HAUSDORFF DIMENSION OF INTRINSICALLY TRANSVERSAL
SOLENOIDAL ATTRACTORS IN HIGH DIMENSIONS

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ABSTRACT. We study the Hausdorff and box-counting dimension of solenoidal attractors that are intrinsically transversal, extending a previous analysis of Bothe to dimensions greater than 3. We prove that if the contraction is sufficiently strong, the expansion is close to conformal and the attractor satisfy a geometrical condition of transversality between its components, then the Hausdorff and box-counting dimension of every stable section of the attractor have the same value, which corresponds to the zero of the topological pressure as in Bowen’s formula, this also allows to calculate the dimension of the attractor.

1. INTRODUCTION

One of the most interesting problems in dynamical systems is the computation of the fractal dimension of hyperbolic sets, what provides information about the complexity of a system originated from chaotic dynamical systems.

This problem was initially solved in the setting of self-similar fractals and conformal repellers given by iterated function systems [5, 10]. Manning and McCluskey [9] calculated the dimension of the hyperbolic sets in surfaces, and their result was extended to conformal hyperbolic sets [1].

Some results were obtained in the non-conformal setting for iterated function systems [6] and for three-dimensional hyperbolic sets [4, 12, 14], although in general the problem of computing the dimension of a non-conformal hyperbolic set is far from complete. For more on this problem, one can check the surveys [2, 15].

An important example of non-conformal hyperbolic set is the Smale-Williams solenoid attractor, given by $T : S^1 \times D^2 \to S^1 \times D^2$, $T(\theta, x, y) = (2\theta \mod 1, \lambda_1 x + \epsilon \cos(2\pi\theta), \lambda_2 y + \epsilon \sin(2\pi\theta))$, with $0 < \lambda_1 \leq \lambda_2 < \min\{\epsilon, 1/2\}$. When $\Delta_T$ is conformal ($\lambda_1 = \lambda_2$), it is easy to calculate the Hausdorff dimension of the solenoid $\Delta_T = \bigcap_{n \geq 0} T^n(S^1 \times D^2)$, which is $1 + \frac{\log 2}{\log \lambda_2}$ [11].

Pesin asked [11] how can we compute the dimension of $\Delta_T$ when it is not conformal ($\lambda_1 < \lambda_2$). Simon [13] proved that its dimension is also $1 + \frac{\log 2}{\log \lambda_2}$ and that the dimension of every stable-section $\Delta_T(\theta) := \Delta_T \cap (\{\theta\} \times D^2)$ of the attractor is equals to $\frac{\log 2}{\log \lambda_2}$, for this he used a result of Bothe [4].

In [4] it is considered a more general family $\mathcal{F}$ of immersions given by $T(t, x, y) = (\varphi(t), \lambda_1(t) \cdot x + z_1(t), \lambda_2(t) \cdot y + z_2(t))$, with $\varphi$ an expanding map of $S^1$ of degree $N$, $\lambda_1, \lambda_2 : S^1 \to (0, 1)$ and $z_1, z_2 : S^1 \to (-1, 1)$ applications of class $C^1$. The attractor $\Delta_T$ is said intrinsically transversal if for every pair of distinct arcs $B_1$ and $B_2$ contained in $\Delta_T$, the arcs $\rho(B_1)$ and $\rho(B_2)$ are transversal (where $\rho(x, y, z) = (x, z)$). It is valid that:
Theorem ([4, Theorem A]). If $\Delta$ is intrinsically transversal and for $i = 1, 2$
\[ \sup \lambda_i < N^{-2} \quad \text{and} \quad \sup \lambda_i < \inf \varphi \sup \varphi^{-4} \log \inf \lambda_i / \log \sup \lambda_i, \]
then $\dim_H(\Delta) = 1 + \max\{p_i\}$ and $\dim_H(\rho(t) \cap (\{t\} \times D)) = p_i$ for all $t \in S^1$, where $p_i$ is the unique number such that the topological pressure $P(p_i \log \lambda_i)$ is 0.

As consequence, it follows the dimension of the Smale-Williams solenoid attractor because it is intrinsically transversal [13].

In this paper, we extend this Theorem to attractors in dimension greater than 3, also computing the dimension of every stable-section.

The Main Theorem of this paper states that if the contraction is sufficiently strong, if the expanding map on the basis is close to conformal and if the attractor is intrinsically transversal, then we can calculate the dimension every cross-section of the attractor as the zero given by the topological pressure of the geometrical potential (Bowen’s formula).

1.1. The Main Theorem. Consider $V = T^l \times E \times F$ and an embedding $T : V \to V$ of class $C^\nu$, $r \geq 2$, given by
\[ T(x, y, z) = (\varphi(x), \nu(x, y), \psi(x, y, z)), \]
where $T^l = \mathbb{R}^l / \mathbb{Z}^l$ is the $l$-dimensional torus, $E \subset \mathbb{R}^p$ and $F \subset \mathbb{R}^d$ are convex open bounded sets. Suppose that $\varphi : T^l \to T^l$ is an expanding map of degree $N \geq 2$, $\nu : T^l \times \mathbb{R}^p \to E$ and $\psi : T^l \times \mathbb{R}^p \times \mathbb{R}^d \to F$ are $C^\nu$ applications that are both contractions and $\psi$ contacts stronger, that is,
\[ 0 < \| (D_x \psi(x, y, z)) \|_x < \| (D_y \nu(x, y)) \|_y < 1. \]
for $Y = \{0\} \times \mathbb{R}^p$, $Z = \{0\} \times \{0\} \times \mathbb{R}^d$ and every $(x, y, z)$. We also suppose that $D_y \nu(x, y) \|_y$ is conformal, that is,
\[ \| D_y \nu(x, y) \|_y = \| (D_y \nu(x, y))^{-1} \|. \]

Let $\mathcal{T}$ be the set of all $C^\nu$ embeddings $T : V \to V$ described as above. For every $T \in \mathcal{T}$, the corresponding attractor is the set $\Delta_T = \bigcap_{n \geq 0} T^n(V)$ and the stable section in $x \in T^l$ is the set $\Delta_T(x) = \Delta_T \cap \mathbb{D}(x)$, where $\mathbb{D}(x) = \{x\} \times E \times F$. We say that $B$ is a component of $\Delta_T$ if it is a $l$-dimensional submanifold contained in $\Delta_T$. Denote $\rho(x, y, z) = (x, y)$ and $\pi(x, y, z) = x$.

Definition 1. The attractor $\Delta_T$ is said intrinsically transversal if for every ball $B \subset T^l$ with radius smaller than $\frac{1}{3}$ and for every two distinct components $B_1, B_2$ of $\Delta_T \cap \pi^{-1}(B)$, it is valid that the submanifolds $\rho(B_1)$ and $\rho(B_2)$ are transversal.

Remind that two submanifolds $S_1, S_2 \subset M$ are said transversal if $T_p S_1 + T_p S_2 = T_p M$ for every $p \in S_1 \cap S_2$. We will ask that $l \geq p$, otherwise this condition would not be satisfied.

The geometrical condition of intrinsic transversality means that the overlaps between $\rho$-projections of distinct components of the attractor do not occupy much space. If the $\rho$-projections of the components of the attractor were disjoint (they are not), it could be easier to prove that the same standard upper bound for the dimension of the attractor is also a lower bound. Intrinsic transversality allows to prove that the overlaps do not affect the calculation of the dimension.
Denote also \( \lambda(x, y) := \|D_y\nu(x, y)\|\), \( \bar{\lambda} = \inf\{\lambda(x, y)\} \), \( \bar{\lambda} = \sup\{\lambda(x, y)\} \), \( \beta = \sup\{\|D\varphi(x)\|\} \) and \( \beta = \sup\{\|D\varphi(x)\|^{-1}\}^{-1} \). Consider the sets

(1.4) \( \mathcal{T}^* = \left\{ T \in \mathcal{T} : \bar{\lambda} < N^{-\max(1, 2)}, \quad \bar{\lambda} < \left( \frac{2 \log(N)}{\log(2)} \right)^{-2} \right\} \)

(1.5) \( \mathcal{E}^* = \{ T \in \mathcal{T} : \beta < N^\frac{1}{2} \beta^p \} \).

