Quantum Chains with $GL_q(2)$ Symmetry

Masoud Alimohammadi

March 28, 2022

Institute for studies in Theoretical Physics and Mathematics
P.O.Box 19395-5746 Tehran, Iran
Department of Physics, Tehran University,
North Karegar, Tehran, Iran
email. alimohmd@irearn.bitnet .

Abstract

Usually quantum chains with quantum group symmetry are associated with representations of quantized universal algebras $U_q(g)$ . Here we propose a method for constructing quantum chains with $C_q(G)$ global symmetry , where $C_q(G)$ is the algebra of functions on the quantum group. In particular we will construct a quantum chain with $GL_q(2)$ symmetry which interpolates between two classical Ising chains. It is shown that the Hamiltonian of this chain satisfies in the generalised braid group algebra.
1 Introduction

Almost all integrable models in two dimensional statistical models, quantum field theories in 1+1 dimensions and quantum chains \cite{1} owe their integrability to some quantum group symmetry. For example in lattice models, if one assigns the local Boltzmann weights of a vertex or an IRF model to be the elements of the R matrix corresponding to a quantum group, then the model will be integrable due to the existence of a one parameter family of commuting transfer matrices. In a sense one can say that local quantum group symmetry ensures integrability. Although global quantum group symmetry does not mean integrability, construction of models with such symmetries may be interesting and important as a first step toward understanding the mechanism of integrability.

Recently new types of 2 and 3 states quantum chains were constructed and shown to possess $U_q(sl(2))$ symmetry \cite{2-4}. The strategy followed in \cite{4} was to define the Hamiltonian as

$$H = \sum_{j=1}^{L} id \otimes \ldots \otimes H_j \otimes id \ldots \otimes id$$

(1)

where $H_j$ acts on sites $j$ and $j+1$ as

$$H_j = (\pi_j \otimes \pi_{j+1})[Q_j(\Delta(C))].$$

(2)

Here $j$ denotes the site of the lattice, $C$ is the quadratic Casimir of $U_q(sl(2))$, $\Delta$ is the coproduct, $\pi_j$ is a typical type $b$ representation \cite{5} of $U_q(sl(2))$ assigned to site $j$, and finally $Q_j$ is a polynomial function of degree $d \leq m$ where the integer $m$ is characterized by the value of $q$ ($q^m = 1$). This Hamiltonian is by construction $U_q(sl(2))$ invariant. The invariance is due to the centrality of the Casimir. For the particular form of the Hamiltonian of the 2-states and 3-states quantum chains see Ref. \cite{4}.

As is well known any quantum group is characterized by two algebras \cite{6,7}. The first being the deformation of the universal enveloping algebra which is denoted by $U_q(g)$ and the second one which is the deformation of the algebra of functions on the group which is denoted by $C_q(G)$. So far everything which has been done concerning the construction of physical models with quantum group symmetry have been based on representation theory of $U_q(g)$. However in the quantum case the second algebra, $C_q(G)$, has also a representation theory which is completely different from that of $U_q(g)$ \cite{8,9}.

The novelty of the representations of $C_q(G)$ is best understood when one considers the classical ($q \rightarrow 1$) limit of $C_q(G)$. In this limit, representations of $U_q(g)$ approach
those of the classical Lie algebra \( g \) while those of \( C_q(G) \) collapse to trivial one dimensional representations, since \( C_q(G) \) will become a commutative algebra. Therefore there is no parallelism between the representation theories in the deformed and the undeformed case. Naively one expects that paying attention to physical models which are \( C_q(G) \) invariant may open up a new road in the study of integrable models. At the present stage this is only a hope and real justification for it will exist if one can somehow gauge a global symmetry of this kind in a particular physical model.

In this letter we construct a quantum chain which has a global \( C_q(GL(2)) \) symmetry, hereafter called \( GL_q(2) \) symmetry for simplicity.

