MEASURES AND INTEGRALS IN CONDITIONAL SET THEORY

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Abstract. The aim of this article is to establish basic results in a conditional measure theory. The results are applied to prove that arbitrary kernels and conditional distributions are represented by measures in a conditional set theory. In particular, this extends the usual representation results for separable spaces.

1. Introduction

A random variable $\xi$ is a measurable function from a probability space $(\Omega, \mathcal{F}, P)$ into a measurable space $(E, \mathcal{E})$. Given a sub-$\sigma$-algebra $\mathcal{G} \subseteq \mathcal{F}$, the conditional distribution of $\xi$ given $\mathcal{G}$, i.e. the quantity $E[1_{\{\xi \in A\}} | \mathcal{G}](\omega)$ for $A \in \mathcal{E}$ and $\omega \in \Omega$, where $E[\cdot | \mathcal{G}]$ denotes conditional expectation, can be computed via a probability kernel $\kappa: \Omega \times E \to [0,1]$ whenever $(E, \mathcal{E})$ is a standard Borel space. In particular, one has

$$E[f(\xi)|\mathcal{G}](\omega) = \int f(x)\kappa(\omega, dx) \quad \text{almost surely},$$

for every measurable function $f: E \to \mathbb{R}$ such that $f(\xi)$ is integrable. By a conditional measure theory, one can extend the previous representation results to random variables with values in an arbitrary measurable space. More precisely, we will show that a conditional distribution corresponds to a real-valued measure in conditional set theory and that $E[f(\xi)|\mathcal{G}]$ can be computed with the help of a conditional Lebesgue integral. Conditional measure theory suggests a natural formalism to study kernels and conditional distributions in a purely measure-theoretic context without unnecessary topological assumptions such as separability.

We will differ from the abstract setting in [9], where conditional set theory is developed relative to an arbitrary complete Boolean algebra. Motivated by the aforementioned applications, we construct a conditional measure theory relative to the complete Boolean algebra obtained from a probability space by quotienting out null sets. In accordance with this aim, we restrict attention to a class of conditional sets which are associated to spaces of random variables stable under countable concatenations. By a straightforward generalization, one can extend all results in this article to the abstract setting in [9].

Our contribution. We establish basic results in conditional measure theory, e.g. a $\pi$-$\lambda$-theorem, a Carathéodory extension theorem, construction of a Lebesgue measure and a Lebesgue integral, see Section 3. These results are applied to connect kernels and conditional distributions with measures in a conditional theory in Section 4. A conditional version of some standard theorems in measure theory is proved in Section 5.

Related literature. A conditional set is an abstract set-like structure, see [9] for a thorough introduction. Conditional set theory is closely related to Boolean-valued models and topos theory, see [15]. Basic results for Riemann integration in a Boolean-valued model are studied in [24, Chapter 2]. See [6, 11, 12, 15, 20, 23] for further results in conditional analysis and conditional set theory. For applications of conditional analysis, we refer to [2, 3, 5, 7, 10, 13, 17, 19]. See [4, 16] for other related results.

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2. Preliminaries

Throughout, we fix a complete probability space \((\Omega, \mathcal{F}, P)\), and identify \(A, B \in \mathcal{F}\) whenever \(P(A \Delta B) = 0\), where \(\Delta\) denotes symmetric difference. In particular, \(A \subseteq B\) is understood in the almost sure (a.s.) sense. Let \(\mathcal{F}_+\) denote the set of all \(A \in \mathcal{F}\) with \(P(A) > 0\). For a function \(x\) on \(\Omega\) with values in an arbitrary set \(E\) and \(A \in \mathcal{F}\), we denote by \(x[A]: A \rightarrow E\) the restriction of \(x\) to \(A\). If \(x, y: \Omega \rightarrow E\) are functions and \(A, B \in \mathcal{F}\), then we always identify \(x[A] \sim y[B]\) whenever \(P(A \Delta B) = 0\) and \(x(\omega) = y(\omega)\) for a.a. \(\omega \in A \cap B\). A partition is a countable family \((A_k)\) of measurable subsets of \(\Omega\) which is pairwise disjoint and such that \(\bigcup_k A_k = \Omega\). Given a sequence of functions \(x_k: \Omega \rightarrow E\) and a partition \((A_k)\), we denote by \(\sum x_k | A_k\) the unique function \(x: \Omega \rightarrow E\) with the property that \(x[A_k] = x_k | A_k\) for all \(k\).

For a set \(X\) of functions \(x: \Omega \rightarrow E\) and \(A, B \in \mathcal{F}\), let \(X[A] := \{x[A]: x \in X\}\), and define \((X[A])B := X[A \cap B]\) for all \(B \in \mathcal{F}\). Notice that \(X[\emptyset] = \{\ast\}\) is a singleton and trivially \(X[\Omega] = \Omega\).

Denote by \(L^0 = L^0(\Omega, \mathcal{F}, P)\) the set of all measurable functions \(x: \Omega \rightarrow [-\infty, \infty]\), and by \(L^0\) its subset of all real-valued functions. On \(L^0\) consider the complete order \(x \leq y\) a.s. Recall that on \(L^0\) this order is Dedekind complete. Let \(L^0_\pm := \{x \in L^0: x \geq 0\}\), \(L^0_+ := \{x \in L^0: x > 0\}\) and \(L^0_{++} := \{x \in L^0: x > 0\}\). For an arbitrary subset \(\mathcal{G} \subseteq \mathcal{F}\), its supremum w.r.t. almost sure inclusion is the measurable set \(A \in \mathcal{F}\) (unique mod a.s. identification) such that \(1_A = \sup\{1_{A'}: A' \in \mathcal{G}\}\). Similarly, we define the infimum w.r.t. almost sure inclusion is the measurable set \(\inf\{A': A' \in \mathcal{G}\}\), respectively. Throughout, all order relations between extended real-valued random variables are understood in the a.s. sense.

**Definition 2.1.** A set \(X\) of functions on \(\Omega\) with values in a fixed image space \(E\) is said to be stable under countable concatenations, or stable for short, if \(X \neq \emptyset\) and \(\sum x_k | A_k \in X\) for every partition \((A_k)\) and each sequence \((x_k)\) in \(X\).

The spaces \(\overline{L}^0_\pm, L^0_\pm, \overline{L}^0_{++}\) and \(L^0_{++}\) are stable sets of functions on \(\Omega\). We provide more examples which are important in the sequel.

**Examples 2.2.**

1) Let \(E\) be a nonempty set, \((x_k)\) a sequence in \(E\) and \((A_k)\) a partition. Let \(\sum x_k | A_k\) denote the function on \(\Omega\) with value \(x_k\) on \(A_k\) for each \(k\) and call it a step function with values in \(E\).

The set of all step functions with values in \(E\) is a stable set of functions on \(\Omega\) and is denoted by \(L^0(E)\). In particular, \(L^0_0(\mathbb{N})\) denotes the set of all step functions with values in the natural numbers.

2) Let \((E, \mathcal{E})\) be an arbitrary measurable space. The set of all measurable functions on \(\Omega\) with values in \(E\) is stable and is denoted by \(L^0(E)\).

3) If \(E\) is a topological space and \(\mathcal{E}\) its Borel \(\sigma\)-algebra. The set of all strongly measurable\(^1\) functions on \(\Omega\) with values in \(E\) is stable and is denoted by \(L^0_m(E)\).

4) Every nonempty order-bounded subset of \(L^p\) for \(0 \leq p \leq \infty\) is stable.

5) Given a nonempty subset \(V\) of a stable set of functions on \(\Omega\), let
\[
st(V) := \left\{ \sum x_k | A_k: (A_k) \text{ partition}, (x_k) \text{ sequence in } V \right\}.
\]

We call \(st(V)\) the stable hull of \(V\).

6) Let \(X\) be a stable set of functions on \(\Omega\), \((V_k)\) a sequence of stable subsets of \(X\) and \((A_k)\) a partition. Then
\[
\sum V_k | A_k := \left\{ \sum x_k | A_k: x_k \in V_k \text{ for each } k \right\}
\]
(2.1)
is a stable subset of \(X\).

7) Let \(X_i\) be a stable set of functions on \(\Omega\) with image spaces \(E_i\) for each \(i\) in an index. Their Cartesian product
\[
\prod X_i := \left\{ x: \Omega \rightarrow \prod E_i: x(\omega) = (x_i(\omega)), x_i \in X_i \text{ for each } i \right\}
\]
is a stable set of functions on \(\Omega\).

**Remark 2.3.** In general, \(L^0(E)\) is strictly smaller than \(L^0_m(E)\) for a topological space \(E\). However, if \((E, \mathcal{E})\) is a standard Borel space, then \(L^0(E)\) and \(L^0_m(E)\) coincide.

We have a close connection between stable sets of functions on \((\Omega, \mathcal{F}, \mathbb{P})\) and conditional sets of the associated measure algebra which will allows us to apply the machinery of conditional set theory \([\mathbb{0}]\). We start by defining an analog of the conditional power set.

\(^1\)Strongly measurable means Borel measurable and essentially separably-valued.
Definition 2.4. Let $X$ be a stable set of functions on $\Omega$. Let $\mathcal{P}(X)$ denote the collection of all conditional subsets $V|A$, where $V$ is a stable subset of $X$ and $A \in \mathcal{F}$. A countable concatenation in $\mathcal{P}(X)$ is defined by

$$
\sum (V_k|B_k)|A_k := (\sum V_k|A_k)| \cup_k (A_k \cap B_k) \in \mathcal{P}(X),
$$

for a sequence $(V_k|B_k)$ and a partition $(A_k)$, where $\sum V_k|A_k$ is defined in \((2.1)\). A subset of $\mathcal{P}(X)$ is called a stable collection if it is nonempty and closed under countable concatenations.

In the present context, the conditional inclusion relation \([9, \text{Definition 2.9}]\) reads as follows.

Definition 2.5. The conditional inclusion relation on $\mathcal{P}(X)$ is the binary relation

$$
V|A \subseteq W|B \text{ if and only if } A \subseteq B \text{ and } V|A \subseteq W|A.
$$

An important result in conditional set theory which will be helpful for the construction of a conditional measure theory is the following theorem which is proved in \([9, \text{Theorem 2.9}]\).

Theorem 2.6. The ordered set $(\mathcal{P}(X), \subseteq)$ is a complete complemented distributive lattice.

Proof. For the sake of completeness, we provide the main constructions of the proof in the present setting. Notice that $X$ is the greatest and $X|\emptyset = \{\ast\}$ the least element in $\mathcal{P}(X)$. Let $(V_i|A_i)$ be a nonempty family in $\mathcal{P}(X)$. We construct its supremum and infimum w.r.t. $\subseteq$. Put $A := \sup_i A_i$ and fix some $x_0 \in X$. Define

$$
V := \left\{ \sum x_k|B_k + x_0|A^c : \text{ for all } k \text{ there is } i \text{ s.t. } B_k \subseteq A_i \text{ and } x_k \in V_i \text{ and } (B_k) \text{ is a partition of } A \right\}.
$$

The conditional subset

$$
\cup_i V_i|A_i := V|A
$$

is the supremum of $(V_i|A_i)$. As for its infimum, let

$$
\mathcal{G} = \left\{ A \in \mathcal{F} : A \subseteq \inf_i A_i \text{ and } \cap_i V_i|A \neq \emptyset \right\}.
$$

Put $A_* = \sup \mathcal{G}$. We show that $A_*$ is attained. Let $(A_k)$ be a sequence in $\mathcal{G}$ such that $A_* = \sup_k A_k$. Define $B_1 = A_1$ and $B_k = A_k \cap (B_1 \cup \ldots \cup B_{k-1})^c$ for $k \geq 2$. Then $(B_k)$ is a partition of $A_*$ and since $\cap_i V_i|B_k \neq \emptyset$ for all $k$, it follows from the stability of the $V_i$’s that $\cap_i V_i|A_* \neq \emptyset$. Fix $x_0 \in X$, and modify each $V_i$ by $W_i = V_i|A_* + \{x_0\}|A^c_i$. Then $\emptyset \neq \cap_i W_i =: W$ is stable, and

$$
\cap_i V_i|A_i := W|A_*
$$

is the infimum of $(V_i|A_i)$.

We provide the construction of the complement. For $V|A \in \mathcal{P}(X)$, define

$$
(V|A)^\mathbb{C} := \cup\{W|B \in \mathcal{P}(X) : W|B \cap V|A = \{\ast\}\}.
$$

By completeness of the ordered set $(\mathcal{P}(X), \subseteq)$, $(V|A)^\mathbb{C}$ is well-defined.

