Discovering and Proving Infinite Pochhammer Sum Identities

Jakob Ablinger
Research Institute for Symbolic Computation, Johannes Kepler University, Linz, Austria

**ABSTRACT**
We consider nested sums involving the Pochhammer symbol at infinity and rewrite them in terms of a small set of constants, such as powers of \( \pi \), \( \log(2) \), or zeta values. In order to perform these simplifications, we view the series as specializations of generating series. For these generating series, we derive integral representations in terms of root-valued iterated integrals or directly in terms of cyclotomic harmonic polylogarithms. Using substitutions, we express the root-valued iterated integrals as cyclotomic harmonic polylogarithms. Finally, by applying known relations among the cyclotomic harmonic polylogarithms, we derive expressions in terms of several constants. We provide an algorithmic machinery to prove identities which so far could only be proved using classical hypergeometric approaches. These methods are implemented in the computer algebra package HarmonicSums.

**KEYWORDS**
binomial sums; Pochhammer symbol; holonomic functions; multiple zeta values

**AMS SUBJECT CLASSIFICATIONS**
05A10; 68W30; 11M32

1. Infinite nested Pochhammer sums
The goal of this article is to find and prove identities of the following form:

\[
\sum_{n=1}^{\infty} \frac{\left(\frac{1}{n}\right)^n}{(n+1)!} \sum_{i=1}^{n} \frac{\sum_{j=1}^{i} \frac{1}{j!}}{j!} = 3\zeta_3,
\]

\[
\sum_{n=1}^{\infty} \frac{\left(\frac{1}{n}\right)^n}{(n+1)!} \sum_{i=1}^{n} \frac{\sum_{j=1}^{i} \frac{1}{j!}}{j!} = \frac{2\pi^3}{3} - 4\zeta_2^2 \frac{\pi^2}{3} + \frac{2}{3} \pi^2 \zeta_3 + 16\zeta_4 - \frac{13\pi^4}{180},
\]

\[
\sum_{n=1}^{\infty} \frac{\left(\frac{1}{n}\right)^n}{(n+1)!} \sum_{i=1}^{n} \frac{\sum_{j=1}^{i} \frac{1}{j!}}{j!} = \frac{7\pi^3}{3} - \frac{16\pi}{3} - 6\zeta_2^2 + 2\pi\zeta_3
\]

\[
\sum_{n=1}^{\infty} \frac{\left(\frac{1}{n}\right)^n}{(n+1)!} \sum_{i=1}^{n} \frac{\sum_{j=1}^{i} \frac{1}{j!}}{j!} = \frac{\pi}{3} - \frac{3}{4} \pi \zeta_3 + \frac{23\pi}{24} \zeta_3
\]

where \( (x)_n \) denotes the Pochhammer symbol, \( l_k := \log(k); \zeta_k := \sum_{n=1}^{\infty} \frac{1}{n^k} \), and \( C \) denotes Catalan’s constant.

Note that similar identities were given in [Ablinger 17, Liu and Wang 19]. Such identities are of interest in physics: In particular, such sums have been studied in order to perform calculations of higher-order corrections to scattering processes in particle physics [Ablinger et al. 14, Davdychev and Kalmykov 01, 04, Fleischer et al. 99, Jegerlehner et al. 03, Kalmykov and Veretin 00, Kalmykov et al. 07, Ogred and Osland 98, Weinziel 04]. Moreover, similar identities were also considered [Borwein et al. 01, Borwein and Lisonek 00, Lehmer 85, Zhi-Wei 11, Zucker 85], and there is a connection to Apéry’s proof of the irrationality of \( \zeta(3) \) (see [Borwein and Borwein 87]) [Weinziel 04, Zhi-Wei 11, Zucker 85].

While [Ablinger 17] basically deals with sums of the form

\[
\sum_{n=1}^{\infty} \frac{\left(\frac{1}{n}\right)^n f(n) }{n!} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{(-\frac{1}{n})^n f(n) }{n!},
\]

we are going to consider a much wider class of sums in the frame of this paper. In addition, we will state a general computer algebra method to evaluate a large class of sums in terms of nested integrals. Moreover, we will be able to prove a structural theorem, about
when such a sum can be expressed in terms of the so-called cyclotomic polylogarithms.

The main purpose of this article is to present methods which can be automated; hence, not all identities presented in this paper are new identities. To make more precise which class of sums we are considering, some definitions are in place. Let \( a_i, c_i \in \mathbb{N} \) and \( b_i \in \mathbb{N}_0 \) for \( 1 \leq i \leq r \) then we call
\[
S_{(a_1, b_1, c_1), \ldots, (a_r, b_r, c_r)}(n) \quad \text{defined as}
\]
\[
S_{(a_1, b_1, c_1), \ldots, (a_r, b_r, c_r)}(n) := \sum_{i=1}^{n} \frac{1}{(a_1 + b_1)^{c_1}} \sum_{i=2}^{n} \frac{1}{(a_2 + b_2)^{c_2}} \cdots \sum_{i=r}^{n} \frac{1}{(a_r + b_r)^{c_r}}
\]
a cyclotomic harmonic sum (compare [Ablinger et al. 13, Ablinger et al. 11, Ablinger et al. 14, Ablinger 13]) of depth \( r \). Note that if \( a_i = 1 \) and \( b_i = 0 \) for \( 1 \leq i \leq r \) we write
\[
S_{c_1, c_2, \ldots, c_r}(n) := S_{(1,0,c_1), (1,0,c_2), \ldots, (1,0,c_r)}(n),
\]
and we call \( S_{c_1, c_2, \ldots, c_r}(n) \) a multiple harmonic sum (see, e.g., [Ablinger et al. 13, Ablinger 13, Blümlein and Kurth 99, Blümlein 00, Vermaseren 99]).

The sums we are considering take the form
\[
\sum_{n=1}^{\infty} \frac{(p)_n}{(an + b)^c(n + d)!} f(n)
\]
where \( a, b, c, d \in \mathbb{N}_0, p \in \mathbb{R}, \) and \( f(n) \) is a cyclotomic harmonic sum. We will refer to (1–3) as Pochhammer sum.

We are going to find representations of these Pochhammer sums in terms of special classes of integrals (that are similar to the iterated integrals in [Ablinger et al. 14] and correspond to the iterated integrals in [Ablinger 17]). These classes of integrals are iterated integrals over hyperexponential functions. More precisely a function \( f(x) \) is called hyperexponential if
\[
\frac{f'(x)}{f(x)} = q(x)
\]
where \( q(x) \) is a rational function in \( x \).

