Local quasi hidden variable modelling and violations of Bell-type inequalities by a multipartite quantum state

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We introduce for a general correlation scenario a new simulation model, a local quasi hidden variable (LqHV) model, where locality and the measure-theoretic structure inherent to an LHV model are preserved but positivity of a simulation measure is dropped. We specify a necessary and sufficient condition for LqHV modelling and, based on this, prove that every quantum correlation scenario admits an LqHV simulation. Via the LqHV approach, we construct analogs of Bell-type inequalities for an $N$-partite quantum state and find a new analytical upper bound on the maximal violation by an $N$-partite quantum state of $S_1 \times \cdots \times S_N$-setting Bell-type inequalities - either on correlation functions or on joint probabilities and for outcomes of an arbitrary spectral type, discrete or continuous. This general analytical upper bound is expressed in terms of the new state dilation characteristics introduced in the present paper and not only traces quantum states admitting an $S_1 \times \cdots \times S_N$-setting LHV description but also leads to the new exact numerical upper estimates on the maximal Bell violations for concrete $N$-partite quantum states used in quantum information processing and for an arbitrary $N$-partite quantum state. We, in particular, prove that violation by an $N$-partite quantum state of an arbitrary Bell-type inequality (either on correlation functions or on joint probabilities) for $S$ settings per site cannot exceed $(2S-1)^{N-1}$ even in case of an infinite dimensional quantum state and infinitely many outcomes.
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I. INTRODUCTION

The seminal papers of Einstein, Podolsky and Rosen\(^1\) (EPR) and Bell\(^2,3\) are still ones of most cited in quantum information. In Ref.\(^4\), Einstein, Podolsky and Rosen argued that locality of measurements performed by spatially separated parties on perfectly correlated
quantum events implies the "simultaneous reality - and thus definite values" of physical quantities described by noncommuting quantum observables. Based on this argument contradicting, however, the quantum formalism and referred to as the EPR paradox, Einstein, Podolsky and Rosen expressed in Ref.\textsuperscript{1} their belief on a possibility of a hidden variable account of quantum measurements.

Analyzing this EPR belief in 1964 - 1966, Bell explicitly constructed\textsuperscript{2} the hidden variable (HV) model reproducing the statistical properties of all quantum observables for a qubit but, however, proved\textsuperscript{3} that, for bipartite measurements on a two-qubit system in the singlet state, a local hidden variable description (LHV) disagrees with the statistical predictions of quantum theory. Based on these results, Bell concluded\textsuperscript{2} that the EPR paradox should be resolved specifically via violation of locality under bipartite quantum measurements and that "...non-locality is deeply rooted in quantum mechanics itself and will persist in any completion".

Ever since 1964, the conceptual and mathematical aspects of the probabilistic description of multipartite quantum measurements have been analyzed in a plenty of papers, see, for example, articles\textsuperscript{4–10} and references therein. Nevertheless, as it has been recently noted by Gisin\textsuperscript{11}, in this field there are still "many questions, a few answers".

It was, for example, proved by Werner\textsuperscript{8} that there exist finite dimensional nonseparable bipartite quantum states admitting an LHV description under all projective bipartite quantum measurements with an arbitrary number of measurement settings at each site. It was also shown in Refs.\textsuperscript{10,12–15} that some nonseparable bipartite quantum states admit an LHV description only under correlation scenarios with specific numbers of measurements at N sites. However, until now it is not still known what state parameter quantitatively determines violation by an N-partite quantum state of Bell-type inequalities\textsuperscript{16} - constraints specifying scenarios admitting an LHV description and named after the seminal result\textsuperscript{3} of Bell.

Nowadays, it is also clear\textsuperscript{10} that though multipartite quantum measurements do not need to be local in the sense of Bell, they are, however, local in the sense meant by Einstein \textit{et al} in Ref.\textsuperscript{4}. The difference between the general nonsignaling condition, the EPR locality and Bell’s locality is analyzed in Ref.\textsuperscript{10}. Thus, the term "a nonlocal quantum state" widespread in quantum information means now only that this state does not admit an LHV description and, therefore, violates some Bell-type inequality.
This takes us back to the EPR locality argument and asks – if it is possible to construct for a quantum correlation scenario a simulation model which would be (i) local in the sense meant by Einstein, Podolsky and Rosen; (ii) similar by its measure-theoretic construction to the concept of an LHV model and (iii) incorporate the latter only as a particular case. This problem is also urgent for all multipartite correlation scenarios (not necessarily quantum) specified not in terms of a single probability space. The latter is one of the main notions of the conventional probability theory.

Apart from the purely theoretical interest, such a local simulation model could also single out a state parameter characterizing quantitatively violations of Bell-type inequalities by a multipartite quantum state - the problem discussed in the literature ever since the seminal result of Tsirelson.

Note that though, for correlation bipartite Bell-type inequalities, quantum violations are upper bounded by the Grothendieck’s constant independently on a dimension of a bipartite quantum state and numbers of settings and outcomes at each site, this is not already the case for bipartite Bell-type inequalities on joint probabilities. Since Bell-type inequalities are now widely used in many quantum information tasks, bounds on quantum violations of Bell-type inequalities have been recently intensively discussed in the literature both computationally and theoretically and it has been found that some tripartite quantum states ”can lead to arbitrarily large” violations of correlation Bell-type inequalities. For an \( N \)-partite quantum state, bounds on violation of a Bell-type inequality of an arbitrary type (either on correlation functions or on joint probabilities) have not been reported in the literature.

In the present paper, we introduce for the probabilistic description of a general correlation scenario a new simulation model, a local quasi hidden variable (LqHV) model, where locality and the measure-theoretic structure inherent to an LHV model are preserved but positivity of a simulation measure is dropped. We prove that every quantum correlation scenario admits the simulation in LqHV terms and construct via the LqHV approach analogs of Bell-type inequalities for an \( N \)-partite quantum state. This allows us to find the new analytical and numerical upper bounds on the maximal violation by an \( N \)-partite quantum state of all Bell-type inequalities – either on correlation functions or on joint probabilities and for outcomes of an arbitrary spectral type, discrete or continuous. The paper is organized as follows.
In section 2, for our consideration in sections 5, 6, we specify some new dilation characteristics of an $N$-partite quantum state and discuss their properties.

In section 3, we introduce for a general $N$-partite correlation scenario with $S_n$ measurements at each $n$-th site the notion of an LqHV model and specify a necessary and sufficient condition for LqHV modelling.

In section 4, we recall for an $S_1 \times \cdots \times S_N$-setting correlation scenario with outcomes of an arbitrary type, discrete or continuous, the general form of all Bell-type inequalities – either on correlation functions or on joint probabilities.

In section 5, we prove that every quantum $S_1 \times \cdots \times S_N$-setting correlation scenario admits an LqHV model and introduce, for an $N$-partite quantum state, the exact analytical upper bound on the state parameter specifying a possibility of its $S_1 \times \cdots \times S_N$-setting LHV description.

In section 6, via the LqHV approach, we construct analogs of Bell-type inequalities for an $N$-partite quantum state and find the new analytical and numerical upper bounds on the maximal violation by an $N$-partite quantum state of all $S_1 \times \cdots \times S_N$-setting Bell-type inequalities. The comparison of our exact general $N$-partite numerical upper estimate specified for $N = 2, 3$ with the bipartite and tripartite numerical estimates reported in the literature is given in section 6.2.

In section 7, we summarize the main results of the present paper.

In appendices A, B, C, we present proofs of some statements formulated in sections 2, 5 and 6, respectively.

II. PRELIMINARIES: SOURCE OPERATORS, TENSOR POSITIVITY, THE COVERING NORM

In this section, for our consideration in sections 5, 6, we specify the notion of a source operator for an $N$-partite state, the notion of tensor positivity and introduce a new norm, the covering norm, on the space of all self-adjoint trace class operators on a tensor product Hilbert space.

For a quantum state $\rho$ on a complex separable Hilbert space $\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_N$ and arbitrary positive integers $S_1, \ldots, S_N \geq 1$, denote by $T^{(\rho)}_{S_1 \times \cdots \times S_N}$ a self-adjoint trace class operator,
defined on \( \mathcal{H}_1^{\otimes S_1} \otimes \cdots \otimes \mathcal{H}_N^{\otimes S_N} \) and satisfying the relation
\[
\text{tr} \left[ T^{(\rho)}_{S_1 \times \cdots \times S_N} \left( \mathbb{I}_{\mathcal{H}_1^{\otimes S_1}} \otimes X_1 \otimes \mathbb{I}_{\mathcal{H}_1^{\otimes (S_1-1-k_1)}} \otimes \cdots \otimes \mathbb{I}_{\mathcal{H}_n^{\otimes k_n}} \otimes X_N \otimes \mathbb{I}_{\mathcal{H}_1^{\otimes (S_N-1-k_N)}} \right) \right] = \text{tr} \left[ \rho \left\{ X_1 \otimes \cdots \otimes X_N \right\} \right],
\]
\[
k_1 = 0, \ldots, (S_1 - 1), \ldots, k_N = 0, \ldots, (S_N - 1),
\]
for all bounded linear operators \( X_1, \ldots, X_N \) on Hilbert spaces \( \mathcal{H}_1, \ldots, \mathcal{H}_N \), respectively. In [1], we set \( \mathbb{I}_{\mathcal{H}_n^{\otimes k}} \otimes X_n \big|_{k=0} = X_n \otimes \mathbb{I}_{\mathcal{H}_n^{\otimes k}} \big|_{k=0} := X_n \). Clearly, \( \text{tr}[T^{(\rho)}_{S_1 \times \cdots \times S_N}] = 1 \) and \( T^{(\rho)}_{1 \times \cdots \times 1} \equiv \rho \).

**Definition 1 (Source operators)\(^{13,14}\)** For a state \( \rho \) on a Hilbert space \( \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_N \) and arbitrary positive integers \( S_1, \ldots, S_N \geq 1 \), we call each of self-adjoint trace class operators \( T^{(\rho)}_{S_1 \times \cdots \times S_N} \) on \( \mathcal{H}_1^{\otimes S_1} \otimes \cdots \otimes \mathcal{H}_N^{\otimes S_N} \) satisfying relation (1) as an \( S_1 \times \cdots \times S_N \)-setting source operator for state \( \rho \).

For a source operator \( T \), its the trace norm
\[
\|T\|_1 := \text{tr}[|T|] = \text{tr} \left[ T^++T^- \right] = 1 + 2\text{tr}[T^-] \geq 1.
\]
Here, \( T^+ \geq 0, T^+T^- = T^-T^+ = 0 \) are positive trace class operators in the spectral decomposition \( T = T^++T^- \) and \( |T| := \sqrt{T^2} = T^++T^- \).

**Proposition 1** For every state \( \rho \) on a Hilbert space \( \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_N \) and arbitrary positive integers \( S_1, \ldots, S_N \geq 1 \), there exists a source operator \( T^{(\rho)}_{S_1 \times \cdots \times S_N} \).

**Proof.** For a bipartite case, this statement has been proved for settings \( 1 \times 2, 2 \times 1 \) by proposition 1 in Ref.\(^{13}\). This proof was further generalized in appendix of Ref.\(^{10}\) for arbitrary \( 1 \times S_2, S_1 \times 1 \). The proof for a general \( N \)-partite case with setting \( S_1 \times \cdots \times S_N \) is presented in appendix A. □

If \( T^{(\rho)}_{S_1 \times \cdots \times S_N} \) is a source operator for state \( \rho \), then each of its reduced \( \left( T^{(\rho)}_{S_1 \times \cdots \times S_N} \right)_{\text{red}} \) on a Hilbert space \( \mathcal{H}_1^{\otimes L_1} \otimes \cdots \otimes \mathcal{H}_N^{\otimes L_N} \), with \( 1 \leq L_n < S_n \), constitutes an \( L_1 \times \cdots \times L_N \)-setting source operator for state \( \rho \) and
\[
1 \leq \left\| \left( T^{(\rho)}_{S_1 \times \cdots \times S_N} \right)_{\text{red}} \right\|_1 \leq \left\| T^{(\rho)}_{S_1 \times \cdots \times S_N} \right\|_1 .
\]

In order to analyze situations where, for a source operator \( T \), relation \( \text{tr}[T \{ X_1 \otimes \cdots \otimes X_m \}] \geq 0 \) holds for arbitrary positive operators \( X_1, \ldots, X_m \), we specify the following general notion.

6
Definition 2 (Tensor positivity) We call a bounded linear operator $Z$ on a Hilbert space $\mathcal{G}_1 \otimes \cdots \otimes \mathcal{G}_m$, $m \geq 1$, as tensor positive and denote it by $Z \geq 0$ if the scalar product
\[(\psi_1 \otimes \cdots \otimes \psi_m, Z \psi_1 \otimes \cdots \otimes \psi_m) \geq 0\]for arbitrary $\psi_1 \in \mathcal{G}_1, ..., \psi_m \in \mathcal{G}_m$.

Remark 1 For space $\mathcal{G}_1 \otimes \mathcal{G}_2$, the notion of tensor positivity is similar by its meaning to "block-positivity" in Ref.\textsuperscript{33}. We, however, consider that, for a tensor product of any number of arbitrary Hilbert spaces, possibly infinite dimensional, our term "tensor positivity" is more suitable.

For $m = 1$, tensor positivity is equivalent to positivity. For $m \geq 2$, positivity implies tensor positivity but not vice versa. For example, operator $V(\psi_1 \otimes \psi_2) := \psi_2 \otimes \psi_1$ on space $\mathcal{H} \otimes \mathcal{H}$ is tensor positive but not positive.

Since on a complex separable Hilbert space, every positive operator is self-adjoint, from \textsuperscript{11} and the spectral theorem it follows that, for a trace class tensor positive operator $W \geq 0$ on a complex separable Hilbert space $\mathcal{G}_1 \otimes \cdots \otimes \mathcal{G}_m$, relation $\text{tr} \{W\{X_1 \otimes \cdots \otimes X_m\}\} \geq 0$ holds for arbitrary positive operators $X_1, ..., X_m$ on spaces $\mathcal{G}_1, ..., \mathcal{G}_m$, respectively. In particular, $\text{tr}[W] \geq 0$, for each trace class $W \geq 0$.

If a trace class operator on $\mathcal{G}_1 \otimes \cdots \otimes \mathcal{G}_m$ is tensor positive, then any of its reduced operators is also tensor positive. The converse of this statement is not true.

Coming back to source operators, we stress that though, for every $N$-partite state, a source operator exists (see proposition 1) for every setting $S_1 \times \cdots \times S_N$, an arbitrary $N$-partite state does not need to have a tensor positive source operator\textsuperscript{34}.

For example, every separable $N$-partite state $\rho = \sum \alpha_i \rho_1^{(i)} \otimes \cdots \otimes \rho_N^{(i)}$, $\alpha_i > 0$, $\sum \alpha_i = 1$, has a positive source operator
\[\sum \alpha_i \left(\rho_1^{(i)}\right)^{\otimes S_1} \otimes \cdots \otimes \left(\rho_N^{(i)}\right)^{\otimes S_N}\]for arbitrary $S_1, ..., S_N \geq 1$. However, a nonseparable state does not need to have a tensor positive source operator even for at least one setting. In Refs.\textsuperscript{10,13–15,32}, we present examples of source operators for some nonseparable bipartite states and single out the state parameters for which these source operators become tensor positive.

Suppose now that we want to decompose a source operator into two tensor positive operators. The following new notion allows us to consider such decompositions.
Definition 3 (Coverings) For a self-adjoint bounded linear operator $Z$ on a Hilbert space $G_1 \otimes \cdots \otimes G_m$, $m \geq 1$, we call a tensor positive operator $Z_{\text{cov}}$ on $G_1 \otimes \cdots \otimes G_m$ satisfying relations

$$Z_{\text{cov}} \pm Z \geq 0$$

(6)

as a covering of $Z$.

If $Z$ is tensor positive, then it, itself, represents one of its coverings. In view of (6), every self-adjoint bounded linear operator $Z$ on $G_1 \otimes \cdots \otimes G_m$ admits the decomposition

$$Z = \frac{1}{2} (Z_{\text{cov}} + Z) - \frac{1}{2} (Z_{\text{cov}} - Z)$$

(7)

via tensor positive operators $Z_{\text{cov}} \pm Z \geq 0$, where $Z_{\text{cov}}$ is any of its coverings.

For a source operator, we are interested in its trace class coverings. Denote by $T_{G_1 \otimes \cdots \otimes G_m}$ the linear space of all trace class operators on a Hilbert space $G_1 \otimes \cdots \otimes G_m$ and by $T_{G_1 \otimes \cdots \otimes G_m}^{(\text{sa})} \subset T_{G_1 \otimes \cdots \otimes G_m}$ – the subspace of all self-adjoint trace class operators.

Proposition 2 For every self-adjoint trace class operator $W$ on a Hilbert space $G_1 \otimes \cdots \otimes G_m$, there exists a trace class covering $W_{\text{cov}}$.

Proof. For an operator $W \in T_{G_1 \otimes \cdots \otimes G_m}^{(\text{sa})}$, consider its spectral decomposition $W = W^+ - W^-$, where $W^\pm \geq 0$, $W^+ W^- = W^- W^+ = 0$. The positive trace class operator $|W| := \sqrt{W^2} = W^+ + W^-$, with $\text{tr}[|W|] := \|W\|_1 < \infty$, constitutes a trace class covering of $W$. This proves the statement. □

From (5) it follows that, for every self-adjoint trace class operator $W$, the relation

$$\text{tr}[W_{\text{cov}}] \geq |\text{tr}[W]| \geq 0$$

(8)

holds for each of its trace class coverings $W_{\text{cov}}$.

