REDUCED WORDS FOR CLANS

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Abstract. Clans are combinatorial objects indexing the orbits of $\text{GL}(C^p) \times \text{GL}(C^q)$ on the variety of flags in $C^{p+q}$. This geometry leads to a partial order on the set of clans analogous to weak Bruhat order on the symmetric group, and we study the saturated chains in this order. We prove an analogue of the Matsumoto-Tits theorem on reduced words in a Coxeter group. We also obtain enumerations of reduced word sets for particular clans in terms of standard tableaux and shifted standard tableaux.

1. Introduction

For $p, q \in \mathbb{N}$, a $(p, q)$-clan is an involution in the symmetric group $S_{p+q}$, each of whose fixed points is labeled either $+$ or $-$, for which

$$(\text{# of fixed points labeled } +) - (\text{# of fixed points labeled } -) = p - q.$$  

We draw clans as partial matchings of $[n] := \{1, 2, \ldots, n\}$ where $n = p + q$:

Example 1.1. The $(1, 2)$-clans are

$\begin{align*}
+--\quad --+\quad ---\quad \cdots\quad \cdots\quad \cdots
\end{align*}$

$\begin{align*}
(1^+)(2^-)(3^-)\quad (1^-)(2^+)(3^-)\quad (1^-)(2^-)(3^+)\quad (1^-)(2^+)(3^-)\quad (1^-)(3^-)\quad (1^-)(2^+)\quad (1^-)(2^-)
\end{align*}$

$\begin{align*}
1^+2^-3^-\quad 1^-2^+3^-\quad 1^-2^-3^+\quad 213^-\quad 1^-3^-\quad 32^-1
\end{align*}$

where the last two lines are cycle notation and one-line notation, respectively. When a clan consists entirely of fixed points, we simplify the one-line notation: $- - +$ instead of $1^-2^-3^+$. 

Our treatment of clans will be combinatorial and algebraic, but their origins are in geometry. A complete flag $F_\bullet$ in a vector space $V$ is a chain of subspaces $F_1 \subseteq F_2 \subseteq \cdots \subseteq F_n = V$ where $\dim F_i = i$. Let $\text{Fl}(V)$ denote the set of complete flags in $V$. The (left) action of $\text{GL}(V)$ on $V$ induces an action on $\text{Fl}(V)$, hence an action of any subgroup of $\text{GL}(V)$ on $\text{Fl}(V)$. Identify $\text{GL}(C^p) \times \text{GL}(C^q)$ with the subgroup of $\text{GL}(C^{p+q})$ consisting of block diagonal matrices with a $p \times p$ block in the upper left and a $q \times q$ block in the lower right. Then the orbits of $\text{GL}(C^p) \times \text{GL}(C^q)$ on $\text{Fl}(C^{p+q})$ are in bijection with the $(p, q)$-clans in a natural way [14, 21].

A closed subgroup $K \subseteq \text{GL}(C^n)$ is spherical if it acts on $\text{Fl}(C^n)$ with finitely many orbits (more generally, one can replace $\text{GL}(C^n)$ with a reductive algebraic group $G$ and $\text{Fl}(C^n)$ with the generalized flag variety of $G$). From the geometry arises a natural partial order on the set of $K$-orbits called weak order [16]. This poset is graded by codimension and has a unique minimal element. The central objects of this paper are the saturated chains containing the minimal element in the case $K = \text{GL}(C^p) \times \text{GL}(C^q)$.

The covering relations in weak order are labelled by integers in $[n - 1]$, so a saturated chain from the minimal element to a clan $\gamma$ can be identified with a word on the alphabet $[n - 1]$, and we call such a word a reduced word for $\gamma$. This is by analogy with the more familiar case where $K$ is the subgroup of lower triangular matrices, in which the $K$-orbits on $\text{Fl}(C^n)$ are in bijection with permutations of $n$, and their closures are the Schubert varieties in $\text{Fl}(C^n)$.
There, weak order is defined by the covering relations $ws_i < w$ whenever $ws_i$ has fewer inversions than $w$, where $s_i$ is the adjacent transposition $(i \ i+1) \in S_n$. The saturated chains from the minimal element to $w$ are then labeled by the reduced words of $w$: the minimal-length words $a_1 \cdots a_\ell$ such that $w = s_1 \cdots s_{a_\ell}$.

**Example 1.2.** Here is the weak order on $\text{Clan}_{1,2}$ (we have labelled the edges by the adjacent transpositions $s_1, \ldots, s_{n-1}$ rather than the integers $1, \ldots, n-1$):

$$
\begin{array}{ccc}
- & + & - \\
\text{s}_2 & \text{s}_1 & \text{s}_2 \\
- & + & - \\
\text{s}_1 & \text{s}_2 & - \\
\end{array}
$$

The reduced words of $-+-$ are $12$ and $21$, while the only reduced word of $--+-$ is $12$.

In $S_n$ (or in any Coxeter group), one can obtain any reduced word for $w$ from any other via simple transformations. Let $\mathcal{R}(w)$ be the set of reduced words of $w \in S_n$.

**Theorem 1.3** (Matsumoto-Tits). Let $\equiv$ be the equivalence relation on the set of words on the alphabet $[n-1]$ defined as the transitive closure of the relations

$$
\cdots i k \cdots \equiv \cdots k i \cdots \quad \text{if } |i-k| > 1
$$

$$
\cdots i j i \cdots \equiv \cdots j i j \cdots \quad \text{if } |i-j| = 1.
$$

Every equivalence class of $\equiv$ either contains no reduced word for any $w \in S_n$, or consists entirely of reduced words. Moreover, the classes containing reduced words are exactly the sets $\mathcal{R}(w)$ for $w \in S_n$.

Example 1.2 shows that an exact analogue of this theorem cannot hold for clans, because different clans can share the same reduced word. However, we do get a similar result by relaxing the constraint that each reduced word set must be a single equivalence class. Let $\mathcal{R}(\gamma)$ be the set of reduced words for a clan $\gamma$, and let $\text{Clan}_{p,q}$ be the set of $(p, q)$-clans.

**Theorem 1.4.** Let $\equiv$ be the equivalence relation on the set of words on $[n-1]$ defined as the transitive closure of the relations $a_1a_2 \cdots a_\ell \equiv (n-a_1)a_2 \cdots a_\ell$ together with the Coxeter relations of Theorem 1.3. Every equivalence class of $\equiv$ either contains no reduced word for any $\gamma \in \text{Clan}_{p,q}$, or consists entirely of reduced words. Also, when restricted to reduced words, $\equiv$ is the strongest equivalence relation for which each $\mathcal{R}(\gamma)$ is a union of equivalence classes. In other words, $a \equiv b$ if and only if

$$
\{ \gamma \in \text{Clan}_{p,q} : a \in \mathcal{R}(\gamma) \} = \{ \gamma \in \text{Clan}_{p,q} : b \in \mathcal{R}(\gamma) \}.
$$

In [17], Stanley defined a symmetric function $F_w$ associated to a permutation $w$ in which the coefficient of a squarefree monomial is the number of reduced words of $w$. For many $w$ of interest (e.g. the reverse permutation $n \cdot \cdots \cdot 21$), the Schur expansion of $F_w$ is simple enough that one obtains enumerations of reduced words in terms of standard tableaux. A formula of Billey-Jockusch-Stanley [2] shows that $F_w$ is a certain limit of Schubert polynomials, which represent the cohomology classes of Schubert varieties in $\text{Fl}(\mathbb{C}^n)$.

We follow a similar approach to prove some enumerative results for reduced words of clans in Section 3. Wyser and Yong [22] define polynomials which represent the cohomology classes of the $\text{GL}(\mathbb{C}^p) \times \text{GL}(\mathbb{C}^q)$-orbit closures on $\text{Fl}(\mathbb{C}^n)$, and a result of Brion [3] implies an analogue
of the Billey-Jockusch-Stanley formula. We define the Stanley symmetric function $F_\gamma$ of a clan $\gamma$ as a limit of the Wyser-Yong polynomials. In particular, the maximal clans in weak order are the matchless clans, those whose underlying involution is the identity permutation, and we show in this case that $F_\gamma$ is the product of two Schur polynomials. This gives a simple product formula for the number of reduced words:

**Theorem 1.5.** Suppose $\gamma \in \text{Clan}_{p,q}$ is matchless with $+$’s in positions $\phi^+ \subseteq [n]$ and $-$’s in positions $\phi^- = [n] \setminus \phi^+$. Then

$$\#R(\gamma) = (pq)! \prod_{i \in \phi^+} \prod_{j \in \phi^-} \frac{1}{|i-j|}.$$ 

The permutation $w \in S_n$ with the most reduced words is the reverse permutation $n \cdots 21$, the unique maximal element in weak order. Similarly, a clan $\gamma \in \text{Clan}_{p,q}$ maximizing $\#R(\gamma)$ must be matchless, but otherwise it is not obvious what these clans are. We investigate this question in Section 4, including connections to work of Pittel and Romik on random Young tableaux of rectangular shape suggested by Theorem 1.5.

The orbits of the orthogonal group $O(\mathbb{C}^n)$ on $\text{Fl}(\mathbb{C}^n)$ are indexed by the involutions in $S_n$, and the resulting weak order on involutions has been studied by various authors. If one forgets the signs of fixed points, clan weak order becomes the opposite of involution weak order suggested by Theorem 1.5. We thank David Speyer and everyone else involved in the program.

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2. **Reduced words for clans**

Let $\text{Clan}_{p,q}$ be the set of $(p,q)$-clans. We usually write $n$ to mean $p + q$ without comment. Let $s_i$ be the adjacent transposition $(i \ i+1)$, and write $i(\gamma)$ for the underlying involution of a clan $\gamma$. We define conjugation of $\gamma$ by $s_i$ as follows: take the underlying involution of $s_i\gamma s_i$ to be $s_i i(\gamma) s_i$, and give the fixed points of $s_i\gamma s_i$, the same signs that they have in $\gamma$ except that the signs of $i$ and $i + 1$ (if any) become the respective signs of $i + 1$ and $i$.

**Example 2.1.**

$$s_2(1 2)(3^-) s_2 = (1 3)(2^-) \text{ and } s_1(1 2)(3^-) s_1 = (1 2)(3^-);$$

$$s_2(1^-)(2^-)(3^+) s_2 = (1^-)(2^+)(3^-).$$

Conjugation preserves the number of $+$’s and $-$’s, hence the set of $(p,q)$-clans. Imagining a clan as an ordered row of unlabeled nodes, each of which has a strand or a sign attached to it (as in Example 1.1), conjugation by $s_i$ simply swaps the $i^{th}$ and $(i + 1)^{th}$ node, with any attached strand or sign being carried along.
Using conjugation we now define a different, partial action of the $s_i$ on clans.

- If $i$ and $i + 1$ are fixed points of $\gamma$ of opposite sign, then $\gamma * s_i$ is $\gamma$ except that $i$ and $i + 1$ are now matched.
- If $i$ and $i + 1$ are matched in $\gamma$, or are fixed points of equal sign, we leave $\gamma * s_i$ undefined.
- If $i$ and $i + 1$ are not fixed points and are not matched with each other, $\gamma * s_i = s_i \gamma s_i$.

The operation $\gamma \mapsto \gamma * s_i$ is not invertible for the same reason that we leave $\gamma * s_i$ undefined in the second case: if $i$ and $i + 1$ are matched in $\gamma$ then they ought to be replaced in $\gamma * s_i$ by $+-$ or $-+$, but there is no reason to choose one over the other.

Let $\ell(w)$ be the Coxeter length of a permutation $w$ (number of inversions).

**Definition 2.2.** The weak order on $\text{Clan}_{p,q}$ is the transitive closure of the relation $\gamma * s_i < \gamma$ if $\ell(\epsilon(\gamma * s_i)) > \ell(\epsilon(\gamma))$.

One should mark the reversal here compared to weak Bruhat order on the symmetric group, which has covering relations $w s_i < w$ whenever $\ell(w s_i) < \ell(w)$. By contrast, the largest elements of $\text{Clan}_{p,q}$ have the fewest inversions when viewed as permutations.

**Definition 2.3.** A clan $\gamma$ is matchless if $\epsilon(\gamma)$ is the identity permutation.

There are $\binom{p+q}{p}$ matchless clans in $\text{Clan}_{p,q}$, and they are exactly the maximal elements in weak order. There is a unique minimal element in weak order on $\text{Clan}_{p,q}$, which we will call $\gamma_{p,q}$: its underlying involution is $(1 \, n)(2 \, n-1) \cdots (m \, n-m+1)$ where $m = \min(p, q)$, and the fixed points $m + 1, m + 2, \ldots, n - m$ are all labeled with the sign of $p - q$.

**Example 2.4.** The minimal element $\gamma_{5,3} \in \text{Clan}_{5,3}$ has $|p - q| = 2$ fixed points, labeled $+$ since $p - q > 0$, and $\min(p, q) = 3$ arcs:

\[ \cdots + + \cdots \]

**Definition 2.5.** A word $a_1 \cdots a_\ell$ with letters in $\mathbb{N}$ is a reduced word for $\gamma \in \text{Clan}_{p,q}$ if there is a saturated chain from the minimal element $\gamma_{p,q} \in \text{Clan}_{p,q}$ to $\gamma$ with edge labels $s_{a_1}, \ldots, s_{a_\ell}$ (in that order, beginning with $\gamma_{p,q}$ and ending with $\gamma$). Let $R(\gamma)$ be the set of reduced words of $\gamma$. Similarly, $R(w)$ denotes the set of reduced words of a permutation $w \in S_n$.

We will use bold for reduced words to distinguish them from permutations.

**Example 2.6.** From Example 1.2 one can see that

\begin{align*}
R(\epsilon) &= \{\epsilon\} \\
R(\epsilon - \epsilon) &= \{1\} \\
R(- - -) &= \{2\} \\
R(-- +) &= \{12\} \\
R(- + -) &= \{12, 21\} \\
R(+ --) &= \{21\}
\end{align*}

where $\epsilon$ is the empty word. Unlike reduced words in Coxeter groups, a word can be a reduced word for more than one clan.

