A note on independence number, connectivity and $k$-ended tree

Pham Hoang Ha*
Department of Mathematics
Hanoi National University of Education
136 XuanThuy Street, Hanoi, Vietnam

Abstract

A $k$-ended tree is a tree with at most $k$ leaves. In this note, we give a simple proof for the following theorem. Let $G$ be a connected graph and $k$ be an integer ($k \geq 2$). Let $S$ be a vertex subset of $G$ such that $\alpha_G(S) \leq k + \kappa_G(S) - 1$. Then, $G$ has a $k$-ended tree which covers $S$. Moreover, the condition is sharp.

Keywords: independence number, connectivity, $k$-ended tree

AMS Subject Classification: 05C05, 05C40, 05C69

1 Introduction

In this note, we only consider finite simple graphs. Let $G$ be a graph with vertex set $V(G)$ and edge set $E(G)$. A subset $X \subseteq V(G)$ is called an independent set of $G$ if no two vertices of $X$ are adjacent in $G$. The maximum size of independent sets in $G$ is denoted by $\alpha(G)$. A graph $G$ is $k$-connected if it has more than $k$ vertices and every subgraph obtained by deleting fewer than $k$ vertices is connected; the connectivity of $G$, written $\kappa(G)$, is the maximum $k$ such that $G$ is $k$-connected. For any $S \subseteq V(G)$, we denote by $|S|$ the cardinality of $S$. We define $\alpha_G(S)$ the maximum cardinality of independent sets of $S$ in $G$, which is called the independence number of $S$ in $G$. For two vertices $x, y$ of $G$, the local connectivity $\kappa_G(x, y)$ is defined to be the maximum number of internally disjoint paths connecting $x$ and $y$ in $G$. We define $\kappa_G(S) := \min\{\kappa_G(x, y) : x, y \in S, x \neq y\}$. Moreover, if $|S| = 1$, $\kappa_G(S)$ is defined to be $+\infty$. When $S = G$, we have $\alpha_G(G) = \alpha(G)$ and by Menger’s theorem we have $\kappa_G(G) = \kappa(G)$. A Hamiltonian cycle (path) is a cycle (path) which passes through all vertices of a graph.

In 1972, Chvátal and Erdős proved the following famous theorem which related to the independence number, connectivity and Hamiltonian cycle (path) of a graph.

**Theorem 1.1 (Chvátal and Erdős)** Let $G$ be a connected graph.

1. If $\alpha(G) \leq \kappa(G)$, then $G$ has a Hamiltonian cycle unless $G = K_1$ or $K_2$.
2. If $\alpha(G) \leq \kappa(G) + 1$, then $G$ has a Hamiltonian path.

*E-mail address: ha.ph@hnue.edu.vn.
Let $T$ be a tree. A vertex of degree one is a leaf of $T$ and a vertex of degree at least three is a branch vertex of $T$. A tree having at most $k$ leaves is called a $k$-ended tree. Then a Hamiltonian path is nothing but a spanning 2-ended tree. In 1979, Win improved the above result by proving the following theorem.

**Theorem 1.2** ([9, Win]) Let $G$ be a graph and let $k$ be an integer ($k \geq 2$). If $\alpha(G) \leq k + \kappa(G) - 1$, then $G$ has a spanning tree with at most $k$ leaves.

On the other hand, when we consider a cycle (path) containing specified vertices of a graph as a generalization of a Hamiltonian cycle (path), many results were invented.

**Theorem 1.3** ([3, Fournier]) Let $G$ be a 2-connected graph, and let $S \subseteq V(G)$. If $\alpha_G(S) \leq \kappa_G(S)$, then $G$ has a cycle covering $S$.

**Theorem 1.4** ([6, Ozeki and Yamashita]) Let $G$ be a 2-connected graph and let $S \subseteq V(G)$. If $\alpha_G(S) \leq \kappa_G(S)$, then $G$ has a cycle covering $S$.

A natural question is whether Win’s result can be improved by giving a sharp condition to show the existence of the $k$-ended tree covering a given subset of $V(G)$. In this note, we give an affirmative answer to this question. In particular, we prove the following theorem.

**Theorem 1.5** Let $G$ be a connected graph and $k$ be an integer ($k \geq 2$). Let $S$ be a subset of $V(G)$ such that $\alpha_G(S) \leq k + \kappa_G(S) - 1$. Then, $G$ has a $k$-ended tree covering $S$.

It is easy to see that if a tree has at most $k$ leaves ($k \geq 2$), then it has at most $k - 2$ branch vertices. Therefore, we immediately obtain the following corollary from Theorem 1.5.

**Corollary 1.6** Let $G$ be a connected graph and $k$ be an integer ($k \geq 2$). Let $S$ be a subset of $V(G)$ such that $\alpha_G(S) \leq k + \kappa_G(S) - 1$. Then, $G$ has a tree $T$ such that $T$ covers $S$ and has at most $k - 2$ branch vertices.

We first show that the conditions of Theorem 1.5 and Corollary 1.6 are sharp. Let $m, k \geq 1$ be integers, and let $K_{m,m+k} = (A, B)$ be a complete bipartite graph with $|A| = m, |B| = m+k$. Set $S = B$. Then we are easy to see that $\alpha_G(S) = k + \kappa_G(S)$ and every tree covering $S$ has at most $k+1$ leaves. Moreover it also has at most $k-1$ branch vertices. Therefore, the conditions of Theorem 1.7 and Corollary 1.6 are sharp.

