Polarizability of 2D monster and light scattering

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The electrostatics of 2D system with complicated inner boundary is studied. The object which we call "monster" is built by an iterative process of multiple conformal mapping of the circle exterior. The procedure leads to the figures built from circle with branching curved cuts (i), to the multiple tangential near-round circles (ii) and the Mandelbrot map (iii). The polarizability of a monster in the homogeneous external field is found. The light scattering cross-section was expressed through the polarizability of a monster.

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**Introduction.** The fractal aggregates appear in many experimental and theoretical situations. Such aggregates are the topic of the diffusion limited aggregation, the theory of percolation and the electric breakdown.

The purpose of the present paper is to study the polarizability of solitary particles with complicated boundaries. We shall refer the complicated particles, built from small subunits, as "monsters", distinguishing them from fractals, which result from the infinite growth. The monsters can be obtained both directly in the growth process, and as a result of Van der Waals aggregation of preproduced spherical nanoclusters into large snowballs. It is natural to expect that in many real situations medium is composed from the monstrous or fractal objects, rarely embedded into matrix.

The optical properties of fractals, including polarizability were studied in Ref. The attention of these studies was mostly concentrated on the localization of high-frequency polar eigenmodes of fractals and corresponding large fluctuations of local electric fields.

The works that are most close to the purpose of the present study, consider the fractal as a molecule built from coupled dipolar monomers. The system of linear equations for dipoles was solved either in self-consistent approximation or by computer simulations.

The self-consistent approximation is valid only if the polarizability of the fractal medium is not large. The large polarizability of the medium leads to the strong screening of the external field. This is the case for metal clusters, studied in the present work.

Unlike the referred papers, we are interested in the situation where the local fields are strongly deviate from the mean value. We shall consider the aggregate interior as a macroscopic medium, neglecting the atomic structure effects. In this approach one should solve the macroscopic Maxwell equation inside and outside of fractal aggregate with some boundary conditions on its surface. We shall deal with aggregates, that are small in comparison with the wavelength. This permits to solve the Laplace equation, instead of the Maxwell one.

The Laplace problem is significantly simplified, if one does not need the solution both inside and outside of the object. It is so, if the dielectric permittivity of the cluster is essentially larger or essentially smaller than that of the external medium. If the cluster medium is metal, then its permittivity is determined by the formula $\frac{1}{\varepsilon} = \frac{1}{\varepsilon_0} - \frac{\omega^2_p}{\omega^2}$, where $\omega$ is the light frequency, and $\omega_p$ is the plasma frequency. The first case corresponds to $\omega \ll \omega_p$, the second one to the frequency near the bulk plasma frequency.

We shall study the 2D case, which allows to use the theory of analytical functions for solution of the Laplace equation. This simplification may be considered as a model for the 3D case. At the same time, it is applicable to the clusters with cylindrical shape $F(x, y) = \text{const}$.

We shall find the electric polarizability of a large aggregate of complicated shape that can be constructed by means of multiple conformal maps. In this case the exact solution of the problem can be found. One kind of such construction of 2D fractal objects was suggested in Ref. The fractal was treated as a result of iterative mapping of a simple domain, for example, exterior of a circle. The elementary step maps the exterior of the circle onto the exterior of a circle with a bump or with a strike. The multiple mapping complicates the form of the domain, producing a bump (strike) for a step of an iteration. This construction permits to simulate growth, guided by so called harmonic measure, when the addition of an element of fixed area to the domain appears with the probability, proportional to the diffusion flow, determined by the solution of the harmonic equation of the diffusion (DLA).

Another approach to constructing fractal is a simple map, for example, square map $z_{n+1} = c - z_n^2$. The multiple superposition of this map separates the complex plane into the repulsion basin, composed from all points $z_0$ for which $z_n \to \infty$. The boundary of that basin is a fractal, called Julia set.

The limited repetition of mapping produces a monster, the complicated but not fractal object. We are interested in the properties of the monster as well as the properties of the limit of the infinite number of repetitions, the fractal. The complexity of the fractal grows linear with the map order. It differs this construction from the square map of
Mandelbrot, where the complexity grows exponentially with the map order.

