Nonlocal Condensates in QCD

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Abstract

In the presence of the nontrivial QCD ground state or vacuum, nonlocal condensates are used to characterize the quark or gluon propagator, or other Green functions of higher order. In this paper, we wish to show that, by taking the large $N_c$ limit (with $N_c$ the number of color) in treating higher-order condensates, a closed set of coupled differential equations may be derived for nonlocal condensates. As a specific example, the leading-order equations for the nonlocal condensates appearing in the quark propagator are derived and explicit solutions are obtained. Some applications of our analytical results are briefly discussed.

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I. Introduction

The problem of strong interaction physics has been around for more than half a century, but the very nature of the problem varies with the so-called “underlying theory” which nowadays is taken universally to be quantum chromodynamics or QCD. Although the asymptotically free nature of QCD allows us to test the candidate theory at high energies, the nonperturbative feature dominates for hadrons or nuclei at low energies. As of today, it remains almost impossible to solve problems related to hadrons or nuclei, except perhaps through lattice simulations but it is unlikely that, through the present-day computation power which is still not quite adequate, simulations would give us most of what we wish to know. However, the need of solving QCD directly has gained tremendous importance over the last decade and, to this end, any fruitful attempt in confronting directly with QCD should be given due attention.

The ground state, or the vacuum, of QCD is known to be nontrivial, in the sense that there are non-zero condensates, including gluon condensates, quark condensates, and perhaps infinitely many higher-order condensates. In such a theory, propagators, i.e. causal Green’s functions, such as the quark propagator

\[ iS_{ij}^{ab}(x) \equiv \langle 0 | T(q_i^a(x)\bar{q}_j^b(0)) | 0 \rangle, \]

carry all the difficulties inherent in the theory. Higher-order condensates, such as a four-quark condensate,

\[ \langle 0 | T(\bar{\psi}(z)\gamma_\mu\psi(z)q_i^a(x)\bar{q}_j^b(0)) | 0 \rangle, \]

with \( \psi(z) \) also labeling a quark field, represent an infinite series of unknowns unless some useful ways for reduction can be obtained. As the vacuum, \( | 0 \rangle \), is highly nontrivial, there is little reason to expect that Wick’s theorem (of factorization), as obtained for free quantum field theories, is still of validity. Thus, we must look for alternative methods in order to obtain useful results.
It is known that the equations for Green functions up to a certain order usually involve Green functions of even higher order, thereby making such hierarchy of equations often useless in practice. In this paper, however, we wish to show that, provided that we may use the large $N_c$ approximation to treat condensates of much higher order, there is in fact a natural way of setting up closed sets of differential equations which govern the inter-related Green functions to a given order. We consider this as an important accomplishment, both because we can always go over to the next level of sophistication in order to improve the approximation and because the large $N_c$ expansion has been shown to yield desirable results for describing hadron physics.

II. Leading-order Equations for Nonlocal Condensates

In light of the nontrivial QCD vacuum, we begin by considering the feasibility of working directly with the various matrix elements such as the quark propagator of Eq. (1). Useful relations may be derived if we regard the equations for interacting fields \[\{i\gamma^\mu(\partial_\mu + ig\frac{\lambda^a}{2}A^a_\mu) - m\} \psi = 0;\] (2)
\[\partial^\nu G^{a\mu\nu}_{\mu\nu} - 2gf^{abc}G^{b\mu\nu}_{\mu\nu}A^c_\nu + g\bar{\psi}\lambda^a\gamma^\mu\psi = 0,\] (3)
as the equations of motion for quantized interacting fields, subject to the standard rule for quantization that the equal-time (anti-)commutators among these quantized interacting fields are identical to those among non-interacting quantized fields. As our basic example, we allow the operator \(\{i\gamma^\mu\partial_\mu - m\}\) to act on the matrix element defined by Eq. (1) and obtain
\[\{i\gamma^\mu\partial_\mu - m\}_{ik}iS^{ab}_{kj}(x) = i\delta^4(x)\delta^{ab}\delta_{ij}i + \langle 0 \mid T\{g\frac{\lambda^n}{2}A^n_\mu\gamma^\mu q(x)\}_{i}q_{j}^{b}(0)\} \mid 0 \rangle.\] (4)

