A Directional Curvature Formula for Convex Bodies in $\mathbb{R}^n$  

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Abstract

For a compact convex set $F \subset \mathbb{R}^n$, with the origin in its interior, we present a formula to compute the curvature at a fixed point on its boundary, in the direction of any tangent vector. This formula is equivalent to the existing ones, but it is easier to apply.

Key words: convex set; curvature; implicit function theorem; tangent vector.

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1 Introduction

In [9] the authors proposed some concepts concerning the geometric structure of a closed convex bounded set $F$, with zero in its interior, in a Hilbert space $H$. Inspired essentially from the geometry of Banach spaces (see [13]), they introduced three moduli of local rotundity for the set $F$, one symmetrical (using the norm of $H$) and two asymmetric (using the ”asymmetric norm” given by the Minkowski functional of $F$). Using the symmetrical modulus the authors defined the concept of strict convexity graduated by some parameter $\alpha > 0$. The main numerical characteristic resulting from these considerations is the curvature (and the respective curvature radius) of $F$, which shows how rotund the set $F$ is near a fixed boundary point $\xi$ watching along a given direction $\xi^*$. Considering the polar set of $F$, $F^o$, they defined also the modulus of local smoothness and the local smoothness of $F^o$. As well-known (see, for example, [12, 13, 15, 16]) the strict convexity of a convex closed bounded set $F$ with zero in its interior is strongly related to the smoothness of $F^o$, but in [9] that relation was quantified. In particular, a local asymmetric version of the Lindenstrauss duality theorem [9, Proposition 4.2] was proved there, which quantitatively establishes the duality between local smoothness and local rotundity. Thus, the curvature of $F$ can be considered also as a numerical characteristic of $F^o$, showing how sleek $F^o$ is in a neighbourhood of a boundary point $\xi^*$ if you look along a direction $\xi$. Applying this theorem, it was obtained a characterization of the curvature of $F$ in terms of the second
derivative of its dual Minkowski functional \cite[Proposition 4.4]{9}. From what we have just said, and not only (for more results see \cite{9,10}), the formula for curvature is, from a theoretical point of view, very useful, but in practice it is very difficult to use even in $\mathbb{R}^2$ as we can see in \cite[Example 8.4]{9}. Then, in this paper, we propose, in some sense and for some kind of convex bodies $F$ (compact convex sets with interior points) in $\mathbb{R}^n$, $n \geq 2$, an equivalent formula to compute its curvature but easier to use. Namely, in Theorem 1 below, given $\xi$ at the boundary of $F$, $\partial F$, near which $\partial F$ is given by an implicit equation, we present a formula for the curvature of $F$ at $\xi$ in the direction of any tangent vector. For this, and for a fixed tangent vector, we will consider the intersection curve between $F$ and a suitable plane, but without using the plane equations or the curve expression. In a few words, we can say that \cite{9} gives us an approximate idea of the shape of $F$ in a global neighbourhood of $\xi$, while in this article the exact shape of $F$ near $\xi$ in each tangent direction is obtained.

Before moving on to the work itself, let us review more precisely what is already done in this area.

A definition for curvature similar to the formula that will be obtained here, and called directional curvature, appears in \cite{1} for a (not necessarily convex) $C^2$-manifold embedded in a Hilbert space.

In \cite[p.14]{5} (see also \cite{2,14}), for a convex body $F$ in $\mathbb{R}^n$ ($n \geq 2$), a smooth point $\xi$ in $\partial F$ (smooth means that at $\xi$ there exists only one supporting hyperplane to $F$), an interior unit normal vector $\xi^*$ of $F$ at $\xi$, and an unit vector $\xi^{**}$ orthogonal to $\xi^*$, H. Busemann considered the 2-dimensional halfplane

$$H(\xi, \xi^*, \xi^{**}) = \{ \eta \in \mathbb{R}^n : \eta = \xi + \lambda \xi^* + \mu \xi^{**} \text{ with } \lambda, \mu \in \mathbb{R} \text{ and } \mu \geq 0 \},$$

which intersects $\partial F$ in a plane convex curve. Denoting by $r_\eta$, for $\eta \in H(\xi, \xi^*, \xi^{**}) \cap \partial F$ near $\xi$, the radius of the circle with centre on the normal line $\xi + \mathbb{R}^+ \xi^*$ containing both $\xi$ and $\eta$, the author defined

$$\rho_l^{\xi^{**}}(\xi) := \liminf_{\eta \to \xi} r_\eta, \quad \rho_u^{\xi^{**}}(\xi) := \limsup_{\eta \to \xi} r_\eta$$

as the lower and upper curvature radius, respectively. If the numbers $\gamma_l^{\xi^{**}}(\xi) := \left( \rho_l^{\xi^{**}}(\xi) \right)^{-1}$ and $\gamma_u^{\xi^{**}}(\xi) := \left( \rho_u^{\xi^{**}}(\xi) \right)^{-1}$ (called lower and upper curvature, respectively) are equal and finite, he says that the curvature of $F$ at $\xi$ in direction $\xi^{**}$ exists and is equal to the common value.

Differential geometry of intersection curves of two (or more) surfaces in $\mathbb{R}^3$ (or higher dimension) were studied by many authors (see, for example, \cite{3,8,17} and the bibliography therein). There are studies in which all the surfaces are defined implicitly, others in which all are parametrically defined, and others in which there are surfaces of both types. For this work, we are only interested in those defined implicitly. At \cite{3,8,17} the authors present formulas (or algorithms) for computing differential geometric properties (such as tangent vector, normal vector, curvatures and torsion) of the intersection curve. In \cite[(5,4)]{8} the author derives a formula for the curvature of the curve defined by the intersection of $n-1$ implicit surfaces in $\mathbb{R}^n$. This formula is laborious to apply when the space has dimension $n \geq 4$, because we need to do
several operations with the gradients of the all functions that implicity define the surfaces. On
the contrary, the formula that will be presented in this paper seems easier, as it uses only the
function that defines implicitly the convex body.

In Section 2 of this paper we introduce some notations, definitions and one example where we
can see that the definition of curvature presented in [9] can give us only an approximate value.
In Section 3, we present the conditions on $F$, the normal cone to $F$ at a convenient $\xi \in \partial F$, the
tangent hyperplane of $F$ at $\xi$, the definition of directional curvature, some its properties and its
relation to the definition of [9]. Section 4 is dedicated to the main result of this paper and its
proof. In Section 5 we relate the directional curvature of $F$ with the radius of a suitable sphere.
The relationship between our formula and Goldman’s one is proved in Section 6. Finally, the
Section 7 is dedicated to the examples.

2 Basic notations and definitions

We will consider in the space $\mathbb{R}^n$, $n \in \mathbb{N}$, $n \geq 2$, with the usual inner product $\langle \cdot, \cdot \rangle$ and the norm $\|\cdot\|$, a compact convex set $F$ with the vector null of $\mathbb{R}^n$ (represented by 0) in its interior $\text{int}F$.
We denote by $F^o$ the polar set of $F$, i.e.,
$$F^o := \{\xi^* \in \mathbb{R}^n : \langle \xi, \xi^* \rangle \leq 1 \ \forall \xi \in F\}.$$ Together with the Minkowski functional $\rho_F (\cdot)$ defined by
$$\rho_F (\xi) := \inf \{\lambda > 0 : \xi \in \lambda F\}$$
we introduce the support function $\sigma_F : \mathbb{R}^n \to \mathbb{R}^+$,
$$\sigma_F (\xi^*) := \sup \{\langle \xi, \xi^* \rangle : \xi \in F\}.$$ Observe that
$$\rho_F (\xi) = \sigma_{F^o} (\xi),$$
and, consequently,
$$\frac{1}{\|F\|} \|\xi\| \leq \rho_F (\xi) \leq \|F^o\| \|\xi\|, \quad \xi \in \mathbb{R}^n,$$
(1)
where $\|F\| := \sup \{\|\xi\| : \xi \in F\}$. The inequalities (1) mean that $\rho_F (\cdot)$ is a sublinear functional "equivalent" to the norm $\|\cdot\|$. It is not a norm since $-F \neq F$ in general.

As usual, we represent by $\partial F$ the boundary of $F$. In what follows we will use the so-called duality mapping $\mathcal{J}_F : \partial F^o \to \partial F$ that associates the set
$$\mathcal{J}_F (\xi^*) := \{\xi \in \partial F : \langle \xi, \xi^* \rangle = 1\}$$
with each $\xi^* \in \partial F^o$. We say that $(\xi, \xi^*)$ is a dual pair when $\xi^* \in \partial F^o$ and $\xi \in \mathcal{J}_F (\xi^*)$.

The normal cone to $F$ at $\xi$, in the sense of Convex Analysis, is given by
$$N_F (\xi) := \{\xi^* \in \mathbb{R}^n : \langle \eta - \xi, \xi^* \rangle \leq 0 \text{ for every } \eta \in F\},$$
and the proximal normal cone to $F$ at $\xi$ is
\[
N_F^P(\xi) := \left\{ \zeta^* \in \mathbb{R}^n : \text{there exists } \sigma \geq 0 \text{ such that } \langle \eta - \xi, \zeta^* \rangle \leq \sigma \| \eta - \xi \|^2 \text{ for every } \eta \in F \right\}.
\]
Since $F$ is closed and convex we have (see [6, Proposition 1.1.10])
\[\tag{2} N_F^P(\xi) = N_F(\xi).\]
It is easy to show that $N_F(\xi) \cap \partial F^o$ is the pre-image of the mapping $J_F(\xi), J_F^{-1}(\cdot)$, calculated at $\xi$.

