GEOMETRIC REALIZATIONS OF TRICATEGORIES

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Abstract. The homotopy theory of higher categorical structures has become a relevant part of the machinery of algebraic topology and algebraic K-theory. This paper contains some contributions to the study of classifying spaces for tricategories, with applications to the homotopy theory of categories, monoidal categories, bicategories, braided monoidal categories, and monoidal bicategories. Any tricategory characteristically has associated various simplicial or pseudo-simplicial objects. This paper explores the relationship amongst three of them: the pseudo-simplicial bicategory so-called Grothendieck nerve of the tricategory, the simplicial bicategory termed its Segal nerve, and the simplicial set called its Street geometric nerve, and it proves the fact that the geometric realizations of all of these possible candidate ‘nerves of the tricategory’ are homotopy equivalent. Any one of these realizations could therefore be taken as the classifying space of the tricategory. Our results provide coherence for all reasonable extensions to tricategories of Quillen’s definition of the classifying space of a category as the geometric realization of the category’s Grothendieck nerve. Many properties of the classifying space construction for tricategories may be easier to establish depending on the nerve used for realizations. For instance, by using Grothendieck nerves we state and prove the precise form in which the process of taking classifying spaces transports tricategorical coherence to homotopy coherence. Segal nerves allow us to obtain an extension to bicategories of the results by Mac Lane, Stasheff, and Fiedorowicz about the relation between loop spaces and monoidal or braided monoidal categories by showing that the group completion of the classifying space of a bicategory enriched with a monoidal structure is, in a precise way, a loop space. With the use of geometric nerves, we obtain genuine simplicial sets whose simplices have a pleasing geometrical description in terms of the cells of the tricategory and, furthermore, we obtain an extension of the results by Joyal, Street, and Tierney about the classification of braided categorical groups and their relationship with connected, simply connected homotopy 3-types, by showing that, via the classifying space construction, bicategorical groups are a convenient algebraic model for connected homotopy 3-types.

1. Introduction and summary

The process of taking classifying spaces of categorical structures has shown its relevance as a tool in algebraic topology and algebraic K-theory, and one of the main reasons is that the classifying space constructions transport categorical coherence to homotopic coherence. We can easily stress the historical relevance of the construction of classifying spaces by recalling that Quillen [35] defines a higher algebraic K-theory by taking homotopy groups of the classifying spaces of certain categories. Joyal and Tierney [26] have shown that Gray-groupoids are a suitable framework for studying homotopy 3-types. Monoidal categories were shown by Stasheff [42] to be algebraic models for loop spaces, and work by May [32] and

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Segal [37] showed that classifying spaces of symmetric monoidal categories provide the most noteworthy examples of spaces with the extra structure required to define an Ω-spectrum, a fact exploited with great success in algebraic K-theory.

This paper contains some contributions to the study of classifying spaces for tricategories \( T = (\mathcal{T}, a, l, r, \pi, \mu, \lambda, \rho) \), introduced by Gordon-Power-Street in [17]. Since our results find here direct applications to monoidal categories, bicategories, braided monoidal categories, or monoidal bicategories, the paper will quite possibly be of special interest to \( K \)-theorists as well as to researchers interested in homotopy theory of higher categorical structures, a theory with demonstrated relevance as a tool for the treatment of an extensive list of subjects of recognized mathematical interest in several mathematical contexts beyond homotopy theory, such as algebraic geometry, geometric structures on low-dimensional manifolds, string field theory, quantum algebra, or topological quantum theory and conformal field theory.

As for bicategories [9], there is a miscellaneous collection of different ‘nerves’ that have been (or might reasonably be) characteristically associated to any tricategory \( T \). This paper explores the relationship amongst three of these nerves: the pseudo-simplicial bicategory called the Grothendieck nerve \( NT = (NT, \chi, \omega) : \Delta^{op} \to \text{Bicat} \), the simplicial bicategory termed the Segal nerve \( ST : \Delta^{op} \to \text{Bicat} \), and the simplicial set called the Street geometric nerve \( \Delta T : \Delta^{op} \to \text{Set} \). Since, as we prove, these three nerve constructions lead to homotopy equivalent spaces, any one of these spaces could therefore be taken as the classifying space \( BT \) of the bicategory. Many properties of the classifying space construction for tricategories, \( T \mapsto BT \), may be easier to establish depending on the nerve used for realizations. Here, both for historical reasons and for theoretical interest, it is appropriate to start with the Grothendieck nerve construction to introduce \( BT \). Let us briefly recall that it was Grothendieck [18] who first associated a simplicial set \( N_C \) to a small category \( C \), calling it its nerve, whose \( p \)-simplices are composable \( p \)-tuples \( x_p \to \cdots \to x_0 \) of morphisms in \( C \). Geometric realization of its nerve is the classifying space of the category, \( BC = |N_C| \). Hence, our first relevant result in the paper shows how Grothendieck nerve construction for categories rises to tricategories, that is, to prove that

"Any tricategory \( T \) defines a normal pseudo-simplicial bicategory

\[
NT = (NT, \chi, \omega) : \Delta^{op} \to \text{Bicat},
\]

whose bicategory of \( p \)-simplices consists of \( p \)-tuples of composable cells,

\[
N_p T = \bigcup_{(x_0, \ldots, x_p) \in \text{Ob}T} T(x_1, x_0) \times \cdots \times T(x_p, x_{p-1}).
\]

If \([q] \xrightarrow{a} [p] \) is any map in the simplicial category \( \Delta \), then the associated homomorphism \( a^* : N_q T \to N_p T \) is induced by the unit \( 1 \to T(x, x) \) and composition \( T(y, z) \times T(x, y) \to T(x, z) \) homomorphisms. The structure pseudo-equivalences \( \chi_{a,b} : b^* a^* \Rightarrow (ab)^* \), for maps \([n] \xrightarrow{b} [q] \xrightarrow{a} [p] \), canonically arise from the structure pseudo-equivalences \( a, l, r \), and \( \pi, \mu, \lambda, \rho \), while the structure invertible modifications \( \chi_{a,b,c} \circ \chi_{b,c} \circ c^* \Rightarrow \chi_{a,b,c} \circ c^* \chi_{a,b,c} \), for maps \([m] \xrightarrow{a} [n] \xrightarrow{b} [q] \xrightarrow{a} [p] \), are canonically provided by the modifications data \( \pi, \mu, \lambda, \rho \) of the tricategory."

Then, heavily dependent on the results by Carrasco-Cegarra-Garzón [10], where an analysis of classifying spaces is performed for lax diagrams of bicategories following the way Segal [37] and Thomason [44] analyzed lax diagrams of categories, we introduce the classifying space \( BT \), of a tricategory \( T \), to be the classifying space of its Grothendieck nerve
$N\mathcal{T}$. Briefly, say that the so-called Grothendieck construction [40] §3.1 on the pseudosimplicial bicategory $N\mathcal{T}$ produces a bicategory $\int \Delta N\mathcal{T}$, whose objects are the pairs $(x,p)$, where $p \geq 0$ is an integer and $x = (x_p \to \cdots \to x_0)$ is an object of the bicategory $N_p\mathcal{T}$, whose hom-categories are

$$\int_\Delta N\mathcal{T}((y,q),(x,p)) = \bigsqcup_{[q] \to [p]} N_q\mathcal{T}(y,a^*x),$$

and whose compositions, identities, and structure constraints are defined naturally. Again, Grothendieck nerve construction on this bicategory, $\int \Delta N\mathcal{T}$, now gives rise to a normal pseudosimplicial category $N(\int \Delta N\mathcal{T}) : \Delta^{op} \to \text{Cat}$, on which once more Grothendieck construction leads to a category, $\int \Delta N(\int \Delta N\mathcal{T})$, whose classifying space is, by definition, the classifying space of the tricategory, that is,

$$\mathcal{B}\mathcal{T} = |N(\int \Delta N(\int \Delta N\mathcal{T}))|.$$ 

The precise behavior of this classifying space construction, $\mathcal{T} \mapsto \mathcal{B}\mathcal{T}$, can be summarized as follows:

"- **Any trihomomorphism between tricategories** $F : \mathcal{T} \to \mathcal{T}'$ **induces a continuous map** $BF : \mathcal{B}\mathcal{T} \to \mathcal{B}\mathcal{T}'$.

- If $F,G : \mathcal{T} \to \mathcal{T}'$ are two trihomomorphisms between tricategories, then any tritransformation, $F \Rightarrow G$, canonically defines a homotopy between the induced maps on classifying spaces, $BF \simeq BG : \mathcal{B}\mathcal{T} \to \mathcal{B}\mathcal{T}'$.

- For any tricategory $\mathcal{T}$, there is a homotopy $B1_\mathcal{T} \simeq 1_{\mathcal{B}\mathcal{T}} : \mathcal{B}\mathcal{T} \to \mathcal{B}\mathcal{T}$ and, for any composable trihomomorphisms $H : \mathcal{T} \to \mathcal{T}'$ and $H' : \mathcal{T}' \to \mathcal{T}''$, there is a homotopy $BH' BH \simeq B(H'H) : \mathcal{B}\mathcal{T} \to \mathcal{B}\mathcal{T}''$.

- **Any triequivalence of tricategories** $\mathcal{T} \to \mathcal{T}'$ **induces a homotopy equivalence on classifying spaces** $\mathcal{B}\mathcal{T} \simeq \mathcal{B}\mathcal{T}'$.

"\n
For instance, we provide a positive answer (long assumed) to the question whether, as a consequence of the coherence theorem for tricategories by Gordon-Power-Street [17], every tricategory is ‘homotopy equivalent’ to a Gray-category. More precisely, the coherence theorem states that, for any tricategory $\mathcal{T}$, there is a Gray-category $G(\mathcal{T})$ with a triequivalence $\mathcal{T} \to G(\mathcal{T})$. Then, it is a result that

"There is an induced homotopy equivalence $\mathcal{B}\mathcal{T} \simeq \mathcal{B}G(\mathcal{T})$.”

To deal with the delooping properties of certain classifying spaces, for any tricategory $\mathcal{T}$, we introduce its Segal nerve $S\mathcal{T}$. This is a simplicial bicategory whose bicategory of $p$-simplices is the bicategory of unitary homomorphic representations of the ordinal category $[p]$ in $\mathcal{T}$ (roughly speaking, trihomomorphisms $[p] \to \mathcal{T}$ satisfying various requirements of normality). Each $S\mathcal{T}$ is a ‘special’ simplicial bicategory, in the sense that the Segal projection homomorphisms on it are biequivalences of bicategories, and thus it is a weak 3-category from the standpoint of Tamsamani [43] and Simpson [38]. When $\mathcal{T}$ is a reduced tricategory (i.e., with only one object), then the simplicial space $BS\mathcal{T} : \Delta^{op} \to \text{Top}$, obtained by replacing the bicategories $S_p\mathcal{T}$ by their classifying spaces $BS_p\mathcal{T}$, is a special simplicial space. Therefore, according to Segal [37], $\Omega|BS\mathcal{T}|$ is a group completion of $BS_1\mathcal{T}$. In our development here, a relevant result is the following:

"For any tricategory $\mathcal{T}$, there is a homotopy equivalence $\mathcal{B}\mathcal{T} \simeq |BS\mathcal{T}|$,”

which we apply to the study of classifying spaces of monoidal bicategories. Recall the aforementioned fact by Stasheff and Mac Lane, that the group completion of the classifying space of a category enriched with a monoidal structure is, in a precise way, a loop space. Any
monoidal bicategory \((\mathcal{B}, \otimes)\) gives rise to a one-object tricategory \(\Sigma(\mathcal{B}, \otimes)\), its ‘suspension’ tricategory following Street’s terminology (or ‘delooping’ in the terminology of Kapranov-Voevodsky \[25\] or Berger \[5\]). Defining the classifying space of a monoidal bicategory \((\mathcal{B}, \otimes)\) to be the classifying space of its suspension tricategory, that is, \(\mathcal{B}(\mathcal{B}, \otimes) = \mathcal{B}\Sigma(\mathcal{B}, \otimes)\), we mainly prove the following extension to bicategories of Stasheff’s result on monoidal categories:

“For any monoidal bicategory \((\mathcal{B}, \otimes)\), the loop space of its classifying space, \(\Omega \mathcal{B}(\mathcal{B}, \otimes)\), is a group completion of the classifying space of the underlying bicategory, \(\mathcal{B}\mathcal{B}\). In particular, if the monoid of connected components \(\pi_0 \mathcal{B}\) is a group, then there is a homotopy equivalence \(\mathcal{B}\mathcal{B} \simeq \Omega \mathcal{B}(\mathcal{B}, \otimes)\).”

If \((\mathcal{C}, \otimes, c)\) any braided monoidal category, then, thanks to the braiding, the suspension of the underlying monoidal category \(\Sigma(\mathcal{C}, \otimes)\), which is actually a bicategory, has a structure of monoidal bicategory. Hence, the double suspension tricategory \(\Sigma^2(\mathcal{C}, \otimes, c) = \Sigma(\Sigma(\mathcal{C}, \otimes), \otimes)\) is defined. Since the classifying space of the braided monoidal category is just the classifying space of its double suspension tricategory, that is, \(\mathcal{B}(\mathcal{C}, \otimes, c) = \mathcal{B}\Sigma^2(\mathcal{C}, \otimes, c)\), the above result implies the existence of a homotopy equivalence \(\Omega \mathcal{B}(\mathcal{C}, \otimes, c) \simeq \mathcal{B}(\mathcal{C}, \otimes)\), between the loop space of classifying space of the braided monoidal category and the classifying space of the underlying monoidal category, a fact recently proved by Carrasco-Cegarra-Garzón \[10\]. Further, we conclude that

“For any braided monoidal category \((\mathcal{C}, \otimes, c)\), the double loop space \(\Omega^2 \mathcal{B}(\mathcal{C}, \otimes, c)\) is a group completion of \(\mathcal{B}\mathcal{C}\),”

and thus a new proof that the group completion of the underlying category of a braided monoidal category is a double loop space, as was noted by Stasheff \[42\] but originally proven by Fiedorowicz \[14\] (see also Berger \[5\] and Baltenau-Fiedorowicz-Schwänzl-Vogt \[4\]). Let us stress that, just because of this double-loop property, braided monoidal categories have been playing a key role in recent developments in quantum theory and related topics.

The process followed for defining the classifying space of a tricategory \(\mathcal{T}\), by means of its Grothendieck nerve \(N\mathcal{T}\), is quite indirect and the CW-complex \(\mathcal{B}\mathcal{T}\) thus obtained has little apparent intuitive connection with the cells of the original tricategory. However, when \(\mathcal{T}\) is a (strict) 3-category, then the space \([\text{diagNNN}]\mathcal{T}\), the geometric realization of the simplicial set diagonal of the 3-simplicial set 3-fold nerve of \(\mathcal{T}\), has usually been taken in the literature as the ‘correct’ classifying space of the 3-category. A result in the paper states that

“For any 3-category \(\mathcal{T}\), there is homotopy equivalence \(\mathcal{B}\mathcal{T} \simeq [\text{diagNNN}]\mathcal{T}\).”

The construction of the simplicial set \(\text{diagNNN}\mathcal{T}\) for 3-categories does not work in the non-strict case since the compositions in arbitrary tricategories are not associative and not unitary, which is crucial for the 3-simplicial structure of the triple nerve \(\text{NNN}\mathcal{T}\), but only up to coherent equivalences or isomorphisms. There is, however, another convincing way of associating a simplicial set to a 3-category through its geometric nerve \(\Delta\mathcal{T}\), thanks to Street \[39\]. He extends each ordinal \([p] = \{0 < 1 < \cdots < p\}\) to a \(p\)-category \(\mathcal{O}_p\), the \(p\)-th-oriental, such that the \(p\)-simplices of \(\Delta\mathcal{T}\) are just the \(3\)-functors \(\mathcal{O}_p \to \mathcal{T}\). Thus, \(\Delta\mathcal{T}\) is a simplicial set whose 0-simplices are the objects (0-cells) \(F_0\) of \(\mathcal{T}\), whose 1-simplices are the 1-cells \(F_{01} : F_1 \to F_0\), whose 2-simplices

\[
\begin{array}{ccc}
F_1 & \xrightarrow{F_{01}} & F_0 \\
\downarrow & \searrow & \downarrow \\
F_2 & \xrightarrow{F_{012}} & F_{02}
\end{array}
\]
Duskin and Street pointed out, and we discuss here in detail. The geometric nerve construction $\Delta T$ in the structure of the tricategory and, furthermore, the cells of its geometric realization have a pleasing geometrical description in terms of the cells of $T$. As a main result in the paper, we state and prove that every connected minimal complex $M$ with $\pi_i M = 0$ for $i \neq 1, 2, 3$, a bicategorical group $(B, \otimes)$ with a simplicial geometric nerve of its suspension tricategory $\Sigma(B, \otimes)$ is any monoidal bicategory, then its geometric nerve, $\Delta(B, \otimes)$, is defined to be the geometric nerve of its suspension tricategory $\Sigma(B, \otimes)$. Then, we obtain the following:

“For any tricategory $T$, there is a homotopy equivalence $BT \simeq |\Delta T|$.”

If $(B, \otimes)$ is any monoidal bicategory, then its geometric nerve, $\Delta(B, \otimes)$, is defined to be the geometric nerve of its suspension tricategory $\Sigma(B, \otimes)$. Then, we obtain the following:

“For any monoidal bicategory $(B, \otimes)$, there is a homotopy equivalence $B(B, \otimes) \simeq |\Delta(B, \otimes)|$.”

For instance, since the geometric nerve of a braided monoidal category $(C, \otimes, c)$ is the geometric nerve of its double suspension tricategory, that is, $\Delta(C, \otimes, c) = \Delta \Sigma^2(C, \otimes, c)$, the existence of a homotopy equivalence $B(C, \otimes, c) \simeq |\Delta(C, \otimes, c)|$ follows, a fact proved in [10].

The geometric nerve $\Delta(B, \otimes)$, of any given monoidal bicategory $(B, \otimes)$, is a Kan complex if and only if $(B, \otimes)$ is a bicategorical group (or weak 3-group, or Gr-bicategory), that is, a monoidal bicategory whose 2-cells are isomorphisms, whose 1-cells are equivalences, and each object $x$ has a quasi-inverse with respect to the tensor product. In other words, a bicategorical group is a monoidal bicategory whose suspension tricategory $\Sigma(B, \otimes)$ is a trigroupoid (or Azumaya tricategory in the terminology of Gordon-Power-Street [17]). The geometric nerve of any bicategorical group $(B, \otimes)$ is then a one vertex Kan complex, whose homotopy groups can be described using only the algebraic structure of $(B, \otimes)$ by

- $\pi_1 \Delta(B, \otimes) = 0$, if $i \neq 1, 2, 3$.
- $\pi_1 \Delta(B, \otimes) = \text{Ob} B / \sim$, the group of equivalence classes of objects in $B$ where multiplication is induced by the tensor product.
- $\pi_2 \Delta(B, \otimes) = \text{Aut}_B(1) / \cong$, the group of isomorphism classes of autoequivalences of the unit object where the operation is induced by the horizontal composition in $B$.
- $\pi_3 \Delta(B, \otimes) = \text{Aut}_B(1, 1)$, the group of automorphisms of the identity 1-cell of the unit object where the operation is vertical composition in $B$.

Hence, its classifying space $B(B, \otimes)$ is a (path) connected homotopy 3-type. In fact, every connected homotopy 3-type can be realized in this way from a bicategorical group, as suggested by the unpublished but widely publicized result of Joyal and Tierney [26] that Gray-groups (called semistrict 3-groups by Baez-Neuchl [1]) model connected homotopy 3-types (see also Berger [5], Lack [28], or Leroy [30]). Recall that, by the coherence theorem for tricategories, every bicategorical group is monoidal biequivalent to a Gray-group. In the last example in the paper, we outline in some detail the proof of the following statement:

“For any connected $CW$-complex $X$ with $\pi_i X = 0$ for $i \geq 4$, there is a bicategorical group $(B, \otimes)$ with a homotopy equivalence $B(B, \otimes) \simeq X$.”

In the proof of this result, we explicitly show how to construct from any connected minimal complex $M$ with $\pi_i M = 0$ for $i \neq 1, 2, 3$, a bicategorical group $(B, \otimes)$ with a simplicial
isomorphism $\Delta(B, \otimes) \cong M$. In the particular case when, in addition, $\pi_3 M = 0$, then the resulting bicategorical group has all its 2-cells identities, that is, it is actually a categorical group $(C, \otimes)$ [24]. While in the particular case where $\pi_1 M = 0$, the bicategorical group has only one object, so that it is actually the suspension of a braided categorical group $(C, \otimes, c)$ [24]. Hence, our proof implicitly covers two relevant particular cases, already well-known from Joyal-Tierney [26] (see also [8]), one stating that categorical groups are a convenient algebraic model for connected homotopy 2-types, and the other that braided categorical groups are algebraic models for connected, simply-connected homotopy 3-types, namely:

“For any path-connected CW-complex $X$, there is categorical group $(C, \otimes)$ and a homotopy equivalence $B(C, \otimes) \simeq X$ if and only if $\pi_i X = 0$ for $i \geq 3$.”

“For any path-connected CW-complex $X$, there is a braided categorical group $(C, \otimes, c)$ and a homotopy equivalence $B(C, \otimes, c) \simeq X$ if and only if $\pi_i X = 0$ for $i = 1$ and $i \geq 3$.”

