Transport in time-dependent random potentials

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The classical dynamics in stationary potentials that are random both in space and time is studied. It can be intuitively understood with the help of Chirikov resonances that are central in the theory of Chaos, and explored quantitatively in the framework of the Fokker-Planck equation. In particular, a simple expression for the diffusion coefficient was obtained in terms of the average power density of the potential. The resulting anomalous diffusion in velocity is classified into universality classes. The general theory was applied and numerically tested for specific examples relevant for optics and atom optics.

I. INTRODUCTION

Potentials that are random both in space in time are models of noise. Often one seeks for methods of suppressing noise, the present work on the other-hand is focused on interesting properties of dynamics generated by noise. It was subject of many sophisticated studies for nearly 100 years [1-4]. The response to forces resulting of such potentials typically differs from ordinary diffusion. Specifically, the diffusion coefficients predicted by such mechanisms sensitively depend on the velocity of the particles. Also, if the potential is time-dependent, the energies of the particles will not be constant. The existence of such ‘anomalous diffusion’ has been demonstrated for classical dynamics with spatially and temporally fluctuating potentials [5-10]. These works claim universal behavior in the sense that for generic random constant. The existence of such anomalous diffusion has been demonstrated for classical dynamics which is random both in space and time is studied. The response to forces resulting of such potentials the diffusion coefficient has a universal power-law dependence on velocity $v$, such that $D(v) \sim |v|^{-3}$ as $|v| \to \infty$. This in turn implies that asymptotically in time the average velocity satisfies $\langle v^2 \rangle \sim t^{2/5}$. The average displacement satisfies $\langle x^2 \rangle \sim t^{12/5}$ for one-dimensional systems [5, 8, 10] (faster than ballistic) and $\langle x^2 \rangle \sim t^2$ (ballistic transport on average) for systems with dimension higher than one [6].

In the present work we will study dynamics of classical particles in potentials which are defined in terms of their Fourier components. Two types of potentials will be explored:

$$V^{(1)}(x, t) = \int \tilde{V}(k, \omega) \exp(ik \cdot x - \omega t) \, dk \, d\omega + c.c. \quad (1)$$

and

$$V^{(2)}(x, t) = \left| V^{(1)}(x, t) \right|^2 \quad (2)$$

where $\tilde{V}(k, \omega)$ is a random field chosen such that the distribution of $V^{(1)}(x, t)$ is stationary both in time and space. For time dependent potentials the velocity of the particles may grow, since energy is not conserved. The dominant mechanism of the growth of the velocity is via Chirikov resonances, namely, resonances between the particle dynamics and the external driving. Those resonances occur when the phases in (1) are stationary. For a given instantaneous velocity $v = \dot{x}$, this happens for

$$k \cdot v = \omega, \quad (3)$$

and a more complicated expression is found for the potential (2). Indeed, transport takes place only in such regions in phase-space where the density of the resonances is non-zero. The Chirikov resonances provide an intuitive picture for the understanding of transport in phase-space. Since, the potentials are given in terms of their Fourier components it is relatively easy to calculate the spectral content or average power spectral density (PSD), $S(k, \omega)$, (see [11, 13]) and
with its help also the diffusion coefficient. The spread in velocity is then calculated using the Fokker-Planck (FP) approximation, assuming that the decay of the potential correlations is sufficiently rapid. The random potentials of the form (1) and (2) are natural in optics realizations since these are formed by a superposition of light beams. In recent experiments in optics [11, 17], light propagates paraxially in a disordered potential, which is produced by utilizing the photo-sensitivity of the medium [14, 15]. In the paraxial approximation a probe beam satisfies a Schrödinger like equation where the refractive index plays the role of the potential, the propagation direction plays the role of time and the dispersion relation is approximately \( \omega(k) = k^2/2 \). Since the index of refraction is proportional to intensity of the electric field of the writing beam, that varies like \( V(1)(x,t) \) of (1), the resulting potential will have the form of \( V(2)(x,t) \) (2). Potentials of this type are also relevant for atom optics. There the electric field induced a dipole moment in a neutral atom which interacts with the electric field and results in a potential which is proportional to the intensity of the electric field. The time dependence is induced by a modulation of the intensity of the electric field, such that it is independent of \( k \).

Although this work was motivated by experiments in optics, where the dynamics is wave dynamics, the classical dynamics of particles in such potentials is a fundamental question by itself. Therefore, in this work we will study the classical dynamics of particles in potentials of the form (1) and (2). Nevertheless, since it is generally believed that for large velocity (or short wave length) wave dynamics is approximated by classical dynamics, we expect at least qualitatively to describe the original wave problem.

We will classify the various systems into universality classes according to the velocity dependence of the diffusion coefficient in large and small velocity limits. The regimes of the validity of the FP approximation will be studied as well, and the crossover between uniform acceleration and diffusive behavior (in velocity) will be studied. The formalism will be demonstrated for specific examples of the form (1) and (2). Our main result is that although the universality of the form \( D(v) \sim v^{-3} \) holds always for dimensions higher than one, for one dimensional systems there is a large variety of novel universality classes. A classification scheme for such classes will be presented and it will be demonstrated how these can be realized.

In Sec. II the time required for the particles to ‘discover’ the fluctuations of the potential will be evaluated. After this time we expect the dynamics to be controlled by the FP equation. An expression for the diffusion coefficient in terms of the spectral content or more precisely the average power spectral density (PSD) is derived in Sec. III, it is computed explicitly and its significance is discussed for specific examples. A scaling property of the FP equation is presented in Sec. IV. A detailed calculation of the diffusion coefficient in velocity for dimensions two and higher is presented in Sec. V A for potentials of the form of (1), and in Sec. V B for potentials of the form (2). In Sec. VI the rich variety of the universality classes found for one-dimensional systems is presented and the asymptotic expansion in powers of \( v^{-1} \) is discussed. The results are summarized and discussed in Sec. VII.

II. TIME DOMAINS

In this section we discuss the typical time scale to observe the anomalous diffusion. For this purpose we assume an initial distribution that is narrow in phase-space. Initially the velocity is growing linearly in time (acceleration), since it takes time for the fluctuations of the potential to become effective. First we define the characteristic length and time scales of the potential. Consider the potential given in (1), let \( l_x \) be the length over which the potential has an almost constant derivative, resulting in a constant force. It satisfies,

\[
\int \langle V^2 \rangle - \langle V \rangle^2
\]

In a similar way we define the characteristic time scale \( l_t \),

\[
\int \langle V^2 \rangle - \langle V \rangle^2
\]

Using,

\[
\int \langle V \rangle = \int \omega_1 \omega_2 \int dk_1 dk_2 \left( \hat{V}(k_1, \omega_2) \hat{V}^*(k_2, \omega_2) \right) \exp i \left( (k_1 - k_2) \cdot x - (\omega_1 - \omega_2) t \right),
\]

\[
= \int \omega \int dk S(k, \omega),
\]

\[
(6)
\]
one finds,
\[
\left\langle |\nabla V|^2 \right\rangle = \int d\omega_1 d\omega_2 \int dk_1 dk_2 k_1 \cdot k_2 \left\langle \hat{V}(k_1, \omega_2) \hat{V}^*(k_2, \omega_2) \right\rangle \exp i \left( (k_1 - k_2) \cdot x - (\omega_1 - \omega_2) t \right),
\]
\[
= \int d\omega \int dk \, k^2 \mathcal{S}(k, \omega),
\]
where $S(k, \omega)$ is the average power spectral density (PSD) defined in (16). Therefore the characteristic length scale, $l_x$, is given by,
\[
l_x^2 = \frac{\int d\omega \int dk \, k \mathcal{S}(k, \omega)}{\int d\omega \int dk \, k^2 \mathcal{S}(k, \omega)},
\]
and similarly the characteristic time scale, $l_t$, is given by,
\[
l_t^2 = \frac{\int d\omega \int dk \, k \mathcal{S}(k, \omega)}{\int d\omega \int dk \mathcal{S}(k, \omega)}.
\]

Note, that $l_t$ and $l_x$ are different from the exponential decay rates of correlations. The difference follows from the fact that we are interested in the typical lengths in both time and space where the force is constant, and not in the asymptotic behavior of the correlation function. For times $t < l_t$, the force resulting from a potential is almost constant, provided that the particle is not displaced more than $l_x$. For a weak force,
\[
F_0 < \frac{l_x}{l_t}.
\]

the particle will travel a distance less than $l_x$ in time $l_t$, therefore the cross-over out of the uniform acceleration regime takes place at time,
\[
t_* \sim l_t.
\]
If the force is strong so that (10) is not satisfied, the particle will reach $l_x$ at a time shorter than $l_t$ and therefore $t_*$ will be,
\[
t_* \sim \sqrt{\frac{l_x}{F_0}}.
\]

In this work we will consider initial conditions of particles with a narrow distribution both in velocity and position, such that $\delta v < l_x/l_t$, where $\delta v$ is the width of the distribution centered at $v = 0$. For these initial conditions uniform acceleration will take place for time $t < l_t$, while for longer time-scales the velocity will exhibit anomalous diffusion. In this work we will assume that the force is weak, such that (10) holds and the crossover to the diffusive regime will take place at (11).

