STANDARD BASES FOR AFFINE PARABOLIC MODULES AND
NONSYMETRIC MACDONALD POLYNOMIALS

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Abstract. We establish a connection between (degenerate) nonsymmetric
Macdonald polynomials and standard bases and dual standard bases of max-
imal parabolic modules of affine Hecke algebras. Along the way we prove a
(weak) polynomiality result for coefficients of symmetric and nonsymmetric
Macdonald polynomials.

Introduction

The symmetric Macdonald polynomials $P_\lambda(q,t)$ are a family of Weyl group
invariant functions depending rationally on parameters $q$ and $t = (t_s, t_t)$, which are
associated to any finite, irreducible root system $\tilde{R}$ and are indexed by the anti–
dominant elements of the weight lattice of $\tilde{R}$. Introduced originally for root sys-
tems of type $A$ as a common generalization of Hall–Littlewood and Jack symmetric
functions it was quickly realized that they have deep properties essentially rooted
in two classical representation–theoretical contexts: the theory of zonal spheri-
cal functions for real (Gel’fand, Harish–Chandra) and $p$–adic (Satake, Macdonald,
Matsumoto) reductive groups. In a more recent development [33, 9], the symmet-
cric Macdonald polynomials were also connected with the representation theory of
affine Kac–Moody groups.

The nonsymmetric Macdonald polynomials $E_\lambda(q,t)$ (indexed now by the full
weight lattice) are of more recent vintage. They were introduced by Heckman,
Opdam [30] (for $t = q^k$ and $q \rightarrow 1$), Macdonald [27] (for $t = q^k$), Cherednik
[2] (general case, reduced root systems) and Sahi [32] (nonreduced root systems).
They turned out to be a crucial tool in all the recent developments in the theory
of orthogonal polynomials, the related combinatorics and the representation the-
ory of double affine Hecke algebras (see, for example [4]). However, they do not
seem to fit easily in a classical representation-theoretical framework, the main ob-
stacle being precisely their non-invariance under the Weyl group. The first hint at

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their representation-theoretical nature came from [33] (type A), [9] (general type): $E_\lambda(q, \infty)$ are Demazure characters of basic representations of affine Kac–Moody groups. Furthermore, $E_\lambda(\infty, \infty)$ are Demazure characters of irreducible representations of simple algebraic groups [9], and $E_\lambda(\infty, t)$ are, for specific values of $t$, matrix coefficients for unramified principal series representations of simple $p$–adic groups [11].

The goal of this paper is add to this list another context in which the nonsymmetric polynomials can be interpreted naturally: the Kazhdan–Lusztig theory. In the symmetric case the connection is well-known: the limits $P_\lambda(\infty, t)$ of the symmetric Macdonald polynomials are (via the Satake transform) the standard basis of the corresponding spherical Hecke algebra. Our main result gives a similar interpretation for the same limit of the nonsymmetric polynomials: they form the standard basis of the maximal parabolic module of the corresponding affine Hecke algebra. It should be noted that the Satake transforms of Kazhdan–Lusztig bases of spherical Hecke algebras are irreducible Weyl characters, a fact which follows almost immediately from the knowledge of the standard bases [13, 22] (see also Theorem 4.10). Similar explicit formulas for Kazhdan-Lusztig bases of maximal parabolic modules are not immediately obtained from similar information, but seem to require new ideas. Conjecture 4.11 gives some indication of what is expected in this situation.

In brief, the content of the paper is the following. Section 1 contains well-known combinatorial properties of affine root systems and their Weyl groups. The main result of Section 2 (stated as Theorem 2.15) is a polynomiality result for certain normalizations of Macdonald polynomials (the normalization factor $e_\lambda$ is a product of factors of the form $(1 - q^a t^b)$ with $a, b$ negative integers).

**Theorem.** If the root system $\hat{R}$ is reduced then,

1. For any weight $\lambda$, the coefficients of $e_\lambda E_\lambda(q, t)$ are polynomials in $q^{-1}, t_s^{-1}, t_\ell^{-1}$ with integer coefficients.
2. For any anti–dominant weight $\lambda$, the coefficients of $e_\lambda P_\lambda(q, t)$ are polynomials in $q^{-1}, t_s^{-1}, t_\ell^{-1}$ with integer coefficients.

If the root system $\hat{R}$ is nonreduced, then

3. For any weight $\lambda$, the coefficients of $e_\lambda E_\lambda(q, t)$ are polynomials in $q^{-1}, t_0^{-1}, t_{t_01}^{-1}, t_{t_01}^{-\frac{1}{2}}, t_{t_03}^{-\frac{1}{2}}, t_{t_01}, t_{t_02}^{-\frac{1}{2}}, t_{t_01}^{-\frac{1}{2}}, t_{t_03}^{-\frac{1}{2}}$ with integer coefficients.

In general, the result improves on what was previously known about the nature of the coefficients [3, Corollary 5.3] (Laurent polynomials in $q, t$), [4] Section 2.3 (polynomials in $t^{-1}$, Laurent polynomials in $q$), but it is still far from being optimal since, as observed in many cases, the normalization factor can be further trimmed down without altering the polynomiality of the coefficients. In fact, in type $A$, stronger results are known for both symmetric and nonsymmetric polynomials [16].
The main technical idea of the proof is to use two affine intertwiners (one dependent of $q$, the other independent) in conjunction. The stronger result in type $A$ was handled similarly taking also advantage of the stability of the relevant polynomials in that case.

Section 3 is concerned with the limit $q \to \infty$ of nonsymmetric polynomials. The results have been already used in [11] to establish a connection between nonsymmetric Macdonald polynomials and matrix coefficients of unramified principal series for reductive $p$-adic groups. Section 4 recalls the basic (maximal parabolic) Kazhdan-Lusztig theory for affine Hecke algebras (in its multi-parameter version [23]) and explains the connection with the Cherednik-Macdonald theory. Our main result (stated as Theorem 4.8) is the following

**Theorem.** The basis $\{\tilde{E}_\lambda(q,t)\}_{\lambda \in P}$ of the parabolic module of the affine Hecke algebra $H^\vee$ is invariant under the Kazhdan–Lusztig involution. Moreover,

1. $\{\tilde{E}_\lambda(\infty,t)\}_{\lambda \in P}$ is the standard basis;
2. $\{\tilde{E}_\lambda(0,t)\}_{\lambda \in P}$ is the dual standard basis.

In type $A$, the result is essentially contained in [16, Corollary 5.3] (see also [17]).

Finally, Section 5 is examining the interplay between the case when all parameters approach infinity and the case when all parameters approach zero as well as a new geometric interpretation (Theorem 5.9) for the polynomials $E_\lambda(0,0)$.

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1. **Preliminaries**

1.1. **Affine root systems.** Let $\tilde{R} \subset \tilde{h}^*$ be a finite, irreducible, not necessarily reduced, root system of rank $n$, and let $\tilde{R}^\vee \subset \tilde{h}$ be the dual root system. We denote by $\{\alpha_i\}_{1 \leq i \leq n}$ a basis of $\tilde{R}$ (whose elements will be called simple roots); the corresponding elements $\{\alpha_i^\vee\}_{1 \leq i \leq n}$ of $\tilde{R}^\vee$ will be called simple coroots. If the root system is nonreduced, let us arrange that $\alpha_n$ is the unique simple root such that $2\alpha_n$ is also a root. Throughout the paper a special role will be played by the root $\theta$, which is defined as the highest short root in $\tilde{R}$ if the root system is reduced, or as the highest root if the root system is nonreduced.

The choice of basis determines a subset $\tilde{R}^+$ of $\tilde{R}$ (positive roots); with the notation $\tilde{R}^- := \tilde{R}^+ \cup \tilde{R}^-$ we have $\tilde{R} = \tilde{R}^+ \cup \tilde{R}^-$. As usual, $\tilde{Q} = \oplus_{i=1}^n \mathbb{Z} \alpha_i$ denotes the root lattice of $\tilde{R}$. Let $\{\lambda_i\}_{1 \leq i \leq n}$ and $\{\lambda_i^\vee\}_{1 \leq i \leq n}$ be the fundamental weights, respectively the fundamental coweights associated to $\tilde{R}^+$, and denote by $P = \oplus_{i=1}^n \mathbb{Z} \lambda_i$ the
weight lattice. An element of \( P \) will be called dominant if it is a linear combination of the fundamental weights with non-negative integer coefficients. Similarly an anti-dominant weight is a linear combination of the fundamental weights with non-positive integer coefficients.

If \( \mathfrak{r} \) is a reduced root system, let \( \mathfrak{g} \) be a simple complex Lie algebra such that \( \mathfrak{h} \) is a Cartan subalgebra and the associated root system is \( \mathfrak{r} \). Also, let \( \mathfrak{b} \supset \mathfrak{h} \) be the Borel subalgebra determined by \( \mathfrak{r}^+ \) and let \( \mathfrak{b}^- \) be the opposite Borel subalgebra. The simply connected complex algebraic group with Lie algebra \( \mathfrak{g} \) is denoted by \( G \) and \( T, B, \) and \( B^- \) denote the subgroups corresponding to \( \mathfrak{h}, \mathfrak{b} \) and \( \mathfrak{b}^- \), respectively.

The real vector space \( \mathfrak{h}^* \) has a canonical scalar product \((\cdot, \cdot)\) which we normalize such that it gives square length 2 to the short roots in \( \mathfrak{r} \) (if there is only one root length we consider all roots to be short); if \( \mathfrak{r} \) is not reduced we normalize the scalar product such that the roots have square length 1, 2 or 4. We will use \( \mathfrak{r}_s \) and \( \mathfrak{r}_l \) to refer to the short and respectively long roots in \( \mathfrak{r} \); if the root system is nonreduced we will also use \( \mathfrak{r}_m \) to refer to the roots of length 2. We will identify the vector space \( \mathfrak{h} \) with its dual using this scalar product. Under this identification \( \alpha^\vee = 2\alpha/(\alpha, \alpha) \) for any root \( \alpha \).

To any finite root system as above we associate an affine root system \( \mathfrak{r} \) as follows. Let \( \text{Aff}(\mathfrak{h}) \) be the space of affine linear transformations of \( \mathfrak{h} \). As a vector space, it can be identified to \( \mathfrak{h}^* \oplus \mathbb{R} \delta \) via \((f + c\delta)(x) = f(x) + c, \text{ for } f \in \mathfrak{h}^*, \ x \in \mathfrak{h} \text{ and } c \in \mathbb{R}\). Assume first that \( \mathfrak{r} \) is reduced, and let \( r \) denote the maximal number of laces connecting two vertices in its Dynkin diagram. Define

\[
R := (\hat{\mathfrak{r}}_s + \mathbb{Z}\delta) \cup (\hat{\mathfrak{r}}_l + r\mathbb{Z}\delta) \subset \mathfrak{h}^* \oplus \mathbb{R}\delta
\]

If the finite root system \( \hat{\mathfrak{r}} \) is nonreduced define

\[
R := (\hat{\mathfrak{r}}_s + \frac{1}{2}\mathbb{Z}\delta) \cup (\hat{\mathfrak{r}}_m + \mathbb{Z}\delta) \cup (\hat{\mathfrak{r}}_l + \mathbb{Z}\delta)
\]

Note that in the latter case \( R \) is itself a nonreduced root system. Let us also consider the reduced root systems

\[
\hat{\mathfrak{r}}_{nd} := \{ \alpha \in \hat{\mathfrak{r}} \mid \alpha/2 \notin \hat{\mathfrak{r}} \} \quad \text{and} \quad \mathfrak{r}_{nd} := \{ \alpha \in R \mid \alpha/2 \notin R \}
\]

\[
\hat{\mathfrak{r}}_{nm} := \{ \alpha \in \hat{\mathfrak{r}} \mid 2\alpha \notin \hat{\mathfrak{r}} \} \quad \text{and} \quad \mathfrak{r}_{nm} := \{ \alpha \in R \mid 2\alpha \notin R \}
\]

The set of affine positive roots \( R^+ \) consists of affine roots of the form \( \alpha + k\delta \) such that \( k \) is non-negative if \( \alpha \) is a positive root, and \( k \) is strictly positive if \( \alpha \) is a negative root. The affine simple roots are \( \{\alpha_i\}_{0 \leq i \leq n} \), where we set \( \alpha_0 := \delta - \theta \) if \( \hat{\mathfrak{r}} \) is reduced and \( \alpha_0 := \frac{1}{2}(\delta - \theta) \) otherwise. In fact, to make some formulas uniform we set \( \alpha_0 := c_0^{-1}(\delta - \theta) \), where \( c_0 \) equals 1 or 2 depending on whether \( \hat{\mathfrak{r}} \) is reduced or not. If \( \alpha_i \) is a simple root, then \( \alpha_i^\vee \) denotes its unique scalar multiple which belongs
to $R_{nm}^*$. Note that $\hat{R}_{nd}$ and $R_{nd}$ have the same basis as $\hat{R}$ and $R$, respectively. Also, a basis for $\hat{R}_{nm}$ and $R_{nm}$ is given by $\{\alpha_i^*\}_{1 \leq i \leq n}$ and $\{\alpha_i^*\}_{0 \leq i \leq n}$, respectively. The root lattice of $R$ is defined as $Q = \oplus_{i=0}^{n} \mathbb{Z} \alpha_i$.

Abstractly, an affine root system is a subset $\Phi \subset \text{Aff}(V)$ of the space of affine-linear functions on a real vector space $V$, consisting of non-constant functions which satisfy the usual axioms for root systems. As in the case of finite root systems, a classification of the irreducible affine root systems is available (see, for example, [28, Section 1.3]). The affine root systems $R$ defined above are just a subset of all the irreducible affine root systems. However, the configuration of vanishing hyperplanes of elements of an irreducible affine root system $\Phi$ coincides with the corresponding configuration of hyperplanes associated to a unique affine root system $R$ as above. The objects we are concerned with here depend in a larger amount on the Weyl group associated to an affine root system rather than on the root system itself, and our restriction reflects that. Moreover, the nonreduced affine root system considered above contains as subsystems all the other nonreduced irreducible affine root systems and all the reduced irreducible affine root systems of classical type of the same rank.