We should always keep in mind that the QCD vacuum \(\mid 0 \rangle\) is a nontrivial ground state which is in general not annihilated by operating on it the annihilation operators.
Eq. (4) can be solved by splitting the propagator into a singular, perturbative part and a nonperturbative part:

\[ iS^{ab}_{ij}(x) = iS^{(0)ab}_{ij}(x) + i\tilde{S}^{ab}_{ij}(x), \tag{5} \]

where

\[ iS^{(0)ab}_{ij}(x) \equiv \int \frac{d^4p}{(2\pi)^4} e^{-ip \cdot x} iS^{(0)ab}_{ij}(p), \tag{6a} \]

\[ iS^{(0)ab}_{ij}(p) = \delta^{ab} \frac{i(\hat{p} + m_{ij})}{p^2 - m^2 + i\epsilon}, \tag{6b} \]

with \( \hat{a} \equiv \gamma^\mu a_\mu \) for a four-vector \( a_\mu \). The nonperturbative part then satisfies the equation:

\[ \{i\gamma^\mu \partial_\mu - m\}_{ik}\tilde{S}^{ab}_{kj}(x) = <0 | T(\{g^n A^n_{\mu} \gamma^\mu q(x)\}^a_{i} \bar{q}^b_{j}(0)) | 0> \tag{7} \]

This would be pretty much the end of the story unless we could find some way to proceed.

We should note that Eq. (4) may also be derived by making use of, e.g., the path-integral formulation, and the issue of how to define renormalized composite operators, i.e. products of field operators, is by no means trivial (and fortunately we need not worry about such problem for the sake of this paper).

As a useful benchmark, we note that, with the fixed-point gauge,

\[ A^n_{\mu}(x) = -\frac{1}{2} G^n_{\mu\nu} x^\nu + \cdots, \tag{8} \]

we may solve the nonperturbative part \( i\tilde{S}^{ab}_{ij}(x) \) as a power series in \( x^\mu \),

\[ i\tilde{S}^{ab}_{ij}(x) = -\frac{1}{12} \delta^{ab} \delta_{ij} <\bar{q}q> + \frac{i}{48} m\tilde{x}_{ij} \delta^{ab} <\bar{q}q> + \frac{1}{192} <\bar{q}g\sigma \cdot Gq> \delta^{ab} x^2 \delta_{ij} + \cdots, \tag{9} \]

The first term is the integration constant which defines the so-called “quark condensate”, while the mixed quark-gluon condensate appearing in the third term arises because of Eqs. (7) and (8). It is obvious that the series (9) is a short-distance expansion, which converges
for small enough $x_\mu$. We note that is just the standard quark propagator cited in most papers in QCD sum rules [2].

The approach which we suggest here [3] is based upon two key elements, namely, the set of interacting field equations plus the rule of canonical quantization (for interacting fields). The equations which we obtain, such as Eq. (5), are much the same as the set of Schwinger-Dyson equations (for the matrix elements). An important aspect in our derivation is that the nontriviality of the vacuum $|0>$ is observed at every step – a central issue in relation to QCD.

As another important exercise, we may split the gluon propagator into the singular, perturbative part and the nonperturbative part. The nonperturbative part is given by [3]

\[
g^2 < 0 | : G^m_{\mu\nu}(x) G^m_{\alpha\beta}(0) : | 0 >
\]

\[
= \frac{\delta^{nm}}{96} < g^2 G^2 > (g_{\mu\alpha} g_{\nu\beta} - g_{\mu\beta} g_{\nu\alpha})
\]

\[
- \frac{\delta^{nm}}{192} < g^3 G^3 > \left\{ x^2 (g_{\mu\alpha} g_{\nu\beta} - g_{\mu\beta} g_{\nu\alpha}) - g_{\mu\alpha} x_\nu x_{\beta} + g_{\mu\beta} x_{\nu} x_{\alpha} - g_{\nu\beta} x_{\mu} x_{\alpha} + g_{\nu\alpha} x_{\mu} x_{\beta} \right\}
\]

\[
+ O(x^4),
\]

(10)

with

\[
< g^2 G^2 > \equiv < 0 | : G^m_{\mu\nu}(0) G^m_{\mu\nu}(0) : | 0 >, \quad (11a)
\]

\[
< g^3 G^3 > \equiv < 0 | : g^3 f^{abc} g^{\mu\nu} G^a_{\mu\alpha}(0) G^b_{\nu\beta}(0) G^c_{\mu\nu}(0) : | 0 > . \quad (11b)
\]

Again, the first term in Eq. (10) is the integration constant for the differential equation satisfied by the gluon propagator. Note that inclusion of the gluon condensate $< g^2 G^2 >$ in Eq. (10) is standard but the triple gluon condensate $< g^3 G^3 >$ is a new entry required by the interacting field equation (3). Our approach indicates when condensates of entirely new types should be introduced as we try to perform operator-product expansions to higher dimensions.

