Entangled Vacuum State for Accelerated Observers

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Received: 28 October 2021 / Accepted: 17 January 2022 / Published online: 10 March 2022 © The Author(s), under exclusive licence to Springer Science+Business Media, LLC, part of Springer Nature 2022

Abstract
The entanglement of Dirac fields has been shown to decrease with increasing acceleration when a quantum state is shared between an inertial and an uniformly accelerated observer. A new form of an entangled vacuum state observed by the accelerated observer is postulated in which it is assumed that entanglement is present between the modes of the quantum field with sharp and opposite momenta defined in two causally disconnected regions of space-time. We find that this assumption does not affect the entanglement of the system.

Keywords Entangled Vacuum · Accelerated Frames

1 Introduction

Quantum entanglement and other correlations are widely used in quantum information theory [1, 2]. The last decade has witnessed a wide array of research efforts focused especially on quantum entanglement between non-inertial systems in quest of a more profound understanding of relativistic quantum information. The features of the systems considered are used in the study of the information paradox of black holes [3, 4] and entropy production in the expanding universe [5–10]. Recently, the quantum decoherence [11, 12], pair production and apparent decoherence [13], quantum teleportation with vacuum-entangled Rindler modes and beyond horizon [14, 15] and relativistic quantum information of anyons [16] have been investigated in non-inertial frames.

For instance, one such interesting feature is that, for a uniformly accelerated observer, a communication horizon exists that does not allow him to access the whole space-time, thereby leading to both information loss and entanglement degradation [17–22].

The Unruh effect is a well-known interesting result of quantum field theory which describes that an uniformly accelerated observer will observe the vacuum as a thermal bath.
In other words, from the viewpoint of the accelerating observer, the vacuum of the inertial observer will look like a state containing many particles in thermal equilibrium. Also, the Schwinger effect can be thought of as vacuum decay in the presence of an electric field. Although the notion of vacuum decay suggests that something is created out of nothing, physical conservation laws are nevertheless obeyed. In fact, the thermodynamical properties of quantum fields have been investigated in flat and curved spacetime. It has been shown that there exists connection between quantum thermodynamic entropy and the entanglement entropy of an expanding universe.

Studies on the entanglement contained in quantum scalar and spinor fields in the flat spacetime case shows that there are strong differences between the behaviour of fermionic and bosonic modes entanglement. It has been shown that there are intrinsic ambiguities related to the formalism employed in the investigations on fermionic entanglement. Also, the differences between boson and fermion fields are also present in the particle creation due to the expansion of the universe. The difference in the quantum correlations present in the particle creation due to the expansion of spacetime in the case of massive bosonic and fermionic fields has been investigated and compared.

The study of entanglement between two modes of a Dirac field for two relatively accelerated observers has shown that entanglement degrades with increasing relative acceleration of observers. In order to measure the entanglement, two maximally entangled modes, \( k_A \) and \( k_R \), of a free Dirac field are considered in a state depicted by Eq. 1 below:

\[
| \phi_{k_A, k_R} \rangle = \frac{1}{\sqrt{2}} \left[ | 0_{k_A} \rangle_M^+ | 0_{k_R} \rangle_M^+ + | 1_{k_A} \rangle_M^+ | 1_{k_R} \rangle_M^+ \right],
\]

where the modes \( k_A \) and \( k_R \) are detected by Alice, the inertial observer, and Rob, the accelerated observer, respectively. We note that the detection process made by Alice means that the Alice’s detector is sensitive to a certain mode. The sign + (or −) is used to indicate the particle (or anti-particle) states and \( M \) is a notational shorthand for Minkowski states. Since one of the observers, Rob, is accelerated, we might expand his states in the Rindler coordinate and study the entanglements between the modes for different values of acceleration. As already mentioned above, a Rindler observer does not have access to the whole space-time; hence, a degradation is expected in the entanglement. In the infinite acceleration limit, entanglement between bosonic modes tends to zero. However, an entanglement of Dirac spinor modes never vanishes altogether.