The symbols $\cup, \cap$ and $\subseteq$, defined in \((2.3), (2.4)\) and \((2.5)\), are called conditional union, conditional intersection and conditional complement, respectively. We give another description of the conditional complement:

$$
(V|A)^\mathbb{C} = Z|B_* + X|A^c,
$$

where

$$
B_* = \sup\{A' \in \mathcal{F} : A' \subseteq A, V|A' \neq X|A'\}
$$

and

$$
Z = \begin{cases} \{x \in X : |x|A' \notin V|A' \text{ for all } A' \in \mathcal{F}_+ \text{ with } A' \subseteq B_*\}, & \text{if } B_* \in \mathcal{F}_+, \\ X, & \text{else}. \end{cases}
$$

By an exhaustion argument (similarly to the construction of the conditional intersection in the previous proof), $B_*$ is attained. It can directly be verified that $Z$ is stable. To see the equality in \((2.6)\), notice that $(Z|B_* + X|A^c) \cap V|A = \{\ast\}$, and therefore $Z|B_* + X|A^c \subseteq (V|A)^\mathbb{C}$. By way of contradiction, suppose that $(Z|B_* + X|A^c) \cap V|A^c \neq \{\ast\}$. We may assume that $B_* \in \mathcal{F}_+$. Since $(V|A)^\mathbb{C} = X|A^c$, there must exist $C \in \mathcal{F}_+$ with $C \subseteq B_*$ and $x \in X$ such that $x(c \in (V|A)^\mathbb{C}$ and $x|c' \notin V|C'$ for all $C' \subseteq C$ with $C' \in \mathcal{F}_+$. However, this contradicts the maximality of $B_*$. As a consequence of elementary results in the theory of Boolean algebras, see e.g. \([22, \text{p. 14}]\), one has:
Corollary 2.7. For a stable set $X$, the structure $(\mathcal{P}(X), \sqcup, \cap, \subseteq, \{\ast\}, X)$ is a complete Boolean algebra.

The importance of the previous result lies in the validity of all Boolean laws in $\mathcal{P}(X)$ which are known from the classical power set algebra. The Boolean laws are fundamental in topology and measure theory. The application of the Boolean laws is usually referred to as Boolean arithmetic, see e.g. [22]. Compared with the usual power set, the conditional power set has additionally the property that its elements can be concatenated which brings some new properties which are specific to the conditional power set and are summarized in the following proposition.

Proposition 2.8. Let $X$ be a stable set, $V|A \in \mathcal{P}(X)$, $(V_i|A_i)_{i \in I}$, $(V_{k,j}|A_{k,j})_{k \in \mathbb{N}, j \in J}$ be families in $\mathcal{P}(X)$, where $I$ and $J$ are arbitrary nonempty index sets. Let $(D_k)$ be a partition of $\Omega$. Then the following are true.

(S1) $\cap_i (V_i|A_i) = (\cap_i V_i)|A$ and $\cup_i (V_i|A_i) = (\cup_i V_i)|A$.
(S2) $(\sum V_k|A_k)|D_k \cap V|A = (\sum V_k|A_k \cap V|A)|D_k$ and $(\sum V_k|A_k)|D_k \cup V|A = (\sum V_k|A_k \cup V|A)|D_k$.
(S3) $\sum (\cap V_{k,j}|A_{k,j})|D_k = \cap (\sum (V_{k,j}|A_{k,j})|D_k)$ and $\sum (\cup V_{k,j}|A_{k,j})|D_k = \cup (\sum V_{k,j}|A_{k,j})|D_k$.
(S4) $(\sum V_k|A_k)|D_k \subseteq (\sum V_k|A_k)|D_k$.

Proof. We prove (S1) and (S4); the remaining two claims can be shown similarly by using the definitions of the conditional set operations.

(S1) We show $\cap_i (V_i|A_i) = (\cap_i V_i)|A$. Suppose $\cap_i (V_i|A_i) = W|A$, and $\cap_i V_i = W'|A$ according to (2.3), and put $C_i = B_i \cap \inf_i A_i$. Then $x|A_i \in W_i|A_i$ for all $i$, and thus $x_i|A_i \in W'|A_i \subseteq W'|C_i$ since $A_i \subseteq C_i$. If now $C_i \setminus A_i \in \mathcal{F}_+$, then this contradicts the maximality of $A_i$, which implies that $W'|C_i \subseteq W'|A$. We show $\cup_i (V_i|A_i) = (\cup_i V_i)|A$. Suppose $\cup_i (V_i|A_i) = W|A$ and $\cup_i V_i = W'|A$ according to (2.3). By stability, it follows that $W|A \subseteq W'|A$. As for the converse, let $(A_{ik})$ be a countable subfamily of $(A_i)$ such that $\cup_i A_{ik} = A$ and $\cup_i V_i = W'$ for some partition $(B_k)$ of $\Omega$ and $x_i \in X_i$ for some $i$ and all $h$. Then $(B_h \cap A_{ik})_{h,k}$ is a partition of $A$, and extending the family $(x_i)$ to a family $(x_{h,k})$ by $x_{h,k} := x_i$ for all $h, k$ we obtain $(\sum x_i|B_i)|A = \sum x_{h,k}|B_{h,k} \subseteq W|A$, which shows that $W'|A \subseteq W|A$.

(S4) According to (2.4), let $(V_k|A_k)|D_k \subseteq U_k|B_{k,*} + X|A_k$ for each $k$, and

$$(\sum V_k|A_k)|D_k \subseteq (\sum V_k|A_k)|D_k \subseteq U_k|B_{k,*} + X|A_k$$

where $A = \cup_k (A_k \cap D_k)$ and the first equality follows from (2.2). On the one hand, we have

$$(\sum U_k|B_{k,*} + X|A_k)|D_k \cup_k (A_k \cap D_k) = X|A_k$$

On the other hand, $U_k|B_{k,*} \cap (A_k \cap D_k) = U_k|B_{k,*} \cap D_k$ for all $k$, which implies

$$U_k|B_k = (\sum U_k|B_{k,*} + X|A_k)|D_k \cup_k (B_{k,*} \cap D_k)$$.

\[\square\]

Remark 2.9. In many concepts and proofs in conditional set theory, it is necessary to construct the largest set $A \in \mathcal{F}$ on which a property is satisfied. For example see the definition of the conditional intersection (2.4) and the representation of the conditional complement (2.5). This can be achieved by an exhaustion argument which requires that the considered property is stable under countable concatenations which follows from the stability of sets and relations. Without further notice, if we write “by an exhaustion argument”, then we mean that an exhaustion can is applied.

Finally, we recall the definition of a stable function, cf. [3] Definition 2.17.

Definition 2.10. Let $X$ and $Y$ be stable sets. A function $f: X \to Y$ is said to be a stable function if $f(\sum x_k|A_k) = \sum f(x_k)|A_k$ for all partitions $(A_k)$ and every sequence $(x_k)$ in $X$. Let $W|A \subseteq Y$ and put

$$C_* := \sup \{A' \in \mathcal{F}: A' \subseteq A \text{ there is } x \in X \text{ such that } f(x)|A' \in W|A'\}.$$ 

By an exhaustion argument, $C_*$ is attained. Let

$$V := \{x \in X: f(x)|C_* \in W|C_*\}.$$
Define the \textit{conditional pre-image} of $W|A$ as
\[
 f^{-1}(W|A) := V|C_v. 
\] (2.7)
For a sequence of stable functions $f_k : X \to Y$ and a partition $(A_k)$, define their concatenation by
\[
 x \mapsto (\sum f_k|A_k)(x) := \sum f_k(x)|A_k,
\] (2.8)
which is a stable function from $X$ to $Y$.

3. \textbf{A conditional version of Carathéodory’s extension theorem and the Lebesgue integral}

The aim of this section is to develop basic results of measure theory in a conditional context. Our focus lies on Carathéodory’s extension and uniqueness theorems and the construction of a Lebesgue integral.

\textbf{Definition 3.1.} Let $X$ be a stable set of functions on $\Omega$. A stable collection $\mathcal{X}$ on $X$ is called a
- \textit{stable ring} whenever $V|A \cap (W|B)^c, V|A \cup W|B \in \mathcal{X}$ for all $V|A, W|B \in \mathcal{X}$;
- \textit{stable Dynkin system} whenever $X \in \mathcal{X}$, $(V|A)^c \in \mathcal{X}$ for all $V|A \in \mathcal{X}$, and $\cup_k (V_k|A_k) \in \mathcal{X}$ for all sequences $(V_k|A_k)$ of pairwise disjoint elements in $\mathcal{X}$;
- \textit{stable $\sigma$-algebra} whenever $X \in \mathcal{X}$, $(V|A)^c \in \mathcal{X}$ for all $V|A \in \mathcal{X}$, and $\cup_k (V_k|A_k) \in \mathcal{X}$ for all sequences $(V_k|A_k)$ in $\mathcal{X}$.

The pair $(X, \mathcal{X})$, where $X$ is a stable $\sigma$-algebra, is called a \textit{stable measurable space}. If $(Y, \mathcal{Y})$ is another stable measurable space, then a stable function $f : X \to Y$ is called \textit{stably measurable} if $f^{-1}(V|A) \in \mathcal{X}$ for all $V|A \in \mathcal{Y}$.

\textbf{Remark 3.2.} Since the intersection of a nonempty family of stable Dynkin systems (stable $\sigma$-algebras) is a stable Dynkin system (stable $\sigma$-algebra), we can define the stable Dynkin system (stable $\sigma$-algebra) \textit{generated} by a collection $\mathcal{E}$ of conditional sets, henceforth denoted by $\mathcal{D}(\mathcal{E}) (\Sigma(\mathcal{E}))$.

\textbf{Examples 3.3.} 1) Let $X$ be a stable set of functions on $\Omega$. Then $\{X|A : A \in \mathcal{F}\}$ and $\mathcal{P}(X)$ are stable $\sigma$-algebras, called the \textit{trivial} and the \textit{discrete} stable $\sigma$-algebra on $X$.
2) Let $L^0(E)$ be the space of all strongly measurable functions with values in a Banach space $E$. The norm $\| \cdot \|$ of $E$ extends to an $L^0$-valued norm on $L^0(E)$ by defining $\|x\|(\omega) := \|x(\omega)\|$ a.s. Let $\mathcal{E}$ be the collection of all \textit{stable open balls} $B_r(x) := \{y \in L^0(E) : \|x - y\| < r\}, x \in L^0(E)$ and $r \in L^0_+$. By straightforward inspection, $\mathcal{E}$ is a stable collection in $\mathcal{P}(L^0(E))$ which is a base for a stable topology on $L^0(E)$, see [9] Section 3. We call the stable $\sigma$-algebra $\Sigma(\mathcal{E})$ the \textit{stable Borel $\sigma$-algebra} on $L^0(E)$, and denote it by $\mathcal{B}(L^0(E))$. The following proposition is an immediate consequence of Boolean arithmetic.

\textbf{Proposition 3.4.} A stable Dynkin system is a stable $\sigma$-algebra if and only if it is closed under finite conditional intersections.

The previous results can be used to establish the following Dynkin’s $\pi$-$\lambda$ type result.

\textbf{Theorem 3.5.} Let $\mathcal{E}$ be a stable collection which is closed under finite conditional intersections. Then $\Sigma(\mathcal{E}) = \mathcal{D}(\mathcal{E})$.

\textbf{Proof.} By Corollary 3.4 it is enough to prove that $\mathcal{D}(\mathcal{E})$ is closed under finite conditional intersections. For $V|A \in \mathcal{D}(\mathcal{E})$, let $F_{V|A} := \{W|B \in \mathcal{P}(X) : V|A \cap W|B \in \mathcal{D}(\mathcal{E})\}$. By (S3) and the stability of $\mathcal{D}(\mathcal{E})$, it follows that $F_{V|A}$ is a stable collection. By Boolean arithmetic, it follows that $F_{V|A}$ is a stable Dynkin system. For every $V|A \in \mathcal{E}$, one has $\mathcal{E} \subseteq F_{V|A}$ from the assumption, and therefore $\mathcal{D}(\mathcal{E}) \subseteq F_{V|A}$. If now $V|A \in \mathcal{E}$ and $W|B \in \mathcal{D}(\mathcal{E})$, then $W|B \in F_{V|A}$, and thus $V|A \in F_{W|B}$. It follows that $\mathcal{E} \subseteq F_{W|B}$ and $\mathcal{D}(\mathcal{E}) \subseteq F_{W|B}$ which shows the claim. \hfill $\square$

\textbf{Remark 3.6.} Let $(X, \mathcal{X})$ and $(Y, \mathcal{Y})$ be stable measurable spaces, $\mathcal{E}$ a collection in $\mathcal{P}(Y)$ generating $\mathcal{Y}$ and $f : X \to Y$ a stable function. Then $f$ is stably measurable if and only if $f^{-1}(V|A) \in \mathcal{X}$ for all $V|A \in \mathcal{E}$. Indeed, by stability of $f$, $\mathcal{P}(X)$ and $\mathcal{P}(Y)$, the collection
\[
 Z = \{V|A \in \mathcal{P}(Y) : f^{-1}(V|A) \in \mathcal{X}\}
\] is a stable collection \footnote{$V|A$ and $W|B$ are said to be disjoint if $V|A \cap W|B = \{\ast\}$}.
is stable. One has \( Y \subseteq Z \), and \( Z \) is closed under conditional complementation and countable conditional unions. Thus \( Y \subseteq Z \) if and only if \( E \subseteq Z \).

