ON THE EFFECT OF HIGHER ORDER DERIVATIVES OF INITIAL DATA ON THE BLOW-UP SET FOR A SEMILINEAR HEAT EQUATION

YOHEI FUJISHIMA

Department of Mathematical and Systems Engineering
Faculty of Engineering, Shizuoka University
3-5-1 Johoku, Hamamatsu 432-8561, Japan

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ABSTRACT. This paper concerns the blow-up problem for a semilinear heat equation

\[
\begin{aligned}
\partial_t u &= \Delta u + u^p, \quad x \in \Omega, \quad t > 0, \\
u(x, t) &= 0, \quad x \in \partial \Omega, \quad t > 0, \\
u(x, 0) &= u_0(x) \geq 0, \quad x \in \Omega,
\end{aligned}
\]  

(P)

where \( \partial_t = \partial / \partial t, \ p > 1, \ N \geq 1, \ \Omega \subset \mathbb{R}^N, \ u_0 \) is a bounded continuous function in \( \Omega \). For the case \( u_0(x) = \lambda \varphi(x) \) for some function \( \varphi \) and a sufficiently large \( \lambda > 0 \), it is known that the solution blows up only near the maximum points of \( \varphi \) under suitable assumptions. Furthermore, if \( \varphi \) has several maximum points, then the blow-up set for (P) is characterized by \( \Delta \varphi \) at its maximum points. However, for initial data \( u_0(x) = \lambda \varphi(x) \), it seems difficult to obtain further information on the blow-up set such that effect of higher order derivatives of initial data. In this paper, we consider another type large initial data \( u_0(x) = \lambda + \varphi(x) \) and study the relationship between the blow-up set for (P) and higher order derivatives of initial data.

1. Introduction. In this paper we are concerned with the blow-up problem for a semilinear heat equation

\[
\begin{aligned}
\partial_t u &= \Delta u + u^p, \quad x \in \Omega, \quad t > 0, \\
u(x, t) &= 0, \quad x \in \partial \Omega, \quad t > 0, \\
u(x, 0) &= u_0(x) \geq 0, \quad x \in \Omega,
\end{aligned}
\]

(1)

where \( \partial_t = \partial / \partial t, \ p > 1, \ N \geq 1, \ \Omega \subset \mathbb{R}^N, \ u_0 \) is a nonnegative initial function. Let \( T(u_0) \) be the maximal existence time of the unique classical solution of (1). If \( T(u_0) < \infty \), then \( \|u(t)\|_{L^\infty(\Omega)} \to \infty \) as \( t \nearrow T(u_0) \). Then we say that the solution of (1) blows up in a finite time, and call \( T(u_0) \) the blow-up time. Furthermore, if \( T(u_0) < \infty \), then we define the set \( B(u_0) \) by

\[
B(u_0) := \left\{ x \in \overline{\Omega} : \text{there exists a sequence } \{(x_n, t_n)\} \subset \overline{\Omega} \times (0, T(u_0)) \right. \\
\left. \text{such that } \lim_{n \to \infty} (x_n, t_n) = (x, T(u_0)) \text{ and } \lim_{n \to \infty} u(x_n, t_n) = \infty \right\},
\]

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and call $B(u_0)$ the blow-up set.

The location of the blow-up set for problem (1) has been studied intensively by many mathematicians. See, for example, [1, 2, 7, 12, 13], [17]–[20] and a survey [11]. In particular, the blow-up set for (1) with small diffusion is studied in [5] and [6]. (See [3] and [4] for more general nonlinearity.) More precisely, they characterized the location of the blow-up set of the solution of

\[ \partial_t u = \epsilon \Delta u + u^p, \quad x \in \Omega, \quad t > 0, \quad u(x, 0) = \varphi(x) \geq 0, \quad x \in \Omega, \]

under the zero Dirichlet boundary condition, where $\epsilon > 0$ is a sufficiently small parameter and $\varphi$ is a nonnegative bounded continuous function. Here we remark that, if $u$ solves (2) together with zero Dirichlet boundary condition, then the function $u_\epsilon(x, t) := \epsilon^{-1/(p-1)}u(x, \epsilon^{-1}t)$ solves (1) with $u_0$ replaced by $\epsilon^{-1/(p-1)}u(x, 0)$. Therefore the heat equation with small diffusion is equivalent to the one with large initial data, and the results obtained in [5] and [6] also hold for (1) replaced by $u_0(x) = \lambda \varphi(x)$ with sufficiently large parameter $\lambda$. This relation together with the results in [5] and [6] gives the location of the blow-up set for problem (1) with $u_0(x) = \lambda \varphi(x)$ for a nonnegative function $\varphi$ and sufficiently large $\lambda$. In fact, the author of this paper and Ishige proved the following results: Let $\lambda > 0$ and $\varphi$ be a nonnegative bounded continuous function on $\overline{\Omega}$ satisfying suitable assumptions, and consider problem (1) with $u_0(x) = \lambda \varphi(x)$. In this case, the blow-up time and the blow-up set is denoted by $T(\lambda \varphi)$ and $B(\lambda \varphi)$ according to the definitions, respectively. Assume that the solution $u$ satisfies

\[ \limsup_{\lambda \to \infty} \sup_{0 < t < T(\lambda \varphi)} (T(\lambda \varphi) - t)^{1/(p-1)} \| u(t) \|_{L^\infty(\Omega)} < \infty. \]

Then,

- for any $\delta > 0$, it holds $B(\lambda \varphi) \subset \{ x \in \overline{\Omega} : \varphi(x) \geq \| \varphi \|_{L^\infty(\Omega)} - \delta \}$ for all sufficiently large $\lambda$;
- if $\varphi \in C^2(\Omega)$ and $a, b \in \Omega$ are maximum points of $\varphi$ satisfying

\[ |(\Delta \varphi)(a)| < |(\Delta \varphi)(b)|, \]

there exists a positive constant $\delta$ such that $B(\lambda \varphi) \cap B(b, \delta) = \emptyset$ for all sufficiently large $\lambda$, that is, the solution $u$ cannot blow-up near the maximum point $b$.

The first result implies that the solution blows up only near the maximum points of $\varphi$ if $\lambda$ is sufficiently large. Moreover, the second result implies that the location of the blow-up set is characterized by $\Delta \varphi$ at maximum points of $\varphi$ if $\varphi$ has several maximum points. In particular, if the set of maximum points of $\varphi$ consists of two points $a$ and $b$, then the solution blows up only near the maximum point $a$.

This paper concerns the case $\Delta u_0(a) = \Delta u_0(b)$, where $a$ and $b$ are maximum points of $\varphi$. Considering the case $\Delta u_0(a) = \Delta u_0(b)$, we try to find the next effect on the location of the blow-up set with large initial data. In order to obtain the location of the blow-up set, we study the profile of the solution near the blow-up time. We need a precise profile of the solution for the case $\Delta u_0(a) = \Delta u_0(b)$ at the time much closer to the blow-up time $T(\lambda \varphi)$ than the one for the case $\Delta u_0(a) \neq \Delta u_0(b)$, and it seems difficult to study the case $\Delta u_0(a) = \Delta u_0(b)$ by using the method used in [6].
In this paper we focus on another “large initial data” of the form $u_0(x) = \lambda + \varphi(x)$ with sufficiently large $\lambda$, that is, we consider

$$
\begin{align*}
&\begin{cases}
  \partial_t u = \Delta u + u^p, & x \in \Omega, \ t > 0, \\
  u(x, t) = 0, & x \in \partial \Omega, \ t > 0, \\
  u(x, 0) = \lambda + \varphi(x), & x \in \Omega.
\end{cases}
\end{align*}
$$

(3)

For this problem, we consider the case $u_0(x) = \lambda + \varphi(x)$ and $\Delta u_0(a) = \Delta u_0(b)$, and study the location of the blow-up set of the solution for problem (1) for a sufficiently large $\lambda$. In particular, we characterize the location of the blow-up set by using $\Delta^2 u_0$ at maximum points of $u_0$.

Before stating our main results, we introduce some notation. For $a \in \mathbb{R}$, let $[a]$ denote the greatest integer that is less than or equal to $a$. For a multi-index $\alpha = (\alpha_1, \ldots, \alpha_N) \in (\mathbb{N} \cup \{0\})^N$, put

$$x^\alpha = x_1^{\alpha_1} \cdots x_N^{\alpha_N}, \quad \nabla^\alpha = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_N}^{\alpha_N}.$$

We denote by $BC_+(\Omega)$ the set of nonnegative bounded continuous functions. For $\varphi \in BC_+(\Omega)$ and $\delta > 0$, put

$$M(\varphi) := \{x \in \Omega : \varphi(x) = \|\varphi\|_{L^\infty(\Omega)}\}, \quad M(\varphi, \delta) := \{x \in \Omega : \varphi(x) \geq \|\varphi\|_{L^\infty(\Omega)} - \delta\}.$$

For any $\varphi \in BC_+(\Omega) \cap C^{2m}(\Omega)$ for some $m \in \mathbb{N}$ and $\eta > 0$, put

$$M_m(\varphi, \eta) := \left\{ \beta \in M(\varphi) : (\Delta^m \varphi)(\beta) \leq \sup_{\alpha \in M(\varphi)} (\Delta^m \varphi)(\alpha) - \eta \right\}.$$

Put $T_\lambda := T(\lambda + \varphi)$ and $B_\lambda := B(\lambda + \varphi)$, that is, we denote the blow-up time and the blow-up set for problem (3) by $T_\lambda$ and $B_\lambda$, respectively. For any $\phi \in L^\infty(\mathbb{R}^N)$, let $e^{t \Delta} \phi$ be the unique bounded solution of

$$\partial_t u = \Delta u \quad \text{in} \ \mathbb{R}^N \times (0, \infty), \quad u(x, 0) = \phi(x) \quad \text{in} \ \mathbb{R}^N.$$

Then we have

$$e^{t \Delta} \phi(x) = (4\pi t)^{-\frac{N}{2}} \int_{\mathbb{R}^N} e^{-\frac{|y|^2}{4t}} \phi(y) \, dy, \quad (x, t) \in \mathbb{R}^N \times (0, \infty).$$

On the other hand, for any functions $f = f(\lambda)$ and $g = g(\lambda)$ from $(0, \infty)$ to $(0, \infty)$, we say that

$$f(\lambda) \lesssim g(\lambda) \quad \text{for all sufficiently large} \ \lambda,$$

if there exist constant $\lambda_* > 0$ and $C > 0$ such that $f(\lambda) \leq C g(\lambda)$ for any $\lambda \geq \lambda_*$. We are ready to state our main results.

**Theorem 1.1.** Let $N \geq 1$, $\Omega$ be a domain in $\mathbb{R}^N$ and $\varphi \in C^4(\Omega) \cap BC_+(\Omega)$ be such that $M(\varphi, \epsilon_0)$ is compact in $\Omega$ for some $\epsilon_0 > 0$, and assume that

$$\Delta \varphi(a) \quad \text{takes the same value for all} \ a \in M(\varphi).$$

Let $p > 1$ and $u$ be the solution of problem (3). Assume that there exists a constant $\lambda_0 > 0$ such that

$$\sup_{\lambda > \lambda_0} \sup_{0 < t < T_\lambda} (T_\lambda - t)^{-\frac{N}{p-1}} \|u(t)\|_{L^\infty(\Omega)} < \infty. \quad (4)$$

Then, for any $\eta > 0$, there exist constants a constant $\delta > 0$ such that

$$B_\lambda \cap \left( \bigcup_{a \in M_2(\varphi, \eta)} B(a, \delta) \right) = \emptyset.$$
for all sufficiently large $\lambda$.

As a corollary of Theorem 1.1, we obtain the following result.

**Corollary 1.** In addition to the same conditions as in Theorem 1.1, assume that $M(\varphi) = \{a, b\}$, $(\Delta \varphi)(a) = (\Delta \varphi)(b)$, $(\Delta^2 \varphi)(a) > (\Delta^2 \varphi)(b)$.

Then

$$\lim_{\lambda \to \infty} \sup_{x \in B_\lambda} |x - a| = 0.$$  

In general, it seems difficult to obtain further results for the case $(\Delta^2 \varphi)(a)$ takes the same value for all $a \in M(\varphi)$. However, if $p$ is close to 1, then we can refine the previous result.

**Theorem 1.2.** Let $N \geq 1$ and $\Omega$ be a domain in $\mathbb{R}^N$. For any $m \geq 3$, there exists a constant $p(m) > 1$ with the following property: Let $1 < p < p(m)$ and $\varphi \in C^{2m}(\Omega) \cap BC_+(\Omega)$ be such that $M(\varphi, \epsilon_0)$ is compact in $\Omega$ for some $\epsilon_0 > 0$. Assume that, for any $k \in \{1, \ldots, m-1\}$, it holds $(\Delta^k \varphi)(a)$ takes the same value for all $a \in M(\varphi)$.

Let $u$ be the solution of problem (3). Assume (4) for some $\lambda_0 > 0$. Then, for any $\eta > 0$, there exists a constant $\delta > 0$ such that

$$B_\lambda \cap \left( \bigcup_{a \in M_m(\varphi, \eta)} B(a, \delta) \right) = \emptyset$$

for all sufficiently large $\lambda$.

