Abstract
This is a tutorial and survey paper on Generative Adversarial Network (GAN), adversarial autoencoders, and their variants. We start with explaining adversarial learning and the vanilla GAN. Then, we explain the conditional GAN and DCGAN. The mode collapse problem is introduced and various methods, including minibatch GAN, unrolled GAN, BourGAN, mixture GAN, D2GAN, and Wasserstein GAN, are introduced for resolving this problem. Then, maximum likelihood estimation in GAN are explained along with f-GAN, adversarial variational Bayes, and Bayesian GAN. Then, we cover feature matching in GAN, InfoGAN, GRAN, LSGAN, energy-based GAN, CatGAN, MMD GAN, LapGAN, progressive GAN, triple GAN, LAG, GMAN, AdaGAN, CoGAN, inverse GAN, BiGAN, ALI, SAGAN, Few-shot GAN, SinGAN, and interpolation and evaluation of GAN. Then, we introduce some applications of GAN such as image-to-image translation (including PatchGAN, CycleGAN, DeepFaceDrawing, simulated GAN, interactive GAN), text-to-image translation (including StackGAN), and mixing image characteristics (including FineGAN and MiXNMATCH).

1. Introduction
Suppose we have a generative model which takes a random noise as input and generates a data point. We want the generated data point to be of good quality; hence, we should somehow judge its quality. One way to judge it is to observe the generated sample and assess its quality visually. In this case, the judge is a human. However, we cannot take derivative of human’s judgment for optimization. Generative Adversarial Network (GAN), proposed in (Goodfellow et al., 2014), has the same idea but it can take derivative of the judgment. For that, it uses a classifier as the judge rather than a human. Hence, we have a generator generating a sample and a binary classifier (or discriminator) to classify the generated sample as a real or generated sample. This classifier can be a pre-trained network which is already trained by some real and generated (fake) data points. However, GAN puts a step ahead and lets the classifier be trained simultaneously with training the generator. This is the core idea of adversarial learning with the classifier, also called the discriminator, and the generator compete each other; hence, they make each other stronger gradually by this competition (Goodfellow et al., 2020).
It is noteworthy that the term “adversarial” is used in two main streams of research in machine learning and they
should not be confused. These two research areas are:

- Adversarial attack, also called learning with adversarial examples or adversarial machine learning. This line of research inspects some examples which can be changed slightly but wisely to fool a trained learning model. For example, perturbation of some specific pixels in the input image may change the decision of learning model. The reason for this can be analyzed theoretically. Some example works in this area are (Huang et al., 2011; Moosavi-Dezfooli et al., 2016; Kurakin et al., 2017a;b; Madry et al., 2018).

- Adversarial learning for generation. This line of research is categorized as generative models (Ng & Jordan, 2002) and/or methods based on that. GAN is in this line of research. This paper focuses on this research area.

Another good tutorial on GAN is (Goodfellow, 2016) but it does not cover most recent methods in adversarial learning. Also, an honorary introduction of GAN, by several main contributors of GAN, is (Goodfellow et al., 2020). Some other existing surveys on GAN are (Wang et al., 2017; Creswell et al., 2018; Gonog & Zhou, 2019; Hong et al., 2019; Pan et al., 2019). This paper is a tutorial and survey on GAN and its variants.

**Required Background for the Reader**

This paper assumes that the reader has general knowledge of calculus, probability, linear algebra, and basics of optimization.

2. Generative Adversarial Network (GAN)

2.1. Adversarial Learning: The Adversarial Game

The original GAN, also called the vanilla GAN, was proposed in (Goodfellow et al., 2014). Consider a \(d\)-dimensional dataset with \(n\) data points, i.e., \(\{x_i \in \mathbb{R}^d\}_{i=1}^n\). In GAN, we have a generator \(G\) which takes a \(p\)-dimensional random noise \(z \in \mathbb{R}^p\) as input and outputs a \(d\)-dimensional generated point \(x \in \mathbb{R}^d\). Hence, it is the mapping \(G : z \rightarrow x\) where:

\[
G(z) = x. \tag{1}
\]

The random noise can be seen as a latent factor on which the generated data point is conditioned. The probabilistic graphical model of generator is a variable \(x\) conditioned on a latent variable \(z\) (see (Goodfellow, 2016, Fig. 13) for its visualization).

Let the distribution of random noise be denoted by \(z \sim p_z(z)\). We want the generated \(\hat{x}\) to be very similar to some original (or real) data point \(x\) in the dataset. We need a module to judge the quality of the generated point to see how similar it is to the real point. This module can be a human but we cannot take derivative of human’s judgment for optimization! A good candidate for the judge is a classifier, also called the discriminator. The discriminator (also called the critic), denoted by \(D : x \rightarrow [0, 1]\), is a binary classifier which classifies the generated point as a real or generated point:

\[
D(x) := \begin{cases} 
1 & \text{if } x \text{ is real,} \\
0 & \text{if } x \text{ is generated (fake).} 
\end{cases} \tag{2}
\]

The perfect discriminator outputs one for real points and zero for generated points. The discriminator’s output is in the range \([0, 1]\) where the output for real data is closer to one and the output for fake data is closer to zero. If the generated point is very good and closely similar to a real data point, the classifier may make a mistake and outputs a value close to one for it. Therefore, if the classifier makes a mistake for the generated point, the generator has done a good job in generating a data point.

The discriminator can be pre-trained but we can make the problem more sophisticated. Let us train the discriminator simultaneously while we are training the generator. This makes the discriminator \(D\) and the generator \(G\) stronger gradually while they compete each other. On one hand, the generator tries to generate realistic points to fool the discriminator and make it a hard time to distinguish the generated point from a real point. On the other hand, the discriminator tries to discriminate the fake (i.e., generated) point from a real point. When one of them gets stronger in training, the other one tries to become stronger to be able to compete. Therefore, there is an adversarial game between the generator and the discriminator. This game is zero-sum because whatever one of them loses, the other wins.

2.2. Optimization and Loss Function

We denote the probability distributions of dataset and noise by \(p_{\text{data}}(x)\) and \(p_z(z)\), respectively. The structure of GAN is depicted in Fig. 1. As the figure shows, the discriminator is trained by real points from dataset as well as generated points from the generator. The discriminator and generator are trained simultaneously. The optimization loss function
for both the discriminator and generator is:

\[
\min_D \max_G V(D, G) := \mathbb{E}_{x \sim p_{\text{data}}(x)} \left[ \log \left( D(x) \right) \right] + \mathbb{E}_{z \sim p_z(z)} \left[ \log \left( 1 - D(G(z)) \right) \right],
\]

where \( \mathbb{E}[\cdot] \) denotes the expectation operator and the loss function \( V(D, G) \) is also called the value function of the game. In practice, we can use the Monte Carlo approximation (Ghojogh et al., 2020) of expectation where the expectations are replaced with averages over the mini-batch.

The first term in Eq. (3) is expectation over the real data. This term is only used for the discriminator while it is a constant for the generator. According to Eq. (2), \( D(x) \) outputs one (the larger label) for the real data; therefore, the discriminator maximizes this term because it assigns the larger label to the real data.

The second term in Eq. (3) is expectation over noise. It inputs the noise \( z \) to the generator to have \( G(z) \). The output of generator, which is the generated point, is fed as input to the discriminator (see Fig. 1) to have \( D(G(z)) \). The discriminator wants to minimize \( D(G(z)) \) because the smaller label is assigned to the generated data, according to Eq. (2). In other words, the discriminator wants to maximize \( 1 - D(G(z)) \). As logarithm is a monotonic function, we can say that the discriminator wants to maximize \( \mathbb{E}_{z \sim p_z(z)} \log(1 - D(G(z))) \) which is the second term in Eq. (3). As opposed to the discriminator, the generator minimizes \( \mathbb{E}_{z \sim p_z(z)} \log(1 - D(G(z))) \) which is the second term in Eq. (3). This is because the generator wants to fool the discriminator to label the generated data as real data.

The Eq. (3) is a minimax optimization problem (Du & Pardalos, 2013) and can be solved using alternating optimization (Ghojogh et al., 2021c) where we optimize over \( D \) and over \( G \) iteratively until convergence (i.e., Nash equilibrium). The original GAN (Goodfellow et al., 2014) uses a step of stochastic gradient descent (Ghojogh et al., 2021c) for updates of each variable in the alternating optimization.

If we denote the loss function in Eq. (3) by \( V(D, G) \), the alternating optimization is done as:

\[
D^{(k+1)} := D^{(k)} + \eta^{(k)} \frac{\partial}{\partial D} V(D, G^{(k)}),
\]

\[
G^{(k+1)} := G^{(k)} - \eta^{(k)} \frac{\partial}{\partial G} V(D^{(k+1)}, G),
\]

where \( k \) is the index of iteration and \( \eta^{(k)} \) is the learning rate at iteration \( k \). Throughout this paper, derivatives w.r.t. \( D \) and \( G \) mean the derivatives w.r.t. the parameters (weights) of \( D \) and \( G \) networks, respectively. Eqs. (4) and (5) are one step of gradient ascent and gradient descent, respectively.

Note that the gradients here are the average of gradients in the mini-batch. Every mini-batch includes both real and generated data. The paper (Goodfellow et al., 2014) suggests that Eq. (4) can be performed for several times before performing Eq. (5); however, the experiments of that paper perform Eq. (4) for only one time before performing Eq. (5). Also note that another way to solve the optimization problem in GAN is simultaneous optimization (Mescheder et al., 2017b) in which Eqs. (4) and (5) are performed at the same time and not one after the other.

**Remark 1** (Minimax versus maximin in GAN (Goodfellow, 2016, Section 5)). We saw in Eq. (3) that the optimization of GAN is a minimax problem:

\[
\min_D \max_G V(D, G).
\]

By changing the order of optimization, one can see GAN as a maximin problem (Goodfellow, 2016):

\[
\max_G \min_D V(D, G).
\]

In fact, under some conditions, Eqs. (6) and (7) are equivalent (Du & Pardalos, 2013).

### 2.3. Network Structure of GAN

In practice, the discriminator and generator are two (deep) neural networks. The structure of GAN is depicted in Fig. 1. The first layer of discriminator network is \( d \)-dimensional and its last layer is one dimensional with scalar output. In the original GAN, maxout activation function (Goodfellow et al., 2013) is used for all layers except the last layer which has the sigmoid activation function to output a probability to model Eq. (2). The closer the output of \( D \) to one, the more probable its input is to be real.

The generator network has a \( p \)-dimensional input layer for noise and a \( d \)-dimensional output layer for generating data. In the generator, a combination of ReLU (Nair & Hinton, 2010) and sigmoid activation functions are used. The space of noise as the input to the generator is called the latent space or the latent factor. Each of the Eqs. (4) and (5) are performed using backpropagation in the neural networks.

### 2.4. Optimal Solution of GAN

**Theorem 1** ((Goodfellow et al., 2014, Proposition 1)). For a fixed generator \( G \), the optimal discriminator is:

\[
D^*(x) = \frac{p_{\text{data}}(x)}{p_{\text{data}}(x) + p_g(x)},
\]

where \( p_{\text{data}}(x) \) is the probability distribution of the real dataset evaluated at point \( x \) and \( p_g(x) \) is the probability distribution of output of generator evaluated at point \( x \).

**Proof.** According to the definition of expectation, the loss
The relation of distributions of input and output of generators is:

\[ V(D, G) = \int_x p_{\text{data}}(x) \log(D(x)) dx + \int_z p_z(z) \log(1 - D(G(z))) dz. \]

According to Eq. (1), we have:

\[ G(z) = x \implies z = G^{-1}(x) \implies dz = (G^{-1})'(x)dx, \]

where \((G^{-1})'(x)\) is the derivative of \((G^{-1})(x)\) with respect to (w.r.t.) \(x\). Hence:

\[ V(D, G) = \int_x p_{\text{data}}(x) \log(D(x)) dx + \int_z p_z(G^{-1}(x)) \log(1 - D(x))(G^{-1})'(x)dx. \]

The relation of distributions of input and output of generator is:

\[ p_g(x) = p_z(z) \times G^{-1}(x) = p_z(G^{-1}(x)) G^{-1}(x), \]

where \(G^{-1}(x)\) is the Jacobian of distribution at point \(x\). Hence:

\[ V(D, G) = \int_x p_{\text{data}}(x) \log(D(x)) dx + \int_z p_g(x) \log(1 - D(x))dx \]

\[ = \int_x \left( p_{\text{data}}(x) \log(D(x)) + p_g(x) \log(1 - D(x)) \right) dx. \]  

(10)

For optimization in Eq. (3), taking derivative w.r.t. \(D(x)\) gives:

\[
\frac{\partial V(D, G)}{\partial D(x)} = \frac{p_{\text{data}}(x) \log(D(x)) + p_g(x) \log(1 - D(x))}{D(x)}
\]

\[ \overset{\text{(a)}}{=} \frac{p_{\text{data}}(x) - p_g(x)}{D(x) - 1 - D(x)} \]

\[ \overset{\text{(b)}}{=} \frac{p_{\text{data}}(x)(1 - D(x)) - p_g(x)D(x)}{D(x)(1 - D(x))} = 0 \]

\[ \Rightarrow p_{\text{data}}(x) - p_{\text{data}}(x)D(x) - p_g(x)D(x) = 0 \]

\[ \Rightarrow D(x) = \frac{p_{\text{data}}(x)}{p_{\text{data}}(x) + p_g(x)}, \]

where (a) is because taking derivative w.r.t. \(D(x)\) considers a specific \(x\) and hence it removes the integral (summation). Q.E.D.

**Theorem 2** ((Goodfellow et al., 2014, Theorem 1)). The optimal solution of GAN is when the distribution of generated data becomes equal to the distribution of data:

\[ p_g^*(x) = p_{\text{data}}(x). \]  

(11)

**Proof.** Putting the optimum \(D^*(x)\), i.e. Eq. (8), in Eq. (10) gives:

\[ V(D^*, G) \]

\[ = \int_x \left( p_{\text{data}}(x) \log(D^*(x)) + p_g(x) \log(1 - D^*(x)) \right) dx \]

\[ \overset{\text{(8)}}{=} \int_x \left[ p_{\text{data}}(x) \log \left( \frac{p_{\text{data}}(x)}{p_{\text{data}}(x) + p_g(x)} \right) + p_g(x) \log \left( \frac{p_g(x)}{p_{\text{data}}(x) + p_g(x)} \right) \right] dx \]

\[ = \int_x \left[ p_{\text{data}}(x) \log \left( \frac{p_{\text{data}}(x)}{2 \times p_{\text{data}}(x) + p_g(x)} \right) + p_g(x) \log \left( \frac{p_g(x)}{2 \times p_{\text{data}}(x) + p_g(x)} \right) \right] dx \]

\[ = \int_x \left[ p_{\text{data}}(x) \log \left( \frac{p_{\text{data}}(x) + p_g(x)}{2} \right) + p_g(x) \log \left( \frac{p_g(x)}{p_{\text{data}}(x) + p_g(x)} \right) \right] dx - \log(4) \]

\[ \overset{\text{(a)}}{=} KL \left( p_{\text{data}}(x) \middle| \frac{p_{\text{data}}(x) + p_g(x)}{2} \right) + KL \left( p_g(x) \middle| \frac{p_{\text{data}}(x) + p_g(x)}{2} \right) - \log(4), \]  

(12)

where (a) is because of the definition of KL divergence. The Jensen-Shannon Divergence (JSD) is defined as (Nielsen, 2010):

\[ \text{JSD}(P || Q) := \frac{1}{2} KL(P || \frac{1}{2}(P + Q)) + \frac{1}{2} KL(Q || \frac{1}{2}(P + Q)), \]

(13)

where \(P\) and \(Q\) denote the probability densities. In contrast to KL divergence, the JSD is symmetric. The obtained \(V(D^*, G)\) can be restated as:

\[ V(D^*, G) = 2 \text{JSD}(p_{\text{data}}(x) || p_g(x)) - \log(4), \]

(14)

According to Eq. (3), the generator minimizes \(V(D^*, G)\). As the JSD is non-negative, the above loss function is minimized if we have:

\[ \text{JSD}(p_{\text{data}}(x) || p_g(x)) = 0 \implies p_{\text{data}}(x) = p_g(x). \]

Q.E.D.
Corollary 1 ((Goodfellow et al., 2014, Theorem 1)). From Eqs. (11) and (14), we conclude that the optimal loss function in GAN is:

\[ V(D^*, G^*) = -\log(4). \]  

(15)

It is noteworthy that one can generalize Eq. (13) in GAN to (Huszár, 2015):

\[
\text{JSD}_\pi(P||Q) := \pi \text{KL}(P||P + (1 - \pi)Q) \\
\quad + (1 - \pi) \text{KL}(Q||P + (1 - \pi)Q),
\]  

(16)

with \( \pi \in (0, 1) \). Its special case is Eq. (13) with \( \pi = 0.5 \).

Corollary 2. From Eqs. (8) and (11), we conclude that at convergence (i.e., Nash equilibrium), the discriminator cannot distinguish between generated and real data:

\[
D^*(x) = 0.5, \quad \forall x \sim p_{\text{data}}(x),
\]

\[
D^*(x) = 0.5, \quad x = G^*(z), \forall z \sim p_z(z).
\]  

(17)

Lemma 1 (Label smoothing in GAN (Salimans et al., 2016, Section 3.4)). It is shown that replacing labels 0 and 1, respectively, with smoother values 0.1 and 0.9 (Szegedy et al., 2016) can improve neural network against adversarial attacks (Hazan et al., 2017). If we smooth the labels of discriminator for real and generated data to be \( \alpha \) and \( \beta \), respectively, the optimal discriminator becomes (Salimans et al., 2016):

\[
D^*(x) = \frac{\alpha p_{\text{data}}(x) + \beta p_g(x)}{p_{\text{data}}(x) + p_g(x)},
\]  

(18)

which generalizes Eq. (8). The presence of \( p_g(x) \) causes a problem because, for an \( x \) with small \( p_{\text{data}}(x) \) and large \( p_g(x) \), the point does not change generator well enough to get close to the real data. Hence, it is recommended to set \( \beta = 0 \) to have one-sided label smoothing. In this case, the optimal discriminator is:

\[
D^*(x) = \frac{\alpha p_{\text{data}}(x)}{p_{\text{data}}(x) + p_g(x)}.
\]  

(19)

Proof (sketch). Using \( \alpha \) and \( \beta \) in the proof of Theorem 1 results in Eq. (18).

2.5 Convergence and Equilibrium Analysis of GAN

Theorem 3 ((Goodfellow et al., 2014, Proposition 2)). If the discriminator and generator have enough capacity and, at every iteration of the alternating optimization, the discriminator is allowed to reach its optimum value as in Eq. (8), and \( p_g(x) \) is updated to minimize \( V(D^*, G) \) stated in Eq. (14), \( p_g(x) \) converges to \( p_{\text{data}}(x) \) as stated in Eq. (11).

Proof. The KL divergences in Eq. (12) are convex functions w.r.t. \( p_g(x) \). Hence, with sufficiently small updates of \( p_g(x) \), it converges to \( p_{\text{data}}(x) \). Note that Eq. (12), which we used here, holds if Eq. (8) holds, i.e., the discriminator is allowed to reach its optimum value. Q.E.D.

The GAN loss, i.e. Eq. (3), can be restated as (Nagarajan & Kolter, 2017):

\[
\min_G \max_{D} V(D, G) := \mathbb{E}_{x \sim p_{\text{data}}(x)} \left[ f(D(x)) \right] \\
\quad + \mathbb{E}_{z \sim p_z(z)} \left[ f\left(-D(G(z))\right) \right],
\]  

(20)

where \( f \) is the negative logistic function, i.e., \( f(x) := -\log(1 + \exp(-x)) \). In fact, the function \( f(\cdot) \) can be any concave function. This formulation is slightly different from the original GAN in the sense that, here, the discriminator \( D \) outputs a real-valued scalar (without any activation function) while the discriminator of Eq. (3) outputs values in the range \((0, 1)\) after a sigmoid activation function. If \( D \) outputs 0.5 and 0, it means that it is completely confused in Eqs. (3) and (20), respectively. The Eq. (20) is a concave-concave loss function in most of the domain of discriminator (Nagarajan & Kolter, 2017, Proposition 3.1).

Theorem 4 ((Nagarajan & Kolter, 2017, Theorem 3.1)). After satisfying several reasonable assumptions (see (Nagarajan & Kolter, 2017) for details), a GAN with loss function of Eq. (20) is locally exponentially stable.

