Secrecy-Achieving Polar-Coding for Binary-Input Memoryless Symmetric Wire-Tap Channels

Eran Hof  Shlomo Shamai
Department of Electrical Engineering
Technion – Israel Institute of Technology
Haifa 32000, Israel

Abstract
A polar coding scheme is introduced in this paper for the wire-tap channel. It is shown that the provided scheme achieves the entire rate-equivocation region for the case of symmetric and degraded wire-tap channel, where the weak notion of secrecy is assumed. For the particular case of binary erasure wire-tap channel, an alternative proof is given. The case of general non-degraded wire-tap channels is also considered.

I. INTRODUCTION
Channel coding via the method of channel polarization is provided by Arikan in [2]. On a binary-input discrete memoryless channel (DMC), polarization ends up with either ‘good bits’, i.e., binary channels whose capacity approaches 1 bit per channel use, or ‘wasted bits’, i.e., channels whose capacity approaches zero. The fraction of the good bits is equal to the mutual information with equiprobable inputs (which equals the capacity for the case of symmetric channels). In a physically degraded setting, as mentioned in [2], an order of polarization is maintained in the sense that ‘good’ bits for the degraded channel, must also be ‘good’ for the better channel.

For a standard single-user channel coding problem, the polar coding scheme is based on transmitting the uncoded information bits over the capacity approaching channels (when we interpret the polarization as a kind of a precoding or pre-processing). At the same time, fixed and predetermined bits are transmitted over the channels whose capacity approaches zero. These predetermined bits are still needed in the successive decoding process, hence they can not be ignored.

A secrecy polar scheme is suggested in this paper for the wire-tap channel. A secret message needs to be transmitted reliably to a legitimate user. At the same time, this message must be kept secret from the eavesdropper. At the first part of this paper, it is assumed that the marginal channel to the eavesdropper is physically degraded with respect to the marginal channel to the legitimate user. The proposed secrecy polar scheme for the degraded case is based on transmitting random bits on the ‘good bits’ of the degraded eavesdropper channel. These random bits are independent of the secret message. At the legitimate receiver, the random bits can be decoded reliably. This is because the ‘good bits’ for the degraded eavesdropper channel are also ‘good’ for the legitimate user. The rest of the ‘good’ bits for the legitimate user are dedicated for the secret message. Additional independent works on this subject are provided in [13] [14] [15].

Transmitting random bits on the ‘good bits’ of the eavesdropper, all the possible information rates that can be detected by the eavesdropper are exhausted. Otherwise, the standard channel capacity could have been beaten. Thus
the ‘good bits’ associated with the secret message for the legitimate channel, must be perfectly secret (at least in the weak sense). Note that this result is satisfied immaterial of whether the eavesdropper adheres to successive decoding or to optimal decoding (as otherwise, its capacity could have been beaten). It is first shown that the provided scheme archives the secrecy capacity for the considered model. The result is then generalized to the entire rate-equivocation region. This result is proved under a weak notion of secrecy. For the particular case of binary erasure wire-tap channel, an alternative proof is provided, based on algebraic arguments. This different notion of proof may contribute to a stronger notion of security.

At the second part of the paper, the secrecy polar scheme is adapted for the general, i.e., non-degraded wire-tap channel. This scheme is based on a conjecture on possible polarization properties of some of the ‘bad’ indices of the eavesdropper. To this end, the polarization of the ‘bad’ indices is concerned while the decoder has perfect knowledge of some of the ‘good’ bits (which are no longer part of the transmitted message, but they are kept predetermined and fixed). The question regarding this aspect is whether the additional information helps in decoding these bits or do they keep their original 'bad' polarization. The original polarization result in [2] does not fully cover this scenario.

This paper is structured as follows: In Section II preliminary introduction is provided. In Section II-A the wire-tap communication model is introduced in addition to some basic definitions and results in information-theoretic security. Polar codes are introduced in Section II-B. The polar secrecy scheme is detailed and studied in Section III. A conjecture on possible polarization properties is stated in Section IV, along with a resulting adaptation of the polar secrecy scheme for non-degraded wiretap channels. A list of possible further generalizations is provided in Section V.

II. PRELIMINARIES

A. The Wire-Tap Communication Model

We consider the communication model in Figure I A coded system is presented which transmits a confidential message $U$ to a legitimate user. The message $U$ is chosen uniformly from a set of size $M$. Next, the message is encoded to a codeword $X$ with a blocklength $n$ over an alphabet $\mathcal{X}$. The resulting code-rate is $R = \frac{1}{n} \log M$. The codeword $X$ is transmitted over a communication channel $P_{Y,Z|X}$ with one input, and two outputs. The transmission is assumed to take place over a DMC $P$, with an input alphabet $\mathcal{X}$, and output alphabets $\mathcal{Y}$ and $\mathcal{Z}$. Let $P(y, z|x)$ denote the probability of receiving the symbols $y \in \mathcal{Y}$, and $z \in \mathcal{Z}$, at the legitimate user and the eavesdropper, respectively, given that the symbol $x \in \mathcal{X}$ is transmitted. Based on the assumption that the channel is memoryless, it follows that

$$P(y, z|x) = \prod_{k=1}^{n} P(y_k, z_k|x_k)$$

where (with some abuse of notation) $P(y, z|x)$ denotes the probability of receiving the symbols $y \in \mathcal{Y}$ and $z \in \mathcal{Z}$, at the legitimate user and the eavesdropper, respectively, given that the symbol $x \in \mathcal{X}$ is transmitted. Moreover, let $G(y|x)$ and $Q(z|x)$ denote the marginal probabilities for receiving the symbols $y \in \mathcal{Y}$ and $z \in \mathcal{Z}$, at the legitimate user and the eavesdropper, respectively, given that the symbol $x \in \mathcal{X}$ is transmitted. Both $G(y|x)$ and $Q(z|x)$ are transition probability laws of DMCs, called the marginal channels of the legitimate user and the eavesdropper, respectively. In addition, the probability to receive the symbol $z \in \mathcal{Z}$ at the eavesdropper, given that the symbol $y \in \mathcal{Y}$ is received at the legitimate user is denoted by $D(z|y)$.

The channel output vectors $Y$ and $Z$, both of length $n$, are received by the legitimate user and the eavesdropper, respectively. The legitimate user decodes the received vector $Y$ resulting in the decoded message $\hat{U}$. The objectives
of the considered coding system is to obtain both secure and reliable communication. These objectives are to be accomplished simultaneously using a single codebook $C_n$. The reliability of the system is measured via the average error probability $P_e(C_n)$ of the decoded message

$$P_e(C_n) = \frac{1}{M} \sum_{m=1}^{M} \Pr \left( \hat{U} \neq m \mid U = m \right).$$

Note that the error probability depends on the blocklength of the coded message. The level of security is measured by the equivocation rate

$$R_e(C_n) \triangleq \frac{1}{n} H(U|Z)$$

where $H(U|Z)$ denotes the conditional entropy of the transmitted message $U$, given the received vector $Z$ at the eavesdropper.

**Definition 1 (Achievable rate-equivocation pair).** A rate-equivocation pair $(R, R_e)$ is achievable if there exists a code sequence $\{C_n\}$ of block length $n$ and rate $R$ such that

$$\lim_{n \to \infty} P_e(C_n) = 0$$

$$R_e \leq \lim_{n \to \infty} R_e(C_n).$$

**Remark 1 (On strong and weak notions of secrecy).** The current discussion considers normalized entropies to measure the level of security (see the definition of equivocation rate in (1)). Therefore, the achieved secrecy notion is referred to as weak secrecy. The strong notion of secrecy considers the unnormalized mutual information between the confidential message and the received vector at the eavesdropper receiver. Strong secrecy guarantees secrecy in the weak sense while the opposite direction does not follow.

**Definition 2 (Secrecy capacity).** The secrecy capacity $C_s$ is the supremum of all the rates $R$, such that the pair $(R, R_e)$ is an achievable rate-equivocation pair.

**Theorem 1 (The secrecy capacity of the wire-tap channel [1]).** The secrecy capacity $C_s$ of the wire-tap channel satisfies:

$$C_s = \max_{P_{UX},P_{YZ|X}} \left( I(U;Y) - I(U;Z) \right)$$

where $U$ is an auxiliary random variable over the alphabet $\mathcal{U}$, satisfying
1) Markov relationship: $U \rightarrow X \rightarrow (Y, Z)$ is a Markov chain.

2) Bounded cardinality: $|U| \leq X + 1$.

Binary-input symmetric wire-tap channels are considered in this paper.

**Definition 3 (Symmetric binary input channels).** A DMC with a transition probability $p$, binary-input alphabet $\mathcal{X}$, and an output alphabet $\mathcal{Y}$ is said to be symmetric if there exists a permutation $\pi$ over $\mathcal{Y}$ such that

1) The inverse permutation $\pi^{-1}$ is equal to $\pi$, i.e.,

$$
\pi^{-1}(y) = \pi(y)
$$

for all $y \in \mathcal{Y}$.

2) The transition probability $p$ satisfies

$$
p(y|0) = p(\pi(y)|1)
$$

for all $y \in \mathcal{Y}$.

**Definition 4 (Symmetric binary-input wire-tap channels).** A binary input discrete memoryless wire-tap channel is symmetric if both of its marginal channels are symmetric.

The particular case of physically degraded channels is studied in this paper.

**Definition 5 (Physically degraded channels).** Let $P$ be a wire-tap channel with an input alphabet $\mathcal{X}$ and output alphabets $\mathcal{Y}$ and $\mathcal{Z}$, at the legitimate and eavesdropper, respectively. Then, $P$ is said to be physically degraded if

$$
P(y, z|x) = G(y|x)D(z|y)
$$

for all $x \in \mathcal{X}$, $y \in \mathcal{Y}$, and $z \in \mathcal{Z}$.