In what follows, we wish to focus on the quark propagator, as specified by Eqs. (1) and (4)-(7). We write

\[
i \hat{S}^{ab}_{ij}(x) = \delta^{ab} \{ \delta_{ij} f(x^2) + i x_{ij} g(x^2) \},
\]

(12)
with \( x^2 \equiv x_0^2 - \mathbf{x}^2 \). \( f(x^2) \) and \( g(x^2) \) are what we refer to as “nonlocal condensates” in connection with the quark propagator. We note that

\[
\langle : \bar{q}(x) q(0) : \rangle = -12 f(x^2),
\]

which is the nonlocal quark condensate in the standard sense. To proceed further, we work only with the leading term in the fixed-point gauge and introduce

\[
\langle g \mathcal{G}_{\mu\nu} q(x) \rangle_i \langle \bar{q}^a_j(0) : \rangle = \delta^{ab} \{ (\gamma_\mu x_\nu - \gamma_\nu x_\mu) A(x^2) + i\sigma_{\mu\nu} B(x^2) + (\gamma_\mu x_\nu - \gamma_\nu x_\mu) \hat{x} C(x^2) + i\sigma_{\mu\nu} \hat{x} D(x^2) \},
\]

with \( \mathcal{G}_{\mu\nu} \equiv (\lambda^2/2)\mathcal{G}_{\mu\nu} \), an antisymmetric operator. The invariant functions \( A(x^2), B(x^2), C(x^2), \) and \( D(x^2) \) are additional nonlocal condensates which we deal with explicitly in this paper.

Under the assumption that we keep only the leading term in the fixed-point gauge (a simplifying assumption which can be removed whenever necessary), we have

\[
\{ i\gamma^\alpha \partial_\alpha - m \}_{ik} \langle g \mathcal{G}_{\mu\nu} q(x) \rangle_i \langle \bar{q}^a_j(0) : \rangle = -\frac{1}{2} x^\beta \langle g^2 \mathcal{G}_{\mu\nu} \mathcal{G}_{\alpha\beta} \gamma^\alpha q(x) \rangle_i \langle \bar{q}^a_j(0) : \rangle
\]

\[
= -\frac{1}{2} \cdot \frac{1}{96} \cdot \frac{1}{3} < g^2 \mathcal{G}^2 > \langle (\gamma_\mu x_\nu - \gamma_\nu x_\mu) q(x) \rangle_i \langle \bar{q}^a_j(0) : \rangle
\]

\[
= -\frac{1}{144} < g^2 \mathcal{G}^2 > \delta^{ab} (\gamma_\mu x_\nu - \gamma_\nu x_\mu) \{ f(x^2) + i\hat{x} g(x^2) \}.
\]

Here the second line, (15a), follows from the field equation and the third line, (15b), is based on the large \( N_c \) approximation that the contribution in which \( \mathcal{G}_{\mu\nu} \) and \( \mathcal{G}_{\alpha\beta} \) do not couple to color-singlet is suppressed by a factor of \( 1/N_c^2 \). Thus, the factorization in the present case is justified to order \( 1/N_c^2 \), rather than just order \( 1/N_c \). We also note that we have used Eq. (10) to leading order in obtaining Eq. (15b), but this approximation can be relaxed if necessary.

Now, we may use Eqs. (7) and (15c) and obtain a closed set of equations:

\[
2f'(x^2) - mg(x^2) = -\frac{3}{2} i(B - x^2 C),
\]

(16a)
\[ 2x^2 g'(x^2) + 4g(x^2) + mf(x^2) = \frac{3}{2} x^2 (A - D), \]  

(16b)

\[ 4iB' - 2iC - 2ix^2 C' - mA = -\frac{1}{144} < g^2 G^2 > f(x^2), \]  

(16c)

\[ 2iA + 2ix^2 D' - mB = 0, \]  

(16d)

\[ -2iA' + 4iD' - mC = -\frac{i}{144} < g^2 G^2 > g(x^2), \]  

(16e)

\[ 2iB' + 2iC - mD = 0, \]  

(16f)

where the derivatives are with respect to the variable \( x^2 \).