Lemma 2 (Nash equilibrium in GAN (Farnia & Ozdaglar, 2020)). Nash equilibrium is the state where no player can improve its gain by choosing a different strategy. At the Nash equilibrium of GAN, we have:

\[
V(D, G^*) \leq V(D^*, G^*) \leq V(D^*, G),
\]  

(21)

which is obvious because we are minimizing and maximizing \( V(G, D) \) by the generator and discriminator, respectively, in Eq. (3).

Empirical experiments have shown that GAN may not reach its Nash equilibrium in practice (Farnia & Ozdaglar, 2020). Regularization can help convergence of GAN to the Nash equilibrium (Mescheder et al., 2018). It is shown in (Mescheder et al., 2018) that A effective regularization for GAN is noise injection (Ghojogh & Crowley, 2019) in which independent Gaussian noise is added to the training data points.

Definition 1 (Proximal equilibrium (Farnia & Ozdaglar, 2020)). We can use the proximal operator (Ghojogh et al., 2021c) in the loss function of GAN:

\[
\min_G \max_{D} \left(V_{\text{prox}}(D, G) := \max_{\tilde{D}}(V(D, G) - \lambda \| \tilde{D} - D \|_2^2) \right),
\]

where \( V(D, G) \) is defined in Eq. (3) and \( \lambda > 0 \) is the regularization parameter. The equilibrium of the game having this loss function is called the proximal equilibrium.

Theorem 5 (Convergence of GAN based on the Jacobian (Mescheder et al., 2017b)). Let the updated solution of GAN optimization at every iteration be obtained by some operator \( F(D, G) \), such as a step of gradient descent. The
convergence of GAN can be explained based on the Jacobian of \( F(D, G) \) with respect to \( D \) and \( G \). If the absolute values of some eigenvalues of the Jacobian are larger than one, GAN will not converge to the Nash equilibrium. If all eigenvalues have absolute values less than one, GAN will converge to the Nash equilibrium with a linear rate \( O(\lambda_{\text{max}}^{-k}) \) where \( \lambda_{\text{max}} \) is the eigenvalue with largest absolute eigenvalue and \( k \) is the iteration index. If all eigenvalues have unit absolute value, GAN may or may not converge.

The readers can refer to (Farnia & Tse, 2018) for duality in GAN, which is not explained here for brevity. Moreover, some papers have specifically combined GAN with game theory. Interested readers can refer to (Oliehoek et al., 2017; Arora et al., 2017; Unterthiner et al., 2018; Tembine, 2019).

2.6. Conditional GAN

As was explained before, in the original GAN, we randomly draw noise \( z \) from a prior distribution \( p_z(z) \) and feed it to the generator. The generator outputs a point \( x \) from the noise \( z \). Assume that the dataset with which GAN is trained has \( c \) number of classes. The original GAN generates points from any class and we do not have control to generate a point from a specific class. Although, note that after GAN is trained, the latent space for \( z \) is meaningful in the sense that every part of the latent space results in generation of some specific points from some class. However, the user cannot choose specifically what class to generate points from.

Conditional GAN (Mirza & Osindero, 2014), also called the conditional adversarial network, gives the user the opportunity to choose the class of generation of points. For the dataset \( \{x_i \in \mathbb{R}^n\}_{i=1}^n \), let the one-hot encoded class labels be \( \{y_i \in \mathbb{R}^c\}_{i=1}^n \). In conditional GAN, we use the following loss function instead of Eq. (3):

\[
\min_G \max_D V_C(D, G) := \mathbb{E}_{x \sim p_{\text{data}}(x)} \left[ \log (D(x|y)) \right] + \mathbb{E}_{z \sim p_z(z)} \left[ \log (1 - D(G(z|y))) \right].
\]

(22)

where the discriminator and generator are both conditioned on the labels. In practice, for implementing the loss function (22), we concatenate the one-hot encoded label \( y \) to the point \( x \) for the input to discriminator. We also concatenate the one-hot encoded label \( y \) to the noise \( z \) for the input to generator. For these, the input layers of discriminator and generator are enlarged to accept the concatenated inputs. In the test phase, the user choose the desired class label and the generator generates a new point from that class.

2.7. Deep Convolutional GAN (DCGAN)

Deep Convolutional GAN (DCGAN), proposed in (Radford et al., 2016), made GAN deeper and generated higher resolution images than GAN. It also showed that it is very hard to train a GAN.

2.7.1. NETWORK STRUCTURE

In DCGAN, we use an all-convolutional network (Springenberg et al., 2015) which replaces the pooling functions with strided convolutions. In this way, the network learns its own spatial downsampling. This network is used for both generator and discriminator. In DCGAN, we also have only convolutional layers in the input layer of generator and output layer of discriminator, without any fully-connected layer. This elimination of fully connected layers is inspired by (Mordvintsev et al., 2015).

We also apply batch normalization (Ioffe & Szegedy, 2015) to all layers except the last layer of generative and the first layer of discriminator. This is because batch normalization makes the mean of input to each neuron zero and its variance one; however, we should learn the mean and variance of data in the first layer of discriminator and the mean and variance of data should be reproduced by the last layer of generator. Batch normalization reduces the problem of mode collapse, which will be introduced later in Section 3 with the price of causing some fluctuations and instability (Radford et al., 2016).

Remark 2 (Virtual batch normalization (Salimans et al., 2016, Section 3.5)). Batch normalization has a problem; it makes the effect of every input \( x \) on network dependent on other inputs in the mini-batch. To not have this problem, Virtual Batch Normalization (VBN) fixes the mini-batches initially once before start of training; these mini-batches are called the reference batches. Every reference batch is normalized by only its own statistics (i.e., mean and covariance). vbn has been found to be effective in training of generator (Salimans et al., 2016).

In DCGAN, the last layer of generator has the hyperbolic tangent activation function and its other layers have the ReLU activation function (Nair & Hinton, 2010). As in GAN, the one-to-last layer of discriminator is flattened and connected to one neuron with the sigmoid activation function. In contrast to GAN, which uses the maxout activation function (Goodfellow et al., 2013) for discriminator activation function (see Section 2.3), DCGAN uses the leaky rectified activation function for discriminator.

2.7.2. VECTOR ARITHMETIC IN LATENT SPACE

DCGAN showed that we can generate images from a specific domain if we train GAN on that domain. For example, bedroom images were generated by DCGAN after being trained on a dataset of bedroom images. DCGAN also showed that the learned latent space is meaningful and we can do vector arithmetic in the latent space. Vector arithmetic in the latent space was previously used for showing the ability of Word2Vec (Mikolov et al., 2013). DCGAN
made it possible to do vector arithmetic in the latent space for images. An example of vector arithmetic by DCGAN is shown in Fig. 2. In the latent space, the latent variables corresponding to man with glasses, man without glasses, and woman without glasses are used. Using one latent point for each of these does not work very well. An average of three latent vectors for each has worked properly in practice. As Fig. 2 shows, vector arithmetic works because “man with glasses” minus “man without glasses” plus “woman without glasses” results in “woman with glasses”.

3. Mode Collapse Problem in GAN

3.1. Mode Collapse Problem

We expect from a GAN to learn a meaningful latent space of \( z \) so that every specific value of \( z \) maps to a specific generated data point \( x \). Also, nearby \( z \) values in the latent space should be mapped to similar but a little different generations. The mode collapse problem (Metz et al., 2017), also known as the Helvetica scenario (Goodfellow, 2016), is a common problem in GAN models. It refers to when the generator cannot learn a perfectly meaningful latent space as was explained. Rather, it learns to map several different \( z \) values to the same generated data point. Mode collapse usually happens in GAN when the distribution of training data, \( p_{\text{data}}(x) \), has multiple modes.

An example of mode collapse is illustrated in Fig. 3 which shows training steps of a GAN model when the training data is a mixture of Gaussians (Metz et al., 2017). In different training steps, GAN learns to map all \( z \) values to one of the modes of mixture. When the discriminator learns to reject generation of some mode, the generator learns to map all \( z \) values to another mode. However, it never learns to generate all modes of the mixture. We expect GAN to map some part, and not all parts, of the latent space to one of the modes so that all modes are covered by the whole latent space.

Another statement of the mode collapse is as follows (Xiao et al., 2018, Fig. 1). Assume \( p_{\text{data}}(x) \) is multi-modal while the latent space \( p_z(z) \) has only one mode. Consider two points \( x_1 \) and \( x_2 \) from two modes of data whose corresponding latent noises are \( z_1 \) and \( z_2 \), respectively. According to the mean value theorem, there is a latent noise with the absolute gradient value \( \|x_2 - x_1\|/\|z_2 - z_1\| \) where \( \|\cdot\| \) is a norm. As this gradient is Lipschitz continuous, when the two modes are very far resulting in a large \( \|x_2 - x_1\| \), we face a problem. In this case, the latent noises between \( z_1 \) and \( z_2 \) generate data points between \( x_1 \) and \( x_2 \) which are not in the modes of data and thus are not valid.

There exist various methods which resolve the mode collapse problem in GAN and adversarial learning. Some of them make the latent space a mixture distribution to imitate generation of the multi-modal training data. Some of them, however, have other approaches. In the following, we introduce the methods which tackle the mode collapse problem.

3.2. Minibatch GAN

One way to resolve the mode collapse problem is minibatch discrimination (Salimans et al., 2016, Section 3.2). In this method, the discriminator considers multiple data points in combination rather than separately. This avoids the mode collapse in generator. Suppose \( f(x_i) \in \mathbb{R}^a \) is the feature vector of one of the intermediate layers, with \( a \) neurons, in the discriminator for the data point \( x_i \). The data point \( x_i \) is either real or generated (fake). We multiply \( f(x_i) \) by a tensor \( T \in \mathbb{R}^{a \times b \times c} \) to obtain \( \mathbb{R}^{b \times c} \ni M_i := (T^T f(x_i))^T \). If there are \( |B| \) points in the mini-batch \( B \), we can calculate \( \{M_i\}_{i=1}^{\vert B \vert} \). Let \((M_i)_L\) denote the \( l \)-th row.
of $M_i$. We define (Salimans et al., 2016):

$$
\mathbb{R} \ni c_l(x_i, x_j) := \exp(-||(M_i)_{l}: -(M_J)_{l}||_1),
$$

$$
\forall l \in \{1, \ldots, b\},
$$

$$
\mathbb{R} \ni o(x_i) := \sum_{j=1}^{[B]} c_l(x_i, x_j), \quad \forall i \in \{1, \ldots, [B]\},
$$

$$
\mathbb{R}^b \ni o(x_i) := [o(x_{i1}), \ldots, o(x_{ib})]^T, \quad \forall i,
$$

$$
\mathbb{R}^{[B] \times b} \ni o(X) := [o(x_1), \ldots, o(x_{[B]})]^T.
$$

For every point $x_i$ within the mini-batch, we concatenate $o(x_i)$ with $f(x_i)$ and feed it to the next layer, rather than feeding merely $f(x_i)$. In other words, the additional features $o(x_i)$ are side information for better training of discriminator (which makes the generator also stronger in the game). This procedure, in the discriminator, is performed for both mini-batches of real and generated data.

### 3.3. Unrolled GAN

Unrolled GAN (Metz et al., 2017) uses $\Delta$ levels of unrolling of discriminator when updating the generator. The alternating optimization in unrolled GAN is:

$$
D^{(k+1)} := D^{(k)} + \eta \frac{\partial}{\partial D} \left( V(D, G^{(k)}) \right),
$$

$$
D_0 := D^{(k+1)},
$$

$$
D^{(k+1)} := D^{(k)} + \eta \frac{\partial}{\partial \delta} \left( V(D, G^{(k)}) \right), \quad \forall \delta \in \{0, \ldots, \Delta - 1\}
$$

$$
V_D(D^{(k+1)}, G) := V(D_{\Delta}, G),
$$

$$
G^{(k+1)} := G^{(k)} - \eta \frac{\partial}{\partial G} \left( V_D(D^{(k+1)}, G) \right),
$$

where Eq. (25) unrolls the parameters of discriminator for $\Delta$ times before using it for updating the generator. The loss function $V(D_{\Delta}, G)$ is called the surrogate loss. As was mentioned in Section 2.2, the original GAN can update the discriminator itself for several time before updating the generator (Goodfellow et al., 2014). Note that Eq. (25) is different from updating the discriminator for several times, as done in GAN, because it does not update the discriminator but it is used in updating the generator in Eq. (26). The gradient for generator in Eq. (26) can be simplified as:

$$
\frac{\partial}{\partial G} \left( V_D(D^{(k+1)}, G) \right) \quad (25)
$$

$$
\frac{\partial}{\partial G} \left( V_D(D_{\Delta}, G) \right) + \frac{\partial}{\partial D_{\Delta}} \left( V_D(D_{\Delta}, G) \right) \frac{\partial D_{\Delta}}{\partial G}.
$$

The gradient for generator in the original GAN has only the first term. The second term in the above equation captures the information of changes of discriminator w.r.t. changes in the generator. This reduces the problem of mode collapse which exists in GAN.

### 3.4. Bourgain GAN (BourGAN)

Bourgain GAN (BourGAN) (Xiao et al., 2018) learns a mixture distribution (Ghojogh et al., 2019) in the latent space to resolve the problem of mode collapse and learn to generate multi-modal data. If the size of dataset, $n$, is large, it samples $m$ points from dataset $\{x_i\}_{i=1}^{n}$ where $m \ll n$. Let $\{\tilde{x}_i\}_{i=1}^{m}$ denote the sampled data. The idea of BourGAN is based on the Bourgain embedding (Bourgain, 1985) which enables embedding a dataset of size $m$ into a $O(\log^2(m))$-dimensional $\ell_2$ norm embedding space with high probability. An improved Bourgain embedding is as follows.

**Theorem 6** ((Xiao et al., 2018, Corollary 2)). For a dataset $\{\tilde{x}_i\}_{i=1}^{m}$ in a space with norm $||.||$, there exists a mapping $f$ from the data space to a $O(\log^2(m))$-dimensional embedding space which preserves the local distances:

$$
||\tilde{x}_i - \tilde{x}_j|| \leq \|f(\tilde{x}_i) - f(\tilde{x}_j)\| \leq \alpha ||\tilde{x}_i - \tilde{x}_j||,
$$

where $\alpha \leq O(\log(m))$. This embedding is achieved by Bourgain embedding (Bourgain, 1985) followed by random projection (Johnson & Lindenstrauss, 1984; Ghojogh et al., 2021b). This combined embedding can be found in (Xiao et al., 2018, Appendix A). This embedding requires computing pairwise distances. The sampling of $m$ points from the $n$ data points is to make this embedding feasible. This sampling does not affect the characteristics of pairwise distances in data if $m$ is sufficiently large (Xiao et al., 2018, Theorem 4). We embed all sampled points to obtain $\{f(\tilde{x}_i)\}_{i=1}^{m}$. The distance characteristics of data is preserved by this embedding (Xiao et al., 2018, Theorem 5).

As the next step in BourGAN, we sample from the embedding points uniformly, i.e., $\mu \sim \{f(\tilde{x}_i)\}_{i=1}^{m}$. Then, we...
sample the latent noise from a multivariate Gaussian distribution with mean $\mu$, i.e., $z \sim \mathcal{N}(\mu, 0.01I)$. This procedure models a multi-modal mixture of Gaussians in the latent space of GAN. Note that it does not imply a mixture of $m$ modes because if several $f(\bar{x}_j)$ are close to each other, they can be considered as one mode. The multi-modality of the latent space is preserved to be the same as the multimodality of data (Xiao et al., 2018, Eq. 6).

The loss function of BourGAN regularizes the cost of generator so that the generator preserves the distances of generated points compared to the distances of corresponding noises. In this way, the multi-modality of latent space will also appear in the distribution of generated data $p_g(x)$.

$$\min_D V_{\text{BourGAN}}(D) := V(D, G),$$

$$\max_G V_{\text{BourGAN}}(G) := V(D, G) + \lambda \mathbb{E}_{z_1, z_2 \sim p_z(z)} \left[ \left( \log(\|G(z_1) - G(z_2)\|) \right) - \log(\|z_1 - z_2\|)^2 \right].$$

where $V(D, G)$ is defined in Eq. (3), $\lambda > 0$ is the regularization parameter, and $\|\cdot\|$ is some norm such as $\ell_2$ norm.

### 3.5. Mixture GAN (MGAN)

Mixture GAN (MGAN) (Hoang et al., 2018) overcomes the mode collapse issue by assuming that the distribution of latent space, from which noise is sampled, is a mixture distribution (Ghojogh et al., 2019) rather than a single distribution. It also enlarges the divergence of latent distributions in the mixture so that each of them covers a different mode for generation of data. The structure of MGAN is shown in Fig. 4. It has $k$ generators $\{G_j\}_{j=1}^k$, a discriminator $D$, and a classifier $C$. In terms of having a classifier, it is similar to triple GAN (Li et al., 2017a) (see Section 5.9). Every generator $G_j$ takes care of the $j$-th latent distribution in the mixture. The difference of MGAN from BourGAN (see Section 3.4) is that MGAN associates a generator to every mode while BourGAN has one generator with a mixture latent distribution.

Let $\pi_j$ be the mixing probability for the $j$-th component in the mixture. We denote the distribution of generated data from the mixture latent distribution and the $j$-th component in the mixture by $p_{g_j}(x)$ and $p_{g_j}(x)$, respectively. The discriminator tries to judge whether a data point is real, $x \sim p_{\text{data}}(x)$, or generated from one of the modes in the mixture, i.e., $x = G_j(z)$. At every iteration of the alternating optimization, the index $j$ of the selected $G_j$ for feeding to discriminator $D$ is sampled from a multinomial distribution $\text{Mult}(\pi)$ where $\pi := [\pi_1, \ldots, \pi_k]^T$. The classifier $C$ classifies which one of the $k$ generators has generated the given data. The loss function of MGAN, as a multi-player minimax game, is:

$$\min_{\{G_j\}_{j=1}^k, C} \max_D V(D, C, G_1, \ldots, G_k) :=$$

$$\mathbb{E}_{x \sim p_{\text{data}}(x)} \left[ \log(D(x)) \right] + \mathbb{E}_{x \sim p_g(x)} \left[ \log(1 - D(x)) \right] - \lambda \sum_{j=1}^k \pi_j \mathbb{E}_{x \sim p_{g_j}(x)} \left[ \log(C_j(x)) \right],$$

where $\lambda > 0$ is the regularization parameter and $C_j(x)$ is the probability that $x$ is generated by $G_j$. The last layer of the classifier $C$ has $k$ neurons with softmax activation function where $C_j$ is the output of $j$-th neuron in classifier after the activation function. The last term in the loss function maximizes the entropy of classification so that, by competition of generators and classifier, the generated data from various generators become separated gradually. In this way, generators will cover different modes of data, resolving the mode collapse problem.

**Theorem 7** (Hoang et al., 2018, Theorem 2). *After convergence (i.e., the Nash equilibrium) of MGAN, we have:

$$p_{g^*}(x) = p_{\text{data}}(x),$$

$$D^*(x) = \frac{p_{\text{data}}(x)}{p_{\text{data}}(x) + p_{g^*}(x)},$$

$$C^*(x) = \frac{\pi_j p_{g_j^*}(x)}{\sum_{l=1}^k \pi_l p_{g_l^*}(x)},$$

$$G^* := \arg\min_G \left( 2 \text{JSD}(p_{\text{data}}\|p_g) - \lambda \text{JSD}_z(p_{G_1}, \ldots, p_{G_k}) \right),$$

where:

$$\text{JSD}_\pi(p_{G_1}, \ldots, p_{G_k}) := \sum_{j=1}^k \pi_j \mathbb{E}_{x \sim p_{g_j}} \left[ \log \left( \frac{\pi_j p_{g_j}(x)}{\sum_{l=1}^k \pi_l p_{g_l}(x)} \right) \right] - \sum_{j=1}^k \pi_j \log(\pi_j).$$
mixtures are non-negative rather than being in range $\alpha, \beta$ where $\alpha, \beta$ are hyperparameters. Alternating optimization (Ghojogh et al., 2021c) between $D_1, D_2$, and $G$ solves the problem.

