$K \to \pi \ell^+ \ell^-$ form factor in the Large-$N_c$ and cut-off regularization method

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Bardeen-Buras-Gérard 1–7 have proposed a large $N_c$ method to evaluate hadronic weak matrix elements to attack for instance the determination of the $\Delta I = 1/2$-rule and Re$(\ell'/\ell)$. Here we test this method to the determination of the form factor parameters $a_+$ and $b_+$ in the decays $K^+ \to \pi^+ \ell^+ \ell^-$ and $K_S \to \pi^0 \ell^+ \ell^-$. The results are encouraging: in particular after a complete treatment of Vector Meson Dominance (VMD).

I. INTRODUCTION

Rare kaon decays play a crucial role in particle physics 8–15, 25 as we will do in this paper. The appearance of chiral unknown constants 23, 24 brings up the crucial question well by NA48/2 16, 19, 22. The form factor in the theoretical framework suggested by the UKQCD Collaborations 21 address this issue. The determination is needed and the recent lattice RBC and corresponding K and thus measures the height of the unitarity triangle. The measurement of this decay may also lead to New Physics test 18. There is also an indirect CP-Violating contribution from $K_S \to \pi^0 e^+ e^-$, the magnitude of which can be obtained from the measured BR for the corresponding $K_S$ decay 19, 20. Also a theoretical determination is needed and the recent lattice RBC and UKQCD Collaborations 21 address this issue. The related $K^+ \to \pi^+ \ell^+ e^-$ decay may help also to this goal: the experimental form factor here has been measured well by NA48/2 16, 19, 22. The appearance of chiral form factor in the decays $K^+ \to \pi^+ e^+ e^-$ and $K^+ \to \pi^+ \mu^+ \mu^-$ separately, the question of Lepton Flavor Universality Violation is also interesting 26, 63.

In this paper we will evaluate the $K^+ \to \pi^+ \ell^+ \ell^-$ form factor in the theoretical framework suggested by Bardeen-Buras-Gerard (BBG) 1–7; the authors of this approach have successfully applied the method to the explanation of the $\Delta I = 1/2$-rule and $\pi^+ - \pi^0$-mass difference: we think it is interesting to apply it here.

The recent lattice result from RBC and UKQCD Collaborations 21 reporting on the $K \to \pi \pi$ matrix element Re$(A_0)$ and Im$(A_0)$ leading to 2–3σ below the experimental world average of Re$(\ell'/\ell)$ has led the authors of Refs. 62, 63 to evaluate the same weak matrix element $B_0$ and $B_s$ in their approach, finding consistency with lattice results and they conclude 63 that New Physics seems required to accommodate the present experimental value of Re$(\ell'/\ell)$. Using Large $N_c$ and Minimal Hadronic Ansatz, Hambye et al. in Ref. 60 still find agreement with experimental values. Final state interaction is not accurately described by lattice (the lattice result 21 for the $I = 0$ phase-shift $\delta_0 = 23.8(4.9)(1.2)^0$ is about $3\sigma$ smaller than the value obtained in dispersive treatments of Ref. 67, 69) and a good theoretical description could lead to agreement with experiments as the approach of Refs. 70, 71; for an alternative solution see Ref. 72.

Nevertheless we think it is interesting to check BBG method in $K^+ \to \pi^0 e^+ e^-$ decay. We dedicate section II and III to model independent discussion, section IV to the BBG method, V to the form factor evaluation, VI to the addition of vectors and VII to the $K_S \to \pi^0 \ell^+ \ell^-$-form factor.

