Inverse Conduction Problem for a Parabolic Equation using a Boundary Integral Method

Christian Daveau ∗ Abdessatar Khelifi † M. Nour. Shamma ‡

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Abstract

In this paper, a boundary integral method is used to solve an inverse linear heat conduction problem in two-dimensional bounded domain. An inverse problem of measuring the heat flux from partial (on part of the boundary) dynamic boundary measurements is considered. An algorithm is given by using the fundamental solution.

Key words. Heat equation, inverse problem, boundary integral method

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1 Introduction

This paper is devoted to an inverse problem for a type of parabolic PDEs in a bounded two-dimensional domain. Here we consider initial boundary value problems for the heat equations by using a boundary integral approaches The inverse heat conduction problem arising in most thermal manufacturing processes has recently attracted much attention [2, 6, 5, 9, 17, 18, 20]. The typical case is the determination of the heat flux on an inaccessible boundary through measurements on an accessible boundary or inside the domain. Similar problems to ours have been studied by many authors, for example in one dimensional space we can refer to an approximate inverse method [9, 10], a boundary element method [16], a fundamental solution method [3, 8] and some other method [19]. In this paper, we use the boundary integral method to solve this problem. This method uses the prescribed initial and boundary data, together with the fundamental solution of a given differential equation defined in some bounded domain, and we construct integral equations on the boundary of . In our case, the solution to the integral equation is a single layer potential. By the boundary integral

∗Département de Mathématiques, Site Saint-Martin II, BP 222, & Université de Cergy-Pontoise, 95302 Cergy-Pontoise Cedex, France (Email: christian.daveau@math.u-cergy.fr).
†Département de Mathématiques & Informatique Faculté des Sciences, 7021 Zarzouna - Bizerte, Tunisia (Email:abdessatar.khelifi@fsb.rnu.tn).
‡Department of mathematics & T. College, Al Gassim University, P. O. Box 53, Al-Rass Province, Kingdom of Saudi Arabia(Email:shamman01@yahoo.com).
equation one can obtain the unknown kernel, and the solution to the given problem will be obtained by integrating the product of the fundamental solution and the unknown kernel over the boundary. The advantage of our approach is that the computation can be limited to the boundary, which reduces the problem from two dimensions to one dimension. As a result of the reduction, we may expect substantial savings in computer time and memory. The work outlined below is based on the use of single layer potentials. Ammari and Kang used boundary integral method to solve inverse conductivity problem and related problems [1]. In [1], both the single layer potential and the double layer potential are used. In this article, our boundary integral method is based on the result of [14], which gives a representation formula for the heat conduction problem with Neumann boundary condition. The equation is assumed to be homogeneous. The outline of our paper is as follows:...

2 Problem formulation

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain of $\mathbb{R}^2$. We denote $\partial \Omega$ the boundary set assumed to be of class $C^1$. We denote by $\nu$ the outward unit normal to $\Omega$ on $\partial \Omega$. Let $T > 0$. We consider the following homogeneous heat equation:

\[
\begin{align*}
\partial_t u - \Delta u &= 0, \quad (x,t) \in \Omega \times [0,T] \\
u(x,0) &= u_0(x), \quad x \in \Omega \\
u(x,t) &= g(x,t), \quad (x,t) \in \partial \Omega \times [0,T] \\
\frac{\partial u}{\partial \nu}(x,t) &= \phi(x,t), \quad (x,t) \in \partial \Omega \times [0,T]
\end{align*}
\]

where $u(x,t)$ is the temperature function, $u_0(x)$ the initial data, $g(x,t)$ is a suitably prescribed function, $\phi(x,t)$ is the unknown heat flux and $\partial_t u = \frac{\partial u}{\partial t}$ is the rate of change of temperature at a point over time. Notice that $\frac{\partial u}{\partial \nu} = \nabla u \cdot \nu$.

Let $\gamma(\zeta) : [0,1] \rightarrow \mathbb{R}^2$ be an analytic, 1-periodic function. We assume that the tangential derivative has the positive length $|\gamma'(\zeta)| > 0$ for all $0 \leq \zeta \leq 1$.