III. ANOMALOUS DIFFUSION IN VELOCITY

In this section the variation of the velocity of particles by potentials $V(x, t)$, random both in space and time will be studied on time scales longer than $t_*$. The equations of motion are,
\[
\frac{dv}{dt} = -\nabla V(x, t),
\]
\[
\frac{dx}{dt} = v,
\]
where $v$ and $x$ are the instantaneous velocity and position. For simplicity we set the mass to be one. We will assume a stationary potential both in space and time and an isotropic distribution of the potential. Therefore, its correlation function is
\[
\left\langle V(x_1, t_1) V(x_2, t_2) \right\rangle = C(x_1 - x_2, t_1 - t_2),
\]
where $C$ represents the correlation function. The characteristic length scale, $l_x$, is given by,
\[
l_x^2 = \frac{\int d\omega \int dk \, k \mathcal{S}(k, \omega)}{\int d\omega \int dk \, k^2 \mathcal{S}(k, \omega)}.
\]
and we may take the average of the potentials, that is a constant, to vanish, namely,
\[
\langle V(x, t) \rangle = 0.
\]  (15)
The angular brackets \(\langle \cdot \rangle\) denote the average over the ensemble of random potentials. The Weiner-Khinchin theorem implies,
\[
C(x, t) = \int d\omega \int dk \, S(k, \omega) \exp i (k \cdot x - \omega t),
\]  (16)
where \(S(k, \omega)\) is the average power spectral density (PSD) of the potential, that is its spectral content.
The Fokker-Planck (FP) equation for the velocity is given by [4],
\[
\frac{\partial P}{\partial t} = \left( \frac{\partial}{\partial v_i} D_{ij} \frac{\partial}{\partial v_j} \right) P,
\]  (17)
where \(P(v, t)\) is the probability density and \(D_{ij}\) is the diffusion tensor, given by the expression,
\[
D_{ij}(v) = \frac{1}{2} \int_{-\infty}^{\infty} \langle F_i(x_\tau, \tau) F_j(0, 0) \rangle \, d\tau,
\]  (18)
where
\[
F = -\nabla V
\]  (19)
is the force. Here one assumes that the correlations decay sufficiently fast and the force is sufficiently weak, so that the variation of the velocity on the time-scale of decay of correlations is negligible, and
\[
x_1 - x_2 = v(t_1 - t_2),
\]  (20)
this is the usual approximation used to derive the Fokker-Planck equation [4]. Here and in the rest of the paper summation over repeated indexes is assumed. We will now calculate the force correlation function,
\[
K_{ij}(x_1 - x_2, t_1 - t_2) = \langle F_i(x_1, t_1) F_j(x_2, t_2) \rangle,
\]  (21)
which can be written using the potential correlation function,
\[
K_{ij}(x_1 - x_2, t_1 - t_2) = \frac{\partial^2}{\partial x_{1,i} \partial x_{2,j}} \langle V(x_1, t_1) V(x_2, t_2) \rangle = -\frac{\partial^2 C(x, t)}{\partial x_i \partial x_j} \big|_{x=x_1-x_2},
\]  (22)
Using (16) one finds,
\[
K_{ij}(x_1 - x_2, t_1 - t_2) = \int d\omega \int dk \, k_i k_j S(k, \omega) \exp i (k \cdot (x_1 - x_2) - \omega (t_1 - t_2)),
\]  (23)
and the diffusion tensor is,
\[
D_{ij}(v) = \frac{1}{2} \int_{-\infty}^{\infty} K_{ij}(v\tau, \tau) \, d\tau.
\]  (24)
Which can be written as,
\[
D_{ij}(v) = \frac{1}{2} \int_{-\infty}^{\infty} d\tau \int d\omega \int dk \, k_i k_j S(k, \omega) \exp i\tau (k \cdot v - \omega).
\]  (25)
Taking the integral over \(\tau\) first, gives a delta function
\[
D_{ij}(v) = \pi \int d\omega \int dk \, k_i k_j S(k, \omega) \delta (k \cdot v - \omega).
\]  (26)
Then the integral over \(\omega\) can be easily performed to give,
\[
D_{ij}(v) = \pi \int dk \, k_i k_j S(k, k \cdot v).
\]  (27)
Since $S(k, \omega)$ is integrable by definition, we will define a function $F(k, \omega)$, so that

$$S(k, \omega) = \frac{\partial^2}{\partial \omega^2} F(k, \omega).$$  \hspace{1cm} (28)

Then, the diffusion tensor can be written as,

$$D_{ij}(v) = \frac{\partial^2}{\partial v_i \partial v_j} I(v),$$  \hspace{1cm} (29)

where

$$I(v) = \pi \int dk \, F(k, k \cdot v),$$ \hspace{1cm} (30)

and we have used the fact that the integral is isotropic. Equation (29) can be decomposed to,

$$D_{ij}(v) = \frac{\partial^2}{\partial v_i \partial v_j} I(v) = \frac{\partial}{\partial v_i} \left( \frac{v_i}{v} \frac{\partial I}{\partial v} \right)$$  \hspace{1cm} (31)

$$= \left( \delta_{ij} - \frac{v_i v_j}{v^2} \right) \frac{\partial I}{\partial v} + v_i v_j \frac{\partial^2 I}{v^2 \partial v^2}.$$

For a general isotropic diffusion tensor of the form,

$$D_{ij}(v) = \left( \delta_{ij} - \frac{v_i v_j}{v^2} \right) f_T(v) + \frac{v_i v_j}{v^2} f_P(v),$$ \hspace{1cm} (32)

the operator on the RHS of (17) is,

$$\hat{L}_0 = \frac{\partial}{\partial v_i} D_{ij}(v) \frac{\partial}{\partial v_j} = v^{-(d-1)} \frac{\partial}{\partial v} v^{(d-1)} f_P(v) \frac{\partial}{\partial v}. \hspace{1cm} (33)$$

We turn now to justify this result. It is convenient to use hyper-spherical variables, such that,

$$\frac{\partial}{\partial v_i} = \frac{\partial}{\partial v_i} + \sum_{j=1}^{d-1} \frac{\partial \phi_j}{\partial v_i} \frac{\partial}{\partial v} = \frac{v_i}{v} \frac{\partial}{\partial v} + \sum_{j=1}^{d-1} \frac{\partial \phi_j}{\partial v_i} \frac{\partial}{\partial \phi_j}, \hspace{1cm} (34)$$

where $d$ is the dimension of space and $\phi_j$ are the angle coordinates. Using (32) and the new variables the operator (33) reduces to,

$$\hat{L}_0 = \frac{\partial}{\partial v_i} \left[ \left( \delta_{ij} - \frac{v_i v_j}{v^2} \right) f_T(v) + \frac{v_i v_j}{v^2} f_P(v) \right] \frac{\partial}{\partial v_j}$$

$$= \frac{\partial}{\partial v_i} \left[ \left( \delta_{ij} - \frac{v_i v_j}{v^2} \right) f_T(v) + \frac{v_i v_j}{v^2} f_P(v) \right] \frac{\partial}{\partial v_j}$$

$$= \frac{\partial}{\partial v_i} \left[ \left( \delta_{ij} - \frac{v_i v_j}{v^2} \right) f_T(v) + \frac{v_i v_j}{v^2} f_P(v) \right] \frac{\partial}{\partial v_j}$$

$$= \frac{\partial}{\partial v_i} \left[ \left( \delta_{ij} - \frac{v_i v_j}{v^2} \right) f_T(v) + \frac{v_i v_j}{v^2} f_P(v) \right] \frac{\partial}{\partial v_j}$$

$$= \frac{\partial}{\partial v_i} \left[ \left( \delta_{ij} - \frac{v_i v_j}{v^2} \right) f_T(v) + \frac{v_i v_j}{v^2} f_P(v) \right] \frac{\partial}{\partial v_j}$$

$$= \frac{\partial}{\partial v_i} \left[ \left( \delta_{ij} - \frac{v_i v_j}{v^2} \right) f_T(v) + \frac{v_i v_j}{v^2} f_P(v) \right] \frac{\partial}{\partial v_j}$$