The tangent cone to $F$ at $\xi$ is the polar of $N_F(\xi)$, since $N_F(\xi)$ is, in fact, a cone, it is given by
\[
\{ u \in \mathbb{R}^n : \langle u, \zeta^* \rangle \leq 0 \text{ for every } \zeta^* \in N_F(\xi) \}.
\]
We will only work with the hyperplane tangent to the set $F$ at the point $\xi$:
\[\tag{3} T_F(\xi) := \{ u \in \mathbb{R}^n : \langle u, \zeta^* \rangle = 0 \text{ for every } \zeta^* \in N_F(\xi) \}.
\]

Following [9, Definition 3.2], for each dual pair $(\xi, \xi^*)$ the modulus of rotundity of $F$ at $\xi$ with respect to (w.r.t.) $\xi^*$ is
\[
\widehat{C}_F(r, \xi, \xi^*) := \inf \{ \langle \xi - \eta, \xi^* \rangle : \eta \in F, \| \xi - \eta \| \geq r \}, \quad r > 0,
\]
and $F$ is said to be strictly convex (or rotund) at $\xi$ w.r.t. $\xi^*$ if
\[\tag{4} \widehat{C}_F(r, \xi, \xi^*) > 0 \quad \text{for all} \quad r > 0.
\]
If (4) is fulfilled then $\xi$ is an exposed point of $F$ and the vector $\xi^*$ exposes $\xi$ in the sense that the hyperplane $\{ \eta \in \mathbb{R}^n : \langle \eta, \xi^* \rangle = \sigma_F(\xi^*) \}$ touches $F$ only at the point $\xi$, or, in other words, $J_F(\xi^*) = \{ \xi \}$. So, in this case, $\xi$ is well defined whenever $\xi^*$ is fixed.

**Definition 1** ([9]) Fix $\xi^* \in \partial F^o$, and let $\xi$ be the unique element of $J_F(\xi^*)$. The set $F$ is said to be strictly convex of order 2 (at the point $\xi$) w.r.t. $\xi^*$ if
\[
\gamma_F(\xi, \xi^*) := \liminf_{(r, \eta, \eta^*) \to (0+, \xi, \xi^*)} \frac{\widehat{C}_F(r, \eta, \eta^*)}{r^2} > 0. \tag{5}
\]

The number
\[
\gamma_F(\xi, \xi^*) = \frac{1}{\| \xi^* \|} \gamma_F(\xi, \xi^*)
\]
is said to be the (square) curvature of $F$ at $\xi \in \partial F$ w.r.t. $\xi^*$.
An example Consider the compact convex set

\[ F := \{ (\xi_1, \xi_2) \in \mathbb{R}^2 : |\xi_2| \leq 1 - \xi_1^4, -1 \leq \xi_1 \leq 1 \} \]

For any arbitrary dual pair \((\xi, \xi^*)\), with \(\xi := (\xi_1, \xi_2)\), by the symmetry of \(F\), just consider the case \(\xi_2 \geq 0\) and \(\xi_1 \leq 0\). Using [9, Example 8.3] we get:

(i) If \(\xi_2 > 0\) then the (unique) normal vector \(\xi^*\) to \(F\) at \(\xi\), such that \(\langle \xi, \xi^* \rangle = 1\) is given by

\[ \xi^* = \frac{1}{1 + 3\xi_1^4} (4\xi_1^3, 1) . \]

After a hard work, we obtained

\[ \hat{\kappa}_F (\xi, \xi^*) = \frac{\hat{\gamma}_F (\xi, \xi^*)}{\|\xi^*\|} \leq \frac{6\xi_1^2}{\sqrt{1 + 16\xi_1^6}} \] (7)

and

\[ \hat{\kappa}_F (\xi, \xi^*) \geq \frac{6\xi_1^2}{\sqrt{1 + 16\xi_1^6}} \Sigma^2 (\xi_1) , \] (8)

where \(\Sigma (\xi_1) := \sqrt{1 + \left( \sum_{k=0}^{3} |\xi_1|^k \right)^2} \). Combining the estimates (7) and (8) we see that the curvature \(\hat{\kappa} (\xi, \xi^*)\) is of order \(O (\xi_1^2)\) (as \(|\xi_1| \to 0\)). In particular, \(\hat{\kappa}_F\) is equal to zero at the points \((0, \pm 1)\).

(ii) If \(\xi := (-1, 0)\) we have

\[ \mathbf{N}_F (\xi) = \{ (v_1, v_2) \in \mathbb{R}^2 : v_1 \leq -4|v_2| \} . \]

For \(\xi^* \in \partial \mathbf{N}_F (\xi)\), by the lower semicontinuity of the function \((\xi, \xi^*) \mapsto \hat{\gamma}_F (\xi, \xi^*)\), we can apply the same reasoning as above, but not for \(\xi^* \in \text{int} \mathbf{N}_F (\xi)\). In this last case we have

\[ \hat{\kappa}_F (\xi, \xi^*) = +\infty \] (see [9, Proposition 3.8]).

Here we got only the estimates (7) and (8), but using the theory developed in this paper we will get an equality (see Example 1).

3 Directional curvature

In everything that follows we consider a compact convex set \(F \subset \mathbb{R}^n, n \geq 2\), with \(0 \in \text{int} F\). Fixed \(\xi \in \partial F\) assume that there are \(\delta > 0\) and \(f : \mathbb{R}^n \to \mathbb{R}\) of class \(C^2\) at \(\xi + \delta \mathbf{B}\) (\(\mathbf{B} \subset \mathbb{R}^n\) represents the open unit ball), such that

\[ F \subset \{ x \in \mathbb{R}^n : f (x) \leq 0 \} , \]

\[ \langle \xi, \nabla f (\xi) \rangle > 0 , \] (9)

and such that, for \(x \in \xi + \delta \mathbf{B}\), we have \(x \in \partial F\) if and only if \(f (x) = 0\).
Remark 1 Thanks to (9) and the continuity of $\nabla f (\cdot)$ at $\xi$ there is $0 < \delta' \leq \delta$ such that
\[
\inf_{\eta \in \xi + \delta' B} \langle \eta, \nabla f (\eta) \rangle > 0.
\] (10)

In particular, we have $\nabla f (\eta) \neq 0$ for any $\eta \in \xi + \delta' B$.

Proposition 1 We have
\[
N_F (\eta) = \bigcup_{\lambda \geq 0} \lambda \nabla f (\eta), \quad \eta \in \partial F \cap (\xi + \delta' B).
\] (11)

Proof. By (2), for an arbitrary $\eta \in \partial F \cap (\xi + \delta' B)$, it’s enough to prove that
\[
N^p_F (\eta) = \bigcup_{\lambda \geq 0} \lambda \nabla f (\eta).
\] (11)

Since $f$ is of classe $C^2$ at $\xi + \delta B$, by [6, Theorem 1.2.5 and Corolary 1.2.6], there are $\sigma, \rho > 0$ such that $(\eta + \rho B) \subset (\xi + \delta' B)$ and
\[
f (y) \geq f (\eta) + \langle \nabla f (\eta), y - \eta \rangle - \sigma \| y - \eta \|^2, \quad \forall y \in \eta + \rho B,
\]
and consequently
\[
\langle \nabla f (\eta), y - \eta \rangle \leq \sigma \| y - \eta \|^2, \quad \forall y \in (\eta + \rho B) \cap F.
\]

Thanks to [6, Proposition 1.1.5] $\nabla f (\eta) \in N^p_F (\eta)$. Since $N^p_F (\eta)$ is a cone we have, in fact, $\lambda \nabla f (\eta) \in N^p_F (\eta), \lambda \geq 0$.

To prove the other inclusion at (11) fix $\zeta \in N^p_F (\eta)$. By [6, Proposition 1.1.5] there is a constant $\sigma > 0$ such that
\[
\langle \zeta, y - \eta \rangle \leq \sigma \| y - \eta \|^2, \quad \forall y \in (\eta + \rho B) \cap F.
\]
whenever $y$ belongs to $\partial F \cap (\xi + \delta B)$. Put another way, this is equivalent to say that the point $\eta$ minimizes the function $y \mapsto \langle -\zeta, y \rangle + \sigma \| y - \eta \|^2$ over all points $y$ satisfying $f (y) = 0$ and $\| y - \xi \| < \delta$. The Lagrange Multiplier Rule of classical calculus provides a scalar $\lambda \geq 0$ such that $\zeta = \lambda \nabla f (\eta)$, which completes the proof.

Consequently, for any $\eta \in \partial F \cap (\xi + \delta' B)$ fixed, $\mathcal{J}^{-1}_{F} (\eta) = N_F (\eta) \cap \partial F^0$ is a singleton, and the unique $\eta^* \in \mathcal{J}^{-1}_{F} (\eta)$ is given by
\[
\eta^* = \frac{1}{\langle \eta, \nabla f (\eta) \rangle} \nabla f (\eta).
\] (12)

This means that $\eta^*$ is well defined whenever $\eta$ is fixed.

On the other hand, $\nabla f (\eta) \neq 0$ implies that there is a first $i \in I := \{1, \ldots, n\}$ such that
\[
f_{x_i} (\eta) := \frac{\partial f}{\partial x_i} (\eta) \neq 0.
\] (13)
Fixed such $i$ the hyperplane tangent to $F$ at $\eta$ is given by (see (3))

$$T_F(\eta) = \{v \in \mathbb{R}^n : \langle v, \nabla f(\eta) \rangle = 0\}$$

$$= \left\{(v_1, \ldots, v_n) \in \mathbb{R}^n : v_i = -\sum_{j=1,j\neq i}^{n} \frac{f_{x_j}(\eta)}{f_{x_i}(\eta)} v_j\right\}. $$

Denote by $u^j(\eta), j \in I \setminus \{i\}$, the vector of $\mathbb{R}^n$ with 1 in the $j$th coordinate, $-\frac{f_{x_j}(\eta)}{f_{x_i}(\eta)}$ in the $i$th coordinate and 0 in the others. Since $f$ is of class $C^2$ at $\xi + \delta B$, $u^j(\eta)$ will be close to $u^j(\xi)$, whenever $\eta$ is close to $\xi$.

For our results we need to introduce the following. Given $\eta \in \xi + \delta B$ and $u(\eta) \in T_F(\eta)$, $u(\eta) \neq 0$, consider the subset of $\mathbb{R}^n$

$$P(\eta, u(\eta)) := \text{span} \{\nabla f(\eta), u(\eta)\} + \eta,$$

where $\text{span} \{\nabla f(\eta), u(\eta)\}$ means the generated space by the vectors $\nabla f(\eta)$ and $u(\eta)$. Note that the vectors $\nabla f(\eta)$ and $u(\eta)$ are linearly independent, so the set $P(\eta, u(\eta))$ is, in fact, a 2-dimensional plane in $\mathbb{R}^n$ (it will simply be called a plane).

