PolyAR: A Highly Parallelizable Solver For Polynomial Inequality Constraints Using Convex Abstraction Refinement

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Abstract: Numerical tools for constraints solving are a cornerstone to control verification problems. This is evident by the plethora of research that uses tools like linear and convex programming for the design of control systems. Nevertheless, the capability of linear and convex programming is limited and is not adequate to reason about general nonlinear polynomials. This limitation calls for new tools that are capable of utilizing the power of linear and convex programming to reason about general multivariate polynomials. In this paper, we propose PolyAR, a highly parallelizable solver for polynomial inequality constraints. PolyAR provides several key contributions. First, it uses convex relaxations of the problem to accelerate the process of finding a solution to the set of the non-convex multivariate polynomials. Second, it utilizes an iterative convex abstraction refinement process which aims to prune the search space and identify regions for which the convex relaxation fails to solve the problem. Third, it allows for a highly parallelizable usage of off-the-shelf solvers to analyze the regions in which the convex relaxation failed to provide solutions. We compared the scalability of PolyAR against Z3 8.9 and Yices 2.6 on control design problems. Finally, we demonstrate the performance of PolyAR on designing switching signals for continuous-time linear switching systems.

Keywords: Polynomial inequalities, Abstraction refinement, convex programming.

1. INTRODUCTION

Advances in constraints programming have opened several avenues for control system synthesis and verification of hybrid systems. For instance, linear programming and convex optimization are heavily used in a multitude of control system design and analysis tools. Recent surveys showed that such numerical tools had changed the control system design philosophy.

Nevertheless, linear and convex programming are limited in their ability to problems with specific structures. In several hybrid system design and verification problems, constraints are neither linear nor convex. This calls for efficient solvers that can reason about general multivariate polynomial constraints. The first CAD algorithm was introduced by [Collins (1975)]. However, the use of CAD is often limited by the number of variables in the input polynomials, a reflection of its worst-case complexity that grows in a doubly exponential fashion in the number of variables [England and Davenport (2016)].

To alleviate the CAD’s doubly exponential issue, we introduce PolyAR, a highly parallelizable solver that uses convex programming and abstraction refinement to solve general multivariate polynomial inequality constraints. The main novel contributions of this work can be summarized as follows:

• PolyAR uses a novel convex abstraction refinement process where the original problem is iteratively relaxed into a series of convex programming problems with the aim to find the solution and prune the search space. Second, it refines such abstraction where it becomes tighter with each iteration of the algorithm. Finally, it examines in parallel all the identified small volume regions left from the abstraction using off-the-shelf solvers (e.g., Z3 and Yices) to search for a solution in these regions.

• We validate our approach by comparing the scalability of the proposed PolyAR solver with respect to the latest versions of state-of-art non-linear real arithmetic solvers, such as Z3 8.9 and Yices 2.6, on synthesizing stabilizing static output feedback controller (SOF) for linear time-invariant (LTI) continuous systems and designing non-parametric controller for the non-linear Duffing oscillator.

• We demonstrate the performance of PolyAR on the problem of designing switching signals for continuous-time linear switching systems.

Related work: The original CAD algorithm that was introduced by Collins (1975) was the first algorithm that solves general polynomial inequality constraints. However, due to Collins CAD’s high time complexity, there have been improvements to this algorithm. Hong (1990) proposed an improvement of the projection operator in Collins CAD. However, the execution time of the modified version of the algorithm is still limited by the number of variables. McCallum (1998) introduced a new projection operator which is a subset of Hong projection operator, removing redundant polynomials. However, McCallum proved that lifting over a sign-invariant CAD with this projection set is not sufficient to guarantee sign-invariance which makes the algorithm prone to error. The AB solver tool proposed by Bauer et al. (2007) leverages a generic nonlinear optimization tool for solving non-linear constraints. However,
2. PROBLEM FORMULATION

2.1 Notation

We denote by \(x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n\) the set of real-valued variables, where \(x_i \in \mathbb{R}\). We denote by \(I_n = [\underline{d}, \overline{d}] \times \cdots \times [\underline{d}, \overline{d}] \subset \mathbb{R}^n\) the \(n\)-dimensional region. We denote the space of polynomials with \(n\) variables and coefficients in \(\mathbb{R}\) by \(\mathbb{R}[x_1, x_2, \ldots, x_n]\). We denote by \(\land\) the Boolean conjunction. A set of the form \(L_0^i(f) = \{(x_1, \ldots, x_n) | f(x_1, \ldots, x_n) \leq 0\}\) and \(U_0^i(f) = \{(x_1, \ldots, x_n) | f(x_1, \ldots, x_n) \geq 0\}\) is called zero sublevel (superlevel) set of \(f\), respectively.

2.2 Main Problem

In this paper, we focus on polynomial inequality constraints with input ranges as closed boxes which are described in the following definition:

**Definition 1.** A polynomial inequality constraint \(F = I^n \land P_m\) consists of:

- a set of interval constraints:
  \[I^n = \bigwedge_{i=1}^{n} x_i \in [\underline{d}_i, \overline{d}_i],\] (1)

- a polynomial constraint:
  \[P_m = \bigwedge_{i=1}^{m} p_i(x_1, \ldots, x_n) \leq 0,\] (2)

where \(p_i(x) = p_i(x_1, \ldots, x_n) \in \mathbb{R}[x_1, x_2, \ldots, x_n]\) is a polynomial over variables \(x_1, \ldots, x_n\). Without loss of generality, \(\bigwedge_{i=1}^{m} p_i(x) \geq 0\) and \(\bigwedge_{i=1}^{m} p_i(x) = 0\) can be encoded in constraint number (2).

