Generalized Loop Groups of Complex Manifolds, Gaussian Quasi-Invariant Measures on them and their Representations.

S.V. Ludkovsky.

25 September 1999

Istituto di Matematica dell’Università di Trieste,
Piazzale Europa, 1
I-34100, Trieste, Italia
Permanent addresses:
Theoretical Department, Institute of General Physics,
Str. Vavilov 38, Moscow, 117942, Russia.

1980 Mathematics Subject Classification (1991 Revision): 22A10, 43A05 and 43A65

Abstract

Loop groups $G$ as families of mappings of the complex manifold $M$ into another complex manifold $N$ preserving marked points $s_0 \in M$ and $y_0 \in N$ are investigated. Quasi-invariant measures $\mu$ on $G$ relative to dense subgroups $G'$ are constructed. These measures are used for the studying of irreducible representations of such groups.

1 Introduction.

Loop groups are very important in differential geometry, algebraic topology and theoretical physics [7, 9, 36, 46], but about Gaussian quasi-invariant differentiable measures on them nothing was known. Only the simplest possible representations associated with path’s integrals were constructed for loop
groups of the circle, that is, for the manifold $M = S^1$ and (real) Riemann manifolds $N$. On the other hand, the quasi-invariant measures may be used for a construction of regular unitary representations. Moreover, the quasi-invariant measures are helpful for an investigation of the group itself. In the previous papers of the author, loop groups of Riemann manifolds $M$ and $N$ were investigated, where either $M = S^n$ were $n$-dimensional real spheres, $n = 1, 2, \ldots$, or $M = S^\infty$ was a unit sphere in a real separable Hilbert space $l^2(\mathbb{R})$. It was a progress in comparison with previous works of others authors, which considered only loop groups for the simplest case $M = S^1$.

This article treats arbitrary complex separable connected metrizable manifolds $M$ and $N$. For example, products of odd-dimensional real spheres $S^{2n-1} \times S^{2m-1}$ may be supplied in different ways by structures of complex manifolds. Another numerous examples of complex manifolds may be found in references therein such as domains in $\mathbb{C}^n$, a complex torus $\mathbb{C}^n/D$, where $D$ is a discrete additive subgroup of $\mathbb{C}^n$ generated by a basis $\tau_1, \ldots, \tau_{2n}$ of $\mathbb{C}^n$ over $\mathbb{R}$; a quotient space $G/D$ of a complex Lie group $G$ by a discrete subgroup $D$, submanifolds of the complex Grassman manifold $G_{p,q}(\mathbb{C})$, also their different products and their submanifolds. In general there are complex compact manifolds, which are not Kähler manifolds. For the construction of loop groups here are used manifolds $M$ with some mild additional conditions. When $M$ is finite-dimensional over $\mathbb{C}$ we suppose that it is compact. This condition is not very restrictive, since each locally compact space has Alexandroff (one-point) compactification (see Theorem 3.5.11 in [13]). When $M$ is infinite-dimensional over $\mathbb{C}$ it is assumed, that $M$ is embedded as a closed bounded subset into the corresponding Banach space $X_M$ over $\mathbb{C}$. This is necessary that to define a group structure on a quotient space of a free loop space.

The free loop space is considered as consisting of continuous functions $f : M \to N$ which are holomorphic on $M \setminus M'$ and preserving marked points $f(s_0) = y_0$, where $M'$ is a closed real submanifold depending on $f$ with a codimension $\text{codim}_RM' = 1$, $s_0 \in M$ and $y_0 \in N$ are marked points. There are two reasons to consider such class of mappings. The first is the need to define correctly compositions of elements in the loop group (see beneath). The second is the isoperimetric inequality for holomorphic loops, which can cause the condition of a loop to be constant on a sufficiently small neighbourhood of $s_0$ in $M$, if this loop is in some small neighbourhood of $w_0$. 


where \( w_0(M) := \{y_0\} \) is a constant loop (see Remark 3.2 in [22]).

In this article loop groups of different classes are considered. Classes analogous to Gevrey classes of \( f : M \setminus M' \to N \) are considered for the construction of dense loop subgroups and quasi-invariant measures. Henceforth, we consider only orientable manifolds \( M \) and \( N \), since for a non-orientable manifold there always exists its orientable double covering manifold (see §6.5 in [1]). Loop commutative monoids with the cancellation property are quotients of families of mappings \( f \) from \( M \) into a manifold \( N \) with \( f(s_0) = y_0 \) by the corresponding equivalence relation. For the definition of the equivalence relation here are not used groups of holomorphic diffeomorphisms because of strong restrictions on their structure caused by holomorphicity (see Theorems 1 and 2 in [5]). Groups are constructed from monoids with the help of A. Grothendieck procedure. These groups are commutative and non-locally compact. They does not have non-trivial local one-parameter subgroups \( \{g^b : b \in (-a, a)\} \) with \( a > 0 \) for an element \( g \) corresponding to a class of a mapping \( f : M \to N, f(s_0) = y_0 \), when \( f \) is such that \( \sup_{y \in N} |\text{card}(f^{-1}(y))| = k < \aleph_0 \), since \( g^{1/p} \) does not exist in the loop group for each prime integer \( p \) such that \( p > k \) (see §2). Therefore, they are not Banach-Lie groups, since in each neighbourhood \( W \) of the unit element \( e \) there are elements which does not belong to any local one-parameter subgroup.

Irreducible components of strongly continuous unitary representations of Abelian locally compact groups are one-dimensional by Theorem 22.17 [20]. In general commutative non-locally compact groups may have infinite-dimensional irreducible strongly continuous unitary representations, for example, infinite-dimensional Banach spaces over \( \mathbb{R} \) considered as additive groups (see §2.4 in [1] and §4.5 [17]).

Quasi-invariant measures are constructed on these loop groups. Then such measures are used for the investigation of irreducible unitary representations. Loop groups, their structure and quasi-invariant measures on them are investigated in §2. Unitary representations of loop groups are given in §3. Irreducibility of unitary representations of a dense subgroup \( G' \) associated with purely Gaussian quasi-invariant infinitely differentiable measures on the entire group \( G \) is investigated using specific properties of a quasi-invariance factor relative to shifts from the dense subgroup \( G' \). Characters and also infinite-dimensional irreducible unitary representations are investigated below. The relation between an equivalence of regular representations...
and equivalence of the corresponding measures is studied.

2 Loop groups.

To avoid misunderstanding we first give our Definitions and notations.

2.1. Definitions and Notes.1. Let $G$ be a Hausdorff topological group, we denote by $\mu : Af(G, \mu) \to [0, \infty) \subset \mathbb{R}$ a $\sigma$-additive measure. Its left shifts $\mu_{\phi}(E) := \mu(\phi^{-1} \circ E)$ are considered for each $E \in Af(G, \mu)$, where $Af(G, \mu)$ is the completion of $Bf(G)$ by $\mu$-null sets, $Bf(G)$ is the Borel $\sigma$-field on $G$, $\phi \circ E := \{ \phi \circ h : h \in E \}$. Then $\mu$ is called quasi-invariant if there exists a dense subgroup $G'$ such that $\mu_{\phi}$ is equivalent to $\mu$ for each $\phi \in G'$. Henceforth, we assume that a quasi-invariance factor $\rho_{\mu}(\phi, g) = \mu_{\phi}(dg)/\mu(dg)$ is continuous by $(\phi, g) \in G' \times G$, $\mu(V) > 0$ for some (open) neighbourhood $V \subset G$ of the unit element $e \in G$ and $\mu(G) < \infty$.

Let $(\mathcal{M}, \mathcal{F})$ be a space $\mathcal{M}$ of measures on $(G, Bf(G))$ with values in $\mathbb{R}$ and $G''$ be a dense subgroup in $G$ such that a topology $\mathcal{F}$ on $\mathcal{M}$ is compatible with $G''$, that is, $\mu \mapsto \mu_{h}$ is the homomorphism of $(\mathcal{M}, \mathcal{F})$ into itself for each $h \in G''$. Let $\mathcal{F}$ be the topology of convergence for each $E \in Bf(G)$. Suppose also that $G$ and $G''$ are real Banach manifolds such that the tangent space $T_{e}G''$ is dense in $T_{e}G$, then $TG$ and $TG''$ are also Banach manifolds. Let $\Xi(G'')$ denotes the set of all differentiable vector fields $X$ on $G''$, that is, $X$ are sections of the tangent bundle $TG''$. We say that the measure $\mu$ is continuously differentiable if there exists its tangent mapping $T_{\phi}\mu(\phi)(X_{\phi})$ corresponding to the strong differentiability relative to the Banach structures of the manifolds $G''$ and $TG''$. Its differential we denote $D_{\phi}\mu(\phi)(E)$, so $D_{\phi}\mu(\phi)(E)(X_{\phi})$ is the $\sigma$-additive real measure by subsets $E \in Af(G, \mu)$ for each $\phi \in G''$ and $X \in \Xi(G'')$ such that $D_{\mu}(E) : TG'' \to \mathbb{R}$ is continuous for each $E \in Af(G, \mu)$, where $D_{\phi}\mu(\phi)(E) = pr_{2} \circ (T\mu)_{\phi}(E), pr_{2} : p \times \mathcal{F} \to \mathcal{F}$ is the projection in $TN, p \in N, T_{p}N = \mathcal{F}$, $N$ is another real Banach differentiable manifold modelled on a Banach space $\mathcal{F}$, for a differentiable mapping $V : G'' \to N$ by $TV : TG'' \to TN$ is denoted the corresponding tangent mapping, $(T\mu)_{\phi}(E) := T_{\phi}\mu(\phi)(E)$. Then by induction $\mu$ is called $n$ times continuously differentiable if $T^{n-1}\mu$ is continuously differentiable such that $T^{n}\mu := T(T^{n-1}\mu), (D^{n}\mu)_{\phi}(E)(X_{1,\phi}, ..., X_{n,\phi})$ are the $\sigma$-additive real measures by $E \in Af(G, \mu)$ for each $X_{1}, ..., X_{n} \in \Xi(G'')$, where $(X_{j})_{\phi} =: X_{j,\phi}$ for each $j = 1, ..., n$ and $\phi \in G''$, $D^{n}\mu : Af(G, \mu) \otimes \Xi(G'')^{n} \to \mathbb{R}$.
2.1.2.1. A canonical closed subset $Q$ of $X = \mathbb{R}^n$ or of the standard separable Hilbert space $X = l_2(\mathbb{R})$ over $\mathbb{R}$ is called a quadrant if it can be given by $Q := \{ x \in X : q_j(x) \geq 0 \}$, where $(q_j : j \in \Lambda_Q)$ are linearly independent elements of the topologically adjoint space $X^*$. Here $\Lambda_Q \subset \mathbb{N}$ (with $\text{card}(\Lambda_Q) = k \leq n$ when $X = \mathbb{R}^n$) and $k$ is called the index of $Q$. If $x \in Q$ and exactly $j$ of the $q_i$'s satisfy $q_i(x) = 0$ then $x$ is called a corner of index $j$. Since the unitary space $X = \mathbb{C}^n$ or the separable Hilbert space $l_2(\mathbb{C})$ over $\mathbb{C}$ as considered over the field $\mathbb{R}$ is isomorphic with $X_{\mathbb{R}} := \mathbb{R}^{2n}$ or $l_2(\mathbb{R})$ respectively, then the above definition also describes quadrants in $\mathbb{C}^n$ and $l_2(\mathbb{C})$ in such sense (see also [37]). In the latter case we also consider generalized quadrants as canonical closed subsets which can be given by $Q := \{ x \in X_{\mathbb{R}} : q_j(x + a_j) \geq 0, a_j \in X_{\mathbb{R}}, j \in \Lambda_Q \}$, where $\Lambda_Q \subset \mathbb{N}$ ($\text{card}(\Lambda_Q) = k \in \mathbb{N}$ when $\dim_{\mathbb{R}} X_{\mathbb{R}} < \infty$).

2.1.2.2. If for each open subset $U \subset Q \subset X$ a function $f : Q \to Y$ for Banach spaces $X$ and $Y$ over $\mathbb{R}$ has continuous Fréchet differentials $D^\alpha f|_U$ on $U$ with $\sup_{x \in U} \| D^\alpha f(x) \|_{L(X^\alpha, Y)} < \infty$ for each $0 \leq \alpha \leq r$ for an integer $0 \leq r$ or $r = \infty$, then $f$ belongs to the class of smoothness $C^r(Q, Y)$, where $0 \leq r \leq \infty$, $L(X^k, Y)$ denotes the Banach space of bounded $k$-linear operators from $X$ into $Y$.

2.1.2.3. A differentiable mapping $f : U \to U'$ is called a diffeomorphism if

1. $f$ is bijective and there exist continuous $f'$ and $(f^{-1})'$, where $U$ and $U'$ are interiors of quadrants $Q$ and $Q'$ in $X$.

In the complex case we consider bounded generalized quadrants $Q$ and $Q'$ in $\mathbb{C}^n$ or $l_2(\mathbb{C})$ such that they are domains with piecewise $C^\infty$-boundaries and we impose additional conditions on the diffeomorphism $f$:

2. $\partial f = 0$ on $U$,

3. $f$ and all its strong (Fréchet) differentials (as multilinear operators) are bounded on $U$, where $\partial f$ and $\overline{\partial} f$ are differential $(1, 0)$ and $(0, 1)$ forms respectively, $d = \partial + \overline{\partial}$ is an exterior derivative. In particular for $z = (z^1, \ldots, z^n) \in \mathbb{C}^n$, $z^j \in \mathbb{C}$, $z^j = x^{2j-1} + ix^{2j}$ and $x^{2j-1}, x^{2j} \in \mathbb{R}$ for each $j = 1, \ldots, n$, $i = (−1)^{1/2}$, there are expressions: $\partial f := \sum_{j=1}^n (\partial f/\partial z^j) dz^j$, $\overline{\partial} f := \sum_{j=1}^n (\partial f/\partial \overline{z}^j) d\overline{z}^j$. In the infinite-dimensional case there are equations: $(\partial f)(e_j) = \partial f/\partial z^j$ and $(\overline{\partial} f)(e_j) = \partial f/\partial \overline{z}^j$, where $\{e_j : j \in \mathbb{N}\}$ is the standard orthonormal base in $l_2(\mathbb{C})$, $\partial f/\partial z^j = (\partial f/\partial x^{2j-1} - i\partial f/\partial x^{2j})/2$, $\partial f/\partial \overline{z}^j = (\partial f/\partial x^{2j-1} + i\partial f/\partial x^{2j})/2$.

Cauchy-Riemann Condition (ii) means that $f$ on $U$ is the holomorphic
mapping.

2.1.2.4. A complex manifold $M$ with corners is defined in the usual way: it is a metric separable space modelled on $X = \mathbb{C}^n$ or $X = l_2(\mathbb{C})$ and is supposed to be of class $C^\infty$. Charts on $M$ are denoted $(U_i, u_i, Q_i)$, that is $u_i : U_i \to u_i(U_i) \subset Q_i$ are $C^\infty$-diffeomorphisms, $U_i$ are open in $M$, $u_i \circ u_j^{-1}$ are biholomorphic from domains $u_j(U_i \cap U_j) \neq \emptyset$ onto $u_i(U_i \cap U_j)$ (that is $u_j \circ u_i^{-1}$ and $u_i \circ u_j^{-1}$ are holomorphic and bijective) and $u_i \circ u_j^{-1}$ satisfy conditions $(i - iii)$ from §2.1.2.3, $\bigcup_j U_j = M$ (see also Note 2.2.3).

A point $x \in M$ is called a corner of index $j$ if there exists a chart $(U, u, Q)$ of $M$ with $x \in U$ and $u(x)$ is of index $\text{ind} M(x) = j$ in $u(U) \subset Q$. The set of all corners of index $j \geq 1$ is called the border $\partial M$ of $M$, $x$ is called an inner point of $M$ if $\text{ind} M(x) = 0$, so $\partial M = \bigcup_{j \geq 1} \partial^j M$, where $\partial^j M := \{x \in M : \text{ind}_M(x) = j\}$.

For the real manifold with corners on the connecting mappings $u_i \circ u_j^{-1} \in C^\infty$ of real charts is imposed only Condition 2.1.2.3(i).

2.1.2.5. A subset $Y \subset M$ is called a submanifold with corners of $M$ if for each $y \in Y$ there exists a chart $(U, u, Q)$ of $M$ centered at $y$ (that is $u(y) = 0$) and there exists a quadrant $Q' \subset \mathbb{C}^k$ or in $l_2(\mathbb{C})$ such that $Q' \subset Q$ and $u(Y \cap U) = u(U) \cap Q'$. A submanifold with corners $Y$ of $M$ is called neat, if the index in $Y$ of each $y \in Y$ coincides with its index in $M$.

Analogously for real manifolds with corners for $\mathbb{R}^k$ and $\mathbb{R}^n$ or $l_2(\mathbb{R})$ instead of $\mathbb{C}^k$ and $\mathbb{C}^n$ or $l_2(\mathbb{C})$.

2.1.2.6. Henceforth, the term a complex manifold $N$ modelled on $X = \mathbb{C}^n$ or $X = l_2(\mathbb{C})$ means a metric separable space supplied with an atlas $\{(U_j, \phi_j) : j \in \Lambda_N\}$ such that:

(i) $U_j$ is an open subset of $N$ for each $j \in \Lambda_N$ and $\bigcup_{j \in \Lambda_N} U_j = N$, where $\Lambda_N \subset \mathbb{N}$;

(ii) $\phi_j : U_j \to \phi_j(U_j) \subset X$ are $C^\infty$-diffeomorphisms, where $\phi_j(U_j)$ are $C^\infty$-domains in $X$;

(iii) $\phi_j \circ \phi_m^{-1}$ is a biholomorphic mapping from $\phi_m(U_m \cap U_j)$ onto $\phi_j(U_m \cap U_j)$ while $U_m \cap U_j \neq \emptyset$. When $X = l_2(\mathbb{C})$ it is supposed, that $\phi_j \circ \phi_m^{-1}$ are Fréchet (strongly) $C^\infty$-differentiable.

2.1.3.1. Let $X$ be either the standard separable Hilbert space $l_2 = l_2(\mathbb{C})$ over the field $\mathbb{C}$ of complex numbers or $X = \mathbb{C}^n$. Let $t \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, $\mathbb{N} := \{1, 2, 3, \ldots\}$ and $W$ be a domain with a continuous piecewise $C^\infty$-boundary $\partial W$ in $\mathbb{R}^{2m}$, $m \in \mathbb{N}$, that is $W$ is a $C^\infty$-manifold with corners and it is a canonical closed subset of $\mathbb{C}^m$, $\text{cl}(\text{Int}(W)) = W$, where $\text{cl}(V)$ denotes the
closure of $V$, $\text{Int}(V)$ denotes the interior of $V$ in the corresponding topological space. As usually $H^1(W,X)$ denotes the Sobolev space of functions $f : W \to X$ for which there exists a finite norm
\[
\|f\|_{H^1(W,X)} := \left( \sum_{|\alpha| \leq 1} \|D^\alpha f\|_{L^2(W,X)}^2 \right)^{1/2} < \infty,
\]
where $f(x) = (f^j(x) : j \in \mathbb{N})$, $f(x) \in \ell_2$, $f^j(x) \in C$, $x \in W$.
\[
\|f\|_{L^2(W,X)}^2 := f_W \|f(x)\|^2 \lambda(dx),\quad \lambda \text{ is the Lebesgue measure on } \mathbb{R}^{2m},
\]
\[
\|z\|_{\ell_2} := \left( \sum_{j=1}^{\infty} |z^j|^2 \right)^{1/2}, \quad z = (z^j : j \in \mathbb{N}) \in \ell_2, \quad z^j \in C. \text{ Then } H^\infty(W,X) := \bigcap_{t \in \mathbb{N}} H^t(W,X) \text{ is the uniform space with the uniformity given by the family of norms } \{ \|f\|_{H^t(W,X)} : t \in \mathbb{N} \}.
\]

2.1.3.2. Let now $M$ be a compact Riemann or complex $C^\infty$-manifold with corners with a finite atlas $At(M) := \{(U_i, \phi_i, Q_i) ; i \in \Lambda_M\}$, where $U_i$ are open in $M$, $\phi_i : U_i \to \phi_i(U_i) \subset Q_i \subset \mathbb{R}^m$ (or it is a subset in $\mathbb{C}^m$) are diffeomorphisms (in addition holomorphic respectively as in §2.1.2.3), $(U_i, \phi_i)$ are charts, $i \in \Lambda_M \subset \mathbb{N}$.

Let also $N$ be a separable complex metrizable manifold with corners modelled either on $X = \mathbb{C}^n$ or on $X = \ell_2(\mathbb{C})$ respectively. Let $(V_i, \psi_i, S_i)$ be charts of an atlas $At(N) := \{(V_i, \psi_i, S_i) : i \in \Lambda_N\}$ such that $\Lambda_N \subset \mathbb{N}$ and $\psi_i : V_i \to \psi_i(V_i) \subset S_i \subset X$ are diffeomorphisms, $V_i$ are open in $N$, $\bigcup_{i \in \Lambda_N} V_i = N$. We denote by $H^t(M,N)$ the Sobolev space of functions $f : M \to N$ for which $f_{i,j} \in H^t(W_{i,j},X)$ for each $j \in \Lambda_M$ and $i \in \Lambda_N$ for a domain $W_{i,j} \neq \emptyset$ of $f_{i,j}$, where $f_{i,j} := \psi_i \circ f \circ \phi_j^{-1}$, and $W_{i,j} = \phi_j(U_j \cap f^{-1}(V_i))$ are canonical closed subsets of $\mathbb{R}^m$ (or $\mathbb{C}^m$ respectively). The uniformity in $H^t(M,N)$ is given by the following base $\{(f,g) \in (H^t(M,N))^2 : \sum_{i \in \Lambda_N,j \in \Lambda_M} \|f_{i,j} - g_{i,j}\|^2_{H^t(W_{i,j},X)} < \epsilon\}$, where $\epsilon > 0$, $W_{i,j}$ are domains of $(f_{i,j} - g_{i,j})$. For $t = \infty$ as usually $H^\infty(M,N) := \bigcap_{t \in \mathbb{N}} H^t(M,N)$.

