On the Kullback-Leibler divergence between discrete normal distributions

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Abstract

Discrete normal distributions are defined as the distributions with prescribed means and covariance matrices which maximize entropy on the integer lattice support. The set of discrete normal distributions form an exponential family with cumulant function related to the Riemann theta function. In this paper, we present several formula for common statistical divergences between discrete normal distributions including the Kullback-Leibler divergence. In particular, we describe an efficient approximation technique for calculating the Kullback-Leibler divergence between discrete normal distributions via the Rényi α-divergences or the projective γ-divergences.

Keywords: Exponential family; discrete normal distribution; lattice Gaussian distribution; theta functions; Siegel half space; Sharma-Mittal divergence; Rényi α-divergences; γ-divergence; Cauchy-Schwarz divergence.

Contents

1 Introduction
1.1 The continuous exponential family of normal distributions 2
1.2 The set of discrete normal distributions as a discrete exponential family 3
1.3 Discrete normal distributions on full-rank lattices 6
1.4 Contributions and paper outline 8

2 Statistical divergences between discrete normal distributions
2.1 Rényi divergences 8
2.2 Kullback-Leibler divergence: Dual natural and moment parameterizations 11
2.3 Sharma-Mittal divergences 15
2.4 Chernoff information on the statistical manifold of discrete normal distributions 16

3 Numerical approximations and estimations of divergences
3.1 Converting numerically natural to moment parameters and vice versa 17
3.2 Some illustrating numerical examples 18
3.3 Approximating the Kullback-Leibler divergence via projective γ-divergences 19

A Code snippet in Julia 25
1 Introduction

1.1 The continuous exponential family of normal distributions

The $d$-variate normal distribution $N(\mu, \Sigma)$ is characterized as the unique continuous distribution defined on the support $\mathcal{X} = \mathbb{R}^d$ with prescribed mean $\mu$ and covariance matrix $\Sigma$ which maximizes Shannon’s differential entropy \cite{13}. Let $\mathcal{P}_d$ denote the open cone of positive-definite matrices and $\Lambda = \{ (\mu, \Sigma) : \mu \in \mathbb{R}^d, \Sigma \in \mathcal{P}_d \}$ the parameter space of the normal distributions. The probability density function (pdf) of a multivariate normal distribution $N(\mu, \Sigma)$ with parameterization $\lambda = (\mu, \Sigma) \in \Lambda$ is

$$q_\lambda(x) = p_{\mu,\Sigma}(x) = \frac{1}{(2\pi)^{\frac{d}{2}} \sqrt{|\Sigma|}} \exp \left( -\frac{1}{2} (x - \mu) \Sigma^{-1} (x - \mu) \right), \quad \lambda \in \Lambda, x \in \mathbb{R}^d,$$

where $|\Sigma|$ denotes the determinant of the covariance matrix.

The set of normal distributions forms an exponential family \cite{35, 6} with pdfs \cite{33} written canonically as

$$q_\rho(x) = \frac{1}{Z_\mathbb{R}(\rho)} \exp \left( x^\top \rho_1 + \text{tr} \left( -\frac{1}{2} xx^\top \rho_2 \right) \right),$$

where $\rho = (\rho_1 = \Sigma^{-1} \mu, \rho_2 = \Sigma^{-1})$ are the natural parameters corresponding to the sufficient statistics $t(x) = (x, -\frac{1}{2} xx^\top)$, and $Z_\mathbb{R}(\rho)$ is the partition function which normalizes the positive unnormalized density:

$$\tilde{q}_\rho(x) = \int_{\mathbb{R}^d} \exp \left( x^\top \rho_1 - \frac{1}{2} xx^\top \rho_2 \right) dx = (2\pi)^{\frac{d}{2}} |\rho_2^{-1}|^{\frac{1}{2}} \exp \left( \frac{1}{2} \rho_1^\top \rho_2^{-1} \rho_1 \right).$$

Notice that we used the invariance of the matrix trace under cyclic permutations to get the last equality of Eq. \cite{3}. The cumulant function\footnote{Also called log-normalizer or log-partition function. The naming “cumulant function” stems from the fact that the cumulant generating function $m_X(u) = E[\exp(u^\top t(x))]$ of the normal is $m_X(u) = F_R(\rho + u) - F_R(\rho)$ for $X \sim q_\rho$.} $F_R(\rho) = \log Z_\mathbb{R}(\rho)$ of the multivariate normal distributions is

$$F_R(\rho) = \frac{1}{2} \left( \rho_1^\top \rho_2^{-1} \rho_1 - \log |\rho_2| + d \log(2\pi) \right).$$

Thus the pdf of a normal distribution writes canonically as the pdf of an exponential family:

$$q_\rho(x) = \exp \left( \underbrace{x^\top \rho_1 - \frac{1}{2} xx^\top \rho_2}^{\langle \rho, t(x) \rangle} - \log Z_\mathbb{R}(\rho) \right),$$

$$q_\lambda(x) = \exp (\langle \rho(\lambda), t(x) \rangle) - \log Z_\mathbb{R}(\rho(\lambda)).$$

where $\langle \rho, \rho' \rangle$ is the following compound vector-matrix inner product between $\rho = (a, B)$ and $\rho' = (a', B')$ with $a, a' \in \mathbb{R}^d$ and $B, B' \in \mathcal{P}_d$:

$$\langle \xi, \xi' \rangle = a^\top a' + \text{tr}(B' B).$$
1.2 The set of discrete normal distributions as a discrete exponential family

Similarly, the \(d\)-variate discrete normal distribution\(^3\) \(^2\) \([29, 27, 2]\) \(N_{Z}(\mu, \Sigma)\) (or discrete Gaussian distribution \([11, 20]\)) is defined as the unique discrete distribution (Theorem 2.5 of \([2]\)) defined on the integer lattice support \(\mathcal{X} = \mathbb{Z}^{d}\) with prescribed mean \(\mu\) and covariance matrix \(\Sigma\) which maximizes Shannon’s entropy. Therefore the set of discrete normal distributions is a discrete exponential family with probability mass function (pmf) which can be written canonically as

\[
p_{\xi}(l) = \frac{1}{Z_{\mathbb{Z}}(\xi)} \exp \left( 2\pi i \left( -\frac{1}{2} l^T \xi_2 l + l^T \xi_1 \right) \right), \quad l \in \mathbb{Z}^d.
\]

The sufficient statistic\(^4\) is \(t(x) = (2\pi x, -\pi xx^T)\) but the natural parameter \(\xi = (\xi_1, \xi_2)\) cannot be written easily as a function of the \(\lambda = (\mu, \Sigma) \in \Lambda\) parameters, where \(\mu := E_{p_{\xi}}[x]\) and \(\Sigma = \text{Cov}_{p_{\xi}}[x] = E_{p_{\xi}}[(x - \mu)(x - \mu)^T]\). It can be shown that the normalizer is related to the Riemann theta function \(\theta_{R}\) (Eq. 21.2.1 of \([25]\)) as follows:

\[
Z_{\mathbb{Z}}(\xi) = \theta_{R}(-i\xi_1, i\xi_2),
\]

where the complex-valued theta function is the holomorphic function defined by its Fourier series as follows:

\[
\theta_{R} : \mathbb{C}^d \times \mathcal{H}_d \to \mathbb{C}
\]

\[
\theta_{R}(z, \Omega) := \sum_{l \in \mathbb{Z}^d} \exp \left( 2\pi i \left( \frac{1}{2} l^T \Omega l + l^T z \right) \right),
\]

where \(\mathcal{H}_d\) denotes the Siegel upper space\(^4\) \([15]\) of symmetric complex matrices with positive-definite imaginary part:

\[
\mathcal{H}_d = \left\{ R \in M(d, \mathbb{C}) : R = R^\top, \text{Im}(R) \in \mathcal{P}_d \right\},
\]

with \(M(d, \mathbb{C})\) denoting the set of \(d \times d\) matrices with complex entries. A matrix \(R \in \mathcal{H}_d\) is called a Riemann matrix. A Riemann matrix can be associated to a plane algebraic curve (loci of the zero of complex polynomial \(P(x, y)\) with \(x, y \in \mathbb{C}\)) via a compact Riemann surface \([18, 47]\).

Remark 1 Notice that the parameterization \(\Lambda = (\mu, \Sigma)\) of continuous normal distribution applied to the discrete normal distribution for the pmf:

\[
p_{\Lambda}(l) \propto \exp \left( -\frac{1}{2} (l - \mu)^\top \Sigma^{-1} (l - \mu) \right), \quad l \in \mathbb{Z}^d
\]

yields in general \(E_{p_{\Lambda}}[X] \neq \Lambda\) and \(\text{Cov}_{p_{\Lambda}}[X] \neq \Sigma\).