**Theorem 9** ([Nguyen et al., 2017b, Proposition 1 and Theorem 2]). **After convergence (i.e., Nash equilibrium) of D2GAN, we have:**

$$
D_1^*(x) = \frac{\alpha p_{data}(x)}{p_g(x)}, \quad D_2^*(x) = \frac{\beta p_g(x)}{p_{data}(x)}.
$$

 (**3.7. Wasserstein GAN (WGAN)**

Wasserstein GAN (WGAN) was proposed in (Arjovsky et al., 2017), developed from (Arjovsky & Bottou, 2017). The Wasserstein-1 or Earth-Mover distance between two distributions $p_{data}(x)$ and $p_g(x)$ is defined as:

$$
W(p_{data}(x), p_g(x)) := \inf_{\gamma \in \Pi(p_{data}(x), p_g(x))} \mathbb{E}_{(x, z) \sim \gamma} \left[ \| x - z \|_1 \right],
$$

where $\Pi(p_{data}(x), p_g(x))$ is the set of all joint distributions whose marginals are $p_{data}(x)$ and $p_g(x)$. By the Kantorovich-Rubinstein duality (Villani, 2009), the Wasserstein-1 distance is equivalent to:

$$
W(p_{data}(x), p_g(x)) = \sup_{\| D \|_{L_1} \leq 1} \left( \mathbb{E}_{x \sim p_{data}(x)} [D(x)] - \mathbb{E}_{z \sim p_g(z)} [D(z)] \right),
$$

where $\| D \|_{L_1} \leq 1$ is the $L_1$-Lipschitz functions $D : \mathcal{X} \rightarrow \mathbb{R}$. The gradient of the Wasserstein-1 distance w.r.t. the parameters of generator $G$ is (Arjovsky et al., 2017, Theorem 3):

$$
\frac{\partial W(p_{data}(x), p_g(x))}{\partial G} = -\mathbb{E}_{z \sim p_g(z)} \left[ \frac{\partial D(G(z))}{\partial G} \right].
$$

The function $D(.)$ plays the role of discriminator in WGAN. In an alternating optimization, we maximize the loss in Eq. (37) for the discriminator and minimize in a gradient descent step by the gradient of Eq. (38) for generator. In other words, the loss function of WGAN is:

$$
\min_G \max_{\| D \|_{L_1} \leq 1} \mathbb{E}_{x \sim p_{data}(x)} [D(x)] - \mathbb{E}_{z \sim p_g(z)} [D(z)],
$$
where $x \sim p_g(x)$ is generated from the generator, i.e., $x = G(z)$ in which $z$ is the latent noise. The weights of discriminator are clipped to $[-0.01, 0.01]$. The constraint $\|D\|_\infty \leq 1$ can be implemented by regularization with a gradient penalty (Gulrajani et al., 2017):

$$\min_G \max_{\|D\|_\infty \leq 1} \mathbb{E}_{x \sim p_{data}(x)}[D(x)] - \mathbb{E}_{x \sim p_g(x)}[D(x)]$$

$$- \lambda (\nabla_\tilde{x} D(\tilde{x}))_2^2 - 1)^2,$$

where $\lambda > 0$ is the regularization parameter and $\tilde{x}$ is uniformly sampled as an interpolation between the real data and the generated data:

$$\tilde{x} := \rho x + (1 - \rho) G(z),$$

in which $\rho \sim U(0, 1)$. Experiments show that WGAN resolves the mode collapse problem (Arjovsky et al., 2017).

### 4. Maximum Likelihood Estimation in GAN

In the following, we introduce the methods which relate GAN and Maximum Likelihood Estimation (MLE).

#### 4.1. Comparison of MLE and GAN

GAN is related to Noise-Contrastive Estimation (NCE) (Gutmann & Hyvärinen, 2010) and MLE, in the sense that they all optimize a distinguishing game value function (Goodfellow, 2015):

$$V(p_c, p_g) := \mathbb{E}_{x \sim p_{data}}[\log(p_c(y = 1|x))] + \mathbb{E}_{x \sim p_g}[\log(p_c(y = 0|x))].$$

(42)

where $p_g$ and $p_{data}$ denote the distributions of generated and real data, respectively, and $p_c(y|x)$ is the output probability of a classifier which judges whether a data point $x$ is generated or real. Let $\theta$ denote the parameters of $p_g(x)$ distribution. If $f(x) := \log(p_c(y = 0|x))$, GAN performs the following optimization for its generator (Goodfellow, 2015):

$$\min_\theta \mathbb{E}_{x \sim p_g}[f(x)]$$

(43)

$$\Rightarrow \frac{\partial}{\partial \theta} \mathbb{E}_{x \sim p_g}[f(x)] = \frac{\partial}{\partial \theta} \int f(x) p_g(x) dx$$

$$= \int f(x) \frac{\partial}{\partial \theta} p_g(x) dx$$

$$= \int f(x) p_g(x) \frac{1}{p_g(x)} \frac{\partial}{\partial \theta} p_g(x) dx$$

$$= \int f(x) p_g(x) \frac{\partial}{\partial \theta} \log(p_g(x)) dx.$$  

(44)

On the other hand, MLE has the following optimization:

$$\max_\theta \mathbb{E}_{x \sim p_g}[p_{data}(x)]$$

$$\Rightarrow \frac{\partial}{\partial \theta} \mathbb{E}_{x \sim p_g}[p_{data}(x)] = \int p_{data}(x) \frac{\partial}{\partial \theta} \log(p_g(x)) dx.$$  

(45)

Eqs. (44) and (45) are for minimization in GAN and maximization in MLE, respectively. By their comparison, we can have MLE in GAN if we set:

$$f(x) = - \frac{p_{data}(x)}{p_g(x)}.$$  

(46)

The discriminator of GAN is modeled as a classifier for $x$ being real and not generated (fake); hence:

$$D(x) = p_c(y = 1|x) = \sigma(D'(x)),$$  

(47)

where $\sigma(.)$ is the sigmoid activation function and $D'(x)$ denotes the discriminator network except the sigmoid function at its last layer.

**Theorem 10** (Goodfellow, 2015; 2016). The loss function for the generator of GAN can be stated as any of the following loss functions:

$$\min_G \mathbb{E}_{z \sim p_{z}}[\log(1 - D(G(z)))],$$  

(48)

$$\min_G -\mathbb{E}_{z \sim p_{z}}[\log(D(G(z))],$$  

(49)

$$\min_G -\mathbb{E}_{z \sim p_{z}}[\sigma^{-1}(D(G(z))],$$  

(50)

where $\sigma(.)$ is the sigmoid function.

**Proof.** Eq. (48) is the generator part of loss function (3). The generator wants to fool the discriminator so it wants $D(G(z))$ to be close to $1$ (see Eq. (2)). Hence, rather than minimizing $\log(1 - D(G(z)))$ in Eq. (48), we can maximize $\log(D(G(z)))$, or minimize its negation, in Eq. (49) (Goodfellow, 2016). The proof of Eq. (50) is as follows (Goodfellow, 2015). Assume the discriminator is optimal for a given generator; hence, according to Eq. (8), we have:

$$p_c(y = 1|x) \overset{(8)}{=} \frac{p_{data}(x)}{p_{data}(x) + p_g(x)} = \frac{1}{1 + \frac{p_g(x)}{p_{data}(x)}}$$

$$\overset{(47)}{=} \sigma(D'(x)) = \frac{1}{1 + \exp(-D'(x))},$$

$$\Rightarrow \frac{p_g(x)}{p_{data}(x)} = \exp(-D'(x))$$

$$\overset{(46)}{=} f(x) = - \exp(D'(x)) \overset{(47)}{=} - \exp(\sigma^{-1}(D(G(z)))).$$

Hence, for the generated data $x = G(z)$, from the latent noise sample $z \sim p_z(z)$, the Eq. (43) becomes Eq. (50). Q.E.D.

#### 4.2. f-GAN

f-GAN (Nowozin et al., 2016) uses f-divergence in the formulation of GAN. The f-GAN computes divergence between two distributions $p(x)$ and $q(x)$ by (Liese & Vajda, 2008).
where the convex so-called generator function $f : \mathbb{R}_+ \to \mathbb{R}$ satisfies $f(1) = 0$. Two special cases of f-GAN are KL-divergence and JL-divergence. We denote the space of data by $\mathcal{X}$.

**Lemma 3** ([Nguyen et al., 2010, Lemma 1]). A lower-bound on the f-divergence is as follows:

$$D_f(P||Q) \geq \sup_{T \in T} \left( \mathbb{E}_{x \sim p(x)}[T(x)] - \mathbb{E}_{x \sim q(x)}[f^*(T(x))] \right),$$  

(52)

where $f^*$ is the convex conjugate of $f$ and $T : \mathcal{X} \to \mathbb{R}$ is an arbitrary class of functions.

**Proof.** The convex conjugate of function $f$ is defined as (Ghojogh et al., 2021c):

$$f^*(t) := \sup_{u \in \text{dom}(f)} (ut - f(u))$$

$$\implies f(u) := \sup_{t \in \text{dom}(f^*)} (tu - f^*(t)).$$  

(53)

We have:

$$D_f(P||Q) \overset{(53)}{=} \int q(x) \sup_{t \in \text{dom}(f^*)} \left( t \frac{p(x)}{q(x)} - f^*(t) \right) dx$$

$$\overset{(a)}{\geq} \sup_{T \in T} \left( \int p(x)T(x)dx - \int q(x)f^*(T(x))dx \right)$$

$$= \sup_{T \in T} \left( \mathbb{E}_{x \sim p(x)}[T(x)] - \mathbb{E}_{x \sim q(x)}[f^*(T(x))] \right),$$

where $(a)$ is because the summation of maximums is greater than or equal to the maximum of summations. Q.E.D.

Variational Divergence Minimization (VDM) ([Nowozin et al., 2016]) optimizes the f-divergence by optimizing the bound in Eq. (52). In this sense, it is similar to variational inference (Ghojogh et al., 2021a). Suppose $p(x) = p_{data}(x)$ and $q(x) = p_{\phi}(x)$ in Eq. (52), where $p_{data}(x)$ and $p_{\phi}(x)$ are the distributions of real and generated data, respectively. Let $T(x) = o_f(V(x))$ where $V : \mathcal{X} \to \mathbb{R}$ is the mapping of network from its input to one output neuron (before activation) and $o_f : \mathbb{R} \to \text{dom}(f^*)$ is the output of activation function. VDM can be used for optimization of various f-divergences. Its loss function is:

$$\min_{G} \max_{V} \mathbb{E}_{x \sim p_{data}(x)}[o_f(V(x))]$$

$$+ \mathbb{E}_{x \sim p_{\phi}(x)}[-f^*(o_f(V(x)))]$$

(54)

The reader can refer to ([Nowozin et al., 2016, Table 2]) for a complete list of expressions for $o_f(v)$ and $f^*$ in different special cases of f-divergence. A special case of VDM is f-GAN in which $o_f(v) = -\log(1 + \exp(-v))$, $f^*(t) = -\log(1 - \exp(-t))$ and the discriminator is the sigmoid function of $V(x)$, i.e., $D(x) = 1/(1+\exp(-V(x)))$. Hence, in f-GAN, we have:

$$o_f(V(x)) = -\log(1 + \exp(-V(x))) = \log(D(x)),$$

$$f^*(o_f(V(x))) = -\log(1 - \exp(\log(D(x))))$$

$$= -\log(1 - D(x)).$$

Putting these in Eq. (54) gives the loss of f-GAN:

$$\min_{G} \max_{D} \mathbb{E}_{x \sim p_{data}(x)}[\log(D(x))]$$

$$+ \mathbb{E}_{x \sim p_{\phi}(x)}[\log(1 - D(x))].$$  

(55)

### 4.3. Adversarial Variational Bayes (AVB)

Adversarial Variational Bayes (AVB) ([Mescheder et al., 2017a]) combines the ideas of variational and adversarial training. Variational inference (Ghojogh et al., 2021a) maximizes an evidence lower bound defined as:

$$\max_{\theta} \min_{\phi} \mathbb{E}_{p(x)}[\mathbb{E}_{q_{\phi}(z|x)}[\log(p(z)) - \log(q_{\phi}(z|x))]]$$

$$+ \log(p_{\theta}(x|z)),$$  

(56)

where $\theta$ and $\phi$ are parameters corresponding to $p_{\theta}(x|z)$ and $q_{\phi}(z|x)$, respectively. On the other hand, adversarial learning uses a discriminator $D(x, z)$ with one output neuron having a sigmoid activation function in variational inference. The loss function for the discriminator is:

$$\max_{D} \mathbb{E}_{p(x)}[\mathbb{E}_{q_{\phi}(z|x)}[\log(D(x, z))]]$$

$$+ \mathbb{E}_{p(x)}[\mathbb{E}_{p_{\theta}(z)}[\log(1 - D(x, z))]],$$  

(57)

whose solution is $D^*(x, z) = -\log(p(z)) + \log(q_{\phi}(z|x))$ ([Mescheder et al., 2017a, Proposition 1]). Therefore, Eq. (56) becomes:

$$\max_{\theta} \min_{\phi} \mathbb{E}_{p(x)}[\mathbb{E}_{q_{\phi}(z|x)}[-D^*(x, z) + \log(p_{\theta}(x|z))]],$$

(58)

which is optimized by backpropagation, after the reparameterization trick (Ghojogh et al., 2021a).

### 4.4. Bayesian GAN (BGAN)

Bayesian GAN (BGAN) ([Saatchi & Wilson, 2017]) models GAN using Bayesian analysis. Let $G$ and $D$ denote the parameters of generator and discriminator, respectively, $x$
be the real data, \( z \) be the noise sample, and \( b \) be the mini-batch size.

\[
p(G|z, D) \propto \left( \prod_{i=1}^{b} D(G(z_i)) \right) p(G),
\]
\[
p(D|z, x) \propto \left( \prod_{i=1}^{b} D(x_i) \right) \left( \prod_{i=1}^{b} (1 - D(G(z_i))) \right) p(D).
\]

We can marginalize these distributions:

\[
p(G|D) = \int p(G, z|D) dz = \int p(G|z, D) p(z|D) dz
\]
\[
\approx \int p(G|z, D) p_z(z) dz \approx \frac{1}{b} \sum_{i=1}^{b} p(G|z_i), \tag{59}
\]

where \( z_i \sim p_z(z) \), \((a)\) is because the noise \( z \) is independent of the discriminator \( D \), and \((b)\) is because of the Monte Carlo approximation (Ghojogh et al., 2020). Similarly, we have:

\[
p(D|G) = \frac{1}{b} \sum_{i=1}^{b} p(D|z_i, x_i). \tag{60}
\]

Sampling from the distributions in Eqs. (59) and (60) will converge to the joint distribution of generator and discriminator, based on Gibbs sampling (Ghojogh et al., 2020). Stochastic Gradient Hamiltonian Monte Carlo (SGHMC) is a technique for training a neural network using the posteriors. The discriminator and generator of BGAN are trained alternatively using this technique and the posteriors in Eqs. (59) and (60). Note that another GAN model with variational inference and Bayesian analysis is the variational Bayesian GAN (Chien & Kuo, 2019).

5. Other Variants of GAN

5.1. Feature Matching in GAN

During training, the layers of discriminator \( D \) are trained to have discriminative features between the real and generated data. Therefore, for better training of the generator \( G \) and fooling the discriminator by it, we can use the features of an intermediate layer of discriminator (Salimans et al., 2016, Section 3.1). We train the generator to match the expected values of the intermediate features for inputs of real and generated data. Hence, the optimization of generator can be:

\[
\min_{G} \| \mathbb{E}_{x \sim p_{data}(x)} [f(x)] - \mathbb{E}_{z \sim p_z(z)} [f(G(z))] \|^2_{z}, \tag{61}
\]

where \( f(x) \) and \( f(G(z)) \) are the features of an intermediate layer of discriminator for inputs \( x \) (real data) and \( G(z) \) (generated data), respectively. The discriminator is trained as in the original GAN, i.e., maximization in Eq. (3).

5.2. InfoGAN

Information maximizing GAN (InfoGAN), proposed in (Chen et al., 2016), is an information-theoretic approach to GAN. It maximizes the mutual information between latent variables and generated data. In InfoGAN, we have two sets of latent variables, i.e., \( z \) and \( c \). The generator gets these two latent variables as input and outputs \( G(z, c) \). The optimization problem in InfoGAN is a regularized problem as:

\[
\min_{G} \max_{D} V_I(D, G) := V(D, G) - \lambda I(c; G(z, c)), \tag{62}
\]

where \( V(D, G) \) is defined in Eq. (3), the \( \lambda > 0 \) is the regularization parameter and \( I(\cdot; \cdot) \) is the mutual information defined as

\[
I(c; G(z, c)) := H(c) - H(c|G(z, c)) \tag{62}
\]

in which \( H(\cdot) \) is the entropy. Note that the added regularization term depends only on \( G \) and not \( D \). The generator maximizes the mutual information \( I(c; G(z, c)) \).

Computing this mutual information is difficult in practice. The mutual information can be simplified as the following by introducing an auxiliary distribution \( Q(c|x) \).

\[
I(c; G(z, c)) := H(c) - H(c|G(z, c))
\]

\[
\approx H(c) - \mathbb{E}_{x \sim p(x)} \mathbb{E}_{c \sim p(c|x)} [\log P(c|x)]
\]

\[
= H(c) + \mathbb{E}_{x \sim p(x)} \mathbb{E}_{c \sim p(c|x)} [\log P(c|x)]
\]

\[
\approx H(c) + \mathbb{E}_{x \sim p(x)} [\mathbb{E}_{c \sim p(c|x)} [\log Q(c|x)]]
\]

\[
= H(c) + \mathbb{E}_{x \sim p(x)} [\mathbb{E}_{c \sim p(c|x)} [\log Q(c'|x)]]
\]

\[
\approx H(c) + \mathbb{E}_{x \sim p(x)} [\mathbb{E}_{c \sim p(c|x)} [\log Q(c'|x)]]
\]

where \((a)\) is because of definition of entropy, \((b)\) is because of the definition of KL divergence, \((c)\) is because the KL divergence is non-negative, \((d)\) is because we define \( L_1(G, Q) \) as that expression. Hence, \( L_1(G, Q) \) is a lower-bound for \( I(c; G(z, c)) \). Using this lower-bound in Eq. (62) gives:

\[
\min_{G} \max_{D} V_I(D, G) := V(D, G) - \lambda L_1(G, Q), \tag{63}
\]

where:

\[
L_1(G, Q) := H(c) + \mathbb{E}_{c \sim p(c|x), x \sim p(x)} [\log Q(c'|x)],
\]

can be calculated by Monte Carlo approximation (Ghojogh et al., 2020).

5.3. Generative Recurrent Adversarial Network (GRAN)

Generative Recurrent Adversarial Network (GRAN) (Im et al., 2016) has been inspired by the Deep Recurrent Attentive Writer (DRAW) (Gregor et al., 2015). DRAW
uses variational inference for drawing images gradually on canvas by passing time. GRAN does the same but using adversarial learning. Therefore, it is a combination of GAN and recurrent networks. In GRAN, the generator $G$ has a recurrent feedback loop whose inputs are a sequence of noise samples $\{z_t\}_{t=1}^T$. The recurrent loop of generator generates a sequence of drawings on canvas, i.e., $\{\Delta C_1, \Delta C_2, \ldots, \Delta C_T\}$. Every recurrent loop, at time $t \in \{1, \ldots, T\}$, is like an autoencoder with encoder $f(.)$ and decoder $g(.)$. The coding layer between the encoder and decoder gives the concatenation of latent coding $h_{z,t}$ and canvas coding $h_{c,t}$. This coding concatenation is fed to the decoder $f$ to result the canvas drawing $\Delta C_t$. In every recurrent loop, at time $t \in \{1, \ldots, T\}$, we have:

\[
\begin{align*}
  z_t &\sim p_z(z), \\
  h_{c,t} &:= g(\Delta C_{t-1}), \\
  h_{z,t} &:= \tanh(W z_t + b), \\
  \Delta C_t &:= f([h_{z,t}, h_{c,t}]^T),
\end{align*}
\]

where $W$ and $b$ are the layer weights and bias weights for the latent variable $z_t$. We use DCGAN (Radford et al., 2016) (see Section 2.7) for the encoder $f$ and decoder $g$ at every recurrent loop, where the canvas drawings $\{\Delta C_t\}_{t=1}^T$ are generated. The total canvas drawing is the summation of drawings at the time slots. We use a logistic function $\sigma(.)$ to scale the drawing to $(0, 1)$ for the sake of pixel visualization:

\[
C = \sigma\left(\sum_{t=1}^T \Delta C_t\right).
\]

5.4. Least Squares GAN (LSGAN)

The GAN loss function has a problem. In the discriminator, the gradient vanishes for the generated data points which fall on the correct side of decision boundary but are still different from the real data. Least Squares GAN (LSGAN) (Mao et al., 2017; 2019) resolves this issue by using least squares cost in the adversarial loss function. For the discriminator $D$ of LSGAN, we use two scalar labels $a$ and $b$ for generated (fake) and real data points. For the generator $G$ of LSGAN, we use the scalar label $c$ which the generator wants the discriminator to believe for in classification. The loss functions in LSGAN are:

\[
\begin{align*}
\min_D V_{\text{LSGAN}}(D) &:= \frac{1}{2} \mathbb{E}_{x \sim p_{\text{data}}(x)} [(D(x) - b)^2] \\
&\quad + \frac{1}{2} \mathbb{E}_{z \sim p_z(z)} [(D(G(z)) - a)^2], \\
\min_G V_{\text{LSGAN}}(G) &:= \frac{1}{2} \mathbb{E}_{z \sim p_z(z)} [(D(G(z)) - c)^2].
\end{align*}
\]

Lemma 4 ((Mao et al., 2019, Proposition 1)). For a fixed generator $G$, the optimal discriminator in LSGAN is:

\[
D^*(x) = \frac{bp_{\text{data}}(x) + ap_G(x)}{p_{\text{data}}(x) + p_G(x)},
\]

where $p_{\text{data}}(x)$ is the probability distribution of real dataset evaluated at point $x$ and $p_G(x)$ is the probability distribution of output of generator evaluated at point $x$.