In Section 2, we will define a semi-conjugation \( h : \Sigma_A \to \Delta_T \) of \( T \) with a bilateral shift \( \sigma \), the geometric potential \( \phi : \Sigma_A \to \mathbb{R} \) by \( \phi(\tilde{\omega}) = \log\|D_y\nu(h(\tilde{\omega}))\| \) and \( d_0 > 0 \) so that \( P(\sigma, d_0 \phi) = 0 \) (Bowen’s equation). Then:

**Main Theorem.** If \( T \in \mathcal{T}^* \cap \mathcal{E}^* \) and \( \Delta_T \) is intrinsically transversal, then:

(1.6) \( \dim_B \rho(\Delta_T(x)) = \dim_H \rho(\Delta_T(x)) = \dim_B(\Delta_T(x)) = \dim_H(\Delta_T(x)) = d_0 \)

for all \( x \in \mathcal{T}^* \). In particular, it follows that:

(1.7) \( \dim_B(\Delta_T) = \dim_H(\Delta_T) = \dim_B(\rho(\Delta_T)) = \dim_H(\rho(\Delta_T)) = l + d_0 \).

Above we denote \( \dim_H(X) \) and \( \dim_B(X) \) for the Hausdorff and box dimension of the set \( X \) (see e.g. [2] for the definitions).

The condition \( T \in \mathcal{T}^* \) means that the contraction is sufficiently strong and \( T \in \mathcal{E}^* \) means that the expanding map is close to conformal. Obvious examples of mappings \( T \in \mathcal{E}^* \) are given when \( l \leq p < 2l \) by expanding linear endomorphisms that are multiples of the identity and their perturbations.

The hypothesis of the Main Theorem are \( C^1 \)-robust because the property of intrinsic transversality is \( C^1 \)-open [3]. The product of intrinsically transversal attractors is also intrinsically transversal [3]. Intrinsic transversality is common for attractors with strong contraction: in dimension 3 it is \( C^1 \)-generic when \( \bar{\lambda} < N^{-2} \) [4 Theorem B]. We believe that it is also \( C^1 \)-generic in any dimension under the condition of \( \bar{\lambda} \) small enough.

In [3], the dimension of \( \Delta_T \) was calculated under weaker conditions on the contracting and expanding maps via an analytical technique that uses potential-theoretic methods, but that analysis only calculated the dimension of almost every stable section. The advantage of the Main Theorem in comparison with [3] is that it gives the dimension of every stable section, for this we make a geometrical approach that analyses more rigidly the structure of the attractor close to the overlaps of its components.

## 2. Codification of the dynamics

Consider \( \mathcal{R} = \{ R_1, R_2, \ldots, R_s \} \) a Markov partition of \( \mathbb{T}^d \) with respect to \( \varphi \) whose diameter \( \gamma = \text{diam}(\mathcal{R}) \) satisfies \( 0 < \gamma < \min\{\frac{1}{2}, \alpha\} \), where \( \alpha \) is an expansivity constant of \( \varphi \) and such that the contractive inverse branches \( \varphi^{-1} : B(x, \gamma) \to \mathbb{T}^d \) are well defined in balls of radius \( \gamma \).

Consider \( I = \{ 1, 2, \ldots, s \} \) and \( \tilde{I}^n \) the set of words with letter in \( I \) and length \( n \), \( 1 \leq n \leq \infty \). Define the subset of admissible words \( I_n := \{ \tilde{a} = (a_1, a_2, \ldots, a_n) \in \tilde{I}^n, \varphi(R_{a_j}) \cap R_{a_{j+1}} \neq \emptyset \ \forall j = 1, \ldots, n - 1 \} \). For each \( \tilde{a} = (a_1, \ldots, a_n) \in I_n \), the set \( T_{\tilde{a}} = \cap_{j \geq 0} \varphi^{-1}(R_{a_{j+1}}) \) is non-empty if and only if \( \tilde{a} \in I_n \). We also denote \( \overline{T_{\tilde{a}}} = T_{\tilde{a}} \).

**Definition 2.** Given \( B \subset \mathbb{T}^d \), the **components** of \( T^k(V) \cap \pi^{-1}(B) \) are the sets \( T^d(\overline{T_{\tilde{a}}} \times E \times F) \cap \pi^{-1}(B) \), for \( \tilde{a} \in I_k \) such that \( \varphi(\overline{T_{\tilde{a}}}) \cap B \neq \emptyset \).
Consider the mapping $\pi_m : \cup_{n \geq m} I_n \rightarrow I_m$, $\pi_m(a_1, a_2, \ldots, a_m) = (a_1, a_2, \ldots, a_m)$. Define also $\tau : I_\infty \rightarrow T^d$ by $\tau(a_1, a_2, \ldots) = \cap_{j \geq 0} \varphi^{-j}(R_{a_{j+1}})$, which is injective and conjugates $\varphi$ with the shift in $I_\infty$.

For fixed $n$, the family $T_n := \{T_n \circ I_n \in I_n\}$ is another Markov partition of $T^d$ and the diameter of each element of $T_n$ satisfies $K_0 \beta^{1-n} \leq \text{diam} T_n \leq \beta^{1-n} \cdot \gamma$, where $K_0 = \min_{1 \leq n \leq \infty} \text{diam}(R_j)$.

For each $x \in T^d$, fix a letter $s(x)$ in $I$ such that $x \in T_{s(x)}$, define the sets $T_n(x) := \{a \in I_n : \varphi^n(T_a) \cap R_{a_n} \neq \emptyset\}$ and $I_\infty(x) = \{a \in I_\infty : [a]_n \in I^n(x), \forall n \geq 1\}$. Given $a \in I^n(x)$, denote $a(x)$ the point $x_0$ of $T_n$ such that $\varphi^n(x_0) = x$, denote also $[a]_j$ the truncation of $a$ of length $j$. For $a \in I_m$, denote $I^n(a) = \{a \in I_n : \varphi^n(T_a) \cap R_{a_n} \neq \emptyset\}$ and $I_\infty(a) = \{a \in I_\infty : [a]_n \in I^n_a, \forall n \geq 1\}$.

Given $a \in I_n$, the image of $T_a \times \{0\} \times \{0\}$ by $T^n$ is the graph of the application $S(\cdot, a) : D(\bar{a}) \subset T^d \rightarrow \mathbb{R}^d \times \mathbb{R}^d$ given by

\begin{equation}
S(x, a) = (\nu(x_1) \circ \nu(x_2) \circ \ldots \circ \nu(x_0), \psi_1(x_1) \circ \psi_2(x_2) \circ \ldots \circ \psi_0(x_0, 0, 0)),
\end{equation}

where $\psi(x, y) = \psi_a(x, y), \nu(x, z) = \nu_a(z)$ and $D(\bar{a}) = \{x \in T^d : \bar{a} \in I^n(x)\}$. Since the inverse branches are well defined in balls of radius $\gamma$ for every $x$, we can extend $S(\cdot, a)$ to a function in $B(x, \gamma)$.

For $a \in I_\infty(x)$, the sequence $(S(x, [a]_n))_{n \geq 1}$ converges uniformly to a function $S(x, a)$ and the sequence of the derivatives of order $j$, $1 \leq j \leq r$, $(D^j S(x, [a]_n))_{n \geq 1}$ converges uniformly to $D^j S(x, a)$. Also, there exists a constant $\kappa > 0$ such that $\|D^j S(x, a)\| \leq \kappa$ for every $a \in I_n$ and $x \in D(\bar{a})$.

**Remark 3.** The graphs of $S(\cdot, a)$ correspond to the components of the attractors, which are unstable manifolds. More precisely:

1. If $(x, y, z) \in \Delta_T$, then $(x, y, z) = (x, S(x, b))$ for some $b \in I_\infty(x)$.
2. If $(x, y, z) \in \Delta_T$ and $a \in I_n$ is such that $(x, y, z) \in Z_a$, where $Z_a$ is the component described by $a$, then there exists $a_\infty \in I_\infty(x)$, with $\pi_n(a_\infty) = a$, and for $n \geq 0$ if we write $x_n = [a_\infty]_n(x)$ then $E^n_u(x_n, y_n, z_n) = \text{graf}(D(\bar{a}) \times \Delta_T \times \Delta_T)$.