Navarro and Ruiz \([30]\) used the parameterization \((a, b)\) to express the univariate pmf as

\[
p_{a,b}(x) = \frac{\exp \left( -\frac{(x-b)^2}{2a^2} \right)}{c(a, b)},
\]

\(^2\)The term “discrete normal distribution” was first mentioned in \([29]\), page 22 (1972).
\(^3\)The canonical decomposition of exponential families is not unique. We may choose \(t_{s}(x) = s t(x)\) and \(\xi_{s} = \frac{1}{s} \xi\) for any non-zero scalar \(s\). The inner product remains invariant: \(\langle t(x), \xi \rangle = \langle t_{s}(x), \xi_{s} \rangle\). Here, we choose \(s = 2\pi\) in order to reveal the Riemann theta function.
\(^4\)Siegel upper space generalizes the Poincaré hyperbolic upper space \([33]\) \(\mathbb{H} = \{ z = x + iy \in \mathbb{C} : y > 0 \} = \mathbb{H}_1\).
Figure 1: Plot of unnormalized discrete normal distributions: Top: \( p_\xi \) on the 1D integer lattice \( \mathbb{Z} \) clipped at \([-10, 10]\) for \( \xi = (0, 0.3) \) (left) and \( \xi = (0.25, 0.15) \) (right). Notice that when \( \xi_1 \in \mathbb{Z} \), the discrete normal is symmetric (left) but not for \( \xi_1 \notin \mathbb{Z} \) (right). Bottom: \( \tilde{p}_\xi \) on the 2D integer lattice \( \mathbb{Z}^2 \) clipped at \([-7, 7] \times [-7, 7] \): (Left) \( \xi_1 = (0, 0) \) and \( \xi_2 = \text{diag}(\frac{1}{10}, \frac{1}{10}) \), (right) \( \xi_1 = (0, 0) \) and \( \xi_2 = \text{diag}(\frac{1}{10}, \frac{1}{2}) \).

where \( c(a, b) := \sum_{x \in \mathbb{Z}} \exp\left(-\frac{(x-b)^2}{2a^2}\right) \). This expression shows that discrete normal distributions are symmetric around the unique mode \( b \): \( p(b-x) = p(b+x) \). Moreover, when \( b \) is an integer, we have \( E_{p_{a,b}}[x](a, b) = b \), and \( \sigma^2(a, b) = \text{Var}_{p_{a,b}}[x](a, b) = a^3 c'(a) \) where \( c(a) = c(a, b) \) for integers \( b \).

In the remainder, Let us denote the partition function of the discrete normal distributions \( N_{\mathbb{Z}}(\xi) \) by

\[
\theta : \mathbb{R}^d \times \mathcal{P}_d \rightarrow \mathbb{R}_+
\]

\[
\xi \rightarrow \theta(\xi) := \theta_R(-i\xi_1, i\xi_2) = \sum_{l \in \mathbb{Z}^d} \exp\left(2\pi \left(\frac{1}{2}l^T \xi_2 l + l^T \xi_1\right)\right),
\]

with the corresponding cumulant function \( F_{\mathbb{Z}}(\xi) = \log \theta(\xi) \). Both the continuous and discrete normal distributions are minimal regular exponential families with open natural parameter spaces and linearly independent sufficient statistic functions \( t_i \)'s. The orders of the \( \mathbb{R} \)-pmf discrete normal distributions and the \( \mathbb{C} \)-pmf discrete normal distributions are \( \frac{d(d+3)}{2} \) and \( d(d+3) \), respectively. By definition, the standard discrete normal distribution has zero mean and unit variance: Its corresponding natural parameters \( \xi_{\text{std}} \) can be approximated numerically as \( \xi_{\text{std}} \simeq (0, 0.1591549 \times I) \) \([2]\), where \( I \) denotes the identity matrix. Observe that it is fairly different from the natural parameter \( \rho_{\text{std}} = (0, I) \) of the continuous normal distribution.
More precisely, we have as a complex-valued probability amplitude \[51\].

Studying more generally the complex-valued discrete normal distributions via properties of the theta function. For example, Agostini and Améndola [2] (Proposition 3.1) proved the quasiperiodicity of the complex discrete normal distribution are complex-valued (C-valued discrete normal distributions). The relationship between univariate discrete normal distributions and the Jacobi function was first reported in [48]. Studying more generally the C-pmf discrete normal distributions using Siegel upper space \(H^\text{right}\_d\) and Riemann theta function \(\theta_R\) allowed to get more easily results on the real-valued discrete normal distributions via properties of the theta function. For example, Agostini and Améndola [2] (Proposition 3.1) proved the quasiperiodicity of the complex discrete normal distributions \(p^C_\zeta(l)\) (C-pmf) when the parameter \(\zeta\) belongs to the set \(C^d \times H^\text{right}\_d \setminus \Theta_0\), where \(H^\text{right}\_d\) is Siegel right half-space (symmetric complex matrices with positive-definite real parts) and \(\Theta_0 = \{(a, B) \in C^d \times H^\text{right}\_d : \theta_R(a, B) = 0\}\), is called the universal theta divisor [15, 2]. The zeros of the Riemann theta function \(\theta_R\) forms an analytic variety of complex dimension \(d - 1\). Notice both the probabilities and the parameter space of the complex discrete normal distribution are complex-valued (C-pmf). For example, consider \(\zeta_1 = (0, 0)\) and \(\zeta_2 = (1 + i)I\) (where \(I\) denotes the identity matrix), then the C-pmf evaluated at \(l = (0, 0)\) is \(\frac{1}{\theta_0(\zeta)} \approx \frac{1}{3 + 2i}\) which is a complex number.

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Table 1 displays the three types of normal distributions handled in this paper.

A key property of Gaussian distributions is that the family is invariant under the action of affine automorphisms of \(R^d\). Similarly, the family of discrete Gaussian distributions is invariant under

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5Complex-valued probabilities have been explored in quantum physics where the wave function can be interpreted as a complex-valued probability amplitude [51].

6When \(d = 1\), the Riemann theta function is called the Jacobi theta function \(\theta(z, \omega) = \sum_{n \in \mathbb{Z}} \exp(2\pi i n z + \pi i n^2 \omega)\). More precisely, we have \(\theta_R(a, b) = \theta_3(\pi a, b)\) where \(\theta_3\) denote the third-type of Jacobian theta function [53].

7By extending \(\theta\) to the Siegel right half-space.

8Namely the Riemann theta function enjoys the following quasiperiodicity property: \(\theta_R(z + u, \Omega) = \theta_R(z, \Omega)\) (periodic in \(z\) with integer periods) and \(\theta_R(z + \Omega v, \Omega) = \exp(-2\pi i (\frac{1}{2} v^\top \Omega v + v^\top z))\theta_R(z, \Omega)\) for any \(u, v \in \mathbb{Z}^d\). The theta function can be generalized to the Riemann theta function with characteristic which involves a non-integer shift in its argument [77].
Figure 2: Top: Two examples of lattices with their basis defining a fundamental parallelepiped: the left one yields a subset of \( \mathbb{Z}^2 \) while the second one coincides with \( \mathbb{Z}^2 \). Bottom: Lattice Gaussian \( N_\Lambda(\xi) \) with \( \Lambda = L\mathbb{Z}^2 \) for \( L = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \), and \( \xi_1 = (0, 0) \) and \( \xi_2 = \text{diag}(0.1, 0.5) \). The lattice points are displayed in blue and the unnormalized pmf values at the lattice points are shown in red.

the action of affine automorphisms of \( \mathbb{Z}^d \) (Proposition 3.5 [2]):

\[
\forall \alpha \in \text{GL}(d, \mathbb{Z}), \quad \alpha X_\xi = X_{\alpha^{-\top} \xi_1, \alpha^{-\top} \xi_2 \alpha^{-1}}.
\]

The parity property of discrete Gaussians follows (Remark 3.7 [2]):

\[
X_{-\xi_1, \xi_2} \sim -X_\xi.
\]

The discrete normal distributions play an important role as the counterpart of the normal distributions in robust implementations on finite-precision arithmetic computers of algorithms in differential privacy [50] [8] and lattice-based cryptography [7]. Recently, the discrete normal distributions have also been used in machine learning for a particular type of Boltzmann machine termed Riemann-Theta Boltzmann machines [10] (RTBMs). RTBMs have continuous visible states and discrete hidden states, and the probability of hidden states follows a discrete multivariate Gaussian.

Let us mention that there exists other definitions of the discrete normal distributions. For example, the discrete normal distribution may be obtained by quantizing the cumulative distribution function of the normal distribution [44]). This approach is also taken when considering mixtures of discrete normal distributions in [31].

### 1.3 Discrete normal distributions on full-rank lattices

Discrete normal distributions can also be defined on a \( d \)-dimensional lattice \( \Lambda \) (also called full-rank lattice Gaussian distributions or lattice Gaussian measures with support not necessarily the integer lattice [22] [28] \( \mathbb{Z}^d \)) by choosing a set of linearly independent basis vectors \( \{l_1, \ldots, l_d\} \) arranged in
a basis matrix $L = [l_1, \ldots, l_d]$ and defining the lattice $\Lambda = L \mathbb{Z}^d = \{ L \times l : l \in \mathbb{Z}^d \}$. The pmf of a random variable $X \sim N_\Lambda(\xi)$ is

$$p_{\xi}(x) = \frac{1}{\theta_\Lambda(\xi)} \exp \left( 2\pi \left( -\frac{1}{2} x^\top \xi_2 x + x^\top \xi_1 \right) \right), \quad x \in \Lambda.$$ 

The above pmf can further be specialized for a random variable $X \sim N_\Lambda(c, \sigma)$ (a lattice Gaussian with variance $\sigma^2$ and center $c$) is $p_{\xi}(l) = \frac{1}{(\sqrt{2\pi}\sigma)^d} \exp(-\frac{\|l-c\|^2}{2\sigma^2})$. For a general lattice $\Lambda = L \mathbb{Z}^d$, we may define the lattice Gaussian distribution $N_\Lambda(\xi)$ with $\xi = (a, B)$ and normalizer

$$\theta_\Lambda(\xi) := \sum_{l \in \Lambda} \exp \left( 2\pi \left( -\frac{1}{2} l^\top \xi_2 l + l^\top \xi_1 \right) \right).$$

