Abstract—We have witnessed the emergence of several controller parameterizations and the corresponding synthesis methods, including Youla, system level, input-output, and many other new proposals. Meanwhile, under the same synthesis method, there are multiple realizations to adopt. Different synthesis methods/realizations target different plants/scenarios. Also, various robust results are proposed to deal with different perturbed system structures. Except for some case-by-case studies, we don’t currently have a unified framework to understand their relationships.

To address the issue, we show that existing controller synthesis methods and realization proposals are all special cases of a simple lemma, the realization-stability lemma. The lemma leads to easier equivalence proofs among existing methods and robust stability conditions for general system analysis. It also enables the formulation of a general controller synthesis problem, which provides a unified foundation for controller synthesis, realization derivation, and robust stability analysis.

Index Terms—Cyber-physical systems, system level synthesis.

I. INTRODUCTION

CONTROLLER synthesis is one of the core missions in control theory. It aims to derive controllers that could stabilize a given plant in the presence of disturbance and/or perturbation. Synthesizing such controllers, the internally stabilizing controllers, is highly non-trivial, especially as the modern systems grow larger, more complex, and involve more input/output signals. Meanwhile, the ever-increasing system complexity also demands sophisticated and efficient controller synthesis techniques and drives the research of controller synthesis theories.

The first challenge for controller synthesis is external disturbance rejection, i.e., the synthesized controller should be able to neutralize the impact of unanticipated external disturbances in the long run. A well-celebrated pioneer work on this direction is by Youla et al. [1], [2], which shows that the set of all internally stabilizing controllers can be parameterized using a coprime factorization approach. One drawback of Youla parameterization is the difficulty of imposing structural constraints – the constraints could only be imposed (while maintaining convexity) in an intricate form admitting quadratic invariant property [3]–[5]. To address this issue, system level parameterization (SLP) [6], [7] proposes to work on the closed-loop system response and the corresponding system level synthesis (SLS) method can easily incorporate multiple structural constraints into a much simpler convex program [8], [9]. The success of SLP triggers the study of affine space parameterization of internally stabilizing controllers. [10] shows that the set of internally stabilizing controllers can also be parameterized in an input-output manner using the input-output parameterization (IOP). Though a recent paper shows that Youla, SLP, and IOP are equivalent [11], there are still new affine space parameterizations found [12].

Besides rejecting external disturbances, more advanced controller synthesis theories study perturbed plants and aim to achieve robust stability, where the controller could still stabilize the underlying plant even when perturbation, or sometimes referred as uncertainty, occurs. There are many robust results proposed in the literature, and they mainly focus on mitigating one of the following two concerns – plant uncertainty or controller perturbation. Due to the estimation precision, dimension limit, or plant nature, the plant model could differ from its true dynamics. To deal with this uncertainty, one category of robust results aims to synthesize a controller that can stabilize a set of plants, such as $\mu$-synthesis [13]–[15], robust primal-dual Youla parameterization [16], and robust input-output parameterization (IOP) [17]. On the other hand, even with an exact plant model, realization of the synthesized controller may still deviate from its desired form because of resolution restriction. As such, another category of robust results ensures a perturbed controller realization can still stabilize the plant, e.g., robust system level synthesis (SLS) [6], [18], [19].

Given the flourishing development of novel parameterizations and their corresponding synthesis methods, we have some natural questions to ask: Have we exhausted all possible parameterizations? Will we discover new synthesis methods? If so, why would they be the way they are? And, perhaps more importantly, how could we find/understand them systematically?

To add to this already puzzling situation, we have seen new results on realizations. Realizations, or block diagrams/implementations,\(^1\) describe how a system can be built from some interconnection of basic blocks/transfer functions. It is well known, also shown by recent studies [20], [22], that the same controller can admit multiple different realizations,\(^1\)

\(^1\)We adopt the terminology in [20], [21] that distinguishes “realizations” from “implementations,” where the former refers to the block diagrams (mathematical expressions) and the latter is reserved for the physical architecture consisting of computation, memory, and communication units.
even under the same parameterization scheme. We would then wonder if we can only handle those realizations individually, or if there is a unified framework to study them.

Similarly, robust controller synthesis results are derived via various analysis tools targeting distinct settings. It is not straightforward to see how they relate with one another and how one method may be applicable for a different setting. As a consequence, most robust results are taught, learned, and applied in a case-by-case manner. Moreover, in addition to the two concerns above, one can easily imagine some compound scenarios where both the plant and controller are subject to perturbation. We would then wonder how to deal with diverse perturbation scenarios systematically. In particular, we are interested in a unified approach to robust controller analysis and synthesis.

A. Contributions and Organization

The main contribution of this paper is a systematic approach to all of the above seemingly unrelated questions through a simple realization-stability lemma that relates closed-loop realizations with internal stability. The lemma enables us to formulate the general controller synthesis problem that can derive all possible parameterizations, thus providing a systematic way to study controller synthesis problems. We show that existing methods on controller synthesis and realization are all special cases of the general formulation. In addition, the lemma reveals that the transformation of external disturbances can be seen as the derivation of an equivalent system. The concept of equivalent systems then enables easy proof of equivalence among synthesis methods.

Also, this paper provides such a unified approach through the robust stability conditions for general systems derived from the realization-stability lemma. As both plant and controller are included in a realization, the uncertainty to any of them is deemed a perturbation of the realization, thereby unifying diverse robustness concerns into one coherent form. We also specialize our result for additively perturbed realizations. The robust stability condition leads to the formulation of the general robust controller synthesis problem, and we demonstrate how to derive existing results in the literature using the condition. In addition, we show how new robust results for output-feedback SLS and IOP can be obtained easily from the condition.

The paper is organized as follows. In Section II, we derive the realization-stability lemma, introduce equivalent systems under transformations, and formulate the general controller synthesis problem. We then extend the lemma to robust settings and derive the robust stability conditions in Section III, along with the corresponding formulation of the general robust controller synthesis problem. Leveraging the realization-stability lemma and the robust stability conditions, we unify existing controller synthesis results in Section IV, realization results in Section V, and robust results in Section VI as corollaries. In addition, we demonstrate how to apply our lemmas to derive new results and discuss the application scope in Section VII. Finally, we conclude the paper in Section VIII.

B. Notation

Let \( \mathcal{R}_p \), \( \mathcal{R}_{sp} \), and \( \mathcal{RH}_\infty \) denote the set of proper, strictly proper, and stable proper transfer matrices, respectively, all defined according to the underlying setting, continuous or discrete. Lower- and upper-case letters (such as \( x \) and \( A \)) denote vectors and matrices respectively, while bold lower- and upper-case characters and symbols (such as \( \mathbf{u} \) and \( \mathbf{R} \)) are reserved for signals and transfer matrices. We denote by \( I \) and \( O \) the identity and all-zero matrices (with dimensions defined according to the context).

II. REALIZATION-STABILITY LEMMA

To begin with, we define the realization and internal stability matrices to derive the realization-stability lemma. We then discuss the transformation of external disturbances and introduce the concept of equivalent systems. Using the realization-stability lemma, we propose the formulation of a general controller synthesis problem.

We remark that the results in this section are general: They apply to both discrete-time and continuous-time systems.

A. Realization and Internal Stability

We consider a closed-loop linear system with internal state \( \eta \) and external disturbance \( d \). The system operates according to the realization matrix \( R \):

\[
\eta = R\eta + d. \tag{1}
\]

\( \eta \) summarizes all signals in the system. For instance, a state-feedback system might have \( \eta = \begin{bmatrix} x \\ \xi \\ u \end{bmatrix} \) where \( x \) is the state, \( \xi \) is the internal state, and \( u \) is the control. For a given signal \( a \), we denote by \( e_a \) the column block that is identity at the rows corresponding to \( a \) in \( \eta \). As a result, \( \eta = \sum a e_a \).

\( R \) describes each signal as a linear combination of the signals in the system. We denote by \( R_{ab} \) the transfer matrix block from signal \( b \) to \( a \) as shown in Fig. 1, and hence given a signal \( a \), we have \( a = \sum b R_{ab} b + d_a \), where \( d_a \) is the external disturbance on \( a \). Notice that all dimensions in the internal state \( \eta \) have their corresponding share in \( d \), thereby avoiding the partial selection issues discussed in [11].

On the other hand, if we deem the external disturbance \( d \) as the input and the internal state \( \eta \) as the output, we can treat the closed-loop system as an open-loop system. We denote by
the internal stability matrix (or stability matrix for short) $S$ the transfer matrix of such an open-loop system:

$$\eta = Sd.$$  \hfill (2)

We define $S_{ab}$ as the transfer matrix block from disturbance on $b$ to the signal $a$, and the columns in $S$ corresponding to $b$ is denoted by $S_{b,}$.

The realization matrix $R$ and the stability matrix $S$ are related by the following lemma.

