On the origin of order from random two-body interactions

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We investigate the origin of order in the low-lying spectra of many-body systems with random two-body interactions. Contrary to the common belief our study based both on analytical as well as on numerical arguments shows that these are the higher \( J \)-sectors whose ground states are more orderly than the ones in the \( J = 0 \)-sector. A predominance of \( J = 0 \) ground states turns out to be the result of putting on together states with different characteristic energy scales from different \( J \)-sectors.

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The nature of mechanism generating order out of randomness constitutes one of the most fundamental issues of the contemporary physics. Theories based on various versions of ensembles of the random matrices provide one possible theoretical frame for studying such effects. In this context the recently identified \([1]\) preponderance of the \( J = 0 \) ground states in strongly interacting Fermi systems, such as atomic nuclei, arising from random two-body interactions came as a surprise since there is no obvious pairing character in the assumed random force. Various possible explanations of this effect have been tested \([2]\) with no success, however. One purpose of the present note to provide a consistent picture of the mechanism generating this effect.

Schematically, indicating nevertheless all the relevant ingredients, the interaction matrix elements \( v_{α,α'}^J \) of good total angular momentum \( J \) in the shell-model basis \( | α \rangle \) can be expressed as follows \([3]\):

\[
v_{α,α'}^J = \sum_{J'} \sum_{\nu} c_{Jα,ν}^{Jα'} g_{iν}^J.
\]

The summation runs over all combinations of the two-particle states \( | ν \rangle \) coupled to the angular momentum \( J' \) and connected by the two-body interaction \( g_{iν}^J \) radial parts of the corresponding two-body matrix elements while \( c_{Jα,ν}^{Jα'} \) globally represent elements of the angular momentum recoupling geometry.

In statistical ensembles of matrices the crucial factor determining the structure of eigenspectrum is the probability distribution \( P_V(v) \) of matrix elements \([3]\). Especially relevant are the tails of such distributions since they prescribe the probability of appearance of the large matrix elements. From the point of view of the mechanism producing the energy gaps they are most effective in generating a local reduction of dimensionality responsible for such effects. In principle, the probability distribution of the shell model matrix elements is prescribed by their general structure expressed by the eq. \([3]\), provided the probability distributions of both \( g_{iν}^J \) and \( c_{Jα,ν}^{Jα'} \) are known. In general terms this structure can be considered to have the following form:

\[
V = V_1 + V_2 + ... + V_N \tag{2}
\]

and each \( V_i \) to be a product of another two variables denoted as \( C_i \) and \( G_i \). By making use of the convolution theorem \([5]\) the probability distribution \( P_V(v) \) that \( V \) assumes a value equal to \( v \) can be expressed as:

\[
P_V(v) = F^{-1}[F(P_{V_1}(v_1)) \cdot F(P_{V_2}(v_2)) \cdot ... \cdot F(P_{V_N}(v_N))], \tag{3}
\]

where \( F \) denotes a Fourier transform, \( F^{-1} \) its inverse and \( P_{V_i}(v_i) \) the probability distributions of individual terms in eq. \([3]\). Taking in addition into account the fact that

\[
P_{V_1}(v_1) = \int dg_iP_{G_i}(g_i)P_{C_i}(v_1/g_i) \frac{1}{g_i} \tag{4}
\]

one can explicitly derive the form of \( P_V(v) \) in several cases. Assuming for instance that all the above constituents are identically Gaussian distributed (then, according to eq. \([3]\), \( P_{V_1}(v_1) = K_0(|v_1|)/\pi \) and thus \( F(P_{V_1}(v_1)) = 1/\sqrt{1+\omega^2} \)) one arrives at

\[
P_V(v) = \frac{|v|^{(N-1)/2}K_{(N-1)/2}(|v|)}{2^{(N-1)/2}\Gamma(N/2)\sqrt{\pi}}, \tag{5}
\]

where \( K \) stands for the modified Bessel function. Asymptotically, for large \( v \), this leads to

\[
P_V(v) \sim \exp(-|v|) |v|^{N/2-1}. \tag{6}
\]

For such a global estimate the identical Gaussian distribution of \( g_{iν}^J \) is consistent both with the Two-Body Random Ensemble (TBRE) \([1]\) and with the Random Quasiparticle Ensemble (RQE) \([4]\). The only anticipated difference originates from the fact that in the second case the variance of the distribution drops down with \( J' \) like the inverse of \( 2J' + 1 \) which is expected to result in a smaller effective \( N \) as compared to TBRE. By contrast, in both versions of the above random ensembles the geometry expressed by \( c_{Jα,ν}^{Jα'} \) enters explicitly. However, the complicated quasi-random coupling of individual spins is believed \([4]\) to result in the so-called geometric chaoticity \([5]\). If applicable indeed then this fact implies validity of the above estimate for \( P_V(v) \). Since this is an important element for the present considerations below we explicitly verify its range of applicability.

