Stochastic discontinuous Galerkin methods for robust deterministic control of convection-diffusion equations with uncertain coefficients

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Abstract
We investigate a numerical behavior of robust deterministic optimal control problem subject to a convection-diffusion equation containing uncertain inputs. Stochastic Galerkin approach, turning the original optimization problem containing uncertainties into a large system of deterministic problems, is applied to discretize the stochastic domain, while a discontinuous Galerkin method is preferred for the spatial discretization due to its better convergence behavior for optimization problems governed by convection dominated PDEs. Error analysis is done for the state and adjoint variables in the energy norm, while the estimate of deterministic control is obtained in the $L^2$-norm. Large matrix system emerging from the stochastic Galerkin method is addressed by the low-rank version of GMRES method, which reduces both the computational complexity and the memory requirements by employing Kronecker-product structure of the obtained linear system. Benchmark examples with and without control constraints are presented to illustrate the efficiency of the proposed methodology.

Keywords PDE-constrained optimization · Uncertainty quantification · Stochastic discontinuous Galerkin · Error estimates · Low-rank approximation

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1 Introduction

In many phenomena in physics or engineering applications, certain parameters of a model are optimized in order to reach the desired target, for instance, the location where the oil is inserted into the medium, the temperature of a melting/heating process, or the shape of the aircraft wings. Such real-world phenomena can be modeled as optimal control problems or optimization problems with PDE constraints. However, in reality, the input parameters of these simulations, such as the wind speed or material properties, are not often known due to the missing information or inherent variability in the problem; see, e.g., [1]. Therefore, in the last decade, the idea of uncertainty quantification, i.e., quantifying the effects of uncertainty on the result of a computation, has become a powerful tool for modeling physical phenomena in the scientific community.

PDE-constraint optimization problems with uncertainty have been studied in various formulations in the literature, such as mean-based control [2, 3], pathwise control [4, 5], average control [6, 7], robust deterministic control [8–13], and robust stochastic control [14–17]. Robust deterministic control is more practical and realistic since randomness cannot be observed during the design of the control. Therefore, we are here interested in the following robust deterministic control problem

\[ \min_{u \in \mathcal{U}^{ad}} J(y, u) := \frac{1}{2} \| y - y^d \|_{\mathcal{X}}^2 + \frac{\gamma}{2} \| \text{std}(y) \|_{\mathcal{Y}}^2 + \frac{\mu}{2} \| u \|_{\mathcal{U}}^2 \]  \hspace{1cm} (1.1)

governed by

\[ S(y(x, \omega)) = f(x) + u(x) \quad \text{in} \quad \mathcal{D} \times \Omega, \]  \hspace{1cm} (1.2a)
\[ y(x, \omega) = y_{DB}(x) \quad \text{on} \quad \partial \mathcal{D} \times \Omega, \]  \hspace{1cm} (1.2b)

where \( S : \mathcal{Y} \rightarrow \mathcal{Y}' \) is a linear operator that contains uncertain parameters, \( \mathcal{D} \subset \mathbb{R}^2 \) is a convex bounded polygonal set with a Lipschitz boundary \( \partial \mathcal{D} \), and \( \Omega \) is a sample space of events. The cost functional including a risk penalization via the standard deviation \( \text{std}(y) \) is denoted by \( J(y, u) \). The first term in (1.1) is a measure of the distance between the state variable \( y \) and the desired state \( y^d \) in terms of expectation of \( y - y^d \). Without loss of generality, we assume that the state \( y \in \mathcal{Y} \) is a random field, whereas the desired state \( y^d \in \mathcal{Y} \) is modeled deterministically. The second term measures the standard deviation of \( y \), which is added since it is desirable to have a control for which the state is more accurately known, leading to a risk averse optimum. The last term corresponds to distributive deterministic control. The constant \( \mu > 0 \) is a positive regularization parameter of the control \( u \), whereas \( \gamma \geq 0 \) is a risk-aversion parameter. Deterministic source function and Dirichlet boundary conditions are denoted by \( f \) and \( y_{DB} \), respectively. We note that the cost functional \( J \) is a deterministic quantity although it contains uncertain inputs. Further, the closed convex admissible set in the control space \( \mathcal{U} \) is defined by

\[ \mathcal{U}^{ad} := \{ u \in \mathcal{U} : u_a \leq u(x) \leq u_b, \ \forall x \in \mathcal{D} \}, \]  \hspace{1cm} (1.3)

where constants \( u_a, u_b \in \mathbb{R} \) with \( u_a \leq u_b \).

Finding an approximate solution for the optimization problems containing uncertainty (1.1)–(1.2) is extremely challenging and requires much more computational
resources than the ones in the deterministic setting. In the literature, there exist various competing methods to solve such kinds of problem, for instance, Monte Carlo (MC) \[5, 18, 19\], stochastic collocation method (SCM) \[13, 17, 20, 21\], and stochastic Galerkin method (SGM) \[9, 12, 13, 22, 23\]. Although the MC method is popular for its simplicity, natural parallelization, and broad applications, it features slow convergence, which does not depend on the number of uncertain parameters \[24, 25\]. For the SCMs, the crucial issue is how to construct the set of collocation points appropriately because the choice of the collocation points determines the efficiency of the method. In contrast to the MC approach and the SCM, the SGM is a nonsampling technique, which transforms the problem into a large system of deterministic problems. As in the classic (deterministic) Galerkin method, the idea behind the SGM is to seek a solution for the model equation such that the residue is orthogonal to the space of polynomials. Since the random process is expressed as an expansion with the help of orthogonal polynomials, the SGM is considered as a variant of the generalized polynomial chaos approximation \[26–28\] as the stochastic collocation method. An important feature of the SGM is the separation of the spatial and stochastic variables, which allows a reuse of established numerical techniques. The results obtained in \[13\] also show that the SGM generally displays superior performance compared to the SCM for the robust deterministic control problems. Within the framework of the aforementioned features, the stochastic Galerkin method is preferred as a stochastic method in this study. On the other hand, for the discretization of the spatial domain, we use a discontinuous Galerkin method due to its better convergence behavior for the optimization problems governed by convection dominated PDEs; see, e.g., \[29–31\]. We also refer to \[32, 33\] and references therein for more details on the discontinuous Galerkin methods.

In spite of these nice properties exhibited by the stochastic discontinuous Galerkin method, the dimension of the resulting linear system increases rapidly, called as the curse of dimensionality. As a remedy, we apply a low-rank variant of generalized minimal residual (GMRES) method \[34\] with a suitable preconditioner. With the help of a Kronecker-product structure of the obtained large matrices, we reduce both the computational complexity and memory requirements; see, e.g., \[35–37\]. Low-rank approximations of the optimal control problems with uncertain terms have also been studied in \[14, 38, 39\] for unconstrained control problems and in \[40\] for control constraint problems. In the aforementioned studies, randomness is generally defined on the diffusion parameter; however, we here consider the randomness on diffusion or convection parameters by applying the discontinuous Galerkin method in the spatial domain. In addition, according to the best of our knowledge, a low-rank approximation of the optimal control problems governed by convection dominated equations containing randomness has not been discussed before in the setting of discontinuous Galerkin discretization in the spatial domain.

We organize our paper by first discussing the existence of the solution in the next section. In Section 3, we reduce the problem into finite-dimensional setting via Karhunen-Loève (KL) expansion, stochastic Galerkin method, and symmetric interior penalty Galerkin method. Error analyses are done in Section 4. In Section 5, we construct the matrix formulation of the underlying optimization problem by proceeding the optimize-then-discretize approach, and then discuss implementation of
the low-rank GMRES solver. Results of the numerical experiments are provided in Section 6 to illustrate the efficiency of the proposed methodology. Finally, we end the paper with some conclusions and discussions in Section 7.

2 Existence and uniqueness of the solution

Let $\Omega$ be a sample space of events, $\mathcal{F} \subset 2^\Omega$ denotes a $\sigma$-algebra, and $\mathbb{P}$ is the associated probability measure that maps the events in $\mathcal{F}$ to probabilities in $[0, 1]$. A generic random field $\eta$ on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is denoted by $\eta(x, \omega): \mathcal{D} \times \Omega \to \mathbb{R}$. For a fixed $x \in \mathcal{D}$, $\eta(x, \cdot)$ is a real-valued square integrable random variable $\eta(x, \cdot) \in L^2(\Omega, \mathcal{F}, \mathbb{P})$, i.e.,

$$L^2(\Omega) := L^2(\Omega, \mathcal{F}, \mathbb{P}) := \{X: \Omega \to \mathbb{R} : \int_\Omega |X(\omega)|^2 \, d\mathbb{P}(\omega) < \infty\}.$$

Then, the mean $\mathbb{E}[\eta]$, the standard deviation $\text{std}(\eta)$, and the corresponding variance $\mathbb{V}(\eta)$ for any random field $\eta$, are given, respectively, by

$$\mathbb{E}[\eta] = \int_\Omega \eta \, d\mathbb{P}(\omega), \quad \text{std}(\eta) = \left[\int_\Omega (\eta - \mathbb{E}[\eta])^2 \, d\mathbb{P}(\omega)\right]^{1/2}, \quad \mathbb{V}(\eta) = [\text{std}(\eta)]^2 = \mathbb{E}[\eta^2] - (\mathbb{E}[\eta])^2.$$

We first note that the mathematical analysis will be done throughout the paper provided that the equation of state has homogeneous boundary condition, i.e., $y_{DB} = 0$. Recalling the tensor-product space $H^k(\mathcal{D}) \otimes L^2(\Omega)$ equipped with the norm

$$\|\eta\|_{H^k(\mathcal{D}) \otimes L^2(\Omega)} := \left(\int_\Omega \|\eta(\cdot, \omega)\|^2_{H^k(\mathcal{D})} \, d\mathbb{P}(\omega)\right)^{1/2} < \infty, \quad (2.1)$$

the state and control spaces are defined as follows, respectively,

$$\mathcal{Y} := H^1_0(\mathcal{D}) \otimes L^2(\Omega) \quad \text{and} \quad \mathcal{U} := L^2(\mathcal{D}).$$

We also set $\mathcal{X} := L^2(\mathcal{D}) \otimes L^2(\Omega)$ and $\mathcal{W} := L^2(\mathcal{D})$.

In order to show existence of the solution, it is assumed that the operator $\mathcal{S}$ satisfies the following conditions:

a) $\mathcal{S}$ is coercive such that $\mathbb{P}$-a.s., $(\mathcal{S}v, v) \geq c \|v\|_{\mathcal{X}}$, $\forall v \in \mathcal{X}$, where $c$ is a positive constant.

b) $(\mathcal{S}u, v) = (u, \mathcal{S}^*v)$ $\forall u, v \in \mathcal{X}$, where $\mathcal{S}^*$ is the adjoint of $\mathcal{S}$.

By following the standard arguments in the theory of optimal control, see, e.g., [41, Theorem 1.3] and [42, Theorem 2.14], the existence and uniqueness of an optimal solution for the optimization problem (1.1)–(1.2) can be proved. With the definitions above, $\mathcal{Y}$ and $\mathcal{U}$ are Hilbert spaces, the functional $\mathcal{J}$ is strictly convex, and the admissible set $\mathcal{U}^{ad}$ is a closed and convex set. Then, according to Lion’s Lemma [41, Theorem 1.3], a unique optimal control $\bar{u} \in \mathcal{U}$ exists and the variational inequality holds

$$\mathcal{J}'(\bar{u}) \cdot (u - \bar{u}) \geq 0, \quad \forall u \in \mathcal{U}^{ad}. \quad (2.2)$$
Now, we can state the first order optimality system of the optimization problem containing uncertain coefficients (1.1)–(1.2).

**Theorem 2.1** A pair \((y, u)\) is a unique solution of the optimization problem (1.1)–(1.2) if and only if there exists an adjoint \(p \in \mathcal{V}\) such that the optimality system holds, \(\mathbb{P}\)-a.s., for the triplet \((y(u), u, p(u)) \in \mathcal{V} \times \mathcal{U}^{ad} \times \mathcal{V}\)

\[
\begin{align*}
S(y(u)) &= f(x) + u(x), \\
S^*(p(u)) &= y^d + \gamma(y(u) - \mathbb{E}[y(u)]), \\
(\mathbb{E}[p(u)] + \mu u, v - u) &\geq 0, \quad v \in \mathcal{U}^{ad}.
\end{align*}
\]

**Proof** Rewrite the objective functional \(J\) as

\[
J(u) = \frac{1}{2} \mathbb{E} \left[ \int_{\Omega} \left( y(u) - y^d \right)^2 \, dx \right] + \frac{\gamma}{2} \mathbb{E} \left[ \int_{\Omega} y(u)^2 \, dx \right] - \frac{\gamma}{2} \int_{\Omega} \mathbb{E}[y(u)]^2 \, dx + \frac{\mu}{2} \int_{\Omega} u^2 \, dx.
\]

By the definition of directional derivative, we obtain that

\[
J'(u) \cdot (v - u) = \mathbb{E} \left[ \int_{\Omega} (y(u) - y^d) y'(u) \cdot (v - u) \, dx \right] + \gamma \mathbb{E} \left[ \int_{\Omega} y(u) y'(u) \cdot (v - u) \, dx \right] - \gamma \mathbb{E} \left[ \int_{\Omega} \mathbb{E}[y(u)] y'(u) \cdot (v - u) \, dx \right] + \mu \int_{\Omega} u \cdot (v - u) \, dx.
\]

By well-posedness of the state (1.2) followed from the Lax-Milgram lemma, one can easily show that the operator \(S\) is invertible so that, by taking directional derivative, one gets

\[
y'(u) \cdot (v - u) = S^{-1} (v - u) = y(v) - y(u).
\]

Thus, (2.4) gives us

\[
J'(u) \cdot (v - u) = \Psi(\gamma) + \mu \int_{\Omega} u \cdot (v - u) \, dx,
\]

where

\[
\Psi(\gamma) = (1 + \gamma) \mathbb{E} \left[ \int_{\Omega} y(u) \cdot (y(v) - y(u)) \, dx \right] - \gamma \mathbb{E} [y(u)] \cdot (y(v) - y(u)) \, dx - \mathbb{E} \left[ \int_{\Omega} y^d \cdot (y(v) - y(u)) \, dx \right].
\]

To guarantee the existence and uniqueness of the solution from Lion’s Lemma [41, Theorem 1.3], we need the following requirement

\[
J'(u) \cdot (v - u) \geq 0.
\]
Next, we introduce the adjoint state \( p(u) \in \mathcal{Y} \) by
\[
S^*(p(u)) = y(u) - y^d + \gamma(y(u) - \mathbb{E}[y(u)]).
\] (2.7)
Multiplying both sides of (2.7) by \( (y(v) - y(u)) \), integrating over \( \mathcal{D} \), and taking the expectation of the resulting system, we obtain
\[
\mathbb{E} \left[ \int_{\mathcal{D}} S^*(p(u)) \cdot (y(v) - y(u)) \, dx \right] = \mathbb{E} \left[ \int_{\mathcal{D}} p(u) \cdot (S(y(v)) - S(y(u))) \, dx \right]
= \mathbb{E} \left[ \int_{\mathcal{D}} p(u) \cdot (v - u) \, dx \right] = \Psi(y). \tag{2.8}
\]
Inserting (2.8) into (2.5) and combining with (2.6) give us
\[
\mathcal{J}'(u) \cdot (v - u) = (\mathbb{E}[p(u)] + \mu u, v - u) \geq 0,
\] (2.9)
which is the desired result.

