POSITIVE GAMES ON RANDOMLY PERTURBED GRAPHS

DENNIS CLEMENS, FABIAN HAMANN, YANNICK MOGGE, AND OLAF PARCZYK

ABSTRACT. Maker-Breaker games are played on a hypergraph \((X, \mathcal{F})\), where \(\mathcal{F} \subseteq 2^X\) denotes the family of winning sets. Both players alternately claim a predefined amount of edges (called bias) from the board \(X\), and Maker wins the game if she is able to occupy any winning set \(F \in \mathcal{F}\). These games are well studied when played on the complete graph \(K_n\) or on a random graph \(G_{n,p}\). In this paper we consider Maker-Breaker games played on randomly perturbed graphs instead. These graphs consist of the union of a deterministic graph \(G_\alpha\) with minimum degree at least \(\alpha n\) and a binomial random graph \(G_{n,p}\). Depending on \(\alpha\) and Breaker’s bias \(b\) we determine the order of the threshold probability for winning the Hamiltonicity game and the \(k\)-connectivity game on \(G_\alpha \cup G_{n,p}\), and we discuss the \(H\)-game when \(b = 1\). Furthermore, we give optimal results for the Waiter-Client versions of all mentioned games.

1. INTRODUCTION

In general, a positional game is a perfect information game played by two players on a hypergraph \((X, \mathcal{F})\). Throughout the game both players occupy elements of the board \(X\) according to some predefined rule, and the winner is determined through the family of winning sets \(\mathcal{F}\). Research of the last decades has generated many interesting results in the area of positional games (see e.g. \([5, 26]\)), where plenty of different types were considered, including Maker-Breaker games (see e.g. \([8, 22, 24, 37, 40]\)), Waiter-Client games (see e.g. \([7, 16, 30]\)), and many more.

In this paper we are mainly interested in positional games where \(X\) is the edge set of some given graph \(G\), and where \(\mathcal{F}\) is the family of all Hamilton cycles of \(G\), all \(k\)-vertex-connected spanning subgraphs of \(G\), or all copies of some fixed graph \(H\) in \(G\), respectively. Before we will discuss such games further, let us first recall what is known about the appearance of these structures in dense graphs, random graphs and randomly perturbed graphs.

1.1. Dense graphs. For a given graph \(G\) let \(\delta(G)\) denote its minimum degree. When we consider all graphs \(G\) on \(n\) vertices, it is quite natural to ask how large \(\delta(G)\) needs to be in order to guarantee the existence of a given structure. Dirac [17] proved that any graph on \(n \geq 3\) vertices with minimum degree at least \(\frac{n}{2}\) is Hamiltonian, with the bound being sharp, since the vertex disjoint union of two cliques of sizes \(\lfloor \frac{n}{2} \rfloor\) and \(\lceil \frac{n}{2} \rceil\) does not contain a Hamilton cycle. Moreover, it is an easy observation, that for any positive integer \(k\), any \(n\)-vertex graph with OP was supported by the DFG (Grant PA 3513/1-1).
minimum degree at least \( \frac{n+k-1}{2} \) is \( k \)-vertex-connected. Again this bound is seen to be sharp by looking at the following example: take the vertex disjoint union of three cliques of sizes \( k-1 \), \( \left\lceil \frac{n-k+1}{2} \right\rceil \) and \( \left\lfloor \frac{n-k+1}{2} \right\rfloor \), respectively, and add all edges between the first clique and both of the other cliques. For the containment of a fixed graph \( H \) a sufficient minimum degree condition can be derived from Turán's Theorem \([46]\) with regularity, or from the following theorem of Erdős and Stone \([20]\): for any fixed graph \( H \) with chromatic number \( r \geq 2 \), any \( n \)-vertex graph with minimum degree \( \left( \frac{r-2}{r-1} + o(1) \right)n \) contains a copy of \( H \) provided that \( n \) is large. This bound is seen to be sharp by considering a Turán graph on \( n \) vertices with \( \left\lceil \frac{n}{r} \right\rceil \) vertex classes, i.e. an \( \left\lceil \frac{n}{r} \right\rceil \)-partite graph with all class sizes differing at most by 1. For bipartite graphs \( H \) any linear minimum degree is sufficient.

1.2. Random graphs. Let \( G_{n,p} \) be the binomial random graph model, i.e. the model of \( n \)-vertex random graphs, where each edge is present with probability \( p \) independently of all others. For simplicity, we will often write \( G_{n,p} \) also for a graph \( G \sim G_{n,p} \) drawn according to this distribution. Given some graph property \( \mathcal{P} \), we are interested in a probability function \( \hat{p} = \hat{p}(\mathcal{P}) \) such that

the following holds: when \( p = \omega(\hat{p}) \), then \( G_{n,p} \) satisfies \( \mathcal{P} \) asymptotically almost surely (a.a.s.), i.e. with probability tending to one as \( n \) tends to infinity. When additionally it holds that \( G_{n,p} \)
a.a.s. does not have property \( \mathcal{P} \) when \( p = o(\hat{p}) \), we call \( \hat{p} \) a threshold probability for the property \( \mathcal{P} \). The existence of such a threshold is known for all monotone graph properties \( \mathcal{P} \) \([12]\). If the same conclusion even holds with \( p \geq (1+\varepsilon)\hat{p} \) and \( p \leq (1-\varepsilon)\hat{p} \) for any constant \( \varepsilon > 0 \), then we call \( \hat{p} \) a sharp threshold probability for the property \( \mathcal{P} \).

Posá \([43]\) and Korshunov \([36]\) independently proved that \( G_{n,p} \) has a sharp threshold for containing a Hamilton cycle at \( \frac{\ln n}{n} \). In fact, there are even more precise results known, e.g. it holds a.a.s. that at the point when \( G_{n,p} \) has minimum degree two it is already Hamiltonian \([13]\). Note that the existence of a Hamilton cycle immediately implies 2-vertex-connectivity. More generally, it follows from an early work of Erdős and Rényi \([19]\), that for any fixed positive integer \( k \) the random graph \( G_{n,p} \) has a sharp threshold for being \( k \)-vertex-connected at \( \frac{\ln n}{n} \). Lastly, consider the containment of a fixed graph \( H \). Let \( d(H) = \frac{e(H)}{v(H)} \) and denote by

\[
m(H) = \max_{H' \leq H, e(H') > 0} d(H')
\]

the maximum subgraph density of \( H \). Bollobás \([12]\) proved that the threshold probability for \( G_{n,p} \) containing a copy of \( H \) lies at \( n^{-1/m(H)} \).

1.3. Randomly perturbed graphs. Note that all of the constructions described above for showing the sharpness of the minimum degree conditions for dense graphs are ‘well-structured’ in a certain way: they contain very dense parts (e.g. large cliques) and very sparse parts (e.g. large independent sets) at the same time, and thus they are far away from the ‘typical shape’ of a
random graph. For that reason, one may hope for improving the minimum degree conditions, that we already discussed for dense graphs, when we allow to extend the given graph slightly by additionally sprinkling a few random edges on top of it. To be more precise, fix any real \( \alpha > 0 \) and let \( G_\alpha \) be a sequence of \( n \)-vertex graphs with minimum degree at least \( \alpha n \). Our goal is to consider the model \( G_\alpha \cup G_{n,p} \) which was first studied by Bohman, Frieze, and Martin \cite{Bohman2001}. In the following we slightly abuse notation and also write \( G_\alpha \cup G_{n,p} \) for \( G_\alpha \cup G \) with \( G \sim G_{n,p} \). Similarly to the discussion of the binomial random graph, we can look for \( \hat{p} = \hat{p}(\mathcal{P}) \) such that the following holds:

- with \( p = \omega(\hat{p}) \) it is true that for any sequence \( G_\alpha \) the desired property \( \mathcal{P} \) holds a.a.s. in \( G_\alpha \cup G_{n,p} \),
- with \( p = o(\hat{p}) \) there exists a sequence \( G_\alpha \) such that the desired property \( \mathcal{P} \) a.a.s. does not hold in \( G_\alpha \cup G_{n,p} \).

Bohman, Frieze, and Martin \cite{Bohman2001} showed that for any \( \alpha > 0 \) there exists a large enough constant \( C \) such that it is sufficient to take \( p \geq \frac{C}{n} \) to ensure that a.a.s. \( G_\alpha \cup G_{n,p} \) is Hamiltonian. This is asymptotically optimal, because with \( G_\alpha = K_{\alpha n,(1-\alpha n)} \) we can use at most \( 2\alpha n \) edges of \( G_\alpha \) for a Hamilton cycle and when \( p = o\left(\frac{1}{n}\right) \) we a.a.s. have that \( G_{n,p} \) only adds \( o(n) \) edges. On one hand, this result shows that any minimum degree linear in \( n \) suffices to ensure that a.a.s. a Hamilton cycle appears when a sparse random graph with \( \omega(n) \) edges is added. On the other hand, we see that starting with a dense graph \( G_\alpha \) we need an edge probability \( p \) for the likely Hamiltonicity of \( G_\alpha \cup G_{n,p} \) which is a ln-factor smaller than the corresponding threshold for the binomial random graph.

Among other things, it was shown by Bohman, Frieze, Krivelevich, and Martin \cite{Bohman2000} that for any fixed positive integer \( k \) the randomly perturbed graph \( G_\alpha \cup G_{n,p} \) is \( k \)-vertex-connected when \( p = \omega\left(\frac{1}{n^2}\right) \); results are also given for the case when \( k \) is not a constant but depending on \( n \).

To discuss the results for a fixed graph \( H \) we first need to define the \textit{\( r \)-partite density} of \( H \), as it was introduced by Krivelevich, Sudakov, and Tetali \cite{Krivelevich2001}. Let

\[
m^{(r)}(H) := \min_{V(H) = \cup_i P_i} \max_i m(H[P_i]),
\]

where we minimise over all possible partitions of the vertex set of \( H \) into \( r \) parts \( P_1, \ldots, P_r \). Then, for any integer \( r \), any \( \alpha \in \left(\frac{r-2}{r-1}, \frac{r-1}{r}\right] \), and any fixed graph \( H \) with \( m^{(r)}(H) > 0 \), the randomly perturbed graph \( G_\alpha \cup G_{n,p} \) a.a.s. contains a copy of \( H \) provided that \( p = \omega\left(\frac{1}{n^{1/m^{(r)}(H)}}\right) \) \cite{Krivelevich2001}. Note that when \( m^{(r)}(H) = 0 \), this implies, that \( \chi(H) \leq r \), and thus, due to Erdős and Stone \cite{Erdos1946}, there already exists a copy of \( H \) in \( G_\alpha \) alone. To see that this is optimal it suffices to consider a complete \( r \)-partite graph on \( n \) vertices with roughly equal parts and to note that in one of the
parts there has to be a copy of some $H' \subseteq H$ with $m(H') \geq m^{(r)}(H)$ fully composed of edges of $G_{n,p}$, which a.a.s. does not appear when $p = o \left( n^{-1/m^{(r)}(H)} \right)$.

1.4. Maker-Breaker games. In our paper we will mainly focus on Maker-Breaker games. A $(1 : b)$ Maker-Breaker game (also referred to as $b$-biased Maker-Breaker game) on some hypergraph $(X, \mathcal{F})$ is played as follows. Maker and Breaker alternate in taking 1 and $b$ unclaimed elements of $X$, respectively (except for maybe the last round where Breaker could take less elements), and the game is Maker’s win if she fully claims an element of $\mathcal{F}$; otherwise Breaker wins. Since increasing the bias $b$ is never a disadvantage for Breaker, it can easily be shown that for any hypergraph $(X, \mathcal{F})$ there must be a threshold bias $b_\mathcal{F}$ such that Maker wins if and only if $b < b_\mathcal{F}$ (see e.g. [26]).

In our setting $X$ will be the edge set of some host graph $G$ and $\mathcal{F}$ will consist of all Hamilton cycles, $k$-vertex-connected subgraphs, or copies of $H$ in $G$, respectively. In the literature, $G$ is usually chosen to be the complete graph $K_n$, or (more recently) the binomial random graph $G_{n,p}$ (see e.g. [27], [44]). But already when playing on $K_n$ the intuition from random graphs plays an important role, since Maker’s subgraph can exhibit similar properties as a subgraph where each edge of $X = E(K_n)$ is taken at random with probability $\frac{1}{b+1}$. Moreover, for many (but not all) families $\mathcal{F}$ the threshold bias in the $(1 : b)$ game on $X = E(K_n)$ is tightly linked to the threshold probability for $G_{n,p}$ to be such that Maker wins the $(1 : 1)$ game with winning sets $\mathcal{F}$.

As a first example, let us look at the Maker-Breaker Hamiltonicity game, where the family of winning sets $\mathcal{F} = \mathcal{H}(G)$ consists of all Hamilton cycles of the host graph $G$ which the game is played on. When playing on $G = K_n$, it was shown by Krivelevich [37] that the threshold bias $b_{\mathcal{H}(K_n)}$ is of size $(1 + o(1))\frac{\ln n}{n}$. One interesting fact about this result is that it supports the probabilistic intuition: if Maker and Breaker would play the $b$-biased game fully at random, Maker’s graph would behave similarly to $G_{n,p}$ with edge probability $p = \frac{1}{b+1}$, which asymptotically equals the threshold $(1 + o(1))\frac{\ln n}{n}$ for containing a Hamilton cycle when $b = b_{\mathcal{H}(K_n)}$.

Furthermore, as shown by Krivelevich, Lee, and Sudakov [38], Maker can even win the $b$-biased Hamiltonicity game on a graph $G$ with $\delta(G) \geq \frac{2}{2}$ provided that $b \leq \frac{c\ln n}{\ln n}$ for some small enough constant $c > 0$. On the other hand [27], $G_{n,p}$ is a.a.s. such that Maker wins the $(1 : 1)$ Maker-Breaker Hamiltonicity game provided that $p \geq (1 + \varepsilon)\frac{\ln n}{n}$ for any fixed $\varepsilon > 0$. This coincides with the appearance of a Hamilton cycle in $G_{n,p}$ as discussed above, and this threshold probability happens to be asymptotically the inverse of the threshold bias $b_{\mathcal{H}(K_n)}$. Moreover, for $p = \omega \left( \frac{\ln n}{n} \right)$ it has been shown that the threshold bias for winning the Hamiltonicity game on $G_{n,p}$ is a.a.s. of size $\Theta \left( \frac{\ln p}{\ln n} \right)$ [21], giving a linear dependency between the edge probability $p$ and the threshold bias $b_{\mathcal{H}(G_{n,p})}$. 
Next let us consider the $k$-vertex-connectivity game where $\mathcal{F} = \mathcal{C}_k(G)$ consists of all spanning $k$-vertex-connected subgraphs of the host graph $G$ which the game is played on. As observed by Krivelevich [37], his approach for the Hamiltonicity game on $K_n$ can be modified to show that $b_{\mathcal{C}_k(K_n)} = (1 + o(1))\frac{n}{\ln n}$, thus again providing an example supporting the probabilistic intuition. Additionally and similarly as above, Maker wins the $(1 : 1)$ Maker-Breaker $k$-vertex-connectivity game on $G_{n,p}$ provided that $p \geq (1 + \varepsilon)\frac{\ln n}{n}$ for any fixed $\varepsilon > 0$ [9]; and for $p = \omega\left(\frac{\ln n}{n}\right)$, the threshold bias for winning the $k$-vertex-connectivity game on $G_{n,p}$ is a.a.s. of size $\Theta\left(\frac{\ln n}{n}\right)$ [21].

Finally, for any fixed graph $H$, let us turn to the Maker-Breaker $H$-game, where $\mathcal{F} = \mathcal{F}_H(G)$ consists of all copies of $H$ that are contained in the host graph $G$ which the game is played on. For any graph $H'$ on at least 3 vertices, let $d_2(H') = \frac{n(H') - 1}{e(H') - 2}$, and denote by

$$m_2(H) = \max_{H' \subseteq H, e(H) > 2} d_2(H')$$

the maximum $2$-density of $H$, where (for convenience) we let $m_2(P_1) = d_2(P_1) = 1$ and $m_2(P_0) = d_2(P_0) = 0$ with $P_1$ being a single edge and $P_0$ being an isolated vertex. For the $H$-game on $K_n$, Bednarska and Łuczak [8] proved that $b_{\mathcal{F}_H(K_n)} = \Theta\left(n^{1/m_2(H)}\right)$, provided that $H$ contains at least 3 non-isolated vertices. Our proofs for the results discussed in the next section will also imply the following for the $(1 : 1)$ Maker-Breaker $H$-game on graphs with large minimum degree.

**Theorem 1.1.** Let $r \geq 2$ be an integer, $\alpha \in \left(\frac{r-2}{r-1}, \frac{r-1}{r}\right)$, let $H$ be a fixed graph with chromatic number $r$, and let $G_\alpha$ be a graph on $n$ vertices with $\delta(G_\alpha) \geq \alpha n$. Then provided that $n$ is large enough the following holds: playing a $(1 : 1)$ Maker-Breaker game on the edges of $G_\alpha$, Maker has a strategy to obtain a copy of $H$.

Now, let $H$ be a graph for which there exists $H' \subseteq H$ such that $d_2(H') = m_2(H)$, $d_2(H^n) < d_2(H')$ for all $H^n \subseteq H'$, and $H'$ is not a tree or triangle. Then the threshold probability $p_{\mathcal{F}_H}$ for winning the Maker-Breaker $H$-game on $G_{n,p}$ turns out to be of the order $n^{-1/m_2(H)}$ [42], and hence, similar to the discussion of the Hamiltonicity game and the $k$-vertex-connectivity game, it is asymptotically the inverse of the order of the threshold bias $b_{\mathcal{F}_H}(K_n)$ for the corresponding game on $K_n$. However, this last observation is not true for general $H$. For instance, when $H = K_3$, the threshold probability $p_{\mathcal{F}_{K_3}}$ is of order $n^{-5/9}$ [44]. Until today, it is still an open problem to find the threshold probability $p_{\mathcal{F}_H}$ for every $H$.

**1.5. Our results - Maker-Breaker games on randomly perturbed graphs.** The main goal of this paper is to study Maker-Breaker games on randomly perturbed graphs where the family of winning sets $\mathcal{F}$ consists of Hamilton cycles, $k$-vertex-connected spanning subgraphs or copies of a fixed graph $H$, respectively. Depending on $\alpha$ and Breaker’s bias $b$, we aim to determine the order of the threshold probability for winning the $(1 : b)$ Maker-Breaker game on $G_\alpha \cup G_{n,p}$ with winning sets $\mathcal{F}$. For the case when $b = 1$, one main question will be, whether
the appearance of the structures from $\mathcal{F}$ in $G_\alpha \cup G_{n,p}$ is roughly sufficient for a Maker’s win in the corresponding game.