**Warning.** We write reduced words starting at the minimal element $\gamma_{p,q} \in \text{Clan}_{p,q}$ by analogy with reduced words for Coxeter groups. However, if $a_1 \cdots a_\ell$ is a reduced word for $\gamma \in \text{Clan}_{p,q}$, then $(\cdots ((\gamma_{p,q} * s_{a_1}) * s_{a_2}) * \cdots) * s_{a_\ell}$ need not be defined (although if it is, then it equals $\gamma$).
Figure 1. Possible local changes in a covering relation in weak order

Rather, one must say that $a_1 \cdots a_\ell$ is a reduced word for $\gamma$ if $(\cdots ((\gamma * s_{a_\ell}) * s_{a_{\ell-1}}) \cdots) * s_{a_1} = \gamma_{p,q}$ and $\ell$ is minimal.

Remark 2.7. The motivation for this definition of weak order on clans comes from geometry. Given any subset $Y \subseteq \text{Fl}(\mathbb{C}^n)$ and $1 \leq i < n$, let $Y * s_i$ be the subset

$\{F : F_1 \subseteq \cdots \subseteq F_{i-1} \subseteq F_i' \subseteq F_{i+1} \subseteq \cdots \subseteq F_n \text{ is in } Y \text{ for some } i\text{-dimensional } F'\}$.

In particular, $Y * s_i$ contains $Y$. Recall from the introduction that the $\text{GL}(\mathbb{C}^p) \times \text{GL}(\mathbb{C}^q)$-orbits on $\text{Fl}(\mathbb{C}^n)$ can be labeled by $(p, q)$-clans. Letting $Y_\gamma$ denote the orbit labeled by $\gamma$, we have $Y_\gamma * s_i = Y_{\gamma * s_i}$ if $\gamma * s_i < \gamma$. This operation is important in Schubert calculus: the Zariski closures $\overline{Y_\gamma}$ and $\overline{Y_{\gamma * s_i}}$ have associated cohomology classes $[\overline{Y_\gamma}]$ and $[\overline{Y_{\gamma * s_i}}]$, and under the Borel isomorphism identifying the cohomology ring $H^*(\text{Fl}(\mathbb{C}^n), \mathbb{Z})$ with a quotient of $\mathbb{Z}[x_1, \ldots, x_n]$, these two classes are related by a divided difference operator; see Section 3. (This can all be phrased more generally for a complex reductive group $G$ and simple generator $s$ of its Weyl group $W$: in passing from $Y$ to $Y * s$, we are first projecting $Y$ from the flag variety of $G$ onto the partial flag variety associated to the parabolic subgroup $\langle s \rangle$ of $W$, and then applying the inverse image of the same projection.)

When passing from $\gamma$ to $\gamma * s_i < \gamma$, only the $i^{\text{th}}$ and $(i+1)^{\text{th}}$ nodes in the matching diagrams change, and it is helpful to have a list of the possible local moves. In Figure 1, we have drawn the $i^{\text{th}}$ and $(i+1)^{\text{th}}$ nodes of $\gamma$ on the left, and those of $\gamma * s_i$ on the right, assuming $\gamma * s_i < \gamma$.

Lemma 2.8 ([16], Lemma 3.16). The reduced word set $\mathcal{R}(\gamma)$ of any $\gamma \in \text{Clan}_{p,q}$ is closed under the Coxeter relations for $S_n$ (cf. Theorem 1.3). That is, $\mathcal{R}(\gamma)$ is closed under the following operations on words:

$\cdots ik \cdots \leadsto \cdots ki \cdots$ if $|i - k| > 1$

$\cdots ij \cdots \leadsto \cdots ji \cdots$ if $|i - j| = 1$.

Moreover, any $a \in \mathcal{R}(\gamma)$ is a reduced word for some permutation.

Definition 2.9. The set of atoms of a $(p, q)$-clan $\gamma$ is the set of permutations $\mathcal{A}(\gamma) \subseteq S_n$ such that $\mathcal{R}(\gamma) = \bigcup_{w \in \mathcal{A}(\gamma)} \mathcal{R}(w)$, guaranteed to exist by Lemma 2.8.
Example 2.10. Example 2.6 shows that $A(-+-) = \{s_1s_2\} = \{231\}$ and $A(---) = \{s_1s_2, s_2s_1\} = \{231, 312\}$. A more interesting example: $A(+--+)=\{4132,3241\}$, because

$$R(+--+)=\{2321,3231,3213,1231,1213,2123\}$$

$$=\{2321,3231,3213\} \cup \{1231,1213,2123\}$$

$$=R(4132) \cup R(3241).$$

Given a word $a$, let $\Gamma(a)$ be the set of clans $\gamma \in \text{Clan}_{p,q}$ such that $a \in R(\gamma)$. If $w$ is a permutation, we also write $\Gamma(w)$ for the set of clans $\gamma \in \text{Clan}_{p,q}$ such that $w \in A(\gamma)$.

Definition 2.11. Let $\sim$ be the strongest equivalence relation on reduced words for members of $\text{Clan}_{p,q}$ with the property that each $R(\gamma)$ for $\gamma \in \text{Clan}_{p,q}$ is a union of equivalence classes. Equivalently, $a \sim b$ if and only if $\Gamma(a) = \Gamma(b)$.

Lemma 2.8 shows that $\sim$ respects the Coxeter relations in the sense that if $a, b \in R(w)$ for some $w \in S_n$, then $a \sim b$. We can therefore simplify the problem of describing $\sim$ by “factoring out” these relations. If $v, w$ are atoms for some members of $\text{Clan}_{p,q}$, write $v \sim w$ if $\Gamma(v) = \Gamma(w)$. Then $a \sim b$ if and only if $a \in R(v), b \in R(w)$ for some atoms $v \sim w$.

To understand this equivalence relation on permutations we need a better understanding of the sets $A(\gamma)$. Given a subset $S \subseteq [n]$ and a clan $\gamma \in \text{Clan}_{p,q}$, call a pair $(i < j) \in S$ valid if either $i$ and $j$ are matched by $\gamma$, or if they are fixed points of opposite sign which are adjacent in the sense that there is no $i' \in S$ with $i < i' < j$. Consider the following algorithm which (nondeterministically) builds a permutation $w \in S_n$ by removing one pair of points from $[n]$ at a time and correspondingly deciding upon two entries of $w$. Set $S := [n]$ to start.

Algorithm 2.12.

(a) Choose a valid pair $(i < j) \in S$ such that $\gamma$ has no matched pair $i', j' \in S$ with $i' < i < j < j'$.

• If $i < j$ are matched by $\gamma$, set $w(i) = s + 1$ and $w(j) = n - s$, where $s = (n - |S|)/2$ (this is the number of pairs deleted from $[n]$ in step (c) so far).

• If $i < j$ are fixed by $\gamma$, set $w(i) = n - s$ and $w(j) = s + 1$, with $s$ as above.

(b) If $S$ consists entirely of fixed points of $\gamma$ of the same sign, fill in the remaining undefined entries of $w$ with the unused entries of $[n]$ in increasing order, and return $w$.

(c) If we did not finish in step (b), then replace $S$ with $S \setminus \{i, j\}$ and go back to (a).

Example 2.13. Here is one way this algorithm can run when $\gamma = (19)(2^+)(3^+)(4^7)(5^-)(6^8)$:

\[
\begin{array}{cccccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
\xrightarrow{++} & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
\xrightarrow{++} & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
\xrightarrow{++} & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
\xrightarrow{++} & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
\xrightarrow{++} & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
\end{array}
\]

$w = 123456789$ After this point no more pairs can be selected in step (a), so the algorithm returns 157236849.

Theorem 2.14 ([4]). $A(\gamma)$ is the set of permutations which can be generated by Algorithm 2.12.

We note that [4] works with the set $W(\gamma) := \{w^{-1} : w \in A(\gamma)\}$ rather than our $A(\gamma)$.

The possible outcomes of Algorithm 2.12 can also be encoded by recording, for each $i$ which is removed in the course of the algorithm, which step it was removed at.

Definition 2.15. A labelled shape for $\gamma$ is a pair $(\omega, F)$ where $\omega$ is the partial function $[n] \rightarrow \mathbb{N}$ obtained from an instance of Algorithm 2.12 by setting $\omega(i) = \omega(j) = k$ if $(i, j)$ is the $k^{th}$ pair deleted from $[n]$ in step (c) of the algorithm, and $F$ is the subset of the domain of $\omega$ consisting of fixed points of $\gamma$. 
We think of a labelled shape \((\omega, F)\) as the edge-labelled partial matching on \([n]\) with an arc labeled \(k\) matching \(i\) and \(j\) for each \(\omega^{-1}(k) = \{i, j\}\), where the arc is marked if \(i, j \in F\). We draw these marked arcs as doubled edges. Given this marking, we will omit \(F\) from the notation since it can be recovered as the set of endpoints of the marked arcs.

**Example 2.16.** The instance of Algorithm 2.12 in Example 2.13 gives the labelled shape

![Diagram](image)

We have drawn the arcs below the baseline to avoid confusion with the matchings in a clan. These diagrams help explain why Theorem 2.14 is true. When following a maximal chain up from \(\gamma_{p,q}\) to \(\gamma\), each matching \((k, n-k+1)\) in \(\gamma_{p,q}\) eventually becomes either a matching in \(\gamma\) or a pair of opposite-sign fixed points, which we record as an arc labeled \(k\) in the labelled shape.

It is not hard to give a direct characterization of the labelled shapes of a clan.

**Proposition 2.17.** Let \(\omega\) be a partial matching on \([n]\) with its \(e\) arcs labelled 1, 2, \ldots, \(e\), where arcs may be marked or unmarked. Then \(\omega\) is a labelled shape for \(\gamma \in \text{Clan}_{p,q}\) if and only if \(e = \min(p, q)\), and for all arcs \(\{i < j\}\) of \(\omega\),

(i) \(i\) and \(j\) are either matched by \(\gamma\) or are a pair of fixed points of opposite sign, according to whether the arc \(\{i < j\}\) is unmarked or marked respectively.

(ii) If \(\{i < j\}\) is marked and \(i < i' < j\), then \(\omega(i')\) is defined and \(\omega(i') < \omega(i) = \omega(j)\).

(iii) If \(\{i' < j'\}\) is an unmarked arc of \(\omega\) with \(i' < i < j < j'\), then \(\omega(i') = \omega(j') < \omega(i) = \omega(j)\).

**Proof.** The number of fixed points remaining in step (b) of Algorithm 2.12 after all possible pairs \(\{i, j\}\) have been removed is

\[ |(\text{# of } +\text{s in } \gamma) - (\text{# of } -\text{s in } \gamma)| = |p - q|. \]

The number of pairs which were removed is therefore \(m = \min(p, q)\), so the image of \(\omega\) is \([m]\) and every \(k \in [m]\) has \(|\omega^{-1}(k)| = 2\).

If the algorithm removes a pair \(i, j\) then it must have already removed all pairs \(i', j'\) matched by \(\gamma\) with \(i' < i < j < j'\), so (iii) is necessary, and if \(i, j\) were matched by \(\gamma\) then this is the only condition needed for \(i, j\) to be removable. To remove a pair \(i, j\) fixed by \(\gamma\) (so \(\{i, j\}\) is marked), one also needs that every \(i'\) with \(i < i' < j\) has already been removed, meaning \(\omega(i') < \omega(i) = \omega(j)\) as demanded by (ii). \(\square\)

Given an atom \(w \in A(\gamma)\), let \(\text{lish}(w)\) be the corresponding labelled shape. Explicitly, the arcs of \(\text{lish}(w)\) are \(\{w^{-1}(k), w^{-1}(n-k+1)\}\) for \(k = 1, 2, \ldots, \min(p, q)\), each arc being marked or unmarked according to whether \(w^{-1}(k) > w^{-1}(n-k+1)\) or \(w^{-1}(k) < w^{-1}(n-k+1)\). Recall that we are trying to characterize the equivalence relation on permutations where \(v \sim w\) if \(\Gamma(v) = \Gamma(w)\). The set \(\Gamma(w)\) is easy to compute from \(\text{lish}(w)\), and in fact the edge labelling on \(\text{lish}(w)\) is not even necessary for this.

**Definition 2.18.** The unlabelled shape \(\text{ush}(w)\) of \(w\) is the pair \((\pi, F)\) where \(\pi\) is the partial matching obtained from \(\text{lish}(w)\) by removing the arc labels, and \(F\) is the set of endpoints of marked arcs in \(\text{lish}(w)\).

As before, we consider \(\text{ush}(w)\) to be a partial matching with some arcs marked and omit mention of \(F\).
Example 2.19. Drawing marked arcs as doubled edges, the unlabelled shape of \( w = 157236849 \) (whose labelled shape is shown in Example 2.16) is

\[
\begin{array}{c}
\bullet \\
\bullet \\
\bullet \\
\bullet
\end{array}
\]

Theorem 2.20. Let \( v \) and \( w \) be atoms for some members of \( \text{Clan}_{p,q} \). Then \( v \sim w \) if and only if \( \text{ush}(v) = \text{ush}(w) \).

Proof. First, \( \Gamma(v) \) depends only on \( \text{ush}(v) \). Indeed, if \( \text{ush}(v) \) has \( e \) marked arcs then \( \Gamma(v) \) consists of the \( 2^e \) clans obtained by:

- Replacing each marked arc \( \{i, j\} \) by fixed points \( i^+, j^- \) or \( i^-, j^+ \);
- Leaving each unmarked arc as a matching;
- Leaving each unmatched point as a fixed point whose sign is the sign of \( p - q \).

This shows that if \( \text{ush}(v) = \text{ush}(w) \) then \( v \sim w \).

Conversely, suppose \( \text{ush}(v) \neq \text{ush}(w) \). If the unmarked arcs of \( \text{ush}(v) \) are different from those of \( \text{ush}(w) \), then by the previous paragraph every clan in \( \Gamma(v) \) has different arcs than every clan in \( \Gamma(w) \), so assume all unmarked arcs are the same. Then there must be, say, a marked arc \( \{i, j\} \) in \( \text{ush}(v) \) such that \( i, j \) are not connected by a marked arc in \( \text{ush}(w) \). But then there are clans in \( \Gamma(w) \) which give the same sign to \( i \) and \( j \), while every clan in \( \Gamma(v) \) gives them opposite signs. In any case, \( \Gamma(v) \neq \Gamma(w) \) so \( v \not\sim w \). \qed

An adjacent transposition \( s_k \) where \( k < \min(p, q) \) acts on a labelled shape \( \omega \) by swapping the labels \( k \) and \( k+1 \), giving a new partial matching \( s_k \omega \) with labelled and possibly marked arcs, although \( s_k \omega \) may not be a valid labelled shape for a clan.

