Efficiency of Ground State Quantum Computer

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The energy gap is calculated for the ground state quantum computer circuit, which was recently proposed by Mizel et al. When implementing a quantum algorithm by Hamiltonians containing only pairwise interaction, the inverse of energy gap $1/\Delta$ is proportional to $N^{4\kappa}$, where $N$ is the number of bits involved in the problem, and $N^\kappa$ is the number of control operations performed in a standard quantum paradigm. Besides suppressing decoherence due to the energy gap, in polynomial time ground state quantum computer can finish the quantum algorithms that are supposed to be implemented by standard quantum computer in polynomial time.

Quantum computer is widely believed to outperform its classical counterpart for some classically difficult problems. Although many schemes have been suggested, the main obstacle is decoherence that makes very difficult the realization of even one qubit or one gate for quantum computing. Ground state quantum computer (GSQC) is a new approach proposed by Mizel et al., which mimics time evolution of a system by space distribution of ground state wavefunction. The advantage of GSQC is that appreciable energy gap suppresses decoherence from environment as long as temperature is low enough.

To analyze the performance of GSQC, the key is to evaluate the scale of $\Delta$, the energy gap between the ground state $|\psi_0\rangle$ and the first excited state $|\psi_1\rangle$, which is related to the time cost. Mizel et al. gave estimations on single qubit and two-qubit circuit in [12]. However, they didn’t extend to arbitrary circuit with boost or projection Hamiltonians applied, while boost and projection Hamiltonians are necessary to make sure that $|\psi_0\rangle$ concentrates on the position corresponding to the final time in standard paradigm so that measurement on GSQC can read out desired information with appreciable probability. In [13] it was proposed to shorten length, for a single qubit, by inserting teleportation circuits, however, it will be shown that there the energy gap scale is incorrect. In the present paper, the scaling of energy gap is analyzed for general circuit that can be used for any known quantum algorithm. I will use boundary condition different from [12], and show that there exist dangerous cases, in which the energy gap shrinks exponentially as the number of qubits increases. In order to prevent such exponential shrink, teleportation circuits are also inserted for multiple interacting qubits. Finally energy gap scaling for general quantum algorithm is presented.

A standard computer is characterized by time dependent state as: $|\psi(t_i)\rangle = U_i|\psi(t_{i-1})\rangle$, where $t_i$ denotes instance of the $i$-th step, and $U_i$ represents for unitary transformation. For GSQC, the time sequence is mimicked by the space distribution of the ground state wavefunction $|\psi_0\rangle$.

As proposed by Mizel et al., a single qubit may be a column of quantum dots with multiple rows, and each row contains a pair of quantum dots. State $|0\rangle$ or $|1\rangle$ is represented by finding electron in one of the two dots. It is important to notice that only one electron exists in a qubit. A GSQC is made up by circuit of multiple interacting qubits, whose ground state is determined by the summation of single qubit unitary transformation Hamiltonian $h(U_j)$, two-qubit interacting Hamiltonian $h(CNOT)$, boost Hamiltonian $h(B, \lambda)$ and projection Hamiltonian $h(\gamma, \lambda)$.

The single qubit unitary transformation Hamiltonian has the form:

$$h^j(U_j) = \epsilon \left[ C^\dagger_{j-1} C_{j-1} + C^\dagger_j C_j \right]$$

$$- \left( C^\dagger_j U_j C_{j-1} + h.c. \right), \quad (1)$$

where $\epsilon$ defines the energy scale of all Hamiltonians, $C^\dagger_j = \left[ c^\dagger_{j,0} c^\dagger_{j,1} \right]$, $c^\dagger_{j,0}$ is the electron creation operator on row $j$ at position 0, and $U_j$ is two dimension matrix representing for unitary transformation from row $j - 1$ to row $j$. The boost Hamiltonian is:

$$h^j(B, \lambda) = \epsilon \left[ C^\dagger_{j-1} C_{j-1} + \frac{1}{\lambda^2} C^\dagger_j C_j \right]$$

$$- \frac{1}{\lambda} \left( C^\dagger_j U_j C_{j-1} + h.c. \right), \quad (2)$$

which amplifies the wavefunction amplitude by large number $\lambda$ compared with previous row in $|\psi_0\rangle$. The projection Hamiltonian is

$$h^j(\gamma, \lambda) = \epsilon \left[ C^\dagger_{j-1,\gamma} C_{j-1,\gamma} + \frac{1}{\lambda^2} C^\dagger_{j,\gamma} C_{j,\gamma} \right]$$

