Random Quantum Billiards
Henrik J. Pedersen and A. D. Jackson
The Niels Bohr Institute
Blegdamsvej 17, DK 2100 Copenhagen Ø
Denmark

Abstract

We present a random matrix model suitable for the quantum mechanical description of a particle confined to move inside a two-dimensional domain. Here, the ensemble average corresponds to an average over domain shapes. Although this approach is formally distinct from that of the familiar Gaussian orthogonal ensemble, there is remarkable agreement in the resulting distributions of level spacings and in the spectral rigidity. We discuss possible applications of this ensemble to the analysis of real data sets.
I. Introduction

Received wisdom suggests that universal features of the spectrum of completely chaotic quantum billiards, such as the distribution of level spacings and the spectral rigidity, are identical to those of the Gaussian orthogonal ensemble (GOE). Both experimental data [1,2] and the results of numerical studies [3,4,5] provide considerable support for this belief. With a proper choice of the zero and the unit of the energy scale, the GOE has become the common standard of comparison. However, in spite of considerable analytical efforts [6,7,8,9], there is still no proof of the Bohigas conjecture [5] that the spectral correlations of quantum systems with chaotic classical analogues are given by random matrix theory. Indeed, some reflection on the physical nature of the two problems suggests that this agreement is surprising. The GOE emerges from the ensemble of real symmetric random matrices in the limit of infinite dimension, $M$. Each matrix, $H$, is drawn from a Gaussian distribution $\exp \left[ -\text{Tr}(H^T H)/2 \right]$. The average value of each matrix element is zero. The quantum billiard problem requires the solution of the Schrödinger equation for a particle which moves freely within a bounded domain. The eigenvalues are typically obtained by expanding the problem in some suitable basis and diagonalizing a matrix of finite dimension. Obviously, all eigenvalues of quantum billiards are positive. This stands in contrast to the fact that the eigenvalues of the GOE are distributed symmetrically about 0 according to the familiar semicircular law, $\rho(x) \sim \sqrt{2M - x^2}$ with support from $-\sqrt{2M}$ to $+\sqrt{2M}$. The eigenvalues are not bounded from below in the thermodynamic limit. Since any given eigenvalue diverges as the dimension of the matrix is increased, questions associated with the convergence of eigenvalues and eigenvectors are largely without meaning. This behaviour is again qualitatively different from that of real quantum billiard problems for which asymptotic power law convergence is expected for both eigenvalues and eigenfunctions. Further, the GOE is invariant under orthogonal transformations, and there is thus no preferred basis. While this represents a considerable technical simplification, it also indicates another difference from quantum billiards. There, we have the expectation of increasing nodal complexity with increasing energy and, thus, a qualitative basis preference.

In spite of these remarks, the intention of this paper is not to advocate rejection of the GOE for the analysis of quantum billiard problems. Rather, we believe it to be desirable to consider the properties of a new random matrix ensemble which more faithfully imitates the properties of quantum billiard problems by respecting the positivity of the spectrum and main-
taining the qualitative basis preference just mentioned. (We note that the
properties of finite ensembles of real “random” billiards have been studied
previously. See, e.g., refs. [10] and [11].) We will show numerically that the
spectral correlations of this model are identical to those of the GOE.

The model and its physical motivation will be introduced in section II. In
section III we will consider the accuracy of eigenvalues calculated with this
model and demonstrate power law convergence in eigenvalues and eigenfunc-
tions with increasing matrix dimension. In section IV we present the level
spacing distribution and spectral rigidity for our random billiard ensemble
(RBE). These results will be found to be statistically indistinguishable from
the GOE. Although we will concentrate on numerical investigation of the
RBE, it has been constructed with sufficient simplicity that its analytic in-
vestigation should also be tractable. In section V we will compare the wave
functions obtained from this model with those of the GOE and of an ensemble
of real quantum billiards. Discussion and conclusions will be relegated
to section VI.