II. MODEL INDEPENDENT ANALYSIS

The decay $K \to \pi \ell \ell$ is dominated by a virtual photon exchange 23, 24,

\[
A[K(k) \to (p)\gamma(q)] = \frac{W_+(z)}{(4\pi)^2} [z(k+p)\mu - (1-r^2)q\mu], \tag{1}
\]

where $r \equiv \frac{M}{M_K}$ and $z \equiv \frac{q^2}{M_K^2}$, with $q^2$ being the photon transferred momentum. With these conventions the decay amplitude takes the form ($\alpha = e^2/4\pi$),

\[
A[K(k) \to \pi(p)\ell^+(p_+)\ell^-(p_-)] = -\frac{\alpha}{4\pi M_K} W_+(z) (k+p)\mu \bar{u}(p_+) \gamma^\mu v(p_-). \tag{2}
\]

The form factor $W_+(z)$ can be decomposed into two parts: one coming from the dominant pion loop contribution $W_\pi^+(z)$, and another one $W_\pi^{pol}(z)$, that accounts for the contributions of higher mass intermediate states (like $K^+ K^-$ for instance and local pieces). $W_\pi^{pol}(z)$ can be well approximated by a linear polynomial for small values of $z$, $W_\pi^{pol}(z) \sim a_+ + b_+ z$. In this way, $W_+(z)$ can be written as 24.

\[
W_+(z) = G_F M_K^2 (a_+ + b_+ z) + W_\pi^+(z), \tag{3}
\]
with a priori unknown low-energy constants contributing to $a_+$ and $b_+$, which have to be experimentally determined

\[ W^{\pi \pi}(z) \] .

$W^{\pi \pi}(z)$ is obtained from the analytic structure of the diagram in Fig. 1 [24].

![Diagram of pion loop contribution to $K^+ \rightarrow \pi^+ \gamma^*$](image)

**FIG. 1.** Pion loop contribution to $K^+ \rightarrow \pi^+ \gamma^*$. The unitary cut used is represented too. The blob represents the $K \rightarrow 3\pi$ vertex.

In Ref. [24], the behaviour of $W_+(z)$ at $z \rightarrow 0$ is entirely fixed up to $W_+(z)$,

\[ W_+(z) \sim G_F M_K^2 a_+ + \left( G_F M_K^2 b_+ + \frac{3r_2^2(\alpha_+ - \beta_+) - \beta_+}{180\pi^5} \right) \] ,

where $\alpha_+ = (-20.6 \pm 0.5) \cdot 10^{-8}$ and $\beta_+ = (-2.6 \pm 1.2) \cdot 10^{-8}$ are the $K \rightarrow 3\pi$ parameters from Ref. [73, 78].

The local counter-term structures at $O(p^4)$ are

\[ a^{(4)}_+ = \frac{G_S}{G_F} \left( \frac{1}{3} - w_+ \right) \] ,

where $w_+$ is given [26] in terms of $N_i$'s [79] and $L_9$ [80] by

\[ w_+ = \frac{64\pi^2}{3} (N_{14} - N_{15}^* + 3L_9) + \frac{1}{3} \ln \frac{\mu^2}{M_K M_\pi} \] .

Since $w_+$ is scale independent, the $\mu$ dependence of the combination among the $N_i$'s and $L_9$ is exactly compensated by the log $\mu^2$ from the chiral loop in eq. [60]. From the loop contribution, one has also,

\[ b^{(4)}_+ = -\frac{G_S}{G_F} \frac{1}{60} \] .

Experimentally [22], we have

\[ a^{\text{exp}}_+ = -0.578 \pm 0.016 \] ,

\[ b^{\text{exp}}_+ = -0.779 \pm 0.066 \] .

As we can see, the experimental values for $a_+$ and $b_+$ are the same order of magnitude. We then have to understand why so. Indeed, $a_+$, $L_9$ and the $N_i$'s have large contributions from the VMD [79, 81, 82], since $b_+$ is mostly a $O(p^3)$ observable, it should have an important enhancement.

The $K_S$ decay is discussed in Sec. VII.