$$- \frac{1}{\lambda} \left( c^\dagger_{j,\gamma} C_{j-1,\gamma} + h.c. \right), \quad (3)$$

where $|\gamma\rangle$ represents for state to be projected to on row $j$ and to be amplified by $\lambda$. The interaction between qubit $\alpha$ and $\beta$ can be represented by $h(CNOT)$:

$$h^\dagger_{\alpha,\beta}(CNOT) = \epsilon C^\dagger_{\alpha,j-1} C_{\alpha,j-1} C^\dagger_{\beta,j} C_{\beta,j}$$

$$+ h^\dagger_{\alpha,\beta}(I) C^\dagger_{\beta,j-1} C_{\beta,j-1} + c^\dagger_{\alpha,j,0} c_{\alpha,j,0} h^\dagger_{\beta}(I)$$

$$+ c^\dagger_{\alpha,j,1} c_{\alpha,j,1} h^\dagger_{\beta}(N). \quad (4)$$
where for $c_{a,b,c}^j$, its subscription $a$ represents for qubit $a$, $b$ for the number of row, $c$ for the state $|c\rangle$. With only $h^j(U_j)$ and $h^j_{\alpha,\beta}(CNOT)$, its ground state is [14]:

$$|\psi_0^j\rangle = \left[1 + c_{a,j,0}^\dagger c_{a,j-1,0} \right] \left[1 + C_{\beta,j}^\dagger C_{\beta,j-1} \right] \times \prod_{a \neq \alpha,\beta} \left(1 + C_{a,j}^\dagger U_{a,j} C_{a,j-1} \right) \langle \psi^{j-1} \rangle.$$

(5)

All above mentioned Hamiltonians are positive semidefinite, and are the same as those in [12]. Only pairwise interaction is considered for Hamiltonians involving multiple qubits.

The input states are determined by the boundary conditions applied upon the first rows of all qubits, which can be Hamiltonian $h^0 = E(I + \sum_i a_i \sigma_i)$ with $\sigma_i$ being Pauli matrix and $\sum_i a_i^2 = 1$. For example, if $h^0 = E(I + \sigma_z)$, then $|\psi_0\rangle$ on the first row is $|1\rangle$; if $h^0 = E(I - \sigma_z)$, then it is $(|0\rangle + |1\rangle)/\sqrt{2}$. Unlike in [12], we set $E > \epsilon > 0$, thus boundary Hamiltonians are not perturbation. If $E$ is large enough, the energy gap is independent of its magnitude. A typical value may be $E \approx 10\epsilon$.

Although one can analytically obtain energy gap for single qubit with $n$ rows [14], it’s difficult to calculate for a complicated circuit. However, we can still manage to find its scale.

Let’s first consider the simplest case: a $n$-row single qubit without projection or boost Hamiltonian. The Hamiltonian $\sum_{i=1}^n h(U_i)$ is in fact the kinetic energy just like a particle in a one-dimension box with length $n$, thus $\Delta \propto \epsilon/n^2$. The unique ground state $|\psi_0\rangle$ is uniform over all rows, and $|\psi_1\rangle$ is orthogonal to $|\psi_0\rangle$ on each row. Due to large value of $E$ in $h^0$, $|\psi_1\rangle$ on the first row is nearly zero, and it is an mono-increasing function that reaches maximum on the final row so as to keep wavefunction smooth. When a boost Hamiltonian $h(B, \lambda)$ is applied to the qubit’s final row, $|\psi_1\rangle$ concentrates there, where the wavefunction amplitude is about $\lambda$ times those at other rows on average, so the wavefunction amplitude on other rows is $O(1/\sqrt{\lambda^2 + n - 1})$. When the qubit length $n \ll \lambda$, $|\psi_1\rangle$ is nearly a linear function of position except for at final row, and its amplitude at the first row is zero. The local kinetic energy is almost constant along the whole qubit, proportional to the square of difference of wavefunction between neighboring rows, hence energy gap scales as:

$$\Delta \propto \frac{\epsilon}{\lambda^2}. \quad (6)$$

If instead of boost Hamiltonian, a projection Hamiltonian $h(|\gamma\rangle, \lambda)$ is applied on the last row of a single qubit, then $|\psi_0\rangle$ on the final row is restricted to state $|\gamma\rangle$. Assuming besides amplitude and phase $|\psi_0\rangle$ on the second last row is at state $|\xi\rangle$, which is normalized, and $\langle \xi|\gamma\rangle$ is appreciable, then $|\psi_0\rangle$ will concentrate on the final row, and $|\psi_1\rangle$ should have little weight there, otherwise $\langle \psi_1|\psi_0\rangle \neq 0$. Thus $\Delta$ remains at about $\epsilon/n^2$, independent of $\lambda$.

