Comparison of classical and path-by-path solutions to SDEs

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Abstract

We consider the Stochastic Differential Equation

\[ X_t = X_0 + \int_0^t b(s, X_s)ds + B_t, \]

in \( \mathbb{R}^d \).

We give an example of a drift \( b \) such that there does not exist a weak solution, but there exists a solution for almost every realization of the Brownian motion \( B \). We also give an explicit example of a drift such that the SDE has a pathwise unique weak solution, but path-by-path uniqueness (i.e. uniqueness of solutions to the ODE for almost every realization of the Brownian motion) is lost. These counterexamples extend the results obtained in [22] to dimension \( d = 1 \).

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1 Introduction

Let \( b : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d \) be measurable and \( B \) be a \( d \)-dimensional Brownian motion. Throughout the paper we always consider Stochastic Differential Equations (SDEs) of the type

\[ X_t = X_0 + \int_0^t b(s, X_s)ds + B_t. \] (1.1)

One can think of different kinds of solutions:

1. Weak and strong solutions to the SDE (2.2).
2. Considering the equation (1.1) as an ODE for each realization \((B_t(\omega))_{0 \leq t \leq T}\) of the Brownian path (“path-by-path” solutions).

The second of the above is only possible as we consider an SDE with constant diffusion coefficient. Therefore we can evaluate the noise term therein for each Brownian path, which would not be possible with a more general diffusion term as this would lead to a stochastic integral. The aim of this paper is to show via two counterexamples that, for any dimension \( d \geq 1 \), the two notions above are not equivalent.

Recall that there are drifts \( b \) such that (1.1) is well-posed, although the equation without additive noise is not. This phenomenon is called regularization by noise (see [11] for a thorough presentation, in particular on PDE models of fluid dynamics). Considering (1.1) in the sense of (1), there is an extensive literature which we will not describe thoroughly. Nevertheless let us mention the important works on existence and uniqueness of solutions by Veretennikov [23] for bounded measurable drift and by Krylov and Röckner [15] for drifts fulfilling an \( L^p - L^q \) condition. For further works on distributional drifts see [3, 10, 12, 13, 16].

The following was shown by Davie [9] when considering (1.1) in the sense of (2): For bounded measurable drift \( b \), there exists a unique solution to (1.1) for almost every realization of Brownian motion, considered as a purely deterministic integral equation (i.e. “path-by-path” uniqueness). After the initial result on path-by-path uniqueness in [9], several extensions have been proven.

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In [20, 21] the author proved Davie’s Theorem using different techniques (in particular the flow property of strong solutions) and also recovered uniqueness for a class of unbounded measurable drifts. For yet another approach, linking (1.1) to a backward SPDE, see [4]. For results on “path-by-path” uniqueness for SPDEs see [4, 5]. There are also results on existence and uniqueness of path-by-path solutions available for SDEs with fractional Brownian noise, allowing for distributional drifts (see [2, 6, 14]).

In both [11] and [1] it was phrased as an open problem whether every solution of type (2) has to arise from an adapted solution. In a higher-dimensional setting this question was answered via a counterexample in [22], making heavy use of the dimension \(d \geq 2\). Hence, it is a natural question whether similar counterexamples can be constructed in a one-dimensional setting. We give a positive answer to this question in Section 3. The construction in Section 3 can easily be extended to the case \(d \geq 2\).

Notations.

- We use the notation \(\mathcal{F} = (\mathcal{F}_t)_{t \geq 0}\) for a filtration. All filtrations are assumed to fulfill the usual conditions.
- For a \(\sigma\)-algebra \(\mathcal{F}\) and a stopping time \(\tau\) we define the stopping time \(\sigma\)-algebra by \(\mathcal{F}_\tau := \{A \in \mathcal{F} : A \cap \{\tau \leq t\} \in \mathcal{F}_t, \forall t \geq 0\}\).
- We call \((B_t)_{t \geq 0}\) an \(\mathcal{F}\)-Brownian motion if \(B\) is a Brownian motion adapted to \(\mathcal{F}\) and for any \(0 \leq s \leq t\), \(B_t - B_s\) is independent of \(\mathcal{F}_s\).

Definitions and previous results.