The following Theorem characterizes the secrecy capacity of a binary-input, memoryless, symmetric and degraded wire-tap channel:

**Theorem 2 ([1]).** Let $P$ be a binary-input, memoryless, symmetric, and degraded wire-tap channel. Denote by $G_{Y|X}$ and $Q_{Z|X}$ the marginal channels to the legitimate user and the eavesdropper, respectively. Then, the secrecy capacity $C_s$ is given by

$$
C_s(P) = C(G_{Y|X}) - C(Q_{Z|X})
$$

where $C(G_{Y|X})$ and $C(Q_{Z|X})$ are the channel capacities of the marginal channel $G_{Y|X}$ and $Q_{Z|X}$, respectively.

**Remark 2 (On the entire rate-equivocation region).** Theorem 2 is a particular case of the rate-equivocation region of less-noisy channels (which is on its own a particular case of the rate-equivocation region of the wire-tap channel). Under the notation in Theorem 1 if $I(U; Y) \geq I(U; Z)$ for every $U$ satisfying the Markov relationship in Theorem 1, then the channel to the legitimate receiver is said to be less noisy than the eavesdropper (the degradation assumption in (2) satisfies the less noisy condition). It can be shown for the case of less-noisy wire-tap channels, that the rate-equivocation region is given by

$$
\bigcup_{P_X P_{YZ|X}} \left\{ (R, R_e) : \begin{array}{l}
0 \leq R \leq I(X; Y) \\
0 \leq R_e \leq R \\
R_e \leq I(X; Y) - I(X; Z)
\end{array} \right\}.
$$

For further details and proof see [1] and references therein. In the particular case of binary-input, memoryless...
symmetric and degraded wire-tap channels as in Theorem 2, the rate-equivocation region is therefore given by

$$\left\{ \begin{align*}
0 \leq R &\leq C(G_{Y|X}) \\
(R, R_e) : &\quad 0 \leq R_e \leq R \\
R_e &\leq C(G_{Y|X}) - C(Q_{Z|X})
\end{align*} \right\}. \quad (3)$$

B. Polar Codes

This preliminary section offers a minimal summary of the basic definitions and results in [2], [3], that are essential to the presentation of the results in Section III.

Let $p$ be a transition probability function of a DMC with a binary input-alphabet $\mathcal{X} = \{0, 1\}$ and an output alphabet $\mathcal{Y}$. The operation of the channel on vectors is also denoted by $p$, that is for $x = (x_1, \ldots, x_n) \in \mathcal{X}^n$, and $y = (y_1, \ldots, y_n) \in \mathcal{Y}^n$, the block transition probability is given by

$$p(y|x) = \prod_{l=1}^{n} p(y_l|x_l).$$

Polar codes are defined in this section using the following recursive construction. At the first step, two independent copies of $p$ are combined to form a new channel $p_2$ over an input alphabet $\mathcal{X}^2$ and output alphabet $\mathcal{Y}^2$. The transition probability function of the combined channel is given by

$$p_2(y_1, y_2|u_1, u_2) = p(y_1|w_1 + w_2)p(y_2|w_2)$$

for all $y_1, y_2 \in \mathcal{Y}$, and $w_1, w_2 \in \mathcal{X}$, where the addition operation is carried modulo 2. At the $i$-th step of the construction, the transition probability function $p_n$, $n = 2^i$, is defined for a combined channel with an input alphabet $\mathcal{X}^n$ and an output alphabet $\mathcal{Y}^n$. The recursive definition of $p_n$ is based on two independent copies of the channel $p_2$ defined at the previous step $(i-1)$. The channel $p_2$ has an input alphabet $\mathcal{X}^2$ and an output alphabet $\mathcal{Y}^2$. The construction of the channel $p_n$ includes the following steps:

1) An input vector $w = (w_1, \ldots, w_n) \in \mathcal{X}^n$ is first transformed to a vector $s = (s_1, \ldots, s_n) \in \mathcal{X}^n$ where

$$s_{2k-1} = w_{2k-1} + w_{2k}$$

and

$$s_{2k} = x_{2k}, \quad 1 \leq k \leq \frac{n}{2}$$

where the addition is carried modulo 2.

2) The vector $s$ is transformed into a vector $v \in \mathcal{X}^n$ where

$$v = (s_1, s_3, \ldots, s_{n-1}, s_2, s_4, \ldots, s_n),$$

i.e., the first $\frac{n}{2}$ elements of $v$, $v_1, \ldots, v_{n/2}$ equal the elements in $s$ with odd indices, and the rest $\frac{n}{2}$ elements of $v$, $v_{n/2+1}, \ldots, v_n$ equal the elements of $s$ with even indices. This operation is called a reverse shuffle operation and can be described by the linear transformation

$$v = sR_n$$

where $R_n$ is an $n \times n$ matrix, called the reverse shuffle operator.
3) $p_n(y|w)$ is given by

$$p_n(y|w) = p_i \left( \left( y_1, y_2, \ldots, y_\frac{n}{2} \right) \mid \left( v_1, v_2, \ldots, v_\frac{n}{2} \right) \right) \cdot p_i \left( \left( y_{\frac{n}{2}+1}, y_{\frac{n}{2}+2}, \ldots, y_n \right) \mid \left( v_{\frac{n}{2}+1}, v_{\frac{n}{2}+2}, \ldots, v_n \right) \right).$$

(5)

The recursive channel-synthesizing operation of $p_n$ is referred to as channel combining, and the channel $p_n$ is referred to as the combined channel. Note that the resulting block length $n$ for this construction must be a power of 2, that is $n = 2^i$ for $i \in \mathbb{N}$. Throughout this paper, all block lengths $n$ are assumed to be integral powers of 2.

The recursive construction of $p_n$ can be equivalently defined using a linear encoding operation. Let

$$F = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

and define the following recursive construction of the $n \times n$ matrices $G_n$:

$$G_1 = I_1$$

$$G_n = \left( I_{\frac{n}{2}} \otimes F \right) R_n \left( I_2 \otimes G_{\frac{n}{2}} \right)$$

(6)

where $I_l$ is the $l \times l$ identity matrix and $\otimes$ denotes the Kronecker product for matrices. The matrix $G_n$ is referred to as the polar generator matrix of size $n$.

**Proposition 1 (2)**. Let $p$ be a DMC, and let $p_n$ be the combined channel with a block length $n$. Then,

$$p_n(y|w) = p(y|wG_n)$$

(7)

for all $y \in Y^n$ and $w \in X^n$, where $p_n$ is the combined channel in (5) and $G_n$ is the $n \times n$ matrix defined in (6).

Denote by $[n] \triangleq \{1, 2, \ldots, n\}$, and let $A_n \subseteq [n]$. In addition, denote by $A_n^c$ the complementary set of $A_n$, that is $A_n^c = [n] \setminus A_n$. Given a set $A_n$, a class of coset codes with a common code-rate $\frac{1}{n}|A_n|$ are formed. Over the indices specified by $A_n$, the components of the input vector $w$ are set according to the information bit vector. The rest of the bits of $w$ are predetermined and fixed according to the particular code design. By setting both the set $A_n$ and the components of $w$ specified by $A_n^c$, a particular coset code is defined. This code can be shown to be a block coset code. The set $A_n$ is referred as the information set. Polar codes are constructed by a specific choice of the information set $A_n$. Moreover, the choice of the information set is tailored to the specific channel over which the communication takes place.

A coset code is defined using a linear block code and a coset vector. Let $G$ be a generator matrix for a binary $(n, k)$ linear block code with block length $n$ and dimension $k$. In addition, let $c \in X^n$ be a binary vector. Then, the coset block code $C(G, c)$ is defined by

$$C(G, c) \triangleq \{ x : x = uG + c, \ u \in X^k \}. $$

(8)

Denote by $G_n(A_n)$ the $|A_n| \times n$ sub-matrix of $G_n$, defined by the rows of $G_n$ whose indices are in $A_n$. Similarly, the matrix $G_n(A_n^c)$ denotes the $|A_n^c| \times n$ sub-matrix of $G_n$ formed by the remaining rows of $G_n$. For each choice of $A_n$ and an arbitrary $n-k$ binary vector $b \in X^{n-k}$, a coset code $C$ is defined according to

$$C = C\left(G_n(A_n), bG_n(A_n^c)\right).$$

(9)

This code coincides with the recursive construction in (7). To see this, plug the information vector $u$ in the information indices, specified by $A_n$, of the input vector $w$ to the recursive construction. In addition, plug the vector $b$ in the rest of the components of $w$. 
Channel splitting is another important operation that is introduced in [2] for polar coding. The split channels \( \{p_n^{(l)}\}_{l=1}^n \), with a binary input alphabet \( \mathcal{X} \) and output alphabets \( \mathcal{Y}^n \times \mathcal{X}^{l-1}, 1 \leq l \leq n \), are defined according to:

\[
p_n^{(l)}(y, w|x) = \frac{1}{2^{n-l}} \sum_{c \in \mathcal{X}^{n-l}} p_n(y|(w, x, c))
\]

where \( y \in \mathcal{Y}^n \), \( w \in \mathcal{X}^{l-1} \), and \( x \in \mathcal{X} \). The channel synthesizing operation in (10) is referred to as channel splitting operation. The Bhattacharyya parameter of \( p_n^{(l)} \) is denoted by:

\[
B(p_n^{(l)}) \triangleq \sum_{y \in \mathcal{Y}^n} \sum_{w \in \mathcal{X}^{l-1}} \sqrt{p_n^{(l)}(y, w|0)p_n^{(l)}(y, w|1)}.
\]

The construction of the sequence of sets of split channels \( \{p_n^{(l)}(y, w|x)\}_{l=1}^n \), \( n = 2^i, i \in \mathbb{N} \), in (10) can be described using the following alternative recursion:

**Proposition 2** ([2]). For all \( i > 0 \), \( 1 \leq l \leq 2^i \),

\[
p_{2^{l+1}}^{(2^l-1)}((y^{(1)}, y^{(2)}), w|w_1) = \sum_{w \in \mathcal{X}} \frac{1}{2}p_{2^l}^{(l)}(y^{(1)}, g(w)|w_1 + w)p_{2^l}^{(l)}(y^{(2)}, e(w)|w)
\]

\[
p_{2^{l+1}}^{(2^l)}((y^{(1)}, y^{(2)}), (w, w_1)|w_2) = \frac{1}{2}p_{2^l}^{(l)}(y^{(1)}, g(w)|w_1 + w_2)p_{2^l}^{(l)}(y^{(2)}, e(w)|w_2)
\]

where \( y^{(1)}, y^{(2)} \in \mathcal{Y}^{2^i} \), \( w = (w_1, \ldots, w_{2^i-2}) \in \mathcal{X}^{2^i-2} \), \( w_1, w_2 \in \mathcal{X} \), the addition operation is carried modulo 2 and \( g = (g_1, \ldots, g_{l-1}) = g(w) \) is a vector in \( \mathcal{X}^{l-1} \) defined according to

\[
g_j = w_{2j-1} + w_{2j}, \quad 1 \leq j \leq l - 1
\]

and

\[
e(w) = (w_2, w_4, \ldots, w_{2^i-2})
\]

is the vector in \( \mathcal{X}^{l-1} \) comprises from the components of \( x \) with even indices.