Throughout this paper we suppose that the closed smooth curve $\partial \Omega$ in $\mathbb{R}^2$ is parameterized by the function $\gamma(\zeta)$ as follows:

\[
\partial \Omega = \{ x = \gamma(\zeta), \quad \zeta \in [0,1] \}.
\]

The present paper proposes a boundary integral method for the numerical solution of the two-dimensional problem defined by (1). Our problem can be stated as follows:

The inverse problem

Let $\Gamma \subset \partial \Omega$ denote a measurable smooth connected part of the boundary $\partial \Omega$ ($\Gamma = \partial \Omega$ or not). The aim of this paper is to determine the heat flux $\phi(x,t) = \frac{\partial u}{\partial \nu}|_{\Gamma}(x,t)$ from measurements of:

\[ u(x,t) \quad \text{on} \quad \Gamma \times [0,T]. \]

For this purpose, we develop a boundary integral method ... as will be described in the next section.
3 Boundary integral method

In this section we consider the basic boundary integral approaches for the solution of the initial boundary value problem (1). As in the time-independent case there are two main types of approaches, namely the direct method based on the representation coming from Green’s formula and the indirect or layer methods (see for example [3]). We begin with the fundamental solution of the heat equation. In several spatial variables, the Green’s function is a solution of the initial value problem (see for example [8]):

\[
\partial_t G - \Delta G = 0, \quad \text{and} \quad G(x, t = 0) = \delta(x)
\]

where \(\delta\) is the Dirac delta function. The solution to this problem in \(\mathbb{R}^n (n \geq 1)\) is the fundamental solution:

\[
G(x, t) = \frac{H(t)}{(4\pi t)^{n/2}} \exp\left(-\frac{|x|^2}{4t}\right),
\]

where \(H(t)\) is the Heaviside function and \(|x| = (x_1^2 + x_2^2 + \cdots + x_n^2)^{1/2}\) for \(x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n\).

Let \(S\) and \(D\) denote the classical single-layer and double-layer heat potentials:

\[
(Sq)(x, t) = \int_0^t \int_{\partial\Omega} G(x - y, t - s)q(y, s) \, d\sigma(y) \, ds,
\]

\[
(D\varphi)(x, t) = \int_0^t \int_{\partial\Omega} \partial_\nu(y)G(x - y, t - s)\varphi(y, s) \, d\sigma(y) \, ds,
\]

and similarly we define \(D'\) as the spatial adjoint of the double-layer \(D\) and \(H\) the hypersingular heat operators as follows:

\[
(D'q)(x, t) = \int_0^t \int_{\partial\Omega} \partial_{\nu(x)}G(x - y, t - s)q(y, s) \, d\sigma(y) \, ds,
\]

\[
(H\varphi)(x, t) = -\int_0^t \int_{\partial\Omega} \frac{\partial^2 G(x - y, t - s)}{\partial_{\nu(x)} \partial_{\nu(y)}} \varphi(y, s) \, d\sigma(y) \, ds,
\]

where \(q\) and \(\varphi\) sufficiently smooth functions.

In terms of the operators \(S\) and \(D\), we introduce the following heat potential

\[
u = Sq - D\varphi \quad \text{in} \quad \Omega \times [0, T],
\]

where \(\nu\) is the solution of (1). Now, by using the well known boundary behavior properties of single-layer and the double-layer heat potential the following result holds.

**Proposition 3.1** Let \(u\) be the solution of the problem (1). Then, in both cases homogeneous or inhomogeneous initial data \(u_0\), the Cauchy data \((\nu|_{\partial\Omega \times [0, T)}, \partial_\nu u|_{\partial\Omega \times [0, T]}\) satisfy:

\[
u = Sq + \left(\frac{1}{2} I - D\right)\varphi \quad \text{in} \quad \partial\Omega \times [0, T]
\]

\[
\partial_\nu u = \left(\frac{1}{2} I + D'\right)q + H\varphi \quad \text{in} \quad \partial\Omega \times [0, T],
\]

where the functions \(q\) and \(\varphi\) are given as in (4).
Proof. For the case of inhomogeneous initial data \( u_0 \equiv 0 \) one can get the above results by using the well known boundary behavior properties of single-layer and the double-layer heat potential together with their normal derivatives. But, for inhomogeneous initial data \( u_0 \neq 0 \), the situation needs more explanations. It can be seen that a function

\[
v(x, t) = \int_0^t \int_{\partial \Omega} G(x - y, t) u_0(y) \, dy
\]

satisfies

\[
v_t - \Delta v = 0
\]

and

\[
\lim_{t \to 0^+} v(x, t) = u_0(x).
\]

