On a stabilized warped brane world without Planck brane

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Abstract

We discuss a stabilized brane world model with two branes, admitting the solution to the hierarchy problem due to the warped extra dimension and possessing a remarkable feature: the strength of gravitational interaction is of the same order on both branes, contrary to the case of the Randall-Sundrum model with a hierarchical difference of gravitational strength on the branes. The solution also admits the existence of two branes with an equal strength of gravitational interaction, which is of interest for treating the matter on the "mirror" brane as dark matter.

1 Introduction

Since in papers [1, 2] it was shown that theories with warped extra dimension could solve the hierarchy problem, such theories are widely discussed in the literature (see, for example, reviews [3, 4]). It turned out that the RS1 model [2] includes the massless radion, which contradicts the experimental data even at the classical level, and thus this model demands a stabilization. The first method was proposed in paper [5], where the size of the extra dimension is defined by the minimum of the effective potential of the five-dimensional scalar field. At the same time the back reaction of the scalar field on the background metric is not taken into account by this mechanism. This problem was solved in the model proposed in [6]. It is necessary to note that in the method proposed in [6] the size of the extra dimension is defined not by the minimum of the effective four-dimensional scalar field potential, but by the boundary conditions on the branes. With the proper choice of the model parameters it is possible to obtain the background solution for the metric which is close to the original Randall-Sundrum solution [7]. It can be considered as the stabilized Randall-Sundrum model.

The main feature of the Randall-Sundrum model is the existence of two branes, which differ significantly in the strength of the four-dimensional gravity. Indeed, the four-dimensional Planck masses, defined by the coupling constant of massless four-dimensional graviton, are [3, 8]

\[ M_{P1}^2 = \frac{M^3}{k} \left( e^{2kL} - 1 \right) \]

for the \( TeV \) brane and

\[ M_{P2}^2 = \frac{M^3}{k} \left( 1 - e^{-2kL} \right) \]

for the Planck brane, where \( M \) is the five-dimensional Planck mass, \( k \) is the inverse anti-de Sitter radius, \( L \) is the size of the extra dimension, \( M \simeq k \sim 1 TeV \) and \( kL \approx 36 \). Obviously, for this choice of the parameters \( M_{P1} \sim 10^{16} \text{TeV} \), whereas \( M_{P2} \sim 1 \text{TeV} \). Thus, the branes are very different from the gravitational point of view, which leads to very different physics on both branes. There arises a question, whether it is possible to construct a stabilized brane world...
model with branes having comparable (or even equal) strength of effective four-dimensional gravity.

In this short paper we discuss a stabilized brane world model, based on the background solution presented in [9, 10]. Although the solution is quite known, we found that it possesses an interesting property, when applied to compact extra dimension. Namely, it allows one to obtain any values of the four-dimensional Planck masses with respect to each other, retaining the main advantages of warped brane worlds – strong five-dimensional gravity and the solution to the hierarchy problem. In other words, there can be two $TeV$ (SM) branes or even the case with one $TeV$ brane and one brane with the gravity much weaker than that on the $TeV$ brane (the notations "Planck brane" and "$TeV$ brane" are used in the same sense as in the Randall-Sundrum model – the Planck brane is the one with the stronger gravity, whereas the $TeV$ brane is the brane with the weaker gravity).

2 The model

To start with, let us denote the coordinates in five-dimensional space-time $E = M_4 \times S^1$ by $\{x^N\} \equiv \{x^\mu, y\}$, $N = 0, 1, 2, 3, 4, \mu = 0, 1, 2, 3$, the coordinate $x^4 \equiv y, -L \leq y \leq L$ parameterizing the fifth dimension with identified points $-y$ and $y$. The branes are located at the points $y = 0$ and $y = L$.

The action of the stabilized brane world model can be written as

$$ S = \int d^4x \int_{-L}^{L} dy \sqrt{\tilde{g}} \left[ 2M^3 R - \frac{1}{2} g^{MN} \partial_M \phi \partial_N \phi - V(\phi) \right] - \int_{y=0}^{y=L} \sqrt{-\tilde{g}} \lambda_1(\phi) d^4x - \int_{y=0}^{y=L} \sqrt{-\tilde{g}} \lambda_2(\phi) d^4x, \quad (1) $$

Here $V(\phi)$ is a bulk scalar field potential and $\lambda_i(\phi), i = 1, 2$, are the brane scalar field potentials, $\tilde{g} = det \tilde{g}_{\mu\nu}$, and $\tilde{g}_{\mu\nu}$ denotes the metric induced on the brane. The signature of the metric $g_{MN}$ is chosen to be $(-, +, +, +, +)$.

