Velocity distribution functions and intermittency in one-dimensional randomly forced Burgers turbulence

Victor Dotsenko

Sorbonne Université, LPTMC, F-75005 Paris, France
L.D. Landau Institute for Theoretical Physics, 119334 Moscow, Russia
E-mail: dotsenko@lptmc.jussieu.fr

Received 3 May 2018
Accepted for publication 30 July 2018
Published 22 August 2018

Online at stacks.iop.org/JSTAT/2018/083302
https://doi.org/10.1088/1742-5468/aad6c8

Abstract. The problem of one-dimensional randomly forced Burgers turbulence is considered in terms of \((1 + 1)\) directed polymers. In the limit of strong turbulence (which corresponds to the zero temperature limit for a directed polymer system), a general explicit expression for the joint distribution function of two velocities separated by a finite distance is derived using the replica technique. In particular, it is shown that at length scales much smaller than the injection length of the Burgers random force, the moments of the velocity increment exhibit typical strong intermittency behavior.

Keywords: cavity and replica method, glasses (colloidal, polymer, etc), interfaces in random media
1. Introduction

In the problem of one-dimensional randomly forced Burgers turbulence, one studies the statistical properties of a velocity field \( v(x, t) \) governed by the Burgers equation [1]

\[
\partial_t v(x, t) + v(x, t) \partial_x v(x, t) = \nu \partial^2_x v(x, t) + f(x, t)
\]

(1)

where the parameter \( \nu \) is the viscosity and \( f(x, t) \) is the Gaussian distributed random force, \( \delta \)-correlated in time, which is characterized by finite correlation length \( R \) in space: \( f(x, t) f(x', t') = u \delta(t - t')  \mathcal{F}[(x - x')/R] \). Here, \( \mathcal{F}(x) \) is a smooth function decaying to zero fast enough at large arguments and the parameter \( u \) is the injected energy density. This problem has been the subject of active investigations for more that six decades (see e.g. [2–4] and references therein).

In the framework of the celebrated Kolmogorov theory [5], one obtains the probability density function (PDF) of the velocity increment \( w = v(x_0 + x, t) - v(x_0, t) \), such that at distances much smaller than the length scale \( R \) of the random stirring force \( f \), one finds simple scaling for the moments \( \langle w^q \rangle \sim x^{\zeta(q)} \) with \( \zeta(q) = n/3 \) (in particular, one can prove that \( \zeta(3) = 1 \)). This prediction is based on the assumption that the statistical properties of the velocity field are locally homogeneous, so that the corresponding PDF of \( w \) depends only on \( r \) and the average rate of the energy dissipation. However, extensive studies during recent decades have convincingly demonstrated that in fact the exponent \( \zeta(q) \) deviates significantly from the \( q/3 \) of Kolmogorov’s law. The physical reason for that is the so called intermittency phenomenon—that is, formation of local coherent structures that drives a strong deviation from the mean fluctuation level of the velocity field [6–14].
In the present paper, using formal equivalence of the above Burgers’ problem, equation (1), with the model of one-dimensional directed polymers in a random potential [3] (see below), we are going to derive an explicit expression for the joint PDF \(P(v, v')\) for two velocities separated by a distance \(x\), as well as the corresponding PDF for the velocity increment \(w = v' - v\). In particular, at distances much smaller than the scale of the stirring force, \(x \ll R\), this allows us to demonstrate the typical intermittency behavior of the exponent \(\zeta(q)\) (see figure 1).

It is well known that the Burgers problem, equation (1), is formally equivalent to that of growing interfaces in a random environment described by the Kardar–Parisi–Zhang (KPZ) equation \([15, 16]\). Indeed, redefining \(v(x, t) = -\partial_x F(x, t)\) and \(f(x, t) = -\partial_x V(x, t)\), and integrating equation (1) once, one gets the KPZ equation for the interface profile \(F(x, t)\),

\[
\partial_t F(x, t) = \frac{1}{2} (\partial_x F(x, t))^2 + \nu \partial_x^2 F(x, t) + V(x, t),
\]

where \(V(x, t)\) is a random potential. On the other hand, let us consider a system of one-dimensional directed polymers which is defined in terms of the Hamiltonian

\[
H[\phi(\tau), V] = \int_0^t d\tau \left\{ \frac{1}{2} [\partial_\tau \phi(\tau)]^2 + V[\phi(\tau), \tau] \right\},
\]

where \(\phi(\tau)\) is a scalar field defined within an interval \(0 \leq \tau \leq t\) and \(V(\phi, \tau)\) is the Gaussian distributed random potential with a zero mean, \(V(\phi, \tau) = 0\), and the correlation function

\[
V(\phi, \tau)V(\phi', \tau') = u \delta(\tau - \tau') U(\phi - \phi').
\]

Here, the parameter \(u\) defines the strength of the disorder and \(U(\phi)\) is the spatial correlation function characterized by the correlation length \(R\). For simplicity, we take

\[
U(\phi) = \frac{1}{\sqrt{2\pi R}} \exp\left\{ -\frac{\phi^2}{2R^2} \right\}.
\]

For a given realization of the random potential \(V[\phi, \tau]\), the partition function of this system is defined as
Velocity distribution functions and intermittency in one-dimensional randomly forced Burgers turbulence

\[ Z(x, t) = \int_{\phi(0)=0}^{\phi(t)=x} \mathcal{D}\phi(\tau) \exp \{-\beta H[\phi(\tau), V]\} = \exp\{-\beta F(x, t)\}, \]  

(7)

where \( \beta \) is the inverse temperature, \( F(x, t) \) is the free energy and the integration is taken over all trajectories \( \phi(\tau) \) with the boundary conditions at \( \phi(\tau = 0) = 0 \) and \( \phi(\tau = t) = x \). One can easily show that the partition function \( Z(x, t) \) defined above satisfies the linear differential equation

\[ \partial_t Z(x, t) = \frac{1}{2\beta} \partial_x^2 Z(x, t) - \beta V(x, t) Z(x, t). \]  

(8)

Substituting \( Z(x, t) = \exp\{-\beta F(x, t)\} \) here, one easily finds that the free energy function \( F(x, t) \) satisfies the KPZ equation (3) with the viscosity parameter \( \nu = \frac{1}{2\beta} \). In other words, the original random force Burgers problem, equation (1), is formally equivalent to the directed polymer system, equations (4)–(7), such that the viscosity parameter \( \nu \) in the Burgers equation is proportional to the temperature in the directed polymer system, \( \nu = \frac{1}{2}T \), and the velocity \( v(x, t) \) in the Burgers equation is the negative spatial derivative of the free energy \( F(x, t) \) of the directed polymer system.

The standard dimensionless parameter which characterizes the level of turbulence of the velocity field in the Burgers problem is called the Reynolds number \( \text{Re} \), and it is defined as the ratio of typical values of the inertial forces to viscous forces. In the present notations it can be defined as \( \text{Re} = \frac{v_0 R}{\nu} \), where \( v_0 \) is the typical flow velocity at the characteristic linear dimension which in the present case is the injection scale of the random force \( R \). Using dimensional arguments, one easily finds that

\[ v_0 \sim \left( \frac{u}{R^2} \right)^{1/3}. \]  

(9)

Indeed, according to equation (2), the dimension of the velocity \( [v_0] = [F]/R \). On the other hand according to equation (4), the dimension of the free energy is \( [F] = [H] = t[V] \). Finally, according to equations (5) and (6), the dimension of the random potential is \( [V] = \sqrt{u/(Rt)} \). Combining all that together, one finds equation (9). Therefore, in terms of the directed polymers notations, the Reynolds number of the Burgers turbulence problem reads

\[ \text{Re} = \frac{v_0 R}{\nu} = 2\beta \left( \frac{u R}{\nu} \right)^{1/3}. \]  

(10)

It is evident that an increasing Reynolds number indicates an increasing turbulence of flow, and that the limit of strongly developed turbulence corresponds to \( \text{Re} \to \infty \). Thus, the strong turbulence Burgers regime corresponds to the zero-temperature limit in the directed polymers system, and it is this limit which will be studied in the present paper.

As the velocity in the Burgers problem is given by the spatial derivative of the free energy of the directed polymer system, it can be expressed in terms of the difference of two free energies:

\[ v(x \cdot t) = -\frac{\partial F(x, t)}{\partial x} = -\lim_{\epsilon \to 0} \frac{F(x + \epsilon, t) - F(x, t)}{\epsilon}. \]  

(11)
In other words, one-point velocity statistics is defined by the joint statistics of two free energies. Correspondingly, if we are going to study the joint statistical properties of two spatially separated velocities, then—in terms of the free energies of the directed polymers—we have to study the four-point spatial object.

In section 2 we describe the general ideas and the main lines of the replica approach which will be used in the further derivations of the probability distribution functions. In section 3, we describe the main points of the zero-temperature limit approach for the directed polymers with finite correlation length of the random potential, equations (5) and (6) (for details, see [17]). The zero temperature limit of the joint probability distribution function of free energies defined at four spatial points is derived in section 4. The explicit expression for the corresponding joint probability density function of two velocities $v$ and $v'$ separated by a distance $x$ is derived in section 5, equations (111)–(113). In section 6, it will be shown that the PDF for the velocity increment $w = v - v'$ has the following form:

$$
P_x(w) = p_0(x/R)\delta\left(w - v_0 \frac{x}{R}\right) + \mathcal{P}_{x/R}(w/v_0) \theta\left(v_0 \frac{x}{R} - w\right)$$  (12)

where $\theta(z)$ is the Heaviside step function, $v_0 \propto (u/R^2)^{1/3}$ (see equation (9)),

$$p_0(x/R) = \int_{-\infty}^{+\infty} \frac{ds}{\sqrt{2\pi}} \frac{\exp\left\{-\frac{1}{2}s^2\right\}}{1 + \frac{s^{3/4}x}{R} \int_0^{+\infty} \frac{d\xi}{\sqrt{2\pi}} \xi \exp\left\{-\frac{1}{2}(s + \xi)^2\right\}}$$  (13)

and

$$\mathcal{P}_{x/R}(w/v_0) = \frac{\zeta_0^{3/4} x}{v_0 R} \int_{-\infty}^{+\infty} \frac{ds}{\sqrt{2\pi}} \int_{0}^{\Delta(w/v_0)} \frac{d\eta}{\sqrt{2\pi}} \frac{\exp\left\{-\frac{1}{2}s^2 - \frac{1}{2}(s - \Delta(w/v_0))^2\right\}}{1 + \frac{s^{3/4}x}{R} \int_0^{+\infty} \frac{d\xi}{\sqrt{2\pi}} \xi \exp\left\{-\frac{1}{2}(\xi + s - \Delta(w/v_0))^2\right\}}$$  (14)

Here, $\zeta_0 \sim 1$ is a number (see section 3) and

$$\Delta(w/v_0) = \zeta_0^{3/4}\left(\frac{x}{R} - \frac{w}{v_0}\right).$$  (15)

The above formulas, equations (12)–(15), constitute the central result of the present research. The distribution function $P_x(w)$ has rather specific structure (see section 6, figure 4). According to equation (12), for a given distance $x$, the values of the velocity increment $w$ are bounded from above: $w \leq \frac{x}{R} v_0$, where $v_0 \propto (u/R^2)^{1/3}$ is the typical flow velocity at the injection scale $R$ of the random force of the strength $u$. Moreover, at $w = \frac{x}{R} v_0$, the distribution function exhibits the $\delta$-function singularity, which means that at a given distance $x$, the difference of two velocities $w = v - v'$ has a finite probability $p_0$, equation (13), to be equal to $\frac{x}{R} v_0$.

