A UNIFIED APPROACH FOR ENERGY SCATTERING FOR FOCUSING NONLINEAR SCHRÖDINGER EQUATIONS

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Abstract. We consider the Cauchy problem for focusing nonlinear Schrödinger equation

\[ i\partial_t u + \Delta u = -|u|^\alpha u, \quad (t,x) \in \mathbb{R} \times \mathbb{R}^N, \]

where \( N \geq 1, \alpha > \frac{4}{N} \) and \( \alpha < \frac{4}{N-2} \) if \( N \geq 3 \). We give a criterion for energy scattering for the equation that covers well-known scattering results below, at and above the mass and energy ground state threshold. The proof is based on a recent argument of Dodson-Murphy [Math. Res. Lett. 25(6):1805–1825, 2018] using the interaction Morawetz estimate.

1. Introduction. We consider the Cauchy problem for a class of focusing inter-critical nonlinear Schrödinger equations

\[
\begin{cases}
  i\partial_t u + \Delta u = -|u|^\alpha u, & (t,x) \in \mathbb{R} \times \mathbb{R}^N, \\
  u(0, x) = u_0(x),
\end{cases}
\]

(1)

where \( u : \mathbb{R} \times \mathbb{R}^N \to \mathbb{C}, u_0 : \mathbb{R}^N \to \mathbb{C}, N \geq 1 \) and \( \frac{4}{N} < \alpha < \alpha^* \) with

\[
\alpha^* := \begin{cases}
  \frac{4}{N-2} & \text{if } N \geq 3, \\
  \infty & \text{if } N = 1, 2.
\end{cases}
\]

It is well-known that (1) is locally well-posed in \( H^1 \) (see e.g. [3]). Moreover, local solutions enjoy the conservation of mass, energy and momentum

\[
\begin{align*}
M(u(t)) &= \int_{\mathbb{R}^N} |u(t,x)|^2 dx = M(u_0), \\
E(u(t)) &= \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u(t,x)|^2 dx - \frac{1}{\alpha + 2} \int_{\mathbb{R}^N} |u(t,x)|^{\alpha+2} dx = E(u_0), \\
P(u(t)) &= \text{Im} \int_{\mathbb{R}^N} \overline{u}(t,x) \nabla u(t,x) dx = P(u_0).
\end{align*}
\]

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The equation (1) has the scaling invariance
\[ u_\lambda(t, x) = \lambda^{\frac{N}{2}} u(\lambda^2 t, \lambda x), \quad \lambda > 0. \] (2)

A direct computation gives
\[ \|u_\lambda(0)\|_{\dot{H}^\gamma} = \lambda^{\gamma+\frac{2}{N} - \frac{2}{2}} \|u_0\|_{\dot{H}^\gamma}. \]

This shows that the scaling (2) leaves the $\dot{H}^\gamma$-norm of initial data invariant, where
\[ \gamma_c := \frac{N}{2} - \frac{2}{\alpha}. \] (3)

Since we are interested in the mass and energy intercritical case, it is convenient to define the exponent
\[ \sigma_c := \frac{1 - \gamma_c}{\gamma_c} = \frac{4 - (N - 2)\alpha}{N\alpha - 4}. \] (4)

The equation (1) admits a global non-scattering solution $u(t, x) = e^{it}Q(x)$, where $Q$ is the unique positive radial solution to the elliptic equation
\[ -\Delta Q + Q - |Q|^\alpha Q = 0. \] (5)

The energy scattering for the equation (1) below the ground state threshold was first proved by Holmer-Roudenko [12] for $N = 3$, $\alpha = 2$ and radially symmetric initial data. Duyckaerts-Holmer-Roudenko [6] later improved this result by removing the radial assumption. This result was extended to any dimension $N \geq 1$ by Cazenave-Fang-Xie [9], Akahori-Nawa [1] and Guevara [11]. More precisely, we have the following result.

**Theorem 1.1** (Scattering below the ground state threshold [1, 9, 6, 11]). Let $N \geq 1$, $\frac{4}{N} < \alpha < \alpha^*$ and $u_0 \in H^1$ satisfy
\[ E(u_0)[M(u_0)]^{\sigma_c} < E(Q)[M(Q)]^{\sigma_c}, \] (6)

and
\[ K(u_0)[M(u_0)]^{\sigma_c} < K(Q)[M(Q)]^{\sigma_c}, \] (7)

where
\[ K(f) := \|\nabla f\|_{L^2}. \] (8)

Then the corresponding solution to (1) exists globally in time and scatters in $H^1$ in both directions, that is, there exist $u^\pm \in H^1$ such that
\[ \lim_{t \to \pm \infty} \|u(t) - e^{it\Delta} u^\pm\|_{H^1} = 0. \]

The energy scattering for (1) at the ground state threshold was studied by Duyckaerts-Roudenko [7] for $N = 3$ and $\alpha = 2$. In particular, they proved the following result.

**Theorem 1.2** (Scattering at the ground state threshold [7]). Let $N = 3$ and $\alpha = 2$. Let $u_0 \in H^1$ satisfy
\[ E(u_0)M(u_0) = E(Q)M(Q) \] (9)

and
\[ K(u_0)M(u_0) < K(Q)M(Q). \] (10)
Then the corresponding solution to (1) exists globally in time. Moreover, it either scatters in $H^1$ in both directions or equals to $Q^-(t)$ up to the symmetries, where $Q^-(t)$ is a radial global solution to (1) satisfying

$$E(Q^-) = E(Q), \quad K(Q^-(0)) = K(Q),$$

$$\exists C > 0: \quad \|Q^-(t) - e^{it} Q\|_{H^1} \leq C e^{-Ct}, \quad \forall t \geq 0,$$

and $Q^-$ scatters in $H^1$ backward in time.

The energy scattering for (1) above the ground state threshold was studied by Duyckaerts-Roudenko [8]. To state their result, we introduce

$$V(t) := \int |x|^2 |u(t,x)|^2 dx.$$

Assuming $V(0) < \infty$, the following virial identities hold:

$$V'(t) = 4 \text{Im} \int x \cdot \nabla u(t,x) u(t,x) dx,$$

$$V''(t) = 8 \int |
abla u(t,x)|^2 dx - \frac{4N\alpha}{\alpha + 2} \int |u(t,x)|^{\alpha + 2} dx.$$

We have the following result.

**Theorem 1.3** (Scattering above the ground state threshold [8]). Let $u_0 \in H^1$ satisfy

$$E(u_0) [M(u_0)]^{\sigma_c} \geq E(Q) [M(Q)]^{\sigma_c},$$  

and

$$V(0) < \infty,$$

$$\frac{E(u_0) [M(u_0)]^{\sigma_c}}{E(Q) [M(Q)]^{\sigma_c}} \left(1 - \frac{(V'(0))^2}{32E(u_0)V(0)}\right) \leq 1,$$

and

$$H(u_0) [M(u_0)]^{\sigma_c} < H(Q) [M(Q)]^{\sigma_c},$$

$$V'(0) \geq 0,$$

where

$$H(f) := \|f\|_{L^{\frac{\alpha + 2}{\alpha}}}^{\alpha + 2}.$$

Then the corresponding solution to (1) exists globally and scatters in $H^1$ forward in time.

The proofs of Theorems 1.1, 1.2 and 1.3 are based on the concentration-compactness-rigidity argument introduced by Kenig-Merle [14]. Dodson-Murphy [5] gave an alternative simple proof for the energy scattering of (1) below the ground state threshold with $N \geq 3$ and $\alpha = \frac{4N - 1}{N - 1}$ that avoids the concentration-compact-rigidity argument. Recently, Gao-Wang [10] gave a unified proof for the energy scattering of the $\dot{H}^{\frac{4}{3}}$-critical Hartree equation with radial data. Inspired by the aforementioned results, we give a unified scattering criterion for (1). More precisely, we have the following result.
Theorem 1.4. Let $N \geq 1$ and $\frac{4}{N} < \alpha < \alpha^*$. Let $u(t)$ be a $H^1$ solution to (1) defined on the maximal forward time interval of existence $[0, T^*)$. Assume that
\[
\sup_{t \in [0, T^*)} H(u(t))[M(u(t))]^{\sigma_c} < H(Q)[M(Q)]^{\sigma_c}.
\]
Then $T^* = \infty$ and $u$ scatters in $H^1$ forward in time.

This result has essentially been given in [8, Theorem 3.7] using the concentration-compactness-rigidity argument of [14] (see also Appendix). In this paper, we give an alternative simple proof in dimensions $N \geq 3$ using the idea of Dodson-Murphy [5].

Suprisingly enough, Theorem 1.4 implies the energy scattering below and above the ground state threshold given in Theorems 1.1 and 1.3 respectively. More precisely, we have the following result.

Lemma 1.5. Let $N \geq 1$ and $\frac{4}{N} < \alpha < \alpha^*$. Let $u_0 \in H^1$ satisfy one of the following conditions:

- (6) and (7);
- (11) and (12)–(15).

Then the corresponding solution to (1) satisfies (17). In particular, the solution exists globally and scatters in $H^1$ forward in time.

Another application of Theorem 1.4 is the following result at the ground state threshold.

Lemma 1.6. Let $N \geq 1$ and $\frac{4}{N} < \alpha < \alpha^*$. Let $u_0 \in H^1$ satisfy (9) and (10).

Then the corresponding solution to (1) exists globally in time. Moreover, either it scatters in $H^1$ forward in time or there exists $t_n \to \infty$ and $(y_n)_{n \geq 1} \subset \mathbb{R}^N$ such that
\[
u(t_n, \cdot + y_n) \to e^{it\theta}Q(\cdot - x_0) \quad \text{strongly in } H^1(\mathbb{R}^N)
\]
for some $\theta \in \mathbb{R}$ and $x_0 \in \mathbb{R}^N$ as $n \to \infty$.

Remark 1. This result is weaker than that of Theorem 1.2. However, it holds for the whole range of the intercritical case and any dimensions.

The paper is organized as follows. In Section 2, we recall some useful estimates including dispersive and Strichartz estimates. In Section 3, we prove criteria for the energy scattering for the equation. Section 4 is devoted to show interaction Morawetz estimates. In Section 5, we give the proof of main results given in Theorem 1.4, Lemma 1.5 and Lemma 1.6. Finally, we gave an alternative proof of Theorem 1.4 using the concentration-compactness-rigidity argument in the Appendix.

2. Useful estimates. In this section, we recall some useful estimates needed in the sequel.

Lemma 2.1 (Dispersive estimates [3]). Let $r \in [2, \infty]$. It holds that
\[
\|e^{it\Delta}f\|_{L^r} \lesssim |t|^{-\frac{N}{2}(1-\frac{2}{r})}\|f\|_{L^r'}
\]
for any $f \in L^{r'}$, where $(r, r')$ is the Hölder’s conjugate pair.

Let $I \subset \mathbb{R}$ be an interval and $q, r \in [1, \infty]$. We define the mixed norm
\[
\|u\|_{L^q(I, L^r)} := \left( \int_I \left( \int_{\mathbb{R}^N} |u(t, x)|^r \, dx \right)^{\frac{q}{r}} \, dt \right)^{\frac{1}{q}}
\]
with a usual modification when either \( q \) or \( r \) are infinity. When \( q = r \), we use the notation \( L^q(I \times \mathbb{R}^N) \) instead of \( L^q(I, L^r) \).

**Definition 2.2.** A pair \((q, r)\) is said to be Schrödinger admissible if

\[
\frac{2}{q} + \frac{N}{r} = \frac{N}{2}, \quad \begin{cases} 
    r \in \left[2, \frac{2N}{N-2}\right] & \text{if } N \geq 3, \\
    r \in [2, \infty) & \text{if } N = 2, \\
    r \in [2, \infty] & \text{if } N = 1.
\end{cases}
\]

**Proposition 1 (Strichartz estimates [3, 13]).** Let \( N \geq 1 \) and \( I \subset \mathbb{R} \) be an interval. There exists a constant \( C > 0 \) independent of \( I \) such that the following estimates hold:

- **Homogeneous estimates**
  \[
  \|e^{it\Delta}f\|_{L^q(I, L^r)} \leq C\|f\|_{L^2}
  \]
  for any \( f \in L^2 \) and any Schrödinger admissible pair \((q, r)\).

- **Inhomogeneous estimates**
  \[
  \left\| \int_0^t e^{i(t-s)\Delta}F(s)ds \right\|_{L^q(I, L^r)} \leq C\|F\|_{L^m(I, L^{n'})}
  \]
  for any \( F \in L^m(I, L^{n'}) \) and any Schrödinger admissible pairs \((q, r), (m, n)\).

We also have the following inhomogeneous Strichartz estimates for non Schrödinger admissible pairs.

**Lemma 2.3 ([4]).** Let \( N \geq 1 \) and \( I \subset \mathbb{R} \) be an interval. Let \((q, r)\) be a Schrödinger admissible pair with \( r > 2 \). Fix \( k > \frac{q}{2} \) and define \( m \) by

\[
\frac{1}{k} + \frac{1}{m} = \frac{2}{q}.
\]

Then there exists \( C > 0 \), depending only on \( N, r \) and \( k \), such that

\[
\left\| \int_0^t e^{i(t-s)\Delta}F(s)ds \right\|_{L^k(I, L^r)} \leq C\|F\|_{L^m(I, L^{n'})}
\]

for any \( F \in L^m(I, L^{n'}) \).

We refer the reader to [4, Lemma 2.1] for the proof of this result.

3. **Scattering criteria.** Let us start with the following nonlinear estimates which follow directly from Hölder’s inequality.

