Weak measurement with a coherent state pointer and its implementation in optomechanical system

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\section*{Abstract}
Weak measurement with a coherent state pointer and in combination with an orthogonal postselection can lead to a surprising amplification effect, and we give a fire-new physical mechanism about the weak measurement in order to understand this effect. Moreover, this physical mechanism is a general result and based on it, we present a scheme of optomechanical system to implement weak measurement amplification on an orthogonal postselection.

\textit{Keywords:} Weak measurement, Coherent states, Optomechanics

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\section*{1. Introduction}
Weak measurement was proposed by Aharonov, Albert and Vaidman in 1988 \cite{1}. Initially the concept of weak measurement gives rise to some controversy, but soon its physical meaning was clarified \cite{2}. Today weak measurement becomes one of the most prospective research tools and it has been applied to various context including foundational questions in quantum mechanics \cite{3,4} and some counter-intuitive quantum paradoxes \cite{5}. In weak measurement, a pointer is weakly coupled to the system to be measured. Contrast to the project quantum measurement, the output signal of the
pointer in a weak measurement can be far beyond the range of the eigenvalues of the system’s observable if the width of the initial wave function of the pointer is quite large. Such a result is attributed to the interference of two or more slightly displaced Gaussian states after a near-orthogonal post-selection on the measured system [1, 6]. Most recently it is revealed that weak measurement can help to measure small physical quantities [7] or enable sensitive estimation of small physical parameters [8, 9] that are difficult to be directly detected by conventional techniques. Most of these weak measurement can be understood using classical wave mechanics [10, 8, 9] with one exception in [11]. Reviews of this filed can be found in [12, 13].

Although weak measurement has many applications, its application in optomechanics [14, 15] is seldom investigated. Optomechanical system consists of an optical cavity and a movable mirror. The photons in the cavity can give rise to a radiation pressure on the mirror and make it produce a displacement. In the weak coupling and when there is only one photon in the cavity, the displacement of the mirror is hard to be detected. The reason is that the displacement of the mirror caused by one photon is much smaller than the width of the mirror wave packet. Recently the standard scenario of weak measurement in [1] can be moved to Fock state space and understood by Fock-state view [6]. And because of this, some of us showed that the displacement of the mirror caused by one photon can be amplified if a weak measurement is used and the initial state of the mirror is assumed to be in the ground state [16]. The result is obtained by retaining the Kerr phase in the Ref. [17].

In the most discussions about the standard weak measurements, the initial state of the pointer is assumed to be the ground state and the interesting features of weak measurements are usually due to the superposition of several pointer states after a near-orthogonal postselection on the measured system which can produce the amplification effect. The relative phases between these pointer states play a key role and they can be adjusted through the postselection (see Appendix A). As we know that coherent states are regarded as classical-like states, naturally we want to know in weak measurement whether there are some new features when the pointer is initially prepared in coherent states. In the present paper, we will first give a general discussion about weak measurements that use a coherent state as the initial state of the pointer. It will be shown that there is a fire-new physical mechanism for weak measurements that have an initially coherent state pointer. It is regarded as a new mechanism because the relative phases between the
pointer states after the postselection, i.e., coherent quantum superposition [18], can be due to the noncommutativity of quantum mechanics induced by a coherent state pointer, which is different from the standard weak measurement [1, 6] where the relative phases are prepared through the postselection (see Appendix A), and this physical mechanism is a general result for weak measurement with a coherent state pointer.

As an example we will consider a weak measurement with a coherent state pointer in optomechanics. The weak measurement we will consider use the same optomechanical model in Ref. [16] and the mirror amplification is discussed. We find there are following new features about the obtained amplification with a coherent state pointer. (1) The maximal amplification of the displacement of the mirror can reach the level of the ground-state fluctuation and occur at time near zero, which is very important for bad optomechanical cavities, i.e., sideband resolution is small, and because of this, the implementation of our scheme is feasible in experiment. (2) The relative phase between two mirror states after an orthogonal in this paper is caused by the noncommutativity of quantum mechanics and the relative phase in Ref. [16] is caused by the Kerr phase generated by the evolution of the Hamiltonian [19, 20]. Therefore, the generation mechanisms of the relative phases in this paper and the Ref. [16] are different. Moreover, this conclusion in the Ref. [16] is a special case of the weak measurement amplification since it can only appear in the optomechanical system.

The structure of our paper is as follows. In Sec. II, we give a general discussion about weak measurement with a coherent state pointer. In Sec. III, we present a scheme for implementation of weak measurement with a coherent state pointer in optomechanics, and In Sec. IV, we give the conclusion about the work.

