On the Zakharov—L’vov Stochastic Model for Wave Turbulence

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Abstract—In this paper we discuss a number of rigorous results in the stochastic model for wave turbulence due to Zakharov—L’vov. Namely, we consider the damped/driven (modified) cubic nonlinear Schrödinger equation on a large torus and decompose its solutions to formal series in the amplitude. We show that when the amplitude goes to zero and the torus’ size goes to infinity the energy spectrum of the quadratic truncation of this series converges to a solution of the damped/driven wave kinetic equation. Next we discuss higher order truncations of the series.

Keywords: wave turbulence, energy spectrum, wave kinetic equation, kinetic limit, nonlinear Schrödinger equation, stochastic perturbation

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1. INTRODUCTION

Proofs of the assertions, stated below without reference, may be found in works [2, 3].

1.1. The Model

Let $L \geq 1$ and $\mathbb{T}^d_L = \mathbb{R}^d / (L \mathbb{Z}^d)$ be a torus of dimension $d \geq 2$. Denote by $H$ the space $L_2(\mathbb{T}^d_L; \mathbb{C})$ with respect to the normalized Lebesgue measure:

$$\|u\|^2 = \|u\|_{L_2(\mathbb{T}^d_L)}^2 = \langle u, u \rangle, \quad \langle u, v \rangle = L^d \mathbb{R} \int_{\mathbb{T}^d_L} u \overline{v} \, dx. \quad (0.1)$$

Consider the modified NLS equation

$$\frac{\partial}{\partial t} u + i\Delta u - i\nu (\|u\|^2 - \|\xi\|^2) u = 0,$$

$$\Delta = (2\pi)^{-2} \sum_{j=1}^d \frac{\partial^2}{\partial x_j^2}, \quad x \in \mathbb{T}^d_L,$$

where $\nu \in \left(0, \frac{1}{2}\right]$. This is a hamiltonian PDE, obtained by modifying the standard cubic NLS equation by another hamiltonian equation whose flow commutes with that of the cubic NLS. The modified NLS can be obtained from the standard cubic equation by the substitution $u = \exp(i\nu \|u\|^2 u')$. This modification is rather often used by people, working on hamiltonian PDEs since it keeps the main features of NLS, reducing some non-critical technicalities. The role of the modification is to remove from the Hamiltonian of the NLS its integrable part (see below the second footnote). If instead the cubic equation we considered the quadratic NLS, corresponding to a three–wave system, the modification would not be needed.

We will write solutions $u$ as $u(t, x) \in \mathbb{C}$ or as $u(t) \in H$. Passing to the slow time $\tau = \nu t$ we re-write the equation as

$$\dot{u} + i\nu^{-1} \Delta u - i\|u\|^2 - \|\xi\|^2 u = 0,$$

$$\dot{u} = \frac{\partial}{\partial \tau} u(\tau, x), \quad x \in \mathbb{T}^d_L. \quad (1.1)$$

The objective of Wave Turbulence (WT) is to study solutions of (1.1) when

$$\nu \to 0, \quad L \to \infty. \quad (1.2)$$
We will write the Fourier series for \( u(x) \) in the form
\[
u(x) = L^{-d/2} \sum_{s \in \mathbb{Z}^d} \nu_s e^{2\pi is \cdot x}, \quad \mathbb{Z}^d = L^{-1} \mathbb{Z}^d,
\]
where \( \nu_s := \hat{u}(s) = L^{-d/2} \int u(x)e^{-2\pi is \cdot x} \, dx \). Then
\[
\| \nu \|^2 = L^{-d} \sum_s |\nu_s|^2 = \| \nu \|^2_{L^2(\mathbb{Z}^d)}.
\]
Denote by \( h \) the Hilbert space \( h = (L_2(\mathbb{Z}^d), \| \cdot \|) \). The Fourier transform defines an isomorphism \( H \to h \), \( u(x) \mapsto (\nu_s = \hat{u}(s)) \).

When studying Eq. (1.1), people from the WT community often talk about “pumping energy to low modes and dissipating it in high modes”. To make this rigorous, Zakharov–L’vov [12] (also see in [4, Section 1.2]) suggested to consider the NLS equation, damped by a (hyper)viscosity and driven by a random force:
\[
\dot{u} + iv^{-1} \Delta u - i\rho \| \nu \|^{2} u = -(-\Delta + 1)^{r} u + \eta^{\gamma}(\tau, x), \quad r_0 > 0,
\]
where \( \rho > 0 \), and the random process \( \eta \) is given by its Fourier series
\[
\eta^{\gamma}(\tau, x) = L^{-d/2} \sum b_s \beta_s^\gamma(\tau)e^{2\pi is \cdot x}.
\]

Here \( \{\beta_s(\tau), s \in Z^d\} \) are standard independent complex Wiener processes, \( b_s > 0 \) fast decay when \( |s| \to \infty \) and are obtained as the restriction to \( \mathbb{Z}^d \) of a positive Schwartz function on \( \mathbb{R}^d \supset \mathbb{Z}^d \). It is known that if \( r_0 \) is sufficiently large then the Cauchy problem for Eq. (1.4) is well-posed. Applying the Ito formula to (1.4) and denoting \( B = L^{-d} \sum b_s^2 \) we get the balance of energy for solutions of Eq. (1.4):
\[
\mathbb{E} \| u(\tau) \|^2 + 2\mathbb{E} \int_0^\tau \| (-\Delta + 1)^{r} u(s) \|^2 \, ds = \mathbb{E} \| u(0) \|^2 + 2B\tau.
\]

We see that \( \mathbb{E} \| u(\tau) \|^2 \) —the averaged energy per volume of a solution \( u \)—is of order one uniformly in \( L \), no matter how big or small \( \rho \) is. Later \( \rho \) will be scaled with \( v \) in such a way that an equation for the distribution of solution’s energy along the spectrum which follows from Eq. (1.4) admits a non-trivial kinetic limit. As we will see this requirement determines the scaling of \( \rho \) uniquely.

