On plane permutations

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Abstract In this paper we generalize permutations to plane permutations. We employ this framework to derive a combinatorial proof of a result of Zagier and Stanley, that enumerates the number of $n$-cycles $\omega$, for which $\omega(1 \cdots n)$ has exactly $k$ cycles. This quantity is 0, if $n - k$ is odd and $\frac{2C(n+1,k)}{n(n+1)}$, otherwise, where $C(n,k)$ is the unsigned Stirling number of the first kind. The proof is facilitated by a natural transposition action on plane permutations which gives rise to various recurrences. Furthermore we study several distance problems of permutations. It turns out that plane permutations allow to study transposition and block-interchange distance of permutations as well as the reversal distance of signed permutations. Novel connections between these different distance problems are established via plane permutations.

Keywords Plane permutation · Hypermap · Permutation factorization · Transposition distance · Reversal distance

1 Introduction

Let $S_n$ denote the group of permutations, i.e. the group of bijections from $[n] = \{1,\ldots,n\}$ to $[n]$, where the multiplication is the composition of maps. We shall discuss the following three representations of a permutation, $\pi$:

two-line form: the top line lists all elements in $[n]$, following the natural order. The
bottom line lists the corresponding images of elements on the top line, i.e.

\[
\pi = \left( \begin{array}{cccccc}
1 & 2 & 3 & \cdots & n-2 & n-1 & n \\
\pi(1) & \pi(2) & \pi(3) & \cdots & \pi(n-2) & \pi(n-1) & \pi(n) \\
\end{array} \right).
\]

**one-line form:** \(\pi\) is represented as a sequence \(\pi = \pi(1)\pi(2)\cdots\pi(n-1)\pi(n)\).

**cycle form:** regarding \(\langle \pi \rangle\) as a cyclic group, we represent \(\pi\) by its collection of orbits (cycles). The set consisting of the lengths of these disjoint cycles is called the cycle type of \(\pi\). We can encode this set into a non-increasing integer sequence \(\lambda = \lambda_1\lambda_2\cdots\), where \(\sum \lambda_i = n\), or as \(1^{a_1}2^{a_2}\cdots n^{a_n}\), where we have \(a_i\) cycles of length \(i\). A cycle of length \(k\) will be called a \(k\)-cycle. A cycle of odd and even length will be called an odd and even cycle, respectively. It is well known that all permutations of a same cycle type forms a conjugacy class of \(S_n\).

Zagier [17] and Stanley [18] studied the following problem: how many permutations \(\omega\) from a fixed conjugacy class of \(S_n\) such that the product \(\omega(12\cdots n)\) have exactly \(k\) cycles?

For the conjugacy class of the cycle type \(n^1\), they both obtained a surprisingly simple formula, i.e., the number of \(\omega\) of the cycle type \(n^1\), for which \(\omega(12\cdots n)\) has exactly \(k\) cycles is 0 if \(n-k\) is odd, and is otherwise equal to \(\frac{2C(n+1,k)}{n(n+1)}\), where \(C(n,k)\) is the unsigned Stirling number of the first kind, i.e., the number of permutations on \([n]\) having \(k\) cycles. Stanley asked for a combinatorial proof for this result [18]. Such proofs were given by Féray in [19] and Schaeffer in [20]. In this paper we give a new combinatorial proof, using the framework of plane permutations.

The transposition action on plane permutations has direct connections to various distances of permutations and signed permutations. This ties to important problem in the context of bioinformatics, in particular the evolution of genomes by rearrangements in DNA as well as RNA. This problem has been extensively studied in [21,25,28] and it has recently been proved to be NP-hard [27]. For more details we refer the reader to [21,25,28] and references therein.

This paper is organized as follows: in Section 2, we introduce plane permutations. Basic concepts and properties related to plane permutations are presented there. In particular, we study a natural action on plane permutations and exceedances. In Section 3, we present the proof of Stanley’s result [18], mentioned above. To this end we establish a new recurrence satisfied by the unsigned Stirling number of the first kind combinatorially by enumerating plane permutations with fixed number of exceedances. Furthermore, by means of a reflection argument, we give a direct proof for the recurrence satisfied by the number of rooted hypermaps with one face and which was recently obtained by Chapuy [6] using bicolored, bipartite maps. In Section 4, we study the transposition distance problem of permutations and derive a lower bound, which implies the lower bound obtained by Bafna and Pevzner [21]. Our formula motivates several optimization problems. In Section 5, we consider the block-interchange distance of permutations and establish the block-interchange distance formula due to Christie [26] employing plane permutations. In Section 6, we study the reversal distance of signed permutations. Encoding properties of the orientational double-cover, we translate the reversal distance of signed permutations into block-interchange distance of permutations with restricted block-interchanges. We
then prove a new formula on the lower bound of the reversal distance based on the computation of the number of cycles. We then observe that this bound is typically equal to the reversal distance.

2 Plane permutations

Definition 1 (Plane permutation) A plane permutation on \([n]\) is a pair \(p = (s, \pi)\) where \(s = (s_i)_{i=0}^{n-1}\) is an \(n\)-cycle and \(\pi\) is an arbitrary permutation. The permutation \(D_p = s \circ \pi^{-1}\) is called the diagonal of \(p\).

The above definition resembles the generalization of partitions of integers into plane partitions of integers [2]. Given \(s = (s_0s_1 \cdots s_{n-1})\), a plane permutation \((s, \pi)\) can be represented by two aligned rows:

\[
(s, \pi) = \left( \begin{array}{cccc}
  s_0 & s_1 & \cdots & s_n-2 & s_n-1 \\
  \pi(s_0) & \pi(s_1) & \cdots & \pi(s_n-2) & \pi(s_n-1)
\end{array} \right)
\]  

Note that \(D_p\) is determined by the diagonals (cyclically) in the two-line representation here, i.e., \(D_p(\pi(s_{i-1})) = s_i\) for \(0 < i < n\), and \(D_p(\pi(s_{n-1})) = s_0\).

In a permutation \(\pi\), \(i\) is an exceedance if \(i < \pi(i)\) and an anti-exceedance, otherwise. Note that \(s\) induces a partial order \(<_s\) (once we write down the top row of \((s, \pi)\)), where \(a <_s b\) if \(a\) appears before \(b\) in \(s\). These concepts then can be generalized to plane permutations as follows

Definition 2 Given a plane permutation \((s, \pi)\), an element \(s_i\) is called an exceedance if \(s_i <_s \pi(s_i)\) and an anti-exceedance, if \(s_i \geq_s \pi(s_i)\).

In the following, we mean by “the cycles of \(p = (s, \pi)\)” the cycles of \(\pi\) and any comparison of elements in \(p\) references \(<_s\).

Obviously, each \(p\)-cycle, contains at least one anti-exceedance as it contains a minimum, \(s_i\), for which \(\pi^{-1}(s_i)\) will be an anti-exceedance. We call these trivial anti-exceedances and refer to a non-trivial anti-exceedance as NTAE. Furthermore, in any cycle of length greater than one, its minimum is an exceedance.

Let \(Exc(p)\) and \(AEx(p)\) denote the number of exceedances and anti-exceedances in \(p\), respectively. For \(D_p\), the quantities \(Exc(D_p)\) and \(AEx(D_p)\) are defined in reference to \(<_s\).

Lemma 1 Given a \(p = (s, \pi)\), we have

\[
Exc(p) = AEx(D_p) - 1.
\]  

Proof By construction of the diagonal permutation \(D_p\), we have

\[
\forall 0 \leq i < n-1, \quad s_i <_s \pi(s_i) \iff \pi(s_i) \geq_s D_p(\pi(s_i)) = s_{i+1}.
\]

Note that \(s_{n-1}\) is always an anti-exceedance of \(p\) since \(s_{n-1} \geq \pi(s_{n-1})\) and that \(\pi(s_{n-1})\), is always an anti-exceedance of \(D_p\), since \(D_p(\pi(s_{n-1})) = s(s_{n-1}) = s_0\) and \(\pi(s_{n-1}) \geq s_0\). Thus we have

\[
Exc(p) = AEx(D_p) - 1,
\]

whence the lemma. □
Proposition 1 Given a plane permutation $\pi = (s, \pi)$ on $[n]$, the sum of the number of cycles in $\pi$ and in $D_p$ is smaller than $n + 2$.

Proof Let $C(\alpha)$ denote the number of cycles in a permutation $\alpha$. Firstly, we have $AEx(\pi) \geq C(\pi)$ and $AEx(D_p) \geq C(D_p)$ and furthermore

$$AEx(\pi) = n - Exc(\pi) = n + 1 - AEx(D_p) \geq C(\pi).$$

Therefore,

$$n + 1 \geq C(\pi) + AEx(D_p) \geq C(\pi) + C(D_p),$$

whence the proposition. □

Remark. Proposition [1] seems not immediately clear without using Lemma [1]

Proposition 2 Given a plane permutation $\pi = (s, \pi)$ on $[n]$, the quantities $C(\pi)$ and $C(D_p)$ satisfy

$$C(\pi) + C(D_p) \equiv n - 1 \pmod{2}. \quad (3)$$

Proof In view of $s = D_p \pi$ the sign of both sides is equal. Since a $k$-cycle can be written as product of $k - 1$ transpositions, the parity of the LHS is $n - 1$ while the parity of the RHS equals $(n - C(\pi)) + (n - C(D_p))$, whence the proposition. □

