Numerical computation of the data-rate limit for state estimation under communication constraints

Sigurdur Hafstein* and Christoph Kawan†

Abstract

In networked control, a fundamental problem is to determine the smallest capacity of a communication channel between a dynamical system and a controller above which a prescribed control objective can be achieved. Often, a preliminary task of the controller, before determining the control input, is to estimate the state with a sufficient accuracy. For time-invariant systems, it has been shown that the smallest channel capacity $C_0$ above which the state can be estimated with an arbitrarily small error, depending on the precise formulation of the estimation objective, is given by the topological entropy or a quantity named restoration entropy, respectively. In this paper, we propose an algorithm that computes rigorous upper bounds of $C_0$, based on estimates of Matveev and Pogromsky.

Keywords: State estimation; nonlinear systems; topological entropy; Lyapunov-type functions; numerical computation

AMS Classification: 37B40, 93C41

1 Introduction

Networked control systems are spatially distributed systems, in which the communication between sensors, controllers and actuators is accomplished through a shared digital communication network. Examples can be found, for instance, in vehicle tracking, underwater communications for remotely controlled surveillance and rescue submarines, remote surgery, space exploration and aircraft design. Another large field of applications can be found in modern industrial systems, where industrial production is combined with information and communication technology (‘Industry 4.0’). A fundamental problem in networked control is to determine the minimal requirements on the communication network for a specified control objective to be achieved.

In the simplest model case, a sensor measures the states of a dynamical system at discrete sampling times and transmits the encoded state measurements through a finite-capacity channel to a controller at a remote location. In this framework, various works characterize or estimate the smallest channel capacity above which a given control objective (usually, stabilization of some sort) can be achieved. An even more fundamental problem is to determine the smallest capacity above which the controller is able to compute an estimate of the state with a given precision. This problem has been studied under various assumptions on the system and the channel. Notably, Savkin [31] characterized the critical capacity by a quantity, which turns out to be infinite if the system is genuinely affected by noise, and otherwise reduces to the topological entropy of the system. A more recent contribution is Matveev and Pogromsky [22], which discusses three estimation objectives of increasing strength and provides constructive methods to obtain upper and lower bounds for the associated critical channel capacities. The contribution of the paper at

*Faculty of Physical Sciences, University of Iceland, Dunhagi 5, IS-107 Reykjavik, Iceland; e-mail: shafstein@hi.is
†Fakultät für Informatik und Mathematik, Universität Passau, Passau, Germany; e-mail: christoph.kawan@uni-passau.de
hand consists in a numerical scheme to compute the upper bounds proposed in \cite{21,22}. Further studies about state estimation under communication constraints include \cite{16,19}.

The systems studied in \cite{22} are of the form $x_{t+1} = \phi(x_t)$ with a differentiable map $\phi : \mathbb{R}^n \to \mathbb{R}^n$. The aim is to generate an accurate estimate $\hat{x}_t$ of the state $x_t$ at a remote location for initial states $x_0$ confined to a compact set $K \subset \mathbb{R}^n$. The only way to transmit information to the estimator is via a noiseless discrete channel. At each time instant $t$, a coder encodes $x_t$ by a symbol $e_t$ from a finite alphabet $\mathcal{M}$. This process can be described by maps $C_t$ so that

$$e_t = C_t(x_0, x_1, \ldots, x_t; \hat{x}_0, \delta), \quad C_t : \mathbb{R}^{t+1} \times \mathbb{R} \times \mathbb{R}_{>0} \to \mathcal{M},$$

where $\mathcal{M}$ denotes the coding alphabet and $\hat{x}_0$ is an initial estimate satisfying $\|x_0 - \hat{x}_0\| < \delta$ for some $\delta > 0$, depending on the aspired accuracy of the estimate. The estimation process similarly can be described by maps $E_t$ so that

$$\hat{x}_t = E_t(e_0, e_1, \ldots, e_t; \hat{x}_0, \delta), \quad E_t : \mathcal{M}^{t+1} \times \mathbb{R} \times \mathbb{R}_{>0} \to \mathbb{R}^n.$$

To allow some flexibility in the information transmission, the number of bits that can be transmitted in any time interval of length $r$ is not fixed, but confined between two numbers $b_- (r) \leq b_+ (r)$, satisfying

$$C := \lim_{r \to \infty} \frac{b_- (r)}{r} = \lim_{r \to \infty} \frac{b_+ (r)}{r},$$

where $C$ by definition is the capacity of the channel. A desirable estimation objective is to obtain an estimate of the form $\|x_t - \hat{x}_t\| \leq \varepsilon$ for all $t \geq 0$, whenever $x_0, \hat{x}_0 \in K$ and $\|x_0 - \hat{x}_0\| \leq \delta$, where $\delta = \delta(\varepsilon)$. Writing $C_0$ for the smallest capacity $C$ above which this can be achieved for any $\varepsilon > 0$, it was shown in \cite{22} that $C_0 \geq h_{\text{top}}(\phi; K)$ and $C_0 = h_{\text{top}}(\phi; K)$ if $K$ is forward-invariant.

To avoid a drastic degradation of accuracy (noting that $\delta$ can be much smaller than $\varepsilon$) and at the same time obtain a coding and estimation scheme that is more robust with respect to perturbations, one may require instead that $\|x_t - \hat{x}_t\| \leq G\delta$ for all $t \geq 0$ with a constant $G > 0$ or even $\|x_t - \hat{x}_t\| \leq G\delta g^t$ with some $g \in (0,1)$. A common upper bound for the smallest channel capacity $C_0$ above which any of these objectives can be achieved can be formulated in terms of the (generalized) singular values of the linearized system. This upper bound, which at the same time is an upper bound on $h_{\text{top}}(\phi; K)$, has been derived earlier by the authors of \cite{22} in their papers \cite{21,28}, both for discrete-time and for continuous-time systems and also for time-varying systems.

In this paper, we consider a flow $(\phi_t)_{t \in \mathbb{R}}$ generated by an ODE $\dot{x} = f(x)$ with a sufficiently smooth vector field $f$ on $\mathbb{R}^n$. Our analysis focuses on the dynamics of $(\phi_t)$ on a compact forward-invariant set $K$. Essentially following an approach used before for the computation of Lyapunov functions \cite{2,13,14,20}, we numerically compute a piecewise affine Riemannian metric on the simplices of a triangulation of $K$, which is then used to produce an upper estimate on $C_0$ in terms of the eigenvalues of the symmetrized derivative of the vector field $f$, computed with respect to that metric. Our algorithm works in two steps. The first one produces a piecewise affine function $P$ on the given triangulation with values in the positive definite $(n \times n)$ symmetric matrices, designed in such a way to minimize the maximum of the largest generalized eigenvalue. The second step produces a Lyapunov-like function, which is used to scale the metric $P$ in order to make the largest generalized eigenvalue even smaller. It needs to be mentioned that the original estimate of Matveev and Pogromsky \cite{22,28} involves not the largest generalized eigenvalue only, but the sum of the $k$ largest generalized eigenvalues, where $1 \leq k \leq n$ is chosen to maximize this sum. Hence, we can only expect good results in low dimensions, where typically $k = 1$.

We apply our algorithm to the well-known Lorenz system with standard parameters on the region which contains the attractor. Using a simplified algorithm which works with a constant function $P$, we are already able to improve former estimates for the topological entropy obtained in \cite{28} by analytical methods.

The paper is organized as follows. In Section 2, we recall the main result of \cite{22,28}, yielding upper estimates on the topological entropy and the critical channel capacity, and we sketch our
algorithm. A detailed description of the algorithm is then presented in Section 3. In Section 4 we apply our algorithm to the Lorenz system. In Section 5, we discuss the theoretical limitations of our method with respect to the approximation of data-rate limits and topological entropy. Finally, in Section 6, we conclude with some further remarks on the usability of our algorithm and its technical limitations.

**Notation:** We denote by $\mathbb{Z}$ the set of integers and write $\mathbb{Z}_+ = \{n \in \mathbb{Z} : n \geq 0\}$. We write $S_n$ for the space of $(n \times n)$ real symmetric matrices and $S_n^+ \subset S_n$ for the space of all positive definite elements of $S_n$. If $\phi(t, x)$ denotes the (local) flow of an ODE $\dot{x} = f(x)$ in $\mathbb{R}^n$ and $v : \mathbb{R}^n \to E$ is a $C^1$-function into a Euclidean space $E$, we write $\dot{v}(x)$ for the orbital derivative of $v$ at $x$, i.e.,

$$\dot{v}(x) = \frac{d}{dt} \bigg|_{t=0} v(\phi(t, x)) \in L(\mathbb{R}, E) \cong E.$$ 

By $I$ we denote the $(n \times n)$ identity matrix for any $n \in \mathbb{N}$. By writing $A \succeq B$ for $A, B \in S_n$, we mean that $A - B$ is positive semi-definite. Furthermore, we write $B_{\varepsilon}(x)$ for the open $\varepsilon$-ball around $x \in \mathbb{R}^n$ (w.r.t. a given norm). Finally, we use the notation $i = 1 : n$ as a short-cut for $i \in \{1, \ldots, n\}$.

## 2 Preliminaries

We first recall the main result of [28], providing an upper bound on topological entropy. In [28], the result is formulated for nonautonomous ODEs. However, we only use the following autonomous version of it.

Consider an ODE of the form

$$\dot{x} = f(x), \quad x \in \mathbb{R}^n \quad (1)$$

with a $C^1$-vector field $f : \mathbb{R}^n \to \mathbb{R}^n$. Since we only consider solutions that evolve within a compact set, we can assume without loss of generality that $f$ is complete. We write $\phi(t, x_0)$ for the unique solution satisfying the initial condition $x(0) = x_0$. For a fixed $t \in \mathbb{R}$, we also write $\phi_t : \mathbb{R}^n \to \mathbb{R}^n$ for the diffeomorphism $x \mapsto \phi(t, x)$. We further assume the existence of a compact forward-invariant set $K \subset \mathbb{R}^n$, i.e., $\phi_t(K) \subset K$ for all $t \geq 0$.

The topological entropy of $\phi$ on $K$, denoted by $h_{\text{top}}(\phi; K)$, can be defined as follows. For $\tau, \varepsilon > 0$, a subset $E \subset \mathbb{R}^n$ $(\tau, \varepsilon)$-spans $K$ if for every $x \in K$ there is $y \in E$ with

$$\max_{0 \leq t \leq \tau} \|\phi(t, x) - \phi(t, y)\| \leq \varepsilon.$$

Writing $r(\tau, \varepsilon, \phi, K)$ for the minimal cardinality of any $(\tau, \varepsilon)$-spanning set for $K$,

$$h_{\text{top}}(\phi; K) := \lim_{\varepsilon \downarrow 0} \lim_{\tau \to \infty} \frac{1}{\tau} \log_2 r(\tau, \varepsilon, \phi, K).$$

### 2.1 Theorem: Let $P : K \to S_n^+$ and $v_d : K \to \mathbb{R}$, $1 \leq d \leq n$, be $C^1$-functions and let $\lambda_1(x) \geq \ldots \geq \lambda_n(x)$ denote the solutions of the algebraic equation

$$\det \left[ Df(x)^T P(x) + P(x) Df(x) + \dot{P}(x) - \lambda P(x) \right] = 0. \quad (2)$$

Let $\Lambda_d \geq 0$, $1 \leq d \leq n$, be constants so that

$$\sum_{i=1}^{d} \lambda_i(x) + \dot{v}_d(x) \leq \Lambda_d \quad \text{for all } x \in K. \quad (3)$$

Then for $\Lambda := \max_{1 \leq d \leq n} \Lambda_d$, the topological entropy of $\phi$ on $K$ satisfies

$$h_{\text{top}}(\phi; K) \leq \frac{\Lambda}{2 \ln 2}.$$
The upper bound provided by this theorem also serves as an upper bound for the smallest channel capacity above which certain estimation objectives can be achieved; see Section 5 for more details. Some remarks about the formulation of the theorem are in order.