The above result allows us to study the behavior of the moments of the velocity increment $\langle w^r \rangle$ at distances $x \ll R$. Introducing the reduced distance parameter $r = \zeta_0^{3/4} x/R$ and the reduced velocity increment $\omega = \zeta_0^{3/4} w/v_0$, in the limit $r \ll 1$ instead of equations (12)–(14), we get

https://doi.org/10.1088/1742-5468/aad6c8
\[ P_r(\omega) \simeq \left(1 - \frac{r}{\sqrt{\pi}}\right)\delta(\omega - r) + \frac{r}{\sqrt{\pi}}(r - \omega) \exp\left\{ -\frac{1}{4}(r - \omega)^2 \right\} \theta(r - \omega). \quad (16) \]

Then, for even moments of the reduced velocity increment we obtain

\[ \langle \omega^{2n} \rangle \simeq r^{2n} + C(n) r, \quad (17) \]

where \( C(n) = 2^{2n+1} \Gamma(1+n) \). The above result can be analytically continued for arbitrary real values \( q \) of the parameter \( 2n \to q \). Then, introducing the exponent \( \zeta(q) \) as \( \langle \omega^q \rangle \simeq r^{\zeta(q)} \), according to equation (17) in the limit \( r \ll 1 \), we recover the typical strong intermittency behavior (see figure 1):

\[ \zeta(q) \simeq \begin{cases} q, & \text{for } q \leq 1; \\ 1, & \text{for } q > 1. \end{cases} \quad (18) \]

It should be stressed that both the structure of the PDF \( P_\omega(w) \), equation (12), and the result for the exponent \( \zeta(q) \), equation (18), are in remarkable agreement with those obtained for the same system many years ago in [3] in the framework of the Gaussian variation method (which, formally, should be valid only at high dimensions). Presumably, it is the one-step RSB (which in the present paper takes place in the replica vector argument of the wave functions (see section 3), while in [3] it appears in the replica matrix of the Gaussian partition function) which is at the core of the same physical phenomena revealed by these two essentially different methods.

2. Replica formalism

In this section, we are going to describe the general scheme of calculations of the statistical properties of the Burgers velocity field \( v(x,t) \), equation (1), in terms of the standard replica approach used for the directed polymers model, equations (4)–(7). Using the relation between \( v(x,t) \) and the free energy \( F(x,t) \) of the corresponding directed polymer model, equation (11), for finite values of the parameter \( \epsilon \) (which should be taken to zero in the final result) we have

\[ \exp\{\beta \epsilon v(x,t)\} = \exp\{-\beta F(x + \epsilon, t) + \beta F(x, t)\} = Z(x + \epsilon, t) \cdot Z^{-1}(x, t). \quad (19) \]

Taking the integer power \( N \) of both sides of the above relation and averaging over the disorder (which in what follows will be denoted by the overline, \( \langle \ldots \rangle \)) we get

\[ \int dv \left\langle P_{\omega,\epsilon}(v) \exp\{\beta N \epsilon v\} \right\rangle = \frac{Z^N(x + \epsilon, t) \cdot Z^{-N}(x, t)}{Z^N(x, t)}; \quad (20) \]

where \( P_{\omega,\epsilon}(v) \) is the PDF of the velocity \( v \). Formally, the above relation can be represented as follows:

\[ \int dv \left\langle P_{\omega,\epsilon}(v) \exp\{\beta N \epsilon v\} \right\rangle = \lim_{M \to 0} Z(M, N, x, \epsilon, t), \quad (21) \]

where
The two-point replica partition function \( Z(M, N, x, \epsilon, t) \) is defined by the relation (21) and can be written as:

\[
Z(M, N, x, \epsilon, t) = Z_N(x + \epsilon, t) \cdot Z_{M-N}(x, t)
\]  

This is the two-point replica partition function.

The general scheme of calculations of the velocity PDF defined by the relation (21) consists of several steps. First, for a given (finite) \( \epsilon \) and integers \( M \) and \( N \), such that \( M > N \), one has to compute the replica partition function \( Z(M, N, x, \epsilon, t) \) as an analytic function of the parameters \( M \) and \( N \). Next, this function should be analytically continued for arbitrary complex values of \( M \) and \( N \), and the limits \( M \to 0 \) and \( N \to 0 \) have to be taken. Next, to take the limit \( \epsilon \to 0 \), one introduces the parameter \( s = \beta \epsilon N \) which has to be kept finite (this implies that, together with the limit \( \epsilon \to 0 \), one simultaneously takes the limit \( N \to \infty \)). Thus, after performing these manipulations (provided all the above limits exist), the relation (21) turns into the bilateral Laplace transform for the velocity PDF \( P_s(v) = \lim_{\epsilon \to 0} \lim_{t \to \infty} P_{x, \epsilon, t}(v) \) (which for finite values of \( x \) in the limit \( t \to \infty \) should be \( x \)-independent):

\[
\int dv P_s(v) \exp\{sv\} = Z_*(s),
\]

where

\[
Z_*(s) = \lim_{\epsilon \to 0} \lim_{t \to \infty} \lim_{M \to 0} Z\left(M, \frac{s}{\beta \epsilon}, x, \epsilon, t\right).
\]

In this way, the PDF \( P_s(v) \) could be recovered by the inverse Laplace transform.

Note that the above type of program has already been successfully implemented for the derivation of the Burgers two-point velocity PDF in the toy (Gaussian) Larking model of random directed polymers [18].

In this paper, we are going to derive the joint PDF of two velocities \( v = v(x/2, t) \) and \( v' = v(-x/2, t) \) at two points separated by a finite distance \( x \). In this case, straightforward generalization of the above replica scheme would require computation of the four-point replica partition function:

\[
\int dv dv' P_{x, \epsilon, t}(v, v') \exp\{\beta N_1 \epsilon v + \beta N_2 \epsilon v'\}
= \lim_{M \to 0} Z_{N_1}(x/2, t) Z_{M-N_1}(x/2 - \epsilon, t) Z_{N_2}(-x/2 + \epsilon, t) Z_{M-N_2}(-x/2, t).
\]

Technically, direct recovery (using the inverse Laplace transformation) of the two-velocity PDF using the above relation turns out to be a rather involved task which still remains to be done. On the other hand, the experience shows that, sometimes, to compute a complicated quantity, first one just has to compute a more general object. In the present case, instead of two velocities PDF let us consider the joint distribution function of three free energy differences. Specifically, for a given four spatial points, \(-x/2, -x/2 + \epsilon, x/2 - \epsilon \) and \( x/2 \), let us define

\[
\begin{align*}
f_1 &= F(x/2, t) - F(-x/2, t) \\
f_2 &= F(x/2 - \epsilon, t) - F(-x/2, t) \\
f_3 &= F(-x/2 + \epsilon, t) - F(-x/2, t).
\end{align*}
\]
In terms of the partition functions, the above relations can be represented as follows:

\[ Z(x/2, t) Z^{-1}(-x/2, t) = \exp\{-\beta f_1\} \]
\[ Z(x/2 - \epsilon, t) Z^{-1}(-x/2, t) = \exp\{-\beta f_2\} \]
\[ Z(-x/2 + \epsilon, t) Z^{-1}(-x/2, t) = \exp\{-\beta f_3\}. \] (27)

As the further considerations will be done in the zero temperature limit (which correspond to the limit of large Reynolds number, equation (10)), it turns out that the simplest way to derive the joint PDF \( P_{x,\epsilon, t}(f_1, f_2, f_3) \) is to use the generating function approach. Specifically, let us introduce the probability function

\[ W_{x,\epsilon, t}(f_1, f_2, f_3) = \int_{-\infty}^{f_1} df_1' \int_{-\infty}^{f_2} df_2' \int_{-\infty}^{f_3} df_3' P_{x,\epsilon, t}(f_1', f_2', f_3'). \] (28)

One can easily see that in the zero temperature limit, this function can be represented in the form of the series

\[ W_{x,\epsilon, t}(f_1, f_2, f_3) = -\lim_{\beta \to \infty} \sum_{N_1=1}^{\infty} \frac{(-1)^{N_1}}{N_1!} \sum_{N_2=1}^{\infty} \frac{(-1)^{N_2}}{N_2!} \sum_{N_3=1}^{\infty} \frac{(-1)^{N_3}}{N_3!} \exp\{\beta N_1 f_1 + \beta N_2 f_2 + \beta N_3 f_3\} \]
\[ \times \left[ Z(x/2, t) Z^{-1}(-x/2, t) \right]^{N_1} \left[ Z(x/2 - \epsilon, t) Z^{-1}(-x/2, t) \right]^{N_2} \left[ Z(-x/2 + \epsilon, t) Z^{-1}(-x/2, t) \right]^{N_3}. \] (29)

Indeed, substituting equation (27) here, we get

\[
W_{x,\epsilon, t}(f_1, f_2, f_3) = -\lim_{\beta \to \infty} \int_{-\infty}^{\infty} df_1' \int_{-\infty}^{\infty} df_2' \int_{-\infty}^{\infty} df_3' P_{x,\epsilon, t}(f_1', f_2', f_3') \left[ \sum_{N_1=1}^{\infty} \frac{(-1)^{N_1}}{N_1!} \exp\{\beta (f_1' - f_1) N_1\} \right] \\
\times \left[ \sum_{N_2=1}^{\infty} \frac{(-1)^{N_2}}{N_2!} \exp\{\beta (f_2' - f_2) N_2\} \right] \left[ \sum_{N_3=1}^{\infty} \frac{(-1)^{N_3}}{N_3!} \exp\{\beta (f_3' - f_3) N_3\} \right] \\
= -\lim_{\beta \to \infty} \int_{-\infty}^{\infty} df_1' \int_{-\infty}^{\infty} df_2' \int_{-\infty}^{\infty} df_3' P_{x,\epsilon, t}(f_1', f_2', f_3') \left\{ \exp\{-\exp[\beta (f_1' - f_1)]\} \right\} - 1 \\
\times \left[ \exp\{-\exp[\beta (f_2' - f_2)]\} \right] - 1 \left[ \exp\{-\exp[\beta (f_3' - f_3)]\} \right] - 1 \\
= \int_{-\infty}^{\infty} df_1' \int_{-\infty}^{\infty} df_2' \int_{-\infty}^{\infty} df_3' P_{x,\epsilon, t}(f_1', f_2', f_3') \theta(f_1 - f_1') \theta(f_2 - f_2') \theta(f_3 - f_3') \] (30)

which coincides with the definition (28).