Lemma 1 (Realization-Stability). Let $R$ be the realization matrix and $S$ be the internal stability matrix, we have

$$(I - R)S = S(I - R) = I.$$  

Proof. Substituting (2) into (1) yields

$$(I - R)\eta = (I - R)Sd = d.$$  

Since $d$ is arbitrary, we have

$$(I - R)S = I.$$  

Given $I - R$ and $S$ are both square matrices, we have

$$S = (I - R)^{-1} \Rightarrow S(I - R) = I,$$

which concludes the proof.

We remark that Lemma 1 does not guarantee the existence of either $R$ or $S$. Rather, it says if both $R$ and $S$ exist, they must obey the relation. When they both exist, a consequence of Lemma 1 is that $R \rightarrow S$ is a bijection map. In other words, if two systems have the same realization $R$ (or $I - R$, equivalently), they have the same internal stability $S$.

B. Disturbance Transformation and Equivalent System

In (1), the external disturbance $d$ affects each signal in the system independently. We can extend (1) and (2) to the cases where the dimensions in $d$ are correlated. In particular, the external disturbance could be a transformation $T$ on a different basis $w$:

$$d = Tw.$$  

When the transformation $T$ is invertible, we have

$$(I - R)\eta = Tw \Rightarrow T^{-1}(I - R)\eta = w = (I - R_{eq})\eta,$$

$$\eta = STw = S_{eq}w.$$  

In other words, the transformation of the disturbance $d = Tw$ can be seen as the derivation of an equivalent closed-loop system with realization $R_{eq}$ and stability $S_{eq}$ based on internal state $\eta$ and external disturbance $w$.

The derivation of an equivalent system is helpful for stability analysis. Since Lemma 1 suggests that there is a bijection map from $R$ to $S$. If there are two systems with realizations $R_1$ and $R_2$ and we can relate them through an (invertible) transformation $T$ by

$$(I - R_2) = T^{-1}(I - R_1),$$

their stability matrices will follow

$$S_2 = S_1T.$$  

C. Controller Synthesis and Column Dependency

Notice that Lemma 1 holds for arbitrary realization/internal stability matrices, e.g., non-causal $R$ and unstable $S$. When synthesizing a controller, we require the closed-loop system to be causal and internally stable. In other words, the transfer functions from one signal to any different signal should be proper, and the transfer functions from the external disturbance $d$ to the internal state $\eta$ should be stable proper, which are written as the following conditions:

$$R_{ab} \in \mathcal{R}_p, \forall a \neq b, \quad S \in \mathcal{RH}_\infty.$$  \hfill (3)

Here, we implicitly require the existence of both $R$ and $S$. Accordingly, general controller synthesis problems (i.e., all possible controller synthesis problems for a system described by some $R$) can be formulated as

$$\min g(R, S)$$

s.t. $$\begin{align*}
(I - R)S &= S(I - R) = I \\
R_{ab} &\in \mathcal{R}_p, \forall a \neq b \\
S &\in \mathcal{RH}_\infty \\
(R, S) &\in C
\end{align*}$$

where $g$ is the objective function and $C$ represents the additional constraints on the realization and internal stability. In the following sections, we will show that the existing controller synthesis methods/realization studies that focus on internal stability are essentially special cases of the feasible set in this general formulation.

A key constraint in the general controller synthesis problem is to enforce $S \in \mathcal{RH}_\infty$. Although we need to enforce all elements in $S$ to be in $\mathcal{RH}_\infty$, we can leverage the linear dependency among the components brought by Lemma 1 to derive some parts automatically without explicit enforcement. In particular, we have Lemma 2.

Lemma 2. Let $a$ be a signal and $R_{aa} = O$, then

$$S_{:,a} = e_a + \sum_{b \neq a} S_{:,b}R_{ba}.$$  

Proof. By Lemma 1, we have $S(I - R) = I$ and hence

$$S_{:,a}(I - R_{aa}) - \sum_{b \neq a} S_{:,b}R_{ba} = e_a.$$  

The lemma follows as $R_{aa} = O$.  

Lemma 2 can greatly reduce the decision variables when synthesizing a controller. For instance, the synthesized control $u$ is usually a function of other signals except for itself, which implies $R_{uu} = O$. Therefore, Lemma 2 gives

$$S_{:,u} = e_u + \sum_{b \neq u} S_{:,b}R_{bu}. $$  \hfill (4)

III. Robust Stability Analysis

In this section, we derive the condition for robust stability analysis using Lemma 1. The condition then allows us to formulate the general robust controller synthesis problem.
A. Robust Stability and Additive Perturbations

Consider a system perturbed according to some uncertain parameter \( \Delta \in \mathcal{D} \) as in Fig. 2(a), where \( \mathcal{D} \) is the uncertainty set. Denote by \( \mathbf{R}(\Delta) \) its realization matrix and by \( \mathbf{S}(\Delta) \) the corresponding stability matrix. By Lemma 1, the perturbed realization and stability matrices satisfy

\[
(I - \mathbf{R}(\Delta))\mathbf{S}(\Delta) = \mathbf{S}(\Delta)(I - \mathbf{R}(\Delta)) = I.
\]

Also, the perturbed system is robustly stable if and only if the open-loop system from the external disturbance \( d \) to the internal state \( \eta \) is stable under all uncertain parameter \( \Delta \in \mathcal{D} \). In other words, we require the stability matrix \( \mathbf{S}(\Delta) \) to obey

\[
\mathbf{S}(\Delta) \in \mathcal{RH}_\infty, \quad \forall \Delta \in \mathcal{D}.
\]

Suppose the system is subject to additive perturbation and its realization matrix \( \mathbf{R}(\Delta) \) can be expressed as

\[
\hat{\mathbf{R}}(\Delta) = \hat{\mathbf{R}} + \Delta
\]

where \( \hat{\mathbf{R}} \) is the nominal realization, as shown in Fig. 2(b). Define the nominal stability \( \mathbf{S} \) as the stability matrix accompanying the nominal realization \( \hat{\mathbf{R}} \), we can express \( \mathbf{S}(\Delta) \) in terms of \( \hat{\mathbf{S}} \) and the perturbation \( \Delta \) as follows.

**Lemma 3 (Stability under Additive Perturbation).** Let \( \hat{\mathbf{R}} \) be the nominal realization matrix and \( \hat{\mathbf{S}} \) be the nominal internal stability matrix. Suppose the system realization \( \hat{\mathbf{R}}(\Delta) \) is subject to additive perturbation \( \Delta \), the corresponding stability \( \mathbf{S}(\Delta) \) is given by

\[
\mathbf{S}(\Delta) = \hat{\mathbf{S}}(I - \Delta \hat{\mathbf{S}})^{-1} = (I - \hat{\mathbf{S}} \Delta)^{-1}\hat{\mathbf{S}}.
\]

**Proof.** Lemma 1 implies \( (I - \hat{\mathbf{R}})\hat{\mathbf{S}} = I \). Therefore,

\[
(I - \mathbf{R}(\Delta))\hat{\mathbf{S}} = (I - \hat{\mathbf{R}} - \Delta)\hat{\mathbf{S}} = I - \Delta \hat{\mathbf{S}}.
\]

As a result, Lemma 1 suggests

\[
(I - \mathbf{R}(\Delta))\hat{\mathbf{S}}(I - \Delta \hat{\mathbf{S}})^{-1} = I
\]

\[
\Rightarrow \quad \mathbf{S}(\Delta) = (I - \hat{\mathbf{S}} \Delta)^{-1}\hat{\mathbf{S}}.
\]

Similarly, we can also derive \( \mathbf{S}(\Delta) = (I - \hat{\mathbf{S}} \Delta)^{-1}\hat{\mathbf{S}} \) from \( \hat{\mathbf{S}}(I - \hat{\mathbf{R}}) = I \), and the lemma follows.

We can interpret the resulting stability matrix \( \mathbf{S}(\Delta) \) in Lemma 3 as the nominal stability \( \hat{\mathbf{S}} \) with a feedback path \( \Delta \) as in Fig. 3. From this perspective, an additive perturbation to the realization results in a feedback path in the stability.

\[\text{(a) Realization perturbed by } \Delta. \quad \text{(b) Additive Perturbation.}\]

**Fig. 2:** We denote by \( \mathbf{R}(\Delta) \) the realization matrix under perturbation \( \Delta \) and by \( \hat{\mathbf{R}} \) the nominal realization matrix without perturbation.