Numerical calculations on single qubit in various situations have confirmed all the above analysis. For example, a 6-row qubit with $h(B, \lambda)$ on final row and with boundary Hamiltonian, $h^0 = 10\epsilon(I - \sigma_z)$, on its first row, has energy gap $\Delta = 0.0782, 0.0174, 1.94 \times 10^{-3}, 1.96 \times 10^{-4}, 1.96 \times 10^{-5} \epsilon$ at $\lambda = 1, \sqrt{10}, 10, \sqrt{1000}, 100$ respectively.

Next we consider two qubits interacting with each other through $(CNOT)$ as shown in Fig. (1a), with the left qubit as control qubit, the right one as target qubit, and both ending with $h(B, \lambda)$. By observing $|\psi_0\rangle$ in Eq. (4), it’s easy to see that it has the form

$$\langle \psi_1|\psi_0\rangle = \left(\langle \psi_{\text{control, upstream}}| + \langle \psi_{\text{control, downstream}}| \right) \langle \psi_{\text{target}}| \langle \psi_{\text{upstream}}| + \langle \psi_{\text{control, downstream}}| \langle \psi_{\text{target}}| \langle \psi_{\text{downstream}}|, \quad (7)$$

where the Hamiltonian $h(CNOT)$ divides both qubits into upstream and downstream parts with $h(B, \lambda)$ in downstream parts. $h(B, \lambda)$ on the final row of the target qubit also raises the amplitude of $|\psi_0\rangle$ on the downstream rows of the control qubit, while $h(B, \lambda)$ on control qubit doesn’t influence the amplitude on the target qubit at all because both parts of states entangled with the downstream state of control qubit. Thus on the control qubit the amplitude of upstream over its final row is $1/\sqrt{\lambda^2}$, and it is $1/\lambda$ on target qubit because the state on final row of control qubit entangles with both upstream and downstream part of wavefunction on target qubit. Numerical calculation on $|\psi_0\rangle$ agrees exactly with this analysis.

The first excited state should be such a state that on the first row of one of the two interacting qubits, $|\psi_1\rangle$ is orthogonal to $|\psi_0\rangle$, and its amplitude is nearly zero due to large on-site potential from boundary Hamiltonian $h^0$.
while on the first row of the other qubit $|\psi_1\rangle$ is at the same state as $|\psi_0\rangle$ with only amplitude modified. If one knows the overall amplitude of wavefunction in upstream part on the qubit whose first row state is orthonormal to $|\psi_0\rangle$,

$$\frac{1}{x} \approx \||\psi_1\rangle_{upstream}||,$$  

then the scale of energy gap can be found by same reasoning for single qubit:

$$\Delta \propto \epsilon/x^2.$$  

If $|\psi_1\rangle$ maintains similar weight distribution to $|\psi_0\rangle$, we find that $|\psi_1\rangle$ should be orthogonal to $|\psi_0\rangle$ on the first row of the control qubit, in which $1/x \propto 1/x^2$, while on target qubit $1/x \propto 1/\lambda$, thus the energy gap $\Delta \propto \epsilon/\lambda^4$. The larger value $1/x$ on target qubit doesn’t determine the energy gap because there at the first row $|\psi_1\rangle$ has appreciable amplitude.

Numerical calculation agrees with above analysis for $|\psi_1\rangle$. We also found that the second excited state $|\psi_2\rangle$ has the same energy gap scaling as $|\psi_1\rangle$. $|\psi_2\rangle$ on the first row of target qubit is orthonormal to $|\psi_0\rangle$ with nearly zero amplitude, while its first row state on control qubit is the same as $|\psi_0\rangle$. $1/x$ on both target qubit and control qubit at $|\psi_2\rangle$ are of the same order $1/x^2$, hence the energy gap is also proportional to $\epsilon/\lambda^4$. The parameter $1/x$ on target qubit in $|\psi_2\rangle$ is different from those in $|\psi_0\rangle$ and $|\psi_1\rangle$ because $|\psi_2\rangle$ at the first row of target qubit has to be nearly zero, according to $h(CNOT)$, the suppression of amplitude of wavefunction requires larger difference over two neighboring rows to supply same amount of kinetic energy. In both $|\psi_1\rangle$ and $|\psi_2\rangle$, the value $1/x$ maintains the same order as ground state on the qubit whose first row state is the same as $|\psi_0\rangle$.