Thus, for every $W \in T_{G_1 \otimes \cdots \otimes G_m}^{(\text{sa})}$, the set $\{W_{\text{cov}} \in T_{G_1 \otimes \cdots \otimes G_m} \|W\| \geq 0 \}$ contains at least one element and $\text{tr}[W_{\text{cov}}] \geq 0$ for each covering $W_{\text{cov}}$. Therefore, we can introduce on space $T_{G_1 \otimes \cdots \otimes G_m}^{(\text{sa})}$ the following function

$$f(W) := \inf_{W_{\text{cov}} \in T_{G_1 \otimes \cdots \otimes G_m}} \text{tr}[W_{\text{cov}}] \geq 0, \quad \forall W \in T_{G_1 \otimes \cdots \otimes G_m}^{(\text{sa})},$$

(9)
and relations

\[ f(W) = 0 \iff W = 0, \]
\[ f(\alpha W) = |\alpha| f(W), \quad \forall \alpha \in \mathbb{R}, \]
\[ f(W_1 + W_2) \leq f(W_1) + f(W_2), \]

hold for all \( W, W_1, W_2 \in \mathcal{T}^{(sa)}_{G_1 \otimes \cdots \otimes G_m} \). The first of these relations follows from property 1 in lemma 1 below. The second relation – from (6), (9). For the proof of the last relation in (10), we note that, for arbitrary trace class coverings \( (W_1)_{\text{cov}}, (W_2)_{\text{cov}} \) of operators \( W_1, W_2 \in \mathcal{T}^{(sa)}_{G_1 \otimes \cdots \otimes G_m} \), operator \( (W_1)_{\text{cov}} + (W_2)_{\text{cov}} \) constitutes a possible trace class covering of \( W_1 + W_2 \). Hence, \( \{(W_1)_{\text{cov}} + (W_2)_{\text{cov}}\} \supseteq \{(W_1)_{\text{cov}}, (W_2)_{\text{cov}}\} \) and, taking this inclusion into the account in infimum (9) specifying \( f(W_1 + W_2) \), we come to the third relation in (10).

In view of relations (10), function (9) constitutes a norm on space \( \mathcal{T}^{(sa)}_{G_1 \otimes \cdots \otimes G_m} \).

**Definition 4** We refer to norm (9) as the covering norm and denote it by

\[ \|W\|_{\text{cov}} := \inf_{W_{\text{cov}} \in \mathcal{T}^{(sa)}_{G_1 \otimes \cdots \otimes G_m}} \text{tr}[W_{\text{cov}}], \quad \forall W \in \mathcal{T}^{(sa)}_{G_1 \otimes \cdots \otimes G_m}. \]  

The following general properties of the covering norm are proved in Appendix A.

**Lemma 1** For a self-adjoint trace class operator \( W \) on a Hilbert space \( G_1 \otimes \cdots \otimes G_m \):

1. \( |\text{tr}[W]| \leq \sup |\text{tr}[W\{X_1 \otimes \cdots \otimes X_m\}]| \leq \|W\|_{\text{cov}} \leq \|W\|_1 \),
   where supremum is taken over all self-adjoint bounded linear operators \( X_1, \ldots, X_m \) with operator norms \( \|X_j\| = 1 \) on spaces \( G_1, \ldots, G_m \), respectively;

2. if \( W \uplus 0 \geq 0 \), then \( \|W\|_{\text{cov}} = \text{tr}[W] \);

3. \( |\text{tr}[W]| \leq \|W_{\text{red}}\|_{\text{cov}} \leq \|W\|_{\text{cov}} \) for each operator \( W_{\text{red}} \) reduced from \( W \).

As an example, consider the self-adjoint operator \( V(\psi_1 \otimes \psi_2) = \psi_2 \otimes \psi_1 \) on \( \mathbb{C}^d \otimes \mathbb{C}^d \). For this operator, the covering norm \( \|V\|_{\text{cov}} = d \) while the trace norm \( \|V\|_1 = d^2 \).

For an \( S_1 \times \cdots \times S_N \)-setting source operator \( T^{(\rho)}_{S_1 \times \cdots \times S_N} \) for a state \( \rho \) on a Hilbert space \( \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_N \), lemma 1 implies

\[ 1 \leq \left\| T^{(\rho)}_{S_1 \times \cdots \times S_N} \right\|_{\text{cov}} \leq \left\| T^{(\rho)}_{S_1 \times \cdots \times S_N} \right\|_1, \quad \text{(12)} \]
\[ T^{(\rho)}_{S_1 \times \cdots \times S_N} \uplus 0 \geq 0 \iff \left\| T^{(\rho)}_{S_1 \times \cdots \times S_N} \right\|_{\text{cov}} = 1. \quad \text{(13)} \]
and
\[ 1 \leq \left\| \left( T^{(\rho)}_{S_1 \times \cdots \times S_N} \right)_{\text{red}} \right\|_{\text{cov}} \leq \left\| T^{(\rho)}_{S_1 \times \cdots \times S_N} \right\|_{\text{cov}} \quad (14) \]
for each source operator \( \left( T^{(\rho)}_{S_1 \times \cdots \times S_N} \right)_{\text{red}} \) reduced from a source operator \( T^{(\rho)}_{S_1 \times \cdots \times S_N} \).

### III. LQHV MODELLING OF A GENERAL CORRELATION SCENARIO

Consider an \( N \)-partite correlation scenario, where each \( n \)-th of \( N \geq 2 \) parties (players) performs \( S_n \geq 1 \) measurements with outcomes \( \lambda_n \in \Lambda_n \) of an arbitrary type and \( \mathcal{F}_{\Lambda_n} \) is a \( \sigma \)-algebra of events \( F_n \subseteq \Lambda_n \) observed at \( n \)-th site. For the general framework on the probabilistic description of multipartite correlation scenarios, see Ref.\(^{10}\).

We label each measurement at \( n \)-th site by a positive integer \( s_n = 1, \ldots, S_n \) and each of \( N \)-partite joint measurements, induced by this correlation scenario and with outcomes \( (\lambda_1, \ldots, \lambda_N) \in \Lambda_1 \times \cdots \times \Lambda_N \) by an \( N \)-tuple \( (s_1, \ldots, s_N) \), where \( n \)-th component refers to a measurement at \( n \)-th site.

For concreteness, we further refer to an \( S_1 \times \cdots \times S_N \)-setting correlation scenario with outcomes in \( \Lambda_1 \times \cdots \times \Lambda_N \) by symbol
\[
\mathcal{E}_{S,\Lambda}, \quad S := S_1 \times \cdots \times S_N, \quad \Lambda := \Lambda_1 \times \cdots \times \Lambda_N, \quad (15)
\]
and denote by \( P^{(\mathcal{E}_{S,\Lambda})}_{(s_1, \ldots, s_N)} \) a probability measure, describing an \( N \)-partite joint measurement \( (s_1, \ldots, s_N) \) of scenario \( \mathcal{E}_{S,\Lambda} \) and defined on the direct product \( (\Lambda_1 \times \cdots \times \Lambda_N, \mathcal{F}_{\Lambda_1} \otimes \cdots \otimes \mathcal{F}_{\Lambda_N}) \) of measurable spaces \( (\Lambda_n, \mathcal{F}_{\Lambda_n}), n = 1, \ldots, N \). Recall\(^{25}\) that the product \( \sigma \)-algebra \( \mathcal{F}_{\Lambda_1} \otimes \cdots \otimes \mathcal{F}_{\Lambda_N} \) is the smallest \( \sigma \)-algebra generated by the set of all rectangles \( F_1 \times \cdots \times F_N \subseteq \Lambda_1 \times \cdots \times \Lambda_N \) with measurable "sides" \( F_n \in \mathcal{F}_{\Lambda_n}, n = 1, \ldots, N \).

In what follows, we consider only standard measurable spaces. In this case, each \( (\Lambda_n, \mathcal{F}_{\Lambda_n}) \) is Borel isomorphic to a measurable space \( (\mathcal{X}_n, \mathcal{B}_{\mathcal{X}_n}) \), where \( \mathcal{X}_n \in \mathcal{B}_{\mathbb{R}} \) is a Borel subset of \( \mathbb{R} \) and \( \mathcal{B}_{\mathcal{X}_n} := \mathcal{B}_{\mathbb{R}} \cap \mathcal{X}_n \) is the trace on \( \mathcal{X}_n \) of the Borel \( \sigma \)-algebra \( \mathcal{B}_{\mathbb{R}} \) on \( \mathbb{R} \).

For a general correlation scenario \( \mathcal{E}_{S,\Lambda} \), let us introduce the following new type of simulation models.

**Definition 5** We say that an \( S_1 \times \cdots \times S_N \)-setting correlation scenario \( \mathcal{E}_{S,\Lambda} \), with joint probability distributions \( P_{(s_1, \ldots, s_N)}^{(\mathcal{E}_{S,\Lambda})}, s_1 = 1, \ldots, S_1, \ldots, s_N = 1, \ldots, S_N, \) and outcomes \( (\lambda_1, \ldots, \lambda_N) \in \)
Λ_1 \times \cdots \times Λ_N := Λ, admits a local quasi hidden variable (LqHV) model if all of its joint probability distributions admit the representation

\[ P_{(s_1, \ldots, s_N)}^{(E,S)}(F_1 \times \cdots \times F_N) = \int_{Ω} P_1^{(s_1)}(F_1|ω) \cdot \cdots \cdot P_N^{(s_N)}(F_N|ω) \, ν_{E,S,Λ}(dω), \]  

(16)

in terms of a single measure space \((Ω, F_Ω, ν_{E,S,Λ})\), with a normalized bounded real-valued measure \(ν_{E,S,Λ}\), and conditional probability measures \(P_n^{(s_n)}(F_n|·) : Ω → [0, 1]\), defined \(ν_{E,S,Λ}\)-a.e. (almost everywhere) on \(Ω\) and such that, for each \(s_n = 1, \ldots, S_n\) and every \(n = 1, \ldots, N\), function \(P_n^{(s_n)}(F_n|·) : Ω → [0, 1]\) is measurable for all \(F_n ∈ F_{Λ_n}\).

**Notation 1** In a triple \((Ω, F_Ω, ν)\) representing a measure space, \(Ω\) is a non-empty set, \(F_Ω\) is a \(σ\)-algebra of subsets of \(Ω\) and \(ν\) is a measure on a measurable space \((Ω, F_Ω)\). A real-valued measure \(ν\) is called normalized if \(ν(Ω) = 1\) and bounded if \(|ν(F)| ≤ M < ∞\) for all \(F ∈ F_Ω\).

We stress that, in an LqHV model \((16)\), measure \(ν_{E,S,Λ}\) has a simulation character and may, in general, depend (via the lower index \(E,S,Λ\)) on measurement settings at all (or some) sites, as an example, see measure \((40)\).

From \((16)\) it follows that a correlation scenario \(E,S,Λ\) admitting an LqHV model satisfies the general nonsignaling condition specified by definition 1 (Eq. (10)) in Ref.\(^{10}\).

If, for a correlation scenario \(E,S,Λ\), there exists representation \((16)\), where a normalized real-valued measure \(ν_{E,S,Λ}\) is positive and, hence, is a probability measure, then this scenario admits an LHV model formulated for a general case by definition 4 (Eq. (26)) in Ref.\(^{10}\).

**Remark 2** Recall\(^{35}\) that a bounded real-valued measure \(ν\) on a measurable space \((Ω, F_Ω)\) admits the Jordan decomposition \(ν = ν^+ − ν^-\) via positive measures

\[ ν^+(F) := \sup_{F' ∈ F_Ω, F' ⊆ F} ν(F'), \quad ν^−(F) := -\inf_{F' ∈ F_Ω, F' ⊆ F} ν(F'), \quad ∀F ∈ F_Ω, \]  

(17)

with disjoint supports. The sum \((ν^+(Ω) + ν^−(Ω))\) coincides with the total variation \(|ν| (Ω)\) of measure \(ν\) on \(Ω\), which is defined by relation

\[ \sup \sum_{i=1}^{m} |ν(F_i)| := |ν| (Ω) ≡ \|ν\|_{\text{var}}, \]  

(18)

where supremum is taken over all finite systems \(\{F_i\}\) of disjoint sets in \(F_Ω\). For a bounded measure \(ν\), its total variation \(\|ν\|_{\text{var}} < ∞\) and \(\|·\|_{\text{var}}\) constitutes a norm, the total variation
norm, on the linear space of all bounded real-valued measures on a measurable space \((\Omega, \mathcal{F}_\Omega)\). Thus, for a bounded real-valued measure \(\nu\), we have

\[
\nu^+(\Omega) + \nu^-(\Omega) = \|\nu\|_{\text{var}}.
\]  

(19)

If a bounded real-valued measure \(\nu\) is normalized, then

\[
\|\nu\|_{\text{var}} = 1 + 2\nu^-(\Omega) \geq 1.
\]

(20)

A normalized bounded real-valued measure \(\nu\) is a probability measure iff \(\|\nu\|_{\text{var}} = 1\). Note that relation

\[
\sup_{F \in \mathcal{F}_\Omega} |\nu(F)| \leq \|\nu\|_{\text{var}} \leq 2 \sup_{F \in \mathcal{F}_\Omega} |\nu(F)|
\]

(21)

holds for every real-valued measure \(\nu\).

From the Jordan decomposition for measure \(\nu_{\mathcal{E}_{S,\Lambda}}\) it follows that if a correlation scenario admits an \(L_q\)HV model (16), then each of its joint probability distributions \(P^{(s_n,\Lambda)}_{(s_1,\ldots,s_N)}\) can be expressed via the affine combination of some LHV distributions \(P^{(s_n,\Lambda)}_{(s_1,\ldots,s_N)}\) that are represented in (16) by the same conditional measures \(P^{(s_n)}_{n} (\cdot | \omega)\). On the other hand, if a correlation scenario with a finite number of outcomes at each site admits the affine model in the sense of Ref. 26, then this scenario admits the special \(L_q\)HV model, where measure \(\nu_{\mathcal{E}_{S,\Lambda}}\) is given by the affine combination of discrete probability measures and each \(P^{(s_n)}_{n} (\cdot | \omega)\) has the particular form \(\chi_{f^{-1}_{n,s_n}(F_n)}(\omega), F_n \in \mathcal{F}_\Lambda_n\), where \(f_{n,s_n}: \Omega \rightarrow \Lambda_n\) is some measurable function, \(f^{-1}_{n,s_n}(F_n) := \{\omega \in \Omega | f_{n,s_n}(\omega) \in F_n\}\) and \(\chi_D(\cdot)\) is the indicator function of a subset \(D \subseteq \Omega\), that is: \(\chi_D(\omega) = 1\) if \(\omega \in D\) and \(\chi_D(\omega) = 0\) if \(\omega \notin D\).

Thus, an \(L_q\)HV model incorporates as particular cases and generalizes in one whole both types of simulation models discussed in the literature – an LHV model and an affine model. Note that the latter model is, in principle, built up on the concept of an LHV model.

We stress that, in an \(L_q\)HV model, locality and the measure-theoretic structure inherent to an LHV model are preserved.

The following general theorem introduces a necessary and sufficient condition for \(L_q\)HV modelling.

**Theorem 1** An \(S_1 \times \ldots \times S_N\)-setting correlation scenario \(\mathcal{E}_{S,\Lambda}\) admits an \(L_q\)HV model (16) if and only if, on the direct product space \((\Lambda_1^{S_1} \times \cdots \times \Lambda_N^{S_N}, \mathcal{F}_{\Lambda_1}^{\otimes S_1} \otimes \cdots \otimes \mathcal{F}_{\Lambda_N}^{\otimes S_N})\), there exists
a normalized bounded real-valued measure

$$
\mu_{E,S,A} \left( d\lambda_1^{(1)} \times \cdots \times d\lambda_1^{(S_1)} \times \cdots \times d\lambda_N^{(1)} \times \cdots \times d\lambda_N^{(S_N)} \right),
$$

(22)

$$
\lambda_n^{(s_n)} \in \Lambda_n, \quad s_n = 1, ..., S_n, \quad n = 1, ..., N,
$$

returning all joint probability distributions $P^{(E,S,A)}_{(s_1, ..., s_N)}$ of scenario $E_{S,A}$ as the corresponding marginals.

**Proof.** Let scenario $E_{S,A}$ admit an LqHV model (16). Then the normalized real-valued measure

$$
\int_{\Omega} \left\{ \prod_{s_n=1, \ldots, S_n, n=1, \ldots, N} P_n^{(s_n)}(d\lambda_n^{(s_n)} | \omega) \right\} \nu_{E,S,A}(d\omega)
$$

(23)
on $(\Lambda_1^{S_1} \times \cdots \times \Lambda_N^{S_N}, \mathcal{F}_{\Lambda_1} \otimes \cdots \otimes \mathcal{F}_{\Lambda_N}^{\otimes S_N})$ returns all distributions $P^{(E,S,A)}_{(s_1, ..., s_N)}$ of scenario $E_{S,A}$ as the corresponding marginals. The total variation of measure (23) is upper bounded by $\|\nu_{E,S,A}\|_{\text{var}} < \infty$, so that, in view of relation (21), this measure is bounded.

In order to prove the sufficiency part of theorem 1, let there exist a normalized bounded real-valued measure $\tilde{\mu}_{E,S,A}$ returning all probability distributions $P^{(E,S,A)}_{(s_1, ..., s_N)}$ of scenario $E_{S,A}$ as the corresponding marginals. This means that the representation

$$
P^{(E,S,A)}_{(s_1, ..., s_N)} (F_1 \times \cdots \times F_N) = \int \chi_{F_1} (\lambda_1^{(s_1)}) \cdots \chi_{F_N} (\lambda_N^{(s_N)}) \tilde{\mu}_{E,S,A} (d\lambda_1^{(1)} \times \cdots \times d\lambda_1^{(S_1)} \times \cdots \times d\lambda_N^{(1)} \times \cdots \times d\lambda_N^{(S_N)}),
$$

(24)

$$
s_1 = 1, ..., S_1, ..., s_N = 1, ..., S_N,
$$

holds for all $F_1 \in \mathcal{F}_{\Lambda_1}, ..., F_N \in \mathcal{F}_{\Lambda_N}$. Representation (24) constitutes a particular case of the LqHV representation (16) specified with

$$
\omega' = \left( \lambda_1^{(1)}, ..., \lambda_1^{(S_1)}, ..., \lambda_N^{(1)}, ..., \lambda_N^{(S_N)} \right),
$$

(25)

$$
\Omega' = \Lambda_1^{S_1} \times \cdots \times \Lambda_N^{S_N}, \quad \mathcal{F}_{\Omega'} = \mathcal{F}_{\Lambda_1} \otimes \cdots \otimes \mathcal{F}_{\Lambda_N}^{\otimes S_N},
$$

$$
\nu'_{E,S,A} = \tilde{\mu}_{E,S,A}, \quad P_n^{(s_n)}(F_n | \omega') = \chi_{F_n} (\lambda_n^{(s_n)}).
$$

This proves the statement. ■

The following corollary of theorem 1 corresponds to the statements (a), (c) of the general theorem 1 on LHV modelling in Ref.10.
Corollary 1  An $S_1 \times \cdots \times S_N$-setting correlation scenario $\mathcal{E}_{S,A}$ admits an LHV model if and only if there exists a probability measure $\mu'_{\mathcal{E}_{S,A}}$ on space $(\Lambda_1^{S_1} \times \cdots \times \Lambda_N^{S_N}, F_{\Lambda_1}^{\otimes S_1} \otimes \cdots \otimes F_{\Lambda_N}^{\otimes S_N})$ returning all joint probability distributions $P_{(s_1,\ldots,s_N)}^{(\mathcal{E}_{S,A})}$ of scenario $\mathcal{E}_{S,A}$ as the corresponding marginals.