When $L = I$ (identity matrix), the lattice Gaussian distributions are the discrete normal distributions but other non-identity matrix basis may also generate $\mathbb{Z}^2$ (see Figure 2). Since $\theta_\Lambda(\xi) := \sum_{l \in \mathbb{Z}^d} \exp \left( 2\pi \left( -\frac{1}{2} (Ll)^\top \xi_2 (Ll) + (Ll)^\top \xi_1 \right) \right)$, we have the following proposition:

**Proposition 1** The normalizer of a lattice normal distribution $N_\Lambda(\xi)$ for $\Lambda = L \mathbb{Z}^d$ and $\xi = (a, B)$ amounts to the following Riemann theta function:

$$\theta_\Lambda(\xi) = \theta_R(-iL^\top a, iL^\top BL).$$

Last, we can translate the lattice $L \mathbb{Z}^d$ by $c \in \mathbb{R}^d$ (i.e., $\Lambda = L \mathbb{Z}^d + c$) so that we have the full generic pmf of a lattice gaussian which can be written for $\xi = (a, B)$ as:

$$p_{\xi}(l) = \frac{1}{\theta_\Lambda(\xi)} \exp \left( 2\pi \left( -\frac{1}{2} l^\top \xi_2 l + l^\top \xi_1 \right) \right), \quad l \in L \mathbb{Z}^d + c,$$

where

$$\theta_\Lambda(\xi) = \sum_{l \in \Lambda = L \mathbb{Z}^d + c} \exp \left( 2\pi \left( -\frac{1}{2} l^\top \xi_2 l + l^\top \xi_1 \right) \right),$$

$$= \sum_{z \in \mathbb{Z}^d} \exp \left( 2\pi \left( -\frac{1}{2} (Lz + c)^\top \xi_2 (Lz + c) + (Lz + c)^\top \xi_1 \right) \right).$$

The normalizer is related to the Riemann Theta functions with characteristics $\alpha, \beta \in \mathbb{R}^d$:

$$\theta_R \begin{bmatrix} \alpha \\ \beta \end{bmatrix} (a, B) := \sum_{l \in \mathbb{Z}^d} e^{2\pi i \left( \frac{1}{2} (l + \alpha)^\top B (l + \alpha) + (l + \alpha)^\top (B + \beta) \right)},$$

$$= e^{2\pi i \left( \frac{1}{2} \alpha^\top B \alpha + \alpha^\top (a + \beta) \right)} \theta_R(a + B\alpha + \beta, B).$$

For example, when $L = I$, $\alpha = c$ and $\beta = 0$. 

7
1.4 Contributions and paper outline

We summarize our main contributions as follows: We report a formula for the Rényi $\alpha$-divergences between two discrete normal distributions in Proposition 3 including related results for the Bhattacharyya divergence, the Hellinger divergence and Amari’s $\alpha$-divergences. We give a formula for the cross-entropy between two discrete normal distributions in Proposition 6 which yields a formula for the Kullback-Leibler divergence (Proposition 5 and Proposition 7). More generally, we extend the formula to Sharma-Mittal divergences in Proposition 8. In Section 3, we show how to implement these formula using numerical approximations of the theta function. We also propose a fast technique to approximate the Kullback-Leibler divergence between discrete normal distributions relying on $\gamma$-divergences [20] (Proposition 9).

2 Statistical divergences between discrete normal distributions

2.1 Rényi divergences

The Rényi $\alpha$-divergence [49] between pmf $r(x)$ to pmf $s(x)$ on support $\mathcal{X}$ is defined for any positive real $\alpha \neq 1$ by

$$D_\alpha[r : s] = \frac{1}{\alpha - 1} \log \left( \sum_{x \in \mathcal{X}} r(x)^\alpha s(x)^{1-\alpha} \right) = \frac{1}{\alpha - 1} \log E_s \left[ \left( \frac{r(x)}{s(x)} \right)^\alpha \right] , \quad \alpha > 0, \alpha \neq 1.$$

When $\alpha = \frac{1}{2}$, Rényi $\alpha$-divergence amounts to twice the symmetric Bhattacharyya divergence [37]: $D_{\frac{1}{2}}[r : s] = 2D_{\text{Bhattacharyya}}[r, s]$ with:

$$D_{\text{Bhattacharyya}}[r, s] := -\log \left( \sum_{x \in \mathcal{X}} \sqrt{r(x)s(x)} \right).$$

The Bhattacharyya divergence can be interpreted as the negative logarithm of the Bhattacharyya coefficient:

$$\rho_{\text{Bhattacharyya}}[r, s] = \sum_{x \in \mathcal{X}} \sqrt{r(x)s(x)}.$$

A divergence related to the Bhattacharyya divergence is the squared Hellinger divergence:

$$D_{\text{Hellinger}}^2[r, s] = \frac{1}{2} \sum_{x \in \mathcal{X}} \left( \sqrt{r(x)} - \sqrt{s(x)} \right)^2 = 1 - \rho_{\text{Bhattacharyya}}[r, s].$$

The squared Hellinger divergence is one fourth of the $\alpha$-divergence for $\alpha = \frac{1}{2}$ [4], where the $\alpha$-divergences are defined by

$$D_{\text{Amari}, \alpha}[r : s] = \frac{1}{\alpha(1-\alpha)} (1 - \rho_{\text{Bhattacharyya}, \alpha}[r : s]).$$

The $\alpha$-divergences can be calculated from the skewed Bhattacharyya coefficients for $\alpha \in \mathbb{R} \setminus \{0, 1\}$:

$$\rho_{\text{Bhattacharyya}, \alpha}[r : s] = \sum_{x \in \mathcal{X}} r(x)^\alpha s(x)^{1-\alpha}.$$
Proposition 5 of [8] upper bounds the Rényi $\alpha$-divergence between discrete normal distributions with same variance $\sigma^2$ as:

$$D_\alpha \left[ N_Z (\mu_1, \sigma^2) : N_Z (\mu_2, \sigma^2) \right] \leq \alpha \frac{(\mu_1 - \mu_2)^2}{2\sigma^2}.$$  

Rényi $\alpha$-divergences are non-decreasing with $\alpha$ [49].

When both pmfs are from the same discrete exponential families with log-normalizer $F(\xi) = \log \theta(\xi)$, the Rényi $\alpha$-divergence [41] amounts to a $\alpha$-skewed Jensen divergence [37] between the corresponding natural parameters:

$$D_\alpha [p_\xi : p_{\xi'}] = \frac{1}{1 - \alpha} J_{F,\alpha}(\xi : \xi'),$$

where

$$J_{F,\alpha}(\xi : \xi') := \alpha F(\xi) + (1 - \alpha) F(\xi') - F(\alpha \xi + (1 - \alpha) \xi').$$

Indeed, let

$$I_{\alpha, \beta}[r : s] = \sum_{x \in \mathcal{X}} r(x)^\alpha s(x)^\beta, \quad \alpha, \beta \in \mathbb{R}.$$  

Then we have the following lemma:

**Proposition 2** For two pmfs $p_\xi$ and $p_{\xi'}$ of a discrete exponential family with log-normalizer $F(\xi)$ with $\alpha \xi + \beta \xi' \in \Xi$, we have

$$I_{\alpha, \beta}[p_\xi : p_{\xi'}] = \exp \left( F(\alpha \xi + \beta \xi') - (\alpha F(\xi) + \beta F(\xi')) \right).$$

**Proof:** We have

$$I_{\alpha, \beta}[p_\xi : p_{\xi'}] = \sum_{x \in \mathcal{X}} \exp(\langle t(x), \alpha \xi \rangle - \alpha F(\xi)) \exp(\langle t(x), \beta \xi' \rangle - \beta F(\xi')), $$

$$= e^{F(\alpha \xi + \beta \xi') - (\alpha F(\xi) + \beta F(\xi'))} \sum_{x \in \mathcal{X}} e^{\langle t(x), \alpha \xi + \beta \xi' \rangle - F(\alpha \xi + \beta \xi')} = 1,$$

since $\sum_{x \in \mathcal{X}} p_{\alpha \xi + \beta \xi'}(x) = 1$ when $\alpha \xi + \beta \xi' \in \Xi$.  

Thus we get the following proposition:

**Proposition 3** The Rényi $\alpha$-divergence between two discrete normal distributions $p_\xi$ and $p_{\xi'}$ for $\alpha > 0$ and $\alpha \neq 1$ is

$$D_\alpha [p_\xi : p_{\xi'}] = \frac{1}{1 - \alpha} \left( \alpha \log \frac{\theta(\xi)}{\theta(\alpha \xi + (1 - \alpha) \xi')} + (1 - \alpha) \log \frac{\theta(\xi')}{\theta(\alpha \xi + (1 - \alpha) \xi')} \right).$$

(9)
Figure 3: Approximating Riemann $\theta_R$ function by summing on the integer lattice points falling inside an ellipsoid $E$: $\theta(\xi) \simeq \tilde{\theta}(\xi; E)$.

Proof: We have

$$D_\alpha[p_\xi : p_{\xi'}] = \frac{1}{1 - \alpha} \left( \alpha \log \theta(\xi) + (1 - \alpha) \log \theta(\xi') - \log (\alpha \xi + (1 - \alpha) \xi') \right).$$

Plugging $\log \theta(\alpha \xi + (1 - \alpha) \xi') = (\alpha + 1 - \alpha) \log \theta(\alpha \xi + (1 - \alpha) \xi')$ in the right-hand-side equation yields the result. Notice that we can express also the Rényi divergences as

$$D_\alpha[p_\xi : p_{\xi'}] = \frac{1}{1 - \alpha} \log \frac{\theta(\xi)^\alpha \theta(\xi')^{1-\alpha}}{\theta(\alpha \xi + (1 - \alpha) \xi')}.$$

See [16, 19, 3] for the efficient numerical approximations of the Riemann theta function. Basically, the infinite theta series $\theta(\eta)$ is approximated by a finite summation over a region $R$ of integer lattice points:

$$\tilde{\theta}(\xi; R) := \sum_{x \in R} \exp \left( 2\pi \left( -\frac{1}{2} x^\top \xi_2 x + x^\top \xi_1 \right) \right).$$

When $R = \mathbb{Z}^d$, we have $\tilde{\theta}(\xi; R) = \theta(\xi)$. The method proposed in [16] consists in choosing the integer lattice points $E_\xi$ falling inside an ellipsoid used to approximate the theta function as illustrated in Figure 3.