1.2. Affine Weyl groups. The scalar product on $\mathfrak{h}^*$ can be extended to a non-degenerate bilinear form on the real vector space

$$\hat{\mathfrak{g}}^* := \mathfrak{h}^* \oplus \mathbb{R} \delta \oplus \mathbb{R} \Lambda_0$$

by requiring that $(\delta, \mathfrak{h}^* \oplus \mathbb{R} \delta) = (\Lambda_0, \mathfrak{h}^* \oplus \mathbb{R} \Lambda_0) = 0$ and $(\delta, \Lambda_0) = 1$. Given $\alpha \in R$ and $x \in \hat{\mathfrak{g}}^*$ let

$$s_\alpha(x) := x - \frac{2(x, \alpha)}{(\alpha, \alpha)} \alpha$$

The affine Weyl group $W$ is the subgroup of $\text{GL}(\hat{\mathfrak{g}}^*)$ generated by all $s_\alpha$ (the simple reflections $s_i = s_{\alpha_i}$ are enough). The finite Weyl group $\hat{W}$ is the subgroup generated by $s_1, \ldots, s_n$. The bilinear form on $\hat{\mathfrak{g}}^*$ is equivariant with respect to the affine Weyl group action. Both the finite and the affine Weyl group are Coxeter groups and they can be abstractly defined as generated by $s_1, \ldots, s_n$, respectively $s_0, \ldots, s_n$, and the following relations:

a) reflection relations: $s_i^2 = 1$;

b) braid relations: $s_i s_j \cdots = s_j s_i \cdots$ (there are $m_{ij}$ factors on each side, $m_{ij}$ being equal to $2, 3, 4, 6$ if the number of laces connecting the corresponding nodes in the Dynkin diagram is $0, 1, 2, 3$ respectively).

The affine Weyl group could also be presented as a semidirect product in the following way: it is the semidirect product of $\hat{W}$ and the lattice $\hat{Q}$ (regarded as an abelian group with elements $\tau_\mu$, where $\mu$ is in $\hat{Q}$), the finite Weyl group acting on
the root lattice as follows
\[ \hat{w}_\tau \hat{w}^{-1} = \tau \hat{w}(\mu) \]
Since the finite Weyl group acts on the weight lattice, we can also consider the extended Weyl group \( W^e \) defined as the semidirect product between \( W \) and \( P \). Unlike the affine Weyl group, \( W^e \) is not a Coxeter group. However, \( W \) is a normal subgroup of \( W^e \) and the quotient is finite.

For \( s \) a real number, \( \hat{g}^*_s = \{ x \in \hat{g} : (x, \delta) = s \} \) is the level \( s \) of \( \hat{g}^* \). We have
\[ \hat{g}^*_s = \hat{g}^*_0 + s \Lambda_0 = h^* + R\delta + s \Lambda_0 . \]
The action of \( W \) preserves each \( \hat{g}^*_s \) and we can identify each level canonically with \( \hat{g}^*_0 \) and obtain an (affine) action of \( W^e \) on \( \hat{g}^*_0 \). If \( s_i \in W \) is a simple reflection, write \( s_i(\cdot) \) for the regular action of \( s_i \) on \( \hat{g}^*_0 \) and \( s_i(\cdot) \) for the affine action of \( s_i \) on \( \hat{g}^*_0 \) corresponding to the level one action. For example, the level zero action of \( s_0 \) and \( \tau_\mu \) is
\[ s_0(x) = s_0(x) + (x, \theta)c_0^{-1}\delta , \]
\[ \tau_\mu(x) = x - (x, \mu)\delta , \]
and the level one action of the same elements is
\[ s_0(x) = s_0(x) + (x, \theta)c_0^{-1}\delta - \alpha_0 , \]
\[ \lambda_\mu(x) = x + \mu - (x, \mu)\delta - \frac{1}{2}|\mu|^2\delta . \]
The level one action on \( \hat{g}^*_0 \) induces an affine action of \( W \) on \( h^* \). As a matter of notation, we write \( w \cdot x \) for this affine action of \( w \in W \) on \( x \in h^* \). For example,
\[ s_0 \cdot x = s_0(x) + c_0^{-1}\theta , \]
\[ \tau_\mu \cdot x = x + \mu . \]
The fundamental alcove is defined as
\[ C := \{ x \in h^* : (x + \Lambda_0, \alpha_i^\vee) \geq 0 , \ 0 \leq i \leq n \} \quad (1) \]
The non-zero elements of \( \mathcal{O}_P := P \cap C \) are the so-called minuscule weights. Note that each orbit of the affine action of \( W \) on \( P \) contains either the origin or a unique fundamental weight \( \lambda_i \) (to keep the notation consistent we set \( \lambda_0 = 0 \)). If we denote \( \Omega := \{ w \in W^e : w \cdot C = C \} \) then, \( W^e = \Omega \ltimes W \). The group \( \Omega \) is finite, of order the size of \( \mathcal{O}_P \). In fact, we can parameterize \( \Omega \) by the elements of \( \mathcal{O}_P \) as follows: for each \( \lambda \in \mathcal{O}_P \) let \( \omega_\lambda \) denote the unique element of \( \Omega \) for which \( \omega_\lambda(0) = \lambda \). It is easy to check that \( \omega_\lambda = \tau_\lambda \hat{w}_\lambda \).

For the definition of \( \hat{w}_\lambda \) see Section 1.4 below.

If we examine the orbits of the level zero action of the affine Weyl group \( W \) on the affine root system \( R \) we find the following:
a) if $\hat{R}$ is reduced there are precisely as many orbits as root lengths;

b) if $\hat{R}$ is nonreduced of rank at least two, then there are five orbits:

$$W(2\alpha_0) = \hat{R}_t + 2\mathbb{Z}\delta + \delta, \quad W(\alpha_0) = \hat{R}_s + \mathbb{Z}\delta + \frac{1}{2}\delta, \quad W(\alpha_1) = \hat{R}_m + \mathbb{Z}\delta$$

$$W(2\alpha_n) = \hat{R}_t + 2\mathbb{Z}\delta \quad \text{and} \quad W(\alpha_n) = \hat{R}_s + \mathbb{Z}\delta$$

$$W(2\alpha_0) = \hat{R}_t + 2\mathbb{Z}\delta + \delta, \quad W(\alpha_0) = \hat{R}_s + \mathbb{Z}\delta + \frac{1}{2}\delta, \quad W(\alpha_1) = \hat{R}_m + \mathbb{Z}\delta$$

c) if $\hat{R}$ is nonreduced of rank one then there are only four orbits: $W(2\alpha_0)$, $W(\alpha_0)$, $W(2\alpha_1)$ and $W(\alpha_1)$.

1.3. The length function. For each $w$ in $W$ let $\ell(w)$ be the length of a reduced (i.e. shortest) decomposition of $w$ in terms of simple reflections. The length of $w$ can be also geometrically described as follows. For any affine root $\alpha$, denote by $H_\alpha$ the affine hyperplane consisting of fixed points of the affine action of $s_\alpha$ on $\hat{\mathfrak{h}}^*$. Then, $\ell(w)$ equals the number of affine hyperplanes $H_\alpha$ separating $C$ and $w \cdot C$. Since the affine action of $W^e$ on $\hat{\mathfrak{h}}^*$ preserves the set $\{H_\alpha\}_{\alpha \in \check{R}}$, we can use the geometric point of view to define the length of any element of $W^e$. For example, the elements of $\Omega$ will have length zero.

For $w$ in $W$ we have $\ell(w) = |\Pi(w)|$, where $\Pi(w) = \{\alpha \in R^+_{nd} \mid w(\alpha) \in R^-_{nd}\}$. We denote $\check{\Pi}(w) := \{\alpha \in R^+_{nd} \mid w(\alpha) \in R^+_{nd}\}$. If $w = s_{j_p} \cdots s_{j_1}$ is a reduced decomposition, then

$$\Pi(w) = \{\alpha^{(i)} \mid 1 \leq i \leq p\},$$

where $\alpha^{(i)} = s_{j_1} \cdots s_{j_{i-1}}(\alpha_{j_i})$. An easy check shows that for any $w$ in $W$ we have

$$w^{-1}(\Pi(w^{-1})) = -\Pi(w) \quad \text{and} \quad w^{-1}(\check{\Pi}(w^{-1})) = \check{\Pi}(w)$$

The following formula is well-known (see, for example, [24]). If $\check{w} \in \hat{W}$ and $\lambda \in P$, then

$$\ell(\check{w}\tau_\lambda) = \sum_{\alpha \in \Pi(\check{w})} |(\lambda, \alpha^\vee)| + 1 + \sum_{\alpha \in -\Pi(\check{w})} |(\lambda, \alpha^\vee)|$$

Let us derive a few immediate consequences which will be useful later on.

Lemma 1.1. Assume that $\lambda$ and $\mu$ are dominant weights and that $\check{w}$ is an element of $\hat{W}$. Then, the following equalities hold:

(1) $\ell(\tau_{\lambda + \mu}) = \ell(\tau_\lambda) + \ell(\tau_\mu)$

(2) $\ell(\check{w}\tau_\lambda) = \ell(\check{w}) + \ell(\tau_\lambda)$

(3) $\ell(\check{w}(\tau_\lambda)) = \ell(\tau_\lambda)$

Proof. The first two claims follow directly from the formula [24] if we keep in mind that the scalar product $\langle \lambda, \alpha^\vee \rangle$ is non-negative if $\lambda$ is dominant and $\alpha$ is a positive
finite root. To prove the third statement note that
\[
\ell(\tau_\lambda) = \sum_{\alpha \in R_{\wedge}^+} |(\lambda, \alpha^\vee)|
= \sum_{\alpha \in P(\hat{w}^{-1})} |(\lambda, \hat{w}^{-1}(\alpha^\vee))| + \sum_{\alpha \in P(\hat{w}^{-1})} |(\lambda, \hat{w}^{-1}(\alpha^\vee))|
= \sum_{\alpha \in P(\hat{w}^{-1})} |(\lambda(\hat{w}), \alpha^\vee)| + \sum_{\alpha \in P(\hat{w}^{-1})} |(\lambda(\hat{w}), \alpha^\vee)|
= \ell(\tau_{\hat{w}(\lambda)})
\]
Along the way we have used the equalities (3). □

1.4. Coset representatives. For each weight \(\lambda\) define \(\lambda_-\), respectively \(\tilde{\lambda}\), to be the unique element in \(\hat{W}(\lambda)\), respectively \(W \cdot \lambda\), which is an anti-dominant weight, respectively an element of \(O(W)\) (i.e. either zero or a minuscule weight), and \(\hat{w}_\lambda^{-1} \in \hat{W}\), respectively \(w_\lambda^{-1} \in W\), to be the unique minimal length elements by which this is achieved. Also, for each weight \(\lambda\) define \(\lambda_+\) to be the unique dominant element in \(\hat{W}(\lambda)\) and denote by \(w_\circ\) the maximal length element in \(\hat{W}\).

Clearly, the element \(\hat{w}_\lambda\) is the minimal length representative in its left coset \(\hat{w}_\lambda \text{Stab}_{\hat{W}}(\lambda_-) \subset \hat{W}\). The element \(w_\lambda\) can be equivalently described as the minimal length representative of the coset \(\tau_\lambda \hat{W} \omega_\lambda^{-1} \subset W\). Similarly, we consider \(v_\lambda\), the unique maximal length representative of the coset \(\tau_\lambda \hat{W} \omega_\lambda^{-1} = w_\lambda \omega_\lambda \hat{W} \omega_\lambda^{-1}\). In fact, the group \(\omega_\lambda \hat{W} \omega_\lambda^{-1}\) is the stabilizer \(\text{Stab}_{\hat{W}}(\tilde{\lambda})\) which will be denoted by \(\hat{W}_\tilde{\lambda}\). Its maximal length element is \(w_{\circ, \tilde{\lambda}} := \omega_\lambda w_{\circ, \lambda} \omega_\lambda^{-1}\) and \(v_\lambda\) and \(w_\lambda\) are related by the formula
\[
v_\lambda = w_\lambda w_{\circ, \tilde{\lambda}} \quad (5)
\]
Moreover,
\[
\ell(v_\lambda) = \ell(w_\lambda) + \ell(w_{\circ, \lambda}) = \ell(w_\lambda) + \ell(w_{\circ}) \quad (6)
\]

Let us recall from [9] the following result.

Lemma 1.2. With the above notation
\[
(1) \quad \Pi(\hat{w}_\lambda^{-1}) = \{\alpha \in R_{\wedge}^+ \mid (\lambda, \alpha) > 0\}
(2) \quad \Pi(w_\lambda^{-1}) = \{\alpha \in R_{\wedge}^+ \mid (\lambda + \Lambda_0, \alpha) < 0\}
\]

The following technical result will be used later.

Lemma 1.3. Let \(\lambda\) be a weight and let \(\beta\) be a root in \(\hat{R}\) such that \(\alpha = \beta + k\delta\) is a positive affine root. If \((\alpha, \lambda + \Lambda_0) < 0\) then \(\hat{w}_\lambda^{-1}(\beta)\) belongs to \(\hat{R}^+\).
Proof. Let us remark that it is enough to prove our result for some positive scaling of the root $\alpha$ and therefore we can safely assume that $\alpha \in R_{nd}^+$. Since $\alpha = \beta + k\delta$ is a positive affine root we have to analyze two possible situations. First, assume that $\beta \in \hat{R}^+$ and $k \geq 0$. In this case, $(\alpha, \lambda + \Lambda_0) < 0$ implies that $(\beta, \lambda) < 0$ and the above Lemma tells us that $\tilde{w}_\lambda^{-1}(\beta) \in \hat{R}^+$. The other possible situation is $\beta \in \hat{R}^-$ and $k > 0$. In this case, $(\alpha, \lambda + \Lambda_0) < 0$ implies that $(-\beta, \lambda) > 0$ and since $-\beta \in \hat{R}^+$ we obtain that $\tilde{w}_\lambda^{-1}(-\beta) \in \hat{R}^-$ or, equivalently, $\tilde{w}_\lambda^{-1}(\beta) \in \hat{R}^+$. □

1.5. The Bruhat order. The Bruhat order is a partial order on any Coxeter group defined in way compatible with the length function. For an element $w$ and a root $\alpha$ we write $w < s_\alpha w$ if and only if $\ell(s_\alpha w) = \ell(w) + 1$. The transitive closure of this relation is called the Bruhat order on $W$. The terminology is motivated by the way this order arises for Weyl groups in connection with inclusions among the closures of the Bruhat cells of a corresponding connected, simple algebraic group. For the basic properties of the Bruhat order we refer to Chapter 5 in [7]. Let us list a few of them (the first two properties completely characterize the Bruhat order):

1. For each $\alpha \in R^+$ we have $s_\alpha w < w$ if and only if $\alpha$ is in $\Pi(w^{-1})$;
2. $w' < w$ if and only if $w'$ can be obtained by omitting some factors in a reduced decomposition of $w$;
3. If $s_i$ is a simple reflection and $w' \leq w$ then either $s_iw' \leq w$ or $s_iw' \leq s_iw$ (or both). For example, if $\ell(s_iw') - \ell(w') \leq \ell(s_iw) - \ell(w)$ then $s_iw' \leq s_iw$.