Passing to the Fourier presentation, we write (1.4) as
\[
\dot{\nu}_s - iv^{-1} |\nu_s|^2 \nu_s + \gamma \nu_s = ipL^{-d} \sum_{j \neq \pm s} \delta_{s, j}^{12} \nu_j \nu_s + b_s \beta_s, \quad s \in \mathbb{Z}^d,
\]
where \( \delta_{s, j}^{12} = (1 + |s|^2)^{r_0} \). Following the tradition of WT we abbreviate \( \nu_j \) to \( \nu_j \), \( \gamma \) to \( \gamma \) etc., abbreviate \( \sum \) to \( \sum_{j \neq \pm s} \) and denote
\[
\delta_{s, j}^{12} = \begin{cases} 1, & \text{if } s_j + s_2 = s_1 + s_2 - s \text{ and } \{s_1, s_2\} \neq \{s_3, s\} \\ 0, & \text{otherwise.} \end{cases}
\]

In view of the factor \( \delta_{s, j}^{12} \), in the double sum above \( s_j \) is a function of \( s_1, s_2, s \), i.e., \( s_3 = s_1 + s_2 - s \). Using the interaction representation
\[
\nu_j(\tau) = \exp(i\nu^{-1} |\nu_s|^2) u_j(\tau), \quad s \in \mathbb{Z}^d,
\]
we re-write the \( \nu \)-equations as \( a \)-equations:
\[
\dot{a}_s + \gamma a_s = ip \beta \gamma a_s(\nu^{-1} |\nu_s|^2) + b_s \beta_s, \quad s \in \mathbb{Z}^d,
\]
\[
\beta \gamma a_s(\nu^{-1} |\nu_s|^2) = L^{-d} \sum_{j \neq s} \delta_{s, j}^{12} a_s a_j \nu_j e^{i2\pi \delta_{s, j}^{12}}.
\]

Here \( \{\beta_s(\tau), s \in Z^d\} \) is another set of standard independent complex Wiener processes and
\[
\omega_{s, j}^{12} = |s|^2 + |s|^2 - |s|^2 - |s|^2 = -2(s_j - s) \cdot (s_2 - s),
\]

(the last equality holds since \( s_3 = s_1 + s_2 - s \) in view of the factor \( \delta_{s, j}^{12} \)). By \( \gamma \beta \gamma a_s(a, a^2; a^3) \) we will denote the natural poly-linear mapping, corresponding to the 3-homogeneous mapping \( \gamma \beta \gamma a_s \), so \( \gamma \beta \gamma a_s(a, a, a; a) \).

1.2. Background

The energy spectrum of a solution \( u(\tau) \) of Eq. (1.4) is the function
\[
\mathbb{E} \| u(\tau) \|^2 = \mathbb{E} \| u(0) \|^2 + 2B\tau.
\]

Traditionally in the center of attention for people, working on WT, is the limiting behaviour of the function \( n_s(\tau) \) and of correlations of solutions \( a_s(\tau) \) under the limit (1.2). One of the main predictions of WT is

\[1\] i.e., \( \beta_s = \beta_s^\prime + \beta_s^\prime \), where \( \{\beta_s^\prime, s \in Z^d, j = 1, 2\} \) are standard independent real Wiener processes.

\[2\] If Eq. (1.1) is replaced by the standard NLS equation, then \( \delta_{s, j}^{12} \) should be modified to \( \delta_{s, j}^{12} \), obtained by dropping in the definition of \( \delta_{s, j}^{12} \) the condition \( \{s_j, s_2\} \neq \{s_3, s\} \). Then the double sum in the \( \nu \)-equation will be modified by adding the “integrable term” \( ipL^{-d} \left( 2v_s \sum_m |v_m|^2 - v_s |v_s|^2 \right) \).

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that under this limit the energy spectrum \( n_\varepsilon(\tau) \) satisfies a wave kinetic equation (WKE). There are plenty of physical works, containing different (but consistent) approaches to the study of the energy spectrum \( n_\varepsilon \) under the limit \((1.2)\) and to the derivation for it the WKE (e.g., see [10, 11, 13] and references therein; see also introduction to the work [11]). None of them was ever rigorously justified despite the strong interest in mathematical community to the questions, addressed by these works.

Exact meaning of the limit \((1.2)\) is not quite clear. It is known (see in [8]) that for \( \rho \) and \( L \) fixed, Eq. \((1.7)\) has a limit as \( \nu \to 0 \), and it was demonstrated in [8] on the physical level of rigour that if we scale has a limit as , and it was demonstrated in [8] on the physical level of rigour that if we scale \( \nu \to 0 \) as \( \bar{\varepsilon} \sqrt{L} \), \( \bar{\varepsilon} \ll 1 \), then the iterated limit \( L \to \infty \) leads to the WKE. Attempts to justify the latter result rigorously so far have failed.