Below we introduce some directional notions, based on the respective notions presented in [9], and already seen here in Section 2. To simplify the notation, in general, we will not refer to the unique $\xi^*$, given by (12).

**Definition 2** The 2-dimensional modulus of strict convexity of $F$ at $\xi \in \partial F$ (with respect to $\xi^*$) in the direction of the vector $u(\xi) \in T_F(\xi) \setminus \{0\}$ is given by

$$\widehat{C}_F(r, \xi, u(\xi)) = \inf\{\langle \xi - \eta, \xi^* \rangle : \eta \in F \cap P(\xi, u(\xi)), \|\xi - \eta\| \geq r\}, \quad r > 0.$$ 

The set $F$ is strictly convex at $\xi$ in the direction of $u(\xi)$ if $\widehat{C}_F(r, \xi, u(\xi)) > 0$ for all $r > 0$.

**Remark 2** Since the set $F \subset \mathbb{R}^n$ is compact and convex we have the equalities

$$\widehat{C}_F(r, \xi, u(\xi)) = \inf\{\langle \xi - \eta, \xi^* \rangle : \eta \in F \cap P(\xi, u(\xi)), \|\xi - \eta\| = r\}$$

$$= \inf\{\langle \xi - \eta, \xi^* \rangle : \eta \in \partial F \cap P(\xi, u(\xi)), \|\xi - \eta\| = r\},$$

for any $r > 0$ and $u(\xi) \in T_F(\xi) \setminus \{0\}$.

**Proposition 2** Let $\xi \in \partial F$, $\xi^* \in \partial F^o$ given by (12) and $u(\xi) \in T_F(\xi)$, $u(\xi) \neq 0$. If $\widehat{C}_F(r, \xi, u(\xi)) > 0$ for all $r > 0$ then $J_F(\xi^*) \cap P(\xi, u(\xi)) = \{\xi\}.$

**Proof.** By construction $\xi \in J_F(\xi^*) \cap P(\xi, u(\xi))$. If there was $\bar{\xi} \in J_F(\xi^*) \cap P(\xi, u(\xi))$ with $\bar{\xi} \neq \xi$, we would have

$$\langle \xi - \bar{\xi}, \xi^* \rangle = 0,$$

and consequently

$$\widehat{C}_F(r, \xi, u(\xi)) = 0,$$

for $r := \|\xi - \bar{\xi}\| > 0$, which is absurd. ■
Definition 3 The 2-dimensional curvature of $F$ at $\xi \in \partial F$ (w.r.t. $\xi^*$) in the direction of $u^j (\xi)$, $j \in I \setminus \{i\}$, is given by

$$\hat{\kappa}_F (\xi, u^j (\xi)) = \frac{1}{\|\xi^*\|} \hat{\kappa}_F (\xi, u^j (\xi)),$$

where

$$\hat{\kappa}_F (\xi, u^j (\xi)) = \liminf_{(r, \eta) \to (0^+ \xi)} \frac{\hat{F}_F (r, \eta, u^j (\eta))}{r^2}.$$

The set $F$ is said to be strictly convex of the second order at $\xi$ in the direction of $u^j (\xi)$ when $\hat{\kappa}_F (\xi, u^j (\xi)) > 0$.

Such at [9, Proposition 3.7] we may extend the concept of directional strict convexity for the case of an arbitrary compact convex solid (do not assuming that $0 \in \text{int} F$). For this, we need to remember that the interior of any convex set $C$ in $\mathbb{R}^n$ relative to its affine hull (the smallest affine set that includes $C$) is the relative interior of $C$, denoted by $\text{rint} C$.

Proposition 3 Let $\xi \in \partial F$, $i \in I$ as above, $j \in I \setminus \{i\}$, $y_1, y_2 \in \text{rint} (F \cap P (\xi, u (\xi)))$ and $\xi^*_1 \in \mathcal{F}^{-1}_{F \cap P_1} (\xi - y_1)$. Then there is an unique $\xi^*_2 \in \mathcal{F}^{-1}_{F \cap P_2} (\xi - y_2)$ colinear with $\xi^*_1$ and such that

$$\frac{1}{\|\xi^*_1\|} \hat{\kappa}_{F \cap P_1} (\xi - y_1, u^j (\xi)) = \frac{1}{\|\xi^*_2\|} \hat{\kappa}_{F \cap P_2} (\xi - y_2, u^j (\xi)). \quad (14)$$

Proof. First, notice that $\xi^*_1$ is unique and, by (12), is given by $\frac{1}{\|\xi - y_1, \nabla f (\xi)\|} \nabla f (\xi)$. As the same reason the unique $\xi^*_2 \in \mathcal{F}^{-1}_{F \cap P_2} (\xi - y_2)$ is given by $\frac{1}{\|\xi - y_2, \nabla f (\xi)\|} \nabla f (\xi)$, and it is colinear with $\xi^*_1$.

Now, let us fix $\eta \in \partial F$ close to $\xi$, and the corresponding vectors $\eta^*_1$ and $\eta^*_2$ (which are close to $\xi^*_1$ and $\xi^*_2$, respectively). Notice that $\eta^*_1 \in \mathcal{F}^{-1}_{F \cap P_1} (\eta - y_1)$ implies $\langle y - y_1, \eta^*_1 \rangle < 1$ for any $y \in \text{int} F$, and we can write

$$\eta^*_2 = \frac{1}{1 + \langle y_1 - y_2, \eta^*_1 \rangle} \eta^*_1.$$

So, from Definition 2 we obtain

$$\frac{1}{\|\eta^*_2\|} \hat{\kappa}_{F \cap P_2} (r, \eta - y_2, u^j (\eta))$$

$$= \frac{1}{\|\eta^*_2\|} \inf \{ \langle \eta - y_2 - \zeta, \eta^*_2 \rangle : \zeta \in (F \cap P (\eta, u (\eta)) - y_2), \|\eta - y_2 - \zeta\| \geq r \}$$

$$= \frac{1}{\|\eta^*_2\|} \inf \{ \langle \eta - y, \eta^*_2 \rangle : y \in F \cap P (\eta, u (\eta)), \|\eta - y\| \geq r \}$$

$$= \frac{1}{\|\eta^*_1\|} \inf \{ \langle \eta - y, \eta^*_1 \rangle : y \in F \cap P (\eta, u (\eta)), \|\eta - y\| \geq r \}$$

$$= \frac{1}{\|\eta^*_1\|} \hat{\kappa}_{F \cap P_1} (r, \eta - y_1, u^j (\eta)).$$
i.e.,
\[
\frac{1}{\|\eta_2\|^2} \mathcal{E}_{r-y_2}(r, \eta - y_2, u^j(\eta)) = \frac{1}{\|\eta_1\|^2} \mathcal{E}_{r-y_1}(r, \eta - y_1, u^j(\eta))
\]
for all \( r > 0 \). Dividing both parts of the last equality by \( r^2 \) and passing to \( \liminf \) as \( r \to 0^+ \), \( \eta \to \xi \) we easily come to (14). \( \square \)

In the last proof we used the known fact that \( N_{F-y} (\xi - y_i) = N_F (\xi) \).

Remember that \( u(\xi) \in T_F(\xi) \setminus \{0\} \) if there are \( n - 1 \) real numbers \( \alpha_j, j \in I \setminus \{i\} \), not simultaneously null, such that
\[
u(\xi) = \sum_{j=1, j \neq i}^{n} \alpha_j u^j (\xi).
\]
This means that \( u(\xi) \) is a vector of \( \mathbb{R}^n \) with
\[-\sum_{j=1, j \neq i}^{n} \alpha_j \frac{f_j(\xi)}{f_i(\xi)}\]
at the \( i \)th coordinate and \( \alpha_j, j \in I \setminus \{i\} \), at the \( j \)th coordinate. If for any \( \eta \in \partial F \) near \( \xi \) we define \( u(\eta) \) as the non-zero vector of \( \mathbb{R}^n \) corresponding to \( u(\xi) \), i.e., the \( j \)th coordinates, \( j \in I \setminus \{i\} \), are the same in both vectors, and the \( i \)th coordinate of \( u(\eta) \) is given by
\[-\sum_{j=1, j \neq i}^{n} \alpha_j \frac{f_j(\eta)}{f_i(\eta)}\]
then it will be possible to put
\[
\hat{\gamma}_F(\xi, u(\xi)) = \frac{1}{\|\xi^*\|^2} \hat{\gamma}_F(\xi, u(\xi)) = \frac{1}{\|\xi^*\|^2} \liminf_{\eta \to \partial F} \frac{\mathcal{E}_F(r, \eta, u(\eta))}{r^2}.
\]
Note that such vector \( u(\eta) \) is, in fact, in \( T_F(\eta) \).

**Proposition 4** Let \( \xi \in \partial F \). If there is \( u(\xi) \in T_F(\xi) \setminus \{0\} \) such that \( \hat{\gamma}_F(\xi, u(\xi)) > 0 \), then we will have \( \exists \eta \cap F(\eta, u(\eta)) = \{\eta\} \) for every \( \eta \) close enough to \( \xi \) (and respective \( \eta^* \) given by (12)).

**Proof.** The condition \( \hat{\gamma}_F(\xi, u(\xi)) > 0 \) means that for some \( \theta > 0 \) and \( \rho > 0 \) the inequality
\[
\mathcal{E}_F(r, \eta, u(\eta)) \geq \theta r^2
\]
takes place whenever \( \|\xi - \eta\| \leq \rho, \eta \in \partial F \) and \( 0 < r \leq \rho \). Thanks to the monotony of the function \( r \mapsto \mathcal{E}_F(r, \eta, u(\eta)) \), decreasing if necessary the constant \( \theta > 0 \), we can assume that (15) is valid for all positive \( r \). In fact, \( \mathcal{E}_F(r, \eta, u(\eta)) = +\infty \) whenever \( r > 2\|F\| \) and for \( \rho \leq r \leq 2\|F\| \) we have
\[
\mathcal{E}_F(r, \eta, u(\eta)) \geq \mathcal{E}_F(\rho, \eta, u(\eta)) \geq \theta \left(\frac{\rho}{2\|F\|}\right)^2 r^2 \geq \theta \left(\frac{\rho}{2\|F\|}\right)^2 r^2.
\]
Hence, \( \hat{\mathcal{C}}_F (r, \eta, u (\eta)) > 0 \), for all \( r > 0 \), and the conclusion follows from Proposition 2. \( \blacksquare \)

In the next proposition \( \hat{\gamma}_F (\xi) \) represents \( \hat{\gamma}_F (\xi, \xi^*) \), given by (8), for the unique \( \xi^* = \frac{\nabla f (\xi)}{\| \nabla f (\xi) \|} \).