Thus, according to equation (29), in terms of the replica technique the probability function, equation (28), can be represented as:

\[ W_{x,\epsilon, t}(f_1, f_2, f_3) = -\lim_{\beta \to \infty} \lim_{M \to 0} \sum_{N_1, N_2, N_3=1}^{\infty} \frac{(-1)^{N_1+N_2+N_3}}{N_1! N_2! N_3!} \\
\exp\{\beta N_1 f_1 + \beta N_2 f_2 + \beta N_3 f_3\} Z_{x,\epsilon, t}(M, N_1, N_2, N_3) \] (31)
where
\[
Z_{x,\epsilon}(M, N_1, N_2, N_3) = Z^{N_1}(x/2, t) Z^{N_2}(x/2 - \epsilon, t) Z^{N_3}(-x/2 + \epsilon, t) Z^{M-N_1-N_2-N_3}(-x/2, t).
\]

(32)

The further program of calculations is as follows. The above four-point replica partition function has to be calculated for an integer \(M > N_1 + N_2 + N_3\) as an analytic function of the parameter \(M\). Then, this function has to be analytically continued for arbitrary real values of \(M\) and the limit \(M \to 0\) has to be taken. Finally, after computing the series in equation (31) (in the limits \(t \to \infty\) and \(\beta \to \infty\)) according to the definition (28) the corresponding PDF \(P_{x,\epsilon}(f_1, f_2, f_3)\) can be obtained as
\[
P_{x,\epsilon}(f_1, f_2, f_3) = \frac{\partial^3}{\partial f_1 \partial f_2 \partial f_3} W_{x,\epsilon}(f_1, f_2, f_3),
\]

(33)

where
\[
W_{x,\epsilon}(f_1, f_2, f_3) \equiv \lim_{\beta \to \infty} \lim_{t \to \infty} W_{x,\epsilon,t}(f_1, f_2, f_3).
\]

(34)

According to the representation (11) and the definitions (26), the velocities \(v \equiv v(x/2, t)\) and \(v' \equiv v(-x/2, t)\) are defined as
\[
v = -\lim_{\epsilon \to 0} \frac{f_1 - f_2}{\epsilon}
\]

(35)

\[
v' = -\lim_{\epsilon \to 0} \frac{f_3}{\epsilon}
\]

(36)

Thus, the corresponding joint PDF of these two velocities, \(P_x(v, v')\), can be obtained as
\[
P_x(v, v') = \lim_{\epsilon \to 0} \left[ e^2 \int_{-\infty}^{+\infty} df_2 P_{x,\epsilon}(f_2 - \epsilon v, f_2, -\epsilon v') \right].
\]

(37)

The above general program of computations will be implemented in the further sections.

3. Zero temperature limit

To compute the replica partition function, equation (32), let us consider a more general object:
\[
\Psi(x_1, x_2, \ldots, x_M; t) = \left( \prod_{a=1}^{M} Z(x_a, t) \right).
\]

(38)

Substituting here equations (7) and (4), and performing simple Gaussian averaging (using equation (5)) we get
\[
\Psi(x_1, x_2, \ldots, x_M; t) = \prod_{a=1}^{M} \left[ \int_{\phi_a(0)=0}^{\phi_a(t)=x_a} D\phi_a(\tau) \right] \exp \left\{ -\beta H_M[\phi_1(\tau), \ldots, \phi_M(\tau)] \right\}
\]

(39)

where

https://doi.org/10.1088/1742-5468/aad6c8
\[ \beta H_M[\phi_1(\tau), \ldots, \phi_M(\tau)] = \int_0^t d\tau \left[ \frac{1}{2} \beta \sum_{a=1}^M (\partial_\tau \phi_a(\tau))^2 - \frac{1}{2} \beta^2 u \sum_{a,b=1}^M U(\phi_a(\tau) - \phi_b(\tau)) \right]; \]

is the replica Hamiltonian with the attractive interaction potential \( U(\phi) \) given in equation (6). One can easily show that the function \( \Psi(x_1, x_2, \ldots, x_M; t) \) is the wave function of one-dimensional quantum bosons which satisfy the imaginary time Schrödinger equation

\[ \beta \frac{\partial}{\partial t} \Psi(x; t) = \frac{1}{2} \sum_{a=1}^M \frac{\partial^2}{\partial x_a^2} \Psi(x; t) + \frac{1}{2} \beta^2 u \sum_{a,b=1}^M U(x_a - x_b) \Psi(x; t) \]

with the initial conditions \( \Psi(x; 0) = \prod_{a=1}^M \delta(x_a) \) (here we have introduced the vector notation \( x \equiv \{x_1, x_2, \ldots, x_M\} \)).

The high temperature limit of the replica problem formulated above is well studied (for a review see e.g. [19] and references therein). It can be shown that in the limit \( \beta \to 0 \), the interaction potential \( U(x) \), equation (6), can be approximated by the \( \delta \)-function, and in this case the generic solution of the Schrödinger equation can be represented in terms of the Bethe ansatz eigenfunctions [20–22]. However, at low temperatures, \( T \lesssim \left( uR \sqrt{2} \pi \right)^{1/3} \), the typical distance between particles (defined by the wave function \( \Psi(x) \)) becomes comparable with the size \( R \) of the interaction potential \( U(x) \), and its approximation by the \( \delta \)-function is no longer valid. The zero temperature limit of the considered system has been studied in [17, 23].

In the limit of low temperatures, it is convenient to redefine the parameters of the system in the following way:

\[ \begin{align*} 
\phi &= R \tilde{\phi} \\
\beta &= T_*^{-1} \tilde{\beta} \\
\tau &= \tau_* \tilde{\tau}
\end{align*} \]

where

\[ T_* = \left( \frac{uR}{\sqrt{2} \pi} \right)^{1/3} \]

\[ \tau_* = \left( \sqrt{2} \pi R^5 u^{-1} \right)^{1/3}. \]

In the new notations, the replica Hamiltonian (40) reads

\[ \beta H_M[\tilde{\phi}] = \int_0^{t/\tau_*} d\tilde{\tau} \left[ \frac{1}{2} \beta \sum_{a=1}^M (\partial_{\tilde{\tau}} \tilde{\phi}_a(\tilde{\tau}))^2 - \frac{1}{2} \tilde{\beta}^2 u \sum_{a,b=1}^M U_0(\tilde{\phi}_a(\tilde{\tau}) - \tilde{\phi}_b(\tilde{\tau})) \right]; \]

where

\[ U_0(\phi) = \exp\left\{ -\frac{1}{2} \phi^2 \right\}. \]

Accordingly, instead of equation (41), we get

https://doi.org/10.1088/1742-5468/aad6c8
\[
\beta \frac{\partial}{\partial t} \Psi(\tilde{x}; t) = \frac{1}{2} \sum_{a=1}^{M} \frac{\partial^2}{\partial \tilde{x}_a^2} \Psi(\tilde{x}; t) + \frac{1}{2} \beta^3 \sum_{a,b=1}^{M} U_0(\tilde{x}_a - \tilde{x}_b) \Psi(\tilde{x}; t),
\]

where \( \tilde{t} = t/\tau_\ast \) and \( \tilde{x} = x/R \). Substituting \( \Psi(\tilde{x}; t) = \psi(\tilde{x}) \exp\{-E\tilde{t}\} \) here, we obtain the following equation for the eigenfunctions \( \psi(\tilde{x}) \) and the eigenvalues (energy) \( E \):

\[
-2\beta E \psi(\tilde{x}) = \sum_{a=1}^{M} \frac{\partial^2}{\partial \tilde{x}_a^2} \psi(\tilde{x}) + \beta^3 \sum_{a,b=1}^{M} U_0(\tilde{x}_a - \tilde{x}_b) \psi(\tilde{x})
\]

which is controlled by the only parameter

\[
\beta = \beta T_\ast = \beta (uR)^{1/3} (2\pi)^{1/6}.
\]

We see that \( T_\ast \), equation (43), is the crossover temperature which separates the high-temperature, \( T \gg T_\ast \), from the low-temperature, \( T \ll T_\ast \), regimes. Note also that the dimensionless inverse temperature parameter \( \beta \), introduced above in equation (49), coincides with the Reynolds number \( \text{Re} \), equation (10), so that the limit of large Reynolds number in the Burgers problem corresponds to the zero temperature limit in the directed polymers model under consideration.

Recently, it has been demonstrated [17] that in the limit \( \beta \to \infty \), the eigenfunction \( \psi(\tilde{x}) \) acquires specific vector replica symmetry breaking (RSB) coordinate structure: specifically, its \( M \) arguments \( \{\tilde{x}_1, \ldots, \tilde{x}_M\} \) split into \( K = M/m \) groups each consisting of \( m \) particles. In other words, to describe the coordinate structure of the eigenfunction \( \psi(\tilde{x}) \), instead of the particle coordinates \( \{\tilde{x}_i\} \) one introduces the coordinates of the centers of mass of the groups \( \{X_\alpha\} \) \( (\alpha = 1, \ldots, K) \) and the deviations \( \{\xi_i^\alpha\} \) \( (i = 1, \ldots, m) \) of the particles of a given group \( \alpha \) from the position of its center of mass:

\[
\tilde{x}_a \to X_\alpha + \xi_i^\alpha; \quad \alpha = 1, \ldots, M/m; \quad i = 1, \ldots, m,
\]

where \( \sum_{i=1}^{m} \xi_i^\alpha = 0 \). It can be shown [17] that in the zero temperature limit, the typical values of deviations inside groups are small, \( \langle (\xi_i^\alpha)^2 \rangle_{\beta \to \infty} \to 0 \), while the typical distance between the groups remains finite. As these two spatial scales are well separated, the wave function \( \psi(\tilde{x}) \) factorizes into the product of two contributions: the ‘external’ wave function, which depends only on the coordinates \( \{X_\alpha\} \) of the center of masses of the groups, and the ‘internal’ wave functions, which depend only on the coordinates \( \{\xi_i^\alpha\} \) of the particles inside the groups:

\[
\psi(\tilde{x}) \to \psi(X_\alpha; \xi_i^\alpha) \simeq \psi_\ast(X_1, \ldots, X_{M/m}) \times \prod_{\alpha=1}^{M/m} \psi_0(\xi_1^\alpha, \ldots, \xi_m^\alpha).
\]

As the values \( \xi_i^\alpha \) are small, the interaction potential, equation (46), between the particles inside groups can be approximated as

\[
U_0(\xi_i^\alpha - \xi_j^\alpha) \simeq 1 - \frac{1}{2} (\xi_i^\alpha - \xi_j^\alpha)^2.
\]

https://doi.org/10.1088/1742-5468/aad6c8
Thus, according to equation (48), the corresponding equation for the ‘internal’ eigenfunction $\psi_0(\xi)$ of any group reads

$$-2\tilde{\beta}E_0 \psi_0(\xi) = \sum_{i=1}^{m} \frac{\partial^2}{\partial \xi_i^2} \psi_0(\xi) + \tilde{\beta}^3 m^2 \psi_0(\xi) - \frac{1}{2} \tilde{\beta}^3 \sum_{i,j=1}^{m} (\xi_i - \xi_j)^2 \psi_0(\xi),$$

where $\xi = \{\xi_1, \xi_2, \ldots, \xi_m\}$. One can easily show that this equation has the following exact (ground state) solution:

$$\psi_0(\xi) = C \exp\left\{ -\frac{1}{4} \tilde{\beta}^2 (\tilde{\beta}m)^{-1/2} \sum_{i,j=1}^{m} (\xi_i - \xi_j)^2 \right\},$$

where $C$ is the normalization constant and

$$E_0 = -\frac{1}{2} (\tilde{\beta}m)^2 + \frac{1}{2} (m - 1) \sqrt{\tilde{\beta}m}$$

is the ground state energy.

