Dual elliptic structures on $\mathbb{CP}^2$

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Abstract. We define the notion of tame elliptic structure $E$ on $\mathbb{CP}^2$, which generalizes an almost complex structure $J$ on which the standard symplectic form is positive. An $E$-curve is a surface in $V$ which is everywhere tangent to $E$ ($J$-holomorphic in the almost complex case), and an $E$-line is an $E$-curve of degree 1. We show that the space $V^*$ of $E$-lines is itself a complex projective plane with a tame elliptic structure $E^*$. Moreover, to each $E$-curve one can associate its dual in $V^*$, which is an $E^*$-curve. This implies that the $E$-curves, and in particular the $J$-curves, satisfy the Plücker formulas, which restricts their possible sets of singularities.

AMS 32Q65, 53C15, 53C42, 53D35, 57R17, 58J60

Keywords: $J$-holomorphic curve, complex projective plane, dual curve, elliptic structure

Introduction

Let $V$ be a smooth oriented 4-manifold, which is a rational homology $\mathbb{CP}^2$ (ie $b_2(V) = 1$), and let $J$ be an almost complex structure on $V$ which is homologically equivalent to the standard structure $J_0$ on $\mathbb{CP}^2$. This means that there is an isomorphism $H^*(V) \to H^*(\mathbb{CP}^2)$ (rational coefficients) which is positive on $H^4$ and sends the Chern class $c_1(J)$ to $c_1(J_0)$.

By definition, a $J$-line is a $J$-holomorphic curve ($J$-curve for short) of degree 1. By the positivity of intersections [McD2], it is an embedded sphere. We denote by $V^*$ the set of $J$-lines.

Now assume that $J$ is tame, ie positive with respect to some symplectic form $\omega$, and also that $V^*$ is nonempty. Then M. Gromov [G] (2.4.A) [cf. also [McD1]) has proved that by two distinct points $x,y \in V$ there passes a unique $J$-line $L_{x,y} \in V^*$, depending smoothly on $(x,y)$; also, for any given $P \in Gr^J_1(TV)$, the Grassmannian of $J$-complex lines in $TV$, there exists a unique $J$-line $L_P \in V^*$ tangent to $P$. Furthermore, $V$ is oriented diffeomorphic to $\mathbb{CP}^2$, $\omega$ is isomorphic to $\lambda \omega_0$ for some positive $\lambda$ so that $J$ is homotopic to $J_0$. Finally, $V^*$ has a natural structure of compact oriented 4-manifold; although it is not explicitly stated in [G], the above properties of $V^*$ imply that it is also oriented diffeomorphic to $\mathbb{CP}^2$.

Later, Taubes [T] proved that the hypothesis that $V^*$ be nonempty is unnecessary, so that all the above results hold when $J$ is tame. We shall call $(V,J)$ a tame almost complex projective plane.

Following [G, 2.4.E], these facts can be extended to the case of an elliptic structure on $V$, ie one replaces $Gr^J_1(TV)$ by a suitable submanifold $E$ of the Grassmannian of oriented 2-planes $Gr_2 TV$. Such a structure is associated to a twisted almost complex structure $J$, which is a fibered map from $TV$ to itself satisfying $J_*^2 = -\text{Id}$ but such that $J_*$ is not necessarily linear.

An elliptic structure on $V$ gives rise to a notion of $E$-curve, ie a surface $S \subset V$ (not necessarily embedded or immersed) whose tangent plane at every point is an element of $E$ (for the precise definitions, see section 2). It will be called tame if there exists a symplectic form $\omega$ strictly positive on each $P \in E$.

In Gromov’s words, “all facts on $J$-curves [proved in [G]] extend to $E$-curves with an obvious change of terminology”. In particular, let $V$ be a rational homology $\mathbb{CP}^2$ equipped with a tame elliptic structure $E$ so that $(V,E)$ is homologically equivalent to $(\mathbb{CP}^2,Gr^C_1(T\mathbb{CP}^2))$. Then one
can define the space $V^*$ of $E$-lines ($E$-curves of degree 1), and prove that all the above properties still hold (see section 3). In particular, $V$ and $V^*$ are oriented diffeomorphic to $\mathbb{CP}^2$.

We shall call $(V,E)$ with the above properties a tame elliptic projective plane. If $C \subset V$ is an $E$-curve, we define its dual $C^* \subset V^*$ by $C^* = \{ T_v C \mid v \in C \}$. A more precise definition is given in section 4; note that one must require that no component of $C$ is contained in an $E$-line. The main new result of this paper is then the following.

**Theorem.** Let $(V,E)$ be a tame elliptic projective plane. Then there exists a unique elliptic structure $E^* \subset \text{Gr}_2 TV^*$ on $V^*$ with the following property: if $C \subset V$ is an $E$-curve, then its dual $C^* \subset V^*$ is an $E^*$-curve.

Furthermore, $(V^*,E^*)$ is again a tame elliptic projective plane. Finally, the bidual $(V^{**},E^{**})$ can be canonically identified with $(V,E)$, and $C^{**} = C$ for every $E$-curve $S$.

If $E$ comes from an almost complex structure, one may wonder if this is also the case for $E^*$, equivalently if the associated twisted almost complex structure $J^*$ is linear on each fiber. We give an example showing that it is not true, which means that $V^*$ has no natural almost complex structure. (Actually I believe that $J^*$ is linear only if $J$ is integrable.)

Theorem 1 enables us to extend to $J$-curves in $\mathbb{CP}^2$ (for a tame $J$) some classical results obtained from the theory of dual algebraic curves. For instance, one immediately obtains the Plücker formulas, which restrict the possible sets of singularities of $J$-curves. Such results could be interesting for the symplectic isotopy problem for surfaces in $\mathbb{CP}^2$ [Sik2]. Maybe also, but this is more hypothetical, for the topology of symplectic 4-manifolds, in view of the result of D. Auroux [A] showing that they are branched coverings of $\mathbb{CP}^2$. The reason why we say “hypothetic” is that the branch locus is not an honest $J$-curve (with nodes and cusps), but may admit negative nodes, although there is some some hope to dispense with them.

In sections 1 and 2, we give the main properties of elliptic structures and $E$-curves in dimension 4. In section 3 we define and study tame elliptic projective planes. Most of the statements and all the ideas in these three sections are already in Gromov’s paper (see especially [G,2.4.E and 2.4.A]), except what regards singularities, where we give more precise results in the vein of [McD2] and [MW].

In section 4 we prove the main result: the structure $E^*$ is defined in one line, but to prove its ellipticity we could not avoid some longish computation. The tameness and the duality property are easy.

In section 5 we give an example of a tame almost complex structure on $V = \mathbb{CP}^2$ such that $V^*$ has no natural almost complex structure.

Finally in section 6 we prove the Plücker formulas for $E$-curves and in particular for $J$-curves.

**Acknowledgment.** This paper would not exist without the obstination of Stepan Orevkov, who asked me many times if something like an almost complex structure on $V^*$ could not exist. I kept saying “no”, until I realized the existence of an elliptic struture, which is just as good!

**Later comment.** After the first version of this paper was posted, Benjamin McKay sent me his PhD thesis (“Duality and integrable systems of pseudoholomorphic curves”, Duke University, 1999), which contains (among many other things) the main result of this paper in a much more general setting. In it one can also find the proof of a strong version of my conjecture at the end of section 5: if two dual structures are almost complex near corresponding points, then they are integrable. It does not however contain the application to singularities.
1. Elliptic structures in dimension 4

1.A. Surface of elliptic type in the Grassmannian $Gr_2\mathbb{R}^4$. Let $T$ be an oriented real vector space of dimension 4. We denote by $G(T) = Gr_2T$ the Grassmannian of oriented 2-planes. Recall that for each $P \in G(T)$, the tangent plane $T_PG(T)$ is canonically identified with $\text{Hom}(P,T/P)$.