**Proposition 5** We have

\[
\hat{\gamma}_F (\xi, u (\xi)) \geq \hat{\gamma}_F (\xi), \quad \forall u (\xi) \in T_F (\xi) \setminus \{0\}.
\]

Furthermore, if \( \hat{\gamma}_F (\xi, u (\xi)) = 0 \) for some \( u (\xi) \in T_F (\xi) \setminus \{0\} \), we have \( \hat{\gamma}_F (\xi) = 0 \) too.

**Proof.** In fact, by (4),

\[
\liminf_{(r, \eta) \to (0^+, \xi)} \frac{\hat{\mathcal{C}}_F (r, \eta, u (\eta))}{r^2} = \liminf_{(r, \eta) \to (0^+, \xi)} \frac{\hat{\mathcal{C}}_F (r, \eta, u (\eta))}{r^2}_{\eta \in \partial F, \eta^* = \frac{1}{\| \nabla f (\eta) \|} \nabla f (\eta)} \geq \liminf_{(r, \eta) \to (0^+, \xi)} \frac{\hat{\mathcal{C}}_F (r, \eta, \eta^*)}{r^2}_{\eta \in \partial F, \eta^* = \frac{1}{\| \nabla f (\eta) \|} \nabla f (\eta)} \geq \liminf_{(r, \eta, \eta^*) \to (0^+, \xi, \xi^*)} \frac{\hat{\mathcal{C}}_F (r, \eta, \eta^*)}{r^2}_{\eta \in \partial F, \eta^* \in \partial F^o},
\]

which implies (16).

Now, recalling that we always have \( \hat{\gamma}_F (\xi) \geq 0 \), if there is \( u (\xi) \in T_F (\xi) \setminus \{0\} \) such that \( \hat{\gamma}_F (\xi, u (\xi)) = 0 \), we will have \( \hat{\gamma}_F (\xi) = 0 \).

When \( n = 2 \), we have \( T_F (\xi) = \text{span} \{ t_F (\xi) \} = \{ \lambda t_F (\xi) : \lambda \in \mathbb{R} \} \), for \( t_F (\xi) := (-f_2 (\xi), f_1 (\xi)) \), and the plane \( P (\xi, \lambda t_F (\xi)) \) coincide with \( \mathbb{R}^2 \) for every \( \lambda \in \mathbb{R} \setminus \{0\} \).

**Proposition 6** For \( n = 2 \) the equality holds at (16) if \( \hat{\gamma}_F (\xi, \lambda t_F (\xi)) > 0 \), for some \( \lambda \in \mathbb{R} \setminus \{0\} \).

**Proof.** Fixed \( \eta \in (\xi + \delta \mathbf{B}) \cap \partial F \) and \( u (\eta) \in T_F (\eta) \setminus \{0\} \), as defined above, we have \( P (\eta, u (\eta)) = \mathbb{R}^2 \), which implies that

\[
\hat{\mathcal{C}}_F (r, \eta, u (\eta)) = \hat{\mathcal{C}}_F (r, \eta, \eta^*), \quad \forall r > 0
\]

(see Definition 2 and 4). Therefore there is an equality in (17).

If \( \lambda \in \mathbb{R} \setminus \{0\} \) is such that \( \hat{\gamma}_F (\xi, \lambda t_F (\xi)) > 0 \), then there exist \( \theta > 0 \) and \( \varepsilon > 0 \) such that

\[
\hat{\mathcal{C}}_F (r, \eta, \lambda t_F (\eta)) \geq \theta r^2,
\]

whenever \( \| \eta - \xi \| \leq \varepsilon, \eta \in \partial F \) and \( 0 < r < \varepsilon \). Now, by Proposition 2 for such \( \eta \) and respective \( t_F (\eta) \) and \( \eta^* \), we have \( \mathcal{F} (\eta^*) = \{ \eta \} \). Therefore, we obtain an equality at (18), and consequently

\[
\hat{\gamma}_F (\xi, \lambda t_F (\xi)) = \hat{\gamma}_F (\xi).
\]

\( \blacksquare \)

The last proposition together with Proposition 5 imply that \( \hat{\gamma}_F (\xi) = \hat{\gamma}_F (\xi, u (\xi)) \), for every \( u (\xi) \in T_F (\xi) \). The last conclusion was already expected, since in \( \mathbb{R}^2 \) there is only one tangent direction.
4 The main result

In this section we prove that, under our conditions, the directional curvature \( \hat{z}_F (\xi, u^j (\xi)) \), \( j \in I \setminus \{ i \} \), can be calculated very easily. For this we need to compute the second derivative of \( f \) at \( \xi, \nabla^2 f (\xi) \), given, as usual, by the \( n \times n \) matrix with \( \frac{\partial^2 f}{\partial x_r \partial x_s} (\xi) \) at the row \( r \) and column \( s \), for every \( r, s \in \{ 1, \ldots, n \} \).

**Theorem 1** Let a compact convex set \( F \subset \mathbb{R}^n \), \( n \geq 2 \), with \( 0 \in \mathbb{R}^n \) in its interior, and a point \( \xi \in \partial F \). Assume that there are \( \delta > 0 \) and \( f : \mathbb{R}^n \to \mathbb{R} \) of class \( C^2 \) at \( \xi + \delta B \), such that

\[
F \subset \{ x \in \mathbb{R}^n : f (x) \leq 0 \},
\]

\[
\langle \xi, \nabla f (\xi) \rangle > 0,
\]

and such that, for \( x \in \xi + \delta B \), we have \( x \in \partial F \) if and only if \( f (x) = 0 \). Then we have

\[
\hat{\gamma}_F (\xi, u^j (\xi)) = \frac{1}{2 \langle \xi, \nabla f (\xi) \rangle ^2} \langle \nabla^2 f (\xi) u^j (\xi), u^j (\xi) \rangle, \quad j \in I \setminus \{ i \}.
\]

**Proof.** By hypothesis \( \langle \xi, \nabla f (\xi) \rangle > 0 \), so let us fix the first \( i \in I \) such that \( f_x (\xi) \neq 0 \) (see Remark [I]). For any \( \eta \in \mathbb{R}^n \), let \( \eta^i \in \mathbb{R}^{n-1} \) the vector \( \eta \) without the \( i \)th coordinate. Thanks to the Implicit Function Theorem there are a neighbourhood \( U := \xi + \delta_1 B \subset \mathbb{R}^{n-1} \), \( 0 < \delta_1 \leq \delta \), and a \( C^2 \) function \( g : U \to \mathbb{R} \) such that:

(i) \( f (\eta^i, g (\eta^i)) = 0 \), for any \( \eta^i \in U \),

(ii) for \( \eta^i \in U \) such that \( f (\eta) = 0 \) we have \( \eta_i = g (\eta^i) \), and

(iii) for any \( \eta^i \in U \) we have

\[
\frac{\partial g}{\partial x_j} (\eta^i) = - \frac{\partial f}{\partial x_j} (\eta^i, g (\eta^i)), \quad (j \in I \setminus \{ i \}),
\]

where \( (\eta^i, g (\eta^i)) \in \mathbb{R}^n \) represents the vector \( \eta \) with \( g (\eta^i) \) instead of \( \eta_i \).

Let \( j \in I \setminus \{ i \} \). By [II] for each \( \varepsilon > 0 \) there is \( 0 < \delta = \delta (\varepsilon) \leq \min \{ \delta', \delta_1 \} \) such that

\[
\left\| \frac{1}{\langle \eta, \nabla f (\eta) \rangle} \nabla f (\eta) - \frac{1}{\langle \xi, \nabla f (\xi) \rangle} \nabla f (\xi) \right\| < \varepsilon
\]

holds for any \( \eta \in \xi + \delta B \) (by the continuity of \( \nabla f (\cdot) \) at \( \xi \)),

\[
\left\| \frac{\langle \nabla^2 f (\xi) v, v \rangle}{\langle \eta, \nabla f (\eta) \rangle} - \frac{\langle \nabla^2 f (\xi) v, v \rangle}{\langle \xi, \nabla f (\xi) \rangle} \right\| < \varepsilon/2,
\]

for any \( \eta, \xi \in \xi + \delta B \) and any \( v \in \mathbb{R}^n \), \( \| v \| = 1 \) (by the continuity of \( \nabla^2 f (\cdot) \) at \( \xi \)), and such that

\[
\left| \langle \nabla^2 f (\xi) \frac{y - \eta}{\| y - \eta \|}, \frac{y - \eta}{\| y - \eta \|} \rangle - \langle \nabla^2 f (\xi) \frac{u^j (\xi)}{\| u^j (\xi) \|}, \frac{u^j (\xi)}{\| u^j (\xi) \|} \rangle \right| < \langle \xi, \nabla f (\xi) \rangle \varepsilon,
\]

11
holds for any \( \eta, y \in \xi + \overline{\mathbb{B}} \), \( y \in P(\eta, u^i(\eta)) \), \( y \neq \eta \), with \( f(y) = f(\eta) = 0 \) (using the continuity of \( \nabla g(\cdot) \) and \( \nabla f(\cdot) \) at \( \xi \) and \( \xi \), respectively, and using the Lagrange Mean Value Theorem).

Let us prove the inequality "\( \geq \)" in (20), assuming that \( \hat{\gamma}_F(\xi, u^j(\xi)) < +\infty \), because in the other case there is nothing to prove.

