S-wave Scattering Length from Effective Positronium-Positronium Interaction for Bose-Einstein Condensates

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The s-wave scattering length for the Positronium-Positronium interaction is estimated semi-phenomenologically with the long-range van der Waals force and the short-range repulsive potential that represents the hard core between two positronium. The obtained value of the scattering length is \( a \sim 0.44 \text{ nm} \), and its stability is also checked for different parametrizations. Using this value, the Gross-Pitaevskii equation can be fixed for the Positronium Bose-Einstein condensates (Ps BEC). The static properties of Ps BEC are studied from the solutions of that equation. The phase-transition temperature shift of the Ps BEC due to the Ps-Ps interaction is also evaluated with the mean-field approximation.

I. INTRODUCTION

Recent realizations of Bose-Einstein condensates for trapped Alkali-atom gas have made a great development in quantum physics of many-body system where the condensation phenomenon is one of the most important concepts and used in a vast area of physics such as atomic, laser, condensed-matter, nuclear, elementary-particle physics and also in cosmology. After the experimental success of the condensation, a lot of physical phenomena has been observed and studied on Alkali-atom BEC that include collective excitations, sound propagations, quantum interference, etc.

The success of the Alkali-atom BEC and the developments of the laser cooling and trapping techniques encouraged the study for the BEC of bose/fermi particles other than Alkali atoms: recently, the BEC of Rb isotope with the sympathetic cooling technique and the potassium fermi condensates have been observed. The bose/fermi mixed condensates are also studied for the potassium isotopes.

The positronium (Ps), the bound state of the electron and the positron, has been also an interesting candidate for the BEC:

- The Ps is the lightest atom with the mass \( m_{\text{Ps}} \sim 2m_e \) where \( m_e \) is the electron mass: \( 10^3 \) times lighter than the hydrogen atom. It leads to the large value of the critical temperature \( T_C \) of the Bose-Einstein phase transition: in free-gas and continuous approximation, \( T_C \) becomes 10 times higher than that of hydrogen BEC for the trapping potential of same size (\( T_C \propto m^{-1/3} \)). The effect of the Ps-Ps interaction for \( T_C \) will be discussed in this paper.
- The Ps has finite life-time and self-annihilates into photons: \( Ps \rightarrow 2\gamma \) with \( \tau_{\text{Ps}} \sim 0.125 \text{ ns} \) for para-Ps and \( Ps \rightarrow 3\gamma \) with \( \tau_{\text{Ps}} \sim 142 \text{ ns} \) for ortho-Ps. Thus, the Ps BEC should be an unstable condensate and decays with gamma-ray radiations, the coincident observation of which will give the precise measurements for position/momentum distributions of the BEC. Those radiated gamma-ray photons are also suggested to show new physical phenomena such as gamma-ray laser by spontaneous amplifications and the superradiance.

However, it should be noted that these very short life-times will make technical difficulties in experimental Ps BEC formations: \( \tau_{\text{Ps}} \) are in the comparable order with the relaxation time \( \tau_{\text{rel}} \sim \text{ ns} \) in atomic phenomena so that the cooling methods that depend on the natural relaxation (like the evaporative cooling) might be less effective for the Ps BEC formation. At present, several faster cooling methods are planned by several experimental groups for Ps-BEC formations, and the preliminary experiments are progressing.

Originally, the Ps BEC was considered as \( e^+e^- \) condensates as a source of gamma-ray laser, the \( e^+e^- \) plasma in the strong magnetic field at the neutron star surface and the storage ring, and also as the source of the gamma-ray burst in astro-physics. The Ps BEC itself has been suggested as a source of the gamma-ray laser, but detailed studies for the properties of Ps BEC including the Ps-Ps interaction effects have not been done.

In the binding structure, the Ps is similar with the hydrogen atom (H), where the main differences are mass of positively charged particle (\( m_{\text{p}}/m_e \sim 2000 \)) (and their stability). The BEC of poralized H has been reported with \( N \sim 10^9 \) atoms. More similar object are excitons in semiconductors: the bound states of electron and hole in
valence band with \( m_h/m_e \approx 0.7 \). The exciton has also finite life-time \( \tau \sim 30 \text{ns} \) and decays into phonon, so that it is quite similar with the Ps. The BEC of excitons in CuO has also been reported [17]. The experimental success of those BEC encourages strongly studies of Ps BEC.