If \( \mathcal{X} \) is a stable collection in \( \mathcal{P}(X) \), then a function \( \mu: \mathcal{X} \to \bar{L}_+^0 \) is said to be a stable set function if
\[
\mu(\sum (V_k|B_k)|A_k) = \sum \mu(V_k|B_k)|A_k
\]
for all sequences \((V_k|B_k)\) in \( \mathcal{X} \) and every partition \((A_k)\), where the concatenation on the l.h.s is defined as in \((2.2)\).

**Definition 3.7.** Let \( X \) be a stable set of functions on \( \Omega \) and \( \mathcal{X} \) a stable ring on \( X \). A function \( \mu: \mathcal{X} \to \bar{L}_+^0 \) is called a stable pre-measure whenever \( \mu \) is a stable function and
\[
\begin{align*}
\mu(V|A) &= \mu(V|A)[A + 0|A]_c \text{ for all } V|A \in \mathcal{X}; \\
\mu(\bigcup_k (V_k|A_k)) &= \sum_{k > 1} \mu(V_k|A_k) \text{ for all sequences } (V_k|A_k) \text{ of pairwise disjoint elements in } \mathcal{P}(X) \\
\text{with } \bigcup_k (V_k|A_k) \in \mathcal{X}. \tag{M2}
\end{align*}
\]

If the domain of a stable pre-measure \( \mu \) is a stable \( \sigma \)-algebra, then \( \mu \) is called a stable measure and the triple \((X, \mathcal{X}, \mu)\) a stable measure space. A stable measure \( \mu \) is said to be
\begin{itemize}
\item finite if \( \mu(X) < \infty \), and a stable probability measure if \( \mu(X) = 1 \);
\item \( \sigma \)-finite if there exists an increasing sequence \((V_k|A_k)\) in \( \mathcal{X} \) with \( \bigcup_k (V_k|A_k) = X \) such that \( \mu(V_k|A_k) < \infty \) for all \( k \).
\end{itemize}

One can see property (M1) as the local analogue of property (M1) for classical measures. We give one simple example of a stable measure and will provide further examples in Section 4.

**Example 3.8.** Let \((X, \mathcal{X})\) be a stable measurable space and \( x \in X \). The stable Dirac measure centered at \( x \) is defined by
\[
\delta_x(V|A) := 1_{|B + 0|B^c},
\]
where \( B = \sup\{B' \in \mathcal{F} : B' \subseteq A, x|B' \in V|B'\} \) which is attained by an exhaustion argument.

Similarly to the classical case, we have the following elementary properties of stable pre-measures.

**Proposition 3.9.** Let \( X \) be a stable set of functions on \( \Omega \), \( X \) a stable ring and \( \mu: \mathcal{X} \to \bar{L}_+^0 \) a stable function satisfying property (M1) and finite additivity (i.e. (M2) for finite sequences). Then the following are true.
\[
\begin{align*}
\mu(V|A \cup W|B) + \mu(V|A \cap W|B) &= \mu(V|A) + \mu(W|B) \text{ for all } V|A, W|B \in \mathcal{X}; \\
\mu(V|A) &\leq \mu(W|B) \text{ whenever } V|A \subseteq W|B; \\
\mu(W|B \cap (V|A)^c) &= \mu(W|B) - \mu(V|A) \text{ whenever } V|A \subseteq W|B \text{ and } \mu(V|A) < \infty; \\
\text{if } \mu \text{ is a stable pre-measure, then } \mu(\bigcup_k V_k|A_k) &\leq \sum_k \mu(V_k|A_k) \text{ for all sequences } (V_k|A_k) \text{ in } \mathcal{X} \text{ with } \bigcup_k V_k|A_k \in \mathcal{X}; \\
\mu \text{ is a stable pre-measure if and only if } \mu(V_k|A_k) &\uparrow \mu(\bigcup_k V_k|A_k) \text{ a.s. for all increasing sequences } (V_k|A_k) \text{ in } \mathcal{X} \text{ with } \bigcup_k V_k|A_k \in \mathcal{X}; \\
\text{if } \mu(V|A) &< \infty \text{ for all } V|A \in \mathcal{X}, \text{ then } \mu \text{ is a stable pre-measure if and only if } \mu(V_k|A_k) \downarrow \mu(\bigcap_k V_k|A_k) \text{ a.s. for all decreasing sequences } (V_k|A_k) \text{ in } \mathcal{X} \text{ with } \bigcap_k V_k|A_k \in \mathcal{X}. 
\end{align*}
\]

**Proof.** Since the verification of the statements is similar to the proof of the respective classical statements, we only provide the main arguments.

(M3) We have
\[
\begin{align*}
\mu(V|A \cup W|B) &= \mu(V|A) + \mu(W|B \cap (V|A)^c), \\
\mu(W|B) &= \mu(V|A \cap W|B) + \mu(W|B \cap (V|A)^c).
\end{align*}
\]
Let \( C = \{\mu(W|B \cap (V|A)^c) < \infty\} \). On \( C \) we get (M3) by adding the previous two identities and subtracting \( \mu(W|B \cap (V|A)^c) \). On \( C^c \) we have \( \mu(V|A \cup W|B) = \mu(W|B) = \infty \) which also implies (M3).

(M4) and (M5) follow from finite additivity.

(M6) can be shown by finite additivity, (M4) and Boolean arithmetic.

(M7) follows from finite additivity and Boolean arithmetic.

(M8) follows from (M7) on applying (M5).

Footnote 3: The infinite series in \( \bar{L}_+^0 \) is understood pointwise a.s.
Definition 3.10. Let $X$ be a stable set of functions on $\Omega$. A stable function $\mu^* : \mathcal{P}(X) \to \bar{L}^0_+$ is called a stable outer measure whenever

(A1) $\mu^*(V | A | B) = \mu^*(V | A) + 0 | B$ for all $V | A \in \mathcal{P}(X)$ and $B \in \mathcal{F}$;

(A2) $\mu^*(V | A) \leq \mu^*(W | B)$ whenever $V | A \subseteq W | B$;

(A3) $\mu^*(\cup_k (V_k | A_k)) \leq \sum_k \mu^*(V_k | A_k)$ for all sequences $(V_k | A_k)$ in $\mathcal{P}(X)$.

An element $V | A \in \mathcal{P}(X)$ is said to be stably $\mu^*$-measurable whenever

$$\mu^*(W | B \cap V | A) + \mu^*(W | B \cap (V | A)^c) = \mu^*(W | B)$$

for all $W | B \in \mathcal{P}(X)$. (3.1)

A stable measure can be obtained from a stable outer measure as follows.

Proposition 3.11. Let $X$ be a stable set of functions on $\Omega$ and $\mu^* : \mathcal{P}(X) \to \bar{L}^0_+$ a stable outer measure. Then $\mu^*$ is a stable measure on the stable collection $\mathcal{X}(\mu^*)$ of all stably $\mu^*$-measurable sets.

Proof. From the stability of $\mu^*$ and (S2), we obtain that $\mathcal{X}(\mu^*)$ is a stable collection. Clearly, $X \in \mathcal{X}(\mu^*)$, and by symmetry, $\mathcal{X}(\mu^*)$ is closed under conditional complementation. Let $V | A, W | B \in \mathcal{X}(\mu^*)$. Then

$$\mu^*(Z | C) = \mu^*(Z | C \cap W | B) + \mu^*(Z | C \cap (W | B)^c)$$

for all $Z | C \in \mathcal{P}(X)$. (3.2)

Replacing $Z | C$ by $Z | C \cap V | A$ and $Z | C \cap (V | A)^c$ in (3.2), respectively, and taking the sum of the two resulting equations, we obtain

$$\mu^*(Z | C) = \mu^*(Z | C \cap V | A \cap W | B) + \mu^*(Z | C \cap V | A \cap (W | B)^c) + \mu^*(Z | C \cap (V | A)^c \cap W | B) + \mu^*(Z | C \cap (V | A)^c \cap (W | B)^c).$$

(3.3)

Replace $Z | C$ by $Z | C \cap (V | A \cup W | B)$ in (3.3), and get from Boolean arithmetic

$$\mu^*(Z | C \cap (V | A \cup W | B)) = \mu^*(Z | C \cap V | A \cap W | B) + \mu^*(Z | C \cap V | A \cap (W | B)^c) + \mu^*(Z | C \cap (V | A)^c \cap W | B).$$

(3.4)

Plugging in (3.4) in (3.3), one has $V | A \cup W | B \in \mathcal{X}(\mu^*)$. By de Morgan’s Law, $\mathcal{X}(\mu^*)$ is also closed under finite intersections. Let $(U_k | D_k)$ be a sequence of pairwise disjoint sets in $\mathcal{X}(\mu^*)$ and $U | D = \cup_k U_k | D_k$. Choosing $V | A = U_1 | D_1$ and $W | B = U_2 | D_2$ in (3.4), one gets

$$\mu^*(Z | C \cap (U_1 | D_1 \cup U_2 | D_2)) = \mu^*(Z | C \cap U_1 | D_1) + \mu^*(Z | C \cap U_2 | D_2)$$

for all $Z | C \in \mathcal{P}(X)$.

By induction,

$$\mu^*(Z | C \cap (\cup_{k \leq n} U_k | D_k)) = \sum_{k \leq n} \mu^*(Z | C \cap U_k | D_k)$$

for all $Z | C \in \mathcal{P}(X)$ and $n \in \mathbb{N}$. (3.5)

From the previous we know that $Y_n | E_n = \cup_{k \leq n} (U_k | D_k) \in \mathcal{X}(\mu^*)$. By (A2), $Z | C \cap (U | D)^c \subseteq Z | C \cap (Y_n | E_n)^c$ implies $\mu^*(Z | C \cap (U | D)^c) \leq \mu^*(Z | C \cap (Y_n | E_n)^c)$ for all $Z | C \in \mathcal{P}(X)$. Therefore, we obtain from (3.5)

$$\mu^*(Z | C) = \mu^*(Z | C \cap Y_n | E_n) + \mu^*(Z | C \cap (Y_n | E_n)^c) \geq \sum_{k \leq n} \mu^*(Z | C \cap U_k | D_k) + \mu^*(Z | C \cap (U | D)^c)$$

for all $Z | C \in \mathcal{P}(X)$ and $n \in \mathbb{N}$. By (A3),

$$\mu^*(Z | C) \geq \sum_{k \geq 1} \mu^*(Z | C \cap U_k | D_k) + \mu^*(Z | C \cap (U | D)^c) \geq \mu^*(Z | C \cap U | D) + \mu^*(Z | C \cap (U | D)^c)$$

(3.6)

for all $Z | C \in \mathcal{P}(X)$, which means $U | D \in \mathcal{X}(\mu^*)$. By Theorem 3.3, $\mathcal{X}(\mu^*)$ is a stable $\sigma$-algebra. To see that $\mu^* : \mathcal{X}(\mu^*) \to \bar{L}^0_+$ is a stable measure, replace $Z | C$ by $U | D$ in (3.6) and note that the reverse inequality follows from (A3).

This leads us to a conditional version of Carathéodory’s extension theorem.

