WEAK PULLBACK MEAN RANDOM ATTRACTORS FOR STOCHASTIC EVOLUTION EQUATIONS AND APPLICATIONS

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ABSTRACT. In this paper, we investigate the existence and uniqueness of weak pullback mean random attractors for abstract stochastic evolution equations with general diffusion terms in Bochner spaces. As applications, the existence and uniqueness of weak pullback mean random attractors for some stochastic models such as stochastic reaction-diffusion equations, the stochastic p-Laplace equation and stochastic porous media equations are established.

1. INTRODUCTION

In this work, we aim to consider the long-term behavior for the following non-autonomous stochastic abstract evolution equation on an open bounded domain \(\mathcal{O} \subseteq \mathbb{R}^n\):

\[
(1.1) \quad du = (A(t, u) + F(u) + g(t, x))dt + \varepsilon G(t, u)dW(t), \quad x \in \mathcal{O}, t > \tau,
\]

with initial-boundary condition, where \(\tau \in \mathbb{R}\), \(F\) is a Lipschitz function, \(A\) and \(G\) are mappings specified later, \(\varepsilon G(t, u)dW\) is a general diffusion term with \(\varepsilon \in (0, 1]\) a constant denoting the intensity of the noise. Originally, such type of equations have a rich mathematical theory and fundamental applications, see e.g. \([27]\) for more details.

Random attractor is the effective tool for capturing the pathwise long-time behavior for random dynamical systems generated by stochastic differential equations. Based on the theory in \([1]\), we know that an Itô-type stochastic ordinary differential equation generates a random dynamical system under natural assumptions on the coefficients. The pullback random attractors theory was systematically established first in \([8, 11, 28]\) and gained its popular applications since the late 1990s, see e.g. \([2, 3, 5, 6, 7, 8, 9, 15, 16, 17, 20, 22, 24, 29, 30, 31, 34]\). The generation of a random dynamical system from an Itô-type stochastic partial differential equations has been a long-standing open problem. Most of the efforts are taken to tackle the special cases when \(G(t, u)\) is linear in \(u\). When \(G(t, u)\) is a general nonlinear function, we even don’t know whether system \((1.1)\) generates a random dynamical system or not. The main reason is that Kolmogorov’s theorem fails for random fields parameterized by infinite-dimensional Hilbert spaces. This fact leads to dynamical aspects such as the asymptotic stability, random attractors, random invariant manifolds have not been fully investigated in generality.

Compared to the pathwise random attractor methods used in \([12, 13, 14, 18, 19, 21, 26]\), one possible theory to deal with this issue above is to replace it by
the mean-square random attractor as suggested in [23]. However, the existence of mean-square random attractor theory is difficulty in dealing with the general nonlinear functions in drift term. Quite recently, a new type of weak mean-square random attractor, i.e., weak pullback mean random attractor in reflective Bochner spaces was introduced in [32, 33] to tackle the stochastic partial differential equations with general nonlinear drift and diffusion terms based on the weak topology. Following this line, we first obtain the dynamic behavior of the abstract system (1.1) when the intensity $\varepsilon$ small enough and then apply to some concrete equations such as stochastic reaction-diffusion equations, the stochastic $p$-Laplace equation and stochastic porous media equations. It seems that this is the first time to consider establishing these results for the non-autonomous stochastic evolution equations based on variational approach. This approach has been used intensively in recent years to analyze stochastic partial differential equations driven by an infinite-dimensional Wiener process. For more details, see e.g. the monographs [25, 27] and the references therein.

The paper is arranged as follows. In the next section, we give some basic concepts and results on the existence of weak pullback mean random attractors for mean random dynamical systems. In section 3, we first define a mean random dynamical system via the solution operators of the abstract non-autonomous stochastic evolution equation, and then prove the existence and uniqueness of weak pullback mean random attractors for this model when the intensity of the noise is small enough. In section 4 some models with different mappings $A$ and appropriate chosen coefficients to guarantee the existence of weak pullback mean random attractors are addressed.

2. Theory of Mean Random Dynamical Systems

We recall some basic concepts and results on the existence of weak pullback mean random attractors for mean random dynamical systems from [1, 23, 32, 33]. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in \mathbb{R}}, \mathbb{P})$ be a complete filtered probability space with $\{\mathcal{F}_t\}_{t \in \mathbb{R}}$ is an increasing right continuous family of sub-$\sigma$-algebras of $\mathcal{F}$ that includes all $\mathbb{P}$-null sets. Denote $X$ be a Banach space with norm $\| \cdot \|_X$. We say a function $\psi : \Omega \to X$ is strongly measurable if there exists a sequence of simple functions $\psi_n : \Omega \to X$, such that

$$\lim_{n \to \infty} \int_{\Omega} \|\psi_n - \psi\|_X d\mathbb{P} = 0,$$

then the Bochner integral of $\psi$ is defined as

$$\int_{\Omega} \psi d\mathbb{P} = \lim_{n \to \infty} \int_{\Omega} \psi_n d\mathbb{P}.$$ 

Now, for $q \geq 1$, we denote $L^q(\Omega, \mathcal{F}; X)$ the Bochner space consisting of all Bochner integrable functions $\psi : \Omega \to X$ such that

$$\|\psi\|_{L^q(\Omega, \mathcal{F}; X)} = \left( \int_{\Omega} \|\psi\|_X^q d\mathbb{P} \right)^{1/q} < \infty.$$

For every $\tau \in \mathbb{R}$, we define the space $L^q(\Omega, \mathcal{F}_\tau; X)$ analogously.
Let $\mathcal{D}$ be a collection of some families of nonempty bounded subsets of $L^q(\Omega, \mathcal{F}_\tau; X)$ parameterized by $\tau \in \mathbb{R}$, that is,

$$
\mathcal{D} = \{ D = \{ D(\tau) \subseteq L^q(\Omega, \mathcal{F}_\tau; X) : D(\tau) \neq \emptyset \text{ bounded, } \tau \in \mathbb{R} \} : D \text{ satisfies some conditions} \}.
$$

(2.1)

Such a collection $\mathcal{D}$ is called inclusion-closed if $D = \{ D(\tau) : \tau \in \mathbb{R} \} \in \mathcal{D}$ implies that every family $\hat{D} = \{ \hat{D}(\tau) : \emptyset \neq D(\tau) \subseteq D(\tau), \forall \tau \in \mathbb{R} \} \in \mathcal{D}$.

