Mesoscopic Fluctuations of Elastic Cotunneling

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We study mesoscopic fluctuations of the conductance through a quantum dot at the wings of the Coulomb blockade peaks. At low temperatures, the main mechanism of conduction is the elastic cotunneling. The conductance strongly fluctuates with an applied magnetic field. The magnetic correlation field is shown to be controlled by the charging energy, and the correlation function has a universal form. The distribution function for the conductance obtained analytically shows a non-trivial crossover between the orthogonal and unitary \((B \to \infty)\) ensembles.

Situation changes drastically if the system is tuned away from the charge degeneracy point, \(E > \Delta \approx T\). In this case the transport is due to the virtual transitions of an electron via excited states of the dot (so-called elastic cotunneling \([3]\)); many levels with energies exceeding \(E\) contribute to the tunneling. The superposition of a large number of tunneling amplitudes changes the properties of the conductance fluctuations, which is studied for the first time in this Letter.

We will show, that the correlation function \(C(\Delta B)\) for the conductance in the cotunneling regime is universal (i.e. all the dependences for different \(E\) can be collapsed to a single curve upon rescaling of the magnetic field); magnetic correlation field \(B_c\) is inversely proportional to \(\sqrt{E}\), and thus the charging energy controls the fluctuations of the elastic cotunneling. Furthermore, the functional form of \(C(\Delta B)\) is entirely different from the known results \([6]\). Finally, we will find the distribution function of the conductance for all values of the magnetic field.

The quantum dot attached to two leads is described by the Hamiltonian:

\[
\hat{H} = \hat{H}_L + \hat{H}_R + \hat{H}_D + \hat{H}_T, \quad (1a)
\]

where the Hamiltonians of the left (L) and right (R) leads are given by

\[
\hat{H}_L = \sum_k \xi_k a_k^\dagger a_k, \quad \hat{H}_R = \sum_k \xi_k b_k^\dagger b_k, \quad (1b)
\]

and \(\xi_k^L, r\) is the one electron energy measured from the Fermi level. Hamiltonian of the dot \(\hat{H}_D\) has the form

\[
\hat{H}_D = \sum_k \xi_k^D c_k^\dagger c_k + E_c (\hat{n} - \mathcal{N})^2, \quad \hat{n} = \sum_k c_k^\dagger c_k, \quad (1c)
\]

where \(\xi_k^D\) describes the one-electron spectrum of the dot, and the second term in \(\hat{H}_D\) corresponds to the charging energy, and \(E_c = e^2/2C\). Here \(C\) is the capacitance of the dot, and \(\mathcal{N}\) is the conventional dimensionless parameter related to the gate voltage \(V_g\) by \(\mathcal{N} = V_g/eC_g\), with \(C_g\) being the gate capacitance. The tunneling Hamiltonian couples the leads with the dot, and it has the form...
\[ \hat{H}_T = \sum_{k,p} t_{kp} a_k^\dagger c_p + \sum_{k,p} t_{kp}^\dagger b_k^\dagger c_p + h.c. \]  

(1d)

Operators \( a, b \) and \( c \) in Eqs. (1c)-(1d) are the corresponding fermionic operators.

If tunneling is weak, the charge of the dot \( \hat{n} \) is quantized. Obviously, degeneracy of the charging energy in Eq. (1c) corresponds to half-integer values \( N_m = m + \frac{1}{2} \) of the dimensionless gate voltage \( \nu \).

If \( V_g \) is tuned away from a degeneracy point, it takes a finite energy \( E \) to add one electron (or hole) to the dot,

\[ E = E_c |N - N_m|, \quad |N - N_m| < 1/2. \]  

(2)

Positive (negative) values of \( N - N_m \) correspond to the electron (hole)-like lowest charged excitations.