The standard ansatz for the metric and the scalar field, which preserves the Poincaré invariance in any four-dimensional subspace $y = const$, looks like

$$ ds^2 = e^{-2A(y)} \eta_{\mu\nu} dx^\mu dx^\nu + dy^2 \equiv \gamma_{MN}(y) dx^M dx^N, \quad \phi(x, y) = \phi(y), \quad (2) $$

$\eta_{\mu\nu}$ denoting the flat Minkowski metric.

For this ansatz the Einstein and the scalar field equations derived from action (1) reduce to the following system:

$$ \frac{dV}{d\phi} + \frac{d\lambda_1}{d\phi} \delta(y) + \frac{d\lambda_2}{d\phi} \delta(y - L) = -4A' \phi' + \phi'' \quad (3) $$

$$ 12M^3 (A')^2 + \frac{1}{2} (V - \frac{1}{2} (\phi')^2) = 0 $$

$$ \frac{1}{2} \left( \frac{1}{2} (\phi')^2 + V + \lambda_1 \delta(y) + \lambda_2 \delta(y - L) \right) = -2M^3 (-3A'' + 6(A')^2). $$

Let us consider a special class of potentials, which can be represented as

$$ V(\phi) = \frac{1}{8} \left( \frac{dW}{d\phi} \right)^2 - \frac{1}{24M^3} W^2(\phi). $$

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Then the solutions of the first order differential equations
\[
\phi'(y) = \text{sign}(y) \frac{1}{2} \frac{dW}{d\phi}, \quad A'(y) = \text{sign}(y) \frac{1}{24M^3} W(\phi)
\] (4)
solve equations (3) in the bulk [6, 11].

Let us consider a linear function \( W(\phi) \) as suggested in papers [9, 10]:
\[
W(\phi) = \alpha \phi, \quad V = \frac{\alpha^2}{8} - \frac{\alpha^2}{24M^3} \phi^2.
\] (5)

Using equations (4) one gets the corresponding background solution, which looks like
\[
\phi = \frac{\alpha}{2} |y| - \frac{\alpha L_1}{4}, \quad A = \frac{\alpha^2}{96M^3} \left[ \left( |y| - \frac{L_1}{2} \right)^2 + C \right],
\] (6)
where \( L_1 \) and \( C \) are integration constants treated as parameters.

In order the equations of motion be valid on the branes too, one can take the brane potentials \( \lambda_i(\phi) \), \( i = 1, 2 \), in the form
\[
\lambda_1(\phi) = W(\phi) + \beta_1^2 (\phi - \phi_1)^2,
\] (7)
\[
\lambda_2(\phi) = -W(\phi) + \beta_2^2 (\phi - \phi_2)^2.
\] (8)

It is easy to check that the equations of motion are satisfied provided
\[
\phi|_{y=0} = \phi_1, \quad \phi|_{y=L} = \phi_2,
\] (9, 10)
which means that
\[
L_1 = -\frac{4\phi_1}{\alpha}, \quad L = \frac{2(\phi_2 - \phi_1)}{\alpha}
\] (11, 12)
(we suppose that \( \phi_1 < 0 \), i.e. \( L_1 > 0 \)). Thus, we see that the size of the extra dimension is fixed. The parameters of the potentials \( \alpha, \phi_{1,2}, \beta_{1,2} \), when made dimensionless by the fundamental five-dimensional energy scale of the theory \( M_5 \), do not contain a hierarchical difference. We note again that fixation of the size of the extra dimension is caused by the boundary conditions on the branes unlike the case discussed in [5], where the size of the extra dimension is defined by the minimum of an effective four-dimensional scalar field potential.

