Weyl points and topological surface states in a three-dimensional sandwich-type elastic lattice

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Abstract
Following the realization of Weyl semimetals in quantum electronic materials, classical wave analogues of Weyl materials have also been theorized and experimentally demonstrated in photonics and acoustics. Weyl points in elastic systems, however, have been a much more recent discovery. In this study, we report on the design of an elastic fully-continuum three-dimensional material that, while offering structural and load-bearing functionalities, is also capable of Weyl degeneracies and surface topologically-protected modes in a way completely analogous to its quantum mechanical counterpart. The topological characteristics of the lattice are obtained by ab initio numerical calculations without employing any further simplifications. The results clearly characterize the topological structure of the Weyl points and are in full agreement with the expectations of surface topological modes. Finally, full field numerical simulations are used to confirm the existence of surface states and to illustrate their extreme robustness towards lattice disorder and defects.

1. Introduction
The search for novel topological states of matter has recently seen many exciting breakthroughs either in quantum or classical wave physics [1–6]. Phenomena such as the quantum Hall effect (QHE), the quantum spin Hall effect (QSHE), and the quantum valley Hall effect (QVHE) have been studied extensively and their nontrivial topological characteristics have been clearly connected to their ability to support back-scattering-immune topological edge states [1, 2]. Although the concept of topological material was discovered and developed in the field of condensed matter physics, the many conceptual and mathematical similarities between different fields of wave physics have led to the development of analogue concepts of topological materials in electromagnetic [3, 4], acoustic [5–10], and elastic [11–19] systems. Recent studies have identified the existence of the so-called Weyl semimetals as three dimensional nontrivial topological materials capable of unidirectional back-scattering-protected surface states [20].

Weyl semimetals are quantum materials whose behavior is characterized in terms of the Weyl Hamiltonian $H (k) = \nu_1 k_x \sigma_x + \nu_2 k_y \sigma_y + \nu_3 k_z \sigma_z$, where $\nu_i$, $k_i$, and $\sigma_i$ represent the components of the group velocity, momentum, and Pauli matrix, respectively. These materials can be thought of as a three-dimensional extension of two-dimensional nontrivial topological materials characterized by Dirac degeneracies such as, for example, systems exhibiting QHE, QSHE, and QVHE [1, 2]. Similar to 2D Dirac materials, Weyl semimetals possess a degenerate nodal point formed by linear intersection of two bands in the three coordinate directions of the three dimensional reciprocal space. This degenerate point, called the Weyl point, carries a nonzero topological charge which confers the material nontrivial topological properties. The charge (i.e. the integral of the Berry curvature flux in a 2D manifold enclosing the Weyl point) can have either a positive or negative sign, hence determining if the degeneracy acts as a quantized source or a sink of Berry curvature flux. The total topological charge associated with the Brillouin zone
Weyl points can be qualitatively interpreted as an extension of Dirac points (typical of 2D lattices) to three-dimensional lattices. However, there are some fundamental qualitative differences that have important implications on both the occurrence and robustness of these degeneracies. Dirac degeneracies are usually classified in either accidental or essential degeneracies. The accidental degeneracies occur as a result of a specific combination of either material or geometric parameters [24, 25]. Accidental degeneracies are lifted as soon as any of these parameters is slightly perturbed from its reference value. On the other hand, the existence of essential Dirac points can be guaranteed purely based on symmetry arguments. This means that lattice structures (such as, for example, graphene or graphene-like unit cells) that are capable of preserving simultaneously both parity-inversion (P) and time-reversal (T) symmetries are guaranteed to exhibit Dirac degeneracies. Such degeneracies cannot be lifted by simply perturbing either geometric or material parameters (like in the case of accidental Dirac degeneracies) but require breaking either P or T-symmetry. Once the degeneracy is lifted, the resulting bandgap is typically topologically non-trivial. While the conditions for the occurrence of Dirac degeneracies in 2D materials are clearly defined, an equivalent set of conditions for the occurrence of Weyl points cannot be easily identified. It follows that the occurrence of these 3D degeneracies cannot be guaranteed a priori exclusively based on symmetry constraints. On the other hand, we can establish a condition under which Weyl points can certainly not exist. For Weyl points, the satisfaction of P and T-symmetries impose contradicting requirements. Assuming a lattice exhibiting a Weyl degeneracy at the \( \mathbf{k} \) high-symmetry point in reciprocal space, T-symmetry would require the topological charges at \( \mathbf{k} \) and \(-\mathbf{k}\) to be of the same sign, while P-symmetry would require them to be opposite. Due to these contradicting requirements, it follows that Weyl points can only be obtained in systems with broken PT-symmetry. A further consequence of this observation is that Weyl degeneracies cannot be lifted by simply breaking symmetry (as is the case of Dirac degeneracies), instead they must be annihilated by combining pair of Weyl points with opposite charges. These more stringent requirements for lifting the Weyl degeneracy are at the foundation of their higher robustness against external perturbations, compared to Dirac points. Also, following the requirements dictated by both P and T-symmetries, the minimum number of Weyl points in a BZ is 4 (2 pairs) when P-symmetry is broken and 2 (1 pair) when T-symmetry is broken [3].

Note also that, although in the current study only Weyl points with unit charge are considered, other studies [26–34] have reported systems possessing Weyl points that have higher charge. They are typically caused by the overlapping of multiple Weyl points of unit charge. Such points can be readily identified from the band structure because the bands do not intersect linearly at these points in the \( k_x\), \( k_y\) plane (although they do intersect linearly along \( k_z\)). Recent studies, both in photonic and phononic systems, have also reported a new kind of Weyl point labeled type-II [35–40]. The degeneracies discussed in the present study are of type-I and represent the more direct analogue to the degeneracies originally studied by Weyl in quantum mechanical systems [41].

While first theorized as the massless solution of the Dirac equation [41], Weyl fermions could be realized and experimentally demonstrated in Weyl semimetals only recently [26–29, 42–46]. The possibility of identifying classical mechanical systems serving as analogue to the Weyl semimetals has opened an intriguing and fertile topic of research spanning many areas of wave physics. As a result, non-quantum-mechanical analogues of the Weyl points were formulated both in photonic [30, 32, 36–38, 47–53] and acoustic [33, 34, 39, 54–56] metamaterial systems. Weyl points were realized and experimentally demonstrated in photonics using double-gyroid structure characterized by broken P-symmetry [47, 48]. Although capable of realizing Weyl points, the double-gyroid structure involves a complex fabrication. Simpler designs based, as an example, on woodpile photonic crystals [32, 52] or on rotated stacked rods in acoustic systems [33, 34] were also explored. A planar fabrication methodology, which outlines a step by step approach to obtain Weyl points, was also presented [30, 51].

