Generalizations of the Jensen functional involving diamond integrals via Abel–Gontscharoff interpolation

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Abstract
In this paper we obtain several refinements of the Jensen inequality on time scales by generalizing Jensen’s functional for \( n \)-convex functions. We also investigate the bounds for the identities related to the new improvements obtained.

Keywords: Time-scales calculus; Abel–Gontscharoff’s interpolation polynomial; Jensen’s inequality; Green function

1 Introduction
Time-scales theory is a well-established theory, where a time scale is a nonempty closed subset of the real numbers. This theory is a unification of discrete and continuous analysis. It was first initiated by Stefen Hilger in 1988 and then several books and research papers appeared, e.g., see [6, 7] for the basic calculus on time scales. Delta and nabla integrals are the basic integrals on time scales. Then, diamond-\( \alpha \) (\( \alpha \in (0, 1) \)) integrals were introduced [18] as a convex combination of delta and nabla integrals. In 2015, diamond integrals were introduced as a generalization of all time-scales integrals including delta, nabla and diamond-\( \alpha \) integrals, see [8]. Therefore throughout this paper we write our results for diamond integrals, but these results are also satisfied for delta, nabla and diamond-\( \alpha \) integrals. Also, note that our results hold for sums and integrals since sums and integrals are specific examples of time-scales integrals.

As we obtain our results for diamond integrals, we assume throughout in this paper that the basic notions of the time scales are understood. Consider the forward jump operator \( \sigma : T \rightarrow \mathbb{R} \) and the backward jump operator \( \rho : T \rightarrow \mathbb{R} \). Then, the gamma function, \( \gamma : T \rightarrow \mathbb{R} \), is defined by

\[
\gamma(b) = \lim_{s \rightarrow b} \frac{\sigma(b) - s}{\sigma(b) + 2b - 2s - \rho(b)}.
\]

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This function is used to define the diamond integrals. Clearly,
\[
\gamma(b) = \begin{cases} \\
\frac{1}{2}, & \text{if } b \text{ is dense;} \\
\frac{\sigma(b) - b}{\tau(b) - \rho(b)}, & \text{if } b \text{ is not dense.}
\end{cases}
\]
and \(0 \leq \gamma(b) \leq 1\).

1.1 Diamond integral

Let \(e, f \in \mathbb{T} (e < f)\). For the function, \(\mathfrak{J} : \mathbb{T} \to \mathbb{R}\), the diamond integral (or \(\Diamond\)-integral) of \(\mathfrak{J}\) from \(e\) to \(f\) (or on \([e, f]_\mathbb{T}\)) is given by
\[
\int_e^f \mathfrak{J}(b) \Diamond db = \int_e^f \gamma(b) \mathfrak{J}(b) \Delta b + \int_e^f (1 - \gamma(b)) \mathfrak{J}(b) \nabla b
\]
provided \(\gamma \mathfrak{J}\) is \(\Delta\)-integrable and \((1 - \gamma) \mathfrak{J}\) is \(\nabla\)-integrable on \([e, f]_\mathbb{T}\). If \(\mathfrak{J}\) is \(\Diamond\)-integrable \(\forall\) \(e, f \in \mathbb{T}\), then we say that it is diamond integrable (or \(\Diamond\)-integrable) on \([e, f]_\mathbb{T}\). Note that if \(\mathbb{T} = \mathbb{R}\), then
\[
\int_e^f \mathfrak{J}(b) \Diamond db = \int_e^f \mathfrak{J}(b) db;
\]
if \(\mathbb{T} = \mathbb{Z}\), then
\[
\int_e^f \mathfrak{J}(b) \Diamond db = \frac{1}{2} \left( \sum_{b=e}^{f-1} \mathfrak{J}(b) + \sum_{b=e+1}^f \mathfrak{J}(b) \right);
\]
if \(\mathbb{T} = h\mathbb{Z}\), where \(h > 0\), then
\[
\int_e^f \mathfrak{J}(b) \Diamond db = \frac{h}{2} \left( \alpha \sum_{b=e/h}^{f/h-1} \mathfrak{J}(bh) + (1 - \alpha) \sum_{b=e/h+1}^{f/h} \mathfrak{J}(bh) \right);
\]
if \(\mathbb{T} = q\mathbb{Z}\), where \(q > 1\), then
\[
\int_e^f \mathfrak{J}(b) \Diamond db = \frac{q-1}{q+1} \left( \sum_{b=\log_q(e)}^{\log_q(f)-1} q^b \mathfrak{J}(q^b) + \sum_{b=\log_q(e)+1}^{\log_q(f)} q^b \mathfrak{J}(q^b) \right).
\]

Throughout mathematics and specifically in mathematical analysis Jensen’s inequality for convex functions has great importance [3, 12–15]. The number of research papers where Jensen’s inequality is used is not countable. Several generalizations and refinements of Jensen’s inequality for time-scales integrals can also be found in the literature (see [1, 5, 6, 11, 16, 18]). For diamond integrals, Jensen’s inequality is generalized in 2017, see [4].

