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Duality in non-linear programming

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Abstract. In this paper we consider duality and converse duality for a programming problem involving convex objective and constraint functions with finite dimensional range. We do not assume any constraint qualification. The dual is presented by reducing the problem to a standard Lagrange multiplier problem.

1. Introduction

Duality is an important concept in both linear and non-linear programming. Dual is used in connection with identifying near optimal solution, sensitivity analysis of the primal problem, convergence of algorithms and so on. More than often, solving a dual problem is easier than solving the original by simple mathematical, geometric and computational structure of the dual.

Many authors have considered optimization without constraint qualification for convex program. Ben Tal, Ben-Israel and Zowe [1] established optimality conditions without constraint qualification with finite number of constraints. Using this Mond and Zlobec proved duality in [7]. Borwein and Wolkowicz [2] established optimality without constraint qualification in a general locally convex space. Based on this Kanniappan[6] proved duality and converse duality with finite dimensional range. Nabaitian[8] has given duality results for a non-smooth multi objective programming problem in general convexity setting. B.D Craven and Mond had established Lagrangean conditions for quasi differentiable optimization[4].

In this paper, we establish duality and converse duality for a convex programming problem. However, the method applied can be applied for locally Lipschitz functions with Clarke [3] derivatives.

2. Preliminaries

We consider the following pair of problems:

(P) Minimize \( f(x) \)
Subject to
\( g(x) \in -C, \ x \in A \)

(D) Maximize \( f(x) + (s^+ \circ g)(x) \)
Subject to
\( 0 \in \partial f(x) + \partial (s^+ \circ g)(x) - \left( D_g(x) \cap \text{cone}(F - a) \right)^\circ \)
Here, $X$ and $Y$ are locally convex linear spaces. $Y$ is a finite dimensional space partially ordered by a closed convex cone $C$. $f : X \to \mathcal{R}$ is a continuous convex functional, $g : X \to Y$ is a continuous $C$-convex operator. i.e., for any $a, b$ in $Y$ and $t \in [0, 1]$

$$tg(a)+(1-t)g(b) - g(ta+(1-t)b) \in C.$$ 

$\partial f(x)$ denote the subgradients of $f$ at $x$. We write $\langle x, x' \rangle$ for the value of the continuous linear functional $x'$ at $x \in X$. $\partial f(x)$ denote the subgradients of $f$ at $x$. We write $\langle x, x' \rangle$ for the value of the continuous linear functional $x'$ at $x \in X$.

$X^*$ and $Y^*$ are the continuous dual spaces of $X$ and $Y$ respectively equipped with $w^*$-topology.

Let $F = \{ x \in X/ g(x) \in -C \}$ be a non-empty subset of domain of $f$ given by $\text{Dom } f \{ x \in X/ f(x) < \infty \}$.

For any non-zero $a \in Y$, let $a^+ \in C^*$ be such that $g(a) = \| a^+ \| = 1$.

Define $K = \{ y \in Y/ \langle y, a^+ \rangle \geq \|a\| \}$ where $0 \leq \alpha \leq 1$.

$K$ is a closed convex cone with interior points. If $Y$ is given the order induced by $K$, $Y$ becomes a partially ordered normed space with order unit norm. As $Y$ is finite dimensional, the norms are equivalent [5]. $K^*$ is well-based with interior points.

Thus, w.l.o.g we can assume that $C$ has interior points.

The one sided directional derivative of $f$ at $x \in \text{cone}$ of $f$ in the direction $d$ denoted by $f^*(x ; d)$ is defined by

$$f^*(x ; d) = \lim_{t \to 0^+} \frac{f(x + td) - f(x)}{t}$$

$\partial f(x) = \{ x \in X' / \langle x, x' \rangle \leq f' (x; d) \}$.

For any $s^+ \in C^*$ $s^* \circ g$ is a convex functional on $X$.

We define $D_g(a) = \{ d \in X/ g'(a; d) \leq 0 \}$

Throughout the paper, we assume that $F^* = \{ x \in F/g(x) \neq 0 \}$ is non-empty.