Proof. If scenario $\mathcal{E}_{S,A}$ admits an LHV model, then there exists representation (16) with some probability measure $\nu'$ and, for this $\nu'$, the constructed normalized measure (23) is a probability one. Conversely, let there exist a probability measure $\mu'_{\mathcal{E}_{S,A}}$ returning all joint probability distributions $P_{(s_1,\ldots,s_N)}^{(\mathcal{E}_{S,A})}$ of scenario $\mathcal{E}_{S,A}$ as the corresponding marginals. Then representation (24) with probability measure $\mu'_{\mathcal{E}_{S,A}}$ constitutes a particular LHV model. ■

IV. BELL-TYPE INEQUALITIES

For an $S_1 \times \cdots \times S_N$-setting correlation scenario $\mathcal{E}_{S,A}$, with joint probability distributions $P_{(s_1,\ldots,s_N)}^{(\mathcal{E}_{S,A})}$ and outcomes $(\lambda_1,\ldots,\lambda_N) \in \Lambda_1 \times \cdots \times \Lambda_N := \Lambda$, consider a linear combination

$$
\sum_{s_1,\ldots,s_N} \left\langle \psi_{(s_1,\ldots,s_N)}(\lambda_1,\ldots,\lambda_N) \right\rangle_{\mathcal{E}_{S,A}}
$$

of averages

$$
\left\langle \psi_{(s_1,\ldots,s_N)}(\lambda_1,\ldots,\lambda_N) \right\rangle_{\mathcal{E}_{S,A}} := \int_{\Lambda} \psi_{(s_1,\ldots,s_N)}(\lambda_1,\ldots,\lambda_N) P_{(s_1,\ldots,s_N)}^{(\mathcal{E}_{S,A})}(d\lambda_1 \times \cdots \times d\lambda_N)
$$

arising under joint measurements $(s_1,\ldots,s_N)$ and specified by a family

$$
\Psi_{S,A} := \left\{ \psi_{(s_1,\ldots,s_N)}, \ s_1 = 1,\ldots, S_1,\ldots,s_N = 1,\ldots, S_N \right\}
$$

of bounded measurable real-valued functions $\psi_{(s_1,\ldots,s_N)}(\lambda_1,\ldots,\lambda_N)$ on the direct product $(\Lambda_1 \times \cdots \times \Lambda_N, F_{\Lambda_1} \otimes \cdots \otimes F_{\Lambda_N})$ of measurable spaces $(\Lambda_n, F_{\Lambda_n})$, $n = 1,\ldots, N$.

If, in (27), function $\psi_{(s_1,\ldots,s_N)}$ has the product form $\phi_{s_1}(\lambda_1) \cdot \cdots \phi_{s_N}(\lambda_N)$, then, depending on a concrete choice of functions $\phi_{s_n}(\lambda_n)$, for a joint measurement $(s_1,\ldots,s_N)$, the average

$$
\int_{\Lambda} \phi_{s_1}(\lambda_1) \cdot \cdots \phi_{s_N}(\lambda_N) P_{(s_1,\ldots,s_N)}^{(\mathcal{E}_{S,A})}(d\lambda_1 \times \cdots \times d\lambda_N)
$$
may refer to either the joint probability
\[
\left\langle \chi_{F_1}(\lambda_1^{(s_1)}) \cdot \ldots \cdot \chi_{F_N}(\lambda_N^{(s_N)}) \right\rangle_{\mathcal{E}_S, \Lambda} = P_{(s_1, \ldots, s_N)}^{(\mathcal{E}_S, \Lambda)}(F_1 \times \ldots \times F_N)
\]  
(30)
of events \(F_1 \in \mathcal{F}_{\Lambda_1}, \ldots, F_N \in \mathcal{F}_{\Lambda_N}\) observed at the corresponding sites or if outcomes are real-valued and bounded – to the expectation value
\[
\left\langle \lambda_{n_1}^{(s_{n_1})} \cdot \ldots \cdot \lambda_{n_M}^{(s_{n_M})} \right\rangle_{\mathcal{E}_S, \Lambda} = \int_{\Lambda} \lambda_{n_1} \cdot \ldots \cdot \lambda_{n_M} \, P_{(s_1, \ldots, s_N)}^{(\mathcal{E}_S, \Lambda)}(d\lambda_1 \times \ldots \times d\lambda_N)
\]  
(31)
of the product \(\lambda_{n_1} \cdot \ldots \cdot \lambda_{n_M}\) of outcomes observed at arbitrary \(M \leq N\) sites \(1 \leq n_1 < \ldots < n_M \leq N\). For \(M \geq 2\), the expectation value (31) is referred to (in quantum information) as a correlation function. A correlation function for an \(N\)-partite joint measurement is called full if, in (31), \(M = N\).

If a correlation scenario \(\mathcal{E}_S, \Lambda\) admits an LHV model, then every linear combination (26) of averages satisfies the tight LHV constraints
\[
B_{\mathcal{E}_S, \Lambda}^{\inf} \leq \sum_{s_1, \ldots, s_N} \left\langle \psi(s_1, \ldots, s_N)(\lambda_1, \ldots, \lambda_N) \right\rangle_{\mathcal{E}_S, \Lambda} \mid_{LHV} \leq B_{\mathcal{E}_S, \Lambda}^{\sup},
\]  
(32)
where the LHV constants \(B_{\mathcal{E}_S, \Lambda}^{\sup}\) and \(B_{\mathcal{E}_S, \Lambda}^{\inf}\) constitute, correspondingly, supremum and infimum of (26) over all LHV scenarios \(\mathcal{E}_{lhv}^{S, \Lambda}\) and have the form:
\[
B_{\mathcal{E}_S, \Lambda}^{\sup} := \sup_{\mathcal{E}_{lhv}^{S, \Lambda}} \sum_{s_1, \ldots, s_N} \left\langle \psi(s_1, \ldots, s_N)(\lambda_1, \ldots, \lambda_N) \right\rangle_{\mathcal{E}_{lhv}^{S, \Lambda}}
\]  
(33)
\[
= \sup_{\lambda_{n_1}^{(s_{n_1})} \in \Lambda_{n_1}, \forall s_n, \forall n} \sum_{s_1, \ldots, s_N} \psi(s_1, \ldots, s_N)(\lambda_1^{(s_1)}, \ldots, \lambda_N^{(s_N)}),
\]

\[
B_{\mathcal{E}_S, \Lambda}^{\inf} := \inf_{\mathcal{E}_{lhv}^{S, \Lambda}} \sum_{s_1, \ldots, s_N} \left\langle \psi(s_1, \ldots, s_N)(\lambda_1, \ldots, \lambda_N) \right\rangle_{\mathcal{E}_{lhv}^{S, \Lambda}}
\]  
(34)
\[
= \inf_{\lambda_{n_1}^{(s_{n_1})} \in \Lambda_{n_1}, \forall s_n, \forall n} \sum_{s_1, \ldots, s_N} \psi(s_1, \ldots, s_N)(\lambda_1^{(s_1)}, \ldots, \lambda_N^{(s_N)}).
\]

Constraints (32) imply
\[
\left| \sum_{s_1, \ldots, s_N} \left\langle \psi(s_1, \ldots, s_N)(\lambda_1, \ldots, \lambda_N) \right\rangle_{\mathcal{E}_S, \Lambda} \right|_{LHV} \leq B_{\mathcal{E}_S, \Lambda},
\]  
(34)
where
\[ B_{\Psi,S,\Lambda} := \max \left\{ \left| B_{\Psi,S,\Lambda}^{\text{sup}} \right|, \left| B_{\Psi,S,\Lambda}^{\text{inf}} \right| \right\} \]

(35)

\[ = \sup_{\lambda^{(s_n)} \in \Lambda_{n,\forall s_n,\forall n}} \sum_{s_1,\ldots,s_N} \psi_{(s_1,\ldots,s_N)}(\lambda^{(s_1)},\ldots,\lambda^{(s_N)}) \].

Inequalities (32) have been introduced in Ref. 17 and represent the general form of all unconditional紧 tight linear LHV constraints on correlation functions and joint probabilities for an \( S_1 \times \cdots \times S_N \)-setting correlation scenario with outcomes of an arbitrary type, discrete or continuous.

Note that some of the LHV constraints (32) may be fulfilled for a wider (than LHV) class of correlation scenarios. This is, for example, the case for those LHV constraints on joint probabilities that follow explicitly from positivity and nonsignaling of probability distributions \( P^{(E_{S,\Lambda})}_{(s_1,\ldots,s_N)} \) and are, therefore, fulfilled for any nonsignaling scenario \( E_{S,\Lambda} \). Moreover, for some \( \Psi_{S,\Lambda} \), the corresponding constraints (32) may be simply trivial – in the sense that these constraints are fulfilled for each scenario \( E_{S,\Lambda} \). For example, if we specify (32) with functions \( \tilde{\psi}_{(s_1,\ldots,s_N)}(\lambda_1,\ldots,\lambda_N) = 1, \forall (\lambda_1,\ldots,\lambda_N) \in \Lambda \), for all joint measurements \( (s_1,\ldots,s_N) \), then

\[ B_{\Psi,S,\Lambda}^{\text{inf}} = \sum_{s_1,\ldots,s_N} \left\langle \tilde{\psi}_{(s_1,\ldots,s_N)}(\lambda_1,\ldots,\lambda_N) \right\rangle_{E_{S,\Lambda}} = B_{\Psi,S,\Lambda}^{\text{sup}} = S_1 \cdot \ldots \cdot S_N \]  

(36)

holds for every scenario \( E_{S,\Lambda} \).

If, however, an LHV constraint may be violated in a non-LHV case, then it is generally named after Bell due to his seminal result in Ref. 3.

\textbf{Definition 6} Each of the tight linear LHV constraints (32) that may be violated under a non-LHV correlation scenario is referred to as a Bell-type (equivalently, Bell) inequality.

As it is discussed in section 3 of Ref. 17, the general form (32) covers in a unified manner all unconditional Bell-type inequalities that were introduced via a variety of methods ever since the seminal publication of Bell\(^3\). Note that the original Bell inequality\(^3\), discussed recently in Ref. 32, constitutes an example of conditional Bell-type inequalities.

\section*{V. LQHV MODELLING OF A QUANTUM CORRELATION SCENARIO}

Let, under an \( S_1 \times \cdots \times S_N \)-setting correlation scenario, each \( N \)-partite joint measurement \( (s_1,\ldots,s_N) \) be performed on a quantum state \( \rho \) on a Hilbert space \( \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_N \) and described
positive operator-valued (POV) measure on the measurable space \((\Lambda_1 \times \cdots \times \Lambda_N, \mathcal{F}_{\Lambda_1} \otimes \cdots \otimes \mathcal{F}_{\Lambda_N})\). Here, each \(M_n^{(s_n)}\) is a normalized positive operator-valued (POV) measure on a measurable space \((\Lambda_n, \mathcal{F}_{\Lambda_n})\) representing on a Hilbert space \(\mathcal{H}_n\) a quantum measurement \(s_n\) at \(n\)-th site. For a POV measure \(M_n^{(s_n)}\), all its values \(M_n^{(s_n)}(F_n), F_n \in \mathcal{F}_{\Lambda_n}\), are positive operators on \(\mathcal{H}_n\) and \(M_n^{(s_n)}(\lambda_n) = I_{\mathcal{H}_n}\).

We specify this quantum \(S_1 \times \cdots \times S_N\)-setting correlation scenario by symbol \(\mathcal{E}_{\rho, M_{S_A}}\), where

\[
M_{S_A} := \{ M_n^{(s_n)}, s_n = 1, \ldots, S_n, n = 1, \ldots, N \}
\]

is a collection of POV measures describing this quantum scenario and denote by

\[
P^{(\mathcal{E}_{\rho, M_{S_A}})}_{s_1, \ldots, s_N}(d\lambda_1 \times \cdots \times d\lambda_N) := \text{tr} \left[ \rho \left\{ M_1^{(s_1)}(d\lambda_1) \otimes \cdots \otimes M_N^{(s_N)}(d\lambda_N) \right\} \right],
\]

\(s_1 = 1, \ldots, S_1, \ldots, s_N = 1, \ldots, S_N\), its joint probability distributions (37).

**Theorem 2** For every state \(\rho\) on a Hilbert space \(\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_N\) and arbitrary positive integers \(S_1, \ldots, S_N \geq 1\), each quantum \(S_1 \times \cdots \times S_N\)-setting correlation scenario \(\mathcal{E}_{\rho, M_{S_A}}\), with joint probability distributions (37) and outcomes of an arbitrary spectral type, discrete or continuous, admits an \(L_qHV\) model.

**Proof.** For a state \(\rho\) on \(\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_N\), let \(T^{(\rho)}_{S_1 \times \cdots \times S_N}\) be an \(S_1 \times \cdots \times S_N\)-setting source operator on space \(\mathcal{H}_1^{\otimes S_1} \otimes \cdots \otimes \mathcal{H}_N^{\otimes S_N}\), see definition 1 in section 2. For each scenario \(\mathcal{E}_{\rho, M_{S_A}}\), the normalized real-valued measure

\[
\mu^{(\rho, M_{S_A})}_{T^{(\rho)}_{S_1 \times \cdots \times S_N}} \left( d\lambda_1^{(S_1)} \times \cdots \times d\lambda_N^{(S_N)} \right) := \text{tr} \left[ T^{(\rho)}_{S_1 \times \cdots \times S_N} \left\{ M_1^{(1)}(d\lambda_1^{(1)}) \otimes \cdots \otimes M_1^{(S_1)}(d\lambda_1^{(S_1)}) \right. \right.
\]

\[
\left. \otimes \cdots \otimes M_N^{(1)}(d\lambda_N^{(1)}) \otimes \cdots \otimes M_N^{(S_N)}(d\lambda_N^{(S_N)}) \right] \]

on the direct product space \((\Lambda_1^{S_1} \times \cdots \times \Lambda_N^{S_N}, \mathcal{F}_{\Lambda_1}^{S_1} \otimes \cdots \otimes \mathcal{F}_{\Lambda_N}^{S_N})\) returns all joint probability distributions \(P^{(\mathcal{E}_{\rho, M_{S_A}})}_{s_1, \ldots, s_N}\) of scenario \(\mathcal{E}_{\rho, M_{S_A}}\) as the corresponding marginals. Due to bound (B10) proved in appendix B and relation (12), the total variation norm of measure (40) is
upper bounded by $\|T^{(p)}_{S_1 \times \cdots \times S_N}\|_1 < \infty$. This and relation (21) imply that the normalized real-valued measure $\mu^{(p,M_{S,A})}_{\bar{S}_1, \ldots, \bar{S}_N}$ is bounded. Thus, for each quantum scenario $E_{p,M_{S,A}}$, the constructed measure $\mu^{(p,M_{S,A})}_{\bar{S}_1, \ldots, \bar{S}_N}$ satisfies the sufficiency condition of theorem 1 on LqHV modelling. This proves the statement.

If, for a state $\rho$, every quantum scenario $E_{\rho,M_{S},\Lambda}$ (i.e. for an arbitrary collection $M_{S,A}$ of POVs measures and an arbitrary outcome set $\Lambda$) admits an LHV model, then, according to our terminology in Ref.10, this state $\rho$ admits the $S_1 \times \cdots \times S_N$-setting LHV description. In the latter case, state $\rho$ admits an $L_1 \times \cdots \times L_N$-setting LHV description for all $L_1 \leq S_1, \ldots, L_N \leq S_N$, but does not need to admit the LHV description whenever at least one $L_n > S_n$.

Via a similar terminology for the LqHV case, theorem 2 reads – every $N$-partite quantum state $\rho$ admits an $S_1 \times \cdots \times S_N$-setting LqHV description for arbitrary numbers $S_1, \ldots, S_N$ of measurements at $N$ sites.

In view of theorems 1, 2, corollary 1 and relation (20) let us introduce, for a quantum correlation scenario $E_{\rho,M_{S,A}}$, the parameter

$$\gamma_{E_{\rho,M_{S,A}}} := \inf_{\mu_{E_{\rho,M_{S,A}}}} \|\mu_{E_{\rho,M_{S,A}}}\|_{\text{var}} \geq 1; \quad (41)$$

where, infimum is taken over all normalized bounded real-valued measures $\mu_{E_{\rho,M_{S,A}}}$, each returning all distributions $P_{(s_1,\ldots,s_N)}^{(E_{\rho,M_{S,A}})}$ of scenario $E_{\rho,M_{S,A}}$ as the corresponding marginals.

The following lemma is proved in appendix B.

Lemma 2 A quantum correlation scenario $E_{\rho,M_{S,A}}$ admits an LHV model if and only if $\gamma_{E_{\rho,M_{S,A}}} = 1$.

