Current-Induced Instability of a Perpendicular Ferromagnet in Spin Hall Geometry

Tomohiro Taniguchi$^1$, Seiji Mitani$^2$, and Masamitsu Hayashi$^2$

$^1$National Institute of Advanced Industrial Science and Technology (AIST), Spintronics Research Center, Tsukuba, Ibaraki 305-8568, Japan

$^2$National Institute for Materials Science, Tsukuba 305-0047, Japan

Abstract—We develop a theoretical formula of spin Hall torque in the presence of two ferromagnets. While the direction of the conventional spin Hall torque always points to the in-plane direction, the present system enables to manipulate the torque direction acting on one magnetization by changing the direction of another magnetization. Based on the diffusion equation of the spin accumulation and the Landauer formula, we derive analytical formula of the spin Hall torque. The present model provides a solution to switch a perpendicular ferromagnet deterministically at zero field using the spin Hall effect.

Index Terms—spintronics, spin Hall effect, perpendicularly magnetized free layer

I. INTRODUCTION

Spin-orbit interaction in a nonmagnetic heavy metal generates pure spin current flowing along the direction normal to an electric current. The phenomenon called spin Hall effect [1-3] has attracted much attention as a new method to excite spin transfer torque [4,5] on a magnetization in a ferromagnet adjacent to the nonmagnetic layer [6-20]. In particular, magnetization switching of a perpendicular ferromagnet is an important issue for practical application.

Unfortunately however, it is difficult to switch the perpendicular magnetization solely by the spin Hall effect due to the following reason. Let us assume that an electric current flows along $x$-direction, while a ferromagnet is set in $z$-direction. Then, the direction of the spin polarization of the pure spin current generated by the spin Hall effect is geometrically determined to be induced in $y$-direction [21]. The spin Hall torque exerted by this pure spin current tries to move the magnetization from the perpendicular ($z$) direction to the $y$-direction, and finally, the magnetization stops its dynamics when it becomes parallel to the $y$-axis. Since the spin Hall torque does not break the symmetry with respect to the film plane, the magnetization does not cross the film plane. Therefore, we cannot switch the magnetization deterministically from one equilibrium direction to the other. To overcome this problem, usually an in-plane field along the $x$-direction is applied [6]. Using a ferromagnet with a tilted anisotropy or non-uniform anisotropy are other solutions [14,16,20].

In this paper, we propose another solution to switch perpendicular ferromagnet deterministically by using the spin Hall effect. As schematically shown in Fig. 1(a), the system consists of two ferromagnets, $F_1$ and $F_2$, and three nonmagnets, $N$, $N_1$, and $N_2$. The two nonmagnets, $N_1$ and $N_2$, show spin-orbit interactions, and inject pure spin current into the $F_1$ and $F_2$ layers via the spin Hall effect. The $N$ layer is just a spacer, and does not show spin-orbit interaction. Current is passed along the $x$-direction and flows within the $N_1$ and $N_2$ layers. The $F_2$ layer is a free layer, and has a perpendicular anisotropy. The spin Hall torque acting on the magnetization of the $F_2$ layer is schematically shown.

Fig. 1. (a) Schematic view of the system consisting of the two ferromagnets and three nonmagnets. The thicknesses of the $N_k$ and $F_k$ ($k = 1, 2$) layers are denoted as $d_{N_k}$ and $d_{F_k}$, respectively. The unit vector pointing in the magnetization direction of the $F_k$ layer is $m_k$. (b) The spin Hall torques acting on the magnetization of the $F_2$ layer are schematically shown.
of the conventional spin Hall system, the directions of $\tau^{(2)}$ and $\tau^{(3)}$ depend on the magnetization of the $F_1$ layer. This is because the longitudinal and transverse spin currents [22-28] have different relaxation length scales; here the longitudinal and transverse correspond to parallel and perpendicular to the local magnetization, respectively. Deterministic switching of the $F_2$ layer is possible when the $F_1$ layer tilts from the $z$-axis. In the following, we derive the explicit forms of the spin Hall torques, $\tau^{(i)}$ ($i = 1, 2, 3$), based on the diffusion equation of the spin accumulation and the Landauer formula.