For the Hamiltonicity game we prove the following.

**Theorem 1.2** (Biased MB Hamiltonicity game). For every real $\alpha > 0$ there exist constants $c, C > 0$ such that the following holds for large enough integers $n$. Let $G_\alpha$ be a graph on $n$ vertices with $\delta(G_\alpha) \geq \alpha n$, let $b \leq \frac{cn}{\ln n}$ be an integer, and let $p \geq \frac{C}{n}$. Then a.a.s. the following holds: playing a $(1 : b)$ Maker-Breaker game on the edges of $G_\alpha \cup G_{n,p}$, Maker has a strategy to claim a Hamilton cycle.

Note that the bound on $b$ is optimal up to the constant factor, since for $b \geq (1 + \varepsilon) \frac{n}{\ln n}$, Breaker can isolate a vertex on any graph having $n$ vertices [14], and hence he wins the Hamiltonicity game. Also, for $\alpha \in (0, \frac{1}{2})$, the bound on $p$ is optimal up to the constant factor. To see this, it is enough to show that for each $\alpha \in (0, \frac{1}{2})$ there exists a sequence of graphs $G_\alpha$ such that Breaker a.a.s. wins the Hamiltonicity game on $G_\alpha \cup G_{n,p}$ for $p \leq \frac{1-2\alpha b}{2(1-\alpha)\ln n}$. For this, consider $G_\alpha$ to be a complete bipartite graph $A \cup B$ with $|A| = \alpha n$ and $|B| = (1-\alpha)n$. Now, every Hamilton cycle in $G_\alpha \cup G_{n,p}$ needs to contain at least $(1-2\alpha)n$ edges within $B$. However, a.a.s. $G_{n,p}$ has less than $\frac{(1-2\alpha)b}{2}$ edges within $B$, and hence, Breaker can ensure that Maker cannot claim $(1-2\alpha)n$ of these edges by occupying $b$ of these edges for himself in each round. Note that in this case we obtain a linear dependency between Breaker’s bias $b$ and the threshold probability $p$ for winning the game, analogously to the Hamiltonicity game on $G_{n,p}$. In the remaining case, when $\alpha \geq \frac{1}{2}$, we can actually choose $p = 0$ and $b \leq \frac{cn}{\ln n}$ by the result of Krivelevich, Lee, and Sudakov [38]. Moreover, note that the above theorem strengthens the result of Bohman, Frieze, and Martin [11] on the containment of Hamilton cycles. When $p \geq \frac{C}{n}$ for some large enough constant $C$, the graph $G_\alpha \cup G_{n,p}$ a.a.s. does not only contain a Hamilton cycle; instead it is so rich of this structure that Maker can win the $(1 : 1)$ Maker-Breaker Hamiltonicity game on it.

For the $k$-vertex-connectivity game we show the following result.

**Theorem 1.3** (Biased MB $k$-vertex-connectivity game). For every real $\alpha > 0$ and every integer $k \geq 1$ there exist constants $C, c > 0$ such that the following holds for large enough integers $n$. Let $G_\alpha$ be a graph on $n$ vertices with $\delta(G_\alpha) \geq \alpha n$, let $b \leq \frac{cn}{\ln n}$ be an integer, and let $p \geq \frac{C}{n^2}$. Then with probability at least $1 - \exp(-cn^2)$ the following holds: playing a $(1 : b)$ Maker-Breaker game on the edges of $G_\alpha \cup G_{n,p}$, Maker has a strategy to claim a spanning $k$-vertex-connected graph.

Note again that this theorem strengthens the result on the $k$-vertex-connectivity of $G_\alpha \cup G_{n,p}$ given in [10], and it also gives a linear dependency between $b$ and the threshold probability $p$. 
The bound on $b$ is optimal up to the constant factor by the same reason as before. For optimality regarding $p$, consider $G_\alpha$ to be a graph consisting of (roughly) $\frac{1}{\alpha}$ vertex disjoint cliques of size (roughly) $\alpha n$. When $p \leq \frac{\alpha}{n^2}$, using Markov’s inequality we see that with probability at least $1 - \varepsilon$ there are less than $b$ edges in the graph $G_{n,p}$ and therefore Breaker can easily ensure that Maker receives at most 1 such edge. However, adding this edge to $G_\alpha$ does not even result in a connected spanning graph and hence Maker cannot occupy such a graph in the game on $G_\alpha \cup G_{n,p}$.

Finally, let us turn to the $H$-game on $G_\alpha \cup G_{n,p}$. For the case when $H$ is a clique we obtain the following result.

**Theorem 1.4** (Unbiased MB Clique game). Let $\gamma > 0$, integers $t, r \geq 2$, and $\alpha \in (\frac{r-2}{r-1}, \frac{r-1}{r}]$ be given. Let $G_\alpha$ be an $n$-vertex graph with $\delta(G_\alpha) \geq \alpha n$, and let $p \geq n^{-2/(t+1)+\gamma}$. Then a.a.s. the following holds: playing a $(1 : 1)$ Maker-Breaker game on the edges of $G_\alpha \cup G_{n,p}$, Maker has a strategy to obtain a copy of $K_t$.

For $t \geq 4$ this is optimal up to the constant $\gamma$ in the exponent, because with $p = o \left( n^{-2/(t+1)} \right)$ and $G_\alpha$ being an $r$-partite Turán graph we need a copy of $K_t$ on at least one of the partite sets, but a.a.s. Breaker has a strategy to prevent Maker from having any copy of $K_t$ in $G_{n,p}$ [42].

To extend this to arbitrary graphs $H$ we introduce the following $r$-partite 2-density, analogously to the $r$-partite density introduced by Krivelevich, Sudakov, and Tetali [39] for studying the appearance of $H$ in $G_\alpha \cup G_{n,p}$. Let

$$m_2^{(r)}(H) := \min_{V(H)=\cup_i P_i} \max_i m_2(H[P_i]),$$

where we minimise over all possible partitions of the vertex set of $H$ into $r$ parts $P_1, \ldots, P_r$. Then the following holds.

**Theorem 1.5** (Unbiased MB $H$-game). Let $\gamma > 0$, $r \geq 2$ be an integer, $\alpha \in (\frac{r-2}{r-1}, \frac{r-1}{r}]$, and let $H$ be a fixed graph with $m_2^{(r)}(H) > 0$. Further, let $G_\alpha$ be a graph on $n$ vertices with $\delta(G_\alpha) \geq \alpha n$, and let $p \geq n^{-1/m_2^{(r)}(H)+\gamma}$. Then a.a.s. the following holds: playing a $(1 : 1)$ Maker-Breaker game on the edges of $G_\alpha \cup G_{n,p}$, Maker has a strategy to obtain a copy of $H$.

Note that $m_2^{(r)}(H) > 0$ if and only if $H$ has chromatic number more than $r$ and, therefore, Theorem 1.1 covers the case $m_2^{(r)}(H) = 0$ with $p = 0$. For many cases of $H$ Theorem 1.5 again is optimal up to the constant $\gamma$ in the exponent. But, similarly to the results for the $H$-game on $G_{n,p}$, we do not have optimality when the relevant density $m_2^{(r)}(H)$ is determined by a subgraph of $H$ isomorphic to $K_3$. We will discuss more details in Section 6 together with some open problems.
1.6. Our results - Waiter-Client games on randomly perturbed graphs. Some of the approaches of our proofs for Maker-Breaker games can be modified to work for Waiter-Client games. A $(1:b)$ Waiter-Client game (also referred to as $b$-biased Waiter-Client game) on some hypergraph $(X,F)$ is played as follows. In every round, Waiter chooses $b+1$ elements of $X$ that have not been chosen before (except for maybe the last round where Waiter could pick less elements), and she offers those to Client. Client then claims one of these offered elements (except for maybe the last round in the case when there is only one element left), while all the other elements go to Waiter. The game is said to be Waiter’s win if Client fully claims an element of $F$; otherwise Client wins. This time, increasing the bias $b$ is never a disadvantage for Client, and hence there must be a threshold bias $b_F$ such that Waiter wins if and only if $b < b_F$ (see e.g. [45]). For the structures discussed above, we prove the following results.

**Theorem 1.6 (Biased WC Hamiltonicity game).** For every real $\alpha > 0$ there exist constants $c, C > 0$ such that the following holds for large enough integers $n$. Let $G_\alpha$ be a graph on $n$ vertices with $\delta(G_\alpha) \geq \alpha n$, let $b \leq cn$ be an integer, and let $p \geq \frac{Cn}{b}$. Then a.a.s. the following holds: playing a $(1:b)$ Waiter-Client game on the edges of $G_\alpha \cup G_{n,p}$, Waiter has a strategy to force Client to occupy a Hamilton cycle.

**Theorem 1.7 (Biased WC $k$-vertex-connectivity game).** For every real $\alpha > 0$ and every integer $k$ there exist constants $C, c > 0$ such that the following holds for large enough integers $n$. Let $G_\alpha$ be a graph on $n$ vertices with $\delta(G_\alpha) \geq \alpha n$, let $b \leq cn$ be an integer, and let $p \geq \frac{Cn}{b}$. Then with probability at least $1 - \exp(-c p n^2)$ the following holds: playing a $(1:b)$ Waiter-Client game on the edges of $G_\alpha \cup G_{n,p}$, Waiter has a strategy to force Client to claim a spanning $k$-vertex-connected graph.

**Theorem 1.8 (Unbiased WC $H$-game).** For any integer $r \geq 2$ let $\alpha \in \left(\frac{r-2}{r}, \frac{r-1}{r}\right)$, and let $H$ be a fixed graph. Further, let $G_\alpha$ be a graph on $n$ vertices with $\delta(G_\alpha) \geq \alpha n$, and let $p = \omega(n^{-1/m^r(H)})$. Then a.a.s. the following holds: playing a $(1:1)$ Waiter-Client game on $G_\alpha \cup G_{n,p}$, Waiter has a strategy to force Client to claim a copy of $H$.

Regarding the edge probability $p$, note that the results for the Hamiltonicity game and the $k$-vertex-connectivity game coincide with the theorems from Maker-Breaker games; the optimality can be explained analogously. However, in contrast to Maker-Breaker games, both Theorem 1.6 and Theorem 1.7 allow the bias to be linear in $n$. For the corresponding games on $K_n$ this was already shown in [7]. Although we use tools for the proof that are similar to the discussion of the corresponding Maker-Breaker games, the difference regarding the bias $b$ requires some new ideas. Moreover, for the $H$-game the threshold on $p$ differs from the discussion of Maker-Breaker games. The bound in Theorem 1.8 is optimal, since for $p \leq o(n^{-1/m^r(H)})$ it is known that a.a.s. $G_\alpha \cup G_{n,p}$ does not contain a copy of $H$ [39].
Organization of the paper. In Section 2 we will summarise some useful tools on probability, regularity and games. In Section 3 we will consider the Hamiltonicity game and prove Theorem 1.2 and Theorem 1.6. Section 4 is devoted to the $k$-vertex-connectivity game, where we show Theorem 1.3 and Theorem 1.7. In Section 5 we continue with the $H$-game. We first start with the discussion of cycle games. Then we use Subsection 5.1 and Subsection 5.2 to prepare our strategy for the general $H$-game. Theorem 1.5 and Theorem 1.8 are proven afterwards in Subsection 5.3. Note that Theorem 1.4 is a corollary of Theorem 1.5, and Theorem 1.1 follows analogously to the proof of Theorem 1.5 (see Remark 5.10). We will finish with a few concluding remarks and open problems in Section 6.

Notation. We use standard graph-theoretic notation, which closely follows [47]. In most of the proofs we will first describe Maker’s or Waiter’s strategy and afterwards discuss why it is possible for the respective player to follow this strategy. We implicitly assume that Maker or Waiter forfeits the game when it is not possible to follow her strategy, while it will follow from the discussion that this does not happen. Additionally, when some Maker-Breaker game is in progress, we emphasise the following. We let $\mathcal{M}$ and $\mathcal{B}$ denote the graphs consisting of Maker’s edges and Breaker’s edges, respectively. Any edge belonging to $\mathcal{M} \cup \mathcal{B}$ is said to be claimed, while all the other edges in play are called free. Analogously, for a Waiter-Client game let $\mathcal{W}$ and $\mathcal{C}$ denote Waiter’s and Client’s graphs, respectively.

2. Preliminaries

2.1. Probabilistic tools. We will extensively use a plethora of tools which come from probability theory or which can be proven by some probabilistic argument. The first tool, which we often use to show that some non-negative random variable very unlikely exceeds a given bound, is Markov’s inequality:

**Lemma 2.1** (Markov’s inequality, see e.g. [32]). Let $X \geq 0$ be a random variable. For every $t \geq 0$ it holds that

$$
\mathbb{P}(X \geq t) \leq \frac{\mathbb{E}(X)}{t}.
$$

We also often use Chernoff’s inequalities (see e.g. [32]) to show that a binomial random variable $X \sim \text{Bin}(n, p)$ is concentrated around its expectation $\mathbb{E}(X) = np$, where $n$ is the number of independent rounds and $p$ is the success probability.

**Lemma 2.2.** If $X \sim \text{Bin}(n, p)$, then

- $\mathbb{P}(X < (1 - \delta)np) < \exp\left(-\frac{\delta^2 np}{2}\right)$ for every $\delta > 0$, and
- $\mathbb{P}(X > (1 + \delta)np) < \exp\left(-\frac{np}{3}\right)$ for every $\delta \geq 1$. 
Lemma 2.3. If $X \sim \text{Bin}(n, p)$ and $k \geq 7\mathbb{E}(X)$, then
\[
\mathbb{P}(X \geq k) \leq \exp(-k) .
\]

We use the following lemma (see e.g. [31]) to bound the value of random variables that are distributed according to the hypergeometric distribution $\text{Hypergeometric}(N, K, n)$, where from $N$ objects, of which $K$ are considered a success, $n$ are drawn without replacement.

Lemma 2.4. If $X \sim \text{Hypergeometric}(N, K, n)$ and $t > 0$, then
\[
\mathbb{P}\left(X < \left(\frac{K}{N} - t\right)n\right) < \exp\left(-2t^2n\right).
\]

Next we focus on random graphs. The following lemma provides some useful properties which are very likely to hold in a random graph $G_{n,p}$. The proof is a standard application of the above concentration bounds.

Lemma 2.5. Let $\beta > 0$ and let $p = p(n) \in (0, 1)$ such that $pn \to \infty$. If we generate a random graph $G \sim G_{n,p}$ on the vertex set $[n]$, then a.a.s. the following properties hold:

1. $e_G(A, B) \geq 0.5p|A||B|$ for every pair of disjoint sets $A, B \subset [n]$ of size $|A|, |B| \geq \beta n$,
2. $d_G(v) \leq \max\{2np, 30\ln n\}$ for every $v \in V(G),$
3. $e(G) \leq n^2p$.

Proof. We prove (1) first. For any specific pair of sets $A, B$ of size $|A| = |B| = \beta n$ it holds that $\mathbb{E}(e_G(A, B)) = p|A||B|$, so applying Lemma 2.2 we obtain
\[
\mathbb{P}(e_G(A, B) < 0.5p(\beta n)^2) < \exp\left(-\frac{1}{8}p(\beta n)^2\right).
\]

Taking union bound over all possible choices of $A$ the claim follows.

Now, to prove (2), observe that $\mathbb{E}(d_G(v)) = (n - 1)p$ for any $v \in V$. If $(n - 1)p \geq 4\ln n$, then Lemma 2.2 yields
\[
\mathbb{P}(d_G(v) > 2np) < \exp\left(-\frac{(n - 1)p}{3}\right) < \exp\left(-\frac{4}{3}\ln n\right).
\]

If otherwise $(n - 1)p \leq 4\ln n$, then Lemma 2.3 gives
\[
\mathbb{P}(d_G(v) > 30\ln n) < \exp(-30\ln n) .
\]

In either case, taking union bound over all possible choices of $v$ the claim follows.

To prove (3) notice that $\mathbb{E}(e(G)) = \binom{n}{2}p < \frac{n^2p}{2}$. Applying Lemma 2.2 once again shows, that
\[
\mathbb{P}(e(G) > n^2p) < \exp\left(-\frac{1}{3}n^2p\right) = o(1).
\]

This proves the lemma. \qed
Lastly, by using a probabilistic argument, we prove two lemmas that help us find two useful partitions of a graph with large minimum degree or large connectivity, respectively.

**Lemma 2.6.** Let $\alpha > 0$, $k \geq 1$ be an integer, and $G$ be a graph on $n$ vertices with $\delta(G) \geq \alpha n$. If $n$ is large enough, there exists a partition $V(G) = U_1 \cup U_2 \cup \ldots \cup U_k$ with

1. $|U_i| = \lfloor \frac{n}{k} \rfloor$ or $|U_i| = \lceil \frac{n}{k} \rceil$, for all $i \in [k]$,
2. $e_G(v, U_i) \geq \frac{2}{3}|U_i|$ for all $v \in V(G)$ and $i \in [k]$.

**Proof.** We show that a random partition of the vertices a.a.s. fulfills the properties, thus proving the statement for large $n$. We choose a partition of $V(G)$ into $k$ sets $U_1, U_2, ..., U_k$ such that

$$\left\lfloor \frac{n}{k} \right\rfloor \leq |U_1| \leq |U_2| \leq \ldots \leq |U_k| \leq \left\lceil \frac{n}{k} \right\rceil$$

uniformly at random among all such partitions. For any $v \in V(G)$ and $i \in [k]$, we thus have

$$\mathbb{P}(e_G(v, U_i) = \ell) = \frac{\binom{d_G(v)}{\ell} \binom{n-d_G(v)}{|U_i|-\ell}}{\binom{n}{|U_i|}}.$$ 

Therefore, $e_G(v, U_i) \sim \text{Hypergeometric}(n, d_G(v), |U_i|)$, and by Lemma 2.4 we have

$$\mathbb{P}\left(e_G(v, U_i) < \frac{\alpha}{2} |U_i|\right) < \mathbb{P}\left(e_G(v, U_i) < \left(\frac{d_G(v)}{n} - \frac{\alpha}{2}\right) |U_i|\right) < \exp\left(-\frac{1}{2} \alpha^2 |U_i|\right).$$

Taking union bound over all possible choices of $v \in V(G)$ and $i \in [k]$, the claim follows. \hfill \Box

**Lemma 2.7.** For every $\beta \in (0, 1)$ there exists a constant $\gamma > 0$ such that the following holds for every large enough $n$: let $G$ be any $\beta n$-vertex-connected graph on at most $n$ vertices. Then there exists a partition $G = G^1 \cup G^2$ such that both parts are $\gamma n$-vertex-connected on $V(G)$.