In this study, we consider \( S \) as the convection-diffusion operator
\[
S := -\nabla \cdot \left( a(x, \omega) \nabla y \right) + b(x, \omega) \cdot \nabla y = f + u \quad \text{in } \mathcal{D} \times \Omega, \tag{2.10}
\]
which turns the state (1.2) into
\[
-\nabla \cdot \left( a(x, \omega) \nabla y \right) + b(x, \omega) \cdot \nabla y = f + u \quad \text{in } \mathcal{D} \times \Omega, \tag{2.11a}
\]
\[
y = y_{DB} \quad \text{on } \partial \mathcal{D} \times \Omega, \tag{2.11b}
\]
where \( a : (D \times \Omega) \rightarrow \mathbb{R} \) and \( b : (D \times \Omega) \rightarrow \mathbb{R}^2 \) are random diffusivity and velocity coefficients, respectively, which is assumed to have continuous and bounded covariance functions. In addition, we make the following assumptions on the uncertain coefficients:

i) \( \exists a_{\min}, a_{\max} \) such that for almost every \( (x, \omega) \in \mathcal{D} \times \Omega, 0 < a_{\min} \leq a(x, \omega) \leq a_{\max} < \infty \). In addition, \( a(x, \omega) \) has a uniformly bounded and continuous first derivatives.

ii) The velocity coefficient \( b \) satisfies \( b(\cdot, \omega) \in \left( L^\infty(\mathcal{D}) \right)^2 \) for a.e. \( \omega \in \Omega \) and \( \nabla \cdot b(x, \omega) = 0 \).

Then, the well-posedness of the state (2.11) can be shown by following the classical Lax-Milgram lemma; see, e.g., [43, 44].

Now, we give the corresponding weak formulation of the optimization problem containing uncertainty (1.1)–(1.2) as follows
\[
\min_{u \in \mathcal{U}^{ad}} \mathcal{J}(u) = \frac{1}{2} \mathbb{E} \left[ \int_{\mathcal{D}} (y(u) - y^d)^2 \, dx \right] + \frac{\gamma}{2} \mathbb{E} \left[ \int_{\mathcal{D}} (y(u) - \mathbb{E}[y(u)])^2 \, dx \right] + \frac{\mu}{2} \int_{\mathcal{D}} u^2 \, dx \tag{2.12}
\]
governed by
\[
a[y, v] + b[u, v] = [f, v], \quad v \in \mathcal{Y}, \tag{2.13}
\]
where
\[ a[y, u] = \mathbb{E} \left[ \int_{\mathcal{D}} (a(x, \omega) \nabla y \cdot \nabla u + b(x, \omega) \cdot \nabla y u) \, dx \right], \quad \forall y, u \in \mathcal{Y}, \]
\[ b[u, v] = -\mathbb{E} \left[ \int_{\mathcal{D}} uv \, dx \right] \quad \text{and} \quad [f, v] = \mathbb{E} \left[ \int_{\mathcal{D}} f v \, dx \right], \quad \forall u \in \mathcal{U}, v \in \mathcal{Y}. \]

Moreover, the optimality system in (2.3) can be stated in the weak formulation as follows:
\[ a[y, u] + b[u, v] = [f, v], \quad \forall v \in \mathcal{Y}, \quad \text{(2.15a)} \]
\[ a[q, p] = [y - y^d, q] + \gamma [y - \mathbb{E}[y], q], \quad \forall q \in \mathcal{Y}, \quad \text{(2.15b)} \]
\[ (\mathbb{E}[p] + \mu u, w - u) \geq 0, \quad \forall w \in \mathcal{U}^{ad}, \quad \text{(2.15c)} \]

where the adjoint \( p \in \mathcal{Y} \) solves the following convection-diffusion equation containing uncertain inputs
\[ -\nabla : (a(x, \omega) \nabla p) - b(x, \omega) \cdot \nabla p = (y - y^d) + \gamma (y - \mathbb{E}[y]) \quad \text{in } \mathcal{D} \times \Omega, \quad \text{(2.16a)} \]
\[ p = 0 \quad \text{on } \partial \mathcal{D} \times \Omega. \quad \text{(2.16b)} \]

In the following, we introduce the techniques, that is, Karhunen-Łóeve (KL) expansion, stochastic Galerkin, and discontinuous Galerkin method, to recast the infinite-dimensional model problem (2.12)–(2.13) into the finite dimensional.

### 3 Finite-dimensional representation

#### 3.1 Finite representation of stochastic fields

To solve (2.12)–(2.13) numerically, it is needed to reduce the stochastic process into finite mutually uncorrelated random variables. Therefore, the coefficients \( a(x, \omega) \) and \( b(x, \omega) \) are approximated by finite uncorrelated components \( \{\xi_i(\omega)\}_{i=1}^{N_{\xi}} \), called as finite-dimensional noise [43, 45]. Introducing the probability density functions of \( \{\xi_i(\omega)\}_{i=1}^{N_{\xi}} \) denoted by \( \rho_i : \Gamma_i \rightarrow [0, 1] \) with a bounded interval \( \Gamma_i = \xi_i(\Omega) \in \mathbb{R} \), the probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \) is replaced by \( (\Gamma, \mathcal{B}(\Gamma), \rho(\xi) \, d\xi) \), where \( \Gamma \) represents the support of such probability density, \( \mathcal{B}(\Gamma) \) is a Borel \( \sigma \)-algebra, and \( \rho(\xi) \, d\xi \) corresponds to the distribution measure of \( \xi \). Moreover, \( \rho(\xi) \) denotes the joint probability density function. Hence, we can state the tensor-product space \( H^k(\mathcal{D}) \otimes L^2(\Gamma) \) endowed with the following norm
\[ \|\eta\|_{H^k(\mathcal{D}) \otimes L^2(\Gamma)} := \left( \int_{\Gamma} \|\eta(\cdot, \xi)\|_{H^k(\mathcal{D})}^2 \rho(\xi) \, d\xi \right)^{1/2} < \infty. \quad \text{(3.1)} \]

Following the well-known KL expansion [46, 47], a random field \( \eta \) having a continuous covariance function as follows
\[ \mathbb{C}_\eta(x, y) := \int_{\Omega} (\eta(x, \cdot) - \bar{\eta}(x))(\eta(y, \cdot) - \bar{\eta}(y)) \, d\mathbb{P}(\omega) \quad \text{(3.2)} \]
admits a proper orthogonal decomposition

$$
\eta(x, \omega) = \overline{\eta}(x) + \kappa \sum_{k=1}^{\infty} \sqrt{\lambda_k} \phi_k(x) \xi_k(\omega),
$$

(3.3)

where \(\overline{\eta}(x)\) and \(\kappa\) are mean and standard deviation of \(\eta\), respectively, and \(\xi := \{\xi_1, \xi_2, \ldots\}\) are uncorrelated random variables. The pair \(\{\lambda_k, \phi_k\}\) is a set of the eigenvalues and eigenfunctions of the corresponding covariance operator \(C_\eta\). Then, we approximate \(\eta(x, \omega)\) by truncating its KL expansion of the form

$$
\eta(x, \omega) \approx \eta_N(x, \omega) := \overline{\eta}(x) + \kappa \sum_{k=1}^{N} \sqrt{\lambda_k} \phi_k(x) \xi_k(\omega).
$$

(3.4)

The truncated KL expansion (3.4) is a finite representation of the random field \(\eta(x, \omega)\) in the sense that the mean-square error of approximation is minimized; see, e.g., [48]. To guarantee the positivity of the truncated KL expansion (3.4) for the diffusivity coefficient \(a(x, \omega)\), it is also assumed that the mean of random coefficient exhibits a stronger dominance; see, e.g., [49].

By the assumption on the finite-dimensional and Doob-Dynkin lemma [50], the solution of (2.11) can be expressed in the finite-dimensional stochastic space, that means, \(y(x, \xi(\omega)) \in \mathcal{Y}_\rho = L^2(H_0^1(D); \Gamma)\) with \(\xi = (\xi_1(\omega), \ldots, \xi_N(\omega))\). Then, setting \(\widetilde{E}[y] = \int_D y \rho(\xi) d\xi\), the optimization problem (2.12)–(2.13) becomes

$$
\min_{u \in \mathcal{U}^d} \mathcal{J}(u) = \frac{1}{2} \widetilde{E} \left[ \int_D (y(u) - y^d)^2 \, dx \right] + \frac{\gamma}{2} \widetilde{E} \left[ \int_D (y(u) - \widetilde{E}[y(u)])^2 \, dx \right]
+ \frac{\mu}{2} \int_D u^2 \, dx
$$

(3.5)

subject to

$$
a[y, v]_\rho + b[u, v]_\rho = [f, v]_\rho, \quad \forall v \in \mathcal{Y}_\rho,
$$

(3.6)

where

$$
a[y, v]_\rho = \int_{\Gamma} \int_D (a(x, \xi) \nabla y \cdot \nabla v + b(x, \xi) \cdot \nabla y v) dx \rho(\xi) d\xi, \quad \forall y, v \in \mathcal{Y}_\rho,
$$

(3.7a)

$$
b[u, v]_\rho = - \int_{\Gamma} \int_D u v dx \rho(\xi) d\xi, \quad \forall u \in \mathcal{U}, \quad \forall v \in \mathcal{Y}_\rho,
$$

(3.7b)

$$
[f, v]_\rho = \int_{\Gamma} \int_D f v dx \rho(\xi) d\xi, \quad \forall v \in \mathcal{Y}_\rho.
$$

(3.7c)

Then, the optimization problem (3.5)–(3.6) has a unique solution pair \((y, u) \in \mathcal{Y}_\rho \times \mathcal{U}^d\) if and only if there is an adjoint \(p \in \mathcal{Y}_p\) such that the following optimality system holds for the triplet \((y, u, p)\):

$$
a[y, v]_\rho + b[u, v]_\rho = [f, v]_\rho, \quad \forall v \in \mathcal{Y}_\rho,
$$

(3.8a)

$$
a[q, p]_\rho = [y - y^d, q]_\rho + \gamma [y - \widetilde{E}[y], q]_\rho, \quad \forall q \in \mathcal{Y}_\rho,
$$

(3.8b)

$$
(\widetilde{E}[p] + \mu u, w - u) \geq 0, \quad \forall w \in \mathcal{U}^d.
$$

(3.8c)
Next, we present the representation of stochastic solutions, i.e., \( y(x, \xi), p(x, \xi) \), by using a polynomial chaos (PC) approximation [26].

### 3.2 Stochastic Galerkin method

The state solution \( y(x, \xi) \in L^2(\Gamma, F, \mathbb{P}) \), as well as the adjoint solution \( p(x, \xi) \in L^2(\Gamma, F, \mathbb{P}) \), can be represented by a finite generalized polynomial chaos (PC) approximation as stated in Cameron-Martin theorem [51],

\[
\begin{align*}
y(x, \omega) & \approx y_j(x, \xi) = \sum_{i=0}^{J-1} y_i(x) \psi_i(\xi), \\
p(x, \omega) & \approx p_j(x, \xi) = \sum_{i=0}^{J-1} p_i(x) \psi_i(\xi),
\end{align*}
\]

where \( y_i(x) \) and \( p_i(x) \) are the deterministic modes of the expansion and the total number of PC basis is determined by the dimension \( N \) of the random vector \( \xi \) and the highest order \( Q \) in the basis set of \( \psi_i \)

\[
J = 1 + \sum_{s=1}^{Q} \frac{1}{s!} \prod_{j=0}^{s-1} (N + j) = \frac{(N + Q)!}{N!Q!}.
\]

By following [49, 52], we then define the stochastic space as

\[
S_k := \text{span}\{\psi_i(\xi) : i = 0, 1, \ldots, J - 1\} \subset L^2(\Gamma).
\]

For simplicity, we only deal with the state equation since the procedure for the adjoint equation is similar to the state ones. By inserting KL expansions (3.4) of the diffusion \( a(x, \omega) \) and the convection \( b(x, \omega) \) coefficients, and the solution expression (3.9) into the variational form of the state (3.6) and projecting onto the space of the PC basis functions, we get a linear system, consisting of \( J \) deterministic convection-diffusion equations for \( j = 0, \ldots, J - 1 \)

\[
\sum_{i=0}^{J-1} \left( - \nabla \cdot (a_{ij} \nabla y_i(x)) + b_{ij} \cdot \nabla y_i(x) \right) = \{\psi_j\} f(x) + \{\psi_j\} u(x),
\]

where

\[
a_{ij} = a(x) \left( \psi_i^2(\xi) \right) \delta_{ij} + \kappa_d \sum_{k=1}^{N} \sqrt{\lambda_k^a \phi_k^a(x)} \langle \xi_k \psi_i(\xi) \psi_j(\xi) \rangle,
\]

\[
b_{ij} = b(x) \left( \psi_i^2(\xi) \right) \delta_{ij} + \kappa_b \sum_{k=1}^{N} \sqrt{\lambda_k^b \phi_k^b(x)} \langle \xi_k \psi_i(\xi) \psi_j(\xi) \rangle
\]

with \( \langle \psi(\xi) \rangle = \int_{\Gamma} \psi(\xi) \rho(\xi) \, d\xi \). Here, we apply the same distribution for both diffusion and convection random coefficients in order to reduce the computational effort. However, it can be possible to use different distributions; see, e.g., [53] for more discussion. We also note that the quantity of interest is the statistical moments of the solution \( y(x, \omega) \) rather than the solution \( y(x, \omega) \).
3.3 Symmetric interior penalty Galerkin (SIPG) method