**Lemma 3.1 (Nonlinear estimates).** Let \( N \geq 1 \), \( \frac{4}{N} < \alpha < \alpha^* \) and \( I \subset \mathbb{R} \) be an interval. Denote

\[
q := \frac{4(\alpha+2)}{N\alpha}, \quad r := \alpha+2, \quad k := \frac{2\alpha(\alpha+2)}{4-(N-2)\alpha}, \quad m := \frac{2\alpha(\alpha+2)}{N\alpha^2 + (N-2)\alpha - 4}.
\]

Then the following estimates hold:

\[
\|u\|_{L^m(I, L^r)} \lesssim \|u\|_{L^k(I, L^r)}^{\alpha+1} \|u\|_{L^k(I, L^r)}^\alpha,
\]

\[
\| \nabla (\|u\|^\alpha u) \|_{L^m(I, L^r)} \lesssim \|u\|_{L^k(I, L^r)}^{\alpha+1} \|u\|_{L^k(I, L^r)}^\alpha \| \nabla u \|_{L^q(I, L^r)}.
\]

**Remark 2.** Let \( q, r, k \) and \( m \) be as in (21). It is easy to check that \((q, r)\) is a Schrödinger admissible pair. Moreover, \( k, m \) and \( q \) satisfy (19), hence (20) holds for such choice of exponents.
Lemma 3.2 (Small data global well-posedness). Let $N \geq 1$ and $\frac{4}{N} < \alpha < \alpha^*$. Let $T > 0$ be such that $\|u(T)\|_{H^1} \leq A$ for some constant $A > 0$. Then there exists $\delta = \delta(A) > 0$ such that if

$$\|e^{(t-T)\Delta} u(T)\|_{L^k([T,\infty),L^r)} < \delta,$$

then there exists a unique global solution to (1) with initial data $u(T)$ satisfying

$$\|u\|_{L^k([T,\infty),L^r)} \leq 2\|e^{(t-T)\Delta} u(T)\|_{L^k([T,\infty),L^r)}$$

and

$$\|\langle \nabla \rangle u\|_{L^q([T,\infty),L^r)} \leq 2C\|u(T)\|_{H^1}.$$ 

Here $k,q$ and $r$ are as in (21).

Proof. Let $q, r, k$ and $m$ be as in (21). Consider

$$X := \{ u : \|u\|_{L^k(I,L^r)} \leq B, \|\langle \nabla \rangle u\|_{L^q(I,L^r)} \leq L \}$$

equipped with the distance

$$d(u,v) := \|u-v\|_{L^k(I,L^r)} + \|u-v\|_{L^q(I,L^r)},$$

where $I = [T, \infty)$ and $B, L > 0$ will be chosen later. We will show that the functional

$$\Phi(u(t)) := e^{i(t-T)\Delta} u(T) + i \int_T^t e^{i(t-s)\Delta} |u(s)|^\alpha u(s) ds$$

is a contraction on $(X,d)$. By Remark 2, (20) and Lemma 3.1,

$$\|\Phi(u)\|_{L^k(I,L^r)} \lesssim \|e^{i(t-T)\Delta} u(T)\|_{L^k(I,L^r)} + C\|u\|_{L^q(I,L^r)}^{\alpha+1}. \tag{24}$$

By Strichartz estimates and Lemma 3.1,

$$\|\langle \nabla \rangle \Phi(u)\|_{L^q(I,L^r)} \lesssim \|u(T)\|_{H^1} + \|u\|_{L^k(I,L^r)} \|\langle \nabla \rangle u\|_{L^q(I,L^r)}.$$ 

We also have

$$\|\Phi(u) - \Phi(v)\|_{L^k(I,L^r)} \lesssim \left(\|u\|_{L^k(I,L^r)} + \|v\|_{L^k(I,L^r)}\right) \|u-v\|_{L^k(I,L^r)}$$

and

$$\|\Phi(u) - \Phi(v)\|_{L^q(I,L^r)} \lesssim \left(\|u\|_{L^q(I,L^r)} + \|v\|_{L^q(I,L^r)}\right) \|u-v\|_{L^q(I,L^r)}.$$ 

There thus exists $C > 0$ independent of $T$ such that for any $u,v \in X$,

$$\|\Phi(u)\|_{L^k(I,L^r)} \leq \|e^{i(t-T)\Delta} u(T)\|_{L^k(I,L^r)} + CB^{\alpha+1},$$

$$\|\langle \nabla \rangle \Phi(u)\|_{L^q(I,L^r)} \leq C\|u(T)\|_{H^1} + CB^{\alpha} L$$

and

$$d(\Phi(u), \Phi(v)) \leq CB^{\alpha} d(u,v).$$

By choosing $B = 2\|e^{i(t-T)\Delta} u(T)\|_{L^k(I,L^r)}$ and $L = 2C\|u(T)\|_{H^1}$. Taking $B$ sufficiently small so that $CB^{\alpha} \leq \frac{1}{2}$, we see that $\Phi$ is a contraction on $(X,d)$. The proof is complete. \hfill \Box

Lemma 3.3 (Small data scattering). Let $N \geq 1$ and $\frac{4}{N} < \alpha < \alpha^*$. Suppose $u(t)$ is a global $H^1$ solution to (1) satisfying

$$\|u\|_{L^\infty(\mathbb{R},H^1)} \leq A$$

for some constant $A > 0$. Then there exists $\delta = \delta(A) > 0$ such that if

$$\|e^{i(t-T)\Delta} u(T)\|_{L^k([T,\infty),L^r)} < \delta$$

is complete.
for some $T > 0$, where $k$ and $r$ are as in (21), then $u$ scatters in $H^1$ forward in time.

Proof. Let $\delta = \delta(A)$ be as in Lemma 3.2. It follows from Lemma 3.2 that the solution satisfies

$$\|u\|_{L^k([T, \infty), L^r)} \leq 2\|e^{i(t-T)\Delta}u(T)\|_{L^k([T, \infty), L^r)},$$

$$\|\nabla u\|_{L^q([T, \infty), L^r)} \leq 2C\|u(T)\|_{H^1} \leq 2CA.$$

Now let $0 < \tau < t < \infty$. By Strichartz estimates, we see that

$$\|e^{-it\Delta}u(t) - e^{-i\tau\Delta}u(\tau)\|_{H^1} \leq \left\| \int_\tau^t e^{-is\Delta}|u(s)|^\alpha u(s)ds \right\|_{H^1}$$

$$\lesssim \|\nabla (|u|^\alpha u)\|_{L^q((\tau, t), L^r)}$$

$$\lesssim \|u\|_{L^k(\tau, t), L^r}\|\nabla u\|_{L^q((\tau, t), L^r)} \to 0$$

as $\tau, t \to \infty$. This shows that $(e^{-it\Delta}u(t))_{t \to \infty}$ is a Cauchy sequence in $H^1$. Thus the limit

$$u^+ := u_0 + i \int_0^\infty e^{-is\Delta}|u(s)|^\alpha u(s)ds$$

exists in $H^1$. By the same reasoning as above, we prove as well that

$$\|u(t) - e^{i\tau\Delta}u^+\|_{H^1} \to 0$$

as $t \to \infty$. The proof is complete. $\square$

Lemma 3.4 (Scattering criteria). Let $N \geq 3$ and $\frac{4}{N} < \alpha < \frac{4}{N-2}$. Suppose $u(t)$ is a global $H^1$ solution to (1) satisfying

$$\|u\|_{L^\infty(R, H^1)} \leq A$$

for some constant $A > 0$. Then there exist $\varepsilon = \varepsilon(A) > 0$ and $T_0 = T_0(\varepsilon, A) > 0$ such that if for any $a > 0$, there exists $t_0 \in (a, a + T_0)$ such that $[t_0 - a, t_0] \subset (a, a + T_0)$ and

$$\|u\|_{L^k([t_0 - a, t_0], L^r)} \lesssim \varepsilon$$

(25)

for some $\sigma > 0$, where $k, r$ are as in (21), then the solution scatters in $H^1$ forward in time.

Proof. By Lemma 3.3, it suffices to show that there exists $T > 0$ such that

$$\|e^{i(t-T)\Delta}u(T)\|_{L^k([T, \infty), L^r)} \lesssim \varepsilon^\mu$$

(26)

for some $\mu > 0$.

To show (26), we write

$$e^{i(t-T)\Delta}u(T) = e^{i\tau\Delta}u_0 + i \int_0^T e^{i(t-s)\Delta}|u(s)|^\alpha u(s)ds.$$

By Strichartz estimates, we have

$$\|e^{i\tau\Delta}u_0\|_{L^k(R, L^r)} \lesssim \|\nabla e^{i\tau\Delta}u_0\|_{L^k(R, L^r)} \lesssim \|u_0\|_{H^1} < \infty,$$

where $\gamma_c$ is as in (3) and

$$l := \frac{2N\alpha(\alpha + 2)}{N\alpha^2 + 4(N-1)\alpha - 8}.$$
Note that \((k, l)\) is a Schrödinger admissible pair. By the monotone convergence theorem, there exists \(T_1 > 0\) sufficiently large such that for any \(T > T_1\),
\[
\| e^{it\Delta} u_0 \|_{L^k((T, \infty), L^r)} \lesssim \varepsilon. \tag{28}
\]
Taking \(a = T_1\) and \(T = t_0\) with \(a\) and \(t_0\) as in (25), we write
\[
i \int_0^T e^{i(t-s)\Delta} |u(s)|^a u(s) ds = i \int I e^{i(t-s)\Delta} |u(s)|^a u(s) ds + i \int J e^{i(t-s)\Delta} |u(s)|^a u(s) ds =: F_1(t) + F_2(t),
\]
where \(I := [0, T - \varepsilon^{-\sigma}]\) and \(J := [T - \varepsilon^{-\sigma}, T]\). By Remark 2, (20), (22) and (25), we see that
\[
\| F_2 \|_{L^k((T, \infty), L^r)} \lesssim \| u \|_{L^{\infty'}(J, L^{r'})}^{a+1} \lesssim \| u \|_{L^k(J, L^r)}^{a+1} \lesssim \varepsilon^{a+1}. \tag{29}
\]
We next estimate \(F_1\). By Hölder’s inequality, we have
\[
\| F_1 \|_{L^k((T, \infty), L^r)} \leq \| F_1 \|_{L^k((T, \infty), L^l)}^{\theta} \| F_1 \|_{L^k((T, \infty), L^l)}^{1-\theta}
\]
where \(l\) is as in (27), \(\theta \in (0, 1)\) and \(n > r\) satisfy
\[
\frac{1}{r} = \frac{\theta}{l} + \frac{1 - \theta}{n}
\]
to be chosen later. Using the fact \((k, l)\) is a Schrödinger admissible pair and
\[
F_1(t) = e^{i(t-T+\varepsilon^{-\sigma})\Delta} u(T - \varepsilon^{-\sigma}) - e^{it\Delta} u_0,
\]
Strichartz estimates imply
\[
\| F_1 \|_{L^k((T, \infty), L^l)} \lesssim 1.
\]
On the other hand, by the dispersive estimates and Sobolev embeddings with the fact \(\| u \|_{L^{\infty'}(\mathbb{R}, H^1)} \leq A\), we have for any \(t \geq T\),
\[
\| F_1(t) \|_{L^n} \lesssim \int_I (t-s)^{-\frac{N}{2}\left(1 - \frac{2}{n}\right)} \| |u(s)|^a u(s) \|_{L^{n'}} ds
\]
\[
= \int_0^{T-\varepsilon^{-\sigma}} (t-s)^{-\frac{N}{2}\left(1 - \frac{2}{n}\right)} \| |u(s)|^{a+1} \|_{L^{n'(a+1)}} ds
\]
\[
\lesssim (t-T + \varepsilon^{-\sigma})^{-\frac{N}{2}(1-\frac{2}{n})+1}
\]
provided
\[
n'(a+1) \in \left[2, \frac{2N}{N-2}\right], \quad \frac{N}{2} \left(1 - \frac{2}{n}\right) - 1 > 0. \tag{30}
\]
It follows that
\[
\| F_1 \|_{L^k((T, \infty), L^n)} \lesssim \left( \int_T^\infty (t-T + \varepsilon^{-\sigma})^{-\frac{N}{2}(1-\frac{2}{n})-1} dt \right)^{\frac{1}{k}} \lesssim \varepsilon^\sigma \left[\frac{N}{2}(1-\frac{2}{n})-1\right]^{\frac{1}{k}}
\]
provided
\[
\frac{N}{2} \left(1 - \frac{2}{n}\right) - 1 - \frac{1}{k} > 0.
\]
We thus obtain
\[
\| F_2 \|_{L^k((T, \infty), L^r)} \lesssim \varepsilon^\sigma \left[\frac{N}{2}(1-\frac{2}{n})-1\right]^{(1-\theta)}. \tag{31}
\]
The above estimate holds true provided
\[
n > r, \quad n'(a+1) \in \left[2, \frac{2N}{N-2}\right], \quad \frac{N}{2} \left(1 - \frac{2}{n}\right) - 1 - \frac{1}{k} > 0.
\]
We will choose a suitable $n$ satisfying the above conditions. By the choice of $r$ and $k$, the above conditions become

$$0 \leq \frac{1}{n} < \frac{1}{\alpha + 2}, \quad \frac{1}{n} \in \left\{ \frac{1 - \alpha}{2}, \frac{N + 2 - (N - 2)\alpha}{2N} \right\}, \quad \frac{1}{n} < \frac{(N - 2)(\alpha^2 + 3\alpha) - 4}{2N\alpha(\alpha + 2)}. \quad (32)$$

In the case $\alpha \geq 1$, we take $\frac{1}{n} = 0$ or $n = \infty$.

In the case $\alpha < 1$, which together with $4 \frac{N}{\alpha} < \alpha < 4 \frac{N - 2}{\alpha}$ imply $N \geq 5$, we take $\frac{1}{n} = \frac{1 - \alpha}{2}$ or $n = \frac{2}{\alpha}$. It is not hard to check that the conditions in (32) are satisfied with this choice of $n$.

Collecting (28), (29) and (31), we prove (26). The proof is complete.

Remark 3. It is easy to see that the last condition in (30) does not hold when $N = 1, 2$. As in [2], a space time estimate (see [2, Proposition 3.1]) is needed to overcome the logarithmic divergence of the dispersive estimate at least in 2D.

