Recurrence on the average on trees

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In this paper we show that all infinite trees which have bounded coordination and whose surface is negligible with respect to the volume in the limit of large distances (so that they can be embedded in a finite-dimensional euclidean space) are recurrent on the average; this has important consequences about the spontaneous symmetry breaking of statistical models defined on such trees.

I. INTRODUCTION

Random walks, naturally connected to the problem of diffusion on discrete structures, are also important tools for studying the properties of inhomogeneous networks. In fact they represent the main link between the geometrical and the physical properties of graphs: a walker needs only the information about local connectivity for the one–step jump probabilities, but for sufficiently long times it samples the whole underlying structure, so that its long time statistics reflects the large scale topology which, in turn, is responsible for many physical properties.

Random walks also represent a connection between mathematics and physics: these are extensively studied in the mathematical literature because of their connections with Markov chains, algebraic graph theory and potential theory (see for example [1]) while, on the other side, many physical problems such as the study of vibrational spectrum [2], the critical behavior of statistical models [3] or simple models of quantum particles [4] on discrete networks can be mapped to random walks. In particular the Type problem, i.e. whether for a certain graph a random walker returns to its starting point with probability one (recurrent graph) or has a nonzero chance to escape (transient graph), is strictly related to the presence of spontaneous symmetry breaking of statistical models defined on the same structure [5].

For inhomogeneous structures the local properties can be different from the average ones, so the Type problem on the average has been introduced [6]: a graph is said to be recurrent on the average (ROA) or transient on the average (TOA) on the basis of the average over all nodes of the returning probability. This new classification has been shown to be the appropriate one in the analysis of statistical models: in particular on ROA graphs, classical $O(n)$ and quantum Heisenberg ferromagnetic spin models cannot show spontaneous magnetization at any finite temperature [7]; on the contrary on TOA ones, $O(n)$ models [8] must have a non–zero magnetization at finite temperature.

In this paper we show that a whole class of trees, which are defined bounded trees in [9], are recurrent on the average. This class is composed by all infinite trees that satisfy two requirements with a natural physical meaning: first the coordination number of the nodes must be bounded (when the graph is embedded in an euclidean space, the number of nearest neighbors of a given site has a geometrical upper bound if the links have bounded length), then the “surface” must be negligible with respect to the “volume” for large distances (this is necessary for the existence of the thermodynamic limit or simply if the graph has to be embedded in a finite–dimensional Euclidean space). The proof is based on the flow criterion: this is a powerful tool for the study of the local Type problem [10] and recently it has been extended to the case on the average [11]. The result obtained in this paper has an important consequence that follows from the previously stated connections between random walk and statistical models defined on graphs: there cannot be spontaneous symmetry breaking for classical $O(n)$ and quantum Heisenberg ferromagnetic models on any bounded tree.

Moreover the recurrence on the average implies that the average spectral dimension $d_s$ of these trees is always smaller than 2; this is a first exact result about the average spectral dimension for bounded trees even if we expect the true upper limit to be 4/3. Indeed this has been conjectured (but not proved) in [12] and the inequality $d_s \leq 4/3$ is satisfied by all known examples of bounded trees.

Another remark about our result is that there are no corresponding results concerning the local recurrence: there are indeed examples of locally transient bounded trees [13].

The outline of the paper is as follows: in next section we recall the definitions about graph averages, recurrence and transience, and the flow criterion; then in section [11] we formally state and prove our result about the recurrence on the average of bounded trees.

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II. DEFINITIONS

Given an infinite graph $G$, let us call $G$ the set of vertices (nodes) and $E(G)$ the set of edges (links). For $x \in G$ the sphere $B(x, r)$ and the spherical shell $S(x, r)$ are defined as:

$$B(x, r) = \{ y \in G : d(x, y) \leq r \}$$

$$S(x, r) = \{ y \in G : d(x, y) = r \}$$

where $d(x, y)$ is the chemical distance on the graph $G$.

Now consider a function $f$ defined on $G$; the limit on the average $L_x(f)$ is defined as [11]:

$$L_x(f) = \lim_{r \to \infty} \frac{1}{|B(x, r)|} \sum_{y \in B(x, r)} f(y)$$  \hspace{1cm} (1)

If $f = \chi_A$, the characteristic function of a subset $A \subseteq G$, we write $L_x(A)$ instead of $L_x(\chi_A)$ and call it measure of $A$. We will consider also the upper and lower limit on the average:

$$\inf L_x(f) = \lim_{r \to \infty} \inf \frac{1}{|B(x, r)|} \sum_{y \in B(x, r)} f(y)$$

$$\sup L_x(f) = \lim_{r \to \infty} \sup \frac{1}{|B(x, r)|} \sum_{y \in B(x, r)} f(y)$$

because these always exist while the limit in equation (1) may not exist.

