Calculation of absolute stability regions for time-varying electromechanical systems

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Abstract. The paper proposes a numerical method of constructing piecewise-linear Lyapunov functions for absolute stability of nonlinear time-varying systems researching. There are examples and results of comparison with classical methods.

1. Introduction
In many real control objects, there is significant nonlinearity and nonstationarity [1]. For electromechanical systems, it can be expressed in a variable total moment of inertia on the motor shaft, in the influence of elastic ties, in external factors and etc. If the characteristics of nonlinear elements and the laws of change parameters in the form of specific formulas are known, then exact or approximate methods can be used, which are described in, for example, [2]. Often, in practice, only boundaries are known, in which nonlinearity and nonstationarity coefficients can be changed, which greatly complicates the analysis. Classical methods of the absolute stability theory, circle criteria and Popov’s criteria, have already become an important step towards the study of these systems. Widespread in the scientific and engineering environment (first of all for simplicity and visibility), they almost never provide necessary and sufficient conditions of stability [3], indeed, where there is more than one nonlinear and/or time-varying element, frequency methods are much more complicated and they loose their visibility. The connection between quadratic Lyapunov functions and frequency criteria was long-established, that is why, if several nonlinear time-varying elements are in the system, it is reasonable to use a numerical algorithm for constructing quadratic Lyapunov functions [4]. The most comprehensive review of this topic is presented in [5]. In this paper, the author will describe the algorithm which is based on the research of necessary and sufficient stability conditions.

2. Formulation of the problem
The paper considers the questions stability of the systems:

\[ \dot{x} = A + \sum_{j=1}^{m} b^j \varphi_j(\sigma_j, t), \]

\[ \sigma_j = (c^j, x) = \sum_{i=1}^{n} c^j_i x_i, \quad \varphi(0, t) \equiv 0, \]

where \( x = (x_1, x_2, ..., x_n)^T \in \mathbb{R}^n \) – \( n \)-dimensional vector of phase states, \( A \) – constant \((n \times n)\) matrix, \( b^j \) and \( c^j \) \((j = 1, m)\) – constant \( n \)-dimensional vectors. It is anticipated that nonlinear time-varying
functions $\varphi_j(\sigma_j, t)$ satisfy sector limitations

$$k_1 \sigma_j^2 \leq \varphi_j(\sigma_j, t) \sigma_j \leq k_2 \sigma_j^2$$

with omnifarious $\sigma_j$ and $t \geq 0$. Electric drives for manipulator and mobile robots, drives for machine tools are described by such equations, under certain conditions, the models like these can be used for analysis aircrafts and etc. In recent decades, the fuzzy systems which are realizing nonlinear laws of control in the channels of PID-controllers, are actively used, which also can be described by equations (1). As shown in [6], the question of stability systems (1) can be reduced to the question of stability equivalent (equivalency is understood within the meaning of coincidence of the sets of the solutions with the same initial conditions) differential inclusion.

$$\dot{x} \in F(x), \; F(x) = \text{conv} \bigcup_{k=1}^{N} A_k x.$$  

(2)

For system (1), matrix $A_k$ looks like

$$A_k = A x + \sum_{j=1}^{m} b_j \lambda_j \sigma_j, \; k = \frac{1}{2}, m,$$

(3)

where scalars $\lambda_j$ are independently accepted values $k_1$ or $k_2$. Let us denote set $A_k$ as $\mathcal{A}$. Single piece-wise linear Lyapunov function

$$v(l, x) = \max_{M \subseteq \mathcal{M}} \{l^T x\},$$

(4)

is introduced in [6] for differential inclusion (2). The level sets of function (4) are convex centrally symmetric polytopes, where $M$ is the number of facets. If the system is asymptotically stable, it is possible to find function (4), which will be decreased in all solutions (2).

One of the first algorithms of construction piecewise linear Lyapunov functions was [7]. Let us briefly describe its idea, as it will be required in the future. Initial differential inclusion (2) is discretized by computing matrix exponential

$$x_{k+1} \in \{e^{A \tau} x, \; A \in \mathcal{A}\},$$

(5)

