GLOBAL KÄHLER-RICCI FLOW ON COMPLETE NON-COMPACT MANIFOLDS

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Abstract. In this paper, we study the global Kähler-Ricci flow on a complete non-compact Kähler manifold. We prove the following result. Assume that $(M, g_0)$ is a complete non-compact Kähler manifold such that there is a potential function $f$ of the Ricci tensor, i.e.,

$$R_{ij}(g_0) = f_{ij}.$$ 

Assume that the quantity $|f|_{C^0} + |
abla_{ar{g}_0} f|_{C^0}$ is finite and the L2 Sobolev inequality holds true on $(M, g_0)$. Then the Kahler-Ricci flow with the initial metric $g_0$ either blows up at finite time or infinite time to Ricci flat metric or exists globally with Ricci-flat limit at infinite time. A related is also discussed.

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1. Introduction

In the study of Ricci flow, the main problem is to understand the global behavior of it [12]. Since one may meet singularities of Ricci flow in finite time, it is interesting to know the singularity structure of the Ricci flow. R.Hamilton conjectures that after suitable normalization, the Ricci flow at the blow-up point looks like a Ricci soliton. In this paper, we study the global behavior of the Kähler-Ricci flow on a complete non-compact manifold with bounded curvature and we show in this case Hamilton’s conjecture is right. We remark that the local existence of the flow has been proved by W.Shi [21] and we use this local flow in this paper.

We prove the following result.

Theorem 1. Assume that $(M^n, g_0)$ $(n = \dim_C M)$ is a complete non-compact Kähler manifold with bounded curvature and there is a potential function $f$ of the Ricci tensor, i.e.,

$$R_{ij}(g_0) = f_{ij}.$$ 

Assume the quantity $|f|_{C^0} + |
abla_{g_0} f|_{C^0}$ is finite. We also assume that the $L^2$ Sobolev inequality holds true on $(M, g_0)$. Then the Kahler-Ricci flow

$$\partial_t g_{ij} = -R_{ij}(g)$$

with the initial metric $g_0$ either blows up at finite time or infinite time to Ricci flat metric or exists globally with Ricci-flat limit at infinite time.

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Our assumptions are nature and as a comparison, one may look the results in [4]. Note that in the statement of our result, we assume $L^2$ Sobolev inequality, namely, there exists a global constant $C > 0$ such that for any $u \in C^1_0(M, g_0)$,

$$C \left( \int_M |u|^{2n/(n-1)} \, dV \right)^{n/(n-1)} \leq C \int_M \left( |\nabla_{g_0} u|^2 + u^2 \right) \, dV,$$

which, plus the bounded curvature assumption, implies the Gross Log-Sobolev inequality and the non-local-collapsing for the metric $g_0$. Hence the W-functional introduced by Perelman [18] makes sense on the Riemannian manifold $(M, g_0)$. Along the Ricci flow, since the curvature is bounded, all the Riemannian metrics are equivalent each other before the first singularity. This implies that the W-functional is well-defined along the Ricci flow. In our proof of Theorem 1, we shall use the equivalent inequality for the W-functional (and see [9] for more discussions).

Without the non-local-collapsing for the metric $g_0$, we have the following result.

**Theorem 2.** Assume that $(M, g_0)$ is a complete non-compact Kähler manifold with bounded curvature and there is a potential function $f$ of the Ricci tensor, i.e.,

$$R_{ij}(g_0) = f_{ij}.$$

Assume only $|\nabla_{g_0} f|_{C^0}$ is finite and the initial metric has non-negative bisectional curvature. There is a global Kähler-Ricci flow with the initial metric $g_0$.

