OPERS AND THE TWISTED BOGOMOLNY EQUATIONS

SIQI HE AND RAFE MAZZEO

Abstract. In this paper, we study the dimensionally reduced twisted Kapustin-Witten equations on the product of a compact Riemann surface $\Sigma$ with $\mathbb{R}_y^+$. The main result is a Kobayashi-Hitchin type correspondence between the space of tilted Nahm pole solutions and the moduli space of Beilinson-Drinfeld opers. This corroborates a prediction of Gaiotto and Witten [10, p.971].

1. Introduction

Let $(M, g)$ denote an oriented Riemannian 4-manifold and $P$ a principal $SU(n)$ bundle over $M$ with adjoint bundle $g_P$. The twisted Kapustin-Witten (TKW) equations [19] are a one-parameter family of equations, parametrized by $t \in (0, \infty)$, for a pair $(A, \Phi)$, where $A$ is a connection and $\Phi$ is a $g_P$-valued 1-form:

$$F_A - \Phi \wedge \Phi + \frac{t-t^{-1}}{2} d_A \Phi + \frac{t+t^{-1}}{2} \ast d_A \Phi = 0,$$

$$d_A \ast \Phi = 0.$$  \hfill (1)

When $t = 1$, these equations have the particularly simple form

$$F_A - \Phi \wedge \Phi + \ast d_A \Phi = 0, \quad d_A \ast \Phi = 0.$$  \hfill (2)

One important case is when $M = X \times \mathbb{R}^+_y$, where $X$ is a 3-manifold. A fascinating proposal of Witten [30] interprets the Jones polynomial of knots in $X$ by counting solutions to (2) which satisfy certain ‘Nahm pole’ singularities at $y = 0$, see [10, 31, 32] for a more detailed explanation, along with [23, 24, 12, 27, 20] for analytic theory related to this program. A similar program using the TKW equations (1) to approach the Jones polynomial is also discussed in [30, 10]; a clearer formulation appears in [25], where the singular boundary conditions appropriate for these equations when $t \neq 1$ are called the tilted Nahm pole boundary condition. We describe these below.

We consider here the dimensionally reduced TKW equations on $\Sigma \times \mathbb{R}^+_y$, where $\Sigma$ is a compact Riemann surface. These are the TKW equations on $S^1 \times \Sigma \times \mathbb{R}^+_y$ for fields which are invariant in the $S^1$ direction. These fields consist of a connection $A$, a $g_P$-valued 1-form $\phi$ and $g_P$-valued 0-forms $A_1$ and $\phi_1$; the corresponding ‘twisted Bogomolny equations’ take the form

$$F_A - \phi \wedge \phi + \frac{t-t^{-1}}{2} d_A \phi + \frac{t+t^{-1}}{2} (\ast d_A \phi_1 + \ast [\phi, A_1]) = 0,$$

$$d_A A_1 - [\phi, \phi_1] + \frac{t-t^{-1}}{2} (d_A \phi_1 + [\phi, A_1]) - \frac{t+t^{-1}}{2} \ast d_A \phi = 0,$$

$$d_A^* \phi - [\phi_1, A_1] = 0.$$  \hfill (3)
Write \( t = \tan(\pi/4 - 3/2 \beta) \). It was observed by Gaiotto and Witten in [10] that with the assumption \( A_1 - \tan \beta \phi_1 = 0 \), the twisted Bogomolny equations have a Hermitian-Yang-Mills structure, leading them to conjecture that there should be a Donaldson-Uhlenbeck-Yau type theorem in this setting. Such a result is now fully understood in the special case \( \beta = 0 \ (t = 1) \), cf. [13, 14, 15]. The results in those two papers give a precise correspondence in this spirit between flat SL(2, \( \mathbb{C} \)) connections over \( \Sigma \) and solutions of the extended Bogomolny equations converging to these connections as \( y \to \infty \) and satisfying certain singular boundary conditions at \( y = 0 \). More precisely, a flat SL(2, \( \mathbb{R} \)) connection in the Hitchin section at infinity corresponds to solutions of the extended Bogomolny equations satisfying the Nahm pole boundary conditions at \( y = 0 \); an arbitrary stable Higgs pair (equivalently, a flat SL(2, \( \mathbb{C} \)) connection), together with a holomorphic line subbundle, corresponds to a solution of the extended Bogomolny equations satisfying the Nahm pole boundary conditions with extra singularities along a divisor determined by the line bundle and the Higgs field. It is the generalization of the former of these two theorems which is the subject of this paper; a full generalization awaits a better understanding of the knot singularities for the tilted Nahm pole boundary conditions.

The study of (3) when \( \beta \neq 0 \ (t \neq 1) \) is motivated by the Atiyah-Floer approach to the Kapustin-Witten equations, see [1, 7] for recent progress. As pointed out in [10, Section 3], this Atiyah-Floer approach has physical obstructions and is unstable for the equations (2) when \( t = 1 \), indicating that it may not be possible to recover the Jones polynomial entirely from that specialization of the equations. The paper [11] contains a more detailed explanation of this.

In any case, we consider here only the cases where \( \beta \neq 0 \). Denote by \( \mathcal{M}_{TBE}^{\beta} \) the space of solutions to the twisted Bogomolny equations with gauge group \( SU(n) \) and with tilted Nahm pole conditions at \( y = 0 \) and a certain boundary condition to be explained later as \( y \to \infty \). We denote by \( \mathcal{M}_{\text{Oper}}^{\beta} \) the twisted oper moduli space with parameter \( \tan \beta \); this is diffeomorphic to the usual oper moduli space of Beilinson-Drinfeld [3], and is defined in Section 3. Using the Hermitian-Yang-Mills structure, Gaiotto and Witten [10] define

\[
I_{\text{Oper}}^{\beta} : \mathcal{M}_{TBE}^{\beta} \rightarrow \mathcal{M}_{\text{Oper}}^{\beta},
\]

explained in Section 4 below, and predict that it is a bijection. We confirm their prediction here.

**Theorem 1.1.** The map \( I_{\text{Oper}}^{\beta} \) is a bijection when the genus \( g \) of \( \Sigma \) is greater than 1.

i) For each element in \( \mathcal{M}_{\text{Oper}}^{\beta} \), there exists a solution to (3) with tilted Nahm pole singularity at \( y = 0 \);

ii) If two solutions satisfy tilted Nahm pole boundary condition and have the same image by \( I_{\text{Oper}}^{\beta} \), then they are gauge equivalent.

There is an identification of \( \mathcal{M}_{\text{Oper}}^{\beta} \) with \( \bigoplus_{i=2}^{n} H^0(K^i) \), where \( K \) is the canonical bundle over \( \Sigma \), which gives a topology and differential structure to this space.

**Theorem 1.2.** The map \( I_{\text{Oper}}^{\beta} \) is a diffeomorphism.

**Acknowledgements.** The first author thanks Victor Mikhaylov for assistance with the computations in the Appendix and Brian Collier, Simon Donaldson, Ciprian
Manolescu and Du Pei for helpful discussions. R.M. has been supported by the NSF grant DMS-1608223.

2. The Twisted Extended Bogomolny Equations

2.1. Hermitian Geometry for the Twisted Bogomolony Equations. We write the product metric on $\Sigma \times \mathbb{R}^+$, where $\Sigma$ is a compact Riemann surface with genus $g \geq 1$, as $g = g_0^2|dz|^2 + dy^2$. Let $E$ be a rank $n$ complex vector bundle on this space with $\det E = 0$. An $SU(n)$ structure on $E$ is determined by a Hermitian metric $H$. The adjoint bundle is denoted $g_E$.

Let $A$ be a connection on $E$ and suppose that $\phi \in \Omega^1(\mathfrak{g}_E)$ and $\phi_1 \in \Omega^0(\mathfrak{g}_E)$. In a unitary gauge defined by $H$, these satisfy $A^* = -A$, $\phi^* = -\phi$, $\phi_1^* = -\phi_1$ where $*$ is the conjugate transpose defined by $H$. Gaiotto and Witten observe in [10] that the twisted extended Bogomolny equations have a Hermitian Yang-Mills structure, and hence there should be a Donaldson-Uhlenbeck-Yau type result as in [8, 28]. Write $z = z_2 + iz_3$ for a local holomorphic coordinate on $\Sigma$ and $y$ for the linear coordinate on $\mathbb{R}^+$. Then

$$d_A = \nabla_2 dx_2 + \nabla_3 dx_3 + D_y dy,$$

and

$$\phi = \phi_2 dx_2 + \phi_3 dx_3 = \frac{1}{2}(\phi_2 dz + \phi_3 d\bar{z}).$$

Following [10, 30], define

$$(4) \quad D_1 = (D_z - \phi_2 \tan \beta)dz, \quad D_2 = (D_z + \phi_2 \cot \beta)dz, \quad D_3 = D_y - i \frac{\phi_1}{\cos \beta},$$

and their adjoints

$$(5) \quad D_1^\dagger = (D_z - \phi_2 \tan \beta)dz, \quad D_2^\dagger = (D_z + \phi_2 \cot \beta)dz, \quad D_3^\dagger = D_y + i \frac{\phi_1}{\cos \beta}.$$

Using this, the twisted extended Bogomolny equations can be written in the particularly elegant form

$$(6) \quad D_i \cdot D_j = 0, \quad i, j = 1, 2, 3,$$

where $[D_i, D_j] = 0$. Using the geometric meaning of a set of holomorphic data satisfying the $G_C$ invariant part of the equations. The main theorem is proved later by finding a solution compatible with any choice of holomorphic data which also solves the other moment map equation.

2.2. Holomorphic data. As in Donaldson and Uhlenbeck-Yau [8, 28, 9], we begin by interpreting the geometric meaning of a set of holomorphic data satisfying the $G_C$ invariant part of the equations. The main theorem is proved later by finding a solution compatible with any choice of holomorphic data which also solves the other moment map equation.
Let \( E_y \) be the restriction of \( E \) to the slice \( \Sigma \times \{y\} \), and consider the \( SL(n, \mathbb{C}) \) connection \( \mathcal{D}_{\Sigma_y} = \mathcal{D}_1 + \mathcal{D}_2 \) on \( E_y \). The commutation relation \([\mathcal{D}_1, \mathcal{D}_2] = 0\) is equivalent to the flatness of \( \mathcal{D}_{\Sigma_y} \), i.e., \( \mathcal{D}_{\Sigma_y}^2 = 0 \). Furthermore, \( \mathcal{D}_3 \) is a covariant derivative in the \( y \) direction, and hence defines a parallel transport in this ‘vertical’ direction. The commutation relations \([\mathcal{D}_1, \mathcal{D}_3] = [\mathcal{D}_2, \mathcal{D}_3] = 0\) identify the flat connections \( \mathcal{D}_{\Sigma_y} \) over the different slices \( E_y \) by parallel transport.

Based on this, we define the holomorphic data as a rank \( n \) bundle \( E \) with \( \det(E) = 0 \), and a system of operators \( \Theta = (\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3) \) acting on sections of \( E \) such that for any smooth function \( f \) and section \( s \) of \( E \), we have

\[
\begin{align*}
\mathcal{D}_1(f s) &= \bar{\partial} f + f \mathcal{D}_1 s, \\
\mathcal{D}_2(f s) &= \partial f + f \mathcal{D}_2 s, \\
\mathcal{D}_3(f s) &= (\partial_y f) s + f \mathcal{D}_3 s,
\end{align*}
\]

\([\mathcal{D}_i, \mathcal{D}_j] = 0\) for all \( i, j \).

The complex gauge group \( G_C \) acts on \((E, \Theta)\) by \( \mathcal{D}_i \to g^{-1} \circ \mathcal{D}_i \circ g \). As above, parallel transport yields the identification

\[
\{ (E, \Theta) \}/G_C \cong \{ \text{flat connections over } \Sigma \}/G_C.
\]

Rather than letting \( G_C \) act on the data set \((E, \Theta)\), it is easier to fix \( \Theta \) and let \( G_C \) act on the Hermitian metric. We thus regard the real moment map in \((6)\) as an equation for the Hermitian metric. Given \((E, \Theta)\), a Hermitian metric determines the adjoints \( \mathcal{D}_i^\dagger \) of the operators \( \mathcal{D}_i \) by

\[
\begin{align*}
\bar{\partial}(H(s, s')) &= H(\mathcal{D}_1 s, s') + H(s, \mathcal{D}_1^\dagger s'), \\
\partial(H(s, s')) &= H(\mathcal{D}_2 s, s') + H(s, \mathcal{D}_2^\dagger s'), \\
\partial_y (H(s, s')) &= H(\mathcal{D}_3 s, s') + H(s, \mathcal{D}_3^\dagger s').
\end{align*}
\]

Now define the unitary operators and forms

\[
\begin{align*}
\mathcal{D}_z &= \sin^2 \beta \mathcal{D}_2 + \cos^2 \beta \mathcal{D}_1^\dagger, \\
\phi_z &= \sin \beta \cos \beta (\mathcal{D}_2 - \mathcal{D}_1^\dagger), \\
\mathcal{D}_y &= \frac{\mathcal{D}_3 + \mathcal{D}_3^\dagger}{2}, \\
\phi_1 &= \frac{i \cos \beta}{2} (\mathcal{D}_3 - \mathcal{D}_3^\dagger).
\end{align*}
\]

We obtain from these a unitary connection \( \nabla_A := \mathcal{D}_z + \mathcal{D}_2 + \mathcal{D}_y \) and a triple \((A, \phi, \phi_1)\) called the Chern connection of \((E, \Theta, H)\).