**Proof.** Set $\gamma = \beta \left(\frac{1}{2}\right)^{4/3+2}$. Let $u, v \in V(G)$ be different vertices. By Menger's Theorem (see e.g. Theorem 4.2.17 in [47]) there exist $\beta n$ internally vertex-disjoint paths between $u$ and $v$. At least $\frac{3n}{2}$ of these paths have length at most $\frac{3}{\beta}$. From now on, fix a family of such $\frac{3n}{2}$ paths and denote it with $P_{[u,v]}$.

In the following we will consider a random partition $G = G^1 \cup G^2$, where each edge of $G$ is either added to $G^1$ or $G^2$ uniformly at random with probability $\frac{1}{2}$ and independently of all other choices. It is enough to show that a.a.s. such a partition has the desired property.

Let $i \in [2]$. A path from $P_{[u,v]}$ is contained in $G^i$ with probability at least $\left(\frac{1}{2}\right)^{4/3}$. Hence, the random variable $X^i_{[u,v]}$, which counts the number of such paths landing in $G^i$, stochastically dominates Bin $\left(|P_{[u,v]}|, \left(\frac{1}{2}\right)^{4/3}\right)$. By an application of Chernoff (Lemma 2.2) and the union bound, it follows a.a.s. that for every $i \in [2]$ and every distinct $u, v \in V(G)$,

$$X^i_{[u,v]} \geq \frac{1}{2} |P_{[u,v]}| \cdot \left(\frac{1}{2}\right)^{4/3} = \gamma n.$$
But this means that a.a.s. for both \( i \in [2] \), \( G^i \) has at least \( \gamma n \) internally vertex-disjoint paths between any pair of vertices, and hence is \( \gamma n \)-vertex-connected. \( \square \)

2.2. **Regularity tools.** We will use standard tools from regularity theory in combination with other techniques as used by Das and Treglown [15] for their perturbed Ramsey results. First, we introduce the relevant terminology. Let a graph \( G \) be given. The *density* of a pair \( (A, B) \) of disjoint subsets of \( V(G) \) is defined as \( d_G(A, B) = \frac{e_G(A, B)}{|A||B|} \). For \( \varepsilon > 0 \) we say that \( (A, B)_G \) is \( \varepsilon \)-regular, if for all subsets \( X \subseteq A \) and \( Y \subseteq B \) with \( |X| \geq \varepsilon |A| \) and \( |Y| \geq \varepsilon |B| \) we have \( |d_G(A, B) - d_G(X, Y)| \leq \varepsilon \). This notion allows us to control the distribution of the edges between \( A \) and \( B \). When passing to smaller sets this property is preserved with adjusted parameters.

**Lemma 2.8** (Regularity slicing, Fact 1.5 in [35]). Let \( (A, B)_G \) be an \( \varepsilon \)-regular pair of density at least \( \delta \) and let \( \eta > \varepsilon \). For any \( X \subseteq A \) and \( Y \subseteq B \) of size at least \( \eta |A| \) and \( |Y| \) the pair \( (X, Y)_G \) is \( \varepsilon' \)-regular with density at least \( \delta' \), where \( \varepsilon' = \max \left\{ \frac{\varepsilon}{\eta}, 2\varepsilon \right\} \) and \( |\delta' - \delta| < \varepsilon \).

We will further use the following consequence of the classical Regularity Lemma [35] together with Turán’s theorem, which allows us to find a family of sets such that all pairs are regular.

**Lemma 2.9** (Corollary 2.4 in [15]). For \( 0 < 3\varepsilon \leq \delta < 1 \) there exists an \( \eta > 0 \) and \( n_0 \) such that for all \( n \geq n_0 \) and \( r \geq 2 \) the following holds. For any \( n \) vertex graph \( G \) of density at least \( \frac{\varepsilon - \frac{3\varepsilon}{r}}{r - 1} + \delta \) there are pairwise disjoint sets \( V_1, \ldots, V_r \subseteq V(G) \) such that \( |V_i| \geq \eta n \) for \( 1 \leq i \leq r \), and \( (V_i, V_j)_G \) is \( \varepsilon \)-regular with density at least \( \frac{\delta}{2} \) for \( 1 \leq i < j \leq r \).

Once the regular pairs are in place we want to use them to combine structures that we can find within each of the sets. For this we ideally want large sets such that all small subsets have many common neighbours. The next lemma follows from the dependent random choice technique [23].

**Lemma 2.10** (Lemma 2.5 in [15]). Given \( r \geq 2, \delta, \beta > 0, \ell \in \mathbb{N} \) there exists a \( \nu > 0 \) such that for \( 0 < \varepsilon < \min\left\{ \frac{1}{2r}, \frac{\delta}{2} \right\} \) there exists an \( m_0 \) such that the following holds for any \( m \geq m_0 \). Let \( V_1, \ldots, V_r \) be disjoint sets of vertices of size \( m \) from a graph \( G \) such that \( (V_1, V_i)_G \) is \( \varepsilon \)-regular with density at least \( \delta \) for \( 2 \leq i \leq r \). Then there exists a subset \( U \subseteq V_1 \) of size at least \( m^{1-\beta} \) such that any \( \ell \) vertices from \( U \) have at least \( \nu m \) common neighbours in each of the sets \( V_i \) for \( 2 \leq i \leq r \).

2.3. **Positional games tools.** Helpful tools for the study of positional games are potential functions. One of the central results in this area that was proven by the use of such a tool, is the following theorem, usually referred to as Beck’s Criterion. Note that in a \( (p : q) \) Maker-Breaker game, in each round Maker and Breaker claim \( p \) and \( q \) elements of the board, respectively.
Theorem 2.11 (Beck’s Criterion, Theorem 1 in [4]). Let \((X, \mathcal{F})\) be a hypergraph satisfying
\[
\sum_{F \in \mathcal{F}} (1 + q)^{-|F|/p+1} < 1
\]
then Breaker has a strategy to win the \((p : q)\) Maker-Breaker game on \((X, \mathcal{F})\), independent of who starts the game.

Although the result guarantees a winning strategy for Breaker it can also be beneficial for the description of strategies for Maker. Here the idea is to redefine a Maker-Breaker game in such a way that the roles of both players are switched. Such an approach is fairly standard and has been applied multiple times in the literature (see e.g. [5, 26]). In order to avoid repetitions of such an argument, we provide a reformulation of Beck’s Criterion which can be applied directly when we study Maker strategies for \((1 : b)\) games throughout the paper. Given a hypergraph \((X, \mathcal{F})\) we define the transversal hypergraph \((X, \mathcal{F}^*)\) as follows:
\[
\mathcal{F}^* := \{S \subset X : (\forall F \in \mathcal{F} : S \cap F \neq \emptyset)\}.
\]

Corollary 2.12. Let \((X, \mathcal{F})\) be a hypergraph satisfying
\[
\sum_{F \in \mathcal{F}} 2^{-|F|/b+1} < 1
\]
then Maker has a strategy to win the \((1 : b)\) Maker-Breaker game on \((X, \mathcal{F}^*)\), independent of who starts the game.

Proof. Maker wins the game on \((X, \mathcal{F}^*)\) if and only if she claims all elements of a transversal \(F \in \mathcal{F}^*\). That is, she needs to prevent Breaker from claiming a set \(F \in \mathcal{F}\) completely. Hence Maker considers playing as \(\mathcal{F}\)-Breaker, playing with bias 1. By Beck’s Criterion she has a strategy for this when \(\sum_{F \in \mathcal{F}} 2^{-|F|/b+1} < 1\). \(\square\)

A result similar to Beck’s Criterion, but for Client-Waiter games, has been proven by Bednarska-Bzdęga [6], and a similar reformulation yields the following criterion for Waiter-Client games.

Theorem 2.13 (Theorem 2.2 in [7]). Let \((X, \mathcal{F})\) be a hypergraph satisfying
\[
\sum_{F \in \mathcal{F}} 2^{-|F|/(2b-1)+1} < 1
\]
then Waiter has a strategy to win the \((1 : b)\) Waiter-Client game on \((X, \mathcal{F}^*)\), independent of who starts the game.

Next to the above winning criteria we will also make use of the trick of fake moves which roughly says that Maker’s situation does not worsen when Breaker’s bias decreases. While the above criteria do not immediately provide fast strategies, the following can be used to obtain suitable upper bounds on the number of rounds needed to claim a winning set.
Lemma 2.14 (Trick of fake moves [5]). Let \((X, \mathcal{F})\) be a hypergraph. Let \(b\) be a positive integer. If Maker has a winning strategy for the \((1 : b)\) Maker-Breaker game on \((X, \mathcal{F})\), then she also has a strategy to win the game within \(\left\lceil \frac{|X|}{b+1} \right\rceil\) rounds even when in each round Breaker is allowed to claim between 0 and \(b\) free elements.

The idea of the above lemma is straightforward: in every move of the new game on \((X, \mathcal{F})\), let Maker (in her mind) give as many additional elements of \(X\) to Breaker as necessary, such that their number together with Breaker’s elements sums up to \(b\). Whenever, in a later round, Breaker claims one of these additional elements, Maker (in her mind) gives another free element to Breaker. Then, applying the winning strategy from the \((1 : b)\) game, the result follows.

Along the lines of the above argument, the following statement for Waiter-Client games can be proven.

Lemma 2.15 (Trick of fake moves [5]). Let \((X, \mathcal{F})\) be a hypergraph. Let \(b\) be a positive integer. If Waiter has a winning strategy for the \((1 : b)\) Waiter-Client game on \((X, \mathcal{F})\), then she also has a strategy to win the game within \(\left\lceil \frac{|X|}{b+1} \right\rceil\) rounds even when in each round she is allowed to offer between 2 and \(b + 1\) free elements.

Moreover, we will make use of an argument which ensures that Maker can claim a spanning graph of a suitable minimum degree. In order to do so, we may use an auxiliary game, called MinBox\((n, D, \gamma, b)\), which was introduced in [22]. This \((1 : b)\) Maker-Breaker game is played on a family of \(n\) disjoint sets (called boxes), each of size at least \(D\), with the board \(X\) being the union of all boxes. Maker is called the winner if she manages to occupy at least \(\gamma|B|\) elements from each box \(B\). The following theorem gives a criterion for Maker to win the game.

Theorem 2.16 (Theorem 2.5 in [22]). Let \(n, D, b\) be positive integers and let \(0 < \gamma < 1\). If \(\gamma < \frac{1}{b+1}\) and \(D > \frac{b(ln n + 1)}{1 - \gamma(b+1)}\), then Maker wins the game MinBox\((n, D, \gamma, b)\).

3. Hamiltonicity game

We will prove Theorem 1.2 and Theorem 1.6 in this section. Depending on the size of the bias \(b\), we will use different arguments. For Maker-Breaker games, the argument in Subsection 3.1 goes through for any \(b = o(\sqrt{n})\) while the argument in Subsection 3.2 works for any \(b = \Omega(ln n)\). Both cannot be significantly improved, because Theorem 3.1 limits the first approach and to improve the second, we would need better estimates for the number of expander graphs with a linear number of edges. Because of the overlap, we will use the restriction \(b \leq n^{0.49}\) in Subsection 3.1. Theorem 1.6 is proven in Subsection 3.3.
3.1. Maker-Breaker Hamiltonicity game with small edge probabilities and biases. In
the following we will prove Theorem 1.2 for $b \leq n^{0.49}$. In order to do so, we will make use of
the following sufficient condition for a graph $G$ to have a Hamilton cycle.

**Theorem 3.1** (Theorem 2.5 in [29]). Let $12 \leq d \leq \sqrt{n}$ and let $G$ be a graph on $n$ vertices such
that the following properties hold:

(i) $|N_G(S)| \geq d|S|$ for every $S \subset V(G)$ of size $|S| \leq \frac{n \ln d}{d \ln n}$,

(ii) $e_G(A, B) > 0$ for every pair of disjoint sets $A, B \subset V(G)$ of size $|A|, |B| \geq \frac{n \ln d}{1035 \ln n}$.

Then, provided that $n$ is large enough, $G$ contains a Hamilton cycle.

**Proof of Theorem 1.2** for $b \leq n^{0.49}$. Let $\alpha$ be given. Choose $\delta = 10^{-4} \alpha$, $C = 10^8 \delta^{-2}$ and $c = 10^{-3} \delta^2 \alpha$. Let $1 \leq b \leq n^{0.49}$ and, by monotonicity, let us assume that $p = \frac{C \delta}{n}$. Let $G_\alpha$ be any $n$-vertex graph with $n$ large enough and minimum degree $\delta(G_\alpha) \geq \alpha n$. Next, we reveal $G_2 \sim G_{n,p}$ on $V(G_\alpha)$. For the remainder of the proof, let us condition on $G_2$ having the properties from
Lemma 2.5 with $\beta = \frac{\delta}{1035}$, and let $G_1 = G_\alpha \backslash G_2$. Then $\delta(G_1) \geq \frac{\alpha n}{2}$.

Given the properties from Lemma 2.5, we will show that Maker has a strategy to occupy a
Hamilton cycle in the $(1:b)$ Maker-Breaker game on $G_1 \cup G_2$. In order to do so, Maker ensures
that her final graph $\mathcal{M}$ will satisfy the following two properties:

(1) $|N_{\mathcal{M}}(S)| \geq n^\delta |S|$ for every $S \subset V(G)$ of size $|S| \leq \delta n^{1-\delta}$,

(2) $e_{\mathcal{M}}(A, B) > 0$ for every pair of disjoint sets $A, B \subset V(G)$ of size $|A|, |B| \geq \beta n$.

Then, according to Theorem 3.1 (applied with $d = n^\delta$), it follows that $\mathcal{M}$ contains a Hamilton
cycle, provided $n$ is large enough.

Before we describe Maker’s strategy, let us fix a partition $V(G_1) = V_1 \cup V_2 \cup V_3 \cup V_4$ such that
for each $i \in [4]$ we have $\frac{n}{4} - 1 \leq |V_i| \leq \frac{n}{4} + 1$ and for every $v \in V_i$ and every $j \neq i$ we have

$$d_{G_i}(v, V_j) > \frac{\alpha}{5} |V_j| > \frac{\alpha}{25} n.$$"
reach the different goals on $G_1$ and $G_2$. We consider each of the boards $E(G_1)$ and $E(G_2)$ separately.

\[1 : 2b\) game on $E(G_1)$: We split $G_1$ into the edge-disjoint subgraphs
\[
G_{1,1} = G_1[V_1, V_2] \cup G_1[V_2, V_3] \cup G_1[V_3, V_4] \cup G_1[V_4, V_1]\quad \text{and}\quad G_{1,2} = G_1[V_1, V_3] \cup G_1[V_2, V_4].
\]

Maker plays on each of the boards $E(G_{1,1})$ and $E(G_{1,2})$ alternately. By the trick of fake moves (Lemma 2.14), we may thus assume that a $(1 : 4b)$ is played on each of the boards.

On $G_{1,1}$ Maker ensures that every vertex in her graph will have degree at least $\frac{an}{2006}$. She can do this as follows: she considers playing the game MinBox($n, D, \gamma, 4b$) with the $n$ disjoint boxes $E_{G_1}(v, V_{i+1})$, for $i \in [4]$ and $v \in V_i$ (where we set $V_5 = V_1$), where each box has size at least $D = \frac{a}{25}n$, and where we set $\gamma = \frac{1}{80}$. By the choice of all parameters, one verifies that
\[
\gamma < \frac{1}{4b + 1} \quad \text{and} \quad \frac{4b(\ln n + 1)}{1 - \gamma(4b + 1)} < n^{0.5} < D
\]
provided $n$ is large enough. Hence, applying Theorem 2.16, Maker wins this auxiliary game. That is, Maker claims at least $\gamma D = \frac{an}{2006}$ elements of every box.

On $G_{1,2}$, Maker ensures to claim an edge in each edge set contained in
\[
\mathcal{F}_1 = \left\{ E_{G_1}(A, B) : \exists i, j \in [4] \text{ with } |i - j| = 2, A \subseteq V_i, B \subseteq V_j, |A| = n^{0.5} \text{ and } |B| = (1 - \frac{a}{10}) |V_j| \right\}. \tag{3.1}
\]

For this, she considers the auxiliary $(1 : 4b)$ Maker-Breaker game with winning sets $\mathcal{F}_1^*$. In order to show that Maker wins this game, it will be enough to show that
\[
\sum_{F \in \mathcal{F}_1} 2^{-|F|/(4b) + 1} = o(1),
\]
according to Corollary 2.12. The latter holds by the following reason: Since $d_{G_1}(v, V_j) > \frac{a}{5}|V_j|$ for every $v \in V_i$ and every $j \neq i$, we see that for every $F = E_{G_1}(A, B) \in \mathcal{F}_1$ it holds that $|F| \geq |A| \cdot \frac{a}{10}|V_j| > n^{1.495}$. Hence, since $b \leq n^{0.49}$,
\[
\sum_{F \in \mathcal{F}_1} 2^{-|F|/(4b) + 1} < (2^n)^2 \cdot 2^{-n^{1.005}/4 + 1} = o(1). \tag{3.2}
\]

It remains to show that, because of the achievements from the games on $G_{1,1}$ and $G_{1,2}$, Maker is able to create a graph with Property (1). For this, let $S$ be any subset of $V(G_1)$ of size $|S| \leq \delta n^{1-\delta}$. Our goal is to show that $|N_M(S)| \geq n^\delta |S|$ holds by the end of the game. If $|S| < n^{0.505}$, then we immediately have
\[
|N_M(S)| \geq \delta |S| > \frac{an}{2006} - |S| > n^\delta |S|
\]
for large $n$. Otherwise, it holds that $n^{0.505} \leq |S| \leq \delta n^{1-\delta}$. In this case we can find a subset $S' \subseteq S \cap V_i$ of size $|S'| = n^{0.5}$ for some $i \in [4]$. If $|N_M(S')| \geq 2\delta n$ holds, then we immediately get

$$|N_M(S)| \geq |N_M(S')| - |S| \geq 2\delta n - \delta n^{1-\delta} > \delta n \geq n^\delta |S|$$

and we are done. So, assume for a contradiction, that $|N_M(S')| < 2\delta n$. Let $j \in [4]$ with $|i - j| = 2$. Then we are able to find a set $B \subseteq V_j$ of size $|B| = |V_j| - 2\delta n$ with $e_M(S', B) = 0$. However, we have $|B| > (1 - \frac{n}{10}) |V_j|$ by the choice of $\delta$, and hence, by the result from the game on $G_{1,2}$, Maker needs to have an edge in $E_{G_1}(S', B)$. This gives the desired contradiction.