2 The quantum group \( GL_q(2) \) and it’s cyclic representations

The quantum group \( GL_q(2) \) is defined by the generators \( 1, a, b, c \) and \( d \), collected in the form of a quantum matrix [7]

\[
T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}
\]

and relations

\[
ab = qba \quad \quad \quad ac = qca \\
bd = qdb \quad \quad \quad cd = qdc \\
bc = cb \\
ad - da = (q - q^{-1})bc
\]

The coproduct which is used in tensor multiplication of representations is defined by:

\[
\Delta \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} a & b \\ c & d \end{pmatrix}
\]

The quantum determinant \( D_q = ad - qbc \) is central and group-like, That is:

\[
\Delta D_q = D_q \otimes D_q
\]

If \( q \) is a root of unity ( \( q^p = 1 \) ), in addition to the determinant, all the elements \( a^p, b^p, c^p \) and \( d^p \) are central. In this case the algebra has a \( p \) dimensional cyclic representation which
is constructed as follows \[^8\]: One first defines a state \(|0\rangle\) which is a common eigenvector of \(b\) and \(c\) with eigenvalues \(\mu\) and \(\nu\) respectively:

\[
|0\rangle > = \mu |0\rangle > \quad \quad \quad |0\rangle > = \nu |0\rangle >
\]

and then builds up the representation space \(V\) as the linear span of vectors \(|n\rangle \equiv d^n|0\rangle > \quad 0 \leq n \leq p - 1\) . It is then easy to show that \(V\) is invariant under the action of \(GL_q(2)\).

\[
\begin{align*}
 d|n >= & |n + 1 >, \quad d|p - 1 >= \eta |0\rangle > \\
 b|n >= & \mu q^n |n >, \quad c|n >= \nu q^n |n > \\
 a|n >= & (\xi + q^{2n-1} \mu \nu)|n - 1 > \\
 a|0 >= & \frac{1}{\eta} (\xi + \mu \nu^{-1}) |p - 1 >
\end{align*}
\]

Here \(\eta\) is the central value of \(d^p\) and \(\xi\) is the value of the q-determinant \(D_q\).

It can be easily checked that the parameters \(\eta, \xi, \mu\) and \(\nu\) are all independent and hence each representation is characterized by the values of these parameters and is denoted by \(\pi(\eta, \xi, \mu, \nu)\).

### 3 Quantum chains with \(GL_q(2)\) symmetry

We construct the quantum chains with \(GL_q(2)\) symmetry as follows: To each site \(1 \leq j \leq L\) , we assign a representation \(\pi_j = \pi(\eta_j, \xi_j, \mu_j, \nu_j)\) , the Hilbert space is the tensor product \(\otimes_{j=1}^{L} V_j\) , where \(V_j\) is the \(p\)-dimensional representation space of \(\pi_j\).

At first glance it seems that the analogue of the construction of Ref.[4] in the case of \(C_q(GL(2))\) is to replace the Casimir \(C\) in eq.(2) by the quantum determinant \(D_q\). However this procedure leads to a trivial Hamiltonian due to the group like property of \(D_q\) (eq.(5)), which makes \(\pi_j \otimes \pi_{j+1}(Q_j(\Delta(D)))\) proportional to the identity. However there is one interesting possibility and it is to define \(H_j\) as:

\[
H_j = \pi_j \otimes \pi_{j+1}(Q_j(\Delta(a^p), \Delta(b^p), \Delta(c^p), \Delta(d^p)))
\]

Here the crucial point is that although in an irreducible representation \(a^p, b^p, c^p\) and \(d^p\) are proportional to the identity, in a tensor product of representations they are not so due to the mixing of the generators in their coproducts (eq.(4)). The Hamiltonian constructed
in this way is \( GL_q(2) \) invariant by construction.

**Two state quantum chains**

Now we restrict ourselves to the case \( p = 2 \) \( (q = -1) \). From eq.(6) we obtain the 2-dimensional cyclic representation of \( GL_q(2) \), which in the explicit matrix notation takes the form:

\[
\pi(a) = \begin{pmatrix} 0 & \gamma/\eta \\ \gamma & 0 \end{pmatrix} \quad \pi(b) = \begin{pmatrix} \mu & 0 \\ 0 & -\mu \end{pmatrix}
\]

\[
\pi(c) = \begin{pmatrix} \nu & 0 \\ 0 & -\nu \end{pmatrix} \quad \pi(d) = \begin{pmatrix} 0 & 1 \\ \eta & 0 \end{pmatrix}
\]