Despite the successful implementation of Weyl points in photonic and acoustic systems, the realization of Weyl points in elastic systems has proven to be more challenging than other classical analogues. Acoustic systems support only longitudinal waves and hence are immune to wave coupling and mode conversion issues, thereby making the task of achieving complete bandgaps easier. While wave coupling and mode conversion between different polarizations occur also in photonic systems (characterized by two transverse modes), the phenomenon is still more challenging in elastic systems given the existence of three possible polarizations; two quasi-shear and one quasi-longitudinal. The quasi-longitudinal elastic mode is acoustically fast (compared to the shear modes) and it does not easily allow for a full topological bandgap (that is a bandgap affecting all modes). While this is not necessarily a problem per se, because a partial
bandgap affecting only shear modes can still exhibit topological properties, it does require additional care in
decoupling longitudinal and shear modes so as to avoid mode conversion from longitudinal to shear. This
latter conversion might corrupt either the bandgap or the topological properties. Mode conversion can be
counteracted by choosing unit cell geometries that reduces mode coupling; this condition typically imposes
constraints on the out-of-plane symmetry of the unit cell. Another complexity in the realization of Weyl
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constraints on the out-of-plane symmetry of the unit cell. Another complexity in the realization of Weyl
elastic systems is associated with fabrication capabilities. To-date, very few elastic systems have actually
reported observation of Weyl points [31, 53, 57]. The study by Wang et al [31] made use of beams and thin
plates with a particular geometry purposely selected to be realized by additive manufacturing. Another
study by Shi et al [57] utilized a truss-like construction made of beam elements. Although elegant and
simple to fabricate, these designs are discrete in nature therefore not well suited for applications where the
load-bearing capabilities in presence of uniformly distributed loads play a critical role (e.g. applications to
structural materials needed to carry aero- or fluid-dynamic loads). In addition, in both the above
mentioned studies, the topological characteristics were obtained based on a tight-binding (TB) formulation
of the Hamiltonian. This approach works well only when the fundamental lattice is composed of weakly
coupled resonant elements. Further, the hopping parameters used in the TB model usually cannot be clearly
connected to the actual geometric and design parameters. For strongly coupled or fully continuous systems,
the TB model is no longer valid. The premise behind the TB model is that the elastic wave is strongly
localized near the periodic resonant sites and only weakly coupled with neighboring ones. Hence, the entire
wavefunction can be expressed as an expansion of eigenmodes on these neighboring sites, and the frequency
is obtained by using a perturbation approach applied in the neighborhood of a single resonant site
frequency. However, in cases involving strong coupling, presence of long-range interactions, and fully
continuous systems, the inference from such perturbation approach cannot hold. We note that, although in
principle one can still build an analogous TB model and obtain the parameters by matching the dispersion
characteristics so as to eventually obtain the effective Hamiltonian, our recent study [18] has shown that
some properties cannot be captured by such low-order approximation, and that direct numerical models
are needed to accurately capture the system’s behavior. Another implementation of an elastic Weyl material
was based on double-gyroid structures [53]. Aside from the fabrication complexity, this design used soft
matter and depended on the application of an external strain field in order to break the symmetry and
obtain the Weyl points. In the same study, the authors employ a direct numerical method to evaluate the
topological charge of the photonic Weyl point. The approach is based on the evaluation of a loop integral of
the Berry connection. The integral that yields the topological charge is undoubtedly gauge invariant.

However, due to the intrinsic phase ambiguity of the Bloch state in momentum space, the Berry connection
field per se is gauge dependent.

In this regard, we propose an elastic analogue of a Weyl semimetal based on a fully continuous
and load-bearing design. Our material is inspired to sandwich composite structures [58] where the classical
honeycomb core is replaced by a network of trusses that satisfies specific symmetry conditions. While the
symmetry of the unit cell is analogous to the design by Xiao et al [55], their system is purely acoustic and
cannot control elastic waves or, equivalently, is not a structural Weyl material.

The interest in structural Weyl materials stems from the need to complement the classical properties of
conventional structural material platforms (primarily their load-bearing capability) with advanced stress
wave control capabilities.

The underlying idea being that if the flow of mechanical energy through the system can be controlled
and energy can be channeled to selected locations, then this energy can be efficiently extracted for either
dissipation or harvesting purposes. The extraction system (e.g. damping materials or transducers) would
implicitly be optimized hence requiring a minimum amount of added components, weight, and overall
complexity.

More specifically, the load-carrying capacity combined with the robust topological surface states for the
control of elastic waves make our design very well suited for vibration control and stress wave shielding. A
variety of applications can be envisioned where this control capability could be of particular interest
including vibration control in aerospace, mechanical, and naval systems. In these applications, distributed
loads must be efficiently carried by the structure without compromising its integrity. At the same time,
these loads inject large amount of mechanical energy that could give rise to high level responses, hence
requiring efficient damping approaches. Other relevant applications include seismic damage prevention
(e.g. in-ground seismic isolation mechanisms or insulating platforms), and insulating pads for heavy
machinery. Applications to structure-radiated noise could also be envisioned. Note also that, once the
vibrational energy is channeled, it could also be efficiently harvested for low power electronics [59, 60].

Since the Weyl points and surface states in this study are purely a result of symmetry breaking, the design
can be scaled or replicated by using different materials in order to achieve the required load-bearing
capacity and the target operating frequency range for each specific application. The topological surface
states are also robust in the sense that as long as the symmetry requirements are met, any amounts of load or deformations in the linear regime do not affect their existence. This further supports their application as load-bearing structures.

We will also present a detailed study of the topological characteristics based on \textit{ab initio} calculations, hence bypassing the limitations of TB approaches when dealing with strongly coupled systems. In our approach, the topological charge is obtained by direct numerical calculation of the surface integral of the gauge invariant Berry curvature flux on a closed manifold enclosing the Weyl point. By calculating and visualizing the Berry curvature on selective planes in momentum space, this approach also provides an intuitive picture of the topological charge (given that the Berry curvature can be considered as the flux emitted by the charge seen as a monopole source).