We are considering the strong Coulomb blockade away from the resonance. Thus, we employ perturbation theory in the tunneling Hamiltonian (1d). The lowest non-vanishing contribution to the conductance \( G \) is

\[ G = \frac{2\pi e^2}{h} \sum_{k,p} |A_{kp}|^2 \delta(\xi_k^p) \delta(\xi_p^k). \]  

(3)

Amplitude \( A_{kp} \) corresponds to the process in which an electron (hole) tunnels from state \( k \) in the left lead into a virtual state in the dot, and then it tunnels out to state \( p \) of the second lead. This amplitude is given by

\[ A_{kp} = \frac{1}{\sqrt{\xi_q^p + E}} \theta(\xi_q^s(N - N_m)). \]  

(4)

The denominator in Eq. (3) corresponds to the energy of virtual state \( \xi_q \) involved in the cotunneling process and the step-function \( \theta(x) \) selects the dominating (electron or hole) channel.

In the most realistic case \([9,10]\) of point contacts, Eqs. (3) and (4) may be further simplified. The tunneling matrix elements \( |t_{kp}^{l,r}|^2 \) do not depend on the indices \( k, q \), and can be related to the conductances of the point contacts, \( G_{ij} = (2\pi e^2/h)\nu_{ij}\nu_{sr}|t_{sr}^{l,r}|^2 \); hence \( \nu_{ij} \) are the ensemble-averaged densities of states in the leads \( (l, r) \) and dot \( (d) \) respectively. Using these definitions, substituting Eq. (4) into Eq. (3), and performing the summation over \( k \) and \( p \) in Eq. (3), we find:

\[ G = \frac{\hbar}{2\pi e^2} G_{ij} G_{r,s} |F(R_i, R_r)|^2. \]  

(5)

The dimensionless function \( F(R_i, R_r) \) contains all the information about elastic cotunneling through the dot between the point contacts located at \( R_i \) and \( R_r \),

\[ F(R_i, R_r) = \frac{1}{\nu d} \sum_q \frac{\psi_q^*(R_i) \psi_q(R_r)}{[\xi_q + E] \theta(\xi_q(N - N_m))}, \]  

(6)

where \( \psi_q \) is the one electron wave function in the closed dot. It is useful to rewrite \( F \) in terms of the retarded and advanced one-electron Green functions \( G^{R,A} \) of the dot,

\[ F(R_i, R_r) = \frac{1}{\nu d} \int \frac{d\omega}{2\pi i} \frac{G^{A}(\omega) - G^R(\omega)}{|\omega| + E} \theta(\omega(N - N_m)), \]  

(7)

where

\[ G_{\omega}^{R,A} = G_{\omega}^{R,A}(R_i, R_r) = \sum_q \frac{\psi_q^*(R_r) \psi_q(R_i)}{\omega - \xi_q \pm i\hbar}. \]  

We put \( \hbar = 1 \) in all the intermediate calculations.

Equations (5) and (6) express the elastic cotunneling conductance in terms of the exact electron wavefunctions of the dot. These functions vary strongly when the magnetic field is applied to the dot or the shape of the dot is changed. Thus, the conductance is a random quantity and one should consider different moments of the conductance distribution function. We will employ the ensemble averaging, which is equivalent to the averaging over applied magnetic field or over the peak index \( m \). According to Eqs. (5) and (6), averaged moments of the conductance are expressed in the terms of the averaged product of the Green functions. It is well known [11], that if the dot in the metallic regime (the transport mean free path or the size of the dot is much larger than the Fermi wavelength), and the relevant energies are much larger than \( \Delta \), these products can be related to the generalized classical correlators – diffusion \( D \) and cooperon \( C \):

\[ \langle G_{\omega_1}^{B_1}(r, s)G_{\omega_2}^{A_2}(s, r) \rangle = 2\pi \nu_{d} D_{\omega_1 - \omega_2}(r, s), \]  

(8a)

\[ \langle G_{\omega_1}^{B_1}(r, s)G_{\omega_2}^{A_2}(r, s) \rangle = 2\pi \nu_{d} C_{\omega_1 - \omega_2}(r, s), \]  