Let us fix \( \varepsilon > 0 \), the corresponding \( \delta > 0 \), \( 0 < r < \frac{\delta}{2} \) and \( \eta \in \left( \xi + \frac{\delta}{2} B \right) \cap \partial F \). We want to prove that

\[
\frac{\tilde{c}_F(r, \eta, u^j(\eta))}{r^2} \geq \frac{1}{2 \langle \xi, \nabla f(\xi) \rangle \|u^j(\xi)\|^2} \langle \nabla^2 f(\xi) u^j(\xi) , u^j(\xi) \rangle.
\]

Recall that \( \eta^* = \frac{1}{\langle \eta, \nabla f(\eta) \rangle} \nabla f(\eta) \). By Remark\(^\circ\) there is \( y \in \partial F \cap P(\eta, u^j(\eta)) \) with \( \|\eta - y\| = r \), such that

\[
\tilde{c}_F(r, \eta, u^j(\eta)) \geq \langle \eta - y, \eta^* \rangle - \frac{\varepsilon}{4} r^2.
\]

Notice that \( \|\eta - y\| = r > 0 \) implies \( y \neq \eta \). Putting

\[
v := \frac{y - \eta}{\|y - \eta\|}.
\]

then \( \eta + rv = y \). Thanks to the Taylor’s formula (see, e.g., [4, p.75])

\[
f(\eta + rv) = f(\eta) + \langle rv, \nabla f(\eta) \rangle + \int_0^r \langle \nabla^2 f(\xi + tv) v, v \rangle (r - \tau) d\tau,
\]

and by the definition of \( v \)

\[
f(y) = f(\eta) + \langle y - \eta, \nabla f(\eta) \rangle + \int_0^r \langle \nabla^2 f(\xi + tv) v, v \rangle (r - \tau) d\tau.
\]

Hence, by using the Mean Value Theorem for integrals and remembering that \( f(\eta) = f(y) = 0 \), we obtain

\[
\langle \eta - y, \nabla f(\eta) \rangle = \int_0^r \langle \nabla^2 f(\xi + tv) v, v \rangle (r - \tau) d\tau
\]

\[
= \frac{r^2}{2} \langle \nabla^2 f(\eta + tv) v, v \rangle,
\]

for some \( \tau = \tau(r, v) \in [0, r[ \). Let us fix such \( \tau \). By \( \frac{24}{24} \), \( \frac{26}{26} \) and \( \frac{26}{26} \), respectively, we have

\[
\tilde{c}_F(r, \eta, u^j(\eta)) \geq \langle \eta - y, \eta^* \rangle - \frac{\varepsilon}{4} r^2
\]

\[
= \frac{1}{\langle \eta, \nabla f(\eta) \rangle} \langle \eta - y, \nabla f(\eta) \rangle - \frac{\varepsilon}{4} r^2
\]

\[
\geq \frac{r^2}{2 \langle \eta, \nabla f(\eta) \rangle} \langle \nabla^2 f(\eta + tv) v, v \rangle - \frac{\varepsilon}{4} r^2.
\]

Since

\[
\|\eta + rv - \xi\| = \left\| \eta + rv - \eta \right\| \|y - \eta\| - \xi \| \leq \|y - \xi\| + \tau < \delta,
\]

12.
by (22) we obtain
\[ \left| \frac{\langle \nabla^2 f (\eta + \tau v) v, v \rangle}{\langle \eta, \nabla f (\eta) \rangle} - \frac{\langle \nabla^2 f (\xi) v, v \rangle}{\langle \xi, \nabla f (\xi) \rangle} \right| < \frac{\varepsilon}{2}, \]
and using (23) and (27) we conclude
\[ \hat{\mathcal{F}} \left( r, n, u^j (\eta) \right) > \frac{1}{2} \frac{\langle \nabla^2 f (\eta + \tau v) v, v \rangle}{\langle \eta, \nabla f (\eta) \rangle} - \frac{\varepsilon}{4} > \frac{1}{2} \frac{\langle \nabla^2 f (\xi) v, v \rangle}{\langle \xi, \nabla f (\xi) \rangle} - \frac{\varepsilon}{2} > \frac{1}{2} \frac{\langle \nabla^2 f (\xi) \frac{u^j (\xi)}{\|u^j (\xi)\|}, \frac{u^j (\xi)}{\|u^j (\xi)\|} \rangle}{\langle \xi, \nabla f (\xi) \rangle} - \varepsilon. \]
Passing to the limit as \( \varepsilon \to 0^+ \) we obtain the desired inequality:
\[ \hat{\gamma}_F (\xi, u^j (\xi)) \geq \frac{1}{2} \frac{\langle \nabla^2 f (\xi) u^j (\xi), u^j (\xi) \rangle}{\langle \xi, \nabla f (\xi) \rangle^2} \geq \frac{1}{2} \frac{\langle \nabla^2 f (\xi) u^j (\xi), u^j (\xi) \rangle}{\|u^j (\xi)\|^2}. \]

In order to show the opposite inequality let us assume that \( \hat{\gamma}_F (\xi, u^j (\xi)) > 0 \) (the case \( \hat{\gamma}_F (\xi, u^j (\xi)) = 0 \) is trivial).

Let us fix \( \varepsilon > 0 \) and \( 0 < \delta = \delta (\varepsilon) \leq \min \{ \delta', \delta_1 \} \) such that (24), (22), (23) and
\[ \hat{\mathcal{E}} \left( r, n, u^j (\eta) \right) > \frac{1}{3} \frac{\langle \nabla^2 f (\eta) v, v \rangle}{\langle \eta, \nabla f (\eta) \rangle} - \varepsilon \]
holds for every \( 0 < r < \frac{\delta}{2} \) and \( \eta \in \partial F \) with \( \| \eta - \xi \| < \delta \).

Let us fix \( 0 < r < \frac{\delta}{2}, \eta \in \left( \xi + \frac{\delta}{2} B \right) \cap \partial F \) and respective \( \eta^* \). We have
\[ \hat{\mathcal{F}} \left( r, n, u^j (\eta) \right) = \inf \{ \langle \eta - y, \eta^* \rangle : y \in \partial F \cap P (\eta, u^j (\eta)) , \| \eta - y \| = r \} \]
\[ = \frac{1}{\langle \eta, \nabla f (\eta) \rangle} \inf \{ \langle \eta - y, \nabla f (\eta) \rangle : y \in \partial F \cap P (\eta, u^j (\eta)) , \| \eta - y \| = r \}. \]
Now let us fix \( y \in \partial F \cap P (\eta, u^j (\eta)) \) with \( \| \eta - y \| = r \) and define \( v \) as in (25). Proceeding as above we obtain (26) for some \( \tau = \tau (r, \eta) \in [0, r] \). Let us fix this \( \tau \). Using (20), (28), (22) and (23), respectively, we obtain
\[ \frac{1}{r^2} \frac{1}{\langle \eta, \nabla f (\eta) \rangle} \langle \eta - y, \nabla f (\eta) \rangle = \frac{1}{2} \frac{\langle \nabla^2 f (\eta + \tau v) v, v \rangle}{\langle \eta, \nabla f (\eta) \rangle} < \frac{1}{2} \frac{\langle \nabla^2 f (\xi) v, v \rangle}{\langle \xi, \nabla f (\xi) \rangle} + \frac{\varepsilon}{4} < \frac{1}{2} \frac{\langle \nabla^2 f (\xi) \frac{u^j (\xi)}{\|u^j (\xi)\|}, \frac{u^j (\xi)}{\|u^j (\xi)\|} \rangle}{\langle \xi, \nabla f (\xi) \rangle} + \frac{\varepsilon}{2} + \frac{\varepsilon}{4} = \frac{1}{2} \frac{\langle \nabla^2 f (\xi) u^j (\xi), u^j (\xi) \rangle}{\langle \xi, \nabla f (\xi) \rangle} + \frac{3\varepsilon}{4}. \]
Consequently (see (30))
\[
\frac{\hat{\mathcal{F}}(r,\eta,u^j(\eta))}{r^2} < \frac{1}{2\langle \xi,\nabla f(\xi) \rangle} - \frac{1}{\|u^j(\xi)\|_2^2}\langle \nabla^2 f(\xi) u^j(\xi),u^j(\xi) \rangle + \frac{3\varepsilon}{4},
\]
and by (29)
\[
\hat{\gamma}_F(\xi,u^j(\xi)) < \frac{1}{2\langle \xi,\nabla f(\xi) \rangle} - \frac{1}{\|u^j(\xi)\|_2^2}\langle \nabla^2 f(\xi) u^j(\xi),u^j(\xi) \rangle + \varepsilon.
\]
Passing to the limit as \(\varepsilon \to 0^+\) we obtain the inequality "\(\leq\)" in (20). 

Remembering the Definition 3 we have
\[
\hat{\mathcal{F}}(\xi,u^j(\xi)) = \frac{1}{2\|\nabla f(\xi)\|} - \frac{1}{\|u^j(\xi)\|_2^2}\langle \nabla^2 f(\xi) u^j(\xi),u^j(\xi) \rangle, \quad j \in I \setminus \{i\}. \tag{31}
\]

Note that, for a fixed \(j \in I \setminus \{i\}\) and \(\lambda \in \mathbb{R} \setminus \{0\}\) we have
\[
\hat{\gamma}_F(\xi,\lambda u^j(\xi)) = \frac{1}{2\|\nabla f(\xi)\|} - \frac{1}{\|u^j(\xi)\|_2^2}\langle \nabla^2 f(\xi) u^j(\xi),u^j(\xi) \rangle, \tag{32}
\]
as it would be expected. Moreover, following the proof of Theorem 1 it is possible to prove that

**Corollary 2** We have
\[
\hat{\mathcal{F}}(\xi,u(\xi)) = \frac{1}{2\|\nabla f(\xi)\|} - \frac{1}{\|u(\xi)\|_2^2}\langle \nabla^2 f(\xi) u(\xi),u(\xi) \rangle,
\]
for any \(u(\xi) \in T_F(\xi) \setminus \{0\}\).

Notice that, by (32), for \(n = 2\), to say that \(\hat{\mathcal{F}}(\xi,u(\xi)) > 0\) for some \(u(\xi) \in T_F(\xi) \setminus \{0\}\), is the same as saying that \(\hat{\gamma}_F(\xi,(\mathcal{J}_2(\xi),f_1(\xi))) > 0\). Therefore, by Proposition 8 if there is \(\lambda \in \mathbb{R} \setminus \{0\}\) such that \(\hat{\mathcal{F}}(\xi,\lambda(\mathcal{J}_2(\xi),f_1(\xi))) > 0\), we will have
\[
\hat{\gamma}_F(\xi,u(\xi)) = \hat{\gamma}_F(\xi), \quad \forall u(\xi) \in T_F(\xi) \setminus \{0\}.
\]

Since \(f\) is of class \(C^2\) in \(\xi + \delta B\), all the conclusions will remain valid if we replace \(\xi\) with any \(\eta \in \partial F \cap (\xi + \delta B)\).