Define the matrix $A = \{a_{ij}\}_{s \times s}$ such that $a_{ij} = 1$ if $\varphi(R_i) \cap R_j \neq \emptyset$ and $a_{ij} = 0$, otherwise. Denote $F = \{1, \ldots, s\}^s$ the set of bilateral sequences $\bar{a} = (\ldots, a_{i-1}, a_i, a_{i+1}, \ldots)$, $\bar{a}_n \in \{1, 2, \ldots, s\}$. Consider $\Sigma_A$ the subset of $F$ of admissible words, that is, $\Sigma_A = \{(\ldots, a_{i-1}, a_i, a_{i+1}, \ldots) \in F : a_{a_{i-1}, a_{i+1}} = 1 \text{ for every } n \in \mathbb{Z}\}$.

Let us see $\Sigma_A$ as a subset of $I_\infty \times I_\infty$, where $I_\infty = \{(\ldots, a_{i-2}, a_i, a_{i+1}, a_0) : a_{a_{i-1}, a_{i+1}} = 1 \text{ for all } n \in \mathbb{N} \cup \{0\}\}$, and we write each element $\bar{a} \in \Sigma_A$ as a pair $\bar{a} = (\bar{a}, \bar{a})$, where $\bar{a} = (\ldots, a_{i-2}, a_{i-1}, a_0)$ and $\bar{a} = (a_0, a_1, a_2, \ldots)$. Define the shift $\sigma : \Sigma_A \rightarrow \Sigma_A$ given by $\sigma((\bar{a})_n)_{n \in \mathbb{Z}} = (\bar{a}_{n+1})_{n \in \mathbb{Z}}$ and the semi-conjugation $h : \Sigma_A \rightarrow I_\infty$ by $h(\bar{a}) = (x, S(x, a))$, where $x$ is such that $\varphi^n(x) \in R_{a_0}$ for every $i \geq 0$ ($x = r(\bar{a})$). We have that $h$ is surjective and $h \circ \sigma = T \circ h$.

Consider the geometric potential geometric potential $\phi : \Sigma_A \rightarrow \mathbb{R}$ given by $\phi(\bar{a}) = \log\|D^j \nu(h(\bar{a}))\|$. The topological pressure of $s\phi$ with respect to $\sigma$ is

\begin{equation}
P(\sigma, s\phi) = \lim_{n \rightarrow \infty} \frac{1}{n} \log \left( \sum_{\bar{a} \in I_n} \exp \left( s \sum_{i=0}^{n-1} \phi(\sigma^i(\bar{a})) \right) \right).
\end{equation}

Since $P(\sigma, s\phi)$ is decreasing with respect to $s$, $P(s, 0) = \log N$ and $\lim_{s \rightarrow +\infty} P(\sigma, s\phi) = -\infty$, so there exists a unique $d_0 > 0$ such that $P(\sigma, d_0\phi) = 0$. 


This number $d_0$ will allow to compute the dimension of the attractor and its stable sections. The upper bounds for the dimensions follows in a standard way:

**Proposition 4** (Theorem 3.3 in [3]). It is valid that:

a) $\dim_H(\rho(\Delta_T(x))) \leq \dim_B(\rho(\Delta_T(x))) \leq d_0$ for every $x \in \mathbb{T}^l$;

b) $\dim_H(\Delta_T(x)) \leq \dim_B(\Delta_T(x)) \leq d_0$ for every $x \in \mathbb{T}^l$;

c) $\dim_H(\Delta_T) \leq \dim_B(\Delta_T) \leq l + d_0$.

The difficult part of the Main Theorem is the lower bounds for the Hausdorff dimensions (that are enough since $\dim_H(X) \leq \dim_B(X)$ for every set $X$).

3. THE GEOMETRIC LEMMA

The main step in the proof of the Main Theorem is the following Geometric Lemma, which is an extension of Lemma 3.1 in [4].

**Lemma 5.** (Geometric Lemma) There is $0 < \mu_0 < \frac{\log \lambda}{2 \log \Delta}$ such that for any $\mu \in (\mu_0, \frac{\log \lambda}{2 \log \Delta})$ and for each integer $n$ large, we can find an integer $k$, with $1 < k < \frac{\pi}{\mu}$, and a proper compact subset $F$ in $\mathbb{T}^l$ which is the union of at least $sN^{n-1} - N^{\mu n}$ sets in $T_n$ such that for any two points $x_1, x_2 \in F$ that satisfy $\varphi^{k-1}(x_1) \neq \varphi^{k-1}(x_2)$, $\varphi^k(x_1) = \varphi^k(x_2)$, we have $\rho(T^k(D(x_1))) \cap \rho(T^k(D(x_2))) = \emptyset$.

This Section is dedicated to the proof of this Lemma.

3.1. OVERLAPS BETWEEN COMPONENTS OF THE ATTRACTOR. Let us define the smallest singular value, that shall be a useful tool to analyze the structure of the attractor close to the intersection between its components.

**Definition 6.** Given a linear transformation $A : \mathbb{R}^m \to \mathbb{R}^n$, with $m \geq n$, the **smallest singular value** of $A$ is

$$m(A) := \sup_{\dim(W)=n} \inf_{v \in W, \|v\|=1} \|A(v)\|,$$

where the supremum is taken over the $n$-dimensional subspaces $W \subset \mathbb{R}^m$.

Two properties of the smallest singular value that shall be useful are:

(P1) A linear mapping $T : \mathbb{R}^m \to \mathbb{R}^n$ is surjective if and only if $m(A) > 0$. For linear mappings $T_i : \mathbb{R}^n \to \mathbb{R}^{m-n}$, $i = 1, 2$, this means that the subspaces $E_i = \text{graph} T_i$ are transversal if and only if $m(T_1 - T_2) > 0$.

(P2) It is valid the following triangular inequality:

$$m(A) \leq m(B) + \|A - B\|.$$

Intrinsic transversality gives a uniform lower bound for $m(\rho DS(x, a) - \rho DS(x, b))$ when $\rho(S(x, a))$ is close to $\rho(S(x, b))$.

**Proposition 7.** There are constants $c_1 > 0$, $\delta_1 > 0$ and $k_1 \geq 1$ large enough for which the following holds for every $k \geq k_1$ (and $k \leq +\infty$): let $a, b \in I^k(x)$ be such that $a_1 \neq b_1$ and $\|\rho(S(x, a)) - \rho(S(x, b))\| \leq \delta_1$, then it is valid that

$$m(\rho DS(x, a) - \rho DS(x, b)) > 2c_1.$$
Proof. Applying the Property (P1) to the graphs of $DS(x, a)$ and $DS(x, b)$, intrinsic transversality implies that $m(\rho DS(x, a) - \rho DS(x, b)) > 0$ for every $(x, a, \infty, b, \infty) \in \tilde{\Sigma} = \{ (x, a, \infty, b, \infty) : \rho S(x, a) = \rho S(x, b), a, b, \infty \in I^\infty(x) \text{ and } |a|_1 \neq |b|_1 |1 \}$. Since $\tilde{\Sigma}$ is compact, there exists a constant $c_1 > 0$ such that
\[
(3.4) \quad m(\rho DS(x, a) - \rho DS(x, b)) > 4c_1
\]
for every $(x, a, \infty, b, \infty) \in \tilde{\Sigma}$.

Define $\tilde{\Sigma}_m = \{ (x, a, \infty, b, \infty) : \|\rho S(x, a) - \rho S(x, b)\| < 1/m, a, b, \infty \in I^\infty(x) \text{ and } |a|_1 \neq |b|_1 |1 \}$. There exists $m_1$ large so that (3.4) is valid for every $(x, a, \infty, b, \infty) \in \tilde{\Sigma}_m, m \geq m_1$. In fact, if for every $m$ there exists $(x_m, a_m, b_m) \in \tilde{\Sigma}_m$ for which (3.4) is not valid, considering $(x_0, a_0, b_0)$ any accumulation point, then by continuity we would have $(x_0, a_0, b_0) \in \tilde{\Sigma}$ and not satisfying (3.4). So it follows the Proposition for $k = \infty$ and $\delta = 1/m_1$.