2. Weak measurement amplification with a coherent state pointer

In the standard scenario of weak measurement, the system to be measured is usually a two-level system and the pointer is a continuous system. The Hamiltonian between the pointer and the system is given in general as

$$\hat{H} = \hbar \chi(t) \hat{\sigma}_z \otimes \hat{p},$$

where $\sigma_z$ is an observable of the system to be measured, $\hat{p}$ is the momentum operator of the pointer and $\chi(t)$ is a narrow pulse function with integration.
\( \chi \). Suppose \( \hat{q} \) is position operator of the pointer that is conjugates to \( \hat{p} \), therefore there is \( [q,p] = i\hbar \). As in Ref. [6], if defining an annihilation operator \( \hat{c} = \frac{1}{\sqrt{2}}\hat{q} + i\frac{\sigma}{\sqrt{2}}\hat{p} \), where \( \sigma \) is the zero-point fluctuation of the pointer ground state, the Hamiltonian of Eq. (1) can be rewritten as

\[
\hat{H} = -i\frac{\hbar^2 \chi(t)}{2\sigma} \hat{\sigma}_z (\hat{c} - \hat{c}^\dagger).
\]

(2)

Here we further considered the initial pointer state is a coherent state \( |\alpha\rangle \) instead of the ground state \( |0\rangle \) in [6]. Suppose the state \( |+\rangle = \frac{1}{\sqrt{2}}(|0\rangle_s + |1\rangle_s) \) is the initial state of the system, where \( |0\rangle_s \) and \( |1\rangle_s \) is eigenstates of \( \hat{\sigma}_z \). Then the time evolution of the total system is given by

\[
e^{-\frac{i}{\hbar} \int \hat{H} dt} |+\rangle |\alpha\rangle = \exp[-i\eta \hat{\sigma}_z (\hat{c} - \hat{c}^\dagger)] |+\rangle |\alpha\rangle
= \frac{1}{\sqrt{2}} (|0\rangle_s D(\eta) D(\alpha) |0\rangle_m + |1\rangle_s D(-\eta) D(\alpha) |0\rangle_m).
\]

(3)

where \( \eta = \frac{\hbar}{2\sigma} \) assumed to be very small, \( D(\eta) = \exp[\eta \hat{c}^\dagger - \eta^* \hat{c}] \) is the displacement operator and we have used \( |\alpha\rangle = D(\alpha) |0\rangle_m \). When the orthogonal postselection \( |-\rangle \) is performed for the system, i.e., \( \langle - | + \rangle = 0 \), then the final state of the pointer becomes

\[
|\psi\rangle_m = \frac{1}{2} (D(\eta) D(\alpha) |0\rangle_m - D(-\eta) D(\alpha) |0\rangle_m)
\]

(4)

For the sake of making the analysis simple, we can displace the state of Eq. (3) to the origin point in phase space, defining \( |\chi\rangle_m = D^\dagger(\alpha) |\psi\rangle_m \) and there is

\[
|\chi\rangle_m = \frac{1}{2} (e^{i\varphi} D(\eta) |0\rangle_m - e^{-i\varphi} D(-\eta) |0\rangle_m),
\]

(5)

where the phases \( e^{i\varphi} \) and \( e^{-i\varphi} \) with \( \varphi = -i\eta (\alpha - \alpha^*) \) are obtained by using the property of the displacement operators \( D(\alpha) D(\beta) = \exp[\alpha \beta^* - \alpha^* \beta] D(\beta) D(\alpha) \), namely, the relative phase \( e^{i2\varphi} \) is caused by the noncommutativity of quantum mechanics. In other words, the relative phase \( e^{i2\varphi} \) is induced by a coherent state pointer since it is a quantum pointer. It is obvious that the superposition state in Eq. (5) is known as "Coherent Quantum Superposition" [18] since there is a relative phase.

For a weak measurement, the pointer states in Eq. (5) overlap significantly. In order to detailedly observe overlap of the pointer states in Eq. (5)
and when $\varphi \ll 1$ and $\eta \ll 1$, we can then perform a small quantity expansion about $\eta$ and $\varphi$ till the first order, then

$$
|\chi\rangle_m \approx \frac{1}{2}((1 - i\varphi)[1 - \eta(\hat{c} - \hat{c}^\dagger)]|0\rangle_m - (1 + i\varphi)[1 + \eta(\hat{c} - \hat{c}^\dagger)]|0\rangle_m)
$$

$$
\approx -i\varphi|0\rangle_m + \eta|1\rangle_m.
$$

(6)

This shows that the unique advantage of the amplification using coherent state pointer in weak coupling regime, where the relative phases caused by the noncommutativity of quantum mechanics is achievable for the supposition of $|0\rangle_m$ and $|1\rangle_m$ of the pointer on orthogonal postselection, in sharp contrast to the fact that only $|1\rangle_m$ of the pointer is generated using the ground state pointer on orthogonal postselection [1, 6] (see Appendix A). Therefore, from the origin of the relative phase and the supposition state of Eq. (6) induced by the relative phase, we can see clearly that the measuring device (the pointer) with a coherent state have an effect on the postselection for the system in weak measurement, which is not studied before.

The average displacement of the pointer position $\hat{q}$ is

$$
\langle \hat{q} \rangle = \frac{\langle \chi_m | \hat{q} | \chi_m \rangle}{\langle \chi | \chi \rangle_m} - \langle 0 | \hat{q} | 0 \rangle_m.
$$

(7)

In a similar way, the average displacement of the pointer momentum $\hat{p}$ is

$$
\langle \hat{p} \rangle = \frac{\langle \chi_m | \hat{p} | \chi_m \rangle}{\langle \chi | \chi \rangle_m} - \langle 0 | \hat{p} | 0 \rangle_m.
$$

(8)

So in this case of the orthogonal postselection, i.e., $\langle -|+ \rangle = 0$, we can find that

$$
\langle \hat{q} \rangle = 0.
$$

(9)

and

$$
\langle \hat{p} \rangle = \frac{\hbar}{2\sigma} \frac{2\varphi\eta}{\varphi^2 + \eta^2}.
$$

(10)

From Eq. (9) and Eq. (10), it can be seen that $\langle \hat{q} \rangle$ is zero in position space and $\langle \hat{p} \rangle$ is non-zero in momentum space since $|\chi\rangle_m$ is a supposition state. Such a result originates from non-zero relative phase $e^{i2\varphi}$ in Eq. (5) caused by the noncommutativity of quantum mechanics. We note that when $\alpha$ is a real number there is $\varphi = 0$ and the displacement of the pointer momentum
is zero, which is the same result as a ground state pointer on non-orthogonal postselection \[1, 6\] (see Appendix A).