$(1 : 2b)$ game on $E(G_2)$: Maker’s goal is to occupy a spanning subgraph of $G_2$ which fulfils Property (2). That is, she aims to claim an element from each of the sets contained in

$$\mathcal{F}_2 := \{E_{G_2}(A, B) : A, B \subseteq V(G_2) \text{ disjoint and } |A| = |B| = \beta n\}. \quad (3.3)$$

By the properties from Lemma 2.5, we have $e_{G_2}(A, B) \geq \frac{1}{2} p |A| |B|$ for every pair of disjoint sets $A, B \subseteq V(G_2)$ of size at least $\beta n$, and hence $|F| \geq \frac{1}{2} p (\beta n)^2$ for any $F \in \mathcal{F}_2$. This yields

$$\sum_{F \in \mathcal{F}_2} 2^{-|F|/(2b)+1} \leq 2^{|\mathcal{F}_2|} \cdot 2^{-p \beta^2 n^2/(4b)+1} = o(1), \quad (3.4)$$

where we use that $\frac{p \beta^2 n^2}{4b} \geq C \beta^2 n > 3n$ by the choice of $p, C$ and $\beta$. Hence, following Corollary 2.12, Maker reaches her goal and is able to occupy a subgraph satisfying Property (2).

3.2. Maker-Breaker Hamiltonicity game with large edge probabilities and biases. In the following we will prove Theorem 1.2 for $b \geq \ln n$. To create a Hamilton cycle in this regime we will combine the approach by Krivelevich [37] for the biased Hamilton cycle game on $K_n$ with ideas of Ferber, Glebov, Krivelevich, and Naor [21] for biased games on random boards.

The following lemma follows from a close inspection of the proof from Gebauer and Szabó [24, Theorem 1.2] on the minimum degree game. A similar statement was observed by Krivelevich [37, Lemma 3] for the game on the complete graph.

At any moment during the game and for any vertex $v$, set $\text{dang}(v) := d_B(v) - 2bd_M(v)$. Consider the following strategy $\mathcal{S}$ for Maker: As long as there exists a vertex $v$ of degree less than 16, she chooses any such vertex which maximises $\text{dang}(v)$, and then she claims an arbitrary free edge incident to $v$. The following statement holds.

**Lemma 3.2.** For every $\alpha > 0$ there exists a constant $c > 0$ and an integer $n_0$ such that for every $n$-vertex graph $G$ with $n \geq n_0$ and minimum degree $\delta(G) \geq \alpha n$ the following holds for $b \leq \frac{\alpha n}{\ln n}$.

If in a $(1 : b)$ Maker-Breaker game on $E(G)$ Maker plays according to strategy $\mathcal{S}$, then Maker’s degree of every vertex $v$ will be 16 at some point during the game, and right at the moment when this degree is reached it holds that $d_B(v) \leq \frac{\alpha n}{2}$. 

We briefly sketch how the argument from [26, Theorem 5.3.6] for $K_n$ can be adapted to $G$.

**Sketch of the proof of Lemma 3.2.** Let $c = \frac{\alpha}{2}$. Without loss of generality, we assume that Breaker starts the game. Let Maker play according to strategy $\mathcal{S}$. For any integer $i \geq 1$, denote with $\mathcal{M}_i$ and $\mathcal{B}_i$ the $i$th move of Maker and Breaker, respectively, and denote with $v_i$ the vertex chosen by Maker in her $i$th move according to strategy $\mathcal{S}$. For both $X \in \{\mathcal{M}_i, \mathcal{B}_i\}$ and any vertex $v \in V(G)$ let $\text{dang}_X(v)$ denote the danger value of $v$ immediately before $X$ happens. Moreover, for any subset $I \subset V(G)$ define
\[
\overline{\text{dang}}_X(I) := \frac{\sum_{v \in I} \text{dang}_X(v)}{|I|}
\]
to be the average danger in $I$ immediately before $X$ happens. Assume Maker fails, i.e. there happens to be a point of the game (say immediately after Breaker’s $g$th move for some $g \in \mathbb{N}$) where there exists a vertex $v_g$ such that $d_{\mathcal{M}}(v_g) < 16$ but $d_{\mathcal{B}}(v_g) > \frac{\alpha n}{2}$. Then, in particular,
\[
\text{dang}_{\mathcal{B}_g}(v_g) > \frac{\alpha n}{2} - b - 2b \cdot 15 = \left(\frac{\alpha}{2} - o(1)\right) n.
\]
Fix $g$ and $v_g$, and set $I_i = \{v_{g-i}, \ldots, v_g\}$ for any $0 \leq i \leq g-1$. Then Corollary 3.3 from [24] still applies as its proof is independent of the graph which the game is played on. Accordingly, we can continue with the same calculations that are given afterwards and based on [24, Corollary 3.3]. In particular, we obtain one of the following two estimates, with $k = \left\lceil \frac{\alpha n}{\ln n} \right\rceil$, $r = |I_g|$ and $H_s = \sum_{j=1}^s \frac{1}{j}$:
\[
\overline{\text{dang}}_{\mathcal{B}_g}(I_{g-1}) \geq \overline{\text{dang}}_{\mathcal{B}_g}(I_0) - bH_k - k \quad \text{or} \quad \overline{\text{dang}}_{\mathcal{B}_g}(I_{g-1}) \geq \overline{\text{dang}}_{\mathcal{B}_g}(I_0) - b(2H_r - H_k) - k.
\]
In any case, since $\overline{\text{dang}}_{\mathcal{B}_g}(I_0) = \text{dang}_{\mathcal{B}_g}(v_g)$, we can conclude that
\[
\overline{\text{dang}}_{\mathcal{B}_g}(I_{g-1}) \geq \left(\frac{\alpha}{2} - 2c - o(1)\right) n > 0
\]
for large enough $n$, where we use that $bH_k \leq \frac{\alpha n}{\ln n}(\ln n + 1) = (c + o(1))n$. However, this is a contradiction, since $\overline{\text{dang}}_{\mathcal{B}_g}(I_{g-1}) = 0$ by its definition. \qed

Krivelevich [37, Lemma 4] observed that we can use the freedom of the choice in the strategy $\mathcal{S}$ to build an expander graph.

**Definition 3.3.** Let $R \in \mathbb{N}$. A graph $G$ is an $R$-expander if $|N_G(A)| \geq 2|A|$ for every $A \subset V(G)$ with $|A| \leq R$.

**Lemma 3.4.** For every $\alpha > 0$ there exist constants $c > 0$, $\varepsilon > 0$ and an integer $n_0$ such that for any $n$-vertex graph $G$ with $n \geq n_0$ and minimum degree $\delta(G) \geq \alpha n$ on the following holds for $b \leq \frac{\alpha n}{\ln n}$. In the $(1:b)$ game on $G$ Maker is able to build an $\varepsilon n$-expander in at most $16n$ rounds.

We repeat the proof from [37] adapted to our setting.
Proof of Lemma 3.4. Given $\alpha > 0$ we let $0 < \varepsilon < \alpha^{8-81}$ and let $c > 0$ and $n_0$ be given by Lemma 3.2. Let $G$ be a graph on $n$ vertices with $n \geq n_0$ and minimum degree $\delta(G) \geq \alpha n$. Further let $b \leq \frac{cn}{\ln n}$.

Maker’s strategy is to play according to strategy $S$ where she chooses the ‘arbitrary edge’ described in that strategy uniformly at random among all suitable edges. After at most $16n$ moves the game stops, since we stop when Maker’s graph $\mathcal{M}$ has minimum degree 16 and since in each of her moves Maker always increases the degree of a vertex of degree less than 16. Suppose that by now $\mathcal{M}$ is not an $\varepsilon n$-expander. This means, that there must be a set $A \subseteq V(G)$ of size $i \leq \varepsilon n$ such that all neighbours of the set $A$ with respect to Maker’s graph are contained in a set $B \subseteq V(G)$ of size $2i - 1$. Since $\delta(\mathcal{M}) \geq 16$, we can assume $i \geq 5$ and that there are at least $8i$ edges between $A$ and $A \cup B$ in Maker’s graph.

In order to finish the proof, it will be enough to show that with probability $o(1)$ there will be sets $A$ and $B$ as described above with at least $8i$ edges between $A$ and $A \cup B$.

Fix any sets $A$ and $B$ with $|A| = i \in [5, \varepsilon n]$ and $|B| = 2i - 1$. First note that at most $16|A \cup B| < 48i$ edges were chosen by dangerous vertices in $A \cup B$. For any edge $e = uv$ that was obtained after Maker chose $v \in A \cup B$ as a dangerous vertex, consider Maker’s possible amount of choices for the vertex $u$. Since $v$ was dangerous, it held that $d_{\mathcal{M}}(v) < 16$, and because of Lemma 3.2 $d_B(v) < \frac{\alpha n}{2}$, which means that there were $d_{G \setminus (\mathcal{M} \cup B)}(v) \geq \frac{\alpha n}{2} - 16$ options for the vertex $u$. Thus, the probability that $u$ also ended up in $A \cup B$ is at most $\frac{3i - 1}{2i - 16} < \frac{8i}{\alpha n}$, and therefore the probability that from the at most $48i$ chosen edges at least $8i$ edges end up between $A$ and $A \cup B$ is bounded from above by $(\frac{48i}{8i}) (\frac{8i}{\alpha n})^{8i}$.

In particular, using a union bound over all choices of $i$, $A$ and $B$, Maker’s randomised strategy to create an $\varepsilon n$-expander fails with probability at most

$$
\sum_{i=5}^{\varepsilon n} \binom{n}{i} \binom{n-i}{2i-1} \left(\frac{48i}{8i}\right) \left(\frac{8i}{\alpha n}\right)^{8i} \leq \sum_{i=5}^{\varepsilon n} \left(\frac{en}{i} \right)^2 2^{48} \left(\frac{8i}{\alpha n}\right)^{8i} \leq \sum_{i=5}^{\varepsilon n} 2^{80} \left(\frac{i}{n}\right)^4 \frac{1}{\alpha^8} \leq \sum_{i=5}^{\varepsilon n} \left(\frac{i}{n}\right)^4 = o(1),
$$

where the third inequality holds by the choice of $\varepsilon$, and where the last estimate holds, since for $i < \sqrt{n}$ we have $\left(\frac{i}{n}\right)^4 \leq \frac{1}{n}$ and since for $i \geq \sqrt{n}$ we have $\left(\frac{i}{n}\right)^i \leq \frac{1}{n^2}$. Now, since Maker creates an $\varepsilon n$-expander with positive probability, there must also exist a deterministic winning strategy for Maker (see e.g. [8, 26]).

Maker will use $G_{n,p}$ to turn this expander into a connected graph and then find many boosters to finish a Hamilton cycle. A booster for a graph $G$ is any non-edge $e \notin E(G)$ such that $G + e$ is either Hamiltonian or the length of the longest path in $G + e$ is larger than in $G$. 


Lemma 3.5 (see e.g. Lemma 8.5 in [12]). If $G$ is a connected non-Hamiltonian $R$-expander, then the set of boosters for $G$ has size at least $R^2/2$.

Now we have everything at hand to proceed with the proof of Theorem 1.2.

Proof of Theorem 1.2 for $\ln n \leq b \leq \frac{cn}{\ln n}$. Given $\alpha > 0$, let $c' > 0$ and $\varepsilon > 0$ be given by Lemma 3.4 on input $\frac{\alpha}{2}$. Then let $c = \frac{c'}{2}$, $C = 10^4\varepsilon^{-2}$, $\ln n \leq b \leq \frac{cm}{\ln n}$, and $p \geq \frac{Cb}{n}$. Let $G_\alpha$ be any $n$-vertex graph with $n$ large enough and minimum degree $\delta(G_\alpha) \geq \alpha n$, and reveal $G_2 \sim G_{n,p}$ on $V(G_\alpha)$. From now on we condition on the properties from Lemma 2.5 (with $\beta = \varepsilon$). In particular, for $G_1 = G_\alpha \setminus G_2$ we then have $\delta(G_1) \geq \frac{\alpha n}{2}$. Moreover, we condition on the following property, for which we will show, analogously to [21, Lemma 2.11], that it holds a.a.s.:

(B) For every non-Hamiltonian connected $\frac{n}{5}$-expander with at least $8n$ and at most $m_0 = \frac{16n}{\varepsilon^2}$ edges there exist at least $\frac{n^3}{100}$ boosters in $G_2$.

In order to see that this property holds a.a.s. fix any non-Hamiltonian connected $\frac{n}{5}$-expander with the mentioned number of edges. By Lemma 3.5, such an expander must have at least $\frac{n^2}{50}$ boosters. The probability that less than $\frac{n^2p}{100}$ of these boosters are edges of $G_2$ is at most $\exp\left(-\frac{n^2p}{400}\right)$, by an application of Chernoff’s inequality (Lemma 2.2). Taking a union bound over all relevant expanders we see that property (B) fails with probability at most

$$
\sum_{m=8n}^{m_0} \binom{n}{m} \exp\left(-\frac{n^2p}{400}\right) \leq \sum_{m=8n}^{m_0} \exp\left(m \ln\left(\frac{en^2}{2m}\right) - n \frac{Cb}{400}\right)
\leq \sum_{m=8n}^{m_0} \exp\left(\frac{16}{\varepsilon^2} n \ln n - \frac{C}{400} n \ln n\right) = o(1)
$$

where in the last step we use the choice of $C$. (Note that estimating the number of non-Hamiltonian connected $\frac{n}{5}$-expanders with $m$ edges as $\binom{n}{m}$ leads to the lower bound of $b = \Omega(\ln n)$.)

We will show that Maker can claim a Hamilton cycle on $G_1 \cup G_2$. Let us state Maker’s strategy.

**Strategy description:** Maker’s strategy consists of two stages. We briefly describe each of these stages here. All further details will be given later in the strategy discussion.

In **Stage I**, which lasts at most $\frac{15n}{\varepsilon}$ rounds, Maker creates a connected $\varepsilon n$-expander. In order to do so, she plays alternately on $G_1$ and $G_2$, always assuming Breaker’s bias to be $2b$ by the trick of fake moves (Lemma 2.14). She only plays on each of these boards until she reaches the following goals. On $G_1$ Maker builds an $\varepsilon n$-expander; on $G_2$ Makers occupies a graph in which between any two sets of size $\varepsilon n$ there is at least one edge.

Afterwards, in **Stage II**, Maker uses the remaining free edges of $G_2$ to claim boosters as long as her graph is not Hamiltonian. This stage lasts less than $n$ rounds.
Strategy discussion: We consider Stage I first. By Lemma 3.4 and as $2b \leq \frac{\varepsilon n}{\ln n}$, Maker can build an $\varepsilon n$-expander in at most $16n$ rounds, playing only on $G_1$. For her goal on $G_2$ it suffices to ensure that Maker claims an edge in each set from $F \in \mathcal{F}$.

Therefore, we consider the transversal game on $(E(G_2), \mathcal{F}^*)$ and, in order to bound the number of rounds for winning this auxiliary game, we let Maker play against a bias of $b' = \frac{\varepsilon^2 n p}{8}$. We observe that, due to the properties from Lemma 2.5, we have $e_{G_2}(A, B) \geq \frac{\varepsilon^2 n^2 p}{2}$ for all disjoint sets $A, B \subseteq V$ with $|A| = |B| = \varepsilon n$. In particular, we obtain

$$\sum_{F \in \mathcal{F}} 2^{-|F|/b'} + 1 \leq 2^{2n} 2^{-\varepsilon^2 n^2 p/(2b')} + 1 = o(1).$$

From this it follows by the transversal version of Beck's Criterion (Corollary 2.12) that Maker wins the $(E(G_2), \mathcal{F}^*)$-game with bias $b'$. Furthermore, with the trick of fake moves (Lemma 2.14) and since $e(G_2) \leq n^2 p$ due to conditioning on Lemma 2.5, it follows that Maker can even win this game against a bias of $2b \leq \frac{2np}{\varepsilon^2} < b'$ in at most $\frac{8n}{\varepsilon^2}$ rounds. Thus, the first stage lasts at most $\frac{8n}{\varepsilon^2} + 16n < \frac{15n}{\varepsilon^2}$ rounds in total.

When Maker enters Stage II, her graph is a connected $\frac{n}{5}$-expander. Indeed, if $A \subseteq V$ is any set with $\varepsilon n \leq |A| \leq \frac{n}{5}$, then, by having a transversal of $\mathcal{F}_2$, there exist less than $\varepsilon n$ vertices in $V \setminus A$ which are not in the neighbourhood of $A$, and hence, $|N_M(A)| \geq n - |A| - \varepsilon n > 2|A|$. Moreover, at this point and until the end of the second stage (which lasts at most $n$ rounds) at most $\frac{16n}{\varepsilon^2}$ rounds were played. As long as Maker’s graph is not Hamiltonian, property (B) provides at least $\frac{n^2 p}{100}$ boosters in $G_2$ and, thus, there are at least

$$\frac{n^2 p}{100} - m_0(b + 1) \geq \left(\frac{C}{200} - \frac{16}{\varepsilon^2}\right) n(b + 1) > 0$$

boosters in $G_2$ that are not covered by the graphs of Maker and Breaker. Maker picks one booster and repeats this argument until she obtains a Hamilton cycle. □

3.3. Waiter-Client Hamiltonicity game.

Proof. Similarly to the the discussion of the Maker-Breaker Hamiltonicity game, we consider the two cases $b \leq n^{0.49}$ and $b \geq \ln n$ separately.

Case A ($b \leq n^{0.49}$): Using the same setup as in the proof of Theorem 1.2 for $b \leq n^{0.49}$ (Subsection 3.1), consider the edge sets of $G_1$ and $G_2$ as two edge disjoint boards. Waiter first plays on $G_1$ in such a way that Client claims a subgraph with property (1). Afterwards, Waiter plays on $G_2$ in such a way that Client claims a subgraph with property (2). As in Subsection 3.1,
it will be enough to show that Waiter can follow the strategy and reach her respective goals on $G_1$ and $G_2$. Again, we consider the two boards $E(G_1)$ and $E(G_2)$ separately.

$(1 : b)$ game on $E(G_1)$: Split $G_1$ into the edge-disjoint subgraphs

$$G_{1,1} = G_1[V_1, V_2] \cup G_1[V_2, V_3] \cup G_1[V_3, V_4] \cup G_1[V_4, V_1] \quad \text{and} \quad G_{1,2} = G_1[V_1, V_3] \cup G_1[V_2, V_4]$$

as before. First Waiter plays on $G_{1,1}$. She ensures that every vertex will have degree at least $\frac{an}{200b}$ in Client’s graph. For this, she plays as follows: as long as there is some $i \in [4]$ and some vertex $v \in V_i$ with $d_C(v) < \frac{an}{200b}$, she offers $(b + 1)$ edges between $v$ and $V_{i+1}$ (where we set $V_5 = V_1$). Afterwards, Waiter plays on $G_{1,2}$. Here she ensures that Client claims an edge in each edge set contained in the family $\mathcal{F}_1$ from (3.1). By the same calculation as in (3.2) we obtain

$$\sum_{F \in \mathcal{F}_1} 2^{-|F|/(2b-1)+1} < \sum_{F \in \mathcal{F}_1} 2^{-|F|/(4b)+1} = o(1)$$

and hence, following Theorem 2.13, Waiter can reach her goal. As before, because of the achievements from the games on $G_{1,1}$ and $G_{1,2}$, Waiter ensures that Client occupies a graph with property (1).