where $\tau$ is some positive scalar. A convex centrally symmetric polytope with non-empty interior is defined. Of all of its vertices, using formula (5), new points are calculate and convex hull of initial polytope and new points are built. Further, the same iteration continues for new points. It is known that the stability of (5) is possible only if, at some stage of iteration process, all new points are going inside of polytope, and instability – if in a process of constructing, the convex hull does not remain an initial polytope. Using the constructed polytope (if the first condition of the previous sentence is done), the estimation of Lyapunov exponent is computed: if upper estimation is negative, then the system is stable. Obviously, that iteration process is dependent on parameter $\tau$: when it is less, the discrete system «is closer» to the continuous system, the more vertices in the resulted polytope, the more accurate the exponent estimation of Lyapunov. Using Lyapunov exponent for stable system (5), we can construct Lyapunov function for the original (2) system. In this way, to make a conclusion about stability of the original continuous differential inclusion, it is necessary to construct a convex centrally symmetric polytope, such that all possible vectors on this polytope will be directed to its interior part. The algorithm that does not use matrix exponential will be described below.

3. The algorithm for construction level sets of Lyapunov function

First of all, it is necessary to notice, that facets of the polytope are convex polytopes, which dimension is lesser per unit. If the normal of facets are normalized and directed to the outside, then the condition of the decrease of Lyapunov function on all solutions on the facet will be equivalent to

$$(A_k x, n_r) = \|A_k x\| \cos \varphi < 0,$$

$$A_k \in \mathcal{A}, \; x \in P,$$

where $\varphi$ – the angle between the normal of facet $n_r$ and vector field $A_k$ at point $x, \; P$ – a set of all facet points. Since $(A_k x, n_r)$ is linear on $x$ functional, the maximum on the convex limited polyhedral set it
reaches in the vertices of facets. As a result, for the estimation of stability, it is necessary to check only vertices of the constructed polytope. With discretization, the facet (connecting \( x_0 \) and \( x_1 \)) «cuts» real trajectory \( e^{At}x_0 \). That is why, at point \( x_0 \), the angle between \( Ax_0 \) and \( n_\tau \) is less than \( \pi/2 \), and that is why it needs relatively low \( \tau \) for good approximation of solution and opportunity to estimate Lyapunov exponent reasonably. Similarly, that approach can form many facets in places, where they are not needed. It is useful to approximate \( e^{At}x_0 \) in direction \( Ax_0 \). In that case, considering errors of computation, angle \( \varphi \) will be equal to \( \pi/2 \). The problem of the choice of the step, that is needed to move in direction \( Ax_0 \), remains.

To qualitatively approximate \( e^{At}x_0 \) by polyline and thus to leave computational complexity in acceptable bounds, it is necessary to add more links to the place where \( e^{At}x_0 \) have a significant bend; and fewer - where \( e^{At}x_0 \) has less bent. In other words, the value of the step is dependent on curvature of solution. From the course of differential geometry [8] it is known that the radius of curvature in multidimensional space is determined by relationship

\[
r = \frac{|\dot{y}(t)|^3}{\sqrt{[\dot{y}(t)]^2|\ddot{y}(t)|^2 - (\dot{y}(t),\dddot{y}(t))^2}},
\]

where \( y(t) \) – parametric equations of the curve. In this case

\[
\dot{y}(t) = \dot{x}(t) = A_kx(t),
\]

\[
\ddot{y}(t) = \dddot{x}(t) = A_k^2x(t).
\]

Now, at each point of the phase space we can find the radius of curvature of the relevant trajectory. Using the formula given below, we derive \( k \) new points for each vertex \( x_0 \).

\[
x^k_1 = x_0 + \frac{A_kx_0}{\|A_kx_0\|}h.
\]

In this paper, the step is taken as equal to \( h = \varepsilon r \), where \( \varepsilon \) – ahead the specified positive number. If \( \varepsilon \) is less, than the approximation is more precise, but it takes more computational resources. For convenience, let us define \( A_kx_0 \) and \( A_k^2x_0 \) as \( \dot{v}_k \) and \( \ddot{v}_k \), respectively. Finally, we get

\[
x^k_1 = x_0 + \frac{\varepsilon \dot{v}_k(\dot{v}_k, \ddot{v}_k)}{\sqrt{(\dot{v}_k, \dot{v}_k)(\dot{v}_k, \ddot{v}_k) - (\dot{v}_k, \ddot{v}_k)}}
\]

(6)

Setting some convex centrally symmetric polytope, let us find \( k \) points for each of its vertices using formula (6) and construct the convex hull. For new points it is necessary to repeat the same operation. If that iteration process for some stage does not give new points, than the system will be stable, and the resulting polytope will be the level set of Lyapunov function for differential inclusion (2).

**Example 1.** There is a structure scheme of the control system in fig. 1. The transfer function of the linear part of the system

\[
W(p) = \frac{1}{p^2 + 0.5p + 1}
\]

corresponds to a DC motor model.