The importance of channel splitting is in its role in the successive cancellation decoding procedure that is provided in [2]. The error performance analysis of this decoding procedure relies on the following two results:

**Theorem 3** ([3]). Let \( p \) be a binary-input symmetric DMC with capacity \( C(p) \), and fix an arbitrary rate \( R < C(p) \) and a positive constant \( \beta < \frac{1}{2} \). Then, there exists a sequence of information sets \( \mathcal{A}_n \subset [n] \), where \( n = 2^i, i \in \mathbb{N} \), such that for large enough blocklengths \( n \) the following properties are satisfied:

1) Rate:

\[|\mathcal{A}_n| \geq nR.\]

2) Performance: The Bhattacharyya parameters in (11) satisfy

\[B(p_n^{(l)}) \leq 2^{-n^\beta}\]

for every \( l \in \mathcal{A}_n \).

**Proposition 3** ([2]). Assume that the vector \( w = (w_1, \ldots, w_n) \in \mathcal{X}^n \) is encoded via the considered recursive construction in (7), and is transmitted over a memoryless and symmetric DMC channel \( p \) with a binary-input alphabet \( \mathcal{X} \) and an output alphabet \( \mathcal{Y} \). Define the event

\[
\mathcal{E}_l(p) \triangleq \left\{ p_n^{(l)}(y, w^{(l-1)}|w_l) \leq p_n^{(l)}(y, w^{(l-1)}|w_l + 1) \right\}
\]

(16)
where $y \in \mathcal{Y}^n$ is the received vector, $w^{(l-1)} = (w_1, \ldots, w_{l-1})$ is the vector comprises of the first $l - 1$ bits of $w$, $p_n^{(l)}$ is the split channel in (10) and the addition is carried modulo 2. Then, the event $E_l$ is independent with the actual input vector $w$ and

$$\Pr(E_l(p)) \leq B(p_n^{(l)})$$

where $B(p_n^{(l)})$ is the Bhattacharyya parameter in (11).

### III. The Proposed Scheme: The Physically Degraded Case

#### A. Polar Coding for Degraded Wire-Tap Channels

**Coset Block Codes**

A polar coding scheme is defined for the wire-tap channel. The proposed scheme is defined using the notion of coset block codes, based on the polar generator matrix $G_n$ introduced in Section II-B. For a given block length $n = 2^i$, $i \in \mathbb{N}$, let $\mathcal{A}_n$ be an arbitrary subset of $[n]$ of size $k$. In addition, let $\mathcal{N}_n$ be an additional arbitrary subset of $\mathcal{A}_n^c$, of size $k^*$, and let $b_n \in \mathcal{X}^{n-k-k^*}$ be a length $n - k - k^*$ binary vector. Denote by $\mathcal{B}_n$ the set of remaining indices in $\mathcal{A}_n^c$, that is

$$\mathcal{B}_n \triangleq \mathcal{A}_n^c \setminus \mathcal{N}_n.$$  \hfill (17)

The sets $\mathcal{A}_n$, $\mathcal{B}_n$, and $\mathcal{N}_n$, the polar generator matrix $G_n$ and the vector $b_n$ are all known to both the legitimate user and the eavesdropper.

Let $u \in \mathcal{X}^k$ be a confidential information bit vector that needs to be transmitted to the legitimate user. The operation of the proposed secrecy scheme is described as follows:

1) A binary vector $b_n^* \in \mathcal{X}^{k^*}$ is chosen uniformly at random.

2) The coset block code $C_n^*$ is chosen according to

$$C_n^* = C(G_n(\mathcal{A}_n), b_n G_n(\mathcal{B}_n) + b_n^* G_n(\mathcal{N}_n)).$$ \hfill (18)

3) The information vector $u$ is encoded into a codeword $x$ using the coset block code $C_n^*$. That is,

$$x = u G_n(\mathcal{A}_n) + b_n G_n(\mathcal{B}_n) + b_n^* G_n(\mathcal{N}_n)$$ \hfill (19)

and it is transmitted over the wire-tap channel.

If the complexity of constructing a random vector is considered as $O(1)$, then the encoding complexity of the proposed scheme equals the encoding complexity of the single-user polar encoding in [2], which is $O(n \log n)$.

For given sets $\mathcal{A}_n$ and $\mathcal{N}_n$, and a vector $b_n$, the resulting coding scheme is denoted by $C_n(\mathcal{A}_n, \mathcal{N}_n, b_n)$. Since symmetric channels are considered, the performance of the provided scheme is shown in the following to be independent with the actual choice of $b_n$. Consequently, the suggested coding scheme is denoted by $C_n(\mathcal{A}_n, \mathcal{N}_n)$.

**Recursive Polar Construction**

An equivalent recursive construction of the proposed scheme is provided. Similarly to the single-user construction in (4), the first step of the recursive construction is the composition of the wiretap channel $P_2$ with an input alphabet from $\mathcal{X}^2$ and an output alphabet from $\mathcal{Y}^2 \times \mathcal{Z}^2$

$$P_2(y_1, y_2, z_1, z_2|w_1, w_2) = P(y_1, z_1|w_1 + w_2)P(y_2, z_2|w_2)$$ \hfill (20)

where $(y_1, y_2) \in \mathcal{Y}^2$, $(z_1, z_2) \in \mathcal{Z}^2$, $(w_1, w_2) \in \mathcal{X}^2$, and the addition is carried modulo 2.
The continuation of the recursive construction follows in a similar manner to the recursion in Section II-B. The transition probability function $P_n$ for a channel with an input alphabet $\mathcal{X}$ and an output alphabet $\mathcal{Y} \times \mathcal{Z}$, is constructed using two independent copies of a channel $P^\mathcal{X}_{\mathcal{Z}}$ with an input alphabet $\mathcal{X} \times \mathcal{Z}$ and an output alphabet $\mathcal{Y} \times \mathcal{Z}$. Note that as in Section II-B all block lengths ($n$) are integral powers of 2. The first part of the recursive step includes the evaluation of the vectors $s, v \in \mathcal{X}^n$. This part is identical to the construction as described in Section II-B (steps 1 and 2). Finally, the transition probability function $P_n(y|x)$ is given by

$$P_n(y, z|x) = P^\mathcal{X}_{\mathcal{Z}}(y_1, y_2, \ldots, y_{\frac{n}{2}}, z_1, z_2, \ldots, z_{\frac{n}{2}} | v_1, v_2, \ldots, v_{\frac{n}{2}}) \cdot P^\mathcal{X}_{\mathcal{Z}}(y_{\frac{n}{2}+1}, y_{\frac{n}{2}+2}, \ldots, y_n, z_{\frac{n}{2}+1}, z_{\frac{n}{2}+2}, \ldots, z_n | v_{\frac{n}{2}+1}, v_{\frac{n}{2}+2}, \ldots, v_n).$$  \hspace{1cm} (21)

The channel $P_n$ in (21) is the combined wire-tap channel.

As in the case of standard polar coding for the single-user model, the recursive construction can be shown to be equivalent to a linear encoding with the polar generator matrix $G_n$:

**Proposition 4.** Let $P$ be a binary memoryless wire-tap channel with an input alphabet $\mathcal{X}$ and output alphabets $\mathcal{Y}$ and $\mathcal{Z}$, for the legitimate user and the eavesdropper, respectively. In addition, let $P_n$ and $G_n$ be the combined wire-tap channel in (21) and the polar generator matrix in (6), respectively. Then,

$$P_n(y, z|w) = P(y, z|wG_n)$$  \hspace{1cm} (22)

for all $w \in \mathcal{X}^n$, $y \in \mathcal{Y}^n$, and $z \in \mathcal{Z}^n$.

**Proof:** The proof of (22) is identical to the proof of (10) in [2], where symbols from the output alphabet of the single user channel are replaced with the corresponding pair of symbols from the composite output alphabet (of the legitimate and the eavesdropper channels).

To obtain the equivalence of the recursive construction of the combined channel $P_n$ in (21) with the encoding operation in (19), the division of the components of $w$ in (22) for information bits, random bits and predetermined and fixed bits, is detailed. This division is defined by the sets $A_n$ and $N_n$ as follows:

1) Over the indices specified by the index set $A_n$, the information bits $u$ are placed.
2) The random bits $b^*_n$ are placed in the indices specified by $N_n$.
3) The predetermined and fixed bits in $b_n$ are left for the remaining indices specified by $B_n$.

Plugging $u$, $b^*_n$, and $b_n$ in $wG_n$, results in the coded message $x$ in (19).