We briefly recall the SIPG discretization following the studies in [30, 54]. A shape-regular simplicial triangulations of $\mathcal{D}$ is denoted by $\{T_h\}$ with $\overline{\mathcal{D}} = \bigcup_{K \in T_h} \overline{K}$. The set of all edges $\mathcal{E}_h$ consists of the interior edges $\mathcal{E}_h^0$ and boundary edges $\mathcal{E}_h^\delta$ such that $\mathcal{E}_h = \mathcal{E}_h^0 \cup \mathcal{E}_h^\delta$. For a fixed realization $\omega$ and the unit outward normal $\mathbf{n}_K$ to $\partial K$, we decompose the boundary edges of an element $K$ into the inflow $\partial K^-$ into the set of all edges $\mathcal{E}_h$ consists of the interior edges $\mathcal{E}_h^0$ and boundary edges $\mathcal{E}_h^\delta$ such that $\mathcal{E}_h = \mathcal{E}_h^0 \cup \mathcal{E}_h^\delta$. For a fixed realization $\omega$ and the unit outward normal $\mathbf{n}_K$ to $\partial K$, we decompose the boundary edges of an element $K$ into the inflow $\partial K^-$ into the flow $\partial K^+$ parts such that $\partial K^+ = \partial K \setminus \partial K^-$. Jump and average operators of $y$ and $\nabla y$ for a common edge $E = K \cap K^e$ are given, respectively, by

$$
\|y\| = y|_E n_K + y^e|_{K^e}, \quad \|\nabla y\| = \nabla y|_E \cdot n_K + \nabla y^e|_E \cdot n_{K^e}, \quad (3.12a)
$$

$$
\{y\} = \frac{1}{2} (y|_E + y^e|_E), \quad \{\nabla y\} = \frac{1}{2} (\nabla y|_E + \nabla y^e|_E), \quad (3.12b)
$$

where $y|_E$ (or $\nabla y|_E$) and $y^e|_E$ (or $\nabla y^e|_E$) are traces from inside $K$ and $K^e$, respectively. For a boundary edge $E \in K \cap \partial \mathcal{D}$, the operators are defined by $\{\nabla y\} = \nabla y$ and $\|y\| = y n$, where $n$ denotes the unit outward normal to $\partial \mathcal{D}$. Further, we set $h = \max_{K \in T_h} h_K$, where $h_K$ is the diameter of an element $K$.

Defining the discrete space as follows

$$
V_h = \{y \in L^2(I) : y|_K \in \mathbb{P}(K) \quad \forall K \in T_h\}, \quad (3.13)
$$

where $\mathbb{P}(K)$ is the set of linear polynomials and following the standard discontinuous Galerkin structure discussed in [32, 33], the (bi)-linear forms for a finite-dimensional vector $\xi$ can be stated as follows:

$$
a_h(y, v, \xi) = \sum_{K \in T_h} \int_K a(\cdot, \xi) \nabla y \cdot \nabla v \, dx - \sum_{E \in \mathcal{E}_h^0 \cup \mathcal{E}_h^\delta} \int_E \{a(\cdot, \xi) \nabla y\} \{v\} \, ds
$$

$$
- \sum_{E \in \mathcal{E}_h^0 \cup \mathcal{E}_h^\delta} \int_E \{a(\cdot, \xi) \nabla v\} \{y\} \, ds + \sum_{E \in \mathcal{E}_h^0 \cup \mathcal{E}_h^\delta} \frac{\sigma}{h_E} \int_E \|y\| \cdot \|v\| \, ds
$$

$$
+ \sum_{K \in T_h} \int_K b(\cdot, \xi) \cdot \nabla y v \, dx + \sum_{K \in T_h} \int_{\partial K \setminus \partial I} b(\cdot, \xi) \cdot n_K (y^e - y) v \, ds
$$

$$
- \sum_{K \in T_h} \int_{\partial K \setminus \partial I} b(\cdot, \xi) \cdot n_K y v \, ds,
$$
where the parameter $\sigma \in \mathbb{R}_0^+$, called as the penalty parameter, should be sufficiently large to ensure the stability of the SIPG scheme; independent of the mesh size $h$. However, as discussed in [33, Sec. 2.7.1], it depends on the degree of polynomials used in the DG discretization and the position of the edge $E$. In our numerical experiments, we choose $\sigma$ as $\sigma = 6$ on the interior edges $E^0_h$ and 12 on the boundary edges $E^\partial_h$.

Then, the (bi)-linear forms of the stochastic discontinuous Galerkin (SDG) for the state equation correspond to

$$a_\xi[y, v] + b_\xi[u, v] = [f, v]_\xi,$$

where

$$a_\xi[y, v] = \int_{\Gamma} a_h(y, v, \xi) \rho(\xi) d\xi, \quad b_\xi[u, v] = \int_{\Gamma} b_h(u, v, \xi) \rho(\xi) d\xi,$$

$$[f, v]_\xi = \int_{\Gamma} l_h(f, v, \xi) \rho(\xi) d\xi.$$

Now, we can state the discrete optimal control problem

$$\min_{u_h \in \mathcal{U}^d_h} \mathcal{J}(u_h) = \frac{1}{2} \int_{\Omega} \left( y_h - y^d \right)^2 \, dx + \frac{\gamma}{2} \int_{\Omega} \left( y_h - \mathbb{E}[y_h] \right)^2 \, dx$$

$$+ \frac{\mu}{2} \int_{\Omega} u_h^2 \, dx$$

(3.14)

governed by

$$a_\xi[y_h, v_h] + b_\xi[u_h, v_h] = [f, v_h]_\xi, \quad \forall v_h \in \mathcal{V}_h = V_h \otimes \mathcal{S}_k,$$

(3.15)

where the discrete admissible set (1.3) is defined by

$$\mathcal{U}^d_h := \{ u_h \in \mathcal{U}_h : u_a \leq u_h(x) \leq u_b, \text{ a.e. } x \in K \subset \mathcal{T}_h \},$$

(3.16)

with $\mathcal{U}^d_h = \mathcal{U}_h \cap \mathcal{U}^d$ and $\mathcal{U}_h = \mathcal{V}_h$. Analogously, a pair $(y_h, u_h) \in \mathcal{V}_h \times \mathcal{U}^d_h$ is a unique solution of the control problem (3.14)–(3.15) if and only if an adjoint $p_h \in \mathcal{V}_h$ exists such that the optimality system holds for $(y_h, u_h, p_h) \in \mathcal{V}_h \times \mathcal{U}^d_h \times \mathcal{V}_h$

$$a_\xi[y_h, v_h] + b_\xi[u_h, v_h] = [f, v_h]_\xi, \quad v_h \in \mathcal{V}_h,$$

(3.17a)

$$a_\xi[q_h, p_h] = [y_h - y^d, q_h]_\xi + \gamma [y_h - \mathbb{E}[y_h], q_h]_\xi, \quad q_h \in \mathcal{V}_h,$$

(3.17b)

$$[p_h + \mu u_h, w_h - u_h]_\xi \geq 0, \quad w_h \in \mathcal{U}_h,$$

(3.17c)
where \([p_h + \mu u_h, w_h - u_h]_\xi = (\mathbb{E}[p_h] + \mu u_h, w_h - u_h)\) since the discrete solution \(u_h\) is deterministic.

Further, by denoting

\[
J_h'(u_h) \cdot w_h = [p_h + \mu u_h, w_h]_\xi, \quad \forall w_h \in \mathcal{U}_h^{ad},
\]

one can easily obtain the following expression for the discrete directional derivative of functional \(J_h(u_h)\):

\[
J_h'(u_h) \cdot (w_h - u_h) \geq 0, \quad \forall w_h \in \mathcal{U}_h^{ad}.
\]

### 4 Error analysis

We provide an a priori error analysis of the optimization problem (2.12)–(2.13), discretized by the stochastic discontinuous Galerkin method. Before deriving the corresponding estimates, we define the associated energy norm on \(\mathcal{D} \times \Gamma\) as

\[
\|y\|_\xi = \left( \int_\Gamma \|y(\cdot, \xi)\|_e^2 \rho(\xi) \, d\xi \right)^{1/2},
\]

where \(\|y(\cdot, \xi)\|_e\) is the energy norm on \(\mathcal{D}\), given as

\[
\|y(\cdot, \xi)\|_e = \left( \sum_{K \in \mathcal{T}_h} \int_K a(\cdot, \xi)(\nabla y)^2 \, dx + \sum_{E \in E_0 \cup E_3} \frac{\sigma}{\ell_E} \int_E \|y\|^2 \, ds \right.
\]

\[
+ \frac{1}{2} \sum_{E \in E_1, E \in E_2} \int_{E \in E_1} b(\cdot, \xi) \cdot n_E y^2 \, ds + \frac{1}{2} \sum_{E \in E_0 \cup E_3} \int_{E \in E_1} b(\cdot, \xi) \cdot n_E (y^e - y)^2 \, ds \right)^{1/2}.
\]

By the standard arguments as done in deterministic case, one can easily show the coercivity and continuity of \(a_\xi(\cdot, \cdot)\) for \(y, v \in \mathcal{Y}_h\)

\[
a_\xi[y, y] \geq c_{cv} \|y\|_\xi^2, \quad a_\xi[y, v] \leq c_{ct} \|y\|_\xi \|v\|_\xi,
\]

where the coercivity constant \(c_{cv}\) depends on \(a_{min}\), whereas the continuity constant \(c_{ct}\) depends on \(a_{max}\).

Next, we state the estimates on the finite-dimensional probability domain \(\Gamma\) and the physical domain \(K \in \mathcal{T}_h\). Let a partition of the support of probability density in finite-dimensional space, i.e., \(\gamma = \prod_{n=1}^N \Gamma_n\), consists of disjoint \(R^N\)-boxes,

\[
\gamma = \prod_{n=1}^N (r_n^\gamma, s_n^\gamma), \text{ with } (r_n^\gamma, s_n^\gamma) \subset \Gamma_n \text{ for } n = 1, \ldots, N \text{ so that the mesh size } k_n \text{ becomes } k_n = \max_{\gamma} |s_n^\gamma - r_n^\gamma|, n = 1 \ldots N. \]  

For the multi-index \(q = (q_1, \ldots, q_N)\), the (discontinuous) finite element approximation space having at most \(q_n\) degree on
each direction $\xi_n$ is denoted by $S^q_k \subset L^2(\Gamma)$. Then, for $v \in H^{q+1}(\Gamma)$, $\varphi \in S^q_k$, we have, see \cite{[43]},

$$\min_{\varphi \in S^q_k} \|v - \varphi\|_{L^2(\Gamma)} \leq \sum_{n=1}^{N} \frac{(k_n)}{2} \frac{q_{n+1} \|\partial^{q_{n+1}}v\|_{L^2(\Gamma)}}{(q_n + 1)!}. \quad (4.3)$$

For $v \in H^2(K)$ and $\tilde{v} \in \mathcal{P}(K)$, where $K \in \mathcal{T}_h$, the following discontinuous Galerkin approximation \cite[Theorem 2.6]{[33]} also holds

$$\|v - \tilde{v}\|_{H^q(K)} \leq C h^{2-q} |v|_{H^2(K)}, \quad 0 \leq q \leq 2, \quad (4.4)$$

where the constant $C$ is not depending on $v$ and $h$.

Further, we define the following projection operators, which are needed in the rest of the paper:

- **$L^2$-projection operators** $\Pi_n : L^2(\Gamma) \to S^q_k$ and $\Pi_h : L^2(D) \to V_h \cap L^2(D)$ are given by

  $$(\Pi_n(\xi) - \xi, \zeta)_{L^2(\Gamma)} = 0, \quad \forall \zeta \in S^q_k, \quad \forall \xi \in L^2(\Gamma), \quad (4.5a)$$

  $$(\Pi_h(v) - v, \chi)_{L^2(D)} = 0, \quad \forall \chi \in V_h, \quad \forall v \in L^2(D) \quad (4.5b)$$

  with the following estimate

  $$\|v - \Pi_h(v)\|_{L^2(L^2(D); \Gamma)} \leq C h \|v\|_{L^2(H^1(D); \Gamma)}. \quad (4.6)$$

  In addition, taking $\xi = \Pi_n(\xi)$ and $\chi = \Pi_h(v)$ in \((4.5a)\) and \((4.5b)\), respectively, it holds that

  $$\|\Pi_n(\xi)\|_{L^2(\Gamma)} \leq C \|\xi\|_{L^2(\Gamma)}, \quad \forall \xi \in L^2(\Gamma), \quad (4.7a)$$

  $$\|\Pi_h(v)\|_{L^2(D)} \leq C \|v\|_{L^2(D)}, \quad \forall v \in L^2(D). \quad (4.7b)$$

- **$H^1$-projection operator** $\mathcal{R}_h : H^1(D) \to V_h \cap H^1(D)$ is stated by

  $$(\mathcal{R}_h(v) - v, \vartheta)_{L^2(D)} = 0, \quad \forall \vartheta \in V_h, \quad \forall v \in H^1(D), \quad (4.8a)$$

  $$(\nabla(\mathcal{R}_h(v) - v), \nabla \vartheta)_{L^2(D)} = 0, \quad \forall \vartheta \in V_h, \quad \forall v \in H^1(D). \quad (4.8b)$$

With the help of the $H^1$-projection operator in \((4.8a)\), the Cauchy-Schwarz inequality, the $L^2$-projection operator in \((4.5a)\), and the approximation in \((4.3)\), we obtain the approximation property \cite[Theorem 3.2]{[54]}: for all $v \in L^2(H^2(D); \Gamma) \cap H^{q+1}(H^1(D); \Gamma)$ and $\tilde{v} \in V_h \times S^q_k$

$$\|v - \tilde{v}\|_{L^2(H^1(D); \Gamma)} \leq C h \|v\|_{L^2(H^2(D); \Gamma)} + \sum_{n=1}^{N} \frac{(k_n)}{2} \frac{q_{n+1} \|\partial^{q_{n+1}}v\|_{L^2(H^1(D); \Gamma)}}{(q_n + 1)!}, \quad (4.9)$$

where the constant $C$ does not depend on $v$, $h$, and $k_n$.