Following the idea of G. Crasta and A. Malusa presented in [7, pg.5749], we have the following result.

**Theorem 3** Let \(\xi \in \partial F\) and \(\hat{\mathcal{F}}(\xi,u^1(\xi)) \leq \cdots \leq \hat{\mathcal{F}}(\xi,u^j(\xi))\) be the curvatures of \(F\) at \(\xi\) in the direction of the \(n - 1\) vectors that generate \(T_F(\xi)\). If
\[
\|\nabla^2 f(\xi)\| := \sup_{u,v \in \mathbb{R}^n} \|\langle \nabla^2 f(\xi) u,v \rangle\| < \infty, \tag{33}
\]
then
\[
\hat{\gamma}_F(\xi,u^1(\xi)) = \min_{u \in U_{\xi}} \hat{\gamma}(u) \quad \text{and} \quad \hat{\gamma}_F(\xi,u^j(\xi)) = \max_{u \in U_{\xi}} \hat{\gamma}(u),
\]
where \(\hat{\gamma}(u) := \frac{1}{2\|\nabla^2 f(\xi)\|} - \frac{1}{\|u\|_2^2}\langle \nabla^2 f(\xi) u,u \rangle\) and \(U_{\xi} := \{v \in T_F(\xi) : \|v\| = 1\}\).
Proof. Assuming (33) it is easy to show that the application \( u \mapsto \hat{\kappa}(u) \) is continuous in 
\( S := \{ x \in \mathbb{R}^n : \| x \| = 1 \} \), and in particular in \( U_\xi \). Hence it admits a maximum and a minimum on \( U_\xi \). Let \( \mathbf{v} \in U_\xi \) be a maximum point. Then, by (31),

\[
\hat{\kappa}(\mathbf{v}) = \max_{u \in U_\xi} \frac{\hat{\kappa}(u^{jn-1}(\xi))}{\| u^{jn-1}(\xi) \|} = \hat{\kappa}_F(\xi, u^{jn-1}(\xi)).
\]

On the other hand, since \( \mathbf{v} \in T_F(\xi) \), then \( \hat{\kappa}_F(\xi, \mathbf{v}) \leq \hat{\kappa}_F(\xi, u^{jn-1}(\xi)) \), and consequently \( \hat{\kappa}(\mathbf{v}) = \hat{\kappa}_F(\xi, u^{jn-1}(\xi)). \) Reasoning as above, if \( v \) is a minimum on \( U_\xi \), we deduce that \( \hat{\kappa}(v) = \hat{\kappa}_F(\xi, u^{j1}(\xi)). \)

5 Directional curvature radius

As in [5, p.14] (see also [14, 2]) we also relate the directional curvature to the radius of some ball.

**Definition 4** The 2-dimensional curvature radius of \( F \) at \( \xi \in \partial F \) (w.r.t. \( \xi^* \)) in the direction of \( u^j(\xi), j \in I \setminus \{i\} \), is given by

\[
\hat{R}_F(\xi, u^j(\xi)) = \frac{1}{2\hat{\kappa}_F(\xi, u^j(\xi))}. \tag{34}
\]

Roughly speaking, the directional curvature \( \hat{\kappa}_F(\xi, u^j(\xi)) \) shows how rotund the boundary \( \partial F \) is in a neighbourhood of \( \xi \) (watching from the end of the vector \( \xi^* \)) when we “cut” \( F \) with the plane \( P(\xi, u^j(\xi)) \). As follows from Proposition 3 it does not depend on the position of the origin in \( \text{int}F \) and can be defined also when \( 0 \notin \text{int}F \). By using (34) we give the following geometric characterization of the directional curvature radius.

**Proposition 7** Fixed \( j \in I \setminus \{i\} \), we have

\[
\frac{\hat{R}_F(\xi, u^j(\xi))}{\|\xi^*\|} = \limsup_{(\varepsilon, \eta) \to (0^+, \xi)} \inf_{\eta \in \partial F} \{ r > 0 : F \cap P(\eta, u^j(\eta)) \cap (\eta + \varepsilon \mathbf{B}) \subset \eta - \rho \eta^* + \rho \| \eta^* \| \mathbf{B} \} \tag{35}
\]

**Proof.** Let us prove first the inequality " \( \leq \) " in (35) assuming without loss of generality that the right-hand side (further denoted by \( R \)) is finite. Taking an arbitrary \( \rho > R \), by the definition of \( \limsup \), we can affirm that for each \( \varepsilon > 0 \) small enough and for each \( \eta \in \partial F \) from a neighbourhood of \( \xi \), the relation

\[
\inf \{ r > 0 : F \cap P(\eta, u^j(\eta)) \cap (\eta + \varepsilon \mathbf{B}) \subset \eta - \rho \eta^* + \rho \| \eta^* \| \mathbf{B} \} < \rho
\]

holds. In particular,

\[
F \cap P(\eta, u^j(\eta)) \cap (\eta + \varepsilon \mathbf{B}) \subset \eta - \rho \eta^* + \rho \| \eta^* \| \mathbf{B},
\]
which implies 
\[ \| \zeta - \eta + \rho \eta^* \|^2 \leq \rho^2 \| \eta^* \|^2, \]
whenever \( \zeta \in F \cap P(\eta, w^j(\eta)) \) with \( \| \zeta - \eta \| = \varepsilon \), or, in another form,
\[ \langle \zeta - \eta, \eta^* \rangle \leq -\frac{\varepsilon^2}{2\rho}. \] 
(36)

If \( w \in F \cap P(\eta, w^j(\eta)) \) is an arbitrary point with \( \| w - \eta \| \geq \varepsilon \) then setting \( \zeta := \lambda w + (1 - \lambda) \eta \) in \( F \cap P(\eta, w^j(\eta)) \), where \( \lambda := \frac{\varepsilon}{\| w - \eta \|} \leq 1 \), we have 
\[ \| \zeta - \eta \| = \| \lambda w + (1 - \lambda) \eta - \eta \| = \lambda \| w - \eta \| = \varepsilon \]
and 
\[ \langle \eta - \zeta, \eta^* \rangle = \lambda \langle \eta - w, \eta^* \rangle. \]

By (36) we obtain 
\[ \frac{\varepsilon^2}{2\rho} \leq \langle \eta - \zeta, \eta^* \rangle = \lambda \langle \eta - w, \eta^* \rangle \leq \langle \eta - w, \eta^* \rangle, \]
so 
\[ \frac{\varepsilon^2}{2\rho} \leq \inf \{ \langle \eta - w, \eta^* \rangle : w \in F \cap P(\eta, w^j(\eta)), \| w - \eta \| \geq \varepsilon \}. \]

Hence, passing to \( \liminf \) as \( \varepsilon \to 0^+ \), \( \eta \to \xi \) and \( \rho \to R^+ \) we conclude the first part of the proof.

In order to show the opposite inequality let us assume that \( R > 0 \) (the case \( R = 0 \) is trivial).
If \( 0 < \rho < R \) then, by the definition of \( \limsup \) there are \( \varepsilon > 0 \) arbitrarily small and \( \eta \in \partial F \) arbitrarily closed to \( \xi \), such that 
\[ \inf \{ r > 0 : F \cap P(\eta, w^j(\eta)) \cap (\eta + \varepsilon B) \subset \eta - r \eta^* + r \| \eta^* \| B \} > \rho. \]

Then the set \( F \cap P(\eta, w^j(\eta)) \cap (\eta + \varepsilon B) \) is not contained in \( \eta - \rho \eta^* + \rho \| \eta^* \| B \), or, in other words, there is \( \zeta \in F \cap P(\eta, w^j(\eta)) \) with \( \| \zeta - \eta \| \leq \varepsilon \) such that 
\[ \| \zeta - \eta + \rho \eta^* \| > \rho \| \eta^* \|. \]

Consequently, setting, \( r := \| \zeta - \eta \| \leq \varepsilon \) we have 
\[ 2\rho \langle \eta - \zeta, \eta^* \rangle < \| \zeta - \eta \|^2 = r^2. \]
So 
\[ \hat{C}_F(r, \eta, w^j(\eta)) = \inf \{ \langle \eta - w, \eta^* \rangle : w \in F \cap P(\eta, w^j(\eta)), \| w - \eta \| \geq r \} \leq \langle \eta - \zeta, \eta^* \rangle < \frac{r^2}{2\rho}. \]

Passing to \( \liminf \) as \( r \to 0^+ \), \( \eta \to \xi \) and then to \( \lim \) as \( \rho \to R^- \) we conclude the proof. 

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6 Relation with the usual curvature formula for implicit space curves

In this section we will compare the formula obtained in Theorem 1 with the usual curvature formula for implicit space curves (that is, curves in $\mathbb{R}^n$ generated by the intersection of $n-1$ implicit equations). More precisely we will compare (31) with the formula obtained by R. Goldman in [8]. To do this we will consider, separately, the cases $n=2$ and $n \geq 3$.

For $n=2$, under our assumptions, near a fixed $\xi \in \partial F$ the curve $\partial F \cap P (\xi, u (\xi))$ is given by $f (\eta) = 0$, and the tangent line at $\xi$ is $T_F (\xi) = \text{span} \{ (-f_2 (\xi), f_1 (\xi)) \}$ (see before Proposition 0). By (32), for any $u (\xi) \in T_F (\xi) \setminus \{0\}$, that is for $u (\xi) = \lambda (-f_2 (\xi), f_1 (\xi))$, with $\lambda \in \mathbb{R} \setminus \{0\}$, we have

$$\kappa_F (\xi, u (\xi)) = \frac{1}{2 \| \nabla f (\xi) \| \| u (\xi) \|^2} \langle \nabla^2 f (\xi) u (\xi), u (\xi) \rangle$$

$$= \frac{1}{2 (f_2^2 (\xi) + f_1^2 (\xi))^{\frac{3}{2}}} \begin{bmatrix} -f_2 (\xi) & f_1 (\xi) \\ f_1 (\xi) & f_2 (\xi) \end{bmatrix} \begin{bmatrix} f_{11} (\xi) & f_{12} (\xi) \\ f_{21} (\xi) & f_{22} (\xi) \end{bmatrix} = \frac{1}{2} k_G (\xi),$$

where $k_G (\xi)$ is the curvature given by R. Goldman in [8, (3.4)]. So, when $n = 2$ the formulas coincide to less than the constant $\frac{1}{2}$. As we will see, we will obtain the same conclusion for $n \geq 3$, but in this case, the formula in [8] is more difficult to apply than ours.