The model to be quantitatively explored here consists, similarly as in ref. \([1]\), of 6 identical particles (all single-particle energies are set to zero) operating in the \( sd \) shell.
From the nuclear spectroscopy point of view this can be identified as $^{22}\text{O}$. Fig. 1 shows distributions of the corresponding geometrical factors $c_{J'ii'}^{J_{\alpha}\alpha'}$ for $\alpha \neq \alpha'$ and for the relevant values of $J$ and $J'$.

which for large $v$ behaves like

$$ P_V(v) \sim \exp(-|v|^2). \quad (8) $$

An explicit calculation of the distribution of the shell model off-diagonal matrix elements for the various $J$-values based on the present model with two-body matrix elements drawn from RQE (TBRE results in similar relations among different $J$-sectors though the distributions are somewhat broader as compared to RQE) confirms the above analytical estimates as is illustrated in Fig. 2.
Indeed, such a distribution in the $J = 0$ sector resembles more a Gaussian and the tails of this distribution drop down faster as compared to the $J \neq 0$-sectors where the large $\tau$ tails drop down slower, as consistent with an exponential asymptotics of eq. (9). At the same time the $J \neq 0$ sectors are dominated by very small matrix elements to a larger degree than $J = 0$. The probability of appearance of a large off-diagonal matrix element which in magnitude overwhelms the remaining ones is thus greater for $J \neq 0$ than for $J = 0$. Such an effective reduction of the rank in the former case is expected to result in a stronger tendency to localization as compared to GOE. The corresponding characteristics can be quantified in terms of the information entropy $K^J_l = -\sum_{\alpha=1}^{M_J} |a^J_{l,\alpha}|^2 \ln |a^J_{l,\alpha}|^2$ of an eigenstate labelled by $l$ from the $J$-sector. The coefficients $a^J_{l,\alpha}$ denote the eigenvector components in the basis $|\alpha\rangle$. Since the definition of $K^J_l$ involves the total number of states $M_J$ which differ for different $J$’s and since the appropriate reference for our considerations is GOE, we normalise $K^J_l$ to the GOE limit of this quantity which reads $K_{GOE}^J = \psi(M_J/2 + 1) - \psi(3/2)$, where $\psi$ is the digamma function. Within our model the so-calculated and RQE ensemble averaged quantity for all the states versus their corresponding energies $E^J_l$ is illustrated in Fig. 3. As anticipated, it is not $J = 0$ whose lowest eigenstate comes out most localised, i.e., most regular. The lowest states for several higher $J$ values deviate much more from GOE. This in particular applies to $J = 2$ and, especially, to $J = 4$. This thus indicates more favorable conditions for the emergence of energy gaps for larger $J$ than for $J = 0$, contrary to the numerical outcome of ref. [1]. That our numerical procedure is equivalent to that of ref. [1] is confirmed by the fact that we can reproduce the results when calculating the same quantities. The question thus arises as how to reconcile such conflicting conclusions.

In fact Fig. 3 provides one hint of what might be the reason. The $J = 0$ states are spread over the broadest energy interval even though the number of states ($M_0 = 14$) is here significantly smaller than for several larger $J$ values ($M_1 = 19, M_2 = 33, M_3 = 29, M_4 = 26$). As a result, the average separation between the states is a factor of few larger for $J = 0$ than for the remaining ones. Putting on the states from different $J$-sectors together, as is done in ref. [1], is thus likely to hide the genuine character of the effect under consideration.

In order to illustrate the consequence of such a procedure in Fig. 4 (dashed line) we show distributions of the ground state ($E^J_1$) gaps $s^J = (E^J_2 - E^J_1)/D^J$, where similarly as in ref. [1] $D^J = <E^J_3 - E^J_2>$, for each $J$ individually.

As it is clearly seen the $J = 0$-sector no longer distinguishes significantly from the remaining ones. From the point of view of our considerations presented above one would however expect an even reduced probability for occurrence of the large ground state energy gaps in this particular sector as compared to the $J > 0$ ones. As the solid lines in Fig. 4 indicate such an effect does indeed take place when $D^J$ in eq. (11) is replaced by $\tilde{D}^J = <E^J_{M_J} - E^J_2> / (M_J - 2)$.

Actually the definition of $D^J$ as specified by eq. (12) looks somewhat arbitrary. It seems more appropriate and more consistent with the above global considerations to relate the ground state energy gap just to the average global level spacing among the remaining states, characteristic for a given $J$.

Finally one may ask a question why this tendency does not extend to even higher $J$-values. In this connection
one has to remember that the off-diagonal matrix elements is not the only relevant element. These are the diagonal matrix elements which constitute the driving term. Irrespective of the value of \( J \) their distribution is always Gaussian-like. This can be observed numerically and is consistent with arguments formulated in terms of eqs. (1 - 8) since the geometrical factors \( c_{ji\alpha}^{J\alpha} \) entering the diagonal matrix elements are always nonnegative. Increasing however \( J \) beyond 4 results in a significant reduction of the variance of \( \mathcal{P}_V(v) \) for the diagonal matrix elements and consequently a larger fraction of the off-diagonal matrix elements becomes effective in mixing the basis states. Thus the effect under investigation results from an interplay between distributions of the diagonal and off-diagonal matrix elements and their relative magnitudes.

![Graph of distributions](image)

**Fig. 4** Distributions of ground state energy gaps \( s^J \) as defined by the eq. (1) for successive \( J \)'s. The dashed line uses \( D^J \) defined by eq. (12) while the solid line the one defined by eq. (13).

In conclusion, the present investigation based both on theoretical as well as on numerical arguments clearly shows that the many-body problems described in terms of various variants of the two-body random ensembles (like RQE or TBRE) develop quantitatively well identified deviations from the GOE. These deviations quantified in terms of localisation or of energy gaps point to the angular momenta between 2 and 4 as those \( J \)-sectors whose ground states are ordered most. From this perspective the current interpretation of the numerical results of ref. [1] is thus an artifact of mixing states with different characteristic energy scales from different \( J \)-sectors. As a side remark it seems also appropriate to notice at this point that the arguments going in parallel to eqs. (1-8) provide a more adequate approach towards understanding the distribution of matrix elements in realistic nuclear shell-model calculations than the ones based on multipole expansion [8].

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