SWITCHING CONTROLS FOR LINEAR STOCHASTIC DIFFERENTIAL SYSTEMS

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ABSTRACT. We analyze the exact controllability problem of switching controls for stochastic control systems endowed with different actuators. The goal is to control the dynamics of the system by switching from an actuator to the other such that, in each instant of time, there are as few active actuators as possible. We prove that, under suitable rank conditions, switching control strategies exist and can be built in a systematic way. The proof is based on building a new functional by the adjoint system whose minimizers are the switching controls.

1. Introduction. Let $T > 0$ and $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in [0,T]}, \mathbb{P})$ be a complete filtered probability space on which a one-dimensional standard Brownian motion $\{W(t)\}_{t \in [0,T]}$ is defined and $\{\mathcal{F}_t\}_{t \in [0,T]}$ is the natural filtration generated by $\{W(t)\}_{t \in [0,T]}$.

Let $k \in \mathbb{N}$. For any $t \in [0,T]$, denote by $L^2_{\mathcal{F}_t}(\Omega; \mathbb{R}^k)$ the Hilbert space of all $\mathcal{F}_t$-measurable random variables $\xi : \Omega \to \mathbb{R}^k$ so that $\mathbb{E}[|\xi|^2_{\mathbb{R}^k}] < \infty$, with the canonical norm. Write $L^2_{\mathcal{F}}(\Omega; C([0,T]; \mathbb{R}^k))$ for the Banach space of all $\mathbb{R}^k$-valued $\{\mathcal{F}_t\}_{t \in [0,T]}$-adapted, continuous stochastic processes $f(\cdot)$, with the following norm

$$|f(\cdot)|_{L^2_{\mathcal{F}}(\Omega; C([0,T]; \mathbb{R}^k))} := \left(\mathbb{E}\sup_{t \in [0,T]} |f(t)|^2_{\mathbb{R}^k}\right)^{1/2}.$$ 

Denote by $L^2_{\mathcal{F}}(0,T; \mathbb{R}^k)$ the Hilbert space of all $\{\mathcal{F}_t\}_{t \in [0,T]}$-adapted, stochastic processes $f(\cdot)$ with the following norm

$$|f(\cdot)|_{L^2_{\mathcal{F}}(0,T; \mathbb{R}^k)} := \left(\mathbb{E}\int_0^T |f(t)|^2_{\mathbb{R}^k} dt\right)^{1/2}.$$ 

Let

$$\Gamma := \{\gamma \in L^2_{\mathcal{F}}(0,T) : \gamma(t) \in \{0,1\} \text{ for a.e. } (t,\omega) \in [0,T] \times \Omega\}.$$ 

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In this paper, we study the exact controllability for the following linear stochastic differential equation with switching controls:

\[
\begin{align*}
    dx &= [Ax + \gamma B_1 u_1 + (1 - \gamma) B_2 u_2]dt \\
    &\quad + [Cx + \hat{\gamma} D_1 v_1 + (1 - \hat{\gamma}) D_2 v_2]dW(t) \quad \text{in } [0, T],
\end{align*}
\]

where

\[
A \in \mathbb{R}^{n \times n}, \quad B_1, B_2 \in \mathbb{R}^{n \times m}, \quad C \in \mathbb{R}^{n \times n}, \quad D_1, D_2 \in \mathbb{R}^{n \times l}, \quad n, m, l \in \mathbb{N},
\]

\[
x_0 \in \mathbb{R}^n, \quad \gamma, \hat{\gamma} \in \Gamma, \quad u_1(\cdot), u_2(\cdot) \in L^2_\Omega(0, T; \mathbb{R}^m) \quad \text{and} \quad v_1(\cdot), v_2(\cdot) \in L^2_\Omega(0, T; \mathbb{R}^l)
\]

are controls and \( x(\cdot) \in L^2_\Omega(0, T; \mathbb{R}^n) \) is the state trajectory.

**Definition 1.1.** System (1) is said to be exact controllable with switching controls if for any \( x_0 \in \mathbb{R}^n \) and \( x_1 \in L^2_\Gamma(\Omega; \mathbb{R}^n) \), there are \( \gamma, \hat{\gamma} \in \Gamma, \) \( u_1(\cdot), u_2(\cdot) \in L^2_\Omega(0, T; \mathbb{R}^m) \) and \( v_1(\cdot), v_2(\cdot) \in L^2_\Omega(0, T; \mathbb{R}^l) \) such that the corresponding solution satisfies that \( x(T) = x_1 \).

**Remark 1.** The motivation to introduce the above notion is as follows:

Generally speaking, one needs very restrictive conditions on controls to get the exact controllability for the control system:

\[
\begin{align*}
    dx &= (Ax + Bu)dt + (Cx + Dv)dW(t) \quad \text{in } [0, T], \\
    x(0) &= x_0.
\end{align*}
\]

For example, in [17], Peng proved that a necessary condition for the exact controllability of (2) is that \( \text{rank}(D) = n \) and \( (A, B) \) fulfills a Kalman-type rank condition. These conditions on the two actuators are too restrictive. But they are necessary. To relax these conditions, we use four actuators. At every time \( t \in [0, T] \), only two of them act on the system. By this, neither \( \text{rank}(D_1) = \text{rank}(D_2) = n \) nor \( (A, B_1) \) or \( (A, B_2) \) should fulfill a Kalman-type rank condition is needed. Instead of this, we assume that \( \text{rank}(D_1, D_2) = n \) and that \( (A, (B_1, B_2)) \) satisfies the Kalman rank condition. This relaxes the restrictions on the actuators.