This paper is organized as follows. In Sec. II, we derive a theoretical formula of the spin torque acting on the magnetization of the $F_2$ layer in the geometry given by Figs. [IIa]. In Sec. [III] we show evidence of the deterministic switching in the $F_2$ layer by solving the Landau-Lifshitz-Gilbert (LLG) equation. The conclusion is summarized in Sec. [IV].

II. Spin Hall Torque Formula

The system we consider is schematically shown in Figs. [IIa] and [IIb]. We denote the thicknesses of the $N_k$ and $F_k$ layers as $d_{N_k}$ and $d_{F_k}$, respectively ($k = 1, 2$). The unit vector pointing in the magnetization direction of the $F_k$ layer is denoted as $m$. The electric field $E_x$ is applied along the $x$-direction, and generates an electric current density in the nonmagnet [21]

$$J_e = \sigma_{N_k} E_x + \frac{\partial N_k \sigma_{N_k} e_y \cdot \partial z m_{N_k}}{2e},$$

where $\sigma_{N_k}$ and $\partial z m_{N_k}$ are the conductivity and the spin Hall angle of the $N_k$ layer, respectively. The spin accumulation in the $N_k$ layer is denoted as $\mu_{N_k}$. Similarly, the spin current density in the nonmagnet flowing along the $z$-direction is [21]

$$J_s = -\frac{\hbar \sigma_{N_k}}{4e^2} \partial x m_{N_k} + \frac{\hbar \partial z m_{N_k} E_x}{2e} e_y,$$

where the vector represents the direction of the spin polarization. The spin accumulation in the nonmagnet obeys the diffusion equation, and its solution can be expressed by a linear combination of $e^{kz/\lambda_{N_k}}$, where $\lambda_{N_k}$ is the spin diffusion length.

The pure spin current injected from the nonmagnet creates a spin accumulation in the ferromagnet. The spin accumulation in a ferromagnet, $\mu_F$, can be decomposed into the longitudinal and transverse components as $\mu_F^L = (m \cdot \mu_F)m$ and $\mu_F^T = m \times (\mu_F \times m) = \mu_F - \mu_F^L$, respectively. The relaxation length scale of the longitudinal spin accumulation, called the spin diffusion length, depends on the spin-flip scattering time, and is on the order of 1-10 nm [22]. The relaxation length of the transverse spin accumulation, called the spin penetration depth of the transverse spin current, depends on both the spin-flip scattering time and precession period of spin around the local magnetization due to the exchange coupling. Since the penetration depth of the transverse spin current is usually shorter than the spin diffusion length [23-28], let us assume that the spin accumulation in the $F$ layer, $\mu_F$, has only the longitudinal component, for simplicity. In other words, the direction of the spin polarization of the electrons becomes parallel to the magnetization direction when it is injected into the ferromagnet. We note that a finite penetration depth of the transverse spin current can be taken into account in the following calculations, as done in the case of spin pumping [26]. It does not however affect the main conclusion significantly, in spite of the fact that it makes the calculations complex. The important point is that the longitudinal and transverse spin currents relax with different length scales, and therefore, the direction of the spin polarization of the spin current is modified by passing through a ferromagnet. The idea is similar to what is used in Ref. [29].

At both ends of the nonmagnets, $N_1$ and $N_2$, the spin currents are zero. On the other hand, the spin current density at the $F/N$ interface flowing from $F$ to $N$ layer is given [23]

$$J_{xF\rightarrow N} = \frac{1}{4\pi S} \left[ \left(1 - \frac{\gamma^2 F/N}{2g_F/N} \right) m \cdot (\mu_F - \mu_N) m - g_F/F \left(m \times (\mu_N \times m)\right) \right],$$

where $S$ is the cross section area of the $xy$-plane (Eq. (3) is applicable to both $F_1/N$ and $F_2/N_k$ interfaces). Here $g = g^t + g^\perp$ and $\gamma = (g^t - g^\perp)/(g^t + g^\perp)$ are the total $F/N$ interface conductance and its spin polarization, respectively. We neglect spin-flip scattering at the interface. The conductance $g$ is related to the $F/N$ interface resistance $r$ via $g = h/(e^2r)$. The real part of the mixing conductance is denoted as $g_r$.

