ISS Property with Respect to Boundary Disturbances for a Class of Riesz-Spectral Boundary Control Systems

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Abstract

This paper deals with the establishment of Input-to-State Stability (ISS) properties for infinite dimensional systems with respect to both boundary and distributed disturbances. First, an ISS estimate is established with respect to finite dimensional boundary disturbances for a class of Riesz-spectral boundary control systems satisfying certain eigenvalue constraints. Second, a concept of weak solutions is introduced in order to relax the disturbances regularity assumptions required to ensure the existence of strong solutions. The proposed concept of weak solutions, that applies to a large class of boundary control systems which is not limited to the Riesz-spectral ones, provides a natural extension of the concept of both strong and mild solutions. Assuming that an ISS estimate holds true for strong solutions, we show the existence, the uniqueness, and the ISS property of the weak solutions.

Key words: Distributed parameter systems; Boundary control systems; Boundary disturbances; Input-to-state stability; Weak solutions.

1 Introduction

The concept of Input-to-State Stability (ISS), originally introduced by Sontag for finite dimensional systems [33], is one of the main tools for assessing the robustness of a system with respect to external disturbances. Specifically, the ISS property provides a quantification of the worst-case perturbation induced by external disturbances on the norm of the system state vector. This notion has been extensively investigated for finite dimensional systems during the last decade. More recently, the possible extension of ISS properties to Partial Differential Equations (PDEs), and more generally to infinite dimensional systems, has attracted much attentions [19,28,29].

For infinite dimensional systems, there exist essentially two distinct types of perturbations. The first type includes distributed perturbations; namely, perturbations acting directly in the state equation. The second type concerns boundary perturbations; namely, perturbations acting on the system state through an algebraic constraint by the means of an unbounded operator. In the case of PDEs, the distributed perturbations are also called in-domain perturbations as they appear directly in the PDEs. In contrast, the boundary perturbations appear in the boundary conditions of the PDEs. This second type of perturbation naturally appears in numerous boundary control problems such as heat equations [6], transport equations [17], diffusion or diffusive equations [2], and vibration of structures [6] with numerous practical applications, e.g., in robotics [12,14], aerospace engineering [3,20,21], and additive manufacturing [10,13].

* This publication has emanated from research supported in part by a research grant from Science Foundation Ireland (SFI) under Grant Number 16/RC/3872 and is co-funded under the European Regional Development Fund. The material in this paper was not presented at any conference.

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While many results have been reported regarding the ISS property with respect to distributed disturbances \([1,7,8,24,25,26,27,30]\), the establishment of ISS properties with respect to boundary disturbances remains challenging \([2,16,17,18]\). The traditional method to study abstract boundary control systems consists of transferring the boundary disturbances into a distributed one by means of a lifting operator. By doing so, the original boundary control system is made equivalent to a standard evolution equation for which efficient analysis tools exist. The main issue with such an approach is that the resulting distributed perturbation involves the time derivative of the boundary perturbation \([6]\). In particular, this induces two main difficulties. First, this approach fails to establish the ISS property with respect to the boundary disturbance, but can only show the ISS property with respect to the first time derivative of the boundary disturbance. Second, in order to ensure the existence of strong solutions, one has to assume that the boundary disturbance is twice continuously differentiable. The relaxation of this regularity assumption requires the introduction of a concept of mild or weak solutions extending the one of strong solutions. However, the explicit occurrence of the time derivative of the boundary perturbation in the evolution equation does not allow a straightforward introduction of such a concept of mild or weak solutions for boundary disturbances that are only assumed, e.g., to be continuous \([11]\).

Inspired by well established finite-dimensional techniques, it has recently been proposed to resort to Lyapunov functions to establish the ISS properties of PDEs \([2,34,37,38]\). An other approach, based on functional analysis tools, has been proposed in \([17]\) for the study of 1-D parabolic equations. In the problem therein, (the negative of) the functions to establish the ISS properties of PDEs \([2,34,37,38]\). An other approach, based on functional analysis tools, inspired by well established finite-dimensional techniques, has recently been proposed to resort to Lyapunov functions to establish the ISS properties of PDEs \([2,34,37,38]\). An other approach, based on functional analysis tools, inspired by well established finite-dimensional techniques, has recently been proposed to resort to Lyapunov functions to establish the ISS properties of PDEs \([2,34,37,38]\). An other approach, based on functional analysis tools, inspired by well established finite-dimensional techniques, has recently been proposed to resort to Lyapunov functions to establish the ISS properties of PDEs \([2,34,37,38]\)

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The contribution of this paper is twofold. First, we establish the ISS property with respect to finite dimensional boundary inputs for a class of Riesz-spectral boundary control systems satisfying certain eigenvalue constraints. A similar problem was investigated in \([15]\) for the case of a one-dimensional input space. The problem was embedded into the extrapolation space \(\mathcal{H}_{-1}\) while invoking admissibility conditions for returning to the original Hilbert space \(\mathcal{H}\). The approach adopted in this paper differs by generalizing the ideas developed first in \([17]\) and then in \([22]\).

Assuming boundary and distributed disturbances of class \(C^2\) and \(C^1\), respectively, the ISS property is established for strong solutions by taking advantage of the projection of the system trajectories over a Riesz basis formed by the eigenvectors of the disturbance free operator. By doing so, the ISS property is derived directly in the original Hilbert space and on the original system, avoiding the occurrence of the time derivative of the boundary perturbation.

The second contribution of this paper deals with the introduction of a concept of weak solutions that allows the relaxation of the perturbations regularity assumptions from \(C^2\) and \(C^1\) to \(C^0\). This approach applies to a general class of boundary control systems that is not limited to Riesz-spectral ones. Inspired by distribution theory, weak solutions are introduced under a variational formulation over an adequate space of test functions within the original Hilbert space \(\mathcal{H}\). In particular, it avoids either the embedding of the original problem into the extrapolation space \(\mathcal{H}_{-1}\) \([11,35]\) or the abstract extension of the mild solutions by pure density arguments \([32]\). First, it is shown that the proposed concept of weak solutions is a natural extension of the concept of both strong and mild solutions. Second, assuming that an ISS estimate holds true with respect to strong solutions, the existence and the uniqueness of the weak solutions are assessed. Finally, it is shown that the ISS estimate satisfied by strong solutions also holds true for weak solutions.

The remainder of this paper is organized as follows. Notations and definitions are introduced in Section 2. The assessment of the ISS property for a class of Riesz-spectral boundary control systems with respect to strong solutions is presented in Section 3. Then, a concept of weak solutions under a variational formulation, its properties, and the derivation of the ISS estimate are presented in Section 4 for a large class of boundary control systems. The obtained results are applied on illustrative examples in Section 5. Concluding remarks are provided in Section 6.

\[\text{More precisely, denoting by } \mathcal{A}_0 \text{ the disturbance free operator, } -\mathcal{A}_0 \text{ is a Sturm-Liouville operator.}\]
2 Notations, Definitions and Problem Setting

2.1 Notations

The sets of non-negative integers, positive integers, integers, real, non-negative real, positive real, negative real, and complex numbers are denoted by \( \mathbb{N} \), \( \mathbb{N}^* \), \( \mathbb{Z} \), \( \mathbb{R} \), \( \mathbb{R}_+ \), \( \mathbb{R}_+^* \), \( \mathbb{R}_- \), and \( \mathbb{C} \), respectively. For any \( z \in \mathbb{C} \), \( \text{Re}(z) \) denotes the real part of \( z \). Throughout the paper, the field \( \mathbb{K} \) is either \( \mathbb{R} \) or \( \mathbb{C} \).

We consider the following classic classes of comparison function:

\[
\mathcal{K} = \left\{ \gamma \in C^0(\mathbb{R}_+; \mathbb{R}_+^*) : \gamma \text{ strictly increasing}, \gamma(0) = 0 \right\},
\]

\[
\mathcal{L} = \left\{ \gamma \in C^0(\mathbb{R}_+; \mathbb{R}_+^*) : \gamma \text{ strictly decreasing}, \lim_{t \to +\infty} \gamma(t) = 0 \right\},
\]

\[
\mathcal{KL} = \left\{ \beta \in C^0(\mathbb{R} \times \mathbb{R}_+; \mathbb{R}_+^*) : \beta(\cdot, t) \in \mathcal{K} \text{ for } t \geq 0, \beta(x, \cdot) \in \mathcal{L} \text{ for } x > 0 \right\}.
\]

For an interval \( I \subset \mathbb{R} \) and a \( \mathbb{K} \)-normed linear space \( (E, \| \cdot \|_E) \), \( C^0(I; E) \) denotes the set of functions \( f : I \to E \) that are \( n \) times continuously differentiable. For any \( a < b \), we endow \( C^0([a, b]; E) \) with the norm \( \| f \|_{C^0([a, b]; E)} \) defined for any \( f \in C^0([a, b]; E) \) by

\[
\| f \|_{C^0([a, b]; E)} = \sup_{t \in [a, b]} \| f(t) \|_E.
\]

We denote by \( L^2(0, 1) \) and \( H^m(0, 1) \) the set of square (Lebesgue) integrable functions over \((0, 1)\) and the usual Sobolev space of order \( m \) over \((0, 1)\), respectively. We also introduce \( H^1_0(0, 1) \equiv \{ f \in H^1(0, 1) : f(0) = 0 \} \) and \( H^1_1(0, 1) \equiv \{ f \in H^1(0, 1) : f(0) = f(1) = 0 \} \).

For a given linear operator \( L, R(L), \ker(L), \) and \( \rho(L) \) denote its range, its kernel, and its resolvent set, respectively. \( L(E, F) \) denotes the set of bounded linear operators from \( E \) to \( F \).

Let \( (\mathbb{K}^m, \| \cdot \|_{\mathbb{K}^m}) \) be a normed space with \( m \in \mathbb{N}^* \). For a given basis \( \mathcal{E} = \{ e_1, e_2, \ldots, e_m \} \) of \( \mathbb{K}^m \), we denote by \( \| \cdot \|_{\infty, \mathcal{E}} \) the infinity norm \( \| \cdot \| \) in \( \mathcal{E} \). By virtue of the equivalence of the norms in finite dimension, we denote by \( c(\mathcal{E}) \in \mathbb{R}_+^* \) the smallest constant such that \( \| \cdot \|_{\infty, \mathcal{E}} \leq c(\mathcal{E}) \| \cdot \|_{\mathbb{K}^m} \).

Finally, we introduce the Kronecker notation: \( \delta_{a,b} = 1 \) if \( a = b \), 0 otherwise. The time derivative of a real-valued differentiable function \( f : I \to \mathbb{R} \) is denoted by \( \dot{f} \). If \( \mathcal{H} \) is a Hilbert space, the time derivative of a \( \mathcal{H} \)-valued differentiable function \( f : I \to \mathcal{H} \) is denoted by \( df/dt \).

2.2 Definitions and related properties

In this paper, we consider the following definition of boundary control systems for finite dimensional boundary inputs [6, Def. 3.3.2].

**Definition 1 (Boundary control system)** Let \( (\mathcal{H}, \langle \cdot, \cdot \rangle_{\mathcal{H}}) \) be a separable Hilbert space over \( \mathbb{K} \) and \( (\mathbb{K}^m, \| \cdot \|_{\mathbb{K}^m}) \) be a \( \mathbb{K} \)-normed space with \( m \in \mathbb{N}^* \). Consider the abstract system taking the form:

\[
\begin{aligned}
\frac{dX}{dt}(t) &= AX(t) + U(t), \quad t \geq 0 \\
BX(t) &= d(t), \quad t \geq 0 \\
X(0) &= X_0
\end{aligned}
\]

with

\[
\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H} \text{ an (unbounded) operator;}
\]

\footnote{Defined as the maximum of the absolute values of the vector components when written in the basis \( \mathcal{E} \).}
We say that \((A,B)\) is a boundary control system, with associated abstract system \((1)\), if

1. the disturbance free operator \(A_0\), defined over the domain \(D(A_0) \triangleq D(A) \cap \ker(B)\) by \(A_0 \triangleq A|_{D(A_0)}\), is the generator of a \(C_0\)-semigroup \(S\) on \(H\);
2. there exists a bounded operator \(B \in \mathcal{L}(\mathbb{K}^m,H)\), called a lifting operator, such that \(R(B) \subset D(A), \ AB \in \mathcal{L}(\mathbb{K}^m,H)\), and \(BB = I_{\mathbb{K}^m}\).

**Remark 1** As the boundary input space \(\mathbb{K}^m\) is finite dimensional, the bounded nature of \(B\) and \(AB\) is immediate. Thus, the existence of the lifting operator \(B\) reduces to the right invertibility of the boundary operator \(B|_{D(A)}\). This condition is equivalent to the surjectivity of \(B|_{D(A)}\).

We then introduce the concepts of Riesz-basis [5] and Riesz-spectral operators [6, Def. 2.3.4].

**Definition 2 (Riesz basis)** A sequence of vectors \(\{\phi_n, \ n \in \mathbb{N}\} \subset H\) from a separable Hilbert space \((H, \langle \cdot, \cdot \rangle_H)\) over \(\mathbb{K}\) is a Riesz basis if

1. \(\{\phi_n, \ n \in \mathbb{N}\}\) is maximal in the sense that \(\operatorname{span} \phi_n = H\), i.e., the closure of the vector space spanned by the vectors \(\phi_n\) coincides with the whole space \(H\);
2. there exist constants \(m_R, M_R \in \mathbb{R}^+\) such that for all \(N \in \mathbb{N}\) and all \(\alpha_0, \ldots, \alpha_N \in \mathbb{K}\),

\[
m_R \sum_{n=0}^N |\alpha_n|^2 \leq \left\| \sum_{n=0}^N \alpha_n \phi_n \right\|_H^2 \leq M_R \sum_{n=0}^N |\alpha_n|^2.
\]

**Definition 3 (Riesz spectral operator)** Let \(A_0 : D(A_0) \subset H \to H\) be a linear and closed operator. For \(n \in \mathbb{N}\), let \(\lambda_n\) be the eigenvalues of \(A_0\) and \(\phi_n \in D(A_0)\) the corresponding eigenvectors. \(A_0\) is a Riesz-spectral operator if

1. \(\{\phi_n, \ n \in \mathbb{N}\}\) is a Riesz basis;
2. the closure of \(\{\lambda_n, \ n \in \mathbb{N}\}\) is totally disconnected, i.e., for any distinct \(a, b \in \{\lambda_n, \ n \in \mathbb{N}\}, [a, b] \subset \{\lambda_n, \ n \in \mathbb{N}\}\).

A subset of the properties satisfied by Riesz-spectral operators that will be useful in the upcoming developments are gathered in the following lemma [6, Lemmas 2.3.2 and 2.3.3].

**Lemma 1** Let \(A_0 : D(A_0) \subset H \to H\) be a Riesz-spectral operator. With the notations of Definition 3, the following are true.