$(1 : b)$ game on $E(G_2)$: Waiter’s goal is to make Client occupy a spanning subgraph of $G_2$ which fulfils property (2). For this, we consider the same family $\mathcal{F}_2$ as in (3.3). Using the calculation from (3.4) we have

$$\sum_{F \in \mathcal{F}_2} 2^{-|F|/(2b-1)+1} = o(1),$$

and hence, following Theorem 2.13, Waiter can ensure that Client claims an edge in each set $F \in \mathcal{F}_2$, which gives property (2).

**Case B** ($b \geq \ln n$): In this case we make use of some ideas from [7]. Given $\alpha > 0$, let $\varepsilon = 10^{-\frac{4}{\alpha}}$, $c = \varepsilon^4$, $C = 10^3\varepsilon^{-4}$. Further let $G_\alpha$ be any graph on $n$ vertices with $n$ large enough and minimum degree $\delta(G_\alpha) \geq \alpha n$, let $\ln n \leq b \leq cn$, and then with $p \geq \frac{Cn}{n}$ reveal $G_2 \sim G_{n,p}$ on $V(G_\alpha)$. From now on we condition on the properties from Lemma 2.5 (with $\beta = \varepsilon$). We set $G_1 = G_\alpha \setminus G_2$ and, as before, we have $\delta(G_1) \geq \frac{an}{2}$. We will show that Waiter has a strategy to force Client to create a Hamilton cycle on $G_1 \cup G_2$. Before we describe Waiter’s strategy, let us fix a partition $V = V_1^{(1)} \cup V_1^{(2)} \cup V_2^{(1)} \cup V_2^{(2)} \cup V_3^{(1)} \cup V_3^{(2)}$ with $\frac{n}{6} - 1 \leq |V_i^{(j)}| \leq \frac{n}{6} + 1$, for every $i \in [3]$ and $j \in [2]$, such that in $G_1$ every vertex has degree at least $\frac{an}{25}$ into each part $V_i^{(j)}$. The existence of such a partition is given by Lemma 2.6.

**Strategy description:** Waiter’s strategy consists of three stages. We briefly describe each of the stages here. All further details will be given later in the strategy discussion.

In **Stage I**, Waiter plays on $G_1$ for at most $\frac{n}{\varepsilon^4}$ rounds. Here, she ensures that Client occupies a graph with the following property:
Afterwards, in Stage II, Waiter plays on $G_2$ for at most $\frac{10n}{\varepsilon}^2$ further rounds. Now she ensures that Client occupies a graph which has an edge between any two disjoint sets of size $\varepsilon n$. We will see later that by the end of the second stage Client’s graph is a connected $\frac{q}{q^2}$-expander.

Finally, in Stage III, by offering only boosters, Waiter turns this expander into a Hamiltonian graph within less than $n$ rounds.

**Strategy discussion:** If Waiter can follow the proposed strategy, it is clear that she forces Client to occupy a Hamilton cycle. Hence, it remains to show that she can indeed do so.

Let us start with the discussion of Stage I. Here we assume that Waiter plays with bias $b' = \varepsilon^2 n > b$, so that we can use the trick of fake moves (Lemma 2.15) later on, in order to obtain a good upper bound on the number of rounds for this stage.

By disjointness Waiter can play on each of the boards $E_{G_i}(V_i^{(1)} \cup V_i^{(2)}, V_{i+1}^{(1)} \cup V_{i+1}^{(2)})$ with $i \in [3]$ separately, and by symmetry it suffices to give a strategy for obtaining (P) with $i = 1$, by playing on the board $E_{G_1}(V_1^{(1)} \cup V_1^{(2)}, V_2^{(1)} \cup V_2^{(2)})$. Waiter’s strategy is as follows.

First, playing only on $G_1[V_1^{(1)} \cup V_1^{(2)}, V_2^{(1)}]$, Waiter ensures that Client occupies a transversal of

$$\mathcal{F}_1 = \left\{ E_{G_1}(A, B): A \subseteq V_1^{(1)} \cup V_1^{(2)}, B \subseteq V_2^{(1)}, |A| = \varepsilon n, \text{ and } |B| = \frac{n}{6} - \frac{an}{100} \right\}.$$

Notice that for any $A$ and $B$ as described in $\mathcal{F}_1$ we have

$$d_{G_1}(v, B) \geq d_{G_1}(v, V_2^{(1)}) - |V_2^{(1)} \setminus B| \geq \frac{an}{25} - \frac{an}{100} - 1 > \frac{an}{50}$$

for all $v \in A$, and hence

$$e_{G_1}(A, B) > \varepsilon an^2 \cdot 50.$$

In particular,

$$\sum_{F \in \mathcal{F}_1} 2^{-|F|/(2n - 1) + 1} \leq 2^{2n^2 - \varepsilon an^2/(100 \varepsilon) + 1} = o(1),$$

by the choice of $b'$ and $\varepsilon$. Thus, Waiter can force a transversal of $\mathcal{F}_1$ according to Theorem 2.13. Note that this already gives $|N_C(A, V_2^{(1)})| \geq \frac{an}{100} - 1 > 2\varepsilon n = 2|A|$ for every $A \subseteq V_1^{(1)} \cup V_1^{(2)}$ of size $\varepsilon n$.

Afterwards, as the second step, consider the inclusion minimal sets $A \subseteq V_1^{(1)} \cup V_1^{(2)}$, such that $|A| \leq \varepsilon n$ and $\left| N_C(A, V_2^{(1)}) \right| < 9|A|$. Denote the family of all these sets $A$ by $\mathcal{A}$, and let $x_1, \ldots, x_r$ be all the vertices contained in these sets. Then $r \leq 2\varepsilon n$ by the following reason: If otherwise $r > 2\varepsilon n$, then we could find sets $A_1, \ldots, A_k \in \mathcal{A}$ for some $k \in \mathbb{N}$ such that $\varepsilon n \leq \sum_{i \in [k]} |A_i| \leq 2\varepsilon n$, and hence a set $A' \subseteq \bigcup_{i \in [k]} A_i$ of size $\varepsilon n$. Then, by having a transversal of $\mathcal{F}_1$ in Client’s graph, $|N_C(A', V_2^{(1)})| \geq \frac{an}{100} - 1 > 20\varepsilon n$; while on the other hand, by the definition of $\mathcal{A}$, $|N_C(A', V_2^{(1)})| \leq \sum_{i \in [k]} |N_C(A_i, V_2^{(1)})| \leq \sum_{i \in [k]} 9|A_i| = 18\varepsilon n$, a contradiction.
For every \( i = 1, \ldots, r \) Waiter now plays \( 9 \) additional rounds where she offers exactly \( 9(b' + 1) \) edges between \( x_i \) and the set

\[
Y_i = \{ y \in V_2^{(2)} : d_C(y, V_1^{(1)} \cup V_1^{(2)}) = 0, \ x_i y \in E(G_1) \}
\]

(which is updated after every move), therefore forcing \( d_C(x_i, V_2^{(2)}) \geq 9 \). This is possible, because at any moment it holds that \( |Y_i| \geq d_{G_1}(x_i, V_2^{(2)}) - 9r \geq \frac{\varepsilon n}{2b} - 9r \geq 9(b' + 1) \).

Afterwards, the vertices \( x_i \) have pairwise disjoint neighbourhoods of size \( 9 \) in \( V_2^{(2)} \), and it follows immediately that after this first stage, property (P) holds for Client’s graph (for \( i = 1 \)). Indeed, let \( A \subseteq V_1^{(1)} \cup V_1^{(2)} \) be any set with \( 1 \leq |A| \leq \varepsilon n \). We can partition \( A \) into inclusion minimal sets \( X_1, \ldots, X_s \) such that \( |N_C(X_i, V_2^{(1)})| < 9|X_i| \) and at most one set \( B \) with \( |N_C(B, V_2^{(1)})| \geq 9|B| \).

Since the vertices \( x \in \bigcup_{i=1}^{s} X_i \) satisfy \( d_C(x_i, V_2^{(2)}) \geq 9 \) and the neighbourhoods are disjoint, we get

\[
N_C(A, V_2^{(1)} \cup V_2^{(2)}) \geq |N_C(B, V_2^{(1)})| + \sum_{i=1}^{s} |N_C(X_i, V_2^{(2)})| \geq 9|B| + \sum_{i=1}^{s} 9|X_i| = 9|A|.
\]

Hence, Waiter can follow the proposed strategy for Stage I and force a graph with property (P). Because she succeeds even when the bias equals \( b' \), it follows by the trick of fake moves (Lemma 2.15), that Stage I with bias \( b \) can be done in less than \( \frac{e(G_1)}{b'} < \frac{n}{2^2} \) rounds.

Moreover, we observe that property (P) implies that Client’s graph is an \( \varepsilon n \)-expander. Indeed, consider any \( S \subseteq V \) of size at most \( \varepsilon n \). Then there is some \( i \in [3] \) such that \( A := S \cap (V_1^{(1)} \cup V_1^{(2)}) \) has size at least \( \frac{|S|}{3} \). By property (P) we then conclude that

\[
|N_C(S)| \geq |N_C(A)| - |S| \geq 9|A| - |S| \geq 3|S| - |S| = 2|S|.
\]

Let us consider **Stage II** now. Here, in order to bound the number of rounds, we will assume that Waiter plays with bias \( b'' = \frac{\varepsilon^2 \varepsilon n}{4} > b \). Consider the family

\[
\mathcal{F}_2 = \{ E_{G_2}(A, B) : A, B \subseteq V \text{ disjoint and } |A| = |B| = \varepsilon n \}.
\]

Waiter again wants Client to occupy a transversal of \( \mathcal{F}_2 \). By the properties from Lemma 2.5 we have \( |F| \geq 0.5p(\varepsilon n)^2 \) for every \( F \in \mathcal{F}_2 \), and therefore

\[
\sum_{F \in \mathcal{F}_2} 2^{-|F|/(2b''-1) + 1} \leq 2^{2n} 2^{-\varepsilon^2 \varepsilon n^2 p/(4b'')} + 1 = o(1)
\]

by the choice of \( b'' \). Thus, Waiter can follow this part of the proposed strategy. Note that this stage lasts at most \( \frac{10n}{\varepsilon^2} \) rounds. Indeed, since Waiter succeeds even when the bias equals \( b'' \), it follows by the trick of fake moves (Lemma 2.15), that Stage II with bias \( b \) can be done in less than \( \frac{e(G_2)}{b'} \leq \frac{10n}{\varepsilon^2} \) rounds, where we use that \( e(G_2) \leq n^2 p \) by the properties from Lemma 2.5.

To finish the discussion of Stage II notice that Client’s graph is a connected \( \frac{n}{b} \)-expander analogously to the discussion of the Maker-Breaker Hamiltonicity game for \( \ln n \leq b \leq \frac{cn}{\ln n} \).
Finally, for **Stage III**, similarly as in the proof of Theorem 1.2 (for \( \ln n \leq b \leq \frac{cn}{\ln n} \)), we conclude that as long as Client’s graph \( C \) is not Hamiltonian, there are at least \( b + 1 \) boosters available. Waiter offers these boosters to Client and, after at most \( n \) additional rounds, there exists a Hamilton cycle in Client’s graph. \( \square \)

4. \( k \)-Connectivity Game

4.1. **Maker-Breaker** \( k \)-vertex-connectivity game. In this subsection we will prove Theorem 1.3. In order to prove the theorem, we will make use of the following two results, the first providing a partition of \( G_\alpha \) into highly connected components, the second providing a necessary condition for a graph to be \( r \)-edge-connected.

**Lemma 4.1** (Lemma 1 in \([10]\)). Let \( G \) be any graph on \( n \) vertices and \( \delta(G) \geq \alpha n \). Then there exists an integer \( s \) and a partition \( V(G) = V_1 \cup V_2 \cup \ldots \cup V_s \) such that the following holds:

(a) \( |V_i| \geq \frac{\alpha}{8} n \) for every \( i \in [s] \), and

(b) \( G[V_i] \) is \( \frac{\alpha^2}{16} n \)-vertex-connected for every \( i \in [s] \).

**Theorem 4.2** (Theorem 6.2 in \([34]\)). There exists a constant \( d > 0 \) such that the following holds: Let \( G \) be an \( r \)-edge-connected graph on \( n \) vertices. Then for every \( t \geq 1 \) there are at most \( dn^{2t} \) cuts of size less than \( rt \) in \( G \).

**Proof of Theorem 1.3.** Let \( \alpha > 0 \) and \( k \geq 1 \) be given. Choose \( C = 10^5 k^3 \alpha^{-3} \ln(8k\alpha^{-1}) \) and \( c = \alpha^4 10^{-5} k^{-2} \), and let \( 1 \leq b \leq \frac{cn}{\ln n} \) and \( p \geq \frac{c^2 k^2}{n^2} \). Before we describe a strategy for Maker, let us observe that \( G \sim G_\alpha \cup G_{n,p} \) satisfies the following properties with probability at least \( 1 - \exp(-cnp^2) \):

(i) there exists an integer \( s \) and a partition \( V(G) = V_1 \cup V_2 \cup \ldots \cup V_s \) as stated in Lemma 4.1,

(ii) for each \( j \in [s] \) there exists a partition \( V_j = V_{j,1} \cup V_{j,2} \cup \ldots \cup V_{j,k} \), such that \( e_G(V_{i,\ell}, V_{s,\ell}) \geq \frac{\alpha^2 n^2 p}{200k^2} \) for every \( i \in [s-1] \) and \( \ell \in [k] \).

Indeed, Lemma 4.1 applied to \( G_\alpha \) gives a partition \( V(G_\alpha) = V_1 \cup V_2 \cup \ldots \cup V_s \) as desired. For any \( j \in [s] \), fix any partition \( V_j = V_{j,1} \cup V_{j,2} \cup \ldots \cup V_{j,k} \) such that \( |V_{j,\ell}| \geq \frac{|V_{j}|}{k} \) \( \geq \frac{\alpha n}{10k} \). Only afterwards reveal the edges of \( G' \sim G_{n,p} \) on \( V(G_\alpha) \), and set \( G = G_\alpha \cup G' \). Adding the edges of \( G' \) to \( G_\alpha \) does not destroy the properties (a) and (b) from Lemma 4.1, and hence \( V(G) = V_1 \cup V_2 \cup \ldots \cup V_s \) stays a partition as required for (i). Moreover, notice that (a) implies \( s \leq \frac{8}{\alpha} \). In order to get (ii), observe that \( e_G(V_{i,\ell}, V_{s,\ell}) \sim \text{Bin}(|V_{i,\ell}||V_{s,\ell}|, p) \) with \( \mathbb{E}[e_G(V_{i,\ell}, V_{s,\ell})] = |V_{i,\ell}| |V_{s,\ell}| p \geq \frac{\alpha^2 n^2 p}{100k^2} \).
Therefore, using Chernoff’s inequality (Lemma 2.2), we obtain
\[
\Pr \left( \exists i \in [s-1], \ell \in [k] : e_G(V_i, \ell, V_s, \ell) < \frac{\alpha^2}{200k^2}n^2p \right) \leq sk \cdot e^{-\alpha^2n^2p/(800k^2)} \leq e^{\ln(8k/\alpha)-(10^{-4}k^{-2}\alpha^{-2}+c)n^2p} \leq e^{-cn^2p},
\]
where the second inequality follows from the choice of \( c \) and since \( s \leq \frac{8}{\alpha} \), and the last inequality follows since \( p \geq \frac{C\beta}{n^2} \geq \frac{C}{n^2} \) and by the choice of \( C \).

From now on we will condition on the properties (i) and (ii). Next, we will first describe a strategy for Maker in a \((1 : b)\) game on \( G \) and then we will show that she can follow that strategy and that it leads to a \( k \)-vertex-connected spanning subgraph of \( G \).

**Strategy description:** In order to describe the strategy, consider the \( 2s-1 \) edge-disjoint boards \( E_G(V_i) \) for all \( i \in [s] \), and \( E_G(V_i, V_s) \) for all \( i \in [s-1] \). Enumerate all these boards in an arbitrary way with the integers \( 0, 1, \ldots, 2s-2 \). In round \( i \), let Maker play on board \( i \mod 2s-1 \). Then, between any two moves on the same board, Breaker claims at most \((2s-1)b =: b_s \) edges. Hence, using the trick of fake moves (see Lemma 2.14), we can assume that on each board we separately play a \((1 : b_s)\) Maker-Breaker game. Now, on each of the boards \( E_G(V_i) \) Maker plays in such a way that she obtains a \( k \)-vertex-connected spanning subgraph of \( G[V_i] \), and on each of the boards \( E_G(V_i, V_s) \) she makes sure to claim a matching of size at least \( k \). All details will be given later in the strategy discussion.

**Strategy discussion:** Notice that, if Maker can follow the strategy, then her final graph \( M \) will be \( k \)-vertex-connected. Indeed, let \( K \subset V(M) \) be any subset of size at most \( k-1 \). Then \( M[V_i \setminus K] \) is connected, since \( M[V_i] \) is \( k \)-vertex-connected, for every \( i \in [s] \), and due to the matchings of size \( k \), there is at least one edge between \( V_s \) and \( V_i \) in the graph \( M - K \), for every \( i \in [s-1] \). So, \( M - K \) is connected.

Now, it remains to show that Maker can indeed follow her strategy. We consider each of the above mentioned boards separately.

\((1 : b_s)\) game on \( E_G(V_i)\): Maker’s goal is to occupy a spanning \( k \)-vertex-connected subgraph of \( G[V_i] \). This part of her strategy is motivated by [2, Theorem 1.6]. Let us define
\[
\mathcal{F} := \{ E_G(S, V_i \setminus (S \cup K)) : K \subset V_i, 0 \leq |K| \leq k-1, S \subset V_i \setminus K \},
\]
If Maker manages to occupy a transversal of \( \mathcal{F} \), then the following holds for her final subgraph \( \mathcal{M}_i \) of \( G[V_i] \): for every subset \( K \subset V_i \) of size at most \( k-1 \), the graph \( \mathcal{M}_i - K \) is connected since there is at least one edge in every cut \((S, V_i \setminus (S \cup K)) \) of \( \mathcal{M}_i - K \). That is, \( \mathcal{M}_i \) is \( k \)-vertex-connected.
Now, according to Corollary 2.12, it suffices to show that

\[ \sum_{F \in \mathcal{F}} 2^{-|F|/b_s + 1} < 1. \]

For this notice that for any \( K \subset V_i \) of size at most \( k - 1 \) the following holds: \( G[V_i] \) is \( \frac{\alpha^2}{16} n \)-vertex-connected and hence the vertex-connectivity of \( G[V_i \setminus K] \) is at least \( \frac{\alpha^2}{16} n - (k - 1) > \frac{\alpha^2}{20} n \).