B. General Formulation for Robust Controller Synthesis

We can now generalize the general controller synthesis problem in [23] to its robust version using condition (5):

\[
\min g(\mathbf{R}(\Delta), \mathbf{S}(\Delta), \mathcal{D})
\]

s.t.

\[
(I - \mathbf{R}(\Delta))\mathbf{S}(\Delta) = \mathbf{S}(\Delta)(I - \mathbf{R}(\Delta)) = I
\]

\[
\forall \Delta \in \mathcal{D}
\]

\[
\mathbf{R}(\Delta)_{ab} \in \mathcal{R}_p \quad \forall \Delta \in \mathcal{D}, a \neq b
\]

\[
\mathbf{S}(\Delta) \in \mathcal{R}H_{\infty} \quad \forall \Delta \in \mathcal{D}
\]

\[
(\mathbf{R}(\Delta), \mathbf{S}(\Delta)) \in \mathcal{C} \quad \forall \Delta \in \mathcal{D}
\]

where \( \mathcal{C} \) represents some additional constraints on the realization and stability. In particular, for a system subject to additive perturbation, the problem can be reformulated as

\[
\min g(\hat{\mathbf{R}}, \hat{\mathbf{S}}, \mathcal{D})
\]

s.t.

\[
(I - \hat{\mathbf{R}})\hat{\mathbf{S}} = \hat{\mathbf{S}}(I - \hat{\mathbf{R}}) = I
\]

\[
\mathbf{R}(\Delta)_{ab} = (\hat{\mathbf{R}} + \Delta)_{ab} \in \mathcal{R}_p \quad \forall \Delta \in \mathcal{D}, a \neq b
\]

\[
\mathbf{S}(\Delta) = (I - \hat{\mathbf{S}} \Delta)^{-1}\hat{\mathbf{S}} \in \mathcal{R}H_{\infty} \quad \forall \Delta \in \mathcal{D}
\]

\[
(\mathbf{R}(\Delta), \mathbf{S}(\Delta)) \in \mathcal{C} \quad \forall \Delta \in \mathcal{D}
\]

This formulation is general as it can describe not only the robust controller synthesis problem for a given uncertainty set \( \mathcal{D} \) but also the stability margin problem like \( \mu \)-synthesis [13]–[15], where \( \mathcal{D} \) is itself a variable to “maximize”.

Despite its generality, solving this general formulation is challenging in general, and the major obstacle is to ensure the perturbed stability for all \( \Delta \in \mathcal{D} \). Except for some computationally tractable cases as those listed in Section ??, enforcing the robust constraints involves dealing with semi-infinite programming when the \( \mathcal{D} \) has infinite cardinality. For those cases, one may instead enforce chance constraints and adopt sampling-based techniques as in [24]–[26].

IV. Corollaries: Controller Synthesis

We use Lemma 1 and condition (3) to derive existing controller synthesis proposals, including Youla [2], input-output
[10], system level [6], [7], mixed parameterizations [12], and
generalized system level synthesis [27], with different
and system level [6], [7], mixed parameterizations [12], and
We then demonstrate a simpler way to obtain the results in [11] using transformations.

A. Youla Parametrization
Youla parameterization is based on the doubly coprime
factorization of the plant \( G \). If \( G \) is stabilizable and detectable, we have
\[
\begin{bmatrix}
M_l & -N_l \\
-V_l & U_l
\end{bmatrix}
\begin{bmatrix}
U_r & N_r \\
V_r & M_r
\end{bmatrix} = I
\]
where both matrices are in \( \mathcal{R}H_\infty \), \( M_l \) and \( M_r \) are both invertible in \( \mathcal{R}H_\infty \), and \( G = M_l^{-1}N_l = N_rM_r^{-1} \) [15, Theorem 5.6]

The following corollary is a modern rewrite of the original
Youla parameterization in [2, Lemma 3] given by [15, Theorem 11.6]:

**Corollary 1.** Let the plant \( G \) be doubly coprime factorizable.
Given \( Q \in \mathcal{R}H_\infty \), the set of all proper controllers achieving
internal stability is parameterized by
\[
K = (V_r - M_rQ)(U_r - N_rQ)^{-1}.
\]

**Proof.** Consider the realization in Fig. 4, which has
\[
R = \begin{bmatrix} O & G \\ K & O \end{bmatrix}, \quad \eta = \begin{bmatrix} x \\ u \end{bmatrix}.
\]

To show that all \( K \) can be parameterized by \( Q \in \mathcal{R}H_\infty \),
we need to show that each \( Q \) is mapped to one valid \( K \) and
vice versa. For mapping \( Q \) to \( K \), we consider the following transformation:
\[
T^{-1} = \begin{bmatrix} I & O \\ -Q & I \end{bmatrix}, \quad T = \begin{bmatrix} I & O \\ Q & I \end{bmatrix}
\]

As such,
\[
I = T^{-1} \begin{bmatrix} M_l & -N_l \\ -V_l & U_l \end{bmatrix} T
= \begin{bmatrix} I & O \\ -Q & I \end{bmatrix} \begin{bmatrix} M_l & O \\ U_l & -QN_l \end{bmatrix} = (I - R)S
\]

where \( S_{uu} \) and \( S_{uy} \) are given by (4) and \( G = M_l^{-1}N_l \):
\[
\begin{bmatrix} S_{uy} \\ S_{uu} \end{bmatrix} = \begin{bmatrix} O \\ I \end{bmatrix} + \begin{bmatrix} U_r & N_r \\ V_r & M_r \end{bmatrix} \begin{bmatrix} Q \\ M_lG \end{bmatrix} = \begin{bmatrix} O \\ I \end{bmatrix} + \begin{bmatrix} U_r & N_r \\ V_r & M_r \end{bmatrix} \begin{bmatrix} Q \\ N_l \end{bmatrix}.
\]

Since \( T \in \mathcal{R}H_\infty \), we have \( S \in \mathcal{R}H_\infty \). Therefore,
\[
-K(U_r - N_rQ) + (V_r - M_rQ) = O,
\]
which leads to the desired \( K \).

On the other hand, for mapping \( K \) to \( Q \), internal stability
of \( K \) implies the corresponding \( S_{ux} \in \mathcal{R}H_\infty \). We compute \( Q \) by
\[
Q = M_l^{-1}(V_r - S_{ux}M_l^{-1}),
\]
which is also in \( \mathcal{R}H_\infty \) as \( M_l \) and \( M_r \) are both invertible in \( \mathcal{R}H_\infty \) (i.e., \( M_l^{-1}, M_r^{-1} \in \mathcal{R}H_\infty \)), and all elements in \( S \) can be expressed in \( Q \) using Lemma 1.

**B. Input-Output Parametrization (IOP)**

Inspired by the system level approach in [7], [10] revisits
the input-output system studied by Youla parameterization and
proposes IOP as follows that does not depend on the doubly
coprime factorization [10, Theorem 1].

**Corollary 2.** For the realization in Fig. 4 with \( G \in \mathcal{R}_{sp} \), the set of all proper internally stabilizing controller is parameterized by \( \{Y, U, W, Z\} \) that lies in the affine subspace defined by the equations
\[
\begin{bmatrix} I & -G \\ Y & W \end{bmatrix} \begin{bmatrix} Y \\ U \end{bmatrix} = \begin{bmatrix} I & O \\ Y & U \end{bmatrix},
\begin{bmatrix} Y & W \\ U & Z \end{bmatrix} \begin{bmatrix} -G \\ I \end{bmatrix} = \begin{bmatrix} O \\ I \end{bmatrix},
\]

\( Y, U, W, Z \in \mathcal{R}H_\infty \),

and the controller is given by \( K = UY^{-1} \).

**Proof.** We can write down the realization matrix in Fig. 4:
\[
R = \begin{bmatrix} O & G \\ K & O \end{bmatrix}, \quad \eta = \begin{bmatrix} y \\ u \end{bmatrix}.
\]

Given \( K \), the derivation of \( \{Y, U, W, Z\} \) is a direct consequence of Lemma 1 and condition (3), which suggest
\[
\begin{bmatrix} I & -G \\ Y & W \end{bmatrix} \begin{bmatrix} I & O \\ Y & U \end{bmatrix} = \begin{bmatrix} I & -G \\ U & Z \end{bmatrix} \begin{bmatrix} I & O \\ -K & I \end{bmatrix},
\]

\( Y, U, W, Z \in \mathcal{R}H_\infty \).

Conversely, given \( \{Y, U, W, Z\} \), (9) implies
\[
U = KY \quad \Rightarrow \quad K = UY^{-1}.
\]

We need to verify that \( Y \) is invertible in \( \mathcal{R}_p \) so that \( K \in \mathcal{R}_p \).
Given \( G \in \mathcal{R}_{sp} \), we know that
\[
Y = I + GU = I + (zI - L)^{-1}J.
\]

for some matrix \( L \) and \( J \in \mathcal{R}_p \). As a result,
\[
Y^{-1} = I + \sum_{k=1}^{\infty} (zI - L)^{-k}J^k \in \mathcal{R}_p,
\]

which concludes the proof.

[27] also provides an alternative proof to the corollary that differs from [10] and the simple proof here.
C. System Level Parametrization/Synthesis (SLP/SLS)

System level synthesis (SLS) uses system level parameterization (SLP) to parameterize internally stabilizing controllers. There are two SLPs: for state-feedback and output-feedback systems, respectively. We discuss them below.

**State-Feedback:** The following state-feedback parameterization is given in [7, Theorem 1] and [6, Theorem 4.1].

**Corollary 3.** For the realization in Fig. 5, the set of all proper internally stabilizing state-feedback controller parameterized by \( \{ \Phi_x, \Phi_u \} \) that lies in the affine space defined by

\[
\begin{bmatrix}
  zI - A & -B \\
  -K & I \\
\end{bmatrix}
\begin{bmatrix}
  \Phi_x \\
  \Phi_u \\
\end{bmatrix} = I,
\]

\( \Phi_x, \Phi_u \in z^{-1}\mathcal{RH}_\infty \),

and the controller is given by \( K = \Phi_u \Phi_x^{-1} \).