**Theorem 1.1** (Jensen’s inequality) Let \(e, f \in \mathbb{T}\) with \(e \leq f\), \(\mathfrak{J} \in C([e, f]_\mathbb{T}, \mathbb{R})\) and \(g \in C([e, f]_\mathbb{T}, E)\). Also, assume \(\phi \in C(E, \mathbb{R})\) is a convex function satisfying \(\int_e^f \mathfrak{J}(b) \Diamond db \neq 0\), where \(E = [o_1, o_2]\) such that \(o_1 = \min_{b \in [e, f]_\mathbb{T}} g(b)\), \(o_2 = \max_{b \in [e, f]_\mathbb{T}} g(b)\). Then, we have
\[
\phi\left(\frac{\int_e^f |\mathfrak{J}(b)| g(b) \Diamond db}{\int_e^f |\mathfrak{J}(b)| \Diamond db}\right) \leq \frac{\int_e^f |\mathfrak{J}(b)| \phi(g(b)) \Diamond db}{\int_e^f |\mathfrak{J}(b)| \Diamond db}. 
\]
From Jensen's inequality, we obtain Jensen's functional:

\[ J(\phi) = \frac{\int_{a}^{b} |\phi| g(b) \phi(b) \, db}{\int_{a}^{b} |\phi| \, db} - \phi \left( \frac{\int_{a}^{b} |\phi| g(b) \phi(b) \, db}{\int_{a}^{b} |\phi| \, db} \right). \]  

(1)

Remark 1.2 For \( T = \mathbb{R} \), Jensen's functional becomes

\[ J(\phi) = \frac{\int_{a}^{b} |\phi| g(b) \, db}{\int_{a}^{b} |\phi| \, db} - \phi \left( \frac{\int_{a}^{b} |\phi| g(b) \, db}{\int_{a}^{b} |\phi| \, db} \right). \]  

(2)

If \( T = \mathbb{Z} \), then

\[ J(\phi) = \frac{\sum_{b \in \mathbb{Z}} |\phi| g(b) + \sum_{b \in \mathbb{Z}} |\phi| g(b)}{\sum_{b \in \mathbb{Z}} |\phi| + \sum_{b \in \mathbb{Z}} |\phi|} - \phi \left( \frac{\sum_{b \in \mathbb{Z}} |\phi| g(b) + \sum_{b \in \mathbb{Z}} |\phi| g(b)}{\sum_{b \in \mathbb{Z}} |\phi| + \sum_{b \in \mathbb{Z}} |\phi|} \right). \]  

(3)

If \( T = \mathbb{Z}/h \mathbb{Z} \), where \( h > 0 \), then

\[ J(\phi) = \frac{\alpha \sum_{b \in \mathbb{Z}/h \mathbb{Z}} |\phi| g(b) + \sum_{b \in \mathbb{Z}/h \mathbb{Z}} |\phi| g(b)}{\sum_{b \in \mathbb{Z}/h \mathbb{Z}} |\phi| + \sum_{b \in \mathbb{Z}/h \mathbb{Z}} |\phi|} - \phi \left( \frac{\alpha \sum_{b \in \mathbb{Z}/h \mathbb{Z}} |\phi| g(b) + \sum_{b \in \mathbb{Z}/h \mathbb{Z}} |\phi| g(b)}{\sum_{b \in \mathbb{Z}/h \mathbb{Z}} |\phi| + \sum_{b \in \mathbb{Z}/h \mathbb{Z}} |\phi|} \right). \]  

(4)

If \( T = q^{\mathbb{Z}_{0}} \), where \( q > 1 \), then

\[ J(\phi) = \frac{\sum_{b \in q^{\mathbb{Z}_{0}}} |\phi| g(b) + \sum_{b \in q^{\mathbb{Z}_{0}}} |\phi| g(b)}{\sum_{b \in q^{\mathbb{Z}_{0}}} |\phi| + \sum_{b \in q^{\mathbb{Z}_{0}}} |\phi|} - \phi \left( \frac{\sum_{b \in q^{\mathbb{Z}_{0}}} |\phi| g(b) + \sum_{b \in q^{\mathbb{Z}_{0}}} |\phi| g(b)}{\sum_{b \in q^{\mathbb{Z}_{0}}} |\phi| + \sum_{b \in q^{\mathbb{Z}_{0}}} |\phi|} \right). \]  

(5)

Remark 1.3 From Jensen's inequality it is clear that \( J(\phi) \geq 0 \) for the family of convex functions and \( J(\phi) = 0 \) for \( \phi(b) = b \) or when \( \phi \) is a constant function.

A function \( \phi : [e,f] \rightarrow \mathbb{R} \) is called \( n \)-convex \( (n \geq 0) \) (see [17]), if for all choices of \( (n + 1) \) different points \( b_{0}, \ldots, b_{n} \in [e,f] \), we have the \( n \)th-order divided difference nonnegative, i.e., \([b_{0}, \ldots, b_{n}; \phi] \geq 0\).