**III. AMPLITUDE ANALYSIS AND SHORT DISTANCE RESULTS**

The behavior of the amplitude in eq. [23] can be studied distinguishing two different contributions: (i) the long-distance (LD) one described by chiral perturbation theory ($\chi$PT) [24, 28], and (ii) the short-distance (SD) one described by an effective four-quark Hamiltonian [83, 85, 87, 88, 91]. The complete description of the amplitude implies then a continuation through both regions.

The dominant $\Delta S = 1$, SD effective four-quark Hamiltonian is given by [83, 91],

\[ H^{\Delta S=1} = -G_F V_{ud}^* V_{us} \left[ C_- (\mu^2) Q_- (\mu^2) + C_7 (\mu^2) Q_7 \right] , \]

where $C_- (\mu^2)$ and $C_7 (\mu^2)$ are the Wilson coefficients (see Appendix C for their expressions) associated to the four-quark operators $Q_- (\mu^2)$ and $Q_7$ respectively, given by

\[ Q_- = 4(\bar{s}_L \gamma^\mu u_L)(\bar{u}_L \gamma^\nu d_L) - 4(\bar{s}_L \gamma^\nu d_L)(\bar{u}_L \gamma^\mu u_L) , \]

\[ Q_7 = 2\alpha(\bar{s}_L \gamma^\nu d_L)(\bar{e}_\gamma \gamma^\mu e) \] .

The SD amplitude then takes the form,

\[ \mathcal{A}(K \rightarrow \pi \ell^+ \ell^-) = -\frac{G_F V_{us}^* V_{ud}}{\sqrt{2}} \times \langle \pi \ell^+ \ell^- \mid C_- (\mu^2) Q_- (\mu^2) + C_7 (\mu^2) Q_7 \mid K \rangle . \]

Both, the Wilson coefficients and the four-quark operators depend on the renormalization scale $\mu$ that separates the two regimes. Nevertheless, the physical amplitude cannot depend on $\mu$. $Q_7$ in eq. [13] is an $\mu^2$-independent operator, so that in order the amplitude to be $\mu^2$ independent, the Wilson coefficient $C_7 (\mu^2)$ has to cancel the $\mu^2$ dependence in $C_- (\mu^2) Q_- (\mu^2)$. Some of the consequences of this SD property will be considered in a model independent form in Ref. [92].

**IV. THE BARDEEN-BURAS-GÉRARD FRAMEWORK**

In Ref. [117], the authors use an order $p^2$ chiral Lagrangian and a physical cut-off $M$ to regularize the contributions beyond tree level instead of the usual local counter-terms (e.g., the $L_i$ and $N_i$ constants). Consequently, their results exhibit a quadratic dependence on the physical cut-off $M$ which according to them is a crucial ingredient in the matching of the meson and quark pictures. They argue that one can obtain a parametrization of non-perturbative QCD effects by matching a low-energy Lagrangian, valid up to the scale $M$, to the logarithmic behaviour of relevant Wilson coefficients at high-energy. In this
work we refer to this computational method as the Bardeen-Burns-Gérard framework (BBG).

In this context, the function \( W_+(z) \) becomes a function of \( q^2 \) and \( M^2 \),

\[
W_+(z) \rightarrow W_+(z, M^2) .
\] (14)

Our goal is to predict the values of the \( a_+ \) and \( b_+ \) coefficients using BBG framework.

At the matching scale \( M \), the description for low and high energy must coincide; this means that the LD quadratic divergence in \( M \) has to be numerically equal to the SD logarithmic divergence. Therefore, at \( \mu^2 = M^2 \) the SD Hamiltonian,

\[
H_{\text{eff},s=1} = - \frac{G_F V_{us} V_{ud}}{\sqrt{2}} 
\times \left[ C_-(M^2)Q_-(M^2) + C_7(M^2)Q_7 \right] ,
\] (15)

must coincide with its chiral representation at LD.