A. Transience and recurrence

Given a graph $G$ and a node $x \in G$, consider the probability $F_x$ for a random walker started from $x$ to ever return to the starting vertex: $G$ is said to be (locally) transient if $F_x < 1$ and (locally) recurrent if $F_x = 1$.

This classification is a property of the graph and does not depend on the choice of starting node $x$ because it is easy to prove, using standard Markov chain properties, that if $F_x = 1$ ($F_x < 1$) for any node $x \in G$ then $F_y = 1$ ($F_y < 1$) for every other node $y \in G$.

B. Transience and recurrence on the average

A graph is said to be transient on the average if, for $x \in G$

$$\inf L_x(F) < 1$$

and recurrent on the average (ROA) if, on the contrary,

$$\inf L_x(F) = 1$$

Important properties of the classification on the average are [11]:

- if there exists $y \in G$ satisfying

$$\sup \frac{|S(y, r + 1)|}{|B(y, r)|} < +\infty$$

then $\inf L_x(F) = 1$ for all $x \in G$ or $\inf L_x(F) < 1$ for all $x \in G$ (the classification does not depend on $x$)

- local recurrence implies recurrence on the average, while this is not true for transience.
C. Flow criterion

If we give an orientation to the links of a graph, i.e. for every \( e \in E(G) \) we write \( e = (e^-, e^+) \) with \( e^-, e^+ \in G \), a flow from \( x_0 \) to infinity with input \( i_0 \) is a function \( u \) defined on \( E(G) \) such that the “current” is conserved at every vertex \( x \):

\[
\sum_{e: e^- = x} u(e) - \sum_{e: e^+ = x} u(e) = \delta_{x,x_0} i_0
\]

and its energy is

\[
\langle u, u \rangle = \sum_{e \in E(G)} u(e)^2
\]

(2)

The existence of finite energy flow is strictly related to transience, as expressed by the following:

- A graph \( G \) is (locally) transient if and only if there exists \( x \in G \) such that it is possible to find a finite energy flow \( u^x \) with non-zero input from \( x \) to infinity [10];

- A graph \( G \) with a bounded coordination is TOA if and only if there exists \( A \subseteq G \) such that \( \sup L_0(A) > 0 \) and for every \( x \in A \) there is a finite energy flow \( u^x \) from \( x \) to \( \infty \) with non-zero input and \( \sup_{x \in A} \langle u^x, u^x \rangle < +\infty \) (theorem 3.10 in [11]).

III. MAIN RESULT

We prove that:

If \( G \) is an infinite tree with bounded coordination and there exists \( x_0 \in G \) such that

\[
\lim_{r \to \infty} \frac{|S(x_0, r)|}{|B(x_0, r)|} = 0
\]

then \( G \) is ROA.

The outline of the proof is the following: first we define the “backbone” of \( G \) and the set \( V \) of the backbone “branching” nodes and we show that it has measure zero; then we prove that for every set \( A \subseteq G \) such that \( \sup L_0(A) > 0 \) and for every \( n \in \mathbb{N} \) there exists \( x \in A \) such that every flow \( u^x \) from \( x \) to \( \infty \) with input \( i_x \) has energy \( \langle u^x, u^x \rangle > \frac{\epsilon^2}{2} n \), so that \( \sup_{x \in A} \langle u^x, u^x \rangle \) is infinite and the graph cannot be TOA.

A. Proof

Consider the subgraph \( R \) (with vertex set \( R \)) obtained from \( G \) by deleting all vertices and links belonging only to finite branches and call it the backbone of \( G \) (fig 1): it can be obtained by recursively deleting all nodes with coordination one together with the links originating from them.

If \( R \) is empty (for example if \( G \) is a half infinite one-dimensional chain, possibly with finite trees attached to some or all nodes) then \( G \) is recurrent (and therefore ROA). In this case, in fact, for every node \( x \in G \) there is only one path that leads to infinity: because of current conservation \( |u^x(e)| \) must be finite and constant on the edges \( e \) belonging to this path and zero on every other edge and this implies that \( \langle u^x, u^x \rangle = +\infty \).

