ON THE IRREDUCIBILITY OF SECANT CONES,
AND AN APPLICATION TO LINEAR NORMALITY

ANGELO LOPEZ AND ZIV RAN

Abstract. Given a smooth subvariety of dimension $\frac{2}{3} (r - 1)$ in $\mathbb{P}^r$, we show that the double locus (upstairs) of its generic projection to $\mathbb{P}^{r-1}$ is irreducible. This implies a version of Zak’s Linear Normality theorem.

A classical, and recently revisited (cf. [GP, L, Pi] and references therein), method for studying the geometry of a subvariety $Y$ in $\mathbb{P}^r$ is to project $Y$ generically to a lower-dimensional projective space, for example so that $Y$ maps birationally to a (singular) hypersurface $\bar{Y} \subset \mathbb{P}^{m+1}$. To make use of this method, it is usually important to have precise control over the singularities of $\bar{Y}$ and in particular over the entire singular (=double) locus $D_Y$ of $\bar{Y}$ and its inverse image $C_Y$ in $Y$. As the dimension of these is easily determined, a natural question is: are $C_Y$ and $D_Y$ irreducible? This question plays an important role, for instance, in Pinkham’s work on regularity bounds for surfaces [Pi]. The purpose of this note is to show that this irreducibility holds provided the codimension of $Y$ is sufficiently small compared to its dimension (see Theorems 1,2 and Corollary 3 below). As an application we give a proof of Zak’s linear normality theorem (in a slightly restricted range, see Corollary 4 below). Indeed the results seem closely related as our argument ultimately depends on having a bound on the dimension of singular loci of hyperplane sections, manifested in the form of the integer $\sigma(Y)$ (see Thm. 1 below), and it is Zak’s theorem on tangencies—also a principal ingredient in other proofs of linear normality—that gives us good control over $\sigma(Y)$.

We begin with some definitions. Let $Y$ denote an irreducible $m-$ dimensional subvariety of $\mathbb{P}^r$. As usual, we mean by a secant line of $Y$ a limit of lines in $\mathbb{P}^r$ spanned by pairs of distinct points of $Y$. The union of all secant lines is denoted by Sec$(Y)$. $Y$ is said to be projectable if Sec$(Y) \subset \mathbb{P}^r$. For any linear subspace $Q \subset \mathbb{P}^r$, we let

$$\pi_Q : \mathbb{P}^r - Q \to \mathbb{P}^{r-\dim Q - 1}$$

denote the associated projection.
For a nondegenerate projective variety $Y$, let $\sigma(Y)$ denote the maximum dimension of a subvariety $Z \subset Y_{\text{smooth}}$ such that

(i) $Z$ contains a generic point of some divisor on $Y$;

(ii) the tangent planes $T_yY$ for all $y \in Z$ are contained in a fixed hyperplane $H$ (i.e. $Z$ is contained in the singular locus of $Y \cap H$).

Note that if $Y$ is nonsingular in codimension 1 then assumption (i) above for a subvariety $Z \subset Y$ already implies that $Z \cap Y_{\text{smooth}}$ is dense in $Z$. Zak’s Tangency theorem (cf. [F, Z1]) implies that if $Y$ is smooth then

\begin{equation}
\sigma(Y) \leq r - m - 1.
\end{equation}

**Theorem 1.** Let $Y \subset \mathbb{P}^r$ be $m$-dimensional, normal, irreducible and non-projectable, and let $Q \subset \mathbb{P}^r$ be a generic (resp. arbitrary) linear subspace disjoint from $Y$. Assume that $\dim Q < r - m - 1$ and that

\begin{equation}
2m > r + \sigma(Y) - 1.
\end{equation}

Then the double locus of $\pi_Q|_Y$ (= locus of points $y \in Y$ such that $\pi_Q^{-1}(\pi_Q(y)) \cap Y \neq \{y\}$ as schemes) is irreducible (resp. connected).