Setting \( w = u - v \), we get

\[
\begin{align*}
\partial_t w - \Delta w &= 0, & (x, t) &\in \Omega \times [0, T] \\
w(x, 0) &= 0, & x &\in \Omega \\
w(x, t) &= g(x, t) - v(x, t), & (x, t) &\in \partial \Omega \times [0, T], \\
\partial w(x, t) &= \varphi(x, t) - \partial v(x, t), & (x, t) &\in \partial \Omega \times [0, T].
\end{align*}
\]

(7)

Since \( w \) is the solution of (4) with homogeneous initial data, it can be solved to find the integral relations (5)-(6). Thus we can solve the problem and get our results by the superpositions of \( v \) and the solution of homogenous initial data. \( \Box \)

Now, we introduce the following notations and the anisotropic Sobolev space to be used in the sequel. A rather comprehensive of the basic presentation of these Sobolev spaces the reader can see [13]. For given \( r, p \geq 0 \), we have the space

\[
H^{r,p}(\Omega \times [0, T]) = L^2([0, T]; H^r(\Omega)) \cap H^p([0, T]; L^2(\Omega)).
\]

The space \( H^{r,p}(\partial \Omega \times [0, T]) \) is defined analogously by replacing \( \Omega \) by \( \partial \Omega \). Moreover the following subspace of \( H^{r,p}(\partial \Omega \times [0, T]) \) is well defined

\[
h^{r,p}(\partial \Omega \times [0, T]) = \{ v = w|_{\partial \Omega \times [0, T]} : w \in H^{r,p}(\Omega \times \mathbb{R}), \ w(\cdot, t) = 0, t < 0 \}.
\]

The norm of the space \( H^{r,p}(\Omega \times [0, T]) \) is denoted by \( \| \cdot \|_{r,p; \Omega \times [0, T]} \) and the norm of \( h^{r,p}(\partial \Omega \times [0, T]) \) is denoted by \( \| \cdot \|_{r,p; \partial \Omega \times [0, T]} \).

Next, introducing the useful spaces

\[
\mathcal{E} = \{ v \in H^{2,1}(\Omega \times [0, T]) : (\Delta + \partial_t)v \in L^2(\Omega \times [0, T]), \partial_v|_{\partial \Omega \times [0, T]} \equiv 0, v(\cdot, T)|_{\Omega} \equiv 0 \},
\]

\[
\mathcal{X} = h^{1/2,1/4}(\partial \Omega \times [0, T]) \quad \text{and the associate dual space } \mathcal{X}' = h^{-1/2,-1/4}(\partial \Omega \times [0, T]).
\]

Then the following definition appears.

**Definition 3.1** For given \( f \in \mathcal{X}' \), we say that \( u \) is a weak solution of (4) if \( u \in H^{1,1/2}(\Omega \times [0, T]) \) and satisfies the following duality product:

\[
\langle u, \Delta \psi + \partial_t \psi \rangle = -\langle f, \psi \rangle_{\partial \Omega \times [0, T]},
\]

for \( \psi \in \mathcal{E} \).

In terms of the last notations, the following mapping properties of the single- and double-layer heat operators holds.
Lemma 3.1 Let the operators \( S \) and \( D \) be defined as in section 2. Then the followings hold:

1. The single-layer heat operator \( S : \mathbb{H}^{r,1/2}((\partial \Omega \times [0,T]) \rightarrow \mathbb{H}^{r+1,1/2(r+1)}(\partial \Omega \times [0,T]) \) is an isomorphism for all \( r \geq -1/2 \).
2. The operator \( \frac{1}{2}I + D : \mathcal{X}' \rightarrow \mathcal{X}' \) is an isomorphism.