If, on the other hand, \( R \) is non-empty let us call “branching” the nodes \( x \in R \) whose coordination in \( R \) is greater than 2; these vertices are important because they are the only ones in which a current can split so that the flow energy can be reduced. Now define \( V \) the set of branching nodes, \( B_V(x_0, r) \) the set of branching nodes inside a ball of radius \( r \) and center \( x_0 \) and \( S_R(x_0, r) \) the vertices of \( S(x_0, r) \) belonging to \( R \):

\[
V = \{ x : x \in R, z^x > 2 \}
\]

\[
B_V(x_0, r) = B(x_0, r) \cap V
\]

\[
S_R(x_0, r) = S(x_0, r) \cap R
\]
If there are no branching vertices ($V$ is empty) $R$ is a linear chain (every node has coordination two) and the flow from any vertex in $G$ has infinite energy so that $G$ is recursive (and therefore ROA). On the other hand even if $V$ is non–empty, it is a zero measure set in $G$ because

$$|S(x_0, r)| \geq |S_R(x_0, r)| = 2 + \sum_{x \in B(x_0, r)} (x^R - 2)$$

$$\geq 2 + |B_V(x_0, r)|$$

$$> |B_V(x_0, r)|$$

so that

$$L_{x_0}(V) = \lim_{r \to \infty} \frac{1}{|B(x_0, r)|} \sum_{x \in B(x_0, r)} \chi_V(x) = \lim_{r \to \infty} \frac{|B_V(x_0, r)|}{|B(x_0, r)|} \leq \lim_{r \to \infty} \frac{|S(x_0, r)|}{|B(x_0, r)|} = 0$$

Now consider a subset $A \subseteq G$ such that $\sup L_{x_0}(A) > 0$, it is easy to see that for every $n$ there exists $x \in A$ such that $d(x, V) > n$. In fact, suppose that there exists $\bar{n}$ such that for every $x \in A$, $d(x, V) \leq \bar{n}$, then

$$A \subseteq M_{\bar{n}} = \{x \in G : d(x, V) \leq \bar{n}\}$$

but since $V$ has zero measure and $G$ has bounded coordination, $M_{\bar{n}}$ has zero measure for every $\bar{n} \in \mathbb{N}$ (lemma 4.8 in [1]) and so $L_o(A) = 0$ contradicting the hypothesis.

Then choose a distance $n \in \mathbb{N}$ and take $x \in G$ such that $d(x, V) > n$; there are two distinct cases: $x \in R$ or $x \notin R$. If $x \in R$, it belongs to finite linear chain in $R$ between two nodes $v_1, v_2 \in V$ or to a semi–infinite chain starting from $v_1 \in V$. In the first case let $n_1 = d(x, v_1)$ and $n_2 = d(x, v_2)$ with, by hypothesis, $n_1 > n$ and $n_2 > n$. Clearly the energy of a flow $u_x$ from $x$ decreases if we restrict the sum in equation (2) to the linear chain between $v_1$ and $v_2$ and all the “restricted”energy of the flows can be parametrized by the fractions $t$ and $1 - t$ of current directed toward the nodes $v_1$ and $v_2$, respectively:

$$\frac{\langle u^x, u^x \rangle}{i_x^2} > \sup_{0 \leq t \leq 1} n_1 t^2 + n_2 (1 - t)^2 = \frac{n_1 n_2}{n_1 + n_2}$$

$$\geq \frac{n^2}{2n} = \frac{n}{2}$$

On the other hand if $x$ belongs to a semi–infinite chain, in order to have a finite energy flow the $|u^x(e)|$ must be constant in the path between $x$ and $v_1$ and zero in the other (infinite) part of the semi–infinite chain; so we have

$$\frac{\langle u^x, u^x \rangle}{i_x^2} > n_1 > \frac{n_1}{2} > \frac{n}{2}$$

(3)
If $x \notin R$ let $y$ be the vertex in $R$ with minimum distance from $x$ and $n_1 = d(x,y)$. In this case $|u^x(e)|$ must be constant for every edge $e$ belonging to the path connecting $x$ to $y$ and then the current can split as in the previous case.

$$\frac{\langle u^x, u^x \rangle}{i^2_x} = n_1 + \frac{\langle u^y, u^y \rangle}{i^2_x} > n_1 + \frac{n - n_1}{2} > \frac{n}{2} \quad (4)$$

Collecting all these results, we can say that for every $x \in G$ such that $d(x,V) > n$

$$\frac{\langle u^x, u^x \rangle}{i^2_x} > \frac{n}{2}$$

This implies that for every subset $A \subseteq G$ such that $\sup L_x(A) > 0$, it must be $\sup_{x \in A} \langle u^x, u^x \rangle = +\infty$, so the graph $G$ is not TOA and must be ROA.

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