To understand the global behavior of the Kähler-Ricci flow established by W. Shi [21], we need to use some invariants for the various curvature quantities. As indicated in the famous work of Chen-Tian [7], the problem of the positivity of the Ricci curvature under the Kähler-Ricci flow is central for understanding the convergence of the Kähler-Ricci flow on Kähler-Einstein manifolds. It is well-known that the Kähler-Ricci flow preserves the positivity of the curvature operator, the positivity of the bisectional curvature and the positivity of the scalar curvature. Phong and Sturm [17] have found that the positivity of the Ricci curvature is preserved by the Kähler-Ricci flow on compact Kähler manifolds of dimension 2, under the assumption that the sum of any two eigenvalues of the traceless curvature operator on traceless $(1,1)$-forms is non-negative. Since their argument only uses the maximum principle, one may easily extend their result to a complete noncompact manifold with the same curvature assumptions above. As applications of our Theorem 1, we can prove that in both cases as above, there are global Kähler-ricci flow on the Kähler manifold with Ricci flat metric as its limit at time infinity.

We remark that one may formulate more convergence results using the invariant set constructed by B. Wilking [22], who finds almost all invariant curvature conditions of Ricci flows [11] [5] [6] [15] [16].

The plan of the paper is below. In section 2, we consider the non-local collapsing result obtained by using Perelman’s W-functional. In section 3, we consider the scalar curvature bound found by R. Hamilton and B. Chow in our Kähler-Ricci flow case. In sections 4 and 5, we prove Theorem 1. We consider some consequences of our Theorem 1 in the last section.
2. NO LOCAL-COLLAPSING AND PERELMAN’S W-FUNCTIONAL

We now recall some facts about Perelman’s W-functional on a closed Riemannian manifold \((M^n, g)\). First G. Perelman [13] defines the F-functional as

\[
F(g, f) = \int_M (R + |\nabla f|^2) dm
\]

where \(dm = e^{-f} dv_g\) is a fixed measure and \(R\) is the scalar curvature of the metric \(g\). This functional is an extension of Hilbert action on the Riemannian metric space on \(M\). Let \(v\) and \(h\) be the variations of \(g\) and \(f\) respectively. Since \(dm\) is fixed, we have the relation

\[
h = \frac{\text{tr}(hv)}{2}.
\]

Then we have

\[
\delta F(v, h) = -\int_M (v, Rc(g) + D^2 f) dm
\]

where \(Rc(g)\) is the Ricci tensor of \(g\) and \(D^2 f\) is the Hessian matrix of the function \(f\).

For any fixed parameter \(\tau > 0\), the W-functional is defined by

\[
W(g, f) = W(g, f, \tau) = \int_M \left[ \tau (R + |\nabla f|^2) + f - n \right] (4\pi \tau)^{-n/2} dm.
\]

That is,

\[
W(g, f) = (4\pi \tau)^{-n/2} \tau F(g, f) + (4\pi \tau)^{-n/2} \int_M (f - n) dm.
\]

The importance of W-functional is that it is a diffeomorphism invariant. Then we have

\[
\delta W(v, h) = -\tau \int_M (v, Rc(g) + D^2 f - \frac{1}{2\tau} g) (4\pi \tau)^{-n/2} dm.
\]

Hence the \(L^2\) gradient flow of \(W(g, f)\) is

\[
\partial_t g = -2(Rc(g) + D^2 f - \frac{1}{2\tau} g)
\]

with

\[
f_t = -\Delta_g f - R + \frac{n}{2\tau}.
\]

Let \(\phi(t)\) be the flow generated by the time-dependent vector-field \(\nabla_g \vec{f}\). Let \(\bar{g}(t) = \phi(t)^* g(t)\) and \(\bar{f}(t) = \phi(t)^* f(t)\).