Lemma 2.21. Let \( w \in \text{A}(\gamma) \) and \( k < \min(p, q) \). Then \( s_k \text{lhs}(w) \) is a labelled shape for \( \gamma \) if and only if \( \ell(s_k s_{n-k} w) = \ell(w) \), and if that holds then \( s_k \text{lhs}(w) = \text{lhs}(s_k s_{n-k} w) \).

Proof. The map \( \text{lhs}^{-1} \) sending a labelled shape to its associated atom makes sense when applied to any partial matching with labelled and marked arcs, though the result may not be an atom. In particular, it sends \( s_k \text{lhs}(w) \) to the permutation \( s_k s_{n-k} w \) regardless of whether the former is a valid labelled shape; here and below, it is helpful to note here that \( s_k s_{n-k} = s_{n-k} s_k \) since \( k < \min(p, q) \).

There are two cases in which \( s_k \text{lhs}(w) \) is not a valid labelled shape:

- Suppose the arc \( \{i < j\} \) of \( \text{lhs}(w) \) labeled \( k+1 \) is nested inside the arc \( \{i' < j'\} \) labeled \( k \), meaning that \( i' < i < j < j' \), and that the arc labeled \( k \) is unmarked. Then \( w \) has the form
  \[
  \cdots \cdots k \cdots k+1 \cdots n-k \cdots n-k+1 \cdots \quad \text{or} \quad \cdots \cdots k \cdots n-k \cdots k+1 \cdots n-k+1 \cdots
  \]
  and \( \ell(s_k s_{n-k} w) = \ell(w) + 2 \).
- Suppose the arc \( \{i < j\} \) of \( \text{lhs}(w) \) labeled \( k+1 \) is marked, and that the arc \( \{i' < j'\} \) labeled \( k \) has \( i < i' < j \) or \( i < j' < j \). If \( \{i' < j'\} \) is marked, the definition of labelled shape forces \( i < i' < j' < j \), so \( w \) has the form
  \[
  \cdots n-k \cdots n-k+1 \cdots k \cdots k+1 \cdots
  \]
  If \( \{i' < j'\} \) is unmarked, then depending on exactly where \( i', j' \) are positioned with respect to \( i, j \), the permutation \( w \) has one of the forms
  \[
  \cdots n-k \cdots k \cdots n-k+1 \cdots k+1 \cdots \\
  \cdots n-k \cdots k \cdots k+1 \cdots n-k+1 \cdots \\
  \cdots k \cdots n-k \cdots n-k+1 \cdots k+1 \cdots \\
  \]
In all of these cases, $\ell(s_k s_{n-k} w) = \ell(w) + 2$ again.

Conversely, suppose $\ell(s_k s_{n-k} w) \neq \ell(w)$, so $\ell(s_k s_{n-k} w) = \ell(w) \pm 2$. If $\ell(s_k s_{n-k} w) = \ell(w) + 2$, then $k$ precedes $k+1$ in the one-line notation of $w$ and $n-k$ precedes $n-k+1$. There are 6 permutations of $k, k+1, n-k, n-k+1$ for which this holds, and they are exactly the 6 cases we considered above in which $s_k \text{lsh}(w)$ is not a valid labelled shape.

So, suppose $\ell(s_k s_{n-k} w) = \ell(w) - 2$. We claim that in this case, $\text{lsh}(w)$ could not have been a valid labelled shape to begin with. Now $k+1$ precedes $k$ in $w$ and $n-k+1$ precedes $n-k$, and the 6 possibilities can be checked directly. If $w$ has one of the forms

$$
\cdots n-k+1 \cdots k+1 \cdots n-k \cdots
\cdots n-k+1 \cdots k+1 \cdots n-k+1 \cdots
\cdots n-k+1 \cdots n-k \cdots k+1 \cdots
\cdots k+1 \cdots n-k+1 \cdots k \cdots n-k \cdots
$$

then the arc $\{i < j\}$ in $\text{lsh}(w)$ labelled $k$ is marked, yet there is $i < i' < j$ such that $i'$ is labelled $k+1$, contradicting Proposition 2.17(ii). If $w$ has the form

$$
\cdots k+1 \cdots n-k+1 \cdots n-k \cdots \text{ or } \cdots k+1 \cdots n-k+1 \cdots k \cdots n-k \cdots
$$

then the arc in $\text{lsh}(w)$ labelled $k$ is nested inside the unmarked arc labelled $k+1$, contradicting Proposition 2.17(iii).

\[\square\]

**Theorem 2.22.** The equivalence relation $\sim$ on atoms for $\text{Clan}_{p,q}$ is the transitive closure of the relations $u \sim s_k s_{n-k} u$ where $\ell(s_k s_{n-k} u) = \ell(u)$ and $k < \min(p,q)$.

**Proof.** Lemma 2.21 implies that if $\ell(s_k s_{n-k} u) = \ell(u)$ where $k < \min(p,q)$, then $u$ and $s_k s_{n-k} u$ have the same unlabelled shape, so $u \sim s_k s_{n-k} u$ by Theorem 2.20.

Conversely, suppose $v \sim w$, so $\text{ush}(v) = \text{ush}(w)$. The labelled shapes $\text{lsh}(v)$ and $\text{lsh}(w)$ are certainly connected by a series of applications of adjacent transpositions, so $v$ and $w$ are connected by transformations $u \mapsto s_k s_{n-k} u$ by Lemma 2.21, but we must see that this can be done in such a way that all of the intermediate steps are valid labelled shapes.

Proposition 2.17 shows that the valid labelings of the unlabelled shape $\text{ush}(w)$ can be thought of as the linear extensions of a poset. The elements of the poset are the arcs of $\text{ush}(w)$, and $\{i' < j'\} \leq \{i < j\}$ if either:

\[\bullet\] $\{i' < j'\}$ is unmarked and $i' < i < j < j'$; or,

\[\bullet\] $\{i < j\}$ is marked and $i < i' < j$ or $i < j' < j$.

Now apply the following general fact: if $P$ is a finite poset and $G$ is the graph whose vertices are the linear extensions $f : P \to [\#P]$ with an edge $(f,g)$ whenever $g = s_i \circ f$, then $G$ is connected.

We can now prove Theorem 1.4, which we restate here.

**Theorem (Theorem 1.4).** Let $\equiv$ be the equivalence relation on the set of words on $[n-1]$ defined as the transitive closure of the relations $a_1 a_2 \cdots a_\ell \equiv (n-a_1) a_2 \cdots a_\ell$ together with the Coxeter relations. Every equivalence class of $\equiv$ either contains no reduced word for any $\gamma \in \text{Clan}_{p,q}$, or consists entirely of reduced words. Moreover, $\equiv$ agrees with $\sim$ when restricted to reduced words for members of $\text{Clan}_{p,q}$.

**Proof.** Suppose $a = a_1 a_2 \cdots a_\ell \in R(\gamma)$ and $b \equiv a$. We know that Coxeter relations preserve $R(\gamma)$, so we can assume $b = (n-a_1) a_2 \cdots a_\ell \equiv a$. We claim $b \in R(\gamma)$ as well. Since $a_1 < \min(p,q)$, it holds that $\gamma_{p,q} \ast s_{a_1}$ is well-defined and equal to $(\cdots (\gamma \ast s_{a_\ell}) \cdots) \ast s_{a_2}$. But also $\gamma_{p,q} \ast s_{a_1} = \gamma_{p,q} \ast s_{a_1}$.
\( \gamma_{p,q} \ast s_{n{-}a_1} \), so
\[
((\cdots (\gamma \ast s_{a_\ell}) \ast \cdots) \ast s_{a_2}) \ast s_{n{-}a_1} = (\gamma_{p,q} \ast s_{a_1}) \ast s_{n{-}a_1} = (\gamma_{p,q} \ast s_{n{-}a_1}) \ast s_{n{-}a_1} = \gamma_{p,q},
\]
which implies that \( b \in \mathcal{R}(\gamma) \). We have \( a \in \mathcal{R}(u) \) and \( b \in \mathcal{R}(s_{n{-}a_1}u) \) for some \( u \in \mathcal{A}(\gamma) \), and \( \ell(s_{n{-}a_1}u) = \ell(u) \) since \( a \) and \( b \) have the same length. Theorem 2.22 shows \( u \sim s_{n{-}a_1}u \), so \( a \sim b \). We also conclude from this if an equivalence class of \( \equiv \) contains a single reduced word, then all its elements are reduced words.

Conversely, suppose \( a \sim b \) where \( a, b \) are reduced words. Then there are atoms \( v \sim w \) with \( a \in \mathcal{R}(v) \) and \( b \in \mathcal{R}(w) \), and applying Theorem 2.22, we can assume that \( w = s_{n{-}k} s_k v \) where \( k < \min(p, q) \). Since \( s_k s_{n{-}k} = s_{n{-}k} s_k \) and \( \ell(v) = \ell(s_{n{-}k} s_k v) \), exactly one of \( s_k \) and \( s_{n{-}k} \) is a left descent of \( v \), say \( s_k \). Then there is \( a' \in \mathcal{R}(v) \) with \( a'_1 = k \), and the equality \( \ell(v) = \ell(s_{n{-}k} s_k v) \) implies that \( b' := (n{-}a'_1) a'_2 \cdots a'_\ell \) is a reduced word of \( s_{n{-}k} s_k v = w \). Since \( a \) is related to \( a' \) and \( b \) to \( b' \) via Coxeter relations (by the Matsumoto-Tits lemma), we see that \( a \equiv a' \equiv b' \equiv b \).

Theorem 1.4 can be interpreted as giving a simple prescription for generating each equivalence class making up \( \mathcal{R}(\gamma) \) beginning with one reduced word. It is also natural to ask for simple transformations relating the equivalence classes to each other. We will think about transformations of unlabelled shapes, since these index the equivalence classes of \( \sim \) by Theorem 2.20. Let \( \text{ush}(\mathcal{A}(\gamma)) \) be the set of unlabelled shapes for \( \gamma \).

**Lemma 2.23.** Given an unlabelled shape for \( \sigma \in \text{ush}(\mathcal{A}(\gamma)) \), suppose one applies to it a transformation of the form
\[
\begin{align*}
\sigma = \cdot \cdot \cdot \alpha \cdot \cdot \cdot \beta \cdot \cdot \cdot \alpha \cdot \cdot \cdot \beta \cdot \cdot \cdot & \quad \rightarrow \quad \sigma' = \cdot \cdot \cdot \alpha \cdot \cdot \cdot \beta \cdot \cdot \cdot \alpha \cdot \cdot \cdot \beta \cdot \cdot \cdot \\
\sigma = \cdot \cdot \cdot \alpha \cdot \cdot \cdot \beta \cdot \cdot \cdot \alpha \cdot \cdot \cdot & \quad \rightarrow \quad \sigma' = \cdot \cdot \cdot \alpha \cdot \cdot \cdot \beta \cdot \cdot \cdot \alpha \cdot \cdot \cdot
\end{align*}
\]
where \( \cdots \) conceals an arbitrary partial matching (with marked/unmarked arcs), \( \cdots \) conceals only a complete matching (no unpaired fixed points allowed), and \( \{\alpha, \beta\} = \{+, -\} \). Then \( \sigma' \in \text{ush}(\mathcal{A}(\gamma)) \), and the directed graph with vertices \( \text{ush}(\mathcal{A}(\gamma)) \) and edges \( \sigma \to \sigma' \) is acyclic.

**Proof.** Proposition 2.17 implies that the unlabelled shapes of \( \gamma \) are those partial matchings of \([n]\) with \( \min(p, q) \) arcs which are either pairs of opposite-sign fixed points or matchings in \( \gamma \), and such that no two marked arcs cross and no unpaired fixed point is underneath a marked arc. From this description it is clear that the transformations in the theorem do preserve \( \text{ush}(\mathcal{A}(\gamma)) \).

Given \( \sigma \in \text{ush}(\mathcal{A}(\gamma)) \), label the right endpoints of the marked arcs 1, 2, \ldots from left to right, and then label each marked arc according to its right endpoint. This is a labelled shape of \( \gamma \); write \( \text{st}(\sigma) \in \mathcal{A}(\gamma) \) for the associated atom. For instance,
\[
\sigma = \begin{array}{c}
\begin{array}{c}
\end{array}
\end{array}
\quad \sim \quad \begin{array}{c}
\begin{array}{c}
1
2
3
\end{array}
\end{array}
\quad \text{so} \quad \text{st}(\sigma) = 461523.
\]
If \( \sigma \to \sigma' \), then \( \text{st}(\sigma) \) and \( \text{st}(\sigma') \) are related by transformations of the form
\[
\cdots n{-}\ell+1 \cdots n{-}k+1 \cdots k \cdots \ell \cdots \to \cdots n{-}k'+1 \cdots k' \cdots n{-}\ell+1 \cdots \ell \cdots \quad (k' \leq k < \ell)
\]
\[
\cdots j \cdots n{-}k+1 \cdots k \cdots \to \cdots n{-}k'+1 \cdots k' \cdots j \cdots \quad (k' \leq \min(p, q) < j < \max(p, q))
\]
In both cases, st(σ′) is lexicographically larger than st(σ), which shows that ush(A(γ)) is acyclic.

Lemma 2.23 gives ush(A(γ)) a poset structure, with a covering relation σ ≪ σ′ when σ → σ′.

Example 2.24. Here are the posets ush(A(γ)) for γ = − − + + + and γ = − − + + + − − − − − :

\[
\begin{align*}
\text{Example 2.24.} & \quad \text{Here are the posets ush(A(γ)) for } \gamma = \ldots + + \ldots \text{ and } \gamma = \ldots + + \ldots - - - - - - - : \\
\end{align*}
\]

Theorem 2.25. The poset ush(A(γ)) has a unique maximal element σ\text{max}, which can be constructed as follows. First, σ\text{max} has an unmarked arc for every matching of γ. Next, find the minimal fixed point i of γ such that the minimal fixed point j > i has opposite sign, and connect i and j by a marked arc in σ\text{max}; repeat this process, ignoring fixed points that have already been connected, until all remaining fixed points have the same sign.