**Proof.** The realization matrix in Fig. 5 is

\[
R = \begin{bmatrix}
  A + (1 - z)I & B \\
  K & O \\
\end{bmatrix}, \quad \eta = \begin{bmatrix} x \\ u \end{bmatrix}.
\]

To derive \( K \) from \( \{ \Phi_x, \Phi_u \} \), Lemma 1 and condition (3) lead to

\[
\begin{bmatrix}
  zI - A & -B \\
  -K & I \\
\end{bmatrix}
\begin{bmatrix}
  \Phi_x \\
  \Phi_u \\
\end{bmatrix} = I,
\]

\( K \in \mathcal{R}_p, \quad \Phi_x, \Phi_u \in \mathcal{RH}_\infty \).

Meanwhile, since

\( (zI - A)\Phi_x = I + B\Phi_u \in \mathcal{RH}_\infty \),

we have \( \Phi_x \in z^{-1}\mathcal{RH}_\infty \). As a result, given \( K \in \mathcal{R}_p \),

\( \Phi_u = K\Phi_x = z^{-1}K(z\Phi_x) \in z^{-1}\mathcal{R}_p \),

we know \( \Phi_u \in z^{-1}\mathcal{R}_p \cap \mathcal{RH}_\infty = z^{-1}\mathcal{RH}_\infty \).

Conversely, given \( \{ \Phi_x, \Phi_u \} \), we can derive \( K = \Phi_u \Phi_x^{-1} \) from Lemma 1. It remains to show that \( S_{xx} \) and \( S_{uu} \) exist whenever \( \{ \Phi_x, \Phi_u \} \) is given. According to (4)

\[
\begin{bmatrix}
  S_{xx} \\
  S_{uu} \\
\end{bmatrix} = \begin{bmatrix} O \\ I \end{bmatrix} + \begin{bmatrix}
  \Phi_x \\
  \Phi_u \\
\end{bmatrix} B \in \mathcal{RH}_\infty,
\]

which concludes the proof.

**Output-Feedback:** The output-feedback SLP below is from [7, Theorem 2] and [6, Theorem 5.1].

**Corollary 4.** For the realization in Fig. 6 with \( D = O \), the set of all proper internally stabilizing output-feedback controller is parameterized by \( \{ \Phi_{xx}, \Phi_{ux}, \Phi_{xy}, \Phi_{uy} \} \) that lies in the affine space defined by

\[
[zI - A - B \begin{bmatrix}
  \Phi_{xx} \\
  \Phi_{ux} \\
\end{bmatrix} \Phi_{xy} = [I \quad O],
\]

\( \Phi_{xx}, \Phi_{xy}, \Phi_{ux}, \Phi_{uy} \in z^{-1}\mathcal{RH}_\infty, \quad \Phi_{uy} \in \mathcal{RH}_\infty \),

and the controller is given by

\[
K = \Phi_{uy} - \Phi_{ux}\Phi_{xx}^{-1}\Phi_{xy}.
\]

In fact, we can extend Corollary 4 to general \( D \).

**Corollary 5.** Given \( \{ \Phi_{xx}, \Phi_{ux}, \Phi_{xy}, \Phi_{uy} \} \) that lies in the affine space in Corollary 4 and an arbitrary \( D \), the proper internally stabilizing output-feedback controller \( K \) is given by

\[
K = K_0 (I + DK_0)^{-1}
\]

where \( K_0 = \Phi_{uy} - \Phi_{ux}\Phi_{xx}^{-1}\Phi_{xy} \).

We prove the more general version – Corollary 5 – below.

**Proof.** The realization matrix in Fig. 6 is

\[
R = \begin{bmatrix}
  A + (1 - z)I & B & O \\
  O & K & O \\
  C & D & O \\
\end{bmatrix}, \quad \eta = \begin{bmatrix} x \\ u \\ y \end{bmatrix}.
\]

Given \( K \) we can directly derive \( \{ \Phi_{xx}, \Phi_{ux}, \Phi_{xy}, \Phi_{uy} \} \) from Lemma 1

\[
I = \begin{bmatrix}
  zI - A - B & O \\
  -C & -D & I \\
\end{bmatrix}
\begin{bmatrix}
  \Phi_{xx} & S_{xx} \\
  \Phi_{ux} & S_{uu} \\
  \Phi_{xy} & S_{xy} \\
\end{bmatrix} = \begin{bmatrix}
  zI - A - B & O \\
  -C & -D & I \\
\end{bmatrix}
\begin{bmatrix}
  \Phi_{ux} & S_{xy} \\
  \Phi_{uy} & S_{yy} \\
\end{bmatrix},
\]

where \( S \in \mathcal{RH}_\infty \) by condition (3). As a result, we have

\[
(zI - A)\Phi_{xx} = I + B\Phi_{ux} \in \mathcal{RH}_\infty,
\]

\[
(zI - A)\Phi_{xy} = B\Phi_{uy} \in \mathcal{RH}_\infty,
\]

\[
\Phi_{ux}(zI - A) = \Phi_{uy}C \in \mathcal{RH}_\infty.
\]

Therefore, \( \Phi_{xx}, \Phi_{ux}, \Phi_{xy} \in z^{-1}\mathcal{RH}_\infty \) and \( \Phi_{uy} \in \mathcal{RH}_\infty \).
Conversely, we can derive $K$ from $\{\Phi_{xx}, \Phi_{ux}, \Phi_{xy}, \Phi_{uy}\}$ as follows. First, we multiply the matrix

$$
\Gamma = \begin{bmatrix}
I & B & O \\
O & I & O \\
O & O & D \\
\end{bmatrix}
$$

at the left of both sides of (12), which leads to

$$
[zI - A - BK] \begin{bmatrix}
\Phi_{xx} \\
S_{yx} \\
\Phi_{xy} \\
\end{bmatrix} = I.
$$

Therefore, as $\Phi_{xx}$ and $S_{yy}$ are both square, taking matrix inverse, we have

$$
I - DK = (S_{yy} - S_{yx} \Phi_{xx}^{-1} \Phi_{xy})^{-1}.
$$

Since $\Phi_{ux} = KS_{yx}$ and $\Phi_{uy} = KS_{yy}$, we know

$$
K_0 = K (S_{yy} - S_{yx} \Phi_{xx}^{-1} \Phi_{xy})
$$

and we can rearrange the equation to obtain

$$
K_0 - KDK_0 = K \Rightarrow K = K_0 (I + DK_0)^{-1}.
$$

The last thing we need to verify is that $S$ exists and is in $\mathcal{RH}_\infty$. By Lemma 1, we know

$$
S_{yx} = C \Phi_{xx} + D \Phi_{ux} \in \mathcal{RH}_\infty,
S_{yy} = C \Phi_{xy} + D \Phi_{uy} + I \in \mathcal{RH}_\infty.
$$

and we can compute the rest by (4)

$$
\begin{bmatrix}
S_{uu} \\
S_{yu} \\
S_{uy} \\
\end{bmatrix} = \begin{bmatrix}
O \\
I \\
O \\
\end{bmatrix} + \begin{bmatrix}
\Phi_{ux} \\
\Phi_{yx} \\
\Phi_{uy} \\
\end{bmatrix} B + \begin{bmatrix}
\Phi_{yy} \\
\Phi_{xy} \\
\Phi_{yu} \\
\end{bmatrix} D \in \mathcal{RH}_\infty,
$$

which concludes the proof. $\square$

### D. Mixed Parameterizations

Letting $G = C(zI - A)^{-1}B + D$, [12, Proposition 3, Proposition 4] provides the following corollaries that have conditions in both SLP and IOP flavors.

