Twin-width of Planar Graphs is at most 8

Petr Hliněný
Masaryk University, Brno, Czech republic

Jan Jedelský
Masaryk University, Brno, Czech republic

Abstract

The structural parameter twin-width was introduced by Bonnet et al. in [FOCS 2020], and already this first paper included an asymptotic argument bounding the twin-width of planar graphs by a non-explicit constant. Quite recently, we have seen first small explicit upper bounds of 183 by Jacob and Pilipczuk [arXiv, January 2022, also WG’22], 583 by Bonnet et al. [arXiv, February 2022], of 37 by Bekos et al. [arXiv, April 2022], and of 9 by the first author [arXiv, June 2022]. We further elaborate on the approach used in the last paper and improve the upper bound to 8. This is already very close to the currently best lower bound of 7 by Král’ and Lamaison [arXiv, September 2022].

2012 ACM Subject Classification
Mathematics of computing \rightarrow Graph theory

Keywords and phrases
twin-width, planar graph

Funding
Supported by the Czech Science Foundation, project no. 20-04567S.

1 Introduction

Twin-width is a new structural width measure of graphs introduced in 2020 by Bonnet, Kim, Thomassé and Watrigant [2].

Consider only simple graphs in the coming definition. A trigraph is a simple graph $G$ in which some edges are marked as red, and with respect to the red edges only, we naturally speak about red neighbours and red degree in $G$. However, when speaking about edges and neighbours without further specification, we count both ordinary and red edges and neighbours. The edges of $G$ which are not red are sometimes called (and depicted) black for distinction. For a pair of (possibly not adjacent) vertices $x_1, x_2 \in V(G)$, we define a contraction of the pair $x_1, x_2$ as the operation creating a trigraph $G'$ which is the same as $G$ except that $x_1, x_2$ are replaced with a new vertex $x_0$ (said to stem from $x_1, x_2$) such that:

- the full neighbourhood of $x_0$ in $G'$ (i.e., including the red neighbours), denoted by $N_{G'}(x_0)$, equals the union of the neighbourhoods of $x_1$ and $x_2$ in $G$ except $x_1, x_2$ themselves, that is, $N_{G'}(x_0) = (N_G(x_1) \cup N_G(x_2)) \setminus \{x_1, x_2\}$, and
- the red neighbours of $x_0$, denoted by $N^r_{G'}(x_0)$, inherit all red neighbours of $x_1$ and of $x_2$ and add those in $N_G(x_1)\Delta N_G(x_2)$, that is, $N^r_{G'}(x_0) = (N^r_G(x_1) \cup N^r_G(x_2) \cup (N_G(x_1)\Delta N_G(x_2))) \setminus \{x_1, x_2\}$, where $\Delta$ denotes the symmetric set difference.

A contraction sequence of a trigraph $G$ is a sequence of successive contractions turning $G$ into a single vertex, and its width is the maximum red degree of any vertex in any trigraph of the sequence. The twin-width of a trigraph $G$ is the minimum width over all possible contraction sequences of $G$.

As shown already in the pioneering paper on this concept [2], the twin-width of planar graphs is bounded (but there was no explicit number given there). In the first half of 2022, first explicit upper bounds on the twin-width of planar graphs have appeared; 183 by Jacob

1 In general, the concept of twin-width is defined for binary relational structures of a finite signature, and so one may either define the twin-width of a multigraph as the twin-width of its simplification, or allow only bounded multiplicities of edges and use the more general matrix definition of twin-width.
Twin-width of Planar Graphs is at most 8

and Pilipczuk [6], 583 by Bonnet et al. [3] (this paper more generally bounds the twin-width of so-called k-planar graphs with an asymptotic exponential function of $O(k)$), and most recently 37 by Bekos et al. [1] (which more generally bounds the twin-width of so-called h-framed graphs, and in particular gives also an upper bound of 80 on the twin-width of 1-planar graphs). It is worth to mention that all three papers [1,3,6], more or less explicitly, use the product structure machinery of planar graphs (cf. [4]) which, however, has its limits towards the exact maximum twin-width of planar graphs.

We have recently developed an alternative decomposition-based approach, leading to a single-digit upper bound of 9 for all planar graphs in [5], followed by an upper bound of 6 on the twin-width of bipartite planar graphs thereafter. However, the approach of [5] seems to be stuck right at 9, too, and new ideas are needed to obtain further improvements, even by 1.

Here we give the following strengthened upper bound, which uses an improved approach over previous [5] and also simplifies some cumbersome technical details of the former:

Theorem 1. The twin-width of any simple planar graph is at most 8, and a corresponding contraction sequence can be found in linear time.

It is worth to note that, very recently, Král’ and Lamaison [7] have found a construction (with a proof) of a planar graph with twin-width 7; hence, a lower bound which is by just one off our new upper bound.

2 Notation and Tools

We start with a few technical definitions and claims needed for the proof of Theorem 1.

BFS layering and related. Let $G$ be a graph and $r \in V(G)$ a fixed vertex. The BFS layering of $G$ determined by $r$ is the vertex partition $\mathcal{L} = (L_0 = \{r\}, L_1, L_2, \ldots)$ of $G$ such that $L_i$ contains all vertices of $G$ at distance exactly $i$ from $r$. A path $P \subseteq G$ is $r$-geodesic if $P$ is a subpath of some shortest path from $r$ to any vertex of $G$ (in particular, $P$ intersects every layer of $\mathcal{L}$ in at most one vertex). Let $T$ be a BFS tree of $G$ rooted at the vertex $r$ as above (that is, for every vertex $v \in V(G)$, the distance from $v$ to $r$ is the same in $G$ as in $T$). A path $P \subseteq G$ is $T$-vertical, or shortly vertical with respect to implicit $T$, if $P$ is a subpath of some root-to-leaf path of $T$. Notice that a $T$-vertical path is $r$-geodesic, but the converse may not be true. Analogously, an edge $e \in E(G)$ is horizontal (with respect to implicit $\mathcal{L}$) if both ends of $e$ are in the same $\mathcal{L}$-layer.

Observe the following:

> Claim 2. For every edge $\{v, w\}$ of $G$ with $v \in L_i$ and $w \in L_j$, we have $|i - j| \leq 1$, and so a contraction of a pair of vertices from $L_i$ may create new red edges only to the remaining vertices of $L_{i-1} \cup L_i \cup L_{i+1}$.

