When the average performance of a prediction model varies significantly with respect to a sensitive attribute (e.g., race or gender), the performance disparity can be expressed in terms of the probability distributions of input and output variables for each sensitive group. In this paper, we exploit this fact to explain and repair the performance disparity of a fixed classification model over a population of interest. Given a black-box classifier that performs unevenly across sensitive groups, we aim to eliminate the performance gap by perturbing the distribution of input features for the disadvantaged group. We refer to the perturbed distribution as a counterfactual distribution, and characterize its properties for popular fairness criteria (e.g., predictive parity, equal FPR, equal opportunity). We then design a descent algorithm to efficiently learn a counterfactual distribution given the black-box classifier and samples drawn from the underlying population. We use the estimated counterfactual distribution to build a data preprocessor that reduces disparate impact without training a new model. We illustrate these use cases through experiments on real-world datasets, showing that we can repair different kinds of disparate impact without a large drop in accuracy.

1. Introduction

A machine learning model has disparate impact when its performance changes across groups defined by a sensitive attribute, such as race or gender. Recent work has shown that many models can exhibit significant performance disparities between groups even when sensitive attributes are omitted (see e.g. Angwin et al., 2016; Buolamwini and Gebru, 2018). Such disparities have led to a plethora of research on fairness in machine learning, focusing on how disparate impact arises (Chen et al., 2018; Datta et al., 2016; Adler et al., 2018), how it can be measured (Žliobaitė, 2017; Pierson et al., 2017; Simoiu et al., 2017; Kilbertus et al., 2017; Kusner et al., 2017; Galhotra et al., 2017), and how it can be mitigated (Feldman et al., 2015; Corbett-Davies et al., 2017; Zafar et al., 2017b; Calmon et al., 2017).

In spite of these developments, disparate impact remains difficult to avoid in a large class of real-world applications where:

- Models are procured from a third-party vendor who has domain expertise or technical expertise required for model development (Diakopoulos, 2014; Guszcza et al., 2018).

- Models are deployed on a population where the data distribution does not reflect the patterns contained in the training data (i.e., due to distribution shift, Sugiyama et al., 2017).

In such settings, disparate impact is challenging to address, let alone understand. Users typically have black-box access to the classifier (e.g., via a prediction API), may not have access to the training data (e.g., due to privacy concerns or intellectual property rights), and may not be able to draw conclusions from the training data (e.g., since disparate impact could arise to distribution shift in the population of interest).
In this paper, we aim to evaluate and mitigate disparate impact in such settings. Our object of interest is a hypothetical distribution of input variables that minimizes disparate impact in a population of interest (i.e., the deployment population). We refer to this distribution as a counterfactual distribution. For example, consider a classifier used to predict criminal recidivism at the moment someone is charged with a crime. The deployment population would be the population of the county where the classifier is used. When the classifier is evaluated over the deployment population, assume that we observe a higher error rate for a given minority group when compared to a baseline group. The counterfactual distribution would then be a new, hypothetical distribution of input features for that minority group that would minimize this disparity.

As we will show, an information-theoretic analysis of counterfactual distributions has much to offer. Given a fixed classifier, disparate impact can be expressed in terms of differences between the distributions of input and output variables between sensitive groups (cf. Figure 1). In turn, one can recover a counterfactual distribution by repeatedly perturbing the distribution of input variables until a specific measure of disparity is minimized in the deployment population. The resulting counterfactual distribution can be used to repair the model so that it no longer exhibits disparate impact over the deployment population (i.e., without training a new model). Moreover, it provides insights on potential sources of disparate impact within the deployment population.

CONTRIBUTIONS

The main contributions of this paper are:

1. We introduce a unified approach to evaluate and mitigate performance disparities for a black-box classification model over a population of interest.

2. We develop machinery to learn counterfactual distributions for a black-box classifier for common fairness criteria given data from the deployment population. Our tools recover a counterfactual distribution using a descent procedure in the simplex of probability distributions. We prove that influence functions can be used to compute a gradient in this setting, and derive closed-form estimators that enable efficient computation.
3. We design pre-processing methods that use counterfactual distributions to resolve disparate impact of a black-box model in a deployment population (without the need to train a new model). The proposed method only randomizes individuals in the target group in a way that improves their outcomes (on average).

4. We validate our procedure by recovering counterfactual distributions for classifiers trained with real-world datasets. Our results suggest that counterfactual distributions can help us avoid disparate impact in real-world applications by building a data preprocessor for the target population. They also illustrate other potential uses, such as the ability to explain discrimination by proxy through contrastive analyses.

5. We provide software tools to recover counterfactual distributions in Python, available online at http://github.com/ustunb/ctfdist.

Related Work
Our approach can mitigate disparate impact from a fixed black-box classifier. It differs from existing pre-processing methods (e.g., Kamiran and Calders, 2012; Calmon et al., 2017) in that it does not require access to the training data, and does not require training a new model.

Here, we develop a theoretical framework that is used as a basis for methods to determine counterfactual distributions in practice. We then use counterfactual distributions to design optimal transport-based pre-processing methods for ensuring fairness. In this regard, the closest work to ours are those of Feldman et al. (2015); Johndrow and Lum (2017), where they also control specific disparate impact metrics by solving an optimal transport problem. These methods differ from ours in that they (i) focus on reducing measures of disparity related to predicted outcomes; (ii) map the input variable distributions across all sensitive groups to a common distribution. The counterfactual distribution-based pre-processing method proposed here only randomizes samples from a given target group in order to improve classification performance.

The term “counterfactual distribution” is commonly used to describe different kinds of hypothetical effects. In the statistics and economics literature (see e.g., DiNardo et al., 1995; Stock, 1989; Fortin et al., 2011; Chernozhukov et al., 2013; Fisher and Kennedy, 2018), a counterfactual distribution refers to a hypothetical distribution of an outcome variable given a specific distribution of input variables (e.g., the distribution of wages (outcome variable) for female workers if female workers had the same qualifications as male workers). The counterfactual distribution in this paper refers to a different kind of effect — a distribution of input variables that minimizes disparate impact — and, consequently, must be derived using a different set of tools.

We discuss additional related work as we present our framework next. All proofs are available in the Appendix.

2. Framework
In this section, we formally define counterfactual distributions and discuss their properties.

2.1 Preliminaries
We consider a standard classification task where the goal is to predict a binary outcome variable $Y \in \{0, 1\}$ using a vector of input variables $X = (X_1, \ldots, X_d) \in \mathcal{X}$ drawn from the probability distribution $P_X$. We examine a black-box classifier $h : \mathcal{X} \rightarrow [0, 1]$, such that $h(x) \in \{0, 1\}$ if the classifier outputs a predicted outcome (e.g., SVM) and $h(x) \in [0, 1]$ if it outputs a predicted probability (e.g., logistic regression).

We evaluate differences in the performance of the classifier with respect to a sensitive attribute $S \in \{0, 1\}$ with distribution $P_S$. We assume that $S$ is not an input variable, as this would violate
groups, respectively. We denote their distributions of input variables as $P_0$ and $P_1$. We provide examples of counterfactual distributions where $P_0$ and $P_1$ are the hypothetical probability distributions of input variables for the target group and the sensitive group, respectively. It is important to note that there exist several ways to resolve the performance disparity of a fixed classification model by perturbing the distributions of input variables for one or more sensitive groups. For example, one could simultaneously perturb the input distributions for all sensitive groups to minimize the performance disparity.

### 2.2 Disparity Metrics

We measure the performance disparity between groups in terms of a disparity metric. 

**Definition 1** (Disparity Metric). Given a classification model $h$ and fixed $P_{Y|X,S}$ (distribution of true outcome given features and sensitive attribute), $P_X$ (distribution of input features over baseline group), and $P_S$ (distribution of the sensitive attribute), a disparity metric is a mapping $M: \mathcal{P} \rightarrow \mathbb{R}$ where $\mathcal{P}$ is the set of probability distributions over $\mathcal{X}$.

We write disparity metrics as $M(P_0)$ since they can be expressed in terms of $P_0$ once the classifier and distributions $P_{Y|X,S}$, $P_X$, and $P_S$ are fixed (see Appendix A.1 for a discussion). It is worth noting that the disparity metrics implicitly depend on the classifier and distributions $P_{Y|X,S}$, $P_X$, and $P_S$. We provide examples of $M(P_0)$ for common fairness criteria in Table 1.

### 2.3 Counterfactual Distributions

A counterfactual distribution is a hypothetical probability distribution of input variables for the target group that minimizes a chosen disparity metric. 

**Definition 2.** For a given disparity metric $M(\cdot)$, a counterfactual distribution $Q_X$ is a distribution of input variables over the target population such that:

$$Q_X \in \arg\min_{Q'_X \in \mathcal{P}} |M(Q'_X)|,$$

where $\mathcal{P}$ is the set of probability distributions over $\mathcal{X}$.

Before deriving properties of counterfactual distributions, we first motivate the above definition. It is important to note that there exist several ways to resolve the performance disparity of a fixed classification model by perturbing the distributions of input variables for one or more sensitive groups. For example, one could simultaneously perturb the input distributions for all sensitive groups.
to a “midpoint” distribution (see e.g., the distribution considered by Feldman et al., 2015; Lum and Isaac, 2016, to resolve statistical parity). While these approaches could be adapted to recover such distributions, we have specifically defined a counterfactual distribution that only affects the input variables for the target group \( S = 0 \) due to the fact that this group attains the less favorable performance. Thus, our definition of counterfactual distribution aims to resolve the performance disparity by having the target group perform better, rather than having the baseline group perform worse. As we discuss later, this decision affects the data requirement to estimate the counterfactual distribution, as well as which individuals are affected by the preprocessor (i.e., this approach only produces a preprocessor that affects individuals where \( S = 0 \)).