Previous study has shown that entanglement is independent of acceleration for the specific state entangled in the helicity part. We generalize the previously introduced postulation of the vacuum state and introduce a new ansatz for the vacuum state observed by Rob. We assume that vacuum is an entangled state with respect to sharp and opposite momenta. This type of vacuum may seem strange at first, but it makes perfect sense for a generalization and can be consistent with the definition of a vacuum. It is also possible to relate this kind of generalization to the problem of thermal correlations of cosmic microwave background (CMB) radiation, which is still not completely well understood problem. Of course, this claim requires extensive and detailed research. It has been shown that one can find a relation between thermodynamic entropy and entanglement entropy of a system. The existence of universal crossover functions connecting the universal parts of the entanglement entropy to the low temperature thermal entropy has been postulated in [31]. High-resolution CMB measurements have opened up the possibility to explore statistical features of the temperature fluctuations down to very small angular scales. Some pseudo-entropy measures which are rotationally invariant measures of entanglement have
been proposed for CMB analysis in [32]. Also, the long-range correlation of CMB temperature fluctuations has been investigated from the holographic entanglement entropy. It has been claimed that using the first law of thermodynamics, one may correlate tiny changes in entanglement entropy with the temperature fluctuations [33]. They argued that the distribution of the entangling region size can be interpreted as the CMB temperature fluctuations and conclude that entanglement might play a role in the quantum aspects of cosmology.

The rest of the paper is organized as follows. In Section 2, we briefly review previous studies of entanglement of Dirac fields and study the Unruh effect. In Section 3, a new form of entangled vacuum state is introduced and the creation and annihilation operators are obtained. Using the density matrix, we evaluate the entanglement entropy and negativity of this system. We will demonstrate that the negativity is decreased by increasing the acceleration. Finally, a summary of our results is provided in Section 4. Purity and relative entropy of coherence are computed in Appendices A and B, respectively.

2 Entanglement of Dirac Fields

Consider a free Minkowski Dirac field, \( \phi \), in \( 1 + 1 \) dimensional flat space-time satisfying the Dirac equation:

\[
i \gamma^\mu \partial_\mu \phi - m \phi = 0,
\]

where, \( \gamma^\mu \) are the Dirac-Pauli matrices and \( m \) is a shorthand for the particle mass. We can expand the field in terms of Minkowski solutions of the Dirac equation:

\[
\Phi = \int dk \left( \hat{a}_k \phi_k^+ + \hat{b}_k^\dagger \phi_k^- \right),
\]

where, \( \phi_k^+ \) and \( \phi_k^- \) are positive and negative energy solutions (fermions and anti-fermions), respectively and \( k \) is the momentum of each mode. The operators \( \hat{a}_k \) and \( \hat{a}_k^\dagger \) are the annihilation and creation operators for the positive solutions of momentum \( k \) while \( \hat{b}_k \) and \( \hat{b}_k^\dagger \) are the annihilation and creation operators for the negative energy solution of Minkowski space. These operators satisfy the usual anti-commutation relations:

\[
\{ \hat{a}_i, \hat{a}_j^\dagger \} = \{ \hat{b}_i, \hat{b}_j^\dagger \} = \delta_{ij}.
\]

The objective of the present work is to study Dirac fields in non-inertial references, for which it is convenient to use Rindler coordinates suited to accelerated reference frames. As can be seen in Fig. 1, two sets of the Rindler coordinate are required for the two different regions, \( I \) and \( II \), in order to cover the whole Minkowski space-time.

\[
\begin{align*}
\text{at} &= e^{ax} \sinh(a \eta), & \text{ax} &= e^{ax} \sinh(a \eta) : I, \\
\text{at} &= -e^{ax} \sinh(a \eta), & \text{ax} &= -e^{ax} \cosh(a \eta) : II,
\end{align*}
\]

where, \( a \) represents proper acceleration. The accelerated observer confined in region \( I \) is called Rob and that in region \( II \) is termed anti-Rob. Since these two regions are causally disconnected, an accelerated observer is restricted to only one of the two Rindler wedges. Therefore, each of the two regions \( I \) and \( II \) has its unique quantization procedure with the corresponding solutions \( \{ \phi_k^{I \pm}, \phi_k^{II \pm} \} \) and \( \{ \phi_k^{I \mp}, \phi_k^{II \mp} \} \). Using these modes, the Dirac field may be expanded in the Rindler coordinate as in Eq. 6 below:

