Strong Decays of Light Vector Mesons

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The vector meson strong decays $\rho \to \pi\pi$, $\phi \to KK$, and $K^* \to \pi K$ are studied within a covariant approach based on the ladder-rainbow truncation of the QCD Dyson-Schwinger equation for the quark propagator and the Bethe–Salpeter equation for the mesons. The model preserves the one-loop behavior of QCD in the ultraviolet, has two infrared parameters, and implements quark confinement and dynamical chiral symmetry breaking. The 3-point decay amplitudes are described in impulse approximation. The Bethe–Salpeter study motivates a method for estimating the masses for heavier mesons within this model without continuing the propagators into the complex plane. We test the accuracy via the $\rho$, $\phi$, and $K^*$ masses and then produce estimates of the model results for $m_{\pi_0}$ and $m_{\eta_0}$ as well as the proposed exotic vector $\pi_1(1400)$.

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I. INTRODUCTION

The study of light-quark pseudoscalar and vector mesons is an important tool for understanding how QCD works in the non-perturbative regime. The pseudoscalars are important because they are the lightest observed hadrons and are the Goldstone bosons associated with dynamical chiral symmetry breaking. The ground state vector mesons are important because, as the lowest spin excitations of the pseudoscalars, they relate closely to hadronic $q\bar{q}$ modes that are electromagnetically excited. The decay of vector mesons to a pair of pseudoscalars proceeds via a P-wave interaction; this probes a different aspect of the meson Bethe–Salpeter [BS] amplitudes than does the electroweak decay constant which is essentially the projection of the relative wavefunction onto the origin of separation.

To calculate these coupling constants, we use an approach based on the Dyson–Schwinger equations [DSEs], which form an excellent tool to study nonperturbative aspects of hadron properties in QCD [1]. The approach is consistent with quark and gluon confinement [1, 2, 3, 4, 5], generates dynamical chiral symmetry breaking [6, 7], and is Poincaré invariant. It is straightforward to implement the correct one-loop renormalization group behavior of QCD [8], and obtain agreement with perturbation theory in the perturbative region. Provided that the relevant Ward–Takahashi identities [WTIs] are preserved in the truncation of the DSEs, the corresponding currents are conserved. Axial current conservation induces the Goldstone nature of the pions and kaons [8]; electromagnetic current conservation produces the correct electric charge of the mesons without fine-tuning. These properties are implemented here within the rainbow truncation of the DSE for the dressed quark propagators together with the ladder approximation for the Bethe–Salpeter equation [BSE] for meson bound states. The model we use has two infrared parameters. Previous work has shown this model to provide an efficient description of the light-quark pseudoscalar and vector mesons [9, 10]. Furthermore, in impulse approximation, the elastic charge form factors of the pseudoscalars [11] and the electroweak transition form factors of the pseudoscalars and vectors [12] are in excellent agreement with data. This suggests that the strong decays of the vector mesons should be well-described in impulse approximation without parameter adjustment.

The Euclidean metric that we employ facilitates the modeling of the gluon 2-point function or more generally the effective quark-quark interaction; it also is the metric within which practical DSE solutions are available in the literature. A complicating element is that solution of the BSE for meson bound states requires an analytic continuation in the meson momentum to reach the mass-shell. The resulting quark $p^2$ in the Bethe–Salpeter integral covers a certain domain in the complex plane that differs from the real spacelike axis by an amount that increases with meson mass. For low mass mesons such as $\pi$ and $K$, the required continuation of DSE solutions for the quark propagator is not difficult. For the ground state vector mesons, the difficulties are much greater but have been overcome directly [10]. For meson masses greater than about 1 GeV$^2$ the numerical requirements of this procedure become burdensome. Furthermore, complex plane singularities of rainbow DSE solutions for propagators limit this direct approach to mesons with masses below about 1.2 GeV [13].

Here we test the reliability of an approximation that uses a Taylor expansion to make the continuation away from the real spacelike $p^2$ axis. Since information about quark propagators is then needed only on the real space-like $p^2$ axis, this approximation is considerably easier to implement. It is similar in spirit to an approximation employed some time ago [14]. If the behavior for real mo-
menta dominates the physics, low orders of a Taylor expansion should be effective. This work is an exploration of that conjecture since we can test against the direct complex plane results for BSE solutions for masses of \( \rho, K^* \) and \( \phi \) within the same model. Independently of how the behavior of the propagators in the complex plane is obtained, the calculation of meson form factors or decays in impulse approximation requires that the relative quark momentum of each vertex amplitude be also continued to the complex plane. In the interests of consistency, we employ a Taylor expansion method for both vertex amplitudes and propagators in the calculation of decays.

The results from the Taylor expansion method are used to motivate a method for estimating the masses and BSE solutions for heavier mesons within this model without continuing the propagators into the complex plane. We test the accuracy of this real-axis approximation for the ground state pseudoscalar and vector mesons, and then produce estimates of the model results for \( m_{\rho_1} \) and \( m_{b_1} \) as well as the proposed exotic \( 1^- \) vector \( \pi_1(1400) \).

In Sec. II we review the formulation that underlies a description of vector meson strong decays within a modeling of QCD through the DSEs and BSEs. We discuss the diagrams needed for a calculation in the impulse approximation for \( \rho \to \pi \pi, \phi \to KK, \) and \( K^* \to K \pi \). In Sec. III we discuss the details of the model and we include a comparison with recent lattice-QCD results for the quark propagator. In Sec. IV we present our numerical results for masses and decays, and discuss the performance of the calculational technique. In Sec. V we the technique for estimating the masses of heavier states is evaluated. A concluding discussion is given in Sec. VI.