In this paper, we derive the Ps-Ps interaction potential with the long-range van der Waals interaction and the short-range cut-off that represents the hard core repulsion of positronium. The s-wave scattering length is estimated from the solution of the Schrödinger equation using this potential. With the given scattering length, the Gross-Pitaevski trapping potential, we study the ground-state properties of Ps BEC. We also estimate the Ps-Ps interaction effects for the critical temperature \( T_C \) in the mean-field approximation, which are shown to be negligible except large particle number BEC.

II. EFFECTIVE PS-PS INTERACTION POTENTIAL

i) van der Waals Potential for Ps-Ps Interaction

We consider the system of two positronium Ps(1) and Ps(2). The spacial coordinates of the electron and positron composing Ps(1) are denoted by \( x_1 \) and \( x_a \) and those of Ps(2) are by \( x_2 \) and \( x_a \), with which the relative coordinates are defined by \( r_{ij} = x_j - x_i \). Especially, for relative coordinates between electron and positron, we denote

\[
\mathbf{r}_1 \equiv \mathbf{r}_{a1}, \quad \mathbf{r}_2 \equiv \mathbf{r}_{a2}.
\]  

The hamiltonian of the system is

\[
H = H_{CM} + H_1 + H_2 + H_{int},
\]

where \( H_{CM} \) is the center-of-mass part that can be neglected in the derivation of the Ps-Ps interaction because of the translational invariance. The \( H_{1,2} \) in (2) are the internal parts of Ps(1,2) with the Coulomb potential:

\[
H_{1,2} = \frac{p_{1,2}^2}{2m_e} - \frac{\alpha \hbar c}{r_{1,2}},
\]

where \( r_{1,2} = |\mathbf{r}_{1,2}| \) and \( \mathbf{p}_{1,2} \) are the momenta conjugate with \( \mathbf{r}_{1,2} \). The \( \alpha \) in (3) is a fine structure constant and \( \alpha = e^2/(4\pi\epsilon_0\hbar c) \sim 1/137 \) in the SI unit system. The interaction part of Ps(1,2) is represented by

\[
H_{int} = \alpha \hbar c \left( \frac{1}{r} \right. \left. + \frac{1}{r_{12}} - \frac{1}{r_{a2}} - \frac{1}{r_{b1}} \right),
\]

where \( r_{ij} = |\mathbf{r}_{ij}| \) and \( r \equiv r_{ab} = |\mathbf{r}_{ab}| \) is the distance between two positrons, Ps(1) and Ps(2).

Let us consider the case where the distance between Ps(1,2) are very large: \( r \gg r_{ab} \geq r_1, r_2 \). In that case, the Ps-Ps interaction potential can be determined by the van der Waals potential [18,19].

Eq. (4) can be expanded by \( 1/r \), and we get the dipole-dipole interaction:

\[
H_{int} \approx -\frac{\alpha \hbar c}{r^3} \left\{ \frac{3}{r^2} \mathbf{r}_1 \cdot \mathbf{r}_2 \mathbf{r}_2 \cdot \mathbf{r}_1 - \mathbf{r}_1 \cdot \mathbf{r}_2 \right\}.
\]

When \( r \gg 0 \), the interaction term \( H_{int} \sim 1/r^3 \) in (3) can be evaluated with perturbation. The unperturbed part is just \( H_0 = H_1 + H_2 \) defined by (3) and the zeroth-order unperturbed wave functions that diagonalize the \( H_0 \) are \( \psi_{nlm}(1) \otimes \psi_{n'm'}(2) \) where \( \psi_{nlm}(i) \) is the eigenstate of Ps(i), \( H_0 \psi_{nlm}(i) = E_n^{(0)} \psi_{nlm}(i) \), with the binding energy:

\[
E_n^{(0)} = -m_e c^2 \alpha^2/(4(n+1)^2).
\]

The \( n, l, m \) are principal, azimuthal and magnetic quantum numbers. In the Ps state function, we dropped the spin angular momentum part and spin quantum numbers \( S = 0, 1 \) and \( S_z \), because the spin-spin/spin-orbital interactions are weak, and give a very small modifications that can be neglected in the present semi-qualitative estimations.