Proof. The unmarked arcs in σ ∈ ush(A(γ)) are determined by the matchings of γ, and play no role in the poset structure. We may therefore ignore them, and assume that γ is matchless. If γ has no pair of fixed points of opposite sign, then ush(A(γ)) has one element, so the theorem is trivially true. Otherwise, let i be minimal such that γ(i) and γ(i + 1) have opposite sign, and define ¯γ ∈ Clan_{p, q - 1} by removing i and i + 1 from the matching diagram of γ.

There is an injection f : ush(A(¯γ)) → ush(A(γ)) which adds the fixed points i, i + 1 back and connects them with a marked arc. For example, if γ = ++ − − + then ¯γ = + −, and f:

\[
\begin{align*}
\end{align*}
\]

By induction, ush(A(¯γ)) has a unique maximal element σ\text{max}' constructed as described in the theorem. Its image f(σ\text{max}') equals the unlabeled shape σ\text{max}, which we must now see is actually the unique maximal element of ush(A(γ)). First, suppose σ ∈ ush(A(γ)) is in the image of f, or equivalently that σ has \{i < i + 1\} as a marked arc. Since f does not add any unpaired fixed points, any transformation which can be performed in ush(A(γ)) can also be performed in ush(A(¯γ)), so f is a poset homomorphism. This implies σ ≤ f(σ\text{max}') = σ\text{max}.

Now suppose σ is not in the image of f; we claim σ cannot be maximal. Consider two cases:

- Suppose σ pairs i with j and i + 1 with j. Then i + 1 < j < j', for otherwise there would be an unpaired fixed point of σ below a marked arc, or else two marked arcs would cross. That is, σ has the form

\[
\begin{align*}
\end{align*}
\]
where \( \{\alpha, \beta\} = \{+, -\} \) and there are no unpaired fixed points in \([i, j']\). But now we can apply the transformation replacing the marked arcs \( \{i < j', \{i+1 < j\} \) by \( \{i < i+1, \{j < j'\} \), so \( \sigma \) is not maximal.

- Suppose one of \( i, i+1 \) is unpaired in \( \sigma \) (they cannot both be unpaired). Then in fact \( i \) must be unpaired, because otherwise it would have to be paired with some \( j > i+1 \), but then the unpaired fixed point \( i+1 \) would be below the marked arc \( \{i < j\} \). So, say \( i+1 \) is paired with \( j \). We must have \( j > i+1 \), because otherwise the unpaired fixed point \( i \) would be below the marked arc \( \{j < i+1\} \). That is, \( \sigma \) has the form

\[
\begin{array}{cccccccccccc}
1 & \cdots & i-1 & i & i+1 & j \\
\alpha & \cdots & \alpha & \alpha & \beta & \cdots & \alpha \\
\end{array}
\]

Now we can apply the transformation replacing the marked arc \( \{i+1 < j\} \) by \( \{i < i+1\} \), so \( \sigma \) is not maximal.

□

Theorem 2.25 gives a prescription for generating all of \( \text{ush}(A(\gamma)) \) from one element \( \sigma_{\max} \) by applying simple transformations. It would be interesting to be able to do this at the level of reduced words: that is, to give a uniform way of beginning with a relation \( \text{ush}(v) \to \text{ush}(w) \) and producing \( a \in R(v) \) and \( b \in R(w) \) which are related in some simple way.

3. Enumerating reduced words for clans

**Definition 3.1.** Let \( a = a_1 \cdots a_\ell \) be a word on the alphabet \( \mathbb{N} \). A compatible sequence for \( a \) is a word \( b \) of length \( \ell \) such that

- \( 1 \leq b_1 \leq \cdots \leq b_\ell \)
- \( b_i \leq a_i \) for each \( i \)
- For each \( i \), if \( a_i < a_{i+1} \), then \( b_i < b_{i+1} \).

We use bold for compatible sequences just as for reduced words.

Let \( \text{comp}(a) \) be the set of compatible sequences for \( a \). For instance, \( \text{comp}(3213) = \{1112, 1113\} \) while \( \text{comp}(3231) \) is empty.

**Definition 3.2.** The Schubert polynomial of a permutation \( w \in S_n \) is

\[
\mathcal{S}_w = \sum_{a \in \mathcal{R}(w)} \sum_{b \in \text{comp}(a)} x_{b_1} \cdots x_{b_\ell}.
\]

The Stanley symmetric function of \( w \) is the formal power series \( F_w = \lim_{m \to \infty} \mathcal{S}_{w+m} \), where \( w+m \) is the permutation defined inductively by \( w^{i+m} = (w^{i+(m-1)})+1 \) and \( w^{i+1} = (w_1 + 1) \cdots (w_n + 1) \) in one-line notation.

It is not hard to check that \( \lim_{m \to \infty} \mathcal{S}_{w+m} \) does exist as a formal power series, so that \( F_w \) is well-defined. The fact that it is actually a symmetric function is rather less obvious, and was proved by Stanley [17].

**Definition 3.3.** The Schubert polynomial of a \((p, q)\)-clan \( \gamma \) is

\[
\mathcal{S}_\gamma = \sum_{a \in \mathcal{R}(\gamma)} \sum_{b \in \text{comp}(a)} x_{b_1} \cdots x_{b_\ell}.
\]

The Stanley symmetric function of \( \gamma \) is \( \lim_{m \to \infty} \mathcal{S}_{\gamma+m} \). Here \( \gamma^{i+m} \) is the \((p+m, q+m)\)-clan defined inductively by \( \gamma^{i+m} = (\gamma^{i+(m-1)})+1 \) and where \( \gamma^{i+1} \) is obtained from \( \gamma \) by shifting all of \( 1, 2, \ldots, n \) up by one and then multiplying by the cycle \((1 \, n+2)\).
Example 3.4. As per Example 2.10, 
\[ R(+--+)=\{3213, 3231, 2321, 1213, 1231, 2123\}. \]
comp(\(a\)) is empty for all \(a \in R(+--+\)) except \(3213\) and \(2123\), while \(\text{comp}(3213) = \{1112, 1113\}\) and \(\text{comp}(2123) = \{1123\}\). Thus \(S_{+--+} = x_1^3x_2 + x_1^3x_3 + x_1^2x_2x_3\). Also,
\[ (+--+)^{+1} = +--+. \]
Since \(R(\gamma) = \bigcup_{w \in A(\gamma)} R(w)\) we have \(S_{\gamma} = \sum_{w \in A(\gamma)} S_w\).

Proposition 3.5. \(A(\gamma^{+1}) = \{w^{+1} : w \in A(\gamma)\}\), and \(F_{\gamma}\) is a well-defined symmetric function equal to \(\sum_{w \in A(\gamma)} F_w\).

Proof. Given a word \(a = a_1 \cdots a_\ell\), let \(a^{+1} = (a_1 + 1) \cdots (a_\ell + 1)\). It is clear that if \(a \in R(\gamma)\), then \(a^{+1} \in R(\gamma^{+1})\).

Conversely, if \(\gamma * s_i < \gamma\), one sees from Figure 1 that the size of the largest cycle in \(\gamma\)—i.e. the maximum of \(|j-i|\) over all 2-cycles \((i,j)\) in \(\gamma\)—is no larger than the size of the largest cycle in \(\gamma * s_i\). This implies that if some \(\delta \in \text{Clan}_{p+1, q+1}\) has a reduced word containing the letter 1 or \(n+1\), then the largest cycle in \(\delta\) has size \(< n + 1\). Since \(\gamma^{+1}\) does have a cycle of size \(n + 1\), its reduced words are supported on the alphabet \(\{2, 3, \ldots, n\}\), and so they must all have the form \(a^{+1}\) for some \(a \in R(\gamma)\). This proves \(A(\gamma^{+1}) = \{w^{+1} : w \in A(\gamma)\}\), and now
\[ F_{\gamma} = \lim_{m \to \infty} \sum_{w \in A(\gamma + m)} S_w = \lim_{m \to \infty} \sum_{w \in A(\gamma)} S_{w+m} = \sum_{w \in A(\gamma)} F_w. \]
\[ \square \]

Proposition 3.6. Letting \(\ell\) be the degree of \(F_{\gamma}\), the coefficient of \(x_1x_2 \cdots x_\ell\) in \(F_{\gamma}\) is \(\#R(\gamma)\).

Proof. If \(m \geq \ell - 1\), then every letter of \(a^{+m}\) for \(a \in R(\gamma)\) is at least \(\ell\), and so \(\text{comp}(a^{+m})\) contains \(12 \cdots \ell\). Proposition 3.5 therefore shows that the coefficient of \(x_1x_2 \cdots x_\ell\) in \(S_{\gamma+m}\) is \(\#R(\gamma^{+m}) = \#R(\gamma)\) for all \(m \geq \ell - 1\).

This proposition holds equally well for Stanley symmetric functions of permutations, which was Stanley’s motivation for defining \(F_w\). One can then use symmetric function techniques to extract coefficients of \(F_w\) and enumerate \(R(w)\). For instance, Stanley showed that \(F_{n \cdots 21}\) is the Schur function \(s_{(n-1, n-2, \ldots, 1)}\), and comparing coefficients of \(x_1x_2 \cdots \) shows that \(\#R(n \cdots 21)\) equals the number of standard tableaux of shape \((n-1, n-2, \ldots, 1)\) [17]. Our intent is to do the same for clans.

Definition 3.7. For \(1 \leq i < n\), the divided difference operator \(\partial_i\) acting on \(R[x_1, \ldots, x_n]\) for a commutative ring \(R\) sends \(f\) to \(\partial_i f = (f - s_i f)/(x_i - x_{i+1})\), where \(s_i\) acts on \(R[x_1, \ldots, x_n]\) by swapping \(x_i\) and \(x_{i+1}\). The isobaric divided difference operator \(\pi_i\) is defined by \(\pi_i(f) = \partial_i(x_i f)\).

Lascoux and Schützenberger defined Schubert polynomials for \(S_n\) by setting \(S_{n \cdots 21} = x_1^{n-1}x_2^{n-2} \cdots x_{n-1}\) and then using the recurrence \(S_{w s_i} = \partial_i S_w\) to define all the other polynomials by induction on weak order [10]. Earlier work of Bernstein, Gelfand, and Gelfand [1] shows that this definition ensures that \(S_w\) represents the cohomology class of the Schubert variety in \(\text{Fl}(\mathbb{C}^n)\) labeled by \(w\). It is a theorem of Billey, Jockusch, and Stanley [2] that this definition is equivalent to Definition 3.2.

Wyser and Yong [22] define clan Schubert polynomials using the same strategy: they give an explicit formula when the clan is matchless, and apply divided difference operators to produce the polynomials for other clans by induction on weak order. Their formulas are given in terms of flagged Schur polynomials, which we now define.
Definition 3.8. Let $X_k$ be the alphabet \{x_1, \ldots, x_k\}. Let $\lambda$ be a partition and $\phi$ a sequence of natural numbers of the same length. The flagged Schur polynomial of shape $\lambda$ with flag $\phi$ is the polynomial $\sum_T x^T$ where $T$ runs over all semistandard tableaux of shape $T$ whose entries in each row $i$ come from $\{1, 2, \ldots, \phi_i\}$, and as usual $x^T$ is the content monomial $\prod_{i \in T} x_i$ of $i$'s in $T$. We write $s_\lambda(X_{\phi_1}, \ldots, X_{\phi_\ell})$ or just $s_\lambda(X_\phi)$ for this polynomial.

Example 3.9.

- $s_{21}(X_1, X_2) = x_1^2 x_2$ is a sum over the single tableau $\begin{array}{|c|}
\hline
1 \\
\hline
2 \\
\hline
\end{array}$
- $s_{21}(X_2, X_1) = x_1^2 x_2 + x_1 x_2^2$ is a sum over the two tableaux $\begin{array}{|c|}
\hline
1 \\
\hline
2 \\
\hline
\end{array}$ and $\begin{array}{|c|}
\hline
2 \\
\hline
1 \\
\hline
\end{array}$

Definition 3.10. Given a matchless $(p,q)$-clan $\gamma$, let $\phi^+(\gamma)$ be the list of positions of the $+$'s in increasing order, and likewise $\phi^-(\gamma)$ the list of positions of the $-$'s. Also define two partitions $\lambda^+(\gamma)$ and $\lambda^-(\gamma)$ by

$$
\lambda^+(\gamma)_i = \# \{ j : \phi^-(\gamma)_j > \phi^+(\gamma)_i \} \quad \text{for } i = 1, \ldots, p \\
\lambda^-(\gamma)_i = \# \{ j : \phi^+(\gamma)_j > \phi^-(\gamma)_i \} \quad \text{for } i = 1, \ldots, q.
$$

The map $\phi^+(\gamma) \mapsto \lambda^+(\gamma)$ is a bijection between $p$-subsets of $[p+q]$ and partitions whose Young diagram is contained in the $p \times q$ rectangle $[p] \times [q]$. Graphically, if one labels the $p+q$ segments of the southwest boundary of the Young diagram of $\lambda^+(\gamma)$ with $1, 2, \ldots, p + q$ from top to bottom, the set of vertical segments is $\phi^+(\gamma)$ and the set of horizontal segments is $\phi^-(\gamma)$.

Example 3.11. Letting $\gamma = + - - + + + -$, we have

$$
\phi^- = (2, 3, 5, 9) \quad \text{and} \quad \lambda^- = (4, 4, 3, 0) \quad \text{and} \quad \lambda^+ = (1, 4, 6, 7, 8, 8, 8, 8, 8, 8, 8, 8, 8).
$$

If $\lambda \subseteq [p] \times [q]$, write $\lambda^\vee$ for the partition whose Young diagram is the complement of $\lambda$ in $[p] \times [q]$ (rotated $180^\circ$). Also let $\lambda^t$ denote the partition conjugate to $\lambda$. For a matchless $(p,q)$-clan $\gamma$, let $\operatorname{rev}(\gamma)$ be the clan obtained by reversing the one-line notation of $\gamma$. Let $\operatorname{neg}(\gamma)$ be the $(q,p)$-clan obtained by switching the signs of all fixed points in $\gamma$. The next proposition is clear from the description above of the map $\gamma \mapsto \lambda^+(\gamma)$ in terms of lattice paths.

Proposition 3.12. $\lambda^+(\gamma)^\vee = \lambda^+(\operatorname{rev}(\gamma))$ and $\lambda^+(\gamma)^t = \lambda^+(\operatorname{rev}(\gamma))$, and therefore $\lambda^-(\gamma) = \lambda^+(\operatorname{rev}(\gamma)) = (\lambda^+(\gamma)^\vee)^t$.