II. The Construction of the Model

Our random matrix model of quantum billiards will be patterned on
the standard conformal mapping method for the solution of the Schrödinger
equation for a point particle in confined to the interior of a simply-connected
domain in the two-dimensional uv-plane [12,13]. It is always possible to find
a new pair of variables, x and y, such that the system is mapped onto some
other convenient simply-connected domain, e.g., the unit disk, for which
the Laplacian is separable and for which its eigenfunctions, \( \phi_i(x,y) \), and
eigenvalues, \( \epsilon_i \), are known. (We order the states so that \( \epsilon_{i+1} \geq \epsilon_i \).
Evidently, \( \epsilon_i > 0 \) for all \( i \). The \( \phi_i \) are orthogonal and normalized in the
xy-plane on the weight 1.) The condition of simple-connectedness is suffi-
cient to ensure that this mapping will be conformal with

\[
\frac{z}{2} + i\frac{v}{2} = w(z),
\]

where \( z = x + iy \). We now expand the desired eigenfunctions, \( \Phi(u,v) \), in the
complete basis \( \phi_i \) with coefficients \( \Phi_i \). Since the transformation is conformal,
the matrix elements of the Hamiltonian are particularly simple:

\[
H_{ij} = \int du \, dv \, \phi_i(x,y) (-\nabla^2_w) \phi_j(x,y)
= \int dx \, dy \, \phi_i(x,y) (-\nabla^2_z) \phi_j(x,y) = \epsilon_i \delta_{ij}.
\]

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The Hamiltonian matrix is thus diagonal with elements \( \epsilon_i \) which are independent of the shape of the initial domain. As a consequence of the separability of the Laplacian in \( z \), the spacings between adjacent levels, \( \epsilon_{i+1} - \epsilon_i \), will have a Poisson distribution. The average level spacing will be independent of energy for a two-dimensional quantum billiard. Since the transformation of variables destroys the orthogonality of the \( \phi_i \), it is necessary to consider the overlap integrals

\[
N_{ij} = \int du \, dv \, \phi_i(x,y) \phi_j(x,y) = \int dx \, dy \, |dw/dz|^2 \phi_i(x,y) \phi_j(x,y)
\]  

(3)

The Schrödinger equation now assumes the form

\[
\epsilon_i \Phi_i = \lambda N_{ij} \Phi_j.
\]  

(4)

All information regarding the shape of the initial domain is carried by the matrix \( N \). One physical property of \( N \) is worth noting. Assume that the mapping is in some sense smooth. The functions \( \phi_i \) will oscillate rapidly on a scale set by the mapping for sufficiently large \( i \). Given the normalization of the \( \phi_i \), the diagonal elements, \( N_{ii} \), will then approach the area of the original domain, which can be set equal to 1.

We can express the elements in \( N \) in a more suggestive fashion. Since the \( \phi_i \) are complete, we can use the Gram-Schmidt procedure to construct a basis of states, \( \psi_i \), which are orthonormal on the weight \( |dw/dz|^2 \). If we write

\[
\phi_i = \sum_{\ell=1}^{i} r_{i\ell} \psi_\ell,
\]  

(5)

the elements of \( N \) are simply

\[
N_{ij} = \vec{r}_i \cdot \vec{r}_j.
\]  

(6)

This structure of \( N \) and the positivity of the \( \epsilon_i \) are sufficient to ensure the positivity of the eigenvalues for every domain. The physical constraint on the diagonal elements of \( N \) assumes the form \( \vec{r}_i \cdot \vec{r}_i \to 1 \) as \( i \to \infty \) for a billiard of unit area.

Everything said so far has been exact. We now use the form of this calculation to design a random matrix model of quantum billiards:

Draw \( \epsilon_1 \) and all subsequent level spacings, \( \epsilon_{i+1} - \epsilon_i \), at random on a Poisson distribution (with variance 1). Draw the elements \( r_{i\ell} \) for \( \ell \leq i \) at random on, e.g., a Gaussian distribution with mean value 0 and variance \( 1/i \) so that \( \vec{r}_i \cdot \vec{r}_i = 1 \) on average. Use these quantities to construct the Schrödinger equation, eqn.(4).
This defines our random billiard ensemble (RBE). The ensemble average over the $\vec{r}_i$ corresponds to an average over billiard shapes. A further average over the $\epsilon_i$ corresponds to averaging over the shape (and basis) chosen in the $z$ plane. Thus, this average is not necessary but can be convenient.