From the second-order perturbation for \( H_{int} \), we get the van der Waals potential:

\[
E^{(2)} = -\alpha^2 \hbar^2 \sum_{n \neq 0,l,m,n' \neq 0,l',m'} \frac{|\langle 0|d_1|\psi_{nlm}(1)\rangle|^2 |\langle 0|d_2|\psi_{n'l'm'}(2)\rangle|^2}{E_n^{(0)} + E_n'^{(0)} - E_0^{(0)} - E_0'^{(0)}},
\]  

where \( d_i \) are the dipole operators.
where \( d_{1,2} = z_{1,2} \) are dipole-moment operators of \( \text{Ps}(1,2) \) and \( |0\rangle = |\psi_{000}\rangle \). Using eq. (3), the denominator of eq. (7) becomes

\[
E_n^{(0)} + E_n'^{(0)} - E_0^{(0)} - E_0'^{(0)} = \frac{\alpha \hbar c}{4a_0} (2 - (n+1)^{-2} - (n' + 1)^{-2}),
\]

where \( a_0 = \hbar c/(am_e c^2) \approx 0.0529 \text{nm} \) is the Bohr radius. The term \((n+1)^{-2}\) in the denominator of eq. (8) is very small for large value of \( n \), so that it can be neglected in the first approximation

\[
E_n^{(0)} + E_n'^{(0)} - E_0^{(0)} - E_0'^{(0)} \sim \frac{\alpha \hbar c}{2a_0}.
\]

Using eq. (8) and the completeness condition: \( \sum_{n,l,m} \langle 0|d_1|\psi_{nlm}\rangle \langle \psi_{nlm}|d_1|0\rangle = \langle 0|z^2|0\rangle \), eq. (7) becomes

\[
E^{(2)} = -\frac{6\alpha \hbar c(2a_0)^5}{r^6} |\langle 0|z^2|0\rangle|^2.
\]

Using \( \langle 0|z^2|0\rangle = (2a_0)^2 \) with the Coulomb wave function \( \psi_{000} \) for the ground state, we get van der Waals potential:

\[
E^{(2)} = -6\alpha \frac{\hbar c(2a_0)^5}{r^6}.
\]

The error for the above approximation \( [1] \) has been evaluated by numerical summation for \( [1] \) for the H-H interaction \( [1] \). When their results are rescaled for Ps potential, they obtained 6.47 instead of 6 in \( [1] \). It suggests that the present results \( [1] \) should be a reasonable estimation.

ii) Effective Ps-Ps interaction potential with the short-range hard core

In the short range part, the real part of the Ps-Ps interaction should be characterized by repulsive hard core at the distance of atomic diameter where two positronium begin to overlap. In the present paper, we treat this hard core potential phenomenologically in two manners: A) the cut-off potential at the distance of atomic diameter where two positronium begin to overlap. In the present paper, we treat this hard core length for nonsinglet channel \( [9] \), the present potential should be applicable also for these channels.

The phenomenological parameters \( r_C \) and \( C \) in \( V^C_{\text{Ps-Ps}} \) and \( V^L_{\text{Ps-Ps}} \) are fixed to reproduce the binding energy \( B_{\text{Ps}_2} \) of the molecular state \( \text{Ps}_2 \), which has not been confirmed experimentally but predicted theoretically/numerically with the hamiltonian \( [2] [4] [8] \). Recent results for \( B_{\text{Ps}_2} \) by elaborate numerical calculations are around 0.435 eV \( [21] [23] \).

To reproduce \( B_{\text{Ps}_2} = 0.435 \text{eV} \) with solving the Schrödinger equation for \( V^C_{\text{Ps-Ps}} \) numerically, we obtain A) \( r_C = 1.78a_0 \) (with the potential depth \( v_0 = -6.13a_0 \hbar c/a_0 \)) for \( V^C_{\text{Ps-Ps}} \), and B) \( C = 2.25 \) (\( v_0 = -2.0a_0 \hbar c/a_0 \)) for \( r_C = 1.90a_0 \) the distance where \( V^L_{\text{Ps-Ps}}(r_C) = 0 \) for \( V^L_{\text{Ps-Ps}} \). It should be noticed that two phenomenological potentials have almost the same hard-core distances \( r_C, r'_C \sim 1.80a_0 \), which are consistent with the diameter of Ps, \( 2a_0 \).

We should comment about the effect of Ps spin state. Because of the Pauli principle, the Ps bound state exists in singlet state, so that the effective Ps-Ps potential using the binding energy of \( \text{Ps}_2 \) as an input parameter should be considered for singlet channel. However, because the \( r_C \) of \( V^C_{\text{Ps-Ps}} \) is very close to \( 2a_0 \) that should be expected to be the hard core length for nonsinglet channel \( [2] \), the present potential should be applicable also for these channels.