Theorem 3.12. Let $X$ be a stable set of functions on $\Omega$, $\mathcal{X}$ be a stable ring on $X$ and $\mu : \mathcal{X} \to \bar{L}^0_+$ be a stable pre-measure. Then there exists a stable measure $\nu : \Sigma(\mathcal{X}) \to \bar{L}^0_+$ which coincides with $\mu$ on $\mathcal{X}$. □
Proof. For each \( V|A \in \mathcal{P}(X) \), let 
\[
B_{V|A} := \sup \{ B' \in \mathcal{F} : \text{there is a sequence } (U_k|C_k) \text{ in } \mathcal{X} \text{ with } (V|A)|B' \subseteq \bigcup_k U_k|C_k \}.
\]
By an exhaustion argument, \( B_{V|A} \) is attained. Let \( \mathcal{U}(V|A) \) be the collection of all sequences \( (U_k|C_k) \) in \( \mathcal{X} \) such that \( (V|A)|B_{V|A} \subseteq \bigcup_k U_k|C_k \). By (S3) and the stability of \( \mathcal{X} \) it follows that \( \mathcal{U}(V|A) \) is a stable collection. Moreover, one has \( \bigcup_{V_k|A_k \in D_k} = \bigcup_k (B_{V_k|A_k} \cap D_k) \) for all sequences \( (V_k|A_k) \) in \( \mathcal{P}(X) \) and partitions \( (D_k) \) of \( \Omega \). Thus \( \mu^* : \mathcal{P}(X) \to \mathbb{L}_+^0 \) defined by
\[
\mu^*(V|A) := \inf \{ \sum_{k \geq 1} \mu(U_k|C_k) : (U_k|C_k) \in \mathcal{U}(V|A) \} |B_{V|A} + \infty|B_{V|A}
\]
is a well-defined stable set function. We want to show that \( \mu^* \) is a stable outer measure. Properties (A1) and (A2) are easy to check. As for (A3), let \( (V_k|A_k) \) be a sequence in \( \mathcal{P}(X) \). Clearly, \( B := \bigcup_k B_{V_k|A_k} = B_{\bigcup_k (V_k|A_k)} \). Fix \( \varepsilon > 0 \). For each \( k \), let \( (U_{k,n}|C_{k,n}) \in \mathcal{U}(V_k|A_k) \) be such that
\[
\sum_{n \geq 1} \mu(U_{k,n}|C_{k,n}) \leq \mu^*(V_k|A_k) + 2^{-\varepsilon} \quad \text{on } B.
\]
Then
\[
\mu^*(\bigcup_k V_k|A_k) \leq \sum_{n,k \geq 1} \mu^*(U_{k,n}|C_{k,n}) \leq \sum_{k \geq 1} \mu^*(V_k|A_k) + \varepsilon \quad \text{on } B,
\]
which proves (A3).

Let \( V|A \in \mathcal{X} \). We want to show that \( V|A \in \mathcal{X}(\mu^*) \), that is,
\[
\mu^*(Z|C \cap V|A) + \mu^*(Z|C \cap (V|A)^c) \leq \mu^*(Z|C), \quad \text{for all } Z|C \in \mathcal{P}(X).
\]
Let \( Z|C \in \mathcal{P}(X) \). On \( B_{Z|C} \) there is nothing to show. On the other hand, we have \( B_{Z|C} \cap \mathcal{V}(V|A) \subseteq B_{Z|C} \). Moreover, for \( (U_k|C_k) \in \mathcal{U}(Z|C) \) it follows from (M2) that
\[
\sum_{k \geq 1} \mu(U_k|C_k) = \sum_{k \geq 1} \mu(U_k|C_k \cap V|A) + \sum_{k \geq 1} \mu(U_k|C_k \cap (V|A)^c).
\]
Hence (3.7) is also satisfied on \( B_{Z|C} \). We have shown \( \mathcal{X} \subseteq \mathcal{X}(\mu^*) \). Now it follows from Proposition 3.11 that the restriction of \( \mu^* \) to \( \Sigma(\mathcal{X}) \), denoted by \( \nu \), is a stable measure. Since \( B_{V|A} = \Omega \) for \( V|A \in \mathcal{X} \), and by (M4) and (M6),
\[
\mu(V|A) = \mu(\bigcup_k (U_k|C_k \cap V|A)) \leq \sum_k \mu(U_k|C_k \cap V|A) \leq \sum_k \mu(U_k|C_k),
\]
for all \( (U_k|C_k) \in \mathcal{U}(V|A) \), it follows that \( \nu \) coincides with \( \mu \) on \( \mathcal{X} \). \( \square \)

As for the uniqueness of the previous extension, we have the following result.

Proposition 3.13. Let \( (X, \mathcal{X}) \) be a stable measurable space and \( \mathcal{E} \) a stable generator of \( \mathcal{X} \) which is closed under finite conditional intersections. For two stable measures \( \mu, \nu \) on \( (X, \mathcal{X}) \), suppose \( \mu(V|A) = \nu(V|A) \) for all \( V|A \in \mathcal{E} \) and there exists a sequence \( (Z_k|C_k) \) in \( \mathcal{E} \) with \( \bigcup_k Z_k|C_k = X \) and \( \mu(Z_k|C_k) = \nu(Z_k|C_k) < \infty \) for all \( k \). Then \( \mu = \nu \).

Proof. The proof follows from a monotone class argument in the present context. Indeed, for \( V|A \in \mathcal{E} \) with \( \mu(V|A) = \nu(V|A) < \infty \), let
\[
\mathcal{D}_{V|A} = \{ W|B \in \mathcal{X} : \mu(V|A \cap W|B) = \nu(V|A \cap W|B) \}.
\]
Then \( X \in \mathcal{D}_{V|A} \). Stability of \( \mathcal{D}_{V|A} \) follows from (S3). By (M5) and Boolean arithmetic, \( \mathcal{D}_{V|A} \) is closed under complementation. By (M2), \( \mathcal{D}_{V|A} \) contains the conditional union of every pairwise disjoint sequence of its elements. We have that \( \mathcal{D}_{V|A} \) is a stable Dynkin system with \( \mathcal{E} \subseteq \mathcal{D}_{V|A} \). By Theorem 3.5 \( \mathcal{D}_{V|A} = \mathcal{X} \). Hence,
\[
\mu(W|B \cap V|A) = \nu(W|B \cap V|A),
\]
for all \( W|B \in \mathcal{X} \) and \( V|A \in \mathcal{E} \) with \( \mu(V|A) = \nu(V|A) \). Therefore,
\[
\mu(W|B \cap Z_k|C_k) = \nu(W|B \cap Z_k|C_k)
\]
for all \( k \), and the claim follows from (M2). \( \square \)
Fix a stable measure space \((X, \mathcal{X}, \mu)\) in the remainder of this section. We construct a conditional Lebesgue integral for stably measurable functions \(f: X \to L^0\), where we endow \(L^0\) with the stable Borel \(\sigma\)-algebra \(\mathcal{B}(L^0)\) defined in Example 3.3 (2). Consider the following stable generators of \(\mathcal{B}(L^0)\). For \(r \in L^0\), let
\[ [r, \infty[ := \{s \in L^0: r \leq s < \infty\}, \quad ]r, \infty[ := \{s \in L^0: r < s < \infty\}, \quad ]-\infty, r] := \{s \in L^0: -\infty < s \leq r\}, \quad ]-\infty, r[ := \{s \in L^0: -\infty < s < r\}. \]
Notice that \([r, \infty[ = -\infty, r]\) and \([r, \infty[ = -\infty, r]\). The collection of all of each type of these intervals is a stable collection in \(\mathcal{P}(L^0)\), and by Boolean arithmetic, a generator of \(\mathcal{B}(L^0)\).

The concatenation of a sequence of stably measurable functions \(f_k: X \to L^0\) along a partition \((A_k)\) of \(\Omega\) is defined in (2.8), and it is a stably measurable function. The sum \(f + g\) and product \(f \cdot g\) of two stably measurable functions \(f, g: X \to L^0\) is defined pointwise, and it can be checked that they are stably measurable functions. Further, we write \(f \leq g\) whenever \(f(x) \leq g(x)\) in \(L^0\) for all \(x \in X\). It follows that \(\max\{f, g\}\) and \(\min\{f, g\}\) are stably measurable. The convergence of a sequence of stably measurable functions \(f_k: X \to L^0\) to a function \(f: X \to L^0\) is defined by
\[ f_k(x) \to f(x) \text{ a.s.} \quad (3.8) \]
for all \(x \in X\), if this limit exists, in which case \(f\) is stably measurable.

We introduce stable indicator functions and stable elementary functions. Let \(V|A \in \mathcal{P}(X)\), and for each \(x \in X\) let
\[ A_x := \sup\{A' \in \mathcal{F}: A' \subseteq A, x|A' \in V|A'\}. \]
By an exhaustion argument, \(A_x\) is attained. Further, \(\sum x_k|A_k = \cup_k(A_{x_k} \cap A_k)\) for every sequence \((x_k)\) in \(X\) and each partition \((A_k)\) of \(\Omega\). Thus the function \(1_{V|A^r}: X \to L^0\) defined by \(x \mapsto 1|A_x + 0|A_x^c\) is well-defined and stable, and called the stable indicator function of \(V|A\). The following properties of stable indicator functions can be directly checked from the definition.

\[ \begin{align*}
(D1) \quad & 1_{V|A} = 1_{V|A + 0|A^c} \quad \text{for all } V|A \subseteq X; \\
(D2) \quad & \sum 1_{V_k|A_k} = \sum 1_{V_k|A_k}|B_k \quad \text{for all sequences } (V_k|A_k) \text{ in } \mathcal{P}(X) \text{ and partitions } (B_k) \text{ of } \Omega; \\
(D3) \quad & \sum 1_{V_k|A_k} = \sum 1_{V_k|A_k} \quad \text{for all sequences } (V_k|A_k) \text{ of pairwise disjoint elements in } \mathcal{P}(X); \\
(D4) \quad & \sum 1_{V|A} = 1 - 1_{(V|A)^c} \quad \text{for all } V|A \subseteq X.
\end{align*} \]

**Definition 3.14.** Let \((r_k)_{k \leq n}\) be a finite family in \(L^0_+\) and \((V_k|A_k)_{k \leq n}\) a finite family of pairwise disjoint elements in \(\mathcal{X}\) with \(\cup_k V_k|A_k = X\). The function \(\sum_{k \leq n} r_k 1_{V_k|A_k}: X \to L^0_+\) is called a stable elementary function.

Without further notice, we identify \(\mathbb{N}\) with a subset of \(L^0_+(\mathbb{N})\) by the embedding \(n \mapsto n1_\Omega\) and \(\mathbb{R}\) with a subset of \(L^0\) by the embedding \(r \mapsto r1_\Omega\). Also \(L^0_+(\mathbb{N})\) can be understood as a subset of \(L^0\).

**Remark 3.15.** We can define \(r \cdot s\) and \(r + s\) for arbitrary \(r, s \in L^0_+\) by considering the conventions \(a \cdot \infty = \infty \cdot a = \infty\) for all \(a \in \mathbb{R}_+\), \(a + \infty = \infty + a = \infty\) for all \(a \in \mathbb{R}_+\) and \(0 \cdot \infty = \infty \cdot 0 = 0\) in a pointwise a.s. sense.

**Definition 3.16.** For a stable elementary function \(f = \sum_{k \leq n} r_k 1_{V_k|A_k}\), we define its stable Lebesgue integral as
\[ \int_X f \, d\mu := \sum_{k \leq n} r_k \mu(V_k|A_k) \in L^0_+. \]
To extend the previous definition to stably measurable functions, we need the following two lemmas.

**Lemma 3.17.** Let \(f: X \to L^0_+\) be a stable elementary function and \((f_n)\) an increasing sequence of stable elementary functions \(f_n: X \to L^0_+\) such that \(f \leq \sup_n f_n\). Then it holds \(\int_X f \, d\mu \leq \sup_n \int_X f_n \, d\mu\).

**Proof.** Suppose \(f = \sum_{k \leq n} r_k 1_{V_k|A_k}\). Fix \(r \in L^0\) with \(0 < r < 1\). For every \(n\), let \(W_n|B_n := (f_n - rf)^{-1}([0, \infty])\). Then \(f_n \geq rf 1_{W_n|B_n}\), and therefore
\[ \int_X f_n \, d\mu \geq r \int_X f 1_{W_n|B_n} \, d\mu \quad (3.9) \]
The following properties of the stable integral can be verified directly: define its best stably measurable. We say that \( f \) be stably measurable. We say that \( f \) be stably measurable. We say that \( f \)

**Definition 3.19.** We have the following version of the monotone convergence theorem.

**Lemma 3.18.** For every stably measurable function \( f : X \to L^0 \) there exists an increasing sequence \((f_n)\) of stable elementary functions such that \( \sup_n f_n(x) = f(x) \) for all \( x \in X \).