We can express these operators in different gauges. A gauge is called parallel if in this trivialization, \( \mathcal{D}_1 = \bar{\partial} \) and \( \mathcal{D}_3 = \partial_y \), and unitary if in this trivialization the matrices \((A, \phi, \phi_1)\) are unitary. As in [2], we record how to relate these two gauge choices.

**Proposition 2.1.** [13] Given \((E, \Theta, H)\) as above, there is a unique triplet \((A, \phi, \phi_y)\) compatible with the unitary and holomorphic structures. In other words, in any unitary gauge, \( A^* = -A, \phi^* = \phi, \phi_1^* = -\phi_1 \), while in every parallel holomorphic gauge, \( \mathcal{D}_1 = \bar{\partial}_E \) and \( \mathcal{D}_3 = \partial_y \), i.e., \((A^{(0,1)}) = A_{\bar{y}} - i\phi_1 = 0.\)

In a local holomorphic trivialization of \( E \), we represent the metric by a Hermitian matrix which we also denote by \( H \). Write \( \mathcal{D}_1 = \bar{\partial}, \mathcal{D}_2 = \partial + \alpha, H = g^\dagger g \) for \( g = H^{\frac{1}{2}} \in G_C \). In holomorphic gauge,

\[
\mathcal{D}_1 = \bar{\partial}, \mathcal{D}_1^\dagger = \partial + H^{-1} \partial H = \partial + g^{-1} (g^\dagger)^{-1} (\partial_z g^\dagger) g + g^{-1} \partial_z g.
\]
Thus \( g \) transforms from holomorphic to unitary gauge. In unitary gauge, we write these operators as \( D^U_i \) and \( (D^i)^U \), and then have
\[
D^U_1 = \bar{\partial} - (\bar{\partial}g)g^{-1}, \quad (D^i)^U = \partial + (g^\dagger)^{-1}\partial_z g^\dagger, \quad (8)
\]
\[
D^U_2 = \partial + g\alpha g^{-1} - (\partial g)g^{-1}, \quad (D^2)^U = \bar{\partial} + (g^\dagger)^{-1}\bar{\partial}g^\dagger - (g^\dagger)^{-1}\alpha^T g^\dagger, \\
D^U_3 = \partial_y - (\partial_y g)g^{-1}, \quad (D_3^U)^U = \partial_y + (g^\dagger)^{-1}\partial_y g^\dagger.
\]

Thus in unitary gauge,
\[
A_z = \sin^2 \beta(g\alpha g^{-1} - (\partial g)g^{-1}) + \cos^2 \beta(g^\dagger)^{-1}\partial_z g^\dagger, \quad A_z = -A_z^\dagger, \\
\phi_z = \sin \beta \cos(g\alpha g^{-1} - (\partial g)g^{-1} - (g^\dagger)^{-1}\partial_z g^\dagger), \quad \phi_z = -(\phi_z^\dagger), \\
A_y = \frac{1}{2}(-(\partial_y g)g^{-1} + (g^\dagger)^{-1}\partial_y g), \quad \phi_1 = \frac{i\cos \beta}{2}(-(\partial_y g)g^{-1} - (g^\dagger)^{-1}\partial_y g^\dagger). \quad (9)
\]

3. Higgs bundles, the Teichmüller component and the space of opers

3.1. The nonabelian Hodge correspondence.

3.1.1. The de Rham moduli space. As always, assume that \( E \) is a bundle of rank \( n \) with \( \det(E) = 0 \) over the Riemann surface \( \Sigma \) and \( \nabla \) a connection on \( E \). This connection is irreducible if there is no parallel subbundle and completely reducible if every \( \nabla \)-invariant subbundle has a \( \nabla \)-invariant complement. If \( C_{\text{flat}} \) denotes the space of flat connections, then its tangent space \( \tau_{\text{flat}} \) consists of the set of \( \sigma \in \Omega^1(\text{End}(E)) \) such that \( \nabla + \sigma \) is flat to first order, i.e., \( T_{\nabla}C_{\text{flat}} := \{ \sigma \in \Omega^1(\text{End}(E)) | \nabla \sigma = 0 \} \). The complex structure on \( \text{SL}(n, \mathbb{C}) \) induces a complex structure \( J \) on \( C_{\text{flat}} \), \( J(\sigma) = i\sigma \).

Complex gauge transformations \( g(\nabla) := g^{-1} \circ \nabla \circ g \) are the smooth automorphisms of \( E \) acting trivially on \( \det(E) \). The de Rham moduli space of flat connections is
\[
M_{\text{flat}} := \{ \nabla \in C_{\text{flat}} | \nabla \text{ is completely reducible} \}/G_{\mathbb{C}}.
\]
The complex structure \( J \) is preserved by the \( G_{\mathbb{C}} \) action, and hence induces a complex structure on \( M_{\text{flat}} \).

3.1.2. The Higgs bundle moduli space. Recall that an \( \text{SL}(n, \mathbb{C}) \)-Higgs bundle over the Riemann surface \( \Sigma \) is a pair \( (\mathcal{E}, \varphi) \), where \( \mathcal{E} \) is a vector bundle \( E \) with holomorphic structure \( \partial_E \) and \( \varphi \) is a traceless holomorphic \((1,0)\)-form, \( \partial_E \varphi = 0 \). We restrict to Higgs bundles with \( \det(E) = 0 \). If \( \mathcal{H} \) denotes the set of \( \text{SL}(n, \mathbb{C}) \)-Higgs bundles on \( E \), then
\[
T_{(\partial_E, \varphi)} \mathcal{H} = \{(a, b) \in \Omega^{0,1}(\text{End}(E)) \oplus \Omega^{1,0}(\text{End}(E)) | \partial_E b + [\varphi, a] = 0 \}.
\]
A complex structure \( I \) on \( \mathcal{H} \) is defined by \( I(a, b) = (ia, ib) \), and the complex gauge group action is \( g(\mathcal{E}, \varphi) := (g^{-1} \circ \partial_E \circ \Phi \circ g, g^{-1} \circ \Phi \circ g) \).

A Higgs pair \( (\mathcal{E}, \varphi) \) with \( \deg \mathcal{E} = 0 \) is called stable if \( \deg(V) < 0 \) for any holomorphic subbundle \( V \) with \( \varphi(V) \subset V \otimes K \), and polystable if it is a direct sum of stable Higgs pairs. We define the Higgs bundle moduli space \( M_{\text{Higgs}} \) by
\[
M_{\text{Higgs}} := \{ (\mathcal{E}, \varphi) \in \mathcal{H} | (\mathcal{E}, \varphi) \text{ is polystable} \}/G_{\mathbb{C}}.
\]
One of the most important features is the Hitchin fibration map
\[
\pi : M_{\text{Higgs}} \to \bigoplus_{i=2}^n H^0(\Sigma, K^i) \quad \pi(\varphi) = (p_2(\varphi), \cdots, p_n(\varphi)), \quad (10)
\]
where \( \det(\lambda - \varphi) = \sum \lambda^{n-j}(-1)^j p_j(\varphi) \). By [16], this map is proper.

3.1.3. The nonabelian Hodge correspondence. Given a Higgs bundle \((E, \varphi)\) the Hitchin equations are equations for the Hermitian metric \(H\)

\[
F_H + [\varphi, \varphi^\dagger] = 0, \quad \bar{\partial}_E \varphi = 0,
\]

where \(F_H\) is the curvature of the Chern-connection, \(\varphi^\dagger\) is the conjugation of \(\varphi\) w.r.t the Hermitian metric \(H\).

**Theorem 3.1.** [18, 6] For any Higgs pair \((E, \varphi)\) on \(\Sigma\), there exists an irreducible solution \(H\) to the Hitchin equations if and only if this pair is stable, and a reducible solution if and only if it is polystable.

To any solution \(H\) to the Hitchin equations is associated a flat \(\text{SL}(n, \mathbb{C})\) connection \(D = \nabla_H + \varphi + \varphi^\dagger\mu\), and hence a representation \(\rho : \pi_1(\Sigma) \to \text{SL}(n, \mathbb{C})\), which is well-defined up to conjugation. Irreducibility of the solution is the same as irreducibility of the representation, while reducibility corresponds to the fact that \(\rho\) is reductive. The map from flat connections back to solutions of the Hitchin system involves finding a harmonic metric which yields a decomposition of \(D = D^{\text{skew}} + D^{\text{Herm}}\) into skew-Hermitian and Hermitian parts, so that \(D^{\text{Herm}} = \varphi + \varphi^* H\) and \(((D^{\text{skew}})^0, 1, \varphi)\) satisfies Hitchin equations. The culmination of the work of Hitchin, Donaldson, Simpson and Corlette is the diffeomorphic equivalence between the spaces of stable Higgs pairs, irreducible solutions of the Hitchin equations and irreducible flat connections; there is a similar equivalence for the polystable/reducible spaces.

In summary, the non-abelian Hodge correspondence says that the map

\[
\mathcal{R}_{\text{NAH}} : \mathcal{M}_{\text{Higgs}} \to \mathcal{M}_{\text{flat}},
\]

\[
\mathcal{R}_{\text{NAH}}(E, \varphi) = A_H + \varphi + \varphi^\dagger\mu,
\]

is a diffeomorphism where \(H\) solves the Hitchin equations and \(A_H\) is the Chern-connection of \(E\) and \(H\).

3.1.4. Hyperkähler structure. The moduli space \(\mathcal{M}_{\text{Higgs}}\) has important geometric structure. The tangent space of \(\mathcal{M}_{\text{Higgs}}\) at each point is a subspace of \(\Omega^{0,1}(\text{End } E) \oplus \Omega^{1,0}(\text{End } E)\) at that point, so it carries a hyperkähler structure as in [18]: if \(a \in \Omega^{0,1}(\text{End } E)\) and \(b \in \Omega^{1,0}(\text{End } E)\), then

\[
I(a, b) = (ia, ib), \quad J(a, b) = (ia^*, -ib^*), \quad K(a, b) = (-a^*, b^*).
\]

3.2. The Hitchin component and opers. In this subsection, we introduce the Hitchin component and the moduli space of opers.
3.2.1. **SL(n)-Hitchin component.** Choose a spin structure $K^{\frac{1}{2}}$; then, for any $q := (q_2, \cdots, q_n)$, define the Higgs bundle $(\mathcal{E}, \varphi_q)$ by

$$
\mathcal{E} := S^{n-1}(K^{-\frac{1}{2}} \oplus K^{\frac{1}{2}}) = K^{-\frac{n-1}{2}} \oplus K^{-\frac{n-1}{2}+1} \oplus \cdots \oplus K^{\frac{n-1}{2}}
$$

(14)

$$
\varphi_q = \begin{pmatrix}
0 & \sqrt{B_1} & 0 & \cdots & 0 \\
0 & 0 & \sqrt{B_2} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & \vdots & \cdots & \sqrt{B_{n-1}} & \\
q_n & q_{n-1} & \cdots & q_2 & 0
\end{pmatrix},
$$

where $B_i = i(n - i)$ and $\sqrt{B_i}$ in the $(i, i + 1)$ entry represents that multiple of the natural isomorphism $K^{-\frac{n-1}{2}+i} \rightarrow K^{-\frac{n-1}{2}+i-1} \otimes K$ and the maps along the bottom row are $H^0(\Sigma, K^{n-i}) \ni q_{n-i} : K^{-\frac{n-1}{2}+i} \rightarrow K^{\frac{n-1}{2}} \otimes K$.

The complex gauge orbit of this family of Higgs bundle,

$$
\{(\mathcal{E} := S^{n-1}(K^{-\frac{1}{2}} \oplus K^{\frac{1}{2}}), \varphi \text{ as in (14)}\} / \mathcal{G}_C.
$$

is called the Hitchin component (or the Hitchin section), and denoted $\mathcal{M}_{Hit}$. Note that when $n$ is odd, only even powers of $K^{1/2}$ appear, so $\mathcal{M}_{Hit}$ is independent of the choice of spin structure in that case. We consider the map $p_{Hit}$ defined as

$$
p_{Hit} : \mathcal{B}_{Hit} := \oplus_{i=2}^n H^0(\Sigma, K^i) \rightarrow \mathcal{M}_{Higgs},$$

(16)

$$
p_{Hit}(q_2, \cdots, q_n) := [(\mathcal{E}, \varphi_q)],$$

with $(\mathcal{E}, \varphi_q)$ as in (14). The space $\mathcal{B}_{Hit}$ is called the Hitchin base, and the image of $p_{Hit}$ is the aforementioned Hitchin component $\mathcal{M}_{Hit}$.

**Theorem 3.2.** [17]

1. Every element in $\mathcal{M}_{Hit}$ is a stable Higgs pair, and $\mathcal{M}_{Hit}$ is parametrized by the vector space $\oplus_{i=2}^n H^0(\Sigma, K^i)$;
2. the map assigning to each element of $\oplus_{i=2}^n H^0(\Sigma, K^i)$ the unique corresponding solution of the Hitchin equations is an equivalence from this vector space to one component of the moduli space of flat completely irreducible $\text{SL}(n, \mathbb{R})$ connections. The restriction of the Hitchin fibration map (10)

$$
\pi|_{\mathcal{M}_{Hit}} : \mathcal{M}_{Hit} \rightarrow \mathcal{B}_{Hit}
$$

is a diffeomorphism.
3. $p_{Hit}$ is a holomorphic embedding into $\mathcal{M}_{Higgs}$ with respect to the holomorphic structure $I$.