Which using the chain rule can be written as,

$$\hat{L}_0 = \left( \frac{v_i v_j - v_i v_j}{v^2} \right) f_T(v) \frac{\partial}{\partial v} + \frac{\partial f_P(v)}{\partial v} \frac{\partial}{\partial v}$$

$$= \frac{v}{v} (d-1) f_P(v) \frac{\partial}{\partial v} + \frac{\partial f_P(v)}{\partial v} \frac{\partial}{\partial v}$$

$$= v^{-(d-1)} \frac{\partial}{\partial v} v^{(d-1)} f_P(v) \frac{\partial}{\partial v}. \hspace{1cm} (36)$$

Therefore, combining (31) and (33) one identifies $f_P(v) = \partial^2 I/\partial v^2$. Therefore, the operator $\hat{L}_0$ is given by,

$$\hat{L}_0 = v^{-(d-1)} \frac{\partial}{\partial v} v^{(d-1)} D(v) \frac{\partial}{\partial v}. \hspace{1cm} (37)$$
The diffusion coefficient \( D(v) \) is found to be,

\[
D(v) = \frac{\partial^2 I}{\partial v^2}. \tag{38}
\]

It can be simplified using the definition (30) and (28),

\[
D(v) = \pi \int dk \ (k \cdot \dot{v})^2 S(k, k \cdot v). \tag{39}
\]

Using (39) and aligning the \( x \)-component of \( k \) with velocity we get,

\[
D(v) = S_d \int dv \int d\theta \int_0^{\infty} dk k^{d+1} \cos^2 \theta S(k, kv \cos \theta), \tag{40}
\]

where \( S_d \) is the surface of the \( d \)-dimensional hyper-sphere. Changing variables to, \( y = v \cos \theta \) we get

\[
D(v) = S_d \int dv \int_{-v}^{v} dy \int_0^{\infty} dk k^{d+1} S(k, ky), \tag{41}
\]

where the factor of two was eliminated due the multiplicity. This expression assumes implicitly that \( d > 1 \). For one dimensional systems there is no integration over the angle (see Eq. (96) in Sec. VI). The different dependence on the velocity of (41) and (96) is the origin of the richness of the one dimensional behavior. For large velocities (41) has the following asymptotic behavior,

\[
D(v) \sim \frac{D_3}{v^5}, \tag{42}
\]

with

\[
D_3 = 2S_d \int_0^{\infty} dy \int_0^{\infty} dk k^{d+1} S(k, ky). \tag{43}
\]

Therefore, for dimensions higher than one the asymptotic behavior of the diffusion coefficient is indeed universal for any choice of the PSD and is given by (42). The diffusion coefficient for zero velocity using (40) is given by,

\[
D(0) = \frac{S_d}{4} \int_0^{\infty} dk k^{d+1} S(k, 0). \tag{44}
\]

If \( D_3 \neq 0 \), as is the case for \( d \geq 2 \), the Fokker-Planck equation for the velocity (17) is asymptotically equivalent to the equation,

\[
\frac{\partial P}{\partial t} = \left( v^{-(d-1)} \frac{\partial}{\partial v} v^{(d-1)} \frac{D_3}{v^5} \frac{\partial}{\partial v} \right) P, \tag{45}
\]

which has the scaling solution

\[
P(v, t) = \frac{1}{t^{d/5}} g \left( \frac{v^5}{t} \right). \tag{46}
\]

The resulting growth of the mean kinetic energy is,

\[
\frac{1}{2} \langle v^2 \rangle \sim t^{2/5}. \tag{47}
\]

This behavior is considered in the literature as universal \([5, 7, 8, 10]\). For dimensions two and higher this is indeed the case, however for one dimensional systems other behaviors are possible, as will be demonstrated in Sec. VI.
IV. SCALING PROPERTIES

Let us assume that the average power spectral density, \( S(k, \omega) \), has natural spatial and temporal frequency scales, \( k_0 \) and \( \omega_0 \), such that,

\[
S(k, \omega) = \frac{V_0^2}{k_0^2 \omega_0} \tilde{S}\left(\frac{k}{k_0}, \frac{\omega}{\omega_0}\right),
\]

where \( V_0 \) is a constant which determines the strength of the potential. Then (39) takes the form,

\[
D(v) = \pi V_0^2 \int \left(\mathbf{k} \cdot \hat{v}\right)^2 \tilde{S}\left(\frac{\mathbf{k}}{k_0}, \frac{\mathbf{k} \cdot v}{\omega_0}\right) d\mathbf{k},
\]

where \( \tilde{S} \) is a dimensionless PSD. Rescaling the variables

\[
k = k'k_0, \quad v = v'v_0, \quad v_0 \equiv \frac{\omega_0}{k_0},
\]

gives

\[
D(v') = \pi V_0^2 \frac{k_0^2}{\omega_0} \int \left(\mathbf{k}' \cdot \hat{v}'\right)^2 \tilde{S}\left(\frac{\mathbf{k}'}{k_0}, \frac{\mathbf{k}' \cdot v'}{\omega_0}\right) d\mathbf{k}'.
\]

The Fokker-Planck equation for the velocity (17) is invariant under the transformation of variables,

\[
v \to v', \quad t \to t' \left(\pi V_0^2 \frac{k_0^2}{\omega_0}\right)^{-1}.
\]

V. SPECIAL POTENTIALS

In this section the spreading in phase-space is studied for specific potentials. Potentials of the form (1) will be studied in subsection V. A while potentials of the form (2) will be studied in subsection V. B. In this section the emphasis will be on the behavior for dimensions higher than one, while one dimensional systems will be analyzed in the next section.

A. Potentials which are a superposition of waves

Natural potentials to consider are potentials which are composed of a superposition of standing waves (c.f., Eq. (1)),

\[
V(x, t) = \int \hat{V}(k) \exp i \left(\mathbf{k} \cdot \mathbf{x} - \omega(k) t\right) d\mathbf{k} + c.c.
\]

with a dispersion relation \( \omega(k) \). We will assume that the amplitudes and the wave numbers are independent random variables. Leading to,

\[
\left\langle \hat{V}(k_1) \hat{V}^*(k_2) \right\rangle = V_0^2 f(k_1) \delta(k_1 - k_2),
\]

where \( \left\langle \left| V(k) \right|^2 \right\rangle = V_0^2 \) and \( f(k) \) is the distribution of the wave numbers. The correlation function of the potential is given by,

\[
C(x_1 - x_2, t_1 - t_2) = V_0^2 \int f(k) \exp i \left(\mathbf{k} \cdot (x_1 - x_2) - \omega(k) (t_1 - t_2)\right) d\mathbf{k} + c.c.
\]

Using (16) the PSD is,
\[ S(q, \omega) = \frac{1}{(2\pi)^{d+1}} \int C(x, t) \exp(-i(q \cdot x - \omega t)) \, dx \, dt \]

\[ = \frac{1}{(2\pi)^{d+1}} V_0^2 \int f(k) \exp((k - q) \cdot x - (\omega(k) - \omega) t) \, dk \, dx \, dt \]

\[ + \frac{1}{(2\pi)^{d+1}} V_0^2 \int f(k) \exp(-i((k + q) \cdot x - (\omega(k) + \omega) t)) \, dk \, dx \, dt. \]  

By making the integrals over \( x \) and \( t \) we get,

\[ S(q, \omega) = V_0^2 f(q) \delta(\omega - \omega(q)) + V_0^2 f(-q) \delta(\omega + \omega(q)). \]  

Starting from (41) and assuming that both \( f(k) \) and \( \omega(k) \) are isotropic gives,

\[ D(v) = \frac{S_d V_0^2}{v^3} \int_{-v}^{v} dy \frac{y^2}{\sqrt{1 - (y/v)^2}} \int_{0}^{\infty} dk \frac{k^{d-2} f(k) \omega^2(k)}{1 - \omega^2(k) / (v^2 k^2)} \hat{H}(v - \omega(k)), \]

where \( \hat{H}(.) \) is a Heaviside step function, which follows from the restriction of the delta function. In the limit of large velocities we have,

\[ D(v) \sim \frac{D_3}{v^4}, \]  

with

\[ D_3 = 2 S_d V_0^2 \int_{0}^{\infty} dk \frac{k^{d-2} f(k) \omega^2(k)}{} \]

For zero velocity using \( (44) \) we have,

\[ D(0) = \frac{S_d V_0^2}{2} \int_{0}^{\infty} dk \frac{k^{d+1} f(k) \delta(\omega(k))}{\omega'(0)} = 0, \]

where we have assumed that

\[ \omega(k) = 0, \]

has only one solution \( k = 0 \), which is a general property of physically relevant dispersion relations.