Then we have

\[
\partial_t \bar{g} = -2(Rc(\bar{g}) - \frac{1}{2\tau} \bar{g})
\]

with

\[
\bar{f}_t = -\Delta_{\bar{g}} \bar{f} - \bar{R} + |\nabla_{\bar{g}} \bar{f}|^2 + \frac{n}{2\tau}.\]

Let \(C(s) = 1 - \frac{s}{\tau}\),

\[
t(s) = -\tau \log C(s).
\]

and

\[
\bar{g}(s) = C(s) \bar{g}(t(s)).
\]
Then we have
\[ \partial_t \bar{g} = -2\text{Rc}(\bar{g}) \]
and
\[ \dot{f} = -\Delta \bar{f} - \bar{R} + |\nabla \bar{f}|^2 + \frac{n}{2\tau C(s)} . \]

Define
\[ \mu(g, \tau) = \inf \{ W(g, f, \tau), \int_M dm = 1 \} . \]

Then for any fixed \( \tau > 0 \), along the Ricci flow, \( \mu(g(t), \tau) \) is non-decreasing.

We assume that the scalar curvature of the Kähler-Ricci flow is uniformly bounded, which can be proved in our situation (see section 3) as in Theorem 1. In a Kähler manifold \((M^n, g)\) (where \(n\) is the complex dimension), we normalize the W-functional as
\[ W(g, f, \tau) = (4\pi \tau)^{-n} \int_M [2\tau(R + |\nabla f|^2) + f - 2n]e^{-f} dV \]
with \((4\pi \tau)^{-n} \int_M e^{-f} dV = 1\) and define
\[ \mu(g, \tau) = \inf_{f \in (4\pi \tau)^{-n} \int_M e^{-f} dV = 1} W(g, f, \tau) . \]

Assume now that \((M, g)\) is a complete noncompact Riemannian manifold. Note that for \( u = e^{-f/2} \), we have
\[ (4\pi \tau)^{-n} \int_M u^2 dV = 1 \]
and
\[ W(g, f, \tau) = (4\pi \tau)^{-n} \int_M [2\tau(Ru^2 + 4|\nabla u|^2) + u^2 \log u^2 - 2nu^2]dV \]
which will be written as \( W(g, u, \tau) \). This latter formulation allows us to define the W-functional for \( u \in C_0^\infty(M) \) (that is \( e^{-f/2} \in C_0^\infty(M) \) and this is noticed by G.Perelman, see Remark 3.2 in [18], see also p.240 of [9]) and this fact is very useful in analyzing the properties of W-functional. Applying the decay estimates of Hamilton [12] and Dai-Ma [10], Li-Yau-Hamilton gradient estimate [9], and Perelman’s Proposition 9.1 [18], we can show that W-functional is also non-decreasing along the Ricci flow on a complete non-compact Riemannian manifold with the initial bounded curvature. Since the argument is well-known to experts, we omit the detail.

The important application about W-functional is the no local collapsing result due to G.Perelman: Using this property and the scalar curvature bound of the Ricci flow on \((0, T)\) with \( T < \infty \), we can get the no local-collapsing at \( T \), and the no collapsing data depends only on the initial metric and the global scalar curvature bound. For the full proof of this result, one may refer to G.Perelman’s paper [18]. See also Sesum-Tian’s paper [20] on compact Kahler manifold. We remark that the argument can be carried out to our complete non-compact case. Since the argument is almost the same, we omit the detail.
3. Uniform Scalar Curvature Bound

In this section, we follow the method of B. Chow [8] to get an uniform bound for the scalar curvature of the global Kahler-Ricci flow. We remark that we can use the maximum principle for the quantities below before the first singularity of the Ricci flow since we always use the local solution obtained by Shi [21] with bounded curvature.

**Theorem 3.** Assume that there is a potential function \( f \) on \( M \) such that

\[
R_{ij}(g_0) = f_{ij}
\]

with bounded gradient \( |\nabla g_0 f|_{C^0} \). Then there is a smooth function \( f = f(t) \) such that

\[
R_{ij}(g(t)) = f(t)_{ij}
\]

and we also have some uniform constant \( C_0 > 0 \) depending only on the initial metric such that

\[
R + |\nabla g(t) f|^2 \leq C_0.
\]

This result on a compact Riemannian manifold is due to B. Chow [8]. Since our manifold is complete and noncompact, we give the full proof.