**Definition 2.1.** A family $\Phi = \{ \phi(t, \tau) : t \in \mathbb{R}^+, \tau \in \mathbb{R} \}$ of mappings is called a mean random dynamical system on $L^q(\Omega, \mathcal{F}; X)$ over $(\Omega, \mathcal{F}, \{ \mathcal{F}_t \}_{t \in \mathbb{R}}, \mathbb{P})$ if for every $\tau \in \mathbb{R}$ and $t, s \in \mathbb{R}^+$,

(i) $\phi(t, \tau)$ maps $L^q(\Omega, \mathcal{F}_\tau; X)$ to $L^q(\Omega, \mathcal{F}_{t+\tau}; X)$;

(ii) $\phi(0, \tau)$ is the identity operator on $L^q(\Omega, \mathcal{F}_\tau; X)$;

(iii) $\phi(t+s, \tau) = \phi(t, \tau+s) \circ \phi(s, \tau)$.

**Definition 2.2.** A family $\mathcal{K} = \{ K(\tau) : \tau \in \mathbb{R} \} \in \mathcal{D}$ is called $\mathcal{D}$-pullback absorbing set for $\Phi$ on $L^q(\Omega, \mathcal{F}; X)$ over $(\Omega, \mathcal{F}, \{ \mathcal{F}_t \}_{t \in \mathbb{R}}, \mathbb{P})$ if for every $\tau \in \mathbb{R}$ and $D \in \mathcal{D}$, there exists $T = T(\tau, D) > 0$ such that

$$
\phi(t, \tau-t)(D(\tau-t)) \subseteq K(\tau) \quad \text{for all } t \geq T.
$$

Furthermore, for every $\tau \in \mathbb{R}$, $K(\tau)$ is a weakly compact nonempty subset of $L^q(\Omega, \mathcal{F}_\tau; X)$, then we call $\mathcal{K} = \{ K(\tau) : \tau \in \mathbb{R} \}$ a weakly compact $\mathcal{D}$-pullback absorbing set for $\Phi$.

**Definition 2.3.** A family $\mathcal{K} = \{ K(\tau) : \tau \in \mathbb{R} \} \in \mathcal{D}$ is called $\mathcal{D}$-pullback weakly attracting set for $\Phi$ on $L^q(\Omega, \mathcal{F}; X)$ over $(\Omega, \mathcal{F}, \{ \mathcal{F}_t \}_{t \in \mathbb{R}}, \mathbb{P})$ if for every $\tau \in \mathbb{R}$, $D \in \mathcal{D}$ and every weak neighborhood $\mathcal{N}^w(K(\tau))$ of $K(\tau)$ in $L^q(\Omega, \mathcal{F}_\tau; X)$, then there exists $T = T(\tau, D, \mathcal{N}^w(K(\tau))) > 0$ such that for all $t \geq T$,

$$
\phi(t, \tau-t)(D(\tau-t)) \subseteq \mathcal{N}^w(K(\tau)).
$$

**Definition 2.4.** A family $\mathcal{A} = \{ \mathcal{A}(\tau) : \tau \in \mathbb{R} \} \in \mathcal{D}$ is called a weak $\mathcal{D}$-pullback mean random attractor for $\Phi$ on $L^q(\Omega, \mathcal{F}; X)$ over $(\Omega, \mathcal{F}, \{ \mathcal{F}_t \}_{t \in \mathbb{R}}, \mathbb{P})$ if for all $\tau \in \mathbb{R}$,

(a) $\mathcal{A}(\tau)$ is a weakly compact subset of $L^q(\Omega, \mathcal{F}_\tau; X)$;

(b) $\mathcal{A}$ is a $\mathcal{D}$-pullback weakly attracting set of $\Phi$;

(c) $\mathcal{A}$ is the minimal one among $\mathcal{D}$ with both properties (a) and (b).

The next result was established to guarantee the existence and uniqueness of weak pullback mean random attractors in $L^q(\Omega, \mathcal{F}; X)$ over filtered probability space $(\Omega, \mathcal{F}, \{ \mathcal{F}_t \}_{t \in \mathbb{R}}, \mathbb{P})$.

**Proposition 2.5** (See [32, 33]). Assume $X$ is a reflexible Banach space and $q \in (1, \infty)$. Let $\mathcal{D}$ be an inclusion-closed collection of some families of nonempty bounded subsets of $L^q(\Omega, \mathcal{F}_\tau; X)$ as given in (2.1) and $\Phi$ be a mean random dynamical system on $L^q(\Omega, \mathcal{F}; X)$ over $(\Omega, \mathcal{F}, \{ \mathcal{F}_t \}_{t \in \mathbb{R}}, \mathbb{P})$. If $\Phi$ has a weakly compact $\mathcal{D}$-pullback absorbing set $\mathcal{K} \in \mathcal{D}$ on $L^q(\Omega, \mathcal{F}; X)$ over $(\Omega, \mathcal{F}, \{ \mathcal{F}_t \}_{t \in \mathbb{R}}, \mathbb{P})$, then $\Phi$ has a unique weak $\mathcal{D}$-pullback mean random attractor $\mathcal{A} \in \mathcal{D}$ with $\mathcal{A}$ given by, for each $\tau \in \mathbb{R}$,

$$
\mathcal{A}(\tau) = \bigcap_{r \geq 0} \bigcup_{t \geq r} \phi(t, \tau-t)(K(\tau-t))^w,
$$

where the closure is taken with respect to the weak topology of $L^q(\Omega, \mathcal{F}_\tau; X)$. 
3. Setup and Main Results

Let

\[ V \subset H \equiv H^* \subset V^* \]

be a Gelfand triple, that is, \( H \) is a separable Hilbert space with inner product \( \langle \cdot, \cdot \rangle_H \) and is identified to its dual \( H^* \) due to the Riesz isomorphism, \( V \) is a reflexive Banach space such that \( V \subset H \) continuously and densely, and denote \( V^*, \langle \cdot, \cdot \rangle_V \) the dualization between \( V^* \) and its dual \( V \). In particular, there is a constant \( \lambda > 0 \) such that

\[ \|v\|_V^2 \geq \lambda \|v\|_H^2 \quad \text{for all} \quad v \in V, \]

which also implies that when \( \alpha > 2 \), there exists constant \( C_\alpha > 0 \) such that

\[ C_\alpha + \|v\|_V^\alpha \geq \lambda \|v\|_H^2. \]

Let \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in \mathbb{R}}, \mathbb{P})\) be a complete filtered probability space enjoying the usual properties, that is, \( \{\mathcal{F}_t\}_{t \in \mathbb{R}} \) is an increasing right continuous family of sub-\( \sigma \)-algebras of \( \mathcal{F} \) that contains all \( \mathbb{P} \)-null sets.