To recognize error contributions emerging from the spatial domain $D$ and the probability domain $\Gamma$, separately, a projection operator $\mathcal{P}_{hn}$ mapping onto the tensor product space $\mathcal{Y}_h$ is given by

$$\mathcal{P}_{hn} Y = \Pi_h \Pi_n Y = \Pi_n \Pi_h Y, \quad \forall Y \in L^2(L^2(D); \Gamma) \quad (4.10)$$
and the decomposition

\[ Y - \mathcal{P}_h Y = (Y - \Pi_h Y) + \Pi_h (I - \Pi_h) Y, \quad \forall Y \in L^2(L^2(\mathcal{D}); \Gamma). \tag{4.11} \]

Then, it follows from (4.7a), (4.7b), and (4.10) that

\[ \| \mathcal{P}_h Y \|_{L^2(L^2(\mathcal{D}); \Gamma)} \leq C \| Y \|_{L^2(L^2(\mathcal{D}); \Gamma)}, \quad \forall Y \in L^2(L^2(\mathcal{D}); \Gamma). \tag{4.12} \]

Before the derivation of a priori error estimate, we state the following auxiliary problem

\[ \mathcal{J}_h'(u) \cdot (w - u) = [p_h(u) + \mu u, w - u], \quad \forall w \in \mathcal{U}^{ad}, \tag{4.13} \]

where \( p_h(u) \in \mathcal{Y}_h \) solves the following auxiliary system:

\[ a_\xi [y_h(u), v_h] + b_\xi [u, v_h] = [f, v_h], \quad v_h \in \mathcal{Y}_h, \tag{4.14a} \]

\[ a_\xi [q_h, p_h(u)] = [y_h(u) - y^d, q_h] + \gamma [y_h(u) - \mathbb{E}[y_h(u)], q_h], \quad q_h \in \mathcal{Y}_h. \tag{4.14b} \]

It is also noted that we prefer to use \( \mathcal{Y}_h \) in the derivation of error estimates instead of \( \| u \|_{L^2(\mathcal{D}); \Gamma} \) for better readability in terms of notation.

**Lemma 4.1** With the definition in (4.13), the following estimate holds:

\[ \left( \mathcal{J}_h'(w) - \mathcal{J}_h'(u) \right) \cdot (w - u) \geq \mu \| w - u \|^2_{L^2(L^2(\mathcal{D}); \Gamma)}. \tag{4.15} \]

**Proof** By (4.13), we have

\[ \left( \mathcal{J}_h'(w) - \mathcal{J}_h'(u) \right) \cdot (w - u) = [p_h(w) - p_h(u), w - u] + \mu [w - u, w - u], \tag{4.16} \]

Now, it follows from (4.14) that

\[ [p_h(w) - p_h(u), w - u] = a_\xi [y_h(w) - y_h(u), p_h(w) - p_h(u)] = (1 + \gamma) [y_h(w) - y_h(u), y_h(w) - y_h(u)] \]

\[ - \gamma [\mathbb{E}[y_h(w) - y_h(u)], y_h(w) - y_h(u)] \tag{4.17} \]

The usage of Cauchy-Schwarz and Young’s inequalities yields

\[ - \gamma [\mathbb{E}[y_h(w) - y_h(u)], y_h(w) - y_h(u)] \geq - \frac{\gamma}{2} \| \mathbb{E}[y_h(w) - y_h(u)] \|_{L^2(L^2(\mathcal{D}); \Gamma)}^2 - \frac{\gamma}{2} \| y_h(w) - y_h(u) \|_{L^2(L^2(\mathcal{D}); \Gamma)}^2. \]

Since all norms are convex functions, Jensen’s inequality \( \| \mathbb{E}[u] \| \leq \mathbb{E}[\| u \|] \) and \( \mathbb{E}[\mathbb{E}[u]] = \mathbb{E}[u] \) give us

\[ - \gamma [\mathbb{E}[y_h(w) - y_h(u)], y_h(w) - y_h(u)] \geq - \gamma \| y_h(w) - y_h(u) \|_{L^2(L^2(\mathcal{D}); \Gamma)}^2. \tag{4.18} \]

Thus, inserting (4.18) into (4.17), it is obtained that

\[ [p_h(w) - p_h(u), w - u] \geq \| y_h(w) - y_h(u) \|_{L^2(L^2(\mathcal{D}); \Gamma)}^2 \tag{4.19} \]

Hence, (4.16) and (4.19) imply that (4.15) holds. \( \square \)
Next, we derive an upper bound for the error between the discrete solutions \((y_h, p_h)\) and the auxiliary solutions \((y_h(u), p_h(u))\).

**Lemma 4.2** Assume that \((y_h, p_h)\) and \((y_h(u), p_h(u))\), respectively, are the solutions of (3.17) and (4.14). Then, the following estimates exist for positive constants \(C_1\) and \(C_2\) independent of \(h\)

\[
\|y_h - y_h(u)\|_\xi \leq C_1 \|u - u_h\|_{L^2(\mathcal{D}; \Gamma)},
\]

\[
\|p_h - p_h(u)\|_\xi \leq C_2 \|u - u_h\|_{L^2(\mathcal{D}; \Gamma)}.
\]

**Proof** By subtracting (4.14a) from (3.17a) and taking \(v_h = y_h - y_h(u)\), we have that

\[
a_\xi [y_h - y_h(u), y_h - y_h(u)] = [u_h - u, y_h - y_h(u)]_\xi.
\]

With the help of the coercivity of \(a_\xi\) (4.2) and the Cauchy-Schwarz inequality, we obtain

\[
c_{cv} \|y_h - y_h(u)\|_\xi^2 \leq a_\xi [y_h - y_h(u), y_h - y_h(u)] \\
\quad \leq \|u_h - u\|_{L^2(\mathcal{D}; \Gamma)} \|y_h - y_h(u)\|_\xi,
\]

which yields the desired result (4.20a).

Analogously, by subtracting (4.14b) from (3.17b) and taking \(v_h = p_h - p_h(u)\), we have that

\[
a_\xi [p_h - p_h(u), p_h - p_h(u)] \\
= (1 + \gamma)[y_h - y_h(u), p_h - p_h(u)]_\xi + \gamma [\overline{E}[y_h(u) - y_h], p_h - p_h(u)]_\xi.
\]

It follows from the coercivity of \(a_\xi\), Cauchy-Schwarz inequality, and Jensen’s inequality that

\[
c_{cv} \|p_h - p_h(u)\|_\xi^2 \leq a_\xi [p_h - p_h(u), p_h - p_h(u)] \\
\quad \leq (1 + 2\gamma) \|p_h - p_h(u)\|_{L^2(\mathcal{D}; \Gamma)} \|y_h - y_h(u)\|_\xi.
\]

We note that the procedure applied in (4.18) is also used in the derivation of (4.21). Hence, by (4.21) and (4.20a), we deduce the desired result (4.20b).

To obtain an upper bound for the control, we divide the domain \(\mathcal{D}\) into pieces by considering the active and inactive parts of the control \(u\) as done in [55, 56]:

\[
\mathcal{D}^+ = \bigcup_K : K \subset \mathcal{D}, \ u_a < u|_K < u_b \bigg\},
\]

\[
\mathcal{D}^\delta = \left\{ \bigcup_K : K \subset \mathcal{D}, \ u|_K = u_a \ \text{or} \ u|_K = u_b \right\},
\]

\[
\mathcal{D}^- = \mathcal{D} \setminus (\mathcal{D}^+ \cup \mathcal{D}^\delta).
\]

It is assumed that these sets are disjoint, \(\mathcal{D} = \mathcal{D}^+ \cup \mathcal{D}^\delta \cup \mathcal{D}^-\), and \(\mathcal{D}^-\) satisfies the following inequality related to the regularity of \(u\) and \(\mathcal{T}_h\)

\[
\text{meas}(\mathcal{D}^-) \leq C h,
\]
which is valid if the boundary of the $\mathcal{D}^d$ is represented by finite rectifiable curves [57]. Further, we define a set such that $\mathcal{D}^+ \subset \mathcal{D}^* = \{ x \in \mathcal{D} : u_a < u(x) < u_b \}$ [58].

**Lemma 4.3** Let $(y, u, p)$ and $(y_h, u_h, p_h)$, respectively, be the solutions of (2.15) and (3.17). Assume that $u \in L^2(W^{1, \infty}(\mathcal{D}); \Gamma)$ with $u|_{\mathcal{D}^+} \in L^2(H^2(\mathcal{D}^+); \Gamma)$. Then, it holds that

$$
\|u - u_h\|_{L^2(L^2(\mathcal{D}); \Gamma)} \\
\leq C \|p - p_h(u)\|_{L^2(L^2(\mathcal{D}); \Gamma)} + C h^{3/2} \|u\|_{L^2(W^{1, \infty}(\mathcal{D}); \Gamma)} \\
+ C \left( h \|p\|_{L^2(H^1(\mathcal{D}); \Gamma)} + \sum_{n=1}^N \left( \frac{k_n}{2} \right)^{q_n+1} \frac{\|\partial_{\xi_n}^{q_n+1} p\|_{L^2(H^1(\mathcal{D}); \Gamma)}}{(q_n + 1)!} \right). (4.24)
$$

**Proof** With the help of Lemma 4.1, (4.13), the standard Lagrangian interpolation $\Pi u$, the assumption $\mathcal{D}^+ \subset \mathcal{D}^*$, and the notation $p_h = p_h(u_h)$, we obtain

$$
\mu \|u - u_h\|_{L^2(L^2(\mathcal{D}); \Gamma)}^2 \\
\leq J_h'(u) \cdot (u - u_h) - J_h'(u_h) \cdot (u - u_h) \\
= \left[ \mu u + p_h(u), u - u_h \right]_\xi - \left[ \mu u + p_h(u), u - u_h \right]_\xi \\
= \left[ \mu u + p, u - u_h \right]_\xi - \left[ p - p_h(u), u - u_h \right]_\xi \\
+ \left[ \mu u_h + p_h, u_h - \Pi u \right]_\xi + \left[ \mu u + p, \Pi u - u \right]_\xi \\
- J_h'(u_h) \cdot (\Pi u - u) \leq 0 \\
\leq [\mu u_h + p_h, \Pi u - u]_\xi + [p_h(u) - p, u - u_h]_\xi. (4.25)
$$

The first term in (4.25) can be rewritten as follows

$$
[\mu u_h + p_h, \Pi u - u]_\xi = [\mu u_h + p_h - \mu u - p, \Pi u - u]_\xi + [\mu u + p, \Pi u - u]_\xi \\
= [\mu u_h - \mu u, \Pi u - u]_\xi + [\mu u + p, \Pi u - u]_\xi \\
+ [p_h - p_h(u), \Pi u - u]_\xi + [p_h(u) - p, \Pi u - u]_\xi. (4.26)
$$

Then, inserting (4.26) into (4.25) and applying Cauchy-Schwarz and Young's inequalities and Lemma 4.2, we obtain

$$
\mu \|u - u_h\|_{L^2(L^2(\mathcal{D}); \Gamma)}^2 \\
\leq c_1 \|p_h(u) - p\|_{L^2(L^2(\mathcal{D}); \Gamma)}^2 + C_2 \|u - u_h\|_{L^2(L^2(\mathcal{D}); \Gamma)}^2 \\
+ c_3 \|u - \Pi u\|_{L^2(L^2(\mathcal{D}); \Gamma)}^2 + [\mu u + p, \Pi u - u]_\xi. (4.27)
$$

Since $\Pi u(x) = u(x)$ for any vertex $x$, $\Pi u \in U_h^{ad}$ and the following estimates hold

$$
\|u - \Pi u\|_{L^2(L^2(\mathcal{D}^+); \Gamma)} \leq C h^2 \|u\|_{L^2(H^2(\mathcal{D}^+); \Gamma)}, (4.28a) \\
\|u - \Pi u\|_{L^2(W^{0, \infty}(\mathcal{D}^-); \Gamma)} \leq C h \|u\|_{L^2(W^{1, \infty}(\mathcal{D}^-); \Gamma)} (4.28b)
$$
for $u \in L^2(W^{1,\infty}(\mathcal{D}; \Gamma))$ and $u|_{\mathcal{D}^\ast} \subset L^2(H^2(\mathcal{D}^\ast); \Gamma)$. Hence

\[
\|u - \Pi u\|_{L^2(L^2(\mathcal{D}); \Gamma)}^2 = \|u - \Pi u\|_{L^2(L^2(\mathcal{D}^+); \Gamma)}^2 + \|u - \Pi u\|_{L^2(L^2(\mathcal{D}^-); \Gamma)}^2 + \|u - \Pi u\|_{L^2(L^2(\mathcal{D}^\ast); \Gamma)}^2 = 0
\]

\[
\leq \|u - \Pi u\|_{L^2(L^2(\mathcal{D}^+); \Gamma)}^2 + C \|u - \Pi u\|_{L^2(W^{0,\infty}(\mathcal{D}^-); \Gamma)}^2 \text{meas}(\mathcal{D}^-)
\]

\[
\leq Ch^4 \|u\|_{L^2(H^2(\mathcal{D}^+); \Gamma)}^2 + Ch^3 \|u\|_{L^2(W^{1,\infty}(\mathcal{D}^-); \Gamma)}^2
\]

\[
\leq Ch^3 \left( \frac{\|u\|_{L^2(H^2(\mathcal{D}^+); \Gamma)}^2 + \|u\|_{L^2(W^{1,\infty}(\mathcal{D}^-); \Gamma)}^2}{2} \right).
\]

By the variational inequality (3.17c) and the definitions of domains (4.22), we have

\[
\mu u + p = 0 \text{ on } \mathcal{D}^+ \quad \text{and} \quad \Pi u - u = 0 \text{ on } \mathcal{D}\delta.
\]

Then,

\[
[\mu u + p, \Pi u - u]_\xi = \left[ \mu u - \Pi_h(\mu u) + \Pi_h(\mu u), \Pi u - u \right]_{\mathcal{T}_1}
\]

\[
+ \left[ p - \mathcal{P}_h(p) + \mathcal{P}_h(p), \Pi u - u \right]_{\mathcal{T}_2}.
\]