There are only a few rigorous works, addressing the limit \((1.2)\). In [6] the authors consider the deterministic NLS equation, take \( d = 2 \) and calculate the limit \((1.2)\) in a regime, when \( L \) goes to infinity much slower than \( \nu^{-1} \). The obtained elegant description of the limit is far from the prediction of WT and rather should be regarded as a kind of averaging. In the recent paper [1] the authors study the deterministic NLS equation with \( d \leq 3 \) and random initial data \( u(0,x) \) and choose the phases \( \arg v_j(0), s \in \mathbb{Z}_L^d \), of the Fourier coefficients of \( u(0,x) \) to be independent uniformly distributed random variables. In the notation of our work they prove that under the limit \((1.2)\), if \( L \) goes to infinity slower than \( \nu^{-1} \) but not too slow, then for the values of the slow time \( \tau \) of order \( L^{-\delta}, \delta > 0 \), the energy spectrum \( n_\varepsilon(\tau) \) approximately satisfies a linearization in time at \( \tau = 0 \) of the WKE, with the wave kinetic integral multiplied by \( \nu \).

Another problem of this kind was rigorously treated in [9], where to achieve a progress in the study of the deterministic NLS equation the authors had to replace the space-domain \( \mathbb{T}^d_L \) by the discrete torus \( \mathbb{Z}^d/(L\mathbb{Z}^d) \) and to modify the discrete Laplacian on \( \mathbb{Z}^d/(L\mathbb{Z}^d) \) to a suitable operator, diagonal in the Fourier basis. The randomization was introduced through the initial data by assuming that \( u(0,x) \) is distributed accordingly to the Gibbs measure of the system, so the solution \( u(t) \) is a stationary in time random process in \( H \). A related problem is treated in [5].

1.3. Results

In this work we specify the limit \((1.2)\) as follows:

\[
\nu \to 0 \quad \text{and} \quad L \geq \nu^{-2-\varepsilon} \quad \text{for some} \quad \varepsilon > 0,
\]

or first \( L \to \infty \), and next \( \nu \to 0 \).

The second option formally corresponds to the first one with \( \varepsilon = \infty \). This well agrees with a postulate widely accepted in the physical community that to get a kinetic limit, \( L \) should go to infinity very fast while \( \nu^{-1} \), not so fast.

We supplement Eq. \((4)–(7)\) with the initial condition

\[
u(-T) = 0 \tag{1.9}
\]

for some \( 0 < T \leq +\infty \), and in the spirit of WT decompose a solution of \((1.7)\), \((1.9)\) to formal series in \( \rho \):

\[
\begin{align*}
\bar{a}(\tau) &= a^{(0)}(\tau) + \rho a^{(1)}(\tau) + \ldots, \\
\bar{a}^{(j)}(\tau) &= a^{(j)}(\tau;\nu,L).
\end{align*}
\]

It can be easily seen that in the case \( T = \infty \) the processes \( a^{(j)} \) are stationary. At first, as in physical works (e.g., cf. [10, Section 6.4]) we retain the quadratic in \( \rho \), and accordingly substitute in the equation 

\[
\rho = \nu^{-1/2} \varepsilon^{1/2},
\]

where \( 0 < \varepsilon \leq 1 \) should be regarded as a small constant. Then \( N_\varepsilon \), may be written as \( N_\varepsilon = N_s^\varepsilon(\tau) + \varepsilon N_s^1(\tau) + O(\varepsilon^2) \), where \( N_s^0, N_s^1 \) are uniformly in \( \nu \) and \( L \). Next in Theorem 2 we prove that for \( L \geq \nu^{-2-\varepsilon} \) (cf. \((1.8)\)), the function \( s \mapsto N_s \) naturally extends to a function on \( \mathbb{R}^d \), which is \( \varepsilon^2 \)-close to a solution \( m_\varepsilon(\tau) \) of the damped/driven wave kinetic equation (WKE)

\[
\begin{align*}
\dot{m}_\varepsilon(\tau) &= -2\gamma m_\varepsilon(\tau) + 2\beta^2_s + \varepsilon K_s(m(\tau)), \\
&\quad s \in \mathbb{R}^d, \\
&\quad m(-T) = 0
\end{align*}
\]

for any \( \tau \geq -T \), where \( K_s \) is the wave kinetic integral (see \((4.1)\)). In the last Section 5 we return to the complete decomposition \((1.10)\) of solutions \( a_\varepsilon(\tau) \). Accordingly, we decompose the energy spectrum of \( a_\varepsilon \) as

\[
\begin{align*}
n_s(\tau) &= n_s^0(\tau) + \rho n_s^1(\tau) + \ldots \tag{1.12}
\end{align*}
\]

and analyze this decomposition, scaling as before 

\( \rho = \nu^{-1/2} \varepsilon^{1/2} \).

Since characteristic time in our system is \( \tau \sim 1 \) and the slow time \( \tau \) is defined as \( \tau = \nu^{-1} t \), under the scaling \( \rho = \nu^{-1/2} \varepsilon^{1/2} \) we have \( t \sim \nu^{-1} \sim \text{(size of nonlinearity)}^{-2} \). This time scale coincides with that usually considered by physicists.

The kinetic limit, presented in Section 4, applies to quasisolutions of Eq. \((1.4)\) and we are not sure that the result remains true for exact solutions of the equation. Still we believe that the result and the method of its proof is valid for exact solutions of some other models.
of WT, and we will clarify this in the nearest future.
In this connection let us emphasize that in physical works, devoted to the subject, the WKE is always deduced for quasisolutions (that is, for energy spectrum corresponding to quadratic truncations of the formal series for solutions in amplitude) but not for the energy spectrum of solutions.

Results of Sections 2–4 are proved in [2], and those of Section 5, in [3]. More discussion of the obtained results may be found in [2]. All constants in this work do not depend on $\nu, L, \rho, \epsilon$ and $\tau, T$, unless the dependence is explicitly indicated.