3.2.2. **Opers.** We begin with the definition of opers from [3].

**Definition 3.3.** An $\text{SL}(n)$-oper on a Riemann surface $\Sigma$ is a triple $(E, F_\bullet, \nabla)$, where $E$ is a rank $n$ holomorphic bundle with $\det(E) = 0$, $\nabla$ is a flat connection and $F_\bullet$ is a complete filtration of $E$ by holomorphic subbundles, $0 = F_0 \subset F_1 \subset \cdots \subset F_n = E$, satisfying

- for any section $s$ of $F_i$, $\nabla s$ is a section of $F_{i+1} \otimes K$;
- the induced map $\nabla : F_i/F_{i-1} \rightarrow F_{i+1}/F_i \otimes K$ is an isomorphism of line bundles, $i = 1, \ldots, n - 1$. 

The complex gauge group acts naturally on the space of opers and we define the oper moduli space

$$\mathcal{M}_{\text{Oper}} := \{(E,F,\nabla)/G\}.$$ 

**Proposition 3.4.** [29] If the genus of $\Sigma$ is greater than 1, then

1. the holonomy representation of an SL($n$)-oper is irreducible;
2. setting $\mathcal{L} \cong E/F_{n-1}$, then $\det(F_j) \cong K^{1-(j+i+1)}$ and $\mathcal{L}^n \cong K^{-n(n-1)}$;
3. the oper structure on $E$ is uniquely determined by $\mathcal{L}$; in particular, the isomorphism class of $\mathcal{L}$ is fixed on every connected component of $\mathcal{M}_{\text{Oper}}$.

**Remark.** There are precisely $n^{2g}$ possibilities for the line bundle $\mathcal{L}$, corresponding to the $n^{2g}$ ways of lifting a monodromy representation in $\text{PSL}(n,\mathbb{C})$ to $\text{SL}(n,\mathbb{C})$; these choices label the different components of $\mathcal{M}_{\text{Oper}}$. We fix one such choice and take $\mathcal{L} = K^{-(n-1)/2}$.

The next result identifies the Hitchin section with the space of opers.

**Theorem 3.5.** [3] Consider a Higgs bundle $(\mathcal{E} = S^{n-1}(K^{-\frac{1}{2}} \oplus K^\frac{1}{2}), \varphi_q) \in \mathcal{M}_{\text{Hit}}$. Let $h_q$ be the solution to the Hitchin equation (11) for the Higgs field $(\mathcal{E}, \varphi_q)$ with $q \in B_{\text{Hit}}$ and Chern connection $D_{h_q}$. Write $F_0 = 0$, $F_i = K^{-\frac{a_i}{n}} \oplus \cdots \oplus K^{-\frac{a_{i+1}}{n}}$ for $1 \leq i \leq n$ and $F_*$ the associated filtration. Define $\nabla_q = D_{h_q} + \varphi_q + \varphi_q^{\dagger h_0}$. Then $\mathcal{M}_{\text{Oper}} \cong \{\nabla_q\}/G\mathbb{C}$. In other words, this family of flat connections exhausts the entire space of opers.

### 3.3. Twistings

Both the Hitchin equations and opers may be ‘twisted’ by a nonzero complex number $w$. These twisted analogues are equivalent to the untwisted ones by a constant gauge transformation, but these are the objects most naturally compatible with the twisted extended Bogomolny equations.

#### 3.3.1. Twisted opers

Define $q_w = (w^2 q_2, w^3 q_3, \cdots, w^n q_n)$ where $q \in B_{\text{Hit}}$ and $w \in \mathbb{C}^\times$. Note that $\varphi_q$ is conjugated to $\varphi_{q_w}$ up to a constant by the constant gauge transformation $g_w = \text{diag}(w^{\frac{a_2}{n}}, w^{\frac{a_3}{n}}, \cdots, w^{\frac{a_n}{n}})$. The associated oper is

$$\nabla^w_q = g_w^{-1} \nabla_q g_w = \nabla_{q_w} = D_{h_q} + w^{-1} \varphi_q + w \varphi_q^{\dagger h_0}.$$ 

It follows directly from Theorem 3.5 that for any oper $\nabla$ and any $w \in \mathbb{C}^\times$, there exists a unique $q$ such that $\nabla = \nabla^w_q$. Just as for the case $w = 1$, we can define the twisted oper moduli space $\mathcal{M}^w_{\text{Oper}}$ carefully below.

#### 3.3.2. The twisted Hitchin equations

Let $(\mathcal{E}, \varphi, H)$ be a solution of the Hitchin equations. Defining $\mathcal{P}_1 := D_z d\bar{z}$, $\mathcal{P}_2 = \varphi = i\phi_d dz$, we can rewrite the Hitchin equations as

$$[\mathcal{P}_1, \mathcal{P}_2] = 0, \quad \Lambda([\mathcal{P}_1, \mathcal{P}_1^{\dagger h}]+[\mathcal{P}_2, \mathcal{P}_2^{\dagger h}]) = 0.$$ 

The map $\mathcal{R}_{\text{NAH}} : \mathcal{M}_{\text{Higgs}} \to \mathcal{M}_{\text{flat}}$ introduced earlier associates to any stable Higgs bundle $(\mathcal{E}, \varphi)$ the flat connection $\nabla_A + \varphi + \varphi^{\dagger h}$, where $H$ is the solution of the Hitchin equations and $A_H$ is its Chern connection $A_H$. However, if $w \in \mathbb{C}^\times$, we can define the twisted operators

$$\mathcal{P}_1^w := (D_z - w\phi_d) d\bar{z}, \quad \mathcal{P}_2^w := (D_z + w^{-1}\phi_d) dz.$$
where \( d_A = D_\omega dz + D_\bar{\omega} d\bar{z} \) and \( \phi = \phi_z dz + \phi_{\bar{z}} d\bar{z} \). Then \( \nabla^w = P_1^w + P_2^w \) is also a flat connection. This gives a family of maps:

\[
\mathcal{R}_{\text{NAH}}^w \colon \mathcal{M}_{\text{Higgs}} \to \mathcal{M}_{\text{flat}}, \quad w \in \mathbb{C}^\times.
\]

The twisted operators \( P_1^w, P_2^w \) lead to a new system of \( w \)-twisted Hitchin equations

\[
[P_1^w, P_2^w] = 0, \quad \Lambda([P_1^w, (P_1^w)^\dagger]) - |w|^2[P_2^w, (P_2^w)^\dagger]) = 0.
\]

As before, given a flat connection \( \nabla := \nabla_1^w + \nabla_2^w \), (19) is an equation for the Hermitian metric, and is equivalent to the standard untwisted Hitchin equations.

**Proposition 3.6.** Define the \( w \)-twisted oper moduli space

\[
\mathcal{M}_\text{Oper}^w := \{ (\nabla, H) \mid \nabla \in \mathcal{M}_\text{Oper}, \ H \text{ solves } (19) \} / \mathcal{G},
\]

i) For any \( \nabla \in \mathcal{M}_\text{Oper}, \) there exists a unique \( H \) solving (19);
ii) \( \mathcal{M}_\text{Oper}^w \) is diffeomorphic to \( \mathcal{M}_\text{Oper} \);
iii) [19, Section 3] \( \mathcal{M}_\text{Oper}^w \) is a holomorphic symplectic submanifold with respect to the complex structure

\[
I_w = \frac{1 - \bar{w} w}{1 + \bar{w} w} I + \frac{i(w - \bar{w})}{1 + \bar{w} w} J + \frac{w + \bar{w}}{1 + \bar{w} w} K.
\]

**Proof.** For i), existence and uniqueness follows from Proposition 3.7 and 3.10, and ii) follows directly from i). For iii), We can understandopers in terms of the hyperkähler structure, cf. [19, Section 3]. Fixing \( w \in \mathbb{C}^\times \), choose the holomorphic coordinates \( A_z + w^{-1} \phi_z, A_{\bar{z}} - w \phi_{\bar{z}} \) for \( \mathcal{M}_\text{Oper}^w \). Then \( \mathcal{M}_\text{Oper}^w \) is holomorphic symplectic with respect to

\[
I_w = \frac{1 - \bar{w} w}{1 + \bar{w} w} I + \frac{i(w - \bar{w})}{1 + \bar{w} w} J + \frac{w + \bar{w}}{1 + \bar{w} w} K.
\]

\( \square \)

3.3.3. Existence and uniqueness. We next consider (19) from the perspective of moment maps: given a flat connection satisfying a suitable stability condition, we wish to find a Hermitian metric solving (19).

**Proposition 3.7.** Suppose that \((\mathcal{E}, \varphi)\) is a stable Higgs bundle and \( H_0 \) the corresponding harmonic metric; write the associated fields as \((A_{H_0}, \phi_{H_0})\). Define \( P_1^w := (D_z - w \phi_z) dz \) and \( P_2^w := (D_{\bar{z}} + w^{-1} \phi_{\bar{z}}) d\bar{z} \). Then \((P_1^w, P_2^w, H_0)\) solves (19).

Conversely, let \((P_1^w, P_2^w, H)\) solve the \( \mathbb{C}^\times \)-Hitchin equations with parameter \( w \in \mathbb{C}^\times \), and define

\[
D_z = \frac{1}{1 + |w|^2} P_1^w + \frac{|w|^2}{1 + |w|^2} (P_2^w)^\dagger, \quad \phi_z = \frac{w}{1 + |w|^2} (P_2^w - (P_1^w)^\dagger).
\]

Then \((D_z, \phi_z, H)\) solves the untwisted Hitchin equations.

The proof follows by unwinding the definitions.

We next recall Corlette’s theorem [6]:
Theorem 3.8. [6] Let $\nabla = \mathcal{P}_1 + \mathcal{P}_2$ be an irreducible flat connection, with $(0, 1)$ and $(1, 0)$ parts $\mathcal{P}_1$ and $\mathcal{P}_2$, respectively. Then there exists a unique Hermitian metric $H$ such that $\bar{\partial}_H \varphi_H = 0$, where

\begin{equation}
\bar{\partial}_H := \frac{1}{2}(\mathcal{P}_1 + \mathcal{P}_2^\dagger), \quad \varphi_H = \frac{1}{2}(\mathcal{P}_2 - \mathcal{P}_1^\dagger).
\end{equation}

The fact that (19) is solvable generalizes this theorem; indeed, the equation in Corlette’s theorem is the $\mathbb{C}^\times$ Hitchin equation with coefficient $w = -i$.

Theorem 3.9. Let $\nabla$ be an irreducible flat connection and fix any $w \in \mathbb{C}^\times$. Then there exists a solution to the $w$-twisted Hitchin equations.

Proof. Write $\nabla = \mathcal{P}_1 + \mathcal{P}_2$, and define, for any $H$,

\begin{equation}
D_z := \frac{1}{1 + |w|^2}\mathcal{P}_1 + \frac{|w|^2}{1 + |w|^2}\mathcal{P}_2^\dagger, \quad \varphi_z := \frac{w}{1 + |w|^2}(\mathcal{P}_2 - (\mathcal{P}_1)^\dagger),
\end{equation}

\begin{equation}
D_z := \frac{1}{1 + |w|^2}(\mathcal{P}_1^\dagger + |w|^2\mathcal{P}_2), \quad \varphi_z := \frac{\bar{w}}{1 + |w|^2}(\mathcal{P}_2^\dagger - \mathcal{P}_1)
\end{equation}

and

\begin{equation}
D' := D_z + D_z + i\varphi_z + i\varphi_z = \frac{1}{(1 + |w|^2)}((1 - i\bar{w})\mathcal{P}_1 + (iw + |w|^2)\mathcal{P}_2 + (1 - iw)\mathcal{P}_1^\dagger + (|w|^2 + i\bar{w})\mathcal{P}_2^\dagger).
\end{equation}

If $s$ is a section such that $D's = 0$, then

\begin{equation}
(1 - i\bar{w})\mathcal{P}_1s + (|w|^2 + i\bar{w})\mathcal{P}_2^\dagger s = 0, \quad (1 - iw)\mathcal{P}_1^\dagger s + (|w|^2 + iw)\mathcal{P}_2s = 0.
\end{equation}

When $w \in \mathbb{C}^\times$, this implies $\mathcal{P}_1s = \mathcal{P}_2s = 0$. As $\nabla$ is irreducible, we obtain that $\text{Ker } D' = 0$, hence $D'$ is irreducible too. Applying Theorem 3.8 to the operator $D'$, there exists a pair $(A_{H_0}, \phi_{H_0})$ with Hermitian metric $H_0$ solves the Hitchin equation by Proposition 3.7, the corresponding $(\mathcal{P}_1^w, \mathcal{P}_2^w, H_0)$ solves the $w$-twisted Hitchin equations. \hfill \Box

Proposition 3.10. [8, 9, 26] Let $\nabla = \mathcal{P}_1^w + \mathcal{P}_2^w$ be irreducible flat connection, then the solution to the $w$-twisted Hitchin equations is unique.