Proof.

\[
V_{\text{LSGAN}}(D) \quad (65) = \frac{1}{2} \mathbb{E}_{x \sim p_{\text{data}}(x)} [(D(x) - b)^2] \\
+ \frac{1}{2} \mathbb{E}_{z \sim p_z(z)} [(D(G(z)) - a)^2]
= \int_x \frac{1}{2} p_{\text{data}}(x) (D(x) - b)^2 dx \\
+ \int_x \frac{1}{2} p_z(z) (D(G(z)) - a)^2 dx
\]

\[
\Rightarrow \int_x \left( p_{\text{data}}(x) (D(x) - b)^2 + p_G(x) (D(x) - a)^2 \right) dx
\]

For optimization in Eq. (65), taking derivative w.r.t. $D(x)$ gives:

\[
\frac{\partial V_{\text{LSGAN}}(D)}{\partial D(x)} \quad (a) = \frac{\partial}{\partial D(x)} \left( \frac{1}{2} (p_{\text{data}}(x) (D(x) - b)^2 + p_G(x) (D(x) - a)^2) \right)
= p_{\text{data}}(x) (D(x) - b) + p_G(x) (D(x) - a) \quad \forall x \geq 0
\Rightarrow D(x) = \frac{bp_{\text{data}}(x) + ap_G(x)}{p_{\text{data}}(x) + p_G(x)},
\]

where (a) is because taking derivative w.r.t. $D(x)$ considers a specific $x$ and hence it removes the integral (summation). Q.E.D.

Theorem 11 ((Mao et al., 2019, Theorem 1)). Optimization of LSGAN is equivalent to minimizing the Pearson $\chi^2$ divergence between $p_{\text{data}}(x) + p_G(x)$ and $2p_G(x)$, if we have:

\[
b - c = 1, \quad b - a = 2.
\]
Proof.

\[
2V_{\text{LSGAN}}(G) \overset{(65)}{=} \mathbb{E}_{x \sim p_{data}(x)} [(D^*(G(z)) - c)^2] \\
\overset{(a)}{=} \mathbb{E}_{x \sim p_{data}(x)} [(D^*(x) - c)^2] \\
\overset{(9)}{=} \mathbb{E}_{x \sim p_{data}(x)} [(D^*(x) - c)^2] \\
\overset{(65)}{=} \mathbb{E}_{x \sim p_{data}(x)} \left[ \left( \frac{b p_{data}(x) + a p_g(x)}{p_{data}(x) + p_g(x)} \right) - c \right]^2 \\
+ \mathbb{E}_{x \sim p_{data}(x)} \left[ \left( \frac{b p_{data}(x) + a p_g(x)}{p_{data}(x) + p_g(x)} \right) - c \right]^2 \\
\overset{(b)}{=} \int_x p_{data}(x) \left[ \left( \frac{b - c + a - c p_g(x)}{p_{data}(x) + p_g(x)} \right) \right]^2 dx \\
\overset{(c)}{=} \int_x \left( \left( \frac{b - c + a - c p_g(x)}{p_{data}(x) + p_g(x)} \right) \right)^2 dx \\
\overset{(67)}{=} \int_x \left( \frac{2 p_g(x) - (p_{data}(x) + p_g(x))}{p_{data}(x) + p_g(x)} \right)^2 dx \\
\overset{(d)}{=} \chi^2(p_{data}(x) + p_g(x) \| 2 p_g(x)),
\]

where \((a)\) is because \(\mathbb{E}_{x \sim p_{data}(x)} [(D^*(x) - c)^2]\) is constant w.r.t. \(G\), \((b)\) is because of the definition of expectation, \((c)\) is because of simplification of terms, and \((d)\) is because of definition of Pearson \(\chi^2\) divergence. Q.E.D.

As we saw, the labels \(a, b, c\) in LSGAN should satisfy Eq. (67). An options for satisfying these conditions is:

\[
a = -1, \quad b = 1, \quad c = 0,
\]

which means the real and fake labels for discriminator are +1 and -1, respectively, while the generator fools the discriminator by label 0. In other words, the generator does not take it very hard on the discriminator and sets the fake label to 0 (some moderate value) rather than 1. Another possible option for the labels is:

\[
a = 0, \quad b = c = 1,
\]

which does not satisfy Eq. (67) but fools the discriminator completely (with more power) by the generator. Experiments have shown both of these options perform equally well in practice (Mao et al., 2017).

5.5. Energy-based GAN (EBGAN)

In energy-based learning (LeCun et al., 2006), a function is learned which maps data points to some energy values where the incorrectly labeled data points are assigned higher energy values. In unsupervised energy-based learning, higher energy is assigned to data points away from the data manifold or data cloud. Energy-based GAN (EBGAN) (Zhao et al., 2017) uses energy-based learning in adversarial learning. The loss functions in EBGAN are:

\[
\min_D V_{\text{EBGAN}}(D) := \mathcal{D}(x) + [m - \mathcal{D}(G(z))]_+,
\]

\[
\min_G V_{\text{EBGAN}}(G) := \mathcal{D}(G(z)),
\]

where \([\cdot ]_+ := \max(\cdot, 0)\) is the standard Hinge loss and \(m > 0\) is the margin. The discriminator minimizes the error of \(D(x)\) while maximizing \(\mathcal{D}(G(z))\) not to be fooled by the generator. The generator minimizes \(\mathcal{D}(G(z))\) to fool the discriminator.

Theorem 12 ((Zhao et al., 2017, Theorem 1)). Let:

\[
Q(D, G) := \int_{x, z} V_{\text{EBGAN}}(D) p_{data}(x) p_z(z) dx dz, \quad (71)
\]

Optimization of EBGAN results in \(p_g(x) = p_{data}(x)\) and \(Q(D^*, G^*) = m\) after convergence (i.e., Nash equilibrium).

Proof.

\[
Q(D, G^*) \overset{(70)}{=} \int_{x, z} \left[ \mathcal{D}(x) + [m - \mathcal{D}(G^*(z))]_+ \right] p_{data}(x) p_z(z) dx dz \\
= \int_x \mathcal{D}(x) p_{data}(x) dx \int_z p_z(z) dz \\
+ \int_x p_{data}(x) dx \int_z [m - \mathcal{D}(G^*(z))]_+ p_z(z) dz \\
\overset{(9)}{=} \int_x \left( p_{data}(x) \mathcal{D}(x) + p_g^*(x) [m - \mathcal{D}(x)]_+ \right) dx.
\]

The function inside the integral is \(at + b|m - t|_+\) whose minimum occurs if \(a < b\). Hence, the minimum of
Using this in Eq. (72) gives:

$$Q(D, G^*)$$ is:

$$Q(D^*, G^*) = m \int_x \mathbb{I}(p_{data}(x) < p_g^*(x)) p_{data}(x) dx$$
$$+ m \int_x \mathbb{I}(p_{data}(x) \geq p_g^*(x)) p_g^*(x) dx$$
$$= m \int_x \left( \mathbb{I}(p_{data}(x) < p_g^*(x)) \right) p_g^*(x) dx$$
$$+ (1 - \mathbb{I}(p_{data}(x) < p_g^*(x))) p_g^*(x) dx$$
$$= m \int_x p_g^*(x) dx \left( \mathbb{I}(p_{data}(x) < p_g^*(x)) \right)$$
$$+ m \int_x \mathbb{I}(p_{data}(x) < p_g^*(x)) (p_{data}(x) - p_g^*(x)) dx.$$  

(73)

As the probability of generated data $p_g^*(x)$ is upper-bounded by the probability of data $p_{data}(x)$, the second term is non-positive. Hence, $Q(D, G) \leq m$. On the other hand, as $p_g^*(x) \leq p_{data}(x)$, we have:

$$\int_x p_g^*(x) D^*(x) dx \leq \int_x p_{data}(x) D^*(x) dx.$$

Using this in Eq. (72) gives:

$$Q(D^*, G^*)$$
$$\geq \int_x \left( p_g^*(x) D^*(x) + p_g^*(x) [m - D^*(x)]_+ \right) dx$$
$$\overset{(a)}{=} \int_x p_g^*(x) D^*(x) dx + \int_x p_g^*(x) (m - D^*(x)) dx$$
$$= m \int_x p_g^*(x) dx = m.$$

where (a) is because $D^*(x) \leq m$ almost everywhere at the Nash equilibrium (since discriminator is trained at the convergence to not violate the margin). We showed that $m \leq Q(D^*, G^*) \leq m$ so $Q(D^*, G^*) = m$. From $Q(D^*, G^*) = m$ and Eq. (73), we have $\int_x \mathbb{I}(p_{data}(x) < p_g^*(x)) dx = 0$. As we have $p_g^*(x) \leq p_{data}(x)$, this only holds when $p_{data}(x) = p_g^*(x)$. Q.E.D.

5.6. Semi-supervised GAN

In the following, we introduce the semi-supervised methods in the GAN literature.

5.6.1. Categorical GAN (CatGAN)

- Unsupervised CatGAN: In Categorical GAN (CatGAN) (Springenberg, 2016), the discriminator classifies $c$ classes (i.e., categories) rather than a binary classification which we had in GAN’s discriminator. Hence, the last layer of discriminator has $c$ neurons with softmax activation functions. Let $D_k(x)$ denote the $k$-th logit, i.e., softmax output.

The conditional probabilities, for the categories, are modeled as follows based on the logits of discriminator:

$$p(y = k \mid x) = \frac{\exp(D_k(x))}{\sum_{k=1}^c \exp(D_k(x))}, \forall k = \{1, \ldots, c\}.$$  

(74)

Note that the dataset is unlabeled (unsupervised) and the categories are just made by our model in the logits of discriminator. The discriminator wants to be certain about classification of real data into the $c$ categories; hence, it should minimize the entropy $H$ of conditional probabilities of real data which is:

$$\mathbb{E}_{x \sim p_{data}(x)}[H(p(y = k \mid x))] \approx \frac{1}{n} \sum_{i=1}^n H(p(y = k \mid x_i))$$

(75)

where $n$ is number of real data points, $(a)$ is because of the Monte Carlo approximation (Ghojogh et al., 2020), and $(b)$ is because of the definition of entropy. We draw $n$ noise samples, $z \sim p_z(z)$, and feed to generator to generate data points $G(z)$. The discriminator wants to be uncertain about classification of generated (fake) data into the $c$ categories; hence, it should maximize its corresponding entropy:

$$\mathbb{E}_{z \sim p_z(z)}[H(p(y = k \mid G(z)))]$$

$$\approx \frac{1}{n} \sum_{i=1}^n H(p(y = k \mid G(z_i))).$$  

(76)

The generator, on the other hand, wants to minimize the above entropy to fool the discriminator. We also assume uniform prior $p(y)$ for categories so we want the discriminator and generator use all categories equally. For that, they should maximize the entropy of marginal category distributions:

$$H_{data}(p(y)) = H\left(\frac{1}{n} \sum_{i=1}^n p(y \mid x_i)\right),$$  

(77)

$$H_g(p(y)) = H\left(\frac{1}{n} \sum_{i=1}^n p(y \mid G(z_i))\right).$$  

(78)

Overall, according to above explanations, the loss functions in CatGAN are:

$$\max_D V(D) := H_{data}(p(y))$$

$$- \mathbb{E}_{x \sim p_{data}(x)}[H(p(y = k \mid x))]$$

(79)

$$+ \mathbb{E}_{z \sim p_z(z)}[H(p(y = k \mid G(z)))]$$.

$$\min_G V(D) := -H_g(p(y))$$

$$+ \mathbb{E}_{z \sim p_z(z)}[H(p(y = k \mid G(z)))].$$

(80)
– **Semi-supervised CatGAN**: The above loss function for CatGAN is used for an unsupervised case. We can extend CatGAN to semi-supervised cases (Springenberg, 2016). Suppose we have \( n_{\ell} \) labeled data points in addition to the \( n \) unlabeled data points. We set \( c \) (i.e., the number of categories) equal to the number of classes of the labeled data. We denote the labeled dataset by \( \mathcal{X}_L := \{ (\mathbf{x}_i^L, y_i^L) \}_{i=1}^{n_{\ell}} \) where \( y_i^L \in \mathbb{R}^c \) is the one-hot encoded label for the \( i \)-th labeled data point. The discriminator should maximize the cross-entropy of the labeled data to be able to discriminate the actual classes in addition to discrimination of the categories. This cross-entropy is:

\[
\text{CE}(y, p(y|x)) := -\sum_{k=1}^{c} y_i \log(p(y = y_i|x)),
\]

where \( p(y = y_i|x) \) is the logit of discriminator for the labeled data input. We regularize this cross-entropy into the loss of discriminator:

\[
\max_{D} V(D) := H_{data}(p(y)) - \mathbb{E}_{x \sim \mathcal{P}_{data}(x)}[H(p(y = k \mid x))] + \text{CE}(y, p(y|x)) + \lambda \text{CE}(y, p(y|x)),
\]

where \( \lambda > 0 \) is the regularization parameter.

### 5.6.2. Generated Data as a New Class

We can consider the generated data to be data with an additional label \((c + 1)\). This idea has appeared in two independent papers which are (Salimans et al., 2016, Section 5) and (Odena, 2016). Here, we explain (Salimans et al., 2016, Section 5). The discriminator \( D \) classifies which class the data point \( x \) has. This is in contrast to the discriminator in the original GAN which has a neuron with sigmoid activation function as its last layer. Here, the last layer of discriminator has \((c + 1)\) neurons with softmax activation function where the \( j \)-th neuron outputs the probability for \( x \) belonging to the \( j \)-th class. The optimization of discriminator is minimization of summation of two cross-entropy costs:

\[
\min_{D} \left( V_{\text{supervised}}(D) + V_{\text{unsupervised}}(D) \right),
\]

where:

\[
V_{\text{supervised}}(D) := \mathbb{E}_{x,y \sim \mathcal{P}_{data}(x,y)} \left[ \log(p_d(y|x)) \right], \forall y < c + 1,
\]

\[
V_{\text{unsupervised}}(D) := -\mathbb{E}_{x \sim \mathcal{P}_{data}(x)} \left[ \log(1 - p_d(y|x)) \right] + \mathbb{E}_{x \sim \mathcal{P}_{data}(x)} \left[ \log(p_d(y|x)) \right], \text{ for } y = c + 1,
\]

where \( p_d(y|x) \) is the output of softmax at the last layer of discriminator. With with cost, discriminator learns to classify the generated (fake) data points as a new class so the generator should try to fool it to not correctly classify it as the new class. The cost of generator is the same as in Eq. (3).

We can subtract a general function from every class label. Hence, we can subtract the output, corresponding to the labels of generated data, from all labels to make the label of fake data zero, \( \ell_{c+1} = 0 \). Hence, the softmax output of generated data becomes \( \exp(\ell_{c+1} = 0) = 1 \). Therefore, according to Eq. (8) and the fact that probabilities are obtained by softmax outputs (in the form of logits), we have:

\[
D(x) = \frac{\sum_{j=1}^{c} \exp(\ell_j(x))}{\sum_{j=1}^{c} \exp(\ell_j(x)) + \exp(\ell_{c+1}(x))} = \frac{\sum_{j=1}^{c} \exp(\ell_j(x))}{\sum_{j=1}^{c} \exp(\ell_j(x)) + 1}.
\]

### 5.7. MMD GAN

MMD GAN (Li et al., 2017b) combines the ideas of moment matching networks (Li et al., 2015) and GAN (Goodfellow et al., 2014) by using adversarial learning in Maximum Mean Discrepancy (MMD). MMD (Gretton et al., 2006) is a measure of divergence of two distributions and it uses distance in the Reproducing Kernel Hilbert Space (RKHS) to measure the difference of moments of two distributions (Ghojogh et al., 2021d). The MMD between two distributions \( p(x) \) and \( q(x) \) is:

\[
M_k(p, q) := \mathbb{E}_{x_i, x_j \sim p(x)}[k(x_i, x_j)] + \mathbb{E}_{x_i, x_j \sim q(x)}[k(x_i, x_j)] - 2\mathbb{E}_{x_i \sim p(x), x_j \sim q(x)}[k(x_i, x_j)],
\]

where \( k(\cdot, \cdot) \) is a kernel function such as the Gaussian kernel. If \( p(x) = \mathcal{P}_{data}(x) \) and \( q(x) = p_g(x) \) are the distributions of real and generated data, respectively, we want to minimize this MMD so that the generated data distribution becomes similar to the real data distribution.

We can find the best kernel, giving the largest MMD for the worst-case scenario, from a set of valid kernel functions \( K \):

\[
\min_{G} \max_{k \in K} M_k(p_{\text{data}}(x), p_g(x)).
\]

However, this optimization is difficult. In MMD GAN, rather than using a fixed kernel such as the Gaussian kernel, we train the kernel function by adversarial learning. We learn a function \( D(\cdot) \) to define the kernel function as:

\[
k_D(x_i, x_j) = \exp(-\|D(x_i) - D(x_j)\|^2).
\]

We use an autoencoder for \( D(\cdot) \) with \( D_e(\cdot) \) and \( D_d(\cdot) \) as encoder and decoder, respectively. This autoencoder plays the role of discriminator in adversarial learning. The generator is denoted by \( G(\cdot) \). This autoencoder should reconstruct both real data, \( x \in \mathcal{X} \), and generated data from latent noise, \( \mathbf{x} \in G(z) \). The loss function of MMD GAN is:

\[
\min_{G} \max_{D} M_{k_D}(p_{\text{data}}(x), p_g(x)) - \lambda \mathbb{E}_{\mathbf{x} \in \mathcal{X}, z \in G(z)}[\|\mathbf{x} - D_d(D_e(z))\|^2],
\]

where \( k_D(\cdot, \cdot) \) is the kernel function used in MMD GAN.
where $\lambda > 0$ is the regularization parameter. Both terms depend on the autoencoder $D$ while the first term depends on the generator $G$. Some theoretical analysis of MMD GAN can be found in (Mroueh & Nguyen, 2021).

5.8. Additive GANs
In the following, we introduce the additive GAN models which have a hierarchical or additive approach.

5.8.1. LAPLACIAN GAN (LapGAN)
Laplacian GAN (LapGAN) (Denton et al., 2015) was one of the first extensions of GAN. It generates higher resolution images compared to GAN and conditional GAN. Inspired by the Laplacian pyramid for image (Burt & Adelson, 1983), LapGAN uses a Laplacian pyramid. The structure of LapGAN for training is illustrated in Fig. 5-a. Let the pyramid have $k$ levels. We start with the image itself at level zero, i.e., $I = I_0$. We downsample the image to $I_1$ by a factor of two, i.e., we halve the rows and columns of image. Then, we upsample $I_1$ to $l_0$ by a factor of two, where $l_0$ is the low-pass (low-resolution) version of $I_0$. We use a conditional GAN (Mirza & Osindero, 2014) (see Section 2.6), denoted by $G_0$, which gets the noise $z_0$ as its input noise and the low-pass $l_0$ as its conditional input. The generator generates $\tilde{h}_0$. Let $h_0 := I_0 - l_0$. We input $h_0$, $\tilde{h}_0$, and $l_0$ to a discriminator $D_0$ whose last layer is a neuron with sigmoid activation function. The discriminator judges whether the image at this level is a real or fake (generated). This procedure is repeated for other levels until the level $(k-1)$. In each of these levels, a conditional GAN is used. In the last level $k$, a GAN (not conditional) is used which gets the noise $z_k$ as input and generates $\tilde{I}_k$. This $\tilde{I}_k$ and the downsampled $I_k$ are input to a discriminator $D_k$ which judges the image at that level.