### A. Amplitude properties

The BBG approach considers only a chiral \( O(p^2) \) effective Lagrangian below the scale \( M \), so that, since the loop calculations are regularized by the cut-off \( M \), higher order Lagrangians (i.e. with \( L_i \) and \( N_i \) constants) do not appear at all. Following their prescriptions, one has then,

\[
\mathcal{A} \left( K \rightarrow \pi \ell^+ \ell^- \right) = - \frac{G_F V_{us}^* V_{ud}}{\sqrt{2}}
\times \left\langle \pi \ell^+ \ell^- \left| C_-(M^2)Q_-(M^2) + C_7(M^2)Q_7 \right| K \right\rangle .
\] (16)

The chiral loop calculation of the matrix element of \( Q_- \) with the \( O(p^2) \) chiral Lagrangian does not provide any quadratic divergences (of course not in dimensional regularization) even in the cut-off regularization (see Appendix B). The \( \ln M^2 \) appearing here at the chiral scale, is cancelled by local counter-terms in eq. (6) in usual \( \chi \)PT. Now, this role is played by \( C_7(\mu^2 = M^2)Q_7 \).

The matching between SD and LD should be around \( 1 \) GeV, then \( C_-(M^2)Q_-(M^2) \) and \( C_7(\mu^2 = M^2)Q_7 \) have to evolve from the chiral scale to 1 GeV. But this evolution violates the mixing between the operators \( Q_- \) and \( Q_7 \) according to RGE [8–15]. In the BBG framework, this mixing is captured by the quadratic divergences [1–7] which in our case can come only from the \( K \rightarrow 3\pi \) vertex (chirally related to \( K \rightarrow 2\pi \) studied by BBG see below). In other words, the authors of Ref. [1–7] have extended the usual renormalization flow of the SD sector (from \( M_W^2 \) to \( M^2 \)) to a flow in the LD sector from \( M^2 \) to 0 (as depicted in Fig. 2) through the relation,

\[
Q_-(M^2) = \mathcal{E}(M^2)Q_-(0) ,
\] (17)

where \( \mathcal{E}(M^2) \) is the evolution operator given by [7],

\[
\mathcal{E}(M^2) \equiv 1 + \frac{3}{16\pi^2} \left[ \frac{M^2}{f^2} + \frac{M_K^2}{f_S^2} \ln \left( 1 + \frac{M^2}{\tilde{m}^2} \right) \right] ,
\] (18)

with \( \tilde{m} \approx 0.3 \) GeV. \( \mathcal{E}(M^2) \) comes from the \( K \rightarrow \pi \pi \) analysis in Ref. [7], and soft-pion theorem tells us that it can be applied to \( K \rightarrow 3\pi \) vertex [8–15]. The amplitude is then given by

\[
C_-(M^2) \left\langle 3\pi \left| Q_-(M^2) \right| K \right\rangle = C_-(M^2)\mathcal{E}(M^2) \left\langle 3\pi \left| Q_-(0) \right| K \right\rangle .
\] (19)

The authors of Ref. [7] find that the range of numerical values for \( M \) that leaves the amplitude invariant is,

\[
0.6 \text{ GeV} \leq M < 1 \text{ GeV} ,
\] (20)

with a preferred value at 0.7 GeV (without vector contribution).

![FIG. 2. The usual Wilson Renormalization flow is represented above the scale \( M^2 \). The extended Renormalization flow defined in eq. (18) is shown below \( M^2 \).](image-url)

### V. DETERMINATION OF \( a_+ \) AND \( b_+ \) (NO VECTORS)

Eq. (14) determines uniquely the coefficients \( a_+ \) and \( b_+ \) as we will see here. Writing

\[
W_+(z, M^2) \sim M_K^2 G_F a_+ + M_K^2 G_F b_+ (M^2)z ,
\] (21)

we identify (the wave function renormalization factors \( Z_\pi \) and \( Z_K \) are given in App. B),

\[
a_+(M^2) = - \frac{V_{us}^* V_{ud}}{\sqrt{2}} \sqrt{Z_\pi Z_K} \left\{ -4\pi C_7(M^2) 
+ C_-(M^2) \left[ -\frac{5}{9} + \frac{1}{3} \ln \frac{M^2}{M_K M_K} \right] \mathcal{E}(M^2) \right\}.
\] (22)