In the case of rare inclusions the effective electrical properties of medium are determined by individual monster properties. For example, the effective dielectric permittivity of the medium, containing monsters with concentration \( n \) and polarizability \( \chi_{ij} \) in the limit of low concentration is

\[ \epsilon_{ij} = \delta_{ij} + 4\pi n \chi_{ij}. \]  

(1)

Similar formula \( \sigma_{ij} = \sigma(\delta_{ij} + 4\pi n \chi_{ij}) \) determines the effective conductivity of the medium with conductivity \( \sigma \), containing insulation inclusions.

The other important property of physical structures is the light scattering cross-section. It is determined by the polarizability, if the wavelength exceeds the diameter of the object. In particular, we shall deal with the polarizability and with the scattering of light on a small metal particle.

**Method.** These properties can be found by solving the Laplace equation \( \Delta \phi (r) = 0 \) for the potential outside of a solitary particle in the external uniform electric field \( E = E_x + i E_y \). For \( \omega \ll \omega_p \) the boundary conditions at the particle surface \( S \) and at the infinity are

\[ \partial_S \phi = 0, \quad \phi|_\infty = -Re(E^*w). \]  

(2)

Here \( w = x + iy \).

The effective conductivity of media with rare dielectric inclusions is determined by the solution of the current flow problem around the inclusion. If \( \omega - \omega_p \ll \omega_p \), the potential \( \psi \) satisfies the condition \( \partial_n \psi|_S = 0 \) at the boundary of the inclusion. According to the Couchi-Rieman conditions \( \phi \) and \( \psi \) are the real and imaginary parts of the complex potential \( \Phi \). This case corresponds to the problem of the current flow in a conducting medium around an insulating inclusion with the same shape.

The idea of the present work is to construct a complicated object from simple one by means of repetition of some conformal map \( f(z) \). The map should transform the exterior of the basic object to the exterior of complicated one and does not move the infinity point. It doesn’t alter the Laplace equation and the surface boundary conditions (with the accuracy to the coefficients). Therefore, the solution of the problem for the source object gives it for the image object too.

Let us consider a multiple map \( F^{(n)} \) composed of the \( n \) basic maps \( f_\lambda(z) \), where \( \lambda \) is a parameter (or a set of parameters) of the map, and some additional map \( G \):

\[ F^{(n)}(w) = G \circ f_{\lambda_1} \circ f_{\lambda_2} \circ \ldots \circ f_{\lambda_n}(w). \]  

(3)

The parameters \( \lambda_i \) may be chosen fixed, random, or depended regulary on \( i \).

If we are interested in the dipole component of electric field only, we should take into account only \( f_1 \) and \( f_{-1} \) coefficients of series expansion of the map \( f_{\lambda_n}(w) = f^{(i)}_{1}(w) + f^{(i)}_{0}(w) + f^{(i)}_{-1}(w) + \ldots \).

The complex potential outside the unite circle in the external uniform electric field in 2D case is

\[ \Phi(w) = -(E^*w - E/w), \]  

(4)

on the complex \( w \)-plane. Let \( G \equiv 1 \). Substitution of the series expansion of the inverse function \( w(z) : F^{(n)}(w(z)) \equiv z \) into (4), gives us the polarizability tensor \( \chi \):

\[ \chi = \frac{A}{2} \begin{pmatrix} Re(B) - \sigma & Im(B) \\ Im(B) & -Re(B) - \sigma \end{pmatrix}, \]  

(5)

where \( \sigma = -1 \) in the case of metal particle and 1 in the case of dielectric one. \( A \) and \( B \) are defined as follows:

\[ A = \prod_{i=1}^{n} f^{(i)}_1 \]  

(6)

\[ B = \frac{1}{A} \left[ -f^{(1)}_{1} f^{(1)}_{-1} + f^{(n)}_1 \sum_{k=2}^{n} f^{(k)}_{-1} \prod_{i=1}^{k-1} (f^{(i)}_1)^2 \right]. \]  

(7)

The principal values of polarizability tensor are

\[ \chi_{1,2} = -\frac{A}{2} (\sigma \pm |B|). \]  

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Examples. In the recent works[10–12] a multiple conformal map (i) was used for constructing of models of fractal objects, in particular, DLA and dielectric breakdown clusters. The map (8) transforms the unit circle to the unit circle with a small bump (or strike) on it (Fig. 1).