Proof. For any Hermitian metric $H$, define $\Xi_H := \Lambda([\mathcal{P}_1^w, (\mathcal{P}_1^w)^\dagger] - w^2[\mathcal{P}_2^w, (\mathcal{P}_2^w)^\dagger])$. If $\bar{K}$ is another Hermitian metric, write $H = \bar{K}e^s$ and consider $\Xi$ as a function of $s$. Define the functional

\begin{equation}
\mathcal{M}(H, K) = \int_0^1 \int_{\Sigma} \langle s, \Xi_{ws} \rangle_K \omega \wedge du,
\end{equation}

where $\omega$ is the area form on $\Sigma$. This functional reveals the variational structure for the $\mathbb{C}^\times$-Hitchin equations. Indeed, writing $H_t = Ke^{ts}$, then

\begin{equation}
\frac{d}{dt} \mathcal{M}(H_t, K) = \int_{\Sigma} \text{Tr}(\Xi_{Ht}, s) \omega, \quad \frac{d^2}{dt^2} \mathcal{M}(H_t, K) = \int_{\Sigma} \cos^2 \beta |\mathcal{P}_1^w s|^2 + \sin^2 \beta |\mathcal{P}_2^w s|^2.
\end{equation}

When $\nabla$ is irreducible, $\frac{d}{dt^2} \mathcal{M}(H_t, K) > 0$, and this strict convexity implies uniqueness. \hfill \Box
3.4. The boundary condition at infinity. In the rest of the paper, we assume that \( w = \tan \beta \) and write \( I_\beta := I_{\tan \beta} = \cos 2\beta I + \sin 2\beta K \), and \( \mathcal{M}_\text{Oper}^\beta := \mathcal{M}_\text{Oper}^{\tan \beta} \) for the corresponding oper moduli space.

The asymptotic boundary condition for twisted extended Bogomolny equations as \( y \to \infty \) corresponds to the requirement that solutions converge to a \( y \)-independent flat twisted connection, and that \( \phi_1 \to 0 \). More explicitly, the boundary conditions for the triple \((A, \phi, \phi_1)\) are that \((A, \phi)\) converges to a flat \( w \)-twisted \(SL(n, \mathbb{C})\) connection \( \nabla^w \) associated to the pair \((A_0, \phi_0)\), and that \( \phi_1 \) converges (exponentially) to 0.

We can phrase this equivalently in terms of the Hermitian metric:

**Definition 3.11.** Suppose that \( A_0 + i\phi_0 \) is an irreducible flat connection, and define the (necessarily commuting) operators

\[
D_1 := (D_{(A_0)_z} - (\phi_0)_z \tan \beta)dz, \quad D_2 := (D_{(A_0)_z} + (\phi_0)_z \cot \beta)dz,
\]

By Proposition 3.7, we obtain a solution \( H_0 \); this is the boundary condition for the Hermitian metric at infinity.

4. The singular boundary Condition and the Model Metric

4.1. The tilted Nahm pole boundary conditions. Fix a principal embedding \( \mathfrak{sl}_2 \hookrightarrow \mathfrak{sl}_n \) and choose a basis \( \mathfrak{sl}_2 = \text{span} \{ e^+, e^-, e^0 \} \) where

\[
[e^+, e^-] = e^0, \quad [e^0, e^-] = -2e^-, \quad [e^0, e^+] = 2e^+.
\]

Set \( B_i = i(n - i) \) and write

\[
(28) \quad e^+ = \begin{pmatrix}
0 & \sqrt{B_1} & 0 & \cdots & 0 \\
0 & 0 & \sqrt{B_2} & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & \sqrt{B_{n-1}}
\end{pmatrix}, \quad e^0 = \begin{pmatrix}
n - 1 & 0 & 0 & \cdots & 0 \\
0 & n - 3 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & -(n - 1)
\end{pmatrix}.
\]

Using local coordinate \((z, y)\) on \( \Sigma \times \mathbb{R}^+ \), we have \( A = A_z dz + A_\bar{z} d\bar{z}, \phi = \phi_z dz + \phi_\bar{z} d\bar{z} \), and the model tilted Nahm pole solution over \( T^2 \times \mathbb{R}^+ \) is

\[
(29) \quad A_z = y^{-1}e^+ \sin \beta, \quad A_\bar{z} = y^{-1}e^- \sin \beta, \\
\phi_z = y^{-1}e^+ \cos \beta, \quad \phi_\bar{z} = y^{-1}e^- \cos \beta, \quad \phi_1 = \frac{i}{2y} e^0 \cos \beta.
\]

**Definition 4.1 ([30]).** The fields \((A, \phi, \phi_1)\) satisfy the tilted Nahm pole boundary conditions if in some local trivialization,

\[
A_z \sim y^{-1}e^+ \sin \beta + \mathcal{O}(y^{-1+\epsilon}), \quad A_\bar{z} \sim y^{-1}e^- \sin \beta + \mathcal{O}(y^{-1+\epsilon}), \\
\phi_z \sim y^{-1}e^+ \cos \beta + \mathcal{O}(y^{-1+\epsilon}), \quad \phi_\bar{z} \sim y^{-1}e^- \cos \beta + \mathcal{O}(y^{-1+\epsilon}), \quad \phi_1 \sim \frac{i}{2y} e^0 \cos \beta + \mathcal{O}(y^{-1+\epsilon}),
\]

as \( y \to 0 \).
We now examine these boundary conditions in a parallel holomorphic gauge. In a
local holomorphic coordinate $z$, we have $\mathcal{D}_1 = \partial_z$ and $\mathcal{D}_3 = \partial_y$, and we also write
$\mathcal{D}_2 = \partial + \alpha$. Suppose now that in some local trivialization,

$$\alpha = \begin{pmatrix}
* & \sqrt{B_1} & 0 & \cdots & 0 \\
* & * & \sqrt{B_2} & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \sqrt{B_{n-1}} & * \\
* & * & \cdots & * & * \\
\end{pmatrix} \, dz,$$

(30)

where all entries two or more ‘levels’ above the main diagonal are zero. Imposing
the singular Hermitian metric $H_0 = \exp(-\log(y \cos \beta) e^0)$ leads to fields satisfying the
tilted Nahm pole boundary condition. This follows directly from the expressions (9)
in unitary gauge. If $H = H_0 e^s$ is another Hermitian metric, where $s$ is a section of
$\text{isu}(E,H_0)$ with $\sup |s| + |y| |ds| \leq C y^\epsilon$, then the corresponding fields $(A_H, \phi_H, (\phi_1)_H)$
also satisfy the tilted Nahm pole boundary conditions.

**Definition 4.2.** If in some local trivialization, $\mathcal{D}_2 = \partial + \alpha$ with $\alpha$ as in (30), and we
set $H_0 = \exp(-\log(y \sin \beta) e^0)$, then any $H = H_0 e^s$ with $|s| + |y| |ds| < C y^\epsilon$ satisfies
the tilted Nahm pole boundary condition.

### 4.2. Holomorphic data from the singular boundary condition.

By (2.1), up to complex gauge transform, a choice of holomorphic data corresponds to a flat con-
nection over $\Sigma$. We now discuss how these singular boundary conditions interact with
this holomorphic data.

As above, in some trivialization near $y = 0$, these fields satisfy

$$\mathcal{D}_1 = \partial_z + A_z = \phi_z \tan \beta + \mathcal{O}(y^{-1+\epsilon}) = \partial_z + \mathcal{O}(y^{-1+\epsilon}),$$

$$\mathcal{D}_2 = \partial_z + \frac{e^+}{y \sin \beta} + \mathcal{O}(y^{-1+\epsilon})$$

$$\mathcal{D}_3 = \partial_y + \frac{e^0}{2y} + \mathcal{O}(y^{-1+\epsilon})$$

(31)

Choose a holomorphic basis $s_1, \ldots, s_{n+1}$ so $s_i$ corresponds to $(0, \cdots, 0, 1, 0, \cdots, 0)^\dagger$. If $s(y)$ is any section of $E$, we can solve the ODE $\mathcal{D}_3 s = 0$ with initial value $s(y)|_{y=1} = \sum_{i=1}^n a_i s_i$, $a_i \in \mathbb{C}$. Then

$$s(y) = \sum_{i=1}^n a_i y^{-\frac{n-i+1}{2}} s_i + \mathcal{O}(y^{-\frac{n-i+1}{2} + \epsilon}) \text{ as } y \searrow 0,$$

for some small $\epsilon > 0$. This determines a filtration $E_\bullet : 0 = E_0 \subset E_1 \subset \cdots \subset E_n = E$
by vanishing rates, where

$$E_i := \{ s \in \Gamma(E) | \mathcal{D}_3 s = 0, \lim_{y \to 0} |sy^{-\frac{n-i+1}{2} + \epsilon}| = 0 \}$$

(32)

for any $0 < \epsilon < 1$. (Note that, by construction, rank $E_i = i$.)

To see that the boundary condition induces an oper structure, we must check the
three points in Definition 3.3. The first two are straightforward. The third follows from the

**Proposition 4.3.** For any section $s$ of $E_i$, both $\mathcal{D}_1 s$ and $\mathcal{D}_2 s$ lie in $E_{i+1} \otimes K$. 
Proof. If $s$ is a smooth section of $E_i$ with $D_3 s = 0$, then $0 = [D_j, D_3] s = D_3 D_j s$, $j = 1, 2$. But then, integrating this singular ODE in the $y$ direction shows that $\lim_{y \to 0} |(D_j s) y^{-\frac{m_j}{2} + i + i'}| = 0$, $j = 1, 2$. □

To conclude the verification that this data gives an oper, observe finally that since the entries of $D_j$, $j = 1, 2$, above the main diagonal are nonzero constants, each $D_2 : E_i / E_{i-1} \to E_{i+1} / E_i \otimes K$ is an isomorphism.

In summary, we have shown that the tilted Nahm pole boundary conditions determine an oper structure on solutions.

4.3. The moduli space and the Gaiotto-Witten map. We can now define the moduli space of solutions to the twisted Bogomolny equations with tilted Nahm pole boundary conditions

$$\mathcal{M}_{\text{TBE}}^\beta := \{ (A, \phi, \phi_1) | \text{TBE}(A, \phi, \phi_1) = 0, (A, \phi, \phi_1) \text{satisfies the tilted Nahm Pole boundary condition at } y = 0, \text{ and } (A, \phi, \phi_1) \text{ converges to a flat SL}(n, \mathbb{C}) \text{ connection as } y \to +\infty \} / G_0;$$

here $G_0$ is the (real) gauge group preserving the Nahm pole boundary condition and decay assumptions.

Proposition 4.4. [10] There is a well-defined Kobayashi-Hitchin map

$$I_{\text{Oper}}^\beta : \mathcal{M}_{\text{TBE}}^\beta \to \mathcal{M}_{\text{Oper}}^\beta.$$

5. Linear Analysis

We now study Fredholm properties of the linearized moment map.

Fix a background Hermitian metric $H_0$ on the bundle $E$ and the three operators $D_i$ satisfying $[D_i, D_j] = 0$. We seek a new Hermitian metric $H = H_0 e^s$ so that the final moment map equation

$$(33) \quad \Omega_H := \frac{i}{2} \Lambda(\cos^2 \beta [D_1, D^\dagger_1] - \sin^2 \beta [D_2, D^\dagger_2]) + [D_3, D^\dagger_3] = 0$$

holds; here $\Lambda$ is contraction by the Kähler form.

Definition 5.1. A Hermitian metric $H_0$ is called admissible if:

- the Chern connection associated to $H_0$ satisfies the Nahm pole boundary conditions;
- the Chern connection converges to an oper as $y \to \infty$, cf. Section 3.4;
- $\Omega_{H_0}$ vanishes to all orders at $y = 0$.