Let \( \mathcal{O} \) be a bounded domain in \( \mathbb{R}^n \) with smooth boundary \( \partial \mathcal{O} \) and the Lebesgue measure of the domain \( |\mathcal{O}| \neq 0 \). For \( t \in \mathbb{R} \), consider the following non-autonomous stochastic evolution equation on \( \mathcal{O} \times (\tau, \infty) \) of type

\[ \begin{cases} \quad du = (A(t, u) + F(u) + g(t, x))dt + \varepsilon G(t, u)dW(t), & x \in \mathcal{O}, \\ u(\tau) = u_0 \in L^2(\mathcal{O}, \mathcal{F}_\tau; H), \end{cases} \]

with \( W(t) \) a two-sided cylindrical \( \mathcal{Q} \)-Wiener process with respect to the filtration \( \{\mathcal{F}_t\}_{t \in \mathbb{R}} \), where \( \mathcal{Q} \) is the identity operator on another separable Hilbert space \( \mathcal{U} \) and with \( G \) taking values in \( L_2(U, H) \) as denoting the Hilbert space consisting of all Hilbert-Schmidt operators from \( \mathcal{U} \) to \( H \), but with \( A \) taking values in \( V^* \). The constant \( \varepsilon \in (0, 1) \), \( g \in L^2_{\text{loc}}(\mathbb{R}, H) \) and the stochastic term is understood in the sense of Itô's integration.

**H0** Assume that \( F : H \to H \) is a Lipschitz continuous function with Lipschitz constant \( \gamma_1 \) and for all \( u \in H \),

\[ \|F(u)\|_H \leq \gamma_2 (1 + \|u\|_H), \quad \text{with} \quad \gamma_2 > 0. \]

Let

\[ A : \mathbb{R} \times V \times \Omega \to V^*, \quad G : \mathbb{R} \times V \times \Omega \to L_2(U, H) \]

are maps such that \( A(\cdot, \cdot; \omega) : \mathbb{R} \times V \to V^* \) and \( G(\cdot, \cdot; \omega) : \mathbb{R} \times V \to L_2(U, H) \) are respectively, \((\mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(V), \mathcal{B}(V^*))\)- and \((\mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(V), \mathcal{B}(L_2(U, H)))\)-measurable for each \( \omega \in \Omega \). We make the following assumptions: for all \( \omega \), \( v, v_1, v_2 \in V \), \( \omega \in \Omega \) and \( t \in \mathbb{R} \),

**H1** (Hemicontinuity) The map

\[ s \mapsto_{V^*} \langle A(t, v_1 + sv_2; \omega), v \rangle_V \]

is continuous on \( \mathbb{R} \).

**H2** (Weak Monotonicity) There exists \( \gamma_3 \in \mathbb{R} \) such that

\[ 2\gamma_1 \|v_1 - v_2\|_H^2 + 2_{V^*} \langle A(t, v_1; \omega) - A(t, v_2; \omega), v_1 - v_2 \rangle_V + \|G(t, v_1) - G(t, v_2)\|^2_{L_2(U, H)} \leq \gamma_3 \|v_1 - v_2\|_H^2. \]

**H3** (Coercivity) There exist \( \alpha \geq 2, \gamma_4 \in \mathbb{R}, \gamma_5 > 0, \) and \( h_1 \in L^2_{\text{loc}}(\mathbb{R}) \) such that

\[ 2_{V^*} \langle A(t, v; \omega), v \rangle_V + \|G(t, v)\|^2_{L_2(U, H)} \leq \gamma_4 \|v\|_H^2 - 3\gamma_5 \|v\|_V^2 + h_1(t). \]
In the sequel, we further assume \( L \) is the solution of system (3.3) with initial data \((u)_{0}\) over \((\Omega, (\Omega, H)) \) and it satisfies, for every \( t \) such that A solution to problem (3.3) if
\[
G(t, v)_{L_{2}(U, H)} \leq \gamma_{4}\|v\|_{H}^{2} + (2\gamma_{6} - 3\gamma_{5})\|v\|_{V}^{2} + 2h_{2}(t)\|v\|_{V} + h_{1}(t)
\]
(3.6) \[
\leq \gamma_{4}\|v\|_{H}^{2} + 2(\gamma_{6} - \gamma_{5})\|v\|_{V}^{2} + C_{\gamma_{6}, \alpha}h_{2}(t)\|v\|_{V} + h_{1}(t)
\]
(3.5) \[
\gamma_{2} + |\gamma_{4}| < \lambda.
\]
Remark 3.1. By (H3) and (H4) we know that for all \( v \in V \),
(3.7) \[
\mathbb{E}\left(\int_{\tau}^{\tau+T}\|u(s)\|_{V}^{2}ds + \sup_{t \in [\tau, \tau+T]}\|u(t)\|_{H}^{2}\right) < \infty.
\]
Now, let \( \phi \) be a mapping from \( \mathbb{R}^{+} \times \mathbb{R} \times L^{2}(\Omega, H) \) to \( L^{2}(\Omega, H) \) given by
(3.8) \[
\phi(t, \tau)(u_{0}) = u(t + \tau, \tau, u_{0}),
\]
where u is the solution of system (3.3) with initial data \( u_{0} \in L^{2}(\Omega, F; H) \). Then, we know that \( \Phi = \{\phi(t, \tau) : t \in \mathbb{R}^{+}, \tau \in \mathbb{R}\} \) is actually a mean random dynamical system on \( L^{2}(\Omega, F; H) \) over \( (\Omega, F, \{F_{t}\}_{t \in \mathbb{R}}, \mathbb{P}) \).