It follows from the inequalities (4.6), (4.7b), and (4.28b), Sobolev embedding theorem, see, e.g., [59], and Young’s inequality that

\[
T_1 = \left[ \mu u - \Pi_h(\mu u), \Pi u - u \right]_{\mathcal{D}} + [\Pi_h(\mu u), \Pi u - u]_{\mathcal{D}}
\]

\[
\leq \mu \left( \|u - \Pi_h u\|_{L^2(L^2(\mathcal{D}^-); \Gamma)} + \|\Pi_h u\|_{L^2(L^2(\mathcal{D}^-); \Gamma)} \right) \|u - \Pi u\|_{L^2(L^2(\mathcal{D}^-); \Gamma)}
\]

\[
\leq \mu \left( \|u - \Pi_h u\|_{L^2(L^2(\mathcal{D}^-); \Gamma)} + C \|u\|_{L^2(L^2(\mathcal{D}^-); \Gamma)} \right) \|u - \Pi u\|_{L^2(L^2(\mathcal{D}^-); \Gamma)}
\]

\[
\times \|u - \Pi u\|_{L^2(L^2(\mathcal{D}^-); \Gamma)}
\]

\[
\leq Ch \|u\|_{L^2(H^1(\mathcal{D}^-); \Gamma)} \|u - \Pi u\|_{L^2(W^{0,\infty}(\mathcal{D}^-); \Gamma)} \text{meas}(\mathcal{D}^-)
\]

\[
\leq Ch^3 \|u\|_{L^2(H^1(\mathcal{D}^-); \Gamma)} \|u\|_{L^2(W^{1,\infty}(\mathcal{D}^-); \Gamma)}
\]

\[
\leq Ch^3 \left( \|u\|_{L^2(H^1(\mathcal{D}^-); \Gamma)}^2 + \|u\|_{L^2(W^{1,\infty}(\mathcal{D}^-); \Gamma)}^2 \right).
\]
Next, with the help of the projector operator in (4.10) and the bounds in (4.3), (4.6), (4.12), and (4.28b), Sobolev embedding theorem, and Cauchy and Young’s inequalities, we find a bound for the second term $T_2$ in (4.30)

$$T_2 = [p - \Pi_h(p), \Pi u - u]_{\mathcal{D}^{-}} + \{p_n(p), \Pi u - u\}_{\mathcal{D}^{-}} + \{\Pi_h(I - \Pi_n)(p), \Pi u - u\}_{\mathcal{D}^{-}}$$

$$\leq \left( \|p - \Pi_h(p)\|_{L^2(\mathcal{D}^{-};\Gamma')} + \|p_n(p)\|_{L^2(\mathcal{D}^{-};\Gamma')} \right) \|\Pi u - u\|_{L^2(\mathcal{D}^{-};\Gamma')} + \|\Pi_h(I - \Pi_n)(p)\|_{L^2(\mathcal{D}^{-};\Gamma')} \|\Pi u - u\|_{L^2(\mathcal{D}^{-};\Gamma')}$$

$$\leq C_1 \left( \|p\|_{L^2(\mathcal{D}^{-};\Gamma')} + \|p_n\|_{L^2(W^{0,\infty}(\mathcal{D}^{-};\Gamma') \text{meas}(\mathcal{D}^{-})))} \right) h^2 \|u\|_{L^2(W^{1,\infty}(\mathcal{D}^{-};\Gamma'))} + C_2 \sum_{n=1}^N \left( \frac{k_n}{2} \right)^{n+1} \frac{\|\partial^{q_n+1} p\|_{L^2(\mathcal{D}^{-};\Gamma')}}{(q_n + 1)!} h^2 \|u\|_{L^2(W^{1,\infty}(\mathcal{D}^{-};\Gamma'))}$$

$$\leq C_1 \left( \frac{h^2}{2} \|p\|_{L^2(\mathcal{D}^{-};\Gamma')} + \frac{h^4}{2} \|u\|_{L^2(W^{1,\infty}(\mathcal{D}^{-};\Gamma'))} \right) + C_2 \left( \frac{1}{2} \sum_{n=1}^N \left( \frac{k_n}{2} \right)^{2n+2} \frac{\|\partial^{q_n+1} p\|_{L^2(\mathcal{D}^{-};\Gamma')}}{(q_n + 1)!^2} + \frac{h^4}{2} \|u\|_{L^2(W^{1,\infty}(\mathcal{D}^{-};\Gamma'))} \right).$$

(4.32)

Combination of (4.31) and (4.32) yields

$$[\mu u + p, \Pi u - u]_{\xi}$$

$$\leq C h^3 \left( \|u\|_{L^2(\mathcal{D}^{-};\Gamma')} + \|u\|_{L^2(W^{1,\infty}(\mathcal{D}^{-};\Gamma'))} \right)$$

$$+ C h^2 \|p\|_{L^2(\mathcal{D}^{-};\Gamma')} + C \sum_{n=1}^N \left( \frac{k_n}{2} \right)^{2n+2} \frac{\|\partial^{q_n+1} p\|_{L^2(\mathcal{D}^{-};\Gamma')}}{(q_n + 1)!^2}.$$

(4.33)

Finally, inserting (4.29) and (4.33) into (4.27), we complete the proof of Lemma 4.3.

\[\square\]

**Lemma 4.4** Assume that $(y, p)$ and $(y_h(u), p_h(u))$, respectively, are the solutions of (2.15) and (4.14). Then, we have

$$\|y - y_h(u)\|_{\xi} \leq C h \|y\|_{L^2(\mathcal{D};\Gamma')}$$

$$+ \sum_{n=1}^N \left( \frac{k_n}{2} \right)^{n+1} \frac{\|\partial^{q_n+1} y\|_{L^2(\mathcal{D};\Gamma')}}{(q_n + 1)!}. \quad (4.34)$$

and

$$\|p - p_h(u)\|_{\xi} \leq C h \left( \|y\|_{L^2(\mathcal{D};\Gamma')} + \|p\|_{L^2(\mathcal{D};\Gamma')} \right)$$

$$+ \sum_{n=1}^N \left( \frac{k_n}{2} \right)^{n+1} \frac{\|\partial^{q_n+1} y\|_{L^2(\mathcal{D};\Gamma')} + \|\partial^{q_n+1} p\|_{L^2(\mathcal{D};\Gamma')}}{(q_n + 1)!}. \quad (4.35)$$
Proof An application of the coercivity and continuity of $a_\xi$ in (4.2), $H^1(D)$-projection $R_h$ in (4.8), $L^2(D)$-projection $\Pi_n$ in (4.5a), and Galerkin orthogonality yields
\[
c_{cv}\|y - y_h(u)\|^2_\xi \leq a_\xi[y - y_h(u), y - y_h(u)] \\
\leq a_\xi[y - y_h(u), y - \Pi_n(R_h(y))] + a_\xi[y - y_h(u), \Pi_n(R_h(y)) - y_h(u)] \\
\leq c_{cl}\|y - y_h(u)\|_\xi\|y - \Pi_n(R_h(y))\|_\xi.
\]
Then, by the approximation property (4.9), we get
\[
\|y - y_h(u)\|_\xi \leq \frac{c_{cl}}{c_{cv}}\|y - \Pi_n(R_h(y))\|_\xi \\
\leq Ch\|y\|_{L^2(H^2(D);\Gamma)} + \sum_{n=1}^N \left(\frac{k_n}{2}\right)^{q_n+1} \frac{\|\partial_{\xi_n}^n y\|_{L^2(H^1(D);\Gamma)}}{(q_n + 1)!},
\]
which is the desired result (4.34). Analogously, we deduce that
\[
c_{cv}\|p - p_h(u)\|^2_\xi \leq a_\xi[p - p_h(u), p - p_h(u)] \\
\leq a_\xi[p - \Pi_n(R_h(y)), p - p_h(u)] + \frac{a_\xi[\Pi_n(R_h(y)) - p_h(u), p - p_h(u)]}{0} \\
= (1 + \gamma)[y - y_h(u), p - \Pi_n(R_h(y))] + \gamma[E[y_h(u) - y, p - \Pi_n(R_h(y))]]_\xi \\
\leq (1 + 2\gamma)\|y - y_h(u)\|_\xi\|p - \Pi_n(R_h(y))\|_{L^2(L^2(D);\Gamma)} \\
\leq \frac{(1 + 2\gamma)}{2}\|y - y_h(u)\|^2_\xi + \frac{(1 + 2\gamma)}{2}\|p - \Pi_n(R_h(y))\|^2_{L^2(L^2(D);\Gamma)},
\]
where the definition of bilinear forms, the procedure applied in (4.18), and Young’s inequality are used. Then, using the approximation property (4.9) and (4.34), we complete the proof of (4.35). \hfill \blacksquare

Now, we finalize the error analysis by combining the findings in Lemmas 4.3 and 4.4.

**Theorem 4.5** Assume that $(y, u, p)$ and $(y_h, u_h, p_h)$, respectively, are the solutions of (2.15) and (3.17). Then, it holds that
\[
\|u - u_h\|_{L^2(L^2(D);\Gamma)} + \|y - y_h\|_\xi + \|p - p_h\|_\xi \\
\leq Ch^{3/2}\|u\|_{L^2(W^{1,\infty}(D);\Gamma)} + Ch\left(\|y\|_{L^2(H^2(D);\Gamma)} + \|p\|_{L^2(H^2(D);\Gamma)}\right) \\
+ C\sum_{n=1}^N \left(\frac{k_n}{2}\right)^{q_n+1} \frac{\|\partial_{\xi_n}^{q_n+1} y\|_{L^2(H^1(D);\Gamma)} + \|\partial_{\xi_n}^{q_n+1} p\|_{L^2(H^1(D);\Gamma)}}{(q_n + 1)!}. \tag{4.36}
\]
Proof From (4.24) and (4.35), we obtain that

\[
\|u - u_h\|_{L^2(L^2(D);\Gamma)} \\
\leq C h^{-3/2} \|u\|_{L^2(W_1,\infty(D);\Gamma)} + C h \left( \|y\|_{L^2(H^2(D);\Gamma)} + \|p\|_{L^2(H^2(D);\Gamma)} \right) \\
+ C \sum_{n=1}^{N} \left( \frac{k_n}{2} \right)^{q_n+1} \left( \frac{\|q_n^{q_n+1}\|_{L^2(H^1(D);\Gamma)} + \|q_n^{q_n+1}\|_{L^2(H^1(D);\Gamma)}}{(q_n + 1)!} \right). \tag{4.37}
\]

Moreover, by Lemmas 4.2 and 4.4, and the bound (4.37), we obtain

\[
\|y - y_h\|_{\xi} + \|p - p_h\|_{\xi} \\
\leq \|y - y_h(u)\|_{\xi} + \|y_h(u) - y_h\|_{\xi} + \|p - p_h(u)\|_{\xi} + \|p_h(u) - p_h\|_{\xi} \\
\leq C \|u - u_h\|_{L^2(L^2(D);\Gamma)} + C h \left( \|y\|_{L^2(H^2(D);\Gamma)} + \|p\|_{L^2(H^2(D);\Gamma)} \right) \\
+ C \sum_{n=1}^{N} \left( \frac{k_n}{2} \right)^{q_n+1} \left( \frac{\|q_n^{q_n+1}\|_{L^2(H^1(D);\Gamma)} + \|q_n^{q_n+1}\|_{L^2(H^1(D);\Gamma)}}{(q_n + 1)!} \right). \tag{4.38}
\]

Thus, by combining (4.37) and (4.38), we deduce the desired result (4.36).

\[
\square
\]

5 Matrix formulation

In this section, we first construct the matrix formulation of the underlying problem (2.12)–(2.13) by employing the “optimize-then-discretize” approach; see, e.g., [42]. In this methodology, one first obtains the optimality system (2.15) of the infinite-dimensional optimization problem, and then discretizes the optimality system by a stochastic discontinuous Galerkin method discussed in Section 3. Later, we propose a low-rank variant of generalized minimal residual (GMRES) method with a suitable preconditioner to solve the corresponding linear system.

5.1 State system

After an application of the discretization techniques discussed in Section 3, one gets the following linear system for the state part of the optimality system (2.15):

\[
\left( \sum_{i=0}^{N} G_i \otimes K_i \right) y - \left( \sum_{i=0}^{N} G_0 \otimes M \right) u = \left( \sum_{i=0}^{N} g_i \otimes f_i \right). \tag{5.1}
\]

where \( y = (y_0, \ldots, y_{J-1})^T \) and \( u = (u_0, \ldots, u_{J-1})^T \) with \( y_i, u_i \in \mathbb{R}^{N_d}, i = 0, 1, \ldots, J - 1 \) and \( N_d \) corresponds to the degree of freedom for the spatial
discretization. The mass matrix $M \in \mathbb{R}^{\mathcal{N}_d \times \mathcal{N}_d}$, the stiffness matrices $K_i \in \mathbb{R}^{\mathcal{N}_d \times \mathcal{N}_d}$, and the right-hand side vectors $f_i \in \mathbb{R}^{\mathcal{N}_d}$ are given, respectively, by

$$M(r, s) = \sum_{K \in T_h} \int \varphi_r \varphi_s \, dx,$$

$$K_0(r, s) = \sum_{K \in T_h} \int (\tilde{a} \nabla \varphi_r \cdot \nabla \varphi_s + \tilde{b} \cdot \nabla \varphi_r \varphi_s) \, dx$$

$$- \sum_{E \in \mathcal{E}^0 \cup \mathcal{E}^a_h} \int \left( \| \tilde{a} \nabla \varphi_r \| \| \varphi_s \| + \| \tilde{a} \nabla \varphi_s \| \| \varphi_r \| \right) \, ds$$

$$+ \sum_{E \in \mathcal{E}^0 \cup \mathcal{E}^a_h} \frac{\sigma}{h_E} \int \| \varphi_r \| \| \varphi_s \| \, ds + \sum_{K \in T_h} \int \tilde{b} \cdot n_E (\varphi^e_r - \varphi_r) \varphi_s \, ds$$

$$- \sum_{K \in T_h} \int \tilde{b} \cdot n_E \varphi_r \varphi_s \, ds,$$

$$K_i(r, s) = \sum_{K \in T_h} \int \left( \left( \kappa_a \sqrt{\lambda_i^a} \phi_i^a \right) \nabla \varphi_r \cdot \nabla \varphi_s + \left( \kappa_b \sqrt{\lambda_i^b} \phi_i^b \right) \cdot \nabla \varphi_r \varphi_s \right) \, dx$$