II. VECTOR MESON STRONG DECAYS

We use the Euclidean metric where \( \{ \gamma_{\mu}, \gamma_{\nu} \} = 2 \delta_{\mu\nu}, \gamma_{\mu}^\dagger = \gamma_{\mu} \) and \( a \cdot b = \sum_{i=1}^4 a_ib_i \). The invariant amplitude for the coupling of a vector state with helicity \( \lambda \) and momentum \( p_1+p_2 \) to a pair of pseudoscalars (or scalars) having momenta \( p_{1/2} \) has the form \( \mathcal{M}^\lambda = \Lambda^\lambda \epsilon^\lambda_\mu \) where \( \epsilon^\lambda_\mu \) is the polarization vector of the vector state, and the vertex which describes the coupling of the Lorentz component of the vector state to the pair of pseudoscalars has the form \( \Lambda^\lambda \mu = g \langle p_1 - p_2 \rangle_\mu \) where \( g \) is the coupling constant. In the case that the vector state and the pseudoscalar states are \( \bar{q}q \) bound states with flavor labels \( \bar{a}b, \bar{c}a \) and \( \bar{b}c \) respectively, the impulse approximation for \( \Lambda^\lambda_{a b, c} \) can be expressed as

\[
\Lambda^\lambda_{a b, c}(P; Q) = \text{tr}_{sc} \int_k \mathcal{M}^\lambda(q) \Gamma_{P \bar{p}}^{a\bar{b}}(q, q_+) S^a(q_+) \times \Gamma_{Q \bar{q}}^{b\bar{c}}(q_+, q) S^b(q),
\]

where \( q = k + P/2 \) and \( q_+ = k - P/2 \pm Q/2 \), and \( \text{tr}_{sc} \) denotes a trace over color and Dirac spin. For convenience we have introduced \( Q = p_1 + p_2 \) as the momentum of the vector, and \( P = (p_1 - p_2)/2 \). We denote the properly normalized BS amplitude at a vertex having an outgoing quark of flavor \( f_1 \) and momentum \( p_1 \) and an incoming quark of flavor \( f_2 \) and momentum \( p_- \) by \( \Gamma_{f_1 f_2}(p_+, p_-) \).

The notation \( \int_k \mathcal{M}^\lambda = \int_k d^4k/(2\pi)^4 \) stands for a translationally invariant regularization of the integral, with \( \Lambda \) being the regularization mass-scale. The regularization can be removed at the end of all calculations, by taking the limit \( \Lambda \to \infty \). It is understood that the same regularization is applied to the calculated dressed quark propagators \( S(q) \) and BS amplitudes appearing in Eq. (1), and that the quark propagators are renormalized at a convenient spacelike momentum scale before taking the limit \( \Lambda \to \infty \).

For mesons with a unique quark flavor content, the physical vertices are given by the \( \Lambda^\lambda_{a b, c}(P; Q) \). For states like \( |q^0> = (|u\bar{u}> - |d\bar{d}>)\sqrt{2} \) linear combinations are required. We have, for example,

\[
\Lambda^\lambda_{\rho} \rho \to \pi^+ \pi^- = \frac{1}{\sqrt{2}}[\Lambda^{uu \bar{d}d}(P; Q) - \Lambda^{dd \bar{u}u}(-P; Q)],
\]

\[
\Lambda^\lambda_{\phi} \phi \to K^+ K^- = \Lambda^{ss \bar{u}d}(P; Q),
\]

\[
\Lambda^\lambda_{K^{*+} \to K^+ \pi^0} = \frac{1}{\sqrt{2}}\Lambda^{us \bar{u}d}(P; Q),
\]

\[
\Lambda^\lambda_{K^{*+} \to K^0 \pi^+} = \Lambda^{ds \bar{d}u}(P; Q).
\]

The vertices for modes that differ from these by electric charge are obtained by quark flavor exchange. For example under isospin exchange of \( u \) and \( d \) flavors, the last two modes become \( K^{*0} \to K^0 \pi^0 \) and \( K^{*0} \to K^+ \pi^- \) respectively. In this work we use isospin symmetry in which \( u \) - and \( d \)-quarks have identical strong interactions and properties except electric charge. Under these circumstances, \( \Lambda^\lambda_{uu \bar{d}d} = \Lambda^\lambda_{ds \bar{d}u} \) and there is only one independent \( K^0 \to K^\pi \) vertex.

The physical coupling constants for each \( v \to pp \) decay are identified from the vertices at the mass shell according to

\[
\Lambda_{\mu \to pp}(P; Q^2 = -m_v^2) = 2P^T_{\mu \bar{v} \to pp},
\]

where \( P^T \) is the component of \( P \) perpendicular to the vector meson momentum \( Q \). The decay width of a vector of mass \( m_v \) is given by

\[
\Gamma = \frac{3}{3} \sum_{\lambda=1}^3 \frac{\hat{\rho}}{2 m_v} |\mathcal{M}^\lambda|^2,
\]

where \( \hat{\rho} \) is the invariant phase space factor, the invariant amplitude is \( \mathcal{M}^\lambda = \Lambda^\lambda \epsilon^\lambda_\mu \), and \( \epsilon^\lambda_\mu \) is the polarization vector. The explicit form is

\[
\Gamma = \frac{g^2}{48 \pi m_v^3} \lambda^{3/2} (m_v^2, m_{p_1}^2, m_{p_2}^2),
\]

where

\[
\lambda(a,b,c) = a^2 + b^2 + c^2 - 2ab - 2ac - 2bc,
\]

and \( m_{p_1} \) and \( m_{p_2} \) are the masses of the pseudoscalar decay products.
A. Dyson–Schwinger Equations

The dressed quark propagator and the meson BS amplitudes are solutions of their respective DSEs, namely

\[ S(p)^{-1} = Z_2 i \not{p} + Z_4 m(\mu) + Z_1 \int_q g^2 D_{\mu\nu}(p-q) \frac{\lambda^i}{2} \gamma_\mu S(q) \Gamma_i^a(q,p), \]  

(10)

and

\[ \Gamma^{ab}(p_+,p_-) = \int_q K(p,q;Q) S^a(q+) \Gamma^{ab}(q_+,q-) S^b(q_-), \]

(11)

where \( D_{\mu\nu}(k) \) is the renormalized dressed-gluon propagator, \( \Gamma_i^a(q,p) \) is the renormalized dressed quark-gluon vertex, and \( K \) is the renormalized \( q\bar{q} \) scattering kernel that is irreducible with respect to a pair of \( q\bar{q} \) lines. The quark momenta are \( q_\pm \); the meson momentum is \( P = q_+ - q_- \) and satisfies \( P^2 = -m^2 \).

