ENTROPY FOR DIQUARKS IN EXOTIC QUARK STATES

David E. Miller$^{1,2,3}$

$^1$ Division of Theoretical Physics, Rudjer Bošković Institute, HR-10002 Zagreb, Croatia
$^2$Fakultät für Physik, Universität Bielefeld, Postfach 100131, D-33501 Bielefeld, Germany
$^3$ Department of Physics, Pennsylvania State University, Hazleton Campus, Hazleton, Pennsylvania 18201, USA

Abstract

We discuss the quantum state structure on the basis of $SU(3)_c$ for some known exotic quark systems using a model which describes these particular states as highly correlated diquarks and antidiquarks. We are then able to calculate for a single colored diquark a finite von Neumann entropy from the quantum reduced density matrix, from which we explicitly evaluate the likelihood of certain arrangements of the quark flavors in a given diquark. These results can be related to some recently experimentally found pentaquark systems as well as a possible model for the scalar mesons.

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$^*$email: dmiller@physik.uni-bielefeld.de
$^\dagger$permanent address, email: om0@psu.edu
1 Introduction

The recent experimental discovery of the pentaquark states [1,2] has brought about new intensity to the search for other exotic quark states beyond the usual meson and baryon singlet quark structures. As well as this experimental work there has been recent numerical work on particular cases involving the diquark structure [3] which provides some information on the nature of the interaction between the quarks within a diquark. In the recent past it has proven itself quite useful to have such information for the case of the scalar meson [4] since the states are experimentally rather difficult to determine their exact physical properties. Furthermore, much earlier there had been important theoretical work based on the bag model which has provided a basis for the study of various exotic quark structures. The work of Jaffe [5] used the bag model to study the dimeson state given in the form $q^2\bar{q}^2$ as a diquark and an antidiquark combination. Further work on the exotic quark formation by Strottman [6] has concentrated particularly on the states $q^4\bar{q}$ and $q^5\bar{q}^2$, which are respectively the pentaquark and the baryon-diquark-antidiquark combined state.

In the present work we want to discuss the role of the colored diquarks [7] as a basic element of the exotic systems. Furthermore, we shall suggest why the diquark is very basic to the structure of baryons, whereby we maintain that we may consider that baryon is a diquark and a quark. An important recent work by Jaffe and Wilczek [8] has shown a significant relation of the diquark structure to the pentaquark states. Because of this proposed model we shall treat the diquark as an important structure in the determination of the exotic states. As we did in our earlier treatment, we shall especially look at the role of the entropy [9] relating to the color structure of the quantum ground state. Quite recently we have looked at the quantum entropy $S_q$ of the color singlet quarks within the hadrons [10] by calculating that of the quarks in the singlet meson and baryon states. Afterwards, this calculation was used to calculate the effect of the contributions of $S_q$ on the equation of state for baryons with the same quark flavors in a bag model situation [11].

The standard model has the color charge carried by the quarks as the fundamental property of the strong nuclear interaction [12]. In clear contrast to the other known charges the color charge cannot be easily isolated and separately measured. In nature it always appears as part of selective states of the $SU(3)_c$, wherein the quarks and antiquarks are placed respectively in the fundamental $3$ and antifundamental $\bar{3}$ representations of this group. These two representations together with the adjoint representation of $SU(3)_c$ make up the symmetry structure of Quantum Chromodynamics(QCD) [13,14]. The two main categories of strong interacting particles (hadrons) are the mesons, which may be written as a product of the fundamental and the antifundamental representations $3 \otimes \bar{3}$ and the baryons, which are a product of three fundamental representations $3 \otimes 3 \otimes 3$. Although the different quarks have other properties like spin, electrical charge, mass as well as a very special property called flavor, which we shall see has a very important role in the determination of the size of the entropy in relation to the diquark structure.
In this work we shall write the quark and antiquark color states as follows: \[ |0\rangle, |1\rangle, |2\rangle \] and \[ |\bar{0}\rangle, |\bar{1}\rangle, |\bar{2}\rangle \]. We shall use this notation to describe the orthonormal bases of the fundamental and the antifundamental representations of \( SU(3)_c \), instead of the more common color names. From these color basis states we can construct a representation for the color exotic state functions— in particular for the dimeson \( \Psi_{dm} \) and pentaquark \( \Psi_{pe} \) groundstates. From these color state functions we are able to construct the corresponding density matrices \( \rho_{dm} \) and \( \rho_{pe} \) for the color states following the usual prescription given by quantum mechanics \([15]\). The general procedure for the following work is that we shall arrive at the single diquark reduced density matrix \( \rho_{dq} \), which is of particular interest in all further calculations. From \( \rho_{dq} \) we can directly calculate the quantum entropy in the sense of von Neumann \([9, 15]\). As in our previous calculations \([10, 11]\), the results of this calculation show a significant contribution of order one to the entropy of the quarks in the exotic states. This value is given as a pure number without physical dimensions when we use the usual high energy units with \( \hbar, c \) and Boltzmann’s constant \( k \) all set to the value one.