Plane graphs and special BFS trees. We will deal with plane graphs, which are planar graphs with a given (combinatorial) embedding in the plane, and one marked outer face (the remaining faces are then bounded). A plane graph is a plane triangulation if every face of its embedding is a triangle. It is easy to turn an embedding of any planar graph into a simple plane triangulation by adding vertices and incident edges into each non-triangular face. Furthermore, twin-width is non-increasing when taking induced subgraphs, and so it suffices to focus on plane triangulations in the proof of Theorem 1

In this planar setting, consider now a plane graph $G$, and a BFS tree $T$ spanning $G$ and rooted in a vertex $r$ of the outer face of $G$, and picture (for clarity) the embedding
\[ G \text{ such that } r \text{ is the vertex of } G \text{ most at the top. For two adjacent vertices } u, v \in V(G), \{u, v\} \in E(G), \text{ we say that } u \text{ is to the left of } v \text{ (wrt. } T) \text{ if neither of } u, v \text{ lies on the vertical path from } r \text{ to the other, and the following holds; if } r' \text{ is the least common ancestor of } u \text{ and } v \text{ in } T \text{ and } P_{r', u} \text{ (resp., } P_{r', v}) \text{ denote the vertical path from } r' \text{ to } u \text{ (resp., } v), \text{ then the cycle } (P_{r', u} \cup P_{r', v}) + uv \text{ has the triple } (r', u, v) \text{ in this counter-clockwise cyclic order.}

A BFS tree } T \text{ of } G \text{ with the BFS layering } \mathcal{L} = (L_0, L_1, \ldots) \text{ is called left-aligned if there is no edge } f = uv \text{ of } G \text{ such that, for some index } i, u \in L_{i-1} \text{ and } v \in L_i, \text{ and } u \text{ is to the left of } v \text{ (an informal meaning is that one cannot choose another BFS tree of } G \text{ which is “more to the left” of } T \text{ in the geometric picture of } G \text{ and } T, \text{ such as by picking the edge } uv \text{ instead of the parental edge of } v \text{ in } T).}

\[ \textbf{Lemma 3.} \text{ Given a simple plane graph } G, \text{ and a vertex } r \text{ on the outer face, there exists a left-aligned BFS tree of } G \text{ and can be found in linear time.}

\textbf{Proof.} \text{ For this proof, we have to extend the above relation of “being left of” to edges emanating from a common vertex of } G. \text{ So, for an arbitrary BFS tree } T \text{ of } G \text{ and edges } f_1, f_2 \in E(G) \text{ incident to } v \in V(G), \text{ such that neither of } f_1, f_2 \text{ is the parental edge of } v \text{ in } T, \text{ we write } f_1 \leq f_2 \text{ if there exist adjacent vertices } u_1, u_2 \in V(G) \text{ such that } u_1 \text{ is to the left of } u_2, \text{ the least common ancestor of } u_1 \text{ and } u_2 \text{ in } T \text{ is } v \text{ and, for } i = 1, 2, \text{ the edge } f_i \text{ lies on the vertical path from } u_i \text{ to } v. \text{ Observe the following; if } f_0 \text{ is the parental edge of } v \text{ in } T \text{ (or, in case of } v = r, f_0 \text{ is a “dummy edge” pointing straight up from } r), \text{ then } f_1 \leq f_2 \text{ implies that the counter-clockwise cyclic order around } v \text{ is } (f_0, f_1, f_2). \text{ In particular, } \leq \text{ can be extended into a linear order on its domain.}

\text{We first run a basic linear-time BFS search from } r \text{ to determine the BFS layering } \mathcal{L} \text{ of } G. \text{ Then we start the construction of a left-aligned BFS tree } T \subseteq G \text{ from } T := \{r\}, \text{ and we recursively (now in a “DFS manner”) proceed as follows:}

- \text{ Having reached a vertex } v \in V(T) \subseteq V(G) \text{ such that } v \in L_i, \text{ and denoting by } X := (N_G(v) \cap L_{i+1}) \setminus V(T) \text{ all neighbours of } v \text{ in } L_{i+1} \text{ which are not in } T \text{ yet, we add to } T \text{ the nodes } X \text{ and the edges from } v \text{ to } X.

- \text{ We order the vertices in } X \text{ using the cyclic order of edges emanating from } v \text{ to have it compatible with } \leq \text{ at } v, \text{ and in this increasing order we recursively (depth-first, to be precise) call the procedure for them.}

\text{The result } T \text{ is clearly a BFS tree of } G. \text{ Assume, for a contradiction, that } T \text{ is not left-aligned, and let } u_1 \in L_{i-1} \text{ and } u_2 \in L_i \text{ be a witness pair of it, where } \{u_1, u_2\} \in E(G) \text{ and } u_1 \text{ is to the left of } u_2. \text{ Let } v \text{ be the least common ancestor of } u_1 \text{ and } u_2 \text{ in } T, \text{ and let } v_1 \text{ and } v_2 \text{ be the children of } v \text{ on the } T\text{-paths from } v \text{ to } u_1 \text{ and } u_2, \text{ respectively. So, by the definition, } v_1 \leq v_2 \text{ at } v, \text{ and hence when } v \text{ has been reached in the construction of } T, \text{ its child } v_1 \text{ has been taken for processing before the child } v_2. \text{ Consequently, possibly deeper in the recursion, } u_1 \text{ has been processed before the parent of } u_2 \text{ and, in particular, the procedure has added the edge } u_1u_2 \text{ into } T, \text{ a contradiction to } u_1 \text{ being to the left of } u_2.

\text{This recursive computation is finished in linear time, since every vertex of } G \text{ is processed only in one branch of the recursion, and one recursive call takes time linear in the number of incident edges to } v.

\textbf{\textit{Notice}} \text{ that we have not assumed } G \text{ to be a triangulation in the previous definition and in Lemma 3 which may be useful for other applications (such as in bipartite planar graphs).}

\textbf{Vertex levels in contraction sequences.} \text{ We are going to work with contraction sequences which, preferably, preserve the BFS layers of } \mathcal{L} \text{ of } G. \text{ However, we do not always preserve the layers, and so we need a notion which is related to the layers of } \mathcal{L}, \text{ but it can differ}
Twin-width of Planar Graphs is at most 8

from these layers when needed — informally, when this “causes no harm at all”. For the graph $G$ itself, we define $\lambda[G](v) = i$ if and only if $v \in L_i \in \mathcal{L}$. If $G'$ is a trigraph along a contraction sequence of $G$, and a vertex $v' \in V(G')$ stems from a set $X \subseteq V(G)$ by (possible) contractions, then $\lambda[G'](v')$ equals the minimum $i$ such that $L_i \cap X \neq \emptyset$. We say that $\lambda[G'](v')$ is the level of $v'$ in $G'$ along the considered contraction sequence of $G$, or simply the level of $v'$ when the particular graph of a sequence is implicit. In other words, we can inductively say that if $v''$ of $G''$ results by the contraction of $u'$ and $v'$ of $G'$, then $\lambda[G''](v'') := \min(\lambda[G'](u'), \lambda[G'](v'))$. If $w'$ does not participate in a contraction along the subsequence from $G'$ to $G''$, then $\lambda[G''](w') := \lambda[G'](w')$.

A partial contraction sequence of $G$ is defined in the same way as a contraction sequence of $G$, except that it does not have to end with a single-vertex graph. A partial contraction sequence of $G$ is level-respecting if every step contracts, in a trigraph $G'$ along the sequence, only a pair $x, y \in V(G')$ such that the following inductively holds; the levels of $x$ and $y$ are the same — $\lambda[G'](y) = \lambda[G'](x)$, or all neighbours of $y$ (red or black) in $G'$ are on the same level as $x$ is on — $\lambda[G'](z) = \lambda[G'](x)$ for all $z$ such that $\{y, z\} \in E(G')$.