**Proof.** We denote \( \{ f < r \} := f^{-1}(-\infty, r) \) and \( \{ r \leq f \} := f^{-1}[r, \infty) \), \( r \in L^0 \). For each \( n \in \mathbb{N} \), let

\[
W_{k,n}|A_{k,n} := \left\{ \begin{array}{ll}
(k2^{-n} \leq f) \cap \{ f < (k+1)2^n \}, & 0 \leq k \leq n2^n - 1, \\
\{ f \geq n \}, & k = n2^n.
\end{array} \right.
\]

Define

\[
f_n := \sum_{1 \leq k \leq n2^n} k2^{-n}1_{W_{k,n}|A_{k,n}}.
\]

Fix \( x \in X \) and \( n \in \mathbb{N} \). For \( k = 0, \ldots, n2^n - 1 \), we have

\[
A_k = \sup\{ A' \in \mathcal{F} : A' \subset A_{k,n}, x|A' \in W_{k,n}|A' \} = \sup\{ A' \in \mathcal{F} : A' \subset A_{2k+1,n+1}, x|A' \in W_{2k+1,n+1}|A' \text{ or } A' \subset A_{2k+1,n+1}, x|A' \in W_{2k+1,n+1}|A' \},
\]

where both suprema are attained by an exhaustion argument. Let

\[
A_{n2^n} = \sup\{ A' \in \mathcal{F} : A' \subset A_{n2^n,n}, x|A \in W_{n2^n,n}|A' \}.
\]

Since \((W_{k,n}|A_{k,n})_{0 \leq k \leq n2^n}\) is a partition of \( X \), \((A_k)_{0 \leq k \leq n2^n}\) is a partition of \( \Omega \). Hence from \( f_n(x) \leq f_{n+1}(x) \) on \( A_k \) for all \( 0 \leq k \leq n2^n \) it follows that \( f_n(x) \leq f_{n+1}(x) \). Thus \((f_n)\) is increasing. By construction, \( \sup_n f_n = f \).

Since \( L^0 \) is a vector lattice, every stably measurable function \( f : X \to L^0 \) can be written as the difference \( f^+ - f^- \) of the stably measurable functions \( f^+ := \max\{f, 0\} \) and \( f^- := \max\{-f(x), 0\} \). It follows from the previous two lemmas that the following is well-defined.

**Definition 3.19.** Let \( f : X \to L^0 \) be stably measurable. Define

\[
\int_X f \, d\mu := \sup_n \int_X f_n \, d\mu \in L^0_+,
\]

where \((f_n)\) is an increasing sequence of stably measurable functions with \( \sup_n f_n = f \). Let \( f : X \to L^0 \) be stably measurable. We say that \( f \) is integrable whenever \( \int_X f^+ \, d\mu, \int_X f^- \, d\mu \in L^0 \). In this case, we define its **stable Lebesgue integral** by

\[
\int_X f \, d\mu := \int_X f^+ \, d\mu - \int_X f^- \, d\mu.
\]

The following properties of the stable integral can be verified directly:

1. \( \int_X \sum f_k \, d\mu = \sum \int_X f_k \, d\mu \) for every sequence \((f_k)\) of integrable functions and every partition \((A_k)\) of \( \Omega \).
2. \( \int_X f \, d\mu \leq \int_X g \, d\mu \) for every pair of integrable functions \( f, g \) with \( f \leq g \).
3. \( \int_X r f + g \, d\mu = r \int_X f \, d\mu + \int_X g \, d\mu \) for all integrable functions \( f, g \) and every \( r \in L^0 \).

We have the following version of the monotone convergence theorem.
Theorem 3.20. Let \((f_n)\) be an increasing sequence of integrable functions \(f_n: X \to L^0_\mu\). Then it holds
\[
\int_X \sup_n f_n \, d\mu = \sup_n \int_X f_n \, d\mu.
\]
In particular,
\[
\int_X \left( \sum_{n=1}^\infty f_n \right) \, d\mu = \sum_{n=1}^\infty \int_X f_n \, d\mu.
\]

Proof. Let \(f := \sup_n f_n\). By (I2), it is enough to build a sequence \((h_n)\) of stable elementary functions with \(\sup_n h_n = f\) and \(h_n \leq f_n\) for all \(n\). By Lemma 3.18 for each \(f_n\) there exists a sequence \((g_{m,n})\) of stable elementary functions such that \(f_n = \sup_m g_{m,n}\). Then \(h_m := \max_k \leq m g_{m,k}\) satisfies the required.

4. Kernels, conditional distributions and stable measures

In this section, we establish a link between kernels and stable measures. Based thereupon, we can extend the representation of conditional distribution of random variables with values in a not necessarily standard Borel space by replacing probability kernels with stable probability measures. We apply this representation to compute the conditional expectation of functions of random variables by means of the stable Lebesgue integral.

Unless mentioned otherwise, \((E, \mathcal{E})\) denotes an arbitrary measurable space throughout this section. Recall that \(L^0_m(E)\) denotes the space of all measurable functions \(x: \Omega \to E\), which are called random variables with values in \(E\). Let \(\mathcal{R}\) be a classical ring of sets generating the \(\sigma\)-algebra \(\mathcal{E}\). For every sequence \((F_k)\) in \(\mathcal{R}\) and each partition \((A_k)\), let
\[
\sum L^0_m(F_k)|A_k := \{x \in L^0_m(E): x|A_k \in F_k \text{ a.s. for all } k\}.
\]

By definition, \(\sum L^0_m(F_k)|A_k\) is a stable subset of \(L^0_m(E)\). Since \(\mathcal{R}\) is a ring, by a straightforward computation,
\[
\mathcal{R} := \{(\sum L^0_m(F_k)|A_k)|A: (F_k) \text{ sequence in } \mathcal{R}, (A_k) \text{ partition of } \Omega, A \in \mathcal{F}\}
\]
is a stable ring on \(L^0_m(E)\). We consider the stable \(\sigma\)-algebra \(\Sigma(\mathcal{R})\) generated by \(\mathcal{R}\). If we replace in (4.2) the ring \(\mathcal{R}\) by the larger \(\sigma\)-algebra \(\mathcal{E}\), then the stable \(\sigma\)-algebra generated by the stable ring with respect to \(\mathcal{E}\) coincides with the one generated by the stable ring with respect to \(\mathcal{R}\). Indeed, this follows from the observation
\[
\bigcup_n L^0_m(F_n) = L^0_m(\cup_n F_n),
\]
\[
L^0_m(F)^c = L^0_m(F^c),
\]
for all \((F_n)\) and \(F\) in \(\mathcal{E}\). In particular, if \(E\) is a separable topological space, by Boolean arithmetic, the stable measurable space \((L^0_m(E), \Sigma(\mathcal{R}))\) is identical with the stable Borel space \((L^0(E), \mathcal{B}(L^0(E)))\), see Examples 3.3.2 and Remark 2.2.

Recall that a kernel on \(E\) is a function \(\kappa: \Omega \times \mathcal{E} \to \mathbb{R}_+\) such that \(\kappa(\omega, F)\) is \(\mathcal{F}\)-measurable in \(\omega \in \Omega\) for fixed \(F \in \mathcal{E}\) and a measure in \(F \in \mathcal{E}\) for fixed \(\omega \in \Omega\). A kernel \(\kappa\) is said to be a probability kernel if \(\kappa(\omega, F)\) is a probability measure in \(F \in \mathcal{E}\) for all \(\omega \in \Omega\). We have the following main result which connects kernels with stable measures.

Theorem 4.1. (i) For every kernel \(\kappa: \Omega \times \mathcal{E} \to \mathbb{R}_+\) there exists a stable measure on \((L^0_m(E), \Sigma(\mathcal{R}))\), denoted by \(\mu_\kappa\), such that
\[
\mu_\kappa(L^0_m(F))(\omega) = \kappa(\omega, F) \text{ a.s. for all } F \in \mathcal{E}.
\]

If \(\kappa\) is a probability kernel, then \(\mu_\kappa\) is the unique stable probability measure satisfying the previous equation.

(ii) Suppose \((E, \mathcal{E})\) is a standard Borel space. For every stable probability measure \(\mu\) on the stable Borel space \((L^0(E), \mathcal{B}(L^0(E)))\) there exists a probability kernel on \(E\), denoted by \(\kappa_\mu\), such that
\[
\mu(L^0(F))(\omega) = \kappa_\mu(\omega, F) \text{ a.s. for all } F \in \mathcal{E}.
\]
In particular, one has the following reciprocity identities:
\[
\mu_n(V|A)(\omega) = \mu(V|A)(\omega) \quad \text{a.s. for all } V|A \in \mathcal{B}(L^0(E)),
\]
\[
\kappa_\mu(\omega, F) = \kappa(\omega, F) \quad \text{a.s. for all } F \in \mathcal{E}.
\]

Proof. (i) For \((\sum L^0_m(F_k)|A_k)|A \in \mathcal{R}\), define
\[
\mu_n((\sum L^0_m(F_k)|A_k)|A) := (\sum \kappa(\cdot, F_k)|A_k)|A + 0|A^c. 
\] (4.3)

If \(\mu_n\) is a stable pre-measure, then the first claim follows from Theorem 3.1.2. (M1) is satisfied by \(\kappa\).\(\] The second claim follows immediately from Proposition 3.13.

(ii) The proof can be carried out similarly to the one of the existence of regular conditional distributions. We will follow the main arguments in [21, Theorem 6.3].

By [21, Theorem A1.2], there exists a Borel isomorphism from \(E\) to a Borel subset \(S\) of \(\mathbb{R}\). Therefore, it is enough to prove the claim for the Borel space \((S, \mathcal{B}(S))\). For each \(q \in \mathbb{Q}\), let \(f_q = f(\cdot, q) : \Omega \rightarrow [0, 1]\) be defined by
\[
f(f, q) = \mu(L^0([\cdot - \infty, q])) \quad \text{a.s.}
\] (4.4)

By (M4), one has \(f(\cdot, p) \leq f(\cdot, q)\) whenever \(p \leq q\). Let \(A\) be the set of all \(\omega \in \Omega\) such that \(\mu(\omega, q)\) is increasing in \(q \in \mathbb{Q}\) with limits 1 and 0 at \(\pm \infty\). Since \(A\) is specified by countably many measurable conditions each of which holds a.s., we have \(A \in \mathcal{F}\). Define
\[
F(\omega, x) := 1_A(\omega) \inf_{q \geq x, q \in \mathbb{Q}} f(\omega, q) + 1_{A^c}(\omega)1_{x \geq 0}, \quad x \in \mathbb{R}, \omega \in \Omega.
\]

From (M4) and (M8) it follows that \(F(\omega, \cdot)\) is a distribution function for every \(\omega \in \Omega\). Hence, by [21, Proposition 2.14], there exists a probability measure \(\kappa(\omega, \cdot)\) such that \(\kappa(\omega, [\cdot - \infty, x]) = F(\omega, x)\) for all \(x \in \mathbb{R}\) and \(\omega \in \Omega\). By a monotone class argument, \(\kappa\) is a probability kernel. Using a monotone class argument based on an a.s. interpretation of (M5) and (M7) shows that \(\kappa(\cdot, F) = \mu(L^0_m(F))\) for all \(F \in \mathcal{B}(\mathbb{R})\). In particular, \(\kappa(\cdot, S^c) = 0\) a.s., and thus \(\kappa_\mu(\cdot, F) = \mu(L^0_m(F))\) for all \(F \in \mathcal{B}(S)\), where \(\kappa_\mu\) is the probability kernel defined by
\[
\kappa_\mu(\omega, \cdot) := \kappa(\omega, \cdot)1_{\{\kappa(\omega, S) = 0\}} + \delta_1{\{\kappa(\omega, S) < 1\}}.
\]

where \(s \in S\) is arbitrary. Uniqueness follows from (1.3) by a monotone class argument. The reciprocity identities are an immediate consequence of the constructions of \(\kappa_\mu\) and \(\mu_n\).

The first part of the previous theorem provides a procedure to construct stable measures from classical ones as follows.