1.1. The organization of the paper. The plan of this paper is, briefly, as follows. After this introductory Section 1, the paper is organized in four sections. Section 2 is long and very technical, but crucial to our discussions. It is dedicated to establishing and proving some needed results concerning the notion of representation of a category in a tricategory, which is at the heart of the several constructions of nerves for tricategories used in the paper. In Section 3, we mainly include the construction of the Grothendieck nerve $\mathcal{N}T : \Delta^{op} \to \text{Bicat}$, for any tricategory $T$, and the study of the basic properties concerning the behavior of the Grothendieck nerve construction, $T \mapsto \mathcal{N}T$, with respect to trihomomorphisms of tricategories and tritransformations between them. Section 4 contains the definition of classifying space $BT$, for any tricategory $T$. The main facts concerning the classifying space construction $T \mapsto BT$ are established here. In this section we also study the relationship between $BT$ and the space realization of the Segal nerve of a tricategory, $ST : \Delta^{op} \to \text{Bicat}$, which, for instance, we apply to prove that the classifying space of any monoidal bicategory is up to group completion a loop space. Finally, Section 5 is mainly dedicated to describing the geometric nerve $\Delta T : \Delta^{op} \to \text{Set}$, of any tricategory $T$, and to proving the existence of homotopy equivalences $BT \simeq |\Delta T|$. Also, by means of the geometric nerve construction for monoidal bicategories, $(B, \otimes) \mapsto \Delta(B, \otimes)$, we show here that bicategorical groups are a convenient algebraic model for connected homotopy 3-types.

1.2. Notations. We refer to Benabou [3] and Street [10] for the background on bicategories. For any bicategory $B$, the composition in each hom-category $B(x, y)$, that is, the vertical composition of 2-cells, is denoted by $v \cdot u$, while the symbol $\circ$ is used to denote the horizontal composition functors $\circ : B(y, z) \times B(x, y) \to B(x, z)$. The identity of an object is written as $1_x : x \to x$, and we shall use the letters $a$, $r$, and $l$ to denote the associativity, right unit, and left unit constraints of the bicategory, respectively. Hereinafter [17] Notation 4.9, the category of bicategories with homomorphisms (or pseudo-functors) between them will be denoted by $\text{Hom}$.

In this paper we use the notion of tricategory $T = (T, a, l, r, \pi, \mu, \lambda, \rho)$ as it was introduced by Gordon, Power, and Street in [17], but with a minor alteration: we require that the homomorphisms of bicategories picking out units are normalized, and then written simply as $1_t \in T(t, t)$. This restriction is not substantive (see Gruski [20] Theorem 4.3.3), but it does slightly reduce the amount of coherence data we have to deal with. For any object $t$ of the tricategory $T$, the arrow $r_1 : 1_t \to 1_t \otimes 1_t$ is an equivalence in the hom-bicategory $T(t, t)$, with the arrow $l_1 : 1_t \otimes 1_t \to 1_t$ an adjoint quasi-inverse (see [20] Lemma A.3.1)].
Hereafter, we suppose the adjoint quasi-inverse of \( r, r^\bullet \vdash r \), has been chosen such that \( r^*1_i = 1_{r_i} \).

As in [17, §5] and [20, §6.3], \textbf{Bicat} denotes the tricategory of small bicategories, homomorphisms, pseudo-transformations and modifications. If \( F, F' : \mathcal{B} \to \mathcal{C} \) are lax functors between bicategories, then we follow the convention of [17] in what is meant by a lax transformation \( \alpha : F \Rightarrow F' \). Thus, \( \alpha \) consists of morphisms \( \alpha_x : Fx \to F'x \), and of 2-cells \( \alpha_a : \alpha_y \circ Fa \Rightarrow F' \circ \alpha_x \), subject to the usual axioms. In the structure of \textbf{Bicat} we use, the composition of pseudo-transformations is taken to be

\[
(C \xrightarrow{G} D)(B \xrightarrow{F} C) = (B \xrightarrow{GF} C) \circ (C \xrightarrow{\beta} D),
\]

where \( \beta \alpha = \beta F \circ G \alpha : GF \Rightarrow G'F \Rightarrow G'F' \), but note the existence of the useful invertible modification

\[
\begin{array}{c}
GF \\
\downarrow \beta F
\end{array} \quad \begin{array}{c}
\Rightarrow \\
\downarrow \beta F
\end{array} \quad \begin{array}{c}
GF' \\
\downarrow \beta F'
\end{array}
\]

whose component at an object \( x \) of \( \mathcal{B} \), is \( \beta_{\alpha x} \), the component of \( \beta \) at the morphism \( \alpha x \).

For the general background on simplicial sets, we mainly refer to [16]. The \textit{simplicial category} is denoted by \( \Delta \), and its objects, that is, the ordered sets \( [n] = \{0, 1, \ldots, n\} \), are usually considered as categories with only one morphism \( (i, j) : j \to i \) when \( 0 \leq i \leq j \leq n \). Then, a non-decreasing map \( [n] \to [m] \) is the same as a functor, so that we see \( \Delta \), the simplicial category of finite ordinal numbers, as a full subcategory of \textbf{Cat}, the category (actually the 2-category) of small categories.

Throughout the paper, Segal’s \textit{geometric realization} [36] of a simplicial (compactly generated topological) space \( X : \Delta^\text{op} \to \textbf{Top} \) is denoted by \( |X| \). By regarding a set as a discrete space, the (Milnor’s) geometric realization of a simplicial set \( X : \Delta^\text{op} \to \textbf{Set} \) is \( |X| \).

2. \textbf{Representations of Categories in Tricategories}

As we will show in the paper, the classifying space of any tricategory can be realized up to homotopy by a simplicial set \( \Delta T \), whose \( p \)-simplices \( \Delta[p] \to \Delta T \) are the different \textit{unitary representations} \( \rho \to \mathcal{T} \), of the category \( \rho \) in the tricategory \( \mathcal{T} \). Hence, we present below a study of these representations of categories in tricategories, and the bicategories they form.

2.1. \textbf{(Unitary, homomorphic) Representations}. Roughly speaking, a representation of a category \( I \) in a tricategory \( \mathcal{T} \) is a lax functor \( I \to \mathcal{T} \), where \( I \) is regarded as tricategory in which the 2-cells and 3-cells are all identities, satisfying various requirements of normality. But noting that the definition given in [17, Definition 3.1] for trihomomorphisms does not work for lax functors between tricategories (the 3-cell \( \pi \) in the equation expressing axiom (HTA1) is not well defined), we establish the following explicit definition:

**Definition 2.1.1.** A representation \( F : I \to \mathcal{T} \), of a category \( I \) in a tricategory \( \mathcal{T} \), is a system of data consisting of: an object \( F_i \in \text{Ob}\mathcal{T} \), for each object \( i \) in \( I \); a 1-cell \( F_a : Fj \to Fi \), for each arrow \( a : j \to i \) in \( I \); 2-cells \( F_{a,b} : Fa \otimes Fb \Rightarrow F(ab) \) and \( F_i : 1_{Fi} \Rightarrow F1_i \), for
for any four composable arrows \( k \xrightarrow{b} j \xrightarrow{a} i \) and each object \( i \in \text{Ob} I \), respectively; and 3-cells

\[
(Fa \otimes Fb) \otimes Fc \xrightarrow{\alpha} Fa \otimes (Fb \otimes Fc)
\]

respectively associated to any three composable arrows \( l \xrightarrow{c} k \xrightarrow{b} j \xrightarrow{a} i \) and any arrow \( j \xrightarrow{a} i \) in the category \( I \). These data are required to satisfy the coherence conditions (CR1) and (CR2) as stated below.

The set of representations of a small category \( I \) in a small tricategory \( T \) is denoted by

\[(2) \quad \text{Rep}(I, T).\]

A representation \( F : I \to T \) is termed unitary or normal whenever the following conditions hold: for each object \( i \) of \( I \), \( F1_i = 1_{F_i} \) and \( F_i = 1_{Fi} \); for each arrow \( a : j \to i \) of \( I \), \( F_a,1_j = r^a : Fa \otimes 1 \to Fa \) and \( F_{1,a} = l : 1 \otimes Fa \to Fa \), and the 3-cells \( F_{a,1,c}, F_{a,c}, \) and \( F_{a,c} \) are the unique coherence isomorphisms. Furthermore, a representation \( F : I \to T \) whose structure 2-cells \( F_{a,b} \) are all equivalences (in the corresponding hom-bicategories of \( T \) where they lie) and whose structure 3-cells \( F_{a,b,c}, F_{a,c}, \) and \( F_{a,c} \) are all invertible is called a homomorphic representation. The subsets of \( \text{Rep}(I, T) \) whose elements are the unitary, homomorphic, and unitary homomorphic representations, are denoted respectively by

\[(3) \quad \text{Rep}_a(I, T), \text{Rep}_h(I, T), \text{Rep}_{uh}(I, T).\]

(CR1): for any four composable arrows in \( I \), \( m \xrightarrow{d} l \xrightarrow{c} k \xrightarrow{b} j \xrightarrow{a} i \), the equation \( A = A' \) on 3-cells in \( T \) holds, where:  

\[
(Fa \otimes Fb) \otimes Fd \xrightarrow{a} (Fa \otimes Fb) \otimes (Fc \otimes Fd) \xrightarrow{a} Fa \otimes (Fb \otimes (Fc \otimes Fd))
\]

\[
(Fa \otimes (Fb \otimes Fc)) \otimes Fd \xrightarrow{a} Fa \otimes ((Fb \otimes Fc) \otimes Fd)
\]

\[
(Fa \otimes F(bc)) \otimes Fd \xrightarrow{a} Fa \otimes (F(bc) \otimes Fd)
\]

\[
F(abc) \otimes Fd \xrightarrow{F_{abc,d}} F(abcd) \xleftarrow{F_{abcd}} Fa \otimes F(bcd)
\]
(CR2): for any two composable arrows $k \xrightarrow{b} j \xrightarrow{a} i$ in $I$, the equations $B = B'$, $C = C'$, and $D = D'$, on 3-cells in $\mathcal{T}$ hold, where:
2.2. Example. Let $A$ be an abelian group, and let $\Sigma^2 A$ denote the tricategory (actually a 3-groupoid) having only one $i$-cell for $0 \leq i \leq 2$ and whose 3-cells are the elements of $A$, with all the compositions given by addition in $A$. Then, for any small category $I$ (e.g., a group $G$ or a monoid $M$), a unitary representation $F : I \to \Sigma^2 A$ is the same as a function $F : N_3 I \to A$ satisfying the equations

$$F(b, c, d) + F(a, bc, d) + F(a, b, c) = F(ab, c, d) + F(a, b, cd),$$

and such that $F(a, b, c) = 0$ whenever any of the arrows $a$, $b$, or $c$ is an identity. Thus $\text{Rep}_a(I, \Sigma^2 A) = Z^3(I, A)$, the set of normalized 3-cocycles of (the nerve $N_I$ of) the category $I$ with coefficients in the abelian group $A$.

2.3. The bicategory of representations. For any category $I$ and any tricategory $\mathcal{T}$, the set $\text{Rep}(I, \mathcal{T})$ of representations of $I$ in $\mathcal{T}$ is the set of objects of a bicategory, the bicategory of representations of $I$ in $\mathcal{T}$, denoted by $\text{Rep}(I, \mathcal{T})$. 
whose 1-cells are a kind of degenerated lax transformations between representations that agree on objects. When $\mathcal{T} = \mathcal{B}$ is a bicategory, that is, when the 3-cells are all identities, these degenerated lax transformations has been considered in \[13, 19, 20\] under the name of \textit{relative to objects lax transformations}, whereas in \[15, 27, 29\] they are termed \textit{icons}, short for ‘identity component oplax natural transformations’. The description of the bicategory $\text{Rep}(I, \mathcal{T})$ is as follows:

- The cells of $\text{Rep}(I, \mathcal{T})$. As we said above, representations $F : I \to \mathcal{T}$ are the 0-cells of this bicategory. For any two representations $F, G : I \to \mathcal{T}$, a 1-cell $\Phi : F \Rightarrow G$ may exists only if $F$ and $G$ agree on objects, that is, $F_i = G_i$ for all $i \in \text{Ob}I$; and is then given by specifying, for every arrow $a : j \to i$ in $I$, a 2-cell $\Phi_a : Fa \Rightarrow Ga$ of $\mathcal{T}$, and 3-cells

$$
\begin{align*}
Fa \otimes Fb & \xrightarrow{a} F(ab) \\
\Phi_a \otimes Fb & \xrightarrow{F_a,b \otimes 1} F(ab) \\
Ga \otimes Gb & \xrightarrow{\Phi_{a,b}} G(ab)
\end{align*}
$$

(5)

respectively associated to each pair of composable arrows $k \xrightarrow{b} j \xrightarrow{a} i$ and each object $i$ of the category $I$, subject to the two coherence axioms (CR3) and (CR4) below.

(CR3): for any two composable arrows triplet of composable morphisms of $I$, $l \xrightarrow{k} b \xrightarrow{j} a \xrightarrow{i}$, the equation $E = E'$ on 3-cells in $\mathcal{T}$ holds, where:

$$
\begin{align*}
E &= (Fa \otimes Fb) \otimes Fc \xrightarrow{a} Fa \otimes (Fb \otimes Fc) \xrightarrow{1 \otimes Fa,c} Fa \otimes F(bc) \\
\phi(a) \otimes \phi(b) \otimes \phi(c) & \xrightarrow{F_{a,b,c} \otimes 1} Fa \otimes F(bc) \\
G(a) \otimes G(b) \otimes G(c) & \xrightarrow{1 \otimes F(a,b,c)} G(ab) \otimes G(bc) \otimes G(ab, bc)
\end{align*}
$$

$$
\begin{align*}
E' &= (Ga \otimes Gb) \otimes Gc \xrightarrow{a} Ga \otimes (Gb \otimes Gc) \xrightarrow{1 \otimes G(a,b,c)} Ga \otimes G(bc) \\
\phi(a) \otimes \phi(b) \otimes \phi(c) & \xrightarrow{Ga \otimes (Gb, Gc) \otimes 1} Ga \otimes G(bc) \\
G(a) \otimes G(b) \otimes G(c) & \xrightarrow{G(a, b, c) \otimes 1} G(ab) \otimes G(bc)
\end{align*}
$$

(CR4): for any morphism of $I$, $j \xrightarrow{a} i$, the following two pasting equalities hold:
A 2-cell \( M : \Phi \Rightarrow \Psi \), for \( \Phi, \Psi : F \Rightarrow G \) two 1-cells in \( \text{Rep}(I, T) \), consists of a family of 3-cells in \( T \), \( M \alpha : \Phi a \Rightarrow \Psi a \), one for each arrow \( a : j \rightarrow i \) in \( I \), subject to the coherence condition \((\text{CR5})\) below.

\((\text{CR5})\): for any object \( i \) and each two composable arrows \( k \rightarrow j 
\rightarrow i \) of \( I \), the diagrams of 3-cells below commute.

### Compositions in \( \text{Rep}(I, T) \)

The vertical composition of a 2-cell \( M : \Phi \Rightarrow \Psi \) with a 2-cell \( N : \Psi \Rightarrow \Gamma \), for \( \Phi, \Psi, \Gamma : F \Rightarrow G \), yields the 2-cell \( N \circ M : \Phi \Rightarrow \Gamma \) which is defined using pointwise vertical composition in the hom-bicategories of \( T \); that is, for each \( a : j \rightarrow i \) in \( I \), \((N \circ M)a = (Na) \cdot (Ma) : \Phi a \Rightarrow \Gamma a : Fa \Rightarrow Ga \). The horizontal composition of 1-cells \( \Phi : F \Rightarrow G \) and \( \Psi : G \Rightarrow H \), for \( F, G, H : I \rightarrow T \) representations, is \( \Psi \circ \Phi : F \Rightarrow H \), where \((\Psi \circ \Phi)a = \Psi a \circ \Phi a : Fa \Rightarrow Ha \), for each arrow \( a : j \rightarrow i \) in \( I \). Its component \((\Psi \circ \Phi)_{a,b} : (\Psi \circ \Phi)a \otimes (\Psi \circ \Phi)b \Rightarrow \Psi(ab) \circ \Phi(ab)\), attached at a pair of composable arrows \( k \rightarrow j \xrightarrow{a} i \) of the category \( I \), is given by pasting in the bicategory \( T(Fk, Fi) \) the diagram

\[
\begin{array}{ccc}
F_a \otimes F_b & \xrightarrow{F_{a,b}} & F(ab) \\
\downarrow \cong & & \downarrow \Phi_{ab} \\
Ga \otimes Gb & \xrightarrow{G_{a,b}} & G(ab)
\end{array}
\]

and its component \((\Psi \circ \Phi)_i : (\Psi \circ \Phi)_1 \circ F_i \Rightarrow H_i\), attached at any object \( i \) of \( I \), is given by pasting in \( T(Fi, Fi) \) the diagram

\[
\begin{array}{ccc}
\Phi_i & \cong & \Psi_i \\
\downarrow F_i & & \downarrow H_i \\
1_{Fi} = G_i = H_i
\end{array}
\]

The horizontal composition of 2-cells \( M : \Phi \Rightarrow \Psi : F \Rightarrow G \) and \( N : \Gamma \Rightarrow \Theta : G \Rightarrow H \) in \( \text{Rep}(I, T) \) is \( N \circ M : \Gamma \Rightarrow \Theta \circ \Psi \), which, at each \( a : j \rightarrow i \) in \( I \), is given by the formula \((N \circ M)a = Na \circ Ma\).

### Identities in \( \text{Rep}(I, T) \)

The identity 1-cell of a representation \( F : I \rightarrow T \) is \( 1_F : F \Rightarrow F \), where \((1_F)a = 1_{Fa}\), the identity of \( Fa \) in the bicategory \( T(Fi, Fi) \), for each \( a : j \rightarrow i \) in \( I \). Its structure 3-cell \((1_F)_{a,b} : F_{a,b} \circ (1_Fa \otimes 1_Fa) \Rightarrow 1_{F(ab)} \otimes F_{a,b}\), attached at each pair of
composable arrows $k \xrightarrow{h} j \xrightarrow{a} i$, is the canonical one obtained from the identity constraints of the bicategory $\mathcal{T}(Fk, Fi)$ by pasting the diagram

$$
\begin{array}{c}
F a \otimes F b \xrightarrow{F_{a,b}} F(ab) \\
1 \otimes 1 \Rightarrow 1 \Rightarrow 1
\end{array}
$$

and its component attached at any object $i$ of $I$ is the left unit constraints of the bicategory $\mathcal{T}(Fi, Fi)$ at $F_i : 1_{Fi} \Rightarrow F_1$, that is, $(1_F)_i = I : 1_{Fi}, \circ F_i \cong F_i$. The identity 2-cell $1_{\Phi}$, of a 1-cell $\Phi : F \Rightarrow G$, is defined at any arrow $a : j \rightarrow i$ of $I$ by the simple formula $(1_{\Phi})a = 1_{\Phi a} : \Phi a \Rightarrow \Phi a$.

- The structure constraints in $\text{Rep}(I, \mathcal{T})$. For any three composable 1-cells $F \overset{\Phi}{\Rightarrow} G \overset{\Psi}{\Rightarrow} H \overset{\Theta}{\Rightarrow} K$ in $\text{Rep}(I, \mathcal{T})$, the component of the structure associativity isomorphism $(\Theta \circ \Psi) \circ \Phi \cong \Theta \circ (\Psi \circ \Phi)$, at any arrow $j \xrightarrow{a} i$ of the category $I$, is provided by the associativity constraint $(\Theta a \circ \Psi a) \circ \Phi a \cong \Theta a \circ (\Psi a \circ \Phi a)$ of the hom-bicategory $\mathcal{T}(Fj, Fi)$. And similarly, the components of structure left and right identity isomorphisms $1_G \circ \Phi \cong \Phi$ and $\Phi \circ 1_F \cong \Phi$, at any arrow $a : j \rightarrow i$ as above, are provided by the identity constraints $1_{Ga} \circ \Phi a \cong \Phi a$, and $\Phi a \circ 1_{Fa} \cong \Phi a$, of the bicategory $\mathcal{T}(Fj, Fi)$, respectively.

2.4. The bicategories of unitary and homomorphic representations. The bicategory of representations of a category $I$ in a tricategory $\mathcal{T}$, $\text{Rep}(I, \mathcal{T})$, contains three sub-bicategories that are of interest in our development:

The bicategory of unitary representations, denoted by

$$\text{Rep}_u(I, \mathcal{T}),$$

whose 0-cells are the unitary representations of $I$ in $\mathcal{T}$; its 1-cells are those $\Phi : F \Rightarrow G$ in $\text{Rep}(I, \mathcal{T})$ that are unitary, in the sense that $\Phi_1 = 1_{I_F}$, and the 3-cell $\Phi_3$ in (5) is the canonical isomorphism $1 \circ 1 \cong 1$, for all objects $i$ of $I$, and it is full on 2-cells $M : \Phi \Rightarrow \Psi$ between such normalized 1-cells.

The bicategory of homomorphic representations, denoted by

$$\text{Rep}_h(I, \mathcal{T}),$$

whose 0-cells are the homomorphic representations, its 1-cells are those $\Phi : F \Rightarrow G$ in $\text{Rep}_u(I, \mathcal{T})$ such that the structure 3-cells $\Phi_{a,b}$ are all invertible, and it is full on 2-cells $M : \Phi \Rightarrow \Psi$ between such 1-cells.