**Remark.** Proposition [1] seems not immediately clear without using Lemma [1].

Given a plane permutation $(s, \pi)$ on $[n]$ and a sequence $h = (i, j, k, l)$, such that $i \leq j < k \leq l$ and $\{i, j, k, l\} \subset [n - 1]$. Let

$$s^h = (s_0, s_1, \ldots, s_{i-1}, s_k, s_{k+1}, \ldots, s_{j-1}, s_j, \ldots, s_{l-1}, s_l, \ldots),$$

i.e. the $n$-cycle obtained by transposing the blocks $[s_i, s_j]$ and $[s_k, s_l]$, which by construction are non-empty. Note that in case of $j + 1 = k$ we have

$$s^h = (s_0, s_1, \ldots, s_{i-1}, s_k, \ldots, s_l, s_j, \ldots).$$

Let furthermore

$$\pi^h = D_p \circ s^h,$$

that is, the derived plane permutation, $(s^h, \pi^h)$, can be represented as

$$
\begin{pmatrix}
\cdots & s_{i-1} & \cdots & s_k & \cdots & s_i & \cdots & s_{j-1} & \cdots & s_j & \cdots & s_{l-1} & \cdots \\
\cdots & \pi(s_{i-1}) & \cdots & \pi(s_k) & \cdots & \pi(s_i) & \cdots & \pi(s_{j-1}) & \cdots & \pi(s_j) & \cdots & \pi(s_{l-1}) & \cdots \\
\end{pmatrix}
$$

and write $\chi_h \circ (s, \pi) = (s^h, \pi^h)$. In particular, in case of $k = j + 1$, we refer to $\chi_h$ as a transpose. Note that the bottom row of the two-row representation of $(s^h, \pi^h)$ is obtained by transposing the blocks $[\pi(s_{i-1}), \pi(s_{j-1})]$ and $[\pi(s_{k-1}), \pi(s_{l-1})]$ of the bottom row of $(s, \pi)$. As a result, we observe
Lemma 2 Given a plane permutation \((s, \pi)\) on \([n]\) and \(h = (i, j, k, l)\) where \(0 < i \leq j < k \leq l < n\). Then \(\pi(s_r) = \pi^h(s_r)\) if \(r \in \{0, 1, \ldots, n-1\} \setminus \{i-1, j, k-1, l\}\), and for \(j + 1 < k\)

\[
\pi^h(s_{j-1}) = \pi(s_{k-1}), \quad \pi^h(s_j) = \pi(s_t), \quad \pi^h(s_{k-1}) = \pi(s_{j-1}), \quad \pi^h(s_l) = \pi(s_j) \tag{4}
\]

holds and for \(j = k - 1\) we have

\[
\pi^h(s_{j-1}) = \pi(s_j), \quad \pi^h(s_j) = \pi(s_t), \quad \pi^h(s_l) = \pi(s_{j-1}).
\]

We shall proceed by analyzing the induced changes of the \(\pi\)-cycles when passing to \(\pi^h\). By Lemma \(2\) only the \(\pi\)-cycles containing \(s_{j-1}, s_j, s_{k-1}, s_l\) will be affected.

Lemma 3 Given \((s, \pi)\) and a transpose \(\chi_h\) where \(h = (i, j, j + 1, l)\) and \(0 < i \leq j < l < n\), then there exist the following six scenarios for the pairs \((\pi, \pi^h)\):

| Case    | \(\pi\)                                      | \(\pi^h\)                                      |
|---------|----------------------------------------------|-----------------------------------------------|
| Case 1  | \((s_{j-1}, v_1', \ldots, v_m'), (s_j, v_1', \ldots, v_m'), (s_t, v_1', \ldots, v_m')\) | \((s_{j-1}, v_1', \ldots, v_m'), (s_j, v_1', \ldots, v_m'), (s_t, v_1', \ldots, v_m')\) |
| Case 2  | \((s_{j-1}, v_1', \ldots, v_m'), (s_j, v_1', \ldots, v_m'), (s_t, v_1', \ldots, v_m')\) | \((s_{j-1}, v_1', \ldots, v_m'), (s_j, v_1', \ldots, v_m'), (s_t, v_1', \ldots, v_m')\) |
| Case 3  | \((s_{j-1}, v_1', \ldots, v_m'), (s_j, v_1', \ldots, v_m'), (s_t, v_1', \ldots, v_m')\) | \((s_{j-1}, v_1', \ldots, v_m'), (s_j, v_1', \ldots, v_m'), (s_t, v_1', \ldots, v_m')\) |
| Case 4  | \((s_{j-1}, v_1', \ldots, v_m'), (s_j, v_1', \ldots, v_m'), (s_t, v_1', \ldots, v_m')\) | \((s_{j-1}, v_1', \ldots, v_m'), (s_j, v_1', \ldots, v_m'), (s_t, v_1', \ldots, v_m')\) |
| Case 5  | \((s_{j-1}, v_1', \ldots, v_m'), (s_j, v_1', \ldots, v_m'), (s_t, v_1', \ldots, v_m')\) | \((s_{j-1}, v_1', \ldots, v_m'), (s_j, v_1', \ldots, v_m'), (s_t, v_1', \ldots, v_m')\) |
| Case 6  | \((s_{j-1}, v_1', \ldots, v_m'), (s_j, v_1', \ldots, v_m'), (s_t, v_1', \ldots, v_m')\) | \((s_{j-1}, v_1', \ldots, v_m'), (s_j, v_1', \ldots, v_m'), (s_t, v_1', \ldots, v_m')\) |

Proof We shall only prove Case 1 and Case 2, the remaining four cases are shown analogously. For Case 1, the \(\pi\)-cycles containing \(s_{j-1}, s_j, s_t\) are

\[
(s_{j-1}, v_1', \ldots, v_m'), (s_j, v_1', \ldots, v_m'), (s_t, v_1', \ldots, v_m').
\]

Lemma \(2\) allows us to identify the new cycle structure inspecting the critical points \(s_{j-1}, s_j\) and \(s_t\). Here we observe that all three cycles merge, forming a single \(\pi^h\)-cycle

\[
(s_{j-1}, \pi^h(s_{j-1}), \pi^h(s_{j-1}), \ldots) = (s_{j-1}, \pi(s_j), \pi^h(s_j), \ldots)
\]

\[
= (s_{j-1}, v_1', \ldots, v_m', s_j, v_1', \ldots, v_m', s_t, v_1', \ldots, v_m').
\]

For Case 2, the \(\pi\)-cycle containing \(s_{j-1}, s_j, s_l\) is

\[
(s_{j-1}, v_1', \ldots, v_m', s_j, v_1', \ldots, v_m', s_l, v_1', \ldots, v_m').
\]
We compute the $\pi^h$-cycles containing $s_{i-1}$, $s_j$ and $s_l$ in $\pi^h$ as

$$(s_{i-1}, \pi^h(s_{i-1}), (\pi^h)^2(s_{i-1}), \ldots) = (s_{i-1}, \pi(s_j), \pi^2(s_j), \ldots) = (s_{i-1}, v_1', \ldots v_{m_{j}}')$$

$$(s_j, \pi^h(s_j), (\pi^h)^2(s_j), \ldots) = (s_j, \pi(s_l), \pi^2(s_l), \ldots) = (s_j, v_1', \ldots v_{m_{l}}')$$

$$(s_l, \pi^h(s_l), (\pi^h)^2(s_l), \ldots) = (s_l, \pi(s_{i-1}), \pi^2(s_{i-1}), \ldots) = (s_l, v_1', \ldots v_{m_{i-1}}')$$

whence the lemma. □

If we wish to express which cycles are impacted by a transpose of scenario $k$ acting on a plane permutation, we shall say “the cycles are acted upon by a Case $k$ transpose”.

We next observe

**Lemma 4** Suppose we are given a plane permutation $p = (s, \pi)$ and a sequence $h$. Then the difference of the number of cycles of $p$ and $p^h = \chi_h \circ p$ is even. Furthermore the difference of the number of cycles, odd cycles, even cycles between $\pi$ and $\pi^h$ is contained in $\{-2,0,2\}$.

**Proof** Lemma 3 implies that the difference of the numbers of cycles of $\pi$ and $\pi^h$ is even. As for the statement about odd cycles, since the parity of the total number of elements contained in cycles containing $s_{i-1}$, $s_j$ and $s_l$ is preserved, the difference of the number of odd cycles is even. Consequently, the difference of the number of even cycles is also even whence the lemma. □

Note that the former part of Lemma 4 holds for any rearrangement of the cycle $s$. The reason is that, by construction, we have $D_p = s \pi^{-1} = \chi_h(\pi^h)^{-1}$. Since the sign of $s$ and $s^h$ are equal, the sign of $\pi$ and $\pi^h$ must be equal, also. Consequently, the difference of number of cycles of $\pi$ and $\pi^h$ is even.

Suppose we are given $h = (i, j, k, l)$, where $j + 1 < k$. Then using the strategy of the proof of Lemma 3 we have

**Lemma 5** Given a plane permutation $(s, \pi)$ and $h = (i, j, k, l)$ where $0 < i < j < k \leq l < n$ and $j + 1 < k$. Then, the difference of the numbers of $\pi$-cycles and $\pi^h$-cycles is contained in $\{-2,0,2\}$. Furthermore, the scenarios, where the number of $\pi^h$-cycles increases by 2, are given by:

| Case  | $\pi$ | $\pi^h$ |
|-------|-------|---------|
| a     | $(s_{i-1}, v_1', \ldots v_{m_i}', s_j, v_1', \ldots v_{m_j}', s_l, v_1', \ldots v_{m_l}')$ | $(s_{i-1}, v_1', \ldots v_{m_i}', (s_j, v_1', \ldots v_{m_j}', s_l, v_1', \ldots v_{m_l}')$ |
| b     | $(s_{i-1}, v_1', \ldots v_{m_i}', s_j, v_1', \ldots v_{m_j}', s_l, v_1', \ldots v_{m_l})$ | $(s_{i-1}, v_1', \ldots v_{m_i}', (s_j, v_1', \ldots v_{m_j}', s_l, v_1', \ldots v_{m_l}')$ |
| c     | $(s_{i-1}, v_1', \ldots v_{m_i}', s_j, v_1', \ldots v_{m_j}', s_l, v_1', \ldots v_{m_l})$ | $(s_{i-1}, v_1', \ldots v_{m_i}', (s_j, v_1', \ldots v_{m_j}', s_l, v_1', \ldots v_{m_l}')$ |
| d     | $(s_{i-1}, v_1', \ldots v_{m_i}', s_j, v_1', \ldots v_{m_j}', s_l, v_1', \ldots v_{m_l})$ | $(s_{i-1}, v_1', \ldots v_{m_i}', (s_j, v_1', \ldots v_{m_j}', s_l, v_1', \ldots v_{m_l}')$ |
| e     | $(s_{i-1}, v_1', \ldots v_{m_i}', s_j, v_1', \ldots v_{m_j}', s_l, v_1', \ldots v_{m_l})$ | $(s_{i-1}, v_1', \ldots v_{m_i}', (s_j, v_1', \ldots v_{m_j}', s_l, v_1', \ldots v_{m_l}')$ |
A plane permutation, \((s, \pi)\) is called planted if it contains a distinguished \(x \in [n]\). Two planted plane permutations \((s, \pi, x)\), \((s^\prime, \pi^\prime, y)\) are equivalent if there exists a permutation \(\alpha\)

\[ s = \alpha s^\prime \alpha^{-1}, \quad \pi = \alpha \pi^\prime \alpha^{-1}, \quad \alpha(y) = x. \]

In the following we shall assume that in any planted plane permutation the distinguished index appears first, i.e. as \(s_0\). With this convention we immediately observe that the planting of plane permutations implies that given two equivalent, planted, plane permutations, \(p = (s, \pi)\) and \(p^\prime = (s^\prime, \pi^\prime)\), we have \(s = s^\prime\) if and only if \(\pi = \pi^\prime\). Clearly, the equation \(s s^\prime \alpha^{-1} = s = s^\prime\) restricts \(\alpha\) to be a shift within the \(n\)-cycle \(s^\prime\) and the latter has to be trivial since the planting implies \(s_0 = s_0^\prime\).

