LOFF Pairing vs. Breached Pairing in Asymmetric Fermion Superfluids

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A general analysis for the competition between breached pairing (BP) and LOFF pairing mechanisms in asymmetric fermion superfluids is presented in the frame of a four fermion interaction model. Two physical conditions which can induce mismatched Fermi surfaces are considered: (1) fixed chemical potential asymmetry $\delta \mu$ and (2) fixed fermion number asymmetry $\alpha$. In case (1), the BP state is ruled out because of Sarma instability, and the LOFF state is thermodynamically stable in a narrow window of $\delta \mu$. In case (2), while the Sarma instability can be avoided and both the BP and LOFF states can survive provided $\alpha$ is less than the corresponding critical value, the BP state suffers magnetic instability and the LOFF state is always thermodynamically stable. While the LOFF window in case (2) is much larger than the one in the conventional case (1), the longitudinal superfluid density of the LOFF state is negative for small $\alpha$ and it suffers magnetic instability too.

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I. INTRODUCTION

The fermion pairing between different species with mismatched Fermi surfaces, which was discussed many years ago, promoted new interest in both theoretic and experimental studies in recent years. The mismatched Fermi surfaces can be realized, for instance, in a superconductor in an external magnetic field or a strong spin-exchange field, an electronic gas with two species of electrons from different bands, a superconductor with overlapping bands, a mixture of two types of fermionic cold atoms with different densities and/or masses, an isospin asymmetric nuclear matter with proton-neutron pairing, and a neutral dense quark matter. In the early studies of superconductivity, Sarma[1] found a spatially isotropic and uniform state where there exist gapless modes, namely the excitation of these quasiparticle modes needs no energy. However, this state is energetically unstable, compared with the fully gapped BCS state. This is the so-called Sarma instability[1]. Recently, this instability was widely discussed in general case[27, 28] and in neutral color superconductivity[20, 21]. It is now widely accepted that the Sarma instability can be avoided through two possible ways, fixed number difference of the two kinds of fermions or a proper momentum structure of the attractive interaction between the two species[27]. The Sarma state is now also called breached pairing (BP) state, since in this state the dispersion relation of one branch of the quasi-particles has two zero points at momenta $p_1$ and $p_2$ where gapless excitations happen, and the superfluid component in the regions $p < p_1$ and $p > p_2$ is breached by a normal component in the region $p_1 < p < p_2$.

The LOFF state which is a spatially anisotropic ground state was proposed in the study of superconductors in a strong spin-exchange field by Larkin, Ovchinnikov, Fulde and Ferrell[22]. In this ground state, the rotational symmetry and/or the translational symmetry of the system are spontaneously broken. In the study of asymmetric nuclear matter, asymmetric atomic fermion gas and color superconductivity, the LOFF phase has been widely investigated[23, 24, 25, 26]. When the chemical potential asymmetry $\delta \mu$ suffers magnetic instability too.

While the Sarma instability can be avoided, new dynamical instability of the BP state which is called magnetic instability in the literatures arises in the study of dense quark matter. It is found that the response to external color magnetic field in neutral dense quark matter in the BP state is paramagnetic, i.e., the squared Meissner masses of some gluons become negative[46, 47, 48, 49]. This instability was also studied in atomic fermion system[51]. A general analysis for asymmetric superconductors shows that this dynamical instability is related to the breached pairing mechanism only and independent of the details of the attractive interactions[52]. It is recently pointed out that, due to the magnetic instability, the free energy of the LOFF state or some other exotic anisotropic states is probably lower than that of the BP state[53, 54, 55, 56, 57, 58, 59, 60]. However, this conclusion is obtained from the study for asymmetric systems with fixed chemical potentials. In many physical cases we are interested in, the chemical potentials are not directly fixed, instead the particle numbers are fixed or some other constraints on the particle numbers are required. For these systems, the magnetic instability is still not clear. In this paper we will give a general and systematic comparison between the BP and LOFF states in asymmetric fermion superfluids under different conditions which can lead to mismatched Fermi surfaces.

The paper is organized as follows. In Section II we briefly review the formalism for the isotropic BCS, BP and anisotropic LOFF states. In Section III we analytically discuss the thermodynamic and dynamical instabil-
ities of the BP state, and show why the LOFF state may be more favored than the BP state. In Sections IV and V we discuss, respectively, the systems with fixed chemical potential asymmetry and fixed density asymmetry. We summarize in Section VI. We use the natural unit of $c = \hbar = 1$ through the paper.

II. MODEL AND FORMALISM

To have a general investigation, we consider the model Lagrangian with two species of fermions interacting with each other via a point interaction,

$$\mathcal{L} = \sum_{i=a,b} \psi_i^* (x) \left[ -\frac{\partial}{\partial \tau} + \frac{\nabla^2}{2m_i} + \mu_i \right] \psi_i (x) + g \psi_i^* (x) \psi_b (x) \psi_a (x)$$

(1)

where $\psi_a (x), \psi_b (x)$ are fermion fields for the two species $a$ and $b$ with space-time $x = (\tau, \vec{x})$, the coupling constant $g$ is positive to keep the interaction attractive, $m_a$ and $m_b$ are the masses of the two species, and $\mu_a$ and $\mu_b$ are the chemical potentials.

The key quantity to describe a thermodynamic system is the partition function which can be defined as

$$Z = \int [d\psi_a] [d\psi_b] [d\psi_b^*] e^{\int_0^\beta d\tau d^3 \vec{x} \mathcal{L}}$$

(2)

in the imaginary time ($\tau$) formalism of finite temperature field theory. According to the standard BCS theory, we introduce the order parameter field $\Phi(x)$ and its complex conjugate $\Phi^*(x)$,

$$\Phi(x) = g \langle \psi_b (x) \psi_a (x) \rangle \ , \ \Phi^*(x) = g \langle \psi_a^* (x) \psi_b^* (x) \rangle$$

(3)

where the symbol $\langle \rangle$ means ensemble average. Introducing the Nambu-Gorkov space defined as

$$\Psi(x) = \left( \begin{array}{cc} \psi_a (x) \\ \psi_b^* (x) \end{array} \right) \ , \ \Psi^*(x) = \left( \begin{array}{cc} \psi_a^* (x) \\ \psi_b (x) \end{array} \right)$$

(4)

the partition function in mean field approximation can be written as

$$Z_{MF} = \int [d\Psi] [d\Psi^*] e^{\int_0^\beta d\tau d^3 \vec{x} (\bar{\Psi} \mathcal{G}^{-1} \Psi + |\Phi|^2 / g)}$$

(5)

with the inverse of the mean field fermion propagator

$$\mathcal{G}^{-1} = \left( \begin{array}{cc} -\frac{\partial}{\partial \tau} + \frac{\nabla^2}{2m_a} + \mu_a & \Phi^*(x) \\ \Phi(x) & -\frac{\partial}{\partial \tau} - \frac{\nabla^2}{2m_b} - \mu_b \end{array} \right)$$

(6)

To have a unified approach for the isotropic BCS, BP states and anisotropic LOFF state, we take in the following a specific ansatz for the order parameter fields

$$\Phi(x) = \Delta e^{2i\vec{q} \cdot \vec{x}}, \quad \Phi^*(x) = \Delta e^{-2i\vec{q} \cdot \vec{x}}$$

(7)