The bicategory of unitary homomorphic representations, denoted by $\text{Rep}_{uh}(I, \mathcal{T})$, which is defined to be the intersection of the above two, that is,

$$\text{Rep}_{uh}(I, \mathcal{T}) = \text{Rep}_u(I, \mathcal{T}) \cap \text{Rep}_h(I, \mathcal{T}).$$

2.5. Example. Let $\Sigma^2 A$ be the strict tricategory defined by an abelian group $A$ as in Example 22 and let $I$ be any category. Then, the bicategory $\text{Rep}_u(I, \Sigma^2 A)$ is actually a 2-groupoid whose objects are normalized 3-cocycles of $I$ with coefficients in $A$. If $F, G : N_3 I \rightarrow A$ are two such 3-cocycles, then a 1-cell $\Phi : F \Rightarrow G$ is a normalized 2-cochain $\Phi : N_2 I \rightarrow A$ satisfying

$$G(a, b, c) + \Phi(b, c) + \Phi(a, bc) = F(a, b, c) + \Phi(ab, c) + \Phi(a, b),$$
that is, \( G = F + \partial \Phi \). And for any two 1-cells \( \Phi, \Psi : F \Rightarrow G \) as above, a 2-cell \( M : \Phi \Rightarrow \Psi \) consists of a normalized 1-cochain \( M : N_{1}I \to A \) such that \( \Psi = \Phi + \partial M \), that is, such that

\[
\Psi(a, b) + M(a) + M(b) = M(ab) + \Phi(a, b).
\]

2.6. Functorial proprieties of \( \text{Rep}_{H}(I, -) \). For a tricategory \( \mathcal{T} \), any functor \( a : J \to I \) induces a homomorphism (actually a 2-functor) \( a^{\ast} : \text{Rep}_{H}(I, \mathcal{T}) \to \text{Rep}_{H}(J, \mathcal{T}) \), and the construction \( (\mathcal{I}, I \mapsto \text{Rep}_{H}(I, \mathcal{T})) \), is functorial on the category \( I \). For a trihomomorphism of tricategories \( H = (H, \chi, i, \omega, \gamma, \delta) : \mathcal{T} \to \mathcal{T}', \) as defined in [17, Definition 3.1], we have the following result:

**Lemma 2.6.1.** Let \( I \) be any given small category.

(i) Every trihomomorphism \( H : \mathcal{T} \to \mathcal{T}' \) gives rise to a homomorphism

\[ H_{\ast} : \text{Rep}_{H}(I, \mathcal{T}) \to \text{Rep}_{H}(I, \mathcal{T}'), \]

which is natural on \( I \), that is, for any functor \( a : J \to I \),

\[ H_{\ast} a^{\ast} = a^{\ast} H_{\ast} : \text{Rep}_{H}(I, \mathcal{T}) \to \text{Rep}_{H}(J, \mathcal{T}'). \]

(ii) If \( H' : \mathcal{T} \to \mathcal{T}' \) and \( H'' : \mathcal{T}' \to \mathcal{T}'' \) are any two composable trihomomorphisms, then there is a pseudo-equivalence \( m : H'_{\ast} H_{\ast} \Rightarrow (H'')_{\ast} \), such that, for any functor \( a : J \to I \), the equality \( m_{\ast} a^{\ast} = a^{\ast} m_{\ast} \) holds.

(iii) For any tricategory \( \mathcal{T} \), there is a pseudo-equivalence \( m : (1_{\mathcal{T}})_{\ast} \Rightarrow 1 \), such that, for any functor \( a : J \to I \), the equality \( m_{\ast} a^{\ast} = a^{\ast} m_{\ast} \) holds.

**Proof.** (i) The homomorphism \( H_{\ast} \) is defined as follows: It carries a homomorphic representation \( F : I \to \mathcal{T} \) to the homomorphic representation \( H_{\ast} F : I \to \mathcal{T}' \), which is defined on objects \( i \) of \( I \) by \( (H_{\ast} F)i = HFi \), and on arrows \( a : j \to i \) by \( (H_{\ast} F)a = HFa : HFj \Rightarrow HFi \).

The 2-cell \( (H_{\ast} F)_{ab} : (H_{\ast} F)a \otimes (H_{\ast} F)b \Rightarrow (H_{\ast} F)(ab) \), for each pair of composable arrows \( k \xrightarrow{b} j \xrightarrow{a} i \), is the composition \( \text{HF}a \otimes \text{HF}b \Rightarrow \text{HF}(ab) \). For each object \( i \), the 2-cell \( (H_{\ast} F)i : 1_{(H_{\ast} F)i} \Rightarrow (H_{\ast} F)i \) is the composite of \( 1_{HF}i \Rightarrow HF1i \Rightarrow HF1i \). The structure 3-cell of \( H_{\ast}F : I \to \mathcal{T}' \) associated to any three composable arrows \( l \xrightarrow{a} k \xrightarrow{b} j \xrightarrow{c} i \), is that obtained by pasting the diagram:

\[
\begin{array}{c}
\xymatrix{ 
(HFa \otimes HFb) \otimes HFc \ar[r]^{a} \ar[d]_{HF_{ab} \otimes 1} & HFa \otimes (HFb \otimes HFc) \ar[d]_{1 \otimes HF_{bc}} \ar[r]^{\omega} & HFa \otimes H(Fb \otimes Fc) \ar[d]_{1 \otimes HF_{bc}} \\
HF(ab) \otimes HFc \ar[r]_{\chi} & H((Fa \otimes Fb) \otimes Fc) \ar[r]_{Ha} & H(Fa \otimes (Fb \otimes Fc)) \ar[r]_{\omega} & H(Fa \otimes F(bc)) \ar[d]_{\chi} \\
& H(F(ab) \otimes Fc) \ar[r]_{F_{ab,c}} & H(Fa \otimes F(bc)) \ar[r]_{\chi} & HFa \otimes HF(bc) \ar[d]_{HF_{ab,c}} \\
& HF(abc) \ar[r]_{HF_{abc}} & HFa(bc) \ar[r]_{HF_{abc}} & \\
& & & \\
}\end{array}
\]
whereas the structure 3-cells of the representation $H_a F$ attached to an arrow $a : j \to i$ of the category $I$, are respectively those obtained by pasting the diagrams below.

\[
\begin{array}{cccc}
H1F_i \otimes HFa & \cong & H(F1i \otimes Fa) & \cong HFa,
\end{array}
\]

If $\Phi : F \Rightarrow G$ is any 1-cell in the bicategory $\text{Rep}_b(I, T)$, then $H_+ \Phi : H_+ F \Rightarrow H_+ G$ is the 1-cell in $\text{Rep}(I, T')$ whose component at an arrow $a : j \to i$ of $I$ is the 2-cell of $T'$ defined by $(H_+ \Phi)a = H\Phi a : HFa \Rightarrow HGa$. For any pair of composable arrows $k : a \Rightarrow b$ and any object $i$ of $I$, the corresponding structure 3-cells $H_i$, $(H_+ \Phi)_{a,b}$ and $(H_+ \Phi)_i$, are respectively given by pasting in

\[
\begin{array}{cccc}
HFa \otimes H1Fj & \cong & H(Fa \otimes F1j) & \Rightarrow HFa.
\end{array}
\]

And a 2-cell $M : \Phi \Rightarrow \Psi$ of $\text{Rep}(I, T)$ is applied by the homomorphism $H_+$ to the 2-cell $H_+ M : H_+ \Phi \Rightarrow H_+ \Psi$ of $\text{Rep}(I, T')$, such that $(H_+ M)a = HMc : HFa \Rightarrow HFa$ for any arrow $a : j \to i$ of the category $I$.

Finally, if $\Phi : F \Rightarrow G$ and $\Psi : G \Rightarrow H$ are any two composable 1-cells in $\text{Rep}(I, T)$, and $F : I \to T$ is any representation, then the constraints $(H_+ \Psi) \circ (H_+ \Phi) \cong H_+(\Psi \circ \Phi)$ and $1_{H_+ F} \cong H_+ 1_F$ are, at any arrow $a : j \to i$ of $I$, the structure isomorphisms $H\Phia \circ H\Phia \cong H(\Psia \circ \Phia)$ and $1 HFa \cong H1Fa$ of the homomorphism $H : T(Fj, Fi) \to H'(HFj, HFi)$, respectively.

(ii) For any homomorphic representation $F : I \to T$, the 2-cell attached by

\[
m = m_F : H'_+ (H_+ F) \Rightarrow (H'H)_+ F
\]

at any arrow $a : j \to i$ of $I$ is the identity, that is, $ma = 1_{H'HFa}$. For any pair of composable arrows $k : a \Rightarrow b$ and any object $i$ of $I$, the corresponding invertible structure 3-cells $M$,

\[
m_{a,b} : (H'_+(H_+ F))_{a,b} \circ (ma \otimes mb) \Rightarrow m(ab) \circ ((H'H)_+ F)_{a,b},
\]

\[
m_i : m_1 \circ (H'_+(H_+ F))_i \Rightarrow ((H'H)_+ F)_i,
\]
are, respectively, given by pasting in the diagrams below.

\[
\begin{array}{c}
H'HF_a \otimes H'HF_b \xrightarrow{\chi} \xrightarrow{\cong} \xrightarrow{\cong} \xrightarrow{\cong} \xrightarrow{\cong} \xrightarrow{1} \xrightarrow{1} \\
H'HF_a \otimes H'HF_b \xrightarrow{\psi} \xrightarrow{\cong} \xrightarrow{\cong} \xrightarrow{\cong} \xrightarrow{\cong} \xrightarrow{1} \xrightarrow{1}
\end{array}
\]

If \( \Phi : F \Rightarrow G \) any 1-cell in \( \text{Rep}(I, T) \), then the invertible naturality 2-cell

\[ m_\Phi : m_G \circ (H'_i(H \cdot \Phi)) \Rightarrow (H'HF) \circ m_F, \]

at any arrow \( a : j \rightarrow i \) of \( I \), is provided by the canonical isomorphism \( 1 \circ H'HFa \cong H'HFa \circ 1 \) in the bicategory \( T'(H'HFj, H'HFj) \).

(iii) For any representation \( F : I \rightarrow T \), the 2-cell attached by

\[ m = m_F : (1_T)_*F \Rightarrow F \]

at any arrow \( a : j \rightarrow i \) of \( I \) is the identity, that is, \( ma = 1_{Fa} \). For any pair of composable arrows \( k \rightarrow j \rightarrow i \) of \( I \), the corresponding invertible structure 3-cells \( \Phi \),

\[ m_{a,b} : F_{a,b} \circ (ma \otimes mb) \Rightarrow m(ab) \circ ((1_T)_*F)_{a,b}, \]

\[ m_i : m_1 \circ ((1_T)_*F)_i \Rightarrow F_i, \]

are, respectively, the canonical isomorphisms in the diagrams below.

\[
\begin{array}{c}
F_a \otimes F_b \xrightarrow{1} \xrightarrow{\cong} \xrightarrow{1} \xrightarrow{1} \\
F_a \otimes F_b \xrightarrow{F_{a,b}} \xrightarrow{1} \xrightarrow{F(ab)} \\
\end{array}
\]

If \( \Phi : F \Rightarrow G \) any 1-cell in \( \text{Rep}(I, T) \), then the invertible naturality 2-cell

\[ m_\Phi : m_G \circ ((1_T)_*\Phi) \Rightarrow \Phi \circ m_F, \]

at any arrow \( a : j \rightarrow i \) of \( I \), is provided by the canonical isomorphism \( 1 \circ \Phi a \cong \Phi a \circ 1 \) in the bicategory \( T(F_j, F_i) \).

\[ 2.7. \text{Representations of free categories.} \]

Let us now replace category \( I \) above by a (directed) graph \( G \). For any tricategory \( T \), there is a bicategory of representations of \( G \) in \( T \), denoted by

\[ \text{Rep}(G, T), \]

where a 0-cell \( f : G \rightarrow T \), consists of a pair of maps that assign an object \( fi \) to each vertex \( i \in G \) and a 1-cell \( fa : fj \rightarrow fi \) to each edge \( a : j \rightarrow i \) in \( G \), respectively. A 1-cell \( \phi : f \Rightarrow g \) may exist only if \( f \) and \( g \) agree on vertices, that is, \( fi = gi \) for all \( i \in G \); and then it consists of a map that assigns to each edge \( a : j \rightarrow i \) in the graph a 2-cell \( \phi a : fa \Rightarrow ga \) of \( T \). And a 2-cell \( m : \phi \Rightarrow \psi \), for \( \phi, \psi : f \Rightarrow g \) two 1-cells as above, consists of a family of 3-cells in \( T \), \( ma : \phi a \Rightarrow \psi a \), one for each arrow \( a : j \rightarrow i \) in \( I \). Compositions in \( \text{Rep}(G, T) \) are defined in
the natural way by the same rules as those stated above for the bicategory of representations of a category in a tricategory.

Suppose now that $I(G)$ is the free category generated by the graph $G$. Then, restriction to the basic graph gives a 2-functor

$$R : \text{Rep}(I(G), T) \to \text{Rep}(G, T),$$

and we shall prove the following auxiliary statement to be used later:

Lemma 2.7.1. Let $I = I(G)$ be the free category generated by a graph $G$. Then, for any tricategory $T$, there are a homomorphism

$$L : \text{Rep}(G, T) \to \text{Rep}(I, T),$$

and a lax transformation

$$v : LR \Rightarrow 1_{\text{Rep}(I, T)},$$

such that the following facts hold:

(a) $RL = 1_{\text{Rep}(G, T)}$, $vL = 1_L$, $Rv = 1_R$.

(b) The image of $L$ is contained into the sub-bicategory $\text{Rep}_{\text{ab}}(I, T) \subseteq \text{Rep}(I, T)$.

(c) The restricted homomorphisms of $L$ and $R$ establish biadjoint biequivalences

$$\text{Rep}(G, T) \xrightarrow{L} \text{Rep}_{\text{ab}}(I, T);$$

$$\text{Rep}(G, T) \xleftarrow{R} \text{Rep}_{\text{ab}}(I, T),$$

whose respective unit is the identity $1 : 1 \Rightarrow RL$, the counit is given by the corresponding restriction of $v : LR \Rightarrow 1$, and whose triangulators are the canonical modifications $1 \sim 1 = vL \circ L1$, $Rv \circ 1R = 1 \circ 1 \sim 1$.

Proof. To describe the homomorphism $L$, we shall use the following useful construction: For any list $(t_0, \ldots, t_p)$ of objects in the tricategory $T$, let

$$\otimes : T(t_1, t_0) \times T(t_2, t_1) \times \cdots \times T(t_p, t_{p-1}) \to T(t_p, t_0)$$

denote the homomorphism recursively defined as the composite

$$\prod_{i=1}^{p} T(t_i, t_{i-1}) = T(t_1, t_0) \times \prod_{i=2}^{p} T(t_i, t_{i-1}) \xrightarrow{1 \times \otimes} T(t_1, t_0) \times T(t_p, t_2) \xrightarrow{\otimes} T(t_p, t_0).$$

That is, $\otimes$ is the homomorphism obtained by iterating composition in the tricategory, which acts on 0-cells, 1-cells and 2-cells of the product bicategory $\prod_{i=1}^{p} T(t_i, t_{i-1})$ by the recursive formula

$$\otimes(x_1, \ldots, x_p) = \begin{cases} x_1 & \text{if } p = 1, \\
 x_1 \otimes (\otimes(x_2, \ldots, x_p)) & \text{if } p \geq 2.
\end{cases}$$

- The definition of $L$ on 0-cells. The homomorphism $L$ takes a representation of the graph in the tricategory, say $f : G \to T$, to the unitary homomorphic representation of the free category

$$L(f) = F : I \to T,$$

such that

$$Fi = fi, \text{ for any vertex } i \text{ of } G (= \text{objects of } I),$$

and $\text{Rep}(G, T) \xrightarrow{\text{Rep}(I, T)} \text{Rep}(I, T)$.
and associates to strings $a : a(p) \rightarrow a(0)$ of adjacent edges in $G$ the 1-cells of $T$

$$Fa = \otimes(fa_1, \ldots, fa_p) : fa(p) \rightarrow fa(0).$$

The structure 2-cells $F_{a,b} : Fa \otimes Fb \Rightarrow F(ab)$, for any pair of strings in the graph, $a = a_1 \cdots a_p$ as above and $b = b_1 \cdots b_q$ with $b(0) = a(p)$, are canonically obtained from the associativity constraint in the tricategory: first by taking $F_{a_1,b} = 1_{F(a_1,b)}$ and then, recursively for $p > 1$, defining $F_{a,b}$ as the composite

$F_{a,b} : Fa \otimes Fb \Rightarrow Fa_1 \otimes (Fa' \otimes Fb) \xrightarrow{1 \otimes F_{a',b}} F(ab),$

where $a' = a_2 \cdots a_p$ (whence $Fa = F_{a_1} \otimes F_{a'}$). And the structure 3-cells $F_{a,b,c}$, for any three strings in the graph $a$, $b$ and $c$ as above with $a(p) = b(0)$ and $b(q) = c(0)$, are the unique isomorphisms constructed from the tricategory coherence 3-cells $\pi$. For a particular construction of these isomorphisms, we can first take each $F_{a_1,b,c}$ to be the canonical isomorphism

$$F(a_1 b) \otimes Fc \xrightarrow{\pi} Fa_1 \otimes (Fb \otimes Fc) \xrightarrow{1 \otimes F_{b,c}} Fa_1 \otimes F(bc),$$

and then, recursively for $p > 1$, take $F_{a,b,c}$ to be the 3-cell canonically obtained from $F_{a_2 \cdots a_p,b,c}$ by pasting the diagram below, where, as above, we write $a'$ for $a_2 \cdots a_p$.

$$\begin{array}{c}
(Fa_1 \otimes (Fa' \otimes Fb)) \otimes Fc \\
\xrightarrow{\pi \\
 \Rightarrow}
(Fa_1 \otimes Fb) \otimes Fc \\
\xrightarrow{\Rightarrow \\
 \Rightarrow}
(Fa_1 \otimes (Fa' \otimes Fb)) \otimes Fc \\
\xrightarrow{1 \otimes F_{b,c} \\
 \Rightarrow}
Fa_1 \otimes (Fb \otimes Fc) \\
\xrightarrow{1 \otimes F_{b,c} \\
 \Rightarrow}
Fa \otimes (Fb \otimes Fc)
\end{array}$$

Note that, since all structure 2-cells $F_{a,b}$ are equivalences in the corresponding hombicategories of $T$ in which they lie, as well as all the structure 3-cells $F_{a,b,c}$ are invertible, the so defined unitary representation $F : I \Rightarrow T$ is actually a homomorphic one; that is, $L(f) = F \in \text{Rep}_{ab}(I, T) \subseteq \text{Rep}(I, T)$.

- The definition of $L$ on 1-cells. Any 1-cell $\phi : f \Rightarrow g$ of $\text{Rep}(G, T)$, is taken by $L$ to the 1-cell in $\text{Rep}_{ab}(I, T)$

$$L(\phi) = \Phi : F \Rightarrow G,$$

consisting of the 2-cells in $T$

$$\Phi_a = \otimes(\phi a_1, \ldots, \phi a_p) : Fa \Rightarrow Ga,$$

attached to the strings $a = a_1 \cdots a_p$ of adjacent edges in the graph. The structure (actually invertible) 3-cells $\Phi_{a,b}$, for any pair of strings in the graph, $a$ and $b$ with $b(0) = a(p)$ as
above, are defined by induction on the length of \( a \) as follows: each \( \Phi_{a_1, b} \) is the canonical isomorphism

\[
\Phi_{a_1, b} : \Phi_{a_1 \otimes \Phi} \cong \Phi_{a_1 b} = \Phi_{a_1 \otimes \Phi}
\]

and then, for \( p > 1 \), each \( \Phi_{a, b} \) is recursively obtained from \( \Phi_{a', b} \), where \( a' = a_2 \cdots a_p \), by pasting

\[
\Phi_{a, b} : \Phi_{a \otimes \Phi} \cong \Phi_{a_1 \otimes \Phi} \otimes \cdots \otimes \Phi = \Phi_{a_1 \otimes \Phi} \otimes \cdots \otimes \Phi
\]

Note that, since all structure 3-cells \( \Phi_{a_1, b} \) are invertible, the thus defined unitary 1-cell \( \Phi \) of \( \text{Rep}_u(I, T) \) is actually a 1-cell of \( \text{Rep}_{uh}(I, T) \), that is, \( L(\phi) \in \text{Rep}_{uh}(I, T) \).

\[
\text{• The definition of } L \text{ on 2-cells . For } \phi, \psi : f \Rightarrow g, \text{ any two 1-cells in } \text{Rep}(G, T), \text{ the homomorphism } L \text{ on a 2-cell } m : \phi \Rightarrow \psi \text{ gives the 2-cell of } \text{Rep}_{uh}(I, T)
\]

\[
L(m) = M : \Phi \Rightarrow \Psi,
\]

consisting of the 3-cells in \( T \)

\[
(19) \quad Ma = \otimes (ma_1, \ldots, ma_p) : \Phi a \Rightarrow \Psi a,
\]

for the strings \( a = a_1 \cdots a_p \) of adjacent edges in the graph \( G \).

\[
\text{• The structure constraints of } L. \text{ If } \phi : f \Rightarrow g \text{ and } \psi : g \Rightarrow h \text{ are 1-cells in } \text{Rep}_u(G, T), \text{ then the structure isomorphism in } \text{Rep}_u(I, T)
\]

\[
L_{\psi, \phi} : L(\psi) \circ L(\phi) \cong L(\psi \circ \phi),
\]

at each string \( a = a_1 \cdots a_p \), as above, is recursively defined as the identity 3-cell on \( \psi a_1 \circ \phi a_1 \) if \( p = 1 \), while, for \( p > 1 \), \( L_{\psi, \phi} a : L(\psi) a \circ L(\phi) a \Rightarrow L(\psi \circ \phi) a \) is obtained from \( L_{\psi, \phi} a' \), where \( a' = a_2 \cdots a_p \), as the composite

\[
L(\psi) a \circ L(\phi) a = (\psi a_1 \otimes L(\psi) a') \circ (\phi a_1 \otimes L(\phi) a') \cong (\psi a_1 \circ \phi a_1) \otimes (L(\psi) a' \circ L(\phi) a')
\]

\[
\otimes L_{\psi, \phi} a' \Rightarrow (\psi a_1 \circ \phi a_1) \otimes L(\psi \circ \phi) a' = L(\psi \circ \phi) a.
\]

And, similarly, the structure isomorphism

\[
L_f : 1_{L(f)} \cong L(1_f)
\]

consists of the 3-cells \( L_f a : 1_{L(f)a} \Rightarrow L(1_f) a \), where \( L_f a_1 = 1 : 1_{fa_1} \Rightarrow 1_{fa_1} \) and, for \( p > 1 \), \( L_f a \) is recursively obtained from \( L_f a' \), \( a' = a_2 \cdots a_p \), as the composite

\[
1_{L(f)a} = 1_{fa_1 \otimes L(f)a'} \cong 1_{fa_1} \otimes 1_{L(f)a'} \otimes L(1_f)a' \Rightarrow 1_{fa_1} \otimes L(1_f)a' = L(1_f)a.
\]

This completes the description of the homomorphism \( L \).

\[
\text{• The definition of lax transformation } v. \text{ The component of this lax transformation at a representation } F : I \rightarrow T, \text{ } v = v(F) : LR(F) \Rightarrow F, \text{ is defined on identities by}
\]

\[
v1_i = F_i : 1_{F_i} \Rightarrow F1_i,
\]
for any vertex i of G, and it associates to each string of adjacent edges in the graph a = a₁⋯aₖ the 2-cell

(20) \( v_a : \otimes (F_{a_1}, \ldots, F_{a_p}) \Rightarrow Fa \),

which is given by taking \( v_a = 1_{F_{a_1}} \) if \( p = 1 \), and then, recursively for \( p > 1 \), by taking \( v_a \) as the composite

\[
\begin{align*}
 v_a : \otimes (F_{a_1}, \ldots, F_{a_p}) &= 1_{\otimes v_{a'}} F_{a_1} \otimes F_{a'} F_{a_1 \cdot \cdot \cdot} F_{a_{p-1} \cdot a'} F_{a_1} \otimes F_{a'}, \\
\end{align*}
\]

where \( a' = a_2 \cdots a_p \).