**Corollary 2.** In addition to the same conditions as in Theorem 1.2, assume that $M(\varphi) = \{a, b\}$, $(\Delta^k \varphi)(a) = (\Delta^k \varphi)(b)$ for $k = 1, \ldots, m-1$, $(\Delta^m \varphi)(a) > (\Delta^m \varphi)(b)$. Assume that the solution $u$ of (3) satisfies (4). Then

$$\lim_{\lambda \to \infty} \sup_{x \in B_\lambda} |x - a| = 0.$$  

We sketch the outline of the proof of main theorems. The proof of Theorem 1.2 is similar to that of Theorem 1.1, and we only explain the outline of the proof of Theorem 1.1. For the proof of Theorem 1.1, we construct comparison functions to obtain the profile of the solution of (3) just before the blow-up time. We first study the profile of the solution at $t = T_\lambda - \lambda^{-2(p-1)-1}$. See Proposition 2. Compared to the argument of [4] and [6], we need more precise profile of the solution at this time. This is one of main difficulties in the proof. In order to obtain precise behavior of the solution, we divide the interval $[0, T_\lambda - \lambda^{-2(p-1)-1}]$ into small intervals. The division of time intervals plays an important role in our argument to obtain precise behavior of the solution. See Lemma 3.3. For construction of comparison functions, we follow the argument of [4] and [6]. For any $\sigma \geq 0$ and $\phi \in BC_+(\mathbb{R}^N)$, put

$$U_\sigma(x, t) := \left( (e^{t\Delta} \phi)(x) - (p-1)(1 + \sigma)t \right)^{-1/(p-1)}.$$  

Consider

$$\partial_t U = \Delta U + U^p \quad \text{in} \quad \mathbb{R}^N \times (0, T), \quad U(x, 0) = \phi(x) \quad \text{in} \quad \mathbb{R}^N. \quad (5)$$

Then one can easily check that

- if $\sigma = 0$, then $U_0$ is a subsolution of (5);
• if \( \inf_{x \in \mathbb{R}^N} \phi(x) > 0 \), then \( U_\sigma \) is a supersolution of (5) provided that

\[
p \left( \inf_{x \in \mathbb{R}^N} \phi(x) \right)^{-2p} U_\sigma(x,t)^{p-1} |\nabla(e^{t\Delta} \phi)(x)|^2 \leq \sigma \quad \text{in} \quad \mathbb{R}^N \times (0,T).
\]

Using these functions, we construct comparison functions on each small time intervals. In this way, we study the profile of the solution at \( t = T_\lambda - \lambda^{-2(p-1)-1} \) for sufficiently large \( \lambda \). Furthermore, by using the profile of the solution at \( t = T_\lambda - \lambda^{-3(p-1)-1} \), we obtain the location of the maximum points of \( u(\cdot, T_\lambda - \lambda^{-3(p-1)-1}) \). See Proposition 3. Once we get maximum points of \( u(\cdot, T_\lambda - \lambda^{-3(p-1)-1}) \), we study the profile of the solution at \( t = T_\lambda - \lambda^{-3(p-1)-2} \), and obtain the location of the maximum points of \( u(\cdot, T_\lambda - \lambda^{-3(p-1)-2}) \). See Proposition 3.

The rest of this paper is organized as follows. In Section 2 we give some preliminary results. In particular, we study short time behavior of the solution for the heat equation and an upper bound of the blow-up time \( T_\lambda \) for sufficiently large \( \lambda \). In Section 3 we study an upper estimate of the solution just before the blow-up time by constructing a family of comparison functions. In Section 4 we prove Theorems 1.1 and 1.2 by applying the results obtained in Sections 2 and 3.

2. Preliminaries. In this section we give some preliminary results. We first study the short time behavior of the solution for the heat equation. For this purpose, we first show a fundamental identity in Lemma 2.1. Using these results, we study an upper bound of the blow-up time \( T_\lambda \) for sufficiently large \( \lambda \). Moreover, we introduce one proposition on the location of the blow-up set for a semilinear heat equation with small diffusion.

We first prove the following fundamental lemma.

**Lemma 2.1.** Let \( m \geq 1 \) and \( \varphi \in C^{2m}(\Omega) \cap BC_+(\overline{\Omega}) \) be such that \( M(\varphi, \epsilon_0) \) is compact in \( \Omega \) for some \( \epsilon_0 > 0 \). Then it holds

\[
(4\pi)^{-\frac{N}{2}} \sum_{k=0}^{2m} \sum_{|\alpha|=k} t^k \frac{(\nabla^\alpha \varphi)(a)}{\alpha!} \int_{\mathbb{R}^N} e^{-\frac{|z|^2}{4}} z^\alpha \, dz = \sum_{k=0}^{m} \frac{(\Delta^k \varphi)(a)}{k!} t^k
\]

for any \( t > 0 \) and \( a \in M(\varphi) \).

**Proof.** Put

\[
I := (4\pi)^{-\frac{N}{2}} \sum_{k=0}^{2m} \sum_{|\alpha|=k} t^k \frac{(\nabla^\alpha \varphi)(a)}{\alpha!} \int_{\mathbb{R}^N} e^{-\frac{|z|^2}{4}} z^\alpha \, dz.
\]

Since

\[
\int_{\mathbb{R}^N} e^{-\frac{|z|^2}{4}} z^{2n-1} \, dz = 0
\]

for all \( n \in \mathbb{N} \) and \( i \in \{1, \ldots, N\} \), we have

\[
I = (4\pi)^{-\frac{N}{2}} \sum_{k=0}^{m} \sum_{|\beta|=k} t^k \frac{(\nabla^{2\beta} \varphi)(a)}{(2\beta)!} \int_{\mathbb{R}^N} e^{-\frac{|z|^2}{4}} z^{2\beta} \, dz.
\]

(6)
For any $n \in \mathbb{N}$ and $i \in \{1, \ldots, N\}$, by integration by parts we have
\[
(4\pi)^{-\frac{N}{2}} \int_{\mathbb{R}^N} e^{-\frac{|z|^2}{4}} z_i^{2n} \, dz = (4\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{4}} x^{2n} \, dx = 2(4\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} \left( e^{-\frac{x^2}{4}} \right)' x^{2n-1} \, dx = 2(2n-1) \int_{-\infty}^{\infty} e^{-\frac{x^2}{4}} x^{2n-2} \, dx = \cdots = 2^n (2n-1)!!.
\]
This implies that
\[
\int_{\mathbb{R}^N} e^{-\frac{|z|^2}{4}} z^{2\beta} \, dz = 2^{\beta_1} (2\beta_1 - 1)!! \cdots (2\beta_N - 1)!!, \quad \beta = (\beta_1, \ldots, \beta_N).
\]
Here we put $(2\beta_i - 1)!! := 1$ for the case where $\beta_i = 0$. Thus, by (6) we have
\[
I = \sum_{k=0}^{m} t^k \sum_{|\beta| = k} \frac{(\nabla_2^2 \varphi)(a)}{(2\beta)!} \cdot 2^{\beta_1} (2\beta_1 - 1)!! \cdots (2\beta_N - 1)!!
= \sum_{k=0}^{m} t^k \sum_{|\beta| = k} \frac{2^{\beta_1} \cdots (2\beta_N)!!}{(2\beta_1)!! \cdots (2\beta_N)!!} \frac{1}{\beta!} (\nabla_2^2 \varphi)(a) = \sum_{k=0}^{m} t^k \sum_{|\beta| = k} \frac{1}{\beta!} (\nabla_2^2 \varphi)(a)
\]
On the other hand, by the multinomial theorem we have
\[
\Delta^k \varphi = \left( \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_N^2} \right)^k \varphi = \sum_{|\beta| = k} \frac{k!}{\beta!} \nabla_2^\beta \varphi,
\]
and by (7) we obtain the desired equality. This completes the proof of Lemma 2.1.

Next, we study the short time behavior of the solution for the heat equation with the aid of Lemma 2.1.

**Lemma 2.2.** Assume the same conditions as in Lemma 2.1. Let $z$ be the solution of
\[
\begin{aligned}
\partial_t z &= \Delta z \quad \text{in} \quad \Omega \times (0, \infty), \\
z(x, t) &= 0 \quad \text{on} \quad \partial \Omega \times (0, \infty), \\
z(x, 0) &= \lambda + \varphi(x) \quad \text{in} \quad \Omega.
\end{aligned}
\]
Then
\[
z(a, t) \geq \lambda + \sum_{k=0}^{m} \frac{(\Delta^k \varphi)(a)}{k!} t^k + o(t^m)
\]
for all $a \in M(\varphi)$, $\lambda > 0$ and all sufficiently small $t > 0$.

**Proof.** Let $\mu > 0$ and fix it. Since $\varphi \in C^{2m}(\Omega) \cap BC_+ (\overline{\Omega})$ and $M(\varphi, \epsilon_0)$ is compact in $\Omega$, by the Taylor theorem we see that there exists a positive constant $\delta > 0$ such that $B(a, \delta) \subset \Omega$ for all $a \in M(\varphi)$ and
\[
\left| \varphi(x) - \sum_{k=0}^{2m} \sum_{|\alpha| = k} \frac{(\nabla_2^\alpha \varphi)(a)}{\alpha!} (x - a)^\alpha \right| \leq \frac{\mu}{2K} |x - a|^{2m}
\]
for all $x \in B(a, \delta)$ and all $a \in M(\varphi)$, where
\[
K := (4\pi)^{-\frac{N}{2}} \int_{\mathbb{R}^N} e^{-\frac{|z|^2}{4}} |z|^{2m} \, dz.
\]
Put

\[ w(x, t) := \int_{B(0, 1)} G(\delta^{-1} x, y; \delta^{-2} t)(\lambda + \varphi(a + \delta y)) \, dy, \]

where \( G(x, y; t) \) denotes the Green function for the heat equation in \( B(0, 1) \) under the zero Dirichlet boundary condition. Then \( w \) satisfies

\[ \begin{cases} 
\frac{\partial_t w}{w} = \Delta w & \text{in } B(0, \delta) \times (0, \infty), \\
w(x, t) = 0 & \text{on } \partial B(0, \delta) \times (0, \infty), \\
w(x, 0) = \lambda + \varphi(a + x) & \text{in } B(0, \delta).
\end{cases} \]

Then the comparison principle together with \( B(a, \delta) \subset \Omega \) yields

\[ z(a + x, t) \geq w(x, t) \quad \text{in } B(0, \delta) \times (0, \infty). \]

Since it follows from [9, Lemma 2.1] that there exist positive constants \( C \) and \( r \in (0, 1) \) such that

\[ G(0, y; t) \geq (1 - e^{-\frac{\delta}{r}})(4\pi t)^{-\frac{n}{2}} e^{-\frac{|y|^2}{4t}} \]

for all \( y \in B(0, r) \) and all sufficiently small \( t > 0 \), we have

\[ z(a, t) \geq w(0, t) = \int_{B(0, 1)} G(0, y; \delta^{-2} t)(\lambda + \varphi(a + \delta y)) \, dy \]

\[ \geq (1 - e^{-\frac{\delta}{r}})(4\pi \delta^{-2} t)^{-\frac{n}{2}} \int_{B(0, 1)} e^{-\frac{|z|^2}{2t \delta^2}} (\lambda + \varphi(a + \delta y)) \, dy \]

\[ \geq (4\pi)^{-\frac{n}{2}} \int_{B(0, \delta^{-1/2} t)} e^{-\frac{|z|^2}{4t}} (\lambda + \varphi(a + t^{1/2} z)) \, dz - e^{-\frac{\delta}{r}(4\pi t)^{-\frac{n}{2}}} (\lambda + \|\varphi\|_{L^\infty(\Omega)}) \]

\[ = I(t) - e^{-\frac{\delta}{r}(4\pi t)^{-\frac{n}{2}}} (\lambda + \|\varphi\|_{L^\infty(\Omega)}) \]

for all \( a \in M(\varphi) \) and all sufficiently small \( t > 0 \). Then, by Lemma 2.1 and (10) we have

\[ I(t) \]

\[ \geq (4\pi)^{-\frac{n}{2}} \int_{B(0, \delta^{-1/2} t)} e^{-\frac{|z|^2}{4t}} \left( \lambda + \sum_{k=0}^{2m} \frac{\sum_{|a|=k} \nabla^a \varphi(a)}{k!} (t \frac{1}{2} z)^a - \frac{\mu}{2K} t^m |z|^{2m} \right) \, dz \]

\[ = (4\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^N} e^{-\frac{|z|^2}{4t}} \left( \lambda + \sum_{k=0}^{2m} \frac{\sum_{|a|=k} \nabla^a \varphi(a)}{k!} (t \frac{1}{2} z)^a - \frac{\mu}{2K} t^m |z|^{2m} \right) \, dz + o(t^m) \]

\[ \geq \lambda + (4\pi)^{-\frac{n}{2}} \sum_{k=0}^{2m} \sum_{|a|=k} t^{\frac{k}{2}} \frac{\sum_{|a|=k} \nabla^a \varphi(a)}{k!} \int_{\mathbb{R}^N} e^{-\frac{|z|^2}{4t}} z^a \, dz - \frac{\mu}{2} t^m + o(t^m) \]

\[ = \lambda + \sum_{k=0}^{m} \frac{(\Delta^k \varphi(a))}{k!} t^k - \frac{\mu}{2} t^m + o(t^m) \]

for all sufficiently small \( t > 0 \). Here we remark that

\[ \left| \int_{\mathbb{R}^N} e^{-\frac{|z|^2}{4t}} z^a \, dz - \int_{B(0, \delta^{-1/2} t)} e^{-\frac{|z|^2}{4t}} z^a \, dz \right| \leq e^{-\frac{\delta^2}{4t} - 1} \int_{\mathbb{R}^N} e^{-\frac{|z|^2}{4t}} |z|^a \, dz = o(t^m) \]
for all $\alpha \in (\mathbb{N} \cup \{0\})^N$ with $|\alpha| \leq 2m$ and all sufficiently small $t > 0$. Thus, by (12) we obtain

$$z(a, t) \geq \lambda + \sum_{k=0}^{m} \frac{(\Delta^k \varphi)(a)}{k!} t^k - \mu t^m$$

for all sufficiently small $t > 0$. This together with arbitrariness of $\mu > 0$ implies (9), and the proof of Lemma 2.2 is complete.