We will use \( D \) to denote the collection of all families of nonempty bounded subsets of \( L^{2}(\Omega, F; H) \), i.e.,
\[
D = \{D(\tau) \subseteq L^{2}(\Omega, F; H) : D(\tau) \neq \emptyset \text{ bounded, } \tau \in \mathbb{R}\} : \lim_{\tau \to -\infty}e^{\lambda_{3}\tau}\|D(\tau)\|_{L^{2}(\Omega,F;H)}^{2} = 0\}.
\]
In the sequel, we further assume
(3.8) \[
\int_{-\infty}^{\tau}e^{\lambda_{3}s}(\|g(s)\|_{H}^{2} + |h_{1}(s)|^{2} + |h_{2}(s)|^{2})ds < \infty, \quad \forall \tau \in \mathbb{R}.
\]
We first derive the following uniform estimates for problem (3.3).

Lemma 3.3. Suppose (H0)-(H5) and (3.8) hold. Then there exists \( \varepsilon_{0} > 0 \) such that for every \( \varepsilon \in (0, \varepsilon_{0}] \) and for every \( \tau \in \mathbb{R} \) and \( D = \{D(t)\}_{t \in \mathbb{R}} \subseteq D \), there exists
Proof. Applying Itô’s formula to \((3.3)\), we obtain for all \(t \geq T\),
\[
\mathbb{E}(\|u(t, \tau - t, u_0)\|^2)
\]
\[
\leq L + Le^{-\lambda \tau} \int_{-\infty}^{r} e^{\lambda \gamma s} (\|g(s)\|^2_H + |h_1(s)| + |h_2(s)|) \frac{\parallel r \|}{\parallel r \|} ds,
\]
where \(u_0 \in D(\tau - t)\) and \(L\) is a positive constant independent of \(\tau\) and \(D\).

Thus, we know that, for almost all \(\tau \geq t\),
\[
\|u(r, \tau - t, u_0)\|^2_H
\]
\[
= \|u_0\|^2_H + 2 \int_{\tau - t}^r V \cdot (A(s, u(s)), u(s, \tau - t, u_0)) ds
\]
\[
+ 2 \int_{\tau - t}^r (F(u(s)), u(s, \tau - t, u_0))_H ds
\]
\[
+ 2 \int_{\tau - t}^r (g(s), u(s, \tau - t, u_0))_H ds
\]
\[
+ \int_{\tau - t}^r \varepsilon^2 \|G(s, u(s, \tau - t, u_0))\|^2_{L_2(U, H)} ds
\]
\[
+ 2 \int_{\tau - t}^r (u(s, \tau - t, u_0), \varepsilon G(s, u(s, \tau - t, u_0))dW(s))_H ds,
\]
which implies that
\[
\mathbb{E}(\|u(r, \tau - t, u_0)\|^2_H)
\]
\[
= \mathbb{E}(\|u_0\|^2_H) + 2 \int_{\tau - t}^r \mathbb{E}(V \cdot (A(s, u(s)), u(s, \tau - t, u_0))_H) ds
\]
\[
+ 2 \int_{\tau - t}^r \mathbb{E}((F(u(s)), u(s, \tau - t, u_0))_H) ds
\]
\[
+ 2 \int_{\tau - t}^r \mathbb{E}((g(s), u(s, \tau - t, u_0))_H) ds
\]
\[
+ \int_{\tau - t}^r \varepsilon^2 \mathbb{E}((G(s, u(s, \tau - t, u_0))_H)_2(U, H)) ds.
\]

Thus, we know that, for almost all \(r \geq \tau - t\),
\[
\frac{d}{dr} \mathbb{E}(\|u(r, \tau - t, u_0)\|^2_H) = 2\mathbb{E}(V \cdot (A(r, u(r)), u(r, \tau - t, u_0))_H)
\]
\[
+ 2\mathbb{E}((F(u(r)), u(r, \tau - t, u_0))_H) + 2\mathbb{E}((g(r), u(r, \tau - t, u_0))_H)
\]
\[
+ \varepsilon^2 \mathbb{E}((G(r, u(r, \tau - t, u_0))_H)_2(U, H)).
\]

First, by \((H_0)\), we see
\[
(F(u), u)_H \leq \|F(u)\|_H^2 \|u\|_H^2 \leq \gamma_2 \|u\|_H^2 + \gamma_2 \|u\|^2_H,
\]
\[
\leq \frac{1}{4} (\lambda \gamma_3 - \gamma_2 - |\gamma_4|) \|u\|^2_H + \frac{1}{\lambda \gamma_5 - \gamma_2 - |\gamma_4|} |O| + \gamma_2 \|u\|^2_H,
\]
which implies that
\[
2\mathbb{E}((F(u(r)), u(r))_H) \leq \frac{1}{4} (\lambda \gamma_5 + 3 \gamma_2 - |\gamma_4|) \mathbb{E}(\|u(r, \tau - t, u_0)\|^2_H)
\]
\[
+ C_{\lambda, \gamma_2, \gamma_4, \gamma_5, |O|}.
\]
By the Young inequality, we get
\[
2\mathbb{E}(\langle g(r), u(r, \tau - t, u_0) \rangle_H) \\
\leq \frac{1}{4} (\lambda \gamma_5 - \gamma_2 - |\gamma_4|) \mathbb{E}(\|u(r, \tau - t, u_0)\|_H^2) + \frac{1}{\lambda \gamma_5 - \gamma_2 - |\gamma_4|} \|g(r)\|_H^2.
\]
(3.11)

Denote
\[
\varepsilon_0 = \min \left\{ 1, \sqrt{\frac{\lambda \gamma_5 - \gamma_2 - |\gamma_4|}{4\lambda \gamma_6 + |\gamma_4|}} \right\}.
\]
(3.12)