The solution of Eq. (11) is renormalized according to

\[ S(p)^{-1} = \gamma \cdot p + m(\mu) \]  

at a sufficiently large spacelike \( \mu^2 \), with \( m(\mu) \) the renormalized quark mass at the scale \( \mu \). In Eq. (11), \( S, \Gamma_i^a, \) and \( m(\mu) \) depend on the quark flavor, although we have not indicated this explicitly. The renormalization constants \( Z_2 \) and \( Z_4 \) depend on the renormalization point and the regularization mass-scale, but not on flavor: in our analysis we employ a flavor-independent renormalization scheme.

To implement Eq. (12) while preserving the constraint \( P = q_+ - q_- \), it is necessary to specify how the total momentum is partitioned between the quark and the antiquark. The general choice is \( q_+ = q + \eta P \) and \( q_- = q - (1-\eta)P \) where \( \eta \) is the momentum partitioning parameter. The choice of \( \eta \) is equivalent to a choice of relative momentum \( q ; \) physical observables should not depend on the choice. This provides us with a convenient check on numerical methods.

The meson BS amplitude \( \Gamma^{ab} \) is normalized according to the canonical normalization condition

\[ 2 P_{\mu} = \text{tr}_{sc} \frac{\partial}{\partial P_{\mu}} \left\{ \int_q \Gamma^{ab}(q', \bar{q}) S^a(q_+) \Gamma^{ab}(\bar{q}, q') S^b(q_-) \right\}, \]

(12)

at the mass shell \( P^2 = Q^2 = -m^2 \), with \( \bar{q} = q + \eta Q \), \( q' = q - (1-\eta)Q \), and similarly for \( k, k' \). Thus the derivative in Eq. (12) acts only upon the propagators \( S \) and the kernel \( K \). For vector mesons, it is understood that one must contract and average over the Lorentz indices of the (transverse) BS amplitudes due to the three independent polarizations.

For pseudoscalar bound states the BS amplitude is commonly decomposed into \( \not{q} \)

\[ \Gamma(q_+, q_-) = \gamma_5 [i E + \not{P} F + \not{q} G + \sigma_{\mu\nu} q_\mu P_\nu H], \]

(13)

where the 4 independent covariants have been constructed from the momentum basis \( q \) and \( P \) rather than from \( q_+ \) and \( q_- \). Hence the invariant amplitudes \( E, F, G \) and \( H \) are Lorentz scalar functions \( f(q^2, q \cdot P; \eta) \) that have an explicit dependence upon the momentum partitioning parameter \( \eta \). The dependence of the amplitudes upon \( q \cdot P \) can be conveniently represented by the following expansion based on Chebyshev polynomials

\[ f(q^2, q \cdot P) = \sum_{i=0}^{\infty} U_i(\cos \theta)(qP)^i f_i(q^2), \]

(14)

where \( \cos \theta = q \cdot P/(qP) \). For charge-parity eigenstates such as the pion, each amplitude \( E, F, G, \) and \( H \) will have a well-defined parity in the variable \( q \cdot P \) if one chooses \( \eta = 1/2 \). In this case, these amplitudes are either entirely even \((E, F, \) and \( H)\) or odd \((G)\) in \( q \cdot P \), and only the even \((E, F, \) and \( H)\) or odd \((G)\) Chebyshev moments \( f_i \) are needed for a complete description.

Since a massive vector meson bound state is transverse, the BS amplitude requires eight covariants for its representation. We choose the transverse projection of the form

\[ \Gamma_{\mu}(q_+, q_-) = \gamma_\mu V_1 + q_\mu \not{q} V_2 + q_\mu P V_3 + \gamma_5 \epsilon_{\mu\nu\alpha\beta} q_\nu P_\beta V_4 + q_\mu V_5 + \sigma_{\mu\nu} q_\nu V_6 + \sigma_{\mu\nu} P_\nu V_7 + q_\mu \sigma_{\alpha\beta} q_\alpha P_\beta V_8. \]

(15)

This form is a variation of that used in Ref. [7] that is simpler and easier to use in many respects. The invariant amplitudes \( V_i \) are Lorentz scalar functions of \( q^2 \) and \( q \cdot P \) and, for charge eigenstates, they are either odd or even in \( q \cdot P \). For the \( 1^- \) rho meson, \( V_3 \) and \( V_6 \) are odd, the other amplitudes are even. To reverse the charge parity, one simply reverses this odd-even property of the amplitudes.