By definition, a surface of elliptic type in $G(T)$ is a smooth, closed, connected and embedded surface $E$ such that for every $P \in E$ one has

$$T_P E \setminus \{0\} \subset \text{Isom}_+(P,T/P).$$

This is equivalent to the existence of (necessarily unique) complex structures on $T_P E$, $P$, $T/P$, such that $\text{im}(\phi)$ is the space of complex morphisms from $P$ to $T/P$ and $\phi$ is a complex morphism.

We shall denote

- $j_{E,P}$ the structure on $T_P E$
- $j_P$ and $j_P^\perp$ the structures on $P$ and $T/P$.

1.B. Surfaces of elliptic type and complex structures. The first example of surface of elliptic type is a Grassmannian $Gr_1^J(T)$ of complex $J$-lines for a positive complex structure $J$ on $T$.

We now prove that every surface of elliptic type is deformable to such a $Gr_1^J(T)$. More precisely, denote by $\mathcal{J}(T)$ the space of positive complex structures, and $\mathcal{E}(T)$ the space of surfaces of elliptic type. Then the embedding $\mathcal{J}(T) \rightarrow \mathcal{E}(T)$ just defined admits a retraction by deformation. In particular, $E$ is always diffeomorphic to $\mathbb{CP}^1$ and thus biholomorphic to $\mathbb{CP}^1$.

To prove this, we fix a Euclidean metric on $V$ and replace $\mathcal{J}(T)$ by the subspace $\mathcal{J}_0(T)$ of isometric structures, to which it retracts by deformation. The space of 2-vectors $\Lambda^2 T$ has a decomposition $\Lambda^2 T = \Lambda^2_+ T \oplus \Lambda^2_- T$ into self-dual and antiaself-dual vectors. The Grassmannian $G(T)$ is identified with $S^2_+ \times S^2_- \subset \Lambda^2_+ T \times \Lambda^2_- T$ by sending a plane $P$ to $(\sqrt{2}(x \wedge y)_+, \sqrt{2}(x \wedge y)_-)$ where $(x,y)$ is any positive orthonormal basis. We denote $P = \phi(u_+, u_-)$ the plane associated to $(u_+, u_-)$.

Identifying $T/P$ with $P^\perp$, the canonical isomorphism

$$T_{u_+} S^2_+ \times T_{u_-} S^2_- \rightarrow \text{Hom}(P, P^\perp)$$

sends $(\alpha_+, \alpha_-)$ to $A$ such that

$$A \xi = \iota_\xi (\alpha_+ + \alpha_-)$$

(interior product). This can be seen by working in a unitary oriented basis of $T$, $(e_1, e_2, e_3, e_4)$ such that $u_\pm = \frac{1}{\sqrt{2}}(e_1 \wedge e_2 \pm e_3 \wedge e_4)$. This leads to unitary oriented bases of $T_{u_+} S^2_+$ and $T_{u_-} S^2_-:

$$v^\pm = \frac{1}{\sqrt{2}}(e_1 \wedge e_3 \mp e_2 \wedge e_4), \ w^\pm = \frac{1}{\sqrt{2}}(e_1 \wedge e_4 \pm e_2 \wedge e_3).$$

Still working in these bases, one gets

$$\det A = -||\alpha_+||^2 + ||\alpha_-||^2.$$
Since $E$ is closed and connected, $p_-$ is a diffeomorphism from $E$ to $S^2_\pm$ and $E$ is the set of points $(a(u), u)$, where $a : S^2_\pm \to S^2_\pm$ is a smooth contraction.

Thus $\mathcal{E}(T)$ is homeomorphic to the space of smooth contractions from $S^2$ to itself. This space retracts by deformation to the space of constant maps: to each map $a$ one associates its unique fixed point $x_a$ and writes $a = \exp_{x_a} A$ with $A : S^2 \to T_{x_a} S^2$; the retraction is then $h_t = \exp_{x_a} (tA)$. Since a constant map $S^2_\pm \to S^2_\pm$ corresponds to a Grassmannian $Gr^I_2(T)$ for some $J \in \mathcal{J}_0(T)$, this gives the retraction by deformation from $\mathcal{E}(T)$ to $\mathcal{J}_0(T)$.

1.C. Twisted complex structure associated to an elliptic surface. Let $E \subset G(T)$ be an elliptic surface. Then we have the

**Proposition.** The space $T \setminus \{0\}$ is the disjoint union $\bigcup_{P \in E} P \setminus \{0\}$.

**Proof.** We use the representation $E = \{P_u \mid u \in S^2_\pm\}$, with $P_u = \phi((a(u), u), a : S^2_\pm \to S^2_\pm$ being a smooth contraction.

Let $\xi \in T \setminus \{0\}$ be given. Then $\xi$ belongs to $P_u$ if and only if $\iota_\xi (a(u) + u) = 0$. We can identify $\xi^\perp \subset T$ with $\Lambda^2_+$ in such a way that $\iota_\xi (a(u) + u) = u \cdot a$. Then $\iota_\xi (a(u) + u) = u + b(u)$, where $b : S^2_\pm \to S^2_\pm$ is a smooth contraction. Thus there exists a unique $u \in S^2_\pm$ such that $-b(u) = u$, i.e. a unique $P = P_u$ in $E$ containing $\xi$.

Thus if $u, v$ are distinct points in $S^2_\pm$, we have $P_u \cap P_v = \{0\}$. In fact $P_u$ and $P_v$ are positively transverse (first occurrence of the positivity of intersection): indeed, the inequality $||a|| < 1$ implies $||a(u) - a(v)||^2 < ||u - v||^2$ i.e.

$$\langle a(u), a(v) \rangle - \langle u, v \rangle > 0,$$

which is precisely the positive transversality of $P_u$ and $P_v$.

This enables us to put together the $j_P, P \in E$, to obtain a map $J : T \to T$ with the following properties:

(i) $J^2 = -\text{Id}$
(ii) $J$ is continuous, and homogeneous of degree 1
(iii) $J$ is smooth away from 0 (by (ii), it is not differentiable at 0 except if it is linear)
(iv) for every $x \in T \setminus \{0\}$, $J(x)$ is linearly independent of $x$, and $J$ is linear on the plane $\langle x, J(x) \rangle$.

Conversely, given $J$ satisfying (i)-(iv), we can define a smooth surface $E \subset G(T)$ by

$$E = \{\langle x, J(x) \rangle \mid x \in T \setminus \{0\}\}.$$

A straightforward computation gives that $E$ is elliptic if and only if

(v) for every $x \in T \setminus \{0\}$ and $\xi \in \langle x, J(x) \rangle^\perp \setminus \{0\}$, $\langle x, J(x), \xi, dJ_x \xi \rangle$ is an oriented basis of $T$.

1.D. Local form of a surface of elliptic type. Let $E$ be a surface of elliptic type in $G(T)$, and fix $P \in E$. Identify $T$ with $\mathbb{C}^2$ such that

(i) $P$ is sent to the horizontal plane $H = \mathbb{C} \times \{0\}$
(ii) the identifications $P \to H$ and $T/P \to \mathbb{C}^2/H$ are complex-linear for the complex structures defined in 1.A.

Then $E$ is given near $P$ by a family of planes of the following form:

$$P_\lambda = \{\langle \delta z, \delta w \rangle \in \mathbb{C}^2 \mid \delta w = \lambda \delta z + h(\lambda) \delta \bar{z}\},$$
where \( h \) is a smooth germ \((C,0) \to (C,0)\) such that \( h(0) = 0 \). The tangent space \( T_{\gamma}E \) is identified with the image of

\[ \xi \in C \mapsto \xi \text{Id} + (dh(\lambda)\xi)\sigma \in \text{End}_{\mathbb{R}}(C), \]

where \( \sigma \) is the complex conjugation. Thus the ellipticity translates to the inequality \( \|dh\| < 1 \). Property (ii) becomes \( dh(0) = 0 \).