The uniqueness of a counterfactual distribution depends on our choice of disparity metric. For a metric such as \( \text{DA}_3 \) with \( \lambda > 0 \) (cf. Table 1), there exists only one counterfactual distribution \( P_1 \). For other disparity metrics of interest, the counterfactual distributions may not be unique (see Example 2 in the Appendix).

At this point, an observant reader may be wondering why the counterfactual distribution for the target group is not simply the distribution of input variables \( P_1 \) over the baseline group. In fact, the distribution of input variables for the baseline group \( P_1 \) is not necessarily a counterfactual distribution when \( P_{Y|X,S=0} \neq P_{Y|X,S=1} \). We illustrate this point in the next example.

Example 1. Recall that \( P_0 \triangleq P_{X|S=0} \) and \( P_1 \triangleq P_{X|S=1} \) are the corresponding input feature distributions across target group \( (S = 0) \) and baseline group \( (S = 1) \). Consider a classification problem where the input variable is given by \( X = (X_1, X_2) \in \{0,1\}^2 \), and the input distributions \( P_0 \) and \( P_1 \) are of the form \( \Pr(X_1 = 1 | S = j) = p_{i,j} \) such that \( (p_{1,0}, p_{2,0}) = (0.9, 0.2) \) and \( (p_{1,1}, p_{2,1}) = (0.1, 0.5) \). Moreover, assume that the (true) outcome variables \( Y \) are drawn from the conditional distributions:

\[
\begin{align*}
P_{Y|X,S=0}(1|x) &= \text{logistic}(2x_1 - 2x_2), \\
P_{Y|X,S=1}(1|x) &= \text{logistic}(2x_1 + 4x_2 - 3).
\end{align*}
\]

In this case, the Bayes optimal classifier for the baseline group is \( h(x) = \mathbb{I}[x_2 = 1] \). Selecting the difference in false positive rate as our disparity metric (cf. FPR in Table 1), this classifier achieves \( M(P_0) = 25.1\% \). Now assume that we change the the distribution \( P_0 \) over the target group to that of the baseline group \( P_1 \), while maintaining \( P_{Y|X,S=0} \) and \( P_{Y|X,S=1} \) given in (2) and the classifier \( h(x) \) fixed. This increases the disparity metric to \( M(P_1) = 43.6\% \). In contrast, we have that \( M(Q_X) = 0.0\% \) under the counterfactual distribution: \( Q_X(0, 0) = 0.50, Q_X(0, 1) = 0.09, Q_X(1, 0) = 0.41, Q_X(1, 1) = 0.00 \).

The previous example demonstrates that, when the underlying conditional distributions between true outcomes \( Y \) and input features \( X \) are different across groups (i.e., \( P_{Y|X,S=0} \neq P_{Y|X,S=1} \)), determining the counterfactual distribution may be non-trivial. Notably, the condition \( P_{Y|X,S=0} \neq P_{Y|X,S=1} \) will always be true when a disparity cannot be eliminated. We formalize and prove this statement in the next proposition.

Proposition 1. If \( M(Q_X) > 0 \) where \( Q_X \) is a counterfactual distribution for a disparity metric in Table 1, then \( P_{Y|X,S=0} \neq P_{Y|X,S=1} \).

This result illustrates how a counterfactual distribution can be used to detect cases where a classifier has an irreconcilable performance disparity between groups (i.e., a disparity that cannot be addressed by distribution of input variables for the target group). Proposition 1 complements a recent set of impossibility results on inevitable trade-offs between groups (see e.g., Lipton et al., 2018; Chouldechova, 2017; Kleinberg et al., 2016; Pleiss et al., 2017), and provides a sufficient testable condition that can indicate when to train different classifiers for different groups (see e.g., the methods of Dwork et al., 2018; Zafar et al., 2017b).
3. Methodology

In this section, we present information-theoretic tools to determine counterfactual distributions from data. We first demonstrate how influence functions play a pivotal role in computing counterfactual distributions by providing a natural “descent direction” for minimizing a disparity metric. We then derive closed-form expressions for the influence functions of the disparity metrics in Table 1. We end with a descent procedure that combines these results to recover a counterfactual distribution from a collection of samples from the deployment population, and prove theoretical performance guarantees for this procedure.

3.1 Measuring the Descent Direction with Influence Functions

In what follows, we describe how to reduce the value of a given disparity metric by perturbing the distribution of input variables over the target group (i.e., $P_0$). Our first step is to formally define local perturbations of input distributions.

**Definition 3.** The perturbed distribution $\tilde{P}_0$ over the target population ($S = 0$) is given by

$$\tilde{P}_0(x) \triangleq P_0(x)(1 + \epsilon f(x)), \quad \forall x \in \mathcal{X}$$

(3)

where $f : \mathcal{X} \to \mathbb{R}$ is a perturbation function from the class of all functions with zero mean and unit variance w.r.t. $P_0$, and $\epsilon > 0$ is a positive scaling constant chosen so that $\tilde{P}_0$ is a valid probability distribution.

Here, $f(x)$ represents a direction in the probability simplex while $\epsilon$ represents the magnitude of perturbation.

As we will see shortly, the direction of steepest descent for disparate impact can be measured using an influence function (see Huber, 2011; Koh and Liang, 2017, for other uses of influence functions in ML).

**Definition 4.** For a disparity metric $M(\cdot)$ (cf. Definition 1), the influence function $\psi : \mathcal{X} \to \mathbb{R}$ is given by

$$\psi(x) \triangleq \lim_{\epsilon \to 0} \frac{M((1-\epsilon)P_0 + \epsilon\delta_x) - M(P_0)}{\epsilon}$$

(4)

where $\delta_x(z) = [z = x]$ is the delta function at $x$.

Intuitively, given a sufficiently large dataset from the deployment population, the influence function approximates the change of a disparity metric when a sample $x \in \mathcal{X}$ from the target group is removed (or added) to the dataset.

In Proposition 2, we show that perturbing the distribution $P_0$ along the direction defined by $-\psi(x)$ results in the largest local decrease of the disparity metric. In other words, $-\psi(x)$ reflects the direction of steepest descent in disparate impact.

**Proposition 2.** For a given disparity metric $M(\cdot)$, we have that

$$\arg\min_{f(x)} \lim_{\epsilon \to 0} \frac{M(\tilde{P}_0) - M(P_0)}{\epsilon} = -\psi(x)$$

(5)

for any influence function $\psi : \mathcal{X} \to \mathbb{R}$ such that $\mathbb{E}[\psi(X)^2|S = 0] \neq 0$.

When disparate impact is measured using a combination of metrics (see e.g., Zafar et al., 2017a), Proposition 3 shows that the influence function for the compound metric is also a linear combination of the influence functions for each metric.
Proposition 3. Given any convex combination of $K$ disparity metrics $M(P_0) = \sum_{i=1}^{K} \lambda_i M_i(P_0)$, the influence function of the compound disparity metric $M(P_0)$ has the form:

$$\psi(x) = \sum_{i=1}^{K} \lambda_i \psi_i(x).$$

Proposition 3 allows us to consider a larger class of disparity measures than those in Table 1. For instance, one can recover a counterfactual distribution to achieve equalized odds (Hardt et al., 2016) by using a convex combination of influence functions for FPR and FNR.

3.2 Computing Influence Functions

We present closed-form expressions for the influence functions of disparity metrics in Table 1. The expressions are cast in terms of three classifiers:

- $h(x)$: the black-box classifier that we aim to audit;
- $\hat{s}(x)$: a classifier that uses the same input variables as $h$ to estimate the probability that an individual belongs to the target group, $P_{\hat{s}|X}(1|x)$.
- $\hat{y}_0(x)$: a classifier that uses the same input variables as $h$ to estimate the true outcome for individuals in the target group, $P_{Y|X,S=0}(1|x)$.

In practice, we train $\hat{s}(x)$ and $\hat{y}_0(x)$ using an auditing dataset $D_{\text{audit}} = \{(x_i, y_i, s_i)\}_{i=1}^n$ drawn from the deployment population. With these models in hand, we can compute influence functions in closed-form using the following proposition.

Proposition 4. The influence functions for the disparity metrics in Table 1 can be expressed as

$$\psi^{SP}(x) = h(x) - \mu_0;$$

$$\psi^{FPR}(x) = \frac{h(x)(1 - \hat{y}_0(x)) - \nu_{0,1} h(x)}{\hat{\mu}_0};$$

$$\psi^{FNR}(x) = \frac{(1 - h(x))\hat{y}_0(x) - \gamma_{0,1} \hat{y}_0(x)}{\mu_0};$$

$$\psi^{FPR}(x) = \frac{h(x)(1 - \hat{y}_0(x)) - \gamma_{1,0} (1 - \hat{y}_0(x))}{(1 - \mu_0)};$$

$$\psi^{DA}(x) = \log \frac{\hat{\mu}_0 (1 - \hat{\mu}_1)}{(1 - \hat{\mu}_0) \hat{\mu}_1} h(x) + \lambda \log \frac{1 - \hat{s}(x)}{\hat{s}(x)} - \hat{\mu}_0 \log \frac{\hat{\mu}_0 (1 - \hat{\mu}_1)}{(1 - \hat{\mu}_0) \hat{\mu}_1} - \lambda \mathbb{E} \left[ \log \frac{1 - \hat{s}(X)}{\hat{s}(X)} \mid S = 0 \right];$$

where $\mu_s, \hat{\mu}_s, \gamma_{a,b},$ and $\nu_{a,b}$ are constants such that,

$$\mu_s \triangleq \Pr(Y = 1|S = s), \quad \hat{\mu}_s \triangleq \Pr(\hat{Y} = 1|S = s),$$

$$\gamma_{a,b} \triangleq \Pr(\hat{Y} = a|Y = b, S = 0),$$

$$\nu_{a,b} \triangleq \Pr(Y = a|\hat{Y} = b, S = 0).$$

In summary, the (local) direction of largest decrease of a disparity metric is given by the negative (normalized) influence function, as shown in Proposition 2. The influence function, in turn, can be estimated from an auditing dataset via Proposition 4. Intuitively, one would expect that a procedure that repeatedly alternates between perturbing the distribution over the target population in the direction of (5) and re-estimating the influence function at the new, perturbed distribution (à la gradient descent) would eventually arrive at a minimum of the disparity metric. This minimum would then approximate the solution of (1) and be achieved at the counterfactual distribution. We make this intuition concrete in the next section.
3.3 Learning Counterfactual Distributions from Data

In Algorithm 1, we present a descent procedure to recover a counterfactual distribution for a given disparity metric $M(\cdot)$. Given a classifier $h$, and a sample of points $\{(x_i, y_i, s_i)\}_{i=1}^n$ from the distribution $P_{X,Y,S}$, the procedure outputs a dataset which follows a counterfactual distribution.