Usefulness of level-respecting contraction sequences lies in the next observation which follows easily by induction from Claim 2 and the definition of levels:

\[
\text{Claim 4. Let a trigraph } G' \text{ result from a level-respecting partial contraction sequence of a graph } G. \text{ Then any vertex } z \in V(G') \text{ may have neighbours (red or black) only on the levels } \lambda[G](z) - 1, \lambda[G](z) \text{ and } \lambda[G](z) + 1. \text{ Moreover, } z \text{ must have some neighbour on the level } \lambda[G](z) - 1.
\]

Informally, we may say that our levels in $G'$ behave analogously to the BFS layers of $G$; the levels form a layering (in the usual sense), albeit not a BFS layering.

### 3 Proof of Theorem 1

#### 3.1 Induction setup

Our main proof proceeds by induction on suitably defined subregions of the assumed plane triangulation $G$. In this subsection, we define the setup of this induction in Lemma 5 and show how it will imply the main result.

For a plane graph $G$ and its cycle $C$, the subgraph of $G$ bounded by $C$, denoted by $G_C$, is the subgraph of $G$ formed by the vertices and edges of $C$ and the vertices and edges of $G$ drawn inside $C$ — formally, in the region of the plane bounded by $C$ and not containing the outer face. Let the vertices in the set $U := V(G_C) \setminus V(C)$ be called the interior vertices of $C$. We call a set $U_0 \subseteq U$ an interior section of $C$ in $G$ if all neighbours of vertices of $U_0$ belong to $U_0 \cup V(C)$ (in other words, $U_0$ is a collection of connected components of $G[U]$).

Consider a now fixed BFS tree $T$ of $G$. Assume that a cycle $C$ of $G$ is formed as $C = (P_1 \cup P_2) + f$, where $P_1$ and $P_2$ are two $T$-vertical paths of length at least 1 with a common end $u \in V(P_1) \cap V(P_2)$ and $f \in E(G)$ is an edge joining the other ends $v_1$ of $P_1$ and $v_2$ of $P_2$. Observe that $u$ is the (unique) vertex of $G_C$ closest to the root $r$ of $T$. Then we say that $C$ is a $V$-separator in $G$ with respect to implicit $T$ ("$V$" as vertical), and we call $u$ the sink of $C$. If the vertices $u, v_1, v_2$ lie on $C$ in this counter-clockwise order (equivalently, if $v_1$ is to the left of $v_2$ with respect to $T$), then we say that $P_1$ is the left path of $C$ and $P_2$ is the right path of $C$ (picture the sink at the top).

\[
\text{Lemma 5. Let } G \text{ be a plane triangulation, and } T \text{ be a left-aligned BFS tree of } G \text{ rooted at a vertex } r \in V(G) \text{ of the outer triangular face and defining the initial levels } \lambda[G](\cdot). \text{ Assume}
\]
that a cycle $C$ of $G$ is a V-separator of $G$, that $G_C$ is the subgraph of $G$ bounded by $C$, and $u$ is the sink of $C$. Let the distance of $u$ from the root $r$ be $\ell$, so $\lambda(G)(u) = \ell$, and the maximum distance from a vertex of $C$ to $r$ be $m$. Let $U \subseteq V(G_C)$ be an interior section of $C$ in $G$, and denote by $W := V(G) \setminus (V(C) \cup U)$ the set of the “remaining” vertices.

Then there exists a level-respecting partial contraction sequence of $G$ which contracts only pairs of vertices that are in or stem from $U$, results in a trigraph $G^*$, and satisfies the following conditions for every trigraph $G'$ along this sequence from $G$ to $G^*$:

(I) For $U' := V(G') \setminus (V(C) \cup W)$ (which are the vertices that are in or stem from $U$ in $G'$), every vertex of $U'$ in $G'$ has red degree at most 8,

(II) every vertex of the left path of $C$ has at most 5 red neighbours and every vertex of the right path of $C$ has at most 3 red neighbours in $U'$,

(III) the sink $u$ of $C$ has no red neighbour in $U'$, and if the least level of a vertex of $U$ in $G$ is $k \geq \ell + 2$, then the vertices of $C$ on levels up to $k - 2$ in $G$ have no red neighbours in $U'$ as well and each of the (two) vertices of $C$ on the level $k - 1$ in $G$ has at most 1 red neighbour in $U'$, and

(IV) at the end of the partial contraction sequence, for the set $U^* := V(G^*) \setminus (V(C) \cup W)$ that stems from $U$, we have that if $z \in U^*$ is of level $i$, then $\ell < i \leq \max(m, \ell + 2)$ and $z$ is the only vertex in $U^*$ of level $i$.

Before proceeding further, we comment on two important things. First, we remark that, in Lemma 5, all vertices of $U$ have the distance from $r$ greater than $\ell$, but on the other hand the distance from $r$ to some vertices in $U$ may be much larger than $m$ (and our coming proof is aware of this possibility). Second, we observe that all vertices of $U$ on level $\ell + 1$ must be adjacent to the sink $u$, since all other potential neighbours of them have the distance from $r$ greater than $\ell$. Consequently, contracting $U$ on level $\ell + 1$ into one vertex within the claimed sequence indeed does not create a red edge to $u$, as long as we do not contract into it from higher levels (which we will explicitly avoid in the proof). We illustrate Lemma 5 in Figure 1.

We also observe that the assumptions and conditions of Lemma 5 directly imply some other properties useful for the coming proofs.
Twin-width of Planar Graphs is at most 8

▷ Claim 6. Respecting the notation and assumptions of Lemma 5 we also have that:

(V) Every red edge in \( G' \) has one end in \( U' \) and the other end in \( U' \cup V(C) \).

(VI) if \( P_1 \) and \( P_2 \) are the left and right paths of \( C \), respectively, and \( v \in V(P_2) \) is of level \( j \) in \( G' \), then there is no edge of \( G' \) (red or black) from \( v \) to a vertex of \( U' \cup (V(P_1) \setminus \{u\}) \) of level \( j-1 \) in \( G' \).

(VII) at the end, that is, in \( G^* \), every vertex of the left path of \( C \) has at most 3 red neighbours and every vertex of the right path of \( C \) has at most 2 red neighbours in \( U^* \).

Proof. Regarding (V), observe that since only vertices that stem from \( U \) participate in contractions, every red edge of \( G' \) must have an end in \( U' \). Furthermore, since \( U \) is an interior section of \( C \), no vertex of \( U \) is adjacent to a vertex of \( W \) in \( G \), and hence no vertex of \( W \) is ever adjacent to a vertex being contracted in our sequence from \( G \) to \( G^* \).