We are now in a position to state the problem that we will consider in this paper.

**Problem 1.** \(\exists x = (x_1, \ldots, x_n)\) subject to \(F = I^n \land P_m\).

3. ABSTRACTION REFINEMENT OF HIGHER ORDER POLYNOMIALS USING QUADRATIC POLYNOMIALS

Traditional techniques for solving Problem 1 focus on finding all the \(n\) roots of the \(m\) polynomials and check all the regions between two successive roots to assign a positive/negative sign for each of these regions. Therefore, solving Problem 1 is known to be a doubly combinatorial problem in \(n\) with a total running time that is bounded by \((md)^{2^n}\) under the maximum degree among polynomials in \(P_m\).

In problems that are doubly exponential in the input space \(n\), it is beneficial to isolate subsets of the search space in which the solution is guaranteed not to exist. Recall that Problem 1 asks for an \(x\) in \(\mathbb{R}^n\) for which all the polynomials are negative. Therefore, a solution does not exist in subsets of \(\mathbb{R}^n\) at which one of the polynomials is always positive. Similarly, isolating regions of the input space for which some of the polynomials are negative is also beneficial to finding the solution faster.

Our tool’s main novelty is to use “convex abstractions” of the polynomials to find subsets of \(L_0^i(p)\) and \(L_0^i(p)\) efficiently. Indeed such “abstractions” may not be able to identify all regions for which the polynomial is positive or negative, which calls for an “abstraction refinement” process in which these “convex abstractions” become tighter with each iteration of the algorithm.

Figure 1(top) visualizes the proposed abstraction refinement process. Starting from a polynomial \(p(x) \in \mathbb{R}[x]\) and an interval \(I_n \subset \mathbb{R}^n\), we compute two quadratic polynomials:

\[O_1^i(x) \geq p(x) \quad \forall x \in I_n,\]
\[U_1^i(x) \leq p(x) \quad \forall x \in I_n,\]

where \(O\) and \(U\) stands for Over-approximate and Under-approximate quadratic polynomials, respectively, and the subscript in \(O_1^i(x)\) and \(U_1^i(x)\) encodes the iteration index of the abstraction refinement process. Computing such upper and lower abstractions can be carried out efficiently using Taylor approximation. Please refer to the example depicted in Figure 1 (top) for a visualization of \(O_1^i(x)\) and \(U_1^i(x)\) for one dimensional higher order polynomial (order \(\geq 3\)) defined in the closed interval \([\underline{d}, \overline{d}] \subset \mathbb{R}\).

The next step is to use the quadratic abstractions to isolate subsets of \(L_0^i(p)\) and \(L_0^i(p)\). It is particularly direct to show that the zero superlevel set of \(U_1^i(x)\) is a subset of \(L_0^i(p)\), i.e., \(L_0^i(U_1^i(x)) \subseteq L_0^i(p)\). Similarly, the zero sublevel set of \(O_1^i(x)\) is a subset of \(L_0^i(p)\), i.e., \(L_0^i(O_1^i(x)) \subseteq L_0^i(p)\). Thanks to the fact that \(O_1^i(x)\) and \(U_1^i(x)\) are quadratic polynomials, finding their zero superlevel and zero sublevel sets, respectively, can be computed efficiently. Referring to the example in Figure 1(top), these zero superlevel and sublevel sets are \(L_0^i(U_1^i(x)) = [\underline{d}, \overline{d}]\) and \(L_0^i(O_1^i(x)) = [\underline{d}, \overline{d}]\), respectively.

It is clear from Figure 1(top) that the abstractions \(O_1^i(x)\) and \(U_1^i(x)\) fail to identify all subsets of \(L_0^i(p)\) and \(L_0^i(p)\). Therefore, the next step is to compute tighter over and under approximations of \(p(x)\). Such a refinement process can be carried out by removing the zero superlevel and the zero sublevel sets, i.e., \(L_0^i(U_1^i(x))\) and \(L_0^i(O_1^i(x))\), identified using the previous abstraction and computing new over and lower approximation, as shown in Figure 1(bottom). The process of abstraction refinement can continue until the remaining subsets of the search space, in which case we call them ambiguous regions, and with some abuse of notation, denoted them by \(L_1^{i,-/+}(p)\), are small enough to be analyzed using off-the-shelf solvers. More details about the proposed abstraction refinement process are given in the next section.

4. ALGORITHM ARCHITECTURE

In this section, we describe the different steps used by our solver PolyAR to solve Problem 1.

Our design methodology for the PolyAR tool aims to reduce the number of the required abstraction refinement and tries to find a solution early on in the process. To that end, the tool starts by computing a set of convex (quadratic or linear) polynomials \(O_0^i(x), i = 1, \ldots, m\), that over approximate the original polynomials. The next
Once a polynomial \(p_i\) is selected, the next step is to apply the abstraction refinement process on \(p_i\) (Abst_Refin, Line 11 in Algorithm 1). The objective of the abstraction refinement process is to identify subsets of the positive regions \(L_0^+(p_i)\) and negative regions \(L_0^-(p_i)\). Indeed, such abstraction refinement may not be able to identify all positive and negative regions, and hence a remaining portion of the search space may not be identified to belong to either \(L_0^+(p_j)\) or \(L_0^-(p_j)\) in which case it belongs to the ambiguous region \(L_0^{\pm/-}(p_i)\). The abstraction refinement process of the polynomial \(p_i\) ensure that the volume of such ambiguous regions are below a certain user defined threshold.