The procedure pairs each point with a weight $w_i$, which reflects its relative frequency in the sample. At each iteration, it then computes the value of the influence function $\psi(x)$ for each sample in the auditing dataset, which reflects the direction in which the distribution for the target group should be perturbed to reduce disparate impact in terms of $M(\cdot)$. The samples are then resampled with weights $1 - \epsilon \psi(x)$ for the target group, where $\epsilon$ corresponds to a user-specified step size parameter. Since perturbing a distribution is equivalent to resampling, the resulting set of resampled points mimics one drawn from the perturbed distribution.

The procedure computes the value $M(\cdot)$ using the set of resampled points to determine whether the classifier still has disparate impact. The procedure repeats until $M(\cdot)$ ceases to decrease. The final outputs of the procedure are a dataset drawn from a counterfactual distribution and a set of sampling weights for each point from the target group, which can be used to draw samples from the counterfactual distribution.

Convergence

Our procedure is analogous to stochastic gradient descent in the space of distributions over $X$, with the resampling at each iteration corresponding to a gradient step and, consequently, it converges (see e.g., Goodfellow et al., 2016).

When influence functions are estimated from data, they are, of course, subject to an estimation error. Next, we provide a probabilistic upper bound of the estimation error in terms of the number of samples and the size of support set.

Proposition 5. If $\hat{s}(x)$ and $\hat{y}_0(x)$ are the empirical conditional distributions obtained from $n$ i.i.d. samples, then, with probability at least $1 - \beta$,

$$\|\hat{\psi}(x) - \psi(x)\|_1 \leq O\left(\sqrt{n^{-1}(|X| - \log \beta)}\right).$$

(7)

Here, $\|f(x) - g(x)\|_1 \triangleq \mathbb{E}[|f(X) - g(X)|]$ denotes the $\ell_1$-norm.

In Figure 2, we show the progress (and convergence) of Algorithm 1 when recovering a counterfactual distribution for a synthetic dataset described in Appendix B.2.

4. Model Repair

In this section, we describe how to use a counterfactual distribution learned from data to repair a classifier that has disparate impact over a deployment population.

Given a classifier $h(x)$ and a counterfactual distribution $Q_X$, we aim to construct a randomized preprocessor $T : X \to X$ that maps a sample drawn from $P_0$ to a new sample in $Q_X$. Using this preprocessor, we produce a repaired classifier $\tilde{h}(\cdot)$ such that:

$$\tilde{h}(x) = \begin{cases} h(T(x)) & \text{if } s = 0, \\ h(x) & \text{otherwise.} \end{cases}$$

(8)

The preprocessor only changes the features for individuals in the target group and it can be obtained by solving an optimal transport problem (also called Monge formulation Villani, 2008) of the form:

$$\min_{\gamma \in \Gamma(P_0, Q_X)} \int \text{cost}(x, \bar{x}) d\gamma(x, \bar{x}).$$

(9)
Algorithm 1 Distributional Descent

Input:
\[ h : \mathcal{X} \to [0, 1] \]
\[ \mathcal{D} = \{(x_i, y_i, s_i)\}_{i=1}^n \]
\[ M(\cdot) \]
\[ \epsilon > 0 \]

Initialize
1: \( I_0 \leftarrow \{i = 1, \ldots, n \mid s_i = 0\} \)
2: \( D_0 \leftarrow \{(x_i, y_i)\} \) for \( i \in I_0 \)
3: \( D_1 \leftarrow \{(x_i, y_i)\} \) for \( i \notin I_0 \)
4: \( w_0 \leftarrow [w_i]_{i \in I_0} \) where \( w_i = 1.0 \)
5: \( M \leftarrow M(D_0 \cup D_1) \)
6: repeat
7: \[ M^{old} \leftarrow M \]
8: \[ \psi_i \leftarrow \psi(x_i) \] for \( i \in I_0 \)
9: \[ w_i \leftarrow (1 - \epsilon \psi_i) \cdot w_i \] for \( i \in I_0 \)
10: \[ \tilde{D}_0 \leftarrow \text{Resample}(D_0, w_0) \]
11: \[ M \leftarrow M(\tilde{D}_0 \cup D_1) \]
12: until \( M \geq M^{old} \)
13: return \( w_0, \tilde{D}_0 \)
14:
15: procedure \text{Resample}(\mathcal{D}, w)
16: \[ \mathcal{D}' \leftarrow \text{collection of } |\mathcal{D}| \text{ new points using } w_i > 0 \text{ as the sampling weight for points } (x_i, y_i, s_i) \]
17: return \( \mathcal{D}' \)

Figure 2: Values of DA₀ for auditing dataset (blue) and holdout dataset (green), respectively, with each iteration in distributional descent for a synthetic dataset. Here, the procedure converges to a counterfactual distribution in 40 iterations. We show additional steps for the sake of illustration.

Here, \( \text{cost} : \mathcal{X} \times \mathcal{X} \to \mathbb{R} \) denotes a user-specified cost function, and \( \Gamma(P_0, Q_X) \) denotes the set of all couplings of \( P_0 \) and \( Q_X \) (i.e., a joint distribution on \( \mathcal{X} \times \mathcal{X} \) with marginals \( P_0 \) and \( Q_X \)). The optimal solution to (9) can be efficiently obtained via linear programming (LP) or other specialized techniques (see e.g., Peyré and Cuturi, 2017; Villani, 2008). We provide further details in Appendix B.

5. Numerical Experiments

In this section, we demonstrate how counterfactual distributions can be used to avoid disparate impact for classifiers trained on real-world datasets. We include all datasets and scripts to reproduce our results at http://github.com/ustunb/ctfdist.
Table 2: Change in disparate impact for classification models for \textit{adult} and \textit{compas} when paired with a randomized preprocessor built to mitigate different kinds of disparity. Each row shows the value of a specific performance metric for the classifier over the target and baseline groups (e.g., FPR, FNR, DA_{0.0} and DA_{0.1}). The target group is defined as the group that attains the less favorable value of the performance metric. The preprocessor aims to reduce to difference in performance metric by randomly perturbing the features for individuals in the target group. We also include AUC to show the change in performance due to the randomized preprocessor. All values are computed using a hold-out sample that is not used to train the model or build the preprocessor.

| Dataset | Metric | Target Group | Without Preprocessor | With Preprocessor | Change in Performance |
|---------|--------|---------------|----------------------|--------------------|----------------------|
|         |        |               | Baseline Value       | Target Value       | Disc. Gap            | Target Value       | Disc. Gap | Target AUC Before | Target AUC After |
| \textit{adult} | FPR     | Male          | 0.016                | 0.089              | \textbf{0.073}       | 0.019               | 0.002     | 0.826              | 0.724              |
| \textit{adult} | FNR     | Female        | 0.508                | 0.119              | \textbf{0.389}       | 0.500               | 0.008     | 0.893              | 0.857              |
| \textit{adult} | DA_{0.0} | Male          | \textbf{0.191}       | 0.000              | 0.826                | 0.129               | 0.002     | 0.733              | 0.729              |
| \textit{compas} | FPR     | Non-white     | 0.152                | 0.057              | \textbf{0.095}       | 0.269               | 0.000     | 0.733              | 0.671              |
| \textit{compas} | FNR     | White         | 0.377                | 0.131              | \textbf{0.246}       | 0.567               | 0.000     | 0.733              | 0.653              |
| \textit{compas} | DA_{0.0} | Non-white     | \textbf{0.131}       | 0.000              | 0.733                | 0.057               | 0.000     | 0.733              | 0.615              |

5.1 Removing disparity

In Table 2, we show the effectiveness of using counterfactual distributions to build a randomized preprocessor for the classifiers trained using \textit{adult} and \textit{compas}. We first resample the target population according to Algorithm 1 in order to mimic samples drawn from the counterfactual distribution. We then build a histogram based on the resampled dataset as an approximation of the counterfactual distribution $Q_X$. This is then used to build the preprocessor by solving (9) with $\text{cost}(x, \tilde{x}) = ||x - \tilde{x}||_1$. As shown, the approach reduces disparate impact in the target group, while having a minor effect on accuracy for decision points across the full ROC curve.
Avoiding Disparate Impact with Counterfactual Distributions

Table 3: Counterfactual distributions obtained using Algorithm 1 for a classifier on adult. We can observe that the recovered counterfactual distributions are different under different metrics. By comparing $P_0$ with the counterfactual distribution, we can identify proxy features and see how to change $P_0$ in order to reduce discrimination.