As is known to all, in standard weak measurement theory \[1, 6\], the relative phase between two different pointer states can be made through proper near-orthogonal postselection if the pointer is initially assumed to be in the ground state (see Appendix A), but this conclusion is not applicable to the condition of orthogonal postselection and in this case the displacement of the pointer is zero. However, our result shows that when the pointer is initially prepared in a coherent state, it is after an orthogonal postselection that the relative phase between two different pointer states, i.e., coherent quantum superposition, can be caused by the noncommutativity of quantum mechanics. It is because of the relative phase caused by the noncommutativity of quantum mechanics that the interference of two different pointer states after an orthogonal postselection can produce a large displacement of the pointer. Therefore this perspective provided here show us a fire-new physical mechanism about the weak measurement compared to the standard weak measurement theory \[1, 6\] and this physical mechanism is a general result for coherent state pointer.

3. Implementation in optomechanics

In the following we will consider an weak measurement model in optomechanical system where the initial state of the pointer (a mirror) is prepared in a coherent state.

3.1. The optomechanical model

We first consider the optomechanical system shown in Fig. 1(a), which evolves under the following Hamiltonian \[10, 20\]

\[
H = \hbar \omega_0 a^\dagger a + \hbar \omega_m c^\dagger c - \hbar g a^\dagger a (c^\dagger + c),
\]

(11)

where \(\hbar\) is Plank’s constant, \(\omega_0\) and \(a\) are angular frequency and annihilation operator of the optical cavity mode, respectively, \(c\) is annihilation operator of mechanical system with angular frequency \(\omega_m\), and the optomechanical coupling strength \(g = \frac{\omega_0}{L} \sigma\), where \(L\) is the length of the cavity, \(\sigma = (\hbar/2m\omega_m)^{1/2}\) is the zero-point fluctuation and \(m\) is the mass of mechanical system.

If the initial state of the mirror is prepared at the coherent state \(|\alpha\rangle\) and one photon is input to the optomechanical cavity, then the mirror state will
evolve as follows \(^{(19, 20)}\):

\[
\psi(\xi, \varphi, t) = e^{i\phi(t)} D(\xi(t)) |\varphi(t)\rangle_m,
\]

where \(e^{i\phi(t)}\) is the Kerr phase of one photon with \(\phi(t) = k^2(\omega_m t - \sin \omega_m t)\), \(D(\xi(t)) = \exp[\xi(t) c^\dagger - \xi^*(t) c]\) is a displacement operator with \(\xi(t) = k(1 - e^{-i\omega_m t})\), \(\varphi(t) = a\e^{-i\omega_m t}\), and \(k = g/\omega_m\) is the scaled coupling parameter. If no photon is input to the optomechanical cavity, the mirror state will be the coherent state \(|\varphi(t)\rangle_m\). The position displacement of the mirror caused by one photon is \(\langle \psi(\xi, \varphi, t) | \hat{q} | \psi(\xi, \varphi, t) \rangle - \langle \varphi(t) | \hat{q} | \varphi(t) \rangle\) with \(\hat{q}\) being the position operator. It can be shown that the displacement can not be bigger than \(4k\sigma\) for any time \(t\). Since \(k = g/\omega_m\) can not be bigger than 0.25 in weak coupling condition \(^{(21)}\), then the position displacement of the mirror caused by one photon can not be bigger than the zero-point fluctuation \(\sigma\). From the literature \(^{(21)}\) we know that if the displacement of the mirror can be detected experimentally it should be not smaller than \(\sigma\). Therefore, the displacement of the mirror caused by one photon can not be detected. In the following we will show how the weak measurement with coherent state in optomechanics can amplify the mirror’s displacement.

### 3.2. The optomechanical model embedded in the interferometer

Now consider a March-Zehnder interferometer which is shown in Fig. 1(b). The optomechanical cavity A is embedded in one arm of the March-Zehnder interferometer and a stationary Fabry-Prot cavity B is placed in another arm. The two beam splitters are both symmetric. The Hamiltonian of optomechanical system is expressed as followed:

\[
H = \hbar \omega_0 (a^\dagger a + b^\dagger b) + \hbar \omega_m c^\dagger c - \hbar ga^\dagger a(c^\dagger + c),
\]

where \(b\) is annihilation operator of the optical cavity B. Other parameters are the same as before. Here it is a weak measurement model where the mirror is used as the pointer to measure the number of photon in cavity A, with \(a^\dagger a\) of the Eq. \(^{(13)}\) corresponding to \(\hat{\sigma}_z\) in Eq. \(^{(11)}\) in the standard scenario of weak measurement and \(c + c^\dagger\) corresponds to \(\hat{p}\).

