Abstract. We consider $G_2$ structures with torsion coupled with $G_2$-instantons, on a compact 7-dimensional manifold. The coupling is via an equation for 4-forms which appears in supergravity and generalized geometry, known as the Bianchi identity. The resulting system of partial differential equations can be regarded as an analogue of the Strominger system in 7-dimensions. We initiate the study of the moduli space of solutions and show that it is finite dimensional using elliptic operator theory. We also relate the associated geometric structures to generalized geometry.

1. Introduction

The study of the moduli space of torsion-free $G_2$ structures on a compact 7-dimensional spin manifold $M$ was initiated by Joyce [24, 25], who proved that the period map defines a local diffeomorphisms to the cohomology group $H^3(M, \mathbb{R})$. Further, there is a natural pseudo-Riemannian metric of Hessian type on the moduli space first defined by Hitchin [22]. In this work we address the question of whether $G_2$ structures with torsion on a compact 7-dimensional manifold arise in moduli and, if so, what is the nature of their moduli space. Our approach to this problem is inspired by physics, and relies on recent developments on the study of the Strominger system of partial differential equations [15]. Here, we consider $G_2$ structures with torsion (cocalibrated, of type $W_3$) coupled with $G_2$ instantons, by means of an equation for 4-forms which appears in supergravity and generalized geometry [34, 23]. The resulting system of equations can be regarded as an analogue of the Strominger system in 7-dimensions. Our main result implies the finite-dimensionality of the moduli space. As a consequence of our method, we also obtain a new proof of the characterization of the infinitesimal moduli space of $G_2$-holonomy metrics in terms of $H^3(M, \mathbb{R})$, as originally observed by Joyce and Hitchin.

Geometry in 6 and 7 dimensions are closely tied by a basic linear algebra fact: the existence of non-degenerate real 3-forms [22]. A single non-degenerate 3-form on a manifold $M$ defines a reduction of the structure group of $M$ to $SL(3, \mathbb{C})$ if dim $M = 6$ — and hence a Calabi-Yau structure —, and a reduction to $G_2$ if dim $M = 7$.

An important part of the classical theory of Calabi-Yau three-folds has evolved recently to the study of $SU(3)$-structures with torsion, and in particular towards the understanding of a complicated system of partial differential equations motivated by physics, known as the Strominger system [34]. The mathematical study of the Strominger system (see [35] for a survey) was initiated by Li and Yau as a natural generalization of the Calabi problem, and in relation to ‘Reid’s fantasy’, on the moduli space of complex 3-folds with trivial canonical bundle and varying topology. Relying on the common geometric features of 6 and 7 dimensions, and the recent progress made in the understanding of the moduli space for the Strominger system [15], it is natural to ask whether there is a similar pattern in 7-dimensions which helps to shed light on the moduli space of $G_2$-structures with torsion.
From the point of view of physics, the Strominger system is a particular instance of a more general system of equations, known as the Killing spinor equations in (heterotic) supergravity. The compactification of the physical theory leads to the study of models of the form $N^k \times M^{10-k}$, where $N^k$ is a $k$-dimensional Lorentzian manifold and $M^{10-k}$ is a Riemannian spin manifold which encodes the extra dimensions of a supersymmetric vacuum. The Killing spinor equations, for a Riemannian metric $g$, a spinor $\Psi$, a function $\phi$ (the dilaton) and a 3-form $H$ (the NS-flux) on $M^{10-k}$, can be written as

\begin{align}
\nabla \Psi &= 0, \\
\left(\frac{d\phi}{4} - \frac{1}{4}H\right) \cdot \Psi &= 0
\end{align}

where $\nabla$ is a $g$-compatible connection with skew-symmetric torsion $H$. Solutions to (1.1) provide rich geometrical structures on $M$. If the torsion $H$ vanishes, the existence of a parallel spinor reduces the holonomy of the Levi-Civita connection on $M$ to $SU(n), Sp(n), G_2$ or $Spin(7)$ according to its dimension. However, the torsion-free condition, often equivalent to the condition $dH = 0$ — the so-called strong solutions —, is very restrictive, as many interesting solutions to the equations arise in manifolds equipped with metric connections with skew-symmetric torsion and holonomy contained in $SU(n), Sp(n), G_2$ or $Spin(7)$.

An interesting relaxation of the notion of strong solution is provided by the Bianchi identity (related to the anomaly cancellation condition in string theory), which requires a correction of $dH$ of the form

\begin{align}
\label{eq:bianchi}
\nabla \Psi &= 0, \\
\left(\frac{d\phi}{4} - \frac{1}{4}H\right) \cdot \Psi &= 0
\end{align}

where $\nabla$ is a $g$-compatible connection with skew-symmetric torsion $H$. Solutions to (1.1) provide rich geometrical structures on $M$. If the torsion $H$ vanishes, the existence of a parallel spinor reduces the holonomy of the Levi-Civita connection on $M$ to $SU(n), Sp(n), G_2$ or $Spin(7)$ according to its dimension. However, the torsion-free condition, often equivalent to the condition $dH = 0$ — the so-called strong solutions —, is very restrictive, as many interesting solutions to the equations arise in manifolds equipped with metric connections with skew-symmetric torsion and holonomy contained in $SU(n), Sp(n), G_2$ or $Spin(7)$.

In a 6-dimensional compact manifold $M$, the combination of the above mentioned equations (1.1), (1.2) and (1.3) leads to the Strominger system. In this paper, we initiate the study of the moduli space of solutions to the Strominger equations in 7-dimensions, that we introduce next.

Consider $M^7$ a compact oriented smooth manifold. Then, the equations (1.1), (1.2) and (1.3) are equivalent to the following system (1.1):

\begin{align}
\label{eq:strominger}
d\omega \wedge \omega &= 0, \\
d\star \omega &= -4d\phi \wedge \star \omega, \\
F_A \wedge \star \omega &= 0, \\
R_{\nabla} \wedge \star \omega &= 0, \\
dH &= \text{tr}(F_A \wedge F_A) - \text{tr}(R_{\nabla} \wedge R_{\nabla}),
\end{align}

where $\omega$ is a positive 3-form that defines a $G_2$ structure on $M$, $-4d\phi$ is the Lee form $\theta_\omega$ of $\omega$, and $H$ is the torsion of the $G_2$-structure, given by

\[ H = -\star (d\omega - \theta_\omega \wedge \omega). \]

The first line of equations in (1.4) characterizes a special type of $G_2$-structures, namely cocalibrated $G_2$ structures of type W3, according to the classification by Fernandez and Gray [8]. Some Riemannian properties of these structures are studied in [11]. The second line of equations in (1.4) is the $G_2$-instanton condition, and has been the subject of important recent progress (see e.g. [6,13,31,33], and the references therein). The last line, the Bianchi identity, is a defining equation for a Courant algebroid, and leads to a new mathematical approach to equations from string theories and supergravity theories using methods from generalized geometry (see e.g. [14,15,18]).