III. LADDER-RAINBOW MODEL

We employ the model that has been developed recently for an efficient description of the masses and decay constants of the light pseudoscalar and vector mesons [7]. This consists of the rainbow truncation of the DSE for the quark propagator and the ladder truncation of the BSE for the pion and kaon amplitudes. The required effective \( q\bar{q} \) interaction is constrained by perturbative QCD in the ultraviolet and has a phenomenological infrared behavior. In particular, the rainbow truncation of the quark DSE, Eq. (12), is

\[ Z_1 g^2 D_{\mu\nu}(k) \Gamma_i^a(q,p) \rightarrow \mathcal{G}(k^2) D_{\mu\nu}^{\text{free}}(k) \gamma_\mu \frac{\lambda^i}{2}, \]

(16)

where \( D_{\mu\nu}^{\text{free}}(k = p - q) \) is the free gluon propagator in Landau gauge. The consistent ladder truncation of the BSE, Eq. (11), is

\[ K(p_+, q_+; P) \rightarrow -\mathcal{G}(k^2) D_{\mu\nu}^{\text{free}}(k) \frac{\lambda^i}{2} \gamma_\mu \otimes \frac{\lambda^i}{2} \gamma_\nu, \]

(17)
and this Landau gauge, rainbow-ladder truncation of the DSEs with renormalization group improvement, can be found in the originating work (9).

where $k = p - q$. These two truncations are consistent in the sense that the combination produces vector and axial-vector vertices satisfying the respective WTs. In the axial case, this ensures that in the chiral limit the ground state pseudoscalar mesons are the massless Goldstone bosons associated with chiral symmetry breaking (7, 8). In the vector case, this ensures electromagnetic current conservation.

The model is completely specified once a form is chosen for the “effective coupling” $\mathcal{G}(k^2)$. The ultraviolet behavior is chosen to be that of the QCD running coupling $\alpha(k^2)$; the ladder-rainbow truncation then generates the correct perturbative QCD structure of the DSE-BSE system of equations. The phenomenological infrared form of $\mathcal{G}(k^2)$ is chosen so that the DSE kernel contains sufficient infrared enhancement to produce an empirically acceptable amount of dynamical chiral symmetry breaking as represented by the chiral condensate (16).

We employ the Ansatz found to be successful in earlier work (6, 7)

$$\frac{\mathcal{G}(k^2)}{k^2} = \frac{4\pi^2 D k^2}{\omega^b} e^{-k^2/\omega^2} + \frac{4\pi^2 \gamma_m \mathcal{F}(k^2)}{1 \ln \left[ \frac{1 + k^2/\Lambda_{QCD}^2}{\tau} \right]^2}, \quad (18)$$

with $\gamma_m = \frac{12}{33 - 2N_f}$ and $\mathcal{F}(s) = (1 - \exp(-s/4m_s^2))/s$. The first term implements the strong infrared enhancement in the region $0 < k^2 < 1$ GeV$^2$ required for sufficient dynamical chiral symmetry breaking. The second term serves to preserve the one-loop renormalization group behavior of QCD. We use $m_\tau = 0.5$ GeV, $\tau = e^2 - 1$, $N_f = 4$, and we take $\Lambda_{QCD} = 0.234$ GeV. The renormalization scale is chosen to be $\mu = 19$ GeV which is well into the domain where one-loop perturbative behavior is appropriate (6, 7). The remaining parameters, $\omega = 0.4$ GeV and $D = 0.93$ GeV$^2$ along with the quark masses, are fitted to give a good description of $\langle q\bar{q}\rangle$, $m_{\pi/K}$ and $f_\pi$. The subsequent values for $f_K$ and the masses and decay constants of the vector mesons $\rho, \phi, K^*$ are found to be within 10% of the experimental data (6), see Tables I and III. A detailed analysis of the relationship between QCD

![FIG. 1: DSE solution (9) for quark propagator amplitudes compared to recent lattice data (17, 18).](image-url)

In Figs. I and II we compare the DSE model (9) propagator amplitudes defined by $S(p) = Z(p^2)[ip + M(p^2)]^{-1}$ with the most recent results in lattice QCD using staggered fermions in Landau gauge (7). These simulations were done with the Asqtad improved staggered quark action, which has lattice errors of order $O(a^4)$ and $O(a^2 g^2)$. In previous such comparisons (14, 20) the lattice data was less reliable. Fig. I shows both $M(p)$ and $Z(p)$ obtained with a bare lattice mass of $m_a = 0.036$ in lattice units, which corresponds to a bare mass of 57 MeV in physical units. The DSE calculations use a current mass value of 75 MeV at $\mu = 1$ GeV to match the lattice mass function around 3 GeV; this current mass is about 0.6 $m_s$. There is agreement in the qualitative infrared structure of the mass function particularly in the way the infrared enhancement sets in. Since the lattice simulation produces the regulated but un-renormalized propagator, the scale of the field renormalization function $Z$ is arbitrary and only the shape is a meaningful comparison. For this reason, we have rescaled the lattice data for $Z$ so that they match the DSE solution above 3 GeV. The ladder-rainbow DSE model typically produces a $Z$ that saturates much slower than does the lattice $Z$; this may signal a deficiency of the bare gluon-quark vertex.

The lattice work (7) also produced a linear extrapolation to the chiral limit mass function $M_0(p)$ and we compare this to the DSE result in Fig II. In principle the leading UV behavior of this mass function is proportional to the chiral condensate. The extraction of this from earlier lattice data has been attempted but there are uncertainties still to be resolved (21). We note here that above 1 GeV there is excellent agreement between the present DSE model and the lattice results.

Recent reviews (1, 22) put this model in a wider perspective. These reviews include a compilation of results...
for both meson and baryon physics with similar models, an analysis how quark confinement is manifest in solutions of the DSEs, and both finite temperature and finite density extensions. The question of the accuracy of the ladder-rainbow truncation has also received some attention; it was found to be particularly suitable for the flavor octet pseudoscalar mesons since the next-order contributions in a quark-gluon skeleton graph expansion, have a significant amount of cancellation between repulsive and attractive corrections [23].