Before analyzing the case $n \geq 3$, we need to introduce a generalization to the cross product from 3-dimensions to $n$-dimensions, called external product.

**Definition 5** [71, p.165] The external product of two vectors in an $n$-dimensional space, $n \geq 3$, spanned by $e_1, \ldots, e_n$ is a vector in a space of dimension $\frac{n(n-1)}{2}$ spanned by a new collection of vectors denoted by $\{ e_i \wedge e_j \}$, where $i < j$. Let $u = u_1 e_1 + \ldots + u_n e_n$ and $v = v_1 e_1 + \ldots + v_n e_n$ then

$$u \wedge v = \sum_{i < j} \det \begin{bmatrix} u_i & u_j \\ v_i & v_j \end{bmatrix} (e_i \wedge e_j).$$

For the next definition, as well as for the rest of the work, we just need to compute the magnitude of the external product, which is given by the formula

$$\| u \wedge v \| = \sum_{i < j} \left( \det \begin{bmatrix} u_i & u_j \\ v_i & v_j \end{bmatrix} \right)^2. \quad (37)$$

Assuming that $e_i$, $i = 1, \ldots, n$, is the vector of $\mathbb{R}^n$ with one in the $i$th position and zero everywhere else, and that $e := (e_1, \ldots, e_n)$ is the canonical basis of $\mathbb{R}^n$, we are in conditions to see the usual curvature formula for implicit space curves (see, for example, [8, (5.4)]).

**Definition 6** [8, (5.4)] The curvature formula for a point $\xi$ on the curve defined by the intersection of $n-1$ implicit hypersurfaces $F_1 (x_1, \ldots, x_n) = 0, \ldots, F_{n-1} (x_1, \ldots, x_n) = 0$ is

$$k_G (\xi) = \frac{\| \Tan (F_1, \ldots, F_{n-1}) (\xi) \wedge \nabla \Tan (F_1, \ldots, F_{n-1}) (\xi) \wedge \Tan (F_1, \ldots, F_{n-1}) (\xi) \|}{\| \Tan (F_1, \ldots, F_{n-1}) (\xi) \|^3}, \quad (38)$$

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where \( \text{Tan}(F_1, \ldots, F_{n-1})(\xi) \) is the tangent to the intersection curve at \( \xi \) given by

\[
\text{Tan}(F_1, \ldots, F_{n-1})(\xi) = \det \begin{bmatrix}
e_F & 0 & \cdots & 0 \\
\nabla F_1(\xi) & e_F & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
\nabla F_{n-1}(\xi) & 0 & \cdots & e_F
\end{bmatrix},
\]

where \( e_F \) represents the product between matrices.

\( \nabla (\text{Tan}(F_1, \ldots, F_{n-1}))(\xi) \) is the \( n \times n \) matrix in which each column is the gradient of the respective component of the row matrix \( \text{Tan}(F_1, \ldots, F_{n-1}) \), with the derivatives calculated at \( \xi \), and \( * \) represents the product between matrices.

To make the desired comparison, we still need to introduce some notation. Fixed \( \xi \in \partial F, i \in I \) (given by (13)) and \( j \in I \setminus \{i\} \), for any \( k \in I \setminus \{i, j\} \) put

\[
a_{kl}(\xi) := -\frac{f_{x_1}(\xi) f_{x_k}(\xi)}{f_{x_i}(\xi) + f_{x_j}(\xi)}, \quad l = i, j,
\]

and

\[
p_{k\xi}(\eta_1, \ldots, \eta_n) := \eta_k - \xi_k + a_{ki}(\xi) (\eta_i - \xi_i) + a_{kj}(\xi) (\eta_j - \xi_j), \quad (\eta_1, \ldots, \eta_n) \in \mathbb{R}^n.
\]

Using the definition of generated space it is easy to show that

\[
P(\xi, w^j(\xi)) = \bigcap_{k \in I \setminus \{i, j\}} \{ \eta \in \mathbb{R}^n : p_{k\xi}(\eta) = 0 \}.
\]

At the neighbourhood \( \xi + \delta' B \) \( (\delta' > 0 \) is given by Remark (1) the curve \( \partial F \cap P(\xi, w^j(\xi)) \) is given by the intersection of \( n - 1 \) implicit equations:

\[
f(\eta) = 0, \quad p_{k\xi}(\eta) = 0, ..., p_{k_{n-2}\xi}(\eta) = 0,
\]

\( k_1, \ldots, k_{n-2} \in I \setminus \{i, j\} \) and \( k_1 < \ldots < k_{n-2} \).

Next we will compute the curvature for this curve in the sense of Definition (6).

**Theorem 4** We have

\[
k_G(\xi) = \frac{\left| f_{x_i x_i}(\xi) f_{x_j}^2(\xi) - 2f_{x_i}(\xi) f_{x_j}(\xi) f_{x_i x_j}(\xi) + f_{x_j x_j}(\xi) f_{x_i}^2(\xi) \right|}{\|\nabla f(\xi)\| \left( f_{x_i}^2(\xi) + f_{x_j}^2(\xi) \right)},
\]

where

\[
f_{x_i x_m}(\xi) := \frac{\partial}{\partial x_m} \left( \frac{\partial f}{\partial x_i} \right)(\xi), \quad m, l \in \{i, j\}.
\]
Proof. For \( k \in I \setminus \{i,j\} \) and \( x := (x_1, ..., x_n) \in \mathbb{R}^n \) fixed, we have
\[
\frac{\partial p_k \xi}{\partial x_m} (x) = \begin{cases}
1, & \text{if } m = k \\
a_{km} (\xi), & \text{if } m = i \text{ or } m = j \\
0, & \text{otherwise}
\end{cases} .
\]
(39)

So the tangent to the intersection curve \( \partial F \cap P (\xi, u^j (\xi)) \) at \( \eta \in (\xi + \delta^i B) \cap \partial F \cap P (\xi, u^j (\xi)) \) is given by
\[
\Tan (f, p_{k_1 \xi}, ..., p_{k_{n-2} \xi}) (\eta) = \det \begin{bmatrix}
\nabla f (\eta) \\
\nabla p_{k_1 \xi} (\eta) \\
\vdots \\
\nabla p_{k_{n-2} \xi} (\eta)
\end{bmatrix}
= \sum_{m=1}^{n} \left( -1 \right)^{1+m} \det A_{m \xi} (\eta) e_m,
\]
where, for each \( m \in I \), \( A_{m \xi} (\eta) \) is the matrix obtained from eliminating the first line and the \( m \)th column. Remembering that we have \( (39) \) for each \( r \in \{1, n-2\} \), then
\[
\left( -1 \right)^{1+m} \det A_{m \xi} (\eta) = \left( -1 \right)^{i+j} \begin{cases}
\sum_{r=1}^{n-2} f_{x_k r} (\eta) a_{k r, j} (\xi) - f_{x_j r} (\eta), & \text{if } i < j \text{ and } m = i \\
f_{x_i (\eta)} - \sum_{r=1}^{n-2} f_{x_k r} (\eta) a_{k i, i} (\xi), & \text{if } i < j \text{ and } m = j \\
f_{x_i (\eta)} - \sum_{r=1}^{n-2} f_{x_k r} (\eta) a_{k i, j} (\xi), & \text{if } i > j \text{ and } m = i \\
\sum_{r=1}^{n-2} f_{x_k r} (\eta) a_{k r, i} (\xi) - f_{x_i (\eta)}, & \text{if } i > j \text{ and } m = j \\
f_{x_j (\eta)} a_{m i, i} (\xi) - f_{x_i (\eta)} a_{m i, j} (\xi), & \text{otherwise}.
\end{cases}
\]

Let \( \omega^j (\xi) \in \mathbb{R}^n \) the vector with \( \left( -1 \right)^{i+j+1} f_{x_j} (\xi) \) in the \( i \)th coordinate, \( \left( -1 \right)^{i+j} f_{x_i} (\xi) \) at the \( j \)th coordinate, if \( i < j \), or with symmetrical values if \( i > j \), and 0 elsewhere. It is easy to show that
\[
\sum_{m=1}^{n} \left( -1 \right)^{1+m} \det A_{m} (\xi) e_m = \frac{\| \nabla f (\xi) \|^2}{f_{x_i}^2 (\xi) + f_{x_j}^2 (\xi)} \omega^j (\xi)
= \left( -1 \right)^{i+j} \frac{\| \nabla f (\xi) \|^2}{f_{x_i}^2 (\xi) + f_{x_j}^2 (\xi)} \frac{f_{x_i} (\xi)}{f_{x_j} (\xi)} u^j (\xi),
\]
that is
\[
\Tan (f, p_{k_1}, ..., p_{k_{n-2}}) (\xi) = \left( -1 \right)^{i+j} \frac{\| \nabla f (\xi) \|^2}{f_{x_i}^2 (\xi) + f_{x_j}^2 (\xi)} \frac{f_{x_i} (\xi)}{f_{x_j} (\xi)} u^j (\xi) .
\]
(40)
After some calculations we conclude that

\[
\Tan \left( f, p_{k_1}, \ldots, p_{k_{n-2}} \right)(\xi) \ast \nabla \left( \Tan \left( f, p_{k_1}, \ldots, p_{k_{n-2}} \right)(\xi) \right) = \frac{\|\nabla f(\xi)\|^2}{f_{x_i}^2(\xi) + f_{x_j}^2(\xi)} M,
\]

where \( M \) is a line matrix. Using (37) we obtain

\[
\left\| \left( \Tan \left( f, p_{k_1}, \ldots, p_{k_{n-2}} \right)(\xi) \ast \nabla \left( \Tan \left( f, p_{k_1}, \ldots, p_{k_{n-2}} \right)(\xi) \right) \right) \right\| = \left( \frac{\|\nabla f(\xi)\|^2}{f_{x_i}^2(\xi) + f_{x_j}^2(\xi)} \right)^5 \left( f_{x_i x_i}(\xi) f_{x_j x_j}(\xi) - 2f_{x_i}(\xi) f_{x_j}(\xi) f_{x_i x_j}(\xi) + f_{x_j x_i}(\xi) f_{x_i}(\xi) f_{x_j}(\xi) \right)^2.
\]