5.2 Understanding Discrimination

Counterfactual distributions provide a flexible tool to understand and explain disparate impact through contrastive analysis. As shown in Table 3, one can visualize the differences between the observed and counterfactual distribution to understand how the input variable distributions could be changed in order to reduce discrimination. The difference between the observed distribution $P_0$ and the counterfactual distribution $Q_X$ can then be used to either identify prototypical samples (see e.g., Bien and Tibshirani, 2011; Kim et al., 2016), or to score features in terms of their ability to discriminate by proxy in the deployment population. We describe these approaches and provide examples in Appendix B.

6. Concluding Remarks

In what follows, we discuss limitations and extensions of our work.

Limitations

Collecting and Predicting Sensitive Attributes: Our approach requires collecting data on sensitive attributes, such as race or gender, which may infringe privacy (though this is difficult to avoid as discussed in Žliobaitė and Custers, 2016). It also requires training a model to detect membership in the target class, which can be problematic if the model is deployed for other purposes (see e.g., Wachter and Mittelstadt, 2018, for a discussion).

Randomized Preprocessor: For the sake of generality, we have presented a way to mitigate discrimination using a randomized preprocessor. While randomization is a common technique used to reduce disparate impact in the literature (see e.g, the randomized post-processing method of Hardt et al. (2016) or the randomized classifiers of Agarwal et al. (2018)), a randomized preprocessor may not be practical in applications such as loan approval since an applicant could achieve a different predicted outcome by applying multiple times. Some effects of randomization can be mitigated by heuristic strategies (e.g., sampling a deterministic mapping to deploy for a fixed window). Given that we have considered counterfactual distributions that improve the performance for the target group, however, our approach does have a benefit in that it randomization will only apply to individuals in the target group and only be applied in a way that will improve their outcomes.

| Feature                    | Baseline | Target | Counterfactual Dist. |
|----------------------------|----------|--------|-----------------------|
|                            | $P_0$    | $P_0$  | $DA_0$ $FNR$         |
| Married                    | 61.6     | 17.5   | 31.9 $FNR$           |
| Immigrant                  | 10.4     | 10.5   | 10.0 $FNR$           |
| HighestDegree_is_HS        | 32.4     | 32.1   | 26.4 $FNR$           |
| HighestDegree_is_A5        | 7.4      | 7.6    | 7.7 $FNR$            |
| HighestDegree_is_BS        | 17.2     | 13.7   | 18.7 $FNR$           |
| HighestDegree_is_MSorPhD   | 7.2      | 5.4    | 8.5 $FNR$            |
| AnyCapitalLoss             | 4.5      | 3.2    | 6.4 $FNR$            |
| Age_leq_30                 | 31.1     | 40.1   | 33.6 $FNR$           |
| WorkHrsPerWeek_lt_40       | 17.6     | 38.5   | 34.8 $FNR$           |
| JobType_is_WhiteCollar     | 18.5     | 33.6   | 35.4 $FNR$           |
| JobType_is_BlueCollar      | 33.8     | 4.6    | 37.5 $FNR$           |
| JobType_is_Specialized     | 21.9     | 22.9   | 26.8 $FNR$           |
| JobType_is_ArmedOrProtective | 2.9     | 0.9    | 1.1 $FNR$            |
| Industry_is_Private        | 69.2     | 70.2   | 66.3 $FNR$           |
| Industry_is_Government     | 12.2     | 15.8   | 18.3 $FNR$           |
| Industry_is_SelfEmployed   | 13.9     | 5.2    | 7.5 $FNR$            |
Extensions

Handling Multiple Sensitive Groups: Our approach can be directly extended to handle problems with multiple sensitive attributes by using a one-vs-all approach. This may provide an interesting alternative to identify subgroups in which discrimination is most pronounced (see e.g., Chouldechova and G’Sell, 2017; Zhang and Neill, 2016). This extension can benefit from further study given that the choice of reference group will affect the reliability with which one can estimate counterfactual distributions and the resulting ability to mitigate discrimination.

Hypothesis Test for Irreconcilable Differences between Sensitive Groups: One can adapt the result in Proposition 1 to design a hypothesis test to identify irreconcilable differences in the disparate impact under distribution shift $H_0 : M(Q_X) = 0$. Such a test could be used to inform when to train different classifiers for different groups as in Dwork et al. (2018); Lipton et al. (2018).

Counterfactual Distributions for Other Supervised Learning Models: Our tools can be adapted to mitigate performance disparities for other kinds of supervised learning models (e.g., multiclass classifiers or ordinal regression models), so long as a appropriate disparity metric are specified.

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Appendix A. Omitted Proofs

A.1 Factorization of Joint Distribution

In this section, we show that the disparity metrics in Table 1 can be expressed in terms of $P_0$ when $P_{Y|X}$, $P_{Y|X,S}$, $P_1$, and $P_S$ are given.

We observe that since our classifier is fixed, the joint distribution $P_{S,X,Y,Y}$ is characterized by the graphical model shown in Figure 3. Accordingly, we can express $P_{S,X,Y,Y}$ as:

$$ P_{S,X,Y,Y} = P_{Y|X} P_{Y|X,S} P_S P_X|S. $$

(10)

Note that $h(x) = P_{Y|X}(1|x)$. In what follows, we use these observations to express each of the disparity metrics in Table 1 as $M(P_0)$ (i.e., a function of $P_0$).

1. **DA.**

$$ D_{KL}(P_{Y|S=0}\|P_{Y|S=1}) + \lambda D_{KL}(P_0\|P_1) = D_{KL}(P_{Y|X} \circ P_0\|P_{Y|X} \circ P_1) + \lambda D_{KL}(P_0\|P_1). $$

(11)

2. **SP.**

$$ P_r(\hat{Y} = 1|S = 0) - P_r(\hat{Y} = 1|S = 1) = E[h(X)|S = 0] - E[h(X)|S = 1] 
= \sum_{x \in \mathcal{X}} h(x)P_0(x) - \sum_{x \in \mathcal{X}} h(x)P_1(x). $$

(12)

3. **FDR.**

$$ P_r(Y = 0|\hat{Y} = 1, S = 0) - P_r(Y = 0|\hat{Y} = 1, S = 1) 
= \frac{P_r(Y = 0, \hat{Y} = 1, S = 0)}{P_r(\hat{Y} = 1, S = 1)} - \frac{P_r(Y = 0, \hat{Y} = 1, S = 1)}{P_r(\hat{Y} = 1, S = 1)} 
= \frac{\sum_{x \in \mathcal{X}} P_{Y|X}(1|x)P_{Y|X,S=0}(0|x)P_0(x)}{\sum_{x \in \mathcal{X}} P_{Y|X}(1|x)P_0(x)} - \frac{\sum_{x \in \mathcal{X}} P_{Y|X}(1|x)P_{Y|X,S=1}(0|x)P_1(x)}{\sum_{x \in \mathcal{X}} P_{Y|X}(1|x)P_1(x)}. $$

(13)

4. **FNR.**

$$ P_r(\hat{Y} = 0|Y = 1, S = 0) - P_r(\hat{Y} = 0|Y = 1, S = 1) 
= \frac{\sum_{x \in \mathcal{X}} P_{Y|X}(0|x)P_{Y|X,S=0}(1|x)P_0(x)}{\sum_{x \in \mathcal{X}} P_{Y|X,S=0}(1|x)P_0(x)} - \frac{\sum_{x \in \mathcal{X}} P_{Y|X}(0|x)P_{Y|X,S=1}(1|x)P_1(x)}{\sum_{x \in \mathcal{X}} P_{Y|X,S=1}(1|x)P_1(x)}. $$

(14)

5. **FPR.**

$$ P_r(\hat{Y} = 1|Y = 0, S = 0) - P_r(\hat{Y} = 1|Y = 0, S = 1) 
= \frac{\sum_{x \in \mathcal{X}} P_{Y|X}(1|x)P_{Y|X,S=0}(0|x)P_0(x)}{\sum_{x \in \mathcal{X}} P_{Y|X,S=0}(0|x)P_0(x)} - \frac{\sum_{x \in \mathcal{X}} P_{Y|X}(1|x)P_{Y|X,S=1}(0|x)P_1(x)}{\sum_{x \in \mathcal{X}} P_{Y|X,S=1}(0|x)P_1(x)}. $$

(15)

Figure 3: Graphical model of the framework.
A.2 Example of Counterfactual Distributions

We show that the counterfactual distributions are not always unique.

Example 2. We use SP as a disparity metric and set \( X | S = 0 \sim \text{Bernoulli}(0.1) \), \( X | S = 1 \sim \text{Bernoulli}(0.2) \). The classifier is chosen as \( h(0) = h(1) = 0.2 \). In this case, any Bernoulli distribution, including \( P_0 \) and \( P_1 \), over \( \{0, 1\} \) is a counterfactual distribution.