- The eigenvalues of the adjoint operator \(A_0^*\) are provided for \(n \in \mathbb{N}\) by \(\mu_n \triangleq \overline{\lambda_n}\) and the associated eigenvectors \(\psi_n \in D(A_0^*)\) can be selected such that \(\{\phi_n, \ n \in \mathbb{N}\}\) and \(\{\psi_n, \ n \in \mathbb{N}\}\) are biorthogonal, i.e., for all \(n,m \in \mathbb{N}\), \(\langle \phi_n, \psi_m \rangle_H = \delta_{n,m}\).
- The sequence of vectors \(\{\psi_n, \ n \in \mathbb{N}\}\) is a Riesz basis.
- For all \((\alpha_n)_{n \in \mathbb{N}} \in \mathbb{K}^\mathbb{N}\),

\[
\sum_{n \in \mathbb{N}} |\alpha_n|^2 < \infty \iff \sum_{n \in \mathbb{N}} \alpha_n \phi_n \in H.
\]

- For all \(z \in H\),

\[
z = \sum_{n \in \mathbb{N}} \langle z, \phi_n \rangle_H \phi_n = \sum_{n \in \mathbb{N}} \langle z, \phi_n \rangle_H \psi_n.
\]

- \(A_0\) is the generator of a \(C_0\)-semigroup \(S\) if and only if

\[
\sup_{n \in \mathbb{N}} \operatorname{Re}(\lambda_n) < \infty.
\]
In this case,
\[ \forall t \in \mathbb{R}_+, \forall z \in \mathcal{H}, \ S(t)z = \sum_{n \in \mathbb{N}} e^{\lambda_n t} \langle z, \psi_n \rangle_{\mathcal{H}} \phi_n, \]
and its growth-bound satisfies:
\[ \omega_0 \triangleq \inf_{t > 0} \frac{\log \| S(t) \|}{t} = \sup_{n \in \mathbb{N}} \Re(\lambda_n). \]

Finally, we introduce the following definition.

**Definition 4 (Riesz-spectral boundary control system)** We say that \((\mathcal{A}, \mathcal{B})\) is a Riesz-spectral boundary control system, with associated abstract system (1), if

1. \((\mathcal{A}, \mathcal{B})\) is a boundary control system with associated abstract system (1);
2. the underlying disturbance free operator \(\mathcal{A}_0\) is a Riesz-spectral operator.

In particular, we will consider the class of Riesz-spectral boundary control systems such that the two following eigenvalue conditions hold:
\[ \omega_0 = \sup_{n \in \mathbb{N}} \Re(\lambda_n) < 0, \quad \zeta \triangleq \sup_{n \in \mathbb{N}} \frac{|\lambda_n|}{|\Re(\lambda_n)|} < \infty. \]

The constraint \(\omega_0 < 0\) is a necessary and sufficient condition for ensuring the exponential stability of the \(C_0\)-semigroup \(S\), i.e., the existence of \(M, \kappa_0 \in \mathbb{R}_+^\ast\) such that \(\| S(t) \| \leq M e^{-\kappa_0 t}\). Introducing \(\xi_n = |\Re(\lambda_n)|/|\lambda_n|\), the damping ratio of the eigenvalue \(\lambda_n\), the second constraint \(\zeta < \infty\) is nothing but the strict positiveness of the infimum of the system damping ratios \(\xi_n\). This second constraint can be slightly relaxed, as discussed latter in Remark 3.

### 2.3 Examples

We present three examples of PDEs that can be written in abstract form as Riesz-spectral boundary control systems satisfying the conditions (6).

#### 2.3.1 1D parabolic PDEs

For \(p \in C^1([0,1]; \mathbb{R})\) and \(q, r \in C^0([0,1]; \mathbb{R})\) such that \(p(x), r(x) > 0\) for all \(x \in [0,1]\), we consider the class of 1D parabolic PDEs given by [17]:
\[ \frac{\partial y}{\partial t} - \frac{1}{r} \frac{\partial}{\partial x} \left( p \frac{\partial y}{\partial x} \right) + \frac{q}{r} y = u, \quad \text{in } \mathbb{R}_+ \times (0,1) \]
\[ \cos(\alpha)y(t,0) - \sin(\alpha) \frac{\partial y}{\partial x}(t,0) = d_1(t), \quad t \in \mathbb{R}_+ \]
\[ \cos(\beta)y(t,1) + \sin(\beta) \frac{\partial y}{\partial x}(t,1) = d_2(t), \quad t \in \mathbb{R}_+ \]
\[ y(0,x) = y_0(x), \quad x \in (0,1) \]
with constants \(\alpha, \beta \in [0,2\pi]\). The functions \(u \in C^0([0,1]; L^2(0,1))\) and \(d_1, d_2 \in C^0(\mathbb{R}_+; \mathbb{K})\) are distributed and boundary perturbations, respectively. The function \(y_0 \in L^2(0,1)\) represents the initial condition. Note that a special case of this problem has been investigated in [17] when \(d_2 = 0, u = 0\), and for initial conditions \(y_0\) and boundary disturbance \(d_1\) of class \(C^2\).

**Remark 2** To place the above discussion in context, this class of 1D parabolic PDEs occurs in the modeling of diffusive phenomena, e.g., the 1D heat equation describing the distribution of heat over a one dimensional region. It also arises, via a change of variable, in 1D transport equations [17].
the distributed parameter system can be written as the abstract system (1) with

$$Af = \frac{1}{r}(pf')' - \frac{q}{r}f$$

defined over the domain $D(A) = H^2(0, 1)$, the boundary operator

$$Bf = \begin{pmatrix} \cos(\alpha)f(0) - \sin(\alpha)f'(0) \\ \cos(\beta)f(1) + \sin(\beta)f'(1) \end{pmatrix}$$

defined over the domain $D(B) = D(A)$, the state vector $X(t) = y(t, \cdot) \in \mathcal{H}$, the initial condition $X_0 = y_0 \in \mathcal{H}$, the boundary disturbance $d = (d_1, d_2) \in C^0(\mathbb{R}_+; \mathbb{K}^2)$, and the distributed disturbance $U = u \in C^0(\mathbb{R}_+; \mathcal{H})$.

The linear operator $B$ defined such that for all $d = (d_1, d_2) \in \mathbb{K}^2$ and for all $x \in [0, 1]$,

$$(Bd)(x) = d_1 (\cos(\alpha) - \sin(\alpha)x) + c_2(d)x^2 + c_3(d)x^3,$$

where in the case $\tan(\beta) \neq 1/2$, $c_3(d) = 0$ and

$$c_2(d) = \frac{d_2 + d_1 [\sin(\beta)\sin(\alpha) - \cos(\beta)(\cos(\alpha) - \sin(\alpha))]}{\cos(\beta) + 2\sin(\beta)},$$

while in the case $\tan(\beta) = 1/2$, $c_2(d) = 0$ and

$$c_3 = \frac{d_2 + d_1 [\sin(\beta)\sin(\alpha) - \cos(\beta)(\cos(\alpha) - \sin(\alpha))]}{\cos(\beta) + 3\sin(\beta)},$$

is a lifting operator associated with $(A, B)$. It is shown in [9] that the disturbance free operator $A_0$ is a Riesz-spectral operator. As $-A_0$ belongs to the class of Sturm-Liouville operators, it is well known [31, Th. 8.97] that all the eigenvalues of $A_0$ are real and form a strictly decreasing sequence $\lambda_0 > \lambda_1 > \ldots > \lambda_n > \ldots$. We deduce from the last point of Lemma 1 that $(A, B)$ is a Riesz-spectral boundary control system. Furthermore, assuming that $p$, $q$, and $r$ are such that $\exists \lambda_0 < 0$, the eigenvalue constraints (6) are satisfied because the growth bound is such that $\omega_0 = \lambda_0 < 0$ and $\zeta = 1 < \infty$.

### 2.3.2 Damped string

PDEs are also used to describe wave propagation and structural vibration. As an example we consider the case of a string with Kelvin-Voigt damping and clamped-free boundary conditions described by [23,36]:

$$\frac{\partial^2 y}{\partial t^2} - \frac{\partial}{\partial x} \left( \alpha \frac{\partial y}{\partial x} + \beta \frac{\partial^2 y}{\partial t \partial x} \right) = u,$$

in $\mathbb{R}_+ \times (0, 1)$

$$y(t, 0) = 0,$$

t $\in \mathbb{R}_+$

$$y(0, x) = y_0(x),$$

$$\frac{\partial y}{\partial t}(0, x) = y_{t0}(x),$$

where $x \in (0, 1)$

where $\alpha, \beta \in \mathbb{R}_+^*$ are constant parameters. The functions $u \in C^0(\mathbb{R}_+; L^2(0, 1))$ and $d \in C^0(\mathbb{R}_+; \mathbb{K})$ are distributed and boundary perturbations, respectively. The functions $y_0 \in H^1_L(0, 1)$ and $y_{t0} \in L^2(0, 1)$ are the initial conditions.

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3 According to [31, Th. 8.97], a sufficient condition is provided by $\tan(\alpha) \geq 0$, $\tan(\beta) \geq 0$, and $\min_{x \in [0, 1]} q(x) > 0$. 

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Introducing the Hilbert space
\[ \mathcal{H} = H^1_0(0, 1) \times L^2(0, 1) \]
with the inner product defined for all \((x_1, x_2), (\hat{x}_1, \hat{x}_2) \in \mathcal{H}\) by
\[ \langle (x_1, x_2), (\hat{x}_1, \hat{x}_2) \rangle_{\mathcal{H}} = \int_0^1 \alpha x'_1(\xi)\hat{x}'_1(\xi) + x_2(\xi)\hat{x}_2(\xi) d\xi, \]
the distributed parameter system can be written as the abstract system (1) with
\[ A(x_1, x_2) = (x_2, (\alpha x'_1 + \beta x'_2)^{\prime}) \]
defined over the domain
\[ D(A) = \{(x_1, x_2) \in \mathcal{H} : x_2 \in H^1_0(0, 1), (\alpha x'_1 + \beta x'_2) \in H^1(0, 1)\}, \]
the boundary operator
\[ B(x_1, x_2) = (\alpha x'_1 + \beta x'_2)(1) \]
defined over the domain \(D(B) = D(A)\), the state vector \(X(t) = (y(t, \cdot), y_0(t, \cdot)) \in \mathcal{H}\), the initial condition \(X_0 = (y_0, y_{00}) \in \mathcal{H}\), and \(U = (0, u) \in C^0(\mathbb{R}_+; \mathcal{H})\). It is shown in [22] that if \(\alpha, \beta \in \mathbb{R}^*_+\) are such that
\[ \frac{2\sqrt{\alpha}}{\pi \beta} - \frac{1}{2} \notin \mathbb{N}, \quad (8) \]
then \((A, B)\) is a Riesz-spectral boundary control system with lifting operator \(B\) defined for any \(d \in \mathbb{K}\) and \(x \in [0, 1]\) by
\[ (Bd)(x) = \left( \frac{d}{\alpha}, 0 \right). \]
Furthermore, the eigenvalue constraints (6) are satisfied since it is shown in [22] that the growth bound is such that \(\omega_0 < 0\) and we have \(\zeta < \infty\) because only a finite number of eigenvalues are such that \(\lambda_n \notin \mathbb{R}\).

### 2.3.3 Damped Euler-Bernoulli beam

An other example of structural vibration described by PDEs is the Euler-Bernoulli beam equation. Specifically, we consider a damped Euler-Bernoulli beam with point torque boundary conditions described by [6]:

\[
\begin{align*}
\frac{\partial^2 y}{\partial t^2} + \frac{\partial^4 y}{\partial x^4} - 2\alpha \frac{\partial^3 y}{\partial t \partial x^2} &= u, & \text{in } \mathbb{R}_+ \times (0, 1) \\
y(t, 0) &= 0, & t \in \mathbb{R}_+ \\
y(t, 1) &= 0, & t \in \mathbb{R}_+ \\
\frac{\partial^2 y}{\partial x^2}(t, 0) &= d_1(t), & t \in \mathbb{R}_+ \\
\frac{\partial^2 y}{\partial x^2}(t, 1) &= d_2(t), & t \in \mathbb{R}_+ \\
y(0, x) &= y_0(x), & x \in (0, 1) \\
\frac{\partial y}{\partial t}(0, x) &= y_{00}(x), & x \in (0, 1)
\end{align*}
\]

where \(\alpha \in \mathbb{R}^*_+ \setminus \{1\}\) is a constant parameter. The functions \(u \in C^0(\mathbb{R}_+; L^2(0, 1))\) and \(d_1, d_2 \in C^0(\mathbb{R}_+; \mathbb{K})\) are distributed and boundary perturbations, respectively. The functions \(y_0 \in H^2(0, 1) \cap H^1_0(0, 1)\) and \(y_{00} \in L^2(0, 1)\) are the initial conditions.

Introducing the Hilbert space
\[ \mathcal{H} = (H^2(0, 1) \cap H^1_0(0, 1)) \times L^2(0, 1) \]
with the inner product defined for all \((x_1, x_2), (\dot{x}_1, \dot{x}_2) \in \mathcal{H}\) by
\[
((x_1, x_2), (\dot{x}_1, \dot{x}_2))_{\mathcal{H}} = \int_0^1 x''_1(\xi)\overline{x''_2(\xi)} + x_2(\xi)\overline{\dot{x}_2(\xi)}d\xi,
\]
the distributed parameter system can be written as the abstract system (1) with
\[
A(x_1, x_2) = (x_2, -x''_1 + 2\alpha x'_2)
\]
defined over the domain
\[
D(A) = (H^4(0, 1) \cap H^1_0(0, 1)) \times (H^2(0, 1) \cap H^1_0(0, 1)),
\]
the boundary operator
\[
B(x_1, x_2) = (x''_1(0), x''_2(1))
\]
defined over the domain \(D(B) = D(A)\), the state vector \(X(t) = (y(t, \cdot), y_1(t, \cdot)) \in \mathcal{H}\), the initial condition \(X_0 = (y_0, y_0) \in \mathcal{H} \), \(d = (d_1, d_2) \in C^0(\mathbb{R}_+; \mathbb{K}^2)\), and \(U = (0, u) \in C^0(\mathbb{R}_+; \mathcal{H})\).

The linear operator \(B\) defined such that for all \(d = (d_1, d_2) \in \mathbb{K}^2\) and for all \(x \in [0, 1]\),
\[
(Bd)(x) = \left(\frac{d_2 - d_1}{6} x^3 + \frac{d_1}{2} x^2 - \frac{2d_1 + d_2}{6} x, 0\right),
\]
is a lifting operator associated with \((A, B)\). Following [6, Exercise 2.23], it can be shown that the disturbance free operator \(A_0\) is a Riesz-spectral operator generating a \(C_0\)-semigroup of contractions. Thus, \((A, B)\) is a Riesz spectral boundary control system. Furthermore, the eigenvalues of \(A_0\) are given by \(\{ -n^2 \pi^2(\alpha \pm i\sqrt{1 - \alpha^2}), n \in \mathbb{N}^* \}\) when \(\alpha \in (0, 1)\) while given by \(\{-n^2 \pi^2(\alpha \pm \sqrt{\alpha^2 - 1}), n \in \mathbb{N}^* \}\) when \(\alpha > 1\). In both cases, the eigenvalue constraints (6) are satisfied with \(\omega_0 = -\alpha \pi^2\) and \(\zeta = \alpha^{-1}\) when \(\alpha \in (0, 1)\) while \(\omega_0 = -(\alpha - \sqrt{\alpha^2 - 1}) \pi^2\) and \(\zeta = 1\) when \(\alpha > 1\).