\[
\Phi = \int dk \left( \hat{c}_k \phi_k^{I \pm} + \hat{c}_k^{\dagger} \phi_k^{II \pm} + \hat{d}_k \phi_k^{I \mp} + \hat{d}_k^{\dagger} \phi_k^{II \mp} \right).
\]
where, \((\hat{c}^\sigma_+, \hat{c}^\sigma_-)\) represent the creation and annihilation operators for the fermions in the region \(\sigma \in \{I, II\}\) of the Rindler space-time and could be determined using the Rindler orthogonality relation \(\hat{c}^\sigma_+ = (\phi^\sigma_+, \Phi)\). In fact, \((\phi^\sigma_+, \Phi)\) is a representation of the inner product whose definition can be found in [18]. The creation and annihilation operators for anti-fermion with negative-frequency modes in region \(II\) of the Rindler space-time are \((\hat{d}^{I\dagger}_k, \hat{d}^{II\dagger}_k)\). Considering Eqs. 3 and 6, one can find a relation as in Eq. 7 below between the creation and annihilation operators of both the Minkowski and Rindler space-times in the single-mode approximation using the Bogoliubov transformation [34–37]:

\[
\hat{a}^\dagger_k = \cos(r)\hat{c}^{I\dagger}_k - \sin(r)e^{i\phi}\hat{d}^{II\dagger}_k,
\]

where, \(\phi\) is a phase that could be absorbed in the operators, \(\omega = (m^2 + k^2)^{1/2}\) is the frequency of the modes observed by Alice and Rob \((\omega_A = \omega_R = \omega)\) and \(r\) is related to the acceleration \(a\) by \(\tan(r) = \exp(-\pi c \omega a)\). In fact, one can write a matrix form for the Bogoliubov transformations between the Minkowski and Rindler modes. This form can be found in [22]. Using the Eqs. (8) and (9) of [22], the inverse transformations is obtained easily. Since acceleration goes from zero to infinity, \(r\) takes values in the interval \([0, \pi/4]\).

In a previous study [21], the Minkowski particle vacuum in terms of the Rindler Fock space is postulated for the mode \(k\) to be of the following form:

\[
|0_k\rangle^+ = \sum_{n=0}^{1} A_n \left| n_k\right|^+_I \left| n_{-k}\right|_{II},
\]

where, the coefficients \(A_n\) can be found using the requirements \(\hat{a}_k |0_k\rangle^+ = 0, \langle0_k | 0_k\rangle^+ = 1\) and will be written explicitly in the following [22]. We should note that Minkowski vacuum is a direct product as \(|0\rangle = \prod_{k,k'} |0_k\rangle^+ |0_{k'}\rangle^-.\) For simplicity, we assume Alice’s
detector is only sensitive to one mode \( k_A \) and Rob’s detector only detects the mode \( k_R \). Now we can focus on \( |0_k\rangle^+ \). As we can see the vacuum state in Eq. 8 has been decomposed as a tensor product of particle modes in region I and antiparticle modes in region II.

It should be noted that Eq. 8 is not entangled with respect to \( k \) and \(-k\). It should be emphasized that the state is entangled in the particle number, but not in the particle momentum; mode number \( k \). The coefficients \( A_n \) in Eq. 8 may be obtained using \( \hat{a}_k |0_k\rangle^+ = 0 \) and the normalization of \( |0_k\rangle^+ \). Thus, the vacuum state for a certain momentum mode of Minkowski space in terms of the Rindler state will be as follows

\[
|0_k\rangle^+ = \cos(r) |0_k\rangle^+_I |0_{-k}\rangle^+_I + e^{-i\phi} \sin(r) |1_k\rangle^+_I |1_{-k}\rangle^+_I.
\]

Using the creation operator, the excited Minkowski state may be found in terms of the Rindler state as in Eq. 10 below:

\[
|1_k\rangle^+ = \hat{a}_k^+ |0_k\rangle^+ = |1_k\rangle^+_I |0_{-k}\rangle^+_I.
\]

This yields the expansions of \( |0_k\rangle^+ \) and \( |1_k\rangle^+ \) in the Rindler coordinate as in Eqs. 9 and 10, respectively, so instead of \( |0_{kR}\rangle^+ \) and \( |1_{kR}\rangle^+ \) in Eqs. 1, 9 and 10 may be employed, respectively. It is, now, straightforward to compute the density matrix and study the entanglement of the system.