We can use the Bruhat order on $W$ to define a partial order on the weight lattice which will also be called the Bruhat order. For any $\lambda, \mu \in P$ we write

$$\lambda < \mu \quad \text{if and only if} \quad \tilde{\lambda} = \tilde{\mu} \quad \text{and} \quad w_\lambda < w_\mu$$

The minimal elements of $P$ with respect to this partial order are the minuscule weights. The next result shows that this partial order relation could have been defined as well using the elements $v_\lambda$ instead of $w_\lambda$.

Lemma 1.4. Let $\lambda$ and $\mu$ be two weights. Then $w_\mu < w_\lambda$ if and only if $v_\mu < v_\lambda$.

Proof. Straightforward from and the third property of the Bruhat order. □

Lemma 1.5. Let $\lambda$ be a weight. We have

$$\{ x \in W \mid x \leq v_\lambda \} = \bigcup_{\mu \leq \lambda} v_\mu W_\lambda$$

Proof. If $\mu$ is a weight such that $\mu \leq \lambda$ then, by the above Lemma, $v_\mu \leq v_\lambda$. Since $v_\mu$ is the maximal element of the coset $v_\mu W_\lambda$, we obtain that $y \leq v_\lambda$ for any element $y$ in $v_\mu W_\lambda$. 

Conversely, let $x \in W$ such that $x \leq v_{\lambda}$. The third property of the Bruhat order together with the definition of $v_{\lambda}$ imply that $z \leq v_{\lambda}$ for any $z \in xW_{\lambda}$. The left coset $xW_{\lambda}$ is of the form $v_{\mu}W_{\lambda}$ for some weight $\mu$ for which $\hat{\mu} = \hat{\lambda}$. Therefore, as claimed, we obtain that $x \in v_{\mu}W_{\lambda}$ and $v_{\mu} \leq v_{\lambda}$. \hfill \Box

The following result can be found in [3].

**Lemma 1.6.** Let $\lambda$ be a weight and $\alpha_i$ be a simple affine root such that $s_i \cdot \lambda \neq \lambda$. The following statements hold:

1. We have, $\hat{w}_{s_i, \lambda} = s_i \hat{w}_{\lambda}$, unless $i = 0$, in which case $\hat{w}_{s_0, \lambda} = s_0 \hat{w}_{\lambda}$.
2. Moreover, $s_i \cdot \lambda > \lambda$ if and only if $(\alpha_i, \lambda + \Lambda_0) > 0$. In particular $\lambda_-$, respectively $\lambda_+$, are the maximal element, respectively the minimal element in $W(\lambda)$ with respect to the Bruhat order.

We close this section with a consequence of the above result.

**Lemma 1.7.** Let $\lambda$ be an anti-dominant weight and $\mu \in W(\lambda)$. Then

1. $w_{\lambda} = \hat{w}_{\mu}^{-1}w_{\mu}$ and $\ell(w_{\lambda}) = \ell(w_{\mu}) + \ell(\hat{w}_{\mu})$
2. $w_{\lambda}\omega_{\lambda} = \tau_{\lambda}$
3. $\tau_{\mu}w_{\mu} = w_{\mu}\omega_{\mu}$ and $\ell(\tau_{\mu}) = \ell(w_{\mu}) + \ell(\hat{w}_{\mu})$

**Proof.** (1) Let us note that if we fix a reduced decomposition $s_{j_p} \cdots s_{j_1}$ for $\hat{w}_{\mu}^{-1}$ then

$$\lambda = s_{j_p} \cdots s_{j_1}(\mu) > \cdots > s_{j_2}s_{j_1}(\mu) > s_{j_1}(\mu) > \mu$$

Indeed, for any $1 \leq i \leq p$ we have

$$(\alpha_{j_i}, s_{j_i-1} \cdots s_{j_1}(\mu)) = (s_{j_i} \cdots s_{j_i-1}(\alpha_{j_i}), \mu)$$

and from equation (2) we know that $s_{j_i} \cdots s_{j_i-1}(\alpha_{j_i})$ belongs to $\Pi(\hat{w}_{\mu}^{-1})$. Furthermore, Lemma 1.6 implies that $(s_{j_i} \cdots s_{j_i-1}(\alpha_{j_i}), \mu) > 0$ and Lemma 1.6 immediately gives us (3). We conclude that $w_{\lambda} = \hat{w}_{\mu}^{-1}w_{\mu}$ and $\ell(w_{\lambda}) = \ell(w_{\mu}) + \ell(\hat{w}_{\mu})$.

(2) By definition, $w_{\lambda}$ is the unique minimal length element in the coset $\tau_{\lambda}\hat{W}\omega_{\lambda}^{-1}$. Since $\omega_{\lambda}$ has length zero it is enough to show that $\ell(\tau_{\lambda}\hat{w}) \geq \ell(\tau_{\lambda})$ for all $\hat{w} \in \hat{W}$. Indeed, $\ell(\tau_{\lambda}\hat{w}) = \ell(\hat{w}^{-1}\tau_{\lambda}^{-1})$ and since the element $-\lambda$ is dominant ($\lambda$ being anti-dominant) Lemma 1.6 implies the desired result.

(3) The statement follows immediately from (1), (2) and Lemma 1.6. \hfill \Box

2. **Nonsymmetric Macdonald polynomials**

2.1. **Parameters and conventions.** Let us introduce a field $\mathbb{F}$ (of parameters) as follows. Let $t = (t_{\alpha})_{\alpha \in R}$ be a set of parameters which is indexed by the set of affine roots and has the property that $t_{\alpha} = t_{\beta}$ if and only if the affine roots $\alpha$ and
β belong to the same orbit under the action of $W$ on $R$. It will be convenient to have also the following convention: if $α$ is not an affine root then $t_α = 1$. Let $q$ be another parameter and let $m$ be the lowest common denominator of the rational numbers $\{(α_j, λ_k) \mid 1 \leq j,k \leq n\}$. The field $F = F_{q,t}$ is defined as the field of rational functions in $q^{1/m}$ and $t_α = (t_α^j)^{1/m}$ with rational coefficients. We will also use the field of rational functions in $t_α = (t_α^j)^{1/m}$ denoted by $F_1$.

As it follows from the discussion at the end of Section 1.2 there are only a small number of distinct parameters. If the root system $R$ is reduced then there are as many distinct parameters $t_α$ as root lengths: at most two, which we denote by $t_s$ (the one corresponding to short roots) and $t_l$ (the one corresponding to long roots). In this case, to avoid unnecessary notational complexity we use $t_i$ to refer to the parameter $t_α$, corresponding to the affine simple root $α_i$.

If $R$ is nonreduced then the action of the affine Weyl group on the affine root system has five orbits $W(2α_0)$, $W(α_0)$, $W(2α_n)$ and $W(α_1)$ (note that the last orbit is empty if $R$ has rank one) and we denote the corresponding parameters by $t_0$, $t_0$, $t_n$ and $t := t_1 = \cdots = t_n$, respectively. The relation with the notation used in [32] is the following: $t_0$, $u_0$, $u_n$ used in [32] are respectively $t_0$, $t_0$, $t_0$ in our notation.

To avoid distinguishing among the reduced and nonreduced case later on we find convenient to define $t_0 = t_0 = t_0 := t_0$ in the reduced case.

2.2. Double affine Hecke algebras. The algebra $R = F[e^λ; λ \in P]$ is the group $F$-algebra of the lattice $P$. Similarly, the algebra $R_t = F_t[e^λ; λ \in P]$ is the group $F_t$-algebra of the lattice $P$. In the discussion that follows we refer to the following group $F$-algebras of the root lattice: $Q_Y := F[Y_μ; μ ∈ Ɛ]$ and $Q_X := F[X_β; β ∈ Ɛ]$. We will also use the following notation: for $μ ∈ Ɛ$ and $k ∈ \frac{1}{m}Z$, let $e^{μ+kδ} := q^{-k}e^μ$, $X_β+δ := q^{-κ}X_β$ and $Y_μ+δ := q^κY_μ$.

In the reduced case, the double affine Hecke algebras were introduced by Cherednik (see, for example, [11]) in his work on affine quantum Knizhnik–Zamolodchikov equations and on Macdonald’s conjectures. In the nonreduced case the definition is due to Sahi [32]. We give here the symmetric definition of the double affine Hecke algebras obtained in [12].

**Definition 2.1.** The double affine Hecke algebra $\tilde{H}$ associated to the root system $R$ is the $F$-algebra described by generators and relations as follows:

**Generators:** One generator $T_α$ for each simple root $α_i$, with the exception of the affine simple root $α_0$ for which we associate three generators $T_{01}$, $T_{02}$ and $T_{03}$.

**Relations:** a) Each pair of generators satisfies the same braid relations as the corresponding pair of simple reflections.
b) If there is a simple root $\alpha$ such that $(\alpha, \alpha') = -2$ then the following relation also holds

$$T_{01}T_{\alpha}^{-1}T_{03}T_{\alpha} = T_{\alpha}^{-1}T_{03}T_{\alpha}T_{01}$$

c) The quadratic relations

$$T_i^2 = (t_i^{\frac{1}{2}} - t_i^{-\frac{1}{2}})T_i + 1, \quad \text{for all} \quad 1 \leq i \leq n, \text{and}$$

$$T_{0j}^2 = (t_{0j}^{\frac{1}{2}} - t_{0j}^{-\frac{1}{2}})T_{0j} + 1, \quad \text{for} \quad 1 \leq j \leq 3$$

c) The relation

$$T_{01}T_{02}T_{03}T_{s_0} = q^{-c_0^1}$$

In the case of a reduced root system the quadratic relations for the elements $T_{0j}$ need not be imposed, since they are a consequence of the other relations. However it is absolutely necessary to impose them for nonreduced root systems. For nonreduced root systems, the relationship between the generators $T_{0j}$ and the notation used in [32] is the following: $T_0, U_0, U_n$ used in [32] are respectively $T_{01}, T_{02}, T_{03}$ in our notation.

The elements $T_1, \ldots, T_n$ generate the finite Hecke algebra $\hat{H}$. There are countably many copies of the affine Hecke algebra associated to the affine root system $R$ inside $\hat{H}$; we will distinguish only two of them: $\mathcal{H}_Y$ which is the subalgebra generated by $T_{01}, T_1, \ldots, T_n$, and $\mathcal{H}_X$ which is the subalgebra generated by $T_{03}, T_1, \ldots, T_n$. There are natural bases of $\mathcal{H}_X$, $\mathcal{H}_Y$ and $\mathcal{H}$: $\{T_w\}_w$ indexed by $w$ in $W$ and in $\tilde{W}$ respectively, where $T_w = T_{i_1} \cdots T_{i_l}$ if $w = s_{i_1} \cdots s_{i_l}$ is a reduced expression of $w$ in terms of simple reflections. Let us recall the well-known result of Bernstein (unpublished) and Lusztig [21] on the structure of affine Hecke algebras as it applies to $\mathcal{H}_Y$ and $\mathcal{H}_X$.

**Proposition 2.2.** With the above notation we have

1. The affine Hecke algebra $\mathcal{H}_Y$ is generated by the finite Hecke algebra $\hat{H}$ and the group algebra $Q_Y$ such that the following relations are satisfied for any $\mu$ in the root lattice and any $1 \leq i \leq n$:

$$Y_\mu T_i - T_i Y_{s_i(\mu)} = (t_i^{\frac{1}{2}} - t_i^{-\frac{1}{2}})Y_\mu - Y_{s_i(\mu)} - \frac{1}{1 - Y_{\alpha_i}} \quad \text{if} \quad 2\alpha_i \notin \hat{R}$$

$$Y_\mu T_n - T_n Y_{s_n(\mu)} = \left(t_n^{\frac{1}{2}} - t_n^{-\frac{1}{2}} + (t_{01}^{\frac{1}{2}} - t_{01}^{-\frac{1}{2}})Y_{\alpha_n}\right) \frac{Y_\mu - Y_{s_n(\mu)}}{1 - Y_{2\alpha_n}} \quad \text{if} \quad 2\alpha_n \in \hat{R}$$

In this description $Y_{-c_0^1} = T_{01}T_{03}$.

2. The affine Hecke algebra $\mathcal{H}_X$ is generated by the finite Hecke algebra $\hat{H}$ and the group algebra $Q_X$ such that the following relations are satisfied for any $\mu$ in the root lattice and any $1 \leq i \leq n$:

$$T_i X_\mu - X_{s_i(\mu)} T_i = (t_i^{\frac{1}{2}} - t_i^{-\frac{1}{2}})X_\mu - X_{s_i(\mu)} - \frac{1}{1 - X_{-\alpha_i}} \quad \text{if} \quad 2\alpha_i \notin \hat{R}$$
If we use the notation \(\omega\) we note that with the above notation the action of \(H\) for any \(T\) using Remark 2.3 the action of \(H\) as the semidirect product of \(\Omega\) and \(T\) algebra by only specifying the action of \(\omega\) is clear that we can equivalently define a representation of the double affine Hecke algebra by only specifying the action of \(T_{00}\), \(T_{01}\), \(T_{02}\) and \(T_{03}\) of Cherednik (in the reduced case) and Sahi (in the nonreduced case) we know that the following formulas define a faithful representation of \(\tilde{\mathcal{H}}\) of \(\Omega\) and \(\check{\nu}\).