The further implications of these results can be brought together with some of our earlier work on the hadronic states \([10, 11]\). We start by combining the known singlet hadronic states. Since the lowest scalar meson, which was formerly called the \( \sigma \)-particle, is known to decay into two pions at very low energies, but also can include two kaons at energies\(^a\) just below \( 1 GeV \). Then the scalar meson can be constructed from the two \( q\bar{q} \) combinations as the direct product state which gives the obvious final state of two pseudoscalar mesons with the necessary positive parity. From this consideration a dimeson is just a simple reordering of the quark and antiquark color states which clearly uphold this singlet structure leading to the colorless state function \( \Psi_{dm} \) with the quark structure of \( q^2\bar{q}^2 \). Similarly, the pentaquark states are built from the color states of the decay into the baryon and meson final states. Here the \( q^4\bar{q} \) are structured as two diquarks and an antiquark. The five color states only maintain their colorlessness with the total baryon color wave function. Then \( \Psi_{pe} \) is colorless even when each single contribution is colored.

In the next section we shall write out these state function explicitly and indicate the forms of the respective density matrices. Afterwards we can find the value of the entropy in the different cases. Finally we conclude with a few remarks on the exotic structures.

### 2 Quark Structure of the Exotic States

The starting point for the investigation of the ground state of the quark structure of the exotic states is the evaluation of the density matrix \([15]\) for the singlet state quark struct-

\(^a\)The actual scalar meson states are discussed in the "Review of Particle Physics" \([16]\) in a "Note on Scalar Mesons" pp. 506 -510 and 522 -526. The actual scalar states go under the names \( f_0(600) \), \( f_0(980) \) and \( a_0(980) \). These states are all known to decay into pairs of pseudoscalar mesons and, of course, pairs of photons.
ture of the mesons and baryons [10,11]. This known color structure may be put together in specific combinations to yield the known final states from the exotic states’ decays.

Here we quickly recall how the singlet meson and baryon color state functions [12] are constructed. The meson is simply the sum of the direct product of quark and anti-quark states of the form \( \frac{1}{\sqrt{3}} \sum_i |i\bar{i}\rangle \), where the numerical forefactor serves as the correct normalization. Similarly the baryons color states are written as a direct product of three colored quark states where the permutation \( P\{ijk\} \) determines the sign in front of each term in the sum. The baryon color singlet state is given by \( \frac{1}{\sqrt{6}} \sum P(-1)^P |ijk\rangle \). The resulting entropy for both the meson and baryon colored states has already been calculated [10] to yield \( S_q = \ln 3 \).