2. SOLUTIONS AS FORMAL SERIES IN $\rho$

As in the introduction, let us decompose a solution $a_s(\tau)$ of (1.7), (1.9) as formal series (1.10). Then

$$\bar{a}^{(0)} + \gamma_s a^{(0)} = b_s \beta_s,$$

so $a^{(0)}$ is the Gaussian process

$$a^{(0)}(\tau) = b_s \int_{-T}^{\tau} e^{-\gamma_s(t-I)} dB_s(l),$$

while $a^{(1)}$ satisfies

$$\bar{a}^{(1)}(\tau) + \gamma_s a^{(1)}(\tau) = \overline{\rho_s}(a^{(0)}(\tau); \nu^{-1}\tau),$$

$$a^{(1)}(\tau) = 0,$$

so

$$a^{(1)}(\tau) = \frac{1}{T} \int_{-T}^{\tau} e^{-\gamma_s(t-I)} \overline{\rho_s}(a^{(0)}(l); \nu^{-1}l) dl$$

(2.1)

is a Wiener chaos of third order. Similar, for $n \geq 2$,

$$a^{(n)}(\tau) + \gamma_s a^{(n)}(\tau) =$$

$$= \sum_{n_1 + n_2 + \cdots + n_{n-1} = n} \overline{\rho_s}(a^{(n_1)}(\tau), a^{(n_2)}(\tau), \ldots, a^{(n_{n-1})}(\tau); \nu^{-1}\tau),$$

so

$$a^{(n)}(\tau) = \frac{1}{T} \sum_{n_1 + n_2 + \cdots + n_{n-1} = n} \int_{-T}^{\tau} e^{-\gamma_s(t-I)} \overline{\rho_s}(a^{(n_1)}(l), a^{(n_2)}(l), \ldots, a^{(n_{n-1})}(l); \nu^{-1}l) dl$$

(2.2)

is a Wiener chaos of order $2n + 1$. We can iterate in the Duhamel integral in the r.h.s. of (2.2) and eventually express $a^{(n)}(l), l \geq -T$ via the processes $a^{(l)}(\tau), l \leq l$.

To analyze the limiting behaviour of correlations of solutions $a_s(\tau)$ and that of the energy spectrum $n_s(\tau)$, written as formal series (1.10) and (1.12), we should analyze the limiting correlations of the processes $a^{(n)}(\tau)$. To give an idea what we should expect there, let us assume for a moment that $T = \infty$ and consider correlations of $a^{(n)}(\tau)$ and $a^{(n')}(\tau)$ with $n, n' \leq 1$. Then obviously,

$$\mathbb{E} a^{(0)}(\tau) a^{(0)}(\tau) = 0,$$

$$\mathbb{E} a^{(0)}(\tau) a^{(0)}(\tau) = \delta_s \epsilon^{-\gamma_s|\tau_T|} \frac{b_{s}^2}{\gamma_s},$$

(2.3)

it also can be shown that $\mathbb{E} a^{(0)}(\tau) a^{(0)}(\tau) = 0$. Denote $B(s) = \frac{b_{s}^2}{\gamma_s}, s \in \mathbb{R}^d$. Then, in view of (2.1), (2.3) and the Wick theorem the correlations of the processes $a^{(1)}(\tau)$ are given by

$$\mathbb{E} a^{(1)}(\tau) a^{(1)}(\tau) = 0,$$

$$\mathbb{E} a^{(1)}(\tau) a^{(1)}(\tau) = \delta_s \epsilon^{-\gamma_s|\tau_T|} \frac{b_{s}^2}{\gamma_s},$$

$$J_s = 2\nu^2 L^{-2d} \sum_{1,2} \gamma_{123} B(s_1, s_2, s_3),$$

(2.4)

where

$$\gamma_{123} = \gamma_1 + \gamma_2 + \gamma_3 + \gamma_s,$$

$$B(s_1, s_2, s_3) = B(s_1)B(s_2)B(s_3)$$

(see [2] for the calculation). The sums $J_s$ may be well approximated by the integrals

$$\mathcal{I}_s = \frac{2\nu^2}{\gamma_s} \int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{\gamma_{123} B(s_1, s_2, s_3)}{4((s_1-s) \cdot (s_2-s))^2 + (\nu \gamma_{123})^2},$$

$$s_3 = s_1 + s_2 - s.$$

Namely,

$$|\mathcal{I}_s - \mathcal{J}_s| \leq C_{s}^2 L^{-2} s^{-2},$$

(2.4)

Here and below by $C_{s}$ we denote various continuous functions of $s$ which decay, as $|s| \to \infty$, faster than any negative degree of $|s|$. Due to (1.8) the r.h.s. is small: it is bounded by $C_{s}^2 s^{-2}$.

The asymptotical behaviour of integrals $\mathcal{I}_s$ is known (see [2, 7]).

**Theorem 2.1.** We have $\mathcal{I}_s = \mathcal{J}_s + O(C_{s}^2 s^{-2})$, where

$$\mathcal{I}_s = \frac{2\nu^2}{\gamma_s} \int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{\gamma_{123} B(s_1, s_2, s_3)}{4((s_1-s) \cdot (s_2-s))^2 + (\nu \gamma_{123})^2},$$

$$s_3 = s_1 + s_2 - s.$$

Here $\Sigma$ is the quadric $\{s_1, s_2, s_3: (s_1-s) \cdot (s_2-s) = 0\}$ and $ds_1 ds_2 l_{\Sigma}$ is the volume element on it, corresponding to the Euclidean structure on $\mathbb{R}^d$.