Treating \( m \) as an expansion parameter,  

\[ F(x^2) = \sum_{k=0}^{\infty} m^k F_k(x^2), \]  

(17)

we may solve the coupled equations, (16a)-(16f), order by order in \( m \). To leading order in \( m \), we obtain  

\[ x^2 f'''_0 + 3f''_0 - \xi_0^2 x^2 f'_0 - 2\xi_0^2 f_0 = 0, \]  

(18)

\[ (x^2)^3 g'''_0 + 5(x^2)^2 g''_0 + \{2x^2 - \xi_0^2 (x^2)^3\} g'_0 - \{2 + 2\xi_0^2 (x^2)^2\} g_0 = 0, \]  

(19)

with \( \xi_0^2 \equiv < g^2 G^2 > /384 \). The equations for \( A_0, B_0, C_0, \) and \( D_0 \) can easily be solved once we obtain \( f_0 \) and \( g_0 \).

Eq. (19) can be simplified considerably by introducing  

\[ g_0(x^2) \equiv (x^2)^{-2} \tilde{g}_0(x^2), \]  

(20a)

which leads to the equation:  

\[ x^2 \tilde{g}'''_0 - \tilde{g}''_0 - \xi_0^2 x^2 \tilde{g}'_0 = 0. \]  

(20b)

Eqs. (17) and (20) can be solved by iteration, leading to the result:

\[ f_0(t) = a_0 \left\{ 1 + \frac{1}{1 \cdot 3} (\xi_0 t)^2 + \frac{1}{1 \cdot 3^2 \cdot 5} (\xi_0 t)^4 + \cdots \right\} \]

\[ + a_1 t \left\{ 1 + \frac{1}{2 \cdot 4} (\xi_0 t)^2 + \frac{1}{2 \cdot 4^2 \cdot 6} (\xi_0 t)^4 + \cdots \right\}, \]  

(21)

\[ \tilde{g}'_0(t) = c_2 t^2 \left\{ 1 + \frac{1}{2 \cdot 4} (\xi_0 t)^2 + \frac{1}{2 \cdot 4^2 \cdot 6} (\xi_0 t)^4 + \cdots \right\}, \]  

(22)
with $t \equiv x^2$ and
\[
a_0 = -\frac{1}{12} < \bar{q}q >, \quad a_1 = \frac{1}{192} < \bar{q}g_c \sigma \cdot Gq >, \quad c_2 = -\frac{g_c^2 < \bar{q}q >^2}{2^5 \cdot 3^4}.
\] (23)

Eq. (23) is obtained by comparing to the well-known series expansion for the quark propagator (see, e.g., [4]). Note that there are two integration constants, $a_0$ and $a_1$, for $f(x^2)$ but there is only one permissible constant for $g(x^2)$ (and $c_2$ is in fact a four-quark condensate taken in the large $N_c$ limit).

For a number of applications, it is useful to obtain analytic expressions for $f_0(t)$ and $g_0(t)$. This turns out to be possible by way of Laplace transforms.

\[
\bar{f}_0(s) \equiv \int_0^\infty dse^{-st} f_0(t), \quad \bar{\tilde{g}}'_0(s) \equiv \int_0^\infty dse^{-st} \tilde{g}'_0(t).
\] (24)

We obtain
\[
\bar{f}_0(s) = -\frac{2a_1}{\xi_0^2} - \frac{a_0}{\xi_0} \frac{s}{\sqrt{s^2 - \xi_0^2}} \sec^{-1} \frac{s}{\xi_0} + \frac{\gamma_0 s}{\sqrt{s^2 - \xi_0^2}},
\] (25)
\[
\bar{\tilde{g}}'_0(s) = \frac{2c_2}{(s^2 - \xi_0^2)^{3/2}},
\] (26)

with $\gamma_0 = 2a_1/\xi_0^2$. Looking up the table for Laplace transforms, we find
\[
\tilde{g}'_0 = \frac{2c_2}{\xi_0^2} \cdot \xi_0 t \cdot I_1(\xi_0 t),
\] (27)

with $I_1(z)$ the modified Bessel function of the first kind, to order one. It is straightforward to show that Eq. (27) yields the series expansion in Eq. (22). Also, the function $I_1(\xi_0 t)$ enters the second series in $f_0(t)$ as in Eq. (21).