This paper is organized as follows: section 2 will introduce the proposed design and the corresponding dispersion properties in connection with Weyl points. Section 3 will present \textit{ab initio} calculations of the topological invariant. Section 4 will focus on the dispersion behavior of a supercell and the corresponding dispersion structure, in order to determine the occurrence of surface states. Finally, section 5 will show full field simulations of the elastic Weyl material in order to illustrate the one-way, backscattering-immune nature of the surface states.

2. Synthesis of the unit cell and dispersion properties

From a general perspective, in order to develop the fundamental unit cell, we select a 2D triangular lattice (assumed in the $xy$-plane) and build the 3D geometry by periodically repeating this unit along the $z$-direction. The sandwich design contains a straight pillar located at the center of the unit cell, connecting the top and bottom skins that are essentially thin metal plates as shown in figure 1(a). The pillar is designed to take compressive and tensile loads and conceptually takes the role of a classical honeycomb core when the composite is subject to compression/tension. The metal plates are an important feature of the structural design because they allow the whole material to carry distributed pressure loads, such as those possibly arising from aero- and fluid-dynamic loads typical of many aerospace, naval, and even civil engineering applications. Note that this latter capability is in contrast with most of the elastic Weyl material designs available in the literature that are mostly based on reticular or discrete geometries with no ability to carry pressure loads. The final result is a layered 3D structure having intact $P$ and $T$-symmetries. Note that, as previously mentioned, starting from a 2D triangular lattice guarantees (due to symmetry arguments) the existence of Dirac degeneracy at the high symmetry points in the $k_x$–$k_y$ momentum space. Periodically repeating this unit in the $z$-direction extends the Dirac degeneracy at virtually any $k_z$ value, thereby generating a line node degeneracy \cite{3, 51} along $k_z$. In order to lift this three-dimensional degeneracy, $P$-symmetry can be broken by the proper introduction of additional structural element that do not respect inversion symmetry. For this purpose, we select slanted beam elements connecting the vertices of the hexagonal structure on two adjacent layers as shown in figure 3. The slanted nature of these beams introduces a chiral coupling, that is a preferential sense of rotation to the unit-cell that breaks the $P$-symmetry and all existing mirror symmetries. The most direct consequence is that the line degeneracy along $k_z$ is reduced to a point, hence giving rise to the Weyl point. Note that the slanted cylinders connecting top and bottom skins also allow carrying both bending loads and in-plane shear loads applied on the skins, hence strengthening the overall load-bearing capability.

In the following section, we first describe the geometry and the dynamic properties of the parity-preserving fundamental unit. Then, we introduce the chiral coupling and investigate its effect compared with the $P$-symmetric unit.
2.1. 3D lattice with intact $P$-symmetry: line node degeneracy

Figures 1(a) and (b) show the fundamental unit cell used to create the lattice. The cell, as mentioned earlier, is made of a vertical cylinder included between two homogeneous, hexagonal-shaped, thin plates. The cylinder has its longitudinal axis aligned with the $z$-direction. The cylinder is made of iron while the plates are made of aluminum. The selection of these materials was motivated by the intent of obtaining marked degeneracies at the high symmetry points in the momentum space. Clearly, several other combinations of structural materials leading to similar conditions could potentially be identified. The dimensions of the unit cell are: $a = 30 \, \text{mm}$, $h_0 = 2 \, \text{mm}$, $r_1 = 5 \, \text{mm}$, $h_1 = 20 \, \text{mm}$, where $a$ is the lattice constant, $h_0$ is the thickness of each homogeneous flat plate, $r_1$ and $h_1$ are the radius and height of the cylinder, respectively. A 2D lattice in the $x$–$y$ plane can be assembled by periodically repeating the unit cell. The final result is a lattice of cylinders in a hexagonal configuration sandwiched between two thin plates. This 2D lattice can be periodically repeated in the $z$-direction to form the final 3D lattice.

The analysis of the band structure of the 3D material can be conveniently performed by applying periodic boundary conditions to the fundamental unit cell. More specifically, periodic boundary conditions can be applied along the sideward faces of the plate in order to create the 2D lattice in the $xy$-plane, while periodic boundary conditions applied on the top and bottom faces of the top and bottom plates will yield the complete 3D lattice. The resulting system is characterized by intact $P$ and $T$-symmetries and the corresponding 3D Brillouin zone is shown in figure 1(c). For a fixed value of the momentum component $k_z$, the 3D BZ simplifies into a 2D BZ typical of a triangular lattice.

The dispersion curves along the boundary of the 2D BZ in the $k_x$--$k_y$ plane at $k_z = 0$, and in the $k_x$--$k_z$ plane at $k_y = 0$ have been calculated using the commercial finite element (FE) package COMSOL Multiphysics and are shown in figure 2. As visible in figure 2(a), four modes intersect linearly at a frequency of approximately 40 kHz and give rise to two degenerate points at $K$. These degenerate points are indicated by the red and green boxes in the inset of figure 2(a). It should be noted that the frequency at which this degeneracy occurs (in this case 40 kHz) is dependent on the geometric as well as the material parameters and would vary as the dimensions or the material is varied. From a high-level perspective, for constant $k_z$, degeneracy occurs (in this case 40 kHz) is dependent on the geometric as well as the material parameters.

The nodal degeneracies at the corner of the BZ (marked in red and green colors in figure 2(b)) exist for all values of $k_z$, thereby resulting in line degeneracies along the $K$--$H$ directions indicated by the red and green lines.

2.2. $P$-symmetry breaking and Weyl points

In order to break $P$-symmetry, a chiral coupling can be introduced by means of straight slanted cylinders connecting top and bottom layers in proximity of the vertices of the hexagonal layer. The resulting unit cell is shown in figure 3 while its 3D BZ remains the same as the one shown in figure 1(c). The slanted cylinders are made of aluminum (as the plates) and have a radius of $r_2 = 2 \, \text{mm}$. The dispersion curves along the boundary of 2D BZ in the $k_x$--$k_y$ plane at $k_z = 0$ and in the $k_x$--$k_z$ plane at $k_y = 0$ are calculated again and shown in figure 4. The degenerate points resulting from the linear intersection of two modes in figure 2(a) are also present in figure 4(a) (marked by the same red and green boxes). However, the line degeneracies...
Figure 3. (a) Schematics of the unit cell with slanted cylinders. The cylinders have radius $r_2$ and are made of aluminum. The resulting unit cell does not preserve $P$-symmetry and mirror symmetry. Periodic boundary conditions are used on the boundaries to create a 3D lattice. (b) Rendered view of a bulk piece of the 3D lattice.