**Channel Splitting and Degradation Properties**

The channel splitting operation in (10) is repeated for the case of wire-tap channels. This procedure can be carried in two different but equivalent options:

1) First performing a channel splitting operation for the wire-tap channel. This operation results in the split wire-tap channels $\{P^{(l)}_n\}_{l=1}^n$ with a binary input alphabet $\mathcal{X}$ and an output alphabet $\mathcal{Y} \times \mathcal{Z} \times \mathcal{X}^{l-1}$:

$$P^{(l)}_n(y, z, w|x) \triangleq \frac{1}{2^{n-1}} \sum_{c \in \mathcal{X}^{n-1}} P_n(y, z|(w, w, c)) \hspace{1cm} (23)$$

where $y \in \mathcal{Y}^n$, $z \in \mathcal{Z}^n$, $w \in \mathcal{X}^{l-1}$, and $w \in \mathcal{X}$. Next, deriving the marginal split channels

$$G^{(l)}_n(y, w|x) \triangleq \sum_{z \in \mathcal{Z}^n} P^{(l)}_n(y, z, w|x) \hspace{1cm} (24)$$
and

\[ Q^{(l)}_n(z, w|w) \triangleq \sum_{y \in \mathcal{Y}^n} P^{(l)}_n(y, z, w|w) \]  \tag{25}

for the legitimate-user and eavesdropper, respectively, where \( y, z, w, \) and \( w \) are as in (23).

2) First deriving the marginal combined channels:

\[ G_n(y|w) \triangleq \sum_{z \in \mathcal{Z}^n} P_n(y, z|w) \]  \tag{26}

and

\[ Q_n(z|w) \triangleq \sum_{y \in \mathcal{Y}^n} P_n(y, z|w) \]  \tag{27}

for the legitimate user and eavesdropper, respectively, where \( y \in \mathcal{Y}^n, z \in \mathcal{Z}^n, \) and \( w \in \mathcal{X}^n. \) Next, split the marginal combined channels in (26) and (27) according to

\[
\frac{1}{2^{n-1}} \sum_{c \in \mathcal{X}^{n-i}} G_n(y|(w, w, c)).
\]  \tag{28}

and

\[
\frac{1}{2^{n-1}} \sum_{c \in \mathcal{X}^{n-i}} Q_n(z|(w, w, c)).
\]  \tag{29}

where \( y, z, w, \) and \( w \) are as in (23).

It is an immediate consequence of the equivalence properties in (7) and (22), that the split channels in (24) and (25) equal to the channels in (28) and (29).

The following proposition considers physically degraded wire-tap channels:

**Proposition 5.** Assume that the wire-tap channel \( P \) is physically degraded. Then, the split channel \( P^{(l)}_n(y, z, w|x) \) in (23) satisfies

\[ P^{(l)}_n(y, z, w|x) = G^{(l)}_n(y, w|x)D(z|y) \]  \tag{30}

where \( G^{(l)}_n \) is the marginal split channel of the legitimate user in (24), \( y = (y_1, \ldots, y_n) \in \mathcal{Y}^n, z = (z_1, \ldots, z_n) \in \mathcal{Z}^n, u \in \mathcal{X}^{l-1}, x \in \mathcal{X}, \) \( D(z|y) \) is a memoryless transition probability law:

\[ D(z|y) = \prod_{l=1}^n D(z_i|y_i) \]

and \( D(z|y) \) is the conditional probability law of receiving a symbol \( z \in \mathcal{Z} \) at the eavesdropper, assuming that the symbol \( y \in \mathcal{Y} \) is received at the legitimate receiver.

**Proof:** The recursion operation in Proposition 2 is valid for the wire-tap channel. Specifically, for all \( i > 0 \) and \( 1 \leq l \leq 2^i \) it follows that

\[
P^{(2l-1)}_{2^{i+1}}((y^{(1)}, y^{(2)}), (z^{(1)}, z^{(2)}), w_1|w_1) = \frac{1}{2} P^{(l)}_2(y^{(1)}, z^{(1)}, g(w_1)|w_1 + w) P^{(l)}_2(y^{(2)}, z^{(2)}, e(w)|w) \]  \tag{31}

\[
P^{(l)}_{2^{i+1}}((y^{(1)}, y^{(2)}), (z^{(1)}, z^{(2)}), (w, w_1)|w_2) = \frac{1}{2} P^{(l)}_2(y^{(1)}, z^{(1)}, g(w_1)|w_1 + w_2) P^{(l)}_2(y^{(2)}, z^{(2)}, e(w)|w_2) \]  \tag{32}

where \( y^{(1)}, y^{(2)} \in \mathcal{Y}^{2^i}, z^{(1)}, z^{(2)} \in \mathcal{Z}^{2^i}, w \in \mathcal{X}^{2i-2}, w_1, w_2 \in \mathcal{X}, \) and \( g(w) \) and \( e(w) \) are as defined in (14) and (15), respectively. The proof of the recursion property in (31) and (32) follows the exact derivation as in (2) (while replacing the output alphabet of the single-user channel with the combined outputs of the legitimate user
and the eavesdropper).

From (26), (31), and (32), a similar recursion follows for the marginal split channel $G_{n}^{(l)}(y, w|x)$ of the legitimate user. To this end, the recursion operations in (12) and (13) are satisfied where $p_{2i+1}^{(2i)}$, $p_{2i}^{(l)}$ and $p_{2i+1}^{(2i)}$ are replaced by $G_{2i+1}^{(2i-1)}$, $G_{2i}^{(l)}$ and $G_{2i+1}^{(2i)}$, respectively.

The proof of the degradation in (30) is accomplished by induction. At the first step, from (31) and (32) it follows that

$$P_{2}^{(l)}((y_{1}, y_{2}), (z_{1}, z_{2})|w_{1}) = \sum_{w \in A} \frac{1}{2} P(y_{1}, z_{1}|w_{1} + w) P(y_{2}, z_{2}|w)$$

(33)

$$P_{2}^{(2)}((y_{1}, y_{2}), (z_{1}, z_{2}), w_{1}|w_{2}) = \frac{1}{2} P(y_{1}, z_{1}|w_{1} + w_{2}) P(y_{2}, z_{2}|w_{2}).$$

(34)

Then, plugging (2) in (33) and (34) concludes the proof for the first step. Next, assume that the split channel $P_{2}^{(l)}$ satisfies the degradation property in (30). That is, assume that

$$P_{2}^{(l)}(y, z, w'|w) = G_{2i}^{(l)}(y, w'|w)D(z|y)$$

(35)

for all $1 \leq l \leq 2^{i}$, $y \in Y^{2i}$, $z \in Z^{2i}$, $w' \in A^{l-1}$, and $w \in A$. Then, from (31) and (35) it follows that

$$P_{2i+1}^{(2i-1)}((y^{(1)}, y^{(2)}), (z^{(1)}, z^{(2)}), w|w_{1}) = \sum_{w \in A} \frac{1}{2} G_{2i}^{(l)}(y^{(1)}, g(w)|w_{1} + w) D(z^{(1)}|y^{(1)})$$

$$= G_{2i+1}^{(2i-1)}((y^{(1)}, y^{(2)}), w|w_{1})$$

where the last step follows using the recursion properties of the marginal split channel for the legitimate user. A similar argument assures the degradation property for $P_{2i+1}^{(2i)}$, which concludes the proof of the proposition.

Successive Cancellation Decoding

The successive cancellation decoding procedure in [2] is applied for the legitimate user. The difference from the standard single-user case is that for the wire-tap channel model the legitimate user needs to decode both the message $u \in A^{k}$ and the noisy vector $b_{n}^* \in A^{k*}$. In terms of information sets, the legitimate receiver operates on the indices specified by both $A_{n}$ and $N_{n}$. Denote by $w = (w_{1}, \ldots, w_{n}) \in A^{n}$ the transmitted vector over the combined channel $P_{n}$, then $w$ is composed from the information vector $u$, the random vector $b_{n}^*$, and the predetermined fixed vector $b_{n}$. It is important not to confuse $w$ with the actual codeword $x$ in (19), which is transmitted over the given wire-tap channel $P$. Both interpretations are equivalent as the coset block code is equivalent to the recursive combining construction. Nevertheless, the decoding rule (and its performance analysis in the following) is characterized in terms of the vector $w$, transmitted over the combined wire-tap channel and received over the marginal split channels for the legitimate user.

The decoding rule operates recursively to compute the length-$n$ decoded vector $\hat{w} = (\hat{w}_{1}, \ldots, \hat{w}_{n}) \in A^{n}$. Let $1 \leq l \leq n$, and assume that the first $l-1$ components of $\hat{w}$, denoted by $\hat{w}^{(l-1)}$, are already evaluated. If $l \notin A_{n}$, where

$$\bar{A}_{n} \triangleq A_{n} \cup N_{n},$$

then the current index $l$ is not in the information index set $A_{n}$ and not in the indices specified in $N_{n}$ for the noisy vector. Consequently, $l \in B_{n}$. Recall that for the indices specified by $B_{n}$, the predetermined vector $b_{n}$ is set. Since
b_n is predetermined and known (both to the legitimate user and the eavesdropper), w_l is known at the receiver and therefore it is possible to set

\[ \hat{w}_l = w_l. \]

If \( l \in \tilde{A}_n \), then the current index is identified either as an information bit in u or as a noisy bit in \( b_n^* \). For this case, the following decoding rule is applied to the marginal split channel \( G_n^{(l)} \) in (24):

\[ \hat{w}_l = \begin{cases} 0 & \text{if } G_n^{(l)}(y, \hat{w}^{(l-1)}|0) \geq G_n^{(l)}(y, \hat{w}^{(l-1)}|1) \\ 1 & \text{else} \end{cases} \]

(36)

The successive cancellation decoding described in this section, is by no mean optimal. This important observation is already noted for the single-user case in [2]. Nevertheless, for an uncoded communication model with a communication channel whose transition probability function is \( G_n^{(l)} \), the detection rule for the single bit \( w_l \) in (36) is optimal, if \( w_l \) is an equiprobable bit.