Basic compact solutions to the 7-dimensional Strominger system (1.1) are provided by torsion-free $G_2$-structures. For this, one sets $K = G_2$ and $P_K$ the bundle of orthogonal frames of a $G_2$-holonomy metric, and defines $\nabla = A$ equal to the Levi-Civita connection. An interesting
open question is whether one can deform a torsion-free solution, whereby defining a torsion solution of the 7-dimensional Strominger system (along the lines of the main result in [28]). The first compact solutions with non-zero torsion (and constant dilaton function \( \phi \)) to the 7-dimensional Strominger system (1.4) have been constructed in [10]. Non-compact solutions to (1.4) have been constructed in [9, 19].

We next state our main result, concerning the 7-dimensional Strominger system (1.4). Let \( P_M \) be the bundle of oriented frames over \( M \). The group \( \tilde{G} := \text{Aut}(P_M \times_M P_K) \) acts naturally on the set of parameters \( (\omega, \phi, \nabla, A) \) for the system (1.4), preserving solutions, and thus defining a natural set

\[
\mathcal{M} = \{ (\omega, \phi, \nabla, A) \text{ satisfying (1.4)} \}/\tilde{G}.
\]

In this paper, we give the first steps towards the construction of a natural structure of smooth manifold on \( \mathcal{M} \). For this, using elliptic operator theory we construct an elliptic complex of differential operators whose cohomology provides an ambient space for the (expected) tangent of \( \mathcal{M} \) at \( (\omega, \phi, \nabla, A) \). More precisely, we construct a finite-dimensional space of infinitesimal deformations of a solution \( (\omega, \phi, \nabla, A) \) of (1.4), modulo the action of \( \tilde{G} \).

**Theorem 1.** Let \( M \) be a 7-dimensional compact manifold. Then the moduli space of solutions to the system of equations (1.4) on \( M \) modulo the \( \tilde{G} \)-action is finite-dimensional.

The Bianchi identity motivates the introduction of generalized geometry to study solutions to (1.4). An equivalent formulation for this system is provided in Section 5 by means of Killing spinors on a fixed Courant algebroid. As a corollary, the system (1.4) turns out to be a natural system of equations in generalized geometry. In Section 5 we provide a relation between the different point of views at the level of moduli spaces.

Further motivation for the study of the moduli space of the system (1.4) comes from physics. In this context, a solution of (1.4) describes a half-BPS domain wall solution of heterotic supergravity with flux in four dimensions, that preserve \( N = 1/2 \) supersymmetry [5, 17]. This type of models is particularly appealing in heterotic string theory, since the resulting vacuum breaks supersymmetry in a controlled way, and the low-energy dynamics still allow some of the methods of an \( N = 1 \) four-dimensional effective field theory.

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### 2. Infinitesimal Moduli of the Strominger System in Dimension 7

Let \( M \) be a 7 dimensional oriented compact manifold. Let \( P \) be a principal bundle over \( M \) with structure group \( G \). Fix a non-degenerate biinvariant pairing

\[
c : g \otimes g \to \mathbb{R}
\]
on the Lie algebra of \( G \). In this section we study the moduli space of solutions to the Strominger system in dimension 7:

\[
d \omega \wedge \omega = 0, \quad d^* \omega = -4d\phi \wedge \ast \omega, \\
-d(\ast (d \omega + 4d\phi \wedge \omega)) = c(F_\theta \wedge F_\theta), \\
F_\theta \wedge \ast \omega = 0,
\]

where \( \omega \in \Omega^3 \) defines a \( G_2 \) structure, \( \phi \in C^\infty(M) \), \( \theta \) is a connection in \( P \), and \( F_\theta \) denotes the curvature of \( \theta \).
The first line of equations in (2.1) implies that $-4d\phi$ is necessarily the Lee form of $\omega$ (see [1, Proposition 1]), and

$$H = -*(d\omega + 4d\phi \wedge \omega)$$

is the associated torsion 3-form. Note that the last equation in (1.4), known as the Bianchi identity, imposes the vanishing of the first Pontryagin class of $P$, calculated using the biinvariant pairing $c$ on $g$

$$p_1(P) = 0.$$ 

As mentioned in Section 1, in physics the principal bundle $P$ is taken to be a product $P_M \times_M P_K$, of the bundle of frames $P_M$ of $M$ by a principal bundle $P_K$ over $M$, with compact structure group $K$. Furthermore, $c$ is taken to be of the form

$$c = 2\alpha'(-\text{tr}_k - c_{gl}),$$

where $\alpha'$ is a positive constant, $-\text{tr}_k$ denotes the Killing form on $k$ and $c_{gl}$ is a non-degenerate invariant metric on $\mathfrak{gl}(7, \mathbb{R})$, which extends the non-degenerate Killing form $-\text{tr}$ on $\mathfrak{sl}(7, \mathbb{R}) \subset \mathfrak{gl}(7, \mathbb{R})$. Hence, in this case the aforementioned topological constraint for the Bianchi identity is equivalent to

$$p_1(\mathcal{P}M) = p_1(\mathcal{P}K).$$

Remark 2.1. An additional condition which appears in the physics literature is that $\theta$ is a product connection $\theta = \nabla \times A$, with induced linear connection $\nabla$ on $M$ satisfying $\nabla g = 0$. This is an additional source of complications for the analysis that we ignore in the present paper, and which can be treated with the methods introduced in [15].

2.1. Main Result. The parameter space $\mathcal{P}$ for the equations (2.1) is:

$$\mathcal{P} = \Omega^3_{>0} \times C^\infty(M) \times \mathcal{A}$$

where $\Omega^3_{>0}$ is the space of positive 3-forms on $M$ [22] and $\mathcal{A}$ is the space of connections $\theta$ on $P$. Let $\text{Diff}$ be the group of diffeomorphisms of $M$ and $\mathcal{G}$ the gauge group of $P$. The group of symmetries that we consider is the extension

$$1 \to \mathcal{G} \to \tilde{\mathcal{G}} \to \text{Diff} \to 1$$

or equivalently, $\tilde{\mathcal{G}} = \text{Aut}(P)$. Note that $\tilde{\mathcal{G}}$ acts on $\mathcal{P}$ preserving solutions of (2.1). Set

$$\mathcal{E} : \mathcal{P} \to \Omega^7 \times \Omega^5 \times \Omega^4 \times \Omega^6(\text{ad}P)$$

(2.3)

$$\begin{array}{rcrc}
(\omega, \phi, \theta) & \mapsto & (E_1, E_2, E_3, E_4) & \\
\end{array}$$

where $E_i$ is the $i$th equation of (2.1). Then

$$\mathcal{M} := \mathcal{E}^{-1}(0)/\tilde{\mathcal{G}}$$

is the moduli space of solutions modulo symmetries. Let $x = (\omega, \phi, \theta) \in \mathcal{P}$ be a solution of $\mathcal{E} = 0$. Denote by $L$ the linearization of the equations $\mathcal{E}$ at $x$, and by $P : \text{Lie}\tilde{\mathcal{G}} \to T_x\mathcal{P}$ the infinitesimal action of $\tilde{\mathcal{G}}$ on $\mathcal{P}$ at $x$. As a first approximation to $\mathcal{M}$, we set

$$H^1(KS_7) := \ker \frac{L}{\text{Im}P}.$$

Theorem 2. The space $H^1(KS_7)$ is finite dimensional.