IV. RESULTS

A. Method of Calculation

In order to identify the meson mass $m$ from the BSE, Eq. (11), in Euclidean metric, it is convenient to first introduce the linear eigenvalue $\lambda(P^2)$ of the kernel, continue the equation in $P^2$ to the timelike region, and then find $m$ such that $\lambda(-m^2) = 1$. The BSE kernel involves terms linear in the 4-vector $P$ and they are given the continuation $P \to (i m, 0)$. Thus with the quark propagators written as $S(p) = -i \sigma \sigma V(p^2) + \sigma \sigma (p^2)$, the kernel of the BSE involves the amplitudes $\sigma (q_{\perp}^2)$, where $\sigma = V, S$ and $q_{\perp}^2 = q^2 - m^2/4 + i q m$. Here $q$ is a direction cosine, and we have adopted equal momentum partitioning $\eta = 1/2$ for convenience. The amplitudes $\sigma (q_{\perp}^2)$ are in principle required to be known in a parabolic domain of the complex $q_{\perp}^2$ plane that includes the positive real axis and that extends symmetrically in the imaginary direction and in the negative (timelike) direction by amounts that grow with $m$. Previous work within the present approach and model has proceeded by use of the quark DSE to make the required analytic continuation of these propagator amplitudes. For $m > 1$ GeV, the difficulty of the necessary numerical methods can outweigh the benefits of solution, and certainly accuracy becomes problematic. For this reason, we explore an approximate method.

The eigenvalue $\lambda(P^2)$ should be real to obtain a real mass (no meson decay mechanisms are present in the ladder BSE kernel) and thus one can speculate that the sub-domain consisting of the positive real axis for $q_{\perp}^2$ might dominate the physics. To explore this systematically, we use a Taylor expansion of $\sigma (q_{\perp}^2)$ to reach the argument values off the positive real axis. Thus the only information needed about the quark propagator amplitudes from solution of the DSE are their values and derivatives on the Euclidean momentum domain. Within the BSE kernel we thus employ

$$\sigma (q_{\perp}^2) = \sigma (q_{\perp}^2) + \Delta_{\pm} (q_{\perp}^2) + \cdots, \quad (19)$$

where $q_{\perp}^2 = \text{max}[0, \Re (q_{\perp}^2)]$ and $q_{\perp}^2 + \Delta_{\pm}^2 = q_{\perp}^2$. For practical reasons we are interested only in low orders of this expansion.

B. Masses and Decay Constants

In Table II, we show the results for the masses and decay constants of $\pi$ and $K$ obtained from the Taylor expansion treatment of $q_{\perp}^2$ in quark propagator amplitudes. The effect of truncation of the 4 Dirac covariants to just the canonical $\gamma_5$ covariant is indicated. Comparison is made with results from use of quark amplitudes along $\Re (q_{\perp}^2)$ only, and with the model-exact results.

|                | $m_\pi$ | $f_\pi$ | $m_K$ | $f_K$ |
|----------------|---------|---------|-------|-------|
| 1st order Taylor |         |         |       |       |
| All 4 ampls     | 0.117   | 0.111   | 0.420 | 0.134 |
| $E$ amp only    | 0.106   | 0.086   | 0.383 | 0.103 |
| 2nd order Taylor |         |         |       |       |
| All 4 ampls     | 0.138   | 0.131   | 0.498 | 0.157 |
| $E$ amp only    | 0.121   | 0.098   | 0.436 | 0.116 |
| 3rd order Taylor |         |         |       |       |
| All 4 ampls     | 0.138   | 0.131   | 0.496 | 0.153 |
| $E$ amp only    | 0.121   | 0.098   | 0.435 | 0.114 |
| Real axis only  | 0.123   | 0.099   | 0.440 | 0.127 |
| Model exact     | 0.138   | 0.131   | 0.497 | 0.155 |
| Experiment      | 0.1385  | 0.131   | 0.496 | 0.160 |

FIG. 2: The chiral limit DSE mass function compared to the lattice chiral extrapolation [17, 18].

TABLE II: Masses and decay constants (in GeV) for $\pi$ and $K$ obtained from the Taylor expansion treatment of $q_{\perp}^2$ in quark propagator amplitudes. The effect of truncation of the 4 Dirac covariants to just the canonical $\gamma_5$ covariant is indicated. Comparison is made with results from use of quark amplitudes along $\Re (q_{\perp}^2)$ only, and with the model-exact results.
evident in the figure at second order in the Taylor expansion method. However, the figure also suggests that at larger timelike $P^2$ a converged behavior is not yet clearly evident at third order.

The results in Table I for $\rho, K^*$ and $\phi$ illustrate this in more detail. At second order the masses and $f_\rho$ are within 1% of the model exact results while $f_{K^*}$ and $f_\phi$ are within 5% and 3% respectively. Since $K^*$ is not a charge conjugation eigenstate, the solution of the BSE is more difficult due the the lack of definite parity in the variable $q \cdot P$. Furthermore, one expects larger errors in the decay constants than in the masses since the former require normalization of the BS amplitudes via Eq. (12) and the derivatives therein add to the numerical difficulty. The third order results for the masses improve slightly for $\rho$ and $K^*$ but $m_\phi$ deteriorates slightly to acquire a 2% error; the decay constants remain within 5% of the exact values. The results do not indicate that the Taylor expansion has yet converged and this correlates with the behavior evident in Fig. 3. Table I also makes clear that a truncation of the eight Dirac covariants to just the dominant one ($\gamma_\mu$) will lead to a persistent error of at least 15-20%. In the case of the $K^*$ this tends to raise the mass to the level where we are unable to determine a converged result.