**Definition 1.1** (Existence). (i) If there exists a filtered probability space \((\Omega, \mathcal{F}, \mathbb{P}, \mathbb{P})\) equipped with a Brownian motion \(B\) and an \(\mathbb{P}\)-adapted process \(X\) such that \((X, B)\) fulfills (1.1) almost surely, we say that \((X, B)\) is a weak solution to (1.1). If the choice of \(B\) is clear from the context, we write that \(X\) is a weak solution.

(ii) We call \(X\) a strong solution if \(X\) is a weak solution and \(X\) is adapted w.r.t. the filtration generated by \(B\).

(iii) Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a filtered probability space on which a Brownian motion \(B\) is defined. We call a mapping \(X: \Omega \to C([0, T])\) a path-by-path solution if there exists a set \(\tilde{\Omega} \subset \Omega\) of full measure such that, for all \(\omega \in \tilde{\Omega}\), \((X(\omega), B(\omega))\) fulfills equation (1.1).

**Definition 1.2** (Uniqueness). (i) We say that pathwise uniqueness for (1.1) holds if for any two weak solutions \((X, B), (\tilde{X}, B)\) defined on the same filtered probability space with the same Brownian motion \(B\) and the same initial condition, \(X\) and \(\tilde{X}\) are indistinguishable.

(ii) We say that path-by-path uniqueness for (1.1) holds if for any probability space on which a Brownian motion \(B\) is defined, there exists a null-set \(N\) such that for \(\omega \notin N\), there exists a unique solution to

\[
X_t(\omega) = X_0(\omega) + \int_0^t b(s, X_s(\omega))ds + B_t(\omega).
\]

(iii) Let \(X\) be a nonnegative weak solution to (1.1) such that all other nonnegative weak solutions defined on the same probability space with the same Brownian motion and the same initial condition are indistinguishable from \(X\). Then we call \(X\) the pathwise unique nonnegative weak solution. In a symmetric manner we define a pathwise unique nonpositive weak solution.

The following implications follow directly from the definitions:

- strong existence \(\rightarrow\) weak existence \(\rightarrow\) path-by-path existence
- path-by-path uniqueness \(\rightarrow\) pathwise uniqueness
Having the proper definitions of different kinds of solutions at hand, we can precisely formulate the initial result on path-by-path uniqueness to (1.1) and the counterexamples mentioned earlier.

For a bounded measurable drift the following is known: Let $B$ be a $d$-dimensional Brownian motion defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, $X_0 \in \mathbb{R}$ and let $b: [0, \infty) \times \mathbb{R}^d \to \mathbb{R}^d$ be a bounded Borel-measurable function. In [9] it was shown that

$$X_t = X_0 + \int_0^t b(r, X_r)dr + B_t$$

(1.2)

has a unique path-by-path solution.

In [22], for dimension $d \geq 2$, drifts $b$ are constructed such that

- there is existence of path-by-path solutions, but there exists no weak solution;
- there exists a pathwise unique weak solution to (1.1), but path-by-path uniqueness is lost.

In Section 3, we will construct drifts $b$ such that the two situations described above happen for Equation (1.1). Recall that by Krylov and Röckner [15] strong existence and pathwise uniqueness for Equation (1.1) hold for $b \in L^p([0, T], L^p(\mathbb{R}^d))$ where $p \in [2, \infty], q \in (2, \infty)$ with

$$\frac{d}{p} + \frac{2}{q} < 1.$$  

(1.3)

Hence, for the first of the above counterexamples, $b$ cannot be in this class of functions. Although a priori this (with the current results on path-by-path uniqueness) does not have to be the case for the second counterexample above, we are not aware of a counterexample such that $b \in L^q([0, T], L^p(\mathbb{R}^d))$ for $p \in [2, \infty], q \in (2, \infty]$ fulfilling (1.3).

2 Preparatory results and Bessel bridges

**Lemma 2.1.** If (1.1) has a weak solution with initial condition $X_0$ fulfilling $\mathbb{P}(X_0 = 0) > 0$, then it also has a weak solution with $X_0 = 0$.