2.1.3.3. For two complex manifolds $M$ and $N$ with corners let $O_T(M,N)$ denotes a space of continuous mappings $f : M \to N$ such that for each $f$ there exists a partition $Z_f$ of $M$ with the help of a real $C^\infty$-submanifold $M'_f$, which may be with corners, such that its codimension over $\mathbb{R}$ in $M$ is $\text{codim}_\mathbb{R} M'_f = 1$ and $M \setminus M'_f$ is a disjoint union of open complex submanifolds $M_{j,f}$ possibly with corners with $j = 1, 2, \ldots$ such that each restriction $f|_{M_{j,f}}$ is holomorphic with all its derivatives bounded on $M_{j,f}$. For a given partition $Z$ (instead of $Z_f$) and the corresponding $M'$ the latter subspace of continuous piecewise holomorphic mappings $f : M \to N$ is denoted by $O(M,N;Z)$. The family $\{Z\}$ of all such partitions is denoted $\Upsilon$. That is $O_T(M,N) = \text{str} - \text{ind}_T(M,N;Z)$. Let also $O(M,N)$ denotes the space of holomorphic
mappings \( f : M \to N \), \( Diff^\infty(M) \) denotes a group of \( C^\infty \)-diffeomorphisms
of \( M \) and \( Diff^\infty_0(M) := \text{Hom}(M) \cap O_T(M, M) \), where \( \text{Hom}(M) \) is a group
of homeomorphisms.

Let \( A \) and \( B \) be two complex manifolds with corners such that \( B \) is a
submanifold of \( A \). Then \( B \) is called a strong \( C^r([0, 1] \times A, A) \)-retract (or
\( C^r([0, 1], O_T(A, A)) \)-retract) of \( A \) if there exists a mapping \( F : [0, 1] \times A \to A \)
such that \( F(0, z) = z \) for each \( z \in A \) and \( F(1, A) = B \) and \( F(x, A) \supset B \)
for each \( x \in [0, 1] := \{ y : 0 \leq y \leq 1, y \in \mathbb{R} \} \), \( F(x, z) = z \) for each \( z \in B \) and \( x \in
[0, 1] \), where \( F \in C^r([0, 1] \times A, A) \) or \( F \in C^r([0, 1], O_T(A, A)) \) respectively,
\( r \in [0, \infty) \), \( F = F(x, z) \), \( x \in [0, 1] \), \( z \in A \). Such \( F \) is called the retraction.
In the case of \( B = \{ a_0 \} \), \( a_0 \in A \) we say that \( A \) is \( C^r([0, 1] \times A, A) \)-contractible
(or \( C^r([0, 1], O_T(A, A)) \)-contractible correspondingly). Two maps \( f : A \to E \)
and \( h : A \to E \) are called \( C^r([0, 1] \times A, E) \)-homotopic (or \( C^r([0, 1], O_T(A, E)) \)-
homotopic ) if there exists \( F \in C^r([0, 1] \times A, E) \) (or \( F \in C^r([0, 1], O_T(A, E)) \)
respectively) such that \( F(0, z) = f(z) \) and \( F(1, z) = h(z) \) for each \( z \in A \),
where \( E \) is also a complex manifold. Such \( F \) is called the homotopy.

Let \( M \) be a complex manifold with corners satisfying the following conditions:

(i) it is compact;

(ii) \( M \) is a union of two closed complex submanifolds \( A_1 \) and \( A_2 \) with
corners, which are canonical closed subsets in \( M \) with \( A_1 \cap A_2 = \partial A_1 \cap \partial A_2 =:
A_3 \) and a codimension over \( \mathbb{R} \) of \( A_3 \) in \( M \) is \( \text{codim}_\mathbb{R} A_3 = 1 \);

(iii) a marked point \( s_0 \) is in \( A_3 \);

(iv) \( A_1 \) and \( A_2 \) are \( C^0([0, 1], O_T(A_j, A_j)) \)-contractible into a marked point
\( s_0 \in A_3 \) by mappings \( F_j(x, z) \), where either \( j = 1 \) or \( j = 2 \).

We consider all finite partitions \( Z := \{ M_k : k \in \Xi_Z \} \) of \( M \) such that \( M_k \)
are complex submanifolds (of \( M \)), which may be with corners and \( \bigcup_{k=1}^s M_k = M, \Xi_Z =\{1, 2, ..., s\}, s \in \mathbb{N} \) depends on \( Z \), \( M_k \) are canonical closed subsets of
\( M \). We denote by \( \text{diam}(Z) := \sup_k (\text{diam}(M_k)) \) the diameter of the partition
\( Z \), where \( \text{diam}(A) = \sup_{x,y \in A} |x - y| \) is a diameter of a subset \( A \) in \( \mathbb{C}^n \),
since each finite-dimensional manifold \( M \) can be embedded into \( \mathbb{C}^n \) with the
corresponding \( n \in \mathbb{N} \). We suppose also that \( M_i \cap M_j \subset M' \) and \( \partial M_j \subset M' \)
for each \( i \neq j \), where \( M' \) is a closed \( C^\infty \)-submanifold (which may be with corners)
in \( M \) with the codimension \( \text{codim}_\mathbb{R}(M') = 1 \) of \( M' \) in \( M, M' = \bigcup_{j \in \Gamma_Z} M' j \),
\( M' j \) are \( C^\infty \)-submanifolds of \( M, \Gamma_Z \) is a finite subset of \( \mathbb{N} \).

We denote by \( H^i(M, N; Z) \) a space of continuous functions \( f : M \to N \)
such that \( f|_{(M \setminus M')} \in H^t(M \setminus M', N) \) and \( f|_{[\text{Int}(M_i) \cup (M_i \cap M')]_j} \in H^t(\text{Int}(M_i) \cup (M_i \cap M')_j, N) \), when \( \partial M_i \cap M'_j \neq \emptyset \), \( h^Z_M : H^t(M, N, Z) \to H^t(M, N; Z') \) are embeddings for each \( Z \leq Z' \) in \( \Upsilon \).

The ordering \( Z \leq Z' \) means that each submanifold \( M'_Z \) from a partition \( Z' \) either belongs to the family \( (M_j : j = 1, \ldots, k) = (M'_j : j = 1, \ldots, k) \) for \( Z \) or there exists \( j \) such that \( M'_Z \subset M'_j \) and \( M'_j \) is a finite union of \( M'_Z \) for which \( M'_Z \subset M'_j \). Moreover, these \( M'_Z \) are submanifolds (may be with corners) in \( M'_j \).

Then we consider the following uniform space \( H^t_p(M, N) \) that is the strict inductive limit \( \text{str} - \text{ind}\{H^t(M, N; Z); h^Z_M; \Upsilon\} \) (the index \( p \) reminds about the procedure of partitions), where \( \Upsilon \) is the directed family of all such \( Z \), for which \( \lim_\Upsilon \text{diam}(Z) = 0 \).

2.1.4. Let now \( s_0 \) be the marked point in \( M \) such that \( s_0 \in A_3 \) (see §2.1.3.3) and \( y_0 \) be a marked point in the manifold \( N \).

(i). Suppose that \( M \) and \( N \) are connected.

Let \( H^t_p(M, s_0; N, y_0) := \{ f \in H^t(M, N)|f(s_0) = y_0\} \) denotes the closed subspace of \( H^t(M, N) \) and \( \omega_0 \) be its element such that \( \omega_0(M) = \{y_0\} \), where \( \infty \geq t \geq m+1, 2m = \text{dim}_RM \) such that \( H^t \subset C^0 \) due to the Sobolev embedding theorem. The following subspace \( \{ f : f \in H^\infty_p(M, s_0; N, y_0), \; \partial f = 0 \} \) is isomorphic with \( O_T(M, s_0; N, y_0) \), since \( f|_{(M \setminus M')} \in H^\infty(M \setminus M', N) = C^\infty(M \setminus M', N) \) and \( \partial f = 0 \).

Let as usually \( A \vee B := A \times \{b_0\} \cup \{a_0\} \times B \subset A \times B \) be the wedge sum of pointed spaces \( (A, a_0) \) and \( (B, b_0) \), where \( A \) and \( B \) are topological spaces with marked points \( a_0 \in A \) and \( b_0 \in B \). Then the wedge combination \( g \vee f \) of two elements \( f, g \in H^t_p(M, s_0; N, y_0) \) is defined on the domain \( M \vee M \) (see also Chapters 0-2 [10] and Example 2.1.3.11 [13]).

The spaces \( O_T(J, A_3; N, y_0) := \{ f \in O_T(J, N) : f(A_3) = \{y_0\}\} \) have the manifold structure and have embeddings into \( O_T(M, s_0; N, y_0) \) due to Condition 2.1.3.3(ii), where either \( J = A_1 \) or \( J = A_2 \). This induces the following embedding \( \chi^* : O_T(M \vee M, s_0 \times s_0; N, y_0) \hookrightarrow O_T(M, s_0; N, y_0) \). Therefore \( g \circ f := \chi^*(f \vee g) \) is the composition in \( O_T(M, s_0; N, y_0) \).

The space \( C^\infty(M, N) \) is dense in \( C^0(M, N) \) and there is the inclusion \( O_T(M, N) \subset H^\infty_p(M, N) \). Let \( M_\mathbb{R} \) be the Riemann manifold generated by \( M \) considered over \( \mathbb{R} \). Then \( Diff^\infty(M_\mathbb{R}) \) is a group of \( C^\infty \)-diffeomorphisms \( \eta \) of \( M_\mathbb{R} \) preserving the marked point \( s_0 \), that is \( \eta(s_0) = s_0 \). There exists the following equivalence relation \( R_\Omega \) in \( O_T(M, s_0; N, y_0) \): \( fR_\Omega h \) if and only if
there exist nets \( \eta_n \in \text{Diff}_{s_0}(M_R) \), also \( f_n \) and \( h_n \in H^\infty_p(M, s_0; N, y_0) \) with \( \lim_n f_n = f \) and \( \lim_n h_n = h \) such that \( f_n(x) = h_n(\eta_n(x)) \) for each \( x \in M \) and \( n \in \omega \), where \( \omega \) is an ordinal, \( f, h \in \mathcal{O}_\gamma(M, s_0; N, y_0) \) and convergence is considered in \( H^\infty_p(M, s_0; N, y_0) \).

The quotient space \( \mathcal{O}_\gamma(M, s_0; N, y_0)/R_\gamma =: (S^M N)_\gamma \) is called the loop semigroup. It will be shown later, that \( (S^M N)_\gamma \) has a structure of topological Abelian monoid with the cancellation property. Applying the A. Grothendieck procedure \( [23, 45] \) to \( (S^M N)_\gamma \) we get a loop group \( (L^M N)_\gamma \).

For the spaces \( H^p_p(M, s_0; N, y_0) \) the corresponding equivalence relations are denoted \( R_{t,H} \), the group semigroups are denoted by \( (S^M R)_\gamma \), the loop groups are denoted by \( (L^M R)_\gamma \).

2.2. Propositions. (1). Let \( f \) be a diffeomorphism satisfying Conditions 2.1.2.3(i – iii), then there are neighbourhoods \( V \) and \( V' \) of bounded generalized quadrants \( Q \) and \( Q' \) such that \( f \) has an extension \( f' \) to the holomorphic diffeomorphism of \( V \) with \( V' \).

(2). If \( f : U \to C \) is a mapping satisfying Conditions 2.1.2.3(ii, iii), then there exists a neighbourhood \( V \) of \( Q \) such that \( f \) has a holomorphic extension onto \( V \).

Proof. (1). From [43, 47] it follows, that \( f \) has a \( C^\infty \)-extension \( h \) onto an open domain \( W \) with a \( C^\infty \)-boundary such that \( W \supset Q \), since due to Condition 2.1.2.3(iii) each partial derivative 
\[
D^\alpha f := \partial^{\alpha_1} \partial(z^1)^{\alpha_1} \cdots \partial(z^n)^{\alpha_n} \partial(\bar{z}_1)^{\beta_1+1} \cdots \partial(\bar{z}_n)^{\beta_n+1}
\]  
has a bounded continuous extension onto \( Q \) due to the line integration of \( D^\beta f \) along \( C^\infty \)-curves, where \( \beta = \alpha + e_j, \alpha = (\alpha_1, \ldots, \alpha_2n), 0 \leq \alpha_j \in \mathbb{Z}, j = 1, \ldots, 2n \), since \( Q \) is simply connected. Therefore, \( \partial f = 0 \) on \( Q \).

If \( F'(z) = f(z) \) and \( f(z) \) does not satisfy the Cauchy-Riemann conditions for \( z \in \partial Q \), then \( F(z) \) also does not satisfy them. In view of Corollary 3.2.3, Conjecture in Exer. 3.2 and Exer. 1.28 [19] (see also references therein) there exists a piecewise holomorphic function \( \Phi \) on a bounded neighbourhood \( V \) of \( Q \) in \( \mathbb{C}^n \) such that on \( \partial Q \) it satisfies the jump condition \( \Phi^-(z) = \Phi^+(z) + g(z) \) for each \( z \in \partial Q \), \( \Phi|_{V \setminus Q} \) and \( \Phi|_{\text{Int}(Q)} \) are holomorphic and bounded together with each partial derivative, where \( (\alpha) g \in H^\infty_p(\partial Q) \), \( (\beta) g|_{S} \in C^\infty(S) \) for each \( C^\infty \)-submanifold \( S \) (without corners) in \( \partial Q \), \( (\gamma) f|_{\partial Q} g \land \partial p = 0 \) for each \( C^{(n,n-2)} \)-form \( p \) in a neighbourhood of \( \partial Q \) in \( \mathbb{C}^n \), restrictions \( \Phi^+(z)|_{\partial Q} \) are taken as limits \( \lim_{a \to z, a \in U} \Phi(a) \) and restrictions \( \Phi^-(z)|_{\partial Q} \) are taken as \( \lim_{a \to z, a \in V \setminus Q} \Phi(a) \), which supposed to be existent, since this is based
on the existence of a solution $u$ of the $\bar{\partial}$-equation $\bar{\partial}u = f$ and the complex conjugate of $\partial u$ is equal to $\bar{\partial}u$ for the corresponding differential forms, where $V \subseteq W$. If $g$ satisfies Conditions $(\alpha, \beta)$ only, then $\Phi|_{V \setminus Q}$ and $\Phi|_{\text{Int}(Q)}$ are of class $C^\infty$ and bounded together with each partial derivative. Indeed, there exists an increasing sequence $W_j$ of $C^\infty$-subregions in $Q$ such that (i) $\sup_k \text{diam}(E_{k,j}) < j^{-2}$; (ii) $\eta_j : \partial Q_j \to \partial W_j$ are $O(\partial)$-diffeomorphisms (that is homeomorphisms of class $O(\partial)$) with $\eta_j|_{\partial Q_j \cap \partial W_j} = \text{id}$; (iii) $g_j$ are $C^\infty$-functions satisfying Condition $(\gamma)$ on $\partial W_j$ (as $g$ on $\partial Q$) and $\lim_{j \to \infty} g_j \circ \eta_j = g$; (iv) $E_{k,j}$ are connected components of $\partial W_j \setminus \partial Q_j$ such that $E_{k,j} \subseteq (\partial^2 Q)^{1/2}$, where 

\[ A^\varepsilon := \{ y \in \mathbb{C}^n : \inf_{z \in A} |y - z| < \varepsilon \} \] 

is an $\varepsilon$-enlargement of a subset $A$ in $\mathbb{C}^n$, $W_j \subseteq W_{j+1}$ for each $j$. In view of Montel Theorem 2.14.1 [13] there exists a sequence of functions $\{ \Phi_j : j = l + 1, l + 2, \ldots \}$ satisfying the same conditions for $(W_j, g_j)$ as $\Phi$ for $(Q, g)$ such that $\Phi_j$ converges uniformly on $\bigcup_{j=l+1}^\infty W_j$ and on $V \setminus Q$, where $V$ is a bounded neighbourhood of $Q$ in $\mathbb{C}^n$. Since $\bigcup_{j=l+1}^\infty W_j \supset (Q \setminus \bar{\partial}^2 Q)$, $Q \cap (\cap_{j=l+1}^\infty (V \setminus \text{Int}(W_j))) = \partial Q$ and due to the solution of the $\bar{\partial}$-equation on $Q$ and on $V \setminus \text{Int}(Q)$ there exists the sequence $\{ \Phi_j : j > l \}$ which converges to the desired $\Phi$ on $V$, where $\bar{\partial} Q := \bigcup_{j \geq p} \partial^j Q$, $p \in \mathbb{N}$ (see §2.1.2.4). In the particular case of a $C^1$-domain $Q$ in $\mathbb{C}$ see also [34, 40]. The line integration of $\Phi$ along $C^\infty$-curves produces a continuous $\Phi$ with the jump condition for $\bar{\partial} \Phi$ instead of $\Phi$ due to Cauchy Integral Theorem in the Stronger Form 6.4.1 [18] and [19], since $Q$ is simply connected (see also §3.5 [33]).

Therefore, there exists $C^\infty$-function $v$ on $V$ such that $v|_{Q} = 0$ and $\bar{\partial}v = -\partial h$ on $V$. That to construct such $v$ let $u = v_1 - v_2 - v_3$, where $v_1|_{Q} = (v_2 + v_3)|_{Q}$, $v_1$ is a $C^\infty$-function on $U = Q \setminus \partial Q$ and on $V \setminus Q$, $v_2$ and $v_3$ are holomorphic on $U$ and $V \setminus Q$, $v_1^+|_{\partial Q} = f^+|_{\partial Q}$, $v_2^+|_{\partial Q} = 0$, $\partial v_2^+|_{\partial Q} = \bar{\partial} f^+|_{\partial Q}$, $v_3^+|_{\partial Q} = f^+|_{\partial Q}$, $v_3^-|_{\partial Q} = 0$. In view of the theorem about maximum principle of the modulus of a holomorphic function on a simply connected domain the condition $v_3^+|_{\partial Q} = 0$ implies $v_3^-|_{\partial Q} = 0$ (see §3.6 [33]). Therefore, $\bar{\partial} v|_{Q} = \bar{\partial} v_1|_{Q}$ and $\bar{\partial} (h + v)|_{V} = 0$ and $(h + v)|_{Q} = f$. Then $u := h + v$ is a holomorphic extension of $f$ on $V$. When $n \geq 2$ Hartogs Theorem 1.2.2 [19] can be used instead of Conjecture in Exer. 3.2, that is for the constructed above $u$ holomorphic on $V \setminus \bar{\partial}^2 Q$ there exists a function $v$ holomorphic on $V$ such that $v|_{V \setminus \bar{\partial} Q} = u|_{V \setminus \bar{\partial} Q}$, since $V \setminus \bar{\partial}^2 Q$ is connected. For $Q \subset \mathbb{C}^4 = X$ there exists a strictly pseudoconvex open set $D \subset \mathbb{C}^2$ such that $X \cap D = Q$, hence by Theorem 3.6.8 [19] there exists $F \in \mathcal{O}(D) \cap C^0(\bar{D})$ such that $F|_{Q} = \bar{\partial}^2 v|_{Q} = 0$. Therefore, $\bar{\partial} v|_{Q} = \bar{\partial} v_1|_{Q}$ and $\bar{\partial} (h + v)|_{V} = 0$.
connected region in $\mathbb{C}$ such that $|E| \xrightarrow{\pi} X \to \mathbb{C}$ appears while treatment of $\prod$ domains in $\mathbb{C}$. Slightly shrinking covering if necessary we can choose $\{A_t\}$ for the closed set $z$ of $f$ follows that $f$ differentiability of $f$ and $J$.

$\phi$ is open in $\mathbb{C}$, consequently, the case of $n = 1$ can be reduced to the case $n = 2$ with a subsequent restriction of a resulting function on $X \cap V$.

Piecewise $\partial Q$ can be written in local coordinates in a neighbourhood $W_z$ of $z \in \partial Q$ as $x^1 = 0$, where $z = (z^1, z^2, \ldots)$, $z^j = x^{2j-1} + ix^{2j}$. Therefore, $f$ has also such extension for $Q \subset l_2(\mathbb{C})$.

On the other hand, the Jacobian $J_f$ is not equal to zero everywhere on the closed set $Q$ in $X = \mathbb{C}^n$ or $X = l_2(\mathbb{C})$. In view of the strong $C^\infty$-differentiability of $f$ there exists an open set $U$ in $X$ such that $Q \subset U \subset V$ and $J_f \neq 0$ everywhere on $U$. Then we take $U' = f(U)$, hence $Q' \subset U'$ and $U'$ is open in $X$.

(2). The proof of this case is analogous to that of Section (1) omitting the last paragraph from it.

2.2.3. Note. From the proof it follows that these propositions are true, when Condition 2.1.2.3(iii) is fulfilled up to the second differentials on $U$, since if $f \in C^1(V, \mathbb{C})$ and $f$ satisfies Cauchy-Riemann conditions on $V$ it follows that $f \in O(V, \mathbb{C})$. These propositions also justify the notion of $f$ to be holomorphic on $Q$ as a restriction $f|_Q$ of $f$ holomorphic on $V$.

2.3.1. Lemma. If $M$ is a complex manifold modelled on $X = \mathbb{C}^n$ or $X = l_2(\mathbb{C})$ with an atlas $At(M) = \{(V_j, \phi_j) : j\}$, then there exists an atlas $At'(M) = \{(U_k, u_k, Q_k) : k\}$ which refines $At(M)$, where $(V_j, \phi_j)$ are usual charts with diffeomorphisms $\phi_j : V_j \to \phi_j(V_j)$ such that $\phi_j(V_j)$ are $C^\infty$-domains in $\mathbb{C}^n$ and $(U_k, u_k, Q_k)$ are charts corresponding to quadrants $Q_k$ in $\mathbb{C}^n$ or $l_2(\mathbb{C})$.