**Remark 2.** In system (1), \( \gamma \) and \( \hat{\gamma} \) decide the switching mode of controls. One needs to choose different \( \gamma \) and \( \hat{\gamma} \) for different \( x_0 \) and \( x_1 \). The way of choosing \( \gamma \) and \( \hat{\gamma} \) can be seen in Section 2.

**Remark 3.** In this paper, the controls in the drift term and the diffusion term are different. It is very interesting to consider the control problem in which the controls are the same. However, this problem is much more difficult and beyond the scope of this paper.

As the development of control theory, controllability problems for stochastic systems drew more and more attention in recent years (see [2, 5, 7, 8, 9, 10, 11, 12, 13, 14, 17, 20, 21, 22, 23] and the rich references therein). Especially, the controllability problem for linear stochastic differential equations are studied in [2, 4, 8, 14, 17, 21]. In [17], it is proved that (2) is exact controllable only if the rank of \( D \) is \( n \). Moreover, a sufficient condition (in the form of rank condition) for exact controllability of (2) is given in [17]. Some generalizations of the results in [17] are obtained in [10]. In [14], it is proved that one can use controls in \( L^r_\Omega(0, T; L^2(\Omega; \mathbb{R}^m)) \) to get the exact controllability of (2) when \( D = 0 \). Moreover, it is shown in [14] that if \( D = 0 \), (2) is not exactly controllable when the control space is \( L^r_\Omega(0, T; L^2(\Omega; \mathbb{R}^m)) \) for any \( r > 1 \). Some generalizations for results in [14] are done in [21]. In [2, 8], the approximate controllability problem for (2) is studied. Particularly, a rank condition
for the approximate controllability of (2) is given in [8]. In [4], a negative result for approximate controllability is given when \( D = 0 \). Furthermore, a characterization of the reachable set of (2) when \( D = 0 \) is given in [4].

Control systems in real applications are often endowed with several actuators. Switching controllers arise in many fields of applications (see [1, 3, 6, 15, 16, 18, 19, 24] for example). In this paper, we consider the problem that, to design switching control strategies to guarantee that, at each instant of time, there are as few activated controls as possible to drive the state of the system to a given destination. Compared with the existing works on the exact controllability of linear stochastic differential equations, the main difference of our results are that we do not need the control matrix in the diffusion term to be full rank. To get this, we construct a way to design the switching mode. More details can be seen in Section 2. We borrow some idea in [24] to design our control. However, since the solution of the adjoint system is not analytic with respect to the time variable, we can not simply use the method in [24].

2. Main results and their proofs. In this section, we present the main results in this paper and their proofs.

Consider the following backward stochastic differential equations

\[
\begin{align*}
    dz &= -(A^\top z + C^\top Z)dt + ZdW(t) \quad \text{in } [0, T]; \\
    z(T) &= z_T,
\end{align*}
\]

where \( z_T \in L^2_{\mathcal{F}_T}(\Omega; \mathbb{R}^n) \).

It is well known that (3) has a unique solution and there is a constant \( L > 0 \) such that

\[
|z|_{L^2_{\mathcal{F}}((0,T);\mathbb{R}^n)} + |Z|_{L^2_{\mathcal{F}}((0,T);\mathbb{R}^n)} \leq L |z_T|_{L^2_{\mathcal{F}}(\Omega;\mathbb{R}^n)}. \tag{4}
\]

For \( x_0 \in \mathbb{R}^n \) and \( x_1 \in L^2_{\mathcal{F}_T}(\Omega; \mathbb{R}^n) \), define a functional

\[
J_{x_0, x_1}(\cdot) : L^2_{\mathcal{F}_T}(\Omega; \mathbb{R}^n) \to \mathbb{R}
\]

as

\[
J_{x_0, x_1}(z_T) = \frac{1}{2} \int_0^T \left( \max \{ \mathbb{E}|B_1^\top z|^2, \mathbb{E}|B_2^\top z|^2 \} + \max \{ \mathbb{E}|D_1^\top Z|^2, \mathbb{E}|D_2^\top Z|^2 \} \right) dt \\
- \mathbb{E}(x_1 \cdot z_T) + x_0 \cdot z(0). \tag{5}
\]

**Theorem 2.1.** Assume that \( \text{rank}(D_1, D_2) = n \) and \( (A, (B_1, B_2)) \) satisfies the Kalman rank condition. Then, for all \( T > 0 \), the functional \( J_{x_0, x_1}(\cdot) \) achieves at least one minimum at some minimizer \( \hat{z}_T \).

**Proof.** Denote by \( \phi(\cdot) \) the fundamental solution of (2) with \( u = 0 \) and \( v = 0 \). Without loss of generality, we can assume that \( x_1 \neq \phi(T)x_0 \). Indeed, when \( x_1 = \phi(T)x_0 \), the null controls \( u_1 = u_2 = 0 \) and \( v_1 = v_2 = 0 \) suffice to drive the initial datum \( x_0 \) to the final one \( x_1 \), and they satisfy the switching condition. Thus, in the sequel, we assume that \( x_0 \) and \( x_1 \) are given such that \( x_1 \neq \phi(T)x_0 \).