Repeating the argument as in [6, Lemma 4.3] and [4, Lemma 2.3], we can obtain an upper bound of the blow-up time by virtue of Lemma 2.2.

**Lemma 2.3.** Let $N \geq 1$ and $\Omega$ be a domain in $\mathbb{R}^N$. Let $p > 1$ and $m \in \mathbb{N}$ satisfy $m \geq 2$. For the case $m \geq 3$, assume further that $1 < p < 1 + 1/(m - 2)$. Let $\varphi \in C^2(\Omega) \cap BC_+(\Omega)$ be such that $M(\varphi, \epsilon_0)$ is compact in $\Omega$ for some $\epsilon_0 > 0$. For $\lambda > 0$, let $T_\lambda$ be the blow-up time of the solution for problem (3). Assume that, for any $k \in \{1, \ldots, m - 1\}$, it holds

$$(\Delta^k \varphi)(a)$$

takes the same value for all $a \in M(\varphi)$.

Then

$$T_\lambda \leq \frac{\lambda^M_m(p-1)}{p-1} + \frac{d}{p-1} \lambda^{-2(p-1)-1} - \sum_{k=2}^{m} \frac{d_k}{k!(p-1)^k} \lambda^{-(k+1)(p-1)-1} + o(\lambda^{-(m+1)(p-1)-1})$$

for all sufficiently large $\lambda$, where

$$M := \|\varphi\|_{L^\infty(\Omega)}, \quad \lambda_M := \lambda + M, \quad d := \inf_{\alpha \in M(\varphi)} ||(\Delta^k \varphi)(a)||, \quad d_k := \sup_{a \in M(\varphi)} (\Delta^k \varphi)(a).$$

**Remark 1.** The assumption $1 < p < 1 + 1/(m - 2)$ for the case $m \geq 3$ is weaker than the condition $1 < p < p(m)$, where $p(m)$ appears in the statement of Theorem 1.2 and will be given in the beginning of Section 3. In fact, if $1 < p < p(m)$ and $m \geq 3$, that is, $p$ satisfies (22), then $1 < p < 1 + 1/(m - 2)$ holds. Therefore the conclusion of Lemma 2.3 holds under the assumption of Theorem 1.2.

**Proof.** Put $\tilde{s}_\lambda := (\lambda_M/2)^{(p-1)/(p-1)}$. Then we see that $\tilde{s}_\lambda$ is the blow-up time of the solution for the ordinary differential equation

$$\zeta' = \zeta^p \quad (t > 0) \quad \text{with} \quad \zeta(0) = \frac{\lambda_M}{2}.$$ 

Let $z$ be the solution of (8). Since $\varphi(a) = \|\varphi\|_{L^\infty(\Omega)} = M$ for $a \in M(\varphi)$, by Lemma 2.2 we have

$$\|z(t)\|_{L^\infty(\Omega)} \geq z(a, t) \geq \lambda_M + \sum_{k=1}^{m} \frac{(\Delta^k \varphi)(a)}{k!} t^k + o(t^m)$$

for all $a \in M(\varphi)$ and all sufficiently small $t > 0$. Since $(\Delta^k \varphi)(a)$ takes the same value for all $a \in M(\varphi)$ and $k = 1, \ldots, m - 1$, by the arbitrariness of $a \in M(\varphi)$ and the definition of $d$ and $d_k$, we obtain

$$\|z(t)\|_{L^\infty(\Omega)} \geq \lambda_M - d \cdot t + \sum_{k=2}^{m} \frac{d_k}{k!} t^k + o(t^m) \quad (13)$$

for all $\lambda > 0$ and all sufficiently small $t > 0$. Note that $\Delta \varphi(a) \leq 0$ for all $a \in M(\varphi)$. Since $\tilde{s}_\lambda \to 0$ as $\lambda \to \infty$, by (13) we have $\|z(\tilde{s}_\lambda)\|_{L^\infty(\Omega)} > \lambda_M/2$, which yields

$$\|z(\tilde{s}_\lambda)\|_{L^\infty(\Omega)}^{(p-1)} < \left(\frac{\lambda_M}{2}\right)^{(p-1)} = (p-1)\tilde{s}_\lambda$$
for all sufficiently large $\lambda$. Therefore there exists a constant $T'_\lambda \in (0, \delta_\lambda)$ such that

$$
\|z(T'_\lambda)\|_{L^\infty(\Omega)}^{-(p-1)} - (p-1)T'_\lambda = 0. 
$$

(14)

By the comparison principle we have $\|z(T'_\lambda)\|_{L^\infty(\Omega)} \leq \|z(0)\|_{L^\infty(\Omega)} = \lambda_M$, so by (14) we obtain

$$
T'_\lambda \geq \frac{1}{p-1} \lambda_M^{-(p-1)}.
$$

(15)

Since the function $u$, defined by

$$
g(x, t) = \left( z(x, t)^{-(p-1)} - (p-1)t \right)^{-\frac{1}{p-1}},$$

is a sub-solution of (1), we have $u(x, t) \geq g(x, t)$ in $\Omega \times (0, T'_\lambda)$. In particular, by (14) we see that $T'_\lambda$ is the blow-up time of $\tilde{g}$ and obtain $T'_\lambda \leq T'_\lambda$ for all sufficiently large $\lambda$. Then, since $T'_\lambda < \delta_\lambda = O(\lambda_M^{-(p-1)})$ and

$$
(1 + s)^{-(p-1)} = 1 - (p-1)s + O(s^2)
$$

(16)

for all sufficiently small $s > 0$, by (13) and (14) we have

$$
T'_\lambda = \frac{1}{p-1} \|z(T'_\lambda)\|_{L^\infty(\Omega)}^{-(p-1)}
$$

$$
\leq \frac{1}{p-1} \left( \lambda_M + O(\lambda_M^{-(p-1)}) \right)^{-(p-1)}
$$

$$
= \frac{1}{p-1} \lambda_M^{-(p-1)} \left( 1 + O(\lambda_M^{-(p-1)}) \right)^{-(p-1)} = \frac{1}{p-1} \lambda_M^{-(p-1)} + O(\lambda_M^{-(2(p-1)-1)})
$$

for all sufficiently large $\lambda$. In particular, by (15) we have

$$
T'_\lambda = \frac{1}{p-1} \lambda_M^{-(p-1)} + O(\lambda_M^{-(2(p-1)-1)})
$$

for all sufficiently large $\lambda$. This together with (14) and (16) implies that

$$
T'_\lambda = \frac{1}{p-1} \|z(T'_\lambda)\|_{L^\infty(\Omega)}^{-(p-1)} \leq \frac{1}{p-1} \left( \lambda_M - dT'_\lambda + \sum_{k=2}^{m} \frac{d_k}{k!}(T'_\lambda)^k + o((T'_\lambda)^m) \right)^{-(p-1)}
$$

$$
= \frac{\lambda_M^{-(p-1)}}{p-1} \left( 1 - \frac{d}{p-1} \lambda_M^{-(p-1)-1} + O(\lambda_M^{-(2(p-1)-2)}) + \sum_{k=2}^{m} \frac{d_k}{k!} \lambda_M^{-(p-1)} + O(\lambda_M^{-(2(p-1)-1)}) + o(\lambda_M^{-(m(p-1)-1)}) \right)
$$

$$
= \frac{\lambda_M^{-(p-1)}}{p-1} \left( 1 + d\lambda_M^{-(p-1)-1} \right)^{-(p-1)}
$$

$$
= \frac{\lambda_M^{-(p-1)}}{p-1} + \frac{d\lambda_M^{2(p-1)-1}}{p-1} - \sum_{k=2}^{m} \frac{d_k}{k!(p-1)^k} \lambda_M^{-(k+1)(p-1)-1} + O(\lambda_M^{-(m+1)(p-1)-1})
$$

for all sufficiently large $\lambda$. Note that $m(p-1) + 1 < 2(p-1) + 2$ by the assumption $1 < p < 1 + 1/(m - 2)$, so $\lambda_M^{-(2(p-1)-1)} = o(\lambda_M^{-(p-1)-1})$ for sufficiently large $\lambda$. 

Then, by $T_\lambda \leq T'_A$ we obtain the desired estimate of $T_\lambda$, and conclude the proof of Lemma 2.3.

In order to study the location of the blow-up set for a semilinear heat equation, we apply the following proposition. For the proof of Proposition 1, see [3, Proposition 3] and [5, Proposition 4.1]. See also [3, Remark 2 (ii)].

**Proposition 1.** Let $p > 1$, $N \geq 1$, $\Omega$ be a domain in $\mathbb{R}^N$, $\epsilon_0 > 0$, $\{ \varphi_\epsilon \}_{0 < \epsilon \leq \epsilon_0}$ be a family of nonnegative functions in $\Omega$ and $u_\epsilon$ be the solution of

$$
\begin{align*}
\partial_t u &= \epsilon \Delta u + u^p, \quad x \in \Omega, \quad t > 0, \\
u(x, t) &= 0, \quad x \in \partial \Omega, \quad t > 0, \\
u(x, 0) &= \varphi_\epsilon(x), \quad x \in \Omega.
\end{align*}
$$

Let $T_\epsilon$ and $B_\epsilon$ be the blow-up time and the blow-up set of the solution $u_\epsilon$, respectively. Assume

$$
\sup_{0 < \epsilon \leq \epsilon_0} \sup_{0 < t < T_\epsilon} (T_\epsilon - t)^{1/(p-1)} \| u_\epsilon(t) \|_{L^\infty(\Omega)} < \infty.
$$

Furthermore, assume that there exists a family $\{ \tilde{\varphi}_\epsilon \}_{0 < \epsilon \leq \epsilon_0} \subset C^1(\Omega) \cap C(\overline{\Omega})$ such that

$$
\begin{align*}
0 &\leq \varphi_\epsilon(x) \leq \tilde{\varphi}_\epsilon(x), \quad x \in \Omega, \quad \epsilon \in (0, \epsilon_0), \\
c_\tilde{\varphi} &:= \liminf_{\epsilon \to 0} \| \tilde{\varphi}_\epsilon \|_{L^\infty(\Omega)} > 0, \\
C_{\tilde{\varphi}} &:= \limsup_{\epsilon \to 0} \| \tilde{\varphi}_\epsilon \|_{L^\infty(\Omega)} < \infty, \\
M(\tilde{\varphi}, \eta_0) \cap \partial \Omega &= \emptyset, \quad \epsilon \in (0, \epsilon_0), \quad \text{for some } \eta_0 > 0.
\end{align*}
$$

Then, for any $\eta_*>0$, if

$$
\sup_{0 < \epsilon \leq \epsilon_0} \left| T_\epsilon - \frac{\| \tilde{\varphi}_\epsilon \|_{L^\infty(\Omega)}}{p - 1} \right| \leq \frac{\eta_*}{10 C_{\tilde{\varphi}}},
$$

it holds $B_\epsilon \subset M(\tilde{\varphi}, \eta_*)$ for all sufficiently small $\epsilon > 0$.

### 3. Upper bound of the solution just before the blow-up time.

In this section we construct suitable supersolutions for problem (1), and study an upper bound of the solution just before the blow-up time. This section plays an important role in the proof of main theorems.

Hereafter, we denote $\| \cdot \|_{L^\infty(\mathbb{R}^N)}$ by $\| \cdot \|_\infty$ for simplicity. Assume the same conditions as in Theorem 1.1 and we use the same notation as in Lemma 2.3. Then we have $d = |\Delta \varphi(\alpha)|$ for any $\alpha \in M(\varphi)$ under the assumptions of Theorem 1.1. For the case $m \geq 3$, we can take a constant $p(m) > 1$ such that, for all $p \in (1, p(m))$, there holds

$$
1 < p < 1 + \frac{\epsilon_*}{m - 2}, \quad \epsilon_* := \frac{1}{2} \left( \frac{[2(p-1) + 1]}{2} - (p - 1) \right) \in [0, 1/4),
$$

where we may assume that $p(2) = \infty$ without loss of generality and $p$ can be taken arbitrarily for the case $m = 2$. Let $\delta$ be a constant satisfying

$$
0 < \delta < \min \left\{ p - 1, \frac{1}{2}, \frac{[2(p-1) + 1]}{4} \epsilon_* \right\}.
$$
Let \( \varphi(x) := \begin{cases} \max \{ \varphi(x), M - \lambda_M^{-(p-1)+\delta} \} & \text{if } x \in \Omega, \\ M - \lambda_M^{-(p-1)+\delta} & \text{if } x \in \mathbb{R}^N \setminus \Omega. \end{cases} \) Then we define the function \( \psi(x) := M - \varphi(x) \) in \( \mathbb{R}^N \).

Since \( \varphi \in C^4(\Omega) \cap BC_+(\overline{\Omega}) \) and \( M(\varphi, \epsilon_0) \) is a compact set in \( \Omega \) for some \( \epsilon_0 > 0 \), we have \( \sup \{ |\nabla^2 \varphi(x)| : x \in M(\varphi, \epsilon_0) \} < \infty \), so we can apply [6, Lemma 4.1] with \( \epsilon = \lambda_M^{-(p-1)+\delta} \) to obtain

\[
\|\nabla \varphi\|_{\infty} \leq \sup \{ |\nabla \varphi(x)| : x \in M(\varphi, \lambda_M^{-(p-1)+\delta}) \} \lesssim \lambda_M^{-(p-1)/2+\delta/2}
\]

for all sufficiently large \( \lambda \). This together with (24) and (25) implies that

\[
\|\psi\|_{\infty} \lesssim \lambda_M^{-(p-1)+\delta}, \quad \|\nabla \psi\|_{\infty} = \|\nabla \varphi\|_{\infty} \lesssim \lambda_M^{-(p-1)/2+\delta/2}
\]

for all sufficiently large \( \lambda \). We first prepare one lemma on the short time behavior of the solution for a heat equation.