Proof. The covering $\{V_j : j\}$ of $M$ has a refinement $\{W_i : l\}$ such that for each $j$ there exists $l = l(j)$ with $W_l \subset V_j$ so that each $\phi_j(W_l)$ is a simply connected region in $\mathbb{C}^n$ or $l_2(\mathbb{C})$ which is not the whole space. We choose $W_i$ such that

(i) either $W_l \cap \partial M = \emptyset$ or $W_l \cap \partial M$ is open in $\partial M$;

(ii) $\{\pi_k(z) = z^k : z \in \phi_j(W_l)\} =: E_{l,k}$, $z \in X$, $\pi_k : X \to \mathbb{C}$ are canonical projections associated with the standard orthonormal base $\{e_j : j\}$ in $X$, $E_{l,k}$ are $C^\infty$-regions in $\mathbb{C}$, $\phi_j(W_l) = \prod_{k=1}^n E_{l,k}$ for $X = \mathbb{C}^n$, or $\pi_j(\phi_j(W_l)) = \prod_{k \in J} E_{l,k}$ for each finite subset $J$ in $\mathbb{N}$ and the corresponding projection $\pi_j : X \to sp_C\{e_k : k \in J\}$. In view of the Riemann Mapping Theorem for each $E_{l,k}$ there exists holomorphic diffeomorphism $\psi_{l,k}$ either onto $B_r := \{z \in \mathbb{C} : |z| < r\}$ or $F_r := \{z \in \mathbb{C} : |z| \geq r, x^1 \geq 0\}$ (see §2.12 in [3]). The latter case appears while treatment of $\pi_k(\phi_j(W_l \cap \partial M)) \neq \emptyset$ (see §10.5.2 [2] and §12 [1]). Slightly shrinking covering if necessary we can choose $\{W_l : l\}$ such that
each $q_{l,k}$ and its derivatives are bounded on $E_{l,k}$. In view of Central Theorem from §6.3 [18] $q_{l,k}$ are boundary preserving maps. In view of Chapter 13 [19, 20] $B_r^-$ and $F_r$ have finite atlases with charts corresponding to quadrants (see §2.1.2.1 and §2.2).

2.3.2 Note. Vice versa there are complex manifolds with corners, which are not usual complex manifolds, for example, canonical closed domains $F$ in $\mathbb{C}^n$ with piecewise $C^\infty$-boundary, which is not of class $C^1$. Since each complex manifold $G$ has $\partial G$ of class $C^\infty$ by Definition 2.1.2.6.

2.4. Lemma. $O_T(M,N)$ from §2.1.3.3 is an infinite-dimensional complex manifold dense in $C^0(M,N)$, when $M$ has $\dim \mathbb{C}M \geq 1$. Moreover, there exists its tangent bundle $T \circ O_T(M,N) = O_T(M,TN)$. If $N = \mathbb{C}^n$ or $N = l_2(\mathbb{C})$, then $O_T(M,N)$ is an infinite-dimensional topological vector space over $\mathbb{C}$.

Proof. The connecting mappings $\phi_j \circ \phi_k^{-1}$ of charts $(U_j, \phi_j)$ and $(U_k, \phi_k)$ with $U_j \cap U_k \neq \emptyset$ are holomorphic on the corresponding domains $\phi_k(U_j \cap U_k)$. For each submanifold $M_j$ in $M$ we have $T(O(M_j,N) = O(M_j,TN)$ [12]. For each $f \in O_T(M,N)$ there exists a partition $Z_f$ of $M$ such that $f|_{M_{j,f}} \in O(M_{j,f},N)$ for each submanifold $M_{j,f}$ with corners defined by $Z_f$. In accordance with §2.1.3.2 and §2.1.4 the topology of $O_T(M,N)$ is the compact-open topology, hence $\phi_j \circ \phi_k^{-1}$ induce connecting mappings $(\phi_j^{-1} \circ \phi_k)^*$ of the corresponding charts in $O_T(M,N)$ such that $(\phi_j^{-1} \circ \phi_k)^*(f(z)) := f \circ (\phi_j^{-1} \circ \phi_k)(z)$ for each $z \in U_j \cap U_k$ such that its Frechét derivatives are the following $[\partial(\phi_j^{-1} \circ \phi_k)^*(f)/\partial f].h = (\phi_j^{-1} \circ \phi_k)^*(h)$ and $[\partial(\phi_j^{-1} \circ \phi_k)^*(f)/\partial f].h = 0$, where $h$ are vectors in $T(\phi_j \circ \phi_k^{-1} \circ \phi_k)(z)$ for each $z \in U_j \cap U_k$ such that its Frechét derivatives are the following $[\partial(\phi_j^{-1} \circ \phi_k)^*(f)/\partial f].h = (\phi_j^{-1} \circ \phi_k)^*(h)$ and $[\partial(\phi_j^{-1} \circ \phi_k)^*(f)/\partial f].h = 0$, where $h$ are vectors in $T(\phi_j \circ \phi_k^{-1} \circ \phi_k)$ for each $z \in U_j \cap U_k$.

In particular $O_T(M,Y)$ is a topological vector space over $\mathbb{C}$ for $Y = \mathbb{C}^n$ or $Y = l_2(\mathbb{C})$. It remains to prove that $O_T(M,N)$ is infinite-dimensional and dense in $C^0(M,N)$. This follows from Corollary 3.2.3, Exer. 1.28 and Conjecture in Exer. 3.2 [15]. Since for each quadrant $Q$ and a given function $s$ on $\partial Q$, which is a restriction $q_{l,k}$ of a holomorphic function $q$ on a neighbourhood of $\partial Q$ in $\mathbb{C}^m$, $m = \dim \mathbb{C}M$, there exists a space of functions $u : W \rightarrow \mathbb{C}^n$ such that $u|_{Int(Q)}$ and $u|_{W \setminus Q}$ are holomorphic and bounded together with each partial derivative, where $W$ is an open ball in $\mathbb{C}^n$ containing $Q$ and such that $u^+(z) - u^-(z) = s(z)$ for each $z \in \partial Q$. Then using Cauchy integration along $C^\infty$-curves we construct a space of continuous functions $f : W \rightarrow \mathbb{C}^n$ holomorphic on $U$ and $W \setminus Q$ with prescribed $(\partial f)^-(z) - (\partial f)^+(z)$ for each $z \in \partial Q$ and analogously for $f \in C^l$ with jump

13
conditions for higher order derivatives. In the case of $n > 1$ there also can be used holomorphic extension of holomorphic functions from proper complex submanifolds $K$ of $\partial Q$ (see Theorems 2(b) and 3(b) in [2] and Theorem 4.1.11 [13], since a space of rational functions $f : K \to T_yN$ such that $f|_K$ are holomorphic is infinite-dimensional, see also Corollaries 3.4 and 3.5 in [21]).

Using charts of the atlas of $M$ we get that $O_T(M, N)$ is infinite-dimensional.

In view of Lemma 2.4 [19], since a space of rational functions is dense in $\mathbb{C}$,

[2.5. Proposition. Let $A$ and $B$ be two compact complex manifolds with corners.

(i). Then $B$ is a strong $C^0([0, 1] \times A, A)$-retract of $A$ if and only if $B$ is a strong $C^0([0, 1], O_T(A, A))$-retract of $A$.

(ii). Two maps $f$ and $h \in O_T(A, B)$ are $C^0([0, 1] \times A, B)$-homotopic if and only if they are $C^0([0, 1], O_T(A, B))$-homotopic.

Proof. Since $C^0([0, 1], O_T(A, B)) \subset C^0([0, 1] \times A, B)$, then from a strong $C^0([0, 1], O_T(A, A))$-retraction (or a $C^0([0, 1], O_T(A, B))$-homotopy) we get a strong $C^0([0, 1] \times A, A)$-retraction (or a $C^0([0, 1] \times A, B)$-homotopy respectively).

It remains to verify the opposite implications.

(ii). Let $F \in C^0([0, 1] \times A, B)$. Consider for each $x \in [0, 1]$ a compact complex submanifold $B_\delta$ in $A$ such that $F(x, A) \subset B_\delta \subset E_\delta$, where $E_\delta(x) = \{z \in A : d(z, F(x, A)) \leq \delta\}$, $d$ denotes a metric in $A$, $0 < \delta(x) < 1 - x$, $\delta(x) < \delta(x')$ for each $x > x'$, $d(z, Y) := \inf_{y \in Y} d(z, y)$ for $Y \subset A$. In view of Lemma 2.4 $C^0([0, 1], O_T(A, C^\infty))$ is dense in $C^0([0, 1] \times A, C^\infty)$. Then for each $0 < \delta$ consider a finite atlas $A_{\delta(M)} = \{ (U_{j, \delta}, \phi_{j, \delta}) : j \}$ of $M$ such that $\sup_{j} \sup_{U_{j, \delta}} (U_{j, \delta}) < \delta$ and $\sup_{j} \sup_{e U_{j, \delta}}(x) < \delta(x)$.

We consider restrictions $\{F(x, z) \} |_{U_{j, \delta}}$ and such $A_{\delta(M)}$ generated by $Z \in T$. Therefore, there exist $E_{\delta(x)}$ and mappings $H(x, *)$ such that $H(x, A) \subset E_{\delta(x)}$ and $H(x, z) \in O_T(A, E_{\delta(x)})$ and $\|H(x, *) - F(x, *)\|_{C^0(A, C^\infty)} < \epsilon(x) < \infty$ for each $x \in [0, 1]$, where $\lim_{x \to 1} \epsilon(x) = 0$. Since $f, h \in O_T(A, B)$ we can choose $H(0, z) = f(z)$ and $H(1, z) = h(z)$, consequently, $f$ and $h$ are $C^0([0, 1], O_T(A, B))$-homotopic.

(i). The proof of (i) is analogous to that of (ii) using the fact that $\cap_{x \in [0, 1]} E_{\delta(x)} = B$ for each $0 < \epsilon < 1$, since $B$ is the complex manifold with corners and $C^0([0, 1], O_T(A, A))$ is complete. For this let us choose a
Cauchy sequence \( H_n(x, z) \) in \( C^0([0, 1], \mathcal{O}_\Gamma(A, A)) \) instead of one \( H \) as in \((ii)\) such that \( H_n(0, z) = z \) for each \( z \in A \) and \( H_n(1, A) = E_{\delta(1), n} \) are complex submanifolds with corners of \( A \) with \( \bigcap_n E_{\delta(1), n} = B \) and \( E_{\delta(1), n+1} \subset E_{\delta(1), n} \) for each \( n \in \mathbb{N} \), \( \lim_{n \to \infty} H_n = H \) is the desired mapping.

2.6. Lemma. Let \( M \) be a manifold from \( \S 2.1.3.3 \) then there exists a mapping \( q : M \to M \) such that

\begin{enumerate}[(i)]  
  \item \( q(A_2) = \{ s_1 \} \).  
  \item \( q : (A_1 \setminus A_3) \to (M \setminus \{ s_1 \}) \) is an \( \mathcal{O}_\Gamma \)-diffeomorphism,  
  \item \( s_1 \in A_2 \setminus A_3 \)  
  \item \( q \) is \( C^0([0, 1], \mathcal{O}_\Gamma(M, M)) \)-homotopic with the identity mapping \( id_M \) on \( M \).
\end{enumerate}

Proof. In view of Lemma 2.5 it is sufficient to construct \( C^0([0, 1] \times M, M) \)-homotopy. Let \( M \) be supplied with an atlas charts of which are diffeomorphic to subregions of quadrants (see Lemma 2.3.1). Then the mapping \( F_2(0, z) \) from Condition 2.1.3.3(iv) has an extension to \( id_M \) and \( F_2(1, z) \) has an extension to mapping satisfying conditions \((i, ii, iv)\) but for \( s_0 \) instead of \( s_1 \). Then due to the Riemann Mapping Theorem and Central Theorem [18] and \( \S \S 2.2-2.4 \) the mapping \( F_2 \) defined on \(([0, 1] \times A_2) \cup \{0, 1\} \times M) \) has an \( C^0([0, 1], \mathcal{O}_\Gamma(M, M)) \)-extension, since \( M \) is connected. Using retraction of \( A_2 \) onto \( \{ s_0 \} \), compositions of homotopies and the fact that \( A_1 \) and \( A_2 \) are simply connected we get a \( C^0([0, 1], \mathcal{O}_\Gamma(M, M)) \)-homotopy of \( q \) with \( id_M \).

2.7. Theorems. (1). The space \( \mathcal{L}^M N \) from \( \S 2.1.4 \) is the complete separable Abelian topological group.

(2). It is non-trivial and non-locally compact.

(3). Moreover, if there are two distinct points \( s_0 \) and \( s_1 \) in \( A_3 \), then two groups \( \mathcal{L}^M N \) defined for \( s_0 \) and \( s_1 \) as marked points are isomorphic.

(4). \( \mathcal{L}^M N \) is the closed proper subgroup in \( \mathcal{L}^M N \).

Proof. At first it is proved that \( \mathcal{S}^M N \) is an Abelian topological monoid with the cancellation property.

For each \( f \in \mathcal{O}_\Gamma(M, s_0; N, y_0) \) the range \( f(M) \) is compact and connected in \( N \), since \( M \) is compact. In view of Lemmas 6.8 and 6.9 [16], \( \S 2.1.4 \) and Propositions 2.2, 2.5 and Lemma 2.4 there exists a countable subfamily \( \{ Z_j : j \in \mathbb{N} \} \) in \( \mathcal{Y} \) such that \( Z_j \subset Z_{j+1} \) for each \( j \) and \( \lim_j diam Z_j = 0 \). Therefore, for each partition \( Z \) there exists \( \delta > 0 \) such that for each partition \( Z'' \in \{ Z_j : j \in \mathbb{N} \} \) with \( \sup_j \inf_j dist(M_i, M''_j) < \delta \) and for \( f \in \mathcal{O}(M, s_0; N, y_0; Z) \), there exists \( f_1 \in \mathcal{O}(M, s_0; N, y_0; Z'') \), such that \( fR_0 f_1 \), where \( \text{dist}(A, B) = \max(\sup_{a \in A} D(a, B), \sup_{b \in B} D(b, A)) \), \( D(a, B) := \inf_{b \in B} d(a, b) \), \( A \) and \( B \) are
subspaces of the metric space $\mathbb{C}^n$ with the metric $d(a, b) = |a - b|_{\mathbb{C}^n}$.

If $h \in cl\{v : f R_0 v\}$, then there exists a net $\{h_\beta \in \{v : f R_0 v\} : \beta \in \alpha\}$ for an ordinal $\alpha$ such that $\lim_\beta h_\beta = h$. In view of the preceding paragraph we can consider sequences instead of general nets and extract from a double sequence $h_{\beta,n}$ a convergent to $h$ subsequence, where $\lim_n h_{\beta,n} = h_\beta$ for each $\beta$ with $h_{\beta,n}(x) = f_{\beta,n}(\eta_{\beta,n}(x))$ for each $x \in M$, $n$ and $\beta$ such that $\eta_{\beta,n} \in Diff_{f_0}(M)$, and $f_{\beta,n} \in \{v : f R_0 v\}$. Therefore, each $R_0$-equivalence class is closed in $O_T(M, s_0; N, y_0)$, since $O_T(M, s_0; N, y_0)$ and $H^\infty_p(M, s_0; N, y_0)$ are complete, $f(M)$ is a compact subset in $N$ and due to the definition of this equivalence relation. Then

(i) $str - ind\{O(M, s_0; N, y_0; Z_j); h_{Z_j}^0; N\}/R_O = (L^M N)_O$ is separable, since each space $O(M, s_0; N, y_0; Z_j)$ is separable where $O(M, s_0; N, y_0; Z) := O_T(M, s_0; N, y_0) \cap O(M, N; Z)$. The continuity of the composition follows from Notes in §2.1.4. The space $str - ind\{O(M, s_0; N, y_0; Z_j); h_{Z_j}^0; N\}$ is complete due to Theorem 12.1.4 [4], each class of $R_0$-equivalent elements is closed in it. Then to each Cauchy sequence in $(S^M N)_O$ there corresponds a Cauchy sequence in $str - ind\{O_T(M, s_0; N, y_0; Z_j); h_{Z_j}^0; N\}$ due to §§2.2.2-5, where $Z_j$ are the corresponding partitions of $M$. Hence $(S^M N)_O$ is complete.

In view of Lemma 2.6 mappings $f$ and $\chi^*(f \lor w_0)$ are $R_0$-equivalent for each $f \in O_T(M, s_0; N, y_0)$, where $w_0(M) = \{y_0\}$, since there exists a sequence $\eta_n \in Diff_{f_0}(M)$ such that $\lim_{n \to \infty} diam(\eta_n(A_2)) = 0$ and $w_n, f_n \in H^\infty_p(M, s_0; N, y_0)$ with $\lim_{n \to \infty} f_n = f$, $\lim_{n \to \infty} w_n = w_0$ and $\lim_{n \to \infty} \chi^*(f_n \lor w_n)(\eta_n^{-1}) = f$ due to $f(s_0) = s_0$ and formulas of differentiation of composite functions (see Theorem 2.5 in [3]). Hence $< w_0 >_O$ is the identity element, where $< f >_O := \{h \in O_T(M, s_0; N, y_0) : h R_O f\}$ denotes the equivalence class. There exists a $O_T$-diffeomorphism $\psi : (M_1 \lor M_2) \to (M_2 \lor M_1)$, where $M_j = M$ for $j = 1$ and $j = 2$. Therefore, $(h \lor f)(\psi(z)) = (f \lor h)(z)$ for each $z \in (M_1 \lor M_2)$ and each $f, h \in O_T(M, s_0; N, y_0)$, consequently, $\chi^*(f \lor h)R_O \chi^*(h \lor f)$, whence $(S^M N)_O$ is commutative. Evidently $\chi^*(f \lor q) = \chi^*(f \lor q)$ is equivalent to $h = q$, where $f, h, q \in O_T(M, s_0; N, y_0)$. Therefore, $< f >_O \lor < h >_O = < f >_O \lor < q >_O$ is equivalent to $< h >_O = < q >_O$, consequently, $(S^M N)_O$ has the cancellation property.

Let $TN$ be the tangent bundle and $\pi : TN \to N$ be the natural projection such that $\pi(x) = z$ for each $x \in T_zN$ with $z \in N$. The tangent bundle $T(S^M N)_O$ is evidently isomorphic with $(O_T(M, s_0; TN, (y_0 \times 0))/R_O) \times T_{y_0}N$, where the equivalence relation $R_O$ is considered in $O_T(M, s_0; TN, (y_0 \times 0))$.  

16
y_0 \in N. If f(M) \neq q(M), then f is not equivalent to q. Therefore, T_ε(S^MN)_O is infinite-dimensional topological vector space, hence (S^MN)_O is not locally compact (see Theorem (5.9.5) in [11]).

If there are two points s_0 and s_1 as in Theorem 2.7.3, then the spaces O_T(M, s_0; N, y_0) and O_T(M, s_1; N, y_0) are isomorphic. On the other hand, χ^*(f ∨ w_0) is in the class of R_O-equivalent elements < f >_O and A_2 is C^0([0, 1] × M, M)-contractible into s_0 and also into s_1. Therefore, the monoid (S^MN)_O does not depend on the choice of a marked point s in A_3.

For a commutative monoid with the cancellation property (S^MN)_O there exists a commutative group (L^MN)_O equal to the Grothendieck group. This group algebraically is the quotient group F/B, where F is a free Abelian group generated by (S^MN)_O and B is a subgroup of F generated by elements [f + g] - [f] - [g], f and g \in (S^MN)_O. [f] denotes an element of F corresponding to f. In view of §9 [23] and [13] the natural mapping γ : (S^MN)_O → (L^MN)_O is injective. We supply F with a topology inherited from the Tychonoff product topology of (S^MN)_O, where each element z of F is z = \sum f n_{f,z} \gamma f, n_{f,z} \in \mathbb{Z} for each f \in (S^MN)_O, \sum_f |n_{f,z}| < \infty. In particular [nf] - n[f] \in B, hence (L^MN)_O is the complete topological group and γ is the topological embedding such that γ(f + g) = γ(f) + γ(g) for each f, g \in (S^MN)_O, γ(e) = e, since (z + B) \in γ(S^MN)_O, when n_{f,z} \geq 0 for each f, so in general z = z^+ - z^-, where (z^+ + B) and (z^- + B) \in γ(S^MN)_O.

The manifold O_T(M, s_0; N, y_0) is the closed submanifold of H^∞_p(M, s_0; N, y_0) and R_O = R^∞_H|_{O_T}, hence < f >_O < f >_H for each f \in O_T(M, s_0; N, y_0), where < f >_H = \{ v : v \in H^1_p(M, s_0; N, y_0), v R t,H f \}. Therefore, there exists embedding of the manifold (S^MN)_O into (S^MN)_R^∞_H such that (S^MN)_O is closed in (S^MN)_R^∞_H, since (S^MN)_O is complete and the uniformity in it inherited from (S^MN)_R^∞_H coincides with the initial one. Then < χ^*(f ∨ h) >_O = < f >_O \circ h >_O and < χ^*(f ∨ h) >_H = < f >_H \circ h >_H, consequently, (S^MN)_O is the submonoid of the monoid (S^MN)_R^∞_H and inevitably (L^MN)_O is the closed subgroup in (L^MN)_R^∞_H. If f(M) \neq q(M), then < f >_H \neq < q >_H, where f, q \in H^∞_p(M, s_0; N, y_0), since inf_{η \in Diff^∞_p(M_R)} ∥ f \circ η - q ∥_{C^0} > 0. If f \in O_T(M, N), then f(M) is a complex submanifold with corners in N, since f is piecewise holomorphic [58, 12]. On the other hand, there exists a family of f \in H^∞_p(M, s_0; N, y_0) such that f(M) is not a complex manifold with corners. Therefore, (L^MN)_R^∞_H \setminus (L^MN)_O has the cardinality c := card(R), since card(H^∞_p(M, s_0; N, y_0)) = c and card(S^MN)_∞_H = c.
2.8. Notes and Definitions. 1. In view of §I.5 [24] a complex manifold $M$ considered over $\mathbb{R}$ admits a Riemann metric $g$. Due to Theorem IV.2.2 [24] there exists the Levi-Civita connection (with vanishing torsion) of $M_{\mathbb{R}}$. Suppose $\nu$ is a measure on $M$ corresponding to the Riemann volume element $w$ (m-form) $\nu(dx) = w(dx)/w(M)$. The Riemann metric $g$ is positive definite and $w$ is non-degenerate and non-negative, since $M$ is orientable.