Let us suppose that we live on the brane at \( y = L \) and \( L \leq L_1 \). In order to have Galilean four-dimensional coordinates on this brane [3, 8], we choose the warp factor such that \( e^{-2A}|_{y=L} = 1 \), i.e. \( C = -\left( L - \frac{L_1}{2} \right)^2 \). Since the wave function of the massless tensor graviton in the extra dimension \( \sim e^{-2A} \) [7], a standard technique (see, for example, [4] for details) gives us an expression for the four-dimensional Planck mass on our brane
\[
M_{Pl}^2 = M^3 \int_{-L}^{L} e^{-2A} dy = M^3 2 e^{\frac{\alpha^2(2L-L_1)^2}{192M^3}} \int_{0}^{L} e^{-\frac{\alpha^2}{48M^3} \left( y - \frac{L_1}{2} \right)^2} dy =
\] (13)
\[
= M^3 2 e^{\frac{\alpha^2(2L-L_1)^2}{192M^3}} \int_{-\frac{L_1}{2}}^{\frac{L_1}{2}} e^{-\frac{\alpha^2}{48M^3} y^2} dy \simeq M^3 2 e^{\frac{\alpha^2(2L-L_1)^2}{192M^3}} \int_{-\infty}^{\infty} e^{-\frac{\alpha^2}{48M^3} y^2} dy =
\] \[
= M^3 \sqrt{\frac{4\pi M^3}{\alpha}} e^{\frac{\alpha^2(2L-L_1)^2}{192M^3}}
\]
Figure 1: Warp factors in the Randall-Sundrum model (dashed line, a) and in the model under discussion (solid line, b). The branes are located at the points $y = 0$, $y = L$ of the extra dimension. The formulas for four-dimensional Planck masses in Galilean coordinates on each brane for both cases are presented (for case b we consider $L_1 = L$). All the parameters are supposed to lie in the $TeV$ range, the hierarchy problem is solved due to the exponential factors $e^{kL} \sim e^{\alpha L^2} \approx 10^{16}$.

and

$$M_{Pl} \approx M \frac{5M^2}{\sqrt{\alpha}} e^{\frac{\alpha^2(2L-L_1)^2}{384M^4}}.$$  

(14)

Let us suppose that all fundamental parameters of the theory lie in the $TeV$ range. To have the hierarchy problem solved, i.e. to have $M_{Pl} \sim 10^{16}TeV$, one should take

$$\frac{\alpha(2L-L_1)}{M^2} = \frac{4\phi_2}{M^2} \approx 120.$$  

(15)

With these values of the parameters (and if $L$ and $L_1$ are of the same order), the approximation used in (13) is very good. Also note that the four-dimensional Planck mass (14) depends mainly on $\phi_2$, not $\phi_1$.

Now let us discuss some properties of this solution. On the second brane

$$e^{-2A}|_{y=0} = e^{-\frac{\alpha^2}{48M^4}[L(L_1-L)]} < 1$$

for $L < L_1$. Let us compare this behavior of the warp factor with that in the RS1 model. Indeed, in the RS1 model (as well as in the stabilized case [6]), if we live on the $TeV$ brane, there exists the Planck brane, where gravity is much stronger. In the case of background solution (6) the gravity on our brane is weak in comparison with the bulk gravity strength, but the gravity on the brane at $y = 0$ is even weaker. Thus, we have no Planck brane as a physical object in this
In the RS1 model the exponent $A(y)$ takes its smallest value at the point, where the Planck brane is located, whereas in the case of solution (6) the analogous point $y = L_1/2$ lies in the bulk, i.e. the massless tensor graviton, whose wave function in the extra dimension is proportional to $e^{-2A}$, is localized in the bulk (for the case $L_1 = L$ see Figure 1). Such situation can lead to interesting consequences for the effect of "mirror" matter (located on the "mirror" brane) on our brane and for the case of universal extra dimensions.

It is not difficult to calculate the four-dimensional Planck mass on the brane at $y = 0$. To this end we pass to Galilean four-dimensional coordinates on that brane, which means that the integration constant in solution (6) for $A(y)$ should be taken such that $A(y)|_{y=0} = 0$, i.e.

$$A = \frac{\alpha^2}{96M^3} \left[ \left( y - \frac{L_1}{2} \right)^2 - \frac{L_1^2}{4} \right]. \tag{16}$$

Carrying out the calculations analogous to those presented above, we obtain

$$M_{Pl}^* \approx M \frac{5M_{Pl}^2}{\sqrt{\alpha}} e^{\frac{\alpha^2 L_1}{384 M^3}}. \tag{17}$$

Thus

$$M_{Pl}^* = M_{Pl} e^{\frac{\alpha^2 L_1 (L_1 - L)}{96 M^3}}. \tag{18}$$

Note that $M_{Pl}$ and $M_{Pl}^*$ were calculated in four-dimensional coordinates, which are Galilean on the branes at $y = L$ and $y = 0$ respectively, not in a four-dimensional coordinate system, which is common for both branes.