Compared to the analysis of \( K \rightarrow 2\pi \) in Ref. [1–7], we have additionally a further cancellation of the log in
\[
\begin{align*}
&\left[ -\frac{9}{8} + \frac{3}{2} \ln \frac{M^2}{M^2_{\pi^+}} \right] \text{ and the log in } C_7(M^2). \text{ This fixes } M \\
& \text{and then } a_+ \text{ as shown in Fig. 3. We have also,}
\end{align*}
\]
\[
\begin{align*}
b_+(M^2) &= -\frac{V_{us}^* V_{ud}}{\sqrt{2}} \frac{Z_\pi Z_K}{60 f_\pi^2} C_-(M^2) E(M^2) \\
& \quad - \frac{1}{M_K^2} \frac{3 r_2^2 (\alpha_+ - \beta_+) - \beta_+}{180 G_F r_\pi^6} .
\end{align*}
\] (23)

In order to find the value of \( M \) where there is a compensation between the LD quadratic dependence (including both terms in eq. (24), constant and the novel logarithmic one) and the SD logarithm, we look for the solution of \( \partial_M a_+ = 0 \). We find that this equation is satisfied when \( M = 0.7 \text{ GeV} \) and numerically one gets
\[
\begin{align*}
a_+ &\left(0.7 \text{ GeV}^2\right) = -0.5 , \quad (24) \\
b_+ &\left(0.7 \text{ GeV}^2\right) = -0.12 . \quad (25)
\end{align*}
\]

Comparing with the experimental values eq. [10], we find a good agreement for \( a_+ \), but not for \( b_+ \). Fig. 2 shows \( a_+ \) as a function of \( M \), together with the contributions coming from \( C_- \) and \( C_7 \), separately. These are the expected behaviours from LD physics. The dashed vertical line corresponds to the scale where \( \partial_{M^2} a_+ = 0 \).

In the following section we study the inclusion of vectors in the BBG approach.

**VI. VECTOR CONTRIBUTIONS IN THE BBG FRAMEWORK**

Vector contributions increase the range of validity of \( M^2 \) and smooth over the transitions between short and long distance continuation [11,12]. We have to consider two counter-term structures from eq. (11) which are shown in Fig. 4.

![Fig. 4](image)

**FIG. 4.** Type (a) diagram represents the VMD contribution to \( L_9 \) (the \( \bullet \) vertex is \( O(p^2) \Delta S = 1 \) vertex). Analogously, type (b) diagram represents the VMD contribution to the \( N_{14} - N_{15} \) one (the \( \oplus \) vertex is \( \Delta S = 1 \) vertex coming from \( Q_- \)).

![Fig. 3](image)

**FIG. 3.** In blue, the variation of \( a_+ \) as a function of \( M \) in GeV. The dotted green curve represents the contribution proportional to \( C_- (M^2) \) and the dashed orange curve the one proportional to \( C_7 (M^2) \). The vertical dashed line stands for the matching scale.

In order to find \( a_+ \) a good approximation for the mixing between \( Q_- \) and \( Q_7 \) in the RGE by adding an extra \( C_- (M^2) \eta_V (M^2, z) \). This contribution is not present in \( K \to \pi \pi \) processes and so does not affect the results in Ref. [10,12]. The complete calculation with vectors can be done using the Hidden Local Symmetry framework [93–97]. We have to be careful, the counting in Large-\( N_c \) must be respected by including all terms up to \( 1/N_c \) corrections with the same argument in Sec. IV.