\[ f_\lambda(w) = w^{1-a} \left\{ \frac{1 + \lambda}{2w} \left( 1 + w \right) \left[ 1 + w + w \left( 1 + \frac{1}{w^2} - \frac{1}{w+1} \right)^{1/2} \right] - 1 \right\}^a, \]  

(8)

\[ f_{\lambda,\theta}(w) = e^{i\theta} f_\lambda(e^{-i\theta} w). \]  

(9)

The shape of the bump is determined by two parameters \( a \) and \( \lambda \). In this work we assume them to be constant. We use the rotated map (9) as the basic for \( F \) and consider simple dependence of the angle of rotation \( \theta_k = k\theta \), where \( \theta = \text{const} \) is the angle incommensurable to \( \pi \). This choice enables us to find the exact equation for the polarizability but still produces the complicated objects. The map, recursively applied to the source unit circle, produces monsters as on the Fig. 1.

The coefficients \( A, B \) of the polarizability tensor in the limit \( n \to \infty \) are:

\[ A = (1 + \lambda)^{2an}, \]

\[ B = a\lambda^2 + (2a - 1)\lambda \frac{e^{2i\theta} - \beta}{(1 + \lambda)^2} \frac{\beta^2 - 2\beta \cos 2\theta + 1}{e^{2i\theta}} \]  

(10)

where \( \beta = (1 + \lambda)^{2a} \).

![Fig. 1. The case (i). The images of the unit circle according to the Eq. 8 after \( n = 50 \) iterations for parameters \( \theta = 2.7, a = 2/3, \lambda = 0.1 \) (case of strikes, figure a) and \( \theta = 2.15, a = 1, \lambda = 0.1 \) (case of bumps, figure b). Images of basic maps applied to the unit circle are shown in the corners (the size of basic map is arbitrary).](image)

Another considered basic map (ii) is the map of the exterior of the unit circle to the exterior of two tangenting unit circles:

\[ f(w) = \frac{i\pi}{\log \left[ \frac{w-1}{w+1} \right]} + 1, \quad f_\theta(w) = e^{i\theta} f(e^{-i\theta} w). \]  

(11)

The example of multiple map is shown on Fig. in the case of \( \theta_k = k\theta \).
FIG. 2. The case (ii). The image of the circle after 10-th map for $\theta_n = n\theta$, $\theta = -2.1$. The basic map is shown also.

In the case of $n \to \infty$ corresponding quantities $A$ and $B$ for polarizability are

$$A = 2 \left(\frac{\pi}{2}\right)^{2n}, \quad B = \frac{\pi^2 (-e^{2i\theta})^{n+1}}{3(4 + \pi^2 e^{2i\theta})}.$$

The third basic map (iii) is the known Mandelbrot map. Resulting transform is

$$F^{(n)}(w) = (1 - (1 - (-w^2)^2)^2)^{2^{-n}}.$$

The additional transform $G(\alpha) = \alpha^{2^{-n}}$ makes the final map single-valued on the exterior. On Fig. 3 the domain is shown, which boundary, so-called Julia set, is the image of a unit circle. The multiple map (iii) produces fractal, which appears, for example, in the hierarchical description of the percolation transition in the complex plane of percolation probability.

The polarizability of the metallic Mandelbrot monster is

$$\chi_{xx,yy} = \pm \frac{1}{2} + \left(\frac{1 + \sqrt{5}}{2}\right)^{2^{-n+1}}, \quad \chi_{xy,yx} = 0.$$

Light scattering. The 2D problem may be considered as a model for the more realistic 3D system. From the other hand, it describes the properties of the cylindrical objects, which size in z-direction exceeds the maximal diameter in (x,y) plane.

In relation to the problem of light scattering we shall consider the geometry, depicted on the Fig. 4. In this case, if the electric vector $\mathbf{E} = |\mathbf{E}|e$ lies in the face plane of the monster, the cross-section is

$$\sigma = \frac{8\pi}{3} k^4 |\chi_{ij} e_j|^2 4 \sin\left(\frac{k_z L}{2}\right) \frac{\sin^2(\frac{k_z L}{2})}{k_z^2},$$

where $e_i$ is the vector of polarization of the electric field.