Proof. The claim (1) can be inspired directly from Theorem 4.3 in [7]. The claim (2) follows by a little modification from [7]. □

Now we're ready to prove the following result.

Theorem 3.1 Assume that \( \phi \in \mathcal{X}' \), Then the function \( u \in H^{1,1/2}(\Omega \times [0,T]) \) is a weak solution of (1) if and only if \( u \) has the representation (4) such that \( \phi \in \mathcal{X}' \) solves the equations (5)-(6).

Proof. Let \( \phi \in \mathcal{X}' \), and let \( \varphi \in \mathcal{X}' \) be the unique solution of (5)-(6). Then, by combining relation (5) and the direct representation (4), we may get the following boundary integral equation of the second kind:

\[
\left( \frac{1}{2}I + D \right) \varphi = S \phi. \tag{8}
\]

Since the set \( D(\partial \Omega \times [0,T]) \) is dense in the space \( \mathcal{X}' \), we can choose a sequence \( \varphi_n \in D(\partial \Omega \times [0,T]) \) such that \( \varphi_n \rightarrow \varphi \) in \( \mathcal{X}' \), and so the following sequence \( \phi_n = S^{-1}\left( \frac{1}{2}I + D \right) \varphi_n \) is well defined. By the mapping properties of \( S \) and of the operator \( \frac{1}{2}I + D \) found in Lemma 3.1, the function \( \phi_n \) is also a smooth function of \( \partial \Omega \times [0,T] \); moreover we have \( \phi_n \rightarrow \phi \) in \( \mathcal{X}' \).

Now, let \( u_n \) be the corresponding classical potential

\[
u_n = S \varphi_n - D \phi_n,
\]

which by the construction satisfies

\[
\partial_t u_n|_{\partial \Omega \times [0,T]} = \phi_n.
\]

Obviously, \( u_n \) is a weak solution of (11) with the Neumann data \( \phi_n \). On the other hand, As done for the Dirichlet-type initial boundary value problem in [7], we can conclude for our problem that the mapping \( \phi \rightarrow u \), is continuous and we have \( u \in H^{1,1/2}(\Omega \times [0,T]) \) such that

\[
\|u\|_{1,1/2,\Omega \times [0,T]} \leq c\|\phi\|_{1/2,1/4,\partial \Omega \times [0,T]}, \quad \text{where } c \text{ is a positive constant.} \tag{9}
\]

Hence, by the continuity (9) we have the convergence \( u_n \rightarrow u \) in \( H^{1,1/2}(\Omega \times [0,T]) \).

Next, we define \( \psi = S \varphi - D \phi \) in \( \Omega \times [0,T] \), then for all \( v \in D(\Omega \times [0,T]) \) it follows that

\[
\langle u, v \rangle = \lim_{n \rightarrow \infty} \langle u_n, v \rangle = \lim_{n \rightarrow \infty} \left( \langle S \varphi_n, v \rangle - \langle D \phi_n, v \rangle \right) = \langle \psi, v \rangle,
\]

which implies that \( u = \psi = S \varphi - D \phi \). □
4 Numerical scheme for the inverse problem

In this section we propose a numerical method to solve our inverse problem. The numerical method is based on the boundary integral equation in Proposition ???. As the measured data for inverse problem, the numerical data obtained by solving the direct problem can be used. The inverse problem is then to solve the following problem:

\[
\begin{align*}
\partial_t u - \Delta u &= 0, \quad (x,t) \in \Omega \times [0,T] \\
 u(x,0) &= 0, \quad x \in \Omega \\
 u(x,t) &= g(x,t), \quad (x,t) \in \partial\Omega \times [0,T] \\
 \frac{\partial u}{\partial \nu}(x,t) &= \phi(x,t), \quad (x,t) \in \partial\Omega \times [0,T].
\end{align*}
\] (10)

We use the following equation given by (4)

\[ u|_{\partial\Omega \times [0,T]} = S_q - D \phi \text{ in } \Omega \times [0,T], \] (11)

together with the relations (5)-(6) to solve the above problem.