![FIG. 3: The BSE eigenvalue in the $\rho$ channel calculated at the indicated orders in the Taylor expansion compared to the model exact result $\mathbf{1}$. The physical point is at $\lambda = 1$.](image)

The evident difficulty in use of the Taylor expansion approach as the mass of the state increases is most likely related to the occurrence of complex conjugate singularities of the employed quark propagator amplitudes $\sigma_\mu(q^2)$. In rainbow approximation, which we employ, the occurrence of complex conjugate logarithmic branch points with timelike values of $\mathbb{M}(q^2)$ is well-known for the fermion propagator in both QED and QCD. For the present model, we find numerically that the non-analytic behavior nearest to the origin is at $q_{\perp}^2 = -0.207 \pm i0.331$ GeV$^2$ for the $u/d$ quark, and at $q_{\perp}^2 = -0.376 \pm i0.602$ GeV$^2$ for the $s$ quark. These points are outside the integration domains needed for the model-exact $\rho$ and $\phi$ states respectively. However the critical bound state masses for which these singular points of the propagators would just begin to enter the integration domain are 1.09 GeV for a $\bar{u}u$ state, and 1.47 GeV for a $\bar{s}s$ state. A Taylor expansion should diverge at these points; it is probably the precursor of this that is beginning to show for the $\rho$ in Fig. 3.

### C. Strong Decays

Since the $\rho$ and $\phi$ appear as resonance poles in the timelike behavior of the charge form factors of the $\pi$ and $K$ mesons, the pole residues involve the coupling constants $g_{\rho\pi\pi}$ and $g_{\phi KK}$. If the charge form factors, or more generally the $\gamma\pi\pi$ and $\gamma KK$ vertex functions, are formulated in impulse approximation, then the values of $g_{\rho\pi\pi}$ and $g_{\phi KK}$ extracted from the pole residues will be in impulse approximation. The quality of the latter values will depend directly upon the quality of the strengths of the $\gamma\pi\pi$ and $\gamma KK$ vertex functions in the relevant timelike region. It is known that the impulse approximation for electromagnetic coupling to mesons conserves the electromagnetic current (and provides the correct charge) independent of model parameters as long as the meson BS amplitudes and photon-quark vertex are in ladder approximation and the quark propagators are in rainbow approximation. We assume that the dynamics which produces the correct strength at the photon point will also produce a good quality strength at the mass shell of the vector mesons and hence render the impulse approximation good for $g_{\rho\pi\pi}$ and $g_{\phi KK}$. The extension of this argument to cover the decay $K^{(*)} \rightarrow K^+\pi^0$ is afforded by the study of the semileptonic $K(l3)$ decay $K^+ \rightarrow l\nu\pi^0$ where $K^{(*)}$ appears as a pole in the $W^+$.

### TABLE III: The vector meson masses and electroweak decay constants (in GeV).

| Mass | $\rho$ | $K^*$ | $\phi$ |
|------|-------|-------|-------|
| All 8 amplitudes | 0.696 | 0.848 | 0.276 |
| $V_1$ only | 0.811 | 0.255 | 0.1202 |
| All 8 amplitudes | 0.732 | 0.949 | 0.253 | 1.064 |
| $V_1$ only | 0.856 | 0.214 | - | 1.235 |
| All 8 amplitudes | 0.735 | 0.934 | 0.253 | 1.054 |
| $V_1$ only | 0.844 | 0.187 | - | 1.182 |
| Model exact | 0.742 | 0.936 | 0.241 | 1.072 |
| Experiment | 0.770 | 0.892 | 0.225 | 1.020 |

The values do not indicate that the Taylor expansion has yet converged and this correlates with the behavior evident in Fig. 3. Table I also makes clear that a truncation of the eight Dirac covariants to just the dominant one ($\gamma_\mu$) will lead to a persistent error of at least 15-20%. In the case of the $K^*$ this tends to raise the mass to the level where we are unable to determine a converged result.
TABLE IV: Vector decay coupling constants calculated with the imaginary parts of amplitudes treated through order two in the Taylor expansion. The dependence on the number of invariant amplitudes employed is indicated. The $K^*$ decay process shown in brackets is related by isospin symmetry to the former process by a factor $1/\sqrt{2}$. The notation $(v,p)$ indicates the number of invariant amplitudes used for the vector meson and for the pseudoscalar mesons respectively.

| $g_{v\rightarrow pp}$ | $(v,p)=(8,4)$ | $(v,p)=(5,4)$ | $(v,p)=(1,1)$ | Expt |
|----------------------|--------------|--------------|--------------|------|
| $g_{v\rightarrow \pi\pi}$ | 5.14 | 4.93 | 8.8 | 6.02 |
| $g_{v\rightarrow KK}$ | 4.25 | 4.06 | 6.91 | 4.64 |
| $g_{K^+\rightarrow K^0\pi^+}$ | 4.81 | 4.56 | - | 4.60 |
| $(g_{K^+\rightarrow K^0\pi^+})$ | (3.40) | (3.22) | - | (3.26) |

TABLE V: The present results for the coupling constants $g_{v\rightarrow pp}$ (calculated with all covariants) compared to values extracted from a pole fit to timelike electroweak form factors, and also to experiment.

| $g_{v\rightarrow pp}$ | this work | pole fit | Expt |
|----------------------|-----------|----------|------|
| $g_{v\rightarrow \pi\pi}$ | 5.14 | 5.2 | 6.02 |
| $g_{v\rightarrow KK}$ | 4.25 | 4.3 | 4.64 |
| $g_{K^+\rightarrow K^0\pi^+}$ | 4.81 | 4.1 | 4.60 |

vertically.

Here the coupling constants for the decays of $\rho, K^*$ and $\phi$ to a pair of pseudoscalar mesons are calculated in impulse approximation according to Eqs. (19). The constraints on the two external Euclidian momenta $P$ and $Q$ needed to satisfy the three mass-shell conditions entail an analytic continuation to a complex value for at least one component. For example, in the $\rho \rightarrow \pi\pi$ case, with the momenta defined in Eq. (19), we have $Q = (im_\rho, 0)$, $P^2 = m_\rho^2/4 - m_\pi^2$ and $P \cdot Q = 0$. A by-product of this is that two of the quark propagators and two of the meson BS amplitudes are needed in domains of the complex $q^2$-plane. In all cases we employ second order Taylor expansions about the closest point on the positive real axis. We also test the dependence upon the number of Dirac covariants retained for the vector and pseudoscalar BS amplitudes. With a maximum of eight for the vector and four each for the pseudoscalar BS amplitudes, there are a maximum 128 distinct quark loop integrals to be performed.

