Hyperon and antihyperon physics

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Abstract. Structure, interactions and decays of hyperons can be studied at an electron-positron collider in annihilations to a hyperon-antihyperon pair. This method already has huge impact on hyperon physics since it was used in a first modern measurement of decay asymmetry in the weak $\Lambda \rightarrow p\pi^-$ decay and the result calls for reinterpretation of all polarization results for the $\Lambda$ hyperon. In the experiment by the BESIII Collaboration nearly half million $e^+e^- \rightarrow J/\psi \rightarrow \Lambda\bar{\Lambda}$ events were used. The $\Lambda\bar{\Lambda}$ pair production process involves two (complex) hadronic form factors. The observed large relative phase $\Delta\Phi$ between the form factors, $(42.4\pm0.6\pm0.5)^\circ$, makes it possible to use both transverse polarization and the spin correlations of the $\Lambda\bar{\Lambda}$ pair to simultaneously measure decay parameters for $\Lambda$ and $\bar{\Lambda}$ hyperons. Of particular importance is the result for the decay asymmetry parameter $\alpha^-$ in $\Lambda \rightarrow p\pi^-$, the most probable $\Lambda$ decay used to determine $\Lambda$ polarization. The new $\alpha^-$ value from BESIII, $0.750\pm0.009\pm0.004$, is 17(3)% larger than the world average of $0.642\pm0.013$ adopted by all experiments between 1978 and 2019 to determine $\Lambda$ polarization from the measured asymmetry in the decay proton distribution. A comparison of the concurrently measured at BESIII $\alpha^-$ and $\alpha^+$, the parameter for the charge conjugated process $\bar{\Lambda} \rightarrow \bar{p}\pi^+$, results in a direct $CP$ symmetry test for the hyperon.

The $e^+e^- \rightarrow \Lambda\bar{\Lambda}$ reaction was studied at BESIII also outside the $J/\psi$ resonance using 66.9 pb$^{-1}$ collected at $\sqrt{s} = 2.396$ GeV. The result is the first complete determination of the time-like elastic form factors $G_M$ and $G_E$ for any baryon including the relative phase $\Delta\Phi = \arg(G_E/G_M)$ of $(37\pm12\pm6)^\circ$. The absolute normalization of the form factors is set by the determined Born cross section of $\sigma_{\text{Born}} = 119.0 \pm 5.3 \pm 5.1$ pb. In a separate run at $\sqrt{s} = 2.2324$ GeV, 1.0 MeV above threshold for $e^+e^- \rightarrow \Lambda\bar{\Lambda}$, a surprisingly large cross section for of $305\pm45^{+66}_{-36}$ pb was measured.

1. Introduction

Hadronic two-body weak decays of hyperons have historically played a significant role in establishing patterns of parity violation [1]. Nowadays the decays are important part of a quest for a $CP$ symmetry violation signal in the baryon sector. They also serve as polarimeters to determine hyperon spin direction in studies of hadronic reactions involving hyperons. The main decay modes of the ground state hyperons are weak transitions into a baryon and a pseudoscalar meson like $\Lambda \rightarrow p\pi^-$ with branching fraction $B \approx 64\%$ [2]. The decay is described by two amplitudes: parity conserving to the $p$ state, and parity violating to the $s$ state. The angular distribution and the polarization of the daughter baryon are described by two decay parameters: the decay asymmetry $\alpha = 2\text{Re}(s^*p)/(|p|^2+|s|^2)$ and the relative phase $\phi = \arg(s/p)$. For example, in the decay $\Lambda \rightarrow p\pi^-$ the $\Lambda$ hyperon polarization vector $P_\Lambda$ can be reconstructed...
from the angular distribution of the daughter protons: $d\Gamma/d\Omega = (1 + \alpha_+ P_\Lambda \cdot \hat{\mathbf{n}})/(4\pi)$, where $\hat{\mathbf{n}}$ is a unit vector along the proton momentum in the $\Lambda$ rest frame and $\alpha_-$ is the decay asymmetry parameter [3]. Correspondingly, one defines the decay asymmetries $\alpha_+$ for $\Lambda \rightarrow \bar{\rho}\pi^0$, $\alpha_0$ for $\Lambda \rightarrow n\pi^0$, and $\bar{\alpha}_0$ for $\Lambda \rightarrow \bar{n}\pi^0$ [2]. The $\Lambda \rightarrow p\pi^+$ decay is commonly used to determine $\Lambda$ polarization in production reactions [4–16] or in studies of baryon decays involving $\Lambda$ [17–19]. All such experiments rely on the precise value of $\alpha_-$. The world average of 0.642 ± 0.013 [20] was based on the results of five experiments [21–25], where the most precise is from 1967 [22] and the most recent is from 1975 [25].