$$- \sum_{E \in \mathcal{E}^0 \cup \mathcal{E}^a_h} \int \left( \| \kappa_a \sqrt{\lambda_i^a} \phi_i^a \| \| \varphi_r \| \right) \| \varphi_s \| + \left( \| \kappa_b \sqrt{\lambda_i^b} \phi_i^b \| \| \varphi_r \| \right) \| \varphi_s \| \, ds$$

$$+ \sum_{E \in \mathcal{E}^0 \cup \mathcal{E}^a_h} \frac{\sigma}{h_E} \int \| \varphi_r \| \| \varphi_s \| \, ds$$

$$+ \sum_{K \in T_h} \int \left( \kappa_b \sqrt{\lambda_i^b} \phi_i^b \right) \cdot n_E (\varphi^e_r - \varphi_r) \varphi_s \, ds$$

$$- \sum_{T \in \mathcal{T}_h} \int \left( \kappa_b \sqrt{\lambda_i^b} \phi_i^b \right) \cdot n_E \varphi_r \varphi_s \, ds,$$

$$f_0(s) = \sum_{K \in T_h} \int f \varphi_s \, dx + \sum_{E \in \mathcal{E}^a_h} \frac{\sigma}{h_E} \int y_{DB} \| \varphi_s \| \, ds - \sum_{E \in \mathcal{E}^a_h} \int y_{DB} \| \tilde{a} \nabla \varphi_s \| \, ds$$

$$- \sum_{K \in T_h} \int \tilde{b} \cdot n_E y_{DB} \varphi_s \, ds,$$

$$f_i(s) = \sum_{E \in \mathcal{E}^a_h} \frac{\sigma}{h_E} \int y_{DB} \| \varphi_s \| \, ds - \sum_{E \in \mathcal{E}^a_h} \int y_{DB} \left( \| \kappa_a \sqrt{\lambda_i^a} \phi_i^a \| \nabla \varphi_s \right) \, ds$$

$$- \sum_{T \in \mathcal{T}_h} \int \left( \kappa_b \sqrt{\lambda_i^b} \phi_i^b \right) \cdot n_E y_{DB} \varphi_s \, ds,$$

where $\{ \varphi_i(x) \}$ corresponds to the set of basis functions for the spatial discretization, i.e., $V_h = \text{span}\{ \varphi_i(x) \}$.  

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On the other hand, for $i = 0, \ldots, N$, the stochastic matrices $G_i \in \mathbb{R}^{J \times J}$ and the stochastic vectors $g_i \in \mathbb{R}^J$ in (5.1) are given, respectively, by

$$G_0(r, s) = \langle \psi_r \psi_s \rangle, \quad G_i(r, s) = \langle \xi_i \psi_r \psi_s \rangle, \quad (5.2a)$$

$$g_0(r) = \langle \psi_r \rangle, \quad g_i(r) = \langle \xi_i \psi_r \rangle. \quad (5.2b)$$

In (5.2), each stochastic basis function $\psi_i(\xi)$ is a product of $N$ univariate orthogonal polynomials, i.e., $\psi_i(\xi) = \psi_{i_1}(\xi) \psi_{i_2}(\xi) \ldots \psi_{i_N}(\xi)$, where the multi-index $i$ is defined by $i = (i_1, i_2, \ldots, i_N)$ with $\sum_{\ell=1}^{N} i_{\ell} \leq Q$. In this paper, Legendre polynomials are chosen as the stochastic basis functions since the underlying random variables have uniform distribution on $[-\sqrt{3}, \sqrt{3}]$ [54]. Then, $G_0$ becomes an identity matrix, whereas $G_k$, $k > 0$, contains at most two nonzero entries per row; see, e.g., [49, 52]. On the other hand, $g_i$ is the first column of $G_i$, $i = 0, 1, \ldots, N$.

### 5.2 Matrix formulation of the optimality system

The discrete optimality system in (3.17) can be represented as a block matrix system including the state, adjoint, and variational equations in the finite-dimensional setting. To solve the underlying block linear system, “the primal-dual active set (PDAS) methodology as a semi-smooth Newton step” is applied; see, e.g., [60] for more details. After a definition of the active sets

$$A^- = \bigcup\{x \in K : -p - \mu u_a < 0, \forall K \in \mathcal{T}_h\},$$

$$A^+ = \bigcup\{x \in K : -p - \mu u_b > 0, \forall K \in \mathcal{T}_h\},$$

and the inactive set

$$\mathcal{I} = \mathcal{T}_h \setminus (A^- \cup A^+),$$

the block formulation becomes

$$A^*y - M_T u = F, \quad (5.3a)$$

$$A^*p - M_y y = -F^d, \quad (5.3b)$$

$$(G_0 \otimes \text{diag}(I_{\mathcal{I}})) p + \mu (G_0 \otimes I) u = (g_0 \otimes \mathbb{1}_{A^-}) \mu u_a + (g_0 \otimes \mathbb{1}_{A^+}) \mu u_b, \quad (5.3c)$$

where

$$M_T := I \otimes M,$$

$$F^d := g_0 \otimes y^d_{(s)} = \sum_{K \in \mathcal{T}_h} \int_K y^d \varphi_s \, dx,$$

$$M_y := (G_0 \otimes M) + y (M_0 \otimes M) \text{ with } M_0 = \text{diag} \left( 0, \langle \psi_1 \rangle^2, \ldots, \langle \psi_{J-1} \rangle^2 \right),$$

and $\mathbb{1}_{A^-}, \mathbb{1}_{A^+},$ and $\mathbb{1}_{\mathcal{I}}$ correspond to the characteristic functions of $A^-$, $A^+$, and $\mathcal{I}$, respectively. Equivalently, $M_y$ can be rewritten as

$$M_y := G_y \otimes M, \quad \text{with} \quad G_y := G_0 + y M_0,$$
where

\[ G_r(r, s) = \begin{cases} 
(\psi_0)^2, & \text{if } r = s = 0, \\
(1 + \gamma) (\psi_r)^2, & \text{if } r = s = 1, \ldots, J - 1, \\
0, & \text{otherwise.}
\]  

(5.4)

Rearranging (5.3) gives us the following linear matrix system

\[
\begin{bmatrix}
\mathcal{M}_r & 0 & -A^* \\
0 & \mu \left( G_0 \otimes I \right) G_0 \otimes \text{diag}(\mathbb{1}_T) & 0 \\
-A & \mathcal{M}_I & 0
\end{bmatrix}
\begin{bmatrix}
y \\
u \\
p
\end{bmatrix}
= \begin{bmatrix}
\mu \left( (g_0 \otimes \mathbb{1}_A^-) u_a + (g_0 \otimes \mathbb{1}_A^+) u_b \right) \\
F_d \\
-F
\end{bmatrix},
\]

(5.5)

which is a saddle point system. We note that since Legendre polynomials are used, \( G_0 = I \), and hence, \( \mathcal{M}_I = \mathcal{M} \).

In practical implementations, the saddle point system (5.5) typically becomes very large, depending on the length of the random vector \( \xi \) and the number of refinements in the spatial discretization. We break this curse of dimensionality by using a low-rank approximation, which reduces both the computational complexity and memory requirements by using a Kronecker-product structure of the matrices defined in (5.5).

### 5.3 Low-rank approach

We first introduce the notation and some basic properties of the low-rank approach. Let \( \Theta = [\theta_1, \ldots, \theta_J] \in \mathbb{R}^{N_d \times J} \) and let the operators \( \text{vec}(.\cdot) \) and \( \text{mat}(\cdot) \) be isomorphic mappings between \( \mathbb{R}^{N_d \times J} \) and \( \mathbb{R}^{N_d J} \) as follows

\[
\text{vec} : \mathbb{R}^{N_d \times J} \rightarrow \mathbb{R}^{N_d J}, \quad \text{mat} : \mathbb{R}^{N_d J} \rightarrow \mathbb{R}^{N_d \times J},
\]

where \( N_d \) and \( J \) are the degrees of freedom for the spatial discretization and the total degree of the multivariate stochastic basis polynomials, respectively. The matrix inner product is defined by \( \langle U, V \rangle_F = \text{trace}(U^T V) \) with \( \| U \|_F = \sqrt{\langle U, U \rangle_F} \). Further, the following relation holds, see, e.g., [36]:

\[
\text{vec}(U \otimes V) = (V^T \otimes U)\text{vec}(\Theta). \tag{5.6}
\]

Now, we can interpret the system (5.5) as follows

\[
\begin{bmatrix}
G_r \otimes M \\
0 \\
-\sum_{i=0}^{N} G_i \otimes K_i^* \\
0 \\
\mu \left( G_0 \otimes I \right) G_0 \otimes \text{diag}(\mathbb{1}_T) \\
-\sum_{i=0}^{N} G_i \otimes K_i \\
G_0 \otimes M
\end{bmatrix}
\begin{bmatrix}
\text{vec}(Y) \\
\text{vec}(U) \\
\text{vec}(P)
\end{bmatrix}
= \begin{bmatrix}
\text{vec}(B_1) \\
\text{vec}(B_2) \\
\text{vec}(B_3)
\end{bmatrix}, \tag{5.7}
\]

where

\[
Y = (y_0, \ldots, y_{J-1}), \quad U = (u_0, \ldots, u_{J-1}), \quad P = (p_0, \ldots, p_{J-1}), \quad B_1 = \text{mat}(F_d), \quad B_2 = \text{mat} \left( \mu \left( (g_0 \otimes \mathbb{1}_A^-) u_a + (g_0 \otimes \mathbb{1}_A^+) u_b \right) \right), \quad B_3 = \text{mat} (-F).
\]
By the identity (5.6), we have

\[
\mathcal{L}\Theta = \text{vec} \left( \begin{bmatrix} \sum_{i=0}^{N} K_i^* P G_i^T - \mu I U Y G_i^T + \text{diag}(\mathbb{I}_T) P G_i^T \\ - \sum_{i=0}^{N} K_i Y G_i^T + M U G_i^T \end{bmatrix} \right) = \text{vec} \left( \begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix} \right). \tag{5.8}
\]

Assuming that the matrices \( \Theta \) and \( B \) have the following low-rank representations, see, e.g., [37, 61, 62],

\[
\begin{align*}
Y &= W_Y V_Y^T, \quad W_Y \in \mathbb{R}^{N_d \times r_Y}, \quad V_Y \in \mathbb{R}^{J \times r_Y}, \\
U &= W_U V_U^T, \quad W_U \in \mathbb{R}^{N_d \times r_U}, \quad V_U \in \mathbb{R}^{J \times r_U}, \\
P &= W_P V_P^T, \quad W_P \in \mathbb{R}^{N_d \times r_P}, \quad V_P \in \mathbb{R}^{J \times r_P}, \\
B_1 &= B_{11} B_{12}^T, \quad B_{11} \in \mathbb{R}^{N_d \times r_{B_1}}, \quad B_{12} \in \mathbb{R}^{J \times r_{B_1}}, \\
B_2 &= B_{21} B_{22}^T, \quad B_{21} \in \mathbb{R}^{N_d \times r_{B_2}}, \quad B_{22} \in \mathbb{R}^{J \times r_{B_2}}, \\
B_3 &= B_{31} B_{32}^T, \quad B_{31} \in \mathbb{R}^{N_d \times r_{B_3}}, \quad B_{32} \in \mathbb{R}^{J \times r_{B_3}}.
\end{align*}
\tag{5.9}
\]

with \( r_Y, r_U, r_P, r_{B_1}, r_{B_2}, r_{B_3} \ll N_d, J \), (5.8) can be stated as follows

\[
\begin{bmatrix}
M W_Y V_Y^T G_Y^T - \sum_{i=0}^{N} K_i^* W_P V_P^T G_i^T \\
\mu I W_U V_U^T G_0^T + \text{diag}(\mathbb{I}_T) W_P V_P^T G_0^T \\
- \sum_{i=0}^{N} K_i W_Y V_Y^T G_i^T + M W_U V_U^T G_0^T
\end{bmatrix} = \begin{bmatrix} B_{11} B_{12}^T \\ B_{21} B_{22}^T \\ B_{31} B_{32}^T \end{bmatrix}, \tag{5.10}
\]

where vec operator is ignored. Moreover, the three block rows in (5.10) can be written as

\[
\begin{align}
&\underbrace{M W_Y - \sum_{i=0}^{N} K_i^* W_P}_{\hat{W}_1} \begin{bmatrix} G_Y V_Y & G_i V_P \end{bmatrix}^T, \tag{5.11a} \\
&\underbrace{\mu I W_U \text{diag}(\mathbb{I}_T) W_P}_{\hat{W}_2} \begin{bmatrix} G_0 V_Y & G_0 V_P \end{bmatrix}^T, \tag{5.11b} \\
&\underbrace{- \sum_{i=0}^{N} K_i W_Y M W_U}_{\hat{W}_3} \begin{bmatrix} G_i V_Y & G_0 V_U \end{bmatrix}^T. \tag{5.11c}
\end{align}
\]

in low-rank formats \( \hat{W}_i \hat{V}_i^T \) for \( i = 1, 2, 3 \). By the usage of (5.11), the low-rank approximate solutions to (5.7) can be obtained; see Algorithm 1 modified from [62] for details of the low-rank implementation of GMRES. Moreover, with the help of the following fact

\[
\text{trace}(A^T B) = \text{vec}(A)^T \text{vec}(B),
\]
the inner products $\langle A, B \rangle_F = \text{trace}(ATB)$ in the iterative low-rank algorithm can be computed efficiently. For instance, the inner product computation in Algorithm 1 denoted by

$$\text{trprod}(A_{11}, A_{12}, A_{21}, A_{22}, A_{31}, A_{32}, B_{11}, B_{12}, B_{21}, B_{22}, B_{31}, B_{32})$$

can be computed as follows

$$\langle A, B \rangle_F = \text{trace} \left( \left( A_{11}A_{12}^T \right)^T \left( B_{11}B_{12}^T \right)^T \right) + \text{trace} \left( \left( A_{21}A_{22}^T \right)^T \left( B_{21}B_{22}^T \right)^T \right)$$

$$+ \text{trace} \left( \left( A_{31}A_{32}^T \right)^T \left( B_{31}B_{32}^T \right)^T \right)$$

$$= \text{trace} \left( A_{11}^T B_{11}A_{12}B_{12} \right) + \text{trace} \left( A_{21}^T B_{21}A_{22}B_{22} \right)$$

$$+ \text{trace} \left( A_{31}^T B_{31}A_{32}B_{32} \right),$$

where

$$A = \text{vec} \left( \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \\ A_{31} & A_{32} \end{bmatrix} \right), \quad B = \text{vec} \left( \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \\ B_{31} & B_{32} \end{bmatrix} \right).$$

During the iteration process, the rank of low-rank factors can increase either via matrix vector products or vector (matrix) additions. Thus, the cost of rank-reduction techniques is kept under control by using truncation based on singular values [36] or truncation based on coarse-grid rank reduction [63]. Our approach is based on the discussion in [37, 64], where a truncated SVD of $U = W^TV \approx \text{diag}(\sigma_1, \ldots, \sigma_r)C^T$ is constructed for the largest $r$ singular values, which are greater than the given truncation tolerance $\epsilon_{\text{trunc}}$. In Algorithm 1, this process is done by the truncation operator $T$. Further, in the numerical simulations, a rather small truncation tolerance $\epsilon_{\text{trunc}}$ is used to represent the full-rank solution as accurate as possible.