The test or sampling phase of the LapGAN is depicted in Fig. 5-b. Like the training phase, all levels except the last level $k$ have conditional GANs while the last level has a GAN. At the $j$-th level, the noise $z_j$ and the low-pass image $l_j$ are fed to generator $G_j$ as its input and conditional input, respectively. The generator generates $\tilde{h}_j$. The generated image at the $j$-th level is obtained as $\tilde{I}_j := \tilde{h}_j + l_j$. The generated image at the level zero, i.e. $\tilde{I}_0$, is the generated image by the LapGAN.

5.8.2. PROGRESSIVE GAN
Progressive GAN (Karras et al., 2018) starts with shallow networks for generator $G$ and discriminator $D$ and increases new layers progressively to the networks. Initially, a small convolutional layer with low spatial resolution exists in $G$ and $D$. This generates a low-resolution image. During training of GAN, we gradually add convolution layers with higher spatial resolutions to $G$ and $D$ so higher resolution images are generated. Training GAN and adding layers occur simultaneously.

5.9. Triple GAN
Triple GAN (Li et al., 2017a) has a discriminator $D$, a classifier $C$, and a generator $G$. In terms of having a classifier, it is similar to MGAN (Hoang et al., 2018) (see Section 3.5). The generator models conditional distribution of data on the label, $p_g(x|y)$, and the classifier mod-

![Figure 5. The structure of LapGAN for (a) training and (b) test, i.e., sampling.](image-url)
els the opposite conditional distribution, i.e., \( p_c(y|x) \). The discriminator judges whether the data-label pair \((x, y)\) is real or generated (fake). The classifier predicts class label \( y \) for the real or generated data \( x \). Let \( p_{\text{data}}(x, y) \) denote the joint distribution of real data and labels. The joint distributions for data-labels in generator and classifier are \( p_g(x, y) = p_g(x)p(y) \) and \( p_c(x, y) = p_c(x)p(y) \), respectively, where \( p(y) \) is the marginal distribution of labels. The generator gets noise \( z \sim p_z(z) \) and label \( y \) as input and generates a data point \( \tilde{x} = G(z, y) \), where \((G(z, y), y) \sim p_g(x, y)\). Triple GAN optimizes the loss function for a three-player game:

\[
\min_{G,C,D} \max_{\gamma} \ V(D, C, G) := E_{(x,y) \sim p_{\text{data}}(x,y)} \left[ \log(D(x, y)) \right] \\
+ \alpha E_{(x,y) \sim p_{\text{data}}(x,y)} \left[ \log(1 - D(x, y)) \right] \\
+ (1 - \alpha) E_{z \sim p_z(z), y \sim p(y)} \left[ \log \left( 1 - D(G(z, y), y) \right) \right] \\
+ E_{(x,y) \sim p_{\text{data}}(x,y)} \left[ -\log(p_c(y|x)) \right],
\]

where \( \alpha \in (0, 1) \) is the regularization parameter (\( \alpha = 0.5 \) is recommended). The last term of loss, in which \( p_c(y|x) \) is the predicted label by classifier, models the KL-divergence between \( p_c(x, y) \) and \( p_{\text{data}}(x, y) \). The discriminator, classifier, and generator get stronger gradually by alternating optimization (Ghojogh et al., 2021c).

**Theorem 13** ((Li et al., 2017a, Lemma 3.1 and Theorem 3.3)). The optimal discriminator of triple GAN is:

\[
D^*(x, y) = \frac{p_c(x, y)}{p(x, y) + (1 - \alpha)p_g(x, y) + \alpha p_c(x, y)}.
\]

After convergence (i.e., Nash equilibrium) of triple GAN, we have:

\[
p_{\gamma^*}(x, y) = p_{c^*}(x, y) = p_{\gamma}(x, y) \overset{(88)}{=} D^*(x, y) = 0.5.
\]

### 5.10. Latent Adversarial Generator (LAG)

Latent Adversarial Generator (LAG) (Berthelot et al., 2020) can generate high-resolution images by taking a corresponding low-resolution image as an input cue. In terms of getting a cue, it can be related to the conditional GAN (see Section 2.6). Let \( z, x, \) and \( \bar{x} \) denote the noise sample, the real data point, and the cue low-resolution data point, respectively. The generator \( G \) takes \( \bar{x} \) and \( z \) as input and generates the high-resolution image \( G(\bar{x}, z) \). The discriminator \( D \) has two parts. First, by a projection operator \( P \), it projects data onto a low-dimensional space, named the perceptual latent space. The operator \( P \) is a nonlinear neural network and gets the high and low dimensional data points as input. Then, by some other layers of network, denoted by the mapping \( F(.) \), the projected data onto the perceptual latent space is mapped to a scalar after the sigmoid activation function. Hence, the discriminator is \( D(x) = F(\Pi(x, \bar{x})) \).

We want to have \( \Pi(G(\bar{x}, z = 0), \bar{x}) \) be similar to \( \Pi(x, \bar{x}) \) so we use a regularization term for it. LAG uses WGAN (see Section 3.7) whose loss is regularized. We regularize Eq. (40) as:

\[
\min_{D} \max_{\gamma} D(x, \bar{x}) - D(G(\bar{x}, z), \bar{x}) \\
- |\nabla_{\bar{x}} D(\bar{x})|_2^2 - 1^2 \\
+ \lambda_2 \|\Pi(G(\bar{x}, z = 0), \bar{x}) - \Pi(x, \bar{x})\|_F^2,
\]

where \( \lambda_1, \lambda_2 > 0 \) are the regularization parameters, \( \| \cdot \|_F \) is the Frobenius norm, and \( \bar{x} \) is defined in Eq. (41).

### 5.11. Ensembles of GAN Models

In the following, we introduce some GAN models which have an ensemble of generators and/or discriminators. Some of them were already introduced, such as MGAN (see Section 3.5) and D2GAN (see Section 3.6). Here, we explain other ensemble GAN methods.

#### 5.11.1. Generative Multi-Adversarial Network (GMAN)

Generative Multi-Adversarial Network (GMAN) (Durugkar et al., 2017) accelerates training of GAN by using several discriminators. Assume we have \( n_d \) discriminators. The loss function of GMAN is:

\[
\max_{D_i} V(D_i, G), \forall i \in \{1, ..., n_d\},
\]

\[
\min_{G} F(V(D_1, G), ..., V(D_{n_d}, G)),
\]

where every \( V(D_i, G) \) is defined in Eq. (3) and the function \( F(.) \) can be an aggregating function such as \( F(.) = \max(\cdot) \) or \( F(.) = \text{mean}(\cdot) \). If \( F(.) \) is the maximum function, generator is trained using the best discriminator at every iteration of the alternating optimization. If \( F(.) \) is the mean function, an average effect of all discriminators are used for training the generator.

#### 5.11.2. ADAAGAN: Boosting GANs

Boosting refers to using weak models additively where every next model gives more weight to the points which were not correctly classified/regressed by the previous model (Ghojogh & Crowley, 2019). One of the most well-known boosting methods for classification and regression is AdaBoost (Freund & Schapire, 1997). AdaGAN (Tolstikhin et al., 2017) is boosting the GAN models for generation of data points. Let \( n \) be the number of data points. We start with the first GAN where the weights of points are all \( 1/n \). Let the generator of the \( j \)-th GAN be denoted by \( G_j \). We have one discriminator \( D \) only as the classifier whose scalar output after sigmoid activation function is \( D(x) \). For the \( j \)-th GAN model, we use a discriminator \( D \) to discriminate between the true data and the generated data \( G_{j-1}(z) \).
Figure 6. The structure of CoGAN.

where $z$ is the latent noise. The weights of points are updated as:

$$w_{i,j} := \frac{1}{n\beta_j} \left[ \lambda - (1 - \beta_j)h(D(x_i)) \right]_+, \quad (93)$$

where $w_{i,j}$ is the weight of $x_i$ for the $j$-th GAN, $[.]_+ := \max(.,0)$, $\beta_j := 1/j$ (or a fixed number in range $[0, 1]$), $h(D(x)) := (1 - D(x))/D(x)$, and $\lambda$ is obtained by iteratively updating:

$$\lambda := \frac{\beta_j}{\sum_{i=1}^k (1/n)} \left( 1 + \frac{1 - \beta_j}{\beta_j} \sum_{i=1}^k (1/n)h(D(x_i)) \right),$$

in which $k$ is the iteration of iterative updating. The generator of $j$-th weak GAN, $G'_i$, is trained by the weighted data points using the updated weights in Eq. (93). Finally, the $j$-th GAN is computed to be the linear combination of $G'_i$ and the previous GAN:

$$G_j := (1 - \beta_j)G_{j-1} + \beta_j G'_i.$$ 

The proofs for the above formulas can be found in (Tolstikhin et al., 2017).

5.11.3. Boosted Generative Model (BGM)

Another similar method for boosting GAN models is the Boosted Generative Model (BGM) (Grover & Ermon, 2018). We briefly introduce its idea here. Again, it starts with equal weights, all $1/n$, for the points. It trains the first generative model $G_1$. For the $j$-th GAN, it uses the lower bound of the f-divergence in Eq. (52) to estimate the next generative model based on the previous model. The formulation is inspired by the AdaBoost (Freund & Schapire, 1997).

5.12. Coupled GAN (CoGAN)

Coupled GAN (CoGAN) (Liu & Tuzel, 2016) is a generative model for several domains, where several data points are generated each of which has a different domain but the data points are related. For example, one domain can be image and another domain can be text where an image and a related caption can be generated. Another example is generation of two related images but from different domains, such as facial and nature images. If the tuples of corresponding data points are available, CoGAN can learn to generate corresponding and related images from different domains; otherwise, it can generate not-necessarily-related data points from the domains.

Assume we have two domains. In this case, CoGAN has two coupled GAN structures as illustrated in Fig. 6. Let $G_1/D_1$ and $G_2/D_2$ denote the generators/discriminators of the first and second GAN structures, respectively. In a generator, the first and last layers of network extract high-level and low-level features, respectively (Liu & Tuzel, 2016). Conversely, in a discriminator, the first and last layers of network extract low-level and high-level features, respectively (Krizhevsky et al., 2012). We want the GAN structures to share their high-level features but their low-level features should differ for capturing each domain’s characteristics. Therefore, as shown in Fig. 6, the first layers of generators and the last layers of discriminators are shared. Let the datasets of the first and second domains be denoted by $p_{\text{data1}}(x)$ and $p_{\text{data2}}(x)$, respectively. The loss function of CoGAN is:

$$\min_{G_1,G_2} \max_{D_1,D_2} \ V(D_1,D_2,G_1,G_2) :=$$

$$\mathbb{E}_{x \sim p_{\text{data1}}(x)} \left[ \log(D(x)) \right] + \mathbb{E}_{z \sim p_z(z)} \left[ \log \left( 1 - D_1(G_1(z)) \right) \right] + \mathbb{E}_{z \sim p_z(z)} \left[ \log \left( 1 - D_2(G_2(z)) \right) \right], \quad (94)$$

subject to the fact that some layers of the generators and some layers of discriminators are shared, as shown in Fig. 6. Note that, although the paper (Liu & Tuzel, 2016) has focused on coupling two GAN structures, the CoGAN can be easily extended to any number of structures and thus any number of domains.
5.13. Inverse GAN Models

We can invert generation of data points in GAN. This refers to generating a latent noise sample \( z \) from some data point \( x \). This latent noise is corresponding to the point \( x \) in the sense that if it is fed to the generator, \( x \) is generated. Some existing methods for inverse in GAN are adversarial autoencoder, BiGAN, ALI, and inverse technique. The adversarial autoencoder will be introduced later in Section 8.1. The other methods are explained in the following.

5.13.1. Bidirectional GAN (BiGAN)

In GAN, the generator gets a latent noise \( z \) and generates data point \( x \). However, the inverse of this process, i.e. outputting a latent variable from the data point \( x \), does not exist in GAN. Bidirectional GAN (BiGAN) (Donahue et al., 2017) is a version of GAN which also includes this inverse. Its structure is depicted in Fig. 7. In BiGAN, the generator \( G \) gets the noise \( z \) as input and generates \( G(z) \). The encoder \( E \), as the inverse of \( G \), gets \( x \) as input and outputs \( E(x) \). Recall that the discriminator of GAN gets the data \( x \) and the generated data \( G(z) \) as input (see Fig. 1). However, the discriminator of BiGAN gets all \( G(z), z, E(x) \), and \( x \) as input and judges whether the generated data \( G(z) \) is real or generated (fake). It assigns label one to each pair \((x, E(x))\) and label zero to each pair \((z, G(z))\). The loss function of BiGAN is:

\[
\min_{G,E} \max_D V(D,G,E) := \\
E_{z \sim p_{\text{data}}(z)} \left[ \log D(z, E(x)) \right] + \lambda \log p_z(z) + \frac{1}{2} \left[ \log D(x, E(x)) \right] + \frac{1}{2} \log \left( 1 - D(x, E(x)) \right).
\]

We use alternating optimization (Ghojogh et al., 2021c) by alternating between optimizing for \( D, G, \) and \( E \).

Theorem 14 (Donahue et al., 2017, Theorem 2)). After convergence (i.e., Nash equilibrium) of BiGAN, the optimal encoder \( E \) and generator \( G \) are inverse of each other:

\[
E^* = (G^*)^{-1}, \quad G^*(E^*(x)) = x, \quad E^*(G^*(z)) = z.
\]

5.13.2. Adversarially Learned Inference (ALI)

Adversarially Learned Inference (ALI) (Dumoulin et al., 2017) is one of the methods for having inverse in GAN. The generator \( G \) of ALI is an autoencoder whose encoder \( G_z(z) \) and decoder \( G_x(x) \) are called the generator network and the inference network, respectively. The generator network \( G_z(z) \) maps latent noise sample \( z \) to a generated data point \( \tilde{x} := G_z(z) \). The inference network \( G_x(x) \) maps a data point \( x \) to its corresponding latent noise sam-

5.13.3. The Inversion Technique

Another approach for having inverse in GAN is the inversion technique (Creswell & Bharath, 2018). For this, after training a GAN model, we find a noise sample which results in the generated data point:

\[
\max_z \mathbb{E} \left[ \log G(z) \right] + \lambda \log p_z(z),
\]

where \( p_z(z) \) is the desired prior distribution of latent space (e.g., \( \mathcal{N}(0, I) \)) and \( \lambda > 0 \) is the regularization parameter. This optimization can be performed using gradient descent.

5.14. Self-Attention GAN (SAGAN)

Attention mechanism (Vaswani et al., 2017) is weighting the features of data in a way that machine attends to the more important features by giving them larger weights (Ghojogh & Ghodsi, 2020). The weights are calculated by measuring the similarity of features with respect to each other using inner product. In self-attention, the similarities of features of every data point with other features of the same data point are calculated. These inner products are implemented within the convolutional layers of network. Self-Attention GAN (SAGAN) (Zhang et al., 2019) uses self-attention mechanism in the networks of both generator and discriminator. For the mathematical details of attention mechanism and SAGAN, refer to (Ghojogh & Ghodsi, 2020) and (Zhang et al., 2019), respectively.

5.15. Few-shot GAN Models

In the following, we introduce the GAN models which learn from few number of training data points.

5.15.1. Transfer Learning in GAN

Consider a GAN \( (G, D) \) which is already trained on some data in a source domain. Few-shot GAN (Ojha et al., 2017)
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...can do transfer learning where the trained GAN on the source domain also generates images from another target domain. In this method, we have an adapted generator $G_{s\rightarrow t}$ which is aimed to generate data points from the target domain. As the target domain has few data points in few-shot learning, it is prone to overfitting (Ghojogh & Crowley, 2019). Hence, we try to preserve the pairwise similarities before and after adaptation. For this, we draw a mini-batch of $(b+1)$ noise samples $\{z_i\}_{i=1}^{b+1}$ from the latent space. We feed these to the generators $G_s$ and $G_{s\rightarrow t}$. At the $\ell$-th layer, we calculate:

$$y_{s,i} := \text{softmax}(\text{sim}(G_s^\ell(z_i), G_s^\ell(z_j))),$$

$$y_{s\rightarrow t,i} := \text{softmax}(\text{sim}(G_{s\rightarrow t}^\ell(z_i), G_{s\rightarrow t}^\ell(z_j))),$$

for all $i \neq j, i, j \in \{1, \ldots, b+1\}$ where sim(.) denotes the cosine similarity. We want the adapted generator to have similar distributions across layers; hence we define the loss:

$$V(G_{s\rightarrow t}, G_s) := \mathbb{E}_{z_i \sim p_z}(\mathbb{E}_{y_i \sim D_{\text{anchor}}}[\sum_{\ell} \sum_i \text{KL}(y_{s,i}^\ell || y_{s\rightarrow t,i}^\ell)]),$$

where KL(.) denotes the KL-divergence.

We then sample $k$ number of random noises and call them the anchor points $Z_{\text{anchor}}$. This anchor space is a subset of the whole latent space $Z$. We have two discriminators which are $D_{\text{image}}$ for judging the whole image and $D_{\text{patch}}$ for judging an image patch. Let:

$$V(D_{\text{image}}, D_{\text{patch}}, G_{s\rightarrow t}) := \mathbb{E}_{x \sim D_t}[\mathbb{E}_{z \sim Z_{\text{anchor}}}[D_{\text{image}}(G_{s\rightarrow t}(z)) - D_{\text{image}}(x))] + \mathbb{E}_{x \sim p_x}(D_{\text{patch}}(G_{s\rightarrow t}(z)) - D_{\text{patch}}(x)),$$

where $D_t$ denotes the target domain. The overall loss function is:

$$\min_{G_{s\rightarrow t}, D_{\text{image}}, D_{\text{patch}}} \max_{G_s} V(D_{\text{image}}, D_{\text{patch}}, G_{s\rightarrow t}) + \lambda V(G_{s\rightarrow t}, G_s),$$

where $\lambda > 0$ is the regularization parameter. In this loss, the first term gives freedom to the structure of patches in the image and the second term takes care of transfer learning.

5.15.2. GAN WITH SINGLE IMAGE (SINGAN)

GAN with Single Image (SinGAN) (Shaham et al., 2019) learns to generate images by being trained on one image only. It generates images which are all related texture-wise to the training image. It learns the distributions of patches within the image in different scales and uses multi-scale adversarial learning. In the sense of using multiple scales in a Laplacian pyramid, it is similar to the LapGAN (Denton et al., 2015) (see Section 5.8.1). Assume we have $(k+1)$ levels $\{0, \ldots, k\}$ in the Laplacian pyramid where the level 0 is the image itself and the image is downsampled in other levels. At every $j$-th level, we have a GAN $(G_j, D_j)$. Training is from the $k$-th to the 0-th level. If $z_j$ is the latent noise at level $j$, the generations are:

$$x_k = G_k(z_k),$$

$$x_j = G_j(z_j, x_{j+1}), \quad \forall j < k,$$

where $x_{j+1}$ is the upsampled version of the generated image $x_{j+1}$. The GANs are trained sequentially and the previously trained GANs are kept fixed while training the next GAN. The loss function is regularized by a reconstruction error to make the model generate better images.

5.16. Training Triplet Network with GAN

A Siamese network (Bromley et al., 1993) is a network composed of multiple networks sharing their weights. If the number of networks is three, the Siamese network is a triplet network. Adversarial learning can be used for training a triplet network (Zieba & Wang, 2017). Consider triplets $(x_a, x_p, x_n)$ where $x_a$ is the anchor point, $x_p$ is the positive point having the same class as anchor, and $x_n$ is the negative point having a different class from anchor. For this, the loss function can be:

$$\min_\theta - \log \left( \frac{\exp(\|x_a - x_p\|^2)}{\exp(\|x_a - x_p\|^2) + \exp(\|x_a - x_n\|^2)} \right) - V(D, G),$$

where $\theta$ is the weights of network, the first term is the Neighborhood Component Analysis (NCA) (Goldberger et al., 2004), and the second term is the adversarial loss function. Paper (Zieba & Wang, 2017) uses Eq. (85) for the discriminator $D$.

6. Sampling and Interpolation in GAN

After training a GAN, we can generate new data points by sampling noise from the latent space and feeding it to the generator. There may exist two problems in sampling from the latent space (White, 2016). First, we should avoid sampling from the locations in the latent space which are highly unlikely. Secondly, as the latent space is usually high dimensional, there often exist some dead-zone locations in the latent space which are not trained during the training (Makhzani et al., 2015). In the following, we introduce some techniques for sampling and interpolation in the latent space. Note that these techniques can also be used for other generative models such as variational autoencoder (Kingma & Welling, 2014; Ghojogh et al., 2021a).