Take $k_1$ large such that $\|\rho S(x, c_0) - \rho S(x, c)\| \leq \delta/3$ and $\|\rho DS(x, c_0) - \rho DS(x, c)\| \leq c_1$ for every $c_0, c \in I^\infty(x), c = [c_0]_k$ and $k \geq k_1$. Proposition 7 follows by triangular inequality and $\delta_1 = \delta/3$.

In what follows, we consider $q$ large fixed so that $\text{diam } T_q < \frac{c_1}{2n}$.

**Definition 8.** Given $m > 1$ and $c \in I_q, q \geq 1$, consider two components of $T^m(V) \cap \pi^{-1}(T_E)$ that are in distinct components of $T(V) \cap \pi^{-1}(T_E)$, that is, there exist words $a, b \in I_m$ with $a_1 \neq b_1$ such that $Z_a = T^m(T_{\Delta E} \times E \times F)$ and $Z_b = T^m(T_{\Delta E} \times E \times F)$. We say that the pair $(Z_a, Z_b)$ is an **overlap** of $T^m(V)$ over $T_E$ if $\rho(Z_a) \cap \rho(Z_b) \cap \pi^{-1}(T_E) \neq \emptyset$. The set $B = \pi(\rho(Z_a) \cap \rho(Z_b)) \subset T^l$ will be called the $\pi$-**projection** of the overlap of the pair $(Z_a, Z_b)$.

**Proposition 9.** For every $k \geq k_1$ it is valid the following: if $D_1$ and $D_2$ are components of $T^k(V) \cap \mathbb{D}(x_0)$ that are in different components of $T(V) \cap \mathbb{D}(x_0)$ and for which $\rho(D_1) \cap \rho(D_2) \neq \emptyset$, then there is an overlap $(Z_a, Z_b)$ in $T^k(V)$ over some $T_{\lambda, c}$ with $c \in I_q, D_1 \subset Z_a$ and $D_2 \subset Z_b$. Moreover, we have
\[
(3.5) \quad m(\rho DS(x, a) - \rho DS(x, b)) > c_1 \quad \forall \ x \in T_{\lambda, c}.
\]

**Proof.** Consider $a, b \in I^l(x_0), [a]_1 \neq [b]_1$, that describe the components $D_1$ and $D_2$, respectively. Consider the corresponding pair $(Z_a, Z_b)$, and $t_0 = \rho(S(x_0, a)) = \rho(S(x_0, b)) \in \rho(\Delta T(x_0))$, in particular $D_1 \subset Z_a$ $D_2 \subset Z_b$ and Proposition 7 implies $m(\rho DS(x_0, a) - \rho DS(x_0, b)) \geq 2c_1$.

Let $c \in I_q$ be so that $x_0 \in T_{\lambda, c}$ by triangular inequality we have:
\[
m(\rho DS(x, a) - \rho DS(x, b)) \geq m(\rho DS(x_0, a) - \rho DS(x_0, b)) - 2c_1||x - x_0|| > c_1
\]
for all $x \in T_{\lambda, c}$. Then $(Z_a, Z_b)$ is an overlap of $T^k(V)$ over $T_{\lambda, c}$ as we want. \hfill \(\Box\)

Let us analyze the $\pi$-projection of the overlap and estimate how small it is.

**Proposition 10.** Given $\omega \in (\lambda, 1)$, there is a constant $K_1 > 0$ and an integer
$k_2$ such that for every $k \geq k_2$ and every overlap in $T^k(V)$, there is a submanifold
$S \subset T^l$ of dimension $l - p$ and diameter at most $K_1$ such that the $\pi$-projection of
this overlap is contained in the tubular neighborhood of $S$ of radius of bounded from
above by $\omega^k$.\hfill \(\Box\)
Proof. Let \((Z_a, Z_b)\) be an overlap of \(T^k(V)\) over some \(T_{c_s} c \in I^q\). Fix \(x_0 \in \pi(\rho(Z_a) \cap \rho(Z_b)), r_0 < \min\{\frac{\lambda}{\mu}, \gamma\}\) and consider \(g : B(x_0, r_0) \to \mathbb{R}^p\) given by
\[
g(x) = \rho S(x, a) - \rho S(x, b),
\]
where \(g\) is defined considering the extensions of \(S(x, a)\) and \(S(x, b)\) to \(B(x_0, \gamma)\). In particular, \(g(x_0) = 0\). Proposition 9 implies that \(g\) is a submersion, so \(S = g^{-1}(0)\) is a submanifold of dimension \(l - p\) with diameter at most \(K_1 > 0\).

Notice that \(\rho(Z_a) \subset \{(x, y) \in \mathbb{T} \times [-1, 1]^p : ||y - \rho S(x, z)|| \leq \lambda^k\}\), so if \(x \in \pi(\rho(Z_a) \cap \rho(Z_b)), \text{then } ||\rho S(x, a) - \rho S(x, b)|| \leq 2\lambda^k\), that is, the \(\pi\)-projection of the overlap \(\pi(\rho(Z_a) \cap \rho(Z_b))\) is contained in \(g^{-1}(B(0, 2\lambda^k)).\)

From the local form of the submersions, we can notice that \(g^{-1}(B(0, 2\lambda^k))\) is contained in a tubular neighborhood of \(S\) of radius bounded from above by \(K_2\lambda^k\). \(K_1\) and \(K_2\) can be taken uniform (independent of \(k, a, b, S, x_0\)) since \(||Dg||\) is uniformly bounded from above and \(m(Dg)\) is uniformly bounded from below for all \((a, b)\) corresponding to overlaps of \(T^k(V)\) with \(k \geq k_1\). The result follows since \(K_2\lambda^k < \omega^k\) for every \(k \geq k_2\) large enough.

\[\square\]

3.2. Proof of the Geometric Lemma.

Proof of the Geometric Lemma. \(T \in T^*\) implies that \(N^2\beta^{-1} \chi < 1\) and

\[
\mu_0 := \frac{l \log \beta - p \log \beta}{2 \log N} - \frac{l \log \beta}{2 \log(\beta^{-1} \chi^p N^2)} < \frac{\log \chi}{2 \log \lambda} < \frac{1}{2},
\]

So \(\mu_0 > \frac{l \log \beta - p \log \beta}{2 \log N} > 0\) since \(T \in E^*, \text{ and for any } \mu > \mu_0 \text{ it is valid that}

\[
\frac{\mu \log N - l \log \beta}{\log(N^2 \beta^{-1} \chi^p)} + \frac{l \log \beta - p \log \beta}{2 \log N} < \frac{\mu}{2},
\]

Consider \(\omega \in (\chi, 1)\) close to \(\chi\) such that \(N^2 \beta^{-p}\omega^p < 1\) and

\[
\frac{\mu \log N - l \log \beta}{\log(N^2 \beta^{-p}\omega^p)} + \frac{l \log \beta - p \log \beta}{2 \log N} < \frac{\mu}{2},
\]

Fix an integer \(k_3 = \max\{k_1, k_2\}\). Given \(k \geq k_3\), consider \(B_1, B_2, ..., B_s^*\) as the \(\pi\)-projections of the overlaps of \(T^k(V)\). The number \(s^*\) of overlaps of \(T^k(V)\) is at most \(s^2 N^{2k}\).

For each \(B_i, i = 1, ..., s^*,\) take \(B_i^*\) such that \(B_i^* = \varphi^{-k}(B_i)\), where \(\varphi^{-k}\) is one of the inverse branches of \(\varphi^k\) corresponding to the overlap of \(T^k(V)\) associated to \(B_i\). Each \(B_i\) is contained in a tubular neighborhood \(V_i\) of some submanifold \(S_i\) of dimension \(l - p\) and radius bounded from above by \(\omega^k\).