The well-defined and simple initial state makes baryon-antibaryon pair production at an electron-positron collider an ideal system to investigate the baryon properties and to test fundamental symmetries in the baryon sector. The spin orientations of the baryon and antibaryon are correlated and, for spin one-half baryons, the pair is produced either with the same or opposite helicities. The transition amplitudes to the respective spin states can acquire a relative phase, $\Delta\Phi$ due to the strong interaction in the final state, leading to a time-reversal-odd observable: a transverse spin polarization of the baryons [26, 27]. The continuum baryon-antibaryon pair production in electron-positron annihilations is related to the time-like baryon form factors and the formalism is well known [28–32] but only low statistics data sets are available. In view of the huge $J/\psi$ samples collected at electron-positron colliders, the decay $J/\psi \rightarrow \Lambda\Lambda$, with branching fraction of $(19.43 \pm 0.03 \pm 0.33) \times 10^{-4}$ [33], is an ideal system to measure $\Lambda$ and $\bar{\Lambda}$ decay parameters precisely. In Ref. [34] we have shown that the same formalism should be used to describe the decays and the continuum process at an electron-positron collider. In addition we have derived explicit expressions for the joint angular distributions, suitable for the maximum loglikelihood fits. The formalism was applied in the BESIII analysis leading to the first observation of $\Lambda$ transverse polarization in $J/\psi \rightarrow \Lambda\Lambda$ decay [35]. A $CP$ symmetry test in $\Lambda/\bar{\Lambda}$ can be defined in terms of asymmetry $A_\Lambda = (\alpha_- + \alpha_+)/\sqrt{\alpha_-^2 + \alpha_+^2}$ [36]. The Cabibbo-Kobayashi-Maskawa (CKM) mechanism predicts the $A_\Lambda$ value of $(1 - 5) \sim 10^{-5}$ [37–39]. It is important to improve sensitivity for the measurement of the $CP$ asymmetries in baryon decays for as many modes as possible [40], to determine whether they are consistent with the CKM mechanism [41].

In the next sections I summarize the formalism, report the results of the BESIII proof of concept analysis and discuss future prospects of the method.

2. Formalism
Consider $e^+e^- \rightarrow B_1\bar{B}_2$ reaction where $B_1\bar{B}_2$ is a baryon-antibaryon pair (both with spin one-half). In general, a quantum state of a fermion pair could be represented by the following spin density matrix:

$$\rho_{1/2,1/2} = \sum_{\mu,\nu=0}^{3} C_{\mu\nu} \sigma_\mu^{B_1} \otimes \sigma_\nu^{B_2},$$  \hspace{1cm} (1)$$

where $\sigma_\mu^B$ is a set of Pauli spin matrices defined in the rest frame of a baryon $B$. Here, as in Ref. [34], we use the same orientations of the spin quantization axes for $B_1$ and $B_2$ with the $z$ direction along $B_1$ momentum in the overall c.m. system of the $B_1\bar{B}_2$ pair. The $y$ direction is given by the vector product of the incoming electron and the outgoing baryon $B_1$ momenta. The coefficients $C_{\mu\nu}$ depend on the angle $\theta$ between the electron and baryon $B_1$. At c.m. energies when electron mass could be neglected, a single photon annihilation process could only proceed if the electron and positron have opposite helicities. The final state baryons can have both $\pm 1/2$ helicities. Due to parity conservation out of the four possible helicity transitions only two are independent: $A_{1/2,1/2} = A_{-1/2,-1/2} = h_1$ and $A_{1/2,-1/2} = A_{-1/2,1/2} = h_2$ [42]. Therefore $e^+e^- \rightarrow B_1\bar{B}_2$ process at fixed c.m. energy is described by two complex form factors. If one is
only interested in the not normalized angular distributions only two real parameters are needed which could be defined as:

\[
\alpha_\psi := \frac{|h_2|^2 - 2|h_1|^2}{|h_2|^2 + 2|h_1|^2}; \quad -1 \leq \alpha_\psi \leq 1 \quad \text{and} \quad \Delta \Phi := \arg(h_1/h_2).
\] (2)