Definition 3.13. The Wyser-Yong Schubert polynomials labeled by the members of Clan_{p,q} are defined by induction on clan weak order using the recurrence

$$
\mathcal{S}_\gamma \phi^+(\gamma) = s_{\lambda^+(\gamma)}(X_{\phi^+(\gamma)}) s_{\lambda^-(\gamma)}(X_{\phi^-(\gamma)}) \quad \text{if } \gamma \text{ is matchless} \\
\mathcal{S}_{\gamma \ast s_i} = \partial_i \mathcal{S}_\gamma \quad \text{if } \gamma \ast s_i < \gamma.
$$

Theorem 3.14 ([23]). Definition 3.13 makes sense: given a fixed $\gamma$, the polynomial $\mathcal{S}_\gamma$ is independent of the choice of matchless clan $\gamma'$ and saturated chain $\gamma < \cdots < (\gamma' \ast s_{a_1}) \ast s_{a_2} < \cdots < s_{a_k} < \gamma'$.

used to compute it. Also, if $\gamma \ast s_i \not< \gamma$, then $\partial_i \mathcal{S}_\gamma = 0$ (this includes the case where $\gamma \ast s_i$ is not defined).
Wyser and Yong also show that $\mathcal{S}_\gamma^{\text{wy}}$ represents the cohomology class $[\mathcal{Y}_\gamma]$. Brion [3] had previously given a formula for $[\mathcal{Y}_\gamma]$ as a sum of Schubert classes, from which one can deduce a formula for $\mathcal{S}_\gamma^{\text{wy}}$ as a sum of Schubert polynomials. In fact, this formula is simply Definition 3.3, so the next lemma is not really new, but we include a self-contained proof because it is not entirely obvious that the summands in Brion’s formula are indeed the $\mathcal{S}_w$ for $w \in A(\gamma)$.

**Remark 3.15.** The last claim in Theorem 3.14, that $\partial_i \mathcal{S}_\gamma^{\text{wy}} = 0$ if $\gamma \neq s_i \neq \gamma$, is not stated explicitly in [22], but it follows from the geometry. Indeed, the geometric interpretation of weak order mentioned in Remark 2.7 is essentially that if $\partial_i [\mathcal{Y}_\gamma]$ is nonzero, then it equals some $[\mathcal{Y}_\gamma']$ and then one takes $\gamma' < \gamma$ to be a covering in weak order labeled by $s_i$.

**Lemma 3.16.** $\mathcal{S}_\gamma = \mathcal{S}_\gamma^{\text{wy}}$ for any clan $\gamma \in \text{Clan}_{p,q}$.

**Proof.** We claim that $\mathcal{S}_\gamma$ satisfies the same recurrence as $\mathcal{S}_\gamma^{\text{wy}}$, namely, if $1 \leq i < n$, then

$$\partial_i \mathcal{S}_\gamma = \begin{cases} 
\mathcal{S}_{\gamma s_i} & \text{if } \gamma s_i < \gamma \\
0 & \text{otherwise}
\end{cases}$$

(1)

Given that $\mathcal{S}_\gamma = \sum_{w \in A(\gamma)} \mathcal{S}_w$ and that ordinary Schubert polynomials satisfy the recurrence

$$\partial_i \mathcal{S}_w = \begin{cases} 
\mathcal{S}_{ws_i} & \text{if } \ell(ws_i) < \ell(w) \\
0 & \text{otherwise}
\end{cases},$$

this claim follows from two simple facts about atoms:

(i) If $\gamma s_i < \gamma$, then $A(\gamma s_i) = \{ws_i : w \in A(\gamma) \text{ and } \ell(ws_i) < \ell(w)\}$.

(ii) If $\gamma s_i \neq \gamma$, then $\ell(ws_i) > \ell(w)$ for all $w \in A(\gamma)$.

If $w \in A(\gamma)$ and $\ell(ws_i) < \ell(w)$, then $w$ has a reduced word ending in $i$, so $\gamma s_i < \gamma$: this proves (ii). As for (i), if $\gamma s_i < \gamma$ then $\mathcal{R}(\gamma s_i) = \{a_1 \cdots a_k : a_1 \cdots a_k i \in \mathcal{R}(\gamma)\}$, which is equivalent to (i).

Now we show that $\mathcal{S}_\gamma = \mathcal{S}_\gamma^{\text{wy}}$ for all $\gamma$ by induction on the rank of $\gamma$ in weak order. The base case is $\mathcal{S}_{\gamma_{p,q}} = \mathcal{S}_{\gamma_{p,q}}^{\text{wy}} = 1$ (the second equality is clear from the geometry, if not from Definition 3.13). Equation (1) and Theorem 3.14 show that for any $i < n$,

$$\partial_i (\mathcal{S}_\gamma - \mathcal{S}_\gamma^{\text{wy}}) = 0,$$

using that $\mathcal{S}_{\gamma s_i} = \mathcal{S}_{\gamma s_i}^{\text{wy}}$ by induction. The kernel of $\partial_i$ on $\mathbb{Z}[x_1, \ldots, x_n]$ consists of those polynomials symmetric in $x_i$ and $x_{i+1}$, so this shows $\mathcal{S}_\gamma - \mathcal{S}_\gamma^{\text{wy}}$ is symmetric in $x_1, \ldots, x_n$. By [22, Proposition 2.9], $\mathcal{S}_\gamma^{\text{wy}}$ is a linear combination of Schubert polynomials $\mathcal{S}_w$ for $w \in S_n$, so the same is true of $\mathcal{S}_\gamma - \mathcal{S}_\gamma^{\text{wy}}$. But it is well-known that the Schubert polynomials $\mathcal{S}_w$ for $w \in S_n$ are linearly independent modulo the ideal in $\mathbb{Z}[x_1, \ldots, x_n]$ generated by symmetric polynomials [13, §2.5.2], so $\mathcal{S}_\gamma - \mathcal{S}_\gamma^{\text{wy}} = 0$.

Our next goal is to leverage the formulas of Wyser and Yong to prove enumerative results for clan words via the Stanley symmetric functions $F_\gamma$. To do this, we must better understand the procedure of passing from $\mathcal{S}_\gamma$ to $F_\gamma$. An important fact about the divided difference operators $\partial_i$ is that they satisfy the braid relations for $S_n$: that is, $\partial_i \partial_k = \partial_k \partial_i$ if $|i - k| > 1$ and $\partial_i \partial_j \partial_i = \partial_j \partial_i \partial_j$ if $|i - j| = 1$. As a consequence, we can define $\partial_a$ as the composition $\partial_{a_n} \cdots \partial_{a_1}$ for a reduced word $a \in \mathcal{R}(w)$, and the resulting operator is independent of the choice of $a$. The same holds for the $\pi_i$.

**Lemma 3.17** ([7], Theorem 3.40; [12], equation (4.25)). Let $w_n = n(n - 1) \cdots 21 \in S_n$. If $f \in \mathbb{Z}[x_1, \ldots, x_n]$ and $N \geq n$, then $\pi_{w_n} f$ is a symmetric polynomial in $x_1, \ldots, x_N$. Moreover, $\lim_{N \to \infty} \pi_{w_n} \mathcal{S}_w = F_w$ for any permutation $w$. 
It follows by linearity that \( \pi_{w_i} \mathfrak{S}_\gamma = F_\gamma \) for \( \gamma \in \text{Clan}_{p,q} \). The result we are working towards is that, if \( \gamma \) is matchless, then \( F_\gamma = s_{\lambda^+ (\gamma)} s_{\lambda^- (\gamma)} \). While it is true that the two Schur functions here are the images under \( \lim_{N \to \infty} \pi_{w_N} \) of the two factors in Definition 3.13, in general \( \pi_{w_N} \) is not a ring homomorphism, so we must work a little harder.

**Lemma 3.18** ([12], equation (3.10)). Suppose \( k \) is such that \( \phi_k \neq \phi_{k'} \) for all \( k' \neq k \). Then
\[
\pi_{\phi_k} s_\lambda (X_{\phi_1}, \ldots, X_{\phi_k}, \ldots, X_{\phi_r}) = s_\lambda (X_{\phi_1}, \ldots, X_{\phi_k+1}, \ldots, X_{\phi_r}).
\]
If \( i \notin \{\phi_1, \ldots, \phi_r\} \), then \( \pi_i s_\lambda (X_{\phi}) = s_\lambda (X_{\phi}) \).

**Proof.** Let us first verify this when \( \lambda = (d) \) has length 1, so \( s_\lambda (X_r) \) is the homogeneous symmetric polynomial \( h_d (X_r) = h_d (x_1, \ldots, x_r) \), and we must see that \( \pi_i h_d (X_r) = h_d (X_{r+1}) \).

This is easy using the generating function
\[
\prod_{i=1}^{r} \frac{1}{1-x_i t} = \sum_{d=0}^{\infty} h_d (X_r) t^d.
\]
The first \( r - 1 \) factors on the left are symmetric in \( x_r \) and \( x_{r+1} \), so commute with \( \pi_r \), so one only needs to verify by direct computation that \( \pi_r (1 - x_r t)^{-1} = (1 - x_r t)^{-1} (1 - x_{r+1} t)^{-1} \).

For general \( \lambda \), we use the Jacobi-Trudi identity for flagged Schur functions [19]:
\[
s_\lambda (X_\phi) = \det (h_{\lambda_i - i+j} (X_{\phi_i}))_{1 \leq i, j \leq \ell (\lambda)}.
\]
This determinant expands as a sum of terms of the form
\[
\pm h_{d_1} (X_{\phi_1}) \cdots h_{d_t} (X_{\phi_t}).
\]  \hspace{1cm} (2)

If \( i \neq r \), then \( h_d (X_r) \) is symmetric in \( x_i \) and \( x_{i+1} \). In particular, the hypothesis \( \phi_{k'} \neq \phi_k \) for \( k' \neq k \) ensures that every factor in the term \( (2) \) is symmetric in \( x_{\phi_k} \) and \( x_{\phi_{k+1}} \) except for \( h_{d_k} (X_{\phi_k}) \). The effect of applying \( \pi_{\phi_k} \) to the term \( (2) \) is therefore the same as the effect of applying it only to the factor \( h_{d_k} (X_{\phi_k}) \), and the previous paragraph shows that this is the same as replacing \( \phi_k \) by \( \phi_k + 1 \). This argument also shows that if \( i \notin \{\phi_1, \ldots, \phi_r\} \), then \( s_\lambda (X_{\phi}) \) is symmetric in \( x_i \) and \( x_{i+1} \), hence fixed by \( \pi_i \).

**Theorem 3.19.** \( F_\gamma = s_{\lambda^+ (\gamma)} s_{\lambda^- (\gamma)} \) for a matchless clan \( \gamma \).

**Proof.** Abbreviate \( \lambda^\pm (\gamma) \) and \( \phi^\pm (\gamma) \) as \( \lambda^\pm \) and \( \phi^\pm \). By Lemma 3.17 and the formulas of Definition 3.13,
\[
F_\gamma = \lim_{N \to \infty} \pi_{w_N} (s_{\lambda^+ (X_{\phi^+})} s_{\lambda^- (X_{\phi^-})}).
\]
Fix \( N \geq n = p+q \). Let \( a^j \) be the word \( i(i+1) \cdots (N-1) \) for \( i < N \). It is not hard to check that \( a^{N-1} \ldots a^2 a^1 \) is a reduced word for \( w_N \), and we will take \( \pi_{w_N} \) to be the specific composition
\[
\pi_{a^{N-1}} \cdots \pi_{a^1}.
\]
Let \( f = s_{\lambda^+ (X_{\phi^+})} s_{\lambda^- (X_{\phi^-})} \).

First consider \( \pi_{N-1} (f) \). If \( N-1 > n \), then \( f \) is symmetric in \( x_{N-1} \) and \( x_N \) (since these variables do not even appear), so \( f \) is fixed by \( \pi_{N-1} \). Otherwise, \( N-1 \) appears in exactly one of \( \phi^- \) and \( \phi^+ \); say \( (\phi^-)^k = N-1 \). The sequences \( \phi^+ \) and \( \phi^- \) are disjoint and have no repeated entries, so it follows from Lemma 3.18 that
\[
\pi_{N-1} (f) = s_{\lambda^+ (X_{\phi^+})} \pi_{N-1} (s_{\lambda^- (X_{\phi^-}))} = s_{\lambda^+ (X_{\phi^+})} s_{\lambda^- (X_{\phi^-})} X_{\phi_{k+1}} \cdots X_{\phi_q};
\]

in words, \( \pi_{n-1} (f) \) is obtained from \( f \) by incrementing the entry \( N-1 \) of \( \phi^- \) to \( N \). This does not alter any entries of \( \phi^\pm \) which are less than \( N-1 \), and so the same argument shows that subsequently applying \( \pi_{N-2}, \pi_{N-3}, \ldots, \pi_1 \) (in that order) has the effect of incrementing every
value in $\phi^-$ and $\phi^+$ which is less than $N$. That is, $\pi_{a\uparrow}(f) = s_{\lambda^+}(X_{\uparrow\phi^+})s_{\lambda^-}(X_{\uparrow\phi^-})$ where for a sequence $\phi$ we define $\uparrow\phi$ as the sequence with

$$(\uparrow\phi)_i = \begin{cases} 
\phi_i + 1 & \text{if } \phi_i < N \\
\phi_i & \text{if } \phi_i \geq N.
\end{cases}$$

Similarly, consider the action of $\pi_{a\downarrow}$. Ignoring entries equal to $N$, the flags $\uparrow\phi^+$ and $\uparrow\phi^-$ are still disjoint with no repeated entries, and so the argument of the last paragraph shows that

$$\pi_{a\downarrow}(\pi_{a\uparrow} f) = \pi_{a\downarrow}(s_{\lambda^+}(X_{\uparrow\phi^+})s_{\lambda^-}(X_{\uparrow\phi^-})) = s_{\lambda^+}(X_{\uparrow\phi^+})s_{\lambda^-}(X_{\uparrow\phi^-}).$$

Continuing in this way, we see that

$$\pi_{w N} f = s_{\lambda^+}(X_{\uparrow N - 1\phi^+})s_{\lambda^-}(X_{\uparrow N - 1\phi^-}) = s_{\lambda^+}(X_N, \ldots, X_N)s_{\lambda^-}(X_N, \ldots, X_N).$$