4. Interaction Morawetz estimates.

4.1. Variational analysis. We recall some properties of the ground state $Q$ related to (5). The ground state $Q$ optimizes the Gagliardo-Nirenberg inequality

$$H(f) \leq C_{opt} [K(f)]^{\frac{2N}{N\alpha}} [M(f)]^{\frac{4-(N-2)\alpha}{N\alpha}}, \quad f \in H^1, \quad (33)$$

that is

$$C_{opt} = H(Q) \div \left[ [K(Q)]^{\frac{2N}{N\alpha}} [M(Q)]^{\frac{4-(N-2)\alpha}{N\alpha}} \right],$$

where $K$ and $H$ are defined in (8) and (16) respectively. Recall that $Q$ satisfies the following Pohozaev’s identities (see e.g. [3])

$$M(Q) = \frac{4 - (N - 2)\alpha}{N\alpha} K(Q) = \frac{4 - (N - 2)\alpha}{2(\alpha + 2)} H(Q).$$

It follows that

$$E(Q) = \frac{N\alpha - 4}{2N\alpha} K(Q) = \frac{N\alpha - 4}{4(\alpha + 2)} H(Q). \quad (34)$$

In particular,

$$E(Q)[M(Q)]^{\sigma_e} = \frac{N\alpha - 4}{2N\alpha} K(Q)[M(Q)]^{\sigma_e} = \frac{N\alpha - 4}{4(\alpha + 2)} H(Q)[M(Q)]^{\sigma_e} \quad (35)$$

and

$$C_{opt} = \frac{2(\alpha + 2)}{N\alpha} (K(Q)[M(Q)]^{\sigma_e})^{\frac{N\alpha - 4}{4}} = \left( \frac{2(\alpha + 2)}{N\alpha} \right)^{\frac{N\alpha - 4}{4}} (H(Q)[M(Q)]^{\sigma_e})^{-\frac{N\alpha - 4}{4}}. \quad (36)$$

We also have the following refined Gagliardo-Nirenberg inequality.

Lemma 4.1. Let $N \geq 1$ and $0 < \alpha < \alpha^*$. Then for any $f \in H^1$ and any $\xi \in \mathbb{R}^N$,

$$H(f) \leq \frac{2(\alpha + 2)}{N\alpha} \left( \frac{K(f)[M(f)]^{\sigma_e}}{K(Q)[M(Q)]^{\sigma_e}} \right)^{\frac{N\alpha - 4}{4}} K(e^{ix \cdot \xi} f). \quad (37)$$
Proof. The proof is essentially given in [5]. For the reader’s convenience, we give some details. Using (36), we have

\[
H(f) \leq \frac{2(\alpha + 2)}{N\alpha} \left( \frac{K(f)|M(f)|^{\sigma_{\epsilon}}}{K(Q)|M(Q)|^{\sigma_{\epsilon}}} \right)^{\frac{N\alpha - 4}{4}} K(f).
\]

Since \( H \) and \( M \) are invariant under the transform \( f \mapsto e^{ix\cdot\xi}f \), we see that

\[
H(f) \leq \frac{2(\alpha + 2)}{N\alpha} \inf_{\xi \in \mathbb{R}^N} \left[ \left( \frac{K(e^{ix\cdot\xi}f)|M(e^{ix\cdot\xi}f)|^{\sigma_{\epsilon}}}{K(Q)|M(Q)|^{\sigma_{\epsilon}}} \right)^{\frac{N\alpha - 4}{4}} K(e^{ix\cdot\xi}f) \right]
\]

\[
= \frac{2(\alpha + 2)}{N\alpha} \inf_{\xi \in \mathbb{R}^N} \left( \frac{K(e^{ix\cdot\xi}f)|M(f)|^{\sigma_{\epsilon}}}{K(Q)|M(Q)|^{\sigma_{\epsilon}}} \right)^{\frac{N\alpha - 4}{4}} \inf_{\xi \in \mathbb{R}^N} K(e^{ix\cdot\xi}f)
\]

which proves the result. \( \square \)

4.2. Interaction Morawetz estimate. Let \( \eta \in (0, 1) \) be a small constant and \( \chi \) be a smooth decreasing radial function which satisfies

\[
\chi(x) = \begin{cases} 
1 & \text{if } |x| \leq 1 - \eta, \\
0 & \text{if } |x| > 1.
\end{cases}
\]

(38)

For \( R > 0 \) large, we define the functions

\[
\phi(x) := \frac{1}{\omega_N R^N} \int \chi^2_R(x-z)\chi^2_R(z) dz
\]

(39)

and

\[
\phi_1(x) := \frac{1}{\omega_N R^N} \int \chi^2_R(x-z)\chi^{\alpha+2}_R(z) dz
\]

(40)

where \( \chi_R(z) := \chi(z/R) \) and \( \omega_N \) is the volume of the unit ball in \( \mathbb{R}^N \). It is easy to see that \( \phi \) and \( \phi_1 \) are radial functions. We next define

\[
\psi(x) = \psi(r) := \frac{1}{r} \int_0^r \phi(\tau) d\tau, \quad \Psi(x) = \Psi(r) := \int_0^r \psi(\tau) d\tau, \quad r = |x|.
\]

(41)

We collect some properties of \( \phi, \psi \) and \( \Psi \) as follows.

Lemma 4.2 ([5, 16]). It holds that for \( j = 1, \cdots, N \),

\[
\partial_j \Psi(x) = x_j \psi(x), \quad \partial^2_{j,k} \Psi(x) = \delta_{jk}\phi(x) + P_{jk}(\psi(x) - \phi(x)),
\]

(42)

where \( P_{jk} := \delta_{jk} - \frac{x_j x_k}{|x|^2} \) with \( \delta_{jk} \) the Kronecker symbol and

\[
\psi(x) - \phi(x) \geq 0, \quad |\nabla \phi(x)| \lesssim \frac{1}{R}, \quad |\phi(x) - \phi_1(x)| \lesssim \eta
\]

(43)

and

\[
|\psi(x)| \lesssim \min \left\{ 1, \frac{R}{|x|} \right\}, \quad |\psi(x) - \phi(x)| \lesssim \min \left\{ \frac{|x|}{R}, \frac{R}{|x|} \right\}, \quad |\nabla \psi(x)| \lesssim \min \left\{ \frac{1}{R}, \frac{R}{|x|^2} \right\}
\]

(44)

for all \( x \in \mathbb{R}^N \).
Lemma 4.3 (Coercivity I). Let $N \geq 1$ and $\frac{4}{N} < \alpha < \alpha^*$. Let $f \in H^1$ satisfy
\[ H(f)[M(f)]^{\sigma_e} \leq (1 - \rho)H(Q)[M(Q)]^{\sigma_e} \] (45)
for some small constant $\rho > 0$. Then there exists $\nu = \nu(\rho) > 0$ such that
\[ K(f) - \frac{N\alpha}{2(\alpha + 2)}H(f) \geq \nu K(f). \] (46)

Proof. We first recall the Gagliardo-Nirenberg inequality
\[ H(f) \leq C_{\text{opt}}[K(f)]^{\frac{N\alpha}{N}} [M(f)]^{\frac{4 - (N - 2)\alpha}{4}}. \] (47)
Multiplying both sides of (47) by $[H(f)]^{\frac{N\alpha - 4}{4}}$ and using (36), it follows from (45) that
\[ [H(f)]^{\frac{N\alpha}{4}} \leq \left( \frac{2(\alpha + 2)}{N\alpha} \right)^{\frac{N\alpha}{4}} \left( \frac{H(f)[M(f)]^{\sigma_e}}{H(Q)[M(Q)]^{\sigma_e}} \right)^{\frac{N\alpha - 4}{4}} [K(f)]^{\frac{N\alpha}{4}} \]
\[ \leq \left( \frac{2(\alpha + 2)}{N\alpha} \right)^{\frac{N\alpha}{4}} (1 - \rho)^{\frac{N\alpha - 4}{4}} [K(f)]^{\frac{N\alpha}{4}}. \]
Since $N\alpha > 4$, we infer that
\[ K(f) - \frac{N\alpha}{2(\alpha + 2)}H(f) \geq \left( 1 - (1 - \rho)^{\frac{N\alpha - 4}{4}} \right) K(f) \]
which shows (46) with $\nu := 1 - (1 - \rho)^{\frac{N\alpha - 4}{4}} > 0$. The proof is complete. \qed

Lemma 4.4 (Coercivity II). Let $N \geq 1$ and $\frac{4}{N} < \alpha < \alpha^*$. Let $u(t)$ be a $H^1$ solution to (1) satisfying (17). Then $T^* = \infty$, and there exists $\nu = \nu(Q) > 0$ such that for any $R > 0$ and any $z, \xi \in \mathbb{R}^N$,
\[ K(\chi_R(\cdot - z)u^\xi(t)) - \frac{N\alpha}{2(\alpha + 2)}H(\chi_R(\cdot - z)u^\xi(t)) \geq \nu K(\chi_R(\cdot - z)u^\xi(t)) \] (48)
for all $t \in [0, \infty)$, where
\[ u^\xi(t, x) := e^{ix\cdot \xi}u(t, x). \] (49)

Proof. It follows from (17), the conservation of mass and energy that
\[ \sup_{t \in [0, T^*)} \| u(t) \|_{H^1} \leq C(u_0, Q) < \infty \] (50)
which by the blow-up alternative implies $T^* = \infty$. By (17), there exists $\rho = \rho(Q) > 0$ such that
\[ \sup_{t \in [0, \infty)} H(\chi_R(\cdot - z)u^\xi(t))[M(u(t))]^{\sigma_e} \leq (1 - \rho)H(Q)[M(Q)]^{\sigma_e}. \]

By the definition of $\chi$ and $u^\xi$, we have
\[ M(\chi_R(\cdot - z)u^\xi(t)) \leq M(u(t)), \quad H(\chi_R(\cdot - z)u^\xi(t)) \leq H(u(t)) \]
for all $t \in [0, \infty)$, all $R > 0$ and all $z, \xi \in \mathbb{R}^N$. We thus get for any $R > 0$ and any $z, \xi \in \mathbb{R}^N$,
\[ \sup_{t \in [0, \infty)} H(\chi_R(\cdot - z)u^\xi(t))[M(\chi_R(\cdot - z)u^\xi(t))]^{\sigma_e} \leq \sup_{t \in [0, \infty)} H(u(t))[M(u(t))]^{\sigma_e} \]
\[ \leq (1 - \rho)H(Q)[M(Q)]^{\sigma_e}. \]
The estimate (48) follows directly from Lemma 4.3. \qed
Let \( u \in C([0, \infty), H^1) \) be a solution to (1). We define the interaction Morawetz quantity
\[
\mathcal{M}_R(t) := 2 \int |u(t,y)|^2 \nabla \Psi(x-y) \cdot \text{Im} \left( \overline{\varpi(t,x)} \nabla u(t,x) \right) \, dx \, dy.
\]

\[\text{(51)}\]

**Lemma 4.5** (Interaction Morawetz identity). Let \( N \geq 1 \) and \( 0 < \alpha < \alpha^* \). Let \( u \in C([0, \infty), H^1) \) be a solution to (1) satisfying
\[
\sup_{t \in [0, \infty)} \| u(t) \|_{H^1} \leq A
\]
for some constant \( A > 0 \). Let \( \mathcal{M}_R(t) \) be as in (51). Then it holds that
\[
\sup_{t \in [0, \infty)} |\mathcal{M}_R(t)| \lesssim R.
\]

Moreover,
\[
\frac{d}{dt} \mathcal{M}_R(t) = -\int \int |u(t,y)|^2 \Delta \Psi(x-y) \Delta(|u(t,x)|^2) \, dx \, dy
\]
\[+ 4 \sum_{j,k} \int \int |u(t,y)|^2 \partial_{jk}^2 \Psi(x-y) \text{Re}(\partial_j u(t,x) \partial_k \varpi(t,x)) \, dx \, dy
\]
\[- 4 \sum_{j,k} \int \int \text{Im}(\varpi(t,y) \partial_j u(t,y)) \partial_{jk}^2 \Psi(x-y) \text{Im}(\varpi(t,x) \partial_k u(t,x)) \, dx \, dy
\]
\[- \frac{2\alpha}{\alpha + 2} \int \int |u(t,y)|^2 \Delta \Psi(x-y)|u(t,x)|^{\alpha + 2} \, dx \, dy\]
\[\text{(57)}\]

for all \( t \in [0, \infty) \).

\[\text{Proof.} \quad \text{The estimate (53) follows directly from (42) and (44), Hölder's inequality and (52). The identities (54)--(57) follow from a direct computation using}
\]
\[
\partial_t(|u|^2) = -2 \sum_j \partial_j [\text{Im}(\varpi \partial_j u)],
\]
\[
\partial_t [\text{Im}(\varpi \partial_j u)] = -\sum_k \partial_k \left[ 2 \text{Re}(\partial_j u \partial_k \varpi) - \frac{1}{2} \delta_{jk} \Delta(|u|^2) \right] + \frac{\alpha}{\alpha + 2} \partial_j (|u|^{\alpha + 2}),
\]

for \( j = 1, \cdots, N \), where \( \delta_{jk} \) is the Kronecker symbol. \[\square\]