Several comments are in order. First, every matrix selected according to this algorithm will have a positive spectrum. Second, it is simple and meaningful to increase the dimension of the matrix in order to study the convergence properties of individual eigenvalues. (Draw a single new $\vec{r}_{M+1}$ and a new $\epsilon_{M+1}$ according to the above rules leaving all preceding $\vec{r}_i$ and $\epsilon_i$ unaltered.) As we shall see below, eigenvalues (and eigenfunctions) will show power law convergence for large $M$. The usual numerical (and experimental) approach to billiard problems is to consider many levels for a given shape. This can lead to the diagonalization of inconveniently large matrices. Here, greater economy is possible. It is possible to obtain impressive statistical accuracy for the usual measures of this ensemble (e.g., the level spacing distribution or the spectral rigidity) by considering a modest number of eigenvalues for each of a large ensemble of “shapes”. As we shall see in section IV, the level spacing distribution, $e_{k+1} - e_k$, converges rapidly (with $k$) to its asymptotic form.

Finally, it is useful to perform a rough counting of the number of free parameters in order to get a sense of those features of real billiards which have been lost in the RBE. For a matrix of dimension $M$, the matrix $N$ contains $O(M^2)$ free parameters. Since the Laplacian is assumed to be separable in the coordinates chosen in $z$, the $M$ basis states $\phi_i$ are the various products of two of the $\sqrt{M}$ independent functions of one variable each. The overlap matrix resulting from the weight function of a conformal map, $|dw/dz|^2$, should thus contain roughly $O(\sqrt{M})$ free parameters. A local (but not conformal) map would lead to $O(M)$ free parameters. By extension, our random matrix model corresponds roughly to choosing a positive but non-local weight function. This is, of course, not consistent with the assumption of the diagonality of $H$, which is correct only when the mapping is conformal. Such inconsistency is a hallmark of random matrix models and is not necessarily cause for alarm.

III. Convergence of the Calculations

Before reporting the results of this model, we wish to discuss the convergence of calculated eigenvalues and eigenfunctions as a function of the
dimension of the matrix, $M$. Given the model of section II, we expect to find (fully converged) energies of $e_k \approx k$ and an average level spacing of 1 (at all energies). Only fully converged eigenvalues are of physical interest, and these will represent a small fraction of the eigenvalues of any given matrix. Nevertheless, useful information about convergence is provided by looking at the change in $e_k$ as $M \to M + 1$ for a broad range of $k$. The logarithm of this change (averaged over ensembles of matrices) as a function of $k/M$ for $M = 200, 500, 700, \text{ and } 1000$ is shown as the heavy band in fig. 1.

Figure 1: Plot of $dE(k)/dM$ as a function of $k/M$. The heavy band represents results for $M = 200, 500, 700, \text{ and } 1000$. The points are the results for a single matrix with $M = 500$.

The surprise here is that $dE^{(M)}_k/dM$ is a function only of the ratio $k/M$. While there are considerable fluctuations about this curve for the eigenvalues of single matrices, these are not so large as to invalidate this average curve. This is indicated by the points in fig. 1, which represent the results for a
single matrix of dimension \( M = 500 \). The data reveal a region of small \( k/M \) (in which the energies are accurate) where \( de_k^{(M)}/dM \) is proportional to \((k/M)^2\). This asymptotic behaviour is followed by a region in which \( de_k^{(M)}/dM \) increases roughly exponentially (and in which the energies are not of useful accuracy). The results of this figure can be integrated to construct the average error in \( e_k \) due to finite matrix size. Clearly, the form of the energy will be given as

\[
e_k^{(M)} = e_k + M f(k/M).
\] (7)

The data show that \( f(k/M) \sim -(k/M)^2 \) in the useful limit of small \( k/M \). This means that we can obtain \( k \) levels which meet any fixed accuracy criterion by allowing \( M \) to grow like \( k^2 \).