The \( C_{\mu\nu} \) 4 \times 4 matrix is given as [34]:

\[
\begin{pmatrix}
1 - \alpha_\psi \cos^2 \theta & 0 & \beta_\psi \sin \theta \cos \theta & 0 \\
0 & \sin^2 \theta & 0 & \gamma_\psi \sin \theta \cos \theta \\
\beta_\psi \sin \theta \cos \theta & 0 & -\alpha_\psi \sin^2 \theta & 0 \\
0 & \gamma_\psi \sin \theta \cos \theta & 0 & \alpha_\psi + \cos^2 \theta
\end{pmatrix},
\]

(3)

where \( \beta_\psi = \sqrt{1 - \alpha_\psi^2 \sin^2 (\Delta \Phi)} \) and \( \gamma_\psi = \sqrt{1 - \alpha_\psi^2 \cos^2 (\Delta \Phi)} \) implying \( \alpha_\psi^2 + \beta_\psi^2 + \gamma_\psi^2 = 1. \)

The polarization vectors \( \mathbf{P} \) of \( B_1 \) and \( B_2 \) has to be in \( y \) direction and the value is given by \( P_y = C_{02}/C_{00} = \beta_\psi \sin \theta \cos \theta/(1 + \alpha_\psi \cos^2 \theta). \)

If a baryon decays weakly, such as the ground state hyperons or charmed baryons, the polarization can be determined using angular distribution of the daughter particles. In a general weak hadronic decay of a spin one-half baryon into a spin one-half baryon and pseudoscalar meson: \( B_A \rightarrow B_B + P \), both the initial an the final states can be represented by a linear combinations of the Pauli spin matrices \( \sigma^B_{\mu} \) and \( \sigma^{BB}_{\mu} \). The weak decay can be therefore expressed by a decay matrix, \( a^{B_A \rightarrow B_B + P}_{\mu,\nu} \), which transforms the base matrices [42]:

\[
\sigma^B_{\mu} \rightarrow \sum_{\nu=0}^{3} a^{B_A \rightarrow B_B + P}_{\mu,\nu} \sigma^{BB}_{\nu}.
\] (4)

In general there are two parameters to describe the \( B_A \rightarrow B_B + P \) decay the decay asymmetry \(-1 \leq \alpha^{B_A \rightarrow B_B + P} \leq 1 \) and the relative phase \(-\pi \leq \phi^{B_A \rightarrow B_B + P} < \pi \). If the polarization of the baryon \( B_B \) is not measured the decay is described only by the \( a^{B_A \rightarrow B_B + P}_{\mu,0} \) elements of the decay matrix and only the \( \alpha^{B_A \rightarrow B_B + P} \) parameter is involved. The joint angular distribution of \( J/\psi \rightarrow \Lambda \bar{\Lambda} (\Lambda \rightarrow f \text{ and } \bar{\Lambda} \rightarrow \bar{f}, f = p\pi^-) \) can be written as the trace of the final proton-antiproton density matrix:

\[
W(\xi; \alpha_\psi, \Delta \Phi, \alpha_-, \alpha_+) := \text{Tr}(\rho_{p\bar{p}}) = \sum_{\mu,\nu=0}^{3} C_{\mu\nu}(\cos \theta; \alpha_\psi, \Delta \Phi) a^{A \rightarrow p\pi^-}_{\mu,0}(\mathbf{n}_1; \alpha_-) a^{\bar{A} \rightarrow \bar{p}\bar{n}^+}_{\nu,0}(\mathbf{n}_2; \alpha_+),
\] (5)