Concerning (VI), if \( v \) were adjacent to \( x \in V(P_1) \) of level \( j-1 \), then this was already true in \( G \); \( \{x,v\} \in E(G) \). If \( v \) were adjacent to \( x' \in U' \) of level \( j-1 \) in \( G' \), then, among the vertices of \( U \) contracted into \( x' \) there had to be \( x \in U \) such that \( \{x,v\} \in E(G) \). By the definition of a level-respecting sequence, possible vertices of level higher than \( j-1 \) contracted into \( x' \) cannot be adjacent to \( v \) of level \( j \), and so \( \lambda(G)(x) = j-1 \), too. Since, in both cases, \( x \) would be to the left of \( v \) in \( G \), this contradicts the assumption that \( T \) is left-aligned.

Finally, (VII) directly follows from Claim 4 and (IV) for the left path of \( C \). For the right path we additionally apply (VI), which for \( x \in V(P_2) \) of level \( j \) says that potential red neighbours of \( x \) are only on levels \( j \) and \( j+1 \).

We also show how Lemma 5 implies the first part of our main result:

Proof of Theorem 4 (the upper bound). We start with a given simple planar graph \( H \), and extend any plane embedding of \( H \) into a simple plane triangulation \( G \) such that \( H \) is an induced subgraph of \( G \). Then we choose a root \( r \) on the outer face of \( G \) and, for some left-aligned BFS tree of \( G \) rooted in \( r \) which exists by Lemma 3, the facial cycle \( C \) of the outer face incident to \( r \), and \( u = r \), we apply Lemma 5.

This way we get a partial contraction sequence from \( G \) to a trigraph \( G^* \) of maximum red degree 8 (along the sequence). Observe by (IV) that the set \( U^* = V(G^*) \setminus V(C) \) contains only two vertices, on levels 1 and 2. In the final phase, we may hence pairwise contract the remaining vertices in an arbitrary order. The restriction of this whole contraction sequence of \( G \) to only \( V(H) \) then certifies that the twin-width of \( H \) is at most 8.

3.2 Decomposing into subregions

Here we, for start, decompose the full general induction step of the proof of Lemma 5 into suitable substeps.

So, let \( C \) be a \( V \)-separator in the plane triangulation \( G \), formed by the left path \( P_1 \), the right path \( P_2 \) and the edge \( f = \{v_1, v_2\} \) where \( v_i \) is an end of \( P_i \). Let \( G_C \) be the subgraph bounded by \( C \) and \( U \subseteq V(G_C) \) be an interior section of \( C \) in \( G \). Moreover, assume that there exists a triangular face in \( G \) incident to \( f \) with the vertices \( v_1, v_2, v_3 \) where \( v_3 \in U \), and that the \( T \)-vertical path from \( v_3 \) to the root \( r \) contains neither \( v_1 \) nor \( v_2 \). In particular, since \( T \) is left-aligned, we have that \( \lambda(G)(v_2) \leq \lambda(G)(v_1) \) and \( \lambda(G)(v_3) \leq \lambda(G)(v_1) \). Under these assumptions, we are going to define the vertical-horizontal division of \( G_C \) as follows.

Let \( P \subseteq T \) be the vertical path connecting \( v_3 \) to the root \( r \) where, by \( r \not\in U \) and planarity of \( G \), we have that \( P \) contains the sink \( u \). Let \( P_3 \subseteq P \) be the subpath of \( P \) from \( v_3 \) to the first vertex \( u_3 \in V(P) \cap V(C) \) shared with the cycle \( C \). We have \( u_3 \neq v_3 \). (It may be that \( u_3 \in V(P_1) \) or \( u_3 \in V(P_2) \) or even \( u_3 = u \); see in Figure 2.) Let \( P_{31} \) denote the subpath of \( P \)
from $v_3$ to the first intersection $x$ with $P_1$ ($x \in \{u_3, u\}$), and $P_{11}$ the subpath of $P_1$ from $v_1$ to $x$. Similarly, let $P_{32}$ denote the subpath of $P$ from $v_3$ to the first intersection $y$ with $P_2$, and $P_{22}$ be the subpath of $P_2$ from $v_2$ till $y$. Let $f_1 = \{v_1, v_3\}$ and $f_2 = \{v_2, v_3\}$. Observe that $P_{31}, P_{32} \subseteq G_C$, and that $C_1 := (P_{11} \cup P_{31}) + f_1$ and $C_2 := (P_{22} \cup P_{12}) + f_2$ are again V-separators in $G$, such that $P_{31}$ is the right path of $C_1$ and $P_{12}$ is the left path of $C_2$.

Furthermore, let $h_1, \ldots, h_a, a \geq 0$, be the collection of all horizontal edges of $G_C$ such that, for $h_i = \{x_i, y_i\}$, we have $x_i \in V(P_1), y_i \in V(P_3) \setminus V(C)$ and $\lambda(G)(x_i) = \lambda(G)(y_i)$, and that $\lambda(G)(z) \leq \lambda(G)(x_i) - 1$ holds for some $z \in U$ (the reason for this strange-looking restriction is property III of Lemma 5). These edges $h_1, \ldots, h_a$ are ordered by their increasing level $\lambda(G)(x_i)$. This is illustrated in Figure 2 (where the ordering of $h_i$’s is top-down). For $i = 1, \ldots, a$, let $C_{1,i-1}$ denote the cycle passing through the sink of $C_1$ (which is $u_3$ or $u$) and formed by relevant subpaths of $P_{11}, P_{31}$ and the edge $h_i$. Let $C_{1,a} = C_1$. Let $U_{1,i}$ denote the set of the interior vertices of $C_{1,0}$ in $G$, and for $i = 1, \ldots, a$, let $U_{1,i} := X \setminus U_{1,i-1}$, where $X$ is the set of the interior vertices of $C_{1,i}$ in $G$. Let $U_2$ denote the set of the interior vertices of $C_2$ in $G$.

The system of the cycles $C_{1,0}, \ldots, C_{1,a}, C_2$ and of the sets $U_{1,0}, \ldots, U_{1,a}, U_2$ is called the vertical-horizontal division of $G_C$. The following is straightforward from the definition:

**Claim 7.** For $i = 0, 1, \ldots, a$, the cycle $C_{1,i}$ is a V-separator in $G$, and each vertex of $U_{1,i}$ has neighbours only in $U_{1,i} \cup V(C_{1,i})$. Hence, $U_{1,i}$ is an interior section of $C_{1,i}$. Consequently, for every $z \in U_{1,i}$ where $i \geq 1$, we have $\lambda(G)(z) \geq \lambda(G)(x_i) + 1$ (where $\{x_i, y_i\} = h_i$ above).