To close our presentation of the explicit solution to leading order in $m$, we note that Eqs. (16) yields
\[
f'_0(t) = -i \frac{3}{4} (tB_0(t))',
\] (28a)
\[
C_0(t) = -B'_0(t),
\] (28b)
\[
tg'_0(t) + 2g_0(t) = -\frac{3}{4} t(tD_0(t))',
\] (28c)
\[
A_0(t) = -tD'_0(t).
\] (28d)
Thus, the functions $A_0$, $B_0$, $C_0$, and $D_0$ can be solved explicitly once $f_0$ and $g_0$ are known.

On the other hand, we may go beyond the leading order in $m$ and obtain, as example,

$$
t f''''_1 + 3 f'''_1 - \xi_0^2 t f'_1 - 2 \xi_0^2 f_1
= \frac{1}{2} t g''_0 + \frac{3}{2} g_0 - \frac{3}{8} \{ t A'_0 + 2 A_0 + t(tD_0)'' + 2(tD_0)' \}; \quad (29a)
$$

$$
t^3 g'''_1 + 5 t g''_1 + (2 t - \xi_0^2 t^3) g'_1 - (2 + 2 \xi_0^2 t^2) g_1
= - \frac{1}{2} (t^2 f''''_0 + t f'''_0 - f_0) + \frac{3t^2}{8} (-i B'_0 + 2 i C_0 + i t C'_0). \quad (29b)
$$

These equations can also be solved by Laplace transforms. In other words, the nonlocal condensate functions $f(x^2)$ and $g(x^2)$ can be analytically solved order by order in $m$.

### III. Discussion and Summary

Thus far, we have described how to obtain a closed set of coupled equations for the nonlocal condensates which are relevant in the description of the quark propagator. We have also shown how these equations can be solved explicitly. Furthermore, some of the assumptions underlying our equations can be relaxed and more elaborate equations may then be obtained. Of course, some of our results are gauge dependent as the quark propagator (1) has been analyzed in a specific gauge (8). Nevertheless, our primary motivation for studying the quark propagator stems from our interest in the method of QCD sum rules [1], which may be regarded as the various approaches in which one tries to exploit the roles played by the quark and gluon condensates for problems involving hadrons. In this regard, results based on our leading-order equations are often adequate and the final answers are in general free from the potential gauge dependent problem.

There are several approaches of QCD sum rules towards hadron physics. As the first approach, we may consider the Belyaev-Ioffe nucleon mass sum rules [5,4], where the short-distance expansion for the quark propagator is needed up to a certain (high) dimension. In this context, our analytical results on nonlocal condensates may not be very useful if
the resultant series converges rapidly, and in general our analytical expressions may be used to perform further analytical analysis of the problems as a way to improve the results obtained via short-distance expansions (in $x_\mu$). The second approach is to consider the response of the QCD vacuum to some external fields, such as the method of QCD sum rules in the presence of an external axial field $Z_\mu(x)$ [6]. In this context, certain induced condensates are introduced (previously as new parameters) but the method offers a simple extension of the first approach in calculating magnetic moments, coupling constants, and other quantities by avoiding a need to treat explicitly the three-point Green’s functions - a need which would still involve some conceptual difficulties. What is of great interest is that our analytical expressions for nonlocal condensates help to determine the induced condensates previously treated as new parameters, thereby making the external-field QCD sum rule method more powerful than what it used to be. To illustrate the point, we consider the external axial field $Z_\mu$ with the interaction,

$$\delta \mathcal{L}(x) = gZ_\mu(x)\bar{q}(x)\gamma_\mu\gamma_5q(x).$$

(30)

For a constant $Z^\mu$ field, there are two major induced condensates [6]:

$$<0 | \bar{q}(0)\gamma_\mu\gamma_5q(0) | 0 >_{Z^\alpha} \quad \text{and} \quad <0 | \bar{q}(0)g_\epsilon \tilde{G}_{\mu\nu}\gamma^\nu q(0) | 0 >_{Z^\alpha}. $$