Here we shall consider explicitly only the color parts of the dimesons’ state functions \( \Psi_{dm} \) and its conjugate \( \Psi_{dm}^* \). The dimesons are constructed from the direct product of the fundamental representations of the constituents in the form \( 3 \otimes 3 \otimes \bar{3} \otimes \bar{3} \). Thus the color singlet state functions for the dimesons are taken from the states arising out of this product representation. We are not able to represent the dynamical process which takes place in the decay to two or more meson singlet states. We notice that each entry is itself color neutral so that in combination these states make up the nine basic contributions to the color singlet dimeson states. Thus we can expect that these are pure physical states even if they may not be easily separated and measured. We write these in the following form:

\[
\Psi_{dm} = \frac{1}{3}( |0 \, 0 \, 0 \, 0 \rangle + |0 \, 1 \, 0 \, 1 \rangle + |1 \, 0 \, 1 \, 0 \rangle + |0 \, 2 \, 0 \, 2 \rangle + |2 \, 0 \, 2 \, 0 \rangle + |1 \, 1 \, 1 \, 1 \rangle + |1 \, 2 \, 2 \, 1 \rangle + |2 \, 1 \, 2 \, 1 \rangle + |2 \, 2 \, 1 \, 2 \rangle ) .
\]

(2.1)

Here we have kept the left to right ordering of the quarks and antiquarks. Also for the corresponding conjugate singlet state function of the dimeson we maintain a similar form\(^b\) that contains the corresponding conjugate color state vectors as previously discussed for the simple meson [10]. Thus we need not explicitly rewrite these terms for the conjugate state here.

Similarly we may write the state function for pentaquarks \( \Psi_{pe} \) coming from the singlet baryon and meson structures in the representation \( 3 \otimes 3 \otimes \bar{3} \otimes \bar{3} \otimes \bar{3} \). Again the same ordering principle is kept for the quark and antiquarks. However, for the pentaquark each state is not colorless but only within the cycles of the permutation of the first three colors, which means that there are three subgroupings of colorless states. Thus we may write down the four quarks and one antiquark terms of the pentaquarks, which can be set out as follows:

\[
\Psi_{pe} = \frac{1}{\sqrt{18}} ( |0 \, 1 \, 2 \, 0 \, 0 \rangle + |1 \, 2 \, 0 \, 0 \, 0 \rangle + |2 \, 1 \, 0 \, 0 \, 0 \rangle - |0 \, 2 \, 1 \, 0 \, 0 \rangle - |1 \, 0 \, 2 \, 0 \, 0 \rangle - |2 \, 1 \, 0 \, 0 \, 0 \rangle + ... - |2 \, 1 \, 0 \, 2 \, 2 \rangle ) .
\]

(2.2)

\(^b\)The order of the representation remains the same although the "ket" states \( |ij..\rangle \) are replaced by the "bra" states \( \langle ij..| \) so that again "i", "j".. are the first, second.. particle.
where the dots indicate the other eleven missing terms arising from the further mesonic color combinations. Here we have written only the six states in the first colorless subgrouping. In an analogous way to that explained for the dimeson we may also express the corresponding conjugate state function in the order of the tensor product for the pentaquarks using the correct conjugate color states.

3 Density Matrix Reduction for the Exotic States

We can now write down the density matrices $\rho$ for the exotic states using the direct product of $\Psi$ and $\Psi^*$ of the color state functions. This gives for the color singlet dimesons $\rho_{dm}$ and pentaquark $\rho_{pe}$ the density matrices in the following general forms:

$$\rho_{dm} = \Psi_{dm} \Psi_{dm}^*$$  \hspace{1cm} (3.1)

and

$$\rho_{pe} = \Psi_{pe} \Psi_{pe}^*.$$  \hspace{1cm} (3.2)