**Proof.** Let $(X, B)$, defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, be a weak solution to (1.1) with $X_0$ fulfilling $\mathbb{P}(X_0 = 0) > 0$. Let $A := \{X_0 = 0\}$. Define $(\tilde{X}, B)$ and $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}, \tilde{\mathbb{P}})$ in the following way: $\tilde{\Omega} = A; \tilde{\mathcal{F}} = \{E \cap A : E \in \mathcal{F}\}; \tilde{\mathcal{F}}_t = \{E \cap A : E \in \mathcal{F}_t\}$; for $C \in \tilde{\mathcal{F}}, \tilde{\mathbb{P}}(C) = \frac{\mathbb{P}(C \cap A)}{\mathbb{P}(A)}$, for $\omega \in A$, $\tilde{X}(\omega) = X(\omega)$ and $\tilde{B}(\omega) = B(\omega)$. First, we check that $\tilde{X}$ is adapted with respect to $\tilde{\mathbb{F}}$. Let $U \subset \mathbb{R}$ be measurable. Then

$$\tilde{X}^{-1}(U) = X^{-1}(U) \cap A \in \tilde{\mathcal{F}}_t.$$

Next, we check that $\tilde{B}$ is an $\tilde{\mathbb{F}}$-Brownian motion. As $X_0$ is $\mathcal{F}_0$-measurable and $B$ is an $\mathbb{F}$-Brownian motion, we know that, for $0 \leq s \leq t$,

$$\tilde{\mathbb{P}}(\tilde{B}_t - \tilde{B}_s \in U) = \frac{\mathbb{P}((B_t - B_s \in U) \cap A)}{\mathbb{P}(A)} = \mathbb{P}(B_t - B_s \in U)$$

and using this we get, for $V \in \tilde{\mathcal{F}}_s$,

$$\tilde{\mathbb{P}}\left(\{\tilde{B}_t - \tilde{B}_s \in U\} \cap V\right) = \frac{\mathbb{P}((B_t - B_s \in U) \cap V)}{\mathbb{P}(A)} = \tilde{\mathbb{P}}(\tilde{B}_t - \tilde{B}_s \in U)\tilde{\mathbb{P}}(V).$$

Hence, $\tilde{B}$ is an $\tilde{\mathbb{F}}$-Brownian motion.

Clearly $(\tilde{X}, \tilde{B})$ fulfills equation (1.1) a.s. □

The following lemma and its proof are similar to [4, Example 7.4] and [8, Example 2.1].
Lemma 2.2. Let $f : \mathbb{R} \to \mathbb{R}$ be measurable. There does not exist a weak solution to
\[
dX_t = b(X_t)dt + dB_t, \quad t \in [0, 1],
\]
with initial condition $X_0$ fulfilling $\mathbb{P}(X_0 \in (-1, 1)) > 0$ and
\[
b(x) := -\mathbb{1}_{\{x \neq 0, |x| \leq 1\}} \frac{1}{2x} + \mathbb{1}_{\{|x| > 1\}}f(x).
\]

Proof. Step 1: First we show that there exists no weak solution to (2.1) on $[0, 1/2]$ with initial condition $X_0 = 0$. Assume there exists a weak solution $(X, B)$. Let $\tau_1 := \inf\{s \geq 0 : |X_s| \geq 1\}$. Then by Itô’s Lemma, we get that
\[
X^2_{t \wedge \tau_1} = \int_0^{t \wedge \tau_1} -\mathbb{1}_{\{X_s \neq 0\}}ds + \int_0^{t \wedge \tau_1} ds + \int_0^{t \wedge \tau_1} 2X_sdB_s
\]
\[
= \int_0^{t \wedge \tau_1} \mathbb{1}_{\{X_s = 0\}}ds + \int_0^{t \wedge \tau_1} 2X_sdB_s.
\]
Define $X^\tau_1 := X_{t \wedge \tau_1}$. Note that $X^\tau_1$ is a continuous semimartingale with quadratic variation $(X^\tau_1)_t = t \wedge \tau_1$, so we can use [19, Chapter 6, Corollary (1.6)] to obtain
\[
\int_0^{t \wedge \tau_1} \mathbb{1}_{\{X_s = 0\}}ds = \int_0^{t \wedge \tau_1} \mathbb{1}_{\{X_s = 0\}}d(X_s) = \int_\mathbb{R} \mathbb{1}_{\{x = 0\}}L^\tau_{t \wedge \tau_1}(X)dx = 0,
\]
where $L(X)$ denotes the local time of $X$. Hence, $(X^\tau_1)^2 = \int_0^{t \wedge \tau_1} 2X_sdB_s$ is a local martingale and as $|X^\tau_1| \leq 1$ it is also a martingale. Note that $X^\tau_1_0 = 0$ and $(X^\tau_1)^2 \geq 0$. This implies that $X^\tau_1$ must be identically 0, which contradicts $(X^\tau_1)_t = t \wedge \tau_1$ as $\tau_1 > 0$ a.s.