3 ISS Assessment For Riesz-Spectral Boundary Control Systems With Respect to Strong Solutions

3.1 Definition of strong solutions and well-posedness

**Definition 5 (Strong solutions)** Let \((A, B)\) be a boundary control system. Let \(X_0 \in D(A), d \in C^0(\mathbb{R}_+; \mathbb{K}^m)\) such that \(BX_0 = d(0)\), and \(U \in C^0(\mathbb{R}_+; \mathcal{H})\) be given. We say that \(X(t)\) is a strong solution of (1) associated with \((X_0, d, U)\) if \(X \in C^0(\mathbb{R}_+; D(A)) \cap C^1(\mathbb{R}_+; \mathcal{H}), X(0) = X_0\), and for all \(t \geq 0\), \((dX/dt)(t) = AX(t) + U(t)\) and \(BX(t) = d(t)\).

The existence of a lifting operator \(B\) (see Definition 1) plays a key role in the well-posedness assessment of boundary control systems when \(d \in C^2(\mathbb{R}_+; \mathbb{K}^m)\) and \(U \in C^1(\mathbb{R}_+; \mathcal{H})\) (see, e.g., [6]). Indeed, introducing \(V = X - Bd\), straightforward computations show that \(X\) is a strong solution of (1) if and only if \(V\) is a strong solution of the boundary disturbance free abstract system:
\[
\begin{align*}
\frac{dV}{dt}(t) &= A_0 V(t) - Bd(t) + ABd(t) + U(t), \quad t \geq 0 \\
V(0) &= X_0 - Bd(0)
\end{align*}
\]
where \(-Bd + ABd \in C^1(\mathbb{R}_+; \mathcal{H})\) can be interpreted as a distributed disturbance resulting from the transfer of the boundary disturbance \(d\) by means of the lifting operator \(B\). In particular, the obtained distributed disturbance involves \(d\), the time derivative of the boundary disturbance \(d\).

As \(A_0\) generates a \(C_0\)-semigroup \(S\) and \(-Bd + ABd + U \in C^1(\mathbb{R}_+; \mathcal{H})\), classic results (see, e.g., [6, Th. 3.1.3]) ensure the well-posedness of (9), yielding the well-posedness of the original boundary control system (1). Furthermore, the system trajectory is explicitly given for all \(t \geq 0\) by:
\[
X(t) = S(t)(X_0 - Bd(0)) + Bd(t) + \int_0^t S(t - \tau) \left\{-B\dot{d}(\tau) + ABd(\tau) + U(\tau)\right\} d\tau.
\]
Assuming the exponential stability of the $C_0$-semigroup $S$, a direct estimation of (10) provides the exponential ISS estimate (in uniform norm) of the strong solutions of (1) with respect to $d$, $\bar{d}$, and $U$. However, this result is not the strict form of ISS due to the presence of the term $\bar{d}$.

3.2 ISS for strong solutions

The main result of this section is the following theorem.

**Theorem 1** Let $(A, B)$ be a Riesz-spectral boundary control system such that the eigenvalue constraints (6) hold. For every initial condition $X_0 \in D(A)$, and every disturbance $d \in C^2(\mathbb{R}_+; \mathbb{K}^m)$ and $U \in \mathcal{C}^1(\mathbb{R}_+; \mathcal{H})$ such that $BX_0 = d(0)$, the abstract system (1) has a unique strong solution $X \in \mathcal{C}^0(\mathbb{R}_+; D(A)) \cap \mathcal{C}^1(\mathbb{R}_+; \mathcal{H})$ associated with $(X_0, d, U)$. Furthermore, the system is exponentially ISS in the sense that there exist $C_0, C_1, C_2 \in \mathbb{R}_+$, independent of $X_0$, $d$, and $U$, such that for all $t \geq 0$,

$$\|X(t)\|_\mathcal{H} \leq C_0 e^{-\kappa_0 t} \|X_0\|_\mathcal{H} + C_1 \|d\|_{C^0([0,t]; \mathbb{K}^m)} + C_2 \|U\|_{C^0([0,t]; \mathcal{H})}$$

(11)

with $\kappa_0 = -\omega_0 < 0$, where $\omega_0$ is the growth bound of the $C_0$-semigroup $S$ generated by the disturbance free operator $\mathcal{A}_0$.

**Proof of Theorem 1.** The existence and uniqueness part directly follows from the classic results discussed in Subsection 3.1. Thus, the proof is devoted to the derivation of the ISS estimate (11). Let $B$ be a lifting operator associated with the boundary control system $(A, B)$ as provided by Definition 1. Let $X_0 \in D(A), d \in C^2(\mathbb{R}_+; \mathbb{K}^m)$, and $U \in \mathcal{C}^1(\mathbb{R}_+; \mathcal{H})$ such that $BX_0 = d(0)$ be given. Let $X$ be the strong solution of (1) associated with $(X_0, d, u)$. Adopting the notations of Definition 3 and Lemma 1, we have from (4) that for all $t \geq 0$,

$$X(t) = \sum_{n \in \mathbb{N}} \langle X(t), \psi_n \rangle_\mathcal{H} \phi_n.$$ 

Introducing for all $n \in \mathbb{N}$ and all $t \geq 0$, $c_n(t) \triangleq \langle X(t), \psi_n \rangle_\mathcal{H}$, then $c_n \in \mathcal{C}^1(\mathbb{R}_+; \mathbb{K})$ with

$$\dot{c}_n(t) = \frac{d}{dt} \langle X(t), \psi_n \rangle_\mathcal{H} = \langle A\{X(t) - Bd(t)\}, \psi_n \rangle_\mathcal{H} + \langle ABd(t), \psi_n \rangle_\mathcal{H} + \langle U(t), \psi_n \rangle_\mathcal{H},$$

where the last equality holds true because $X(t) \in D(A)$ and $R(B) \subset D(A)$, providing $X(t) - Bd(t) \in D(A)$. Furthermore, $B\{X(t) - Bd(t)\} = d(t) - d(t) = 0$. Thus $X(t) - Bd(t) \in D(A_0)$, yielding

$$\langle A\{X(t) - Bd(t)\}, \psi_n \rangle_\mathcal{H} = \langle A_0\{X(t) - Bd(t)\}, \psi_n \rangle_\mathcal{H} = \langle X(t) - Bd(t), A_0^* \psi_n \rangle_\mathcal{H} = \langle X(t) - Bd(t), \lambda_n \psi_n \rangle_\mathcal{H} = \lambda_n \langle X(t) - Bd(t), \psi_n \rangle_\mathcal{H}.$$

We get for all $t \geq 0$,

$$\dot{c}_n(t) = \lambda_n c_n(t) - \lambda_n \langle Bd(t), \psi_n \rangle_\mathcal{H} + \langle ABd(t), \psi_n \rangle_\mathcal{H} + \langle U(t), \psi_n \rangle_\mathcal{H}.$$

(12)

As all the terms involved in (12) are continuous over $\mathbb{R}_+$, a straightforward integration gives for all $t \geq 0$,

$$c_n(t) = e^{\lambda_n t} c_n(0) - \lambda_n \int_0^t e^{\lambda_n (t-\tau)} \langle Bd(\tau), \psi_n \rangle_\mathcal{H} d\tau + \int_0^t e^{\lambda_n (t-\tau)} \langle ABd(\tau), \psi_n \rangle_\mathcal{H} d\tau + \int_0^t e^{\lambda_n (t-\tau)} \langle U(\tau), \psi_n \rangle_\mathcal{H} d\tau.$$

(13)
Note that
\[ X(t) = \sum_{k \in \mathbb{N}} c_n(t) \phi_n, \]
\[ S(t)X_0 = \sum_{k \in \mathbb{N}} e^{\lambda_n t} c_n(0) \phi_n, \]
\[ \int_0^t S(t - \tau)ABd(\tau)\,d\tau = \sum_{n \in \mathbb{N}} \left( \int_0^t S(t - \tau)ABd(\tau)\,d\tau, \psi_n \right)_\mathcal{H} \phi_n \]
\[ = \sum_{n \in \mathbb{N}} \int_0^t \left( \int_0^t e^{\lambda_n (t-\tau)} \langle ABd(\tau), \psi_\lambda \rangle_\mathcal{H} \phi_\lambda, \psi_n \right)_\mathcal{H} \,d\tau \phi_n \]
\[ = \sum_{n \in \mathbb{N}} \int_0^t \int_0^t e^{\lambda_n (t-\tau)} \langle ABd(\tau), \psi_\lambda \rangle_\mathcal{H} \phi_\lambda, \psi_n \rangle_\mathcal{H} \,d\tau \phi_n \]
\[ = \sum_{n \in \mathbb{N}} \int_0^t e^{\lambda_n (t-\tau)} \langle ABd(\tau), \psi_n \rangle_\mathcal{H} \,d\tau \phi_n, \]

and, similarly,
\[ \int_0^t S(t - \tau)U(\tau)d\tau = \sum_{n \in \mathbb{N}} \int_0^t e^{\lambda_n (t-\tau)} \langle U(\tau), \psi_n \rangle_\mathcal{H} \,d\tau \phi_n. \] (14)

Thus, introducing
\[ \alpha_n(t) \triangleq \lambda_n \int_0^t e^{\lambda_n (t-\tau)} \langle Bd(\tau), \psi_n \rangle_\mathcal{H} \,d\tau, \] (15)
we deduce from (3) and (13) that \( (\alpha_n(t))_{n \in \mathbb{N}} \) is a square summable sequence for all \( t \geq 0 \) and that
\[ \alpha(t) \triangleq \sum_{n \in \mathbb{N}} \alpha_n(t) \phi_n \in \mathcal{H}. \]

Therefore, multiplying both sides of (13) by \( \phi_n \) and summing over \( n \in \mathbb{N} \) yields
\[ X(t) = S(t)X_0 - \alpha(t) + \int_0^t S(t - \tau) \{ ABd(\tau) + U(\tau) \} \,d\tau. \]

Thus, for all \( t \geq 0 \), we have
\[ \|X(t)\|_\mathcal{H} \leq \|S(t)X_0\|_\mathcal{H} + \|\alpha(t)\|_\mathcal{H} + \left\| \int_0^t S(t - \tau) \{ ABd(\tau) + U(\tau) \} \,d\tau \right\|_\mathcal{H}. \] (16)

Let \( m_r, M_R \in \mathbb{R}_+^* \) be the constants associated with the inequality (2) for the Riesz basis formed by the eigenvectors \( \phi_n \) of \( A_0 \). Introducing \( \kappa_0 = -\omega_0 > 0 \) where \( \omega_0 \) is the growth bound of \( S \), it is easy to see based on (2) and (5) that
\[ \|S(t)X_0\|_\mathcal{H} \leq \sqrt{m_r \frac{M_R}{m_r}} e^{-\kappa_0 t} \|X_0\|_\mathcal{H}. \] (17)

Similarly,
\[ \left\| \int_0^t S(t - \tau) \{ ABd(\tau) + U(\tau) \} \,d\tau \right\|_\mathcal{H} \leq \sqrt{m_r \frac{M_R}{m_r}} \int_0^t e^{-\kappa_0 (t-\tau)} \|ABd(\tau) + U(\tau)\|_\mathcal{H} \,d\tau \]
\[ \leq \frac{1}{\kappa_0} \sqrt{m_r \frac{M_R}{m_r}} \left\{ \|AB\|_{\mathcal{L}(K^m, \mathcal{H})} \|d\|_{C^0([0,t]; K^m)} + \|U\|_{C^0([0,t]; \mathcal{H})} \right\}. \] (18)
It remains to evaluate \( \| \alpha(t) \|_{\mathcal{H}} \). To do so, consider a basis \( \mathcal{E} = (e_1, e_2, \ldots, e_m) \) of \( \mathbb{K}^m \). We introduce \( d_1, d_2, \ldots, d_m \in C^2(\mathbb{R}^+; \mathbb{K}) \) such that
\[
d = \sum_{k=1}^{m} d_k e_k.
\]
Based on this projection, one can get for all \( \tau \in [0, t] \),
\[
| \langle Bd(\tau), \psi_n \rangle_{\mathcal{H}} | = \left| \sum_{k=1}^{m} d_k(\tau) \langle B e_k, \psi_n \rangle_{\mathcal{H}} \right| \\
\leq \| d(\tau) \|_{\infty, \mathcal{E}} \sum_{k=1}^{m} | \langle B e_k, \psi_n \rangle_{\mathcal{H}} | \\
\leq c(\mathcal{E}) | d(\tau) | | \mathcal{K}^{\infty} \sum_{k=1}^{m} | \langle B e_k, \psi_n \rangle_{\mathcal{H}} | \\
\leq c(\mathcal{E}) | d | | C^0([0, t]; \mathbb{K}^m) \sum_{k=1}^{m} | \langle B e_k, \psi_n \rangle_{\mathcal{H}} |.
\]
Thus, for all \( t \geq 0 \), we have
\[
| \alpha_n(t) | \leq |\lambda_n| \int_{0}^{t} e^{Re\lambda_n(t-\tau)} | \langle Bd(\tau), \psi_n \rangle_{\mathcal{H}} | d\tau \\
\leq \frac{\lambda_n}{Re\lambda_n} c(\mathcal{E}) | d | | C^0([0, t]; \mathbb{K}^m) \sum_{k=1}^{m} | \langle B e_k, \psi_n \rangle_{\mathcal{H}} | \int_{0}^{t} -Re\lambda_n \cdot e^{Re\lambda_n(t-\tau)} d\tau \\
\leq \zeta c(\mathcal{E}) | d | | C^0([0, t]; \mathbb{K}^m) \sum_{k=1}^{m} | \langle B e_k, \psi_n \rangle_{\mathcal{H}} |,
\]
where the eigenvalue constraints (6) have been used. We deduce that for all \( t \geq 0 \),
\[
| \alpha_n(t) |^2 \leq m \zeta^2 c(\mathcal{E})^2 | d |^2 | C^0([0, t]; \mathbb{K}^m) \sum_{k=1}^{m} | \langle B e_k, \psi_n \rangle_{\mathcal{H}} |^2.
\]
From (2),
\[
\forall k \in \{1, \ldots, m\}, \quad \sum_{n \in \mathbb{N}} | \langle B e_k, \psi_n \rangle_{\mathcal{H}} |^2 \leq \frac{1}{m_R} \| B e_k \|_{\mathcal{H}}^2,
\]
which gives,
\[
\sum_{n \in \mathbb{N}} | \alpha_n(t) |^2 \leq \frac{m}{m_R} \zeta^2 c(\mathcal{E})^2 | d |^2 | C^0([0, t]; \mathbb{K}^m) \sum_{k=1}^{m} \| B e_k \|_{\mathcal{H}}^2.
\]
Using again (2), we finally obtain for all \( t \geq 0 \),
\[
\| \alpha(t) \|_{\mathcal{H}} \leq \zeta c(\mathcal{E}) \sqrt{\frac{m}{m_R} \sum_{k=1}^{m} \| B e_k \|_{\mathcal{H}}^2 | d | C^0([0, t]; \mathbb{K}^m)}. \quad (19)
\]
Substituting inequalities (17-19) into (16), we obtain the desired result (11) with
\[
C_0 = \sqrt{\frac{m}{m_R}}, \quad (20a)
\]
\[
C_1 = \sqrt{\frac{M_R}{m_R}} \left\{ \frac{1}{\kappa_0} \| AB \|_{C(\mathbb{K}^m, \mathcal{H})} + \zeta c(\mathcal{E}) \sqrt{\frac{m}{m_R} \sum_{k=1}^{m} \| B e_k \|_{\mathcal{H}}^2} \right\}, \quad (20b)
\]
11
\[ C_2 = \frac{1}{\kappa_0} \sqrt{\frac{M_R}{m_R}}. \] 

(20c)

This concludes the proof. 