2.2 Remark: The functions \( P \) and \( v_d \) in \([23]\) depend on three variables, i.e., \( P = P(t, s, x_0) \) and \( v_d = v_d(t, s, x_0) \), where \( t \geq s \) are time variables. Such functions can be obtained from the above formulation by putting 

\[
\hat{P}(t, s, x_0) := P(\phi(t - s, x_0)), \quad \hat{v}_d(t, s, x_0) := v_d(\phi(t - s, x_0)),
\]

and it is easy to verify that the so-defined functions satisfy the requirements of \([23, \text{Thm. 3.2}]\).

2.3 Remark: The function \( P \) can be interpreted as a Riemannian metric on \( K \), defined by 

\[
\langle v, w \rangle_x := \langle P(x)v, w \rangle \quad \text{for all } x \in K.
\]

Indeed, let \( X(\cdot) \) denote the solution to the following initial value problem corresponding to the variational equation of \([\text{I}]\):

\[
\dot{Y}(t) = Df(\phi(t))Y(t), \quad Y(0) = I.
\]

Let \( \alpha_1(t) \geq \ldots \geq \alpha_n(t) \) denote the singular values of \( X(t) \) w.r.t. the metric \( \langle \cdot, \cdot \rangle(t) \), i.e., the eigenvalues of the self-adjoint operator \( \sqrt{X(t)^*X(t)} \), where \( X(t)^* \) is defined by \( \langle X(t)v, w \rangle_{\phi(t)} = \langle v, X(t)^*w \rangle_{x_0} \). Then, according to \([23, \text{Prop. 8.6}]\),

\[
\alpha_1(t)\alpha_2(t)\cdots\alpha_d(t) \leq \exp \left( \frac{1}{2} \int_0^t [\lambda_1(\phi(s, x_0)) + \cdots + \lambda_d(\phi(s, x_0))]ds \right), \quad 1 \leq d \leq n. \tag{4}
\]

We expect that the number \( d \), where the maximum \( \max_{1 \leq d \leq n} \Lambda_d \) is attained, is more or less fixed for a given system under any reasonable choice of the functions \( v_1, \ldots, v_n \) (it is something like the number of positive Lyapunov exponents). If this is the case, we can also incorporate the function \( v_d \) into the metric by putting 

\[
\langle v, w \rangle_x := \langle e^{v_d(x)/d}P(x)v, w \rangle \quad \text{for all } x \in K.
\]

Then \([2]\) is equivalent to 

\[
\det \left[ \mathcal{D}f(x)^T P(x) + P(x)\mathcal{D}f(x) + \hat{P}(x) - \left( \lambda - \frac{1}{d} \hat{v}_d(x) \right) P(x) \right] = 0,
\]

and thus, the sum \( \sum_{i=1}^d \lambda_i(x) \) with the solutions of \([2]\) becomes \( \sum_{i=1}^d \lambda_i(x) + \hat{v}_d(x) \).

The functions \( v_1, \ldots, v_n \) in Theorem 2.1 have some similarity with Lyapunov functions. Instead of \( \hat{v}_1 < 0 \) we have the inequalities \([3]\). In the rest of the paper, we call these functions Lyapunov-type functions.

Our algorithm aims at the computation of the functions \( P \) and \( v_i \) by solving two optimization problems. Starting with a triangulation \( T \) of \( K \) (or some larger set), the first optimization problem delivers a piecewise affine function \( P \) on \( K \), affine on each simplex of the triangulation \( T \), with values in \( \Sigma_n \). This is accomplished by solving a semidefinite feasibility problem with linear matrix inequality constraints at each vertex and extension to the whole domain by affine interpolation of the values obtained at the vertices. The decisive quantity in this problem is a positive parameter \( \mu \), so that the solution \( P \) (if it exists) satisfies \( \lambda_{\max}(x) \leq \mu \) for all \( x \in K \), where \( \lambda_{\max}(x) \) denotes the largest generalized eigenvalue of the pair \((A(x), P(x))\) with 

\[
A(x) := P(x)\mathcal{D}f(x) + \mathcal{D}f(x)^T P(x) + (w_{ij}' \cdot f(x))_{i,j=1:n},
\]
where $w_{ij}^\nu$ stands for the gradient of the $(i,j)$-th entry of $P$ on the simplex $S_{\nu}$ satisfying $x \in S_{\nu}$.

If the algorithm yields a feasible solution for one parameter $\mu_1$, it can be run again for a smaller parameter $\mu_2 < \mu_1$ to check if there is still a feasible solution. Repeating this procedure, the maximum of the largest generalized eigenvalues over $K$ can be minimized.

The second optimization problems takes as an input a feasible solution $P$ of the first problem and an upper bound $\tilde{m}$ on the number of positive generalized eigenvalues of the matrix pairs $(A(x), P(x))$. It delivers a piecewise affine real-valued function $\tilde{V}$, affine on each simplex of a triangulation $T^\ast$, which is a refinement of $T$, and another real-valued function $\mu$. This is done by solving a semidefinite problem with linear matrix inequality constraints at each vertex of $T^\ast$ and extending again by affine interpolation. The optimization minimizes $Q$ so that for all $x \in K$,

$$A(x) - \mu(x)P(x) \preceq 0 \quad \text{and} \quad \tilde{V}(x) + \tilde{m}\mu(x) \leq Q.$$ 

A detailed description of these two steps is given in the following section.

### 3 Description of the algorithm

In this section, we describe the algorithm for computing the upper bounds provided by Theorem 2.1, which is split into two optimization problems.

#### 3.1 The semidefinite optimization problem

Given vectors $x_0, x_1, \ldots, x_n \in \mathbb{R}^n$ that are affinely independent, i.e. the vectors $x_1 - x_0, x_2 - x_0, \ldots, x_n - x_0$ are linearly independent, the convex hull

$$\mathcal{S} = \text{co}(x_0, x_1, \ldots, x_n) := \left\{ \sum_{k=0}^{n} \lambda_k x_k : \lambda_k \in [0, 1] \text{ and } \sum_{k=0}^{n} \lambda_k = 1 \right\}$$

is called an $n$-simplex or simply a simplex. A set

$$\text{co}(x_{k_0}, x_{k_1}, \ldots, x_{k_j}) := \left\{ \sum_{i=0}^{j} \lambda_{k_i} x_{k_i} : \lambda_{k_i} \in [0, 1] \text{ and } \sum_{i=0}^{j} \lambda_{k_i} = 1 \right\}$$

with $0 \leq k_0 < k_1 < \ldots < k_j \leq n$ and $0 \leq j < n$ is called a $j$-face of the simplex $\mathcal{S}$.

**3.1 Definition: (Triangulation)** We call a finite set $\mathcal{T} = \{S_{\nu}\}_\nu$ of $n$-simplices $S_{\nu}$ a triangulation in $\mathbb{R}^n$ if two simplices $S_{\nu}, S_{\mu} \in \mathcal{T}$, $\mu \neq \nu$, intersect in a common face or not at all and the interior $D_{\nu}^\ast$ of $D_{\mathcal{T}} := \bigcup_{S_{\nu} \in \mathcal{T}} S_{\nu}$ is connected.

**3.1 Optimization Problem** Given is a system $\dot{x} = f(x)$, $f \in C^3(\mathbb{R}^n; \mathbb{R}^n)$, a triangulation $\mathcal{T}$ in $\mathbb{R}^n$, and a parameter $\mu \geq 0$. The optimization problem is a semidefinite feasibility problem with linear matrix inequality constraints.

**Constants:** The constants used in this problem are

1. $\epsilon_0 > 0$ – lower bound on the matrix $P(x)$
2. The diameter $h_{\nu}$ of each simplex $S_{\nu} \in \mathcal{T}$:

$$h_{\nu} := \text{diam}(S_{\nu}) = \max_{x, y \in S_{\nu}} \|x - y\|_2$$

3. Upper bounds $B_{\nu}$ on the second-order derivatives of the components $f_k$ of $f$ on each simplex $S_{\nu} \in \mathcal{T}$:

$$B_{\nu} \geq \max_{i,j,k=0:1} \left| \frac{\partial^2 f_k(x)}{\partial x_i \partial x_j} \right|$$  \hfill (5)
4. Upper bounds $B_{3,\nu}$ on the third-order derivatives of the components $f_k$ of $f$ on each simplex $S_\nu \in T$:

$$B_{3,\nu} \geq \max_{x \in S_\nu} \left| \frac{\partial^3 f_i(x)}{\partial x_i \partial x_j \partial x_k} \right|$$

**Variables:** The variables of the problem are

1. $P_{ij}(x_k) \in \mathbb{R}$ for all $1 \leq i \leq j \leq n$ and all vertices $x_k$ of all simplices $S_\nu = \text{co}(x_0, \ldots, x_n) \in T$. For $1 \leq i \leq j \leq n$ the variable $P_{ij}(x_k)$ is the $(i, j)$-th entry of the $(n \times n)$ matrix $P(x_k)$. The matrix $P(x_k)$ is assumed to be symmetric and therefore these components determine it.

2. $C_\nu \in \mathbb{R}_0^+$ for all simplices $S_\nu \in T$ – upper bound on $P$ in $S_\nu$.

3. $D_\nu \in \mathbb{R}_0^+$ for all simplices $S_\nu \in T$ – upper bound on the derivative of $P_{ij}$ in $S_\nu$.

**Objective:** The objective function of the optimization problem is not needed because it is a feasibility problem, but one can, e.g., minimize $\max_{S_\nu \in T} C_\nu$.

**Constraints:**

1. **Positive definiteness of $P$**
   For each simplex $S_\nu = \text{co}(x_0, \ldots, x_n) \in T$ and each vertex $x_k$ of $S_\nu$:
   $$P(x_k) \succeq \epsilon_0 I$$

2. **Upper bound on $P$**
   For each simplex $S_\nu = \text{co}(x_0, \ldots, x_n) \in T$ and each vertex $x_k$ of $S_\nu$:
   $$P(x_k) \preceq C_\nu I$$

3. **Bound on the derivative of $P$**
   For each simplex $S_\nu \in T$ and all $1 \leq i \leq j \leq n$:
   $$\|w_{ij}^\nu\|_1 \leq D_\nu$$
   Here $w_{ij}^\nu = \nabla P_{ij}(x_k)$ for all $x \in S_\nu$. See Remark 3.2 for details.

4. **Bounds on the largest generalized eigenvalue**
   For each simplex $S_\nu = \text{co}(x_0, \ldots, x_n) \in T$ and each vertex $x_k$ of $S_\nu$:
   $$0 \geq A(x_k) - \mu P(x_k) + h_\nu^2 E_\nu I$$
   Here
   $$A(x_k) := P(x_k)Df(x_k) + Df(x_k)^T P(x_k) + (w_{ij}^\nu \cdot f(x_k))_{i,j=1:n},$$
   where $Df(x_k)$ is the Jacobian matrix of $f$ at $x_k$, $(w_{ij}^\nu \cdot f(x_k))_{i,j=1:n}$ denotes the symmetric $(n \times n)$-matrix with entries $w_{ij}^\nu \cdot f(x_k)$ and $w_{ij}^\nu$ is defined as in (6) and is the same vector for all vertices in one simplex. Further,
   $$E_\nu := n^2[(1 + 4\sqrt{n})B_\nu D_\nu + 2nB_{3,\nu}C_\nu].$$


3.2 \textbf{Remark:} In Constraints 3 and 4 in Optimization Problem 3.1 the gradient $w_{ij}''$ of the affine function $P_{ij}|_{\mathcal{S}_\nu}$ on the simplex $\mathcal{S}_\nu = \text{co}(x_0, \ldots, x_n)$, i.e. $\nabla P_{ij}|_{\mathcal{S}_\nu} = w_{ij}''$, is given by the expression

$$w_{ij}'' := X_\nu^{-1}
\begin{pmatrix}
P_{ij}(x_1) - P_{ij}(x_0) \\
\vdots \\
P_{ij}(x_n) - P_{ij}(x_0)
\end{pmatrix}
\in \mathbb{R}^n,
$$

(6)

where $X_\nu = (x_1 - x_0, x_2 - x_0, \ldots, x_n - x_0)^T \in \mathbb{R}^{n \times n}$ is the so-called shape-matrix of the simplex $\mathcal{S}_\nu$. For a proof of this fact and, moreover, that the definition is independent of the choice of the vertex $x_0$, see [14, Rem. 2.9].