Step 2: Assume that there exists a weak solution $(X, B)$ to (2.1) (defined on some filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F})$) with initial condition $X_0$ fulfilling $\mathbb{P}(X_0 \in (-1, 1)) > 0$. We assume w.l.o.g. that $\mathbb{P}(X_0 \in [0, 1)) > 0$ as $\mathbb{P}(X_0 \in (-1, 0)) > 0$ can be dealt with in an entirely symmetric way. Let $a \in (0, 1)$ such that $\mathbb{P}(X_0 \in [0, a]) > 0$. Let $\tau_0 := \inf\{t \geq 0 : X_t = 0\}$ and
\[
C := \{X_0 \in [0, a]\} \cap \{\sup_{t \in [0, 1/2]} B_t < 1 - a, B_{1/2} \leq -a\}.
\]
Then $\{\tau_0 \leq 1/2\} \supset C$. To see this let $\omega \in C$ and assume that $\inf_{t \in [0, 1/2]} X_t(\omega) > 0$. Then
\[
X_{1/2}(\omega) = X_0(\omega) - \int_0^{1/2} \frac{1}{2X_s(\omega)}ds + B_{1/2}(\omega) \leq a + B_{1/2}(\omega) \leq 0,
\]
which gives a contradiction. Hence $\mathbb{P}(\tau_0 \leq 1/2) > 0$ by continuity of $X$ and as $\mathbb{P}(C) > 0$. Knowing this, below we construct a solution to (2.1) on $[0, 1/2]$ starting from 0 with positive probability. This gives a contradiction to Step 1 due to Lemma 2.1.

Let $\tilde{\tau}_0 := \tau_0 \wedge 1/2$, $\tilde{B}_t := B_{\tilde{\tau}_0 + t} - B_{\tilde{\tau}_0}$ and $\tilde{X}_t := X_{\tilde{\tau}_0 + t}$. Then $(\tilde{X}, \tilde{B})$ is a weak solution to (2.1) on $[0, 1/2]$ with initial condition $X_{\tilde{\tau}_0}$ which fulfills $\mathbb{P}(X_{\tilde{\tau}_0} = 0) > 0$. First, for $t \in [0, 1/2],$
\[
\tilde{X}_t = X_{\tilde{\tau}_0 + t} = X_{\tilde{\tau}_0} + \int_{\tilde{\tau}_0}^{\tilde{\tau}_0 + t} b(X_s)ds + B_{\tilde{\tau}_0 + t} - B_{\tilde{\tau}_0}
\]
\[
= X_{\tilde{\tau}_0} + \int_0^t b(\tilde{X}_s)ds + \tilde{B}_t.
\]

Furthermore, $\tilde{B}$ is an $\mathbb{F}^{\tilde{\tau}_0}$-Brownian motion for $\mathcal{F}_t^{\tilde{\tau}_0} := \mathcal{F}_{\tilde{\tau}_0 + t}$ by [18, page 23, Theorem 32] and $\tilde{X}$ is clearly adapted to $\mathbb{F}^{\tilde{\tau}_0}$. □

In the following lemma we consider two SDEs and present some properties of solutions to these SDEs. Both have been investigated thoroughly in the literature. The lemma is stated to give a clearer presentation of the counterexamples in Section 3.
Lemma 2.3. Let $y > 0$. Consider the following equations on $[0, 1]$:

\[
\begin{align*}
    dX_t &= \mathbb{1}_{\{X_t > 0\}} \left( \frac{y - X_t}{1 - t} + \frac{1}{X_t} \right) dt + \mathbb{1}_{\{X_t < 0\}} \left( -\frac{y - X_t}{1 - t} + \frac{1}{X_t} \right) dt + dB_t, \quad X_0 = 0, \\
    dX_t &= \mathbb{1}_{\{X_t \neq 0\}} \frac{1}{X_t} dt + dB_t, \quad X_0 \in \mathbb{R}.
\end{align*}
\]

(a) Any weak solution to (2.2) satisfies $|X_1| = y$ a.s. and does not change its sign on the interval $(0, 1]$ a.s. Moreover, there exists a pathwise unique nonnegative strong solution and a pathwise unique nonpositive strong solution. We will call these solutions nonnegative/nonpositive Bessel bridge.