The structure 3-cell

(21) \( v_{a,b} : F_{a,b} \circ (v_a \otimes v_b) \Rightarrow v(ab) \circ LR(F)_{a,b} \),

for any pair of composable morphisms in I, is defined as follows: when \( a = 1_i \) or \( b = 1_j \) are identities, then \( v_{1,1} \) and \( v_{a_1,b} \) are respectively given by pasting the diagrams

And, for strings \( a \) and \( b \) in the graph with \( b(q) = a(0) \), \( v_{a,b} \) is defined by induction on the length of \( a \) by taking \( v_{a,b} \) to be the canonical isomorphism

\[
\begin{align*}
 F_{a_1} \otimes LR(F) b &= 1_{LR(F)(a_1 b)} \\
 v_{a_1,b} : F_{a_1} \otimes Fb &= F_{a_1,b} \\
\end{align*}
\]

and then, for \( p > 1 \), \( v_{a,b} \) is recursively obtained from \( v_{a',b} \), where \( a' = a_2 \cdots a_p \), by pasting

And the structure 3-cell

(22) \( v_i : v_{1_i} \circ LR(F)_i \Rightarrow F_i \),

for any vertex \( i \) of the graph, is the canonical isomorphism \( F_i \circ 1 \cong F_i \).

The naturality component of \( v \) at a 1-cell \( \Phi : F \Rightarrow G \) in \( \text{Rep}_a(I, T) \),

(23) \( v_{\Phi} : v(G) \circ LR(\Phi) \Rightarrow \Phi \circ v(F) \),
is given on identities by

\[
\begin{array}{c}
1_F 
\overset{1}{\Rightarrow}
 F_i 
\downarrow 
\Phi_i \\
\downarrow G_i 
\downarrow 
\Phi_i \\
1_{F_1} 
\overset{1}{\Rightarrow} 
G_{i_1}
\end{array}
\]

and it is recursively defined at each string of adjacent edges in the graph \(a = a_1 \cdots a_p\), by the 3-cells \(v_\Phi a\) where, if \(p = 1\), then

\[
\begin{array}{c}
F_{a_1} 
\overset{1}{\Rightarrow} 
F_{a_1} \\
v_{\Phi a_1} 
\Rightarrow 
\Phi_{a_1} \\
G_{a_1} 
\overset{1}{\Rightarrow} 
G_{a_1},
\end{array}
\]

is the canonical isomorphism, and then, when \(p > 1\), \(v_\Phi a\) is obtained from \(v_\Phi a'\), where \(a' = a_2 \cdots a_p\), by pasting

\[
\begin{array}{c}
F_{a_1} \otimes LR(F)(a') 
\overset{1 \otimes v(F)(a')}{\Rightarrow} 
F_{a_1} \otimes F_a' 
\overset{F_{a_1}a'}{\Rightarrow} 
F_a \\
v_{\Phi a} : 
\Phi_{a_1} \otimes LR(F)(a') 
\Rightarrow 
\Phi_{a_1} \otimes \Phi_a' 
\Rightarrow 
\Phi_a \\
G_{a_1} \otimes LR(G)(a') 
\overset{1 \otimes v(G)(a')}{\Rightarrow} 
G_{a_1} \otimes G_{a'} 
\overset{G_{a_1}a}{\Rightarrow} 
G_a.
\end{array}
\]

We are now ready to complete the proof of the lemma. That the equalities \(RL = 1\), \(vL = 1\), and \(Rv = 1\) hold only requires a straightforward verification, and then part (a) follows. Moreover, (b) has already been shown by construction of the homomorphism \(L\).

- The proof of (c). Suppose that \(F : I \to T\) is any homomorphic representation. This means that all structure 2-cells \(F_{a,b}\) and \(F_i\) are equivalences, and 3-cells \(F_{a,b,c}\), \(\tilde{F}_a\), and \(\tilde{F}_a\) are isomorphisms in the hom-bicategories of \(T\) in which they lie. Then, directly from the construction given, it easily follows that all the 2-cells \(v(F)a\) in (20) are equivalences in the corresponding hom-bicategories, and that all the 3-cells \(v(F)_{a,b}\) in (21), and \(v_i\) in (22) are invertible. Hence, each \(v(F) : LR(F) \Rightarrow F\), for \(F : I \to T\) any homomorphic representation, is an equivalence in the bicategory \(\text{Rep}_h(I, T)\). Moreover, if \(\Phi : F \Rightarrow G\) is any 1-cell in \(\text{Rep}_h(I, T)\), so that every 3-cell \(\Phi_{a,b}\) is an isomorphism, then we see that the component (23) of \(v\) at \(\Phi\) consists only of invertible 3-cells \(v_\Phi a\), whence \(v_\Phi\) is invertible itself. Therefore, when \(v\) is restricted to \(\text{Rep}_h(I, T)\), it actually gives a pseudo-equivalence between \(LR\) and \(1\), the identity homomorphism on the bicategory \(\text{Rep}_h(I, T)\). The claimed biadjoint biequivalence (13) is now an easy consequence of all the already parts proved. Finally, it is clear that the biadjoint biequivalence (13) gives by restriction the biadjoint biequivalence (14).

\[\square\]

3. The Grothendieck Nerve of a Tricategory

Let us briefly recall that it was Grothendieck \[15\] who first associated a simplicial set

(24)

\[NC : \Delta^{op} \to \text{Set}\]

to a small category \(C\), calling it its nerve. The set of \(p\)-simplices

\[N_pC = \coprod_{(c_0, \ldots, c_p)} C(c_1, c_0) \times C(c_2, c_1) \times \cdots \times C(c_p, c_{p-1})\]
consists of length \( p \) sequences of composable morphisms in \( C \). Geometric realization of its nerve is the **classifying space** of the category, \( BC \). A main result here shows how the Grothendieck nerve construction for categories rises to tricategories.

### 3.1. The pseudo-simplicial bicategory nerve of a tricategory

When a tricategory \( \mathcal{T} \) is strict, that is, a 3-category, then the nerve construction (24) actually works by giving a simplicial 2-category (see Example 4.3). However, for an arbitrary tricategory, the device is more complicated since the compositions of cells in a tricategory is in general not associative and not unitary (which is crucial for the simplicial structure in the construction of \( NT \) as above), but it is only so up to coherent isomorphisms. This ‘defect’ has the effect of forcing one to deal with the classifying space of a nerve of \( \mathcal{T} \) that is not simplicial but only up to coherent isomorphisms, that is, a pseudo-simplicial bicategory as stated in the theorem below. Pseudo-simplicial bicategories, and the tricategory they form (whose 1-cells are pseudo-simplicial homomorphisms, 2-cells pseudo-simplicial transformations, and 3-cells pseudo-simplicial modifications) are treated in [10], to which we refer the reader.

**Theorem 3.1.1.** Any tricategory \( \mathcal{T} \) defines a normal pseudo-simplicial bicategory, called the nerve of the tricategory,\( \text{(25) } NT = (NT, \chi, \omega) : \Delta^{\text{op}} \to \text{Bicat}, \)

whose bicategory of \( p \)-simplices, for \( p \geq 1 \), is
\( \text{(26) } N_pT = \bigsqcup_{(t_0, \ldots, t_p) \in \text{Ob}{\mathcal{T}}^{p+1}} T(t_1, t_0) \times T(t_2, t_1) \times \cdots \times T(t_p, t_{p-1}), \)

and \( N_0T = 0b\mathcal{T} \), as a discrete bicategory. The face and degeneracy homomorphisms are defined on 0-cells, 1-cells and 2-cells of \( N_pT \) by the ordinary formulas
\( \text{(27) } d_i(x_1, \ldots, x_p) = \begin{cases} (x_2, \ldots, x_p) & \text{if } i = 0, \\ (x_1, \ldots, x_i \otimes x_{i+1}, \ldots, x_p) & \text{if } 0 < i < p, \\ (x_1, \ldots, x_{p-1}) & \text{if } i = p, \end{cases} \)

\( s_i(x_1, \ldots, x_p) = (x_1, \ldots, x_i, 1, x_{i+1}, \ldots, x_p). \)

Indeed, if \( a : [q] \to [p] \) is any map in the simplicial category \( \Delta \), then the associated homomorphism \( N_a : N_pT \to N_qT \) is induced by the composition \( T(t', t) \times T(t'', t') \overset{\otimes}{\to} T(t'', t) \) and unit \( 1_t : 1 \to T(t, t) \) homomorphisms. The structure pseudo-equivalences
\( \text{(28) } \)

\[
\begin{array}{c}
N_pT \\
\downarrow_{\chi_{a,b}} \\
\downarrow_{\omega_{a,b,c}} \\
N_nT \\
\end{array}
\]

for each pair of composable maps \([n] \overset{b}{\to} [q] \overset{a}{\to} [p] \) in \( \Delta \), and the invertible modifications
\( \text{(29) } \)

\[
\begin{array}{c}
N_pT \\
\downarrow_{\chi_{a,b}} \\
\downarrow_{\omega_{a,b,c}} \\
N_nT \\
\end{array}
\]

\[
\begin{array}{c}
N_pT \\
\downarrow_{\chi_{a,b,c}} \\
N_nT \\
\end{array}
\]

respectively associated to triplets of composable arrows \([n] \overset{b}{\to} [q] \overset{a}{\to} [p] \), canonically arise all from the structure pseudo equivalences and modifications data of the tricategory.
We shall prove Theorem 3.1.1 simultaneously with Theorem 3.1.2 below, which states the basic properties concerning the behavior of the Grothendieck nerve construction, $\mathcal{T} \mapsto N\mathcal{T}$, with respect to trihomomorphisms of tricategories.

**Theorem 3.1.2.** (i) Any trihomomorphism between tricategories $H : \mathcal{T} \to \mathcal{T}'$ induces a normal pseudo-simplicial homomorphism

$$NH = (NH, \theta, \Pi) : N\mathcal{T} \to N\mathcal{T}'$$

which, at any integer $p \geq 0$, is the evident homomorphism $N_pH : N_p\mathcal{T} \to N_p\mathcal{T}'$ defined on any cell $(x_1, \ldots, x_p)$ of $N_p\mathcal{T}$ by

$$N_pH(x_1, \ldots, x_p) = (Hx_1, \ldots, Hx_p).$$

The structure pseudo-equivalence

$$\xymatrix{ N_p\mathcal{T} \ar[rr]^{N_pH} \ar[d]_{N\mathcal{T}} & & N_p\mathcal{T}' \ar[d]^{N\mathcal{T}'} \cr N_q\mathcal{T} \ar[rr]_{N_H} & & N_q\mathcal{T}' }$$

for each map $a : [q] \to [p]$ in $\Delta$, and the invertible modifications

$$\xymatrix{ N_nH \ar@{=>}[r]^{\theta_{N\mathcal{T}}}_N \ar@{=>}[d]_{N\mathcal{T}'} & \ar@{=>}[d]^\theta \cr N_n\mathcal{T} \ar@{=>}[r]_{N_H} & N_n\mathcal{T}' }$$

respectively associated to pairs of composable arrows $[n] \to [q] \to [p]$, canonically arise all from the structure pseudo equivalences and modifications data of the trihomomorphism $H$ and the involved tricategories $\mathcal{T}$ and $\mathcal{T}'$.

(ii) For any pair of composable trihomomorphisms $H : \mathcal{T} \to \mathcal{T}'$ and $H' : \mathcal{T}' \to \mathcal{T}''$, there is a pseudo-simplicial pseudo-equivalence

$$NH' \circ NH \Rightarrow N(H'H).$$

(iii) For any tricategory $\mathcal{T}$, there is a pseudo-simplicial pseudo-equivalence

$$N1_{\mathcal{T}} \Rightarrow 1_{N\mathcal{T}}.$$

**Proof of Theorems 3.1.1 and 3.1.2.** Let us note that, for any integer $p \geq 0$, the category $[p]$ is free on the graph

$$\mathcal{G}_p = (p \to \cdots \to 1 \to 0),$$

and that $N_p\mathcal{T} = \text{Rep}(\mathcal{G}_p, \mathcal{T})$. The existence of a biadjoint biequivalence

$$L_p \dashv R_p : N_p\mathcal{T} \rightleftarrows \text{Rep}_n([p], \mathcal{T})$$

follows from Lemma 2.7.1, where $R_p$ is the 2-functor defined by restricting to the basic graph $\mathcal{G}_p$ of the category $[p]$, such that $R_pL_p = 1$, whose unity is the identity, and whose counit $v_p : L_pR_p \Rightarrow 1$ is a pseudo-equivalence satisfying the equalities $v_pL_p = 1$ and $R_pv_p = 1$. 


Then, if \( a : [q] \rightarrow [p] \) is any map in the simplicial category, the associated homomorphism \( N_a : N_p \mathcal{T} \rightarrow N_q \mathcal{T} \), is defined to be the composite

\[
N_p \mathcal{T} \xrightarrow{L_p} \operatorname{Rep}_h([p], \mathcal{T}) \xrightarrow{a^*} \operatorname{Rep}_h([q], \mathcal{T}).
\]

Observe that, thus defined, the homomorphism \( N_a \) maps the component bicategory of \( N_p \mathcal{T} \) at \( (t_0, \ldots, t_p) \) into the component at \( (t_{a(0)}, \ldots, t_{a(q)}) \) of \( N_q \mathcal{T} \), and it acts on 0-cells, 1-cells, and 2-cells of \( N_p \mathcal{T} \) by the formula

\[
N_a(x_1, \ldots, x_p) = (y_1, \ldots, y_q)
\]

where, for \( 0 \leq k < q \), (see (15) for the definition of \( \otimes \))

\[
y_{k+1} = \begin{cases} \otimes (x_{a(k)+1}, \ldots, x_{a(k+1)}) & \text{if } a(k) < a(k+1), \\ 1 & \text{if } a(k) = a(k+1).
\end{cases}
\]

Whence, in particular, the formulas (27) for the face and degeneracy homomorphisms.

The pseudo natural equivalence (28) is

\[
N_b \xrightarrow{N_b \circ L_p} \operatorname{Rep}_h([p], \mathcal{T}) \xrightarrow{\alpha^*} \operatorname{Rep}_h([q], \mathcal{T}) = N_b \xleftarrow{R_q \circ R_p} N_q \mathcal{T},
\]

and the invertible modification (29) is

\[
\omega_{a,b,c} = R_m \circ \omega'_b \circ \alpha^* \circ L_p,
\]

where \( \omega'_b \) is the canonical modification.

Thus defined, \( N \mathcal{T} \) is actually a normal pseudo-simplicial bicategory. Both coherence conditions for \( N \mathcal{T} \) (i.e., conditions (CC1) and (CC2) in \([10]\), with the modifications \( \gamma \) and \( \delta \) the unique unity coherence isomorphisms \( 1 \circ 1 \cong 1 \)) follow from the equalities \( R_p \circ L_p = 1 \), \( v_p \circ L_p = 1 \), and \( R_p \circ v_p = 1 \), by employing the useful Fact 3.1.3 below. This proves Theorem 3.1.1.

**Fact 3.1.3.** Let \( \alpha : F \Rightarrow F' : \mathcal{B} \rightarrow \mathcal{C} \) be a lax transformation between homomorphisms of bicategories. Then, for any 2-cell in \( \mathcal{B} \)

\[
\begin{array}{c}
\xymatrix{\cdots x_n \ar[r]^{a_n} & \text{ } \ar[r]^{a_n} & \cdots x_1 \ar[r]^{x_1} & x \ar[r]^{u} & x' \ar[r]^{x'} & \cdots x_m \ar[r]^{b_m} & \cdots x_1' \ar[r]^{b_0} & \text{ } \ar[r]^{b_0} & \cdots x_n'}
\end{array}
\]
the following equality holds:

\[
\begin{array}{c}
\begin{array}{c}
F_{a_0} F_{x_1} \cdots F_{x_n} F_{a_n} \\
\cdots \\
F_{a_m} \rightarrow \cdots \rightarrow F_{x_m} F_{a_m}
\end{array}
\end{array}
\]

And when it comes to Theorem 3.1.2 first, let us note that the homomorphisms \(N_p H : N_p T \rightarrow N_p T', p \geq 0\), make commutative the diagrams

\[
\begin{array}{c}
\begin{array}{c}
N_p T \xrightarrow{L=L''_p} \Rep_h([p], T) \xrightarrow{H_r} \Rep_h([p], T'),
\end{array}
\end{array}
\]

where \(H_r\) is the induced homomorphism by the trihomomorphism \(H : T \rightarrow T'\) (see Lemma 2.6.1 (i)). Then, the pseudo-equivalence \(\theta_a\), is provided by the pseudo-equivalences \(\nu : LR \Rightarrow 1\) and their adjoint quasi-inverses \(\nu^* : 1 \Rightarrow LR\) (which we can choose such that \(R\nu^* = 1\) and \(\nu^* L = 1\)); that is, \(\theta_a = Ra^* \circ \nu^* H_s L \circ RH_a L\),

\[
\begin{array}{c}
R H_a^* L = Ra^* H_s L
\end{array}
\]

\(N_q H N_0 T = RH_s LRa^* L \xrightarrow{\theta_a} Ra^* LRH_s L = N_0 T' N_p H.\)

And, for \([n] \xrightarrow{\sim} [g] \xrightarrow{\sim} [p]\), any two composable arrows of \(\Delta\), the structure invertible modification \(\Pi_{a,b}\), is the modification obtained by pasting the diagram

\[
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
RH_s Lb^* LRa^* L \\
\cdots \\
Rb^* LRH_s L
\end{array}
\end{array}
\end{array}
\]

where the isomorphism labelled \((A)\) is given by the adjunction invertible modification \(\nu \circ \nu^* \cong 1\).

The coherence conditions for \(NH : NT \rightarrow NT'\) (i.e., conditions (CC3) and (CC4) in [10], with the modifications \(\Gamma\) the coherence isomorphisms \(1 \circ 1 \cong 1\)), are easily verified by using Fact 3.1.3.
Suppose now $\mathcal{T} \xrightarrow{H} \mathcal{T}' \xrightarrow{H'} \mathcal{T}''$ two composable trihommorphisms. Then, the pseudo-simplicial pseudo-equivalence $\alpha_N : \text{Rep}_h([p], \mathcal{T}) \to \text{Rep}_h([p], \mathcal{T}'')$ is defined by the family of pseudo-equivalences

\[
\begin{array}{c}
\alpha_N^* : \text{Rep}_h([q], \mathcal{T}) \to \text{Rep}_h([q], \mathcal{T}')
\end{array}
\]

where the pseudo-equivalence $m : H'_*H_* \Rightarrow (H'H)_* : \text{Rep}_h([p], \mathcal{T}) \to \text{Rep}_h([p], \mathcal{T}'')$ is that given in Lemma 2.6.1 (ii). The naturality component of $\alpha$ at any map $a : [q] \to [p]$,

\[
\begin{array}{c}
\alpha_N^* : \text{Rep}_h([q], \mathcal{T}) \to \text{Rep}_h([q], \mathcal{T}')
\end{array}
\]

is provided by the invertible modification obtained by pasting in

\[
\begin{array}{c}
\alpha_N^* : \text{Rep}_h([q], \mathcal{T}) \to \text{Rep}_h([q], \mathcal{T}')
\end{array}
\]

where the isomorphism labelled (A) is given by the adjunction invertible modification $\text{vol}^* \cong 1$.