Figure 4. Dispersion curves for the 3D lattice with broken $P$-symmetry. (a) Dispersion along the 2D BZ in the $k_x$–$k_y$ plane at $k_z = 0$. The nodal degeneracies from figure 2(a) are still present, despite a slight shift in frequency. (b) Dispersion curves of the same 3D lattice along the 2D BZ in the $k_z$–$k_y$ plane at $k_x = 0$. The line degeneracies in figure 2(b) are lifted (magnified image of the green modes is shown in the inset). The resulting nodal degeneracies at $K$ and $H$ are Weyl points, marked by red and green circles. (c) Dispersion curves along the 2D BZ in the $k_x$–$k_z$ plane at $k_y = 0$. The line degeneracies in figure 2(b) are lifted (magnified image of the green modes is shown in the inset). The resulting nodal degeneracies at $K$ and $H$ are Weyl points, marked by red and green circles. These degeneracies are the Weyl points that are formed by the linear intersection of the two (initially degenerate) modes along all three directions. Figure 4(a) shows the linear intersection in the $k_x$–$k_y$ plane while figure 4(b) shows the linear intersection in the $k_z$–$k_y$ plane.

As a result, the degeneracy marked in figure 4(a) is lifted for nonzero values of $k_z$ (with $k_z \neq n\pi/az$ for integer $n$) hence giving rise to a topological bandgap as shown in figure 4(c). The band structure shown in figure 4(c) corresponds to $k_z = \pm \frac{0.1\pi}{az}$, where $a_x = h_1 + 2h_0 = 24$ mm is the total height of unit cell. The subscripts $\pm$ of the high symmetry points’ labels indicate $k_z = \pm \frac{0.1\pi}{az}$, as also depicted in figure 1(c). The bandgap size varies with $k_z$ as shown in figure 4(b). Thus, $k_z$ can be treated as a parameter that controls either the opening or closing of the topologically nontrivial bandgap, provided that the system remains periodic along the $z$-direction.

3. Ab initio calculations of the topological properties

3.1. Topological charge

In order to assess the topological significance of the degeneracies identified in the chiral 3D lattice and, consequently, the existence of Weyl points, this section presents a numerical investigation into the calculation of the topological charge. The charge can be calculated by integrating the Berry curvature flux on a 2D manifold $S$ enclosing the Weyl point in the reciprocal space, that is

$$ C = \frac{1}{2\pi} \oint_S \Omega(k) \cdot dS $$

where $C$ is the topological monopole charge, i.e., the Chern number, $dS$ is the vector surface element aligned with its local normal direction, and $\Omega$ is the Berry curvature (vector field) given by

$$ \Omega(k) = \nabla_k \times \langle u(k) | i\nabla_k | u(k) \rangle $$

where $k$ is the wavevector, and $u_n(k)$ is the displacement eigenstate as a function of $k$. Consider the Weyl point at $K$, at a frequency of approximately 40 kHz, as indicated by the red dashed box in figure 4(a). This
Figure 5. The topological charge at the \( K \) point can be calculated by integrating the outward Berry curvature flux of a 2D manifold enclosing the \( K \) point. In this case, the surfaces of a triangular prism are chosen.

Figure 6. (a) and (b) The \( z \)-Berry curvature distribution of the 16th and 17th bands, respectively, on the \( k_z = 0.1 \pi/a \) plane. The two red dashed triangles in (a) indicate the top faces of the chosen prismatic manifolds enclosing the Weyl points, \( K \) or \( K' \) points, respectively. The total integral of the Berry curvature flux on the vertical side walls of the prism vanishes due to the \( C_6 \) and translation symmetry. (c) and (d) Zoomed-in view of the Berry curvature corresponding to the two triangular areas in (a). Each yields the outward flux integral of \( \pi \), which shows the Weyl points \( K \) and \( K' \) are of +1 Chern number.

Point corresponds to the intersection of the 16th and the 17th bands. To calculate the corresponding topological charge, we can integrate the Berry curvature flux threading the outer surface area (on the top, bottom, and side faces) of a fictitious prismatic manifold enclosing the \( K \) point (see figure 5). We anticipate, and show later, that the flux on the side faces is zero if a certain triangular prism is chosen as prototypical 2D manifold.

Without loss of generality, we can select \( k_z = \pm \frac{0.1 \pi}{a} \) as the top and bottom faces of the prism. At these planes, we have \( K_+ \) and \( K_- \) with the same \((k_x, k_y)\) coordinate as the \( K \) point. The \( z \)-component of the Berry curvature on the top plane is then calculated using equation (2) (see figures 6(a) and (b)) for the 16th and 17th bands, respectively. Note that, at the two distinct points (i.e. they cannot be transformed into one another by translation symmetry operations), \( K_+ \) and \( K'_+ \) points exhibit the same Berry curvature pattern. This is a direct result of the \( C_6 \) symmetry of the lattice. Therefore, if the 2D manifold is chosen to be the triangular prism with the right red dashed triangle in figure 6(a) as its top face, we can assure that the flux integral on the side faces of the prism is zero due to symmetry in the reciprocal space. In fact, the \( C_6 \) symmetry guarantees that the outward flux on the side face \( 1 \) equals the inward flux on the side face \( 2 \) the same cyclic symmetry along with the translation symmetry (from the left to the right, for example) forces the flux integral on the entire side surface \( 3 \) to vanish.

It follows that we only need to integrate the Berry curvature flux on the top and bottom faces. The bottom face which is around the \( K_- \) point, is actually the time reversed counter-part of the triangular face around the \( K'_+ \) point (inversion of \((k_x, k_y, k_z)\)) as shown in the left red dashed triangle in figure 6(a).
agreement with the degeneracies in pairs of points with opposite charge. Figure 7 shows all the Weyl points within the half BZ (in the $k_z$ face is oriented in the momentum space around the numerical approach, the topological charge at the $k_z$-symmetric nature of the Weyl points which requires the appearance of the monopole charge of $H$ and $K$ both with charge +1 (that is sources of Berry curvature), while on the $k_z = \pi/a_z$ plane, each of the two Weyl points (H and H') are of charge −1 (that is sinks of Berry curvature).