We know that iterative methods such as GMRES exhibit a better convergence in terms of the number of iterations when they are used with a suitable preconditioner. The low-rank variants also display the same behavior so that we use a block diagonal mean-based preconditioner of the form

$$P_0 = \begin{bmatrix} \mathcal{M}_Y & 0 & 0 \\ 0 & \mu (\mathcal{G}_0 \otimes I) & 0 \\ 0 & 0 & \widetilde{S} \end{bmatrix},$$

where $\widetilde{S} = \left( \mathcal{G}_0 \otimes \mathcal{K}_0 \right)M^{-1}_Y \left( \mathcal{G}_0 \otimes \mathcal{K}_0 \right)^T$ corresponds to the approximated Schur complement with $\mathcal{K}_0 = \mathcal{K}_0 + \sqrt{1+\gamma} \mu M \text{diag}(\mathbb{1}_\Omega)$; see, e.g., [14, 49].

### 6 Numerical results

This section contains a set of numerical experiments to illustrate the performance of proposed discretization techniques and a low-rank variant of GMRES approach. All numerical simulations are done in MATLAB R2021a on an Ubuntu Linux machine.
Input: Coefficient matrix $\mathcal{L} \in \mathbb{R}^{3N_d \times J}$, inverse of the preconditioner matrix $\mathcal{P}_0^{-1} \in \mathbb{R}^{3N_d \times J}$, and right-hand side matrix $B$ in the low-rank formats. Truncation operator $\mathcal{T}$ with given tolerance $\epsilon_{trunc}$.

Output: Matrix $\Theta \in \mathbb{R}^{3N_d \times J}$ satisfying $\|\mathcal{L}(\Theta) - B\|_F / \|B\|_F \leq \epsilon_{tol}$.

1: Choose initial guess $\Theta_{01}^{(0)}, \Theta_{12}^{(0)}, \Theta_{21}^{(0)}, \Theta_{22}^{(0)}, \Theta_{31}^{(0)}, \Theta_{32}^{(0)}$.
2: $(\tilde{\Theta}_{11}, \tilde{\Theta}_{12}, \tilde{\Theta}_{21}, \tilde{\Theta}_{22}, \tilde{\Theta}_{31}, \tilde{\Theta}_{32}) = \mathcal{L}(\Theta_{01}^{(0)}, \Theta_{12}^{(0)}, \Theta_{21}^{(0)}, \Theta_{22}^{(0)}, \Theta_{31}^{(0)}, \Theta_{32}^{(0)})$. 
3: $R_{11}^{(0)} = \{B_{11}, -\tilde{\Theta}_{11}\}$, $R_{12}^{(0)} = \{B_{12}, \tilde{\Theta}_{12}\}$.
4: $R_{21}^{(0)} = \{B_{21}, -\tilde{\Theta}_{21}\}$, $R_{22}^{(0)} = \{B_{22}, \tilde{\Theta}_{22}\}$.
5: $R_{31}^{(0)} = \{B_{31}, -\tilde{\Theta}_{31}\}$, $R_{32}^{(0)} = \{B_{32}, \tilde{\Theta}_{32}\}$.
6: $\|R_i^{(0)}\| = \left\| \text{trprod}(R_{11}^{(0)}, \ldots, R_{11}^{(0)}) \right\|_F$.
7: $V_{11}^{(0)} = R_{11}^{(0)} / \|R_i^{(0)}\|_F$, $V_{11}^{(0)} = R_{12}^{(0)}$.
8: $V_{21}^{(0)} = R_{21}^{(0)} / \|R_i^{(0)}\|_F$, $V_{22}^{(0)} = R_{22}^{(0)}$.
9: $V_{31}^{(0)} = R_{31}^{(0)} / \|R_i^{(0)}\|_F$, $V_{32}^{(0)} = R_{32}^{(0)}$.
10: $y = [y_1, 0, \ldots, 0]$, $y_1 = \sqrt{\text{trprod}(V_{11}^{(0)}, \ldots, V_{11}^{(0)})}$.
11: while $i < \text{maxit}$ do
12: $(Z_{11}, Z_{12}^{(0)}, Z_{21}^{(0)}, Z_{22}^{(0)}, Z_{31}^{(0)}, Z_{32}^{(0)}) = \mathcal{P}_0^{-1}(V_{11}^{(0)}, V_{12}^{(0)}, V_{21}^{(0)}, V_{22}^{(0)}, V_{31}^{(0)}, V_{32}^{(0)})$.
13: $(W_{11}, W_{12}, W_{21}, W_{22}, W_{31}, W_{32}) = \mathcal{L}(Z_{11}, Z_{12}^{(0)}, Z_{21}^{(0)}, Z_{22}^{(0)}, Z_{31}^{(0)}, Z_{32}^{(0)})$.
14: for $j = 1, \ldots, i$ do
15: $m_{j,i} = \sqrt{\text{trprod}(W_{11}, \ldots, V_{11}^{(j)})}$.
16: $W_{11} = \{W_{11}, -m_{j,i}V_{11}^{(j)}\}$, $W_{12} = \{W_{12}, V_{12}^{(j)}\}$.
17: $W_{21} = \{W_{21}, -m_{j,i}V_{21}^{(j)}\}$, $W_{22} = \{W_{22}, V_{22}^{(j)}\}$.
18: $W_{31} = \{W_{31}, -m_{j,i}V_{31}^{(j)}\}$, $W_{32} = \{W_{32}, V_{32}^{(j)}\}$.
19: end for
20: $m_{i+1,i} = \sqrt{\text{trprod}(W_{11}, \ldots, W_{11}^{(i)})}$.
21: $V_{11}^{(i+1)} = W_{11}/m_{i+1,k}$, $V_{12}^{(i+1)} = W_{12}$.
22: $V_{21}^{(i+1)} = W_{21}/m_{i+1,k}$, $V_{22}^{(i+1)} = W_{22}$.
23: $V_{31}^{(i+1)} = W_{31}/m_{i+1,k}$, $V_{32}^{(i+1)} = W_{32}$.
24: Perform Givens rotations for the $i$th column of $m$:
25: for $j = 1, \ldots, i$ do
26: $m_{j+1,i,j} = \begin{bmatrix} c_j & s_j \\ -s_j & c_j \end{bmatrix} \begin{bmatrix} m_{j,i} \\ m_{j+1,i} \end{bmatrix}$.
27: end for
28: Compute $i$th Givens rotation, and perform for $y$ and last column of $m$.
29: $y_{i+1} = \begin{bmatrix} y_i \\ y_{i+1} \end{bmatrix}$.
30: $m_{i,i} = c_1m_{1,i} + s_1m_{1+i,1}$, $m_{i+1,i} = 0$.
31: if $|y_{i+1}| \leq \epsilon_{tol}$ then
32: Compute $y$ from $My = \xi$, where $(M)_{j,i} = m_{j,i}$.
33: $Y_{11} = \{y_1V_{11}^{(i)}, \ldots, y_kV_{11}^{(i)}\}$, $Y_{12} = \{V_{12}^{(i)}, \ldots, V_{12}^{(i)}\}$.
34: $Y_{21} = \{y_1V_{21}^{(i)}, \ldots, y_kV_{21}^{(i)}\}$, $Y_{22} = \{V_{22}^{(i)}, \ldots, V_{22}^{(i)}\}$,
35: $Y_{31} = \{y_1V_{31}^{(i)}, \ldots, y_kV_{31}^{(i)}\}$, $Y_{32} = \{V_{32}^{(i)}, \ldots, V_{32}^{(i)}\}$.
36: $(\tilde{Y}_{11}, \tilde{Y}_{12}, \tilde{Y}_{21}, \tilde{Y}_{22}, \tilde{Y}_{31}, \tilde{Y}_{32}) = \mathcal{P}_0^{-1}(Y_{11}, Y_{12}, Y_{21}, Y_{22}, Y_{31}, Y_{32})$.
37: $\Theta_{11} = \{(\Theta_{11}^{(0)}), \tilde{Y}_{11}\}$, $\Theta_{12} = \{(\Theta_{12}^{(0)}, \tilde{Y}_{12}\}$.
38: $\Theta_{21} = \{(\Theta_{21}^{(0)}, \tilde{Y}_{21}\}$, $\Theta_{22} = \{(\Theta_{22}^{(0)}, \tilde{Y}_{22}\}$.
39: $\Theta_{31} = \{(\Theta_{31}^{(0)}, \tilde{Y}_{31}\}$, $\Theta_{32} = \{(\Theta_{32}^{(0)}, \tilde{Y}_{32}\}$.
40: end if
41: end while

Algorithm 1 Low-rank preconditioned GMRES (LRPGMRES).
Table 1  Descriptions of the parameters used in the simulations

| Parameter | Description |
|-----------|-------------|
| $N_d$     | Degrees of freedom for the spatial discretization |
| $N$       | Truncation number in KL expansion |
| $Q$       | Highest order of basis polynomials for the stochastic domain |
| $\mu$     | Regularization parameter of the control $u$ |
| $\gamma$  | Risk-aversion parameter |
| $\nu$     | Viscosity parameter |
| $\xi$     | Correlation length |
| $\kappa$  | Standard deviation |

with 32 GB RAM. Iterative approaches are ended when the residual becomes smaller than the given tolerance value $\epsilon_{tol} = 5 \times 10^{-3}$ or the maximum iteration number ($\#\text{iter}_{\text{max}} = 250$) is reached. The truncation tolerance $\epsilon_{\text{trunc}} = 10^{-8}$ is chosen, such that $\epsilon_{\text{trunc}} \leq \epsilon_{\text{tol}}$; otherwise, one would iterate the noise during the low-rank process.

In the numerical experiments, the random coefficient $\eta$ is described by the following covariance function

$$\mathbb{C}_\eta(x, y) = \kappa^2 \prod_{n=1}^{2} e^{-|x_n - y_n|/\ell_n}, \quad \forall (x, y) \in D$$  \hspace{1cm} (6.1)

with the correlation length $\ell_n$. Linear elements are used to generate discontinuous Galerkin basis, whereas Legendre polynomials are taken as the stochastic basis functions since the underlying random variables have uniform distribution over $[-3, 3]$, that is, $\xi_j \sim \mathcal{U}[-3, 3]$, $j = 1, \ldots, N$. Explicit eigenpairs $(\lambda_j, \phi_j)$ of the covariance function (6.1) can be found in [44]. Further, all parameters used in the simulations are described in Table 1.

### 6.1 Unconstrained problem with random diffusion parameter

As a first benchmark problem, we consider an unconstrained optimal control problem, that is, $\mathcal{U}^{\text{ad}} = \mathcal{U}$, having a random diffusion coefficient defined on $D = [-1, 1]^2$ with the source function $f(x) = 0$, the convection parameter $b(x) = (0, 1)^T$, and the Dirichlet boundary condition

$$y_{DB}(x) = \begin{cases} y_{DB}(x_1, -1) = x_1, & y_{DB}(x_1, 1) = 0, \\ y_{DB}(-1, x_2) = -1, & y_{DB}(1, x_2) = 1. \end{cases}$$

The random diffusion parameter is chosen as $a(x, \omega) = \nu \eta(x, \omega)$, where the random field $\eta(x, \omega)$ has the unity mean with the corresponding covariance function (6.1) and $\nu$ is the viscosity parameter. The desired state (or target) $y^d$ corresponds to the stochastic solution of the forward model by taking $u(x) = 0$. We note that the desired state exhibits exponential boundary layer near $x_2 = 1$, where the solution changes in a dramatic manner [54].
Table 2 Example 6.1: Computational values of the cost functional $\mathcal{J}(u_h)$ and tracking term $\|y_h - y^d\|_X^2$ obtained by $\mathcal{L}\backslash \mathcal{B}$ with $N_d = 6144$, $N = 3$, $Q = 3$, $\ell = 1$, $\kappa = 0.5$, and $\gamma = 1$ for varying values of the viscosity parameter $\nu$ and the regularization parameter $\mu$.

| $\nu$ | $\mu$ | $\mathcal{J}(u_h)$ | $\|y_h - y^d\|_X^2$ | $\mathcal{J}(u_h)$ | $\|y_h - y^d\|_X^2$ | $\mathcal{J}(u_h)$ | $\|y_h - y^d\|_X^2$ | $\mathcal{J}(u_h)$ | $\|y_h - y^d\|_X^2$ |
|-------|-------|-----------------|----------------|-----------------|----------------|----------------|----------------|----------------|----------------|
| $1$   | $1$   | $1.2393e-05$    | $5.4679e-06$   | $1.4349e-05$    | $1.3120e-05$   | $1.3675e-05$    | $1.5285e-05$   | $1.0257e-06$   | $1.9753e-07$   |
|       | $10^{-2}$ | $2.5034e-06$    | $7.1725e-07$   | $8.3285e-06$    | $4.1452e-06$   | $1.1798e-06$    | $4.3924e-07$   | $3.2113e-08$   | $1.6931e-09$   |
|       | $10^{-4}$ | $1.0257e-06$    | $9.7533e-07$   | $7.2067e-07$    | $9.0731e-08$   | $3.9380e-07$    | $1.1422e-07$   | $1.6931e-09$   | $6.0683e-07$   |
|       | $10^{-6}$ | $9.7533e-07$    | $1.6931e-09$   | $7.2067e-07$    | $3.9380e-07$   | $1.1422e-07$   | $8.3896e-09$   | $1.6931e-09$   | $6.0683e-09$   |

Table 2 shows the values of the cost functional $\mathcal{J}(u_h)$ and tracking term $\|y_h - y^d\|_X^2$ obtained by $\mathcal{L}\backslash \mathcal{B}$ for various values of the viscosity parameter $\nu$ and the regularization parameter $\mu$. We observe that the tracking term and the objective functional become smaller as $\mu$ decreases. Moreover, Table 3 exhibits that the peak values of states’ variance can be reduced by increasing the value of the parameter $\gamma$.