For simplicity, we neglect the imaginary part of the mixing conductance, which is usually negligible in a metallic current-perpendicular-to-plane system [23,24] but might be large in spin Hall geometry [11-16]. The spin Hall torque acting on the magnetization of the $F_k$ layer is given by

$$\frac{dm_k}{dt} = \frac{\gamma_0}{M_k S d_{F_k}} m_k \times \left( J_{xF\rightarrow N}^k + J_{xF\rightarrow N}^k \right),$$

where $\gamma_0$ is the gyromagnetic ratio of the $F_k$ layer.

We assume that the $N$ layer sandwiched by the two ferromagnets is sufficiently thin compared to its spin diffusion length, and therefore, the spin current in the $N$ layer is conserved, i.e., $J_{xF\rightarrow N}^1 + J_{xF\rightarrow N}^2 = 0$. Solving the diffusion equations of the $N_k$ and $F_k$ layers, then substituting the solutions to Eq. (3), and using the relation $J_{xF\rightarrow N}^1 + J_{xF\rightarrow N}^2 = 0$, Eq. (4) becomes

$$\frac{dm_k}{dt} = (-1)^k \tau_{k}^{(1)} m_k \times (e_y \times m_k)$$

$$- (-1)^{k+1} \left[ \tau_{k}^{(2)} m_{k_y} m_1 \cdot m_2 - \tau_{k}^{(3)} m_{k_y} \right] m_k \times (m_r \times m_k)$$

$$\frac{1}{1 - \lambda_k^{(2)} m_1 \cdot m_2)^2},$$

where $(k, \ell) = (1, 2)$ or $(2, 1)$. The following notations are introduced:

$$\tau_{k}^{(1)} = \frac{\gamma_0 \hbar \sigma_{N_k} g_{r(F_k/N_k)} \sigma_{N_k} E_x}{2e g_{sd(N_k)} M_k d_{F_k}} \tanh \left( \frac{d_{N_k}}{2\lambda_k} \right),$$

$$\tau_{k}^{(2)} = \frac{\gamma_0 \hbar \sigma_{N_k} g_{r(F_k/N_k)} g_{F_k/N_k} \sigma_{N_k} E_x}{2e g_{r(F_k/N_k)} + g_{F_k/N_k} M_k d_{F_k}},$$

$$\tau_{k}^{(3)} = \frac{\gamma_0 \hbar \sigma_{N_k} g_{r(F_k/N_k)} g_{F_k/N_k} \sigma_{N_k} E_x}{2e g_{r(F_k/N_k)} + g_{F_k/N_k} M_k d_{F_k}}.$$
\[
\lambda_k = \frac{g_t(F_k/N) - g'_{F_k/N}}{g_t(F_k/N) + g'_{F_k/N}},
\]
where
\[
\frac{1}{g'_{F_k/N}} = \frac{1}{g_t(F_k/N)} + \frac{1}{g_{sd}(N_k) \tanh(d_{N_k}/\lambda_{N_k})},
\]
\[
\frac{1}{g_{F_k/N}} = \frac{2}{g_{sd}(F_k)} + \frac{1}{g_{sd}(F_k) \tanh(d_{F_k}/\lambda_{F_k})} + \frac{1}{g_{sd}(N_k) \tanh(d_{N_k}/\lambda_{N_k})},
\]
\[
\frac{1}{g_{sd}(F_k)} = \frac{2}{g_{sd}(F_k)} + \frac{1}{g_{sd}(F_k) \tanh(d_{F_k}/\lambda_{F_k})} - \frac{g_{sd}(F_k) \sinh(d_{F_k}/\lambda_{F_k})}{g_{sd}(F_k) \sinh(d_{F_k}/\lambda_{F_k})},
\]
\[
g_{sd}(F_k) = \frac{h(1 - \gamma^2_{F_k})}{2e^2 \rho_{F_k} \lambda_{F_k}}, \quad g_{sd}(N_k) = \frac{hS}{2e^2 \rho_{N_k} \lambda_{N_k}},
\]
and \(\rho = 1/\sigma\). The spin polarization of the conductivity is denoted as \(\beta = (\sigma^1 - \sigma^\perp)/(\sigma^1 + \sigma^\perp)\). A calculation of spin Hall torque in a system including two ferromagnets with a different geometry was developed in Ref. [30].