**Lemma 3.1.** Let \( \psi \) be the function defined by (25). Let \( m \in \mathbb{N} \) with \( m \geq 2 \). Assume that, for any \( k \in \{1, \ldots, m-1\} \), it holds

\[
(\Delta^k \varphi)(a) \text{ is independent of } a \in M(\varphi).
\]

Then there hold the following:

(i) Let \( C > 0 \). Then there exist constants \( \kappa_* \) and \( \gamma \) such that, if \( \kappa \geq \kappa_* \), then

\[
\sup_{x \in \Omega}(e^{t\Delta} \varphi)(x) \leq M - \gamma \kappa \lambda_M^{-(p-1)}
\]

for all \( t \in [0, C \lambda^{-1}(p-1)] \) and all sufficiently large \( \lambda \), where

\[
\Omega_\lambda := \left\{ x \in \overline{\Omega} : \varphi(x) < M - \kappa \lambda_M^{-(p-1)} \right\}.
\]

(ii) Let \( C > 0 \). Then

\[
\sup_{x \in \mathbb{R}^N}(e^{t\Delta} \varphi)(x) \leq M - dt + \sum_{k=2}^m \frac{t^k}{k!} d_k + o(\lambda_M^{-m(p-1)})
\]

for all \( t \in [0, C \lambda^{-1}(p-1)] \) and all sufficiently large \( \lambda \).

(iii) For any \( \eta > 0 \), there exists a constant \( \delta(\eta) > 0 \) such that

\[
\sup_{\alpha \in M(\varphi, \eta)}(\Delta^m \varphi)(x) \leq \sup_{\alpha \in M(\varphi)}(\Delta^m \varphi)(x) - \frac{\eta}{2}.
\]

(iv) For any \( \eta > 0 \), let \( \delta(\eta) > 0 \) be the constant given in assertion (ii). Let \( C > 0 \). Then

\[
\sup_{\alpha \in M(\varphi, \eta)}(e^{t\Delta} \varphi)(x) \leq M - dt + \sum_{k=2}^m \frac{t^k}{k!} d_k - \frac{\eta t^m}{2m!} + o(\lambda_M^{-m(p-1)})
\]

for all \( t \in [0, C \lambda^{-1}(p-1)] \) and all sufficiently large \( \lambda \).
Proof. We apply the argument as in [6, Lemma 4.2]. Assertion (i) is proved by the same argument as in [6, Lemma 4.2]. Furthermore, assertion (iii) can be easily derived from $C^{2m}$ regularity of $\varphi$. Put
\[ U_\lambda := \left\{ x \in \Omega : \varphi(x) \geq M - \lambda_M^{-(p-1)\delta} \right\}, \]
\[ V_\lambda := \left\{ x \in \mathbb{R}^N : B(x, \lambda_M^{-(p-1)/2}\delta/2) \subset U_\lambda \right\}. \]
Let $C > 0$. Since $\varphi_\lambda = \varphi$ in $U_\lambda$ and $V_\lambda \subset \Omega$, for any $\mu > 0$, by the Taylor theorem we obtain
\[
\left| \varphi_\lambda(y + x) - \sum_{k=0}^{2m} \sum_{|\alpha| = k} \frac{(\nabla^\alpha \varphi)(x)}{\alpha!} y^\alpha \right| \leq \frac{\mu}{2C^{m\delta}K} |y|^{2m}, \quad |y| \leq \lambda_M^{-(p-1)/2+\delta/2},
\]
for all $x \in V_\lambda$ and all sufficiently large $\lambda$, where $K$ is the constant given by (11). This implies that
\[
\begin{align*}
(e^t \Delta \varphi_\lambda)(x) &= (4\pi t)^{-\frac{N}{2}} \int_{\mathbb{R}^N} e^{-\frac{|x-y|^2}{4t}} \varphi_\lambda(y) \, dy \\
&= (4\pi t)^{-\frac{N}{2}} \int_{B(0, \lambda_M^{-(p-1)/2})} e^{-\frac{|y|^2}{4t}} \varphi_\lambda(x + y) + O \left( e^{-\frac{\lambda_M^{-(p-1)+\delta}}{8t}} \right) \\
&= (4\pi t)^{-\frac{N}{2}} \int_{B(0, \lambda_M^{-(p-1)/2})} e^{-\frac{|y|^2}{4t}} \varphi_\lambda(x) + O \left( e^{-\frac{\lambda_M^\delta}{8t}} \right) \\
&\leq (4\pi t)^{-\frac{N}{2}} \int_{B(0, \lambda_M^{-(p-1)/2})} e^{-\frac{|y|^2}{4t}} \\
&\quad \times \left[ \sum_{k=0}^{2m} \sum_{|\alpha| = k} \frac{(\nabla^\alpha \varphi)(x)}{\alpha!} y^\alpha + \frac{\mu|y|^{2m}}{2C^{m\delta}K} \right] \, dy + O \left( e^{-\frac{\lambda_M^\delta}{8t}} \right) \\
&= (4\pi t)^{-\frac{N}{2}} \int_{\mathbb{R}^N} e^{-\frac{|y|^2}{4t}} \sum_{k=0}^{2m} \sum_{|\alpha| = k} \frac{(\nabla^\alpha \varphi)(x)}{\alpha!} y^\alpha + \frac{\mu|y|^{2m}}{2C^{m\delta}K} \, dy + O \left( e^{-\frac{\lambda_M^\delta}{8t}} \right) \\
&= \sum_{k=0}^{m} \frac{(\Delta^k \varphi)(x)}{k!} t^k + \frac{\mu}{2C^{m\delta}} + O \left( e^{-\frac{\lambda_M^\delta}{8t}} \right) \leq \sum_{k=0}^{m} \frac{(\Delta^k \varphi)(x)}{k!} t^k + \mu \lambda_M^{-m(p-1)}
\end{align*}
\]
for all $(x, t) \in V_\lambda \times (0, C \lambda_M^{-(p-1)})$ and all sufficiently large $\lambda$. See also the proof of Lemma 2.2. Once we get (28), since $\mu > 0$ is arbitrary, by similar argument as in the proof of [6, Lemma 4.2] we can obtain assertions (ii) and (iv). Therefore we complete the proof of Lemma 3.1. \(\square\)

In order to construct supersolutions for problem (1), we consider the problem
\[
\begin{cases}
\partial_t U = \Delta U + U^p, & x \in \mathbb{R}^N, \ t > 0, \\
U(x, 0) = \lambda_M - \psi_\lambda(x), & x \in \mathbb{R}^N.
\end{cases}
\]
Then, by (24) and (25) we have $U(x, 0) \geq \lambda + \varphi(x)$ in $\Omega$, and see that the solution $U$ of (1) is a supersolution of (1). For any $\sigma \geq 0$, we define the function
\[
U(x, t) := \left( \left( e^{t \Delta U(0)}(x) \right)^{(p-1) - (p-1)(1+\sigma)t} e^{-\frac{1}{t}} \right).\]
We divide the argument into several steps. In the first step, we study the profile of $C$ constant. Assume the same conditions as in Theorem 3.2. It is easy to check that, if $U$ satisfies
\[
p\left(\inf_{x \in \mathbb{R}^N} U(x, 0)\right)^{-2p} U(x, t)^{p-1} |\nabla (e^{t\Delta} U(0))(x)|^2 \leq \sigma \quad \text{in} \quad (x, t) \in \mathbb{R}^N \times (0, T)
\] (30)
for some $T > 0$, then $U$ is a supersolution of (29) in $\mathbb{R}^N \times (0, T)$. Using above functions, we study the profile of $U(\cdot, t)$ at $t = S_\lambda$ and obtain an upper estimate of $u$, where
\[
S_\lambda := \frac{1}{p-1} \lambda^{-2(p-1)-1} + \frac{1}{p-1} d \lambda^{-2(p-1)-1} - \frac{\lambda^{2(p-1)-1}}{p-1}.
\] (31)
We divide the argument into several steps. In the first step, we study the profile of the solution $u$ at $t = s^{(1)}_\lambda$, where
\[
s^{(1)}_\lambda := \frac{1}{p-1} \lambda^{-2(p-1)-1} + \frac{1}{p-1} d \lambda^{-2(p-1)-1} - \frac{\lambda^{2(p-1)-1}}{p-1}.
\]
For simplicity, we put $\alpha := p-1 > 0$.

**Lemma 3.2.** Assume the same conditions as in Theorem 1.1. Then there exists a constant $C_1 > 0$ such that the function $\varphi_1 \in C^1(\mathbb{R}^N)$, defined by
\[
\varphi_1(x) = \left(\lambda^{-\frac{\alpha}{2}} - d \lambda^{-2\alpha+1} + (p-1)\lambda^{-\alpha-1} z_1(x, s^{(1)}_\lambda) - C_1 \lambda^{-3\alpha+\frac{3}{2}+\delta}\right) \leq \frac{1}{\lambda},
\]
satisfies
\[
U(x, s^{(1)}_\lambda) \leq \varphi_1(x), \quad |\nabla \varphi_1(x)| = O(\lambda^{\frac{\alpha}{2}+\frac{3}{2}+\delta}),
\]
for all $x \in \mathbb{R}^N$ and all sufficiently large $\lambda$.

**Proof.** Put $z_1(x, t) := (e^{t\Delta} \psi_\lambda(x))$ and $E_1 := \mathbb{R}^N \times [0, s^{(1)}_\lambda]$. Let $\sigma_1 := A_1 \lambda^{-2\alpha-3/2+\delta}$, where $A_1$ is a constant to be chosen later and put
\[
U_1(x, t) := (|e^{t\Delta} U(0)(x)|)^{-\alpha} \leq (1 + \sigma_1) t^{-\frac{1}{\alpha}} \quad \text{in} \quad E_1.
\]
By (26) and (29) we have
\[
m_1 := \inf_{x \in \mathbb{R}^N} U(x, 0) \geq \frac{\lambda M}{2}
\] (32)
for all sufficiently large $\lambda$. Furthermore, by (26) we have
\[
(\lambda M)^{-\alpha} |(e^{t\Delta} U(0)(x)|^{-\alpha} = \lambda^{-\alpha} + \alpha \lambda^{-\alpha} z_1(x, t) + O(\lambda^{-3\alpha+2\delta-2})
\]
for all sufficiently large $\lambda$. This together with (23) implies that
\[
[(e^{t\Delta} U(0))(x)|^{-\alpha} - \alpha t \geq \lambda^{-\alpha} - \alpha t + \alpha \lambda^{-\alpha} z_1(x, t) + O(\lambda^{-3\alpha+2\delta-2})
\]
for all $(x, t) \in E_1$ and all sufficiently large $\lambda$. Thus we obtain
\[
\sup_{(x, t) \in E_1} U_1(x, t) \lesssim \lambda^{-\frac{3}{2}}
\] (33)
for all sufficiently large \( \lambda \). Then, in view of (26) and (32), there exists a constant \( c_1 > 0 \) such that

\[
(\alpha + 1) m_1^{-2a-2} U_1(x, t)\alpha |\nabla (e^{t\Delta U(0)}(x))|^2 \leq c_1 \lambda_1^{-\alpha - \frac{1}{2} + \delta} \tag{34}\]

for all sufficiently large \( \lambda \). Note that \( p = \alpha + 1 \). Putting \( A_1 = c_1 \), by (30) and (34) we see that the function \( U_1 \) is a supersolution of (29), and we can take a constant \( C_1 > 0 \) such that

\[
U(x, s_\lambda^{(1)}) \leq \left( \lambda_1^{-\alpha - \frac{1}{2}} - d\lambda_1^{-2a-1} + \alpha \lambda_1^{-a-1} z_1(x, s_\lambda^{(1)}) + O(\lambda_1^{-3a-3} + \delta) \right)^{-\frac{1}{\alpha}} - \lambda_1^{-\alpha} \lambda_1^{-1} = \varphi_1(x) \tag{27}\]

for all \( x \in \mathbb{R}^N \) and all sufficiently large \( \lambda \). Then, repeating the argument as (33), we have \( \|\varphi_1\|_{\alpha} \lesssim \lambda_1^{\alpha + \frac{1}{2}} \) for all sufficiently large \( \lambda \), and by the definition of \( \varphi_1 \) and (26) we obtain

\[
|\nabla \varphi_1(x)| = \varphi_1(x)^{\alpha} \cdot \lambda_1^{-\alpha} |\nabla z_1(x, s_\lambda^{(1)})| \lesssim \lambda_1^{\alpha} \cdot \|\nabla \varphi_1\|_{\alpha} \lesssim \lambda_1^{\alpha + \frac{1}{2} + \frac{1}{\alpha}} \]

for all \( x \in \mathbb{R}^N \) and all sufficiently large \( \lambda \). Therefore we obtain the desired inequalities, and conclude the proof of Lemma 3.2.

In the second step, we construct a family of supersolutions of (29) in order to study an upper bound of the solution at \( t = \lambda_1^{-\alpha}/\alpha + d\lambda_1^{-2a-1}/\alpha - \lambda_1^{-\alpha - [2a+1]/\alpha} \). For \( i \in \mathbb{N} \) with \( i \geq 2 \), put

\[
s_\lambda^{(i)} := \frac{\lambda_1^{-\alpha - 1/2}}{\alpha} = \frac{\lambda_1^{-\alpha}}{\alpha}, \quad E_i := \mathbb{R}^N \times [0, s_\lambda^{(i)}], \tag{35}\]

and let \( U_i \) be the solution of

\[
\begin{cases}
\partial_t U_i = \Delta U_i + U_i^{\alpha+1}, & x \in \mathbb{R}^N, \ t > 0, \\
U_i(x, 0) = \varphi_{i-1}(x), & x \in \mathbb{R}^N,
\end{cases} \tag{36}\]

where \( \varphi_i \) for \( i \geq 2 \) will be constructed inductively in Lemma 3.3 and \( \varphi_1 \) is the function given in Lemma 3.2. We put \( U_1 := U \), which is the solution of (29), for convenience. Furthermore, we define the function \( z_i \) by \( z_i(x, t) := (e^{t\Delta z_1}(x, s_\lambda^{(i-1)})) \) inductively, where \( z_1 \) is given in the proof of Lemma 3.2. Then we prove the following lemma.