**Lemma 6** Given two equivalent planted plane permutations \(p = (s, \pi)\) and \(p^\prime = (s^\prime, \pi^\prime)\), we have

\[ \text{Exc}(p) = \text{Exc}(p^\prime). \]  \(5\)

**Proof** Assume

\[ s = \alpha s^\prime \alpha^{-1}, \quad \pi = \alpha \pi^\prime \alpha^{-1}, \quad \alpha(s_0) = s_0. \]

Since conjugation by \(\alpha\) is equivalent to relabeling according to \(\alpha, a <_p b\), is equivalent to \(\alpha(a) <_\pi \alpha(b)\). Therefore, an exceedance of \(p^\prime\) will uniquely correspond to an exceedance of \(p\), whence the lemma. \(\square\)

Let \(U_{D_p}\) denote the set of planted plane permutations having \(D_p\) as diagonal. In the following we refer to the cycle-type of \(D_p\) as the cycle-type of \(p\). Since \(D_p = s o \pi^{-1}\), the set of planted plane permutations having diagonal, \(D_p\), enumerates the ways to write \(D_p\) as a product of an \(n\)-cycle with an arbitrary permutation as well as the rooted hypermaps having one face. A rooted hypermap is a triple of permutations \((\alpha, \beta_1, \beta_2)\), such that \(\alpha = \beta_1 \beta_2\). The cycles in \(\alpha\) are called faces, the cycles in \(\beta_1\) are called (hyper)edges, and the cycles in \(\beta_2\) are called vertices. If \(\beta_1\) is an involution without fixed points, the hypermap is an ordinary map. Since \(s = D_p \pi\), our scenario corresponds to hypermaps having one face. We refer to [3–6, 8–11, 13–15] and references therein for an in depth-study of hypermaps and maps.

In particular, there is a bijection between rooted hypermaps with one face and the so called bicolored bipartite maps with one face given by Walsh [4]. Accordingly, the enumeration of hypermaps (and hence the enumeration of planted plane permutations) can be transformed into the enumeration of maps by means of [4]. Chapuy [6] combinatorially obtained a recurrence to count bicolored bipartite maps with one face (hence a recurrence for hypermaps with one face), which reads, reformulated in terms of plane permutations:

**Proposition 3** [6] Let \(\xi_{l,k}(n)\) denote the number of planted plane permutation \(p = (s, \pi) \in U_{D_p}\), where \(\pi\) has \(k\) cycles and \(D_p\) has \(l\) cycles. Then, we have

\[ (n + 1 - k - l) \xi_{l,k}(n) = \sum_{i=1}^{\lfloor n/k \rfloor} \binom{k+2i}{k-1} \xi_{i,k+2i}(n) + \sum_{i=1}^{\lfloor n/l \rfloor} \binom{l+2i}{l-1} \xi_{i+l,2i}(n). \]  \(6\)
Remark. Note from Proposition 2 if $k + l$ has different parity than $n - 1$, then $\xi_{l,k}(n) = 0$.

Setting in eq. 6 $l = 1$, we obtain

Corollary 1 For $k \geq 1, n \geq 1$ and $n - k$ is even,

$$ (n + 1 - k)\xi_{1,k}(n) = \sum_{i=1}^{\left\lfloor n/2 \right\rfloor} \binom{k+2i}{k-1} \xi_{1,k+2i}(n) + C(n,k). \quad (7) $$

Proof For $l = 1$ in eq. 6, we have

$$ (n - k)\xi_{1,k}(n) = \sum_{i=1}^{\left\lfloor n/2 \right\rfloor} \binom{k+2i}{k-1} \xi_{1,k+2i}(n) + \sum_{i=1}^{\left\lfloor n/2 \right\rfloor} \binom{1+2i}{0} \xi_{1+2i,k}(n). $$

We compute the second term on the RHS of the above equation as

$$ \sum_{i=1}^{\left\lfloor n/2 \right\rfloor} \binom{1+2i}{0} \xi_{1+2i,k}(n) = \sum_{j \in \text{odd}} \xi_{j,k}(n) - \xi_{1,k}(n) = C(n,k) - \xi_{1,k}(n). $$

Indeed, $\xi_{j,k}$ counts the number of factorizations of a given permutation $D_{\pi}$ with $k$ cycles into an $n$-cycle $s$ and a permutation $\pi^{-1}$ with $j$ cycles, i.e., $D_{\pi} = s\pi^{-1}$. In view of

$$ D_{\pi} = s\pi^{-1} \iff (12\cdots n) = s\pi y^{-1} = cy\pi^{-1}, $$

where $\gamma$ is unique if $\gamma(s_0) = 1$, we observe that this number is equal to the number of factorizations of $(12\cdots n)$ into a permutation with $k$ cycles and a permutation with $j$ cycles. Furthermore, if $n - k$ is even, i.e., $k$ and $n - 1$ have different parity, Proposition 2 implies that $(12\cdots n)$ can be only factorized into a permutation with $k$ cycles and a permutation with $j$ cycles for some odd $j$. Therefore, summing over all these odd indices, $j$, is equivalent to sum over all $\sum_{j \in \text{odd}} \xi_{j,k}(n) = C(n,k)$. \qed

3 A combinatorial proof for Zagier and Stanley’s result

Zagier [17] and Stanley [18] studied the following problem: how many permutations $\omega$ from a fixed conjugacy class of $S_n$ such that the product $\omega(12\cdots n)$ have exactly $k$ cycles?

Both authors employed the character theory of the symmetric group in order to obtain certain generating polynomials for the number of $\omega$ from a fixed conjugacy class, for which $\omega(12\cdots n)$ has exactly $k$ cycles. By evaluating these polynomials for a specific conjugacy class, Zagier obtained an explicit formula for the number of rooted maps with one face (i.e., the conjugacy class being involutions without fixed points), and both, Zagier as well as Stanley obtained the following surprisingly simple formula in case of the conjugacy class being $n!$: the number of $\omega$ for which $\omega(12\cdots n)$ has exactly $k$ cycles is $0$ if $n - k$ is odd, and is otherwise equal to $\frac{2C(n+1,k)}{m(n+1)}$.
where $C(n,k)$ is the unsigned Stirling number of the first kind, i.e., the number of permutations on $n$ with $k$ cycles.

Note the number of these $\omega$ is equal to $\bar{\xi}_{1,k}(n)$. Our idea to prove Stanley’s result is to combinatorially show that $\frac{2}{n(n+1)}C(n+1,k)$ satisfies the recurrence of eq. (7) and that $\frac{2}{n(n+1)}C(n+1,k) = \bar{\xi}_{1,k}(n)$ for $n \geq 1$.

To this end, we study the statistics on the number of anti-exceedances and exceedances of permutations via the framework of planted plane permutations. Any such element exists by construction and we have $\min_{i} m_{i} = \min_{i} n_{i} - 1$, this follows from Lemma 6.

Furthermore, its diagonal is equal to $\gamma^{2} \gamma^{-1}$ which is of cycle type $\lambda$.

Now, we shall enumerate plane permutations $p = (s, \pi) \in U_{D_{n}}$ having $k$ cycles, $a$ exceedances, and a diagonal $D_{p}$ of cycle-type $\lambda$.

**Lemma 7** Given a planted plane permutation $(s, \pi)$ and two $\pi$-cycles, $C_1$ and $C_2$, such that $\min\{C_1\} < \min\{C_2\}$. Suppose we have some Case 2 transpose on $C_2$, splitting $C_2$ into the three $\pi$-cycles $C_{21}, C_{22}, C_{23}$ in $(\pi' , \pi'')$. Then

$$\min\{C_1\} < \min\{\min\{C_{21}\}, \min\{C_{22}\}, \min\{C_{23}\}\}.$$  (8)

**Proof** Note that any Case 2 transpose on the cycle $C_2$ will not change the cycle $C_1$. Furthermore, it will only impact the relative order of elements larger than $\min\{C_2\}$ whence the proof.  □

Let $Y_1$ denote the set of pairs $(p, \varepsilon)$, such that $p \in U_{D_{n}}$ has $b$ cycles and $\varepsilon$ is a NTAE in $p$. Let furthermore $Y_2$ denote the set of $p' \in U_{D_{n}}$ in which there are 3 labeled cycles among $b + 2$ $\pi'$-cycles and finally let $Y_3$ denote the set of plane permutations $p' \in U_{D_{n}}$ where there are 3 labeled cycles among $b + 2$ $\pi'$-cycles and a distinguished NTAE contained in the labeled cycle that contains the largest minimum element.

The following proposition establishes a bijection based on Case 1 and Case 2 of Lemma 3 and is a reformulation of a bijection of Chapuy [6] in terms of plane permutations.