(8b)

where \( \langle \ldots \rangle \) stand for the ensemble averaging perfomed under the fixed magnetic fields \( B_1, B_2 \). The averages of the type \( (G^{R}G^{R}) \) and \( (G^{A}G^{A}) \) are much smaller and can be neglected. If the sample is dirty, so that the motion of electrons in the dot is diffusive, the diffuson and cooperon \( D \) and \( C \) satisfy the equations

\[ -i\omega + D \left(-i\nabla_r + \frac{e}{c} A^-(r)\right)^2 D_{\omega_1}^{B_1, B_2} = \delta(r - s), \]  

(9a)

\[ -i\omega + D \left(-i\nabla_r + \frac{e}{c} A^+(r)\right)^2 C_{\omega_1}^{B_1, B_2} = \delta(r - s), \]  

(9b)

where \( D \) is the diffusion constant, and \( A^\pm \) is the vector potential due to the magnetic field, \( \nabla \times A^\pm = B_1 \pm B_2 \). For the dot in the ballistic regime, the diffusion operator in the l.h.s. of Eqs. (9) should be replaced with the Liouvillean operator. Solution of Eqs. (9) with the condition of vanishing normal component of the gauge invariant current at the boundary of the dot will enable us to find all the relevant correlation functions of the conductance and we are turning to this calculation now.

The averaged cotunneling conductance is obtained immediately by the averaging of Eq. (7) with the help of Eqs. (5) and (6). The result is
The term in the second line of Eq. (11) corresponds to the square of the average conductance, and the last two terms describe the conductance fluctuations. The term in the third line of Eq. (11) depends only on $\Delta B$, [cf. Eq. (5a)] and it is present both for the orthogonal ($B = 0$) and for the unitary ($B \to \infty$) ensembles. To the contrary, the last term in Eq. (11) dies out for the unitary ensemble.

It is easily seen from Eq. (11) that the fluctuations are always of the order of the conductance itself. This may be understood from the following qualitative consideration. There are $N \sim E/\Delta \gg 1$ contributions corresponding to different eigenstates in the cotunneling amplitude $\mathcal{C}$. Assume that the phases of these contributions are completely random. Conductance is proportional to the modulus squared of the sum of these contributions, and thus there are $N^2$ terms in the conductance. Among those, $N$ terms do not fluctuate, and the rest $N^2 - N$ are random. These random terms, however, do contribute to the fluctuation $\langle \delta G^2 \rangle$, and the number of non-vanishing terms in it is $N^2 - N$. Therefore, the average conductance is proportional to $N$, and its r.m.s. fluctuation is $\sim \sqrt{N^2 - N} \approx N$. Thus, conductance in the cotunneling regime is not a self-averaging quantity despite a naive expectation that a large number of virtual states participating in the cotunneling may decrease the fluctuations.

Equations (10) and (11) are quite general, i.e. they are valid for an arbitrary relation between the energy of charged excitation $E$ and Thouless energy $E_T \equiv \hbar D/L^2$ (here $L$ is the linear size of the dot). In the most interesting regime, $E < E_T$, the correlation function of conductance fluctuations $C(\Delta B)$ acquires a universal form, as will be shown below.

Because $E < E_T$, the characteristic frequency $\omega$ in Eqs. (3) is much smaller than the lowest non-zero eigenvalue of the diffusion operator (which is of the order of $D/L^2$). Therefore, only the zero frequency mode can be retained in the solutions of Eqs. (3). This mode corresponds to the probability density homogeneously distributed over the dot, and the solution has the form:

$$\delta \omega^0 \left( \frac{1}{B_1}, \frac{1}{B_2} \right) c_{\omega^0}^2 = \Omega_{+} \frac{S^{-1} - i \omega + \Omega_{-}}{\Phi_0^2}, \quad \Omega_{\pm} = E_T S (\frac{B_1 \pm B_2}{2})^2,$$

where $S$ is the area of the dot, $\Phi_0 = 2\pi \hbar c/e$ is the flux quantum, and the Thouless energy is given by $E_T = \alpha h D/S$, with shape-dependent coefficient $\alpha$ of the order of unity. Equation (12) holds also for ballistic cavities; the only difference is that the expression for the Thouless energy changes to $E_T \approx h/\tau f$, with $\tau f$ being the time of flight of an electron across the dot. Thouless energy can be independently measured by studying the correlation function of mesoscopic fluctuations for the same dot but with the contacts adjusted to the ballistic regime.