Remark 3 In the proof of Theorem 1, the eigenvalue constraint \( \zeta < \infty \) can be weakened to

\[ \forall k \in \{1, \ldots, m\}, \sum_{n \geq 0} \left| \frac{\lambda_n}{\text{Re}\lambda_n} \right|^2 |(Be_k, \psi_n)_\mathcal{H}|^2 < \infty. \] 

(21)

It is easy to see that the condition above does not depend on a specific selection of either the lifting operator \( B \) (when \( \omega_0 < 0 \) or the basis \( \mathcal{E} = (e_1, e_2, \ldots, e_m) \) of \( \mathbb{K}^m \). In this case, the constant \( C_1 \) is given by

\[ C_1 = \frac{1}{\kappa_0} \sqrt{\frac{M_R}{m_R}} \|AB\|_{\mathcal{L}(\mathbb{K}^m, \mathcal{H})} + c(\mathcal{E}) \sqrt{mM_R \sum_{k=1}^{m} \sum_{n \geq 0} \left| \frac{\lambda_n}{\text{Re}\lambda_n} \right|^2 |(Be_k, \psi_n)_\mathcal{H}|^2}. \] 

(22)

3.3 An energy-based interpretation for the constant related to the boundary perturbation in the ISS estimate

The obtained expression (20b) of the constant \( C_1 \) depends on the selected lifting operator \( B \). However, the lifting operator provided by Definition 1 is not unique. The objective of this subsection is to provide a constructive definition of a constant \( C_1 \), independent of a specific selection of the lifting operator \( B \), such that the ISS estimate (11) holds true.

Lemma 2 Let \( (A, B) \) be a boundary control system such that \( A_0 \) is injective. For any \( e \in \mathbb{K}^m \), there exists a unique \( X_e \in D(A) \) such that \( AX_e = 0 \) and \( BX_e = e \). Furthermore, if \( B \) is a lifting operator associated with \( (A, B) \), then

\[ X_e = Be - A_0^{-1}ABe. \]

Proof of Lemma 2. The uniqueness part follows from the injectivity of \( A_0 \). Indeed, by linearity it is sufficient to check that \( AX = 0 \) and \( BX = 0 \) implies \( X = 0 \). But \( BX = 0 \) implies \( X \in D(A_0) \) whence \( A_0X = 0 \). As \( A_0 \) is invertible, this yields \( X = 0 \).

For the existence part, let \( B \) be a lifting operator associated with the boundary control system \( (A, B) \) as provided by Definition 1. Consider \( X_e = X_e - Be \). Then for \( X_e \in D(A) \),

\[ AX_e = 0 \text{ and } BX_e = e \]

\[ \Leftrightarrow A\dot{X}_e = -ABe \text{ and } B\dot{X}_e = 0 \]

\[ \Leftrightarrow A_0\dot{X}_e = -ABe \text{ and } \dot{X}_e \in D(A_0) \]

\[ \Leftrightarrow \dot{X}_e = -A_0^{-1}ABe. \]

Thus, \( X_e = Be - A_0^{-1}ABe \) is the unique solution. 

Remark 4 The stationary trajectory \( X(t) = X_e \) provided by Lemma 2 is the strong solution of the abstract boundary control system (1) associated with the initial condition \( X_0 = X_e \), the constant boundary disturbance \( \lambda(t) = e \), and the zero distributed disturbance \( U = 0 \).

Theorem 2 Let \( \mathcal{E} = (e_1, e_2, \ldots, e_m) \) be a basis of \( \mathbb{K}^m \). Under the assumptions of Theorem 1, the conclusion of the theorem holds true with constants \( C_0, C_1, C_2 > 0 \) involved in the ISS estimate (11) given by

\[ C_0 = \frac{1}{\kappa_0} \sqrt{\frac{M_R}{m_R}}. \]
\[ C_1 = \zeta c(\mathcal{E}) \frac{M_P}{\kappa_0} \frac{m}{m_R} \sum_{k=1}^{m} \|X_{e,k}\|^2_{\mathcal{H}}, \]

\[ C_2 = \frac{1}{\kappa_0} \frac{M_P}{m_R}, \]

where \( X_{e,k} \in D(A) \) is, for all \( k \in \{1, \ldots, m\} \), the unique solution of \( AX_{e,k} = 0 \) and \( BX_{e,k} = e_k \).

**Proof of Theorem 2.** Consider the proof of Theorem 1 up to Equation (12) included. For the basis \( \mathcal{E} = (e_1, e_2, \ldots, e_m) \) of \( \mathbb{K}^m \), let \( d_1, d_2, \ldots, d_m \in C^2(\mathbb{R}^+; \mathbb{K}) \) be such that

\[ d = \sum_{k=1}^{m} d_k e_k. \]

As \( A_0 \) is a Riesz-spectral operator with \( \omega_0 < 0 \), we have \( 0 \in \rho(A_0) \). Based on Lemma 2, let, for any \( k \in \{1, \ldots, m\} \), \( X_{e,k} \in D(A) \) be the unique solution of \( AX_{e,k} = 0 \) and \( BX_{e,k} = e_k \). In particular, an explicit expression is given by \( X_{e,k} = Be_k - A_0^{-1}ABe_k \). Introducing \( \tilde{X}_{e,k} \triangleq X_{e,k} - Be_k = -A_0^{-1}ABe_k \), it yields \( \tilde{X}_{e,k} \in D(A_0) \) and that for all \( \tilde{X}_{e,k} \)

\[ \lambda_n \langle X_{e,k}, \psi_n \rangle_{\mathcal{H}} = \langle \tilde{X}_{e,k}, \tilde{A}_0^* \psi_n \rangle_{\mathcal{H}} + \langle Be_k, A_0^* \psi_n \rangle_{\mathcal{H}} + \langle A_0 \tilde{X}_{e,k}, \psi_n \rangle_{\mathcal{H}} + \lambda_n \langle Be_k, \psi_n \rangle_{\mathcal{H}} = -\lambda_n \langle A_0 \tilde{X}_{e,k}, \psi_n \rangle_{\mathcal{H}} + \lambda_n \langle Be_k, \psi_n \rangle_{\mathcal{H}}. \]

Introducing that

\[ D = \sum_{k=1}^{m} d_k X_{e,k} \in C^0(\mathbb{R}^+; D(A)) \cap C^2(\mathbb{R}^+; \mathcal{H}), \]

it follows that

\[ -\lambda_n \langle Bd(t), \psi_n \rangle_{\mathcal{H}} + \langle ABd(t), \psi_n \rangle_{\mathcal{H}} = \sum_{k=1}^{m} d_k(t) \{ -\lambda_n \langle Be_k, \psi_n \rangle_{\mathcal{H}} + \langle ABe_k, \psi_n \rangle_{\mathcal{H}} \} = -\lambda_n \sum_{k=1}^{m} d_k(t) \langle X_{e,k}, \psi_n \rangle_{\mathcal{H}} = -\lambda_n \langle D(t), \psi_n \rangle_{\mathcal{H}}. \]

Thus, we get from (12) that for all \( t \geq 0 \),

\[ \dot{c}_n(t) = \lambda_n c_n(t) - \lambda_n \langle D(t), \psi_n \rangle_{\mathcal{H}} + \langle U(t), \psi_n \rangle_{\mathcal{H}}, \]

and that for all \( t \geq 0 \),

\[ c_n(t) = e^{\lambda_n t} c_n(0) - \lambda_n \int_0^t e^{\lambda_n (t-\tau)} \langle D(\tau), \psi_n \rangle_{\mathcal{H}} \ d\tau + \int_0^t e^{\lambda_n (t-\tau)} \langle U(\tau), \psi_n \rangle_{\mathcal{H}} \ d\tau. \]

The first and third terms on the right hand side of the equation above have been estimated in the proof of Theorem 1. This procedure yields the the constants \( C_0 \) and \( C_2 \). The second term can be treated with the same procedure that one employed for (15) via the projection (23). This also provides the estimate of \( C_1 \). \( \square \)

**Remark 5** From the uniqueness part of Lemma 2, the constant \( C_1 \) given by Theorem 2 is independent of the chosen lifting operator \( B \). Instead, it depends on the energy \( \|X_{e,k}\|_{\mathcal{H}} \) of \( m \) linearly independent stationary solutions \( X_{e,k}\). \footnote{Directly follows from the definition of \( X_{e,k} \) and the fact that \( \mathcal{E} \) is a basis.}
Remark 6: Definition 6 is relevant because for any \( z \) only the zero function. Indeed, the function based on the two identities above, we introduce the following definition.

4 Concept of Weak Solutions and ISS Property

Assume that a given boundary control system \((A, B)\) satisfies an ISS estimate with respect to strong solutions corresponding to any initial condition \( X_0 \in D(A_0) \) and any disturbances \( d \in C^2(\mathbb{R}_+; \mathbb{K}^m) \) and \( U_0 \in C(\mathbb{R}_+; H) \). The objective of this section is to extend such an ISS estimate to every initial condition \( X_0 \in H \) and every disturbance \( d \in C^0(\mathbb{R}_+; \mathbb{K}^m) \) and \( U \in C([0, T]; H) \). To do so, we need to introduce a concept of weak solutions that extends the notion of strong solutions to initial conditions and disturbances exhibiting relaxed regularity assumptions.

4.1 Definition of weak solutions

To motivate the definition of weak solutions, let us consider first \( X \) a strong solution associated with \((X_0, d, U)\). Let \( T > 0 \) and a function \( z \in C^0([0, T]; D(A_0^*) \) be arbitrarily given. As \( X(t) - Bd(t) \in D(A_0) \), we obtain for all \( t \in [0, T] \),

\[
\langle \frac{dX}{dt}(t), z(t) \rangle_H = \langle AX(t) + U(t), z(t) \rangle_H \\
= \langle A_0 \{ X(t) - Bd(t) \}, z(t) \rangle_H + \langle ABd(t), z(t) \rangle_H + \langle U(t), z(t) \rangle_H \\
= \langle X(t), A_0^* z(t) \rangle_H - \langle Bd(t), A_0^* z(t) \rangle_H + \langle ABd(t), z(t) \rangle_H + \langle U(t), z(t) \rangle_H.
\]

Assuming now that \( z \in C^0([0, T]; D(A_0^*) \cap C^1([0, T]; H) \), integration by parts gives

\[
\int_0^T \langle \frac{dX}{dt}(t), z(t) \rangle_H dt = \langle X(T), z(T) \rangle_H - \langle X_0, z(0) \rangle_H - \int_0^T \langle X(t), \frac{dz}{dt}(t) \rangle_H dt.
\]

Based on the two identities above, we introduce the following definition.

Definition 6 (Weak solutions) Let \((A, B)\) be a boundary control system. For \( X_0 \in H \) and disturbances \( d \in C^0(\mathbb{R}_+; \mathbb{K}^m) \) and \( U \in C(\mathbb{R}_+; H) \), we say that \( X \in C^0([0, T]; H) \) is a weak solution of the abstract boundary control system \((1)\) associated with \((X_0, d, U)\) if for all \( T > 0 \) and for all \( z \in C^0([0, T]; D(A_0^*) \cap C^1([0, T]; H) \) such that \( A_0^* z \in C^0([0, T]; H) \) and \( z(T) = 0 \) (such a function \( z \) is called a test function over \([0, T]\)), the following equality holds true:

\[
\int_0^T \langle X(t), A_0^* z(t) + \frac{dz}{dt}(t) \rangle_H dt = -\langle X_0, z(0) \rangle_H + \int_0^T \langle Bd(t), A_0^* z(t) \rangle_H dt - \int_0^T \langle ABd(t), z(t) \rangle_H dt - \int_0^T \langle U(t), z(t) \rangle_H dt,
\]

where \( B \) is an arbitrary lifting operator associated with \((A, B)\).

Remark 6: Definition 6 is relevant because for any \( T > 0 \), the set of test functions over \([0, T]\) is not reduced to only the zero function. Indeed, the function \( z \) defined for all \( t \in [0, T] \) by \( z(t) = (t - T)z_0 \) with \( z_0 \in D(A_0^*) \) is obviously a non zero test function over \([0, T]\).

Remark 7: The definition of a weak solution for the abstract system \((1)\) is compatible with the notion of strong solution. Indeed, the developments preliminary to the introduction of Definition 6 show that a strong solution is also a weak solution.

\(^5\) Not necessarily a Riesz-spectral one.

\(^6\) In the sense provided later in Theorem 3.
**Remark 8** At first sight, the right hand side of (24) depends on the selected lifting operator $B$, making it necessary to specify the selected lifting operator $B$ when saying that a trajectory $X$ is a weak solution associated with $(X_0, d, U)$. However, Definition 6 implicitly claims that a weak solution is actually independent of the selected lifting operator $B$. It directly follows from the fact that if $B$ and $\tilde{B}$ are two lifting operators associated with $(A, B)$, then $\tilde{B} \cong B - \tilde{B}$ satisfies $R(\tilde{B}) \subset D(A)$ and $BB - B\tilde{B} = I_{\mathbb{R}^m} - I_{\mathbb{R}^m} = 0$, i.e., $R(\tilde{B}) \subset D(A)$. Thus we obtain,

$$\left\langle \dot{B}d(t), A_0^*z(t) \right\rangle_H = \left\langle A_0\dot{B}d(t), z(t) \right\rangle_H = \left\langle A\dot{B}d(t), z(t) \right\rangle_H,$$

from which we deduce that

$$\left\langle Bd(t), A_0^*z(t) \right\rangle_H - \left\langle ABd(t), z(t) \right\rangle_H = \left\langle \dot{B}d(t), A_0^*z(t) \right\rangle_H - \left\langle A\dot{B}d(t), z(t) \right\rangle_H.$$

This equality shows that the right hand side of (24) remains unchanged when switching between different lifting operators $B$ associated with $(A, B)$. So, it indeed makes sense to discuss about weak solutions without mentioning a particular lifting operator $B$ associated with $(A, B)$.

**Remark 9** Note that the concept of weak solution does not require that the initial condition satisfies the boundary condition $BX_0 = d(0)$. Such an algebraic condition is not even well defined when $X_0 \notin D(B)$.

4.2 Properties of weak solutions

When defining a notion of a weak solution, it is generally desirable to preserve the uniqueness of the solution.