To give our numerical method, we introduce the following result.

Lemma 4.1 Let \( \varphi = [u]|_{\partial\Omega \times [0,T]} \), \( u \) solution of (10). In term of the hyper-singular heat operator the heat flux is solution of:

\[ \mathcal{H} \varphi = \phi, \] (12)

where \( [u]|_{\partial\Omega \times [0,T]} \) means the jump of the function \( u \) via the boundary \( \partial\Omega \times [0,T] \).

The boundary integral equation of the first kind given in Lemma 4.1 is deduced from a normal derivative applied to the double layer representation which itself given by managing relation (4) into relations given in Proposition 3.1.

To proceed with our numerical scheme, we may follows two cases, measure on the boundary \( \partial\Omega \) and measure on a smooth connected subset of the boundary.

4.1 Measure on the boundary \( \partial\Omega \)

The aim of this section is to reconstruct \( \phi(x,t) \) from measurements of \( u(x,t) \) on the boundary \( \partial\Omega \times (0,T) \). For this purpose, we develop the subdivision of \([0,1]\):

\[ \zeta_0 = 0, \zeta_i = \zeta_0 + ih, i = 1,2,\ldots,N, \quad \text{where } N \text{ is an integer and } h = 1/N. \] (13)

Analogously, we assume the subdivision of \([0,T]\):

\[ t_0 = 0, t_j = t_0 + jh', j = 1,2,\ldots,N' \quad \text{where } N' \text{ is an integer and } h' = T/N'. \] (14)

Then, by using Lemma 4.1 the following main result follows.

Theorem 4.1 Let \( g \in \mathcal{X} \) be a given function and \( u \in H^{1.1/2}(\Omega \times [0,T]) \) be the solution of (10). Assume that we have the subdivisions (13)-(14). Suppose that the heat flux of the problem (10) is continuous up to the inner side of the inaccessible boundary \( \partial\Omega \), then the unknown data \( \phi \) can be recovered by the following discrete scheme

\[
\phi(\gamma(\zeta),t_j) = \frac{hh'}{4} \sum_{k=1}^{N} \sum_{l=1}^{N'} \frac{g(\gamma(\zeta_k),t_l) \gamma'(\zeta_k)}{(t_j-t_l)^2} \frac{\gamma'(\zeta_l)}{|\gamma'(\zeta_l)|} \cdot [-\gamma'(\zeta_k) + \cdots] \] (15)
Let $u$ be the solution of the inverse heat problem (10) and inserting the expression of the hyper-singular operator $\mathcal{H}$ into relation (12), we get

$$
\phi(x, t) = - \int_0^t \int_{\partial \Omega} \frac{\partial^2 G(x - y, t - s)}{\partial \nu(x) \partial \nu(y)} g(y, s) \, d\sigma(y) \, ds, \quad \text{for} \ (x, t) \in \partial \Omega \times [0, T].
$$

By change of variable $x = \gamma(\zeta), \zeta \in [0, 1]$, we write

$$
\phi(\gamma(\zeta), t) = - \int_0^t \int_0^1 \frac{\partial^2 G(\gamma(\zeta) - (\gamma(\zeta'), t - s) g(y, s) |\gamma'(\zeta')| \, d\zeta' \, ds, \quad \text{for} \ (\zeta, t) \in [0, 1] \times [0, T].
$$

Then according to (13)-(14), we can discretize relation (16) as follows:

$$
\phi(\gamma(\zeta_i), t_j) = -hh' \sum_{k=1}^N \sum_{l=1}^{N'} \frac{\partial^2 G(\gamma(\zeta_i) - (\gamma(\zeta_k), t_j - t_l) g(\gamma(\zeta_k), t_l) |\gamma'(\zeta_k)|. \quad (17)
$$

On the other hand the normal derivative of the fundamental solution $G$ of (3) in two dimensional space is:

$$
\partial_{\nu(y)} G(x - y, t - s) = -\nu(y) \cdot \frac{(y - x)}{4(t - s)^2} \exp\left(-\frac{|x - y|^2}{(t - s)}\right).
$$