Remark 2.2. Using Theorem 2 and following Kuranishi’s work [27], it is possible to build a local slice to the $\tilde{\mathcal{G}}$-orbits in $\mathcal{P}$ through a point $x \in \mathcal{P}$ solution to (2.1). Then, the local moduli space of solutions around $x$ will be in correspondence with an analytical subset of the slice, quotiented by the action of the isotropy group of $x$. 
2.2. **Strategy of the proof.** The proof of Theorem 2 relies on elliptic operator theory. First, we note that at the level of symbols, the equations (2.1) form a system of uncoupled equations for the parameters \((\omega, \phi)\) on one hand and the parameter \(\theta\) on the other hand. This fact is also true for the infinitesimal action of \(\tilde{G}\): at the level of symbols, this action is the product action of \(\text{Diff}\) on one hand and \(G\) on the other hand. Thus, the proof of Theorem 2 reduces to the separate study of infinitesimal variations of cocalibrated \(G_2\) structures of type \(W_3\) modulo the \(\text{Diff}\)-action and of infinitesimal variations of the \(G_2\) instanton equation for a fixed cocalibrated \(G_2\) structures of type \(W_3\).

3. \(G_2\)-holonomy metrics revisited

Let \(M\) be a 7 dimensional oriented compact manifold. In this section we study cocalibrated \(G_2\) structures of type \(W_3\), in the special case that torsion 3-form \(H\) is closed. In other words, we consider the Strominger system (2.1) in the case of trivial structure group \(G = \{1\}\)

\[
d\omega \wedge \omega = 0, \quad d*\omega + 4d\phi \wedge \omega = 0, \\
d*(d\omega + 4d\phi \wedge \omega) = 0.
\]

(3.1)

This case study will be used in Section 4 to prove the finiteness of the infinitesimal moduli for the Strominger system in seven dimensions (1.4).

Despite the complicated conditions in (3.1), its solutions correspond essentially to \(G_2\)-holonomy metrics.

**Proposition 3.1.** A pair \((\omega, \phi)\) is a solution of (3.1) on a compact 7-manifold \(M\) if and only if \(\phi\) is constant and \(\omega\) is torsion-free, that is, \(d\omega = 0\) and \(d^*\omega = 0\).

This fact is well-known in the physics literature (see e.g. [16]). We give a short proof based on two methods for calculating the scalar curvature of a solution of the system (3.1), one coming from the relation between Killing spinors in 7 dimensions and conformally coclosed \(G_2\)-structures, and the other specifically considering the equations of motion in heterotic string theory implied by (3.1) (see [21]).

**Proof of Proposition 3.1.** From Theorem 1.1 of [21] a solution of the system (3.1) for \(\omega\) a positive 3-form determines a metric \(g\) with Ricci curvature given by, for the above convention on the Lee form \(\theta = -4d\phi\),

\[
\text{Ric}^g_{ij} = \frac{1}{4}H_{mn}H^m_j + 4\nabla_i \nabla_j \phi.
\]

Taking the trace with respect to the metric \(g\) we obtain

\[
S^g = \frac{1}{4}|H|^2 - 4\Delta \phi
\]

where \(\Delta = d^*d\) is the Laplacian with positive spectrum. A different expression is calculated as Equation (1.5) of [11], without the assumption that \(dH = 0\),

\[
S^g = 32|d\phi|^2 - \frac{1}{12}|H|^2 - 12\Delta \phi.
\]

Combining these equations we obtain

\[
32|d\phi|^2 - \frac{1}{3}|H|^2 - 8\Delta \phi = 0.
\]

However, for any \(\alpha \in \mathbb{R}\), and for \(\Delta = d^*d\),

\[
\Delta(e^{\alpha\phi}) = -\alpha^2 e^{\alpha\phi} |d\phi|^2 + \alpha e^{\alpha\phi} \Delta \phi.
\]
The tuples \( \text{Lemma 3.3.} \)

where for \( (3.3) \)

and

\[ \text{Diff} \]

\[ \text{Lie(Diff)} \]

The sequence:

\[ \text{Proposition 3.4.} \]

and so

\[-8\Delta(e^{\alpha\phi}) + 8(4\alpha - \alpha^2)e^{\alpha\phi}|d\phi|^2 - \frac{1}{3}\alpha e^{\alpha\phi}|H|^2 = 0.\]

We can take \( \alpha = 5 \) and integrate over \( M \) to conclude that \( H = 0 \) and \( d\phi = 0 \), which imply that \( d\omega = 0 \) and \( d^*\omega = 0 \) by \( (3.1) \). \( \square \)

Next, we consider the deformation problem for solutions of \( (3.1) \), and characterize the space of infinitesimal deformations of this system. By Proposition 3.1 we recover with different methods classical results by Joyce and Hitchin about the infinitesimal moduli of \( G_2 \) holonomy metrics \( [25, 22] \). As mentioned earlier, our calculations here will be used in Section 4 to prove the finiteness of the infinitesimal moduli for the Strominger system in seven dimensions \( [14] \).

Let \( \Omega^3_{>0} \) denote the space of positive 3-forms on \( M \) \( [22] \). Consider the following parameter space \( \mathcal{P}_M \) for the deformation problem for the equations \( (2.1) \)

\[ \mathcal{P}_M = \Omega^3_{>0} \times C^\infty(M). \]

Let \( \text{Diff} \) be the group of diffeomorphisms of \( M \). Then \( \text{Diff} \) acts on \( \mathcal{P}_M \) on the left by push-forward preserving solutions of \( (3.1) \).

Let \( (\omega, \phi) \) be a solution of \( (3.1) \). Let \( L_M \) be the linearisation of the operator corresponding to the left hand side of equations \( (3.1) \) at the point \( (\omega, \phi) \), and let \( P_M \) be the infinitesimal action of \( \text{Diff} \) at this point. Using Proposition 3.1 we have explicit formulae:

\[ \begin{align*}
\text{P}_M : & \quad \Omega^0(T) \rightarrow \Omega^3 \times C^\infty(M) \\
& V \mapsto (dV \omega, 0)
\end{align*} \]

and

\[ \begin{align*}
\text{L}_M : & \quad \Omega^3 \times C^\infty(M) \rightarrow \Omega^7 \times \Omega^5 \times \Omega^4 \\
& (\dot{\omega}, \dot{\phi}) \mapsto \begin{cases} 
\dot{d}\omega \wedge \omega \\
\dot{d} \ast J\dot{\omega} + 4\dot{d}\phi \wedge *\omega \\
d\ast(d\omega + 4\dot{d}\phi \wedge \omega)
\end{cases}
\end{align*} \]

where for \( l = 3, 4 \), \( J : \Omega^l \rightarrow \Omega^l \) is defined by

\[ (3.4) \quad J(\xi) = \frac{4}{3} \pi_1(\xi) + \pi_7(\xi) - \pi_{27}(\xi) \]

and \( \pi_k \) denotes the projection onto the \( k \) dimensional component of \( \Omega^l \) (see e.g. \( [25] \) for the variation of \( *\omega \)).