2 Generalization of Jensen's inequality

Let \( n, u \in \mathbb{N}, n \geq 2, 0 \leq u \leq n - 1, \phi \in C^{u}([a_{1}, a_{2}]) \) and \( G_{n}(r,q) \) be the Green function defined as,

\[ G_{n}(r,q) = \begin{cases} 
\frac{1}{(n-1)!} \left( \sum_{k=1}^{n-u} (r - a_{1})^{k} (a_{1} - q)^{n-k-1}, & 0 \leq q \leq r, \\
- \sum_{k=1}^{n-u} (r - a_{1})^{k} (a_{1} - q)^{n-k-1}, & r \leq q \leq a_{2}. 
\end{cases} \]
Then, for $0_1 \leq r, q \leq 0_2$, the following inequalities hold (see [2])

\[ (-1)^{n-u-1} \frac{\partial^j G_n(r, q)}{\partial r^j} \geq 0, \quad 0 \leq j \leq u, \]
\[ (-1)^{n-u} \frac{\partial^j G_n(r, q)}{\partial r^j} \geq 0, \quad u + 1 \leq j \leq n - 1. \]  

Assume

\[
T_{n-1}(0_1, 0_2, r; \phi) = \sum_{j=0}^{u} \frac{(r - 0_1)^j}{j!} \phi^{(j)}(0_1) + \sum_{d=0}^{n-u-2} \sum_{j=0}^{d} \frac{(-1)^{d+j} (0_1 - 0_2)^{d-j} \phi^{(u+1+j)}(0_2)}{(u + 1 + j)(d-j)!}.
\]

is the Abel–Gontscharoff interpolation polynomial of degree $n - 1$, then we have

\[ \phi(r) = T_{n-1}(0_1, 0_2, r; \phi) + R(r; \phi) \quad (7) \]

[2] (see also [10, 19–22]), where the remainder is given by

\[ R(r; \phi) = \int_{0_1}^{0_2} G_n(r, q)\phi^n(q) \, dq. \]

By using Abel–Gontscharoff’s interpolating polynomial, we establish the following identity.

**Theorem 2.1** Let $n, u \in \mathbb{N}, n \geq 2, 0 \leq u \leq n - 1$. If $g \in C([e, f]_T, E)$, $\phi \in C^n(E, \mathbb{R})$ is a convex function, and $3 \in C([e, f]_T, \mathbb{R})$ such that $\int_{e}^{f} 3(b) \diamond b > 0$, then we have

\[ J(\phi(g)) = \sum_{j=2}^{u} \frac{\phi^{(j)}(0_1)}{j!} f(g(b) - 0_1)^j 
\]

\[ + \sum_{d=0}^{n-u-2} \sum_{j=0}^{d} \frac{(-1)^{d+j} (0_2 - 0_1)^{d-j} \phi^{(u+1+j)}(0_2)}{(u + 1 + j)(d-j)!} \]

\[ \times f(g(b) - 0_1)^{(u+1+j)} \]

\[ + \int_{0_1}^{0_2} J(G_n(g(b), q))\phi^n(q) \, dq. \quad (8) \]

where

\[ J(G_n(g(b), q)) = \frac{\int_{e}^{f} |3(b)| G_n(g(b), q) \diamond b}{\int_{e}^{f} |3(b)| \diamond b} - G_n\left( \frac{\int_{e}^{f} |3(b)| g(b) \diamond b}{\int_{e}^{f} |3(b)| \diamond b} \right). \]
Proof. By replacing $r$ with $g$ in equation (7) we obtain

\[
\phi(g(b)) = \sum_{j=2}^{u} \frac{(g(b) - o_1)^j}{j!} \phi^j(o_1)
\]

\[
+ \sum_{d=0}^{n-u-2} \sum_{j=0}^{d} \frac{(g(b) - o_1)^{u+1+j}(o_1 - o_2)^{d-j}}{(u+1+j)!(d-j)!} \phi^{(u+1+j)}(o_2)
\]

\[
+ \int_{o_1}^{o_2} (G_n(g(b), q)) \phi^n(q) dq.
\]

Now, (8) is followed by substituting (9) into (1) and using the linearity of $J(\cdot)$. □

Remark 2.2 For different time scales, special cases of the inequality (8) can be deduced. For example, when $T = \mathbb{R}$ the inequality (8) holds with Jensen’s functional (2). Similarly, for $T = \mathbb{Z}$ the inequality (8) holds with Jensen’s functional (3); for $T = h\mathbb{Z}$, where $h > 0$, the inequality (8) holds with Jensen’s functional (4); for $T = q^\mathbb{N}$, where $q > 1$, the inequality (8) holds with Jensen’s functional (5).

The next theorem gives the generalization of Jensen’s inequality for $n$-convex functions.