Introduce also the state parameters

$$\Upsilon_{\rho}^{(\rho,A)} := \sup_{M_{S,A}} \gamma_{E_{\rho,M_{S,A}}} \geq 1, \quad (42)$$

$$\Upsilon_\rho := \sup_{\Lambda} \Upsilon_{\rho}^{(\rho,A)} \geq 1. \quad (43)$$

Proposition 3 (a) For a state $\rho$ on $\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_N$, each quantum $L_1 \times \cdots \times L_N$-setting scenario $E_{\rho,M_{L,A}}$, with $L_1 \leq S_1, \ldots, L_N \leq S_N$ and an outcome set $\Lambda$, admits an LHV model if and only if $\Upsilon_{\rho}^{(\rho,A)} = 1$;

(b) A state $\rho$ on $\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_N$ admits an $S_1 \times \cdots \times S_N$-setting LHV description if and only if $\Upsilon_{\rho}^{(\rho)} = 1$.
Proof. If each scenario $\mathcal{E}_{\rho,M_{L,A}}$, where $L_1 \leq S_1, \ldots, L_N \leq S_N$, admits an LHV model, then, by lemma 2, $\gamma_{\mathcal{E}_{\rho,M_{L,A}}} = 1$ for all collections $M_{L,A}$ of POV measures, where $L_1 \leq S_1, \ldots, L_N \leq S_N$. Hence, due to its definition (42), parameter $\Upsilon_{\rho}^{(A)} = 1$ for all collections $M_{L,A}$ of POV measures. By lemma 2, this implies LHV modelling of every quantum correlation scenario $\mathcal{E}_{\rho,M_{L,A}}$. By proposition 3 in Ref.10, the latter, in turn, implies LHV modelling of each quantum correlation scenario $\mathcal{E}_{\rho,M_{L,A}}$ with settings $L_1 \leq S_1, \ldots, L_N \leq S_N$. This proves the sufficiency part of statement (a). Statement (b) is proved quite similarly. ■

Thus, for an $N$-partite state $\rho$, it is specifically the state parameter $\Upsilon_{\rho}^{(A)}$ that determines quantitatively a possibility of an LHV description of all quantum scenarios (39) with settings up to setting $S_1 \times \cdots \times S_N$ and outcomes of an arbitrary type.

**Proposition 4** For a state $\rho$ on a Hilbert space $\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_N$ and arbitrary positive integers $S_1, \ldots, S_N \geq 1$,

$$1 \leq \Upsilon_{S_1 \times \cdots \times S_N}^{(A)} \leq \Upsilon_{S_1 \times \cdots \times S_N}^{(A)} \leq \inf_{T_{S_1 \times \cdots \times S_N}} \| T^{(A)}_{S_1 \times \cdots \times S_N} \|_{cov},$$

(44)

where (i) infimum is taken over all source operators $T^{(A)}_{S_1 \times \cdots \times S_N}$ with only one setting at $n$-th site and over all $n = 1, \ldots, N$; (ii) $\| \cdot \|_{cov}$ is the covering norm (see definition 4 in section 2).

Proof. Inequalities (44) follow from (42), (43) and the upper bound

$$\gamma_{\mathcal{E}_{\rho,M_{S,A}}} \leq \inf_{T_{S_1 \times \cdots \times S_N}} \| T^{(A)}_{S_1 \times \cdots \times S_N} \|_{cov}$$

(45)

constituting relation (B18) of lemma 5 in appendix B. ■

Propositions 3, 4 imply the following general statements on an $S_1 \times \cdots \times S_N$-setting LHV description of an $N$-partite quantum state.

**Proposition 5** (a) Every $N$-partite quantum state $\rho$ admits an $1 \times \cdots \times 1 \times S_n \times 1 \times \cdots \times 1$-setting LHV description;

(b) If, for a state $\rho$ on a Hilbert space $\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_N$, there exists a tensor positive source operator $T^{(A)}_{S_1 \times \cdots \times S_N}$ for some $n$, then $\rho$ admits the $S_1 \times \cdots \times \tilde{S}_n \times \cdots \times S_N$-setting LHV description.
description for an arbitrary number $S_n \geq 1$ of settings at this $n$-th site;

(c) If, for a state $\rho$ on a Hilbert space $H_1 \otimes \cdots \otimes H_N$, there exists a tensor positive source operator $T_{S_1 \times \cdots \times S_N}^{(\rho)}$, then $\rho$ admits the $S_1 \times \cdots \times S_n \times \cdots \times S_N$-setting LHV description for an arbitrary number $S'_n$ of measurements at every $n$-th site.

**Proof.** Since $T_{1 \times \cdots \times 1}^{(\rho)} = \rho$ and $\|\rho\|_{\text{cov}} = 1$, from bound (44) it follows that $\Upsilon_{1 \times \cdots \times S_n \times \cdots \times 1}^{(\rho)} \geq 0$ for some $n$, then from relation (13) it follows that, for this source operator, the covering norm $\|T_{S_1 \times \cdots \times S_n \times \cdots \times 1}^{(\rho)}\|_{\text{cov}} = 1$. In view of bound (44), the latter implies $\Upsilon_{S_1 \times \cdots \times S'_n \times \cdots \times S_N}^{(\rho)} = 1$ for any number $S'_n \geq 1$ of settings at this $n$-th site. By proposition 3, this proves statement (b).

Let an $N$-partite state $\rho$ have a tensor positive source operator $T_{S_1 \times \cdots \times S_N}^{(\rho)} \geq 0$. Then, for each $n = 1, \ldots, N$, the operator on $H_{S_1} \otimes \cdots \otimes H_n \otimes \cdots \otimes H_{S_N}$ reduced from $T_{S_1 \times \cdots \times S_N}^{(\rho)}$ constitutes a tensor positive source operator $T_{S_1 \times \cdots \times 1 \times \cdots \times S_N}^{(\rho)}$ for state $\rho$ and, therefore, statement (c) follows from statement (b). 

Statement (a) of proposition 5 agrees with proposition 2 of Ref. 10 on the LHV description of a general correlation scenario with setting $S \times 1 \times \cdots \times 1$.

Specified for a bipartite case ($N = 2$), statements (b), (c) of proposition 5 are consistent in view of note 34 with theorems 1, 2 in Ref. 12.

**VI. QUANTUM VIOLATIONS OF BELL-TYPE INEQUALITIES**

Consider a linear combination (26) of averages (27), arising under a quantum $S_1 \times \cdots \times S_N$-setting correlation scenario $E_{\rho, M_{S,A}}$ and specified by a family $\Psi_{S,A} = \{\psi_{(s_1,\ldots,s_N)}\}$ of bounded measurable real-valued functions $\psi_{(s_1,\ldots,s_N)} : \Lambda_1 \times \cdots \times \Lambda_N \to \mathbb{R}$.

By theorem 2, every quantum scenario $E_{\rho, M_{S,A}}$ admits an LqHV model and, by theorem 1, the latter is equivalent to the existence of a bounded real-valued measure $\mu_{E_{\rho,M_{S,A}}}$ returning all joint probability distributions $P_{(s_1,\ldots,s_N)}^{(E_{\rho,M_{S,A}})}$ of scenario $E_{\rho, M_{S,A}}$ as the corresponding marginals. Therefore, for a quantum scenario $E_{\rho, M_{S,A}}$, a linear combination (26) of averages
takes the form

\[
\sum_{s_1, \ldots, s_N} \langle \psi_{s_1, \ldots, s_N}(\lambda_1, \ldots, \lambda_N) \rangle_{\mathcal{E}_{\rho, M, S, A}} = \int \sum_{s_1, \ldots, s_N} \psi_{s_1, \ldots, s_N}(\lambda_1^{s_1}, \ldots, \lambda_N^{s_N}) \mu_{\mathcal{E}_{\rho, M, S, A}} \left( d\lambda_1^{(1)} \times \cdots \times d\lambda_1^{(S_1)} \times \cdots \times d\lambda_N^{(1)} \times \cdots \times d\lambda_N^{(S_N)} \right).
\]  

(46)

Substituting the Jordan decomposition (see remark 2) of measure \(\mu_{\mathcal{E}_{\rho, M, S, A}}\) into (46) and taking into the account (19), (20), we derive:

\[
\begin{align*}
\mathcal{B}^{\inf}_{\Psi, S, A} & \leq \sum_{s_1, \ldots, s_N} \langle \psi_{s_1, \ldots, s_N}(\lambda_1, \ldots, \lambda_N) \rangle_{\mathcal{E}_{\rho, M, S, A}} \\
& \leq \mathcal{B}^{\sup}_{\Psi, S, A} + \frac{\| \mu_{\mathcal{E}_{\rho, M, S, A}} \|_{\text{var}} - 1}{2} (\mathcal{B}^{\sup}_{\Psi, S, A} - \mathcal{B}^{\inf}_{\Psi, S, A}),
\end{align*}
\]

(47)

where \(\mathcal{B}^{\sup}_{\Psi, S, A}, \mathcal{B}^{\inf}_{\Psi, S, A}\) are the LHV constants (33) and \(\| \mu_{\mathcal{E}_{\rho, M, S, A}} \|_{\text{var}} \geq 1\) is the total variation norm of measure \(\mu_{\mathcal{E}_{\rho, M, S, A}}\).

Since inequalities (47) hold for each measure \(\mu_{\mathcal{E}_{\rho, M, S, A}}\) returning all joint probability distributions \(P_{(s_1, \ldots, s_N)}^{(\mathcal{E}_{\rho, M, S, A})}\) of scenario \(\mathcal{E}_{\rho, M, S, A}\) as the corresponding marginals, we have:

\[
\begin{align*}
\mathcal{B}^{\inf}_{\Psi, S, A} & - \frac{\gamma_{\mathcal{E}_{\rho, M, S, A}}}{2} (\mathcal{B}^{\sup}_{\Psi, S, A} - \mathcal{B}^{\inf}_{\Psi, S, A}) \\
& \leq \sum_{s_1, \ldots, s_N} \langle \psi_{s_1, \ldots, s_N}(\lambda_1, \ldots, \lambda_N) \rangle_{\mathcal{E}_{\rho, M, S, A}} \\
& \leq \mathcal{B}^{\sup}_{\Psi, S, A} + \frac{\gamma_{\mathcal{E}_{\rho, M, S, A}}}{2} (\mathcal{B}^{\sup}_{\Psi, S, A} - \mathcal{B}^{\inf}_{\Psi, S, A}),
\end{align*}
\]

(48)

where \(\gamma_{\mathcal{E}_{\rho, M, S, A}} = \inf_{\mu_{\mathcal{E}_{\rho, M, S, A}}} \| \mu_{\mathcal{E}_{\rho, M, S, A}} \|_{\text{var}} \geq 1\) is the scenario parameter (41).

Maximizing (48) over all possible scenarios \(\mathcal{E}_{\rho, M, S, A}\), performed on a quantum state \(\rho\) and with outcomes in a set \(\Lambda\), and taking into the account that \(\sup_{M, S, A} \gamma_{\mathcal{E}_{\rho, M, S, A}} = \gamma_{\mathcal{E}_{\rho, M, S, A}}^{(\rho, \Lambda)}\) is the state parameter (42), for an \(N\)-partite quantum state \(\rho\) and a function collection...
\( \Psi_{S,A} = \{ \psi_{(s_1, \ldots, s_N)} \} \), we derive the following analogs

\[
\begin{align*}
B^\inf_{\Psi_{S,A}} - \frac{\tau_{S_1 \times \ldots \times S_N} - 1}{2} (B^\sup_{\Psi_{S,A}} - B^\inf_{\Psi_{S,A}}) \\
\leq \sum_{s_1, \ldots, s_N} \langle \psi_{(s_1, \ldots, s_N)} (\lambda_1, \ldots, \lambda_N) \rangle_{E^\rho, M_{S,A}} \\
\leq B^\sup_{\Psi_{S,A}} + \frac{\tau_{S_1 \times \ldots \times S_N} - 1}{2} (B^\sup_{\Psi_{S,A}} - B^\inf_{\Psi_{S,A}})
\end{align*}
\]

of the LHV constraints (32). Since inequalities (19) are non-trivial only for those \( \Psi_{S,A} \) that correspond via (32) to Bell-type inequalities, we refer to (49) as the analogs of Bell-type inequalities for an \( N \)-partite quantum state \( \rho \).

From (49) it follows

\[
\left| \sum_{s_1, \ldots, s_N} \langle \psi_{(s_1, \ldots, s_N)} (\lambda_1, \ldots, \lambda_N) \rangle_{E^\rho, M_{S,A}} \right| \leq \tau_{S_1 \times \ldots \times S_N} B_{\Psi_{S,A}},
\]

where \( B_{\Psi_{S,A}} \) is the LHV constant (35).

**Remark 3** The quantum constraints (49), (50) are equivalent if \( B^\inf_{\Psi_{S,A}} = -B^\sup_{\Psi_{S,A}} \). For an arbitrary function collection \( \Psi_{S,A} \), (49) \( \Rightarrow \) (50) but not vice versa. In order to see a difference between these two types of quantum constraints for an arbitrary \( \Psi_{S,A} \), let us take \( \tilde{\Psi}_{S,A} \) for which \( B^\sup_{\tilde{\Psi}_{S,A}} = 0 \). In this case, the left-hand and the right-hand sides of (49) are equal to

\[
-\frac{\tau_{S_1 \times \ldots \times S_N} + 1}{2} |B^\inf_{\Psi_{S,A}}|, \quad \frac{\tau_{S_1 \times \ldots \times S_N} - 1}{2} |B^\inf_{\Psi_{S,A}}|,
\]

respectively, whereas the right hand side of (50) is given by \( \tau_{S_1 \times \ldots \times S_N} |B^\inf_{\Psi_{S,A}}| \). Note that, specifically for bipartite Bell-type inequalities with \( B^\sup_{\Psi_{S,A}} = 0 \), the maximal violations by two-qubit states have been analyzed numerically in Ref. 24.

The following statement (proved in appendix C) shows that the quantum constraints (49), (50) are tight in the sense that the state parameter \( \tau_{S_1 \times \ldots \times S_N} \) represents the maximal violation by state \( \rho \) of all Bell-type inequalities (either on correlation functions or on joint probabilities) for a given outcome set \( \Lambda \) and settings \( L_1 \times \ldots \times L_N \) with \( L_1 \leq S_1, \ldots, L_N \leq S_N \).

**Lemma 3** In (57), parameter \( \tau_{S_1 \times \ldots \times S_N} = \sup_{M_{S,A}, \Psi_{S,A}} \gamma_{E^\rho, M_{S,A}} \) is otherwise expressed by

\[
\tau_{S_1 \times \ldots \times S_N} = \sup_{M_{S,A}, \Psi_{S,A} \neq 0} \left| \frac{1}{B_{\Psi_{S,A}}} \sum_{s_1, \ldots, s_N} \langle \psi_{(s_1, \ldots, s_N)} (\lambda_1, \ldots, \lambda_N) \rangle_{E^\rho, M_{S,A}} \right|,
\]

where
where supremum is taken over all non-trivial \((B_{\Psi S,\Lambda} \neq 0)\) families \(\Psi_{S,\Lambda} = \{ \psi_{(s_1,\ldots,s_N)} \}\) of bounded measurable real-valued functions on \(\Lambda = \Lambda_1 \times \cdots \times \Lambda_N\) and over all possible families \(M_{S,\Lambda} = \{ M_n \} \) of POV measures on spaces \((\Lambda_n, \mathcal{F}_\Lambda_n)\).

From lemma 3 it follows that the state parameter
\[
\Upsilon_{(\rho)}^{(S_1 \times \cdots \times S_N)} := \sup_{\Lambda, M_{S,\Lambda}, \Psi_{S,\Lambda}, B_{\Psi S,\Lambda} \neq 0} \left| \frac{1}{B_{\Psi S,\Lambda}} \sum_{s_1,\ldots,s_N} \left\langle \psi_{(s_1,\ldots,s_N)}(\lambda_1,\ldots,\lambda_N) \right\rangle E_{\rho, M_{S,\Lambda}} \right|
\]
and, therefore, represents the maximal violation by state \(\rho\) of all Bell-type inequalities on correlation functions and joint probabilities for settings up to setting \(S_1 \times \cdots \times S_N\) and an arbitrary outcome set \(\Lambda\).

**Proposition 6** For a state \(\rho\) on a Hilbert space \(\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_N\), the following statements are mutually equivalent:

(a) State \(\rho\) admits the \(S_1 \times \cdots \times S_N\)-setting LHV description;

(b) Parameter \(\Upsilon_{(\rho)}^{(S_1 \times \cdots \times S_N)} = 1\);

(c) State \(\rho\) does not violate any Bell-type inequality with settings \(L_1 \leq S_1,\ldots, L_N \leq S_N\) and outcomes in an arbitrary outcome set \(\Lambda\).

**Proof.** Equivalence (a)⇔(b) follows from proposition 3. Implication (a) ⇒ (c) follows from definition 6 of a Bell-type inequality. Let (c) hold. Then from (53) it follows \(\Upsilon_{(\rho)}^{(S_1 \times \cdots \times S_N)} = 1\), so that (c) ⇒ (b). Thus, we have proved (a) ⇔ (b), (a) ⇒ (c), (c) ⇒ (b). These implications prove the mutual equivalence of statements (a), (b), (c).\(\blacksquare\)

The following theorem introduces a general analytical upper bound on the maximal violation \(\Upsilon_{(\rho)}^{(S_1 \times \cdots \times S_N)}\) by state \(\rho\) of all \(S_1 \times \cdots \times S_N\)-setting Bell-type inequalities – the maximal \(S_1 \times \cdots \times S_N\)-setting Bell violation for state \(\rho\), for short.

**Theorem 3** For every quantum state \(\rho\) on a Hilbert space \(\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_N\) and arbitrary positive integers \(S_1,\ldots,S_N \geq 1\), the maximal \(S_1 \times \cdots \times S_N\)-setting Bell violation \(\Upsilon_{(\rho)}^{(S_1 \times \cdots \times S_N)} \geq \ldots\)
1 is upper bounded by

\[
\Upsilon_{S_1 \times \cdots \times S_N}^{(\rho)} \leq \inf_{T^{(\rho)}_{S_1 \times \cdots \times S_N}, \forall n} \|T^{(\rho)}_{S_1 \times \cdots \times S_N} \|_{\text{cov}},
\]

where infimum is taken over all source operators \(T^{(\rho)}_{S_1 \times \cdots \times S_N}\) with only one setting at \(n\)-th site and over all \(n = 1, \ldots, N\) and \(\| \cdot \|_{\text{cov}}, \| \cdot \|_1\) mean the covering norm and the trace norm, respectively.