**Remark.** Denote by \( J \) the complex structure on \( T. \) Properties (i) and (ii) imply that \( P \in Gr_{1}^{J}T \) and \( T_{\gamma}Gr_{1}^{J}(TV) = T_{\gamma}E \), cf. \([G, 2.4.E]\).

### 1.E. Elliptic structure on a 4-manifold; twisted almost complex structure.

Let \( V \) be an oriented 4-manifold, and let \( G = Gr_{2}TV \) be the Grassmannian of oriented tangent 2-planes, which is fibered over \( V \) with fiber \( G_{v} = Gr_{2}(T_{v}V) \).

By definition, an *elliptic structure* on \( V \) is a smooth compact submanifold \( E \subset G \) of dimension 6, transversal to the fibration \( G \to V \), such that each fiber \( E_{v} \) is a surface of elliptic type in \( Gr_{2}T_{v}V \).

Denote \( \mathcal{E}(TV) \) the space of elliptic structures on \( V \) and \( \mathcal{J}(TV) \) the space of positive almost complex structures on \( V \). The map \( J \mapsto Gr_{1}^{J}(TV) \) gives a natural embedding from \( \mathcal{J}(TV) \) to \( \mathcal{E}(TV) \). These are both spaces of sections of a bundle on \( V \), with respective fibers \( \mathcal{E}(T_{v}V) \) and \( \mathcal{J}(T_{v}V) \). Since \( \mathcal{E}(T_{v}V) \) retracts by deformation to \( \mathcal{J}(T_{v}V) \), \( \mathcal{E}(TV) \) retracts by deformation to \( \mathcal{J}(TV) \). In particular, every elliptic structure defines a unique homotopy class of almost complex structures on \( V \). Thus the Chern class \( c_{1}(E) = c_{1}(TV, J) \in H^{2}(V, \mathbb{Z}) \) is well defined.

Finally, the twisted structures \( J_{v}, v \in V \), can be put together to give a *twisted almost complex structure* on \( V \), ie a fiber-preserving map \( J : TV \to TV \) such that all the \( J_{v} \) have the properties (i)-(v) of 1.C. It is continuous on \( TV \) [in fact locally Lipschitz], and smooth away from the zero section. Conversely, a map \( J \) with all these properties clearly defines an elliptic structure.

### 2. Solutions of \( E \), \( E \)-maps and \( E \)-curves

In this section we consider an oriented 4-manifold \( V \) equipped with an elliptic structure \( E \subset G = Gr_{2}TV \). If \( S \) is an oriented surface and \( f : S \to V \) is an immersion, we denote \( \gamma_{f} : S \to G \) the associated Gauss map.

#### 2.A. Immersed solutions. Local equation as a graph

**Definition.** An immersed *solution of \( E \)* is a \( C^{1} \) immersion \( f : S \to V \) where \( S \) is an oriented surface and \( \gamma_{f}(S) \subset E \).

Let \( v \in V \) and \( P \in E_{v} \) be fixed. We describe a local equation for germs of immersed solutions of \( E \) which are tangent to \( P \) at \( v \), or more generally which have a tangent close enough to \( P \). Choose a local chart \((V,v) \to (C^{2},0)\) such that the properties of 1.D are satisfied for \( E_{v} \) and \( P \). Then the elliptic structure on \( E \) near \( H \) is given by a family of planes

\[
P_{z,w,\lambda} = \{(\delta z, \delta w) \in C^{2} \mid \delta w = \lambda \delta z + h(z, w, \lambda) \delta z \}.
\]

Here \((z, w, \lambda)\) belongs to a neighbourhood of 0 in \( C^{3} \), and \( P_{z,w,\lambda} \) represents a plane tangent at the point of coordinates \((z,w)\). The map \( h \) is a smooth germ \((C^{3},0) \to (C,0)\) such that

\[
\begin{align*}
||D_{3}h(z,w,\xi)|| &< 1 \quad (\forall(z,w,\xi)) \\
D_{3}h(0,0,0) & = 0.
\end{align*}
\]

(1)
A germ of surface $S \subset V$ passing through $P$ with a tangent plane close enough to $P$, can be written as a graph $w = f(z)$, where $f : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$ satisfies

\begin{equation}
\frac{\partial f}{\partial z} = h(z, f(z), \frac{\partial f}{\partial z}).
\end{equation}

Remark. This equation with the property $||D_3 h|| < 1$ is the general (resolved) form of an elliptic equation $C \to C$, cf. [V]. It implies the existence of a local immersed solution of $E$ with any given tangent plane, and even with an arbitrary “compatible” $k$-jet (an easy proof can be given by a suitable implicit function theorem, modifying slightly the proofs given in Chapter V or VI of [AL]), and also that each solution is of class $C^\infty$.

2.B. Conformal parametrization, $E$-maps. Let $f : S \to V$ be an immersed solution of $E$. Since every tangent plane $P_z = df_z(T_z S)$ has a well-defined complex structure $j_{P_z}$, this induces a canonical almost complex structure $j_f$ on $S$, i.e. a natural structure of Riemann surface. In other words, every (immersed) solution of $E$ admits a natural conformal parametrization.

If $S$ is a Riemann surface, we say that an immersion $f : S \to V$ is a conformal solution of $E$ if it is a solution and $j_f$ is the canonical almost structure on $S$. This is equivalent to the equation

\begin{equation}
df_z \circ i = J_{f(z)} \circ df_z.
\end{equation}

We can now eliminate the immersion condition and define an $E$-map as a $C^1$ map $f : S \to V$, where $S$ is a Riemann surface, which is a solution of (3).

Note that since $J$ is only Lipschitz, the fact that $E$-maps are smooth is not completely obvious at this stage. But the arguments of [AL, chap. V or VI] imply that if $f$ is a nonconstant local $E$-map, then $df$ has only isolated zeros and the Gauss map $\gamma_f$ can be extended continuously at these zeros. And also that there exist $E$-immersions with an arbitrary given 2-jet, satisfying suitable compatibility conditions.

2.C. $E$-maps as pseudoholomorphic maps. We prove here that $E$-maps can be considered as pseudoholomorphic maps. For this, we define the 4-dimensional holonomy distribution $\Theta$ on $E$ by setting $\Theta_P = d\pi_P^{-1}(P)$, where $\pi : G \to V$ is the natural projection. It is characterized by the property that every Gauss map $\gamma_f : S \to E$ associated to an $E$-map (not locally constant), is tangent to $\Theta$. The construction of [G],1.4 can be generalized to give the

**Proposition.** There exists a unique almost complex structure $\tilde{J}$ on $\Theta$, such that every Gauss map $\gamma : S \to E$ associated to an $E$-map is $\tilde{J}$-holomorphic (or is a $\tilde{J}$-map), i.e. it is $C^1$ (or $C^\infty$), tangent to $\Theta$ and satisfies

\begin{equation}
d\gamma_z \circ i = \tilde{J}_{\gamma(z)} \circ d\gamma_z.
\end{equation}

Moreover, the differential $d\pi_P : T_P G \to P$ is complex linear on $\Theta_P$ and the subbundle $F$ tangent to the fibers is $\tilde{J}$-invariant, and $\tilde{J}_{P|F_P} = j_{E^c, P}$ (notation of 1.A).

Conversely, a $\tilde{J}$-map not locally contained in a fiber is the Gauss map associated to an $E$-map.