(b) For $X_0 \geq 0$, there exists a pathwise unique nonnegative strong solution to (2.3). For $X_0 > 0$,\pathwise uniqueness holds.

Proof. (a): Weak existence follows by [17, page 274, Equation (29)]. Pathwise uniqueness and strong existence follow by the same arguments as in the proof of Proposition 1 in [22], which is stated for $y = 1$.

(b): Special case of Theorem 3.2 in [7].

\[\square\]

Lemma 2.4. Let $X$ be the nonnegative Bessel bridge on $[0, 1]$ with $X_0 = 0$ and $X_1 = 1$. Then there exists $\varepsilon > 0$ small enough such that $\{\sup_{t\in[0,1]} X_t < 2\} \supset \{\sup_{s\in[0,1]} |B_s| < \varepsilon\}$ and therefore

\[\mathbb{P}(\sup_{t\in[0,1]} X_t < 2) > 0.\]

Proof. Recall that $X$ satisfies

\[X_t = \int_0^t \mathbb{1}_{\{X_s > 0\}} \left( \frac{1 - X_s}{1 - s} + \frac{1}{X_s} \right) ds + B_t.\]

Let $\varepsilon \in (0, 1/6)$ and consider the set $A := \{\sup_{s\in[0,1]} |B_s| < \varepsilon\}$ fulfilling $\mathbb{P}(A) > 0$. Let $\omega \in A$. Assume that there exists $\tau_2 \in [0, 1]$ such that $X_{\tau_2}(\omega) = 2$. Let $\tau_{5/3} := \sup\{s \in [0, \tau_2] : X_s = 5/3\}$. By continuity $X_{\tau_{5/3}} = 5/3$. Then

\[X_{\tau_2} = \frac{5}{3} + \int_{\tau_{5/3}}^{\tau_2} \left( \frac{1 - X_s}{1 - s} + \frac{1}{X_s} \right) ds + B_{\tau_2} - B_{\tau_{5/3}} \leq \frac{5}{3} + \int_{\tau_{5/3}}^{\tau_2} \left( -\frac{2}{3} + \frac{3}{5} \right) ds + 2\varepsilon < 2,\]

which gives a contradiction. Hence, $\sup_{t\in[0,1]} X_t(\omega) < 2$ for $\omega \in A$.

\[\square\]

Remark 2.5. Let $X$ be the Bessel bridge with $X_0 = 2$ and $X_1 = 1$. By symmetry and a space shift the above lemma gives that $\{\inf_{t\in[0,1]} X_t > 0\} \supset \{\sup_{t\in[0,1]} |B_t| < \varepsilon\}$ for small enough $\varepsilon > 0$ and therefore

\[\mathbb{P}(\inf_{t\in[0,1]} X_t > 0) > 0.\]

3 Counterexample

3.1 Path-by-path existence but no weak existence

In the following proposition we construct an SDE without weak solutions, but with multiple path-by-path solutions. The construction is done in a one-dimensional setting, but can easily be extended to multiple dimensions.
Proposition 3.1. Consider the SDE
\[ dX_t = b(t, X_t)dt + dB_t, \quad X_0 = 0, \quad t \in [0, 3], \]  
where
\[ b(t, x) = \begin{cases} 0 & \text{if } 0 \leq t < 1, \\ \mathbb{1}_{\{x > 0\}} \left( \frac{1 - x}{1 - t} + \frac{1}{x} \right) + \mathbb{1}_{\{x < 0\}} \left( \frac{-1 - x}{1 - t} + \frac{1}{x} \right) & \text{if } 1 \leq t < 2, \\ \mathbb{1}_{\{x \neq 0, \vert x \vert \leq 1\}} \frac{-1}{2x} + \mathbb{1}_{\{x > 1\}} \frac{1}{x - 1} + \mathbb{1}_{\{x < -1\}} \frac{1}{x + 1} & \text{if } 2 \leq t \leq 3. \end{cases} \]

There does not exist a weak solution to (3.1), but there exist path-by-path solutions.