On the other hand, the ‘external’ wave function $\psi_*(X)$ (with $X = \{X_1, \ldots, X_{M/m}\}$) is defined by the equation

$$-2(\tilde{\beta}m) E_* \psi_*(X) = \sum_{\alpha=1}^{M/m} \frac{\partial^2}{\partial X_\alpha^2} \psi_*(X) + \frac{1}{2} (\tilde{\beta}m)^3 \sum_{\alpha \neq \alpha'} U_0(X_\alpha - X_{\alpha'}) \psi_*(X).$$

In terms of the replica approach, the parameter $m$ of the RSB ansatz described above is an integer such that $1 \leq m \leq M$ (so that $M/m$ is also an integer). In the framework of the standard replica technique, after computing the corresponding partition function and its analytic continuation for arbitrary (non-integer) values of $M$ and $m$, in the limit $M \to 0$ the parameter $m$ takes continuous (real) values at the interval $0 \leq m \leq 1$. Its actual physical value $m(\tilde{\beta})$ is fixed by the condition of the maximum of the total (linear in time $t \to \infty$) replica free energy. It can be shown [17] that in the limit $\tilde{\beta} \to \infty$ the value $m(\tilde{\beta})$ is defined by the relation

$$\tilde{\beta} m = \zeta_0,$$

where $\zeta_0$ is a number of the order of one (such that $m(\tilde{\beta}) \to 0$ as $\tilde{\beta} \to \infty$). The exact value of $\zeta_0$ is yet to be computed, as it is defined by the exact solution of the ‘external’ problem, equation (56), which at present is not known.

In terms of this RSB ansatz in the zero temperature limit, the replica partition function of the system considered here, equation (32), factorizes into two parts:

$$Z_{x,\epsilon,\tilde{t}}(M, N_1, N_2, N_3) \sim Z_*([\tilde{\beta}m], M/m, \tilde{t}) \times Z_0(M, m, \tilde{\beta}, N_1, N_2, N_3, x, \epsilon),$$

where $Z_*$ is the ‘external’ replica partition function

$$Z_* = \prod_{\alpha=1}^{M/m} \left[ \int_{\varphi_0(0) = 0} \mathcal{D}\varphi_0(\tau) \right] \exp \left\{ -\frac{1}{2} \int_0^{\tilde{t}} d\tau \left[ (\tilde{\beta}m) \sum_{\alpha=1}^{M/m} (\partial_\tau \varphi_\alpha)^2 - (\tilde{\beta}m)^2 \sum_{\alpha \neq \alpha'} U_0(\varphi_\alpha - \varphi_{\alpha'}) \right] - \frac{M}{m} E_0 \right\},$$

where $E_0$ is given in equation (55). Note that in the limit $\tilde{t} \to \infty$, this partition function becomes independent of $x$ and $\epsilon$, as these parameters do not scale with $\tilde{t}$. The above
‘external’ partition function $Z_*$ defines the extensive-in-$\tilde{\ell} \to \infty$ part of the directed polymer free energy and fixes the value of the parameter $m = m(\tilde{\beta})$, equation (57), and it is in this way that the parameters of the large-scale random potential influence the small-scale statistics defined by the ‘internal’ partition function (see below), which also depends on the value of $m(\tilde{\beta})$. On the other hand, by definition,

$$
\lim_{M \to 0} Z_*((\tilde{\beta}m), M/m, \tilde{\ell}) = 1,
$$

and therefore, except for fixing the value of the replica parameter $m(\tilde{\beta})$, this part of the total partition function does not contribute to the probability function $W_{x,\epsilon}(f_1, f_2, f_3)$, equations (34) and (31). This probability function is defined only by the ‘internal’ (independent of $\tilde{\ell}$) partition function

$$
Z_0(N_1, N_2, N_3; x, \epsilon) = \lim_{\tilde{\beta} \to \infty} \lim_{M \to 0} Z_0(M, m, \tilde{\beta}, N_1, N_2, N_3, x, \epsilon)
$$

$$
= \lim_{\tilde{\beta} \to \infty} \lim_{M \to 0} \left[ \sum_{\{\xi^a\}} \prod_{\alpha=1}^{M/m} \psi_0(\tilde{\xi}^a_1, \ldots, \tilde{\xi}^a_m) \right]_{\{\xi^a_i\} = (\tilde{x}/2, \tilde{x}/2 - \epsilon, -\tilde{x}/2 + \epsilon, -\tilde{x}/2)},
$$

where the explicit expression for $\psi_0(\tilde{\xi})$ is given in equation (54), and where we have to sum over all possible distributions of $M$ particle coordinates $\{\tilde{\xi}^a_i\}$ ($\alpha = 1, \ldots, M/m$; $i = 1, \ldots, m$) over four end-points $\tilde{x}/2$, $\tilde{x}/2 - \epsilon$, $-\tilde{x}/2 + \epsilon$ and $-\tilde{x}/2$ with $\tilde{x} = x/R$ and $\xi^a_i = \xi^a_i/R$.

4. Free energy probability distribution function

Substituting equations (54) and (49), as well as $\tilde{x} = x/R$ and $\tilde{\xi}^a_i = \xi^a_i/R$, into equation (61), we get

$$
Z_0(N_1, N_2, N_3; x, \epsilon) = \lim_{\tilde{\beta} \to \infty} \lim_{M \to 0} \left[ \sum_{\{\xi^a\}} \prod_{\alpha=1}^{M/m} \exp\left\{ -\frac{1}{4} \gamma^2 - \sum_{\alpha=1}^{m} \frac{1}{4} (\xi^a_\alpha - \xi^a_j)^2 \right\} \right]_{\{\xi^a_i\} = (x/2, x/2 - \epsilon, -x/2 + \epsilon, -x/2)},
$$

where

$$
\gamma = \frac{T_*}{R(\beta m)^{1/4}}
$$

and $T_*$ is given in equation (43). Note that the normalization factor $C$ of the wave function (54) can be dropped out in equation (62), as $\lim_{M \to 0} C^{M/m} = 1$. According to the definition, equation (32), in the summation over various distributions of $M$ end-points $\xi^a_i$ over four spatial points, the total number of $\xi^a_i$ values attached to $x/2$, $x/2 - \epsilon$, $-x/2 + \epsilon$ and $-x/2$ is equal to $N_1$, $N_2$, $N_3$ and $(M - N_1 - N_2 - N_3)$ correspondingly. Let us denote the number of values of the group $\xi^a_i$ attached to the points $x/2$, $x/2 - \epsilon$, $-x/2 + \epsilon$ and $-x/2$ by $k_1^a$, $k_2^a$, $k_3^a$ and $k_4^a$. As the total number of particles in each group is equal to $m$, by definition,
Velocity distribution functions and intermittency in one-dimensional randomly forced Burgers turbulence

\[ k_1^\alpha + k_2^\alpha + k_3^\alpha + k_4^\alpha = m \]

and

\[
\begin{align*}
\sum_{\alpha=1}^{M/m} k_1^\alpha &= N_1 \\
\sum_{\alpha=1}^{M/m} k_2^\alpha &= N_2 \\
\sum_{\alpha=1}^{M/m} k_3^\alpha &= N_3 \\
\sum_{\alpha=1}^{M/m} k_4^\alpha &= M - N_1 - N_2 - N_3.
\end{align*}
\]

(Figure 2. Schematic representation of the replica structure of the partition function in equations (62)–(66).

The above replica structure of the partition function (62) is represented schematically in figure 2. Accordingly, the factors \((\xi_i^\alpha - \xi_j^\alpha)^2\) in equation (62) can take four possible values: \(\epsilon^2\), \((x - \epsilon)^2\), \((x - 2\epsilon)^2\) and \(x^2\). Simple combinatoric considerations yield

\[
Z_0(N_1, N_2, N_3; x, \epsilon) = \lim_{\beta \to \infty} \lim_{M \to 0} \left\{ \frac{N_1! N_2! N_3! (M - N_1 - N_2 - N_3)!}{M!} \right. \\
\times \left. \prod_{\alpha=1}^{M/m} \left[ \left( \prod_{i=1}^{4} \sum_{k_i^\alpha=0}^{m} \frac{m!}{k_1^\alpha! k_2^\alpha! k_3^\alpha! k_4^\alpha!} \delta \left( \sum_{i=1}^{4} k_i^\alpha, m \right) \exp \left\{ -\frac{1}{4}\beta^2 \gamma^2 \sum_{i,j=1}^{4} D_{ij} k_i^\alpha k_j^\alpha \right\} \right] \right. \\
\times \delta \left( \sum_{\alpha=1}^{M/m} k_1^\alpha, N_1 \right) \delta \left( \sum_{\alpha=1}^{M/m} k_2^\alpha, N_2 \right) \delta \left( \sum_{\alpha=1}^{M/m} k_3^\alpha, N_3 \right) \right\},
\]

where \(\delta(p, q)\) is the Kronecker symbol and

\[
\hat{D} = \begin{pmatrix}
0 & \epsilon^2 & (x - \epsilon)^2 & x^2 \\
\epsilon^2 & 0 & (x - 2\epsilon)^2 & (x - \epsilon)^2 \\
(x - \epsilon)^2 & (x - 2\epsilon)^2 & 0 & \epsilon^2 \\
x^2 & (x - \epsilon)^2 & \epsilon^2 & 0
\end{pmatrix}.
\]

Note that the last constraint in equation (65) can be dropped out of the expression (66), as it is automatically fulfilled due to the previous three together with the condition (64).

Substituting the matrix (67) into equation (66), we get

\[
\text{https://doi.org/10.1088/1742-5468/aad6c8}
\]
\[ Z_0(N_1, N_2, N_3; x, \epsilon) = \lim_{\beta \to \infty} \lim_{M \to 0} \left\{ \frac{N_1! N_2! N_3! (M - N_1 - N_2 - N_3)!}{M!} \right\} \times \exp \left\{ -\frac{1}{2} \beta N_1 (\beta m) \gamma^2 x^2 - \frac{1}{2} \beta N_2 (\beta m) \gamma^2 (x - \epsilon)^2 - \frac{1}{2} \beta N_3 (\beta m) \gamma^2 \epsilon^2 \right\} \times \prod_{\alpha=1}^{M/m} \left[ \sum_{k_1^\alpha, k_2^\alpha, k_3^\alpha = 0}^{m} C_{k_1^\alpha, k_2^\alpha, k_3^\alpha}^m \exp \left\{ \frac{1}{2} \beta^2 \gamma^2 (x k_1^\alpha + (x - \epsilon) k_2^\alpha + \epsilon k_3^\alpha)^2 \right\} \right] \times \delta \left( \sum_{\alpha=1}^{M/m} k_1^\alpha, N_1 \right) \delta \left( \sum_{\alpha=1}^{M/m} k_2^\alpha, N_2 \right) \delta \left( \sum_{\alpha=1}^{M/m} k_3^\alpha, N_3 \right), \tag{68} \]

where

\[ C_{k_1^\alpha, k_2^\alpha, k_3^\alpha}^m = \frac{m!}{k_1^\alpha! k_2^\alpha! k_3^\alpha! (m - k_1^\alpha - k_2^\alpha - k_3^\alpha)!}. \tag{69} \]

Using the standard integral representation of the Kronecker symbol,
\[ \delta(p, q) = \oint \frac{dz}{2\pi i z} z^{p-q} \tag{70} \]