Here $\Delta$ is a real quantity. Obviously, we recover the isotropic state with $\vec{q} = 0$ and obtain the anisotropic LOFF state with $\vec{q} \neq 0$. After a transformation of the fermion fields $\chi_a (x) = e^{i\vec{q} \cdot \vec{x}} \psi_a (x)$ and $\chi_b (x) = e^{i\vec{q} \cdot \vec{x}} \psi_b (x)$, we can directly evaluate the Gaussian path integral and obtain the thermodynamic potential

$$\Omega(\Delta, \vec{q}) = \frac{\Delta^2}{g} - T \sum_n \int d^3 \vec{p} \left( \frac{2\pi}{3} \right)^3 \text{Tr ln} \mathcal{G}^{-1}(i\omega_n, \vec{p})$$

(8)

in terms of the inverse fermion propagator in the momentum and frequency space

$$\mathcal{G}^{-1}(i\omega_n, \vec{p}) = i \frac{\omega_n - \epsilon_a}{\Delta} - \frac{\omega_n + \epsilon_b}{\Delta}$$

(9)

with $\epsilon_a = (\vec{p} + \vec{q})^2 - \mu_a$ and $\epsilon_b = (\vec{p} - \vec{q})^2 - \mu_b$. The explicit form of the propagator can be written as

$$\mathcal{G}(i\omega_n, \vec{p}) = \left( \begin{array}{cc} \mathcal{G}_{11}(i\omega_n, \vec{p}) & \mathcal{G}_{12}(i\omega_n, \vec{p}) \\ \mathcal{G}_{21}(i\omega_n, \vec{p}) & \mathcal{G}_{22}(i\omega_n, \vec{p}) \end{array} \right)$$

(10)

with the elements

$$\mathcal{G}_{11}(i\omega_n, \vec{p}) = \frac{i\omega_n - \epsilon_- + \epsilon_+}{(i\omega_n - \epsilon_-)^2 - \epsilon_+^2},$$

$$\mathcal{G}_{22}(i\omega_n, \vec{p}) = \frac{i\omega_n - \epsilon_- + \epsilon_+}{(i\omega_n - \epsilon_-)^2 - \epsilon_+^2},$$

$$\mathcal{G}_{12}(i\omega_n, \vec{p}) = \mathcal{G}_{21}(i\omega_n, \vec{p}) = \frac{-\Delta}{(i\omega_n - \epsilon_-)^2 - \epsilon_+^2},$$

(11)

where the quantities $\epsilon_+$, $\epsilon_-$ and $\epsilon_\Delta$ are defined as

$$\epsilon_+ = \frac{\epsilon_a + \epsilon_b}{2}, \quad \epsilon_- = \frac{\epsilon_a - \epsilon_b}{2}, \quad \epsilon_\Delta = \sqrt{\epsilon_+^2 + \Delta^2}.$$  

(12)

Using the identity $\text{Tr} \ln = \ln \det$, the thermodynamic potential can be evaluated as

$$\Omega(\Delta, \vec{q}) = \frac{\Delta^2}{g} - T \sum_n \int d^3 \vec{p} \left( \frac{2\pi}{3} \right)^3 \left[ \left( \frac{\epsilon_A}{2} + T \ln(1 + e^{-\epsilon_A}) \right) + \left( \frac{\epsilon_B}{2} + T \ln(1 + e^{-\epsilon_B}) \right) - \epsilon_+ \right],$$

(13)

where $\epsilon_A$ and $\epsilon_B$ are the quasiparticle energies

$$\epsilon_A = \epsilon_- + \epsilon_\Delta, \quad \epsilon_B = \epsilon_- - \epsilon_\Delta.$$  

(14)

The gap equation which determines the order parameter $\Delta$ self-consistently is related to the off-diagonal element of the propagator matrix,

$$\Delta = gT \sum_n \frac{d^3 \vec{p}}{(2\pi)^3} \mathcal{G}_{12}(i\omega_n, \vec{p}),$$

(15)

which is equivalent to the stationary condition of the thermodynamic potential, $\partial \Omega / \partial \Delta = 0$. After the summation over the fermion frequency, it becomes

$$\Delta(1 - gI_\Delta) = 0.$$  

(16)
Completing the Matsubara frequency summation, we obtain
\[
0 = \int \frac{d^3\vec{p}}{(2\pi)^3} \left[ f(-\epsilon_B) - f(-\epsilon_A) \right] \frac{\partial \epsilon_+}{\partial q} - \left( 1 - \frac{\epsilon_+}{\epsilon_A} + f(\epsilon_A) \frac{\epsilon_+}{\epsilon_A} \right) \frac{\partial \epsilon_+}{\partial q}. \tag{18}
\]

It is easy to see that \( q = 0 \) is a trivial solution of the equation, which corresponds to the isotropic BCS or BP state.

The occupation numbers are determined through the diagonal elements of the fermion propagator,
\[
n_a(\vec{p}) = T \lim_{\eta \to 0} \sum_n G_{11}(i\omega_n, \vec{p}) e^{i\omega_n \eta},
\]
\[
n_b(\vec{p}) = -T \lim_{\eta \to 0} \sum_n G_{22}(i\omega_n, \vec{p}) e^{-i\omega_n \eta}. \tag{19}
\]

Completing the Matsubara frequency summation, we obtain
\[
n_a(\vec{p}) = u^2_p f(\epsilon_A) + v^2_p f(-\epsilon_B),
\]
\[
n_b(\vec{p}) = u^2_p f(-\epsilon_B) + v^2_p f(-\epsilon_A) \tag{20}
\]
with the definition \( u^2_p = (1 + \epsilon_+ / \epsilon_A) / 2 \) and \( v^2_p = (1 - \epsilon_+ / \epsilon_A) / 2 \). At zero temperature, they are reduced to
\[
n_a(\vec{p}) = v^2_p \theta(\epsilon_A - \epsilon_+) + u^2_p \theta(-\epsilon_A - \epsilon_-),
\]
\[
n_b(\vec{p}) = v^2_p \theta(\epsilon_A + \epsilon_-) + u^2_p \theta(-\epsilon_A + \epsilon_+). \tag{21}
\]
The particle number densities \( n_a \) and \( n_b \) can be calculated from
\[
n_a = \int \frac{d^3\vec{p}}{(2\pi)^3} n_a(\vec{p}), \quad n_b = \int \frac{d^3\vec{p}}{(2\pi)^3} n_b(\vec{p}). \tag{22}
\]
For a free Fermi gas, we can define the Fermi momenta \( p_F^a \) and \( p_F^b \) for the two species, and the particle number densities are then expressed as
\[
n_a = \frac{(p_F^a)^3}{6\pi^2}, \quad n_b = \frac{(p_F^b)^3}{6\pi^2}. \tag{23}
\]
For convenience, we define total Fermi momentum \( p_F \), Fermi energy \( \epsilon_F \) and Fermi velocity \( v_F \) through the total number density \( n = n_a + n_b \) and the reduced mass \( m = 2m_a m_b / (m_a + m_b) \), \( p = p_F^a / (3\pi^2), \) \( \epsilon_F = p_F^a / (2m) \) and \( v_F = p_F / m \).