Thus we have the following proposition:

**Proposition 4** The squared Hellinger distance between two discrete normal distributions $p_\xi$ and $p_{\xi'}$ is

$$D_{\text{Hellinger}}^2[p_\xi, p_{\xi'}] = 1 - \frac{\theta \left( \frac{\xi + \xi'}{2} \right)}{\sqrt{\theta(\xi) \theta(\xi')}}.$$
We can also write

\[
D_\alpha[p_\xi : p_{\xi'}] = \frac{1}{\alpha - 1} \log E_{p_{\xi'}} \left[ \left( \frac{p_\xi}{p_{\xi'}} \right)^\alpha \right],
\]

\[
= \frac{1}{\alpha - 1} \left( \alpha \log \frac{\theta(\xi')}{\theta(\xi)} + E_{p_{\xi'}} \left[ \left( \frac{\tilde{p}_\xi(x)}{\tilde{p}_{\xi'}(x)} \right)^\alpha \right] \right),
\]

\[
= \frac{\alpha}{\alpha - 1} \log \frac{\theta(\xi')}{\theta(\xi)} + \frac{1}{\alpha} \sum_{l \in \mathbb{Z}^d} \tilde{p}_{\xi'}(l) \left( \frac{\tilde{p}_\xi(l)}{\tilde{p}_{\xi'}(l)} \right)^\alpha.
\]

This last expression can be numerically estimated.

The Bhattacharyya divergence between two discrete normal distributions \( p_\xi \) and \( p_{\xi'} \) can be expressed as an equivalent Jensen divergence between its natural parameters:

\[
D_{\text{Bhattacharyya}}[p_{\xi'}, p_\xi] = J_F(\xi, \xi'),
\]

where

\[
J_F(\xi : \xi') := \frac{F(\xi) + F(\xi')}{2} - F \left( \frac{\xi + \xi'}{2} \right).
\]

Thus we have \( D_{\text{Bhattacharyya}}[p_\xi, p_{\xi'}] = \log \sqrt{\frac{\theta(\xi) \theta(\xi')}{\theta(\xi + \xi')}} \). We can also express the Bhattacharyya divergence using the unnormalized pmfs:

\[
D_{\text{Bhattacharyya}}[p_{\xi'}, p_\xi] = \log \sqrt{\theta(\xi) \theta(\xi')} - \log \left( \sum_{l \in \mathbb{Z}^d} \tilde{p}_\xi(l) \tilde{p}_{\xi'}(l) \right).
\]

Consider the transformations \( \tau \) that leaves the \( \theta \) function invariant: \( \theta(\tau(\xi)) = \theta(\xi) \). Then the Rényi \( \alpha \)-divergences simplifies to the following formula:

\[
D_\alpha[p_\xi : p_{\tau(\xi)}] = \frac{1}{1 - \alpha} \log \frac{\theta(\xi)}{\theta(\alpha \xi + (1 - \alpha)\tau(\xi))},
\]

(10)

For example, consider \( \xi_1 = \xi'_1 \in \mathbb{Z}^d \) and \( \xi_2 = \text{diag}(b_1, \ldots, b_d) \) and \( \xi'_2 = \text{diag}(\sigma(b_1, \ldots, b_d)) \) for a permutation \( \sigma \in S_d \). Then we have \( \theta(\xi') = \theta(\xi) \), and formula of Eq. (10) applies.

### 2.2 Kullback-Leibler divergence: Dual natural and moment parameterizations

When \( \alpha \to 1 \), the Rényi \( \alpha \)-divergences tend asymptotically to the Kullback-Leibler divergence (KLD). The KLD between two pmfs \( r(x) \) and \( s(x) \) defined on the support \( \mathcal{X} \) is defined by

\[
D_{\text{KL}}[r : s] = \sum_{x \in \mathcal{X}} r(x) \log \frac{r(x)}{s(x)}.
\]

In general, the KLD between two pmfs of a discrete exponential family amounts to a reverse Bregman divergence between their natural parameters [34]:

\[
D_{\text{KL}}[p_\xi : p_{\xi'}] = B_F^*(\xi : \xi') = B_F(\xi' : \xi),
\]
where the Bregman divergence with generator \( F(\xi) \) is defined by:

\[
B_F(\xi' : \xi) = F(\xi') - F(\xi) - \langle \xi' - \xi, \nabla F(\xi) \rangle,
\]
where \( \langle \xi, \xi' \rangle \) is the following compound vector-matrix inner product between \( \xi = (a, B) \) and \( \xi' = (a', B') \) with \( a, a' \in \mathbb{R}^d \) and \( B, B' \in \mathcal{P}_d \):

\[
\langle \xi, \xi' \rangle = a^\top a' + \text{tr}(B' B).
\]

The gradient \( \nabla F(\xi) = \nabla_\theta F(\xi) \) defines the dual parameter \( \eta \) of an exponential family: \( \eta = \nabla F(\xi) \).

This dual parameter is called the moment parameter (or the expectation parameter) because we have \( \mathbb{E}_{p_\xi}[t(x)] = \nabla F(\xi) \) and therefore \( \eta = \mathbb{E}_{p_\xi}[t(x)] \). A discrete normal distribution can thus be parameterized either by its ordinary parameter \( \lambda = (\mu, \Sigma) \), its natural parameter \( \xi \), or its dual moment parameter \( \eta \). We write the distributions accordingly: \( \mathcal{N}(\lambda), \mathcal{N}(\xi), \) and \( \mathcal{N}(\eta) \) with corresponding pmfs: \( p_\lambda(x), p_\xi(x), \) and \( p_\eta(x) \).

There exists a bijection between the space of natural parameters and the space of moment parameters induced by the Legendre-Fenchel transformation of the cumulant function:

\[
F^*(\eta) = \sup_{\xi \in \Xi} \langle \xi, \eta \rangle - F(\xi),
\]
where \( \Xi = \mathbb{R}^d \times \mathcal{P}_d \). Function \( F^* \) is called the convex conjugate and induces a dual Bregman divergence so that we have \( B_F(\xi' : \xi) = B_{F^*}(\eta' : \eta) \) with \( \eta' = \nabla F(\xi') \). The dual parameters are linked as follows: \( \eta = \nabla F(\xi), \xi = \nabla F^*(\eta) \), and therefore we get:

\[
F^*(\eta) = \langle \xi, \eta \rangle - F(\xi).
\]

The convex conjugate of the cumulant function \( F(\xi) \) is called the negentropy because it can be shown \([6, 39]\) that we have

\[
F^*(\eta) = -H[p_\xi] = \sum_{x \in \mathcal{X}} p_\xi(x) \log p_\xi(x),
\]
where \( H[p_\xi] = -\sum_{x \in \mathcal{X}} p_\xi(x) \log p_\xi(x) \) denotes Shannon’s entropy of the random variable \( X \sim p_\xi \).

The maximum likelihood estimator (MLE) of a density of an exponential family from \( n \) identically and independently distributed samples \( x_1, \ldots, x_n \) is given by \([6]\):

\[
\hat{\eta} = \frac{1}{n} \sum_{i=1}^n t(x_i).
\]

It follows from the equivariance property of the MLE that we have \( \hat{\xi} = \nabla F^*(\hat{\eta}) \). We get the following MLE for the discrete normal family:

\[
\hat{\eta}_1 = \frac{2\pi}{n} \sum_{i=1}^n x_i = 2\pi \hat{\mu},
\]
\[
\hat{\eta}_2 = -\pi \sum_{i=1}^n x_i x_i^\top = -\pi (\hat{\Sigma} + \hat{\mu} \hat{\mu}^\top).
\]
The Fenchel-Young inequality for convex conjugates $F(\xi)$ and $F^*(\eta)$ is

$$ F(\xi) + F^*(\eta') \geq \langle \xi, \eta' \rangle, $$

with equality holding if and only if $\eta' = \nabla F(\xi)$. The Fenchel-Young inequality induces a Fenchel-Young divergence:

$$ Y_{F,F^*}(\xi : \eta') := F(\xi) + F^*(\eta') - \langle \xi, \eta' \rangle = Y_{F,F^*}(\eta' : \xi) \geq 0, $$

such that $Y_{F,F^*}(\xi : \eta') = B_F(\xi : \xi')$. Thus the Kullback-Leibler divergence between two pmfs of a discrete exponential family can be expressed in the following equivalent ways using the natural-/moment parameterizations:

$$ D_{KL}[p_\xi : p_{\xi'}] = B_F(\xi' : \xi) = B_{F^*}(\eta' : \eta) = Y_{F,F^*}(\eta' : \xi') = Y_{F,F^*}(\xi' : \eta). \tag{11} $$

Thus using the fact that the KLD amounts to a reverse Bregman divergence for the cumulant function $F(\xi) = \log \theta(\xi)$, we get the following proposition:

**Proposition 5**  The Kullback-Leibler divergence between two discrete normal distributions $p_\xi$ and $p_{\xi'}$ with natural parameters $\xi$ and $\xi'$ is

$$ D_{KL}[p_\xi : p_{\xi'}] = \log \frac{\theta(\xi')}{\theta(\xi)} - \frac{1}{\theta(\xi)} \langle \xi' - \xi, \nabla \theta(\xi) \rangle. $$

Some software packages for the Riemann theta function can numerically approximate both the theta function and its derivatives \[3\]. Using the periodicity property of the theta function for $\xi' = (\xi_1 + u, \xi_2)$ with $u \in \mathbb{Z}^d$, we have $\theta(\xi') = \theta(\xi)$, and therefore $D_{KL}[p_\xi : p_{\xi'}] = \frac{1}{\theta(\xi)} \langle \xi - \xi', \nabla \theta(\xi) \rangle.$

For the discrete normal distributions, we can express the moment parameter for the discrete normal distributions using the ordinary mean-covariance parameters $\lambda = (\mu, \Sigma)$. Since the sufficient statistics is $2\pi(x, xx^\top)$, we have $\eta_1(\xi) = E_{p_\xi}[2\pi x] = 2\pi \mu$ and $\eta_2(\xi) = E_{p_\xi}[-\pi xx^\top] = -\pi(\Sigma + \mu \mu^\top)$.