(where the contour of integration in the complex plane is a circle around zero) the partition function, equation (68), can be represented as follows:

\[ Z_0(N_1, N_2, N_3; x, \epsilon) = \lim_{\beta \to \infty} \lim_{M \to 0} \left\{ \frac{N_1! N_2! N_3! (M - N_1 - N_2 - N_3)!}{M!} \exp \left\{ -\beta N_1 f_{01} - \beta N_2 f_{02} - \beta N_3 f_{03} \right\} \times \frac{1}{(2\pi i)^3} \oint \frac{dz_1}{z_1} z_1^{-N_1} \oint \frac{dz_2}{z_2} z_2^{-N_2} \oint \frac{dz_3}{z_3} z_3^{-N_3} \times \left[ \left\langle \left( 1 + z_1 \exp \{ \beta \gamma x \xi \} + z_2 \exp \{ \beta \gamma (x - \epsilon) \xi \} + z_1 \exp \{ \beta \gamma \epsilon \xi \} \right)^m \right\rangle_\xi \right]^{M/m} \right\} \tag{71} \]

where

\[ f_{01} = \frac{1}{2} (\beta m) \gamma^2 x^2 \]
\[ f_{01} = \frac{1}{2} (\beta m) \gamma^2 (x - \epsilon)^2 \]
\[ f_{03} = \frac{1}{2} (\beta m) \gamma^2 \epsilon^2 \tag{72} \]

and \( \langle \ldots \rangle_\xi \) denotes the Gaussian average over the variable \( \xi \):
\[ \langle \ldots \rangle_\xi \equiv \int_{-\infty}^{+\infty} \frac{d\xi}{\sqrt{2\pi}} \ldots \exp \left\{ -\frac{1}{2} \xi^2 \right\}. \tag{73} \]

The expression for the replica partition function, equation (71), can now be analytically continued for arbitrary non-integer values of the parameter \( M \); in particular, the factorial prefactor
\[
\frac{(M - N_1 - N_2 - N_3)!}{M!} \xrightarrow{\text{Using the Gamma function relation,}} \frac{\Gamma(M - N_1 - N_2 - N_3 + 1)}{\Gamma(M + 1)}.
\]

Using the Gamma function relation,
\[
\Gamma(-z) = -\frac{\pi}{\Gamma(z + 1)\sin(\pi z)}
\]
for integer and positive values of \(N_{1,2,3}\), this prefactor can be represented as follows:
\[
\left.\frac{\Gamma(M - N_1 - N_2 - N_3 + 1)}{\Gamma(M + 1)}\right|_{M \to 0} \xrightarrow{\text{Using (77) and (78)}} \frac{(-1)^{-N_1+N_2+N_3-1}}{M \Gamma(N_1 + N_2 + N_3 + 1)}. 
\]

On the other hand, for the last factor in the expression (71), we find
\[
\left.\left[\ldots\right]^{M/m}\right|_{M \to 0} \xrightarrow{\text{Taking into account that for any non-zero integer } N,} 1 + \frac{M}{m}\ln\left[\ldots\right].
\]

Substituting these expressions into equations (31) and (34) for the free energy probability distribution function we obtain

\[
Z_0(N_1, N_2, N_3; x, \epsilon) = \lim_{\beta \to \infty} \left\{ \frac{(-1)^{N_1+N_2+N_3-1}}{m \Gamma(N_1 + N_2 + N_3)} \right. 
\times \exp\{-\beta N_1 f_{01} - \beta N_2 f_{02} - \beta N_3 f_{03}\} 
\times \frac{1}{(2\pi)^3} \int \frac{d\xi}{z_1^{N_1}} \int \frac{d\xi}{z_2^{N_2}} \int \frac{d\xi}{z_3^{N_3}} 
\times \ln \left[\left\langle 1 + z_1 \exp\{\beta \gamma x \xi\} + z_2 \exp\{\beta \gamma (x - \epsilon) \xi\} + z_1 \exp\{\beta \gamma \epsilon \xi\}\right\rangle \right] \left.\right\}.
\]

Substituting this expression into equations (31) and (34) for the free energy probability distribution function we obtain

https://doi.org/10.1088/1742-5468/aad6c8
Velocity distribution functions and intermittency in one-dimensional randomly forced Burgers turbulence

\[
W_{x, \epsilon}(f_1, f_2, f_3) = \lim_{\beta \to \infty} \left\{ \sum_{N_1, N_2, N_3=1}^{\infty} \frac{\exp\{\beta N_1(f_1 - f_{01}) + \beta N_2(f_1 - f_{02}) + \beta N_3(f_1 - f_{03})\}}{m \Gamma(N_1 + N_2 + N_3)} \times \frac{1}{(2\pi)^3} \iint_{\mathbb{R}^3} \frac{dz_1}{z_1} z_1^{-N_1} \iint_{\mathbb{R}^3} \frac{dz_2}{z_2} z_2^{-N_2} \iint_{\mathbb{R}^3} \frac{dz_3}{z_3} z_3^{-N_3} \times \ln \left[ \left( 1 + z_1 \exp\{\beta \gamma x \xi \} + z_2 \exp\{\beta \gamma(x - \epsilon) \xi \} + z_1 \exp\{\beta \gamma \epsilon \xi \} \right)^m \right] \right\}. \tag{81}
\]

The limit \( \beta \to \infty \) is somewhat tricky: on one hand, according to equation (57) in the zero temperature limit \( m \propto 1/\beta \to 0 \); and on the other hand, we have several exponential factors in the above expression which are formally divergent in this limit. To take the limit \( m \propto 1/\beta \to 0 \), the expression under the logarithm in equation (81) can be represented as follows:

\[
\left\langle \left( 1 + z_1 \exp\{\beta \gamma x \xi \} + z_2 \exp\{\beta \gamma(x - \epsilon) \xi \} + z_1 \exp\{\beta \gamma \epsilon \xi \} \right)^m \right\rangle_\xi = 1 + \sum_{k_1 + k_2 + k_3 \geq 1} C_{k_1 k_2 k_3}^m z_1^{k_1} z_2^{k_2} z_3^{k_3} \left\langle \exp\{\beta k_1 \gamma x \xi + \beta k_2 \gamma(x - \epsilon) \xi + \beta k_3 \gamma \epsilon \xi \} \right\rangle_\xi, \tag{82}
\]

where

\[
C_{k_1 k_2 k_3}^m = \frac{\Gamma(m + 1)}{\Gamma(k_1 + 1) \Gamma(k_2 + 1) \Gamma(k_3 + 1) \Gamma(m - k_1 - k_2 - k_3 + 1)}. \tag{83}
\]

In the limit \( m \to 0 \), we get (see equations (76) and (77))

\[
C_{k_1 k_2 k_3}^m \bigg|_{m \to 0} \approx m \frac{(-1)^{k_1 + k_2 + k_3 - 1}}{k_1 + k_2 + k_3} C_{k_1 k_2 k_3}^0, \tag{84}
\]

where

\[
C_{k_1 k_2 k_3}^0 = \frac{\Gamma(k_1 + k_2 + k_3 + 1)}{\Gamma(k_1 + 1) \Gamma(k_2 + 1) \Gamma(k_3 + 1)}. \tag{85}
\]

Substituting equations (82), (84) and (85) into equation (81), and expanding the logarithm term after integrations over \( z_1, z_2 \) and \( z_3 \), we obtain

\[
W_{x, \epsilon}(f_1, f_2, f_3) = \lim_{\beta \to \infty} \left\{ \frac{1}{(\beta m)} \sum_{n=1}^{\infty} \frac{(-1)^{m-1}}{n} \left( \sum_{k_1 + k_2 + k_3 \geq 1} (\beta m)^n \frac{(-1)^{k_1 + k_2 + k_3 - 1}}{\beta(k_1 + k_2 + k_3)} C_{k_1 k_2 k_3}^n \left\langle \exp\{\beta k_1 \gamma x \xi + \beta k_2 \gamma(x - \epsilon) \xi + \beta k_3 \gamma \epsilon \xi \} \right\rangle_\xi \right) \right\} \times \prod_{a=1}^{n} \left[ \frac{\beta(N_1 + N_2 + N_3)}{\beta(N_1 + N_2 + N_3 + 1)} \exp\{\beta N_1(f_1 - f_{01}) + \beta N_2(f_1 - f_{02}) + \beta N_3(f_1 - f_{03})\} \right] \times \delta\left(\sum_{n=1}^{N_1} k_1^n, N_1\right) \delta\left(\sum_{n=1}^{N_2} k_2^n, N_2\right) \delta\left(\sum_{n=1}^{N_3} k_3^n, N_3\right). \tag{86}
\]

https://doi.org/10.1088/1742-5468/aad6c8
Substituting $\beta m = \tilde{\beta} m / T_\star = \zeta_0 / T_\star$ (see equations (57), (42) and (43)) and resolving the Kronecker symbols in the summations over $N_1$, $N_2$ and $N_3$, we get

\[
\mathcal{W}_{x,\epsilon}(f_1, f_2, f_3) = \frac{T_\star}{\zeta_0} \lim_{\beta \to \infty} \left\{ \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \left( \frac{\zeta_0}{T_\star} \right)^n \prod_{\alpha=1}^{n} \left[ \sum_{k_1^\alpha + k_2^\alpha + k_3^\alpha \geq 1} \frac{(-1)^{k_1^\alpha + k_2^\alpha + k_3^\alpha - 1}}{\beta(k_1^\alpha + k_2^\alpha + k_3^\alpha)} C_0^{k_1^\alpha k_2^\alpha k_3^\alpha} \right] \times \left\{ \exp \left\{ \beta k_1^\alpha (\gamma x \epsilon + f_1 - f_{01}) + \beta k_2^\alpha (\gamma (x - \epsilon) \xi + f_2 - f_{02}) + \beta k_3^\alpha (\gamma \epsilon \xi + f_3 - f_{03}) \right\} \right\} \bigg[ \sum_{\alpha=1}^{n} (k_1^\alpha + k_2^\alpha + k_3^\alpha) \right] \frac{\theta \left( \sum_{\alpha=1}^{n} (k_1^\alpha - 1) \right) \theta \left( \sum_{\alpha=1}^{n} (k_2^\alpha - 1) \right) \theta \left( \sum_{\alpha=1}^{n} (k_3^\alpha - 1) \right)}{\Gamma \left[ \sum_{\alpha=1}^{n} (k_1^\alpha + k_2^\alpha + k_3^\alpha) + 1 \right]} \bigg] \right\},
\]

where the symbol $\theta(p - 1)$ (the ‘discrete step function’) indicates that $p \geq 1$. Note, however, that according to equation (87), the contributions with $\sum_{\alpha=1}^{n} k_\alpha = 0$ (such that all $k_1^\alpha = k_2^\alpha = \ldots = k_3^\alpha = 0$) are independent of the corresponding free energy parameter $\beta$. On the other hand, the probability density function $P_{x,\epsilon}(f_1, f_2, f_3)$ which we are aiming to derive is given by the derivatives of the above probability function $W$ over all three variables $f_1, f_2$ and $f_3$ (see equation (33)). Therefore, as far as the PDF $P_{x,\epsilon}(f_1, f_2, f_3)$ is concerned, the restrictions imposed by the last three ‘discrete step function’ in equation (87) can be omitted.