Finally, the pseudo-simplicial pseudo-equivalence $\beta_N : N_1\mathcal{T} \to 1_N\mathcal{T}$, is defined by the family of pseudo-equivalences

\[
\begin{array}{c}
\beta_N : N_1\mathcal{T} \to 1_N\mathcal{T}
\end{array}
\]

where $m : (1_N\mathcal{T})_* \Rightarrow 1 : \text{Rep}_h([p], \mathcal{T}) \to \text{Rep}_h([p], \mathcal{T})$ is the pseudo-equivalence in Lemma 2.6.1 (iii). The naturality invertible modification attached at any map $a : [q] \to [p]$,

\[
\begin{array}{c}
\beta_N : N_1\mathcal{T} \to 1_N\mathcal{T}
\end{array}
\]
is that obtained by pasting the diagram

\[
\begin{array}{c}
R(1_\tau)_* LRa^* L & \xrightarrow{RmL(Ra^*)} & RLRa^* L = Ra^* L \\
R(1_\tau)_* va^* L & \xrightarrow{1=Ra^* L} & \equiv 1 \\
R(1_\tau)_* a^* L & \xrightarrow{Rma^* L} & Ra^* L \\
Ra^* v^* (1_\tau)_* L & \xrightarrow{Ra^* v^* L} & \equiv 1 \\
Ra^* LR(1_\tau)_* L & \xrightarrow{Ra^* LRmL} & Ra^* LRa^* L = Ra^* L.
\end{array}
\]

The coherence conditions \((\text{CC5})\) and \((\text{CC6})\) in \([10]\), for both \(\alpha\) and \(\beta\), are plainly verified.

\[\square\]

4. The classifying space of a tricategory

4.1. Preliminaries on classifying spaces of bicategories. When a bicategory \(\mathcal{B}\) is regarded as a tricategory all of whose 3-cells are identities, the nerve construction \(\text{Nerve}\) on it actually produces a normal pseudo-simplicial category

\[\text{NB} = (\text{NB}, \chi) : \Delta^{op} \rightarrow \text{Cat},\]

which is called in \([9, \S3]\) the pseudo-simplicial nerve of the bicategory. The classifying space of the bicategory, denoted by \(\mathcal{B}\mathcal{B}\), is then defined to be the ordinary classifying space of the category obtained by the Grothendieck construction \([19]\) on the nerve of the bicategory, that is, \(\mathcal{B}\mathcal{B} = \mathcal{B}\text{Nerve}\mathcal{B}\) \([9, \text{Definition 3.1}]\). The following facts, concerning classifying spaces of bicategories, are proved in \([9, (30)\) and Theorem 7.1]:

**Fact 4.1.1.** Each homomorphism between bicategories \(F : \mathcal{B} \rightarrow \mathcal{C}\) induces a continuous cellular map \(BF : \mathcal{B}\mathcal{B} \rightarrow \mathcal{C}\mathcal{B}\). Thus, the classifying space construction, \(\mathcal{B} \mapsto \mathcal{B}\mathcal{B}\), defines a functor from the category \(\text{Hom}\) of bicategories to CW-complexes.

**Fact 4.1.2.** If \(F, F' : \mathcal{B} \rightarrow \mathcal{C}\) are two homomorphisms between bicategories, then any lax (or oplax) transformation, \(F \Rightarrow F'\), canonically defines a homotopy between the induced maps on classifying spaces, \(BF \simeq BF' : \mathcal{B}\mathcal{B} \rightarrow \mathcal{C}\mathcal{B}\).

**Fact 4.1.3.** If a homomorphism of bicategories has a left or right biadjoint, the map induced on classifying spaces is a homotopy equivalence. In particular, any biequivalence of bicategories induces a homotopy equivalence on classifying spaces.

Furthermore, we should recall that the classifying space of any pseudo-simplicial bicategory \(\mathcal{F} : \Delta^{op} \rightarrow \text{Bicat}\) is defined \([10, \text{Definition 5.4}]\) to be the classifying space of its bicategory of simplices \(\int_{\Delta} \mathcal{F}\), also called the Grothendieck construction on \(\mathcal{F}\) \([10, \S3.1]\). That is, the bicategory whose objects are the pairs \((x, p)\), where \(p \geq 0\) is an integer and \(x\) is an object of the bicategory \(\mathcal{F}_p\), and whose hom-categories are

\[\int_{\Delta} \mathcal{F}((y, q), (x, p)) = \bigsqcup_{[q] \rightarrow [p]} \mathcal{F}_q(y, a^* x),\]

where the disjoint union is over all arrows \(a : [q] \rightarrow [p]\) in the simplicial category \(\Delta\); compositions, identities, and structure constraints are defined in the natural way. We refer the reader to \([10, \S3]\) for details about the bicategorical Grothendieck construction trihomomorphism

\[\int_{\Delta} : \text{Bicat}^{\Delta^{op}} \rightarrow \text{Bicat},\]
from the tricategory of pseudo-simplicial bicategories to the tricategory of bicategories. The following facts, concerning classifying spaces of pseudo-simplicial bicategories, are proved in [10, (42), (43), Proposition 5.5, and Theorem 5.7] :

**Fact 4.1.4.** (i) If $F, G : \Delta^\text{op} \to \text{Bicat}$ are pseudo-simplicial bicategories, then each pseudo-simplicial homomorphism $F \to G$ induces a continuous map $\int_\Delta F : \int_\Delta F \to \int_\Delta G$.

(ii) For any pseudo-simplicial bicategory $F : \Delta^\text{op} \to \text{Bicat}$, there is a homotopy $\int_\Delta 1_F \simeq 1_{\int_\Delta F} : \int_\Delta F \to \int_\Delta F$.

(iii) For any pair of composable pseudo-simplicial homomorphism $F \to G, G \to H$, there is a homotopy $\int_\Delta G \int_\Delta F \simeq \int_\Delta (GF) : \int_\Delta F \to \int_\Delta H$.

**Fact 4.1.5.** Any pseudo-simplicial transformation $F \Rightarrow G : F \to G$ induces a homotopy $\int_\Delta F \simeq \int_\Delta G : \int_\Delta F \to \int_\Delta G$.

**Fact 4.1.6.** If $F : \Delta^\text{op} \to \text{Hom} \subset \text{Bicat}$ is a simplicial bicategory, then there is a natural homotopy equivalence $\int_\Delta F \simeq |BF|$, where the latter is the geometric realization of the simplicial space $BF : \Delta^\text{op} \to \text{Top}$, obtained by composing $F$ with the classifying space functor $B : \text{Hom} \to \text{Top}$.

4.2. The classifying space construction for tricategories. We are now ready to set the following definition, which recovers the more traditional way through which a classifying space is assigned in the literature to certain specific kinds of tricategories, such as 3-categories, bicategories, monoidal categories, or braided monoidal categories (see Examples 4.3, 4.6, and 4.7 below, also [4, 10] or [23] and references therein).

**Definition 4.2.1.** The classifying space $BT$, of a tricategory $\mathcal{T}$, is the classifying space of its bicategorical pseudo-simplicial Grothendieck nerve $(25), N\mathcal{T} : \Delta^\text{op} \to \text{Bicat}$, that is,

$$BT := \int_\Delta N\mathcal{T}.$$  

Let us remark that the classifying space of a tricategory $\mathcal{T}$ is then realized as the classifying space of a category canonically associated to it, namely, as

$$\int_\Delta N\mathcal{T} = |\int_\Delta N(\int_\Delta N\mathcal{T})|.$$  

The next two theorems deal with the basic properties concerning with the homotopy behavior of the classifying space construction, $\mathcal{T} \mapsto BT$, with respect to trihomomorphisms of tricategories.

**Theorem 4.2.2.** (i) Any trihomomorphism between tricategories $H : \mathcal{T} \to \mathcal{T}'$ induces a continuous map $BH : BT \to BT'$.

(ii) For any pair of composable trihomomorphisms $H : \mathcal{T} \to \mathcal{T}'$ and $H' : \mathcal{T}' \to \mathcal{T}''$, there is a homotopy $BH' BH \simeq B(H'H) : BT \to BT''$.  

(iii) For any tricategory $T$, there is a homotopy $B1_T \simeq 1_{B{T}} : B{T} \to B{T}$.

Proof. (i) By Theorem 3.1.2 (i), any trihomomorphism $H : T \to T'$ gives rise to a pseudo-simplicial homomorphism on the corresponding Grothendieck nerves $NH : NT \to NT'$, which, by Fact 4.1.4 (i), produces the claimed continuous map $B_H = B\int \Delta N : B{T} \to B{T'}$.

(ii) Suppose $T \xrightarrow{H'} T' \xrightarrow{H} T''$ are trihomomorphisms. By Theorem 3.1.2 (ii), there is a pseudo-simplicial pseudo-equivalence $N'H \Rightarrow N(H'H)$, which, by Fact 4.1.5, induces a homotopy $B\int \Delta (N'H) \simeq B\int \Delta N(H'H') = B(H'H)$.

Then, the result follows since, by Fact 4.1.4 (iii), there is a homotopy $B\int \Delta (N'H) \simeq B\int \Delta N' \Rightarrow B\int \Delta N = B'H'B$. 

(iii) By Theorem 3.1.2 (iii), there is a pseudo-simplicial pseudo-equivalence $N1_T \Rightarrow 1_{NT}$, which, by Fact 4.1.5, induces a homotopy $B1_T = B\int \Delta N1_T \simeq B\int \Delta 1_{NT}$. Since, by Fact 4.1.4 (ii), there is a homotopy $B\int \Delta 1_{NT} \simeq B1_{\Delta NT} = 1_{B{T}}$, the result follows. □

**Theorem 4.2.3.** If $F, G : T \to T'$ are two trihomomorphisms between tricategories, then any tritransformation, $F \Rightarrow G$, canonically defines a homotopy between the induced maps on classifying spaces, $BF \simeq BG : B{T} \to B{T'}$.

Proof. Suppose $\theta = (\theta, \Pi, M) : F \Rightarrow G : T \to T'$ is a tritransformation. There is a trihomomorphism $H : T \times [1] \to T'$ making the diagram commutative

$$
\begin{array}{c}
T \times [0] \cong T \\
\downarrow 1 \times \delta_0 \\
T \times [1] \\
\downarrow 1 \times \delta_1 \\
T \times [0] \cong T \\
\end{array}
$$

such that, for any objects $p, q$ of $T$, the homomorphism

$$
H = H_{(p,1)}(q,0) : T \times [1]((p,1),(q,0)) \to T'(Fp,Gq)
$$

is the composite of

$$
T(p,q) \times \{(1,0)\} \cong T(p,q) \xrightarrow{G} T'(Gp,Gq) \xrightarrow{T'(\theta_p,1)} T'(Fp,Gq).
$$

For objects $p, q, r$ of $T$, the pseudo-equivalence

$$
\begin{array}{c}
(T \times [1])((q,0),(r,0)) \xrightarrow{(T \times [1])(p,1)} (T \times [1])(p,1),(q,0)) \\
\downarrow \otimes \\
T \times [1](p,1),(r,0)) \xrightarrow{H} T'(Fp,Gq)
\end{array}
$$

$$
\begin{array}{c}
\otimes \\
\otimes \\
\end{array}
$$

$$
\begin{array}{c}
(T \times [1])((q,0),(r,0)) \xrightarrow{H \times H} T'(Gq,Gr) \times T'(Fp,Gq) \\
\end{array}
$$

$$
\begin{array}{c}
\otimes \\
\otimes \\
\end{array}
$$

$$
\begin{array}{c}
\otimes \\
\otimes \\
\end{array}
$$

$$
\begin{array}{c}
\otimes \\
\otimes \\
\end{array}
$$

$$
\begin{array}{c}
\otimes \\
\otimes \\
\end{array}
$$

$$
\begin{array}{c}
\otimes \\
\otimes \\
\end{array}
$$
is obtained by pasting the diagram

\[
\begin{array}{c}
\mathcal{T}(q,r) \times \mathcal{T}(p,q) \xrightarrow{G \times G} \mathcal{T}'(Gq,Gr) \times \mathcal{T}'(Gp,Gq) \\
\odot \quad \odot \quad \odot \quad \odot \\
\mathcal{T}(p,r) \xrightarrow{G} \mathcal{T}'(Gp,Gr) \xrightarrow{\mathcal{T}'(\theta_p,1)} \mathcal{T}'(Fp,Gr),
\end{array}
\]

and the pseudo-equivalence

\[
(\mathcal{T} \times [1])((q,1),(r,0)) \times (\mathcal{T} \times [1])((p,1),(q,1)) \xrightarrow{H \times H} \mathcal{T}'(Fq,Gr) \times \mathcal{T}'(Fp,Fq)
\]

by pasting in

\[
\begin{array}{c}
\mathcal{T}(q,r) \times \mathcal{T}(p,q) \xrightarrow{G \times F} \mathcal{T}'(Gq,Gr) \times \mathcal{T}'(Fp,Fq) \xrightarrow{\mathcal{T}'(\theta_q,1) \times 1} \mathcal{T}'(Fq,Gr) \times \mathcal{T}'(Fp,Fq) \\
\odot \quad \odot \quad \odot \quad \odot \\
\mathcal{T}(p,r) \xrightarrow{F \times F} \mathcal{T}'(Fp,Fr) \xrightarrow{\mathcal{T}'(1,\theta_r) \times 1} \mathcal{T}'(Fp,Gr) \xrightarrow{\mathcal{T}'(\theta_p,1)} \mathcal{T}'(Fp,Gr).
\end{array}
\]

For \(p,q,r,s\) any objects of \(\mathcal{T}\), the component of the invertible modification \(\omega^H\) at the triples of composable 1-cells of \(\mathcal{T} \times [1]\)

\[
(p,1) \xrightarrow{(z,1,0)} (q,0) \xrightarrow{(y,1,0)} (r,0) \xrightarrow{(x,1,0)} (s,0),
\]

are canonically provided by the 3-cells (46), (47) and (48) below.

\[
\begin{array}{c}
((Gx \otimes Gy) \otimes Gz) \otimes \theta_p \xrightarrow{\chi^G \otimes \theta_p} G(x \otimes y) \otimes Gz \otimes \theta_p \\
\odot \odot \quad \odot \quad \odot \\
G(x \otimes (G(y \otimes z)) \otimes \theta_p \xrightarrow{\chi^G \otimes \theta_p} G(x \otimes (y \otimes z)) \otimes \theta_p.
\end{array}
\]

(46)

\[
\begin{array}{c}
(Gx \otimes (Gy \otimes Gz)) \otimes \theta_p \xrightarrow{(1 \otimes \chi^G) \otimes \theta_p} G(x \otimes (G(y \otimes z)) \otimes \theta_p \\
\odot \odot \quad \odot \quad \odot \\
G(x \otimes (y \otimes z)) \otimes \theta_p.
\end{array}
\]

(47)
(48)\[
Gx \otimes ((\theta_r \otimes Fy) \otimes Fz) \xrightarrow{1 \otimes (\delta \otimes 1)} Gx \otimes ((Gy \otimes \theta_q) \otimes Fz) \xrightarrow{1 \otimes \alpha} Gx \otimes (Gy \otimes (\theta_q \otimes Fz)) \xleftarrow{1 \otimes (1 \otimes \theta)} Gx \otimes (Gy \otimes (Gz \otimes \theta_p)) \xrightarrow{1 \otimes (1 \otimes 1)} Gx \otimes (G(y \otimes z) \otimes \theta_p) \xrightarrow{1 \otimes (\chi \otimes 1)} Gx \otimes ((Gy \otimes Gz) \otimes \theta_p)
\]

To finish the description of the homomorphism $H$, say that the component of the invertible modification $\delta^H$ at any morphism $(x, (1, 0)) : (p, 1) \to (q, 0)$ is canonically obtained from the 3-cells $1 \otimes M$ and $\delta^G \otimes 1$ below, while the component of $\gamma^H$ is provided by 3-cell $\gamma^G \otimes 1$.

\[
Gx \otimes \theta_p \xrightarrow{1 \otimes r} Gx \otimes (\theta_p \otimes 1) \xrightarrow{1 \otimes (1 \otimes \delta)} Gx \otimes (\theta_p \otimes F1) \xrightarrow{1 \otimes 1^*} Gx \otimes (\theta_p \otimes (G1 \otimes \theta_q)),
\]

\[
Gx \otimes (1 \otimes \theta_p) \xrightarrow{1 \otimes (\chi \otimes 1)} Gx \otimes (G(y \otimes z) \otimes \theta_p) \xrightarrow{1 \otimes a^*} Gx \otimes ((Gy \otimes Gz) \otimes \theta_p).
\]

We are now ready to complete the proof of the theorem: Applying the classifying space construction to diagram (48), we obtain a diagram of maps

\[
\begin{array}{ccc}
B T \times B[0] & \cong & B T
\\
\downarrow 1 \times \delta_0 & & \downarrow B F
\\
B T \times B[1] & \xrightarrow{B H} & B T',
\end{array}
\]

where, by Theorem (12.2) (ii), both triangles are homotopy commutative. Since $B[1] = [0, 1]$, the unity interval, the result follows.

As a relevant consequence for triequivalences between tricategories [17, Definition 3.5], we have the following:

**Theorem 4.2.4.** (i) If $F : T \to T'$ is any trihomomorphism such that there are a trihomomorphism $G : T' \to T$ and tritransformations $FG \Rightarrow 1_{T'}$ and $1 \Rightarrow GF$, then the induced map $BF : B T \to B T'$ is a homotopy equivalence.

(ii) Any triequivalence of tricategories induces a homotopy equivalence on classifying spaces.

**Proof.** (i) Given any trihomomorphism $F : T \to T'$ in the hypothesis, by Theorem (12.2) (ii), there is a homotopy $BF BG \approx B(FG)$. By Theorem (12.3) the existence of a homotopy $B(FG) \approx B1_{T'}$ follows. Since, by Theorem (12.2) (iii), there is a homotopy $B1_{T'} \approx B1_{T'}$, we conclude the existence of a homotopy $BF BG \approx B1_{T'}$. Analogously, we can prove that $1_{B T} \approx BG BF$, which completes the proof.
Part (ii) clearly follows from part (i).

\[\square\]

4.3. Example: Classifying spaces of 3-categories. In [36], Segal observed that, if \(C\) is a topological category, then its Grothendieck nerve [24] is, in a natural way, a simplicial space, that is, \(NC : \Delta^\text{op} \rightarrow \text{Top}\). Then, he defines the classifying space of a topological category \(C\), the realization of this simplicial space.

This notion given by Segal provides, for instance, the usual definition of classifying spaces of strict tricategories, or 3-categories (hence of categories and 2-categories, which can be regarded as special 3-categories). A 3-category \(\mathcal{T}\) is just a category enriched in the category of 2-categories and 2-functors, that is, a category \(\mathcal{T}\) endowed with 2-categorical hom-sets \(\mathcal{T}(t', t)\), in such a way that the compositions \(\mathcal{T}(t', t) \times \mathcal{T}(t'', t') \rightarrow \mathcal{T}(t'', t)\) are 2-functors. By replacing the hom 2-categories \(\mathcal{T}(t', t)\) by their classifying spaces, we obtain a topological category, say \(\mathcal{C}_{\mathcal{T}}\), with discrete space of objects and whose hom-spaces are \(\mathcal{C}_{\mathcal{T}}(t', t) = B\mathcal{T}(t', t)\). The classifying space of this topological category \(|NC_{\mathcal{T}}|\) is, by definition, the classifying space of the 3-category \(\mathcal{T}\).

**Theorem 4.3.1.** For any 3-category \(\mathcal{T}\) there are homotopy equivalences
\[
|NC_{\mathcal{T}}| \simeq B\mathcal{T} \simeq |\text{diag} \text{NNN}_{\mathcal{T}}|,
\]
where \(\text{NNN}_{\mathcal{T}} : \Delta^{\text{op}} \times \Delta^{\text{op}} \times \Delta^{\text{op}} \rightarrow \text{Set}\), \([(p), [q], [r]] \mapsto N_p N_q N_r \mathcal{T}\), is the trisimplicial set triple nerve of the 3-category, and \(\text{diag} \text{NNN}_{\mathcal{T}} : \Delta^{\text{op}} \rightarrow \text{Set}\), \([p] \mapsto N_p N_q N_r \mathcal{T}\), its diagonal simplicial set.

**Proof.** Note that, for any 3-category \(\mathcal{T}\), the equality of simplicial spaces \(NC_{\mathcal{T}} = B\mathcal{T}\) holds, where the latter is the simplicial space obtained by composing its (actually simplicial 2-category) nerve [24], \(\mathcal{N}_{\mathcal{T}} : \Delta^{\text{op}} \rightarrow \text{2-Cat}\), with the classifying functor \(B : \text{2-Cat} \rightarrow \text{Top}\). Then, by Fact 4.1.7 there is a natural homotopy equivalence \(B\mathcal{T} \simeq |NC_{\mathcal{T}}|\).

Furthermore, an iterated use of the natural homotopy equivalences \(B\mathcal{T} \simeq |B\mathcal{T}|\) (which, of course, also work both for 2-categories and categories) give the following chain of homotopy equivalences, for any 3-category \(\mathcal{T}\),
\[
B\mathcal{T} \simeq |[p] \mapsto BN_p \mathcal{T}| \simeq |[p] \mapsto |[q] \mapsto BN_q N_p \mathcal{T}||
\]
\[
\simeq |[p] \mapsto |[q] \mapsto |[r] \mapsto N_q N_p N_r \mathcal{T}|| \simeq |\text{diag} \text{NNN}_{\mathcal{T}}|,
\]
where the last homeomorphism above is a consequence of Quillen’s Lemma [35 page 86]. \(\square\)

4.4. Example: Classifying spaces of bicategories. When a bicategory \(\mathcal{B}\) is viewed as a tricategory whose 3-cells are all identities, then its classifying space as a tricategory, according to Definition 3.1, coincides with the classifying space of the bicategory, \(\mathcal{B}\mathcal{E}\), as defined in [9 Definition 3.1].