The above formalism is general for any system with \( \mu_a \neq \mu_b \) and \( m_a \neq m_b \). In the following we will consider only the case without mass difference between the two species, \( m_a = m_b = m \), which is relevant for the study of cold atomic fermion gas, isospin asymmetric nuclear matter, and two flavor color superconducting quark matter. For convenience, we define the average of the chemical potentials \( \mu_a \) and \( \mu_b \) and their difference,
\[
\mu = \frac{\mu_a + \mu_b}{2}, \quad \delta \mu = \frac{\mu_b - \mu_a}{2}. \tag{24}
\]

In the weak coupling BCS region, the chemical potentials \( \mu_a \) and \( \mu_b \) are approximately the corresponding Fermi energies \( \epsilon_F^a = (p_F^a)^2 / (2m) \) and \( \epsilon_F^b = (p_F^b)^2 / (2m) \), and the pairing gap \( \Delta \) and the Fermi surface mismatch \( \delta \mu \) are much smaller than the average chemical potential, \( \Delta \sim \delta \mu \ll \mu \).

### A. Isotropic Superfluid with \( \vec{q} = 0 \)

We discuss firstly the possible isotropic superfluid state, i.e., the case with \( \vec{q} = 0 \). The dispersion relations of the quasiparticle excitations are expressed as
\[
\epsilon_A = \delta \mu + \sqrt{\left( \frac{p^2}{2m} - \mu \right)^2 + \Delta^2},
\]
\[
\epsilon_B = \delta \mu - \sqrt{\left( \frac{p^2}{2m} - \mu \right)^2 + \Delta^2}. \tag{25}
\]

Without losing generality, we set \( \delta \mu \geq 0 \). It is easy to see from the equation \( \epsilon_B = 0 \) that only in the case with \( \Delta < \delta \mu \), there are gapless excitation at momenta
\[
p_1 = \sqrt{2m \left( \mu - \sqrt{\delta \mu^2 - \Delta^2} \right)},
\]
\[
p_2 = \sqrt{2m \left( \mu + \sqrt{\delta \mu^2 - \Delta^2} \right)}. \tag{26}
\]
In the gapless state, we have \( n_a(\vec{p}) = n_b(\vec{p}) = v^2_p \) for \( p < p_1 \) and \( p > p_2 \) and \( n_a(\vec{p}) = 0, n_b(\vec{p}) = 1 \) for \( p_1 < p < p_2 \), the superfluidity state with fermion pairing in the regions \( p < p_1 \) and \( p > p_2 \) is breached by the normal fermion gas in the region \( p_1 < p < p_2 \). This is the reason why people call the gapless state as breached pairing (BP) state. In the case with \( \Delta > \delta \mu \), there is no gapless excitation, and the system is in BCS phase with \( n_a(\vec{p}) = n_b(\vec{p}) = v^2_p \) for all \( p \).

### B. Anisotropic Superfluid with \( \vec{q} \neq 0 \)

We now discuss the anisotropic superfluid state with \( \vec{q} \neq 0 \). The dispersion relations of the elementary excita-
tions can be written as
\[ \epsilon_A = \delta \mu + \frac{pq}{m} \cos \theta + \sqrt{\left( \frac{p^2 + q^2}{2m} - \mu \right)^2 + \Delta^2}, \]
\[ \epsilon_B = \delta \mu + \frac{pq}{m} \cos \theta - \sqrt{\left( \frac{p^2 + q^2}{2m} - \mu \right)^2 + \Delta^2}, \quad (27) \]
where \( \theta \) is the angle between the momenta \( \vec{p} \) and \( \vec{q} \). In many of the previous investigations, the term \( pq/2m \) was neglected and the term \( pq \cos \theta/m \) was approximated by \( v_F q \cos \theta \) since \( q/p_F \ll 1 \) in weak coupling limit. Taking this approximation, in the case of \( \delta \theta = \delta \mu + v_F q \cos \theta > \Delta \), the gapless excitation happens at
\[ p_1(\theta) = \sqrt{2m \left( \mu - \sqrt{\delta_0^2 - \Delta^2} \right)}, \]
\[ p_2(\theta) = \sqrt{2m \left( \mu + \sqrt{\delta_0^2 - \Delta^2} \right)}. \quad (28) \]

For the following numerical research, we will do full calculation without employing this approximation. We will see that the anisotropic LOFF state possesses always unequal particle numbers for the two species, and hence the LOFF superfluid is gapless.

Since the gap equation for \( \Delta \) suffers ultraviolet divergence, a regularization scheme is needed. We consider now two regularization schemes. In the first scheme, we simply employ a hard cutoff \( p_A \) for the three momentum. Such a regularization scheme is related to the study of color superconductivity at moderate baryon density and isospin asymmetric nuclear matter. To avoid a specific choice of model parameters, such as the mass \( m \), we express the coupling \( g \) in terms of the s-wave scattering length \( a_s \) as \( g = \frac{4\pi a_s}{m} \) with \( a_s = |a_s|^2 \). Once this is done, the two original parameters \( p_A \) and \( g \) in the model are replaced by two dimensionless parameters \( \Lambda = (p_A/p_F)^2 \) and \( p_F a \). In the weak coupling BCS region \( 0 < p_F a < 1 \) and for not very large cutoff \( 1 < \Lambda < 2 \), the solution of the gap equation can be well described by
\[ \Delta_0 \simeq \frac{8}{e^2} \epsilon_F \sqrt{\Lambda^2 - 1} e^{-\pi \Lambda/2} \quad (29) \]
in the symmetric case with \( \delta \mu = 0 \).

In the second regularization scheme, we replace the bare coupling \( g \) by the low energy limit of the two-body \( T \)-matrix [31]
\[ \frac{m}{4\pi a_s} = -\frac{1}{g} + \int \frac{d^3 \vec{p}}{(2\pi)^3} \frac{m}{p^2}. \quad (30) \]
Such a regularization scheme is often used in the study of atomic fermion gas. In this scheme, it is not necessary to introduce a momentum cutoff, and all physical results depend only on the dimensionless coupling \( p_F a \). This can be seen explicitly by scaling all the quantities with energy dimension by the Fermi energy \( \epsilon_F \) and all the quantities with momentum dimension by the Fermi momentum \( p_F \),
\[ z = \left( \frac{p}{p_F} \right)^2, \quad \nu = \left( \frac{\epsilon}{\epsilon_F} \right)^2, \quad \nu_a = \frac{\mu_a}{\epsilon_F}, \quad \nu_b = \frac{\mu_b}{\epsilon_F}, \quad \nu = \frac{\mu}{\epsilon_F}. \quad (31) \]
With these dimensionless quantities, we obtain the scaled gap equations,
\[ 0 = \delta \left[ \int dz \sqrt{z} \int_{-1}^{1} dx \frac{f(\epsilon_{z,x}) - f(\epsilon_{x})}{\sqrt{(z + z_q - \nu)^2 + \delta^2} - \frac{2\pi}{p_F a}} \right], \]
\[ 0 = \delta \left[ \int dz \sqrt{z} \int_{-1}^{1} dx \left[ \frac{\epsilon_{z,x}}{2} + t \ln(1 + e^{-\frac{\epsilon_{z,x}}{T}}) \right] + \frac{\epsilon_{z,x}}{2} + t \ln(1 + e^{-\frac{\epsilon_{z,x}}{T}}) \right] \]
\[ \frac{\pi \delta^2}{2p_F a} \quad (33) \]
and the scaled number densities,
\[ \frac{n_a}{n_0} = \frac{1}{2} \int dz \sqrt{z} \int_{-1}^{1} dx \left[ u^2 f(\epsilon_{z,x}) + v^2 f(\epsilon_{z,x}) \right], \quad (34) \]
\[ \frac{n_b}{n_0} = \frac{1}{2} \int dz \sqrt{z} \int_{-1}^{1} dx \left[ 1 - u^2 f(\epsilon_{z,x}) - v^2 f(\epsilon_{z,x}) \right], \]
where the normalization constants \( n_0 \) and \( E_0 \) are chosen to be \( n_0 = p_F^2/(4\pi^2) \) and \( E_0 = p_F^2/(8\pi^2 m) \), the quasiparticle energies and the functions \( u, v \) are reexpressed in terms of the dimensionless quantities,
\[ \epsilon_{z,x} = \delta \nu + 2 \sqrt{z + z_q - \nu} \sqrt{(z + z_q - \nu)^2 + \delta^2}, \quad (35) \]
\[ \epsilon_{z,x} = \delta \nu + 2 \sqrt{z + z_q - \nu} \sqrt{(z + z_q - \nu)^2 + \delta^2}, \]
\[ u = \frac{1}{2} \left( 2 - (z + z_q - \nu)/\sqrt{(z + z_q - \nu)^2 + \delta^2} \right), \quad (36) \]
\[ v = \frac{1}{2} \left( 2 - (z + z_q - \nu)/\sqrt{(z + z_q - \nu)^2 + \delta^2} \right). \]
In weak coupling limit, the solution of the gap equation is
\[ \Delta_0 \simeq \frac{8}{e^2} \epsilon_F e^{-\pi \Lambda/2} \quad (36) \]
in the symmetric case with \( \delta \mu = 0 \). It is obvious that if we take \( \Lambda = \sqrt{2} \), the results in the two regularization schemes are the same in symmetric case and will
not make significant difference in asymmetric case in the region $0 < p_T a < 1$.