**Lemma 3.3.** Assume the same conditions as in Theorem 1.1. Let \( i \in \{1, \ldots, [2a+1]\} \). Then there exists a constant \( C_i > 0 \) such that the function \( \varphi_i \in C^1(\mathbb{R}^N) \), defined by

\[
\varphi_i(x) = \left( \lambda_1^{-\alpha - 1/2} - d\lambda_1^{-2a-1} + \alpha \lambda_1^{-a-1} z_i(x, s_\lambda^{(i)}) - C_i \lambda_1^{-3a-3} + 2\alpha \right)^{-\frac{1}{\alpha}}, \tag{37}\]

satisfies

\[
U_i(x, s_\lambda^{(i)}) \leq \varphi_i(x), \tag{38}\]

\[
|\nabla \varphi_i(x)| = O(\lambda_1^{(\alpha+1)/2a - \alpha - \frac{1}{2}}), \tag{39}\]

for all \( x \in \mathbb{R}^N \) and all sufficiently large \( \lambda \).
Proof. The proof is by induction. Let $i \geq 2$ and assume that $\varphi_{i-1}$ is of the form (37) and satisfies (39). In view of Lemma 3.2, the function $\varphi_1$ given in Lemma 3.2 satisfies (38) and (39) with $i = 1$. Put

$$J(x, t) := \left[1 - d\lambda^{-\alpha - 1 + \frac{i-1}{2}} + \alpha \lambda^{-1 + \frac{i-1}{2}} z_i(x, t)\right]^{-\frac{1}{2}}$$

in $\mathbb{R}^N \times [0, \infty)$. Then we have

$$\partial_t J - \Delta J = -(\alpha + 1)\lambda^{i-3} J^{2\alpha + 1} |\nabla z_i|^2 \quad \text{in} \quad \mathbb{R}^N \times (0, \infty)$$

and obtain

$$\|J(t) - e^{t\Delta} J(0)\|_\infty \leq (\alpha + 1)\lambda^{i-3} \int_0^t \|e^{(t-s)\Delta} J(s)^{2\alpha + 1} |\nabla z_i(s)|^2\|_\infty ds. \quad (40)$$

By (26) and the definitions of $z_i$ and $J$ we have

$$\|J(t)\|_\infty = O(1), \quad \|\nabla z_i(t)\|_\infty^2 = O(\lambda^{-\alpha + \delta}), \quad (41)$$

for all sufficiently large $\lambda$. This together with (35) and (40) yields

$$\sup_{0 \leq t \leq \delta_0(1)} \|J(t) - e^{t\Delta} J(0)\|_\infty \lesssim \lambda^{i-3} \cdot \lambda^{-\alpha + \delta} \cdot \delta_0(1) \lesssim \lambda^{-2\alpha + \frac{\delta}{2} - \frac{1}{2} + \delta} \quad (42)$$

for all sufficiently large $\lambda$. Then, by the definition of $z_i$ and (37) we have

$$\varphi_{i-1}(x) = \left(\lambda^{-\alpha - 1 + \frac{i-1}{2}} - d\lambda^{-2\alpha - 1} + \alpha \lambda^{-\alpha - 1} z_i(x, 0) - C_i \lambda^{-3a - \frac{1}{2} + 2\delta}\right)^{-\frac{1}{2}}$$

$$= \lambda^{1 + \frac{i-1}{2\alpha}} (J(x, 0)^{-\alpha} + O(\lambda^{-2\alpha + \frac{1}{2} - 2 + 2\delta}))^{-\frac{1}{2}} \quad (43)$$

$$= \lambda^{1 + \frac{i-1}{2\alpha}} (J(x, 0) + O(\lambda^{-2\alpha + \frac{1}{2} - 2 + 2\delta})) \gtrsim \lambda^{1 + \frac{i-1}{2\alpha}}$$

for all $x \in \mathbb{R}^N$ and all sufficiently large $\lambda$, and by (42) we obtain

$$(e^{t\Delta} \varphi_{i-1})(x) = \lambda^{1 + \frac{i-1}{2\alpha}} (J(x, t) + O(\lambda^{-2\alpha + \frac{1}{2} - 2 + 2\delta})) \quad (44)$$

for all $(x, t) \in \mathbb{R}^N \times [0, \infty)$ and all sufficiently large $\lambda$.

Let $A_i$ be a constant to be chosen later and put $\sigma_i := A_i \lambda^{-2\alpha + i/2 - 2 + \delta}$. We define the function $\mathcal{U}_i$ and the constant $m_i$ by

$$\mathcal{U}_i(x, t) := [(e^{t\Delta} \varphi_{i-1})(x)]^{-\alpha} - \alpha (1 + \sigma_i) t^{-1/\alpha}, \quad m_i := \inf_{x \in \mathbb{R}^N} \varphi_{i-1}(x). \quad (45)$$

Since $J(x, t) = 1 + o(1)$ for all $(x, t) \in \mathbb{R}^N \times (0, \infty)$ and all sufficiently large $\lambda$, by (44) we have

$$[(e^{t\Delta} \varphi_{i-1})(x)]^{-\alpha} = \lambda^{-\alpha - 1 + \frac{i-1}{2}} J(x, t)^{-\alpha} \left(1 + O(\lambda^{-2\alpha + \frac{1}{2} - 2 + 2\delta})\right)$$

$$= \lambda^{-\alpha - 1 + \frac{i-1}{2}} \left[1 - d\lambda^{-\alpha - 1 + \frac{i-1}{2}} + \alpha \lambda^{-1 + \frac{i-1}{2}} z_i(x, t) + O(\lambda^{-2\alpha + \frac{1}{2} - 2 + 2\delta})\right]$$
for all \((x, t) \in \mathbb{R}^N \times [0, \infty)\) and all sufficiently large \(\lambda\). This together with \((35)\) implies that
\[
[(e^{t\Delta} \varphi_{i-1})(x)]^{-\alpha} - \alpha(1 + \sigma_i) t \\
= \lambda_M^{-\alpha - \frac{1}{2} - \frac{1}{\alpha}} - \alpha t - d\lambda_M^{-2\alpha - 1} + \alpha \lambda_M^{-\alpha - 1} z_i(x, t) + O(\lambda_M^{-3\alpha - \frac{3}{2} + 2\delta}) + O(\sigma_i s_{\lambda}^{(i)}) \\
\geq \lambda_M^{-\alpha - \frac{2}{\alpha} - 2\alpha - 1} + \alpha \lambda_M^{-\alpha - 1} z_i(x, t) + O(\lambda_M^{-3\alpha - \frac{3}{2} + 2\delta}) \\
= \lambda_M^{-\alpha - \frac{2}{\alpha} - 1} (1 + o(1))
\]
for all \((x, t) \in E_i\) and all sufficiently large \(\lambda\). Here we have used the fact \(i \leq [2\alpha + 1]\).

Thus we have
\[
\sup_{(x, t) \in E_i} \mathcal{U}_i(x, t)^{\alpha} \lesssim \lambda_M^{\alpha + \frac{2}{\alpha}}
\]
for all sufficiently large \(\lambda\), and by \((39)\) and \((43)\) we can find a constant \(c_i > 0\) such that
\[
(\alpha + 1)m_i^{-2\alpha - 2} \mathcal{U}_i(x, t)^{\alpha} |\nabla (e^{t\Delta} \varphi_{i-1})(x)|^2 \leq c_i \lambda_M^{-2\alpha - 2 + \frac{2}{\alpha} + \delta}
\]
for all \((x, t) \in E_i\) and all sufficiently large \(\lambda\). Putting \(A_i = c_i\), by \((30)\) and \((47)\) we see that the function \(\mathcal{U}_i\) is a supersolution of \((36)\), and by \((46)\) we have
\[
\mathcal{U}_i(x, s_{\lambda}^{(i)}) \leq \left( \lambda_M^{-\alpha - \frac{2}{\alpha} - 2\alpha - 1} + \alpha \lambda_M^{-\alpha - 1} z_i(x, s_{\lambda}^{(i)}) + O(\lambda_M^{-3\alpha - \frac{3}{2} + 2\delta}) \right)^{-\frac{1}{\alpha}}
\]
for all sufficiently large \(\lambda\). Therefore we can find a constant \(C_i > 0\) such that the function \(\varphi_i\) defined by \((37)\) satisfies \((38)\). We can easily derive \((39)\) for \(\varphi_i\) from \((37)\) by similar calculation as in the proof of Lemma 3.2. Thus we complete the proof of Lemma 3.3.

Finally, using supersolutions constructed above, we study an upper bound of the solution of \((1)\) at the time \(t = S_{\lambda}\). To this end, we consider the problem
\[
\begin{cases}
\partial_t V = \Delta V + V^{\alpha + 1}, & x \in \mathbb{R}^N, \ t > 0, \\
V(x, 0) = \varphi_{2\alpha + 1}(x), & x \in \mathbb{R}^N,
\end{cases}
\]
and prove the following lemma. Put
\[
s_{\lambda} := \frac{\lambda_M^{-\alpha - \frac{2\alpha + 1}{\alpha}}}{\alpha} - \frac{\lambda_M^{-2\alpha - 1}}{\alpha}.
\]

**Lemma 3.4.** Assume the same conditions as in Theorem 1.1. Then there exists a constant \(C > 0\) such that the function \(\Phi \in C^1(\mathbb{R}^N)\) defined by
\[
\Phi(x) = \left( \lambda_M^{-2\alpha - 1} - d\lambda_M^{-2\alpha - 1} + \alpha \lambda_M^{-\alpha - 1} (e^{s_{\lambda} \Delta z_{2\alpha + 1}})(x) - C \lambda_M^{-3\alpha - 1 - \epsilon_*} \right)^{-1/\alpha},
\]
satisfies
\[
V(x, s_{\lambda}) \leq \Phi(x)
\]
for all \(x \in \mathbb{R}^N\) and all sufficiently large \(\lambda\), where \(\epsilon_*\) is the constant given by \((22)\).

**Proof.** For simplicity, put
\[
Z(x, t) := (e^{t\Delta z_{2\alpha + 1}})(x), \quad E := \mathbb{R}^N \times [0, s_{\lambda}].
\]
Let \(A\) be a constant to be chosen later and \(\sigma := A \lambda_M^{-\alpha - 1 + \delta}\) and define the function \(\mathcal{V}\) by
\[
\mathcal{V}(x, t) := (\{e^{t\Delta \varphi_{2\alpha + 1}}(x)\}^{-\alpha} - \alpha(1 + \sigma) t)^{-1/\alpha}, \quad m := \inf_{x \in \mathbb{R}^N} \varphi_{2\alpha + 1}(x).
\]
Put
\[ K(x, t) := \left[ 1 - d\lambda_M^{-\alpha - 1 + \frac{2\alpha + 1}{2}} + \alpha \lambda_M^{-\frac{\alpha - 2\alpha - 1}{2}} Z(x, t) \right]^{-1/\alpha}. \]

Then, by the similar argument as in the proof of Lemma 3.3 we have

\[ \sup_{0 \leq t \leq s\lambda} \| K(t) - e^{t\Delta} K(0) \|_\infty \lesssim \lambda_M^{-2\alpha + \frac{2\alpha + 1}{2} - 2 + \delta} \] (52)

for all sufficiently large \( \lambda \). Since \( K(x, t) = 1 + o(1) \) for all \( x \in \mathbb{R}^N \times [0, \infty) \) and all sufficiently large \( \lambda \), by Lemma 3.3 we have

\[ \varphi_{[2\alpha + 1]}(x) = \lambda_M^{1 + \frac{2\alpha + 1}{2}} \left( K(x, 0) + O(\lambda^{-2\alpha + \frac{2\alpha + 1}{2} - \frac{3}{2} + 2\delta}) \right), \quad m \gtrsim \lambda_M^{1 + \frac{2\alpha + 1}{2}} \] (53)

for all \( x \in \mathbb{R}^N \) and all sufficiently large \( \lambda \). Since \( z_i(x, t) = (e^{t\Delta} z_{i-1}(s_{\lambda}^{(i-1)}))(x) = z_{i-1}(x, t + s_{\lambda}^{(i-1)}) \) and \( z_1(x, t) = M - (e^{t\Delta} \varphi_{\lambda})(x) \), we have

\[ Z(x, t) = z_1 \left( x, \frac{\lambda_M - \alpha}{\alpha} + \frac{d\lambda_M^{2\alpha - 1}}{\alpha} - \frac{\lambda_M^{\alpha - 2\alpha + 1/2}}{\alpha} + t \right) \]

and by Lemma 3.1 we obtain

\[ Z(x, t) \geq d \left( \frac{\lambda_M - \alpha}{\alpha} + \frac{d\lambda_M^{2\alpha - 1}}{\alpha} - \frac{\lambda_M^{\alpha - 2\alpha + 1/2}}{\alpha} + t \right) + O(\lambda^{-2\alpha}) = \frac{d\lambda_M^{-\alpha}}{\alpha} + O(\lambda^{-2\alpha}) \]

for \( (x, t) \in E \) and all sufficiently large \( \lambda \). This together with (52) and (53) yields

\[ [(e^{t\Delta} \varphi_{[2\alpha + 1]})(x)]^{-\alpha} - \alpha(1 + \sigma)t \]

\[ = \lambda_M^{-\alpha - 2\alpha + \frac{2\alpha + 1}{2}} K(x, t)^{-\alpha} \left( 1 + O(\lambda^{-2\alpha + \frac{2\alpha + 1}{2} - \frac{3}{2} + 2\delta}) \right) - \alpha t + O(\sigma s\lambda) \]

\[ = \lambda_M^{-\alpha - 2\alpha + \frac{2\alpha + 1}{2}} - \alpha t - d\lambda_M^{-2\alpha + 1} + \alpha \lambda_M^{-\alpha - 2\alpha + 1} Z(x, t) \]

\[ + O(\lambda^{-3\alpha - \frac{3}{2} + 2\delta}) + O(\lambda^{-2\alpha - \frac{[2\alpha + 1]}{2} - 1 + \delta}) \]