The induction step in the proof of Lemma 5 will start with the next lemma:

**Lemma 8.** Assume the notation and assumptions of Lemma 5 for the graph $G$, cycle $C$ and set $U$, and consider the vertical-horizontal division of the subgraph $G_C$ as defined above; that is, the cycles $C_{1,0}, \ldots, C_{1,a}, C_2$ and the sets $U_{1,0}, \ldots, U_{1,a}, U_2$. Then the following hold:

a) Each cycle $C^1 \in \{C_{1,0}, \ldots, C_{1,a}, C_2\}$ and the corresponding set $U^1 \in \{U_{1,0}, \ldots, U_{1,a}, U_2\}$ satisfy the assumptions of Lemma 5 (in the place of $C$ and $U$).

b) Let $\tau_{1,i}, i = 0, \ldots, a$, denote the level-respecting partial contraction sequence of $G$ claimed by Lemma 5 for the input as in (a) $C^1 := C_{1,i}$ and $U^1 := U_{1,i}$, and likewise, $\tau_2$ be that for the input $C^1 := C_2$ and $U^1 := U_2$. Then the concatenated partial contraction sequence

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure2.png}
\caption{Three schematic cases of the vertical-horizontal division of $G_C$ discussed in Section 3.2.}
\end{figure}
\( \sigma_0 := \tau_2 \cdot \tau_{1,0} \cdots \tau_{1,a} \), i.e., one starting with \( \tau_2 \) and ending with \( \tau_{1,a} \), again satisfies the properties (I), (III) and (IV) of Lemma 5.

**Proof.** Part (a) immediately follows from the definition and Claim 7.

In part (b), we first argue that the concatenation \( \sigma_0 \) is well-founded; that contractions in one of the subsequences of \( \sigma_0 \) have no effect on vertices being contracted in another of the subsequences. This is since, by Claim 7, neighbours of contracted pairs of one subsequence are only in the interior sections of the same subsequence or on the bounding cycles which together form \( V(C) \cup V(P_3) \) (and the latter set is not participating in the contractions of \( \sigma_0 \)).

Following on the previous argument, we have that the only vertices of \( G_C \) that may potentially receive red edges from more than one of the subsequences forming \( \sigma_0 \), are those of \( V(C) \cup V(P_3) \). See Figure 2. For all other vertices of \( G_C \), we have that (I) is true for them along whole \( \sigma_0 \) since it has been true for them along their subsequences of \( \sigma_0 \).

For a vertex \( z \in V(P_3) \setminus V(C) \), we see that \( z \) belongs to \( C_2 \) and to each of \( C_{1,i}, \ldots, C_{1,a} \) for some \( i \in \{0, \ldots, a\} \). However, even if \( i \leq a - 2 \), vertices of \( U_{1,i+2} \) cannot be neighbours of \( z \) in \( G_C \) due to a combination of Claim 4 and Claim 7. Therefore, \( z \) may have neighbours (and so can get red edges from by contractions) only in the interior sections \( U_2 \) and \( U_{1,i} \), and possibly in \( U_{1,i+1} \) if \( z \) is an end of the horizontal edge \( h_{i+1} \). Recall also that \( z \) belongs to the right path of \( C_{1,i} \). Along the sequence \( \sigma_0 \), but before \( \tau_{1,i+1} \), the vertex \( z \) has red degree at most \( 5 + 3 = 8 \) by (III) applied to \( U_2 \) and \( U_{1,i} \). After \( \tau_{1,i} \) is finished, \( z \) has red degree at most \( 3 + 2 = 5 < 8 \) by (VII) of Claim 6. So, along the rest of \( \sigma_0 \), (I) stays true for \( z \) with red degree at most \( 5 + 1 = 6 \) by (III) applied possibly to \( U_{1,i+1} \).

For the vertex \( u \) itself, (III) is true automatically. For \( z \in V(P_3) \setminus \{u\} \) and \( i \in \{0, \ldots, a\} \) being the least index such that \( z \in V(C_{1,i}) \), we get that the properties are true along \( \tau_{1,i} \) and before by (I), and since \( \tau_{1,i} \) ends, the vertex \( z \) has at most 3 red neighbours in \( U_{1,i} \) by (VII). Additionally, \( z \) may get at most 1 red neighbour in \( U_{1,i+1} \) (and none in \( U_{1,i+2}, \ldots \)) by (III), altogether at most 4, satisfying (I). In the special case of \( z \in V(P_1) \setminus \{u\} \) covered by property (III) with respect to \( C \), that is when all vertices belonging to the interior of \( C_1 \) are on levels higher than \( \lambda(G)(z) \), we get that this property is satisfied by (III) with respect to \( C_{1,i} \), and there are no more red neighbours of \( z \) from elsewhere.

Finally, for \( z \in V(P_2) \setminus \{u\} \), the conditions are simply true by (I) and possibly (III) for \( \tau_2 \) and then along the whole sequence \( \sigma_0 \).

### 3.3 Finishing the proof

Now we get to the proof of core Lemma 5 which will conclude our main result.

**Proof of Lemma 5.** We first resolve several special cases. If \( U = \emptyset \), we are immediately done with the empty partial contraction sequence. So, assume \( U \neq \emptyset \).

Recall the edge \( f = \{v_1, v_2\} \in E(G) \) connecting the other ends of the left path \( P_1 \) and the right path \( P_2 \) of \( C \). See again Figure 2. If \( v_1 \) has no neighbour in \( U \), then \( \{v_2, v_3\} \in E(G) \) where \( v_3 \) is the neighbour of \( v_1 \) on \( P_1 \). In such case, we simply apply Lemma 5 inductively to \( P_1 - v_1 \) and \( P_2 \), while the rest of the assumptions remain the same. The symmetric argument is applied when \( v_2 \) has no neighbour in \( U \).

Otherwise, let \( v_3 \in U \) be the vertex (unique in \( U \)) such that \( \{v_1, v_2, v_3\} \) bound a triangular face of \( G \). Let \( P \subseteq T \) be the vertical path connecting \( v_3 \) to the root \( r \). If \( v_1 \in V(P) \), then \( P \supseteq P_1 \) and we (similarly as above) apply Lemma 5 inductively to the V-separator \( C^1 = P \cup P_2 \), while the rest of the assumptions again remain the same. Note that in this case, \( \lambda(G)(v_1) = \lambda(G)(v_3) - 1 = \lambda(G)(v_2) \) since \( T \) is left-aligned. In the resulting trigraph \( G^1 \), we have the set \( U^1 := V(G^1) \setminus (V(C^1) \cup W) \) that stems by contractions from the interior
of \( C^1 \). There is no vertex in \( U^1 \) of level higher than \( \lambda[G](v_3) \geq 2 \) and at most one of level equal to \( \lambda[G](v_3) \), by (IV) of Lemma 5. We contract the latter vertex with \( v_3 \), and then with the vertex of \( U^1 \) of the previous level \( \lambda[G](v_1) \) unless \( v_1 \) is a neighbour of \( u \) (cf. the special case in (IV)). This clearly does not exceed red degree 8 there, and does not add new potential red neighbours to the vertices of \( C \). Since (IV) is now satisfied, too, we are done. If \( v_2 \in V(P) \), we solve the case similarly by induction applied to the V-separator \( C^1 = P_1 \cup P_2 \).