The factor $\beta \sum_{\alpha=1}^{n} (k_1^\alpha + k_2^\alpha + k_3^\alpha)$ in the numerator of the last term in equation (87) can be obtained by taking the derivatives $\left( \frac{\partial}{\partial f_1} + \frac{\partial}{\partial f_2} + \frac{\partial}{\partial f_3} \right)$ of $\mathcal{W}_{x,\epsilon}(f_1, f_2, f_3)$. On the other hand,

\[
\frac{1}{\beta(k_1^\alpha + k_2^\alpha + k_3^\alpha)} = \int_{0}^{+\infty} dy \exp \left\{ -\beta \left( k_1^\alpha + k_2^\alpha + k_3^\alpha \right) y \right\}.
\]

Substituting this into equation (87) and then substituting the expression thus obtained into equation (33) for the PDF $P_{x,\epsilon}(f_1, f_2, f_3)$, we get

\[
P_{x,\epsilon}(f_1, f_2, f_3) = \frac{T_\star}{\zeta_0} \frac{\partial^3}{\partial f_1 \partial f_2 \partial f_3} \left( \frac{\partial}{\partial f_1} + \frac{\partial}{\partial f_2} + \frac{\partial}{\partial f_3} \right) \lim_{\beta \to \infty} \left\{ \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \left( \frac{\zeta_0}{T_\star} \right)^n \right\}
\]

\[
\times \prod_{\alpha=1}^{n} \int_{0}^{+\infty} dy \sum_{k_1^\alpha + k_2^\alpha + k_3^\alpha \geq 1} \frac{(-1)^{k_1^\alpha + k_2^\alpha + k_3^\alpha - 1}}{\beta(k_1^\alpha + k_2^\alpha + k_3^\alpha)} C_0^{k_1^\alpha k_2^\alpha k_3^\alpha}
\]

\[
\times \left\{ \exp \left\{ \beta k_1^\alpha (\gamma x \epsilon + f_1 - f_{01} - y) + \beta k_2^\alpha (\gamma (x - \epsilon) \xi + f_2 - f_{02} - y) + \beta k_3^\alpha (\gamma \epsilon \xi + f_3 - f_{03} - y) \right\} \right\}
\]

\[
\times \frac{1}{\Gamma \left[ \sum_{\alpha=1}^{n} (k_1^\alpha + k_2^\alpha + k_3^\alpha) + 1 \right]} \bigg] \right\}.
\]

In the limit $\beta \to \infty$, the summation the series over $k_\alpha$ in the above expression can be done using the their integral representation. Specifically, let us consider the series of a general type

https://doi.org/10.1088/1742-5468/aad6c8
Velocity distribution functions and intermittency in one-dimensional randomly forced Burgers turbulence

\[ R(\beta) = \sum_{k=0}^{\infty} (-1)^{k-1} \Phi(\beta k; k), \]  

(90)

where \( \Phi(z, z') \) is a ‘good’ analytic function in the complex plane. One can easily see that the summation in equation (90) can be changed by the integration in the complex plane:

\[ R(\beta) = \frac{1}{2i} \int_{C} \frac{dz}{\sin(\pi z)} \Phi(\beta z; z), \]  

(91)

where the integration goes over the contour \( C \) shown in figure 3, and it is assumed that the function \( \Phi \) is such that its integration at infinity gives no contribution. Indeed, due to the sign alternating contributions of simple poles at integer \( z = 1, 2, \ldots \), equation (91) reduces to equation (90). Then, redefining \( z \to z/\beta \) we get

\[ \lim_{\beta \to \infty} R(\beta) = \frac{1}{2\pi i} \int_{C} \frac{dz}{z} \lim_{\beta \to \infty} \Phi(z; z/\beta). \]  

(92)

In terms of the above integral representation, changing \( k_i^a \to z_i^a/\beta \) for the Gamma function factors in equation (89), we have

\[ \Gamma \left[ \sum_{\alpha=1}^{n} (k_1^a + k_2^a + k_3^a) + 1 \right] \to \Gamma \left[ \sum_{\alpha=1}^{n} (z_1^a + z_2^a + z_3^a)/\beta + 1 \right] \bigg|_{\beta \to \infty} \to 1 \]  

(93)

and (see equation (85))

\[ C_{k_1^a k_2^a k_3^a}^{0} \to \frac{\Gamma(z_1^a/\beta + z_2^a/\beta + z_3^a/\beta + 1)}{\Gamma(z_1^a/\beta + 1)\Gamma(z_2^a/\beta + 1)\Gamma(z_3^a/\beta + 1)} \bigg|_{\beta \to \infty} \to 1. \]  

(94)

Thus, after extracting the contributions with \( k_1^a = k_2^a = k_3^a = 0 \), the expression in equation (89) reduces to

\[ P_{x,\xi}(f_1, f_2, f_3) = \frac{T_s}{\xi_0} \frac{\partial^3}{\partial f_1 \partial f_2 \partial f_3} \left( \frac{\partial}{\partial f_1} + \frac{\partial}{\partial f_2} + \frac{\partial}{\partial f_3} \right) \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \left( \frac{\xi_0}{T_s} \right)^n \]  

\[ \times \left[ \int_{0}^{+\infty} dy \left( \int_{C} \frac{dz_1}{2\pi i z_1} \exp \left\{ z_1(\gamma x_1 + f_1 - f_{01} - y) \right\} \right) \right] \times \left[ \int_{C} \frac{dz_2}{2\pi i z_2} \exp \left\{ z_2(\gamma (x - \epsilon)\xi + f_2 - f_{02} - y) \right\} \right] \]  

\[ \times \left[ \int_{C} \frac{dz_3}{2\pi i z_3} \exp \left\{ z_3(\gamma \epsilon \xi + f_3 - f_{03} - y) \right\} + 1 \right] \right]\left|_{\xi}^{n} \right. \]  

(95)

One easily finds that

\[ \int_{C} \frac{dz}{2\pi i z} \exp \{ \lambda z \} = -\theta(-\lambda). \]  

(96)

Indeed, in the case \( \lambda > 0 \), the contour of integration \( C \) can be shifted to \(-\infty\) so that due to the factor \( \exp \{ \lambda z \} \) the integral turns into zero, while in the case \( \lambda < 0 \) the contour
of integration $C$ can be shifted to $+\infty$ so that the only non-zero contribution to this integral comes from the integration around the pole at $z = 0$ which is equal to $-1$. Substituting equation (97) into equation (96), we obtain

$$P_{x,\epsilon}(f_1, f_2, f_3) = \frac{T_0}{\zeta_0} \frac{\partial^3}{\partial f_1 \partial f_2 \partial f_3} \left( \frac{\partial}{\partial f_1} + \frac{\partial}{\partial f_2} + \frac{\partial}{\partial f_3} \right) \ln \left[ 1 + S(f_1, f_2, f_3) \right]$$

or

$$P_{x,\epsilon}(f_1, f_2, f_3) = \frac{\partial^3}{\partial f_1 \partial f_2 \partial f_3} \left[ \left( 1 + S(f_1, f_2, f_3) \right)^{-1} G(f_1, f_2, f_3) \right],$$

where

$$S(f_1, f_2, f_3) = \frac{\zeta_0}{T_0} \int_0^{+\infty} dy \int_{-\infty}^{+\infty} \frac{d\xi}{\sqrt{2\pi}} \exp \left\{ -\frac{\xi^2}{2} \right\} \times \left[ 1 - \theta(y + f_{01} - f_1 - \gamma x \xi) \theta(y + f_{02} - f_2 - \gamma(x - \epsilon) \xi) \theta(y + f_{03} - f_3 - \gamma \epsilon \xi) \right]$$

and

$$G(f_1, f_2, f_3) = \frac{T_0}{\zeta_0} \left( \frac{\partial}{\partial f_1} + \frac{\partial}{\partial f_2} + \frac{\partial}{\partial f_3} \right) S(f_1, f_2, f_3) \nonumber$$

$$= \int_{-\infty}^{+\infty} \frac{d\xi}{\sqrt{2\pi}} \exp \left\{ -\frac{\xi^2}{2} \right\} \left[ 1 - \theta(f_{01} - f_1 - \gamma x \xi) \theta(f_{02} - f_2 - \gamma(x - \epsilon) \xi) \theta(f_{03} - f_3 - \gamma \epsilon \xi) \right].$$

5. Two velocity probability density function

In this section, using the general result for the directed polymer three-point free energy distribution function, equations (98)–(100), we are going to derive two velocity probability density functions $P_{x}(v, v')$ of the corresponding randomly forced Burgers problem. According to the discussion of section 2, equations (35)–(37),

https://doi.org/10.1088/1742-5468/aad6c8
Explicitly, the expression for the function $P_{x,\epsilon}(f_1, f_2, f_3)$, equation (98), reads

$$P_{x,\epsilon}(f_1, f_2, f_3) = (1 + S)^{-1} G''_{123} - (1 + S)^{-2} \left[ S'_1 G''_{23} + S'_2 G''_{13} + S'_3 G''_{12} + S''_{12} G'_3 + S''_{13} G'_2 + S''_{23} G'_1 + S''_{123} G \right] + 2(1 + S)^{-3} \left[ S'_1 S'_2 G'_3 + S'_1 S'_3 G'_2 + S_2 S'_3 G'_1 + (S'_1 S''_{23} + S'_2 S''_{13} + S'_3 S''_{12}) G \right] - 6(1 + S)^{-4} S'_1 S'_2 S'_3 G,$$

(102)

where we have introduced the notations $\Phi'_i \equiv \frac{\partial}{\partial f_i} \Phi$ and the functions $S = S(f_1, f_2, f_3)$ and $G = G(f_1, f_2, f_3)$ are given in equations (99) and (100). Substituting this expression into equation (101), we find that the only non-zero contributions in the limit $\epsilon \to 0$ come from two terms in the r.h.s. of equation (102): $(1 + S)^{-1} G''_{123}$ and $-(1 + S)^{-2} S''_{12} G'_3$ (both of which $\propto 1/\epsilon^2$), where (see appendix)

$$G''_{123} \bigg|_{\epsilon \to 0} = \frac{x}{\epsilon^2 \gamma \sqrt{2\pi}} \exp\left\{ -\frac{1}{2\gamma^2} (v')^2 \right\} \delta(f_2 - f_0 + x v') \delta(x v' - x v - 2 f_0)$$

(103)

$$G'_3 \bigg|_{\epsilon \to 0} = \frac{1}{\epsilon \gamma \sqrt{2\pi}} \exp\left\{ -\frac{1}{2\gamma^2} (v')^2 \right\} \theta(f_0 - f_2 - x v')$$

(104)

$$S''_{12} \bigg|_{\epsilon \to 0} = -\frac{\zeta_0}{\epsilon \gamma T_s \sqrt{2\pi}} \exp\left\{ -\frac{1}{2\gamma^2 x^2} (x v + 2 f_0)^2 \right\} \theta(x v + f_0 + f_2).$$

(105)

Here, according to equations (72), (63), (57) and (49),

$$f_0 \equiv f_{01} = \frac{1}{2} \left( \beta m \right)^{-2} = \frac{1}{2} \sqrt{\zeta_0} T_s \cdot \frac{x^2}{R^2}$$

(106)

$$\gamma = \zeta^{-1/4} \frac{T_s}{R}$$

(107)

and $T_s$ is given in equation (43). Using explicit expression for $S(f_1, f_2, f_3)$, equation (99), one finds

$$\lim_{\epsilon \to 0} S(f_2 - \epsilon v, f_2, -\epsilon v') = \frac{\zeta_0}{\gamma T_s x} \int_0^{\infty} \frac{d\xi}{\sqrt{2\pi}} \xi \exp\left\{ -\frac{1}{2\gamma^2 x^2} (\xi + f_0 - f_2)^2 \right\}.$$
Velocity distribution functions and intermittency in one-dimensional randomly forced Burgers turbulence

\[ P_x(v, v') = \int_{-\infty}^{+\infty} \mathrm{d}f_2 \left\{ \frac{x}{\sqrt{2\pi}} \exp\left\{ -\frac{1}{4\gamma^2} (v')^2 \right\} \delta\left( f_2 - f_0 + x v' \right) \delta\left( x v' - x v - 2 f_0 \right) \right. \]
\[ + \left. \frac{\zeta_0}{2\pi \gamma^2 T_x} \exp\left\{ -\frac{1}{4\gamma^2} (x v + 2 f_0)^2 - \frac{1}{2\gamma^2} (v')^2 \right\} \theta(x v + f_0 + f_2) \theta(f_0 - f_2 - x v') \right\}. \]