The process of using the convex solver to find the solution and abstracting one polynomial continues. Since a solution of Problem 1 needs to lie in a negative region for all the polynomials, we confine the tool attention to the negative regions identified by the abstraction refinement in the previous iterations (Line 16 in Algorithm 1) to accelerate the process of searching for the solution.

While excluding the positive regions identified in previous iterations does not affect the tool (since a solution is guaranteed not to exist in such regions), excluding the ambiguous regions from the next iterations may affect the correctness of the tool. Therefore, the last step in the PolyAR tool is to examine all the identified ambiguous regions using off-the-shelf solvers (e.g., Z3 and Yices) to search for a solution in these regions (Solver_Parallel, Line 20 in Algorithm 1). Because the volume of these ambiguous regions is smaller than a user-defined threshold, the execution time of running off-the-shelf tools on such small volume regions is shorter than solving the original problem. This reflects that the number of roots for each polynomial is limited in small regions. Moreover, searching for a solution in these ambiguous regions can be highly parallelized, leading to an extra level of efficiency. This process is summarized in Algorithm 1 and Figure 2. We describe in detail each block algorithm that constitutes Algorithm 1 in the next subsections.

**Algorithm 1** PolyAR\((F)\)

**Input:** \(F = F^0 \land P_m\)

**Output:** STATUS, \(x_{Sol}\)

1. \(Neg = \{L_0^2\}\)
2. \(Ambig = \{\}\)
3. \(List\_pols = \{p_1, \ldots, p_m\}\)
4. while \(List\_pols \neq \emptyset\) do
5. \(x_{Sol} := Conv\_Solver(Neg, List\_pols)\)
6. if \(x_{Sol} \neq None\) then
7. \(STATUS := SAT\)
8. return STATUS, \(x_{Sol}\)
9. end if
10. \(p_j = Select\_Poly(List\_pols)\)
11. \(L_0^+(p_j), L_0^-(p_j), L_0^{\pm/-}(p_j) := Abst\_Refin(Neg, p_j)\)
12. \(Ambig.add\left(L_0^{\pm/-}(p_j)\right)\)
13. if \(L_0^+(p_j) == \emptyset\) then
14. break
15. end if
16. \(Neg = L_0^-(p_j)\)
17. \(List\_pols = List\_pols \setminus p_j\)
18. end while
19. if \(List\_pols \neq \emptyset\) then
20. \(STATUS, x_{Sol} := Solver\_Parallel(Ambig, P_m)\)
21. return STATUS, \(x_{Sol}\)
22. else
23. \(STATUS := SAT\)
24. \(x_{Sol} = center(Neg)\)
25. return STATUS, \(x_{Sol}\)
26. end if

### 4.1 Early Termination Using Conv\_Solver:

The objective of the Conv\_Solver (Algorithm 2) is to search for a solution to Problem 1 using the information...
Algorithm 2 Conv_Solver \((Neg, \text{List}_{\text{pols}})\)

**Input:** \(Neg, \text{List}_{\text{pols}}\)

**Output:** \(x_{\text{Sol}}\)

1. for region \(\in Neg\) do
2. for \(i \in \text{List}_{\text{pols}}\) do
3. if Taylor_{over} \((p_i, x), 2\) is convex then
4. \(O^p_i(x) = \text{Taylor}_{over}(p_i, x), 2\)
5. else
6. \(O^p_i(x) = \text{Taylor}_{over}(p_i, x), 1\)
7. end if
8. end for
9. \(x_{\text{Sol}} := \arg \min_{x \in \text{region}} 1 \ s.t. \ O^p_i(x) \leq 0 \) (see eq. (5))
10. return \(x_{\text{Sol}}\)
11. end for

of (i) a set of closed convex regions \(Neg\) identified by the previous iterations of the abstraction refinement process and (ii) a list of polynomials \(\text{List}_{\text{pols}}\) that have not yet been processed by the abstraction refinement process.

Our approach is to compute a convex over-approximation of the polynomials in \(\text{List}_{\text{pols}}\) using Taylor approximation. To that end, we recall the definition of Taylor polynomials:

**Definition 2.** Let \(f : \mathbb{R}^n \rightarrow \mathbb{R}\) be a two times differentiable in open interval around a point \(a \in \mathbb{R}^n\), then \(f(x)\) can be written in terms of first and second order Taylor polynomials, \(T_1(x)\) and \(T_2(x)\), around the neighborhood of \(a\), as follows:

\[
\begin{align*}
T_1(x) &= (x-a)^T D_f(a) + R_1(c_1) \\
T_2(x) &= (x-a)^T D_f(a) + \frac{1}{2} (x-a)^T H_f(a)(x-a) + R_2(c_2)
\end{align*}
\]

where \(T_1(x) = (x-a)^T D_f(a)\) and \(T_2(x) = (x-a)^T D_f(a) + \frac{1}{2} (x-a)^T H_f(a)(x-a)\). \(D_f(a)\) and \(H_f(a)\) denote the Gradient vector and the Hessian matrix of \(f\) at the point \(a\). \(R_1(c_1)\) and \(R_2(c_2)\) are reminders that depend on \(a\) and two points \(c_1 \in \mathbb{R}^n\) and \(c_2 \in \mathbb{R}^n\) that are located in the neighborhood of \(a\).