Since $\lambda^-$ and $\lambda^+$ have length at most $n \leq N$ by definition, $s_{\lambda^\pm}(X_N, \ldots, X_N)$ is simply the ordinary Schur polynomial $s_{\lambda^\pm}(x_1, \ldots, x_N)$. Thus, $\lim_{N \to \infty} \pi_{w N} f = s_{\lambda^+} s_{\lambda^-}$. 

\[\square\]

**Corollary 3.20.** Let $f^\lambda$ be the number of standard tableaux of shape $\lambda$. Then for a matchless clan $\gamma \in \Clan_{p,q}$,

$$\#\mathcal{R}(\gamma) = \left(\frac{|\lambda^+| + |\lambda^-|}{\lambda^+, |\lambda^-|}\right) f^{\lambda^+} f^{\lambda^-} = (pq)! \prod_{i \in \phi^+, j \in \phi^-} \frac{1}{|i - j|}.$$  

\[\text{Proof.}\] The first equality follows from Theorem 3.19 by comparing coefficients of $x_1 x_2 \cdots$, as per Proposition 3.6. For the second, apply the hook length formula. The hook lengths of $\lambda^+$ are exactly the distances from each $+ \in \gamma$ to some following $-$. To be precise, the hook with corner $(i, j)$ in $\lambda^+$ has size $\phi^+_i - \phi^-_{j+1}$. This statement and the corresponding statement for $\lambda^-$ imply via the hook length formula that

$$f^{\lambda^+} f^{\lambda^-} = |\lambda^+|! \prod_{i \in \phi^+, j \in \phi^- \atop i < j} \frac{1}{|i - j|} |\lambda^-|! \prod_{i \in \phi^+, j \in \phi^- \atop i > j} \frac{1}{|i - j|}.$$  

Since $|\lambda^+| + |\lambda^-|$ is the total number of pairs of a $+$ and following a $-$ or vice versa, i.e. $pq$, the second equality follows. 

\[\square\]

4. Maximizing the number of reduced words

Any reduced word for a permutation in $S_n$ is a prefix of at least one reduced word for $w_n$, so $\# \mathcal{R}(w)$ is maximized when $w = w_n$. For the same reason, the clan $\gamma \in \Clan_{p,q}$ with the most reduced words must be matchless, but it is not immediately clear which matchless clans maximize $\# \mathcal{R}(\gamma)$. In the smallest case $q \geq p = 1$, Corollary 3.20 says that $\# \mathcal{R}(\gamma) = \binom{q}{\phi^+_1 - 1}$, confirming the natural guess that $\# \mathcal{R}(\gamma)$ is maximized when $\gamma$ has its single $+$ as close to the middle of its one-line notation as possible. To proceed further, it is helpful to rewrite the formula of Corollary 3.20.

**Proposition 4.1.** For real numbers $1 \leq \phi_1 < \cdots < \phi_m \leq p + q = n$ let

$$f(\phi_1, \ldots, \phi_m) = \prod_{1 \leq k < \ell \leq m} \frac{1}{(\phi_k - \phi_\ell)^2} \prod_{k=1}^m \Gamma(\phi_k) \Gamma(n - \phi_k).$$

If $\gamma \in \Clan_{p,q}$ is matchless, then $\# \mathcal{R}(\gamma) = (pq)! f(\phi^+(\gamma)) = (pq)! f(\phi^- (\gamma)).$
Proof. Regroup the factors in Corollary 3.20:

\[ \# \mathcal{R}(\gamma) = (pq)! \prod_{i \in \phi^+} \left( \frac{1}{i-j} \prod_{j \in \phi^- : j<i} 1 \prod_{j \in \phi^- : j>i} 1 \right) . \]

Rewriting

\[ \prod_{j \in \phi^- : j<i} \frac{1}{i-j} = \frac{1}{(i-1)!} \prod_{k \in \phi^+} (i-k) \quad \text{and} \quad \prod_{j \in \phi^- : j>i} \frac{1}{j-i} = \frac{1}{(n-i)!} \prod_{k \in \phi^+} (k-i) \]
gives

\[ \# \mathcal{R}(\gamma) = (pq)! \prod_{i \in \phi^+} \left( \frac{1}{(i-1)!(n-i)!} \prod_{k \in \phi^+ : k \neq i} (k-i)^2 \right) = \frac{(pq)!}{f(\phi^+)} . \]

The same argument works with the roles of \( \phi^+ \) and \( \phi^- \) reversed. \( \square \)

Although there is not a unique maximizer of \( \# \mathcal{R}(\gamma) \), as for instance \( \# \mathcal{R}(++-+) = \# \mathcal{R}(++++) \), the next lemma provides a weaker uniqueness statement.

Lemma 4.2. On the domain \( 1 \leq \phi_1 < \cdots < \phi_p \leq p + q = n \), the function \( \log f(\phi_1, \ldots, \phi_p) \) is strictly convex in each variable. In particular, \( f \) has a unique global minimum \( \phi^* = (\phi_1^*, \ldots, \phi_p^*) \), and any minimizer of \( f \) restricted to the integer lattice \( \mathbb{Z}^p \) is one of the \( 2^p \) points obtained from \( \phi^* \) by rounding each coordinate either up or down.

Proof. Fixing \( \phi_2, \ldots, \phi_p \), we have

\[ \log f(\phi_1, \ldots, \phi_p) = \log \Gamma(\phi_1) + \log \Gamma(n+1-\phi_1) - 2 \sum_{k=1}^{p} \log(\phi_k - \phi_1) + C \]

for some constant \( C \). By the Bohr-Mollerup theorem, \( \log \Gamma(\phi_1) \) is a convex function of \( \phi_1 \), and taking second derivatives shows the same is true of \( \log \Gamma(n+1-\phi_1) \) and \(-\log(\phi_k - \phi_1)\). In fact, \(-\log(\phi_k - \phi_1)\) is strictly convex, so the sum \( \log f(\phi_1, \ldots, \phi_p) \) is also strictly convex in \( \phi_1 \), and in every \( \phi_i \) by symmetry.

Strict convexity implies that \( \log f \) (hence \( f \)) has at most one global minimum. To see that it does have one, observe that \( f(\phi) \to \infty \) as \( \phi \) approaches the boundary of the domain of \( f \) where \( \phi_i = \phi_{i+1} \) for some \( i \), so that for sufficiently small \( \varepsilon > 0 \), the global minimum of \( f \) on the compact set where \( 1 \leq \phi_i \leq \phi_{i+1} - \varepsilon \leq n \) for each \( i \) will also be a global minimum of \( f \) on its whole domain.

Finally, using the convexity of \( \log f \) in each variable individually, the claim about the minimizer of \( \log f \) restricted to \( \mathbb{Z}^n \) reduces to the fact that if \( g : [a, b] \to \mathbb{R} \) is a convex function with global minimum \( x^* \), then \( g \) is decreasing on \([a, x^*]\) and increasing on \([x^*, b]\). \( \square \)

We can work out almost exactly which \((2, q)\)-clans maximize \( \# \mathcal{R}(q) \). The minimizer \( \phi^* \) of \( f \) from Lemma 4.2 must be invariant under the transformation

\[ (\phi_1^*, \ldots, \phi_p^*) \mapsto (n+1-\phi_p^*, \ldots, n+1-\phi_1^*) , \]
since \( f \) itself is. When \( \phi = \phi^+(\gamma) \) this transformation corresponds to reversing the one-line notation of \( \gamma \). In particular, \( \phi^* \) is determined by the one parameter \( \phi_1^* \) for \( p = 2 \), or equivalently by the distance between \( \phi_1^* \) and \( \phi_2^* \). The next theorem shows that \( \# \mathcal{R}(\gamma) \) is maximized for \( \gamma \in \text{Clan}_{2,q} \) when \( \gamma \) is (essentially) invariant under reversal and its two + signs are separated by distance \( \sqrt{n} \).
Theorem 4.3. When \( p = 2 \), the minimizer \( \phi^* = (\phi^*_1, \phi^*_2) \) of \( f \) satisfies
\[
\left| \phi^*_1 - \left( \frac{n+1}{2} - \frac{1}{2} \sqrt{n} \right) \right| \leq \frac{33}{64},
\]
(3)
Setting \( \alpha_1 = \frac{n+1}{2} - \frac{1}{2} \sqrt{n} \) and \( \alpha_2 = \frac{n+1}{2} + \frac{1}{2} \sqrt{n} \), the clans \( \gamma \in \text{Clan}_{2,q} \) maximizing \( \# R(\gamma) \) have
\[
\phi^+(\gamma)_1 \in \{ [\alpha_1] - 1, [\alpha_1], [\alpha_1] + 1 \}
\]
\[
\phi^+(\gamma)_2 \in \{ [\alpha_2] - 1, [\alpha_2], [\alpha_2] + 1 \}.
\]

Proof. Lemma 4.2 shows that \( \phi^+(\gamma)_1 \) is one of the two closest integers to \( \phi^*_1 \), so if it is known that \( |\phi^*_1 - \alpha_1| < 1 \), then \( \phi^+(\gamma)_1 \) must be one of \( [\alpha_1] - 1, [\alpha_1], [\alpha_1] + 1 \). The analogous fact for \( \phi^+(\gamma)_2 \) holds by symmetry of \( \phi^* \). Thus, it suffices to prove the bound (3).

Since \( \phi_2^* = n+1 - \phi_1^* \), we may as well minimize the single variable function \( f(\phi_1, n+1-\phi_1) \) on the domain \( \phi_1 \in [1, \frac{n}{2}] \). It is helpful to let \( m = \frac{n+1}{2} \) and use the new coordinate \( x = m - \phi_1 \):
\[
\log f(\phi_1, n+1-\phi_1) = 2 \log \Gamma(n+1-\phi_1) + 2 \log \Gamma(\phi_1) - 2 \log(n+1-2\phi_1)
\]
\[
= 2 \log \Gamma(m+x) + 2 \log \Gamma(m-x) - 2 \log(2x).
\]
Set \( g(x) = \log \Gamma(m+x) + \log \Gamma(m-x) - \log(2x) \). Then
\[
g'(x) = \Psi(m+x) - \Psi(m-x) - \frac{1}{x},
\]
where \( \Psi(y) = \frac{d}{dy} \log \Gamma(y) \). The inequalities \( \log(\frac{1}{2}) < \Psi(y) < \log(y) \) for \( y > \frac{1}{2} \) and \( \log(1+y) \leq y \) for \( y > -1 \) imply
\[
- \log \frac{m-x}{m+x-\frac{1}{2}} - \frac{1}{x} < g'(x) < \log \frac{m+x}{m-x-\frac{1}{2}} - \frac{1}{x}
\]
\[
\implies \quad 2x - \frac{1}{x} < g'(x) < \frac{2x+1}{m-x-\frac{1}{2}} - \frac{1}{x}.
\]
The positive zeroes of the lower and upper bounds here are, respectively,
\[
\frac{3}{8} + \frac{1}{2} \sqrt{2m - \frac{7}{16}} \quad \text{and} \quad \frac{3}{8} + \frac{1}{2} \sqrt{2m - \frac{7}{16}},
\]
or \( \pm \frac{3}{8} + \frac{1}{2} \sqrt{n + \frac{n}{16}} \). It follows that \( g \) has a critical point \( x^* \) in \( (-\frac{3}{8} + \frac{1}{2} \sqrt{n + \frac{n}{16}}, \frac{3}{8} + \frac{1}{2} \sqrt{n + \frac{n}{16}}} \), and strict convexity of \( g \) forces \( x^* \) to be its unique global minimizer.

Using these bounds on \( x^* \) and the inequality \( \sqrt{n + 9/16 - \sqrt{m}} = \frac{9/16 + \sqrt{m}}{\sqrt{n + 9/16 + \sqrt{m}}} \leq \frac{9}{32} \) for \( n \geq 1 \) gives
\[
\frac{3}{8} < x^* - \frac{1}{2} \sqrt{n} < \frac{3}{8} + \frac{9}{64} = \frac{33}{64}.
\]
Since \( \phi^*_1 = \frac{n+1}{2} - x^* \), the bound (3) follows. \( \square \)

Let \( R(\gamma) \) be the number of reduced words of \( \gamma \). Although we do not have a general description of the clans maximizing \( R \), we can prove a sort of continuity result showing that a maximizer of \( R \) in \( \text{Clan}_{p,q} \) cannot be very different from a maximizer in \( \text{Clan}_{p,q+1} \). Define a partial order \( \preceq \) on matchless \( (p,q) \)-clans by declaring \( \gamma' \preceq \gamma \) if \( \phi^+(\gamma')_i \leq \phi^+(\gamma)_i \) for \( i = 1, \ldots, p \). This partial order is a lattice, with
\[
\phi^+(\gamma \vee \gamma')_i = \max(\phi^+(\gamma)_i, \phi^+(\gamma')_i) \quad \text{and} \quad \phi^+(\gamma \wedge \gamma')_i = \min(\phi^+(\gamma)_i, \phi^+(\gamma')_i).
\]
Write \( \gamma- \) and \( -\gamma \) for the clans obtained by appending or prepending a \( - \) to the one-line notation of \( \gamma \).
Lemma 4.4. If \( \gamma' \prec \gamma \) and \( R(\gamma') \leq R(\gamma) \), then \( R(\gamma'-) < R(\gamma-) \).

Proof. Abbreviate \( \phi^+(\gamma) \) and \( \phi^+(\gamma'-) \) as \( \phi \) and \( \phi' \). Proposition 4.1 shows

\[
R(\gamma) = (pq)! \prod_{1 \leq i < j \leq p} (\phi_j - \phi_i)^2 \prod_{i=1}^{p} \frac{1}{(\phi_i - 1)!/(n - \phi_i)!}.
\]

Thus,

\[
\frac{R(\gamma-)}{R(\gamma'-)} = \prod_{i=1}^{p} \frac{(n + 1 - \phi'_i)!}{(n + 1 - \phi_i)!} = \frac{R(\gamma)}{R(\gamma')} \prod_{i=1}^{p} \frac{n - \phi'_i + 1}{n - \phi_i + 1}
\]

and the last expression strictly exceeds \( R(\gamma)/R(\gamma') \geq 1 \) because \( \gamma' \prec \gamma \). \( \square \)

Theorem 4.5. Suppose \( \gamma \in \text{Clan}_{p,q} \) and \( \delta \in \text{Clan}_{p,q+1} \) are maximizers of \( R \). Then all entries of the vector \( \phi^+(\delta) - \phi^+(\gamma) \) are either 0 or 1.