Asymptotic similar to that in Theorem 2.1 can be obtained for arbitrary $-\infty < T < 0$. Substituting $s_1 = s + x, s_2 = s + y$ and denoting $z = (x, y)$ we re-write $\mathcal{I}_s$ as
\[ \frac{\pi}{\gamma_s} \int_{|z|} B(s + x, s + y, s + x + y) \, dz, \]

\[ \Sigma = \{z = (x, y) : x \cdot y = 0\}. \]

Denote \( F(z) := x \cdot y = -\frac{1}{2} \omega_{\delta}(x+y)^2 \). Then \( |\nabla F(z)| = |z| \), so the integral above is exactly the integral \( \int B\delta(F) \) of the function \( B \) against the delta-function of \( F \), see [13, p. 67]. Since \( F = -\frac{1}{2} \omega_{\delta}(x+y)^2 = -\frac{1}{2} \omega_{\delta}(x+y)^2 \), then neglecting the minus-sign we write \( I_{sL}^\nu \) as

\[ \frac{\pi}{\gamma_s} \int B\delta(F) \, dx \cdot dy = \frac{\pi}{\gamma_s} \int B\delta(B^3(1 + x + y)) \, dx \cdot dy = \frac{\pi}{\gamma_s} \int B\delta(\omega_{\delta}(1 + x + y)) \, dx \cdot dy = \frac{\pi}{\gamma_s} \int B\delta(\omega_{\delta}(1 + x + y)) \, dx \cdot dy. \]

Regarding \( |z|^{-1} \, dz \) as a measure in the space \( \mathbb{R}^{2d} \), supported by the quadric \( \Sigma \), we show in [2] that it may be disintegrated as \( |x|^{-1} \, dx \cdot y \), where \( d_x \) is the Lebesgue measure on the space \( x^{-1} = \{y : y \cdot x = 0\} \).

That is, for a function \( f(z) \) we have

\[ \int f(z) \, dz = \int |x|^{-1} \left( \int f(x, y) \, dx \cdot y \right) \, dx. \tag{2.5} \]

This representation is instrumental to work with integrals of the form \( \int f\delta(F) \) and is used below.

### 3. QUASISOLUTIONS

As it was announced in Introduction, we start with investigating the quadratic in \( \rho \) part of (1.10) which we call a quasisolution \( \Lambda(\tau) = (\Lambda_1(\tau; \nu, L), s) \in \mathbb{R}^{2d} \):

\[ \Lambda_1(\tau) = a_{sL}^0(\tau) + \rho a_{sL}^0(\tau) + \rho^2 a_{sL}^2(\tau). \]

Consider the energy spectrum of \( \Lambda \), \( N_1(\tau; \nu, L) = \mathbb{E} \left| \Lambda_1(\tau) \right|^2 \), and decompose it in series in \( \rho \):

\[ N_1(\tau; \nu, L) = n_{1L}^0(\tau) + \rho n_{1L}^1(\tau) + \rho^2 n_{1L}^2(\tau) + \rho^3 n_{1L}^3(\tau) + \rho^4 n_{1L}^4(\tau), \tag{3.1} \]

where \( n_{1L}(\tau) = n_{1L}(\tau; \nu, L) \). Here \( n_{1L}^0 = \mathbb{E} \left| a_{sL}^0(\tau) \right|^2 - C_s^2 \), and it is easy to see that \( n_{1L}^0 \equiv 0 \). Next, \( n_{1L}^2 = \mathbb{E} \left| a_{sL}^0(\tau) + 2\Re \mathbb{E} a_{sL}^0(\tau) \right|^2 \) which is of order \( \nu \) and is given by Theorem 1 if \( T = \infty \); the second one is similar. So

\[ n_{1L}^0 \sim C_s^2, \quad n_{1L}^1 \equiv 0, \quad n_{1L}^2 \sim C_s^2 \nu. \tag{3.2} \]

It turns out that

\[ |n_{1L}^1|, |n_{1L}^2| \leq C_s^2 \nu^2, \tag{3.3} \]

if \( L \geq \nu^{-2 - \epsilon} \) (see (1.8)).

For any \( \tau \geq -T \), any \( \nu \), and any \( k = 0, \ldots, 4 \) the function \( \tau \mapsto n_{1L}^k(\tau) \) naturally extends to a Schwartz function on \( \mathbb{R}^{d} \). The limit

\[ n_{1L}^k(\tau; \nu, \infty) = \lim_{L \to \infty} n_{1L}^k(\tau; \nu, L) \]

exists, is a Schwartz function of \( s \in \mathbb{R}^{d} \) and satisfies (3.2), (3.3). Accordingly, the limiting energy spectrum of a quasisolution, \( N_1(\tau; \nu, \infty) = \lim_{L \to \infty} N_1(\tau; \nu, L) \), also exists and is a Schwartz function of \( s \in \mathbb{R}^d \).

Relations (3.2) and (3.3) suggest that the right scaling for \( \rho \) is \( \rho \sim \nu^{1/2} \), and we choose \( \rho \) to be of the form

\[ \rho = \nu^{-1/2} \epsilon^{1/2}, \quad \epsilon \in (0, 1]. \tag{3.4} \]

With this choice of \( \rho \) the process \( N_1(\tau) \) is an \( \epsilon \)-small perturbation of the linear process \( n_{1L}^0 \) and does not converge to \( n_{1L}^0 \) under the limit (1.8): decomposition (3.1) takes the form \( N_1 = n_{1L}^0 + \epsilon N_1^1 + O(\epsilon^2) \), where \( N_1^1 = \nu^{-1} n_{1L}^2 C_s^2 \).