(8)

where \( \gamma = \xi - \mu \nu \). This represent a continuous 4-parameter family of two dimensional representations for \( GL_q(2) \) \( (q = -1) \). If \( t \) stands for \( a, b, c \) or \( d \) , then a straightforward calculation shows that:

\[
(\pi_j \otimes \pi_{j+1}) \Delta t^2 = l_t 1 \otimes 1 + n_t \sigma_x \otimes \sigma_x + p_t \sigma_y \otimes \sigma_y - iq_t \sigma_x \otimes \sigma_y - ir_t \sigma_y \otimes \sigma_x
\]

(9)

where

\[
n_t = \frac{1}{4}(1 - \eta_j - \eta_{j+1} + \eta_j \eta_{j+1})m_t
\]

\[
p_t = -\frac{1}{4}(1 + \eta_j + \eta_{j+1} + \eta_j \eta_{j+1})m_t
\]

\[
q_t = \frac{1}{4}(-1 + \eta_j - \eta_{j+1} + \eta_j \eta_{j+1})m_t
\]

\[
r_t = \frac{1}{4}(-1 - \eta_j + \eta_{j+1} + \eta_j \eta_{j+1})m_t
\]

(10)

and

\[
m_a = \frac{2\gamma_j \gamma_{j+1} \mu_j \nu_{j+1}}{\eta_j \eta_{j+1}} \quad , \quad m_b = -\frac{2\mu_j \mu_{j+1} \gamma_j}{\eta_j}
\]

\[
m_c = -\frac{2\nu_j \nu_{j+1} \gamma_{j+1}}{\eta_{j+1}} \quad , \quad m_d = 2\mu_{j+1} \nu_j
\]

(11)

The explicit expressions of \( l_t \)’s are not necessary in this stage. As the simplest choice for the polynomial (in eq.(7)) we set :

\[
Q_0 = \alpha_a a^2 + \alpha_b b^2 + \alpha_c c^2 + \alpha_d d^2
\]

(12)
where $\alpha_t$'s are arbitrary constants. Combination of eqs.(9) and (12) leads to the following Hamiltonian:

$$
H_j = A\sigma^i_x\sigma^{i+1}_x + B\sigma^i_y\sigma^{i+1}_y + -iC\sigma^i_x\sigma^{i+1}_y - iD\sigma^i_y\sigma^{i+1}_x
$$

(13)

where

$$
A = \beta_j(1 - \eta_j - \eta_{j+1} + \eta_j\eta_{j+1})
$$

$$
B = -\beta_j(1 + \eta_j + \eta_{j+1} + \eta_j\eta_{j+1})
$$

$$
C = \beta_j(-1 + \eta_j - \eta_{j+1} + \eta_j\eta_{j+1})
$$

$$
D = \beta_j(-1 - \eta_j + \eta_{j+1} + \eta_j\eta_{j+1})
$$

(14)

and $\beta_j = \alpha_a m_a + \alpha_b m_b + \alpha_c m_c + \alpha_d m_d$. If the factor $\beta_j$ is site independent, then modulo a constant overall factor the Hamiltonian becomes:

$$
H_0 = \sum_j \{(1 - \alpha)\sigma^j_x\sigma^{j+1}_x - (1 + \alpha)\sigma^j_y\sigma^{j+1}_y + \sqrt{1 - \alpha^2}(\sigma^j_x\sigma^{j+1}_y - \sigma^j_y\sigma^{j+1}_x)\}
$$

(15)

Now the condition of Hermiticity of the Hamiltonian restricts the parameters $\eta_j$ and $\eta_{j+1}$ to the following form:

$$
\eta_j = \alpha + i\sqrt{1 - \alpha^2}
$$

$$
\eta_{j+1} = \eta_j^*
$$

(16)

where $\alpha$ is a real parameter. Under this condition the Hamiltonian takes the following simple form:

$$
H_0 = \sum_j \{(1 - \alpha)\sigma^j_x\sigma^{j+1}_x - (1 + \alpha)\sigma^j_y\sigma^{j+1}_y + \sqrt{1 - \alpha^2}(\sigma^j_x\sigma^{j+1}_y - \sigma^j_y\sigma^{j+1}_x)\}
$$

(17)

This is the desired Hamiltonian with $GL_q(2)$ symmetry.