**Corollary 6.** For the realization in Fig. 6, the set of all proper internally stabilizing output-feedback controller is parameterized by $\{\Phi_{yx}, \Phi_{ux}, \Phi_{yy}, \Phi_{uy}\}$ that lies in the affine space defined by

$$
\begin{bmatrix}
I & -G \\
\Phi_{yx} & \Phi_{xy} \\
\Phi_{ux} & \Phi_{uy} \\
\end{bmatrix} C(zI - A)^{-1} I = 0,
\Phi_{yx}, \Phi_{ux}, \Phi_{yy}, \Phi_{uy} \in \mathcal{RH}_\infty,
$$

and the controller is given by

$$
K = \Phi_{uy} \Phi_{yy}^{-1}.
$$

**Corollary 7.** For the realization in Fig. 6, the set of all proper internally stabilizing output-feedback controller is parameterized by $\{\Phi_{yx}, \Phi_{uy}, \Phi_{ux}, \Phi_{uu}\}$ that lies in the affine space defined by

$$
\begin{bmatrix}
zI - A & -B \\
\Phi_{yx} & \Phi_{xy} \\
\Phi_{ux} & \Phi_{uy} \\
\end{bmatrix} = 0,
\Phi_{yx}, \Phi_{uy}, \Phi_{ux}, \Phi_{uu} \in \mathcal{RH}_\infty,
$$

**Proof.** Lemma 1 gives

$$
I = (I - R)S = \begin{bmatrix}
I & O & O \\
O & I & O \\
C(zI - A)^{-1} & I \\
\end{bmatrix},
\Gamma_1 = \begin{bmatrix}
I & O & O \\
O & I & O \\
0 & 0 & 1 \\
\end{bmatrix},
\Gamma_2 = \begin{bmatrix}
I & (zI - A)^{-1} & 0 \\
0 & I & 0 \\
0 & 0 & 1 \\
\end{bmatrix}.
$$

Analogous to the proof of Corollary 5, Corollary 6 can be derived from the following conditions and condition (3).

$$
\Gamma_1 (I - R) S = \Gamma_1, \quad S (I - R) I = I.
$$

Similarly, we derive Corollary 7 from condition (3) and

$$
(I - R) S = I, \quad S (I - R) \Gamma_2 = \Gamma_2.
$$

$\square$

### E. Generalized System Level Synthesis

Fig. 4 consists of the state/measurement $x/y$ and the control $u$. [27] generalizes the setting to study how a controller could also stabilize some output $z$ under an additional external disturbance $w$ as shown in Fig. 7. The following corollary is a slight rewrite of the proposed generalized system level synthesis in [27, Theorem 2].

**Corollary 8.** For the realization in Fig. 7 with $G \in \mathcal{R}_{sp}$, the set of all proper internally stabilizing controller is parameterized by $\Psi$ where

$$
\Psi = \begin{bmatrix}
\Psi_{yy} & \Psi_{yu} & \Psi_{yw} \\
\Psi_{uy} & \Psi_{uu} & \Psi_{uw} \\
\Psi_{zy} & \Psi_{zu} & \Psi_{zw} \\
\end{bmatrix} \in \mathcal{RH}_\infty
$$

$^3$Although bearing the name “generalized system level synthesis,” Corollary 8 actually generalizes input-output parameterization (Corollary 2).
and $\Psi$ lies in the affine space defined by
\[
\begin{bmatrix}
\Psi \\
\Psi \\
\Psi
\end{bmatrix}
\begin{bmatrix}
G & P_{yw} \\
I & O \\
O & I \\
P_{zu} & P_{zw}
\end{bmatrix}
= 
\begin{bmatrix}
0 & 0 \\
I & 0 \\
O & 0
\end{bmatrix},
\]
and the controller is given by $u = -Ky$ where
\[
K = -\Psi_{uy}^{-1}\Psi_{yy}.
\]

We prove Corollary 8 using Lemma 1 as follows.

Proof. The realization matrix in Fig. 7 is
\[
R = 
\begin{bmatrix}
O & -G & O & -P_{yw} \\
-K & O & O & O \\
O & -P_{zu} & O & -P_{zw} \\
O & O & O & O
\end{bmatrix},
\end{bmatrix}
\]
\[
\eta = 
\begin{bmatrix}
y \\
u \\
z \\
w
\end{bmatrix}.
\]

Lemma 1 implies
\[
I = (I - R)S
\]
\[
= 
\begin{bmatrix}
I & G & O & P_{yw} \\
K & I & O & O \\
O & P_{zu} & I & P_{zw} \\
O & O & O & O
\end{bmatrix},
\end{bmatrix}
\]
\[
= 
\begin{bmatrix}
\Psi_{yy} & \Psi_{yu} & S_{yz} & \Psi_{yw} \\
\Psi_{uy} & \Psi_{uu} & S_{uz} & \Psi_{uw} \\
\Psi_{zy} & \Psi_{zu} & S_{zs} & \Psi_{zw} \\
\Psi_{sy} & \Psi_{sw} & S_{sw} \\
O & O & O & O
\end{bmatrix}.
\]

By the 3rd column and the 4th row of $I - R$, we know
\[
S =
\begin{bmatrix}
\Psi_{yy} & \Psi_{yu} & O & \Psi_{yw} \\
\Psi_{uy} & \Psi_{uu} & O & \Psi_{uw} \\
\Psi_{zy} & \Psi_{zu} & I & \Psi_{zw} \\
O & O & O & I
\end{bmatrix}.
\]

Accordingly, we can take $S$ except for its last row multiplied by the 2nd and the 4th columns of $I - R$ to derive
\[
\begin{bmatrix}
\Psi \\
\Psi \\
\Psi
\end{bmatrix}
\begin{bmatrix}
G & P_{yw} \\
I & O \\
O & I \\
P_{zu} & P_{zw}
\end{bmatrix}
= 
\begin{bmatrix}
0 & 0 \\
I & 0 \\
O & 0
\end{bmatrix},
\]
after rearranging the columns of $S$. Similarly, we can derive
\[
\begin{bmatrix}
I & G & O & P_{yw} \\
O & P_{zu} & I & P_{zw} \\
O & O & O & I
\end{bmatrix}
= 
\begin{bmatrix}
I & O & O \\
O & 0 & 0
\end{bmatrix}
\]
from multiplying the 1st and 3rd rows of $I - R$ by $S$ except for the 3rd column.

Lastly, we can derive $K$ from the 2nd row of $I - R$ multiplied by the 1st column of $S$, which yields
\[
K\Psi_{yy} + \Psi_{uy} = 0.
\]

Below we show that $\Psi_{yy}$ is invertible in $R_p$, which allows us to derive $K$ as desired. Similar to the proof of Corollary 2, since $G \in R_{sp}$ and
\[
\Psi_{yy} + G\Psi_{uy} = I,
\]
we can write
\[
\Psi_{yy} = I + (zI - \Lambda)^{-1}J
\]
for some matrix $\Lambda$ and $J \in R_p$. As a result, $\Psi_{yy}$ is invertible in $R_p$ as
\[
\Psi_{yy}^{-1} = I + \sum_{k=1}^{\infty} (zI - \Lambda)^{-k}J^k \in R_p.
\]

Notice that if $\Psi_{uu}$ is invertible, we can also alternatively derive $K = -\Psi_{uu}^{-1}\Psi_{uy}$ from multiplying the 2nd row of $S$ by the 1st column of $I - R$.

F. Equivalence among Synthesis Methods

The parameterizations above are shown equivalent in [11] through careful calculations. Here we demonstrate how Lemma 1 and transformations lead to more straightforward derivations of equivalent components.

Lemma 1 implies that $R \rightarrow S$ is a one-to-one mapping. Therefore, to show the equivalence among different synthesis methods, we can simply find a transformation $T$ such that the equivalent system has the same realization as the other system. As such, Lemma 1 suggests that the stability matrices are the same, and we just need to compare the elements correspondingly.

When comparing a state-feedback system with an output-feedback system in the following analyses, we assume that the state $x$ is taken as the measurement $y$.

Youla parameterization and IOP: Youla parameterization and IOP share the same realization in Fig. 4 (except for changing $x$ to $y$). Therefore,
\[
\begin{bmatrix}
Y \\
U \\
Z
\end{bmatrix} = 
\begin{bmatrix}
U_r & N_r \\
V_r & M_r \\
0 & 1 - Q N_r
\end{bmatrix},
\begin{bmatrix}
I \\
0
\end{bmatrix},
\begin{bmatrix}
M_l & 0 \\
U_l & -QN_l
\end{bmatrix}
\]
\[
= 
\begin{bmatrix}
U_r - N_r Q M_l \\
V_r - M_l Q N_l
\end{bmatrix},
\begin{bmatrix}
M_l \\
I + (V_r - M_l Q) N_l
\end{bmatrix}.
\]

IOP and SLP: We then show the equivalence between IOP and SLP. For state-feedback SLP with realization in Fig. 5, we perform the transformation
\[
T^{-1} = 
\begin{bmatrix}
(zI - A)^{-1} & O \\
O & I
\end{bmatrix},
\]
which leads to
\[
T^{-1} 
\begin{bmatrix}
zI - A & -B \\
-K & I
\end{bmatrix} = 
\begin{bmatrix}
I & -G \\
-K & I
\end{bmatrix}.
\]

Accordingly, the stability matrix becomes
\[
\begin{bmatrix}
Y \\
U \\
Z
\end{bmatrix} = 
\begin{bmatrix}
\Phi_x & \Phi_x B \\
\Phi_u & I + \Phi_u B
\end{bmatrix} T
\]
\[
= 
\begin{bmatrix}
\Phi_x (zI - A) & \Phi_x B (zI - A) \\
\Phi_u & I + \Phi_u B
\end{bmatrix}.
\]