Consider \(\hat{V}_i\) the neighborhood of \(V_i\) of radius \(\beta^{-(n-k)}\) and \(V_i^{**} = \varphi^{-k}(\hat{V}_i)\). Let \(R_{i1}, R_{i2}, ..., R_{im}\) be the rectangles of \(T_n\) such that \(R_{ij} \cap B_i^* \neq \emptyset, \text{ then} \)

\[
R_{ji} \cap B_i^* \neq \emptyset \Rightarrow \varphi^k(R_{ji}) \cap B_i \neq \emptyset \Rightarrow \varphi^k(R_{ji}) \subset \hat{V}_i \Rightarrow R_{ji} \subset \varphi^{-k}(\hat{V}_i) = V_i^{**}.
\]
This implies that \( \bigcup_{j=1}^{m} R_{j_i} \subset V_{i}^{**} \) and \( \sum_{j=1}^{m} \text{vol}(R_{j_i}) \leq \text{vol}(V_{i}^{**}) \).

Notice that \( \bar{V}_i \) is contained in a tubular neighborhood of a submanifold with diameter uniformly bounded and with radius at most \( \omega + \beta^{-(n-k)} \), so we have the following estimates:

\[
\text{vol}(\bar{V}_i) \leq K_3 (\omega^k + \beta^{-(n-k)})^p
\]

for some constant \( K_2 \). We will also use that \( \text{vol}(V_{i}^{**}) \leq \beta^{-k} \text{vol}(\bar{V}_i) \).

For \( R \in \mathcal{T}_n \), we have \( K_0 \beta^{-n} \leq \text{diam} R \leq \beta^{-n} \). So \( mK_0 \beta^{-ln} \leq \sum_{j=1}^{m} \text{vol}(R_{j_i}) \), what implies:

\[
m \leq K_0^{-1} \beta^{ln} \text{vol}(V_{i}^{**}) \leq K_3 \beta^{ln} \beta^{-kl} (\omega^k + \beta^{-(n-k)p})
\]

for some \( K_3 > 0 \).

Considering \( B^* = \bigcup_{i=1}^{n^*} B_i^* \), the number \( M \) of rectangles in \( \mathcal{T}_n \) that intersect \( B^* \) is bounded from above by

\[
m \leq sN^{n-1} \leq K_4 \beta^{ln} \beta^{-kl} \omega^{kp} N^{2k} + \beta^{ln} \beta^{-kl} (n-k)p N^{2k}
\]

for constant \( K_4 > 0 \). Therefore, the number of rectangles in \( \mathcal{T}_n \) that do not intersect \( B^* \) is at least

\[
sN^{n-1} - K_4 \beta^{ln} \beta^{-kl} \omega^{kp} N^{2k} + \beta^{ln} \beta^{-kl} (n-k)p N^{2k}.
\]

Let \( \mathbf{F} \) be the closure of the union of rectangles in \( \mathcal{T}_n \) that do not intersect \( B^* \). We have that \( \mathbf{F} \) is compact and that for any \( x_1, x_2 \in F \) it holds that if \( \varphi^{-1}(x_1) \neq \varphi^{-1}(x_2) \) and \( \varphi(x_1) = \varphi(x_2) \), then \( \rho(T^k(D(x_1))) \cap \rho(T^k(D(x_2))) = \emptyset \), otherwise we would have \( \varphi^k(x_1) \) in some \( B_i \).

Now to conclude the Lemma, we need to show that for sufficiently large \( n \) we can have \( k < \frac{n}{2} \) for which it holds

\[
K_3 \beta^{ln} \beta^{-kl} \omega^{kp} N^{2k} + K_4 \beta^{ln} \beta^{-kl} (n-k)p N^{2k}.
\]

Since \( N^2 \beta^{-l} \omega^p < 1 \), we have:

\[
K_4 \beta^{ln} \beta^{-kl} \omega^{kp} N^{2k} < \frac{N^{n\mu}}{2} \iff k > n \left( \frac{\mu \log N - l \log \beta}{\log(N^2 \beta^{-l} \omega^p)} - \frac{\log(2K_4)}{\log(N^2 \beta^{-l} \omega^p)} \right)
\]

and

\[
K_4 \beta^{ln} \beta^{-kl} (n-k)p N^{2k} < \frac{N^{n\mu}}{2} \iff k < n \left( \frac{\mu}{2} - \frac{(l \log \beta - p \log \beta)}{2 \log N - (l - p) \log \beta} \right) - K_5
\]

for \( K_5 = \frac{\log(2K_4)}{2 \log N - (l - p) \log \beta} \).

Therefore, (3.13) is valid if we consider \( n \) and \( k \) with

\[
n \left( \frac{\mu \log N - l \log \beta}{\log(N^2 \beta^{-l} \omega^p)} - \frac{\log(2K_4)}{\log(N^2 \beta^{-l} \omega^p)} \right) < k < n \left( \frac{\mu}{2} - \frac{(l \log \beta - p \log \beta)}{2 \log N - (l - p) \log \beta} \right) - K_5.
\]

By taking \( n \) large enough, the difference between the right-hand expression and the left-hand expression above is greater than 1. So there exist \( k \) satisfying this inequality and \( k < n/2 \).

□
4. Consequences of the Geometric Lemma

The rest of the proof of the Main Theorem is similar to the proof of [4, Theorem A], we describe it in this Section for completeness.

The Geometric Lemma allows us to construct sub-attractors that we know how to estimate a lower bounded of its Hausdorff dimension, and the restriction of the unstable holonomy to the \( \rho \)-projections of these sub-attractors is locally bi-Lipschitz, this will allow to extend the lower bound for every sable section.

4.1. Subsets with large dimension. Consider \( m \) a large integer so that Lemma 6 holds for a fixed value \( \mu \in (\mu_0, \frac{\log x}{2\log x}) \) and \( n = 2m \), it gives an integer \( k = k(m) < m \) and a compact set \( F = F(m) \subset T^d \).

Let \( \bar{a}_1, \bar{a}_2, ..., \bar{a}_k \) be the words in \( \mathcal{I}_n \) for which \( \mathcal{T}_{\bar{a}_1}, \mathcal{T}_{\bar{a}_2}, ..., \mathcal{T}_{\bar{a}_k} \in \mathcal{T}_n \) are not in \( F \). Since \( F \) is the union of at least \( sN^{n-1} - N^\mu \) sets in \( \mathcal{T}_n \), it follows that \( 1 \leq t \leq N^\mu = N^{2\mu} \).

**Definition 11.** Given a word \( \bar{a} = (a_1, a_2, ..., a_m) \in \mathcal{I}_m \), we say that \( \bar{a} \) appears in \( \bar{u} \in \mathcal{I}_n \) if \( n \geq m \) and exists \( j_0 \in \{0, 1, ..., n - m\} \) such that \( \bar{a}_{j_0+j} = a_j \) for \( j = 1, ..., m \).

Define \( \mathcal{I}_m = \{ \bar{a} \in \mathcal{I}_m : \bar{a} \) does not appear in \( \bar{u}_j, j = 1, 2, ..., t \} \).

For every \( \bar{a}_j, j = 1, 2, ..., t \), we cannot have all \( \pi_m(\bar{a}_j), \pi_m(\sigma(\bar{a}_j)), ..., \pi_m(\sigma^m(\bar{a}_j)) \) in \( \mathcal{I}'_m \), so the number \( r = r(m) = \#\mathcal{I}'_m \) satisfies \( sN^{m-1} - (m + 1)t \leq r < sN^{m-1} \). Changing \( \mu \) by a slightly smaller number and \( m \) large, we can suppose that

\[
(4.1) \quad sN^{m-1} - N^{2\mu} < r < sN^{m-1}.
\]

For each \( u = 1, 2, ..., \) consider \( \mathcal{I}_{u,m} \) as subset of \( \mathcal{I}_m \times \mathcal{I}_m \times ... \times \mathcal{I}_m \) and \( \mathcal{I}'_{u,m} = (I'_m \times \cdots \times I'_m) \cap I_{u,m} \).

Consider \( \mathcal{I}'_{\infty,m} := \{ \bar{a} \in \mathcal{I}_\infty : \pi_m(\bar{a}) \in \mathcal{I}'_{u,m}, \ u = 1, 2, ... \} \) and notice that if \( \bar{a} \in \mathcal{I}'_{\infty,m} \), then none of the words \( \bar{a}_1, \bar{a}_2, ..., \bar{a}_k \) appear in \( \bar{a} \).