While the values of $\Delta, \delta \mu$ and $q$ depend on the cutoff $\Lambda$, we emphasize that the scaled quantities such as $\Delta/\Delta_0, \delta \mu/\Delta_0$ and $v_F q/\delta \mu$ are regularization scheme independent in the weak coupling region.\cite{10, 52, 53, 62}.

III. THERMODYNAMIC AND DYNAMICAL INSTABILITIES IN BP SUPERFLUID

In this section, we discuss some possible thermodynamic and dynamical instabilities of the BP state. From the analytic analysis, we will see that there are two types of instabilities in the BP phase. One is the Sarma dynamic and dynamical instabilities of the BP state. From the analytic analysis, we will see that there are two types of instabilities in the BP phase. One is the Sarma instability.

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In the weak coupling region with $\Delta \sim \delta \mu \ll \mu$, the integral over the $\delta$-function in (38) disappears automatically, $\kappa_\Delta = \frac{\partial^2 \Omega}{\partial \Delta^2} = \int_0^\infty \frac{d\pi \Delta^2}{2\pi^2} \left[ \frac{\theta(\epsilon_\Delta - \delta \mu)}{\epsilon_\Delta} - \delta(\epsilon_\Delta - \delta \mu) \right].$

In weak coupling region with $\Delta \sim \delta \mu \ll \mu$, it can be approximated as

$$\kappa_\Delta \approx \frac{m v_F}{\pi^2} \left[ 1 - \frac{\delta \mu}{\sqrt{\delta \mu^2 - \Delta^2}} \right]. \quad (39)$$

For the BCS state which satisfies the constraint $\Delta > \delta \mu$, the $\delta$-function in (38) disappears automatically, $\kappa_\Delta$ is obviously positive, and the state is stable. For the BP state with $\Delta < \delta \mu$, the integral over the $\delta$-function becomes nonzero and dominates the susceptibility. In this case, we have $\kappa_\Delta < 0$, and the state is unstable. This is the so-called Sarma instability.

For the systems with fixed chemical potentials $\mu_a$ and $\mu_b$ or fixed total chemical potential $\mu$ and chemical potential difference $\delta \mu$, we can directly compare $\delta \mu$ with $\Delta$ and see if the state is stable. However, for a systems under some constraints on particle numbers, the grand canonical ensemble should be replaced by canonical ensemble, and the essential quantity to describe the thermodynamics of the systems is the free energy $F$ instead of the thermodynamic potential $\Omega$. In this case, the chemical potentials are no longer independent thermodynamic variables, $\Delta$ and $\delta \mu$ should be determined self-consistently via solving the coupled set of equations for the gap parameter and the constraints on the particle numbers. Therefore, it is not convenient to judge the instability of the BP state through $\kappa_\Delta$. On the other hand, when the number difference between the two species is fixed, the BCS solution is ruled out, and only the BP state survives.\cite{20, 21}. A direct way for the instability analysis is to compare the thermodynamic potentials or free energies themselves in BP and LOFF states, and see which one is more stable.

We now turn to discuss the instability induced by a small superfluid velocity $\vec{v}_s$. When the superfluid moves with a uniform velocity $\vec{v}_s$, the condensate transforms like $\Phi \to 2i \pi m \vec{v}_s \cdot \vec{x}$ and $\Phi^* \to e^{-i2i \pi m \vec{v}_s \cdot \vec{x}}$, and the supercurrent $\vec{j}_s$ and the superfluid density $\rho_s$ are defined via the thermodynamic potential

$$\Omega(\vec{v}_s) = \Omega(\vec{v}_s = 0) + \vec{j}_s \cdot \vec{v}_s + \frac{1}{2} \rho_s \vec{v}_s^2 + \cdots. \quad (40)$$

When the density $\rho_s$ is negative, the superfluid is dynamically unstable. This dynamical instability can be related to the thermodynamic instability induced by pair momentum fluctuations. We perturb the BP state with an infinitely small LOFF momentum $q$ and expand the thermodynamic potential around $q = 0$.

$$\Omega(\vec{v}_s = 0) + \frac{\partial \Omega}{\partial q} \bigg|_{q = 0} q + \frac{\partial^2 \Omega}{\partial q^2} \bigg|_{q = 0} q^2 + \cdots. \quad (41)$$

The linear term vanishes due to the gap equation for the pair momentum, and we can easily observe the relations between the superfluid density $\rho_s$, the Meissner mass squared $m_A^2$, and the pair momentum susceptibility $\kappa_q = \frac{\partial^2 \Omega}{\partial q^2} \big|_{q = 0}$, $\rho_s = m^2 \kappa_q$, $m_A^2 = e^2 \kappa_q$.\cite{34}

Using the result in (42), at zero temperature we have

$$\rho_s = mn - \frac{1}{6\pi^2} \int_0^\infty d\pi \Delta^2 \delta(\epsilon_\Delta - \delta \mu). \quad (43)$$

In the weak coupling region, $\epsilon_\Delta - \delta \mu = 0$ has two possible roots $p_1$ and $p_2$, and $\rho_s$ is reduced to

$$\rho_s = mn \left[ 1 - \frac{\delta \mu}{\sqrt{\delta \mu^2 - \Delta^2}} \right]. \quad (44)$$

with the coefficient $\eta = \left( p_1^2 + p_2^2 \right) / (6\pi^2 n)$. Since the total fermion density $n$ at $T = 0$ can be expressed as

$$n = \frac{p_1^2 + p_2^2}{6\pi^2} + \frac{1}{\pi^2} \left( \int_0^{p_1} d\pi v_p^2 + \int_{p_2}^{\infty} d\pi v_p^2 \right), \quad (45)$$

and the two integrals in the bracket are very small in weak coupling region and thus can be neglected, the coefficient $\eta$ is approximately equal to 1, and $\rho_s$ can be expressed as

$$\rho_s \approx mn \left[ 1 - \frac{\delta \mu}{\sqrt{\delta \mu^2 - \Delta^2}} \right]. \quad (46)$$
It is easy to see that in the BP phase with $\Delta < \delta \mu$, $\rho_s$ becomes negative. This is the so called magnetic instability, since it is directly related to the negative Meissner mass squared, if the fermions are charged. This dynamical instability implies a lower free energy of the LOFF state in comparison with the BP state.

