A UNIFIED GENERATING FUNCTION OF THE $q$-GENOCCHI POLYNOMIALS WITH THEIR INTERPOLATION FUNCTIONS

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Abstract. The purpose of this paper is to construct of the unification $q$-extension Genocchi polynomials. We give some interesting relations of this type of polynomials. Finally, we derive the $q$-extensions of Hurwitz-zeta type functions from the Mellin transformation of this generating function which interpolates the unification of $q$-extension of Genocchi polynomials.

1. Introduction, Definitions and Notations

Recently, many mathematician have studied to unification Bernoulli, Genocchi, Euler and Bernstein polynomials (see [20,21]). Ozden [20] introduced $p$-adic distribution of the unification of the Bernoulli, Euler and Genocchi polynomials and derived some properties of this type of unification polynomials.

In [20], Ozden constructed the following generating function:

$$\sum_{n=0}^{\infty} y_{n,\beta}(x; k, a, b) \frac{t^n}{n!} = 2^{1-k} k e^{xt} \frac{\beta^b}{e^t - a^b}, \quad |t + b \ln \left(\frac{\beta}{a}\right)| < 2\pi$$

where $k \in \mathbb{N} = \{1, 2, 3, ...\}$, $a, b \in \mathbb{R}$ and $\beta \in \mathbb{C}$. The polynomials $y_{n,\beta}(x; k, a, b)$ are the unification of the Bernoulli, Euler and Genocchi polynomials.

Ozden showed, $\beta = b = 1$, $k = 0$ and $a = -1$ into (1.1), we have

$$y_{n,1}(x; 0, -1, 1) = E_n(x),$$

where $E_n(x)$ denotes classical Euler polynomials, which are defined by the following generating function:

$$\frac{2e^{xt}}{e^t + 1} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}, \quad |t| < \pi,$$

In [5,20], classical Genocchi polynomials defined as follows:

$$\frac{2te^{xt}}{e^t + 1} = \sum_{n=0}^{\infty} G_n(x) \frac{t^n}{n!}, \quad |t| < \pi,$$

From (1.2) and (1.3), we easily see,

$$G_n(x) = nE_{n-1}(x),$$

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For a fixed real number $|q| < 1$, we use the notation of $q$-number as
\[ [x]_q = \frac{1 - q^x}{1 - q}, \] (see [1-4,6-24]).

Thus, we note that $\lim_{q \to 1} [x]_q = x$.

In [1,7,8,10], $q$-extension of Genocchi polynomials are defined as follows:
\[ G_{n+1,q} (x) = (n + 1) [2]_q \sum_{l=0}^{\infty} (-1)^l q^l [x + l]^n. \]

In [2], $(h,q)$-extension of Genocchi polynomials are defined as follows:
\[ G_{n+1,q} (x) = (n + 1) [2]_q \sum_{l=0}^{\infty} (-1)^l q^{(h-1)l} [x + l]^n. \]

In this paper, we shall construct unification of $q$-extension of the Genocchi polynomials, however we shall give some interesting relationships. Moreover, we shall derive the $q$-extension of Hurwitz-zeta type functions from the Mellin transformation of this generating function which interpolates.

### 2. Novel Generating Functions of $q$-extension of Genocchi polynomials

**Definition 1.** Let $a, b \in \mathbb{R}$, $\beta \in \mathbb{C}$ and $k \in \mathbb{N} = \{1, 2, 3, \ldots \}$, Then the unification of $q$-extension of Genocchi polynomials defined as follows:

\begin{equation}
F_{\beta, q} (t, x | k, a, b) = \sum_{n=0}^{\infty} S_{n, \beta, q} (x | k, a, b) \frac{t^n}{n!},
\end{equation}

and

\begin{equation}
F_{\beta, q} (t, x | k, a, b) = - [2]_q^{1-k} \frac{t^k}{k} \sum_{m=0}^{\infty} \beta^m a^{-b m - b} e^{[m+x]_q t}.
\end{equation}

where into (2.1) substituting $x = 0$, $S_{n, \beta, q} (0 | k, a, b) = S_{n, \beta, q} (k, a, b)$ are called unification of $q$-extension of Genocchi numbers.