Theorem 2.3 Assume that all the conditions of Theorem 2.1 are satisfied and

\[ J(G_n(g(b), q)) \geq 0, \quad \text{for all } q \in [o_1, o_2] \text{ and } n \geq 2. \]

If $\phi$ is $n$-convex such that $\phi^{n-1}$ is absolutely continuous, then

\[ J(\phi(g)) \geq \sum_{j=2}^{u} \frac{\phi^{(j)}(o_1)}{j!} J(g(b) - o_1)^j \]

\[
+ \sum_{d=0}^{n-u-2} \sum_{j=0}^{d} \frac{(-1)^{d-j}(o_2 - o_1)^{d-j} \phi^{(u+1+j)}(o_2)}{(u+1+j)!(d-j)!} \int_{o_1}^{o_2} (G_n(g(b), q)) \phi^n(q) dq.
\]

Proof. Since $\phi^{n-1}$ is absolutely continuous on $[o_1, o_2]$, $\phi^n$ exists almost everywhere. Also, $\phi$ is $n$-convex, therefore $\phi^n(b) \geq 0$ for all $b \in [o_1, o_2]$ (see [17, page 16]). Hence, the inequality (10) follows from (8). □

Theorem 2.4 Suppose that all the suppositions of Theorem 2.1 are satisfied such that $\phi$ is $n$-convex.

(i) If $u$ is odd and $n$ is even, or $n$ is odd and $u$ is even, then the inequality (10) is satisfied.

(ii) If the inequality (10) holds and the function

\[ V(g(b)) = \sum_{j=0}^{u} \frac{(g(b) - o_1)^j}{j!} \phi^j(o_1) \]

\[
+ \sum_{d=0}^{n-u-2} \sum_{j=0}^{d} \frac{(g(b) - o_1)^{u+1+j}(o_1 - o_2)^{d-j}}{(u+1+j)!(d-j)!} \phi^{(u+1+j)}(o_2)
\]

is convex, then the right-hand side of (10) is nonnegative and we have $J(\phi(g)) \geq 0$. 

Proof

(i) By using (6) for $a_1 \leq r, q \leq a_2$, we have

$$(-1)^{n-u-1} \frac{\partial^2 G_n(r, q)}{\partial r^2} \geq 0.$$ 

Clearly, $\frac{\partial^2 G_n(r, q)}{\partial r^2} \geq 0$ if $u$ is odd and $n$ is even, or $n$ is odd and $u$ is even. In this case, $G_n$ is convex with respect to the first variable and hence the inequality (11) is followed by Theorem 2.1.

(ii) Since $J(\phi)$ is linear, we can restate the right-hand side of (10) as $J(V(g(b)))$ and hence by Remark 1.3 we obtain the nonnegativity of the right-hand side of (10).

In order to obtain more generalizations of Jensen’s inequality, we also use the following Green function $G : [a_1, d_2] \times [a_1, a_2] \to \mathbb{R}$ such that $a_1, a_2 \in \mathbb{R}, a_1 \neq a_2$.

$$G(q, r) = \begin{cases} (q - a_2)(r - a_1) - a_2 - a_1, & a_1 \leq r \leq q; \\ (r - a_2)(q - a_1) - a_2 - a_1, & q \leq r \leq a_2. \end{cases}$$

The function $G$ has continuity and convexity with respect to $r$ and $q$. It is well known that (see [17]) for any convex function $\phi \in C^2([a_1, a_2])$, we have

$$\phi(b) = \frac{a_2 - b}{a_2 - a_1} \phi(a_1) + \frac{b - a_1}{a_2 - a_1} \phi(a_2) + \int_{a_1}^{a_2} G(b, r) \phi''(r) dr. \quad (12)$$

**Theorem 2.5** Let $n, u \in \mathbb{N}, n \geq 4, 0 \leq u \leq n - 1$. If $g \in C([e, f] \cap E), \phi \in C^n(E, \mathbb{R})$ is a convex function, and $\exists \in C([e, f] \cap E)$ such that $\int_{e}^{f} \exists(b) \phi(b) > 0$, then we have

$$J(\phi(g)) = \sum_{\delta = 0}^{u} \frac{\phi^{(\delta u)}(a_1)}{\delta !} \int_{a_1}^{a_2} J(G(g(b), r))(r - a_1)^{\delta} dr$$

$$+ \sum_{d = 0}^{n-u-4} \left( \sum_{\delta = 0}^{d} \frac{(u+1)!(d-\delta)!}{(u+1+\delta)!(d-\delta)!} \right) \int_{a_1}^{a_2} J(G(g(b), r))(r - a_1)^{u+1+\delta} dr$$

$$\times \int_{a_1}^{a_2} J(G(g(b), r))(r - a_1)^{u+1+\delta} dr + \int_{a_1}^{a_2} \int_{a_1}^{a_2} J(G(g(b), r))G_{n-2}(r, q)\phi''(q) dq dr. \quad (14)$$