In particular, \( e_G(S, V_i \setminus (S \cup K)) \geq \frac{\alpha^2}{20} n \) for every \( S \subset V_i \setminus K \). Now, applying Theorem 4.2 (with \( r = \frac{\alpha^2}{20} n \) and \( t = \left\lfloor \frac{j+1}{r} \right\rfloor \)), we obtain

\[
\sum_{F \in \mathcal{F}} 2^{-|F|/b_s + 1} \leq \sum_{K \subseteq V_i, |K| \leq k-1} \sum_{j = \frac{\alpha^2}{20} n}^{n^2} |\{ S \subseteq V_i \setminus K : e_G(S, V_i \setminus (S \cup K)) = j\}| \cdot 2^{-j/b_s + 1}
\]

\[
\leq \sum_{K \subseteq V_i, |K| \leq k-1} \sum_{j = \frac{\alpha^2}{20} n}^{n^2} d(V_i \setminus K)^{2(j+1)/r} \cdot 2^{-j/(2b_s)}
\]

\[
\leq n^k \sum_{j = \frac{\alpha^2}{20} n}^{n^2} n^{4j/r} \cdot 2^{-j/2b_s}
\]

\[
\leq n^k \sum_{j = \frac{\alpha^2}{20} n}^{n^2} \exp \left( \frac{4}{r} \cdot \ln n - \frac{\ln 2}{2b_s} \cdot j \right)
\]

\[
\leq n^k \sum_{j = \frac{\alpha^2}{20} n}^{n^2} \exp \left( -\frac{100k}{\alpha^2} \cdot \frac{\ln n}{n} \cdot j \right)
\]

\[
\leq n^k \sum_{j = \frac{\alpha^2}{20} n}^{n^2} \exp (-5k \ln n) = o(1)
\]

where the fifth inequality holds by the choice of \( c \) and since \( b_s < 2sb \leq \frac{16c}{\alpha} \cdot \frac{n}{\ln n} \).

(1 : \( b_s \)) game on \( E_G(V_i, V_s) \): In order for Maker to claim a matching of size \( k \) between \( V_i \) and \( V_s \), it is enough to claim one edge between \( V_{i,\ell} \) and \( V_{s,\ell} \) for every \( \ell \in [k] \). As this takes \( k \) rounds in total, at most \( k(b_s + 1) \) edges can be claimed in the meantime. Thus, Maker succeeds easily if \( e_G(V_{i,\ell}, V_{s,\ell}) > k(b_s + 1) \) for every \( \ell \in [k] \). The latter is the case since by (ii) we obtain

\[
e_G(V_{i,\ell}, V_{s,\ell}) \geq \frac{\alpha^2}{200k^2} n^2 p \geq \frac{\alpha^2 C b}{200k^2} \geq 4skb > k(b_s + 1),
\]

where the third and the last inequality follow by the definition of \( C \) and \( b_s \), respectively. \( \square \)
4.2. Waiter-Client $k$-vertex-connectivity game. In this subsection we will prove Theorem 1.7.

Proof of Theorem 1.7. Let $\alpha > 0$ and $k \geq 1$ be given. We set $\beta = \frac{\alpha^2}{80k}$ and let $\gamma$ be returned by Lemma 2.7. Now we choose $\varepsilon = 10^{-\omega k}$, $c = 10^{-3} \alpha^4 k^{-2} \varepsilon^3$ and $C = 10^6 k^3 \alpha^{-3} \ln(8k\alpha^{-1})$.

Before we describe Waiter’s strategy we split the board into suitable subboards. As a first step, fix a partition $V(G) = U_1 \cup U_2 \cup \ldots \cup U_k$ such that

1. $\frac{n}{k} - 1 \leq |U_i| \leq \frac{n}{k} + 1$ for all $i \in [k]$,
2. $e_{G_\alpha}(v, U_i) \geq \frac{\alpha}{2} |U_i|$ for all $v \in V(G)$ and $i \in [k]$.

Such a partition exists by Lemma 2.6. Next, we additionally split each of the sets $U_i$, $i \in [k]$, to obtain a partition $U_i = U_{i,1} \cup U_{i,2} \cup \ldots \cup U_{i,s_i}$ such that

(a) $|U_{i,j}| \geq \frac{\alpha}{80k} n$, and
(b) $G_\alpha[U_{i,j}]$ is $\beta n$-vertex-connected

for every $j \in [s_i]$. Such a partition can be found by Lemma 4.1. For every $i \in [k]$ and $j \in [s_i]$, set $G_{i,j} := G_\alpha[U_{i,j}]$. Applying Lemma 2.7 we can find a partition $G_{i,j} = G_{i,j}^1 \cup G_{i,j}^2$ such that both parts are $\gamma n$-vertex-connected graphs on $U_{i,j}$. Only afterwards, we reveal the edges of $G' \sim G_{n,p}$ and observe that with probability at least $1 - \exp(-cn^2 \varepsilon)$ the following holds:

(c) $e_{G'}(U_{i,j_1}, U_{i,j_2}) \geq \frac{\alpha^2}{80k} n^2 p$ for every $i$ and $j_1 \neq j_2$.

The proof of (c) is analogous to the discussion of (ii) in the proof of Theorem 1.3. From now on, we will condition on (c) being satisfied, and show that Waiter has a strategy to force a $k$-vertex-connected spanning subgraph of $G = G_\alpha \cup G'$.

Strategy description: Waiter’s strategy consists of four stages. We briefly describe each of these stages here by mentioning the board and the goal of the stage. All further details will be given later in the strategy discussion.

In **Stage I**, Waiter plays on the board $G_I := \bigcup_{i,j} E(G_{i,j}^1)$. Here she ensures that Client creates an $\varepsilon n$-expander on $V(G)$. If Waiter succeeds, then immediately afterwards each component in Client’s graph has size at least $\varepsilon n$ and hence there are at most $\varepsilon^{-1}$ such components. Furthermore, each of these components must be a subset of some set $U_{i,j}$ with $i \in [k]$ and $j \in [s_i]$. In **Stage II**, Waiter plays on the board $G_{II} := \bigcup_{i,j} E(G_{i,j}^2)$ where she makes sure that Client’s graph becomes connected on each of the sets $U_{i,j}$. Next, in **Stage III**, Waiter plays on the board $G_{III} := \bigcup_{i,j_1,j_2} E_{G'}(U_{i,j_1}, U_{i,j_2})$. She forces Client to make $U_i$ a connected component in her graph for every $i \in [s]$. In **Stage IV**, Waiter considers the board $G_{IV} := \bigcup_{i_1,i_2} E_{G_\alpha}(U_{i_1}, U_{i_2})$. She ensure, that by the end of this stage Client’s graph $\mathcal{C}$ satisfies the following: $e_{\mathcal{C}}(v, U_{i_2}) > 0$ for every $i_1 \neq i_2$ and $v \in U_{i_1}$.
Strategy discussion: If Waiter can follow the proposed strategy, Client’s graph will be $k$-vertex-connected by the end of the game. This can be seen as follows. Let $K \subset V(G)$ be any set of size at most $k - 1$. Then there exists some $i \in [k]$ such that $U_i \cap K = \emptyset$. By Stage III, $U_i$ is a connected component in Client’s graph; lastly by Stage IV, every other vertex in $V \setminus (K \cup U_i)$ has a neighbour in $U_i$; i.e. $C - K$ is connected.

Hence, it remains to be shown that Waiter can follow the proposed strategy. We will discuss each stage separately. However, before doing so, observe that the four different boards $G_I, \ldots, G_{IV}$ are pairwise disjoint, i.e. Waiter can play on these boards one after the other.

Stage I. Since $G^1_{i,j}$ is $\gamma n$-vertex-connected for every $i, j$, we have $\delta(G_I) \geq \gamma n$. Waiter follows the strategy from Stage I in Case B of the proof of Theorem 1.6 and thus forces an $\varepsilon n$-expander. Note that the mentioned strategy of Case B only used the fact that the game was played on a graph with minimum degree at least $\frac{\alpha n}{2}$ (which is replaced here with $\gamma n$); the strategy worked for a bias $b' = \varepsilon^2 n$ (now $\varepsilon$ is chosen depending on $\gamma$ instead of $\alpha$) and the same strategy can be used for any smaller bias by the trick of fake moves (Lemma 2.15).

Stage II. When Waiter enters Stage II, Client’s graph consists of at most $\varepsilon^{-1}$ components, each of which has size at least $\varepsilon n$ and is contained in one of the sets $U_{i,j}$. Let $i \in [k]$ and $j \in [s_i]$ be fixed. Waiter plays on $E(G^2_{i,j})$ as follows. As long as Client’s graph on $U_{i,j}$ is not connected, Waiter looks for two components $A, B \subset U_{i,j}$ such that there are at least $b + 1$ free edges in $G^2_{i,j}[A, B]$. Waiter then offers these edges to Client and thus reduces the number of components by 1; she repeats this step until $U_{i,j}$ is a connected component in Client’s graph.

In order to show that Waiter can indeed follow this strategy, it remains to be shown that we can always find such sets $A, B$ as described above. We do this as follows. Since Client’s graph is assumed to be disconnected on $U_{i,j}$ there must be two vertices $v, w \in U_{i,j}$ which belong to different components. Now $G^2_{i,j}$ is $\gamma n$-vertex-connected, and hence in this graph there must be $\gamma n$ internally vertex-disjoint paths between $u$ and $v$. Each of these paths must have at least one edge connecting two of Client’s components. Since there are at most $\varepsilon^{-1}$ components, there must be some pair $A, B$ of components, between which $G^2_{i,j}$ has at least $\varepsilon^2 \gamma n \geq b + 1$ edges. Waiter can offer $b + 1$ of these edges and thus follow her strategy. Note that this way she does not offer edges between any other pair of components, which makes it possible to repeat this argument until $U_{i,j}$ is a connected component.

Stage III. When Waiter enters Stage III, Client’s components are the sets $U_{i,j}$. Now, for each $i \in [k]$ and distinct $j_1, j_2 \in [s_i]$, Waiter plays exactly one round on $E_{G'}(U_{i,j_1}, U_{i,j_2})$ offering $b + 1$
arbitrary edges. This is possible by (c) and since
\[ \frac{\alpha^2 n^2 p}{800k^2} \geq \frac{\alpha^2 Cb}{800k^2} > b + 1 \]
by the choice of C. One can easily check that this makes Client’s graph connected on each set \( U_i \) with \( i \in [k] \).

**Stage IV.** Let distinct \( i_1, i_2 \in [k] \) be fixed. Waiter plays on the graph \( G_\alpha[U_{i_1}, U_{i_2}] \) to make sure that every vertex \( v \in U_{i_1} \) gets a neighbour in \( U_{i_2} \) in Client’s graph, and vice versa. For this, note that by (1) and (2) \( G_\alpha[U_{i_1}, U_{i_2}] \) has minimum degree at least \( \frac{\alpha n}{3k} \). We can split this graph into two subgraphs \( H_1 \) and \( H_2 \) on the same vertex set \( U_{i_1} \cup U_{i_2} \), each having minimum degree at least \( \frac{\alpha n}{10k} \). Indeed, a random partition of the edges of \( G_\alpha[U_{i_1}, U_{i_2}] \) can be used to prove the existence of \( H_1 \) and \( H_2 \). Using only the edges of \( H_1 \), Waiter ensures that every vertex in \( U_{i_1} \) gets a neighbour in \( U_{i_2} \). This is possible as \( d_{H_1}(v, U_{i_2}) \geq \frac{\alpha n}{10k} > b + 1 \) for every \( v \in U_{i_1} \). Then, Waiter repeats the same with \( H_2 \) and every vertex \( v \in U_{i_2} \). This finishes the proof. \( \square \)

## 5. Unbiased \( H \)-games

For the study of the \( H \)-game we will use regularity tools. For a first example we will prove the following proposition.

**Proposition 5.1.** Let \( H \) be an odd cycle of length at least 5, \( \alpha > 0 \), \( G_\alpha \) be a graph on \( n \) vertices with \( \delta(G_\alpha) \geq \alpha n \), and \( p = \omega(n^{-2}) \). Then a.a.s. the following holds: playing an unbiased Maker-Breaker game on \( G_\alpha \cup G_{n,p} \), Maker has a strategy to claim a copy of \( H \).

**Proof.** Let \( H = C_{2\ell+1} \) with \( \ell \geq 2 \). An application of Lemma 2.9 with \( \varepsilon = \frac{\alpha^2}{10p} \) (and \( \delta = \alpha \)) leads to a constant \( \eta > 0 \). Now, by Lemma 2.9 we can find an \( \varepsilon \)-regular pair \( (V_1, V_2) \) in \( G_\alpha \), with density at least \( \frac{\eta}{2} \) and such that \( |V_1| \geq |V_2| \geq \eta n \). Let \( G = G_\alpha[V_1, V_2] \) be the bipartite subgraph of \( G_\alpha \) with vertex classes \( V_1 \) and \( V_2 \). Let \( V_i' := \{ v \in V_i : d_G(v, V_{3-i}) \geq \frac{\eta}{3} |V_{3-i}| \} \). Then, by \( \varepsilon \)-regularity of \((V_1, V_2)_G\), we get \( |V_i'| \geq (1 - \varepsilon)|V_i| \) for \( i \in [2] \). Then for any \( v \in V_i' \) it holds that
\[
\frac{1}{3}d_G(v, V_{3-i}) - \varepsilon |V_{3-i}| \geq \frac{\alpha}{4} |V_{3-i}| .
\]
Next, we reveal the edges of \( G' \sim G_{n,p} \) on \( V(G_\alpha) \). Then a.a.s. there exists an edge, say \( xy \), in \( V_i' \). Indeed, by Chernoff (Lemma 2.2) and by the size of \( V_i' \), we obtain that
\[
\mathbb{P}(e_{G'}(V_i') = 0) \leq \exp \left( -\frac{1}{3} \mathbb{E}(e_{G'}(V_i')) \right) \leq e^{-\omega(1)} .
\]
From now on, we will condition on the existence of such an edge \( xy \). We will describe a strategy for Maker, when playing on \( G_\alpha \cup G' \), and show that she can follow that strategy until a copy of \( H \) is created.
Strategy description: Maker’s strategy consists of two stages. 

Stage I lasts exactly $\frac{\alpha}{16}|V_2| + 1$ rounds. In the first round, Maker claims the edge $xy$. In the proceeding $\frac{\alpha}{16}|V_2|$ rounds, Maker claims arbitrary edges from $E_G(y, V_2')$. At the end of this stage, let $v_1 = x$, $N_y = N_M(y, V_2')$ and set $V_1^* = \{v \in V_1' : d_G(v, N_y) \geq \frac{\alpha}{8}|N_y|\}$. Afterwards, Stage II lasts exactly $2\ell - 1$ rounds. For any $k \leq 2\ell - 1$, Maker does her $k$th move in Stage II as follows:

- If $v_k \in V_1$ then she fixes an arbitrary vertex $z \in \begin{cases} V_2' & \text{if } k = 1, \\ N_y & \text{if } k \neq 1 \end{cases}$ such that $d_G(z) \leq \sqrt{n}$ and $v_kz \in G\setminus B$.

- If $v_k \in V_2$ then she fixes an arbitrary vertex $z \in V_1^*$ such that $d_B(p, z) \leq \sqrt{n}$ and $v_kz \in G\setminus B$.

Maker then claims the edge $v_kz$ and sets $v_{k+1} = z$.

Strategy discussion: If Maker can follow her strategy, then $(v_1, v_2, \ldots, v_{2\ell+1})$, with $v_{2\ell+1} = y$, forms a cycle of length $2\ell + 1$ in her graph. Hence, it remains to be shown that Maker can follow the proposed strategy. 

Consider Stage I first. Claiming $xy$ in round 1 is not a problem since Maker starts the game. Since $y \in V_1'$, we have $d_G(y, V_2') \geq \frac{\alpha}{4}|V_2|$. Hence, Maker can easily claim $\frac{\alpha}{16}|V_2|$ edges from $E_G(y, V_2')$ in the beginning of the game. 

At the end of Stage I, observe that, according to the slicing lemma (Lemma 2.8), the pair $(V_1', N_y)$ is $\varepsilon'$-regular in $G$ with $\varepsilon' = \frac{10\varepsilon}{\alpha}$ and density at least $\frac{\alpha}{4}$. In particular, less than $\varepsilon'|V_1'|$ vertices from $V_1'$ have less than $\frac{\alpha}{8}|N_y|$ neighbours in $N_y$. Hence, 

$$|V_1^*| \geq (1 - \varepsilon')|V_1'| > (1 - 2\varepsilon')|V_1|.$$

Next, let us look at Stage II. Throughout this stage, we have the bound

$$e(B) \leq 1 + \frac{\alpha}{16}|V_2| + (2\ell - 1) < \frac{\alpha}{8}|V_1|$$

for $i \in [2]$. Assume that Maker can follow her strategy until she reaches round $k$ of Stage II. We explain now why she can follow the strategy for the $k$th move in Stage II.

If $k = 1$, then $v_k = x \in V_1'$ and hence

$$d_{G\setminus B}(v_k, V_2') \geq d_G(v_k, V_2') - e(B) \geq \frac{\alpha}{3}|V_1| - \frac{\alpha}{8}|V_1| > \frac{\alpha}{5}|V_1| > e(B).$$

Therefore, there exists a vertex $z$ as required for the strategy, even with $d_B(z) = 0$. 

If $k \neq 1$ is odd, then by the strategy from the previous round, we know that $v_k \in V_1^*$ and $d_B(v_k) \leq \sqrt{n} + 1$. In this case we have

$$d_{G\setminus B}(v_k, N_y) \geq d_G(v_k, N_y) - d_B(v_k) \geq \frac{\alpha}{8} |N_y| - (\sqrt{n} + 1) > \frac{\alpha}{10} |N_y|$$

for large enough $n$. Hence, there are more than $\frac{\alpha}{10} |N_y|$ choices for the desired vertex $z$ if we ignore the constraint $d_B(z) \leq \sqrt{n}$. However, there must be a vertex fulfilling this constraint since otherwise we have

$$e(B) > \frac{\alpha}{10} |N_y| \cdot \sqrt{n} \geq \frac{\alpha^2 \eta}{160} n^{\frac{5}{2}} > \frac{\alpha}{8} |V_2|$$

for large enough $n$, in contradiction to the upper bound on $e(B)$ described earlier.