Now the case where $\dim Q > 0$ (no other hypotheses needed but non-projectability) is an easy and well known consequence, due to Franchetta and Mumford, of Bertini’s Theorem (see [Mo], p.115 or [Pi] or below), so the only new conclusion is when $Q$ is a point and as usual, the case $Q$ arbitrary follows easily by connectedness principles from the case $Q$ generic. For this case, it is convenient to shift our viewpoint slightly, as follows.

Let $Y^{[2]}$ denote the normalization of the blow-up of $Y \times Y$ along the diagonal $\Delta_Y$, with exceptional divisor $E_Y$. Let $I_Y$ denote the tautological $\mathbb{P}^1$-bundle (or ’incidence variety’) over $Y^{[2]}$ (= pullback of analogous object over $(\mathbb{P}^r)^{[2]}$). Being a $\mathbb{P}^1$ bundle over a normal variety, $I_Y$ is also normal and we have a diagram

\begin{equation}
\begin{array}{ccc}
I_Y & \to & \mathbb{P}^r \\
\pi \downarrow & & \downarrow \\
Y^{[2]}, & & \\
\end{array}
\end{equation}

where non-projectability means $f$ is surjective. Now it follows easily from the classical trisecant Lemma (most secants are not multisecant) that for a generic linear subspace $Q$, $f^{-1}(Q)$ is birational to the double locus of $\pi_Q|_Y$. If $\dim(Q) > 0$, then $f^{-1}(Q)$ is automatically irreducible by Bertini’s theorem, which already proves Theorem 1 for this case. This result is due originally to Franchetta in the case of surfaces (cf. [Fr], [En]); the foregoing argument is due to Mumford and is given in [Mo].

Let us say that $Y$ has the **irreducible secant cone** (ISC) property if a generic fibre of $f$ is irreducible. Then the remaining case $\dim(Q) = 0$ of Theorem 1 follows from (indeed, is equivalent to) the following
Theorem 2. Hypotheses as in Theorem 1, \( Y \) has the ISC property.

In view of Zak’s Tangency theorem, Theorem 2 implies the following result (which will shortly be improved below):

Corollary 3(temp). If \( Y \subset \mathbb{P}^r \) is smooth, non-projectable with

\[
\dim(Y) > \frac{2}{3}(r - 1),
\]

then \( Y \) has the ISC property.

As another application, we obtain a proof of a version of Zak’s linear normality theorem (cf. [Z2], Thm. II.2.14):

Corollary 4. Let \( X \subset \mathbb{P}^N \) be irreducible, nondegenerate, and set \( b = 0 \) if \( X \) is smooth and otherwise \( b = \dim \text{Sing}(X) \). Assume that

\[
\dim(X) > \frac{1}{3}(2N + b - 1).
\]

Then \( X \) is linearly normal, i.e. not the image of the bijective projection of a nondegenerate subvariety of \( \mathbb{P}^{N+1} \).

Proof that Corollary 3(temp) \( \Rightarrow \) Corollary 4. We use induction on \( \dim(X) \). By generically projecting, we may assume \( X \) is nonprojectable. Assume for contradiction that \( X \) is the bijective projection of a nondegenerate variety \( \tilde{X} \subset \mathbb{P}^{N+1} \) from a point \( Q \in \mathbb{P}^{N+1} - \tilde{X} \). Let \( M \subset \mathbb{P}^N \) be a generic \( \mathbb{P}^{N-b-1} \) and \( Y = X \cap M \). Thus \( Y \) is smooth and spans \( M \). Assume to begin with that \( Y \) is nonprojectable within \( M \). By Corollary 3(temp), the secant variety \( \text{Sec}(Y) \) coincides with \( M \) with multiplicity 1, in the sense that, for a generic linear \( \mathbb{P}^{b+1}, L \subset \mathbb{P}^N \), the scheme-theoretic inverse image \( f^{-1}(L) \) is a reduced irreducible subvariety of codimension \( N - b - 1 \) lying over a single point of \( L \) (viz. \( L \cap M \)); i.e. \( f^{-1}(L) \) coincides with the fibre of \( I_Y \rightarrow \text{Sec}(Y) = M \) over (the general point) \( L \cap \text{Sec}(Y) \).