2.1 Unruh Effect

We study the Unruh effect with this vacuum state, and find the Unruh temperature. The Unruh effect is an amazing result of quantum field theory that predicts that a uniformly accelerated observer surprisingly witnessing a thermal bath with a temperature related to the acceleration as follows [18, 38, 39]:

\[ T = \frac{\hbar a}{2\pi k_B c}, \]

where, \( \hbar \) is the reduced Planck constant, \( k_B \) is the Boltzmann constant and \( c \) is the speed of light. In fact, when the observer accelerates in region I, his detector finds a number of particles that may be evaluated as follows:

\[
n_k^+ = + \left\langle 0_k | \hat{a}_k^+ \hat{c}_I^+ |0_k\right\rangle^+ = \sin^2(r) = \frac{1}{e^{\hbar \omega / k_B T} + 1},
\]

where, \( T \) is the Unruh temperature as defined in Eq. 11. Equation 12 is a thermal Fermi-Dirac (FD) distribution of particles detected by the accelerated observer. In fact, the Minkowski vacuum for an accelerated observer is unstable and decays to particle and antiparticle states. Tracing out over the anti-particle modes in region II, remains a thermal particles in region I. A comprehensive discussion about the Unruh effect in non-inertial frames can be found in [22, 40].

3 New Entangled Vacuum State for Dirac Fields

As mentioned earlier, the entanglement between the modes of a Dirac field observed by Alice and Rob degrades when the relative acceleration increases. This is because Rob lacks access to the whole space-time; more precisely, he is confined to region I and has no access to region II. Now we would like to see what will happen for new entangled state of Dirac
fields. Here, instead of postulating the previous vacuum state (Eq. 8), we might postulate that the vacuum is an entangled state with respect to modes $k$ and $-k$ as captured by Eq. 13 below:

$$|0_k\rangle^+ = \sum_{n=0}^{1} \left[A_n |n_k\rangle_I^+ |n_{-k}\rangle_{II} + B_n |n_{-k}\rangle_I^+ |n_k\rangle_{II}\right].$$  \hspace{1cm} (13)

This state is an entangled state with respect to modes $k$ and $-k$.

In Eq. 13, $A_n$ and $B_n$ are two coefficients which will be obtained later. Here we have more than one single mode $k$, and we use the following form of the creation operator, which is a generalized form of the earlier creation operator in Eq. 7.

$$\hat{a}_k^\dagger = \sum_{k'} \cos(r) \hat{c}_{k',I}^\dagger - e^{i\phi} \sin(r) \hat{d}_{-k,II},$$  \hspace{1cm} (14)

We note that $\sum_k$ indicate that the summation runs on the direction of each mode. It means that $k$ in both side of above equation is a free indices and the summation runs on the sharp and opposite momenta modes. In the above equation, the $k=0$ mode is not considered because the sharp and opposite momenta modes do not exist. This satisfies the usual anti-commutation rule. Using the relations $\hat{a}_k |0_k\rangle^+ = 0$, $\langle 0_k | 0_k\rangle^+ = 1$, and Eq. 14, the coefficients $A_n$ and $B_n$ in Eq. 13 could be found. Therefore,

$$|0_k\rangle^+ = \alpha \left[\cos(r) |0_k\rangle_I^+ |0_{-k}\rangle_{II} + e^{-i\phi} \sin(r) |1_k\rangle_I^+ |1_{-k}\rangle_{II}\right]$$

$$+ \beta \left[\cos(r) |0_{-k}\rangle_I^+ |0_k\rangle_{II} + e^{-i\phi} \sin(r) |1_{-k}\rangle_I^+ |1_k\rangle_{II}\right],$$

where, $\alpha$ and $\beta$ are two coefficients added for normalization and $\alpha^2 + \beta^2 = 1$. When $\alpha = 1$ and $\beta = 0$, the previous case is obtained as in Eq. 9. It should be noted that this postulation is different from that of the beyond single mode analysis [41].