Firstly, we will prove that the functional \( J_{x_0, x_1} : L^2_{\mathcal{F}_T}(\Omega; \mathbb{R}^n) \to \mathbb{R} \) is continuous. Let \( \tilde{z}_T, \hat{z}_T \in L^2_{\mathcal{F}_T}(\Omega; \mathbb{R}^n) \). Denote by \((\tilde{z}, \tilde{Z})\) and \((\hat{z}, \hat{Z})\) the solution of (3) with the final data \( \tilde{z}_T \) and \( \hat{z}_T \), respectively. Then
where \((\hat{\xi}, \hat{Z})\) is the solution of (3) with the final data \(\hat{\xi}_T - \hat{\xi}_T\). Without loss of generality, we assume that \(|\hat{\xi}_T - \hat{\xi}_T|_{L^2_x(\Omega; \mathbb{R}^n)} < 1\). From (4), we know that

\[ |J_{x_0, x_1}(\hat{\xi}_T) - J_{x_0, x_1}(\hat{\xi}_T) | \leq L(B_1|_{\mathbb{R}^n}) + |B_2|_{\mathbb{R}^n} + |D_1|_{\mathbb{R}^n} + |D_2|_{\mathbb{R}^n} + |x_1|_{L^2_x(\Omega; \mathbb{R}^n)} + |x_0|_{\mathbb{R}^n} \]

Hence, for given \(\varepsilon > 0\), by taking

\[ \delta = \varepsilon L(B_1|_{\mathbb{R}^n}) + |B_2|_{\mathbb{R}^n} + |D_1|_{\mathbb{R}^n} + |D_2|_{\mathbb{R}^n} + |x_1|_{L^2_x(\Omega; \mathbb{R}^n)} + |x_0|_{\mathbb{R}^n} \]

we have that for any \(\tilde{\xi}_T \in L^2_{\tilde{x}_T}(\Omega; \mathbb{R}^n)\) satisfying

\[ |\tilde{\xi}_T - \hat{\xi}_T|_{L^2_{\tilde{x}_T}(\Omega; \mathbb{R}^n)} < \min\{1, \delta\}, \]

it holds

\[ |J_{x_0, x_1}(\tilde{\xi}_T) - J_{x_0, x_1}(\hat{\xi}_T) | < \varepsilon. \]

Thus, \(J_{x_0, x_1}\) is continuous.

Next, we will prove that \(J_{x_0, x_1}\) is convex. Let \(\lambda \in (0, 1)\) and assume that \(\hat{\xi}_T \neq \tilde{\xi}_T\). Then

\[ J_{x_0, x_1}(\lambda \hat{\xi}_T + (1 - \lambda) \tilde{\xi}_T) = \frac{1}{2} \int_0^T \left( \max \left\{ \mathbb{E}|\lambda B_1 \tilde{\xi}|^2 + (1 - \lambda) B_1 \hat{\xi}^2, \mathbb{E}|\lambda B_2 \tilde{\xi}|^2 + (1 - \lambda) B_2 \hat{\xi}^2 \right\} \right) dt \]

\[ -\lambda \mathbb{E}(x_1 \cdot \hat{\xi}_T) - (1 - \lambda) \mathbb{E}(x_1 \cdot \tilde{\xi}_T) + \lambda x_0 \cdot \hat{\xi}(0) + (1 - \lambda) x_0 \cdot \tilde{\xi}(0) \]

\[ \leq \frac{1}{2} \int_0^T \left( \max \left\{ \mathbb{E}|B_1 \tilde{\xi}|^2 + (1 - \lambda) \mathbb{E}|B_1 \hat{\xi}|^2, \mathbb{E}|B_2 \tilde{\xi}|^2 + (1 - \lambda) \mathbb{E}|B_2 \hat{\xi}|^2 \right\} \right) dt \]

\[ -\lambda \mathbb{E}(x_1 \cdot \hat{\xi}_T) - (1 - \lambda) \mathbb{E}(x_1 \cdot \tilde{\xi}_T) + \lambda x_0 \cdot \hat{\xi}(0) + (1 - \lambda) x_0 \cdot \tilde{\xi}(0) \]

\[ \leq \frac{1}{2} \int_0^T \left( \max \left\{ \mathbb{E}|B_1 \tilde{\xi}|^2, \mathbb{E}|B_2 \tilde{\xi}|^2 \right\} + \max \left\{ \mathbb{E}|D_1 \tilde{\xi}|^2, \mathbb{E}|D_2 \tilde{\xi}|^2 \right\} \right) dt \]

\[ +\frac{1}{2} (1 - \lambda) \int_0^T \left( \max \left\{ \mathbb{E}|B_1 \hat{\xi}|^2, \mathbb{E}|B_2 \hat{\xi}|^2 \right\} + \max \left\{ \mathbb{E}|D_1 \hat{\xi}|^2, \mathbb{E}|D_2 \hat{\xi}|^2 \right\} \right) dt \]

\[ -\lambda \mathbb{E}(x_1 \cdot \hat{\xi}_T) - (1 - \lambda) \mathbb{E}(x_1 \cdot \tilde{\xi}_T) + \lambda x_0 \cdot \hat{\xi}(0) + (1 - \lambda) x_0 \cdot \tilde{\xi}(0) \]

\[ = \lambda J_{x_0, x_1}(\hat{\xi}_T) + (1 - \lambda) J_{x_0, x_1}(\tilde{\xi}_T). \]

Hence, \(J_{x_0, x_1}\) is convex.
Next, we will prove that $J_{x_0,x_1}$ is coercive. It is sufficient to show that there exists a positive constant $L > 0$ such that

$$
E|z_T|^2 \leq L \int_0^T \left( \max\{E|B_1^T z|^2, E|B_2^T z|^2\} + \max\{E|D_1^T Z|^2, E|D_2^T Z|^2\} \right) dt.
$$

Noting that

$$
E|B_1^T z|^2 + E|B_2^T z|^2 + E|D_1^T Z|^2 + E|D_2^T Z|^2
\leq 2\left( \max\{E|B_1^T z|^2, E|B_2^T z|^2\}, \max\{E|D_1^T Z|^2, E|D_2^T Z|^2\} \right),
$$

we only need to prove that there is a constant $L > 0$ such that

$$
E|z_T|^2 \leq LE \int_0^T \left( |B_1^T z|^2 + |B_2^T z|^2 + |D_1^T Z|^2 + |D_2^T Z|^2 \right) dt. \tag{6}
$$