**Proposition 4** [3] We have

$$|Y_1| = |Y_2| + |Y_3|.$$  (9)

**Proof** Given $(p, \varepsilon) \in Y_1$ where $p = (s, \pi)$. We consider the NTAE $\varepsilon$ and identify a Case 2 transpose $\chi_{m, h} = (i, j, j + 1, l)$ as follows: assume $\varepsilon$ is contained in the cycle

$$C = (s_{j-1}, v_{1}'_{m_{j}}, s_{j}, v_{1}'_{m_{j}}, \ldots, v_{1}'_{m_{j}}, s_{j}, \ldots, v_{1}'_{m_{j}}),$$

where $s_{j-1} = \min\{C\}$, $v_{m_{j}}' = \varepsilon$, $s_{j} = \pi(\varepsilon)$ and $s_{j}$ has the property $s_{j} < s_{j}$ and there exists no element in the segment $v_{1}'_{m_{j}}, \ldots, v_{m_{j}}'$ which is larger than $s_{j}$ but smaller than $s_{j}$. Such an element exists by construction and we have $s_{j-1} < s_{j} < s_{j} \leq \varepsilon$. 


Let \( p^h = (s^h, \pi^h) = \chi_h \circ p \), we have

\[
(s, \pi) = \begin{pmatrix}
\cdots & s_{i-1} & s_i & s_j & s_{j+1} & \cdots & s_l & \cdots & e & \cdots \\
\cdots & \pi(s_{i-1}) & \pi(s_i) & \pi(s_j) & \pi(s_{j+1}) & \cdots & \pi(s_l) & \cdots & \pi(e) & \cdots
\end{pmatrix}
\]

\[
(s^h, \pi^h) = \begin{pmatrix}
\cdots & s_{i-1} & s_i & s_j & s_{j+1} & \cdots & s_l & \cdots & e & \cdots \\
\cdots & \pi(s_{i-1}) & \pi(s_i) & \pi(s_j) & \pi(s_{j+1}) & \cdots & \pi(s_l) & \cdots & \pi(e) & \cdots
\end{pmatrix}
\]

Then, \( s_{i-1} <_h s_j <_h s_l \). According to Lemma 3, \( s_{i-1}, s_j, s_l \) will be contained in three distinct cycles of \( \pi^h \), namely

\[ (s_{i-1}, v_{1}^l, \ldots, v_{m}^l), \ (s_j, v_{1}'^l, \ldots, v_{m}'^l), \ (s_l, v_{1}^l, \ldots, v_{m}^l). \]

It is clear that \( s_{i-1} \) is still the minimum element w.r.t. \( <_h \) in its cycle. By construction we have

\[ \{v_{1}^l, \ldots, v_{m}^l\} \subset ]s_{i-1}, \ s_j[ \cup ]s_j, \ s_l[ \quad \text{and} \quad \{v_{1}'^l, \ldots, v_{m}'^l\} \subset ]s_{i-1}, \ s_j[ \cup ]s_j, \ s_l[ \]

in \( s \). After transposing \([s_i, s_j] \) and \([s_{j+1}, s_l] \), all elements contained in \([s_{i-1}, s_j] \) will be larger than \( s_l \) in \( s^h \) and all elements of \([s_j, s_l] \) remain in \( s^h \) to be larger than \( s_l \). This implies that all elements in the segment \( v_{1}'^l, \ldots, v_{m}'^l \) will be larger than \( s_l \) in \( s^h \). Accordingly, \( s_j \) is the minimum element in the cycle \( (s_j, v_{1}'^l, \ldots, v_{m}'^l) \).

It remains to inspect \( (s_j, v_{1}'^l, \ldots, v_{m}'^l) \). We find two scenarios:

1. If \( s_j \) is the minimum (w.r.t. \( <_h \)), then \( v_{1}'^l, \ldots, v_{m}'^l \) contains no element of \([s_j, s_{j+1}] \) in \( s \). We claim that in this case there is a bijection between the pairs \((p, e)\) and the set \( Y_2 \). It suffices to specify the inverse: given an \( Y_2 \)-element, \( p' = (s', \pi') \) with three labeled cycles \( (s'_{i-1}, u_{1}'^l, \ldots, u_{m}'^l), \ (s'_j, u_{1}'^{l'}, \ldots, u_{m}'^{l'}) \) and \( (s'_l, u_{1}'^l, \ldots, u_{m}'^l) \) we consider a Case 1 transpose determined by the three minimum elements, \( s'_{i-1} <_f s'_j <_f s'_l \) in the respective three cycles. This generates a planted plane permutation \((s, \pi)\) together with a distinguished NTAE, \( e \), obtained as follows: after transposing the three cycles merge into

\[
C = (s'_{i-1}, u_{1}'^l, \ldots, u_{m}'^l, s'_j, u_{1}'^{l'}, \ldots, u_{m}'^{l'}, s'_l, u_{1}'^l, \ldots, u_{m}'^l),
\]

where \( s'_{i-1} <_e s'_j <_e s'_l \). Since elements contained in \( u_{1}'^l, \ldots, u_{m}'^l \) are by construction larger than \( s'_j \) w.r.t. \( <_f \) and these elements will not be moved by the transpose, \( u_{m}'^l >_s s'_j \), i.e., \( e = u_{m}'^l \) is the NTAE. In case of \( \{u_{1}'^l, \ldots, u_{m}'^l\} = \emptyset \) we have \( e = s'_j \).
The following diagram illustrates the situation

\[
\begin{align*}
\text{Case 1: } & \quad (s_1, \ldots, s_p) (s_j, \ldots, s_{j+1}) (s_k, \ldots, s_{k+1}) \\
\text{Case 2: } & \quad (s_1, \ldots, s_p) (s_j, \ldots, s_{j+1}) (s_k, \ldots, s_{k+1})
\end{align*}
\]

where \( \cdots \) denotes the sequence \( v_1, \ldots, v_{m_i} \).

2. If \( s_j \) is not the minimum, then \( \{v_1', \ldots, v_{m_i}'\} \neq \emptyset \) and \( \epsilon = v_{m_i}' \). Since by construction, \( \epsilon \in [s_1, s_p] \) in \( s \), it will not be impacted by the transposition and we have \( s_j < v_1 \). Therefore, \( \epsilon \) persists to be a NTAE in \( p' \). We furthermore observe

\[
\epsilon > s_1 s_j > s_i \min \{s_j, v_1', \ldots, v_{m_i}'\} > s_j s_i > \cdots > s_i-1,
\]

where \( \min \{s_j, v_1', \ldots, v_{m_i}'\} > s_j \) is due to the fact that, in \( s \), we have

\[
\{v_1', \ldots, v_{m_i}'\} \subset [s_{i-1}, s_j] \cup [s_i, s_n].
\]

After transposing \([s_i, s_j]\) and \([s_{j+1}, s_i]\), all elements in \( \{v_1', \ldots, v_{m_i}'\} \) will be larger than \( s_j \) following \( <v_1 \). We claim that there is a bijection between such pairs \((p, \epsilon)\) and the set \( Y_3 \). To this end we specify its inverse: given an element in \( Y_3 \), \( p' = (s', \pi') \) with three labeled cycles

\[
(s_{i-1}', u_{i}', \ldots, u_{m_i}'), (s_j', u_1', \ldots, u_{m_i}'), (s_j', u_1', \ldots, u_{m_i}'),
\]

where \( \epsilon = u_{m_i}' \) is the distinguished NTAE. Then a Case 1 transpose w.r.t. the two minima \( s_{i-1}' \) and \( s_j' \), and \( s_j' \) generates a planted plane permutation, \( p \), in which \( \epsilon \) remains as a distinguished NTAE.

This completes the proof of the proposition. \( \Box \)

Combining Lemma[7] and Proposition[8] we can conclude that each planted plane permutation with \( k \) cycles and a distinguished NTAE is in one-to-one correspondence with a planted plane permutation having \( 2I + 1 \) labeled cycles among its \( k + 2I \) cycles for some \( i > 0 \).

Let \( p_i^k(n) \) denote the number of \( p \in U_{D_k} \) having \( k \) cycles and for which \( D_p \) is of cycle-type \( \lambda \). Let \( p^k_{a,b}(n) \) denote the number of \( p \in U_{D_k} \) where \( p \) has \( k \) cycles, \( \text{Exc}(p) = a \) and \( D_p \) is of type \( \lambda \). Then we have \( n - a \) anti-exceedances, exactly \( k \) of which are trivial.
Proposition 5

\[ \sum_{a \geq 0} (n - a - k)p_{a,k}^\lambda(n) = \sum_{i=1}^{\left\lfloor \frac{n-k}{2} \right\rfloor} \left( \frac{k + 2i}{k - 1} \right) p_{k+2i}^\lambda(n). \tag{10} \]

Proof Using the notation of Proposition 4 and recursively applying Lemma 7 and Proposition 4, we have

\[ |Y_1| = \sum_{a \geq 0} (n - a - k)p_{a,k}^\lambda(n) \]

\[ = |Y_2| + |Y_3| = \left( \frac{k + 2}{3} \right) p_{k+2}^\lambda(n) + |Y_3| \]

\[ = \left( \frac{k + 2}{3} \right) p_{k+2}^\lambda(n) + \left( \frac{k + 4}{5} \right) p_{k+4}^\lambda(n) + \ldots \]

\[ = \left( \frac{k + 2}{3} \right) p_{k+2}^\lambda(n) + \ldots + \left( \frac{k + 2\left\lfloor \frac{n-k}{2} \right\rfloor}{k - 1} \right) p_{k+2\left\lfloor \frac{n-k}{2} \right\rfloor}^\lambda(n), \]

whence the proposition. \( \square \)

Remark. The exact number of terms of the RHS of eq. (10) depends on the cycle type \( \lambda \), i.e., some of terms \( p_{a,k}^\lambda(n) \) are actually 0.