One can evaluate this contribution as
\[
\eta_V(M^2, z) = 4\pi \left[ \frac{f_\pi^2}{M_V^2} - \frac{z}{2} \frac{M_K^2}{M_V^2} \ln \frac{M^2}{M_K^2} \right] .
\] (29)

One gets therefore,
\[
\begin{align*}
a_+ (M^2, M_V^2) &= \\
& \quad - \frac{V_{us}^* V_{ud}}{\sqrt{2}} \frac{Z_\pi Z_K}{M_{\pi^+}M_K} \left\{ -4\pi C_7 (M^2) \\
& \quad + C_- (M^2) \left[ \frac{5}{9} + \frac{1}{3} \ln \frac{M^2}{M_{\pi^+}M_K} \right] E(M^2, M_V^2) \right. \\
& \quad + C_- (M^2) 4\pi \frac{f_\pi^2}{M_V^2} \left. \right\} ,
\end{align*}
\] (30)
\[ b_+(M^2, M_V^2) = \frac{M_V^2}{M_V^2 a_+(M^2, M_V^2)} \left\{ \begin{array}{l}
\frac{3r^2_0(\alpha_+ - \beta_+) - \beta_+}{8\pi M_V^2} - \frac{8\pi M_V^2}{3 M_V^2} \ln \frac{M^2}{M_K^2} \left( \frac{M^2}{M_K^2} \right) \right. \\
\left. - \frac{V_s V_{ud}}{\sqrt{2}} \sqrt{Z_0 Z_K C_-(M^2)} \right\}.
\]

In the same manner as before, we evaluate the scale \( M \) by requiring \( \partial^2 a_+ = 0 \) in eq. (30), and obtain that for \( M = 0.7 \) GeV

\[ a_+ ((0.7 \text{ GeV})^2, (0.775 \text{ GeV})^2) = -0.54, \]

\[ b_+ ((0.7 \text{ GeV})^2, (0.775 \text{ GeV})^2) = -0.72. \]

The interplay between strong amplitudes (\( L_3 \)) with external weak transitions (diagrams (b) in Fig. 4) have been already noticed by the authors of Ref. 98 for the VMD \( \mathcal{O}(p^6) \) contribution to \( K_L \to \pi^0\gamma\gamma \).

We show in Fig. 5 \( a_+ \) as a function of \( M \) in the three different scenarios: ‘BBG no vect.’ is the framework where no vectors are included at all and ‘BBG(vect)(a)’ is the one where only the diagrams (a) in Fig. 4 are considered. We refer to ‘BBG(vect) (a) + (b)’ as the last case where all kinds of diagrams in Fig. 4 have been included.

![Graph](image)

**FIG. 5.** \( a_+ \) as a function of \( M \) in the three different frameworks: ‘BBG no vect.’ where vectors are not included, ‘BBG(vect)(a)’ represents the contribution coming only from diagrams (a) in Fig. 4 and ‘BBG(vect) (a) + (b)’ is the case where both (a) and (b) diagrams were included. The vertical line indicates the value \( M = 0.7 \) GeV.

Following Buchalla et al. in Ref. 99, we investigate our predictions of what the authors call \( a_+^{\text{VMD}} \) and \( a_+^{\text{VMD}} \). Under the general hypothesis that the \( b_+ \) term in eq. (3) is generated by the expansion of a vector-meson propagator, \( W_+(z) \) can be written as,

\[ W_+(z) = G_F M_K^2 \left( a_+^{\text{VMD}} + a_+^{\text{VMD}} + a_+^{\text{VMD}} \frac{M_K^2}{M_V^2} \right), \]

where \( a_+^{\text{VMD}} \) denotes \( z \)-independent non-VDM contributions. The introduction of the \( \eta \) contribution is necessary to recover this separation between \( a_+^{\text{VMD}} \) and \( a_+^{\text{VMD}} \). Indeed, we find,

\[ a_+^{\text{VMD}} = \frac{M_V^2}{M_K^2} \left( b^{\text{BBG(vect)}}(a)+(b)-b^{\text{BBG}} \right) \]

\[ = \frac{M_V^2}{M_K^2} [-0.72 - (-0.12)] = -1.5, \]

which is in good agreement with \( a_+^{\text{VMD}} = \frac{M_V^2}{M_K^2} \text{exp} = -1.6 \pm 0.1 \) 99.