Suppose that one photon is input into the interferometer, the state of the photon after the first beam splitter becomes

\[
|\psi_i\rangle = \frac{1}{\sqrt{2}} (|1\rangle_A |0\rangle_B + |0\rangle_A |1\rangle_B),
\]

\(7\)
and after interacting weakly with the mirror prepared at coherent state $|\alpha\rangle$, $\alpha = |\alpha|e^{i\theta}$, where $|\alpha|$ and $\theta$ are real numbers called the amplitude and phase of the state, respectively, the state of the total system is

$$\psi_{om}(t) = \frac{1}{\sqrt{2}}(|1\rangle_A|0\rangle_B|\psi(\xi, \varphi, t)\rangle_m + |0\rangle_A|1\rangle_B|\varphi(t)\rangle_m),$$

(15)

When the photon is detected at dark port, i.e., in the language of weak measurement [16] the postselected state of the single-photon is

$$\psi_f = \frac{1}{\sqrt{2}}(|1\rangle_A|0\rangle_B - |0\rangle_A|1\rangle_B),$$

(16)

which is orthogonal to $|\psi_i\rangle$. Then the final state of the mirror becomes

$$\psi_{os}(t) = \frac{1}{2}(|\psi(\xi, \varphi, t)\rangle_m - |\varphi(t)\rangle_m).$$

(17)

For the sake of making the analysis simple, we can displace the state of Eq. (17) to the origin point in phase space, defining $\chi_{os}(t) = D^\dagger(\varphi(t))|\psi_{os}(t)\rangle$.
and there is

$$|\chi_{\alpha}(t)\rangle = \frac{1}{2}(\exp[i\phi(t) + i\phi(\alpha,t)]|\xi(t)\rangle_m - |0\rangle_m),$$

(18)

where the phase $e^{i\phi(\alpha,t)}$ with $\phi(\alpha,t) = -i[\langle (\xi(t)\varphi^*(t) - \xi^*(t)\varphi(t)) \rangle]$ is a relative phase between the coherent state $|\xi(t)\rangle$ and the ground state $|0\rangle$, and it is obtained through the property of the displacement operators $D(\alpha)D(\beta) = \exp[\alpha\beta^* - \alpha^*\beta]D(\beta)D(\alpha)$, i.e., the noncommutativity of quantum mechanics. And by the way, in the Ref. [16] the relative phase between two mirror states after an orthogonal postselection is caused by the Kerr phase due to the evolution of the Hamiltonian [19, 20] and leads to the amplification. This can only appear in the optomechanical system and it can be said to be a special case of the amplification. But the amplification caused by the phase due to the noncommutativity of quantum mechanics is a general result for coherent state pointer.

Next, we will show how the amplification of the displacement of the mirror is generated via this relative phase $e^{i\phi(\alpha,t)}$. 
3.3. Amplification about position variable $q$ via coherent state pointer

The average displacement of pointer variable $\hat{q}$ of the mirror is

$$\langle \hat{q} \rangle = \frac{\langle \Psi_{os}(t)|\hat{q}|\Psi_{os}(t) \rangle}{\langle \Psi_{os}(t)|\Psi_{os}(t) \rangle} - \langle \varphi(t)|\hat{q}|\varphi(t) \rangle,$$

which will be

$$\langle \hat{q} \rangle = \frac{Tr(|\chi_{os}(t)\rangle\langle\chi_{os}(t)|\hat{q})}{Tr(|\chi_{os}(t)\rangle\langle\chi_{os}(t)|)} - Tr(0\langle 0|m|0\rangle_m\hat{q}).$$

It is clearly that through Eq. (18) we can obtain the average displacement of the mirror’s position $q$ and as a result we have

$$\langle q(t) \rangle = \sigma[\xi(t) + \xi^*(t) - e^{-\frac{|\xi(t)|^2}{2}}(e^{i\phi(t)+i\phi(\alpha,t)}\xi(t) + e^{-i\phi(t)+i\phi(\alpha,t)}\xi^*(t))/[2 - e^{-\frac{|\xi(t)|^2}{2}}(e^{i\phi(t)} + e^{-i\phi(t)+i\phi(\alpha,t))}],$$

where the phase $e^{i\phi(\alpha,t)}$ is the relative phase from the noncommutativity of quantum mechanics we mentioned in the previous subsection.

The average displacement $\langle q(t) \rangle / \sigma$ of the mirror is shown in Fig. 2 as a function of time $\omega_m t$ with $k = 0.005$ for different coherent states $|\alpha = |\alpha|e^{i\theta}\rangle$. It is clear from Fig. 2 that the amplification occurring near $\omega_m t = 0$ can reach the strong-coupling limiting (the level of the ground-state fluctuation) $\langle q \rangle = \sigma$ or $-\sigma$. This result is very important for bad optomechanical cavities where its decay rate is very large so that the photon will have a large probability to leak out from the cavity within a very short time. And the time interval where there are large amplification around $\omega_m t = 0$ is broad for some coherent states and therefore can be easier to be detected, which is contrast to the time interval of amplification around $\omega_m t = 0$ in Ref. [16] where it is very narrow. Note that the maximal displacement of the mirror caused by one photon in the cavity (see Fig. 1(a)) is $4k\sigma$ and the displacement achieved here can be $\sigma$ or $-\sigma$ using weak measurement, therefore the amplification factor can be $Q = \pm 1/4k$ which is $\pm 50$ when $k = 0.005$.

