An interesting track for the Brachistochrone

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Abstract

If a particle has to fall first vertically 1 m from A and then move horizontally 1 m to B, it takes a time \( t = \tau_1 + \tau_2 = \tau_3 = \frac{3}{\sqrt{2g}} = 0.67 \) s. Under gravity and without friction, if it sides down on a linear track inclined at 45\(^{0}\) between two points A and B of 1 m height, it takes time \( t = \tau_4 = \frac{2}{\sqrt{g}} = 0.63 \) s. Between these two extremes, historically, Bernoulli (1718) proved that the fastest track between these points A and B is cycloid with the least time of descent \( t = \tau_B = 0.58 \) s.

Apart from other interesting cases, here we study the frictionless motion of a particle/bead on an interesting track/wire between A and B given by \( y(x) = (1 - x^\nu)^{1/\nu} \). For \( \nu > 1 \) the track becomes convex and \( t >> \tau_4 \), and when \( \nu > 1.22 \), the motion with zero initial speed is not possible. We find that when \( \nu \in (0.09653, 0.31749) \), \( \tau_4 < t < \tau_3 \) and when \( \nu \in (0.31749, 1) \), \( \tau_B < t < \tau_4 \). But most remarkably, the concave curve becomes very steep/deep if \( \nu \in (0, \nu_c = 0.09653) \), then \( t = 0.2258 \) s \( < \tau_B \), this is as though a particle would travel 1 meter horizontally with a speed equal \( \sqrt{2g} \) m/sec to take the time \( (\frac{1}{\sqrt{2g}} = \tau_2) < \tau_B \). The function \( t(\nu) \) suffers a jump discontinuity at \( \nu = \nu_c \), we offer some resolution.

Ignoring friction and taking the acceleration due to gravity \( g = 9.8 \) m/s\(^2\), let us first appreciate the various times a particle would take in the following motions under gravity:

1. in free falling from a height of 1 m, \( t = \tau_1 = \sqrt{\frac{2}{g}} = 0.4517 \) s.
2. in moving a distance of 1 m after falling, \( t = \tau_2 = \sqrt{\frac{1}{2g}} = 0.2258 \) s. Let \( \tau_3 = \tau_1 + \tau_2 = \frac{3}{\sqrt{2g}} = 0.6776 \) s.
3. in falling 1 meter on an inclined straight line (Fig.1) at 45\(^{0}\), \( t = \tau_4 = \frac{2}{\sqrt{g}} = 0.6338 \) s. Bernoulli (1718) found that the classical Brachistochrone (least time) problem gave the cycloid (like \( b \) in Fig.1) as the fastest descent curve between two points and the time \( t = \tau_B = \frac{1}{\sqrt{2g}} \int_0^{\pi/2} \sqrt{\csc x} \, dx = 0.5822 \) s \[1\] which is less than \( \tau_4 \). The distance between the end points of this cycloid is \( AB = \sqrt{2} \) m.

The sliding of a small bead under gravity on an inclined and curved (Fig.1) tracks (wires) is what makes most simple model for Brachistochrone. This problem is well discussed in the textbooks \[2\] using the simple
FIG. 1: For various values of $\nu$ the tracks given by (7). a: $\nu = 0.2$, b: $\nu = 0.5$, d: $\nu = 1.2$, f: $\nu = 2$. The track d denotes one of the family of $\mu$-curves given by (7). These $\mu$-tracks are essentially vertical at both points A and B. From various tracks given by $y(x) = 1 - x^\lambda$, $y(x) = \sqrt{1 - x^\mu}$ and $y(x) = (1 - x^\nu)^{1/\nu}$ between two fixed points A(0,1) and B(1,0). It is the third $\nu$-track that gives a surprising result that yields $t$ lesser than the so far acclaimed least value $\tau_B$, when $\nu \in (0.0, 0.09653)$.