**Proof.** The statement follows from relation (53), proposition 4 and bound (12).

### A. Numerical estimates

In this section, via the analytical upper bound (54) we estimate the maximal \(S_1 \times \cdots \times S_N\)-setting Bell violation \(\Upsilon_{S_1 \times \cdots \times S_N}^{(\rho)}\) in terms of numerical characteristics of quantum correlation scenarios such as a number \(N \geq 2\) of sites, a number \(S_n \geq 1\) of measurements and the Hilbert space dimension \(d_n := \dim \mathcal{H}_n\) at each \(n\)-th of \(N\) sites.

Let us first evaluate \(\Upsilon_{S_1 \times \cdots \times S_N}^{(\rho)}\) for some concrete quantum states generally used in quantum information processing.

For the two-qubit singlet \(\psi_{\text{singlet}} = \frac{1}{\sqrt{2}}(e_1 \otimes e_2 - e_2 \otimes e_1)\), the analytical upper bound (54) and relation (A23) imply

\[
\Upsilon_{2 \times 2}^{(\rho_{\text{singlet}})} \leq \sqrt{3}, \ \forall S \geq 2.
\] (55)

Note that, due to Tsirelson’s bound\(^5\) and the analysis of Fine\(^7\), violation by a bipartite state \(\rho\) of an arbitrary \(2 \times 2\)-setting Bell-type inequality (either on correlation functions or on joint probabilities) for two settings and two outcomes per site cannot exceed \(\sqrt{2}\) – in our notation \(\Upsilon_{2 \times 2}^{(\rho;\{\lambda_1, \lambda_2\}^2)} \leq \sqrt{2}\). The maximal violation by the two-qubit singlet of all correlation Bell-type inequalities is given\(^3\) by the Grothendieck’s constant \(K_G(3)\) of order 3 and it is known\(^19,20\) that \(\sqrt{2} \leq K_G(3) \leq 1.5164\).

Consider also the maximal Bell violations for the \(N\)-qudit Greenberger- Horne - Zeilinger (GHZ) state

\[
\psi_d = \frac{1}{\sqrt{d}} \sum_{j=1}^{d} |j\rangle^{\otimes N} \in (\mathbb{C}^d)^{\otimes N}
\] (56)
and the generalized $N$-qubit GHZ state
\[
\psi_{\mathrm{gen}}^{(2)} = \sin \varphi |1\rangle^\otimes N + \cos \varphi |2\rangle^\otimes N \in (\mathbb{C}^2)^\otimes N, \tag{57}
\]
where $|j\rangle$, $j = 1, \ldots, d$, are mutually orthogonal unit vectors in $\mathbb{C}^d$. For each of these states, the trace norm of the source operator (A20) is upper bounded by
\[
\|\tilde{T}(\rho_{S_1 \times \cdots \times S_N})\|_1 \leq 1 + 2^{N-1}(d-1),
\]
\[
\|\tilde{T}(\rho_{S_1 \times \cdots \times S_N}^{(\mathrm{gen})})\|_1 \leq 1 + 2^{N-1} |\sin \varphi \cos \varphi|,
\]
and, in view of relations (58), (A9), the analytical upper bound (54) implies
\[
\Upsilon(\rho_{S_1 \times \cdots \times S_N}) \leq \min\{ (2S - 1)^{N-1}, 1 + 2^{N-1}(d-1) \}
\]
\[
\leq 1 + 2^{N-1} \left[ \min\{S^{N-1}, d\} - 1 \right]
\]
and
\[
\Upsilon(\rho_{S_1 \times \cdots \times S_N}^{(\mathrm{gen})}) \leq \min\{ (2S - 1)^{N-1}, 1 + 2^{N-1} |\sin 2\varphi| \}
\]
\[
\leq 1 + 2^{N-1} |\sin 2\varphi|.
\]
The second lines in (59), (60) are due to relation $(2S - 1)^{N-1} \leq 2^{N-1}(S^{N-1} - 1) + 1$ that can be easily proved by induction.

For an arbitrary $N$-partite state $\rho$, the general analytical upper bound (54) implies the following new numerical upper estimate.

**Theorem 4** For every state $\rho$ on a Hilbert space $\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_N$ and arbitrary positive integers $S_1, \ldots, S_N \geq 1$, the maximal $S_1 \times \cdots \times S_N$-setting Bell violation satisfies relation
\[
\Upsilon(\rho_{S_1 \times \cdots \times S_N}) \leq \min\{ \xi_N, \theta_N, \}
\]
\[
\leq 1 + 2^{N-1} \left[ \min\left\{ \frac{d_1 \cdot \ldots \cdot d_N}{\max_n d_n}, \frac{S_1 \cdot \ldots \cdot S_N}{\max_n S_n} \right\} - 1 \right],
\]
where $d_n = \dim \mathcal{H}_n$, $n = 1, \ldots, N$, and
\[
\xi_N = 1 + 2^{N-1} \left( \frac{d_1 \cdot \ldots \cdot d_N}{\max_n d_n} - 1 \right),
\]
\[
\theta_N = (-1)^{N-1} + \min_{\{n_1, \ldots, n_{N-1}\} \subset \{1, \ldots, N\}} \sum_{k=0}^{N-2} (-1)^k 2^{N-1-k} \sum_{n_j \neq \#n_{j-1} - k, \atop n_j \in \{n_1, \ldots, n_{N-1}\}} S_{n_1} \cdot \ldots \cdot S_{n_{N-1}}.
\]
If, in particular, \( d_n = d, S_n = S, \forall n \), then

\[
\Upsilon_{S \times \ldots \times S}^{(\rho)} \leq \min \left\{ (2S - 1)^{N-1}, 2^{N-1}(d^{N-1} - 1) + 1 \right\} \tag{63}
\]

\[
\leq 1 + 2^{N-1} \left[ \left( \min\{S, d\} \right)^{N-1} - 1 \right].
\]

**Proof.** From bound (54) it follows

\[
\Upsilon_{S_1 \times \ldots \times S_N}^{(\rho)} \leq \inf_{\tau_{S_1 \times \ldots \times S_N}^{(\rho)}} ||T_{S_1 \times \ldots \times S_N}^{(\rho)}||_{\text{cov}}
\]

\[
\leq \inf_{\tau_{S_1 \times \ldots \times S_N}^{(\rho)}} ||T_{S_1 \times \ldots \times S_N}^{(\rho)}||_{1}
\]

\[
\leq \min\{||\tau_{S_1 \times \ldots \times S_N}^{(\rho)}||_{\text{cov}}, ||\tau_{S_1 \times \ldots \times S_N}^{(\rho)}||_{\text{cov}}, n = 1, \ldots, N\},
\]

where \( \tau^{(\rho)}_{S_1 \times \ldots \times S_N} \) and \( \tilde{\tau}^{(\rho)}_{S_1 \times \ldots \times S_N} \) are the specific source operators constructed for an arbitrary \( N \)-partite state \( \rho \) in appendix A. Taking into the account the upper bounds (A9), (A21) for these source operators and also relation \((2S - 1)^{N-1} \leq 2^{N-1}S^{N-1} - 2^{N-1} + 1\), we come to relation (61), implying, in turn, (63).

Note that estimate (61) implies that \( \Upsilon_{S_1 \times \ldots \times S_n}^{(\rho)} = 1 \) for every \( N \)-partite state \( \rho \). In view of statement (b) of proposition 3, this result agrees with statement (a) of proposition 5.

Theorem 4 implies.

**Corollary 2** (a) For an arbitrary \( N \)-partite quantum state, violation of a Bell-type inequality (either on correlation functions or on joint probabilities) for \( S \) settings per site cannot exceed \((2S - 1)^{N-1}\) even in case of an infinite dimensional state and infinitely many outcomes.

(b) For an arbitrary state \( \rho \) on \((\mathbb{C}^d)^{\otimes N}\), violation of a Bell-type inequality (either on correlation functions or on joint probabilities) is upper bounded by \(2^{N-1}(d^{N-1} - 1) + 1\) independently on a number of settings and a number of outcomes at each site.

Let us now specify the general \( N \)-partite upper estimate (62) for \( N = 2, 3 \).
Corollary 3  For every bipartite state $\rho$ and arbitrary positive integers $S_1, S_2 \geq 1$,

$$\Upsilon^{(\rho)}_{S_1 \times S_2} \leq 2 \min \{S_1, S_2, d_1, d_2\} - 1. \tag{65}$$

If, in particular, $d_1 = d_2 = d$, $S_1 = S_2 = S$, then

$$\Upsilon^{(\rho)}_{S \times S} \leq 2 \min \{S, d\} - 1. \tag{66}$$

For every tripartite state $\rho$ and arbitrary positive integers $S_1, S_2, S_3 \geq 1$,

$$\Upsilon^{(\rho)}_{S_1 \times S_2 \times S_3} \leq \min \left\{ \frac{\min \{S_1 S_2 S_3\}}{\max_n S_n}, \frac{d_1 d_2 d_3}{\max_n d_n} \right\} - 3. \tag{67}$$

If, in particular, $d_n = d$, $S_n = S, \forall n$, then

$$\Upsilon^{(\rho)}_{S \times S \times S} \leq \min \left\{ (2S - 1)^2, 4d^2 - 3 \right\} \leq 4 \left( \min \{S, d\} \right)^2 - 3. \tag{68}$$

We stress that, in contrast to the bipartite and tripartite numerical estimates found in Refs.\textsuperscript{25,27–29} up to unknown universal constants, our bipartite and tripartite numerical upper estimates (65) - (68) are exact.

**B. Discussion**

For bipartite and tripartite correlation scenarios with a finite number of outcomes at each site, the numerical estimates on the maximal Bell violations have been recently presented in Refs.\textsuperscript{25–29}. The results of corollary 3 indicate.

- Our exact bipartite upper estimate \textsuperscript{(65)} improves the approximate bipartite estimate $\leq \min \{d, S\}$ found in Ref.\textsuperscript{29} (theorem 6.8) up to an unknown universal constant.

- The bipartite upper estimates (in our notation)

$$\Upsilon^{(\rho, \Lambda)}_{S_1 \times S_2} \leq 2K_G + 1, \quad \text{if} \quad |A| = |B| = 2, \tag{69}$$

$$\Upsilon^{(\rho, \Lambda)}_{S_1 \times S_2} \leq 2 |A| |B| (K_G + 1) - 1, \quad \forall |A|, |B|,$$
derived in theorem 22 of Ref. 26 for arbitrary \( S_1, S_2 \geq 1 \) and numbers \(|A|, |B|\) of outcomes at Alice’s and Bob’s sites, are improved by our bipartite upper estimate if

\[
\min\{d_1, d_2\} < K_G + 1, \tag{70}
\]

\[
\min\{d_1, d_2\} < |A||B| (K_G + 1),
\]

respectively. Here, \( K_G \) is the Grothendieck constant, and it is known\(^{19,20}\) that \( K_G \in [1.676, 1.783] \).

- From our exact tripartite upper estimates (67), (68) it follows that violation by a tripartite quantum state of a Bell-type inequality for \( S \) settings per site cannot exceed \((2S - 1)^2\). Therefore, the tripartite lower estimate \( \geq \sqrt{d} \), found in theorem 1 of Ref. 25 for violation of some correlation Bell-type inequality by some tripartite state on \( \mathbb{C}^d \otimes \mathbb{C}^D \otimes \mathbb{C}^D \), is meaningful if only a number \( S \) of settings per site needed for such a violation in the corresponding Bell-type inequality obeys relation

\[
(2S - 1)^2 \geq \sqrt{d}. \tag{71}
\]

Thus, for an arbitrarily large tripartite violation argued in Ref. 25 to be reached, not only a Hilbert space dimension \( d \) but also a number \( S \) of settings per site in the corresponding tripartite Bell-type inequality must be large and the required growth of \( S \) with respect to \( d \) is given by (71).

VII. CONCLUSIONS

In the present paper, for the probabilistic description of a general correlation scenario, we have introduced (definition 5) a new simulation model, a local quasi hidden variable (LqHV) model, where locality and the measure-theoretic structure inherent to an LHV model are preserved but positivity of a simulation measure is dropped.

We have specified (theorem 1) a necessary and sufficient condition for LqHV modelling and, based on this, proved (theorem 2) that every quantum correlation scenario admits an LqHV simulation.

Via the LqHV approach, we have constructed analogs (Eq. 49) of Bell-type inequalities for an \( N \)-partite quantum state and found (theorem 3) a new analytical upper bound on the
maximal violation by an $N$-partite state of all $S_1 \times \cdots \times S_N$-setting Bell-type inequalities – either on correlation functions or on joint probabilities and for outcomes of an arbitrary spectral type, discrete or continuous.

This analytical upper bound is based on the new state dilation characteristics (definitions 1-4) introduced in the present paper and this allows us:

- to trace (propositions 5, 6) $N$-partite quantum states admitting an $S_1 \times \cdots \times S_N$-setting LHV description;

- to find the exact numerical upper estimates (Eqs. (55), (59), (69)) on the maximal $S_1 \times \cdots \times S_N$-setting Bell violations for some concrete $N$-partite quantum states used in quantum information processing;

- to find (theorem 4) the exact numerical upper estimate on the maximal $S_1 \times \cdots \times S_N$-setting Bell violation for an arbitrary $N$-partite quantum state, in particular, to show (corollary 2) that violation by an $N$-partite quantum state of a Bell-type inequality (either on correlation functions or on joint probabilities) for $S$ settings per site is upper bounded by $(2S - 1)^{N-1}$ even in case of an infinite dimensional quantum state and infinitely many outcomes.

Specified (corollary 3) for $N = 2, 3$, our exact $N$-partite numerical upper estimate (Eq. (61)) improves the bipartite numerical upper estimates in Refs. 26, 29 and clarifies the range of applicability of the approximate tripartite numerical lower estimate in Ref. 25.

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**Appendix A: proofs for section 2**

In this appendix, we prove proposition 1 on the existence of source operators for an $N$-partite quantum state and introduce some source operators which are different from those constructed in the proof of proposition 1 and are needed for our consideration in section 2 of appendix B. We also prove lemma 1 on the properties of the covering norm.
1. Proof of proposition 1

For a state $\rho$ on a Hilbert space $\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_N$, consider its decomposition

$$
\rho = \sum \eta_{mm_1 \cdots k_1} |e^{(1)}_m\rangle \langle e^{(1)}_m| \otimes \cdots \otimes |e^{(N)}_{k_1}\rangle \langle e^{(N)}_{k_1}|,
$$

where

$$
\eta_{mm_1 \cdots k_1} = \sum_i \alpha_i^* s^{(i)}_{m \cdots k}, \quad \alpha_i > 0, \quad \sum_i \alpha_i = 1, \quad \sum_{m, \cdots, k} s^{(i)}_{m \cdots k} = 1,
$$

in orthonormal bases $\{e^{(n)}_m \in \mathcal{H}_n\}$, $n = 1, \ldots, N$.

Let $N = 2$. For a bipartite state $\rho$, denote by $\rho_n$ the reduced state on $\mathcal{H}_n$, $n = 1, 2$, and introduce on a Hilbert space $\mathcal{H}_1^{\otimes S_1} \otimes \mathcal{H}_2^{\otimes S_2}$ the self-adjoint operator

$$
\tau^{(\rho)}_{S_1 \times S_2} = \sum \eta_{mm_1 k_1} \left[ |e^{(1)}_m\rangle \langle e^{(1)}_{m_1}| \otimes \sigma_1^{\otimes(S_1-1)} \right]_{\text{sym}} \otimes \left[ |e^{(2)}_{k_1}\rangle \langle e^{(2)}_{k_1}| \otimes \sigma_2^{\otimes(S_2-1)} \right]_{\text{sym}}
$$

\[ - (S_2 - 1)\rho_1^{\otimes S_1} \otimes \rho_2^{\otimes S_2} - (S_1 - 1)\sigma_1^{\otimes S_1} \otimes \rho_2^{\otimes S_2}
\]

\[ - (S_1 - 1)(S_2 - 1)\sigma_1^{\otimes S_1} \otimes \sigma_2^{\otimes S_2},
\]

where $\sigma_n$ is a state on $\mathcal{H}_n$ and notation $[\cdot]_{\text{sym}}$ means symmetrization on $\mathcal{H}_n^{\otimes S_n}$. For example, $[X_1 \otimes X_2]_{\text{sym}} := X_1 \otimes X_2 + X_2 \otimes X_1$. It is easy to verify that (A2) represents an $S_1 \times S_2$-setting source operator for state (A1) specified with $N = 2$ and the trace norm of this source operator satisfies relation

$$
1 \leq \left\| \tau^{(\rho)}_{S_1 \times S_2} \right\|_1 \leq 2S_1S_2 - 1.
$$

Let $N = 3$. For a tripartite state $\rho$, in addition to the above notation $\rho_n$ for the reduced state on $\mathcal{H}_n$, denote by $\rho_{1,n}$ the reduced state on $\mathcal{H}_1 \otimes \mathcal{H}_n$. For short of notation, we further take one of settings to be equal only to 1, say $S_1 = 1$. Introduce on a Hilbert space $\mathcal{H}_1 \otimes \mathcal{H}_2^{\otimes S_2} \otimes \mathcal{H}_3^{\otimes S_3}$ the self-adjoint operator

$$
\tau^{(\rho)}_{1 \times S_2 \times S_3} = \sum \eta_{mm_1 j_1 k_1} \left[ |e^{(1)}_m\rangle \langle e^{(1)}_{m_1}| \otimes \left[ |e^{(2)}_{j_1}\rangle \langle e^{(2)}_{j_1}| \otimes \sigma_3^{\otimes(S_3-1)} \right]_{\text{sym}}
\right.
$$

\[ \left. \otimes \left[ |e^{(3)}_{k_1}\rangle \langle e^{(3)}_{k_1}| \otimes \sigma_3^{\otimes(S_3-1)} \right]_{\text{sym}} - (S_3 - 1)\tau^{(\rho_{1,2})}_{1 \times S_2} \otimes \sigma_3^{\otimes S_3}
\]

\[ - (S_2 - 1)\sqrt{\sigma_2^{\otimes S_2}} \otimes \sqrt{\sigma_3^{\otimes S_3}} - (S_2 - 1)(S_3 - 1)\rho_1^{\otimes S_1} \otimes \sigma_2^{\otimes S_2} \otimes \sigma_3^{\otimes S_3},
\]

where: (i) $\tau^{(\rho_{1,n})}_{1 \times S_n}$ is the $1 \times S_n$-source operator (A2) specified for the reduced state $\rho_{1,n}$ on $\mathcal{H}_1 \otimes \mathcal{H}_n$; (ii) notation in the third line means operator derived by insertion of term $\otimes \sigma_2^{\otimes S_2}$ into each term of the tensor product decomposition of the source operator $\tau^{(\rho_{1,3})}_{1 \times S_3}$ for state.
\( \rho_{1,3} \) on \( \mathcal{H}_1 \otimes \mathcal{H}_3 \). It is easy to verify that (A4) represents an \( 1 \times S_2 \times S_3 \)-setting source operator for state (A1) specified with \( N = 3 \). Substituting (A2) into (A4), taking into the account (2) and evaluating the negative part of the self-adjoint operator (A4), we derive

\[
\left\| \tau_{1 \times S_2 \times S_3}(\rho) \right\|_1 \leq 1 + 2(S_2 - 1)S_3 + 2(S_3 - 1)S_2 \]
\[
= 4S_2S_3 - 2(S_2 + S_3) + 1 \]
\[
\leq 4S_2S_3 - 3. \tag{A5}
\]