**Sketch of proof.** For every $X \in \Theta_P \setminus F_P$ there is an $E$-map $f : S \to V$ and a vector $u \in T_z S$ such that $d\gamma_z(u) = X$ where $\gamma$ is the Gauss map. Thus necessarily $\tilde{J}_{P|X} = d\gamma_z(iu)$, which implies the uniqueness.
The existence of $\tilde{J}$ can be proved in local coordinates. The equation (3) becomes

$$\frac{\partial f}{\partial y} = J_f(z)\left(\frac{\partial f}{\partial x}\right),$$

and the associated Gauss map is $\gamma = (f, s)$ with

$$s(z) = \frac{\frac{\partial f}{\partial x} \wedge (J_f(z)\left(\frac{\partial f}{\partial x}\right))}{||\frac{\partial f}{\partial x} \wedge (J_f(z)\left(\frac{\partial f}{\partial x}\right))||}.$$

By differentiating (3) and after some (straightforward but tedious) computation, one can obtain an equation of the form $\frac{\partial \gamma}{\partial y} = \tilde{J}$ and check that $\tilde{J}$ has all the stated properties. The uniqueness implies that it is independent of the chart.

We know already that $\gamma$ is $C^\infty$ away from singularities of $f$. If $z$ is such a singularity, then since $\gamma$ satisfies (4) away from $z$ and is continuous up to $z$, it is $C^\infty$ up to $z$. Note that it implies that every $E$-map is $C^\infty$.

Finally, if $\gamma : S \to E$ is a $C^1$ map, tangent to $\Theta$ and satisfying (4), the composed map $f = \pi \circ \gamma$ satisfies (3). If $\gamma$ is not locally contained in a fiber, $f$ is not locally constant, thus its Gauss map $\gamma_f$ is well-defined and one has $\gamma_f = \gamma$.

Remarks

(i) One can view $\tilde{J}$-map as an ordinary pseudoholomorphic map by extending $\tilde{J}$ arbitrarily to $\hat{J}$ defined on $TE$ (using Riemannian metric for instance). Then a $\tilde{J}$-map is a $\hat{J}$-map which is tangent to $\Theta$.

(ii) If $F$ was totally real instead of being complex, the triple $(\Theta, J, F)$ would be (adding some partial integrability properties) what F. Labourie [L] calls a Monge-Ampère geometry. The main difference is that his compactness theorem requires to add “curtain surfaces”, ie $\tilde{J}$-curves satisfying $\dim(TS \cap F) \equiv 1$.

2.D. General $E$-curves, compactness theorem. Using the conformal parametrization, we can now define an $E$-curve as a $J$-curve $C$ in $E$ which is “almost transverse” to $F$. To make this definition precise, we would have

(i) to choose a definition of $\tilde{J}$-curve, for instance a stable $\tilde{J}$-curve in the sense of Kontsevich.

(ii) to say what “almost transverse” means: essentially that no nonconstant component has an image contained in a fiber, or that it is transverse to $F$ except on a finite subset.

We would then obtain topological spaces of $E$-curves. We shall not give any details here since the only spaces of $E$-curves we shall consider will consist of curves which are embedded (in $V$!). The space of embedded $E$-curves will be considered as a subspace of the space of smooth surfaces in $V$: recall that this is a Fréchet smooth manifold whose tangent space at $S$ is the space of sections of the normal bundle $N(S, V) = T_S V / TS$.

We shall also use the following notions of individual $E$-curves:

- a primitive (or irreducible) $E$-curve is the image $C = f(S)$ where $S$ is a closed and connected Riemann surface, $f$ is an $E$-map which does not factor $f = f_1 \circ \pi$ with $\pi$ a nontrivial holomorphic covering. As in the case of $J$-curves, the image determines $(S, f)$ up to isomorphism

- an $E$-cycle (à la Barlet) $C = \sum n_i C_i$ where the $C_i$ are distinct primitive $E$-curves and the $n_i$ are positive integers
One expects a compactness theorem for $E$-curves, analogous to the one for pseudoholomorphic curves: roughly speaking, it should say that (if $V$ is compact) a set of $E$-curves is relatively compact if their areas in $V$ are uniformly bounded. If one replaces “areas in $V”$ by “areas in $E$ of the Gauss maps”, then such a result follows from

- the compactness theorem for $\hat{J}$ on $TE$ (cf. the remark (i) in 2.C)
- the fact that the conditions “tangent to $\Theta”$ and “almost transverse to $F”$ are closed conditions.

However, it is not clear that an area bound in $V$ gives an area bound in $E$. Gromov [G, 2.4.E] says that the Schwarz lemma is still valid for $E$-curves under an area bound in $V$, but I do not understand the proof. Anyhow, here we shall only need the compactness theorem for $E$-lines, which we shall prove in section 3.

2.E. Singularities of $E$-curves and positivity of intersections

Here we extend to $E$-curves the result of M. Micallef and B. White [MW] (see also [Sik1]): we prove that a $E$-curve, possibly non reduced, is $C^1$-equivalent to a germ of standard holomorphic curve in $C^2$. It implies the positivity of intersections for $E$-curves, in particular the genus and intersection formulas.

Such a result could be proved by showing that such a surface is quasiminimizing in the sense of [MW], but we prefer to use more complex-analytic arguments as in [Sik1].

We use the chart of 2.A to write the equation in intrinsic form, i.e., as a graph over the tangent space. If the curve is non singular, this is the equation (2) where $h$ satisfies (1). Now, consider a germ of non-immersed $E$-map $F : (C,0) \rightarrow (C^2,0)$ with horizontal tangent at the origin. Then equation (3) and the similarity principle [Sik1] (proposition 2; cf. also [McD2]) give the existence of $a \in C^*$ and $k \in N$, $k \geq 2$, such that

$$f(z) = (az^k,0) + O_{2,1}(z^{k+1}).$$

Here we use some notation from [MW], [Sik1]: $g(z) = O_{2,1}(z^{k+1})$ means

$$\begin{cases} g(z) = O(z^{k+1}), \quad dg(z) = O(z^k), \quad d^2g(z) = O(z^{k-1}) \\ (\forall \alpha < 1) d^2g is \alpha-Hölder with Hölder constant $O(|z|^{k-1-\alpha})$. \end{cases}$$

Thus we can reparametrize the curve by setting $pr_1 \circ F(z) = t^k$, where $z \mapsto t$ is a $C^1$ local diffeomorphism. We obtain a map $t \mapsto (t^k, F(t))$ where $F$ is of class $C^{2,1-} = \cap_{\alpha < 1} C^{1,\alpha}$, with $F(t) = O_{2,1}(t^{k+1})$ (cf. [Sik1], proposition 4). The fact that the image, viewed locally as a graph satisfies (1), means that $F$ satisfies

$$\frac{\partial F}{\partial t} = k t^{k-1} h(t^k, F(t)), \quad \frac{1}{kt^{k-1}} \frac{\partial F}{\partial t}.$$ 

Note that this makes sense also at the origin.

Next, we show that the “difference” of two such germs satisfies the similarity principle. More precisely, we have the following generalization of [Sik1], prop. 5:
Proposition. Assume that $F$ and $G$ satisfy (6) with the same value of $k$, and are not identical as germs. Then there exists $a \in \mathbb{C}^*$ and $k \in \mathbb{N}^*$ such that

$$F(t) - G(t) = at^k + O_{1,1-}(t^{k+1}).$$

Proof. Set $u = F - G$, and take the difference of the two equations on $F$ and $G$. Using Taylor’s integral formula and setting

$$\gamma(t, s) = (t^k, G(t) + su(t), \frac{1}{k!}t^{k-1}\frac{\partial G}{\partial t} + s\frac{\partial u}{\partial t}),$$

we get

$$\frac{\partial u}{\partial t} = A(t).u(t) + B(t).\frac{\partial u}{\partial t},$$

where

$$\begin{cases}
A(t) = \int_0^1 k!t^{k-1}D_2h(\gamma(t, s))
ds \\
B(t) = \int_0^1 t^{k-1}D_3h(\gamma(t, s))
ds.
\end{cases}$$

The properties of $h$, $F$ and $G$ imply that $A$ and $B$ are of class $C^{1,1-}$, and $\|B\|_{L^\infty} < 1$. Then the proposition follows from a variant of proposition 2 in [Sik1].