Substitution of Eq. (12) into Eq. (10) immediately yields the known result for the averaged conductance

$$\langle G \rangle = \frac{DG_G}{2\pi \nu e^2} \frac{\Delta}{E}.$$  

However, the fluctuations $\delta G = G - \langle G \rangle$ are large. We find from Eq. (11) with the help of Eqs. (12):

$$\frac{\langle \delta G(B) \delta G(B + \Delta B) \rangle}{\langle G \rangle^2} = \Lambda \left( \frac{\Delta B}{B_c} \right) + \Lambda \left( \frac{2B + \Delta B}{B_c} \right),$$

where the scaling function $\Lambda(x)$ is given by

$$\Lambda(x) = \frac{1}{\pi x^2} \left[ \ln x^2 (1 + x^4) + \pi \arctan x^2 + \frac{1}{2} \text{Li}_2(-x^4) \right],$$

with $\text{Li}_2(x)$ being the second polylogarithm function [12]. The asymptotic behavior of function $\Lambda$ is $\Lambda(x) = 1 + (2x^2 \ln x^2)/\pi$, for $x \ll 1$ and $\Lambda(x) = (\pi x^2)^{-2} \ln^4 x^2$ for $x \gg 1$.

The correlation magnetic field $B_c$ in Eq. (14) is controlled by the charging energy

$$B_c = \frac{\Phi_0}{S} \sqrt{\frac{E}{E_T}}.$$  

It is worth noticing from Eq. (3) and Eq. (11) that the correlation magnetic field $B_c$ drops with approaching a charge degeneracy point (in agreement with the recent experiment [4]), whereas the quantity $\langle G \rangle B_c^2$ remains invariant. This invariance can be easily checked experimentally.

Let us present also the expression for the experimentally measurable correlation function of the conductance
fluctuations \( C(\Delta B) = \langle \delta G(B)\delta G(B + \Delta B) \rangle / \langle \delta G(B)^2 \rangle \). For both orthogonal and unitary ensembles we obtain from Eq. (14)

\[
C(\Delta B) = \Lambda \left( \Delta B/B_c \right),
\]

function \( \Lambda(x) \) is defined by Eq. (15). We emphasize that the functional form of \( C(\Delta B) \) is different from the results for the peak heights fluctuations \([6]\), see Fig. 1.

\[
\Lambda \left( \frac{\Delta B}{B_c} \right) = \sqrt{\frac{\Delta}{B_c}} \left( \frac{\Delta}{B_c} \right)^{-1/2}
\]

for both orthogonal and unitary ensembles respectively.

So far we considered the elastic cotunneling only. It dominates over the inelastic processes \([3]\) at \( T < \sqrt{E\Delta} \), which is the typical regime for the modern experiments with semiconductor quantum dots \([10]\). At higher temperatures, the main conduction mechanism switches to the inelastic cotunneling. Nevertheless, the fluctuations are still determined by the elastic mechanism for \((E\Delta)^{1/2} \lesssim T \lesssim (E^2\Delta)^{1/4} \). At even higher temperatures, the inelastic contribution dominates also in the fluctuations. Their relative magnitude, however, is small, \(< \delta G_{in}^2 > / < G_{in} >^2 \approx \Delta/T \). The correlation magnetic field is controlled by the temperature rather than by the charging energy and therefore is independent on the gate voltage.

In conclusion, we studied the statistics of mesoscopic fluctuations of the elastic cotunneling. We showed that the correlation magnetic field is controlled by the charging energy and the correlation function of the conductance is universal.

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