**Proof** By using (12) and the linearity of $J$ we obtain

$$J(\phi(g)) = \int_{a_1}^{a_2} J(G(g(b), r))\phi''(r) dr. \quad (15)$$
From (7), $\phi''(r)$ becomes

$$
\phi''(r) = \sum_{j=0}^{u} \frac{(r - o_1)^j}{j!} \phi^{(j+2)}(o_1)
$$

$$
+ \sum_{d=0}^{n-u-4} \left[ \sum_{j=0}^{d} \frac{(r - o_1)^{u+1+j}(o_1 - o_2)^{d-j}}{(u + 1 + j)! (d - j)!} \phi^{(u+3+d)}(o_2) \right]
$$

$$
+ \int_{\sigma_1}^{\sigma_2} G_{n-2}(r, q) \phi''(q) \, dq.
$$

(16)

Using (16) in (15), we obtain (13).

\[ \square \]

**Theorem 2.6** Let

$$
\int_{\sigma_1}^{\sigma_2} f(G(g(b), r))(G_{n-2}(r, q)) \, dr \geq 0, \quad \forall q \in [o_1, o_2],
$$

with the assumptions of Theorem 2.5. If $\phi$ is $n$-convex such that $\phi^{n-1}$ is absolutely continuous, then

$$
f(\phi(g)) \geq \sum_{j=0}^{u} \frac{\phi^{(j+2)}(o_1)}{j!} \int_{\sigma_1}^{\sigma_2} f(G(g(b), r))(r - o_1)^j \, dr
$$

$$
+ \sum_{d=0}^{n-u-4} \left[ \sum_{j=0}^{d} \frac{(-1)^{d-j}(o_2 - o_1)^{d-j} \phi^{(u+3+d)}(o_2)}{(u + 1 + j)! (d - j)!} \right]
$$

$$
\times \int_{\sigma_1}^{\sigma_2} f(G(g(b), r))(r - o_1)^{u+1+j} \, dr.
$$

**Proof** We can prove this theorem like Theorem 2.3, except here we use Theorem 2.5 instead of Theorem 2.1.

\[ \square \]

**Theorem 2.7** Suppose that all the suppositions of Theorem 2.1 are satisfied such that $\phi$ is $n$-convex.

(i) If $(n = odd, l = even)$ or $(l = odd, n = even)$, then

$$
f(\phi) \geq \sum_{j=0}^{u} \frac{\phi^{(j+2)}(o_1)}{j!} \int_{\sigma_1}^{\sigma_2} f(G(g(b), r))(r - o_1)^j \, dr
$$

$$
+ \sum_{d=0}^{n-u-4} \left[ \sum_{j=0}^{d} \frac{(-1)^{d-j}(o_2 - o_1)^{d-j} \phi^{(u+3+d)}(o_2)}{(u + 1 + j)! (d - j)!} \right]
$$

$$
\times \int_{\sigma_1}^{\sigma_2} f(G(g(b), r))(r - o_1)^{u+1+j} \, dr.
$$

(17)

Moreover, suppose that $\phi^{(j+2)}(o_1) \geq 0$ if $j = 0, 1, 2, \ldots, u$ and $\phi^{(u+3+d)}(o_2) \geq 0$ if $d - j$ is odd, and $\phi^{(u+3+d)}(o_2) \leq 0$ if $d - j$ is odd, for $j = 0, \ldots, d$ and $d = 0, \ldots, n - u - 4$.

Then, the right-hand side of the inequality (17) becomes nonnegative.

(ii) If both $n$ and $l$ are odd or even simultaneously, then the reverse of (17) is valid.

Further, if $\phi^{(j+2)}(o_1) \leq 0$ for $j = 0, 1, 2, \ldots, u$ and $\phi^{(u+3+d)}(o_2) \leq 0$ if $d - j$ is even, and
\( \phi^{(u+3+d)}(o_2) \geq 0 \), if \( d - 3 \) is odd, for \( 3 = 0, \ldots, d \) and \( d = 0, \ldots, n - u - 4 \), then the right-hand side of the reverse inequality becomes nonpositive.

**Proof** We can prove this theorem like Theorem 2.4. \( \square \)

**Remark 2.8** As mentioned in Remark 2.2, we can deduce special cases for the results of this section for different time scales.

### 3 Bounds concerning the identities for generalization of a Jensen-type inequality

For our results of this section, we use the following two theorems given by Cerone and Dragomir [9].