We upper-bound \(R_1(c_1)\) and \(R_2(c_2)\) by \(M_1 \in \mathbb{R}_+\) and \(M_2 \in \mathbb{R}_+\), i.e., \(|R_1(c_1)| \leq M_1\) and \(|R_2(c_2)| \leq M_2\). Hence:

\[
\begin{align*}
T_1(x) - M_1 &\leq f(x) \leq T_1(x) + M_1 \\
T_2(x) - M_2 &\leq f(x) \leq T_2(x) + M_2
\end{align*}
\]

Next, we check the convexity of the obtained second Taylor approximation and use it to compute the over-approximation function \(O^p_i\) whenever it is convex (Line 4 in Algorithm 2). Otherwise, we use the first Taylor approximation instead (Line 6 in Algorithm 2). Finally, for each \(region\) in the set of negative regions \(Neg\), we solve the following convex feasibility problem:

\[
x_{\text{Sol}} := \arg \min_{x \in \text{region}} 1 \ s.t. \ O^p_i(x) \leq 0, \ i \in \text{List}_{\text{pols}}\)
\]

4.2 Abstraction Refinement Using \text{Abst_Refin}:

Given the set of negative regions \(Neg\) identified by the previous abstraction refinement process along with the polynomial \(p_i\) selected by the \text{Select_Poly} algorithm, the objective of the \text{Abst_Refin} algorithm is to find subsets of the zero sublevel sets of \(p_i\) that lie inside \(Neg\). The output of this algorithm are subsets of \(L^+_0(p_i)\) and \(L^-_0(p_i)\). The remainder of \(Neg\) is then considered to be part of the ambiguous regions \(L^+_0/\neg(p_i)\). To do so, for every region in \(Neg\), the tool initiates a list of ambiguous regions \(\text{List}_{\text{Ambig}}\), which will contain all the ambiguous regions from the abstraction refinement (Line 4 in Algorithm 3). Next, it selects one element from these ambiguous regions (Line 5 in Algorithm 3) and performs the abstraction refinement on this region iteratively until the volume of the remaining ambiguous region is smaller than a user-defined threshold (Line 6 in Algorithm 3). During the iterative abstraction refinement, all the identified zero sublevel and superlevel subsets are stored in the sets \(L^+_0(p_j)\) and \(L^-_0(p_j)\), respectively.

While the zero sublevel (superlevel) sets of the quadratic over-approximation (under-approximation) are ellipsoid or hyperboloid in general, we opt to represent all the subsets of \(L^+_0(p_j)\) and \(L^-_0(p_j)\) as \(n\)-dimensional hypercubes. This choice reflects the fact that off-the-shelf solvers (e.g., Z3 and Yices) can exploit the geometry of hypercubes to accelerate their computations. The process of finding these hypercubes can be summarized as follows:

1. **Step 1:** Compute the largest polytope inside the ellipsoid or hyperboloid representing the zero sublevel (superlevel) sets of the quadratic over-approximation (under-approximation) of \(p_j\). To that end, we use a set of user-defined templates for the polytope.

2. **Step 2:** The previous step uses user-defined templates to find the polytope, such templates may fail and return an infeasible solution. In such scenarios, we split the ambiguous region into two (along the longest dimension) until a polytope is found.

3. **Step 3:** Finally, we under approximate the computed polytope with hypercubes.

This process is visualized in Figure 3. The details of each of these steps are given in the following subsections.

![Fig. 3. Polytopic under-approximation of a 2-dimensional ellipse sublevel set \(L^+_0(p_j)\). \(p_N^N\) presents the under-approximate polytope inscribed in \(L^+_0(p_j)\), and \(B^N\) represents the axis-aligned box of maximum volume inscribed in \(p_N^N\).](image-url)

**Step 1: Computing the largest polytope subset of \(L^+_0(p_j)\) and \(L^-_0(p_j)\)**

Given the over-approximation \(O^p_i\) computed using Taylor polynomials (detailed in Section 4.1) and a convex ambiguous region \((\text{Ambig}_{reg})\), we start by computing a set of \(n+1\) vertices \(v_1^N, \ldots, v_{n+1}^N\) that are inscribed in the ambiguous region \(\text{Ambig}_{reg}\). Each vertex can be computed by solving the following convex optimization problem:

\[
v_i^N = \arg \min_{v_i \in \text{Ambig}_{reg}} (l_i^T v_i) \ s.t. \ O^p_i(v_i) \leq 0,
\]

where \(l_i\) is a user defined normal vector (or template) (see Figure 3 for graphical representation of such normal
Algorithm 3 Abst_Refin(Neg, p_j)