As discussed earlier, the number of \( p = (s, \pi) \in U_{D_p} \) that have \( a \) exceedances, \( k \) cycles and \( D_p \) is of cycle type \( \lambda \) is equal to the number of planted plane permutations such that \( s = (1, 2, \ldots, n) \) having \( a \) exceedances, \( k \) cycles and the cycle type of its diagonal is \( \lambda \). Note, that summing over all diagonals, we obtain the number of planted plane permutations such that \( s = (1, 2, \ldots, n) \), that contain \( a \) exceedances and \( k \) cycles, i.e. the number of permutations containing \( a \) exceedances and \( k \) cycles. The latter equals also the sum of all \( p_{a,k}^\lambda(n) \) taken over all different \( \lambda \) (for each fixed \( \lambda \), we consider only one fixed \( D_p \)). Analogously, summing the \( p_{a,k}^\lambda(n) \) over all different \( \lambda \) equals the number permutations containing \( k \) cycles, \( C(n, k) \). Therefore, we have

Corollary 2 Let \( p_{a,k}(n) \) denote the number of permutations on \([n]\) containing \( a \) exceedances and \( k \) cycles. Then,

\[ \sum_{a \geq 0} (n - a - k)p_{a,k}(n) = \sum_{i=1}^{\left\lfloor \frac{n-k}{2} \right\rfloor} \left( \frac{k + 2i}{k - 1} \right) C(n, k + 2i). \tag{11} \]

In particular, \( p_{0,0}(n) = 1 \), \( p_{1,n-1}(n) = \binom{n}{2} \).

Proof Summing over all possible cycle types on both sides of eq. (10) implies the corollary. \( \square \)

Clearly, we have \( \sum_{a} p_{a,k}(n) = C(n, k) \) and furthermore \( \sum_{a} a p_{a,k}(n) \) counts the total number of exceedances in all permutations with \( k \) cycles. Hence, reformulating eq. (11), we have the following corollary:
Corollary 3 The total number of exceedances in all permutations on \([n]\) with \(k\) cycles is given by

\[
\sum_a ap_{a,k}(n) = (n-k)C(n,k) - \sum_{i=1}^{\lfloor n/2 \rfloor} \binom{k+2i}{k-1} C(n,k+2i).
\] (12)

However, it is easy to compute the total number of exceedances as shown below.

Proposition 6 The total number of exceedances in all permutations on \([n]\) with \(k\) cycles is \(\binom{\lfloor n/2 \rfloor}{2} C(n-1,k)\).

Proof Note the total number of exceedances in all permutations on \([n]\) with \(k\) cycles is equal to the total number of the set \(X\) of permutations \(\pi\) on \([n]\) with \(k\) cycles and with one pair \((i, \pi(i))\) distinguished, where \(i\) is an exceedance in \(\pi\). Let \(Y\) denote the set of pairs \((\tau, \alpha)\), where \(\tau\) is a subset of \([n]\) having 2 elements and \(\alpha\) is a permutation on \([n-1]\) having \(k\) cycles. We will show that there is a bijection between \(X\) and \(Y\).

Given \((\pi, (i, \pi(i))) \in X\), we obtain \((\tau, \alpha) \in Y\) as follows: set \(\tau = \{i, \pi(i)\}\) and \(\alpha'\) on \([n]\ \backslash \{\pi(i)\}\) as \(\alpha'(j) = \pi(j)\) if \(j \neq i\) while \(\alpha'(i) = \pi^2(i)\). Now we obtain \(\alpha\) from \(\alpha'\) by substituting \(x-1\) for every number \(x > \pi(i)\). Conversely, given \((\tau, \alpha) \in Y\), where \(\tau = \{a, b\}\) and \(a < b\). Define \(\alpha'\) from \(\alpha\) by substituting \(x+1\) for every number \(x \geq b\). Next we define \(\pi\) from \(\alpha'\) in the following way: \(\pi(j) = \alpha'(j)\) if \(j \neq a, b\) while \(\pi(a) = b\) and \(\pi(b) = \alpha'(a)\). Note that by construction \(a\) is an exceedance in \(\pi\) and clearly, \(|Y| = \binom{\lfloor n/2 \rfloor}{2} C(n-1,k)\), whence the proposition. □

Corollary 3 and Proposition 6 give rise to a new recurrence for the unsigned Stirling numbers of the first kind.

Theorem 1 For \(n \geq 1, k \geq 1\), we have

\[
C(n+1,k) = \sum_{i \geq 1} \binom{k+2i}{k-1} \frac{C(n+1,k+2i)}{n+1-k} + \binom{n+1}{2} \frac{C(n,k)}{n+1-k}.
\] (13)

Reformulating eq. (13), we obtain

\[
\frac{2C(n+1,k)}{n(n+1)} = \sum_{i \geq 1} \binom{k+2i}{k-1} \frac{1}{n+1-k} \frac{2C(n+1,k+2i)}{n(n+1)} + \frac{C(n,k)}{n+1-k}.
\] (14)

Comparing eq. (6) and eq. (14), we observe that \(\frac{2}{m(n+1)} C(n+1,k)\) and \(\xi_{1,n}(n)\) satisify the same recurrence. Furthermore, the initial value \(\xi_{1,n}(n)\) is equal to the number of different ways to factorize an \(n\)-cycle into an \(n\)-cycle and a permutations with \(n\) cycles. Since only the identity map has \(n\) cycles, we have \(\xi_{1,n}(n) = 1\). On the other hand, \(C(n+1,n)\) is the number of permutations on \([n+1]\) with \(n\) cycles. Such permutations have cycle type \(1^{n-1}2^1\). It suffices to determine the 2-cycle, which is equivalent to choose 2 elements from \([n+1]\). Therefore, the initial value \(\frac{2}{m(n+1)} C(n+1,n) = \frac{2}{m(n+1)} \binom{n+1}{2} = 1\). Thus, \(\frac{2}{m(n+1)} C(n+1,k)\) and \(\xi_{1,n}(n)\) agree on the initial values.
Proposition 7 (Zagier [17], Stanley [18]) For $k \geq 1$, $n \geq 1$ and $n - k$ even, we have

$$\xi_{1,k}(n) = \frac{2}{n(n+1)} C(n+1,k).$$

(15)

**Remark.** Proposition 2 implies that $\xi_{1,k}(n) = 0$ if $n - k$ is odd.

In the following we study Proposition 3 in more detail, observing that it can be refined, eventually allowing to clear the parameter $a$. To this end we enumerate plane permutations, $p = (s, \pi) \in U_{D_p}$, having $a$ exceedances, in which $D_p$ has cycle-type $\lambda$ and $\pi$ has cycle type $\eta$. Let us denote this set as $U_{D_p}^{n,\lambda,\eta}$.

Let $\mu, \eta$ be partitions of $n$. We write $\mu \triangleright 2n+1 \eta$ if $\mu$ can be obtained by splitting one $\eta$-block into $(2i+1)$ non-zero parts. Let furthermore $\kappa_{\mu, \eta}$ denote the number of different ways to obtain $\eta$ from $\mu$ by merging $\ell(\mu) - \ell(\eta) + 1$ $\mu$-blocks into one, where $\ell(\mu)$ and $\ell(\eta)$ denote the number of blocks in the partitions $\mu$ and $\eta$, respectively.

Theorem 2 Let $f_{\eta, \lambda}(n) = |U_{D_p}^{n,\lambda,\eta}|$. Then, we have

$$f_{\eta, \lambda}(n) = \frac{\sum_{i=1}^{\ell(n)} \sum_{\mu \triangleright 2i+1 \eta} \kappa_{\mu, \eta} f_{\mu, \lambda}(n) + \sum_{i=1}^{\ell(\lambda)} \sum_{\mu \triangleright 2i+1 \eta} \kappa_{\mu, \eta} f_{\mu, \lambda}(n)}{n+1 - \ell(\eta) - \ell(\lambda)}.$$  

(16)

**Proof.** Let $f_{\eta, \lambda}(n, a)$ denote the number of $p \in U_{D_p}^{n,\lambda,\eta}$ having $a$ exceedances.

**Claim**

$$\sum_{a \geq 0} (n-a-\ell(\eta)) f_{\eta, \lambda}(n, a) = \sum_{i=1}^{\ell(n)} \sum_{\mu \triangleright 2i+1 \eta} \kappa_{\mu, \eta} f_{\mu, \lambda}(n),$$

(17)

Given $p = (s, \pi)$ where the cycle-type of $\pi$ is $\eta$, a Case 2 transpose will result in a $p^h = (s^h, \pi^h)$, where $\pi^h$ has cycle-type $\mu$ and $\mu \triangleright 2n+1 \eta$. Refining the proof of Proposition 3 each pair $(p = (s, \pi), \varepsilon)$ for which $p \in U_{D_p}^{n,\lambda,\eta}$ and $\varepsilon$ is a NTAE, uniquely corresponds to a planted plane permutation $p^h = (s^h, \pi^h)$ with $2i+1$ labeled cycles for some $i > 0$, and $\pi^h$ has cycle-type $\mu$ with $\mu \triangleright 2i+1 \eta$. Conversely, suppose we have $p^h = (s^h, \pi^h)$ where $\pi^h$ has cycle-type $\mu$ with $\mu \triangleright 2i+1 \eta$. If there are $\kappa_{\mu, \eta}$ ways to obtain $\eta$ by merging $2i+1$ $\mu$-blocks into one, then we can label $2i+1$ cycles of $p^h$ in $\kappa_{\mu, \eta}$ different ways, which corresponds to $\kappa_{\mu, \eta}$ pairs $(p = (s, \pi), \varepsilon)$ where the cycle-type of $\pi$ is $\eta$. This proves the Claim.