A quite peculiar case is $L = L_1$. For this choice of the parameters, the warp factor has equal values on both branes, and we have two equal branes from the gravitational point of view! At the same time the hierarchy problem is solved in this case also because of the quadratic behavior of the function $A(y)$ and the corresponding behavior of the warp factor, see Figure 1. The peculiar feature of the model is that the hierarchy problem appears to be solved for both branes, contrary to the case of the RS1 model, in which the hierarchy problem is solved only for brane at $y = L$.

The branes in this case are not only gravitationally equivalent, but also have the same negative tension (energy density), which is characteristic of the TeV brane in the RS1 model. If not only the gravitational constants on both branes are of the same order, but the density of the "mirror" matter is also of the same order as that of the ordinary matter on our brane, the "mirror" matter of WIMP type may play, in principle, the role of dark matter. This possibility deserves a more detailed investigation.

Now let us discuss stability of this background solution under small fluctuations of the fields. To this end we should consider the linearized theory. The physical degrees of freedom in five-dimensional brane world models stabilized by the scalar field were described in [7] for the general case of stabilizing scalar field potential. It has been shown that if there exists a background solution of form (2) to field equations (3) and if the size of the extra dimension is fixed by boundary conditions on the branes, the tensor sector of Kaluza-Klein excitations does not contain tachyons or fields with the wrong sign of the kinetic term (ghosts). As for the scalar sector, it does not contain tachyons, ghosts and massless (from the four-dimensional point of view) modes, if [7]

$$\left( \frac{1}{2} \frac{d^2 \lambda_1}{d \phi^2} - \frac{\phi''}{\phi'} \right) |_{y=0+\epsilon} > 0, \quad \left( \frac{1}{2} \frac{d^2 \lambda_2}{d \phi^2} + \frac{\phi''}{\phi'} \right) |_{y=L-\epsilon} > 0. \tag{19}$$
Note that conditions (19) do not involve the bulk potential \( V(\phi) \), and it may be unbounded from below as in equation (5).

One can easily find that for the scalar field configuration satisfying equation (4) and potentials given by (7), (8) conditions (19) reduce to \( \beta_{1,2}^2 > 0 \) and are satisfied. Thus, the model under consideration is indeed stable, at least perturbatively.

It is also worth mentioning that in the case of gravitationally equivalent branes there exists another interesting possibility to stabilize the size of the extra dimension. There is a good reason to guess that the branes are equivalent initially, i.e. the scalar field potentials on the branes are of the same form. Namely, the (fine-tuned) brane potentials can be chosen to be non-polynomial

\[
\lambda_{1,2}(\phi) = -3 \left( \frac{\rho \alpha^2}{4} \right)^{\frac{1}{3}} + \frac{\rho}{\phi^2}
\]

with \( \rho > 0 \). From the boundary conditions on the branes, which follow from equations (3), one can easily obtain

\[
L = L_1 = 4 \left( \frac{2 \rho}{\alpha^4} \right)^{\frac{1}{3}}.
\]

The stability conditions (19) are fulfilled for the choice of the potentials in (20).

As for the wave functions, coupling constants and masses of the tensor and scalar Kaluza-Klein modes, it seems that it is impossible to solve the corresponding equations of motion analytically for the case of background solution (3) (except for the tensor zero mode, which is proportional to \( e^{-2A} \)). But it is quite obvious that masses of the lowest excitations are of the order of \( 1/L \) (since the model is stabilized, there is no massless radion), and the corresponding coupling constants should also be expressed through the fundamental parameters of the theory. Thus, all these parameters should lie in the \( \text{TeV} \) energy range, as it usually happens in the brane world models.

### 3 Conclusion

In this paper we have discussed the stabilized brane world model, admitting the solution to the hierarchy problem on both branes, contrary to the case of the Randall-Sundrum-type models, in which solution to the hierarchy problem can be obtained only for one brane. The stability of the model under fluctuations of metric and scalar field is provided by the fulfilment of conditions (19), which are valid for any brane world model with the action of the form (11), stabilized by a scalar field [7]. We also show that if one assumes the branes to have an equal structure, namely, the brane potential to be the same on both branes, there also exists a solution with fixed size of the extra dimension. In this case the effective four-dimensional Planck masses on both branes appear to be equal.

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