It is possible to understand this behaviour by re-expressing eqns.(4) and (6) in the orthonormal basis of eqn.(5).

\[
h \Psi = R^T \frac{1}{\lambda} R \Psi = \frac{1}{\lambda} \Psi,
\] (8)

where \( R \) is the lower triangular matrix whose elements are the \( r_{i\ell} \). The eigenvalues and eigenfunctions of eqn.(8) are identical to those of eqns.(4) and (6). Given the structure of \( R \) and the diagonality of \( H \), we can write \( h \) (in the Gram-Schmidt basis) as

\[
h_i^{(M)} = \sum_{k=\text{max}(i,j)}^{M} \frac{r_{ki} r_{kj}}{\epsilon_k}.
\] (9)

Imagine that we have determined the eigenvalues and eigenfunctions of \( h^{(M-1)} \). Working in this basis, we consider the change in \( \delta h = h^{(M)} - h^{(M-1)} \), which results from increasing the dimension by 1. Every element of \( \delta h \) will be of \( \mathcal{O}(M^{-2}) \). Since the difference between \( 1/e_k^{(M-1)} \) and \( 1/e_k^{(M-1)} \) (which equals 0) will be \( \mathcal{O}(1/k) \) for \( M >> k \), it is legitimate to estimate the change in eigenvalues using perturbation theory. Making use of the fact that \( \epsilon_k \approx k \) on average, zeroth-order perturbation theory yields

\[
\left\langle \frac{1}{\epsilon_k^{(M)}} \right\rangle - \left\langle \frac{1}{\epsilon_k^{(M-1)}} \right\rangle = \frac{1}{M^2}
\] (10)

when an ensemble average is performed over the elements of \( \delta h \). Summing these changes over \( M \) yields the consistent result that \( \left\langle 1/\epsilon_k \right\rangle \to 1/k \) as
\( M \to \infty \) and that \( \langle e_k \rangle \) will be approximately \( k \) as expected. Making the tentative replacement of \( \langle 1/e_k \rangle \) by \( 1/\langle e_k \rangle \), this suggests that

\[
\langle d e_k / dM \rangle \approx -\frac{k^2}{M^2},
\]

which is the asymptotic result of fig. 1. We note that it is possible to provide a rigorous proof, valid for all \((k/M)\), of the scaling behaviour indicated in fig.1 and eqn.(7). This will be described elsewhere.

The ensemble average of every off-diagonal matrix element of \( \delta h \) is 0, and the variance is \( 1/M^4 \). Thus, first-order perturbation theory then tells us that the probability of finding the state \( \psi_M \) in the \( k \)-th eigenfunction should be proportional to \( k^2/M^4 \) for \( M \gg k \). (Given the symmetry of first-order perturbation theory, this is also the probability of finding the state \( \psi_k \) in the \( M \)-th eigenfunction for \( k \ll M \).) We shall see in the next section that this behaviour of eigenfunctions is correct.

Since the dominant effect of increasing the size of the basis can be estimated from zeroth-order perturbation theory, it is easy to accelerate the convergence of these calculations significantly. Specifically, the upper limit of the sum in eqn.(9) can be extended to some number much larger than the size of the largest matrix to be considered. This matrix can be truncated to dimension \( M - 1 \) as before. Now, the effect of increasing the dimension to \( M \) now is the addition of a single row and column. The original matrix of dimension \( M - 1 \) is unaltered, and zeroth-order perturbation theory will not change the energy of state \( k \). To the extent that this eigenfunction is dominated by a single state in the Gram-Schmidt basis, first-order perturbation theory gives a correction of order \((k/M^3)\) to \( 1/\langle e_k \rangle \). This is to be compared with the result of eqn.(10) above. Except where otherwise noted, we have use this method to accelerate the convergence of the numerical calculations reported below.

The above estimates indicate slow but adequate (power law) convergence of eigenvalues as a function of \( M \) and completely acceptable convergence of eigenfunctions.

We note that the combination of power law and exponential behaviour indicated in fig. 1 makes it easy to draw misleading conclusions regarding the convergence of these calculations. Specifically, study of the error for fixed \( k \) and large \( M \) would reveal only the explicit power-law dependence on \( M \). The \( M \) dependence of the exponential is easily missed. However, study for fixed \( M \) as a function of \( k \) (relatively small) readily reveals an exponential divergence with \( k \). The naive combination of these results can lead to the
erroneous conclusion that the size of the matrix must grow exponentially with $k$ in order to meet a fixed accuracy criterion. As already indicated, this is not correct.