Then, an iterated integral over the hyperexponential functions \( f_1(x), f_2(x), \ldots, f_k(x) \) is defined recursively by
\[
G(f_1(\tau), f_2(\tau), \ldots, f_k(\tau), \tau) := \int_0^\tau f_1(\tau_1) G(f_2(\tau), \ldots, f_k(\tau), \tau_1) d\tau_1,
\]
with the special case \( G(x) = 1 \). Since some letters might have a non-integrable singularity at the base point \( x = 0 \) we consistently define
\[
G(f(\tau), x) := \int_0^\tau \left( f(t) - \frac{c}{t} \right) dt + c \log(x),
\]
where \( c \) takes the unique value such that the integrand on the right hand side is integrable at \( t = 0 \). It is important to note that this definition preserves the derivative \( \frac{d}{dx} G(f(\tau), x) = f(x) \). In general, we set
\[
G(f_1(\tau), \ldots, f_k(\tau), x) := \int_0^\tau f_1(t) G(f_2(\tau), \ldots, f_k(\tau), t) - \sum_{i=0}^{k-1} c_i \log(t)^i dt, \quad \int_0^\tau \frac{c_i}{t} dt,
\]
\[
+ \sum_{i=0}^{k-1} \frac{c_i}{i+1} \log(x)^{i+1},
\]
where \( k \) and \( c_0, \ldots, c_k \) are chosen to remove any non-integrable singularity. Again the result is unique and retains
\[
\frac{d}{dx} G(f_1(\tau), \ldots, f_k(\tau), x) = f_1(x) G(f_2(\tau), \ldots, f_k(\tau), x).
\]

In the following, we will define a subclass of iterated integrals (compare [Ablinger et al. 11]). For \( a \in \mathbb{N} \) and \( b \in \mathbb{N}_0, b < \phi(a) \), where \( \phi \) denotes Euler's totient function, we define
\[
f_a^b : (0, 1) \mapsto \mathbb{R}
\]
\[
f_a^b(x) = \begin{cases} \frac{1}{x}, & \text{if } a = b = 0 \\ \frac{x}{\Phi_a(x)}, & \text{otherwise,} \end{cases}
\]
where \( \Phi_a(x) \) denotes the \( a \)th cyclotomic polynomial, e.g., the first cyclotomic polynomials are given by
\[
\Phi_1(x) = x - 1 \\
\Phi_2(x) = x + 1 \\
\Phi_3(x) = x^2 + x + 1 \\
\Phi_4(x) = x^2 + 1 \\
\Phi_5(x) = x^4 + x^3 + x^2 + x + 1 \quad \text{etc.}
\]

Now, let \( m_i = (a_i, b_i) \in \mathbb{N}^2, b_i < \phi(a_i) \); for \( x \in (0, 1) \), we define cyclotomic polylogarithms recursively as follows (compare, e.g., [Ablinger et al. 11]):
\[
H(x) = 1,
\]
\[
H_{m_1, \ldots, m_k}(x) = \begin{cases} \frac{1}{k!} (\log x)^k, & \text{if } m_i = (0, 0) \\ \int_0^x f_a^b(y) H_{m_2, \ldots, m_k}(y) dy, & \text{otherwise.} \end{cases}
\]

We call \( k \) the weight of a cyclotomic polylogarithm and in case the limit exists we extend the definition to \( x = 1 \) and write
\[
H_{m_1, \ldots, m_k}(1) := \lim_{x \to 1} H_{m_1, \ldots, m_k}(x).
\]

Note that restricting the alphabet to the letters \( (0, 0), (1, 0), \) and \( (2, 0) \) leads to harmonic polylogarithms [Remiddi and Vermaseren 00].
The proposed strategy to prove and find Pochhammer sum identities reads as follows and follows the method proposed in [Ablinger 17]:

**Step 1:** Rewrite the sums in terms of nested integrals.
**Step 2:** Rewrite the integrals in terms of cyclotomic polylogarithms (see [Ablinger 17, Section 4]).
**Step 3:** Provide a sufficiently strong database to eliminate relations among these cyclotomic polylogarithms and find reduced expressions (see Section 4).

This article focuses on Step 1, and we will present three different possibilities to find integral representations of Pochhammer sums. In order to accomplish this task, we view infinite sums as specializations of generating functions [Ablinger 17, Ablinger et al. 14]. Namely, if we are given an integral representation of the generating function [Ablinger 17, Ablinger et al. 14], let's consider three different possibilities to find integral representations of the constants given above, we refer to the accompanying Mathematica notebook.

In the following sections, we will use different methods to compute integral representations of the generating function. In Section 2, we will use holonomic closure properties, while in sections 3 and 4 we will use rewrite rules. In Section 4, we will consider a subclass of Pochhammer sums, for which we can directly find representations in terms of cyclotomic polylogarithms, i.e., we do not have to deal with Step 2 of the proposed strategy.

2. Using closure properties of holonomic functions to derive generating functions

In the following, we repeat important definitions and properties (compare [Ablinger 16, Ablinger et al. 14, Kauers and Paule 11]). Let \( \mathbb{K} \) be a field of characteristic 0. A function \( f = f(x) \) is called holonomic (or D-finite) if there exist polynomials \( p_d(x), p_{d-1}(x), \ldots, p_0(x) \in \mathbb{K}[x] \) (not all \( p_i \) being 0) such that the following holonomic differential
equation holds:

\[ p_d(x)f^{(d)}(x) + \cdots + p_1(x)f'(x) + p_0(x)f(x) = 0. \]  

(2-1)

We emphasize that the class of holonomic functions is rather large due to its closure properties. Namely, if we are given two such differential equations that contain holonomic functions \( f(x) \) and \( g(x) \) as solutions, one can compute holonomic differential equations that contain \( f(x) + g(x), \ f(x)g(x), \) or \( \int_0^x f(y)dy \) as solutions. In other words, any composition of these operations over known holonomic functions \( f(x) \) and \( g(x) \) is again a holonomic function \( h(x) \).

In particular, if for the inner building blocks \( f(x) \) and \( g(x) \), the holonomic differential equations are given, also the holonomic differential equation of \( h(x) \) can be computed.

Of special importance is the connection to recurrences. A sequence \((f_n)_{n \geq 0}\) with \( f_n \in \mathbb{K} \) is called holonomic (or \( P \)-finite) if there exist polynomials \( p_d(n), p_{d-1}(n), \ldots, p_0(n) \in \mathbb{K}[n] \) (not all \( p_1 \) being 0) such that the holonomic recurrence

\[ p_d(n)f_{n+d} + \cdots + p_1(n)f_{n+1} + p_0(n)f_n = 0 \]  

(2-2)

holds for all \( n \in \mathbb{N} \) (from a certain point on). In the following, we utilize the fact that holonomic functions are precisely the generating functions of holonomic sequences: if \( f(x) \) is holonomic, then the coefficients \( f_n \) of the formal power series expansion

\[ f(x) = \sum_{n=0}^{\infty} f_n x^n \]

form a holonomic sequence. Conversely, for a given holonomic sequence \((f_n)_{n \geq 0}\), the function defined by the above sum (i.e., its generating function) is holonomic (this is true in the sense of formal power series, even if the sum has a zero radius of convergence). Note that given a holonomic differential equation for a holonomic function \( f(x) \), it is straightforward to construct a holonomic recurrence for the coefficients of its power series expansion. For a recent overview of this holonomic machinery and further literature, we refer to [Kauers and Paule 11].

Since cyclotomic sums are holonomic sequences with respect to \( n \) and the iterated integrals we consider are holonomic functions with respect to \( x \), we can use holonomic closure properties to derive integral representations of Pochhammer sums: Given a Pochhammer sum

\[ \sum_{n=0}^{\infty} \frac{(-\frac{1}{2})^n S_1(n)}{(3+n)(n-1)!}, \]

(2-3)

will deal as a representative example to illustrate all three different methods that are presented in this

\[ f_n := \frac{(p_n)}{(an+b)^c(n+d)!}g(n) \]

and try to find an iterated integral representation of

\[ f(x) := \sum_{n=1}^{\infty} x^nf_n \]

using the following steps:

1. Compute a holonomic recurrence equation for \((f_n)_{n \geq 0}\).
2. Compute a holonomic differential equation for \(f(x)\).
3. Compute initial values for the differential equation.
4. Solve the differential equation to get a closed form representation for \(f(x)\).