B. A Secrecy Achieving Property for Degraded Channels

**Theorem 4.** Let \( P \) be a binary-input, memoryless, degraded and symmetric wire-tap channel with a secrecy capacity \( C_s(P) \). Fix an arbitrary positive \( \beta < \frac{1}{2} \), and \( R < C_s(P) \). Then, there exist sequences of sets \( A_n \) and \( N_n \) such that the secrecy coding scheme \( C_n(A_n, N_n) \) satisfies the following properties:

1) Rate: For a sufficiently large block length \( n \)

\[ R \leq \frac{1}{n}|A_n|. \]

(37)

2) Security: The equivocation rate \( R_e(C_n(A_n, N_n)) \) satisfies

\[ \lim_{n \to \infty} R_e(C_n(A_n, N_n)) \geq R. \]

(38)

3) Reliability: The average block error probability under successive cancellation decoding \( P_e(C_n(A, N)) \) satisfies

\[ P_e(C_n(A_n, N_n)) = o\left(2^{-n^\beta}\right). \]

Proof:

The proof comprises of three parts: A code construction part where the construction of the sets \( A_n \) and \( N_n \) is described in details, along with the derivation of the coding rate property in (37). An analysis of the equivocation rate is provided in the second part of the proof. Finally, in the third part an upper bound on the block error probability at the legitimate receiver is provided under successive cancellation decoding.

**Part I: The code construction**

Fix some \( r^* = C(P_Z|X) - \epsilon \), and \( r = C(P_Y|X) - \epsilon \), where \( C(P_Y|X) \) and \( C(P_Z|X) \) are the channel capacities of the marginal channels for the legitimate user and the eavesdropper, and \( \epsilon > 0 \) is determined later. According to Theorem [3] there exists a sequence of index sets \( \tilde{N}_n \subset [n] \), satisfying:

1) The cardinality of the index set \( \tilde{N}_n \) satisfies

\[ |\tilde{N}_n| \geq |nr^*|. \]

(39)

2) For all \( l \in \tilde{N} \), the Bhattacharyya parameter \( B(Q_n^{(l)}) \) of the split channel \( Q_n^{(l)} \) of the eavesdropper in (25), is upper bounded by

\[ B(Q_n^{(l)}) \leq 2^{-n^\beta}. \]

(40)
The index set $\mathcal{N}_n$ of size $\lceil nr^* \rceil$ is chosen arbitrary from $\tilde{\mathcal{N}}_n$.

Next, Theorem 3 is applied for the marginal channel of the legitimate user. Accordingly, there exists a sequence of index sets $\tilde{\mathcal{A}}_n \subset [n]$, satisfying:

1) The cardinality of the index set $\tilde{\mathcal{A}}_n$ satisfies
   \[ |\tilde{\mathcal{A}}_n| \geq \lceil nr \rceil. \tag{41} \]

2) For all $l \in \tilde{\mathcal{A}}_n$, the Bhattacharyya parameter $B(G_n^{(l)})$ of the split channel $G_n^{(l)}$ of the legitimate user in (28), is upper bounded according to
   \[ B(G_n^{(l)}) \leq 2^{-n^s}. \tag{42} \]

For each $n$, the information index set $\mathcal{A}_n$ of size $\lceil nr \rceil - \lceil nr^* \rceil$ is chosen from $\tilde{\mathcal{A}}_n \setminus \mathcal{N}_n$. As $|\mathcal{N}_n| = \lceil nr^* \rceil$ and $|\tilde{\mathcal{A}}_n| \geq \lceil nr \rceil$, the set $\tilde{\mathcal{A}}_n \setminus \mathcal{N}_n$ is of sufficient size. The specific choice of $\mathcal{A}_n$ may be carried arbitrarily. Nevertheless, the best choice is to pick the indices in $\tilde{\mathcal{A}}_n \setminus \mathcal{N}_n$ whose corresponding marginal split-channels for the legitimate-user have the lowest Bhattacharyya parameters.

The code rate of the resulting scheme satisfies
\[
\frac{1}{n} |\mathcal{A}_n| \geq \frac{r - 1}{n} - \frac{r^* - 1}{n}
= C(P_{Y|X}) - C(P_{Z|X}) - 2\epsilon - \frac{2}{n}
= C_s(P) - 2\epsilon - \frac{2}{n} \tag{43}
\]
where the last equality follows since the message bit vector is of length $|\mathcal{A}_n|$ and equiprobable. Using the chain rule of mutual information
\[
I(U, X; Z) = I(U; Z) + I(X; Z|U)
= I(X; Z) + I(U; Z|X).
\]
Consequently,

\[ I(U; Z) = I(X; Z) + I(U; Z|X) - I(X; Z|U) \]

\[ \leq (a) I(X; Z) - I(X; Z|U) \]

\[ \leq nC(P_{Z|X}) - I(X; Z|U) \]  

(45)

where \((a)\) follows since \(U \rightarrow X \rightarrow Z\) is a Markov chain which implies that \(Z\) and \(U\) are statistically independent given \(X\), and \(C(P_{Z|X})\) is the channel capacity of the marginal channel to the eavesdropper. The conditional mutual information \(I(X; Z|U)\) is given by

\[ I(X; Z|U) = H(X|U) - H(X|U, Z) \]

\[ \geq (a) |\mathcal{N}_n| - H(X|U, Z) \]

\[ \geq n(C(P_{Z|X}) - \epsilon) - 1 - H(X|U, Z) \]  

(46)

where \((a)\) follows since the binary vector \(b^*\) is chosen uniformly at random and it is independent with the confidential message, and \((b)\) follows since \(|\mathcal{N}_n| = \lfloor nr^* \rfloor\) and \(r^* = C(P_{Z|X}) - \epsilon\).

Let \(P_{e|U}\) denote the error probability of a decoder that needs to decode \(X\) while having access to both the eavesdropper observation vector \(Z\), the confidential message vector \(U\), and the predetermined vector \(b_n\) (which is fixed, predetermined, and known to all the users in the model). Note that if both the confidential message \(U\) and the predetermined vector \(b_n\) are known at the receiver, then the remaining uncertainty in the codeword \(X\) relates only to the random vector \(b_n^*\) of size \(N_n\). Using Fano’s inequality (see, e.g., [5]), the conditional entropy \(H(X|U, Z)\) is bounded according to

\[ H(X|U, Z) \leq h_2(P_{e|U}) + P_{e|U} \log(2^{|\mathcal{N}_n|} - 1) \]

\[ \leq h_2(P_{e|U}) + nr^*P_{e|U} \]  

(47)

where \(h_2(x) \triangleq -x \log x - (1 - x) \log(1 - x)\) is the binary entropy function. From (44)-(47) it follows that

\[ R_e(\mathcal{C}(A, \mathcal{N})) \geq \frac{1}{n} |\mathcal{A}_n| - \epsilon - \frac{1}{n} h_2(P_{e|U}) + nr^*P_{e|U} \]

\[ \geq R - \frac{1}{n} h_2(P_{e|U}) + nr^*P_{e|U} \]  

(48)

where the last inequality follows from (16) for a sufficiently small \(\epsilon\) and a sufficiently large \(n\). The error probability \(P_{e|U}\) in (49) can be upper bounded by the error probability under the suboptimal successive cancellation decoder in (2), which is fully informed with both the predetermined vector \(b_n\) and the confidential message vector \(U\). It follows from (3) that

\[ P_{e|U} \leq o(2^{-n^a}) \]

which concludes the proof of (38).

**Part III: The error performance at the legitimate decoder**

The successive cancellation decoding procedure at the legitimate receiver is analyzed. First, fix a vector \(w = (w_1, \ldots, w_n) \in \mathcal{X}^n\) comprises of the information message \(u \in \mathcal{X}^k\), the randomly chosen vector \(b^* \in \mathcal{X}^{k^*}\), and the predetermined vector \(b \in \mathcal{X}^{n-k-k^*}\). The conditional block error probability is denoted by \(P_{e|w}\). That is, \(P_{e|w}\) is the probability of a block error event given that the input vector is \(w\). Denote by \(w^{(l)} = (w_1, \ldots, w_l)\) the first \(l\)
bits of \( \mathbf{w} \), and by \( \hat{\mathbf{w}}^{(l)} = (\hat{w}_1, \ldots, \hat{w}_l) \) the first \( l \) decoded bits. The event
\[
\mathcal{F}_l \triangleq \left\{ \mathbf{w}^{(l-1)} = \hat{\mathbf{w}}^{(l-1)}, \ w_l \neq \hat{w}_l \right\}
\]
corresponds to the case where the first \( l - 1 \) bits of \( \mathbf{w} \) are decoded correctly and the first decoding error is in the \( l \)-th bit. Notice that
\[
\mathcal{F}_l \subset \mathcal{E}_l(G_n^{(l)})
\]
where \( \mathcal{E}_l \) is the event defined in (16), and \( G_n^{(l)} \) is the marginal split channel in (24). Consequently, it follows using the union bound that
\[
P_{e|w} = \Pr\left( \bigcup_{l=1}^n \mathcal{F}_l \mid \mathbf{w} \right) \leq \sum_{l \in \mathcal{A}_n} \Pr\left( \mathcal{E}_l(G_n^{(l)}) \mid \mathbf{w} \right). \tag{50}
\]