**Proposition 2** (Interaction Morawetz estimate). Let \( N \geq 1 \) and \( \frac{4}{N} < \alpha < \alpha^* \). Let \( u(t) \) be a \( H^1 \) solution to (1) satisfying (17). Define \( \mathcal{M}_R(t) \) as in (51). Then for \( \varepsilon > 0 \) sufficiently small, there exist \( T_0 = T_0(\varepsilon) \), \( J = J(\varepsilon) \), \( R_0 = R_0(\varepsilon, u_0, Q) \) sufficiently large and \( \eta = \eta(\varepsilon) > 0 \) sufficiently small such that for any \( a \in \mathbb{R} \),
\[
\frac{1}{JT_0} \int_a^{a + T_0} \int_{R_0}^{R_0 e^J} \int \int |\chi(x-z)u(t,y)|^2 \times |\nabla \left[ \chi(x-z)u(t,x) \right]|^2 \, dx \, dy \, dz \, dt \lesssim \varepsilon,
\]
\[\text{(58)}\]

where \( \chi_R(x) = \chi(x/R) \) with \( \chi \) as in (38), and \( u^\varepsilon \) is as in (49) with some \( \xi = \xi(t,z,R) \in \mathbb{R}^N \).
Proof. By integration by parts and using the fact
\[ \Delta \Psi = N\phi + (N - 1)(\psi - \phi), \]
we have
\[ (54) = -\sum_k \iint |u(t, y)|^2[N\phi + (N - 1)(\psi - \phi)](x - y)\partial_k^2(\|u(t, x)\|^2)dx dy \]
\[ = \sum_k \iint |u(t, y)|^2\partial_k[N\phi + (N - 1)(\psi - \phi)](x - y)\partial_k(\|u(t, x)\|^2)dx dy. \] (60)
By (42), we have
\[ (55) = 4 \sum_{j,k} \iint |u(t, y)|^2\delta_{jk}\phi(x - y) \text{Re}(\partial_j u(t, x)\partial_k \bar{\pi}(t, x))dx dy \]
\[ + 4 \sum_{j,k} \iint |u(t, y)|^2 P_{jk}(\psi - \phi)(x - y) \text{Re}(\partial_j u(t, x)\partial_k \bar{\pi}(t, x))dx dy. \] (61)
We also have
\[ (56) = -4 \sum_{j,k} \iint \text{Im}(\bar{\pi}(t, y)\partial_j u(t, y))\delta_{jk}\phi(x - y) \text{Im}(\bar{\pi}(t, x)\partial_k u(t, x))dx dy \]
\[ -4 \sum_{j,k} \iint \text{Im}(\bar{\pi}(t, y)\partial_j u(t, y))P_{jk}(\psi - \phi)(x - y) \text{Im}(\bar{\pi}(t, x)\partial_k u(t, x))dx dy. \] (63)
We see that
\[ (62) + (64) = 4 \iint |u(t, y)|^2(\nabla y u(t, x))^2(\psi - \phi)(x - y)dx dy \]
\[ -4 \iint \text{Im}(\bar{\pi}(t, y)\nabla x u(t, y)) \cdot \text{Im}(\bar{\pi}(t, x)\nabla y u(t, x)) \cdot (\psi - \phi)(x - y)dx dy, \]
where
\[ \nabla y u(t, x) := \nabla u(t, x) - \frac{x - y}{|x - y|} \left( \frac{x - y}{|x - y|} \cdot \nabla u(t, x) \right) \]
is the angular derivative centered at \( y \), and similarly for \( \nabla x u(t, y) \). By Cauchy-Schwarz inequality and the fact \( \psi - \phi \) in non-negative, we deduce
\[ (62) + (64) \geq 0. \] (65)
We next have
\[ (61) + (63) = 4 \iint \phi(x - y) \left[ |u(t, y)|^2|\nabla u(t, x)|^2 \right. \]
\[ \left. - \text{Im}(\bar{\pi}(t, y)\nabla u(t, y)) \cdot \text{Im}(\bar{\pi}(t, x)\nabla u(t, x)) \right] dx dy. \]
By rewriting
\[ \phi(x - y) = \frac{1}{\omega_N R^N} \int \chi_R^2(x - y - z)\chi_R^2(z)dz = \frac{1}{\omega_N R^N} \int \chi_R^2(x - z)\chi_R^2(y - z)dz, \]
we get
\[ (61) + (63) = \frac{4}{\omega_N R^N} \iint \chi_R^2(x - z)\chi_R^2(y - z) \]
\[ \times \left[ |u(t, y)|^2|\nabla u(t, x)|^2 - \text{Im}(\bar{\pi}(t, y)\nabla u(t, y)) \cdot \text{Im}(\bar{\pi}(t, x)\nabla u(t, x)) \right] dx dy dz. \]
For fixed \( z \in \mathbb{R}^N \), we consider the quantity defined by
\[
\int\int \chi_R^2(x - z) \chi_R^2(y - z) \left[ |u(t, y)|^2 |\nabla u(t, x)|^2 - \text{Im}(\overline{\pi}(t, y) \nabla u(t, y)) \cdot \text{Im}(\overline{\pi}(t, x) \nabla u(t, x)) \right] dx dy.
\]
We claim that this quantity is invariant under the Galilean transformation
\( u(t, x) \mapsto u^\xi(t, x) := e^{ix \cdot \xi} u(t, x) \)
for any \( \xi = \xi(t, z, R) \). Indeed, one has
\[
|u^\xi(y)|^2 |\nabla u^\xi(x)|^2 - \text{Im}(\overline{\pi}^\xi(y) \nabla u^\xi(y)) \cdot \text{Im}(\overline{\pi}^\xi(x) \nabla u^\xi(x)) = |u(y)|^2 |\nabla u(x)|^2 - \text{Im}((\overline{\pi}(y) \nabla u(y)) \cdot \text{Im}(\overline{\pi}(x) \nabla u(x))
\]
\[
+ \xi \cdot |u(y)|^2 \text{Im}(\overline{\pi}(x) \nabla u(x)) - \xi \cdot |u(x)|^2 \text{Im}(\overline{\pi}(y) \nabla u(y))
\]
and hence the claim follows by symmetry of \( \chi \) and a change of variable. We now define \( \xi = \xi(t, z, R) \) so that
\[
\int \chi_R^2(x - z) \text{Im}(\overline{\pi}^\xi(t, x) \nabla u^\xi(t, x)) dx = 0.
\]
In particular, we can achieve this by choosing
\[
\xi(t, z, R) = -\int \chi_R^2(x - z) \text{Im}(\overline{\pi}(t, x) \nabla u(t, x)) dx \div \int \chi_R^2(x - z) |u(t, x)|^2 dx
\]
provided the denominator is non-zero (otherwise \( \xi \equiv 0 \) suffices). For this choice of \( \xi \), we have
\[
(61) + (63) = \frac{4}{\omega_N R^N} \int\int\int \chi_R^2(x - z) \chi_R^2(y - z) |u(t, y)|^2 |\nabla u^\xi(t, x)|^2 dx dy dz. \quad (66)
\]
Next, by (59), we have
\[
(57) = -\frac{2N\alpha}{\alpha + 2} \int\int |u(t, y)|^2 |\phi(x - y)| |u(t, x)|^{\alpha + 2} dx dy
\]
\[
- \frac{2(N - 1)\alpha}{\alpha + 2} \int\int |u(t, y)|^2 (\psi - \phi)(x - y) |u(t, x)|^{\alpha + 2} dx dy
\]
\[
= -\frac{2N\alpha}{\alpha + 2} \int\int |u(t, y)|^2 (\phi - \phi_1)(x - y) |u(t, x)|^{\alpha + 2} dx dy \quad (67)
\]
\[
- \frac{2(N - 1)\alpha}{\alpha + 2} \int\int |u(t, y)|^2 (\psi - \phi)(x - y) |u(t, x)|^{\alpha + 2} dx dy \quad (68)
\]
\[
- \frac{2N\alpha}{\alpha + 2} \int\int |u(t, y)|^2 \phi_1(x - y) |u(t, x)|^{\alpha + 2} dx dy, \quad (69)
\]
where \( \phi_1 \) is as in (40). We will consider (67) and (68) as error terms. Moreover, we use the fact
\[
\phi_1(x - y) = \frac{1}{\omega_N R^N} \int \chi_R^2(x - y - z) \chi_R^{\alpha + 2}(z) dz = \frac{1}{\omega_N R^N} \int \chi_R^2(y - z) \chi_R^{\alpha + 2}(x - z) dz
\]
to write
\[
(69) = -\frac{2N\alpha}{(\alpha + 2)\omega_N R^N} \int\int\int \chi_R^2(y - z) \chi_R^{\alpha + 2}(x - z) |u(t, y)|^2 |u(t, x)|^{\alpha + 2} dx dy dz. \quad (70)
\]
Collecting (60), (65), (66), (67), (68) and (70), we obtain
\[
\frac{d}{dt} M_R(t) \geq \iint |u(t, y)|^2 \nabla [N \phi + (N - 1)(\psi - \phi)](x - y) \cdot \nabla(|u(t, x)|^2) dxdy \tag{71}
\]
\[
+ \frac{4}{\omega_N\alpha} \iint \int |\chi_R(y - z)u(t, y)|^2 |\chi_R(x - z)\nabla \xi(t, x)|^2 dxdydz \tag{72}
\]
\[
- \frac{2N\alpha}{\alpha + 2} \iint |u(t, y)|^2 (\phi - \phi_1)(x - y)|u(t, x)|^{\alpha + 2} dxdy \tag{73}
\]
\[
- \frac{2(N - 1)\alpha}{\alpha + 2} \iint |u(t, y)|^2 (\psi - \phi)(x - y)|u(t, x)|^{\alpha + 2} dxdy 
\]
\[
- \frac{2N\alpha}{\alpha + 2} \omega_N \iint \iint \chi_R(y - z)u(t, y)|^2 |\chi_R(x - z)u(t, x)|^{\alpha + 2} dxdydz. 
\]

It follows that
\[
\frac{4}{\omega_N\alpha} \iint \int |\chi_R(y - z)u(t, y)|^2 \left[ |\chi_R(x - z)\nabla \xi(t, x)|^2 - \frac{N\alpha}{2(\alpha + 2)} |\chi_R(x - z)u(t, x)|^{\alpha + 2} \right] dxdydz 
\]
\[
\leq \frac{d}{dt} M_R(t) + \frac{2N\alpha}{\alpha + 2} \iint |u(t, y)|^2 (\phi - \phi)(x - y)|u(t, x)|^{\alpha + 2} dxdy 
\]
\[
+ \frac{2(N - 1)\alpha}{\alpha + 2} \iint |u(t, y)|^2 (\psi - \phi)(x - y)|u(t, x)|^{\alpha + 2} dxdy 
\]
\[
- \iint |u(t, y)|^2 \nabla [N \phi + (N - 1)(\psi - \phi)](x - y) \cdot \nabla(|u(t, x)|^2) dxdy. 
\]

Let us estimate the terms appeared in the second, third and fourth lines. By (53), we see that
\[
\left| \frac{1}{JT_0} \int_a^{a + T_0} \int_{R_0}^{R_0 e^{t'}} \frac{d}{dt} M_R(t) \cdot \frac{dR}{R} \right| \leq \frac{1}{JT_0} \int_{R_0}^{R_0 e^{t'}} \sup_{t \in [a, a + T_0]} |M_R(t)| \cdot \frac{dR}{R} 
\]
\[
\lesssim \frac{1}{JT_0} \int_{R_0}^{R_0 e^{t'}} dR 
\]
\[
\lesssim \frac{R_0 e^{t'}}{JT_0}. 
\]

Using (44), we see that
\[
\left| \frac{1}{JT_0} \int_a^{a + T_0} \int_{R_0}^{R_0 e^{t'}} \iint |u(t, y)|^2 (\psi - \phi)(x - y)|u(t, x)|^{\alpha + 2} dxdy \cdot \frac{dR}{R} dt \right| 
\]
\[
\leq \frac{1}{JT_0} \int_a^{a + T_0} \int_{R_0}^{R_0 e^{t'}} \iint |u(t, y)|^2 \min \left\{ \frac{|x - y|}{R}, \frac{R}{|x - y|} \right\} |u(t, x)|^{\alpha + 2} dxdy \cdot \frac{dR}{R} dt 
\]
\[
\leq \frac{1}{JT_0} \int_a^{a + T_0} \iint |u(t, y)|^2 |u(t, x)|^{\alpha + 2} \left( \int_{R_0}^{R_0 e^{t'}} \min \left\{ \frac{|x - y|}{R}, \frac{R}{|x - y|} \right\} dR \right) dxdydt 
\]
\[
\lesssim \frac{1}{JT}. 
\]