To define an action of \(\tilde{\mathcal{H}}\) one needs only to define the action of the generators \(T_{0j}\), \(1 \leq j \leq 3\). However, from Remark 2.3 and Proposition 2.2 it is clear that we can equivalently define a representation of the double affine Hecke algebra by only specifying the action of \(T_{00}, T_{01}, 1 \leq i \leq n\) and \(\mathbb{Q}X\). From the work of Cherednik (in the reduced case) and Sahi (in the nonreduced case) we know that the following formulas define a faithful representation of \(\tilde{\mathcal{H}}\) on \(\mathcal{R}\):

\[
X_{\mu} \cdot e^\lambda = e^{\lambda + \mu} \quad \text{for } \mu \in \tilde{\mathcal{Q}}
\]

\[
T_{i} \cdot e^\lambda = t_{i} e^{s_{i}(\lambda)} + (t_{i}^{-1} - t_{i}) e^{s_{i}(\lambda)} e^{-s_{i}(\lambda)} \quad \text{if } 1 \leq i \leq n\text{ and } 2\alpha_{i} \not\in \tilde{R}
\]

\[
T_{n} \cdot e^\lambda = t_{n} e^{s_{n}(\lambda)} + (t_{n} - t_{n}^{-1}) e^{s_{n}(\lambda)} e^{-s_{n}(\lambda)} \quad \text{if } 2\alpha_{n} \in \tilde{R}
\]

\[
T_{00} \cdot e^\lambda = t_{00} e^{s_{0}(\lambda)} + (t_{00}^{-1} - t_{00}) e^{s_{0}(\lambda)} e^{-s_{0}(\lambda)} \quad \text{if } 2\alpha_{00} \not\in R
\]

\[
T_{00} \cdot e^\lambda = t_{00} e^{s_{0}(\lambda)} + (t_{00} - t_{00}^{-1}) e^{s_{0}(\lambda)} e^{-s_{0}(\lambda)} \quad \text{if } 2\alpha_{0} \in R
\]

Using Remark 2.3, the action of \(T_{02}\) is easily computable. Let us list the results:

\[
T_{02} \cdot e^\lambda = t_{02} e^{s_{0}(\lambda)} + (t_{02}^{-1} - t_{02}) e^{s_{0}(\lambda)} e^{-s_{0}(\lambda)} \quad \text{if } 2\alpha_{00} \not\in R
\]

\[
T_{02} \cdot e^\lambda = t_{02} e^{s_{0}(\lambda)} + (t_{02}^{-1} - t_{02}) e^{s_{0}(\lambda)} e^{-s_{0}(\lambda)} \quad \text{if } 2\alpha_{0} \in R
\]

We also need to consider the extended affine Hecke algebra \(\mathcal{H}_{X}^{e}\) which is defined as the semidirect product of \(\Omega\) and \(\mathcal{H}_{X}\). The action of \(\Omega\) on \(\mathcal{H}_{X}\) is induced from the action of \(\Omega\) on the affine Weyl group: if \(\omega \in \Omega\) and \(w \in W\) then \(\omega T_{w}^{-1} \omega = T_{\omega w^{-1}}\).

If we use the notation \(T_{w} \cdot e^\lambda = t_{w} e^\lambda\), a basis for \(\mathcal{H}_{X}^{e}\) is given by \(\{T_{w}\}_{w \in \check{W}}\). The action of \(\mathcal{H}_{X}\) on \(\mathcal{R}\) described above can be extended to an action of \(\mathcal{H}_{X}^{e}\) by defining

\[
\omega \cdot e^\mu = e^{\lambda T_{w^{-1}} \cdot e^\mu}
\]

for any \(\lambda \in \mathcal{Q}\). It is important to note that for any dominant weights \(\nu_{1}\) and \(\nu_{2}\) the element \(X_{\nu_{1} - \nu_{2}} := T_{\nu_{1}}^{-1} T_{\nu_{2}}^{-1}\) acts on \(\mathcal{R}\) as multiplication by \(e^{\nu_{1} - \nu_{2}}\).
2.3. The Cherednik scalar product. The involution of $\mathbb{F}$ which inverts each of the parameters $q$, $\{t_\alpha\}_{\alpha \in R}$ extends to an involution $\cdot$ on the algebra $\mathcal{R}$ which sends each $e^\lambda$ to $e^{-\lambda}$. Following Cherednik let us define

$$K(q,t) = \prod_{\alpha \in R^+_m} \frac{1 - e^\alpha}{(1 - t_\alpha^+ t_\alpha^- e^\alpha)(1 + t_\alpha^+ t_\alpha^- e^\alpha)}$$

which should be seen as a formal series in the elements $e^\alpha$ with coefficients in $Z[t^{-1}][q^{-1}]$ (power series in $q^{-1}$ with coefficients polynomials in $t^{-1}$). Recall that we agreed to set $t_\alpha^= 1$ if $\alpha/2$ is not a root. If $K_0$ denotes the coefficient of $e^0$, then the Cherednik kernel is by definition

$$C(q,t) := \frac{K(q,t)}{K_0}$$

It is a formal series in the elements $e^\alpha$ with coefficients rational functions in $q$ and $t$ (see, for example, [28, (5.1.10)]). Moreover, $C(q,t)$ it is fixed by the above involution.

A scalar product on $\mathcal{R}$ can be defined as follows

$$\langle f, g \rangle_{q,t} := CT(f \bar{g} C(q,t))$$

where $CT(\cdot)$ denotes the constant term (i.e. the coefficient of $e^0$) of the expression inside the parenthesis. The scalar product is Hermitian with respect to the involution $\cdot$:

$$\langle g, f \rangle_{q,t} = \overline{\langle f, g \rangle_{q,t}}$$

and the above representation becomes unitary with respect to $\langle \cdot, \cdot \rangle_{q,t}$.

2.4. Macdonald polynomials. If $\gamma$ is an element of $\mathfrak{h}^* \oplus \frac{\Delta}{e_0} \mathbb{Z}$ we denote by $q^{(\gamma,\vec{\lambda})}$ the element of $\mathbb{F}$

$$q^{(\gamma,\lambda+\Lambda_0)}(t_n^+ t_n^-)^{-\frac{1}{2}(\gamma,\bar{w}_\lambda(\lambda^\vee))} \prod_{i=1}^{n-1} e_i^{-(\gamma,\bar{w}_\lambda(\lambda^\vee))}$$

where $t_n^*$ equals $t_n$ if $R$ is reduced or $t_{01}$ if $R$ is nonreduced. Under the same conditions let

$$t^{(\gamma,\vec{\lambda})} := (t_n^+ t_n^-)^{-\frac{1}{2}(\gamma,\bar{w}_\lambda(\lambda^\vee))} \prod_{i=1}^{n-1} e_i^{-(\gamma,\bar{w}_\lambda(\lambda^\vee))}$$

In particular, we have $q^{(\gamma,\lambda+\Lambda_0)} = q^{(\gamma,\vec{\lambda})} t^{(\gamma,\vec{\lambda})}$.

For each $\lambda \in P$ we can construct a $\mathbb{F}$-algebra morphism $ev(\lambda) : Q_\mathcal{Y} \to \mathbb{F}$, which sends $Y_\mu$ to $q^{(\mu,\vec{\lambda})}$. If $f(Y)$ is an element of $Q_\mathcal{Y}$ we will write $f(\lambda)$ for $ev(\lambda)(f)$.

For every weight $\lambda$ define

$$\mathcal{R}_\lambda = \{ f \in \mathcal{R} \mid Y_\mu \cdot f = q^{(\mu,\vec{\lambda})} f \text{ for any } \mu \in \hat{Q} \}.$$
Definition 2.4. Given a weight $\lambda$ the nonsymmetric Macdonald polynomial $E_\lambda(q,t)$ is the unique element in $R_\lambda$ in which the coefficient of $e^\lambda$ is 1. If $k \in \mathbb{Z}$ denote

$$E_{\lambda+k\delta}(q,t) := q^{-k}E_\lambda(q,t)$$

The nonsymmetric Macdonald polynomials form a basis of $R$ orthogonal with respect to the scalar product $\langle \cdot, \cdot \rangle_{q,t}$. They are also triangular with respect to the Bruhat order on the weight lattice. Since the minimal elements for this order relation are the minuscule weights we immediately obtain the following

Proposition 2.5. If $\lambda$ is a minuscule weight, then $E_\lambda(q,t) = e^\lambda$.

For any anti–dominant weight $\lambda$ we write $R_\lambda$ for the subspace of $R$ spanned by \{E_\mu \mid \mu \in \hat{W}(\lambda)\}. The relationship with the symmetric Macdonald polynomials is the following.

Definition 2.6. Given an anti–dominant weight $\lambda$ the symmetric Macdonald polynomial $P_\lambda(q,t)$ can be characterized as the unique $\hat{W}$-invariant element in $R_\lambda$ for which the coefficient of $e^\lambda$ equals 1.

In fact, the coefficients $a_\mu$ in the expansion

$$P_\lambda(q,t) = \sum_{\mu \in \hat{W}(\lambda)} a_\mu E_\mu(q,t)$$

can be computed explicitly (see, for example, [9] or [8, Theorem 3.20, Theorem 4.11]). The formulas for the coefficients $a_\mu$ in the case of a nonreduced root system are slightly more complicated and we will not list them here. We only stress that the Proposition 3.5 is valid for all root systems.

Proposition 2.7. If the affine root system $R$ is reduced, then

$$a_\mu = \prod_{\alpha \in R^+_{\text{aff}}, (\alpha,\mu) > 0} \frac{t_{\alpha}^{-1} - q^{-(\alpha,\mu)}}{1 - q^{-(\alpha,\mu)}}$$

Originally, the definition of symmetric Macdonald polynomials, due to Macdonald in the reduced setup and to Koornwinder in the nonreduced setup, preceded that of the nonsymmetric ones. Let us note that when the root system $R$ is nonreduced the associated polynomials are called in the literature Koornwinder polynomials.

2.5. The normalization factor. Given a weight $\lambda$ let us define the following normalization factor

$$e_\lambda := \prod_{\alpha \in R^+_{\text{aff}}, (\alpha,\lambda + \Lambda_0) < 0} (1 - q^{(\alpha,\lambda)})$$
Let us remark that if $\alpha$ is a positive affine root such that $(\alpha, \lambda + \Lambda_0) < 0$ then $q^{(\alpha, \lambda)}$ is a monomial in $q^{-1}, t_1^{-1}, \ldots, t_{n-1}^{-1}$ and $(t_1^{*}t_n)^{-\frac{1}{2}}$. This fact easily follows from the definition of $q^{(\alpha, \lambda)}$ and Lemma 1.3. Therefore, we can state the following

**Lemma 2.8.** Let $\lambda$ be a weight. The element $e_\lambda$ is a polynomial in the variables $q^{-1}, t_1^{-1}, \ldots, t_{n-1}^{-1}$ and $(t_1^{*}t_n)^{-\frac{1}{2}}$. This fact easily follows from the definition of $q^{(\alpha, \lambda)}$ and Lemma 1.3. Therefore, we can state the following

**Definition 2.9.** For all weights $\lambda$ and for all anti-dominant $\mu$, the polynomials $e_\lambda E_\lambda(q, t)$, respectively $e_\mu P_\mu(q, t)$, will be called here the normalized non-symmetric, respectively symmetric, Macdonald polynomials.

These normalized polynomials do not seem to have any particularly interesting properties except for those stated in Theorem 2.15. In type $A$, the normalization factor is much larger than the one used to define the integral forms $J_\lambda(x; q, t)$ of symmetric Macdonald polynomials [26, VI. §8], and the same is true for the integral form of the non-symmetric polynomials [16, Corollary 5.2].

The following result explains the relationship between the normalization factors associated to weights in the same $\tilde{W}$-orbit.

**Lemma 2.10.** Let $\lambda$ be an anti-dominant weight and let $\mu$ be an element of $\tilde{W}(\lambda)$. Then

$$e_\lambda = e_\mu \prod_{\alpha \in \Pi(w_\mu^{-1} \cap \Pi(w_\mu^{-1} \cap (\alpha, \mu) > 0)} (1 - q^{(\alpha, \lambda)})$$

**Proof.** The affine root system $R_{nm}$ is reduced and hence there will be no loss of generality if we assume $R$ to be reduced. Let us note first that our hypothesis and Lemma 1.7 imply that $w_\lambda = \tilde{w}_\mu^{-1} w_\mu$ and $\ell(w_\lambda) = \ell(\tilde{w}_\mu) + \ell(w_\mu)$. Hence, from (2) we obtain that

$$\Pi(w_\lambda^{-1}) = \Pi(\tilde{w}_\mu^{-1} \cup \tilde{w}_\mu^{-1} \cap (\alpha, \lambda))$$

Also, the condition $\alpha \in R^+, (\alpha, \lambda + \Lambda_0) > 0$ is equivalent, by Lemma 1.2, to the condition $\alpha \in \Pi(w_\lambda^{-1})$. Therefore,

$$e_\lambda = \prod_{\alpha \in \Pi(w_\mu^{-1})} (1 - q^{(\tilde{w}_\mu^{-1}(\alpha, \lambda))}) \prod_{\alpha \in \Pi(w_\mu)} (1 - q^{(\alpha, \lambda)})$$

$$e_\lambda = \prod_{\alpha \in \Pi(w_\mu^{-1})} (1 - q^{(\alpha, \lambda)}) \prod_{\alpha \in \Pi(\tilde{w}_\mu^{-1} \cap (\alpha, \lambda))} (1 - q^{(\alpha, \lambda)})$$

$$e_\mu = e_\mu \prod_{\alpha \in \Pi(\tilde{w}_\mu^{-1})} (1 - q^{(\tilde{w}_\mu^{-1}(\alpha, \lambda))})$$

$$e_\mu = e_\mu \prod_{\alpha \in \Pi(\tilde{w}_\mu^{-1})} (1 - q^{-(\alpha, \lambda)})$$

and our statement is proved. □
Lemma 2.11. Let $\mu$ be a weight and $s_i$ an affine simple reflection such that $s_i, \mu > 0$. Then,

$$e_{s_i, \mu} = (1 - q^{-(\alpha_i, \lambda)})e_{\mu}$$

Proof. As above, the affine root system $R_{nm}$ is reduced and we can safely assume that $R$ is reduced. Denote $\lambda := s_i, \mu$. From Lemma 1.6 we know that $(\mu + \Lambda_0, \alpha_i) > 0$, $s_i w_\mu = w_\lambda$ and $\ell(w_\mu) + 1 = \ell(w_\lambda)$. Therefore, if we choose a reduced decomposition $s_{j_p} \cdots s_{j_1}$ of $w_\mu^{-1}$, then $s_{j_p} \cdots s_{j_1}$ is a reduced decomposition of $w_\lambda^{-1}$ and formula (2) implies that

$$\Pi(w_\lambda^{-1}) = \{\alpha_i\} \cup s_i(\Pi(w_\mu^{-1}))$$

Now, from Lemma 2.2 we obtain that

$$\{\alpha \in R^+ \mid (\alpha, \lambda + \Lambda_0) < 0\} = \{\alpha_i\} \cup \{s_i(\alpha) \mid \alpha \in R^+ \text{ and } (\alpha, \mu + \Lambda_0) < 0\}$$

Therefore,

$$e_{\lambda} = (1 - q^{(\alpha, \lambda)}) \prod_{(\alpha, \mu + \Lambda_0) < 0} (1 - q^{(s_i(\alpha), \lambda)})$$

Note that $(\alpha_i, \lambda + \Lambda_0) = -(\alpha_i, \mu + \Lambda_0)$ and $(\alpha_i, \tilde{w}_\lambda(\lambda_j^\vee)) = -(\alpha_i, \tilde{w}_\mu(\lambda_j^\vee))$ (we have used here the first part of Lemma 1.6). This implies that

$$q^{(\alpha, \lambda)} = q^{-(\alpha, \lambda)}$$

The same argument shows that

$$\prod_{(\alpha, \mu + \Lambda_0) < 0} (1 - q^{(s_i(\alpha), \lambda)}) = e_{\mu}$$

which finishes the proof. \hfill \qedsymbol

2.6. Intertwiners. The main technical tools used in this paper are the intertwining operators of double affine Hecke algebras defined by Cherednik [3]. Our notation and normalization of the intertwiners differ slightly from [3], but are consistent with [3]. The novelty is the intertwiner denoted below by $\tilde{G}_{0, \lambda}$.