Proof. Suppose \( \varepsilon \in \text{Clan}_{p,q+1} \) is such that \( \gamma - \not\prec \varepsilon \). We will show that \( \varepsilon \) then does not maximize \( R \), so that necessarily \( \gamma - \leq \delta \) and (by a symmetric argument) \( -\gamma \leq \delta \), which together imply the theorem.

It is clear that the one-line notation of \( \varepsilon \land \gamma- \) ends in \( - \), so let \( \zeta \in \text{Clan}_{p,q} \) be such that \( \zeta- = \varepsilon \land \gamma- \). Then \( R(\zeta) \leq R(\gamma) \) by the choice of \( \gamma \), and the (strict!) inequality \( \varepsilon \land \gamma- \prec \gamma- \) implies \( \zeta \prec \gamma \). Therefore \( R(\zeta-) = R(\varepsilon \land \gamma-) < R(\gamma-) \) by Lemma 4.4.

Now, using the formula of Proposition 4.1,

\[
\frac{R(\varepsilon \land \gamma-)R(\varepsilon \lor \gamma-)}{R(\gamma-)} \leq \left[ \prod_{1 \leq i < j \leq p} \frac{\phi_j^+(\varepsilon \land \gamma-) - \phi_i^+(\varepsilon \land \gamma-)}{\phi_j^+(\gamma-) - \phi_i^+(\gamma-) \phi_j^+(\varepsilon \lor \gamma-) - \phi_i^+(\varepsilon \lor \gamma-)} \right]^2,
\]

which is at least 1 by the general inequality

\[
(a_2 - a_1)(b_2 - b_1) \leq (\min(a_2, b_2) - \min(a_1, b_1))(\max(a_2, b_2) - \max(a_1, b_1))
\]

when \( a_1 < a_2 \) and \( b_1 < b_2 \). Having shown \( R(\varepsilon \land \gamma-) < R(\gamma-) \) in the previous paragraph, this implies \( R(\varepsilon \lor \gamma-) > R(\varepsilon) \), so \( \varepsilon \) does not maximize \( R \). \( \square \)

We conclude this section by describing connections to work of Pittel and Romik on random Young tableaux of rectangular shape, although we do not attempt to prove any precise results. The uniqueness in Lemma 4.2 shows that if \( \gamma^* \in \text{Clan}_{p,q} \) maximizes \( R(\gamma^*) \), then \( \gamma^* \) and \( \text{rev}(\gamma^*) \) should be effectively equal (to be precise, \( \phi^+(\gamma^*) \) and \( \phi^+(\text{rev}(\gamma^*)) \)) differ by a vector with entries from \( \{0, 1, -1\} \). Proposition 3.12 then implies \( \lambda^+(\gamma^*) \approx \lambda^+(\gamma^*)' = \lambda^-(\gamma^*)' \), so

\[
R(\gamma^*) \approx \left( \frac{pq}{pq/2} \right) f^{\lambda^+(\gamma^*)} f^{\lambda^-(\gamma^*)} = \left( \frac{pq}{pq/2} \right) f^{\lambda^+(\gamma^*)} f^{\lambda^+(\gamma^*)'} = \left( \frac{pq}{pq/2} \right) f^{\lambda^+(\gamma^*)} f^{\lambda^+(\gamma^*)'}.
\]

Thus, maximizing \( R \) is equivalent to maximizing \( f^\lambda f^\lambda' \) over \( \lambda \subseteq [p] \times [q] \) with \( |\lambda| \approx |pq/2| \).

Let \( \text{SYT}(\lambda) \) be the set of standard tableaux of shape \( \lambda \), and write \( (q^p) \) for the \( p \times q \) rectangular partition. For any fixed \( 0 \leq k \leq pq \), there is a bijection

\[
\text{SYT}(q^p) \rightarrow \bigcup_{\lambda \subseteq [p] \times [q]} \text{SYT}(\lambda) \times \text{SYT}(\lambda'),
\]

which sends \( T \in \text{SYT}(q^p) \) to \( (T_1, T_2) \) where \( T_1 \) is the subtableau of \( T \) containing \( 1, 2, \ldots, k \), and \( T_2 \) is the complement of \( T_1 \) in \( T \) rotated 180° and with the entries \( pq, pq - 1, \ldots, k + 1 \) replaced by \( 1, 2, \ldots, pq - k \). It follows that \( f^\lambda f^{\lambda'} / f^{(q^p)} \) is the probability that the entries in \( [11|\lambda|] \) of a uniformly random member of \( \text{SYT}(q^p) \) form a subtableau of shape \( \lambda \). By the previous paragraph we would like to maximize this probability over \( \lambda \) with \( |\lambda| = |pq/2| \).
In [15], Pittel and Romik describe a “typical” random standard tableau of shape \((q^p)\) when \(p, q\) are large (and in a fixed ratio). To be precise, given \(T \in \text{SYT}(q^p)\) let \(S_T : [0, 1) \times [0, p/q)\) be the function

\[
S_T(x, y) = \frac{1}{pq} T([qy] + 1, [qx] + 1),
\]

where \(T(i, j)\) is the entry of \(T\) in row \(i\) and column \(j\). That is, we think of \(T\) as a surface whose height above the \(xy\)-plane is given by the entries of \(T\), rescaled so that the maximum height is 1 and the surface lies above the rectangle \([0, 1) \times [0, p/q)\). It is helpful to picture \(T\) in the French style here, so that 1 is in its lower-left corner at \((0, 0)\) and \(pq\) is in its upper-right corner.

**Theorem 4.6** ([15], Theorem 5). Fix \(\theta \in (0, 1]\), and suppose \(p_1, p_2, \ldots\) is a sequence of integers such that \(\lim_{q \to \infty} p_i/q = \theta\). There is an explicit function \(L_\theta : [0, 1] \times [0, \theta] \to [0, 1]\) such that for all \(\varepsilon > 0\) and all \((x, y) \in [0, 1) \times [0, \theta]\),

\[
\lim_{q \to \infty} P(|S_T(x, y) - L_\theta(x, y)| > \varepsilon : T \in \text{SYT}(q^p)) \text{ uniformly random} = 0.
\]

In particular, for large \(q\), a random \(T \in \text{SYT}(q^p)\) has its entries 1, 2, \ldots, \(\lfloor p/q \rfloor\) contained in a subtableau whose shape resembles the region in \([0, 1) \times [0, \theta]\) below the level curve \(\{(x, y) : L_\theta(x, y) = \frac{1}{2}\}\). We expect that if a matchless clan \(\gamma\) is chosen as the top element of a uniformly random maximal chain in \(\text{Clan}_{p_i, q}\) with \(q\) large, then \(\lambda^+(\gamma)\) should resemble this same limiting shape, and \(\gamma\) should be close to a maximizer of \(R\) with high probability.

It is natural to describe the resulting “limit clan” by a density function \(f : [0, 1] \to \mathbb{R}\), so that for \(t \in [0, 1]\),

\[
\text{# of } +'s \text{ among } \gamma_1, \gamma_2, \ldots, \gamma_{\ell(p+q)} \approx p \int_0^t f(t') \, dt'.
\]

Write \(C(t) = \int_0^t f(t') \, dt'\). If (4) holds, then

\[
C\left(\frac{\phi^+}{p+q}\right) = C\left(\frac{q - \lambda^+}{p+q}\right) \approx \frac{i}{p}
\]

for \(i \in [p]\), by definition of \(\phi^+(\gamma)\). Letting \(p, q \to \infty\) (with \(p/q \to \theta\)) and replacing \(i/p\) with \(t \in [0, 1]\), equation (5) becomes

\[
C\left( \frac{1 - x(t) + \theta t}{1 + \theta} \right) = t.
\]

where \(x(t)\) is such that \(L_\theta(x(t), \theta t) = \frac{1}{2}\). Using the explicit formulas from [15], one finds

\[
C'(t) = f(t) = \begin{cases} 
\frac{1+\theta}{2\theta} \left[ 1 - \frac{2}{\pi} \sin^{-1} \left( \frac{1-\theta}{\sqrt{\theta(1-\theta)}} \right) \right] & \text{if } |t - \frac{1}{2}| < \frac{\sqrt{\theta}}{\pi+1} \\
0 & \text{otherwise}
\end{cases}
\]

5. Connections to involution words

Let \(I_n\) be the set of involutions in \(S_n\). Given \(z \in I_n\) and an adjacent transposition \(s_i\), define

\[
z * s_i = \begin{cases} 
z s_i & \text{if } s_i z = z s_i \\
s_i z s_i & \text{otherwise}
\end{cases}
\]

Note that \(z * s_i\) is again an involution. The **weak order** on \(I_n\) is the transitive closure of the relations \(z * s_i < z\) when \(\ell(z * s_i) < \ell(z)\) [4, 7, 6, 9, 16].
Definition 5.1. A reduced involution word for $z \in \mathcal{I}_n$ is the sequence of labels along a saturated chain in weak order from the identity involution to $z$. Equivalently, it is a minimal-length word $a_1 \cdots a_k$ such that

$$z = (\cdots ((1 \ast s_{a_1}) \ast s_{a_2}) \ast \cdots) \ast s_{a_k}.$$ 

To avoid confusion with usual reduced words for $z$, we write $\hat{\mathcal{R}}(z)$ for the set of reduced involution words of $z$.

Example 5.2. The weak order on $\mathcal{I}_3$, with involutions drawn as partial matchings of \{1, 2, 3\}:

\[
\begin{array}{c}
\bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet
\end{array}
\]

The reduced involution words of the maximal element (13) are 12 and 21.

Just as in weak Bruhat order on $S_n$, the operation $z \mapsto z \ast s_i$ moves up or down in involution order according to whether $i$ is an ascent or descent of $z$.

Proposition 5.3 ([9], Lemma 3.8). If $z(i) > z(i + 1)$ then $z \ast s_i < z$, and if $z(i) < z(i + 1)$ then $z \ast s_i > z$.

Write $\kappa(z)$ for the number of 2-cycles in an involution $z$, and define $\mathcal{I}_{p,q} = \{z \in \mathcal{I}_{p+q} : \kappa(z) \leq \text{min}(p, q)\}$. Let $w_{p,q}$ be the involution $(1 \ n)(2 \ n-1) \cdots (m \ n-m+1)$ where $m = \text{min}(p, q)$, so $w_{p,q} = \iota(\gamma_{p,q})$.

Lemma 5.4. The set $\mathcal{I}_{p,q}$ has $w_{p,q}$ as its unique maximal element in involution weak order.

Proof. Set $m = \text{min}(p, q)$. First we prove the lemma with $\mathcal{I}_{p,q} = \{z \in \mathcal{I}_n : \kappa(z) \leq m\}$ replaced by $\{z \in \mathcal{I}_n : \kappa(z) = m\}$; call the latter set $J$. Suppose $z$ is maximal in $J$. By Proposition 5.3 this is equivalent to the condition that if $z(i) < z(i + 1)$, then $z \ast s_i \notin J$, which can only happen if $z \ast s_i$ has $\kappa(z) + 1$ cycles, i.e. if $z(i) = i$ and $z(i + 1) = i + 1$. Letting $i$ and $j$ be such that

$$z(1) > \cdots > z(i-1) > z(i) < z(i+1) < \cdots < z(j-1) > z(j),$$

it follows that $i, i+1, \ldots, j-1$ are all fixed points. We have $z(j) < z(j-1) = j-1$, and $z(j)$ is none of $\{z(j-1), \ldots, z(i)\} = \{j-1, \ldots, i\}$, so it must be one of $i-1, \ldots, 2, 1$ (assuming $z(j)$ exists). But the one-line notation of $z$ must end with $(i-1) \cdots 21$: otherwise, $z$ would have an ascent beginning with one of $1, 2, \ldots, i-1$, which would contradict maximality of $z$ since those are not fixed points. This completely determines $z$:

$$z = n(n-1) \cdots (n-i+2)(i+1) \cdots (n-i+1)(i-1) \cdots 21$$

$$= (1 \ n)(2 \ n-1) \cdots (i-1 \ n-i+2)$$

$$= (1 \ n)(2 \ n-1) \cdots (m \ n-m+1) = w_{p,q} \quad \text{(given that } \kappa(z) = m)$$

Now let us see that $w_{p,q}$ is also the unique maximal element of $\{z \in \mathcal{I}_n : \kappa(z) \leq m\}$. By induction on $m$, we can assume $y := (1 \ n)(2 \ n-1) \cdots (m-1 \ n-m+2)$ is the unique maximal element of $\{z \in \mathcal{I}_n : \kappa(z) \leq m-1\}$, and it is enough to show that $y < w_{p,q}$. But $y \ast s_m = (1 \ n)(2 \ n-1) \cdots (m-1 \ n-m+2)(m \ m+1)$ is in $J$, so $y < y \ast s_m \leq w_{p,q}$ by the previous paragraph. \qed
Proposition 5.5.
(a) If $\gamma * s_i$ is defined, then $\iota(\gamma * s_i) = \iota(\gamma) * s_i$.
(b) If $\gamma * s_i < \gamma$, then $\iota(\gamma * s_i) > \iota(\gamma)$.
(c) Let $\gamma \in \text{Clan}_{p,q}$ and suppose $i$ is a descent of $\iota(\gamma)$.
   • If $\iota(\gamma) * s_i = s_i\iota(\gamma)s_i$, there is a unique $\gamma' > \gamma$ in $\text{Clan}_{p,q}$ such that $\gamma' * s_i = \gamma$;
   • If $\iota(\gamma) * s_i = \iota(\gamma)s_i$, there are exactly two such $\gamma'$.
(d) $\iota : \text{Clan}_{p,q} \to \mathcal{I}_n$ is an order-reversing map with image $\mathcal{I}_{p,q}$.