This procedure is implemented in the packages HarmonicSums and can be called by

\[
\text{ComputeGeneratingFunction}
\left[\text{Pochhammer}[p,n]/(an+b)^c(n+d)!\right] g[n], x, \{n, 1, \infty\}.
\]

We will succeed in finding a closed form representation for \(f(x)\) in terms of iterated integrals, if we can find a full solution set of the derived differential equation. The command \text{ComputeGeneratingFunction} internally uses the differential solver implemented in HarmonicSums, which finds all solutions of holonomic differential equations that can be expressed in terms of iterated integrals over hyperexponential alphabets [Ablinger 16, Ablinger et al. 14, Bronstein 92, Petkovsek 92, Hendriks and Singer 99]; these solutions are called d’Alembertian solutions [Abramov and Petkovsek 94], in addition for differential equations of order two it finds all solutions that are Liouvillian [Ablinger 17a, Kovacic 86, Hendriks and Singer 99].

If we succeed in finding a closed form representation for \(f(x)\) in terms of iterated integrals, we proceed with Step 2 and Step 3 of the proposed strategy. Hence, we send \( x \to 1 \) and try to transform these iterated integrals to expression in terms of cyclotomic polylogarithms and finally we use relations between cyclotomic polylogarithms at one to derive an expression in terms of known constants.

The Pochhammer sum

\[
\sum_{n=0}^{\infty} \frac{(-\frac{1}{2})^n S_1(n)}{(3+n)(n-1)!}.
\]
article. First, we work out the sum using the method presented above.

**Example 1.** We consider the sum \((2-3)\) and start to derive a recurrence for

\[
 f_n := \frac{(-\frac{1}{2})^n S_1(n)}{(3+n)^2(n+1)!};
\]

we find:

\[
 (2+n)(4+n)^2(1+2n)(3+2n)f_n-2(1+n)
\]

\[
 (5+n)(5+2n)(5+2n)f_{n+1}
\]

+ 4(1+n)(2+n)(3+n)(6+n)^2f_{n+2} = 0.

Using the closure properties of holonomic functions, we find the following differential equation

\[
 96f(x) + 3(-250 + 343x)f'(x) + 3(144 - 590x + 481x^2)f''(x)
\]

\[
 + x(352 - 942x + 599x^2)f^{(3)}(x) + 8x^2(9 - 20x + 11x^2)f^{(4)}(x)
\]

\[
 + 4(-1 + x)^2x^3f^{(5)}(x) = 0,
\]

satisfied by

\[
 \sum_{n=1}^{\infty} x^n \frac{(-\frac{1}{2})^n S_1(n)}{(3+n)^2(n+1)!}.
\]

We can solve this differential equation for example using the differential equation solver implemented in HarmonicSums:

**SolveDE** \([96f[x] + 3(-250 + 343x)f'[x]
\]

\[
 + 3(144 - 590x + 481x^2)f''[x]
\]

\[
 + x(352 - 942x + 599x^2)f^{(3)}[x]
\]

\[
 + 8x^2(9 - 20x + 11x^2)f^{(4)}[x]
\]

\[
 + 4(-1 + x)^2x^3f^{(5)}[x] == 0, f[x], x].
\]

By checking initial values we find

At this point we send \(x \rightarrow 1\) and use the command SpecialGLToH in HarmonicSums to derive an expression in terms of cyclotomic polylogarithms (compare [Ablinger 17, Section 3]). This leads to

\[
 \frac{\frac{1}{7350} - \frac{32}{35}H_{(0,0)}(1)}{\frac{3675}{7350}} + \frac{8}{35}H_{(0,0)}(1)^2 - \frac{64}{35}H_{(2,0)}(1)
\]

\[
 - \frac{32}{35}H_{(2,0),(0,0)}(1) + \frac{16}{35}H_{(0,0),(0,0)}(1) - \frac{32}{35}H_{(0,0),(1,0)}(1)
\]

\[
 - \frac{32}{35}H_{(2,0),(0,0)}(1) + \frac{64}{35}H_{(2,0),(1,0)}(1) + \frac{64}{35}H_{(2,0),(2,0)}(1).
\]

Finally, we can use relations between cyclotomic polylogarithms at one (compare [Ablinger 17, Section 4]) to derive

\[
 \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n S_1(n) \frac{1}{(n+1)!} = -\frac{9367 + 560n^2 + 6720l_2 - 128l_2}{7350}.
\]

(2–5)

Note that in the last step of this example we are actually only dealing with harmonic polylogarithms (see [Remiddi and Vermaseren 00]).

Let us now list several identities that could be computed using this method:

\[
 \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n S_{1,1,1}(n) \frac{1}{(n+1)!} = 18\zeta_3 - \frac{\pi^3}{\sqrt{3}},
\]

\[
 \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n S_2(n) \frac{1}{(n+1)!} = \frac{5\pi^2}{16} + \frac{27h_1}{8} + \frac{3}{8} \sqrt{3}\pi l_3 - \frac{27l_2}{16},
\]

\[
 \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n S_3(n) \frac{1}{(n+1)!} = -\frac{16C}{3} + \frac{7\pi^2}{18} - 6l_2^2 + 2\pi l_2,
\]

\[
 \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n S_{2,1,1}(n) \frac{1}{(2n+1)!} = \frac{1}{4} \pi l_2(3l_2-2),
\]

\[
 \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n S_{2,1,1}(2n+1)^2 \frac{1}{n!} = \frac{1}{96} \pi(4\pi^2 l_2 - 72l_2^2 + 56l_2^3 - 9\zeta_3).
\]

(2–4)
Several formulas that can be found in [Liu and Wang 19] can be also discovered and proved using the described method. Here, we are going to list some of them:

\[
\sum_{n=0}^{\infty} \left( \frac{1}{2} \right)_n \left( S_1(n)^2 - S_2(n) \right) (n + 1)! = 8 \zeta_2^2 + \frac{2\pi^2}{3},
\]

\[
\sum_{n=0}^{\infty} \left( \frac{1}{2} \right)_n \left( S_1(n)^3 - 3S_1(n)S_2(n) + 2S_3(n) \right) (n + 1)! = 24\zeta_3 + 16\zeta_2^3 + 4\pi^2 l_2,
\]

\[
\sum_{n=0}^{\infty} \left( \frac{1}{2} \right)_n \left( S_1(n)^3 - 3S_1(n)S_2(n) + 2S_3(n) \right) (n + 1)! = -96\zeta_2 + 16\pi C + 72\zeta_3 + 36l_2^2 - 18\pi l_2
\]
\[
+ 13\pi^2 l_2^2 - \frac{9\pi^3}{2},
\]

\[
\sum_{n=0}^{\infty} \left( \frac{1}{2} \right)_n \left( S_1(n)^2 - S_2(n) \right) (n + 1)! = 288\zeta_2 + 48\pi C + 216\zeta_3 + 108l_2^3 + 54\pi l_2
\]
\[
+ 39\pi^2 l_2 + \frac{27\pi^3}{2},
\]

\[
\sum_{n=0}^{\infty} \left( \frac{1}{2} \right)_n \left( S_1(n)^2 - S_2(n) \right) (n + 1)! = -32C + 12l_2^5 - 4\pi l_2 + \frac{13\pi^2}{9},
\]

\[
\sum_{n=0}^{\infty} \left( \frac{1}{2} \right)_n \left( S_1(n)^2 - S_2(n) \right) (n + 1)! = 32C + 36l_2^5 + 12\pi l_2 + \frac{13\pi^2}{3},
\]

\[
\sum_{n=1}^{\infty} \left( \frac{1}{2} \right)_{n-1} \left( S_1(n)^2 - 2S_1(n) + S_2(n) \right) n! = 8.
\]