For any weight $\lambda$ and any $1 \leq i \leq n$ define the operator $G_{i, \lambda} = G_{i, \lambda}(q, t)$ as follows. If $2\alpha_i \notin R$ then

$$G_{i, \lambda} := (1 - q^{-(\alpha_i, \lambda)})t_i^{-\frac{1}{2}}T_i + q^{-(\alpha_i, \lambda)}(1 - t_i^{-1})$$

If $2\alpha_n \in R$ then

$$G_{n, \lambda} := (1 - q^{-(\alpha_i, \lambda)})t_n^{-\frac{1}{2}}T_n + q^{-(\alpha_i, \lambda)}(1 - t_n^{-1}) + q^{-(\alpha_n, \lambda)}t_n^{-\frac{1}{2}}(t_n^2 - t_0^2)$$

(11)

The operator $G_{0, \lambda}$ is defined by the first formula below if $2\alpha_0$ is not a root or by the second formula otherwise

$$G_{0, \lambda} := q^{-(\alpha_0, \lambda) + \Lambda_0} \left( (1 - q^{-(\alpha_0, \lambda)})t_0^{-\frac{1}{2}}T_0 + q^{-(\alpha_0, \lambda)}(1 - t_0^{-1}) \right)$$

(12)
Proof. The formula for the action of $G_{0,\lambda}$ involves $T_{02}$, but thanks to Remark 2.3, we can express $T_{02}$ in terms of $T_{03}$ as follows

$$T_{02} = T_{02}^{-1} + t_{02}^{-\frac{1}{2}} - t_{02}^{-\frac{1}{2}} = T_{03}Y_{\alpha_0} + t_{02}^{-\frac{1}{2}} - t_{02}^{-\frac{1}{2}}$$

Note that $T_{03}Y_{\alpha_0} \cdot E_\lambda(q,t) = q^{(\alpha_0,\alpha_0)}T_{03} \cdot E_\lambda(q,t)$ and therefore

$$T_{02} \cdot E_\lambda(q,t) = q^{(\alpha_0,\alpha_0)}T_{03} \cdot E_\lambda(q,t) + (t_{02}^{-\frac{1}{2}} - t_{02}^{-\frac{1}{2}})E_\lambda(q,t)$$

Our claim is an immediate consequence of this formula.
2.7. A weak polynomiality result. Generically, the coefficients of Macdonald polynomials are rational functions in $q$ and $t$. Therefore, when assigning specific values to parameters one has to make sure that the coefficients are well defined for those values. One particular instance was considered in [9, Section 2.3] where it was shown that the limit $E_\lambda(q, \infty) := \lim_{t \to \infty} E_\lambda(q, t)$ is well defined for any weight. This a fact is a consequence of the following more precise description of the coefficients: they are quotients of polynomials in $q, q^{-1}$ and $t^{-1}$ and the denominators approach 1 when $t \to \infty$.

The technique used to show such a result was introduced by Knop and Sahi [19] (for Jack polynomials), Knop [14] (in type $A$) and Cherednik [3] (general case). The idea is to analyze the action of intertwiners on nonsymmetric polynomials and prove the statement by induction. In the general case, the first result on the nature of the coefficients of $e_\lambda E_\lambda(q, t)$ is due to Cherednik [3, Corollary 5.3]: they are Laurent polynomials in $q, t$. A slight improvement (obtain merely by revisiting Cherednik’s argument) appeared in [9, Section 2.3]: the coefficients are polynomials in $q, q^{-1}, t^{-1}$. However, since here we are interested in the limit $q \to \infty$ a stronger version of these results is needed. The general lines of the argument are the same, but the key new ingredient is the operator $\tilde{G}_{0, \lambda}$ whose action, unlike that of $G_{0, \lambda}$, is virtually independent on $q$.

In type $A$, stronger results are known [19 Corollary 5.2], [31] as the polynomiality of the coefficients is obtained for a normalization of the nonsymmetric polynomials by a significantly smaller factor. The stronger results are obtained along the same lines as below but taking advantage of the additional stability of these polynomials in type $A$. However, for our present purposes the following result will be sufficient.

**Theorem 2.15.** If the root system $\hat{R}$ is reduced then,

1. For any weight $\lambda$ the coefficients of $e_\lambda E_\lambda(q, t)$ are polynomials in $q^{-1}, t_s^{-1}, t_\ell^{-1}$ with integer coefficients.

2. For any anti–dominant weight $\lambda$, the coefficients of $e_\lambda P_\lambda(q, t)$ are polynomials in $q^{-1}, t_s^{-1}, t_\ell^{-1}$ with integer coefficients.

If the root system $\hat{R}$ is nonreduced, then

3. For any weight $\lambda$, the coefficients of $e_\lambda E_\lambda(q, t)$ are polynomials in $q^{-1}, t_{01}^{-1}, t_{02}^{-1}, t_{03}^{-1}, t_n^{-1}, t_0^{-1}, t_n^{-\frac{1}{2}}, t_0^{-\frac{1}{2}}, t_n^{\frac{1}{2}}, t_0^{\frac{1}{2}}, t_n^{\frac{1}{2}}, t_0^{\frac{1}{2}}, t_n^{-\frac{1}{2}}, t_0^{-\frac{1}{2}}, t_n^{\frac{1}{2}}, t_0^{\frac{1}{2}}$ with integer coefficients.

**Proof.** (1) Let us note first that the proof of part (3) of our statement follows by precisely the same argument presented below by only keeping in mind that the elements $T_n, T_{01}, T_{02}, G_{0, \lambda}$ and $\tilde{G}_{0, \lambda}$ act in a slightly different way. In what follows let us assume that $R$ is a reduced root system.
The statement will be proved by induction on the Bruhat order of $P$. The minimal elements with respect to the Bruhat order are the minuscule weights $\lambda \in \mathcal{O}_P$. For such an element we have that $E_\lambda(q, t) = e^\lambda$, by Proposition 2.5. Moreover, $\lambda$ being an element of the affine fundamental chamber $C$ it satisfies $(\lambda + \Lambda_0, \alpha_i) \geq 0$ for all affine simple roots $\alpha_i$ and in consequence $(\lambda + \Lambda_0, \alpha) \geq 0$ for all affine positive roots. In conclusion, the normalizing factor $e_\lambda$ equals 1 and it clear that in this case $e_\lambda E_\lambda(q, t)$ has the predicted properties.

Assume now that $\lambda$ is an arbitrary non-minuscule weight and that our statement is true for all weights $\mu < \lambda$. Since the weight $\lambda$ does not belong to the affine fundamental chamber we can find an affine simple root $\alpha_i$ such that $(\lambda + \Lambda_0, \alpha_i) < 0$.

Let us consider the weight $\mu = s_i \cdot \lambda$. It is clear that $(\mu + \Lambda_0, \alpha_i) > 0$ and therefore Lemma 1.6 implies $\lambda = s_i \cdot \mu > \mu$.

In particular, the induction hypothesis applies and we have that $e_\mu E_\mu(q, t)$ has coefficients which satisfy the conclusion of the Theorem. Moreover, Corollary 2.14 implies that $G_{i, \mu} \cdot e_\mu E_\mu(q, t) = e_\lambda E_\lambda(q, t)$.

From the fact that $(1 - q^{-(\alpha, \overline{\alpha})})$ appears as a factor in $e_\lambda$ and from Proposition 2.8 we deduce that $q^{-(\alpha, \overline{\alpha})}$ is a monomial in $q^{-1}$, $t_s^{-1}$, $t_\ell^{-1}$. If $i \neq 0$, it can be seen directly from the the formula (10) that the action of $G_{i, \mu}$ involves only $t_i^{-1}$ and $q^{-(\alpha, \overline{\alpha})}$. In conclusion, since the coefficients of $e_\mu E_\mu(q, t)$ are polynomials in $q^{-1}$, $t_s^{-1}$, $t_\ell^{-1}$ with integer coefficients, the coefficients of $e_\lambda E_\lambda(q, t)$ have the same property.

If $i = 0$, the action of $t_0^{-\frac{1}{2}} T_{02}$ involves $q$ in highly nontrivial manner but, nevertheless, it involves only $t_0^{-1}$, so we can deduce that the coefficients of $e_\lambda E_\lambda(q, t)$ are polynomials in $q^{\pm 1}$, $t_s^{-1}$, $t_\ell^{-1}$ with integer coefficients.

However, from Proposition 17 we also obtain that $\widetilde{G}_{0, \mu} \cdot e_\mu E_\mu(q, t) = e_\lambda E_\lambda(q, t)$.

From Proposition 2.2 we deduce that $T_{03} = X_0 T_{s_0}^{-1}$ and therefore its action involves the parameters $t_j^{\pm \frac{1}{2}}$ in a complicated way but it does not involve the parameter $q$ at all. From this fact and from formula (15) we obtain that the coefficients of $e_\lambda E_\lambda(q, t)$ are polynomials in $q^{-1}$, $t_s^{\pm \frac{1}{2}}$, $t_\ell^{\pm \frac{1}{2}}$ with integer coefficients.

Combining he conclusions of the previous two paragraphs we can conclude that the coefficients of $e_\lambda E_\lambda(q, t)$ must be polynomials in $q^{-1}$, $t_s^{-1}$, $t_\ell^{-1}$ with integer coefficients.
(2) From Proposition 2.7 we deduce that

$$e_{\lambda}P_{\lambda}(q, t) = \sum_{\mu \in W(\lambda)} b_{\mu}e_{\mu}E_{\mu}(q, t)$$

(19)

with the coefficients

$$b_{\mu} = a_{\mu} e_{\lambda} = \prod_{\alpha \in \Pi(w_{\mu}^{-1})} (t_{\alpha}^{-1} - q^{-(\alpha, \mu)})$$

Above we have used Lemma 2.10 and the definition of $a_{\mu}$. We can conclude, using Lemma 1.2, that $b_{\mu}$ is a polynomial in $q^{-1}, t_{\mu}^{-1}, t_{\ell}^{-1}$ with integer coefficients and therefore by part (1) we obtain the desired result. 

□

3. The $p$–adic degeneration

3.1. The limit $q \to \infty$. In recognition of their interpretation within the framework of the representation theory of $p$–adic reductive groups we collect in this section a few results regarding the limit of nonsymmetric Macdonald polynomials as $q \to \infty$.

One important consequence of Theorem 2.15 is that for any weight $\lambda$ the coefficients of $E_{\lambda}(q, t)$ are rational functions in $q$ and $t$ which can be written as quotients of two polynomials in $q^{-1}$ and $t^{-1}$ (if the root system in question is nonreduced the statement about the polynomiality in $t^{-1}$ should be altered in accordance with Theorem 2.15). Moreover, the denominator $e_{\lambda}$ approaches 1 when $q \to \infty$. Therefore, all coefficients of $E_{\lambda}(q, t)$ have finite limits as $q \to \infty$. Essentially the same argument shows that $P_{\lambda}(\infty, t)$ is well defined.

The limit of the nonsymmetric Macdonald polynomials $E_{\lambda}(q, t)$ as the parameter $q$ approaches infinity will be denoted by $E_{\lambda}(\infty, t)$ and will be referred to as the $p$–adic degeneration of nonsymmetric Macdonald polynomials. The terminology is motivated by the fact that for specific values of the parameters $t_{\mu}$ they do have an interpretation as Satake transforms of some matrix coefficients in unramified principal series representations of simple $p$–adic groups [11].

The symmetric polynomials $P_{\lambda}(\infty, t)$ are in fact already familiar objects in the representation theory of $p$–adic groups: up to a scalar factor they are the polynomials that give the values of zonal spherical functions on a simple algebraic group $G$ (defined over a $p$–adic field $k$), relative to a special maximal compact subgroup $K$, such that the affine root system associated to $G$ is the dual affine root system $R^\vee$ or $R$ depending on whether $G$ does or does not split over the unramified closure of $k$. The parameters $t_{\mu}$ represent here specific integer powers of the cardinality of the residue field of $k$. If $\hat{R}$ is the root system of type $A_n$ the polynomials $P_{\lambda}(\infty, t)$ are also known as the Hall–Littlewood polynomials. For reduced root systems, by further specializing the parameters $t \to \infty$, we obtain that $P_{\lambda}(\infty, \infty)$ is the irreducible Weyl character with lowest weight $\lambda$ for the simple complex Lie algebra $g$.
with root system $\tilde{R}$ (or, equivalently, for the simple, simply-connected, compact Lie group with root system $\tilde{R}$).

Since we specialized the parameter $q$, from now on we will assume that the affine Hecke algebra $\mathcal{H}_X$ is defined over $F_t$ rather than over $F$.

**Theorem 3.1.** Let $\lambda$ be a weight and let $\alpha_i$ be a simple affine root such that $(\lambda + \Lambda_0, \alpha_i) > 0$. If $i \neq 0$ then

$$T_i \cdot E_\lambda(\infty, t) = t_i^{\frac{1}{2}} E_{s_i \cdot \lambda}(\infty, t)$$

If $i = 0$ then

$$T_{03} \cdot E_\lambda(\infty, t) = t_{01}^{\frac{1}{2}} t^{-c_0^{-1}(\theta, \widetilde{\lambda})} E_{s_{0} \cdot \lambda}(\infty, t)$$

**Proof.** The action of the operators $G_{s_i, \lambda}$ for $i \neq 0$ and $\tilde{G}_{0, \lambda}$ admit limits as $q \to \infty$. The above remarks, Theorem 2.12 and Proposition 2.13 imply the desired result. □

Let $\xi : \mathcal{H}_X \to F_t$ be the $\mathbb{Q}$-algebra morphism of which acts as identity on the parameters $t_i$, 1 ≤ $i$ ≤ $n$, sends $t_{03}$ to $t_{01}$ and

$$\xi(T_i) = t_i^{\frac{1}{2}}, \ i \neq 0 \text{ and } \xi(T_{03}) = t_{01}^{\frac{1}{2}}$$

(22)

We will abuse notation and write $\xi(w)$ to refer to $\xi(T_w)$ for $w$ in $W$. If we set all the parameters equal $t_1 = \cdots = t_n = t_01 =: t$, then

$$\xi(w) = t^{(w)/2}$$

We will abuse notation and write $\xi(w)$ to refer to $\xi(T_w)$ for $w$ in $W$. If we set all the parameters equal $t_1 = \cdots = t_n = t_01 =: t$, then

$$\xi(w) = t^{(w)/2}$$

(23)

Given a weight $\lambda$ define the following normalization factor

$$f_\lambda := \xi(w_\lambda) t^{(\lambda, \theta)}$$

(24)

**Proposition 3.2.** Let $\lambda$ be a weight and let $\alpha_i$ be a simple affine root such that $(\lambda + \Lambda_0, \alpha_i) > 0$. Then

$$T_i \cdot f_\lambda E_\lambda(\infty, t) = f_{s_i \cdot \lambda} E_{s_i \cdot \lambda}(\infty, t)$$

**Proof.** From Theorem 3.1 it is clear that we only have to check that under our hypothesis $f_{s_i \cdot \lambda} = t_i^{\frac{1}{2}} f_\lambda$ if $i \neq 0$ and $f_{s_0 \cdot \lambda} = t_{01}^{\frac{1}{2}} t^{-c_0^{-1}(\theta, \widetilde{\lambda})} f_\lambda$ if $i = 0$.