Here we have used the fact that \( \sup_{t \in [0, \infty)} \|u(t)\|_{H^1} \leq C(u_0, Q) < \infty \) (see (50)) and
\[
\int_{R_0}^{R_0 e^{t'}} \min \left\{ \frac{|x - y|}{R}, \frac{R}{|x - y|} \right\} dR \lesssim 1. 
\]
Using (43), we have
\[
\int_{a+T_0}^{a+T_0} \int_{R_0}^{R_0 e^f} \int |u(t,y)|^2 |(\phi - \phi_1)(x-y)| u(t,x) |^{\alpha + 2} dxdy dR \, dt \\
\lesssim \int_{a}^{a+T_0} \int_{R_0}^{R_0 e^f} \eta \frac{dR}{R} \, dt \\
\lesssim \eta.
\]
By (44), we have
\[
\int |\nabla[N\phi + (N-1)(\psi - \phi)](x)| \lesssim \min \left\{ \frac{1}{R} \frac{R}{|x|^2} \right\} < \frac{1}{R}
\]
which implies
\[
\int_{a+T_0}^{a+T_0} \int_{R_0}^{R_0 e^f} \int |u(t,y)|^2 \nabla[N\phi + (N-1)(\psi - \phi)](x) \cdot \nabla(|u(t,x)|^2) dxdy dR \, dt \\
\lesssim \int_{a}^{a+T_0} \int_{R_0}^{R_0 e^f} \|u(t)\|_{L^2}^2 \|\nabla u(t)\|_{L^2}^2 \frac{dR}{R^2} \, dt \\
\lesssim \int_{a}^{a+T_0} \int_{R_0}^{R_0 e^f} \frac{dR}{R^2} \, dt \\
\lesssim \frac{1}{JR_0}.
\]
Combining (71), (72), (73), (75) and (76), we obtain
\[
\int_{a+T_0}^{a+T_0} \int_{R_0}^{R_0 e^f} \frac{1}{R^N} \int \int |\chi_R(y-z)u(t,y)|^2 \\
\times \left[ |\chi_R(x-z)\nabla u^\xi(t,x)|^2 - \frac{N\alpha}{2(\alpha + 2)} |\chi_R(x-z)u(t,x)|^{\alpha + 2} \right] dxdydz \frac{dR}{R} \, dt \\
\lesssim \frac{R_0 e^f}{JT_0} + \frac{1}{J} + \eta + \frac{1}{JR_0}.
\]
Now, for fixed $z, \xi \in \mathbb{R}^N$, we use the fact
\[
\int |\nabla(f)|^2 \, dx = \int \chi^2 |\nabla f|^2 \, dx - \int \chi \Delta |f|^2 \, dx, \quad f \in H^1
\]
to have
\[
\int |\chi_R(x-z)\nabla u^\xi(t,x)|^2 \, dx = \|\nabla[\chi_R(\cdot-z)u^\xi(t)]\|^2_{L^2} + O(R^{-2}\|u(t)\|^2_{L^2}).
\]
It follows that from the conservation of mass and (48) that for any $R > 0$ and any $z, \xi \in \mathbb{R}^N$,
\[
\int |\chi_R(x-z)\nabla u^\xi(t,x)|^2 \, dx = \frac{N\alpha}{2(\alpha + 2)} \int |\chi_R(x-z)u(t,x)|^{\alpha + 2} \, dx \\
= \|\nabla[\chi_R(\cdot-z)u^\xi(t)]\|^2_{L^2} - \frac{N\alpha}{2(\alpha + 2)} \|\chi_R(\cdot-z)u^\xi(t)\|_{L^{\alpha+2}}^{\alpha+2} + O(R^{-2}) \\
\geq \nu \|\nabla[\chi_R(\cdot-z)u^\xi(t)]\|^2_{L^2} + O(R^{-2}).
\]
The term $O(R^{-2})$ can be treated as in (76). We thus infer from (77) that
\[
\left| \frac{1}{JT_0} \int_{a}^{a+T_0} \int_{R_0}^{e^J} \frac{1}{R^\infty} \int \int \left| \chi_R(y-z)u(t,y) \right|^2 \left| \nabla \chi_R(x-z)\hat{u}(t,x) \right|^2 dx dy dz dR dt \right| \\
\leq \frac{R_0 e^J}{JT_0} + \frac{1}{J} + \frac{1}{JR_0}.
\]
This proves (58) by taking $\eta = \epsilon, J = \epsilon^{-2}, R_0 = \epsilon^{-1}$ and $T_0 = \epsilon^{-2}$.

\section{Energy scattering.} In this section, we give the proof of Theorem 1.4 and Lemma 1.5. Let us start with the proof of Lemma 1.5.

\textbf{Proof of Lemma 1.5.} Let us consider $u_0 \in H^1$ satisfying (6) and (7). We will show that there exists $\rho = \rho(u_0, Q) > 0$ such that
\[
K(u(t))[M(u(t))]^{\sigma_c} \leq (1 - \rho)K(Q)[M(Q)]^{\sigma_c}
\]
for all $t \in [0, T^*)$. Assume it for the moment, let us prove (17). By the Gagliardo-Nirenberg inequality (33) and (78), we have
\[
H(u(t))[M(u(t))]^{\sigma_c} \leq C_{opt}[K(u(t))]^{\frac{N\alpha}{4}} [M(u(t))]^{\frac{4-(N-2)\alpha}{4}} [M(u(t))]^{\sigma_c} \\
\leq C_{opt} (K(u(t))[M(u(t))]^{\sigma_c})^{\frac{N\alpha}{4}} \\
< C_{opt}(1 - \rho)^{\frac{N\alpha}{4}} K(Q)[M(Q)]^{\sigma_c}.
\]
It follows from (36) that
\[
H(u(t))[M(u(t))]^{\sigma_c} < (1 - \rho)^{\frac{N\alpha}{4}} \frac{2(\alpha + 2)}{N\alpha} K(Q)[M(Q)]^{\sigma_c} = (1 - \rho)^{\frac{N\alpha}{4}} H(Q)[M(Q)]^{\sigma_c}
\]
for all $t \in [0, T^*)$ which proves (17). Let us now prove (78). By the Gagliardo-Nirenberg inequality, we have
\[
E(u(t))[M(u(t))]^{\sigma_c} \\
= \frac{1}{2} K(u(t))[M(u(t))]^{\sigma_c} - \frac{1}{\alpha + 2} H(u(t))[M(u(t))]^{\sigma_c} \\
\geq \frac{1}{2} K(u(t))[M(u(t))]^{\sigma_c} - \frac{C_{opt}}{\alpha + 2} [K(u(t))]^{\frac{N\alpha}{4}} [M(u(t))]^{\frac{4-(N-2)\alpha}{4} + \sigma_c} \\
= F(K(u(t))[M(u(t))]^{\sigma_c},
\]
where
\[
F(\lambda) = \frac{1}{2} \lambda - \frac{C_{opt}}{\alpha + 2} \lambda^{\frac{N\alpha}{4}}.
\]
Using (36), we see that
\[
F(K(Q)[M(Q)]^{\sigma_c}) = \frac{4}{2N\alpha} K(Q)[M(Q)]^{\sigma_c} = E(Q)[M(Q)]^{\sigma_c}.
\]
It follows from (6), (79), the conservation of mass and energy that
\[
F(K(u(t))[M(u(t))]^{\sigma_c} \leq E(u_0)[M(u_0)]^{\sigma_c} < E(Q)[M(Q)]^{\sigma_c} = F(K(Q)[M(Q)]^{\sigma_c})
\]
for all $t \in [0, T^*)$. By (7), the continuity argument implies that
\[
K(u(t))[M(u(t))]^{\sigma_c} < K(Q)[M(Q)]^{\sigma_c}
\]
for all $t \in [0, T^*)$. Next, by (6), there exists $\theta = \theta(u_0, Q) > 0$ such that
\[
E(u_0)[M(u_0)]^{\sigma_c} < (1 - \theta)E(Q)[M(Q)]^{\sigma_c}.
\]
Using the fact
\[ E(Q)[M(Q)]^{\sigma_c} = \frac{N\alpha - 4}{2N\alpha}K(Q)[M(Q)]^{\sigma_c} = \frac{N\alpha - 4}{2(\alpha + 2)}C_{\text{opt}}(K(Q)[M(Q)]^{\sigma_c})^{\frac{N\alpha}{4}}, \]
we infer from (79) and (81) that
\[ \frac{N\alpha}{N\alpha - 4} K(u(t))[M(u(t))]^{\sigma_c} - \frac{4}{N\alpha - 4} \left( \frac{K(u(t))[M(u(t))]^{\sigma_c}}{K(Q)[M(Q)]^{\sigma_c}} \right)^{\frac{N\alpha}{4}} \leq 1 - \theta. \quad (82) \]
By (80), we consider the function
\[ G(\lambda) := \frac{N\alpha}{N\alpha - 4} \lambda - \frac{4}{N\alpha - 4} \lambda^{\frac{N\alpha}{4}}, \quad 0 < \lambda < 1. \]
We see that \( G \) is strictly increasing on \((0, 1)\) with \( G(0) = 0 \) and \( G(1) = 1 \). It follows from (82) that there exists \( \rho = \rho(\theta) > 0 \) such that \( \lambda < 1 - \rho \) which proves (78).

\[ \bullet \] We next consider the case \( u_0 \in H^1 \) satisfies (11) and (12)–(15). We will prove (17) by following the argument of [8]. It is done by several steps.

**Step 1. Some preliminary estimates.** We first recall the following estimate (see [8, Lemma 2.1]): for \( f \in H^1 \cap L^2([x^2dx]), \)
\[ \left( \text{Im} \int x \cdot \nabla f f dx \right)^2 \leq \int |x|^2 |f|^2 dx \left( K(f) - [C_{\text{opt}}]^{-\frac{4}{N\alpha}} [M(f)]^{-\frac{4-(N-2)\alpha}{N\alpha}} [H(f)]^{\frac{4}{N\alpha}} \right). \quad (83) \]
Using the fact
\[ V''(t) = 8K(u(t)) - \frac{4N\alpha}{\alpha + 2} H(u(t)) \]
\[ = 16E(u(t)) - \frac{4(N\alpha - 4)}{\alpha + 2} H(u(t)) \]
\[ = 4N\alpha E(u(t)) - 2(N\alpha - 4) K(u(t)), \]
we infer that
\[ H(u(t)) = \frac{\alpha + 2}{4(N\alpha - 4)} (16E(u(t)) - V''(t)), \]
\[ K(u(t)) = \frac{1}{2(N\alpha - 4)} (4N\alpha E(u(t)) - V''(t)). \]
Note that since \( H(u(t)) \geq 0 \), we have \( V''(t) \leq 16E(u(t)) \). From these identities and (83), we see that
\[ (V'(t))^2 \leq 16V(t) \left[ \frac{1}{2(N\alpha - 4)} (4N\alpha E(u(t)) - V''(t)) \right. \]
\[ \left. - [C_{\text{opt}}]^{-\frac{4}{N\alpha}} [M(u(t))]^{-\frac{4-(N-2)\alpha}{N\alpha}} \left( \frac{\alpha + 2}{4(N\alpha - 4)} (16E(u(t)) - V''(t)) \right)^{\frac{4}{N\alpha}} \right] \]
which implies
\[ (z'(t))^2 \leq 4g(V''(t)), \quad (84) \]
where
\[ z(t) := \sqrt{V(t)} \]
and
\[ g(\lambda) := \frac{4N\alpha E(u) - \lambda}{2(N\alpha - 4)} - [C_{\text{opt}}]^{-\frac{4}{N\alpha}} [M(u)]^{-\frac{4-(N-2)\alpha}{N\alpha}} \left( \frac{\alpha + 2}{4(N\alpha - 4)} (16E(u) - \lambda) \right)^{\frac{4}{N\alpha}} \]
with \( \lambda \leq 16E(u) \). Since \( N\alpha > 4 \), we see that \( g(\lambda) \) is decreasing on \((-\infty, \lambda_0)\) and increasing on \((\lambda_0, 16E(u))\), where \( \lambda_0 \) satisfies

\[
\frac{N\alpha}{2(\alpha + 2)} = \left[ C_{opt}^{-\frac{\alpha}{N\alpha}}[M(u)]^{-\frac{\alpha + 2}{4(N\alpha - 4)}} \left( \frac{\alpha + 2}{4(N\alpha - 4)} (16E(u) - \lambda_0) \right)^{\frac{4-N\alpha}{N\alpha}} \right].
\]  

(85)

This implies that

\[
g(\lambda_0) = (16E(u) - \lambda_0) \left[ \frac{1}{2(N\alpha - 4)} - C_{opt}^{-\frac{\alpha}{N\alpha}} [M(u)]^{-\frac{\alpha + 2}{4(N\alpha - 4)}} \times \left( \frac{\alpha + 2}{4(N\alpha - 4)} (16E(u) - \lambda_0) \right)^{\frac{4-N\alpha}{N\alpha}} \right] + 2E(u)
\]

\[
= \left( \frac{16E(u) - \lambda_0}{8} \right) + 2E(u)
\]

\[
= \frac{\lambda_0}{8}.
\]

By (35), we have

\[
C_{opt} = \frac{2(\alpha + 2)}{N\alpha} \left[ \frac{2N\alpha}{N\alpha - 4} E(Q)[M(Q)]^{\sigma_c} \right]^{\frac{N\alpha - 4}{4}}.
\]

Inserting it to (85), we get

\[
1 = \left( \frac{16E(Q)[M(Q)]^{\sigma_c}}{(16E(u) - \lambda_0)[M(u)]^{\sigma_c}} \right)^{\frac{N\alpha - 4}{N\alpha}}
\]

or

\[
\frac{E(u)[M(u)]^{\sigma_c}}{E(Q)[M(Q)]^{\sigma_c}} \left( 1 - \frac{\lambda_0}{16E(u)} \right) = 1.
\]

(86)

Consequently, the assumption (11) is equivalent to

\[
\lambda_0 \geq 0
\]

(87)

and the assumption (13) is equivalent to

\[
(V'(0))^2 \geq 2V(0)\lambda_0
\]

or

\[
(z'(0))^2 \geq \frac{\lambda_0}{2} = 4g(\lambda_0).
\]

(88)

Moreover, the assumption (15) is equivalent to

\[
z'(0) \geq 0
\]

(89)

and the assumption (14) is equivalent to

\[
V''(0) > \lambda_0.
\]

(90)
In fact, to see (90), we use (14) to have
\[
V''(0) = 16E(u) - \frac{4(N\alpha - 4)}{\alpha + 2}H(u_0)
\]
\[
> 16E(u) - \frac{4(N\alpha - 4)}{\alpha + 2} [H(Q)][M(u_0)]^\sigma_c
\]
\[
= 16 \left( E(u) - \frac{E(Q)[M(Q)]^\sigma_c}{[M(u_0)]^\sigma_c} \right)
\]
\[
= 16E(u) \left( 1 - \frac{E(Q)[M(Q)]^\sigma_c}{E(u_0)[M(u_0)]^\sigma_c} \right)
\]
\[
= \lambda_0,
\]
where the last equality comes from (86).

**Step 2. Lower bound of** \(V''(t)\) **implies** (17). We will prove (17) by assuming the following lower bound: there exists \(\delta_0 > 0\) small such that for all \(t \in [0, T^*)\),
\[
V''(t) \geq \lambda_0 + \delta_0.
\]
(91)

In fact, we have
\[
H(u(t))[M(u)]^\sigma_c = \frac{\alpha + 2}{4(N\alpha - 4)}(16E(u) - V''(t))[M(u)]^\sigma_c
\]
\[
\leq \frac{\alpha + 2}{4(N\alpha - 4)}(16E(u) - \lambda_0 - \delta_0)[M(u)]^\sigma_c
\]
\[
= \frac{4(\alpha + 2)}{N\alpha - 4}E(Q)[M(Q)]^\sigma_c - \frac{\alpha + 2}{4(N\alpha - 4)}\delta_0[M(u_0)]^\sigma_c
\]
\[
= (1 - \rho)H(Q)[M(Q)]^\sigma_c
\]
(92)
for all \(t \in [0, T^*)\), where \(\rho := \frac{(\alpha + 2)\delta_0[M(u_0)]^\sigma_c}{4(N\alpha - 4)H(Q)[M(Q)]^\sigma_c}\). Here we have used (86) to get the third line. This shows (17).