We also have the following.

**Theorem 4.** Assume the same condition for the global Kahler-Ricci flow \( g(t) \) with the potential function \( f \) being a bound function. Then we also have some uniform constant \( C_1 > 0 \) depending only on the initial metric and the dimension \( n \) such that

\[
R + |\nabla g(t) f|^2 \leq C_1/t.
\]

WE now prove Theorem 3.

**Proof:** Since

\[
R_{ij} = -\partial_i \partial_j \log \text{det}(g_{k\bar{l}}) = \partial_i \partial_j f,
\]

we have

\[
R = g^{ij} \nabla_i \nabla_j f = \Delta f
\]

and

\[
\partial_t R_{ij} = -\nabla_i \nabla_j (g^{kl} \partial_k g_{ij}) = \nabla_i \nabla_j R.
\]

The latter is equivalent to

\[
\nabla_i \nabla_j f_t = \nabla_i \nabla_j \Delta f.
\]

That is,

\[
\partial_t \partial_j (\partial_i - \Delta) f = 0.
\]

We let

\[
\phi = -\int_0^t (\partial_i - \Delta) f.
\]

Then we have

\[
\phi(0) = 0, \; \phi_t = -(\partial_i - \Delta) f,
\]

and

\[
\partial_t \partial_i \partial_j \phi = \partial_i \partial_j \phi_t = 0.
\]
Hence,
\[ \partial_t \partial_j \phi = 0, \]
\[ R_{ij} = \partial_i \partial_j (f + \phi), \quad R = \Delta (f + \phi), \]
and
\[ (\partial_t - \Delta)(f + \phi) = 0, \quad \text{on } M \times [0, T). \]

Starting from now, we replace \( f \) by \( f + \phi \), which is still denoted by \( f \) for simplicity.

Using
\[ R = \Delta (f + \phi) = \partial_t (f + \phi), \]
we can write
\[ (f + \phi)(t) = f(0) + \int_0^t R, \]
which has bounded gradient in any finite time.

Recall that the scalar curvature evolution equation is
\[ (\partial_t - \Delta) R = |R_{ij}|^2. \]
By this, we have
\[ (\partial_t - \Delta) R \geq \frac{1}{n} R^2. \]
Applying the maximum principle we get the lower bound \( R \geq -\frac{4}{n} \).

Recall that
\[ |\nabla f|^2 = g^{ij} f_i f_j. \]
Then we have
\[ \partial_t |\nabla f|^2 = R^{ij} f_i f_j + g^{ij} (\Delta f)_i f_j + g^{ij} (\Delta f)_f. \]
Using the Bochner formula we then get
\[ \partial_t |\nabla f|^2 = \Delta |\nabla f|^2 - |f_i|^2 - |f_j|^2. \]
Using \( R_{ij} = f_{ij} \) again, we then get
\[ (2) \quad \partial_t |\nabla f|^2 = \Delta |\nabla f|^2 - |f_i|^2 - |R_{ij}|^2 \]
Adding (1) and 2(2) we obtain
\[ \partial_t (2|\nabla f|^2 + R) = \Delta (2|\nabla f|^2 + R) - 2|f_i|^2 - |R_{ij}|^2 \leq \Delta (2|\nabla f|^2 + R). \]
Applying the maximum principle we know that
\[ 2|\nabla f|^2 + R \leq \sup_M [2|\nabla f|^2(0) + R(0)]. \]
In particular, we have
\[ R \leq C_0 := \sup_M [2|\nabla f|^2(0) + R(0)]. \]

This finished the proof of Theorem 3. \( \square \)

The proof of Theorem 4 is similar and for completeness, we present its proof in full detail.
Proof. We now recall the following Bernstein type estimate. Note that from
\[(\partial_t - \Delta) |\nabla f|^2 \leq 0,\]
we can deduce that
\[(\partial_t - \Delta)(t|\nabla f|^2) \leq |\nabla f|^2.\]

Consider the equation
\[(\partial_t - \Delta)f^2 = 2|\nabla f|^2\]
Then
\[(\partial_t - \Delta)(t|\nabla f|^2 + f^2) \leq 0.\]
Using the maximum principle we get that
\[t|\nabla f|^2 + f^2 \leq C\]
where \(C\) is some uniform constant depending only on the initial data \(f_0\). In particular, we have \(|\nabla f|^2 \leq \frac{C}{t}\). We shall use this fact in B.Chow’s argument.