Let us assume first that $i \neq 0$. Then, Lemma 1.4 implies that $\xi(w_{s_i \cdot \lambda}) = t_i^{\frac{1}{2}} \xi(w_\lambda)$ and $\tilde{w}_{s_i \cdot \lambda} = s_i \tilde{w}_\lambda$ which prove the above claim. If $i = 0$ then by the same result

$$\xi(w_{s_0 \cdot \lambda}) = t_{01}^{\frac{1}{2}} \xi(w_\lambda)$$

and $\tilde{w}_{s_0 \cdot \lambda} = s_0 \tilde{w}_\lambda$. In particular, for any $x \in \hat{h}^*$

$$(s_0 \cdot \lambda, \tilde{w}_{s_0 \cdot \lambda}(x)) = (s_0(\lambda) + c_0^{-1}(\theta, \tilde{w}_\lambda(x)))$$

$$(\lambda, \tilde{w}_\lambda(x)) = (\lambda, \tilde{w}_\lambda(x))$$

Therefore, $t^{(s_0 \cdot \lambda, \tilde{w}_{s_0 \cdot \lambda})} = t^{-c_0^{-1}(\theta, \widetilde{\lambda})} t^{(\lambda, \theta)}$ and the proof is completed. □

**Corollary 3.3.** Let $\lambda$ be a weight. Then

$$T_{w_\lambda} : f_\lambda e^\lambda = f_\lambda E_\lambda(\infty, t)$$
Corollary 3.4. For any weight $\lambda$, the parameters $t_i$, $t_{01}$ appear in $\xi(w_\lambda) f_\lambda/f_{\bar{\lambda}}$ with integer exponents. If all the parameters are equal (and denoted by $t$) then

$$f_\lambda = t^{-\ell(w_\lambda)/2}$$

Proof. The first claim follows immediately from the fact that the monomials $f_\mu$ are obtained inductively as in the proof of the above Proposition and the fact that $c_0^{-1}(\theta, \tilde{w}_\mu(\lambda^\vee))$ are integers.

For the second claim, note that by using the first part of Lemma 1.6 and formula (4) we obtain that

$$t(\lambda, \lambda) = t(\lambda-1, \lambda-1) = t^{-\ell(\tau_\lambda)/2}$$

and our statement immediately follows. \qed

The next result expresses the relationship between the $p$–adic degeneration of the symmetric and nonsymmetric Macdonald polynomials.

Proposition 3.5. Let $\lambda$ be an anti–dominant weight. Then,

$$P_\lambda(\infty, t) = \sum_{\mu \in \tilde{W}(\lambda)} \xi(\tilde{w}_\mu)^{-2} E_\mu(\infty, t)$$

Proof. Straightforward from Proposition 2.7 and the definition of $\xi(\tilde{w}_\mu)$. We only note that although we only stated Proposition 2.7 for reduced root systems, a similar fact holds for nonreduced root systems \cite[Theorem 4.11]{8} and the limit of the corresponding coefficients $a_\mu$ in the limit $q \to \infty$ equals also $\xi(\tilde{w}_\mu)^{-2}$. \qed

Proposition 3.6. Assume $\lambda$ is a dominant weight. Then, $E_\lambda(\infty, t) = e^\lambda$.

Proof. First, note that Corollary 3.3 allows us to write

$$T_{w_\lambda} \cdot f_\lambda e^\lambda = f_\lambda E_\lambda(\infty, t) \quad (25)$$

Second, from Lemma 1.7 we know that $\ell(\tau_\lambda) = \ell(w_\lambda) + \ell(\tilde{w}_\lambda^{-1})$. Hence, $T_{\tau_\lambda} = T_{w_\lambda} \omega_\lambda T_{\tilde{w}_\lambda^{-1}}$. The weight $\lambda$ being dominant we have that $X_\lambda = T_{\tau_\lambda}$ and consequently,

$$T_{w_\lambda} X_\lambda = X_\lambda T_{\tilde{w}_\lambda^{-1}} T_{\tilde{w}_\lambda^{-1}}$$

Therefore,

$$T_{w_\lambda} \cdot f_\lambda e^\lambda = f_\lambda X_\lambda T_{\tilde{w}_\lambda^{-1}}^{-1} T_{\tilde{w}_\lambda^{-1}} \cdot 1 = f_\lambda \xi(\tilde{w}_\lambda)^{-1} \xi(\tilde{w}_\lambda) e^\lambda$$

Also, the coefficient of $e^\lambda$ in $E_\lambda(\infty, t)$ is 1 and the claim follows. \qed
One immediate consequence of the above computation is that
\[ f_\lambda/f_\tilde{\lambda} = \xi(\tilde{w}_\lambda)^{-1}/\xi(\tilde{w}_{\tilde{\lambda}})^{-1} \] (26)
for \( \lambda \) anti-dominant, but keeping in mind (see the proof of Corollary \[ \text{Corollary 3.4} \]) that \( t(\mu, \mu) \) is constant for \( \mu \in \hat{W}(\lambda) \) and the Lemma \[ \text{Lemma 1.7 (1)} \) we deduce that (26) is true for any weight \( \lambda \).

**Corollary 3.7.** Let \( \lambda \) be a weight. Then,
\[ E_\lambda(\infty, t) = \xi(\tilde{w}_\lambda w_\circ)^{-1}T_{\tilde{w}_\lambda w_\circ} \cdot e^{\lambda^+} \]
In particular, if \( \hat{R} \) nonreduced the polynomials \( E_\lambda(\infty, t) \) are free of the variables \( t_{01}, t_{02} \).

As anticipated in \[ \text{Corollary 3.7} \) we have the following

**Corollary 3.8.** Assume \( \hat{R} \) is reduced. For any weight \( \lambda \)
\[ E_\lambda(\infty, \infty) := \lim_{t \to \infty} E_\lambda(\infty, t) \]
is the Demazure character associated to the irreducible representation of \( \mathfrak{g} \) with highest weight \( \lambda^+ \) and extremal weight \( \lambda \). In particular, if \( \lambda \) is anti-dominant, then \( E_\lambda(\infty, \infty) \) is the Weyl character of the irreducible representation of \( \mathfrak{g} \) with lowest weight \( \lambda \). Moreover,
\[ E_\lambda(\infty, \infty) = \lim_{t \to \infty} \lim_{q \to \infty} E_\lambda(q, t) = \lim_{q \to \infty} \lim_{t \to \infty} E_\lambda(q, t) \] (27)

**Proof.** The fact that \( E_\lambda(\infty, \infty) \) is the Demazure character associated to the irreducible representation of \( \mathfrak{g} \) with highest weight \( \lambda^+ \) and extremal weight \( \lambda \) is an immediate consequence of Corollary \[ \text{Corollary 3.7} \) and of the Demazure character formula \[ \text{Proposition 3.5} \). The equation (27) follows by combining this result with Theorem 3 in \[ \text{Corollary 3.7} \)  \( \square \)

It is also clear that \( E_\lambda(\infty, 1) = e^{\lambda} \) for all weights \( \lambda \) and therefore the polynomials \( E_\lambda(\infty, t) \) interpolate between monomials \( e^{\lambda} \) and Demazure characters associated to the irreducible representation of \( \mathfrak{g} \) with highest weight \( \lambda^+ \) and extremal weight \( \lambda \). This property is the nonsymmetric analogue of the corresponding fact regarding the symmetric polynomial \( P_\lambda(\infty, t) \) which is known to interpolate between the symmetrized monomial \( \sum_{\mu \in \hat{W}(\lambda)} e^\mu \) and the Weyl character of the irreducible representation of \( \mathfrak{g} \) with lowest weight \( \lambda \) (note that here \( \lambda \) is an anti-dominant weight).

**Corollary 3.9.** Let \( \lambda \) be an anti-dominant weight. Then,
\[ P_\lambda(\infty, t) = \xi(w_\circ)^{-1} \sum_{\mu \in \hat{W}(\lambda)} \xi(\tilde{w}_\mu)^{-1}T_{\tilde{w}_\mu w_\circ} \cdot e^{\lambda^+} \] (28)

**Proof.** Straightforward from Proposition \[ \text{Proposition 3.5} \) and Corollary \[ \text{Corollary 3.7} \) \( \square \)
Remark that if $\hat{R}$ is a reduced root system and $t \to \infty$ the equation \(28\) becomes precisely the Demazure character formula for the irreducible representation of $g$ with highest weight $\lambda_+$. In the light of the connection between $P_\lambda(\infty, t)$ and spherical functions on simple groups over $p$-adic fields, the equation \(28\) could be seen as a counterpart of Demazure’s formula for this type of spherical functions. The above result also follows from equation (5.4) and Lemma 4.2 in [18] together with Macdonald’s formula for the Satake transforms of the elements $N_\lambda$ in [18].

It is natural to introduce the following normalization of the nonsymmetric Macdonald polynomials.

**Definition 3.10.** For any weight $\lambda$ define

$$\tilde{E}_\lambda(q, t) := \xi(\hat{w}_\lambda)^{-1}E_\lambda(q, t)$$

We close this section with a reformulation of Corollary 3.3 in terms of the above normalization. For roots systems of type $A$ the result was proved by Knop [16, Corollary 5.3].

**Corollary 3.11.** Let $\lambda$ be a weight. Then

$$T_{w_\lambda} \cdot \tilde{E}_\lambda(\infty, t) = \tilde{E}_\lambda(\infty, t)$$

**Proof.** Clearly,

$$T_{w_\lambda} \cdot \tilde{E}_\lambda(\infty, t) = (\xi(\hat{w}_\lambda)^{-1}/f_\lambda)T_{w_\lambda} \cdot f_\lambda e^{\tilde{\lambda}} = (\xi(\hat{w}_\lambda)^{-1}/f_\lambda)E_\lambda(\infty, t) = \tilde{E}_\lambda(\infty, t)$$

In the last step \(26\) was used. \qed

### 3.2. Normalized intertwiners

For any weight $\lambda$ and any $0 \leq i \leq n$ define the following normalized versions of the intertwiners. The second formula below defines $I_{0,\lambda}$ for reduced root systems and the third formula defines it for nonreduced root systems

$$I_i,\lambda := t_i^{\frac{1}{2}}G_i,\lambda/(1 - q^{-(\alpha_i^*, \lambda)})$$

$$I_{0,\lambda} := t_0^{\frac{1}{2}}G_0,\lambda/(t^{(\theta, \lambda)} - q^{-(\alpha_0, \lambda + \Lambda_0)})$$

$$I_{0,\lambda} := t_0^{\frac{1}{2}}G_0,\lambda/(t^{(\theta, \lambda)} - q^{-(\alpha_0, \lambda + \Lambda_0)})$$

Fix a weight $\lambda$ and a reduced decomposition $s_{j_\ell} \cdots s_{j_1}$ of $w_\lambda$. Denote $\lambda(1) = \tilde{\lambda}$ and $\lambda(i) = s_{j_{i-1}} \cdots s_{j_1} \cdot \tilde{\lambda}$ for $2 \leq i \leq \ell$. With this notation define

$$I_{w_\lambda} := I_{j_\ell, \lambda(1)} \cdots I_{j_1, \lambda(1)}$$
**Theorem 3.12.** Let $\lambda$ be a weight. Then

$$I_w \cdot \tilde{E}_\lambda(q, t) = \tilde{E}_\lambda(q, t)$$

**Proof.** First, remark that Theorem 3.1 holds for $I_i$ replacing $T_i$ and $E_\lambda(q, t)$ replacing $E_\lambda(\infty, t)$ which then implies the conclusions of Proposition 3.2 and Corollary 3.3 (under the same substitutions). The proof can be concluded following exactly the same line as the proof of Corollary 3.11. □

4. Bases for maximal parabolic modules

4.1. The Kazhdan-Lusztig involution. In this section we begin to explore the connection between the nonsymmetric Macdonald polynomials and the Kazhdan–Lusztig theory (in its parabolic version [4]). We start by recalling the construction of the Kazhdan–Lusztig basis [14] in its multi-parameter version [23] and some other basic facts.

From now on we will work over the field obtained from $\mathbb{F}$ by specializing the parameters $t_{01}, t_{02}$ (if present) to 1. As stated in the Corollary 3.7 the $p$–adic limit is independent of these variables hence unaffected by the specialization. To avoid introduction new notation we will use the old notation for the fields and polynomials under consideration. From the point of view of the Kazhdan–Lusztig theory a new feature is the introduction of $q$ as a parameter. As we will see below, this is completely harmless in regard to the general theory, but it will allow us to draw some conclusions regarding the interpretation of the nonsymmetric polynomials within this framework.

Let $\chi : \mathcal{H}_X \to \mathbb{F}$ be the $\mathbb{F}$–algebra map which sends each of the generators $T_i, T_{03}$ to the square root of the corresponding parameter.

The Kazhdan–Lusztig involution $\kappa$ is the involution of the algebra $\mathcal{H}_X$ which inverts the parameters $q, \{t_\alpha\}_{\alpha \in \mathcal{R}}$ and the generators $T_i, T_{03}$. On a standard basis element it acts as follows

$$\kappa(T_w) = T_w^{-1} \quad (29)$$

In fact, we can extend $\kappa$ to $\mathcal{H}_X'$ (as an algebra map) by letting it act as identity on $\Omega$. The formula (29) is then valid for any $w \in W^e$.

Recall from [28] Proposition 2, [25] Theorem 5.2] the following result.