**Step 3. Proof of lower bound of** \(V''(t)\). It remains to show (91). By (90), we take \(\delta_1 > 0\) so that
\[
V''(0) \geq \lambda_0 + 2\delta_1.
\]
(93)
By the continuity argument, we have
\[
V''(t) > \lambda_0 + \delta_1, \quad \forall t \in [0, t_0)
\]
(93)
for some \(t_0 > 0\) sufficiently small. By taking \(t_0\) smaller if necessary, we can assume that
\[
z'(t_0) > 2\sqrt{g(\lambda_0)}.
\]
(94)
In fact, if \(z'(0) > 2\sqrt{g(\lambda_0)}\), then we have (94) by the continuity argument. Otherwise, if \(z'(0) = 2\sqrt{g(\lambda_0)}\), then using the fact
\[
z''(t) = \left( \frac{V''(t)}{z(t)^2} - (z'(t))^2 \right)
\]
(95)
and (90), we have \(z''(0) > 0\). This shows (94) by taking \(t_0 > 0\) sufficiently small.

Now, let \(\epsilon_0 > 0\) be a small constant so that
\[
z'(t_0) \geq 2\sqrt{g(\lambda_0)} + 2\epsilon_0.
\]
(96)
We will prove by contradiction that
\[ z'(t) > 2\sqrt{g(\lambda_0)} + \epsilon_0, \quad \forall t \geq t_0. \]  
(97)

Assume that it does not hold and let
\[ t_1 := \inf \left\{ t \geq t_0 : z'(t) \leq 2\sqrt{g(\lambda_0)} + \epsilon_0 \right\}. \]

By (96), \( t_1 > t_0 \). By continuity,
\[ z'(t_1) = 2\sqrt{g(\lambda_0)} + \epsilon_0 \]  
(98)

and
\[ z'(t) \geq 2\sqrt{g(\lambda_0)} + \epsilon_0, \quad \forall t \in [t_0, t_1]. \]  
(99)

By (84),
\[ \left(2\sqrt{g(\lambda_0)} + \epsilon_0\right)^2 \leq (z'(t))^2 \leq 4g(V''(t)), \quad \forall t \in [t_0, t_1]. \]  
(100)

As consequence, \( g(V''(t)) > g(\lambda_0) \) for all \( t \in [t_0, t_1] \), thus \( V''(t) \neq \lambda_0 \) and by continuity, \( V''(t) > \lambda_0 \) for all \( t \in [t_0, t_1] \).

We prove that there exists a universal constant \( D > 0 \) such that
\[ V''(t) \geq \lambda_0 + \frac{\sqrt{\epsilon_0}}{D}, \quad \forall t \in [t_0, t_1]. \]  
(101)

Indeed, by the Taylor expansion of \( g \) near \( \lambda_0 \) with the fact \( g'(\lambda_0) = 0 \), there exists \( a > 0 \) such that
\[ g(\lambda) \leq g(\lambda_0) + a(\lambda - \lambda_0)^2, \quad \forall \lambda : |\lambda - \lambda_0| \leq 1. \]  
(102)

If \( V''(t) \geq \lambda_0 + 1 \), then (101) holds by taking \( D \) large. If \( \lambda_0 < V''(t) \leq \lambda_0 + 1 \), then by (100) and (102), we get
\[ \left(2\sqrt{g(\lambda_0)} + \epsilon_0\right)^2 \leq (z'(t))^2 \leq 4g(V''(t)) \leq 4g(\lambda_0) + 4a(V''(t) - \lambda_0)^2 \]
thus
\[ 4\epsilon_0 \sqrt{g(\lambda_0)} + \epsilon_0^2 \leq 4a(V''(t) - \lambda_0)^2. \]
This shows (101) with \( D = \sqrt{a[g(\lambda_0)]^{-\frac{1}{2}}} \).

However, by (95) and (101), we have
\[
\begin{align*}
z''(t_1) &= \frac{1}{z(t_1)} \left( \frac{V''(t_1)}{2} - (z'(t_1))^2 \right) \\
&\geq \frac{1}{z(t_1)} \left( \frac{\lambda_0}{2} + \sqrt{\epsilon_0} - \left(2\sqrt{g(\lambda_0)} + \epsilon_0\right)^2 \right) \\
&\geq \frac{1}{z(t_1)} \left( \sqrt{\epsilon_0} - 4\epsilon_0 \sqrt{g(\lambda_0)} - \epsilon_0^2 \right) > 0
\end{align*}
\]
provided \( \epsilon_0 \) is taken small enough, thus, contradicting (98) and (99). This proves (97). Note that we have also proved (101) for all \( t \in [t_0, T^\ast] \). This together with (93) imply (91) with \( \delta_0 = \min \{ \delta_1, \sqrt{\epsilon_0} \} \).

Collecting the above cases, we finish the proof. \( \square \)

Before giving the proof of Lemma 1.6, let us recall the following compactness of optimizing sequence for the Gagliardo-Nirenberg inequality due to [15].
Lemma 5.1 ([15]). Let \( N \geq 1 \) and \( \frac{4}{N} < \alpha < \alpha^* \). Let \((f_n)_{n \geq 1}\) is a sequence of \(H^1\) functions satisfying

\[
M(f_n) = M(Q), \quad E(f_n) = E(Q), \quad \lim_{n \to \infty} K(f_n) = K(Q).
\]

Then there exists a subsequence still denoted by \((f_n)_{n \geq 1}\) and a sequence \((y_n) \subset \mathbb{R}^N\) such that

\[
f_n(\cdot + y_n) \to e^{i\theta}Q(\cdot - x_0) \quad \text{strongly in } H^1
\]

for some \( \theta \in \mathbb{R} \) and \( x_0 \in \mathbb{R}^N \) as \( n \to \infty \).

Proof. For the reader’s convenience, we give some details. Since \((f_n)_{n \geq 1}\) is a bounded sequence in \(H^1\) satisfying \(M(f_n) = M(Q)\) for all \( n \geq 1 \), we use the concentration-compactness lemma of Lions [15] to have: there exists a subsequence still denoted by \((f_n)_{n \geq 1}\) satisfying one of the following three possibilities:

- **Vanishing:**
  \[
  \lim_{n \to \infty} \sup_{y \in \mathbb{R}^N} \int_{B(y, R)} |f_n(x)|^2 \, dx = 0
  \]
  for all \( R > 0 \).

- **Dichotomy:** There exist \( \mu \in (0, M(Q)) \) and sequences \((f_1^n)_{n \geq 1}, (f_2^n)_{n \geq 1}\) bounded in \(H^1\) such that
  \[
  \begin{aligned}
  &\|f_n - f_1^n - f_2^n\|_{L^r} \to 0 \quad \text{as} \quad n \to \infty \quad \text{for any} \quad 2 \leq r < 2^*; \\
  &M(f_1^n) \to \mu, \quad M(f_2^n) \to M(Q) - \mu \quad \text{as} \quad n \to \infty; \\
  &\text{dist}(\text{supp}(f_1^n), \text{supp}(f_2^n)) \to \infty \quad \text{as} \quad n \to \infty; \\
  &\liminf_{n \to \infty} K(f_n) - K(f_1^n) - K(f_2^n) \geq 0,
  \end{aligned}
  \]

  where \( 2^* := \frac{2N}{N-2} \) if \( N \geq 3 \) and \( 2^* = \infty \) if \( N = 1, 2 \).

- **Compactness:** There exists a sequence \((y_n)_{n \geq 1} \subset \mathbb{R}^N\) such that for all \( \varepsilon > 0 \), there exists \( R(\varepsilon) > 0 \) such that for all \( n \geq 1 \),
  \[
  \int_{B(y_n, R(\varepsilon))} |f_n(x)|^2 \, dx \geq M(Q) - \varepsilon.
  \]

It was shown in [15] that: if the vanishing occurs, then \( f_n \to 0 \) strongly in \(L^r\) for any \( 2 < r < 2^* \); and if the compactness occurs, then up to a subsequence, \( u_n(\cdot + y_n) \to f \) strongly in \(L^r\) for any \( 2 \leq r < 2^* \) for some \( f \in H^1\).

We see that the vanishing cannot occur since \( H(f_n) = \|f_n\|_{L^{\alpha+2}}^{\alpha+2} \to \|Q\|_{L^{\alpha+2}}^{\alpha+2} = H(Q) > 0 \).

If the dichotomy occurs, then, by the Gagliardo-Nirenberg inequality, we have

\[
H(f_1^n) \leq C_{\text{opt}}[K(f_1^n)]^{\frac{\alpha n}{\alpha + 2}} [M(f_1^n)]^{\frac{4 - (N - 2)\alpha}{4}} < C_{\text{opt}}[K(f_1^n)]^{\frac{\alpha n}{\alpha + 2}} [M(Q)]^{\frac{4 - (N - 2)\alpha}{4}}.
\]

Similarly,

\[
H(f_2^n) < C_{\text{opt}}[K(f_2^n)]^{\frac{\alpha n}{\alpha + 2}} [M(Q)]^{\frac{4 - (N - 2)\alpha}{4}}.
\]
By (103) and the fact $\frac{Nα}{τ} > 1$, we see that
\[
H(Q) = \lim_{n→∞} H(f_n) = \lim_{n→∞} H(f_n^1) + H(f_n^2)
\]
\[
< C_{opt} \lim_{n→∞} \left( [K(f_n^1)]^{\frac{2α}{Nα}} + [K(f_n^2)]^{\frac{2α}{Nα}} \right) [M(Q)]^{\frac{4−(N−2)α}{4}}
\]
\[
\leq C_{opt} \lim_{n→∞} \left( K(f_n^1) + K(f_n^2) \right)^{\frac{2α}{Nα}} [M(Q)]^{\frac{4−(N−2)α}{4}}
\]
\[
\leq C_{opt} \lim_{n→∞} [K(f_n)]^{\frac{2α}{Nα}} [M(Q)]^{\frac{4−(N−2)α}{4}}
\]
\[
= C_{opt} [K(Q)]^{\frac{2α}{Nα}} [M(Q)]^{\frac{4−(N−2)α}{4}}
\]
which is a contradiction.

Thus, the compactness must occurs, then there exist a subsequence still denoted by $(f_n)_{n≥1}$, a function $f ∈ H^1$ and a sequence $(y_n)_{n≥1} ⊂ \mathbb{R}^N$ such that $f_n(· + y_n) → f$ strongly in $L^r$ for any $2 ≤ r < 2^*$ and weakly in $H^1$. We have
\[
M(f) = \lim_{n→∞} M(f_n(· + y_n)) = M(Q)
\]
and
\[
H(f) = \lim_{n→∞} H(f_n(· + y_n)) = H(Q)
\]
and
\[
K(f) ≤ \liminf_{n→∞} K(f_n(· + y_n)) = K(Q).
\]

On the other hand, by the sharp Gagliardo-Nirenberg inequality, we have
\[
[K(f)]^{\frac{2α}{Nα}} ≥ \frac{H(f)}{C_{opt}[M(f)]^{\frac{4−(N−2)α}{4}}} = \frac{H(Q)}{C_{opt}[M(Q)]^{\frac{4−(N−2)α}{4}}} = [K(Q)]^{\frac{2α}{Nα}}
\]
hence $K(f) ≥ K(Q)$, so $K(f) = \lim_{n→∞} K(f_n(· + y_n)) = K(Q)$. Thus, $f_n(· + y_n) → f$ strongly in $H^1$ and $f$ is an optimizer for the sharp Gagliardo-Nirenberg inequality. By the characterization of ground state (see e.g. [15]) and taking into account that $M(f) = M(Q)$, we have $f(x) = e^{iθ}Q(x - x_0)$ for some $θ ∈ \mathbb{R}$ and $x_0 ∈ \mathbb{R}^N$. The proof is complete.

\[\square\]

\textbf{Proof of Lemma 1.6.} Let $u_0 ∈ H^1$ satisfies (9) and (10). We first note that (9) and (10) are invariant under the scaling
\[
u_0^\lambda(x) := \lambda^\frac{2}{τ} u_0(λx), \quad λ > 0.
\]
By choosing $λ > 0$ so that $λ^2 τc = M(u_0) M(Q)\), we can assume that
\[
M(u_0) = M(Q), \quad E(u_0) = E(Q).\quad (104)
\]
Thus (10) becomes $K(u_0) < K(Q)$. We will show that
\[
K(u(t)) < K(Q)\quad (105)
\]
for all $t ∈ [0, T^*)$, where $T^*$ is the maximal forward time of existence. In fact, assume by contradiction that there exists $t_0 ∈ [0, T^*)$ such that $K(u(t_0)) = K(Q)$. It follows from (34) that
\[
H(u(t_0)) = (α + 2) \left( E(u(t_0)) - \frac{1}{2} K(u(t_0)) \right) = \frac{2(α + 2)}{Nα} K(Q).
\]
This shows that $u(t_0)$ is an optimizer for Gagliardo-Nirenberg inequality (33). By the characterization of ground state, we have $u(t_0) = Q$ up to symmetries. By the uniqueness of solution to (1), we have $u(t) = e^{iθ}Q$ which contradicts (10). We thus
prove (105), and, by the blow-up alternative, $T^* = \infty$. We are now consider two cases.

**Case 1.** If $\sup_{t \in [0, \infty)} K(u(t)) < K(Q)$, thus there exists $\rho > 0$ such that for all $t \in [0, \infty)$,

$$K(u(t)) \leq (1 - \rho)K(Q).$$

By (104), we see that (78) holds. Arguing as in the proof of the first item in Lemma 1.5, we see that the corresponding solution scatters in $H^1$ forward in time.