3.4. Small quantity expansion about time for amplification

To study the amplification effects occurring near $\omega_m t = 0$, for Eq. (18) we can then make a small quantity expansion about time 0 till the second
order. Suppose that \( \omega_m t \ll 1 \), and \( k \ll 1 \), then

\[
|\chi_{\text{os}}(t)| \approx \frac{1}{2}|i k \omega_m t|_m |1>_m + i 2 k|\alpha|\left|\frac{\omega_m t}{2}\right| \sin \theta \\
+ \omega_m t \cos \theta|0>_m, \tag{22}
\]

which is a superposition of \( |0>_m \) and \( |1>_m \) and the amplitude of \( |0>_m \) is due to the relative phase \( e^{i \phi(\alpha, t)} \) caused by noncommutativity of quantum mechanics when we use a non-zero coherent state \( |\alpha| \neq 0 \). The superposition of \( |0>_m \) and \( |1>_m \) is the key to obtain amplification. From Appendix A, we know that when

\[
k \omega_m t = 2 k|\alpha|\left|\frac{\omega_m t}{2}\right| \sin \theta + \omega_m t \cos \theta \tag{23}
\]

the displacement of the mirror can reach the maximal value \( \sigma \), and when

\[
k \omega_m t = -2 k|\alpha|\left|\frac{\omega_m t}{2}\right| \sin \theta + \omega_m t \cos \theta \tag{24}
\]

the displacement of the mirror can reach the minimal value \( -\sigma \). Hence, the mirror state achieving the maximal positive amplification is \( \frac{1}{\sqrt{2}}(|0>_m + |1>_m) \) and the one achieving the maximal negative amplification is \( \frac{1}{\sqrt{2}}(|0>_m - |1>_m) \).

We emphasis that the superposition of \( |0>_m \) and \( |1>_m \) achieved here is through an orthogonal postselection that is impossible in the standard weak measurement with an orthogonal postselection (see Appendix A). The superposition achieved here is due to the noncommutativity of quantum mechanics and the superposition in the standard weak measurement is due to the non-orthogonal postselection (see Appendix A), therefore it is a new mechanism for weak measurement. Substituting Eq. \( \text{(22)} \) into Eq. \( \text{(20)} \), the average value of displacement operator \( q \) is given by

\[
\langle q(t) \rangle_{\omega_m t \ll 1} = \sigma [4 k^2|\alpha|\left|\frac{\omega_m t}{2}\right| \cos \theta + \left(\frac{\omega_m t}{2}\right)^3 \sin \theta)]/[k^2(\omega_m t)^2 + 4 k^2|\alpha|^2(\omega_m t \cos \theta + \left(\frac{\omega_m t}{2}\right)^2 \sin \theta)^2], \tag{25}
\]

which is plotted in Fig. 3 for different coherent state \( |\alpha| = |\alpha|e^{i \theta} \) with \( k = 0.005 \) and \( \omega_m t = 0.001 \).
3.5. Amplification about position variable \( q \) based on the dissipation.

Taking into account of dissipation, the master equation of the mechanical system \([20]\) is given by

\[
\frac{d\rho(t)}{dt} = -\frac{i}{\hbar}[H, \rho(t)] + \frac{\gamma_m}{2}[2c\rho(t)c^\dagger - c^\dagger c\rho(t) - \rho(t)c^\dagger c],
\]

(26)

where \( \gamma_m \) is the damping constant. And the above master equation have an analytical solution for the evolution of the mechanical system, which is obtained by the method in Ref. \([20]\).

If steps corresponding to those of Eqs. \([14] - [16]\) are carefully carried out in this case, then the final state of mirror becomes

\[
\rho_m(t) = \frac{1}{4} [\left| \varphi_1(\gamma, t) \right>_m \left\langle \varphi_1(\gamma, t) \right|_m
- e^{i\varphi(t)+i\varphi(\alpha,t)/2-D(\gamma,t)} \left| \varphi_1(\gamma, t) \right>_m \left\langle \varphi_0(\gamma, t) \right|_m
- e^{-i\varphi(t)-i\varphi(\alpha,t)/2-D(\gamma,t)} \left| \varphi_0(\gamma, t) \right>_m \left\langle \varphi_1(\gamma, t) \right|_m
+ \left| \varphi_0(\gamma, t) \right>_m \left\langle \varphi_0(\gamma, t) \right|_m],
\]

(27)

where \( \gamma = \gamma_m/\omega_m \) and \( \varphi_n(\gamma, t) = \alpha e^{-(i+\gamma/2)\omega_m t} + \frac{ikn}{i+\gamma/2}(1 - e^{-(i+\gamma/2)\omega_m t}) \) (n =
Figure 4: For $|\alpha| = \frac{1}{\sqrt{2}}$ and $\theta = \pi/4$, (a) the average displacement $\langle q(t) \rangle / \sigma$ is as a function of $\omega_m t$ with $k = 0.005$, $\gamma = 0$ (solid line) and $\gamma = 0.005$ (dashed line). (b) the average displacement $\langle p(t) \rangle / \hbar(2\sigma)^{-1}$ is as a function of $\omega_m t$ with $k = 0.005$, $\gamma = 0$ (solid line) and $\gamma = 0.005$ (dashed line).