**Examples 4.2.** Let \(\lambda\) be a \(\sigma\)-finite measure on \((E, \mathcal{E})\), and view \(\mu\) as a constant kernel on \(\Omega \times \mathcal{E}\). Let \(\mu_\lambda\) be the induced stable measure on \((L^0_\lambda(E), \Sigma(\mathcal{R}))\) as in Theorem 3.1.1. Let \(\{F_n\}\) be an increasing sequence in \(\mathcal{E}\) such that \(\sum F_n = E\) and \(\lambda(F_n) < \infty\) for all \(n\). Then \(\lim F_n = L^0(E)\) and \(\mu_\lambda(L^0_m(F_n)) < \infty\) for all \(n\), which implies that \(\mu_\lambda\) is a \(\sigma\)-finite stable measure. By Proposition 3.1.3, \(\mu_\lambda\) is the unique stable measure extending the classical measure \(\lambda\) from \((E, \mathcal{E})\) to \((L^0_\lambda(E), \Sigma(\mathcal{R}))\). In particular, if \((\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d)), \lambda\) is the classical Borel space with \(\lambda\) the \(d\)-dimensional Lebesgue measure, then the extension \(\mu_\lambda\) to the stable Borel space \(((L^0)^d, \mathcal{B}((L^0)^d)))\) is called the stable \(d\)-dimensional Lebesgue measure.

Next, we study the link to the conditional distribution of random variables. For the remainder of this section, let \(\mathcal{G} \subseteq \mathcal{F}\) be a sub-\(\sigma\)-algebra and \(\xi : \Omega \rightarrow E\) a random variable. Recall that a regular conditional distribution of \(\xi\) given \(\mathcal{G}\) is a version of the function \(P[\xi \in \cdot | \mathcal{G}]\) on \(\Omega \times \mathcal{E}\) which is a probability kernel. It is well known that such a representing probability kernel exists whenever \((E, \mathcal{E})\) is a standard Borel space, see e.g. [21, Theorem 6.3]. In view of Theorem 4.1, we can extend this representation result to arbitrary spaces as follows.
Example 4.3. All objects will be defined relative to the probability space \((\Omega, \mathcal{G}, \mathbb{P})\), e.g. \(\mathcal{G}\)-stable sets, \(\mathcal{G}\)-stable conditional power sets, \(\mathcal{G}\)-stable conditional set operations, etc. Let \(L^0_{m, \mathcal{G}}(E)\) denote the space of all \(\mathcal{G}\)-measurable functions \(x: \Omega \to E\). For a sequence \((F_k)\) in \(\mathcal{G}\), a partition \((A_k)\) with 

\[
\sum L^0_{m, \mathcal{G}}(F_k)[A_k] := \{x \in L^0_{m, \mathcal{G}}(E): x|_{A_k}(\omega) \in F_k \text{ a.s. for all } k\}.
\]

Let \(R_\mathcal{G}\) be the \(\mathcal{G}\)-stable ring on \(L^0_{m, \mathcal{G}}(E)\) consisting of all \((\sum L^0_{m, \mathcal{G}}(F_k)[A_k])A\), where \((F_k)\) is a sequence in \(\mathcal{G}\). \((A_k)\) is a partition with \(A_k \in \mathcal{G}\) for all \(k\) and \(A \in \mathcal{G}\). We denote by \(\Sigma_\mathcal{G}(R_\mathcal{G})\) the smallest \(\mathcal{G}\)-stable \(\sigma\)-algebra on \(L^0_{m, \mathcal{G}}(E)\) including \(R_\mathcal{G}\). Then there exists a unique \(\mathcal{G}\)-stable probability measure on \((L^0_{m, \mathcal{G}}(E), \Sigma_\mathcal{G}(R_\mathcal{G}))\), denoted by \(\nu = \nu_{\xi, \mathcal{G}}\), such that

\[
P[\xi \in F|\mathcal{G}](\omega) = \nu(L^0_{m, \mathcal{G}}(F))(\omega) \text{ a.s. for all } F \in \mathcal{G}.
\]

Indeed, put

\[
\nu((\sum L^0_{m, \mathcal{G}}(F_k)[A_k])[A]) := \{\mathbb{P}[\xi \in F|\mathcal{G}][A_k][A + 0]A^c, \quad (\sum L^0_{m, \mathcal{G}}(F_k)[A_k])[A] \in R_\mathcal{G},
\]

and proceed similarly to the first part of the proof of Theorem 4.4.

We want to compute the conditional expectation of \(f(\xi)\) with the help of the stable Lebesgue integral on applying the representation \((4.5)\). Fix a Borel measurable function \(f: E \to \mathbb{R}\) such that \(\mathbb{E}[|f(\xi)|] < \infty\). Recall that the distribution of \(\xi\) is defined by the pushforward measure \(\hat{\nu}_\xi := P \circ \xi^{-1}\) on \((E, \mathcal{G})\). By the transformation theorem, it holds

\[
\mathbb{E}[f(\xi)] = \int_E f \, d\hat{\nu}_\xi.
\]

If \((E, \mathcal{G})\) is a standard Borel space, then we have the following conditional analogue of \((4.6)\):

\[
\mathbb{E}[f(\xi)|\mathcal{G}](\omega) = \int f(x) \kappa_\xi(\omega, dx) \text{ a.s.,}
\]

where \(\kappa_\xi\) is a regular conditional distribution of \(\xi\) given \(\mathcal{G}\). We can extend the representation \((4.7)\) to spaces which are not necessarily standard Borel in the following way.

**Theorem 4.4.** Let \(f: E \to \mathbb{R}\) be a Borel measurable function with \(\mathbb{E}[|f(\xi)|] < \infty\). Then \(f\) extends to a \(\mathcal{G}\)-stable integrable function \(\hat{f}: L^0_{m, \mathcal{G}}(E) \to L^0_\mathcal{G}\) such that

\[
\mathbb{E}[f(\xi)|\mathcal{G}](\omega) = \left(\int_{L^0_{m, \mathcal{G}}(E)} \hat{f} \, d\nu_{\xi, \mathcal{G}}\right)(\omega) \text{ a.s.}
\]

**Proof.** By inspection, \(\hat{f}: L^0_{m, \mathcal{G}}(E) \to L^0_\mathcal{G}\) given by \(\hat{f}(x) := f \circ x\) is well-defined and \(\mathcal{G}\)-stable. We show that \(\hat{f}\) satisfies \((4.8)\). We can assume w.l.o.g. that \(f \geq 0\). First, suppose \(f = \sum_{k \leq n} x_k 1_{F_k}\). Then \(\hat{f}\) is equal to the \(\mathcal{G}\)-stable elementary function \(\sum_{k \leq n} x_k 1_{L^0_{m, \mathcal{G}}(F_k)}\). It follows from (13) and \((4.5)\) that

\[
\mathbb{E}[f(\xi)|\mathcal{G}](\omega) = \sum_{k \leq n} x_k \nu(L^0_{m, \mathcal{G}}(F_k))[A_k](\omega) = \left(\int_{L^0_{m, \mathcal{G}}(E)} \hat{f} \, d\nu_{\xi, \mathcal{G}}\right)(\omega) \text{ a.s.}
\]

Second, suppose \(f \geq 0\). Let \((f_n)\) be an increasing sequence of simple functions such that \(f = \lim f_n\). Then \((f_n)\) is an increasing sequence of stable elementary functions such that \(\hat{f}\) is the pointwise limit of \((f_n)\). By the monotone convergence theorem for conditional expectations and Theorem 3.20, we obtain from \((4.9)\)

\[
\mathbb{E}[f(\xi)|\mathcal{G}](\omega) = \lim_n \mathbb{E}[f_n(\xi)|\mathcal{G}](\omega) = \lim_n \left(\int_{L^0_{m, \mathcal{G}}(E)} \hat{f}_n \, d\nu_{\xi, \mathcal{G}}\right)(\omega) = \left(\int_{L^0_{m, \mathcal{G}}(E)} \hat{f}_n \, d\nu_{\xi, \mathcal{G}}\right)(\omega) \text{ a.s.}
\]

\[\square\]
5. A conditional version of Fubini’s theorem, the Radon-Nikodým theorem, the Daniell-Stone theorem and the Riesz representation theorem

In this section, we establish four important theorems in measure theory for stable measure spaces. We start with Fubini’s theorem and the Radon-Nikodým theorem, and close with the Daniell-Stone theorem from which we derive two Riesz type representation and regularity results.

5.1. Fubini’s theorem. Throughout this subsection, let \((X, \mathcal{X}, \mu)\) and \((Y, \mathcal{Y}, \nu)\) be two stable probability spaces. On the stable set \(X \times Y\) of functions on \(\Omega\) consider the conditional rectangles \(V|A \times W|B := V \times W|A \cap B\) where \(V|A \in \mathcal{X}\) and \(W|B \in \mathcal{Y}\). Inspection shows

\[
V_1|A_1 \times W_1|B_1 \cap V_2|A_2 \times W_2|B_2 = V_1|A_1 \cap V_2|A_2 \times W_1|B_1 \cap W_2|B_2,
\]

which implies that the collection

\[
\mathcal{E} := \{V|A \times W|B : V|A \in \mathcal{X}, W|B \in \mathcal{Y}\}
\]

is a stable collection in \(\mathcal{P}(X \times Y)\) closed under finite conditional intersections. We define the stable product \(\sigma\)-algebra of \(\mathcal{X}\) and \(\mathcal{Y}\) to be the stable \(\sigma\)-algebra generated by \(\mathcal{E}\), and denote it by \(\mathcal{X} \otimes \mathcal{Y}\). Let \(V|A \subseteq X \times Y\) and \(x \in X\). The conditional \(x\)-section of \(V|A\) is defined to be the conditional set

\[
(V|A)_x := W|D_x,
\]

where

\[
D_x := \sup\{A' \in \mathcal{F} : A' \subseteq A\}, \text{ there is } y \in Y \text{ such that } (x, y)|A' \in V|A',\]

\[
W := \{y \in Y : (x, y)|D_x \in V|D_x\}.
\]

Since \(D_x\) is attained by an exhaustion argument, \(W\) is a stable subset of \(Y\) by stability of \(V\). Similarly, we define the conditional \(y\)-section of \(V|A\) for some \(y \in Y\). For a function \(f : X \times Y \to E\), we denote by \(f_x\) and \(f_y\) its \(x\)- and \(y\)-section, respectively, which are defined classically as \(f_x(y) = f_y(x) = f(x, y)\). If \(f\) is a stable function, then \(f_x\) and \(f_y\) are stable functions as well. We have the following useful properties which we state for (conditional) \(x\)-sections; analogous properties hold for \(y\)-sections.

**Proposition 5.2.** Let \(Z|C \in \mathcal{P}(X \times Y)\), \((Z|C_i)_{i \in I}\) and \((Z|A_k)_{k \in \mathbb{N}}\) be families in \(\mathcal{P}(X \times Y)\), where \(I\) is an arbitrary nonempty index set. Let \(f : X \times Y \to L^0\) be a function. Let \(x \in X\), \((x_k)\) be a sequence in \(X\) and \((D_k)\) a partition. Then the following holds true.

1. \((\sum(Z|A_k)|D_k)_x = \sum(Z|A_k)_x|D_k\); \((5.1)\)
2. \((Z|C)|\sum x|D_k = \sum(Z|C)_x|D_k\); \((5.2)\)
3. \((\sqcup i|Z_i|C_i|) = \sqcup i|Z_i|C_i|_x\); \((5.3)\)
4. \((\sqcup i|Z_i|C_i|_x)^\ast = ((\sqcup i|Z_i|C_i|)^\ast)_x\); \((5.4)\)
5. if \(Z|C \in \mathcal{X} \otimes \mathcal{Y}\), then \((Z|C)_x \in \mathcal{Y}\); \((5.5)\)
6. if \(f\) is conditionally \(X \otimes \mathcal{Y}\)-measurable, then \(f_x\) is conditionally \(\mathcal{Y}\)-measurable. \((5.6)\)
7. for \(Z|C \in \mathcal{X} \otimes \mathcal{Y}\), the function \(x \mapsto \nu ((Z|C)_x)\) is conditionally \(\mathcal{X}\)-measurable. \((5.7)\)

**Proof.** The statements (F1) and (F2) can easily be verified from the definitions.

(F3) Suppose \(\sqcup i|Z_i|C_i = Z|C\) where \(C = \sqcup i|C_i\). Let

\[
D_x = \sup\{D' \in \mathcal{F} : D' \subseteq C\}, \text{ there is } y \in Y \text{ such that } (x, y)|D' \in Z|D',\]

\[
D_x = \sup\{D' \in \mathcal{F} : D' \subseteq C_i\}, \text{ there is } y \in Y \text{ such that } (x, y)|D' \in Z|D_i'.
\]

By definition of the conditional union (see (4.3)), one has \(D_x \subseteq \sqcup i|D_i'\). To see the converse inclusion, let \((D_k)\) be such that \(x|D_k = \sup_k D_k\). Let \((y_k)\) be a sequence in \(Y\) such that \((x, y_k)|D_k \in Z_i|D_i_k\) for all \(k\). Let \(C_k = D_k \cap \cap (C_1 \cup \ldots \cup C_{i-1})^c\) for \(k \geq 2\). For \(y = \sum y_k|C_k + y_0|\sup_i D_k\) \((2)\), where \(y_0 \in Y\) is arbitrary, one has \((x, y)|D_k \in Z|\sup_k D_k\), which shows \(\sqcup i|D_i = \sup_k D_k \subseteq D\). We conclude that \((Z|C)_x = \sqcup i|Z_i|C_i|_x\).