We now have

$$<0 | \bar{q}(0)\gamma_\mu\gamma_5q(0) | 0 >_{Z^\alpha} = i \int d^4xgZ_\alpha(x) <0 | T(\bar{q}(x)\gamma_\alpha\gamma_5q(x)\bar{q}(0)\gamma_\mu\gamma_5q(0)) | 0 >$$

$$= i \int d^4xgZ_\alpha(x) \{ Tr[iS^{(0)}(-x)\gamma_\alpha\gamma_5iS^{(0)}(x)\gamma_\mu\gamma_5] + Tr[i\tilde{S}(-x)\gamma_\alpha\gamma_5i\tilde{S}(x)\gamma_\mu\gamma_5]$$

$$+ Tr[iS^{(0)}(-x)\gamma_\alpha\gamma_5\tilde{S}(x)\gamma_\mu\gamma_5] + Tr[i\tilde{S}(-x)\gamma_\alpha\gamma_5\tilde{S}(x)\gamma_\mu\gamma_5]\}.$$

(31)
The first term is the one-loop result which can be regularized (e.g., in $d$ dimensions) and, as expected for a perturbative contribution, its finite part is small compared to the second and third terms. The last term, which can be treated numerically, involves products of two condensates and it is higher than the second or third term by at least three dimensions (and is likely of less numerical significance).

Analogously, we have

$$<0 | \bar{q}(0) g_c \tilde{G}_{\mu\nu} \gamma^\nu q(0) | 0 > Z^\alpha$$

$$= i \int d^4x gZ^\alpha(x) <0 | T(\bar{q}(x) \gamma_\alpha \gamma_5 q(x) \bar{q}(0) g_c \tilde{G}_{\mu\nu} \gamma^\nu q(0)) | 0 >$$

$$= igZ^\alpha g_c \frac{1}{2} \epsilon_{\mu\nu\lambda\eta} \int d^4x Tr\{\gamma_\alpha \gamma_5 i S^{ba}(x) \gamma^\nu <\{G^\lambda q(0)\}^a q^b(x) :>\}. \quad (32)$$

Thus, our analytical results on $A_0$, $B_0$, $C_0$, and $D_0$ [Cf. Eqs. (14) and (28)] may be used to evaluate Eq. (32).

Without going into the details (which shall be presented elsewhere together with detailed discussions), we mention that, upon Wick’s rotation on the time integration ($\int_{-\infty}^{\infty} \rightarrow \int_{i\infty}^{i\infty}$), our analytical solutions can be employed explicitly. The final results for the above two induced condensates are recorded immediately below:

$$<0 | \bar{q}(0) \gamma_\mu \gamma_5 q(0) | 0 > \equiv g\chi Z_\mu <\bar{q}q >, \quad (33a)$$

$$<0 | \bar{q}(0) g_c \tilde{G}_{\mu\nu} \gamma^\nu q(0) | 0 > \equiv g\kappa Z_\mu <\bar{q}q >, \quad (33b)$$

$$\chi^{(1)} <\bar{q}q > = \frac{m^2}{2\pi^2} (\ln \frac{\pi m^2}{\mu^2} + \gamma_E), \quad (33c)$$

$$\chi^{(2)} = \frac{\pi m}{2} \frac{1}{\xi_0} - \frac{1}{16} \frac{mm_0^2}{\xi_0^2} + \frac{g_0^2}{216} \frac{1}{\xi_0^2}, \quad (33d)$$

$$\kappa = \frac{m}{8} (\pi + 2 \ln 2 + 1) - \frac{1}{32} \frac{mm_0^2}{\xi_0}. \quad (33e)$$

Numerically, we find $\chi a \approx 0.15 GeV^2$ and $\kappa a \approx 6 \times 10^{-4} GeV^4$. (Here we have used the updated value on the gluon condensate [7], the current quark mass, and other input parameters as adopted previously [4].) Such values for the induced condensates, albeit
somewhat smaller than the commonly adopted ones, are not unexpected since the leading contributions are linear in the “current” quark mass $m$.