**Theorem 3.1** Let \( k: [o_1, o_2] \rightarrow \mathbb{R} \) be a Lebesgue integrable function and \( l: [o_1, o_2] \rightarrow \mathbb{R} \) be an absolutely continuous function with \( (o_2 - o_1)[l'(b)]^2 \in L[o_1, o_2] \). Then, we have

\[
|\varphi(k, l)| \leq \frac{1}{\sqrt{2}} \left[ \psi(k, k) \right]^{\frac{1}{2}} \frac{1}{\sqrt{o_2 - o_1}} \left[ \int_{o_1}^{o_2} (b - o_1)(o_2 - b)[l'(b)]^2 \, db \right]^{\frac{1}{2}}, \tag{18}
\]

where

\[
\psi(k, l) = \frac{1}{o_2 - o_1} \int_{o_1}^{o_2} k(q)l(b) \, db - \frac{1}{o_2 - o_1} \int_{o_1}^{o_2} k(b) \, db \cdot \frac{1}{o_2 - o_1} \int_{o_1}^{o_2} l(b) \, db. \tag{19}
\]

In the inequality (18), the constant \( \frac{1}{\sqrt{2}} \) is the most suitable option.

**Theorem 3.2** Assume that \( k: [o_1, o_2] \rightarrow \mathbb{R} \) is monotonic nondecreasing on \([o_1, o_2]\) and \( l: [o_1, o_2] \rightarrow \mathbb{R} \) is absolutely continuous with \( k' \in L_\infty[o_1, o_2] \). Then, we have the inequality

\[
|\varphi(k, l)| \leq \frac{1}{2(o_2 - o_1)} \|k'\|_\infty \int_{o_1}^{o_2} (b - o_1)(o_2 - b) \, dl(b).
\]

The constant \( \frac{1}{2} \) in the above equation is the most suitable option.

The following notations are used throughout this section,

\( \xi(q) = f(G_n(b, q)) \quad q \in [o_1, o_2], \)

and

\[
\varphi(\xi, \xi) = \frac{1}{o_2 - o_1} \int_{o_1}^{o_2} \xi^2(q) \, dq - \left( \frac{1}{o_2 - o_1} \int_{o_1}^{o_2} \xi(q) \, dq \right)^2.
\]

By using the Čhebyšev functional (19) we obtain the following identity.
Theorem 3.3 If \((\cdot, -\xi)(0_2, 0_2)[\phi^{n*1}]^2 \in L[0_1, 0_2]\) with the assumptions of Theorem 2.1, then we have

\[
J(\phi(g)) = \sum_{j=2}^{n} \frac{\phi^{(j)}(0_1)}{j!} J(g(b) - 0_1)^j + \sum_{d=0}^{n-1} \left[ \sum_{j=0}^{d} \frac{(-1)^{d-j}(0_2 - 0_1)^{d-j} \phi^{(u+1+j)}(0_2)}{(u + 1 + j)(d-j)!} \right] J(g(b) - 0_1)^{(u+1+j)}
\]

\[
+ \frac{\phi^{n-1}(0_2) - \phi^{n-1}(0_1)}{0_2 - 0_1} \int_{0_1}^{0_2} \xi(q) dq + \kappa_n(0_1, 0_2; \phi),
\]

(20)

where

\[
|\kappa_n(0_1, 0_2; \phi)| \leq \sqrt{\frac{0_2 - 0_1}{2}} \left[ \psi(\xi, \xi) \right] \frac{1}{\sqrt{0_2 - 0_1}} \left| \int_{0_1}^{0_2} (q - 0_1)(0_2 - q)[\phi^{(n*1)}(q)]^2 dq \right|^{\frac{1}{2}}.
\]

(21)

Proof By replacing \(k\) with \(\xi\) and \(l\) with \(\phi^n\) in Theorem 3.1, we obtain

\[
\left| \frac{1}{0_2 - 0_1} \int_{0_1}^{0_2} \xi(q) \phi^n(q) dq - \frac{1}{0_2 - 0_1} \int_{0_1}^{0_2} \xi(q) dq \phi^n(q) dq \right| \leq \frac{1}{\sqrt{0_2 - 0_1}} \left| \int_{0_1}^{0_2} (q - 0_1)(0_2 - q)[\phi^{(n*1)}(q)]^2 dq \right|^{\frac{1}{2}}.
\]

Therefore,

\[
\int_{0_1}^{0_2} \xi(q) \phi^n(q) dq = \frac{\phi^{n-1}(0_2) - \phi^{n-1}(0_1)}{0_2 - 0_1} \int_{0_1}^{0_2} \xi(q) dq + \kappa_n(0_1, 0_2; \phi),
\]

where the estimation (21) is satisfied by the remainder \(\kappa_n(0_2, 0_1; \phi)\). Now, from identity (8) we obtain (20).

Our next theorem gives a Grüss–type inequality for diamond integrals.