We next clear the parameter $a$ in the LHS of eq. (17). To this end, since each $p = (s, \pi) \in U_{D_p}^{n,\lambda,\eta}$ satisfies $s = D_p \pi$, taking the inverse produces $s^{-1} = \pi^{-1} D_p^{-1}$. Accordingly, there is a bijection between $U_{D_p}^{n,\lambda}$ and the set of $p' = (s', \pi')$ where $\pi'$ is fixed, has cycle-type $\lambda$ and $D_p'$ has cycle-type $\eta$. Namely, for $p \in U_{D_p}^{n,\lambda,\eta}$, we obtain $s' = (s_0, s_{n-1}, \cdots, s_1)$, $\pi' = D^{-1}_p$ and $D_p' = \pi^{-1}$. Note $\pi^{-1}$ has cycle type $\eta$ and $D_p^{-1}$ has cycle type $\lambda$. Thus, we have $|U_{D_p'}^{n,\eta}| = |U_{D_p}^{n,\lambda}|$. Eq. (17) can be employed in order
to enumerate $U_{D_p}^{\lambda, \eta}$ and we obtain

$$\sum_{a \geq 0} (n - a - \ell(\lambda))f_{\lambda, \eta}(n, a) = \sum_{i=1}^{\ell(\lambda)} \sum_{\mu \supseteq 2i+1} \kappa_{\mu, \lambda} f_{\mu, \eta}(n). \quad (18)$$

We immediately observe that if $\pi$ has $a$ exceedances, then Lemma 4 guarantees that $D_p$ has $n - (a + 1) = n - 1 - a$ exceedances w.r.t. $\prec_s$. Since an exceedance in $\alpha$ is a strict anti-exceedance (i.e., strictly decreasing) in $\alpha^{-1}$, $\pi' = D_p^{-1}$ has $n - 1 - a$ anti-exceedances w.r.t. $\prec_s$. W.r.t. $\prec_{s'}$, any anti-exceedance of $\prec_s$ of $D_p^{-1}$ the image of which not being $s_0$, will become an exceedance.

Thus, if there exists a none strict anti-exceedance having $s_0$ as image, i.e., $s_0$ is a fixed point, $\pi' = D_p^{-1}$ has $n - 1 - a$ exceedances. If $s_0$ is not a fixed point, the strict anti-exceedance having $s_0$ as image remains as a strict anti-exceedance in $\pi' = D_p^{-1}$.

Furthermore, $s_0$ must be an exceedance of $\pi' = D_p^{-1}$ (w.r.t. $\prec_s$), and it remains to be an exceedance w.r.t. $\prec_{s'}$. In this case, there are also $(n - 1 - a - 1) + 1 = n - 1 - a$ exceedances in $\pi' = D_p^{-1}$.

Thus, each $p = (s, \pi) \in U_{D_p}^{\eta, \lambda}$ contributes a multiplicity of $n - a - \ell(\eta)$ on the LHS of eq. (17) and $(s', \pi')$ contributes a multiplicity of $n - (n - 1 - a) - \ell(\lambda)$ on the LHS of eq. (18). Finally, summing up the LHS and the RHS of eq. (17) and eq. (18), respectively and using

$$n - a - \ell(\eta) + (n - (n - 1 - a) - \ell(\lambda)) = n + 1 - \ell(\eta) - \ell(\lambda),$$

completes the proof. \qed

**Remark.** Note if $\ell(\eta) + \ell(\lambda) > n + 1$, $f_{\eta, \lambda}(n) = 0$. This fact also follows from Proposition 4.

Summing over all $\eta$ with $\ell(\eta) = k$, we obtain

**Corollary 4**

$$p_k^\lambda(n) = \sum_{i=1}^{\ell(\mu)} \sum_{\mu \supseteq 2i+1} \kappa_{\mu, \lambda} p_{\mu}^{\lambda}(n) + \sum_{i=1}^{\ell(\lambda)} \sum_{\mu \supseteq 2i+1} \kappa_{\mu, \lambda} p_{\mu}^{\lambda}(n), \quad (19)$$

**Proof.** Note any $\mu$ with $\ell(\mu) = k + 2i$ satisfies $\mu \supseteq 2i+1 \eta$ for some $\eta$ with $\ell(\eta) = k$. Thus, $\sum_{\ell(\eta) = k} \kappa_{\mu, \eta} = \binom{k+2i}{2i+1}$. Furthermore, $\sum_{\ell(\mu) = k+2i} f_{\mu, \lambda}(n) = p_{k+2i}^{\lambda}(n)$. Therefore,

$$\sum_{\ell(\eta) = k} \sum_{i=1}^{\ell(\eta)} \sum_{\mu \supseteq 2i+1} \kappa_{\mu, \eta} f_{\mu, \lambda}(n) = \sum_{i=1}^{\ell(\mu)} \sum_{\ell(\mu) = k+2i} \kappa_{\mu, \eta} f_{\mu, \lambda}(n) = \sum_{i=1}^{\ell(\mu)} \binom{k+2i}{k-1} p_{k+2i}^{\lambda}(n).$$
We also have
\[
\sum_{\ell(\eta)=k} \sum_{i=1}^{\lfloor n - \ell(\eta) \rfloor} \kappa_{\mu, \lambda} f_{\mu, \eta}(n) = \sum_{i=1}^{\lfloor n - \ell(\lambda) \rfloor} \sum_{\mu>2i+1 \lambda} \kappa_{\mu, \lambda} \sum_{\ell(\eta)=k} f_{\mu, \eta}(n) = \sum_{i=1}^{\lfloor n - \ell(\lambda) \rfloor} \sum_{\mu>2i+1 \lambda} \kappa_{\mu, \lambda} p_{\mu, \lambda}^b(n),
\]
whence the corollary. □

Analogously, summing over all \( \eta \) with \( \ell(\eta) = k \) and \( \lambda \) with \( \ell(\lambda) = l \), we obtain Proposition 3.

4 Transposition distance

In the following, we will show that plane permutation represent a simple framework for studying the sorting of permutations. The main idea is to utilize various block-transposition actions on plane permutations, motivated by the study of transposition actions on the boundary component of fatgraphs [24], where a topological framework for studying reversal distance of signed permutations is presented.

In this section, we shall represent permutations in one-line, i.e., we consider them to be sequences. Given a sequence on \([n]\)
\[
s = a_1 \cdots a_{i-1} a_j a_{j+1} \cdots a_k a_{k+1} \cdots a_n.
\]
A transpose action on \(s\) means to change \(s\) into
\[
s' = a_1 \cdots a_{i-1} a_{j+1} \cdots a_k a_i \cdots a_{k+1} a_j \cdots a_n,
\]
for some \(1 \leq i \leq j < k \leq n\). Let \(e_n = 123\cdots n\). The transposition distance of a sequence \(s\) on \([n]\) is the minimum number of transpositions needed to sort \(s\) into \(e_n\).

Denote this distance as \(td(s)\).

Let \(C(\pi), C_{\text{odd}}(\pi)\) and \(C_{\text{ev}}(\pi)\) denote the number of cycles, the number of odd cycles and the number of even cycles in \(\pi\), respectively. Furthermore, let \([n]^* = \{0, 1, \ldots, n\}\), and
\[
\hat{e}_n = (0123\cdots n), \quad \hat{s} = (0, a_1, a_2, \ldots, a_n), \quad p_t = (n, n-1, \ldots, 1, 0).
\]

**Theorem 3**

\[
td(s) \geq \max \left\{ \frac{\max_{\gamma_1} \{ |C(p_t \delta_\gamma s) - C(\gamma_1)| \}^2}{2}, \frac{\max_{\gamma_1} \{ |C_{\text{odd}}(p_t \delta_\gamma s) - C_{\text{odd}}(\gamma_1)| \}^2}{2}, \frac{\max_{\gamma_1} \{ |C_{\text{ev}}(p_t \delta_\gamma s) - C_{\text{ev}}(\gamma_1)| \}^2}{2} \right\}, \tag{20}
\]
where \(\gamma_1, \gamma_2, \gamma_3\) ranges over all permutations on \([n]^*\).
Proof For an arbitrary permutation \( \gamma_1 \) on \([n]\), \( p = (\delta, \gamma_1) \) is a plane permutation. By construction, each transposition of the sequence \( s \) induces a transpose on \( p \). If \( s \) changes to \( e_n \) by transposes, we have, for some \( \beta \), that \( p \) changes into the plane permutation \((\delta_n, \beta)\). By construction, we have
\[
D_p = s\gamma_1^{-1} = \delta_n\beta^{-1},
\]
and accordingly
\[
\beta = \gamma_1\delta^{-1}e_n.
\]
Since each transpose changes the number of cycles by at most 2 according to Lemma 4, \( n \) transposes are needed from \( \gamma_1 \) to \( \beta \). The same argument applies for deriving the lower bounds in terms of odd and even cycles, respectively. \( \Box \)

The most common model used to study transposition distance is cycle-graph proposed by Bafna and Pevzner [21]. Given a permutation \( s = s_1s_2\cdots s_n \) on \([n]\), the cycle graph \( G(s) \) of \( s \) is obtained as follows: add two additional elements \( s_0 = 0 \) and \( s_{n+1} = n+1 \). The vertices of \( G(s) \) are numbers in \([n+1]\). Draw a directed black edge from \( i+1 \) to \( i \), and draw a directed gray edge from \( s_i \) to \( s_{i+1} \), we then obtain \( G(s) \). An alternating cycle in \( G(s) \) is a directed cycle, where its edges alternate in color. An alternating cycle is called odd if the number of black edges is odd. Bafna and Pevzner obtained lower and upper bound for \( td(s) \) in terms of the number of cycles and odd cycles of \( G(s) \) [21].

By examining the cycle graph model \( G(s) \) of a permutation \( s \), it turns out the cycle graph \( G(s) \) is actually the directed graph representation of the product \( p\delta \), if we identify the two auxiliary points 0 and \( n+1 \). The directed graph representation of a permutation \( \pi \) is the directed graph by drawing an directed edge from \( i \) to \( \pi(i) \). If we color the directed edge of \( \delta \) gray and the directed edge of \( p \) black, an alternating cycle then determines a cycle of the permutation \( p\delta \). Therefore, the number of cycles and odd cycles in \( p\delta \) is equal to the number of cycles and odd cycles in \( G(s) \), respectively. As result, Theorem 3 immediately implies:

**Corollary 5 (Pevzner, [21])**

\[
\begin{align*}
\text{td}(s) & \geq \frac{n + 1 - C(p\delta)}{2}, \\
\text{td}(s) & \geq \frac{n + 1 - \text{C}_{\text{odd}}(p\delta)}{2}.
\end{align*}
\]

Proof Setting \( \gamma_1 = (p, \delta)^{-1} = p_\gamma^1 \) in Theorem 3 implies the result. \( \Box \)

In view of Theorem 3 employing appropriate \( \gamma \), it is possible to improve the lower bound. This motivates the following problems: given a permutation \( \pi \), what is the maximum number of \( |C(\pi \gamma) - C(\gamma)| \) (resp. \( |\text{C}_{\text{odd}}(\pi \gamma) - \text{C}_{\text{odd}}(\gamma)|, |\text{C}_{\text{ev}}(\pi \gamma) - \text{C}_{\text{ev}}(\gamma)| \)), where \( \gamma \) ranges over a set of permutations.