6.1. Interpolation in the Latent Space

For showing that the GAN model has not memorized the training data and the latent space is meaningful for the trained GAN, we can traverse different locations in the latent space and see what data points are generated from the
sampled noises. Traversing different locations in the latent space with some step is usually called interpolation in the latent space. A problem with linear interpolation, which has fixed step size, is that we traverse some highly unlikely priors. This can result in strange generated data points. Therefore, rather than the linear interpolation, we can use spherical linear interpolation (White, 2016), called slerp, to traverse a path on a \( p \)-dimensional hypersphere in the \( p \)-dimensional latent space. Assume we want to sample noises between locations \( z_1 \) and \( z_2 \) in the latent space. The interpolated locations are obtained as (Shoemake, 1985):

\[
\text{slerp}(z_1, z_2, \mu) := \frac{\sin((1 - \mu)\theta)}{\sin(\theta)} z_1 + \frac{\sin(\mu\theta)}{\sin(\theta)} z_2,
\]

where \( \mu \) is swept in range \([0, 1]\) and \( \theta := \cos^{-1}(z_1^T z_2) \).

We can have generated data points from the sampled noises by interpolation in the latent space. If we do interpolation across two perpendicular axes in the latent space, we can put the generations in a two dimensional table. An example for linear interpolation is shown in Fig. 8-a. Interpolation shows how the latent space is covering generation of various data points and what the shared features are between data points.

### 6.2. Manifold Interpolated Neighbor Embedding

Rather than reporting the generated data points from the sampled latent vectors in interpolation, we can find the nearest neighbor of the generated point among the training data points. The nearest neighbors for the generated points are then shown in a two dimensional grid. This is called the Manifold Interpolated Neighbor Embedding (MINE) (White, 2016). An example grid for MINE is shown in Fig. 8-b.

### 6.3. Analogy and J-Diagram

We can have vector arithmetic in the latent space (see Section 2.7.2). The vector arithmetic shows analogy relation between vectors. Let \( a, b, c, \) and \( d \) be the latent vectors associated with four generated data points by the generator.
We want to find the vector \( \mathbf{d} \) to satisfy the analogy relation:
\[
\mathbf{a} : \mathbf{b} :: \mathbf{c} : \mathbf{d} \implies (\mathbf{b} - \mathbf{a}) = (\mathbf{d} - \mathbf{c}).
\] (102)

In the natural language processing models, a famous analogy relation is “man : king :: woman : queen” (Mikolov et al., 2013). J-diagram (White, 2016) is a J-shape diagram whose top left corner, top right corner, bottom left corner, and bottom right corner are the generated images for the source vector \( \mathbf{a} \), analogy target vector \( \mathbf{b} \), analogy target vector \( \mathbf{c} \), and the result vector \( \mathbf{d} \), respectively. The other images inside the diagram are obtained by linear or slerp interpolation between these vectors. This diagram shows how an image is obtained from another by changing its features. An example J-diagram, for a GAN trained on the CelebA dataset (Liu et al., 2015), is shown in Fig. 8-c. As can be seen, moving along an axis changes some specific features of generated images. In this figure, the vertical axis takes care of gender and the horizontal axis is responsible for hair color, hair type, and facial pose.

### 6.4. Attribute Vector

We can obtain attribute vectors for an embedding space as follows (White, 2016). For example, a smile vector (Larsen et al., 2016) can be obtained by subtracting the latent vector for a neutral face from the latent vector for the smiling face of the same person. The resulted vector can be considered as the latent vector for smiling. Other attribute vectors can be obtained similarly. An attribute vector can be used to change a neutral image to an image having that attribute. For example, we can add the smiling latent vector, denoted by \( \mathbf{z}_s \in \mathbb{R}^p \), to the latent vector of a (neutral) face, denoted by \( \mathbf{z}_{n} \in \mathbb{R}^p \), to obtain a new latent vector which results in generation of a smiling face of that person, after being fed to the generator. Let \( \eta \in \mathbb{R} \) be the weight for smiling. The vector \( \mathbf{z}_n + \eta \mathbf{z}_s \) is the latent vector for face with different levels of smiling. A negative \( \eta \) makes a smiling face neutral. An example of traversal along the smile vector is shown in Fig. 8-d.

### 6.5. Evaluation of Generated Images

**Remark 3** (The Inception score (Salimans et al., 2016, Section 4)). A score, named the Inception score, can be used to assess the quality of generated images by GAN models. For this, we feed the generated images \( \mathbf{x} \) to the Inception network (Szegedy et al., 2016) which outputs predicted labels \( \hat{p}(y|\mathbf{x}) \) where \( y \) is the label. On one hand, we desire this conditional label distribution to have low entropy. On the other hand, we want the generator to generate various images; hence, the marginal \( p(y) = \int p(y|\mathbf{x})d\mathbf{x} \) for \( \mathbf{x} = G(\mathbf{z}) \) should be large. The Inception score combines these two as:
\[
\text{Inception score} = \exp\left( E_{\mathbf{x}} \left( \text{KL}(p(y|\mathbf{x})||p(y)) \right) \right). \tag{103}
\]

The higher this score, the more quality the generated image has. It has been observed that this score is very similar to human’s evaluation of the generated images (Salimans et al., 2016).

Note that there exists another method for quantitative analysis of GAN results (Wu et al., 2017) which is based on the annealed importance sampling (Neal, 2001).

### 7. Applications of GAN

We already saw that GAN can be used for data generation for any data type such as image. In the following, we introduce some other applications of GAN.

#### 7.1. Image-to-Image Translation by GAN

There exist some methods, based on GAN, for image-to-image translation where an image is generated corresponding to an input image. The correspondence can be any relation in different applications. In the following, we introduce these methods.

**7.1.1. PatchGAN**

PatchGAN (Isola et al., 2017) uses conditional GAN (Mirza & Osindero, 2014) (see Section 2.6) with a regularized loss function. It uses \( \ell_1 \) norm between data and generated data for regularization because \( \ell_1 \) norm encourages less blurring compared to \( \ell_2 \) norm. The loss is:
\[
\min_D \max_G V_C'(D, G) + \lambda \mathbb{E}_{\mathbf{x}, \mathbf{z}, \mathbf{y}} \left[ ||\mathbf{x} - G(\mathbf{z}, \mathbf{y})||_1 \right],
\] (104)

where \( \lambda > 0 \) is the regularization parameter and \( V_C'(D, G) \) is a slightly modified version of Eq. (22):
\[
V_C'(D, G) := \mathbb{E}_{\mathbf{x}, \mathbf{y}} \left[ \log(D(\mathbf{x}, \mathbf{y})) \right] + \mathbb{E}_{\mathbf{z}, \mathbf{y}} \left[ \log \left(1 - D(G(\mathbf{z}, \mathbf{y}), \mathbf{y})\right) \right],
\] (105)

in which \( \mathbf{x} \) is the data, \( \mathbf{y} \) is the label of data, and \( \mathbf{z} \sim p_z(\mathbf{z}) \) is the noise. The generator \( G \) takes the noise \( \mathbf{z} \) and label \( \mathbf{y} \) as input and generates data denoted by \( G(\mathbf{z}, \mathbf{y}) \). The discriminator takes the data point \( \mathbf{x} \) and its label \( \mathbf{y} \) as input. It judges whether the data point \( \mathbf{x} \) is real or generated.

For the generator \( G \), PatchGAN uses skips or connections between every layer \( \ell \) and layer \( (L - \ell) \) where \( L \) is the number of layers. This is inspired by the structure of U-Net (Ronneberger et al., 2015). Moreover, the \( \ell_1 \) norm, used in Eq. (104), takes care of the low-frequency features of generated image (Isola et al., 2017). Therefore, the discriminator should take care of the high-frequency features. For this, the discriminator \( D \) classifies the image patch-wise rather than the whole image. Every patch is judged to be whether it is real or generated (fake). We average the judgments of patches to have model averaging for classifying the whole image. This patch-wise classification of an image models the image as a Markov random field because
it assumes that every patch of pixels is independent of other patches.

The PatchGAN has been used for image-to-image translation without the need to pairs of training images (in contrast to PatchGAN which needs pairs of images). Let the two domains of image translation be $X$ and $Y$. In cycleGAN, we have two generators $G : X \rightarrow Y$ and $F : Y \rightarrow X$. Two discriminators also exist; one is $D_X$ for judging images in $X$ and $F(Y)$ and the other is $D_Y$ for judging images in $Y$ and $G(X)$. Hence, we have two GAN losses:

$$V(D_Y, G, X, Y) := \mathbb{E}_{y \sim \text{Plan}}(y) \left[ \log(D_Y(y)) \right] + \mathbb{E}_{x \sim \text{Plan}}(x) \left[ \log(1 - D_Y(G(x))) \right],$$

$$V(D_X, F, X, Y) := \mathbb{E}_{x \sim \text{Plan}}(x) \left[ \log(D_X(x)) \right] + \mathbb{E}_{y \sim \text{Plan}}(y) \left[ \log(1 - D_X(F(y))) \right].$$

We also define the following cycle consistency loss to have $F(G(x)) \approx x$ and $G(F(y)) \approx y$:

$$V_{\text{cycle}}(G, F) := \mathbb{E}_{x \sim \text{Plan}}(x) \left[ \|F(G(x)) - x\|_1 \right] + \mathbb{E}_{y \sim \text{Plan}}(y) \left[ \|G(F(y)) - y\|_1 \right].$$

The overall loss function of CycleGAN is:

$$\min G, F \max D_X, D_Y \quad V(D_Y, G, X, Y) + V(D_X, F, X, Y) + \lambda V_{\text{cycle}}(G, F),$$

where $\lambda > 0$ is the regularization parameter. A result of CycleGAN is shown in Fig. 9-e.

### 7.1.3. Deep Face Drawing

DeepFaceDrawing (Chen et al., 2020) generates high-quality facial images from input sketches of faces. For training data, automatic sketches have been created using the Canny edge detection (Canny, 1986). DeepFaceDrawing has three modules. The first one is the component embedding module which takes different facial patches as input and learns embedding vectors for them. Then, these vectors are fed to the feature mapping module which transform the embedding vectors to 2D facial features patches. These feature patches are then fed to the image synthesis module which is a conditional GAN (see Section 2.6), generating facial images from the feature patches. A result of DeepFaceDrawing is shown in Fig. 9-f.

### 7.1.4. Simulated GAN (SimGAN)

Simulated GAN (SimGAN) (Shrivastava et al., 2017) is an unsupervised method for transforming simulated images to real-world images while preserving the annotation information of images, such as image landmarks and pose of image. This transformation is performed by a refiner $R(.)$. Let $y_i$, $x_i$, and $\bar{x}_i$ denote the training unlabeled real-world images, the training simulated images, and the transformation of the training simulated images to real world, i.e., $\bar{x}_i = R(x_i)$. In SimGAN, we train a discriminator $D$ by minimizing the loss:

$$\min_D \left[ - \sum_i \log(D(\bar{x}_i)) - \sum_j \log(1 - D(y_j)) \right],$$

Figure 9. Image-to-image translation: (a) coloring a sketch, (b) changing daylight to night darkness in image, (c) changing an aerial image to a map, (d) coloring a black-and-white image, (e) transforming zebra to horse and vice versa, and (f) generating a facial image from a facial sketch. Transformations in (a), (b), (c), and (d) are by PatchGAN whose credits are for (Isola et al., 2017). Transformations in (e) and (f) are by CycleGAN (credit: (Zhu et al., 2017)) and DeepFaceDrawing (credit: (Chen et al., 2020)), respectively.
so $D$ generates labels close to one and zero for the real-world and simulated images, respectively. After the discriminator is trained, we use it in the loss function of refiner. The refiner acts like the generator in GAN so it tries to confuse the discriminator; hence, the loss of refiner is:

$$\min_{\hat{R}} - \sum_i \log(1 - D(R(x_i))) + \lambda \| \psi(R(x_i)) - \psi(x_i) \|_1,$$

where $\lambda > 0$ is the regularization parameter, $\psi(.)$ is a mapping from the pixel space to a feature space, and the second term tries to minimize the reconstruction error in the feature space.

### 7.1.5. Interactive GAN (iGAN)

Interactive GAN (iGAN) (Zhu et al., 2016) allows users to edit the image interactively while the edited image remains realistic. In iGAN, we first project the image onto the manifold of latent noise in GAN. This projection is done by finding the closest latent noise which can generate the image:

$$z^* := \arg \min_z \| G(z) - x \|^2_2.$$  

In this sense, this projection is similar to the approach of inverse GAN models (see Section 5.13). Then, we edit the projected image, i.e., $z^*$, by different brushing and editing tools. Then, we add back the geometric and color changes to re-obtain the image, but edited this time.

### 7.2. Text-to-Image Generation

There exist several methods for text-to-image generation where an image is generated from some descriptive caption. Some of these methods are (Reed et al., 2016a; b; Zhang et al., 2017; Reed et al., 2017; Nguyen et al., 2017a; Zhang et al., 2018). Here, we introduce Stacked GAN (StackGAN) (Zhang et al., 2017) for text-to-image generation.

In StackGAN, we first generate embedding of texts by a pre-trained autoencoder. Let the text and the embedding of text be denoted by $t$ and $\phi_t$, respectively. We have a stack of two stages of adversarial learning where the first stage generates a low-resolution image by drawing merely the shapes and colors. The loss function of the first stage is:

$$\min_D \mathbb{E}_{x,t} \sim p_{data}(x,t)[\log(D(x, \phi_t))] + \mathbb{E}_{t \sim p_{data}(t), z \sim p_z(z)}[\log(1 - D(G(z), \phi_t))],$$

$$\min_G \mathbb{E}_{t \sim p_{data}(t), z \sim p_z(z)}[\log(1 - D(G(z), \phi_t))] + \lambda KL(q_z(z) || p_z(z)),  \tag{108}$$

where $q_z(z)$ is the distribution of the latent code from the encoder of an autoencoder and $p_z(z)$ is the prior on the latent noise. The next stage takes the low-resolution generated image from the first stage, denoted by $s$, as well as the text embedding as input and generates a high-resolution image. The adversarial loss of the second stage is the same as Eqs. (107) and (108) but it has $G(s)$ rather than $G(z)$ because the low-resolution image is fed to its generator. Some results of StackGAN are shown in Fig. 10. An improved version of StackGAN is StackGAN++ (Zhang et al., 2018).

### 7.3. Mixing Image Characteristics

#### 7.3.1. FineGAN

FineGAN (Singh et al., 2019) is an unsupervised GAN model which disentangles the features of the generated image to background, shape, and color/texture. For this, we have three separate latent noise samples, i.e., the background code $b$, the parent code $p$, and the child code $c$, responsible for the background, shape, and color/texture, respectively. We assume we have $n_b$, $n_p$, and $n_c$ unknown categories (classes) for the background, shape, and color/texture, respectively, which will be learned by the FineGAN. The priors for the latent codes are categorical distribution where the probability of every class is $1/n_b$, $1/n_p$, and $1/n_c$, respectively. As every shape of some object may have several various textures in different images, we take $n_p < n_c$.

FineGAN generates an image hierarchically. It starts with generating the background. For training data, we use a pre-trained detector to detect the background patches. We also use a continuous latent code $z_b$ which controls the background details within every category of background. The generator $G_b$ takes both $b$ and $z_b$ as input and $D_b$ is the discriminator for judging the generated background. We also use another discriminator $D'_b$ which is a binary classifier to two classes of foreground and background. This discriminator is pre-trained by cross entropy on the background and background. The discriminator for judging the generated background. We also use another discriminator $D'_b$ which is a binary classifier to two classes of foreground and background. This discriminator is pre-trained by cross entropy on the background and foreground.
foreground patches. The loss of the background stage is:
\[
\min_{G_b, D_b} \mathbb{E}_x [\log(D_b(x))] \\
+ \mathbb{E}_{b, z_b} [\log(1 - D_b(G(b, z_b)))] \\
+ \lambda \mathbb{E}_{b, z_b} [\log(1 - D_b'(G(b, z_b)))]
\]
(109)
where \( \lambda > 0 \) is the regularization parameter.

In the parent stage, we have two generators \( G_{p,m} \) and \( G_{p,f} \) generating the mask and initial texture of the object, respectively. A network \( G_p \) takes the categorical \( p \) and continuous \( z_p \) as input and outputs \( z_p' \), which is the input code for \( G_{p,m} \) and \( G_{p,f} \). The \( z_p \) controls the initial texture. The two generations of \( G_{p,m} \) and \( G_{p,f} \) are glued together to obtain the shape of object with some initial texture, which we denote by \( x_p \). Then, we stitch it to the background obtained before. If the discriminator of this stage is \( D_p \), the loss of this stage maximizes the mutual information between \( p \) and \( x_p \) as:
\[
\max_{D_p, G_{p,m}, G_{p,f}} \mathbb{E}_{p, z_p} [\log(D_p(p|x_p))].
\]
(110)

In the child stage, we have two generators \( G_{c,m} \) and \( G_{c,f} \) generating the mask and color/texture of the object, respectively. A network \( G_c \) takes \( c \) and \( z_p' \) as input and outputs \( z_c' \), which is the input code for \( G_{c,m} \) and \( G_{c,f} \). The two generations of \( G_{c,m} \) and \( G_{c,f} \) are glued together to obtain the shape of object with color/texture, which we denote by \( x_c \). Then, we stitch it to \( x_p \) obtained before, to have the final generated image \( x_f \). The loss of the background stage is:
\[
\min_{G_c} \mathbb{E}_x [\log(D_c(x))] + \mathbb{E}_{c, p, z_p} [\log(1 - D_c(x_f))] \\
+ \max_{D_c, G_{c,m}, G_{c,f}} \mathbb{E}_{b, z_b} [\log(1 - D_c'(c|x_c))]
\]
(111)
where the first two terms are for adversarial learning and the last term is for maximizing the mutual information. Some results of FineGAN are illustrated in Fig. 11-a.

7.3.2. MIXNMATCH

MixNMatch (Li et al., 2020) is built upon FineGAN introduced in Section 7.3.1. It gives the user the opportunity to choose the background, shape, and color/texture from three pictures and it generates an image with the chosen characteristics. For this, we need an encoder network \( E(x) \) which gets three images for their background, shape, and color/texture characteristics and outputs the three latent codes \( b, p, \) and \( c \). These codes are then fed to FineGAN.

In MixNMatch, we use the idea of inverse in GAN (see Section 5.13) to have the input of the encoder and FineGAN networks. The input/output pair of encoder is \((x \sim p_{data}(x), \tilde{y} = E(x))\) where \( \tilde{y} \) is the codes \( b, p, \) and \( c \). The output/input pair of the FineGAN is \((\tilde{x} = G(y), y \sim p_{code}(y))\) where \( G(\cdot) \) denotes the FineGAN and \( p_{code}(y) \) is the prior distribution of the latent codes \( b, p, \) and \( c \). We have a discriminator \( D \) which takes an image-code pair and judges whether it is the pair of encoder or the FineGAN. The loss of MixNMatch is:
\[
\min_{G, E} \max_D \mathbb{E}_{x \sim p_{data}(x)} [\mathbb{E}_{y \sim E(x)} [\log(D(x, \tilde{y}))]] \\
+ \mathbb{E}_{y \sim p_{code}(y)} [\mathbb{E}_{\tilde{x} \sim G(y)} [\log(1 - D(\tilde{x}, y))]]
\]
(112)

MixNMatch lets the user choose image characteristics from the same domain and the generated image is from that domain. An example result of MixNMatch is shown in Fig. 11-b. Recently, an improved version of MixNMatch (Ojha et al., 2021b) can take the characteristics from multiple domains and generate a new image having those characteristics. An example result of this version is also shown in Fig. 11-c.

7.4. Other Applications

There are some other applications for GAN. One of the applications is inpainting some lost parts of image with GAN (Pathak et al., 2016). GAN learns to inpaint the lost part based on the available pixels in the image. A medical application of GAN is generating histopathology images which can give insight into cancer diagnosis from pathology whole slide images (Levine et al., 2020). GAN has also been used for NLP (Li et al., 2018; Wang et al., 2019), speech processing (Pascual et al., 2017; Sriram et al., 2018), network embedding (Dai et al., 2018), logic (Nagisetty et al., 2021), and sketch retrieval (Creswell & Bharath, 2016).
8. Autoencoders Based on Adversarial Learning

Previously, variational Bayes was used in an autoencoder setting to have variational autoencoder (Kingma & Welling, 2014; Ghojogh et al., 2021a). Likewise, adversarial learning can be used in an autoencoder setting (Makhzani, 2018b). Several adversarial-based autoencoders exist which we introduce in the following.