We will proceed by calculating the diffusion coefficient for a particular choice of the dispersion relation, \( \omega(k) = k^2/2 \), and a uniform density of wave numbers,

\[ f(k) = \begin{cases} \frac{1}{V_d k_R^d} & 0 < k < k_R, \\ 0 & k > k_R, \end{cases} \]

where \( V_d \) is the volume of the \( d- \) dimensional unit hyper sphere. Therefore, we have

\[ D(v) = \frac{d V_0^2}{2 k_R^d v^3} \begin{cases} \int_{0}^{2v} dk \frac{k^{d+2}}{\sqrt{1-k^2/(4v^2)}} & v < k_R/2, \\ \int_{0}^{k_R} dk \frac{k^{d+2}}{\sqrt{1-k^2/(4v^2)}} & v > k_R/2. \end{cases} \]
Figure 1: A log-log plot of average squared velocity as a function of time for a two dimensional system with the potential \((53)\) and a distribution of wave-numbers \((64)\). The blue dots represent the result of the Monte-Carlo solution of \((13)\) averaged over 20 realizations and the black solid line is the numerical solution of the Fokker-Planck equation for the velocity \((17)\). The dashed black and red lines are guides for the eye with the corresponding slopes of 2 and 2/5. The initial condition was a narrow distribution of velocities for the Fokker-Planck and \(x = v = 0\) for the Monte-Carlo calculation. The parameters used for this simulation are, \(V_0 = 10^{-2}\), \(k_R = 0.1\).

where we have used the fact \(S_d/V_d = d\). Taking the integrals gives,

\[
D(v) = \frac{dV_0^2}{2k_R^d v^3} \begin{cases} 
2^{d+2} \sqrt{\pi} \frac{\Gamma(d+3)}{\Gamma(2+d)} v^{d+3} & v < k_R/2 \\
\frac{\pi}{2} \Gamma \left( \frac{d+3}{2} \right) \, _1F_2 \left( \frac{1}{2}, \frac{d+3}{2}, \frac{k_R^2}{v^2} \right) & v > k_R/2
\end{cases}
\]

where \(_1F_2(x)\) is Gauss’s hypergeometric function. For the special case of \(d = 2\),

\[
D(v) = V_0^2 \begin{cases} 
6\pi \left( \frac{v}{k_R} \right)^2 & v < k_R/2 \\
3 \left( \frac{v}{k_R} \right)^2 \csc^{-1} \left( \frac{2v}{k_R} \right) - \left( \frac{2v}{k_R} \right)^3 \left( 2 + 3 \left( \frac{2v}{k_R} \right)^2 \right) \sqrt{1 - \left( \frac{4v}{k_R} \right)^2} & v > k_R/2
\end{cases}
\]

The prediction of the FP equation was compared to the Monte-Carlo simulation and is presented in Fig. 1. It is found that uniform acceleration takes place for times \(v < k_R/2\) and the FP prediction clearly fails in this regime. This can be expected since from \((67)\) \(D(v) \sim v^2\), the assumption that the velocity is constant \((20)\) cannot hold, rendering the FP approximation inconsistent. Nevertheless, for \(v > k_R/2\) the FP predictions are satisfied. Note that, for any potential discussed in this section the scaling properties of Sec. [IV] hold.

Numerically, it is found that for large velocities \((v > k_R/2)\) the average square position is growing ballistically, \(\langle x^2 \rangle \sim t^2\), as expected from previous studies [5, 6, 8, 10], while for small velocities \((v < k_R/2)\), \(\langle x^2 \rangle \sim t^4\), as expected for uniform acceleration.

B. Potentials proportional to the intensity of a superposition of waves

In this subsection the diffusion coefficient for potentials of the form \((2)\) will be calculated. These potentials appear in experiments with neutral atoms and in some experiments in optics, where the refractive index (which plays the role of the potential) of a photosensitive material is proportional to the intensity of light [14, 15]. The complex field, \(U\), which represents the superposition pattern of waves is given by,

\[
U(x, t) = \int \text{d}k \hat{U}(k) \exp i(k \cdot x - \omega(k) t),
\]

where we have used the fact \(S_d/V_d = d\). Taking the integrals gives,

\[
D(v) = \frac{dV_0^2}{2k_R^d v^3} \begin{cases} 
2^{d+2} \sqrt{\pi} \frac{\Gamma(d+3)}{\Gamma(2+d)} v^{d+3} & v < k_R/2 \\
\frac{\pi}{2} \Gamma \left( \frac{d+3}{2} \right) \, _1F_2 \left( \frac{1}{2}, \frac{d+3}{2}, \frac{k_R^2}{v^2} \right) & v > k_R/2
\end{cases}
\]

where \(_1F_2(x)\) is Gauss’s hypergeometric function. For the special case of \(d = 2\),

\[
D(v) = V_0^2 \begin{cases} 
6\pi \left( \frac{v}{k_R} \right)^2 & v < k_R/2 \\
3 \left( \frac{v}{k_R} \right)^2 \csc^{-1} \left( \frac{2v}{k_R} \right) - \left( \frac{2v}{k_R} \right)^3 \left( 2 + 3 \left( \frac{2v}{k_R} \right)^2 \right) \sqrt{1 - \left( \frac{4v}{k_R} \right)^2} & v > k_R/2
\end{cases}
\]

The prediction of the FP equation was compared to the Monte-Carlo simulation and is presented in Fig. 1. It is found that uniform acceleration takes place for times \(v < k_R/2\) and the FP prediction clearly fail in this regime. This can be expected since from \((67)\) \(D(v) \sim v^2\), the assumption that the velocity is constant \((20)\) cannot hold, rendering the FP approximation inconsistent. Nevertheless, for \(v > k_R/2\) the FP predictions are satisfied. Note that, for any potential discussed in this section the scaling properties of Sec. [IV] hold.

Numerically, it is found that for large velocities \((v > k_R/2)\) the average square position is growing ballistically, \(\langle x^2 \rangle \sim t^2\), as expected from previous studies [5, 6, 8, 10], while for small velocities \((v < k_R/2)\), \(\langle x^2 \rangle \sim t^4\), as expected for uniform acceleration.
where $\hat{U}(k)$ are random Fourier coefficients, and $\omega(k)$ is some dispersion relation. Following the analysis of the previous subsection we will assume that $U$ is composed of plane-waves with independent phases and amplitudes, leading to

$$\langle \hat{U}(k) \rangle = 0 \tag{69}$$
$$\langle \hat{U}(k_1) \hat{U}^*(k_2) \rangle = I_0 \ f(k_1) \delta(k_1 - k_2)$$
$$\langle \hat{U}(k_1) \hat{U}(k_2) \rangle = 0,$$

where $\langle |\hat{U}(k)|^2 \rangle = I_0$ and $f(k)$ is the probability density of the wave numbers $k$. The potential is proportional to the intensity of the $U$,

$$V(x,t) = |U(x,t)|^2 = \int d{k_1} d{k_2} \hat{U}(k_1) \hat{U}^*(k_2) \exp i ((k_1 - k_2) \cdot x - (\omega(k_1) - \omega(k_2)) t), \tag{70}$$

Using the assumptions (69) we get that the potential is constant on average,

$$\langle V(x,t) \rangle = \int d{k_1} d{k_2} \left\langle \hat{U}(k_1) \hat{U}^*(k_2) \right\rangle \exp i ((k_1 - k_2) \cdot x - (\omega(k_1) - \omega(k_2)) t)$$
$$= I_0 \int d{k} f(k) = I_0. \tag{71}$$

The correlation function of the potential is,

$$\langle V(x_1,t_1) V(x_2,t_2) \rangle = \int d{k_1} d{k_2} d{k_3} d{k_4} \left\langle \hat{U}(k_1) \hat{U}^*(k_2) \hat{U}(k_3) \hat{U}^*(k_4) \right\rangle$$
$$\times \exp i ((k_1 - k_2) \cdot x_1 - (\omega(k_1) - \omega(k_2)) t_1)$$
$$\times \exp i ((k_3 - k_4) \cdot x_2 - (\omega(k_3) - \omega(k_4)) t_2). \tag{72}$$

Since we have assumed that the complex amplitudes of the plane-waves are independent random variables we can decompose,

$$\left\langle \hat{U}(k_1) \hat{U}^*(k_2) \hat{U}(k_3) \hat{U}^*(k_4) \right\rangle = \left\langle \hat{U}(k_1) \hat{U}^*(k_2) \right\rangle \left\langle \hat{U}(k_3) \hat{U}^*(k_4) \right\rangle$$
$$+ \left\langle \hat{U}(k_1) \hat{U}^*(k_1) \right\rangle \left\langle \hat{U}^*(k_2) \hat{U}(k_3) \right\rangle$$
$$= I_0^2 f(k_1) f(k_3) \delta(k_1 - k_2) \delta(k_3 - k_4)$$
$$+ I_0^2 f(k_1) f(k_2) \delta(k_1 - k_4) \delta(k_2 - k_3). \tag{73}$$