**Lemma 3** Let $(A, B)$ be a boundary control system such that the disturbance free operator $A_0$ is injective. Then, for any given $X_0 \in \mathcal{H}$ and disturbances $d \in C^0(\mathbb{R}_+; \mathbb{K}^m)$ and $u \in C^0(\mathbb{R}_+; \mathcal{H})$, there exists at most one weak solution associated with $(X_0, d, U)$ of the abstract system (1).

The proof of Lemma 3 is in Annex A. We deduce from Remark 7 and Lemma 3 the following result.

**Corollary 1** Let $(A, B)$ be a boundary control system such that the disturbance free operator $A_0$ is injective. For any initial condition $X_0 \in D(A)$ and disturbances $d \in C^2(\mathbb{R}_+; \mathbb{K}^m)$ and $U \in C^1(\mathbb{R}_+; \mathcal{H})$ such that $BX_0 = d(0)$, the concepts of strong and weak solutions coincide. More specifically, the two following statements are equivalent:

1. $X$ is a strong solution associated with $(X_0, d, U)$;
2. $X$ is a weak solution associated with $(X_0, d, U)$.

When $X$ is a strong solution of the abstract boundary control system (1) associated with $(X_0, d, U)$, we have by Definition 5 that $X(0) = X_0$. Such an initial condition is not explicitly imposed in the Definition 6 of a weak solution. However, it is a consequence of (24) as shown by the following lemma.

**Lemma 4** Let $(A, B)$ be a boundary control system. Under the terms of Definition 6, assume that $X$ is a weak solution associated with $(X_0, d, U)$. Then $X(0) = X_0$.

**Proof of Lemma 4.** Let $z \in C^0(\mathbb{R}_+; D(A_0^*)) \cap C^1(\mathbb{R}_+; \mathcal{H})$ such that $A_0^*z \in C^0(\mathbb{R}_+; \mathcal{H})$. Then for all $T > 0$, $\tilde{z}_T \triangleq z - z(T)$ is a test function over $[0, T]$. As $dz(T)/dt = 0$, (24) gives for all $T > 0$,

$$\frac{1}{T} \int_0^T \left\langle X(t), A_0^*z(t) - A_0^*z(T) \right\rangle_H \, dt + \frac{1}{T} \int_0^T \left\langle X(t), \frac{dz}{dt}(t) \right\rangle_H \, dt$$

$$= \left\langle X_0, \frac{z(T) - z(0)}{T} \right\rangle_H + \frac{1}{T} \int_0^T \left\langle Bd(t), \{A_0^*z(t) - A_0^*z(T)\} \right\rangle_H \, dt$$

$$- \frac{1}{T} \int_0^T \left\langle ABd(t), z(t) - z(T) \right\rangle_H \, dt - \frac{1}{T} \int_0^T \left\langle U(t), z(t) - z(T) \right\rangle_H \, dt.$$

(25)
It is straightforward to show that for any \( f, g \in C^0(\mathbb{R}_+; \mathcal{H}) \),
\[
\lim_{T \to 0^+} \frac{1}{T} \int_0^T \langle f(t), g(t) - g(T) \rangle_{\mathcal{H}} \, dt = 0.
\]

Thus, based on the regularity assumptions given in Definition 6, the first integral term on the left hand side and the three integral terms on the right hand side of (25) converge to zero as \( T \to 0^+ \). We deduce, by letting \( T \to 0^+ \) in (25),
\[
\langle X(0), \frac{dX}{dt}(0) \rangle_{\mathcal{H}} = \langle X_0, \frac{dX}{dt}(0) \rangle_{\mathcal{H}}.
\]
Taking in particular \( z(t) = tz_0 \) with \( z_0 \in D(A_0^\alpha) \), we get that \( \langle X(0) - X_0, z_0 \rangle_{\mathcal{H}} = 0 \) holds true for all \( z_0 \in D(A_0^\alpha) \). As \( D(A_0^\alpha) = \mathcal{H} \), we deduce that \( X(0) = X_0 \). \( \square \)

Thus, for a weak solution \( X \) associated with \((X_0,d,u)\), it makes sense to say that \( X_0 \) is the initial condition of the system trajectory.

### 4.3 Existence of weak solutions and extension of ISS estimates

We can now introduce the following theorem.

**Theorem 3** Let \((\mathcal{A}, \mathcal{B})\) be a boundary control system. Assume that there exist \( \beta \in \mathcal{KL} \) and \( \gamma_1, \gamma_2 \in \mathcal{K} \) such that for any initial condition \( X_0 \in D(\mathcal{A}) \) and any disturbances \( d \in C^2(\mathbb{R}_+; \mathbb{R}^m) \) and \( U \in C^1(\mathbb{R}_+; \mathcal{H}) \) such that \( \mathcal{B}X_0 = d(0) \), the strong solution \( X \) of the abstract boundary control system (1) associated with \((X_0,d,U)\), satisfies for all \( t \geq 0 \),
\[
\|X(t)\|_{\mathcal{H}} \leq \beta(\|X_0\|_{\mathcal{H}}, t) + \gamma_1(\|d\|_{C^0([0,t];\mathbb{R}^m)}) + \gamma_2(\|U\|_{C^0([0,t];\mathcal{H})}). \tag{26}
\]

Then, for any initial condition \( X_0 \in \mathcal{H} \), and any disturbances \( d \in C^0(\mathbb{R}_+; \mathbb{R}^m) \) and \( U \in C^0(\mathbb{R}_+; \mathcal{H}) \):

1. the abstract boundary control system (1) has a unique weak solution \( X \in C^0(\mathbb{R}_+; \mathcal{H}) \) associated with \((X_0,d,U)\);
2. this weak solution satisfies the ISS estimate (26) for all \( t \geq 0 \).

The proof of Theorem 3, which is provided in Annex B, is essentially technical and relies on density arguments. We directly deduce from Theorem 3 the following extension of Theorems 1 and 2 for Riesz-spectral operators.

**Corollary 2** Let \((\mathcal{A}, \mathcal{B})\) be a Riesz-spectral boundary control system such that the eigenvalue constraints (6) hold true. For every initial condition \( X_0 \in \mathcal{H} \), and every disturbance \( d \in C^0(\mathbb{R}_+; \mathbb{R}^m) \) and \( U \in C^0(\mathbb{R}_+; \mathcal{H}) \), the abstract system (1) has a unique weak solution \( X \in C^0(\mathbb{R}_+; \mathcal{H}) \) associated with \((X_0,d,U)\). Furthermore, \( X \) satisfies the ISS estimate (11) with constants \( \kappa_0, C_0, C_1, C_2 \) given by either Theorem 1 or 2.

Based on the existence and uniqueness of weak solutions, and by linearity of (24), we can state the following linearity result.

**Corollary 3** Assume that the assumptions of Theorem 3 hold true. For \( i \in \{1,2\} \), let \( X_i \) be the weak solution associated with \((X_{i,0},d_i,U_i)\). Then, for all \( \alpha, \beta \in \mathbb{K} \), \( \alpha X_1 + \beta X_2 \) is the unique weak solution associated with \((\alpha X_{1,0} + \beta X_{2,0}, \alpha d_1 + \beta d_2, \alpha U_1 + \beta U_2)\).

### 4.4 Mild solutions, semigroup property, and vanishing disturbances

#### 4.4.1 Compatibility with the concept of mild solutions

When the boundary disturbance satisfies the additional regularity assumption \( d \in C^1(\mathbb{R}_+; \mathbb{R}^m) \), a more intuitive approach for extending the concept of strong solutions is to define \( X \) provided by (10) as the mild solution associated with \((X_0,d,U)\). The following result shows that such an approach is compatible with the concept of weak solutions as introduced by Definition 6 in the more general case \( d \in C^0(\mathbb{R}_+; \mathbb{R}^m) \).

\footnote{The constraint \( \zeta < \infty \) can be relaxed to (21). In that case, the constant \( C_1 \) is given by (22).}
Theorem 4 Let \((A, B)\) be a boundary control system such that the assumptions of Theorem 3 hold true. Under the terms of Definition 6, let \(X\) be the weak solution associated with \((X_0, d, U)\). Assume that the boundary disturbance satisfies the extra regularity assumption \(d \in \mathcal{C}^1(\mathbb{R}^+; \mathbb{K}^m)\). Then \(X\) is also a mild solution in the sense that (10) holds true for all \(t \geq 0\).

Proof of Theorem 4. Let \(X_0 \in \mathcal{H}, d \in \mathcal{C}^1(\mathbb{R}^+; \mathbb{K}^m), U \in \mathcal{C}^0(\mathbb{R}^+; \mathcal{H}),\) and \(T > 0\) be arbitrarily given. We denote by \(X\) the weak solution associated with \((X_0, d, U)\). As \(\mathcal{C}^2([0, T]; \mathbb{K}^m)\) is dense in \(\mathcal{C}^1([0, T]; \mathbb{K}^m)\) endowed with the usual norm \(\|f\|_{\mathcal{C}^1([0, T]; \mathbb{K}^m)} = \|f\|_{\mathcal{C}^2([0, T]; \mathbb{K}^m)} + \|\dot{f}\|_{\mathcal{C}^2([0, T]; \mathbb{K}^m)}\), we can select, similarly to the proof of Theorem 3, approximating sequences \((X_{0,n})_n \in D(A)^N, (d_n)_n \in \mathcal{C}^2([0, T]; \mathbb{K}^m)^N,\) and \((U_n)_n \in \mathcal{C}^1([0, T]; \mathcal{H})^N\) such that \(BX_{0,n} = d_n(0), X_{0,n} \xrightarrow{n \to +\infty} X_0,\)

\[
\|d_n - d\|_{\mathcal{C}^2([0, T]; \mathbb{K}^m)} \xrightarrow{n \to +\infty} 0, \\
\|d_n - d\|_{\mathcal{C}^2([0, T]; \mathbb{K}^m)} \xrightarrow{n \to +\infty} 0, \\
\|U_n - U\|_{\mathcal{C}^2([0, T]; \mathcal{H})} \xrightarrow{n \to +\infty} 0.
\]

We denote by \(X_n\) the unique strong solution of the abstract system (1) over \([0, T]\) associated with \((X_{0,n}, d_n, U_n)\). From (10), we obtain for all \(t \in [0, T]\) and all \(n \in \mathbb{N},\)

\[
X_n(t) = S(t)(X_{0,n} - Bd_n(0)) + Bd_n(t) + \int_0^t S(t - \tau) \left\{-Bd_n(\tau) + ABd_n(\tau) + U_n(\tau)\right\} d\tau. 
\]

Furthermore, from the proof of Theorem 3, we know that

\[
\|X_n - X\|_{\mathcal{C}^0([0, T]; \mathcal{H})} \xrightarrow{n \to +\infty} 0.
\]

Thus, by letting \(n \to +\infty\) in (27), we obtain that (10) holds true for all \(t \in [0, T].\) As \(T > 0\) has been arbitrarily chosen, this concludes the proof. □

4.4.2 Semigroup property

It is well known that the strong solutions of the abstract system (1) satisfy the semigroup property in the sense that if \(X\) is the strong solution associated with \((X_0, d, U)\), then \(X(\cdot + t_0)\) is the strong solution associated with \((X(t_0), d(\cdot + t_0), U(\cdot + t_0))\) for any \(t_0 > 0\). The following result shows that this semigroup property extends to the concept of weak solutions.

Theorem 5 Let \((A, B)\) be a boundary control system such that the assumptions of Theorem 3 hold true. Let \(X\) be the weak solution associated with an initial condition \(X_0 \in \mathcal{H},\) a boundary disturbance \(d \in \mathcal{C}^0(\mathbb{R}^+; \mathbb{K}^m),\) and a distributed disturbance \(U \in \mathcal{C}^0(\mathbb{R}^+; \mathcal{H}).\) Then, for any \(t_0 > 0,\) \(X(\cdot + t_0)\) is the weak solution associated with \((X(t_0), d(\cdot + t_0), U(\cdot + t_0))\).

Proof of Theorem 5 Note first that, based on the developments preliminary to the introduction of Definition 6, we have for any strong solution \(X\) associated with \((X_0, d, U)\), any \(z \in \mathcal{C}^0([0, T]; D(A_0^z)) \cap \mathcal{C}^1([0, T]; \mathcal{H})\) such that\(^8\) \(A_0^z \in \mathcal{C}^0([0, T]; \mathcal{H})\), and any \(T > 0,\)

\[
\int_0^T \left\langle X(t), A_0^z(t) + \frac{dz}{dt}(t) \right\rangle_{\mathcal{H}} dt = \langle X(T), z(T) \rangle_{\mathcal{H}} - \langle X_0, z(0) \rangle_{\mathcal{H}} + \int_0^T (Bd(t), A_0^z(t))_{\mathcal{H}} dt - \int_0^T \langle Abd(t), z(t) \rangle_{\mathcal{H}} dt - \int_0^T \langle U(t), z(t) \rangle_{\mathcal{H}} dt. 
\]

By resorting to the same density argument as the one employed in Step 4 of the proof of Theorem 3, it yields that any weak solution \(X\) associated with \((X_0, d, U)\) also satisfies (28) for all \(T \geq 0\) and all \(z \in \mathcal{C}^0([0, T]; D(A_0^z)) \cap \mathcal{C}^1([0, T]; \mathcal{H})\) such that \(A_0^z \in \mathcal{C}^0([0, T]; \mathcal{H}).\)

\(^8\) We do not impose here the condition \(z(T) = 0\) as in the case of the test functions.
Now, let $X$ be the weak solution associated with $(X_0, d, U)$. Let $t_0, T > 0$ and a test function $\hat{z} \in C^0([0, T]; D(\mathcal{A}_0^\ast)) \cap C^1([0, T]; \mathcal{H})$ over $[0, T]$ be arbitrarily given. We define the test function $\hat{z} \in C^0([0, t_0 + T]; D(\mathcal{A}_0^\ast)) \cap C^1([0, t_0 + T]; \mathcal{H})$ as $\hat{z} = \varphi(-t_0)\vert_{[0,t_0+T]}$ where $\varphi \in C^0(\mathbb{R}; D(\mathcal{A}_0)) \cap C^1(\mathbb{R}; \mathcal{H})$ is given for all $t \in [0, T]$ and all $k \in \mathbb{Z}$ by $\varphi(t + 2kT) = \hat{z}(t) - 2k\hat{z}(0)$ and $\varphi(t + (2k + 1)T) = -\hat{z}(T - t) - 2k\hat{z}(0)$. In particular we have $\hat{z}(t + T) = \varphi(T) = \hat{z}(T) = 0$ and for all $t \in [0, T]$, $\hat{z}(t + t_0) = \varphi(t) = \hat{z}(t)$. By using (28) once with $z = \hat{z}\vert_{[0,t_0]}$ and once with $z = \hat{z}$, we obtain after a change of variable:

$$
\int_0^T \left\langle X(t + t_0), \mathcal{A}_0^\ast \hat{z}(t) + \frac{d\hat{z}}{dt}(t) \right\rangle_{\mathcal{H}} dt = -\langle X(t_0), \hat{z}(t_0) \rangle_{\mathcal{H}} + \int_0^T \langle Bd(t + t_0), \mathcal{A}_0^\ast \hat{z}(t) \rangle_{\mathcal{H}} dt
$$

$$
- \int_0^T \langle ABd(t + t_0), \hat{z}(t) \rangle_{\mathcal{H}} dt - \int_0^T \langle U(t + t_0), \hat{z}(t) \rangle_{\mathcal{H}} dt.
$$

As $T$, $\hat{z}$ and $t_0$ have been arbitrarily selected, it follows from Definition 6 that for all $t_0 > 0$, $X(\cdot + t_0)$ is the weak solution associated with $(X(t_0), d(\cdot + t_0), U(\cdot + t_0))$. □

4.4.3 Vanishing disturbances

A direct consequence of Theorems 3 and 5 is the following corollary regarding vanishing disturbances.