3.2. Bandgap–Chern number
An alternative way to obtain the topological charge of the Weyl point is to use a topological invariant called the bandgap–Chern number $C_g$. This invariant is obtained by summing the Chern numbers of all the bands below the gap of interest. In the following, we consider $k_z$ as a parameter and focus on the bandgap that forms in the $k_x$–$k_y$ plane. $C_g$ remains constant with varying $k_z$ except at $K$ and $H$ where the bandgap closes and reopens and $C_g$ experiences a sudden change in its value. This change is equal to the topological charge.

$T$-symmetry requires that $\Omega(-k) = -\Omega(k)$ which results in the $z$-component of the Berry curvature on the bottom face to have the same strength but opposite sign. Also, given that the dS element on the bottom face is oriented in the $-k_z$ direction, it still contributes a positive outward flux. As a result, the topological charge is the sum of the integrals of the Berry curvature calculated over the two red dashed triangles in figure 6(a).

The Bloch eigenmodes were solved numerically using the FE model. For this calculation, the momentum space around the $K_+$ point was spanned by using a mesh of grid points (having $(k_x, k_y)$ coordinates and placed on the top surfaces of the triangular prism) distributed according to an adaptive resolution scheme (that is, getting finer near the $K_+$ point). The eigenmodes returned by the FE analysis contained some uncontrollable phase ambiguities. This is not an issue as long as such phase ambiguities are smooth enough across different grid points, so that the derivative operators in equation (2) can be calculated. To ensure such smoothness, we pre-processed the eigenmodes obtained from the FE analysis with a $U(1)$ gauge transformation $|u(k)\rangle \rightarrow e^{-i\theta(k)}|u(k)\rangle$ throughout the surface; $\theta(k)$ is a reference phase of the $z$-component of the displacement associated with a reference point in the unit cell (in our simulations this point was chosen as one of the unit cell vertices). It follows that the resulting eigenmodes all have this reference point as phase datum; this approach provides sufficient smoothness on the selected surfaces. The $z$-component of the Berry curvature is then calculated based on equation (2), and the partial derivatives are obtained via central differences as,

\[
\partial_{k_x} |u(k_x, k_y)\rangle \rightarrow \frac{|u(k_x + \Delta k_x, k_y)\rangle - |u(k_x - \Delta k_x, k_y)\rangle}{2\Delta k_x},
\]

\[
\partial_{k_y} |u(k_x, k_y)\rangle \rightarrow \frac{|u(k_x, k_y + \Delta k_y)\rangle - |u(k_x, k_y - \Delta k_y)\rangle}{2\Delta k_y}.
\]

The Berry curvature integral on the selected surface $P$ is also replaced by the discrete summation throughout the $l$ grid points,

\[
\int_P \Omega_d dk_x dk_y \rightarrow \sum_l (\Omega_d \Delta k_x \Delta k_y).
\]
of the Weyl points on that plane because Weyl points act as either sources or sinks of Berry curvature flux [52]. This principle can be used to calculate the charge of the Weyl points in the band structure. Particularly, we focus on the degeneracies occurring at approximately 40 kHz. It is found that the Berry curvature integrals of the first 15 bands cancel each others, therefore \( C_g \) reduces to the integral of Berry curvature in the 2D BZ for the 16th mode only.

Figure 8 shows the plot of \( C_g \) as a function of \( k_x \) for the Weyl points at 40 kHz. It clearly shows that \( C_g \) has a value of \(-1\) for \( k_x < 0 \) and \(+1\) for \( k_x > 0 \) which results in a net difference \( \Delta C_g = +2 \). This is consistent with the previous analysis that showed the \( k_x = 0 \) plane to host two Weyl points of equal charge at \( \mathbf{K} \) and \( \mathbf{K}' \) due to intact \( T \)-symmetry. The two Weyl points at \( \mathbf{K} \) and \( \mathbf{K}' \) therefore have a topological charge of \(+1\) each. In the same figure, there is also a change in the value of \( C_g \) at \( k_x = \pm \pi/2 \). This indicates that the charge of the Weyl point at \( \mathbf{H} (\mathbf{H}') \) is \(-1\). Once again, this is expected and consistent with the fact that Weyl points always occur in pairs with opposite topological charge. The nonzero value of charge also confirms that the nodal degenerate points under consideration are in fact Weyl points with nontrivial topological significance. In light of the bulk-surface correspondence [2], a nonzero \( C_g \) indicates that topological surface modes can exist within the target bandgap while the magnitude of \( C_g \) indicates how many surface modes should be expected. In our specific example, \( C_g = \pm 1 \) for \( k_x \in (0, \pi/a) \) or \( k_x \in (-\pi/a, 0) \), respectively. Compared with trivial materials (such as vacuum), there is a \( k_x \)-locked net difference of \( \pm 1 \) in \( C_g \). Therefore, one surface mode should be expected either in case of positive or negative \( k_x \). The corresponding surface states propagate unidirectionally either in the clockwise or counter-clockwise direction along the structure border (depending on the sign of \( C_g \)), as shown later on in the numerical simulations. This result further confirms that the selected geometry leads to the formation of Weyl points and it is capable of supporting topological surface states.

We note that the nonzero Chern number and the Berry curvature distribution for, as an example, \( k_z = 0.1\pi/a \) (figure 6) is reminiscent of 2D topological materials manifesting quantum Hall effect (QHE) [61]. In QHE materials, \( T \)-symmetry is broken due to the use of an external field. In the present material, the chiral structure breaks the \( z \)-mirror symmetry therefore a mode propagating in the \( z \)-direction in the infinite lattice is expected to exhibit \( k_z \)-locked unidirectional surface states. In fact, at steady state, the role of temporal and the spatial (in this case \( z \)) variables can be interchanged. A similar idea was also presented in studies of 3D Floquet topological insulators [62]. This latter aspect will be further clarified in the next section.

4. Supercell dispersion and surface modes

Weyl points with opposite topological charges are connected by nontrivial topological surface states that exist only on the boundary of the medium. In this section, we consider the physical response of a periodic supercell made out of the 3D unit cell described above. Consider a supercell having a finite dimension of 38 units in the \( y \) direction and infinite dimensions (obtained by means of periodic boundary conditions) in the \( x \) and \( z \) directions. Figure 9(a) shows the actual domain used for the numerical calculations.