Then for all \( \varepsilon \in (0, \varepsilon_0) \), by (H3) and (3.6) we get
\[
2\mathbb{E}(\langle \langle A(r, u(r)), u(r, \tau - t, u_0) \rangle \rangle_V) + \varepsilon^2 \|G(r, u(r, \tau - t, u_0))\|_{L^2(u, H)}^2 \\
\leq -2\gamma_5 \mathbb{E}(\|u(r, \tau - t, u_0)\|_V^2) + \gamma_4 \|u(r, \tau - t, u_0)\|_H^2 \\
+ \varepsilon^2 \mathbb{E}(\|u(r, \tau - t, u_0)\|_V^2) + 2\varepsilon^2 \gamma_6 \mathbb{E}(\|u(r, \tau - t, u_0)\|_V^2) \\
+ C_{\gamma_5, \gamma_6} |h_2(r)|^{\frac{\alpha}{\sigma}} + 2|h_1(r)|
\]
\[
\leq -2\gamma_5 \mathbb{E}(\|u(r, \tau - t, u_0)\|_V^2) + \frac{2(\lambda \gamma_5 - \gamma_2 - |\gamma_4|) \gamma_6}{4\lambda \gamma_6 + |\gamma_4|} \mathbb{E}(\|u(r, \tau - t, u_0)\|_V^2) \\
+ \left( \frac{4\gamma_6 + |\gamma_4|}{4\gamma_6 + |\gamma_4|} + |\gamma_4| \right) \mathbb{E}(\|u(r, \tau - t, u_0)\|_H^2) \\
+ C_{\gamma_5, \gamma_6} |h_2(r)|^{\frac{\alpha}{\sigma}} + 2|h_1(r)|,
\]
which along with (3.9), (3.11) and (3.2) shows that for almost all \( r \geq \tau - t \),
\[
\frac{d}{dr} \mathbb{E}(\|u(r, \tau - t, u_0)\|_H^2) \\
+ \left( 2\gamma_5 - \frac{2(\lambda \gamma_5 - \gamma_2 - |\gamma_4|) \gamma_6}{4\lambda \gamma_6 + |\gamma_4|} \right) \lambda \mathbb{E}(\|u(r, \tau - t, u_0)\|_H^2) \\
\leq \frac{1}{2} \left( \lambda \gamma_5 + \gamma_2 + |\gamma_4| + \frac{\lambda \gamma_5 - \gamma_2 - |\gamma_4| |\gamma_4|}{4\lambda \gamma_6 + |\gamma_4|} \right) \mathbb{E}(\|u(r, \tau - t, u_0)\|_H^2) \\
+ C_{\lambda, \gamma_5, \gamma_4, \alpha, |\gamma_4|} + C_{\lambda, \gamma_2, \gamma_4, \gamma_5} \mathbb{E}(\|g(r)\|_H^2) + C_{\gamma_5, \gamma_6} |h_2(r)|^{\frac{\alpha}{\sigma}} + 2|h_1(r)|.
\]
(3.13)

Obviously, we see
\[
\left( 2\gamma_5 - \frac{2(\lambda \gamma_5 - \gamma_2 - |\gamma_4|) \gamma_6}{4\lambda \gamma_6 + |\gamma_4|} \right) \lambda \\
- \frac{1}{2} \left( \lambda \gamma_5 + \gamma_2 + |\gamma_4| + \frac{\lambda \gamma_5 - \gamma_2 - |\gamma_4| |\gamma_4|}{4\lambda \gamma_6 + |\gamma_4|} \right) = \lambda \gamma_5 .
\]

Now, by (3.13) we obtain, for almost all \( r \geq \tau - t \),
\[
\frac{d}{dr} \mathbb{E}(\|u(r, \tau - t, u_0)\|_H^2) + \lambda \gamma_5 \mathbb{E}(\|u(r, \tau - t, u_0)\|_H^2) \\
\leq C_{\lambda, \gamma_2, \gamma_4, \gamma_5, \alpha, |\gamma_4|} + C_{\lambda, \gamma_2, \gamma_4, \gamma_5} \mathbb{E}(\|g(r)\|_H^2) + C_{\gamma_5, \gamma_6} |h_2(r)|^{\frac{\alpha}{\sigma}} + 2|h_1(r)|.
\]
(3.14)

By multiplying (3.11) by \( e^{\lambda \gamma_5 r} \) and integrating over \( (\tau - t, \tau) \) with \( t > 0 \) to show
\[
\mathbb{E}(\|u(\tau, \tau - t, u_0)\|_H^2) \leq e^{-\lambda \gamma_5 t} \mathbb{E}(\|u_0\|_H^2) + C_{\lambda, \gamma_2, \gamma_4, \gamma_5, \alpha, |\gamma_4|} \\
+ C_{\lambda, \gamma_2, \gamma_4, \gamma_5} \int_{\tau - t}^{\tau} \mathbb{E}(\|g(r)\|_H^2) + |h_2(r)|^{\frac{\alpha}{\sigma}} + |h_1(r)|)dr.
\]
(3.15)
Since \( u_0 \in D(\tau - t) \) and \( D = \{ D(t) \}_{t \in \mathbb{R}} \in \mathcal{D} \) we see
\[
e^{-\lambda_5 \tau} e^{\lambda_5 (\tau - t)} \mathbb{E}(\|u_0\|^2_H) \leq e^{-\lambda_5 \tau} e^{\lambda_5 (\tau - t)} \|D(\tau - t)\|^2_{L^2(\Omega, \mathcal{F}; H)} \to 0
\]
as \( t \to \infty \),
which implies that there exists \( T_1 = T_1(\tau, D) > 0 \) such that for all \( t \geq T_1 \),
\[
e^{-\lambda_5 \tau} e^{\lambda_5 (\tau - t)} \mathbb{E}(\|u_0\|^2_H) \leq 1. \tag{3.16}
\]

By (3.15)- (3.16) we have, for all \( t \geq T_1 \),
\[
\mathbb{E}(\|u(\tau, \tau - t, u_0)\|^2_H) \leq 1 + C_{\lambda, \gamma_2, \gamma_4, \gamma_5, \alpha, |\mathcal{C}|} + C_{\lambda, \gamma_2, \gamma_4, \gamma_5, \alpha} e^{-\lambda_5 \tau} \int_{-\infty}^{T} (\|g(r)\|^2_H + |h_2(r)|^{\alpha_5} + |h_1(r)|) \, dr.
\]

The proof is complete. \( \Box \)

Based on Lemma 3.3, we establish the existence of weakly compact \( \mathcal{D} \)-pullback absorbing set for (3.3).