The Christoffel symbols $\Gamma^k_{ij}$ of the Levi-Civita derivation (see §1.8.12 [23]) are of class $C^\infty$ for $M$. Then the equivalent uniformity in $H^t(M,N)$ for $0 \leq t < \infty$ is given by the following base $\{(f,g) \in (H^t(M,N))^2 : ||(\psi_j \circ f - \psi_j \circ g)||^2_{H^t(M,N)} < \epsilon, \text{ where } D^\alpha = \partial^{|\alpha|}/\partial(x^1)^{\alpha_1}...\partial(x^{2n})^\alpha, \epsilon > 0, ||(\psi_j \circ f - \psi_j \circ g)||^2_{H^t(M,N)} := \sum_{|\alpha| \leq t} \int_M |D^\alpha(\psi_j \circ f(x) - \psi_j \circ g(x))|^2 \nu(dx)\}$, $j \in \Lambda_N$, $X$ is the Hilbert space over $\mathbb{C}$ either $\mathbb{C}^n$ or $l_2(\mathbb{C})$, $x$ are local normal coordinates in $M_{\mathbb{R}}$ (see also §2.1). Let now $(Z_j : j \in \mathbb{N})$ be the sequence from the proof of Theorem 2.7. We consider submanifolds $M_{l_k}$ and $M_{l_k}'$ for each partition $Z_k$ as in §2.1.3.3 (with $Z_k$ instead of $Z$), $i \in \Xi_{Z_k}$, $j \in \Gamma_{Z_k}$, where $\Xi_{Z_k}$ and $\Gamma_{Z_k}$ are finite subsets of $\mathbb{N}$. We supply $H^\gamma(M, X; Z_k)$ with the following metric $p_{k,\gamma}(y) := \sum_{i \in \Xi} ||y_{M_{l_k}}||_{\gamma_{i,k}}^2$ for $y \in H^\gamma(M, X; Z_k)$ and $p_{k,\gamma}(y) = +\infty$ in the contrary case, where $\Xi = \Xi_{Z_k}$, $\gamma > t \geq \gamma \in \mathbb{N}$, $\gamma \geq m + 1$, $||y_{M_{l_k}}||_{H^\gamma(M, X)}$ is given analogously to $||y||_{H^\gamma(M, X)}$ but with $f_{M_{l_k}}$ instead of $f_M$.

Let $Z^\gamma(M, X)$ be the completion of $str-\text{ind}\{H^\gamma(M, X; Z_j); h_{Z_j}^{Z_k}; \mathbb{N}\} =: Q$ relative to the following norm $||y||'_{\gamma} := \inf_k p_{k,\gamma}(y)$, as usually $Z^{\infty}(M, X) = \cap_{\gamma \in \mathbb{N}} Z^\gamma(M, X)$. Let $Y^{\gamma}(M, X) := \{f : f \in Z^{\infty}(M, X), \partial f_j |_{M_{l_k}} = 0 \text{ for each } k\}$, where $f \in Z^{\infty}(M, X)$ implies $f = \sum_j f_j$ with $f_j \in H^{\infty}(M, X; Z_j)$ for each $j \in \mathbb{N}$.

For a domain $W$ in $\mathbb{C}^m$, which is a complex manifold with corners, let $Y^{r,a}(W, X)$ (and $Z^{r,a}(W, X)$) be a subspace of those $f \in Y^{\gamma}(W, X)$ (or $f \in Z^{\gamma}(W, X)$) respectively for which

$$||f||_{Y^{r,a}} := \left(\sum_{j=0}^{\infty} (||f||^2_{j})^2/(j!)^a\right)^{1/2} < \infty,$$

where $(||f||^2_{j})^2 := (||f||'_{j})^2 - (||f||_{j-1})^2$ for $j \geq 1$ and $||f||^2_0 = ||f||'_{0}, 0 < a < \infty$.

Using the atlases $At(M)$ and $At(N)$ as in §2.1 for $M$ and $N$ of class of smoothness $Y^{r,b}$ with $\infty > a \geq b$ we get the uniform space $Y^{r,a}(M, s_0; N, y_0)$ (and also $Z^{r,a}(M, s_0; N, y_0)$) of mappings $f : M \to N$ with $f(s_0) = y_0$.

18
such that $\psi_j \circ f \in Y^T,\alpha(M, X)$ (or $\psi_j \circ f \in Z^T,\alpha(M, X)$ respectively) for each $j$, where $\sum_{p \in \Lambda_M, j \in \Lambda_N} \| f_{p,j} - (w_0)_{p,j} \|^2_{Y^T,\alpha(W_{p,j}, X)} < \infty$ for each $f \in Y^T,\alpha(M, s_0; N, y_0)$ is satisfied with $w_0(M) = \{ y_0 \}$, since $M$ is compact. To each equivalence class $\{ g : gR_0f \} = f > 0$ there corresponds an equivalence class $f > 0\ll\alpha := cl(< f >_0 \cap Y^T,\alpha(M, s_0; N, y_0))$ (or $f > 0\ll\beta := cl(< f >_0 \cap Z^T,\alpha(M, s_0; N, y_0))$, where the closure is taken in $Y^T,\alpha(M, s_0; N, y_0)$ (or $Z^T,\alpha(M, s_0; N, y_0)$ respectively). This generates equivalence relations $R_{\alpha,\beta}$ and $R_{\beta,\alpha}$ respectively. We denote the quotient spaces $Y^T,\alpha(M, s_0; N, y_0)/R_{\alpha,\beta}$ and $Z^T,\alpha(M, s_0; N, y_0)/R_{\beta,\alpha}$ by $(S^M N)_{\alpha,\beta}$ and $(S^M N)_{\beta,\alpha}$ correspondingly. Using the A. Grothendieck construction we get the loop groups $(L^M N)_{\alpha,\beta}$ and $(L^M R)_{\alpha,\beta}$ respectively.

2.8.2. Let $M$ be infinite-dimensional complex $Y^\xi$-manifold with corners modelled on $l_2(C)$ such that

(i) there is the sequence of the canonically embedded complex submanifolds $n_{m+1} : M_m \to M_{m+1}$ for each $m \in N$ and to $s_{0,m}$ in $M_m$ it corresponds $s_{0,m+1} = n_{m+1}(s_{0,m})$ in $M_{m+1}$, $dim_CM_m = n(m)$, $0 < n(m) < n(m+1)$ for each $m \in N$, $\bigcup_m M_m$ is dense in $M$;

(ii) $M$ and $At(M)$ are foliated, that is, $\alpha_i \circ u_{i-1}|_{u_j(U_i \cap U_j)} \to l_2$ are of the form: $u_i \circ u_{i-1}(\{z^i : l \in N\}) = (\alpha_i,\xi,\gamma_i,\epsilon_{i,m}(z^i : l > n(m)))$ for each $m$, when $M$ is without a boundary. If $\partial M \neq \emptyset$ then

(β) for each boundary component $M_0$ of $M$ and $U_j \cap M_0 \neq \emptyset$ we have $\phi_j : U_j \cap M_0 \to H_{l,Q}$, moreover, $\partial M_m \subset \partial M$ for each $m$, where $H_{l,Q} := \{ z \in Q_l : x^l \in \mathbb{R}, z^l \in C \}$ (see also §2.1.4);

(iii) $M$ is embedded into $l_2$ as a bounded closed subset;

(iv) each $M_m$ satisfies conditions 2.1.3.3(i − iv).

Let $W$ be a bounded canonical closed subset in $l_2(C)$ with a continuous piecewise $C^\infty$-boundary and $H_m$ an increasing sequence of finite-dimensional subspaces over $C$, $H_m \subset H_{m+1}$ and $dim_CH_m = n(m)$ for each $m \in N$. Then there are spaces $P^\infty_{\alpha,\beta}(W, X) := str-ind_m Y^T,\alpha(W_m, X)$, where $W_m = W \cap H_m$ and $X$ is a separable Hilbert space over $C$.

Let $Y^\xi(W, X)$ be the completion of $P^\infty_{\alpha,\beta}(W, X)$ relative to the following norm

$$
\| f \| := \left( \sum_{m=1}^\infty \| f|_{W_m} \|^2_{Y^T,\alpha(W_m, X)} / (n(m))! \right)^{1/2},
$$
where $0 < c < \infty$ and $\xi = (\Upsilon, a, c)$. Let $M$ and $N$ be the $Y^{\Upsilon, a', c'}$-manifolds with $0 < a' < a$ and $0 < c' < c$.

If $N$ is finite-dimensional complex $Y^{\Upsilon, a}$-manifold, then it is also $Y^{\Upsilon, a', c'}$-manifold. There exists the strict inductive limit of loop groups $(L^M N)_Y =: L^m$, since there are natural embeddings $L^m \hookrightarrow L^{m+1}$, such that each element $f \in Y^{\Upsilon, a}(M, s_0, m; N, y_0)$ is considered in $Y^{\Upsilon, a}(M_{m+1}, s_0, m+1; N, y_0)$ as independent from $(z^{n(m)+1}, \ldots, z^{n(m+1)-1})$ in the local normal coordinates $(z^1, \ldots, z^{n(m+1)})$ of $M_{m+1}$. We denote it $\text{str} - \text{ind}_m L^m =: (L^M N)_Y$ and also $\text{str} - \text{ind}_m Y^{\Upsilon, a}(M; N) =: Q^m$, where $Q^m := Y^{\Upsilon, a}(M_m, s_0, m; N, y_0)$.

Then with the help of charts of $\text{At}(M)$ and $\text{At}(N)$ the space $Y^\xi(W, X)$ induces the uniformity $\tau$ in $Q^m_Y(N, y_0)$ and the completion of it relative to $\tau$ we denote by $Y^\xi(M, s_0; N, y_0)$, where $\xi = (\Upsilon, a, c)$ and $\sum_{p \in \Lambda_M, j \in \Lambda_N} \|f_{p,j} - (w_0)_{p,j}\|^2 < \infty$ for each $f \in Y^\xi(M, s_0; N, y_0)$ is supposed to be satisfied with $w_0(M) = \{y_0\}$, since each $M_m$ is compact. Therefore, using classes of equivalent elements from $Q^m_Y(N, y_0)$ and their closures in $Y^\xi(M, s_0; N, y_0)$ as in §2.8.1 we get the corresponding loop monoids which are denoted $(S^M N)_\xi$. With the help of A. Grothendieck construction we get loop groups $(L^M N)_\xi$. Substituting spaces $Y^{\Upsilon, a}$ over $C$ onto $Z^{\Upsilon, a}$ over $R$ with respective modifications we get spaces $Z^{\Upsilon, a, c}(M, N)$ over $R$, loop monoids $(S^M R)_\xi$ and groups $(L^M R)_\xi$ for the multi-index $\xi = (\Upsilon, a, c)$.

Let $\exp : T^N \to N$ be the exponential mapping, where $T^N$ is a neighbourhood of $N$ in $TN$ [23].

2.9. Theorems. (1) The space $(L^M N)_\xi =: G$ for $\xi = (\Upsilon, a)$ or $\xi = (\Upsilon, a, c)$ from §2.8 is the complete separable Abelian topological group. Moreover, $G$ is the dense subgroup in $(L^M N)_0$ for $\xi = (\Upsilon, a)$; $G$ is non-discrete non-locally compact and locally connected.

(2) The space $X^\xi(M, N) := T_e(L^M N)_\xi$ is Hilbert for each $1 \leq m = \text{dim}_C M \leq \infty$.

(3) Let $N$ be a complex Hilbert $Y^{\xi'}$-manifold with $\infty > a > a' > 0$ and $\infty > c > c' > 0$ for $\xi' = (\Upsilon, a')$ or $\xi' = (\Upsilon, a', c')$ respectively, then there exists a mapping $E : T(L^M N)_\xi \to (L^M N)_\xi$ defined by $E_\eta(v) = \exp_\eta(s) \circ v_\eta$ on a neighbourhood $V_\eta$ of the zero section in $T_\eta(L^M N)_\xi$ and it is a $C^\infty$-mapping for $Y^{\xi'}$-manifold $N$ by $v$ onto a neighbourhood $W_\eta = W_e \circ \eta$ of $\eta$ in $(L^M N)_\xi$; $E$ is the uniform isomorphism of uniform spaces $V_\eta$ and $W_\eta$, where $s \in M$, $e$ is the unit element in $G$, $v \in V_\eta$, $1 \leq m \leq \infty$.  

20
(4). \((L^M N)_\xi\) is the closed proper subgroup in \((L^N R)_\xi\).

**Proof.** There are true analogs of Propositions 2.2, 2.5 and Lemmas 2.3.1, 2.4 for the considered here classes of smoothness \(Y^\xi\) defined with the help of Gevrey classes due to Theorem VI.9 \([8]\) and \([19]\), using the standard procedure of an increasing sequence of \(C^\infty\)-domains \(W_n\), in a quadrant \(Q\) with \(dim_C Q < \infty\) such that \(cl(\bigcup W_n) = Q\).

For \(\xi = (Y, a)\) or \(\xi = (Y, a, c)\), classes \(< f >_\xi\) are closed due to \(\$\$2.3, 2.7, 2.8\) and the corresponding analog of Lemma 2.4 for the considered class of smoothness, since the uniform spaces \(Y^\xi(M, s_0; N, y_0)\) are complete. The space \(T_e(L^M N)_\xi\) is linear, where \(e\) is the unit element of the group \((L^M N)_\xi\), then in particular \(X^\xi(M_m, N)\) is the Banach space with \(\|f\|_{X^\xi(M_m, N)} = \inf_{y \in f} \|y\|_\xi\), where \(f = < y >_\xi, y \in Y^\xi(M, s_0; X, 0)\). On the other hand, \(X^\xi(M_m, N)\) is isomorphic with the completion of \(T_e(L^M N)_\xi\) by the norm \(\|f\|_{X^\xi(M_m, N)}\). Then \((\rho_{k, \gamma}(y^1 + y^2))^2 = 2(\rho_{k, \gamma}(y^1))^2 + (\rho_{k, \gamma}(y^2))^2\) for each \(y^1, y^2 \in H^\gamma(M, X; Z_k)\) due to the choices of \(\nu\) and \(\rho_{k, \gamma}\). If \(y \in H^\gamma(M, X; Z_k)\) then \(\rho_{k, \gamma}(y) = \rho_{l, \gamma}(y)\) for each \(l > k\), since \(\nu(M') = 0\) and \(y \in H^\gamma(M, X; Z_k)\). For each \(y^1, y^2 \in H^\gamma_p(M, s_0; X, 0)\) there exists \(Z \in \gamma\) such that \(y^1, y^2 \in H^\gamma(M, X; Z)\). Therefore, from Equality (i) in \(\S 2.7\) it follows that \(\|f_1 + f_2\|_{X^\xi(M_m, N)}^2 + \|f_1 - f_2\|_{X^\xi(M_m, N)}^2 = 2[\|f_1\|_{X^\xi(M_m, N)}^2 + \|f_2\|_{X^\xi(M_m, N)}^2]\) for each \(f_1, f_2 \in X^\xi(M_m, N)\). Then \(\|f_1 + f_2\|_{X^\xi(M_m, N)}^2 + \|f_1 - f_2\|_{X^\xi(M_m, N)}^2 = 2(\|f_1\|_{X^\xi(M_m, N)}^2 + \|f_2\|_{X^\xi(M_m, N)}^2)\) for each \(0 \leq k \in Z\) and each \(f_1, f_2 \in Q_{\gamma, a}^\infty(X)\) of \(\S 2.8\), consequently, \(\|f_1 + f_2\|_{X^\xi(M_m, N)}^2 + \|f_1 - f_2\|_{X^\xi(M_m, N)}^2 = 2[\|f_1\|_{X^\xi(M_m, N)}^2 + \|f_2\|_{X^\xi(M_m, N)}^2]\). Hence the formula \(4(f_1, f_2) := \|f_1 + f_2\|_{X^\xi(M_m, N)}^2 - \|f_1 - f_2\|_{X^\xi(M_m, N)}^2 = i\|f_1 + i f_2\|^2_{X^\xi(M_m, N)} + i\|f_1 - i f_2\|^2_{X^\xi(M_m, N)}\) gives the scalar product \((f_1, f_2)\) in \(X^\xi(M, N)\) and this is the Hilbert space over \(C\).

The spaces \(Y^\xi(M, s_0; N, y_0)\) and \(X^\xi(M, N)\) are complete, consequently, \(G\) is complete. The space \(X^\xi(M, N)\) is separable and \(\Lambda_N \subset \text{N}\), consequently, \(G\) is separable. The composition and the inversion in \((L^M N)_\xi\) induces these operations in \(G\), that are continuous due to \(\S\S 2.3, 2.7, 2.8\), consequently, \(G\) is the Abelian topological group.

Let \(\beta : M_m \to N\) be a \(Y^\xi\)-mapping such that \(\beta(s_0) = y_0\). If \(C_0\) is the connected component of \(y_0\) in \(N\) then \(\beta(M_m) \subset C_0\). On the other hand, \(N\) was supposed to be connected. In view of Theorems about extensions of functions of different classes of smoothness \([13, 17]\) (see also \(\S 2.2\)) and using completions in the described above spaces there exists a neighbourhood \(W\) of \(w_0\) such that for each \(f : M_m \times \{0, 1\} \to N\) of class \(Y^\xi\) with \(f(M_m, 0) = -21\).
{y_0} and \( f(s, 1) = \beta(s) \) for each \( s \in M \) there exists its \( Y^\xi \)-extension \( f : M_m \times [0, 1] \to N \), where \( \{0, 1\} := \{0\} \cup \{1\} \), \( \beta \in W \), since there exists a neighbourhood \( V_0 \) of \( y_0 \) in \( N \) such that it is \( C^0([0, 1] \times V_0, N) \)-contractible into a point. Hence for each class \( \beta > \xi \) in a sufficiently small (open) neighbourhood of \( e \) there exists a continuous curve \( h : [0, 1] \to G \) such that \( h(0) = e \) and \( h(1) = \beta > \xi \).

By Theorem 2.7 the tangent space \( T_e G \) is infinite-dimensional over \( C \), consequently, \( G \) is not locally compact, where \( e \) is the unit element in \( G \).

Let \( \nabla \) be a covariant differentiation in \( N \) corresponding to the Levi-Civita connection in \( N \) due to Theorem 5.1 \([15]\). This is possible, since \( N \) is the Hilbert manifold and hence has the partition of unity \([24]\). Therefore, there exists the exponential mapping \( exp : TN \to N \) such that for each \( z \in N \) there are a ball \( B(T_z N, 0, r) := \{y \in T_z N : \|y\|_{T_z N} \leq r\} \) and a neighbourhood \( S_z \) of \( z \) in \( N \) for which \( exp_z : B(T_z N, 0, r) \to S_z \) is the homeomorphism, since \( \phi \) is in the class of smoothness \( Y^\xi \) due to Theorem IV.2.5 \([23]\), where \( exp_z w = \phi(1), \phi(q) \) is a geodesic, \( \phi : [0, 1] \to N, d\phi(q)/dq|_{q=0} = w, \phi(0) = z, w \in B(T_z N, 0, r) \) \([23]\).

In view of Theorems 5.1 and 5.2 \([12]\) a mapping

(i) \( E : T Y^\xi(M_m, s_0; N, y_0) \to Y^\xi(M_m, s_0; N, y_0) \) is a local isomorphism, since \( a > b \), so \( \sum_{k=1}^\infty k^2(k!)^{b-a} < \infty \), where

(ii) \( E_g(h) := exp_g(s) \circ h, s \in M_m, h \in TY^\xi(M_m, s_0; N, y_0), h \in T_g Y^\xi(M_m, s_0; N, y_0), g \in Y^\xi(M_m, s_0; N, y_0) \). In view of \([10, 12]\) the tangent bundle \( T Y^\xi(M_m, s_0; N, y_0) \) is isomorphic with \( Y^\xi(M_m, s_0; TN, y_0 \times \{0\}) \times T_{y_0} N \). Then \( E \) induces \( \tilde{E} \) with the help of factorisation by \( R_\xi \) and the subsequent A. Grothendieck construction. This mapping \( \tilde{E} \) is of class of smoothness \( C^\infty \) as follows from equations for geodesics (see §IV.3 \([23]\)), since \( TTN \) is the \( Y^\xi \)-manifold with \( a > a'' > 0 \) and \( c > c'' > 0 \). Indeed, this construction at first may be applied for \((L^{M_n} N)_0\) and then using the completion to \((L^M N)_\xi\).

The last statement is proved analogously to that of Theorem 2.7 using classes of smoothness \( Y^\xi \) and \( Z^\xi \).

2.10. Notes and Definitions. Let \( l_{2,\epsilon} \) be the Hilbert space of sequences \( x = (x^j : x^j \in C; j \in N) \) such that \( \|x\|_{l_{2,\epsilon}} := \{\sum_{j=1}^\infty |x^j|^2 j^{2\epsilon}\}^{1/2} < \infty \). For \( \epsilon = 0 \) we omit it as the index. Suppose that in the either \( Y^\xi_{b, d'} \)-Hilbert or \( Y^\xi \)-manifold \( N \) modelled on \( l_2 \) (see §2.1 and §2.8) there exists a dense \( Y^\xi_{b', \epsilon} \) or \( Y^\xi_{b', d', \epsilon} \)-Hilbert submanifold \( N' \) modelled on \( l_{2,\epsilon} \), where

1. \( \infty > a > b > b' > 0 \) and \( \infty > c > d' \) and either
(2) \( \infty > \epsilon > 1 \) and \( d' \geq d'' > 0 \) or
(3) \( \infty > \epsilon \geq 0 \) and \( d' > d'' > 0 \) correspondingly.