The Constraints 3 are indeed linear and can be implemented using the auxiliary variables $D^k_\nu$ and the constraints

$$-D^k_\nu \leq [w_{ij}''|_k \leq D^k_\nu \text{ for } k = 1 : n,$$

where $[w_{ij}''|_k$ is the k-th component of the vector $w_{ij}''$, and setting $D_\nu = \sum_{k=1}^{n} D^k_\nu$. Similarly, the constraints $\|\nabla \mu\|_\infty \leq D^\mu_\xi$ in Optimization Problem 3.2 can be implemented as

$$-D^\mu_\xi \leq [\nabla \mu|_k \leq D^\mu_\xi \text{ for } k = 1 : n.$$

3.3 \textbf{Remark:} The constraints above are easily transferred into the standard form $\sum_{i=1}^{m} F_i y_i - F_0 \geq 0$, $F_0, F_1, \ldots, F_m \in \mathbb{R}^{n \times c}$ constant matrices and $y_1, y_2, \ldots, y_m \in \mathbb{R}$ the variables, for semidefinite programming (SDP) with linear matrix inequality (LMI) constraints. See, e.g., [13, Rem. 4.10] for a similar transfer.

3.2 \textbf{Feasible solution to Optimization Problem 3.1}.

A feasible solution of the Optimization Problem 3.1 returns a matrix $P(x_k) = (P_{ij}(x_k))_{i,j=1:n}$ at each vertex $x_k$ of the triangulation $\mathcal{T}$ and values $C_\nu$ and $D_\nu$ for each simplex $\mathcal{S}_\nu \in \mathcal{T}$. From these we can easily obtain $A(x_k)$ and $E_\nu$ as in Constraints 4 at each vertex $x_k$ and for each simplex $\mathcal{S}_\nu$, respectively.

We define the CPA (continuous piecewise affine) metric $M$ by affine interpolation on each simplex.

3.4 \textbf{Definition: (CPA interpolation)} Let $\mathcal{T}$ be a triangulation in $\mathbb{R}^n$ with $\mathcal{D}_\mathcal{T} = \bigcup_{\mathcal{S}_\nu \in \mathcal{T}} \mathcal{S}_\nu$. Let $P_{ij}(x_k)$ be fixed by a feasible solution to the Optimization Problem 3.1. An $x \in \mathcal{S}_\nu = \text{co}(x_0, \ldots, x_n) \in \mathcal{T}$ can be written uniquely as $x = \sum_{k=0}^{n} \lambda_k x_k$ with $\lambda_k \in [0,1]$ and $\sum_{k=0}^{n} \lambda_k = 1$ and we define

$$P_{ij}(x) := \sum_{k=0}^{n} \lambda_k P_{ij}(x_k)$$

and

$$P(x) := \begin{pmatrix}
P_{11}(x) & P_{12}(x) & \cdots & P_{1n}(x) \\
P_{21}(x) & P_{22}(x) & \cdots & P_{2n}(x) \\
\vdots & \vdots & \ddots & \vdots \\
P_{n1}(x) & P_{n2}(x) & \cdots & P_{nn}(x)
\end{pmatrix}.$$

We refer to the functions $P_{ij}$ and $P$ as the CPA interpolations of the values $P_{ij}(x_k)$ and $P(x_k)$, respectively, where the $x_k$ are the vertices of the simplices in $\mathcal{T}$. Furthermore, we write CPA[\mathcal{T}] for the space of all piecewise affine functions on $\mathcal{D}_\mathcal{T}$ defined in this way by interpolation of the values on the vertices of $\mathcal{T}$.

The following lemma can be proved exactly as [13, Lem. 4.13].

3.5 \textbf{Lemma:} The matrix $P(x)$ from Definition 3.4 is symmetric and positive definite for all $x \in \mathcal{D}_\mathcal{T}$.
3.6 Definition: (Orbital derivative) Let $P(x)$ be as in Definition 3.4 and fix a point $x \in D^p_T$. As shown in the proof of [13, Lem. 4.7], there exists a $\mathcal{S}_\nu = \{x(0), \ldots, x_n\} \in T$ and a number $\theta^* > 0$ such that $x + \theta f(x) \in \mathcal{S}_\nu$ for all $\theta \in [0, \theta^*)$. Then $x = \sum_{k=0}^n \lambda_k x_k$ with $\lambda_k \in [0,1]$, $\sum_{k=0}^n \lambda_k = 1$, and we define the orbital derivative $\dot{P}_{ij}(x)$ of $P_{ij}$ at $x$ as

$$
\dot{P}_{ij}(x) := w^\nu_{ij} \cdot f(x).
$$

Our definition of the orbital derivative is natural, because with $t \mapsto \phi(t, x)$ as the solution to $\dot{x} = f(x)$ crossing $x$ at time $t = 0$ and for any locally Lipschitz function $g : \mathbb{R}^n \to \mathbb{R}$, we have (cf. [20, Thm. 1.17])

$$
\lim_{h \to 0^+} \frac{g(\phi(h, x)) - g(x)}{h} = \lim_{h \to 0^+} \frac{g(x + hf(x)) - g(x)}{h}
$$

and with $\mathcal{S}_\nu$ chosen for $x$ as in Definition 3.6 we have

$$
\lim_{h \to 0^+} \frac{P_{ij}(x + hf(x)) - P_{ij}(x)}{h} = w^\nu_{ij} \cdot f(x).
$$

Note that $P_{ij} |_{\mathcal{S}_\nu}$ is an affine function and its gradient $w^\nu_{ij}$ was defined in Constraints 3 of Optimization Problem 3.1 and is the same vector for all points $x \in \mathcal{S}_\nu$.

Before proceeding to prove the implications of Optimization Problem 3.1, let us first recall a few elementary results about matrix norms. For an $A \in \mathbb{R}^{n \times n}$ we define

$$
\|A\|_\infty := \max_{i=1:n} |a_{ij}| \quad \text{and} \quad \|A\|_p := \max_{\|x\|_p = 1} \|Ax\|_p \quad \text{for} \quad p = 1, 2, \infty.
$$

The following relations hold:

$$
\|A\|_\infty \leq \|A\|_2 \leq n\|A\|_\infty, \quad \|A\|_2 \leq \sqrt{n} \|A\|_1, \quad \text{and} \quad \|A\|_2 \leq \sqrt{n} \|A\|_\infty.
$$

For a symmetric and positive definite $A$, the largest singular value $\lambda_{\max}$ of $A$, which equals $\|A\|_2$ and is the largest of its eigenvalues, is the smallest number such that $A \leq \lambda_{\max} I$. Further,

$$
\|A\|_1 = \max_{j=1:n} \sum_{i=1}^n |a_{ij}| = \|A^T\|_\infty.
$$

We will now relate $\dot{P}(x)$ to $\dot{P}(x_k)$, as well as $P(x)Df(x)$ to $P(x_k)Df(x_k)$. For the proof we will need the following auxiliary result, see [2, Prop. 4.1 and Cor. 4.3]. The notation is as in Optimization Problem 3.1.

3.7 Lemma: Denoting the Hessian of $f$ by $H(x) := \left(\frac{\partial^2 f(x)}{\partial x_i \partial x_j}\right)_{ij}$, we have

$$
\left\| f(x) - \sum_{k=0}^n \lambda_k f(x_k) \right\|_\infty \leq \max_{x \in \mathcal{S}_\nu} \|H(x)\|_2 h^2_{\nu} \leq n B_{\nu} h^2_{\nu}.
$$

3.8 Lemma: Consider a feasible solution to Optimization Problem 3.1 and let $P$ be defined as in Definition 3.4. Fix a point $x \in D^p_T$ and a corresponding simplex $\mathcal{S}_\nu = \{x(0), x_1, \ldots, x_n\} \in T$ as in Definition 3.6. Set

$$
A(y) := P(y)Df(y) + Df(y)^T P(y) + (w^\nu_{ij} \cdot f(y))_{i,j=1:n}
$$

for all $y \in \mathcal{S}_\nu$. Then we have the following estimate:

$$
\left\| A(x) - \sum_{k=0}^n \lambda_k A(x_k) \right\|_2 \leq h^2_{\nu} E_{\nu}.
$$

(7)
Proof: We show this in several steps:

**Step 1: Entry-wise bounds on \( \tilde{P}(x) \)**

The estimate

\[
\left| w^\nu_{ij} \cdot f(x) - \sum_{k=0}^{n} \lambda_k w^\nu_{ij} \cdot f(x_k) \right| \leq nB_\nu D_i h^2_\nu, \tag{8}
\]

follows by Hölder’s inequality, Constraints 3, and Lemma 3.7.

\[
\left| w^\nu_{ij} \cdot \left( f(x) - \sum_{k=0}^{n} \lambda_k f(x_k) \right) \right| \leq \| w^\nu_{ij} \|_1 \left\| f(x) - \sum_{k=0}^{n} \lambda_k f(x_k) \right\|_\infty \leq D_\nu nB_\nu h^2_\nu.
\]

**Step 2: Entry-wise bounds on \( P(x)Df(x) \) and \( Df(x)^T P(x) \)**

We show that

\[
\left| [P(x)Df(x)]_{ij} - \sum_{k=0}^{n} \lambda_k [P(x)Df(x)]_{ij} \right| \leq nh^2_\nu (2\sqrt{n}B_\nu D_\nu + nB_{3,\nu} C_\nu). \tag{9}
\]

Consider two scalar-valued functions \( g, h \in C^2(\mathbb{S}_\nu). \) We apply Lemma 3.7 to \( gh \), yielding

\[
\left| g(x)h(x) - \sum_{k=0}^{n} \lambda_k g(x_k)h(x_k) \right| \leq \max_{y \in \mathbb{S}_\nu} \| H(y) \|_y h^2_\nu, \tag{10}
\]

where the matrix \( H(y) \) is defined by \( [H(y)]_{rs} := \frac{\partial^2 (gh)(y)}{\partial x_r \partial x_s}. \) Set \( g(y) := P_{il}(y). \) Since \( P_{il}(y) = w^\nu_{il} \cdot (y - x_0) + P_{il}(x_0) \), we get \( \frac{\partial g}{\partial x_r}(y) = [w^\nu_{il}]_s \) and \( \frac{\partial^2 g}{\partial x_r \partial x_s}(y) = 0 \) for all \( y \in \mathbb{S}_\nu. \) Hence,

\[
\frac{\partial g}{\partial x_s} \cdot h = \frac{\partial g}{\partial x_s} h + g \frac{\partial h}{\partial x_s} = [w^\nu_{il}]_s h + g \frac{\partial h}{\partial x_s}
\]

and then

\[
\frac{\partial^2 g}{\partial x_r \partial x_s} \cdot h = [w^\nu_{il}]_r \frac{\partial h}{\partial x_r} + [w^\nu_{il}]_s \frac{\partial h}{\partial x_s} + g \frac{\partial^2 h}{\partial x_r \partial x_s}.
\]