If projection Hamiltonians are applied instead of boost Hamiltonians, and at $|\psi_0\rangle$ $\langle\xi|\gamma\rangle$ is appreciable, where $|\xi\rangle$ is the state on the row before $h(B, \lambda)$, then $\Delta \propto \epsilon/\lambda^2$ due to the same reason as for single qubit. If there is one boost Hamiltonian and one projection Hamiltonian on the two interacting qubits, energy gap is still $\Delta \propto \epsilon/\lambda^4$. It is interesting to note that by projecting the target qubit into one state on its last row and boost this state, one can manipulate the state on the final row of the control qubit, and this can be observed in Eq.(7).

Numerical calculations confirm our analysis. For example, the energy gap for two 4-row qubits ended with $h(B, \lambda)$, as shown in Fig.(11) with $h_1^B = 10\epsilon(I - \sigma_x)$, $h_2^B = 10\epsilon(I + \sigma_x)$, is $\Delta = 0.0574$, $2.43 \times 10^{-3}$, $3.05 \times 10^{-5}$, $3.12 \times 10^{-7}$, $3.13 \times 10^{-9}\epsilon$ at $\lambda = 1$, $\sqrt{10}$, $10$, $\sqrt{1000}$, 100, respectively.

In the works by Mizel et.al. [14], when $h(B, \lambda)$ are applied to two interacting qubits, their lower bound energy gap agrees with result in the present paper, however, they didn’t elaborate further for multiple interacting qubits.

$|\psi_0\rangle$ shows how to use teleportation to increase $\Delta$ for single qubit evolution, and there $\Delta$ is proportional to $\epsilon/\lambda^2$, which is incorrect because even two interacting qubits will give energy gap of $\epsilon/\lambda^4$ as shown in our analysis and numerical calculation for two qubit case, while the teleportation circuit involves more than two qubits interacting with each other. However, continuous single qubit evolution can always be combined into one unitary transformation, there is no need to evaluate this situation.

With multiple qubits interacting with each other, we need to evaluate on the top part of each qubit the parameter $1/x$ defined in Eq.(5) assuming that only on the first row of that qubit the excited state is orthonormal to $|\psi_0\rangle$ while states on all other qubits remains the same as corresponding ground state with only magnitude changed. Then the minimum $1/x$ gives the energy gap scale as $\epsilon(1/x)^2_{\min}$. Thus the energy gap depends on the detail of the circuit.

In principle, when all qubits end with either projection or boost Hamiltonian containing same $\lambda$, at ground state and first excited state, the boost Hamiltonian or projection Hamiltonian raises the amplitude of wavefunction on the final row by $\lambda$ compared without boost or projection Hamiltonian. When estimating $1/x$ of $|\psi_1\rangle$ for any qubit, say qubit $A$, the boost Hamiltonian, not the projection Hamiltonian, on qubit $A$ itself contributes $\lambda$ to $x$, and projection or boost Hamiltonians from its control operation mates contribute $\lambda$ to its $x$ value; contribution from those qubits not directly interacting with qubit $A$ can be analyzed according to ground state wavefunction distribution, Eq.(7).

For example, in Fig.(11a) qubit 1 $CNOT$ controls qubit 2, and in its downstream part, qubit 2 $CNOT$ controls qubit 3, so on to qubit 5. When all qubits end with $h(B, \lambda)$, we find that, for $\lambda \gg 5$ (5 is the typical length of qubit) the parameter in Eq.(5) on qubit 5 has contribution only from $h(B, \lambda)$’s on itself and qubit 4, those on qubit 3, 2 and 1 doesn’t make contribution because on qubit 5 its upstream part can entangle with the final rows of other qubits except for qubit 4. We get $x \propto \lambda^2$, $\lambda^3$, $\lambda^4$, $\lambda^5$ for qubit 5, 4, 3, 2 and 1 respectively. The energy gap is determined by value of $1/x$ on qubit 1 or 2, hence $\Delta \propto \epsilon/\lambda^{10}$. As number of qubits $N$ in such a chain increases, the energy gap shrinks as $\epsilon/\lambda^{2N}$.