Consider the following stochastic differential equation:

$$
\begin{cases}
dy = -(A^T y + C^T g)dt + gdW(t) & \text{in } [0, T], \\
y(0) = y_0.
\end{cases} \tag{7}
$$

In (7), $y$ is the solution and $g$ is the nonhomogeneous term. For a given $z_T \in L^2_{\mathcal{F}_T}(\Omega; \mathbb{R}^n)$ and the corresponding solution $(z, Z)$, clearly, $z$ is a solution of (7) with the initial data $y(0) = z(0)$ and nonhomogeneous term $g = Y$. Hence, we only need to prove that there is a constant $L > 0$ such that for any $y_0 \in \mathbb{R}^n$ and $g \in L^2(0, T; \mathbb{R}^n)$,

$$
E|y(T)|^2_{\mathbb{R}^n} \leq LE \int_0^T \left( |B_1^T y|^2 + |B_2^T y|^2 + |D_1^T g|^2 + |D_2^T g|^2 \right) dt. \tag{8}
$$

From the wellposedness result for stochastic differential equations, we know that there is $L > 0$ such that for any $y_0 \in \mathbb{R}^n$ and $g \in L^2(0, T; \mathbb{R}^n)$,

$$
E|y(T)|^2_{\mathbb{R}^n} \leq L(|y(0)|^2_{\mathbb{R}^n} + |g|^2_{L^2(0, T; \mathbb{R}^n)}). \tag{9}
$$

Since the rank of $(D_1, D_2)$ is $n$, there is a constant $L > 0$ such that

$$
|g|^2_{L^2(0, T; \mathbb{R}^n)} \leq LE \int_0^T \left( |D_1^T g|^2 + |D_2^T g|^2 \right) dt. \tag{10}
$$

Let $\bar{y} = Ey$. Then $\bar{y}$ is the solution of

$$
\begin{cases}
d\bar{y} = -(A^T \bar{y} + C^T \bar{E} g)dt & \text{in } [0, T]; \\
\bar{y}(0) = y_0.
\end{cases} \tag{11}
$$

The equation (11) is an ordinary differential equation. Since $(A, (B_1, B_2))$ satisfies the Kalman rank condition, there is a constant $L > 0$ such that for all $y_0 \in \mathbb{R}^n$ and $g \in L^2(0, T; \mathbb{R}^n)$,

$$
|y_0|^2_{\mathbb{R}^n} \leq L \int_0^T \left( |B_1^T \bar{y}|^2 + |B_2^T \bar{y}|^2 + |\bar{E} g|^2 \right) dt. \tag{12}
$$

From (9), (10) and (12), we get (8). Therefore, we find that $J_{x_0,x_1}$ is coercive. By the above argument, we see that $J_{x_0,x_1}$ is continuous, convex and coercive. Hence, it has at least a minimizer. \qed
Let $\tilde{z}$ be a minimizer of $J_{\alpha_0, \alpha_1} (\cdot)$ and $(\tilde{z}, \tilde{Z})$ be the corresponding solution of (3). Put

$$I_1 := \{ t \in [0, T] : \mathbb{E}|B_1^T \tilde{z}|^2 > \mathbb{E}|B_2^T \tilde{z}|^2 \},$$

$$I_2 := \{ t \in [0, T] : \mathbb{E}|B_1^T \tilde{z}|^2 < \mathbb{E}|B_2^T \tilde{z}|^2 \},$$

$$I_3 := \{ t \in [0, T] : \mathbb{E}|D_1^T \tilde{Z}|^2 > \mathbb{E}|D_1^T \tilde{Z}|^2 \},$$

$$I_4 := \{ t \in [0, T] : \mathbb{E}|D_1^T \tilde{Z}|^2 < \mathbb{E}|D_1^T \tilde{Z}|^2 \}.$$

**Theorem 2.2.** If

$$m(I_1 \cup I_2) = m(I_3 \cup I_4) = T,$$

then by choosing the switching controller

$$\begin{cases}
  u_1(t) = B_1^T \tilde{z}, & \text{if } t \in I_1, \\
  u_2(t) = B_1^T \tilde{z}, & \text{if } t \in I_2,
\end{cases} \quad \begin{cases}
  v_1(t) = D_1^T \tilde{Z}, & \text{if } t \in I_3, \\
  v_2(t) = D_1^T \tilde{Z}, & \text{if } t \in I_4,
\end{cases}$$

we have the solution of (1) with $x(0) = x_0$ satisfying that $x(T) = x_1$ in $L^2_{\mathcal{F}_T} (\Omega; \mathbb{R}^n)$.

**Proof.** Let $\tilde{z}_T$ be a minimizer of $J_{\alpha_0, \alpha_1} (\cdot)$ and $(\tilde{z}, \tilde{Z})$ the corresponding solution of (3). For any $\psi_T \in L^2_{\mathcal{F}_T} (\Omega; \mathbb{R}^n)$, we have

$$J(\tilde{z}_T) \leq J(\tilde{z}_T + h \psi_T), \quad \forall h \in \mathbb{R}.$$ 