In all other cases, we have got a vertical-horizontal division of the subgraph \( G_C \), with \( P_3 \neq \{ v_3, u \} \), with the horizontal edges \( h_1, \ldots, h_a, h_i = \{ x_i, y_i \} \), the cycles \( C_{1,0}, \ldots, C_{1,a}, C_2 \) and the interior sets \( U_{1,0}, \ldots, U_{1,a}, U_2 \), and we apply Lemma 8 to it. This way we get a level-respecting partial contraction sequence \( \sigma_0 \), which satisfies the properties (I), (II) and (III) of Lemma 5. Let \( G^0 \) denote the trigraph which results from \( G \) by \( \sigma_0 \), and let \( U^0_1, U^0_2 \) denote the vertex sets of \( G^0 \) that stem from \( U_{1,0}, U_{1,1}, U_{2,0}, U_{2,1} \), respectively.

We first consider a subcase, that \( P_3 \) consists of a single edge \( \{ v_3, u_3 \} \) and there is no vertex \( z \in U \) in \( G \) such that \( \lambda[G](z) \leq \lambda[G](u_3) \). This case has to be treated specially to fulfill (III) of Lemma 5. Then \( a = 0 \) in the vertical-horizontal division of \( G_C \), and \( \lambda[G](v_1) \leq \lambda[G](v_2) + 1 = \lambda[G](u_3) + 2 \). Each of the sets \( U^0_1, U^0_2 \) hence contains vertices at most on the levels \( \lambda[G](v_3) \) and \( \lambda[G](v_3) + 1 \), by (IV) of Lemma 5. We finish the desired partial contraction sequence from \( G^0 \) in this subcase by firstly contracting the two (if existing) vertices of \( U^0_1 \cup U^0_2 \) on the level \( \lambda[G](v_3) + 1 \) and, secondly by contracting each of the vertices of \( U^0_1 \cup U^0_2 \) on the level \( \lambda[G](v_3) \) with \( v_3 \). If \( u_3 = u \) is the sink of \( C \), then the only vertex on the level \( \lambda[G](v_3) + 1 = \lambda[G](u) \) in \( G_C \) is \( u \), and so every vertex of \( U^0_1 \cup U^0_2 \) on the level \( \lambda[G](v_3) \) must be adjacent to \( u \) (cf. Claim 3), and this is by a black edge due to an inductive invocation of (III). Therefore, the contractions into \( v_3 \) do not create a red edge to \( u \). If \( u_3 \neq u \), then let \( u'_3 \) denote the other vertex of \( P_1 \cup P_2 \) on the level \( \lambda[G](u_3) \). Analogously to the previous, one of the contractions into \( v_3 \) does not create a red edge to \( \{ v_3, u'_3 \} \) and the other contraction can do so, but at most one red edge to each of \( u_3, u'_3 \). Therefore, (IV) is true here, and the remaining properties of Lemma 5 are fulfilled easily.

In the remaining cases, we possibly add the following bit in a sequence \( \sigma_1 \) after \( \sigma_0 \) (while this bit has not been possible in the special subcase above): If \( u_3 \in V(P_1) \setminus \{ u \} \) and \( v_3 \) is a neighbour of both \( x_1, y_1 \) (of \( h_1 \)), then \( U^0_1 \) by (IV) consists of at most two vertices, which we contract into one vertex in \( \sigma_1 \) – this move adds one red edge incident to \( u_3 \). Analogously, if \( u_3 \in V(P_2) \setminus \{ u \} \) and \( u_3 \) is a neighbour of both \( v_2, v_3 \) (of \( f_2 \)), then we contract the at most two vertices of \( U^0_2 \) into one within \( \sigma_1 \). Although this contraction in \( \sigma_1 \) does not preserve levels, it is level-respecting by Claim 3 since \( U_{1,0} \) and \( U_2 \) were interior sections of the triangles \( C_{1,0} \) and \( C_2 \), respectively. In both cases, the added red edge incident to \( u_3 \) does not violate the properties of Lemma 5; this follows from the bounds in (VII) of Claim 6 which are by at least one lower than the bounds in (III) of Lemma 5 and property (III) is void for \( u_3 \) unless we have got the previous special subcase. Otherwise, we leave \( \sigma_1 = \emptyset \).

After applying \( \sigma_1 \) to \( U^0_1, U^0_2 \) in \( G^0 \), we get the trigraph \( G^1 \) and the sets \( U^1_1, U^1_2 \) (which are identical to the former ones except possibly \( U^0_1 \) or \( U^0_2 \)). See Figure 3.

In the next steps, we are going to define level-preserving partial contraction sequences \( \sigma_{2,a}, \sigma_{2,a-1}, \ldots, \sigma_{2,0} \) which, when concatenated after \( \sigma_0 \cdot \sigma_1 \), give the desired outcome. If \( a = 0 \), the sequence \( \sigma_{2,0} \) is going to contract the sets \( U^1_{1,0} \) with \( S_0 := V(P_3) \setminus \{ u_3 \} \) and \( U^1_{2,0} := U_2 \). If \( a > 0 \), the sequence \( \sigma_{2,a} \) is going to contract \( U^1_{2,a} \) with \( S_a := V(P_3) \setminus \{ u_3 \} \) and \( U^1_{1,a} := U_1 \), where \( S_a \subseteq V(P_3) \setminus \{ u_3 \} \) and \( U^1_{1,a} \subseteq U_2 \) are both the subsets of those vertices on levels greater than \( \lambda[G](y_a) \). The sequence \( \sigma_{2,a} \) for \( 0 \leq i < a \) is going to contract \( U^1_{1,i} \) with the
10 Twin-width of Planar Graphs is at most 8

![Figure 3](image)

**Figure 3** Proof of Lemma 5, a schematic picture of the situation after the parts of the depicted vertical-horizontal division of $G_C$ have been recursively contracted (right before the $\sigma_2$-contractions start). The cases of $u_3$ on the left and right paths are not symmetric in general.

sets $S_i$ and $U^2_{i,j}$, where $S_i \subseteq V(P_i) \setminus \{u_3\}$ and $U^2_{i,j} \subseteq U_2$ are the subsets of those vertices on levels greater (if $i \geq 1$) than $\lambda(G)\{y_i\}$ and not greater than $\lambda(G)\{y_{i+1}\}$. Of course, some of these sets may be empty, and hence some contractions may not happen.

Specifically, for $i \in \{0, \ldots, a\}$ let $p = \max_{z \in C_{1,1}} \lambda(G)(z)$ and $q = 1 + \min_{z \in C_{1,1}} \lambda(G)(z)$. Observe that there is no vertex in $U^1_{i,1} \cup U^2_{i,2}$ of level lower than $q$ or greater than $p$. This follows from an inductive invocation of property (IV) of Lemma 5 and from the sequence $\sigma_1$. So, the union $U^1_{i,1} := U^1_{1,0} \cup \ldots \cup U^1_{1,a}$ has at most one vertex on each level. Likewise, each of the sets $V(P_3)$ and $U^1_{3,1}$ has at most one vertex on each level. The sequence $\sigma_2$, first runs over $j = p, p-1, \ldots, q$ in this order, and contracts the pair of vertices of $S_i \cup U^1_{i,1}$ of the equal level $j$ in $G^1$ (or nothing if there is at most one such vertex there). In its second round, $\sigma_2$, again runs over $j = p, p-1, \ldots, q$ in this order, and contracts the vertex of level $j$ that stems from $S_i \cup U^1_{i,1}$ in the first round, with the vertex of $U^1_{i,1}$ of equal level $j$ in $G^1$.