A.3 Proof of Proposition 1

Proof. First, the counterfactual distributions under DA or SP always achieve zero of the disparity metric. Hence, \( M(Q_X) > 0 \) happens only if the disparity metric is neither DA nor SP. We assume that \( P_{Y|X,S=0} = P_{Y|X,S=1} \) and \( M(Q_X) > 0 \). In particular, \( |M(P_1)| \geq M(Q_X) > 0 \). Note that the disparity metrics in Table 1 except DA are the form of the discrepancies of performance metrics between two groups. Here the performance metrics for each group only depend on \( P_{Y|X,S=i} \), \( P_{X|S=i} \), and \( \hat{P}_Y|X \). If we assume that \( P_{Y|X,S=0} = P_{Y|X,S=1} \) and set the distribution of target group as \( P_1 \), then the performance metrics achieve the same values for two groups. Hence, \( M(P_1) = 0 \) which contradicts the assumption, so \( P_{Y|X,S=0} \neq P_{Y|X,S=1} \). \( \square \)

A.4 Proof of Proposition 2

Proof. First, we define

\[
\Delta(f) \triangleq \lim_{\epsilon \to 0} \frac{M(\tilde{P}_0) - M(P_0)}{\epsilon},
\]

where \( \tilde{P}_0(x) \) is the perturbed distribution defined in (3). Then we prove that

\[
\Delta(f) = \mathbb{E}[f(X)\psi(X)|S=0].
\]

Note that an alternative way (see e.g., Huber, 2011) to define influence functions is in terms of the Gâteaux derivative:

\[
\sum_{x \in X} \psi(x)P_0(x) = 0,
\]

and

\[
\lim_{\epsilon \to 0} \frac{1}{\epsilon} (M((1-\epsilon)P_0 + \epsilon Q) - M(P_0)) = \sum_{x \in X} \psi(x)Q(x), \ \forall Q \in \mathcal{P}.
\]

In particular, we can choose

\[
Q(x) = \left( \frac{1}{M_U}f(x) + 1 \right) P_0(x),
\]

where \( M_U \triangleq \sup\{|f(x)| \mid x \in X\} + 1 \). Then

\[
(1-\epsilon)P_0(x) + \epsilon Q(x) = P_0(x) + \frac{\epsilon}{M_U}f(x)P_0(x).
\]
For simplicity, we use $P_0 + \epsilon f P_0$ and $P_0 + \frac{\epsilon}{M_U} f P_0$ to represent $P_0(x) + \epsilon f(x) P_0(x)$ and $P_0(x) + \frac{\epsilon}{M_U} f(x) P_0(x)$, respectively. Then

$$
\Delta(f) = \lim_{\epsilon \to 0} \frac{1}{\epsilon} (M(P_0 + \epsilon f P_0) - M(P_0))
$$

$$
= \lim_{\epsilon \to 0} \frac{M_U}{\epsilon} \left( M \left( P_0 + \frac{\epsilon}{M_U} f P_0 \right) - M(P_0) \right)
$$

$$
= M_U \lim_{\epsilon \to 0} \frac{1}{\epsilon} (M((1 - \epsilon)P_0 + \epsilon Q) - M(P_0))
$$

$$
= M_U \sum_{x \in X} \psi(x) Q(x)
$$

$$
= \sum_{x \in X} \psi(x) \left( \frac{1}{M_U} f(x) + 1 \right) P_0(x)
$$

$$
= \sum_{x \in X} \psi(x) f(x) P_0(x)
$$

$$
= \mathbb{E} [f(X) \psi(X)|S = 0].
$$

Following from Cauchy-Schwarz inequality,

$$
\mathbb{E} [f(X) \psi(X)|S = 0] \geq -\sqrt{\mathbb{E} [f(X)^2|S = 0]} \sqrt{\mathbb{E} [\psi(X)^2|S = 0]} = -\sqrt{\mathbb{E} [\psi(X)^2|S = 0]}.
$$

Here the equality can be achieved by choosing

$$
f(x) = \frac{-\psi(x)}{\sqrt{\mathbb{E} [\psi(X)^2|S = 0]}}.
$$

\[ \Box \]

**A.5 Proof of Proposition 3**

Proof. When the disparity metric is a linear combination of $K$ different disparity metrics:

$$
M(P_0) = \sum_{i=1}^{K} \lambda_i M_i(P_0),
$$

the influence function, following from Definition 4, is

$$
\psi(x) = \lim_{\epsilon \to 0} \frac{M((1 - \epsilon)P_0 + \epsilon \delta_x) - M(P_0)}{\epsilon}
$$

$$
= \sum_{i=1}^{K} \lambda_i \lim_{\epsilon \to 0} \frac{M_i((1 - \epsilon)P_0 + \epsilon \delta_x) - M_i(P_0)}{\epsilon}
$$

$$
= \sum_{i=1}^{K} \lambda_i \psi_i(x).
$$

\[ \Box \]

**A.6 Proofs of Proposition 4**

We prove the closed-form expressions of influence functions provided in Proposition 4 in this section. Again, we view the classifier $h(x)$ as a conditional distribution $P_{Y\mid X}(1|x)$. 

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When we perturb the distribution $P_0$, the classifier $h(x)$ and $Pr(\hat{Y} = 1|S = 1)$ do not change. Therefore,

$$\psi(x) = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left( \sum_{x' \in \mathcal{X}} h(x')((1 - \epsilon)Pr(x') + \epsilon \delta_x(x')) - \sum_{x' \in \mathcal{X}} h(x')P_0(x') \right)$$

$$= h(x) - Pr(\hat{Y} = 1|S = 0).$$

**Influence function for FNR.** Next, we compute the influence function of FNR. Similar analysis holds for FPR and FDR. Due to the factorization of the joint distribution (see Appendix A.1), we have

$$Pr(\hat{Y} = 0|Y = 1, S = 0) = \frac{\sum_{x' \in \mathcal{X}} P_{\hat{Y}|X}(0|x')P_{Y|X,S=0}(1|x')P_0(x')}{\sum_{x' \in \mathcal{X}} P_{Y|X,S=0}(1|x')P_0(x')}.$$

We denote $r_1(x) \triangleq P_{\hat{Y}|X}(0|x)P_{Y|X,S=0}(1|x)$ and $r_2(x) \triangleq P_{Y|X,S=0}(1|x)$. Then

$$Pr(\hat{Y} = 0|Y = 1, S = 0) = \frac{\sum_{x' \in \mathcal{X}} r_1(x')P_0(x')}{\sum_{x' \in \mathcal{X}} r_2(x')P_0(x')} = \frac{E[r_1(X)|S = 0]}{E[r_2(X)|S = 0]},$$

which implies

$$M((1 - \epsilon)P_0 + \epsilon \delta_x)$$

$$= \frac{\sum_{x' \in \mathcal{X}} r_1(x')((1 - \epsilon)P_0(x') + \epsilon \delta_x(x'))}{\sum_{x' \in \mathcal{X}} r_2(x')((1 - \epsilon)P_0(x') + \epsilon \delta_x(x'))} - Pr(\hat{Y} = 0|Y = 1, S = 1)$$

$$= \frac{E[r_1(X)|S = 0] + \epsilon (r_1(x) - E[r_1(X)|S = 0])}{E[r_2(X)|S = 0] + \epsilon (r_2(x) - E[r_2(X)|S = 0])} - Pr(\hat{Y} = 0|Y = 1, S = 1).$$

Therefore,

$$\psi(x) = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left( M((1 - \epsilon)P_0 + \epsilon \delta_x) - M(P_0) \right)$$

$$= \frac{Pr(Y = 1|S = 0)r_1(x) - Pr(\hat{Y} = 0, Y = 1|S = 0)r_2(x)}{Pr(Y = 1|S = 0)^2}$$

$$= \frac{P_{\hat{Y}|X}(0|x)P_{Y|X,S=0}(1|x) - Pr(\hat{Y} = 0|Y = 1, S = 0)P_{Y|X,S=0}(1|x)}{Pr(Y = 1|S = 0)).$$
**Influence function for DA.** Following from the definition of influence functions, we start with computing $D_{\text{KL}}((1 - \epsilon)P_0 + \epsilon \delta_x||P_1)$.

$$
D_{\text{KL}}((1 - \epsilon)P_0 + \epsilon \delta_x||P_1) = \sum_{x' \in X} ((1 - \epsilon)P_0(x') + \epsilon \delta_x(x')) \log \frac{(1 - \epsilon)P_0(x') + \epsilon \delta_x(x')}{P_1(x')} = \sum_{x' \in X} (P_0(x') + \epsilon (\delta_x(x') - P_0(x'))) \times \left( \log \frac{P_0(x')}{P_1(x')} + \log \left( 1 + \frac{\epsilon (\delta_x(x') - P_0(x'))}{P_0(x')} \right) \right) = \sum_{x' \in X} (P_0(x') + \epsilon (\delta_x(x') - P_0(x'))) \times \left( \log \frac{P_0(x')}{P_1(x')} + \frac{\epsilon (\delta_x(x') - P_0(x'))}{P_0(x')} + O(\epsilon^2) \right) = D_{\text{KL}}(P_0||P_1) + \epsilon \sum_{x' \in X} (\delta_x(x') - P_0(x')) \log \frac{P_0(x')}{P_1(x')} + O(\epsilon^2)
$$

Hence,

$$
\lim_{\epsilon \to 0} \frac{1}{\epsilon} (D_{\text{KL}}((1 - \epsilon)P_0 + \epsilon \delta_x||P_1) - D_{\text{KL}}(P_0||P_1)) = \log \frac{P_0(x)}{P_1(x)} - \mathbb{E} \left[ \log \frac{P_0(X)}{P_1(X)} \middle| S = 0 \right]. \quad (20)
$$

Similarly, we have

$$
\lim_{\epsilon \to 0} \frac{1}{\epsilon} \left( D_{\text{KL}}((1 - \epsilon)P_{Y|S=0} + \epsilon P_{Y|X} \circ \delta_x||P_{Y|S=1}) - D_{\text{KL}}(P_{Y|S=0}||P_{Y|S=1}) \right) = \sum_{y \in \{0,1\}} \log \frac{P_{Y|S=0}(y)}{P_{Y|S=1}(y)} P_{Y|X}(y|x) - \mathbb{E} \left[ \log \frac{P_{Y|S=0}(\hat{Y})}{P_{Y|S=1}(Y)} \middle| S = 0 \right]. \quad (21)
$$

Combining (20) with (21), we have

$$
\psi(x) = \sum_{y \in \{0,1\}} \log \frac{P_{Y|S=0}(y)}{P_{Y|S=1}(y)} P_{Y|X}(y|x) - \mathbb{E} \left[ \log \frac{P_{Y|S=0}(\hat{Y})}{P_{Y|S=1}(Y)} \middle| S = 0 \right] + \lambda \left( \log \frac{P_0(x)}{P_1(x)} - \mathbb{E} \left[ \log \frac{P_0(X)}{P_1(X)} \middle| S = 0 \right] \right).
$$