Proposition 4.4 of \[2\] reports the entropy of $p_\xi$ as

$$ H[p_\xi] = \log \theta(\xi) - 2\pi \xi_1^\top \mu + \pi \text{tr}(\xi_2(\Sigma + \mu \mu^\top)). $$

We can rewrite the entropy as minus the convex conjugate function of the cumulant function:

$$ H[p_\xi] = -F^*(\eta) = F(\xi) - \langle \xi, \eta \rangle. $$

Thus we have the convex conjugate which can be expressed as

$$ F^*(\eta) = -\log \theta(\xi) + 2\pi \mu^\top \xi_1 - \pi \text{tr}(\xi_2(\Sigma + \mu \mu^\top)). \tag{12} $$

The entropy of $p_\xi$ can be calculated using the unnormalized pmf as follows:

$$ H[p_\xi] = -\sum_{l \in \mathbb{Z}^d} p_\xi(l) \log p_\xi(l) = -E_{p_\xi}[\log p_\xi(l)] = \log \theta(\xi) - \frac{1}{\theta(\xi)} \sum_{l \in \mathbb{Z}^d} \tilde{p}_\xi(l) \log \tilde{p}_\xi(l) > 0. $$

13
The cross-entropy between two pmfs \( r(x) \) and \( s(x) \) defined over the support \( \mathcal{X} \) is

\[
H[r : s] = -\sum_{x \in \mathcal{X}} r(x) \log s(x).
\]

Entropy is self cross-entropy: \( H[r] = H[r : r] \). The formula for the cross-entropy of a density of an exponential family \([39]\) can be written as:

\[
H[p_{\xi} : p_{\xi'}] = F(\xi') - \langle \xi', \nabla F(\xi) \rangle = F(\xi') - \langle \xi', \eta \rangle.
\]

Thus we get the following proposition:

**Proposition 6** The cross-entropy between two discrete normal distributions \( p_{\xi} \sim N_{\mathbb{Z}}(\mu, \Sigma) \) and \( p_{\xi'} \sim N_{\mathbb{Z}}(\mu', \Sigma') \) is

\[
H[N_{\mathbb{Z}}(\mu, \Sigma) : N_{\mathbb{Z}}(\mu', \Sigma')] = \log \theta(\xi') - 2\pi \mu^\top (\xi'_1 - \xi_1) + \pi \text{tr}(\Sigma + \mu \mu^\top)). \tag{13}
\]

Notice that the cross-entropy can be written using the unnormalized pmf as

\[
H[p_{\xi} : p_{\xi'}] = -E_{p_{\xi}}[\log p_{\xi'}(x)] = \log \theta(\xi') - \frac{1}{\theta(\xi)} \sum_{l \in \mathbb{Z}^d} \tilde{p}_{\xi}(l) \log \tilde{p}_{\xi'}(l).
\]

The KLD can be expressed as the cross-entropy minus the entropy (and henceforth its other name is relative entropy):

\[
D_{KL}[p_{\xi} : p_{\xi'}] = H[p_{\xi} : p_{\xi'}] - H[p_{\xi}].
\]

It follows that we can compute the KLD between two discrete normal distributions as follows:

**Proposition 7** The Kullback-Leibler divergence between two discrete normal distributions \( p_{\xi} \sim N_{\mathbb{Z}}(\mu, \Sigma) \) and \( p_{\xi'} \sim N_{\mathbb{Z}}(\mu', \Sigma') \) is:

\[
D_{KL}[p_{\xi} : p_{\xi'}] = \log \frac{\theta(\xi')}{\theta(\xi)} - 2\pi \mu^\top (\xi'_1 - \xi_1) + \pi \text{tr}((\xi'_2 - \xi_2)(\Sigma + \mu \mu^\top)). \tag{14}
\]

Notice that we use mixed \((\xi, \lambda)\)-parameterizations in the above formula. In practice, we estimate discrete normal distributions and then calculate the corresponding natural parameters by solving a gradient system explained in \([3]\).

Notice that the KLD between normal distributions can be decomposed as a sum of a squared Mahalanobis distance and a matrix Burg divergence (see Eq. 5 of \([14]\)). For discrete normal distributions, when \( \xi = (a, B) \) with \( a \in \mathbb{Z} \) and \( \xi' = (a + m, B) \) with \( m \in \mathbb{Z} \), we have \( \theta(\xi) = \theta(\xi') \) and \( \mu = \mu' \), so that the KLD simplifies to the following formula:

\[
D_{KL}[p_{\xi} : p_{\xi'}] = \frac{1}{\theta(\xi)} \langle (-m, B), \nabla \theta(\xi) \rangle.
\]
Notice that the MLE $\hat{\xi}_n$ of $n$ samples $x_1, \ldots, x_n \sim_{\text{i.i.d.}} p_\xi$ can be interpreted as a KL divergence minimization problem:

$$\hat{\xi}_n = \arg\min_{\xi \in \Xi} D_{\text{KL}}[p_\xi : p_\xi],$$

where $p_e(x) = \frac{1}{n} \sum_{i=1}^n \delta(x-x_i)$ denotes the empirical distribution with $\delta(x)$ the Dirac’s distribution: $\delta(x) = 1$ if and only if $x = 0$.

Notice that when $\alpha \to 1$, we have $J_{F,\alpha}(\xi : \xi') \to B_F(\xi : \xi')$ [37], and $D_{\alpha}[p_\xi : p'_\xi] \to D_{\text{KL}}[p_\xi : p'_\xi]$.

### 2.3 Sharma-Mittal divergences

The Sharma-Mittal divergence [40] $D_{\alpha,\beta}[p : q]$ between two pmfs $p(x)$ and $q(x)$ defined over the discrete support $\mathcal{X}$ unifies the Rényi $\alpha$-divergences ($\beta \to 1$) with the Tsallis $\alpha$-divergences ($\beta \to \alpha$):

$$D_{\alpha,\beta}[p : q] := \frac{1}{\beta - 1} \left( \left( \sum_{x \in \mathcal{X}} p(x)^\alpha q(x)^{1-\alpha} \right)^{\frac{1-\beta}{1-\alpha}} - 1 \right), \quad \forall \alpha > 0, \alpha \neq 1, \beta \neq 1.$$

Moreover, we have $D_{\alpha,\beta}[p : q] \to D_{\text{KL}}[p : q]$ when $\alpha, \beta \to 1$.

For two pmfs $p_\xi$ and $p'_\xi$ belonging to the same exponential family [40], we have:

$$D_{\alpha,\beta}[p_\xi : p'_\xi] = \frac{1}{\beta - 1} \left( e^{\frac{1-\beta}{1-\alpha} J_{F,\alpha}(\xi : \xi') - 1} \right).$$

Thus we get the following proposition:

### Proposition 8

The Sharma-Mittal divergence $D_{\alpha,\beta}[p_\xi : p'_\xi]$ between two discrete normal distributions $p_\xi$ and $p'_\xi$ is:

$$D_{\alpha,\beta}[p_\xi : p'_\xi] = \frac{1}{\beta - 1} \left( \left( \frac{\theta_d(\xi)^\alpha \theta_d(\xi')^{1-\alpha}}{\theta_d(\alpha \xi + (1 - \alpha) \xi')} \right)^{\frac{1-\beta}{1-\alpha}} - 1 \right),$$

$$= \frac{1}{\beta - 1} \left( \frac{\theta_d(\xi)}{\theta_d(\alpha \xi + (1 - \alpha) \xi')} \right)^{\frac{\alpha(\beta-1)}{1-\alpha}} \left( \frac{\theta_d(\xi')}{{\theta_d(\alpha \xi + (1 - \alpha) \xi')}} \right)^{\beta - 1}. \quad (15)$$