FIG. 3. The case (iii). The Mandelbrot fractal, obtained by the map of the unite circle.

FIG. 4. The geometry of the light scattering problem on a cylindrical object with complicated section. $k$ is the wave vector. The plane of polarization coincide with the cross-section plane.
Discussion. The polarizability of the round metal particle in 2D is roughly proportional to the second power of its radius. For a strongly extended asymmetric particle the components of polarizability tensor are determined by the sizes of the particle in the direction of the field.

In general, the monsters in the cases (i) and (ii) have no symmetry, so each component of tensor $\chi$ is finite. In the case (iii) the fractal has $x$ and $y$ axes of symmetry resulting in $\chi_{xy} = 0$. The monsters (i) and (ii) grow exponentially with the number of iteration. This leads to the same growth of the polarizability.

In the case (iii) the sizes of monster are limited, as is the polarizability tensor. The limiting ratio of its components roughly corresponds to the ratio of sizes. The fractal structure of this system manifests itself in the exponential law of approaching to the limit.

The purpose of the present work was to find the polarizability of metal clusters. However, these results are more widely applicable. They describe the effective conductivity of a medium with insulating or perfectly conducting inclusions. In accord with the Einstein relation, the formulae similar to (1) determine the effective diffusion coefficient of some particles in the medium with inclusions, non-penetrable for diffusing particles.

We have used some ways of constructing monsters and fractals by means of multiple conformal maps. The basic map may be more simple as the Mandelbrote square map (iii) or more complicated as the bump map (i) or the tangential circle map (ii). The advantage of complicated maps (i) and (ii) is that they permit to adjoin unites of fixed form to a cluster, the disadvantage is that the complexity of monster grows linearly with the iteration number, while the complexity of square map (iii) (and more general one, $z_{n+1} = c - z_n^2$) grows exponentially.

We have considered the sequent maps with fixed parameters only. In the cases (i) and (ii) the external parts of monsters are essentially larger than internal ones. The fine details of the monster are inside of it and the exterior of the monster is not fractal. The polarizability of the monster depends generally on its external shape, scaling like the square of its diameter. This differs situation from one, considered in Ref. 10–12, where the fractal shape of a growing cluster is achieved by the selection of the map parameters $\lambda_i$ in such a way, that adjoining bumps have equal sizes. Our results may be easily generalized to this case also and to the other cases of random, or some regular dependence of parameters on the map order.

The specific shape of monsters (i) is similar to the branching traces of electric breakdown and hence is applicable for description of electric properties of medium after electric breakdown. The monsters (ii) have the shape of coupled droplets in emulsion. By selection of parameters this case may be generalized to describe clusters of near round droplets with equal sizes.

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1. P. M. Hui and D. Stroud, Phys. Rev. B, 49, 11729 (1994).
2. F. Claro, R. Fuchs, Phys. Rev. B, 44, 4109 (1991).
3. F. Brouers, D. Rauw, J.P. Clerc and G. Giraud, Phys. Rev. B, 49, 14582 (1994).
4. V. M. Shalaev, M. I. Stockman, JETP 65, 287 (1987).
5. M. V. Entin and G.M. Entin, JETP Lett. 64, 427 (1996).
6. E. M. Baskin, M. V. Entin, A. K. Sarychev, A. A. Snarskii, Physica A, 242, 49 (1997).
7. V. M. Shalaev, R. Botet, A. V. Butenko, Phys. Rev. B, 48, 6662 (1993).
8. V. A. Markel, V. M. Shalaev, E. B. Stechel, W. Kim and R. L. Armstrong, Phys. Rev. B 53, 2425 (1996).
9. V. M. Shalaev, E. Y. Poliakov, and V. A. Markel, Phys. Rev. B 53, 2437 (1996).
10. M. B. Hastings and L. S. Levitov, Physica D 116, 244 (1998).
11. B. Davidovich, I. Procaccia, Report No. chao-dyn/9812026.
12. B. Davidovich, H. Hentschel, Z. Olami, I. Procaccia, L. Sander, E. Somfai, Phys. Rev. E 59, 1368 (1999). chao-dyn/9812020.
13. H.-O. Peitgen, P. H. Richter, The Beauty of Fractals. Images of Complex Dynamical Systems, Springer-Verlag, 1986.