Which implies

\[
\left\| \left( \Tan \left( f, p_{k_1}, \ldots, p_{k_{n-2}} \right)(\xi) \ast \nabla \left( \Tan \left( f, p_{k_1}, \ldots, p_{k_{n-2}} \right)(\xi) \right) \right) \right\| = \left( \frac{\|\nabla f(\xi)\|^5}{f_{x_i}^2(\xi) + f_{x_j}^2(\xi)} \right) \left( f_{x_i x_i}(\xi) f_{x_j x_j}(\xi) - 2f_{x_i}(\xi) f_{x_j}(\xi) f_{x_i x_j}(\xi) + f_{x_j x_i}(\xi) f_{x_i}(\xi) f_{x_j}(\xi) \right),
\]

and consequently (see (31))

\[
k_G(\xi) = \frac{\|\nabla f(\xi)\|^5}{(f_{x_i}^2(\xi) + f_{x_j}^2(\xi))^2} \frac{\left| f_{x_i x_i}(\xi) f_{x_j x_j}(\xi) - 2f_{x_i}(\xi) f_{x_j}(\xi) f_{x_i x_j}(\xi) + f_{x_j x_i}(\xi) f_{x_i}(\xi) f_{x_j}(\xi) \right|}{\left( \frac{\|\nabla f(\xi)\|^2|f_{x_i}(\xi)|}{f_{x_i}^2(\xi) + f_{x_j}^2(\xi)} \|u^j(\xi)\| \right)^3} = \frac{\left| f_{x_i x_i}(\xi) f_{x_j x_j}(\xi) - 2f_{x_i}(\xi) f_{x_j}(\xi) f_{x_i x_j}(\xi) + f_{x_j x_i}(\xi) f_{x_i}(\xi) f_{x_j}(\xi) \right|}{\|\nabla f(\xi)\|^2(f_{x_i}^2(\xi) + f_{x_j}^2(\xi))}.
\]

\[\blacksquare\]

Remembering (31), and that \( \hat{\zeta}_F(\xi, u^j(\xi)) \geq 0 \) because \( \hat{\mathcal{C}}_F(r, \eta, u^j(\xi)) \geq 0 \) for every \( r > 0 \) and every \( \eta \in \partial F \) close enough to \( \xi \), it is easy to show the next result.

**Corollary 5** We have

\[
k_G(\xi) = 2\hat{\zeta}_F(\xi, u^j(\xi)). \quad (41)
\]

Furthermore, following the proof of Theorem 4 it is easy to see that we have \( k_G(\eta) = 2\hat{\zeta}_F(\eta, u^j(\eta)) \), for any \( \eta \in \partial F \) close enough to \( \xi \).

So, we have the equality (41) for every \( n \in \mathbb{N} \) and for every \( \eta \in \partial F \) close enough to \( \xi \). For \( n \geq 3 \), if our conditions are verified, as we can easily see, the formula (38) is harder to apply than formula (31), that is, it is more laborious to calculate \( k_G(\eta) \) than \( \hat{\zeta}_F(\eta, u^j(\eta)) \). Furthermore, the difficulty increases as \( n \) increases, and with (31) it is easy to calculate the curvature in different directions, while using (38) this would be very laborious, since we would have to repeat all the calculations made in the proof of Theorem 4 for each new \( u^j \).

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7 Examples

1. Consider the compact convex set \( F \subset \mathbb{R}^2 \), with \((0, 0)\) in its interior,
\[
F = \left\{ (x_1, x_2) \in \mathbb{R}^2 : |x_2| \leq 1 - x_1^4, \ -1 \leq x_1 \leq 1 \right\}.
\]
Close to \( \xi = (\xi_1, \xi_2) \in \partial F \) with \( \xi_2 > 0 \) (the case \( \xi_2 < 0 \) is analogous) we have \( f(x_1, x_2) := x_2 - 1 + x_1^4 \),
\[
\nabla f(\xi) = (4\xi_3^3, 1), \quad \nabla^2 f(\xi) = \begin{bmatrix} 12\xi_1^2 & 0 \\ 0 & 0 \end{bmatrix},
\]
and
\[
T_F(\xi) = \text{span} \left\{ (1, -4\xi_3^3) \right\}.
\]
Consequently, for any \( u(\xi) \in T_F(\xi) \setminus \{(0, 0)\} \),
\[
\hat{\kappa}_F(\xi, u(\xi)) = \frac{12\xi_1^2}{2\sqrt{(4\xi_3^3)^2 + 1}} \geq \frac{6\xi_1^2}{(16\xi_1^6 + 1)^{\frac{3}{2}}}.
\]
Recalling Proposition 6 we can see that Theorem 1 allows us to obtain the following equality
\[
\hat{\kappa}_F(\xi) = \frac{6\xi_1^2}{(16\xi_1^6 + 1)^{\frac{3}{2}}},
\]
whereas in [9, Example 8.3] we had obtained only the inequalities
\[
\frac{6\xi_1^2}{\sqrt{1 + 16\xi_1^6 \Sigma^2(\xi_1)}} \leq \hat{\kappa}_F(\xi) \leq \frac{6\xi_1^2}{\sqrt{1 + 16\xi_1^6}},
\]
where \( \Sigma(\xi_1) := \sqrt{1 + \left( \sum_{k=0}^{3} |\xi_1|^k \right)^2} \).

Note that at \( \xi = (0, \pm 1) \) we have \( \hat{\kappa}_F(\xi, u(\xi)) = 0 \), as would be expected. Here we can’t calculate the curvature at \( \xi = (\pm 1, 0) \) because there isn’t a \( C^2 \) function \( f \) checking our conditions, but in [9, Example 8.3] there is an estimate for the curvature at such points.

2. Let \( F \) a sphere in \( \mathbb{R}^n \)
\[
\left\{ x = (x_1, \ldots, x_n) : \sum_{t=1}^{n} x_t^2 \leq R^2 \right\}.
\]
Consider \( f(x) = \sum_{t=1}^{n} x_t^2 - R^2 \) for \( x \) near a fixed \( \xi \in \partial F \). We have
\[
\nabla f(\xi) = 2\xi, \quad \nabla^2 f(\xi) = 2I_n,
\]
where \( I_n \) is the identity matrix of the order \( n \). Fix the first \( i \in I := \{1, \ldots, n\} \) such that \( f_{x_i}(\xi) = 2\xi_i \neq 0 \), then \( T_F(\xi) \) is spanned by \( n - 1 \) vectors \( u^j(\xi) \in \mathbb{R}^n, \ j \in I \setminus \{i\} \), with 1
in the \( j \)th coordinate, \(-\frac{\xi_j}{\xi_i}\) in the \( i \)th coordinate and 0 in the others. Therefore

\[
\hat{\kappa}_F (\xi, u^j (\xi)) = \frac{2 \left( \frac{\xi_j}{\xi_i} \right)^2 + 1}{4 \|\xi\| \|u^j (\xi)\|^2} = \frac{1}{2R},
\]

which means that the curvature at any point of the boundary of the sphere, in the direction of any vector of its hyperplane tangent, is equal to \( \frac{1}{2R} \).

3. Consider a cylinder

\[
F_{a,b} = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_3^2 \leq a^2, \ |x_2| \leq b\}, \ a, b \in \mathbb{R}^+.
\]

Near \( \xi = (\xi_1, \xi_2, \xi_3) \in \partial F_{a,b} \), with \( \xi_1^2 + \xi_3^2 = a^2, \ |\xi_2| < b \), put \( f (x_1, x_2, x_3) := x_1^2 + x_3^2 - a^2. \)

We have

\[
\nabla f (\xi) = 2 (\xi_1, 0, \xi_3), \quad \nabla^2 f (\xi) = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix},
\]

and, fixed the first \( i \in \{1, 3\} \) such that \( f_{x_i} (\xi) \neq 0 \), we have

\[
T_{F_{a,b}} (\xi) = \left\{ (v_1, v_2, v_3) : v_i = -\frac{\xi_j}{\xi_i} v_j, \ j \in \{1, 3\} \setminus \{i\}, \ v_2 \in \mathbb{R} \right\}.
\]

Then

\[
\langle \nabla^2 f (\xi) u^j (\xi), u^j (\xi) \rangle = \begin{cases} 0, & \text{if } j = 2 \\ \frac{1}{2a}, & \text{if } j \in \{1, 3\} \setminus \{i\} \end{cases}
\]

and consequently

\[
\hat{\kappa}_{F_{a,b}} (\xi, u^j (\xi)) = \begin{cases} 0, & \text{if } j = 2 \\ \frac{1}{2a}, & \text{if } j \in \{1, 3\} \setminus \{i\} \end{cases}.
\]

If we consider \( u (\xi) = \alpha u^2 (\xi) + \beta u^j (\xi) \), for any \( \alpha, \beta \in \mathbb{R} \) we will obtain

\[
\hat{\kappa}_{F_{a,b}} (\xi, u (\xi)) = \frac{\beta^2 \alpha}{2 (a^2 \beta^2 + \xi_i^2 \alpha^2)} \in \left[ 0, \frac{1}{2a} \right].
\]

Now fix \( \xi = (\xi_1, \xi_2, \xi_3) \in \partial F_{a,b} \), with \( \xi_1^2 + \xi_3^2 < a^2 \) and \( \xi_2 = b \) (the case \( \xi_2 = -b \) is analogous).

Near \( \xi \) we have \( f (x_1, x_2, x_3) := x_2 - b, \)

\[

\nabla f (\xi) = (0, 1, 0), \quad T_{F_{a,b}} (\xi) = \text{span} \{(1, 0, 0), (0, 1, 0)\}
\]

and \( \nabla^2 f (\xi) \) is the zero matrix of the order \( n \). So

\[
\hat{\kappa}_{F_{a,b}} (\xi, u (\xi)) = 0, \quad \forall u (\xi) \in T_{F_{a,b}} (\xi) \setminus \{(0, 0, 0)\}.
\]
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