**Case 2.** If $\sup_{t \in [0, \infty)} K(u(t)) = K(Q)$, then there exists a time sequence $(t_n)_{n \geq 1} \subset [0, \infty)$ such that

$$M(u(t_n)) = M(Q), \quad E(u(t_n)) = E(Q), \quad \lim_{n \to \infty} K(u(t_n)) = K(Q).$$

We see that $t_n \to \infty$. In fact, if it is not true, then up to a subsequence $t_n \to t_0$ as $n \to \infty$. By continuity, $u(t_n) \to u(t_0)$ strongly in $H^1$. It follows that $u(t_0)$ is an optimizer for the sharp Gagliardo-Nirenberg inequality, and we have a contradiction. We now apply Lemma 5.1 to $f_n = u(t_n)$ to get (up to a subsequence), there exists $(y_n)_{n \geq 1} \subset \mathbb{R}^N$ such that

$$u(t_n, \cdot + y_n) \to e^{i\theta}Q(\cdot - x_0) \quad \text{strongly in } H^1$$

for some $\theta \in \mathbb{R}$ and $x_0 \in \mathbb{R}^N$ as $n \to \infty$. The proof is complete. □

**Proof of Theorem 1.4.** By Lemma 4.4, we have $T^* = \infty$. It remains to show the scattering. Thanks to the scattering criteria given in Lemma 3.4, it suffices to show that there exists $T_0 = T_0(\varepsilon) > 0$ such that for any $a > 0$, there exists $t_0 \in (a, a + T_0)$ such that $I_0 := [t_0 - \varepsilon^{-\sigma}, t_0] \subset (a, a + T_0)$ and

$$\|u\|_{L^k(I_0, L^r)} \lesssim \varepsilon$$

for some $\sigma > 0$, where $k$ and $r$ are as in (21). By (58), there exist $T_0 = T_0(\varepsilon), J = J(\varepsilon), R_0 = R_0(\varepsilon, u_0, Q)$ and $\eta = \eta(\varepsilon)$ such that

$$\frac{1}{T_0} \int_a^{a + T_0} \int_{R_0}^{R_0 e^{\theta'}} \int \int |\chi_R(y-z)u(t,y)|^2 |\nabla \chi_R(x-z)u^k(t,x)|^2 \, dx \, dy \, dz \, dt \lesssim \varepsilon.$$

It follows that there exists $R_1 \in [R_0, R_0 e^{\theta'})$ such that

$$\frac{1}{T_0} \int_a^{a + T_0} \frac{1}{R_1^N} \int \int |\chi_{R_1}(y-z)u(t,y)|^2 |\nabla \chi_{R_1}(x-z)u^{k}(t,x)|^2 \, dx \, dy \, dz \, dt \lesssim \varepsilon$$

hence

$$\frac{1}{T_0} \int_a^{a + T_0} \frac{1}{R_1^N} \int |\chi_{R_1}(\cdot - z)u(t)||^2_{L^2} |\nabla \chi_{R_1}(\cdot - z)u^{k}(t)||^2_{L^2} \, dz \, dt \lesssim \varepsilon.$$

By the change of variable $z = \frac{R_1}{4}(w + \theta)$ with $w \in \mathbb{Z}^N$ and $\theta \in [0,1]^N$, we deduce that there exists $\theta_1 \in [0,1]^N$ such that

$$\frac{1}{T_0} \int_a^{a + T_0} \sum_{w \in \mathbb{Z}^N} \left| \chi_{R_1}\left(\cdot - \frac{R_1}{4}(w + \theta_1)\right)u(t) \right|^2_{L^2} \left| \nabla \chi_{R_1}\left(\cdot - \frac{R_1}{4}(w + \theta_1)\right)u^{k}(t) \right|^2_{L^2} \, dt \lesssim \varepsilon.$$

Let $\sigma > 0$ to be chosen later. By dividing the interval $[a + \frac{T_0}{\sigma}, a + \frac{2T_0}{\sigma}]$ into $T_0\varepsilon^{-\sigma}$ intervals of length $\varepsilon^{-\sigma}$, we infer that there exists $t_0 \in [a + \frac{T_0}{\sigma}, a + \frac{2T_0}{\sigma}]$ such that $I_0 = [t_0 - \varepsilon^{-\sigma}, t_0] \subset (a, a + T_0)$ and

$$\int_{I_0} \sum_{w \in \mathbb{Z}^N} \left| \chi_{R_1}\left(\cdot - \frac{R_1}{4}(w + \theta_1)\right)u(t) \right|^2_{L^2} \left| \nabla \chi_{R_1}\left(\cdot - \frac{R_1}{4}(w + \theta_1)\right)u^{k}(t) \right|^2_{L^2} \, dt \lesssim \varepsilon^{1-\sigma}.$$
This together with the Gagliardo-Nirenberg inequality
\[ \|u\|^{4}_{\frac{2N}{2N-2}, \frac{L^{2N}}{L^{\frac{2N}{2N-2}(I_{0} \times \mathbb{R}^{N})}}} \lesssim \|u\|^{2}_{\frac{L^{2}}{2}, \|\nabla u\|^{2}_{\frac{L^{2}}{2}}} \]
imply that
\[ \int_{t_{0}} \sum_{w \in \mathbb{Z}^{N}} \left\| \chi_{R_{1}} \left( \cdot - \frac{R_{1}}{4}(w + \theta_{1}) \right) u(t) \right\|^{4}_{\frac{L^{2N}}{2N-2}} dt \lesssim \varepsilon^{1-\sigma}. \tag{107} \]

On the other hand, by Hölder’s inequality, Cauchy-Schwarz inequality and Sobolev embedding, we have
\[ \sum_{w \in \mathbb{Z}^{N}} \left\| \chi_{R_{1}} \left( \cdot - \frac{R_{1}}{4}(w + \theta_{1}) \right) u(t) \right\|^{2}_{\frac{2N}{2N-1}} \]
\[ \leq \sum_{w \in \mathbb{Z}^{N}} \left\| \chi_{R_{1}} \left( \cdot - \frac{R_{1}}{4}(w + \theta_{1}) \right) u(t) \right\|_{L^{2}} \left\| \chi_{R_{1}} \left( \cdot - \frac{R_{1}}{4}(w + \theta_{1}) \right) u(t) \right\|_{L^{\frac{2N}{2N-2}}} \]
\[ \leq \left( \sum_{w \in \mathbb{Z}^{N}} \left\| \chi_{R_{1}} \left( \cdot - \frac{R_{1}}{4}(w + \theta_{1}) \right) u(t) \right\|^{2}_{L^{2}} \right)^{1/2} \]
\[ \times \left( \sum_{w \in \mathbb{Z}^{N}} \left\| \chi_{R_{1}} \left( \cdot - \frac{R_{1}}{4}(w + \theta_{1}) \right) u(t) \right\|^{2}_{L^{\frac{2N}{2N-2}}} \right)^{1/2} \]
\[ \lesssim \|u(t)\|_{L^{2}} \|\nabla u(t)\|_{L^{2}} \lesssim 1. \tag{108} \]

Combining (107) and (108), we get from the property of \( \chi_{R_{1}} \) that
\[ \|u\|^{\frac{2N}{2N-2}, \frac{L^{2N}}{L^{\frac{2N}{2N-2}(I_{0} \times \mathbb{R}^{N})}}} \lesssim \int_{t_{0}} \sum_{w \in \mathbb{Z}^{N}} \left\| \chi_{R_{1}} \left( \cdot - \frac{R_{1}}{4}(w + \theta_{1}) \right) u(t) \right\|^{\frac{2N}{2N-1}}_{L^{\frac{2N}{2N-2}}} dt \]
\[ \lesssim \int_{t_{0}} \left( \sum_{w \in \mathbb{Z}^{N}} \left\| \chi_{R_{1}} \left( \cdot - \frac{R_{1}}{4}(w + \theta_{1}) \right) u(t) \right\|^{4}_{\frac{L^{2N}}{2N-2}} \right)^{\frac{1}{4}} dt \]
\[ \times \left( \sum_{w \in \mathbb{Z}^{N}} \left\| \chi_{R_{1}} \left( \cdot - \frac{R_{1}}{4}(w + \theta_{1}) \right) u(t) \right\|^{2}_{L^{\frac{2N}{2N-2}}} \right)^{\frac{N-2}{2N-1}} dt \]
\[ \lesssim \left( \int_{t_{0}} \sum_{w \in \mathbb{Z}^{N}} \left\| \chi_{R_{1}} \left( \cdot - \frac{R_{1}}{4}(w + \theta_{1}) \right) u(t) \right\|^{4}_{\frac{L^{2N}}{2N-2}} dt \right)^{\frac{1}{4}} \]
\[ \times \left( \int_{t_{0}} \sum_{w \in \mathbb{Z}^{N}} \left\| \chi_{R_{1}} \left( \cdot - \frac{R_{1}}{4}(w + \theta_{1}) \right) u(t) \right\|^{2}_{L^{\frac{2N}{2N-2}}} dt \right)^{\frac{N-2}{2N-1}} \]
\[ \lesssim \varepsilon^{\frac{1-N}{N-1}} \varepsilon^{-\frac{(N-2)(\sigma)}{N-1}} = \varepsilon^{\frac{1}{N-1}} \sigma \]
which implies that
\[ \|u\|_{L^{\frac{2N}{2N-1}(I_{0} \times \mathbb{R}^{N})}} \lesssim \varepsilon^{\left(\frac{1}{N-1}\right) \sigma \frac{N-1}{2N}}. \tag{109} \]

By interpolation, we see that
\[ \|u\|_{L^{\kappa}(I_{0}, L^{r})} \leq \|u\|^{\kappa}_{L^{\frac{2N}{2N-1}(I_{0} \times \mathbb{R}^{N})}} \|u\|^{1-\kappa}_{L^{\infty}(I_{0}, L^{n})}, \]
where
\[
\vartheta := \frac{N[4 - (N - 2)\alpha]}{(N - 1)\alpha(\alpha + 2)} \in (0, 1),
\]
\[
n := \frac{2[(N - 1)\alpha^2 + (N^2 - 2)\alpha - 4N]}{(N - 1)(N\alpha - 4)}.
\]

It is not hard to check that \(2 < n < \frac{2N}{N-2}\) for \(4 < \alpha < \frac{4N}{N-2}\). The Sobolev embedding then implies
\[
\|u\|_{L^\infty(I_0, L^\infty)} \lesssim \|u\|_{L^\infty(I_0, H^1)} < \infty.
\]
Finally, we obtain
\[
\|u\|_{L^k(I_0, L^r)} \lesssim \varepsilon \left(\frac{1}{N - 1 - \sigma}\right)^{\frac{4 - (N - 2)\alpha}{N\alpha(\alpha + 2)}}
\]
which proves (106) by choosing \(0 < \sigma < \frac{1}{N-1}\). The proof is complete. \(\square\)

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**Appendix A. An alternative proof via the concentration-compactness-rigidity argument.** In this appendix, we briefly describe the proof of the energy scattering in Theorem 1.4 using the concentration-compactness-rigidity argument of Duyckaerts-Roudenko [8]. Before entering more details, let us give some preliminaries needed in the proof.

**Lemma A.1** (Profile decomposition [6]). Let \(N \geq 1\) and \(0 < \alpha < \alpha^*\). Let \((\phi_n)_{n \geq 1}\) be a uniformly bounded sequence in \(H^1\). Then for each integer \(J \geq 1\), there exists a subsequence, still denoted by \(\phi_n\), and

- for each \(1 \leq j \leq J\), there exists a fixed profile \(\psi^j \in H^1\);
- for each \(1 \leq j \leq J\), there exists a sequence of time shifts \((t^j_n)_{n \geq 1} \subset \mathbb{R}\);
- for each \(1 \leq j \leq J\), there exists a sequence of space shifts \((x^j_n)_{n \geq 1} \subset \mathbb{R}^N\);
- there exists a sequence of remainders \((W^j_n)_{n \geq 1} \subset H^1\);

such that
\[
\phi_n(x) = \sum_{j=1}^J e^{-it^j_n\Delta} \psi^j(x - x^j_n) + W^j_n(x).
\]
The time and space shifts have a pairwise divergence property, i.e. for \(1 \leq j \neq k \leq J\), we have
\[
\lim_{n \to \infty} |t^j_n - t^k_n| + |x^j_n - x^k_n| = \infty. \tag{110}
\]
The remainder has the following asymptotic smallness property
\[
\lim_{J \to \infty} \lim_{n \to \infty} \|e^{it\Delta} W^j_n\|_{L^k(\mathbb{R}, L^r)} = 0, \tag{111}
\]
where $k, r$ are as in (21). Moreover, for fixed $J$, we have the asymptotic Pythagorean expansions

$$
\|\phi_n\|_{H^\gamma}^2 = \sum_{j=1}^{J} \|\psi_j\|_{H^\gamma}^2 + \|W_j\|_{H^\gamma}^2 + o_n(1), \quad \forall \gamma \in [0, 1],
$$

(112)

$$
E(\phi_n) = \sum_{j=1}^{J} E(e^{-it\Delta}\psi_j) + E(W_j) + o_n(1).
$$

(113)

**Lemma A.2.** Let $N \geq 1$ and $\frac{4}{N} < \alpha < \alpha^*$. Let $f \in H^1$ satisfy

$$
H(f)[M(f)]^{\sigma_c} \leq A < H(Q)[M(Q)]^{\sigma_c}
$$

for some constant $A > 0$. Then there exists $\nu = \nu(A, Q) > 0$ such that

$$
K(f) - \frac{N\alpha}{2(\alpha + 2)}H(f) \geq \nu K(f),
$$

(114)

$$
E(f) \geq \frac{\nu}{2} K(f).
$$

(115)

**Proof.** By writing

$$
A = (1 - \rho)H(Q)[M(Q)]^{\sigma_c}
$$

for some $\rho = \rho(A, Q) > 0$, we have

$$
H(f)[M(f)]^{\sigma_c} \leq (1 - \rho)H(Q)[M(Q)]^{\sigma_c}.
$$

The estimate (114) follows directly from Lemma 4.3. The estimate (115) follows from (114) and

$$
E(f) = \frac{1}{2} \left( K(f) - \frac{N\alpha}{2(\alpha + 2)}H(f) \right) + \frac{N\alpha - 4}{4(\alpha + 2)} H(f) \geq \frac{1}{2} \left( K(f) - \frac{N\alpha}{2(\alpha + 2)}H(f) \right).
$$

\[\Box\]

Let $A > 0$ and $\delta \in \mathbb{R}$. We define

$$
S(A, \delta) := \sup \{ \|u\|_{L^1([0, \infty), L^r)} : u \text{ is a } H^1 \text{ solution to (1) satisfying (117)} \},
$$

(116)

where $k, r$ are as in (21) and

$$
\sup_{t \in [0, \infty)} H(u(t))[M(u(t))]^{\sigma_c} \leq A, \quad E(u)[M(u)]^{\sigma_c} \leq \delta.
$$

(117)

**Proof of Theorem 1.4.** By Lemma 4.4, $H^1$ solutions to (1) satisfying (17) exist globally in time and satisfy $\|u\|_{L^\infty([0, \infty), H^k)} \leq C(u_0, Q) < \infty$. By (116), we see that Theorem 1.4 is reduced to show the following result.