For output-feedback SLP, we consider the transformation
\[
T^{-1} = 
\begin{bmatrix}
I & O & O \\
O & I & 0 \\
\frac{C(zI - A)^{-1}}{O I}
\end{bmatrix},
\]
\[
T = 
\begin{bmatrix}
I \\
O \\
\frac{-C(zI - A)^{-1}}{O I}
\end{bmatrix}.
\]
which leads to
\[
T^{-1}(I - R) = T^{-1} \begin{bmatrix}
  zI - A & -B & 0 \\
  O & I & -K \\
-\Phi & D & -G & I
\end{bmatrix}
\]
\[
= \begin{bmatrix}
  zI - A & -B & 0 \\
  O & I & -K \\
-\Phi & 0 & -G & I
\end{bmatrix}, \quad \eta = \begin{bmatrix}
x \\
u \\
\eta
\end{bmatrix}.
\]
And the transformed stability matrix is
\[
ST = \begin{bmatrix}
\Phi_{xx} & S_{xx} & \Phi_{xy} \\
\Phi_{ux} & S_{uu} & \Phi_{uy} \\
S_{yx} & S_{yu} & S_{yy}
\end{bmatrix} T
\]
\[
= \begin{bmatrix}
\Phi_{xx} - \Phi_{xy} C(zI - A)^{-1} S_{xx} & \Phi_{xy} \\
\Phi_{ux} - \Phi_{uy} C(zI - A)^{-1} S_{uu} & \Phi_{uy} \\
S_{yx} - S_{yy} C(zI - A)^{-1} S_{yu} & S_{yy}
\end{bmatrix}.
\]
Comparing the corresponding elements and we have
\[
\begin{bmatrix}
Y \\
W \\
U \\
Z
\end{bmatrix} = \begin{bmatrix}
S_{yy} & S_{yu} \\
\Phi_{uy} & S_{uu}
\end{bmatrix}
\begin{bmatrix}
C\Phi_{xy} + D\Phi_{uy} + I \\
\Phi_{uy} \Phi_{ux} B + \Phi_{uy} D + I
\end{bmatrix} S_{yu}
\]
where
\[
S_{yu} = (C\Phi_{xx} + D\Phi_{ux})B + (C\Phi_{xy} + D\Phi_{uy} + I)D.
\]
Our result extends the \( D = O \) case in [11] to general \( D \).

**SLP and mixed parameterizations:** SLP and mixed parameterizations share the same realization Fig. 6. Therefore, they also share the same stability matrix according to Lemma 1, i.e., \( \Phi_{ux}, \Phi_{uu}, \Phi_{yx}, \) and \( \Phi_{yy} \) can be found in (13) and (14).

### V. Corollaries: Realizations

The same parameterization could admit multiple different realizations\(^4\). In this section, we consider the original state-feedback SLS realization and two alternative realization proposals for SLS. We show that the realizations can be derived from Lemma 1 through transformations.

#### A. Original State-Feedback SLS Realization

SLP parameterizes all internally stabilizing controller \( K \) for the state-feedback system in Fig. 5. Using the resulting \( \{\Phi_x, \Phi_u\} \), SLS proposes to implement the controller as in Fig. 8.

In other words, given \( R \) as in (10) and \( S \) satisfying Lemma 1, we can realize the closed-loop system by
\[
R_r = \begin{bmatrix}
A + (1 - z)I & B & 0 \\
O & O & z\Phi_u \\
O & I & I - z\Phi_x
\end{bmatrix}, \quad \eta = \begin{bmatrix}
x \\
u \\
\delta
\end{bmatrix}.
\]
To show that, we augment (10) with a dummy node
\[
I = (I - R_{aug}) S_{aug}
\]
\[
= \begin{bmatrix}
zI - A & -B & 0 \\
-K & I & O \\
O & O & I
\end{bmatrix} \begin{bmatrix}
\Phi_x & S_{xx} & \Phi_{xx}K \\
\Phi_u & S_{uu} & \Phi_{uu}K \\
z^{-1} & z^{-1} B & I - z^{-1}A
\end{bmatrix} \in \mathcal{RH}_\infty.
\]

#### B. Simpler Realization for Deployment

The original SLS realization in Fig. 8 needs to perform two convolutions \( I - z\Phi_x \) and \( z\Phi_u \), which are expensive to implement in practice. Therefore, [20] proposes a new realization in Fig. 9 that replaces one convolution by two matrix multiplications through the following corollary [20, Theorem 1].

**Corollary 9.** Let \( A \) be Schur stable, the dynamic state-feedback controller \( u = Kx \) realized via
\[
\delta[t] = x[t] - Ax[t - 1] - Bu[t - 1],
\]
\[
u[t] = \sum_{\tau \geq 1} \Phi_u[\tau] \delta[t + 1 - \tau]
\]
is internally stabilizing.

Proof. We first write the controller realization in frequency domain:

\[ \delta = (I - z^{-1}A)x - Bu, \]
\[ u = z\Phi_u\delta. \]

Together with the system, the realization is shown in Fig. 9.
Essentially, the corollary says that given \( R \) as in (10) and \( S \) satisfying Lemma 1, we can realize the closed-loop system by

\[
R_r = \begin{bmatrix} A + (1 - z)I & B & O \\ O & O & z\Phi_u \\ z^{-1}(I - A) & -z^{-1}B & O \end{bmatrix}, \quad \eta_r = \begin{bmatrix} x \\ u \\ \delta \end{bmatrix}.
\]

Again, we consider the augmented system in (15) and transform it to achieve the desired realization by

\[
T^{-1} = \begin{bmatrix} I & O & O \\ S_{uu} & I & z\Phi_u \\ z^{-1} & O & I \end{bmatrix}, \quad \begin{bmatrix} x \\ u \\ \delta \end{bmatrix} = \begin{bmatrix} T^{-1} & 0 & 0 \\ \Phi_u & 0 & -z\Phi_u \\ z^{-1} & 0 & I \end{bmatrix} \begin{bmatrix} x \\ u \\ \delta \end{bmatrix}.
\]

As such, \( (I - R_r) = T^{-1}(I - R_{aug}) \) and the resulting stability matrix is

\[
S_r = S_{aug}T = \begin{bmatrix} \Phi_x & S_{ux} & S_{uu}^{-1}zS_{uu}S_{uu}^{-1}\Phi_u \\ \Phi_u & I & z\Phi_u \\ z^{-1} & O & I \end{bmatrix}.
\]

Since \( S_{uu} = I + \Phi_uB \) is invertible, \( (zI - A)S_{uu} = BS_{uu}, \) and \( A \) is Schur stable, we have

\[
S_{uu}^{-1} = (zI - A)^{-1}B \in \mathcal{RH}_\infty,
\]

and hence the stability matrix is in \( \mathcal{RH}_\infty. \)

In [20], the authors substitute \( u \) into \( \delta \) before analyzing the internal stability, which is simply another (linear) transformation of \( d \) and the resulting stability matrix is still internally stable.

C. Closed-Loop Design Separation

Instead of directly adopting the realization in Fig. 8, [22] found that it is possible to use much simpler transfer matrices to realize the same controller. The following corollary is from [22, Theorem 2].

**Corollary 10.** For the causal realization in Fig. 10 and a given \( \{\Phi_x, \Phi_u\} \) that satisfies Corollary 3, \( \{P_c, M_c\} \) realizes \( \{\Phi_x, \Phi_u\} \) if and only if they satisfy

\[
\begin{bmatrix} \Phi_x \\ \Phi_u \end{bmatrix} \begin{bmatrix} zI - A & -B \\ I \end{bmatrix} = \begin{bmatrix} P_c \\ M_c \end{bmatrix}.
\]

(16)

Proof. The corollary says that for the realization

\[
R = \begin{bmatrix} A + (1 - z)I & B & O \\ O & O & zM_c \\ I & O & I - zP_c \end{bmatrix}, \quad \eta = \begin{bmatrix} x \\ u \\ \delta \end{bmatrix}
\]

and the given \( \{\Phi_x, \Phi_u\} \), there exists a solution

\[
S = \begin{bmatrix} \Phi_x & S_{ux} & S_{x\delta} \\ \Phi_u & S_{uu} & S_{u\delta} \\ S_{\delta x} & S_{\delta u} & S_{\delta\delta} \end{bmatrix}
\]

satisfying Lemma 1 if and only if (16) holds.
If such an \( S \) exists, Lemma 1 suggests

\[
S(I - R) = I,
\]

and we have

\[
(I - R) \begin{bmatrix} zP_c \\ zM_c \\ I \end{bmatrix} = \begin{bmatrix} zI - A & -B & O \\ O & O & O \\ I \end{bmatrix} \begin{bmatrix} zP_c \\ zM_c \\ I \end{bmatrix} = \begin{bmatrix} zP_c \\ zM_c \\ I \end{bmatrix}.
\]

Therefore,

\[
S(I - R) = \begin{bmatrix} zP_c \\ zM_c \\ I \end{bmatrix} = \begin{bmatrix} zP_c \\ zM_c \\ I \end{bmatrix}.
\]

and (16) follows from dividing both sides by \( z \).