The following adapts a result in [14] to the present setting:

Proposition 5.2. Defining $H = H_0 e^s$, then

$$(34) \quad \Omega_H = \Omega_{H_0} + \gamma(-s) \mathcal{L}_{H_0} s + Q(s),$$

where

$$\mathcal{L}_{H_0} s := \frac{i}{2} \Lambda(\cos^2 \beta D_1 D^\dagger_{1H_0} - \sin^2 \beta D_2 D^\dagger_{2H_0}) s + D_3 D^\dagger_{3H_0} s,$$
and

\[ Q(s) := \frac{i}{2} \Lambda (\cos^2 \beta \mathcal{D}_1 (\gamma(-s)) \mathcal{D}_1^* \gamma s - \sin^2 \beta \mathcal{D}_2 (\gamma(-s)) \mathcal{D}_2^* \gamma s) + \mathcal{D}_3 \gamma(-s) \mathcal{D}_3^* \gamma s, \]

where \( \gamma(s) := \frac{e^{-\Lambda s}}{\text{ad}_s}. \) Furthermore,

\[
\langle \Omega_H - \Omega_{H_0}, s \rangle_{H_0} = \Delta |s|_{H_0}^2 + |v(s) \nabla s|^2
\]

where \( v(s) = \sqrt{\gamma(-s)} = \sqrt{\frac{1}{1 - e^{-\Lambda s}}}, \Delta = \Delta_S - \nabla^2 \text{ and } (36) \]

\[
|v(s) \nabla s|^2 := \frac{1}{4} \cos^2 \beta (|v(s) \mathcal{D}_1 s|^2 + |v(s) \mathcal{D}_1^* s|^2) + \frac{1}{4} \sin^2 \beta (|v(s) \mathcal{D}_2 s|^2 + |v(s) \mathcal{D}_2^* s|^2) \\
+ \frac{1}{2} (|v(s) \mathcal{D}_3 s|^2 + |v(s) \mathcal{D}_3^* s|^2).
\]

**Proof.** First we have \( \mathcal{D}_1^* = \mathcal{D}_1^* + e^{-s} (\mathcal{D}_1^* e^s) \). Next, if \( X(w) \) is any smooth family of Hermitian metrics, then

\[
(37) \partial_w e^X = e^X \gamma(-X) \partial_w X = \gamma(X) \partial_w X e^X,
\]

where \( \partial_w \) denotes a ‘generic’ derivative, so e.g. could be \( \mathcal{D}_1 \) or \( \mathcal{D}_1^* \). Using these, we now compute

\[
\Omega_H = \Omega_{H_0} + \frac{i}{2} \Lambda (\cos^2 \beta \mathcal{D}_1 (e^{-s} \mathcal{D}_1^* e^s) - \sin^2 \beta \mathcal{D}_2 (e^{-s} \mathcal{D}_2^* e^s)) + \mathcal{D}_3 (e^{-s} \mathcal{D}_3^* e^s)
\]

\[
= \Omega_{H_0} + \frac{i}{2} \Lambda (\cos^2 \beta \mathcal{D}_1 (\gamma(-s) \mathcal{D}_1^* \gamma s) - \sin^2 \beta \mathcal{D}_2 (\gamma(-s) \mathcal{D}_2^* \gamma s)) + \mathcal{D}_3 (\gamma(-s) \mathcal{D}_3^* \gamma s)
\]

\[
= \Omega_{H_0} + \gamma(-s) \mathcal{L}_{H_0} s + Q(s).
\]

The identity (35) requires an analysis of the expression

\[
\langle \Omega_H - \Omega_{H_0}, s \rangle_{H_0}
\]

\[
= \langle \frac{i}{2} \Lambda (\cos^2 \beta \mathcal{D}_1 (\gamma(-s) \mathcal{D}_1^* \gamma s) - \sin^2 \beta \mathcal{D}_2 (\gamma(-s) \mathcal{D}_2^* \gamma s)) + \mathcal{D}_3 (\gamma(-s) \mathcal{D}_3^* \gamma s), s \rangle_{H_0}.
\]

We consider the summands in turn.

For the first, we compute

\[
\langle \frac{i}{2} \Lambda \mathcal{D}_1 (\gamma(-s) \mathcal{D}_1^* \gamma s), s \rangle = \frac{i}{2} \Lambda \mathcal{D}_1 (\gamma(-s) \mathcal{D}_1^* \gamma s, s) + \frac{1}{2} \langle \gamma(-s) \mathcal{D}_1^* \gamma s, \mathcal{D}_1^* \gamma s \rangle
\]

\[
= \frac{i}{2} \Lambda \mathcal{D}_1 (\gamma(-s) s, \mathcal{D}_1^* \gamma s) + \frac{1}{2} |v(-s) \mathcal{D}_1^* s|^2
\]

\[
= \frac{i}{2} \Lambda \mathcal{D}_1 (\gamma(-s) s, \mathcal{D}_1^* \gamma s) + \frac{1}{2} |v(-s) \mathcal{D}_1^* s|^2 + \frac{1}{4} |v(-s) \mathcal{D}_1 s|^2.
\]
The second term follows from

\[
\langle \frac{i}{2} \Lambda D_2 (\gamma(-s) D_2^{\dagger H_0}s) , s \rangle = \frac{i}{2} \Lambda \partial (\gamma(-s) D_2^{\dagger H_0}s, s) - \frac{1}{2} \langle \gamma(-s) D_2^{\dagger H_0}s, D_2^{\dagger H_0}s \rangle \\
= \frac{i}{2} \Lambda \partial (D_2^{\dagger H_0}s, \gamma(-s)s) - \frac{1}{2} |v(-s)D_2^{\dagger H_0}s|^2 \\
= i \Lambda \partial \partial |s|^2 - \frac{1}{4} |v(-s)D_2^{\dagger H_0}s|^2 \\
= - \frac{1}{2} \Delta \Sigma |s|^2 - \frac{1}{4} |v(-s)D_2^{\dagger H_0}s|^2 - \frac{1}{4} |v(-s)D_2s|^2.
\]

Finally, calculating as previously, the third term equals

\[
\langle D_3 (\gamma(-s) D_3^{\dagger H_0}s) , s \rangle = \partial_y \langle \gamma(-s) D_3^{\dagger H_0}s, s \rangle + \langle \gamma(-s) D_3^{\dagger H_0}s, D_3^{\dagger H_0}s \rangle \\
= - \partial_y^2 |s|^2 + \frac{1}{2} (|v(s)D_3s|^2 + |v(s)D_3^{\dagger H_0}|^2).
\]

Combining these yields the desired identity.

Next recall the Kähler identities [26, Lemma 3.1]:

\[
i[\Lambda, D_1] = (D_1^{\dagger H})^*, \quad i[\Lambda, D_1^{\dagger H}] = -(D_1)^*
\]

\[
i[\Lambda, D_2] = -(D_2^{\dagger H})^*, \quad i[\Lambda, D_2^{\dagger H}] = (D_2)^*
\]

Noting also that \((D_3^{\dagger H})^* = D_3\), we conclude

**Corollary 5.3.** \(L_{H_0} = \frac{1}{2}(\cos^2 \beta (D_1^{\dagger H})^* D_1^{\dagger H} + \sin^2 \beta (D_2^{\dagger H})^* D_2^{\dagger H}) + D_3 D_3^{\dagger H_0}\).

**Proposition 5.4.** Setting \(\nabla_1 = D_1 + D_1^{\dagger}, \nabla_2 = D_2 + D_2^{\dagger}\), then there are Weitzenböck formulae

\[
(D_1^*)^* D_1 = \frac{1}{2} \nabla_1^* \nabla_1 + \frac{i}{2} \Lambda [D_1, D_1^*], \quad (D_1)^* D_1 = \frac{1}{2} \nabla_1^* \nabla_1 - \frac{1}{2} \Lambda [D_1, D_1^*],
\]

\[
(D_2^*)^* D_2 = \frac{1}{2} \nabla_2^* \nabla_2 + \frac{i}{2} \Lambda [D_2, D_2^*], \quad (D_2)^* D_2 = \frac{1}{2} \nabla_2^* \nabla_2 - \frac{1}{2} \Lambda [D_2, D_2^*],
\]

and also

\[
D_3 D_3^* = -(D_3^2 + \frac{\phi_1^2}{\cos^2 \beta}) + \frac{1}{2} [D_3, D_3^*], \quad D_3 D_3^* = -(D_3^2 + \frac{\phi_1^2}{\cos^2 \beta}) - \frac{1}{2} [D_3, D_3^*].
\]

**Proof.** We compute

\[
\nabla_1^* \nabla_1 = D_1^* D_1 + (D_1^*)^* D_1^* = -i \Lambda D_1^* D_1 + i \Lambda D_1 D_1^*
\]

\[
= 2i \Lambda D_1 D_1^* - i \Lambda [D_1, D_1^*] = -2i \Lambda D_1 D_1^* + i \Lambda [D_1, D_1^*].
\]

For \(D_2\), we have

\[
\nabla_2^* \nabla_2 = D_2^* D_2 + (D_2^*)^* D_2^* = -i \Lambda D_2^* D_2 + i \Lambda D_2 D_2^*
\]

\[
= -2i \Lambda D_2 D_2^* + i \Lambda [D_2, D_2^*] = 2i \Lambda D_2 D_2^* - i \Lambda [D_2, D_2^*].
\]

Finally, to obtain (41), the formulas \(D_3 = D_y - i \frac{\phi_1}{\cos \beta}\) and \(D_3^* = -D_y - i \frac{\phi_1}{\cos \beta}\) lead to \((D_3 - D_3^*)^2 = 4D_y^2, (D_3)^2 + (D_3^*)^2 = 2D_y^2 - \frac{2 \phi_1^2}{\cos^2 \beta}\). In addition,

\[
(D_3 - D_3^*)^2 = D_3^2 + (D_3^*)^2 - D_3 D_3^* - D_3^* D_3.
\]
so altogether

$$2(D_y^2 + \frac{\phi_1^2}{\cos^2 \beta}) = -D_3 D_3^\dagger - D_3^\dagger D_3 = -[D_3, D_3^\dagger] - 2D_3^\dagger D_3 = [D_3, D_3^\dagger] - 2D_3 D_3^\dagger.$$ 

This leads to a simpler expression for $\mathcal{L}_H$:

**Corollary 5.5.**

$$\mathcal{L}_H = \frac{1}{4} (\cos^2 \beta \nabla_1^\dagger \nabla_1 + \sin^2 \beta \nabla_2^\dagger \nabla_2) - (D_y^2 + \frac{\phi_1^2}{\cos^2 \beta}) + \frac{1}{2} \Omega_H,$$

where $\phi_1^2 = [\phi_1, [\phi_1, \cdot]].$

### 5.1. Indicial Roots

The mapping properties and regularity of solutions for the operator $\mathcal{L}_H$ in (44) rely on the determination of the indicial roots of this operator. Note that the final term in $\mathcal{L}_H$ which involves $\Omega_H$ is absent since we work at an exact solution.

First some notation. Recall the component operators

$$D_1 = (D_z - \phi_1 \tan \beta) dz, \quad D_2 = (D_z + \phi_1 \cot \beta) dz, \quad D_3 = D_y - i \frac{\phi_1}{\cos \beta},$$

and the model tilted Nahm singularities near $y = 0$

$$A_z \sim y^{-1} e^+ \sin \beta + \mathcal{O}(y^{-1+\epsilon}), \quad A_z \sim y^{-1} e^- \sin \beta + \mathcal{O}(y^{-1+\epsilon}),$$

$$\phi_z \sim y^{-1} e^+ \cos \beta + \mathcal{O}(y^{-1+\epsilon}), \quad \phi_z \sim y^{-1} e^- \cos \beta + \mathcal{O}(y^{-1+\epsilon}), \quad \phi_1 \sim \frac{i}{2y} e^0 \cos \beta + \mathcal{O}(y^{-1+\epsilon}).$$

Putting these together, we have

$$\cos \beta D_1 = \cos \beta D_{A_z} - \sin \beta \phi_z \sim \mathcal{O}(1), \quad \sin \beta D_2 = \sin \beta D_{A_z} - \cos \beta \phi_z \sim \frac{e^+}{y},$$

$$D_3 = D_y - \frac{i \phi_1}{\cos \beta} \sim \partial_y + \frac{1}{2y} e^0.$$

The operator $\mathcal{L}_H$ is singular at $y = 0$, and its behavior there is well-approximated by the so-called normal operator, cf. [22]:

$$\mathcal{N}(\mathcal{L}_{H_0}) := \Delta_{\mathbb{R}^2} - \frac{1}{2} ([e^+, [e^-, s]] + [e^-, [e^+, s]]) - \frac{1}{4} [e^0, [e^0, s]].$$

This is both translation-invariant in $z$ and dilation-invariant in $(z, y)$ jointly, and is identified with the linearization of the nonlinear equations at the global model solution on $\mathbb{R}^3_+$. A key invariant of $\mathcal{L}_H$, or equivalently of its normal operator $\mathcal{L}_{H_0}$, is its associated set of indicial roots. These are the formal rates of growth or decay of solutions at $y = 0$. By definition, $\lambda$ is an indicial root of $\mathcal{L}_H$ at $(z_0, 0)$ if there exists a section $s$ defined in a neighborhood of this point such that

$$\mathcal{L}_{H_0}(y^\lambda s_0) = 0, \quad \text{where } s_0 = s|_{z = z_0, y = 0}.$$

Using that $\mathcal{L}_H(y^\lambda s) = \mathcal{N}(\mathcal{L}_{H_0})(y^\lambda s_0) + \mathcal{O}(y^{\lambda-1})$, we see that $\lambda$ is an indicial root if there exists some $s$ such that $\mathcal{L}_H(y^\lambda s) = \mathcal{O}(y^{\lambda-1})$, in contrast to the expected rate $\mathcal{O}(y^{\lambda-2}).$
From the explicit form of this operator, we see that $\lambda$ is an indicial root if

$$\lambda(\lambda - 1)s = \frac{1}{2}([e^+, [e^-, s]] + [e^-, [e^+, s]]) + \frac{1}{4}[e^0, [e^0, s]].$$

The operator on the right is the Casimir operator for $\mathfrak{sl}_2$,

$$\Delta_{\text{Cas}}s := \frac{1}{2}([e^+, [e^-, s]] + [e^-, [e^+, s]]) + \frac{1}{4}[e^0, [e^0, s]],$$

so $\lambda$ is an indicial root for $\mathcal{L}$ if and only if $\lambda(\lambda - 1)$ is an eigenvalue for $\Delta_{\text{Cas}}$.

Using the known values of this Casimir spectrum for $\mathfrak{sl}_n$, we arrive at a result mirroring the result and calculations in [23]:

**Proposition 5.6.** The set of indicial roots of the twisted extended Bogomolny equations with tilted Nahm pole boundary conditions is $\{-(n - 1), \ldots, -1, 2, \ldots, n\}$.