Denote by $(\psi, \Psi)$ the solution of (3) with the final data $\psi_T$. Then

$$\mathbb{E}(x_1 \cdot \psi_T) - \mathbb{E}(x_0 \cdot \psi(0))$$

$$\leq \lim_{h \to 0^+} \frac{1}{2h} \int_0^T \left[ \max\{\mathbb{E}|B_1^T (\tilde{z} + h \psi)|^2, \mathbb{E}|B_2^T (\tilde{z} + h \psi)|^2\} 
- \max\{\mathbb{E}|B_1^T \tilde{z}|^2, \mathbb{E}|B_2^T \tilde{z}|^2\} 
+ \max\{\mathbb{E}|D_1^T (\tilde{Z} + h \Psi)|^2, \mathbb{E}|D_1^T (\tilde{Z} + h \Psi)|^2\} 
- \max\{\mathbb{E}|D_1^T \tilde{Z}|^2, \mathbb{E}|D_1^T \tilde{Z}|^2\}\right] dt. \quad (15)$$

We claim that the limit on the right hand side of (15) is

$$\mathbb{E} \int_{I_1} B_1^T \tilde{z} \cdot B_1^T \psi dt + \mathbb{E} \int_{I_2} B_2^T \tilde{z} \cdot B_2^T \psi dt + \mathbb{E} \int_{I_3} D_1^T \tilde{Z} \cdot D_1^T \Psi dt + \mathbb{E} \int_{I_4} D_2^T \tilde{Z} \cdot D_2^T \Psi dt. \quad (16)$$

Indeed, it is easy to see that, pointwise a.e. in $(0, T)$,

$$\frac{1}{2h} \left[ \max\{\mathbb{E}|B_1^T (\tilde{z} + h \psi)|^2, \mathbb{E}|B_2^T (\tilde{z} + h \psi)|^2\} - \max\{\mathbb{E}|B_1^T \tilde{z}|^2, \mathbb{E}|B_2^T \tilde{z}|^2\} \right]$$

$$= \begin{cases}
  \mathbb{E}(B_1^T \tilde{z} \cdot B_1^T \psi), & \text{a.e. } t \in I_1, \\
  \mathbb{E}(B_2^T \tilde{z} \cdot B_2^T \psi), & \text{a.e. } t \in I_2,
\end{cases}$$

and

$$\frac{1}{2h} \left[ \max\{\mathbb{E}|D_1^T (\tilde{Z} + h \Psi)|^2, \mathbb{E}|D_2^T (\tilde{Z} + h \Psi)|^2\} - \max\{\mathbb{E}|D_1^T \tilde{Z}|^2, \mathbb{E}|D_2^T \tilde{Z}|^2\} \right]$$

$$= \begin{cases}
  \mathbb{E}(D_1^T \tilde{Z} \cdot D_1^T \Psi), & \text{a.e. } t \in I_3, \\
  \mathbb{E}(D_2^T \tilde{Z} \cdot D_2^T \Psi), & \text{a.e. } t \in I_4.
\end{cases}$$
as \( h \to 0 \). To get the limit (16) it is then sufficient to apply the dominated convergence theorem. To do that it suffices to show that

\[
\frac{1}{2h} \left[ \max \{ E[B^T_1 (\bar{z} + h\psi)]^2, E[B^T_2 (\bar{z} + h\psi)]^2 \} - \max \{ E[B^T_1 \bar{z}]^2, E[B^T_2 \bar{z}]^2 \} \right] \leq f_1(t), \quad \text{a.e. } t \in (0, T)
\]

and

\[
\frac{1}{2h} \left[ \max \{ E[D^T_1 (\bar{Z} + h\Psi)]^2, E[D^T_2 (\bar{Z} + h\Psi)]^2 \} - \max \{ E[D^T_1 \bar{Z}]^2, E[D^T_2 \bar{Z}]^2 \} \right] \leq f_2(t), \quad \text{a.e. } t \in (0, T),
\]

where \( f_1(\cdot), f_2(\cdot) \in L^1(0, T) \). We only prove (17) here. The proof of (18) is very similar.

The only difficulty of proving the uniform bound arises on the set where the two maximizers are not taken over the same component. Indeed, when both maximizers are taken over the same component, for instance, if

\[
\max \{ E[B^T_1 (\bar{z} + h\psi)]^2, E[B^T_2 (\bar{z} + h\psi)]^2 \} = E[B^T_1 (\bar{z} + h\psi)]^2
\]

and

\[
\max \{ E[B^T_1 \bar{z}]^2, E[B^T_2 \bar{z}]^2 \} = E[B^T_1 \bar{z}]^2,
\]

then the quotient in (17) can be bounded above by \( 2E(|B^T_1 \bar{z}| |B^T_1 \psi|) \), which is in \( L^1(0, T) \) since both \( \bar{z} \) and \( \psi \) belong to \( L^2(0, T) \).

Let us then consider the remaining case where, for instance,

\[
\max \{ E[B^T_1 (\bar{z} + h\psi)]^2, E[B^T_2 (\bar{z} + h\psi)]^2 \} = E[B^T_1 (\bar{z} + h\psi)]^2
\]

and

\[
\max \{ E[B^T_1 \bar{z}]^2, E[B^T_2 \bar{z}]^2 \} = E[B^T_2 \bar{z}]^2.
\]

In that case, the quotient in (17) coincides with

\[
\frac{1}{2h} \left[ E[B^T_1 (\bar{z} + h\psi)]^2 - E[B^T_2 \bar{z}]^2 \right] = E \left\{ \left| B^T_1 (\bar{z} + h\psi) \right| + \left| B^T_2 \bar{z} \right| \right\} \frac{1}{2h} \left| B^T_1 (\bar{z} + h\psi) - B^T_2 \bar{z} \right|
\]

It is then sufficient to get an upper bound on

\[
\frac{1}{2h} \left| B^T_1 (\bar{z} + h\psi) - B^T_2 \bar{z} \right| \leq \frac{1}{2h} \left| B^T_1 \psi \right| + \left| B^T_1 \bar{z} \right| - B^T_2 \bar{z} \leq \frac{1}{2h} \left| B^T_1 \bar{z} \right| - B^T_2 \bar{z} \right|
\]

(21)

To do this, the only difficulty is to get an upper bound on

\[
\frac{1}{2h} \left| B^T_1 \bar{z} \right| - B^T_2 \bar{z} \right|
\]

(22)

But, obviously, for (19) and (20) to hold we need that

\[
|h||B^T_1 \psi| + |h||B^T_2 \psi| \geq |B^T_1 \bar{z}| - |B^T_1 \bar{z}| = ||B^T_2 \bar{z}| - |B^T_1 \bar{z}||,
\]

which guarantees the uniform boundedness of (22).