On the other hand, when (16) holds and \( \{\Phi_x, \Phi_u\} \) satisfies Corollary 3, the stability matrix can be derived from \( S(I - R) = I \) as

\[
S = \begin{bmatrix} \Phi_x & \Phi_xB & \Phi_x(zI - A) - I \\ \Phi_u & I + \Phi_uB & \Phi_u(zI - A) \\ S_{\delta x} & S_{\delta u} & S_{\delta\delta} \end{bmatrix}
\]

where, by (17),

\[
S_{\delta x} = z^{-1}\Delta_c^{-1} = z^{-1} \left[ \begin{bmatrix} zI - A & -B \end{bmatrix} \begin{bmatrix} P_c \\ M_c \end{bmatrix} \right]^{-1}.
\]

\( S \) exists if \( S_{\delta x} \) exists. In other words, we have to show that \( \Delta_c \) is invertible. Since the system is causal, \( I - zP_c \) and \( zM_c \) are both in \( \mathcal{R}_p \). Therefore,

\[
\Delta_c = zP_c - (AP_c + BM_c) = I - z^{-1}J
\]

where \( J \in \mathcal{R}_p \), and hence

\[
\Delta_c^{-1} = I + \sum_{k=1}^{\infty} z^{-k}J^k \in \mathcal{R}_p,
\]

which concludes the proof. \( \square \)

---

\( ^5 \)To avoid the confusion with the realization matrix \( R \), we write \( P_c \) here instead.
We remark that Corollary 10 does not guarantee that $S \in \mathcal{RH}_\infty$, and hence the authors in [22] propose to perform a posteriori stability check. According to the proof of Corollary 10, we can easily guarantee $S \in \mathcal{RH}_\infty$ by requiring $S_{66} = S_{65}(zI - A) \in \mathcal{RH}_\infty$. This is one benefit resulting from the analysis using Lemma 1 and condition (3).

VI. COROLLARIES: EXISTING ROBUST RESULTS

We unite the existing robust results under Lemma 1 and Lemma 3, including $\mu$-synthesis [13]–[15], primal-dual Youla parameterization [16], input-output parameterization [17], and system level synthesis [6], [18], [19]. In the following derivations, we use hatted symbols to denote nominal signals/matrices.

A. $\mu$-Synthesis

$\mu$-synthesis studies the quantity $\det(I - M\Delta)$ [13]–[15], where the matrix $M \in \mathcal{RH}_\infty$ maps some external disturbance $w$ to some output $z$. The $\mu$ function, or the structured singular value, is the inverse size of the “smallest” structured perturbation that destabilizes $(I - M\Delta)^{-1}$. Further, the original papers claim that it is always possible to choose $M$ so that $\Delta$ is block-diagonal. As to why $\det(I - M\Delta)$ is critical and why $\Delta$ could be formed block-diagonal, Lemma 3 provides intuitive answers.

Suppose the realization is subject to additive perturbation $\Delta$. We first show that $M$ is a wrapped nominal stability matrix $\hat{S}$. Since $\eta$ contains all the internal signals, the output $z$ can be expressed as a linear map $F_z$ of $\eta$ and the external input $w$. Meanwhile, $w$ can be mapped to the external disturbance $d$ through a transformation $T$. Therefore, the matrix $M$ can be expressed as in Fig. 11. Here the $\hat{S}(\Delta)$ is expanded by Lemma 3 as in Fig. 3.

Since $\Delta$ is an additive perturbation on the realization, we can create dummy signals to the realization so that each signal has at most one perturbed input/output. Equivalently, $\Delta$ is structured such that there is at most one perturbation block at each row/column. Hence, $\Delta$ can be permuted to take a block-diagonal form.

On the other hand, when $w = d$ and $z = \eta$, we have $M = \hat{S} \in \mathcal{RH}_\infty$. By Lemma 3, the perturbed stability is

$$S(\Delta) = \hat{S}(I - \hat{S}\Delta)^{-1} = M(I - M\Delta)^{-1},$$

which is not stable if $I - M\Delta$ is not invertible, or $\det(I - M\Delta) = 0$.

B. Robust Primal-Dual Youla Parameterization

[28, Chapter 3, Theorem 4.2] generalizes the primal Youla parameterization $Q$ in Section IV-A to also parameterize the stabilizable plants $G$ using dual Youla parameter $P$.

Here we present a modern rewrite of [28, Chapter 3, Theorem 4.2]:

**Corollary 11.** For the realization in Fig. 4 and the doubly coprime factorization (8), let $G$ and $K$ be parameterized by

$$G = (N_r - U_rP)(M_r - V_rP)^{-1}$$

$$K = (V_r - M_rQ)(U_r - N_rQ)^{-1}$$

where $P, Q \in \mathcal{RH}_\infty$ are the dual and primal Youla parameters. The system is internally stable if and only if the following $(P, Q)$ realization is internally stable:

$$\begin{pmatrix} -P & \beta \\ P & -Q \end{pmatrix}$$

**Proof.** Similar to the approach in [23], we transform the coprime factorization (8) by

$$T^{-1} = \begin{bmatrix} (M_l - PV_l)^{-1} & O \\ O & (U_l - QN_l)^{-1} \end{bmatrix} \begin{bmatrix} I & P \\ Q & I \end{bmatrix}$$

$$T = \begin{bmatrix} I & P \\ Q & I \end{bmatrix}^{-1} \begin{bmatrix} M_l - PV_l & O \\ O & U_l - QN_l \end{bmatrix}$$

where

$$\begin{bmatrix} I & P \\ Q & I \end{bmatrix}^{-1} = \begin{bmatrix} I & -P \\ -Q & I \end{bmatrix} (I - PQ)^{-1} \begin{bmatrix} O & O \\ 0 & (I - QP)^{-1} \end{bmatrix}.$$

The transformation leads to

$$T^{-1} = \begin{bmatrix} M_l & -N_l \\ -V_l & U_l \end{bmatrix} \begin{bmatrix} U_r & N_r \\ V_r & M_r \end{bmatrix} T$$

$$= \begin{bmatrix} I & -G \\ -K & I \end{bmatrix} \begin{bmatrix} U_r & N_r \\ V_r & M_r \end{bmatrix} T = I.$$

By Lemma 1, the stability matrix of the system is

$$\begin{bmatrix} U_r & N_r \\ V_r & M_r \end{bmatrix} T$$

which should be in $\mathcal{RH}_\infty$ to ensure internal stability. By assumption, the coprime factorization matrix, $P$, and $Q$ are all in $\mathcal{RH}_\infty$. We then only need to ensure $T \in \mathcal{RH}_\infty$, which requires

$$\begin{bmatrix} I & P \\ Q & I \end{bmatrix}^{-1} \in \mathcal{RH}_\infty.$$

Treating the matrix inverse as a stability matrix, its corresponding realization is exactly the $(P, Q)$ realization. Consequently, the whole system is internally stable if and only if the $(P, Q)$ realization is internally stable, which concludes the proof. \(\square\)

\(\text{To avoid the confusion with the stability matrix } S, \text{ we express the dual Youla parameter by } P \text{ instead.}\)
Leveraging the dual Youla parameterization, [16] proposes to embed the perturbation $\Delta$ on the plant $G$ into $P$ as such
\[
G(\Delta) = (N_r - U_r P(\Delta))(M_r - V_r P(\Delta))^{-1}. \tag{18}
\]
[16] then suggests following result.

**Corollary 12.** Consider a perturbed Youla parameterization as in Corollary 11 with the plant (18). The system is internally stable if
\[
(I - QP(\Delta))^{-1} \in RH_{\infty}. \tag{19}
\]

**Proof.** As shown in the proof of Corollary 11, the system is internally stable if and only if
\[
\begin{bmatrix}
I & P(\Delta) \\
Q & I
\end{bmatrix}^{-1} = \begin{bmatrix}
I & -P(\Delta)
\end{bmatrix} \begin{bmatrix}
(I - P(\Delta)Q)^{-1} & 0 \\
0 & (I - QP(\Delta))^{-1}
\end{bmatrix}
\]
is in $RH_{\infty}$. Since $P(\Delta), Q \in RH_{\infty}$ and
\[
(I - P(\Delta)Q)^{-1} = I + P(I - QP(\Delta))^{-1} Q,
\]
we know $(I - QP(\Delta))^{-1} \in RH_{\infty}$ implies $(I - P(\Delta)Q)^{-1} \in RH_{\infty}$, which concludes the proof. \hfill \Box

**C. Robust Input-Output Parameterization (IOP)**

[17, Theorem C.2] presents the following robust result for IOP in Section IV-B.