If \( N \) is finite-dimensional let \( N' = N \). Evidently, each \( Y^{\mathcal{T}.b} \)-manifold is the complex \( C^\infty \)-manifold. Certainly we suppose, that a class of smoothness of a manifold \( N' \) is not less than that of \( N \) and classes of smoothness of \( M \) and \( N \) are not less than that of a given loop group for it as in \( \subseteq 2.8 \) and of \( G' \) as below. Let \( G' := (L^M N')_{\xi'} \) be a dense subgroup of \( G = (L^M N)_{\xi} \) with

(a) \( \xi' = (\Upsilon, a'' \xi') \) such that \( \infty > a'' > b \) for \( \xi = O \) and the \( Y^{\mathcal{T}.b} \)-manifolds

(b) \( \xi' = (\Upsilon, a'' \xi') \) such that \( a > a'' > b \) for \( \xi = (\Upsilon, a) \);

(c) \( \xi' = (\Upsilon, a'', c'' \xi') \) for \( \xi = (\Upsilon, a, c) \) and \( \text{dim}_{\mathcal{C}} M = \infty \) such that \( b < a'' < a \) and \( d' < c'' < c \) and either (2) \( \infty > \epsilon > 1 \) with \( 0 < d'' \leq d' \) or (3)

\[ \infty > \epsilon \geq 0 \) \( 0 \leq d'' \leq d' \) where \( M \) and \( N \) are \( Y^{\mathcal{T}.b,d'} \)-manifolds, \( N' \) is the \( Y^{\mathcal{T}.b,d'} \)-manifold, \( 1 \leq \text{dim}_{\mathcal{C}} M =: m < \infty \) in the cases \( a - b \). For the corresponding pair \( G' := (L^M N')_{\xi'} \) and \( G := (L^M N)_{\xi} \) let indices in (1 - 3) and \( a - c \) be the same with substitution of \( \xi = O \) on \( \xi = (\infty, H) \).

2.11. Theorem. On the group \( G \) there exists a probability quasi-invariant measure \( \mu \) relative to a dense subgroup \( G' \) (see \( \subseteq 2.8 \) and \( \subseteq 2.10 \)). Moreover, this measure can be chosen \( \infty \)-continuously differentiable relative to \( G' \).

Proof. (I). From the conditions imposed on a manifold \( M \) it follows, that there exists a partition \( Z \) of it into complex submanifolds with corners. Then there exists \( Z \) such that the covering induced by it refines the covering of \( At(M) \) by charts. There exist two mappings \( f_1 \) (and \( f_2 \), when \( \partial \mathcal{M} \neq \emptyset \)) in \( \mathcal{O}_{\mathcal{T}}(M, \mathcal{C}) \) such that \( A_3 = f^{-1}_1(0) \) (and \( \partial \mathcal{M} = f_2^{-1}(0) \) respectively). The manifold \( M \) is modelled on a Hilbert space \( X \) over \( \mathcal{C} \), so for \( A_1 \) and \( A_2 \) there are quadrants \( Q_1 \) and \( Q_2 \) in \( X \) and surjective mappings \( \kappa_j : Q_j \rightarrow A_j \) such that \( \kappa_j \in \mathcal{O}_{\mathcal{T}}(Q_j, A_j) \), moreover, \( \kappa_j \) are homeomorphisms of \( \text{Int}(Q_j) \) onto \( \text{Int}(A_j) \) and \( \kappa_j(\partial Q_j) = \partial A_j \), where \( j = 1 \) or \( j = 2 \). On the other hand, \( \partial A_j = A_3 \cup (A_j \cap \partial \mathcal{M}) \). The combination of two \( \kappa_j \) and \( Q_j \) produces a quadrant \( Q \) which is \( \mathcal{O}_{\mathcal{T}} \)-diffeomorphic to \( Q_1 \cup Q_2 \) in \( X \) and a submanifold \( S \) in \( Q \) with \( \text{codim}_{\mathcal{R}} S = 1 \) such that \( S \subset \partial Q_1 \cup \partial Q_2 \), \( Q_1 \cap Q_2 = \partial Q_1 \cap \partial Q_2 \), \( \kappa_j(\partial Q_1 \cap \partial Q_2) = \partial A_j \) with the corresponding embeddings of \( Q_j \) into \( X \) and there exists a surjective \( \mathcal{O}_{\mathcal{T}} \)-mapping \( \kappa : Q \rightarrow M \) such that \( \kappa : \text{Int}(Q \setminus S) \rightarrow M \setminus (A_3 \cup \partial \mathcal{M}) \) is a homeomorphism, where \( j = 1 \) or \( j = 2 \). Let \( v = \kappa(\zeta) \) denote points in \( M \) for each \( \zeta \in Q \).

(II). At first we consider case (b) for \( G = (L^M N)_{\xi} \) and \( G' = (L^M N')_{\xi'} \).
For the compact manifold \( M \) we can take \( Q \) up to \( O_\tau \)-diffeomorphism equal to \([0,1]^{2m} \) with 0 corresponding to a marked point \( s_0 \) in \( M \). Then the measure \( \nu \) of a subset \( V_M := \{ v : v \in M, \text{card}(\kappa^{-1}(v)) > 1 \} \) is equal to zero (see §2.8.1). There exists a continuous mapping

\[(i) K : O_\tau(M,N) \times O_\tau(M,A^k N) \to H^\infty_p(M,B^k M) \]
given by the following formula:

\[(ii) K(F, w)(v) := \int_{0}^{1} ... \int_{0}^{2m} (F^* w) := \int_{F_0} w, \text{where } A^k N = \bigoplus_{j+l=0}^{k} \Lambda^{(j,l)} N \]
and \( \Lambda^{(j,l)} N \) denotes the space of differential \((j,l)\)-forms \( w \) on \( N \),
\[w = \sum_{|I| = i,j} w_{i,j} dz^I \wedge \bar{dz}^J, \text{where } dz^I = dz^{i_1} \wedge ... \wedge dz^{i_n} \text{ for a multi-index } I = \{i_1, ..., i_n\}, n \in \mathbb{N}, |I| = i_1 + ... + i_n, 0 \leq i_j \in \mathbb{Z}, w_{i,j} : M \to C, \]
\[B^N := \bigoplus_{j=0}^{k} \Lambda^j N. \]
Here manifolds \( A^k N \) and \( B^k N \) are considered to be of classes of smoothness \( \mathcal{O} \) and \( C^\infty \) respectively, where \( \Lambda^j N \) is for the manifold \( N_\mathbb{R} \), that is the manifold \( N \) which is considered over \( \mathbb{R} \), \( \Lambda^j N \) is the space of differential forms \( w \) on \( N_\mathbb{R} \) such that \( w_I : N \to C, w = \sum w_l dx^I \) (see also Proposition 1.6.4.2 and §1.6.5 [24]). It is correct, since \( M \) satisfies conditions 2.1.3.3(i - iv) and in local coordinates \( v \in M \)
\[(iii) (F^* w)_{v,...,v+s}(v^1, ..., v^m) = \sum_{l=1}^{l=2m} w_{L,J} (\partial z^l / \partial v^I) ... (\partial z^l / \partial v^{l+n}) \text{, where } z = (z^1, z^2, ...) = F(v^1, ..., v^m), \text{since points } v \in M \]
are parametrized with the help of \((\zeta^1, ..., \zeta^{2m}) = \zeta \in Q, v = \kappa(\zeta), z = (z^1, z^2, ...) \text{ are local complex coordinates in } N; \text{we also write simply } F_T := \{ F \circ \kappa(\tau) : 0 \leq \tau \leq T, j = 1, ..., 2m \} \text{ is a set, } F \in O_\tau(M,N), w \in O_\tau(M,A^k N), F^* \text{ is the pull back generated by } F \text{ (see also [24],[24],[24]). We take this mapping } K, \text{when dim}_C M \leq \text{dim}_C N. \text{ When dim}_C M > \text{dim}_C N \text{ we take } w \in O_\tau(M,A^k(N^s)) \text{ instead of } O_\tau(M,A^k N), \text{where } N^s = N_1 \times ... \times N_s \text{ with } N_j = N \text{ for each } j = 1, ..., s \text{ such that } s \geq \text{dim}_C M / \text{dim}_C N, s \in \mathbb{N}. \text{ A mapping } F : M \to N \text{ generates a mapping } F^{\otimes s} := (F, ..., F) : M \to N^s \text{ and the pull back } (F^{\otimes s})^* \text{ which is also denoted simply by } F^*, \text{ where } F^* w \text{ is a piecewise } C^\infty \text{-mapping for the considered here classes of mappings, } (F, ..., F) \text{ is an } s\text{-tuple.}

From the definition of \( Y^{\tau,a} \) in §2.8 it follows that \( K \) has a continuous extension \( K : Y^{\tau,a}(M,N) \times Y^{\tau,a}(M,A^k(N^s)) \to Z^{\tau,a^\prime}(M,B^k M) \) for each \( 0 < a < a'' < \infty \) given by formula (ii), where \( M \) and \( B^k M \) for \( Z^{\tau,a^\prime} \) are considered over \( \mathbb{R} \), since \( C \) over \( \mathbb{R} \) is isomorphic with \( \mathbb{R}^2 \). This is due to the Lebesgue Theorem about the differentiation \( d/dz \) of the Lebesgue indefinite integral \( \int_{0}^{1} f(x) \)dx. Let \( \hat{K} \) be defined on tangent spaces to these with the help of the composition of the local diffeomorphism \( E \) given by Formulas 2.9(i, ii) and \( K \) as above. The tangent spaces over \( C \) can also be
Considered over $\mathbf{R}$. Then $\tilde{K}$ is continuously strongly differentiable such that $(D\tilde{K}(F, w)).(\eta, \psi) = \tilde{K}(\eta, w) + \tilde{K}(F, \psi) + \tilde{K}(F, L_\eta w)$, since $T_{w_0}Y^\xi(M, N) \subset TTY^\xi(M, N) = Y^\xi(M, TN)$ for $w_0(M) = \{y_0\}$, also $T_{w_0}Y^\xi(M, A^k(N^s)) \subset Y^\xi(M, A^k(T(N^s)))$ for $w_0(M) = \{y_0 \times 0\}$, such that $y_0 \times 0 \in A^k(N^s)$, where $F, \eta \in U_N \subset T_{w_0}Y^\xi(M, N)$ for $\xi = (\Upsilon, a)$ and $w, \psi \in U_k \subset T_{w_0}Y^\xi(M, A^k(N^s))$, $U_N$ and $U_k$ are the corresponding neighbourhoods of zero sections, $L_\eta w$ is the Lie derivative of $w$ along $\eta$, moreover, $\tilde{K}(F, w) \in T_{w_0}Z^T.a(M, B^k M)$.

In view of the formula of integration on manifolds \cite{[1]} : $\int_M f^* w = \int_M h^* w$ for each differential form $w$ on $N$ and maps $f, h \in \mathcal{O}_\Upsilon(M, s_0; N, y_0)$, when $f \leq h > 0$, since a subspace of smooth functions from $M$ to $N$ considered over $\mathbf{R}$ is dense in $\mathcal{O}_\Upsilon(M, N)$ and $\int_M f^* w$ is the continuous functional on $\mathcal{O}_\Upsilon(M, N)$ for each given $w$, where $f^*$ is defined $\nu$-almost everywhere on $M$, $M$ and $N$ are orientable (see Introduction). If $F(s_0) = y_0$ it does not imply such restriction for $K(F, w)(v)|_{v = \kappa(\zeta), \zeta = (1, \ldots, 1)}$. For each continuous $w \neq 0$ there exists $F_0 \in Y^\xi(M, N)$ with its closed support $\text{supp}(F_0)$ such that $s_0 \not\in \text{supp}(F_0)$ and $K(F_0, w)(v)|_{v = \kappa(\zeta), \zeta = (1, \ldots, 1), s_0} = (s_0; 1, \ldots, 1)$, since there exists $s \in M$, $s \neq s_0$, such that $w(s) \neq 0$. Thus for each $w$ there are $F_0$ and a mapping $\tilde{A}_w := \tilde{A} : \mathbf{R}^d \times Y^\xi(M, N) \to Y^\xi(M, N)$ such that

$(iv) \ K(\tilde{A}(c, F_0), w)(v)|_{v = \kappa(\zeta), \zeta = (1, \ldots, 1)} = c$, where $c := K(F, w)(v)|_{v = \kappa(\zeta), \zeta = (1, \ldots, 1)}$.

$d = \sum_{l=0}^{k} \binom{2m}{l} \dim_{\mathbf{R}}B_{s_0}^k$, \quad \binom{m}{k} = m!/(k!(m-k)!)$ are binomial coefficients (see also \cite{[1]} \cite{[10]} and Hodge Decomposition Theorems 7.5.3, 7.5.5 in \cite{[1]}). On the other hand, $A^k(L^M N) = (L^M A^k N) \times A^k_{y_0} N$ with the marked point $(y_0 \times 0)$ in $A^k N$, in particular we consider $A^k(L^M N)_{\xi} = (L^M A^k M)_{\xi} \times A^k_{y_0} M$ which is an infinite-dimensional complex manifold even for $k = m = 1$ due to Theorem 2.9.2, since $\dim C A^k M > m$ for each $1 \leq k \leq m$, where traditionally $TM := \bigcup_{x \in M} T_x M$, such that $\dim C TM = 2 \times \dim C M$ and $TU^i = U^i \times X$ for $U^i \subset M$ corresponding to the chart in $M$, $X$ is the Banach space on which $M$ is modelled \cite{[22]}.

The space $Y^\xi(M, N) \times Y^\xi(M, A^k(N^s))$ is isomorphic with $Y^\xi(M, N \times A^k(N^s))$, hence $E^{-1} \circ K \circ E = \tilde{K}$ is defined on a neighbourhood of the zero section in $TTY^\xi(M, N \times A^k(N^s))$ into a neighbourhood of the zero section in $TZ^\xi(M, B^k M)$ for $\xi = (\Upsilon, a)$. The restriction of the latter mapping $\tilde{K}$ on the corresponding neighbourhood of the zero section in $TTY^\xi(M, N \times A^k(N^s), y_0 \times (y_0, 0))$ and then the factorization by the equivalence relation $R_\xi$ and the usage of $A$. Grothendieck construction produces the mapping $K_1$ from the corresponding neighbourhood of the zero section in $T(L^M N)_{\xi} \times$
$T(L^M A^k(N^s))_ξ$ into a neighbourhood of the zero section in $T(L^M B^k M)_ξ \times \mathbb{R}^d$, where $s \geq \text{dim}_CM/\text{dim}_CN$.

Therefore, using $E$ we get a mapping $K_1$ from $T_e(L^M N)_ξ \times T_e(L^M A^k(N^s))_ξ$ into $T_e(L^M B^k M)_ξ \times \mathbb{R}^d$ respectively such that it is continuously strongly differentiable with $(DK_1(f, <w>)).(η, <ψ>) = K_1(η, <w>) + K_1(f, <ψ>) + K_1(f, L_0 <w>)$, where $f, η \in V_N \subset T_e(L^M N)_ξ$, and $<w >, <ψ > \in V_k \subset T_e(L^M A^k(N^s))_ξ$, $V_N$ and $V_k$ are the corresponding neighbourhoods of zero sections. In view of the existence of the mapping $E$ in §2.9 for $T(L^M N)_ξ$ there exists the continuous mapping $K : W_e \times V_e \to V'_0$ induced by $E$ and $K_1$, where $W_e$ is a neighbourhood of $e$ in $(L^M N)_ξ$, $V_e$ is a neighbourhood of the zero section in $T_e(L^M A^k(N^s))_ξ$ for the unit element $e$ in $(L^M A^k(N^s))_ξ$, $V'_0$ is a neighbourhood of zero in the Hilbert space $T_e(L^M B^k M)_ξ \times \mathbb{R}^d$ over $R$. On the other hand, we can use the mapping $χ^*$ from §2.1.4. This mapping $χ^*$ induces the tangent mapping $Tχ^* : TY_ξ((M)_1 \cup (M)_2, s_0; N, y_0) \to TY_ξ(M, s_0; N, y_0)$ such that $χ^*$ is in the class of smoothness $C^\infty$. Therefore, there is the linear mapping (differential) $Dχ^*(h) : T_hY_ξ((M)_1 \cup (M)_2, s_0; N, y_0) \to F$ for each $h \in Y_ξ((M)_1 \cup (M)_2, s_0; N, y_0)$, where $F$ is the Hilbert space such that $T_2Y_ξ(M, s_0; N, y_0) = \{z\} \times F$ for each $z \in Y_ξ(M, s_0; N, y_0)$, in particular for $z = χ^*(h)$ (see (23)).

Then we define by induction the following mapping

$(v) \big(\Psi_{1_{-1},N}(f, <w>) := \Psi_{1,M,N}(\Psi_{1,M,N}(id_M, K(f, <w>)))$, where $\Psi_{1,M,N}(f, <w>) := K(f, <w>)$, $\Psi_{1,M,N}$ is defined analogously to $\Psi_{1,M,N}$, but with $M$ over $R$ instead of $N$ over $C$, that is $\Psi_{1,M,N} : (L^M R^k M)_ξ \times T_e(L^M B^k M)_ξ \times \mathbb{R}^{d-1}d \to T_e(L^M B^k M)_ξ \times \mathbb{R}^d$ due to Formula $(iv)$, $id_M : M \to M$ is the identity mapping, $id_M(z) = z$ for each $z \in M$.

For a sequence of Hilbert spaces $P_q$ all over either $C$ or $R$ with $q \in J \subset N$ let $l_2,δ(\{P_q : q \in J\}) := \{x = (x^q : x^q \in P_q, q \in J); \|x\|^2_{l_2,δ(\{P_q : q \in J\})} := (\sum_{q \in J}q^δx^q)^{1/2} < \infty\}$ be a new Hilbert space, where $\infty > δ \geq 0$, the index $δ$ is omitted for $δ = 0$. If $J$ is finite then $l_2,δ(\{P_q : q \in J\})$ is isomorphic with $\bigotimes_{q \in J}P_q$. On the other hand, $l_2$ and $l_2(\{C_q : q \in N\})$ are isomorphic with equivalent norms, since $\|x\|^2_{l_2} = \sum_{j=1}|x_j|^2 = \sum_{n=0}^\infty\sum_{j=1}^d|x_n+j|^2 = \|x\|^2_{l_2(\{C_q : q \in N\})}$ for each $x \in C, j \in N \in l_2$. The Hilbert space $l_2,ε$ is isomorphic with $l_2,δ(\{C_q : q \in N\})$. Their norms are equivalent, since $(d)^2\|x\|^2_{l_2,δ(\{C_q : q \in N\})} \geq \|x\|^2_{l_2,ε} \geq \|x\|^2_{l_2,ε(\{C_q : q \in N\})}$ for each $x \in l_2,ε$.

We choose $P_q = T_e(L^M R^k B^k M)_ξ \times \mathbb{R}^d$ for each $q$, where either $J = N$ for infinite-dimensional $N$ or $J = \{1\}$ for finite-dimensional $N$. Let $l \geq 2$ be
fixed for \( \Psi_{l,M,N} \). Let also \( e_q \in l_2(\{ P_q : q \in J \}) \) for each \( q \in J \) such that \( \pi_q : P_q \rightarrow l_2(\{ P_q : q \in J \}) \) are the natural embeddings with \( e_q \in \pi_q(P_q) \) and \( \| e_q \|_{l_2((P_q,q \in J))} = 1 \) for each \( q \in J \). Therefore, due to formulas \((ii - v)\) there exists a family \( \{ < w^{i,q} : i = 1, \ldots, d; q \in J \} \subseteq T_v(L^M A^k(N^*)^\#)_{\gamma,\beta}^\# \) for \( 0 < b < \beta' < a' < \infty \), where \( k = 2m = \dim_\mathbb{R} M \), \( < w^{i,q} > \) are the corresponding classes of equivalent elements, such that the mapping

\[
(vi) \Psi_l(f) := \sum_{q \in J} \sum_{i=1}^d \Psi_{l,M,N}(f, < w^{i,q} >) e_q \in l_2(\{ P_q : q \in J \})
\]

is injective. Due to Theorem 2.9(3) about properties of \( \tilde{E} \) and the open mapping Theorem (14.4.1) \[ \] the mapping \( \Psi_l \) is the diffeomorphism of a suitable neighbourhood \( U_v \) of the unit element \( e \in (L^M N)_\xi \) (considered as the manifold over \( \mathbb{R} \)) onto a neighbourhood \( V_0 \) of 0 in the corresponding Hilbert subspace \( K_0 \) in \( l_2(\{ P_q : q \in J \}) \). Let the image \( V_0 \) of \( U_v \) be supplied with the strongest uniformity relative to which \( \Psi_l \) is uniformly continuous, that produces the Hilbert space \( K_0 = \cup_{j \in \mathbb{N}} jV_0 \).