where \( \xi := (\cos \theta, \mathbf{n}_1, \mathbf{n}_2) \) is the complete set of kinematical variables describing the event configuration in the five dimensional phase space and \( \mathbf{n}_1 \) (\( \mathbf{n}_2 \)) is the unit vector in the direction of the nucleon (antinucleon) in the rest frame of \( \Lambda \) (\( \bar{\Lambda} \)). It can be written explicitly as [34]:

\[
W(\xi; \alpha_\psi, \Delta \Phi, \alpha_-, \alpha_+) = C_{00} + C_{02}(\alpha_- n_{1y} + \alpha_+ n_{2y}) + \alpha_+ \alpha_- \{ C_{11}n_{1x}n_{2x} + C_{22}n_{1y}n_{2y} + C_{33}n_{1z}n_{2z} + C_{13}(n_{1x}n_{2z} + n_{1z}n_{2x}) \}.
\] (6)

The terms multiplied by \( \alpha_- \alpha_+ \) represent the contribution from the \( \Lambda \bar{\Lambda} \) spin correlations, while the terms multiplied by \( \alpha_- \) and \( \alpha_+ \) separately represent the contribution from the polarizations. If all three contributions in Eq. (6) are non-zero an unambiguous determination of the parameters \( \alpha_\psi \) and \( \Delta \Phi \) and the decay asymmetries \( \alpha_- \), \( \alpha_+ \) is possible. One should stress that the inclusive measurement of the proton momenta from the \( \Lambda \rightarrow p\pi^- \) decay is not sufficient to determine uniquely \( \alpha_- \). Integrating Eq. (6) over the unmeasured \( \bar{p} \) direction \( \mathbf{n}_2 \) one gets:

\[
4\pi\left(1 + \alpha_\psi \cos^2 \theta + \alpha_- \sqrt{1 - \alpha_\psi^2 \sin^2 (\Delta \Phi)} n_{1y} \sin \theta \cos \theta\right).
\] (7)

It is therefore clear that only the product \( \alpha_- \cdot \sin (\Delta \Phi) \) can be determined from such inclusive analysis.
Table 1. Summary of the BESIII results on $J/\psi \rightarrow \Lambda \bar{\Lambda}$: angular distribution parameter $\alpha_\psi$, the phase $\Delta \Phi$, the asymmetry parameters for the $\Lambda \rightarrow p\pi^-$ ($\alpha_-$), $\bar{\Lambda} \rightarrow \bar{p}\pi^+$ ($\alpha_+$) and $\Lambda \rightarrow \bar{n}\pi^0$ ($\bar{\alpha}_0$) decays, the $CP$ asymmetry $A_\Lambda$, and the ratio $\bar{\alpha}_0/\alpha_+$.

| Parameters | BESIII |
|------------|--------|
| $\alpha_\psi$ | $0.461 \pm 0.006 \pm 0.007$ |
| $\Delta \Phi$ | $(42.4 \pm 0.6 \pm 0.5)^\circ$ |
| $\alpha_-$ | $0.750 \pm 0.009 \pm 0.004$ |
| $\alpha_+$ | $-0.758 \pm 0.010 \pm 0.007$ |
| $\bar{\alpha}_0$ | $-0.692 \pm 0.016 \pm 0.006$ |
| $A_\Lambda$ | $-0.006 \pm 0.012 \pm 0.007$ |
| $\bar{\alpha}_0/\alpha_+$ | $0.913 \pm 0.028 \pm 0.012$ |