Let \( N = 4 \). For a quadripartite state \( \rho \), denote by \( \rho_n, \rho_{n_1,n_2}, \rho_{n_1,n_2,n_3} \) the reduced states on \( \mathcal{H}_n, \mathcal{H}_{n_1} \otimes \mathcal{H}_{n_2} \) and \( \mathcal{H}_{n_1} \otimes \mathcal{H}_{n_2} \otimes \mathcal{H}_{n_3} \), respectively and consider on \( \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3 \otimes \mathcal{H}_4 \) the self-adjoint operator

\[
\tau_{1 \times S_2 \times S_3 \times S_4}^{(p)} = \sum \eta_{m_{\text{sym}}l_{\text{sym}}l_kl_{\text{sym}}} c^{(1)}_m | e_n^{(1)} \rangle | e_{n_1}^{(1)} \rangle \otimes \left[ | e_j^{(2)} \rangle | e_{n_1}^{(2)} \rangle \otimes \sigma_2^{(S_2 - 1)} \right]_{\text{sym}} \otimes \left[ | e_k^{(4)} \rangle | e_{k_4}^{(4)} \rangle \otimes \sigma_4^{(S_4 - 1)} \right]_{\text{sym}}
\]
\[
- (S_4 - 1) \tau_{1 \times S_2 \times S_3 \times S_4}^{(p_1,2,3)} \otimes \sigma_4^{S_4} - (S_2 - 1) \sqrt{\otimes} \sigma_2^{S_2} \otimes \sqrt{\otimes}
\]
\[
- (S_3 - 1) \sqrt{\otimes} \otimes \sigma_3^{S_3} \otimes \sqrt{\otimes}
\]
\[
- (S_3 - 1)(S_4 - 1) \tau_{1 \times S_2 \times S_3}^{(p_1,2)} \otimes \sigma_3^{S_3} \otimes \sigma_4^{S_4} \tag{A6}
\]
\[
- (S_2 - 1)(S_3 - 1) \sqrt{\otimes} \sigma_2^{S_2} \otimes \sigma_3^{S_3} \otimes \sqrt{\otimes}
\]
\[
- (S_2 - 1)(S_4 - 1) \sqrt{\otimes} \sigma_2^{S_2} \otimes \sigma_4^{S_4} \tag{\rho_1,4}
\]
\[
- (S_2 - 1)(S_3 - 1)(S_4 - 1) \rho_{1,3}^{S_1} \otimes \sigma_2^{S_2} \otimes \sigma_3^{S_3} \otimes \sigma_4^{S_4},
\]

where \( \tau_{1 \times S_n}^{(p_1,n)} \) is the \( 1 \times S_n \)-setting source operator (A2) specified for the reduced state \( \rho_{1,n} \) and \( \tau_{1 \times S_{n_1} \times S_{n_2}}^{(p_1,n_1,n_2)} \) is the source operator (A5) for the reduced state \( \rho_{1,n_1,n_2} \). It is easy to verify that (A6) represents an \( 1 \times S_2 \times S_3 \times S_4 \)-setting source operator for state (A1) specified with \( N = 4 \). Substituting (A2), (A4) into (A6) and evaluating the negative part of the
self-adjoint operator \((\text{A6})\), we derive

\[
\left\| \tau^{(\rho)}_{1 \times S_2 \times S_3 \times S_4} \right\|_1 \leq 1 + 2(S_2 - 1)S_3S_4 + 2(S_3 - 1)S_2S_4
\]

\[
+ 2(S_4 - 1)S_2S_3 + 2(S_2 - 1)(S_3 - 1)(S_4 - 1)
\]

\[
= 8S_2S_3S_4 - 4(S_2S_3 + S_2S_4 + S_3S_4)
\]

\[
+ 2(S_2 + S_3 + S_4) - 1.
\]

Thus, in view of \((\text{A3})\), \((\text{A5})\), \((\text{A7})\),

\[
\left\| \tau^{(\rho)}_{1 \times S_2} \right\|_1 \leq 2S_2 - 1, \tag{A8}
\]

\[
\left\| \tau^{(\rho)}_{1 \times S_2 \times S_4} \right\|_1 \leq 4S_2S_3 - 2(S_2 + S_3) + 1 \leq 4S_2S_3 - 3,
\]

\[
\left\| \tau^{(\rho)}_{1 \times S_2 \times S_3 \times S_4} \right\|_1 \leq 8S_2S_3S_4 - 4(S_2S_3 + S_2S_4 + S_3S_4)
\]

\[
+ 2(S_2 + S_3 + S_4) - 1 \leq 8S_2S_3S_4 - 7.
\]

All these bounds are tight in the sense they imply \(\left\| \tau^{(\rho)}_{1 \times \ldots \times 1} \right\|_1 = 1\). The generalization to an \(N\)-partite case with setting \(S_1 = 1\) is straightforward and gives

\[
\left\| \tau^{(\rho)}_{1 \times S_2 \times \ldots \times S_N} \right\|_1 \leq \sum_{k=0}^{N-2} (-1)^k 2^{N-1-k} \sum_{n_1 \neq \ldots \neq n_{N-1-k}, n_j = 2, \ldots, N} S_{n_1} \cdot \ldots \cdot S_{n_{N-1-k}} \tag{A9}
\]

\[
+ (-1)^{N-1},
\]

\[
\left\| \tau^{(\rho)}_{1 \times \ldots \times S} \right\|_1 \leq (2S - 1)^{N-1}.
\]

The constructed source operators \((\text{A2}), (\text{A4}), (\text{A6}), (\text{A9})\) prove the statement of proposition 1.

\[a. \quad \text{Other examples of source operators}\]

For our consideration in section 2 of appendix B, let us also construct source operators of a type \(\tilde{\tau}\) different from type \(\tau\) in Eqs. \((\text{A2}), (\text{A4}), (\text{A6}), (\text{A9})\). Denote \(d_n := \dim \mathcal{H}_n, n = 1, \ldots, N\), and assume that \(\max_n d_n = d_1\).

Let \(N = 2\) and \(\psi \in \mathcal{H}_1 \otimes \mathcal{H}_2\). For a pure bipartite state \(|\psi\rangle\langle\psi|\), consider its Schmidt decomposition

\[
|\psi\rangle\langle\psi| = \sum \xi_j \xi_j |g_j^{(1)}\rangle \langle g_j^{(1)}| \otimes |g_j^{(2)}\rangle \langle g_j^{(2)}|, \quad \xi_j > 0, \sum \xi_j^2 = 1, \tag{A10}
\]
where the sum is taken over $j, j_1 = 1, \ldots, d_2$ and $\{g^{(n)}_j\}$ is an orthonormal base in $\mathcal{H}_n$, $n = 1, 2$.

Introduce the self-adjoint operator

$$\tilde{\tau}^{(\psi)}_{1 \times S_2} = \sum_j \xi_j^2 |g^{(1)}_j\rangle \langle g^{(1)}_j| \otimes (|g^{(2)}_j\rangle \langle g^{(2)}_j|) \otimes S_2 + \sum_{j \neq j_1} \xi_j \xi_{j_1} |g^{(1)}_{j_1}\rangle \langle g^{(1)}_{j_1}| \otimes W^{(2,S_2)}_{jj_1},$$  \hspace{1cm} (A11)

where

$$2W^{(2,S_2)}_{jj_1} = \frac{(|g^{(2)}_j + g^{(2)}_{j_1}\rangle \langle g^{(2)}_j + g^{(2)}_{j_1}|)^{\otimes S_2} - (|g^{(2)}_j - g^{(2)}_{j_1}\rangle \langle g^{(2)}_j - g^{(2)}_{j_1}|)^{\otimes S_2}}{2^{S_2}}$$  \hspace{1cm} (A12)

$$- i \frac{(|g^{(2)}_j + ig^{(2)}_{j_1}\rangle \langle g^{(2)}_j + ig^{(2)}_{j_1}|)^{\otimes S_2} + i (|g^{(2)}_j - ig^{(2)}_{j_1}\rangle \langle g^{(2)}_j - ig^{(2)}_{j_1}|)^{\otimes S_2}}{2^{S_2}},$$

$$\left(W^{(2,S_2)}_{jj_1}\right)^* = W^{(2,S_2)}_{j_1 j}.$$  

It is easy to verify that (A11) represents an $1 \times S_2$-setting source operator for state (A10) and

$$\left\| \tilde{\tau}^{(\psi)}_{1 \times S_2} \right\|_1 \leq 1 + 2 \sum_{j \neq j_1} \xi_j \xi_{j_1} = 2 \left( \sum \xi_j \right)^2 - 1$$  \hspace{1cm} (A13)

$$\leq 2d_2 - 1$$

for any $S_2 \geq 1$. By convexity, for an arbitrary bipartite state $\rho = \sum_i \alpha_i |\psi_i\rangle \langle \psi_i|$, operator

$$\tilde{\tau}^{(\rho)}_{1 \times S_2} = \sum_i \alpha_i \tilde{\tau}^{(\psi_i)}_{1 \times S_2}$$

represents an $1 \times S_2$-setting source operator and

$$\left\| \tilde{\tau}^{(\rho)}_{1 \times S_2} \right\|_1 \leq 2d_2 - 1, \hspace{1cm} \forall S_2 \geq 1.$$  \hspace{1cm} (A14)

Let $N = 3$. For state (A1) with $N = 3$ introduce the self-adjoint operator

$$\tilde{\tau}^{(\rho)}_{1 \times S_2 \times S_3} = \sum_{i,m,n,k,k_1,k_2,k_3} \eta_{mm_1,jj_1 \ldots, k_1 k_2 k_3} |e^{(1)}_m\rangle \langle e^{(1)}_m| \otimes W^{(2,S_2)}_{jj_1} \otimes W^{(3,S_3)}_{kk_1 k_2 k_3},$$  \hspace{1cm} (A15)

where $W^{(n,S_n)}_{il} := (|e^{(n)}_i\rangle \langle e^{(n)}_i|)^{\otimes S_n}$ and operator $W^{(n,S_n)}_{l \neq i}$ is defined by (A12) via replacements $2 \rightarrow n, S_2 \rightarrow S_n, g^{(2)}_j \rightarrow e^{(n)}_j$. It is easy to verify that (A15) is an $1 \times S_2 \times S_3$-setting source operator for state (A1) with $N = 3$. Splitting (A15) into four sums

$$\tilde{\tau}^{(\rho)}_{1 \times S_2 \times S_3} = \sum_{i,j,k} \alpha_i (\beta^{(i)}_{jk})^2 |\phi^{(i)}_{jk}\rangle \langle \phi^{(i)}_{jk}| \otimes (|e^{(2)}_j\rangle \langle e^{(2)}_j|) \otimes S_2 \otimes (|e^{(3)}_k\rangle \langle e^{(3)}_k|) \otimes S_3$$  \hspace{1cm} (A16)

$$+ \sum_{i \neq j_1, k} \alpha_i (\beta^{(i)}_{jk}) (\beta^{(i)}_{j_1 k}|\phi^{(i)}_{jk}\rangle \langle \phi^{(i)}_{j_1 k}| \otimes W^{(2,S_2)}_{jj_1} \otimes (|e^{(3)}_k\rangle \langle e^{(3)}_k|) \otimes S_3$$

$$+ \sum_{i,j,k \neq k_1} \alpha_i (\beta^{(i)}_{jk}) (\beta^{(i)}_{jk}|\phi^{(i)}_{jk}\rangle \langle \phi^{(i)}_{jk}| \otimes (|e^{(2)}_j\rangle \langle e^{(2)}_j|) \otimes S_1 \otimes W^{(3,S_3)}_{kk_1}$$

$$+ \sum_{i \neq j_1, k \neq k_1} \alpha_i (\beta^{(i)}_{jk}) (\beta^{(i)}_{j_1 k}|\phi^{(i)}_{jk}\rangle \langle \phi^{(i)}_{j_1 k}| \otimes W^{(2,S_2)}_{jj_1} \otimes W^{(3,S_3)}_{kk_1},$$

33
where, for any index $i$,

$$\phi_{jk}^{(i)} = \frac{1}{\beta_{jk}^{(i)}} \sum_m \epsilon_{mjk}^{(i)} e_m, \quad \|\phi_{jk}^{(i)}\| = 1, \quad (A17)$$

$$\beta_{jk}^{(i)} = \left( \sum_m |\epsilon_{mjk}^{(i)}|^2 \right)^{1/2}, \quad \sum_j (\beta_{jk}^{(i)})^2 = 1,$$

and taking into the account that

$$\|\phi_{jk}^{(i)}\rangle\langle \phi_{jk}^{(i)}\|_1 = 1, \quad \sum_j \beta_{jk}^{(i)} \leq \sqrt{d_2 d_3}, \quad (A18)$$

$$\|W_{ll}^{(n,S_n)}\|_1 = 1, \quad \|W_{l\neq l_1}^{(n,S_n)}\|_1 \leq 2, \quad n = 2, 3,$$

we derive

$$\left\|\hat{T}^{(\rho)}_{1 \times S_2 \times S_3}\right\|_1 \leq 4d_2 d_3 - 3 \quad (A19)$$

for any $S_2, S_3 \geq 1$.

The generalization of (A15), (A19) for $N \geq 4$ is straightforward and gives the source operator

$$\hat{T}^{(\rho)}_{1 \times S_2 \times \cdots \times S_N} = \sum \eta_{mm_1 jj_1 \cdots kk_1} |e_m^{(1)}\rangle \langle e_m^{(1)}| \otimes W_{jj_1}^{(2,S_2)} \otimes \cdots \otimes W_{kk_1}^{(N,S_N)} \quad (A20)$$

with the trace norm

$$\left\|\hat{T}^{(\rho)}_{1 \times S_2 \times \cdots \times S_N}\right\|_1 \leq 2^{N-1}(d_2 \cdot \cdots \cdot d_N - 1) + 1. \quad (A21)$$

### b. Source operator for the singlet

For the two-qubit singlet $\psi_{\text{singlet}} = \frac{1}{\sqrt{2}} (e_1 \otimes e_2 - e_2 \otimes e_1)$, the self-adjoint operator

$$T_{1 \times 2}^{(\rho_{\text{singlet}})} = \frac{1}{2} |e_1\rangle \langle e_1| \otimes |e_2\rangle \langle e_2| + \frac{1}{2} |e_2\rangle \langle e_2| \otimes |e_1\rangle \langle e_1|$$

$$- \frac{1}{2} |e_1\rangle \langle e_2| \otimes |e_2\rangle \langle e_1| + \frac{1}{2} |e_2\rangle \langle e_1| \otimes |e_1\rangle \langle e_2|$$

on $(\mathbb{C}^2)^{\otimes 3}$ represents an $1 \times 2$-setting source operator. Calculating the eigenvalues of this source operator, we derive $\lambda_{1,2,3,4} = 0$, $\lambda_{5,6} = \frac{1 - \sqrt{3}}{4}$ and $\lambda_{7,8} = \frac{1 + \sqrt{3}}{4}$. Hence,

$$\left\|T_{1 \times 2}^{(\rho_{\text{singlet}})}\right\|_1 = \sqrt{3}. \quad (A23)$$
2. Proof of lemma 1

Property 1. The first left-hand side inequality of property (1) is trivial. The last right-hand side inequality is due to the fact that, for each $W \in \mathcal{T}^{(sa)}_{G_1 \otimes \cdots \otimes G_m}$, operator $|W|$ is one of its coverings, so that, by (11) and definition of the trace norm, we have

$$
\|W\|_{cov} \leq \text{tr}[|W|] := \|W\|_1. \tag{A24}
$$

In order to prove

$$
\sup_{X_j = X_j^*, \|X_j\|_1 = 1, \forall j} |\text{tr}[W\{X_1 \otimes \cdots \otimes X_m]\}| \leq \|W\|_{cov}, \tag{A25}
$$

where supremum is taken over all self-adjoint bounded linear operators $X_j$, $j = 1, \ldots, N$, with the operator norm $\|X_j\|_1 = 1$, let us represent operator $W \in \mathcal{T}^{(sa)}_{G_1 \otimes \cdots \otimes G_m}$ via decomposition (7) with an arbitrary a trace class covering $W_{cov}$. We have

$$
\sup_{X_j = X_j^*, \|X_j\|_1 = 1, \forall j} |\text{tr}[W\{X_1 \otimes \cdots \otimes X_m]\}| \leq \frac{1}{2} \sup_{X_j = X_j^*} |\text{tr}[(W_{cov} + W)\{X_1 \otimes \cdots \otimes X_m]\}| + \frac{1}{2} \sup_{X_j = X_j^*} |\text{tr}[(W_{cov} - W)\{X_1 \otimes \cdots \otimes X_m]\}|. \tag{A26}
$$

Applying to each term in (A26) the spectral theorem

$$
X_k = \int \lambda \mathbb{E}_{X_k}(d\lambda), \tag{A27}
$$

where $\mathbb{E}_{X_k}$ is the spectral (projection-valued) measure on $(\mathbb{R}, \mathcal{B}_\mathbb{R})$ for each self-adjoint bounded linear operator $X_k$, $k = 1, \ldots, m$, and taking into the account that $W_{cov} \pm W \succeq 0$ and spectrum $\sigma(X_k) \subseteq [-1, 1]$ for each $k$, we derive

$$
\sup_{X_j = X_j^*} |\text{tr}[(W_{cov} \pm W)\{X_1 \otimes \cdots \otimes X_m]\}] \leq \sup_{X_j = X_j^*} \left| \int \lambda_1 \cdot \ldots \cdot \lambda_m \text{tr} [(W_{cov} \pm W) \{\mathbb{E}_{X_1}(d\lambda_1) \otimes \cdots \otimes \mathbb{E}_{X_m}(d\lambda_m)\}] \right| \leq \text{tr}[W_{cov} \pm W]. \tag{A28}
$$

Substituting (A28) into (A26), we have

$$
\sup_{X_j = X_j^*, \|X_j\|_1 = 1, \forall j} |\text{tr}[W\{X_1 \otimes \cdots \otimes X_m]\}] | \leq \text{tr}[W_{cov}] \tag{A29}
$$

for each trace class covering $W_{cov}$ of an operator $W \in \mathcal{T}^{(sa)}_{G_1 \otimes \cdots \otimes G_m}$. This relation and definition (11) of the covering norm imply inequality (A25).
Property 2. If a self-adjoint trace class operator $W$ is tensor positive, then $\text{tr}[W] \geq 0$ and $W$ is itself one of its possible coverings. Therefore, by (11), $\|W\|_{\text{cov}} \leq \text{tr}[W]$. This and property (1) of lemma 1 prove property (2).