Finally, one proceeds exactly as in [Sik1] (inspired by [MW]) to deduce from this proposition the

Proposition. Let $E$ be a germ of elliptic structure on $\mathbb{C}^2$ near 0 such that the horizontal plane $H = \mathbb{C} \times \{0\}$ belongs to $E_0$. Let $f_i : (\mathbb{C}, 0) \to (\mathbb{C}^2, 0)$, $i = 1, \ldots, r$, be germs of $E$-maps, all tangent to $H$ at 0. Then there exist

- a local $C^1$ diffeomorphism $\phi : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$, with support in an arbitrarily small sector

$$S_\epsilon = \{(x, y) \in \mathbb{C}^2 \mid |y| \leq \epsilon|x|\}$$

- local diffeomorphisms $u_i : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$, tangent to the identity,

such that all the maps $\phi \circ f_i \circ u_i$ are holomorphic.

If the tangents to the $f_i$ are not the same, we cannot in general expect to find a differentiable chart on $V$ in which the image become holomorphic: there is an obstruction already at the linear algebraic level. However, by superposing the diffeomorphisms given by the proposition we easily obtain a Lipschitz chart:

Theorem. Let $f_i : (\mathbb{C}, 0) \to (V, v)$, $i = 1, \ldots, r$, be germs $E$-maps through the same point. Then there exists a germ of Lipschitz oriented homeomorphism $\phi : (V, v) \to (\mathbb{C}^2, 0)$ such that all the maps $\phi \circ f_i \circ u_i$ are holomorphic.

Proof. We can assume that $(V, v) = (\mathbb{C}^2, 0)$. Let $(P_j)$, $j = 1, \ldots, r$, be the different tangent planes to the $f_i$ at $v$, and let $I_j \subset \{1, \ldots, r\}$ be the indices corresponding to the branches with tangent $P_j$. The $P_j$ are not complex linear in general, but there exist a Lipschitz oriented homeomorphism $h$ of $\mathbb{C}^2$ such that the $h(P_j)$ are complex linear, thus there exist a family of complex linear $A_j$ such that

$$(\forall j) \ A_j h(P_j) = \mathbb{C} \times \{0\}.$$
Furthermore, we can assume that $h$ is smooth [even linear] on a sector $S_j$ around $P_j$.

Then we can apply the proposition to $A_j h f_i$, $i \in I_j$: there exists a $C^1$ diffeomorphism $\phi_j : C^2 \to C^2$, such that $\phi_j A_j h f_i$ is holomorphic with horizontal tangent for $i \in I_j$. Moreover, we can assume that the support of $\phi_j$ is contained in $A_j h(S_j)$.

Then the desired homeomorphism is given by

$$
\phi = \begin{cases} 
A_j^{-1} \circ \phi_j \circ A_j \circ h & \text{on } S_j \\
h & \text{elsewhere.}
\end{cases}
$$

From this theorem, one deduces the positivity of intersections. More precisely, one can define

- a local intersection index $(C, C')_v \in \mathbb{N}^*$ for two germs $C$ and $C'$ of $E$-cycles at the point $v$ without common component. It is equal to 1 if and only if $C$ and $C'$ are smooth at $v$, with distinct tangents

- a local self-intersection number $\delta_v(C) \in \mathbb{N}^*$ (number of double points in a generic deformation equitopological at the source). It is equal to 1 if and only if $C$ is smooth at $v$.

One then has the intersection and genus (or “adjunction”) formulas

**Theorem.**

(i) If $C$ and $C'$ are two $E$-cycles without common components, then $C \cap C'$ (intersection of the supports) is finite and the homological intersection is given by

$$
C.C' = \sum_{v \in C \cap C'} (C, C')_v.
$$

(ii) If $C$ is an irreducible $E$-curve of genus $g$, then it has a finite number of singularities, and its genus is given by

$$
g = \frac{C.C - c_1(E).C}{2} + 1 - \sum_v \delta_v.
$$

Assume that $(V, E)$ is homologically equivalent to $(\mathbb{CP}^2, E_0)$. Thus there is a well defined degree map $H_2(V; \mathbb{Z}) \to \mathbb{Z}$ (an isomorphism modulo torsion, but not necessarily an isomorphism at this stage). Define an $E$-line as a primitive $E$-curve $C \subset V$ of degree 1. Then Since an $E$-line satisfies $C.C = 1$ and $c_1(E).C = 3$, we get the

**Corollary.** Every $E$-line is an embedded sphere, and two distinct $E$-lines intersect transversely in one unique point.

2.F. Linearization of the equation of $E$-curves; automatic genericity. We consider here embedded $E$-curves, i.e. smooth surfaces $S \subset V$ satisfying $\tilde{S} \subset E$ where $\tilde{S}$ is the Gaussian lift. Following [G, 2.4.E] we linearize this “equation” at $S$, obtaining an equation $\nabla_E f = 0$ where $\nabla_E$ acts on sections $f : S \to N = T_S V/T S$, the normal bundle, with values in $\Omega^{0,1}_j(S, N)$. Here $J = j^+_{TS}$ (cf. 1.A) is the natural complex structure on $N$.

One can obtain explicitly this equation by using the equation (2) in local coordinates. The compatibility of the complex structure on $C^2$ with the structure on $N$ means that $D_3 h(z, 0, 0) = 0$. Thus the linearization of (2) has locally the form

$$
\frac{\partial f}{\partial z} - D_2 h.f = 0,
$$

10
ie the operator $\bar{\partial}_E$ has the form $\bar{\partial}_E = \bar{\partial} + R$ where $\bar{\partial}$ is associated to a holomorphic structure on $N$ and $R$ is of order 0. Thus one can apply to it the arguments of [G, 2.1.C] (cf. also [HLS]).

**Proposition.** If $c_1(N) > 2g - 2$, i.e. $c_1(E) \cdot S > 0$, then $\bar{\partial}_E$ is onto. Thus the space $M_A$ of connected embedded $E$-curves in the class $A \in H_2(V; \mathbb{Z})$, if nonempty, is a smooth manifold if $c_1(E) \cdot A > 0$. Its real dimension is

$$2(c_1(N) \cdot S + 1 - g) = 2A \cdot A + (c_1(E) \cdot A - A \cdot A) = A \cdot A + c_1(E) \cdot A.$$ 

Also, $M$ is oriented since the homotopy $\ker(\bar{\partial} + R_i)$ gives it a natural homotopy class of almost complex structure (not more than a homotopy class, as we prove in section 5!).

Now assume that $(V, E)$ is homologically equivalent to $(\mathbb{CP}^2, E_0)$. Then the space of $E$-lines is the disjoint union of the $M_A$ for all $A \in H_2(V; \mathbb{Z})$ of degree 1. For such classes, we have $c_1(E) \cdot A = 3$ thus the proposition applies:

**Corollary.** The space of $E$-lines $V^*$, if nonempty, is naturally a smooth oriented 4-manifold.

**Extensions.** Since $c_1(E) \cdot A = 3$ we can still impose on $S$ a condition of complex codimension 1 or 2, and keep the automatic genericity (cf. for instance [B]). In particular:

(i) Let $\mathcal{L}_v^+ \times \ell$ be the space of $E$-lines through a given $v \in V$: it is an oriented smooth surface in $V^*$ when nonempty. Note that it can be identified with an open subset of the projective line $G_1^1(T_vV)$. Also, $\mathcal{L}_v^+$ depends smoothly on $v$.

(ii) Let $\mathcal{L}_{v,w}^*$ be the space of $E$-lines through two given points $v, w \in V$: when nonempty, it is a point $L_{v,w}$ which depends smoothly on $(v, w)$. This is the case for some open subset $U_1 \subset V \times V \setminus \Delta_V$.

(iii) Let $\mathcal{L}_P^* = \ell$ be the space of $E$-lines with a given tangent plane $P \in E$: again, when nonempty, it is a point $L_P$ which depends smoothly on $P$. This is the case for some open subset $U_2 \subset E$.