Note that this method can also be used to compute integral representations of sums of the form

\[
\sum_{n=1}^{\infty} \frac{x^n(3)_nS_3(n)}{n^2n!}.
\]

Here, we find

\[
\sum_{n=1}^{\infty} \frac{x^n(3)_nS_3(n)}{n^2n!} = H_{(0,0),(1,0),(0,0),(0,0),(1,0)}(x)
\]

\[
= \frac{3Li_2(x)^2}{4} - \frac{Li_3(x)}{2(-1 + x)} - \frac{3}{2} \log (1 - x) Li_3(x) + \frac{3Li_4(x)}{2} + Li_5(x).
\]

and sending for instance \( x \to \frac{1}{2} \) we get:

\[
\sum_{n=1}^{\infty} \frac{\left( \frac{1}{2} \right)_n(3)_nS_3(n)}{n^2n!} = -\frac{\pi^4}{192} - \frac{\pi^2 l_2}{12} + \frac{\pi^4 l_2}{288} - \frac{1}{16} \pi^2 l_2^2
\]

\[
+ \frac{l_2^3}{6} - \frac{5}{72} \pi^2 l_2^2 + \frac{l_4}{16} + \frac{11l_2^5}{120}
\]

\[
+ \frac{3p_4}{2} + 3l_2p_4 + 4p_5 + \frac{7\zeta_3}{8}
\]

\[
- \frac{7\pi^2 \zeta_3}{48} + \frac{21l_2 \zeta_3}{16} + \frac{7}{8} l_2^2 \zeta_3 - \frac{81l_2 \zeta_5}{64}.
\]

Finally, we consider

\[
\sum_{n=1}^{\infty} \left( \frac{1}{2} \right)_n S_{11}(n) (n + 1)!,
\]

proceeding as proposed, we find a differential equation of order 16:
430080f(x) + 210(−4096 + 1592275xf′(x)
+ 42(−33554432 − 6356812x + 407269601x^2)f''(x)
+ (671088640 − 33746963856x − 8037305736x^2 + 192200072453x^3)f^{(3)}(x)
+ x(11047661360 − 20499450032x − 61653276602x^2 + 771941124781x^3)f^{(4)}(x)
+ 13x^2(3812823056 − 38280317036x + 13991732902x^2 + 109483643797x^3)f^{(5)}(x)
+ 26x^3(3555308396 − 22952549314x − 986689087x^2 + 53652573053x^3)f^{(6)}(x)
+ 572x^4(152216474 − 697858881x − 344085550x^2 + 1394066246x^3)f^{(7)}(x)
+ 143x^5(323583896 − 1123610312x − 623388464x^2 + 1975831409x^3)f^{(8)}(x)
+ 143x^6(103854560 − 285705072x − 175728306x^2 + 451597351x^3)f^{(9)}(x)
+ 286x^7(10492016 − 23636810x − 15928139x^2 + 34105982x^3)f^{(10)}(x)
+ 3x^8(130094536 − 246156812x − 18008400x^2 + 328091581x^3)f^{(11)}(x)
+ x^9(32842216 − 53242200x − 41920782x^2 + 66163633x^3)f^{(12)}(x)
+ x^{10}(1762640 − 2487876x − 2095294x^2 + 2904173x^3)f^{(13)}(x)
+ 2x^{11}(28900 − 35986x − 32239x^2 + 39703x^3)f^{(14)}(x)
+ 4x^{12}(262 − 291x − 276x^2 + 305x^3)f^{(15)}(x)
+ 8(−1 + x)^2x^{13}(1 + x)f^{(16)}(x) = 0.

Solving this differential equation is possible but takes quite some time, so this indicates that we might look for more feasible methods to find generating function representations for Pochhammer sums of that kind. In the following sections, we will introduce rewrite rules, which will allow us to compute generating function representations of Pochhammer sums without having to solve differential equations.

3. Using rewrite rules to derive generating functions

In this section, we are going to state rewrite rules which will allow us to find integral representations of the generating functions of Pochhammer sums without having to solve differential equations. We will summarize these rewrite rules in the following lemmas. We start with the base cases where there is no inner sum present:

Lemma 2. Let $\mathbb{K}$ be a field of characteristic 0. Then, the following identities hold in the ring $\mathbb{K}[[x]]$ of formal power series with $a, c \in \mathbb{N}$ and $b, d \in \mathbb{Z}$:

$$
\sum_{n=1}^{\infty} x^n \frac{(p)^n}{n}! = \frac{(1-x)^{-d}}{x^d} (p)^{-d}, \quad d < 0,
$$

(3-1)

$$
\sum_{n=1}^{\infty} x^n \frac{(p)^n}{(n+d)!} = (1-x)^{-p} - 1,
$$

(3-2)

$$
\sum_{n=1}^{\infty} x^n \frac{(p)^n}{(n+d)!} = (1-x)^{-d-p} \frac{p}{d!x^d} \int_0^x (1-t)^{p-d-1} t^{d-1} dt, \quad d > 0,
$$

(3-3)

$$
= \frac{x^{-d}}{a} \int_0^x \sum_{n=1}^{\infty} t^n \frac{1}{(a+n+b)^{e-1}} (n+d)! dt.
$$

(3-4)

In case an inner sum is present, we will make use of the following three lemmas.

Lemma 3. Let $\mathbb{K}$ be a field of characteristic 0 and let $f: \mathbb{N} \rightarrow \mathbb{K}$. Then the following identity holds in the ring $\mathbb{K}[[x]]$ of formal power series with $d < 0$:

$$
\sum_{n=1}^{\infty} x^n \frac{(p)^n}{(n+d)!} = \left( \frac{1-x}{x^d} (p)^{-d} \right) \frac{1}{d!x^d} \int_0^x (1-t)^{p-d-1} t^{d-1} - \int_0^x \sum_{i=1}^{\infty} t^n \frac{(p)^n}{(n+d-1)!} f(n) dt.
$$

(3-5)
Proof. Both sides satisfy the following initial value problem for \( y(x) \), which has a unique solution near \( x = 0 \):
\[
y'(x) - \frac{p x - d}{(1 - x)x} y(x) = \frac{1}{(1 - x)x} \sum_{n=1}^{\infty} x^n \frac{(p)_n}{(n + d - 1)!} f(n),
\]
\[
y(0) = 0.
\]

Lemma 4. Let \( \mathbb{K} \) be a field of characteristic 0 and let \( f : \mathbb{N} \to \mathbb{K} \). Then the following identity holds in the ring \( \mathbb{K}[x] \) of formal power series with \( a > 0 \):
\[
\sum_{n=1}^{\infty} x^n \frac{(p)_n}{(n + d)!} \sum_{i=1}^{n} f(i) = \frac{(1-x)^{d-p}}{x^d} \int_{0}^{x} d^{d-1} (1-t)^{p-d-1} \sum_{n=1}^{\infty} x^n \frac{(p)_n}{(n + d - 1)!} f(n) dt.
\]
\[(3-6)\]