3. Observation of polarization in $e^+e^- \rightarrow J/\psi \rightarrow \Lambda \bar{\Lambda}$ at BESIII [35]

The BESIII Collaboration has carried out an analysis of $e^+e^- \rightarrow J/\psi \rightarrow \Lambda \bar{\Lambda}$ based on $1.31 \times 10^9$ $J/\psi$ events collected with the BESIII detector [35]. The $\Lambda$ hyperons are reconstructed using their $p\pi^-$ decays and the $\bar{\Lambda}$ hyperons using their $\bar{p}\pi^+$ decays. The sizes of the final data samples are 420,593 and 47,009 events with an estimated background of 399 $\pm$ 20 and 66.0 $\pm$ 8.2 events for the $p\pi^-\bar{p}\pi^+$ and $p\pi^-\bar{n}\pi^0$ final states, respectively. The background contribution is determined from Monte Carlo (MC) simulation including all known $J/\psi$ decays. For each event the full set of the kinematic variables $\xi$ is reconstructed. The free parameters describing the angular distributions for the two data sets — $\alpha_\psi$, $\Delta \Phi$, $\alpha_-$, $\alpha_+$, and $\bar{\alpha}_0$ — are determined from a simultaneous unbinned maximum likelihood fit using angular distribution given by Eq. (6). A clear polarization signal, strongly dependent on the $\Lambda$ direction, $\cos \theta$, is observed for $\Lambda$ and $\bar{\Lambda}$. In Fig. 1(a) the moment $\mu(\cos \theta) = (m/N) \sum_k N_k (n_{1,g}^{(i)} - n_{2,g}^{(i)})$ related to the polarization is calculated for $m = 50$ bins in $\cos \theta$. $N$ is the total number of events in the data sample and $N_k$ is the number of events in $k$-th $\cos \theta$ bin. The expected angular dependence is $\mu(\cos \theta) \sim (\alpha_- - \alpha_+) \cdot C_02$ for the acceptance corrected data (compare Eq. (6)). The phase between helicity flip and helicity conserving transitions is determined to be $\Delta \Phi = (42.4 \pm 0.6 \pm 0.5)^\circ$. This value of the phase corresponds to the transverse polarization $P_y$ as shown in Fig. 1(b) reaching maximum of
The observed non-zero phase is a clear demonstration that the EFFs cannot be described by

\[ \Delta \Phi = \arg(\alpha) \]

as deduced from the results in Table 1, the average value of \( \alpha \) deviations from the world average value of \( \alpha_{\text{PDG78}} \) = 0.642 ± 0.013 established in 1978 [20]. We note that the two most precise results [22, 24] included in the average were obtained by measuring the asymmetry in the secondary scattering of the polarized protons from \( \Lambda \) decays on a Carbon target. The \( \alpha_- \) value was then determined using a compilation of the polarized proton scattering data on Carbon which is no longer in use [43]. In addition, the uncertainty of \( \alpha_{\text{PDG78}} \) does not include a systematical uncertainty of 5% mentioned in Ref. [22] which points to the need of a critical reevaluation of the value. Our value implies that all published measurements of the \( \Lambda/\bar{\Lambda} \) polarization should be reduced by \((17 \pm 3)\%\). Since the correlation coefficient between \( \alpha_- \) and \( \alpha_+ \) is large \((0.82 \text{ for statistical uncertainties and 0.84 for systematic, as deduced from the results in Table 1})\) the average value of \( \alpha_- \) assuming CP conservation is very precise: \( \langle \alpha \rangle \equiv (\alpha_- - \alpha_+)/2 = 0.754 \pm 3_{\text{stat}} \pm 2_{\text{syst}} \).

4. The \( e^+e^- \rightarrow \Lambda\bar{\Lambda} \) process at the continuum [44, 45]

In a run at \( \sqrt{s} = 2.396 \text{ GeV} \) with integrated luminosity of 66.9 pb\(^{-1} \) a data sample of 555 \( e^+e^- \rightarrow \Lambda\bar{\Lambda} (\Lambda \rightarrow p\pi^- \text{ and } \bar{\Lambda} \rightarrow \bar{p}\pi^+) \) candidate events was collected. The estimated background is 14±4. This allows measurement of \( \Lambda \) hyperon form factors, the first complete determination of time-like elastic form factors (EFFs) for any baryon. In the analysis the joint angular distribution given by Eq. (6) is used. The helicity non-flip and flip amplitudes are identified with the electric and magnetic form factors \( G_E \) and \( G_M \), respectively. The complete information about the form factors is represented by the Born cross-section, the ratio \( R = |G_E/G_M| \) and the phase \( \Delta \Phi = \arg(G_E/G_M) \). The obtained values are: \( R = 0.96 \pm 0.14 \pm 0.02 \) and \( \Delta \Phi = (37 \pm 12 \pm 6) \)°. The observed non-zero phase is a clear demonstration that the EFFs cannot be described by rational functions of the four-momentum transfer squared [27]. The Born cross section of the process is measured to be 119.0 ± 5.3 ± 5.1 pb.