**Theorem 4.1.** For any element $w$ of the affine Weyl group $W$ there is a unique element $C'_w$ of $\mathcal{H}_X$ which satisfies the properties

(a) $\kappa(C'_w) = C'_w$

(b) $C'_w = \sum_{y \leq w} P_{y, w}^*(t)y$, where $P_{y, w}^*(t) = 1$ and, if $y < w$, $P_{y, w}^*(t)$ are polynomials in $\{t_{-\frac{1}{2}}\}_{\alpha \in \mathcal{R}}$ with integer coefficients and no constant term.
Moreover, \( P_{y, w}(t) := \chi(y)^{-1}\chi(w)P^{*}_{y, w}(t) \in \mathbb{Z}[t_{\alpha} | \alpha \in R] \).

Lusztig’s result is in fact valid for any Coxeter group. The polynomials \( P_{y, w}(t) \) are Kazhdan–Lusztig polynomials (for the affine Weyl group \( W \)). For equal parameters the polynomials \( P_{y, w}(t) \) have non-negative coefficients. This fact follows from a beautiful cohomological interpretation [15, Theorem 5.5] in terms of the Deligne–Goresky–MacPherson middle intersection cohomology.

We also need the following basic facts [23, (4.2), (4.3)].

**Proposition 4.2.** Let \( s_{i} \) be a simple reflection and \( w \) be an element of \( W \) such that \( s_{i}w < w \). Then,

\[
(T_{i} - t_{i}^{1/2})C'_{w} = 0
\]

**Lemma 4.3.** Let \( s_{i} \) be an affine simple reflection and let \( x, y \) let elements of \( W \) such that \( x < y, x < xs_{i} \) and \( ys_{i} < y \). Then,

\[
P^{*}_{x, y}(t) = t_{i}^{-1/2}P^{*}_{xs_{i}, y}(t)
\]

**Corollary 4.4.** Let \( \mu \) and \( \lambda \) be two weights such that \( \mu \leq \lambda \). Then,

\[
P^{*}_{v_{\mu}, v_{\lambda}}(t) = \chi(y)^{-1}P^{*}_{v_{\lambda}, v_{\lambda}}(t)
\]

for all \( y \) in \( W_{\lambda} \).

The elements \( C'_{v_{\lambda}} \) can be factorized as follows.

**Lemma 4.5.** Let \( \lambda \) be a weight. Then

\[
C'_{v_{\lambda}} = \left( \sum_{\mu \leq \lambda} P^{*}_{v_{\mu}, v_{\lambda}}(t)T_{w_{\mu}, \omega_{\lambda}} \right) \left( \chi(w_{0})^{-1} \sum_{x \in W} \chi(x)T_{x} \right) \omega_{\lambda}^{-1}
\]

**Proof.** Let us remark first that from Lemma 1.5

\[
\{ y \in W \ | \ y \leq v_{\lambda} \} = \bigcup_{\mu \leq \lambda} v_{\mu}W_{\lambda}
\]

and from the above Corollary

\[
P^{*}_{v_{\mu}, v_{\lambda}}(t) = \chi(y)^{-1}P^{*}_{v_{\lambda}, v_{\lambda}}(t)
\]

for all \( y \) in \( W_{\lambda} \) and \( \mu \leq \lambda \). Hence, the element \( C'_{v_{\lambda}} \) takes the form

\[
C'_{v_{\lambda}} = \sum_{\mu \leq \lambda} P^{*}_{v_{\mu}, v_{\lambda}}(t) \left( \sum_{y \in W_{\lambda}} \chi(y)^{-1}T_{v_{\mu}, y} \right)
\]

Since \( v_{\mu}W_{\hat{\mu}} = w_{\lambda}W_{\hat{\mu}} \) and \( \ell(v_{\mu}, y) = \ell(v_{\mu}) - \ell(y) = \ell(w_{\mu}) + \ell(w_{0, \lambda}) - \ell(y) \) for any \( y \) in \( W_{\hat{\mu}} \) we get that

\[
\sum_{y \in W_{\lambda}} \chi(y)^{-1}T_{v_{\mu}, y} = \chi(w_{0, \lambda})^{-1}T_{w_{\mu}} \sum_{x \in W_{\lambda}} \chi(x)T_{x}
\]
Now, $W_\lambda = \omega_\lambda \tilde{W} \omega_\lambda^{-1}$ and $\chi$ is invariant under the conjugation action of $\Omega$. Our claim now immediately follows. □

For the following result recall the notation in Section 3.2 and that we work under the assumption $t_{01} = t_{02} = 1$.

**Lemma 4.6.** Let $\lambda$ be a weight. Then $I_w \omega_\lambda$ is fixed by $\kappa$.

**Proof.** It is a straightforward check that each factor of $I_w \omega_\lambda$ is fixed by $\kappa$. □

### 4.2. The parabolic module.

Restricting $\chi$ to $\tilde{H}$ we obtain $(\chi_{|\tilde{H}}, \mathbb{F}_t)$ a one dimensional representation of $\tilde{H}$. The induced representation

$$\text{ind}_{\tilde{H}}^{H_X}(\chi) := H_X \otimes_{\tilde{H}} \mathbb{F}_t$$

is a left module for $H_X$. In general, there are several standard maximal parabolic subgroups of $W$ isomorphic to $\tilde{W}$ (as many as the order of $\Omega$) and one can construct in the same manner the corresponding induced representation of the affine Hecke algebra $H_X$. These, however, are all isomorphic to the one defined above. One can consider all of them together by constructing the $H_X^c$-module

$$\text{ind}_{\tilde{H}}^{H_X^c}(\chi) := H_{X^c} \otimes_{\tilde{H}} \mathbb{F}_t$$

Following Knop [17] we call the above module the (maximal) parabolic module of $H_{X^c}$. By Proposition 2.2 the parabolic module has a basis given by $\{X_\lambda \otimes 1\}_{\lambda \in P}$ and it is isomorphic as a $H_X^c$-module to $R_t$ ($X_\lambda \otimes 1$ and $e^\lambda$ correspond under the isomorphism).

It is a standard fact (see [4]) that there exists an involution (still called the Kazhdan–Lusztig involution and denoted by $\kappa$)

$$\kappa : R \to R, \quad f \mapsto f^\kappa$$

compatible with the one on $H_X^c$ in the following sense

$$(H \cdot f)^\kappa = \kappa(H) \cdot f^\kappa$$  \hspace{2cm} (30)

for any $H \in H_X^c$ and $f \in R$. In our case, however, everything can be made quite explicit. Note that for $\lambda$ dominant $X_\lambda = T_{\tau_\lambda}$ and therefore

$$\kappa(X_\lambda) = T_{\tau_\lambda}^{-1} = T_{w_0} T_{\tau_{w_0(\lambda)}} T_{w_0}^{-1} = T_{w_0} X_{w_0(\lambda)} T_{w_0}^{-1}$$

The map $\kappa$ being an algebra morphism we obtain that

$$\kappa(X_\lambda) = T_{w_0} X_{w_0(\lambda)} T_{w_0}^{-1}$$
for any weight $\lambda$. Applying now (30) for $H = X_{\lambda}$ and $f = 1$ we obtain that

$$\kappa(e^{\lambda}) = \chi(w_o)^{-1}T_{w_o} \cdot e^{w_o(\lambda)}$$

Of course, $\kappa$ acts on $F$ by inverting the parameters.

Keeping in mind that $\{w_{\lambda}^\omega \}_{\lambda \in P}$ is the set of minimal coset representatives for $W^e/\dot{W}$ we obtain the following bases of $R$ which are induced from elements of $H^F_{\lambda}$.

**Definition 4.7.** Define the following are bases for $R$:

(a) the standard basis: $\{T_{w_\lambda^\omega} \cdot 1 \}_{\lambda \in P}$;
(b) the dual standard basis: $\{T^{-1}_{w_\lambda^\omega} \cdot 1 \}_{\lambda \in P}$;
(c) the canonical basis $\{C_{\lambda}' := t^\ell(\dot{w})/2 \dot{W}(t)^{-1}C_{\lambda}' \cdot 1 \}_{\lambda \in P}$.

Above we denoted by $\dot{W}(t) := \sum_{\dot{w} \in \dot{W}} t^\ell(\dot{w})$ the Poincaré polynomial of $\dot{W}$.

The coefficients of the expansion of the canonical basis in the standard basis are the parabolic Kazhdan–Lusztig polynomials (for the maximal parabolics $\dot{W}_\lambda \subset W$) of Deodhar [6]. We are now ready to establish one connection between the non-symmetric Macdonald polynomials and the (parabolic) Kazhdan–Lusztig theory.

**Theorem 4.8.** The basis $\{\tilde{E}_\lambda(q,t)\}_{\lambda \in P}$ of the parabolic module of the affine Hecke algebra $H^F_{\lambda}$ is invariant under the Kazhdan–Lusztig involution. Moreover,

1. $\{\tilde{E}_\lambda(\infty,t)\}_{\lambda \in P}$ is the standard basis;
2. $\{\tilde{E}_\lambda(0,t)\}_{\lambda \in P}$ is the dual standard basis.

**Proof.** Let $\lambda$ be a weight and fix a reduced decomposition of $w_\lambda$. By Lemma 4.6 the elements $I_{w_\lambda^\omega}$ are fixed by $\kappa$ and therefore, using (30) for $H = I_{w_\lambda^\omega}$ and $f = 1$, we obtain that

$$I_{w_\lambda^\omega} \cdot 1 = I_{w_\lambda^\omega} \cdot \tilde{E}_\lambda(q,t) = \tilde{E}_\lambda(q,t)$$

is fixed by $\kappa$. Also, (1) is exactly Corollary 3.11. To explain (2) note that

$$\tilde{E}_\lambda(q,t) = \chi(w_o)^{-1}T_{w_o} \cdot w_o(\tilde{E}_\lambda(q^{-1}, t^{-1}))$$

As the limit as $q \to 0$ of the right hand side exists (see the discussion at the beginning of Section 3.1) the limit of the left hand side also exists and

$$\tilde{E}_\lambda(0,t) = \tilde{E}_{\lambda}^{\infty}(\infty,t)$$

In conclusion, $\tilde{E}_\lambda(0,t)$ is an element of the dual standard basis. □

The expansion of the canonical basis in terms of the standard basis takes the following form.
Proposition 4.9. Let $\lambda$ be a weight. Then

$$C'_\lambda = \sum_{\mu \leq \lambda} P_{v_\mu, v_\lambda}^* (t) \tilde{E}_{\mu}(\infty, t)$$  \hspace{1cm} (31)

Proof. Straightforward from Lemma 4.5. \hfill \Box

It is clear from (31) that the coefficients of $C'_\lambda$ are polynomials in $\{t^{-\frac{1}{2}}\}_{\alpha \in \hat{R}}$ with integer coefficients. For reduced root systems and $\lambda$ anti-dominant $C'_\lambda$ were shown (originally in [22], later reproved by several authors) to be Weyl characters of the irreducible representation of $g$ with lowest weight $\lambda$. For completeness, we also give a proof here. The argument follows the idea used in [13].

Theorem 4.10. Assume $\hat{R}$ to be a reduced root system and let $\lambda$ be an anti-dominant weight. Then, $C'_\lambda$ is the Weyl character of the irreducible representation of $g$ with lowest weight $\lambda$.

Proof. Let $s_i$ be a simple reflection. The condition $v_\lambda > s_i v_\lambda$ in Proposition 4.2 translates into $\lambda \geq s_i \cdot \lambda$ and this certainly holds for any $1 \leq i \leq n$ (an anti-dominant weight is the highest in its $\hat{W}$ orbit). Therefore,

$$(T_i - t^{\frac{1}{2}})C'_\lambda = 0, \quad \text{for all } 1 \leq i \leq n$$

which is equivalent to $C'_\lambda$ being $\hat{W}$–invariant. On $\hat{W}$–invariant elements, $\kappa$ has only the effect of inverting the parameters. Thus, $C'_\lambda$ being fixed by $\kappa$ and with coefficients polynomials in $\{t^{-\frac{1}{2}}\}_{\alpha \in \hat{R}}$ forces it to be free of parameters. Therefore, we may safely take all the parameters to infinity in (31) without altering $C'_\lambda$. The only term on the right hand side which survives this process in $\tilde{E}_{\lambda}(\infty, \infty)$ which by Corollary 3.8 is the specified Weyl character. \hfill \Box

Explicit formulas or representation-theoretical interpretations of the elements $C'_\lambda$ beyond the case described above seem to be unknown. However, in the equal parameter case a few special properties are expected.

For the following remarks assume that $\hat{R}$ is a reduced root system and the parameters are equal $t_s = t_\ell =: t$. Computational evidence suggests the following

Conjecture 4.11. For any weight $\lambda$ the polynomial $C'_\lambda$ is the $T$–character of a graded $B$–module. In particular, the positive integers $P_{v_\mu, v_\lambda}(1)$ represent weight multiplicities in $B$–modules.

Given that the the polynomials $\tilde{E}_{\lambda}(q, t)$ interpolate between the standard and the dual standard basis the expansion of the canonical basis in terms of them is especially intriguing. One case in particular draws attention: the expansion of the canonical basis (for $\hat{R}$ reduced, equal parameters) in the polynomials $\tilde{E}_{\lambda}(q, t)$ (for $q = t, t_s = t, t_\ell = t^\ell$) seems to characterized by a support condition which in turn
reduces the problem of computing all the elements of the canonical basis (infinitely many) for a fixed root system to a finite computation. We will report on these investigations elsewhere.

4.3. Orthogonality. We now revert back to the multi-parameter situation. Since the polynomials $E_\lambda(q,t)$ form a basis of $R$ orthogonal with respect to the scalar product $\langle \cdot, \cdot \rangle_{q,t}$. It is natural to ask if such a property holds for the polynomials $\bar{E}_\lambda(\infty,t)$ with respect to the space $R_t$ and a suitable degeneration of the scalar product $\langle \cdot, \cdot \rangle_{q,t}$ as $q \to \infty$. Unfortunately, the definition of the Cherednik scalar product involves the involution $\bar{\cdot}$ on $R$, which inverts the parameter $q$ and it is therefore inconsistent with the process of taking the limit $q \to \infty$. However, we can try to examine the limit as $q \to \infty$ of

$$\langle E_\lambda(q,t), E_\mu(q,t) \rangle_{q,t} = CT \left( E_\lambda(q,t)E_\mu(q,t)C(q,t) \right)$$

(32)

Although it is clear that the limit as $q$ approaches infinity of $E_\lambda(q,t)$ and $C(q,t)$ exists (and equals $E_\lambda(\infty,t)$ and, respectively, $C(\infty,t)$) it is not clear what happens to $\bar{E}_\mu(q,t)$ in the limit. Before stating a result of Cherednik which will allow us to perform such a computation we need to introduce some notation.

Let $\varsigma$ be the involution of $R$ which fixes the parameters $q$ and $t$ and, for any weight $\lambda$, sends $e^\lambda$ to $e^{-w_0(\lambda)}$. Also, let

$$\iota : R \to R, \quad \iota = \chi(w_0)T_{w_0}^{-1}$$

We will also use the notation $f^\iota := \iota(f)$ for any element $f$ of $R$.