There are other problems for which our analytical results may be very useful. For example, the amplitude given by

$$T_{\mu\nu}(q^2, p \cdot q) = i \int d^4 x e^{-iq \cdot x} < \pi^+(p) | T(J_\mu(x)J_\nu(0)) | \pi^+(p) > \quad (34a)$$

$$\equiv (-g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2}) T_1(q^2, p \cdot q)$$

$$+ \frac{1}{p^2} (p_\mu - \frac{p \cdot q}{q^2} q_\mu)(p_\nu - \frac{p \cdot q}{q^2} q_\nu) T_2(q^2, p \cdot q), \quad (34b)$$

characterizes the forward Compton scattering off $\pi^+$ (a Goldstone boson) and also the parton distributions of $\pi^+$. Applying the soft-pion theorem (together with current algebra), we find, in the limit of $p_\mu \to 0$,

$$T_{\mu\nu}(q^2, 0) = \frac{i}{f_{\pi}^2} \int d^4 x e^{-iq \cdot x} < 0 | T(\{A^1_\mu(x) - iA^2_\mu(x)\}\{A^1_\nu(0) + iA^2_\nu(0)\} - 2V^3_\mu(x)V^3_\nu(0)) | 0 > .$$

$$= \frac{i}{f_{\pi}^2} \int d^4 x e^{-iq \cdot x} Tr\{iS^b_a(x)\gamma_\mu\gamma_5 iS^a_b(x)\gamma_\nu\gamma_5$$

$$- \frac{1}{2} iS^b_a(x)\gamma_\mu iS^a_b(x)\gamma_\nu - \frac{1}{2} iS^b_d(x)\gamma_\mu iS^a_d(x)\gamma_\nu\} \quad (35)$$

The structure functions $W_i(q^2, p \cdot q)$ (in the description of deep inelastic scattering off the $\pi^+$ target) is the imaginary part of $T_i(q^2, p \cdot q)$ divided by the factor $\pi$. Eq. (35) suggest that our analytical expressions for the nonlocal condensates may be useful for analyzing properties of Goldstone pions. We find, as a soft-pion limit,

$$W_1(q^2, p \cdot q) = \frac{1}{\pi} \text{Im} T_1(q^2, p \cdot q)$$

$$\longrightarrow \frac{m <q\bar{q}>}{2 f_{\pi}^2 \xi_0}, \quad \text{as} \quad p_\mu \to 0 \quad \text{and} \quad q_\mu \to 0. \quad (36)$$

This limit is derived making use of our analytical expressions on the nonlocal condensates, again with Wick’s rotation on the time integration.
In summary, we have in this paper suggested a specific way of obtaining a closed set of coupled differential equations for nonlocal condensates. Specifically, the leading-order equations for the nonlocal condensates in relation to the quark propagator are obtained and explicit analytical solutions are obtained. We believe that such results could be very useful for a large number of problems in hadron physics.

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References

1. For notations, see, e.g., T.-P. Cheng and L.-F. Li, *Gauge Theory of Elementary Particle Physics* (Clarendon Press, Oxford, 1984).
2. M. A. Shifman, A.J. Vainshtein, and V.I. Zakharov, Nucl. Phys. 147, 385, 448 (1979).
3. W.-Y. P. Hwang, Preprint hep-ph/9601219 & MIT-CTP-2498, Z. Physc. C, accepted for publication.
4. K.-C. Yang, W.-Y. P. Hwang, E.M. Henley, and L.S. Kisslinger, Phys. Rev. D47, 3001 (1993).
5. B. L. Ioffe, Nucl. Phys. B188, 317 (1981); (E) B191, 591 (1981); V. M. Belyaev and B. L. Ioffe, Zh. Eksp. Teor. Fiz. 83, 876 (1982) [Sov. Phys. JETP 56, 493 (1982)].
6. V. M. Belyaev and Ya. I. Kogan, Pis’ma Zh. Eksp. Teor. Fiz. 37, 611 (1983) [JETP Lett. 37, 730 (1983); C. B. Chiu, J. Pasupathy, and S.J. Wilson, Phys. Rev. D32, 1786 (1985); E. M. Henley, W.-Y. P. Hwang, and L.S. Kisslinger, Phys. Rev. D46, 431 (1992); Chinese J. Phys. (Taipei) 30, 529 (1992).
7. S. Narison, Phys. Lett. B 387, 162 (1996).