Previously we had only considered the ordinary hadronic states as being made out of the simple singlet combinations of the quark and antiquark states [10, 11]. Now we shall extend this study in order to construct more general combinations of the quarks and antiquarks in the formulation of the resulting density matrices. We know that for the hadrons the wavefunctions yield pure states [15]. Since the exotic states are built up from a direct product of the pure hadronic states into which the decay, the resulting density matrices also again represent pure states. However, the diquarks are a constituent subsystem making up the complete exotic state. For the calculation of the internal quark structure we look at the diquark reduced density matrices, which give the actual statistical state of the individual diquark within the total system. In order to get the reduced density matrices for the diquark in the dimesons, we project out all the antidiquark states, where colors are specified by $i$ and $j$ in the diquark color pair $[ij]$, $\langle [ij] | \bar{\psi} \rangle$ and $\langle [\bar{i}j] | \bar{\psi} \rangle$ by using the orthonormality and the completeness properties. Similarly for the pentaquark structures we project out first the antiquark state $\langle [\bar{i}] | \bar{\psi} \rangle$ and $\langle [\bar{i}] | \bar{\psi} \rangle$. Then in the second reduction of the density matrices we project $\langle [ij] | \bar{\psi} \rangle$ and $\langle [\bar{i}j] | \bar{\psi} \rangle$ onto the other two diquark states to the right. These diquark states result in two contributions for each color. After carrying out these reductions, we find that both the reduced density matrices from the dimeson and the pentaquark for the diquark states take on the same general diagonal form:

$$\rho_{dq} = \frac{1}{N} \sum_{[ij]} \{ [ij] \}.$$  \hspace{1cm} (3.3)

This is the reduced density matrix for the diquarks in the colored states. It always yields a completely mixed state with the same eigenvalue $\lambda_i = 1/N$ for each diquark contribution.
In order to determine the exact value of \( N \) in \( \rho_{dq} \), we must specify the diquark more exactly. It depends upon the flavors in the diquark as well as from where it came. In all cases if the two quarks in the diquark have the same flavor, the the value of \( N = 3 \). Two quarks with same color and flavor act as identical particles for which the Pauli exclusion principle automatically applies up to the spin. However, with different flavors in the diquark the two quarks are clearly distinguishable. This is essential to the diquark states in the density matrix. When the flavors are different the state \( |01\rangle \) is different from the state \( |10\rangle \). Then the projection in the density matrix, \( \langle 10|01 \rangle \) is zero for different flavors, but is one for the same flavors. This fact eliminates the nondiagonal projections for diquarks with different flavors within thereby changing the eigenvalues. After we have carried out the reduction of the density matrix for the same flavors\(^c\), we have found that the value \( N = 3 \) is then the same as in the case of the single quark reduced density matrix \( \rho_q \) for quarks in the hadronic singlet state \([10]\). However, for the dimeson with different flavors the value of the normalization \( N = 9 \) since there are nine different diquark states which contribute. Nevertheless, for the pentaquark state the baryonic states determine the nature of the diquarks. Thus we find after the second reduction that the value for \( N = 6 \) when there are different flavors. In some ways this result for the pentaquark is surprising since it has many more states initially than the dimeson. In the pentaquark case the reduction of the density matrix depends on the baryon, which with different flavors in the diquarks also yields \( N = 6 \). However, the dimeson has the diagonal structure of the meson. When there are different flavors for the diquarks in the reduced density there are nine different states.

### 4 Entropy for Exotic Quark States

We can calculate the entropy \( S \) of the quantum states using the prescription of von Neumann \([9,15]\), which makes direct use of the density matrix \( \rho \). It is simply written as

\[
S = -\text{Tr}(\rho \ln \rho),
\]

where the trace \( \text{"Tr"} \) is taken over the quantum states. When, as is presently the case, the eigenvectors are known for \( \rho \), we may write this form of the entropy in terms of the sum of the eigenvalues \( \lambda_i \) as follows:

\[
S = -\sum_i \lambda_i \ln \lambda_i.
\]

It is obviously important to have positive eigenvalues. For a zero eigenvalue we use the fact that \( x \ln x \) vanishes in the small \( x \) limit. Then for the density matrix \( \rho \) we may interpret

\(^c\)Here the diquark reduction of the exotic state is analogous to the first reduction in the baryon with the same flavors. Then the first reduction leaves the same diquark structure which is then reduced to the diagonal singlet density matrix all with the probability 3.
\( \lambda_i \) as the probability of the state \( i \) or \( p_i \). This meaning demands that \( 0 < p_i \leq 1 \). Thus the orthonormality condition for the given states results in the trace condition

\[
\text{Tr} \, \rho = \sum_i p_i = 1. \quad (4.3)
\]

This is a very important condition for the entropy.