**VII. ANALYSIS OF \( K_S \to \pi^0\ell^+\ell^- \)**

The analysis of \( K_S \to \pi^0\ell^+\ell^- \) can be directly deduced from the previous one, \( K^+ \to \pi^+\ell^+\ell^- \). Indeed,

\[ \langle \pi^0\gamma^*(q) \mid Q_-(0) \mid K_S \rangle = -\langle \pi^0\gamma^*(q) \mid Q_-(0) \mid K^+ \rangle \bigg|_{M_+ = M_K} . \]

And, in this case, the local counter-term structures at \( \mathcal{O}(p^4) \) are 24

\[ a_s^{(4)} = \frac{G_S}{G_F} \left( \frac{1}{3} - w_s \right), \]

where \( w_s = 23, 79, 80 \)

\[ w_S = \frac{32\pi^2}{3} (2N_r^r + N_l^r) + \frac{1}{3} \ln \frac{M^2}{M_K^2}. \]

Given the decay \( K_S \to 3\pi (\Delta I = 1/2 \text{ transitions}) \) is not allowed (\( \Delta I = 3/2 \text{ transitions are permitted} \), only kaons are present in the loop (see Appendix [4]). Using the same identification as in eq. (31) and following the same procedure as in the case of the decay \( K^+ \to \pi^+\ell\ell \), we find that \( a_S = 1.2 \) \( (a_S^{\exp} = |1.08|\pm0.26 \) 22 \) for the same scale \( M = 0.7 \) GeV established from eq. (30). This value is in agreement with the fitted \( w_S \) value obtained in Ref. 100.

**VIII. CONCLUSION**

We have evaluated the \( K^+ \to \pi^+\ell^+\ell^- \) form factor parameters \( a_+ \) and \( b_+ \) in the BBG framework. Regarding \( a_+ \) the theoretical dependence/uncertainty in this framework on the matching scale seems small,
see Fig. 3: comparison with phenomenology seems very successful, see eq. (24). Consistency with the full chiral structure of the weak counter-terms has required a more general discussion on vector contributions (see section VI in the context of $K \rightarrow 2\pi$. This extension met nicely with the experimental values [22]. We have applied our method to $K \rightarrow \pi^0 e^+ e^-$ in section VII and found a good agreement with experimental results too.

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Appendix A: The evaluation of scalar integrals

The loop integral with a cut-off $M^2$ has the form,

$$I(\alpha, R^2, M^2) = -i \int \frac{d^4 \ell}{(2\pi)^4} \frac{1}{(\ell^2 - R^2)^\alpha}, \quad (A1)$$

and gives

$$I(\alpha, R^2, M^2) = \frac{M^4 R^{-2\alpha}}{6(4\pi)^2} \mathcal{F}_1 \left( \frac{\alpha, 2}{3} - \frac{M^2}{R^2} \right), \quad (A2)$$

$\mathcal{F}_1$ is the Gauss' hypergeometric function. In the one loop case for example, the integral is given by

$$A_0 (m^2) = I(1, m^2, M^2) = \frac{1}{3(4\pi)^2} \left[ M^2 - m^2 \ln \left( 1 + \frac{M^2}{m^2} \right) \right]. \quad (A3)$$

All the scalar integrals can be evaluated using eq. (A2).