As well as, from (2.1) and (2.2) Ozden’s constructed the following generating function, namely, we obtain \(2.1\),

\[
\lim_{q \to 1} F_{\beta, q} (t, x | k, a, b) = \frac{2^{1-k} t e^{xt}}{\beta^b e^t - a^b}.
\]

By (2.2), we see readily,
\[
\sum_{n=0}^{\infty} S_{n, \beta, q} (x | k, a, b) \frac{t^n}{n!} = - [2]_q^{1-k} \frac{t^k}{k} \sum_{m=0}^{\infty} \beta^m a^{-b m - b} e^{[m+x]_q t}
\]
\[
= \frac{e^{[x]_q t}}{q^x} \left( - [2]_q^{1-k} (q^x t)^k \sum_{m=0}^{\infty} \beta^m a^{-b m - b} e^{(q^x t)[m]_q} \right)
\]
\[
= q^{-kx} \left( \sum_{n=0}^{\infty} [x]_q^n \frac{t^n}{n!} \right) \left( \sum_{n=0}^{\infty} q^n x^n S_{n, \beta, q} (k, a, b) \frac{t^n}{n!} \right)
\]

(2.3)
From \( 23 \) by using Cauchy product we get
\[
\sum_{n=0}^{\infty} S_{n, \beta, q} (x | k, a, b) \frac{x^n}{n!} = q^{-kx} \sum_{n=0}^{\infty} \left( \sum_{i=0}^{n} \binom{n}{i} q^{ix} S_{i, \beta, q} (k, a, b) [x]_{q}^{n-i} \right) \frac{x^n}{n!}
\]

By comparing coefficients of \( \frac{x^n}{n!} \) in the both sides of the above equation, we arrive at the following theorem:

**Theorem 1.** For \( a, b \in \mathbb{R} \), \( \beta \in \mathbb{C} \) which \( k \) is positive integer. We obtain
\[
S_{n, \beta, q} (x | k, a, b) = \sum_{l=0}^{n} \binom{n}{l} q^{lx} S_{l, \beta, q} (k, a, b) [x]_{q}^{n-l}
\]

As well as, we obtain corollary 1:

**Corollary 1.** For \( a, b \in \mathbb{R} \), \( \beta \in \mathbb{C} \) which \( k \) is positive integer. We obtain,
\[
(2.4) \quad S_{n, \beta, q} (x | k, a, b) = \left( S_{\beta, q} (k, a, b) + [x]_{q} \right)^n
\]
with usual the convention about replacing \( (S_{\beta, q} (x | k, a, b))^n \) by \( S_{n, \beta, q} (x | k, a, b) \).

By applying the definition of generating function of \( S_{n, \beta, q} (x | k, a, b) \), we have
\[
\sum_{n=0}^{\infty} S_{n, \beta, q} (x | k, a, b) \frac{x^n}{n!} = -[2]_{q}^{1-k} \sum_{m=0}^{\infty} \beta^{bm} a^{-bm} \left( \sum_{n=0}^{\infty} [m + x]_{q} \frac{x^n}{n!} \right) \frac{x^n}{n!}
\]
\[
= \sum_{n=0}^{\infty} \left( -[2]_{q}^{1-k} \sum_{m=0}^{\infty} \beta^{bm} a^{-bm} [m + x]_{q} \right) \frac{x^n}{n!}
\]

So we derive the Theorem 2 which we state hear:

**Theorem 2.** For \( a, b \in \mathbb{R} \), \( \beta \in \mathbb{C} \) which \( k \) is positive integer. We obtain,
\[
(2.5) \quad S_{n, \beta, q} (x | k, a, b) = -\frac{n! [2]_{q}^{1-k}}{(n-k)!} \sum_{m=0}^{\infty} \beta^{bm} a^{-bm} [m + x]_{q}^{n-k}
\]

With regard to \( 25 \), we see after some calculations
\[
S_{n, \beta, q} (x | k, a, b) = -\frac{n! [2]_{q}^{1-k}}{a^{b} (n-k)!} \left( \frac{1}{1-q} \right)^{n-k} \sum_{l=0}^{n-k} \binom{n-k}{l} (-1)^{l} q^{lx} \sum_{m=0}^{\infty} \binom{\beta^{b} q^m}{a^{b}} \frac{m!}{a^{b} (n-k)!} \left( \frac{1}{1-q} \right)^{n-k-n} \sum_{l=0}^{n-k} \binom{n-k}{l} (-1)^{l} q^{lx} \frac{1}{\beta^{b} q^{l-k} - a^{b}}
\]
\[
(2.6) \quad = \frac{k! [2]_{q}^{1-k}}{a^{b}} \left( \frac{1}{1-q} \right)^{n-k} \sum_{l=0}^{n-k} \binom{n-k}{l} (-1)^{l} q^{lx} \frac{1}{\beta^{b} q^{l-k} - a^{b}}
\]