0, 1) are the amplitude of the coherent states of the mirror and

$$D(\gamma, t) = \frac{k^2 \gamma}{2(1 + \gamma^2/4)} [\omega_m t + \frac{1 - e^{-\gamma \omega_m t}}{\gamma} - \frac{e^{(i-\gamma/2)\omega_m t} - 1}{i - \gamma/2} + \frac{e^{-(i-\gamma/2)\omega_m t} - 1}{i + \gamma/2}] + \frac{1}{\sqrt{2}} \omega_m t + \frac{1}{\sqrt{2}} \omega_m t.$$

(28)

The relative phase $e^{i\phi(t) + i\phi(\alpha, t)/2}$ between the coherent states $|\varphi_1(\gamma, t)\rangle$ and $|\varphi_0(\gamma, t)\rangle$ is composed of the Kerr phase $e^{i\phi(t)}$ and the phase $e^{i\phi(\alpha, t)/2}$ obtained by interchanging two displacement operators.

Substituting Eq. (27) into Eq. (20) using $\rho_m(t)$ and $|\varphi_0(\gamma, t)\rangle$ instead of $|\chi_{0s}(t)\rangle$ and $|0\rangle$, respectively. As a result, we have

$$\langle q(t) \rangle = \sigma [\varphi_1(\gamma, t) + \varphi_0^* (\gamma, t) - e^{-|\varphi_1(\gamma, t) - \varphi_0(\gamma, t)|^2} \times (e^{i\phi(t) + i\phi(\alpha, t)/2 + i\tau - D(\gamma, t)} (\varphi_1(\gamma, t) + \varphi_0^*(\gamma, t)) + e^{-(i\phi(t) + i\phi(\alpha, t)/2 + i\tau - D(\gamma, t)} (\varphi_1^*(\gamma, t) + \varphi_0(\gamma, t)) + \varphi_0(\gamma, t)) + \varphi_0(\gamma, t) + \varphi_0^*(\gamma, t))] / 2$$

$$- e^{-|\varphi_1(\gamma, t) - \varphi_0(\gamma, t)|^2} \times (e^{i\phi(t) + i(\alpha, t)/2 + i\tau - D(\gamma, t)} (\varphi_1(\gamma, t) + \varphi_0^*(\gamma, t)) + e^{-(i\phi(t) + i(\alpha, t)/2 + i\tau - D(\gamma, t)} (\varphi_1^*(\gamma, t) + \varphi_0(\gamma, t))].$$

(29)

where $\tau = \text{Re}\varphi_0(t)\text{Im}\varphi_1(t) - \text{Im}\varphi_0(t)\text{Re}\varphi_1(t)$.

The average displacement $\langle q(t) \rangle / \sigma$ of the mirror is shown in Fig. 4(a) as a function of $\omega_m t$ with $k = 0.005$, $\gamma = 0$ (solid line) and $\gamma = 0.005$ (dashed line).
line). It can be seen from Fig. 4(a) that all the amplification values in the presence of the damping are reduced (dashed line), but the actual $\gamma$ can be very small ($\gamma = 5 \times 10^{-7}$ in [17]). The result for $\gamma = 5 \times 10^{-7}$ is almost the same as the one for $\gamma = 0$.

### 3.6. Amplification about momentum variable $p$ via coherent state pointer

Next, without loss of generality, we would like to discuss the amplification of the momentum variable $p$ of the mirror in presence of damping. Substituting Eq. (27) into Eq. (20) using $p$ instead of $q$, then we have

$$
\langle p(t) \rangle = -i \frac{\hbar}{2\sigma} [\varphi_1(\gamma, t) - \varphi_1^*(\gamma, t)]
- e^{-\frac{|\varphi_1(\gamma, t) - \varphi_0(\gamma, t)|^2}{2}}
\left[ e^{i\phi(t) + \phi(\alpha, t)+\tau - D(\gamma, t)} \left( \varphi_1(\gamma, t) - \varphi_0(\gamma, t) \right)
+ e^{-(i\phi(t) + \phi(\alpha, t)+\tau - D(\gamma, t))} \left( \varphi_1^*(\gamma, t) - \varphi_0^*(\gamma, t) \right)
+ \varphi_0(\gamma, t) - \varphi_0^*(\gamma, t) \right]/[2 - e^{-\frac{|\varphi_1(\gamma, t) - \varphi_0(\gamma, t)|^2}{2}}]
\left[ e^{i(\phi(\alpha, t)+\tau) - D(\gamma, t)} + e^{-i(\phi(\alpha, t)+\tau) - D(\gamma, t)} \right].
$$

(30)

The average displacement $\langle p(t) \rangle/\hbar(2\sigma)^{-1}$ of the mirror momentum is shown in Fig. 4(b) as a function of time $\omega_m t$ with $k = 0.005$, $\gamma = 0$ (solid line) and $\gamma = 0.005$ (dashed line). It can be seen that the maximal amplifications occur at time $\omega_m t = (2n + 1)\pi$ ($n = 0, 1, 2, \ldots$) and can reach the level of the ground state fluctuation [21] $\langle p \rangle = -\hbar/2\sigma$. So the amplification of the
displacement of the mirror momentum caused by one photon can be detected in principle. We can also see from Fig. 4(b) that all the amplification values are reduced (dashed line) in the presence of damping. But because the actual damping $\gamma$ ($\gamma = 5 \times 10^{-7}$ in [17]) in optomechanical cavity is very small, so the amplifications is barely affected by the damping.