If otherwise $k$ is even, then by the strategy from the previous round, we know that $v_k \in N_y \subset V'_2$ and $d_B(v_k) \leq \sqrt{n} + 1$. In this case we have

$$d_{G\setminus B}(v_k, V_1^*) \geq d_{G\setminus B}(v_k, V_1) - |V_1 \setminus V_1^*| \geq d_G(v_k, V_1) - d_B(v_k) - |V_1 \setminus V_1^*|$$

$$> \frac{\alpha}{3} |V_1| - (\sqrt{n} + 1) - 2\varepsilon' |V_1| > \frac{\alpha}{5} |V_1|$$

for large enough $n$. Analogously to the previous case, using the upper bound on $e(B)$, we can conclude that there must exist a vertex $z \in N_{G\setminus B}(v_k, V_1^*)$ such that $d_B(z) \leq \sqrt{n}$. Hence, in any case, Maker can follow the proposed strategy.

The previous result covers all odd cycles except for $C_3$. For the sake of completeness, we will discuss $C_3$ with the following proposition. A book with $t$ pages consists of $t$ triangles overlapping in a single edge.

**Proposition 5.2.** Let $H$ be a book, $\alpha > 0$, $G_\alpha$ be a graph on $n$ vertices with $\delta(G_\alpha) \geq \alpha n$, and $p = \omega(n^{-2})$. Then a.a.s. the following holds: playing an unbiased Maker–Breaker game on $G_\alpha \cup G_{n,p}$, Maker has a strategy to claim a copy of $H$.

**Proof.** Let $H$ be a book with $t$ pages, $t \in \mathbb{N}$. For our strategy consider the following graph $F$: take $2$ vertex-disjoint matchings $M_1, M_2$ of size $k = 12t$; set $V(F) = V(M_1) \cup V(M_2)$ and

$$E(F) = E(M_1) \cup E(M_2) \cup \{vw : v \in V(M_1) \text{ and } w \in V(M_2)\}.$$

Then $m^{(2)}(F) = \frac{1}{2}$. Hence, using Theorem 2.1 from [39] we know that for $p = \omega(n^{-2})$ a.a.s. $G \sim G_\alpha \cup G_{n,p}$ contains a copy of $F$. In the following we will show that, playing only on $F$, Maker has a strategy to occupy a copy of $H$. At first Maker claims $\frac{k}{3}$ edges of each of the matchings $M_1$ and $M_2$. As Breaker in the meantime can claim at most $\frac{2k}{3}$ edges, Maker can easily do so. Denote with $M'_1 = \{e_1, \ldots, e_{\frac{k}{2}}\}$ and $M'_2 = \{f_1, \ldots, f_{\frac{k}{2}}\}$ the submatchings of $M_1$ and $M_2$ claimed by Maker. Afterwards, consider the $\frac{k^2}{9}$ edge-disjoint boards $E_{i,j} = \{vw : v \in V(e_i) \text{ and } w \in V(f_j)\}$ with $i, j \in \left[\frac{k}{2}\right]$. Since so far only $\frac{2k}{3}$ rounds have been played, at least $\frac{k^2}{9} - \frac{2k}{3} \geq \frac{k^2}{18}$ of these boards are free of Breaker’s edges. On each of these boards, Maker now ensures to claim two
adjacent edges by a simple pairing strategy. This way, Maker creates at least $\frac{k^2}{18}$ triangles, each of which contains one of the edges from $M'_1 \cup M'_2$. By a simple averaging argument we conclude that at least one of these edges needs to be contained in at least $\frac{k}{12} = t$ Maker’s triangles, hence leading to a copy of $H$. □

Both propositions are optimal in terms of $p$. However, when $\alpha > \frac{1}{2}$, then playing on $G_\alpha$ is sufficient and we can set $p = 0$, c.f. Theorem 1.1. For the proof of this theorem we first prove a useful lemma that allows Maker to maintain regularity in her graph. We will also require this in the proof of Theorem 1.5 and, in fact, Theorem 1.1 will easily follow from the proof of Theorem 1.5.

5.1. Maintaining regular pairs. One of the ingredients for Maker’s strategy in the $(1:1)$ Maker-Breaker $H$-game on $G_\alpha \cup G_{n,p}$ is the following lemma. It roughly states, that if a game is played on the edge set of some graph $G$, in which a given pair $(A, B)$ of disjoint subsets of vertices is regular, then Maker can ensure to claim a subgraph for which the pair $(A, B)$ is still regular.

Lemma 5.3. For every reals $0 < \varepsilon < \alpha < 1$ with $\alpha > 8\varepsilon$ the following holds provided $n$ is large enough. Let $G = (A \cup B, E)$ be a bipartite graph with $|A| = |B| = n$ such that $(A, B)_G$ is $\varepsilon$-regular and has density $\alpha$. Then Maker has a strategy to ensure that in the $(1:1)$ Maker-Breaker game on $E(G)$ she occupies a subgraph $M \subset G$ such that $(A, B)_M$ is $4\varepsilon$-regular and has density $\frac{\alpha}{2}$.

The above statement will follow easily from the following more general statement on Discrepancy games due to Hefetz, Krivelevich, and Szabó [28]. The general setup of such a game is as follows. Let a bias $b$, a constant $\varepsilon > 0$ and some hypergraph $(X, \mathcal{H})$ be given. The $(b, 1, \mathcal{H})$ $\varepsilon$-Discrepancy game is played by two players, called Balancer and Unbalancer, who alternately claim previously unclaimed elements of $X$. Balancer, starting the game, claims $b$ elements of $X$ in every round (except for maybe the last round when there are less then $b$ elements left), while Unbalancer always claims 1 such element. Denote with $B$ the set of all elements claimed by Balancer by the end of the game. Then Balancer is called the winner if and only if

$$|B \cap A| - \frac{b}{b+1}|A| < \varepsilon|A|$$

(5.1)

holds for every $A \in \mathcal{H}$. The following theorem provides a general winning criterion for Balancer.

Theorem 5.4 (Theorem 1.5 in [28]). Let $(X, \mathcal{H})$ be a $k$-uniform hypergraph. If

$$b < \frac{1}{3} \sqrt{\frac{k}{\ln(|\mathcal{H}|k)}}$$

and

$$\varepsilon > 3 \sqrt{\frac{\ln(|\mathcal{H}|k)}{kb}}$$

while $k$ is sufficiently large, then Balancer has a winning strategy for the $(b, 1, \mathcal{H})$ $\varepsilon$-Discrepancy game.
With the above result in our hands, let us now turn to the proof of Lemma 5.3.

**Proof of Lemma 5.3.** First note, that if \((A, B)_G\) has density \(\alpha\), then \((A, B)_M\) will have density \(\frac{\alpha}{2}\) since Maker claims half of all edges. Now, for every \(S \subset A, T \subset B\) of size \(\varepsilon n\) we have \(\alpha - \varepsilon < d_G(S, T) < \alpha + \varepsilon\) and hence

\[
(\alpha - \varepsilon)\varepsilon^2 n^2 < e_G(S, T) < (\alpha + \varepsilon)\varepsilon^2 n^2.
\]

For every such pair \((S, T)\) fix an arbitrary subset \(H(S, T) \subseteq e_G(S, T)\) of size \((\alpha - \varepsilon)\varepsilon^2 n^2\), and let

\[
\mathcal{H} = \left\{ H(S, T) : S \subset A, T \subset B \text{ of size } \varepsilon n \right\}.
\]

Maker plays as Balancer on the hypergraph \((X, \mathcal{H})\), where \(X = \bigcup_{H(S, T) \in \mathcal{H}} H(S, T)\). That is, whenever Breaker claims an edge belonging to \(X\), Maker (as Balancer) claims an edge according to the strategy for the \((1, 1, \mathcal{H})\) \(\varepsilon\)-Discrepancy game, given by Theorem 5.4, for \(b = 1\) and \(k = (\alpha - \varepsilon)\varepsilon^2 n^2\). Whenever Breaker claims an edge not in \(X\), Maker does the same.

In order to see that Theorem 5.4 can be applied, observe that

\[
|H(S, T)| \leq \left(\frac{n}{\varepsilon n}\right)^2 < 4^n,
\]

and hence

\[
\frac{k}{\ln(|\mathcal{H}|)} = \Omega(n) \quad \text{and} \quad \frac{\ln(|\mathcal{H}|)}{kb} = O\left(\frac{1}{n}\right),
\]

which yields

\[
b < \frac{1}{3} \sqrt[3]{\frac{k}{\ln(|\mathcal{H}|)}} \quad \text{and} \quad \varepsilon > 3 \sqrt{\frac{\ln(|\mathcal{H}|)}{kb}}
\]

provided \(n\) is large enough. Now, let \(M\) denote Maker’s graph at the end of the game. As a result of Maker’s strategy, Maker ensures that

\[
\left| |E(M) \cap H(S, T)| - \frac{1}{2} |H(S, T)| \right| < \varepsilon |H(S, T)| \quad \text{for every } H(S, T) \in \mathcal{H},
\]

see inequality (5.1). From this, we obtain for every \(S \subset A, T \subset B\) of size \(\varepsilon n\) that

\[
e_M(S, T) \geq |E(M) \cap H(S, T)| \geq \left(\frac{1}{2} - \varepsilon\right) |H(S, T)| \geq \left(\frac{1}{2} - \varepsilon\right) (\alpha - \varepsilon)\varepsilon^2 n^2
\]

\[
\Rightarrow d_M(S, T) \geq \left(\frac{1}{2} - \varepsilon\right) (\alpha - \varepsilon) \geq \frac{1}{2} \alpha - 2\varepsilon, \quad \text{and}
\]

\[
e_M(S, T) \leq |E(M) \cap H(S, T)| + |E_G(S, T) \setminus H(S, T)| \leq \left(\frac{1}{2} + \varepsilon\right) (\alpha - \varepsilon)\varepsilon^2 n^2 + 2\varepsilon^3 n^2
\]

\[
\Rightarrow d_M(S, T) \leq \left(\frac{1}{2} + \varepsilon\right) (\alpha - \varepsilon) + 2\varepsilon \leq \frac{1}{2} \alpha + 3\varepsilon,
\]

i.e. \(|d_M(S, T) - d_M(A, B)| < 4\varepsilon\). By a simple averaging argument the latter extends to all subsets \(S, T\) of size at least \(\varepsilon n\). That is, \((A, B)_M\) is \(4\varepsilon\)-regular with density at least \(\frac{\alpha}{2}\). \(\square\)
Together with Lemma 2.9 this is sufficient to prove Theorem 1.1. For Theorem 1.5 Maker also needs to find many copies of small graphs within the random graph such that they can be combined using the regular pairs.

5.2. Creating many $H$-copies on a random graph. It is known (Theorem 16 in [42]) that for any graph $H$, which contains a cycle, there is a constant $C > 0$ such that a.a.s. the following holds when $p \geq C n^{-1/m_2(H)}$: in the $(1 : 1)$ Maker-Breaker game on $G_{n,p}$, Maker has a strategy to occupy a copy of $H$.

Another ingredient for Maker’s strategy is to show that, if $p$ is slightly larger than mentioned above, Maker a.a.s. has a strategy to occupy a copy of $H$ on every vertex set of almost linear size.

**Lemma 5.5.** Let $H$ be any graph. Then for every $\gamma > 0$ there exists $\beta > 0$ such that with $p \geq n^{-1/m_2(H)} + \gamma$ a random graph $G \sim G_{n,p}$ a.a.s. satisfies the following property: playing a $(1 : 1)$ Maker-Breaker game on $G$, Maker has a strategy to occupy a subgraph of $G$ that has a copy of $H$ on every vertex set of size $n^{1-\beta}$.

The proof of the above lemma will follow mostly the argument given in the papers of Bednarska and Łuczak [8], as well as Stojaković and Szabó [44]. We start with the following lemma. For this, note that $G(n, M)$ denotes the random graph model, where we pick a graph on $n$ vertices and with exactly $M$ edges uniformly at random.

**Lemma 5.6** (Lemma 4 in [8]). Let $H$ be any graph containing a cycle, then there exist constants $\beta > 0$ and $c > 0$ such that for every large enough $n$ the following holds: if $M = 2n^{2-1/m_2(H)}$ then with probability at least $1 - \exp(-cM)$ each subgraph of $G \sim G(n, M)$ with $(1 - \beta)M$ edges contains a copy of $H$.

Note that in [8, Lemma 4] the probability of the good event happening is stated to be at least $\frac{2}{3}$; however with a closer look at the proof one actually sees that this probability is at least $1 - \exp(-cM)$ for some positive constant $c$. Further note that if we increase the number of edges of the random graph, but still delete at most the same number $2\beta n^{2-1/m_2(H)}$ of edges, then it does not become less likely to find copies of $H$. In particular, we obtain the following.

**Corollary 5.7.** Let $H$ be any graph containing a cycle, then there exist constants $\beta > 0$ and $c > 0$ such that for every large enough $n$ the following holds: if $M \geq 2n^{2-1/m_2(H)}$ then with probability at least $1 - \exp(-cn^{2-1/m_2(H)})$ each subgraph of $G \sim G(n, M)$ with $M - \beta n^{2-1/m_2(H)}$ edges contains a copy of $H$.

In fact, if the number of edges is increased slightly in the order of magnitude, then we can even find copies of $H$ on every vertex set of almost linear size.
Corollary 5.8. Let $H$ be any graph containing a cycle. Then for every $\gamma > 0$ there exists a constant $\beta > 0$ such that the following holds a.a.s. in $G \sim G(n, M)$ with $M \geq n^{2-1/m_2(H)+\gamma/4}$, for every vertex subset $A \subseteq [n]$ of size $n^{1-\beta}$ and every subgraph $F \subseteq G$ with $e(F) \leq n^{2-1/m_2(H) - \gamma/3}$ it holds that $(G \setminus F)[A]$ contains a copy of $H$.

Proof. Let constants $\beta$ and $c$ be chosen such that Corollary 5.7 can be applied and such that $0 < \beta < \frac{\gamma}{12}$ and $(1 - \beta) \left(2 - \frac{1}{m_2(H)}\right) > 1 + \beta$. To see that the latter is possible, note that $m_2(H) > 1$ by the assumption on $H$. From now on, whenever necessary, assume $n$ to be large enough.

Let $A \subseteq [n]$ be any vertex subset of size $n^{1-\beta}$, and let $\mathcal{E}_A$ be the event that, when generating $G \sim G(n, M)$, there exists a subgraph $F$ with $e(F) \leq n^{2-1/m_2(H) - \gamma/3}$ such that $(G \setminus F)[A]$ does not contain a copy of $H$. We will show in the following that

$$\mathbb{P}(\mathcal{E}_A) \leq 2e^{-n^{1+\beta}}; \quad (5.2)$$

with a union bound over all choices for $A$ (the number of which is bounded by $2^n$) the corollary then follows. In order to prove inequality (5.2), let us observe first that

$$(1 - \beta)Mn^{-2\beta} \leq e_G(A) \leq (1 + \beta)Mn^{-2\beta} \quad (5.3)$$

holds with probability at least $1 - \exp(-n^{1+\beta})$. Indeed, the random variable $e_G(A)$ has hypergeometric distribution with expectation

$$\mathbb{E}(e_G(A)) = M \cdot \binom{|A|}{2} = (1 - o(1))M \cdot \frac{|A|^2}{n^2} = (1 - o(1))Mn^{-2\beta}.$$

Then, with Lemma 2.4, we get

$$\mathbb{P}\left(|e_G(A) - Mn^{-2\beta}| > \beta Mn^{-2\beta}\right) \leq e^{-\beta^2 Mn^{-2\beta}/4} < e^{-n^{1+\beta}},$$

where for the last inequality we use that $2 - \frac{1}{m_2(H)} > 1$ and $\frac{\gamma}{4} - 2\beta > \beta$. Next, if we condition on (5.3), we obtain that $G[A]$ is distributed according to $G(N, M^*)$ with $N = n^{1-\beta}$ and

$$M^* \geq (1 - \beta - o(1))Mn^{-2\beta} \geq N^{2-1/m_2(H)+\beta},$$

where for the last inequality we use that $\frac{\gamma}{4} - 2\beta > \beta$ and $n > N$. It follows from Corollary 5.7 that with probability at least $1 - \exp(-cN^{2-1/m_2(H)}) \geq 1 - \exp(-n^{1+\beta})$ each subgraph of $G[A]$ with $e_G(A) - \beta N^{2-1/m_2(H)}$ edges contains a copy of $H$. Since $n^{2-1/m_2(H) - \gamma/3} < \beta N^{2-1/m_2(H)}$ holds for large $n$ by having $\beta < \frac{\gamma}{12}$, this in particular means that $(G \setminus F)[A]$ contains a copy of $H$ for every graph $F$ satisfying $e(F) \leq n^{2-1/m_2(H) - \gamma/3}$. Hence, inequality (5.2) is proven.

If $H$ does not contain a cycle, we can get the same conclusion.
Corollary 5.9. Let $H$ be any graph with $m_2(H) = 1$. Then for every $\gamma > 0$ there exists a constant $\beta > 0$ such that the following holds \textit{a.a.s.} in $G \sim G(n, M)$ with $M \geq n^{1+\gamma/4}$: for every vertex subset $A \subseteq [n]$ of size $n^{1-\beta}$ and every subgraph $F \subseteq G$ with $e(F) \leq n^{1-\gamma/3}$ it holds that $(G \setminus F)[A]$ contains a copy of $H$.

Proof. Let $H$ be any graph with $m_2(H) = 1$ and let $\gamma > 0$. We construct a graph $H'$ by adding a disjoint cycle of length $\lceil \frac{3}{\gamma} + 2 \rceil$ to $H$ and note that $m_2(H') \leq 1 + \frac{2}{8}$ and, therefore, $2 - \frac{1}{m_2(H')} \leq 1 + \frac{2}{8}$. Now we let $\beta > 0$ be given by Corollary 5.8 with input $H'$ and $\frac{2}{8}$. Then \textit{a.a.s.} in $G(n, M)$ with $M \geq n^{1+\gamma/4} \geq n^{2-1/m_2(H') + \gamma/8}$ for every vertex subset $A \subseteq [n]$ of size $n^{1-\beta}$ and every subgraph $F \subseteq G$ with $e(F) \leq n^{2-1/m_2(H') - \gamma/6}$ it holds that $(G \setminus F)[A]$ contains a copy of $H'$ and, thus, also of $H$. Since $n^{1-\gamma/3} \leq n^{2-1/m_2(H') - \gamma/6}$ the statement follows. \hfill \Box

Using Corollaries 5.8 and 5.9 we finally can prove Lemma 5.5.

Proof of Lemma 5.5. The proof idea for the lemma is similar to the one given by Bednarska and Łuczak [8], or Stojaković and Szabó [44] for the discussion of a Maker’s strategy in the $H$-game. We will prove that \textit{a.a.s.} Maker has a strategy for occupying a graph as desired when playing on a random graph $G' \sim G(n, M')$ where $M' = p(n_2)$. The result then follows for $G_{n,p}$ as the property we are looking for is monotone increasing (see e.g. Proposition 1.12 in [32]).