**Idea of the proof.**

The drift is constructed in a way such that for a solution \( X \) the following must hold: \( X_1 \) is forced to be equal to 1 or \(-1\). On the time interval \([2, 3]\) the drift is constructed in a way such that there exists no adapted solution if \( X_2 \in (-1, 1) \). However if \( \vert X_2 \vert > 1 \), then \( X \) can be extended to the interval \([0, 3]\) with \( \vert X_t \vert > 1 \) for all \( t \in [2, 3] \) while still being adapted. Hence, any adapted solution must avoid taking a value in \((-1, 1)\) at time 2. If \( B_2 - B_1 = X_2 - X_1 \in (0, 2) \), this can only be achieved if one can choose \( X_1 = 1 \). Similarly, if \( B_2 - B_1 = X_2 - X_1 \in (-2, 0) \), one must be allowed to choose \( X_1 = -1 \). This necessity of “looking into the future” prohibits the existence of weak solutions, but not the construction of path-by-path solutions.

**Proof. No weak solution.**

Assume that there exists a weak solution \( X \) defined on some filtered probability space \((\Omega, \mathcal{F}, \mathbb{P})\). By Lemma 2.3(a), \( \mathbb{P}(X_1 = 1 \text{ or } X_1 = -1) = 1 \). Assume first that \( \mathbb{P}(X_1 = 1) > 0 \). Let \( A_1 := \{X_1 = 1\} \) and \( A_2 := \{B_2 - B_1 \in (-2, 0)\} \). Then we have \( X_2(\omega) \in (-1, 1) \) for \( \omega \in A_1 \cap A_2 \). As \( A_1 \) is \( \mathcal{F}_1 \)-measurable and \( B_2 - B_1 \) is independent of \( \mathcal{F}_1 \), \( \mathbb{P}(A_1 \cap A_2) > 0 \). Hence, after a time shift, we would have a weak solution to the SDE
\[ dX_t = \left( -\mathbb{1}_{\{x \neq 0, \vert x \vert < 1\}} \frac{1}{2X_t} + \mathbb{1}_{\{x > 1\}} \frac{1}{X_t - 1} + \mathbb{1}_{\{x < -1\}} \frac{1}{X_t + 1} \right) dt + dB_t, \]
on \([0, 1]\) with \( \mathbb{P}(X_0 \in (-1, 1)) > 0 \), which contradicts Lemma 2.2. Hence, \( X_1 = -1 \) a.s. By the same arguments as above we must have \( X_1(\omega) = 1 \) for \( \omega \in A_2 \), where \( A_2 := \{B_2 - B_1 \in (0, 2)\} \), which gives a contraction as \( \mathbb{P}(A_2) > 0 \).

**Existence of a path-by-path solution.**

Let \( X^+, X^- : \Omega \to C([0, 1]) \) be the nonnegative and nonpositive Bessel bridge with terminal value 1, respectively \(-1\). Let \( C_1 := \{B_2 - B_1 > 0\} \) and \( C_2 := \{B_2 - B_1 < 0\} \). Define \( X : \Omega \to C([0, 2]) \) such that, for \( t \in [0, 2] \),
\[ X_t(\omega) := \begin{cases} X_{t \uparrow 1}^+(\omega) + B_{t \uparrow 1}(\omega) - B_1(\omega) & \text{if } \omega \in C_1, \\ X_{t \uparrow 1}^-(\omega) + B_{t \uparrow 1}(\omega) - B_1(\omega) & \text{if } \omega \in C_2. \end{cases} \]

Hence \( \vert X_2(\omega) \vert > 1 \) for \( \omega \in C_1 \cup C_2 \). By Lemma 2.3(b) (after a shift of the space variable), we can uniquely extend \( X \) to \([0, 3]\) fulfilling (3.1) so that \( \vert X_t \vert > 1 \) for all \( t \in [2, 3] \). As \( C_1 \cup C_2 \) has full measure, \( X \) is indeed a path-by-path solution.