### 2.4 Chernoff information on the statistical manifold of discrete normal distributions

Chernoff information stems from the characterization of the error exponent in Bayesian hypothesis testing (see §11.9 of [13]). The Chernoff information between two pmfs $r(x)$ and $s(x)$ is defined by

$$D_{\text{Chernoff}}[r, s] := - \min_{\alpha \in [0,1]} \log \left( \sum_{x \in \mathcal{X}} r^\alpha(x) s^{1-\alpha}(x) \right),$$
where $\alpha^*$ denotes the best exponent: $\alpha^* = \arg\min_{\alpha \in [0,1]} \sum_{x \in X} r^\alpha(x)s^{1-\alpha}(x)$. When $r(x) = p_\xi(x)$ and $s(x) = p_{\xi'}(x)$ are pmfs of a discrete exponential family with cumulant function $F(\xi)$, we have (Theorem 1 of [32]):

$$D_{\text{Chernoff}}[p_\xi, p_{\xi'}] = B_F(\xi : \xi^*) = B_F(\xi' : \xi^*),$$

where $\xi^* := \alpha^* \xi + (1 - \alpha)\xi'$. Thus calculating Chernoff information amounts to first find the best $\alpha^*$ and second compute $D_{\text{KL}}[p_\xi^* : p_{\xi}]$ or equivalently $D_{\text{KL}}[p_{\xi'}^* : p_{\xi'}]$. By modeling the exponential family as a manifold $M = \{ p_\xi : \xi \in \Xi \}$ equipped with the Fisher information metric (a Hessian metric expressed in the $\xi$-coordinate system by $\nabla^2 F(\xi)$ so that the length element $ds$ appears in the Taylor expansion of the KL divergence: $D_{\text{KL}}[p_\xi+ds : p_\xi] = \frac{1}{2}ds^2 = \frac{1}{2}d\xi^\top \nabla^2 F(\xi)d\xi$), we can characterize geometrically the exact $\alpha^*$ (Theorem 2 of [32]) as the unique intersection of an exponential geodesic $\gamma_{\xi,\xi'}$ with a mixture bisector $\text{Bi}(\xi, \xi')$ where

$$\gamma_{\xi,\xi'} := \{ p_\lambda\xi + (1-\lambda)\xi' : \lambda \in (0,1) \},$$

$$\text{Bi}(\xi, \xi') := \{ p_\omega \in M : D_{\text{KL}}[p_\omega : p_\xi] = D_{\text{KL}}[p_\omega : p_{\xi'}] \}.$$  

Thus we have $p_{\xi^*} = \gamma_{\xi,\xi'} \cap \text{Bi}(\xi, \xi')$. This geometric characterization yields a fast numerical approximation bisection technique to obtain $\alpha^*$ within a prescribed precision error. Since the discrete normal distributions form an exponential family, we can apply the above technique derived from information geometry\footnote{Information geometry is the field which considers differential-geometric structures of families of probability distributions. Historically, Hotelling [24] first introduced the Fisher-Rao manifold. The term “information geometry” occurred in a paper of Chentsov [11] in 1978.} to calculate numerically the Chernoff information. Various statistical inference procedures like estimators in curved exponential families and hypothesis testing can be investigated using the information-geometric dually flat structure of $M$, called a statistical manifold (see [4, 34] for details).

**Remark 2** The Fisher information matrix of the univariate normal distributions is $I(\xi) = (\log \theta(\xi))'' = \left( \frac{\theta'(\xi)}{\theta(\xi)} \right)^2$, where $\theta'$ and $\theta''$ are the derivative and second derivatives of the Jacobi function $\theta$.

Knowing that the KL divergence between two discrete normal distributions amounts to a Bregman divergence is helpful for a number of tasks like clustering [21]: The left-sided KL centroid of $n$ discrete normal distributions $p_{\xi_1}, \ldots, p_{\xi_n}$ amounts to a right-sided Bregman centroid which is always the center of mass of the natural parameters [5]:

$$\xi^* = \arg\min_\xi \sum_{i=1}^n \frac{1}{n}D_{\text{KL}}[p_\xi : p_{\xi_i}] = \arg\min_\xi \sum_{i=1}^n \frac{1}{n}B_F(\xi_i : \xi) \Rightarrow \xi^* = \frac{1}{n} \sum_{i=1}^n \xi_i.$$

3 Numerical approximations and estimations of divergences

Although conceptually very similar as maximum entropy distributions to the continuous normal distributions, discrete normal distributions are mathematically very different to handle. On one hand, the normal distributions are exponential families with all parameter transformations and convex conjugates $F_R(\rho)$ and $F_R^*(\tau)$ available in closed-form [36] (where $\tau = E_{\rho}(t(x))$). On the other hand, the discrete normal distributions with source parameters $\lambda = (\mu, \Sigma)$ can be converted
from/back the moment parameters, but the conversions between natural parameters $\xi$ and expectation parameters $\eta = \nabla F(\xi)$ are not available in closed-form, nor the cumulant function $F(\xi) = \log \theta(\xi)$ and its convex conjugate $F^*(\eta)$.

### 3.1 Converting numerically natural to moment parameters and vice versa

In practice, we can approximate the conversion procedures $\xi \leftrightarrow \eta$ as follows:

- **Given natural parameter $\xi$,** we may approximate the dual moment parameter $\eta = \nabla F(\xi) = E_{p_\xi}[t(x)]$ where $x_1, \ldots, x_m$ are independently and identically sampled from $N_{d}(\xi)$. Sampling uniformly from discrete normal distributions can be done exactly in 1D [8] (requiring average constant time) but requires sampling heuristics in dimension $d > 1$. Two common sampling heuristics approximating for handling discrete normal distributions are:
  
  - $H_1$: Draw a variate $x \sim q_{\mu, \Sigma}$ from the corresponding normal distribution $q_{\mu, \Sigma}$, and round or choose the closest integer lattice point $\tilde{x}$ of $Z^d$ with respect to the $\ell_1$-norm (i.e.,
    $$\tilde{x} = \arg \min_{l \in Z^d} \|l - x\|_1 = \sum_{i=1}^{d} |l^i - x^i|$$
    where $(l^1, \ldots, l^d)$ and $(x^1, \ldots, x^d)$ denote the coordinates of $l$ and $x$, respectively.
  
  - $H_2$: Consider the integer lattice points $E_\xi$ falling inside the ellipsoid region [16] used for approximating $\theta(\xi)$ by $\hat{\theta}(\xi; E_\xi)$ (Figure 3), draw uniformly an integer lattice point $l$ from $E_\xi$ and accept it with probability $p_\xi(l)$ (acceptance-rejection sampling described in [10]).

- **Given the moment parameter $\eta$,** we may approximate $\theta = \nabla F^*(\eta)$ by solving a gradient system. Since the moment generating function (MGF) of an exponential family [6] is $m_X(u) := E_X[\exp(u^\top X)] = \exp(F(\xi + u) - F(\xi))$, we deduce that the MGF of the discrete normal distributions $X \sim p_\xi$ is

  $$m_\xi(u) = \frac{\theta(\xi + u)}{\theta(\xi)}.$$

The non-central moments of the sufficient statistics (also called raw moments or geometric moments) of an exponential family can be retrieved from the partial derivatives of the MGF. For the discrete normal distributions, Agostini and Améndola [2] obtained the following gradient system:

$$\begin{align*}
\eta_1 &= E_{p_\xi}[t_1(x)] = \frac{1}{2\pi} \frac{1}{\theta(\xi)} \nabla \xi \theta(\xi), \\
\eta_2 &= E_{p_\xi}[t_2(x)] = -\frac{1}{2\pi} \frac{1}{\theta(\xi)} \left( \nabla \xi \theta(\xi) + \text{diag}(\nabla \xi \theta_d(\xi)) \right).
\end{align*}$$

In practice, this gradient system can be solved up to arbitrary machine precision using software packages (initialization can be done from the closed-form conversion of the moment parameter to the natural parameter for the continuous normal distribution). For example, one way to solve the gradient system is by using the technique described in [52] that we summarize as follows:
First, let us choose the following canonical parameterization of the densities of an exponential family:

\[ p_\psi(x) := \exp \left( -\sum_{i=0}^{D} \psi_i t_i(x) \right). \]

That is, \( \psi_0 = F(\psi) \) and \( \psi_i = -\xi_i \) for \( i \in \{1, \ldots, D\} \) (i.e., parameter \( \psi \) is an augmented natural parameter which includes the log-normalizer in its first coefficient).

Let \( K_i(\psi) := E_{p_\psi}[t_i(x)] = \eta_i \) denote the set of \( D + 1 \) non-linear equations for \( i \in \{0, \ldots, D\} \). The method of [52] converts iteratively \( p^0 \) to \( p_\psi \). We initialize \( \psi(0) \) and calculate numerically \( \psi(0)_0 = F(\psi(0)) \).

At iteration \( t \) with current estimate \( \psi(t) \), we use the following first-order Taylor approximation:

\[ K_i(\psi) \approx K_i(\psi(t)) + (\psi - \psi(t)) \nabla K_i(\psi(t)). \]

Let \( H(\psi) \) denote the \( (D + 1) \times (D + 1) \) matrix:

\[ H(\psi) := \left[ \frac{\partial K_i(\psi)}{\partial \psi_j} \right]_{ij}. \]

We have

\[ H_{ij}(\psi) = H_{ji}(\psi) = -E_{p_\psi}[t_i(x)t_j(x)]. \]  \( (16) \)

We update as follows:

\[ \psi(t+1) = \psi(t) + H^{-1}(\psi(t)) \left[ \begin{array}{c} \eta_0 - K_0(\psi(t)) \\ \vdots \\ \eta_D - K_D(\psi(t)) \end{array} \right]. \]  \( (17) \)

When implementing this method, we need to approximate \( H_{ij} \) of Eq. 16 using the theta ellipsoid points. For \( d \)-variate discrete normal distributions with \( D = \frac{d(d+3)}{2} \), we have \( t_i(x) = x_1, \ldots, t_d(x) = x_d, t_{d+1}(x) = -\frac{1}{2}x_1x_1, t_{d+2}(x) = -\frac{1}{2}x_1x_2, \ldots, t_D(x) = -\frac{1}{2}x_dx_d \).