8.1. Adversarial Autoencoder (AAE)

8.1.1. UNSUPERVISED AAE

Adversarial Autoencoder (AAE) was proposed in (Makhzani et al., 2015). In contrast to variational autoencoder (Kingma & Welling, 2014; Ghojogh et al., 2021a) which uses KL divergence and evidence lower bound, AAE uses adversarial learning for imposing a specific distribution on the latent variable in its coding layer. The structure of AAE is depicted in Fig. 12. Each of the blocks $B_1$, $B_2$, and $B_3$ in this figure has several network layers with nonlinear activation functions. AAE has an encoder (i.e., block $B_1$) and a decoder (i.e., block $B_2$). The input of encoder is a real data point $x \in \mathbb{R}^d$ and the output of decoder is the reconstructed data point $\hat{x} \in \mathbb{R}^d$. One of the low-dimensional middle layers is the latent (or code) layer, denoted by $z \in \mathbb{R}^p$, where $p \ll d$. The encoder and decoder model conditional distributions $p(z|x)$ and $p(x|z)$, respectively. Let the distribution of the latent variable in the autoencoder be denoted by $q(z)$. This is the posterior distribution of latent variable. The blocks $B_1$ and $B_3$ are the generator $G$ and discriminator $D$ of adversarial network, respectively. We also have a prior distribution, denoted by $p(z)$, on the latent variable which is chosen by the user. This prior distribution can be a $p$-dimensional normal distribution $\mathcal{N}(0, I)$, for example. The encoder of the autoencoder (i.e., block $B_1$) is the generator $G$ which generates the latent variable from the posterior distribution:

$$G(x) = z \sim q(z).$$ (113)

The discriminator $D$ (i.e., block $B_3$) has a single output neuron with sigmoid activation function. It classifies the latent variable $z$ to be a real latent variable from the prior distribution $p(z)$ or a generated latent variable by the encoder of autoencoder:

$$D(z) := \begin{cases} 1 & \text{if } z \text{ is real, i.e., } z \sim p(z) \\ 0 & \text{if } z \text{ is generated, i.e., } z \sim q(z). \end{cases}$$ (114)

As was explained, the block $B_1$ is shared between the autoencoder and the adversarial network. This adversarial learning makes both autoencoder and adversarial network stronger gradually because the autoencoder tries to generate the latent variable which is very similar to the real latent variable from the prior distribution. In this way, it tries to fool the discriminator. The discriminator, on the other hand, tries to become stronger not to be fooled by the encoder of autoencoder.

In AAE, we have alternating optimization (Ghojogh et al., 2021c) where reconstruction and regularization phases are repeated iteratively. In the reconstruction phase, the mean squared error is minimized between the data $x$ and the reconstructed data $\hat{x}$. In the regularization phase, the discriminator and generator are updated using the GAN approach. For each of these updates, we use stochastic gradient descent (Ghojogh et al., 2021c) with backpropagation. Overall, the two phases are performed as:

$$B_1', B_2^{(k+1)} := \arg \min_{B_1, B_2} \|\hat{x} - x\|_2^2,$$ (115)

$$\begin{align*}
B_3^{(k+1)} & := B_3^{(k)} - \eta^{(k)} \frac{\partial}{\partial B_3} \left(V(B_3, B_1')\right), \\
B_1^{(k+1)} & := B_1' - \eta^{(k)} \frac{\partial}{\partial B_1} \left(V(B_3^{(k+1)}, B_1)\right),
\end{align*}$$ (116)

where $B_1 = G$ and $B_3 = D$ (see Fig. 12). Eq. (115) is the reconstruction phase and Eq. (116) is the regularization phase.

8.1.2. SAMPLING THE LATENT VARIABLE

There are several approaches for sampling the latent variable $z$ from the coding layer of autoencoder with posterior $q(z)$. In the following, we explain these approaches (Makhzani et al., 2015):

- Deterministic approach: the latent variable is the output of encoder directly, i.e., $z_i = B_1(x_i)$. The stochasticity in $q(z)$ is in the distribution of dataset, $p_{data}(x)$.

- Gaussian posterior: this approach is similar to what we have in variational autoencoder (Kingma & Welling, 2014; Ghojogh et al., 2021a). The encoder outputs the mean $\mu$ and covariance $\Sigma$ and the latent variable is sampled from the Gaussian distribution, i.e., $z_i \sim \mathcal{N}(\mu(x_i), \Sigma(x_i))$. The stochasticity in $q(z)$ is in both $p_{data}(x)$ and the Gaussian distribution as output of encoder.

- Universal approximator posterior: we concatenate the data point $x$ and some noise $\eta$, with a fixed dis-
distribution such as Gaussian, as input to the encoder. Hence, the latent variable is \( z_i = B_1(x_i, \eta_i) \) where \( \eta_i \sim \mathcal{N}(0, I) \). The stochasticity in \( q(z) \) is in both \( p_{\text{data}}(x) \) and the noise \( \eta \).

8.1.3. Supervised AAE

We have two variants for supervised AAE (Makhzani et al., 2015) where the class labels are utilized. These two structures are illustrated in Fig. 13. Let \( c \) denote the number of classes. In the first variant, we feed the one-hot encoded label \( y \in \mathbb{R}^c \) to the discriminator, i.e. block \( B_3 \), by concatenating it to the latent variable \( z \). In this way, the discriminator learns the label of point \( x \) as well as discrimination of real and generated latent variables. This makes the generator or the encoder to generate the latent variables corresponding to the label of point for fooling the discriminator.

In the second variant of supervised AAE, the one-hot encoded label \( y \) is fed to the decoder, i.e. block \( B_2 \), by concatenating it to the latent variable \( z \). In this way, the decoder learns to reconstruct the data point by using the label of point. This also makes the encoder, which is also the generator, generate the latent variable \( z \) based on the label of point. Hence, the discriminator also gets stronger for competing the generator, in adversarial learning. Note that the two variants can also be combined, i.e., we can feed the one-hot encoded label can be fed to both the discriminator and the decoder.

8.1.4. Semi-supervised AAE

Consider a partially labeled dataset. The labeled part of data has \( c \) number of classes. AAE can be used for semi-supervised learning with partially labeled dataset. The structure for semi-supervised AAE is depicted in Fig. 14. This structure includes an autoencoder (blocks \( B_1 \) and \( B_2 \)), an adversarial learning for generating latent variable (blocks \( B_1 \) and \( B_3 \)), and an adversarial learning for generating class labels (blocks \( B_1 \) and \( B_4 \)). The encoder generates both label \( y \in \mathbb{R}^c \) and latent variable \( z \in \mathbb{R}^p \). The last layer of encoder for label has softmax activation function to output a \( c \)-dimensional vector whose entries sum to one (behaving as probability). The last layer of encoder for latent variable has linear activation function.

It has three phases which are reconstruction, regularization, and semi-supervised classification. In the reconstruction phase, we minimize the reconstruction error. The regularization phase trains the discriminator and generator for generating the latent variable \( z \). The semi-supervised classification phase generates the one-hot encoded class label \( y \) for the point \( x \). If the point \( x \) has a label, we use its label for training \( B_1 \) and \( B_4 \). However, if the point \( x \) does not have any label, we randomly sample a label \( y \in \mathbb{R}^c \) from a categorical distribution, i.e., \( y \sim \text{Cat}(y) \). This categorical distribution gives a one-hot encoded vector where the prior probability of every class is estimated by the proportion of class’s population to the total number of labeled points. An iteration of the alternating optimization for semi-supervised learning is:

\[
B'_1, B_2^{(k+1)} := \arg\min_{B_1, B_2} \| \hat{x} - x \|^2, \quad (117)
\]

\[
\begin{align*}
B_3^{(k+1)} &:= B_3^{(k)} - \eta^{(k)} \frac{\partial}{\partial B_3} \left( V_z(B_3, B_1') \right), \\
B_1' &:= B_1' - \eta^{(k)} \frac{\partial}{\partial B_1} \left( V_y(B_3^{(k+1)}, B_1) \right), \\
B_4^{(k+1)} &:= B_4^{(k)} - \eta^{(k)} \frac{\partial}{\partial B_4} \left( V_y(B_4, B_4') \right), \\
B_1^{(k+1)} &:= B_1'' - \eta^{(k)} \frac{\partial}{\partial B_1} \left( V_y(B_4^{(k+1)}, B_1) \right),
\end{align*}
\]

where \( V_z(D, G) \) and \( V_y(D, G) \) are the loss functions defined in Eq. (3) in which the generated variables are the latent variable \( z \) and the one-hot encoded label \( y \), respectively.

8.1.5. Unsupervised Clustering with AAE

We can use the structure of Fig. 14 for clustering but rather than the classes, we assume we have \( c \) number of clusters.
We do not have a partially labeled part of dataset. All points are unlabeled and the cluster indices are sampled randomly by the categorical distribution. The cluster labels and the latent code are both trained in the three phases which were explained in Section 8.1.4.

8.1.6. Dimensionality Reduction with AAE

The AAE can be used for dimensionality reduction and representation learning. The structure of AAE for this purpose is depicted in Fig. 15. The encoder generates both label \( y \in \mathbb{R}^c \) and latent variable \( z \in \mathbb{R}^p \) where \( p \ll d \). Everything is similar to what we had before but a network layer \( W \in \mathbb{R}^{c \times p} \) is added after the generated label by the encoder. The low-dimensional representation \( \tilde{x} \in \mathbb{R}^p \) is obtained as:

\[
\mathbb{R}^p \ni \tilde{x} = W^\top y + z, \tag{120}
\]

where \( z \) is the latent variable generated by the encoder. The three phases explained in Section 8.1.4 trains the AAE for dimensionality reduction.

8.2. PixelGAN Autoencoder

In variational inference (Ghojogh et al., 2021a), the Evidence Lower Bound (ELBO) can be restated as (Hoffman & Johnson, 2016):

\[
\begin{align*}
\mathbb{E}_{x \sim p_{data}}[\log(p(x))] & > \quad -\mathbb{E}_{x \sim p_{data}}[\mathbb{E}_{q(z|x)}[-\log(p(x|z))]] \\
& - KL(q(z)||p(z)) - I(z; x), \tag{121}
\end{align*}
\]

where \( I(.; .) \) denotes the mutual information. The first and second terms in this lower bound are the reconstruction error and the marginal KL divergence on the latent space. The PixelGAN autoencoder (Makhzani & Frey, 2017) uses this lower bound but ignores its third term which is the mutual information because optimization of that term makes \( z \) be independent of \( x \). The reconstruction error is minimized in a reconstruction phase of training and the KL divergence part is taken care of by an adversarial learning.

The structure of PixelGAN is shown in Fig. 16. The block \( B_1 \) is the encoder which gets the data point \( x \) added with some noise \( n \) as input and outputs the latent code \( z \sim q(z|x) \). The block \( B_2 \) is the decoder which is a PixelCNN network (Oord et al., 2016) from which PixelGAN has borrowed its name. This decoder outputs the reconstructed data \( \hat{x} \). The generated latent code \( z \) is used as the adaptive biases of layers in the PixelCNN. The blocks \( B_1 \) and \( B_3 \) are the generator and discriminator of adversarial learning, respectively, where we try to make the distribution of the generated latent code \( z \) similar to some prior distribution \( p(z) \). In summary, blocks \( B_1 \) and \( B_2 \) are used for the reconstruction phase and blocks \( B_1 \) and \( B_3 \) are used for the adversarial learning phase.

8.3. Implicit Autoencoder (IAE)

In variational inference (Ghojogh et al., 2021a), the Evidence Lower Bound (ELBO) can be restated as (Makhzani, 2018a):

\[
\mathbb{E}_{x \sim p_{data}}[\log(p(x))] \geq - KL(q(x, z)||q(\bar{x}, z)) - KL(q(z)||p(z)) - H_{data}(x), \tag{122}
\]
where $H_{\text{data}}(x)$ is the entropy of data, $\hat{x}$ is the reconstructed data, and $z$ is some latent factor. The proof is straightforward and can be found in (Makhzani, 2018a, Appendix A). The first and second terms are the reconstruction and regularization terms, respectively. The Implicit Autoencoder (IAE) (Makhzani, 2018a) implements the above distributions in Eq. (122), implicitly using networks. The structure of IAE is shown in Fig. 17. The block $B_1$ is the encoder which takes data $x$ and noise $n_1$ as input and outputs the latent code $z \sim q(z)$. The block $B_2$ takes the generated latent code $z$ as well as some noise $n_2$ and outputs the reconstructed data $\hat{x}$. The blocks $B_1$ and $B_3$ are the generator $G_1$ and discriminator $D_1$ of the first adversarial learning used for making the distribution of latent code $z$ similar to some prior distribution $p(z)$. The blocks $B_1$ and $B_4$ are the generator $G_2$ and discriminator $D_2$ of the second adversarial learning used for making the distribution of reconstructed data $\hat{x}$ similar to data $x$. The inputs of $B_4$ are the pairs $(x, z)$ and $(\hat{x}, z)$ to model the distributions $q(x, z)$ and $q(\hat{x}, z)$ in Eq. (122). In summary, three phases of training are performed which are the reconstruction phase and the two adversarial learning phases.

9. Conclusion

This was a tutorial and survey paper on GAN, adversarial learning, adversarial autoencoder, and their variants. We covered various aspects and theories of the methods as well as applications of GAN.

References

Arjovsky, Martin and Bottou, Léon. Towards principled methods for training generative adversarial networks. In International Conference on Machine Learning, 2017.

Arjovsky, Martin, Chintala, Soumith, and Bottou, Léon. Wasserstein generative adversarial networks. In International conference on machine learning, pp. 214–223, 2017.

Arora, Sanjeev, Ge, Rong, Liang, Yingyu, Ma, Tengyu, and Zhang, Yi. Generalization and equilibrium in generative adversarial nets (GANs). In International Conference on Machine Learning, pp. 224–232, 2017.

Berthelot, David, Milanfar, Peyman, and Goodfellow, Ian. Creating high resolution images with a latent adversarial generator. arXiv preprint arXiv:2003.02365, 2020.

Bourgain, Jean. On Lipschitz embedding of finite metric spaces in Hilbert space. Israel Journal of Mathematics, 52(1-2):46–52, 1985.

Bromley, Jane, Bentz, James W, Bottou, Léon, Guyon, Isabelle, LeCun, Yann, Moore, Cliff, Säckinger, Eduard, and Shah, Roopak. Signature verification using a “siamese” time delay neural network. International Journal of Pattern Recognition and Artificial Intelligence, 7(04):669–688, 1993.

Burt, Peter J and Adelson, Edward H. The Laplacian pyramid as a compact image code. IEEE Transactions on Communications, 31(4):532–540, 1983.

Canny, John. A computational approach to edge detection. IEEE Transactions on Pattern analysis and machine intelligence, (6):679–698, 1986.

Chen, Shu-Yu, Su, Wanchao, Gao, Lin, Xia, Shihong, and Fu, Hongbo. DeepFaceDrawing: Deep generation of face images from sketches. ACM Transactions on Graphics (TOG), 39(4):72–1, 2020.

Chen, Xi, Duan, Yan, Houthooft, Rein, Schulman, John, Sutskever, Ilya, and Abbeel, Pieter. InfoGAN: Interpretable representation learning by information maximizing generative adversarial nets. In Proceedings of the 30th International Conference on Neural Information Processing Systems, pp. 2180–2188, 2016.

Chien, Jen-Tzung and Kuo, Chun-Lin. Variational Bayesian GAN. In 2019 27th European Signal Processing Conference (EUSIPCO), pp. 1–5. IEEE, 2019.

Creswell, Antonia and Bharath, Anil Anthony. Adversarial training for sketch retrieval. In European Conference on Computer Vision, pp. 798–809. Springer, 2016.
Creswell, Antonia and Bharath, Anil Anthony. Inverting the generator of a generative adversarial network. *IEEE transactions on neural networks and learning systems*, 30(7):1967–1974, 2018.

Creswell, Antonia, White, Tom, Dumoulin, Vincent, Arulkumaran, Kai, Sengupta, Biswa, and Bharath, Anil A. Generative adversarial networks: An overview. *IEEE Signal Processing Magazine*, 35(1):53–65, 2018.

Dai, Quanyu, Li, Qiang, Tang, Jian, and Wang, Dan. Adversarial network embedding. In *Proceedings of the AAAI Conference on Artificial Intelligence*, volume 32, 2018.

Denton, Emily, Chintala, Soumith, Szlam, Arthur, and Fergus, Rob. Deep generative image models using a laplacian pyramid of adversarial networks. *arXiv preprint arXiv:1506.05751*, 2015.

Donahue, Jeff, Krähenbühl, Philipp, and Darrell, Trevor. Adversarial feature learning. In *International Conference on Learning Representations*, 2017.

Du, Ding-Zhu and Pardalos, Panos M. *Minimax and applications*, volume 4. Springer Science & Business Media, 2013.

Dumoulin, Vincent, Belghazi, Ishmael, Poole, Ben, Mastropietro, Olivier, Lamb, Alex, Arjovsky, Martin, and Courville, Aaron. Adversarially learned inference. In *International Conference on Learning Representations*, 2017.

Durugkar, Ishan, Gemp, Ian, and Mahadevan, Sridhar. Generative multi-adversarial networks. In *International Conference on Learning Representations*, 2017.

Farnia, Farzan and Ozdaglar, Asuman. Do GANs always have Nash equilibria? In *International Conference on Machine Learning*, pp. 3029–3039, 2020.

Farnia, Farzan and Tse, David. A convex duality framework for GANs. In *Advances in neural information processing systems*, volume 31, 2018.

Freund, Yoav and Schapire, Robert E. A decision-theoretic generalization of on-line learning and an application to boosting. *Journal of computer and system sciences*, 55(1):119–139, 1997.

Ghojogh, Benyamin and Crowley, Mark. The theory behind overfitting, cross validation, regularization, bagging, and boosting: tutorial. *arXiv preprint arXiv:1905.12787*, 2019.

Ghojogh, Benyamin and Ghodsi, Ali. Attention mechanism, transformers, BERT, and GPT: Tutorial and survey. 2020.

Ghojogh, Benyamin, Ghojogh, Aydin, Crowley, Mark, and Karray, Fakhri. Fitting a mixture distribution to data: tutorial. *arXiv preprint arXiv:1901.06708*, 2019.

Ghojogh, Benyamin, Nekoei, Hadi, Ghojogh, Aydin, Karray, Fakhri, and Crowley, Mark. Sampling algorithms, from survey sampling to Monte Carlo methods: Tutorial and literature review. *arXiv preprint arXiv:2011.00901*, 2020.

Ghojogh, Benyamin, Ghodsi, Ali, Karray, Fakhri, and Crowley, Mark. Factor analysis, probabilistic principal component analysis, variational inference, and variational autoencoder: Tutorial and survey. *arXiv preprint arXiv:2101.00734*, 2021a.

Ghojogh, Benyamin, Ghodsi, Ali, Karray, Fakhri, and Crowley, Mark. Johnson-Lindenstrauss lemma, linear and nonlinear random projections, random Fourier features, and random kitchen sinks: Tutorial and survey. *arXiv preprint arXiv:2108.04172*, 2021b.

Ghojogh, Benyamin, Ghodsi, Ali, Karray, Fakhri, and Crowley, Mark. Reproducing kernel Hilbert space, Mercer’s theorem, eigenfunctions, Nyström method, and use of kernels in machine learning: Tutorial and survey. *arXiv preprint arXiv:2106.08443*, 2021d.

Goldberger, Jacob, Hinton, Geoffrey E, Roweis, Sam, and Salakhutdinov, Russ R. Neighbourhood components analysis. *Advances in neural information processing systems*, 17, 2004.

Gonog, Liang and Zhou, Yimin. A review: Generative adversarial networks. In *2019 14th IEEE Conference on Industrial Electronics and Applications (ICIEA)*, pp. 505–510. IEEE, 2019.

Goodfellow, Ian. On distinguishability criteria for estimating generative models. In *International Conference on Learning Representations, Workshop track*, 2015.

Goodfellow, Ian. NIPS 2016 tutorial: Generative adversarial networks. In *Advances in neural information processing systems, Tutorial track*, 2016.

Goodfellow, Ian, Warde-Farley, David, Mirza, Mehdi, Courville, Aaron, and Bengio, Yoshua. Maxout networks. In *International conference on machine learning*, pp. 1319–1327, 2013.
Goodfellow, Ian, Pouget-Abadie, Jean, Mirza, Mehdi, Xu, Bing, Warde-Farley, David, Ozair, Sherjil, Courville, Aaron, and Bengio, Yoshua. Generative adversarial nets. In *Advances in neural information processing systems*, volume 27, 2014.

Goodfellow, Ian, Pouget-Abadie, Jean, Mirza, Mehdi, Xu, Bing, Warde-Farley, David, Ozair, Sherjil, Courville, Aaron, and Bengio, Yoshua. Generative adversarial networks. *Communications of the ACM*, 63(11):139–144, 2020.