Theorem 3.4 Let \(0 \leq u \leq n - 1\) and \(\phi \in C^n([0_1, 0_2])\), where \(\phi^{n*1} \geq 0\) on \([0_1, 0_2]\). Then, (20) is satisfied with

\[
|\kappa_n(0_1, 0_2; \phi)| \leq (0_2 - 0_1) \| \xi \|_{\infty} \left\{ \frac{\phi^{n-1}(0_2) + \phi^{n-1}(0_1)}{2} - \frac{\phi^{n-2}(0_2) - \phi^{n-2}(0_1)}{0_2 - 0_1} \right\}.
\]

(22)

Proof By replacing \(k\) with \(\xi\) and \(l\) with \(\phi^n\) in Theorem 3.2, we obtain

\[
\frac{1}{0_2 - 0_1} \int_{0_1}^{0_2} \xi(q) \phi^n(q) dq - \frac{1}{0_2 - 0_1} \int_{0_1}^{0_2} \xi(q) dq \phi^n(q) dq \leq \frac{1}{2(0_2 - 0_1)} \| \xi \|_{\infty} \int_{0_1}^{0_2} (q - 0_1)(0_2 - q)[\phi^{n*1}](q) dq.
\]

(23)
Since
\[
\int_{o_1}^{o_2} (q - o_1)(o_2 - q)\phi^{n+1}(q) \, dq
= \int_{o_1}^{o_2} [2q - (o_1 + o_2)]\phi^n(q) \, dq
= (o_2 - o_1)\left[\phi^{n+1}(o_2) + \phi^{n+1}(o_1)\right] - 2\left[\phi^{n-2}(o_2) - \phi^{n-2}(o_1)\right],
\]
we deduce (22) by using the identity (8) and the inequality (23).

\[\square\]

In the following theorem, we obtain an Ostrowski-type inequality for diamond integrals to generalize Jensen's inequality.

**Theorem 3.5** Let \(h, z \in [1, \infty)\) such that \(1/h + 1/z = 1\). If \(|\phi^n|^{h} : [o_1, o_2] \to \mathbb{R}\) is integrable for some \(n \in \mathbb{N}\) and \(n \geq 2\) with the assumptions of Theorem 2.1, then we have

\[
J(\phi) - \sum_{3=0}^{u} \frac{\phi^3(o_1)}{3!} J(q - o_1)^3
- \sum_{d=0}^{n-u-2} \left[\sum_{3=0}^{d} \frac{(-1)^{d-3}(o_2 - o_1)^{d-3}\phi^{(u+1+d)}(o_2)}{(u+1+3)!d!(d-3)!}\right] J(q - o_1)^{u+1+3}
\leq ||\phi^{(n)}||_{h} \left(\int_{o_1}^{o_2} |J(G_n(b, q))|^z \, dq\right)^{\frac{1}{z}}.
\]

The constant on the right-hand side of (24) is sharp for \(1 < h \leq \infty\) and the best possible for \(h = 1\).

**Proof** Utilizing identity (8) and after application of Hölder's inequality, we obtain

\[
J(\phi) - \sum_{3=0}^{u} \frac{\phi^3(o_1)}{3!} J(g(b) - o_1)^3
+ \sum_{d=0}^{n-u-2} \left[\sum_{3=0}^{d} \frac{(-1)^{d-3}(o_2 - o_1)^{d-3}\phi^{(u+1+d)}(o_2)}{(u+1+3)!d!(d-3)!}\right] J(g(b) - o_1)^{u+1+3}
\leq \left(\int_{o_1}^{o_2} \xi(q)\phi^n(q) \, dq\right) \leq ||\phi^{(n)}||_{h} \left(\int_{o_1}^{o_2} |\xi(q)|^z \, dq\right)^{\frac{1}{z}}.
\]

To show the sharpness and exactness of the constant \(||\xi(q)||^z \, dq\)^{\frac{1}{z}}, let us discover a function \(\phi\) for which the inequality in (24) is gained. For \(1 < h < \infty\) take \(\phi\) such that

\[\phi^n(q) = \text{sgn} \xi(q)|\xi(q)|^{\frac{1}{h-1}}.\]

For \(h = \infty\) take \(\phi^n(q) = \text{sgn} \xi(q)\), for \(h = 1\) we prove that

\[
\left(\int_{o_1}^{o_2} \xi(q)\phi^n(q) \, dq\right) \leq \max_{q \in [o_1, o_2]} |\xi(q)| \left(\int_{o_1}^{o_2} |\phi^n(q)| \, dq\right)
\]

(25)
is the most suitable inequality. Assume that \(|\xi(q)|\) acquires a maximum at \(q_0 \in [o_1, o_2]\).