Consider also the sets:

- \( C' = C'(m) = \bigcap_{u=1}^{\infty} (\bigcup_{\bar{a} \in \mathcal{I}'_{u,m}} \mathcal{T}_{\bar{a}}) \subset T^d \);
- \( V' = V'(m) = C'(m) \times E \times F = \sigma^{-1}(C'(m)) \);
- \( \Delta'_T = \Delta'_T(m) = \bigcap_{u=1}^{\infty} T^m(\mathcal{V'}) \subset \Delta_T \cap \Delta' \);
- \( \Delta'_T(x) = \Delta'_T(m, x) = \Delta'_T(m) \cap \mathcal{D}(x) \).

It is valid that \( \varphi^m(C') = C' \) and, since every \( \bar{a}_j \) does not appear in any \( \bar{a} \in \mathcal{I}'_{\infty,m} \), it follows that \( \varphi^i(C') \subset F \) for all \( i \geq 0 \). Notice also that \( \tau(\mathcal{I}_{\infty,m}) = C'(m) \) and \( \sigma^m(\mathcal{I}_{\infty,m}) = \mathcal{I}_{\infty,m} \).

**Theorem 12.** It is valid that \( \liminf_{m \to \infty} \inf_{x' \in C'(m)} \left( \dim_H \rho(\Delta'_T(x')) \right) \geq d_0 \). Moreover, the restriction \( \rho|_{\Delta'_T(x)} \) is injective for every \( x \in C'(m) \).

Given \( x' = \tau(\bar{a}') \in C'(m), \) with \( \bar{a}' \in \mathcal{I}'_{\infty,m} \), denote

\[
(4.2) \quad \mathcal{I}'_{u,m}(x') = \{ \bar{a} \in \mathcal{I}'_{u,m} : \rho \bar{a}' \in \mathcal{I}'_{\infty,m} \}.
\]

For \( \bar{a} \in \mathcal{I}_{u,m}(x') \), denote also

\[
(4.3) \quad D_{x, \bar{a}} := \rho T_{\bar{a}}^{\mathcal{I}'_{u,m}(x')}(\rho(\mathcal{D}(\rho(\mathcal{I}_{u,m}(x')))))
\]

and define the positive real numbers \( d(x, m) \) and \( d^*(x, m) \) by the equalities

\[
\sum_{\bar{a} \in \mathcal{I}_m(x')} \text{diam}(D_{x, \bar{a}}) d(x, m) = 1 \quad \text{and} \quad \sum_{\bar{a} \in \mathcal{I}'_{u,m}(x')} \text{diam}(D_{x, \bar{a}}) d^*(x, m) = 1.
\]
Remark 13. It is valid that \( d(x, m) \geq d^*(x, m) \) and \( d_0 = \lim_{m \to +\infty} d(x, m) \) for every \( x \in T^d \).

Remark 14. A simple counting argument gives that \( \#I_{um}(x') = N^{um} \) and
\[
\#I'_{u,m}(x') \geq (N^m - N^{2um})^u
\]
since for each word \( b \in I_m \) there exist exactly \( N^m \) words \( a \in I_m \) such that \( ab \in I_{2m} \). Analogously, for each word \( b \in I'_{u,m}(x') \) there exist at least \( (N^m - N^{2um})^{u-1} \) words \( a \in I'_{u,m} \) such that \( ab \in I'_{u,m}(x') \).

Proposition 15. It is valid that \( \lim_{m \to \infty} d^*(x, m) = d_0 \).

Proof. We prove by contradiction. Suppose that \( d_0 - d^*(x, m) > \alpha > 0 \) for infinitely many integers \( m \), then for these integers \( m \):
\[
1 = \sum_{a \in I_m(x')} \text{diam}(D_{x,a})d(x, m) + \sum_{a \in I_m(x')-I_{u,m}} \text{diam}(D_{x,a})d(x, m) \\
\leq N^m \alpha + N^{2um}m \text{d}(x, m).
\]
Since \( \mu < \frac{\log N}{\log A} \), we have that \( d_0 = \limsup_{m \to +\infty} d(x, m) \leq \frac{2\mu \log N}{- \log A} < \frac{\log N}{- \log A} \).

On the other hand,
\[
1 \geq (N^m - (N^m - N^{2um})) A^m d^*(x, m) = N^{2um} A^m d^*(x, m),
\]
what implies that: \( \liminf_{m \to +\infty} d^*(x, m) \geq \frac{\log N}{- \log A} \).

Therefore \( \frac{\log N}{- \log A} \geq \liminf_{m \to +\infty} d^*(x, m) \gg d_0 + \alpha \gg \frac{\log N}{- \log A} \), what is a contradiction.

In the following, we need some estimates of bounded distortion for \( D_y \nu \). Denote \( T^n(x, y) = (\varphi^n(x), \nu^n(x, y), \psi^n(x, y, z)) \) and \( T^n_x(y) = \nu^n(x, y) \).

Lemma 16. There exists a constant \( K_6 > 0 \) such that for every \( n \geq 1, \ x \in T^d \) and \( \varphi \in I^\infty(x) \), it is valid that: if \( \varphi^{n-k}(x, y_k) \) and \( \varphi^{n-k}(x, y_k') \) are in the convex hull of \( \tilde{T}^n_\varphi(x) (\mathbb{D}(\varphi(x))) \) for every \( k = 0, 1, ..., n-1 \), then:
\[
K_6^{-1} \leq \prod_{k=0}^{n-1} \frac{\lambda(\varphi^{n-k}(x, y_k)}{\lambda(\varphi^{n-k}(x, y_k'))} \leq K_6.
\]

Proof. By conformality, we have that \( \lambda(x, y) = \|D_y \nu(x, y)\|_{\mathcal{Y}} \| \det D_y \nu(x, y) \|_{\mathcal{Y}}^{1/p} \), so the standard estimates of distortion for \( \det D_y \nu \) are also valid for \( \lambda \).

Proposition 17. For every \( m \), there exist \( \delta_1 = \delta_1(m) \) so that: for every \( x_1, x_2 \in C'(m) \) with \( \varphi^m(x_1) = \varphi^m(x_2) \), the distance between \( \rho(T^m(\Delta^+_{x_1} )) \) and \( \rho(T^m(\Delta^+_{x_2} )) \) is at least \( \delta_1 \). Moreover, if \( x \in C' \) then \( \rho(\Delta^+_{x}) \) is injective.

Proof. Consider \( k \) and \( F \) as in Lemma 3 so for every \( x \in T^d \) and any two components \( D_1, D_2 \) of \( T(V) \cap D(x) \) we have \( \rho(D_1 \cap T^k(\pi^{-1}(F))) \cap \rho(D_2 \cap T^k(\pi^{-1}(F))) = \emptyset \).

Therefore we can take \( \delta = \delta(m) \) so that
\[
d(\rho(T^k(D(x_1))), \rho(T^k(D(x_2)))) \geq \delta
\]
for every $x_1, x_2 \in F$ such that $\varphi^{k-1}(x_1) \neq \varphi^{k-1}(x_2)$ and $\varphi^k(x_1) = \varphi^k(x_2)$. We can also suppose that
\begin{equation}
(4.7) \quad d(\rho(D_1 \cap T^k(\pi^{-1}(F))), \rho(D_2 \cap T^k(\pi^{-1}(F)))) \geq \delta
\end{equation}
for any $D_1, D_2$ distinct components of $T(V) \cap D(x)$, $x \in T'$.  