The energy gap of a GSQC circuit may be exponentially small depending on detail of circuits. The chain in Fig.(11b) is the most dangerous circuit giving the least energy gap, the other circuit that also suffers exponential shrink of energy gap $\Delta \propto \epsilon/\lambda^{2N}$ is a qubit interacting with $N$ other qubits as the control qubit. These two kinds of circuits are quite common in quantum algorithm, such as quantum Fourier transform.

In order to break such exponential shrink of $\Delta$ like in Fig.(11b), the circuit is modified by inserting teleportation boxes for all qubits between two control Hamiltonians, as in Fig.(12), which shows how the circuit in the dotted box of Fig.(11b) is modified. A teleportation box teleports quantum state of a qubit to the other one, and it can be implemented in both standard paradigm.
While projection or boost Hamiltonian on qubit 8 or 10 doesn't contribute to 1/x on qubit 3. By inserting teleportation boxes, we find that when the amplifying factor of all boost and projection Hamiltonians is \( \lambda \), the energy gap for arbitrary GSQC circuit is always
\[
\Delta \propto \epsilon/\lambda^8.
\] (10)
Hence there is no chain action on 1/x like in Fig. (1b) that leads to exponential shrink of energy gap. In fact, the upper right and the lower left teleportation boxes are not needed concerning on circuit like Fig. (1b), however, I keep them for more general consideration.

For arbitrary circuit, it can always be modified similar to Fig. (2), and the energy gap will be kept at the scale of \( \epsilon/\lambda^8 \). The price of this modification is that the total number of qubits increases, thus from measurement concern, it requires larger value of \( \lambda \) to make sure appreciable probability to find all electrons on the final row's of all qubits. However, this price is worthy because it may change energy gap from exponentially small to polynomially small, while the value of \( \lambda \) is only polynomially increased. This can be demonstrated in the following example.

Now we can check the energy gap when a quantum algorithm is implemented by GSQC. The most powerful quantum algorithm up to now is quantum Fourier transform, which is used in factorization, period finding, etc. The detail of quantum Fourier transform, more precisely, the inverse quantum Fourier transform, can be found in many literatures, such as [10]. It takes \( O(N^2) \) control operations \( h(CR_k^1) \) to carry out inverse quantum Fourier transform by standard time dependent approach, where \( N \) is the number of qubits involved in the problem, and \( R_k^1 \) is the unitary transformation shifting phase of state \( |1⟩ \) by \(-2\pi i/k^2 \). \( h(CR_k^1) \) can be obtained by replacing NOT operator by \( R_k^1 \) in Eq. (4). One can simply replace the time evolution by circuit array and calculate the energy gap of the circuit. By inserting teleportation boxes between control operations, when all qubits end either by \( h(|0⟩, \lambda) \) or \( h(B, \lambda) \), the energy gap is \( \Delta \propto \epsilon/\lambda^8 \). Because after experiencing a control operation each qubit is teleported to a new qubit, the total number of qubits is of the order \( N^2 \) instead of \( N \). With each qubit ended with boost or projection Hamiltonian, at \( |ψ_0⟩ \) the probability of finding electron on the final row on any qubit is larger than \( (1-C/\lambda^2) \) with \( C \leq 8 \) being the number of rows on the qubit. In order to find all electrons on final rows with appreciable probability \( P \geq (1-C/\lambda^2)^{FN^2} \), it requires \( \lambda \approx \sqrt{D/N} \), here we assume there are totally \( FN^2 \) qubits involved, and \( D \) is a constant that could be tuned. Thus the probability is \( P \approx e^{-FC/D} \) as \( N \) being large number, the energy gap is \( \Delta \propto \epsilon/(D^4N^8) \), noting that standard paradigm needs only \( O(N^2) \) steps.

We conclude that the energy gap of ground state quantum computer is determined by the number of control operation, and any quantum algorithm implemented by standard paradigm can be implemented by GSQC, whose energy gap is \( \Delta \propto 1/N^{4k} \), where \( N \) is the number of bits.
in the problem, and $N^k$ is the number of control operations needed in standard quantum paradigm. There are various approaches to make GSQC stay at its ground state, and the time cost is determined by different approaches. Estimation of time can be obtained by adiabatic approach, in which we increase $\lambda$ slowly from 1 to $O(N^k/2)$ in all boost and projection Hamiltonians on final row of all qubits, GSQC will stay at ground state and time cost is of the order $N^{8k}$, determined by inverse of square of minimum energy gap. By local adiabatic approach, the time cost is even shorter.$[4]$.

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