When a particle of mass $m$ sides under gravity on the track given by $y(x)$, the conservation of energy leads to

$$\frac{m}{2} [x^2 + y^2] + mgy(x) = mgy(x_1) \quad (1)$$

Introducing differential length of the curve $(ds)^2 = (dx)^2 + (dy)^2$, we can write

$$\frac{dt}{dx} = \sqrt{\frac{1 + y'^2(x)}{2g(y(x_1) - y(x))}} \quad (2)$$

By taking the initial point as $x_1 = 0$ and the final point as $x_2 = 1$. Here the distances are in meters and $g = 9.8 \text{ m/s}$, we get

$$t = \tau_2 \int_0^1 \sqrt{\frac{1 + y'^2(x)}{y(0) - y(x)}} \, dx, \quad \tau_2 = \frac{1}{\sqrt{2g}} \quad (3)$$

For the linear track-c (Fig.1) $y = a(1 - x)$, where $a = \tan \theta > 0$, we get

$$t = \frac{1}{\sqrt{g}} \sqrt{\frac{1 + a^2}{2a}} \int_0^1 \frac{dx}{\sqrt{x}} \geq \frac{2}{\sqrt{g}} = \tau_4 \quad (4)$$

As $(1 - a)^2 \geq 0 \implies 1 + a^2 \geq 2a$ and the equality holds for $a = 1 \implies \theta = \pi/4$. This proves that among all the linear inclined tracks starting from A and ending at $x = 1$, the one with 45° angle gives the least time of descent, Galileo is known to have pointed out this fact.
This parametric regime in a limiting way represents motion of the particle vertically down from point A, followed by the horizontal motion with speed $\sqrt{2g}$ m/s towards the point B. When $\lambda \in (.07, 1)$ the track-b is concave, $\tau_B < t(\lambda) < \tau_4$. Next, when $\lambda \in (1, 2)$, see the convex track-e, we find that $t(\lambda) >> \tau_4$. The motion is forbidden, when $\lambda \geq 2$, $t(\lambda) (6)$ diverges: the convexity of the track-f does not allow the motion with zero initial velocity.

We can have wavvy tracks (see Fig. 2) that are given by $y(x) = 1 - xf(x), 1 - \sqrt{xf(x)}, 1 - x^{3/2}f(x)$. Here we choose $f(x) = e^{\epsilon \cos^2(7\pi x/2)}, \epsilon = \pm 0.06$ to give a mild and smooth modulation to the tracks. These small modulations in the case of concave or convex tracks do not change the inequality of the time of descent $t$, with respect to $\tau_4$. The essence of this study is that if initial part of a track is concave for a good length like the lower one, then $t = 0.6373$ is lesser than for the upper track $t = 0.6478$, even though its length is a little less, but its initial part is convex. However, in both the cases $t > \tau_4$.

The family of tracks given by

$$y(x) = \sqrt{1 - x^{\mu}}, \mu > 0$$

are vertical at $x = 0$ and $x = 1$, see track-d for $\mu = 0.5$. We find that when $\mu \in (0, 0.0161)$, $\tau_4 < t(\mu) < \tau_3$, for $\mu \in (0.0161, 0.615), t(\mu) < \tau_4$, for $\mu \in (0.615, 1.23), t(\mu) >> \tau_4$. For $\mu \in (1.23, 1.61)$, we get $t(\mu) >> \tau_3$ with a warn-
ing of accuracy. For \( \mu > 1.61 \) the integral (4) starts diverging and the convexity of the track does not allow the motion with zero initial speed. For \( \mu = 0.5 \), see the track-d, \( t = 0.6211 < \tau_4 \), here the initial part of the track is concave followed by a convex part.

We would like to remark that for both \( \lambda \) and \( \mu \) tracks, the numerical integration yields: \( \tau_B < t(0^+) \leq \tau_3 \) with only a warning of accuracy.