As a consequence of this analysis, the Euler-Lagrange equation associated to the minimization of \( J \) takes the form

\[
E \int_{t_1}^{t_2} B^T_1 \bar{z} \cdot B^T_1 \psi dt + E \int_{t_2}^{t_3} B^T_2 \bar{z} \cdot B^T_2 \psi dt + E \int_{t_3}^{t_4} D^T_1 \bar{Z} \cdot D^T_1 \Psi dt
\]

\[
+ E \int_{t_4}^{t_5} D^T_2 \bar{Z} \cdot D^T_2 \Psi dt - E(x_1 \cdot \psi_T) + x_0 \cdot \psi(0) = 0
\]

(23)
for all $\psi_T \in L^2_{\mathcal{F}_T}(\Omega; \mathbb{R}^n)$. In view of (23) we conclude that the following switching controllers

$$
\begin{align*}
    u_1(t) &= B_1^T \tilde{z}(t)1_{I_1}(t), \\
    u_2(t) &= B_2^T \tilde{z}(t)1_{I_2}(t), \\
    v_1(t) &= D_1^T \tilde{Z}(t)1_{I_1}(t), \\
    v_2(t) &= D_2^T \tilde{Z}(t)1_{I_2}(t),
\end{align*}
$$

where $1_{I_j}(j = 1, 2, 3, 4)$ stands for the characteristic function of the sets $I_j(j = 1, 2, 3, 4)$ are such that the switching condition (2.4) holds and the solution of (2.1) satisfies the final requirement (2.2).

Now we discuss the construction of the control when $m(I_1 \cup I_2) < T$. We make the following assumption:

$$
B_1, B_2 \in \mathbb{R}^{n \times 1}. \tag{25}
$$

First, we define a functional $\tilde{J}_{x_0, z_1}(\cdot) : L^2_{\mathcal{F}_T}(\Omega; \mathbb{R}^n) \to \mathbb{R}$ as follows:

$$
\tilde{J}_{x_0, z_1}(z_T) = \frac{1}{2} \int_0^T \left( \max \{ \mathbb{E}|B_1^T z|^2, \mathbb{E}|B_2^T z|^2 \} + \max \{ \mathbb{E}|D_1^T Z|^2, \mathbb{E}|D_2^T Z|^2 \} \right) dt - \mathbb{E}(x_1 \cdot z_T) + x_0 \cdot z(0). \tag{26}
$$

Let $(\tilde{z}, \tilde{Z})$ be the minimizer of (26). Set

$$
\begin{align*}
    Q_1 &= \{(t, \omega) \in [0, T] \times \Omega : |B_1^T \tilde{z}| > |B_2^T \tilde{z}|\}, \\
    Q_2 &= \{(t, \omega) \in [0, T] \times \Omega : |B_1^T \tilde{z}| < |B_2^T \tilde{z}|\}, \\
    Q_3 &= \{(t, \omega) \in [0, T] \times \Omega : B_1^T \tilde{z} = B_2^T \tilde{z}\}, \\
    Q_4 &= \{(t, \omega) \in [0, T] \times \Omega : B_1^T \tilde{z} = -B_2^T \tilde{z}\}.
\end{align*} \tag{27}
$$

The following holds:

**Theorem 2.3.** If $m(I_3 \cup I_4) = T$, then by choosing the controller

$$
\begin{align*}
    u_1(t) &= B_1^T \tilde{z}, & \text{if } (t, \omega) \in Q_1, \\
    u_2(t) &= B_2^T \tilde{z}, & \text{if } (t, \omega) \in Q_2, \\
    u_1(t) &= \frac{1}{2} B_1^T \tilde{z}(t), & \text{if } t \in Q_3 \cup Q_4, \\
    v_1(t) &= D_1^T \tilde{Z}, & \text{if } t \in I_3, \\
    v_2(t) &= D_2^T \tilde{Z}, & \text{if } t \in I_4,
\end{align*} \tag{28}
$$

we have the solution of (1) with $x(0) = x_0$ satisfying that $x(T) = x_1$ in $L^2_{\mathcal{F}_T}(\Omega; \mathbb{R}^n)$.

**Remark 4.** Note that if the measure of $I \setminus (I_1 \cup I_2)$ is positive, then both controls $u_1$ and $u_2$ are active on this set.