As we mentioned above, however, for the systems with fixed density asymmetry instead of chemical potential asymmetry, the essential thermodynamic function is the free energy $F$ instead of $\Omega$. For such systems, the above argument fails and we need to calculate the free energy directly and then determine the thermodynamically stable state.

We end this section with a qualitative discussion on the instability analysis in strong coupling case. For such systems, the condition $\mu >> \delta \mu$ does not hold and even $\mu < 0$ happens in the BEC region, the node $p_1$ becomes imaginary, and there is only one gapless node $p_2$. In this case, it has been argued that the gapless state is automatically free from the Sarma and magnetic instabilities[42, 43, 44, 64], and therefore, the LOFF state becomes less favored than the BP state. In fact, some recent studies have shown that the BP state will be the ground state of asymmetric fermion gas when the coupling is strong[42, 43, 44, 64].

IV. FIXED CHEMICAL POTENTIAL ASYMMETRY

We now solve the gap equations to find the thermodynamically stable state among the normal, BCS, BP and LOFF states. The four states are defined through the solutions of the gap equations,

1) normal state with $\Delta = 0$,
2) BCS state with $\Delta \neq 0$, $q = 0$ and $\Delta > \delta \mu$,
3) LOFF state with $\Delta \neq 0$, $q = 0$ and $\Delta < \delta \mu$,
4) LOFF state with $\Delta \neq 0$, $q \neq 0$.

In this section we consider the case of fixed chemical potentials $\mu_a$ and $\mu_b$ with $\mu_a \neq \mu_b$. For example, if a superfluid with cooper pair composed of spin up and down fermions is placed in an external magnetic field $H$, there will be an effective chemical potential asymmetry $\delta \mu = \mu_B H$ with the fermion magneton $\mu_B$. When the external field is fixed, the chemical potential asymmetry is naturally fixed. In this case, the system is connected to reservoirs of each species and the particle numbers of the two species are not conserved. Fig.1 shows the condensate $\Delta$, scaled by its value $\Delta_0$ in the corresponding symmetric system with $\delta \mu = 0$, as a function of the chemical potential difference $\delta \mu$ scaled by $\Delta_0$ too at $p_F a = 0.4$ and $T = 0$. In a wide range of coupling $0 < p_F a < 1$, we found that the results are almost the same for the scaled quantities. We compare first the three uniform phases. The solution $\Delta = 0$ is the trivial solution of the gap equations which describes the normal phase without superfluidity, the BCS solution does not change with increasing $\delta \mu$ and keeps its symmetric value until $\delta \mu = \Delta_0$, and the BP solution starts to appear at $\delta \mu = 0.5\Delta_0$ and satisfies $\Delta < \delta \mu$ in the region $0.5\Delta_0 < \delta \mu < \Delta_0$. The BP solution can be expressed analytically as $\Delta = \sqrt{\Delta_0(2\delta \mu - \Delta_0)}[1]$. From the analysis in Section III the BP state is thermodynamically unstable. This can be seen clearly from the thermodynamic potential as a function of the scaled condensate $\Delta/\Delta_0$ for three values of scaled chemical potential difference $\delta \mu/\Delta_0$, shown in Fig.2. For $\delta \mu < 0.5\Delta_0$, there are a maximum at $\Delta = 0$ and a minimum at $\Delta = \Delta_0$, corresponding, respectively, to the local phase and BCS phase, and for $0.5\Delta_0 < \delta \mu < 0.707\Delta_0$ the BCS state is the true ground state.

We now include the anisotropic LOFF state. The scaled LOFF gap $\Delta/\Delta_0$ and scaled LOFF momentum $v_F q/\delta \mu$ are shown in Figs.4 and 5 as functions of the scaled chemical potential difference $\delta \mu/\Delta_0$ at $p_F a = 0.4$ and $T = 0$. The LOFF state starts at $\delta \mu = 0.635\Delta_0$ and there are two branches called LOFF1 and LOFF2. To find the true ground state, we compared the thermodynamic potentials for the four phases in Fig.4. It is easy to see that, below $\delta \mu \sim \Delta_0/\sqrt{2} = 0.707\Delta_0$ the LOFF1 and LOFF2 are unstable and the system is in BCS state, above $\delta \mu \sim 0.754\Delta_0$ the normal state becomes stable, and in between $\delta \mu$ and $\delta \mu_2$ the LOFF2 state is the true ground state. In this LOFF window, the gap $\Delta$ starts from $0.24\Delta_0$ at $\delta \mu = \delta \mu_1$ and decreases monotonically to 0 at $\delta \mu = \delta \mu_2$, the scaled LOFF momentum $v_F q/\delta \mu$ is almost a constant, $1.2 < v_F q/\delta \mu < 1.3$ in the window. The critical values $\delta \mu_1$ and $\delta \mu_2$ for the LOFF state agree with the analytical and numerical results obtained.
FIG. 2: The scaled thermodynamic potential $\Omega/E_0$ as a function of the scaled gap parameter $\Delta/\Delta_0$ at $p_F a = 0.4, T = 0$ and $q = 0$ for three chemical potential values, $0 < \delta \mu/\Delta_0 < 1/2, 1/2 < \delta \mu/\Delta_0 < \sqrt{2}$ and $\delta \mu/\Delta_0 > \sqrt{2}$. We have chosen the thermodynamic potential of the BCS state to be zero.

FIG. 3: The scaled LOFF momentum $v_F q/\delta \mu$ as a function of the scaled chemical potential difference $\delta \mu/\Delta_0$ at $p_F a = 0.4$ and $T = 0$ for LOFF1 and LOFF2.

FIG. 4: The scaled thermodynamic potential $\Omega/E_0$ as a function of the scaled chemical potential difference $\delta \mu/\Delta_0$ at $p_F a = 0.4$ and $T = 0$ for normal, BCS, BP, LOFF1, and LOFF2 states. We have chosen the thermodynamic potential of the normal state to be zero.

in other studies in weak coupling limit $[2, 3, 29, 33, 54]$. In Fig. 4 we showed $\delta \mu_1/\Delta_0$ and $\delta \mu_2/\Delta_0$ as functions of $p_F a$. In small coupling region, they are almost constants and agree well with the analytical result, while when the coupling is strong enough, the LOFF window becomes small. This behavior agrees with the result in crystalline color superconductivity $[24]$.

FIG. 5: The starting and ending chemical potential difference $\delta \mu_1/\Delta_0$ and $\delta \mu_2/\Delta_0$ for the LOFF window as functions of $p_F a$.