Property 3. Let $W_{\text{red}} \in \mathcal{T}_{G_1 \otimes \cdots \otimes G_{kj}}^{(sa)}$, $1 \leq k_1 < \cdots < k_j \leq m$, $j < m$, be the self-adjoint trace class operator reduced from a self-adjoint trace class operator $W$ on a Hilbert space $G_1 \otimes \cdots \otimes G_m$. The left-hand side inequality in property (3) follows from property (1) and relation $\text{tr}[(W_{\text{cov}})_{\text{red}}] = \text{tr}[W_{\text{cov}}]$. Further, if $(W_{\text{cov}})_{\text{red}} \in \mathcal{T}_{G_{k_1} \otimes \cdots \otimes G_{kj}}^{(sa)}$ is the operator reduced from a trace class covering $W_{\text{cov}}$ of operator $W \in \mathcal{T}_{G_1 \otimes \cdots \otimes G_m}^{(sa)}$, then $(W_{\text{cov}})_{\text{red}}$ is one of trace class coverings of operator $W_{\text{red}} \in \mathcal{T}_{G_{k_1} \otimes \cdots \otimes G_{kj}}^{(sa)}$. Therefore, for each $W \in \mathcal{T}_{G_1 \otimes \cdots \otimes G_m}^{(sa)}$, we have the following inclusion

\[
\left\{(W_{\text{cov}})_{\text{red}} \in \mathcal{T}_{G_{k_1} \otimes \cdots \otimes G_{kj}}^{(sa)} \mid W_{\text{cov}} \in \mathcal{T}_{G_1 \otimes \cdots \otimes G_m}\right\} \subseteq \left\{(W_{\text{red}})_{\text{cov}} \in \mathcal{T}_{G_{k_1} \otimes \cdots \otimes G_{kj}}^{(sa)}\right\}.
\]

In view of definition (11) of the covering norm and relation $\text{tr}[(W_{\text{cov}})_{\text{red}}] = \text{tr}[W_{\text{cov}}]$, inclusion (A30) implies

\[
\|W_{\text{red}}\|_{\text{cov}} = \inf_{(W_{\text{cov}})_{\text{red}} \in \mathcal{T}_{G_{k_1} \otimes \cdots \otimes G_{kj}}^{(sa)}} \text{tr}[(W_{\text{cov}})_{\text{red}}] \leq \inf_{W_{\text{cov}} \in \mathcal{T}_{G_1 \otimes \cdots \otimes G_m}} \text{tr}[(W_{\text{cov}})_{\text{red}}] = \inf_{W_{\text{cov}} \in \mathcal{T}_{G_1 \otimes \cdots \otimes G_m}} \text{tr}[W_{\text{cov}}] = \|W\|_{\text{cov}}
\]

for each $W_{\text{red}}$ reduced from an operator $W \in \mathcal{T}_{G_1 \otimes \cdots \otimes G_m}^{(sa)}$. This proves property (3).

Appendix B: proofs for section 5

In this appendix we prove lemma 2 and find also some upper bounds needed for our presentation in section 5.

1. Proof of lemma 2

For short, we further omit the below indices $\rho$, $M_{S,A}$ at notation $\mathcal{E}_{\rho,M_{S,A}}$ and denote by $\mathcal{M}_\mathcal{E} := \{\mu\}$ the set of all normalized bounded real-valued measures $\mu$, each returning all
distributions \(P_{(s_1,\ldots,s_N)}^{(E)}\) of scenario \(E\) as the corresponding marginals and defined on the direct product space \((\Lambda^S_1 \times \cdots \times \Lambda^S_N, \mathcal{F}^S_{\Lambda_1} \otimes \cdots \otimes \mathcal{F}^S_{\Lambda_N}) := (\Lambda', \mathcal{F}')\).

If scenario \(E\) admits an LHV model, then, by corollary 1, in set \(\mathcal{M}_E\), there is a probability measure. Since the total variation norm of any probability measure is equal to 1, parameter \(\gamma_E = \inf_{\mu \in \mathcal{M}_E} ||\mu||_{\text{var}} = 1\).

Conversely, let \(\gamma_E = 1\). As it is shown in section 2 of this appendix, for a quantum scenario, set \(\mathcal{M}_E\) contains more than one element, so that, in view of convexity of \(\mathcal{M}_E\), this set is infinite. From relation
\[
\gamma_E = \inf\{||\mu||_{\text{var}} \mid \mu \in \mathcal{M}_E\} = 1 \quad (\text{B1})
\]
it follows that, in set \(\mathcal{M}_E\), there is a sequence \(\{\mu_m\}\), for which \(||\mu_m||_{\text{var}} \to 1\) as \(m \to \infty\) and which is bounded in norm \(||\cdot||_{\text{var}}\). Note that, equipped with the total variation norm \(||\cdot||_{\text{var}}\), the linear space \(\mathfrak{F}(\Lambda', \mathcal{F}')\) of all bounded real-valued measures on the measurable space \((\Lambda', \mathcal{F}')\) is Banach. Therefore, there exists a subsequence \(\{\mu_{k_m}\} \subseteq \{\mu_m\}\) and a measure \(\tilde{\mu} \in \mathfrak{F}(\Lambda', \mathcal{F}')\) such that
\[
\lim_{m \to \infty} \int f(\lambda') \mu_{k_m}(d\lambda') = \int f(\lambda') \tilde{\mu}(d\lambda'), \quad ||\tilde{\mu}||_{\text{var}} \leq 1, \quad (\text{B2})
\]
for all Borel measurable bounded real-valued functions \(f\) on \((\Lambda', \mathcal{F}')\). Denote by \(\tilde{P}_{(s_1,\ldots,s_N)}(d\lambda^{(s_1)}_1 \times \cdots \times d\lambda^{(s_N)}_N)\) the corresponding marginal of measure \(\tilde{\mu}(d\lambda^{(1)}_1 \times \cdots \times d\lambda^{(s_1)}_1 \times \cdots \times d\lambda^{(1)}_N \times \cdots \times d\lambda^{(s_N)}_N)\). Specifying \(\text{(B2)}\) with the indicator function \(\chi_F((\lambda^{(s_1)}_1, \ldots, \lambda^{(s_N)}_N))\), \(F \in \mathcal{F}_1 \otimes \cdots \otimes \mathcal{F}_N\), and taking into the account that \(\mu_{k_m} \in \mathcal{M}_E, \forall k_m\), we have
\[
P_{(s_1,\ldots,s_N)}^{(E)}(F) = \tilde{P}_{(s_1,\ldots,s_N)}(F) \quad (\text{B3})
\]
for all sets \(F \in \mathcal{F}_1 \otimes \cdots \otimes \mathcal{F}_N\) and all tuples \((s_1, \ldots, s_N)\). Thus, \(\tilde{P}_{(s_1,\ldots,s_N)} = P_{(s_1,\ldots,s_N)}^{(E)}\) for all joint measurements \((s_1, \ldots, s_N)\) of scenario \(E\). The latter means that the bounded real-valued measure \(\tilde{\mu}\) is normalized and returns all probability distributions \(P_{(s_1,\ldots,s_N)}^{(E)}\) of scenario \(E\) as the corresponding marginals. Hence, \(\tilde{\mu}\) belongs to set \(\mathcal{M}_E\). In view of relation \(\text{(20)}\) and the second relation in \(\text{(B2)}\), the normalized bounded real-valued measure \(\tilde{\mu} \in \mathcal{M}_E\) is a probability measure. By corollary 1, this proves lemma 2.

2. Some upper bounds

For the evaluation in proposition 4 of the state parameters \(\text{(12)}, \text{(13)}\), in addition to measure \(\text{(40)}\), let us also consider another example of possible measures \(\mu_{\mathcal{E},\rho,M_{S,\Lambda}}\) in \(\text{(11)}\).
Let $T_{1 \times S_2 \times \cdots \times S_N}^{(\rho)}$ be an $1 \times S_2 \times \cdots \times S_N$-setting source operator for state $\rho$ on $\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_N$ (see definition 1 in section 2). Introduce the following collection of normalized bounded real-valued measures

$$\text{tr}[T_{1 \times S_2 \times \cdots \times S_N}^{(\rho)} \{M_1^{(s_1)}(d\lambda_1^{(s_1)}) \otimes M_2^{(1)}(d\lambda_2^{(1)}) \otimes \cdots \otimes M_2^{(S_2)}(d\lambda_2^{(S_2)}) \} \otimes \cdots \otimes M_N^{(S_N)}(d\lambda_N^{(S_N)})] \quad (B4)$$

where each $s_1$-th measure returns as the corresponding marginals all joint distributions

$$P_{(s_1, \ldots, s_N)}^{(\mathcal{E}_{\rho,M_{S,A}})}$$

of scenario $\mathcal{E}_{\rho,M_{S,A}}$ with measurement $s_1$ at site $n = 1$.

For an arbitrary trace class covering $(T_{1 \times S_2 \times \cdots \times S_N}^{(\rho)})_{\text{cov}}$ of a source operator $T_{1 \times S_2 \times \cdots \times S_N}^{(\rho)}$ (see definition 3 in section 2), decomposition (7) implies the following representation

$$\frac{1}{2} \text{tr} \left[ \tau^+ \{ M_1^{(s_1)}(\cdot) \otimes M_2^{(1)}(\cdot) \otimes \cdots \} \right] - \frac{1}{2} \text{tr} \left[ \tau^- \{ M_1^{(s_1)}(\cdot) \otimes M_2^{(1)}(\cdot) \otimes \cdots \} \right] \quad (B5)$$

of each $s_1$-th measure (B4) via two positive real-valued measures, where

$$\tau^{\pm} = (T_{1 \times S_2 \times \cdots \times S_N}^{(\rho)})_{\text{cov}} \pm T_{1 \times S_2 \times \cdots \times S_N}^{(\rho)} \geq 0 \quad (B6)$$

are tensor positive trace class operators.

As it is discussed in the proof of theorem 2 in Ref.\textsuperscript{10}, for a positive measure $\nu$ on a direct product space $(\Lambda_1, \mathcal{F}_{\Lambda_1}) \times (\Lambda_2, \mathcal{F}_{\Lambda_2})$, each positive measure $\nu(B_1 \times \cdot)$, $B_1 \in \mathcal{F}_{\Lambda_1}$, on $(\Lambda_2, \mathcal{F}_{\Lambda_2})$ is absolutely continuous\textsuperscript{39} with respect to the marginal measure $\nu(\Lambda_1 \times \cdot)$ on $(\Lambda_2, \mathcal{F}_{\Lambda_2})$. Hence, for each of positive measures in decomposition (B5), the Radon-Nykodim theorem\textsuperscript{35} implies representation

$$\text{tr} \left[ \tau^{\pm} \{ M_1^{(s_1)}(d\lambda_1^{(s_1)}) \otimes M_2^{(1)}(d\lambda_2^{(1)}) \otimes \cdots \} \right] = \alpha^{(\pm)}_{s_1}(d\lambda_1^{(s_1)} \mid \lambda_2^{(1)}, \ldots) \text{tr} \left[ \tau^{\pm} \{ \mathbb{I}_{\mathcal{H}_1} \otimes M_2^{(1)}(d\lambda_2^{(1)}) \otimes \cdots \} \right] \quad (B7)$$

via conditional probability measure $\alpha^{(\pm)}_{s_1}(\cdot \mid \lambda_2^{(1)}, \ldots)$ and marginal

$$\text{tr} \left[ \tau^{\pm} \{ \mathbb{I}_{\mathcal{H}_1} \otimes M_2^{(1)}(d\lambda_2^{(1)}) \otimes \cdots \} \right]. \quad (B8)$$

From (B4) - (B7) it follows that the normalized bounded real-valued measure

$$\mu_{T_{1 \times S_2 \times \cdots \times S_N}^{(\rho)}(\mathcal{E}_{\rho,M_{S,A}})}^{(\rho,M_{S,A})} \{ (d\lambda_1^{(1)} \times \cdots \times d\lambda_1^{(S_1)} \times \cdots \times d\lambda_N^{(1)} \times \cdots \times d\lambda_N^{(S_N)}) \} \quad (B9)$$

is defined by

$$\frac{1}{2} \left\{ \prod_{s_1=1}^{S_1} \alpha^{(+)}_{s_1}(d\lambda_1^{(s_1)} \mid \lambda_2^{(1)}, \ldots) \right\} \text{tr} \left[ \tau^+ \{ \mathbb{I}_{\mathcal{H}_1} \otimes M_2^{(1)}(d\lambda_2^{(1)}) \otimes \cdots \} \right]$$

and

$$\frac{1}{2} \left\{ \prod_{s_1=1}^{S_1} \alpha^{(-)}_{s_1}(d\lambda_1^{(s_1)} \mid \lambda_2^{(1)}, \ldots) \right\} \text{tr} \left[ \tau^- \{ \mathbb{I}_{\mathcal{H}_1} \otimes M_2^{(1)}(d\lambda_2^{(1)}) \otimes \cdots \} \right]$$
Lemma 4. For arbitrary source operators $T_{S_1 \times \cdots \times S_N}^{(\rho)}$, $T_{1 \times S_2 \times \cdots \times S_N}^{(\rho)}$ for state $\rho$, the total variation norms of measures \((B9), (39)\) satisfy relations

\[
1 \leq \left\| \mu_{T_{S_1 \times \cdots \times S_N}^{(\rho)}}^{(\rho) M_{S, \Lambda}} \right\|_{\text{var}} \leq \left\| T_{S_1 \times \cdots \times S_N}^{(\rho)} \right\|_{\text{cov}} \tag{B10}
\]

and

\[
1 \leq \inf_{(T_{1 \times S_2 \times \cdots \times S_N})_{\text{cov}}} \left\| \mu_{T_{S_1 \times \cdots \times S_N}^{(\rho)}}^{(\rho) M_{S, \Lambda}} \right\|_{\text{var}} \leq \left\| T_{1 \times S_2 \times \cdots \times S_N}^{(\rho)} \right\|_{\text{cov}} \tag{B11}
\]

for every collection $M_{S, \Lambda}$ of POV measures on $\Lambda$ and an arbitrary outcome set $\Lambda$. Here, infimum is taken over all trace class coverings $(T_{1 \times S_2 \times \cdots \times S_N}^{(\rho)})_{\text{cov}}$ of a source operator $T_{1 \times S_2 \times \cdots \times S_N}^{(\rho)}$ and $\| \cdot \|_{\text{cov}}$ is the covering norm (see definition 4 in section 2).

Proof. In view of \((7)\), consider the decomposition of a source operator

\[
T_{S_1 \times \cdots \times S_N}^{(\rho)} = \frac{1}{2} \left\{ (T_{S_1 \times \cdots \times S_N}^{(\rho)})_{\text{cov}} + T_{S_1 \times \cdots \times S_N}^{(\rho)} \right\} - \frac{1}{2} \left\{ (T_{S_1 \times \cdots \times S_N}^{(\rho)})_{\text{cov}} - T_{S_1 \times \cdots \times S_N}^{(\rho)} \right\} \tag{B12}
\]

via tensor positive operators $(T_{S_1 \times \cdots \times S_N}^{(\rho)})_{\text{cov}} \pm T_{S_1 \times \cdots \times S_N}^{(\rho)}$, where $(T_{S_1 \times \cdots \times S_N}^{(\rho)})_{\text{cov}}$ is an arbitrary trace class covering of $T_{S_1 \times \cdots \times S_N}^{(\rho)}$, see definition 3 of section 2.

Substituting \((B12)\) into \((40)\), we represent measure $\mu_{T_{S_1 \times \cdots \times S_N}^{(\rho)}}^{(\rho) M_{S, \Lambda}}$ as the difference of two positive measures and, for each of these measures, the total variation \((18)\) is upper bounded by

\[
\frac{1}{2} \text{tr} \left[ \left( (T_{S_1 \times \cdots \times S_N}^{(\rho)})_{\text{cov}} \pm T_{S_1 \times \cdots \times S_N}^{(\rho)} \right) \right]. \tag{B13}
\]

Therefore, for the total variation \((18)\) of the normalized measure $\mu_{T_{S_1 \times \cdots \times S_N}^{(\rho)}}^{(\rho) M_{S, \Lambda}}$, we have

\[
1 \leq \left\| \mu_{T_{S_1 \times \cdots \times S_N}^{(\rho)}}^{(\rho) M_{S, \Lambda}} \right\|_{\text{var}} \leq \text{tr} \left[ \left( T_{S_1 \times \cdots \times S_N}^{(\rho)} \right)_{\text{cov}} \right] \tag{B14}
\]

for every trace class covering $(T_{S_1 \times \cdots \times S_N}^{(\rho)})_{\text{cov}}$ of a source operator $T_{S_1 \times \cdots \times S_N}^{(\rho)}$. Relations \((B14), (B11)\) imply bound \((B10)\).