Note that

$$
\log \frac{P_0(x)}{P_1(x)} = \log \frac{P_{X,S}(x,0)}{P_{X,S}(x,1)} + \log \frac{P_{S}(1)}{P_{S}(0)} = \log \frac{P_{S|X}(0|x)}{P_{S|X}(1|x)} + \log \frac{P_{S}(1)}{P_{S}(0)}.
$$
Hence,
\[
\log \frac{P_0(x)}{P_1(x)} - E \left[ \log \frac{P_0(X)}{P_1(X)} \right]_{S = 0} = \log \frac{P_{S|X}(0|x)}{P_{S|X}(1|x)} - E \left[ \log \frac{P_{S|X}(0)}{P_{S|X}(1)} \right]_{S = 0}
\]
\[
= \log \frac{1 - P_{S|X}(1|x)}{P_{S|X}(1|x)} - E \left[ \log \frac{1 - P_{S|X}(1)}{P_{S|X}(1)} \right]_{S = 0}.
\]

Next,
\[
\sum_{y \in \{0,1\}} \log \frac{P_{Y|S=0}(y)}{P_{Y|S=1}(y)} P_{Y|X}(y|x) - E \left[ \log \frac{P_{Y|S=0}(\hat{Y})}{P_{Y|S=1}(\hat{Y})} \right]_{S = 0}
\]
\[
= \log \frac{P_{Y|S=0}(1)}{P_{Y|S=1}(1)} P_{Y|X}(1|x) + \log \frac{P_{Y|S=0}(0)}{P_{Y|S=1}(0)} (1 - P_{Y|X}(1|x))
\]
\[
- E \left[ \log \frac{P_{Y|S=0}(\hat{Y})}{P_{Y|S=1}(\hat{Y})} \right]_{S = 0}
\]
\[
= \log \frac{P_{Y|S=0}(1)P_{Y|S=1}(0)}{P_{Y|S=1}(1)P_{Y|S=0}(0)} P_{Y|X}(1|x)
\]
\[
+ \log \frac{P_{Y|S=0}(0)}{P_{Y|S=1}(0)} - \log \frac{P_{Y|S=0}(0)}{P_{Y|S=1}(0)} P_{Y|S=0}(0) - \log \frac{P_{Y|S=0}(1)}{P_{Y|S=1}(1)} P_{Y|S=0}(1)
\]
\[
= \left( \log \frac{P_{Y|S=0}(1)P_{Y|S=1}(0)}{P_{Y|S=1}(1)P_{Y|S=0}(0)} \right) P_{Y|X}(1|x) - \left( \log \frac{P_{Y|S=0}(1)P_{Y|S=1}(0)}{P_{Y|S=1}(1)P_{Y|S=0}(0)} \right) P_{Y|S=0}(1).
\]

Therefore, we have
\[
\psi(x) = \left( \log \frac{P_{Y|S=0}(1)P_{Y|S=1}(0)}{P_{Y|S=1}(1)P_{Y|S=0}(0)} \right) (P_{Y|X}(1|x) - P_{Y|S=0}(1))
\]
\[
+ \lambda \left( \log \frac{1 - P_{S|X}(1|x)}{P_{S|X}(1|x)} - E \left[ \log \frac{1 - P_{S|X}(1)}{P_{S|X}(1)} \right]_{S = 0} \right).
\]

\[\square\]

**A.7 Proofs of Proposition 5**

**Lemma 1.** Let \(\hat{\psi}(x)\) and \(\psi(x)\) be the estimated influence function and the true influence function, respectively. If the given disparity metric is \(DA\),
\[
\left\| \hat{\psi}(x) - \psi(x) \right\|_p \leq O \left( \left\| P_{S|X}(1|x) - P_{S|X}(1|x) \right\|_p \right).
\]

For all other disparity metrics in Table 1,
\[
\left\| \hat{\psi}(x) - \psi(x) \right\|_p \leq O \left( \left\| P_{Y|X,S=0}(1|x) - P_{Y|X,S=0}(1|x) \right\|_p \right).
\]

Here, \(\left\| f(x) - g(x) \right\|_p \triangleq \left( \mathbb{E} [ |f(X) - g(X)|^p | S = 0 ] \right)^{1/p}\) denotes the \(\ell_p\)-norm for \(p \geq 1\).
Proof. We denote \( \hat{P} \) and \( \hat{\Pr} \) as estimated probability distribution and probability, respectively. Then we assume that

\[
\| \hat{P}_{X|S=0} - P_{X|S=0} \|_p \lesssim \| \hat{P}_{Y|X,S=0}(1|x) - P_{Y|X,S=0}(1|x) \|_p; \quad (22)
\]

\[
\left| \hat{\Pr}(Y = 1|S = 0) - \Pr(Y = 1|S = 0) \right| \lesssim \| \hat{P}_{Y|X,S=0}(1|x) - P_{Y|X,S=0}(1|x) \|_p; \quad (23)
\]

\[
\left| \hat{\Pr}(\hat{Y} = 0|Y = 1, S = 0) - \Pr(Y = 0|Y = 1, S = 0) \right| \lesssim \| \hat{P}_{Y|X,S=0}(1|x) - P_{Y|X,S=0}(1|x) \|_p, \quad (24)
\]

where \( \| \hat{P}_{X|S=0} - P_{X|S=0} \|_p \triangleq \left( \sum_{x \in X} \left| \hat{P}_{X|S=0}(x) - P_{X|S=0}(x) \right|^p \right)^{1/p} \). We make similar assumptions for \( \hat{P}_{S|X}(1|x) \) (i.e., the \( \ell_p \) distance between \( \hat{P}_{S|X}(1|x) \) and \( P_{S|X}(1|x) \) upper bounds the left-hand side of (22), (23), (24)). These assumptions are reasonable in practice since estimating conditional distribution is usually harder than estimating marginal distribution which is harder than estimating the distribution of Bernoulli random variable.

1. **SP.** The influence function under SP is

\[
\psi(x) = h(x) - \Pr(\hat{Y} = 1|S = 0).
\]

In order to compute the influence function under SP, we only need to estimate \( \Pr(\hat{Y} = 1|S = 0) \) since the classifier \( h(x) \) is given. Estimating the distribution of a Bernoulli random variable is more reliable than estimating the conditional distribution so

\[
\| \hat{\psi}(x) - \psi(x) \|_p \lesssim \| P_{Y|X,S=0}(1|x) - \hat{P}_{Y|X,S=0}(1|x) \|_p.
\]

2. **Class-Based Error Metrics.** Next, we present a proof of the generalization bound for FNR. Similar proofs hold for other class-based error metrics such as FDR and FPR.

The influence function under FNR is

\[
\psi(x) = \frac{E[r_2(X)|S = 0]r_1(x) - E[r_1(X)|S = 0]r_2(x)}{\Pr(Y = 1|S = 0)^2},
\]

where \( r_1(x) = P_{Y|X}(0|x)P_{Y|X,S=0}(1|x) \) and \( r_2(x) = P_{Y|X,S=0}(1|x) \). Note that

\[
E[r_2(X)|S = 0] = \Pr(Y = 1|S = 0),
\]

\[
E[r_1(X)|S = 0] = \Pr(\hat{Y} = 0,Y = 1|S = 0).
\]

Hence, the influence function under FNR has the following equivalent expression.

\[
\psi(x) = \frac{\Pr(Y = 1|S = 0)P_{Y|X}(0|x) - \Pr(\hat{Y} = 0,Y = 1|S = 0)}{\Pr(Y = 1|S = 0)^2}P_{Y|X,S=0}(1|x)
\]

\[
= \frac{P_{Y|X}(0|x) - \Pr(\hat{Y} = 0|Y = 1,S = 0)}{\Pr(Y = 1|S = 0)}P_{Y|X,S=0}(1|x).
\]

(25)

The estimated influence function under FNR is

\[
\hat{\psi}(x) = \frac{P_{Y|X}(0|x) - \hat{\Pr}(\hat{Y} = 0|Y = 1,S = 0)}{\Pr(Y = 1|S = 0)}\hat{P}_{Y|X,S=0}(1|x).
\]

(26)
Combining (27) and (28) with the assumptions (23) and (24), we have, for FNR,

\[ \|\psi(x) - \hat{\psi}(x)\|_p \lesssim \| P_{Y|X,S=0}(1|x) - \hat{P}_{Y|X,S=0}(1|x) \|_p. \]
3. **DA.** The influence function under DA is

\[
\psi(x) = \left( \log \frac{P_{Y|S=0}(1)P_{Y|S=1}(0)}{P_{Y|S=1}(1)P_{Y|S=0}(0)} \right) \left( P_{\hat{Y}|X}(1|x) - P_{Y|S=0}(1) \right)
+ \lambda \left( \log \frac{1 - P_{S|X}(1|x)}{P_{S|X}(1|x)} - E \left[ \log \frac{1 - P_{S|X}(1|X)}{P_{S|X}(1|X)} \right| S = 0 \right). \]

Since \( h(x) = P_{Y|X}(1|x) \) is a given classifier, estimating

\[
\left( \log \frac{P_{Y|S=0}(1)P_{Y|S=1}(0)}{P_{Y|S=1}(1)P_{Y|S=0}(0)} \right) \left( P_{\hat{Y}|X}(1|x) - P_{Y|S=0}(1) \right)
\]

is more reliable than estimating

\[
\psi_r(x) = \log \frac{1 - P_{S|X}(1|x)}{P_{S|X}(1|x)} - E \left[ \log \frac{1 - P_{S|X}(1|X)}{P_{S|X}(1|X)} \right| S = 0
- \sum_{x \in X} P_{X|S=0}(x) \log \frac{1 - P_{S|X}(1|x)}{P_{S|X}(1|x)} = 0 \right). \tag{29}
\]