Therefore, the correlation function reduces to,

$$C(x_1 - x_2, t_1 - t_2) = I_0^2$$
$$+ I_0^2 \int d{k_1} d{k_2} f(k_1) f(k_2)$$
$$\times \exp i ((k_1 - k_2) \cdot (x_1 - x_2) - (\omega(k_1) - \omega(k_2))(t_1 - t_2)). \tag{74}$$

The PSD is just,

$$S(q,\omega) = \frac{1}{(2\pi)^{d+1}} \int dx \int dt C(x,t) \exp -i (q \cdot x - \omega t)$$
$$= I_0^2 \delta(q) \delta(\omega)$$
$$+ \frac{I_0^2}{(2\pi)^{d+1}} \int dx \int dt \int d{k_1} d{k_2} f(k_1) f(k_2)$$
$$\times \exp i ((k_1 - k_2 - q) \cdot x - [\omega(k_1) - \omega(k_2) - \omega] t). \tag{75}$$

Taking the integral with respect to $x$ and $t$ we get,

$$S(q,\omega) = I_0^2 \delta(q) \delta(\omega) + I_0^2 \int d{k_1} d{k_2} f(k_1) f(k_2) \delta(k_1 - k_2 - q) \delta [\omega - (\omega(k_1) - \omega(k_2))] \tag{76},$$

$$S(q,\omega) = I_0^2 \delta(q) \delta(\omega) + I_0^2 \int d{k_1} d{k_2} f(k_1) f(k_2) \delta(k_1 - k_2 - q) \delta [\omega - (\omega(k_1) - \omega(k_2))], \tag{77}$$
or
\[
S(q, \omega) = I_0^2 \delta(q) \delta(\omega) + I_0^2 \int dk f(k) f(k - q) \delta[\omega - (\omega(k) - \omega(|k - q|))].
\]  
(78)

The diffusion coefficient is given by (see, 89)
\[
D(v) = \pi I_0^2 \int dqdk (q \cdot \hat{v})^2 f(k) f(k - q) \delta(q \cdot \hat{v} - (\omega(k) - \omega(|k - q|))).
\]  
(79)

Changing the variables to,
\[
q' = k - q, \\
k' = k,
\]  
(80)

and suppressing the primes for simplicity gives,
\[
D(v) = \pi I_0^2 \int dqdk ((k - q) \cdot \hat{v})^2 f(k) f(q) \delta((k - q) \cdot \hat{v} - (\omega(k) - \omega(q))).
\]  
(81)

Choosing the \(x\)-components of \(k\) and \(q\) such that they are aligned with the velocity \(v\) yields,
\[
D(v) = \pi I_0^2 \left(\frac{S_d}{2\pi}\right)^2 \int_0^\infty dq \int_0^{2\pi} d\theta_q \int_0^\infty dk \int_{-\pi}^{\pi} d\theta_k (qk)^{d-1}
\times (k \cos \theta_k - q \cos \theta_q)^2 f(k) f(q) \delta(v (k \cos \theta_k - q \cos \theta_q) - (\omega(k) - \omega(q))).
\]  
(82)

Changing variables to, \(y_k = v \cos \theta_k\) and \(y_q = v \cos \theta_q\) in (82) we have the Jacobian,
\[
J = \frac{1}{v^2 \sqrt{(1 - (y_k/v)^2)(1 - (y_q/v)^2)}}.
\]  
(83)

In these variables the diffusion coefficient is,
\[
D(v) = 4\pi I_0^2 \left(\frac{S_d}{2\pi}\right)^2 \frac{1}{v^4} \int_0^\infty dq \int_0^\infty dk \int_{-\pi}^{\pi} dy_k \int_{-\pi}^{\pi} dy_q (qk)^{d-1}
\times \frac{(ky_k - qy_q)^2}{\sqrt{(1 - (y_k/v)^2)(1 - (y_q/v)^2)}} f(k) f(q) \delta((ky_k - qy_q) - (\omega(k) - \omega(q))).
\]  
(84)

For large velocities this gives,
\[
D(v) \sim \frac{D_3}{v^2},
\]  
(85)

with
\[
D_3 = \frac{4}{\pi} I_0^2 S_d^2 \int_0^\infty dq q^{d-1} \int_0^\infty dk k^{d-2} (\omega(k) - \omega(q))^2 f(k) f(q).
\]  
(86)

Since \(f(k)\) is non-negative the integral for \(D_3\) does not vanish and for an appropriate choice of \(f(k)\) it is convergent. The diffusion coefficient for zero velocity can be calculated from (82) by setting \(v = 0\),
\[
D(0) = \pi I_0^2 \left(\frac{S_d}{2\pi}\right)^2 \int_0^\infty dq \int_0^{2\pi} d\theta_q \int_0^\infty dk \int_0^{2\pi} d\theta_k (qk)^{d-1}
\times (k \cos \theta_k - q \cos \theta_q)^2 f(k) f(q) \delta(\omega(k) - \omega(q)).
\]  
(87)

Assuming that \(\omega(k)\) is a monotonic function we have,
\[
\delta(\omega(k) - \omega(q)) = \frac{1}{|\omega'(q)|} \delta(k - q),
\]  
(88)
Figure 2: Same as Fig. 1 but with the potential (70) and a uniform distribution of wave-numbers given by (64). The dashed black lines are guides for the eye with the corresponding slopes of 2 and 2/5. The parameters used for this simulation are, $I_0 = 10^{-4}$, $k_R = 0.1$.

and therefore,

$$D(0) = \pi I_0^2 \left( \frac{S_d}{2\pi} \right)^2 \int_0^{2\pi} d\theta_q \int_0^{2\pi} d\theta_k \left( \cos \theta_k - \cos \theta_q \right)^2 \times \int_0^{\infty} dq q^{2d} f^2(q) \left| \frac{\omega'}{\omega'(q)} \right|.$$  

(89)

After integrating over the angles,

$$D(0) = \pi I_0^2 S_d^2 \int_0^{\infty} dq q^{2d} \frac{f^2(q)}{\left| \omega'(q) \right|},$$  

(90)

that is non-vanishing. Therefore, the potentials described in this section are generic, namely, produce the universal behavior both for small and large velocities. We will now demonstrate the calculation for a particular choice of dispersion relation, $\omega(k) = k^2/2$ and a uniform distribution function $f(k)$ on a disk given by (64) the integrals in (82) cannot be taken explicitly, however we have calculated them numerically (see Fig. 5) and the resulting dynamics is demonstrated in Fig. 2. Also here we found numerically that $\langle x^2 \rangle \sim t^2$ for large $t$ and $\langle x^2 \rangle \sim t^4$ for small $t$.

For same dispersion relation and a Gaussian distribution of wave numbers,

$$f(k) = \frac{1}{(2\pi k_R^2)^{d/2}} e^{-\frac{k^2}{2k_R^2}},$$  

(91)

using (81) we have,

$$D(v) = \frac{\pi I_0^2}{(2\pi k_R^2)^d} \frac{1}{2\pi} \int_{-\infty}^{\infty} d\tau \int dq dk \left( (k - q) \cdot \dot{v} \right)^2 e^{-\frac{q^2}{2k_R^2}} \exp i\tau \left[ (k - q) \cdot v - \frac{1}{2} (k^2 - q^2) \right],$$  

(92)

where we have used the exponential representation of the delta function. Noticing that,

$$D(v) = \frac{\pi I_0^2}{(2\pi k_R^2)^d} \frac{1}{2\pi} \left[ \int_{-\infty}^{\infty} \frac{d\tau}{\pi^2} \int dq dk e^{-\frac{q^2}{2k_R^2}} \frac{\partial^2}{\partial v^2} \exp i\tau \left[ (k - q) \cdot v - \frac{1}{2} (k^2 - q^2) \right] \right]$$  

(93)
and taking the Gaussian integrals gives,

\[
D(v) = \frac{\pi I_0^2}{(2\pi k^2 R)^d} \frac{1}{2\pi} \left[ \int_{-\infty}^{\infty} \frac{d\tau}{\tau^2} \frac{(2\pi k^2 R)^d}{(1 + k^4 R^2)^{d/2}} \partial^2 \frac{e^{-\left(\frac{k^2 R^2}{1 + k^4 R^2}\right)\tau^2}}{\tau^2} - \left(\frac{x^2}{1 + x^2}\right)^2 \right],
\]

(94)

or

\[
D(v) = \pi I_0^2 \left[ \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{d\tau}{\tau^2} \frac{1}{(1 + k^4 R^2)^{d/2}} \partial^2 \frac{e^{-\left(\frac{k^2 R^2}{1 + k^4 R^2}\right)\tau^2}}{\tau^2} \right].
\]

(95)

The dynamics following from (95) is rather similar to that demonstrated in Fig. 2 and therefore it is not presented here.