We are now able to show that the use of our new ansatz for the vacuum state recovers the well-known Unruh effect. In fact, when the observer in region $I$ accelerates, his detector detects a number of particles that may be evaluated as follows:

$$n_k^+ = +\langle 0_k | \hat{c}_{k,I}^\dagger \hat{c}_{k,I} + \hat{c}_{-k,I}^\dagger \hat{c}_{-k,I} | 0_k\rangle^+$$

$$= (\alpha^2 + \beta^2) \sin^2(r) = \frac{1}{e^{\hbar\omega/K_BT} + 1},$$  \hspace{1cm} (16)

where, $T$ is temperature as defined in Eq. 11. Again, Eq. 16 is a thermal Fermi-Dirac (FD) distribution of particles detected by the accelerated observer similar to the previous cases and is known as the Unruh effect.

Applying $\hat{a}_k^\dagger$ to Eq. 9 again yields Eq. 10. However, we apply it to Eq. 15 to obtain the expansion of one-particle fermion state in the Minkowski vacuum in terms of the Rindler states as follows:

$$|1_k\rangle^+ = \alpha |1_k\rangle_I^+ |0_{-k}\rangle_{II} + \beta |1_{-k}\rangle_I^+ |0_k\rangle_{II},$$  \hspace{1cm} (17)

which is an entangled state of a fermionic mode in two causally disconnected regions, $I$ and $II$. Now, we can study the entanglement of the system.

### 3.1 Density Matrix for Fermions in Non-Inertial Frames

In order to measure the entanglement of fermions in non-inertial frames in these protocols, we consider a Bell state in the form of Eq. 1 and replace Rob’s state’s, $|0_{kR}\rangle$ and $|1_{kR}\rangle$, by
those that obtained in Eqs. 15 and 17, respectively. In fact, we consider an initially Bell state between two inertial observers namely Alice and Bob. There is no confusing assumption in this case. Then we suppose that the second observer accelerates and we call him Rob. The density matrix of Alice, Rob, and anti-Rob can be defined as \( \rho_{A,I,II} = | \phi_{kA,kR} \rangle \langle \phi_{kA,kR} | \).

As regions I and II are causally disconnected, we need to trace out over the modes in region II in order to find the reduced density matrix connecting Alice and Rob, \( \rho_{A,I} \):

\[
\rho_{A,I} = Tr_{II}(\rho_{A,I,II}) = \frac{\beta^2 \cos(r)}{2} |0_A,0_{-k}\rangle \langle 1_A,1_{-k}| + \frac{\beta^2 \cos(r)}{2} |0_A,1_{-k}\rangle \langle 0_A,0_{-k}| \\
+ \frac{\alpha^2 \cos^2(r)}{2} |0_A,0_k\rangle \langle 0_A,0_k| + \frac{\beta^2 \cos^2(r)}{2} |0_A,0_{-k}\rangle \langle 0_A,0_{-k}| \\
+ \frac{\alpha^2 \sin^2(r)}{2} |0_A,1_k\rangle \langle 0_A,1_k| + \frac{\beta^2 \sin^2(r)}{2} |0_A,1_{-k}\rangle \langle 0_A,1_{-k}| \\
+ \frac{\alpha^2 \cos(r)}{2} |1_A,1_k\rangle \langle 0_A,0_k| + \frac{\alpha^2 \cos(r)}{2} |0_A,0_k\rangle \langle 1_A,1_k| \\
+ \frac{\beta^2}{2} |1_A,1_{-k}\rangle \langle 1_A,1_{-k}| + \frac{\alpha^2}{2} |1_A,1_{k}\rangle \langle 1_A,1_{k}|.
\]  

(18)

It is mentionable that |0_A\rangle and |1_A\rangle represent the vacuum and one-particle state in a specified mode for Alice in Minkowski spacetime in non-inertial frame.