3.2 Some illustrating numerical examples

To compute numerically the theta functions and its derivatives, we may use the following software packages (available in various programming languages): abelfunctions in SAGE [16], algcurves in Maple® [17], Theta in Python [9], Riemann of jTEM (Java Tools for Experimental Mathematics) in Java [23] (see also [16]), or Theta.jl in Julia [3].

For our experiments, we used Java™ (in-house implementation) and Julia (with the package Theta.jl [3]). We consider the following two discrete normal distributions \( p_\xi \) and \( p_{\xi'} \) with the following parameters:

\[ \xi = \left( (-0.2, -0.2), \text{diag}(0.1, 0.2) \right), \]
\[ \xi' = \left( (0.2, 0.2), \text{diag}(0.15, 0.25) \right). \]

These bivariate discrete normal distributions are plotted in Figure 4.
Figure 4: Two bivariate discrete normal distributions used to calculate statistical divergences.

We implemented the statistical divergences between discrete normal distributions using an in-house Java™ software and Julia Theta.jl[3] package (see Appendix A for a code snippet).

For the above discrete normal distributions, we calculated:

\[
D_{\text{Bhattacharyya}}[p_\xi, p_{\xi'}] = \frac{1}{2} D_{\text{KL}}[p_\xi : p_{\xi'}] \simeq 1.626,
\]

and approximated the KL divergence by the Rényi divergence for \(\alpha_{\text{KL}} = 1 - 10^{-5} = 0.99999:\)

\[
D_{\text{KL}}[p_\xi : p_{\xi'}] \simeq D_{\alpha_{\text{KL}}}[p_\xi : p_{\xi'}] = \frac{1}{1 - \alpha_{\text{KL}}} J_{F,\alpha_{\text{KL}}} (\xi : \xi') \simeq 7.84.
\]

Implementing these formula required to calculate \(F(\xi),\) i.e., to evaluate the logarithm of theta functions. The following section describes another efficient method based on a projective divergence, i.e., a divergence which does not require pmfs to be normalized.

3.3 Approximating the Kullback-Leibler divergence via projective \(\gamma\)-divergences

The \(\gamma\)-divergences \[20\] \[12\] between two pmfs \(p(x)\) and \(q(x)\) defined over the support \(\mathcal{X}\) for a real \(\gamma > 1\) is defined by:

\[
D_\gamma[p : q] := \frac{1}{\gamma - 1} \log \left( \frac{\left( \sum_{x \in \mathcal{X}} p^\gamma(x) \right) \left( \sum_{x \in \mathcal{X}} q^{\gamma}(x) \right)^{\gamma - 1}}{\left( \sum_{x \in \mathcal{X}} p(x) q^{\gamma - 1}(x) \right)^{\gamma}} \right), \quad (\gamma > 1).
\]

The \(\gamma\)-divergences are projective divergences, i.e., they satisfy the following identity:

\[
D_\gamma[p : p'] = D_\gamma[\lambda p : \lambda' p'], \quad (\forall \lambda, \lambda' > 0).
\]

Thus let us rewrite \(p(x) = \tilde{p}(x) \frac{Z_p}{Z_{\tilde{p}}}\) and \(q(x) = \tilde{q}(x) \frac{Z_q}{Z_{\tilde{q}}}\) where \(\tilde{p}(x)\) and \(\tilde{q}(x)\) are computationally tractable unnormalized pmfs, and \(Z_p\) and \(Z_q\) their respective computationally intractable normalizers. Then we have

\[
D_\gamma[p : p'] = D_\gamma[\tilde{p} : \tilde{p'}].
\]
Let us define
\[ I_\gamma[p : q] := \sum_{x \in \mathcal{X}} p(x) q(x)^{\gamma - 1}. \]

Then the $\gamma$-divergence can be written as:
\[ D_\gamma[p : q] = D_\gamma[\tilde{p} : \tilde{q}] = \frac{1}{\gamma(\gamma - 1)} \log \left( \frac{I_\gamma[\tilde{p} : \tilde{q}]^{\gamma - 1}}{I_\gamma[\tilde{p} : \tilde{q}]^\gamma} \right). \]

Consider $p = p_\xi$ and $q = p_{\xi'}$ two pmfs belonging to the lattice Gaussian exponential family, and let
\[ \tilde{I}_\gamma(\xi : \xi') = I_\gamma[\tilde{p}_\xi : \tilde{p}_{\xi'}]. \]

Provided that $\xi + (\gamma - 1)\xi' \in \Xi$, we have following the proof of Proposition 2 that
\[ \tilde{I}_\gamma(\xi : \xi') = \exp(F_\Lambda(\xi + (\gamma - 1)\xi')) \frac{p_\xi + (\gamma - 1)\xi'}{\sum_{\xi} p_\xi + (\gamma - 1)\xi'}, \]

where $F_\Lambda(\xi) = \log \theta_\Lambda(\xi)$ denotes the cumulant function of the Gaussian distributions on lattice $\Lambda$. That is, we have
\[ \tilde{I}_\gamma(\xi : \xi') = \theta_\Lambda(\xi + (\gamma - 1)\xi'), \]

and therefore, we can express the $\gamma$-divergences as
\[ D_\gamma[p_\xi : p_{\xi'}] = \frac{1}{\gamma(\gamma - 1)} \log \left( \frac{\theta_\Lambda(\gamma\xi) \theta_\Lambda(\gamma\xi')^{\gamma - 1}}{\theta_\Lambda(\xi + (\gamma - 1)\xi')^\gamma} \right). \]  

Notice that the exact values of the infinite summations $\tilde{I}_\gamma(\xi : \xi')$ depend on the Riemannian theta function.

Now, the $\gamma$-divergences tend asymptotically to the Kullback-Leibler divergence between normalized densities when $\gamma \to 1$ [20, 12]: \[ \lim_{\gamma \to 1} D_\gamma[\tilde{p} : \tilde{q}] = D_{\text{KL}}[\tilde{p}_\xi : \tilde{q}_{\xi'}]. \]

Let us notice that the KLD is not a projective divergence, and that for small enough $\gamma > 1$, we have $\xi + (\gamma - 1)\xi'$ always falling inside the natural parameter space $\Xi$. Moreover, we can approximate the infinite summation using a finite region of integer lattice points $R_{\xi,\xi'}$:
\[ \tilde{I}_{\gamma, R_{\xi,\xi'}}(\xi : \xi') := \sum_{x \in R_{\xi,\xi'}} \tilde{p}_\xi \tilde{p}_{\xi'}(x)^\gamma. \]

For example, we can use the theta ellipsoids [16] $E_\xi$ and $E_{\xi'}$ used to approximate $\theta(\xi)$ and $\theta(\xi')$, respectively (Figure 3): We choose $R_{\xi,\xi'} = (E_\xi \cup E_{\xi'}) \cap \mathbb{Z}^d$. In practice, this approximation of the $I_\gamma$ summations scales well in high dimensions. Overall, we get our approximation of the KLD between two lattice Gaussian distributions summarized in the following proposition:
Proposition 9 The Kullback-Leibler divergence between two lattice Gaussian distributions \( p_\xi \) and \( p_{\xi'} \) can be efficiently approximated:

\[
D_{\text{KL}}[p_\xi : p_{\xi'}] \approx D_\gamma[p_\xi : p_{\xi'}] = \frac{1}{\gamma(\gamma - 1)} \log \left( \frac{\bar{I}_{\gamma, R_\xi}(\xi : \xi') \bar{I}_{\gamma, R_{\xi'}}(\xi' : \xi')^{-1}}{\bar{I}_{\gamma, R_\xi}(\xi' : \xi')^{\gamma-1}} \right),
\]

for \( \gamma > 1 \) close to 1 (say, \( \gamma = 1 + 10^{-5} \)), where \( R_\xi \) and \( R_{\xi'} \) denote the integer lattice points falling inside the theta ellipsoids \( E_\xi \) and \( E_{\xi'} \) used to approximate the theta functions \( \theta_\Lambda(\xi) \) and \( \theta_\Lambda(\xi') \), respectively.

Table 2 summarizes the various closed-formula obtained for the statistical divergences between lattice Gaussian distributions considered in this paper.

Other statistical divergences like the projective H"older divergences \([42]\) between lattice Gaussian distributions can be obtained similarly in closed-form:

\[
D_{\alpha,\gamma}^{\text{H"older}}[r : s] := \log \left( \frac{\sum_{x \in X} r(x)^{\gamma/\alpha} s(x)^{\gamma/\beta}}{r(x)^{1/\alpha} s(x)^{1/\beta}} \right), \quad \gamma > 0, \frac{1}{\alpha} + \frac{1}{\beta} = 1
\]
The Hölder divergences include the Cauchy-Schwarz divergence \[25\] for \(\gamma = \alpha = \beta = 2:\)

\[
D_{CS}[r : s] := -\log \frac{\sum_{x \in X} r(x)s(x)}{\sqrt{\left(\sum_{x \in X} r^2(x)\right)\left(\sum_{x \in X} s^2(x)\right)}}.
\]

Since the natural parameter space \(\Xi\) is a cone \[42\], we get:

\[
D_{H,\gamma}^{\alpha,\gamma}[p_\xi : p_{\xi'}] = \left| \log \frac{\theta_\Lambda(\gamma \xi)\frac{1}{\gamma} \theta_\Lambda(\gamma \xi')}{\theta_\Lambda(\frac{\alpha}{\gamma} \xi + \frac{\beta}{\gamma} \xi')} \right|.
\]

Thus we get the following closed-form for the Cauchy-Schwarz divergence between two lattice Gaussian distributions:

\[
D_{CS}[p_\xi : p_{\xi'}] = \log \frac{\sqrt{\theta_\Lambda(2\xi)\theta_\Lambda(2\xi')}}{\theta_\Lambda(\xi + \xi')}.
\]

**References**

[1] Divesh Aggarwal, Daniel Dadush, Oded Regev, and Noah Stephens-Davidowitz. Solving the shortest vector problem in \(2^n\) time using discrete Gaussian sampling. In *Proceedings of the forty-seventh annual ACM symposium on Theory of computing*, pages 733–742, 2015.