3. Tame elliptic projective planes

By definition, a tame elliptic projective plane $(V, E)$ is a 4-manifold equipped with a tame elliptic structure, homologically equivalent to $(\mathbb{CP}^2, E_0)$. Note that we do not require a priori $V$ to be diffeomorphic to $\mathbb{CP}^2$.

3.A. Proposition. Let $(V, E)$ be a tame elliptic projective plane. Then

(i) by two distinct points $x, y \in V$ there passes a unique $E$-line $L_{v,w}$, and for any given $P \in E$ there exists a unique $E$-line $L_P$ tangent to $P$.

(ii) $V$ is oriented diffeomorphic to $\mathbb{CP}^2$, $E$ is homotopic to $E_0$, and any taming $\omega$ is isomorphic to $\lambda \omega_0$ for some $\lambda > 0$.

**Proof**

(i) It suffices to prove that $V^*$ is compact: this will imply that that the open sets $U_1$ and $U_2$ defined at the end of 2.F are also closed, so $U_1 = V \times V \setminus \Delta_V$ and $U_2 = E$, which proves (i).

The compactness of $V^*$ will follow from the compactness theorem for $\tilde{J}$. First, there exists a taming $\Omega$ for $\tilde{J}$: as usual in the theory of symplectic bundles, we set $\Omega = \pi^* \omega + \alpha$ where $\alpha$ is a closed 2-form on $E$ which is positive on very fiber. Such a form exists since $H^2(E) \rightarrow H^2(V)$ is
onto: this is true since it holds for in the standard case $V = \mathbb{CP}^2$, $E = E_0$ and that our case is homologically standard.

Furthermore, let $A \in H_2(E;\mathbb{Z})$ be the homology class of the Gauss lift of $E$-lines. Then $A$ is $\Omega$-indecomposable, ie not equal to a sum $A = A_1 + A_2$ with $\omega(A_j) > 0$. This can again be seen in the standard situation in that case, a holomorphic curve $C$ in the class $A$ is always a section $s(L)$ over a line $L$ in $\mathbb{CP}^2$; if $C$ is not the Gaussian lift of $L$, then there exists $v_0 \in \mathbb{CP}^2 \setminus L$ such that $s(v)$ is the tangent to the line $[v_0 v]$ for every $v \in L$.

The $\Omega$-indecomposability and the compactness theorem of [G] imply that the space $M$ of rational $\tilde{J}$-curves in the class $A$ is compact, and since $V^*$ is homeomorphic to a closed subset of $M$ it is also compact.

(ii) Fix three lines $L_0, L_1, L_\infty$. We deform $E$ so that it remains tamed by $\omega$, the $L_i$ are still $E$-lines and $E$ comes from a complex structure isomorphic to the standard one near $L_\infty^0$. This is possible, using Darboux-Givental and the contraction $E_\omega \to J_\omega$.

We shall find a diffeomorphism $\phi : V \to \mathbb{CP}^2$ which sends them to the $x$-axis $L_0^0$, the $y$-axis $L_1^0$ and the line at infinity $L_\infty^0$.

Denote $v_0, v_1$ the intersections $L_0 \cap L_\infty, L_1 \cap L_\infty$. Let $v \in V \setminus L_\infty$. Then the $E$-lines $v_0 v$ and $v_1 v$ meet $L_0$ and $L_1$ in $x(v)$ and $y(v)$ respectively. Identifying $L_0$ with $L_0^0$, $L_1$ with $L_1^0$, we define $\phi(v)$ to be the intersection of $v_0 x(v)$ and $v_1 y(v)$. We obtain thus a smooth map $\phi : V \setminus L_\infty \to \mathbb{CP}^2 \setminus L_\infty^0$.

Exchanging the roles of $V$ and $\mathbb{CP}^2$, we obtain $\psi : V \setminus L_\infty^0 \to \mathbb{CP}^2 \setminus L_\infty^0$ which is the inverse of $\psi$. Since everything is standard near $L_\infty$, one can extend $\phi$ to $L_\infty$ and $\psi$ to $L_\infty^0$.

Finally, the fact that $\omega$ is isomorphic to $\lambda \omega_0$ results from Moser’s lemma.

3.B. Conversely, as shown by Gromov, one has the

**Proposition [G, 2.4.A’].** Assume that $V^*$ is compact and nonempty. Then there exists a taming symplectic form $\omega$.

**Proof.** Using a positive volume form $\nu$ on $V^*$ (identified with a smooth measure $d\nu$), define a 2-form $\omega$ by Crofton’s formula:

$$\int_S \omega = \int_{V^*} \text{Int}(S, L) \, d\nu(L)$$

for every oriented surface $S \subset V$. Here, $\text{Int}(S, L)$ is the algebraic intersection number, which is defined for almost all $L \in V^*$.

Let us give a more explicit definition of $\omega$. First, fix $v \in V$ and denote by $\mathcal{L}_v^* \subset V^*$ the subset of $E$-lines containing a given $v \in V$, which is a submanifold diffeomorphic to $\mathbb{CP}^1$.

**Proposition.** There is a canonical isomorphism between normal bundles, $\nu : N_v L \to N_L \mathcal{L}_v^*$.

**Proof.** Choose an $E$-line $L^\perp$ different from $L$ at $v$. If $\delta L \in T_L V^*$, let $(L_t)$ be a path such that $\frac{d}{dt}_{t=0} L_t = \delta L$, and set

$$\phi(\delta L) = \left(\frac{d}{dt}\right)_{t=0} (L_t \cap L^\perp) \in T_v V.$$

Dually, choose a point $w \in L$ different from $v$. If $\delta v \in T_v V$, let $(v_t)$ be a path such that $\frac{d}{dt}_{t=0} v_t = \delta v$, and set

$$\psi(\delta v) = \left(\frac{d}{dt}\right)_{t=0} L_{v_t, w} \in T_L V^*.$$
Then clearly, $\phi$ induces the desired isomorphism $\nu$ and $\psi$ its inverse.

We can now define a morphism

$$s : T_v V \to \Gamma(L_v^*, N(V^*, V))$$

by composing

$$T_v V \to N_v L \to N_v L_v^* : L \in L_v^*.$$ 

Thus for $X, Y \in T_v V$ and $L \in V^*_x$, $s(X)(L)$ and $s(Y)(L)$ are elements of $N_v L_v^*$. Lifting them to $\tilde{X}, \tilde{Y} \in T_v V^*$, we see that

$$\nu(\tilde{X}, \tilde{Y}) = \iota_{\tilde{X}} \iota_{\tilde{Y}} \nu \in \Lambda^2 T_v^* V^*$$

(interior products) is independent of the lifts. Varying $L \in L_v^*$, we obtain a 2-form on $L_v^*$ which we denote by $\nu(s(X), s(Y))$, and we set

$$\omega(X, Y) = \int_{L_v^*} \nu(s(X), s(Y)).$$

It is easy to see that it is positive on $E$ and satisfies Crofton’s formula.

4. Proof of the main result

4.A. Definition of $E^*$. We set

$$E^* = \{T_L L_v^* \mid v \in V, L \in L_v^*\}.$$ 

This is clearly a submanifold fibered over $V^*$, the fiber at $L$ being

$$E_L^* = \{T_L L_v^* \mid v \in L\}.$$ 

It is equipped with a natural distribution of codimension 2, $\Theta_P = d\pi^{-1}(P^*)$. Note also that $E^*$ is naturally diffeomorphic to $E$ via $\phi : T_{v,L} \to T_{L_v^*,L}$, in fact both are naturally diffeomorphic to the incidence variety

$$I = \{(v, L) \in V \times V^* \mid v \in L\} = \{(v, L) \in V \times V^* \mid L \in L_v^*\}.$$ 

This variety is equipped with two natural fibrations $p : I \to V$, $p^* : I \to V^*$. It also has one natural distribution. Indeed, by differentiating the condition $(v(t) \in L(t))$, one obtains the

Proposition. If $(\delta v, \delta L) \in T_{v,L} I$, then $(\delta v \in T_v L \Leftrightarrow \delta L \in T_L L_v^*)$.