VI. ANOMALOUS DIFFUSION IN ONE DIMENSIONAL SYSTEMS

In this section we will consider one dimensional systems and explore the different possibilities of asymptotic behavior of the velocity distribution. In particular, we will show that the asymptotic behavior of the form (42), is not universal since some possibilities were not considered in previous studies [5, 8, 10]. In the first subsection we will present some representative examples of the new universality classes, and in the second subsection we will explain when the asymptotic expansion of [5, 8, 10] is not effective, and provide a prescription for designing transport properties using the PSD.

A. New universality classes

Following from (99) the diffusion coefficient for a one dimensional system takes the form,

\[
D(v) = \pi \int k^2 S(k, kv) dk.
\]

(96)

For a PSD with the property,

\[
S(k, \omega) = 0 \quad \omega/k > v_{\max},
\]

(97)

implies that the diffusion coefficient is zero for large velocities,

\[
D(v) = 0 \quad v > v_{\max},
\]

(98)

as demonstrated in [10, 18]. The correlation function corresponding to the PSD with the property (97) may be infinitely differentiable, nevertheless the diffusion coefficient vanishes for large velocities as is clear from (98), a possibility which was overlooked in [5, 8, 10].

We will now present specific examples of this new universality class using the specific potentials of the form (1) and (2). The PSD of potentials of the type (1) is,

\[
S(k, \omega) = V_0^2 f(k) \left[ \delta(\omega - \omega(k)) + \delta(\omega + \omega(k)) \right].
\]

(99)

Taking the dispersion relation \( \omega = k^2/2 \), and using (96) and (99) yields,

\[
D(v) = \pi V_0^2 \int k^2 f(k) \left[ \delta(kv - k^2/2) + \delta(kv + k^2/2) \right] dk
\]

\[
= \pi V_0^2 \int k^2 f(k) \left[ \frac{\delta(k)}{|v|} + \frac{\delta(k - 2v)}{|v|} + \frac{\delta(k + 2v)}{|v|} \right] dk,
\]

(100)

which after integrating over \( k \) gives,

\[
D(v) = 8\pi V_0^2 |v| f(2v).
\]

(101)
Any asymptotic behavior is possible as the diffusion coefficient is proportional to \( f(v) \) and therefore clearly may decay faster than \( v^{-3} \). The correlation function for these potentials can be obtained for example for a Gaussian distribution of wave-numbers (91). Using (16) and (99) we have,

\[
C(x, t) = 2V_0^2 \cos \left[ \frac{k_0^2 \xi^2}{2(1+k_R^2t^2)} - \frac{1}{2} \text{arg} \left( 1 + ik_R^2t \right) \right] \exp \left[ -\frac{k_R^2 x^2}{2(1+k_R^2t^2)^{1/4}} \right].
\] (102)

Note, that this correlation function is an infinitely differentiable function over the whole \((x, t)\) plane. It has fast decaying spatial correlations compared to a very slow temporal correlations.

For potentials of the type (2) the PSD is given by,

\[
S(k, \omega) = I_0^2 \int dq \, f(q) \, f(q-k) \, \delta[\omega - (\omega(q) - \omega(q-k))].
\] (103)

As in the preceding example, taking a quadratic dispersion relation, \( \omega = k^2/2 \), gives

\[
D(v) = \pi I_0^2 \int k^2 f(q) f(q-k) \delta[kv - \frac{1}{2} \left(q^2 - (q-k)^2\right)] \, dq \, dk
= \pi I_0^2 \int k^2 f(q) f(q-k) \left[ \frac{\delta(k)}{|v-q|} + \frac{\delta(k-2(q-v))}{|v-q|} \right] \, dq \, dk
= 4\pi I_0^2 \int |q-v| f(q) f(2v-q) \, dq.
\] (104)

Changing the variables to \( q = q' + v \) and suppressing the prime we have,

\[
D(v) = 4\pi I_0^2 \int |q| f(q+v) f(q-v) \, dq.
\] (105)

And the decay of the diffusion coefficient with the velocity is also dictated by the decay of \( f(q) \). An explicit expression may be obtained for example for a uniform distribution of wave-numbers (64), giving (14) - (18),

\[
D(v) = \begin{cases} \frac{\pi I_0^2}{k_R} (k_R - |v|)^2 & |v| < k_R \\ 0 & |v| > k_R \end{cases}.
\] (106)

The resulting dynamics is demonstrated in Fig. 3. The regime of a unit slope corresponds to regular diffusion in velocity and the asymptotic regime corresponds to an absence of diffusion in velocity. We have verified numerically that the asymptotic regime corresponds to a ballistic growth in position, \( \langle x^2 \rangle \sim t^2 \).

Another explicit expression may be obtained for a Gaussian distribution of wave-numbers (91). The diffusion coefficient is given by,

\[
D(v) = 2I_0^2 e^{-\frac{v^2}{4k_R^2}}.
\] (107)

The predictions of this equation are compared with Monte-Carlo simulation and the results are presented in Fig. 4.

The correlation function for this potentials can be obtained using (16),

\[
C(x, t) = I_0^2 \frac{1}{\sqrt{1 + k_R^2t^2}} \exp \left( -\frac{k_R^2 x^2}{1+k_R^2t^2} \right).
\] (108)

Similarly, to the correlation function (102) this correlation function is infinitely differentiable, decaying fast in position and slowly decaying in time.

This existence of a dispersion relation between the spatial and temporal frequencies is not a necessary condition for the new universality class, as is demonstrated by the following PSD,

\[
S(k, \omega) = \frac{eV_0^2}{4\pi k_0 \omega_0} e^{-\left(\frac{k_0}{\omega_0}\right)^2} e^{-\left(\frac{\omega^2}{2\pi^2} + \frac{\omega^2}{2\pi^2}\right)},
\] (109)

where the normalization factor was chosen such that

\[
\int S(k, \omega) \, dk \, d\omega = V_0^2.
\] (110)
Using (96) and defining $v_0 = \omega_0/k_0$ yields,

$$D(v) = \frac{eV_0^2 k_0}{\sqrt{\pi}} \frac{k_0}{v_0} \left[ \frac{1}{\left(1 + (v/v_0)^2\right)^{3/2}} + \frac{1}{\left(1 + (v/v_0)^2\right)^3} \right] e^{-\sqrt{1+(v/v_0)^2}},$$  

(111)

which is exponentially decaying for a large velocity. We could not obtain an analytical expression for the correlation.
function, but it can be expressed as an integral,
\[
C(x, t) = \frac{eV_0^2}{4\pi k_0^2\omega_0} \int e^{-\left(\frac{k_x^2}{2k_0^2} + \frac{k^2}{2}\right)} \exp i (kx - \omega t) \, dk \, d\omega,
\]
\[
= \frac{eV_0^2}{2k_0\sqrt{2\pi}} e^{-\frac{x^2}{2k_0^2}} \int \exp \left(-\frac{k^2}{2k_0^2} - \frac{k^2}{k_0^2}\right) e^{ikx} \, dk.
\]
(112)

The temporal correlations are rapidly decaying and since the function in the integrand is infinitely differentiable and integrable following from the Riemann–Lebesgue lemma the spatial correlations will also decay faster than any power law.

In summary, in this subsection we have presented a few examples of potentials, which give diffusion coefficients with various asymptotic behavior for large velocities different from (12). We have also calculated their corresponding correlation functions and showed that they are smooth and differentiable. Therefore, this class of potentials spans a new universality class, which is qualitatively different from the universal behavior described in previous studies [5, 8, 10]. We have also demonstrated that the conditions for appearance of this class are not necessarily connected to long range correlations or the special form of the potentials (1) and (2). In particular, the existence of a dispersion relation is not required. In the next subsection we will show that the conclusions drawn here are valid for any physical dispersion relation, and present a classification of the potentials, based on their PSDs, into different universality classes.

B. Classification of potentials into different universality classes

In the present work we have chosen to work with the PSD and not the correlation function, since in relevant experiments [11, 17, 19] the PSD rather than the correlation function is naturally controlled. For the purpose of clarity we will repeat here the asymptotic expansion of the diffusion coefficient, which was done in [8, 10] and connect it to our representation using the PSD.