More interestingly, the tracks given by

\[
y(x) = (1 - x^\nu)^{1/\nu}, \nu > 0 \tag{8}
\]

From (3), we get \( t(\nu) = \tau_2 I(\nu) \) in terms of an interesting integral

\[
I(\nu) = \int_0^1 \frac{x^{2(\nu-1)} + (1-x^\nu)^2(1-\nu)/\nu}{x^{2(1-\nu)/\nu}[1-(1-x^\nu)^{1/\nu}]} dx \tag{9}
\]

The simple interesting cases are \( I(0) = 1, I(1) = 2\sqrt{2} \) and

\[
I(\nu) = \int_0^1 \frac{dx}{x^{(3\nu-2)/2}} < \infty \text{, if } \nu < 4/3 \tag{10}
\]

NIntegrate of Mathematica gives \( I(0 \leq \nu < \nu_c) = 1 \) and \( I(\nu_c < \nu < \nu_s) = 3 \) without a warning for any type of error and shows \( I(\nu > \nu_s) \) as divergent, where \( \nu_s = 1.22 \). \( I(\nu) \) for \( \nu \in (0, 1.22) \) has a jump discontinuity at \( \nu = \nu_c \), see Fig. 2.

For \( \nu > 1 \) the track becomes convex and \( t >> \tau_1 \), and when \( \nu > 1.22 \), the motion with zero initial speed is not possible as the integral (3) diverges. We find that when \( \nu \in (0.09653, 0.31749) \), \( \tau_4 < t < \tau_3 \) and when \( \nu \in (0.31749, 1) \), \( \tau_B < t(\nu) < \tau_3 \). But most remarkably, if \( \nu \in (0, 0.09653 = \nu_c) \), the track is concave but deep and steep then \( t = \tau_2 = 0.2258 \text{ sec} < \tau_B \), this is the time (= \( \frac{1}{\sqrt{2g}} = \tau_2 \)) that a particle would take to travel 1 meter horizontally with a speed equal \( \sqrt{2g} \text{ m/sec} \) which is acquired at the origin.

The function \( t(\nu) \) suffers a jump discontinuity at \( \nu = \nu_c \), see Fig. 3.

Though, the numerical computation underlying the result \( t(0 < 0.09653) < \tau_B \) in fig. 3 have been done using “NIntegrate” of Mathematica that worked error-free yet we decided to re-calculate \( t(\nu) \) (6) by splitting it as

\[
t(\nu) = \int_0^{\varepsilon(\nu)} F(x, y, y')dx + \int_{\varepsilon(\nu)}^1 F(x, y, y')dx,
\]

\[
\varepsilon(\nu) = 10^{-n(\nu)}, n(\nu) \in N. \tag{11}
\]

The first part in above, is very interesting.

FIG. 3: \( t(\nu) \) for the track (8). In the domains \( \nu = (0.09653, 0.361), (0.361, 1), (1, 1.22) \), the time of descent \( t(\nu) \) is \( \tau_2, (\tau_4, \tau_3), (\tau_B, \tau_4), \gg \tau_4 \). The discontinuity is at \( \nu = \nu_c = 0.09653 \). Motion is not possible with zero speed from the point A, when \( \nu > 1.22 \).
where the $x$ could be extremely small in a small domain but the range of $F(x)$ consisting of large numbers. We find that for every $\nu \geq 0.0014$ one can choose $\varepsilon(\nu)$ or $n(\nu)$ such that $I(\nu) = 3$. For instance, when $\nu = 0.01$, $n = 27$. For $\nu < 0.003$ $n = 98$. For $\nu = 0.002$, $n = 148$. For $\nu = 0.0015$, $n = 198$. For $\nu = 0.0014$, $n = 212$. We failed to find a suitable value of $n$ for $\nu < 0.0014$ to get $I(\nu) = 3$, hence $t = \tau_2 < \tau_B$. We would like to assert that $\nu = 0.0014$ defines the actual critical value $\nu_c = 0.0014$ below which the time of descent remains less than $\tau_B$, however the discontinuity in $t(\nu)$ in Fig. 3 has been pushed towards $\nu = 0$. One may wonders if $I(\nu)$ is discontinuous at $\nu = 0$ such that $I(0) = 1, I(0^+) = 3$ or equivalently

$$t(0) = \tau_2, \quad t(0^+) = 3\tau_2 > \tau_B,$$

(12)

is expected physically.

