemically reacting gaseous mixture. *J. Differ. Equ.*, 253:3471–3500, 2012.

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