**Proposition 3.** Let $N \geq 1$ and $\frac{4}{N} < \alpha < \alpha^*$. If $A < H(Q)[M(Q)]^{\sigma_c}$, then for all $\delta > 0$, $S(A, \delta) < \infty$.

The proof of Proposition 3 is done by several steps.

**Step 1. Small data theory.** By (115), we have

$$
\|u_0\|_{H^{2\sigma_c}}^{\frac{2\sigma_c}{2\sigma_c}} \leq K(u_0)[M(u_0)]^{\sigma_c} \leq \frac{2E(u_0)}{\nu}[M(u_0)]^{\sigma_c} \leq \frac{2\delta}{\nu}.
$$

\[1\text{Note that, by Lemma A.2, } E(u) > 0.\]
By taking $\delta > 0$ sufficiently small, we see that $\|u_0\|_{H^\infty} < 1$ which, by the small data theory, implies $S(A, \delta) < \infty$.

**Step 2. Existence of a critical solution.** Assume by contradiction that $S(A, \delta) = \infty$ for some $\delta > 0$. By Step 1, the quantity

$$\delta_c = \delta_c(A) := \inf \{ \delta > 0 : S(A, \delta) = \infty \}$$

is well-defined and positive. By the definition of $S$, we have:

(i) If $u$ a $H^1$ solution to (1) satisfying

$$\sup_{t \in [0, \infty)} H(u(t))[M(u(t))]^{\sigma_c} \leq A, \quad E(u)[M(u)]^{\sigma_c} < \delta_c,$$

then $\|u\|_{L^k([0, \infty), L^r)} < \infty$ and the solution scatters forward in time.

(ii) There exists a sequence of $H^1$ solutions $u_n$ to (1) with initial data $u_{n,0}$ such that

$$\sup_{t \in [0, \infty)} H(u_n(t))[M(u_n(t))]^{\sigma_c} \leq A, \quad E(u_n)[M(u_n)]^{\sigma_c} \searrow \delta_c \text{ as } n \to \infty,$$

$$\|u_n\|_{L^k([0, \infty), L^r)} = \infty \text{ for all } n.$$

We will prove that there exists a $H^1$ solution $u_c$ to (1) with initial data $u_{c,0}$ such that

$$M(u_c) = 1,$$

$$\sup_{t \in [0, \infty)} H(u_c(t)) \leq A,$$

$$E(u_c) = \delta_c,$$

$$\|u_c\|_{L^k([0, \infty), L^r)} = \infty.$$}

To this end, we consider the sequence $u_{n,0}$. Using the scaling (2), we may assume that $M(u_{n,0}) = 1$ for all $n$. Note that this scaling does not affect $E(u_n)[M(u_n)]^{\sigma_c}$ and $\sup_{t \in [0, \infty)} H(u_n(t))[M(u_n(t))]^{\sigma_c}$. After this scaling, we have

$$M(u_{n,0}) = 1, \quad \sup_{t \in [0, \infty)} H(u_{n,0}(t)) \leq A, \quad E(u_{n,0}) \searrow \delta_c, \quad \|u_n\|_{L^k([0, \infty), L^r)} = \infty.$$

We apply the profile decomposition to $u_{n,0}$ (which is now uniformly bounded in $H^1$) to have

$$u_{n,0}(x) = \sum_{j=1}^{J} e^{-it_n^j \Delta} \psi_j(x - x_n^j) + W_n^f(x)$$

(122)

together with the properties (110)–(113), where $J$ will be taken large later. We will distinguish two cases: more than one non-zero profile and only one non-zero profile.

**Case 1. More than one non-zero profiles.** By passing to a subsequence if necessary, we can assume that

$$t_n^j \to -\infty \quad \text{or} \quad t_n^j \to +\infty \quad \text{or} \quad t_n^j \to 0 \quad \text{as } n \to \infty.$$ 

We now define the nonlinear profile $v^j$ associated to $\psi_j$ and $t_n^j$ as follows:

\footnotetext{2If $t_n^j \to T < \infty$, we can write $e^{-it_n^j \Delta} \psi_j = e^{-iT \Delta} \psi_j$, $t_n^j = t_n^j - T \to 0$, $\psi^j = e^{-iT \Delta} \psi_j \in H^1$.}
• If \( t_n^j \to -\infty \), then \( \nu^j \) is the maximal lifespan solution to (1) that scatters backward in time to \( e^{it_0^j} \psi^j \);
• If \( t_n^j \to \infty \), then \( \nu^j \) is the maximal lifespan solution to (1) that scatters forward in time to \( e^{it_0^j} \psi^j \);
• If \( t_n^j \to 0 \), then \( \nu^j \) is the maximal lifespan solution to (1) with data \( \nu^j(0) = \psi^j \).

By definition and the continuity of the linear flow, we have

\[
\| \nu^j(-t_n^j) - e^{-it_n^j} \psi^j \|_{H^1} \to 0 \quad \text{as} \ n \to \infty. \tag{123}
\]

We thus can rewrite (122) as

\[
u_{n,0}(x) = \sum_{j=1}^{J} \nu^j(-t_n^j, x - x_n^j) + \tilde{W}_n^J(x), \tag{124}
\]

where

\[
\tilde{W}_n^J(x) = \sum_{j=1}^{J} e^{-it_n^j} \psi^j(x - x_n^j) - \nu^j(-t_n^j, x - x_n^j) + W_n^J(x).
\]

It follows from (111) and (123) that

\[
\lim_{J \to \infty} \left[ \lim_{n \to \infty} \| e^{it_0^j} \tilde{W}_n^J \|_{L^k(\mathbb{R}, L^r)} \right] = 0. \tag{125}
\]

Denote \( NLS(t) \sigma \) the solution to (1) with initial data \( f \). We have the following Pythagorean expansions along the bounded NLS flow (see e.g. [11, Lemma 3.9]).

**Lemma A.3** ([11]). Let \( T \in (0, \infty) \) be a fixed time. Assume that \( u_n(t) \equiv NLS(t)u_{n,0} \) exists up to time \( T \) for all \( n \geq 1 \) and

\[
\lim_{n \to \infty} \sup_{t \in [0, T]} K(u_n(t)) < \infty.
\]

Consider the profile decomposition (124). Denote \( \tilde{W}_n^J(t) = NLS(t)\tilde{W}_n^J \). The for all \( 1 \leq j \leq J \), the nonlinear profiles \( \nu^j(t) \) exist up to time \( T \) and for all \( t \in [0, T] \),

\[
K(u_n(t)) = \sum_{j=1}^{J} K(\nu^j(t - t_n^j)) + K(\tilde{W}_n^J(t)) + o_n(1),
\]

\[
H(u_n(t)) = \sum_{j=1}^{J} H(\nu^j(t - t_n^j)) + H(\tilde{W}_n^J(t)) + o_n(1),
\]

where \( o_n(1) \to 0 \) uniformly on \( 0 \leq t \leq T \).

By (119), we have

\[
\sup_{n \geq 1} \sup_{t \in [0, \infty)} K(u_n(t)) \leq C(A, \delta_c) < \infty.
\]

Thanks to Lemma A.3, we see that for \( n \) sufficiently large, \( K(\nu^j(t - t_n^j)) \) and \( K(\tilde{W}_n^J(t)) \) are uniformly bounded which imply that \( \nu^j \) and \( NLS(t)\tilde{W}_n^J \) exist globally in time.

Since there are more than one non-zero profiles, we have

\[
M(\nu^j(t - t_n^j)) = M(\psi^j) < 1, \quad \forall j \geq 1. \tag{126}
\]

Moreover, by (121), Lemma A.3 and (126), we have

\[
\sup_{t \in [0, \infty)} H(\nu^j(t - t_n^j))M(\nu^j(t - t_n^j))^{\sigma_c} < A, \quad E(\nu^j(t - t_n^j))M(\nu^j(t - t_n^j))^{\sigma_c} < \delta_c
\]
which, by Item (i), implies
\[ \|v_j^j(t - t_n^k)\|_{L^k([0, \infty), L^r)} < \infty, \quad \forall j \geq 1. \]

Moreover, by (24), (123) and (125), we have
\[ \lim_{j \to \infty} \left( \lim_{n \to \infty} \| \text{NLS}(t) \tilde{W}_n \|_{L^k([0, \infty), L^r)} \right) = 0. \]

Using these estimates, we can approximate
\[ u_n(t, x) = \sum_{j=1}^{J} v^j(t - t_n^j, x - x_n^j) \]
using the long time perturbation argument (see e.g. [12]) and get for \( J \) sufficiently large,
\[ \|u_n\|_{L^k([0, \infty), L^r)} < \infty \]
which is a contradiction.

**Case 2. Only one non-zero profile.** We now have only one non-zero profile, namely
\[ u_{n,0}(x) = e^{-it_n^k \Delta} \psi^1(x - x_n^1) + W_n(x), \quad \lim_{n \to \infty} e^{it_n^k \Delta} W_n \|_{L^k([0, \infty), L^r)} = 0. \]

As above, passing to a subsequence, we may assume that
\[ t_n^1 \to -\infty, \quad t_n^k \to \infty, \quad t_n^1 \to 0 \quad \text{as} \quad n \to \infty. \]

We claim that \( t_n^1 \not\to -\infty \). Indeed, suppose \( t_n^1 \to -\infty \) and fix \( \epsilon > 0 \). We have for \( n \) large,
\[ \| e^{it_n^k \Delta} u_{n,0} \|_{L^k([0, \infty), L^r)} \leq \| e^{it_n^k \Delta} \psi^1 \|_{L^k([-t_n^1, \infty), L^r)} + \| e^{it_n^k \Delta} W_n \|_{L^k([0, \infty), L^r)} \leq \epsilon. \]

By (24), the bootstrap argument yields \( \| u_n \|_{L^k([0, \infty), L^r)} < \infty \) for \( n \) sufficiently large which is a contradiction.

Let \( v^1 \) be the corresponding nonlinear profile associated to \( \psi^1 \) (see Case 1), namely
\[ u_{n,0}(x) = v^1(-t_n^1, x - x_n^1) + \tilde{W}_n(x). \]

Arguing as above, we have \( v^1 \) and NLS(\( t \))\( \tilde{W}_n \) are defined globally in time and satisfies
\[ M(v^1(t - t_n^1)) \leq 1, \quad \sup_{t \in [0, \infty)} H(v^1(t - t_n^1)) \leq A, \quad E(v^1(t - t_n^1)) \leq \delta_c \]
and
\[ \lim_{n \to \infty} \| \text{NLS}(t) \tilde{W}_n \|_{L^k([0, \infty), L^r)} = 0. \]

We infer that
\[ M(v^1(t - t_n^1)) = 1, \quad E(v^1(t - t_n^1)) = \delta_c. \]

In fact, if \( M(v^1(t - t_n^1)) < 1 \), then
\[ \sup_{t \in [0, \infty)} H(v^1(t - t_n^1))[M(v^1(t - t_n^1))]^{\sigma_c} < A, \quad E(v^1(t - t_n^1))[M(v^1(t - t_n^1))]^{\sigma_c} < \delta_c. \]

By Item (i), we have
\[ \|v^j(t - t_n^j)\|_{L^k([0, \infty), L^r)} < \infty \]
which, by the long time perturbation argument, implies
\[ \|u_n\|_{L^k([0, \infty), L^r)} < \infty, \]
We get a contradiction. We now define $u_c$ the solution to (1) with initial data $v^1(0)$. We see that $u_c$ satisfies (120). In fact, we have

$$
M(u_c) = M(\text{NLS}(t)v^1(0)) = M(v^1(t)) = M(v^1(t - t^1_n)) = 1,
$$

$$
E(u_c) = E(\text{NLS}(t)v^1(0)) = E(v^1(t)) = E(v^1(t - t^1_n)) = \delta_c
$$

and

$$
sup_{t \in [0, \infty]} H(u_c(t)) = sup_{t \in [0, \infty]} H(v^1(t)) = sup_{t \in [t^1_n, \infty]} H(v^1(t - t^1_n)) \leq A.
$$

By the definition of $\delta_c$, we must have $\|u_c\|_{L^k([0, \infty), L^r)} = \infty$.

**Step 3. Exclusion of the critical solution.** By the standard compactness lemma (see e.g. [6]), there exists a continuous path $x(t) \in \mathbb{R}^N$ such that

$$
K := \{u_c(t, \cdot - x(t)) : t \in [0, \infty)\} \subset H^1
$$

is precompact in $H^1$. Using this compactness result, the well-known rigidity argument using localized virial estimates and Lemma A.2 shows that $u_c \equiv 0$ which contradicts (120). We refer the readers to [6] for more details. The proof is complete. 

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