Proof. Both sides satisfy the following initial value problem for \( y(x) \), which has a unique solution near \( x = 0 \):
\[
y'(x) - \frac{p x - d}{(1 - x)x} y(x) = \frac{1}{(1 - x)x} \sum_{n=1}^{\infty} x^n \frac{(p)_n}{(n + d - 1)!} f(n),
\]
\[
y(0) = 0.
\]

Lemma 5. Let \( \mathbb{K} \) be a field of characteristic 0 and let \( f : \mathbb{N} \to \mathbb{K} \). Then the following identity holds in the ring \( \mathbb{K}[x] \) of formal power series with \( a, c \in \mathbb{N} \) and \( b \in \mathbb{Z} \):
\[
\sum_{n=1}^{\infty} x^n \frac{(p)_n}{(a + n + b)^c (n + d)!} \sum_{i=1}^{n} f(i) = \frac{x^{-1}}{a} \int_{0}^{x} t^{d-1} \sum_{n=1}^{\infty} t^n \frac{(p)_n}{(a + n + b)^c (n + d)!} \sum_{i=1}^{n} f(i) dt.
\]
\[(3-7)\]

Example 6. We again look for a closed form representation in terms of iterated integrals of
\[
\sum_{n=1}^{\infty} x^n \frac{(-\frac{1}{2})_n S_1(n)}{(3 + n)^2 (n - 1)!}.
\]

We start by using Lemma 5 twice:
\[
\sum_{n=1}^{\infty} x^n \frac{(-\frac{1}{2})_n S_1(n)}{(3 + n)^2 (n - 1)!} = x^{-3} \int_{0}^{x} t^{d-1} \sum_{n=1}^{\infty} t^n \frac{S_1(n)(-\frac{1}{2})_n}{(3 + n)(n - 1)!} dt
\]
\[
= x^{-3} \int_{0}^{x} t^{d-1} \int_{0}^{u} \frac{u^3}{\sqrt{1 - u}} \left( \int_{0}^{u} \frac{1}{\sqrt{1 - v}} \sum_{n=1}^{\infty} v^n \frac{(-\frac{1}{2})_n}{n(n - 2)!} dv - \frac{1}{2} \right) du dt
\]
\[
= x^{-3} \int_{0}^{x} t^{d-1} \int_{0}^{u} \frac{u^3}{\sqrt{1 - u}} \left( \int_{0}^{u} \frac{1}{\sqrt{1 - v}} \sum_{n=1}^{\infty} w^n \frac{(-\frac{1}{2})_n}{(n - 2)!} dw - \frac{1}{2} \right) du dt
\]
\[
= x^{-3} \int_{0}^{x} t^{d-1} \int_{0}^{u} \frac{u^3}{\sqrt{1 - u}} \left( \int_{0}^{u} \frac{1}{\sqrt{1 - v}} \sum_{n=1}^{\infty} \frac{(-\frac{1}{2})_n}{w^{n+1}} dwdv - \frac{1}{2} \right) du dt
\]
\[
= x^{-3} \int_{0}^{x} t^{d-1} \int_{0}^{u} \frac{u^3}{\sqrt{1 - u}} \left( \int_{0}^{u} \frac{1}{\sqrt{1 - v}} \sum_{n=1}^{\infty} \frac{(-\frac{1}{2})_n}{w^{n+1}} dwdv - \frac{1}{2} \right) du dt.
\]

Note that formulas related to the previous lemmas concerning binomial sums can be found in [Ablinger et al. 14].

Let us now, for the second time, consider (2–3) and illustrate how the previous lemmas can be used as rewrite rules to find integral representations of Pochhammer sums.
At this point, we rewrite the expression in terms of iterated integrals (this can be done by hand or by using the command GLIntegrate of HarmonicSums) and arrive again at (2–4) and hence we can proceed as in Example 1 to arrive at
\[
\sum_{n=1}^{\infty} \frac{(-\frac{1}{2})_n S_1(n)}{(3+n)^2(n-1)!} = \frac{-9367 + 560\pi^4 + 6720\pi^2 - 128\pi^2}{7350}.
\]

Note that this method is implemented in the package HarmonicSums using the command PochhammerSumToGL. Calling
\[
PochhammerSumToGL\left[\frac{\text{Pochhammer}[-\frac{1}{2}, n] S[1, n]}{(3+n)^2(n-1)!}, x, \{n, 1, \infty\}\right]
\]
will immediately (after regrouping) give (2–4).

Reconsidering (2–6) we find
\[
\sum_{n=1}^{\infty} \frac{\left(\frac{1}{2}\right)_n S_1(n)}{(n+1)!} = -\frac{\pi^4}{32\sqrt{3}} + \frac{9}{16} \sqrt{3}\pi h_1 - \frac{15\pi^2 l_3}{32}
\]
\[
- \frac{81h_1 l_3}{16} - \frac{9}{32} \sqrt{3}\pi^2 l_3
\]
\[
+ \frac{27l_3^2}{32} + 6\zeta_3,
\]

(3–8)
\[
\sum_{i=1}^{\infty} \frac{\left(\frac{1}{2}\right)_i S_{1,1,1,1}(n)}{(n+1)!} = 60\zeta_3,
\]

(3–9)
\[
\sum_{i=1}^{\infty} \frac{\left(\frac{1}{2}\right)_i S_{1,1,1,1,1}(n)}{(n+1)!} = 180\zeta_5 - \frac{\pi^5}{\sqrt{3}},
\]

(3–10)

\[
\sum_{n=1}^{\infty} \frac{\left(\frac{1}{2}\right)_n S_4(n)}{m(n+1)!} = -\frac{\pi^4}{20} + \frac{2\pi^2 l_2 - 4\pi^2 l_3 + 4l_4 + 8l_5}{3} + 16\rho_4
\]
\[
+ 16\rho_5 - \frac{7\pi^2 \zeta_3}{12} + 8\zeta_3 + 7l_3 \zeta_5 - \frac{63\zeta_5}{8},
\]

(3–11)
To conclude this section we consider the sum

$$\sum_{n=1}^{\infty} \frac{(\frac{1}{2})_n S_{3,1}(n)}{(n+1)!n^3} = \frac{13\pi^4}{180} + \frac{\pi^6}{189} + \frac{199\pi^6 l_2}{7560} + \frac{2}{3} \pi^2 l_2^2 - \frac{4}{27} \pi^4 l_2^3 - \frac{2 l_2}{3} + \frac{8}{45} \pi^2 l_5^2 
- 16p_4 + \frac{16}{3} \pi^2 l_2 p_4 + \frac{16\pi^2 p_5}{3} + 24s_1 + \frac{200l_2 s_1}{7} + 136s_2 + \frac{3}{2} l_2^2 - \frac{3\pi^2 s_3}{2} + \frac{29\pi^2 r_{s_3}}{16} + 4l_2 r_{s_3} - 3\pi^2 l_2 s_3 + \frac{5}{6} \pi^3 l_2^2 s_3 
- \frac{3}{2} l_2^2 - 36p_4 s_3 + \frac{9e^2}{8} - \frac{243}{7} l_2 s_3 + \frac{75s_5}{4} - \frac{2935\pi^2 r_{s_5}}{168} 
- 9l_2 s_5 + \frac{111}{2} l_2^2 s_5 + \frac{12685s_7}{112};$$