We investigated also systems under condition of fixed total number $n$ and chemical potential asymmetry $\delta \mu$. The behavior of the gap parameter, pair momentum and the free energy $F = \Omega + \mu n$ is almost the same as what we showed here. The reason may be the assumption $m_a = m_b$ considered in this paper. With a large mass difference, the two cases will be quite different $[62]$.

V. FIXED DENSITY ASYMMETRY

For many asymmetric fermion systems, the fixed quantity is the number density asymmetry instead of the chemical potential asymmetry. For example, in cold atom gas of $^6Li$ the number of the atoms in each hyperfine state can be directly adjusted, in isospin asymmetric nuclear matter the density difference between neutron and proton is fixed, and in two flavor dense quark matter, the density difference between u and d quarks is required to satisfy charge neutrality. In this section, we assume that the number densities of the two species are fixed. In this case, the chemical potentials $\mu_a$ and $\mu_b$ are not thermodynamic variables, they should be determined by the known number densities $n_a$ and $n_b$. As a result, the essential quantity to describe the thermodynamics of the system is the free energy defined as

$$ F = \Omega + \mu_a n_a + \mu_b n_b. $$

In general case of $n_a \neq n_b$, instead of $n_a$ and $n_b$, we can equivalently describe the system by the total number $n = n_a + n_b$ and the quantity

$$ \alpha = \frac{n_b - n_a}{n_b + n_a} $$

which controls the degree of asymmetry between the two species and satisfies $0 < \alpha < 1$. Obviously, once $n_a \neq n_b$ or $\alpha \neq 0$, the BCS pairing mechanism is ruled out, only the normal phase, BP phase and LOFF phase are left as candidates of ground states.
Considering the coupled set of equations for the gap parameter and the particle numbers

\[ n_a = \frac{1 - \alpha}{2} n, \quad n_b = \frac{1 + \alpha}{2} n, \quad (49) \]

the gap parameter \( \Delta \) and chemical potential difference \( \delta \mu \) in the BP state as functions of the asymmetry degree \( \alpha \) are shown in Fig.1 at \( T = 0 \), where the scaling parameter \( \alpha_c \) is the critical asymmetry where the BP superfluidity disappears. While both \( \Delta \) and \( \delta \mu \) drop down with increasing degree of asymmetry, they always satisfy the condition for the BP state, \( \Delta < \delta \mu \), and the difference between them becomes larger and larger. Finally, they end at the critical value \( \alpha_c \). From the comparison with Fig.1 where \( \delta \mu \) is fixed, the BP gap parameter here still starts with \( \Delta = 0 \) at \( \delta \mu = 0.5 \Delta_0 \) and ends with \( \Delta = \Delta_0 \) at \( \delta \mu = \Delta_0 \). The \( \alpha \)-dependence of \( \Delta \) and \( \delta \mu \) and the critical value \( \alpha_c \) in our calculation agree well with the analytic results obtained in weak coupling region[65, 66],

\[ \frac{\Delta(\alpha)}{\Delta_0} = \sqrt{1 - \frac{\alpha}{\alpha_c}}, \quad \frac{\delta \mu(\alpha)}{\Delta_0} = 1 - \frac{1}{2} \frac{\alpha}{\alpha_c}. \quad (50) \]

The question is which of the normal and BP states is relatively stable. From the direct calculation of the free energy as a function of the gap parameter for a specific asymmetry \( \alpha = 0.05 \) and coupling \( p_F a = 0.8 \) at \( T = 0 \), shown in Fig.7 the trivial solution \( \Delta = 0 \) is the maximum but the BP solution \( \Delta \neq 0 \) is the minimum of the free energy. We have checked this result for other values of \( \alpha \) and \( p_F a \), the qualitative behavior does not change. Therefore, the Sarma instability is avoided and the BP state becomes a stable solution in the family of isotropic phases. It is necessary to emphasize that the point to avoid the Sarma instability is the constraint of unequal numbers of the two species.

The LOFF solution is obtained through solving the equations for the gap \( \Delta \) and LOFF momentum, together with the known particle number densities \( n_a \) and \( n_b \). At zero temperature, the scaled gap parameter and LOFF momentum are plotted as functions of the scaled degree of asymmetry \( \alpha/\alpha_c \) in Figs.8 and 9. Here \( \alpha_c \) is the critical asymmetry where the LOFF superfluidity disappears. We have checked that for any \( \alpha < \alpha_c \), the LOFF solution obtained here is thermodynamically stable against a small perturbation of \( \Delta \), i.e., it corresponds to the minimum of \( F \) as a function of \( \Delta \). Exactly the same as that shown in Fig.1 the two branches LOFF1 and LOFF2 meet at \( \delta \mu = 0.635 \Delta_0 \), LOFF1 ends at \( \Delta = \Delta_0 \) and \( \delta \mu = \Delta_0 \), and LOFF2 ends at \( \Delta = 0 \) and \( \delta \mu \approx 0.754 \Delta_0 \). If we denote the asymmetry corresponding to the meeting point of LOFF1 and LOFF2 as \( \alpha_0 \), we found \( \alpha_0 \approx 0.55 \alpha_c \). The scaled LOFF momentum can vary in the range \( 0 < v_F q/\delta \mu < 1.3 \). In Fig.10 we showed the critical asymmetry for BP and LOFF states as a function of coupling strength \( p_F a \). We found that the critical asymmetry of LOFF state is larger than that of BP state, and the difference between the two increases with increasing coupling strength. It is necessary to note that the condition for the BP gap parameter, \( \Delta < \delta \mu \), is no longer necessary for the LOFF gap parameter, because of the term \( pq \cos \theta/m \) in the dispersion relation (27).

Now the question left is to extract the real ground state from the isotropic and anisotropic phases. Since the BCS phase is already kicked out, we need to com-
FIG. 9: The scaled LOFF momentum $v_F q / \delta \mu$ as a function of the scaled degree of asymmetry $\alpha / \alpha_c$ at $T = 0$. The dashed and solid lines correspond, respectively, to the branches LOFF1 and LOFF2.

FIG. 10: The critical asymmetry $\alpha_c$ as a function of the interaction strength $p_F a$. The solid and dashed lines correspond, respectively, to the LOFF and BP states.

FIG. 11: The scaled free energy $F/E_0$ as a function of the scaled pair momentum $q/p_F$ at $T = 0$, $\alpha = 0.05$ and $p_F a = 0.8$. The maximum at zero pair momentum and the minimum at finite pair momentum correspond, respectively, to the BP and LOFF states.

FIG. 12: The scaled free energies for the normal, BP and LOFF states as functions of the degree of asymmetry $\alpha$ at $T = 0$ and $p_F a = 0.6$ and 0.8. We have chosen the free energy of the normal state to be zero.

FIG. 13: The logitudinal superfluid density, scaled by $mn$, as a function of $\Delta / \Delta_0$.
fixed chemical potential asymmetry $\delta \mu$, both LOFF1 and LOFF2 are thermodynamically stable in the case with fixed density asymmetry $\alpha$. The conventional LOFF window $0.707 \Delta_0 < \delta \mu < 0.754 \Delta_0$ corresponds to the highly asymmetric region $0.93 \alpha_c < \alpha < \alpha_c$. If the density asymmetry is smaller than $0.93 \alpha_c$, the LOFF solution will not locate in the conventional LOFF window. In a recent study of two flavor color superconductivity, it is found that the LOFF solution satisfying electric charge neutrality is located at the LOFF1 branch.

