The magnetic flux dynamics in critical state of one-dimensional discrete superconductor

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We give a theoretical description of avalanche-like dynamics of magnetic flux in the critical state of “hard” type-II superconductors using a model of a one-dimensional multijunction SQUID that well reproduces the main magnetic properties of these objects. We show that the system under consideration demonstrates the self-organized criticality. The avalanches of vortices manifest themselves as jumps of the total magnetic flux in the sample. The sizes of these jumps have a power-law distribution. Our results are in qualitative agreement with experiments.

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I. INTRODUCTION

About three decades ago Charles P. Bean proposed a model describing a critical state in “hard” type-II superconductors. In the frame of this model, the critical state, arising in a superconductor as a result of interplay of magnetic and pinning forces, is a non-equilibrium state, in which the density of Abrikosov vortices varies linearly with the distance from the sample surface. Later Pierre G. de Gennes considered an interesting analogy between the critical behaviour of magnetic flux in superconductors and evolution of a sand pile. De Gennes compared the magnetic flux dynamics in the critical state with the process that is observed in an over-critical sand pile where sand slides from the top of the pile forming avalanches. According to this brilliant concept, when the superconductor becomes over-critical, the vortices begin to move in form of avalanches. Each avalanche includes a simultaneous motion of a large number of vortices.

Although this picture of the magnetic dynamics in superconductors was formulated in 60-th, the experimental techniques, allowing observation of vortex avalanches and studying their size distribution, was developed only in the last decade (see, for example, Ref. 3). These experiments demonstrated that in certain conditions slow changes of an applied magnetic field result in an avalanche-like redistribution of vortices. It turns out that the sizes of avalanches have a power-law distribution. Such a distribution is, as is well known, a characteristic feature of systems with a self-organized criticality (SOC).

The concept of SOC was formulated in 1987 in order to explain a behavior of giant dissipative dynamical systems consisting of a large number of interacting elements. According to the main principles of this concept, such systems are able to accumulate small external perturbations and evolve into a self-reproducing critical state. This state is an ensemble of various metastable states. The critical system migrates from one metastable state to the other by means of avalanches. The avalanches are initiated by small external perturbations and may have various sizes. The distribution of avalanche sizes is a power-law function with an exponent close to 1. The paradigm of SOC is a sandpile. This is why the corresponding mathematical model is called the sandpile model.

Recently, considering the similarity between the critical magnetic dynamics in superconductors and SOC, we showed that employing of dynamical equations provides a promising way to correlate these two phenomena. In these works, we used the fact that the main features in the behavior of a “hard” type-II superconductor may well be reproduced by considering a so called discrete superconductor, i.e., a lattice of Josephson junctions (multijunction SQUID). An additional advantage of this approach is that the corresponding equations are not difficult for the analysis.

It is known that magnetic properties of a lattice of Josephson junctions may be characterized by a parameter $V \sim j_c a^2/\Phi_0$ ($j_c$ is the critical current density of the junction, $\Phi_0$ is the magnetic flux quantum, and $a$ is the interjunction distance). In the case of $V \ll 1$, the system can be considered as a single sandwich-like Josephson junction without pinning. If the condition $V \ll 1$ is not satisfied, a multijunction SQUID, due to its discreteness, is able to pin magnetic flux. In the case of $V \gg 1$, we have a system with strong pinning.

It was also shown theoretically and by computer simulations that the critical state of two-dimensional (2D) and one-dimensional (1D) multijunction SQUID’s with $V \gg 1$ is self-organized. Avalanches in such systems manifest themselves as pulses of voltage across the junctions. It was demonstrated that the total voltage integrated over the time of the avalanche is an analog of the avalanche size in the sandpile model. For some way of perturbation, this quantity has a power-law distribution. It was also proven that for $V \gg 1$ the dynamical equations for the superconductor are equivalent to algorithms describing the sandpile model.

The quantities usually measured in the sandpile experiment are the mass of the pile and its variations. The direct analog for the multijunction SQUID is the total magnetic flux $\Phi$ in the sample and its fluctuations. This characteristic was studied experimentally, for instance, in Refs. 4 and 6. This is why, in this work, we concentrate on calculation of $\Phi$.