### 4. THE WAVE KINETIC EQUATION

For a real function \( s \in \mathbb{R}^d \), consider the corresponding cubic wave kinetic integral:

\[ K_s(y) = 2\pi \int_{\mathbb{R}^d} \frac{ds_1 ds_2 ds_3 y_1 y_2 y_3 y_4}{\sqrt{|s_1 - s|^2 + |s_2 - s|^2}} \times \left( \frac{1}{y_1} + \frac{1}{y_3} - \frac{1}{y_2} - \frac{1}{y_4} \right), \tag{4.1} \]

where for \( j = 1, 2, 3 \) we denote \( y_j = y_j^s \) and where \( s_j = s_1 + s_2 - s \). Noting the notation, evoked after Theorem 2.1, the integral above may be written as

\[ 4\pi \int_{\mathbb{R}^d} \frac{ds_1 ds_2 ds_3 y_1 y_2 y_3 y_4}{\sqrt{|s_1 - s|^2 + |s_2 - s|^2}} \times \left( \frac{1}{y_1} + \frac{1}{y_3} - \frac{1}{y_2} - \frac{1}{y_4} \right) \delta(s_{1L}^2 s_{2L}^2 s_3 s_4) ds_1 ds_2 ds_3 ds_4. \]

This integral exactly coincides with the kinetic integral, used by physicists to describe WT for the 4–waves interaction, see [13, p. 71; 10, p. 91].

Consider the function spaces \( C_s(\mathbb{R}^d) = \{s \in C(\mathbb{R}^d) : |x| = \sup \{1 + |s| \, |x(s)| < \infty\} \} \). The representation (2.5) for the measure \( |x|^{-1} \, dz \mid_\Sigma \) implies that the wave kinetic integral \( K \) defines a continuous operator

\[ K \]
$K: C_r(\mathbb{R}^d) \to C_{r+1}(\mathbb{R}^d), \; y_s \mapsto K_y(y),$ provided that $r > d.$ Now let us consider the wave kinetic equation:

$$\dot{m}_s(\tau) = -2\gamma_t m_s(\tau) + 2h^2 + \varepsilon K_s(m(\tau)), \quad s \in \mathbb{R}^d. \tag{4.2}$$

There exists $\varepsilon_\ast > 0$ such that if $0 \leq \varepsilon \leq \varepsilon_\ast,$ then (4.2) has a unique solution $m^*(\tau),$ vanishing at $\tau = -T$ and defining a bounded continuous curve $m^*: [-T, \infty) \mapsto C_r(\mathbb{R}^d),$ in any space $C_r(\mathbb{R}^d).$ It may be written as

$$m^*_s(\tau) = m_s^m(\tau) + \varepsilon m_s^1(\tau, \varepsilon), \quad m_s^m(-T) = m_s^m(\tau) = 0,$$

where $m_s^m, m_s^1 \leq 1,$ $m_s^m(\tau)$ equals $n_s^m(\tau)$ and satisfies the linear equation

$$\dot{m}_s^m(\tau) = -2\gamma_t m_s^m(\tau) + 2h^2. \tag{4.3}$$

Everywhere below $\varepsilon$ is a fixed small constant, independent from $\nu$ and $L,$ satisfying $\varepsilon \in (0, \varepsilon_\ast].$ The parameter $\varepsilon$ should be interpreted as the squared amplitude of a quasisolution $A(\tau),$ written in a right scaling. The following theorem is the main result of [2].

**Theorem 4.1.** Let in (1.4) $L \geq \nu^{-2-\varepsilon}$ and $\rho = \nu^{-1/2} \varepsilon^{1/2}.$ Then the energy spectrum $N_s(\tau, \nu, L)$ of a quasisolution $A_s$ is $\varepsilon^2$-close to the solution $m^*$ of (4.2) in the sense that for any $r$

$$|m^s - N_s(\tau)| \leq C_r \varepsilon^2, \quad \forall \tau \geq -T,$$

with some $C_r > 0,$ provided that $0 < \nu \leq \nu_{\epsilon,r}$ for a suitable $\nu_{\epsilon,r} > 0.$ The limiting energy spectrum $N_s(\tau, \nu, \infty)$ also satisfies the estimate above for any $r$ and for $0 < \nu \leq \nu_{\epsilon,r}.$

Equation (4.3) has the unique steady state $m^0,$ $m^0_s = \frac{b^2}{\gamma_t},$ which is asymptotically stable. By the implicit function theorem, for $\varepsilon$ sufficiently small Eq. (4.2) has a unique steady state $m^0$ close to $m^0,$ which is asymptotically stable. Decreasing $\varepsilon_\ast$ if needed we may assume that the unique $m^0$ exists for $\varepsilon \leq \varepsilon_\ast.$

Jointly with Theorem 4.1 this result describes the asymptotic in time behaviour of the energy spectrum $N_s(\tau);$ for any $r > d,$

$$|m^e - N_e(\tau)| \leq C_r (|m^e| e^{-\gamma t} + \varepsilon^2), \quad \forall \tau \geq -T. \tag{4.4}$$

Due to Theorem 2.1 together with (2.4) and some modifications of these results, the iterated limit

$$\lim_{\nu \to 0} \lim_{L \to \infty} \nu^{-1/2} \varepsilon^{1/2}$$

exists and $\nu$ is the limiting kinetic energy spectrum.