Gregor, Karol, Danihelka, Ivo, Graves, Alex, Rezende, Danilo, and Wierstra, Daan. DRAW: A recurrent neural network for image generation. In *International Conference on Machine Learning*, pp. 1462–1471, 2015.

Gretton, Arthur, Borgwardt, Karsten, Rasch, Malte, Schölkopf, Bernhard, and Smola, Alex. A kernel method for the two-sample-problem. *Advances in neural information processing systems*, 19:513–520, 2006.

Grover, Aditya and Ermon, Stefano. Boosted generative models. In *Proceedings of the AAAI Conference on Artificial Intelligence*, volume 32, 2018.

Gulrajani, Ishaan, Ahmed, Faruk, Arjovsky, Martin, Dumoulin, Vincent, and Courville, Aaron. Improved training of Wasserstein GANs. In *Advances in neural information processing systems*, 2017.

Gutmann, Michael and Hyvärinen, Aapo. Noise-contrastive estimation: A new estimation principle for unnormalized statistical models. In *Proceedings of the thirteenth international conference on artificial intelligence and statistics*, pp. 297–304. JMLR Workshop and Conference Proceedings, 2010.

Hazan, Tamir, Papandreou, George, and Tarlow, Daniel. Adversarial perturbations of deep neural networks. 2017.

Hoang, Quan, Nguyen, Tu Dinh, Le, Trung, and Phung, Dinh. MGAN: Training generative adversarial nets with multiple generators. In *International Conference on Learning Representations*, 2018.

Hoffman, Matthew D and Johnson, Matthew J. ELBO surgery: yet another way to carve up the variational evidence lower bound. In *Workshop in Advances in Approximate Bayesian Inference, NIPS*, 2016.

Hong, Yongjun, Hwang, Uiwon, Yoo, Jaeyoon, and Yoon, Sungroh. How generative adversarial networks and their variants work: An overview. *ACM Computing Surveys (CSUR)*, 52(1):1–43, 2019.

Huang, Ling, Joseph, Anthony D, Nelson, Blaine, Rubinstein, Benjamin IP, and Tygar, J Doug. Adversarial machine learning. In *Proceedings of the 4th ACM workshop on Security and artificial intelligence*, pp. 43–58, 2011.

Huszár, Ferenc. How (not) to train your generative model: Scheduled sampling, likelihood, adversary? *arXiv preprint arXiv:1511.05101*, 2015.

Im, Daniel Jiwoong, Kim, Chris Dongjoo, Jiang, Hui, and Memisevic, Roland. Generating images with recurrent adversarial networks. *arXiv preprint arXiv:1602.05110*, 2016.

Ioffe, Sergey and Szegedy, Christian. Batch normalization: Accelerating deep network training by reducing internal covariate shift. In *International conference on machine learning*, pp. 448–456, 2015.

Isola, Phillip, Zhu, Jun-Yan, Zhou, Tinghui, and Efros, Alexei A. Image-to-image translation with conditional adversarial networks. In *Proceedings of the IEEE conference on computer vision and pattern recognition*, pp. 1125–1134, 2017.

Johnson, William B and Lindenstrauss, Joram. Extensions of Lipschitz mappings into a Hilbert space. *Contemporary mathematics*, 26, 1984.

Karras, Tero, Aila, Timo, Laine, Samuli, and Lehtinen, Jaakko. Progressive growing of GANs for improved quality, stability, and variation. In *International Conference on Learning Representations*, 2018.

Kingma, Diederik P and Welling, Max. Auto-encoding variational Bayes. In *International Conference on Learning Representations*, 2014.

Krizhevsky, Alex, Sutskever, Ilya, and Hinton, Geoffrey E. Imagenet classification with deep convolutional neural networks. *Advances in neural information processing systems*, 25:1097–1105, 2012.

Kurakin, Alexey, Goodfellow, Ian, and Bengio, Samy. Adversarial machine learning at scale. In *International Conference on Learning Representations*, 2017a.

Kurakin, Alexey, Goodfellow, Ian, Bengio, Samy, et al. Adversarial examples in the physical world. In *International Conference on Learning Representations, Workshop Track*, 2017b.

Larsen, Anders Boesen Lindbo, Sønderby, Søren Kaae, Larochelle, Hugo, and Winther, Ole. Autoencoding beyond pixels using a learned similarity metric. In *International conference on machine learning*, pp. 1558–1566, 2016.
LeCun, Yann, Chopra, Sumit, Hadsell, Raia, Ranzato, M, and Huang, F. A tutorial on energy-based learning. *Predicting Structured Data*, 1, 2006.

Levine, Adrian B, Peng, Jason, Farnell, David, Nursey, Mitchell, Wang, Yiping, Naso, Julia R, Ren, Hezhen, Farahani, Hossein, Chen, Colin, Chiu, Derek, et al. Synthesis of diagnostic quality cancer pathology images by generative adversarial networks. *The Journal of pathology*, 252(2):178–188, 2020.

Li, Changliang, Su, Yixin, and Liu, Wenju. Text-to-text generative adversarial networks. In *2018 International Joint Conference on Neural Networks (IJCNN)*, pp. 1–7. IEEE, 2018.

Li, Chongxuan, Xu, Kun, Zhu, Jun, and Zhang, Bo. Triple generative adversarial nets. In *Advances in neural information processing systems*, 2017a.

Li, Chun-Liang, Chang, Wei-Cheng, Cheng, Yu, Yang, Yiming, and Póczos, Barnabás. MMD GAN: Towards deeper understanding of moment matching network. In *Advances in neural information processing systems*, 2017b.

Li, Yuheng, Singh, Krishna Kumar, Ojha, Utkarsh, and Lee, Yong Jae. MixNMatch: Multifactor disentanglement and encoding for conditional image generation. In *Proceedings of the IEEE/CVF Conference on Computer Vision and Pattern Recognition*, pp. 8039–8048, 2020.

Li, Yujia, Swersky, Kevin, and Zemel, Rich. Generative moment matching networks. In *International Conference on Machine Learning*, pp. 1718–1727. PMLR, 2015.

Liese, Friedrich and Vajda, Igor. On divergences and informations in statistics and information theory. *IEEE Transactions on Information Theory*, 52(10):4394–4412, 2006.

Liu, Ming-Yu and Tuzel, Oncel. Coupled generative adversarial networks. *Advances in neural information processing systems*, 29:469–477, 2016.

Liu, Ziwei, Luo, Ping, Wang, Xiaogang, and Tang, Xiaoou. Deep learning face attributes in the wild. In *Proceedings of the IEEE international conference on computer vision*, pp. 3730–3738, 2015.

Madry, Aleksander, Makelov, Aleksandar, Schmidt, Ludwig, Tsipras, Dimitris, and Vladu, Adrian. Towards deep learning models resistant to adversarial attacks. In *International Conference on Learning Representations*, 2018.

Makhzani, Alireza. *Unsupervised representation learning with autoencoders*. PhD thesis, University of Toronto, 2018b.

Makhzani, Alireza and Frey, Brendan. PixelGAN autoencoders. In *Advances in neural information processing systems*, 2017.

Makhzani, Alireza, Shlens, Jonathon, Jaitly, Navdeep, Goodfellow, Ian, and Frey, Brendan. Adversarial autoencoders. *arXiv preprint arXiv:1511.05644*, 2015.

Mao, Xudong, Li, Qing, Xie, Haoran, Lau, Raymond YK, Wang, Zhen, and Paul Smolley, Stephen. Least squares generative adversarial networks. In *Proceedings of the IEEE international conference on computer vision*, pp. 2794–2802, 2017.

Mao, Xudong, Li, Qing, Xie, Haoran, Lau, Raymond YK, Wang, Zhen, and Smolley, Stephen Paul. On the effectiveness of least squares generative adversarial networks. *IEEE Transactions on Pattern Analysis & Machine Intelligence*, 41(12):2947–2960, 2019.

Mescheder, Lars, Nowozin, Sebastian, and Geiger, Andreas. Adversarial variational Bayes: Unifying variational autoencoders and generative adversarial networks. In *International Conference on Machine Learning*, pp. 2391–2400, 2017a.

Mescheder, Lars, Nowozin, Sebastian, and Geiger, Andreas. The numerics of GANs. In *Advances in neural information processing systems*, 2017b.

Mescheder, Lars, Geiger, Andreas, and Nowozin, Sebastian. Which training methods for GANs do actually converge? In *International conference on machine learning*, pp. 3481–3490. PMLR, 2018.

Metz, Luke, Poole, Ben, Pfau, David, and Sohl-Dickstein, Jascha. Unrolled generative adversarial networks. In *International Conference on Learning Representations*, 2017.

Mikolov, Tomas, Sutskever, Ilya, Chen, Kai, Corrado, Greg S, and Dean, Jeff. Distributed representations of words and phrases and their compositionality. In *Advances in neural information processing systems*, pp. 3111–3119, 2013.

Mirza, Mehdi and Osindero, Simon. Conditional generative adversarial nets. *arXiv preprint arXiv:1406.2661*, 2014.

Moosavi-Dezfooli, Seyed-Mohsen, Fawzi, Alhussein, and Frossard, Pascal. DeepFool: a simple and accurate method to fool deep neural networks. In *Proceedings of the IEEE conference on computer vision and pattern recognition*, pp. 2574–2582, 2016.
Mordvintsev, Alexander, Olah, Christopher, and Tyka, Mike. Inceptionism: Going deeper into neural networks. Google AI Blog, 2015.

Mroueh, Youssef and Nguyen, Truyen. On the convergence of gradient descent in GANs: MMD GAN as a gradient flow. In International Conference on Artificial Intelligence and Statistics, pp. 1720–1728, 2021.

Nagarajan, Vaishnavh and Kolter, J Zico. Gradient descent GAN optimization is locally stable. In Advances in neural information processing systems, 2021.

Nair, Vinod and Hinton, Geoffrey E. Rectified linear units improve restricted Boltzmann machines. In International Conference on Machine Learning, 2010.

Neal, Radford M. Annealed importance sampling. Statistics and computing, 11(2):125–139, 2001.

Ng, Andrew Y and Jordan, Michael I. On discriminative vs. generative classifiers: A comparison of logistic regression and naive Bayes. In Advances in neural information processing systems, pp. 841–848, 2002.

Nguyen, Anh, Clune, Jeff, Bengio, Yoshua, Dosovitskiy, Alexey, and Yosinski, Jason. Plug & play generative networks: Conditional iterative generation of images in latent space. In Proceedings of the IEEE Conference on Computer Vision and Pattern Recognition, pp. 4467–4477, 2017a.

Nguyen, Tu Dinh, Le, Trung, Vu, Hung, and Phung, Dinh. Dual discriminator generative adversarial nets. Advances in neural information processing systems, 2017b.

Nguyen, XuanLong, Wainwright, Martin J, and Jordan, Michael I. Estimating divergence functionals and the likelihood ratio by convex risk minimization. IEEE Transactions on Information Theory, 56(11):5847–5861, 2010.

Nielsen, Frank. A family of statistical symmetric divergences based on Jensen’s inequality. arXiv preprint arXiv:1009.4004, 2010.

Nowozin, Sebastian, Cseke, Botond, and Tomioka, Ryota. f-GAN: Training generative neural samplers using variational divergence minimization. In Proceedings of the 30th International Conference on Neural Information Processing Systems, pp. 271–279, 2016.

Odena, Augustus. Semi-supervised learning with generative adversarial networks. In International conference on machine learning, Data Efficient Machine Learning workshop, 2016.

Ojha, Utkarsh, Li, Yijun, Lu, Jingwan, Efros, Alexei A, Lee, Yong Jae, Shechtman, Eli, and Zhang, Richard. Few-shot image generation via cross-domain correspondence. In Proceedings of the IEEE/CVF Conference on Computer Vision and Pattern Recognition, pp. 10743–10752, 2021a.

Ojha, Utkarsh, Singh, Krishna Kumar, and Lee, Yong Jae. Generating furry cars: Disentangling object shape & appearance across multiple domains. In International Conference on Learning Representations, 2021b.

Oliehoek, Frans A, Savani, Rahul, Gallego-Posada, Jose, Van der Pol, Elise, De Jong, Edwin D, and Groß, Roderich. GANGs: Generative adversarial network games. arXiv preprint arXiv:1712.00679, 2017.

Oord, Aaron van den, Kalchbrenner, Nal, Vinyals, Oriol, Espeholt, Lasse, Graves, Alex, and Kavukcuoglu, Koray. Conditional image generation with PixelCNN decoders. In Advances in neural information processing systems, pp. 4790–4798, 2016.

Pan, Zhaoqing, Yu, Weijie, Yi, Xiaokai, Khan, Asifullah, Yuan, Feng, and Zheng, Yuhui. Recent progress on generative adversarial networks (GANs): A survey. IEEE Access, 7:36322–36333, 2019.

Pascual, Santiago, Bonafonte, Antonio, and Serra, Joan. SEGAN: Speech enhancement generative adversarial network. In Conference of the International Speech Communication Association (INTERSPEECH), 2017.

Pathak, Deepak, Krahenbuhl, Philipp, Donahue, Jeff, Darrell, Trevor, and Efros, Alexei A. Context encoders: Feature learning by inpainting. In Proceedings of the IEEE conference on computer vision and pattern recognition, pp. 2536–2544, 2016.

Radford, Alec, Metz, Luke, and Chintala, Soumith. Unsupervised representation learning with deep convolutional generative adversarial networks. In International Conference on Learning Representations, 2016.

Reed, Scott, Akata, Zeynep, Mohan, Santosh, Tenka, Samuel, Schiele, Bernt, and Lee, Honglak. Learning what and where to draw. Advances in neural information processing systems, 29:217–225, 2016a.

Reed, Scott, Akata, Zeynep, Yan, Xinchen, Logeswaran, Lajanugen, Schiele, Bernt, and Lee, Honglak. Generative adversarial text to image synthesis. In International Conference on Machine Learning, pp. 1060–1069, 2016b.
Reed, Scott, van den Oord, Aäron, Kalchbrenner, Nal, Bapst, Victor, Botvinick, Matt, and De Freitas, Nando. Generating interpretable images with controllable structure. In *International Conference on Learning Representations, Workshop track*, 2017.

Ronneberger, Olaf, Fischer, Philipp, and Brox, Thomas. U-net: Convolutional networks for biomedical image segmentation. In *International Conference on Medical image computing and computer-assisted intervention*, pp. 234–241. Springer, 2015.

Saatci, Yunus and Wilson, Andrew. Bayesian GAN. In *Advances in neural information processing systems*, pp. 3624–3633, 2017.

Salimans, Tim, Goodfellow, Ian, Zaremba, Wojciech, Cheung, Vicki, Radford, Alec, and Chen, Xi. Improved techniques for training GANs. *Advances in neural information processing systems*, 29:2234–2242, 2016.

Shaham, Tamar Rott, Dekel, Tali, and Michaeli, Tomer. SinGAN: Learning a generative model from a single natural image. In *Proceedings of the IEEE/CVF International Conference on Computer Vision*, pp. 4570–4580, 2019.

Shoemake, Ken. Animating rotation with quaternion curves. In *Proceedings of the 12th annual conference on Computer graphics and interactive techniques*, pp. 245–254, 1985.

Shrivastava, Ashish, Pfister, Tomas, Tuzel, Oncel, Susskind, Joshua, Wang, Wenda, and Webb, Russell. Learning from simulated and unsupervised images through adversarial training. In *Proceedings of the IEEE conference on computer vision and pattern recognition*, pp. 2107–2116, 2017.

Singh, Krishna Kumar, Ojha, Utkarsh, and Lee, Yong Jae. FineGAN: Unsupervised hierarchical disentanglement for fine-grained object generation and discovery. In *Proceedings of the IEEE/CVF Conference on Computer Vision and Pattern Recognition*, pp. 6490–6499, 2019.

Springenberg, Jost Tobias. Unsupervised and semi-supervised learning with categorical generative adversarial networks. In *International Conference on Learning Representations*, 2016.

Springenberg, Jost Tobias, Dosovitskiy, Alexey, Brox, Thomas, and Riedmiller, Martin. Striving for simplicity: The all convolutional net. In *International Conference on Learning Representations, Workshop Track*, 2015.

Sriram, Anuroop, Jun, Heewoo, Gaur, Yashesh, and Satheesh, Sanjeev. Robust speech recognition using generative adversarial networks. In *2018 IEEE international conference on acoustics, speech and signal processing (ICASSP)*, pp. 5639–5643. IEEE, 2018.

Szegedy, Christian, Vanhoucke, Vincent, Ioffe, Sergey, Shlens, Jon, and Wojna, Zbigniew. Rethinking the inception architecture for computer vision. In *Proceedings of the IEEE conference on computer vision and pattern recognition*, pp. 2818–2826, 2016.

Tembine, Hamidou. Deep learning meets game theory: Bregman-based algorithms for interactive deep generative adversarial networks. *IEEE transactions on cybernetics*, 50(3):1132–1145, 2019.

Theis, Lucas, Oord, Aäron van den, and Bethge, Matthias. A note on the evaluation of generative models. In *International Conference on Learning Representations*, 2016.

Tolstikhin, Ilya, Gelly, Sylvain, Bousquet, Olivier, Simon-Gabriel, Carl-Johann, and Schölkopf, Bernhard. AdaGAN: Boosting generative models. *arXiv preprint arXiv:1701.02386*, 2017.

Unterthiner, Thomas, Nessler, Bernhard, Seward, Calvin, Klambauer, Günter, Heusel, Martin, Ramsauer, Hubert, and Hochreiter, Sepp. Coulomb GANs: Provably optimal Nash equilibria via potential fields. In *International Conference on Learning Representations*, 2018.

Vaswani, Ashish, Shazeer, Noam, Parmar, Niki, Uszkoreit, Jakob, Jones, Llion, Gomez, Aidan N, Kaiser, Łukasz, and Polosukhin, Illia. Attention is all you need. In *Advances in neural information processing systems*, pp. 5998–6008, 2017.

Villani, Cédric. *Optimal transport: old and new*, volume 338. Springer, 2009.

Wang, Kunfeng, Gou, Chao, Duan, Yanjie, Lin, Yilun, Zheng, Xinhua, and Wang, Fei-Yue. Generative adversarial networks: introduction and outlook. *IEEE/CAA Journal of Automatica Sinica*, 4(4):588–598, 2017.

Wang, William Yang, Singh, Sameer, and Li, Jiwei. Deep adversarial learning for nlp. In *Proceedings of the 2019 Conference of the North American Chapter of the Association for Computational Linguistics: Tutorials*, pp. 1–5, 2019.

White, Tom. Sampling generative networks. In *Advances in neural information processing systems*, 2016.

Wu, Yuhuai, Burda, Yuri, Salakhutdinov, Ruslan, and Grosse, Roger. On the quantitative analysis of decoder-based generative models. 2017.

Xiao, Chang, Zhong, Peilin, and Zheng, Changxi. BourGAN: Generative networks with metric embeddings.
In *Advances in neural information processing systems*, 2018.

Zhang, Han, Xu, Tao, Li, Hongsheng, Zhang, Shaoting, Wang, Xiaogang, Huang, Xiaolei, and Metaxas, Dimitris N. StackGAN: Text to photo-realistic image synthesis with stacked generative adversarial networks. In *Proceedings of the IEEE international conference on computer vision*, pp. 5907–5915, 2017.

Zhang, Han, Xu, Tao, Li, Hongsheng, Zhang, Shaoting, Wang, Xiaogang, Huang, Xiaolei, and Metaxas, Dimitris N. StackGAN++: Realistic image synthesis with stacked generative adversarial networks. *IEEE transactions on pattern analysis and machine intelligence*, 41(8):1947–1962, 2018.

Zhang, Han, Goodfellow, Ian, Metaxas, Dimitris, and Odena, Augustus. Self-attention generative adversarial networks. In *International conference on machine learning*, pp. 7354–7363, 2019.

Zhao, Junbo, Mathieu, Michael, and LeCun, Yann. Energy-based generative adversarial network. In *International Conference on Learning Representations*, 2017.

Zhu, Jun-Yan, Krähenbühl, Philipp, Shechtman, Eli, and Efros, Alexei A. Generative visual manipulation on the natural image manifold. In *European conference on computer vision*, pp. 597–613. Springer, 2016.

Zhu, Jun-Yan, Park, Taesung, Isola, Phillip, and Efros, Alexei A. Unpaired image-to-image translation using cycle-consistent adversarial networks. In *Proceedings of the IEEE international conference on computer vision*, pp. 2223–2232, 2017.

Zieba, Maciej and Wang, Lei. Training triplet networks with GAN. In *International Conference on Learning Representations, Workshop track*, 2017.