We now apply these definitions to the entropy for the quark states. It is clear that the original hadron states are pure colorless states which possess zero entropy. For the dimeson it is immediately obvious since each colored diquark state has the opposing colored antidiquark state for the resulting colorless singlet state. The sum of all the cycles determine the colorlessness of the pentaquark singlet state thereby also yielding no entropy. However, the reduced density matrix for the individual quarks (antiquarks) \( \rho_q \) or \( \rho_{\bar{q}} \) has a finite entropy. For \( SU(3)_c \), all the eigenvalues \( \lambda_i \) in Equation (3.2) have the same value \( 1/N \). Thus we find for all the quarks (antiquarks) in these exotic singlet states

\[
S_{dq} = \ln N. \quad (4.4)
\]

Our further consideration involves the above mentioned different cases. When both the quarks in the diquark have the same flavor, then, as stated above, the diquark reduces to the single quark with \( N = 3 \). It follows that \( S_{dq} = \ln 3 \) is the same as \( S_q \). Thus a diquark in the singlet color state with no specified spin orientations for its constituent quarks only can exist with mutually different flavors. In the pentaquarks with different flavors in the diquarks we have found that \( N = 6 \). This is actually the same result we get for the usual baryon [10] after carrying out the first reduction of the density matrix with different quark flavors, which yields for \( S_{dq} \) just \( \ln 6 \). The dimeson with each diquark and antidiquark containing internally different flavors has \( N = 9 \). Thereupon, the color entropy \( S_{dq} \) of a diquark with different flavors within has the same value for the diquark entropy as the meson with the total quark and antiquark entropy of just \( \ln 9 \).

Hereupon, we may discuss the entropy in some more detail for the main examples of the colorless exotic ground states—the dimesons and the pentaquarks. As we have discussed above for the density matrix, all the dimesons consist of a diquark-antidiquark pair bound together as a sum of all the three colors. Since each single quark or antiquark state is equally weighted in the reduced density matrix, therefore each state possesses equal probability of \( 1/3 \). Thus we easily get the entropy of \( \ln 3 \). The baryon has the doubly reduced density matrix for each single quark state appearing twice so that with the normalization factor of \( 1/6 \) the probability of each colored quark state is again \( 1/3 \), which yields the same result for the entropy, \( \ln 3 \). This value gives the maximal entropy for a completely mixed state. For the pentaquark with different flavors we can carry out the two reductions—first the antiquark reduction and then the second diquark reduction. After these reductions there are exactly six diquark states which all remain in a diagonal form.
5 Conclusions

Our objective in this work was to investigate the diquark structure of the color states for the quark and antiquark systems involving exotic quark states. We chose to discuss two obvious examples of these exotic states both from their simplicity and their physical relevance. Although there has been recently a considerable amount of both experimental and theoretical work on both these cases, it has generally investigated the flavor, charge or spin structure, but to our knowledge not explicitly the color states. It is clear that the experiments do not have the direct access to the color states which is the case for the flavor and spin. However, the symmetry rules and the probability factors arise out of the color states. Thus we have seen that the diquark structure depends upon the relation of the colors to the flavors present. It is the entropy \( S_{dq} = \ln N \) which determines the relation of particular diquark configurations to the number of nontrivial eigenstates of the reduced density matrix. This fact allows us to determine the more favorable diquark combinations for any given exotic quark state.

Here we have used a known method in quantum statistical physics to examine a situation in high energy particle physics. It can be well carried out for the ground state of known particle quark structures in the color basis states. Thus the dependence on type of the quark color structure in relation to the different flavors is the extension of this approach of the three basis states of the fundamental representations of \( SU(3)_c \). The results of our calculation are consistent with the expected diquark structure which consist of colored quarks with different flavors [8].

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