On the other hand, note that \( M \) is the projection of a unique codimension-\((b + 1)\) linear subspace of \( \mathbb{P}^{N+1} \) containing \( Q \), say \( A \), and \( Y \simeq \tilde{X} \cap A \). Consequently, \( Y \) can be viewed as a specialization of a smooth subvariety \( Y' \subset X \), which is the (isomorphic) projection of a generic codimension-\((b + 1)\) linear space section \( \tilde{X} \cap A' \subset \mathbb{P}^{N+1} \), with \( Q \not\in A' \). Note that \( Y' \) spans a \( \mathbb{P}^{N-b} \) which we denote by \( M' \). By semi-continuity, similar assertions as for \( f \) must hold also for the analogously-defined map \( f_{Y'} \); thus \( f_{Y'}^{-1}(L) \) is reduced,
irreducible and of codimension $N - b - 1 = \text{codim}(L)$. This implies firstly that $\text{Sec}(Y')$ is $(N - b - 1)$-dimensional; then since $f^{-1}_Y(L)$ has a component over each point of $L \cap \text{Sec}(Y')$, the only way $f^{-1}_Y(L)$ can be irreducible is if $\text{Sec}(Y')$ is a linear $\mathbb{P}^{N - b - 1}$, which contradicts the fact that $Y'$ spans $M'$ of dimension $N - b$.

Finally, suppose $Y$ is projectable within $M$ and let $Y', M'$ be as above. Then the nondegenerate subvariety

$$Y' \subset M' = \mathbb{P}^{N - b}$$

of dimension $\dim(X) - b - 1$ is projectable to $\mathbb{P}^{N - b - 2}$, which contradicts our induction hypothesis. □

Remarks. 1. For $X$ smooth, Zak’s linear normality theorem covers the larger range $\dim(X) > \frac{1}{3}(2N - 2)$.

2. The basic idea of the foregoing argument goes back to [R], and a similar idea was recently used by Brandigi [B] to prove linear normality in the range $\dim(X) \geq \frac{3}{4}N$. In fact, this argument proves the following fact of independent interest: if $X \subset \mathbb{P}^N$ irreducible, nondegenerate and its general hyperplane section is smooth, nonprojectable and has the ISC property, then $X$ is linearly normal (in the above sense).

3. Corollary 4 is sharp: to see this let $Z$ be a smooth Severi variety (cf. [Z2]) in $\mathbb{P}^r$, that is $Z$ is projectable and $\dim(Z) = \frac{2}{3}(r - 2)$. Set $N = r + b$ and let $Z' \subset \mathbb{P}^{N + 1}$ be the cone over $Z$ with vertex $\mathbb{P}^{b}$, $X \subset \mathbb{P}^N$ its generic (isomorphic) projection. Then $\dim(X) = \frac{1}{3}(2N + b - 1)$ and $\dim \text{Sing}(X) = b$.

Given Corollary 4, we can sharpen slightly the statement of Corollary 3(temp):

**Corollary 3.** Let $Y \subset \mathbb{P}^r$ be smooth nondegenerate with

$$\dim(Y) > \frac{2}{3}(r - 1).$$

Then $Y$ is non-projectable and has the ISC property.

**Proof.** By Corollary 3(temp), it suffices to prove that $Y$ is non-projectable. If not, apply Corollary 4 to the generic projection of $Y$ to $\mathbb{P}^{r-1}$ to deduce a contradiction. □

**Remark.** Again Corollary 3 is sharp: for this let $Y \subset \mathbb{P}^r$ be the generic projection of a Severi variety (cf. Remark 3 following Corollary 4). Then $Y$ is smooth, non-projectable and does not have the ISC property (e.g. because the cone on $Y$ is not linearly normal).