More generally, we can study the distribution function
\[
\sum_{\gamma \in A} 2^{C(\pi \gamma) - C(\gamma)}, \sum_{\gamma \in A} 2^{\text{C}_{\text{odd}}(\pi \gamma) - \text{C}_{\text{odd}}(\gamma)}, \sum_{\gamma \in A} 2^{\text{C}_{\text{ev}}(\pi \gamma) - \text{C}_{\text{ev}}(\gamma)},
\]

\( (23) \)
where $A$ is a conjugacy class of permutations. We remark that Stanley has studied
the particular case where $\pi$ is a $n$-cycle and $\gamma$ is a conjugate class [18]. In this pa-
per, we will later solve this optimization problem w.r.t. the number of cycles, i.e.,
$max_{\gamma}\{C(\pi\gamma) - C(\gamma)\}$, where $\pi$ is an arbitrary, even permutation.

### 5 Block-interchange distance of permutations

Lemma 5 facilitates the study of more general transpositions, namely, where the
two blocks are not adjacent. This is referred to as the block-interchange problem
in Christie [26]. The minimum number of block-interchanges needed to sort $s$ into $e_n$
is accordingly called the block-interchange distance of $s$ and denoted as $bid(s)$.

**Lemma 8** Given $p = (\bar{s}, \pi)$ on $[n]^*$ where $D_p = p_1^{-1}$ and $\bar{s} \neq \hat{e}_n$. Then, there exist
$\bar{s}_{i-1} \leq \bar{s}_j < \bar{s}_{k-1} \leq \bar{s}_l$ such that
$$\pi(\bar{s}_{i-1}) = \bar{s}_{k-1}, \quad \pi(\bar{s}_j) = \bar{s}_l.$$

**Proof** Since $\bar{s} \neq \hat{e}_n$, there exists $x < \bar{s}_k$. Find the largest integer such that $x = \bar{s}_{k-1}$
and let $\bar{s}_l = x + 1$. Then, $\pi(\bar{s}_{i-1}) = x = \bar{s}_{k-1}$ since $D_p(\pi(\bar{s}_{i-1})) = \pi(\bar{s}_{i-1}) + 1 = x + 1$.
Between $\bar{s}_{i-1}$ and $x$, find the largest integer, which is larger than $x$. Since $x + 1$ is such an integer, this maximum exists and we denote it by $y$. Then we have by construction
$$\bar{s}_{i-1} < \bar{s}_j = y < \bar{s}_k < \bar{s}_{k-1} < \bar{s}_l + 1.$$
Therefore, $\pi(\bar{s}_j) = D_p^{-1}(y + 1) = y = \bar{s}_j$, whence the lemma. \qed

Now we can derive the block-interchange distance formula obtained by Christie [26].

**Theorem 4** (Christie, [26])

$$bid(s) = \frac{n + 1 - C(p\bar{s})}{2}. \quad (24)$$

**Proof** According to Lemma 8 if $\bar{s}_{i-1} < \bar{s}_j < \bar{s}_{k-1} < \bar{s}_l$, we either have
$$(\bar{s}_{i-1}, \bar{s}_{k-1}, \ldots, \bar{s}_l, \bar{s}_j, \ldots) \quad \text{or} \quad (\bar{s}_{i-1}, \bar{s}_{k-1}, \ldots, \bar{s}_l, \bar{s}_j, \ldots).$$
Then, $\chi_0$ determined by $\bar{s}_{i-1} < \bar{s}_j < \bar{s}_{k-1} < \bar{s}_l$ is either Case $c$ or Case $e$ of Lemma 5.
If $\bar{s}_{k-1} = \bar{s}_l$, we have the situation $(\bar{s}_{i-1}, \bar{s}_{k-1}, \bar{s}_j, \ldots, \bar{s}_l)$. In this case, $\chi_0$, determined
by $\bar{s}_{i-1} < \bar{s}_j < \bar{s}_l$ is a Case 2 transpose of Lemma 5. Consequently, arguing as in
Theorem 3 we can always find a block-interchange to increase the number of cycles
by 2, whence the theorem. \qed

In view of Theorem 3 and Theorem 4 we are now in position to answer one of
the optimization problem mentioned earlier.

**Theorem 5** Given an even permutation $\alpha$ on $[n]$ and $n \geq 1$. Then we have
$$\max_{\gamma}\{|C(\alpha\gamma) - C(\gamma)|\} = n - C(\alpha), \quad (25)$$
where $\gamma$ ranges over all permutations on $[n]$. 

Proof Claim. for arbitrary \( s \), we have

\[
\max_{\gamma} |C(p_t \bar{s} \gamma) - C(\gamma)| = n + 1 - C(p_t \bar{s}),
\]

(26)

where \( \gamma \) ranges over all permutations on \([n]^*\).

To prove the Claim, we argue as in Theorem 3, that

\[
\text{bid}(s) \geq \frac{1}{2} \max_{\gamma} |C(p_t \bar{s} \gamma) - C(\gamma)|
\]

holds. On the other hand Theorem 4 guarantees

\[
\text{bid}(s) = \frac{n + 1 - C(p_t \bar{s})}{2} = \frac{|C(p_t \bar{s} \gamma) - C(\gamma)|}{2}_{\gamma = (p_t \bar{s})^{-1}},
\]

which means the maximum is achieved for \( \gamma = (p_t \bar{s})^{-1} \), whence the Claim.

We now use the fact that any even permutation \( \alpha' \) on \([n]^*\) has a factorization into two \((n + 1)\)-cycles. Assume \( \alpha' = \beta_1 \beta_2 \) where \( \beta_1, \beta_2 \) are two \((n + 1)\)-cycles, and \( p_t = \theta \beta_1 \theta^{-1} \). Suppose in the following that \( \gamma \) is an arbitrary permutation

\[
\max_{\gamma} |C(\alpha' \gamma) - C(\gamma)| = \max_{\gamma} |C(p_t \theta \beta_2 \theta^{-1} \gamma \theta^{-1}) - C(\theta \gamma \theta^{-1})|
\]

\[
= \max_{\gamma} |C(p_t \theta \beta_2 \theta^{-1} \gamma \theta^{-1}) - C(\theta \gamma \theta^{-1})|
\]

\[
= n + 1 - C(p_t \theta \beta_2 \theta^{-1})
\]

\[
= n + 1 - C(\beta_1 \beta_2) = n + 1 - C(\alpha'),
\]

whence the theorem.  \( \square \)

Furthermore Theorem 4 implies

**Corollary 6** Let \( \text{bid}_k(n) \) denote the number of sequences \( s \) on \([n]\) such that \( \text{bid}(s) = k \). Then,

\[
\text{bid}_k(n) = \frac{2C(n + 2, n + 1 - 2k)}{(n + 1)(n + 2)}
\]

(27)

Proof Let

\[
k = \text{bid}(s) = \frac{n + 1 - C(p_t \bar{s})}{2},
\]

The number of \( s \) such that \( \text{bid}(s) = k \) is equal to the number of permutation \( \bar{s} \) such that \( C(p_t \bar{s}) = n + 1 - 2k \). Clearly, the latter is equal to \( \xi_{n+1-2k}(n+1) \).  \( \square \)
6 Reversal distance for signed permutations

In this section, we consider the reversal distance for signed permutations, a problem extensively studied in the context of genome evolution [22][23][29] and references therein. Pevzner et al. [22, 23] obtained a lower bound for reversal distance based on the breakpoint graph model and their lower bound was formulated in two parameters: the number of break points \( b \) and the number of cycles \( c \) in the breakpoint graph.

In our framework the reversal distance problem can be expressed as a block-interchange distance problem. Our lower bound only depends on the number of cycles of a permutation.

Let \( \mathbb{n}^- = \{-1, -2, \ldots, -n\} \).

**Definition 4** A signed permutation on \( [n] \) is a pair \((a, w)\) where \( a \) is a sequence on \([n]\) while \( w \) is a word of length \( n \) on the alphabet set \( \{+, -\} \).

Usually, a signed permutation is represented by a single sequence \(a_w = a_{w,1}a_{w,2} \cdots a_{w,n}\) where \( a_{w,k} = w_k a_k \), i.e., each \( a_k \) carries a sign determined by \( w_k \).

Given a signed permutation \( a = a_1a_2 \cdots a_{i-1}a_i a_{i+1} \cdots a_{j-1}a_j a_{j+1} \cdots a_n \) on \([n]\), a reversal \( \rho_{i,j} \) acting on \( a \) will change \( a \) into
\[
a' = \rho_{i,j} \circ a = a_1a_2 \cdots a_{i-1}(-a_j)(-a_{j-1}) \cdots (-a_{i+1})(-a_i)a_{j+1} \cdots a_n.
\]
The reversal distance \( d_r(a) \) of a signed permutation \( a \) on \([n]\) is the minimum number of reversals needed to sort \( a \) into \( e_n = 12 \cdots n \).