Thus, we can derive a gain by $\nu(x)$ to get throw the formula $\nabla(A \cdot B) = A \times (\nabla \times B) + B \times (\nabla \times A) + (A \cdot \nabla) B + (B \cdot \nabla) A$ that:

$$
\frac{\partial^2 G}{\partial \nu(x) \partial \nu(y)} (x - y, t - s) = -\frac{1}{4(t - s)^2} \nu(x) \cdot [(y - x) \times (\nabla \times \nu(y))

+(\nu(y) \cdot \nabla)(y - x) + ((y - x) \cdot \nabla) \nu(y) + 2(\nu(y) \cdot (y - x))(y - x) \frac{y - x}{t - s} \exp\left(-\frac{|x - y|^2}{(t - s)}\right).
$$

Therefore to achieve the proof, we insert the last formula into relation (17) by taking $x = \gamma(\zeta_i)$ and $y = \gamma(\zeta_k)$.

Now, one can give an approximation to the solution $u(x, t)$ of the problem (10) by inserting the discretized formula of the heat flux given by Theorem 4.1 into the representation (4) as:

$$
uu(x_i, t_j) := hh' \sum_{k=1}^N \sum_{l=1}^{N'} G(\gamma(\zeta_i) - (\gamma(\zeta_k), t_j - t_l) \phi(\gamma(\zeta_k), t_l) |\gamma'(\zeta_k)|)
$$

$$
-hh' \sum_{k=1}^N \sum_{l=1}^{N'} \frac{\partial G}{\partial \nu(y)}(\gamma(\zeta_i) - (\gamma(\zeta_k), t_j - t_l) g(\gamma(\zeta_k), t_l) |\gamma'(\zeta_k)|.
$$
4.2 Measure on a smooth subset of the boundary

Let \( \Gamma \subset \partial \Omega \) denote a measurable smooth connected part of the boundary \( \partial \Omega \) and \( \Gamma_c \) denotes \( \partial \Omega \setminus \Gamma \). Introduce the trace space

\[
\mathcal{X} = \left\{ v \in X, v \equiv 0 \text{ on } \Gamma_c \times (0,T) \right\}.
\]

Here and in the sequel we identify \( g \) defined only on \( \Gamma \) with its extension by 0 to all \( \partial \Omega \).

The aim of this section is then to identify the heat flux \( \phi \) from the local measure on the Cauchy data \( g \in \mathcal{X} \). To do this, we may assume that there exists \( \zeta_* \in (0,1) \) such that

\[
\Gamma := \{ \gamma(\zeta) : \zeta \in [0,\zeta_*], \quad \zeta_* << 1 \}.
\]

As done in last section, we introduce the subdivision of \([0,\zeta_*] \):

\[
\zeta_0 = 0, \zeta_i = \zeta_0 + ir, i = 1, 2, \ldots, M \quad \text{where } M \text{ is an integer and } r = \zeta_*/M.
\] (19)

Similarly for \([0,T]\),

\[
t_0 = 0, t_j = t_0 + jT', j = 1, 2, \ldots, M' \quad \text{where } M' \text{ is an integer and } T' = T/M'.
\] (20)

As done in Theorem 4.1 for the local measurement we have the main result.

**Corollary 4.1** Let \( g \in \mathcal{X} \) be a given function and \( u \in H^{1,1/2}(\Omega \times [0,T]) \) be the solution of (10). Assume that we have the subdivision (19)-(20). Suppose that the heat flux satisfy the hypothesis in Theorem 4.1. Then the data \( u \) can be measured on \( \Gamma \) as :

\[
\phi(\gamma(\zeta_i), t_j) = -rT' \sum_{k=1}^{M} \sum_{l=1}^{M'} \frac{\partial^2 G}{\partial x \partial y}(\gamma(\zeta_l) - \gamma(\zeta_k), t_j - t_l)g(\gamma(\zeta_k), t_l) |\gamma'(\zeta_k)|
\]

\[
\quad = \frac{T\zeta_*}{4MM'} \sum_{k=1}^{N} \sum_{l=1}^{N'} \frac{g(\gamma(\zeta_k), t_l) \gamma'(\zeta_l)}{(t_j - t_l)^2} |\gamma'(\zeta_l)| \left[ -\gamma'(\zeta_k) + 2(\gamma'(\zeta_k) \cdot (\gamma(\zeta_k) - \gamma(\zeta_l))) \right] \exp\left(-\frac{|\gamma(\zeta_l) - \gamma(\zeta_k)|^2}{(t_j - t_l)^2}\right),
\]

for \( i = 1, 2, \ldots, M; \quad j = 1, 2, \ldots, M' \) and \( \gamma \) is given by Section 2.