Next, we display the performance of $\mathcal{L}\backslash \mathcal{B}$ in terms of total CPU times (in seconds) and storage requirements (in KB) in Table 4. However, we could not report some numerical results since the simulation is ended with “out of memory”, which we have denoted as “OoM”. To handle the curse of dimensionality and so increase the value of truncation number $N$, we need effective numerical approaches or solvers such as a low-rank variant of GMRES iteration with a mean-based preconditioner discussed in Section 5.3.

Table 5 reports the results of the simulations by considering various data sets in the low-rank format. By keeping other parameters fixed, we show results for varying truncation number $N$ in KL expansion and regularization parameter $\mu$ for $\kappa = 0.5$ in Table 5. When $N$ increases, the complexity of the problem increases in terms of the number of rank, memory, and CPU time. Another key observation is that the relative residual decreases independently of the value of $\mu$ while increasing $N$.

Next, we investigate the effect of the standard deviation parameter $\kappa$ on the numerical simulations. Figure 1 displays the behaviors of the cost functional $\mathcal{J}(u_h)$, the tracking term $\|y_h - y^d\|_X^2$, and the relative residual for various values of $\kappa$. We
Table 4 Example 6.1: Total CPU times (in seconds) and memory (in KB) for \( N_d = 6144 \), \( Q = 3 \), \( \ell = 1 \), \( \mu = 10^{-2} \), \( \gamma = 1 \), and \( \kappa = 0.5 \)

|   | \( \nu = 10^0 \)      | \( \nu = 10^{-2} \)        | \( \nu = 10^{-4} \)        |
|---|-----------------------|-----------------------------|-----------------------------|
| 1 | CPU (Memory)          | CPU (Memory)                | CPU (Memory)                |
| 2 | 116.0 (2880)          | 116.1 (2880)                | 117.5 (960)                 |
| 3 | 779.6 (5760)          | 787.7 (5760)                | 813.0 (5760)                |
| 4 | OoM                   | OoM                         | OoM                         |

observe that the values of \( J(u_h) \) and \( \| y_h - y_d \|^2 \) decrease monotonically as the value of \( \kappa \) increases. Moreover, the low-rank variant of preconditioned GMRES method yields convergence behavior for all values of \( \kappa \). Lastly, Fig. 2 shows that the speed of convergence of relative residual decreases by increasing the value of risk-aversion parameter \( \gamma \) in the beginning of the iteration.

Table 5 Example 6.1: Total number of iterations, total rank of the truncated solutions, total CPU times (in seconds), relative residual, and memory demand of the solution (in KB) with \( N_d = 6144 \), \( Q = 3 \), \( \ell = 1 \), \( \kappa = 0.5 \), \( \nu = 1 \), \( \gamma = 0 \), and the mean-based preconditioner \( \mathcal{P}_h \) for varying values of \( N \) and \( \mu \)

|   | \( \mu = 1 \) | \( \mu = 10^{-2} \) | \( \mu = 10^{-4} \) |
|---|--------------|---------------------|---------------------|
| 4 | \#iter       | 250                 | 250                 | 250                 |
|   | Rank         | 51                  | 51                  | 51                  |
|   | CPU          | 40126.1             | 40017.9             | 39950.4             |
|   | Resi.        | 3.5759e-02          | 3.3521e-02          | 3.7375e-02          |
|   | Memory       | 2461.9              | 2461.9              | 2461.9              |
| 5 | \#iter       | 250                 | 250                 | 250                 |
|   | Rank         | 84                  | 84                  | 84                  |
|   | CPU          | 91366.2             | 90544.0             | 90021.2             |
|   | Resi.        | 2.1672e-02          | 2.3056e-02          | 3.1080e-02          |
|   | Memory       | 4068.8              | 4068.8              | 4068.8              |
| 6 | \#iter       | 250                 | 250                 | 250                 |
|   | Rank         | 126                 | 126                 | 126                 |
|   | CPU          | 208643.4            | 207964.0            | 207464.4            |
|   | Resi.        | 1.8357e-02          | 1.8064e-02          | 2.0494e-02          |
|   | Memory       | 6130.7              | 6130.7              | 6130.7              |
| 7 | \#iter       | 250                 | 250                 | 250                 |
|   | Rank         | 180                 | 180                 | 180                 |
|   | CPU          | 355115.9            | 355167.5            | 355652.3            |
|   | Resi.        | 1.1208e-02          | 1.3833e-02          | 1.4914e-02          |
|   | Memory       | 8808.8              | 8808.8              | 8808.8              |
Example 6.1: Behaviors of the cost functional \( J(u_h) \) (left), the tracking term \( \| y_h - y^d \|_{L^2}^2 \) (middle), and the relative residual (right) with \( N_d = 6144, Q = 3, \ell = 1, v = 1, \mu = 10^{-2}, \gamma = 0 \), and the mean-based preconditioner \( P_0 \) for varying values of \( \kappa \).

### 6.2 Unconstrained problem with random convection parameter

Our second example is an unconstrained optimal control problem containing random velocity input parameter. To be precise, we set the deterministic diffusion parameter \( a(x, \omega) = \nu > 0 \), the deterministic source function \( f(x) = 0 \), and homogeneous Dirichlet boundary conditions on the spatial domain \( D = [-1, 1]^2 \). On the other hand, the random velocity field \( b(x, \omega) \) is defined as \( b(x, \omega) = (\eta(x, \omega), \eta(x, \omega))^T \), where the random input \( \eta(x, \omega) \) has the unity mean, i.e., \( \mathbb{E}[\eta(x)] = 1 \). Further, the desired state \( y^d \) is given by

\[
y^d(x) = \exp \left[ -64 \left( \left( x_1 - \frac{1}{2} \right)^2 + \left( x_2 - \frac{1}{2} \right)^2 \right) \right].
\]

Figures 3 and 4 display, respectively, the mean of state \( \mathbb{E}[y_h] \) and the control \( u_h \) for varied values of the regularization parameter \( \mu \) obtained by solving the full-rank system \( \mathcal{L} \setminus \mathcal{B} \). As the previous example, we observe that the state \( y_h \) becomes closer to the target solution \( y^d \) while \( \mu \) decreases.

Fig. 2 Example 6.1: Convergence of LRPGMRES with \( N_d = 6144, N = 5, Q = 3, \ell = 1, \mu = 1, v = 1 \) for varying \( \kappa \) and \( \gamma \).
**Example 6.2:** Simulations of the mean of state $\mathbb{E}[y_h]$ obtained by $L^\dagger B$ with $N_d = 6144$, $N = 3$, $Q = 3$, $\ell = 1$, $\kappa = 0.05$, $v = 1$, and $\gamma = 0$ for varying $\mu = 1$, $10^{-2}$, $10^{-4}$, $10^{-6}$ and the desired state $y^d$

**Example 6.2:** Simulations of the control $u_h$ obtained solving by $L^\dagger B$ with $N_d = 6144$, $N = 3$, $Q = 3$, $\ell = 1$, $\kappa = 0.05$, $v = 1$, and $\gamma = 0$ for varying regularization parameter $\mu = 1$, $10^{-2}$, $10^{-4}$, $10^{-6}$

**Example 6.2:** Behaviors of the cost functional $J(u_h)$ (left), the tracking term $||y_h - y^d||_C^2$ (middle), and the relative residual (right) with $N_d = 6144$, $N = 3$, $Q = 3$, $\kappa = 0.05$, $\ell = 1$, $v = 1$, $\gamma = 0$, and the mean-based preconditioner $P_0$ for varying $\mu$

**Table 6** Example 6.2: Simulation results showing total number of iterations, ranks of the truncated solutions, total CPU times (in seconds), relative residual, and memory demand of the solution (in KB) with $N_d = 6144$, $N = 3$, $Q = 3$, $\ell = 1$, $v = 1$, $\mu = 10^{-6}$, and the mean-based preconditioner $P_0$ for varying $\gamma$

| $\gamma$       | #iter | Rank | CPU    | Resi.   | Memory |
|-----------------|-------|------|--------|---------|--------|
| 0               | 250   | 29   | 24468.2| 2.1733e-01| 1396.5 |
| $10^{-6}$       | 250   | 30   | 19383.2| 2.6663e-01| 1444.7 |
| $10^{-4}$       | 250   | 30   | 17382.0| 4.0428e-01| 1444.7 |
| $10^{-2}$       | 250   | 30   | 17422.8| 6.9542e-01| 1444.7 |
| 1               | 250   | 21   | 17797.1| 9.1911e-01| 963.2 |
Example 6.2: Behavior of the differences \( \| y_f - y^d \|_{\mathcal{X}}^2 \) (left), \( \| y_i - y^d \|_{\mathcal{X}}^2 \) (middle), and \( \| y_f - y_i \|_{\mathcal{X}}^2 \) (right), where the full-rank and low-rank solutions are denoted by \( y_f \) and \( y_i \), respectively, computed by solving the full-rank and low-rank systems with \( N_d = 6144, N = 3, Q = 3, \ell = 1, \mu = 10^{-6}, \gamma = 0, \nu = 1 \), and \( \kappa = 0.05 \) for varying values of the mean of random input \( \eta(x) \).

Next, we compare the full-rank solutions obtained by solving the system \( \mathcal{L} \setminus \mathcal{B} \) with the low-rank ones. Figure 5 exhibits behaviors of the cost functionals \( \mathcal{J}(u_k) \) (left), the tracking term \( \| y_h - y^d \|_{\mathcal{X}}^2 \) (middle), and the relative residual (right) for varying values of the regularization parameter \( \mu \). The key observation is that the low-rank solutions display the same pattern with the full-rank solutions as \( \mu \) increases. Moreover, Table 6 reports the results of the simulations by considering various values of the risk-aversion parameter \( \gamma \). As the previous example, the relative residual becomes smaller as decreasing the value of \( \gamma \).

Last, we investigate the effect of the mean of random input \( \eta(x) \) on both full-rank and low-rank solutions. Denoting the full-rank solution and the low-rank solution by \( y_f \) and \( y_i \), respectively, the behavior of the differences \( \| y_f - y^d \|_{\mathcal{X}}^2 \), \( \| y_i - y^d \|_{\mathcal{X}}^2 \), and \( \| y_f - y_i \|_{\mathcal{X}}^2 \) computed by solving the full-rank and low-rank systems is displayed in Fig. 6. As increasing the mean of random input \( \eta(x) \), the difference between the full-rank and low-rank solutions becomes smaller.

Example 6.3: Simulations of the desired state \( y^d \), the mean of state \( \mathbb{E}[y_h] \), and the control \( u_h \) (from left to right) obtained by \( \mathcal{L} \setminus \mathcal{B} \) with \( N_d = 6144, N = 3, Q = 3, \ell = 1, \kappa = 0.05 \), and \( \nu = 1 \).
Table 7  Example 6.3: Simulation results showing the memory demand of the solution (in KB), the objective function $J(u_h)$, the tracking term $\|y_h - y^d\|_{L^2}^2$, the difference of the full-rank and low-rank $\|y_f - y_l\|_{L^2}$, ranks of the truncated solutions, and the relative residual with $N_d = 6144$, $Q = 3$, $e = 1$, $v = 1$, and the mean-based preconditioner $P_0$.

| $N$ | Memory | $J(u_h)$ | $\|y_h - y^d\|_{L^2}^2$ | $\|y_f - y_l\|_{L^2}$ | Rank | Res. |
|-----|--------|----------|----------------|----------------|------|------|
| 3   | 5744.0 | 5.508e-04| 6.031e-04      |                | 60   | 9.232e-01 |
| 4   | 1444.7 | 1.046e-02| 2.091e-02      | 1.802e-02      | 30   | 9.161e-01 |
| 5   | 2461.9 | 1.029e-02| 2.056e-02      | 1.769e-02      | 51   | 9.161e-01 |
| 6   | 4068.8 | 9.996e-03| 1.996e-02      | 1.713e-02      | 84   | 9.042e-01 |
| 7   | 6130.7 | 9.616e-03| 1.919e-02      | 1.642e-02      | 126  | 8.895e-01 |

### 6.3 Constrained problem with random convection parameter

Last, we consider a constrained optimal control problem containing a random velocity parameter. Except from the set up of Example 6.2, we have an upper bound for the control variable such as $u_h = 100$. Taking the results in the previous example into account, the regularization and risk-averse parameters are chosen as $\mu = 10^{-6}$ and $\gamma = 0$, respectively.

Figure 7 displays the desired state $y^d$, the mean of state $E[y_h]$, and the control $u_h$ obtained by $L \setminus B$. We observe that the upper bound of the control constrained is satisfied. In Table 7, we compare the low-rank solutions with the full-rank ones. As increasing the truncation number $N$, we obtain better results as expected.

### 7 Conclusions

In this paper, we have numerically studied the statistical moments of a robust deterministic optimal control problem subject to a convection-diffusion equation having random coefficients. With the help of the stochastic discontinuous Galerkin method, we transform the original problem into a large system consisting of deterministic optimal control problems for each realization of the random coefficients. However, we could not obtain some numerical results when increasing the value of truncation number $N$. Therefore, to reduce computational time and memory requirements, we have used a low-rank variant of GMRES iteration with a mean-based preconditioner (LRPGMRES). It has been shown in the numerical simulations that LRPGMRES can be an alternative to solve such large systems. As a future study, randomness can be considered in different forms, for instance, in boundary conditions, desired state, or geometry. Moreover, to handle curse of dimensionality, reduced order models, see, e.g., [65, 66], can be an alternative to the low-rank approximations.

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Declarations

Conflict of interest The authors declare no competing interests.

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