Here, we fail to transform the iterated integrals in terms of cyclotomic polylogarithms; however, since the integrals are simple enough, we are able to perform the integrals in (3.12) for example by using Mathematica and find the result

$$\sum_{n=1}^{\infty} \frac{(\frac{1}{2})_n S_{3,1}(n)}{(n+1)!n} = -\frac{4\pi^{3/2}}{27\sqrt{3} \Gamma\left(\frac{1}{3}\right) \Gamma\left(\frac{2}{6}\right)}.$$

We find that it equals

$$1 - 2G\left(\frac{\sqrt{1-\tau}}{1-\sqrt{\tau}}; 1\right) + 2G\left(\frac{\sqrt{1-\tau}}{1+\sqrt{\tau}}; \tau; 1\right) - 33G\left(\sqrt{1-\tau}; 1\right) - 2G\left(\frac{\sqrt{1-\tau}}{1+\sqrt{\tau}}; 1\right) \frac{1}{45}.$$

(3.14)
Another example where we fail to transform the iterated integrals in terms of cyclotomic polylogarithms but still can do the integrals is
\[ \sum_{n=1}^{\infty} \frac{(\frac{1}{2})_n S_{(3.1.2)}(n) - S_{(3.2.1)}(n)}{(n+1)!} = -\frac{3}{4} + \pi \sqrt{3} \Gamma(\frac{3}{4}) \frac{\sqrt{3} \Gamma(\frac{3}{4})}{2} \]

In the following section, we will consider a subclass of Pochhammer sums, for which we will always be able to derive a representation in terms of cyclotomic polylogarithms.

4. Using rewrite rules to directly derive generating functions in terms of cyclotomic polylogarithms

In this section, we will deal with a subclass of the Pochhammer sums, namely we restrict the inner sum to be a multiple harmonic sum and we set \( p = 1/q \) with \( q \in \mathbb{Z} \setminus \{0\} \) and \( a = 1 \) in (1–3), i.e., we are considering sums of the form
\[ \sum_{n=1}^{\infty} \frac{(p)_n}{(n+b)^a(n+d)!} S_{c_1, c_2, \ldots, c_r}(n), \quad (4-1) \]
where \( c, c_i \in \mathbb{N}, b, d \in \mathbb{Z} \) and \( p = \frac{1}{q} \) with \( q \in \mathbb{Z} \setminus \{0\} \). Considering a Pochhammer sum in this subclass we could again use the rewrite rules presented in Section 3 to find an integral representation, however we can also use the following lemmas. These new rewrite rules will directly lead to cyclotomic polylogarithms. We again start with the base cases where no inner sum is present (compare Lemma 2):

**Lemma 7.** Let \( \mathbb{K} \) be a field of characteristic 0. Then, the following identities hold in the ring \( \mathbb{K}[[x]] \) of formal power series with \( c \in \mathbb{N} \) and \( b, d \in \mathbb{Z} \):
\[ \sum_{n=1}^{\infty} x^n \frac{(p)_n}{(n+b)^a(n+d)!} = -\frac{(1-x)^{d-p} x^{-d}}{|p|} \int_1^{(1-x)^p} (1-t^{-1})^d dt, d > 0, \]
\[ \sum_{n=1}^{\infty} x^n \frac{(p)_n}{(n+b)^a(n+d)!} = -\frac{1}{|p|} \int_1^{(1-x)^p} \frac{(1-t^{-1})^b}{t^{1-\frac{1}{p}}} \sum_{n=1}^{\infty} \frac{(1-t^d)^n}{(n+b)^c(a+n+d)!} dt. \]
\[ (4-2) \]
\[ (4-3) \]

In the cases where there is an inner multiple harmonic sum present we can refine the Lemmas 3, 4, and 5 and get the following result.

**Lemma 8.** Let \( \mathbb{K} \) be a field of characteristic 0 and let \( f : \mathbb{N} \to \mathbb{K} \). Then the following identities hold in the ring \( \mathbb{K}[[x]] \) of formal power series with \( c \in \mathbb{N}, b, d \in \mathbb{Z} \) and \( S_{m_1, \ldots, m_r}(n) \) a multiple harmonic sum:
\[ c = 0, d < 0 : \]
\[ \sum_{n=1}^{\infty} x^n \frac{(p)_n}{(n+b)^a(n+d)!} S_{m_1, \ldots, m_r}(n) = \]
\[ \frac{(1-x)^{d-p}}{|p|} \int_1^{(1-x)^p} \frac{(1-t^{-1})^d}{t^{1-\frac{1}{p}}} \sum_{n=1}^{\infty} \frac{(1-t^d)^n}{(n+b)^c(n+b+n+d)!} s(n) dt; \]
\[ c > 0 : \]
\[ \sum_{n=1}^{\infty} x^n \frac{(p)_n}{(n+b)^a(n+d)!} S_{m_1, \ldots, m_r}(n) = \]
\[ -\frac{x^{-b}}{|p|} \int_1^{(1-x)^p} \frac{(1-t^{-1})^b}{t^{\frac{1}{p}-1}} \sum_{n=1}^{\infty} \frac{(1-t^d)^n}{(n+b)^c(n+b+n+d)!} s(n) dt. \]
Here, we use the abbreviations $s(n) := s_{m_1, \ldots, m_k}(n)$ and $\tilde{s}(n) := s_{m_2, \ldots, m_k}(n)$.

**Proof.** For all these equalities, it is possible to find an initial value problem, which has a unique solution near $x = 0$ and is satisfied by both sides of the respective equation.

Note that the polynomials arising in the left hand sides of the equations in Lemmas 7 and 8 are of the form $t^k$ or $(1-t)^k$ for $i, k \in \mathbb{Z}$, hence integrating over these integrands will lead to cyclotomic polylogarithms. Therefore, the Pochhammer sums of the form (4–1) will be expressible in terms of cyclotomic polylogarithms, and we can state the following structural theorem.

\[ \sum_{n=1}^{\infty} x^n \frac{(-\frac{1}{2})^n S_1(n)}{(3 + n)^2(n - 1)!} = -\frac{2}{3^3} \int_{-1}^{\frac{1}{2}} \int_{0}^{1} t(1-t^2)^{\frac{1}{2}} \left(1 - u^2 \right)^3 \left(2 \int_{1}^{n} \frac{(1-u^2)^n}{(1-v^2)^{n-2}} \right) du dt \]

**Theorem 9.** Any sum of the form

\[ \sum_{n=1}^{\infty} \frac{\left(\frac{1}{n}\right)^n}{(n+b)^c(n+d)!} S_{c_1, c_2, \ldots, c_k}(n), \tag{4–7} \]

where $c, c_i \in \mathbb{N}, b, d \in \mathbb{Z}$ and $q \in \mathbb{Z} \setminus \{0\}$, can be expressed in terms of cyclotomic polylogarithms at one.

Let us now, for the third time, consider (2–3) and illustrate how the previous lemmas can be used as rewrite rules to directly find a representation in terms of cyclotomic polylogarithms.