IV. Level Densities, the Level Spacing Distribution, and the Spectral Rigidity

In this section, we wish to present a number of numerical results for the random billiard ensemble introduced in the section II. These include the one-body level density, the level spacing distribution, and the spectral rigidity, $\Delta_3(L)$. Fig. 2 shows the one-body level density obtained as an average over 300 matrices of dimension $M = 300$.

Given the error estimates of the preceding section, we expect that only the first 50 states (with energies from 0 to roughly 50) are quantitatively useful. The resulting density of states is nearly constant over this limited energy range and equal to 1, as expected for a family of two-dimensional billiards of unit area. As indicated in section III, this range of constant level density grows as $M$ increases. The one-body level density for larger energies has necessarily been constructed from eigenvalues which are not of quantitative accuracy. As expected, the average spacing between these states is too large and the average level density is less 1.

In order to combine data for level spacing distributions from various parts of the spectrum, it is first necessary to scale the energies with the local average level spacing (i.e., to “unfold” the spectrum). This is not essential here since we always work with converged eigenvalues and since the average level spacing is always very close to 1 for the levels considered. Nevertheless, we performed such a scaling by first binning pairs of levels according to their mean energy and then determining the average level spacing within each bin. Since our model averages over billiard shapes, it is easy to consider the distribution of level spacings between states $k + 1$ and $k$ for any fixed value of $k$. Level spacing distributions obtained for small $k$ contain unwanted edge effects. (For example, the distribution for $k = 1$ shows a significantly enhanced probability for small level spacings.) Fortunately, the level spacing distribution reaches a stable asymptotic form rapidly with increasing $k$. 

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Figure 2: The level density for the RBE as obtained for 300 matrices of dimension $M = 300$.

Thus, fig. 3 shows the RBE level spacing distribution of $1.5 \times 10^5$ levels with $19 \leq k \leq 30$ obtained from matrices with $M = 200$. We also show the exact level spacing distribution for the GOE. (The familiar Wigner surmise is not adequate for our purposes.) Cursory inspection suggests a fully quantitative fit.

This conclusion is supported by the average value of $\chi^2 = 1.28$ per datum over the range $0 \leq s \leq 4.0$. (This value of $\chi^2$ can be made closer to 1 by diagonalizing larger matrices and using larger values of $k$.) As a measure of the sensitivity of this comparison, we note that attempting to fit our data with the Wigner surmise leads to a $\chi^2$ of 4.6 per datum. Given that our data sample includes 80 bins, this represents a remarkably strong preference for the exact GOE level spacing distribution.

A somewhat longer-range characterization of the spectrum is provided by
the spectral rigidity, $\Delta_3(L)$, which is a measure of the fluctuations of a given sequence of unfolded levels about the best straight line fit [14]. Consider the level density for a given sequence of levels $x_i$ (with average spacings 1) which lie in the range $-L/2 \leq x \leq +L/2$:

$$\rho(x) = \sum \delta(x - x_i) \quad (12)$$

Using this density, define the “staircase” function

$$N(x) = \int_{-L/2}^{x} dx' \rho(x') \quad . \quad (13)$$

Determine the mean square deviation relative to the best straight line fit for

Figure 3: The histogram shows the RBE level spacing distribution of $1.5 \times 10^5$ level spacings using levels $k = 19$ to 30 obtained from the diagonalization of matrices of dimension 200. The solid line represents the exact level spacing distribution of the GOE.
each such sequence and perform an ensemble average to obtain $\Delta_3(L)$ as

$$\Delta_3(L) = \langle \min \frac{1}{L} \int_{-L/2}^{+L/2} dx (N(x) - A - Bx)^2 \rangle,$$

(14)

where the minimization is with respect to the constants $A$ and $B$. We have constructed $\Delta_3(L)$ for our model of random quantum billiards over the range $0 \leq L \leq 30$. The results shown in fig. 4 were obtained from an ensemble of 500 matrices of dimension 300 using levels 16 to 60.