We have shown that the LOFF state is thermodynamically stable in the whole region $0 < \alpha < \alpha_c$, where the gap can vary in the region $0 < \Delta < \Delta_0$ and the pair momentum can vary in the region $0 < v_F q < 1.3 \delta \mu$. Now we turn to check whether the LOFF state is dynamically stable in the whole region. Since the rotational symmetry $O(3)$ is broken down to $O(2)$, the superfluid density becomes a tensor $\rho_{ij}$. We decompose $\rho_{ij}$ into a transverse and a longitudinal part due to the residue symmetry $O(2)$,

$$\rho_{ij} = \rho_T (\delta_{ij} - \frac{\hat{q} \cdot \hat{q}_j}{\hat{q} \cdot \hat{q}_i}) + \rho_L \delta_{ij}$$  \hspace{0.5cm} (51)$$

with $\hat{q} \equiv \frac{q}{|q|}$. Neglecting the terms proportional to $q/p_F$ and $(q/p_F)^2$ and replacing $pq \cos \theta/m$ by $v_F q \cos \theta$ under the assumption of small $q$, we get at zero temperature

$$\rho_T = mn \left[ 1 - \frac{3}{4} \int_1^1 d \cos \theta \sin^2 \theta \eta_0 \frac{(\delta \mu - \Delta)}{\sqrt{\delta \mu - \Delta^2}} \right]$$

$$\rho_L = mn \left[ 1 - \frac{3}{2} \int_1^1 d \cos \theta \cos^2 \theta \eta_0 \frac{(\delta \mu - \Delta)}{\sqrt{\delta \mu - \Delta^2}} \right]$$  \hspace{0.5cm} (52)$$

with $\eta_0$ defined as

$$\eta_0 = \frac{\eta_3^3(\theta) + \eta_3^2(\theta)}{6\pi^2n}.$$  \hspace{0.5cm} (53)$$

Note that the expression $\rho_T$ is the same as the Meissner mass tensor for the 8th gluon in two flavor LOFF color superconductor, except for a pre-factor $mn$, if we employ the good approximation $\eta_0 \approx 1$. Using the numerical result for the LOFF solution, we calculated $\rho_T$ and $\rho_L$ and found that $\rho_T$ is almost zero, which is consistent with the analytical result in [54]. The longitudinal superfluid density $\rho_L$ scaled by $mn$ is shown in Fig. 13 as a function of the scaled gap parameter, it is positive in the region $0 < \Delta < 0.83 \Delta_0$ and negative for $0.83 \Delta_0 < \Delta < \Delta_0$. Hence, the LOFF state with $0.83 \Delta_0 < \Delta < 1$, i.e., in the small asymmetry region $0 < \alpha < 0.2 \alpha_c$ or small LOFF momentum region $0 < v_F q/\delta \mu < 0.4$, is dynamically unstable. For the BP state, the superfluid density is absolutely negative due to the fact that the manifold of the gapless excitations in momentum space is very large. This negative superfluid density is generally cured in the LOFF state since the manifold of gapless excitations is suppressed by anisotropy. However, in the region $0.83 \Delta_0 < \Delta < 1$, the scaled LOFF momentum $v_F q/\delta \mu$ which reflects the degree of anisotropy becomes very small, the superfluid density will still be negative. When $\Delta/\Delta_0 \to 1$, $v_F q/\delta \mu \to 0$, the anisotropy disappears, and the superfluid density becomes divergent.

VI. SUMMARY

The key theoretic problem in the study of asymmetric fermion superfluid is the pairing mechanism. The familiar mechanisms include isotropic BCS and BP states and anisotropic LOFF state. The question is which of them is the real ground state for a system under some physical condition which can lead to mismatched Fermi surfaces of the two fermions.

We have investigated the competition between the normal, BCS, BP and LOFF phases in asymmetric fermion superfluids in the frame of a general four fermion interaction model in weak coupling region. The masses of the two species $m_a$ and $m_b$ are set to be equal, but their chemical potentials $\mu_a$ and $\mu_b$ can be different. Two constraints on the system are discussed: (1) fixed chemical potential asymmetry and (2) fixed fermion number asymmetry.

In BP phase, there are two kinds of instabilities, one is the Sarma instability which was found many years ago, and the other is the magnetic instability which was recently pointed out in the study of interior gap superfluidity and gapless color superconductivity. While the magnetic instability gives us a strong hint that the LOFF phase may be stabler than the BP phase, to confirm this statement needs the comparison of the free energies for the normal, BCS, BP and LOFF states, when the chemical potentials of the two species are not directly fixed. By solving the gap equations, we obtained all possible solutions, and then by comparing the thermodynamic potentials $\Omega$ or free energies $F$ for the four phases, we determined the real ground state which corresponds to the lowest minimum of $\Omega$ or $F$. For case (1), the BP state is always unstable due to the Sarma instability, the system is in BCS phase when the chemical potential difference is small and in normal phase when the difference is large enough, and the LOFF state can exist only in a narrow window of asymmetry. For case (2), the Sarma instability of the BP state can be avoided due to the disappearance of the BCS phase, if $n_a$ and $n_b$ are set to be unequal. However, due to the magnetic instability of the BP state, in the coexistence region of BP and LOFF, the LOFF state is always the stable one. In addition, the LOFF state can even survive when the BP disappears in the highly asymmetric region. We emphasis that the LOFF windows in case (1) and (2) are quite different. In case (1), the LOFF state is thermodynamically stable only in a narrow window, i.e., $0.707 \Delta_0 < \delta \mu < 0.754 \Delta_0$ in weak coupling. In this window, the gap $\Delta$ can vary in the region $0 < \Delta < 0.24 \Delta_0$ and the LOFF momentum is almost a constant, $1.2 < v_F q/\delta \mu < 1.3$. However, in case (2), the LOFF state is thermodynam-
ically stable in a large region $0 < \alpha < \alpha_c$, which corresponds to $0.635 \Delta_0 < \delta \mu < \Delta_0$, $0 < \Delta < \Delta_0$ and $0 < \nu \rho < 1.3 \delta \mu$. Only in the small $\alpha$ region corresponding to $0.83 \Delta_0 < \Delta < \Delta_0$ the superfluid density is negative, which means that the LOFF state in this region is dynamically unstable.

We considered in this paper only the asymmetric systems with equal masses of the two kinds of fermions. However, a system with mass difference is of great interest, since it can be realized in the atomic fermion gas with a mixture of different fermionic ions such as $^6\text{Li}$ and $^{40}\text{K}$. As we have shown in [52], the breached pairing state can be free from magnetic instability when the mass ratio of the two species becomes very large. The true ground state of these systems is still unknown and needs further investigation. The superfluidity we considered here is non-relativistic, which can be applied to the study of electronic system, atomic fermion gas, and nuclear matter, and can be extended to relativistic fermion superfluids if the broken symmetry is Abelian. In the study of color superconductor where the broken color symmetry is non-Abelian, the problem of (chromo)magnetic instability is more complicated. Some approximate calculations in this direction are presented recently [57, 58]. Nevertheless, the mixed phase or phase separation is not included in our discussion. We have planned to give a more complete work including the important mixed phase [57].