Proof.
(a) The cases in which $\gamma * s_i$ is defined are: (1) if $i$ and $i + 1$ are fixed points of $\gamma$ of opposite sign, then $\gamma * s_i$ is obtained from $\gamma$ by making $i$ and $i + 1$ matched; (2) if $\{i, i + 1\}$ is not $\iota(\gamma)$-invariant, then $\gamma * s_i = s_i\gamma s_i$. In case (1) $\iota(\gamma * s_i) = \iota(\gamma)s_i$ and $s_i$ commutes with $\iota(\gamma)$, while in case (2) $\iota(\gamma * s_i) = s_i\iota(\gamma)s_i \neq \iota(\gamma)$, so in either case we get $\iota(\gamma * s_i) = \iota(\gamma) * s_i$.
(b) The relation $\gamma * s_i < \gamma$ implies $\ell(\iota(\gamma * s_i)) > \ell(\iota(\gamma))$, and by part (a) this is the same as $\ell(\iota(\gamma) * s_i) > \ell(\iota(\gamma))$, which means $\iota(\gamma) * s_i > \iota(\gamma)$ in weak order on $\mathcal{I}_n$.
(c) Suppose $\gamma'$ is such that $\gamma' * s_i = \gamma$. Because $i$ is a descent of $\iota(\gamma)$, Proposition 5.3 implies that $\ell(\iota(\gamma')) = \ell(\iota(\gamma)) + 1$ (using part (a)), so that $\gamma' > \gamma$.
   If $\iota(\gamma) * s_i = s_i\iota(\gamma)s_i$ then $i, i + 1$ are not matched by $\gamma$ and they are not both fixed points, so the same is true of $\gamma'$. In that case, $\gamma' * s_i$ is defined as $s_i\gamma s_i$, forcing $\gamma' = s_i\gamma s_i$.
   If $\iota(\gamma) * s_i = \iota(\gamma)s_i$, then $i$ and $i + 1$ are matched by $\gamma$ (they cannot be fixed points since $i$ is a descent of $\iota(\gamma)$). Thus $\gamma'$ and $\gamma$ agree on $[n] \setminus \{i, i + 1\}$, and $i, i + 1$ must be fixed points of $\gamma'$ labeled $-$ or $+$ in order to have $\gamma' * s_i = \gamma$.
(d) Part (b) shows that $\iota$ is order-reversing. An involution $z \in S_n$ has $n - 2\kappa(z)$ fixed points, and constructing $\gamma \in \text{Clan}_{p,q}$ with $\iota(\gamma) = z$ is equivalent to choosing $a$ of those fixed points to label $+$ and $b$ of them to label $-$, subject to the constraints $a + b = n - 2\kappa(z)$ and $a - b = p - q$. This gives $a = p - \kappa(z)$ and $b = q - \kappa(z)$, so $z \in \iota(\text{Clan}_{p,q})$ if and only if $\kappa(z) \leq \min(p, q)$. In fact, we get the stronger result that
\[
\# \{ \gamma \in \text{Clan}_{p,q} : \iota(\gamma) = z \} = \binom{n - 2\kappa(z)}{p - \kappa(z)} = \binom{n - 2\kappa(z)}{q - \kappa(z)}.
\]
□

Lemma 5.6. Suppose $C$ is a saturated chain $1 = z^0 < z^1 < z^2 < \cdots < z^r = z$ in $\mathcal{I}_{p,q}$.
(a) For a fixed $\gamma \in \text{Clan}_{p,q}$ with $\iota(\gamma) = z$, there are exactly $2^{\kappa(z)}$ saturated chains in $\text{Clan}_{p,q}$ with minimal element $\gamma$ whose image under $\iota$ is $C$.
(b) The total number of saturated chains in $\text{Clan}_{p,q}$ with image $C$ is
\[
\binom{n - 2\kappa(z)}{p - \kappa(z)} 2^{\kappa(z)} = \binom{n - 2\kappa(z)}{q - \kappa(z)} 2^{\kappa(z)}.
\]

Proof. Part (b) follows from (a) because the number of $\gamma \in \text{Clan}_{p,q}$ such that $\iota(\gamma) = z$ is $\binom{n - 2\kappa(z)}{p - \kappa(z)}$, as per the proof of Proposition 5.5(d).
As for part (a), let $k$ be the number of covering relations $z^j < z^{j+1}$ in the chain $z^0 < z^1 < \cdots < z^r = z$ for which $z^{j+1} = z^js_i$ for some $i$ (as opposed to $z^{j+1} = z^is_j$). Proposition 5.5(c,a) show that the number of saturated chains in $\text{Clan}_{p,q}$ with image $C$ and minimal element $\gamma$ is $2^k$. But the number $k$ is $\kappa(z)$ for any saturated chain from 1 to $z$, because
\[
k(z * s_i) = \begin{cases} 
\kappa(z) & \text{if } z * s_i = s_izs_i \\
\kappa(z) + 1 & \text{if } z * s_i = zs_i \text{ and } \ell(zs_i) > \ell(z).
\end{cases}
\]
□
Because \( i \) is order-reversing, Lemma 5.6 does not in general relate reduced words for \( \gamma \in \text{Clan}_{p,q} \) to reduced involution words for \( i(\gamma) \). However, it does when \( z = w_{p,q} \) is maximal in \( I_{p,q} \).

**Corollary 5.7.** The number of maximal chains in the poset \( \text{Clan}_{p,q} \) is \( 2^{\min(p,q)} \# \hat{\mathcal{R}}(w_{p,q}) \).

We can go further using known results for involution words.

**Definition 5.8.** The *involution Stanley symmetric function* of \( z \in I_n \) is

\[
\hat{F}_z = \lim_{m \to \infty} \sum_{a \in \hat{R}(z \times m)} \sum_{b \in \text{comp}(a)} x_{b_1} \cdots x_{b_k}.
\]

Just as for clans, the set \( \hat{\mathcal{R}}(z) \) is closed under the Coxeter relations for \( S_n \) [16, 3.16], so can be written as a disjoint union \( \bigcup_{w \in A(\gamma)} \hat{\mathcal{R}}(w) \) over some set \( A(\gamma) \subseteq S_n \). This implies that \( \hat{F}_z = \sum_{w \in A(\gamma)} F_w \), so \( \hat{F}_z \) is indeed a symmetric function.

**Definition 5.9.** A partition \( \lambda \) is *strict* if \( \lambda_1 > \lambda_2 > \cdots > \lambda_\ell \), and the *shifted shape* of a strict \( \lambda \) is the set of boxes \( \{(i, j) : 1 \leq i \leq \ell(\lambda) \text{ and } i \leq j \leq i + \lambda_1 - 1\} \) in matrix coordinates. A filling of a shifted shape by the alphabet \( \{1' < 2' < 2 < \cdots\} \) is a *marked shifted semistandard tableau* if:

- Its entries are weakly increasing across rows and down columns;
- No unprimed (resp. primed) number appears twice in a column (resp. row);
- There are no primed numbers on the main diagonal.

The *Schur P-function* of shifted shape \( \lambda \) is \( P_\lambda = \sum_T x^T \) where \( T \) runs over marked shifted semistandard tableaux of shape \( \lambda \). Here \( x^T \) is the monomial in which the power of \( x_i \) is the number of entries \( i \) and \( i' \) in \( T \). The *Schur Q-function* \( Q_\lambda \) is then defined to be \( 2^{\ell(\lambda)} P_\lambda \). These are both symmetric functions [11, III §8].

**Example 5.10.** Here is a marked shifted semistandard tableau of shifted shape \( (5, 4, 1) \):

\[
\begin{array}{cccc}
1 & 2' & 2 & 2' & 6' \\
2 & 1 & 4 & 6 & \\
5 & & & & \\
\end{array}
\]

**Lemma 5.11.** Suppose \( z = (1 b_1)(2 b_2) \cdots (k b_k) \) where \( b_1 > \cdots > b_k > k \). Then \( \hat{F}_z = P_\lambda \) where \( \lambda = (b_1 - 1, b_2 - 2, \ldots, b_k - k) \).

**Proof.** For \( y \in I_n \), let \( D(y) = \{(i, j) : j > i, z(j) < z(i), \text{ and } z(j) \leq i\} \), thought of as a subset of \( [n] \times [n] \) in matrix coordinates. Let \( \mu \) be the partition whose parts are the row lengths of \( D(y) \). By [8, Corollary 4.42], \( \hat{F}_y \) is a nonnegative integer combination of Schur \( P \)-functions, whose leading term in dominance order is \( P_\mu t^v \), where \( v^t \) is the partition conjugate to \( \mu \). For \( z \) as defined above, one checks that \( D(z) \) is the transpose of the shifted Young diagram of \( \lambda = (b_1 - 1, \ldots, b_k - k) \), so the leading term of \( \hat{F}_z \) is \( P_\lambda \).

Equivalently, the leading term of the Schur \( Q \)-function expansion of \( 2^{\ell(z)} \hat{F}_z \) is \( 2^{\ell(z)} P_\lambda = 2^k P_\lambda = Q_\lambda \). By [8, Theorem 4.67], \( 2^{\ell(y)} \hat{F}_y \) equals a single Schur \( Q \)-function with coefficient 1 if and only if \( y \) is a 2143-avoiding permutation, i.e. there are no \( a < b < c < d \) such that \( y(b) < y(a) < y(d) < y(c) \). This condition holds for \( z \), so \( 2^{\ell(z)} \hat{F}_z = Q_\lambda \), or \( \hat{F}_z = P_\lambda \). \( \square \)

**Theorem 5.12.** Assume \( p \geq q \) without loss of generality. Then

\[
\sum_{\gamma \in \text{Clan}_{p,q}} \sum_{\gamma \text{ matchless}} F_\gamma = 2^q P_{(n-1, n-3, \ldots, n-2q+1)} = Q_{(n-1, n-3, \ldots, n-2q+1)}.
\]
The number of maximal chains in Clan\textsubscript{p,q} is
\[2^{pq}(pq)! \prod_{i=1}^{q} \frac{(p+q-2i)!(p+q-2i+1)!}{(p-i)!(q-i)!}^{-1} = 2^{pq} \left( \frac{pq}{n-1, n-3, \ldots, n-2q+1} \right) \prod_{i=1}^{q} \frac{(p+q-2i)!(p-1, q-1)!}{(p-i, q-i)!}^{-1} \cdot \]

Proof. Lemma 5.6 gives a 2\textsuperscript{g}-to-1 correspondence between maximal chains in Clan\textsubscript{p,q} and reduced involution words of \(w_{p,q}\) which preserves the labeling of covering relations,
\[\sum_{\gamma \in \text{Clan}_{p,q}} F_{\gamma} = 2^{q} \hat{F}_{w_{p,q}} = 2^{q} P_{(n-1, n-3, \ldots, n-2q+1)}, \quad (6)\]
where the second equality holds by Lemma 5.11.

Let \(g^{\lambda}\) denote the number of unmarked standard shifted tableaux of shape \(\lambda\): fillings of the shifted shape of \(\lambda\) by 1, 2, \ldots, \(|\lambda|\) which are strictly increasing across rows and down columns. As in Proposition 3.6, the coefficient of \(x_{1} x_{2} \cdots x_{pq}\) on the lefthand side of (6) is the number of maximal chains in Clan\textsubscript{p,q}, while the coefficient on the right side is \(2^{q} \gamma(\lambda) g^{\lambda} = 2^{pq} g^{\lambda}\).

The shifted hook length formula [18] computes \(g^{\lambda}\), as follows. The doubled shape \(\tilde{\mu}\) of a strict partition \(\mu\) is obtained by placing a copy of the shifted shape of \(\mu\) to the right of its transpose so that their main diagonals are adjacent (but are not identified):
\[
\mu = \begin{array}{c|c|c}
\hline
& & \\
& & \\
\hline
\end{array}
\quad \sim \quad \tilde{\mu} = \begin{array}{c|c|c}
\hline
& & \\
& & \\
\hline
\end{array}
\]
where \(\cdot\) marks the new boxes. The shifted hook length formula is then \(g^{\mu} = |\mu|! \prod_{(i,j) \in \mu} h_{ij}\), where \(h_{ij}\) is the usual hook length of box \((i,j)\) in \(\tilde{\mu}\), but \((i,j)\) only runs over those boxes corresponding to the original shifted shape \(\mu\) in the example above, \(g^{\mu} = 8!/(7 \cdot 5 \cdot 4 \cdot 2 \cdot 4 \cdot 3 \cdot 1)\).

When \(\lambda = (p+q-1, p+q-3, \ldots, p-q+1)\), this formula gives
\[2^{pq} g^{\lambda} = 2^{pq}(pq)! \prod_{i=1}^{q-1} \frac{(p+q-2i)!(p+q-2i+1)!}{(p-q)!(q-i)!}^{-1} \cdot \]
where the \(i\text{th}\) factor in the first product is the product of the hook lengths in row \(i\) and columns \(1, \ldots, q-1\), and the \(i\text{th}\) factor in the second product is the product of the remaining hook lengths in row \(i\).

As a corollary of Theorem 5.12 we obtain an interesting symmetric function identity, which also appears in [5, §4.6] and [20, §7] in a slightly different form.

Corollary 5.13.
\[
\sum_{\lambda \subseteq \mu \times \nu} s_{\lambda} s_{\lambda^{\gamma}} = Q_{\lambda^{\gamma}_{p+q-1, p+q-3, \ldots, p-q+1}}.
\]

Proof. When \(\gamma\) is matchless, \(F_{\gamma} = s_{\lambda^{+}(\gamma)} s_{\lambda^{-}(\gamma)}\) by Theorem 3.19, where \(\lambda^{+}(\gamma)\) is the number of \(\textit{+}\)'s following the \(i\text{th}\) \(\textit{+}\) in \(\gamma\), and \(\lambda^{-}(\gamma)\) is the number of \(\textit{+}\)'s following the \(i\text{th}\) \(-\). Now,
\[
\lambda^{-}(\gamma)_{j} = \#\{i : \lambda^{-}(\gamma)_{i} \geq j\} \quad \text{number of \(-\)'s followed by at least \(\textit{+}\)}
\]
\[= q - (\text{number of \(-\)'s followed by at most \((j-1)\textit{+}\)}
\]
\[= q - (\text{number of \(-\)'s following the \((p-j+1)\textit{+}\)}
\]
\[= q - \lambda^{+}(\gamma)_{p-j+1} = \lambda^{+}(\gamma)_{j}.
\]
That is, \( \lambda^- (\gamma) = \lambda^+ (\gamma)^\vee \). The map \( \gamma \mapsto \lambda^+ (\gamma) \) is a bijection between matchless \((p,q)\)-clans and partitions contained in \([p] \times [q]\), so the corollary follows from Theorem 5.12.

\[ \square \]

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