First, we suppose that \(\xi(q_0) \geq 0\). For \(\epsilon\) small enough, we determine \(\phi_\epsilon(q)\) by

\[
\phi_\epsilon(q) := \begin{cases} 
0, & o_1 \leq q \leq q_0, \\
\frac{1}{\epsilon n}(q - q_0)^n, & q_0 \leq q \leq q_0 + \epsilon, \\
\frac{1}{\epsilon n}(q - q_0)^{n-1}, & q_0 + \epsilon \leq q \leq o_2.
\end{cases}
\]

Then, for small enough \(\epsilon\),

\[
\left|\int_{o_1}^{o_2} \xi(q) \phi_\epsilon^n(q) \, dq\right| = \left|\int_{q_0}^{q_0 + \epsilon} \xi(q) \, dq\right| = \frac{1}{\epsilon} \int_{q_0}^{q_0 + \epsilon} \xi(q) \, dq.
\]

Now, using inequality (25), we have,

\[
\frac{1}{\epsilon} \int_{q_0}^{q_0 + \epsilon} \xi(q) \, dq \leq \xi(q_0) \int_{q_0}^{q_0 + \epsilon} \frac{1}{\epsilon} \, dq = \xi(q_0).
\]

Since

\[
\lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_{q_0}^{q_0 + \epsilon} \xi(q) \, dq = \xi(q_0);
\]

this statement follows \(\xi(q_0) \leq 0\)

\[
\phi_\epsilon(q) := \begin{cases} 
\frac{1}{\epsilon n}(q - q_0 - \epsilon)^{n-1}, & o_1 \leq q \leq q_0, \\
-\frac{1}{\epsilon n}(q - q_0 - \epsilon)^n, & q_0 \leq q \leq q_0 + \epsilon, \\
0, & q_0 + \epsilon \leq q \leq o_2.
\end{cases}
\]

and the rest of the proof also follows the same steps as above. \(\Box\)

For further results we denote

\[
\zeta(q) = \int_{o_1}^{o_2} f(G(g(b), r))G_{n-2}(r, q) \, dr \geq 0, \quad q \in [o_1, o_2],
\]

and

\[
\varphi(\zeta, \zeta) = \frac{1}{o_2 - o_1} \int_{o_1}^{o_2} \zeta^2(q) \, dq - \left(\frac{1}{o_2 - o_1} \int_{o_1}^{o_2} \zeta(q) \, dq\right)^2.
\]

**Theorem 3.6** If \((.-o_1)(o_2 - .)[\phi^{n+1}]^2 \in L[o_1, o_2] with the assumptions of Theorem 2.5, then we have

\[
J(\phi(g)) = \sum_{j=0}^{u} \frac{\phi^{(j+2)}(o_1)}{j!} \int_{o_1}^{o_2} f(G(g(b), r)) r^{-1} \, dr
\]

\[
+ \sum_{d=0}^{n-u-4} \left[ \sum_{n=0}^{d} \frac{(-1)^{d-n}(o_2 - o_1)^{d-3} \phi^{(u+3+d)}(o_2)}{(u + 1 + d)! (d - 3)!} \right]
\]

where
Theorem 3.7 If \( \phi^{n+1} \geq 0 \) on \([a_1, b_2]\) with the assumptions of Theorem 2.5, then equation (26) and the remainder \( \kappa_n(a_2, a_1; \phi) \) satisfies the condition

\[
|\kappa_n(a_2, a_1; \phi)| \leq (b_2 - a_1) \| \zeta \|_\infty \left\{ \frac{\phi^{n-1}(a_2) + \phi^{n-1}(a_1)}{2} - \frac{\phi^{n-2}(a_2) - \phi^{n-2}(a_1)}{b_2 - a_1} \right\}.
\]

Proof The inequality (26) can be obtained in a similar way as the inequality (20).

Theorem 3.8 Let \( b, z \in [1, \infty) \) such that \( \frac{1}{b} + \frac{1}{z} = 1 \). If \( |\phi^n| : [a_1, b_2] \to \mathbb{R} \) is integrable for some \( n \in \mathbb{N} \) and \( n \geq 4 \) and the assumptions of Theorem 2.5 hold, then we obtain

\[
J(\phi) - \sum_{j=0}^{n} \frac{\phi^{(j+2)}(a_1)}{j!} J(G(g(b), r))(r - a_1)^j dr
\]

\[
- \sum_{d=0}^{n-4} \sum_{l=0}^{d-3} \frac{(-1)^d}{[b_2 - a_1]^{d-3} \phi^{(l+3)}(a_2)(u + 3)! (d - 3)!} \times \int_{a_1}^{b_2} J(G(g(b), r))(r - a_1)^{(u+1)+3} dr
\]

\[
\leq \| \phi \|_b \left( \int_{a_1}^{b_2} J(G(g(b), r)) G_{n-2}(r, q) dr \right)^{\frac{1}{2}}.
\]

The constant of (28) in the above equation on the right-hand side is sharp for \( 1 \leq b \leq \infty \) and the better estimate for \( b = 1 \).

Proof The inequality (28) can be obtained in a similar way as the inequality (24).

4 Conclusion

The purpose of this paper is to obtain new refinements of Jensen’s inequality and functionals on time scales. As Jensen’s inequality is considered an important tool for producing classical inequalities, many classical inequalities can be improved by using the refined Jensen inequality. Also, new inequalities can be rewritten for several particular cases of time-scales integrals. Moreover, a similar method can be applied for the functionals obtained from the converse of Jensen’s inequality and the Jensen–Mercer inequality.

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The authors declare that they have no competing interests.

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