The aim of this work is a theoretical investigation and a
computer simulation of the magnetic flux dynamics and the avalanche statistics in the critical state of a multijunction SQUID. We use a model of a 1D multijunction SQUID with intrinsic spatial randomness, which was introduced in Ref. 11. This model takes into account that real superconductors are disordered systems. A spatial randomness, introduced in our model, is an equivalent of the existence of randomly distributed pinning centers. We consider here not only the case of \( V > 1 \), studied earlier, but also \( V \lesssim 1 \). We show that for all values of \( V \) and for a fixed degree of disorder, the critical state of the system may be represented as a set of metastable states transforming to each other due to avalanches. An avalanche in such a system results in penetration of magnetic flux into superconducting sample and its redistribution between the system cells. We demonstrate that for all considered values of \( V \) the probability density of magnetic flux jumps has a power-law distribution. Thus, using the simplest model of a superconductor, we obtain the results that are in qualitative agreements with experiments.

The paper is organized as follows. The second section is devoted to description of our model of 1D multijunction SQUID with intrinsic spatial randomness. We analyze the dynamical equations and demonstrate that disorder of the system is a key factor for realization of the complex critical state. In the third section, we discuss the structure of the critical state for various values of the SQUID-parameter \( V \). Single avalanches are described in the forth section. In the fifth section, we analyze the statistics of magnetic flux jumps and discuss the possibility of the realization of the self-organized critical state in the system under consideration. The results of calculations are compared with experiments. In Conclusions we summarize the main results of the work.

II. BASIC EQUATIONS

The one-dimensional multijunction SQUID, which we consider here, may be represented as two infinitely long Josephson junctions, as is shown in Fig. 1. All junctions have the same length \( l \) along the \( x \)-axis. The distance between the \( i \)-th and the \( i + 1 \)-th junctions we denote as a random variable \( b_i \). The system is placed in a slowly increasing external magnetic field \( H_{\text{ext}} \), aligned along the \( y \)-axis. Using the resistive model of a Josephson junction and neglecting by thermal fluctuations, we can write the current density \( j_i \) as:

\[
  j_i = j_c \sin \varphi_i + \frac{\Phi_0}{2\pi \rho} \frac{\partial \varphi_i}{\partial t},
\]

where \( j_c \) is the critical current density, \( \varphi_i \) is the gauge-invariant phase difference across the \( i \)-th junction, \( \rho \) is the junction resistance per unit area. The current density \( j_i \) is connected with the magnetic field \( H_i \) in the neighboring cells by the following expression (we numerate the

\[
  4\pi j_i = \frac{H_i - H_{i-1}}{l} = \left( \frac{\Phi_i}{S_i} - \frac{\Phi_{i-1}}{S_{i-1}} \right) \frac{1}{l},
\]

where \( \Phi_i = H_i S_i \) is the magnetic flux in the \( i \)-th cell, \( S_i = 2\lambda b_i \) is the cell area, \( \lambda \) is the magnetic field penetration depth. Taking into account that

\[
  \Phi_i(t) = \frac{\Phi_0}{2\pi} [\varphi_{i+1}(t) - \varphi_i(t)],
\]

we obtain the system of differential equations for the gauge-invariant phase differences:

\[
  V \sin \varphi_i + \tau \frac{\partial \varphi_i}{\partial t} = J_i \varphi_{i+1} - (J_i + J_{i-1}) \varphi_i + J_{i-1} \varphi_{i-1}; \quad i \neq 1, N;
\]

\[
  V \sin \varphi_1 + \tau \frac{\partial \varphi_1}{\partial t} = [J_1 \varphi_2 - J_1 \varphi_1] - 2\pi h_{\text{ext}};
\]

\[
  V \sin \varphi_N + \tau \frac{\partial \varphi_N}{\partial t} = [-J_N \varphi_N + J_{N-1} \varphi_{N-1}] + 2\pi h_{\text{ext}};
\]

\[
  V = \frac{16\pi^2 a l \lambda j_c}{\Phi_0} \quad \tau = \frac{8 \pi a l \lambda}{\Phi_0} \quad J_i = \frac{a}{b_i} \quad h_{\text{ext}} = \frac{2\lambda a}{\Phi_0} H_{\text{ext}},
\]

where \( a = \langle b_i \rangle \) is the average interjunction distance.