Quite similarly, for measure \((B9)\),

\[
1 \leq \left\| \mu_{T_{1 \times S_2 \times \cdots \times S_N}^{(\rho)}}^{(\rho) M_{S, \Lambda}} \right\|_{\text{var}} \leq \text{tr} \left[ \left( T_{1 \times S_2 \times \cdots \times S_N}^{(\rho)} \right)_{\text{cov}} \right] \tag{B15}
\]

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for each trace class covering \((T^{(\rho)}_{1 \times S_2 \times \ldots \times S_N})_{\text{cov}}\) of a source operator \(T^{(\rho)}_{1 \times S_2 \times \ldots \times S_N}\). From (B15), (B11) we derive

\[
1 \leq \inf_{(T^{(\rho)}_{1 \times S_2 \times \ldots \times S_N})_{\text{cov}}} \| \mu^{(\rho, M_{S, A})}_{T^{(\rho)}_{1 \times S_2 \times \ldots \times S_N}}(T^{(\rho)}_{1 \times S_2 \times \ldots \times S_N})_{\text{cov}} \|_{\text{cov}} \leq \| T^{(\rho)}_{1 \times S_2 \times \ldots \times S_N} \|_{\text{cov}},
\]

representing relation (B11). ■

Generalizing measure (B9) to the case of a source operator \(T^{(\rho)}_{S_1 \times \ldots \times S_n}\) with \(S_n = 1\) at an arbitrary \(n\)-th site, similarly to bound (B11), we have

\[
1 \leq \inf_{(T^{(\rho)}_{S_1 \times \ldots \times S_n})_{\text{cov}}} \| \mu^{(\rho, M_{S, A})}_{T^{(\rho)}_{S_1 \times \ldots \times S_n}}(T^{(\rho)}_{S_1 \times \ldots \times S_n})_{\text{cov}} \|_{\text{cov}} \leq \| T^{(\rho)}_{S_1 \times \ldots \times S_n} \|_{\text{cov}},
\]

(B17)

Bounds (B13), (B17) allow us to evaluate the scenario parameter \(\gamma_{E_{\rho, M_{S, A}}}\) defined by relation (B11).

**Lemma 5** For each \(S_1 \times \cdots \times S_N\)-setting correlation scenario \(E_{\rho, M_{S, A}}\) on a state \(\rho\) on a Hilbert space \(\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_N\),

\[
\gamma_{E_{\rho, M_{S, A}}} \leq \inf_{T^{(\rho)}_{S_1 \times \cdots \times S_n} \forall n} \| T^{(\rho)}_{S_1 \times \cdots \times S_n} \|_{\text{cov}},
\]

(B18)

**Proof.** From (B11), (B10), (B17), it follows

\[
\gamma_{E_{\rho, M_{S, A}}} \leq \inf \left\{ \| T^{(\rho)}_{S_1 \times \cdots \times S_n} \|_{\text{cov}} \text{,} \| T^{(\rho)}_{S_1 \times \cdots \times S_n} \|_{\text{cov}} \text{,} n = 1, \ldots, N \right\},
\]

(B19)

where infimum is taken over all source operators \(T^{(\rho)}_{S_1 \times \cdots \times S_n}\) and over all \(n = 1, \ldots, N\). For each \(n\), the set \(\{T^{(\rho)}_{S_1 \times \cdots \times S_n}\}\) of all source operators \(T^{(\rho)}_{S_1 \times \cdots \times S_n}\) for state \(\rho\) includes the set of all source operators on \(\mathcal{H}_{1}^{(\otimes S_1)} \otimes \cdots \mathcal{H}_{n}^{(\otimes S_n)} \otimes \cdots \mathcal{H}_{N}^{(\otimes S_N)}\), each reduced from some \(T^{(\rho)}_{S_1 \times \cdots \times S_n}\), as a particular subset. This inclusion and relation (B14) imply

\[
\inf_{T^{(\rho)}_{S_1 \times \cdots \times S_n}} \| T^{(\rho)}_{S_1 \times \cdots \times S_n} \|_{\text{cov}} \leq \inf_{T^{(\rho)}_{S_1 \times \cdots \times S_n}} \| T^{(\rho)}_{S_1 \times \cdots \times S_n} \|_{\text{cov}}
\]

for each \(n\). Taking this into the account in (B19), we prove (B18). ■
Appendix C: proof of lemma 3 in section 6

Constraints (48) imply

$$\sup_{\Psi \in \Psi_{S,A}} \left| \frac{1}{B_{\Psi,S,A}} \sum_{s_1, \ldots, s_N} \left( \sum_{\{s_1, \ldots, s_N\}} \langle \psi_{s_1, \ldots, s_N}, \pi_{1, \ldots, n} \rangle \right) \right| \leq \gamma_{\varepsilon_{S,A}},$$

where supremum is taken over all non-trivial (i.e. $B_{\Psi,S,A} \neq 0$) function collections $\Psi_{S,A}$. For short of notation, we further replace $E_{\rho,M_S} \rightarrow E_{S,A}$. In order to prove

$$\sup_{\Psi \in \Psi_{S,A}} \left| \frac{1}{B_{\Psi,S,A}} \sum_{s_1, \ldots, s_N} \left( \sum_{\{s_1, \ldots, s_N\}} \langle \psi_{s_1, \ldots, s_N}, \pi_{1, \ldots, n} \rangle \right) \right| = \gamma_{\varepsilon_{S,A}},$$

we note that, by introducing variables $\xi = \mu_{\varepsilon_{S,A}}(\Omega) \geq 0$, the parameter $\gamma_{\varepsilon_{S,A}}$, defined by (41), can be otherwise expressed as

$$\gamma_{\varepsilon_{S,A}} = \inf \{ \xi^+ + \xi^- | \xi^+ \geq 0, \xi^+ - \xi^- = 1, \exists E_{\rho,M_S} : P_{(s_1, \ldots, s_n)} - \xi^- P_{(s_1, \ldots, s_n)} \}, \forall s_n, \forall n \}.$$ 

As it is specified in section 3, we consider only standard measurable spaces. In this case, $(\Lambda_n, F_{\Lambda_n})$ is Borel isomorphic to some measurable space $(\mathcal{X}_n, \mathcal{B}_{\mathcal{X}_n})$, where $\mathcal{X}_n \in \mathcal{B}_{\mathcal{R}}$ is a Borel subset of $\mathbb{R}$ and $\mathcal{B}_{\mathcal{X}_n} := \mathcal{B}_{\mathcal{R}} \cap \mathcal{X}_n$ is the trace on $\mathcal{X}_n$ of the Borel $\sigma$-algebra $\mathcal{B}_{\mathcal{R}}$ on $\mathbb{R}$. We have two major cases.

(a) **Discrete case.** Let, for a correlation scenario $E_{S,A}$, each outcome set be finite: $\Lambda_n = \{\lambda_n^{(k_n)}, k_n = 1, \ldots, K_n < \infty\}$. Then

$$\sum_{s_1, \ldots, s_N} \langle \psi_{s_1, \ldots, s_N}, \pi_{1, \ldots, n} \rangle = \sum_{s_1, \ldots, s_N, k_1, \ldots, k_N} \beta_{(s_1, \ldots, s_N)}^{(k_1, \ldots, k_N)} P_{(s_1, \ldots, s_n)} \left( \{\lambda_n^{(k_1)}\} \times \cdots \times \{\lambda_n^{(k_N)}\} \right),$$

where $\beta_{(s_1, \ldots, s_N)}^{(k_1, \ldots, k_N)}$ are real numbers. Hence, in (C1), supremum over $\Psi_{S,A}$ reduces to supremum over families $\{\beta_{(s_1, \ldots, s_N)}^{(k_1, \ldots, k_N)}\}$ of real numbers and equality (C2) follows from (C1) and (C3) by the linear programming (LP) duality. This proof is similar to the proof of theorem 17 in Ref.20 for a bipartite case with a finite number of outcomes at each site.

Let now, for a correlation scenario $E_{S,A}$, every outcome set $\Lambda_n$ be infinite but countable: $\Lambda_n = \{\lambda_n^{(k_n)}, k_n = 1, \ldots, K_n, \ldots\}$. Consider collections $\Psi_{S,A}^B_{(\beta_{(k_n, K)})}$, of bounded measurable real-
valued functions

\[ \psi^{(\beta K, K)}_{s_1, \ldots, s_N}(\lambda_1, \ldots, \lambda_N) := \sum_{k_1, \ldots, k_N} \beta^{(k_1, \ldots, k_N)}(s_1, \ldots, s_N) \chi^{(k_1)}(\lambda_1) \cdots \chi^{(k_N)}(\lambda_N), \]  

specified via tuples \( K := (K_1, \ldots, K_N) \) of positive integers \( K_1, \ldots, K_N < \infty \) and families \( \beta_K := \{\beta^{(k_1, \ldots, k_N)}(s_1, \ldots, s_N) \in \mathbb{R}, s_n = 1, \ldots, S_n, k_n = 1, \ldots, K_n \} \) of real numbers. For each of these function collections, the expression

\[ \sum_{s_1, \ldots, s_N} \left( \frac{1}{B_{\Psi_{s,A}}} \sum_{k_1, \ldots, k_N} \beta^{(k_1, \ldots, k_N)}(s_1, \ldots, s_N) \phi_{(s_1, \ldots, s_N)}(\lambda_1^{(k_1)} \times \cdots \times \lambda_N^{(k_N)}) \right)_{\xi, \Lambda} \]

is similar by its form to (C4), so that relation (C1) and the proof in the above case imply

\[ \gamma_{\xi, \Lambda} \geq \sup_{\xi, \Lambda, \xi, \Lambda \neq 0} \left| \frac{1}{B_{\Psi_{s,A}}} \sum_{s_1, \ldots, s_N} \left( \psi^{(\beta K, K)}_{s_1, \ldots, s_N}(\lambda_1, \ldots, \lambda_N) \right)_{\xi, \Lambda} \right| \]

\[ \geq \sup_{\beta_K, K} \left| \frac{1}{B_{\Psi_{s,A}}} \sum_{s_1, \ldots, s_N} \left( \psi^{(\beta K, K)}_{s_1, \ldots, s_N}(\lambda_1, \ldots, \lambda_N) \right)_{\xi, \Lambda} \right| \]

\[ = \sup_K \gamma^{(K)}_{\xi, \Lambda}, \]

where, for each \( K = (K_1, \ldots, K_N) \),

\[ \gamma^{(K)}_{\xi, \Lambda} := \inf \{ \xi^+ + \xi^- | \xi^+ \geq 0, \xi^+ - \xi^- = 1, \exists \xi, \Lambda : \bar{\xi, \Lambda} : \]

\[ P_{(s_1, \ldots, s_N)}^{(\xi, \Lambda)}(\lambda_1^{(k_1)} \times \cdots \times \lambda_N^{(k_N)}) = \xi^+ P_{(s_1, \ldots, s_N)}^{(\xi, \Lambda)}(\lambda_1^{(k_1)} \times \cdots \times \lambda_N^{(k_N)}) \]

\[ - \xi^- P_{(s_1, \ldots, s_N)}^{(\xi, \Lambda)}(\lambda_1^{(k_1)} \times \cdots \times \lambda_N^{(k_N)}) \}, \]

\[ k_1 = 1, \ldots, K_1, s_1 = 1, \ldots, S_1, \ldots, k_N = 1, \ldots, K_N, s_N = 1, \ldots, S_N \} \].

From (C3), (C8) it follows that \( \gamma^{(K)}_{\xi, \Lambda} \leq \gamma^{(L)}_{\xi, \Lambda} \leq \gamma_{\xi, \Lambda} \), if \( K_1 \leq L_1, \ldots, K_N \leq L_N \), and

\[ \lim_{K_1, \ldots, K_N \to \infty} \gamma^{(K)}_{\xi, \Lambda} = \gamma_{\xi, \Lambda}. \]

Hence, \( \sup_K \gamma^{(K)}_{\xi, \Lambda} = \gamma_{\xi, \Lambda} \). Taking this into the account in (C7), we derive

\[ \gamma_{\xi, \Lambda} \geq \sup_{\xi, \Lambda, \xi, \Lambda \neq 0} \left| \frac{1}{B_{\Psi_{s,A}}} \sum_{s_1, \ldots, s_N} \left( \psi^{(\beta K, K)}_{s_1, \ldots, s_N}(\lambda_1, \ldots, \lambda_N) \right)_{\xi, \Lambda} \right| \]

\[ \geq \gamma_{\xi, \Lambda}. \]

This proves equality (C2) if set \( \Lambda_0 \) is infinite and countable.
(b) **Continuous case.** Let, for a correlation scenario $\mathcal{E}_{S,\Lambda}$, each outcome set $\Lambda_n$ be infinite and uncountable. For positive integers $K_n \geq 1$, introduce partitions

$$D_{K_n} = \{ D_{K_n}^{(k_n)} \in \mathcal{F}_{\Lambda_n} \mid D_{K_n}^{(k_n)} \neq \emptyset, \quad D_{K_n}^{(k_n)} \cap D_{K_n}^{(k'_n)} = \emptyset, \quad \forall k_n \neq k'_n; \quad (C10) \}$$

$$\cup_k D_{K_n}^{(k_n)} = \Lambda_n, \quad k_n = 1, \ldots, 2^{K_n} \}$$

of each set $\Lambda_n$, such that

$$D_{K_n+1}^{(2k_n-1)} \cup D_{K_n+1}^{(2k_n)} = D_{K_n}^{(k_n)}; \quad k_n = 1, \ldots, 2^{K_n}, \quad \forall K_n \in \mathbb{N}. \quad (C11)$$

For some partitions $D_{K_1}, \ldots, D_{K_N}$, of sets $\Lambda_1, \ldots, \Lambda_N$, respectively, consider collections $\tilde{\Psi}^{(\beta, K)}_{S,\Lambda}$ of real-valued measurable functions

$$\tilde{\Psi}^{(\beta, K)}_{(s_1, \ldots, s_N)}(\lambda_1, \ldots, \lambda_N) := \sum_{k_1, \ldots, k_N} \beta^{(k_1, \ldots, k_N)}(E_k) \chi_{D_k}^{(k_n)}(\lambda_1) \cdots \chi_{D_k}^{(k_N)}(\lambda_N), \quad (C12)$$

specified by tuples $K := (K_1, \ldots, K_N)$ and collections $\beta_K := \{ \beta^{(k_1, \ldots, k_N)}(s_1, \ldots, s_N) \in \mathbb{R}, s_n = 1, \ldots, S_n, k_n = 1, \ldots, 2^{K_n} \}$ of real numbers. Similarly to the derivation of $(C7)$, we have:

$$\gamma^{(K)}_{S,\Lambda} \geq \sup_{\psi_{S,\Lambda} \in \psi_{S,\Lambda} \neq 0} \left| \frac{1}{B_{\psi_{S,\Lambda}}} \sum_{s_1, \ldots, s_N} \left\langle \tilde{\psi}_{(s_1, \ldots, s_N)}(\lambda_1, \ldots, \lambda_N) \right\rangle_{\mathcal{E}_{S,\Lambda}} \right| \geq \sup_{\beta_K, K} \left| \frac{1}{B_{\psi_{(\beta, K)}^{(\beta, K)}}} \sum_{s_1, \ldots, s_N} \left\langle \tilde{\psi}_{(s_1, \ldots, s_N)}(\lambda_1, \ldots, \lambda_N) \right\rangle_{\mathcal{E}_{S,\Lambda}} \right| = \sup_K \gamma^{(K)}_{S,\Lambda}, \quad (C13)$$

where

$$\gamma^{(K)}_{S,\Lambda} := \inf \{ \xi^+ + \xi^- \mid \xi^+ \geq 0, \quad \xi^+ - \xi^- = 1, \quad \exists \xi_{S,\Lambda}, \bar{\xi}_{S,\Lambda} : \quad (C14) \}$$

$$P^{(\xi_{S,\Lambda})}_{(s_1, \ldots, s_N)} \left( D_K^{(k_1)} \times \cdots \times D_K^{(k_N)} \right) = \xi^+ P^{(\xi_{S,\Lambda})}_{(s_1, \ldots, s_n)} \left( D_K^{(k_1)} \times \cdots \times D_K^{(k_N)} \right) - \xi^- P^{(\bar{\xi}_{S,\Lambda})}_{(s_1, \ldots, s_n)} \left( D_K^{(k_1)} \times \cdots \times D_K^{(k_N)} \right), \quad k_n = 1, \ldots, 2^{K_n}, \quad s_n = 1, \ldots, S_n \}, \quad \text{for each } K = (K_1, \ldots, K_N), \quad K_n \in \mathbb{N}. \text{ From (C3), (C4) and the special construction (C11) of partitions } D_K, \text{ it follows that } \gamma^{(K)}_{S,\Lambda} \leq \gamma^{(L)}_{S,\Lambda} \leq \gamma^{(L)}_{S,\Lambda}; \quad \text{if } K_1 \leq L_1, \ldots, K_N \leq L_N, \text{ and } \lim_{K_1, \ldots, K_N \to \infty} \gamma^{(K)}_{S,\Lambda} = \gamma_{S,\Lambda}. \text{ Therefore, } \sup_K \gamma^{(K)}_{S,\Lambda} = \gamma_{S,\Lambda}. \text{ Substituting this into (C13), we prove equality (C2) in case of uncountable sets } \Lambda_n. \text{ Coming back to notation } \mathcal{E}_{S,\Lambda} \to \mathcal{E}_{\rho, M_{S,\Lambda}} \text{ and taking supremum of the left-hand and the right-hand sides of (C2) over all collections } M_{S,\Lambda} \text{ of POV measures, we prove relation (52).} \quad 43
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31. We shall consider only complex separable Hilbert spaces and henceforth suppress terms “complex” and “separable” with respect to a Hilbert space.

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34. Symmetric ($S_1$, $S_2$) quasi-extensions introduced for a bipartite state in Ref.12 correspond to a particular type of tensor positive $S_1 \times S_2$-setting source operators.

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36. Throughout this paper, for a measure $\mu$ on the direct product of measurable spaces, we often use notation $\mu(d\lambda \times \cdots \times d\lambda')$ outside of an integral since this allows us: (i) to specify visually the structure of different marginals of measure $\mu$; (ii) to facilitate the description of measures constructed via tensor products of POV measures and the computation of their total variation norms.

37. On the general form of conditional LHV constraints, see Ref.17.

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