(54)

\[ \geq \lambda_M^{-2\alpha + 1} + o(\lambda^{-2\alpha + 1}) + O(\lambda^{-2\alpha - \frac{[2\alpha + 1]}{2} - 1 + 2\delta}) \]

for all \( (x, t) \in E \) and all sufficiently large \( \lambda \). Note that

\[ \max \left\{ -3\alpha - \frac{3}{2} + 2\delta, -2\alpha - \frac{2\alpha + 1}{2} - 1 + \delta \right\} \leq -2\alpha - \frac{2\alpha + 1}{2} - 1 + 2\delta \]

and

\[ -\frac{[2\alpha + 1]}{2} + 2\delta < 0 \]

by (23). This implies that

\[ \sup_{(x, t) \in E} V(x, t)^\alpha \lesssim \lambda_M^{2\alpha + 1} \]

for all sufficiently large \( \lambda \). Therefore, by (39) with \( i = [2\alpha + 1] \) and (53) we find a constant \( c \) such that

\[ (\alpha + 1)m^{-2\alpha - 2} V(x, t)^\alpha |\nabla(e^{t\Delta} \varphi_{[2\alpha + 1]})(x)|^2 \leq c\lambda_M^{-\alpha - 1 + \delta} \]
for all sufficiently large \( \lambda \). Putting \( A = c \), by (30) we see that the function \( \mathcal{V} \) is a supersolution of (48), and by (54) we have
\[
V(x, s_\lambda) \leq \left( \lambda_M^{-2\alpha-1} + \lambda_M^{-\alpha-1} (e\phi(x) - d\lambda_M^{-\alpha}) + O(\lambda_M^{-3\alpha-1-2\epsilon}) \right)^{-1/\alpha}
\]
for all \( x \in \mathbb{R}^N \) and all sufficiently large \( \lambda \). Therefore we can take a constant \( C > 0 \) such that the function \( \Phi \) defined by (49) satisfies (50), and the proof of Lemma 3.4 is complete.

As a consequence of above lemmas, we get the following proposition.

**Proposition 2.** Assume the same conditions as in Theorem 1.1. Then there exists a constant \( C > 0 \) such that the function \( \Phi \in C^1(\mathbb{R}^N) \) defined by
\[
\Phi(x) = \left( \lambda_M^{-2\alpha-1} + \lambda_M^{-\alpha-1} \left[ e^{s_\lambda \Delta \psi_\lambda} - d\lambda_M^{-\alpha} \right] - C\lambda_M^{-3\alpha-1-\epsilon} \right)^{-1/\alpha},
\]
for all \( x \in \mathbb{R}^N \) and all sufficiently large \( \lambda \).

**Proof.** Since \( U(x, s^{(1)}_\lambda + t) \) solves
\[
\partial_t u = \Delta u + u^p, \quad x \in \mathbb{R}^N, \quad t > 0, \quad u(x, 0) = U(x, s^{(1)}_\lambda), \quad x \in \mathbb{R}^N,
\]
by Lemma 3.2 and (36) we have \( U(x, s^{(1)}_\lambda + t) \leq U_2(x, t) \in \mathbb{R}^N \times [0, s^{(2)}_\lambda] \). This together with Lemma 3.3 implies that \( U(x, s^{(1)}_\lambda + s^{(2)}_\lambda) \leq U_2(x, s^{(2)}_\lambda) \leq \varphi_2(x) \). Since \( U(x, s^{(1)}_\lambda + \cdots + s^{[2\alpha+1]}_\lambda + t) \) solves
\[
\partial_t u = \Delta u + u^p, \quad x \in \mathbb{R}^N, \quad t > 0, \quad u(x, 0) = U(x, s^{(1)}_\lambda + \cdots + s^{[2\alpha+1]}_\lambda), \quad x \in \mathbb{R}^N,
\]
repeating above argument, by Lemma 3.3 and (48) we obtain
\[
U(x, s^{(1)}_\lambda + \cdots + s^{[2\alpha+1]}_\lambda + t) \leq U_2(x, s^{(2)}_\lambda + \cdots + s^{[2\alpha+1]}_\lambda + t) \leq \cdots \leq V(x, t)
\]
in \( \mathbb{R}^N \times [0, \lambda] \) for all sufficiently large \( \lambda \). Since \( e^{s_\lambda \Delta \psi_{[2\alpha+1]}} = e^{s_\lambda \Delta \psi_\lambda} \), by (50) and (55) we obtain
\[
U(x, s_\lambda) = U(x, s^{(1)}_\lambda + \cdots + s^{[2\alpha+1]}_\lambda + s_\lambda) \leq V(x, s_\lambda) \leq \Phi(x)
\]
for all \( x \in \mathbb{R}^N \) and all sufficiently large \( \lambda \). This yields (56) and completes the proof of Proposition 2.

The rest of this section is devoted to the study of an upper bound of the solution of problem 1 at the time
\[
t = \frac{\lambda_M^{-\alpha}}{\alpha} + \frac{d\lambda_M^{-2\alpha-1}}{\alpha} - \sum_{k=2}^{\lambda} \frac{d_k \lambda_M^{-(k+1)\alpha-1}}{k! \alpha^k} - \frac{\lambda_M^{-(m+1)\alpha-1}}{\alpha}.
\]
For this purpose, consider
\[
\left\{ \begin{array}{ll}
\partial_t W = \Delta W + W^p, & x \in \mathbb{R}^N, \quad t > 0, \\
W(x, 0) = \Phi_*(x), & x \in \mathbb{R}^N,
\end{array} \right.
\]
where
\[
\Phi_*(x) := \left( \lambda_M^{-2\alpha-1} + \lambda_M^{-\alpha-1} \psi(x) - C\lambda_M^{-3\alpha-1-\epsilon} \right)^{-1/\alpha}
\]
and
\[
\psi(x) := \inf \left\{ \alpha (e^{s_\lambda \Delta \psi_\lambda}(x) - d\lambda_M^{-\alpha} \lambda_M^{-2\alpha+\delta}) \right\}.
\]
By (55) and the definition of Ψ we have
\[ \Phi(x) \leq \Phi_*(x) \] (58)
in \( \mathbb{R}^N \). Since \( \Psi(x) \leq \lambda_M^{-2\alpha+\delta} \) for all \( x \in \mathbb{R}^N \) and all sufficiently large \( \lambda \), we have
\[ m_* := \inf_{x \in \mathbb{R}^N} \Phi_*(x) \geq \lambda_M^{2\alpha+1} \]
for all sufficiently large \( \lambda \). Furthermore, by (24), (25), (31) and Lemma 3.1 we have
\[ \inf_{x \in \Omega} \Psi(x) \geq (\alpha\gamma \kappa - d) \lambda_M^{-\alpha} \]
for all sufficiently large \( \lambda \), where \( \gamma \), \( \kappa \), \( \eta \) and \( \delta(\eta) \) are the positive constants given in Lemma 3.1. Here we have used the assumption \( -2\alpha - 1 < -m\alpha \) derived from the assumption \( \alpha < 1 + \epsilon_*/(m - 2) \). Then we have:

**Proposition 3.** Assume the same conditions as in Theorem 1.2. In particular, assume the same conditions as in Theorem 1.1 for the case \( m = 2 \). Then there exists a constant \( C > 0 \) such that the function \( \tilde{\varphi}_\lambda \in C^1(\mathbb{R}^N) \) defined by
\[ \tilde{\varphi}_\lambda(x) = \left( 1 + \lambda_M^{m\alpha} \Psi(x) + \sum_{k=2}^m \frac{d_k \lambda_M^{(m-k)\alpha}}{k!\alpha^{k-1}} - C\lambda_M^{(m-2)\alpha - \epsilon_*} \right)^{-1/\alpha} \] (63)
satisfies
\[ U \left( x, S_\lambda + \frac{\lambda_M^{-2\alpha-1}}{\alpha} - \sum_{k=2}^m \frac{d_k \lambda_M^{-(k+1)\alpha-1}}{k!\alpha} \right) \leq \lambda_M^{(m+1)\alpha+1} \tilde{\varphi}_\lambda(x) \]
\[ |\nabla \tilde{\varphi}_\lambda(x)| \leq C\lambda_M^{(m-1)\alpha+\delta/2} \] (65)
for all \( x \in \mathbb{R}^N \) and all sufficiently large \( \lambda \).

In order to prove Proposition 3, we prepare the following lemma on the estimate of \( \|\nabla \Psi\|_\infty \).

**Lemma 3.5.** Assume the same conditions as in Theorem 1.1. Then there exists a constant \( C > 0 \) such that
\[ \sup_{\lambda > \lambda_*} \|\nabla \Psi\|_\infty \leq C\lambda_M^{-\alpha+\delta/2} \]
for all sufficiently large \( \lambda \).
Proof. We modify the argument in the proof of [6, Lemma 4.1]. Let
\[ \Omega_1 := \left\{ x \in \mathbb{R}^N : \psi_\lambda(x) \leq C_1 \lambda_M^{-\alpha + \frac{\delta}{4}} \right\}, \quad \Omega_2 := \left\{ x \in \mathbb{R}^N : \psi_\lambda(x) \leq 2C_1 \lambda_M^{-\alpha + \frac{\delta}{4}} \right\}, \]
where \( C_1 > 0 \) is a constant to be chosen later. By (24) and (25) we have \( \psi_\lambda(x) = M - \varphi(x) \) in \( \Omega_2 \), which yields
\[ \sup_{x \in \Omega_2} |\nabla^2 \psi_\lambda(x)| \leq \| \nabla^2 \varphi \|_\infty \quad (66) \]
for all sufficiently large \( \lambda \). Let \( x \in \Omega_1 \) and \( |z| \leq \lambda_M^{\delta/8} \). Then, by the mean value theorem and (26) we have
\[
\psi_\lambda(x + S^{1/2}_\lambda z) \leq \psi_\lambda(x) + \| \nabla \psi_\lambda \|_{\infty} S^{1/2}_\lambda |z| \leq C_1 \lambda_M^{-\alpha + \frac{\delta}{4}} + O(\lambda_M^{-\alpha/2} \cdot \lambda_M^{-\delta/8}) \leq 2C_1 \lambda_M^{-\alpha + \frac{\delta}{4}}
\]
for all sufficiently large \( \lambda \). This implies that
\[ x + S^{1/2}_\lambda z \in \Omega_2. \quad (67) \]

Since
\[
\partial^2_{x_i, x_j}(e^{S_\lambda \Delta \psi_\lambda})(x) = \partial_{x_i} \left[ \frac{1}{2S^{1/2}_\lambda} (4\pi S_\lambda)^{-\frac{N}{2}} \int_{\mathbb{R}^N} \frac{y_j - x_j}{S^{1/2}_\lambda} e^{-\frac{|x - y|^2}{4S_\lambda}} \psi_\lambda(y) \, dy \right]
\]
\[ = \partial_{x_i} \left[ \frac{1}{2S^{1/2}_\lambda} (4\pi)^{-\frac{N}{2}} \int_{\mathbb{R}^N} z_j e^{-\frac{|z|^2}{S_\lambda}} \psi_\lambda(x + S^{1/2}_\lambda z) \, dz \right]
\]
\[ = \frac{1}{2S^{1/2}_\lambda} (4\pi)^{-\frac{N}{2}} \int_{\mathbb{R}^N} z_j e^{-\frac{|z|^2}{S_\lambda}} (\partial_{x_j} \psi_\lambda)(x + S^{1/2}_\lambda z) \, dz
\]
\[ = -\frac{1}{S^{1/2}_\lambda} (4\pi)^{-\frac{N}{2}} \int_{|z| \leq \lambda_M^{\delta/8}} \partial_{x_j} \left( e^{-\frac{|z|^2}{S_\lambda}} \right) (\partial_{x_j} \psi_\lambda)(x + S^{1/2}_\lambda z) \, dz
\]
\[ \quad + \frac{1}{2S^{1/2}_\lambda} (4\pi)^{-\frac{N}{2}} \int_{|z| \geq \lambda_M^{\delta/8}} z_j e^{-\frac{|z|^2}{S_\lambda}} (\partial_{x_j} \psi_\lambda)(x + S^{1/2}_\lambda z) \, dz
\]
for all \( i, j \in \{1, \ldots, N\} \), by (26), (66) and (67) we have
\[ \sup_{x \in \Omega_1} |(\nabla^2 e^{S_\lambda \Delta \psi_\lambda})(x)| \leq O \left( \lambda_M^{\frac{\delta}{4}} \cdot \left[ e^{-\frac{|z|^2}{S_\lambda}} \cdot |\partial B(0, \lambda_M^{\delta/8})| + e^{-\frac{|z|^2}{S_\lambda}} \cdot \lambda_M^{-\delta/4 + \frac{\delta}{4}} \right] \right)
\]
\[ \quad + (4\pi)^{-\frac{N}{2}} \int_{|z| \leq \lambda_M^{\delta/8}} e^{-\frac{|z|^2}{S_\lambda}} \sup_{x \in \Omega_2} |\nabla^2 \psi_\lambda(x)| \, dz \leq 2\| \nabla^2 \varphi \|_{\infty}
\]
(68)
for all sufficiently large \( \lambda \), where \( |\partial B(0, \lambda_M^{\delta/8})| \) denote the surface area of the sphere \( \partial B(0, \lambda_M^{\delta/8}) \). On the other hand, if \( \psi_\lambda(x) \geq 2\lambda_M^{-\alpha + \frac{\delta}{4}} \) and \( |z| \leq \lambda_M^{\delta/8} \), then we have
\[ \psi_\lambda(x + S^{1/2}_\lambda z) \geq \psi_\lambda(x) - \| \nabla \psi_\lambda \|_{\infty} S^{1/2}_\lambda |z| \geq 2\lambda_M^{-\alpha + \frac{\delta}{4}} + O(\lambda_M^{-\alpha + \frac{\delta}{4}})
\]
for all sufficiently large \( \lambda \). Then we obtain
\[
(e^{S_\lambda \Delta \psi_\lambda})(x) \geq (4\pi)^{-\frac{N}{2}} \int_{|z| \leq \lambda^{-\delta/8}} e^{-\frac{|z|^2}{2}} \psi_\lambda(x + S_\lambda^{1/2}z) \, dz
\]
\[
\geq (4\pi)^{-\frac{N}{2}} \int_{\mathbb{R}^N} e^{-\frac{|y|^2}{2}} \psi_\lambda(x + S_\lambda^{1/2}z) \, dz + O \left( e^{-\frac{\delta^2}{4}} \right)
\]
\[
\geq 2\lambda_M^{-\alpha + 3\delta/4} + O(\lambda_M^{-\alpha + 5\delta/8}) + O \left( e^{-\frac{\delta^2}{4}} \right) > \lambda_M^{-\alpha + 3\delta/4}
\]
for all sufficiently large \( \lambda \), that is, \( \Psi(x) = \lambda_M^{-2\alpha + \delta} \). Furthermore, if \( x \in \Omega_1 \), then by (67) we have
\[
(e^{S_\lambda \Delta \psi_\lambda})(x) = (4\pi)^{-\frac{N}{2}} \left( \int_{|z| \leq \lambda^{-\delta/8}} + \int_{|z| \geq \lambda^{-\delta/8}} \right) e^{-\frac{|z|^2}{2}} \psi_\lambda(x + S_\lambda^{1/2}z) \, dz
\]
\[
\leq 2C_1 \lambda_M^{-\alpha + 3\delta/4} + O \left( e^{-\frac{\delta^2}{4}} \right) < \frac{1}{\alpha} \lambda_M^{-2\alpha + \delta}
\]
for all sufficiently large \( \lambda \), that is,
\[
\Psi(x) = \alpha(e^{S_\lambda \Delta \psi_\lambda})(x) - d\lambda_M^{-\alpha}
\]
for all sufficiently large \( \lambda \).