The reduced density matrix can be written in the basis |0_A,0_k\rangle, |0_A,0_{-k}\rangle, |0_A,1_k\rangle, |0_A,1_{-k}\rangle, |1_A,0_k\rangle, |1_A,0_{-k}\rangle, |1_A,1_k\rangle, |1_A,1_{-k}\rangle as follows

\[
\rho_{A,I} = \begin{pmatrix}
\frac{\alpha^2 \cos^2(r)}{2} & 0 & 0 & 0 & 0 & 0 & \frac{\alpha^2 \cos(r)}{2} & 0 \\
0 & \frac{\beta^2 \cos^2(r)}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{\alpha^2 \sin^2(r)}{2} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{\beta^2 \sin^2(r)}{2} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{\alpha^2 \cos(r)}{2} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{\alpha^2 \cos(r)}{2} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{\beta^2}{2} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\beta^2}{2}
\end{pmatrix}.
\]  

(19)

The eigenvalues of \( \rho_{A,I} \) are

\[
\lambda(\rho_{A,I}) = \{0, 0, 0, 0, \frac{(1-\alpha^2)}{2} \sin^2(r), \frac{\alpha^2}{2} \sin^2(r), \frac{\alpha^2}{4}(3+\cos(2r)), \frac{(1-\alpha^2)}{4}(3+\cos(2r))\},
\]  

(20)

where we used \( \beta^2 = 1 - \alpha^2 \). Thus, the entanglement between the mode II and (modes A and I) is depicted in Fig. 2 since \( S(\rho(II)) = S(\rho(A,I)) = -\sum_{i=1}^{8} \lambda_i log_2 \lambda_i \).

As we can see in Figs. 2 and 3, the entanglement entropy will have its maximum value when \( \alpha = \beta = \frac{1}{\sqrt{2}} \). It can be argued that, when \( \alpha = \beta = \frac{1}{\sqrt{2}} \), both |n_k\rangle_I |n_{-k}\rangle_{II} and |n_{-k}\rangle_I |n_k\rangle_{II} modes exist so the information from each mode will be less than the case in which \( \alpha = 1 \) (or \( \alpha = 0 \)) where we only have the mode |n_k\rangle_I |n_{-k}\rangle_{II} (or |n_{-k}\rangle_I |n_k\rangle_{II}). Thus, as expected, the entanglement entropy for the maximally entangled case \( \alpha = \beta = \frac{1}{\sqrt{2}} \) is maximum. As we can see in Fig. 3, the entanglement entropy increases with rising acceleration. The reason is that entanglement entropy is not a suitable measure to use in mixed states [42]. Of course, it has been shown that the entanglement entropy and the thermodynamic entropy are connected to each other [8, 25]. Therefore, calculation of
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Fig. 2 Entanglement entropy as a function of $\alpha$ and $r$

Entanglement entropy is not worthless, although it is not a good measure of entanglement. We therefore, need to use other measures to explore entanglement of the system.

3.2 Negativity

There is a necessary condition stating that if there exists at least one negative eigenvalue of the partial transpose of a bipartite density matrix, the density matrix is entangled [43]. So, negative eigenvalues of the partial transpose of a density matrix can be a good indicator of how a system is entangled. Based on this criterion, we may introduce the negativity (Eq. 21 below) as an entanglement measure distinguishing between the different bi-partitions of a system [44]

$$N_{AB} = \sum_i \frac{|\lambda_i| - \lambda_i}{2} = -\sum_{\lambda_i<0} \lambda_i,$$

where, $\lambda_i$s are the eigenvalues of a partial transpose of the bi-partite density matrix $\rho_{AB}$.