By direct computation, we have
\[(\partial_t - \Delta)[t(|\nabla f|^2 + R)] \leq -\frac{2t}{n}R^2 + R + 2|\nabla f|^2.\]
Note that
\[\frac{t}{n}R^2 = \frac{t}{n}(R + 2|\nabla f|^2)^2 - \frac{4t|\nabla f|^2}{n}(R + |\nabla f|^2).\]
Then we get from (3) that
\[(\partial_t - \Delta)[t(|\nabla f|^2 + R)] \leq -\frac{t}{n}(R + 2|\nabla f|^2)^2 - \frac{2t}{n}R^2 + R + 2|\nabla f|^2 + \frac{4t|\nabla f|^2}{n}(R + |\nabla f|^2).\]
Assume that \(R \geq 0\), using \(|\nabla f|^2 \leq \frac{C}{t}\), we obtain from (4) that
\[(\partial_t - \Delta)[t(|\nabla f|^2 + R)] \leq -\frac{t}{n}(R + 2|\nabla f|^2)^2 - \frac{2t}{n}R^2 + C(R + 2|\nabla f|^2)\]
Note that the term \(R + 2|\nabla f|^2\) is bounded by \(C_0\), and if \(R \geq 0\) and \(\frac{t}{n}(R + 2|\nabla f|^2) \geq 2CC_0\), we then have
\[(\partial_t - \Delta)[t(|\nabla f|^2 + R)] \leq 0.\]
As in [21], any \(T > 0\), we construct a distance like positive function \(h(x)\) such that for any \(t \in [0, T]\),
\[\Delta_{g(t)} h \leq 1,\]
and
\[h(x) \to \infty\]
as \(d(x, \bar{x}) \to \infty\), where \(\bar{x}\) is a fixed point in \(M\). Define the domain
\[\Omega_T = \{(x, t) \in M \times (0, T], R \geq 0, \frac{t}{n}(R + 2|\nabla f|^2) \geq 2CC_0\}.\]
Note that in \(\Omega_T\), for \(\epsilon > 0\),
\[\epsilon t \nabla f|^2 + R + 2\epsilon t - \epsilon h(x)\]
\[\leq \epsilon t |\nabla f|^2 + R - 2\epsilon + \epsilon \Delta_{g(t)} h(x)\]
\[\leq -\epsilon < 0.\]
Since the region $U_\epsilon$ for the function $t(\|\nabla f\|^2 + R) - 2et - eh(x) \geq 2CC_0$ is compact and using the maximum principle in $U_\epsilon$, we know that

$$t(\|\nabla f\|^2 + R) \leq 2CC_0 + 2et + eh(x), \quad (x, t) \in U_\epsilon \cap \Omega_T.$$  

Note that $\Omega_T = \bigcup_{\epsilon > 0} U_\epsilon$ and $U_\epsilon \subset U_{\epsilon'}$ for $\epsilon > \epsilon'$. Then for any $(x_0, t_0) \in \Omega_T$, we have some $\epsilon_0 > 0$ such that

$$(x_0, t_0) \in U_\epsilon$$

for any $0 < \epsilon < \epsilon_0$ and

$$t(\|\nabla f\|^2 + R)(x_0, t_0) \leq 2CC_0 + 2et_0 + eh(x_0).$$

Letting $\epsilon \to 0$, we have

$$t(\|\nabla f\|^2 + R)(x_0, t_0) \leq 2CC_0,$$

That is to say, in $\Omega_T$, we have

$$t(\|\nabla f\|^2 + R) \leq 2CC_0.$$

Therefore, for any $T > 0$, we always have $|R| \leq 2\max(CC_0, n)/t$ in $M \times (0, T)$. This completes the proof of Theorem 4.