Let $\sigma_2$ be the concatenation of the described sequences, $\sigma_2 := \sigma_{2,a} \cdot \sigma_{2,a-1} \cdot \ldots \sigma_{2,0}$ in this order, and $G^2$ denote the trigraph which results from $G^1$ by applying $\sigma_2$. Let $U^2 := V(G^2) \setminus (V(C) \cup W)$ denote the contracted vertices in the interior of $C$ in $G^2$. Then $G^2$ and $U^2$ satisfy property (IV) of Lemma 5 (in the place of $G^*$ and $U^*$), which is immediate from the previous definition of $\sigma_2$. It thus remains to verify the properties (III) and (II) of Lemma 5 along the sequence $\sigma_2$ from $G^1$ to $G^2$, that is, for every trigraph $G'$ along $\sigma_2$.

Denote by $U' := V(G') \setminus (V(C) \cup W)$ all interior vertices of $C$ in $G'$, and by $U'' := U' \setminus V(G^1)$ the (new) interior vertices that stem by $\sigma_2$-contractions from $G^1$ to $G'$, and recall (from Section 3.2) that $P_{31} \supseteq P_3$ is the right path of $C_1$ and $P_{32} \supseteq P_3$ is the left path of $C_2$ in $G_C$.

We start with verification of property (III) which has already been in parts addressed above. Regarding the sink vertex $u$, it has got red edges neither from the sequence $\sigma_0$ by an inductive invocation of property (III), nor from the sequence $\sigma_1$. The vertices of $C$ on levels up to $k - 2$ as in property (III) do not have any neighbour in $U'$ by Claim 4. Consider the vertices $z, z' \in V(C)$ on the level $k - 1$ as in property (III) (if $k \geq \ell + 2$ there). If $\lambda(G)\{u_3\} \geq k$, then no contraction on the level $k$ happens within $\sigma_2$ (Figure 2), and so $z$ and $z'$ have each at most one red edge to $U''$ by an inductive invocation of property (III). Otherwise, up to symmetry, $z' = u_2$. Similarly as argued earlier in this proof, $u_3$ then has a black edge to $U^1_{1,0}$ (if $u_3 \in V(P_i)$) or
to $U_2^1$ (if $u_3 \in V(P_2)$) in $G^1$, and so the contraction on the level $k$ incident with this black edge does not create a new red edge to either of $z, u_3$. At the same time, each of $z, u_3$ has at most one red edge in $G^1$ by an inductive invocation of (III), and this stays true also (with the set $U'$) during and after contractions on the level $k$ within $\sigma_2$.

We move towards verification of (I). Let $z \in U'$ for the rest. If $z \in U_2^1$, then no $\sigma_2$-contraction has touched $z$ so far. In this case $z$ may have red edges to up to 3 vertices of $V(P_2) \cup U_2^1 \cup V(P_{32})$ of level $\lambda(G^1)(z) - 1$, to 2 vertices of $V(P_2) \cup V(P_{32})$ of level $\lambda(G^1)(z)$, and to 2 vertices of $V(P_{32}) \cup U_2^1 \cup U'''$ of level $\lambda(G^1)(z) + 1$, altogether at most 7. Note that there is no edge from $z$ to the vertex of $P_2$ of level $\lambda(G^2)(z) + 1$ by (VI) of Claim 6. If $z \in V(P_3) \setminus \{u_3\}$, then similarly, $z$ may have red edges to up to 2 + 2 vertices of $U_1^1 \cup U_2^1$ on the levels $\lambda(G^1)(z) - 1$ and $\lambda(G^1)(z)$, and to up to 2 vertices of $U_1^1 \cup U_2^1 \cup U'''$ on the level $\lambda(G^1)(z) + 1$. The case of $z \in U_1^1$ (not-yet touched by a $\sigma_2$-contraction) is similarly easy.

Assume now that $z \in U''$ has been created in $G'$ by a contraction of $z_2 \in U_2^1$ and $z_3 \in V(P_3) \setminus \{u_3\}$ (i.e., within the first round of some $\sigma_{2,i}$ above), but $z$ is not contracted with a vertex of $U_1^1$ yet. Let $t \in V(P_1) \setminus V(P_3)$ denote the possible (unless equal to $u_3$) vertex of $P_1$ of level $\lambda(G^1)(z_3) - 1$. Then there is no edge in $G^1$ from $t$ to $z_2$ by planarity, and no from $t$ to $z_3$ by (VI) of Claim 6. The same applies to the possible vertex $t' \in U_1^1$ of level $\lambda(G^1)(z_3) - 1$. Consequently, $z$ may have red edges to up to 3 vertices of $V(P_2) \cup U_2^1 \cup V(P_{32})$ of level $\lambda(G^1)'(z) - 1$, to 3 vertices of $V(P_2) \cup V(U_1^1) \cup V(P_1)$ of level $\lambda(G^1)'(z)$, and to up to 3 vertices of $U''' \cup (U_1^1) \cup V(P_1)$ of level $\lambda(G^1)'(z) + 1$, again using (VI). This sums to $3 + 3 + 3 = 9$, but we are going to show that this maximum of 9 cannot be achieved. Let $z_1 \in V(P_1)$ be such that $\lambda(G^1)'(z_1) = \lambda(G^1)'(z)$. If $\{z_1, z_3\} \not\in E(G)$, then no red edge $\{z_1, z\}$ is created by the current contraction and the sum is at most 8, as needed. If $\{z_1, z_3\} = h_i \in E(G)$, then the sequence $\sigma_{2,i}$ has already contracted $S_i \cup U_{1,i}^1$ into $U''$, and so there are only 2 red neighbours of $z$ in $U''' \cup V(P_1)$ on the level $\lambda(G^1)'(z) + 1$, again summing to at most 8. If $\{z_1, z_3\} \in E(G)$, but $z_1, z_3$ has not been chosen as any $h_i$ in the vertical-horizontal division above, then there are no vertices in $U_2^1 \cup (V(P_3) \setminus \{u_3\})$ on the level $\lambda(G^1)'(z) - 1$, and so the sum is at most 8.