### III. S-Wave Scattering Length by Effective Ps-Ps Interaction

For the Ps-Ps s-wave scattering, the relative wave function of two positronium becomes \( \Psi(r) = r\chi(r) \) that satisfies the Schrödinger equation:

\[
-\frac{\hbar^2}{2\mu} \frac{d^2}{dr^2}\chi(r) + V_{\text{Ps-Ps}}(r)\chi(r) = E\chi(r),
\]

where \( \mu \) is a reduced mass \( \mu = m_{\text{Ps}}/2 \sim m_e \).

The s-wave scattering length \( a \) is obtained from the asymptotic form of the \( E = 0 \) solution for eq. (13) as
\[ \chi(r \to \infty) \sim 1 - \frac{r}{a}. \] (14)

In fig. 4, we show the numerical results for the \( E = 0 \) wave function \( \chi(r) \) of eq. (13) with \( V_{\text{P}S_{\text{Ps}}}^{\text{CO},LJ} \) defined in the previous section. The one peak at \( r \sim 3.5a_0 \) shows that (only) one weak bound state exists for this potential and it corresponds to the \( \text{Ps}_2 \) molecular state whose binding energy has been used as an input for \( V_{\text{Ps}_{\text{Ps}}} \). As seen in fig. 4, for \( r \gtrsim 5a_0 \), \( \chi(r) \) in fig. 4 are in the asymptotic linear region, so that the scattering length \( a \) can be read off by \( \chi(a) = 0 \):

\[
\begin{align*}
\text{A)} & \quad a = 0.440 \text{ nm} \quad \text{for } V_{\text{Ps}_{\text{Ps}}}^{\text{CO}}, \\
\text{B)} & \quad a = 0.437 \text{ nm} \quad \text{for } V_{\text{Ps}_{\text{Ps}}}^{LJ}.
\end{align*}
\] (15)

Both results are almost consistent, and it means that the \( s \)-wave scattering length (the low-energy limit) do not depend on the detailed form of the short-range potential. Existence of only one weakly bound state in \( \text{Ps}_{\text{Ps}} \) system is also essential for the rigidness of \( a \) given with the effective potential.

For the illustration of the effects by bound states for the scattering length, the \( s \)-wave scattering state function \( \chi(r) \) is shown in fig. 5 for the effective interaction potential fitted with \( \text{H-H} \) binding energy. That potential has the larger \( v_0 \) than the \( \text{Ps}_{\text{Ps}} \) potential and produces many excited molecular states that can be seen as the oscillating behavior of \( \chi(r) \) for \( r \gtrsim 5a_0 \). Its behavior depends on the short-range part of the potential given only in phenomenological manner in the effective potential, so that it leads to the impreciseness of the connected long-range behavior of \( \chi(r) \) from which the scattering length \( a \) is given. For the \( \text{Ps}_{\text{Ps}} \) case, it should also be too rough approximation to treat the molecular bound state \( \text{Ps}_2 \) itself as that of the effective potential given in this paper, but the long-range part of the scattering state should be more reliable because its short-range behavior is very simple and the potential has only one bound state that was used for the input parameter. The consistency between the scattering lengths given with \( V_{\text{Ps}_{\text{Ps}}}^{\text{CO}} \) and \( V_{\text{Ps}_{\text{Ps}}}^{LJ} \) also supports it.

Currently-used \( s \)-wave scattering lengths for Alkali atoms are, 5.77 nm for \( ^{87}\text{Rb} \) [24], 2.75 nm for \( ^{23}\text{Na} \) [22], and \(-1.45\) nm for \( ^{7}\text{Li} \) [20]. The resultant \( \text{Ps}_{\text{Ps}} \) \( s \)-wave scattering lengths in [13] are \( a \sim 0.44 \) nm, which is smaller than those for Alkali atoms in one order but larger than that for polarized \( \text{H} \) atom (triplet), \( a = 0.065 \) nm [25]. It shows that the \( \text{Ps} \)-gas behaves more like perfect gas than the Alkali atom gas. The rough estimation for the \( \text{Ps}_{\text{Ps}} \) scattering length can also be made directly from the binding energy of \( \text{Ps}_2 \) [28]: \( a \sim (2m_e B_{\text{Ps}_2})^{-1/2} \sim 0.3 \) nm [2], which is also consistent with the present result.