It is amusing, perhaps, to translate the irreducibility conclusion of Corollary 3 into cohomology (taking for granted the nonprojectability conclusion). Let $F$ denote a general fibre of $f$. Then $F$ is smooth, nonempty and
(2 \dim(Y) + 1 - r)\text{-dimensional. Clearly irreducibility (i.e. connectedness) of } F \text{ is equivalent, provided } q(Y) = h^1(O_Y) = 0, \text{ to the vanishing }

(4) \quad H^1(I_Y, I_F) = 0,

where } I_F \text{ denotes the ideal sheaf of } F; \text{ in the dimension range in question, } q(Y) = 0 \text{ automatically by the Barth-Larsen Theorem. Pulling back the Koszul resolution of the ideal sheaf of a point in } \mathbb{P}^r \text{ and using standard vanishing results (e.g. [SS], Thm. 7.1) which imply that }

H^i(I_Y, f^*(O(-j))) = 0, \forall i < r, j > 0,

we see easily that (4) is equivalent to the vanishing }

(5) \quad H^r(I_Y, f^*(O(-r))) = 0.

Let } E \text{ denote the tautological subbundle on } Y^{[2]}, \text{ so that } I_Y = \mathbb{P}(E). \text{ Then by standard computations the vanishing (5) reduces to the vanishing on } Y^{[2]}:

(6) \quad H^{r-1}(Y^{[2]}, \text{Sym}^{r-2}(E^v) \otimes \det(E^v)) = 0.

\textbf{Corollary 5.} \textit{With hypotheses as in Corollary 3, the vanishing (6) holds.}

Trying to find a direct proof of Corollary 5 might seem like a promising route to a cohomological proof of Corollary 3, but we were unable to find such a direct proof. This still looks like an intriguing, though difficult problem.

We now give the proof of Theorem 2, letting notations and assumptions be as there. The basic idea is the following. Consider a Stein factorization of } f:

I_Y \rightarrow Z \xrightarrow{g} \mathbb{P}^r,

where } Z \text{ is normal and } g \text{ is generically finite and surjective. Now it is a general fact that if } h : W \rightarrow T \text{ is a morphism of irreducible varieties and } W \text{ is normal, then so is a general fibre of } h: \text{ this can be seen, e.g. using Serre’s criterion, or alternatively, use [G], Thm 12.2.4, which says, in the scheme-theoretic context, that the set } N(h) \text{ of points } t \in T \text{ such that } h^{-1}(t) \text{ is normal is open; when } W \text{ is normal, } N(h) \text{ contains the generic point (in the scheme-theoretic sense) of } T, \text{ hence also an open set of closed points. In our case, since } I_Y \text{ is normal, it follows that so is a generic fibre of } f, \text{ therefore the irreducible and connected components of this fibre coincide (cf. [E], Thm. 18.12). Consequently the degree of } g \text{ coincides with the number of irreducible (=connected) components of the general fibre of } f, \text{ so the Theorem’s assertion is that } g \text{ is birational.
Then there is a Zariski open $U \subset \mathbb{P}^r$ such that $\mathbb{P}^r - U$ has codimension $> 1$ and $g^{-1}(U) \to U$ is finite, and we may assume $g^{-1}(U)$ is smooth as well. Since $U$, like $\mathbb{P}^r$, is simply connected, it follows that if $\deg(g) > 1$ then $g$, hence $f$, is ramified in codimension 1, i.e. there is a prime divisor $F \subset I_Y$ such that $f(F) \subset \mathbb{P}^r$ is a divisor and $f$ is ramified on $F$. We proceed to show that the latter conclusion leads to a contradiction.

Now it is an easy consequence of the Fulton-Hansen Connectedness Theorem (cf. [FL], Corollary 5.5) that in our case we have

$$f(\pi^{-1}(E_Y)) = \mathbb{P}^r,$$

hence $F \neq \pi^{-1}(E_Y)$, and therefore a general point of $F$ is of the form $(x, y, z)$ where $x, y \in Y$ are distinct and

$$z \in < x, y >$$

($< x, y >$ denotes the line spanned by $x, y$). Now a standard computation known as Terracini’s Lemma [FR] says that

$$\text{im } df_{(x,y,z)} = < T_xY, T_yY >,$$

and in particular this image is independent of $z \in < x, y >$. It follows that $F$ is the pullback of a divisor $D \subset Y[2]$, where a general point $(x, y) \in D$ has the property that $x \neq y$ and