For the given signed permutation \( a \), we associate the sequence \( s = s(a) \) as follows
\[
s = s_0s_1s_2 \cdots s_{2n} = 0a_1a_2 \cdots a_n(-a_n)(-a_{n-1}) \cdots (-a_2)(-a_1),
\]
i.e., \( s_0 = 0 \) and \( s_k = -s_{2n+1-k} \) for \( 1 \leq k \leq 2n \). Furthermore, such sequences will be referred to as skew-symmetric since we have \( s_k = -s_{2n+1-k} \). \( s \) is exact if there exists some \( s_i \) that is negative, for \( 1 \leq i \leq n \). The reversal distance of \( a \) is equal to the block-interchange distance of \( s \) into
\[
e_n^\circ = 012 \cdots n(-n+1) \cdots (-2)(-1),
\]
where only certain block-interchanges are allowed, i.e., only the actions \( \chi_h, h = (i, j, 2n+1-j, 2n+1-i) \) are allowed where \( 1 \leq i \leq j \leq n \). Hereafter, we will denote these particular block-interchanges on \( s \) reversals, \( \rho_{i,j} \).

Let
\[
\tilde{s} = (s) = (0, a_1, a_2, \ldots, a_{n-1}, a_n, -a_n, -a_{n-1}, \ldots, -a_2, -a_1)
\]
\[
pr = (-1, -2, \ldots, -n+1, -n, n-1, \ldots, 2, 1, 0)
\]
A planted plane permutation of the form \((\tilde{s}, \pi)\) will be called skew-symmetric.

According to Theorem\[3\] we have

**Corollary 7**
\[
d_r(a) \geq \frac{2n+1 - C(pr, \tilde{s})}{2}.
\]
Proof Since reversals are restricted block-interchanges, the reversal distance will be bounded by block-interchange distance without restriction. Theorem 3 then implies the corollary.

Our approach gives rise to the question of how potent the restricted block-interchanges are. Is it difficult to find a block-interchange increasing the number of cycles by 2 that is a reversal (2-reversal)?

A planted plane permutation $(\pi, \pi)$ will be called exact, skew-symmetric if $\pi$ is exact and skew-symmetric. The following lemma will show there is almost always such a 2-reversal.

Lemma 9 Given an exact, skew-symmetric $p = (\pi, \pi)$ on $[n]^+ \cup [n]^{-}$, where $D_p = p^{-1}_r$. Then, there always exist $i - 1$ and $2n - j$ such that

$$\pi(s_{i-1}) = s_{2n-j},$$

where $0 \leq i - 1 \leq n - 1$ and $n + 1 \leq 2n - j \leq 2n$. Furthermore, we have the following cases

(a) If $s_{i-1} < s_j < s_{2n-j} < s_{2n+1-i}$, then

$$\pi(s_{i-1}) = s_{2n-j}, \quad \pi(s_j) = s_{2n+1-i}.$$  

(b) If $s_j < s_{i-1} < s_{2n+1-i} < s_{2n-j}$, then

$$\pi(s_{i-1}) = s_{2n-j}, \quad \pi(s_j) = s_{2n+1-i}.$$

Proof We first prove $\pi(s_{i-1}) = s_{2n-j}$. Consider the smallest negative element in the subsequence $s_1 s_2 \cdots s_n$. Assume $s_i$ is the smallest negative element, where $1 \leq i \leq n$. If $s_i = -n$, then we have $s_{2n+1-i} = -s_i = n$ by symmetry. Since $D_p = p^{-1}_r$, for any $k$, $s_{k+1} = p^{-1}_r(s_k) = s_k + 1$ where $n + 1$ is interpreted as $-n$. Thus, $\pi(s_{i-1}) = D^{-1}_p(s_i) = D^{-1}_p(-n) = n = s_{2n+1-i}$. Let $2n - j = 2n + 1 - i$, then $2n - j \geq n + 1$ and we are done.

If $s_i > -n$, then we have $\pi(s_{i-1}) = D^{-1}_p(s_i) = s_i - 1 \geq -n$. Since $s_i$ is the smallest negative element among $s_i$ for $1 \leq i \leq n$, if $s_{2n-j} = s_i - 1 < s_i$, then $2n - j \geq n + 1$, whence $\pi(s_{i-1}) = s_{2n-j}$.

Using $D_p = p^{-1}_r$ and the skew-symmetry $s_k = -s_{2n+1-k}$, we have in case of (a) the following situation in $p$ (only relevant entries are illustrated)

$$\begin{pmatrix}
(i-1) & i & \cdots & j & j+1 & \cdots & 2n-j & 2n+1-j & \cdots & 2n+1-i \\
\pi(s_{i-1}) & (s_{2n-j} + 1) & \cdots & s_j & -s_{2n-j} & \cdots & s_{2n-j} & -s_j & \cdots & (-s_{2n-j} - 1) \\
\pi(s_{2n-j}) & \diamond & \cdots & (-s_{2n-j} - 1) & \diamond & \cdots & \diamond & \diamond & \cdots & \diamond
\end{pmatrix}$$

Therefore, we have

$$\pi(s_{i-1}) = s_{2n-j}, \quad \pi(s_j) = -s_{2n-j} - 1 = s_{2n+1-i}.$$  

Analogously we have in case of (b) the situation

$$\begin{pmatrix}
j & j+1 & \cdots & i-1 & i & \cdots & 2n+1-i & 2n+2-i & \cdots & 2n-j \\
\pi(s_j) & -s_{2n-j} & \cdots & s_{i-1} & s_{2n-j} + 1 & \cdots & -s_{2n-j} - 1 & -s_{i-1} & \cdots & s_{2n-j}
\end{pmatrix}$$

$$\begin{pmatrix}
-s_{2n-j} - 1 & \diamond & \cdots & s_{2n-j} & \diamond & \cdots & \diamond & \diamond & \cdots & \diamond
\end{pmatrix}$$
Therefore, we have
\[ \pi(s_{i-1}) = s_{2n-j}, \quad \pi(s_j) = -s_{2n-j} - 1 = s_{2n+1-i}, \]
whence the lemma.  

**Remark.** The pair \( s_{i-1} \) and \( s_{2n-j} \) such that \( \pi(s_{i-1}) = s_{2n-j} \) is not unique. For instance, assume the positive integer \( 1 \leq k \leq n-1 \) is not in the subsequence \( s_1 s_2 \cdots s_n \) but \( k+1 \) is, then \( \pi^{-1}(k) \) and \( k = D_p^{-1}(k+1) \) form such a pair.

Inspection of Lemma 5 and Lemma 9 shows that there is almost always a 2-reversal for signed permutations. The only critical cases, not covered in Lemma 9 are

- The signs of all elements in the given signed permutation are positive.
- Exact signed permutation which for \( 1 \leq i \leq n \) and \( n+1 \leq 2n-j, \pi(s_{i-1}) = s_{2n-j} \) iff \( 2n-j = 2n + 1 - i \).

We proceed to analyze the latter case. Then, since \( \pi(s_{i-1}) = s_{2n+1-i} = -s_i \), we have
\[
\begin{pmatrix}
    s_{i-1} & s_i & s_{i+1} & s_{2n+1-i} & s_{2n+2-i} \\
\end{pmatrix} = \begin{pmatrix}
    s_{i-1} & s_i & s_{i+1} & \cdots & s_{2n+1-i} & s_{2n+2-i} \\
    -s_i & -s_{i+1} & \cdots & \cdots & \cdots & \cdots \\
\end{pmatrix}.
\]

Due to \( D_p \), \( D_p(-s_i) = s_i = -s_i + 1 \) (note that \( n+1 \) is interpreted as \(-n\)). The only situation satisfying this condition is that \( s_i = -n \), i.e., the sign of \( n \) in the given signed permutation is negative. Then, we have \( \pi(s_{i-1}) = s_{2n-j} = s_{2n+1-i} = n \). We believe that in this case Lemma 5 (instead of Lemma 9) provides 2-reversal. Namely, \( s_{i-1} \) (i.e., the preimage of \( n \)), \( s_n \) and \( n = s_{2n+1-i} \) will form a Case 2 transpose in Lemma 9 which will be true if \( n \) and \( s_n \) are in the same cycle of \( \pi \), i.e., \( \pi \) has a cycle \((s_{i-1}, s_{2n+1-i}, \ldots, s_n, \ldots)\). In order to illustrate this we consider

**Example 1**

\[
\begin{pmatrix}
    0 & -3 & 1 & 2 & -4 & 4 & -2 & -1 & 3 \\
    -4 & 0 & 1 & 4 & 3 & -3 & -2 & 2 & -1 \\
\end{pmatrix} = \begin{pmatrix}
    0 & -4 & 3 & -1 & 2 & 4 & -3 \\
    1 & 4 & -2 & 2 & -4 & 0 & 3 & -3 & -1 \\
\end{pmatrix} = \begin{pmatrix}
    0, -4, 3, -1, 2, 4, -3 \\
    1, 4, -2, -4, -2, 1, 2, 4, -3 \\
\end{pmatrix} = (0, -4, 3, -1, 2, 4, -3)(1)(-2)
\]

We inspect, that in the first case \( s_{i-1} = 2, s_n = -4 \) and \( n = 4 \) form a Case 2 transpose of Lemma 3. In the second case \( s_{i-1} = 2, s_n = 3 \) and \( n = 4 \) form again a Case 2 transpose of Lemma 3.

Therefore, we conjecture

**Conjecture 1** Given an exact, skew-symmetric \( p = (\pi, \sigma) \) on \([n]^+ \cup [n]^\circ\) where \( D_p = p_t^{-1} \) and suppose \( \pi(s_{i-1}) = s_{2n+1-i} \), where \( 1 \leq i \leq n \). Then, \( n \) and \( s_n \) are in the same cycle of \( \pi \).

**Lemma 9** Conjecture 1 and an analysis of the preservation of exactness under 2-reversals suggest, that for a random signed permutation, it is likely to be possible to transform \( \pi \) into \( \pi_t \) via a sequence of 2-reversals. In fact many examples, including Table 3.2 in Braga [30] indicate that the lower bound of Theorem 7 equals the exact reversal distances. Note, that the lower bound obtained by Pevzner et al. 23 also provides the exact reversal distance for most of signed permutations.

At last, we present the following generalization of Conjecture 1:
Conjecture 2 Given a skew-symmetric \( \mathbf{p} = (\tilde{s}, \pi) \) on \( [n]^+ \cup [n]^− \) where \( D_{\mathbf{p}} = p_r \). Then, \( n \) and \( s_n \) are in the same cycle of \( \pi \).

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