Now set \( h(y) := |Df(y)|_{ij}. \) Then \( \frac{\partial h}{\partial x_r} = \frac{\partial^2 f}{\partial x_r \partial x_s} \) and \( \frac{\partial^2 h}{\partial x_r \partial x_s} = \frac{\partial^3 f}{\partial x_r \partial x_s \partial x_j} \) and thus

\[
[H(y)]_{rs} = \left| \frac{\partial^2 (gh)(y)}{\partial x_r \partial x_s} \right| \leq \| [w^\nu_{il}]_s B_\nu + [w^\nu_{il}]_r B_\nu + |P_{il}(y)| B_{3,\nu}. \]

Using in succession for any \( H_1, H_2, H_3 \in \mathbb{R}^{n \times n} \) that

\[
\| H_1 + H_2 + H_3 \|_2 \leq \| H_1 \|_2 + \| H_2 \|_2 + \| H_3 \|_2
\]

and

\[
\| H_{1,2} + H_{3,2} \|_2 \leq \sqrt{n} \| H_{1,2} \|_2 + \sqrt{n} \| H_{3,2} \|_2
\]

this delivers

\[
\| H(y) \|_2 \leq \sqrt{n} \| w^\nu_{il} \|_2 B_\nu + \sqrt{n} \| w^\nu_{il} \|_2 B_\nu + nB_{3,\nu} \max_{x \in \mathbb{S}_\nu} \max_{1 \leq i \leq n} |P_{il}(x)| \leq 2\sqrt{n}B_\nu D_\nu + nB_{3,\nu} C_\nu, \tag{11}
\]

because we have \( |P_{il}(y)| \leq \| P(y) \|_2 \leq C_\nu \) by Constraints 2.
Since the generalized eigenvalue problem for two symmetric matrices is the largest generalized eigenvalue of the pair \((A, B)\), then
\[
\lambda_{\max} := \frac{\|A\|_2}{\|B\|_2} \text{ is the largest generalized eigenvalue of the pair } (A, B). \]

Step 3: Bounds on matrices

From the definition of \(A(y)\) we get
\[
\left\| A(x) - \sum_{k=0}^{n} \lambda_k A(x_k) \right\|_2 \leq \left\| P(x) D f(x) - \sum_{k=0}^{n} \lambda_k P(x_k) D f(x_k) \right\|_2 \]
\[
+ \left\| D f(x)^T P(x) - \sum_{k=0}^{n} \lambda_k D f(x_k)^T P(x_k) \right\|_2 \]
\[
+ \left\| (w_{i,j} \cdot f(x))_{i,j=1:n} - \sum_{k=0}^{n} \lambda_k (w_{i,j} \cdot f(x_k))_{i,j=1:n} \right\|_2.
\]

The first two norms on the right-hand side are equal because \(P\) is symmetric and therefore the matrices in the norms are conjugate. The entry-wise bounds \(11\) and \(12\) together with \(\|H\|_2 \leq n\|H\|_{\max}\) for any \(H \in \mathbb{R}^{n \times n}\) now deliver \(10\).

Using Lemma 3.8 we can now establish, that the parameter \(\mu\) in Optimization Problem 3.1 is an upper bound on the generalized eigenvalues of the matrix pair \((A(x), P(x))\) for all \(x \in D^n\). For completeness, we first give a short description of generalized eigenvalues as needed here.

The generalized eigenvalue problem for two symmetric matrices \(A, B \in \mathbb{R}^{n \times n}, B > 0\), is to find values \(\lambda_i \in \mathbb{R}\) and corresponding nonzero vectors \(x_i \in \mathbb{R}^n\) such that \(A x_i = \lambda_i B x_i\) for \(i = 1 : n\). Since \(B > 0\), we can define \(D^\frac{1}{2} := O^T D^{-\frac{1}{2}} O\), where \(B = O^T D O\) is the spectral decomposition of \(B\), i.e. \(O\) is orthogonal and \(D\) is a diagonal matrix with strictly positive entries on the diagonal. \(D^\frac{1}{2}\) is then canonically defined as the diagonal matrix with the square-roots of the entries of \(D\) on the diagonal. Further, \(B^{-\frac{1}{2}} := (B^\frac{1}{2})^{-1}\). The matrix \(C = B^{-\frac{1}{2}} A B^{-\frac{1}{2}}\) is then symmetric and if \(\lambda \in \mathbb{R}\) is an eigenvalue of \(C\) with corresponding eigenvector \(y \in \mathbb{R}^n\), then \(B^{-\frac{1}{2}} A B^{-\frac{1}{2}} y = \lambda y\). From this \(B^{-\frac{1}{2}} y = \lambda B^{-\frac{1}{2}} y\) or \(A x = \lambda B x\) with \(x = B^{-\frac{1}{2}} y\), i.e. \(\lambda\) is a generalized eigenvalue for the matrix pair \(A\) and \(B\) and \(x\) is a corresponding generalized eigenvector. With \(\{y_i\}\) as an orthonormal set of eigenvectors of \(C\), the generalized eigenvectors \(x_i = B^{-\frac{1}{2}} y_i\) are thus a basis of \(\mathbb{R}^n\) and \(x_i^T B x_j = y_i^T B^{-\frac{1}{2}} B^{-\frac{1}{2}} y_j = y_i^T y_j = \delta_{i,j}\). It follows from an easy calculation that if \(\lambda_{\max}\) is the largest generalized eigenvalue of the pair \((A, B)\), then
\[
0 \geq A - \mu B \quad \text{if and only if} \quad \mu \geq \lambda_{\max}.
\]

Also note that \(0 \geq A - \mu B + \alpha I\) for an \(\alpha \geq 0\) clearly implies \(0 \geq A - \mu B\). Since \(B > 0\), the smallest eigenvalue of \(B\) is given by \(\|B^{-1}\|_2^{-1}\) and then \(\|B^{-1}\|_2^{-1} I \preceq B\) and with \(\alpha \geq 0\) we get
\[
A - \mu B + \alpha I \preceq A - (\mu - \alpha \|B^{-1}\|_2) B \preceq 0 \quad \text{if} \quad \mu - \alpha \|B^{-1}\|_2 \geq \lambda_{\max}.
\]
3.9 Lemma: Assume that we have a feasible solution to Optimization Problem 3.1 with parameter $\mu \geq 0$ and let $P(x)$ be defined from the feasible solution as in Definition 3.4 and define for every $x \in D^\nu_T$ the matrix

$$A(x) := P(x)Df(x) + Df(x)^TP(x) + (w^\nu_i \cdot f(x))_{i,j=1:n}.$$ 

Denote for every $x \in D^\nu_T$ by $\lambda_{\text{max}}(x)$ the largest generalized eigenvalue of the matrix pair $(A(x), P(x))$. Then $\mu \geq \lambda_{\text{max}}(x)$ for every $x \in D^\nu_T$.

**Proof:** Let $x \in D^\nu_T$ be arbitrary and fix an $\mathcal{S}_\nu \in \mathcal{T}$ as in Definition 3.6 such that $x \in \mathcal{S}_\nu$. By Lemma 3.8 we have

$$\left\| A(x) - \sum_{k=0}^{n} \lambda_k A(x_k) \right\|_2 \leq h^2 \nu E_\nu,$$

from which

$$A(x) - \sum_{k=0}^{n} \lambda_k A(x_k) \leq \left\| A(x) - \sum_{k=0}^{n} \lambda_k A(x_k) \right\|_2 \leq h^2 \nu I$$

follows. But then

$$A(x) - \mu P(x) \leq \left\| A(x) - \sum_{k=0}^{n} \lambda_k A(x_k) \right\|_2 + \sum_{k=0}^{n} \lambda_k A(x_k) - \mu \sum_{k=0}^{n} \lambda_k P(x_k)$$

$$\leq \sum_{k=0}^{n} \lambda_k [A(x_k) - \mu P(x_k) + h^2 \nu I] \leq 0$$

by Constraints 4. The assertion of the lemma now follows by the discussion on generalized eigenvalues above. \(\square\)

After we have found a parameter $\mu \geq 0$ and a triangulation $\mathcal{T}$ such that Optimization Problem 3.1 has a feasible solution, we can use this solution as the input to another optimization problem to get bounds on $h_{\text{top}}(\phi; K)$ as in Theorem 2.4. We now use $P(x)$ computed by Optimization Problem 3.1 to compute two functions, $\mu, V : D^\tau_T \to \mathbb{R}$. The function $\mu$, satisfying $0 \leq \lambda_{\text{max}}(x) \leq \mu(x) \leq \mu$, is a local upper bound on the largest generalized eigenvalue $\lambda_{\text{max}}(x)$ of $(A(x), P(x))$ and $V$ is a Lyapunov-type function as used in Theorem 2.4.

3.2 Optimization Problem Given is a feasible solution to Optimization Problem 3.1 for a system $\dot{x} = f(x), f \in C^3(\mathbb{R}^n; \mathbb{R}^n)$, with a triangulation $\mathcal{T}$, a parameter $\mu \geq 0$, and an upper bound $m$ on the number of positive generalized eigenvalues of the matrix pairs $(A(x), P(x))$ from the feasible solution. Further, a refined triangulation $\mathcal{T}^*$ of $\mathcal{T}$ is given, i.e., $\mathcal{T}^*$ is a triangulation as in Definition 3.1 $D_{\mathcal{T}^*} = D_{\mathcal{T}}$, and each simplex $\mathcal{S}_\nu \in \mathcal{T}$ is the union of simplices $\mathcal{S}_\xi$ in $\mathcal{T}^*$.

The optimization problem is a semidefinite problem with linear matrix inequality constraints.

**Constants** The constants used in problem are:

1. The diameter $h_\xi$ of each simplex $\mathcal{S}_\xi \in \mathcal{T}^*$:

$$h_\xi := \text{diam}(\mathcal{S}_\xi) = \max_{x,y \in \mathcal{S}_\xi} \|x - y\|_2$$

2. $D_\nu$ and $E_\nu$ for each simplex $\mathcal{S}_\nu \in \mathcal{T}$ — delivered by the feasible solution to Optimization Problem 3.1

3. Upper bounds $B^*_\xi$ on the second-order derivatives of the components of $f$ on each simplex $\mathcal{S}_\xi \in \mathcal{T}^*$, just as in (5) but for the simplices in $\mathcal{T}^*$. For example, one can set $B^*_\xi := B_\nu$ for every $\mathcal{S}_\xi \in \mathcal{T}^*$ fulfilling $\mathcal{S}_\xi \subset \mathcal{S}_\nu \in \mathcal{T}$. 

11
We additionally use the functions $A, P : D^*_T \to \mathbb{R}^{n \times n}$ from the feasible solution to Optimization Problem 3.1.

**Variables** The variables of the semidefinite feasibility problem are:

1. $\mu(x_k) \in \mathbb{R}$ for all vertices $x_k$ of all simplices $\mathcal{S}_\nu = \text{co}(x_0, \ldots, x_n) \in T^*$ – upper bound on the largest generalized eigenvalue.
2. $D^\mu_\xi \in \mathbb{R}^+_0$ for all simplices $\mathcal{S}_\xi \in T^*$ – upper bound on the gradient of $\mu$.
3. $V(x_k) \in \mathbb{R}$ for all vertices $x_k$ of all simplices $\mathcal{S}_\nu = \text{co}(x_0, \ldots, x_n) \in T^*$ – value of the Lyapunov-type function at $x_k$.
4. $D^V_\xi \in \mathbb{R}$ for all simplices $\mathcal{S}_\xi \in T^*$ – upper bound on the gradient of $V$.
5. $Q \in \mathbb{R}^+_0$ – the quantity to be minimized.