Figure 4: The points show $\Delta_3(L)$ for the RBE using levels $k = 16$ to 60 obtained from 500 matrices of dimension $M = 300$.

Again, we find quantitative agreement with the GOE; the small visible differences can be eliminated by working with higher levels (and larger matrices). In both cases, $\Delta_3(L)$ approaches $\ln(2\pi L)/\pi^2$ in the limit of large $L$. The correlations inherent in these spectra result in fluctuations which are much smaller than those of a random spectrum (with a Poisson distribution of level spacings) for which $\Delta_3(L) = L/15$. 
V. Eigenfunctions and Real Billiards

Having seen such striking agreement between RQB and GOE results, it is of interest to find some differences. The most natural place to seek them is in the structure of eigenfunctions. Thus, we consider the expansion coefficients of a given eigenfunction in the orthonormal basis of states defined in eqn.(5),

$$\Psi_k = \sum d_{\ell}^{(k)} \psi_{\ell},$$

so that the $[d_{\ell}^{(k)}]^2$ can be regarded as probabilities. As discussed in section III, we expect that the ensemble average of $[d_{\ell}^{(k)}]^2$ should grow like $\ell^2$ for $\ell << k$ and vanish like $1/\ell^4$ for $\ell >> k$. This expectation is confirmed by fig. 5, which shows the $[d_{\ell}^{(k)}]^2$ for eigenvector $k = 10$ averaged over 200 matrices of dimension 400. (Here, we have not accelerated the convergence of the calculation.) As expected, this result stands in sharp contrast to the eigenfunctions of the GOE. Given the basis-independence of the GOE, the ensemble averaged probability should be precisely $1/N$ for each state. One might expect to find some small variation if the diagonal elements of these matrices are first ordered in increasing value. There is little sign of such dependence for states in the middle of the spectrum.

Having found a difference between RQB and the GOE, it is appropriate to consider the behaviour of real billiards. Any real billiard will have a weight function which will be smooth on some suitably small distance scale. Above, this led us to expect that the diagonal elements $N_{ii}$ will approach the area of the billiard in the limit of large $i$. (This physically motivated constraint was built into our model so that the ensemble average of $r_{i\ell}^2$ is $1/i$ (for $\ell < i$). Local smoothness of the weight has other consequences. When the (two-dimensional) quantum numbers of states $\phi_i$ and $\phi_j$ are different, their product will oscillate over the domain of integration. The magnitude of $N_{ij}$ should thus decrease with some power of the mismatch in the quantum numbers of the states. Thus, we expect that the $[d_{\ell}^{(k)}]^2$ for real billiards should decrease when $k$ is very different from $\ell$ (and the two-dimensional quantum numbers are necessarily different). The hull of maximum probabilities should show a power law decay away from the peak. Given the fact that two states of approximately the same energy can have grossly different quantum numbers, the actual probabilities should show large local fluctuations.

In order to demonstrate these effects, we have consider an ensemble of randomly selected real billiards. Specifically, we have mapped a simply con-
connected domain of unit area onto a square of length 1 with one corner at the origin. This problem can be treated using the conformal mapping method of section I, and the basis functions are now simple sines. All information regarding the original shape now resides in the analytic function $dw/dz$, which we have arbitrarily chosen to be a polynomial of order 3 to 6. This polynomial is determined by its zeros which must lie outside the unit square. Subject to this constraint, we have drawn these zeros at random uniformly on a square of side 2.6 centered at the origin and performed an ensemble average over 200 real billiards. (There is, on the one hand, no guarantee that the periodic orbits for these billiards are always of measure zero. On the other, there is no assurance that the corresponding randomly selected mappings are univalent. These matters should not be of importance for our limited considerations.) The appropriate Gram-Schmidt basis was constructed for each shape drawn, and the $|d_ℓ^{(k)}|^2$ were determined. The value of the $|d_ℓ^{(k)}|^2$ for eigenvector $k = 10$ is also shown in fig. 5. (The eigenvectors were obtained from the diagonalization of matrices of dimension 300.)