5. ENERGY SPECTRA OF SOLUTIONS $a_s(\tau),$ WRITTEN AS FORMAL SERIES IN $\rho$

Let us come back to the decomposition (10). As it was mentioned in Section 2, iterating the integral (2.2) we may express each $a_s^{(n)}(l_0)$ via the processes $a_s^{(0)}(l),$ $l \leq l_0.$ Then $a_s^{(n)}$ represents as the sum

$$a_s^{(n)}(l_0) = \sum_{\beta \in \Gamma(n)} I_s(l_0; n, \beta), \tag{5.1}$$

where the meaning of the summation index $\beta$ is explained below and $I_s(\beta) := I_s(l_0; n, \beta)$ is an iterated integral of the form

$$I_s(\beta) = \int \ldots \int L^{-n} \sum_{l_n \ldots l_0} (\ldots) dl_1 \ldots dl_n. \tag{5.2}$$

The integrating zone in (5.2) is a convex polyhedron in $[-T, l_0]^n.$ The summation is taken over vectors $s_1, \ldots, s_n \in \mathbb{Z}^2_\nu$ which are subject to the linear relations, following from the factor $\partial_3^2 \nu^3$ in the definition of $Y_s.$ The summand $(\ldots)$ in (5.2) is a product of functions $e^{-\gamma_t (l_j - l_i)},$ $\exp(\pm i\nu \partial_{33}^3)$ and the processes $[a_s^{(0)}(l_i)]^\ast,$ where $a^\ast$ is either $a$ or $a^\nu,$ with various indices $k, j, r$ and various $s, s', s''.$ Taken from the set $\{s_1, \ldots, s_n\}.$ It is of degree $2n + 1$ with respect to the process $a_s^{(0)}.$ Each integral $I_s(l_0; n, \beta)$ corresponds to an oriented rooted tree $T$ from a class $\Gamma(n)$ of trees with the root at $a_s^{(n)}$ with random variables $[a_s^{(0)}(l_i)]^\ast$ at its leaves, and with vertices labeled by symbols $[a_s^{(n)}(l_i)]^\ast$ with $1 \leq n' < n$ (see Fig. 1). To each vertex enters one edge of the tree.
and three edges outgo from it. For a vertex, labeled by some \( a_i^{(\eta)}(l) \), \( \eta \geq 1 \), the three edges outgo to the vertices, corresponding to a choice of the three terms \( a_i^{(n_1)} \), \( a_i^{(n_2)} \), \( a_i^{(n_3)} \) in the decomposition (2.2) of \( a_i^{(n)}(\tau) := a_i^{(\eta)}(l) \).

Accordingly we write the energy spectrum of a solution \( a \) as formal series (1.12). There \( n_0^1 = 1, n_0^2 = 0 \), and \( n_k^2 \) are the same as in (3.1), but \( n_k^0 \) and \( n_k^1 \) are different; this small ambiguity should not cause a problem. The new coefficients \( n_k^0 \) and \( n_k^1 \) still meet (3.3) (see below). Let us consider any \( n_k^0, n_k^1 \). It equals

\[
\sum_{k+\bar{k} = k} R_{j\bar{j}}(\tau) \quad \text{for} \quad d \geq 3.
\]

Here given by the finite sum (5.1), parametrized by the trees \( \mathcal{I} \in \Gamma(k) \). Let us consider any \( \tau \). It equals the new coefficients \( \tau \) and \( \tau \) still meet (3.3)

\[
\sum_{k+\bar{k} = k} (d)^{k+\bar{k}} \mathcal{I}(\tau ; k, \mathcal{I}),
\]

ferent; this small ambiguity should not cause a prob-

and \( \tau \) are the same as in (3.1), but \( \tau \) and \( \tau \) are dif-

where \( \mathcal{I} \) is a product of the terms in the brackets from the

As for \( \tau \) the Gaussian variables \( \tau \) and \( \tau \) are
different, then the summation in (5.3) is

uncorrelated, then the summation \( s \cdots s \) in (5.3) is taken only over those vectors \( (s_1, \ldots, s_k) \) for which all Wick-paired variables \( a_i^{(0)} \) and \( \mathcal{A}_i^{(0)} \) are uncorrelated, then the summation \( s \cdots s \) in (5.3) is taken only over those vectors \( (s_1, \ldots, s_k) \) for which all Wick-paired variables \( a_i^{(0)} \) and \( \mathcal{A}_i^{(0)} \) have equal indices \( s = s' \). Thus, in every Feynman diagram for (5.3) each leaf \( a_i^{(0)}(l) \) of \( \mathcal{T}_1 \) is paired either with a leaf \( \mathcal{A}_i^{(0)}(l) \) of \( \mathcal{T}_1 \), or with a leaf \( \mathcal{A}_i^{(0)}(l) \) of \( \mathcal{T}_2 \), etc. We have seen that

\[
n_k^i(\tau) = \sum_{\mathcal{I} \in \Gamma(k)} \mathcal{F}_k(\tau, k, \mathcal{I}), \quad (5.4)
\]

where the sum is taken over the set \( \mathcal{H}(k) \) of Feynman diagrams, associated to all possible pairings of the trees \( \mathcal{T}_1 \in \Gamma(k_1) \) and \( \mathcal{T}_2 \in \Gamma(k_2) \), \( k_1 + k_2 = k \), via their leaves.