Next, the summation in (50) is split to two summations: a summation over the indices in \( \mathcal{A}_n \) and a summation over the indices in \( \mathcal{N}_n \). For an index \( l \in \mathcal{A}_n \), it follows from Proposition 3 that for all \( \mathbf{w} \in \mathcal{A}_n \)
\[
\Pr\left( \mathcal{E}_l(G_n^{(l)}) \mid \mathbf{w} \right) \leq B(G_n^{(l)}) \tag{51}
\]
where \( B(G_n^{(l)}) \) is the Bhattacharyya parameter in (11). To address the probability of the event \( \mathcal{E}_l(G_n^{(l)}) \) where \( l \in \mathcal{N}_n \), notice that at the output of the marginal split channel, the decoding rule for \( w_l \) in (36) is optimal\(^1\). Recall the degradation property in Proposition 5. According to Proposition 5 the marginal split channel of the eavesdropper is physically degraded with respect to the marginal split channel of the legitimate user. Consequently, it is clearly suboptimal to first degrade the observations at the split channel of the legitimate user, and only then to detect the bit \( w_l \) over the corresponding marginal split channel of the eavesdropper. Specifically, \( w_l \) is detected according to
\[
\hat{w}_l = \begin{cases} 0 & \text{if } Q_n^{(l)}(z, \hat{w}^{(l-1)}|0) \geq Q_n^{(l)}(z, \hat{w}^{(l-1)}|1) \\ 1 & \text{else} \end{cases}
\]
where \( z \in \mathcal{Z}^n \) is a degraded version of \( y \in \mathcal{Y}^n \), randomly picked according to the probability law \( D(z|y) \) in (30). This detection rule is inferior with respect to (36). Hence, based on Proposition 5 the upper bound
\[
P_{e|w} \Pr\left( \mathcal{E}_l(G_n^{(l)}) \mid \mathbf{w} \right) \leq B(Q_n^{(l)}) \tag{52}
\]
holds for all \( l \in \mathcal{N}_n \). From (50), (51), and (52), it follows that the average block error probability is upper bounded by
\[
P_e(\mathcal{C}_n(\mathcal{A}, \mathcal{N})) \leq \sum_{l \in \mathcal{A}_n} B(G_n^{(l)}) + \sum_{l \in \mathcal{N}_n} B(Q_n^{(l)}).
\]
The proof concludes using the bound on the polarization rate of the Bhattacharyya parameter in Theorem 3 and the specific choice of the sets \( \mathcal{A}_n \) and \( \mathcal{N}_n \). \( \blacksquare \)

**Remark 3** *(On communicating with full capacity).* The noisy bits \( b^*_n \), defining the coset block code \( C^*_n \) based on the noisy index set \( \mathcal{N}_n \) (see eq. (18)), are reliably detected by the legitimate user. It is therefore suggested to utilize these bits in order to communicate with the legitimate user. That is, instead of setting the bits in \( b^*_n \) to noisy random bits, non-secret information bits are suggested to be set on \( b^*_n \). The non-secret information bits must be statistically independent and equiprobable. In addition, the non-secret information must be statistically independent

\(^1\text{As stated, this optimality is only under the setting of the split channel, and by no means implies optimality of the complete procedure (which is clearly suboptimal).} \)
with the secret-information. These statistical properties allows the non-secret information bits to act as if they are noisy bits (where the eavesdropper is concerned). As a result of the cardinality of the index set $A_n$ (41), the overall rate, including secret and non-secret information, is arbitrarily close the full (marginal) channel capacity of the legitimate user $C(P_{Y|X})$.

Remark 4 (The noisy bits must not be fixed). It is important to note that the bits in $b_n^*$ must be chosen at random for each block transmission. To see this, first note (based on the data processing inequality) that

$$\frac{1}{n}I(b_n^*; Z) \leq \frac{1}{n}I(X; Z)$$

for all $n > 0$. Assuming that (53) is satisfied with equality. It follows that both the legitimate user and the eavesdropper can reliably decoded the vector $b_n^*$. Considering the current setting as if it is a broadcast communication problem over the given channel, a broadcast scheme is therefore provided where we can reliably communicated with the legitimate user at a rate arbitrarily close to its marginal capacity $C(P_{Y|X})$ and at the same time with the eavesdropper at a (common) rate which is arbitrarily close to $\frac{1}{n}I(X; Z)$. This violates the fundamental limit imposed by the capacity region of the degraded broadcast channel (see, e.g., [5]). Consequently, it follows that

$$\frac{1}{n}I(b_n^*; Z) < \frac{1}{n}I(X; Z)$$

for all $n > 0$. Next, since there is a one-to-one correspondence between the transmitted codeword $X$ and the vector pair which is comprised of the random bits $b^*$ and the confidential message $U$ (the vector $b$ is predetermined and fixed), it follows that

$$\frac{1}{n}I(X; Z) = \frac{1}{n}I(U, b^*; Z)$$

$$= \frac{1}{n}I(b^*; Z) + \frac{1}{n}I(U; Z|b^*)$$

for all $n > 0$, where (a) follows by the chain rule of mutual information. Hence it is observed from (54) and (55) that

$$\frac{1}{n}I(U; Z|b^*) > 0$$

for all $n$. This assures that if the vector $b^*$ is known to the eavesdropper, for example by choosing a fixed $b^*$, perfect secrecy can not be established, not even in the weak sense.

It is observed in [6], that if $(R_1, R_1)$ is an achievable rate-equivocation pair and in addition, an additional information rate $R_2$ is achievable without secrecy (that is, in the ordinary notion of reliable communication), then the $(R_1 + R_2, R_1)$ is also an achieved rate-equivocation pair. The other direction is also provided in [6, p. 411]. Following Remark 4 which suggests the option of communicating in full rate, and the observations in [6], it is expected that the entire rate-equivocation region is obtained with polar coding. This result is provided in the following corollary:

Corollary 1 (The entire rate-equivocation region is achievable with polar codes). Under the assumptions and notation in Theorem 4 the entire rate-equivocation region is achievable with polar coding.

**Proof:** Take a rate-equivocation pair $(R, R_e)$ in the rate-equivocation region defined in (3). Define $R_1 = R_e$, and $R_2 = R - R_1$. Note that $R_2 \geq 0$ as $R_e \leq R$. Consider the coset block code in (18). Since $R_e \leq C_s(P)$, the rate $R_1$ is achievable via the index set $A_n$. It is further detailed in the proof of Theorem 4 that the information transmitted via the indices in $A_n$ is secure. Specifically, it follows from (48) that the equivocation rate is arbitrarily
close to \( \frac{1}{n} |A_n| \). As explained in Remark 3, reliable communication (not necessarily secure) of an additional rate of up to the capacity \( C(P_Y|X) \) of the marginal channel to the legitimate user, is achievable. Therefore, the additional rate \( R_2 \), is achievable either via the remaining indices in \( A_n \) and the vector \( b_n^* \) corresponding to the indices in \( N_n \).

C. Secrecy Achieving Properties for Erasure Wiretap Channels

In this section, a particular case of binary erasure wiretap channel is considered. Specifically, it is assumed that the channel to the legitimate user is noiseless, and the channel to the eavesdropper is a binary erasure channel (BEC) with an erasure probability \( \delta \), is considered. Recall that the set sequence \( N_n \) of the indices that correspond to “good” split channel to the eavesdropper, is chosen as to achieve the capacity to the eavesdropper. As the channel to the legitimate user is noiseless, that is \( y = x \), the set sequence \( A_n \) and is set according to

\[
A_n \triangleq [n] \setminus N_n. \tag{56}
\]

Note that for this particular case \( B_n = \emptyset \). The resulting coding scheme is then a particular case of the coset coding scheme in [12] where the base code is determine by the generator matrix \( G_n(N_n) \) and the actual coset is determined by \( uG_n(A_n) \) where \( u \) is the transmitted information bits (the secret message) and \( G_n \) is the polar generator matrix for a block length \( n \). Specifically, the codeword \( x \) is given, based on (19), by

\[
x = uG_n(A_n) + b_n^*G_n(N_n). \tag{57}
\]

The rate and reliability properties in this particular case follows immediately as a result of Theorem 4. That is, the rate approaches the secrecy capacity, which in this case equals \( \delta \), and the legitimate user obviously can decode the transmitted message. As in the second part of the proof of Theorem 4, the confidential message vector, the transmitted codeword, and the received vector at the eavesdropper are denoted by the random vectors \( U \), \( X \), and \( Z \), respectively. The following lemma address the entropy measure \( H(U|Z) \).

**Lemma 1.** Under the assumption and notation for the consider binary erasure wiretap channel, the entropy \( H(U|Z) \) satisfies

\[
H(U|Z) \geq n\delta(1 - c2^{-n^3})
\]

where \( \delta \) is the erasure probability of the wiretap channel, and \( c > 0 \).

**Proof:** Let us fix a particular realization of the channel erasure sequence \( \mathcal{D} \). Denote by \( \mathcal{D} \) the set of \( \mu \) indices which are not erased. That is, the eavesdropper received the bits \( X_i \) for every \( i \in \mathcal{D} \), and erasure symbols for every index in \( \mathcal{D}^c \triangleq [n] \setminus \mathcal{D} \). Consider the \( |N_n| \times n \) matrix \( \{G_n(N_n)\} \). As the generator matrix \( G_n \) for the polar construction has a full rank (for every \( n \) in the construction), the matrix \( G_n(N_n) \) has a rank \( N_n \). Therefore, it is a generator matrix for a binary linear block code of dimension \( N_n \). This code has a parity check matrix of size \( |A_n| \times n \), denoted by \( H_n \) (recall that \( A \) is given by (56)). Since all the information bits are equiprobable, and all the noisy bits are also equiprobable, the codeword \( X \), given by (19), id uniformly distributed over all possible binary vectors in \( \{0, 1\}^n \). Consequently, all the bits in \( X \) are independent and identically distributed uniform binary random variables. Hence, \( H(X|Z) = n - \mu \). In addition, note that if the codeword \( X \) is known, then information

\footnotetext{2}{This case is studied in [12], and some parts of the provided proof are based on proper presentation of the techniques developed in [12] for the case at hand.}
bits $U$ are fully determined for the considered polar coding scheme. It follows that

$$H(U|Z) = H(U|X, Z) + H(X|Z) − H(X|U, Z)$$  \hfill (58)

$$= m − \mu − H(X|U, Z).$$  \hfill (59)

Note that (58) is a restatement of [12, Eq. (5)], and (59) is a restatement of [12, Eq. (6)].