4.5. The Segal nerve of a tricategory. Several theoretical interests suggest dealing with the so-called Segal nerve construction for tricategories. This associates to any tricategory \(\mathcal{T}\) a simplicial bicategory, denoted by \(\mathcal{S}\mathcal{T}\), which can be thought of as a `rectification’ of the pseudo-simplicial Grothendieck nerve of the tricategory \(\mathcal{N}\mathcal{T}\) [24], since both are bi-equivalent in the tricategory of pseudo-simplicial bicategories and therefore model the same homotopy type. Furthermore, \(\mathcal{S}\mathcal{T}\) is a weak 3-category under the point of view of Tamsamani [13] and Simpson [38], in the sense that it is a special simplicial bicategory, that is, a simplicial bicategory \(S : \Delta^{\text{op}} \rightarrow \text{Hom}\) satisfying the following two conditions:

(i) \(S_0\) is discrete (i.e., all its 1- and 2-cells are identities).
(ii) for \( n \geq 2 \), the Segal projection homomorphisms (see [37, Definition 1.2])

\[
p_n = \prod_{k=1}^{n} d_0 \cdots d_{k-2} d_{k+1} \cdots d_n: S_n \rightarrow S_1 d_0 \times d_1 S_1 d_0 \times \cdots \times d_n \times d_1 S_1
\]

are biequivalences of bicategories.

For a reduced special simplicial bicategories \( S \) as above, that is, with \( S_0 = 1 \), the one-object discrete bicategory, the simplicial space \( BS: \Delta^\op \rightarrow \text{Top} \), obtained by replacing each bicategory by its classifying space, satisfies hypothesis (i) and (ii) of Segal’s Proposition 1.5 in [37] (see also [33]): \( BS_0 \) is contractible, and the maps \( BP_n: BS_n \rightarrow B(S^n_1) = (BS_1)^n \) are homotopy equivalences. Then, \( BS_1 \) becomes an \( H \)-space with multiplication induced by the composite homomorphism \( BS_1 \times BS_1 \xrightarrow{\circ} BS_2 \xrightarrow{d_1} BS_1 \), and we have the following useful result:

**Lemma 4.5.1.** If \( S \) is any reduced special simplicial bicategory, then the loop space \( \Omega|BS| \) is a group completion of \( BS_1 \). Then, if \( \pi_0 S_1 \) is a group, there is a homotopy equivalence \( BS_1 \simeq \Omega|BS| \).

Let us recall that, for a given tricategory \( T \), the construction given in [8] of the bicategory of unitary homomorphic representations of any small category \( I \) in the tricategory \( T \), \( I \mapsto \text{Rep}_{uh}(I, T) \), is clearly functorial on the small category \( I \), and it leads to the definition below.

**Definition 4.5.2.** The Segal nerve of a tricategory \( T \) is the simplicial bicategory

\[
ST: \Delta^\op \rightarrow \text{Hom} \subset \text{Bicat}, \quad [p] \mapsto SP_T = \text{Rep}_{uh}( ][p][ ]; T).
\]

We should remark that, when \( T = B \) is a bicategory, that is, when its 3-cells are all identities, then the Segal nerve \( SB \) is introduced in [9, Definition 5.2], although it was first studied by Lack and Paoli in [29] under the name of ’2-nerve of \( B \)’. However, this may be a confusing terminology for \( SB \) since, for example, the so-called ‘geometric nerve’ of a 2-category \( B \) [6, 9] is also called the ’2-nerve of \( B \’) in [45].

**Theorem 4.5.3.** Let \( T \) be a tricategory. Then, the following statements hold:

(i) There is a normal pseudo-simplicial homomorphism

\[
L: NT \rightarrow ST,
\]

such that, for any \( p \geq 0 \), the homomorphism \( L_p: N_p T \rightarrow S_p T \) is a biequivalence of bicategories.

(ii) The simplicial bicategory \( ST \) is special.

**Proof.** Let us recall the explicit construction of \( NT = (NT, \chi, \omega) \) given in the proof of Theorem [31.1.1] particularly the constructions of the bicategories \( N_p T \) in [29], of the homomorphisms \( N_p T \) in [38], of the pseudo-equivalences \( \chi_{a,b} \) in [40], and of the modifications \( \omega_{a,b,c} \) in [41]. Furthermore, recall from Lemma [27.1.1] (c) that every biadjoint biequivalence \( SB \) restricts by giving a biadjoint biequivalence

\[
L_p \dashv R_p : N_p T \rightleftarrows S_p T.
\]

The normal pseudo-simplicial homomorphism \( (L, \theta, \Pi) : NT \rightarrow ST \), is then defined by the homomorphisms \( L_p: N_p T \rightarrow S_p T \), \( p \geq 0 \). For any map \( a: [q] \rightarrow [p] \) in the
simplicial category, the structure pseudo-equivalence
\[ N_pT \xrightarrow{L_p} S_pT \]
\[ N_aT \xrightarrow{\theta_a \Rightarrow \theta} a^* \]
\[ N_qT \xrightarrow{L_q} S_qT, \]
is provided by the counit pseudo-equivalence \( v_q : L_q R_q \Rightarrow 1_{S_qT} \); that is,
\[ (54) \]
\[ L_q N_aT = L_q R_q a^* L_p \xrightarrow{\theta = v_q a^* L_p} \]
\[ a^* L_p. \]

For \([n] \xrightarrow{b} [q] \xrightarrow{a} [p]\), any two composable arrows of \( \Delta \), the structure invertible modification
\[ \Pi_a,b \xrightarrow{\Pi_{a,b} \Rightarrow \Pi} \]
\[ b^* L_q N_aT \xrightarrow{b \theta} b^* a^* L_p \xrightarrow{1} (ab)^* L_p. \]
is directly provided by the canonical modification \( (42) \), \( \Pi_{a,b} = \omega^a_{b} a^* L_p. \)

The coherence conditions for \( L \) (i.e., conditions \( CC3 \) and \( CC4 \) in \( [10] \), with the modifications \( \Gamma \) the coherence isomorphisms \( 1 \circ 1 \cong 1 \)), are easily verified by using Fact \( 3.1.3 \). This complete the proof of part (i).

And when it comes to part (ii), that is, that \( ST \) is a special simplicial bicategory, we have the following (quite obvious) identifications between bicategories:
\[ (55) \]
\[ S_0T = \text{Rep}_{uh}([0], T) = \text{Ob}T = N_0T, \]
\[ (56) \]
\[ S_1T = \text{Rep}_{uh}([1], T) = \bigsqcup_{(t_0, t_1)} T(t_1, t_0) = N_1T, \]
and, for any integer \( p \geq 2 \),
\[ S_iT_{d_0 \times d_1 \cdots d_i} = \bigsqcup\limits_{(t_0, t_1)} T(t_1, t_0) \times \bigsqcup\limits_{(t_2, t_1)} T(t_2, t_1) \times \cdots \times \bigsqcup\limits_{(t_p-1, t_p)} T(t_p, t_{p-1}) \]
\[ = \bigsqcup\limits_{(t_0, \ldots, t_p)} T(t_1, t_0) \times T(t_2, t_1) \times \cdots \times T(t_p, t_{p-1}) = N_pT. \]

Through these identifications we see that, for any integer \( p \geq 2 \), the Segal projection homomorphism \( (50) \) is precisely the biequivalence \( R_p : S_pT \to N_pT \) in \( (53) \) which, recall, is defined by restricting it to the basic graph \( (36) \) of the category \([p]\). Whence the simplicial bicategory \( ST \) is special.

The following theorem states that the classifying space of a tricategory \( T \) can be realized, up to homotopy equivalence, by its Segal nerve \( ST \). This fact will be relevant for our later discussions on loop spaces. Let
\[ \text{BST} : \Delta^\text{op} \to \text{Top} \]
be the simplicial space obtained by composing \( ST : \Delta^\text{op} \to \text{Hom} \subset \text{Bicat} \) with the classifying functor \( B : \text{Hom} \to \text{Top} \) (recall Fact \( 4.1.1 \)).

\textbf{Theorem 4.5.4.} \textit{For any tricategory \( T \), there is a homotopy equivalence \( BT \cong |\text{BST}| \).}
Proof. Let us consider the pseudo-simplicial homomorphism (52), $L : NT \to ST$. Since, for every integer $p \geq 0$, the homomorphism $L_p : N_pT \to S_pT$ is a biequivalence, it follows from Fact 4.1.3 that the induced cellular map $B L_p : BN_pT \to BS_pT$ is a homotopy equivalence. Then, by Fact 4.1.3 the induced map $B \int L : B \int NT \to B \int ST$ is a homotopy equivalence. Since, by definition, $BT = B \int NT$, whereas, by Fact 4.1.7 there is a homotopy equivalence $B \int ST \simeq |BST|$, the claimed homotopy equivalence follows. □

4.6. Example: Classifying spaces of monoidal bicategories. Any monoidal bicategory $(\mathcal{B}, \otimes) = (\mathcal{B}, \otimes, I, a, l, r, \pi, \mu, \lambda, \rho)$ can be viewed as a tricategory

$$\Sigma(\mathcal{B}, \otimes)$$

with only one object, say $\ast$, whose hom-bicategory is the underlying bicategory. Thus, $\Sigma(\mathcal{B}, \otimes)(\ast, \ast) = \mathcal{B}$, and it is the composition given by the tensor functor $\otimes : \mathcal{B} \times \mathcal{B} \to \mathcal{B}$ and the identity at the object is $1_\ast = I$, the unit object of the monoidal bicategory. The structure pseudo-equivalences and modifications $a, l, r, \pi, \mu, \lambda, \rho$ for $\Sigma(\mathcal{B}, \otimes)$ are just those of the monoidal bicategory, respectively. Call this tricategory the suspension, or delooping, tricategory of the bicategory $\mathcal{B}$ induced by the monoidal structure given on it, and call its corresponding Grothendieck nerve (25) the nerve of the monoidal bicategory, hereafter denoted by $N(\mathcal{B}, \otimes)$. Thus,

$$N(\mathcal{B}, \otimes) = N \Sigma(\mathcal{B}, \otimes) : \Delta^{op} \to \text{Bicat}, \quad [p] \mapsto \mathcal{B}^p,$$

is a normal pseudo-simplicial bicategory, whose bicategory of $p$-simplices is the $p$-fold power of the underlying bicategory $\mathcal{B}$, with face and degeneracy homomorphisms induced by the tensor homomorphism $\otimes : \mathcal{B} \times \mathcal{B} \to \mathcal{B}$ and unit object $I$, following the familiar formulas (27), in analogy with those of the reduced bar construction on a topological monoid, and with structure pseudo-equivalences and modifications canonically arising from the data of the monoidal structure on $\mathcal{B}$. The general Definition 4.2.1 for classifying spaces of tricategories leads to the following:

Definition 4.6.1. The classifying space of the monoidal bicategory, denoted by $B(\mathcal{B}, \otimes)$, is defined to be the classifying space of its delooping tricategory $\Sigma(\mathcal{B}, \otimes)$. Thus,

$$B(\mathcal{B}, \otimes) = B \Sigma(\mathcal{B}, \otimes) = B \int N(\mathcal{B}, \otimes).$$

If $(C, \otimes) = (C, \otimes, I, a, l, r)$ is a monoidal category, and we regard $C$ as a bicategory all of whose 2-cells are identities, then the suspension tricategory $\Sigma(C, \otimes)$ is actually a bicategory, called in [25, 2.10] the delooping bicategory of the category induced by its monoidal structure. The nerve of $\Sigma(C, \otimes)$ then becomes the pseudo-simplicial category

$$N(C, \otimes) : \Delta^{op} \to \text{Cat}, \quad [p] \mapsto C^p,$$

used by Jardine in [23, §3] to define the classifying space of the monoidal category just as above: $B(C, \otimes) = \int N(C, \otimes)$ (see also [7], [11], or [22] Appendix, for more references). Thus,

$$B(C, \otimes) = B \Sigma(C, \otimes),$$

as above for arbitrary monoidal bicategories.

It is a well-known fact by Stasheff [22] that the classifying space of a monoidal category $(C, \otimes)$ is, up to group completion, a loop space. More precisely, it is a fact that

"the loop space $\Omega B(C, \otimes)$ is a group completion of $BC$"

(see [23, Propositions 3.5 and 3.8] or [7, Corollary 4]). Next theorem extends Stasheff’s result to bicategories, by showing that the group completion of the classifying space of a bicategory enriched with a monoidal structure is homotopy equivalent to a loop space.
Theorem 4.6.2. For any monoidal bicategory \((\mathcal{B}, \otimes)\), the loop space of its classifying space, \(\Omega \mathcal{B}(\mathcal{B}, \otimes)\), is a group completion of the classifying space of the underlying bicategory, \(\mathcal{B}\). In particular, if the monoid of connected components \(\pi_0 \mathcal{B}\) is a group, then there is a homotopy equivalence \(\mathcal{B} \simeq \Omega \mathcal{B}(\mathcal{B}, \otimes)\).

Proof. By Theorem 4.6.1 \(\mathcal{B}(\mathcal{B}, \otimes)\) is homotopy equivalent to \(|\mathcal{B} \Sigma \mathcal{B}(\mathcal{B}, \otimes)|\), the geometric realization of the simplicial space obtained by taking classifying spaces on the simplicial bicategory \(\mathcal{S} \Sigma (\mathcal{B}, \otimes)\), Segal nerve of the suspension tricategory of the monoidal bicategory. By Theorem 4.5.1, \(\mathcal{S} \Sigma (\mathcal{B}, \otimes)\) is a special simplicial bicategory. Furthermore, since the tricategory \(\Sigma(\mathcal{B}, \otimes)\) has only one object, \(\mathcal{S} \Sigma (\mathcal{B}, \otimes)\) is reduced (see (55)). Hence, the result follows from Lemma 4.3.1 since \(S \Sigma(\mathcal{B}, \otimes) = \mathcal{B}\), by the identification (56). \(\Box\)

4.7. Example: Classifying spaces of braided monoidal categories. Let \((\mathcal{C}, \otimes, c)\) be a braided monoidal category as in [24]. Thanks to the braidings \(c : x \otimes y \rightarrow y \otimes x\), the given tensor product on \(\mathcal{C}\) defines in the natural way a tensor product homomorphism on the suspension bicategory of the underlying monoidal category, \(\otimes : \Sigma(\mathcal{C}, \otimes) \times \Sigma(\mathcal{C}, \otimes) \rightarrow \Sigma(\mathcal{C}, \otimes)\). Thus \(\Sigma(\mathcal{C}, \otimes)\) is a monoidal bicategory. The corresponding suspension tricategory,

\[
\Sigma^2(\mathcal{C}, \otimes, c) = \Sigma(\Sigma(\mathcal{C}, \otimes) \otimes \mathcal{C}, \otimes)
\]

is called the double suspension, or double delooping, of the underly ing category \(\mathcal{C}\) associated to the given braided monoidal structure on it (see [14, 4.2], [25, 4.2] or [17, 7.9]). This is a tricategory with only one object, say \(*\), only one arrow \(* \rightarrow *\), the objects of \(\mathcal{C}\) are the 2-cells, and the morphisms of \(\mathcal{C}\) are the 3-cells. The hom-bicategory is \(\Sigma^2(\mathcal{C}, \otimes, c)(*, *) = \Sigma(\mathcal{C}, \otimes)\), the suspension bicategory of the underlying monoidal category \((\mathcal{C}, \otimes)\), the composition is also (as the horizontal one in \(\Sigma(\mathcal{C}, \otimes)\)) given by the tensor functor \(\otimes : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}\) and the interchange 3-cell between the two different composites of 2-cells is given by the braiding. The most striking instance is for \((\mathcal{C}, \otimes, c) = (A, +, 0)\), the strict braided monoidal category with only one object defined by an abelian group \(A\), where both composition and tensor product are given by the addition \(+\) in \(A\); in this case, the double suspension tricategory \(\Sigma^2 \mathcal{A}\) is precisely the 3-category treated in Examples 2.2 and 2.4.

For any braided monoidal category \((\mathcal{C}, \otimes, c)\), the Grothendieck nerve [25] of the double suspension tricategory \(\Sigma^2(\mathcal{C}, \otimes, c)\) coincides with the pseudo-simplicial bicategory called in [10] the nerve of the braided monoidal category, and denoted by \(N(\mathcal{C}, \otimes, c)\). Thus,

\[
N(\mathcal{C}, \otimes, c) = N\Sigma^2(\mathcal{C}, \otimes, c) : \Delta^\text{op} \rightarrow \text{Bicat}, \quad [p] \mapsto (\Sigma(\mathcal{C}, \otimes))^p = \Sigma(\mathcal{C}^p, \otimes),
\]

is a normal pseudo-simplicial one-object bicategory whose bicategory of \(p\)-simplicies is the suspension bicategory of the monoidal category \(\mathcal{C}\). Since the classifying space of the monoidal category \([\mathcal{C}, \otimes]\) is a double loop space. Hence, the following:

Theorem 4.7.1. The classifying space of a braided monoidal category coincides with the classifying space of its double suspension tricategory, \(\mathcal{B}(\mathcal{C}, \otimes, c) = B \Sigma^2(\mathcal{C}, \otimes, c)\).
as was noticed by J. D. Stasheff in [12], but originally proven by Z. Fiedorowicz in [14] (other proofs can be found in [4, 5]). Below we show a new proof of Stasheff-Fiedorowicz’s result, in the more precise form stated in [10, Theorem 6.10]:

**Theorem 4.7.2.** For any braided monoidal category \((C, \otimes, c)\) there is a homotopy equivalence \(B(C, \otimes) \simeq \Omega B(C, \otimes, c)\).

**Proof.** By Theorem 4.7.1, the classifying space of any braided monoidal category \((C, \otimes)\) is the same as the classifying space of the monoidal bicategory \(\Sigma(\Sigma(C, \otimes))\). Therefore, \(\Omega B(C, \otimes, c) = \Omega B \Sigma(\Sigma(C, \otimes))\). Since \(\Sigma(C, \otimes)\) has only one object, it is obvious that its monoid of connected component \(\pi_0 \Sigma(C, \otimes) = 1\), the trivial group. Then, by Theorem 4.6.2 there is a homotopy equivalence \(B \Sigma(C, \otimes) \simeq \Omega B \Sigma(\Sigma(C, \otimes))\). Since, by (61), \(B(C, \otimes) = B \Sigma(C, \otimes)\), the result follows.

**Corollary 4.7.3.** For any braided monoidal category \((C, \otimes, c)\), the double loop space of its classifying space, \(\Omega^2 B(C, \otimes, c)\), is a group completion of the classifying space of the underlying category, \(BC\).

**Proof.** By Theorem 4.6.2 \(\Omega B(C, \otimes)\) is a group completion of \(BC\). By Theorem 4.7.2 above, there is a homotopy equivalence \(\Omega B(C, \otimes) \simeq \Omega^2 B(C, \otimes, c)\), whence the result.

**Remark 4.7.4.** When each suspension bicategory \(\Sigma(C^p, \otimes)\) is replaced in (63) by its nerve, that is, by \(N(C^p, \otimes) : \Delta^{op} \to \text{Cat}\), then one has a pseudo-bisimplicial category

\[
\Delta^{op} \times \Delta^{op} \to \text{Cat}, \quad ([p], [q]) \mapsto C^{pq},
\]

which (for \((C, \otimes)\) strict) is taken in [4] to construct a double delooping space for the classifying space of the underlying category \(BC\). That double delooping space is homotopy equivalent to the classifying space of the braided monoidal \(B(C, \otimes, c)\).

5. **The Street nerve of a tricategory**

With the notion of the classifying space of a tricategory \(T\) given above, the resulting CW-complex \(BT\) thus obtained has many cells with little apparent intuitive connection with the cells of the original tricategory, and they do not enjoy any proper geometric meaning. This leads one to search for any simplicial set realizing the space \(BT\) and whose cells give a logical geometric meaning to the data of the tricategory. With the definition below (which, up to minor changes and terminology, is essentially due to Street [39, 41]), we give a convincing natural response for such a simplicial set.

5.1. **The geometric nerve of a tricategory.** For any given tricategory \(T\), the construction \(I \mapsto \text{Rep}_u(I, T)\) given in [3], which carries each category \(I\) to the set of unitary representations of \(I\) in \(T\), is clearly functorial on the small category \(I\), whence we have the following simplicial set:

**Definition 5.1.1.** The geometric nerve of a tricategory \(T\) is the simplicial set

\[
\Delta T : \Delta^{op} \to \text{Set}, \quad [p] \mapsto \text{Rep}_u([p], T),
\]

whose \(p\)-simplices are unitary representations of the category \([p]\) in \(T\).