**Example 10.** We seek a closed form representation in terms of cyclotomic polylogarithms of

\[ \sum_{n=1}^{\infty} x^n \frac{(-\frac{1}{2})^n S_1(n)}{(3 + n)^2(n - 1)!}, \]

so we use (4–6) twice:

\[ \sum_{n=1}^{\infty} x^n \frac{(-\frac{1}{2})^n S_1(n)}{(3 + n)^2(n - 1)!} = -\frac{2}{3^3} \int_{-1}^{\frac{1}{2}} \int_{0}^{1} t(1-t^2)^{\frac{1}{2}} \left(1 - u^2 \right)^3 \left(2 \int_{1}^{n} \frac{(1-u^2)^n}{(1-v^2)^{n-2}} \right) du dt \]

Now, we can send $x \to 1$ and rewrite this expression directly in terms of cyclotomic harmonic polylogarithms (again this can be done by hand or by using the command GLIntegrate of HarmonicSums) and arrive again at

\[ \frac{9367}{7350} - \frac{64H_{(2,0)}(1)}{3675} - \frac{32}{35} H_{(0,0), (1,0)}(1) \]

\[ \frac{32}{35} H_{(2,0), (0,0)}(1) + \frac{64}{35} H_{(2,0), (1,0)}(1). \tag{4–8} \]

Finally, we can again use relations between cyclotomic polylogarithms at one to derive (2–5).

Note that this is implemented in the command PochhammerSumToH, so calling

**PochhammerSumToH**

\[ \frac{\text{Pochhammer}[\frac{1}{2}, \frac{1}{n} S[1, n]}{(n+3)^2(n-1)!}, \{x, n, 1, \infty\} \]

will immediately give (after pretty printing)
-11856 + 2552x + 537x^2 - 600x^3 + 4\sqrt{1-x}(1482 + 107x + 75x^3) \\
\frac{7350x^3}{3675x^3} + \frac{64}{3675x^3} + \frac{4\sqrt{1-x}(1276 + 218x + 111x^2 + 75x^3)}{3675x^3} \right) H_1(\sqrt{1-x}) \\
- \frac{4\sqrt{1-x}(1276 + 218x + 111x^2 + 75x^3)H_0(\sqrt{1-x})}{3675x^3} \\
+ H_1(1) \left( - \frac{64}{3675x^3} - \frac{4\sqrt{1-x}(1276 + 218x + 111x^2 + 75x^3)}{3675x^3} + \frac{64H_1(\sqrt{1-x})}{35x^3} \right) \\
- \frac{96H_{1,0}(1)}{35x^3} + \frac{32H_{1,0}(\sqrt{1-x})}{35x^3} - \frac{64H_{0,1}(1)}{35x^3} + \frac{32H_{0,1}(\sqrt{1-x})}{35x^3}.

Sending \( s \to 1 \) will give (4–8).

To conclude we are going to list several identities that could be computed with the help of this command (note that these identities could have also be computed using the methods presented in the previous sections):

\[
\sum_{n=1}^{\infty} \frac{\left( \frac{1}{2} \right)_n S_1(n)}{(n+1)!} = \frac{25h_6}{4}, \quad (4-9)
\]
\[
\sum_{n=1}^{\infty} \frac{\left( \frac{1}{2} \right)_n S_2(n)}{(n+1)!} = \frac{875h_6^2}{48} + \frac{125}{12}\sqrt{5}h_6^2 + \frac{125h_7}{16} + \frac{125h_8}{8}, \quad (4-10)
\]
\[
\sum_{n=1}^{\infty} \frac{\left( \frac{1}{2} \right)_n S_{2,2}(n)}{(n+1)!} = \frac{2\pi^6}{189} - \frac{9e_2^3}{4} - \frac{15l_2\zeta_5}{2}, \quad (4-11)
\]
\[
\sum_{n=1}^{\infty} \frac{\left( \frac{1}{2} \right)_n S_{2,2}(n)}{(n+2)!} = \frac{2\pi^2}{9} + \frac{\pi^4}{45} - \frac{8l_2^2}{3} - \zeta_3 - 2l_2\zeta_3, \quad (4-12)
\]
\[
\sum_{n=1}^{\infty} \frac{\left( \frac{1}{2} \right)_n S_{2,2}(n)}{(n+3)!} = \frac{\pi^2}{9} - \frac{2\pi^4}{45} + \frac{4l_2^2}{3} + \frac{\zeta_3}{2} + 4l_2\zeta_3 + \frac{15\zeta_5}{16}, \quad (4-13)
\]
\[
\sum_{n=1}^{\infty} \frac{\left( \frac{1}{2} \right)_n S_{2,2}(n)}{(n+3)!} = \frac{9}{320} + \frac{3\sqrt{3}\pi}{32} + \frac{51\pi^2}{320} + \frac{11\pi^3}{128\sqrt{3}} + \frac{91\pi^4}{5760} + \frac{1377h_1}{1280} \\
- \frac{27}{128} \sqrt{3}\pi h_1 + \frac{135\pi h_1}{128} + \frac{243h_2}{64} + \frac{27}{64} \sqrt{3}\pi h_2 + \frac{1539h_3}{128} \\
- \frac{4}{16} \sqrt{3}\pi h_2 + \frac{153\sqrt{3}\pi h_3}{128} + \frac{11\pi^3 h_3}{128\sqrt{3}} - \frac{27}{128} \sqrt{3}\pi h_3 + \frac{243h_3}{64} \sqrt{3}\pi h_3 + \frac{1377l_2^2}{2560} + \frac{81\zeta_3}{64} + \frac{39}{64} \sqrt{3}\pi \zeta_3 + \frac{81l_2\zeta_3}{64}. \quad (4-14)
\]

**Acknowledgments**

The author would like to thank C. Schneider for useful discussions.

**Funding**

This study was supported by the Austrian Science Fund (FWF) grant SFB F50 (F5009-N15), by the strategic program "Innovatives OÖ 2020" by the Upper Austrian Government and by the bilateral project DNTS-Austria 01/3/2017 (WTZ BG 03/2017), funded by Bulgarian National Science Fund and OeAD (Austria).

**References**

[Ablinger 17] J. Ablinger. “Discovering and Proving Infinite Binomial Sums Identities.” *J. Exp. Math.* 26:1 (2017), 62–71. arXiv: 1507.01703
[Ablinger 17a] J. Ablinger. “Computing the Inverse Mellin Transform of Holonomic Sequences using Kovacic’s Algorithm.” In PoS RADCOR2017, 069, 2017. arXiv: 1801.01039

[Ablinger 16] J. Ablinger. “Inverse Mellin Transform of Holonomic Sequences.” PoS LL. 067 (2016), 2016. arXiv: 1606.02845

[Ablinger 14] J. Ablinger. “The Package HarmonicSums: Computer Algebra and Analytic Aspects of Nested Sums.” In Loops and Legs in Quantum Field Theory – LL, 2014. arXiv: 1407.6180

[Ablinger and Blümlein 13] J. Ablinger, and J. Blümlein. “Harmonic Sums, Polylogarithms, Special Numbers, and their Generalizations.” In Computer Algebra in Quantum Field Theory: Integration, Summation and Special Functions, Texts & Monographs in Symbolic Computation, edited by C. Schneider and J. Blümlein, pp. 132. Wien: Springer, 2013. arXiv: 1304.7071

[Ablinger et al. 13] J. Ablinger, J. Blümlein, and C. Schneider. “Analytic and Algorithmic Aspects of Generalized Harmonic Sums and Polylogarithms.” J. Math. Phys. 54:8 (2013), 082301. arXiv: 1302.0378

[Ablinger et al. 14] J. Ablinger, J. Blümlein, and C. Schneider. “Generalized Harmonic, Cyclotomic, and Binomial Sums, their Polylogarithms and Special Numbers.” J. Phys. Conf. Ser. 523:1 (2014), 012060. arXiv: 1310.5645

[Ablinger 2013] J. Ablinger. “Computer Algebra Algorithms for Special Functions in Particle Physics.” PhD diss., Johannes Kepler University Linz, 2013. arXiv: 1310.0687.