**Lemma 3.4.** Suppose \((H0)-(H5)\) and (3.3) hold. Then there exists \( \varepsilon_0 > 0 \) such that for every \( \varepsilon \in (0, \varepsilon_0] \), the random dynamical system \( \Phi \) for problem (3.3) possesses a weakly compact \( \mathcal{D} \)-pullback absorbing set \( \mathcal{K} = \{ K(\tau) : \tau \in \mathbb{R} \} \in \mathcal{D} \), which is given by, for each \( \tau \in \mathbb{R} \),
\[
K(\tau) = \{ u \in L^2(\Omega, \mathcal{F}_\tau; H) : \mathbb{E}(\|u\|^2_H) \leq R(\tau) \}, \tag{3.17}
\]
where
\[
R(\tau) = L + Le^{-\lambda_5 \tau} \int_{-\infty}^{T} e^{\lambda_5 s}(\|g(r)\|^2_H + |h_2(\tau)|^{\alpha_5} + |h_1(\tau)|) \, ds,
\]
with \( L \) being the same constant as in Lemma 3.3.

**Proof.** Since for each \( \tau \in \mathbb{R} \), \( K(\tau) \) defined in (3.17) is a bounded closed convex subset of \( L^2(\Omega, \mathcal{F}_\tau; H) \), and hence it is weakly compact in \( L^2(\Omega, \mathcal{F}_\tau; H) \). Moreover, by Lemma 3.3 we know that for every \( \tau \in \mathbb{R} \) and \( D = \{ D(t) \}_{t \in \mathbb{R}} \in \mathcal{D} \), there exists \( T = T(\tau, D) > 0 \) such that for all \( t \geq T \) and \( 0 < \varepsilon \leq \varepsilon_0 \),
\[
\Phi(t, \tau - t, D(\tau - t)) \subseteq K(\tau).
\]

Obviously by (3.3), we can verify \( \mathcal{K} \in \mathcal{D} \). Therefore, \( \mathcal{K} \) is a weakly compact \( \mathcal{D} \)-pullback absorbing set for \( \Phi \). \( \Box \)

Now, by Lemma 3.4 and Proposition 2.5 we obtain the main result of the existence of weak \( \mathcal{D} \)-pullback mean random attractor for (3.3).

**Theorem 3.5.** Suppose \((H0)-(H5)\) and (3.3) hold. Then there exists \( \varepsilon_0 > 0 \) such that for every \( \varepsilon \in (0, \varepsilon_0] \), the random dynamical system \( \Phi \) for problem (3.3) possesses a unique weak \( \mathcal{D} \)-pullback mean random attractor \( \mathcal{A} = \{ \mathcal{A}(\tau) : \tau \in \mathbb{R} \} \in \mathcal{D} \) in \( L^2(\Omega, \mathcal{F}_\tau; H) \), over \( (\Omega, \mathcal{F}, \{ \mathcal{F}_\tau \}_{\tau \in \mathbb{R}}, \mathcal{P}) \).
4. Some Applications

In this section, we present some examples of concrete stochastic evolution equations. Here, we solely focus on $A$ independent of $t$ and take $G \equiv 0$. The latter we do because of the fact in [27, Exercise 4.1.2]. We also note that from here examples for $A$ dependent on $(t,\omega)$ are then immediate. We always consider $\mathcal{O} \subset \mathbb{R}^n$ as an open bounded domain with smooth boundary $\partial \mathcal{O}$ and the external forcing term $g \in L^2_{\text{loc}}(\mathbb{R}, H)$.

Example 4.1 \((H_1) \subset L^2 \subset (H_1)^* \) and \(A = \Delta\). Consider the following triple
\[ V := H^1_0(\mathcal{O}) \subset L^2(\mathcal{O}) := H \subset (H^1_0(\mathcal{O}))^* = H^{-1}(\mathcal{O}) := V^* \]
and the stochastic reaction-diffusion equation
\[ \begin{aligned}
&\frac{du}{dt} = (\Delta u + F(u) + g(x,t))dt + G(u)dW, \quad x \in \mathcal{O}, \ t > t_0, \\
&u(x, t_0) = 0, \quad x \in \partial \mathcal{O}, \ t > t_0, \\
&u(x, \tau) = u_0 \in L^2(\Omega, \mathcal{F}_\tau; L^2(\mathcal{O})).
\end{aligned} \tag{4.1} \]

Let \(A(u) = \Delta u\). Similarly to [27, Example 4.1.7], we can extend $A$ with initial domain $C_0^\infty(\mathcal{O})$ to a bounded linear operator $A : V \to V^*$. Since $A : V \to V^*$ is linear, \((H_1)\) is obviously satisfied. In this case, we don't need \((3.6)\) due to
\[ 2\lambda A(u),u)_V = -\|\nabla u\|^2_{L^2(\mathcal{O})} \leq 0. \]

So the conditions given in \((3.5)\) and \((3.12)\) should be adjusted as
\[ \frac{\gamma_2}{\gamma_5} < \lambda, \quad \tilde{\varepsilon}_0 = \min \left\{ 1, \sqrt{\frac{\lambda \gamma_5 - \gamma_2}{4\lambda \gamma_6 + |\gamma_4|}} \right\}. \tag{4.2} \]

Indeed, the $\lambda$ defined in \((3.1)\) can be chosen as
\[ \|v\|^2_{L^2(\mathcal{O})} \leq \frac{\|\mathcal{O}\|}{\mu_n} \|\nabla v\|^2_{L^2(\mathcal{O})} \quad \text{for all} \quad v \in H^1_0(\mathcal{O}), \]
where $\mu_n$ is the volume of unit ball in $\mathbb{R}^n$ in terms of the Gamma function $\Gamma$:
\[ \mu_n = \frac{2\pi^{n/2}}{\Gamma(\frac{n}{2})}. \]

Now, take $\gamma_1 \geq 0, \ 0 < \gamma_2 < \frac{2}{\gamma_5}, \ \gamma_3 = 2\gamma_1, \ \gamma_4 = 2, \ \gamma_5 = \frac{2}{\gamma_5}, \ \gamma_6 = 1$ and $\alpha = 2$, then all the conditions in \((H_2)-(H_4)\) and \((4.2)\) are satisfied. Then, we have