Introducing the notation (see equation (9))

\[ v_0 = \zeta_0^{3/4} \gamma = \sqrt{\zeta_0} \frac{T_s}{R} = \left( \frac{\zeta_0^3}{2\pi} \right)^{1/6} \left( \frac{u}{R^2} \right)^{1/3} \]

and changing the integration variables, \( f_2 \to f_0 - x v_0 \eta, \xi \to x \xi \), we eventually get the following result for the joint probability density function of two velocities at the distance \( x \):

\[ P_x(v, v') = p_0(v, x) \delta(v' - v - v_0 \frac{x}{R}) + P_x(v, v') \theta(v + v_0 \frac{x}{R} - v'), \tag{111} \]

where

\[ p_0(v, x) = \frac{\zeta_0^{3/4}}{v_0 \sqrt{2\pi}} \exp\left\{ -\frac{1}{2} \zeta_0^{3/2} \left( \frac{v}{v_0} + \frac{x}{R} \right)^2 \right\} \]

\[
\frac{1 + \zeta_0^{3/4} \frac{x}{R} \int_0^\infty \frac{d\xi}{\sqrt{2\pi}} \exp\left\{ -\frac{1}{2} \left[ \xi + \zeta_0^{3/4} \left( \frac{v}{v_0} + \frac{x}{R} \right)^2 \right] \right\}}{\int_{v/v_0}^{v/v_0 + x/R} \int_0^{\infty} \frac{d\eta}{\sqrt{2\pi}} \exp\left\{ -\frac{1}{2} \left( \xi + \zeta_0^{3/4} \eta \right)^2 \right\}}. \tag{112} \]

\[ P_x(v, v') = \frac{\zeta_0^3 \frac{x}{2\pi v_0^2 R}}{\int_{v/v_0}^{v/v_0 + x/R} \mathrm{d}\eta} \exp\left\{ -\frac{1}{2} \zeta_0^{3/2} \left[ \left( \frac{v}{v_0} + \frac{x}{R} \right)^2 + \left( \frac{v'}{v_0} \right)^2 \right] \right\} \]

\[
\frac{1 + \zeta_0^{3/4} \frac{x}{R} \int_0^\infty \frac{d\xi}{\sqrt{2\pi}} \exp\left\{ -\frac{1}{2} \left( \xi + \zeta_0^{3/4} \eta \right)^2 \right\}}{\int_{v/v_0}^{v/v_0 + x/R} \mathrm{d}\eta} \exp\left\{ -\frac{1}{2} \left( \xi + \zeta_0^{3/4} \eta \right)^2 \right\} \right\}^2. \tag{113} \]

6. Probability distribution function of the velocity difference

Using the joint distribution function \( P_x(v, v') \) of two velocities \( v \) and \( v' \) at distance \( x \) derived above, equations (111)–(113), the probability density function of the velocity difference \( w = v' - v \) can be obtained as follows:

\[ P_x(w) = \int_{-\infty}^{+\infty} \mathrm{d}v P_x(v, v + w). \tag{114} \]

Substituting equations (111)–(113) here, we get

\[ P_x(w) = p_0(x) \delta(w - v_0 \frac{x}{R}) + P_x(w) \theta(v_0 \frac{x}{R} - w), \tag{115} \]

where

https://doi.org/10.1088/1742-5468/aad6c8
Velocity distribution functions and intermittency in one-dimensional randomly forced Burgers turbulence

\[ p_0(x) = \frac{\zeta_0^{3/4}}{v_0 \sqrt{2\pi}} \int_{-\infty}^{+\infty} dv \frac{\exp\left\{ -\frac{1}{2} \left( \frac{v}{v_0} + \frac{x}{R} \right)^2 \right\}}{1 + \zeta_0^{3/4} \frac{x}{R} \int_0^\infty \frac{d\xi}{\sqrt{2\pi}} \xi \exp\left\{ -\frac{1}{2} \left( \xi + \zeta_0^{3/4} \left( \frac{v}{v_0} + \frac{x}{R} \right) \right)^2 \right\}} \] (116)

and

\[ P_r(w) = \frac{\zeta_0^{3/4}}{2\pi v_0^2 R} \int_{-\infty}^{+\infty} dv \int_{v/v_0 + x/R}^{v/v_0 + x/R} d\eta \frac{\exp\left\{ -\frac{1}{2} \left( \frac{v}{v_0} + \frac{x}{R} \right)^2 + \left( \frac{z}{v_0} + \frac{x}{R} \right)^2 \right\}}{1 + \zeta_0^{3/4} \frac{x}{R} \int_0^\infty \frac{d\xi}{\sqrt{2\pi}} \xi \exp\left\{ -\frac{1}{2} (\xi + \zeta_0^{3/4} \eta)^2 \right\}} \] (117)

Changing the integration variables, \( v = -\frac{x}{R} v_0 + \zeta_0^{-3/4} v_0 s \) and \( \eta = (v + w)/v_0 + \zeta_0^{-3/4} z \), and introducing rescaled (dimensionless) distance

\[ r \equiv \frac{\zeta_0^{3/4} x}{R} \] (118)

and rescaled (dimensionless) velocity difference

\[ \omega \equiv \frac{\zeta_0^{3/4} w}{v_0} = \frac{\zeta_0^{3/4} (v' - v)}{v_0} , \] (119)

we obtain the following final result for the corresponding probability density function \( P_r(\omega) \):

\[ P_r(\omega) = p_0(r) \delta(\omega - r) + P_r(\omega) \theta(r - \omega) \] (120)

where

\[ p_0(r) = \int_{-\infty}^{+\infty} ds \frac{\exp\left\{ -\frac{1}{2} s^2 \right\}}{\sqrt{2\pi} \left[ 1 + r \int_0^\infty \frac{d\xi}{\sqrt{2\pi}} \xi \exp\left\{ -\frac{1}{2} (\xi + s)^2 \right\} \right]} \] (121)

and

\[ P_r(\omega) = r \int_{-\infty}^{+\infty} ds \int_{r-\omega}^{r} dz \frac{\exp\left\{ -\frac{1}{2} s^2 - \frac{1}{2} (s + \omega - r)^2 \right\}}{\sqrt{2\pi} \left[ 1 + r \int_0^\infty \frac{d\xi}{\sqrt{2\pi}} \xi \exp\left\{ -\frac{1}{2} (\xi + z + s + \omega - r)^2 \right\} \right]} . \] (122)

It is evident that this function is positively defined and it can be easily checked that for any value of \( r \) it is normalized:

\[ \int_{-\infty}^{+\infty} d\omega P_r(\omega) = 1 . \] (123)

We see that the distribution function \( P_r(\omega) \) has rather specific structure (see figure 4). According to equation (120), for a given (rescaled) distance \( r \), possible values of the (rescaled) velocity difference \( \omega \) are bounded from above: \( \omega \leq r \), or in terms of the original values, \( (v' - v) \leq \frac{x}{R} v_0 \). In other words, at a given distance \( x \) between two points at which we measure two velocities \( v \) and \( v' \), their difference cannot be bigger than \( v_0 x/R \)—where, according to equation (110), \( v_0 \propto \left(u/R^2\right)^{1/3} \) is the typical flow velocity.
Velocity distribution functions and intermittency in one-dimensional randomly forced Burgers turbulence

Let us investigate the statistical properties of the velocity difference at small distances: \( x \ll R \) or \( r \ll 1 \). In the limit of small values of the parameter \( r \), the probability density function \( P_r(\omega) \), equations (120)–(122), takes a much more simple form:

\[
P_r(\omega) \simeq \left(1 - \frac{1}{\sqrt{\pi}} r\right) \delta(\omega - r) + \frac{1}{\sqrt{\pi}} r (r - \omega) \exp\left\{-\frac{1}{2}(r - \omega)^2\right\} \theta(r - \omega).
\]  

For even moments of the velocity difference \( \langle \omega^{2n} \rangle \), we find

\[
\langle \omega^{2n} \rangle = \int_{-\infty}^{+\infty} d\omega \omega^{2n} P_r(\omega) \simeq r^{2n} + C(n) r,
\]

where \( C(n) = 2^{2n+1} \Gamma(1+n) \). Then, the analytic continuation of the above result for arbitrary real values of the parameter \( 2n \to q \), in the limit \( r \ll 1 \) yields

\[
\langle \omega^q \rangle \simeq r^q + C(q/2) r \simeq \begin{cases} r^q, & \text{for } q \leqslant 1; \\ C(q/2) r, & \text{for } q > 1. \end{cases}
\]

Finally, introducing the exponent \( \zeta(q) \) according to the definition \( \langle \omega^q \rangle = r^{\zeta(q)} \), we recover the typical strong intermittency behavior [3] (see figure 1):

\[
\zeta(q) \simeq \begin{cases} q, & \text{for } q \leqslant 1; \\ 1, & \text{for } q > 1. \end{cases}
\]

7. Conclusions

In this paper, we have studied the statistical properties of the velocity field \( v(x, t) \) in one-dimensional randomly forced Burgers turbulence—equation (1). This system is
known to be equivalent to the model of directed polymers in a random potential—equations (4)–(7)—such that the viscosity parameter $\nu$ in the Burgers equation is proportional to the temperature in the directed polymer system, $\nu = \frac{1}{2} T$, and the velocity $v(x,t)$ in the Burgers equation is the negative spatial derivative of the free energy $F(x,t)$ of the directed polymers. The parameter which characterizes the level of turbulence of the velocity field in the Burgers problem is the Reynolds number $Re$, which in terms of the directed polymer notation is expressed as $Re = 2(uR)^{1/3}/T$, where $R$ and $u$ are the correlation length and the strength of the random potential—equations (5) and (6). Thus, the strong turbulence regime where $Re \to \infty$ corresponds to the zero-temperature limit in the directed polymer system. In this limit, a general expression for the joint distribution function of two velocities $v(-x/2,t)$ and $v(x/2,t)$ separated by a finite distance $x$ has been derived in terms of the replica technique—equations (111)–(113). Besides this, we have obtained an explicit expression for the probability density function for the corresponding velocity increment $w = v(-x/2,t) - v(x/2,t)$—equations (120)–(122)—which has been shown to exhibit a rather specific structure. Specifically, for any given distance $x$, the values of the velocity increment $w$ are bounded from above: $w \leq \frac{x}{R} v_0$, where $v_0 \propto (u/R^2)^{1/3}$ is the typical flow velocity at the injection scale $R$ of the random potential. Moreover, at $w = \frac{x}{R} v_0$, the distribution function exhibits the $\delta$-function singularity, which means that at a given distance $x$, the difference of two velocities $w = v(-x/2,t) - v(x/2,t)$ has a finite probability to be equal to $\frac{x}{R} v_0$.

Using this distribution function at length scales much smaller than the injection length of the random potential, $x \ll R$, we have computed the moments of the velocity increment $\langle \omega^q \rangle$—equation (126). Introducing the exponent $\zeta(q)$ according to the definition $\langle \omega^q \rangle \simeq r^{\zeta(q)}$, we have demonstrated that the function $\zeta(q)$ exhibits behavior typical of strong intermittency phenomena—equation (127), figure 1.