**Proposition 4.12.** Let $\lambda$ be a weight. Then,

1. $E_\lambda(q,t) = \chi(\tilde{w}_\lambda)^{-2}\chi(w_0)T_{w_0}^{-1}, E_{-w_0(\lambda)}(q,t)$
2. $\varsigma(E_\lambda(q,t)) = E_{-w_0(\lambda)}(q,t)$
3. $E_\lambda(q,t) = \bar{E}_\lambda(q,t)$

The first claim was proved by Cherednik [4, Proposition 3.3] and the second claim follows along exactly the same lines. Although it is not explicitly stated in [4] it is implicitly used at several places. The third claim is simply a combination of the previous two. As an immediate consequence we have the following

**Corollary 4.13.** For any weight $\lambda$, the limit of $E_\lambda(q,t)$ as $q$ approaches infinity exists and equals $\chi(\tilde{w}_\lambda)^{-2}E_\lambda(\infty,t)$. 

Using this result, it is clear that we can take the limit $q \to \infty$ directly on the right hand side of equation (32) and obtain that

$$\lim_{q \to \infty} \langle E_\lambda(q,t), E_\lambda(q,t) \rangle_{q,t} = \chi(\tilde{w}_\lambda)^{-2}CT \left( E_\lambda(\infty,t)E_\lambda(\infty,t)C(\infty,t) \right)$$

(33)
On the other hand, $E_{\lambda}(q,t)$ are orthogonal with respect to the scalar product $\langle \cdot, \cdot \rangle_{q,t}$ and their norms are explicitly known (see, for example, [2] for formulas for reduced root systems or [28] for completely general results). By examining these norms it is easy to see that

$$\lim_{q \to \infty} \langle E_{\lambda}(q,t), E_{\lambda}(q,t) \rangle_{q,t} = 1$$

It is therefore natural to define, following [4, Corollary 4.3], the following symmetric scalar product on $R_t$. For $f$ and $g$ in $R_t$ let

$$\langle f, g \rangle_t := CT(fg' C(\infty, t))$$

(34)

We have proved the following:

**Proposition 4.14.** The polynomials $\widehat{E}_{\lambda}(\infty, t)$ form a basis of $R_t$ which is orthonormal with respect to the scalar product $\langle \cdot, \cdot \rangle_t$.

Therefore, the natural scalar product in the Kazhdan–Lusztig theory (the one for which the canonical basis is orthonormal) can be seen as a degenerate version of the Cherednik scalar product. One immediate consequence of these considerations is that parabolic Kazhdan–Lusztig polynomials can be obtained as

$$\langle C'_\lambda, \widetilde{E}_\mu(\infty, t) \rangle_t = P^{v_\mu, v_\lambda}_t(t)$$

For root systems of type $A$ this is Lemma 11.3 in [17]. Similarly, from

$$\langle \widetilde{E}_{\lambda}(0, t), \widetilde{E}_\mu(\infty, t) \rangle_t = R^{v_\mu, v_\lambda}_t(t)$$

we obtain the parabolic $R$–polynomials (the notation is consistent with [23]).

5. RELATING TWO LIMITING CASES

5.1. The 0–Hecke algebra. The 0–Hecke algebra discussed here is a suitable degeneration of the Hecke algebra $\hat{H}$ as the parameters $t$ are specialized to zero.

**Definition 5.1.** The 0–Hecke algebra $\hat{N}$ associated to $\hat{R}$ is the $\mathbb{Q}$–algebra described by generators and relations as follows:

**Generators:** One generator $N_i$ for each simple root $\alpha_i$.

**Relations:** a) Each pair of generators satisfies the same braid relations as the corresponding pair of simple reflections.

b) The quadratic relations

$$N_i^2 = -N_i, \quad 1 \leq i \leq n$$

Since the generators $N_i$ satisfy the braid relations a standard basis of $\hat{N}$ is given by the elements $\{N_w\}_{w \in \hat{W}}$ where, as usual, $N_w = N_{i_1} \cdots N_{i_l}$ if $w = s_{i_1} \cdots s_{i_l}$ is a
reduced expression of $w$ in terms of simple reflections. The 0–Hecke algebra has a linear action on $R$ described by

$$N_i \cdot e^\lambda = \frac{e^\lambda - e^{s_i(\lambda)}}{1 - e^{-\alpha_i}}, \quad 1 \leq i \leq n$$

It is straightforward to check that the action of $t_i^2 T_i$ degenerates to the action of $N_i$ if we specialize the parameter $t_i$ to zero. In general, $\chi(w)T_w$ will degenerate to $N_w$. In fact, by degenerating $t_i^2 T_i^{-1}$ we obtain another set of generators

$$N'_i := N_i + 1$$

satisfying the same braid relations and the quadratic relations $N'_i^2 = N'_i$. Of course, $\chi(w)T_w^{-1}$ will degenerate to $N'_w$. For this reason we call $\{N'_w\}_{w \in \dot{W}}$ the dual standard basis of $\dot{N}$. The operators $N_w$ are closely related to the Demazure operators $\Delta_w$, where $\Delta_i$ act as

$$\Delta_i \cdot e^\lambda = e^{s_i(\lambda)} + \frac{e^\lambda - e^{s_i(\lambda)}}{1 - e^{-\alpha_i}}, \quad 1 \leq i \leq n$$

The relationship is

$$w_0 N'_w w_0 = \Delta_w$$

The generators $N'_i$ act on the standard basis as follows

$$N'_i N_w = N_w + N_{s_i w}, \quad \text{if } s_i w > w$$

$$N'_i N_w = 0, \quad \text{if } s_i w < w$$

The following results are certainly well-known.

**Lemma 5.2.** Let $w$ be an element of $\dot{W}$ and let $s_i$ be a simple reflection such that $s_i w > w$. Then,

$$N'_i \sum_{x \leq w} N_x = \sum_{y \leq s_i w} N_y$$

**Proof.** Using the above two formulas we obtain

$$N'_i \sum_{x \leq w} N_x = \sum_{x \leq w, x < s_i x} (N_{s_i x} + N_x)$$

$$= \sum_{y \leq s_i w} N_y$$

The last equality followed from the third property of the Bruhat order. \qed

**Corollary 5.3.** Let $w$ be an element of $\dot{W}$. Then,

$$N'_w = \sum_{x \leq w} N_x$$

**Proof.** Apply the previous Lemma repeatedly. \qed
5.2. The limit $q \to 0$. In this section we collect some immediate consequences of Theorem 4.8 regarding the limit $q \to 0$.

**Corollary 5.4.** Let $\lambda$ be a weight. Then,

$$\widetilde{E}_\lambda(0, t) = T_{\hat{w}_\lambda} \cdot e^{\lambda^-}$$

In consequence, the coefficients of $E_\lambda(0, t)$ are polynomials in $\{t_\alpha\}_{\alpha \in R}$ with integer coefficients.

**Proof.** From Lemma 1.7 (2) we obtain that $X_{\lambda^-} = T_{\hat{w}_\lambda}^{-1} \omega_{\hat{\lambda}}^-$. Hence, Theorem 4.8 implies

$$\widetilde{E}_{\lambda^-}(0, t) = e^{\lambda^-}$$

From Lemma 1.7 (1) we deduce that $T_{\hat{w}_\lambda}^{-1} = T_{\hat{w}_\lambda} T_{\hat{w}_\lambda}^{-1}$. Now, Theorem 4.8 and the above formula give the desired result. \(\square\)

**Corollary 5.5.** Let $\lambda$ be a weight. Then,

$$E_\lambda(0, 0) = N_{\hat{w}_\lambda} \cdot e^{\lambda^-}$$

**Proof.** The conclusion follows by sending the parameters $\{t_\alpha\}_{\alpha \in R}$ to zero in

$$E_\lambda(0, t) = \chi(\hat{w}_\lambda) T_{\hat{w}_\lambda} \cdot e^{\lambda^-}$$

and keeping in mind that $\chi(\hat{w}_\lambda) T_{\hat{w}_\lambda}$ degenerate to $N_{\hat{w}_\lambda}$. \(\square\)

Next, we explain the relationship between the $q \to 0$ limit and the $q \to \infty$ limit.

**Proposition 5.6.** Let $\lambda$ be a weight. Then,

$$w_o \cdot E_\lambda(\infty, t^{-1}) = \chi(w_o \hat{w}_\lambda) T_{(w_o \hat{w}_\lambda)^{-1}}^{-1} \cdot e^{\lambda^-}$$

**Proof.** Let us argue first that

$$T_{w_o}^{-1} T_{\hat{w}_\lambda}^{-1} = T_{(w_o \hat{w}_\lambda)^{-1}}^{-1} T_{\hat{w}_\lambda}^{-1}$$

Indeed, from Lemma 1.7 (1) we get $T_{w_\lambda} = T_{\hat{w}_\lambda} T_{w_\lambda}^{-1}$. Keeping in mind that $T_{w_o} = T_{w_o w_\lambda} T_{\hat{w}_\lambda}$ we obtain

$$T_{w_o} T_{w_\lambda} = T_{w_o \hat{w}_\lambda} T_{\hat{w}_\lambda}$$

The claim follows by applying the Kazhdan-Lusztig involution to this identity.

By Theorem 4.8, $\widetilde{E}_\lambda(\infty, t)$ and $\widetilde{E}_\lambda(0, t)$ are interchanged by $\kappa$. This fact can be expressed as

$$w_o \cdot \widetilde{E}_\lambda(\infty, t^{-1}) = \chi(w_o) T_{w_o}^{-1} \cdot \widetilde{E}_\lambda(0, t)$$
Now,
\[
T_{w_0}^{-1} \cdot \tilde{E}_\lambda(0, t) = T_{w_0}^{-1} T_{w_0^{-1}}^{-1} \cdot \tilde{E}_\lambda(0, t)
\]
\[
= T_{(w_0 \cdot w_0^{-1})^{-1}}^{-1} \cdot \tilde{E}_\lambda(0, t)
\]
\[
= T_{(w_0 \cdot w_0^{-1})^{-1}}^{-1} \cdot e^\lambda.
\]

The conclusion immediately follows. □

Specializing further all the remaining parameters to 0 we obtain again the Demazure character formula.

**Corollary 5.7.** Let \( \lambda \) be a weight. Then
\[
w_0 \cdot E_\lambda(\infty, \infty) = N_{w_0} \cdot e^{\lambda_-}
\]

To see that this is indeed equivalent to Demazure’s formula
\[
E_\lambda(\infty, \infty) = \Delta_{w_0} \cdot e^{\lambda_+}
\]

note that
\[
w_0 N_{w_0} w_0 = \Delta_{w_0} w_0
\]

The relationship between the limits \( t \to 0, \infty \) is described in the following

**Corollary 5.8.** Let \( \lambda \) be a weight. Then
\[
w_0 \cdot E_\lambda(\infty, \infty) = \sum_{w_0(\lambda) \leq \mu, \mu \leq \lambda_-} E_\mu(0, 0)
\]

**Proof.** From Corollary 5.3 we know that
\[
N_{w_0} = \sum_{x \leq w_0 \cdot w_0} N_x
\]

Now, keep in mind that \( N_i \cdot e^{\lambda_-} = 0 \) if \( s_i \) fixes \( \lambda_- \) and apply Corollary 5.4 to obtain the desired result. □

### 5.3. A geometric interpretation.

We first recall from [9] the geometric interpretation of the polynomials \( E_\lambda(\infty, \infty) \). For \( w \in W \) let \( S_w \), respectively \( S_w^- \), be the closure of the Bruhat cell \( BwB/B \), respectively of \( B^-wB/B \), inside the flag variety \( G/B \). For \( \lambda \) a weight, \( L_\lambda \) denotes the corresponding line bundle over \( G/B \). We will use the same notation for the restriction of this line bundle to any subvariety \( S_w \) or \( S_w^- \).

On the algebraic side, let \( \lambda \) be a weight and let \( V_\lambda^+ \) the irreducible \( G \)-module with highest weight \( \lambda_+ \). By \( V_\lambda^+(\lambda) \) we denote the (one dimensional) weight space of weight \( \lambda \). The Demazure module corresponding to \( \lambda \) is defined as \( D_\lambda := B \cdot V_\lambda^+(\lambda) \).

The connection with geometry is the following: if \( \lambda = w(\lambda_+) \), the Demazure module
corresponding to $\lambda$ and the dual of the space of global sections of $L_{-\lambda_+}$ over $S_w$ are isomorphic as $B$–modules

$$H^0(S_w, L_{-\lambda_+})^* \cong D_\lambda$$

and the $T$–character of $D_\lambda$ is $E_\lambda(\infty, \infty)$.

Equivalently, let $D^-_\lambda := B^- \cdot V_{\lambda_+}(\lambda)$ and let $w$ such that $w(\lambda_+) = \lambda$. Then,

$$H^0(S_w^-, L_{-\lambda_+})^* \cong D^-_\lambda$$

as $B^-$–modules and the $T$–character of $D^-_\lambda$ is

$$w_\circ E_{w_\circ(\lambda)}(\infty, \infty) = \sum_{\lambda \leq \mu \leq \lambda} E_\mu(0, 0) \quad (35)$$

With the above notation, let $K_\lambda$ be the kernel of the restriction map

$$H^0(S_w^-, L_{-\lambda_+}) \to H^0(\bigcup_{w < y} S_y^-, L_{-\lambda_+})$$

(keep in mind that $S_y^- \subset S_w^-$ for $w \leq y$). The dual of $K_\lambda$ is isomorphic to the co-kernel of the inclusion map

$$\bigcup_{\lambda < \mu \leq \lambda_-} D^-_\mu \to D^-_\lambda$$

By (35) the $T$-character of $\bigcup_{\lambda < \mu \leq \lambda_-} D^-_\mu$ equals

$$\sum_{\lambda < \mu \leq \lambda_-} E_\mu(0, 0)$$

and therefore the character of $K_\lambda^*$ is $E_\lambda(0, 0)$.

**Theorem 5.9.** Let $w$ be an element of $\tilde{W}$ and let $\lambda = w(\lambda_+)$. Then, $E_\lambda(0, 0)$ is the character of the dual space of sections of $L_{-\lambda_+}$ which are supported on

$$S_w^- - \bigcup_{w < y} S_y^-$$

In consequence, they are polynomials with non-negative integer coefficients.

As a terminological coincidence, in type $A$ the earliest reference to the polynomials $E_\lambda(0, 0)$ seems to go back to the work of Lascoux and Schützenberger [21, Theorem 3.8] where they form their “standard basis”.

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