Since $\varphi^m(C') = C'$, there exist $y_1, y_2 \in C'$ such that $\varphi^m(y_1) = x_1$ and $\varphi^m(y_2) = x_2$. Then we have $\varphi^{2m}(y_1) = \varphi^{2m}(y_2)$, by (4.6) and Lemma 16 there exists $\delta_1 > 0$ such that $d(\rho(T^{2m}(D(y_1))), \rho(T^{2m}(D(y_2)))) \geq \delta_1$, so it follows that:
\begin{equation}
(4.8) \quad d(\rho(T^m(\Delta_T(x_1))), \rho(T^m(\Delta_T(x_2)))) \geq \delta_1.
\end{equation}

Now, given $u_1 \neq u_2$ in $\Delta'(x)$, with $x \in C'$. Consider $j_0$ the smallest integer so that $x_1 = \pi(T^{-j_0}u_1) = \pi(T^{-j_0}u_2) = x_2$. Then $\varphi^m(x_1) = \pi(T^{-j_0}u_1) = \pi(T^{-j_0}u_2) = \varphi^m(x_2)$. It follows that
\begin{equation}
(4.9) \quad d(\rho(T^{-j_0}u_1), \rho(T^{-j_0}u_2)) \geq \delta_1,
\end{equation}
and in particular $\rho(T^{-j_0}u_1) \neq \rho(T^{-j_0}u_2)$.

The map $T^{-j_0}m : \rho(\Delta_T(\varphi^m(x_1))) \to \rho(\Delta_T(\varphi^m(x_2)))$, given by $\hat{T}^{j_0}m(\rho(u)) = \rho(T^{-j_0}m(u))$, is a bijection. So $\rho(T^{-j_0}u_1) \neq \rho(T^{-j_0}u_2)$ implies that $\rho(u_1) \neq \rho(u_2)$.

\begin{proposition}
If $\underline{a}_1$ and $\underline{a}_2$ are in $I'_{u,m}(x')$, with $u \geq 2$, and $\sigma^{j-1}m(\underline{a}_1) \neq \sigma^{j-1}m(\underline{a}_2)$, $\sigma^{jm}(\underline{a}_1) = \sigma^{jm}(\underline{a}_2)$ for some $j \in \{2, ..., u\}$, then
\begin{equation}
(4.10) \quad d(I_{\underline{a}_1}, I_{\underline{a}_2}) \geq \delta_2 \text{diam}(I_{\sigma^{jm}(\underline{a}_1)})
\end{equation}
for some constant $\delta_2 = \delta_2(m) > 0$.
\end{proposition}

\begin{proof}
It is clear that $I_{\sigma^{jm}(\underline{a}_1)} = I_{\sigma^{jm}(\underline{a}_2)} = \rho(T^{(u-j)m}(\pi(\sigma^{jm}(\underline{a}_1), \underline{a}_1'))).$ Consider $u_1, u_2 \in \mathbb{D}(\pi(\sigma^{jm}(\underline{a}_1), \underline{a}_1'))$ such that the diameter of $I_{\sigma^{jm}(\underline{a}_1)}$ is equal to $d(\rho(T^{(u-j)m}(u_1), \rho(T^{(u-j)m}(u_2))).$ Then
\begin{equation}
(4.11) \quad \text{diam}(I_{\sigma^{jm}(\underline{a}_1)}) \leq \left( \prod_{k=1}^{(u-j)m-1} \lambda(t_k) \right) d(u_1, u_2) \leq 2 \prod_{k=1}^{(u-j)m-1} \lambda(t_k)
\end{equation}
for some $t_k \in \rho(T^{(u-j)m-k}(\mathbb{D}(\pi(\sigma^{jm}(\underline{a}_1), \underline{a}_1'))).$

Now, consider $x_1 = \pi(\mathbb{D}(\pi(\sigma^{(j-1)m}(\underline{a}_1), \underline{a}_1'))$ and $x_2 = \pi(\mathbb{D}(\pi(\sigma^{(j-1)m}(\underline{a}_2), \underline{a}_2'))).$ As $\sigma^{(j-1)m}(\underline{a}_1) \neq \sigma^{(j-1)m}(\underline{a}_2)$, we have that $x_1 \neq x_2$ and as $\sigma^{jm}(\underline{a}_1) = \sigma^{jm}(\underline{a}_2)$, we have $\varphi^m(x_1) = \varphi^m(x_2).$ Then, applying (4.6), we can conclude that
\begin{equation}
(4.12) \quad d(\rho(T^{m}(\mathbb{D}(\pi(\sigma^{(j-1)m}(\underline{a}_1), \underline{a}_1'))), \rho(T^{m}(\mathbb{D}(\pi(\sigma^{(j-1)m}(\underline{a}_2), \underline{a}_2'))))) \geq \delta_1.
\end{equation}
This gives that $d(\rho(T^{jm}(\mathbb{D}(\pi(\underline{a}_1), \underline{a}_1'))), \rho(T^{jm}(\mathbb{D}(\pi(\underline{a}_2), \underline{a}_2')))) \geq \delta_1.$

On the other hand, we have that
\begin{equation}
(4.13) \quad d(I_{\underline{a}_1}, I_{\underline{a}_2}) \geq \delta_1 \prod_{k=1}^{(u-j)m-1} \lambda(w_k) \geq \delta_1 \frac{K_{\underline{a}_1}^{-1} \text{diam}(I_{\sigma^{jm}(\underline{a}_1)})}{2}.
\end{equation}
\end{proof}
Lemma 19. For each \( \mathbf{a} \in I_{u,m}'(x') \) and \( x \in C'(m) \), suppose that the closed subset \( D_{x,\mathbf{a}} \) of \( \rho(\mathcal{D}(x)) \) satisfy:

1. \( D_{x,\mathbf{a}} \subset D_{x,\mathbf{b}} \), \( \forall \mathbf{a} \in I_{u,m}'(x') \);
2. For some \( t_k \) and \( u_k \) in \( \rho(T^{um-k}(\mathcal{D}(\mathbf{a}))) \):

\[
\prod_{k=1}^{um} \lambda(t_k) \leq \frac{\text{diam}(D_{x,\mathbf{a}})}{\text{diam}(\rho(\mathcal{D}(\mathbf{a})))} \leq \prod_{k=0}^{um} \lambda(u_k);
\]
3. There is \( \delta > 0 \) such that for any \( \mathbf{a} = (a_1, ..., a_u) \), \( \mathbf{b} = (b_1, ..., b_u) \in I_{u,m}'(x') \), with \( u \geq 2 \), if the greatest index \( j \) such that \( (a_1, ..., a_{j}) = (b_1, ..., b_{j}) \) is less than or equal to \( u - 2 \), so \( D_{x,\mathbf{a}} \cap D_{x,\mathbf{b}} = \emptyset \) and

\[
d(D_{x,\mathbf{a}}, D_{x,\mathbf{b}}) \geq \delta \text{diam}(D_{x,\pi_m(a)}).
\]

Denote \( C_*(x) = \bigcap_{u \geq 1, \mathbf{a} \in I_{u,m}'(x')} D_{x,\mathbf{a}} \), then for every \( \epsilon > 0 \) it is valid that \( \dim_H(C_*(x)) \geq d^*(x,m) - \epsilon \) for every \( m \) sufficiently large.

Proof of the Lemma. For every \( \mathbf{a} \in I_{u,m}'(x') \) we consider closed and subsets \( J_*^x \subset \rho(\mathcal{D}(x)) \) such that \( \text{diam}(J_*^x) = \text{diam}(D_{x,\mathbf{a}}) \). We obtain the sets \( J_*^x \) by a translation of the \( D_{x,\mathbf{a}} \) so that they are disjoint.

For every \( u \), we define recursively the sets \( J_*^x \) by a translation of the sets \( D_{x,\mathbf{a}} \). In the step \( u+1 \), for each \( \mathbf{a} = (a_1, ..., a_u) \in A_*^u \), we define the set \( J_*^u_{\mathbf{a},a_{u+1}} \) making a translation of the set \( D_{x,\mathbf{a}} \) so that \( J_*^u_{\mathbf{a},a_{u+1}} \subset J_*^x \) and \( J_*^u_{\mathbf{a},a_{u+1}} \cap J_*^u_{\mathbf{a},a_{u+2}} = \emptyset \). Since they are obtained from translations, we have that \( J_*^x \) is closed and \( \text{diam}(J_*^x) = \text{diam}(D_{x,\mathbf{a}}) \) for every \( \mathbf{a} \in I_{u,m}'(x') \).