4. global flow and finite time blow-up is Ricci flat

In this section we shall prove Theorem 2.

Using our recent result about extension result of Ricci flow [13], which extends the beautiful result of N.Sesum on compact manifold, we know that if the Kähler-Ricci flow blows up at first finite time $T$, then the Ricci curvature must blow up at time $T$. Using the scalar curvature bound of the Kähler-Ricci flow in previous section, we can show the Ricci-flatness of the blow up limit of the flow.

**Theorem 5.** Assume the conditions as in Theorem 1. If the Kahler-Ricci flow exists only in finite time, then its blow-up limit at the maximal time exists and the limit is Ricci flat.

We remark that the the blow-up limit or just limit in the results above and below is only for some subsequence, not for full blow-up sequence. To prove the result above, we need to recall Hamilton’s Cheeger-Gromov convergence theorem for Ricci flows.

**Theorem 6.** (Compactness theorem for Ricci flows) Assume that $(M^n_j, g_j(t), x_j), t \in (a, b) \ni 0$, is a sequence of complete pointed solutions of Ricci flows satisfying the following two conditions.

1. (Uniform bounded curvature condition)

$$|Rm(g_j(t))|_{g_j(t)} \leq C, \quad M^n_j \times (a, b)$$

2. (Uniform injectivity radius bound at $(x_j, 0)$)

$$\inf_{(x_j, 0)}(x_j) \geq \delta > 0.$$
Then there exists a subsequence of the solutions \((M^n_j, g_j(t), x_j)\) which converges as \(j \to \infty\) to a pointed complete Ricci flow \((M^n_\infty, g_\infty(t), x_\infty)\), \(t \in (a, b) \ni 0\), with above two conditions.

The process to prove Theorem 5 is below. First we note that by using Perelman’s non-local-collapsing result at any finite time, we can applying Hamilton’s Cheeger-Gromov convergence theorem to take the blow-up limit \(g_\infty(t)\). Then we use the uniform scalar curvature bound of the original Ricci flow to obtain \(R(g_\infty) = 0\), which implies the Ricci flatness of the limit metric \(g_\infty\).

**Proof.** (for Theorem 5). Let \(T < \infty\) be the first blow-up time of the Ricci flow \((M, g(t))\). Choose some constants \(C > 0\), the finite time \(t_j \to T^-\) and take points \(x_j\) such that

\[
\sup_{M \times [t_j - C^{-1}K_j^{-1}, t_j]} |Rm(g(t))| \leq CK_j
\]

where \(K_j = |Rm(g(t_j))(x_j)| \to \infty\). Define

\[
g_j(t) := K_j g(t_j + K_j^{-1}t).
\]

Clearly we have \(|Rm(g_j(t))| \leq C\) and \(|R(g_j(t))| \leq C/K_j \to 0\). Using Perelman’s non-local collapsing result we know that the injectivity radius bound condition in Theorem 6 is satisfied. By Theorem 6 we get the limit Ricci flow \(g_\infty(t)\) of \(g_j(t)\) with \(|Rm(g_\infty(t))| \leq C\) and \(R(g_\infty(t)) = 0\) for \(t \in (-\infty, \omega)\). Using the scalar curvature equation for Ricci flow

\[
(\partial_t - \Delta g) R = |Rc(g)|^2,
\]

we get that \(Rc(g_\infty(t)) = 0\). This completes the proof of Theorem 5. \(\Box\)

We now prove Theorem 2. Assume that we have a local Kähler-Ricci flow with initial metric \(g_0\) with bounded curvature and non-negative bisectional curvature. Using the result of Bando-Mok, we know that the non-negativity bisectional curvature is preserved along the flow. Hence the Ricci curvature is non-negative along the Ricci flow. Using the scalar curvature bound and non-negativity Ricci curvature condition, we get via the use of the elementary relation \(R_{ij} \leq R_{gij}\) the uniform bound for the Ricci curvature. Then using Theorem 1.4 [13], we know that the Kähler-Ricci flow exists globally with bounded curvature.