The PSD is the Fourier transform of the correlation function of the potential, namely,
\[
S(k, \omega) = \frac{1}{(2\pi)^2} \int C(x, t) \exp -i (kx - \omega t) \, dx \, dt.
\]
(113)

Therefore using (96),
\[
D(v) = \frac{1}{4\pi} \int k^2 C(x, t) \exp -ik (x - vt) \, dx \, dt \, dk,
\]
(114)

which could be written as,
\[
D(v) = -\frac{1}{4\pi} \int C(x, t) \frac{\partial^2}{\partial x^2} \exp -ik (x - vt) \, dx \, dt \, dk.
\]
(115)

Integrating by parts twice and taking into the account that \(C(x, t)\) and its derivatives vanish in the limit of \(x \to \pm \infty\), we get,
\[
D(v) = -\frac{1}{4\pi} \int \frac{\partial^2 C}{\partial x^2} (x, t) \exp -ik (x - vt) \, dx \, dt \, dk.
\]
(116)

Finally integrating over \(k\) gives,
\[
D(v) = -\frac{1}{2} \int \frac{\partial^2 C}{\partial x^2} (x, t) \delta (x - vt) \, dx \, dt.
\]
(117)

This is the same expression for the diffusion coefficient, which was obtained in previous studies (see Eq. 7 of [8]). For simplicity of notation we will set,
\[
W(x, \tau) = \frac{\partial^2 C}{\partial x^2} (x, \tau).
\]
(118)
Changing variables to \( t = t'/v \) we get,

\[
D(v) = -\frac{1}{2v} \int W \left( x, \frac{t'}{v} \right) \delta (x - t') \, dx \, dt' \\
= -\frac{1}{2v} \int W \left( t' \frac{t'}{v} \right) \, dt'.
\] (119)

Therefore, if \( W \) is sufficiently differentiable, the asymptotic expansion of \( D(v) \) is,

\[
D(v) = \frac{D_1}{v} + \frac{D_2}{v^2} + \frac{D_3}{v^3} \cdots,
\]

where

\[
D_n = -\frac{1}{2n!} \int x^{n-1} \frac{\partial^{n-1} W}{\partial \tau^{n-1}} (x, \tau) \big|_{\tau=0} \, dx \quad n = 1, 2, \ldots.
\] (121)

The first term vanishes, since

\[
D_1 = -\frac{1}{2} \int W(x, 0) \, dx = -\frac{1}{2} \int \frac{\partial^2 C}{\partial x^2} (x, 0) \, dx = 0,
\]

and all the even terms vanish due to the symmetry, \( x \rightarrow -x \) (following from the fact that the potential is translationally invariant),

\[
D_{2n} = -\frac{1}{2(2n)!} \int x^{2n-1} \frac{\partial^{2n-1} W}{\partial \tau^{2n-1}} (x, \tau) \big|_{\tau=0} \, dx = 0.
\] (123)

Therefore, the leading term in the asymptotic expansion of the diffusion coefficient is,

\[
D(v) \sim \frac{D_3}{v^3},
\]

where

\[
D_3 = -\frac{1}{12} \int x^2 \frac{\partial^2 W}{\partial \tau^2} (x, \tau) \big|_{\tau=0} \, dx.
\] (125)

In [5, 8, 10] it was claimed that if the correlation function is at least twice differentiable in both arguments then the asymptotic behavior (124) follows, and furthermore it is universal. However, as was demonstrated in the previous subsection exceptions are possible, since some or even all of the terms \( D_{n>3} \) may vanish, giving a qualitatively different asymptotic behavior. If the diffusion coefficient vanishes or decreases faster than any power law for large velocities the asymptotic expansion in powers of \( v^{-1} \) clearly vanishes. Thus the transport in such case is sub-diffusive for large velocity and the growth of the moments of the velocity distribution is slower than any power law. In what follows we will demonstrate that for the potentials of type (11) and (12) the asymptotic expansion of \( D(v) \) vanishes, for any choice of the dispersion relation, provided that the distribution of the wave-numbers decays faster than any power law.

Taking (99) and the relation (113) we can calculate,

\[
W(x, t) = -V_0^2 \int k^2 f(k) \left[ \delta (\omega - \omega (k)) + \delta (\omega + \omega (k)) \right] \exp i (kx - \omega t) \, dk \, d\omega
\]

\[
= -V_0^2 \int k^2 f(k) \left[ \exp i (kx - \omega (k) t) + \exp i (kx + \omega (k) t) \right] \, dk.
\] (126)

Using the expression for the coefficients of the asymptotic expansion (121) we have,

\[
D_n = \frac{V_0^2}{n!} \int k^2 f(k) (-i \omega (k))^{n-1} t^{n-1} e^{ikt} \, dk \, dt'
\]

\[
= \frac{V_0^2}{n!} \int k^2 f(k) (\omega^{n-1} (k)) \frac{\partial^{n-1}}{\partial k^{n-1}} e^{ikt} \, dk \, dt'.
\] (127)
Integrating by parts \( n - 1 \) times, and assuming that \( f(k) \) decreases faster than any power law, so that the boundary terms may be disregarded, we obtain,

\[
D_n = \frac{(-1)^{2n-1} V_0}{n!} \int e^{ikt'} \frac{\partial^{n-1}}{\partial k^{n-1}} \left( k^2 f(k) \omega^{n-1}(k) \right) dk dt' \\
= 2\pi \frac{(-1)^{2n-1} V_0}{n!} \left[ \frac{\partial^{n-1}}{\partial k^{n-1}} \left( k^2 f(k) \omega^{n-1}(k) \right) \right]_{k=0}.
\]

For a physical dispersion relation which has the property \( \omega(0) = 0 \) we will have \( D_n = 0 \) for all \( n \). As was shown in (101) the diffusion coefficient in this case is not zero, but a function which decays with velocity faster than any power law, rendering the expansion (120) useless. For a potential with a PSD given by (103) we have,

\[
W(x, t) = -I_0^2 \int k^2 f(q) f(k-q) \delta(\omega - (\omega(q) - \omega(q-k))) \exp(ikx - \omega t) dq dk d\omega \\
= -I_0^2 \int k^2 f(q) f(k-q) \exp(ikx - (\omega(q) - \omega(q-k))) dq dk.
\]

The coefficients of the asymptotic expansion are therefore given by,

\[
D_n = \frac{I_0^2}{2n!} \int (-i(\omega(q) - \omega(q-k)))^{n-1} k^2 f(q) f(k-q) t^{n-1} e^{ikt'} dt' dq dk \\
= \frac{I_0^2}{2n!} \int (-i(\omega(q) - \omega(q-k)))^{n-1} k^2 f(q) f(k-q) \left( -i \frac{\partial}{\partial k} \right)^{n-1} e^{ikt'} dt' dq dk.
\]

Integrating by parts and assuming \( f(k) \) decays faster than any power law gives,

\[
D_n = \frac{I_0^2}{2n!} \int e^{ikt'} \frac{\partial^{n-1}}{\partial k^{n-1}} \left( [\omega(q) - \omega(q-k)]^{n-1} k^2 f(q) f(k-q) \right) dt' dq dk \\
= 2\pi I_0^2 \frac{(-1)^{2n-1} \partial^{n-1}}{2n!} \left[ \int [\omega(q) - \omega(q-k)]^{n-1} k^2 f(q) f(k-q) dq \right]_{k=0}.
\]

These coefficients vanish for any dispersion relation, and the diffusion coefficient is different from zero, and was obtained explicitly for a specific choice of the dispersion relation in (105).

In experiments one usually does not control the diffusion coefficient directly, and therefore we will classify the potentials leading to slow transport using the PSD, which can be controlled. Starting from a relation for the diffusion coefficient (96) we change the variable to \( k = k'/v \) and obtain,

\[
D(v) = \frac{\pi}{v^3} \int k'^2 S\left( \frac{k'}{v}, k' \right) dk'.
\]

If the PSD is not singular (note, that this is not the case for the potentials of type (11) and (22) discussed earlier) and differentiable on the line \( k = 0 \) (where \( k \) is the first argument of \( S(k, \omega) \)) we can expand,

\[
D(v) = \frac{D_3}{v^3} + \frac{D_5}{v^5} + \cdots,
\]

where

\[
D_n = \frac{1}{(n-3)!} \int d\omega \omega^{n-1} \frac{\partial^{n-3} S}{\partial k^{n-3}} (k, \omega) \big|_{k=0} \quad n \geq 3.
\]

Note, that in this representation the leading order of the expansion is directly \( v^{-3} \), since representation using the PSD already contains the assumption that the potential is stationary. If the first \( n_{\text{max}} \) derivatives, \( \partial^n S/\partial k^n \), vanish on the line \( k = 0 \), where \( n_{\text{max}} \) is the first derivative that is different from zero, then the resulting asymptotic behavior is \( D(v) \sim D_{n_{\text{max}}+3}/v^{n_{\text{max}}+3} \). All the derivatives vanish only if \( S(k, \omega) \) is non-analytic on the line \( k = 0 \). The non-analytic behavior may result either from a function, which is strictly zero on some finite strip around the line \( k = 0 \), as in (97) or due to an essential singularity of \( S(k, \omega) \) on this line, which will be demonstrated below. The first case will lead to a diffusion coefficient that will vanish for large velocities, for example (101) (with finite support \( f(k) \)) or (109). The second case will result in a decay faster than any power law, for example (107). In experiments
with a good control of the PSD, by controlling the amount of vanishing derivatives of the PSD on the line \( k = 0, \) one can vary the asymptotic behavior of the velocity distribution. The range of variation is from \( D(v) \sim v^{-3} \), through \( D(v) \sim v^{-n} \) (with \( n > 3 \)), sub-exponential, exponential, super-exponential and up-to \( D(v) = 0 \) (for \( v > v_{\text{max}} \)).