In Eq. (5), the first torque proportional to \(\tau^{(1)}_k\) originates from the pure spin current injected from the \(N_k\) layer via the spin Hall effect, and points to the \(y\)-direction, as in the case of the conventional spin Hall torque. The second term proportional to \(\tau^{(2)}_k\) originates from the electrons passing from the \(N_k\) layer through the \(F_k\) layer and being reflected from the \(N_\ell\) layer \((\ell \neq k)\) This torque becomes zero when \(m_{ky} = 0\) because the spin current from the \(N_k\) layer is completely absorbed at the \(F_k/N_k\) interface, and therefore the electrons passing through the \(F_k\) layer and diffusing between the \(N_1\) and \(N_2\) layers do not have net spin polarization. Similarly, this torque also becomes zero when \(m_1 \cdot m_2 = 0\) because in this case the spin current is completely absorbed at the \(F_2/N_2\) or \(F_2/N_1\) interface. The third term proportional to \(\tau^{(3)}_k\) originates from the pure spin current injected from the \(N_\ell\) layer. Due to the absorption of the transverse spin current in the \(N_\ell\) layer, this torque is zero when \(m_{k\ell y} = 0\).

The directions of the second and third torques acting on the \(F_k\) layer, \(\mathbf{m}_k \times (\mathbf{m}_k \times \mathbf{m}_k)\), can be changed by controlling the magnetization direction of the other layer, \(\mathbf{m}_\ell\). This is because the direction of the spin polarization is modified by passing through the ferromagnet due to the different relaxation lengths between the longitudinal and transverse spin currents. Now let us assume that the magnetization in the \(F_1\) layer, \(\mathbf{m}_1\), tilts from the \(z\)-axis. Then the second and third torques acting on the magnetization of the \(F_2\) layer remain finite even when \(\mathbf{m}_2\) arrives at the film-plane, contrary to the first torque which becomes zero when \(\mathbf{m}_2\) becomes parallel to the \(y\)-axis, as in the case of the conventional spin Hall torque. Depending on the direction of the electric field \(E_x\), the second and third torques move the magnetization \(\mathbf{m}_2\) to the positive or negative \(z\)-direction. Therefore, a deterministic switching of \(\mathbf{m}_2\) can be expected. A finite tilted angle of \(\mathbf{m}_1\) is necessary to break the symmetry with respect to the film-plane and induces the deterministic switching. If \(\mathbf{m}_1 \parallel e_z\), and when \(\mathbf{m}_2\) arrives at the film-plane, the second and third torques in Eq. (5) become zero. Then, the magnetization will be stopped at the plane, as in the case of the conventional spin Hall system.

III. LLG Equation

Using Eq. (5), the LLG equation of the \(F_k\) layer becomes
\[
\frac{d\mathbf{m}_k}{dt} = -\gamma_0 \mathbf{m}_k \times \mathbf{H}_k + \tau_k + \alpha \mathbf{m}_k \times \frac{d\mathbf{m}_k}{dt},
\]
where the spin Hall torque \(\tau_k\) is given by Eq. (3). As mentioned above, we assume that the magnetization of the \(F_1\) layer is fixed, while that of the \(F_2\) layer changes its direction according to Eq. (15). The magnetic field acting on \(\mathbf{m}_2\), \(\mathbf{H}_2 = H_K m_{2z} e_z\), consists of the perpendicular anisotropy field \(H_K\). The Gilbert damping constant of the \(F_2\) layer is denoted as \(\alpha\). In the absence of the electric field \(E_x\), the \(F_2\) layer has two stable states, \(\mathbf{m}_2 = \pm e_z\). In this calculation, we assume that the magnetization \(\mathbf{m}_2\) initially points to the positive \(z\)-direction.