In a separate analysis the Born cross section for the process \( e^+e^- \rightarrow \Lambda\bar{\Lambda} \) was measured at \( \sqrt{s} = 2.2324 \text{ GeV} \) [45], i.e. 1.0 MeV above threshold, to be 305 ± 45 ± 36 pb. Such large value is not expected since for neutral baryons the cross section is not enhanced by the Coulomb factor and should vanish at threshold.

5. Discussion and outlook

The BESIII result for \( \alpha_- \) is now incorporated in the 2019 on-line update of Particle Data Group review [2] and it overrules all previous results for the parameter. It implies that all decay asymmetries for decays with a \( \Lambda \) hyperon in the final state are to be revised too. For example the new values for the \( \Xi^0 \rightarrow \Lambda n^0 \) and \( \Xi^- \rightarrow \Lambda\pi^- \) decay asymmetry parameters are \(-0.347(10)\) and \(-0.392(8)\), respectively. The important role of the \( \alpha_- \) parameter requires that the new value should be verified by independent measurements. In particular it would be reassuring to repeat the old experiments where the decay proton polarization was measured using a dedicated detector. Other possibilities include studies of exclusive reactions with polarized beams or targets where polarization of \( \Lambda \) is tagged, such as \( \gamma p \rightarrow \Lambda K^+ \) and \( \bar{p}p \rightarrow \Lambda\bar{\Lambda} \). In fact some clues at PS185 experiment with polarized proton target [46] indicated that the \( \alpha_- \) value might be significantly larger. A recent reanalysis of CLAS spin data on \( \gamma p \rightarrow \Lambda K^+ \) is reported in Ref. [47] and \( \alpha_- \) of 0.721(6)(5) is extracted, much larger than \( \alpha_{\text{PDG78}} \) but significantly different from the BESIII \( \langle \alpha \rangle \) result. There are also several independent ways to verify the result using the full data sample of \( 10^{10} J/\psi \) events at BESIII. A simple example is decay of the pseudoscalar charmonium \( \eta_c \) into the \( \Lambda\bar{\Lambda} \) pair with \( B(\eta_c \rightarrow \Lambda\bar{\Lambda}) = 1.04(24) \times 10^{-3} \). Here \( \eta_c \) is produced in the radiative decay \( J/\psi \rightarrow \eta_c \gamma \) with the branching fraction of 1.7(4)%. The angular distribution in the \( \eta_c \) decay is extremely simple, \( \mathcal{W} = (1 - \alpha_- \alpha_+ (n_{1,x} n_{2,x} + n_{1,y} n_{2,y} + n_{1,z} n_{2,z})) \). Even if the
expected number of event is much smaller, the product $\alpha_-\alpha_+$ is the only parameter to be fitted. The most interesting scenario for BESIII is cascade-anticascade production, $e^+e^- \rightarrow J/\psi \rightarrow \Xi \Xi$, involving two chains of sequential weak decays $\Xi \rightarrow \pi (\Lambda \rightarrow p \pi^-) + c.c.$, with branching fractions $B(J/\psi \rightarrow \Xi^- \Xi^+) = 10.40(6) \times 10^{-4}$ [48] and $B(J/\psi \rightarrow \Xi^0 \Xi^0) = 11.65(4) \times 10^{-4}$ [49]. The application of Eqs. (1) and (4) leads to the following joint angular distribution in the nine dimensional phase space [42]:

$$W(\xi, \omega) = \text{Tr}(\rho_{p,\bar{p}}) = \sum_{\mu,\nu=0}^{3} C_{\mu
u} \sum_{\mu',\nu'=0}^{3} a_{\mu,\mu'}^{\Xi \rightarrow \Lambda \pi} a_{\nu,\nu'}^{\Xi \rightarrow \Lambda \pi} a_{\mu',0}^{\Lambda \rightarrow p \pi^-} a_{\nu',0}^{\Lambda \rightarrow p \pi^+}. \quad (8)$$