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[1] G.Sarma, J.Phys.Chem.Solid 24, 1029 (1963).
[2] A.I.Larkin and Yu.N.Ovchinnikov, Sov.Phys. JETP 20 (1965).
[3] P.Fulde and R.A.Ferrell, Phys. Rev. A135, 550 (1964).
[4] S.Takada and T.Izuyama, Prog.Theor.Phys.41, 635 (1969).
[5] W.V.Liu and F.Wilczek, Phys. Rev. Lett.90, 047002 (2003).
[6] H.H.Oehl, B.T.Mathias and L.R.Walker, Phys.Rev.Lett.3, 552 (1959).
[7] J.Kondo, Prog.Theor.Phys.29, 1 (1963).
[8] J.I.Cirac and P.Zoller, Physics Today 57, 38 (2004).
[9] P.F.Bedaque, H.Caldas and G.Rupak, Phys. Rev. Lett. 91, 247002 (2003).
[10] H.Caldas, Phys. Rev. A69, 063602 (2004).
[11] M.W.Zwielein, A.Schirotzek, C.H.Schunck and W.Ketterle, Science 311 (5760), 492 (2006); cond-mat/0511197.
[12] G.B.Partridge, W.Li, R.I.Kamar, Y.Liao and R.G.Hulet, Science, 23 December 2005 (10.1126/science.1122876); cond-mat/0511172.
[13] A.Sedrakian and U.Lombardo, Phys. Rev. Lett. 84, 602 (2000).
[14] T.Schäfer and F.Wilczek, Phys. Rev. D60, 074014 (1999).
[15] M.Alford, J.Berbeges, and K.Rajagopal, Nucl. Phys. B558, 219 (1999).
[16] K.Rajagopal and F.Wilczek, Phys. Rev. Lett. 86, 3492 (2001).
[17] A.W.Steiner, S.Reddy, and M.Paraksh, Phys. Rev. D66, 094007 (2002).
[18] M.Alford and K.Rajagopal, JHEP 06, 031 (2002).
[19] M.Huang, P.Zhuang, and W.Chao, Phys. Rev. D67, 065015 (2003).
[20] I.Shovkovy and M.Huang, Phys. Lett. B564, 205 (2003).
[21] M.Huang and I.Shovkovy, Nucl. Phys. A729, 835 (2003).
[22] M.Alford, C.Kouvaris and K. Rajagopal, Phys. Rev. Lett. 92, 222001 (2004).
[23] H.Abuki, M.Kitazawa, and T.Kunihiro, Phys.Lett. B615, 102 (2005); H.Abuki, T.Kunihiro, Nucl.Phys. A768, 118 (2006).
[24] S.B.Ruster, V.Werth, M.Buballa, I.Shovkovy and D.H.Rischke, Phys. Rev. D72, 034004 (2005).
[25] D.Blaschke, S.Fredriksson, H.Grigorian, A.M.Oztas, and F.Sandlin, Phys.Rev. D72, 065020 (2005).
[26] S.Lawley, W.Bentz, A.W.Thomas, Phys.Lett. B632, 495 (2006).
[27] M.M.Forbes, E.Gubankova, W.Vincent Liu, F.Wilczek, Phys. Rev. Lett. 94, 017001 (2005).
[28] J.Liao and P.Zhuang, Phys. Rev. D68, 114016 (2003).
[29] M.Alford, J.Bowers and K.Rajagopal, Phys.Rev. D63, 074016 (2001).
[30] J.Bowers, J.Kundu, K.Rajagopal and E.Shuster, Phys.Rev. D64, 014024 (2001).
[31] A.K. Leibovich, K.Rajagopal and E.Shuster, Phys.Rev. D64, 094005 (2006).
[32] R.Casalbuoni, R.Gatto, M.Mannarelli and G.Nardulli, Phys.Lett. B511, 218 (2001).
[33] R.Casalbuoni and G.Nardulli, Rev.Mod.Phys. 76, 263 (2004).
[34] R.Casalbuoni, M.Ciminale, M.Mannarelli, G.Nardulli, M.Ruggieri, R.Gatto, Phys.Rev. D70, 054004 (2004).
[35] A.Sedrakian, Phys.Rev. C63, 025801 (2001).
[36] H.Muether and A.Sedrakian, Phys. Rev. C67, 015802 (2003).
[37] K. Machida and H. Nakanishi, Phys.Rev.B30, 122 (1984).
[38] A.Sedrakian, J.Mur-Petit, A.Polls and H.Mtcher, Phys.Rev. A72, 013613 (2005).
[39] P.Castorina, M.Grasso, M.Oertel, M.Urban, D.Zappala, Phys.Rev. A72, 025601 (2005).
[40] T.Mizushima, K.Machiida and M.Ichioka, Phys.Rev.Lett.94, 060404 (2005); Phys. Rev. Lett. 95, 117003 (2005).
[41] Kun Yang, Phys.Rev.Lett.95, 218903 (2005).
[42] D.T.Son and M.A.Stephanov, cond-mat/0507558.
[43] D.E.Sheehy and L.Radzihovsky, Phys.Rev.Lett. 96, 064001 (2006).
[44] Kun Yang, cond-mat/0508484.
[45] J.Dukelsky, G.Ortiz and S.M.A. Rombouts, cond-mat/0510635.
[46] M.Huang and I.Shovkovy, Phys.Rev. D70, R051501 (2004).
[47] M. Huang and I. Shovkovy, Phys. Rev. D70, 094030 (2004).
[48] R. Casalbuoni, R. Gatto, M. Mannarelli, G. Nardulli and M. Ruggieri, Phys. Lett. B605, 362 (2005).
[49] M. Alford, Q. Wang, J. Phys. G31, 719 (2005).
[50] K. Fukushima, [hep-ph/0506080]
[51] S. T. Wu and S. Yip, Phys. Rev. A67, 053603 (2003).
[52] L. He, M. Jin and P. Zhuang, Phys. Rev. B73, 024511 (2005).
[53] I. Giannakis and H. Ren, Phys. Lett. B611, 137 (2005).
[54] I. Giannakis and H. Ren, Nucl. Phys. B723, 255 (2005).
[55] I. Giannakis, D. Hou and H. Ren, Phys. Lett. B631, 16 (2005).
[56] M. Huang, Phys. Rev. D73, 045007 (2006).
[57] D. Hong, [hep-ph/0506097]
[58] E. V. Gorbar, M. Hashimoto and V. A. Miransky, Phys. Lett. B632, 305 (2006).
[59] E. V. Gorbar, M. Hashimoto and V. A. Miransky, Phys. Rev. Lett. 96, 022005 (2006).
[60] R. Casalbuoni, R. Gatto, N. Ippolito, G. Nardulli and M. Ruggieri, Phys. Lett. B627, 89 (2005).
[61] C. H. Pao, S. Wu and S. K. Yip, Phys. Rev. B73, 132506 (2006).
[62] E. Gubankova, W. V. Liu, F. Wilczek, Phys. Rev. Lett. 91, 032001 (2003).
[63] M. Kitazawa, D. Rischke and A. Shovkovy, [hep-ph/0602065]
[64] J. Carlson and S. Reddy, Phys. Rev. Lett. 95, 060401 (2005).
[65] J. Mur-Petit, A. Polls and H.-J. Schulze, Phys. Lett. A 290, 317 (2001).
[66] U. Lombardo, P. Nozieres, P. Schuck, H.-J. Schulze and A. Sedrakian, Phys. Rev. C 67, 015802 (2003).
[67] L. He, M. Jin and P. Zhuang, in progress.