The hull of the maximum probabilities (as well as the behavior of the local geometric mean probability) is similar to our RBE results. We note, in particular, the power-law rise to the peak and a decay which is somewhat faster than the $1/k^4$ behaviour of fig. 5. (If required, this decay could be simulated more accurately in the RBE by drawing the elements $r_{ki}$ with a variance which decreases smoothly as $i$ ranges from $k$ to 1.)

This figure also reveals the expected large local fluctuations which persist in spite of the ensemble average. (Basis states with a significant mismatch in quantum numbers will tend to have small probabilities independent of the details of the mapping.) It would, of course, be possible to construct a random matrix model which mimics these violent local fluctuations. However, the underlying physical origin of these fluctuations (i.e., similarities or gross differences in the nodal structure of the basis functions) implies the existence of global correlations between matrix elements. These are beyond the scope of simple random matrix models. We believe that the present model is as successful in reproducing the average behaviour of real billiard wave functions as can reasonably be expected.

It is possible to study these differences in wave functions empirically by considering the parametric motion of energy levels. Experimentally, one studies eigenvalues as some external variable (e.g., the temperature or the shape of the billiard) is varied continuously [15]. Measurements of the velocity autocorrelation function have already been reported in the literature [16], and curvature distributions are anticipated shortly [17]. In the GOE,
Figure 5: Average probabilities, $[d_{ik}(k)]^2$, for the state $k = 10$. The upper curve was obtained from the RBE using 200 matrices of dimension $M = 400$. The lower curve was obtained for the ensemble of 200 real billiards as described in the text.

for example, one considers the eigenvalues of matrices $\cos \lambda H_1 + \sin \lambda H_2$ with $H_1$ and $H_2$ elements of the GOE. Thus, it is meaningful to consider $de_i/d\lambda$. An ensemble average over $H_2$ (for any choice of $H_1$ and $\lambda$) leads to a Gaussian distribution of slopes with a variance which is independent of $H_1$ and $\lambda$. A similar calculation in the RBE would involve the replacement of the matrix $R$ of eqn.(8) by $\cos \lambda R_1 + \sin \lambda R_2$. The ensemble average over $R_2$ also gives a Gaussian distribution of slopes. In this case, however, the variance depends on $R_1$ and $\lambda$. (This is also true for an ensemble of real billiards.) This represents a clear distinction between RBE (and real billiards) and the GOE. It can be studied with existing data. What remains to be seen is whether the distribution of these variances is sufficiently general to
be of interest.
VI. Discussion and Conclusions

Despite considerable experimental and numerical success, there are no compelling arguments establishing a relationship between the properties of real quantum billiard systems and random matrix ensembles. This led Ott to observe [18] that “[w]hile there are some suggestive theoretical results supporting the random matrix conjecture for quantum chaos, . . . the validity of this conjecture and its range of applicability, if valid, remain unsettled.” In our view, the most disturbing aspects of the Gaussian orthogonal ensemble are (i) that its spectrum is unbounded from below, (ii) that, as a consequence, there is little meaning in considering the convergence of individual eigenvalues or eigenfunctions, and (iii) that the technical simplification of basis independence is physically unreasonable. We have presented a random matrix model of quantum billiards with a much clearer connection to genuine billiard problems which meets each of these concerns. The spectrum is necessarily positive. Eigenvalues and eigenfunctions show power-law convergence as the dimension of the matrix is increased. None the less, numerical investigation of the usual spectral measures (i.e., the level spacing distribution and the spectral rigidity) are in fully quantitative agreement with GOE results. In this sense, the RBE can be thought of as an “interpolating” ensemble which serves to make the connection between real billiards and the GOE more plausible. We believe that it represents significant new evidence in support of the random matrix conjecture for quantum chaos.

The most striking differences between the present results and those of the GOE are in the behaviour of eigenfunctions. Here, our model is in good agreement with results obtained for an ensemble average over real quantum billiards. We have noted that existing experimental data (e.g., the parametric motion of individual levels) can reveal shortcomings of the GOE. It remains to be seen whether “more realistic” random matrix models such as the one considered here can extend the range of applicability of random matrix techniques in quantum chaos.

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