The results are summarized in Table IV in comparison with experimental values obtained from decay widths. With just the dominant covariant employed for each meson ($\gamma_5$ for pseudoscalars, $\gamma_\mu$ for vectors, i.e, $(v,p)=(1,1)$ in Table IV) the coupling constants that can be obtained are about 50% larger than experiment. It is known from work on the vector meson BSE that a very efficient truncation for soft physics can be obtained with a particular set of 5 covariants [19]. With use of this truncation together with all 4 covariants for each pseudoscalar, the results in Table IV indicate that the dominant physics has been captured; the $K^*$ decay is within 1%, the $\phi$ is within 12% and the $\rho$ decay is 18% less than experiment. The addition of the remaining three transverse vector covariants leads to modest improvement giving a deviation from experiment of between 5% and 10% with the error being larger if the vector meson is lighter.

Since the width of the $\rho$ is almost 20% of its mass while the widths of the $\phi$ and $K^*$ are significantly less important, we expect the ladder approximation for the BSE kernel (which omits the strong channels $\pi\pi, KK$ and $K\pi$ respectively) to be less accurate for the $\rho$ than for the $\phi$ and $K^*$. Accordingly we speculate that this is largely the reason why the result for $g_{\rho\rightarrow \pi\pi}$ in Table IV deviates from experiment twice as much (15%) as do the other decay constants.

The results shown in Table V compare well with coupling constants extracted from the behavior of electroweak form factors calculated, independently and in the same model, at timelike momentum near the vector mesons poles [28, 29]. The general behavior of a pseudoscalar meson charge form factor $F_P(Q^2)$ in this domain is [28]

$$F_P(Q^2) \rightarrow \frac{g_{v\rightarrow pp} \, m_v^2}{g_v \, (Q^2 + m_v^2 - i m_v \, \Gamma_v)}.$$ (20)

Here $m_v^2/g_v$ is the $v\gamma\gamma$ coupling strength calculated from the $v \rightarrow e^+ e^- \gamma$ decay, and $\Gamma_v$ is the vector meson width associated with the mass-pole. In the present work the ladder approximation to the BSE produces real mass-poles. The vector decay coupling constants obtained in this way from a fit to the calculated $F_\pi(Q^2)$ and $F_K(Q^2)$ form factors and from the form factor for the semileptonic $K(l3)$ decay $K^+ \rightarrow l\nu\pi^0$ are shown in Table V. These results from a pole fit employ all relevant covariants for BS calculations and employ complex plane continuations where necessary. From this perspective the comparison with the left-most column of Table V provides a rough measure of the effectiveness of the Taylor expansion method for BSE solutions.

V. ESTIMATE OF HEAVIER STATES

In the present formulation, the nearby singularities discussed in Sec. VLB hinder a direct application of our rainbow-ladder model for masses above about 1.2 GeV. In particular, it is of interest to go beyond the present S-wave states to consider orbital excitations such as the axial vectors $a_1$ and $b_1$. One can however provide an indirect estimate as follows. Although the quark propagator amplitudes $\sigma_q(q^2)$ are complex-valued in the complex $q^2$-plane, the ladder BSE eigenvalue $\lambda(P^2)$ is real and so is the mass. The evident cancellations may allow the real axis behavior of the quark amplitudes to dominate the outcome. As a test, we introduce a real-axis approximation by setting the imaginary part of the momentum argument of the quark propagator amplitudes $\sigma_q(q^2)$ to zero in the BSE kernel. Then, instead of Eq. (19), one
has

\[ \sigma_\alpha(q^2) \approx \sigma_\alpha(q^2 - m^2/4), \]  

where \( \alpha = v, s \) and \( q^2 > 0 \). This is not quite identical to zeroth order in the Taylor expansion of Eq. (19) because the latter expands about the nearest point on the positive real axis, while in Eq. (21) the interval \([-m^2/4, \infty]\) is used. Thus the required domain of the timelike real axis is treated exactly and this can be important for the heavier mesons.

To implement this approximation, we use the quark DSE to continue the propagator solutions to the timelike (negative real) \( p^2 \) axis. In this respect, it is advantageous to enhance numerical stability of the present model by modifying the infrared behavior of the factor \( F(k^2) \) in the small second term of the effective coupling, given in Eq. (18), so that, like the first term, it is identically zero at \( k^2 = 0 \). Its UV limit and range should remain much the same. To achieve this we use

\[ \tilde{F}(s) = \frac{1 - e^{-(s/4m^2)^2}}{s}. \]  

With the real axis approximation of Eq. (21), the resulting modified model yields mass estimates that tend to become more accurate as the mass increases. From Table I, \( m_{\pi/K} \) are underestimated by 10% and \( f_{\pi/K} \) are underestimated by 20% compared to the model-exact values. However, as shown in Table II, the produced ground state vector meson masses \( m_\rho, m_{K^*} \) and \( m_\phi \) are within 4%, 7% and 1% of the model-exact values respectively; the decay constants deviate by 5%, 0% and 11% respectively. We therefore expect this real-axis approximation to provide mass estimates in the 1-2 GeV range with an error of about 5-10%.