### 5.2. Function spaces and Mapping Properties.

Using the results of the last subsection and invoking the theory in [22], we now state the Fredholm theory for the operator $\mathcal{L}_H$ acting on a family of weighted Hölder spaces adapted to the degeneracy of the operator.

**Definition 5.7.** Define $\mathcal{C}^{k,\alpha}_{\text{ie}}(\Sigma \times \mathbb{R}^+)$ to be the space of all functions $u$ on $\Sigma_x \times \mathbb{R}^+$ such that

i) In the region $\{y \leq 1\}$,

$$||u||_{L^\infty} + \sup_{i + |\beta| \leq k} [(y\partial_y)^i (y\partial_x)^\beta u]|_{\text{ie};0,\alpha} < \infty,$$

where

$$[v]|_{\text{ie};0,\alpha} := \sup_{(y,x)\neq(y',x')} \frac{|u(y,x) - u(y',x')|(y + y')^\alpha}{|y - y'|^\alpha + |x - x'|^\alpha}.$$

ii) Away from all boundaries we require that $u$ lies in the ordinary Hölder space $\mathcal{C}^{k,\alpha}$ on each slab $\Sigma \times [L, L + 1]$, uniformly for $L \geq 1$.

Fixing $0 < \alpha < 1$, for any $\mu, \delta \in \mathbb{R}$, define

$$\mathcal{X}^k_{\mu,\delta}(\Sigma \times \mathbb{R}^+; i\mathfrak{su}(E, H_0)) = y^\mu e^{y\delta} \mathcal{C}^{k,\alpha}_{\text{ie}}(\Sigma \times \mathbb{R}^+) = \{u = y^\mu e^{y\delta} v : v \in \mathcal{C}^{k,\alpha}_{\text{ie}}\}.

**Theorem 5.8.** [23, 14] Suppose $\mu \in (-1, 2)$ and $\delta > 0$; then for any $k \geq 0$ and $0 < \alpha < 1$,

$$\mathcal{L}_H : \mathcal{X}^{k+2}_{\mu,-\delta}(\Sigma \times \mathbb{R}^+; i\mathfrak{su}(E, H_0)) \rightarrow \mathcal{X}^k_{\mu-2,-\delta}(\Sigma \times \mathbb{R}^+; i\mathfrak{su}(E, H_0)).$$

is a Fredholm operator.

**Proof.** The operator $\mathcal{L}_H$ is the same as the operator considered in [14, Theorem 5.8] up to a compact operator, so this result follows directly from that one. □

### 6. Construction of approximate solutions

Given an oper $(E, F_\bullet, \nabla)$, we now construct an admissible Hermitian metric satisfying the tilted Nahm pole boundary conditions.

**Proposition 6.1.** For every element $(E, F_\bullet, \nabla) \in \mathcal{M}_{\text{oper}}$, there exists an Hermitian metric $H_0$ satisfying Nahm pole boundary conditions such that in unitary gauge relative to $H_0$, $\Omega_{H_0} = O(y^\infty)$. 
Proof. By Theorem 3.5, given an oper $\nabla^\beta_q \in \mathcal{M}_{\text{oper}}$, we can write

$$\nabla^\beta_q := \bar{\partial} + \bar{\partial}^h_0 + \varphi_q + \varphi_0^t,$$

with respect to the trivialization $\mathcal{E} = K^{-\frac{n-1}{2}} \oplus K^{-\frac{n}{2}+1} \oplus \cdots \oplus K^{-\frac{n-1}{2}}$.

Now define $\mathcal{D}_{1,cpx} := (\nabla^\beta_q)^{0,1} = \bar{\partial} - \varphi_0^t$, $\mathcal{D}_{2,cpx} := (\nabla^\beta_q)^{1,0} = \partial^h_0 + \varphi_q$, where the label "cpx" means we are working in a complex gauge.

Indeed, suppose that we have found a Hermitian metric $\gamma$ such that $\Omega_{H_0^{(0)}} = \exp(-\log(y \sin \beta) e_0)$ and $g = \sqrt{H_0^{(0)}} = \text{diag}(\lambda_1, \lambda_2, \cdots, \lambda_n)$. By the definition of $e_0$, $\lambda_i = (y \sin \beta)^{-\frac{n}{2}+i-1}$ as an element of $\text{End}(K^{-\frac{n}{2}+i-1}, K^{-\frac{n}{2}+i-1})$.

Now set $\mathcal{D}_{i,\text{app}} := g \mathcal{D}_{i,cpx} g^{-1}$ (where "app" indicates that this is an approximate solution). We compute these operator explicitly.

We have $\mathcal{D}_{1,\text{app}} := g \mathcal{D}_{1,cpx} g^{-1} = g^{-1} \bar{\partial} g + g^{-1} \varphi_0^t g$. Since $g$ is a constant section, $g^{-1} \circ \bar{\partial} \circ g = \bar{\partial}$. By [4], $h_0$ is diagonal, so if $(\varphi_0^t)_{ij}$ denotes the $(i, j)$-component of $\varphi_0^t$, then $(\varphi_0^t)_{ij} = 0$ for $j \neq i - 1$. As $(g \varphi_0^t g^{-1})_{ij} = \lambda_i (\varphi_0^t)_{ij} \lambda_j^{-1}$, we have $g \varphi_0^t g^{-1} \sim \mathcal{O}(y)$.

Next, we compute $\mathcal{D}_{2,\text{app}} = g \mathcal{D}_{2,cpx} g^{-1} = \partial^h_0 + g \varphi_q g^{-1}$. As before, $g \varphi_q g^{-1} = (\lambda_i \lambda_j^{-1} \varphi_{ij})$, where $\varphi_q = (\varphi_{ij})$. We can decompose this as $\mathcal{D}_2 = \partial_z + \phi_z^{\text{mod}} + b$, where

$$\phi_z^{\text{mod}} = (y \sin \beta)^{-1} \begin{pmatrix} 0 & \sqrt{B_1} & 0 & \cdots & 0 \\ 0 & 0 & \sqrt{B_2} & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \sqrt{B_{n-1}} & 0 \end{pmatrix}$$

and

$$b = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & y^n q_n & y^{n-2} q_{n-1} & \cdots & y q_2 \end{pmatrix}.$$

Since $b = \mathcal{O}(y)$, then in the gauge defined by $g$,

$$\Omega_{H_0^{(0)}} = [\phi_z^{\text{mod}}, b^t] + [b, (\phi_z^{\text{mod}})^t] = \mathcal{O}(1);$$

even more specifically, the right hand side can be written $F_{H_0^{(0)}} + \mathcal{O}(y)$. Clearly $\Omega_{H_0^{(0)}}$ depends continuously on $q$.

We now add correction terms to make this error vanish to higher and higher order. Indeed, suppose that we have found a Hermitian metric $H_0^{(j)}$ such that $\Omega_{H_0^{(j)}} = F_j y^j + \mathcal{O}(y^{j+1})$ for some $j \geq 0$ (so $F_0 = F_{H_0^{(0)}}$ above), and define $H_0^{(j+1)} = H_0^{(j)} e^s$. Using (34), we see that in order to show that $\Omega_{H_0^{(j+1)}} = F_{j+1} y^{j+1} + \mathcal{O}(y^{j+2})$, it suffices to solve the equation

$$\gamma(-s) \mathcal{L}_{H_0^{(j)}} s = -F_j y^j \mod \text{order } y^{j+1-\epsilon},$$

But $\gamma(-s_j y^j) = \text{Id} + \mathcal{O}(y)$, and $\mathcal{L}_{H_0^{(j)}}$ equals the normal operator $N(\mathcal{L}_{H_0})$ to leading order, so we must solve $N(\mathcal{L}_{H_0}) s_j y^j = -F_j y^j$, where $s_j$ is an element of $\mathfrak{isu}(E, H_0)$. 


This linear algebraic equation is solvable at least when $j$ is not an indicial root, and the solution depends continuously on $q$; in the exceptional cases where $j$ is an indicial root, one must replace $s_j y^j$ by $\tilde{s}_j y^j \log y$ to obtain a solution. (The possibility of these extra log factors is why the error has been written as $O(y^{j+1-\epsilon})$.) In any case, we can carry out this inductive procedure and then take a Borel sum to obtain a Hermitian endomorphism

$$s \sim \sum_{j=0}^{\infty} s_{j\ell} y^j (\log y)^\ell$$

(with $s_{0\ell} = 0$ for $\ell > 0$, and with only finitely many $s_{j\ell}$ nonzero for each $j$), such that if we set $H_0 = H_0^{(0)} e^s$, then $\Omega_{H_0} = O(y^N)$ for every $N \geq 0$.

In summary, we obtain the

Theorem 6.2. For any $(E, F_{\bullet}, \nabla_q) \in M_{\text{Oper}}^\beta$, there exists an admissible Hermitian metric $H_0$; the approximate solution $\Omega_{H_0}(q)$ depends continuously on $q$.

7. Continuity Method

We now solve the extended Bogomolny equations with these boundary conditions using the standard method of continuity; this argument is close to the one in [14], so our treatment is brief.

7.1. Method of continuity. Given an oper $(E, F_{\bullet}, \nabla) \in M_{\text{Oper}}^\beta$, fix an admissible approximate solution $H_0$ to the equation $\Omega_H = 0$.

Let $\text{isu}(E, H_0)$ be the subspace of Hermitian endormorphisms in $\text{End}(E)$ preserving $H_0$. For any $s \in \text{isu}(E, H_0)$, define the new Hermitian metric $H = H_0 e^s$ and the family of maps

$$N_t(s) := \text{Ad}(e^{\frac{t}{2}^s})\Omega_H + ts = 0.$$  

Note that $\Omega_H \in \text{isu}(E, H)$ and $\text{Ad}(e^{\frac{t}{2}^s}) : \text{isu}(E, H) \to \text{isu}(E, H_0)$ is a bundle isomorphism satisfying

$$\langle \text{Ad}(e^{\frac{t}{2}^s})f, \text{Ad}(e^{\frac{t}{2}^s})g \rangle_H = \langle f, g \rangle_{H_0} \text{ where } f, g \text{ are sections of } \text{isu}(E, H).$$

Define

$$I := \{ t \in [0, 1] : N_t(s) = 0 \text{ has a solution } s \in \mathcal{X}_{2-\epsilon, -\delta} \};$$

we show that $I$ is nonempty, open and closed, so that $I = [0, 1]$, and the problem is solved.

7.2. $I$ is nonempty.

Proposition 7.1. There exists an admissible Hermitian metric $H_0$ and section $s$ such that $N_1(s) = 0$.

Proof. Following [21], pick up any admissible Hermitian metric $H_{-1}$, write $\kappa = \Omega_{H_{-1}}$ and define $H_0 = H_{-1} e^s$. Then if we set $s = -\kappa$, we have that $N_1(-\kappa) = \text{Ad}(e^{-\frac{t}{2}^s})\Omega_{H_0 e^{-\kappa}} - \kappa = \Omega_{H_{-1}} - \kappa = 0$.  \qed
7.3. Openness. We next study the linearization more closely. Assume $s$ satisfies $N_t(s) = 0$ for some $t$ and define

$$L_{t,s}(s') := \frac{d}{du}|_{u=0}N_t(s + us').$$

Using the computations in [14], we have the

**Proposition 7.2.** [14, Proposition 6.2, 6.4] Suppose that $N_t(s) = 0$. Then for any sections $s_1, s_2$ of $\mathcal{X}^{k+2}_{\mu, -\delta}(\Sigma \times \mathbb{R}^+; i\mathfrak{su}(E, H_0))$, we have

i) $L_{t,s}(s_1) = \text{Ad}(e^{\hat{\tau}})L_H s_1 + ts_1$;

ii) $\langle L_{t,s} s_1, \text{Ad}(e^{\hat{\tau}})s_1 \rangle_{H_0} = \int \sum_{i=1}^3 |D^i_1 u|_H^2 + t \left| \text{Ad}(e^{\hat{\tau}})s_1 \right|_{H_0}^2$;

iii) $\langle L_{t,s} s_1, s_2 \rangle_{H_0} = \langle s_1, \text{Ad}(e^{\hat{\tau}})L_{t,s} (\text{Ad}(e^{-\hat{\tau}})s_2) \rangle$.

(Note that part i) is purely algebraic and does not require any assumptions on the decay of $s_1$.)

**Proposition 7.3.** $L_{t,s} : \mathcal{X}^{k+2}_{\mu, -\delta}(\Sigma \times \mathbb{R}^+; i\mathfrak{su}(E, H_0)) \longrightarrow \mathcal{X}^k_{\mu, -\delta}(\Sigma \times \mathbb{R}^+; i\mathfrak{su}(E, H_0))$ is an isomorphism.