In order to take into account a spatial disorder of real physical systems, we consider the distances between the junctions as random. As may be seen from Eqs. (4), in our model this randomness is equivalent to a scatter of the coefficients \( J_i \). In the following, we assume that the values of these coefficients are distributed uniformly in the interval \([1, 1 + \Delta J]\).

As was stated above, there must be a large number of metastable states in the system in order to expect a self-organized behavior. In our case, such a situation can be realized only if \( \Delta J \neq 0 \). This is illustrated in Fig. 2 which shows the dimensionless current.
The dependence of the dimensionless current $J$ (upper panel) and $\Delta z$ (lower panel) scattering of the coefficients $J_i$, which remain unchanged during the simulation process. Starting from the state with $\varphi_i = 0$, we perturb the system by increasing the external field from $h_{ext}$ to $h_{ext} + \Delta h_{ext}$. In our simulations, we use $\Delta h_{ext} = 1$ for $V = 40$ and $\Delta h_{ext} = 0.1$ for small and transient values of $V$. Then we allow the system to relax and, as was mentioned above, we assume that the value of $h_{ext}$ is constant during the relaxation process. We take that the system has already reached its metastable state if $d\varphi_i/dt < 10^{-7}$ for all $i$. When the dynamics stops we perturb the system again and so on.

First, we analyze the distribution of magnetic field inside our SQUID for various values of $V$. The magnitude of the dimensionless magnetic field in the $i$-th cell, measured at the end of $n$-th avalanche, may be calculated as:

$$h_i^{(n)} = \frac{2\lambda_0 \Phi_i^{(n)}}{S \Phi_0}.$$

Note that for all values of $V$ and $\Delta J$ the system demonstrates irreversible magnetic behavior. Remanent magnetization becomes zero only for negligible value of the parameter $V$ ($V = 0.06$) and for $\Delta J_i = 0$.

Fig. 3 shows the profiles of magnetic field inside the SQUID for one of metastable states for two different values of $V$ and $\Delta J = 0.1$. We see that for $V = 40$, when every cell acts as pinning center, the profile is similar to the result of Bean’s model. In the case of $V = 0.3$, the pinning centers are represented by groups of cells and a number of peaks may be seen. We note that the peak amplitudes are different for different metastable states.

Now we consider the dynamics of the total magnetic flux in the sample and its dependence on the external magnetic field $h_{ext}$. The total magnetic flux may be calculated as

$$\Phi(n) = \sum_{i=1}^{N-1} \frac{\Phi_0}{2\pi} \left( \varphi_i^{(n)} - \varphi_1^{(n)} \right).$$

Here $\varphi_i^{(n)}$ denotes the value of the phase for the final moment of $n$-th avalanche. The variation of the total magnetic flux due to the $n$ avalanche

$$\Delta \Phi^{(n)} = \Phi^{(n)} - \Phi^{(n-1)}.$$
FIG. 3: The magnetic field profiles $h_i^{(n)}$ inside the SQUID for $V = 40$ (upper panel) and $V = 0.3$ (lower panel), $N = 257$.

FIG. 4: The total magnetic flux in the SQUID (a–b) and the magnitude of the flux jumps (c–d) as functions of the external magnetic field. The SQUID size $N = 65$ and $\Delta J = 0.1$.

FIG. 5: $\Delta \Phi$ as a function of time from Ref. 5. The data presented in this figure come from series of nine experiments and different experimental runs are separated by vertical arrows.

The magnetic flux jumps was also observed in an artificial 2D lattice of Josephson junctions. In these experiments, the external magnetic field was changed continuously but slowly. This allows to neglect the change of the magnetic field during the relaxation process. Fig. 6 shows the hysteresis loops of the total magnetic moment of the lattice from Ref. 6. It may be seen that the avalanches, which are represented by jumps of the magnetic moment, have different sizes and include hundreds of magnetic flux quanta. The upper trace shows several superposed hysteresis loops for the same experimental conditions. Due to this superposition, we can see the randomness in jumps. The random jumps with different sizes may be considered as a manifestation of SOC. As may be seen comparing Figs. 4 and 6, the results of our simulation are in qualitative agreement with experiments.