**Existence of other path-by-path solutions.**

Other path-by-path solutions can be constructed. Indeed on the set \( C_3 := \{B_2 - B_1 > 2\} \) we can choose freely if \( X \) coincides with the nonnegative or nonpositive Bessel bridge on \([0, 1]\) since, for \( \omega \in C_3 \), \( X_2(\omega) > 1 \). Then, we can proceed the same way as before.

\[ \square \]

### 3.2 Pathwise unique weak solution, no path-by-path uniqueness

The following proposition gives an example of a one-dimensional SDE with a pathwise unique weak solution, but path-by-path uniqueness does not hold. Again, the construction can easily be extended to multiple dimensions.
Proposition 3.2. Consider the SDE

\[ dX_t = b(t, X_t)dt + dB_t, \quad X_0 = 0, \quad t \in [0, 4], \]

where

\[
b(t, x) = \begin{cases} 
\mathbb{1}_{\{x > 0\}} \left( \frac{2 - x}{1 - t} + \frac{1}{x} \right) + \mathbb{1}_{\{x < 0\}} \left( \frac{-2 - x}{1 - t} + \frac{1}{x} \right) & \text{if } 0 \leq t < 1, \\
\mathbb{1}_{\{x < 0\}} \frac{1}{x} + \mathbb{1}_{\{x > 2\}} \left( \frac{3 - x}{2 - t} + \frac{1}{x} \right) + \mathbb{1}_{\{0 < x < 2\}} \left( \frac{1 - x}{2 - t} + \frac{1}{x - 2} \right) & \text{if } 1 \leq t < 2, \\
\mathbb{1}_{\{x < 0\}} \frac{1}{x} & \text{if } 2 \leq t < 3, \\
\mathbb{1}_{\{x < 0\}} \frac{1}{x} - \mathbb{1}_{\{x \neq 2, 0 \leq x \leq 1\}} \frac{1}{2(x - 2)} + \mathbb{1}_{\{x > 3\}} \frac{1}{x - 3} + \mathbb{1}_{\{0 < x < 1\}} \frac{1}{x - 1} & \text{if } 3 \leq t \leq 4.
\end{cases}
\]

There exists a pathwise unique weak solution to (3.2), but path-by-path uniqueness does not hold.

**Idea of the proof.** On the time interval \([0, 1]\) the drift is the one of an SDE solved by a Bessel bridge. Hence, for \(t \in [0, 1]\) there exists a pathwise unique nonnegative weak solution and a pathwise unique nonpositive weak solution. On \([1, 4] \times \mathbb{R}_-\) the drift is constructed such that the nonpositive Bessel bridge on \([0, 1]\) can be extended in a unique way to a weak solution on the time interval \([0, 4]\). On \([1, 4] \times \mathbb{R}_+\) the drift is constructed so that we can extend the nonnegative Bessel bridge on the time interval \([0, 1]\) to path-by-path solutions on \([0, 4]\), but not to a weak solution. Hereby solutions are allowed to enter the negative half plane (which is no problem as the drift there ensures the existence of nonpositive solution), but with positive probability a situation as in Proposition 3.1 occurs; i.e. weak solutions \(X\) are forced to fulfill \(X_3 \in (1, 3)\) with positive probability and on the time interval \([3, 4]\) the drift is constructed such that these solutions cannot be extended to \([0, 4]\).

**Proof.** Existence of a pathwise unique nonnegative weak solution.

Note that there exists a pathwise unique nonpositive weak solution \(\tilde{X}\) on \([0, 1]\) to (3.2) with \(\tilde{X}_1 = -2\) by Lemma 2.3(a). We can extend this solution in a pathwise unique way to \([0, 4]\) by Lemma 2.3(b).

**No other weak solution.**

Assume that there exists another weak solution \(X\) to (3.2). Then we must have \(\mathbb{P}(X_1 = 2) > 0\).

Let

\[ A_1 := \{ X_1 = 2 \}, A_2 := \{ \inf_{t \in [1, 2]} X_t > 0, X_2 = 1 \}, A_3 := \{ \inf_{t \in [2, 3]} (B_t - B_2) > -1, B_3 - B_2 \in (0, 2) \}. \]

First assume that \(\mathbb{P}(X_2 = 1 \mid X_1 = 2) = 1\). Then by assumption and by Remark 2.5, for \(\varepsilon > 0\) small enough,

\[
\mathbb{P}(A_1 \cap A_2) \geq \mathbb{P}(A_1 \cap \{ \sup_{t \in [1, 2]} |B_t - B_1| < \varepsilon \}) = \mathbb{P}(A_1) \mathbb{P}(\sup_{t \in [1, 2]} |B_t - B_1| < \varepsilon) > 0.
\]

Hence \(\mathbb{P}(\bigcap_{i=1}^{3} A_i) > 0\) as \(A_3\) is independent of \(A_1 \cap A_2\). Note that \(X_3(\omega) \in (1, 3)\) for \(\omega \in \bigcap_{i=1}^{3} A_i\). This gives a contradiction to Lemma 2.2 as after a space shift we would have a weak solution to Equation (2.1) with initial condition \(X_0 \in (-1, 1)\) > 0.