$$\rho := \dim < T_xY, T_yY > < r.$$

We may assume that the projection map $p_1 : D \to Y$ is surjective, and let $D_x \subset Y$ denote the image of its general fibre under $p_2$, which is a divisor on $Y$. Setting $W = T_xY$, note that a general $y \in D_x$ is smooth on $Y$ and we have

$$\rho - 1 \leq \dim < T_yD_x, W > \leq \rho.$$

Now consider the following diagram, with vertical arrows only rational maps induced by projection from $W$:

$${\begin{array}{c}
S_{x,v} \subset D_x \subset \mathbb{P}^r \\
\downarrow \quad \downarrow \quad \downarrow \\
v \in V_x \subset \mathbb{P}^{c - 1}.
\end{array}}$$

Here $c = r - m$, $V_x$ is the (closure of the) image of $D_x$, $v \in V_x$ is a general point and $S_{x,v} = \pi^{-1}_W(v)$, which we may assume contains a general point of $D_x$. By (4), the dimension of $V_x$ is either $\rho - m - 1$ or $\rho - m - 2$, and
in these respective cases we have \( \dim S_{x,v} = 2m - \rho \) (resp. \( 2m - \rho + 1 \)). Though not essential for our purposes, it is interesting to note that when \( x \) is viewed as variable, a general hyperplane in \( \mathbb{P}^{c-1} \) corresponds in \( \mathbb{P}^{r} \) to a general tangent hyperplane to \( Y \), i.e. a general element of the dual variety \( Y^* \).

Now suppose that \( V_x \) is of dimension \( \rho - m - 1 \), so that \( S_{x,v} \) is of dimension \( 2m - \rho \). Note that by (4) this implies that for general \( y \in D_x \),

\[
\langle T_y Y, W \rangle = \langle T_y D_x, W \rangle,
\]

which projects modulo \( W \) to \( T_v V_x, v = \pi_W(y) \). Now for a linear subspace \( U \subset \mathbb{P}^{c-1} \), we denote by \( \pi_W^*(U) \) the unique linear subspace of \( \mathbb{P}^{r} \) which contains \( W \) and projects to \( U \) (this is uniquely determined by \( U \)). Then we conclude that for general \( y \in S_{x,v} \), we have

\[
T_y Y \subset \pi_W^*(T_v V_x).
\]

Thus the linear space \( \pi_W^*(T_v V_x) \) of dimension \( \rho \leq r - 1 \) is tangent to \( Y \) along a locus of dimension at least \( 2m - \rho \geq 2m - r + 1 \), contradicting (2).

Suppose now that \( V_x \) is of dimension \( \rho - m - 2 \), so \( S_{x,v} \) is of dimension \( 2m - \rho + 1 \). Then for \( y \in S_{x,v} \), the projection of \( T_y Y \) to \( \mathbb{P}^{c-1} \) is a \( \mathbb{P}^{\rho-m-1} \) containing \( T_v V_x = \mathbb{P}^{\rho-m-2} \), and the set of all these linear subspaces is a \( \mathbb{P}^{r-\rho} \), so imposing such a subspace to stay fixed is \( r - \rho \) conditions. Thus, pulling back to \( \mathbb{P}^{r} \), we can find a subvariety \( T \) of codimension at most \( r - \rho \) in \( S_{x,v} \), containing a general point of \( S_{x,v} \) (hence of \( D_x \)), such that \( T_y Y \) is contained in a fixed \( \mathbb{P}^{\rho} \) for all \( y \in T \). Since

\[
\dim T \geq 2m - \rho + 1 - (r - \rho) = 2m - r + 1,
\]

this again contradicts (2). \( \square \)