This follows from the consideration of a space \( Z^\gamma_q(M,X) \) for \( \dim_{\mathbb{C}} M < \infty \) which is defined to be the completion of a subspace \( f \in Z^{\gamma+qm}(M,X) \) for which \( D^\alpha f|\iota \) there exists \( j \) with \( w = s_j^\alpha \) relative to the following norm

\[
\| f \|_{Z^\gamma_q(M,X)} := (\sum_{\alpha=(\alpha^1,\ldots,\alpha^m):0 \leq \alpha^j \leq q} \| D^\alpha f \|^2_{Z^{\gamma+qm}(M,X)} \right)^{1/2},
\]

where \( 0 \leq q \in \mathbb{Z} \). In this class of smoothness analogously to \( \S \) we get spaces \( Z^\gamma_q(M,N) \), such that \( Z^\gamma_q(M,s_0;N,y_0) = \{ f \in Z^\gamma_q(M,N) : f(s_0) = y_0 \} \) and \( < f >^{R}_{\gamma,q} := \text{cl}(< f >^{R}_{\gamma,\alpha}) \) with the closure in \( Z^\gamma_q(M,s_0;N,y_0) \), where \( < f >^{R}_{\gamma,q} \) are classes of equivalent elements in \( Z^\gamma_q(M,s_0;N,y_0) \). Then we consider a loop group \( (L^M N)^\#_\gamma \) constructed from the loop monoid \( (S^M N)^\#_\gamma \) := \( Y^\gamma_q(M,s_0;N,y_0)/R^\gamma_q \) and a Hilbert space \( X^\gamma_q(M,N) := T_v(L^M N)^\#_\gamma \), where \( R^\gamma_q \) is the equivalence relation generated by classes \( < f >^{R}_{\gamma,q} \).

Then the mapping \( \Psi_l : U_v \rightarrow K_0 \) given by Formula \((vi)\) is continuously strongly differentiable. There exist neighbourhoods \( V' \ni e \in G' \) and \( U_\xi \ni e \in G \) such that \( V' \ni U_\xi \subset U_e \). We consider next a mapping \( \Psi_0(v) := \Psi_l \circ L_\phi \circ \Psi^{-1}_0(v) - v \) with \( v \in V_\xi \), \( \phi \in V' \), where \( V_\xi = \Psi_l(U_\xi) \) and \( L_\phi(f) := \phi( f ) \) denotes an operator of the left shift in \( G \). Then either \( \alpha \) \( S_\phi(V_\xi) \subset l_2,e(\{ P': q \in J \}) \) for each \( \phi \) \( V' \), where \( P'_q = T_v(L^M B^k M)_{\gamma,\alpha'} \times \mathbb{R}^d_e \) with \( 1 < \epsilon < \infty \) and \( 0 < d'' \leq d \) for each \( q \in J \); or \( \beta \) \( S_\phi(V_\xi) \subset l_2,e(\{ P''_q : q \in J \}) \) for each \( \phi \) \( V' \) with \( 0 \leq \epsilon < \infty \) and \( 0 < d'' < d' \), where \( P''_q = P'_q \) and


\[ \|f\|_{P^q} = \|f\|_{P^q(q!)}^{d^* - d^*} \] is the relation between norms of \( f \) in \( P^q \) and \( P^q \) respectively for each \( q \).

For an open interval \( J \subset \mathbb{R} \) and two Banach spaces \( E, F \) and open subset \( U, U \subset E \), if \( f : J \times U \to F \) is continuous and \( D_2 f(\phi, z) \) is continuous in \( J \times U \), if also \( \alpha \) and \( \beta \) are two continuously differentable mappings of \( U \) into \( J \), then \( g(z) := \int_{\alpha(z)}^{\beta(z)} f(\phi, z) d\phi \) is continuously differentiable in \( U \) such that \( Dg(z) \) is the linear mapping \( Dg(z)(h) = (\int_{\alpha(z)}^{\beta(z)} D_2 f(\phi, z) d\phi) h + (D\beta(z)(h)) f(\beta(z), z) - (D\alpha(z)(h)) f(\alpha(z), z) \). From Formulas (ii - vii) it follows that \( S_\phi(v) \) is strongly continuously differentiable by \( v \in V_0 \) for each \( \phi \in V' \). Moreover, \( \partial S_\phi(v)/\partial v = \hat{P}_1 \hat{P}_1^{-1} \), where \( \hat{P}_n \) for sufficiently large \( n \in \mathbb{N} \) and \( \hat{P}_1 \) are operators of trace class for each \( \phi \in V' \) and \( v \in V_\xi \) such that \( \hat{P}_1 : K_0 \to X', \hat{P}_2 : X' \to l_2, \{P_q : q \in J\} \), \( X' \) is the corresponding Hilbert space, \( \hat{P}_1(K_0) = X' \). It follows from the fact that \( Y^{\alpha, \beta} (M, N) \supset Y^{\alpha, \beta} (M, N) \supset Y^{\alpha, \beta} (M, N') \) for each \( 0 < \beta' < b < b' < a < \infty \) such that the corresponding operators of embeddings are of trace class, since \( \sum_{j \in \mathbb{N}} j^b a^j < \infty \) and either \( \sum_{j \in \mathbb{N}} j^{-\epsilon} < \infty \) for \( \epsilon > 1 \) or \( \sum_{j=1}^\infty j! a^j b^j < \infty \). In the case of \((L^M_N)_{\alpha, \beta}\) using cylindrical functions (see below) the desired measure can be induced from the corresponding Hilbert subspace \( X' \) either \( (\alpha) \) in \( l_2, \alpha^- (\{P_q : q \in J\}) \) such that \( \hat{P}_q = T e(L^M_N)_{\alpha, \beta} \times R^d \) for each \( q \), where \( 0 < d^* \leq d' \) and \( 1 < \epsilon - \delta' < \epsilon \); or \( (\beta) \) in \( l_2, \beta^- (\{P_q : q \in J\}) \) with \( 0 < d' \leq d' \) and \( 0 \leq \epsilon < \infty \), where \( \hat{P}_q = \hat{P}_q \) such that \( \|f\|_{P_q} = \|f\|_{\hat{P}_q(q!)}^{d^* - d^*} \) for each \( f \in \hat{P}_q \) and each \( q \in J \).

There exists a Gaussian quasimeasure \( \lambda \) on \( X' \). It induces a Gaussian probability measure \( \nu \) on \( K_0 \) with the help of an operator of trace class \( \hat{P}_1 : K_0 \to X' \). This measure induces a measure \( \hat{\mu} \) on \( U_\xi \) with the help of \( \Psi \) such that \( \hat{\mu}(A) = \nu(\Psi_1(A)) \) for each \( A \in Bf(U_\xi) \), since \( \nu(V_0) > 0 \). The groups \( G \) and \( G' \) are separable and metrizable, hence there exist locally finite coverings \( \{\phi_i \circ W_i : i \in \mathbb{N}\} \) of \( G \) and \( \{\phi_i \circ W'_i : i \in \mathbb{N}\} \) of \( G' \) with \( \phi_i \in G' \) such that \( W_i \) are open subsets in \( U_\xi \), \( W'_i \) are open subsets in \( V' \), where \( \phi_1 = e \) and \( W_1 = U_\xi, e \in W'_i \subset W_i \) for each \( i \), that is \( \bigcup_{i \in \mathbb{N}} \phi_i \circ W_i = G \) and \( \bigcup_{i \in \mathbb{N}} \phi_i \circ W'_i = G' \). Then \( \hat{\mu} \) can be extended onto \( G \) by the following formula \( \hat{\mu}(A) := (\sum_{i=1}^{\infty} \hat{\mu}((\phi_i^{-1} \circ A) \cap W_i)2^{-i})/(\sum_{i=1}^{\infty} \hat{\mu}(W_i)2^{-i}) \) for each \( A \in Bf(G) \). In view of Theorems 26.1 and 26.2 this \( \mu \) can be chosen quasi-invariant on \( G \) relative to \( G' \).

Therefore, to verify differentiability of \( \mu \) it is sufficient to consider \( \mu \) on \( W_1 \) and with \( \phi \in V'' \) for open \( V'' \) in \( G' \), \( e \in V'' \subset V' \), such that
\[ \|D_\phi S_\phi(v)(X_\phi)\| < \tilde{c} \times \|v\|_{K_\phi} \times \|X_\phi\|_{T_\phi G'} \] for each \( \phi \in V'' \), where \( 0 < \tilde{c} = \text{const} < 1 \). From the above construction and the formula for the quasi-invariance factor \( \rho_\mu(\phi, g) \) with the help of \( \Psi_t \) we get that \( \mu \) is \( \infty\)-continuously differentiable relative to \( G' \), since \( \Psi_t \) is also \( \infty\)-continuously strongly differentiable due to Theorems 2.9. Indeed, the facts that \( \det \nu \) and since \( \nu \) differentiable by \( \phi \) and \( G \) continuously differentiable relative to \( \rho \) = \( O \) quite analogously to the complex case substituting \( F \) we can take define the following subsets of open \( W \).

(III). Let now \( G = (L^M_R N)_{\infty, b} \) and \( G' = (L^M_R N)^{\infty, a} \). Let \( \infty > a > a'' > b \) and the measure (denoted by \( \tilde{\mu} \) on \( (L^M_R N)^{\infty, a} \)) be from §2.11.(II) quite analogously to the complex case substituting \( O_{\infty, a} \) on \( H^\infty_p(M, N) \) and \( O_{\infty, a} \) on \( H^\infty_p(M, B^k(N^a)) \). Since by the corresponding analog of Theorem 2.9(1) \( G_1 \) is the dense subgroup of \( G \), the measure \( \tilde{\mu} \) on \( G_1 \) induces the measure \( \mu \) on \( G \) such that \( \mu(Q) = \tilde{\mu}(Q) \), where \( Q := [x \in G_1 : (h_1(x), ..., h_s(x)) \in R] \), \( Q := [x \in G : (h_1(x), ..., h_s(x)) \in R] \), \( R \in Bf(R^s) \), \( h_i \in \{ h : G \to R, h \text{ are continuous} \} \), the real field \( R \) is considered with the standard norm. Indeed, the minimal \( \sigma\)-fields over \( G_1 \) and \( G \) generated by such \( \tilde{Q} \) and \( Q \) coincide with the Borel \( \sigma\)-fields \( Bf(G_1) \) and \( Bf(G) \) respectively. If \( Q_1 \cap Q_2 = \emptyset \) for such \( Q_j \) then \( Q_1 \cap Q_2 = \emptyset \), hence \( \mu \) is additive and has \( \sigma\)-additive extension on \( Bf(G) \), since the uniformity in \( G_1 \) is stronger than in \( G \).

Then the spaces \( C^0_b(G, R) \) and \( C^0_b(G_1, R) \) of bounded continuous functions \( h : G \to R \) and \( h : G_1 \to R \) are separable such that \( \|h\|_{C^0_b(G, R)} := \sup_{x \in G} |h(x)| < \infty \). There exists a countable family \( \{h_j : j \in N\} =: F \) which is dense in \( C^0_b(G_1, R) \) and in \( C^0_b(G, R) \), since if \( f \in C^0_b(G, R) \) then its restriction \( f|_{G_1} \in C^0_b(G_1, R) \). The groups \( G \) and \( G_1 \) are separable, consequently, we can take \( F \) separating points in \( G \) and in \( G_1 \), since each \( h \in C^0_b(G, R) \) is entirely defined by its values on a countable dense subset of \( G \). We may define the following subsets of open \( W_1 \) in \( G_1 \) and open \( W \) in \( G \), such that
such that $\chi^m \leq c$ and $W_1(k; c; f) := [g \in W : \rho(k; g, f) \leq c]$ and $W_1(k; c; f) := [g \in W_1 : \rho(k; g, f) \leq c]$, where $\infty > c > 0$, $k \in \mathbb{N}$, $f \in W_1$, the mappings $\rho(k; k'; g; f) := \sum_{h \in F(k,k')} |h(g) - h(f)|$; $F(k, k') := \{h_j \in F : j = k, \ldots, k'\}$ for each $k' > k$; $\rho(k; g, f) := \rho(1; k; f, g)$, so $W(k + 1, c; f) \subset W(k; c; f)$ for each $k \in \mathbb{N}$. Therefore, $\cap \{W(k, 1/k; f) : k \in \mathbb{N}\} = \{f\}$, whence the least $\sigma$-field $\mathcal{A}$ generated by the following family $\mathcal{V} := \{W(k, c; f) : c > 0, k \in \mathbb{N}, f \in W, W \subset G, W$ is open} is such that $\mathcal{A} \supset Bf(G)$. Moreover, $\cap_{k=1}^\infty (\cap_{m=1}^\infty \cup_{n>m} W(k, 1/k; f_n)) = \{f\}$ for each $f \in W$ and each sequence $\{f_n\} \subset W$ converging to $f$. Hence $\mu(W(k, c; f)) := \tilde{\mu}(W_1(k, c; f))$ for each $c > 0$, $f \in W_1, k \in \mathbb{N}$. From the Definition of $\tilde{\mu}$ it follows that $\mu(\cap_{k=1}^\infty \cap_{m=1}^\infty \cup_{n>m} W(k, 1/k; f_n))] = 0$, consequently, $\mu$ is countably additive on $Bf(G)$.

Then with the help of such cylindrical subsets we get that a derivative $D^k_{\phi}\tilde{\mu}(W_1(j, 1/j; f_n))(X_1, \phi, \ldots, X_k, \phi)$ for each $\phi \in \mathcal{G}'$ and $X_i, \phi \in \Xi(\mathcal{G}')$ induces a measure on the family of cylindrical subsets of $G$ coinciding with $D^k_{\phi}\mu(W(j, 1/j; f_n))(X_1, \phi, \ldots, X_k, \phi)$ and it has the extension up to the $\sigma$-additive measure on $\mathcal{A}$. Therefore, $\mu$ on $G$ is quasi-invariant and $\infty$-continuously differentiable relative to $\mathcal{G}'$; analogously for $G = (L^M N)_\xi$ and $\mathcal{G}' = (L^M N')_{\xi',\eta}$.

(IV). Consider now the case of $dim_{\xi}M = \infty$ for $G = (L^M N)_\xi$ and $\mathcal{G}' = (L^M N')_{\xi'}$. If $f \in Y^{a,a}(M_m, s_0; X, 0)$ then $f$ is independent from local complex coordinates $v^l$ for each $l > n(m) = dim_{\xi}M_m$, consequently, $D^a f = 0$ if there is $\alpha^j > 0$ for $l > j > n(m)$, where $\alpha = (\alpha^1, \ldots, \alpha^l)$. Let $\nu_m$ denotes the measure $\nu$ defined on $M_m$ in §2.8, where $\nu_m(M_m) = 1$. Hence $f \in Y^{a,a}(X, 0)$. Let $A^\infty N$ be a $Y^{a,a}(c, X, 0)$-submanifold of $\bigoplus_{i=0}^\infty A^{(j,i)} N$ and $B^\infty M$ be a $Y^{a,a}(c, X, 0)$-submanifold of $\bigoplus_{i=0}^\infty B^{(j,i)} M$ such that there are natural embeddings $A^k N \hookrightarrow A^\infty N$ and $B^k M \hookrightarrow B^\infty M$ for each $k \in \mathbb{N}$.

There is a mapping $K_\infty : Q^{\infty}_{Y^a}(N) \times Q^{\infty}_{Y^a}(A^\infty N) \to \bigoplus_{m \in \mathbb{N}} Z^{\infty}_{Y^a}(M_m, B^{2n(m)} M_m)$ such that $K_\infty(F, w) := \{K_m(F|_{M_m}, \omega|_{M_m}) : m \in \mathbb{N}\}$, $K_m$ are defined for each $M_m$ by formula (ii) of §2.11, (II), where $\tilde{N} = N$ for $dim_{\xi}N = \infty$ and $\tilde{N}$ is a submanifold of $N^{\infty} := \bigotimes_{j=1}^\infty N_j$ with $N_j = N$ for each $j$ modelled on $l_{2,d'}(\{S_j : j \in \mathbb{N}\})$, $S_j = T_y N$ for each $j$, $0 < d' < \infty$ (see §§2.8 and 2.10, about compositions of functions of such classes analogous to Gevrey see [8, 173]). There are the embeddings $\eta_m : M_m \hookrightarrow M$. Then the atlases of $M$ and of each $M_m$ can be chosen consistent. Hence there are the embeddings $\chi_m$ of $Z^{\infty}_{Y^a}(M_m, B^{2n(m)} M_m)$ into $Z^{\infty}_{Y^a}(M, B^\infty M)$. We can choose $\chi_m$ such that $\chi_m(Z^{\infty}_{Y^a}(M_m, B^{2n(m)} M_m)) \cap \chi_i(Z^{\infty}_{Y^a}(M_i, B^{2n(i)} M_i)) = \{0\}$ for each
n \neq l$, since $M$ and $B^\infty M$ are the Hilbert manifolds. Let $t - a < (a - a^\nu)/2$ and $q - c < (c - c^\nu)/2$. Therefore, $K_\infty$ generates the following continuous operator $K^\Sigma: Y^\xi(M; N) \times Y^\xi(M; A^\infty \tilde{N}) \to Z^\Sigma t.q(M; B^\infty M)$ for $a \leq t < \infty$ and $c < q < \infty$ given by the following formula

$K^\Sigma(F, w) := \sum_{m=1}^\infty \chi_m(K_m(F|_{M_m}, w|_{M_m}))$, since for the corresponding $K_m$ to $K_m$ and $\bar{K}_m$ to $K^\Sigma$ on the tangent spaces $\|\bar{K}_m\| \leq [\sum_{m=1}^\infty \|K_m\|^2(n(m)!)^{c-q}]^{1/2} < \infty$.

where $M$ and $B^\infty M$ for $Z^\Sigma t.q$ are considered over $\mathbb{R}$. Suppose $z = (z^m \in \mathbb{R}^{d(m)} : m \in \mathbb{N})$, where $z^m = K_m(F|_{M_m}, w|_{M_m})|_{v=\kappa(i), \zeta=(1,\ldots,1)}$, $d(m) = dim \mathbb{R} B_{m,m}^{2n(m)} M_m$, $z^m = (z^m_j : j = 1, \ldots, d(m))$. Then

$$\|z\|_q^2 := \sum_{m=1}^\infty (\|z^m\|^2_m/(n(m)!))^{1/2} < \infty,$$

where $\|z\|_n^2 := \sum_{j=1}^{d(m)} |z^m_j|^2$. The Hilbert space of such sequences we denote by $\mathbb{N}_q$.

Then we use $\tilde{E}$ (or $E$) as in §2.11(II,III) and the natural quotient mappings $\tilde{\zeta}_{\infty,L}: Y^\xi(M, s_0; \mathbb{N}, y_0) \to (S^M N)_\xi$ corresponding to the classes of equivalent elements (see §2.8). Therefore, we get from $K^\Sigma$ the continuous mapping $\tilde{K}: T_e(L^M N)_\xi \times T_e(L^M A^\infty \tilde{N})_\xi \to T_e(L^M B^\infty M)_{\gamma,t,q} \times \mathbb{N}_q$, where the Hilbert space $\mathbb{N}_q$ appears from taking into account the operators $\bar{A}$ for each $M_m$ from §2.11.(II). Then $\tilde{K}$ and again $\tilde{E}$ (or $E$) generate the mapping $\tilde{K}: W_e \times V_e \to V'_0$, where $W_e$ is a neighbourhood of $e$ in $G$, $V_e$ is a neighbourhood of the zero section in $T_e(L^M A^\infty \tilde{N})_\xi$, $V'_0$ is a neighbourhood of zero in the Hilbert space $H_{t,q} := T_e(L^M B^\infty M)_{\gamma,t,q} \times \mathbb{N}_q$. Let $l_{2,\delta}(\{P_q : q \in \mathbb{N}\})$ be the same as in §2.11(II) and $P_q$ be equal to $H_{t,q}$ for each $q \in \mathbb{N}$. From the definition of $\tilde{K}$ and the consideration of $Q^\infty_{\gamma,\beta'}(N'_0, y_0)$ dense in $Y^{\gamma,\beta',\gamma'}(M, s_0; N'_0, y_0)$ it follows the existence of a family $\{<w_i,q,m> : i = 1, \ldots, d(m) ; m \in \mathbb{N} ; q \in \mathbb{N}\} \subset T_e(L^M A^\infty \tilde{N})_{\gamma,\beta',\gamma'}$ for $b < \beta' < \alpha$, $d' < \gamma' < c''$ such that the mapping

$$\Psi_1(f) := \sum_{q=1}^{\infty} \sum_{m \in \mathbb{N}} \sum_{i=1}^{d(m)} \bar{K}(f, w_i,q,m) e_q \in l_2(\{P_q : q \in \mathbb{N}\})$$

is injective, where $A^\infty \tilde{N}'$ is a $Y^{\gamma,\beta',\gamma'}$-submanifold of $\bigoplus_{j+t=0}^\infty A^{(j,l)} \tilde{N}'$ such that there are natural embeddings $A^{k} \tilde{N}' \hookrightarrow A^\infty \tilde{N}'$ for each $k \in \mathbb{N}$; $w_i,q,m$ are independent from local coordinates $(z^{(n(m)+1)}_{(m)+1}, z^{(n(m)+2)}_{(m)+2}, \ldots)$ for each $i$, $q$ and $m$. 

31
m. Let \( V_0 = \Psi_1(U_e) \) be supplied with the strongest uniformity relative to which \( \Psi_1 \) is uniformly continuous, where \( U_e \) is a neighbourhood of \( e \) in \( G \) such that \( U_e \subset W_e \). This gives the Hilbert space \( K_0 = \bigcup_{j \in \mathbb{N}} jV_0 \). There are a neighbourhood \( V' \ni e \) in \( G' \) and \( \tilde{U}_e \ni e \) in \( G \) such that \( V' \circ \tilde{U}_e \subset U_e \). Let \( S_\phi(v) := \Psi_1 \circ L_\phi \circ \Psi_1^{-1}(v) - v \) with \( v \in V_\xi \), \( \phi \in V' \), where \( V_\xi = \Psi_1(U_\xi) \). Then either \( (a) \) \( S_\phi(V_\xi) \subset l_{2,\epsilon}(\{P'_{\phi,q} : q \in \mathbb{N}\}) \) for each \( \phi \in V' \), where \( P'_{\phi,q} = H_{\beta', \gamma'} \) with \( 1 \leq \epsilon < \infty \), \( a'' < \beta' < a \) and \( c'' < \gamma' < c \); or \( (b) \) \( S_\phi(V_\xi) \subset l_{2,\epsilon}(\{P''_{\phi,q} : q \in \mathbb{N}\}) \) for each \( \phi \in V' \), where \( a'' < \beta' < a - \delta \), \( c'' < \gamma' < c - \delta \), \( 0 \leq \epsilon < \infty \), \( P''_{\phi,q} = P'_{\phi,q} \) and \( \|f\|_{P''_{\phi,q}} = \|f\|_{P'_{\phi,q}}(q)\delta \) for each \( q \), \( 0 < \delta < \min(a - a'', c - c'')/2 \). Moreover, \( \partial S_\phi(v)/\partial v = \tilde{P}_1 \tilde{P}_2 \), where \( \tilde{P}_j \) are operators of trace class for each \( \phi \in V' \) and \( v \in V_\xi \) and \( j \in \{1, 2\} \). The final part of the proof is analogous to that of \( \S 2.11.2 \). The remaining cases of \( G = (L^1_kN)_\xi \) and \( G' = (L^1_kN')_{\xi'} \) are analogous with substitutions of \( Y_\xi \) on \( Z_\xi \) and \( A^kN \) on \( B^kN \) for each \( 1 \leq k \leq \infty \).