Let $r \in F^*$. Following the construction as above, we have a closed convex cone $K$ containing $-g(r)$ as an interior point and a $y^*$ in $C^* \cap K^*$ such that $y^*$ is an interior point of $K^*$. Let $B$ be a base for $K^*$. Let $M = B \cap C^*$. Let $S = M^*$. Then $S$ contains both $K$ and $C$. ($P$) can be rewritten as

Minimize $f(x)$

Subject to

$$g(x) \in -S, \ x \in F.$$

This satisfies Slater's constraint qualification

3. Duality Theorems

Lemma 3.1

If $(a, s^+)$ is feasible for $D$, then $\langle g(a), s^+ \rangle \leq 0$.

Proof:

Given $(a, s^+)$ is feasible for $(D)$.

$$\langle (g(x) - \langle g(a), s^+ \rangle, 0) \rangle < 0 \text{ for all } x \in \text{cone } (F - a)$$

Hence, $\langle (g^*(a, d), s^+ \rangle < 0 \text{ for all } d \in \text{cone } (F - a)$

Also, $f' (a + d) + \langle g'(a, d), s^+ \rangle \geq 0 \text{ for all } d \in \text{cone } (F - a)$

It follows that $a$ is optimal for $(P)$

Define $T : (F - a) \to \mathcal{R} \times \mathcal{R}$ by $T(x) = (f'(a, x), \langle g'(a, d), s^+ \rangle)$

There is no $x \in F - a$ such that $T(x) < 0$.

By Hahn Banach separation theorem, We have $\langle f'(a, x), \langle g'(a, d), s^+ \rangle \rangle \geq 0 \text{ for all } x \in F - a$

For $x = 0$, we have $\langle g'(a, d), s^+ \rangle \geq 0$
which is a contradiction.
This contradiction proves the lemma.

**Lemma 3.2**
If \((a, s^+)\) is feasible for \((D)\) and \(b\) is feasible for \((P)\), then
\[ f(b) \geq f(a) + \langle g(a), s^+ \rangle \]
Proof:
Suppose not,
\[ f(b) < f(a) + \langle g(a), s^+ \rangle < 0 \] by Lemma 3.1
\[ f(b) - f(a) < \langle g(a), s^+ \rangle < 0 \]
as \(a\) is feasible for \((P)\)
Let \(c = b - a\)
\[
\therefore f(b) = f(a+c) - f(a) < 0
\]
which implies \(f'(a;d) < 0\) for some \(d\) contradicting the feasibility of \((a, s^+)\) for \((D)\).
This contradiction proves the result.

**Theorem 3.1**
Let \(f\) be a continuous convex functional from a locally convex space \(X\), \(g\) be a continuous convex operator from \(X\) to a partially ordered finite dimensional locally convex space ordered by a closed convex cone \(C\). If \(a\) is optimal for \((P)\), then there exists \(s^+ \in C^*\) such that
\[ 0 \in \partial f(a) + \partial (s^+ \circ g)(a) - (D_g(a) \cap \text{cone}(F - a))^\dagger \]
\[ \langle g(a), s^+ \rangle = 0. \]
This can be proved using basic theorem of the alternative.

Based on the above characterization, we formulate a dual problem
\[ (D) \quad \text{Maximize } f(x) + \langle g(x), s^+ \rangle \]
Subject to
\[ 0 \in \partial f(x) + \partial (s^+ \circ g)(x) - (D_g(x) \cap \text{cone}(F - a))^\dagger, \]
\(x \in A\)

**Theorem 3.2** (Duality)
Let us assume the hypothesis of Theorem 3.1. If \(a_0 \in A\) is optimal for \((P)\), then there exists \(s_0^+ \in S^* \subseteq C^*\) such that \((a_0, s_0^+)\) is optimal for \((D)\) and the two problems have the same extremal values.
Proof:
As \(a_0\) is optimal for \((P)\), there exists \(s_0^+ \in S^* \subseteq C^*\) such that \((a_0, s_0^+)\) is feasible for \((D)\) and
\[ \langle g(a_0, s_0^+) \rangle = 0 \]
\[ \therefore f(a_0) = f(a_0) + \langle g(a_0, s_0^+) \rangle \] - (1)
From Lemma 3.2 and (1), we have \((a_0, s_0^+)\) is an optimal solution of \((D)\) and the optimal solution of \((P)\) and \((D)\) are equal.