We focus the following analysis on the topological bandgap that develops at the approximate frequency of 40 kHz and that is marked by the red box in figure 4(c). Figure 9(b) shows the dispersion curves for the supercell at \( k_y = \frac{0.1\pi}{a} \). It is evident from the visual inspection of the dispersion that the supercell admits two additional modes, the surface modes, that exist within the topological bandgap. Bulk modes are in grey color while the surface modes are indicated in green and red. The green modes correspond to the lower
edge of the supercell while red modes correspond to upper edge. This correspondence is also reflected in the plot of the eigenvectors on the supercell shown in figures 9(c) and (d). The dispersion of the surface modes corresponding to the upper edge has a positive slope (i.e. positive group velocity), which means that these modes are expected to travel in the positive x-direction. Similarly, the surface modes defined on the lower edge have negative slope and therefore are expected to travel in the negative x-direction.

Further, it can be observed that the supercell in figure 9 is obtained by terminating the triangular lattice along the zigzag edge. However, triangular lattices exhibit also another edge type named the armchair edge. We explored the possible propagation and existence of topological modes also on this edge type. For this purpose, a finite size supercell terminated along the x-direction was developed, as shown in figure 10(a). The supercell extended indefinitely in the y and z directions (periodic boundary conditions were used to simulate the infinite size). The corresponding band structure at $k_z = \frac{\pi}{a}$ is shown in figure 10(b). The bulk modes are marked in grey color, the surface modes corresponding to the right edge of the supercell are indicated in green color, and those corresponding to the left edge are indicated in red color. The eigenvectors corresponding to the surface states on both the right and left edges are shown in figures 10(c) and (d), respectively. Similar to the zigzag edge, the dispersion of the surface modes corresponding to the left edge has a positive slope (i.e. positive group velocity), which means that these modes are expected to travel in the positive y-direction. The surface modes defined on the right edge have negative slope and therefore are expected to travel in the negative y-direction. Therefore, if we assemble an extended 2D domain in the plane x–y we would expect to see the surface mode to propagate in a clockwise direction along the boundary of the lattice.
Figure 11. Dispersion curves of the supercell shown in figure 9(a) for different values of the radius of the center cylinder. Values of \( r_1 \) equal to (a) 4 mm (b) 5 mm (c) 6 mm (d) 7 mm (e) 8 mm (f) 9 mm are considered. It is seen that the group velocity of the topological surface modes decreases with increasing \( r_1 \) and that the surface states evolve from a single-valley to an inter-valley dominated behavior.

For \( kz = \frac{0.1\pi}{a} \), the dispersion curves remain unaltered but the direction of propagation of the surface modes is reversed. The surface mode corresponding to the upper edge acquires a negative group velocity (propagation in the negative \( x \)-direction) while the surface mode corresponding to the lower edge acquires a positive group velocity (propagation in the positive \( x \)-direction). Similarly, the surface mode corresponding to the left edge acquires a negative group velocity (propagation in the negative \( y \)-direction) while the surface mode corresponding to the right edge acquires a positive group velocity (propagation in the positive \( y \)-direction). An extended 2D domain in the \( x-y \) plane would then show a wave propagating in the anti-clockwise sense along the boundaries.

The combined behavior illustrated above is well consistent with those of dynamical systems characterized by Weyl points and corresponding topologically nontrivial modes. Results also suggest that, upon injecting a wave into the system, the direction of propagation can be controlled by properly choosing the sign of \( k_z \). This latter aspect will be further clarified by means of full field simulations in the next section.

Another interesting feature of the proposed geometry is that the group velocity of the surface modes can be controlled simply by changing the radius of the center vertical cylinder \( r_1 \). Figure 11 shows the dispersion curve of the supercell (a) for various values of \( r_1 \). The group velocity of the surface modes decreases with increasing radius \( r_1 \) of the center cylinder. This reduction of the group velocity can be explained by observing that by increasing \( r_1 \) the homogenized mass density of the lattice increases, but without significantly increasing the rigidity. Hence, given that the eigenmode does not involve significant compressive or bending motion of the center cylinder, the surface modes tend to slow down. By tuning the radius \( r_1 \), the surface states can gradually evolve from the single valley mechanism (similar to the 2D model discussed by Raghu [61]) to the inter-valley mechanisms (similar to the model discussed in Kane [63]). In either case, the unidirectional gapless surface states connecting the lower and upper bulk bands are protected by the topological nature of the band structure and are guaranteed to be present. Note that, while a similar variation in the group velocity of the topological edge modes was earlier observed in 2D chiral materials [62], this tuning ability of the surface states in 3D lattice was never observed and documented in earlier studies.

5. Full field numerical simulations

Based on the 3D unit cell developed above, a complete domain in the \( x-y \) plane was also simulated in order to observe both the direction of propagation and the scattering behavior in presence of defects. The domain used is shown in figure 12(a). The domain is rectangular and finite along both the \( x \)- and \( y \)-directions. All
edges except the bottom one were set to free boundaries. The boundaries normal to the z-axis were assigned periodic boundary conditions so as to render the lattice infinite in this direction. Perfectly matched layers (PML) were used at the bottom edge to absorb the incoming wave and allow a clear assessment of the direction of propagation of the surface state when performing a steady state analysis (i.e. it avoids the surface mode to propagate all around the boundary and get back to the source).

The domain was excited by a harmonic point force acting in the z-direction at the location indicated by the red star marker in figure 12(a). A harmonic excitation at a frequency of 40 kHz was selected because it lies right within the target topological bandgap where surface states exist. At $k_z = \frac{n\pi}{L_z}$, a PML boundary was applied along the bottom edge of the domain to absorb the incoming wave in steady-state analyses. The wave was injected by means of a harmonic point force applied at the location marked by the red star at a frequency of 40 kHz. (b) Steady-state response showing the propagation of the surface state when excited with $k_z = \frac{0.1\pi}{L_z}$. 99.41% of power is injected into the right branch. (c) Steady-state response showing the propagation of the surface state when excited with $k_z = \frac{-0.1\pi}{L_z}$. These results clearly show the unidirectional nature of the topological surface modes. 99.93% of power is injected into the left branch.

Figure 12. Full field simulations demonstrating unidirectional propagation of the surface state. (a) Domain used for full field simulations. The domain is finite in size along the x- and y-directions but it employs periodic boundary conditions along the z-direction. A PML boundary was applied along the bottom edge of the domain to absorb the incoming wave in steady-state analyses. The wave was injected by means of a harmonic point force applied at the location marked by the red star at a frequency of 40 kHz. (b) Steady-state response showing the propagation of the surface state when excited with $k_z = \frac{0.1\pi}{L_z}$. 99.41% of power is injected into the right branch. (c) Steady-state response showing the propagation of the surface state when excited with $k_z = \frac{-0.1\pi}{L_z}$. These results clearly show the unidirectional nature of the topological surface modes. 99.93% of power is injected into the left branch.