3. Unitary representations of loop groups.

3.1. Theorem. Let \( \mu \) be a quasi-invariant relative to \( G' \) measure on \( (G,Bf(G)) \) as in Theorem 2.11. Assume also that \( H := L^2(G, \mu, C) \) is the standard Hilbert space of equivalence classes of square-integrable (by \( \mu \)) functions \( f : G \rightarrow C \). Then there exists a strongly continuous injective homomorphism \( T^\mu : G' \rightarrow U(H) \), where \( U(H) \) is the unitary group on \( H \) in a topology induced from a Banach space \( L(H) \) of continuous linear operators \( A : H \rightarrow H \) supplied with the operator norm.

Proof. Let \( f \) and \( h \) be in \( H \), their scalar product is given by \((f,h) := \int_G \bar{h}(g)f(g)\mu(dg)\), where \( f \) and \( h : G \rightarrow C \), \( \bar{h} \) denotes complex conjugated \( h \). There exists the regular representation \( T := T^\mu : G' \rightarrow U(H) \) defined by the following formula (see \( \S 2.1.1 \)):

\[
T^\mu(z)f(g) := [\rho_\mu(z,g)]^{1/2}f(z^{-1}g).
\]

For each fixed \( z \) the quasi-invariance factor \( \rho_\mu(z,g) \) is continuous by \( g \), hence \( T(z)f(g) \) is measurable, if \( f(g) \) is measurable (relative to \( Af(G, \mu) \) and \( Bf(C) \)). Therefore, \((T(z)f(g),T(z)h(g)) = \int_G \bar{h}(z^{-1}g)f(z^{-1}g)\rho_\mu(z,g)\mu(dg) = (f,h)\), consequently, \( T \) is unitary. From \( \mu_{z'z}(dg)/\mu(dg) = \rho_\mu(z',z) = \rho_\mu(z, (z')^{-1}g)\rho_\mu(z',g) = [\mu_{z'z}(dg)/\mu_{z'}(dg)][\mu_{z'}(dg)/\mu(dg)] \) it follows that \( T(z')T(z) = T(z'z) \) and \( T(id) = I, T(z^{-1}) = T^{-1}(z) \).
The embedding of $T, G'$ into $T, G$ is the operator of trace class. The measure $\mu$ on $G$ is induced by the Gaussian measure on the corresponding separable Hilbert space $K_0$ over $\mathbb{R}$. In view of Theorems 26.1 and 26.2 [14] for each $\delta > 0$ and $\{f_1, ..., f_n\} \subset H$ there exists a compact subset $B$ in $G$ such that $\sum_{i=1}^n \int_{G \setminus B} |f_i(g)|^2 \mu(dg) < \delta^2$. Therefore, there exists an open neighbourhood $W'$ of $e$ in $G'$ and an open neighbourhood $S$ of $e$ in $G$ such that $\rho_\mu(z, g)$ is continuous and bounded on $W' \times W' \circ S$, where $S \subset W' \circ S \subset G$. In view of this, Theorems 2.9 and 2.11 and the Hölder inequality $\lim_{j \to \infty} \sum_{n=1}^\infty \| (T(z^j) - I) f_i \|_H = 0$ for each sequence $\{z^j : j \in \mathbb{N}\}$ converging to $e$ in $G'$, $\lim_{j \to \infty} z^j = e$, where $I$ is the unit operator on $H$. Indeed, for each $\nu > 0$ and a continuous function $f : G \to \mathbb{C}$ with $\|f\|_H = 1$ there is an open neighbourhood $V$ of $id$ in $G'$ (in the topology of $G'$), such that $|\rho_\mu(z, g) - 1| < \nu$ for each $z \in V$ and each $g \in F$ for some open $F$ in $G$, $id \in F$ with $\mu_\nu^{\ast}(G \setminus F) < \nu$. At first this can be done analogously for the corresponding Banach space from which $\mu$ was induced, where $f \in \{f_1, ..., f_n\}, n \in \mathbb{N}$.

In $H$ continuous functions $f(g)$ are dense, hence $\|f(g) - f(z^{-1}g)\|_H = 0$ for each finite family $\{f_i\}$ with $\mu(\{f_i\}) = 1$ and $z \in V' = V \cap V''$, where $V''$ is an open neighbourhood of $id$ in $G'$ such that $\|f(g) - f(z^{-1}g)\|_H < \nu$ for each $z \in V''$, $0 < \nu < 1$, consequently $T$ is strongly continuous (that is, $T$ is continuous relative to the strong topology on $U(H)$ induced from $L(H)$, see its definition in [14]).

Moreover, $T$ is injective, since for each $g \neq id$ there is $f \in C^0(G, \mathbb{C}) \cap H$, such that $f(id) = 0$, $f(g) = 1$, and $\|f\|_H > 0$, so $T(f) \neq I$.

3.2. Note. In general $T$ is not continuous relative to the norm topology on $U(H)$, since for each $z \neq id \in G'$ and each $1/2 > \nu > 0$ there is $f \in H$ with $\|f\|_H = 1$, such that $\|f - T(z)f\|_H > \nu$, when $\text{supp}(f)$ is sufficiently small with $(z \circ \text{supp}(f)) \cap \text{supp}(f) = \emptyset$.

3.3. Theorem. Let $G$ be a loop group with a real probability quasi-invariant measure $\mu$ relative to a dense subgroup $G'$ as in Theorem 2.11. Then $\mu$ may be chosen such that the associated regular unitary representation of $G'$ is irreducible.

Proof. Let a measure $\nu$ on the Hilbert space $K_0$ be of the same type as in the proof of Theorem 2.11. Let a $\nu$-measurable function $f : H \to \mathbb{C}$ be such that $\nu(\{x \in K_0 : f(x + y) \neq f(x)\}) = 0$ for each $y \in X_0$ with $f \in L^1(H, \nu, \mathbb{C})$, where $\nu$ is quasi-invariant relative to shifts from a dense linear subspace $X_0$ in $K_0$. Let also $P_k : \ell_2 \to L(k)$ be projectors such that $P_k(x) = \ldots$
$x^k$ for each $x = (\sum_{j \in \mathbb{N}} x^j e_j)$, where $x^k := \sum_{j=1}^k x^j e_j$, $x^k \in L(k)$, $L(k) := sp_{\mathbb{R}}(e_1, \ldots, e_k)$, $sp_{\mathbb{R}}(e_j : j \in \mathbb{N}) := \{y : y \in l_2; y = \sum_{j=1}^n x^j e_j; x^j \in \mathbb{R}; n \in \mathbb{N}\}$.

Since $K_0$ is isomorphic with $l_2$, then each finite-dimensional subspace $L(k)$ is complemented in $K_0$ \[11\]. From the proof of Proposition II.3.1 \[10\] in view of the Fubini Theorem there exists a sequence of cylindrical functions $f_k(x) = f_k(x^k) = \int_{K_0 \cap L(k)} f(P_kx + (I - P_k)y)\nu_{l-P_k}(dy)$ which converges to $f$ in $L^1(K_0, \nu, C)$, where $\nu = \nu_{L(k)} \otimes \nu_{l-P_k}$, $\nu_{l-P_k}$ is the measure on $K_0 \cap L(k)$.

Each cylindrical function $f_k$ is $\nu$-almost everywhere constant on $K_0$, since $L(k) \subset X_0$ for each $k \in \mathbb{N}$, consequently, $f$ is $\nu$-almost everywhere constant on $K_0$. Let $A := \Psi_I : U_0 \to V_0$ be the same as in \S 2.11. From the construction of $G'$ and $\mu$ with the help of the local diffeomorphism $A$ and $\nu$ it follows that, if a function $f \in L^1(G, \mu, C)$ satisfies the following condition $f^h(g) = f(g)$ (mod $\mu$) by $g \in G$ for each $h \in G'$, then $f(x) = \text{const (mod $\mu$)}$, where $f^h(g) := f(hg), g \in G$.

Let $f(g) = Ch_{l'}(g)$ be the characteristic function of a subset $U, U \subset G, U \in Af(G, \mu)$, then $f(hg) = 1 \Leftrightarrow g \in h^{-1}U$. If $f^h(g) = f(g)$ is true by $g \in G$ $\mu$-almost everywhere, then $\mu\{(g \in G : f^h(g) \neq f(g))\} = 0$, that is $\mu((h^{-1}U) \Delta U) = 0$, consequently, the measure $\mu$ satisfies the condition $(P)$ from \S VIII.19.5 \[14\], where $A \Delta B := (A \setminus B) \cup (B \setminus A)$ for each $A, B \subset G$.

For each subset $E \subset G$ the outer measure is bounded, $\mu^*(E) \leq 1$, since $\mu(G) = 1$ and $\mu$ is non-negative \[3\], consequently, there exists $F \in Bf(G)$ such that $F \supset E$ and $\mu(F) = \mu^*(E)$. This $F$ may be interpreted as the least upper bound in $Bf(G)$ relative to the latter equality. In view of the Proposition VIII.19.5 \[14\] the measure $\mu$ is ergodic, that is for each $U \in Af(G, \mu)$ and $F \in Af(G, \mu)$ with $\mu(U) \times \mu(F) \neq 0$ there exists $h \in G'$ such that $\mu((h \circ E) \cap F) \neq 0$.

From Theorem I.1.2 \[10\] it follows that $(G, Bf(G))$ is a Radon space, since $G$ is separable and complete. Therefore, a class of compact subsets approximates from below each measure $\mu^f$, $\mu^f(dg) := |f(g)|\mu(dg)$, where $f \in L^2(G, \mu, C) =: H$. Due to the Egorov Theorem II.1.11 \[14\] for each $\epsilon > 0$ and for each sequence $f_n(g)$ converging to $f(g)$ for $\mu$-almost every $g \in G$, when $n \to \infty$, there exists a compact subset $K$ in $G$ such that $\mu(G \setminus K) < \epsilon$ and $f_n(g)$ converges on $K$ uniformly by $g \in K$, when $n \to \infty$. In each Hilbert space $L^2(\mathbb{R}^n, \lambda, \mathbb{R})$ the linear span of functions $f(x) = \text{exp}[(b, x) - (ax, x)]$ is dense, where $b$ and $x \in \mathbb{R}^n$, $a$ is a real symmetric positive definite $n \times n$ matrix, $(\ast, \ast)$ is the standard scalar product in $\mathbb{R}^n$ and $\lambda$ is the Lebesgue measure on $\mathbb{R}^n$. If
a non-linear operator $U$ on $K_0$ satisfies conditions of Theorem 26.1 [14], then
\[ \nu^U(dx)/\nu(dx) = |det(U(U^{-1}(x))\nu(x-U^{-1}(x)), x), \]
where $\nu^U(B) := \nu(U^{-1}B)$ for each $B \in Bf(K_0)$, $\rho_{\nu}(z, x) = \exp\{\sum_{i=1}^{\infty}[2(z, e_i)(x, e_i) - (z, e_i)^2]/\lambda_i\}$ by
Theorem 26.2 [14], where $\lambda_i$ and $e_i$ are eigenvalues and eigenfunctions of the
correlation operator $\hat{\rho}_1 : K_0 \to X'$ enumerated by $l \in N, z \in X_0, \rho_{\nu}(z, x) :=
\nu_z(dx)/\nu(dx), \nu_z(B) := \nu(B - z)$ for each $B \in Bf(K_0)$. Hence in view of
the Stone-Weierstrass Theorem A.8 [14] an algebra $V(Q)$ of finite pointwise
products of functions from the following space $sp_C(\psi(g) := (\rho(h, g))^{1/2} : h \in
G') =: Q$ is dense in $H$, since $\rho_{\mu}(e, g) = 1$ for each $g \in G$ and $L_h : G \to G$
are diffeomorphisms of the manifold $G, L_h(g) = hg$.

For each $m \in N$ there are $C^\infty$-curves $\phi_j^m \in G' \cap W$, where $j = 1, \ldots, m$
and $b \in (-2, 2) := \{a : -2 < a < 2; a \in R\}$ is a parameter, such that
$\phi_j^m|_{b=0} = e$ and $\phi_j := \phi_j^1$ and vectors $(\partial \phi_j^m/\partial b)|_{b=0}$ for $j = 1, \ldots, m$ are linearly
independent in $T_eG'$. Then the following condition $det(\Psi(g)) = 0$ defines a
submanifold $\Psi \Psi$ in $H$ of codimension over $R$.

(i) $\text{codim}_R G_{\Psi} \geq 1$, where $\Psi(g)$ is a matrix dependent from $g \in G$
with matrix elements $\Psi_{l,j}(g) := D_{\phi_j}(\rho(\phi_j,g))^{1/2}$. If $f \in H$ is such that
$(f(g), (\rho(\phi,g))^{1/2})_H = 0$ for each $\phi \in G' \cap W$, then differentials of these
scalars products by $\phi$ are zero. But $V(Q)$ is dense in $H$ and in view of condition (i) this means that $f = 0$, since for each $m$ there are $\phi_j \in G' \cap W$
such that $det(\Psi(g)) \neq 0 \mu$-almost everywhere on $G, g \in G$. If $\|f\|_H > 0$, then
$\mu(supp(f)) > 0$, consequently, $\mu(G'\text{supp}(f)) = 1$, since $G'U = G$ for each
open $U$ in $G$ and for each $\epsilon > 0$ there exists an open $U, U \supset \text{supp}(f)$, such
that $\mu(U \setminus \text{supp}(f)) < \epsilon$.

This means that the vector $f_0$ is cyclic, where $f_0 \in H$ and $f_0(\phi) = 1$
for each $\phi \in G$. From the construction of $\mu$ it follows that for each $f_{1,j}$ and $f_{2,j} \in H$
with $j = 1, \ldots, n, n \in N$ and each $\epsilon > 0$ there exists $h \in G'$ such that
$|(T_h f_{1,j}, f_{2,j})_H| \leq \epsilon|(f_{1,j}, f_{2,j})_H|$, when $|(f_{1,j}, f_{2,j})_H| > 0$, since $G$
is the Radon space by Theorem I.1.2 [14] and $G$ is not locally compact. This means that
there is not any finite-dimensional $G'$-invariant subspace $H'$ in $H$ such that
$T_h H' \subset H'$ for each $h \in G'$ and $H' \neq \{0\}$. Hence if there is a $G'$-invariant
closed subspace $H' \neq 0$ in $H$ it is isomorphic with the subspace $L^2(V, \mu, C)$,
where $V \in Bf(G)$ with $\mu(V) > 0$.

Let $A_G$ denotes a $*$-subalgebra of $L(H)$ generated by the family of unitary
operators $\{T_h : h \in G'\}$. In view of the von Neumann double commutator
Theorem (see §VI.24.2 [14]) $A_G$ coincides with the weak and strong operator
closures of $A_G$ in $L(H)$, where $A_G'$ denotes the commuting algebra of $A_G$ and $A_G'' = (A_G')'$.

We suppose that $\lambda$ is a probability Radon measure on $G'$ such that $\lambda$ has not any atoms and $\text{supp}(\lambda) = G'$. In view of the strong continuity of the regular representation there exists the S. Bochner integral $\int c_{\text{cyclic}} f(g) \mu(dg)$ for each $f \in H$, which implies its existence in the weak (B. Pettis) sense. The measures $\mu$ and $\lambda$ are non-negative and bounded, hence $H \subset L^1(G, \mu, C)$ and $L^2(G', \lambda, C) \subset L^1(G', \lambda, C)$ due to the Cauchy inequality. Therefore, we can apply below the Fubini Theorem (see §II.16.3). Let $f \in H$, then there exists a countable orthonormal base $\{f_j : j \in N\}$ in $H \otimes C_f$. Then for each $n \in N$ the following set $B_n := \{q \in L^2(G', \lambda, C) : (f_j, f)_H = \int_{G'} q(h)(f_j, T_h f_0)_H \lambda(dh) \text{ for } j = 0, \ldots, n\}$ is non-empty, since the vector $f_0$ is cyclic, where $f_0 := 1$. There exists $\infty > R > \|f\|_H$ such that $B_n \cap B^R =: B_n^R$ is non-empty and weakly compact for each $n \in N$, since $B^R$ is weakly compact, where $B^R := \{q \in L^2(G', \lambda, C) : \|q\| \leq R\}$ (see the Alaoglu-Bourbaki Theorem in §(9.3.3)). Therefore, $B_n^R$ is a centered system of closed subsets of $B^R$, that is, $\cap_{n=1}^m B_n^R \neq \emptyset$ for each $m \in N$, hence it has a non-empty intersection, consequently, there exists $q \in L^2(G', \lambda, C)$ such that

$$(ii) \quad f(g) = \int_{G'} q(h) T_h f_0(g) \lambda(dh)$$

for $\mu$-a.e. $g \in G$. If $F \in L^\infty(G, \mu, C)$, $f_1$ and $f_2 \in H$, then there exist $q_1$ and $q_2 \in L^2(G', \lambda, C)$ satisfying Equation $(ii)$. Therefore,

$$(iii) \quad (f_1, F f_2)_H =: c = \int_G \int_G \int_{G'} \int_{G'} \bar{q}_1(h_1) q_2(h_2) \rho^{1/2}(h_1, g) \rho^{1/2}(h_2, g) F(g) \lambda(dh_1) \lambda(dh_2) \mu(dg).$$

Let $\xi(h) := \int_G \int_{G'} \bar{q}_1(h_1) q_2(h_2) \rho^{1/2}(h_1, g) \rho^{1/2}(hh_2, g) \lambda(dh_1) \lambda(dh_2) \mu(dg)$. Then there exists $\beta(h) \in L^2(G', \lambda, C)$ such that

$$(iv) \quad f \xi(h) = (\beta(h) \xi(h) \lambda(dh) = c.$$ To prove this we consider two cases. If $c = 0$ it is sufficient to take $\beta$ orthogonal to $\xi$ in $L^2(G', \lambda, C)$. Each function $q \in L^2(G', \lambda, C)$ can be written as $q = q^1 - q^2 + iq^3 - iq^4$, where $q^j(h) \geq 0$ for each $h \in G'$ and $j = 1, \ldots, 4$, hence we obtain the corresponding decomposition for $\xi$, $\xi = \sum_{j,k} b_{j,k} \xi_{j,k}^{i,j,k}$, where $\xi_{j,k}^{i,j,k}$ corresponds to $q^1$ and $q^2$, where $b_{j,k} \in \{1, -1, i, -i\}$. If $c \neq 0$ we can choose $(j_0, k_0)$ for which $\xi_{j_0,k_0} \neq 0$ and

$$(v) \quad \beta \text{ is orthogonal to others } \xi_{j,k} \text{ with } (j, k) \neq (j_0, k_0).$$

Otherwise, if $\xi_{j,k} = 0$ for each $(j, k)$, then $q^j(h) = 0$ for each $(l, j)$ and $\lambda$-a.e.
$h \in G'$, since $\xi(0) = \int_G \mu(dx)(\int_G \tilde{q}_1(h_1)\rho_\mu^{1/2}(h_1, g)\lambda(dh_1))\int_G \tilde{q}_2(h_2)\rho_\mu^{1/2}(h_2, g)\lambda(dh_2) = 0$ and this implies $c = 0$, which is the contradiction with the assumption $c \neq 0$. Hence there exists $\beta$ satisfying conditions (iv, v).

Let $a(x) \in L^\infty(G, \mu, \mathbf{C})$, $f$ and $g \in H$, $\beta(h) \in L^2(G', \lambda, \mathbf{C})$. Since $L^2(G', \lambda, \mathbf{C})$ is infinite-dimensional, then for each finite family of $a \in \{a_1, ..., a_m\} \subset L^\infty(G, \mu, \mathbf{C})$, $f \in \{f_1, ..., f_m\} \subset H$ there exists $\beta(h) \in L^2(G', \lambda, \mathbf{C})$, $h \in G'$, such that $\beta$ is orthogonal to $\int_G \tilde{f}_s(g)(\int_f (\rho_\mu(h, g))^{1/2} - f_j(g))\mu(dx)$ for each $s, j = 1, ..., m$. Hence each operator of multiplication on $a_j(g)$ belongs to $A_{G''}$, since due to Formula (iv) and Condition (v) there exists $\beta(h)$ such that $(f_s, a_j f_i) = \int_G \int_G f_s(g)\beta(h)(\rho_\mu(h, g))^{1/2} f_i(h^{-1}g)\lambda(dh)\mu(dx)$, $\int_G \tilde{f}_s(g) a_j(f_i)\mu(dx) = \int_G \int_G \tilde{f}_s(g)\beta(h)(T_h f_i(g))\lambda(dh)\mu(dx)$, $\int_G \tilde{f}_s(g) a_j f_i \mu(dx) = \int_G \int_G \tilde{f}_s(g)\beta(h) f_i (g)\lambda(dh)\mu(dx) = (f_s, a_j f_i)$. Hence $A_{G''}$ contains subalgebra of all operators of multiplication on functions from $L^\infty(G, \mu, \mathbf{C})$.