Next, we bound the generalization error of estimating \( \psi_r(x) \). Its estimator is

\[
\hat{\psi}_r(x) = \log \frac{1 - \hat{P}_{S|X}(1|x)}{\hat{P}_{S|X}(1|x)} - \sum_{x \in X} \hat{P}_{X|S=0}(x) \log \frac{1 - \hat{P}_{S|X}(1|x)}{\hat{P}_{S|X}(1|x)}. \tag{30}
\]

Note that, for \( a, b > 0 \),

\[
\left| \log \frac{a}{b} \right| \leq \frac{|a - b|}{\min\{a, b\}}. \tag{31}
\]

Then

\[
\left| \log \frac{1 - \hat{P}_{S|X}(1|x)}{\hat{P}_{S|X}(1|x)} - \log \frac{1 - P_{S|X}(1|x)}{P_{S|X}(1|x)} \right|
\leq |\hat{P}_{S|X}(1|x) - P_{S|X}(1|x)| \left( \frac{1}{\min\{\hat{P}_{S|X}(1|x), P_{S|X}(1|x)\}} + \frac{1}{\min\{1 - \hat{P}_{S|X}(1|x), 1 - P_{S|X}(1|x)\}} \right)
\leq |\hat{P}_{S|X}(1|x) - P_{S|X}(1|x)| \frac{2}{m_X}, \tag{32}
\]

where \( m_X \) is a constant number:

\[
m_X \triangleq \min \left\{ \left\{ \hat{P}_{S|X}(1|x) | x \in X \right\} \cup \left\{ P_{S|X}(1|x) | x \in X \right\} \cup \left\{ 1 - \hat{P}_{S|X}(1|x) | x \in X \right\} \cup \left\{ 1 - P_{S|X}(1|x) | x \in X \right\} \right\}.
\]

Also of note, for any \( x \in X \),

\[
\left| \log \frac{1 - \hat{P}_{S|X}(1|x)}{\hat{P}_{S|X}(1|x)} \right| \leq \frac{1 - 2\hat{P}_{S|X}(1|x)}{\min\{\hat{P}_{S|X}(1|x), 1 - \hat{P}_{S|X}(1|x)\}} \leq \frac{1}{m_X}. \tag{33}
\]
Combining (29) and (30) with (32) and (33), we have

\[
\left| \hat{\psi}_r(x) - \psi_r(x) \right| \\
\leq \frac{2}{m_X} \left| \hat{P}_{S|X}(1|x) - P_{S|X}(1|x) \right| + \frac{1}{m_X} \sum_{x \in X} \left| \hat{P}_{X|S=0}(x) - P_{X|S=0}(x) \right| \\
+ \frac{2}{m_X} \sum_{x \in X} \left| \hat{P}_{S|X}(1|x) - P_{S|X}(1|x) \right| P_{X|S=0}(x)
\]

\[
= \frac{2}{m_X} \left| \hat{P}_{S|X}(1|x) - P_{S|X}(1|x) \right| + \frac{1}{m_X} \left\| \hat{P}_{X|S=0} - P_{X|S=0} \right\|_1 + \frac{2}{m_X} \left\| \hat{P}_{S|X}(1|x) - P_{S|X}(1|x) \right\|_1.
\]

Therefore,

\[
\left\| \hat{\psi}_r(x) - \psi_r(x) \right\|_p \\
\leq \frac{2}{m_X} \left\| \hat{P}_{S|X}(1|x) - P_{S|X}(1|x) \right\|_p + \frac{1}{m_X} \left\| \hat{P}_{X|S=0} - P_{X|S=0} \right\|_1 + \frac{2}{m_X} \left\| \hat{P}_{S|X}(1|x) - P_{S|X}(1|x) \right\|_1.
\]

Based on the assumption: \( \left\| \hat{P}_{X|S=0} - P_{X|S=0} \right\|_1 \leq \left\| \hat{P}_{S|X}(1|x) - P_{S|X}(1|x) \right\|_1 \), we have

\[
\left\| \hat{\psi}_r(x) - \psi_r(x) \right\|_p \lesssim \left\| \hat{P}_{S|X}(1|x) - P_{S|X}(1|x) \right\|_1 + \left\| \hat{P}_{S|X}(1|x) - P_{S|X}(1|x) \right\|_1 \\
\lesssim \left\| \hat{P}_{S|X}(1|x) - P_{S|X}(1|x) \right\|_p.
\]

Hence, for DA,

\[
\left\| \hat{\psi}(x) - \psi(x) \right\|_p \lesssim \left\| \hat{P}_{S|X}(1|x) - P_{S|X}(1|x) \right\|_p.
\]

Proposition 5 follows from Lemma 1 and the following large deviation results by Weissman et al. (2003). For all \( \epsilon > 0 \),

\[
\Pr \left( \| \hat{P} - P \|_1 \geq \epsilon \right) \leq (2^M - 2) \exp \left( -n \tilde{\phi}(\pi_P) \epsilon^2 / 4 \right),
\]

where \( P \) is a probability distribution on the set \([M]\), \( \hat{P} \) is the empirical distribution obtained from \( n \) i.i.d. samples, \( \pi_P \triangleq \max_{M \subseteq \text{[M]}} \min(P(M), 1 - P(M)) \),

\[
\tilde{\phi}(p) \triangleq \begin{cases} \\
\frac{1 - 2p}{4} \log \frac{1-p}{p} & p \in [0, 1/2), \\
2 & p = 1/2,
\end{cases}
\]

and \( \| \hat{P} - P \|_1 \triangleq \sum_{x \in X} |\hat{P}(x) - P(x)| \). Note that \( \tilde{\phi}(\pi_P) \geq 2 \) which implies that

\[
\Pr \left( \| \hat{P} - P \|_1 \geq \epsilon \right) \leq \exp(M) \exp(-n \epsilon^2 / 2).
\]

(34)

Hence, by taking \( P = P_{Y,X|S=0} \), \( M = |Y| |X| = 2 |X| \) and \( \epsilon = \sqrt{\frac{2}{n} (M - \log \beta) } \), Inequality (34) implies that, with probability at least \( 1 - \beta \),

\[
\left\| \hat{P}_{Y,X|S=0} - P_{Y,X|S=0} \right\|_1 \leq \sqrt{\frac{2}{n} (2 |X| - \log \beta)},
\]

(35)

25
where \( \hat{P}_{Y,X|S=0} \) is the empirical distribution obtained from \( n \) i.i.d. samples. Similarly, with probability at least \( 1 - \beta \),

\[
\left\| \hat{P}_{S,X} - P_{S,X} \right\|_1 \leq \sqrt{\frac{2}{n} (2|X| - \log \beta)}.
\]

(36)

Let \( \hat{P}_{Y|X,S=0} = \hat{P}_{Y,X|S=0} \hat{P}_{X|S=0} \) be the empirical conditional distribution obtained from \( n \) i.i.d. samples. Then, for the disparity metrics in Table 1 except DA, with probability at least \( 1 - \beta \),

\[
\left\| \hat{\psi}(x) - \psi(x) \right\|_1 \lesssim \left\| \hat{P}_{Y|X,S=0}(1|x) - P_{Y|X,S=0}(1|x) \right\|_1
\]

\[
\lesssim \left\| \hat{P}_{Y|S=0} - P_{Y|S=0} \right\|_1
\]

\[
\lesssim \sqrt{\frac{2}{n} (2|X| - \log \beta)}.
\]

Here the second inequality holds true because

\[
\left\| \hat{P}_{Y|X,S=0}(1|x) - P_{Y|X,S=0}(1|x) \right\|_1
\]

\[
= \sum_{x \in X} P_{X|S=0}(x) \left| \hat{P}_{Y|X,S=0}(1|x) - P_{Y|X,S=0}(1|x) \right|
\]

\[
\leq \left\| \hat{P}_{Y|X,S=0} - P_{Y|X,S=0} \right\|_1 + \sum_{x \in X} \hat{P}_{Y|X,S=0}(1|x) \left| \hat{P}_{X|S=0}(x) - P_{X|S=0}(x) \right|
\]

\[
\leq \left\| \hat{P}_{Y|X,S=0} - P_{Y|X,S=0} \right\|_1 + \left\| \hat{P}_{X|S=0} - P_{X|S=0} \right\|_1 \lesssim \left\| \hat{P}_{Y|X,S=0} - P_{Y|X,S=0} \right\|_1.
\]

Similar analysis also holds for DA.

\[ \square \]

Appendix B. Supporting Experimental Results

B.1 Further Details on Model Repair

We provide more details on how to solve the optimal transport problem when the probability distributions \( P_0 \) and \( Q_{X} \) are discrete with finite support set. We assume w.l.o.g. that \( P_0 \) and \( Q_{X} \) take values over \( \{1, \cdots, M\} \)\(^1\) and denote

\[
p \triangleq (P_0(1), \cdots, P_0(M)) \in \Delta_M,
\]

(37)

\[
q \triangleq (Q_{X}(1), \cdots, Q_{X}(M)) \in \Delta_M,
\]

(38)

\[
\Delta_M \triangleq \left\{ x \in \mathbb{R}^M \mid \sum_{i=1}^M x_i = 1, x_i \geq 0 \right\}.
\]

(39)

In practice, \( p \) and \( q \) can be empirically estimated from data. We then represent the cost function by a ground metric \( C \in \mathbb{R}^{M \times M} \) with \( C_{i,j} \triangleq \text{cost}(i, j) \).