Theorem 4.2. Assume that $\gamma_1 \geq 0, \ 0 < \gamma_2 < \frac{2}{\gamma_5}, \ \gamma_3 = 2\gamma_1, \ \gamma_4 = 2, \ \gamma_5 = \frac{2}{\gamma_5}, \ \gamma_6 = 1$ and $\alpha = 2$ in \((H_0)-(H_5)\) and \((3.8)\) hold. Then there exists $\tilde{\varepsilon}_0 = \min \left\{ 1, \sqrt{\frac{2\lambda \gamma_5 - \gamma_2}{4\lambda \gamma_6 + |\gamma_4|}} \right\}$ such that for every $\varepsilon \in (0, \tilde{\varepsilon}_0)$, the random dynamical system $\Phi$ for problem \((1.1)\) possesses a unique weak $D$-pullback mean random attractor $\mathcal{A} = \{ \mathcal{A}(\tau) : \tau \in \mathbb{R} \} \in \mathcal{D}$ in $L^2(\Omega, \mathcal{F}_\tau; L^2(\mathcal{O}))$ over $(\Omega, \mathcal{F}, \{ \mathcal{F}_\tau \})_{\tau \in \mathbb{R}, \mathbb{P}}$.

We remark that the similar result has been established in [32].

Example 4.3 \([L^p \subset L^2 \subset L^{p/p-1} \) and \(A(u) := -u|u|^{p-2}\). For $p \geq 2$, consider the following triple
\[ V := L^p(\mathcal{O}) \subset L^2(\mathcal{O}) := H \subset (L^p(\mathcal{O}))^* = L^{\frac{p}{p-2}}(\mathcal{O}) := V^* \]
and the stochastic evolution equation

\begin{equation}
\begin{aligned}
&\begin{cases}
  du = (-u|u|^{p-2} + F(u) + g(t,x))dt + G(u)dW, & x \in \mathcal{O}, \ t > \tau, \\
  u(x,0) = 0, & x \in \partial \mathcal{O}, \ t > \tau, \\
  u(x,\tau) = u_0 \in L^2(\Omega, \mathcal{F}_\tau; L^2(\mathcal{O})).
\end{cases}
\end{aligned}
\tag{4.3}
\end{equation}

Define \( A : V \to V^* \) by

\[ A(u) := -u|u|^{p-2}, \quad u \in V. \]

Indeed, \( A \) takes values in \( V^* \) for all \( u \in V \). Furthermore, \( A \) satisfies \((H1)-(H4)\). This can be verified by the same steps as in [27, Example 4.1.9] with

- \( \gamma_3 := 2\gamma_1 \) in \((H2)\);
- \( \alpha := p, \ \gamma_4 = 0 \) and \( \gamma_5 = \frac{2}{3} \) in \((H3)\);
- \( \alpha := p \) and \( \gamma_6 = 1 \) in \((H4)\).

From (3.1), we know that for \( p > 2 \),

\[ \|v\|_{L^2(\mathcal{O})}^2 \leq \|v\|_{L^p(\mathcal{O})}^p \] for all \( v \in L^p(\mathcal{O}) \),

which implies that we can choose \( \gamma_2 > 0 \) given in (3.4) such that

\[ \frac{\gamma_2 + |\gamma_4|}{\gamma_5} = \frac{3}{2} \gamma_2 < \lambda_0 \Rightarrow \gamma_2 < \frac{2}{3} \lambda_0, \]

where \( \lambda_0 \) depends on \( |\mathcal{O}| \) and \( p \).

Then, we have the existence of weak \( \mathcal{D} \)-pullback mean random attractor for \((4.3)\).

**Theorem 4.4.** Assume that \( \gamma_1 \geq 0, \ 0 < \gamma_2 < \frac{2}{3} \lambda_0, \ \gamma_3 = 2\gamma_1, \ \gamma_4 = 0, \ \gamma_5 = \frac{2}{3}, \ \gamma_6 = 1, \ \alpha = p \) in \((H0)-(H5)\) and (3.3) hold. Then there exists \( \bar{\varepsilon}_0 = \min \left\{ 1, \sqrt{\frac{\lambda_0 - \gamma_2}{2 \lambda_0}} \right\} \) such that for every \( \varepsilon \in (0, \bar{\varepsilon}_0] \), the random dynamical system \( \Phi \) for problem \((4.3)\) possesses a unique weak \( \mathcal{D} \)-pullback mean random attractor \( \mathcal{A} = \{ \mathcal{A}(\tau) : \tau \in \mathbb{R} \} \in \mathcal{D} \) in \( L^2(\Omega, \mathcal{F}_\tau; L^2(\mathcal{O})) \) over \( (\Omega, \mathcal{F}, \{\mathcal{F}_\tau\}_{\tau \in \mathbb{R}}, \mathbb{P}) \).

**Example 4.5** \((W_0^{1,p} \subset L^2 \subset (W_0^{1,p})^* \) and \( A = p \)-Laplacian). Again, we take \( p \geq 2 \), consider the following triple

\[ V := W^{1,p}_0(\mathcal{O}) \subseteq L^2(\mathcal{O}) := H \subseteq (W^{1,p}_0(\mathcal{O}))^* := V^* \]

and the stochastic evolution equation

\begin{equation}
\begin{aligned}
&\begin{cases}
  du = \text{div}(\nabla u|u|^{p-2}\nabla u) + F(u) + g(t,x))dt + G(u)dW, & x \in \mathcal{O}, \ t > \tau, \\
  u(x,0) = 0, & x \in \partial \mathcal{O}, \ t > \tau, \\
  u(x,\tau) = u_0 \in L^2(\Omega, \mathcal{F}_\tau; L^2(\mathcal{O})).
\end{cases}
\end{aligned}
\tag{4.4}
\end{equation}

Define \( A : V \to V^* \) by

\[ A(u) := \text{div}(\nabla u|u|^{p-2}\nabla u), u \in V. \]