**Theorem 3.3** (Unicity)
Let us assume the hypothesis of theorem 3.1. If \((a_0, s_0^+)\) is an optimal solution of the problem \((D)\), then \(a_0\) is an optimal solution of the problem \((P)\) and the two problems have the same extremal values.
Proof:
\((a_0, s_0^+)\) is optimal for \((D)\). Hence
\[ 0 \in \partial f(a) + \partial (s^+ \circ g)(a) - (D_g(a) \cap \text{cone}(F - a))^\dagger \]
It follows that
\[ f'(a_0;x) + \langle g'(a_0;x), s_0^+ \rangle \geq 0, x \in [D_g(a_0) \cap \text{cone}(F-a_0)] \]

Then  \( f'(a_0;x) \geq 0 \), for all \( x \) as otherwise the hypothesis that \( (a_0, s_0^+) \) is optimal for (D) is violated. The convexity of \( f \) implies that \( a_0 \) is optimal for (P).

That is, \( (a_0, 0) \) is feasible for (D).

Hence \( f(x) \geq f(a_0) \) for all \( x \) which are feasible for (P).

As \( (a_0, s_0^+) \) is optimal for (D),

\[ f(a_0) \geq f(a_0) + \langle g(a_0, s_0^+), x \rangle \]

which are feasible for (D).

Thus, \( f(a_0) \geq f(a_0) + \langle g(a_0, s_0^+), x \rangle \) which implies

\[ \langle g(a_0, s_0^+), s_0^+ \rangle = 0. \]

Thus (P) and (D) have the same extremal values.

**Remark:**

When \( \gamma = \mathbb{R}^m \), \( C = \mathbb{R}_+^m \)

Problem (D) reduces to problem (D_1)

Maximize \( f(x) + < y^+, g(x) > \)

Subject to

\[ y \geq 0, x \in F \text{ and } 0 \in \partial f(x) + \sum_{i=1}^m y_i g_i(x) + N(x/F) \]

and we get Schechter’s [9] duality theorem.

4. **Conclusion**

Many authors have established duality results for non-linear and non-convex programming problems. Though we have considered convex problems, the method of proof can be applied to locally Lipchitz and quasi-convex optimization problems with corresponding subgradients theory.

**References**

[1] Ben-Tal, Ben-Israel and Zowe, Characterization of Optimality in convex programming without Constraint qualification, J.Opt.Theory and Appl. 20,(1978) 417-437

[2] J.M.Borewein and H.Wolkowicz, Characterization of Optimality for the abstract convex program with finite dimensional range, J.Aust.Math.Soc. Ser.A, A-30 (1981), 390-411

[3] F.H.Clarke, Optimization and Non-smooth Analysis, John-Wiley & Sons, New York(1983).

[4] B.D.Craven and Mond, Lagrangean conditions for Quasi-differentiable Optimization, Survey of Mathematical Programming, Vol. 1, Proceedings of the 9th Symposium, North-Holland Publishing Company, (1976).

[5] G. Jameson, Ordered Linear Spaces, Lecture Notes in Mathematics, Vol.141, Springer – Verlag.

[6] P. Kanniappan, Duality Theorems for Convex Programming without constraint qualification, J.Australian Mathematical Society (Series A). 36 (1984), 253-266

[7] B.Mond and Zlobec, Duality for Non-differentiable Convex Programming, Utilitas Math., 15, (1974), 291-302

[8] S.Nobathitan, Duality without Constraint qualification in Non-Smooth Optimization, International Journal of Mathematics and Mathematical Sciences, VII, (2006), 1-11

[9] M.S. Schechter, A sub-gradient duality theorem, J.Math.Anal.Appl. 61. (1977), 850-855