IV. THE AVALANCHE STRUCTURE IN THE CRITICAL STATE

As was demonstrated above, an increase of the external magnetic field can launch an avalanche. In this way the system migrates form one metastable state to the other. An avalanche in our multijunction SQUID is a simultaneous penetration of a considerable number of vortices in the sample and a redistribution of the corresponding magnetic flux inside the system. As a result of such a process, the system reaches a next metastable state, the
total magnetic flux increases and the values of the magnetic flux in some cells change. In this section, we consider the process of the avalanche development.

At every moment $t_n$ during the $n$-th avalanche we calculate the magnetic flux $\Phi_{i}^{(n)}(t_n)$ using the expression $\Phi_{i}^{(n)}(t_n) = [\Phi_{i}^{(n)}(t_{0_n}) - \Phi_{i}^{(n)}(t_{0_n})]$ where $t_{0_n}$ is the initial moment of the avalanche. We emphasize that all the results in this section are related to a single avalanche and the external magnetic field is assumed to be constant.

Fig. 7 demonstrates the process of penetration of magnetic flux inside the SQUID and its redistribution between the cells during the avalanche. The results presented in Fig. (a) correspond to a regular SQUID with $\Delta J = 0$. The figure shows the magnetic flux distributions corresponding to several different moments in time during the avalanche. We see that at the beginning of the avalanche the magnetic flux penetrates into the boundary cells and only with time the penetration reaches the central part of the SQUID. The analogous results for a disordered system are presented in Fig. (b). In this case, the magnetic flux demonstrates the same dynamics as for a regular system. However, due to the randomness the magnetic flux profile is much less regular than in the previous case.

If $V \lesssim 1$, each flux quantum may be considered as distributed between several neighboring cells. In some cases, motion of such extended vortices results in negative values of $\Delta \Phi$, as may be seen in Fig. This is true for both regular and disordered SQUID’s.

V. THE AVALANCHE STATISTICS IN THE CRITICAL STATE AND THE SELF-ORGANIZED CRITICALITY

In experimental works on the critical state of superconductors, the authors often points out on the similarity of the system behavior and the phenomenon of self-organized criticality. Besides the avalanche-like dynamics, it is a power-law distribution of magnetic flux jumps. For instance, Fig. shows the distribution of avalanche-like dynamics of magnetic flux in a NbTi tube was studied. Three distribution functions correspond to three fixed values of the external magnetic field. The external magnetic field varies in the interval of 30 Oe centered at one of the fixed values with the rate of 5 Oe/s.

Fig. demonstrates the results of our calculations for the probability densities of jumps of the total magnetic flux for multijunction SQUID for three different values.
of the parameter $V$. As was shown earlier\,\,\footnote{\textsuperscript{11}} the case of $V = 40$ for this degree of disorder demonstrates the self-organized behavior. As may be seen in Fig. 10 there are rather extended intervals of power-law distributions for all considered values of $V$.

At the same time, from the point of view of classical interpretation of SOC the magnitude of flux jumps is not a direct analog of the avalanche size.\,\footnote{\textsuperscript{12}} However, just as the classical self-organized critical state, the critical state in real superconductors, as well as in our model, is self-reproducing and consists of a large number of metastable states that transform to each other by means of avalanches. Thus, we can conclude that here we are dealing with a more general type of a self-organized critical state, in which the total magnetic flux plays a role of the main characteristic of the system.

\section*{VI. CONCLUSIONS}

Based on dynamical equations describing the simplest model of a discrete superconductor (1D disordered multijunction SQUID), we present a theoretical description of the avalanche-like dynamics of the magnetic flux in "hard" type-II superconductors. Different values of the SQUID-parameter $V \sim J_c a^3/\Phi_0$ are considered. For all values of $V$, including $V \lesssim 1$, the critical state in the multijunction SQUID can be considered as a generalized type of a self-organized critical state. In contrast to the classical definition of SOC,\,\footnote{\textsuperscript{13}} the main characteristic of generalized critical state is a size of magnetic flux jumps. This quantity demonstrates a power-law distribution if some degree of disorder is introduced into the system. Our results are in qualitative agreement with experiments.
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