Assume now that \(\mathbb{P}(X_2 = 1 \mid X_1 = 2) < 1\) and therefore \(\mathbb{P}(X_2 = 3 \mid X_1 = 2) > 0\). Let \(\tilde{A}_1 := A_1, \tilde{A}_2 := \{ X_2 = 3 \}\) and

\[
\tilde{A}_3 := \{ \inf_{t \in [2, 3]} (B_t - B_2) > -3, B_3 - B_2 \in (-2, 0) \}.
\]
Then by assumption $\mathbb{P}(\cap_{i=1}^{3} \hat{A}_i) > 0$ and $X_3(\omega) \in (1, 3)$ for $\omega \in \cap_{i=1}^{3} \hat{A}_i$ and again this leads to a contradiction to Lemma 2.2.

**Construction of another path-by-path solution.**

By separately considering the sets $C_1 := \{B_3 - B_2 > 0\}$ and $C_2 := \{B_3 - B_2 < 0\}$ we construct an additional path-by-path solution on the set $C_1 \cup C_2$ of full measure. Let $X : C_1 \rightarrow \mathcal{C}([0, 2])$ coincide with the Bessel bridges such that $X_1 = 2$ and $X_2 = 3$. Let $\tau := \inf\{t \geq 2 : B_t - B_2 = -3\}$. Then, by Lemma 2.3(b), for $\omega \in C_1 \cap \{\tau > 3\}$ we can extend $X$ to the time interval $[0, 4]$ fulfilling equation (3.2). Now consider $\omega \in C_1 \cap \{\tau \leq 3\}$. We can clearly extend $X$ to the time interval $[0, \tau]$ by adding the Brownian increment $B_t - B_2$. Then we have that $X_\tau(\omega) = 0$. Hence, for $\omega \in C_1 \cap \{\tau \leq 3\}$ and $t \in [\tau, 4]$, we can choose $X$ to be the pathwise unique nonpositive solution to

$$dX_t = \frac{1}{X_t} dt + dB_t, \ t \in [\tau, 4]$$

and therefore we can construct $X : C_1 \rightarrow \mathcal{C}([0, 4])$ fulfilling equation (3.2).

Let

$$\hat{b}(t, x) := b(t, x) - \mathbb{1}_{\{t \geq 1, x < 0\}} \frac{1}{x} + \mathbb{1}_{\{t > 3, x < 0\}} \frac{1}{x - 1}.$$

For $\omega \in C_2$ and $t \in [0, 4]$, let $\hat{X}$ fulfill

$$\hat{X}_t(\omega) = \int_0^t \hat{b}(s, \hat{X}_s(\omega))ds + B_t(\omega)$$

such that $\hat{X}$ coincides with the two Bessel bridges on $[0, 1]$ and $[1, 2]$ so that $\hat{X}_1 = 2$ and $\hat{X}_2 = 1$. This is possible by the same arguments as in the proof of Proposition 3.1. Let $\tau_0 := \inf\{t > 1 : X_t = 0\} \wedge 4$. On $[0, \tau_0] \cap [0, 4]$ let $X := \hat{X}$. Then $X$ is a solution to (3.2) on $[0, \tau_0]$ as, for $x > 0$, $b(t, x) = \hat{b}(t, x)$. On $[\tau_0, 4]$, we choose $\hat{X}$ to be the pathwise unique nonpositive solution to

$$dX_t = \frac{1}{X_t} dt + dB_t, \ t \in [\tau_0, 4].$$

As $\mathbb{P}(C_1 \cup C_2) = 1$, we can construct a path-by-path solution $Y : \Omega \rightarrow \mathcal{C}([0, 4])$ by setting

$$Y(\omega) = \begin{cases} X(\omega) & \text{if } \omega \in C_1, \\ \hat{X}(\omega) & \text{if } \omega \in C_2. \end{cases}$$

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