In order to find the partial transpose of the density matrix of our system $\rho_{A,I}$ (Eq. 19), we need to interchange Alice’s qubits ($|a_A, b_I\rangle\langle c_A, d_I| \longrightarrow |c_A, b_I\rangle\langle a_A, d_I|$). The negative

Fig. 3 (Color online) Bipartite pure state entanglement entropy as a function of $r$ for some values of $\alpha$. Dashed curve (Green) for $\alpha = 1$, $\beta = 0$. Thick solid curve (Blue) for $\alpha = \beta = \frac{1}{\sqrt{2}}$. Dotted curve (Red) for $\alpha = 0$, $\beta = 1$
The eigenvalues of the partial transpose of $\rho_{A,I}$ are $\lambda_i = \left\{ -\frac{\beta^2(1+\cos(2r))}{4}, -\frac{\alpha^2(1+\cos(2r))}{4} \right\}$. Therefore, the negativity may be evaluated along the following lines:

$$N(\rho_{A,I}) = \frac{1 + (\alpha^2 + \beta^2) \cos(2r)}{4} = \frac{\cos^2(r)}{2}.$$  \hspace{1cm} (22)

The negativity does not depend on $\alpha$ and is a decreasing function of $r$. The negativity is plotted for different values of $r$ in Fig. 4. Similar to Refs. [21, 22], the negativity is a monotonically decreasing function with respect to the acceleration. However, the negativity of bosonic modes vanishes asymptotically at infinite acceleration limit [21] while for the fermionic modes, it tends to a non zero minimum value [22].

\section*{4 Conclusion}

Considering a maximally entangled Bell state with two modes detected by Alice and Rob, we proposed a new form of the vacuum state for Rob which has an entanglement between the momentum modes leading to an excited state as in Eq. 17 and we studied the entanglement of this system. We realize that the entanglement as a function of acceleration of non-inertial observer does not depend on the amount of initial entanglement of the vacuum state but it decreases with increasing the acceleration.

In this paper, we just studied the entanglement between the modes $A$ and $I$. Other cases like entanglement between the modes $A$ and $II$ while those of $I$ and $II$ could also be considered. The results for entanglement between regions $I$ and $II$ are similar to previous works [22], and so we did not go into the details of these calculations. A future extension would be to quantify other correlation quantifiers such as quantum discord and mutual information.

\section*{Appendix A: Purity}

It was shown that the initial entanglement between the $k$ and $-k$ modes does not affect the negativity as a function of acceleration. We explored the potential effects of this proposal on other correlations of the system. For this purpose, we computed the two other measures.
Purity is a measure representing the degree of mixedness of states [1]. The concept is defined as follows:

\[ P = Tr[\rho^2]. \]  

(23)

As expected, purity should be equal to unity for pure states and minimum for maximally mixed states (Fig. 5). Compared to other values of \( \alpha \) in the case of \( \alpha = 1/\sqrt{2} \), we have a more stable value for purity when acceleration changes. Suppose two cases \( \alpha = 0 \) and \( \alpha = 1 \). For these states, the purity is decreased by increasing the acceleration. For a special case \( \alpha = 1/\sqrt{2} \), the purity is still a decreasing function with respect to the acceleration. However, the rate of decreasing is less than the other cases and the purity is almost constant.

### Appendix B: Relative Entropy of Coherence

Coherence is an important subject in optics and quantum mechanics and plays a central role in interference. In fact, coherence describes all the correlation features between two waves [45]. In quantum mechanics, we can find the coherence of a state based on its density matrix. One such quantifier is relative entropy of coherence [46]:

\[ C_r(\rho) := S(\rho_{diag}) - S(\rho), \]  

(24)

where, \( \rho_{diag} = \sum \rho_{ii} |i\rangle\langle i| \) which is obtained by removing off-diagonal elements of \( \rho \). The state is incoherent when \( C_r(\rho) = 0 \). Using Eqs. 19 and 24, the relative entropy of coherence is obtained and plotted in Fig. 6.

It is seen in the Fig. 6 that the relative entropy of coherence is nonzero, which means that this system is always coherent. For the initial state which corresponds to the case of \( a = 0 \), the coherence is maximum. As we can see the coherence has the same value for all values of \( \alpha \) and it decreases with increasing the acceleration. In fact, the coherence is independent of the value of \( \alpha \).
Acknowledgements  We would like to thank Mrs Zeynab Harsij for her comments.

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