**Example.** By Corollary 3, any smooth 3-fold in \( \mathbb{P}^{5} \) has the ISC property. On the other hand, if \( Y \) is a smooth surface in \( \mathbb{P}^{4} \), Theorem 2 says that \( Y \) has the ISC property unless \( \sigma(Y) = 1 \), i.e. unless \( Y \) admits a hyperplane section with a multiple component. For example, the projected Veronese surface \( Y \subset \mathbb{P}^{4} \) admits multiple hyperplane sections, and indeed does not have the ISC property; in fact, the double curve of its generic projection to \( \mathbb{P}^{3} \) consists of 3 conics on \( Y \) mapping 2:1 to 3 lines on \( \bar{Y} \subset \mathbb{P}^{3} \). See [GH], pp. 628-635 for this and other interesting examples. As we mentioned above, Franchetta proved that for any smooth nondegenerate surface in \( \mathbb{P}^{5} \) or higher, other than the Veronese, the double curve of its generic projection to \( \mathbb{P}^{3} \) is irreducible.

**Acknowledgment.** This work was begun when the second author was visiting the Mathematics Department at Università di Roma Tre. He would like to thank the department, especially Edoardo Sernesi and Sandro Verra, the Director, for their assistance and their hospitality.
References

[B] C. Brandigi, *On quadratic and higher normality of small codimension projective varieties* [arXiv:math.AG/0002140].

[E] D. Eisenbud, *Commutative algebra. With a view toward algebraic geometry*, Springer-Verlag, 1995.

[En] F. Enriques, *Le Superficie Algebriche*, Zanichelli, 1949.

[F] W. Fulton, *On the topology of algebraic varieties*, Algebraic Geometry (Bowdoin 1985) (S. J. Bloch, ed.), vol. 1, Amer. Math. Soc., 1987, pp. 15-46.

[FL] R. Lazarsfeld, *Connectivity and its applications in algebraic geometry*, Algebraic Geometry, Chicago Circle 1980 (A. Libgober, ed.), Lect. Notes Math., vol. 862, Springer-Verlag, 1981, pp. 26-92.

[Fr] A. Franchetta, *Sulla curva doppia della proiezione di una superficie generale dell’ $S_4$, da un punto generico su un $S_3* (Rend. Cl. Sci. Fis. Mat. Nat. 2 (1941), 282-288.

[FR] T. Fujita, J. Roberts, *Varieties with small secant varieties*, Amer. J. Math. 103 (1981), 953-976.

[G] A. Grothendieck, *Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas IV*, Inst. Hautes Études Sci. Publ. Math. 32 (1967).

[GH] P. Griffiths, J. Harris, *Principles of Algebraic Geometry*, Wiley, 1978.

[GP] L. Gruson, C. Peskine, *Space curves: complete series and speciality*, Space curves (Rocca di Papa 1985), Lect. Notes in Math., vol. 1266, Springer, 1987, pp. 108-123.

[L] R. Lazarsfeld, *A sharp Castelnuovo bound for smooth surfaces*, Duke Math. J. 55 (1987), 423-429.

[Lv] A. van de Ven, *Topics in the geometry of projective space. Recent work of F. L. Zak. With an addendum by Zak*, DMV Seminar, vol. 4, Birkhäuser Verlag, 1984.

[Mo] B.G. Moishezon, *Complex surfaces and connected sums of complex projective planes. With an appendix by R. Livné*, Lect. Notes Math. vol.603, Springer, 1977.

[Pi] H.C. Pinkham, *A Castelnuovo bound for smooth surfaces*, Inventiones Math. 83 (1986), 321-332.

[R] Z. Ran, *Systèmes linéaires complets de sections hypersurfaces sur les variétés projectives de codimension 2*, C. R. Acad. Sci. Paris 298 (1984), 111-112.

[SS] B. Shiffman, A.J. Sommese, *Vanishing theorems on complex manifolds*, Birkhäuser, 1985.

[Z1] F.L. Zak, *Projections of algebraic varieties (English transl.)*, Math USSR-Sb. 116(158) (1981), 593-602, 608.

[Z2] , *Tangents and secants of algebraic varieties*, Transl. Math. Monogr., vol. 127, Amer. Math. Soc., 1993.

Univiersità di Roma Tre
E-mail address: lopez@matrm3.mat.uniroma3.it

University of California, Riverside
E-mail address: ziv@math.ucr.edu