The simplicial set \(\Delta T\) encodes the entire tricategorical structure of \(T\) and, as we will prove below, faithfully represent the classifying space of the tricategory \(T\), up to homotopy. We shall stress here that the simplices of the geometric nerve \(\Delta T\) have the following pleasing
geometric description: The vertices of $\Delta T$ are points labelled with the objects $F_0$ of $T$. The 1-simplices are paths labelled with the 1-cells

$$F_{01} : F_1 \to F_0.$$  

The 2-simplices are oriented triangles

$$F_{i2} \xleftarrow{F_{0i2}} F_{i1} \xrightarrow{F_{0i}} F_{0},$$

with objects $F_i$ placed on the vertices, 1-cells $F_{ij} : F_j \to F_i$ on the edges, and labelling the inner as a 2-cell $F_{012} : F_{01} \otimes F_{12} \Rightarrow F_{02}$. For $p \geq 3$, a $p$-simplex of $\Delta T$ is geometrically represented by a diagram in $T$ with the shape of the 3-skeleton of an oriented standard $p$-simplex whose 3-faces are oriented tetrahedrons

$$F_{ikl} \xrightarrow{F_{ijl}} F_{ij}, \quad F_{ik} \xrightarrow{F_{ijk}} F_{ij}, \quad F_{jk} \xrightarrow{F_{ijk}} F_{ij}.$$  

are geometric 2-simplices as above, and

$$F_{ikk} \xrightarrow{F_{ikk}} F_{ijk}, \quad F_{ikk} \xrightarrow{F_{ikk}} F_{ijk}.$$  

is a 3-cell of the tricategory that labels the inner of the tetrahedron. For $p \geq 4$, these data are required to satisfy the coherence condition (CR1), as stated in Section 2, that is, for each $0 \leq i < j < k < l < m \leq p$, the following diagram commutes:

$$
\begin{align*}
F_i & \Rightarrow F_{ijk} \\
F_k & \Rightarrow F_{ijkl} \\
F_{ijkl} & \Rightarrow F_{ijkm}
\end{align*}
$$

(The fourth Street's oriental.)

The simplicial set $\Delta T$ is coskeletal in dimensions greater than 4. More precisely, for $p \geq 4$, a $p$-simplex $F : \Delta[p] \to T$ of $\Delta T$ is determined uniquely by its boundary $\partial F = \cdots$
\[(Fd^0, \ldots, Fd^p)\]
\[
\partial \Delta[p] \xrightarrow{\partial F} \Delta T, \quad \Delta[p] \xrightarrow{F} \Delta T,
\]
and, for \(p \geq 5\), every possible boundary of a \(p\)-simplex, \(\partial \Delta[p] \rightarrow \Delta T\), is actually the boundary \(\partial F\) of a geometric \(p\)-simplex \(F\) of the tricategory \(T\).

5.2. **Example: Geometric nerves of bicategories.** When a bicategory \(B\) is regarded as a tricategory, all of whose 3-cells are identities, then a unitary representation of any category \(I\) in it is the same as a unitary lax functor \(I \rightarrow B\). Hence, the simplicial set \(\Delta B\) is precisely the unitary geometric nerve of the bicategory, as it is called in [9] (but denoted by \(\Delta^u B\)) where, in Theorem 6.1, the following fact is proved:

**Fact 5.2.1.** For any bicategory \(B\), there is a homotopy equivalence \(BB \simeq |B|\).

The construction of the geometric nerve for a bicategory was first given in the late eighties by J. Duskin and R. Street (see [40], pag. 573). In [13], Duskin gave a characterization of the unitary geometric nerve of a bicategory if and only if it satisfies the coskeletal conditions above as well as supporting appropriate sets of ‘abstractly invertible’ 1- and 2-simplices (see Gurski [21], for an interesting new approach to this subject).

5.3. **Geometric nerves realize classifying spaces of tricategories.** We now state and prove a main result of the paper, namely:

**Theorem 5.3.1.** For any tricategory \(T\), there is a homotopy equivalence \(B_T \simeq |T|\).

**Proof.** Let us consider, for any given tricategory \(T\), the simplicial bicategory
\[
\Delta T : \Delta^{op} \rightarrow \text{Hom} \subset \text{Bicat}, \quad [q] \mapsto \text{Rep}_u([q], T),
\]
whose bicategories of \(q\)-simplices are the bicategories of unitary representations \(\text{Rep}_u\) of \([q]\) in \(T\). In this simplicial bicategory, the homomorphism induced by any map \(a : [q] \rightarrow [p]\), \(a^* : \Delta_p T \rightarrow \Delta_q T\), is actually a 2-functor. Hence, the bisimplicial set
\[
\Delta \Delta T : \Delta^{op} \times \Delta^{op} \rightarrow \text{Set}, \quad ([p],[q]) \mapsto \Delta_p \Delta_q T = \text{Rep}_u([p], \text{Rep}_u([q], T))
\]
is well defined, since the geometric nerve construction \(\Delta\) is functorial on unitary homomorphisms between bicategories. The plan is to prove the existence of homotopy equivalences
\[
B_T \simeq |\text{diag} \Delta \Delta T|, \quad |T| \simeq |\text{diag} \Delta \Delta T|,
\]
whence the theorem follows.

- **The homotopy equivalence \(72\).** The Segal nerve of the tricategory \([\Delta]\) is a simplicial sub-bicategory of \(\Delta T\). Let \(L : NT \rightarrow \Delta T\) be the pseudo-simplicial homomorphism obtained by composing the pseudo simplicial homomorphism \(\Delta\), equally denoted by \(L : NT \rightarrow ST\), with the simplicial inclusion \(ST \subseteq \Delta T\). Let us now observe that, at any degree \(p \geq 0\), the homomorphism \(L_p : N_p T \rightarrow \Delta_p T\) is precisely the homomorphism \(\text{Rep}_u([p], \text{Rep}_u([q], T)) \rightarrow \text{Rep}_u([p], T)\), corresponding with the basic graph \(G_p = (p \rightarrow \cdots \rightarrow 1 \rightarrow 0)\) of the category \([p]\). Then, by Lemma 2.7.1, we have a homomorphism \(\text{Rep}_u([p], T) \rightarrow \text{Rep}_u([p], T)\), such that \(R_p L_p = 1_{N_p T}\), and a lax transformation \(v_p : L_p R_p \Rightarrow 1_{\Delta_p T}\). It follows from Fact 4.1.2
that every induced map $B\Delta_p : BN_p \mathcal{T} \to B\Delta_q \mathcal{T}$ is a homotopy equivalence. Then, by Fact 4.1.6, the induced map $B\int_L : B\int_L N\mathcal{T} \to B\int_L \Delta \mathcal{T}$ is a homotopy equivalence. Let $B\Delta \mathcal{T} : \Delta^{op} \to \textbf{Top}$ be the simplicial space obtained by composing $\Delta \mathcal{T}$ with the classifying functor $B : \textbf{Hom} \to \textbf{Top}$ (see Fact 4.1.1). Since, by definition, $B\mathcal{T} = B\int_L N\mathcal{T}$, whereas, by Fact 4.1.7, there is a homotopy equivalence $B\Delta \mathcal{T} \simeq |B\Delta \mathcal{T}|$, we have a homotopy equivalence $B\mathcal{T} \simeq |B\Delta \mathcal{T}|$. Furthermore, by Fact 5.2.1 in Example 5.2, we have a homotopy equivalence $B\Delta \mathcal{T} \simeq |B\Delta \mathcal{T}|$.

The homotopy equivalence $B\Delta \mathcal{T}$ is the simplicial set of objects of the simplicial bicategory $B\Delta \mathcal{T}$, that is, $B\Delta \mathcal{T} = \Delta_B \mathcal{T}$. Therefore, if we regard $B\Delta \mathcal{T}$ as a simplicial discrete bicategory (i.e., all 1-cells and 2-cells are identities), then $\Delta_B \mathcal{T}$ becomes a bisimplicial set that is constant in the horizontal direction, and there is a natural bisimplicial map $\Delta_B \mathcal{T} \to \Delta_B \mathcal{T}$, which is, at each horizontal level $p \geq 1$, the composite simplicial map

$$\Delta_B \mathcal{T} = \Delta_0 \Delta_B \mathcal{T} \simeq \Delta_1 \Delta_B \mathcal{T} \hookrightarrow \cdots \hookrightarrow \Delta_{p-1} \Delta_B \mathcal{T} \simeq \Delta_p \Delta_B \mathcal{T}.$$  

Next, we prove that the simplicial map $\Delta_B \mathcal{T} \to \text{diag} \Delta_B \mathcal{T}$, induced on diagonals, is a weak homotopy equivalence, whence the announced homotopy equivalence in (73). It suffices to prove that every induced map $B\Delta \mathcal{T} \to \text{diag} \Delta_B \mathcal{T}$, induced on diagonals, is a weak homotopy equivalence and, in fact, we will prove more: Every simplicial map $s_{p-1}^b : \Delta_B \mathcal{T} \to \Delta_B \mathcal{T}$, $p \geq 1$, embeds the simplicial set $\Delta_B \mathcal{T}$ into $\Delta_B \mathcal{T}$ as a simplicial deformation retract. Since $d_{p-1}^b s_{p-1}^b = 1$, it is enough to exhibit a simplicial homotopy $h : s_{p-1}^b \Rightarrow 1 : \Delta_B \mathcal{T} \to \Delta_B \mathcal{T}$.

To do so, we shall use the following notation for the bisimplices in $\Delta_B \mathcal{T}$. Since such a bisimplex of bidegree $(p, q)$, say $F \in \Delta_B \mathcal{T}$, is a unitary representation of the category $[p]$ in the bicategory of unitary representations $\textbf{Rep}_u([q], \mathcal{T})$, it consists of unitary representations $F^u : [q] \to \mathcal{T}$, 1-cells $F^{u,v} : F^v \Rightarrow F^u$, and 2-cells $F^{u,v,w} : F^{u,v} \circ F^{v,w} \Rightarrow F^{u,w}$ in the bicategory $\textbf{Rep}_u([q], \mathcal{T})$, for $0 \leq u \neq v \neq w \leq p$, such that the diagrams

$$\begin{array}{ccc}
(F^u \circ F^{v,w}) \circ F^{w,t} & \xrightarrow{\alpha} & F^v \circ (F^{u,w} \circ F^{w,t}) \\
F^{u,v} \circ F^{v,w} \circ F^{w,t} & \xrightarrow{\circ} & F^{u,v,w} \circ F^{v,t}
\end{array}$$

commute for $u < v < w < t$. Hence, such a $(p, q)$-simplex is described by a list of cells of the tricategory $\mathcal{T}$

$$F = (F_i, F_{i,j}, F_{i,j,k}, F_{i,j,k,l}, F_{i,j,k,l}^{u,v}, F_{i,j,k}^{u,v,w})$$

with $0 \leq i < j < k < l \leq q$, where each $F_i$ is an object, the $F_{i,j} : F_j \to F_i$ are 1-cells, the $F_{i,j,k} : F_{i,j} \otimes F_{i,j,k} \Rightarrow F_{i,j,k}$, and the $F_{i,j,k,l} : F_{i,j,k} \Rightarrow F_{i,j,k,l}$ are 2-cells, and the remaining are 3-cells as in

$$\begin{array}{ccc}
(F^{u,v} \otimes F_{i,j}^{u,v}) \otimes F_{i,j,k}^{u,v} & \xrightarrow{\alpha} & F_{i,j}^{u,v} \otimes (F_{i,j,k}^{u,v} \otimes F_{i,j,k,l}^{u,v}) \\
F_{i,j,k,l}^{u,v} \otimes F_{i,j,k,l}^{u,v} & \xrightarrow{\otimes} & F_{i,j,k,l}^{u,v} \otimes F_{i,j,k,l}^{u,v}
\end{array}$$
satisfying the various conditions. The horizontal faces and degeneracies of such a bisimplex are given by the simple rules \(d^h_m F = (F_i, F^d_{i,j}, \ldots)\) and \(s^h_m F = (F_i, F^s_{i,j}u, \ldots)\), whereas the vertical ones are given by \(d^v_m F = (F^h_{d,m}, F^u_{d,m,j}, \ldots)\) and \(s^v_m F = (F^v_{s,m}, F^u_{s,m,j}, \ldots)\).

Then, we have the following simplicial homotopy \(h : s^h_{p-1}d^h_p \Rightarrow 1 : \Delta_p \Delta_q \mathcal{T} \rightarrow \Delta_p \Delta_{q+1} \mathcal{T}\). For each \(0 \leq m \leq q\), the map \(h_m : \Delta_p \Delta_q \mathcal{T} \rightarrow \Delta_p \Delta_{q+1} \mathcal{T}\) takes a \((p, q)\)-simplex \(\mathfrak{u}\) of \(\Delta \Delta \mathcal{T}\) to the \((p, q+1)\)-simplex \(h_m \mathfrak{u}\) defined by

- \((h_m \mathfrak{u})_1 = F^u_{s,1}\), for \(0 \leq i \leq q + 1\),
- \((h_m \mathfrak{u})_{i,j} = F^u_{s,m_i,s_mj}\) if \(u < p\) or \(j \leq m\),
- \((h_m \mathfrak{u})_{i,j}^p = F^p_{s,m_{i,j}-1}\) if \(m < j\),
- \((h_m \mathfrak{u})_{i,j,k}^u = F^u_{s,m_{i,j}u,m_{j,k}}\) if \(u < p\) or \(k \leq m\),
- \((h_m \mathfrak{u})_{i,j,k}^p = F^p_{s,m_{i,j}p,m_{j,k}p}\) if \(m < j\),
- \((h_m \mathfrak{u})_{i,j,k,l}^p\), for \(j \leq m < k\), is the 2-cell obtained by the composition

\[
(F^p_{i,j} \otimes F^p_{j,k-1} \otimes F^p_{k-1,l-1} \otimes F^p_{l-1,k-1}) \otimes F^p_{i,j} \otimes F^p_{j,k-1} \otimes F^p_{k-1,l-1} \otimes F^p_{l-1,k-1} \Rightarrow F^p_{i,j} \otimes F^p_{j,k-1} \otimes F^p_{k-1,l-1} \otimes F^p_{l-1,k-1},
\]

- \((h_m \mathfrak{u})_{i,j,k,l}^u = F^u_{s,m_i,s_mj,s_mk,s_mm}\) if \(u < p\) or \(l \leq m\),
- \((h_m \mathfrak{u})_{i,j,k,l}^p = F^p_{s,m_{i,j},p,m_{j,k},p,m_{k,l},p}\) if \(m < j\),
- \((h_m \mathfrak{u})_{i,j,k,l}^p\), for \(j \leq m < k\), is the 3-cell obtained by pasting the diagram
- \((h_mF)_{i,j,k,l}^p\) for \(k \leq m < l\), is the 3-cell obtained by pasting

\[
\begin{array}{c}
F_{i,j}^p \otimes F_{j,k}^p \otimes F_{k,l}^p \otimes F_{l,i}^p \ar[rr]_(a) & & F_{i,j}^p \otimes (F_{j,k}^p \otimes F_{k,l}^{p-1}) \otimes F_{l,i}^p \ar[d] \\
F_{i,j}^p \otimes F_{j,k}^p \otimes F_{k,l}^{p-1} \otimes F_{l,i}^p \ar[d] & & F_{i,j}^p \otimes F_{j,k}^p \otimes F_{k,l}^{p-1} \otimes F_{l,i}^p \ar[d] \\
F_{i,j}^p \otimes F_{j,k}^{p-1} \otimes F_{k,l}^{p-1} \otimes F_{l,i}^p \ar[d] & & F_{i,j}^p \otimes F_{j,k}^{p-1} \otimes F_{k,l}^{p-1} \otimes F_{l,i}^p \\
F_{i,j}^p \otimes F_{j,k}^{p-1} \otimes F_{k,l}^{p-1} \otimes F_{l,i}^p \ar[d] & & F_{i,j}^p \otimes F_{j,k}^{p-1} \otimes F_{k,l}^{p-1} \otimes F_{l,i}^p \\
F_{i,j}^p \otimes F_{j,k}^{p-1} \otimes F_{k,l}^{p-1} \otimes F_{l,i}^p \ar[d] & & F_{i,j}^p \otimes F_{j,k}^{p-1} \otimes F_{k,l}^{p-1} \otimes F_{l,i}^p \\
\end{array}
\]

- \((h_mF)_{i,j}^v \cdot F_{s_{m,i}s_{m,j}}^{u,v}\) if \(v < p \) or \(j \leq m\),

- \((h_mF)_{i,j}^v \cdot F_{s_{m,i}s_{m,j}}^{u,v}\) if \(j > m\),

- \((h_mF)_{i,j,k}^v \cdot F_{s_{m,i}s_{m,j}s_{m,k}}^{u,v}\) if \(v < p \) or \(k \leq m\),

- \((h_mF)_{i,j,k}^v \cdot F_{s_{m,i}s_{m,j}s_{m,k}}^{u,v}\) if \(m < j\),

- \((h_mF)_{i,j,k}^v \cdot F_{s_{m,i}s_{m,j}s_{m,k}}^{u,v}\) for \(j \leq m < k\), is the 3-cell is obtained by pasting in

\[
\begin{array}{c}
F_{i,j}^p \otimes F_{j,k}^{p-1} \otimes F_{k,l}^{p-1} \otimes F_{l,i}^p \ar[rr]_(a) & & F_{i,j}^p \otimes F_{j,k}^{p-1} \otimes F_{k,l}^{p-1} \otimes F_{l,i}^p \\
F_{i,j}^p \otimes F_{j,k}^{p-1} \otimes F_{k,l}^{p-1} \otimes F_{l,i}^p \\
F_{i,j}^p \otimes F_{j,k}^{p-1} \otimes F_{k,l}^{p-1} \otimes F_{l,i}^p \\
\end{array}
\]

- \((h_mF)_{i,j}^v \cdot F_{s_{m,j}}^{u,v}\) if \(v < p \) or \(i \leq m\),

- \((h_mF)_{i,j}^v \cdot F_{s_{m,j}}^{u,v}\) if \(i < j\),

- \((h_mF)_{i,j}^v \cdot F_{s_{m,j}}^{u,v}\) if \(w < p \) or \(j \leq m\),

- \((h_mF)_{i,j}^v \cdot F_{s_{m,j}}^{u,v}\) if \(m < j\).

So defined, a straightforward (though quite tedious) verification shows that \(h : s_h^{p-1} \Rightarrow 1\) is actually a simplicial homotopy, and this completes the proof. \(\Box\)

5.4. Example: Geometric nerves of \(n\)-categories. On a small category \(C\), viewed as a tricategory in which all 2-cells and 3-cells are identities, both the geometric and the Grothendieck nerve constructions can be identified: \(\Delta C = NC\), since, for any integer \(p \geq 0\), we have \(\text{Rep}_d([p], C) = \text{Func}([p], C) \cong \text{Rep}(\mathcal{G}_p, C)\). For instance \(\Delta[n]\) is the simplicial standard \(n\)-simplex whose \(p\)-simplices are the maps \([p] \to [n]\) in \(\Delta\).

The case of geometric nerves of 2-categories was dealt with by Street in \cite{ST}. In \cite{ST}, but under the name of ‘2-nerve of 2-categories’, the geometric nerve construction \(\Delta : 2\text{-Cat} \to \text{Simpl.Set}\) was considered for proving that the category \(2\text{-Cat}\), of small 2-categories and
2-functors, has a Quillen model structure such that the functor

$$Ex^2 \Delta : \mathbf{2-Cat} \to \mathbf{Simpl.Set}$$

is a right Quillen equivalence of model categories, where $Ex$ is the endofunctor in $\mathbf{Simpl.Set}$ right adjoint to the subdivision $Sd$ (see [16], for example). In [10], it was first proved that, for any 2-category $B$, there is a natural homotopy equivalence $BB \simeq |\Delta B|$. It follows that the correspondence $B \mapsto BB$ induces an equivalence between the corresponding homotopy category of 2-categories and the ordinary homotopy category of CW-complexes. In [11], generalizations are given of both Quillen’s Theorem B and Thomason’s Homotopy Colimit Theorem to 2-functors.

In [39], Street gave a precise notion of nerve for $n$-categories. He extended each graph $G_p = (p \to \cdots \to 1 \to 0)$ to a “free” $\omega$-category $O_p$ (called the $p^{\text{th}}$-oriental) such that, for any $n$-category $X$, the $p$-simplices of its nerve, are just $n$-functors $O_p \to X$, from the underlying $n$-category of the $p^{\text{th}}$-oriental to $X$. In the case when $n = 3$, Street’s nerve construction on a 3-category $T$ just produces, up to some directional changes, its geometric nerve $\Delta T$, as stated in Definition 5.1.1. From the homotopy equivalences in (49) and Theorem 5.3.1, we get the following new result (see Example 4.3 for a discussion about the notion of classifying space of a 3-category):

**Theorem 5.4.1.** For any 3-category $T$, there are homotopy equivalences

$$|\text{diagNNN}T| \simeq BT \simeq |\Delta T|.$$  

5.5. **Example: Geometric nerves of braided monoidal categories.** If $A$ is any abelian group, then the braided monoidal category with only one object it defines, $(A, +, 0)$, has as double suspension the tricategory $\Sigma^2 A$, treated in Examples 2.2 and 2.5. For any integer $p \geq 0$, we have

$$\text{Rep}_a([p], \Sigma^2 A) \overset{[\Sigma^2 A]}{\longrightarrow} Z^3([p], A) = Z^3(\Delta [p], A) = K(A, 3)_p,$$

whence $\Delta \Sigma^2 A = K(A, 3)$, the minimal Eilenberg-Mac Lane complex. Hence, from Theorems 4.7.1 and 5.3.1 it follows that $B(A, +, 0) = [K(A, 3)]$.