![Figure 1](image-url)

**Figure 1.** A structure scheme of the control system

Element \( \phi(\sigma, t) \) characterized nonlinearity and nonstationarity in the channel of control.

\[
0 \leq \phi(\sigma, t)\sigma \leq \mu\sigma^2
\]
If circle criteria is applied, then \( \mu \leq 1.25 \) [7]. Using formula (3), the differential inclusion is as follows:

\[
A_1 = \begin{pmatrix} 0 & 1 \\ -1 & -0.5 \end{pmatrix}
\]

\[
A_2 = \begin{pmatrix} 0 & 1 \\ -(1 + \mu) & -0.5 \end{pmatrix}.
\]

In accordance with the proposed algorithm, it is necessary for different \( \mu \) to find such coefficient \( \epsilon \), in which the system is stable. The results of numerical experiments are in table 1. As shown in table, this approach allows increasing \( \mu \) almost per unit. This is important when synthesising control laws, because the rise of total system gain coefficient provides the increase of the precision in a steady mode.

| \( \mu \) | Facets count | \( \epsilon \) |
|---|---|---|
| 1.25 | 44 | 0.180 |
| 1.4 | 56 | 0.145 |
| 1.55 | 70 | 0.115 |
| 1.7 | 94 | 0.085 |
| 1.85 | 144 | 0.055 |
| 2.0 | 268 | 0.030 |
| 2.15 | 1610 | 0.005 |

As indicated in [6], approaching the bound of stability, the number of Lyapunov function level set facets increases unlimitedly. Vice versa, if the system is «far» from the bound of stability, the number of facets is relatively small. If matrix \( A \) eigenvalues of ordinary linear system \( \dot{x} = Ax \) are really different and negative, then in order to form vector \( l^\nu \) in (4), it is sufficient to take eigenvectors transposed matrix \( A^T \) (and also eigenvectors with an inverse sign), in this case \( M = 2n \). If two-dimensional linear systems have complex eigenvalues with a zero real part, then phase trajectories will be circles, and the best Lyapunov function will be the one where level sets coincide with phase circles. In this case, the system will be stable, but not asymptotically. An infinite number of facets for the approximation circle by the polytope are needed.

This effect is fair for the considered example. Really, if \( \mu = 1.25 \) needs only 44 facets, \( \mu = 2.15 \) needs already 1610 facets. By analyzing this sequence, it can be assumed, that the system will be on the border of stability if \( \mu \approx 2.2 \).

There is the level set of Lyapunov function in fig. 2 if \( \mu = 2.15 \), \( \epsilon = 0.005 \).

![Figure 2. The level set of Lyapunov function for the 2-dimensional system.](image)
Example 2. The linear part of the transfer function is described by equation

\[ W(p) = \frac{p + 1}{p^3 + 1.5p^2 + 3p + 2}. \]

Particularly, such transfer functions correspond to the systems of positional control by the electric drive. As it was mentioned earlier, in the structure scheme (fig. 1), the nonlinear time-varying element satisfies the following sector bounds:

\[ 0 \leq \varphi(\sigma, t) \sigma \leq \mu \sigma^2. \]

Using circle criteria, we obtain \( \mu \leq 2.28 \), maximal value \( \mu = 3.82 \) [3]. Let us apply formula (3).

\[ A_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -3 & -4 & -1.5 \end{pmatrix} \]

\[ A_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -(3 + \mu) & -(4 + \mu) & -1.5 \end{pmatrix} \]

As in the previous example, let us make a table of experiments.

**Table 2.** Dependence of the facets count on parameter \( \mu \)

| \( \# \) | Facets count | \( \mu \) | \( \epsilon \) |
|-------|-------------|--------|--------|
| 1     | 532         | 1.0    | 0.235  |
| 2     | 1128        | 2.0    | 0.145  |
| 3     | 8762        | 3.0    | 0.055  |

The level set of Lyapunov function is in fig. 3 for case \( \mu = 3.0 \).

**Figure 3.** The level set of Lyapunov function for a 3-dimensional system.

4. **Conclusion**

The suggested algorithm allows significantly improved estimations of the stability regions control’s system compared with classical frequency criteria. Evaluation of stability, based on constructing piecewise linear Lyapunov functions, is quite convenient for solving tasks of nonlinear time-varying system’s synthesis. At the same time, the task of optimization time cost needs further research, especially in the areas closest to the border of stability.

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