Note that \( L_M \) is a multi-degree differential operator \( [7] \). Given \( v \in T^* \setminus M \), we have the following:

**Lemma 3.2.** The tuples \( t_P = (1, 1) \) and \( s_P = (0, 0) \) form a system of orders for \( P_M \). The \( (t_P, s_P) \) principal symbol of the linearisation of the infinitesimal action of \( \text{Diff} \) is

\[ \sigma_{P_M}(v)(V) = (v \wedge t_P \omega, 0) \]

**Lemma 3.3.** The tuples \( t_L = (2, 2) \) and \( s_L = (1, 1, 0) \) form a system of orders for \( L_M \). The \( (t_L, s_L) \) principal symbol of \( L_M \) is

\[ \sigma_{L_M}(v)(\dot{\omega}, \dot{\phi}) = (v \wedge \dot{\omega} \wedge \omega, v \wedge (\ast J\dot{\omega} + \dot{\phi} \wedge *\omega), v \wedge (v \wedge (\dot{\omega} + \dot{\phi} \wedge \omega))) \]

Set \( \mathcal{R}_M = \Omega^7 \times \Omega^5 \times \Omega^4 \). We can now prove the main result of this section.

**Proposition 3.4.** The sequence:

\[ \text{Lie(Diff)} \xrightarrow{P_M} T_{(\omega, 0)} \mathcal{P}_M \xrightarrow{L_M} \mathcal{R}_M \]
is elliptic in the middle. Furthermore,

\[(3.8) \quad \ker L_M \simeq \mathcal{H}^3(M, \mathbb{R}) \times \mathbb{R} \]

where \( \mathcal{H}^i(M, \mathbb{R}) \) is the space of harmonic \( i \)-forms on \((M, g)\).

**Proof.** We start proving the isomorphism \((3.8)\). Let \((\hat{\omega}, \hat{\phi}) \in T_{(\omega, \phi)}P_M\) be in the kernel of \(L_M\). From

\[d((d\hat{\omega} + 4d\hat{\phi} \wedge \omega)) = 0 \quad \text{and} \quad d\omega = 0\]

we deduce

\[d^*d(\hat{\omega} + 4\hat{\phi}\omega) = 0,\]

and thus \(\hat{\omega} + 4\hat{\phi}\omega\) is closed. By Hodge decomposition, there exists a two form \(\beta \in \Omega^2\) and a harmonic 3-form \(h\) such that

\[(3.9) \quad \hat{\omega} + 4\hat{\phi}\omega = h + d\beta.\]

By Proposition \(3.1\), \(\omega\) is a \(G_2\) metric, and therefore the decomposition of forms given by the \(G_2\) representation commutes with the Hodge Laplacian. Thus \(*\) and \(J\) preserve harmonic forms. Recall that (see e.g. [1])

\[\Omega^2 = \Omega_7^2 \oplus \Omega_{14}^2\]

and decompose \(\beta\) into 7th and 14th components, \(\beta = \beta_7 + \beta_{14}\). Equation

\[d\ast J\hat{\omega} + 4d\hat{\phi} \wedge \ast\omega = 0\]

then implies

\[(3.10) \quad d\ast Jd\beta_7 + d\ast Jd\beta_{14} - \frac{4}{3}d\hat{\phi} \wedge \ast\omega = 0.\]

On the other hand, using \(\beta_7 = \ast(\alpha \wedge \ast\omega)\) for some one form \(\alpha\), and Bryant’s work [1, Prop. 3], we obtain

\[d\ast Jd\beta_7 = 0 \quad \text{and} \quad \pi_7(d\ast Jd\beta_{14}) = 0.\]

Thus \((3.10)\) is equivalent to

\[\pi_{14}(d\ast Jd\beta_{14}) - \frac{4}{3}d\hat{\phi} \wedge \ast\omega = 0.\]

Using type decomposition on \(\Omega^5\), and [1, Prop. 3], we deduce:

\[(3.11) \quad \pi_{14}(d\ast Jd\beta_{14}) = \Delta\beta_{14} - \frac{3}{2}d_7^2d_{14}d_{14}\beta_{14} = 0\]

and

\[d\hat{\phi} = 0.\]

Thus \(\hat{\phi}\) is constant. Moreover, as \(\beta_{14} \in \Omega_7^2\) is equivalent to \(\beta_{14} \wedge \omega = -\ast \beta_{14}\), we have

\[d_7^2d_{14}d_{14}\beta_{14} = \pi_{14}(d\ast d(\beta_{14} \wedge \omega)) = -\pi_{14}(dd^* \beta_{14}).\]

Using the fact that

\[\pi_{14} = \frac{2}{3}Id - \frac{1}{3} \ast(\cdot \wedge \omega)\]

equation \((3.11)\) becomes:

\[\Delta\beta_{14} + dd^* \beta_{14} + \frac{1}{2}d^* (d^*(\beta_{14} \wedge \omega)) = 0.\]

Using Hodge decomposition into orthogonal components, we deduce that \(d^* \beta_{14} = 0\) and \(d\beta_{14} = 0\). Returning to \(\hat{\omega}\), and by definition of \(\Omega_7^2\), there exists \(V \in \Omega^6(T)\) such that

\[\hat{\omega} + 4\hat{\phi}\omega = h + d\beta_7 = h + dV\omega.\]
Thus we can define a map
\[
\ker L_M \rightarrow \mathcal{H}^3 \times \mathbb{R}
\]
(\dot{\omega}, \phi) \mapsto (h - 4\phi \omega, \phi).

This map is well defined, surjective, and has kernel the image of \(P_M\), thus proving (3.8).

Using now the same argument at the level of symbols, combined with Lemma 3.2 and Lemma 3.3 gives a proof of the ellipticity of (3.7). More details are given in the proof of Proposition 4.1.

4. Proof of Theorem 2

Let \(x = (\omega, \phi, \theta) \in \mathcal{P}\). Then the tangent of \(\mathcal{P}\) at \(x\) is:
\[
T_x \mathcal{P} = \Omega^3 \times C^\infty(M) \times \Omega^1(\text{ad}P).
\]

Set \(\mathcal{R} := \Omega^7 \times \Omega^5 \times \Omega^4 \times \Omega^6(\text{ad}P)\). To prove Theorem 2 it is enough to show that the sequence
\[
\text{Lie}(\tilde{G}) \xrightarrow{P} T_x \mathcal{P} \xrightarrow{L} \mathcal{R}
\]
is elliptic at a point \(x \in \mathcal{P}\) which is a solution to \(\mathcal{E}(x) = 0\).