To prove Theorem 1.5 we prove a slightly stronger following result.

**Theorem 1.7** Let $G$ be a connected graph and $k$ be an integer ($k \geq 2$). Let $S$ be a subset of $V(G)$. Then either $G$ has a $k$-ended tree $T$ covering $S$, or there exists a $k$-ended tree $T$ in $G$ such that

$$\alpha_G(S - V(T)) \leq \alpha_G(S) - \kappa_G(S) - k + 1.$$
2 Proof of Theorem 1.7

By using the same technique in [4], Yan in [8] proved the following result. It needs for the proof of Theorem 1.7.

Lemma 2.1 ([8, Corollary 1]) Let $G$ be a connected graph and $S \subseteq V(G)$. Then either the vertices of $S$ can be covered by one path of $G$, or there exists a path $P$ of $G$ such that

$$\alpha_G(S - V(P)) \leq \alpha_G(S) - \kappa_G(S) - 1.$$ 

Next, we prove Theorem 1.7 by induction on $k(\geq 2)$.

For $k = 2$, by Lemma 2.1 the theorem holds.

Assume that the theorem holds for some $k = t \geq 2$, that is, either the vertices of $S$ can be covered by one $t$-ended tree of $G$, or there exists a $t$-ended tree $T$ of $G$ such that

$$\alpha_G(S - V(T)) \leq \alpha_G(S) - \kappa_G(S) - t - 1. \quad (2.1)$$

If there exists a $(t + 1)$-ended tree such that it covers $S$ then the theorem holds for $k = t + 1$. Otherwise, every $(t + 1)$-ended tree of $G$ does not cover $S$. In particular, $S$ can not be covered by any $t$-ended tree of $G$. By the induction hypothesis, there exists a $t$-ended tree $T$ of $G$ such that (2.1) is correct. Let $S_1, ..., S_m$ be all subsets of $S - V(T)$ such that $|S_i| = \alpha_G(S - V(T))$ for all $i \in \{1, ..., m\}$. For each vertex $s \in \cup_{i=1}^{m} S_i$, since $G$ is connected, there exists some path joining $s$ to $T$. Denote by $P[s, T]$ the set of such paths in $G$. We choose a maximal path $P_0$ in $\{P[s, T] | s \in \cup_{i=1}^{m} S_i\}$. Assume that $P_0$ joins the vertex $s_0 \in \cup_{i=1}^{m} S_i$ to $T$. Now, we prove that $P_0 \cap S_i \neq \emptyset$ for all $i \in \{1, ..., m\}$. Indeed, otherwise, there exists some $j$ such that $P_0 \cap S_j = \emptyset$. By $|S_j| = \alpha_G(S - V(T))$ and $s_0 \in S - V(T)$, there exists some vertex $s_j \in S_j$ such that $s_0s_j \in E(G)$. We consider the path $P' = P_0 + s_0s_j$. Then $P'$ joins $s_j$ to $T$ and $|P'| > |P_0|$, which implies a contradiction with the maximality of $P_0$. Therefore we conclude that $P_0 \cap S_i \neq \emptyset$ for all $i \in \{1, ..., m\}$. Now, we set $T' = T + P_0$. Then $T'$ has at most $(t + 1)$ leaves. On the other hand, because $P_0 \cap S_i \neq \emptyset$ and $|S_i| = \alpha_G(S - V(T))$ for all $i \in \{1, ..., m\}$, we obtain $\alpha_G(S - V(T')) \leq \alpha_G(S - V(T)) - 1 \leq \alpha_G(S) - \kappa_G(S) - t$. So

$$\alpha_G(S - V(T')) \leq \alpha_G(S - V(T)) - 1 \leq \alpha_G(S) - \kappa_G(S) - t.$$ 

This implies that the theorem holds for $k = t + 1$.

Therefore, the theorem holds for all $k \geq 2$ by the principle of mathematical induction. Hence we complete the proof of Theorem 1.7.

Acknowledgements. The research is supported by the NAFOSTED Grant of Vietnam (No. 101.04-2018.03).

References

[1] V. Chvátal and P. Erdős, A note on Hamiltonian circuits, Discrete Math. 2 (1972), 111-113.

[2] S. Chiba, R. Matsubara, K. Ozeki and M. Tsugaki, A $k$-tree containing specified vertices, Graphs Comb. 26 (2010), 187-205.
[3] I. Fournier, *Thèse d’Etat, annexe G*, 172-175, L.R.I., Université de Paris-Sud (1985).

[4] M. Kouider, *Cycles in graphs with prescribed stability number and connectivity*, J. Comb. Theory Ser. B 60 (1994), 315-318.

[5] V. Neumann-Lara and E. Rivera-Campo, *Spanning trees with bounded degrees*, Combinatorica 11 (1991), 55-61.

[6] K. Ozeki and T. Yamashita, *A degree sum condition concerning the connectivity and the independence number of a graph*, Graphs Comb. 24 (2008), 469-483.

[7] K. Ozeki and T. Yamashita, *Spanning trees: A survey*, Graphs Combin., 22 (2011), 1-26.

[8] Z. Yan, *Independence number and k-trees of graphs*, Graphs Combin., 33 (2017), 1089-1093.

[9] S. Win, *On a conjecture of Las Vergeas concerning certain spanning trees in graphs*, Results Math. 2 (1979), 215–224.