Consider \( C_*(x) = \bigcap_{u \geq 1, \mathbf{a} \in I_{u,m}'(x')} J_*^x \).

It follows that \( \dim_H(C_*(x)) \geq d^*(x,m) - \epsilon \) for every \( m \) sufficiently large, since it is a Moran symbolic construction of a conformal fractal induced by a subshift of finite (see e.g. [8, Theorem 1]).

Let us define a surjective transformation \( h^* : C_*^*(x) \to C_*(x) \) that is Lipschitz continuous. This will imply that

\[
\dim_H(C_*(x)) \geq \dim_H(h^*(C_*(x))) = \dim_H(C^*(x)) \geq d^*(x,m).
\]

Given \( c_* \in C_*(x) \), for every \( u = 1, 2, ... \) there is \( \mathbf{a} \in I_{u,m}'(x') \) such that \( c_* \in D_{x,\mathbf{a}} \). Define \( h^*(c_*) = \bigcap_{u=1}^{\infty} J_*^x \), which is well defined because \( \{J_*^x\} \) is a decreasing sequence of compact sets and \( \lim_{u \to \infty} \text{diam}(J_*^x) = 0 \). Moreover, \( h^* \) is a bijection between \( C_*(x) \) and \( C^*(x) \).

For \( c_* \neq c^* \in C_*(x) \), consider \( u_0 \) the greatest index such that \( h^*(c_*) \) and \( h^*(c^*) \) are in the same \( J_*^u_{\mathbf{a},a_{u+1}} \). So, \( d(h^*(c_*), h^*(c^*)) \leq \text{diam}(J_*^u_{\mathbf{a},a_{u+1}}) \). By (4.14), we have that \( d(c_*, c^*) \geq \delta \text{diam}(D_{x,\pi_m(a)}) = \delta \text{diam}(J_*^u_{\mathbf{a},a_{u+1}}) \). Therefore \( d(h^*(c_*), h^*(c^*)) \leq \delta^{-1} d(c_*, c^*) \), what completes this proof.

Now we can finish the proof of Theorem.

Proof of Theorem. For any arbitrary \( x'_m = \tau(\mathbf{a}_{u,m}) \in C'(m) \), \( \mathbf{a}'_m \in I'_{u,m} \), the sets \( D_{\mathbf{a}} = \rho(T^{um}(\mathcal{D}(\tau(\mathbf{a}, \mathbf{a}_{u,m})))) \) satisfy Properties (1) and (2) above, and Proposition.
implies Property (3). Since \( \rho(\Delta_T'(x'_m)) = \bigcap_{n=1}^{\infty} \bigcup_{l=n}^{\infty} D_{g_l} \), Lemma [13] implies that \( \dim_H(\rho(\Delta_T'(x'_m))) \geq d^*(x, m) \) for \( m \) sufficiently large. Then
\[
\lim_{m \to \infty} \inf \dim_H(\rho(\Delta_T'(x'_m))) \geq \lim_{m \to \infty} \inf d^*(x, m) = d_0.
\]

\[\square\]

4.2. Unstable holonomy. Consider an integer \( m \) large so that Theorem [12] is valid. Given \( x' \in C'(m) \) and any \( x \in T^l \), denote \( x_c = x_c(x', x) \) the middle point of the geodesic segment joining \( x' \) to \( x \). For each \( t' \in \Delta_T'(x') \), take \( t_c \in \Delta_T(x_c) \cap W^u(t') \) the closest point to \( t' \) in \( W^u(t') \). For \( t^* \in V \) and \( L \geq 0 \), denote
\[
W^u_{\mathcal{H}}(t^*) := \{ t \in W^u(t^*) : d(\pi(T^{-n}(t)), \pi(T^{-n}(t^*))) \leq L, \forall n \geq 0 \}.
\]
We have that \( t' \in W^u_{\mathcal{H}}(t_c) \) for \( d = d(x', x) \), therefore for each \( t' \in \Delta_T'(x') \) we can associate a single \( t \in W^u_{\mathcal{H}}(t_c) \cap \Delta_T(x) \). This defines the unstable holonomy
\[
\tilde{h} : \Delta_T'(x') \to \Delta_T(x)
\]
given by \( \tilde{h}(t') = t \in W^u_{\mathcal{H}}(t_c) \cap \Delta_T(x) \). By Theorem [12] we have that \( \rho|_{\Delta_T'(x')} \) injective for each \( t' \in C'(m) \). So the application \( h := \rho \circ \rho^{-1} : \rho(\Delta_T'(x')) \to \rho(\Delta_T(x)) \) is well defined. For this \( h \), we have the following result:

**Theorem 20.** For every \( x' \in C'(m) \), with \( m \) large enough, there is a finite partition of \( \rho(\Delta_T'(x')) \) in disjoint compact subsets \( E_1, E_2, ..., E_q \) such that \( h|_{E_i}, i = 1, 2, ..., q \), is injective and has a continuous Lipschitz inverse.

**Proof.** Consider \( m, x' \in C'(m), x \in T^l \), \( d, \tilde{h}, \) and \( h \) as before. By Proposition [17] for any \( x^* \in C'(m) \) and \( t_1, t_2 \in \Delta_T'(x^*) \), with \( \pi(T^{-m}(t_1)) \neq \pi(T^{-m}(t_2)) \), it is valid that
\[
d(\rho(W^u_{\Delta_1}(t_1)), \rho(W^u_{\Delta_1}(t_2))) \geq \delta_1,
\]
because \( W^u_{\Delta_1}(t_1) \subset T^m(\Delta_T'(\pi(T^{-m}(t_1)))) \) and \( W^u_{\Delta_1}(t_2) \subset T^m(\Delta_T'(\pi(T^{-m}(t_2)))) \).

Consider \( B = B(x, d/2) \) and \( k_3 \) positive integer large enough such that the diameter of every component of \( \varphi^{-k_3m}(B) \) is at most \( \delta_1 \).

Suppose that \( \{ x_1, x_2, ..., x_q \} = \varphi^{-k_3m}(C'(m)) \cap C'(m) \). Define \( E_j = \rho(T^{k_3m}(\Delta_T'(x_j))) \), \( j = 1, ..., q \). We have that \( \rho(\Delta_T'(x^*)) \) is the union of all \( E_j \), \( j = 1, ..., q \), are pairwise disjoint.

Now the proof that the restriction \( h|_{E_j}, j = 1, ..., q \), is injective and that its inverse is Lipschitz continuous is the same of the proof of Lemma 3.B in [4].

\[\square\]

4.3. Proof of the Main Theorem.

**Proof of the Main Theorem.** For every \( \epsilon > 0 \) we can consider \( m \) large, \( C'(m) \subset T^l, \Delta_T'(m) \subset \pi^{-1}(C'(m)) \cap \Delta_T \) and \( \Delta_T'(x') \) for some \( x' \in C'(m) \), such that \( \dim_H(\rho(\Delta_T'(x'))) \geq d_0 - \epsilon \) for every \( x' \in C' = C'(m) \). For every \( x \in T^l \), consider the applications \( \tilde{h} : \Delta_T'(x') \to \Delta_T(x) \) and \( h := \rho \circ \rho^{-1} : \rho(\Delta_T'(x')) \to \rho(\Delta_T(x)) \).

Partitioning \( \rho(\Delta_T'(x')) = E_1 \cup \cdots \cup E_r \) as in Theorem 20, we have \( \dim_H E_i \geq d_0 - \epsilon \) for some \( i \), then \( \dim_H \rho(\Delta_T(x)) \geq \dim h(E_i) \geq d_0 - \epsilon \). Since \( \rho \) is Lipschitz continuous, we have
\[
d_0 - \epsilon \leq \dim_H \rho(\Delta_T(x)) \leq \dim_H(\Delta_T(x)) \leq d_0
\]
for every $x$ and $\epsilon$. Therefore, $\dim_H(\Delta_T(x)) = d_0$ for every $x \in T^l$, what implies the first part of the Main Theorem.

The lower bound $l + d_0$ for the dimension of the attractor $\Delta_T$ is an immediate consequence of the lower bound $d_0$ for $\Delta_T(x)$ for every $x \in T^l$. □

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