**Objective**

minimize $Q$

**Constraints**

1. **Bound on the gradient of $\mu$**
   For each simplex $\mathcal{S}_\xi = \text{co}(x_0, \ldots, x_n) \in T^*$:
   \[ \|\nabla \mu_\xi\|_\infty \leq D^\mu_\xi \]
   See Remark 3.2 for details.

2. **$\mu(x)$ an upper bound on the generalized eigenvalues**
   For each simplex $\mathcal{S}_\xi = \text{co}(x_0, \ldots, x_n) \in T^*$ and each vertex $x_k$ of $\mathcal{S}_\xi$:
   \[ A(x_k) - \mu(x_k)P(x_k) + h^2_\xi(E_\nu + 2n\sqrt{n}D_\nu D^\mu_\xi)I \preceq 0 \] (13)
   $E_\nu$ and $D_\nu$ correspond to the simplex $\mathcal{S}_\nu \in T$ such that $\mathcal{S}_\xi \subset \mathcal{S}_\nu$.

3. **Bound on the gradient of $V$**
   For each simplex $\mathcal{S}_\xi = \text{co}(x_0, \ldots, x_n) \in T^*$:
   \[ \|\nabla V_\xi\|_1 \leq D^V_\xi \]
   Here $\nabla V_\xi := \nabla V \big|_{\mathcal{S}_\xi}(x)$ for all $x \in \mathcal{S}_\xi$. It is constructed exactly as the vectors $w^\nu_{ij}$ in Optimization Problem 3.1, see also Remark 3.2.

4. **Upper bound on the sum of generalized eigenvalues**
   For each simplex $\mathcal{S}_\xi = \text{co}(x_0, \ldots, x_n) \in T^*$ and each vertex $x_k$ of $\mathcal{S}_\xi$:
   \[ \nabla V_\xi \cdot f(x_k) + h^2_\xi \cdot nB^*_\xi D^V_\xi + \tilde{m}\mu(x_k) \leq Q \] (14)

The orbital derivative of $V$ is defined exactly as in Definition 3.4 but with the triangulation $T^*$ of course.

**3.10 Lemma:** Consider a feasible solution to Optimization Problem 3.2 and let $\mu(x)$ and $V(x)$ be constructed as in Definition 3.4 Then for every $x \in D^*_T$, we have

\[ A(x) - \mu(x)P(x) \preceq 0 \]

and

\[ V'(x) + \tilde{m}\mu(x) \leq Q. \]
Proof: Fix a point \( x \in D^p_\mathcal{T} \) and a corresponding simplex \( \mathcal{S}_\xi = \text{co}(x_0, x_1, \ldots, x_n) \in \mathcal{T}^* \) as in Definition 5.6. Further, denote by \( \mathcal{S}_\nu \) the simplex in \( \mathcal{T} \) such that \( \mathcal{S}_\xi \subset \mathcal{S}_\nu \). Now

\[
\left\| A(x) - \mu(x)P(x) - \sum_{k=0}^{n} \lambda_k [A(x_k) - \mu(x_k)P(x_k)] \right\|_2 \\
\leq \left\| A(x) - \sum_{k=0}^{n} \lambda_k A(x_k) \right\|_2 + \left\| \mu(x)P(x) - \sum_{k=0}^{n} \lambda_k \mu(x_k)P(x_k) \right\|_2.
\]

(15)

Just as in the proof of Lemma 5.8, we can show that

\[
\left\| A(x) - \sum_{k=0}^{n} \lambda_k A(x_k) \right\|_2 \leq h^2\nu.
\]

(16)

For the second norm on the right-hand side of (15) consider two scalar-valued functions \( g, h : \mathcal{S}_\xi \to \mathbb{R} \), where \( g, h \in C^2 \). We apply Lemma 5.7 to \( gh \), yielding

\[
\left| g(x)h(x) - \sum_{k=0}^{n} \lambda_k g(x_k)h(x_k) \right| \leq \max_{y \in \mathcal{S}_\xi} \| H(y)\|_2 h^2 \nu,
\]

where the matrix \( H(y) \) is defined by \( [H(y)]_{rs} := \frac{\partial^2 (gh)(y)}{\partial x_r \partial x_s} \). Set \( g(y) := P_{il}(y) \) and \( h(y) := \mu(y) \). Since \( \mathcal{S}_\xi \subset \mathcal{S}_\nu \), we have \( P_{il}(y) = w_{il}' \cdot (y - x_0) + P_{il}(x_0) \) and because \( \mu(y) \) is defined as a CPA interpolation, we have \( \mu(y) = [\nabla \mu_\xi]|_{(y-x_0)+\mu(x_0)} \). Thus, we have \( \frac{\partial g}{\partial x_r}(y) = [w_{il}'], \frac{\partial^2 g}{\partial x_r \partial x_s}(y) = 0, \)\( \frac{\partial h}{\partial x_r}(y) = [\nabla \mu_\xi]_s \), and \( \frac{\partial^2 h}{\partial x_r \partial x_s}(y) = 0 \) for all \( y \in \mathcal{S}_\nu \). Hence,

\[
\frac{\partial}{\partial x_s} gh = \frac{\partial g}{\partial x_s} h + \frac{\partial h}{\partial x_s} = [w_{il}']_s h + g[\nabla \mu_\xi]_s
\]

and

\[
\frac{\partial^2}{\partial x_r \partial x_s} gh = [w_{il}']_s \frac{\partial h}{\partial x_r} + \frac{\partial g}{\partial x_r} [\nabla \mu_\xi]_s = [w_{il}']_s [\nabla \mu_\xi]_r + [w_{il}']_r [\nabla \mu_\xi]_s.
\]

Thus

\[
\| [H(y)]_{rs} \|_{\mathcal{S}_\xi} = \| [w_{il}']_s [\nabla \mu_\xi]_r + [w_{il}']_r [\nabla \mu_\xi]_s \| \leq \| [w_{il}']_s [\nabla \mu_\xi]_r \| + \| [w_{il}']_r [\nabla \mu_\xi]_s \|.
\]

Using that for any \( H_1, H_2 \in \mathbb{R}^{n \times n} \) we have

\[
\| H_1 + H_2 \|_2 \leq \| H_1 \|_2 + \| H_2 \|_2 \leq \sqrt{n} \| H_1 \|_1 + \sqrt{n} \| H_2 \|_\infty,
\]

we get

\[
\| H(y) \|_2 \leq 2\sqrt{n} \| w_{il}'' \|_1 \| \nabla \mu_\xi \|_\infty \leq 2\sqrt{n} D_\nu D_\xi^\mu,
\]

because \( \| w_{il}'' \|_1 \leq D_\nu \) by Constraints 3 in Optimization Problem 5.1 and \( \| \nabla \mu_\xi \|_\infty \leq D_\xi^\mu \) by Constraints 1 in Optimization Problem 5.2.

We have shown that

\[
\left\| \mu(x)P(x) - \sum_{k=0}^{n} \lambda_k \mu(x_k)P(x_k) \right\|_{\text{max}} \leq h^2 \cdot 2\sqrt{n} D_\nu D_\xi^\mu
\]

and it follows that

\[
\left\| \mu(x)P(x) - \sum_{k=0}^{n} \lambda_k \mu(x_k)P(x_k) \right\|_2 \leq h^2 \cdot 2n\sqrt{n} D_\nu D_\xi^\mu.
\]

(17)
Thus, we have by (15), (16), (17), and Constraints 2 of Optimization Problem 3.2

\[ A(x) - \mu(x)P(x) \leq \sum_{k=0}^{n} \lambda_k [A(x_k) - \mu(x_k)P(x_k)] + \left\| A(x) - \sum_{k=0}^{n} \lambda_k A(x_k) \right\|_2 I \\
+ \left\| \mu(x)P(x) - \sum_{k=0}^{n} \lambda_k \mu(x_k)P(x_k) \right\|_2 I \\
\leq \sum_{k=0}^{n} \lambda_k [A(x_k) - \mu(x_k)P(x_k)] + h^2(\epsilon_0 + 2n\sqrt{n}D\varepsilon D^V_\xi)I \leq 0 \]

and the estimate (18) follows and, as before, it also follows that \( \mu(x) \) is an upper bound on the largest generalized eigenvalue of the matrix pair \( (A(x), P(x)) \) for every \( x \in D^\tau_\tau \).

Let \( x \in D_\tau \) and \( \mathcal{G}_\xi \) be as above. We now show the implications of Constraints 4. By Hölder’s inequality and Lemma 3.7 we get

\[ \nabla V_\xi \cdot f(x) \leq \sum_{k=0}^{n} \lambda_k \nabla V_\xi \cdot f(x_k) + \left\| \nabla V_\xi \cdot f(x) - \sum_{k=0}^{n} \lambda_k \nabla V_\xi \cdot f(x_k) \right\|_1 \]

\[ \leq \sum_{k=0}^{n} \lambda_k \nabla V_\xi \cdot f(x_k) + \left\| \nabla V_\xi \right\|_1 \left\| f(x) - \sum_{k=0}^{n} \lambda_k f(x_k) \right\|_\infty \]

\[ \leq \sum_{k=0}^{n} \lambda_k \nabla V_\xi \cdot f(x_k) + h^2 \cdot nB^V_\xi D^V \]

and then by Constraints 4 of Optimization Problem 3.2

\[ \nabla V_\xi \cdot f(x) + \bar{m}_\mu(x) \leq \sum_{k=0}^{n} \lambda_k \left[ \nabla V_\xi \cdot f(x_k) + h^2 \cdot nB^V_\xi D^V + \bar{m}_\mu(x_k) \right] \leq \sum_{k=0}^{n} \lambda_k Q = Q, \]

i.e., the estimate (19).

The following lemma shows that the piecewise affine functions \( P, V \) and \( \mu \) computed by our algorithm can be approximated by smooth functions asymptotically satisfying the same inequalities. Hence, we do not get into trouble because of the differentiability assumptions in Theorem 2.1.

For any subset \( D \subset \mathbb{R}^n \) and \( \epsilon > 0 \) define \( D_{-\epsilon} := \{ x \in D : B_\epsilon(x) \subset D \} \). Define \( \phi : \mathbb{R}^n \to \mathbb{R}_+, \)

\[ \phi(x) := C \exp(-1/(1 - \|x\|_2)) \quad \text{for} \quad \|x\|_2 < 1 \quad \text{and} \quad \phi(x) := 0 \quad \text{otherwise and choose the constant} \quad C \quad \text{such that} \quad \int_{\mathbb{R}^n} \phi(y) \, dy = 1. \]

For an \( \epsilon > 0 \) define

\[ \tilde{\phi}_\epsilon(x) := \frac{\phi(x/\epsilon)}{\epsilon^n}. \]

For a locally integrable \( g : D \to \mathbb{R}, D \subset \mathbb{R}^n \), define the function \( g_\epsilon := g * \tilde{\phi}_\epsilon \), i.e., \( g_\epsilon(x) = \int_D g(y)\tilde{\phi}_\epsilon(x-y) \, dy \).

It is well-known that \( g_\epsilon, \tilde{\phi}_\epsilon \in C^\infty(\mathbb{R}^n) \) and if \( g \) is continuous on \( D \subset \mathbb{R}^n \) and \( K \subset D^p \) is compact, then the functions \( g_\epsilon \) approximate \( g \) uniformly on \( K \), i.e. \( \max_{x \in K} |g_\epsilon(x) - g(x)| \to 0 \) as \( \epsilon \to 0+ \).