[Ablinger et al. 14a] J. Ablinger, J. Blümlein, C. G. Raab, and C. Schneider. “Iterated Binomial Sums and their Associated Iterated Integrals.” J. Math. Phys. Compt. 55:11 (2014), 1–57. arXiv: 1407.1822

[Ablinger et al. 11] J. Ablinger, J. Blümlein, and C. Schneider. “Harmonic Sums and Polylogarithms Generated by Cyclotomic Polynomials.” J. Math. Phys. 52:10 (2011), 102301. arXiv: 1105.6063

[Abramov and Petkovšek 94] S. A. Abramov and M. Petkovšek. “D’Alembertian Solutions of Linear Differential and Difference Equations.” In proceedings of ISSAC’94, ACM Press, 1994.

[Baker 66] A. Baker. “Linear Forms in the Logarithms of Algebraic Numbers.” I. Mathematika. 13:2 (1966), 204–216. D60 014018.

[Blümlein and Kurth 99] J. Blümlein and S. Kurth. “Harmonic Sums and Mellin Transforms up to Two-loop Order.” Phys. Rev D 60:1 (1999), 014018. arXiv: hep-ph/9810241v2

[Blümlein 00] J. Blümlein. “Analytic Continuation of Mellin Transforms up to Two-loop Order.” Comput. Phys. Commun. 133:1 (2000), 76–104. arXiv: 0003100

[Blümlein et al. 10] J. Blümlein, D. J. Broadhurst, and J. A. M. Vermaseren. “The Multiple Zeta Value Data Mine.” Comput. Phys. Commun. 181:3 (2010), 582–625. arXiv: math-ph/0907.2557

[Borwein and Borwein 87] J. M. Borwein and P. B. Borwein. Pi and the AGM: A Study in Analytic Number Theory and Computational Complexity. New York: Wiley, 1987. Reprinted 1998.

[Borwein et al. 01] J. M. Borwein, D. J. Broadhurst, and J. Kamnitzer. “Central Binomial Sums, Multiple Clausen Values, and Zeta Values.” Exp. Math. 10:1 (2001), 25–34. arXiv: hep-th/0004153

[Borwein and Lisoněk 00] J. M. Borwein and P. Lisoněk. “Applications of Integer Relation Algorithms.” Discrete Math. 217:1–3 (2000), 65–82.

[Bronstein 92] M. Bronstein. “Linear Ordinary Differential Equations: Breaking through the Order 2 Barrier.” In proceedings of ISSAC’92, ACM Press, 1992.

[Davydychev and Kalmykov 01] A. I. Davydychev and M. Y. Kalmykov. “New Results for the Epsilon-expansion of Certain One-, Two- and Three-loop Feynman Diagrams.” Nucl. Phys. B. 605:1–3 (2001), 266–318. arXiv: hep-th/0012189

[Davydychev and Kalmykov 04] A. I. Davydychev and M. Y. Kalmykov. “Massive Feynman Diagrams and Inverse Binomial Sums.” Nucl. Phys. B. 699:1–2 (2004), 3–64. arXiv: hep-th/0303162

[Fleischer et al. 99] J. Fleischer, A. V. Kotikov, and O. L. Veretin. “Analytic Two Loop Results for Selfenergy Type and Vertex Type Diagrams with One Nonzero Mass.” Nucl. Phys. B. 547:1–2 (1999), 343–374. arXiv: hep-ph/9808242

[Jegerlehner et al. 03] F. Jegerlehner, M. Y. Kalmykov, and O. Veretin. “MS Versus Pole Masses of Gauge Bosons II: Two-Loop Electroweak Fermion Corrections.” Nucl. Phys. B. 658:1–2 (2003), 49–112. arXiv: hep-ph/0212319

[Kalmykov and Veretin 00] M. Y. Kalmykov and O. Veretin. “Single Scale Diagrams and Multiple Binomial Sums.” Phys. Lett. B. 483:1–3 (2000), 315–323. arXiv: hep-th/0004010

[Kalmykov et al. 07] M. Y. Kalmykov, B. F. L. Ward, and S. A. Yost. “Multiple (Inverse) Binomial Sums of Arbitrary Weight and Depth and the All-order \( e \)-expansion of Generalized Hypergeometric Functions with One Half-integer Value of Parameter.” J. High Energy Phys. 2007:10 (2007), 048. arXiv: 0707.3654

[Kauers and Paule 11] M. Kauers and P. Paule. The Concrete Tetrahedron, Texts and Monographs in Symbolic Computation. Wien: Springer, 2011.

[Kovacic 86] J. J. Kovacic. “An Algorithm for Solving Second Order Linear Homogeneous Differential Equations.” J. Symbolic Comput. 2:1 (1986), 3–43.

[Lehmer 85] D. H. Lehmer. “The Multiple Zeta Value Data Mine.” Amer. Math. Monthly 92:7 (1985), 449–457.

[Liu and Wang 19] H. Liu and W. Wang. “Gauss’s Theorem and Harmonic Number Summation Formulae with Certain Mathematical Constants.” J. Difference Equations Appli. 25:3 (2019), 1–18. doi: 10.1080/10236198.2019.1572127

[M. Y. Kalmykov 98] M. Y. Kalmykov. “Hypergeometric Solutions of Linear Recurrences with Polynomial Coefficients.” J. Symbolic Comput. 14:2–3 (1992), 243–264

[Remiddi and Vermaseren 00] E. Remiddi and J. A. M. Vermaseren. “Harmonic Polylogarithms.” Int. J. Mod. Phys. A. 15:05 (2000), 725–754. arXiv: hep-ph/9905237
[Hendriks and Singer 99] P. A. Hendriks and M. F. Singer. “Solving Difference Equations in Finite Terms.” *J. Symbolic Comput.* 27:3 (1999), 239–259.

[Vermaseren 99] J. A. M. Vermaseren. “Harmonic Sums, Mellin Transforms and Integrals.” *Int. J. Mod. Phys. A.* 14:13 (1999), 2037–2076. arXiv: 9806280v1

[Weinzierl 04] S. Weinzierl. “Expansion Around Half Integer Values, Binomial Sums and Inverse Binomial Sums.” *J. Math. Phys.* 45:7 (2004), 2656–2673. arXiv: hep-ph/0402131

[Zhi-Wei 11] S. Zhi-Wei. “List of Conjectural Series for Powers of π and Other Constants.” 2011. arXiv: 1102.5649.

[Zucker 85] I. J. Zucker. “On the Series $\sum_{k=1}^{\infty} \binom{2k}{k}^{-1}k^{-a}$ and Related Sums.” *J. Number Theory.* 20:8 (1985), 92–102.