Let \( \lambda_1 > 1 \) and \( C_3 > 0 \) be constants such that \( \|\nabla \psi_\lambda\|_\infty \leq C_3 \lambda_M^{-\alpha/2 + \delta/2} \),
\[
\inf_{x \in \mathbb{R}^N} \Psi(x) \geq -C_3 \lambda_M^{-2\alpha}, \quad C_3 \lambda_M^{-2\alpha} < \frac{\alpha}{2} \lambda_M^{-2\alpha + \delta},
\]
and (66)–(70) hold for all \( \lambda \in (\lambda_1, \infty) \). Assume that there exists a constant \( \lambda \in (\lambda_1, \infty) \) and a point \( x \in \mathbb{R}^N \) such that \( |\nabla \Psi(x)| > 2\alpha(2\|\nabla^2 \varphi\|_\infty)^{1/2} \lambda_M^{-\alpha + \delta/2} \). If \( \psi_\lambda(x) \geq 2\lambda_M^{-\alpha + 3\delta/4} \), then by (69) we have \( \Psi(x) = \lambda_M^{-\alpha + \delta} \) near \( x \), so we obtain \( \nabla \Psi(x) = 0 \). Hence we have \( \psi_\lambda(x) \leq 2\lambda_M^{-\alpha + 3\delta/4} \). Put
\[
z := x - (2\|\nabla^2 \varphi\|_\infty)^{-1/2} \lambda_M^{-\alpha/2 + \delta/2} \frac{\nabla \Psi(x)}{|\nabla \Psi(x)|}.
\]
Then, by the mean value theorem we have
\[
\psi_\lambda(z) \leq \psi_\lambda(x) + \|\nabla \psi_\lambda\|_\infty |z - x| \leq \left[ 2 + C_3(2\|\nabla^2 \varphi\|_\infty)^{-1/2} \lambda_M^{-\alpha/2 + \delta/2} \right] \lambda_M^{-\alpha + 3\delta/4}.
\]
Since \( \lambda_M > \lambda_1 > 1 \) and \( \delta < \alpha \) by (23), we have \( \lambda_M^{-\alpha/2 + \delta/4} < 1 \). Therefore, choosing a constant \( C_1 > 0 \) satisfying \( C_1 > C_2 \) and \( C_1 > 2 + C_3(2\|\nabla^2 \varphi\|_\infty)^{-1/2} \), we see that \( x, z \in \Omega_1 \), and by the Taylor theorem, (68), (70) and (71) we obtain
\[
\Psi(z) = \alpha(e^{S_\lambda \Delta \psi_\lambda})(z) - d\lambda_M^{-\alpha}
\]
\[
\leq \alpha(e^{S_\lambda \Delta \psi_\lambda})(x) - d\lambda_M^{-\alpha} + \nabla \Psi(x) \cdot (z - x) + \frac{\alpha}{2} |z - x|^2 \sup_{y \in \Omega_1} |(\nabla^2 e^{S_\lambda \Delta \psi_\lambda})(y)|
\]
\[
\leq \alpha \lambda_2^{-2\alpha + \delta} - 2\alpha \lambda_M^{-2\alpha + \delta} + \frac{\alpha}{2} \lambda_M^{-2\alpha + \delta} = -\frac{\alpha}{2} \lambda_M^{-2\alpha + \delta} < -C_3 \lambda_M^{-2\alpha}.
\]
This contradict (71). Thus we complete the proof of Lemma 3.5. \( \square \)

**Proof of Proposition 3.** Put
\[
\sigma_\lambda := \frac{\lambda_M^{-2\alpha - 1}}{\alpha} - \sum_{k=2}^m \frac{d_k}{k!} \lambda_M^{-(k+1)\alpha - 1} - \frac{\lambda_M^{(m+1)\alpha - 1}}{\alpha}.
\]
By the definition of $\Phi_\ast$ and $\Psi$ we have
\[
\|\Phi_\ast\|_\infty \lesssim \lambda_M^{2\alpha+1} \tag{72}
\]
for all sufficiently large $\lambda$. See also (60). Then, by Lemma 3.5 we have
\[
\|\nabla\Phi_\ast\|_\infty \leq \frac{1}{\alpha}\|\Phi_\ast\|_\infty^{\alpha+1} \cdot \lambda_M^{-\alpha-1} \|\nabla\Psi\|_\infty \lesssim \lambda_M^{(2\alpha+1)(\alpha+1)-2\alpha-1+\frac{\alpha}{2}} \tag{73}
\]
for all sufficiently large $\lambda$. Then, since $L(0) = 1 + o(1)$ and
\[
\|L(t) - e^{t\Delta}L(0)\|_\infty \leq \frac{\alpha+1}{\alpha^2}\lambda_M^{2\alpha} \int_0^t \|e^{(t-s)\Delta}L(s)^{2\alpha+1}\|_\infty ds 
\]
for all sufficiently large $(x, t) \in \mathbb{R}^N \times (0, \infty)$ and all sufficiently large $\lambda$, by Lemma 3.5 we have
\[
\sup_{0 \leq t \leq \sigma_\lambda} \|L(t) - e^{t\Delta}L(0)\|_\infty \lesssim \lambda_M^{2\alpha} \cdot \sigma_\lambda \cdot \lambda_M^{-2\alpha+\delta} \lesssim \lambda_M^{-2\alpha-1+\delta} 
\]
for all sufficiently large $\lambda$. Furthermore, since
\[
(e^{t\Delta}\Psi)(x) - \Psi(x) = (4\pi)^{-\frac{N}{2}} \int_{\mathbb{R}^N} e^{-\frac{|z|^2}{4}}(\Psi(x + t^{1/2}z) - \Psi(x)) \, dz,
\]
we obtain
\[
\sup_{0 \leq t \leq \sigma_\lambda} \|\lambda_M e^{t\Delta}\Psi - \lambda_M^\alpha \Psi\|_\infty \lesssim \lambda_M^{\alpha+1/2} \|\nabla\Psi\|_\infty \lesssim \lambda_M^{-\alpha-1/2+\delta/2} 
\]
for all sufficiently large $\lambda$. These imply that
\[
[(e^{t\Delta}\Phi_\ast)(x)]^{-\alpha} = \lambda_M^{-2\alpha-1} \left[\left(e^{t\Delta}(L(0) + O(\lambda_M^{-\alpha-\epsilon}))(x)\right)\right]^{-\alpha} 
= \lambda_M^{-2\alpha-1} \left[\lambda_M^{-2\alpha-1}L(x, t) + O(\lambda_M^{-\alpha-\epsilon}) + O(\lambda_M^{-2\alpha-1+\delta})\right]^{-\alpha} 
= \lambda_M^{-2\alpha-1}L(x, t)^{-\alpha} + O(\lambda_M^{-\alpha-\epsilon}) \tag{74}
\]
for all $(x, t) \in \mathbb{R}^N \times [0, \sigma_\lambda]$ and all sufficiently large $\lambda$. Note that $-\alpha - 1/2 + \delta/2 < -\alpha - \epsilon$ by (22) and (23).

Let $A$ be a positive constant to be chosen later and put
\[
\mathcal{W}(x, t) := \left((e^{t\Delta}\Phi_\ast)(x)\right)^{-\alpha} - \alpha(1 + A\lambda_M^{-\alpha-1+\delta+\epsilon})t \right)^{-1/\alpha}.
\]
By (60) and (74) we have

\[
[(e^{tA\Phi_\ast})(x)]^{-\alpha} - \alpha(1 + A\lambda_M^{-1+\delta+\epsilon_\ast})t \\
= \lambda_M^{-2\alpha - 1}L(x,0)^{-\alpha} - \alpha t + O(\sigma\lambda\lambda_M^{-1+\delta+\epsilon_\ast}) + O(\lambda_M^{-3\alpha - 1-\epsilon_\ast}) \\
= \lambda_M^{-2\alpha - 1} + \lambda_M^{-\alpha - 1}\Psi(x) - \alpha t + O(\lambda_M^{-3\alpha - 2+\delta+\epsilon_\ast}) + O(\lambda_M^{-3\alpha - 1-\epsilon_\ast}) \\
\geq \lambda_M^{-2\alpha - 1} + \lambda_M^{-\alpha - 1} \left(-\alpha \sum_{k=2}^{m} \frac{d_k\lambda_M^{-(k+1)\alpha - 1}}{k!\alpha^k} + o(\lambda_M^{-m\alpha}) \right) \\
- \alpha \left(\frac{\lambda_M^{-2\alpha - 1}}{\alpha} - \sum_{k=2}^{m} d_k\lambda_M^{-(k+1)\alpha - 1} \right) - \lambda_M^{-\alpha - 1} \left(1 + o(1) + O(\lambda_M^{-2\alpha - \epsilon_\ast}) \right) \\
= \lambda_M^{-(m+1)\alpha - 1} \left(1 + o(1) + O(\lambda_M^{-2\alpha - \epsilon_\ast}) \right)
\]

for all \((x, t) \in \mathbb{R}^N \times [0, \sigma_\lambda]\) and all sufficiently large \(\lambda\). Note that \(-3\alpha - 2 + \delta + \epsilon_\ast < -3\alpha - 1 - \epsilon_\ast\) by (22) and (23). Since \(p < 1 + \epsilon_\ast/(m - 2)\) for the case \(m \geq 3\) by (22), we have

\[(m - 2)\alpha - \epsilon_\ast < 0. \tag{76}\]

This together with (75) implies that

\[\sup_{(x, t) \in \mathbb{R}^N \times [0, \sigma_\lambda]} W(x, t)^\alpha \lesssim \lambda_M^{(m+1)\alpha - 1}\]

for all sufficiently large \(\lambda\). Therefore, by (59), (73) and (76) we have

\[(\alpha + 1)m_\ast^{-2\alpha - 2}\hat{W}(x, t)^\alpha |\nabla (e^{tA\Phi_\ast})(x)|^2 \leq C_\ast \lambda_M^{(m+3)\alpha - 1+\delta} < C_\ast \lambda_M^{-\alpha - 1+\delta + \epsilon_\ast},\]

for all \((x, t) \in \mathbb{R}^N \times [0, \sigma_\lambda]\) and all sufficiently large \(\lambda\), where \(C_\ast > 0\) is a constant. Putting \(A = C_\ast\), by (30) we see that the function \(W\) is a supersolution of (57), and by (63) we can take a constant \(C > 0\) such that

\[W(x, \sigma_\lambda) \leq \left(\lambda_M^{-(m+1)\alpha - 1} + \lambda_M^{-\alpha - 1}\Psi(x) + m \sum_{k=2}^{m} \frac{d_k\lambda_M^{-(k+1)\alpha - 1}}{k!\alpha^k} + O(\lambda_M^{-3\alpha - 1-\epsilon_\ast}) \right)^{-\frac{1}{\alpha}} \leq \lambda_M^{-(m+1)\alpha - 1} \phi_\lambda(x) \]

for all \(x \in \mathbb{R}^N\) and all sufficiently large \(\lambda\). Thus we obtain (64) since \(U(x, S_\lambda + \sigma_\lambda) \leq W(x, \sigma_\lambda)\). (65) follows from a direct calculation as in Lemma 3.2. This completes the proof of Proposition 3. \(\square\)

4. **Proof of Theorems 1.1 and 1.2.** In this section we prove main theorems of this paper with the aid of the results obtained in Section 3.