Finally, a few remarks about the status of the obtained results. First of all, as the considerations have been made in the framework of the heuristic replica method, the proposed derivation can be considered as rigorous. Moreover, at the moment it is also difficult to say whether the obtained results are exact or not: on one hand, no approximation has been used in the calculations performed; but on the other hand, the derivation considered is based on the unproved crucial assumption about the vector replica symmetry breaking structure of the $N$-particle bosonic wave function in the zero-temperature limit—which, in particular, contains the undefined numerical factor $\zeta_0$; equation (57) ([17], section 3). Presumably, the most direct way to check whether the theoretical construction with vector RSB ansatz proposed here does make sense would be to perform numerical simulations for the PDF $P_\epsilon(w)$ and to confirm (or to reject) the result shown in figure 4. In any case, further systematic study of the problem is required.

**Acknowledgments**

I am grateful to Kostya Khanin for numerous useful discussions.

I would like to thank the mathematical research institute MATRIX in Australia where part of this research was performed.
Appendix

In this technical appendix, the derivatives of the functions $S(f_1, f_2, f_3)$ and $G(f_1, f_2, f_3)$, equations (99) and (100), will be calculated in the limit $\epsilon \to 0$. Redefining

\[ \xi = \frac{1}{\gamma x} \xi \]

\[ \epsilon = \tilde{\epsilon} x \]

and introducing velocities $v$ and $v'$ instead of $f_1$ and $f_3$,

\[ f_1 = \tilde{\epsilon} x v + f_2 \]

\[ f_3 = \tilde{\epsilon} x v' \]

we get

\[ \frac{\partial}{\partial f_1} = -\frac{1}{\epsilon x} \frac{\partial}{\partial v} \]

\[ \frac{\partial}{\partial f_3} = -\frac{1}{\epsilon x} \frac{\partial}{\partial v'} \]

According to equations (72), (63), (57) and (49), in the first order in $\epsilon \to 0$ we have

\[ f_{01} = \frac{1}{2} (\beta m) \frac{\gamma^2}{2} x^2 = \frac{1}{2} \sqrt{\zeta_0 T_\ast} \left( \frac{x}{R} \right)^2 \equiv f_0 \]

\[ f_{01} = \frac{1}{2} (\beta m) \frac{\gamma^2}{2} (x - \epsilon)^2 \simeq f_0 + 2 \epsilon f_0 \]

\[ f_{03} = \frac{1}{2} (\beta m) \frac{\gamma^2}{2} \epsilon^2 \to 0. \]

Substituting equations (A.1)–(A.9) into equations (99) and (100), we find

\[ S(v, f_2, v') = \frac{\zeta_0}{\gamma x R} \int_{-\infty}^{+\infty} dy \int_{-\infty}^{+\infty} \frac{d\xi}{\sqrt{2\pi}} \exp\left\{ -\frac{\xi^2}{2\gamma^2 x^2} \right\} \]

\[ \times \left[ 1 - \theta(y + f_0 - f_2 + \epsilon xv - \xi) \theta(y + f_0 - f_2 - 2\epsilon f_0 + \tilde{\epsilon} \xi - \xi) \theta(y + \epsilon xv' - \xi) \right] \]

and

\[ G(v, f_2, v') = \frac{1}{\gamma x} \int_{-\infty}^{+\infty} \frac{d\xi}{\sqrt{2\pi}} \exp\left\{ -\frac{\xi^2}{2\gamma^2 x^2} \right\} \left[ 1 - \theta(f_0 - f_2 + \epsilon xv - \tilde{\xi}) \theta(f_0 - f_2 - 2\epsilon f_0 + \tilde{\epsilon} \xi - \xi) \theta(\epsilon xv' - \tilde{\xi}) \right] \]

The calculation of the derivatives of these functions is straightforward. For example, for the derivative $G'_1 \equiv -\frac{1}{\epsilon x} \frac{\partial}{\partial v} G$, we get

\[ G'_1 = \frac{1}{\gamma x} \int_{-\infty}^{+\infty} \frac{d\xi}{\sqrt{2\pi}} \exp\left\{ -\frac{\xi^2}{2\gamma^2 x^2} \right\} \delta(f_0 - f_2 + \epsilon xv - \tilde{\xi}) \theta(f_0 - f_2 - 2\epsilon f_0 + \tilde{\epsilon} \xi - \xi) \theta(\epsilon xv' - \tilde{\xi}). \]
Taking the limit $\epsilon \to 0$, we find
\[
\lim_{\epsilon \to 0} G'_1 = \frac{1}{\gamma x \sqrt{2\pi}} \exp\left\{ -\frac{(f_2 - f_0)^2}{2\gamma^2 x^2} \right\} \theta(f_2 - f_0 + x') \theta(-f_0 - f_2 - xv).
\]

We obtain the rest of the derivatives in a similar way:

\[
\lim_{\epsilon \to 0} G'_2 = \frac{1}{\gamma x \sqrt{2\pi}} \exp\left\{ -\frac{(f_2 - f_0)^2}{2\gamma^2 x^2} \right\} \theta(f_2 - f_0 + x') \theta(f_0 + f_2 + xv).
\]

\[
\lim_{\epsilon \to 0} G'_3 = \frac{1}{\gamma \epsilon \sqrt{2\pi}} \exp\left\{ -\frac{(\epsilon')^2}{2\gamma^2 \epsilon^2} \right\} \theta(f_0 - f_2 - xv).
\]

\[
\lim_{\epsilon \to 0} G''_{12} = -\frac{1}{\gamma \epsilon \sqrt{2\pi}} \exp\left\{ -\frac{(\epsilon')^2}{2\gamma^2 \epsilon^2} \right\} \theta(xv' - x - 2f_0) \delta(f_2 - f_0 + xv')
\]

\[
\lim_{\epsilon \to 0} G''_{13} = -\frac{1}{\gamma \epsilon \sqrt{2\pi}} \exp\left\{ -\frac{(\epsilon')^2}{2\gamma^2 \epsilon^2} \right\} \theta(xv - xv' + 2f_0) \delta(f_2 - f_0 + xv')
\]

\[
\lim_{\epsilon \to 0} G''_{123} = \frac{1}{\gamma \epsilon^2 \sqrt{2\pi}} \exp\left\{ -\frac{(\epsilon')^2}{2\gamma^2 \epsilon^2} \right\} \delta(xv' - x - 2f_0) \delta(f_2 - f_0 + xv')
\]

\[
\lim_{\epsilon \to 0} S'_1 = \frac{\zeta_0}{T \gamma x \sqrt{2\pi}} \int_0^\infty \exp\left\{ -\frac{(y - f_2 + f_0)^2}{2\gamma^2 x^2} \right\} \theta(y - f_2 - f_0 - xv)
\]

\[
\lim_{\epsilon \to 0} S'_2 = \frac{\zeta_0}{T \gamma x \sqrt{2\pi}} \int_0^\infty \exp\left\{ -\frac{(y - f_2 + f_0)^2}{2\gamma^2 x^2} \right\} \theta(f_2 + f_0 + xv - y)
\]

\[
\lim_{\epsilon \to 0} S'_3 = \frac{\zeta_0}{T \gamma x \sqrt{2\pi}} \int_0^\infty \exp\left\{ -\frac{(y + xv')^2}{2\gamma^2 x^2} \right\} \theta(f_0 - f_2 - x' - y) \theta(f_0 - f_2 - xv')
\]

\[
\lim_{\epsilon \to 0} S''_{12} = -\frac{\zeta_0}{T \gamma x \sqrt{2\pi}} \exp\left\{ -\frac{(xv + 2f_0)^2}{2\gamma^2 x^2} \right\} \theta(xv + f_0 + f_2)
\]

\[
\lim_{\epsilon \to 0} S''_{13} = -\frac{\zeta_0}{T \gamma x \sqrt{2\pi}} \exp\left\{ -\frac{(f_2 - f_0)^2}{2\gamma^2 x^2} \right\} \theta(-f_2 - f_0 - xv) \theta(f_0 - f_2 - xv')
\]

\[
\lim_{\epsilon \to 0} S''_{23} = -\frac{\zeta_0}{T \gamma x \sqrt{2\pi}} \exp\left\{ -\frac{(f_2 - f_0)^2}{2\gamma^2 x^2} \right\} \theta(f_2 + f_0 + xv) \theta(f_0 - f_2 - xv')
\]
Velocity distribution functions and intermittency in one-dimensional randomly forced Burgers turbulence

\[ \lim_{\epsilon \to 0} S_{123}^{m} = \frac{\zeta_0}{T^* \gamma^2 \sqrt{2\pi}} \exp\left\{ - \frac{(xv + 2f_0)^2}{2\gamma^2 x^2} \right\} \theta(xv + 2f_0 - xv'). \quad (A.26) \]

We see that in the limit \( \epsilon \to 0 \), according to the above expressions—equations (A.13)–(A.26)—the only non-zero contributions to the function \( P_x(v, v') \) in equations (101)–(102) are given by the two terms \( G_{123}^{m} \propto 1/\epsilon^2 \), equation (A.19), and the product \( G_{3}^{m} S_{12}^{m} \propto 1/\epsilon^2 \), equations (A.15) and (A.23).

References

[1] Burgers J M 1974 *The Nonlinear Diffusion Equation* (Dordrecht: Reidel)
[2] Sinai Y G 1992 *Commun. Math. Phys.* **148** 601
[3] Bouchaud J P, Mezard M and Parisi G 1995 *Phys. Rev. E* **52** 3656
[4] Bec J and Khanin K 2007 *Phys. Rep.* **447** 1
[5] Kolmogorov A 1941 *C. R. Acad. Sci. USSR* **30** 301
[6] Kolmogorov A 1962 *J. Fluid Mech.* **13** 82
[7] Obukhov A M 1962 *J. Fluid Mech.* **13** 77
[8] Frisch U, Sulem P L and Nelkin M 1978 *J. Fluid Mech.* **87** 719
[9] Argoul F *et al* 1989 *Nature* **338** 51
[10] Castaing B, Gagne Y and Hopfield E J 1990 *Physica D* **46** 177
[11] Benzi R *et al* 1993 *Phys. Rev. E* **48** R29
[12] She Z-S and Leveque E 1994 *Phys. Rev. Lett.* **72** 336
[13] Dubrulle B 1994 *Phys. Rev. Lett.* **73** 959
[14] Arneodo A, Bacry E and Muzy J F 1995 *Physica A* **213** 232
[15] Kardar M, Parisi G and Zhang Y-C 1986 *Phys. Rev. Lett.* **56** 889
[16] Halpin-Healy T and Zhang Y-C 1995 *Phys. Rep.* **254** 215
[17] Dotsenko V 2016 *J. Stat. Mech.* **123304
[18] Dotsenko V 2015 *J. Stat. Mech.* P02016
[19] Dotsenko V 2018 Statistical properties of one-dimensional directed polymers in a random potential Order, Disorder and Criticality vol 5 (Singapore: World Scientific)
[20] Lieb E H and Liniger W 1963 *Phys. Rev.* **130** 1605
[21] McGuire J B 1964 *J. Math. Phys.* **5** 622
[22] Yang C N 1968 *Phys. Rev.* **168** 1920
[23] Agoritsas E, Bustingorry S, Lecomte V, Schehr G and Giamarchi T 2012 *Phys. Rev. E* **86** 031144
Agoritsas E, Lecomte V and Giamarchi T 2013 *Phys. Rev. E* **87** 042406
Agoritsas E, Lecomte V and Giamarchi T 2013 *Phys. Rev. E* **87** 062405

https://doi.org/10.1088/1742-5468/aad6c8 28