**Proof.** Similar to the arguments developed in the proof of Theorem 2.2, we can find the controls on the set $Q_1 \cup Q_2$ in the form of (28). Thus, we only need to analyze the behavior of the functional within the set $Q_3 \cup Q_4$. It is easy to see that if $\tilde{z}_T$ is a minimizer of $\tilde{J}_{x_0, z_1}$, then it is also a minimizer of the modified functional

$$
\tilde{J}_{x_0, z_1}(z_T) = \frac{1}{2} \int_{Q_1 \cup Q_2} \left( \max \{ \mathbb{E}|B_1^T z|^2, \mathbb{E}|B_2^T z|^2 \} \right) dP dt + \frac{1}{4} \int_{Q_3 \cup Q_4} \left( |B_1^T z|^2 + |B_2^T z|^2 \right) dP dt + \int_0^T \max \{ \mathbb{E}|D_1^T Z|^2, \mathbb{E}|D_2^T Z|^2 \} dt - \mathbb{E}(x_1 \cdot z_T) + x_0 \cdot z(0) \tag{29}
$$
over the set of solutions \((z(\cdot), Z(\cdot))\) such that
\[
\mathcal{X} = \{(z(\cdot), Z(\cdot)) : (B_1^T - B_2^T)z(t, \omega) = 0 \text{ in } Q_3, (B_1^T + B_2^T)z(t, \omega) = 0 \text{ in } Q_4\}.
\]
Thus, there exist Lagrange multipliers \(\lambda_3\) and \(\lambda_4\) such that the Euler-Lagrange equation reads:
\[
\begin{align*}
&\int_{Q_1} B_1^T \ddot{z} \cdot B_1^T \psi dt + \int_{Q_2} B_2^T \ddot{z} \cdot B_2^T \psi dt \\
+ \ &\frac{1}{2} \int_{Q_3 \cup Q_4} (B_1^T \ddot{z} \cdot B_1^T \psi + B_2^T \ddot{z} \cdot B_2^T \psi) dt \\
+ \ &E \int_{I_3} D_1^T \dot{Z} \cdot D_1^T \Psi dt + E \int_{I_4} D_2^T \dot{Z} \cdot D_2^T \Psi dt - E(x_1 \cdot \psi_T) + x_0 \cdot \psi(0) \\
= \ &\lambda_3 \int_{Q_3} (B_1^T - B_2^T) \ddot{z} \cdot (B_1^T - B_2^T) \psi dt + \lambda_4 \int_{Q_4} (B_1^T + B_2^T) \ddot{z} \cdot (B_1^T + B_2^T) \psi dt.
\end{align*}
\]
This implies that by the control defined as
\[
\begin{align*}
u_1(t, \omega) &= B_1^T \ddot{z}(t, \omega) \quad \text{in } Q_1, \\
u_2(t, \omega) &= B_2^T \ddot{z}(t, \omega) \quad \text{in } Q_2, \\
u_1(t, \omega) &= \left(\frac{1}{2} - \lambda_3\right) B_1^T \ddot{z}(t, \omega) + \lambda_3 B_2^T \ddot{z}(t, \omega) \quad \text{in } Q_3, \\
u_2(t, \omega) &= \left(\frac{1}{2} - \lambda_3\right) B_2^T \ddot{z}(t, \omega) + \lambda_3 B_1^T \ddot{z}(t, \omega) \quad \text{in } Q_3, \\
u_1(t, \omega) &= \left(\frac{1}{2} - \lambda_4\right) B_1^T \ddot{z}(t, \omega) - \lambda_3 B_2^T \ddot{z}(t, \omega) \quad \text{in } Q_4, \\
u_2(t, \omega) &= \left(\frac{1}{2} - \lambda_4\right) B_2^T \ddot{z}(t, \omega) - \lambda_3 B_1^T \ddot{z}(t, \omega) \quad \text{in } Q_4, \\
v_1(t, \omega) &= D_1^T \ddot{Z}(t, \omega) \quad \text{in } I_3 \times \Omega, \\
v_2(t, \omega) &= D_2^T \ddot{Z}(t, \omega) \quad \text{in } I_4 \times \Omega,
\end{align*}
\]
we get \(x(T) = x_1\) in \(L^2_{\text{pp}}(\Omega; \mathbb{R}^n)\).

By the fact that
\[
B_1^T \ddot{z}(t, \omega) = B_2^T \ddot{z}(t, \omega) \quad \text{in } Q_3
\]
and
\[
B_1^T \ddot{z}(t, \omega) = -B_2^T \ddot{z}(t, \omega) \quad \text{in } Q_4,
\]
we have that, on the set \(Q_3 \cup Q_4\), the controls take the form
\[
u_1(t, \omega) = \frac{1}{2} B_1^T \ddot{z}(t, \omega), \quad \nu_2(t, \omega) = \frac{1}{2} B_2^T \ddot{z}(t, \omega) \quad \text{in } Q_3 \cup Q_4.
\]
This completes the proof of Theorem 2.3. \(\square\)

We have built switching controllers by minimizing functionals \(J_{x_0, x_1}(\cdot)\) of the form (5) or its variants. Now we show that the controls obtained this way are also optimal in some sense. More precisely, we have the following result:

**Theorem 2.4.** Let the assumptions of Theorem 2.2 be satisfied. Then, the controls \((\tilde{u}_1, \tilde{u}_2, \tilde{v}_1, \tilde{v}_2)\) obtained by minimizing the functional \(J_{x_0, x_1}\) are of minimal norm. More precisely,
\[
E \int_0^T (|\tilde{u}_1|^2 + |\tilde{u}_2|^2 + |\tilde{v}_1|^2 + |\tilde{v}_2|^2) dt \leq E \int_0^T (|u_1|^2 + |u_2|^2 + |v_1|^2 + |v_2|^2) dt.
\]
for all other control pair \((u_1, u_2, v_1, v_2)\) satisfying the final requirement \(x(T) = x_1\).