Next, fix a realization $Z = z \in \{0, 1\}^n$ and $U = u \in \{0, 1\}^{|A_n|}$. From (57), it follows that the erased bits $\{X_i\}_{i \in Dc}$ satisfies the linear equations

$$\sum_{i \in D} X_i (H_n)_i = H_n u G_n (A_n) + \sum_{i \in Dc} X_i (H_n)_i$$  \hfill (60)

where $(H_n)_i$ is the $i$-th column of the parity check matrix $H_n$. The number of solutions to (60) is given by

$$2^{n−\mu−d\left(\{(H_n)_i\}_{i \in D}\right)}$$

where $d\left(\{(H_n)_i\}_{i \in D}\right)$ is the dimension of the linear space spanned by the the column vectors in $\{(H_n)_i\}_{i \in D}$. Since all the solutions for the erasures $X_i, i \in D,$ are equally likely, it follows that

$$H(X|U = u, Z = z) = n − \mu − d\left(\{(H_n)_i\}_{i \in D}\right).$$  \hfill (61)

From (59) and (61), it follows that

$$H(U|Z) = Ed\left(\{(H_n)_i\}_{i \in D}\right).$$  \hfill (62)

As the information indices $N_n$ for the eavesdropper are chosen such that it can decode the noisy bits $b^*$ with an error probability of $O(2^{−n\delta})$, it follows that

$$H(U|Z) \geq E\left(d\left(\{(H_n)_i\}_{i \in D}\right), \text{correct decoding}\right)\left(1 − c2^{−n\delta}\right)$$  \hfill (63)

$$= n\delta\left(1 − c2^{−n\delta}\right)$$  \hfill (64)

where $c > 0$ and $\delta$ is the erasure probability of the eavesdropper channel.

\begin{remark}
\textbf{(All coset must be equally likely).} In the current discussion, the secrecy polar coding scheme is applied with $B_n = \emptyset$. This fact is crucial for the proof of Lemma [1] It is conjectured that this choice may be crucial to achieve the entire secrecy capacity under the strong secrecy condition.
\end{remark}

\begin{remark}
\textbf{(On possible stronger notion of secrecy).} Consider the conditions in Theorem [3] In particular, not that the rate $R < C(p)$ is kept fixed for the polarization structure of the code. If, it be possible to construct the sequence of polar codes, with a sequence of blocklength dependent rates $R_n$ having the property that

$$R_n \geq C(p) − \frac{\alpha}{n^\gamma}$$  \hfill (65)

where $\alpha > 0$ and $\gamma > 1$ are arbitrarily fixed parameters. Then, it will follow as a corollary of Lemma [1] that a strong notion of secrecy is guaranteed. That is, the entropy $H(U|Z)$ is arbitrarily close to $H(U).$ To see this, note that if polarization is possible while satisfying (65), it follows that

$$|N_n| \geq n\left(1 − \delta − \frac{\alpha}{n^\gamma}\right).$$

Consequently,

$$H(U) = |A_n| = n − |N_n| = n\delta + \frac{\alpha}{n^{1−\gamma}}.$$
Hence $H(U|Z)$ is lower bounded by a quantity which is arbitrarily close $H(U)$ as the blocklength increases. For the particular case of the BEC, it follows from [2, Eq. (34)-(35)], that the considered question requires the analysis of the following sequence

$$\{|i \in [n] : Z_n^i \leq C e^{n\delta}\}$$

where $\{Z_n^i\}_{i \in [n]}$ is a sequence, generated recursively according to

$$Z_{2k}^{(2i-1)} = 2Z_k^{(i)} - \left(Z_k^{(i)}\right)^2$$

$$Z_{2k}^{(2i)} = \left(Z_k^{(i)}\right)^2,$$

where $i \in [k]$ and $Z_1^{(1)} = \delta$.

IV. AN OPEN POLARIZATION PROBLEM AND THE GENERAL WIRETAP CHANNEL

An open polarization problem is presented in addition to a conjecture which suggests a possible solution. A polar secrecy scheme for non-degraded wiretap channels is provided based on suggested conjecture.

A. On the polarization of the ‘bad’ indices

Let $W = (W_1, \ldots, W_n)$ be a random vector, where $\{W_i\}_{i=1}^n$ are statistically independent and equiprobable $\Pr(W_i = 0) = \Pr(W_i = 1) = \frac{1}{2}$ for all $i \in [n]$. The random vector $W$ is polar encoded to a codeword $X = G_n W$, where $G_n$ is the polar generator matrix of size $n$. The codeword $X$ is transmitted over a binary input DMC $p$, whose output alphabet is $Y$. The received vector is denoted by $Y = (Y_1, \ldots, Y_n)$. For a given vector $W$ and a set $A \subseteq [n]$, the following notation is used

$$W_A \triangleq (W_{i_1}, \ldots, W_{i_{|A|}})$$

where $i_1 < i_2 < \ldots < i_{|A|}$ and $i_k \in A$ for all $k \in [|A|]$. Define the following quantities of mutual information

$$I_i \triangleq I(W_i; W_{[i-1]}, Y), \quad i \in [n].$$

(66)

The following polarization of mutual information is the key result in [2], [3]:

**Theorem 5 (On the polarization of mutual information [2])**. Assume that $p$ is a binary-input output-symmetric DMC whose capacity is $C(p)$, and fix $0 < \delta < 1$. Then,

$$\lim_{n \to \infty} \left(\frac{1}{n} \left|\left\{i \in [n] : I_i \in (1-\delta, 1]\right\}\right|\right) = C(p)$$

$$\lim_{n \to \infty} \left(\frac{1}{n} \left|\left\{i \in [n] : I_i \in [0, \delta)\right\}\right|\right) = 1 - C(p).$$

Denote by $A_n$ the set of indices for which the corresponding mutual information quantities $I_i$, $i \in A_n$, are arbitrarily close to 1 bit (for a sufficiently large $n$). The set $A_n$ is called the information index set. This is the very same index set in Theorem [3] of ‘good’ split channels whose corresponding Bhattacharyya constants approach 0. Let $A'_n \subset A_n$ and let $S_n \subset A'_n$. We define the index sets

$$D_n \triangleq A'_n \cup S_n$$
and

\[ D^{(i)}_n \triangleq \{ j \in D_n : j < i \}, \quad i \in [n]. \]

A problem of interest lies in the \(|D_n|\) quantities of mutual information:

\[ J_i \triangleq I(W_i; W_{D^{(i)}_n}, Y), \quad i \in D_n. \] (67)

For the indices in \(A'_n\) a straightforward answer is provided:

**Lemma 2 (on the indices of ‘good’ split channels).** Fix a \(0 < \delta < 1\) and an index \(i \in A'_n\). For sufficiently large \(n\)

\[ J_i \geq 1 - \delta. \]

**Proof:** As the mutual information \(I_i\) in (66) includes a subset of the random variables in \(J_i\) in (67), it follows that

\[ J_i \geq I_i. \]

The proof concludes using Theorem 5 as \(A'_n \subset A_n\). \(\blacksquare\)

According to Lemma 2, ‘good’ indices for which the mutual information quantities \(I_i\) approach 1 bit, remain ‘good’ with respect to the mutual information \(J_i\). The characterization of the ‘bad’ indices seems at this point to be a greater challenge. A conjecture for possible polarization properties of the mutual information quantities \(J_i\) in (67) is provided for the (‘bad’) indices in \(S_n\). Two possible polarization properties are considered:

**Conjecture 1 (On possible polarization dichotomy).** Fix a \(0 < \delta < 1\). There exists a partition of \(S_n\) to two sets \(S'_n\) and \(S''_n = S_n \setminus S'_n\), such that for a sufficiently large \(n\)

\[ J_i < \delta, \quad \text{for all } i \in S'_n \]

\[ J_i > 1 - \delta, \quad \text{for all } i \in S''_n. \] (68) (69)

**Remark 7 (On degenerated and non-degenerated possible partitions).** One of the possible option resulting from Conjecture 1 is that \(S'_n = S_n\). In case where this degenerated partition is proved to be correct, then it follows that the additional information provided by the bits in \(W_{D_n}\) do not alter the known polarization of the mutual information quantities \(I_i\) in (66). The non-degenerated partition of \(S_n\) offers (in the case it is proven to be correct) a dichotomy of the indices in \(S_n\). Accordingly, either the former polarization remains or alternatively the knowledge of the bits in \(W_{D_n}\) completely changes the orientation of the polarization. The size of \(S''_n \cup A'_n\) must satisfy

\[ |S''_n \cup A'_n| \overset{(a)}{=} |A'_n| + |A''_n| \overset{(b)}{\leq} nC(p). \] (70)

Equality (a) in (70) is obvious as the sets \(A'_n\) and \(S_n\) are disjoint. Violating the inequality (b) in (70) results in violating the coding theorem for a DMC as the input bits to the split channels specified by the set \(S''_n \cup A'_n\) can be reliably decoded (This can be shown in a similar fashion as in [2]).

**Remark 8 (On a particular trivial case where Conjecture 1 is true).** There exists an option where Conjecture 1 is trivially proved as a particular application of Theorem 5. Specifically, assume that for every index \(i \in D_n\), it follows that

\[ j < i \quad \forall j \in D'_n. \]

In that case, the degenerated partition in Remark 7 follows as an immediate particular case of Theorem 5.
B. A polar secrecy scheme

In this section, a polar secrecy scheme is provided assuming that Conjecture 1 is true. The same notation and definitions of the coset code defined in Section III-A are assumed. The transmitted codeword $x$ is defined in (19). This definition is based on the index sets $A_n$ and $N_n$. The secure information bits are considered as if they are being transmitted over the split channels whose indices are in $A_n$. Over the split channels whose indices are in $N_n$, noisy bits are attributed. The polar secrecy scheme is provided in Section III by a proper choice of the sets $A_n$ and $N_n$. The degradation property in Section III assures that the indices which correspond to split channels which polarize to ‘good channels’ for the eavesdropper, also polarize for ‘good channels’ for the legitimate user. This clearly does not necessarily follow for the general not-degraded case.