Input: Neg, p_j
Output: L_0^-(p_j), L_0^+(p_j), L_0^N^p \ (p_j)
1: L_0^-(p_j) = \{ \}, L_0^+(p_j) = \{ \}, L_0^N^p \ (p_j) = \{ \}
2: for region \in Neg do
3: vertices^N = \{ \}, vertices^p = \{ \}
4: List\_Ambig\_reg = \{region\}
5: Ambig\_reg = Select\_region(List\_Ambig\_reg)
6: while Volume(Ambig\_reg) > Vol\_threshold do
7: List\_Ambig\_reg = List\_Ambig\_reg \backslash Ambig\_reg
8: for i \in (1, \ldots , n+1) do
9: v_i^N = \arg\min_{v_i \in Ambig\_reg} \left( \frac{1}{\|v_i\|} \right) \ s.t. \ O^p(v_i) \leq 0.
10: if v_i^N \neq \text{None} then
11: vertices^N.add(v_i^N)
12: end if
13: v_i^p = \arg\min_{v_i \in Ambig\_reg} \left( \frac{1}{\|v_i\|} \right) \ s.t. \ U^p(v_i) \leq 0.
14: if v_i^p \neq \text{None} then
15: vertices^p.add(v_i^p)
16: end if
17: end for
18: if (vertices^N == \emptyset \ \text{and} \ \text{vertices}^p == \emptyset) then
19: Ambig\_eq1, Ambig\_eq2 := Half\_Div(Ambig\_reg)
20: List\_Ambig\_reg.add(Ambig\_eq1, Ambig\_eq2)
21: else if (vertices^N \neq \emptyset \ \text{and} \ \text{vertices}^p \neq \emptyset) then
22: \mathcal{P}^N = Convex\_Hull(vertices^N)
23: \mathcal{P}^P = Convex\_Hull(vertices^p)
24: B^N = Box(\mathcal{P}^N); B^P = Box(\mathcal{P}^P)
25: L_0^-(p_j).add(B^N); L_0^+(p_j).add(B^P)
26: Ambig\_reg = Ambig\_reg \backslash (B^N \cup B^P)
27: List\_Ambig\_reg.add(Ambig\_reg)
28: end if
29: Ambig\_reg = Select\_region(List\_Ambig\_reg)
30: end while
31: L_0^N^p \ (p_j).add(List\_Ambig\_reg)
32: end for
33: return L_0^-(p_j), L_0^+(p_j), L_0^N^p \ (p_j)

vectors). Using these vertices, we obtain the polytope \mathcal{P}^N as:

\[
\mathcal{P}^N = \text{Convex\_Hull}(v_1^N, \ldots , v_{n+1}^N).
\]

Thanks to the constraints in the optimization problem (6) along with the convexity of \(L_0^-(p_j)\), it is direct to conclude that the polytope \(\mathcal{P}^N \subseteq L_0^-(p_j)\). We compute the polytope \(\mathcal{P}^P \subseteq L_0^+(p_j)\) in a similar fashion using the under-approximation \(U^p\) (Line 23 in Algorithm 3).

Step 3: Under approximate the polytopes with axis aligned boxes: To compute the largest axis-aligned hypercube \(B^N\) inscribed inside the polytope \(\mathcal{P}^N\), we solve the following convex optimization problem Behroozi (2019):

\[
\begin{align*}
& \text{arg}\max_{(l_1^N, u_1^N, \ldots , l_n^N, u_n^N) \in \mathbb{R}^n} \sum_{k=1}^{n} \log (u_k^N - l_k^N) \\
& \text{s.t.} \sum_{k=1}^{n} (p_k^N + u_k^N - p_k^N - l_k^N) \leq c_i^N, \ i = 1, \ldots , n_p,
\end{align*}
\]

where \((l_1^N, u_1^N, \ldots , l_n^N, u_n^N) \in \mathbb{R}^n\) is the representation of the box \(B^N\) with \((l_k, u_k)\) is the lower/upper limit of the box in the kth dimension, \(p_k^N = \max\{p_k^N, 0\}\), \(p_k^N = \max\{-p_k^N, 0\}\), and \(c_i^N\) are the rows of the half-space matrix/vector representation of the polytope \(\mathcal{P}^N\).

4.3 Highly Parallelizable Analysis of Ambiguous Regions using Solver\_Parallel

Once all the ambiguous regions are identified, the next step is to analyze all of them using off-the-shelf solvers. In particular, PolyAR supports the use of the latest versions Z3 8.9 and Yices 2.6 solvers. Thanks to the fact that all the ambiguous regions are hypercubes, both these solvers can exploit the geometry of the region to accelerate their computations. Also, thanks to the fact that the volume of all ambiguous regions is lower than a user-defined threshold, the CAD algorithm can run efficiently. To that end, PolyAR tool runs multiple instances of Z3 or Yices to analyze all these ambiguous regions in parallel as summarized in Algorithm 4.

5. EXTENSION TO SMT SOLVING

We extend the PolyAR solver described in the previous sections to account for combinations of Boolean and Polynomial inequality constraints of the form:

\[
\exists(x_1, \ldots , x_n) \in \mathbb{R}^n, \ 
\text{subject to:}
\]

\[
\begin{align*}
& p_i(x_1, \ldots , x_n) \leq 0, \ i = 1, \ldots , m \quad (8) \\
& x_k \in [d_k, \bar{d}_k], \ k = 1, \ldots , n \quad (9) \\
& \varphi_1(b_1, \ldots , b_n) \leftrightarrow \text{TRUE}, \ j = 1, \ldots , r \quad (10) \\
& b_l \leftrightarrow (p_{l+1}(x_1, \ldots , x_n) \leq 0), \ l = 1, \ldots , h \quad (11)
\end{align*}
\]

where \(\varphi_1(b_1, \ldots , b_n)\) is any combinations of Boolean and pseudo-Boolean predicates.