IV. PS BEC WITH PS-PS INTERACTION

The dynamical behavior of BEC is represented by the ground-state wave function (order parameter) \( \psi(x) \) and, for weak-interacting bose gas in the harmonic oscillator trapping potential at \( T = 0 \), it satisfies the Gross-Pitaevski (GP) equation [3]:

\[
i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m_{\text{Ps}}} \nabla^2 \psi + \frac{1}{2} m_{\text{Ps}} \omega_{\text{HO}}^2 r^2 \psi + g N |\psi|^2 \psi,
\] (16)

where \( \omega_{\text{HO}} \) is the angular frequency of the harmonic oscillator potential, and \( N \) is the condensed bose particle number: the normalization condition for \( \psi \) is taken as

\[
\int d^3x |\psi(x)|^2 = 1.
\] (17)

For low-density bose gas, the boson-boson interaction is dominated by the low-energy \( s \)-wave scattering, and the interaction constant \( g \) in [16] is given with the \( s \)-wave scattering length \( a \) [2]:

\[
g = \frac{4\pi \hbar^2}{m_{\text{Ps}}} a.
\] (18)

Using the dimensionless variables \( \tilde{r} = r/a_{\text{HO}} \), \( \tilde{\omega} = \omega_{\text{HO}} \) and \( \tilde{\psi} = a_{\text{HO}}^{3/2} \psi \) with the harmonic oscillator length \( a_{\text{HO}} = \sqrt{\hbar/(m_{\text{Ps}} \omega_{\text{HO}})} \), the GP equation (14) becomes

\[
i\frac{\partial \tilde{\psi}}{\partial \tilde{t}} = -\frac{\hbar^2}{2} \tilde{\nabla}^2 \tilde{\psi} + \frac{\hbar^2}{2} \tilde{\psi} + \tilde{g} N |\tilde{\psi}|^2 \tilde{\psi},
\] (19)
where the dimensionless constant $\tilde{g}N$ is

$$
\tilde{g}N = \frac{4\pi a}{a_{HO}}N.
$$

(20)

Thus, the solutions of GP equation are similar for the same value of $\tilde{g}N$. It should be noticed that the interaction effects come into eq. (19) through $\tilde{g}N$-term, so that they are more effective for large value of $\tilde{g}N$.

In fig. 6, the solutions of eq. (16) are shown with $a = 0.437$ nm in (15) and $\omega = 3.1$ sec$^{-1}$, for several boson numbers: $N = 2.6 \times 10^7$ (curve 2), $2.0 \times 10^8$ (curve 3), $7.9 \times 10^8$ (curve 4), $2.5 \times 10^9$ (curve 5). With $a(PS) = 0.44$ nm for Ps, we obtain $a(PS)/a_{HO} = 2.16$ eV$^{-1/2} \sqrt{\hbar \omega_{HO}}$, which is much smaller than $a(Rb)/a_{HO} = 7062$ eV$^{-1/2} \sqrt{\hbar \omega_{HO}}$ for Rb atom: this large difference comes not only from weak Ps-Ps interaction $a(PS)/a(Rb) \sim 1/130$ but from the mass difference $m_{Ps}/m_{Rb} \sim 1/281$. Because of the small $a/a_{HO}$ for the Ps-Ps interaction, for large particle number $N = 2.6 \times 10^7$ (curve 2 in fig. 6), the calculated $|\psi(r)|$ changes only $\sim 16\%$ from that for the ideal gas $a = 0$ (curve 1 in fig. 6). To realize 70\% change for $|\psi(0)|$, the Ps of $N = 2.5 \times 10^9$ should condensate (curve 5 in fig. 6). For Rb BEC, because of large scattering length $a = 5.77$ nm, $N = 6.7 \times 10^9$ is enough for realization of the profile 5 in fig. 6.

For the ideal gas with no interactions, the critical temperature $T_C^0$ can be calculated in thermodynamical continuum limit [3]:

$$
k_B T_C^0 = 0.94 \hbar \omega_{HO} N^{1/3},
$$

(21)

where $k_B$ is a Boltzmann constant. The particle interaction has an effect to shift the critical temperature: $T_C = T_C^0 + \delta T_C$. The $\delta T_C$ can be evaluated with the mean-field and semiclassical approximation [29]:

$$
\frac{\delta T_C}{T_C^0} = -1.33 \frac{a}{a_{HO}} N^{1/6}.
$$

(22)

For Alkali atom BEC, eq. (22) is consistent with experimentally observed temperature shift [2] ($\sim 2\%$ shift for Rb BEC of $N \sim 10^4$). In the case of Ps BEC, eq. (22) becomes

$$
\frac{\delta T_C}{T_C^0} = -2.87 eV^{-1/2} \sqrt{\hbar \omega_{HO}} N^{1/6}.
$$

(23)

It gives very small shift for $T_C$ for Ps BEC, and can be neglected in the first approximation except BEC of large $N$.