(F4) By (F3),

\[
Y = (X \times Y)_x = (Z|C \sqcup (Z|C)^\ast)_x = (Z|C)_x \sqcup ((Z|C)^\ast)_x.
\]

Suppose \((Z|C)^\ast \in W|B\). Let

\[
D_x = \sup\{D' \in \mathcal{F} : D' \subseteq C\}, \text{ there is } y \in Y \text{ such that } (x, y)|D' \in Z|D',\]

\[
D_x = \sup\{D' \in \mathcal{F} : D' \subseteq B\}, \text{ there is } y \in Y \text{ such that } (x, y)|D' \in W|D'\).
Lemma 5.3. There exists a unique stable probability measure \( \lambda \) on \( \mathcal{X} \otimes \mathcal{Y} \) such that

\[
\lambda(V|A \times W|B) = \mu(V|A)\nu(W|B)
\]

for all \( V|A \in \mathcal{X} \) and \( W|B \in \mathcal{Y} \). Moreover, for all \( Z|C \in \mathcal{X} \otimes \mathcal{Y} \), one has

\[
\lambda(Z|C) = \int_X s_{Z|C}^X d\mu = \int_Y s_{Z|C}^Y d\nu,
\]

where \( s_{Z|C}^X(x) := \nu((Z|C)_x), x \in X, \) and \( s_{Z|C}^Y(y) := \mu((Z|C)_y), y \in Y \).

We call \( \mu \otimes \nu := \lambda \) the stable product measure of \( \mu \) and \( \nu \) and the triple \( (X \times Y, \mathcal{X} \otimes \mathcal{Y}, \mu \otimes \nu) \) the stable product probability space.

Proof. For \( Z|C \in \mathcal{X} \otimes \mathcal{Y} \), define

\[
\lambda(Z|C) := \int_X s_{Z|C}^X d\mu.
\]

By (F2), the stability of \( \nu \) and (I1), \( \lambda \) is a stable function on \( \mathcal{X} \otimes \mathcal{Y} \) satisfying (M1). That \( \lambda \) satisfies (M2) follows from (F3) and Theorem 3.20. For \( V|A \in \mathcal{X} \) and \( W|B \in \mathcal{Y} \), one has

\[
\lambda(V|A \times W|B) = \nu(V|A)\mu(W|B).
\]

By analogous arguments, the mapping

\[
\mathcal{X} \otimes \mathcal{Y} \ni Z|C \mapsto \lambda'(Z|C) := \int_Y s_{Z|C}^Y d\nu
\]

is a stable probability measure with the above property. Since \( \mathcal{E} \) as defined in (5.1) is a stable generator of \( \mathcal{X} \otimes \mathcal{Y} \) closed under finite conditional intersections, the claim follows from Proposition 3.13.

We have the following version of Fubini’s theorem, which can easily be extended to \( \sigma \)-finite stable measure spaces.

Theorem 5.4. Let \( f : X \times Y \to L^0 \) be a stable integrable function with respect to \( (X \times Y, \mathcal{X} \otimes \mathcal{Y}, \mu \otimes \nu) \). Then the functions

\[
x \mapsto \int_Y f_x d\nu \quad \text{and} \quad y \mapsto \int_X f_y d\mu
\]

are stable integrable functions, and one has

\[
\int_{X \times Y} f d\mu \otimes \nu = \int_X \left( \int_Y f_x d\nu \right) d\mu = \int_Y \left( \int_X f_y d\mu \right) d\nu.
\]
Proof. First, suppose that \( f \) is a stable elementary function of the form \( \sum_{k \leq n} r_k 1_{A_k} \). Then \( f = \sum_{k \leq n} r_k 1_{A_k} \) is stably measurable due to (F6). By (F7),

\[
x \mapsto \int_Y f_x d\nu = \sum_{k \leq n} r_k \nu(A_k)
\]

is stably measurable and \( L^0 \)-bounded. By Lemma 5.3,

\[
\int_X \left( \int_Y f_x d\nu \right) d\mu = \sum_{k \leq n} r_k \mu(A_k) = \int_X f d\mu \otimes \nu.
\]

Second, let \( f \) be a non-negative stable integrable function, and choose with the help of Lemma 3.18 an increasing sequence \((f_n)\) of stable elementary functions such that \( f = \sup_n f_n \). By Theorem 3.20 (12) and the first step,

\[
\int_{X \times Y} f d\mu \otimes \nu = \int_X \left( \int_Y f_x d\nu \right) d\mu.
\]

Finally, for an arbitrary stable integrable function \( f \), the previous equality follows from the identities \( |f| = (f^+) + (f-) \) and \( f_x = (f^+)_x - (f^-)_x \), (I3) and the first and second step. Analogously, one can prove

\[
\int_{X \times Y} f d\mu \otimes \nu = \int_Y \left( \int_X f_y d\mu \right) d\nu,
\]

which finishes the proof. \(\square\)

Remark 5.5. A stable function \( K: X \times Y \to L^0 \) is said to be a stable Markov kernel if

(i) \( K(x, \cdot) \) is a stable probability measure for all \( x \in X \),

(ii) \( K(\cdot, V|A) \) is stably measurable for all \( V|A \in \mathcal{Y} \).

We can extend Fubini’s theorem to stable Markov kernels. We provide the statement of this extension in the following, but will omit a proof as it can be worked out with a similar strategy as above. Let \( \mu \) be a stable probability measure on \((X, \mathcal{X})\). Then \( K \otimes \mu(V|A) := \int_X K(x, (V|A)_x) d\mu \) defines a stable probability measure on \( \mathcal{X} \otimes \mathcal{Y} \). If \( f: X \times Y \to L^0 \) is stably integrable w.r.t. \( K \otimes \mu \), then

\[
\int_{X \times Y} f dK \otimes \mu = \int_X \int_Y f(x, y) K(x, dy) d\mu.
\]

5.6. Radon-Nikodým theorem. Throughout this subsection, fix a stable measurable space \((X, \mathcal{X})\), and let \( \mu \) be a stable probability measure on \((X, \mathcal{X})\). If \( f \geq 0 \) is a stable integrable function with \( \int_X f d\mu = 1 \), then from (D1)-(D3) and Theorem 3.20 we have that

\[
\nu(V|A) := \int_X 1_{V|A} f d\mu, \quad V|A \in \mathcal{X},
\]

is a stable probability measure absolutely continuous w.r.t. \( \mu \), where a stable probability measure \( \nu \) is said to be absolutely continuous w.r.t. \( \mu \) whenever \( V|A \in \mathcal{X} \) and \( \mu(V|A) = 0 \) implies \( \nu(V|A) = 0 \). In this subsection, we prove the following converse statement.

Theorem 5.7. If \( \nu \) is a stable probability measure absolutely continuous relative to \( \mu \), then there exists a stable integrable function \( f: X \to L^0 \) such that

\[
\nu(V|A) = \int_X 1_{V|A} f d\mu,
\]

for all \( V|A \in \mathcal{X} \).

By a straightforward extension, a Radon-Nikodým theorem can also be established in the case where \( \mu \) and \( \nu \) are \( \sigma \)-finite. The proof is based on the following auxiliary result.

Lemma 5.8. Let \( \mu_1 \) and \( \mu_2 \) be finite stable measures on \((X, \mathcal{X})\) and define \( \mu_3 := \mu_2 - \mu_1 \). Then there exists \( X_0|A_0 \in \mathcal{X} \) such that

(i) \( \mu_3(X) \leq \mu_3(X_0|A_0) \);

(ii) \( 0 \leq \mu_3(V|A) \) for all \( V|A \in \mathcal{X} \) with \( V|A \subseteq X_0|A_0 \).

Proof. First, we establish the weaker statement: For all \( r \in L^0_{++} \) there is \( X_r|A_r \in \mathcal{X} \) such that
We prove Theorem 5.7.

We can recursively choose $B_i$ and $\mu_i$ which contradicts the finiteness of $\mu$ and we can repeat the previous procedure. If this procedure does not yield the desired after finitely many steps, we obtain a sequence $(B_n)_{n\geq 0}$ of pairwise disjoint elements in $\mathcal{F}$ and a sequence $(V_n|A_n)_{n\geq 1}$ of pairwise disjoint sets in $\mathcal{X}$ such that

\begin{align*}
\mu_3((V_1|A_1)\cup \ldots \cup V_n|A_n)^c &\subseteq (B_0 \cup B_1 \cup \ldots \cup B_n)^c \\
+ (V_1|A_1 \cup \ldots \cup V_{n-1}|A_{n-1})^c &\subseteq B_n + \ldots + (V_1|A_1)^c &|B_2 + X|B_1 + \{\ast\}|B_0) \\
\mu_3(V_n|A_n) &\geq \mu_3(X),
\end{align*}

and

$$
\mu_3((B_0 \cup B_1 \cup \ldots \cup B_n)^c \subseteq -\tau|(B_0 \cup B_1 \cup \ldots \cup B_n)^c
$$

for all $n \geq 1$. If $B := \cap_{n\geq 0}B_n \in \mathcal{F}$, then by (M2),

$$
\mu_3(\cup_n V_n|A_n \cap B) = \sum_n \mu_3(V_n|A_n)|B + 0|B^c = -\infty|B + 0|B^c
$$

which contradicts the finiteness of $\mu_3$. Thus $(B_n)_{n\geq 0}$ is a partition of $\Omega$ and $X_r|A_r := \sum W_n|B_n$ where

$$
W_0 := \{\ast\}, W_1 := X, W_n := (V_1|A_1 \cup \ldots \cup V_{n-1}|A_{n-1})^c, n \geq 2,
$$

satisfies (i)' and (ii)'.

Second, we apply the previously established weaker statement to prove the claim. For every $n \in \mathbb{N}$, we can recursively choose $X_{1/n}|A_{1/n}$ such that $\mu_3(X_{1/n}|A_{1/n}) \leq -1/n < \mu_3(V|A)$ for all $V|A \in \mathcal{X}$ with $V|A \subseteq X_{1/n}|A_{1/n}$ and such that $X_{1/(n+1)}|A_{1/(n+1)} \subseteq X_{1/n}|A_{1/n}$. Then $X_0|A_0 := \cap_{n\geq 0}X_{1/n}|A_{1/n}$ fulfills (i)' and (ii)' due to (M8).

We prove Theorem 5.7.

**Proof.** Let $\mathcal{H}$ be the collection of all stable integrable functions $f : X \to L_+^0$ such that

$$
\int_X 1_{V|A}f \, d\mu \leq \nu(V|A)
$$

for all $V|A \in \mathcal{X}$ with $V|A \subseteq X_{1/n}|A_{1/n}$ and such that $X_{1/(n+1)}|A_{1/(n+1)} \subseteq X_{1/n}|A_{1/n}$. Then $X_0|A_0 := \cap_{n\geq 0}X_{1/n}|A_{1/n}$ fulfills (i)' and (ii)' due to (M8).
for all $V|A \in \mathcal{X}$. Inspection shows that $\mathcal{H}$ is upward directed. Let $r = \sup_{f \in \mathcal{H}} \int_X f \, d\mu \leq 1$. There exists an increasing sequence $(f_k)$ in $\mathcal{H}$ such that $\sup_k \int_X f_k \, d\mu = r$. By Theorem 3.20, we have $\int_X f \, d\mu = r$ for $f := \sup_k f_k$. By another application of Theorem 3.20 we see that

$$\tilde{\nu}(V|A) := \int_X 1_{V|A} f \, d\mu \leq \nu(V|A) \quad \text{for all } V|A \in \mathcal{X}.$$ 

It remains to show that $\lambda := \nu - \tilde{\nu} = 0$. By contradiction, suppose that $C := \{\lambda(X) > 0\} \in \mathcal{F}_+$. W.l.o.g. assume $C = \Omega$. Let $s := \lambda(X)/2$. Applying Lemma 5.8 to $\mu_2 = \lambda$ and $\mu_1 = s\mu$ supplies us with $X_0|A_0 \in \mathcal{X}$ such that $\lambda(X_0|A_0) - s\mu(X_0|A_0) \geq \lambda(X) - s\mu(X) > 0$ and $\lambda(V|A) \geq s\mu(V|A)$ for all $V|A \in \mathcal{X}$ with $V|A \subseteq X_0|A_0$. It can be checked that $f := f + s1_{X_0|A_0} \in \mathcal{H}$ with $\int_X \tilde{f} \, d\mu > r$ which is a contradiction. $\square$

5.9. Daniell-Stone theorem and Riesz representation theorem. The Daniell-Stone theorem and the Riesz representation theorem provide sufficient conditions under which a positive linear functional on a vector lattice is an integral. In this last section, we establish a conditional version of both theorems which state when a positive $L^0$-linear function on a stable vector lattice is a stable integral. The proof of the Daniell-Stone theorem in the present setting is an adaptation of a proof of the classical statement by using Carathéodory’s extension theorem, and therefore we omit its proof. We prove Riesz type representation and regularity results for positive $L^0$-linear functions on the stable vector lattice of all stable sequentially continuous functions $f : (L^0)^d \rightarrow L^0$ and its stable subspace consisting of functions with stably compact conditional support, respectively.