Maker’s strategy is to play randomly. That is, in each of her moves, Maker takes an edge from $G'$ uniformly at random from all the edges she has not taken in previous rounds. If this edge is free, then she claims it; otherwise she declares her move as a failure and simply skips her move. We will show that against any fixed Breaker’s strategy this random strategy \textit{a.a.s.} leads to a subgraph $H$. From this it then follows that \textit{a.a.s.} there must also exist a deterministic strategy for winning the $H$-game (see e.g. [8, 26]).

In order to prove that Maker succeeds \textit{a.a.s.} let us consider only the first $M = n^{2-1/m_2(H)+\gamma/4}$ rounds of the game. At the end of the $M$-th round we have that all edges taken by Maker (but not necessarily being claimed by her due to a failure) form a graph distributed from $G(n, M)$. Hence, using Corollaries 5.8 and 5.9, it is enough to prove that \textit{a.a.s.} we have at most $n^{2-1/m_2(H) - \gamma/3} = n^{-7\gamma/12}M$ failures by the end of round $M$. In order to see that the number of failures can be bounded this way, notice that the board size is $e(G') = M' \geq \frac{4}{8}n^{2-1/m_2(H)+\gamma}$ and hence, until round $M$, each round happens to be a failure with probability at most

$$\frac{M}{M' - M} \leq 4n^{-3\gamma/4}.$$

Therefore, the expected number of failures up to round $M$ can be bounded from above by $4n^{-3\gamma/4}M$. Applying Markov’s inequality (Lemma 2.1) we thus obtain that with probability at most $4n^{-\gamma/6}$ there happen to be more than $n^{-7\gamma/12}M$ failures. This proves the lemma. \hfill \Box
5.3. A general result for \( H \)-games. The goal of this subsection is to prove Theorem 1.5.

Proof of Theorem 1.5. Let \( \gamma > 0 \) and \( r \geq 2 \) be any integer, let \( \alpha \in (\frac{r-2}{r-1}, \frac{r-1}{r}) \), \( H \) be a fixed graph with \( m_2^r(H) > 0 \), and let \( G_\alpha \) be any \( n \)-vertex graph with minimum degree at least \( \alpha n \) and 
\[
p \geq n^{-1/m_2^r(H)+\gamma}.
\]

By the definition of \( m_2^r(H) \), we find a partition \( P_1, \ldots, P_r \) of \( V(H) \) such that \( m_2^r(H[P_i]) \leq m_2^r(H) \) for every \( 1 \leq i \leq r \). For short, let \( H_i = H[P_i] \) for \( 1 \leq i \leq r \), \( H^{(i)} = H[P_1 \cup \cdots \cup P_i] \) for \( 0 \leq i \leq r \), and set \( \ell = \max_i |P_i| \). Then \( H^{(0)} \) is the empty graph and \( H^{(r)} = H \); and, moreover, for \( 1 \leq i \leq r-1 \), \( H^{(i+1)} \) is contained in the graph that we get when we take the vertex disjoint union of \( H^{(i)} \) and \( H_i \) and add all edges between both graphs.

We apply Lemma 5.5 with \( \frac{\alpha}{2} \) for each \( H_i \) such that \( v(H_i) \geq 2 \), resulting in some output \( \beta_i = \beta(H_i) \), and set \( \beta_i = 1 \) if \( v(H_i) = 1 \). Then we set \( \beta = \min_{i \in [r]} \beta_i \). Further, we choose \( \delta > 0 \) such that \( \alpha \geq \frac{r-2}{r-1} + \delta \). From Lemma 2.10 with input \( r, \frac{\delta}{2}, r, \frac{\beta}{2} \), and \( \ell \) we obtain a constant \( \nu \leq \frac{1}{2} \). We then choose a positive constant \( \varepsilon \leq \min\left(\frac{\nu \delta}{2m}, 2\ell \right) \), obtain \( m_0 \) from Lemma 2.10 with input \( \varepsilon \), and obtain \( \eta \) and \( n_0 \) from Lemma 2.9 with inputs \( \varepsilon \) and \( \delta \). We let \( n \) be large enough for Lemma 5.3 and for the application of the other lemmas, to ensure that \( n \geq n_0, \nu^r \eta n \geq m_0 \), and \( (\nu^r \eta n)^{1-1/2} \geq n^{1-\beta} \).

Before revealing \( G \sim G_{n,p} \), we apply Lemma 2.9 to the graph \( G_\alpha \), and we find pairwise disjoint sets \( V_1, \ldots, V_r \) of size at least \( \eta n \) such that \( (V_i, V_j) \) is \( \varepsilon \)-regular with density equal to some constant \( \delta_{i,j} \geq \frac{\delta}{2} \), for every \( 1 \leq i < j \leq r \). We denote by \( G_1 \) the \( r \)-partite subgraph of \( G_\alpha \) with classes \( V_1, \ldots, V_r \). Finally, revealing the edges of \( G \sim G_{n,p} \), we let \( G_2 \) denote the union of all graphs \( G[V_i] \) with \( 1 \leq i \leq r \). Then a.a.s. we have that each \( G[V_i] \) satisfies the conclusion of Lemma 5.5 applied to the graph \( H_i \), since \( p \geq |V_i|^{-1/m_2(H_i)+\gamma/2} \) for large enough \( n \). From now on we will condition on these conclusions. Next we will describe a strategy for Maker in an unbiased game on \( G_\alpha \cup G_{n,p} \), and show that she can follow that strategy, leading to a copy of \( H \) in her final graph.

**Strategy description:** Maker plays as follows. Consider the edge-disjoint boards \( E_{G_2}(V_i) \) for all \( i \in [r] \), and \( E_{G_1}(V_i, V_j) \) for all \( 1 \leq i < j \leq r \). Maker always plays on the same board as Breaker. On each of the boards \( E_{G_1}(V_i, V_j) \) Maker follows the strategy guaranteed by Lemma 5.3 and thus occupies a subgraph \( \mathcal{M}_1 \subseteq G_1 \) such that \( (V_i, V_j, \mathcal{M}_1) \) is \( 4\varepsilon \)-regular with density \( \frac{\delta_i}{4} \) for every \( 1 \leq i < j \leq r \). On each of the boards \( E_{G_2}(V_i) \) Maker follows the strategy guaranteed by Lemma 5.5 and thus obtains a subgraph \( \mathcal{M}_2 \subseteq G_2 \) such that for every \( 1 \leq i \leq r \) and any \( U \subset V_i \) of size at least \( n^{1-\beta} \) there is a copy of \( H_i \) in \( \mathcal{M}_2[U] \).

**Strategy discussion:** Maker can follow her strategy by the conclusions of Lemma 5.3 and Lemma 5.5. Hence, it remains to show that, by following the strategy, Maker obtains a copy of \( H \).
We will build this copy inductively in the order $H^{(0)}, H^{(1)}, \ldots, H^{(r)}$. For $0 \leq s \leq r$ we want to find a copy of $H^{(s)}$ in $(\mathcal{M}_1 \cup \mathcal{M}_2)[V_1 \cup \cdots \cup V_s]$ such that in the common neighbourhood (with respect to $\mathcal{M}_1$) of the vertices of this copy there are pairwise disjoint sets $V_{s+1}^{(s)} \subset V_{s+1}, \ldots, V_r^{(s)} \subset V_r$ of size $\nu^s \eta n$ such that $(V_i^{(s)}, V_j^{(s)})_{\mathcal{M}_1}$ is $4\varepsilon \nu^{-s}$-regular with density at least $\frac{\delta}{2^{s+1}}$ for $s+1 \leq i < j \leq r$. Observe, that the above already holds for $s = 0$ when we choose $V_i^{(0)} = V_i$ for $1 \leq i \leq r$. We are finished when we arrive at $s = r$, since we then have a copy of $H^{(r)} = H$ in $\mathcal{M}_1 \cup \mathcal{M}_2$.

Assume the above holds for some $0 \leq s \leq r-1$. We already have a copy of $H^{(s)}$ with the described properties. Next we apply Lemma 2.10 to obtain a set $U \subseteq V_r^{(s)}$ of size $(\nu^s \eta n)^{1-\beta/2} \geq n^{1-\beta}$ such that any $\ell$ vertices from $U$ have at least $\nu^{s+1} \eta n$ common neighbours in each of the sets $V_i^{(s)}$ for $s+2 \leq i \leq r$, which we denote by $V_i^{(s+1)}$. Then by Lemma 2.8 and the choice of $\varepsilon$ we obtain that all pairs $(V_i^{(s+1)}, V_j^{(s+1)})_{\mathcal{M}_1}$ are $4\varepsilon \nu^{-s-1}$-regular with density at least $\frac{\delta}{2^{s+1}}$ for $s+2 \leq i < j \leq r$. Moreover, by the game on $E_{G_2}(V_{s+1})$ there is a copy of $H_{s+1}$ in $\mathcal{M}_2[U]$. By construction together with the copy of $H^{(s)}$ this gives a copy of $H^{(s+1)}$ with the desired properties.

\begin{remark}
The proof of Theorem 1.1 follows analogously. Indeed, when $H$ has chromatic number $r$, we have $m_2^{(r)}(H) = 0$ and all $H_i$ consist of isolated vertices. Then Maker only needs to play on $G$ to obtain a copy of $H$, exactly as described above.
\end{remark}

\begin{remark}
We do not need the $\gamma$ in Theorem 1.5 when there is only a single graph $H'$ in $H_1, \ldots, H_k$ that satisfies $m_2(H') = m_2^{(r)}(H)$. This is because in our construction we can find this copy in the last set $V_r$ using a version of Lemma 5.5 that only requires $p \geq Cn^{-1/m_2(H)}$ and guarantees a copy of $H'$ in every set of size $\beta n$ for some not too small constant $\beta > 0$.

We also do not need the $\gamma$ when $G_\alpha$ is the $r$-partite Turán-graph. Intuitively the behaviour should be the same as in the general case, and therefore we believe that $\gamma$ is not needed in any case.
\end{remark}

Finally, the Waiter-Client result for the $H$-game can be proven fairly easily.

\begin{proof}[Proof of Theorem 1.8]
For any graph $G$ set $k(G) = 2^{e(G)}$. Let $F$ be the vertex disjoint union of $k(H)$ copies of the graph $H$. It holds that $m^{(r)}(F) = m^{(r)}(H)$. Hence, following Theorem 2.1 from [39], we know that a.a.s. a graph $G \sim G_\alpha \cup G_{n,p}$ contains a copy of $F$.

It thus remains to show that playing on $F$, Waiter has a strategy to force a copy of $H$ in Client’s graph. This can be done by induction on $e(H)$. Let $e$ be any edge in $E(H)$, and denote with $e_1, \ldots, e_k(H)$ the copies of $e$ in the copies of $H$ in $F$. Then Waiter can offer these edges in pairs, until Client claimed $\frac{k(H)}{2} = k(H - e)$ such edges. Immediately afterwards, there are $k(H - e)$ copies of $H$ in which Client already claimed the copy of $e$ and in which Waiter does not occupy any edge yet. Thus the problem is reduced to force a copy of $H - e$ on the disjoint union of $k(H - e)$ copies of $H - e$, which can be shown by induction.
\end{proof}
6. Concluding remarks

6.1. Optimality of Theorem 1.5. In this paper we proved optimal results for the $k$-vertex-connectivity and Hamiltonicity Maker-Breaker games in randomly perturbed graphs. It remains to discuss when Theorem 1.5 is optimal up to the $\gamma$. More precisely, the question is when $p \leq cn^{-1/m_2^{(r)}(H)}$ is enough to ensure that a.a.s. Breaker has a winning strategy in the $H$-game on $G_\alpha \cup G_{n,p}$ for some choice of $G_\alpha$. For this, consider $G_\alpha$ to be the $r$-partite Turán graph on $n$ vertices, with vertex classes $V_1, \ldots, V_r$. If Maker wants to create a copy of $H$ in the game on $G_\alpha \cup G_{n,p}$, then she must have a strategy for creating some subgraph $H' \subseteq H$ with $m_2(H') \geq m_2^{(r)}(H)$ on one of the sets $V_i$. Therefore, it suffices for Breaker to ensure that, when playing on $G_{n,p}$, Maker loses the Maker-Breaker $\mathcal{H}$-game, where the family $\mathcal{H}$ of winning sets consists of all copies of all such subgraphs $H'$. A.a.s. Breaker has a winning strategy if $\mathcal{H}$ does not contain any exceptional graph $H'$, for which the Maker-Breaker $H'$-game threshold is not known, or is not of order $n^{-1/m_2(H')}$, or $H' = K_4$. This can be proven analogously to the Maker-Breaker $H$-game on random graphs [42, Theorem 2]. The reason that $K_4$ is excluded is because this case is treated separately in [41, Lemma 2.1] and there is no immediate way to combine the proofs. When the threshold of $H'$ is not of the order $n^{-1/m_2(H')}$, e.g. when $H$ is a tree or a triangle, the bound from Theorem 1.5 can be significantly improved as we have demonstrated in Proposition 5.1 and 5.2. It would be interesting to see if, more generally, $p = \omega(n^{-2})$ is always sufficient when there exists an edge $e \in E(H)$ such that $m_2^{(r)}(H - e) = 0$. This also generalises to many other graphs $H$ with $m_2^{(r)}(H) = 1$ and we give more details for one example in the next section.

6.2. $H$-game for small graphs. $H = K_4$ is the smallest graph for which we do not know the threshold probability for winning the $H$-game on $G_\alpha \cup G_{n,p}$ with $\alpha \in (0, \frac{1}{2})$. Note that it follows from [39] that a.a.s. in $G_\alpha \cup G_{n,p}$ there is a copy of $K_4$, provided that $p = \omega(n^{-2})$. However, even when $p = o(n^{-5/4})$, there is a.a.s. an easy strategy for Breaker to ensure that Maker does not get a copy of $K_4$. Indeed, in this case, let $G_\alpha$ be any bipartite graph with $\delta(G_\alpha) \geq \alpha n$ and with partition classes $A$ and $B$, and note that in $G_{n,p}$ a.a.s. all components are trees on at most 4 vertices. Conditioning on this event and assuming that all edges in $G[A]$ and $G[B]$ already belong to Maker, it is a simple case distinction to check that Breaker has a strategy on the edges of $G[A,B]$ to prevent Maker from claiming a copy of $K_4$. It is plausible that with a more involved strategy for Breaker, also responding on the edges inside $A$ and $B$, the bound on $p$ can be increased further. On the other hand, there also is a simple strategy for Maker when $p = \omega(n^{-8/7})$. Here, we can use Theorem 2.1 from [39] to show that $G_\alpha \cup G_{n,p}$ will a.a.s. contain a graph created from a complete bipartite graph by adding to both partition classes many vertex-disjoint copies of stars.
with 7 edges. Maker can easily claim 4 edges from such a star, when she is the first player to
claim an edge. By claiming multiple stars with 4 edges in each partition class, she can ensure
that she claims a pair of such stars such that all edges between the two stars are still free. Then
she can restrict the game to the complete bipartite graph between those two stars, and again
it is an easy case distinction to show that Maker can complete a copy of $K_4$ within the next 5
moves.

It is unclear if this strategy can be significantly improved or how far the Breaker argument can
be pushed.

6.3. Maker-Breaker $K_t$-factor game. In an $n$-vertex graph with $t|n$ a $K_t$-factor is the disjoint
union of $\frac{n}{k}$ copies of $K_t$. In this section we implicitly assume $k|n$. Recently, Allen, Böttcher,
Kohayakawa, Naves, and Person [1] determined the threshold bias for the Maker-Breaker $K_t$-
factor game on $K_n$, for $t \in \{3, 4\}$, up to a logarithmic factor. Even more recently, Liebenau and
Nenadov [40] determined the threshold for every $t \geq 3$. They proved that for $t \geq 3$ there are
constants $c, C > 0$ such that Breaker wins if $b < cn^{2/(t+2)}$ and Maker wins if $b > Cn^{2/(t+2)}$.

We briefly summarise what is known about the appearance of a $K_t$-factor in the other models that
we discussed. Hajnal and Szemerédi [18] proved that any $n$-vertex graph with minimum degree at
least $(1 - \frac{1}{t}) n$ contains a $K_t$-factor. Johansson, Kahn, and Vu [33] showed that $n^{-2/t} \ln^{2/(t^2-t)} n$
gives the threshold for the containment of a $K_t$-factor in $G_{n,p}$. In the perturbed model $G_{\alpha} \cup G_{n,p}$ it
was shown by Balogh, Treglown, and Wagner [3] that for any $\alpha > 0$ a probability of $p = \omega(n^{-2/t})$
is a.a.s. sufficient, while for large $\alpha$ more precise results are known [25].

The $\ln$-term in [33] is needed to ensure that every vertex is contained in a copy of $K_t$, which,
of course, is immediate in the perturbed model. Similarly, when we consider the Maker-Breaker
$K_t$-factor game on $G_{n,p}$, Maker needs to ensure that in her graph every vertex is contained in
a copy of $K_t$. Since each vertex has roughly $np$ neighbours and there has to be a copy of $K_{t-1}$
in each neighbourhood, we thus require that Maker wins the $K_{t-1}$-game in $G_{np,p}$. If $k \geq 5$ this
implies that we need $p \geq C(np)^{-2/t}$ and, thus, $p \geq C' n^{-2/(t+2)}$. Note that this probabilistic
intuition aligns with the threshold bias in [40] discussed above.

While it is not known if the threshold probability for the $K_t$-factor game in $G_{n,p}$ is $n^{-2/(t+2)}$, it
would be interesting to investigate this in the perturbed model. More precisely, we ask which
lower bound on $p$ is sufficient such that Maker a.a.s. has a winning strategy in the Maker-Breaker
$K_t$-factor game on $G_{\alpha} \cup G_{n,p}$. When $p = \omega(n^{-2/(t+1)})$ Maker a.a.s. has a strategy such that every
set of linear size in $G_{n,p}$ contains a copy of $K_t$ (similar as in Corollary 5.8). Together with the
deterministic graph this could be sufficient to ensure that Maker is able to create a $K_t$-factor.
This is particularly interesting, because this probability differs significantly from the one needed
for the $K_t$-factor game on $G_{n,p}$.
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(DC, FH, YM) Hamburg University of Technology, Institute of Mathematics, Am Schwarzenberg-Campus 3, 21073 Hamburg, Germany

E-mail address: dennis.clemens@tuhh.de, fabian.hamann@tuhh.de, yannick.mogge@tuhh.de

(OP) London School of Economics, Department of Mathematics, London, WC2A 2AE, UK.

E-mail address: o.parczyk@lse.ac.uk