We can create a Satisfiability Modulo Theory (SMT) solver by combining a SAT solver for Boolean and pseudo-Boolean constraints and a theory solver (PolyAR) for interval and polynomial constraints on real numbers by following the lazy SMT paradigm Barrett and Tinelli (2018). The SAT solver solves the combination of Boolean and pseudo-Boolean constraints using the David-Putnam-Logemann-Loveland (DPLL) algorithm and suggests satisfying assignments for the Boolean variables \(b\) and thus suggesting which polynomial constraints should jointly satisfied (or unsatisfied). The theory solver (PolyAR) checks the validity of the given assignments and provides an explanation of the conflict, i.e., an UNSAT certificate, whenever a conflict is found. Each certificate is a new Boolean constraint that will be used by the SAT solver to prune the search space.

While in the lazy SMT paradigm, the PolyAR solver needs to be executed multiple times with a different set of polynomial constraints, we modify the PolyAR solver to perform all the abstraction refinement for all the polynomials as a pre-processing step. This eliminates the
need to re-compute the same abstraction refinement every time the PolyAR solver is executed.

Whenever the SAT solver assigns one of the Boolean variables \( b_i \) in (11) to zero, then the PolyAR solver needs to guarantee that the corresponding polynomial \( p_{l+m} \) satisfy \( p_{l+m}(x) > 0 \) or equivalently \( -p_{l+m}(x) \leq 0 \). To eliminate the need to apply the convex abstraction refinement process for both \( p_{l+m}(x) \) and \( -p_{l+m}(x) \), the PolyAR solver computes the negative and positive boxes \( (B^N, B^P) \) only for \( p_{l+m}(x) \) and flips their usage for \( -p_{l+m}(x) \).

6. NUMERICAL RESULTS

In this section, we compare the performance of PolyAR to the state-of-the-art solvers Z3 8.9 and Yices 2.6. The objective of this comparison is to study the performance on:

- Problems that appear naturally in parametric controller synthesis. In particular, we focus on the problem of designing stabilizing SOF controllers for LTI systems Bahavarnia et al. (2020).
- Problems that appear in non-parametric controller synthesis for non-linear systems. In particular, we focus on the problem of designing a controller for the nonlinear Duffing oscillator Fiotou et al. (2006).
- Additionally, we demonstrate the performance of PolyAR on designing a hybrid switching system; a problem which state-of-the-art tools are incapable of handling.

All the experiments were executed on an Intel Core i7 2.6-GHz processor with 16 GB of memory.

6.1 Static Output Feedback Controller Synthesis for Linear Time Invariant Systems

In this subsection, we assess the scalability of the PolyAR solver compared to state-of-the-art solvers on control synthesis problems. In particular, we consider the problem of synthesizing a parametric controller for the following continuous LTI system:

\[
\dot{x} = Ax + Bu, \quad y = Cx,
\]

where \( x \in \mathbb{R}^{n_x} \) is the system state, \( u \in \mathbb{R}^{n_u} \) is the system control input, \( y \in \mathbb{R}^{n_y} \) is the system output, and the matrices \( A \in \mathbb{R}^{n_x \times n_x} \), \( B \in \mathbb{R}^{n_y \times n_u} \) and \( C \in \mathbb{R}^{n_y \times n_x} \) are the system matrices. We are interested in designing a static output feedback controller of the form:

\[
u = Ky,
\]

such that the resulting closed loop system:

\[
\dot{x} = (A + BK)Cx,
\]

is stable, i.e., the matrix \( A + BK \) is Hurwitz.

We follow the steps detailed in Bahavarnia et al. (2020) to pose the problem of designing the static output feedback controller as a set of polynomial constraints using the Routh-Hurwitz stability criteria. The Routh-Hurwitz stability criteria result in a set of \( n_A \) polynomials in the elements of the controller matrix \( K \). We consider five instances of the controller synthesis problem with the following parameters:

- **Example 1**: \( n_A = 3, n_B = 4, n_C = 4 \) which results in 3 polynomial constraints with 16 variables and maximum polynomial order of 4. We restrict the elements of the controller matrix to be inside \([-4, 7]\).
- **Example 2**: \( n_A = 3, n_B = 6, n_C = 5 \) which results in 3 polynomial constraints with 25 variables and maximum polynomial order of 3. We restrict the elements of the controller matrix to be inside \([-0.5, 1]\).
- **Example 3**: \( n_A = 2, n_B = 6, n_C = 6 \) which results in 2 polynomial constraints with 36 variables and maximum polynomial order of 3. We restrict the elements of the controller matrix to be inside \([0, 5]\).

We compare the execution times of these four solvers with Z3 8.9 and Yices 2.6. Table 1 shows the execution time for all the solvers. As evident by the results in Table 1, off-the-shelf solvers are incapable of solving all the five examples and they time out after one hour. On the other hand, and thanks to the abstraction refinement process, the PolyAR solver is able to solve all the instances in a few seconds, leading to 240X speed up in the total execution time in the PolyAR+Yices (max threads) case, evidence of the scalability of the proposed approach.