**Proof.** If $s \in \text{Ker} L_{t,s}$, then by ii) of this last Proposition, $\text{Ad}(e^{\hat{\tau}})s = 0$ and hence $s = 0$. Part iii) shows that the range of $L_{t,s}$ is dense. Since this operator is Fredholm, its range is closed, and hence it is an isomorphism. □

**Proposition 7.4.** $I$ is open.

**Proof.** The nonlinear map

$$N_t : \mathcal{X}^{k+2}_{\mu, -\delta}(\Sigma \times \mathbb{R}^+; i\mathfrak{su}(E, H_0)) \longrightarrow \mathcal{X}^k_{\mu, -\delta}(\Sigma \times \mathbb{R}^+; i\mathfrak{su}(E, H_0)),$$

is well-defined and smooth, and it linearization $L_{t,s}$ at $s$ is an isomorphism. The statement now follows from the implicit function theorem. □

7.4. A priori estimates and closeness. To prove that $I$ is closed, we must show that if $H_j$ is a sequence of solutions corresponding to $t_j \in I$, and if $t_j \to \tilde{t}$, then $H_j$ also has a limit. The analytic steps are essentially the same as [14] except for the initial step, which is the $C^0$ estimate, so we concentrate on this.

Denote by $\mathcal{C}^{k,\alpha}_D(\Sigma \times \mathbb{R}^+)$ the space of sections which are uniformly in $\mathcal{C}^{k,\alpha}$ on every strip $\sigma \times [t, t + 1]$, and which also vanish at $y = 0$ (the subscript ‘$D$’ stands for Dirichlet), and also set $\mathcal{C}^{k,\alpha}_{D, -\delta} = e^{-y\delta}\mathcal{C}^{k,\alpha}_D$. Fix $\chi \in \mathcal{C}^\infty(\Sigma \times \mathbb{R}^+)$ with $\chi \geq 0$, $\chi(y) = 1$ for $y \geq 2$ and $\chi(y) = 0$ for $y \leq 1$.

**Proposition 7.5.** [14, Proposition 8.1] If $\Delta$ is the scalar Laplacian, then

$$\Delta : \mathcal{C}^{k+2,\alpha}_{D, -\delta}(\Sigma \times \mathbb{R}^+) \oplus \mathbb{R} \longrightarrow \mathcal{C}^{k,\alpha}_{-\delta}(\Sigma \times \mathbb{R}^+)$$

$$(u, A) \longmapsto \Delta u + A \Delta(\chi)$$

is an isomorphism.

We now obtain a $C^0$ estimate, cf. [14]:
Proposition 7.6. If \( s \) is a Hermitian endomorphism satisfying \( N_1(s) = 0 \), then there exist a constant \( C \) depending only on \( H_0 \) such that
\[
|s|_{C^0(\Sigma \times \mathbb{R}^+)} \leq C.
\]

Proof. Taking the inner product of (34) with \( s \), where \( H = H_0 e^s \), gives
\[
\Delta |s|^2 + |v(s) \nabla s|^2 + t|s|^2 + \langle \Omega_{H_0}, s \rangle = 0;
\]
here \( \Delta = -\partial_y^2 + \Delta_\Sigma \) and \( |v(s) \nabla s| \) is as in (36). Let \( M := \sup |s| \), then
\[
\Delta |s|^2 + t|s|^2 \leq -\langle \Omega_{H_0}, s \rangle \implies \Delta |s|^2 \leq M |\Omega_{H_0}|.
\]
By Proposition 7.5, there exists \( u \in C_{D,-\delta}^{2,\alpha} \) and \( A \in \mathbb{R} \) such that \( \Delta(u - A \chi) = |\Omega_{H_0}| \), hence \( \Delta(|s|^2 - Mu + AM \chi) \leq 0 \), i.e., \( |s|^2 - Mu + AM \) is a subsolution. Since both \( s \) and \( u \) decay as \( y \to \infty \) and vanish at \( y = 0 \), and \( \chi \) is bounded, we conclude that \( |s|^2 - Mu + AM \chi \leq 0 \). This gives that \( |s|^2 \leq M^2 \leq M \sup(|u| + |A|) \), which gives the desired bound since \( u \) and \( A \) depend only on \( \Omega_{H_0} \). \( \square \)

Theorem 7.7. Let \( N_1(s) = 0 \), and suppose that \( H_0 \) satisfies the tilted Nahm pole boundary condition. Then for any \( k \in \mathbb{N} \) and \( \alpha \in (0,1) \),
\[
[s]_{g^k_{C^0_0}} \leq C,
\]
where \( C \) depends only on \( k, \alpha \) and \( \Omega_{H_0} \), but not on \( t \).

The decay estimate is exactly the same as in [14], and leads to

Proposition 7.8. Assuming that \( ||s||_{L^\infty} + ||e^{-\delta y} \Omega_{H_0}||_{C^k} \leq C_k \) for any \( k \geq 0 \), then for all \( k, ||e^{-\delta y} s||_{C^k} \leq C'_k \).

Theorem 7.9. Suppose that \( N_1(s) = 0 \) and \( H_0 \) has a Nahm pole but no knot singularities. Let \( \kappa \) be the first positive indicial root of \( \mathcal{L}_{H_0} \). Then for all \( k \in \mathbb{N} \), there is an a priori estimate
\[
[s]_{\lambda^k_{\kappa'} \delta} \leq C
\]
for any \( 0 < \kappa' < \kappa \), where \( C \) depends on \( k, l, \alpha \) and \( \Omega_{H_0} \), but not on \( t \).

An immediate corollary is

Corollary 7.10. \( I \) is closed in \([0,1]\).

Theorem 7.11. The maps \( I^\beta_{\text{Oper}} : \mathcal{M}^\beta_{\text{TBE}} \to \mathcal{M}^\beta_{\text{Oper}} \) is surjective.

8. Uniqueness and Properness

8.1. Uniqueness. Uniqueness of the solution is proved using convexity of the Donaldson functional. For any two Hermitian metrics \( K \) and \( H = Ke^s \), with \( \text{Tr}(s) = 0 \), write
\[
\Omega_{H,K} := \frac{i}{2} \Lambda(\cos^2 \beta |\mathcal{D}_1, \mathcal{D}_1^\dagger| - \sin^2 \beta |\mathcal{D}_2, \mathcal{D}_2^\dagger|) + |\mathcal{D}_3, \mathcal{D}_3^\dagger|,
\]
where \( \mathcal{D}_i^\dagger \) is the conjugate with respect to \( H \) defined in Section 2; the subscript \( K \) emphasizes that when \( K \) is fixed, \( \Omega_{H,K} = 0 \) is an equation for \( s \).
Define a Donaldson functional for the twisted Bogomolny equations in analogy with the well-known Donaldson functional for the Hermitian-Yang-Mills equations \[8, 9, 26\]:

\[
\mathcal{M}(H, K) = \int_0^1 \int_{\Sigma \times \mathbb{R}^+} \langle s, \Omega(K e^{i s}, K) \rangle_K \omega \wedge dy \wedge du,
\]

where \(\omega\) is the volume form of \(\Sigma\). This functional reveals the variational structure for the extended Bogomolny equations. Indeed, setting \(H_t = Ke^{ts}\),

\[
\frac{d}{dt} \mathcal{M}(H_t, K) = \int_{\Sigma \times \mathbb{R}^+} \text{Tr}(\Omega_{H_t, K}s)\omega \wedge dy,
\]

\[
\frac{d^2}{dt^2} \mathcal{M}(H_t, K) = \int_{\Sigma \times \mathbb{R}^+} \cos^2 \beta |\mathcal{D}_1 s|^2 + \sin^2 \beta |\mathcal{D}_2 s|^2 + |\mathcal{D}_3 s|^2
\]

\[
+ \int_{\Sigma \times \mathbb{R}^+} \cos^2 \beta \bar{\partial} \text{Tr}(\mathcal{D}_1^\dagger s \wedge s) + \sin^2 \beta \partial \text{Tr}(\mathcal{D}_2^\dagger s \wedge s) + \partial_y \text{Tr}(\mathcal{D}_3^\dagger s \wedge s).
\]

We now use this to prove injectivity of the maps \(I^\beta_{\text{TBE}}\).

**Proposition 8.1.** Given any element in \(\mathcal{M}^\beta_{\text{TBE}}\), suppose \(H, K\) are two solutions to the twisted Bogomolny equations with the same singularity type and corresponding to this same set of holomorphic data. Then \(H = K\).

**Proof.** Write \(H = Ke^s\) and \(H_t = Ke^{ts}\). By the indicial root computations for \(\mathcal{L}\), the order of vanishing of \(s\) in \(y\) is greater than 1, hence the boundary terms in (56) vanish. Furthermore, the Higgs pair associated to \((\mathcal{D}_1, \mathcal{D}_2)\) is stable, so \(\text{Ker} \mathcal{D}_1 \cap \text{Ker} \mathcal{D}_2 = \emptyset\). Hence if we set \(m(t) := \mathcal{M}(H_t, K)\), then \(m'(0) = 0\) and \(m'' > 0\) if \(s \not\equiv 0\). However, since \(m(0) = m(1) = 0\), we see that \(m \equiv 0\), so \(H \equiv K\) after all. \(\square\)

**Corollary 8.2.** The maps \(I^\beta_{\text{TBE}} : \mathcal{M}^\beta_{\text{TBE}} \to \mathcal{M}^\beta_{\text{Oper}}\) is injective.

We have now established the main result, that the maps \(I^\beta_{\text{Oper}}\) is a bijection.

8.2. **Properness.** We finally consider the properness of the maps \(I^\beta_{\text{Oper}}\). We first define the topologies on \(\mathcal{M}^\beta_{\text{Oper}}\) and \(\mathcal{M}^\beta_{\text{TBE}}\).

By Theorem 3.5, we can write \(\mathcal{M}^\beta_{\text{Oper}} = \{\nabla q\},\) where \(q = (q_1, \cdots, q_n) \in \bigoplus_{i=2}^n H^0(K^i)\). Fix a metric on \(K\) and use the \(C^0\) norm on \(\bigoplus_{i=2}^n H^0(K^i)\) to define the topology of \(\mathcal{M}^\beta_{\text{Oper}}\). The moduli space \(\mathcal{M}^\beta_{\text{TBE}}\) consists of pairs \(\mathcal{A} := (A, \phi, \phi_1)\) satisfying the tilted Nahm pole boundary conditions. If \(\mathcal{A}_1\) and \(\mathcal{A}_2\) are two solutions, define

\[
\|\mathcal{A}_1 - \mathcal{A}_2\|_{\mathcal{V}} := \{\sup_k \|\mathcal{A}_1 - \mathcal{A}_2\|_{X^k_{\mu, \delta}} < \infty \text{ for all } k\},
\]

for any fixed \(\mu \in (-1, 1)\) and \(\delta \in (0, 1)\).

**Proposition 8.3.** If \(X \subset \mathcal{M}^\beta_{\text{Oper}}\) is a compact subset, set \(Y := (I^\beta_{\text{Oper}})^{-1}X\). Then any sequence \(\{y_n\}\) in \(Y\) has a convergent subsequence \(\{y_{n_k}\}\).

**Proof.** Set \(x_n = I^\beta_{\text{Oper}}(y_n) \in X\). It is possible to construct model approximate solutions \(\Omega_n\) uniformly over compact subsets of \(X\). Let us also fix a solution \(\mathcal{A}_0\) to the twisted Bogomolny equations. Since \(X\) is compact, then by Theorem 6.2, we
have \( |\Omega_n|_{C^0} \leq C \) where \( C \) is independent of \( n \). By Theorem 7.9, we obtain that 
\( \|y_n - \mathcal{A}_0\|_{\mathcal{X}^k} \leq C_k \) for any \( k \in \mathbb{N} \). Hence there exists a subsequence \( y_{n_i} - \mathcal{A}_0 \) such that \( y_{n_i} - \mathcal{A}_0 \) converges in the norm \( \| \cdot \|_{\mathcal{X}} \), so \( y_{n_i} \) also convergent. \( \square \)

**Theorem 8.4.** The map \( I_{\text{Oper}}^\beta : \mathcal{M}_{\text{TBE}}^\beta \to \mathcal{M}_{\text{Oper}}^\beta \) is a diffeomorphism.

**Proof.** Implicit in the discussion above is the fact that the spaces \( \mathcal{M}_{\text{TBE}}^\beta \) and \( \mathcal{M}_{\text{Oper}}^\beta \) are both smooth manifolds. The properness of \( I_{\text{Oper}}^\beta \) is Proposition 8.3, and it is bijective by Theorem 7.11 and Proposition 8.1. Following through the construction in Section 4, we see that \( I_{\text{Oper}}^\beta \) is not only continuous, but actually a diffeomorphism. \( \square \)

**Appendix:** The twisted Bogomolny equations

In this Appendix, we discuss the tilted Nahm pole boundary condition for the twisted Kapustin-Witten equations, as well as the Hermitian-Yang-Mills structure for its reduction, the twisted Bogomolny equations. We refer to [10, 25] for more detailed explanations.