4.1. Linearisation and symbols. The infinitesimal action \(P\) and the linearisation \(L\) of \(\mathcal{E}\) are given by:
\[
P : \Omega^0(T) \times \Omega^0(\text{ad}P) \rightarrow \Omega^3 \times C^\infty(M) \times \Omega^1(\text{ad}P)
\]
(\(V, r\) \(\mapsto \) (\(L_V \omega, L_V \phi, d^V r + t_V F_\theta\))

and
\[
L : \Omega^3 \times C^\infty(M) \times \Omega^1(\text{ad}P) \rightarrow \Omega^7 \times \Omega^5 \times \Omega^4 \times \Omega^6(\text{ad}P)
\]
(\(\dot{\omega}, \dot{\phi}, \dot{\theta}\) \(\mapsto \)
\[
\begin{align*}
L_1 &= d\dot{\omega} \wedge \omega + d\omega \wedge \dot{\omega} \\
L_2 &= d \ast J \dot{\omega} + 4d\dot{\phi} \wedge \ast \omega + 4d\phi \wedge \ast J \dot{\omega} \\
L_3 &= -d(*(d\dot{\omega} + 4d\dot{\phi} \wedge \omega)) - d(*((d\omega + 4d\phi \wedge \omega))) \\
&- d(*((d\phi \wedge \omega))) - 2d(e(\dot{\theta}, F_\theta)) \\
L_4 &= d^\theta \dot{\theta} \wedge \ast \omega + F_\theta \wedge \ast J \dot{\omega}
\end{align*}
\]
where \(d^\theta\) denotes the extension of the covariant derivative associated to \(\theta\) on \(\Omega^*(\text{ad}P)\); and recall \(J\) is defined by (3.3). We will use the theory of linear multi-degree elliptic differential operators [20, 30]. At the level of symbols, only the highest order operators will appear. Thus, the symbols of \(L\) and \(P\) are the same as the symbols of \(L_h\) and \(P_h\) defined by:
\[
P_h : \Omega^0(T) \times \Omega^0(\text{ad}P) \rightarrow \Omega^3 \times C^\infty(M) \times \Omega^1(\text{ad}P)
\]
(\(V, r\) \(\mapsto \) (\(dV \omega, t_V d\phi, d^\theta r\))

and
\[
L_h : \Omega^3 \times C^\infty(M) \times \Omega^1(\text{ad}P) \rightarrow \Omega^7 \times \Omega^5 \times \Omega^4 \times \Omega^6(\text{ad}P)
\]
(\(\dot{\omega}, \dot{\phi}, \dot{\theta}\) \(\mapsto \)
\[
\begin{align*}
d\dot{\omega} \wedge \omega & \& \\
&+ d \ast J \dot{\omega} + 4d\dot{\phi} \wedge \ast \omega \\
&- d(*((d\phi \wedge \omega))) \\
&= d^\theta \dot{\theta} \wedge \ast \omega
\end{align*}
\]
Note that for the later equations, the parameters \((\omega, \phi)\) and \(\theta\) are uncoupled. We next construct two elliptic complexes
\[
\text{Lie}(\text{Diff}) \xrightarrow{P_M} T_{(\omega, \phi)} \mathcal{P} \xrightarrow{L_M} \mathcal{R}_M
\]
(4.7) \[
\text{Lie}(G) \xrightarrow{p_7} T_\theta A \xrightarrow{\text{Lie}} R_P
\]
so that \( L_h = L_M \times L_P \) and \( P_h = P_M \times P_P \) (and hence at the level of symbols \( \sigma_{L_h} = \sigma_{L_M} \times \sigma_{L_P} \)
and \( \sigma_{P_h} = \sigma_{P_M} \times \sigma_{P_P} \). From our previous considerations, Theorem 2 follows from the ellipticity of (4.6) and (4.7), that we prove respectively in Proposition 4.1 and Proposition 4.2.

4.2. Ellipticity of (4.6). The first complex (4.6) is closely related to the linearisation of the system of equations (3.1). However we do not assume that \((\omega, \phi)\) is a solution to (3.1). We set \( P_M \) and \( L_M \) to be the operators defined in (4.2) and (4.3), with a slight modification: \( P_M(V) = (dv \omega, \iota V d\phi) \), as \( \phi \) might not be constant.

**Proposition 4.1.** Let \((\omega, \phi) \in P_M\). Then the sequence

(4.8) \[
\text{Lie}(\text{Diff}) \xrightarrow{P_M} T_{\omega, \phi} P_M \xrightarrow{\text{Lie}} R_M
\]
is elliptic in the middle.

**Proof.** Let \( v \in T^* \setminus M \). Note that for the same choice of tuples as in Lemmas 3.2 and 3.3 the principal symbols for \( P_M \) and \( L_M \) are the one given in equations (3.5) and (3.6). Assume that \( \sigma_{L_M}(v)(\hat{\omega}, \hat{\phi}) = (0, 0, 0, 0) \).

We want to show that \( \dot{\phi} = 0 \) and \( \hat{\omega} = v \wedge \iota_V \omega \) for some \( V \in T \), or equivalently \( \hat{\omega} = v \wedge \beta \) for some \( \beta \in \Lambda_7^2 \). The proof follows the one of Proposition 3.4 at the symbol level. From

\[
v \wedge \ast(v \wedge (\hat{\omega} + \hat{\phi} \omega)) = 0
\]
we deduce

\[
0 = \langle v \wedge \ast(v \wedge (\hat{\omega} + \hat{\phi} \omega)), \ast(\hat{\omega} + \hat{\phi} \omega) \rangle = \langle v \wedge (\hat{\omega} + \hat{\phi} \omega), v \wedge (\hat{\omega} + \hat{\phi} \omega) \rangle
\]
and thus

\[
v \wedge (\hat{\omega} + \hat{\phi} \omega) = 0.
\]

There exists \( \beta_7 \in \Lambda_7^2 \) and \( \beta_{14} \in \Lambda_{14}^2 \) such that \( \hat{\omega} + \hat{\phi} \omega = v \wedge \beta_7 + v \wedge \beta_{14} \).

Together with the equation

\[
v \wedge (\ast J \hat{\omega} + \hat{\phi} \ast \omega) = 0
\]
we obtain

(4.9) \[
v \wedge \ast(J(v \wedge \beta_7)) + v \wedge \ast(J(v \wedge \beta_{14})) = \frac{4}{3} \phi v \wedge \ast \omega.
\]

Now, note that if we don’t assume the 3-form \( \omega \) to be torsion free, the formulas in [1, Proposition 3] are only modified by lower order terms. As noticed in the proof of Proposition 3.4 these formulas imply that for 2-forms \( \beta_7 \in \Omega_7^2 \) and \( \beta_{14} \in \Omega_{14}^2 \), if \( \omega \) is torsion-free, then

\[
d \ast J d \beta_7 = 0 \quad \text{and} \quad \pi_7(d \ast J d \beta_{14}) = 0.
\]

In general, these operators might not vanish. However, they are operators of degree less or equal to one. This implies that the quadratic part of the symbol of \( d \ast J d \) restricted to \( \Omega_7^2 \) vanishes, as well as the quadratic part of the symbol of \( \pi_7(d \ast J d) \) restricted to \( \Omega_{14}^2 \), even if \( \omega \) has torsion. Thus, \( v \wedge J(v \wedge \beta_7) = 0 \) and \( \pi_7(v \wedge \ast J(v \wedge \beta_{14})) = 0 \).