In the following, we give the stabilizing controller matrices \( K_1, K_2, K_3, K_4, K_5 \), and the two Boolean variables \( b_1 \) and \( b_2 \) that PolyAR (Yices) returned for Examples 1, 2, 3, 4, and 5:

| Example | \( K_1 \) | \( K_2 \) | \( K_3 \) | \( K_4 \) | \( K_5 \) |
|---------|----------|----------|----------|----------|----------|
| Example 1 | \[\begin{bmatrix} -4 & -2 & -1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 4 & 1 \end{bmatrix} \] | \[\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \] | \[\begin{bmatrix} 1 & 3 & 3 & 2 \\ 2 & 2 & 2 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \] | \(-8, i = 1, j = 1, -5, 2 \leq i, j \leq 7, 5\) | \[\begin{bmatrix} -2 & 4 & -2 \\ 5 & 6 & 1 \\ 1 & 6.99 & 1 \end{bmatrix} \] |

\( b_1 = True, b_2 = True. \)
It is easily to note that the solutions given by PolyAR (Yices) satisfies the two Boolean constraints, i.e., $k_{21} \times k_{22} \times k_{23} = -3.5 < 0$ and $k_{21} + k_{22} + k_{23} = -1.5 < -1$.

In conclusion, PolyAR+Yices (max thread) solver outperforms all the other solvers due to the effectiveness of Yices in reasoning about problems with small volumes.

6.2 Non-Linear Controller Design for a Duffing Oscillator

In this subsection, we assess the scalability of PolyAR solver compared to state-of-the-art solvers on synthesizing a non-parametric controller for a Duffing oscillator reported by Fotiou et al. (2006). The dynamics of the oscillator is given by the higher-order differential equation:

$$y^{(n)}(t) + \cdots + y^{(2)}(t) + 2\zeta y^{(1)}(t) + y(t) + y(t)^3 = u(t), \quad (12)$$

where $y \in \mathbb{R}$ is the continuous state variable and $u \in \mathbb{R}$ is the control input. The parameter $\zeta$ is the damping coefficient. The objective of the control is to regulate the state to the origin. To derive the discrete-time model, forward difference approximation is used (with sampling period of $h = 0.05$ time units). The resulting state space model with discrete state vector $x = [x_1, x_2, \ldots, x_n]^T = [y, y^{(1)}, \ldots, y^{(n-1)}]^T \in \mathbb{R}^{n-1}$ and input $u \in \mathbb{R}$ is:

$$\begin{bmatrix} x_1^{(1)} \\ x_2 \\ \vdots \\ x_n \\ \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & h & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -h & -2h & \cdots & \cdots & -h_1 \\ \end{bmatrix} \begin{bmatrix} x_1^{(0)} \\ x_2^{(0)} \\ \vdots \\ x_n^{(0)} \\ \end{bmatrix} + \begin{bmatrix} 0 \\ u \\ \vdots \\ 0 \\ \end{bmatrix}. \quad (13)$$

The previous equation is written in the form of $x(k+1) = Ax(k) + Bu(k) + E(x)$, which includes a nonlinear term $E(x) = [0, \ldots, -h_1^3(x(k))]^T$. Our objective is to design a non-parametric controller. To that end, we encode the controller as the solution of a feasibility problem of several constraints that capture the system dynamics, state/input constraints, and stability constraints as discussed below.

First, to ensure the stability of the resulting non-parametric controller, we consider the candidate quadratic Lyapunov function $V(x) = x^TPx$ with the symmetric positive definite matrix $P$ is a solution of the discrete-time Lyapunov equation $APA^T + P + Q = 0$ and is a positive definite matrix. Thanks to the fact that $E(x)$ satisfies $\lim_{x \to 0} \frac{E(x)}{\|x\|} = 0$ along with the Lyapunov's indirect method in Khalil (2002), one can directly conclude that $V(x)$ is indeed a Lyapunov function. For simplicity, we pick $Q = I_n$, where $I_n$ is the identity matrix of size $n$.

Moreover, to ensure the smoothness of the resulting controller signals, we add additional filters in the form of high order polynomial $L(x, u) \leq 0$. In addition, we consider the state-constraints of the form $\|x(k)\|_\infty \leq 0.6$.

The final non-parametric controller is then encoded as the solution of the following feasibility problem:

$$\exists x_1(k), \ldots, x_n(k), x_1(k+1), \ldots, u_n(k+1), u(k)$$

subject to:

$$x(k+1) = Ax(k) + Bu(k) + E(x),$$

$$V(x(k+1)) - V(x(k)) \leq -\epsilon,$$

$$L(x(k), u(k)) \leq 0,$$

$$\|x(k)\|_\infty \leq 0.6. \quad (14)$$

Since the PolyAR solver only handles polynomial inequalities, hence, we transform the equality constraint $x(k+1) = Ax(k) + Bu(k) + E(x)$ above into two inequalities $x(k+1) - Ax(k) + Bu(k) + E(x) \leq \epsilon$ and $x(k+1) - Ax(k) + Bu(k) + E(x) \geq -\epsilon$, where $\epsilon \in \mathbb{R}$ is a small value.

We consider three instances of the controller synthesis problem for the Duffing oscillator with the following parameters:

- $n = 2, \quad \zeta = 0.3, \quad x(0) = [0.4, 0.1]^T, \quad L(x(k), u(k)) = x_1^{11}(k) + x_2^{11}(k) - u_1^{10}(k)$, which results in 6 polynomial constraints with 3 variables and max polynomial order of 11.