From \( 26 \) and well known identity \( \binom{n}{l} \binom{n-k}{k-l} = \binom{n}{k} \binom{k+l}{k} \), we obtain the following theorem:

**Theorem 3.** For \( a, b \in \mathbb{R} \), \( \beta \in \mathbb{C} \) which \( k \) is positive integer. We obtain
\[
(2.7) \quad S_{n, \beta, q} (x | k, a, b) = \frac{k! [2]_{q}^{1-k}}{a^{b}} \left( \frac{1}{1-q} \right)^{n-k} \sum_{l=0}^{n-k} \binom{n-k}{l} (-1)^{l-k} q^{(l-k)x} \frac{1}{\beta^{b} q^{l-k} - a^{b}}
\]
We put \( x \to 1 - x, \beta \to \beta^{-1}, q \to q^{-1} \) and \( a \to a^{-1} \) into (2.7), namely,

\[
S_{n,\beta^{-1},q^{-1}}(1-x|k,a^{-1},b) = \frac{k! [2]_{q^{-1}}^{1-k}}{a^{-b}} \bigg( \frac{1}{1-q^{-1}} \bigg)^{n-k} \sum_{l=k}^{n} \binom{n}{l} \binom{l}{k} (-1)^{l-k} q^{-(l-k)(1-x)} \frac{1}{\beta^{-b} q^{-(l-k)} - a^{-b}}
\]

\[
= (-1)^{n-k} q^{k-1} q^{n-k} [2]_{a}^{1-k} \frac{(1-q)}{1-q} \sum_{l=k}^{n} \binom{n}{l} \binom{l}{k} (-1)^{l-k} q^{k-1} q^{(l-k)x} \frac{\beta^{b} q^{l-k} a^{b}}{a^{b} - \beta^{b} q^{l-k}}
\]

\[
= (-1)^{n-k} q^{-n-1} a^{3b} \beta^{b} \left( \frac{k! [2]_{a}^{1-k}}{a^{b}} \right) \left( \frac{1}{1-q} \right)^{n-k} \sum_{l=k}^{n} \binom{n}{l} \binom{l}{k} (-1)^{l-k} q^{x(l-k)} \frac{1}{\beta^{b} q^{l-k} - a^{b}}
\]

So, we obtain symmetric properties of \( S_{n,\beta,q}(x|k,a,b) \) as follows:

**Theorem 4.** For \( a, b \in \mathbb{R}, \beta \in \mathbb{C} \) which \( k \) is positive integer. We obtain

\[
S_{n,\beta^{-1},q^{-1}}(1-x|k,a^{-1},b) = (-1)^{n-k-1} q^{a^{-1}-1} a^{3b} \beta^{b} S_{n,\beta,q}(x|k,a,b).
\]

By (2.7), we see

\[
\frac{\beta}{a} F_{\beta,q}(t,1|k,a,b) - F_{\beta,q}(t,0|k,a,b) = \sum_{n=0}^{\infty} \left( \frac{\beta}{a} \right) S_{\beta,q}(1|k,a,b) - S_{n,\beta,q}(k,a,b) = \frac{t^{n}}{n!} = \frac{[2]_{q}^{1-k}}{a^{b}} t^{k}
\]

From (2.8), we obtain the following theorem:

**Theorem 5.** For \( a, b \in \mathbb{R}, \beta \in \mathbb{C} \) which \( k \) is positive integer. We obtain

\[
S_{n,\beta,q}(k,a,b) - \left( \frac{\beta}{a} \right) S_{\beta,q}(1|k,a,b) = \left\{ \begin{array}{ll} 0, & n \neq k \\ \frac{[2]_{q}^{1-k}}{a^{b}} k!, & n = k \end{array} \right.
\]

From (2.8) and (2.9), we obtain corollary as follows:

**Corollary 2.** For \( a, b \in \mathbb{R}, \beta \in \mathbb{C} \), which is \( k \) positive integer. We get

\[
S_{n,\beta,q}(k,a,b) - \left( \frac{\beta}{a q^{k}} \right) (q S_{\beta,q}(k,a,b) + 1)^{n} = \left\{ \begin{array}{ll} 0, & n \neq k \\ \frac{[2]_{q}^{1-k}}{a^{b}} k!, & n = k \end{array} \right.
\]

with the usual convention about replacing \((S_{\beta,q}(k,a,b))^{n}\) by \(S_{n,\beta,q}(k,a,b)\).
From (9), now, we shall obtain distribution relation for unification $q$-extension of Genocchi polynomials, after some calculations, namely,

\[
S_{n,\beta,q}(x|k,a,b) = -\frac{n!}{a^k(n-k)!} \sum_{m=0}^{\infty} \left(\frac{\beta}{a}\right)^{bm} \frac{q}{m + x}^{n-k}
\]

\[
= -\frac{n!}{a^k(n-k)!} \sum_{l=0}^{d-1} \sum_{m=0}^{\infty} \left(\frac{\beta}{a}\right)^{b(l+md)} \frac{q}{m + x}^{n-k}
\]

\[
= [d]_{q}^{n-k} \sum_{l=0}^{d-1} \left(\frac{\beta}{a}\right)^{bl} \frac{q}{m + x}^{n-k} S_{n,\beta^d,q^d} \left(\frac{x + l}{d}|k, a^d, b\right)
\]

Therefore, we obtain the following theorem:

**Theorem 6.** (Distribution formula for $S_{n,\beta,q}(x|k,a,b)$) For $a,b \in \mathbb{R}$, $\beta \in \mathbb{C}$ which $k$ is positive integer. We obtain,

\[
S_{n,\beta,q}(x|k,a,b) = \frac{[2]_{q}^{1-k}}{[2]_{q^d}^{1-k}} [d]_{q}^{n-k} \sum_{l=0}^{d-1} \left(\frac{\beta}{a}\right)^{bl} S_{n,\beta^d,q^d} \left(\frac{x + l}{d}|k, a^d, b\right)
\]

3. Interpolation function of the polynomials $S_{n,\beta,q}(x|k,a,b)$

In this section, we give interpolation function of the generating functions of $S_{n,\beta,q}(x|k,a,b)$ however, this function is meromorphic function. This function interpolates $S_{n,\beta,q}(x|k,a,b)$ at negative integers.

For $s \in \mathbb{C}$, by applying the Mellin transformation to (2.2), we obtain

\[
\mathfrak{S}_{\beta,q}(s;x,a,b) = \frac{(-1)^{k+1}}{\Gamma(s)} \int_{t} t^{-k-1} F_{\beta,q}(-t, x|k,a,b) dt
\]

\[
= [2]_{q}^{1-k} \sum_{m=0}^{\infty} \beta^{bm} a^{-bm-b} \frac{1}{\Gamma(s)} \int_{0}^{\infty} t^{s-1} e^{-t[m+x]} dt
\]

So, we have

\[
\mathfrak{S}_{\beta,q}(s;x,a,b) = [2]_{q}^{1-k} \sum_{m=0}^{\infty} \beta^{bm} a^{-bm-b} \frac{1}{\Gamma(s)} \int_{0}^{\infty} t^{s-1} e^{-t[m+x]} dt
\]

We define $q$-extension Hurwitz-zeta type function as follows theorem:

**Theorem 7.** For $a,b \in \mathbb{R}$, $\beta,s \in \mathbb{C}$ which $k$ is positive integer. We obtain,

\[
\mathfrak{S}_{\beta,q}(s;x,a,b) = [2]_{q}^{1-k} \sum_{m=0}^{\infty} \beta^{bm} a^{-bm-b} \frac{1}{\Gamma(s)} \int_{0}^{\infty} t^{s-1} e^{-t[m+x]} dt
\]

for all $s \in \mathbb{C}$. We note that $\mathfrak{S}_{\beta,q}(s;x,a,b)$ is analytic function in the whole complex $s$-plane.

**Theorem 8.** For $a,b \in \mathbb{R}$, $\beta \in \mathbb{C}$ which $k$ is positive integer. We obtain,

\[
\mathfrak{S}_{\beta,q}(-n;x,a,b) = \frac{(n-k)!}{n!} S_{n,\beta,q}(x|k,a,b).
\]
Proof. Let \( a, b \in \mathbb{R} \), \( \beta \in \mathbb{C} \) and \( k \in \mathbb{N} \) with \( k \in \mathbb{N} = \{1, 2, 3, \ldots\} \). \( \Gamma(s) \), has simple poles at \( z = -n = 0, -1, -2, -3, \ldots \). The residue of \( \Gamma(s) \) is
\[
\text{Re}s \left( \Gamma(s), -n \right) = \frac{(-1)^n}{n!}.
\]
We put \( s \to -n \) into (3.1) and using the above relations, the desired result can be obtained. \( \square \)

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