Formula (4.9) becomes

\[
\pi_{14}(v \wedge \ast(J(v \wedge \beta_{14}))) = \frac{4}{3} \phi v \wedge \ast \omega.
\]

As \( v \wedge \omega \) is in \( \Lambda_7^2 \), we have \( \hat{\phi} = 0 \) and thus \( v \wedge \ast(J(v \wedge \beta_{14})) = 0 \). Then, as in the proof of Proposition 3.4 we obtain

\[
\sigma_\Delta(v) \beta_{14} + \sigma_{dd^*}(v) \beta_{14} + \frac{1}{2} \sigma_1(v) \beta_{14} = 0,
\]
where \( l = d^*(d^* \cdot \omega) \). We conclude similarly that \( v \wedge \beta_{14} = 0 \), which ends the proof. \( \square \)

4.3. \( G_2 \)-instantons. The second complex \([14, 27]\) corresponds to a system parameterizing \( G_2 \) instantons on \( P \) modulo the gauge group \( G \), for a cocalibrated \( G_2 \) structures of type \( W3 \). For the proof, we can indeed assume that \( \omega \) is an arbitrary \( G_2 \) structure. The equation \( E_P \) we consider on \( \mathcal{A} \) is:

\[
(F_\theta \wedge \ast \omega = 0)
\]

Set as before \( L_P \) the linearisation of \( E_P \) at \( \theta \in \mathcal{A} \), \( P_P \) the infinitesimal action of \( G \) on \( \mathcal{A} \) at \( \theta \) and

\[
\mathcal{R}_P = \Omega^6(\text{ad}P).
\]

**Proposition 4.2.** Assume \((\theta)\) is a solution to \((4.10)\). Then the sequence

\[
\text{Lie}(\mathcal{G}) \xrightarrow{\mathcal{P}_\omega} T_\theta \mathcal{A} \xrightarrow{L_P} \mathcal{R}_P
\]

is elliptic at the middle term.

This is to say that the image of the symbol \( \sigma(P_P) \), which equals \((\ker \sigma(P_P))^{\perp}\), is equal to the kernel of \( \sigma(L_P) \). This is equivalent to the fact that the symbol of the differential operator \( D_\theta = L_P + P_P^* \) is injective.

**Lemma 4.3.** The symbol of \( D_\theta = L_P + P_P^* \) given by

\[
\sigma(D_\theta)(\xi) a = (\ast \omega \wedge (\xi \wedge a), \iota_{\xi} \iota_{\omega} a)
\]

is injective.

**Proof.** \( D_\theta \) is given by

\[
D_\theta a = (\ast \omega \wedge d^\theta a, (d^\theta)^* a)
\]

so the expression for the symbol is clear. To see that the map is injective we neglect the coefficient bundle \( \text{ad}P \) and observe that if \( \iota_{\xi} \iota_{\omega} a = 0 \) then \( \xi, a \in \Lambda^1 \) are orthogonal. \( G_2 \) acts transitively on the set of ordered pairs of orthogonal vectors in \( \mathbb{R}^7 \) so we suppose that \( \xi = e^1 \) and \( a = \lambda e^2 \) while \( \omega = e^{123} + e^1 \wedge (e^{45} - e^{67}) + e^2 \wedge (e^{46} - e^{75}) + e^3 \wedge (e^{47} - e^{56}) \). Then \( \ast \omega \wedge \xi \wedge a = 0 \) if and only if \( \xi \wedge a \in \Lambda^3_{14} \). The \( \Lambda^3 \) component of \( e^1 \wedge \lambda \epsilon^2 \) is given by \( \pi_7(e^1 \wedge \lambda \epsilon^2) = \lambda/3(e^1 \wedge \epsilon^2 - e^1 \wedge \epsilon^7 + e^5 \wedge \epsilon^6) \) which is non-zero whenever \( \lambda \neq 0 \). \( \square \)

**Remark 4.4.** The usual method for studying the moduli problem for \( G_2 \)-instantons is to consider the \( G_2 \)-monopole equation \( F_\theta \wedge \ast \omega + \ast d^\theta \Phi = 0 \) for \( \theta \) a connection and \( \Phi \in \Omega^1(\text{ad}P) \) a Higgs-type field. The monopole equation arises as the dimensional reduction of the \( \text{Spin}_7 \)-instanton equation and so defines an elliptic system. As observed by Walpuski [36], the Bianchi identity for \( \theta \) implies that on a closed manifold \((M, \omega)\) with \( G_2 \)-structure satisfying \( d \ast \omega = 0 \) a \( G_2 \)-monopole must satisfy \( F_\theta \wedge \ast \omega = 0 \).

Under the condition that \( d \ast \omega = -4d\phi \wedge \ast \omega \), we can define a new positive 3-form \( \tilde{\omega} = e^{3\phi} \omega \). Then \( \omega \)-instantons coincide exactly with \( \tilde{\omega} \)-instantons. The form \( \tilde{\omega} \) also satisfies \( d \tilde{\omega} = 0 \) so we can conclude that \( \omega \)-instantons coincide with \( \tilde{\omega} \)-monopoles. The latter, modulo gauge, is a well-determined elliptic condition. We avoid this framework however, to avoid introducing the extraneous Higgs term.
Killing spinors in generalized geometry were introduced in [15] and studied on 6 dimensional manifolds. In this section we show that on a 7 dimensional manifold the Killing spinor equations on a suitable transitive Courant algebroid are equivalent to the system (5.1). Furthermore, following [15] we introduce the space of infinitesimal variations of a solution of the Killing spinor equations modulo inner symmetries of the Courant algebroid. We relate this space to $H^1(KS_7)$.