We will now turn to examine the correlation function of the potential, \( C(x, t) \), for potentials, which belong to the new universality classes. For simplicity we will consider the behavior along the line,

\[
c(t) \equiv C(0, t) = \int \left( \int S(k, \omega) \, dk \right) e^{-i\omega t} \, d\omega
\]

and,

\[
c(x) \equiv C(x, 0) = \int \left( \int S(k, \omega) \, d\omega \right) e^{ikx} \, dk.
\]

The moments of the correlation function along these lines are,

\[
\langle t^n \rangle = \int t^n c(t) \, dt
\]

and

\[
\langle x^n \rangle = \int x^n c(x) \, dx.
\]

For the PSD (99) we have,

\[
\langle t^n \rangle = V_0^2 \int t^n f(k) \left[ \delta(\omega - \omega(k)) + \delta(\omega + \omega(k)) \right] e^{-i\omega t} \, dk \, d\omega \, dt
\]

\[
= V_0^2 \int f(k) \, t^n \left( e^{-i\omega(k)t} + e^{i\omega(k)t} \right) \, dk \, dt.
\]

We will now calculate the second moment, which will contain all the representative features. Using the fact that,

\[
\langle t^2 \rangle = 2V_0^2 \int f(k) \left( -\frac{\partial^2}{\partial \omega^2} \right) \cos \omega t \, dk \, d\omega \, dt,
\]

and

\[
\frac{\partial}{\partial \omega} = \frac{dk}{d\omega} \frac{\partial}{\partial k},
\]

we have,

\[
\langle t^2 \rangle = 2V_0^2 \int f(k) \left( \frac{dk}{d\omega} \frac{\partial}{\partial k} \frac{\partial}{\partial k} \frac{\partial}{\partial \omega} \right) \cos \omega t \, dk \, d\omega.
\]

Integrating by parts twice gives,

\[
\langle t^2 \rangle = 2V_0^2 \int \left( \frac{\partial}{\partial k} \frac{dk}{d\omega} \frac{\partial}{\partial k} \frac{\partial}{\partial \omega} \right) f(k) \, \cos \omega t \, dk \, d\omega
\]

\[
= 4\pi V_0^2 \left[ \left( \frac{\partial}{\partial k} \frac{dk}{d\omega} \frac{\partial}{\partial k} \frac{dk}{d\omega} \right) f(k) \right] \mid_{k=0},
\]

where we have used the assumption that \( \omega(k) = 0 \) if and only if \( k = 0 \). Note, that for any increasing (near \( k = 0 \)) dispersion relation, except \( \omega = |k|c \), this moment diverges. For the later dispersion relation the derivative, \( dk/d\omega \) does not exist at \( k = 0 \). Therefore, the temporal correlation function \( c(t) \) for these potentials is slowly decreasing. Nevertheless, for the spatial correlation function, we have,

\[
h(k) = \int S(k, \omega) \, d\omega
\]

\[
= 2V_0^2 f(k).
\]
Figure 5: Log-Log plot of the diffusion coefficients as a function of the velocity for a two dimensional system and different potentials. The dots stand for potentials of the form (1), while the solid lines stand for potentials of the form (2). The blue (light) line and dots are for Gaussian distribution of wave-numbers while the black line and dots are for uniform distribution of wave-numbers over a disc of unit radius. The dashed black lines are guides to the eye with the slopes, 2 and −3.

If one chooses the function $f(k)$ to be infinitely differentiable then following from the Riemann–Lebesgue lemma the spatial correlation function, $c(x)$ will decay faster than any power law. A similar calculation could be repeated for the potentials of the type (2), with the PSD (103), leading to a similar result. Therefore, a natural question to ask is whether a necessary condition for the new universality class is a slowly decaying correlation function. The answer to this question is negative as was exemplified after (109).

In this subsection we have demonstrated that potentials of the type (1) and (2), with any dispersion relation and a fast decaying distribution of wave-numbers will result in a transport slower than (124). Those potentials will have short correlations in position and long correlations in time. For potentials with smooth PSDs we have suggested a classification scheme based on their PSD properties.

VII. DISCUSSION

In the present work the diffusion coefficient for the velocity is presented in terms of the spectral content, namely, the average power spectral density (PSD), (see Eq. (39)) for potentials that are random both in space and time. This representation is very natural for potentials generated as a superposition of waves used in optics and atom optics. It is of particular use in the case where the potential is a stationary process both in time and space. The simplicity of the expression enabled to explore the properties of the diffusion coefficient, to establish a scaling form of the Fokker-Planck equation, and to discover new universality classes. In particular, we were able to calculate explicitly the diffusion coefficient for representative examples, relevant for applications both in optics and atom optics (Gaussian and uniform distributions of wave-vectors). We have shown that for dimensions larger than one only one universality class is possible in the framework of the FP approximation, $D(v) \sim v^{-3}$. However for one dimensional systems new possibilities for large velocity asymptotics were also found.

In the limit of small velocities the diffusion coefficient vanishes only when $S(k, 0) \sim \delta(k)$, which is not the typical behavior. However, for potentials described in Sec. VA $D(0) = 0$ and moreover for small velocity the diffusion coefficient grows faster than $v$, invalidating the Fokker-Planck approximation for those potentials, in this regime.

There is an important case were the waves composing the potential satisfy a dispersion relation given by $\omega(k) \sim k^2$. In this case we found that the diffusion coefficient depends on the characteristic scale of the potential, $k_R$. In Fig. 5 $D(v)$ is presented for similar values of the characteristic scale of the potential, $k_R$.

The main result of the paper is the classification of universality classes in one dimensional systems presented in Sec. VII In the past it was found that in the large velocity limit the diffusion coefficient depends on the velocity as, $D(v) \sim v^{-3}$ [5, 6, 8, 10]. In the present work we have shown that this is always the case for dimensions larger than...
one, and explained the mechanism of this behavior (see (11)). However for one dimensional systems this is only one of the possibilities. Generally it depends on the asymptotic expansion of \( D(v) \) in powers of \( v^{-1} \):

1. The first term of the asymptotic expansion of \( D(v) \) is non-zero, \( D(v) \sim v^{-3} \).

2. The first non-vanishing term in the asymptotic expansion of \( D(v) \) is the \( n \)-th term then, \( D(v) \sim v^{-n} \).

3. All terms in the asymptotic expansion of \( D(v) \) are zero, \( D(v) \leq v^{-\alpha} \), for any \( \alpha > 0 \). In particular, the diffusion coefficient may be zero for \( v > v_{\text{max}} \) (e.g. (108)) or non-zero but decreasing faster than any power law (e.g. (107)).

All these possibilities can be realized in experiments with control over the PSD or more precisely \( f(k) \), as is clear from (101) or (105). Unlike previous studies (2, 8, 10) the classification into universality classes does not rely on the differentiability of the correlation function of the potential. Additionally, as was demonstrated in (112), the new universality classes does not depend on the range of the correlation function. The general behavior can be understood heuristically in terms of the Chirikov resonances (20).

We have also shown that initially particles will experience a uniform acceleration and at some later stage, on a longer time-scale, the velocity will follow diffusion as predicted by the Fokker-Planck equation. Eventually, it will reach the asymptotic long-time behavior predicted by the Fokker-Planck approximation.

In the present research the spreading of the velocity distribution is studied in the framework of the Fokker-Planck equation, therefore an obvious question to study is what happens when the Fokker-Planck approximation fails. Another important issue, which was not addressed in this work, is how to obtain analytically the spreading in position. Here this spreading was studied numerically for the potentials (1) and (2). Since in the experiments that have motivated this work, the relevant dynamics is of waves, rather than particles, an obvious question to explore is the correspondence between the classical and wave dynamics.

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