This is much more complicated formula than for the $J/\psi \rightarrow (\Lambda \rightarrow p \pi^-) (\bar{\Lambda} \rightarrow \bar{p} \pi^+) \text{ case}$ and it involves eight global parameters ($\omega$). Two parameters describe the $e^+e^- \rightarrow J/\psi \rightarrow \Xi \Xi$ reaction spin density matrix ($\alpha_\psi$ and $\Delta \Phi$), two are needed to specify each of the $a_{\mu,\mu'}^{\Xi \rightarrow \Lambda \pi}$ and $a_{\nu,\nu'}^{\Xi \rightarrow \Lambda \pi}$ decay matrices (since the polarization of $\Lambda(\bar{\Lambda})$ is measured), $\alpha_-$ and $\alpha_+$. One can rewrite the angular distributions from Eq. (5) and Eq. (8) as a sum of terms consisting of products of functions of the global parameters ($\omega$) and the kinematic variables ($\xi$):

$$W(\xi, \omega) = \sum_{k=1}^{M} g_k(\omega) \cdot h_k(\xi). \quad (9)$$

Such representation for the angular distribution in Eq. (8) requires $M = 72$ unique functions $g_k(\omega)$ of the global parameters while Eq. (5) only $M = 7$. If $\Delta \Phi = 0$ the number of such terms reduces to $M = 56$ in Eq. (8). Instead of discussing simplified cases or projections of such multidimensional angular distributions one can study likelihood function defined as:

$$L(\omega) = \prod_{i=1}^{N} P(\xi_i, \omega) \text{ with } P(\xi_i, \omega) = \frac{W(\xi_i, \omega)}{W(\xi, \omega)}d\xi, \quad (10)$$

where $N$ is the number of events in the final selection and $\xi_i$ is the full set of kinematic variables describing $i$-th event. The asymptotic expression for the $kl$ element of the inverse covariance matrix $V^{-1}$ between parameters $\omega_k$ and $\omega_l$ of the vector parameter $\omega$ is:

$$V^{kl}_{-1} = E\left(-\frac{\partial^2 \ln L}{\partial \omega_k \partial \omega_l}\right) = N \int \frac{1}{P} \frac{\partial P}{\partial \omega_k} \frac{\partial P}{\partial \omega_l} d\xi, \quad (11)$$

where $E(h)$ denotes the expectation value of a random variable $h(\xi)$. This method allows to estimate statistical uncertainties and correlations between the parameters. This will reveal structure of the distributions e.g if $V^{-1}$ matrix is singular some of the parameters are fully correlated and cannot be determined separately. The appealing feature of the formula (11) is that it only requires to calculate a limited number of integrals over the angular distributions and to invert the resulting low rank matrix. This procedure effectively replaces the method of moments for the multi-parameter cases. Using this approach we have shown in Ref. [50] that in $e^+e^- \rightarrow J/\psi \rightarrow \Xi \Xi$ all parameters can be determined even if $\Delta \Phi = 0$ and the $\alpha_-$ statistical uncertainty for the fixed number of the reconstructed events is more than twice better than in the $e^+e^- \rightarrow J/\psi \rightarrow \Lambda \bar{\Lambda}$ process. This can be qualitatively understood by recalling that even if $\Xi^-$ is unpolarized the $\Lambda$ from the weak decay will have longitudinal polarization equal to $|\alpha_\xi| \approx 40\%$, while in $e^+e^- \rightarrow J/\psi \rightarrow \Lambda \bar{\Lambda}$ the average length of the $\Lambda$ polarization vector is only $11\%$. In addition the $e^+e^- \rightarrow J/\psi \rightarrow \Xi \Xi$ process allows for $CP$ test using comparison of the weak decay phases in $\Xi \rightarrow \Lambda \pi$ and c.c. decays.

Finally the modular formalism was extended to the case of spin-3/2 baryons [42]. It can be used to describe e.g. the reaction $e^+e^- \rightarrow \psi' \rightarrow \Omega^- \bar{\Omega}^+$ with the subsequent decays. The production of higher spin baryon-antibaryon pairs in electron-positron annihilations introduces additional spin polarization effects.
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