**Tilted Nahm pole boundary condition.** Let \( P \) denote a \( G \)-bundle over \( M^4 \), and \( A \) a connection and \( P \) a 1-form over \( P \). Then the twisted Kapustin-Witten equations [19] are

\[
F_A - \Phi \wedge \Phi + \frac{t - t^{-1}}{2} d_A \Phi + \frac{t + t^{-1}}{2} \ast d_A \Phi = 0, \quad d_A \ast \Phi = 0.
\]

We focus on the setting where \( M = X \times \mathbb{R}_y^+ \) where \( X \) is a 3-manifold and \( \mathbb{R}_y^+ = (0, \infty) \) with coordinate \( y \). Let \( \mathfrak{g} \) be the Lie algebra of \( G \). We first consider the boundary condition on \( X \times \{0\} \subset M \). Given a principle embedding \( \rho : \mathfrak{su}(2) \to \mathfrak{g} \otimes \mathbb{C} \), given \( x \in X \), let \( \{e_a\}_{a=1,2,3} \) be an orthonormal basis of \( T^*X \) and \( \{t_i\} \) sections of the adjoint bundle \( \mathfrak{g}_P \) lying in the image of \( \rho \) such that \( [t_a, t_b] = \epsilon_{abc} t_c \). We write the dreibein form \( e := \sum_{i=1}^3 t_i e_i \), so \( e \) gives an endomorphism \( TX \to \mathfrak{g}_P \). The definition of \( e \) depends on the choice of \( \rho \).

**Definition 8.5.** For each \( t = \tan(\frac{\pi}{4} - \frac{3\beta}{2}) \), a solution \( (A, \Phi) \) to (1) over \( M \) is a tilted Nahm pole solution if for any point \( x \in X \), there exist \( \{e_a\}, \{t_i\} \) as above such that

\[
A = \frac{\epsilon}{y} \sin \beta + \mathcal{O}(y^{-1+\epsilon}), \quad \Phi = \frac{\epsilon}{y} \cos \beta + \mathcal{O}(y^{-1+\epsilon}), \quad \text{as } y \to 0,
\]

for some constant \( \epsilon > 0 \).

**Remark.** For each \( t \) there exist three possible corresponding values of \( \beta \), and hence three different boundary conditions. For example, when \( t = 1 \), we have \( A \sim \mathcal{O}(1), \Phi \sim \frac{\epsilon}{y} \), but the other two possibilities are \( A \sim \pm \frac{\sqrt{3}}{2} \frac{\epsilon}{y}, \Phi \sim \frac{1}{2} \frac{\epsilon}{y} \).

**The Dimensional Reduction.** Now consider the 4-manifold \( \mathbb{R}_{x_1} \times \Sigma_2 \times \mathbb{R}_y^+ \), with coordinates \( (x_1, z, y) \). We write \( \tilde{A} = A + A_1 dx_1 \) and \( \tilde{\Phi} = \phi + \phi_1 dx_1 \) and consider \( \mathbb{R}_{x_1} \) invariant solutions. Fixing the orientation \( dx_1 \wedge d\Sigma \wedge dy \), the twisted Kapustin-Witten
equations reduce to
\[
F_A - \phi \wedge \phi + \frac{t - t^{-1}}{2} d_A \phi - \frac{t + t^{-1}}{2}(\ast d_A \phi_1 + \ast [\phi, A_1]) = 0,
\]
(59)
\[
d_A A_1 - [\phi, \phi_1] + \frac{t - t^{-1}}{2}(d_A \phi_1 + [\phi, A_1]) - \frac{t + t^{-1}}{2} \ast d_A \phi = 0,
\]
\[
d_A^* \phi - [\phi_1, A_1] = 0,
\]
where \(\ast\) is the Hodge star operator on \(\Sigma \times \mathbb{R}^+\). When \(t = 1\) and \(A_1 = 0\), this recovers the previous extended Bogomolny equations and there is a linear relationship between \(\phi_1\) and \(A_1\).

Introduce the condition \(\phi_1 = 0\) and write \(d_A = D_z + D_{\bar{z}} + D_y\). Using the local coordinate \(\Sigma \times \mathbb{R}^+\), the first equation in (59) becomes
\[
F_{z\bar{z}} - [\phi_z, \phi_{\bar{z}}] + \frac{t - t^{-1}}{2}(D_z \phi_z - D_{\bar{z}} \phi_{\bar{z}}) - \frac{t + t^{-1}}{2} D_y \phi_1 = 0,
\]
(60)
\[
F_{y\bar{z}} + \frac{t - t^{-1}}{2} D_y \phi_{\bar{z}} - \frac{t + t^{-1}}{2} i(D_z \phi_1 + [\phi_z, A_1]) = 0.
\]

We write the second equation in (59) in local coordinates:
\[
\partial_y A_1 + \frac{t - t^{-1}}{2} D_y \phi_1 - \frac{t + t^{-1}}{2}(-2i)(D_z \phi_z - D_{\bar{z}} \phi_{\bar{z}}) = 0
\]
(61)
\[
D_z A_1 - [\phi_z, \phi_1] + \frac{t - t^{-1}}{2}(D_z \phi_1 + [\phi_z, A_1]) - \frac{t + t^{-1}}{2} iD_y \phi_z) = 0.
\]

Finally, the third equation in (59) becomes
\[
D_z \phi_z + D_{\bar{z}} \phi_{\bar{z}} = 0.
\]

Setting \(t = \tan(\frac{\pi}{4} - \frac{3}{2} \beta)\), we compute that \(\frac{t - t^{-1}}{2} = -\tan(3\beta)\), \(\frac{t + t^{-1}}{2} = \frac{1}{\cos(3\beta)}\).

We also impose the asymptotic boundary condition \(\phi_1 \to 0\) as \(y \to \infty\). This system still reduces to the Hitchin equation in that limit, and hence determines a flat \(SL(n, \mathbb{C})\) connection.

Remark. Any \(t\) corresponds to three different values of \(\beta\). Since the vanishing condition \(A_1 - \tan \beta \phi_1 = 0\) depends on \(\beta\), there are actually three different equations (63) corresponding to the same value of \(t\).
Now define the covariant derivatives

\[ D_1 = D \bar{z} - \phi \tan \beta \]
\[ D_2 = D z + \phi \cot \beta \]
\[ D_3 = D y - i \frac{\phi_1}{\cos \beta} \]

We compute their commutators and the moment map equations:

\[ I_{12} = -\sin 2\beta [D_1, D_2] = \sin 2\beta f_{zz} + \cos 2\beta (D_z \phi_z - D \bar{z} \phi_z) - (D_z \phi_z + D \bar{z} \phi_z), \]
\[ I_{13} = [D_1, D_3] = F_{yz} - \tan \beta D_y \phi_z - \frac{i}{\cos \beta} D_z \phi_1 + \frac{i \sin \beta}{\cos^2 \beta} [\phi_z, \phi_1], \]
\[ I_{23} = [D_2, D_3] = F_{yz} + \cot \beta D_y \phi_z + \frac{i}{\cos \beta} D_z \phi_1 + \frac{i \sin \beta}{\cos \beta} [\phi_z, \phi_1], \]
\[ I_{mm} = -\cos^2 \beta [D_1, D_1^\dagger] + \sin^2 \beta [D_2, D_2^\dagger] + \frac{1}{4} [D_3, D_3^\dagger] \]
\[ = \cos 2\beta f_{zz} - \sin 2\beta (D_z \phi_z - D \bar{z} \phi_z) - \frac{i}{2 \cos \beta} D_y \phi_1. \]

It is straightforward to check that (65) and (63) are equivalent.

**Corollary 8.6.** If \( \phi_1 = 0 \), then the \( y \)-independent solution of (65) satisfies the Hitchin equations.

**Proof.** By (63), if \( \phi_1 = 0 \), the \( y \)-independent solution satisfies

\[ f_{zz} = -\cot(2\beta)(D_z \phi_z - D \bar{z} \phi_z), \]
\[ D_z \phi_z - D \bar{z} \phi_z = 0, \]
\[ D_z \phi_2 + D \bar{z} \phi_2 = 0, \]

which is equivalent to the Hitchin system (11). \( \square \)

**References**

[1] Mohammed Abouzaid and Ciprian Manolescu. A sheaf-theoretic model for \( SL(2; \mathbb{C}) \) Floer homology. arXiv preprint arXiv:1708.00289, 2017.
[2] Michael Atiyah. Geometry of Yang-Mills fields. In *Mathematical problems in theoretical physics (Proc. Internat. Conf., Univ. Rome, Rome, 1977)*, volume 80 of *Lecture Notes in Phys.*, pages 216–221. Springer, Berlin-New York, 1978.
[3] Alexander Beilinson and Vladimir Drinfeld. Opers. arXiv preprint math/0501398, 2005.
[4] Brian Collier and Qiongling Li. Asymptotics of certain families of Higgs bundles in the Hitchin component. arXiv preprint arXiv:1405.1106, 2014.
[5] Brian Collier and Richard Wentworth. Conformal limits and the bialynicki-birula stratification of the space of lambda-connections. arXiv preprint arXiv:1808.01622, 2018.
[6] Kevin Corlette. Flat \( G \)-bundles with canonical metrics. *J. Differential Geom.*, 28(3):361–382, 1988.
[7] Laurent Cote and Ciprian Manolescu. A sheaf-theoretic model for \( SL(2; \mathbb{C}) \) Floer homology for knots. arXiv preprint arXiv:1811.07000, 2018.
[8] Simon K. Donaldson. Anti-self-dual Yang-Mills connections over complex algebraic surfaces and stable vector bundles. *Proc. London Math. Soc. (3)*, 50(1):1–26, 1985.
[9] Simon K. Donaldson. Infinite determinants, stable bundles and curvature. *Duke Math. J.*, 54(1):231–247, 1987.
[10] Davide Gaiotto and Edward Witten. Knot invariants from four-dimensional gauge theory. *Advances in Theoretical and Mathematical Physics*, 16(3):935–1086, 2012.
[11] Sergei Gukov, Du Pei, Pavel Putrov, and Cumrun Vafa. BPS spectra and 3-manifold invariants. arXiv preprint arXiv:1701.06567, 2017.
[12] Siqi He. A gluing theorem for the Kapustin-Witten equations with a Nahm pole. *arXiv preprint arXiv:1707.06182*, 2017.

[13] Siqi He and Rafe Mazzeo. The extended Bogomolny equations and generalized Nahm pole boundary condition. *arXiv preprint arXiv:1710.10645*, 2017.

[14] Siqi He and Rafe Mazzeo. The extended Bogomolny equations with generalized Nahm pole boundary conditions, II. *arXiv preprint arXiv:1806.06314*, 2018.

[15] Siqi He and Rafe Mazzeo. Classification of Nahm pole solutions of the Kapustin-Witten equations on $S^1 \times \Sigma \times \mathbb{R}^+$. *arXiv preprint arXiv:1901.00274*, 2019.

[16] Nigel Hitchin. Stable bundles and integrable systems. *Duke mathematical journal*, 54(1):91–114, 1987.

[17] Nigel Hitchin. Lie groups and Teichmüller space. *Topology*, 31(3):449–473, 1992.

[18] Nigel J. Hitchin. The self-duality equations on a Riemann surface. *Proc. London Math. Soc. (3)*, 55(1):59–126, 1987.

[19] Anton Kapustin and Edward Witten. Electric-magnetic duality and the geometric Langlands program. *Commun. Number Theory Phys.*, 1(1):1–236, 2007.

[20] Naichung Conan Leung and Ryosuke Takahashi. Energy bound for kapustin-witten solutions on $S^3 \times \mathbb{R}^+$. *arXiv preprint arXiv:1801.04412*, 2018.

[21] Lubke Martin and Teleman Andrei. *The Kobayashi-Hitchin correspondence*. World Scientific, 1995.

[22] Rafe Mazzeo. Elliptic theory of differential edge operators. I. *Comm. Partial Differential Equations*, 16(10):1615–1664, 1991.

[23] Rafe Mazzeo and Edward Witten. The Nahm pole boundary condition. *The influence of Solomon Lefschetz in geometry and topology*. Contemporary Mathematics, 621:171–226, 2013.

[24] Rafe Mazzeo and Edward Witten. The KW equations and the Nahm pole boundary condition with knot. *arXiv preprint arXiv:1712.00855*, 2017.

[25] Victor Mikhaylov. Teichmüller TQFT vs. Chern-Simons theory. *Journal of High Energy Physics*, 2018(4):85, 2018.

[26] Carlos T. Simpson. Constructing variations of Hodge structure using Yang-Mills theory and applications to uniformization. *J. Amer. Math. Soc.*, 1(4):867–918, 1988.

[27] Clifford Taubes. Sequences of Nahm pole solutions to the $SU(2)$ Kapustin-Witten equations. *arXiv preprint arXiv:1805.02773*, 2018.

[28] Karen Uhlenbeck and S.-T. Yau. On the existence of Hermitian-Yang-Mills connections in stable vector bundles. *Comm. Pure Appl. Math.*, 39(S, suppl.):S257–S293, 1986. Frontiers of the mathematical sciences: 1985 (New York, 1985).

[29] Richard A Wentworth. Higgs bundles and local systems on riemann surfaces. *arXiv preprint arXiv:1402.4203*, 2014.

[30] Edward Witten. Fivebranes and knots. *Quantum Topol.*, 3(1):1–137, 2012.

[31] Edward Witten. Two lectures on the Jones polynomial and Khovanov homology. *arXiv preprint arXiv:1401.6996*, 2014.

[32] Edward Witten. Two lectures on Gauge theory and Khovanov homology. *arXiv preprint arXiv:1603.03854*, 2016.

**Simons Center for Geometry and Physics, Stony Brook, NY, 11794 USA**

*E-mail address: sihe@scgp.stonybrook.edu*

**Department of Mathematics, Stanford University, Stanford, CA 94305 USA**

*E-mail address: rmazzeo@stanford.edu*