Finally, let us resolve the curious domain $\nu \in (0, 0.00114)$ for the $\nu$-track (8). In this case, the track becomes practically discontinuous as

$$y(x = 0) = 1, \quad y(x = 0 + \varepsilon) = 0,$$

(13)

but in the energy conservation condition (1), the particle has already been assumed to have the initial potential energy equal to $mgy(0)$. Next, the integral (6) determining time of descent, inherently starts integration from $x = 0 + \varepsilon$, leaving out $x = 0$. Consequently, the particle executes only the linear horizontal motion from $x = 0 + \varepsilon$ to $x = 1$ with the conditioned initial speed $v = \sqrt{2gy(0)}$ and takes time $t = \tau_2 < \tau_B$. One may call this trivial case as a mathematical Brachistochrone (MB). In this domain of $\nu$, the deep and steep $\nu$-track would lie below even the track-\textbf{f} in Fig.1. When the $\nu > 0.00114$ steepness of the track reduces, the particle performs almost vertical plus horizontal motion and we get $\tau_4 < t \leq \tau_3$, so on and so forth (see Fig. 3).

The next question is how the optimization of $t$ in the earlier treatments of Brachistochrone specially in textbooks [2] have ignored this trivial yet interesting track $y(x) = 0$ (13). In this regard, our discussion differs slightly from textbooks as we take the starting point as (0,0) but ours the point (0,1). So in our eq. (2), $1 - y(x)$ occurs instead of $y(x)$ as in books. Optimization of $t = \int_0^1 F(x, y, y')dx$ is done using Euler-Lagrange equation, where $F = \sqrt{1 + y'^2}/(1 - y(x))$.

$$\frac{\partial F}{\partial y} - \frac{d}{dx} \frac{\partial F}{\partial y'} = 0.$$  

(14)

If we multiply (9) by $y'$ on both sides, $y'$ must be non-zero and then we get

$$\frac{d}{dx} \left( F - y' \frac{\partial F}{\partial y'} \right) = 0 \Rightarrow F - y' \frac{\partial F}{\partial y'} = C$$

(15)

In text books one uses (15) instead of (14), to get the ordinary differential equation (ODE) as $(1 - y)(1 + y'^2) = C$, solving which in various ways one gets the acclaimed cycloid. Hence the condition that $y'(x) \neq 0$ rules out the trajectory $y(x) = c$ from the optimization
of the time of descent, without even a mention, though we can see it as mathematically consistent case giving $t < \tau_B$.

We have discussed the time of descent of a bead/particle on various tracks/wire on several tracks from point A to the point B. We found that even convex tracks can allow the side of the bead downwards with zero-initial speed up to a limited initial curvature of the track with $t >> \tau_3$. Concave tracks take time in the interval $(\tau_4, \tau_3)$. Between two slightly wavy tracks the one that has initial part as concave is faster. The mathematical Brachistochrone $y = 0$ pushed by an initial speed $v(x = 0) = 0 = \sqrt{2g}$ m/s is faster than even the cycloid track $\tau_2 < \tau_B$. The Numerical quadrature for $x \in (0,1)$ gives a surprising result wherein $t$ turned out to be lesser than $\tau_B$, but by splitting it in two parts (11), we could rescue this unphysical result to some extent. For both $\lambda$ and $\mu$ tracks, numerically $t(0^+)$ is found to be in $(\tau_4, \tau_3)$, with the warning of inaccuracy. However, for the $\nu$-track $t = \tau_2 < \tau_B$ when $0 < \nu < 0.0014$, we believe that in this domain the track collapses to the trivial yet interesting mathematical track $y = 0$ (13) and we have $t(0^+) = \tau_2$ instead of $3\tau_2 > \tau_B$. It could be challenging to develop a numerical quadrature for the integral (9) that gives $t(0^+) = 3\tau_2$.

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