It should be noted that while the states propagate with the same intensity along the x- and y-aligned edges, they are associated with different eigenstates. This can be clearly observed in figures 12(b) and (c), and it is due to the different structure of the edges (i.e. zigzag versus armchair).

In order to put this results in perspective, we also show the performance in approximately equivalent non-topological waveguide modes. Figures 13(a) and (e) show the geometry and the response of an elastic half-space (simulated by a finite domain with PML) subject to a harmonic source located at the top surface. As expected, non-leaky Rayleigh surface wave propagates along the surface. Figures 13(b) and (f) show the geometry and the response of a similar waveguide when a sharp corner is inserted on its path. The interference pattern evident in the response along the top edge and the very attenuated propagation along the vertical edge provide concrete evidence of the strong back-scattering occurring at the corner. Also, some degree of scattering into the bulk is clearly visible. It is also well-known that waveguides can be built based on the concept of a phononic bandgap. In figures 13(c) and (g), we adopt the 2D phononic waveguide proposed by Sun and Wu [64] which was based on an array of steel rods embedded in a uniform epoxy
matrix. The phononic waveguide is assumed infinite in the $z$-direction, therefore resembling a surface-like waveguide. We also assume $k_z = 0.1\pi/a$, similar to the above conditions used in the simulations of the topological surface state. Figures 13(d) and (h) show the geometry and the response of the waveguide in presence of a sharp corner. Albeit no leakage is visible into the bulk, the reflection taking place at the corner is still severe, as visible by a standing-wave-like pattern in the horizontal branch and by the sharp decrease in wave amplitude in the vertical branch. Studies have explored the use of additional scatterers placed near the corner [65] in order to improve the transmission, however a transmission coefficient of 72% (52% power) was achieved; still very limited compared to topological designs.

The immunity to back-scattering was also assessed by performing full field simulations on domains with three different types of either disorder or defect. Figures 14(a) and (c) show the geometry and the steady-state response of the reference configuration having neither edge defects nor disorder. The power transmission coefficient evaluated at the bottom-right end is 98.87%. Figures 14 (b) and (f) show the geometry and the steady-state response of a domain having randomly shifted center pillars in each of the 24 unit cells near the top free boundary. The inset in (b) shows the magnified image of the shifted pillars. Each of them is shifted away from its original position of a random distance between $\pm 0.15a$ in both the $x$- and $y$-directions. The power transmission coefficient evaluated at the bottom-right end is 98.85%. Figures 14 (c) and (g) show the geometry and the steady-state response of a domain with center pillars removed in the 24 unit cells near the top free boundary. The inset in (c) shows the magnified image of the missing pillars. The power transmission coefficient evaluated at the bottom-right end is 99.53%. Finally, figures 14 (d) and (h) show the geometry and the steady-state response of a domain with a rectangular cut on the top edge. The power transmission coefficient evaluated at the bottom-right end is 97.64%. It is found that less than 1.23% (i.e. $-0.05$ dB) of power transmission loss occurred in the three cases; part of this loss also due to computational aspects as previously discussed. However, by direct comparison with the reference configuration, these are found to be minor differences in the transmission coefficients, hence confirming the robustness of the edge states against various types of in-plane disorder and defect.

We note that the origin of the robustness can be connected to the $k_z$-locking of the uni-directional surface states. In other words, with positive (negative) $k_z$ fixed, there is only one uni-directional clockwise (counterclockwise) surface state that can propagate along the boundary. Therefore, the surface state is backscattering immune in the presence of all $k_z$-preserving perturbations, including those characterized by strong disorder/defect on the $xy$-plane, as far as they preserve the periodicity along the $z$-axis. In presence of disturbances destroying the $k_z$ protection, such as strong short-range perturbation along the $z$-axis (e.g. missing center pillars in one layer along the $z$-axis), the $z$-momentum will be altered and the $k_z$ could potentially change sign, therefore resulting in backscattering.
6. Conclusions

Weyl points and topological surface modes in elastic solid media are still in their early stages of formulation and design. The very few implementations proposed to-date are based on discrete designs that are not conducive to use in practical applications, particularly those requiring load-bearing designs. Examples might include stress wave management for vibration control or energy harvesting in structural systems subject to distributed loads (such as those associated with aerothermal forces). This study presented a fully continuous, load-bearing, elastic material capable of unidirectionally-propagating and topologically-protected surface states. The design of the 3D unit cell, inspired to sandwich composites, consisted in a layered prismatic lattice with hexagonal cross section in which the layers were spaced by solid cylindrical elements and by slanted circular beams connecting consecutive faces of the prismatic unit cell. The cylinders were used to set the necessary in-plane symmetry conditions and to provide high load carrying capacity. The slanted beams were used to achieve chirality and allowed achieving the necessary P-symmetry breaking condition.

The analysis of the lattice dynamics highlighted the existence of Weyl points following the breaking of the z-mirror-symmetry and the P-symmetry of the lattice. To gain insight into the mechanism leading to the formation of these degenerate points, we evaluated the topological invariants using ab initio calculations and without employing any further simplifications. Results were very well aligned with the expected integer values. The relationship between the surface states and the topological invariants was also explored from different perspectives. k_z-locked unidirectional propagating surface elastic states were predicted on the external surface of the 3D lattice and their existence was further confirmed by full field numerical simulations. Numerical results also confirmed the extreme robustness of these states against strong lattice defects. This study may serve as a basis to develop composite-like structures having advanced elastic waveguiding capabilities for applications in fields such as vibration control, energy harvesting, structural health monitoring, and on-chip telecommunication signal processing.