3.11 Lemma: Assume \( g \in CPA[\mathcal{G}_\nu] \to \mathbb{R} \) (cf. Definition 3.4), \( \epsilon > 0 \), and denote by \( w_\nu \) the gradient of \( g \) on \( \mathcal{G}_\nu \). That is, \( g(x) = w_\nu \cdot x + b_\nu \) on \( \mathcal{G}_\nu \). Then for every \( x \in (D_\tau)_{-\epsilon} \) we have

\[ \nabla g_\epsilon(x) = \sum_{\nu} \alpha^{x,\epsilon}_\nu w_\nu, \quad \text{where} \quad \alpha^{x,\epsilon}_\nu := \int_{\mathcal{G}_\nu \cap B_\epsilon(x)} \tilde{\phi}_\epsilon(x-y) \, dy. \]

Especially, the nonnegative numbers \( \alpha^{x,\epsilon}_\nu \) only depend on \( x \) and \( \epsilon > 0 \) and not on the function \( g \) and they sum to one.
Lemma: Given the same assumption as in Lemma 3.10, let \( \delta > 0 \). Then there exist smooth \( P, \mu : (D_T)_- \delta \to \mathbb{R}^{n \times n} \) and \( V, \mu : (D_T)_- \delta \to \mathbb{R} \), such that for every \( x \in (D_T)_- \delta \) we have

\[
A_\varepsilon(x) - \mu_\varepsilon(x) P(x) \leq \delta I
\]

and

\[
V_\varepsilon(x) + \tilde{\mu}\mu_\varepsilon(x) \leq Q + \delta,
\]

where

\[
A_\varepsilon(x) := P_\varepsilon(x) D f(x) + D f(x)^T P_\varepsilon(x) + \dot{P}_\varepsilon(x)
\]

and \( Q \) is the same constant as in Lemma 3.10.

Proof: Let \( P, V, \) and \( \mu \) be defined as in Lemma 3.10 and set

\[
G := \max \left\{ \max_{i, j, x, y} \| w_{ij}^x \|_2, \max_{\xi \in T} \| \nabla V_\xi \|_2 \right\}.
\]

Fix \( 0 < \varepsilon < \delta \) so small that for all \( x \in (D_T)_- \delta \) and all \( y \) satisfying \( \| x - y \|_2 < \varepsilon \) we have

\[
|(B_\varepsilon)_{ij}(x) - B_{ij}(x)| < \frac{\delta}{3n},
\]

\[
|B_{ij}(x) - B_{ij}(y)| < \frac{\delta}{3n},
\]

\[
\| f(x) - f(y) \|_2 < \frac{\delta}{3nG},
\]

\[
|V(x) - V(y)| < \frac{\delta}{2},
\]

\[
|\mu(x) - \mu(y)| < \frac{\delta}{3M},
\]

\[
|\mu_\varepsilon(x) - \mu(x)| < \frac{\delta}{3},
\]

where the mollified functions with \( \varepsilon \) in the subscript are defined as in Lemma 3.11.

\[
B(x) := P(x) D f(x) + D f(x)^T P(x) - \mu(x) P(x)
\]

and

\[
B_\varepsilon(x) := P_\varepsilon(x) D f(x) + D f(x)^T P_\varepsilon(x) - \mu_\varepsilon(x) P_\varepsilon(x).
\]

Fix \( x \in (D_T)_- \delta \). For each \( \alpha_{\varepsilon, \gamma}^+ > 0 \), cf. Lemma 3.11 for a \( \mathcal{G}_\nu \in T \) select an \( x_\nu \) in the interior of \( \mathcal{G}_\nu \cap B_\varepsilon(x) \) and for each \( \alpha_{\varepsilon, \gamma}^- > 0 \) for a \( \mathcal{G}_\xi \in T^* \) select an \( x_\xi \) in the interior of \( \mathcal{G}_\xi \cap B_\varepsilon(x) \).
Now for all \( i, j = 1 : n \) we have by the estimates above, Lemma 3.11 and the Cauchy-Schwarz inequality
\[
(B_\varepsilon)_{ij}(x) + \nabla (P_\varepsilon)_{ij}(x) \cdot f(x) < B_{ij}(x) + \frac{\delta}{3n} + \sum_\nu \alpha_{\nu}^{x, \varepsilon} w_{ij}^{\nu} \cdot f(x)
\]
\[
< \sum_\nu \alpha_{\nu}^{x, \varepsilon} \left( B_{ij}(x_\nu) + \frac{\delta}{3n} + w_{ij}^{\nu} \cdot f(x_\nu) + \|w_{ij}^{\nu}\|_2 \frac{\delta}{3nG} \right) + \frac{\delta}{3n}
\]
Hence,
\[
A_\varepsilon(x) - \mu_\varepsilon(x) P_\varepsilon(x) = B_\varepsilon(x) + \hat{P}_\varepsilon(x)
\]
\[
\leq \sum_\nu \alpha_{\nu}^{x, \varepsilon} \left( B(x_\nu) + \hat{P}(x_\nu) \right) + \frac{\delta}{n} (1)_{ij}
\]
\[
= \sum_\nu \alpha_{\nu}^{x, \varepsilon} (A(x_\nu) - \mu(x_\nu) P(x_\nu)) + \frac{\delta}{n} (1)_{ij} \leq \delta I.
\]
Similarly,
\[
\dot{V}_\varepsilon(x) + \tilde{m} \mu_\varepsilon(x) = \sum_\xi \alpha_\xi^{x, \varepsilon} (\nabla V_\xi \cdot f(x) + \tilde{m} \mu(x)) + \frac{\delta}{3}
\]
\[
= \sum_\xi \alpha_\xi^{x, \varepsilon} \left( \nabla V_\xi \cdot f(x_\xi) + \tilde{m} \mu(x_\xi) \right) + \frac{\delta}{3n} + \frac{\delta}{3} + \frac{\delta}{3} \leq Q + \delta.
\]

Note that the Constraints 1 and 2 in Optimization Problem 3.2 are not very strongly coupled to the Constraints 3 and 4. Constraints 2 balance the values \( \mu(x_k) \) and the gradient \( D_\xi^\mu \), whereas Constraints 4 do not have to take the gradient of \( \mu \) into account. Since the gradient is multiplied by \( h_\xi^2 \), which is small for small simplices \( \mathcal{G}_\xi \in \mathcal{T}^* \), the gradient can be rendered less important in Constraints 2 by using smaller simplices. It is thus tempting to split Optimization Problem 3.2 into two optimization problems, the first with Constraints 1 and 2 and some objective function that makes the collection of the \( \mu(x_k) \) small in some sense, and then consecutively run an optimization problem with Constraints 3 and 4, where the \( \mu(x_k) \) from a solution to the first optimization problem are constants. This is especially tempting, because SDP solvers have not reached the maturity of linear programming solvers and are sometimes not able to deliver solutions to moderately sized feasible problems or worse, deliver solutions that are quite far from being feasible.

Further, if we use Optimization Problem 3.1 to find a constant matrix \( P(x) \), the optimization problem is much smaller and easier to solve. From such a solution the Optimization Problem 3.2 can be naturally split into two optimization problems as described above without any disadvantage, because the coupling between Constraints 1 and 2 on the one hand and Constraints 3 and 4 on the other hand vanishes completely. Even better, since \( P(x) \) is constant, its gradient is zero and therefore \( D_\nu = 0 \). Thus, the gradient of \( \mu \) plays no role, because its upper bound \( D_\xi^\mu \) is multiplied by \( D_\nu \) in Constraints 2. We can thus drop Constraints 1 and compute the optimal \( \mu(x_k) \) directly. First, for each vertex \( x_k \) we find the minimum \( \mu(x_k) \) such that (13) is fulfilled for every \( \nu \) such that \( x_k \) is a vertex of \( \mathcal{G}_\nu \). Then we minimize \( Q \) under the linear constraints (14). This is described in more detail in the next section.

### 3.3 Simplified procedure

In the simplified procedure we restrict our search for a matrix \( P(x) \) in Optimization Problem 3.1 to a constant matrix and then split Optimization Problem 3.2 into two simpler problems. In
In Optimization Problem 3.1 we set \( P_{ij}(x_k) := P_{ij} \) and then \( P(x_k) := (P_{ij}) \) for all vertices \( x_k \) of all simplices of \( T \). Then clearly we can set \( D_\nu := 0 \) and \( C := C_\nu \) for all \( \xi_\nu \in T \) and the constraints simplify to:

\[
e_0 I \preceq P \preceq CI
\]

and for each simplex \( \xi_\nu = \text{co}(x_0, \ldots, x_n) \in T \) and each vertex \( x_k \) of \( \xi_\nu \):

\[
0 \succeq A(x_k) - \mu P + h_\xi^2 \cdot 2n^3 B_{3,\nu} CI,
\]

where

\[
A(x_k) = PD_f(x_k) + D_f(x_k)^T P,
\]

because \( w_{ij}^\nu \), the gradient of \( \mu \)

An even simpler version of this optimization problem is obtained by defining

\[
B_3^\nu := \max_{\xi_\nu \in T} B_{3,\nu}
\]

and solving:

For each vertex \( x_k \) of the triangulation \( T^* \) maximize the value \( \mu(x_k) \) under the constraints

\[
0 \succeq A(x_k) - \mu(x_k) P + h_\xi^2 \cdot 2n^3 B_{3,\nu} CI
\]

for every \( \nu \) such that \( x_k \in \xi_\nu \in T \).

An even simpler version of this optimization problem is obtained by defining

\[
B_3^\nu := \max_{\xi_\nu \in T} B_{3,\nu}
\]

and solving:

For each vertex \( x_k \) of the triangulation \( T^* \) maximize the value \( \mu(x_k) \) under the constraints

\[
0 \succeq A(x_k) - \mu(x_k) P + h_\xi^2 \cdot 2n^3 B_{3,\nu} CI
\]

for small \( h_\xi > 0 \) good estimates on these optimal \( \mu(x_k) \) for the Optimization Problem 3.2 can be directly computed using (12). Just set

\[
\mu(x_k) := \lambda_{\text{max}}(x_k) + h_\xi^2 \cdot 2n^3 B_{3,\nu} x_z C \| P^{-1} \|_2
\]

where \( \lambda_{\text{max}}(x_k) \) is the largest generalized eigenvalue of the matrix pair \( (A(x_k), P) \). Since \( C \succeq \| P \|_2 \), this formula can be further simplified to

\[
\mu(x_k) := \lambda_{\text{max}}(x_k) + h_\xi^2 \cdot 2n^3 B_{3,\nu} \kappa_2(P),
\]

using the condition number \( \kappa_2(P) := \| P \|_2 \| P^{-1} \|_2 \) of \( P \).

After this being done, we can minimize \( Q \) under the Constraints 3 and 4 of Optimization Problem 3.2 and this is a linear programming problem, for which much more mature solvers exist.

4 An example: the Lorenz system

We consider the Lorenz system

\[
\frac{dx}{dt} = \begin{pmatrix}
-x - \sigma y \\
\sigma x + y - xz \\
xz - by + x y
\end{pmatrix} = g(x, y, z).
\]
For our approach it is advantageous to scale the system such that its attractors are contained in a smaller set. For this purpose, define $S := \text{diag}(s_x, s_y, s_z)$ for constants $s_x, s_y, s_z > 0$ and consider the system $\mathbf{x} = f(\mathbf{x})$ with $f(\mathbf{x}) = S^{-1}g(S\mathbf{x})$, i.e. the system

$$
\frac{d}{dt} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix}
-\sigma x + \frac{\rho_x y s_x}{s_y} x \\
\frac{\rho_y x}{s_x} y - y - \frac{s_x}{s_y} x z \\
-\rho_z + \frac{s_x}{s_y} xy
\end{pmatrix}.
$$

Clearly one can take

$$B_\nu = s_x \cdot \max \left\{ \frac{s_y}{s_x}, \frac{s_z}{s_y} \right\}
$$

in the optimization problems for all $\mathcal{G}_\nu$, independent of the triangulation $\mathcal{T}$, because the right-hand side is a global bound on the second-order derivatives of $f$, and we can set $B_{3,\nu} := 0$ for all $\mathcal{G}_\nu$, because the components of $f$ are second-order polynomials. We use the scaling parameters $s_x = 24.5$ and $s_y = s_z = 100$ and thus $B_\nu = 24.5$ in what follows.