**Corollary 4** Let $(A, B)$ be a boundary control system such that the assumptions of Theorem 3 hold true. Let an initial condition $X_0 \in \mathcal{H}$, and disturbances $d \in C^0(\mathbb{R}_+; \mathbb{K}^m)$ and $U \in C^0(\mathbb{R}_+; \mathcal{H})$ such that

$$
\lim_{t \to +\infty} \|d(t)\|_{\mathbb{K}^m} = \lim_{t \to +\infty} \|U(t)\|_{\mathcal{H}} = 0.
$$

Then, the weak solution $X$ associated with $(X_0, d, U)$ satisfies:

$$
\lim_{t \to +\infty} \|X(t)\|_{\mathcal{H}} = 0.
$$

5 Applications

For a function $f : \mathbb{R}_+ \to L^2(0, 1)$, we denote, with a slight abuse of notation, $f(t, \xi) \equiv [f(t)](\xi)$. When $f \in C^1(\mathbb{R}_+; \mathcal{H})$, we denote $\frac{df}{dt}(t, \xi) \equiv \left[\frac{df}{dt}(t)\right](\xi)$. Finally, when $f : \mathbb{R}_+ \to H^1(0, 1)$, we denote $f'(t, \xi) \equiv [f(t)]'(\xi)$.

5.1 1D parabolic PDEs

We consider the class of 1D parabolic PDEs introduced in Subsection 2.3.1. Assuming that $\tan(\alpha) \geq 0$, $\tan(\beta) \geq 0$, and $\min q(x) > 0$, we have $\omega_0 < 0$. From the well-known fact that Sturm-Liouville operators are self-adjoint (see, e.g., [31]), we get $\mathcal{A}_0^\ast = \mathcal{A}_0$. Thus, for an initial condition $y_0 \in L^2(0, 1)$ and disturbances $d = (d_1, d_2) \in C^0(\mathbb{R}_+; \mathbb{K}^2)$ and $u \in C^0(\mathbb{R}_+; L^2(0, 1))$, $X = x \in C^0(\mathbb{R}_+; L^2(0, 1))$, $X = x \in C^0([0, T]; \mathcal{H})$ and $z(T) = 0$, the following equality is satisfied:

$$
\int_0^T \int_0^1 x(t, \xi) \left\{ (pz')' - qz + r \frac{dz}{dt} \right\} (t, \xi) d\xi dt = -\int_0^1 r(\xi) y_0(\xi) z(0, \xi) d\xi
$$

$$
+ \int_0^T \int_0^1 (Bd)(t, \xi) \times \{ (pz')' - qz \} (t, \xi) d\xi dt
$$

$$
- \int_0^T \int_0^1 \{(p \times (Bd)')(t, \xi) - q \times Bd \} (t, \xi) \times \hat{z}(t, \xi) d\xi dt
$$

$$
- \int_0^T \int_0^1 r(\xi)u(t, \xi) \hat{z}(t, \xi) d\xi dt,
$$

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where the lifting operator $B$ is given by (7) and is such that for any $t \geq 0$, $x \to (Bd)(t, x)$ is a polynomial function of degree no greater than 3.

Based on the properties of the Sturm-Liouville operator \cite{31}, we have: 1) $m_R = M_R = 1$ because the eigenvectors of $\mathcal{A}_0$ form a Hilbert Basis; 2) $\zeta = 1$ since all the eigenvalues are real; 3) $\kappa_0 = -\omega_0 = \min_{x \in [0, 1]} q(x)/ \max_{x \in [0, 1]} r(x)$. The boundary disturbance evolves in the two dimensional ($m = 2$) space $(\mathbb{K}^2, \| \cdot \|_2)$ endowed with the usual euclidean norm. By selecting $\mathcal{E} = \{e_1, e_2\}$ as the canonical basis of $\mathbb{K}^2$, we obtain $c(\mathcal{E}) = 1$.

We deduce that the ISS constants provided by Theorems 1 and 2 are such that
\[
C_0 = 1, C_2 = \kappa_0^{-1} \leq \max_{x \in [0, 1]} r(x) / \min_{x \in [0, 1]} q(x),
\]
and
\[
C_1 = \min \left( \sqrt{2} \sqrt{\|X_{e,1}\|_{\mathcal{H}}^2 + \|X_{e,2}\|_{\mathcal{H}}^2} + \frac{1}{\kappa_0} \|AB\|_{L(\mathbb{K}^m, \mathcal{H})} + \sqrt{2} \sqrt{\|Be_1\|_{\mathcal{H}}^2 + \|Be_2\|_{\mathcal{H}}^2} \right).
\]

In general, the constant $C_1$ provided by Theorem 2 is difficult to estimate as soon as $p, q,$ and $r$ are nonconstant functions\footnote{We refer to \cite{17} for a detailed example of such a configuration when $d_1 = 0$. The extension to the general case $d_1 \neq 0$ is straightforward based on the results of the present paper.}. Indeed, it requires to solve explicitly a second order ordinary differential equation with non constant coefficients. On the other hand, at the price of a certain conservatism, the constant $C_1$ provided by Theorem 1 can be used to provide the following estimate:
\[
C_1 \leq \max_{x \in [0, 1]} \frac{r(x)}{\min_{x \in [0, 1]} q(x)} (\|ABe_1\|_{\mathcal{H}} + \|ABe_2\|_{\mathcal{H}}) + \sqrt{2} \sqrt{\|Be_1\|_{\mathcal{H}}^2 + \|Be_2\|_{\mathcal{H}}^2},
\]
where the quantities $\|ABe_1\|_{\mathcal{H}}$ and $\|Be_i\|_{\mathcal{H}}$ can be computed (or at least numerically estimated) based on the explicit knowledge of the functions $p, q,$ and $r$.

### 5.2 Damped string

We consider the clamped-free damped string introduced in Subsection 2.3.2. The adjoint operator $\mathcal{A}_0^*$ is defined over the domain
\[
D(\mathcal{A}_0^*) = \{(x_1, x_2) \in \mathcal{H} : x_2 \in H^1(0, 1), (\alpha x_1' - \beta x_2') \in H^1(0, 1), (\alpha x_1' - \beta x_2')(1) = 0\},
\]
by
\[
\mathcal{A}_0^*(x_1, x_2) = (-x_2, -(\alpha x_1' - \beta x_2')).
\]
Thus, for initial conditions $y_0 \in H^1_0(0, 1)$ and $y_0' \in L^2(0, 1)$, and disturbances $d \in C^0(\mathbb{R}_+; \mathbb{K})$ and $u \in C^0(\mathbb{R}_+; L^2(0, 1))$, $X = (x_1, x_2) \in C^0(\mathbb{R}_+; \mathcal{H})$ is the weak solution of the abstract boundary control system (1) associated with $((y_0, y_0'), d, (0, u))$ if for all $T > 0$ and for all test function $z = (z_1, z_2) \in C^0([0, T]; D(\mathcal{A}_0^*) \cap C^1([0, T]; \mathcal{H})$ such that $\mathcal{A}_0^* z \in C^0([0, T]; \mathcal{H})$ and $z(T) = 0$, the following equality is satisfied:
\[
\int_0^T \int_0^1 \alpha x_1(t, \xi) \left\{ -z_2' + \left[ \frac{dz_2}{dt} \right]' \right\} (t, \xi) \, dt \, d\xi + \int_0^T \int_0^1 x_2(t, \xi) \left\{ (-\alpha z_1' + \beta z_2')' + \frac{dz_2}{dt} \right\} (t, \xi) \, dt \, d\xi = -\int_0^1 \alpha y_0'(\xi) z_2(0, \xi) \, d\xi - \int_0^1 y_0(\xi) z_2(0, \xi) \, d\xi - \int_0^T d(t) z_2(t, 1) \, dt - \int_0^T u(t, \xi) z_2(t, \xi) \, d\xi.
\]

Noting that $AB = 0$, the constants $C_1$ provided by both Theorems 1 and 2 are identical. Introducing $k_0 \in \mathbb{N}$ defined by
\[
k_0 = \left\lfloor \frac{2 \sqrt{\alpha}}{\pi \beta} - \frac{1}{2} \right\rfloor \geq 0,
\]
where \([\cdot]\) denotes the ceiling function, it follows from [22] that

\[
\kappa_0 = \begin{cases} 
\min \left( \frac{\beta \pi^2}{8}, \frac{\alpha}{\beta} \right) & \text{if } k_0 \geq 1; \\
\frac{\alpha}{\beta} & \text{if } k_0 = 0,
\end{cases}
\]

\[
\zeta = \begin{cases} 
\frac{4\sqrt{\alpha}}{\beta \pi} & \text{if } k_0 \geq 1; \\
1 & \text{if } k_0 = 0,
\end{cases}
\]

\[m_R = 1 - C, \quad M_R = 1 + C\] with \(C \in (0, 1)\) given by

\[
C = \begin{cases} 
\max \left( \frac{4\sqrt{\alpha}}{(2k_0 + 1)\beta \pi}, \frac{(2k_0 - 1)\beta \pi}{4\sqrt{\alpha}} \right) & \text{if } k_0 \geq 1; \\
\frac{4\sqrt{\alpha}}{\beta \pi} & \text{if } k_0 = 0.
\end{cases}
\]

Furthermore, as the boundary disturbance evolves in the one dimensional \((m = 1)\) space \([K, |\cdot|]\), by selecting the basis \(e = 1 \in K\) we obtain \(c(\mathcal{E}) = 1\) and \(\|Be\|_{\mathcal{H}} = 1/\sqrt{\alpha}\). Thus, the constants of the ISS estimate are given by

\[
C_0 = \sqrt{\frac{1 + C}{1 - C}}, \quad C_1 = \frac{\zeta}{4\sqrt{\alpha}} \sqrt{\frac{1 + C}{1 - C}}, \quad C_2 = \frac{1}{\kappa_0} \sqrt{\frac{1 + C}{1 - C}}.
\]

The evolution of the convergence rate \(\kappa_0\) and the constants \(C_0, C_1,\) and \(C_2\) in function of \(2\sqrt{\alpha}/(\pi \beta)\) for different values of the parameter \(\alpha\) are depicted in Figure 1. Note that since \(C_0\) can be expressed uniquely in function of the parameter \(2\sqrt{\alpha}/(\pi \beta)\) regardless of the selected value of \(\alpha\), the corresponding curves depicted in Fig. 1(b) are on top of each other.

Fig. 1. Parameters of the ISS estimate for the clamped-free damped string

It can be seen that constants \(C_0, C_1,\) and \(C_2\) diverge to \(+\infty\) when the quantity \(2\sqrt{\alpha}/(\pi \beta) - 1/2\) converges to a non-negative integer. Such a behaviour directly follows from the fact that the underlying disturbance free operator \(A_0\) is a Riesz-spectral operator if and only if \(2\sqrt{\alpha}/(\pi \beta) - 1/2 \notin \mathbb{N}\). In particular, \(C \to 1\) when \(2\sqrt{\alpha}/(\pi \beta) - 1/2\) converges to an element of \(\mathbb{N}\), yielding \(m_R \to 0\).
For a given (frozen) value of $\alpha > 0$, it is interesting to study the evolution of the parameters of the ISS estimate when the damping parameter $\beta \to +\infty$. Based on the analytical expressions of the different constants, one can see that $C_0$ and $C_1$ are decreasing functions of $\beta$ for $\beta > 4\sqrt{\alpha}/\pi$. Furthermore, we have $C_0 \to 1$ and $C_1 \to 1/\sqrt{\alpha}$.

However, at the same time, $\kappa_0 \to 0$ and, consequently, $C_2 \to +\infty$. It is a direct consequence of the fact that certain eigenvalues of the disturbance free operator $A_0$ (and so the growth bound $\omega_0 = -\kappa_0$) converge to zero when $\beta \to 0$. Globally, it shows that the increase of the damping factor (for $\beta > 4\sqrt{\alpha}/\pi$) ensures a reduced impact of both initial condition and boundary disturbances on the system trajectory. Nevertheless, it degrades the decay rate as well as the bound on the impact of distributed disturbances on the system trajectories.

The divergent behaviour of $C_2$ is not induced by a tightness issue of the constants provided by Theorems 1 and 2 but is a property of the abstract boundary control system when the distributed disturbance $U$ evolves in the full space $H$. Indeed, for a given (frozen) $\alpha > 0$, consider an arbitrary $\beta > 4\sqrt{\alpha}/\pi$. Then, assumption (8) is satisfied and we have $k_0 = 0$. Let $C_2^\beta$ be any constant (possibly different from the one provided by Theorems 1 and 2) such that the ISS estimate (11) holds true. The superscript "$\beta$" indicates explicitly the dependency over the parameter $\beta$. Based on [22], we can sort the eigenvalues such that

$$\lambda_0^\beta = \frac{-\beta \pi^2}{8} + \frac{\pi}{4} \beta^2 \pi^2 - 4\alpha$$

with

$$\psi_0^\beta = \frac{1}{\lambda_0^\beta \rho_0^\beta} \left( -\sin \left( \frac{\pi}{2} \right), \lambda_0^\beta \sin \left( \frac{\pi}{2} \right) \right),$$

where

$$\rho_0^\beta = -\frac{\pi}{2\sqrt{2}} \sqrt{\frac{\beta^2 \pi^2 - 16\alpha}{4|\lambda_0^\beta|^2 + \alpha \pi^2}}.$$ 

In particular, we have the following asymptotic behaviours

$$\lambda_0^\beta \sim_{\beta \to +\infty} -\frac{\alpha}{\beta}, \quad \rho_0^\beta \sim_{\beta \to +\infty} -\frac{\beta \pi}{2\sqrt{2} \alpha}. \quad (29)$$

Let $X^\beta$ be the strong solution associated with the initial condition $X_0 = 0$, the boundary disturbance $d = 0$, and the (constant) distributed disturbance $U(t) = U_0 = \left( \frac{2}{\pi} \sqrt{\frac{2}{\alpha}} \sin \left( \frac{\pi}{2} \right), 0 \right) \in H$ which is such that $\|U(t)\|_H = 1$. Using the projection in the Riesz basis $\{\phi_n : n \in \mathbb{N}\}$ and (2), the trajectory satisfies for all $t \geq 0$,

$$\|X^\beta(t)\|_H = \left\| \int_0^t S^\beta(t-\tau)U_0 d\tau \right\|_H \geq \sqrt{m_R^\beta} \left| \langle U_0, \psi_0^\beta \rangle_H \right| \left| \int_0^t e^{\lambda_0^\beta \tau} d\tau \right| \geq \sqrt{m_R^\beta} \left| \langle U_0, \psi_0^\beta \rangle_H \right| \left| \frac{1 - e^{\lambda_0^\beta t}}{\lambda_0^\beta} \right|.$$ 

As $\left| \langle U_0, \psi_0^\beta \rangle_H \right| = \sqrt{\alpha \pi}/[2\sqrt{2\lambda_0^\beta \rho_0^\beta}]$, evaluating the equation above at $t = -1/\lambda_0^\beta > 0$ provides

$$C_2^\beta \geq \frac{\|X^\beta(-1/\lambda_0^\beta)\|_H}{\|U\|_{C^0([-1,-1/\lambda_0^\beta];H)}} \geq \sqrt{m_R^\beta} \frac{\sqrt{\alpha \pi(1 - e^{-1})}}{2\sqrt{2}[|\lambda_0^\beta|^2 \rho_0^\beta]}.$$ 

As the construction above is valid for any $\beta > 4\sqrt{\alpha}/\pi$, it follows from $m_R^\beta \to 1$ and (29) that $C_2^\beta \to +\infty$. 