If $(C, \otimes, c)$ is a braided monoidal category, then a unitary representation of a category $I$ in the double suspension tricategory, $I \to \Sigma^2 (C, \otimes, c)$, is what was called in [10, Definition 6.6] and [12, §4] a (normal) 3-cocycle of $I$ with coefficients in the braided monoidal category. Therefore, the geometric nerve $\Delta \Sigma^2 (C, \otimes, c)$ coincides with the simplicial set [10, Definition 6.7]

$$Z^3(C, \otimes, c) : \Delta^{op} \to \mathbf{Set}, \quad [p] \mapsto Z^3([p], (C, \otimes, c)),$$

whose $p$-simplices are the 3-cocycles of $[p]$ in the braided monoidal category. The geometric nerve $Z^3(C, \otimes, c)$ is then a 4-coskeletal 1-reduced (one vertex, one 1-simplex) simplicial set whose 2-simplices are the objects $F_{012}$ of $C$. The 3-simplices are morphisms of the form

$$F_{0123} : F_{123} \otimes F_{013} \longrightarrow F_{012} \otimes F_{023},$$

and, for $p \geq 4$, the $p$-simplices are families of 3-simplices

$$F_{ijkl} : F_{jkl} \otimes F_{ijk} \to F_{ijk} \otimes F_{ikl},$$
connected CW-complex $X$-type can be realized in this way from a bicategorical group. That is, for any given path-

Homotopy Hypothesis that supports the sense that the homomorphisms $x$ a bicategorical group is a monoidal bigroupoid \((\cdot \otimes \cdot)\)

\(\alpha \) product; that is, there is an object \(B(B, \otimes)\) as the geometric nerve of its suspension 3-category \(\Sigma(B, \otimes)\). Then, Theorem 5.6.1.

For any monoidal bicategory \(B(B, \otimes)\), there is a homotopy equivalence \(B(B, \otimes) \simeq |\Sigma(B, \otimes)|\).

5.6. Example: Geometric nerves of monoidal bicategories. If \((B, \otimes)\) is any monoidal bicategory, then we define its geometric nerve, denoted by \(\Delta(B, \otimes)\), as the geometric nerve of its suspension 3-category \(\Sigma(B, \otimes)\). That is,

\[
\Delta(B, \otimes) : \Delta^{op} \to \text{Simpl.Set}, \quad [p] \mapsto \text{Rep}_u([p], \Sigma(B, \otimes)).
\]

Then, Theorem 5.6.1 particularizes to monoidal bicategories stating the following:

**Theorem 5.6.1.** For any monoidal bicategory \((B, \otimes)\), there is a homotopy equivalence
\[
B(B, \otimes) \simeq |\Delta(B, \otimes)|.
\]

5.7. Example: Bicategorical groups and homotopy 3-types. A **bicategorical group**, also called a (weak) 3-group, or **Gr-bicategory**, is a monoidal bicategory \((B, \otimes)\), in which every 2-cell is an isomorphism; that is, all the hom-categories \(B(x, y)\) are groupoids, every morphism \(u : x \to y\) is an equivalence, that is, there exist a morphism \(u' : y \to x\) and 2-cells \(u \circ u' \Rightarrow 1_y\) and \(1_x \Rightarrow u' \circ u\), and each object \(x\) has a quasi-inverse with respect to the tensor product; that is, there is an object \(x'\) with 1-cells \(1 \to x \otimes x'\) and \(x' \otimes x \to 1\). In other words, a bicategorical group is a monoidal bigroupoid \((B, \otimes)\) in which every object \(x\) is regular, in the sense that the homomorphisms \(x \otimes - : B \to B\) and \(- \otimes x : B \to B\) are biequivalences.

If \((B, \otimes)\) is any monoidal bicategory, then its geometric nerve \(\Delta(B, \otimes)\) is a 4-coskeletal reduced (one vertex) simplicial set, which satisfies the Kan extension condition if and only if \((B, \otimes)\) is a bicategorical group. In such a case, the homotopy groups of its geometric realization \(\pi_i B(B, \otimes) = \pi_i \Delta(B, \otimes)\) are plainly recognized to be

- \(\pi_1 B(B, \otimes) = 0\), if \(i \neq 1, 2, 3\).
- \(\pi_1 B(B, \otimes) = \text{Ob}B/\sim\), the group of equivalence classes of objects in \(B\) where multiplication is induced by the tensor product.
- \(\pi_2 B(B, \otimes) = \text{Aut}_B(1)/\cong\), the group of isomorphism classes of autoequivalences of the unit object where the operation is induced by the horizontal composition in \(B\).
- \(\pi_3 B(B, \otimes) = \text{Aut}_B(1_1)\), the group of automorphisms of the identity 1-cell of the unit object where the operation is vertical composition in \(B\).

Thus, bicategorical groups arise as algebraic path-connected homotopy 3-types, a fact that supports the **Homotopy Hypothesis** of Baez [2]. Indeed, every path-connected homotopy 3-type can be realized in this way from a bicategorical group. That is, for any given path-connected CW-complex \(X\) for which \(\pi_i X = 0\) for \(i \geq 4\), there is a bicategorical group
(\mathcal{B}, \otimes) such that \text{B}(\mathcal{B}, \otimes) is homotopy equivalent to \text{X}, as we show below (cf. Berger [5], Joyal-Tierney [26], Lack [22], or Leroy [30] for alternative approaches to this issue).

Given \text{X} as above, let \text{M} \subset \text{S}(\text{X}) be a minimal subcomplex that is a deformation retract of the total singular complex of \text{X}, so that |\text{M}| \simeq \text{X}. Taking into account the Postnikov \text{k}-invariants, this minimal complex \text{M} can be described (see [16] VI. Corollary 5.13]), up to isomorphism,

\begin{equation}
\text{M} = \text{K}(\text{B}, 3) \times \text{h}(\text{K}(\text{A}, 2) \times \text{h}\text{K}(\text{G}, 1)),
\end{equation}

by means of the group \text{G} = \pi_1 \text{X}, the \text{G}-modules \text{A} = \pi_2 \text{X} and \text{B} = \pi_3 \text{X}, and two maps,

\begin{align*}
h : \text{G}^3 & \to \text{A}, \quad t : \text{A}^6 \times \text{G}^4 \to \text{B},
\end{align*}

defining normalized cocycles \text{h} \in \text{Z}^3(\text{G}, \text{A}) and \text{t} \in \text{Z}^4(\text{K}(\text{A}, 2) \times \text{h}\text{K}(\text{G}, 1), \text{B}). That is, \text{M}

\begin{align*}
\text{tr}_4 \text{M} = \text{B}^4 \times \text{A}^6 \times \text{G}^4 \quad \xrightarrow{d_0} \quad \text{B} \times \text{A}^3 \times \text{G}^3 \quad \xrightarrow{d_0} \quad \text{A} \times \text{G}^2 \quad \xrightarrow{d_0} \quad \text{G},
\end{align*}

whose face and degeneracy operators are given by \((\sigma_i \in \text{G}, x_j \in \text{A}, u_k \in \text{B})\)

\begin{align*}
d_i(x_1, \sigma_1, \sigma_2) = \begin{cases}
\sigma_2 & i = 0, \\
\sigma_1 \sigma_2 & i = 1, \\
\sigma_1 & i = 2.
\end{cases}

\end{align*}

\begin{align*}
d_i(u_1, x_1, x_2, x_3, \sigma_1, \sigma_2, \sigma_3) = \begin{cases}
(x_2 + x_3, \sigma_1 \sigma_2 \sigma_3) & i = 0, \\
(x_1 + x_2, \sigma_1 \sigma_2 \sigma_3) & i = 1, \\
(x_1 - h(\sigma_1, \sigma_2, \sigma_3), \sigma_1, \sigma_2, \sigma_3) & i = 2.
\end{cases}

\end{align*}

\begin{align*}
d_i(u_1, u_2, u_3, u_4, x_1, x_2, x_3, x_4, x_5, x_6, \sigma_1, \sigma_2, \sigma_3, \sigma_4) = \begin{cases}
(u_1, x_1, x_2, x_3, \sigma_1, \sigma_2) & i = 0, \\
(u_1 + u_2, x_1, x_2, x_3, x_4, x_5, x_6, \sigma_1, \sigma_2, \sigma_3, \sigma_4) & i = 1, \\
(u_1 + u_2, x_1, x_2, x_3, x_4, x_5, x_6, \sigma_1, \sigma_2, \sigma_3, \sigma_4) & i = 2, \\
(u_1 + u_2, x_1, x_2, x_3, x_4, x_5, x_6, \sigma_1, \sigma_2, \sigma_3, \sigma_4) & i = 3, \\
(u_1 + u_2, x_1, x_2, x_3, x_4, x_5, x_6, \sigma_1, \sigma_2, \sigma_3, \sigma_4) & i = 4.
\end{cases}

\end{align*}

where \(\bar{u}_1 = u_1 - t(x_1, x_2, x_3, x_4, x_5, x_6, \sigma_1, \sigma_2, \sigma_3, \sigma_4), \bar{x}_1 = x_1 - h(\sigma_1, \sigma_2, \sigma_3, \sigma_4) + h(\sigma_1, \sigma_2, \sigma_3), \bar{x}_2 = x_2 - h(\sigma_1, \sigma_2, \sigma_3, \sigma_4) + \sigma_1 h(\sigma_2, \sigma_3, \sigma_4), and \bar{x}_3 = x_3 - \sigma_1 h(\sigma_2, \sigma_3, \sigma_4).

\begin{align*}
s_i(\sigma_1) = \begin{cases}
(0, 1, \sigma_1) & i = 0, \\
(0, 1, \sigma_1) & i = 1.
\end{cases}

\end{align*}

\begin{align*}
s_i(x_1, \sigma_1, \sigma_2) = \begin{cases}
(0, 0, 0, x_1, 1, \sigma_1, \sigma_2) & i = 0, \\
(0, 0, 0, x_1, \sigma_1, 1, \sigma_2) & i = 1, \\
(0, 0, 0, x_1, 0, \sigma_1, \sigma_2) & i = 2.
\end{cases}

\end{align*}

\begin{align*}
s_i(u_1, x_1, x_2, x_3, \sigma_1, \sigma_2, \sigma_3) = \begin{cases}
(0, 0, 0, u_1, 0, 0, x_1, x_2, x_3, 1, \sigma_1, \sigma_2, \sigma_3) & i = 0, \\
(0, 0, u_1, 0, 0, x_1, x_2, 0, 0, x_3, 1, \sigma_1, \sigma_2, \sigma_3) & i = 1, \\
(0, u_1, 0, x_1, 0, x_2, 0, x_3, 0, 0, \sigma_1, \sigma_2, \sigma_3) & i = 2, \\
(u_1, 0, 0, x_1, x_2, 0, x_3, 0, 0, \sigma_1, \sigma_2, \sigma_3) & i = 3.
\end{cases}

\end{align*}

Then, a bicategorical group \((\mathcal{B}, \otimes)\) with a simplicial isomorphism \(\Delta(\mathcal{B}, \otimes) \cong \text{M}\) is defined as follows: a 0-cell of \(\mathcal{B}\) is an element \(\sigma \in \text{G}\). If \(\sigma \neq \tau\) are different elements of \(\text{G}\), then \(\mathcal{B}(\sigma, \tau) = \emptyset\), that is, there is no 1-cell between them, whereas if \(\sigma = \tau\), then 1-cell \(x : \sigma \to \sigma\) is an element \(x \in \text{A}\). Similarly, there is no any 2-cell in \(\mathcal{B}\) between two 1-cells \(x, y : \sigma \to \sigma\).
if } x \neq y, \text{ whereas, when } x = y, \text{ a 2-cell } u : x \Rightarrow x \text{ is an element } u \in B. \text{ The vertical composition of 2-cells is given by addition in } B, \text{ that is,}

\[(x \xrightarrow{u} x) \cdot (x \xrightarrow{v} x) = (x \xrightarrow{u+v} x)\].

The horizontal composition of 1-cells and 2-cells is given by addition in } A \text{ and } B \text{ respectively, that is,

\[(\sigma \xrightarrow{u} \sigma) \circ (\sigma \xrightarrow{v} \sigma) = (\sigma \xrightarrow{u+v} \sigma)\].

The associativity isomorphism is

\[
\sigma \xrightarrow{(x+y)+z} \sigma, \quad a = t(x, y, z, 0, 0, 0, \sigma, 1, 1, 1),
\]

and the 0 of } A \text{ gives the (strict) unit on each } \sigma, \text{ that is, } 1_{\sigma} = 0 : \sigma \to \sigma.

The (strictly unitary) tensor homomorphism } \otimes : B \times B \to B \text{ is given on cells of } B \text{ by

\[(\sigma \xrightarrow{u} \sigma) \otimes (\tau \xrightarrow{v} \tau) = (\sigma \otimes \tau \xrightarrow{u+v} \sigma \otimes \tau),\]

and the structure interchange isomorphism, for any 1-cells } \sigma \xrightarrow{x} \sigma \text{ and } \tau \xrightarrow{y} \tau,

\[
\sigma \xrightarrow{(x+y)+z+y'} \sigma, \quad (x+z+y') = (x+y)+(y+z),
\]

is that obtained by pasting in the bigroupoid } B \text{ the diagram

\[
\begin{array}{c}
\sigma \xrightarrow{\eta_{y+y'}} \sigma \xrightarrow{\eta_{x}} \sigma \xrightarrow{\eta_{x'}} \sigma \xrightarrow{\eta_{y}} \sigma \xrightarrow{\eta_{x'+y}} \\
\sigma \xrightarrow{\eta_{x+y'+z}} \sigma \xrightarrow{\eta_{x+y}} \sigma \xrightarrow{\eta_{x+y'+z}} \sigma \xrightarrow{\eta_{x+y}} \sigma \xrightarrow{\eta_{x+y'+z}}
\end{array}
\]

where

\[
\chi = -t(0, 0, 0, \sigma_{y}, \sigma_{y}', 0, \sigma, \tau, 1, 1),
\]

\[
c = t(0, x, 0, 0, \sigma_{y}, 0, \sigma, \tau, 1, 1) - t(0, 0, 0, \sigma_{y}, 0, \sigma, \tau, 1, 1) - t(x, 0, 0, 0, \sigma_{y}, 0, \sigma, \tau, 1, 1) - t(0, 0, \sigma_{x}, 0, 0, 0, \sigma, \tau, 1, 1),
\]

\[
\chi = -t(0, x, x', 0, 0, \sigma, \tau, 1, 1) + t(x, 0, x', 0, 0, 0, \sigma, 1, \tau, 1) - t(x, x, 0, 0, 0, \sigma, 1, \tau, 1).
\]

The associativity pseudo-equivalence } (\sigma \otimes -) \otimes - = \sigma (\tau \otimes -) : B^{3} \to B \text{ is defined by the 1-cells}

\[
h(\sigma, \tau, \gamma) : (\sigma \tau) \gamma \Rightarrow \sigma (\tau \gamma).
\]

The naturality component of } a \text{, at any 1-cells } \sigma \xrightarrow{x} \sigma, \tau \xrightarrow{y} \tau \text{ and } \gamma \xrightarrow{z} \gamma,

\[
\sigma \xrightarrow{(x+y)+z} \sigma \xrightarrow{(x+y)+z} \sigma \xrightarrow{(x+y)+z} \sigma \xrightarrow{(x+y)+z} \sigma
\]
is given by pasting in \( \mathcal{B} \) the diagram

\[
\begin{array}{c}
\xymatrix{ 
& (\sigma \tau) \gamma \ar[rd]_{\psi} & \\
(\sigma \tau) \gamma \ar[rr]^x & & (\sigma \tau) \gamma \ar[ld]_{\psi} \\
(\sigma \tau) \gamma \ar[rr]_{\sigma y} & & (\sigma \tau) \gamma \ar[ld]_{\psi} \\
(\sigma \tau) \gamma \ar[rr]_{\sigma z} & & (\sigma \tau) \gamma \ar[ld]_{\psi} \\
& (\sigma \tau) \gamma \ar[ru]_{\sigma z} & \\
}\end{array}
\]

where

\[
\Phi = t(0, h, 0, 0, \sigma \tau z, 0, \sigma, \tau \gamma, 1, 1) - t(h, 0, 0, 0, 0, \sigma \tau z, 0, \sigma, \tau \gamma, 1, 1),
\]

\[
\Psi = t(h, 0, 0, 0, \sigma y, 0, \sigma, \tau, 1, \gamma) - t(h, 0, 0, 0, 0, \sigma y, 0, \sigma, \tau, 1, 1) + t(0, h, 0, 0, 0, \sigma y, 0, \sigma, \tau, 1, 1)
\]

\(-t(0, h, 0, 0, \sigma y, 0, \sigma, \tau \gamma, 1, 1),
\]

\[
\Omega = -t(x, h, 0, 0, 0, 0, \sigma, \tau, 1, \gamma) + t(h, x, 0, 0, 0, 0, \sigma, \tau, 1, \gamma) - t(h, x, 0, 0, 0, \sigma, \tau, 1, 1)
\]

\(+t(0, h, x, 0, 0, 0, 0, \sigma, \tau, 1, 1) - t(0, h, 0, 0, 0, 0, \sigma, \tau, 1, 1),
\]

The structure modification \( \pi \), at any objects \( \sigma, \tau, \gamma, \delta \in G \), is

\[
\begin{array}{c}
\xymatrix{ 
(\sigma \tau) \gamma \ar@/_/[rr]_{\psi} & \\
(\sigma \tau) \gamma \ar[rr]_{\sigma z} & & (\sigma \tau) \gamma \\
(\sigma \tau) \gamma \ar[rr]_{\sigma z} & & (\sigma \tau) \gamma \ar[ru]_{\sigma z} \\
(\sigma \tau) \gamma \ar[rr]_{\sigma z} & & (\sigma \tau) \gamma \ar[rd]_{\psi} \\
& (\sigma \tau) \gamma & \\
}\end{array}
\]

\[
\pi = t(h_3, h_1 - h_0, 0, h_0, 0, 0, \sigma, \tau, \gamma, \delta) - t(h_2, h_4, 0, 0, 0, 0, \sigma, \tau, \gamma, 1, \delta)
\]

\(+t(h_2, 0, h_4, 0, 0, 0, \sigma, \tau, \gamma, 1, \delta) - t(h_3, 0, h_1 - h_0, 0, h_0, 0, \sigma, \tau, \gamma, \delta, 1)
\]

\(+t(0, h_3, h_1 - h_0, 0, h_0, 0, 0, \sigma, \tau, \gamma, \delta, 1, 1) - t(0, h_2 + h_4, 0, 0, 0, \sigma, \tau, \gamma, \delta, 1, 1)
\]

\(-t(0, h_2, h_4, 0, 0, 0, \sigma, \tau, \gamma, \delta, 1, 1). \]

This completes the description of bicategorical group \( (\mathcal{B}, \otimes) \), whose geometric nerve is recognized to be isomorphic to the minimal complex \( M \) in \( [77] \) by means of the simplicial map \( \varphi : \Delta(\mathcal{B}, \otimes) \to M \) which, in dimensions \( \leq 4 \),

\[
\begin{array}{c}
\xymatrix{ 
\Delta_4(\mathcal{B}, \otimes) & \Delta_3(\mathcal{B}, \otimes) & \Delta_2(\mathcal{B}, \otimes) & \Delta_1(\mathcal{B}, \otimes) & 1, \\
& \varphi & \varphi & \varphi & \varphi, \\
\xymatrix{ 
B^4 \times A^6 \times G^4 & B \times A^3 \times G^3 \ar[ll] & A \times G^2 \ar[ll] & G \ar[ll] & 1, \\
}\end{array}
\]

is defined as follows (keeping the notations stated in \([66] - [70]\)): It carries a (unitary) representation \( F : [1] \to \Sigma(\mathcal{B}, \otimes) \) to \( \varphi(F) = F_{01} \), a representation \( F : [2] \to \Sigma(\mathcal{B}, \otimes) \) to \( \varphi(F) = \langle -F_{012}, F_{01}, F_{12} \rangle \), a representation \( F : [3] \to \Sigma(\mathcal{B}, \otimes) \) to \( \varphi(F) = \langle -F_{0123}, -F_{01}F_{123} + F_{023}, -F_{01}F_{123}, -F_{012}, F_{012}, F_{12}, F_{23}, \rangle \), and a representation \( F : [4] \to \Sigma(\mathcal{B}, \otimes) \) to \( \varphi(F) = \langle u_1, u_2, u_3, u_4, x_1, x_2, x_3, x_4, x_5, x_6, F_{01}, F_{12}, F_{23}, F_{34} \rangle \),

where

\[
\begin{align*}
\varphi(F) &= \langle u_1, u_2, u_3, u_4, x_1, x_2, x_3, x_4, x_5, x_6, F_{01}, F_{12}, F_{23}, F_{34} \rangle \\
\varphi(F) &= \langle -F_{0123}, -F_{01}F_{123} + F_{023}, -F_{01}F_{123}, -F_{012}, F_{012}, F_{12}, F_{23}, \rangle \\
\end{align*}
\]

To finish, we shall remark on two particular relevant cases of the demonstrated relationship between monoidal bicategories and path-connected homotopy 3-types. Since categorical
groups [24] §3] are the same thing as bicategorical groups in which all 2-cells are identities, then categorical groups are algebraic models for path-connected homotopy 2-types. This fact goes back to Whitehead (1949) and Mac Lane-Whitehead (1950) since every categorical group is equivalent to a strict one, and strict categorical groups are the same as crossed modules. On the other hand, if $(C, \otimes, c)$ is any braided categorical group [24], then its classifying space $B(C, \otimes, c)$ is the classifying space of its suspension bicategorical group $(\Sigma(C, \otimes, \otimes))$ (see Examples 4.7 and 5.5), which is precisely a one-object bicategorical group. Therefore, we conclude from the above discussion that braided categorical groups are algebraic models for path-connected simply connected homotopy 3-types, a fact due to Joyal and Tierney [26], but also proved in [8] and, implicitly, in [24].

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