Indeed, \( A \) takes values in \( V^* \) for all \( u \in V \). Similarly to [27, Example 4.1.9], \((H1)\) is obvious. We also have

\[ \tilde{\lambda} \|v\|_{L^p(\mathcal{O})} \leq \|\nabla v\|_{L^p(\mathcal{O})} \] for all \( v \in W^{1,p}_0(\mathcal{O}), \)

where \( \tilde{\lambda} = \sqrt{\frac{\lambda_0}{|\mathcal{O}|}} \). Now, by taking

- \( \gamma_1 \geq 0, \ 0 < \gamma_2 < \frac{2}{3} \min \{1, \tilde{\lambda} \} \) in \((H0)\);
- \( \gamma_3 := 2\gamma_1 \) in \((H2)\);
- \( \alpha := p, \ \gamma_4 = 0 \) and \( \gamma_5 = \frac{1}{6} \min \{1, \tilde{\lambda} \} \) in \((H3)\);
Theorem 4.6. Assume that \( \gamma_1 \geq 0, 0 < \gamma_2 < \frac{1}{6} \min\{1, \tilde{\lambda} \}, \gamma_3 = 2\gamma_1, \gamma_4 = 0, \gamma_5 = \frac{1}{6} \min\{1, \tilde{\lambda} \}, \gamma_6 = 1, \alpha = p \) in (H0)-(H5) and \((\ref{eq:3.8})\) hold. Then there exists \( \varepsilon_0 = \min \left\{ 1, \sqrt{\frac{\gamma_1}{2}} \right\} \) such that for every \( \varepsilon \in (0, \varepsilon_0) \), the random dynamical system \( \Phi \) for problem \((\ref{eq:4.4})\) possesses a unique weak \( \mathcal{D} \)-pullback mean random attractor \( \mathcal{A} = \{ \mathcal{A}(\tau) : \tau \in \mathbb{R} \} \) in \( L^2(\Omega, \mathcal{F}_\tau; L^2(O)) \) over \( (\Omega, \mathcal{F}, \{\mathcal{F}_\tau\}_{\tau \in \mathbb{R}}, \mathbb{P}) \).

Example 4.7 \( (L^p \subset (H^1_0)^* \subset (L^p)^*) \) and \( A = \) porous medium operator. For \( p \geq 2 \), consider the following triple
\[
V := L^p(O) \subseteq (H^1_0(O))^* := H \subseteq (L^p(O))^* = L^{2(p-1)}(O) := V^* \]
and the stochastic evolution equation
\[
\begin{cases}
    du = \Delta \Psi(u) + F(u) + g(t, x) dt + G(u) dW, & x \in O, \ t > \tau, \\
    u(x, t) = 0, & x \in \partial O, \ t > \tau, \\
    u(x, \tau) = u_0 \in L^2(\Omega, \mathcal{F}_\tau; H^{-1}(O)).
\end{cases}
\]
(4.5)
Here, the function \( \Psi : \mathbb{R} \rightarrow \mathbb{R} \) possesses the following properties:

\begin{itemize}
  \item (\Psi1) \( \Psi \) is continuous.
  \item (\Psi2) For all \( s, t \in \mathbb{R} \),
    \[
    (t-s)(\Psi(t) - \Psi(s)) \geq 0.
    \]
  \item (\Psi3) There exist \( p \geq 2, \beta_1 > 0, \beta_2 \geq 0 \) such that for all \( s \in \mathbb{R} \),
    \[
    s\Psi(s) \geq \beta_1|s|^p - \beta_2.
    \]
  \item (\Psi4) There exist \( \beta_3, \beta_4 > 0 \) such that for all \( s \in \mathbb{R} \),
    \[
    |\Psi(s)| \leq \beta_3|s|^{p-1} + \beta_4.
    \]
\end{itemize}

Define the porous medium operator \( A : L^p(O) \rightarrow (L^p(O))^* \) by
\[
A(u) := \Delta \Psi(u), \quad u \in L^p(O).
\]
Then the operator is well-defined, see e.g. \( [27] \) Example 4.1.11. Also, (H1) holds. By (\Psi2), \( \gamma_3 := 2\gamma_1 \) in (H2). (H3) is satisfied with \( \gamma_4 = 0, \gamma_5 = \frac{\gamma_1}{3} \), \( \alpha = p \) and \( h_1(t) := 2\beta_2|O| \). Take \( \alpha = p, \gamma_6 = \beta_3 \) and \( h_2(t) := \beta_4|O|^{p/p-1} \), then (H4) is satisfied. Furthermore, the \( \gamma_2 \) can be chosen such that
\[
\gamma_2 < \frac{2}{3} \beta_1 \tilde{\lambda},
\]
where \( \tilde{\lambda} \) defined in
\[
\|v\|^2_{L^p(O)} \geq \tilde{\lambda} \|v\|^2_{H^{-1}(O)} \quad \text{for all} \quad v \in L^p(O).
\]
Then we have the main result:

Theorem 4.8. Assume that \( \gamma_1 \geq 0, 0 < \gamma_2 < \frac{2}{3} \beta_1 \tilde{\lambda}, \gamma_3 = 2\gamma_1, \gamma_4 = 0, \gamma_5 = \frac{2}{3} \beta_1, \gamma_6 = \beta_3, \alpha = p, h_1(t) := 2\beta_2|O|, h_2(t) := \beta_4|O|^{p/p-1} \) in (H0)-(H5) and \((\ref{eq:3.8})\) hold. Then there exists \( \varepsilon_0 = \min \left\{ 1, \sqrt{\frac{2\beta_2 \gamma_2}{3 \beta_1 \tilde{\lambda}}} \right\} \) such that for every \( \varepsilon \in (0, \varepsilon_0) \), the random dynamical system \( \Phi \) for problem \((\ref{eq:4.5})\) possesses a unique weak \( \mathcal{D} \)-pullback mean random attractor \( \mathcal{A} = \{ \mathcal{A}(\tau) : \tau \in \mathbb{R} \} \in \mathcal{D} \) in \( L^2(\Omega, \mathcal{F}_\tau; H^{-1}(O)) \) over \( (\Omega, \mathcal{F}, \{\mathcal{F}_\tau\}_{\tau \in \mathbb{R}}, \mathbb{P}) \).
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