\textsuperscript{10}I.e., $U = (u_1, u_2)$ with possibly $u_1 \neq 0$. 
Nevertheless, in the original problem as introduced in Subsection 2.3.2, the distributed disturbance does not evolve in the full space \( \mathcal{H} \) but in the subspace \( \{0\} \times L^2(0,1) \), i.e., \( U = (0,u) \) with \( u(t) \in L^2(0,1) \). As, Theorems 1 and 2 deal with full distributed perturbations, it can be established, by evaluating the term (14), a tighter version of the constant \( C^2_2 \) when \( U(t) \in \{0\} \times L^2(0,1) \). Specifically, considering again an arbitrary \( \beta > 4\sqrt{\alpha}/\pi \), assumption (8) is satisfied and we have \( k_0 = 0 \). In this configuration, it is shown in [22] via a direct evaluation of (14) that a constant \( C^2_2 \) such that the ISS estimate (11) holds true is given by\(^{11}\) \( C^2_2 = \gamma^2 \sqrt{M_H^2/2} \) where

\[
\gamma^2 = \frac{\sum_{n \in \mathbb{N}} (\gamma_{n,\epsilon}^2)^2}{\epsilon \in \{-1, +1\}} < \infty
\]

with for all \( n \in \mathbb{N} \) and \( \epsilon \in \{-1, +1\} \),

\[
\gamma_{n,\epsilon}^2 = 4\sqrt{2} \frac{1 + (2n + 1)^2 \alpha \pi^2}{(2n + 1)^2 \beta \pi^2} \sqrt{1 - \frac{16\alpha}{(2n + 1)^2 \beta^2 \pi^2}},
\]

and \( \{\lambda_{n,\epsilon}^2 : n \in \mathbb{N}, \epsilon \in \{-1, +1\} \} \) denotes the set of the eigenvalues of the disturbance free operator \( A_0 \). Using the fact that for all \( n \in \mathbb{N} \) and \( \epsilon \in \{-1, +1\} \), \( |\lambda_{n,\epsilon}| \geq \alpha/\beta \), we obtain:

\[
(\gamma_{n,\epsilon}^2)^2 \leq \frac{\beta^2 \pi^2}{\beta^2 \pi^2 - 16\alpha} \left( \frac{32}{\beta^2 \pi^2 (2n + 1)^4} + \frac{8}{\alpha \pi^2 (2n + 1)^2} \right).
\]

This yields\(^{12}\)

\[
C^2_2 \leq \frac{\beta \pi}{\sqrt{\beta^2 \pi^2 - 16\alpha}} \sqrt{\frac{1}{\alpha} \left( 1 + \frac{\alpha}{3\beta^2} \right) \left( 1 + \frac{4\sqrt{\alpha}}{\beta \pi} \right)},
\]

providing the asymptotic behavior:

\[
\lim_{\beta \to +\infty} C^2_2 \leq 1/\sqrt{\alpha}
\]

To summarize: the consideration of distributed perturbations \( U \) evolving in the full space \( \mathcal{H} \) leads to a constant \( C^2_2 \) that necessarily diverges when the damping parameter \( \beta \to +\infty \). In contrast, for distributed perturbations \( U = (0,u) \) evolving in the subspace \( \{0\} \times L^2(0,1) \), the constant \( C^2_2 \) can be selected such that it is asymptotically bounded when \( \beta \to +\infty \).

5.3 Damped Euler-Bernoulli beam

We consider the damped Euler-Bernoulli beam introduced in Subsection 2.3.3. The adjoint operator \( A_0^* \) is defined over the domain \( D(A_0^*) = D(A_0) \) by

\[
A_0^*(x_1, x_2) = (-x_2, x_1''' + 2\alpha x_2').
\]

Thus, for initial conditions \( y_0 \in H^2(0,1) \cap H^1_0(0,1) \) and \( y_0 \in L^2(0,1) \), and disturbances \( d = (d_1, d_2) \in C^0(\mathbb{R}_+; \mathbb{K}^2) \) and \( u \in C^0(\mathbb{R}_+; L^2(0,1)) \), \( X = (x_1, x_2) \in C^0(\mathbb{R}_+; \mathcal{H}) \) is the weak solution of the abstract boundary control system (1) associated with \((y_0, y_0), d, (0, u))\) if for all \( T > 0 \) and for all test function \( z = (z_1, z_2) \in C^0([0,T]; D(A_0^*)) \cap C^1([0,T]; \mathcal{H}) \) such that \( A_0^* z \in C^0([0,T]; \mathcal{H}) \) and \( z(T) = 0 \), the following equality is satisfied:

\[
\int_0^T \int_0^1 x_1''(t, \xi) \left\{ -z_2'' + \frac{d z_1}{d t} \right\} (t, \xi) d \xi dt + \int_0^T \int_0^1 x_2(t, \xi) \left\{ z_1''' + 2\alpha z_2' + \frac{d z_2}{d t} \right\} (t, \xi) d \xi dt
\]

\(^{11}\) The version provided in [22] is \( C^2_2 = \gamma^2 \sqrt{3M_H^2/2} \). The tighter version used in this paper is obtained by using (16).

\(^{12}\) We used \( \sum_{n \geq 0} \frac{1}{2n + 1} = \frac{\pi^2}{8} \) and \( \sum_{n \geq 0} \frac{1}{(2n + 1)^2} = \frac{\pi^4}{96} \).
while taking into account the sparse structure $U$

Similarly to the previous study, the divergent behavior of $\alpha > C$ when $\kappa \rightarrow C$.

As the chosen lifting operator satisfies $AB = 0$, the constants $C_1$ provided by both Theorem 1 and 2 are identical. Following [6, Exercise 2.23] and using the same approach that the one used in [22] for establishing the constants $m_R$ and $M_R$ related to the Riesz basis, one can show that:

1. Case $\alpha \in (0, 1)$: $\kappa_0 = \alpha \pi^2$, $\zeta = \alpha^{-1}$, $m_R = 1 - \alpha$, $M_R = 1 + \alpha$;
2. Case $\alpha > 1$: $\kappa_0 = (\alpha - \sqrt{\alpha^2 - 1})\pi^2$, $\zeta = 1$, $m_R = 1 - \alpha^{-1}$, $M_R = 1 + \alpha^{-1}$.

The boundary disturbance evolves into the two dimensional ($m = 2$) space $(\mathbb{K}^2, \| \cdot \|_2)$ endowed with the usual euclidean norm. By selecting $\mathcal{E} = \{e_1, e_2\}$ as the canonical basis of $\mathbb{K}^2$, we obtain $c(\mathcal{E}) = 1$ and $\|Be_1\|_\mathcal{H} = \|Be_2\|_\mathcal{H} = 1/\sqrt{3}$. Thus, the constants of the ISS estimate are given by

$$C_0 = \sqrt{\frac{1 + \alpha}{1 - \alpha}}, \quad C_1 = \frac{2}{\alpha \sqrt{\beta}} \sqrt{\frac{1 + \alpha}{1 - \alpha}}, \quad C_2 = \frac{1}{\alpha \pi^2} \sqrt{\frac{1 + \alpha}{1 - \alpha}}$$

when $\alpha \in (0, 1)$, while

$$C_0 = \sqrt{\frac{\alpha + 1}{\alpha - 1}}, \quad C_1 = \frac{2}{\sqrt{3}} \sqrt{\frac{\alpha + 1}{\alpha - 1}}, \quad C_2 = \frac{1}{(\alpha - \sqrt{\alpha^2 - 1})\pi^2} \sqrt{\frac{\alpha + 1}{\alpha - 1}}$$

when $\alpha > 1$. The evolution of the ISS constants in function of the damping parameter $\alpha$ is depicted in Figure 2.

We observe that constants $C_0$, $C_1$, and $C_2$ diverge to $+\infty$ when the quantity $\alpha \rightarrow 1$. It follows from the fact that for $\alpha > 0$, the underlying disturbance free operator $A_0$ is a Riesz-spectral operator if and only if $\alpha \neq 1$. In particular, $C_1 \rightarrow 1$ which yields $m_R \rightarrow 0$.

The impact of an increased damping term $\alpha$ is similar to the case of the clamped-free damped string. Indeed, $C_0$ and $C_1$ are decreasing functions of $\alpha > 1$ and we have $C_0 \rightarrow 1$ and $C_1 \rightarrow 2/\sqrt{3}$. Furthermore, we also observe that $\kappa_0 \rightarrow 0$ and, consequently, $C_2 \rightarrow +\infty$.

Similarly to the previous study, the divergent behavior of $C_2$ is induced by the fact that Theorems 1 and 2 deal with distributed perturbations evolving in the full space $\mathcal{H}$. This approach does not take into account the fact that the distributed disturbance evolves in the subspace $\{0\} \times L^2(0, 1)$. To provide a tighter constant $C_2$, we directly estimate (14) while taking into account the sparse structure $U = (0, u)$. For $\alpha > 1$, the eigenvalues of the disturbance free...
operator $A_0$ are given by $\lambda_{n,\varepsilon} = -n^2\pi^2(\alpha + \varepsilon\sqrt{\alpha^2 - 1}) \in \mathbb{R}^+$ where $n \in \mathbb{N}^*$ and $\varepsilon \in \{-1, +1\}$. The corresponding eigenvectors are given by

$$
\phi_{n,\varepsilon} = \frac{1}{n^2\pi^2\sqrt{\alpha (\alpha + \varepsilon\sqrt{\alpha^2 - 1})}} \left( \begin{array}{c} \sin(n\pi \cdot) \\ \lambda_{n,\varepsilon} \sin(n\pi \cdot) \end{array} \right).
$$

The eigenvalues of the adjoint operator $A_0^*$ are given by $\mu_{n,\varepsilon} = \lambda_{n,\varepsilon}$ and the corresponding eigenvectors, scaled such that $\langle \phi_{n_1,\varepsilon_1}, \psi_{n_2,\varepsilon_2} \rangle_{H^2} = \delta((n_1,\varepsilon_1),(n_2,\varepsilon_2))$, are given by

$$
\psi_{n,\varepsilon} = \frac{2\sqrt{\alpha (\alpha + \varepsilon\sqrt{\alpha^2 - 1})}}{n^2\pi^2 (1 - (\alpha + \varepsilon\sqrt{\alpha^2 - 1}))} \left( \begin{array}{c} \sin(n\pi \cdot) \\ -\lambda_{n,\varepsilon} \sin(n\pi \cdot) \end{array} \right).
$$

We denote by $\psi_{n,\varepsilon}^2$ the second component of $\psi_{n,\varepsilon}$. Based on 1) the sparse structure $U = (0, u)$ of the distributed disturbance, 2) equations (2) and (14), and 3) the Cauchy-Schwartz inequality, we obtain the following estimate.

$$
\left\| \int_0^t S(t - \tau)U(\tau) d\tau \right\|_{H^2}^2 \leq M_R \sum_{n \in \mathbb{N}^*} \left\| e^{\lambda_{n,\varepsilon}^* (t - \tau)} \int_0^1 u(\tau) \overline{\psi_{n,\varepsilon}^2} d\xi \right\|^2 d\tau \\
\leq M_R \left\| \sum_{n \in \mathbb{N}^*} \sum_{\varepsilon \in \{-1, +1\}} \frac{\|\psi_{k,\varepsilon}^2\|^2_{L^2(0,1)}}{|\lambda_{k,\varepsilon}|^2} \|u\|^2_{C^0([0,t];L^2(0,1))} \right\| \|u\|^2_{C^0([0,t];L^2(0,1))}.
$$

Straightforward computations show that\(^\text{13}\)

$$
\sum_{n \in \mathbb{N}^*} \sum_{\varepsilon \in \{-1, +1\}} \frac{\|\psi_{k,\varepsilon}^2\|^2_{L^2(0,1)}}{|\lambda_{k,\varepsilon}|^2} = \frac{1}{45} \times \frac{\alpha^2}{\alpha^2 - 1},
$$

providing for $\alpha > 1$ the following new version of the constant $C_2$ for the ISS estimate:

$$
C_2 = \frac{1}{3\sqrt{5}} \sqrt{\frac{\alpha}{\alpha - 1}},
$$

which is such that $C_2 \xrightarrow{\alpha \rightarrow +\infty} 1/(3\sqrt{5})$.

\section{Conclusion}

This paper established the Input-to-State Stability (ISS) property for a class of Riesz-spectral boundary control system with respect to both boundary and distributed perturbations. By projecting the system trajectories over an adequate Riesz basis, it was shown that the ISS property holds true for strong solutions associated with sufficiently regular disturbances. Then, in order to relax the regularity assumptions required for assessing the existence of strong solutions, a concept of weak solution that applies for a large class of boundary control systems (which is not limited to Riesz-spectral ones) has been introduced under a variational formulation. Various properties of the weak solutions were derived, including their existence and uniqueness, as well as their ISS property.