**Proof.** Consider that \( u \) is the solution of the inverse heat problem (10), then by (12), we can write

\[
\phi(x,t) = -\int_0^t \int_{\Gamma \cup \Gamma_c} \frac{\partial^2 G}{\partial x \partial y}(x - y, s - t)g(y, s) \, d\sigma(y) \, ds, \quad \text{for } (x,t) \in \Gamma \times [0,T].
\]

The fact that \( g \in \mathcal{X} \) we can reduce

\[
\phi(x,t) = -\int_0^t \int_{\Gamma} \frac{\partial^2 G}{\partial x \partial y}(x, s - t)g(y, s) \, d\sigma(y) \, ds, \quad \text{for } (x,t) \in \Gamma \times [0,T].
\]

By change of variables and by considering the subdivisions (19)-(20) one can deduce from the proof of Theorem 4.1 that

\[
\phi(\gamma(\zeta_i), t_j) = -rT' \sum_{k=1}^{M} \sum_{l=1}^{M'} \frac{\partial^2 G}{\partial x \partial y}(\gamma(\zeta_l) - \gamma(\zeta_k), t_j - t_l)g(\gamma(\zeta_k), t_l) |\gamma'(\zeta_k)|.
\]

Then, the proof achieves by inserting in last relation the possible normal derivations of the function \( G \). \( \square \)
4.3 Numerical examples

Numerical evaluations of the heat flux $\phi(x,t)$ are now obtained by solving the integral equations (11) and (12) with the specific parameterization and discretisation of the boundary.

In this section, we assume that the curve is parameterized by $\gamma(\zeta) = (\cos(2\pi\zeta), \sin(2\pi\zeta))$ and we suppose that $T = 10$, $\zeta_* = 10^{-2}$.

4.3.1 Measure on $\partial \Omega \times [0, T]$

We consider the following direct problem:

$$\begin{align*}
\partial_t u - \Delta u &= 0, & (x, t) \in \Omega \times (0, T) \\
u(x, 0) &= 0, & x \in \Omega \\
u(x, t) &= 2|x| \cos(3t), & (x, t) \in \partial \Omega \times (0, T) \\
\frac{\partial \nu}{\partial \nu}(x, t) &= \phi(x, t), & (x, t) \in \partial \Omega \times (0, T).
\end{align*}$$

We solve this problem by using Theorem 4.1 given by boundary integral method which is presented in sections 3 and 4. The parameter is chosen as $N = 50$, $N' = 100$. A numerical result is shown in Figures.......

4.3.2 Measure on $\Gamma \times [0, T]$

For the case of smooth subset $\Gamma$ we consider the following problem:

$$\begin{align*}
\partial_t u - \Delta u &= 0, & (x, t) \in \Omega \times (0, T) \\
u(x, 0) &= 0, & x \in \Omega \\
u(x, t) &= 2|x| \cos(3t), & (x, t) \in \Gamma \times (0, T) \\
\frac{\partial \nu}{\partial \nu}(x, t) &= \phi(x, t), & (x, t) \in \Gamma \times (0, T).
\end{align*}$$

This problem can be solved by using Corollary 4.1. The parameter is chosen as $M = 50$, $M' = 100$. A numerical result is shown in Figures.......

4.4 Conclusion

A boundary integral method for the two-dimensional inverse heat conduction problem is considerably discussed. We presented a numerical scheme for the inverse problem. The heat flux was measured from the whole boundary and from a smooth subset of this boundary. In this paper we restricted our selves to homogeneous conduction problem, but the case with an external source may be considered in a forthcoming work.

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