In [5 §2.2] it is shown that if $\sigma \geq 1$ and $b \geq 2$ in (21), then it is dissipative in the sense of Levinson and the region $\mathcal{D}$ of dissipation fulfills

$$\mathcal{D} \subset \left\{ (x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + (z - [\sigma + r])^2 \leq \frac{b}{2} (\sigma + r)^2 \right\},$$

(23)

$$\mathcal{D} \subset \left\{ (x, y, z) \in \mathbb{R}^3 : 2x^2 + y^2 + (z - [\sigma + r])^2 \leq \left[ 1 + \frac{(b-2)^2}{4(b-1)} \right] (\sigma + r)^2 \right\},$$

(24)

$$\mathcal{D} \subset \left\{ (x, y, z) \in \mathbb{R}^3 : y^2 + (z - r)^2 \leq \frac{b^2 r^2}{4(b-1)} \right\},$$

(25)

and

$$\mathcal{D} \subset \left\{ (x, y, z) \in \mathbb{R}^3 : z \geq 0 \right\}.$$ 

(26)

All attractors of (21) are inside $\mathcal{D}$. For the common parameters $\sigma = 10$, $r = 28$, and $b = 8/3$ in (21), it suffices to compute our metric $P$ and Lyapunov-type function $V$ on the set

$$\mathcal{K} := [-1,1] \times [-0.29,0.29] \times [0,0.57],$$

because with these parameters (25) implies

$$|y| \leq \frac{4}{\sqrt{15}} r \leq 29 = s_y \cdot 0.29$$

and (24) and (26) imply

$$0 \leq z \leq \left( 1 + \frac{4}{\sqrt{15}} \right) r \leq 57 = s_z \cdot 0.57,$$

which in turn with (24) implies that

$$|x| \leq \sqrt{\frac{1}{2} \left( \frac{16}{15} (\sigma + r)^2 - \left( \frac{4}{\sqrt{15}} r - \sigma \right)^2 \right)} \leq 24.5 = s_x \cdot 1.$$

Thus, all the attractors of (21) are inside of $\mathcal{K}$.

For our example we used the simplified procedure from Section 3.3 and the computer we used has an i9-7900X CPU. First we compute a constant metric $P(x) = \mathring{P}$ on the set $\mathcal{K}$ using a triangulation with the vertices

$$\left( 1.0 \cdot \frac{i_x}{12}, 0.29 \cdot \frac{i_y}{6}, 0.57 \cdot \frac{i_z}{10} \right) \text{ for } i_x = -12:12, i_y = -6:6, \text{ and } i_z = 0:10.$$

The triangulation we used is a so-called standard-triangulation as in [15 §4], but with different scaling along the different axes, i.e. $\rho$ in [15 Def. 4.8] is $\rho_x = 1/12$ along the $x$-axis, $\rho_y = 0.29/6$.
along the $y$-axis, and $\rho_z = 0.57/10$ along the $z$-axis. We set $\epsilon_0 := 0.1$ in Optimization Problem 3.1 and the constraints become

$$0.1I \preceq P \preceq CI$$

for each simplex $\mathcal{S}_\nu \in \mathcal{T}$ and for each vertex $x_k$ of $\mathcal{S}_\nu$:

$$0 \preceq Pf(x_k) + Df(x_k)^T P - \mu P.$$

The optimization problem is especially simple because $B_{3,\nu} = 0$ for all $\mathcal{S}_\nu$ and therefore we do not even have to repeat the second constraints for all $\mathcal{S}_\nu$, i.e. we can replace for each simplex $\mathcal{S}_\nu \in \mathcal{T}$ and for each vertex $x_k$ of a simplex in $\mathcal{T}$.

By trying out a few different $\mu$s, we obtained a feasible solution with $\mu = 27$. Investigation gave us that there is only one positive generalized eigenvalue for all $x$, so we can take $\tilde{m} = 1$. Writing the problem took a few seconds with our software and solving the optimization problem took 140 sec. using the solver PENSDP 2.2 [18]. The problem has 7 variables and 69,122 matrix constraints.

The matrix computed is

$$P = \begin{pmatrix}
0.1008469737786 & -0.01415360101927 & 0 \\
-0.01415360101927 & 0.3361537095909 & 0 \\
0 & 0 & 0.3139832543019
\end{pmatrix}$$

which has $\|P^{-1}\|_2 = 0.1$ as the smallest and $C = \|P\|_2 = 0.3370007906512$ as the largest eigenvalue. If we only used these results in the formula in [28, Thm. 3.2] for the upper bound on the topological entropy, i.e. set $V(x) = \text{const.}$, this $P$ delivers $\lambda/(2 \ln(2)) \approx 19.4764$ as an upper bound for the topological entropy.

The formula (13) in [28] delivers the upper bound

$$\frac{1}{2 \ln(2)} \left( \sqrt{(\sigma - 1)^2 + 4r \sigma} - (\sigma + 1) \right) \approx 17.0638$$

(27)

for our parameters.

For computing the Lyapunov-type function we used the triangulation $\mathcal{T}^*$, which is constructed exactly as the triangulation $\mathcal{T}$ above, but with the vertices

$$\left(1.0 \cdot \frac{i_x}{N_x}, 0.29 \cdot \frac{i_y}{N_y}, 0.57 \cdot \frac{i_z}{N_z}\right)$$

for $i_x = -N_x : N_x$, $i_y = -N_y : N_y$, and $i_z = -1 : N_z$.

for some $N_x, N_y, N_z \in \mathbb{N}$. We tried a few different sets of parameter values, expecting lower upper bounds on the topological entropy for larger values of $N_x, N_y, N_z$. We used the state of the art solver GUROBI, which is free for academic use, to solve the LP problems using the barrier method.

The first set of parameters was $N_x = 30$, $N_y = 14$, and $N_z = 28$ and with those the LP problem with 930,032 variables and 3,800,122 constraints was written in 11 sec. with our software and solved in 291 sec. with the optimal value of $Q = 23.9094$ which delivers the upper bound 17.247 on the topological entropy, which is slightly worse than in (27).

The second set of parameters was $N_x = 42$, $N_y = 14$, and $N_z = 28$ and with those the LP problem with 1,301,696 variables and 5,320,176 constraints was written in 28 sec. with our software and solved in 889 sec. with the optimal value of $Q = 21.7011$ which delivers the upper bound 16.970 on the topological entropy, which is slightly better than in (27).

Because we are using such a simple axially parallel triangulation, one can use somewhat less conservative bounds on the second-order derivatives of $f$ in the LP problems, compare [20, Lem. 4.16]. Using these less conservative bounds in the LP problems gave notably better results. Using the first set of parameters, the LP problem was solved in 437 sec. with the optimal value of $Q = 22.5403$,
which delivers the upper bound 16.260 on the topological entropy, and using the second set of parameters the LP problem was solved in 508 sec. with the optimal value of $Q = 22.094$, which delivers the upper bound 15.937 on the topological entropy.

For the third set of parameters we took $N_x = 50$, $N_y = 18$, and $N_z = 32$ and only used the less conservative bounds on the second-order derivatives of $f$. The LP problem had 2,265,460 variables and 9,266,354 constraints, was written in 1.9 sec. with our software and solved in 877 sec. with the optimal value of $Q = 21.701$ which delivers the upper bound 15.654 on the topological entropy.

The fourth and final set of parameters was $N_x = 70$, $N_y = 22$, and $N_z = 40$ and only used the less conservative bounds on the second-order derivatives of $f$. The LP problem had 4,812,572 variables and 19,699,632 constraints, was written in 61 sec. with our software and solved in 5,801 sec. with the optimal value of $Q = 21.311$ which delivers the upper bound 15.373 on the topological entropy.

Thus, the best estimate we got with our method was the upper bound 15.373 on the topological entropy, which is considerably better than 17.064 given by formula (13) in [28].

Running the full Optimization algorithms 3.1 and 3.2 and thus computing a nonconstant matrix $P$ would be very interesting, but is hardly possible for examples of interest with today’s SDP problems solvers. It remains interesting to see if these solvers mature enough in the near future for this to change and how much the upper bound on the topological entropy decreases using our fully fledged method.

### 5 Discussion

In this section, we discuss the usability of our algorithm with respect to the computation of data-rate limits and topological entropy.

#### 5.1 Computation of data-rate limits

In this subsection, we provide more details about the relation of our estimates to data-rate limits for state estimation problems.

Assume that a sensor measures the states of the dynamical system $(\phi_t)$ at discrete sampling times $t_k$, $k \in \mathbb{N}$, for initial states in a compact forward-invariant set $K$. Based on these measurements, a coder generates a symbol $q_k$ from a finite coding alphabet (possibly of time-varying length) at time $t_k$ and transmits $q_k$ over a noiseless digital channel to an estimator. At each time $t > 0$, the estimator tries to generate an approximation $\hat{x}(t)$ of $x(t)$, based on the information it has received through the channel up to this time. A fundamental problem in this context is to determine the smallest rate of information above which the estimator is able to generate an estimate of a given accuracy $\varepsilon > 0$.

The works [22, 23] studied this problem for three estimation criteria of increasing strength. By [22, Thm. 8], the critical channel capacity for the objective of (eventually) bounding the estimation error by some arbitrarily small $\varepsilon$, is given by the topological entropy on $K$. However, if one wants to reproduce and maintain an initial estimation accuracy (at time $t = 0$) somewhere in the future or even improve it exponentially, topological entropy is no longer the appropriate figure of merit. In fact, in these cases the critical capacity is measured by a quantity named *restoration entropy* in [23], which by [23, Thm. 11] (under mild assumptions) can be computed as

$$h_{\text{res}}(f; K) = \lim_{t \to \infty} \frac{1}{t} \max_{x \in K} \log_2 \|D\phi_t(x)\|.$$  

By the considerations in the next subsection this quantity can be strictly larger than the topological entropy on $K$. At this point, it is open question whether it can be approximated (and not only bounded) by the bounds given in Theorem 2.1. We hope to answer this question in a future work.
We further note that a constant matrix $P$, as delivered by the simplified procedure outlined in Subsection 3.3 and used in the example, is of practical advantage with respect to the state estimation problem. Indeed, the functions $P$ and $v$ can be used to practically implement a coding and estimation policy which achieves the estimation objective via a channel of capacity given by the associated bound, see [22, App. E]. If the matrix $P$ is not constant, this estimation scheme will become considerably more difficult to implement, and might even become infeasible.

5.2 Theoretical limitations with respect to the approximation of topological entropy

In the following, we discuss how close an approximation of $h_{\text{top}}(\phi; K)$ can possibly be achieved by Theorem 2.1, and hence by our algorithm. We first recall two well-known results on the topological entropy of maps. The first result is due to Przytycki [29] and reads as follows.

5.1 Theorem: Let $g$ be a $C^{1+\epsilon}$-diffeomorphism of a compact Riemannian manifold $M$ and let $\mu$ be the Riemannian volume measure on $M$. Then

$$h_{\text{top}}(g) \leq \limsup_{k \to \infty} \frac{1}{k} \log_2 \int_M \|Dg^k(x)\| d\mu(x).$$

(28)

The second result is due to Kozlosvki [17] and shows that for $C^{\infty}$-maps, the bound (28) is tight.