Thus one can define the distribution $D \subset TI$ by

$$D_{v,L} = dp^{-1}(T_v L) = dp^*^{-1}(T_L L_v^*).$$

We then have a commutative triangle

$$\begin{array}{ccc}
(I, D) & \xrightarrow{\gamma} & (E, \Theta) \\
\phi \downarrow & \swarrow \gamma^* & \\
(E, \Theta) & \xrightarrow{\phi} & (E^*, \Theta^*)
\end{array}$$
where \( \gamma(v, L) = T_v L \) and \( \gamma^*(v, L) = T_L L_v^* \).

**4.B. Proof that \( E^* \) is elliptic**

Let us fix \((v, L)\) such that \( v \in L \), and denote
\[
P = T_v L, \quad P^* = T_L L_v^*.
\]
Then there are natural embeddings
\[
i : T_P E_v \to \text{Hom}(T_v L, N_v L)
i^* : T_{P^*} E_L^* \to \text{Hom}(T_L L_v^*, N_L L_v^*).
\]
By the ellipticity of \( E \), \( i(p) \) is an oriented isomorphism if \( p \neq 0 \). We want to prove the same property for \( i^*(p^*) \). This will follow from
- the existence of canonical isomorphisms \( T_P E_v \approx P^* \), \( T_{P^*} E_L^* \approx P, N_v L \approx N_L L_v^* \) (this last we know already); thus \( i \) and \( i^* \) become morphisms \( P^* \to \text{Hom}(P, N) \) and \( P \to \text{Hom}(P^*, N) \)
- the formula
\[
i^*(p)(p^*) = i(p^*)(p).
\]
To prove this, we define local charts on \( V, V^* \) and \( G \):

1) We start with a chart \( \Psi : V \to T_v L \times N_v L \), such that
\[
\begin{align*}
\Psi(L) &= T_v L \times \{0\} \\
\Psi(L^\perp) &= \{0\} \times N_v L \\
\text{pr}_1 \circ d\Psi_v|_{T_v L} &= \text{Id}
\end{align*}
\]
\[
\text{pr}_2 \circ d\Psi_v = \text{natural projection}.
\]

2) We define a chart \( \Psi^* : V^* \to T_L L_v^* \times N_L L_v^* \) such that \( \Psi^{-1}(0, 0) \) passes through \( v \) and \( \Psi^{-1}(\alpha, \beta) \) is “horizontal”. More precisely, \( \Psi^{-1}(\alpha, \beta) \) is given in the chart \( \Psi \) by an equation
\[
y = f_{\alpha, \beta}(x), \quad x \in T_v L, \quad y \in N_v L,
\]
such that
\[
\begin{align*}
f_{\alpha, \beta}(0) &= \nu(\beta) \\
f_{\alpha, 0}(0) &= 0 \\
\frac{\partial f_{\alpha, \beta}}{\partial x}(0) &= i(\alpha).
\end{align*}
\]
In the last equation, \( \alpha \in T_L L_v^* = P^* \) is interpreted as an element of \( T_P E_v \), so that \( i(\alpha) \in \text{Hom}(T_v L, N_v L) \).

3) Finally, let \( P^* \in Gr_2 T_v L V^* \) close to \( T_v L V^* \), we define \( \chi(P^*) \in \text{Hom}(T_L L_v^*, N_L L_v^*) \) as the unique \( h \) such that
\[
d\Psi^*(P^*) = \text{graph}(h).
\]
**End of the proof.** Let \( w = \Psi^{-1}(x) \) be an element of \( L \) close to \( v \). Then \( \Psi^{-1}(\alpha, \beta) \in V_w^* \) if and only if \( f_{\alpha, \beta}(x) = 0 \), thus \( T_L V_w^* \) is given by
\[
\{(\delta \alpha, \delta \beta) \mid \frac{\partial f_{\alpha, 0}}{\partial \alpha}(x). \delta \alpha + \frac{\partial f_{0, \beta}}{\partial \beta}(x). \delta \beta = 0\},
\]
5. Inexistence of a natural almost complex structure on \( V^* \)

In other words

\[
\frac{\partial f_{\alpha,0}}{\partial \alpha}(x) \cdot \delta \alpha + \nu(\delta \beta) = 0.
\]

Thus, the tangent space of \( E^*_L \) at \( T_L \mathcal{L}^*_v \) is identified with the image of the morphism

\[
i^* = \nu^{-1} \circ \frac{\partial^2 f_{\alpha,0}}{\partial x \partial \alpha}(0) : T_v L \to \text{Hom}(T_L \mathcal{L}^*_v, N_L \mathcal{L}^*_v).
\]

Since \( \frac{\partial f}{\partial x}(\alpha,0,0) = i(\alpha) \), one has

\[
i^*(\xi)(\alpha) = i(\alpha)(\xi), \quad (\xi, \alpha) \in T_v L \times T_L \mathcal{L}^*_v.
\]

Since \( E_v \) is elliptic, \( i(\alpha) \) is invertible and orientation-preserving if \( \alpha \neq 0 \). Thus one can identify the oriented planes \( T_L \mathcal{L}^*_v, T_v L \) and \( N_L \mathcal{L}^*_v \) with \( \mathbb{C} \) so that \( i(\alpha) \) is the multiplication by \( \alpha \). Then \( i^*(\xi) \) is the multiplication by \( \xi \), thus it is invertible and orientation-preserving if \( \xi \neq 0 \), which means that \( E^*_L \) is elliptic.

4.C. Proof that \( V^* \) is oriented diffeomorphic to \( \mathbb{CP}^2 \). One could prove it similarly to the proof for \( V \). The simplest proof however is to remark that the space of elliptic structures on \( V = \mathbb{CP}^2 \) which are tamed by \( \omega_0 \) is contractible. For each \( E \) in this space, we obtain an oriented manifold \( V_E \) which varies smoothly with \( E \), thus keeps the same oriented diffeomorphism type. Since for \( E \) associated to \( J_0 \) one has \( V_E^* = \mathbb{CP}^{2*} \) the standard dual projective plane, this proves the result.

4.D. Tameness of \( E^* \) and identification \( (V^*)^* = V \). For each \( v \in V \), the surface \( \mathcal{L}^*_v \subset V^* \) is an \( E^* \)-curve of degree 1, ie an \( E^* \)-line. Moreover, for two distinct points \( L, L' \in V^* \) there exists a unique \( v \in L \cap L' \), equivalently a unique \( \mathcal{L}^*_v \) containing \( L \) and \( L' \): this means that the \( E^* \)-lines are precisely the \( \mathcal{L}^*_v \), and thus that \( E^* \) is tame and \( V^{**} = V \). The equivalence \( (v \in L \Leftrightarrow L \in \mathcal{L}^*_v) \) implies that \( E^{**} \) is identified to \( E \).

4.E. Dual curves. Let \( J \) be the restriction to \( T(I) \) of \( (J,J^*) \), where \( J \) and \( J^* \) are the twisted almost complex structures associated to \( E \) and \( E^* \): it is an almost complex structure, whose images by \( \gamma \) and \( \gamma^* \) (notations of 4.A) are \( \tilde{J} \) and \( \tilde{J}^* \), the complex structures on \( \Theta \) and \( \Theta^* \) associated to \( E \) and \( E^* \). Thus the map \( \phi : (E, \Theta) \to (E^*, \Theta^*) \) is a \((\tilde{J}, \tilde{J}^*)\)-biholomorphism.