Resolving all the restrictions, imposed on the indi-

where \( \alpha^\mathcal{F} = (\alpha^\mathcal{F}) \) is a skew-symmetric (constant) matrix

without zero lines and rows. Its rank is at least two, and for some diagrams \( \mathcal{F} \) it equals two. Moreover, each function \( s \mapsto \mathcal{F}_s(\tau, k, \mathcal{F}) \) naturally extends to a Schwartz function on \( \mathbb{R}^d \), and after this extension (5.5) holds for every \( s \in \mathbb{R}^d \). Consequently, for any fixed \( \nu > 0 \),

\[
\lim_{L \to \infty} \mathcal{F}_s(\tau, k, \mathcal{F}) = J_s(\tau, k, \mathcal{F})
\]

\[
\forall \tau \geq -T, \quad s \in \mathbb{R}^d, \quad k \geq 1, \quad \mathcal{F} \in \mathcal{H}(k).
\]

Then in view of (5.4) we have

\[
n_k^i(\tau, \nu, \infty) := \lim_{L \to \infty} n_k^i(\tau, \nu, L) = \sum_{\mathcal{I} \in \mathcal{H}(k)} J_s(\tau, k, \mathcal{F}),
\]

for all \( k, s, \) and \( \tau \).

Relations (3.2) and (3.3) suggest to assume that

\[
|n_k^i| \leq C^\eta(k)^{\nu 2^{i/2}} \quad \text{for} \quad \forall \nu \quad (5.6)
\]
for every \( k \), if \( L \) is sufficiently big in terms of \( \nu^{-1} \) and \( k \).
In this direction we have the two theorems below.

**Theorem 5.1.** For each \( k \) and each \( \mathcal{F} \in \hat{\mathcal{F}}(k) \),
\[
|\mathcal{J}_s(\tau,k,\mathcal{F})| \leq C_s^u(k)\nu^{\min(k/2,d)} \quad \forall \tau \geq -T,
\]
(5.7)
where \( \lceil k/2 \rceil \) is the smallest integer \( \geq k/2 \).

By (5.5) if \( L \) is so big that
\[
L^{-2}\nu^{-2} \leq \nu^{\min(k/2,d)},
\]
(5.8)
then \( \mathcal{J}_s \) also satisfies (5.7), and in view of (5.4) \( n^k_s(\tau) \) is bounded by the r.h.s. of (5.7), multiplied by \( |\hat{\mathcal{F}}(k)| \). So (5.6) holds true for \( k \leq 4 \) since \( d \geq 2 \). But for \( k \) large in terms of \( d \) the upper estimate (5.7) is worse then the desired bound (5.6), and our next result shows that estimate (5.7) is sharp in the sense that in the exponent in the r.h.s. of (5.7), \( \min(\lceil k/2 \rceil,d) \) cannot be replaced by \( \lceil k/2 \rceil \).

Let \( \hat{\mathcal{F}}_s(k) \subset \hat{\mathcal{F}}(k) \) be a set of Feynman diagrams \( \mathcal{F} \),
for which the matrix \( a^\beta \) from (5.5) has exactly one nonzero row and column. This set is not empty.

**Theorem 5.2.** If \( k > 2d \), then for any \( \mathcal{F} \in \hat{\mathcal{F}}_s(k) \) we have
\[
|\mathcal{J}_s(\tau,k,\mathcal{F})| \sim \nu^d C_s^u(k) \gg \nu^{\lceil k/2 \rceil} C_s^u(k). \quad \text{But in the same time,}
\]
\[
\sum_{\mathcal{F} \in \hat{\mathcal{F}}_s(k)} |\mathcal{J}_s(\tau,k,\mathcal{F})| \leq \nu^{k-1} C_s^u(k) \ll \nu^{\lceil k/2 \rceil} C_s^u(k).
\]
(5.9)

It is plausible that the cancellation, leading to the validity of (5.9), is a general fact, and we suggest the following problem.

**Problem 5.1.** Prove that
\[
\sum_{\mathcal{F} \in \hat{\mathcal{F}}_s(k)} |\mathcal{J}_s(\tau,k,\mathcal{F})| \leq C_s^u(k)\nu^{k/2} \quad \text{for all } k \text{ and all } \nu. \quad \text{In particular,}
\]
\[
|\mathcal{J}_s(\tau,k,\mathcal{F})| \leq C_s^u(k)\nu^{k/2} \text{ and (5.6) holds if } L \text{ is sufficiently big in terms of } \nu^{-1} \text{ and } k.
\]

If this conjecture is true, then under the substitution (3.4) the limiting decomposition \( n_1(\tau;\nu,\infty) = n^0_1(\tau;\nu,\infty) + \nu n^1_1(\tau;\nu,\infty) + \ldots \) becomes a formal series in \( \sqrt{\epsilon} \), uniformly in \( \nu \). So for any \( M \geq 2 \) its truncation of order \( M \) is \( n_{1,M}(\tau;\nu,\infty) = n^0_1(\tau;\nu,\infty) + \ldots + \nu^M n^M_1(\tau;\nu,\infty), \) is \( \epsilon^2\)-close to \( N(\tau;\nu,\infty) \) and also meets the assertion of Theorem 4.1. It is unclear for us if for a large \( M \) the truncation \( n_{1,M}(\tau) \) satisfies Eq. (4.2) with an accuracy, better than \( \epsilon^3 \).

On the contrary, if the conjecture in Problem 5.1 is wrong in the sense that
\[
\sup_{\tau \geq -7} \|n^k_s(\tau;\nu,L)\| \geq C \nu^{k/2-k}, \quad \kappa > 0,
\]
then (1.12) with \( \rho = \nu^{-1/2} \epsilon^{-1/2} \) is not a formal series in \( \sqrt{\epsilon} \) uniformly in \( \nu \). We do not rule out this possibility since NLS equations appear in physics as models for small oscillations in various media, obtained by neglecting in the exact equations terms of high order in the amplitude. So it is not impossible that the kinetic limit holds for the energy spectra of quasisolutions, but not for the exact energy spectrum or for the energy spectrum of high order in \( \rho \) truncations of the series (1.10).

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