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References

[1] Ren Y, Qiao Z and Niu Q 2016 Rep. Prog. Phys. 79 066501
[2] Hasan M Z and Kane C L 2010 Rev. Mod. Phys. 82 3045
[3] Lu L, Joannopoulos J D and Soljačić M 2014 Nat. Photon. 8 821
[4] Ozawa T et al 2019 Rev. Mod. Phys. 91 015006
[5] Zhang X, Xiao M, Cheng Y, Lu M-H and Christensen J 2018 Commun. Phys. 1 97
[6] Ma G, Xiao M and Chan C 2019 Nat. Rev. Phys. 1 281–94
[7] Lu J, Qiu C, Ke M and Liu Z 2016 Phys. Rev. Lett. 116 093901
[8] Lu J, Qiu C, Ye L, Fan X, Ke M, Zhang F and Liu Z 2016 Nat. Phys. 13 369
[9] Liu Y, Lian C-S, Li Y, Xu Y and Duan W 2017 Phys. Rev. Lett. 119 255901
[10] He C, Ni X, Ge H, Sun X-C, Chen Y-B, Lu M-H, Liu X-P and Chen Y-F 2016 Phys. Rev. 110 1124
[11] Pal R K and Ruzzene M 2017 New J. Phys. 19 025001
[12] Vila J, Pal R K and Ruzzene M 2017 Phys. Rev. B 96 134307
[13] Zhou Y, Bandaru P R and Sievenpiper D F 2018 New J. Phys. 20 123011
[14] Mousavi S H, Kaniakae A B and Wang Z 2015 Nat. Commun. 6 8682
[15] Liu T-W and Semperottili F 2018 Phys. Rev. Appl. 9 014001
[16] Zhu H, Liu T-W and Semperottili F 2018 Phys. Rev. B 97 174301
[17] Liu T-W and Semperottili F 2019 Phys. Rev. Appl. 11 014040
[18] Liu T-W and Semperottili F 2019 Phys. Rev. B 100 214110
[19] Ganti S S, Liu T-W and Semperottili F 2020 J. Sound Vib. 466 115060
[20] Yan B and Fleber C 2017 Annu. Rev. Condens. Matter Phys. 8 337
[21] Nielsen H B and Ninomiya M 1981 Nucl. Phys. B 193 173
[22] Nielsen H B and Ninomiya M 1983 Phys. Lett. B 130 389
[23] Yang K-Y, Lu Y-M and Rand Y 2011 Phys. Rev. B 84 073129
[24] Mei J, Wu Y, Chan C T and Zhang Z-Q 2012 Phys. Rev. B 86 035141
[25] Zhu H and Semperottili F 2017 Phys. Rev. Appl. 8 064031
[26] Xu S-Y et al 2015 Science 349 613
[27] Lv B et al 2015 Phys. Rev. X 5 031013
[28] Xu S-Y et al 2015 Nat. Commun. 6 11748
[29] Fang C, Gilbert M J, Dai X and Bernevig B A 2012 Phys. Rev. Lett. 108 266802
[30] Chen W-J, Xiao M and Chan C T 2016 Nat. Commun. 7 13038
[31] Wang Y-T and Tsai Y-W 2018 New J. Phys. 20 083031
[32] Chang M-L, Xiao M, Chen W-J and Chan C T 2017 Phys. Rev. B 95 125136
[33] Chen T-G, Jiao J-R, Dai H-Q and Yu D-J 2018 Phys. Rev. B 98 214110
[34] Liu T, Zheng S, Dai X, Yu D and Xia B 2018 arXiv:1803.04284
[35] Soluyanov A A, Gesch G, Wang Z, Wu Q, Troyer M, Dai X and Bernevig B A 2015 Nature 527 495
[36] Xiao M, Lin Q and Fan S 2016 Phys. Rev. Lett. 117 057401
[37] Yang B et al 2017 Nat. Commun. 8 97
[38] Noh J, Huang S, Leykam D, Chong Y D, Chen K P and Rechtsman M C 2017 Nat. Phys. 13 611
[39] Yang Z and Zhang B 2016 Phys. Rev. Lett. 117 224301
[40] Xie B, Liu H, Cheng H, Liu Z, Chen S and Tian J 2019 Phys. Rev. Lett. 122 104302
[41] Weyl H 1929 Z. für Physik A Hadrons Nucl. 56 330
[42] Wann X, Turner A M, Vishwanathan A and Savrasov S Y 2011 Phys. Rev. B 83 205101
[43] Singh B, Sharma A, Lin H, Hasen M, Prasad R and Bansil A 2012 Phys. Rev. B 86 115208
[44] Bulmash D, Liu C-X and Qi X-L 2014 Phys. Rev. B 89 081106
[45] Huang S-M et al 2015 Nat. Commun. 6 7373
[46] Weng H, Fang C, Fang Z, Bernevig B A and Dai X 2015 Phys. Rev. X 5 011029
[47] Lu L, Wang Z, Ye D, Ran L, Fu L, Joannopoulos J D and Soljačić M 2015 Science 349 622
[48] Lu L, Fu L, Joannopoulos J D and Soljačić M 2013 Nat. Photon. 7 294
[49] Gao W, Yang B, Lawrence M, Fang F, Béri B and Zhang S 2016 Nat. Commun. 7 12455
[50] Yang B et al 2018 Science 359 1013
[51] Bravo-Abad J, Lu L, Fu L, Buljan H and Soljačić M 2015 2D Mater. 2 034013
[52] Takahashi S, Ono S, Iwamoto S, Hatsugai Y and Arakawa Y 2018 J. Phys. Soc. Japan 87 123401
[53] Fruchart M, Jon S-Y, Hur K, Cheianov V, Wiesner U and Vitelli V 2018 Proc. Natl Acad. Sci. 115 E3655
[54] Li F, Huang X, Lu J, Ma J and Liu Z 2018 Nat. Phys. 14 30
[55] Xiao M, Chen W-J, He W-Y and Chan C T 2015 Nat. Phys. 11 920
[56] Ge H, Ni X, Tian Y, Gupta S K, Lu M-H, Lin X, Huang W-D, Chan C and Chen Y-F 2018 Phys. Rev. Appl. 10 014017
[57] Shi X, Chaunisli R, Li F and Yang J 2019 Phys. Rev. Appl. 12 024058
[58] Birman V and Kardomeas G A 2018 Composites 42 221
[59] Zhao L, Conlon S C and Semperottili F 2014 Smart Mater. Struct. 23 065021
[60] Zhao L and Semperottili F 2017 J. Sound Vib. 388 42
[61] Raghu S and Haldane F D M 2008 Phys. Rev. A 78 033834
[62] Rechtsman M C, Zeuner J M, Plotnik Y, Lumer Y, Podolsky D, Dreisow F, Nolte S, Segev M and Szameit A 2013 Nature 496 196
[63] Kane C L and Mele E J 2005 Phys. Rev. Lett. 95 146802
[64] Sun J-H and Wu T-T 2005 Phys. Rev. B 71 174303
[65] Sun J-H and Wu T-T 2006 Phys. Rev. B 74 174305