5. **global Kähler-Ricci flow and its limit at \(t = \infty\)**

In this section, we prove Theorem 1. Note that there is no local collapsing for the Ricci flow at any finite time according to the result of perelman mentioned in section two. If the curvature of the flow blows up at infinite time, we then can take the similar blow-up limit as in the finite time case [13] and show that its blow-up limit is Ricci flat. Otherwise, the curvature of the flow is uniformly bounded and by Theorem 4, we can simply take the limit, which is scalar flat at first. Using the evolution equation

\[
(\partial_t - \Delta g(t)) R = |Rc(g(t))|^2,
\]

again we know that the limit is also Ricci flat.

This completes the proof of Theorem 1.
6. CONSEQUENCES OF THEOREM [1]

We give two applications of Theorem [1]. One is for the metric $g_0$ with non-negative bisectional curvature, which implies the non-negativity of the Ricci curvature. Note that the non-negativity of the bisectional curvature property is preserved by Kahler-Ricci flow as proved by Bando-Mok as mentioned in the introduction. Note that in this case, the scalar curvature dominates other curvatures. Since we have global bound of the scalar curvature of the Ricci flow, we must have the global Kahler-Ricci flow with bounded curvature. Hence by Theorem [1] the flow is global and has its limit at $t = \infty$ the Ricci flat metric. Then we have the following result.

**Theorem 7.** Assume that $(M, g)$ is a complete non-compact Kähler manifold with bounded curvature such that there is a potential function $f$ of the Ricci tensor, i.e.,

$$R_{ij}(g_0) = f_{ij}.$$  

Assume the quantity $|f|_{C^0} + |
abla g_0 f|_{C^0}$ is finite. Suppose the initial metric has non-local-collapsing and the $L^2$ Sobolev inequality holds true on $(M, g_0)$. Assume also that the initial metric has non-negative bisectional curvature. Then the Kahler-Ricci flow with the initial metric $g_0$ exists globally with Ricci-flat limit at infinite time.

The other is for complex dimension two, where we assume that $(M, g)$ is a complete non-compact Kähler surface with the bounded curvature, positive Ricci curvature, and positive partial curvature operator $S$ in the case handled by Phong-Sturm [17]. We make this precise and define the partial curvature operator

$$S_{ijkl} = R_{ijkl} - \frac{1}{n} (S_{ijl} \delta_{kl} + \delta_{ij} S_{kl}) + \frac{1}{n^2} R \delta_{ij} \delta_{kl}$$  

acting on the space of traceless $(1, 1)$ forms on $M$. Here, $R_{ijkl}$ is the curvature tensor of $g$, $S_{ij} = R_{ij} - \frac{1}{n} R g_{ij}$. Then we have

**Theorem 8.** Assume that $(M, g)$ is a complete non-compact Kähler surface with bounded curvature such that there is a potential function $f$ of the Ricci tensor, i.e.,

$$R_{ij}(g_0) = f_{ij}.$$  

Suppose the quantity $|f|_{C^0} + |
abla g_0 f|_{C^0}$ is finite. Assume that the initial metric has non-local-collapsing and the $L^2$ Sobolev inequality holds true on $(M, g_0)$. Assume also that the initial metric has non-negative non-trivial Ricci curvature and the non-negative sum of the two lowest eigenvalues of the operator $S$ acting on the space of traceless $(1, 1)$ forms on $M$. Then the Kahler-Ricci flow with the initial metric $g_0$ exists globally with Ricci-flat limit at infinite time.

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