Appendix B: Amplitudes formulae

1. $K^+ \rightarrow \pi^+ \gamma^*$

The form factor defined in eq. (14) is obtained from

$$W_+(z, M^2) = \frac{M^2 G_{FV} V^*_u V_d}{\sqrt{2}} \frac{\sqrt{Z_K Z_\pi}}{\sqrt{Z_K Z_\pi}} \times$$

$$\left[ C_-(M^2) \langle \pi^+ \gamma^*(q) | Q_-(M^2) | K^+ \rangle + 4\pi C_7(M^2) \right]. \quad (B1)$$

where,

$$\langle \pi^+ \gamma^*(q) | Q_-(M^2) | K^+ \rangle = \mathcal{E}(M^2) \langle \pi^+ \gamma^*(q) | Q_-(0) | K^+ \rangle, \quad (B2)$$

and

$$\sqrt{Z_K Z_\pi} = 1 + \frac{1}{16\pi^2} \left[ \frac{M^2}{f^2} - \frac{5 M^2}{12} \ln \left( 1 + \frac{M^2}{M_K^2} \right) - \frac{1}{8} \frac{M^2}{f^2} \ln \left( 1 + \frac{M^2}{M_M^2} \right) \right]. \quad (B3)$$

From a pure $\chi$PT loop calculation using the cut-off prescription in eq. (A2), one has

$$\langle \pi^+ \gamma^*(q) | Q_-(0) | K^+ \rangle = \chi \left( \frac{z}{r^2} \right) + \chi(z) - \frac{5}{9} + \frac{1}{3} \ln \frac{M^2}{M_K M_M}. \quad (B4)$$

The $\chi$ function is the one defined in Ref. [24] and it is related to the $\Phi$ in Ref. [23] as $\chi(z) = \Phi(z)/z + 1/6$. Numerically the kaon loop contribution, the $\chi(z)$ term in eq. (B4), is negligible. The extra constant term and $\ln(M^2)$ in (B4) come from the cut-off regularization. It is from this formula that one can extract the expressions for $a_+(M^2)$ and $b_+(M^2)$.

2. $K_S \rightarrow \pi^0 \gamma^*$

For this decay, the form factor $W_S(z, M^2)$ is,

$$W_S(z, M^2) = \frac{M^2 G_{FV} V^*_u V_d}{\sqrt{2}} \sqrt{Z_K Z_\pi} \times$$

$$\left[ C_-(M^2) \langle \pi^0 \gamma^*(q) | Q_-(M^2) | K_S \rangle - 4\pi C_7(M^2) \right]. \quad (B5)$$

The evolution operator in eq. (18) is exactly the same as in the $K_S$ case, so,

$$\langle \pi^0 \gamma^*(q) | Q_-(M^2) | K_S \rangle = \mathcal{E}(M^2) \langle \pi^0 \gamma^*(q) | Q_-(0) | K_S \rangle, \quad (B6)$$

where,

$$\langle \pi^0 \gamma^*(q) | Q_-(0) | K_S \rangle = 2\chi(z) - \frac{5}{9} + \frac{1}{3} \ln \frac{M^2}{M_K}. \quad (B7)$$

Appendix C: Expressions for $C_-(\mu^2)$ and $C_7(\mu^2)$

The expressions for $C_-(\mu^2)$ and $C_7(\mu^2)$ are [91],

$$C_-(\mu^2) = \frac{1}{2} \left[ \frac{\alpha_s(\mu^2)}{\alpha_s(M^2, 3)} \right] \left[ \frac{\alpha_s(M^2, 4)}{\alpha_s(M^2, 4)} \right] \quad (C1)$$
and

\[ C_T(\mu^2) = \frac{16}{99\alpha_s(M_w^2, 3)} \times \left\{ \frac{\alpha_s(M_t^2, 4)}{\alpha_s(M_w^2, 4)} \left[ 1 - \left( \frac{\alpha_s(\mu^2, 3)}{\alpha_s(M_t^2, 3)} \right)^{\frac{11}{25}} \right] \right\}, \] (C2)

where

\[ \alpha_s(\mu^2, n) = \frac{12\pi}{33 - 2n} \ln \left( \frac{n!}{\mu^2 \gamma (\gamma)} \right). \] (C3)

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