5.2 Theorem: Let $g$ be a $C^\infty$-map of a compact Riemannian manifold $M$ and let $\mu$ be the Riemannian volume measure on $M$. Then

$$h_{\text{top}}(g) = \lim_{k \to \infty} \frac{1}{k} \log_2 \int_M \|Dg^k(x)\| d\mu(x).$$

In these results, $Dg^k(x)^\wedge$ denotes the linear map between the exterior algebras of the tangent spaces $T_xM$ and $T_{g^k(x)}M$, induced by $Dg^k(x)$. The norm of this map can be expressed as

$$\|Dg^k(x)^\wedge\| = \max_{1 \leq d \leq n} \prod_{i=1}^{d} \lambda_i^{(k)}(x),$$

where $n = \dim M$ and $\lambda_1^{(k)}(x) \geq \ldots \geq \lambda_n^{(k)}(x)$ denote the singular values of $Dg^k(x)$.

Now let $g$ be the time-1 map of the flow $\phi$. Then, by (21) (putting $v_1 = v_2 = \ldots = v_n = 0$),

$$\|Dg^k(x)^\wedge\| \leq \max_{1 \leq d \leq n} \exp \left( \frac{1}{2} \int_0^k \left[ \lambda_1(\phi(s, x_0)) + \ldots + \lambda_d(\phi(s, x_0)) \right] ds \right) \leq e^{(k\Lambda)/2}.$$

Hence, from (28) we also obtain the estimate

$$h_{\text{top}}(\phi; K) = h_{\text{top}}(\phi_1; K) \leq \frac{\Lambda}{2 \ln 2}.$$

Since the inequality (28) can be strict if $g$ is not $C^\infty$, we see that in general we cannot approximate the topological entropy by our upper bounds.

Besides a lack of differentiability, there is another reason why the upper bounds obtained from Theorem 2.1 might not approach $h_{\text{top}}(\phi; K)$. Assuming w.l.o.g. that $\mu$ is a probability measure, observe that

$$\int_K \|D\phi_k(x)^\wedge\| d\mu(x) \leq \max_{x \in K} \|D\phi_k(x)^\wedge\| \leq e^{(k\Lambda)/2}.$$
In general, however, the right-hand side of (28) can be strictly smaller than
\[ \lim_{k \to \infty} \frac{1}{k} \max_{x \in K} \log_2 \| D\phi_k(x) \|. \] (29)

To see this, first observe that the map \( \alpha : (k, x) \mapsto \log_2 \| D\phi_k(x) \| \), \( \alpha : \mathbb{Z}_+ \times K \to \mathbb{R} \), defines a continuous subadditive cocycle over \( \phi_1 \), i.e.,
\[ \alpha_{n+m}(x) \leq \alpha_n(x) + \alpha_m(\phi(n, x)). \]

The general theory of such cocycles, in particular, shows that the limit (29) exists and is equal to
\[ \max_{\nu \in \mathcal{M}(\phi|_K)} \lim_{k \to \infty} \frac{1}{k} \int_K \alpha_k(x) d\nu(x), \] (30)

where \( \mathcal{M}(\phi|_K) \) denotes the set of all Borel probability measures on \( K \), invariant under \( \phi_1 \) (cf. [26, App. A]). The variational principle [24] together with the Margulis-Ruelle inequality [30] yield
\[ h_{\text{top}}(\phi; K) = \sup_{\nu \in \mathcal{M}(\phi|_K)} h_{\nu}(\phi_1) \quad \text{and} \quad h_{\nu}(\phi_1) \leq \lim_{k \to \infty} \frac{1}{k} \int_K \alpha_k(x) d\nu(x), \]

where \( h_{\nu}(\phi_1) \) denotes the measure-theoretic entropy of \( \phi_1 \) w.r.t. \( \nu \). It is well-known that the inequality here is an identity if and only if \( \nu \) is an SRB (Sinai-Ruelle-Bowen) measure or the right-hand side is zero. Hence, it is unlikely that the right-hand side of (28) will coincide with (29). Indeed, even if an SRB measure and an entropy-maximizing measure exist, they typically do not coincide.\(^6\) Assuming that \( \nu \) is an entropy-maximizing measure on \( K \), we hence obtain
\[ h_{\text{top}}(\phi; K) = h_{\nu}(\phi_1) < \lim_{k \to \infty} \frac{1}{k} \int_K \alpha_k(x) d\nu(x) \leq \lim_{k \to \infty} \frac{1}{k} \max_{x \in K} \log_2 \| D\phi_k(x) \| \leq \frac{\Lambda}{2 \ln 2}. \]

6 Final remarks

Some remarks on other methods to estimate topological entropy. The numerical approximation of topological entropy is not an easy task. Most approaches are restricted to one-dimensional maps (see, e.g., [1, 3, 6, 7, 10, 25, 11, 8]), because they make use of tools such as kneading theory or the entropy formulas for piecewise monotone maps proved by Misiurewicz and Szołc [23]. Newhouse and Pignataro [27] study an algorithm applicable to smooth maps, based on estimating the exponential length growth rates of suitably chosen curves under the dynamics of the map. The theoretical background of this algorithm suggests that it provides sharp estimates when the state space is two-dimensional (in the general case) or three-dimensional (in the case of time-t-maps of differential equations). The algorithm presented by Chen et al. [9] provides lower bounds for the entropy of chaotic attractors and chaotic saddles, based on successive preimages. Other approaches in higher dimensions are based on constructing partitions of the state space adapted well to the dynamics and considering associated shift spaces, see D’Alessandro et al. [11] for an application of this method to the Hénon map. The paper by Froyland et al. [12] provides a very general method for obtaining rigorous upper bounds of \( h_{\text{top}}(f) \), assuming nothing but continuity of the map \( f \). The idea of this algorithm is to choose a partition \( \mathcal{A} \) of the state space, which in the best case is generating, to construct an associated sofic shift whose entropy is not less than the ‘counting entropy’ of \( f \) on \( \mathcal{A} \), and finally to evaluate the entropy of the sofic shift. One drawback of this algorithm is that it requires the clever choice of a partition, which should not contain too many elements, because this significantly affects the computation time. However, this method does not have theoretical limitations as the ones described in Section 5.

\(^6\)In the Axiom A case, SRB measures do not maximize entropy, but pressure with respect to the potential \( v(x) = -\log |\det Dg(x)|_{E^s(x)}| \).
References

[1] S. L. Baldwin, E. E. Slaminka. Calculating topological entropy. J. Statist. Phys. 89 (5/6) (1997), 1017–1033.

[2] R. Baier, L. Grüne, S. Hafstein. Linear programming based Lyapunov function computation for differential inclusions. Discrete Contin. Dyn. Syst. Ser. B 17(1) (2012), 33–56.

[3] N. Balmforth, E. Spiegel, C. Tresser. Topological entropy of one-dimensional maps: approximations and bounds. Phys. Rev. Lett. 72 (1) (1994), 80–83.

[4] L. Block, J. Keesling, S. Li, K. Peterson. An improved algorithm for computing topological entropy. J. Statist. Phys. 55 (5/6) (1989), 929–939.

[5] V. Boichenko, G. Leonov, V. Reitmann. Dimension Theory for Ordinary Differential Equations. Teubner, 2005.

[6] L. Block, J. Keesling. Computing the topological entropy of maps of the interval with three monotone pieces. J. Statist. Phys. 66 (3/4) (1992), 755–774.

[7] A. Boyarsky, P. Gora. Computing the topological entropy of general one-dimensional maps. Trans. Am. Math. Soc. 323 (1) (1991), 39–49.

[8] J. S. Cánovas, M. M. Guillermo. Computing the topological entropy of continuous maps with at most three different kneading sequences with applications to Parrondo’s paradox. Chaos Soliton Fractals 83 (2016), 1–17.

[9] Q. Chen, E. Ott, L. Hurd. Calculating topological entropies of chaotic dynamical systems. Phys. Lett. A 156 (1991), no. 1–2, 48–52.

[10] P. Collet, J. Crutchfield, J. Eckmann. Computing the topological entropy of maps. Commun. Math. Phys. 88 (1983), 257–262.

[11] G. D’Alessandro, P. Grassberger, S. Isola, A. Politi. On the topology of the Hénon map. J. Phys. A 23 (1990), no. 22, 5285–5294.

[12] G. Froyland, O. Junge, G. Ochs. Rigorous computation of topological entropy with respect to a finite partition. Phys. D 154 (2001), no. 1–2, 68–84.

[13] P. Giesl, S. Hafstein. Construction of a CPA contraction metric for periodic orbits using semidefinite optimization. Nonlinear Anal. 86 (2013), 114–134.

[14] P. Giesl, S. Hafstein. Revised CPA method to compute Lyapunov functions for nonlinear systems. J. Math. Anal. Appl. 410 (2014), 292–306.

[15] P. Giesl, S. Hafstein. Computation and Verification of Lyapunov Functions. SIAM J. Appl. Math. 14 (4) (2015), 1663–1698.

[16] C. Kawan, S. Yüksel. On optimal coding of non-linear dynamical systems. Submitted for publication, 2017. arXiv:1711.06600 [math.DS].

[17] O. S. Kozlovski. An integral formula for topological entropy of $C^\infty$ maps. Ergodic Theory Dynam. Systems 18 (1998), no. 2, 405–424.

[18] M. Kocvara, M. Stingl. PENNON: Software for Linear and Nonlinear Matrix Inequalities. In: Handbook on Semidefinite, Conic and Polynomial Optimization. Springer 2012, pp. 755-794.

[19] D. Liberzon, S. Mitra. Entropy and minimal bit rates for state estimation and model detection. IEEE Trans. Automat. Control (2018) (in press).
[20] S. Marínósson. *Stability Analysis of Nonlinear Systems with Linear Programming: A Lyapunov Functions Based Approach*. PhD thesis: Gerhard-Mercator-University Duisburg, Duisburg, Germany, 2002.

[21] A. Matveev, A. Pogromsky. *A topological entropy approach for observation via channels with limited data rate*. IFAC Proceedings Volumes 44.1 (2011), 14416–14421.

[22] A. Matveev, A. Pogromsky. *Observation of nonlinear systems via finite capacity channels: constructive data rate limits*. Automatica J. IFAC 70 (2016), 217–229.

[23] A. Matveev, A. Pogromsky. *Observation of nonlinear systems via finite capacity channels. Part II: Restoration entropy and its estimates*. Submitted, 2017.

[24] M. Misiurewicz. *Topological entropy and metric entropy*. Ergodic theory (Sem., Les Plans-sur-Bex, 1980) (French), pp. 61–66, Monograph. Enseign. Math., 29, Univ. Genève, Geneva, 1981.

[25] M. Misiurewicz, W. Szlenk. *Entropy of piecewise monotone mappings*. Stud. Math. 67 (1980), 45–63.

[26] I. Morris. *Mather sets for sequences of matrices and applications to the study of joint spectral radii*. Proc. Lond. Math. Soc. (3) 107 (2013), no. 1, 121–150.

[27] S. Newhouse, T. Pignataro. *On the estimation of topological entropy*. J. Statist. Phys. 72 (1993), no. 5–6, 1331–1351.

[28] A. Pogromsky, A. Matveev. *Estimation of topological entropy via the direct Lyapunov method*. Nonlinearity 24 (2011), no. 7, 1937–1959.

[29] F. Przytycki. *An upper estimation for topological entropy of diffeomorphisms*. Invent. Math. 59 (1980), no. 3, 205–213.

[30] D. Ruelle. *An inequality for the entropy of differentiable maps*. Bol. Soc. Brasil. Mat. 9 (1978), no. 1, 83–87.

[31] A. V. Savkin. *Analysis and synthesis of networked control systems: topological entropy, observability, robustness and optimal control*. Automatica J. IFAC 42 (2006), no. 1, 51–62.