Based on Eq. (18), we can also make a small quantity expansion about time $\omega_m t = \pi$ till the second order. Suppose that $|\omega_m t - \pi| \ll 1$, and $k \ll 1$, then

$$|\chi_{\cos}(t)\rangle \approx \frac{1}{2} [2k|1\rangle_m + i2k|\alpha\rangle(2\sin\theta - (\omega_m t - \pi) \cos \theta)]|0\rangle_m],$$

where the first term of Eq. (31) are generated by the amplitude $\xi(t)$ of coherent state $|\xi(t)\rangle$, while the second term is due to the relative phase $e^{i\phi(\alpha,t)}$ caused by the noncommutativity of quantum mechanics. Based on Eq. (31), when

$$\pm 2k = 2k|\alpha\rangle(2\sin\theta - (\omega_m t - \pi) \cos \theta),$$

we can obtain the maximal value $\frac{\hbar}{2\sigma}$ and minimal value $-\frac{\hbar}{2\sigma}$, respectively. So the key to understand the amplification is the superposition of the vacuum state and one phonon state of the mirror, which is due to the relative phase $e^{i\phi(\alpha,t)}$ caused by the noncommutativity of quantum mechanics.

Fig. 5(a) shows $\langle p(t)\rangle/\hbar$ for the amplitude $|\alpha| = 4$, the phase $\theta = 0$ (black-dotted line) and $\pi$ (red-solid line). We can see from Fig. 5(a) that the maximal amplification value occur around time $\omega_m t = (2n + 1)\pi$ ($n = 0, 1, 2, \cdots$). It is clear that the maximal amplification around time $\omega_m t = (2n + 1)\pi$ ($n = 0, 1, 2, \cdots$) can be explained by the expression of Eq. (32). Fig. 5(b) shows $\langle p(t)\rangle/\hbar$ for the amplitude $|\alpha| = 1/2$, the phase $\theta = \pi/2$ (black-dotted line) and $3\pi/2$ (red-solid line). In Fig. 5(b), we find that the maximal amplification value occur at time $\omega_m t = (2n + 1)\pi$ ($n = 0, 1, 2, \cdots$). In the same way, it can also be explained by the result of Eq. (32). In one word, the maximal amplification is caused by the equal superposition of $|0\rangle$ and $|1\rangle$, due to the presence of the relative phase caused by the noncommutativity of quantum mechanics.

3.7. Discussion

For the feasibility of the proposed scheme, we consider the experimental experiments from two aspects. First, the mechanical oscillator (mirror)
of our device is initially prepared in coherent state. Coherent state of the mechanical oscillator has been prepared with itinerant microwave fields $\text{[22]}$.

We shall show that how the photon arrival rate density varies with time after the single photon is detected at the dark port. The probability density of a photon being released from the optomechanical cavity after time $\omega_m t$ is

$$\kappa \exp(-\kappa t),$$

where $\kappa$ is the decay rate of the cavity. The probability of a successful post-selection being released after $\omega_m t$ is $\frac{1}{2} (1 - \exp[-\frac{1}{2} |\xi(t)|^2] \cos[\phi(t) + \phi(\alpha, t)])$. For $k \ll 1$, this is approximately

$$\frac{1}{4} [\langle t \rangle^2 + \phi(\alpha, t)^2].$$

Let us multiply Eq. (33) and Eq. (34), then we obtain the photon arrival rate density in optomechanical cavity

$$\frac{\kappa}{4P} \exp(-\kappa t)(|\xi(t)|^2 + \phi(\alpha, t)^2),$$

where $P$ is overall single photon probability of the state in Eq. (18):

$$P = \frac{1}{4} \int_0^\infty \kappa \exp(-\kappa t)(|\xi(t)|^2 + \phi(\alpha, t)^2) dt$$

$$= \frac{k^2 \omega_m^2}{2} (2\kappa^2 + 5\omega_m^2)/(\kappa^4 + 5\kappa^2 \omega_m^2 + 4\omega_m^4)$$

$\text{(36)}$
when $|\alpha| = \frac{1}{2}$ and $\theta = 0$.

Fig. 6 show that the photon arrival rate density. It can be seen clearly that in the bad-cavity limit $\kappa > \omega_m$ and as the decay rate of the cavity $\kappa$ increases, such as $\kappa = 10\omega_m$, the photon arrival rate density increasingly distributed mainly at time $t$ near 0 where it is very narrow. Because of the photon arrival rate density concentrating near the zero time (blue line) in Fig. 6 and the maximal amplification occurring at time $t$ near 0 (blue solid line) in Fig. 2, for a repeated experimental setup with identical conditions, the "average" position displacement of the pointer is given by

\[
\langle q(t) \rangle = \frac{\kappa}{4P} \int_0^\infty \exp(-\kappa t)(|\xi(t)|^2 + \phi(\alpha, t)^2)\langle q(t) \rangle dt \approx 0.98 \sigma, \tag{37}
\]

where $\langle q(t) \rangle$ is the same as $\langle q(t) \rangle$ of Eq. (21). In principle, this result can be experimentally detected since it is almost close to the strong coupling limit $\sigma$. In principle, this result can be experimentally detected since it is almost close to the strong coupling limit $\sigma$.