- $n = 3, \quad \zeta = 1.0, \quad x(0) = [0.1, 0.1, 0.1]^T, \quad L(x(k), u(k)) = x_1^{10}(k) + x_2^{10}(k) + x_3^{10}(k) + u^3(k)$, which results in 8 polynomial constraints with 4 variables and max polynomial order of 5.

- $n = 4, \quad \zeta = 1.75, \quad x(0) = [0.1, 0.1, 0.1, 0.1]^T, \quad L(x(k), u(k)) = x_1^{10}(k) + x_2^{10}(k) + x_3^{10}(k) + x_4^{10}(k) - u^4(k)$, which results in 10 polynomial constraints with 5 variables and max polynomial order of 4.

We feed the resultant polynomial inequality constraint to PolyAR+Yices, PolyAR+Z3, Yices, and Z3. We solve the feasibility problem for $n = 2, n = 3, n = 4$. We set the timeout to be 1s. Figure 4 (left) shows the state-space evolution of the controlled Duffing oscillator for different solvers for number of variables $n$ of 2, 3, and 4. Figure 4 (right) shows the evolution of the execution time of the solvers during the 20 seconds. As it can be seen from Fig. 4, our solver PolyAR+Yices succeeded to find a control input $u$ that regulates the state to the origin for all $n$. However, off-the-shelf solvers are incapable of solving all the three instances and they early time out after one second out of the simulated 20 seconds.

6.3 Designing Switching Signals for Continuous-Time Linear Switching Systems

In this subsection, we show how to use the PolyAR solver to successfully design a controller for a continuous-time linear switching system. In particular, we consider the following switching dynamics:

$$\dot{x} = A_{\sigma(t)}x, \quad \sigma(t) = \{1, 2, 3\},$$
In our experiments, we pick the horizon $x$ modes switching signal switching times and the associated system mode that leads of time $A$ the matrices $σ$. Obstacle $j$ Goal $exp (\cdot)$ that can steer the state of the system to the goal these Boolean variables to encode the problem of designing subject to:

$$exp\left(\begin{array}{c} t_1 = 0.391, t_2 = 0.5, \text{ and } t_3 = 0.25 \text{ that ensures that } x(3) \text{ reaches a Goal while the intermediate states } x(2), x(1) \text{ avoid Obstacles.} \\
\end{array}\right)$$

In addition, we remark that the trajectory between $x(0)$ and $x(2)$ is making its way to the equilibrium point $[0,0]$. This is explained by the fact that the matrices $A_1$ and $A_2$ are stables. Our solver computes the necessary time $t_3$ that ensures that the final state $x(3) \in \text{Goal}$ and does not converge to the equilibrium point.

Fig. 5. The trajectory that starts from an initial state $x(0) = [40,30]^T$ and reaching a final state $x(3) \in \text{Goal}$ while avoiding the obstacles. The goal and the obstacles are represented with a red and yellow rectangle, respectively.

with $x(t) \in X \subset \mathbb{R}^2$ is the system state at time $t$ and the matrices $A_1, A_2,$ and $A_3 \in \mathbb{R}^{2 \times 2}$ represents three modes for the switching system. Consider the state space in Figure 5. The objective is to design a switching signal $σ(t)$ that can steer the state of the system to the goal set $\text{Goal} \subset X$ while avoiding entering the obstacle set $\text{Obstacle} \subset X$. For simplicity, we confine our attention to step-wise switching signals $σ(t)$. That is, we assume the switching signal $σ(t)$ will be constant for some amount of time $t_1, t_2, \ldots, t_L$. Our objective is then to design the switching times and the associated system mode that leads to the satisfaction of the reach-avoid specifications. To that end, we define a set of Boolean variables $b_{ij}$ such that $b_{ij}$ is equal to 1 whenever the $j$th mode is active during $t_i$. Given the initial condition of the system $x(0)$, we can use these Boolean variables to encode the problem of designing the switching signal as the following SMT constraints:

$$b_{11}, b_{12}, \ldots, b_{L1}, b_{L2}, x(1), \ldots, x(L), t_1, \ldots, t_L$$

subject to:

$$b_{11} \rightarrow x(1) = \exp(A_1t_1)x(0),$$
$$b_{12} \rightarrow x(1) = \exp(A_2t_1)x(0),$$
$$b_{13} \rightarrow x(1) = \exp(A_3t_1)x(0),$$
$$b_{11} + b_{12} + b_{13} = 1,$$
$$b_{L1} \rightarrow x(L) = \exp(A_1t_L)x(L-1),$$
$$b_{L2} \rightarrow x(L) = \exp(A_2t_L)x(L-1),$$
$$b_{L3} \rightarrow x(L) = \exp(A_3t_L)x(L-1),$$
$$b_{L1} + b_{L2} + b_{L3} = 1,$$
$$x(1), \ldots, x(L-1) \notin \text{Obstacle},$$

$$x(L) \in \text{Goal},$$

This is explained by the fact that the matrices $A_1$ and $A_2$ are stables. Our solver computes the necessary time $t_3$ that ensures that the final state $x(3) \in \text{Goal}$ and does not converge to the equilibrium point.

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