For a u/d quark axial vector state, the required general form of BS amplitude is simply \( \gamma_5 \) times Eq. (13). For the \( a_1 \) (1++), amplitudes \( V_5, V_4 \) and \( V_6 \) will be odd in \( q \cdot P \) and the rest will be even; for the opposite C-parity state \( b_1 \) (1--) the opposite is true. The real axis approximation gives the following axial vector estimates: \( m_{a_1} \approx 0.891 \) GeV and \( m_{b_1} \approx 0.775 \) GeV. Hence the orbital excitation energy \( m_{a_1} - m_\rho \) is 0.150 GeV and thus a factor of 3 too small in this model. Indications from estimates attempted using the complex plane information are similar, as are recent results from a related study [31]. Models of the present rainbow-ladder type are significantly too attractive for these orbital excitations. Other studies of \( a_1 \) and \( b_1 \) based on the DSEs have used a separable approximation where the quark propagators are the phenomenological instruments [31, 32, 33] and these studies find more acceptable masses in the vicinity of 1.3 GeV. The manner in which the effective interaction is modeled is clearly important. Preliminary results from an extension of the present rainbow-ladder level through a 1-loop dressing of the quark-gluon vertex, while preserving the vector and axial vector WTI, indicate that the separation of the axial vector states from the vector states increases significantly and becomes quite acceptable [33].

Recently there has been interest in meson states with quantum numbers that are called exotic in the sense that they cannot be produced as \( \bar{q}q \) bound states within static quantum mechanics where C-parity is given by \( C = (-1)^{L+S} \). The \( \pi_0(1400) \) state, formerly called the \( \bar{q}q(1405) \), is a resonance with \( J^{PC} = 1^{-+} \) that is supported by evidence from \( \pi d \) scattering and \( \bar{p}d \) annihilation [34]. In a covariant field theory treatment of bound states via the BSE, the extra degree of freedom represented by relative time, or relative energy, allows states to have an additional classification in terms of an associated time-parity quantum number \( \kappa \) and C-parity is given by the more flexible form \( C = (-1)^{L+S+\kappa} \). The states with \( \kappa = \text{odd} \) have no static quantum mechanical \( \bar{q}q \) analog and in this sense the exotic \( J^{PC} \) obtainable from the ladder BSE are evidently a realization of the multi-particle features of covariance. Although a full understanding within field theory of the norm of these odd solutions of the BSE remains to be achieved [33], examples from otherwise realistic models are of interest in connection within on-going experimental searches.

A covariant separable BSE model, closely connected with the present model, has recently produced [33] \( m_{\pi_0} = 1.439 \) GeV, i.e., about 100 MeV above the \( a_1 \), which in that model is \( m_{a_1} = 1.337 \) GeV. To estimate whether this is consistent with the present study, we use the real axis approximation to search for a \( J^{PC} = 1^{-+} \) solution. The general form for such a BS amplitude is the same as Eq. (13) for the 1-- amplitude except that one inverts the odd-even property of the invariant amplitudes in \( q \cdot P \). That is, \( V_5 \) and \( V_6 \) are to be even, the rest are to be odd. This produces a \( \pi_1 \) solution 112 MeV above the \( a_1 \) within the same approach. In this respect we agree with Ref. [33]. We speculate that if the problem of the ground state orbital excitations being about 300 MeV too low is remedied, then the \( \pi_1 \) state in this model should rise by a similar amount to be around 1.3-1.4 GeV.

VI. DISCUSSION

We have studied vector meson strong decays within a Euclidean space model of QCD based on the Dyson–Schwinger equations truncated to ladder-rainbow level. The infrared structure of the ladder-rainbow kernel is described by two parameters; the ultraviolet behavior is fixed by the one-loop renormalization group behavior of QCD. Within the u/d and s quark sector we have obtained the coupling constants for the strong decays: \( \rho \to \pi \pi, K^* \to K \pi, \) and \( \phi \to KK \). The deviation from experiment is between 5% and 10% with the error being larger if the vector meson is lighter.

Our method of calculation employed a Taylor expansion to continue the dressed quark propagators from the real spacelike \( p^2 \) axis into the domain of the complex plane required by the mass-shell condition of the meson bound states. We used this both for the solution of the
BSE and for calculation of decays in the impulse approximation. The benefit is that knowledge of the dressed quark propagators is needed only along the real space-like $p^2$ axis and this simplifies the numerical requirements considerably. The vector and pseudoscalar meson masses compared well with the model-exact results obtained previously by direct continuation of the dressed quark propagators into the complex plane via the DSE. In particular, the masses and electroweak decay constants at second order are all within 5% of the model-exact results. The increased deviation observed at third order and for heavier masses is attributed to nearby complex conjugate singularities in the propagators that are known to occur in rainbow solutions of the DSE.

For mesons heavier than the $\phi$, these singularities hinder the applicability of Euclidean space implementations of the DSEs as presently formulated. The Taylor expansion method cannot lessen this difficulty. We therefore explored a real-axis approximation that exploited the cancellations of imaginary parts evident in the way the complex-valued quark amplitudes off the real axis enter the BSE kernel to produce real eigenvalues and masses. The real-axis approximation avoids complex plane singularities and, based on its performance for the pseudoscalar and vector mesons, we conclude it can be used for mass estimates in the 1-2 GeV region with an accuracy of about 5-10%. We then produced estimates of $m_{\pi_1}$ and $m_{\pi_2}$ for orbital excitations as well as $m_{\pi_3}$ for the proposed exotic $1^{-+}$ vector state. We mentioned evidence that an extension beyond ladder-rainbow level to include dressing of the quark-gluon vertex could address our finding that the present model is too attractive for states just above the ground state vectors. With this, our estimate for $m_{\pi_3}$ is 1.3-1.4 GeV.

Work in progress [35] on the case of propagators having explicit complex conjugate poles, indicates that a reformulation produces well-defined integrals and integral equations above the critical masses. Beyond rainbow-ladder approximation, very little is known about the singularity structure. In one simplified model that includes a fully dressed quark-gluon vertex in a way that has allowed exploration of analytic properties, the non-analytic behavior is relegated to essential singularities at infinity [3].

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