The second, for the above analysis, we discuss experimental requirements for the optomechanical device. Here we need $k$ is high enough so that the probability of successful postselection is common which depends on the dark count rate of the detector and the stability of the setup. As shown in Eq. (36), the probability of successful postselection in an optomechanical device with $\kappa = 10\omega_m$ is approximately $0.01k^2$. The window in which the detectors will need to be open for photons is approximately $1/\kappa$, leading to a requirement that the dark count rate be lower than $0.01k^2\kappa$. Because the best silicon avalanche photodiode have dark count rate of $\sim 2$ Hz, so we get $k \geq 0.0033$ for a 450 kHz device with $\kappa = 10\omega_m$. Therefore, for Proposed device no. 2 [17], we need to change some parameters, including mechanical frequency $f_m = 450$ kHz and sideband resolution $\kappa = 10\omega_m$, implying that the optical finesse $F$ in Proposed device no. 2 is reduced to $3.33 \times 10^2$. Other parameters do not change. Therefore, the implementation of the scheme provided here is feasible in experiment.

4. Conclusion

In conclusion, we have investigated the weak measurement amplification with a coherent state pointer. It is regarded as a fire-new mechanism because
the relative phases between the pointer states after the postselection can be due to the noncommutativity of quantum mechanics, which is different from the standard weak measurement [1, 6] where the relative phases are prepared through the postselection. We find that the maximal amplification of the displacement of the mirror’s position in optomechanical system can occur near $\omega_m t = 0$, which can’t be achieved if the mirror is initially prepared in the ground state [16]. This result is very important for bad optomechanical cavities, and because of this, the implementation of our scheme is feasible in experiment. So these results extend application of weak measurement in optomechaical system and also deepen our understanding of the weak measurement.

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Appendix A. Weak measurement with a ground state pointer

In Ref. [6], they consider the standard weak measurement model considered in Section II but the initial state of the pointer is assumed to be the ground state $|0\rangle_m$. Suppose the state $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle_s + |1\rangle_s)$ is the initial state of the system to be measured, where $|0\rangle_s$ and $|1\rangle_s$ is eigenstates of $\hat{\sigma}_z$. According to the Hamiltonian of Eq. (2), then the time evolution of the total system is given by

$$e^{-\frac{i}{\hbar}\int \hat{H}dt}|+\rangle |0\rangle_m = \exp[-\eta \hat{\sigma}_z (\hat{c} - \hat{c}^\dagger)] |+\rangle |0\rangle_m$$

$$= \frac{1}{\sqrt{2}} (|0\rangle_s D(\eta)|0\rangle_m$$

$$+ |1\rangle_s D(-\eta)|0\rangle_m),$$

(A.1)

where $D(\eta) = \exp[\eta \hat{c}^\dagger - \eta^* \hat{c}]$ with $\eta = \frac{\hbar \chi}{2\sigma}$ is a displacement operator and $\eta \ll 1$. In the weak measurement regime [1] the post-selected state of the system is closely orthogonal to the initial state of the system which is usually
chosen as $\varepsilon|+\rangle + |-\rangle$, where $|\varepsilon| \ll 1$. After postselection the final state of the pointer became

$$|\psi\rangle_m = \frac{1}{\sqrt{2}}[(1 + \varepsilon)D(\eta)|0\rangle_m - (1 - \varepsilon)D(-\eta)|0\rangle_m].$$

(A.2)

When $|\varepsilon| \ll 1$ and $\eta \ll 1$, there is

$$|\psi\rangle_m \approx \frac{1}{2}[(1 + \varepsilon)(1 - \eta \hat{\sigma}_z(\hat{c} - \hat{c}^\dagger))|0\rangle_m - (1 - \varepsilon)(1 - \eta \hat{\sigma}_z(\hat{c} - \hat{c}^\dagger))|0\rangle_m] \approx \varepsilon|0\rangle + \eta|1\rangle.$$

(A.3)

Noted that the tiny relative phase $\varepsilon$ arise from a near-orthogonal postselection on the system. Using the expression of the pointer’s displacement

$$\langle \hat{q} \rangle = \frac{\langle \psi|_m \hat{q} |\psi\rangle_m}{\langle \psi| \psi \rangle_m} - \langle 0|_m \hat{q} |0\rangle_m,$$

(A.4)

and

$$\langle \hat{p} \rangle = \frac{\langle \psi|_m \hat{p} |\psi\rangle_m}{\langle \psi| \psi \rangle_m} - \langle 0|_m \hat{p} |0\rangle_m.$$

(A.5)

Hence in this case of the near-orthogonal postselection, i.e., $\langle -|+ \rangle \neq 0$, we can find that

$$\langle \hat{q} \rangle = \frac{2\varepsilon\eta}{\varepsilon^2 + \eta^2}\sigma$$

(A.6)

and

$$\langle \hat{p} \rangle = 0.$$

(A.7)

When $\varepsilon = \pm \eta$ we will have the largest displacement $\pm \sigma$ in position space and when $\varepsilon = 0$, indicating that the post-selected state of the system is absolutely orthogonal to the initial state of the system, i.e., $\langle -|+ \rangle = 0$, the displacement of pointer position is zero. However, the displacement of the pointer is always zero in momentum space.

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