Let us remind the following. A Banach bundle $B$ over a Hausdorff space $G'$ is a bundle $< B, \pi >$ over $G'$, together with operations and norms making each fiber $B_h$ $(h \in G')$ into a Banach space such that conditions $BB(i)$ is satisfied: $BB(i) x \mapsto \|x\|$ is continuous on $B$ to $\mathbf{R}$; $BB(ii)$ the operation + is continuous as a function on $(\{ (x, y) \in B \times B : \pi(x) = \pi(y) \})$ to $B$; $BB(iii)$ for each $\lambda \in \mathbf{C}$, the map $x \mapsto \lambda x$ is continuous on $B$ to $B$; $BB(iv)$ if $h \in G'$ and $\{ x^i \}$ is any net of elements of $B$ such that $\|x^i\| \to 0$ and $\pi(x^i) \to h$ in $G'$, then $x^i \to 0$ in $B$, where $\pi : B \to G'$ is a bundle projection, $B_h := \pi^{-1}(h)$ is the fiber over $h$ (see §II.13.4 [14]). If $G'$ is a Hausdorff topological group, then a Banach algebraic bundle over $G'$ is a Banach bundle $B = < B, \pi >$ over $G'$ together with a binary operation $\bullet$ on $B$ satisfying conditions $AB(i - v)$: $AB(i) \quad \pi(b \bullet c) = \pi(b)\pi(c)$ for $b$ and $c \in B$; $AB(ii)$ for each $x$ and $y \in G'$ the product $\bullet$ is bilinear on $B_x \times B_y$ to $B_{xy}$; $AB(iii)$ the product $\bullet$ on $B$ is associative; $AB(iv) \quad \|b \bullet c\| \leq \|b\| \times \|c\|$ $(b, c \in B)$; $AB(v)$ the map $\bullet$ is continuous on $B \times B$ to $B$ (see §VIII.2.2 [4]). With $G'$ and a Banach algebra $A$ the trivial Banach bundle $B = A \times G'$ is associative, in particular let $A = \mathbf{C}$ (see §VIII.2.7 [4]).

The regular representation $T$ of $G'$ gives rise to a canonical regular $H$-projection-valued measure $\mathcal{P}$: $\mathcal{P}(W)f = Ch_W f$, where $f \in H$, $W \in Bf(G)$, $Ch_W$ is the characteristic function of $W$. Therefore, $T_h \mathcal{P}(W) = \mathcal{P}(h \circ W) T_h$ for each $h \in G'$ and $W \in Bf(G)$, since $\rho_\mu(h, h^{-1} \circ g)\rho_\mu(h, g) = 1 = \rho_\mu(e, g)$ for each $(h, g) \in G' \times G$, $Ch_W (h^{-1} \circ g) = Ch_{h \circ W} (g)$ and $T_h (\mathcal{P}(W) f(g)) = \mathcal{P}(h \circ W) f(h^{-1} \circ g)$. Thus $< T, \mathcal{P} >$ is a system of imprimitivity for $G'$ over $G$, which is denoted $\mathbb{T}^\mu$. This means that conditions $SI(i - iii)$
are satisfied: \(SI(i)\) \(T\) is a unitary representation of \(G'\); \(SI(ii)\) \(P\) is a regular \(H\)-projection-valued Borel measure on \(G\) and \(SI(iii)\) \(T_h\bar{P}(W) = \bar{P}(h \circ W)T_h\) for all \(h \in G'\) and \(W \in \mathcal{B}(G)\).

For each \(F \in \mathcal{L}^\infty(G, \mu, C)\) let \(\bar{\alpha}_F\) be the operator in \(\mathcal{L}(H)\) consisting of multiplication by \(F\): \(\bar{\alpha}_F(f) = Ff\) for each \(f \in H\). The map \(F \to \bar{\alpha}_F\) is an isometric \(*\)-isomorphism of \(\mathcal{L}^\infty(G, \mu, C)\) into \(\mathcal{L}(H)\) (see §VIII.19.2[14]). Therefore, Propositions VIII.19.2,5[14] (using the approach of this particular case given above) are applicable in our situation.

If \(\bar{p}\) is a projection onto a closed \(T^\mu\)-stable subspace of \(H\), then \(\bar{p}\) commutes with all \(\bar{P}(W)\). Hence \(\bar{p}\) commutes with multiplication by all \(F \in \mathcal{L}^\infty(G, \mu, C)\), so by §VIII.19.2[14] \(\bar{p} = \bar{P}(V)\), where \(V \in \mathcal{B}(G)\). Also \(\bar{p}\) commutes with all \(T_h\), \(h \in G'\), consequently, \((h \circ V) \setminus V\) and \((h^{-1} \circ V) \setminus V\) are \(\mu\)-null for each \(h \in G'\), hence \(\mu((h \circ V) \Delta V) = 0\) for all \(h \in G'\). In view of ergodicity of \(\mu\) and Proposition VIII.19.5[14] either \(\mu(V) = 0\) or \(\mu(G \setminus V) = 0\), hence either \(\bar{p} = 0\) or \(\bar{p} = I\), where \(I\) is the unit operator. Hence \(T\) is the irreducible unitary representation.

3.4. **Theorem.** There exists a bounded intertwining operator \(V : \mathcal{L}^2(G, \mu, C) \to \mathcal{L}^2(G, \mu', C)\) such that \(VT^\mu(\psi) = T^\mu(\psi)V\) for each \(\psi \in G'\) if and only if \(\mu\) and \(\mu'\) are equivalent, where \(\mu\) and \(\mu'\) are quasi-invariant measures on \(G\) relative to \(G'\) and \(T^\mu\) is the associated regular representation of \(G'\) from Theorems 3.1 and 3.3.

**Proof.** If \(\mu\) is equivalent with \(\mu'\), then \(\mu(dg)/\mu'(dg) := \phi(g)\) is \(\mu\)-a.e positive, which produces an intertwining operator \(V\), which is an isomorphism \(V : \mathcal{L}^2(G, \mu, C) \to \mathcal{L}^2(G, \mu', C)\) given by the following formula: \(f(g) \mapsto f(g)\phi(g)\).

It remains to verify the reverse implication. In view of Theorem 3.3 representations \(T^\mu\) are irreducible. It was proved in §3.3 that

(i) the weak closure of subalgebra generated by the family \(\{T^\mu(h) : h \in G'\}\) in the algebra of bounded linear operators \(\mathcal{L}(H)\) contains all operators of multiplication on functions from the space \(\mathcal{L}^\infty(G, \mu, C)\), where \(H := \mathcal{L}^2(G, \mu, C)\). If measures \(\mu\) and \(\mu'\) are singular, then

(ii) either \(\sup_{g \in G} |\mu'(dg)/\mu(dg)| = \infty\) or \(\sup_{g \in G} |\mu(dg)/\mu'(dg)| = \infty\), where \(\mu'(dg)/\mu(dg) := \lim_{\mu(B) \to 0, \|g\|_B \to 0} \mu'(B)/\mu(B) \in [0, \infty], [0, \infty] := ([0, \infty) \cup \{\infty\}), [0, \infty) := \{x : x \in \mathbb{R}, 0 \leq x\}, B \in \mathcal{B}(G)\). In view of the existence of the intertwining operator \(V\) of \(T^\mu\) with \(T^\mu\) there exists an isomorphism of Hilbert spaces \(\tau : \mathcal{L}^2(G, \mu, C) \to \mathcal{L}^2(G, \mu', C)\), which has a continuous extension to an isomorphism of Banach spaces \(\tau : \mathcal{L}^\infty(G, \mu, C) \to \mathcal{L}^\infty(G, \mu', C)\).
for each \(j\) cylinder subalgebra induced by the projection of \(T\) differ by their supports. 

\[ \forall \text{ multilinear functionals } \sigma \] 

**3.5. Note.** It follows from \([11]\), that on \(K_0\) there is a family \(P\) of orthogonal Gaussian measures of cardinality \(\text{card}(P) = \text{card}(R) =: c\), which induce quasi-invariant measures on \(G\) relative to \(G'\) and have continuous quasi-invariance factor on \(G' \times G\). Therefore, there are \(c\) non-equivalent unitary representations \(T^\mu\) of \(G'\) in \(L^2(G, \mu, C)\) due to Theorems 3.3 and 3.4.

**3.6. Theorem.** On the loop group \(G = (L^M N)_\xi\) and \(G = (L^1 N)_\xi\) from \(\S 2.1.4\) and \(\S 2.8\) there exists a family of continuous characters \(\{\Xi\}\), which separate points of \(G\).

**Proof:** Since \(N\) is either finite-dimensional or the separable Hilbert manifold, then \(N\) has a countable locally finite covering subordinated to the covering of \(N\) induced by the exponential mapping \(\exp : TN \to N\) from a neighbourhood \(\tilde{T}N\) of \(N\) in \(TN\) such that \(\exp_y : V_y \to W_y\) are local diffeomorphisms of the corresponding neighbourhoods \(V_y\) and \(W_y\) of the zero section in \(T_y N\) and of \(y \in N\). Let \(\lambda\) be equivalent with a Gaussian probability \(\sigma\)-additive measure either on the entire \(T_y N\) or on its Hilbert subspace \(P\). Each such \(\lambda\) induces a family of probability measures \(\nu\) on \(Bf(N)\) or its cylinder subalgebra induced by the projection of \(T_y N\) onto \(P\), which may differ by their supports.

Let \(T_y N_R := L\) be an infinite-dimensional separable Banach space over \(R\), so there exists a topological vector space \(L^N := \prod_{j=1}^{\infty} L_j\), where \(L_j = L\) for each \(j \in N\) \([14]\). Consider a subspace \(\Lambda^\infty\) of a space of continuous \(\infty\)-multilinear functionals \(w : L^N \to R\) such that \(w(x + y) = w(x) + w(y)\), \(w(\sigma x) = (-1)^{||\sigma||} w(x), w(x) = \lambda w(z)\) for each \(x, y \in L^N\), \(\sigma \in S_\infty\) and \(\lambda \in R\), where \(x = \{x^j : x^j \in L, j \in N\} \in L^N\), \(z^j = x^j\) for each \(j \neq k_0\) and \(\lambda z^{k_0} = x^{k_0}\). \(S_\infty\) is a group of all bijections \(\sigma : N \to N\) such that \(\text{card}\{j : \sigma(j) \neq j\} < \aleph_0, ||\sigma|| = 1\) for \(\sigma = \sigma_1...\sigma_n\) with odd \(n \in N\) and pairwise transpositions \(\sigma_t = I\), that is \(\sigma_t(j_1) = j_2, \sigma_t(j_2) = j_1\) and \(\sigma_t[N_{\{j_1, j_2\}}] = I\) for the corresponding \(j_1 \neq j_2, ||\sigma|| = 2\) for even \(n\) or \(\sigma = I\). Then \(\Lambda^\infty\) (or \(\Lambda^j\)) induces a vector bundle \(\Lambda^\infty N_R\) (or \(\Lambda^j N_R\)) on a manifold \(N_R\) of \(\infty\)-multilinear
There exists its pull back \( f \) respectively) as the additive group (see \([20]\)). Therefore, \( \Xi(\bigoplus_k \mathcal{E} \mathcal{R}_\mathcal{R}) \) is the vector bundle of differential \( \infty \)-forms on \( \mathcal{R}_\mathcal{R} \). Then there exist a subfamily \( \Lambda_N^\infty \mathcal{R}_\mathcal{R} \) of differential forms \( w \) on \( \mathcal{N} \) induced by the family \( \{\nu\} \).

Let \( \overline{B}^\infty \mathcal{N} := (\bigoplus_{0 \leq j \in \mathbb{Z}} \Lambda^j \mathcal{E} \mathcal{R}_\mathcal{R}) \oplus \Lambda^\infty_N \mathcal{R}_\mathcal{R} \) for \( \dim \mathcal{R}_\mathcal{R} = \infty \) and \( \overline{B}^k \mathcal{N} = \bigoplus_{j=0}^k \Lambda^j \mathcal{E} \mathcal{R}_\mathcal{R} \) for each \( k \in \mathbb{N} \). We choose \( w \in \overline{B}^k \mathcal{N} \), where \( k = \min(\dim \mathcal{R}_\mathcal{R}, \dim \mathcal{R}_\mathcal{M}_\mathcal{R}) \).

There exists its pull back \( f^*w \) for each \( f \in \mathcal{Y}^\xi(\mathcal{M}, \mathcal{N}) \) (see \([23]\) and \(\S 2.11\)). Let \( \overline{E}_j : \mathcal{S}_j \to \mathcal{P} \) be a family of continuous linear operators from Banach spaces \( \mathcal{S}_j \) into a Banach space \( \mathcal{P} \), then there exists a continuous linear operator \( E : l_{2,d'}(\{S_j : j \in \mathbb{N}\}) \to \mathcal{P} \) such that \( E\mathcal{x} = \sum_{j=1}^{\infty} \overline{E}_j x^j \), where \( \mathcal{x} = \{x^j : x^j \in \mathcal{S}_j, j \in \mathbb{N}\} \in l_{2,d'}(\{S_j : j \in \mathbb{N}\}) \).

Moreover, to \( f^*w \) a measure \( \mu_{w,f} \) on \( \mathcal{M} \) corresponds. Then \( F_w(f) := \int_{\mathcal{M}} f^*w = \int_{\mathcal{M}} (f \circ \psi)^*w \) for each \( f \in \mathcal{Y}^\xi(\mathcal{M}, s_0; \mathcal{N}, y_0) \) and \( \psi \in \mathcal{D}iff_{s_0}^\infty(\mathcal{M}_\mathcal{R}, \mathcal{M}_\mathcal{R}) \) due to \(\S 26\) \([24]\) and \([1]\). Therefore, \( F_w \) is continuous and constant on each class \( f > \xi \) due to \(\S 2.9 \) and \(\S 2.11\). If \( f^*w = 0 \) for each \( w \) as above, then \( \mathcal{D}f = 0 \). In view of \( f(s_0) = y_0 \) this implies that \( f(\mathcal{M}) = \{y_0\} \).

Hence for each \( f > \xi \neq e \) there exists \( w \) such that \( F_w(f) \neq 0 \).

Let \( \Xi : \mathcal{C} \to \mathcal{S}^1 \) be a continuous character of \( \mathcal{C} \) (or \( \Xi : \mathcal{R} \to \mathcal{S}^1 \) respectively) as the additive group (see \([20]\)). Therefore, \( \Xi(g) := \Xi(F_w(f)) \) is a continuous character on \( \mathcal{G} \) for each \( g := < f > \xi \in \mathcal{G} \).

**References**

[1] R. Abraham, J.E. Marsden, T. Ratiu. ”Manifolds, Tensor Analysis and Applications” (Addison-Wesley, London, 1983).

[2] K. Adachi, H.R. Cho. ”\( H^p \) and \( L^p \) Extensions of Holomorphic Functions from Subvarieties to Certain Convex Domains”, Math. J. Toyama Univ., 20 (1997), 1-13.

[3] V.I. Averbukh, O.G. Smolyanov. ”The Theory of Differentiation in Linear Topological Spaces”, Usp. Mat. Nauk. 22 (1967), 201-260 (N 6).
[4] W. Banaszczyk. "Additive Subgroups of Topological Vector Spaces" (Springer, Berlin, 1991).

[5] S. Bochner, D. Montgomery. "Groups on Analytic Manifolds", Annals of Mathem. 48 (1947), 659-668.

[6] N. Bourbaki. "Integration", Ch. 1-9 (Nauka, Moscow, 1970 and 1977).

[7] J.L. Brylinski. "Loop Spaces, Characteristic Classes and Geometric Quantisation." Progr. in Math. V. 107 (Birkhäuser, Boston, 1993).

[8] J. Chaumat, A.-M. Chollet. "Classes de Gevrey non Isotropes et Application à Interpolation", Annali della Scuola Norm. Sup. di Pisa, Ser. IV, 15 (1988), 615-676.

[9] K.T. Chen. "Iterated Integrals of Differential Forms and Loop Space Homology", Ann. Math. Ser. 2, 97(1973), 217-246.

[10] Yu.L. Dalecky, S.V. Fomin. "Measures and Differential Equations in Infinite-Dimensional Space" (Kluwer, Dordrecht, 1991).

[11] D.G. Ebin, J. Marsden. "Groups of Diffeomorphisms and the Motion of Incompressible Fluid", Ann. of Math. 92(1970),102-163.

[12] H.I. Eliasson. "Geometry of Manifolds of Maps", J.Difer. Geom. 1(1967),169-194.

[13] R. Engelking. "General Topology" (Mir, Moscow, 1986).

[14] J.M.G. Fell, R.S. Doran. "Representations of ∗-Algebras, Locally Compact Groups, and Banach ∗-Algebraic Bundles" V. 1 and V. 2. (Acad. Press, Boston, 1988).

[15] P. Flaschel, W. Klingenberg. "Riemannsche Hilbermmannigfaltigkeiten. Periodische Geodäische" (Springer, Berlin, 1972).

[16] E. Getzler, J.D.S. Jones, S. Petrack. "Differential Forms on Loop Spaces and the Cyclic Bar Complex", Topology 30 (1991), 339-371.

[17] I.M. Gelfand, N.Ya. Vilenkin. "Generalised Functions. V. 4. Some Applications of Harmonic Analysis. Rigged Hilbert Spaces" (Nauka, Moscow, 1961).
[18] M. Heins. "Selected Topics in the Classical Theory of Functions of a
Complex Variable" (Holt, Rinehard and Winston, New York, 1962).

[19] G. Henkin, J. Leiterer. "Theory of Functions on Complex Manifolds"
(Birkhauser, Basel, 1984).

[20] E. Hewitt, K.A. Ross. "Abstract Harmonic Analysis" (Springer, Berlin,
1979).

[21] P.-C. Hu, C.-C. Yang, "Applications of Volume Forms to Holomorphic
Mappings", Complex Variab. 30 (1996), 153-168.

[22] C. Hummel. "Gromov’s Compactness Theorem for Pseudo-Holomorphic
Curves", Ser. Progr. in Math. v. 151 (Bikhäuser, Basel, 1997).

[23] W. Klingenberg. "Riemannian Geometry" (Walter de Gruyter, Berlin,
1982).

[24] S. Kobayashi, K. Nomizu. "Foundations of Differential Geometry" v. 1
and v. 2 (Interscience, New York, 1963).

[25] S. Lang. "Differential Manifolds" (Springer, Berlin, 1985).

[26] S. Lang. "Algebra" (Addison-Wesley, New York, 1965).

[27] J.J. Loeb, M. Nicolau, "On the Complex Geometry of a Class of non-
Kählerian Manifolds", Isr. J. Math. 110 (1999), 371-379.

[28] J.J. Loeb, M. Nicolau, "Holomorphic Flows and Complex Structures on
Products of Odd-Dimensional Spheres", Mathem. Annalen, 306 (1996),
781-817.

[29] S.V. Ludkovsky. "Measures on Groups of Diffeomorphisms of non-
Archimedean Banach Manifolds", Usp. Mat. Nauk. 51 (1996), 169-170
(N 2).

[30] S.V. Ludkovsky. "Quasi-Invariant Measures on non-Archimedean Loop
Semigroups", Usp. Mat. Nauk. 53 (1998), 203-204 (N 3).

[31] S.V. Ludkovsky. "Measures on Groups of Diffeomorphisms of non-
Archimedean Manifolds, Representations of Groups and their Applications",
Theor. and Math. Phys., 119 (1999), 381-396.
[32] S.V. Ludkovsky, "Gaussian Quasi-Invariant Measures on Loop Groups and Semigroups for Real Manifolds and their Representations”. Institut des Hautes Études Scientifiques (Bures-sur-Yvette, France) Preprint, IHES/M/97/95, Décembre, 1997, 32 pages.

[33] S.V. Ludkovsky, Dokl. Acad. Nauk. (Russ.), "Quasi-Invariant Measures on Loop Groups of Riemann Manifolds", is accepted to publication.

[34] S.V. Ludkovsky, "Quasi-Invariant Measures on a Group of Diffeomorphisms of an Infinite-Dimensional Real Manifold and Induced Irreducible Unitary Representations", Rendiconti dell’Istituto di Matematica dell’Università di Trieste. Nuova Serie, 31, 1999.

[35] S. L. de Medrano, A. Verjovsky. "A New Family of Complex, Compact, non-Symplectic Manifolds”, Bolet. da Sociede Brasil. de Math. 28 (1997), 1-20.

[36] M.B. Mensky. "The Paths Group. Measurement. Fields. Particles” (Nauka, Moscow, 1983).

[37] P.W. Michor. "Manifolds of Differentiable Mappings” (Shiva, Boston, 1980).

[38] G. Moretti. "Functions of a Complex Variable” (Prentice Hall, Inc., Englewood Cliffs, N.J., 1964).

[39] A.S. Mshimba, "On the Riemann Boundary Value Problem for Holomorphic Functions in the Sobolev Space $W_{1,p}(D)$”, Complex Variab. 14 (1990), 237-242.

[40] A.S. Mshimba, "The Riemann Boundary Value Problem for Nonlinear Systems in Sobolev Space $W_{1,p}(D)$”, Complex Variab. 14 (1990), 243-249.

[41] L. Narici, E. Beckenstein. "Topological Vector Spaces” (Marcel Dekker Inc., New York, 1985).

[42] G. Sansone, J. Gerretsen. "Lectures on the Theory of Functions of a Complex Variable” v. II (Wolters Noordhoff, Groningen, 1969).
[43] R.T. Seeley. "Extensions of $C^\infty$ Functions Defined in a Half Space", Proceed. Amer. Math. Soc. 15 (1964), 625-626.

[44] A.V. Skorohod. "Integration in Hilbert space" (Springer, Berlin, 1974).

[45] R.C. Swan. "The Grothendieck Ring of a Finite Group", Topology 2 (1963), 85-110.

[46] R.M. Switzer. "Algebraic Topology - Homotopy and Homology" (Springer, Berlin, 1975).

[47] J.C. Tougeron. "Ideaux de Fonctions Differentiables" (Springer, Berlin, 1972).