For the general wiretap channel, indices that are ‘good’ for the eavesdropper may not be ‘good’ for the legitimate user and vice-versa. A binary-input symmetric wiretap channel is assumed. As in the construction detailed in Part I of the proof of Theorem 4 the sets $\tilde{A}_n$ and $\tilde{N}_n$ of ‘good indices’ are considered. The sets $\tilde{A}_n$ and $\tilde{N}_n$ include the indices for which the Bhattacharyya parameters of the corresponding split channels approach zero as the block length approach infinity. Specifically, fixing $r < C(P_Y|X)$ and $r^* < C(P_Z|X)$, the conditions in (59)-(42) follow.

Define the index set $S_n \triangleq \tilde{A}_n \setminus \tilde{N}_n$ of indices which are ‘good’ for both the legitimate user and the eavesdropper. According to Conjecture 1 the set $S_n$ can be partitioned into two index sets $S'_n$ and $S''_n$, satisfying the polarization properties in (63)-(69) where $A_n$ is replaced by $\tilde{N}_n$, and $A'_n$ is replaced by $\tilde{A}_n \cap \tilde{N}_n$. Next, the set $N_n$ is defined according to

$$N_n \triangleq (\tilde{A}_n \cap \tilde{N}_n) \cup S''_n$$

and the set $A_n$ is defined to be the remaining indices in $S_n$, that is

$$A_n \triangleq S'_n.$$ 

As explained in Remark 7 the term $\frac{1}{n}|N_n|$ can not exceed the capacity of the eavesdropper marginal channel. Consequently, the size of $S'_n$ can be chosen such that $\frac{1}{n}|S'_n|$ is arbitrarily close to $C(P_Y|X) - C(P_Z|X)$.

Next, the same coset coding scheme defined in (19) is applied to the case at hand (with the new construction of the sets $A_n$ and $N_n$). As the information rate $\frac{1}{n}|A_n|$ of the considered scheme may be chosen arbitrarily close to $C(P_Y|X) - C(P_Z|X)$, the same coding rate as in Theorem 4 is obtained. The decoding reliability at the legitimate user is clear and follows the same proof as for the degraded case (note that all the noisy bits in the considered scheme are ‘transmitted’ over the split channels that are ‘good’ for the legitimate user). It is left to establish that the equivocation rate can approach the information rate of the considered scheme.

C. Analysis of the equivocation rate

As explained in Section III-A the bits $b_n$ corresponding to the indices in $B_n$ are predetermined and fixed. These bits are known both to the eavesdropper and the legitimate user. For each blocklength $n$, consider the ensemble of coset codes corresponding for all the possible selection of fixed bits $b_n$. An analysis of the equivocation rate where the coset code is chosen in random is considered. Specifically, it is assumed that the actual code is chosen from the ensemble by picking the bits in $b_n$ in random. The random selection of the bits in $b_n$ is carried independently and identically. Each bit is picked at random with an equiprobable probability, $Pr(0) = Pr(1) = \frac{1}{2}$. In addition, it is assumed that the random selection of $b_n$ is independent with the random noisy bits in $b_n^*$ and the secret message. It is important to distinguish between the ransom selection of a code and the noisy bits $b^*$. The random selection of code is part of our analysis, this selection (i.e., the bits in $b_n$) is known to both the legitimate and the eavesdropper. In contrast, the random noisy bits $b^*$ are immanent part of the encoding procedure and they are unknown to both
the legitimate user and the receiver. The noisy bits \( b^* \) are picked randomly, each independent with the others, and with a uniform probability. The information bits are also assumed to be independent and equiprobable.

The secrecy properties of the suggested scheme is considered in the following proposition:

**Proposition 6.** Consider the polar secrecy scheme in Section [IV-B] whose transmissions take place over a binary-input memoryless symmetric wiretap channel. Then, there exists a bit vector \( b_n \) for which the equivocation rate satisfy the secrecy condition in (38).

**Proof:** Denote by \( W \) the random binary vector comprises the random bits in \( b_n \), \( b^*_n \), and \( u \) in the encoding procedure (19), and by \( Z \) the random vector received at the eavesdropper. According to the considered assumptions, all the bits in \( W \) are independent and equiprobable. It follows using the chain rule of mutual information that

\[
I(W_{N_n}, W_{A_n}; W_{B_n}, Z) = I(W_{A_n}; W_{B_n}, Z) + I(W_{N_n}; W_{B_n} | W_{A_n})
\]

\[
= I(W_{A_n}; W_{B_n}) + I(W_{N_n}; Z | W_{A_n}) + I(W_{N_n}; Z | W_{A_n}; W_{B_n})
\]

where the last equality follows since \( W_{A_n}, W_{N_n}, \) and \( W_{B_n} \) are independent. As the set \( N_n \) comprises indices of split channels which polarize to perfect channels, the bits in \( W_{N_n} \) can be reliably decoded at the eavesdropper based on perfect knowledge of the remaining bits and the received vector (this is shown in a similar fashion to [2]). Hence, the decoding error probability \( P_e(W_{N_n}) \) of the bits in \( W_{N_n} \) based on the received vector and the remaining bits \( W_{N_n}^c \), can be made arbitrarily low. As a consequence of Fano’s inequality it follows that

\[
|N_n| \geq I(W_{N_n}; Z | W_{A_n}, W_{B_n})
\]

\[
= H(W_{N_n} | W_{A_n}, W_{B_n}) - H(W_{N_n} | Z, W_{A_n}, W_{B_n})
\]

\[
> H(W_{N_n}) - h_2(P_e(W_{N_n})) - |N_n| P_e(W_{N_n})
\]

where \( h_2 \) is the binary entropy function. For a sufficiently large block length \( n \), the expected decoding error probability approaches zero. Consequently, \( \frac{1}{n} I(W_{N_n}; Z | W_{A_n}, W_{B_n}) \) can be made arbitrarily close to \( \frac{1}{n} |N_n| \). It follows from (72) and (73) that

\[
\frac{1}{n} H(W_{A_n} | Z, W_{B_n}) \geq \frac{|A_n|}{n} + \frac{|N_n|}{n} - \frac{1}{n} I(W_{N_n}; W_{A_n}; W_{B_n}; Z)
\]

where \( \epsilon_n \geq 0 \) and approaches zero as \( n \) grows.

Based on Conjecture [1] the mutual information \( \frac{1}{n} I(W_{N_n}, W_{A_n}; W_{B_n}, Z) \) can be shown to be arbitrarily close to \( \frac{1}{n} |N_n| \). Using the chain rule of mutual information it follows that

\[
I(W_{N_n}, W_{A_n}; W_{B_n}, Z) = \sum_{i \in N_n} I(W_i; W_{B_n}, Z | W_{N_n}^{(i)}, W_{A_n}^{(i)})
\]

\[
+ \sum_{i \in A_n} I(W_i; W_{B_n}, Z | W_{N_n}^{(i)}, W_{A_n}^{(i)})
\]

\[
= \sum_{i \in N_n} I(W_i; W_{N_n}^{(i)}, W_{A_n}^{(i)}, W_{B_n}, Z)
\]

\[
+ \sum_{i \in A_n} I(W_i; W_{N_n}^{(i)}, W_{A_n}^{(i)}, W_{B_n}, Z)
\]

where the last equality follows as all the bits in \( W \) are independent. For every index \( i \in N_n \), it follows from
Lemma 2 and Conjecture 1 that

\[ I(W_i; W_{N_n}, W_{A_n}, W_{B_n}, Z) > 1 - \delta. \]  

(76)

In addition, for all the indices \( i \in A_n \) it also follows from Conjecture 1 that

\[ I(W_i; W_{N_n}, W_{A_n}, W_{B_n}, Z) < \delta. \]  

(77)

From (75), (76) and (77) it follows that

\[ \frac{1}{n} I(W_{N_n}, W_{A_n}, W_{B_n}, Z) \leq \frac{|N_n|}{n} + \frac{\delta |A_n|}{n} \]
\[ \leq \frac{|N_n|}{n} + \delta. \]  

(78)

Hence, based on (74) and (78) we end up with

\[ \frac{1}{n} H(W_{A_n} | Z, W_{B_n}) \geq \frac{1}{n} |A_n| - \epsilon_n - \delta. \]

As \( \delta \) can be fixed arbitrarily small, and \( \epsilon_n \) approaches zero, the equivocation rate can be made arbitrarily close to \( \frac{1}{n} |A_n| \) which assures the secrecy property of the provided scheme.

\[ \square \]

V. Summary and Conclusions

A polar secrecy scheme is provided in this paper for the two-user, memoryless, symmetric and degraded wiretap channel. The provided polar codes are shown to achieve the entire rate-equivocation region. Our polar coding scheme is based on the channel polarization method originally introduced by Arikan for single-user setting. For the particular case of binary erasure channel, the secrecy capacity is shown to achieve the secrecy capacity under the strong notion of secrecy.

Proving (disproving, or finding a counter example) Conjecture 1 is the main interest in the continuation of the research discussed in this paper. The following generalizations are of additional possible interest:

1) Non-binary settings: In light of the recent results by Sasoglu et al. [7], a generalization to the non-binary setting may be a straightforward generalization.
2) Secrecy polar schemes for non-symmetric wiretap channels, based on the non-binary polarization provided in [7].
3) Polar coding for a broadcast channel with confidential messages. The particular case of degraded message sets over a degraded channel is first considered.
4) Strong secrecy properties: As noted, the provided scheme is shown to provide weak secrecy. It is of great interest to find out if this scheme can also provide strong secrecy.
5) Generalized polar secrecy-schemes based on the ideas in [4], [8]-[10].
6) Combining the polar scheme with the MAC approach for the wiretap channel (see, e.g., [11]).

Acknowledgment

The authors are grateful to Prof. Emre Telatar for reviewing an early version of this paper in October 2009.

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