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\(^\text{13}\) We used $\sum_{n \geq 1} \frac{1}{n^4} = \frac{\pi^4}{90}$.
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A Proof of Lemma 3

By linearity, we must show that if $X \in C^0(\mathbb{R}_+; \mathcal{H})$ satisfies

$$
\int_0^T \left\langle X(t), A_0^* z(t) + \frac{dz}{dt}(t) \right\rangle_{\mathcal{H}} \, dt = 0 \tag{A.1}
$$

for all $T > 0$ and for all $z \in C^0([0,T]; \mathcal{D}(A^*_0)) \cap C^1([0,T]; \mathcal{H})$ such that $A^*_0 z \in C^0([0,T]; \mathcal{H})$ and $z(T) = 0$, then $X = 0$. Denoting by $S$ the $C_0$-semigroup generated by $A_0$, then $S^*$ is the $C_0$-semigroup generated by $A^*_0$ (see, e.g., [6, Thm 2.2.6]). Let $z_0 \in \mathcal{D}(A^*_0)$ and $\alpha > 0$ be arbitrarily given. For any given $T > 0$, we consider the function $z_{z_0,\alpha,T}$ defined for any $t \in [0,T]$, by $z_{z_0,\alpha,T}(t) = S^*(\alpha t)z_0 - S^*(\alpha T)z_0$. As $z_0 \in \mathcal{D}(A^*_0)$, we obtain that for any $t \geq 0$, $S^*(\alpha t)z_0 \in \mathcal{D}(A^*_0)$, $A^*_0 S^*(\alpha t)z_0 = S^*(\alpha t)A^*_0 z_0$, and

$$
\frac{dz_{z_0,\alpha,T}}{dt}(t) = \alpha A^*_0 S^*(\alpha t)z_0 = \alpha S^*(\alpha t)A^*_0 z_0.
$$

Thus, $z_{z_0,\alpha,T}$ is a test function over $[0,T]$ and (A.1) shows:

$$
(\alpha + 1) \int_0^T \langle X(t), S^*(\alpha t)A^*_0 z_0 \rangle_{\mathcal{H}} \, dt = \int_0^T \langle X(t), S^*(\alpha T)A^*_0 z_0 \rangle_{\mathcal{H}} \, dt.
$$

Using the definition of the adjoint operator, the fact that $S(\alpha T) \in \mathcal{L}(\mathcal{H})$, and the properties of the Bochner integral, the equation above is equivalent to

$$
\left( \alpha + 1 \right) \int_0^T S(\alpha t)X(t)dt - S(\alpha T)\int_0^T X(t)dt, A^*_0 z_0 \right\rangle_{\mathcal{H}} = 0.
$$

Because $\mathcal{H}$ is a Hilbert space with $A_0$ closed and densely defined, we have that $\ker(A_0)^\perp = \mathcal{R}(A^*_0)$ (see, e.g., [4, Rem. 1.7]). Since $A_0$ is assumed to be injective, it yields that $\mathcal{R}(A^*_0) = \mathcal{H}$. Consequently, we obtain that the following equality holds true for all $\alpha > 0$ and $T \geq 0$

$$
(\alpha + 1) \int_0^T S(\alpha t)X(t)dt = S(\alpha T)\int_0^T X(t)dt. \tag{A.2}
$$

This implies that for any $h > 0$,

$$
\frac{\alpha + 1}{h} \int_T^{T+h} S(\alpha t)X(t)dt = S(\alpha(T + h)) \left\{ \frac{1}{h} \int_T^{T+h} X(t)dt \right\} + \frac{S(\alpha h) - I_{\mathcal{H}}}{\alpha h} \left\{ \alpha S(\alpha T) \int_0^T X(t)dt \right\}.
$$
As $t \to X(t)$ and $t \to S(\alpha t)X(t)$ are continuous over $\mathbb{R}_+$, we obtain by the strong continuity property of the $C_0$-semigroups that
\[
\lim_{h \to 0^+} \frac{S(\alpha h) - I_H}{\alpha h} \left\{ S(\alpha T) \int_0^T X(t)dt \right\} = \frac{\alpha + 1}{\alpha} S(\alpha T)X(T) - \frac{1}{\alpha} S(\alpha T)X(T) = S(\alpha T)X(T).
\]
Thus we have $S(\alpha T) \int_0^T X(t)dt \in D(A_0)$ and
\[
A_0 \left\{ S(\alpha T) \int_0^T X(t)dt \right\} = S(\alpha T)X(T).
\]
From (A.2), we deduce that $\int_0^T S(\alpha t)X(t)dt \in D(A_0)$ and
\[
(\alpha + 1)A_0 \int_0^T S(\alpha t)X(t)dt = S(\alpha T)X(T)
\]
for all $\alpha > 0$ and $T \geq 0$. Introducing $y_\alpha(t) = \int_0^T S(\alpha t)X(t)dt$ and noting that $t \to S(\alpha t)X(t)$ is continuous over $\mathbb{R}_+$, we obtain that $y_\alpha$ satisfies over $\mathbb{R}_+$ the differential equation $\frac{dy_\alpha}{dt} = (\alpha + 1)A_0y_\alpha$ with the initial condition $y_\alpha(0) = 0$. As $A_0$ generates the $C_0$-semigroup $S$ and $\alpha + 1 > 0$, we deduce that $(\alpha + 1)A_0$ generates the $C_0$-semigroup $S((\alpha + 1)\cdot)$. Thus we have $y_\alpha = S((\alpha + 1)\cdot)y_\alpha(0) = 0$. By taking the time derivative of $y_\alpha$, we deduce that $S(\alpha t)X(t) = 0$ for all $\alpha > 0$ and $t \geq 0$. From the strong continuity property of the $C_0$-semigroups, we obtain by letting $\alpha \to 0^+$ that $X(t) = 0$ for all $t \geq 0$. □

## B Proof of Theorem 3

To establish the uniqueness part, we only need to show that $A_0$ is injective. In that case, the conclusion will follow from the application of Lemma 3. Let $x_0 \in \ker(A_0)$ be arbitrarily given. Introducing $X(t) = x_0$ for all $t \geq 0$, $X_0 = x_0$, $d = 0$, and $U = 0$, one has for all $t \geq 0$, $\frac{dX}{dt}(t) = 0 = A_0x_0 = AX(t) + U(t)$, $BX(t) = Bx_0 = 0 = d(t)$, and $X(0) = x_0$. Thus $X$ is the strong solution associated with $(X_0, d, U) = (x_0, 0, 0)$. The ISS estimate (26) gives
\[
\|x_0\|_H \leq \beta(\|x_0\|_H, t) \underset{t \to +\infty}{\longrightarrow} 0.
\]
This yields $x_0 = 0$, ensuring the injectivity of $A_0$.

To show the existence part, let an initial condition $X_0 \in H$, and disturbances $d \in C^0(\mathbb{R}_+; \mathbb{K}^m)$ and $U \in C^0(\mathbb{R}_+; H)$ be arbitrarily given. We also consider an arbitrarily given lifting operator $B$ associated with $(A, B)$.

**Step 1: Construction of a weak solution candidate $X \in C^0([0, T]; H)$ by density arguments.**

Let $T > 0$ be arbitrarily given. As $C^2([0, T]; \mathbb{K}^m)$ and $C^1([0, T]; H)$ are dense in $C^0([0, T]; \mathbb{K}^m)$ and $C^0([0, T]; H)$, respectively, there exist $(d_n)_n \in C^2([0, T]; \mathbb{K}^m)^N$ and $(U_n)_n \in C^1([0, T]; H)^N$ such that
\[
\|d_n - d\|_{C^0([0, T]; \mathbb{K}^m)} \underset{n \to +\infty}{\longrightarrow} 0,
\]
\[
\|U_n - U\|_{C^0([0, T]; H)} \underset{n \to +\infty}{\longrightarrow} 0.
\]
Now, as $D(A_0) = H$, there exists $(\tilde{X}_{0,n})_n \in (D(A_0))^N$ such that $\tilde{X}_{0,n} \underset{n \to +\infty}{\longrightarrow} X_0 - Bd(0)$. Introducing $X_{0,n} = \tilde{X}_{0,n} + Bd_n(0) \in D(A)$, the bounded nature of $B$ provides $X_{0,n} \underset{n \to +\infty}{\longrightarrow} X_0$. Recalling that $D(A_0) \subset \ker(B)$ and $BB = I_C$, we get $BX_{0,n} = B\tilde{X}_{0,n} + BBd_n(0) = d_n(0)$. 27
For any $n \in \mathbb{N}$, let $X_n \in C^0([0, T]; D(\mathcal{A})) \cap C^1([0, T]; \mathcal{H})$ be the unique strong solution of the abstract system (1) over $[0, T]$ associated with $(X_{0,n}, d_n, U_n)$. For any $n, m \in \mathbb{N}$, by linearity, $X_n - X_m$ is the unique strong solution of the abstract system (1) over $[0, T]$ associated with $(X_{0,n} - X_{0,m}, d_n - d_m, U_n - U_m)$. Thus the ISS estimate for strong solutions (26) yields for all $n, m \in \mathbb{N}$,

$$
\|X_n - X_m\|_{C^0([0,T]; \mathcal{H})} \leq \beta \left( \|X_{0,n} - X_{0,m}\|_{\mathcal{H}}, 0 \right) + \gamma_1 \left( \|d_n - d_m\|_{C^0([0,T]; \mathcal{K} m)} \right) + \gamma_2 \left( \|U_n - U_m\|_{C^0([0,T]; \mathcal{H})} \right).
$$

Since $(X_{0,n})_n$, $(d_n)_n$, and $(U_n)_n$ are Cauchy sequences, so is $(X_n)_n$. As $C^0([0, T]; \mathcal{H})$ is a Banach space, there exists $X \in C^0([0, T]; \mathcal{H})$ such that $X_n \to X$ uniformly on $[0, T]$. Writing the ISS estimate (26) for each strong solution $X_n$ and letting $n \to +\infty$ shows that (26) holds true for $X$ for all $t \in [0, T]$.

**Step 2:** The obtained $X \in C^0([0, T]; \mathcal{H})$ is independent of the chosen approximating sequences of $X_0$, $d$, and $U$.

For a given $T > 0$, we show that the construction of Step 1 provides a $X \in C^0([0, T]; \mathcal{H})$ that uniquely depends on $(X_0, d, u)$ in the sense that it is independent of the employed approximation sequences. Assume that, following the construction of Step 1, $(X_{1,0,n})_n \in D(\mathcal{A})^N$, $(d_{1,n})_n \in C^2([0, T]; \mathbb{K} m)^N$, $(U_{1,n})_n \in C^1([0, T]; \mathcal{H})^N$ and $(X_{2,0,n})_n \in D(\mathcal{A})^N$, $(d_{2,n})_n \in C^0([0, T]; \mathcal{K} m)^N$, $(U_{2,n})_n \in C^1([0, T]; \mathcal{H})^N$ converge to $X_0$, $d|_{[0, T]}$, $U|_{[0, T]}$, respectively. For any $n \in \mathbb{N}$ and $i \in \{1, 2\}$, let $X_{i,n}$ be the unique strong solution associated with $(X_{i,n,0}, d_{i,n}, U_{i,n})$ over $[0, T]$. We know from Step 1 that $X_{i,n} \to X_i \in C^0([0, T]; \mathcal{H})$ when $n \to +\infty$. By linearity $X_{1,n} - X_{2,n}$ is the unique strong solution associated with $(X_{1,n} - X_{2,0,n}, d_{1,n} - d_{2,n}, U_{1,n} - U_{2,n})$. Thus the ISS estimate for strong solutions (26) yields for all $n \in \mathbb{N}$,

$$
\|X_{1,n} - X_{2,n}\|_{C^0([0,T]; \mathcal{H})} \leq \beta \left( \|X_{1,0,n} - X_{2,0,n}\|_{\mathcal{H}}, 0 \right) + \gamma_1 \left( \|d_{1,n} - d_{2,n}\|_{C^0([0,T]; \mathcal{K} m)} \right) + \gamma_2 \left( \|U_{1,n} - U_{2,n}\|_{C^0([0,T]; \mathcal{H})} \right).
$$

Letting $n \to +\infty$, it gives $\|X_1 - X_2\|_{C^0([0,T]; \mathcal{H})} = 0$, i.e., $X_1 = X_2$.

**Step 3:** Definition of a weak solution candidate $X \in C^0(\mathbb{R}_+; \mathcal{H})$.

Let $0 < T_1 < T_2$ be arbitrarily given and let $X_1 \in C^0([0, T_1]; \mathcal{H})$ and $X_2 \in C^0([0, T_2]; \mathcal{H})$ as provided by the construction of Step 1. It is easy to see that, by restricting the approximation sequences of $d|_{[0, T_1]}$ and $u|_{[0, T_1]}$ from $[0, T_2]$ to $[0, T_1]$ and by resorting to the uniqueness result of Step 2, that $X_1 = X_2|_{[0, T_1]}$. Therefore, we can define $X \in C^0(\mathbb{R}_+; \mathcal{H})$ such that for any $T > 0$, $X|_{[0,T]} \in C^0([0, T]; \mathcal{H})$ is the result of the construction of Step 1. As (26) holds true for all $t \in [0, T]$ and for all $T > 0$ with functions $\beta, \gamma_1, \gamma_2$ that are independent of $T$, then (26) holds true for the built function $X \in C^0(\mathbb{R}_+; \mathcal{H})$ for all $t \geq 0$.

**Step 4:** The obtained candidate $X \in C^0(\mathbb{R}_+; \mathcal{H})$ associated with the unique weak solution associated with $(X_0, d, U)$.

Let $T > 0$ be arbitrarily given. Let $(X_{0,n})_n \in D(\mathcal{A})^N$, $(d_{n})_n \in C^0([0, T]; \mathbb{K} m)^N$, and $(U_{n})_n \in C^0([0, T]; \mathcal{H})^N$ be approximating sequences, compliant with the procedure of Step 1, converging to $X_0$, $d|_{[0, T]}$, and $U|_{[0, T]}$, respectively. Thus, the corresponding sequence of strong solutions $(X_n)_n$ converges to $X|_{[0,T]}$. Based on Corollary 1, $X_n$ is also a weak solution for all $n \in \mathbb{N}$. Thus, we have for all $z \in C^0([0, T]; D(\mathcal{A}_0^*)) \cap C^1([0, T]; \mathcal{H})$ such that $\mathcal{A}_0^* z \in C^0([0, T]; \mathcal{H})$ and $z(T) = 0$,

$$
\int_0^T \left\langle X_n(t), A_0^* z(t) + \frac{dz}{dt}(t) \right\rangle_{\mathcal{H}} dt = - \langle X_{n,0}, z(0) \rangle_{\mathcal{H}} + \int_0^T \langle Bd_n(t), A_0 z(t) \rangle_{\mathcal{H}} dt \tag{B.1}

- \int_0^T \langle ABd_n(t), z(t) \rangle_{\mathcal{H}} dt - \int_0^T \langle U_n(t), z(t) \rangle_{\mathcal{H}} dt.
$$

From $X_{0,n} \to X_0$, it is immediate by the Cauchy-Schwarz inequality that $\langle X_{n,0}, z(0) \rangle_{\mathcal{H}} \to \langle X_0, z(0) \rangle_{\mathcal{H}}$. Again, by the Cauchy-Schwarz inequality,

$$
\left| \int_0^T \left\langle X_n(t) - X(t), A_0^* z(t) + \frac{dz}{dt}(t) \right\rangle_{\mathcal{H}} dt \right| \leq \int_0^T \|X_n(t) - X(t)\|_{\mathcal{H}} \left\| A_0^* z(t) + \frac{dz}{dt}(t) \right\|_{\mathcal{H}} dt.
$$
\[ \leq T \left\| A_0^* z + \frac{dz}{dt}\right\|_{\mathcal{C}^0([0,T];\mathcal{H})} \|X_n - X\|_{\mathcal{C}^0([0,T];\mathcal{H})} \]
\[ \xrightarrow{n \to +\infty} 0. \]

Applying a similar procedure to the three integral terms on the right hand side of (B.1), one can show their convergence when \( n \to +\infty \). Thus, letting \( n \to +\infty \) in (B.1), we obtain that \( X \) satisfies (24) for all \( T > 0 \) and all test function \( z \) over \([0,T]\). Thus, \( X \) is the unique weak solution associated with \((X_0, d, U)\). \( \square \)