Assume that $z \in U''$ has already been created in $G''$ by a contraction of all vertices in $U_2^1 \cup V(P_2) \cup U_1^1$ of the same level. Let this contraction be part of $\sigma_{2,i}$, for some $0 \leq i \leq a$. Then $z$ may have red edges to up to 2 vertices of $V(P_2) \cup V(P_1)$ of level $\lambda(G^1)'(z)$, to 2 vertices of $U''' \cup V(P_1)$ of level $\lambda(G^1)'(z) + 1$ (but not to $V(P_2)$ due to (VI)), and to vertices of $V(P_2) \cup U_2^1 \cup V(P_3) \cup V(U_1^1) \cup U''' \cup V(P_1) =: Y$ of level $\lambda(G^1)'(z) - 1$. However, at most 4 of the vertices of $Y$ are potential red neighbours of $z$ (so summing to at most 8), as we now show. If contractions on the level $\lambda(G^1)'(z) - 1$ are part of $\sigma_{2,i}$, too, then red neighbours of $z$ in $Y$ of level $\lambda(G^1)'(z) - 1$ actually belong to $V(P_2) \cup U''' \cup V(U_1^1) \cup V(P_1)$ with an upper bound of 4. Otherwise, if contractions on the level $\lambda(G^1)'(z) - 1$ are part of $\sigma_{2,i-1}$, then there is no edge from $z$ to a vertex of $U_{1,i-1}^1$, or $U_{1,i-1}^1$ on the level $\lambda(G^1)'(z) - 1$ has already been contracted into $U'''$, too. Then red neighbours of $z$ in $Y$ of level $\lambda(G^1)'(z) - 1$ belong to $V(P_2) \cup U_2^1 \cup V(P_3) \cup V(P_1)$ or to $V(P_2) \cup U''' \cup V(P_1)$, and we again get a bound of 4.

Finally, we want to verify (II) of Lemma 5. Consider $z \in V(P_2) \setminus \{u\}$. By the definition of $\sigma_2$, the vertex $z$ may have at most one red neighbour of each level in $U''$ (at any moment of $\sigma_2$). Then the bound of at most 3 red edges from $z$ to $U''$ follows immediately in view of Claim 5. Consider now $z \in V(P_1) \setminus \{u\}$, which is a bit more complicated case. On each of the levels $\lambda(G^1)'(z) - 1$, $\lambda(G^1)'(z)$ and $\lambda(G^1)'(z) + 1$ of $U''$, there are clearly at most 2 red neighbours of $z$. Although, we now show that the maximum sum of 6 cannot be achieved. If $z = u_3$, then there is actually at most red neighbour of $z$ on the level $\lambda(G^1)'(z) - 1$. Otherwise, we denote the following vertices of $G^1$ of level $\lambda(G^1)'(z) + 1$ by $z_1, z_2, z_3$ such that $z_1 \in U_1^1$,
12 Twin-width of Planar Graphs is at most 8

$z_2 \in U_1^1$, $z_3 \in V(P_3)$, and $\lambda(G^1)(z_1) = \lambda(G^1)(z_2) = \lambda(G^1)(z_3) = \lambda(G^1)(z) + 1$. Then $z_3$ has no edge to $z \neq u_3$ by (VI) of Claim 6 and $z_2$ has no edge to $z$ by planarity. If $z_3 \in U'$ (i.e., not contracted yet), then only $z_1$ may be a red neighbour of $z$. If $z_2$ and $z_3$ have already been contracted in $G'$, but $z_1 \in U'$, then the new vertex again has no edge to $z$. Finally, if all of $z_1, z_2, z_3$ have been contracted in $G'$, then $U'$ has only (this) one vertex of level $\lambda(G^1)(z) + 1$.

In any case, $z$ has at most 5 red neighbours in $U'$.

We have verified all conditions of Lemma 5 for the partial contraction sequence $\sigma_0 \cdot \sigma_1 \cdot \sigma_2$, and so we can set $G^* := G^2$ and the proof is done.

Proof of Theorem 1 (the algorithmic part). We can construct a simple plane triangulation $G \supseteq H$ in linear time using standard planarity algorithms, and then construct a left-aligned BFS tree $T \subseteq G$ again in linear time by Lemma 3. In the rest, we straightforwardly implement the recursive vertical-horizontal division of $G$ as used in the proof of Lemma 5 and construct the contraction sequence of $H$ on return from the recursive calls as defined in the proof. Note that we do not need at all to construct the intermediate trigraphs along the constructed contraction sequence, and so the construction of the sequence is very easy—each recursive call returns just a simple list of the vertices which stem from the recursive contractions, indexed by the levels. Then these (up to) two lists are easily in linear time “merged” together with the dividing path $P_3$, as specified by the proof of Lemma 5 into the resulting list of this call.

We may account total runtime in the “division part” of the algorithm to the edge(s) of $v_3$ into $v_1$ or $v_2$ and the edges of the path $P_3$ starting in $v_3$ in each call of the recursion, and these edges are not counted multiple times in different branches of the recursion. Likewise, runtime of the “merging” part of each recursive call can be counted to the individual steps of the resulting contraction sequence, which is of linear length. Hence, altogether, the algorithm runs in linear time.

4 Conclusion

We have further improved by one the previous best upper bound [5] on the twin-width of planar graphs. This seemingly small improvement has required a careful reconsideration of the previous method and several new ideas, and although our new approach has simplified some cumbersome technical details in [5], new technical difficulties emerged which makes some parts of the proof again quite technical. This is probably to be expected since we are now very close to the currently best lower bound of 7 on the twin-width of planar graphs [2].

The problem to determine the exact maximum value of twin-width over planar graphs is still open, and we have no good guess about which of the two possible values, 7 or 8, is the right answer.

Furthermore, [2] has also proved an upper bound of 6 on the twin-width of bipartite planar graphs. Some of the ideas of this paper may be used to significantly simplify that proof from [5], but we are not yet able to improve the bound of 6 for bipartite planar graphs further down.

References

1 Michael A. Bekos, Giordano Da Lozzo, Petr Hlinéný, and Michael Kaufmann. Graph product structure for h-framed graphs. CoRR, abs/2204.11495, 2022. Accepted to ISAAC 2022. arXiv:2204.11495v1.

2 Édouard Bonnet, Eun Jung Kim, Stéphan Thomassé, and Rémi Watrigant. Twin-width I: tractable FO model checking. J. ACM, 69(1):3:1–3:46, 2022.
Édouard Bonnet, O-joung Kwon, and David R. Wood. Reduced bandwidth: a qualitative strengthening of twin-width in minor-closed classes (and beyond). *CoRR*, abs/2202.11858, 2022. [arXiv:2202.11858](https://arxiv.org/abs/2202.11858).

Vida Dujmovic, Gwenaël Joret, Piotr Micek, Pat Morin, Torsten Ueckerdt, and David R. Wood. Planar graphs have bounded queue-number. *J. ACM*, 67(4):22:1–22:38, 2020.

Petr Hliněný. Twin-width of planar graphs is at most 9, and at most 6 when bipartite planar. *CoRR*, abs/2205.05378, 2022. [arXiv:2205.05378](https://arxiv.org/abs/2205.05378).

Hugo Jacob and Marcin Pilipczuk. Bounding twin-width for bounded-treewidth graphs, planar graphs, and bipartite graphs. *CoRR*, abs/2201.09749, 2022. Accepted to WG 2022. [arXiv:2201.09749](https://arxiv.org/abs/2201.09749).

Daniel Král and Ander Lamaison. Planar graph with twin-width seven. *CoRR*, abs/2209.11537, 2022. [arXiv:2209.11537](https://arxiv.org/abs/2209.11537).