Imposing the condition of site independence on $\beta_j$ in eq.(14), the $m_t$'s are restricted to be site independent, as $\alpha_t$'s are arbitrary constants. Solving these conditions and using eq.(16), results the following relation between the parameters of the different representations of the sites:

$$
\gamma_j = \gamma_{j+2}, \quad \nu_j = \nu_{j+2}, \quad \mu_j = \mu_{j+2}
$$

$$
\frac{\gamma_j}{\eta_j} = \frac{\gamma_{j+1}}{\eta_{j+1}}, \quad \frac{\mu_j}{\nu_j} = \frac{\mu_{j+1}}{\nu_{j+1}}.
$$

(18)
So the whole representations of the sites specify only by four complex parameters \( \nu_1, \mu_1, \nu_2, \gamma_1 \) and one real parameter \( \alpha \).

There are several observations on the above Hamiltonian (eq.(17)):

1) Instead of the original continuous parameter \( q \), the Hamiltonian depends on the continuous parameter \( \alpha \), which comes from the representation.

2) In the two limits \( \alpha = 1 \) and \( \alpha = -1 \), this Hamiltonian degenerates into a exactly solvable chain, that is \( \sum_{j=1}^{L} \sigma_i^j \sigma_i^{j+1} \). So our \( GL_q(2) \) invariant Hamiltonian interpolates between two \( \mathbf{xx} \) and \( \mathbf{yy} \) classical Ising chains, when \( \alpha \) is changed continuously from \(-1\) to \(1\).

3) It is crucial to note that the Hamiltonian (17) is not equivalent to an \( \hat{n} \hat{n} \) chain where \( \hat{n} \) is a new unit vector in the \( x-y \) plane. That is the transformations \( \sigma_x \rightarrow a\sigma_x + b\sigma_y \) and \( \sigma_y \rightarrow c\sigma_x + d\sigma_y \) can not diagonalize the Hamiltonian.

4) If one defines \( U_i = 2 + H_i \), then there exists the following interesting relation between \( U_i \)'s:

\[
U_i^2 = 4U_i
\]

\[
(U_iU_{i\pm1}U_i - U_{i\pm1}U_iU_{i\pm1})(U_i - U_{i\pm1}) = 64(1 - \alpha^2).
\]

The above equation is the generalised braid group algebra \[^2\].

5) The simplest situation that \( H_0 \) can be solved exactly is the case of 2-site lattice. In this case there are four states with eigenvalues 2, 2, -2 and -2. It can be shown that the degenerate orthonormal states \(| 2 >_\pm \) and \(| -2 >_\pm \), are two-dimensional representations of \( \Delta(t) \)'s.

**Higher state quantum chains**

Choosing \( p=3 \), one may expect to obtain a 3-state \( GL_q(2) \) invariant quantum chain. However, for \( q^3 = 1 \) it is seen by computation that:

\[
\Delta a^3 = a^3 \otimes a^3 + b^3 \otimes c^3
\]

\[
\Delta b^3 = a^3 \otimes b^3 + b^3 \otimes d^3
\]

\[
\Delta c^3 = c^3 \otimes b^3 + d^3 \otimes d^3
\]

\[
\Delta d^3 = c^3 \otimes a^3 + d^3 \otimes c^3
\]
which means that the Hamiltonian (7) is proportional to the identity. This phenomena may occur for all odd integers $p$.

**Acknowledgement** I would like to thank V. Karimipour for useful discussions.

**References**

1. For a review see H. Saluer and J. B. Zuber : ” Integrable Lattice Models and Quantum Groups ” Proceedings of the Trieste Spring School (1990), World Scientific, Singapore

2. H. Hinrichsen and V. Rittenberg : Phys. Lett. B 275 (1992) 350

3. H. Hinrichsen and V. Rittenberg : Phys. Lett. B 304 (1993) 115

4. D. Arnaudon and V. Rittenberg : Phys. Lett. B 306 (1993) 86

5. P. Roche and D. Arnaudon: Lett. Math. Phys. 17 (1989) 295

6. V. G. Drinfeld ” Quantum Groups ”, Proc. Internat. Congr. Math. (Berkeley) , vol. 1, Academic Press , New York, 1986

7. N. Yu. Reshetikhin , L.A. Takhtajan , and L.D. Faddeev ; Lenningrad Math. Journal Vol. 1 (1990) 193

8. M. L. Ge, X. F. Liu , and C. P. Sun : J. Math. Phys. 38 (7) 1992

9. V. Karimipour; Lett. Math. Phys. 28 1993, 207

  V. Karimipour; J. Phys. A: Math. Gen. 26 (1993) 6277