We assume that the parameters of the \(N_1\) and \(N_2\) layers, as well as those of the \(F_1\) and \(F_2\) layers, are identical. We define the current density \(j\) as \(j = \sigma_{N_1} E_x = \sigma_{N_2} E_x\). Linearizing Eq. (15) around the initial state, we found that the magnetization of the free (\(F_2\)) layer is destabilized when the current magnitude is larger than a critical value,
\[
j_c = \frac{2e N d_2}{\hbar \mathcal{P}} H_K,
\]
where \(\mathcal{P}\) is given by
\[
\mathcal{P} = \frac{g_{F_k/N} g'_{F_k/N}}{2g_k [g_{F_k/N} + g'_{F_k/N}]} \times \left[ 2(1 - \lambda'^2_1 \lambda'^2_2) + \frac{\lambda'^2_1}{1 - \lambda'^2_1 \lambda'^2_2} \right] m_{1y} m_{1z}
\]

We confirm the deterministic switching of \(\mathbf{m}_2\) from \(\mathbf{m}_2 = +e_z\) to \(\mathbf{m}_2 = -e_z\) by numerically solving Eq. (15). The values of parameters are taken from typical experiments in CoFeB/ TaN heterostructure [16] or similar metallic F/N multilayer [22-26] as \(\lambda_F = 12\) nm, \(\rho_F = 1600\) \(\Omega\)\(\text{mm}, \beta = 0.56\), \(d_F = 1\) nm, \(\lambda_N = 25\) nm, \(\rho_N = 3750\) \(\Omega\)\(\text{mm}, d_N = 4\) nm, \(\varphi = 0.1, r = 0.28\) \(\text{k}\Omega\)\(\text{mm}^2, \gamma = 0.70, g_r/S = 15\) \(\text{nm}^{-2}, \gamma_0 = 1.764 \times 10^7\) rad/(Oe s), \(\alpha = 0.005, M_s = 1250\) emu/cm\(^3\), and \(H_K = 450\) Oe, where, we assume that the interface conductances, \(g_r\) and \(g_s\), at the \(F_2/N_2\) and \(F_2/N_1\) interfaces are the same. We assume that the magnetization of the \(F_1\) layer points to the direction \(\mathbf{m}_1 = (\sin \theta_p, \cos \varphi_p, \sin \theta_p \sin \varphi_p, \cos \theta_p)\) where \(\theta_p = 30^\circ\) and \(\varphi_p = 90^\circ\). The critical current density estimated by Eq. (16) is \(1.98 \times 10^6\) A/cm\(^2\). Figure 2 shows the time evolution of \(m_{2z}\) obtained from Eq. (15) with these parameters and several current densities. The dynamics becomes relatively slow around \(m_{2z} \sim 0.6 - 0.8\) where \(\mathbf{m}_2\) becomes almost parallel to \(\mathbf{m}_1\), and therefore, the second and third torques in
When the current density is larger than $j$, the spin Hall effect, whose spin polarization points to the in-plane ($\langle\rangle$ direction. Passing through a ferromagnet however, the direction of the spin polarization is modified due to the different relaxation length scales of the longitudinal and transverse spin currents. Consequently, a tilt of the magnetization in one ferromagnet breaks the symmetry of the system, and the spin Hall torque excited in such geometry enables to switch a perpendicular magnetization in another ferromagnet deterministically. The idea was confirmed by deriving the spin Hall torque formula and solving the LLG equation numerically.

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