\(^1\) When the input variables \( X = (X_1, \cdots, X_d) \) are multi-dimension (i.e., \( d > 1 \)), we can map the probability distributions \( P_0 \) and \( Q_{X} \) to vectors \( p \) and \( q \), respectively, of dimension \( \prod_{i=1}^d |X_i| \).
The optimal transport problem (9) takes $p$, $q$, and $C$ as input and outputs a coupling (i.e., a joint distribution matrix $P^*$) by solving an LP:

$$
P^* = \arg\min_{P \in \mathbb{R}^{M \times M}} \sum_{i=1}^{M} \sum_{j=1}^{M} C_{i,j} P_{i,j},
\text{s.t. } P_{i,j} \geq 0 \forall i,j \in \{1, \cdots, M\},
\sum_{j=1}^{M} P_{i,j} = p_i \forall i \in \{1, \cdots, M\},
\sum_{i=1}^{M} P_{i,j} = q_j \forall j \in \{1, \cdots, M\}.
$$

Given a coupling $P^*$ that achieves the minimal cost, we can construct a randomized preprocessor $T(\cdot)$ that takes a sample $x$ in and returns a perturbed sample $\tilde{x} \in \{1, \cdots, M\}$ with probability $P_{x,\tilde{x}}/p_x$.

### B.2 Experiments on Synthetic Datasets

**Joint Proxies**

**Setup:** We consider a simple experiment to show that the preprocessor mitigates discrimination while removing a single proxy variable does not. We consider a setting where $X = (X_1, X_2, X_3) \in \{-1,1\}^3$ and choose the joint distribution matrices of $(X_1, X_2)$ for $S = 0$ and $S = 1$ as

$$
P_0 = \begin{pmatrix}
0.60 & 0.00 \\
0.25 & 0.15
\end{pmatrix},
P_1 = \begin{pmatrix}
0.05 & 0.00 \\
0.20 & 0.75
\end{pmatrix}.
$$

Then we choose $X_3$ to be independent of $(X_1, X_2)$ with $\Pr(X_3 = 1|S = i) = 0.3$ for $i = 0, 1$. We draw the values of $Y$ according to $P_{Y|X,S=i}(1|x) = \text{logistic}(6x_1x_2 + x_3)$ for $i = 0, 1$, and fit a logistic regression using 50k samples.

**Results:** The value of $\text{DA}_0$ is 14.0%. In this case, both $X_1$ and $X_2$ are proxy variable. We remove $X_1$ from dataset and retrain a logistic regression as a classifier. It turns out that the value of $\text{DA}_0$ becomes larger: 24.8%. This is because the pair $(X_1, X_2)$ is a joint proxy and, consequently, removing one of them could not reduce discrimination.

Next, we apply Algorithm 1 and the proposed preprocessor to decrease discrimination. For the sake of example, we randomly draw 12.5k new samples for the auditing dataset and 12.5k samples for the holdout dataset, and apply the descent procedure in Algorithm 1 under $\text{DA}_0$. At each step, the influence function is computed on the auditing dataset, and applied to both the auditing and the holdout set. We show the values of $\text{DA}_0$ with each iteration in Figure 2. Then we use the preprocessor to map samples from $S = 0$ to new samples and $\text{DA}_0$ becomes 0.0%.

**Descent Procedure**

**Setup:** We consider a toy problem with 3 binary features $X = (X_1, X_2, X_3)$. We define $p_i = \Pr(X_i = 1|S = 0)$ and $q_i = \Pr(X_i = 1|S = 1)$, and assume that:

$$(p_1, p_2, p_3) = (0.9, 0.2, 0.2)$$

$$(q_1, q_2, q_3) = (0.1, 0.5, 0.5)$$

Given any value of $X$, we draw the value of $Y$ for using the same distribution for each group, namely:

$$P_{Y|X,S=0}(1|x) = P_{Y|X,S=1}(1|x) = \text{logistic}(5x_1 - 2x_2 - 2x_3).$$

We train a logistic regression model over 50k samples. We randomly draw 12.5k samples for the auditing dataset and 12.5k samples for the holdout dataset, and apply the descent procedure in
Algorithm 1 for the FPR metric. At each step, the influence function is computed on the auditing dataset, and applied to both the auditing and the holdout set.

Results: As shown in Figure 4, the procedure converges to a counterfactual distribution after around 40 iterations (we show additional steps for the sake of illustration). In practice, a stopping rule can be designed to stop the descent procedure based on number of iterations or a target discrimination gap value.

Then we use the proposed preprocessor to map samples from $S = 0$ to new samples. Then the value of FPR decreases from 29.1% to 4.1%.

B.3 Experiments on Real-World Datasets

We show additional experimental results on real-world datasets in this section.

Understanding Discrimination

Identifying Proxy Scores. Counterfactual distributions can be used to score proxy variables.

Definition 5. Let the input to the classifier be given by $X = (X_1, \ldots, X_d)$. For a given disparity metric and a counterfactual distribution $Q_X$, the proxy score for feature $X_j$ is defined as

$$\gamma_j \triangleq \text{dist} \left( P_{X_j}, Q_{X_j} \right),$$

where $P_{X_j}$ and $Q_{X_j}$ are the marginal distributions of $X_j$ w.r.t. $P_0$ and $Q_X$, respectively, and $\text{dist}(\cdot, \cdot)$ is a measure of how two probability distributions are different (e.g., the KL-divergence or $|P_{X_j}(1) - Q_{X_j}(1)|$ when the features are binary).

Prototypes. Given a counterfactual distribution $Q_X$, one can identify prototypes to highlight the characteristics that must change the least or the most in order to mitigate disparities. Prototypes can be chosen in terms of the value of $Q_X(x)/P_0(x)$. These examples can be chosen to reflect to maximize or minimize this score.

Local Influence. Our interpretation of influence functions motivates their use as a way to score proxy features and prototypes. Since influence functions represent the change in discrimination due to the addition or removal of a point from a dataset (by definition), prototypes that maximize the influence function $\psi(x)$ result in the largest local reduction in discrimination. Since an influence function reflects the direction of steepest descent with respect to a disparity metric (see Proposition 2), proxy features can be scored in terms of $\gamma_i$ (see Definition 6). The benefit of using influence functions in this
Avoiding Disparate Impact with Counterfactual Distributions

Figure 5: Proxy scores for variables in the adult dataset (top) and compas dataset (bottom). We show scores computed using the influence functions for DA₀ (left) and FPR (right).
setting is that they are unique and can be computed directly. However, they reflect local information with respect to the discrimination. Even as these metrics may not appear to differ significantly, small differences in these initial directions can lead to different counterfactual distributions.

**Definition 6.** Let the input to the classifier be given by \( X = (X_1, \ldots, X_d) \). For a given disparity metric, the proxy score for feature \( X_1 \) is defined as

\[
\gamma_1 \triangleq \sum_{x_2, \ldots, x_d} \delta_\psi(x_2, \ldots, x_d) \Pr(X_2 = x_2, \ldots, X_d = x_d | S = 0),
\]

where the function \( \delta_\psi(x_2, \ldots, x_d) \) measures the change of the influence function \( \psi \) with respect to the first feature. The proxy score for the remaining variables \( X_2, \ldots, X_d \) is defined equivalently. It can also be generalized to measure the variation of the influence functions with respect to more than one given feature.

For example, one can choose \( \delta_\psi(x_2, \ldots, x_d) = \max_{x_1, x_1' \in X_1} |\psi(x_1, \ldots, x_d) - \psi(x_1', \ldots, x_d)| \) with \( X_1 \) the support set of \( X_1 \) or, alternatively, \( \delta_\psi(x_2, \ldots, x_d) = \psi(1, \ldots, x_d) - \psi(0, \ldots, x_d) \) when the features are binary.

We compute the values of proxy scores for all input variables and show the results in Figure 5. We compute the argmax of the influence functions and present the results in Table 4.

|                  | DA_0 | FNR |
|------------------|------|-----|
| Married          | 0    | 1   |
| Immigrant        | 0    | 0   |
| HighestDegree_is_HS | 0 | 0   |
| HighestDegree_is_AS | 0 | 0   |
| HighestDegree_is_BS | 1 | 0   |
| HighestDegree_is_MSorPhD | 0 | 1   |
| AnyCapitalLoss   | 0    | 0   |
| Age_leq_30       | 1    | 0   |
| WorkHrsPerWeek_lt_40 | 0 | 1   |
| JobType_is_WhiteCollar | 0 | 0   |
| JobType_is_BlueCollar | 1 | 0   |
| JobType_is_Specialized | 0 | 0   |
| JobType_is_ArmedOrProtective | 0 | 0   |
| Industry_is_Private | 1 | 0   |
| Industry_is_Government | 0 | 0   |
| Industry_is_SelfEmployed | 0 | 0   |

|                  | DA_0 | FPR |
|------------------|------|-----|
| age_leq_24       | 1    | 0   |
| age_25_to_45     | 0    | 1   |
| age_geq_46       | 0    | 0   |
| female           | 0    | 0   |
| n_priors_eq_0    | 0    | 0   |
| n_priors_geq_1   | 1    | 1   |
| n_priors_geq_2   | 1    | 1   |
| n_priors_geq_5   | 1    | 1   |
| n_juvenile_misdemeanors_eq_0 | 0 | 1   |
| n_juvenile_misdemeanors_geq_1 | 1 | 0   |
| n_juvenile_misdemeanors_geq_2 | 0 | 0   |
| n_juvenile_misdemeanors_geq_5 | 0 | 0   |
| n_juvenile_felonies_eq_0 | 0 | 0   |
| n_juvenile_felonies_eq_1 | 1 | 1   |
| n_juvenile_felonies_eq_2 | 1 | 1   |
| n_juvenile_felonies_eq_5 | 1 | 0   |
| charge_degree_eq_M | 0 | 0   |

Table 4: We show the features of target entries in adult dataset (left) and compas dataset (right) that attain the maximum values of the influence function.