**Proof.** Let \((u_1, u_2, v_1, v_2)\) be any other pair of switching controls. Denote by \((\tilde{z}, \tilde{Z})\) the solution of the adjoint system associated to the minimizer of \(J_{x_0, x_1}\). By Itô's formula, and noting that both for the controls \((u_1, u_2, v_1, v_2)\) and \((\tilde{u}_1, \tilde{u}_2, \tilde{v}_1, \tilde{v}_2)\), \(x(T) = x_1\), we deduce that:

\[
\mathbb{E}(x_1 \cdot \tilde{v}_T) - x_0 \cdot \tilde{v}(0) = \mathbb{E} \int_0^T \{ [B_1 u_1(t) + B_2 u_2(t)] \cdot \tilde{z}(t) + [D_1 v_1(t) + D_2 v_2(t)] \cdot \tilde{Z}(t) \} dt, \tag{32}
\]

and

\[
\mathbb{E}(x_1 \cdot \tilde{v}_T) - x_0 \cdot \tilde{v}(0) = \mathbb{E} \int_0^T \{ [B_1 \tilde{u}_1(t) + B_2 \tilde{u}_2(t)] \cdot \tilde{z}(t) + [D_1 \tilde{v}_1(t) + D_2 \tilde{v}_2(t)] \cdot \tilde{Z}(t) \} dt. \tag{33}
\]

For the switching controls \((\tilde{u}_1, \tilde{u}_2, \tilde{v}_1, \tilde{v}_2)\), we have that

\[
\mathbb{E} \int_0^T [B_1 \tilde{u}_1(t) + B_2 \tilde{u}_2(t)] \cdot \tilde{z}(t) dt = \mathbb{E} \int_0^T (|\tilde{u}_1|^2 + |\tilde{u}_2|^2) dt \tag{34}
\]

and

\[
\mathbb{E} \int_0^T [D_1 \tilde{v}_1(t) + D_2 \tilde{v}_2(t)] \cdot \tilde{Z}(t) dt = \mathbb{E} \int_0^T (|\tilde{v}_1|^2 + |\tilde{v}_2|^2) dt. \tag{35}
\]

Since \(\mathbb{E}|u_1(t)|^2 \mathbb{E}|u_2(t)|^2 = 0\) and \(\mathbb{E}|\tilde{u}_1(t)|^2 \mathbb{E}|\tilde{u}_2(t)|^2 = 0\) for every \(t \in [0, T]\), we have

\[
\mathbb{E} \int_0^T [B_1 u_1(t) + B_2 u_2(t)] \cdot \tilde{z}(t) dt
\]

\[
= \mathbb{E} \int_0^T [u_1(t) \cdot B_1^T \tilde{z}(t) + u_2(t) \cdot B_2^T \tilde{z}(t)] dt
\]

\[
\leq \int_0^T [(\mathbb{E}|u_1(t)|^2)^{\frac{1}{2}} + (\mathbb{E}|u_2(t)|^2)^{\frac{1}{2}}] \max \left[(\mathbb{E}|B_1^T \tilde{z}(t)|^2)^{\frac{1}{2}}, (\mathbb{E}|B_2^T \tilde{z}(t)|^2)^{\frac{1}{2}}\right] dt
\]

\[
\leq \int_0^T [(\mathbb{E}|u_1(t)|^2)^{\frac{1}{2}} + (\mathbb{E}|u_2(t)|^2)^{\frac{1}{2}}] \max \left[(\mathbb{E}|\tilde{u}_1(t)|^2)^{\frac{1}{2}}, (\mathbb{E}|\tilde{u}_2(t)|^2)^{\frac{1}{2}}\right] dt
\]

\[
\leq \int_0^T \mathbb{E}(|u_1(t)|^2 + |u_2(t)|^2) dt \left[ \int_0^T \mathbb{E}(|\tilde{u}_1(t)|^2 + |\tilde{u}_2(t)|^2) dt \right]^{\frac{1}{2}}
\]

\[
\leq \frac{1}{2} \int_0^T \mathbb{E}(|u_1(t)|^2 + |u_2(t)|^2) dt + \frac{1}{2} \int_0^T \mathbb{E}(|\tilde{u}_1(t)|^2 + |\tilde{u}_2(t)|^2) dt.
\]

Similarly, we can obtain that

\[
\mathbb{E} \int_0^T [D_1 v_1(t) + D_2 v_2(t)] \cdot \tilde{Z}(t) dt
\]

\[
\leq \frac{1}{2} \int_0^T \mathbb{E}(|v_1(t)|^2 + |v_2(t)|^2) dt + \frac{1}{2} \int_0^T \mathbb{E}(|\tilde{v}_1(t)|^2 + |\tilde{v}_2(t)|^2) dt. \tag{37}
\]

From (32)–(37), we get

\[
\mathbb{E} \int_0^T (|\tilde{u}_1|^2 + |\tilde{u}_2|^2 + |\tilde{v}_1|^2 + |\tilde{v}_2|^2) dt \leq \mathbb{E} \int_0^T (|u_1|^2 + |u_2|^2 + |v_1|^2 + |v_2|^2) dt \]
Remark 5. Theorems 2.1–2.4 can be generated to the infinite dimensional setting. For example, the three key points in the proof of Theorem 2.1 is as follows:

1. the wellposedness of (3) and the inequality (4), which guarantee the continuity of the functional $J_{u_0,x_1}$;
2. the introduction of stochastic differential equation (7), which reduces the observability estimate of the backward stochastic differential equation (3) to the wellposedness of (7) and the observability estimate of an ordinary differential equation.

Both 1 and 2 above can be generalized to the infinite dimensional setting. The details is beyond the scope of this paper.

Remark 6. In this paper, we consider the exact controllability problem. It is also very interesting to study the null/approximate controllability problem with switching controls. Of course, under the conditions of Theorems 2.2 and 2.3, we know that system (1) is null/approximately controllable. However, whether one can find some weaker conditions to get the null/approximate controllability of (1) is unknown. In fact, according to some recent results in [4], the null/approximate controllability problem for stochastic differential equations is much more complex than the exact controllability problem.

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