5.1. Courant algebroids, generalized metrics and Killing spinors.

Definition 5.1. A Courant algebroid $(E, ⟨·, ·⟩, [·, ·], π)$ over a manifold $M$ consists of a vector bundle $E → M$ together with a non-degenerate symmetric bilinear form $⟨·, ·⟩$ on $E$, a (Dorfman) bracket $[·, ·]$ on the sections $Ω^0(E)$, and a bundle map $π : E → TM$ such that the following properties are satisfied, for $e, e′, e'' ∈ Ω^0(E)$ and $ϕ ∈ C^∞(M)$:

\begin{align*}
(D1): \quad & [e, [e′, e'']] = [[e, e′], e''] + [e′, [e, e'']], \\
(D2): \quad & π([e, e′]) = [π(e), π(e′)], \\
(D3): \quad & π(ϕe′) = π(e)(ϕ)e′ + ϕ(e, e′), \\
(D4): \quad & π(e)(e′, e'′) = ⟨[e, e′], e''⟩ + ⟨e′, [e, e'′]⟩, \\
(D5): \quad & [e′, e] + [e′, e′] = 2π^*d(e, e′).
\end{align*}

We are interested in a particular class of Courant algebroids, that can be constructed as follows. Let $M$ be a smooth manifold of dimension $n$. Let $G$ be a Lie group and $P$ be a principal $G$-bundle over $M$. As in Section 2 we fix a non-degenerate biinvariant pairing $c$ on the Lie algebra $\mathfrak{g}$ of $G$. Consider the vector bundle

\[ E = T ⊕ adP ⊕ T^* \]

endowed with the symmetric pairing

\[ ⟨X + r + ξ, Y + t + η⟩ = \frac{1}{2}(η(X) + ξ(Y)) + c(r, t), \]

and the canonical projection

\[ π : E → T. \]

Given 3-form $H_0$ on $M$ and a connection $θ_0$ on $P$ with curvature $F_0$, we can endow $Ω^0(E)$ with a bracket

\[ [X + r + ξ, Y + t + η] = [X, Y] + L_Xη - i_Ydξ + i_ξdY H_0 \]

\[ - [r, t] - F_0(X, Y) + d_Xt - d_Yr \]

\[ + 2c(d^θ_0 r, t) + 2c(F_0(X, ·), t) - 2c(F_0(Y, ·), r). \]

Following [3], it can be checked that the tuple $(E, ⟨·, ·⟩, [·, ·], π)$ satisfies the axioms of Definition 5.1 if and only if the following Bianchi identity is satisfied

\[ dH_0 = c(F_0 ∧ F_0). \]

Let $(t, s)$ be the signature of the pairing on the Courant algebroid $E$. A generalized metric of signature $(p, q)$ is given by a subbundle

\[ V_+ ⊂ E \]

such that the restriction of the metric on $E$ to $V_+$ is a non-degenerate metric of signature $(p, q)$. We denote by $V_-$ the orthogonal complement of $V_+$ on $E$.

Definition 5.2 ([14]). A metric $V_+$ of arbitrary signature is admissible if

\[ V_+ ∩ T^* = \{0\} \quad \text{and} \quad \text{rank } V_+ = \text{rank } E - \dim M. \]
A generalized connection $D$ (or simply, a connection) on $E$ is a first order differential operator
\[ D : \Omega^0(E) \to \Omega^0(E^* \otimes E) \]
satisfying the Leibniz rule $D(e\phi') = eD\phi + \pi(e)(\phi)\phi'$, for $e, e' \in \Omega^0(E)$ and $\phi \in C^\infty(M)$ and compatible with the inner product on $E$, that is, satisfying
\[ \pi(e)(\langle e', e'' \rangle) = \langle D(e')e'' + (e', D(e'') \rangle). \]

Given an admissible metric and a smooth function $\phi \in C^\infty(M)$, one can associate a torsion-free, compatible connection $D^\phi$ \cite{14, 15}, constructed from the Gualtieri-Bismut connection \cite{13}. The connection $D^\phi$ induces differential operators
\[ D^\phi_+ : V_- \to V_- \otimes (V_+)^*, \]

From $D^\phi_+ \text{ and } D^\phi_-$ we get differential operators on spinors
\[ D^\phi_+ : S_+(V_-) \to S_+(V_-) \otimes (V_+)^* \]

and the associated Dirac operator
\[ \Psi^\phi : S_+(V_-) \to S_-(V_-). \]

**Definition 5.3.** Given a generalized metric $V_+$ and $\phi \in C^\infty(M)$, the Killing spinor equations for a spinor $\eta \in S_+(V_-)$ are
\[ D^\phi_+ \eta = 0, \]
\[ \Psi^\phi \eta = 0. \]

Specifying now the previous construction to dimension $n = 7$, we obtain a characterization of the Strominger system \cite{2, 11}.

**Theorem 3.** Let $M$ be 7-dimensional oriented spin manifold, endowed with the Courant algebroid $(E, \langle \cdot, \cdot \rangle, [\cdot, \cdot], \pi)$ determined by a pair $(H_0, \theta_0)$ satisfying \cite{5, 3}. A solution $(V_+, \phi, \eta)$ of the Killing spinor equations \cite{4, 5} is equivalent to a tuple $(\omega, \phi, \theta)$ satisfying the Strominger system \cite{2, 11}, where $\omega$ is a $G_2$-structure on $M$, with torsion
\[ -*(d\omega - \theta_\omega \wedge \omega) = H_0 + db + 2c(a, F_0) + c(a, d^\phi a) + \frac{1}{3}c(a, [a, a]), \]
for $a := \theta - \theta_0 \in \Omega^1(adP)$ and a suitable 2-form $b$ on $M$.

**Proof.** The result follows combining the proof of \cite{11, Lemma 5.1} (which can be easily generalized to arbitrary dimension of $M$) with the proof of \cite{11, Theorem 1.2}. For the convenience of the reader, we give a sketch of the argument. Following \cite{11}, an admissible metric is equivalent to a metric $g$ on $M$, together with an isotropic splitting of the anchor map $\pi : E \to T$, which determines a 2-form $b \in \Omega^2$ and a 1-form $a \in \Omega^1(adP)$ with values in $adP$. Define a 3-form $H$ and a connection $\theta$ on $P$ by the formulae
\[ H = H_0 + db + 2c(a, F_0) + c(a, d^\phi a) + \frac{1}{3}c(a, [a, a]), \]
\[ \theta = \theta_0 + a, \]
such that the Bianchi identity $dH = c(F_0 \wedge F_0)$ is satisfied. Then, a solution to the Killing spinor equations in generalized geometry \cite{5, 4} gives a tuple $(g, \theta, H, \eta, \phi)$, where $\eta$ is a spinor with respect to $g$ and $\phi$ is the dilaton function. Arguing as in the proof of \cite{11, Lemma 5.1}, it follows that tuple satisfies the Killing spinors equations \cite{11}, jointly with the instanton condition \cite{15} and the Bianchi identity \cite{5, 3}. Finally, following the proof of \cite{11, Theorem 1.2}, $(\omega, \eta)$ determines a $G_2$ structure $\omega$ with torsion $H$, and $(\omega, \phi, \theta)$ is a solution to \cite{2, 11}. Conversely, given a solution $(\omega, \phi, \theta)$ of \cite{2, 11} satisfying \cite{5, 3} for a 2-form $b$ on $M$, we can associate a generalized metric $V_+$ on the fixed Courant algebroid $E$, determined by $g = g_\omega$, $a = \theta - \theta_0$ and $b$. Then, considering the spinor $\eta$ determined by $\omega$, it follows that $(V_+, \phi, \eta)$ provides a solution of \cite{5, 4}. \hfill \square
Note that the condition \((5.5)\) in Theorem 3 can be expressed more invariantly by the following equivalent condition in the equivariant cohomology of \(P\):
\[
[p^*H_0 - CS(\theta_0)] = [p^*H - CS(\theta)] \in H^3(P, \mathbb{R})^G,
\]
where \(p: P \to M\) is the canonical projection and \(CS(\theta) \in \Omega^3(\text{ad}P)^G\) is the \((G\text{-invariant})\) Chern-Simons 3-form of the connection \(\theta\). The class \([p^*H_0 - CS(\theta_0)] \in H^3(P, \mathbb{R})^G\) can be regarded as the isomorphism class of a \(G\text{-equivariant} \text{ (exact) Courant algebroid} \) on the total space of \(P\), from which the \((\text{transitive})\) Courant algebroid \(E\) is obtained by reduction \([14]\).