Consider on $(L^0)^d = L^0(\mathbb{R}^d)$ the $L^0$-valued Euclidean norm, see Example 3.35, and endow it with the stable Borel $\sigma$-algebra $\mathcal{B}^d := \mathcal{B}((L^0)^d)$, see Example 3.32. A conditional subset $V|A \subseteq (L^0)^d$ is said to be

- $L^0$-bounded if there exists $M \in L^0$ such that $\|x|A\| = \|x\||A| \leq M|A|$ for all $x|A \in V|A$;
- sequentially closed if $V|A$ includes the limit of every a.s. convergent sequence in $V|A$;
- stably compact if $V|A$ is $L^0$-bounded and sequentially closed

Let $X$ be a stable set of functions on $\Omega$. Recall that for a sequence $(f_k)$ of stable functions $f_k : X \rightarrow L^0$ and a partition $(A_k)$, we define $\sum f_k|A_k : X \rightarrow L^0$ by $(\sum f_k|A_k)(x) := \sum f_k(x)|A_k$, $x \in X$. We call a space $\mathcal{L}$ of functions $f : X \rightarrow L^0$ stable if $\mathcal{L} \neq \emptyset$ and $\sum f_k|A_k \in \mathcal{L}$ for all sequences $(f_k)$ in $\mathcal{L}$ and every partition $(A_k)$ of $\Omega$.

Definition 5.10. Let $X$ be a stable set of functions on $\Omega$. A set $\mathcal{L}$ of stable functions $f : X \rightarrow L^0$ is said to be a stable Stone vector lattice if $\mathcal{L}$ is stable and $f + rg, \min\{f, g\}, \min\{f, 1\} \in \mathcal{L}$ for all $f, g \in \mathcal{L}$ and $r \in L^0$.

Examples 5.11. Let $cl(V|A)$ denote the sequential closure of $V|A$. Let $f : (L^0)^d \rightarrow L^0$ be a stable function. Its conditional support is defined by

$$\text{supp}(f) := cl(f^{-1}(\{0\}^c)),$$

and $f$ is said to have stably compact conditional support if supp$(f)$ is stably compact. A function $f : (L^0)^d \rightarrow L^0$ is said to be sequentially continuous if $f(x_n) \rightarrow f(x)$ a.s. whenever $x_n \rightarrow x$ a.s. Let $\mathcal{L}$ denote the space of all stable and sequentially continuous functions, and $\mathcal{C}_c$ denote its subspace of functions with stably compact conditional support. Then both $\mathcal{C}$ and $\mathcal{C}_c$ are stable Stone vector lattices.

Given a stable Stone vector lattice $\mathcal{L}$, a function $L : \mathcal{L} \rightarrow L^0$ is said to be

- stable if $L(\sum f_k|A_k) = \sum L(f_k)|A_k$ for all sequences $(f_k)$ in $\mathcal{L}$ and every partition $(A_k)$;
- $L^0$-linear if $L(f + rg) = L(f) + rL(g)$ for all $f, g \in \mathcal{L}$ and $r \in L^0$;
- continuous from above if $L(f_n) \downarrow 0$ a.s. whenever $f_n(x) \downarrow 0$ a.s. for all $x \in X$.

We have the following conditional version of the Daniell-Stone theorem.

Theorem 5.12. Let $\mathcal{L}$ be a stable Stone vector lattice and $L : \mathcal{L} \rightarrow L^0$ stable, $L^0$-linear, continuous from above and such that $L(f) \geq 0$ whenever $f \geq 0$. Then there exists a stable measure $\mu$ on $\Sigma(\mathcal{L}) := \Sigma(\{f^{-1}(V|A) : V|A \in B(L^0), f \in \mathcal{L}\})$.

\footnote{Stable compactness refers to the interpretation of conditional compactness in a classical context. The characterization, which is stated above, is based on a classical version of the Heine-Borel theorem, see [2] Theorem 4.6.}

\footnote{Here, 1 denotes the function with the constant value $1 = 1_\Omega \in L^0$.}
such that
\[ L(f) = \int_X fd\mu \quad \text{for all } f \in \mathcal{L}. \]

In the following proofs of the conditional Riesz representation results it is necessary to pass from the sequential continuity of a sequence of stable functions to its sequential uniform continuity. This can be achieved by the following conditional version of Dini’s theorem. In the statement of the next lemma, we use the following extension of a sequence of stable functions. If \((f_n)\) is a classical sequence of stable functions, we can extend \((f_n)\) to a stable net parametrized by \(L^0_+(\mathbb{N})\) by defining \(f_n := \sum n_k |A_k| \) for \(n = \sum n_k |A_k| \in L^0_+(\mathbb{N})\).

**Lemma 5.13.** Let \(X \subset (L^0)^d\) be stably compact and \((f_n)\) a decreasing sequence of stable and sequentially continuous functions \(f_n: X \to L^0\). Let \(f: X \to L^0\) be stably sequentially continuous such that \(f_n(x) \to f(x)\) a.s. for all \(x \in X\). Then, for every \(r \in L^0_+\) there exists \(n_0 \in L^0_+(\mathbb{N})\) such that \(\sup_{x \in X} |f_n(x) - f(x)| < r\) for all \(n \geq n_0\).

**Proof.** The statement can be proved similarly to the respective classical statement, see e.g. [11 Theorem 2.66], by using the characterization of stable compactness in terms of stable open coverings, see [9]. □

We introduce the following regularity conditions for stable measures on \((L^0)^d\).

**Definition 5.14.** A stable measure \(\mu\) on \(B^d\) is called

- conditionally closed regular whenever
  \[ \mu(V|A) = \sup\{\mu(W|B) : W|B \subseteq V|A \text{ sequentially closed}\} \]
  for all \(V|A \in B^d\);

- conditionally regular whenever
  \[ \mu(V|A) = \sup\{\mu(W|B) : W|B \subseteq V|A \text{ stably compact}\} \]
  for all \(V|A \in B^d\).

**Theorem 5.15.** Let \(L : C \to L^0\) be stable, \(L^0\)-linear and such that \(L(f) \geq 0\) whenever \(f \geq 0\). Then there exists a conditionally regular finite stable measure \(\mu\) on \(B^d\) such that
\[ L(f) = \int_{(L^0)^d} fd\mu \quad \text{for all } f \in C. \]

**Proof.** Let \((f_l)\) be a sequence of stable functions in \(C\) such that \(f_l \downarrow 0\). Let \(V_k := \{x \in (L^0)^d : \|x\| \leq k\}\) for \(k \in \mathbb{N}\), and notice that \(V_k\) is stably compact. By triangle inequality, \(d(\cdot, V_k) := \inf\{\|\cdot - y\| : y \in V_k\}\) is sequentially continuous, and by [6 Theorem 4.4], the infimum is attained since it is also stable. Define \(g_k := \max(1 - d(\cdot, V_k), 0)\) and \(h_{k,l} := (1 - g_k)f_l + (1 - g_k)f_l\|/\|2k\) for all \(k, l \in \mathbb{N}\). By definition, \(f_l \leq h_{k,l}\) for all \(k, l\). Fix \(r \in L^0_+\). By changing, if necessary, to a stable net, let \(k \in L^0_+(\mathbb{N})\) be large enough such that \(1/(2k)L((f_l(1 - g_k)) \| ) < r/2\). By Lemma 5.13 choose \(l \in L^0_+(\mathbb{N})\) sufficiently large such that \(L(g_k f_l) < r/2\). We obtain
\[ L(f_l) \leq L(g_k f_l) + 1/(2k)L((1 - g_k)f_l \| ) < r. \]

By Theorem 5.2 there exists a finite stable measure \(\mu\) on \(B^d\) such that
\[ L(f) = \int_{(L^0)^d} fd\mu \quad \text{for all } f \in C. \]

As for the conditional tightness of \(\mu\), let
\[ \mathcal{G} := \{V|A \in B^d : V|A \text{ and } (V|A)^\subset \text{ are conditionally closed regular}\}. \]

It follows from (M5) that \((L^0)^d \in \mathcal{G}\), and thus from (S4) and the stability of \(\mu\) that \(\mathcal{G}\) is a stable collection. By definition, \(\mathcal{G}\) is closed under conditional complementation. Let \((W_n|B_n)\) be a sequence in \(\mathcal{G}\). Fix \(r \in L^0_+\). Let \(Z_n|C_n \subseteq W_n|B_n\) be sequentially closed with \(\mu(W_n|B_n \cap (Z_n|C_n)^\subset ) < r/2^n\) for each \(n\). By (M6) and Boolean arithmetic, we obtain \(\mu((\sqcup_n W_n|B_n) \cap (\sqcup_n (Z_n|C_n)^\subset )) < r\). Since \(r \in L^0_+\) is arbitrary, we have \(\sqcup_n W_n|B_n \in \mathcal{G}\). It follows from (M5) that also \((\sqcup_n W_n|B_n)^\subset \in \mathcal{G}\). It remains to show that every stable open ball is conditionally closed regular. Let \(B_r(x)\) be a stable open ball. Define \(U_n := \{x \in (L^0)^d : d(x, B_r(x)) \geq 1/n\}\). Since \((U_n)\) is a sequence of stable sequentially closed sets with \(\sqcup_n U_n = B_r(x)\), it follows from (M7) that \(B_r(x)\) is conditionally closed regular. Finally,
for every $V|A \in \mathcal{B}^d$ we have $\mu(V|A \cap V_n) \uparrow \mu(V|A)$ due to (M7), where $V_n := \{x \in (L^0)^d; \|x\| \leq n\}$ for $n \in \mathbb{N}$. Moreover, for every $n$ there exists a sequentially closed set $W_n|B_n \subseteq V|A \cap V_n$ such that $\mu(W_n|B_n) \geq \mu(V|A \cap V_n) - 1/n$. Since a stable sequentially closed subset of a stably compact set is stably compact (see [9] Proposition 3.27), it follows that $\mu$ is conditionally tight.

\begin{theorem}
Let $L : C_c \to L^0$ be stable, $L^0$-linear and such that $L(f) \geq 0$ whenever $f \geq 0$. Then there exists a stable measure $\mu$ on $\mathcal{B}^d$ such that
\[ L(f) = \int_{(L^0)^d} f d\mu \quad \text{for all } f \in C_c. \]
Moreover, it holds
\begin{itemize}
  \item $\mu([x, y)) < \infty$ for all stably compact intervals $[x, y) := \{z \in (L^0)^d; x \leq z \leq y\}$
  \item $\mu(V|A) = \sup \{\mu(W|B); W|B \subseteq V|A \text{ stably compact}\}$ for all $V|A \in \mathcal{B}^d$ with $\mu(V|A) < \infty$.
\end{itemize}
\end{theorem}

\begin{proof}
In order to obtain the assumptions of Theorem 5.12, given a sequence $(f_n)$ of stable functions in $C_c$ such that $f_n \downarrow 0$, apply Lemma 5.13 to the sequence $(\text{supp}(f_n))$. We have $\mu([x, y]) \leq \int_{(L^0)^d} f d\mu = L(f) < \infty$, where $f = \max \{1 - d([x, y]), 0\}$. The regularity condition can be shown similarly to Theorem 6.13.
\end{proof}

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\footnote{We consider on $(L^0)^d$ the order $(x_1, \ldots, x_d) \leq (y_1, \ldots, y_d)$ if and only if $x_i \leq y_i$ for all $i = 1, \ldots, d$.}
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