**Proof of Theorems 1.1 and 1.2.** In the proof of main theorems, we always assume that \(p \in (1, p(m))\), so \(1 < p < 1 + \epsilon_\ast/(m - 2)\) for the case \(m \geq 3\). We do not need any restriction on \(p\) for the case \(m = 2\). See (22). We make use of the results in Section 3. By (24), (25) and (29) we have \(u(x, t) \leq U(x, t)\). This together with Proposition 3 implies that

\[T_\lambda > \frac{1}{\alpha} \lambda_M^{\alpha} + \frac{1}{\alpha} d\lambda_M^{-2\alpha - 1} - \sum_{k=2}^{m} \frac{d_k\lambda_M^{-(k+1)\alpha - 1}}{k!\alpha^k} - \frac{\lambda_M^{-(m+1)\alpha - 1}}{\alpha} =: t_\lambda \tag{77}\]
for all sufficiently large \( \lambda \), where \( \alpha = p - 1 \). Put
\[
v(x, \tau) := \epsilon_\lambda^{1/\alpha} u(x, t_\lambda + \epsilon_\lambda \tau), \quad \epsilon_\lambda := \lambda_\lambda^{-\alpha(1+\alpha-1)}.
\]
(78)

Then the function \( v \) satisfies
\[
\partial_\tau v = \epsilon_\lambda \Delta v + \epsilon^{\alpha+1} \quad \text{in} \quad \Omega \times (0, \tau_\lambda), \quad v(x, \tau) = 0 \quad \text{on} \quad \partial \Omega \times (0, \tau_\lambda),
\]
(79)

where \( \tau_\lambda = \epsilon_\lambda^{-1}(T_\lambda - t_\lambda) \) is the blow-up times of \( v \). By (1), (4) and the definition of \( v \) we have
\[
\sup_{0 < \tau < \tau_\lambda} (\tau_\lambda - \tau)^{1/\alpha} \| v(\tau) \|_{L^\infty(\Omega)} = \sup_{t_\lambda < t < T_\lambda} \left[ \epsilon_\lambda \left( \frac{T_\lambda - t}{\epsilon_\lambda} \right) \right]^{1/\alpha} \| u(t) \|_{L^\infty(\Omega)} \quad \text{for all sufficiently large} \quad \lambda
\]
(80)

By (60) and (63) we have
\[
\epsilon_\lambda^{1/2} \| \nabla \hat{\varphi}_\lambda \|_{\infty} \lesssim \lambda_\lambda^{-(\alpha(1+\alpha-1)/2)} \cdot \lambda_\lambda^{(m-1)/2 + \epsilon} = \lambda_\lambda^{-(\alpha(1+\alpha-1)/2) + \epsilon}
\]
for all sufficiently large \( \lambda \). Since \( (m-3)\alpha + 1 + \epsilon < \epsilon_\star - \alpha + \delta - 1 < 0 \) by (22), (23) and (76), we obtain
\[
\lim_{\lambda \to \infty} \epsilon_\lambda^{1/2} \| \nabla \hat{\varphi}_\lambda \|_{\infty} = 0.
\]
(82)

The following lemma gives estimation of \( \| \hat{\varphi}_\lambda \|_{\infty} \) for sufficiently large \( \lambda \).

**Lemma 4.1.** Let \( \hat{\varphi}_\lambda \) be the function defined by (63). Then
\[
\| \hat{\varphi}_\lambda \|_{\infty}^{-\alpha} = 1 + o(1)
\]
for all sufficiently large \( \lambda \).

**Proof.** By (60) and (63) we have
\[
\hat{\varphi}_\lambda(x) \leq (1 + o(1))^{-1/\alpha}
\]
for all \( x \in \mathbb{R}^N \) and all sufficiently large \( \lambda \). Thus we have \( \| \hat{\varphi}_\lambda \|_{\infty}^{-\alpha} \geq 1 + o(1) \) for all sufficiently large \( \lambda \).

In order to obtain an upper estimate of \( \| \hat{\varphi}_\lambda \|_{\infty}^{-\alpha} \), we make use of a sub-solution of (1). Let \( z \) be the solution of (8) and \( u \) be the function defined by
\[
u(x, t) := (z(x, t)^{-\alpha} - \alpha t)^{-1/\alpha}.
\]

Then it is well-known that \( u \) is a sub-solution of (1). For any \( a \in M(\varphi) \), by Lemma 2.2 we have
\[
z(a, t_\lambda) \geq \lambda + \sum_{k=0}^{m} \frac{\Delta_k \varphi(a)}{k!} t_\lambda^k + o(t_\lambda^k) = \lambda_M + \sum_{k=1}^{m} \frac{\Delta_k \varphi(a)}{\alpha k!} \lambda_\lambda^{-k\alpha} + o(\lambda_\lambda^{-m\alpha})
\]
for all sufficiently large $\lambda$. Note that $\lambda_M = \lambda + \|\varphi\|_{L^\infty(\Omega)} = \lambda + \varphi(a)$. Since $\frac{m\alpha + 1}{2} > 2\alpha + 2$ for $1 < p < P(m)$, $\Delta \varphi(a) \leq 0$ for all $a \in M(\varphi)$, $(\Delta^k \varphi)(a)$ takes the same value for all $a \in M(\varphi)$ and $k \in \{1, \ldots, m - 1\}$ and (77) implies that

$$
\alpha t_\lambda = \lambda_{-\alpha}^\alpha + d\lambda_{-2\alpha}^\alpha - \sum_{k=2}^{m} \frac{d \lambda_{-k(\alpha-1)}^\alpha}{k!} - \lambda_{-(m+1)\alpha}^\alpha,
$$

by (16) we have

$$
z(a, t_\lambda)^{-\alpha} - \alpha t_\lambda
\leq \lambda_{M}^{-\alpha} \left( 1 + \sum_{k=1}^{m} \frac{\Delta^k \varphi(a)}{\alpha k!} \lambda_{-k\alpha}^\alpha + o(\lambda_{-m\alpha}^\alpha) \right)^{-\alpha} - \alpha t_\lambda
= \lambda_{M}^{-\alpha} \left( 1 - \sum_{k=1}^{m} \frac{\Delta^k \varphi(a)}{\alpha k!} \lambda_{-k\alpha}^\alpha + o(\lambda_{-m\alpha}^\alpha) \right) - \alpha t_\lambda
= \left( \lambda_{M}^{-\alpha} + |\Delta \varphi(a)| \lambda_{-2\alpha}^\alpha - \sum_{k=2}^{m} \frac{\Delta^k \varphi(a)}{\alpha k!} \lambda_{-k(\alpha-1)}^\alpha - \alpha t_\lambda \right) + o(\lambda_{-m\alpha}^\alpha)
= \lambda_{M}^{-\alpha} \left( 1 - \frac{d_m - \Delta^m \varphi(a)}{\alpha m!} \right) \lambda_{-m(\alpha-1)}^\alpha
= \lambda_{M}^{-\alpha} \left( 1 + o(1) \right)
$$

for all sufficiently large $\lambda$, and we obtain

$$
\inf_{a \in M(\varphi)} z(a, t_\lambda)^{-\alpha} - \alpha t_\lambda \leq \lambda_{M}^{-\alpha} \left( 1 + o(1) \right)
$$

(83)

for all sufficiently large $\lambda$. Since the comparison principle yields

$$
\|u(t_\lambda)\|_{L^\infty(\Omega)}^\alpha \leq \inf_{a \in M(\varphi)} u(a, t_\lambda)^{-\alpha} = \inf_{a \in M(\varphi)} z(a, t_\lambda)^{-\alpha} - \alpha t_\lambda,
$$

by (78), (81) and (83) we obtain

$$
\|\varphi_\lambda\|_{\infty}^{-\alpha} \leq \|v(0)\|_{L^\infty(\Omega)}^{-\alpha} = \lambda_{M}^{(m+1)\alpha-1} \|u(t_\lambda)\|_{L^\infty(\Omega)}^{-\alpha} \leq 1 + o(1)
$$

for all sufficiently large $\lambda$. Thus we obtain an upper estimate of $\|\varphi_\lambda\|_{\infty}^{-\alpha}$, and complete the proof of Lemma 4.1.

Combining Lemmas 2.3 and 4.1, we have

$$
T_\lambda \leq t_\lambda + \lambda_{M}^{-(m+1)\alpha-1} \|\varphi_\lambda\|_{\infty}^{-\alpha} \leq 1 + o(1)
$$

for all sufficiently large $\lambda$. This together with the definitions of $\epsilon_\lambda$ and $\tau_\lambda$ implies that

$$
\tau_\lambda \leq \frac{\|\varphi_\lambda\|_{\infty}^{-\alpha}}{\alpha} + o(1)
$$

(84)

for all sufficiently large $\lambda$. On the other hand, since the function

$$
\zeta(t) := \left( \|v(0)\|_{L^\infty(\Omega)}^{-\alpha} - \alpha t \right)^{-1/\alpha}
$$

satisfies $\partial_t \zeta = \zeta^{\alpha+1}$ and $\zeta(0) = \|v(0)\|_{L^\infty(\Omega)} \geq v(x, 0)$ for all $x \in \Omega$, by (79) and the comparison principle we have $u(x, t) \leq \zeta(t)$ in $\Omega \times (0, \|v(0)\|_{L^\infty(\Omega)}/\alpha)$,
where \( \|v(0)\|_{L^\infty(\Omega)}/\alpha \) is the blow-up time of \( \zeta \). In particular, by (81) we have 
\[
\tau_\lambda \geq \|v(0)\|_{L^\infty(\Omega)}/\alpha \geq \|\tilde{\varphi}_\lambda\|_{\infty}/\alpha 
\]
for all sufficiently large \( \lambda \). Then we obtain 
\[
\lim_{\lambda \to \infty} \left| \tau_\lambda - \frac{\|\tilde{\varphi}_\lambda\|_{\infty}}{\alpha} \right| = 0. \tag{85}
\]

Next we study the location of the maximum points of \( \tilde{\varphi}_\lambda \). For \( \eta > 0 \), let \( \delta(\eta) \) be the positive constant given in assertion (ii) of Lemma 3.1. Let \( \mu > 0 \) and \( \eta_* > 0 \) be small constants and \( \kappa \geq \kappa_* \) be a large constant satisfying
\[
(1 + \frac{\alpha \eta}{2m!} - \mu)^{-1/\alpha} < (1 + \mu)^{-1/\alpha} - \eta_*^-, \tag{86}
\]
\[
(\alpha \gamma \kappa - d) \geq 2, \tag{87}
\]
where \( \kappa_* \) is the constant given in Lemma 3.1. By (61), (62), (63), (86) and (87) we have
\[
\sup_{x \in \Omega} \tilde{\varphi}_\lambda(x) \leq \left(1 + 2\lambda^{(m-1)\alpha} + \sum_{k=2}^{m} \frac{d_k \lambda^{(m-k)\alpha}}{k! \alpha^{k-1}} - C \lambda^{(m-2)\alpha - \epsilon_*}\right)^{-1/\alpha} \leq 2^{-1/\alpha} \lambda^{-(m-1)} (1 + o(1)) < (1 + \mu)^{-1/\alpha} - \eta_*^-, \tag{88}
\]
and
\[
\sup_{a \in M_m(\varphi, \eta)} \sup_{x \in B(a, \delta(\eta))} \tilde{\varphi}_\lambda(x) \leq \left(1 + \frac{\alpha \eta}{2m!} + o(1)\right)^{-1/\alpha} \leq \left(1 + \frac{\alpha \eta}{2m!} - \mu\right)^{-1/\alpha} \leq (1 + \mu)^{-1/\alpha} - \eta_*^- \tag{89}
\]
for all sufficiently large \( \lambda \). On the other hand, by Lemma 4.1 we have
\[
\|\tilde{\varphi}_\lambda\|_{\infty} = (1 + o(1))^{-1/\alpha} \geq (1 + \mu)^{-1/\alpha} \tag{90}
\]
for all sufficiently large \( \lambda \). Therefore, by (88)–(90) we have
\[
\max \left\{ \sup_{x \in \Omega} \tilde{\varphi}_\lambda(x), \sup_{a \in M_m(\varphi, \eta)} \sup_{x \in B(a, \delta(\eta))} \tilde{\varphi}_\lambda(x) \right\} < \|\tilde{\varphi}_\lambda\|_{\infty} - \eta_*^-, \tag{91}
\]
and obtain
\[
M(\tilde{\varphi}_\lambda, \eta_*) \cap \left( \Omega \cup \bigcup_{a \in M_m(\varphi, \eta)} B(a, \delta(\eta)) \right) = \emptyset \tag{91}
\]
for all sufficiently large \( \lambda \). In particular, since \( M(\varphi, \epsilon_0) \) is a compact set in \( \Omega \), we have \( \partial \Omega \subset \Omega_\lambda \) and obtain
\[
M(\tilde{\varphi}_\lambda, \eta_*) \cap \Omega_\lambda = \emptyset. \tag{92}
\]
Therefore, by (80), (81), (82), (85), (91) and (92) we can apply Proposition 1 to problem (79), and obtain \( B_\lambda \subset M(\tilde{\varphi}_\lambda, \eta_*) \) for all sufficiently large \( \lambda \). This together with (91) implies that
\[
M(\tilde{\varphi}_\lambda, \eta_*) \cap \left( \bigcup_{a \in M_m(\varphi, \eta)} B(a, \delta(\eta)) \right) = \emptyset
\]
for all sufficiently large \( \lambda \). \( \Box \)
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E-mail address: fujishima@shizuoka.ac.jp