5.2. Infinitesimal moduli for the Killing spinor equations in dimension 7. We now describe the space of infinitesimal solutions to the Killing spinors equations \((5.4)\) on a fixed Courant algebroid modulo inner symmetries of the algebroid.

Let \((E, \langle \cdot, \cdot \rangle, [\cdot, \cdot], \pi)\) be the Courant algebroid determined by a solution \(x = (\omega, \phi, \theta)\) of \((5.4)\), as in Section 5.1 (setting \(H_0\) to be the torsion of \(\omega\) and \(\theta_0 = \theta\)). The group of symmetries \(\text{Aut}(E)\) of this transitive Courant algebroid has been described in \([15, \text{Proposition 4.3}]\), following \([32]\). In particular, the space \(\Omega^3(E)\), sections of the bundle \(E\), naturally embeds into the Lie algebra of inner symmetries of the Courant algebroid via the map
\[
\Omega^0(E) \to \text{Lie}(\text{Aut}(E)) \quad e \mapsto [e, \cdot].
\]
More explicitly, this is described by
\[
\Omega^0(E) = \Omega^0(T) \times \Omega^0(\text{ad}P) \times \Omega^1 \to \text{Lie}(\text{Aut}(E)) \quad (V, r, \xi) \mapsto (V, r, -d\xi - \iota_V H + 2c(r, F)),
\]
\((5.6)\)

Note also
\[
\Omega^0(E) = \text{Lie}(\tilde{G}) \times \Omega^1.
\]

An admissible generalized metric is equivalent to a metric \(g\) on \(M\) together with an isotropic splitting of \(E \to T\). Such isotropic splitting are locally modelled on \(\Omega^1(\text{ad}P) \times \Omega^2\). Thus the space of infinitesimal variations of generalized metrics is modeled on \(S^2T^* \times \Omega^1(\text{ad}P) \times \Omega^2\). We add the dilaton \(\phi\) as a parameter defining the connection \(D^\phi\) of a generalized metric. We also consider \(G_2\) metrics \(g\) defined by a 3-form \(\omega\). Then the tangent space to the space of parameters for the Killing spinor equations \((5.4)\) in dimension 7 is:
\[
T_x \tilde{P} := \Omega^3 \times C^\infty(M) \times \Omega^1(\text{ad}P) \times \Omega^2.
\]

Note that
\[
T_x P = T_x P \times \Omega^2.
\]

We can consider the infinitesimal action \(P\) of \(\Omega^0(E)\) on the space of parameters:
\[
P : \quad \Omega^0(E) \to T_x \tilde{P} \quad (V, r, \xi) \mapsto (P(V, r), d\xi + iv(H) - 2c(r, F)),
\]
\((5.7)\)

where we recall the operator \(P\) is defined in Section 4.1. Note that from the naturality of the equations \((5.4)\), solutions to these equations form orbits under the action of \(\text{Aut}(E)\).

We set now \(L\) to be the linearisation of the Killing spinors equations \((5.4)\) in dimension 7 on the fixed Courant algebroid \((E, \langle \cdot, \cdot \rangle, [\cdot, \cdot], \pi)\). Fixing the Courant algebroid structure amounts to allow variations of the 3-form \(H\) by exact terms only, and thus replace the Bianchi identity equation by a primitive equation:

\[
L : \quad \Omega^3 \times C^\infty(M) \times \Omega^1(\text{ad}P) \times \Omega^2 \to \Omega^5 \times \Omega^3 \times \Omega^6(\text{ad}P)
\]
\((5.8)\)

\[
(\dot{\omega}, \dot{\phi}, \dot{\theta}, b) \quad \mapsto
\begin{cases}
\dot{L}_1 = d\dot{\omega} \wedge \omega + d\omega \wedge \dot{\omega} \\
\dot{L}_2 = d \star J^\omega + 4d\dot{\phi} \wedge \ast \omega + 4d\phi \wedge \ast J \dot{\omega} \\
\dot{L}_3 = \dot{T} - 2(c(\dot{\theta}, F_\theta)) - db \\
\dot{L}_4 = d^b \dot{\theta} \wedge \ast \omega + F_\theta \wedge \ast J \dot{\omega}
\end{cases}
\]

where \( \dot{T} \) stands for the infinitesimal variation of the torsion term \(- (d\omega + 4d\phi \wedge \omega)\) with respect to the variation \((\dot{\omega}, \dot{\phi})\). Then define a sequence of differential operators
\[
(5.9) \quad \Omega^0(E) \xrightarrow{\mathcal{P}} T_x \hat{\mathcal{P}} \xrightarrow{\mathcal{L}} \Omega^7 \times \Omega^5 \times \Omega^3 \times \Omega^6 (\text{ad} \mathcal{P})
\]
A straightforward calculation shows that (5.9) is a complex of differential operators. Then, relying on Theorem 2, one easily shows:

**Theorem 4.** The complex (5.9) is elliptic, and thus the space
\[
H^1(\hat{KS}_7) := \ker \mathcal{L} / \text{Im} \mathcal{P}
\]
is finite dimensional.

To relate \(H^1(\hat{KS}_7)\) to the space \(H^1(KS_7)\) from Theorem 2, we follow [15] and introduce the flux map:
\[
(5.10) \quad \delta : H^1(KS_7) \to H^3(M) \quad \left[ (\dot{\omega}, \dot{\phi}, \dot{\omega}) \right] \mapsto [\dot{T} - 2\epsilon(\dot{\theta}, F)],
\]
motivated by the flux quantization condition in Heterotic string theory. We note that this map carries information on the infinitesimal variation of the Courant algebroid structure. The kernel \(\ker \delta\) of the flux map parametrizes infinitesimal variations of solutions to (5.4) modulo (a subgroup of) symmetries \(\text{Aut}(E)\). Then, following [15], we obtain:

**Proposition 5.4.** The group \(H^1(\hat{KS}_7)\) is given by an extension:
\[
(5.11) \quad 0 \to H^2(M) \to H^1(\hat{KS}_7) \to \ker \delta \to 0
\]
The space \(H^1(\hat{KS}_7)\) is closer to the physical moduli space of 3-dimensional compactifications of the heterotic string (preserving \(N = 1/2\) supersymmetry), as its elements are compatible with the flux quantization principle.

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