It should be noted that the concept of $T_C$ is exact for equilibrium (thermodynamical) system. For real experimental situations, the Ps BEC might be performed as nonequilibrium system due to the small Ps life-time that is almost the same order with relaxation time. In that case, the $T_C$ evaluated here should be considered as estimation for the energy scale.

V. SUMMARY AND DISCUSSIONS

In the present paper, we studied the effective Ps-Ps interaction potentials with interpolating the long-range van der Waals interaction and the short-range repulsive hard core. The latter was taken phenomenologically in two kinds of parametrizations: A) sudden cut-off and B) Lenard-Jones (and cut-off) types. The Ps-Ps s-wave scattering length was calculated from the solutions of the Schrödinger equation with those effective potentials, and we obtained $a \sim 0.44$ nm. It is ($2 \sim 10$)-times smaller than that of the Alkali atoms, and larger than that of the polarized hydrogen atom. The stability of $a$ for the different parametrizations of the short-range interaction was also checked. Using that value of the scattering length, we studied the ground-state wave functions for Ps BEC with solving the Gross-Pitaevskii equation. The Ps BEC is shown to be more perfect-gas-like than Alkali-atom BEC because of weaker interaction. The critical temperature shift $\delta T_C$ due to the Ps-Ps interaction was also estimated and shown to be very small because of the weak interaction and the small mass of Ps atom.

As commented in the last section, the inclusion of the dissipative effects to the Gross-Pitaevskii equation is essentially important for further studies of Ps BEC physics discussed in the introduction. It will be discussed in a future publication [30].

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FIG. 1. Relative coordinates of $e^+$ and $e^-$ in Ps-Ps system.

FIG. 2. Effective Ps-Ps interaction potential $V_{CO}^{Ps-Ps}$ against $r/a_0$. The $a_0 = 0.053$ nm is the Bohr radius. The $r_C$ is the cut-off length that represents the hard-core repulsive interaction. The $v_0$ in figure represents the potential depth.

FIG. 3. Effective Ps-Ps interaction potential $V_{LJ}^{Ps-Ps}$ in unit of $\alpha \hbar c/a_0$ eV against $r/a_0$. The $a_0 = 0.053$ nm is the Bohr radius. The short-range part is parametrized by the Lenard-Jones-type $r^{-12}$-potential. The cut-off $r'_C$ is defined by $V_{LJ}^{Ps-Ps}(r'_C) = 0$, and the $v_0$ in figure represents the potential depth.

FIG. 4. The profile of the s-wave Ps-Ps scattering solution $\chi(r)$ for $V_{LJ}^{Ps-Ps}$ in unit of $\alpha \hbar c/a_0$ eV against $r/a_0$ ($a_0 = 0.053$ nm). The s-wave scattering length $a$ can be read off at $\chi(a) = 0$, and $a = 0.437$ nm. For $V_{CO}^{Ps-Ps}$, we obtain almost the same result, and $a = 0.440$ nm.

FIG. 5. The profile of the s-wave H-H scattering solution $\chi(r)$ for $V_{LJ}^{Ps-Ps}$ against $r/a_0$ ($a_0 = 0.053$ nm).
FIG. 6. The space-distribution $|\psi(r)|$ for Ps BEC at $T = 0$ with the trapping harmonic oscillator potential ($\omega_{HO} = 3.1 \text{s}^{-1}$) for several condensed Ps numbers: $N = 2.6 \times 10^7$ (curve 2), $2.0 \times 10^8$ (curve 3), $7.9 \times 10^8$ (curve 4), $2.5 \times 10^9$ (curve 5). The curve 1 with gaussian profile corresponds to the BEC with no interaction. The normalization condition of $\psi$ is given by (17).
Fig. 1
Fig. 2

Fig. 3
$\chi(r^2)$

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig4}
\caption{Fig. 4}
\end{figure}

$r/a$

$a = 0.437 \text{nm}$
Fig. 5
Fig. 6