Now let \( C = f(S) \subset V \) be an irreducible \( E \)-curve (or an irreducible germ) not contained in an \( E \)-line. Let \( \gamma : S \to E \) be the Gauss map, which is \( \tilde{J} \)-holomorphic. Then \( \gamma^* = \phi \circ \gamma : S \to E^* \) is \( \tilde{J}^* \)-holomorphic and not locally constant, thus it is the Gauss map of an \( E^* \)-map \( f^* : S \to V^* \). By definition, \( C^* = f^*(S) \) is the dual curve of \( C \): it is again an irreducible \( E^* \)-curve (or germ), not contained in an \( E^* \)-line, and of course one has \( C^{**} = C \).

5. Inexistence of a natural almost complex structure on \( V^* \)

Here we construct a tame almost complex structure \( J \) on \( V = \mathbb{CP}^2 \) such that the twisted almost complex structure \( J^* \) on \( V^* \) is non linear.

We impose on \( J \) the following properties:
- it is standard outside

\[ U_0 = (\Delta(2) \setminus \Delta(1)) \times \Delta(2) \subset \mathbb{C}^2 \subset \mathbb{CP}^2 \]

- it is \( \omega_0 \)-positive

- for \( \alpha \in \mathbb{C} \) small enough, the \( J \)-line \( L(\alpha, \alpha) \) passing through the point \( (0, \alpha) \) with the slope \( \alpha \) has an intersection with \( U_0 \) given by the equation

\[ y = f_{\alpha, \alpha}(x) = \alpha + \alpha x + \frac{1}{5} \rho(x) \bar{x} x^2, \]

where \( \rho : \mathbb{C} \to [0, 1] \) takes the value 1 on \( U_1 = \Delta(1) \times \Delta(1) \) and 0 outside \( U_0 \). Note that the factor \( \frac{1}{5} \) guarantees that \( \alpha \mapsto f_{\alpha, \alpha}(x) \) is an embedding for \( |x| < 2 \) near 0.

One can find such a \( J \) under the form

\[ J(x, y) = \begin{pmatrix} i & 0 \\ b(x, y) \sigma & i \end{pmatrix} \]

where \( b(x, y) \in \mathbb{C} \) and \( \sigma \) is the complex conjugation. Then \( L(\alpha, \alpha) \cap U_0 \) is \( J \)-holomorphic if and only if

\[ b(x, f_{\alpha, \alpha}(x)) = \frac{\partial f_{\alpha, \alpha}}{\partial \bar{x}}(x). \]

Since \( \alpha \mapsto f_{\alpha, \alpha}(x) \) is an embedding near 0 for \( |x| < 2 \) and the second member vanishes for \( x \) close to 2, one can find a smooth \( b(x, y) \) with support in \( U_0 \), satisfying the above equality for \( |x| < 2 \) and \( \alpha \) small enough.

We now prove that \( J^*_L \) is not linear. Note that on \( U_1 \), we have \( J = J_0 \) and \( L(\alpha, \alpha) \cap U_1 \) is given by

\[ y = f_{\alpha, \alpha}(x) = \alpha + \alpha x + \frac{1}{5} \bar{x} x^2. \]

Let \( L \) be the \( J \)-line \( L(0, 0) \), which is the \( x \)-axis. Recall that for each \( v \in L \) the subspace \( T_L \mathcal{L}^*_v \subset T_L V^* \) is preserved by \( J^* - L \), which is linear on it. The global linearity of \( J^*_L \) is equivalent to the following:

\[ (\forall \xi, \eta \in T_L V^*) \quad \xi + \eta \in T_L \mathcal{L}^*_v \Rightarrow J^*_L(\xi) + J^*_L(\eta) \in T_L \mathcal{L}^*_v. \]

Consider on \( L \) the points \( v_0 = 0 \) and \( v_1 = \infty \). Then we have a direct sum \( T_L V^* = T_L \mathcal{L}^*_0 \oplus T_L L^*_\infty \). Then fix \( \alpha \neq 0 \) and consider the path \( t \in [0, 1] \mapsto \gamma(t) = L(t \alpha, t \alpha) \in V^* \), and write its derivative at \( t = 0 \) as

\[ \dot{\gamma} = \xi + \eta, \xi \in T_L \mathcal{L}^*_0, T_L \mathcal{L}^*_\infty. \]

It belongs to \( T_L \mathcal{L}^*_v \), where \( v = \lim_{t \to 0}(\gamma(t) \cap L) \). Identifying \( L \) with \( \mathbb{CP}^1 \), this means that \( v \) is the solution of the equation

\[ \alpha + \alpha v + \frac{1}{5} \bar{v} v^2 = 0. \]

If we change \( \alpha \) to \( i \alpha \), \( \xi \) and \( \eta \) are changed to \( J^*(\xi) \) and \( J^*(\eta) \) (essentially since \( J_0 \) is standard near 0 and \( \infty \), thus \( J^*_L(\xi) + J^*_L(\eta) \in T_L \mathcal{L}^*_w \)) where \( w \) is the solution of

\[ \alpha + \alpha w - \frac{1}{5} \bar{w} w^2 = 0. \]

Thus \( w \neq v \), which means that \( J^*_L \) is not linear.
Conjecture. Let $(V,J)$ be a tame almost complex projective plane. Assume that the elliptic structure $E^*$ on $V^*$ comes from an almost complex structure $J^*$ on $V^*$. Then $J$ is integrable, thus $(V,J)$ is biholomorphic to $(\mathbb{CP}^2,J_0)$.

More precisely: if $J^*_L$ is linear, then the Nijenhuis torsion of $J$ vanishes on $L$.

6. Plücker formulas for $E$-curves

We follow the classical topological method in algebraic geometry, cf. for instance [GH] p.279.

Let $C = f(S) \subset V$ be an irreducible $E$-curve, not contained in an $E$-line, and let $C^* \subset V^*$ be its dual. We compute the degree $d^*$ of $C^*$, which is the number of intersection points of $C^*$ with an $E^*$-line, ie the number of points of $C$ such that the tangent line $L_v C$ contains $v$. This number is to be interpreted algebraically, but for a generic $v$ it is equal to the set-theoretic number.

Let $L$ be an $E$-line disjoint from $v$, then the central projection $V \setminus \{v\} \to L$ along $E$-lines through $v$ induces an “almost holomorphic” branched covering $C \to L$ of degree $d$, in the sense that each singularity has a model $z \to z^k$: this is a consequence of the positivity of intersections. Let $S$ be the normalization of $C$, then the number of branch points of the induced covering $S \to L$ is $d^* + \kappa$ where $\kappa$ is the algebraic number of cusps, ie the algebraic number of zeros of $df$ if $f$ is a parametrization of $C$. Thus we have the Hurwitz formula $2 - 2g = 2d - (d^* + \kappa)$, where $g$ is the genus of $C$, ie

$$d^* = 2d + 2g - 2 - \kappa.$$ 

In particular, if $C$ has only $\delta$ nodes and $\kappa$ cusps, we have $2g - 2 = d(d - 3) - 2\delta - 2\kappa$ thus we get the first Plücker formula

$$d^* = d(d - 1) - 2\delta - 3\kappa.$$ 

As in the classical case, the other Plücker formulas follow from this and the genus formula, with the fact that an ordinary bitangent (resp. flex) of $C$ corresponds to a node (resp. cusp) of $C^*$.

This implies restrictions on the possible sets of singularities going beyond the genus formula. For instance, if $C$ has only nodes and cusps, then another form of Plücker formula is

$$\kappa = 2g - 2 + 2d - d^*.$$ 

If $d = 5$ and $g = 0$ we get $\kappa = 8 - d^*$, and since $d^* \geq 3$ we have $\kappa \leq 5$: not all 6 nodes of a generic rational curve can be transformed to cusps.

In general, if $C$ is rational with only nodes and cusps, we get $\kappa = 2d - 2 - d^* < 3d$, which implies that the space of rational $J$-curves is, at the point $C$, a smooth manifold of the expected dimension (equal to $d(d + 3)$ over $\mathbb{R}$): this follows from the generalization of the automatic genericity proved in [B]. This could be interesting for the isotopy problem of symplectic surfaces [Sik2].

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