[2] Daniele Agostini and Carlos Amédola. Discrete Gaussian distributions via theta functions. *SIAM Journal on Applied Algebra and Geometry*, 3(1):1–30, 2019.

[3] Daniele Agostini and Lynn Chua. Computing theta functions with Julia. *Journal of Software for Algebra and Geometry*, 11(1):41–51, 2021.

[4] Shun-ichi Amari. *Information geometry and its applications*, volume 194. Springer, 2016.

[5] Arindam Banerjee, Srujana Merugu, Inderjit S Dhillon, and Joydeep Ghosh. Clustering with Bregman divergences. *Journal of machine learning research*, 6(10), 2005.

[6] Ole Barndorff-Nielsen. *Information and exponential families in statistical theory*. John Wiley & Sons, 2014.

[7] Alessandro Budroni and Igor Semaev. New Public-Key Crypto-System EHT. *arXiv preprint arXiv:2103.01147*, 2021.

[8] Clément L Canonne, Gautam Kamath, and Thomas Steinke. The discrete Gaussian for differential privacy. *arXiv preprint arXiv:2004.00010*, 2020.

[9] S Carrazza and D Krefl. Theta: A Python library for Riemann-Theta function based machine learning. *https://doi.org/10.5281/zenodo.1120325*

[10] Stefano Carrazza and Daniel Krefl. Sampling the Riemann-Theta Boltzmann machine. *Computer Physics Communications*, 256:107464, 2020.

[11] N. N. Čencov. Algebraic foundation of mathematical statistics. *Statistics: A Journal of Theoretical and Applied Statistics*, 9(2):267–276, 1978.
[12] Andrzej Cichocki and Shun-ichi Amari. Families of alpha-beta-and gamma-divergences: Flexible and robust measures of similarities. *Entropy*, 12(6):1532–1568, 2010.

[13] Thomas M Cover. *Elements of information theory*. John Wiley & Sons, 1999.

[14] Jason V. Davis and Inderjit Dhillon. Differential entropic clustering of multivariate gaussians. *Advances in Neural Information Processing Systems*, 19:337, 2007.

[15] Robin De Jong. Theta functions on the theta divisor. *The Rocky Mountain Journal of Mathematics*, pages 155–176, 2010.

[16] Bernard Deconinck, Matthias Heil, Alexander Bobenko, Mark Van Hoeij, and Marcus Schmies. Computing Riemann theta functions. *Mathematics of Computation*, 73(247):1417–1442, 2004.

[17] Bernard Deconinck and Matthew S Patterson. Computing with plane algebraic curves and Riemann surfaces: the algorithms of the Maple package “algcurves”. In *Computational approach to Riemann surfaces*, pages 67–123. Springer, 2011.

[18] Bernard Deconinck and Mark Van Hoeij. Computing Riemann matrices of algebraic curves. *Physica D: Nonlinear Phenomena*, 152:28–46, 2001.

[19] Jörg Frauendiener, Carine Jaber, and Christian Klein. Efficient computation of multidimensional theta functions. *Journal of Geometry and Physics*, 141:147–158, 2019.

[20] Hironori Fujisawa and Shinto Eguchi. Robust parameter estimation with a small bias against heavy contamination. *Journal of Multivariate Analysis*, 99(9):2053–2081, 2008.

[21] Vincent Garcia and Frank Nielsen. Simplification and hierarchical representations of mixtures of exponential families. *Signal Processing*, 90(12):3197–3212, 2010.

[22] Craig Gentry, Chris Peikert, and Vinod Vaikuntanathan. Trapdoors for hard lattices and new cryptographic constructions. In *Proceedings of the fortieth annual ACM symposium on Theory of computing*, pages 197–206, 2008.

[23] Tim Hoffmann and Markus Schmies. jReality, jtem, and oorange—a way to do math with computers. In *International Congress on Mathematical Software*, pages 74–85. Springer, 2006.

[24] Harold Hotelling. Spaces of statistical parameters. *Bull. Amer. Math. Soc*, 36:191, 1930. First mention hyperbolic geometry for Fisher-Rao metric of location-scale family.

[25] Robert Jenssen, Jose C Principe, Deniz Erdogmus, and Torbjørn Eltoft. The Cauchy–Schwarz divergence and Parzen windowing: Connections to graph theory and Mercer kernels. *Journal of the Franklin Institute*, 343(6):614–629, 2006.

[26] Angshuman Karmakar, Sujoy Sinha Roy, Oscar Reparaz, Frederik Vercauteren, and Ingrid Verbauwhede. Constant-time discrete Gaussian sampling. *IEEE Transactions on Computers*, 67(11):1561–1571, 2018.

[27] Adrienne W Kemp. Characterizations of a discrete normal distribution. *Journal of Statistical Planning and Inference*, 63(2):223–229, 1997.
[28] Cong Ling and Jean-Claude Belfiore. Achieving AWGN channel capacity with lattice Gaussian coding. *IEEE Transactions on Information Theory*, 60(10):5918–5929, 2014.

[29] JHC Lisman and MCA Van Zuyl. Note on the generation of most probable frequency distributions. *Statistica Neerlandica*, 26(1):19–23, 1972.

[30] J Navarro and JM Ruiz. A note on the discrete normal distribution. *Advances and Applications in Statistics*, 5(2):229–245, 2005.

[31] Eric Nichols and Christopher Raphael. Automatic transcription of music audio through continuous parameter tracking. In *International Society for Music Information Retrieval (ISMIR)*, pages 387–392, 2007.

[32] Frank Nielsen. An information-geometric characterization of Chernoff information. *IEEE Signal Processing Letters*, 20(3):269–272, 2013.

[33] Frank Nielsen. On the Jensen–Shannon symmetrization of distances relying on abstract means. *Entropy*, 21(5):485, 2019.

[34] Frank Nielsen. An elementary introduction to information geometry. *Entropy*, 22(10):1100, 2020.

[35] Frank Nielsen. The Siegel–Klein Disk: Hilbert Geometry of the Siegel Disk Domain. *Entropy*, 22(9):1019, 2020.

[36] Frank Nielsen. On a Variational Definition for the Jensen-Shannon Symmetrization of Distances Based on the Information Radius. *Entropy*, 23(4):464, 2021.

[37] Frank Nielsen and Sylvain Boltz. The Burbea-Rao and Bhattacharyya centroids. *IEEE Transactions on Information Theory*, 57(8):5455–5466, 2011.

[38] Frank Nielsen and Vincent Garcia. Statistical exponential families: A digest with flash cards. *arXiv preprint arXiv:0911.4863*, 2009.

[39] Frank Nielsen and Richard Nock. Entropies and cross-entropies of exponential families. In *2010 IEEE International Conference on Image Processing*, pages 3621–3624. IEEE, 2010.

[40] Frank Nielsen and Richard Nock. A closed-form expression for the Sharma–Mittal entropy of exponential families. *Journal of Physics A: Mathematical and Theoretical*, 45(3):032003, 2011.

[41] Frank Nielsen and Richard Nock. On Rényi and Tsallis entropies and divergences for exponential families. *arXiv preprint arXiv:1105.3259*, 2011.

[42] Frank Nielsen, Ke Sun, and Stéphane Marchand-Maillet. On Hölder projective divergences. *Entropy*, 19(3):122, 2017.

[43] Frank WJ Olver, Daniel W Lozier, Ronald F Boisvert, and Charles W Clark. *NIST handbook of mathematical functions*. Cambridge university press, 2010.

[44] Dilip Roy. The discrete normal distribution. *Communications in Statistics-theory and Methods*, 32(10):1871–1883, 2003.
A Code snippet in Julia

The Julia language can be freely downloaded from [https://julialang.org/](https://julialang.org/).

The ellipsoids used to approximate the theta function $\theta_R$ are stored in the `RiemannMatrix` structure of the `Theta.jl` Julia package:

```
M = [0.1 0; 0 0.2];
v1 = [-0.2; -0.2];
```

Executing the code below gives the following result:

```
julia> BhattacharyyaDistance(v1,M1,v2,M2)
1.6259948590224578
```

```
julia> KLDivergence(v1,M1,v2,M2)
7.841371347366552
```

# in Julia 1.4.2
using Theta

```
M1=[0.1 0; 0 0.2];
v1 = [-0.2; -0.2];
```
M2 = [0.15 0; 0 0.25];
v2 = [0.2; 0.2];

# cumulant function of the discrete normal family
function F(v, M)
    R = RiemannMatrix(im*M);
    log(real(theta(-im*v, R)))
end

# Renyi divergence between two discrete normal distributions
function RenyiDivergence(alpha, v1, M1, v2, M2)
    M12 = alpha*M1 + (1-alpha)*M2;
    v12 = alpha*v1 + (1-alpha)*v2;
    (1/(1-alpha)) * (alpha*F(v1, M1) + (1-alpha)*F(v2, M2) - F(v12, M12))
end

function BhattacharyyaDistance(v1, M1, v2, M2)
    (1/2)*RenyiDivergence(1/2, v1, M1, v2, M2)
end

function KLDivergence(v1, M1, v2, M2)
    alpha = 0.9999999999;
    RenyiDivergence(alpha, v1, M1, v2, M2)
end

BhattacharyyaDistance(v1, M1, v2, M2)
KLDivergence(v1, M1, v2, M2)