**Corollary 13.** For the realization in Fig. 4, let $K_r$ be the set of robustly stabilizing controllers defined by
\[
K_r = \{ K : K \text{ internally stabilizes } G(\Delta_G) \forall \Delta_G \in D_r \}
\]
where $G(\Delta_G) = G + \Delta_G$ and $D_r = \{ \Delta_G : \|\Delta_G\|_{\infty} < \epsilon \}$. Then $K_r$ is parameterized by $\{ Y, W, U, Z \}$ that satisfies Corollary 2 and
\[
\begin{bmatrix}
\hat{U} \\
\hat{Y}
\end{bmatrix}_I \leq \epsilon^{-1},
\]
and the controller is given by $\hat{K} = \hat{U}Y^{-1}$

**Proof.** Let
\[
\hat{S} = \begin{bmatrix}
\hat{Y} & \hat{W} \\
\hat{U} & \hat{Z}
\end{bmatrix}, \quad \Delta = \begin{bmatrix}
\Delta_G & 0 \\
0 & \Delta_G
\end{bmatrix}.
\]

Lemma 3 implies that the system is internally stable if and only if
\[
S(\Delta) = \hat{S}(I - \Delta \hat{S})^{-1} \in RH_{\infty}
\]
for all $\Delta \in D_r$. Since $O \in D_r$, we need $\hat{S} \in RH_{\infty}$ (which leads to the first three conditions, see [23]) and
\[
(I - \Delta \hat{S})^{-1} = \begin{bmatrix}
I - \Delta_G \hat{U} & -\Delta_G \hat{Z} \\
0 & I
\end{bmatrix}^{-1} = \begin{bmatrix}
(I - \Delta_G \hat{U})^{-1} & (I - \Delta_G \hat{U})^{-1} \Delta_G \hat{Z} \\
0 & I
\end{bmatrix} \in RH_{\infty}.
\]
Since $\Delta_G$ has a bounded $H_{\infty}$ norm, we know $\Delta_G \in RH_{\infty}$. By the small gain theorem, $(I - \Delta_G \hat{U})^{-1} \in RH_{\infty}$ for all $\Delta_G \in D_r$ if and only if $\|\hat{U}\|_{\infty} \leq \epsilon^{-1}$, which concludes the proof. \hfill \Box

**D. Robust System Level Synthesis (SLS)**

In addition to the classic SLS/SLP as in Section IV-C, Recent studies investigate robust controller synthesis problem with perturbed SLP under both state-feedback and output-feedback settings. We show below how to derive those results using Lemma 3.

**State-Feedback:** The following robust result from [18, Theorem 2] and [6, Theorem 4.3] examines the perturbed state-feedback SLP.

**Corollary 14.** For the system with realization as in Fig. 5, let $(\hat{\Phi}_x, \hat{\Phi}_u, \Delta_{SLS})$ be a solution to
\[
[zI - A - B] \begin{bmatrix}
\hat{\Phi}_x \\
\hat{\Phi}_u
\end{bmatrix} = I + \Delta_{SLS}, \quad \hat{\Phi}_x, \hat{\Phi}_u \in z^{-1}RH_{\infty}.
\]

Then, the controller realization
\[
\hat{\delta}_x = x - \hat{x}, \quad u = \hat{z} x \hat{\delta}_x, \quad \hat{x} = (z \hat{z} - I) \hat{\delta}_x
\]
internally stabilizes the system if and only if $(I + \Delta_{SLS})^{-1}$ is stable. Furthermore, the actual system responses achieved are given by
\[
\begin{bmatrix}
x \\
u
\end{bmatrix} = \begin{bmatrix}
\hat{\Phi}_x \\
\hat{\Phi}_u
\end{bmatrix} (I + \Delta_{SLS})^{-1} d_x.
\]

**Proof.** By definition, $(\hat{\Phi}_x, \hat{\Phi}_u, \Delta_{SLS})$ is also a solution to
\[
[zI - A - B] \begin{bmatrix}
\hat{\Phi}_x \\
\hat{\Phi}_u
\end{bmatrix} = I + \Delta_{SLS} \begin{bmatrix}
O \\
I
\end{bmatrix}.
\]

Therefore, by Lemma 1, the stability matrix for the realization in Fig. 5 is
\[
\begin{bmatrix}
\hat{\Phi}_x \\
\hat{\Phi}_u
\end{bmatrix} \begin{bmatrix}
S_{xu} \\
S_{u u}
\end{bmatrix} = \begin{bmatrix}
[I + \Delta_{SLS}]^{-1} \\
[\hat{\Phi}_u (I + \Delta_{SLS})^{-1}]
\end{bmatrix},
\]

which derives the system response. As shown in [23], the nominal controller $K$ is internally stabilizing if and only if
\[
\hat{\Phi}_x (I + \Delta_{SLS})^{-1}, \hat{\Phi}_u (I + \Delta_{SLS})^{-1} \in z^{-1}RH_{\infty}.
\]

In other words, since $\hat{\Phi}_x, \hat{\Phi}_u \in z^{-1}RH_{\infty}$, $K$ is internally stabilizing if and only if $(I + \Delta_{SLS})^{-1} \in RH_{\infty}$. The internal stability of the proposed controller can then be showed by transforming the realization in Fig. 5 to the one in Fig. 8 as done in [23]. \hfill \Box

**Output-Feedback:** For output-feedback systems as in Fig. 6, the two corollaries below from [19, Lemma 4.3, Corollary 4.4] concern the controller resulting from perturbed affine space in Corollary 4.

**Corollary 15.** For the system $G$ with realization as in Fig. 6 with $D = O$, let $(\hat{\Phi}_{xx}, \hat{\Phi}_{ux}, \hat{\Phi}_{xy}, \hat{\Phi}_{uy})$ satisfy (11a), (11b), and
\[
\begin{bmatrix}
\hat{\Phi}_{xx} & \hat{\Phi}_{xy} \\
\hat{\Phi}_{ux} & \hat{\Phi}_{uy}
\end{bmatrix} \begin{bmatrix}
zI - A \\
-C
\end{bmatrix} = \begin{bmatrix}
I + \Delta_{SLS} \\
\Delta_{SLS}
\end{bmatrix} \tag{20}
\]


and let the system response be given by

\[
\begin{bmatrix}
\Phi_{xx}(\Delta_{SLS}) \\
\Phi_{ux}(\Delta_{SLS}) \\
\Phi_{uy}(\Delta_{SLS})
\end{bmatrix}
\begin{bmatrix}
\Phi_{xx} \\
\Phi_{ux} \\
\Phi_{uy}
\end{bmatrix}
\]

\[
= \begin{bmatrix}
(I + \Delta_{1}^{SLS})^{-1} O \\
-\Delta_{2}^{SLS}(I + \Delta_{1}^{SLS})^{-1} I \\
\end{bmatrix}
\begin{bmatrix}
\Phi_{xx} \\
\Phi_{ux} \\
\Phi_{uy}
\end{bmatrix}
\]

where by assumption \((I + \Delta_{1}^{SLS})^{-1}\) exists and is in \(\mathcal{R}\mathcal{H}_{\infty}\). Then \(\{\Phi_{xx}(\Delta_{SLS}), \Phi_{ux}(\Delta_{SLS}), \Phi_{xy}(\Delta_{SLS}), \Phi_{uy}(\Delta_{SLS})\}\) satisfies Corollary 4 for \(G\). Furthermore, suppose \(C\) is subject to an additive disturbance \(\Delta_{C}\), i.e.,

\[C(\Delta_{C}) = C + \Delta_{C},\]

and

\[\Delta_{1}^{SLS} = -\Phi_{xy}\Delta_{C}, \quad \Delta_{2}^{SLS} = -\hat{\Phi}_{xy}\Delta_{C} .\]

We then have \(\{\hat{\Phi}_{xx}, \hat{\Phi}_{ux}, \hat{\Phi}_{xy}, \hat{\Phi}_{uy}\}\) also satisfies Corollary 4 for the perturbed system.

**Corollary 16.** Suppose \(\{\hat{\Phi}_{xx}, \hat{\Phi}_{ux}, \hat{\Phi}_{xy}, \hat{\Phi}_{uy}\}\) satisfies the conditions in Corollary 15. Then, the controller

\[K = \hat{\Phi}_{uy} - \hat{\Phi}_{ux}\hat{\Phi}_{xx}^{-1}\hat{\Phi}_{xy}\]

stabilizes the system and achieves the closed-loop system response if and only if \((I + \Delta_{1}^{SLS})^{-1}\) exists in \(\mathcal{R}\mathcal{H}_{\infty}\).

We now prove these two corollaries by investigating \(S(\Delta)\).

**Proof.** By definition, \(\{\hat{\Phi}_{xx}, \hat{\Phi}_{ux}, \hat{\Phi}_{xy}, \hat{\Phi}_{uy}\}\) is also a solution to

\[
\begin{bmatrix}
\Phi_{xx} \\
\Phi_{ux} \\
\Phi_{uy}
\end{bmatrix}
\begin{bmatrix}
S_{xx} & S_{xy} & S_{yx} \\
S_{uu} & S_{uy} & S_{yu} \\
S_{yy}
\end{bmatrix}
\begin{bmatrix}
\Phi_{xx} \\
\Phi_{ux} \\
\Phi_{uy}
\end{bmatrix}
\]

\[
= \begin{bmatrix}
(I + \Delta_{SLS}^{SLS}) O \\
\Delta_{2}^{SLS} I O \\
O O I
\end{bmatrix}
\begin{bmatrix}
O O O \\
O O O \\
O O O
\end{bmatrix}
\]

where \(D = O\). Therefore, using the same transformation technique in [23], we can derive the nominal controller \(K\) in Corollary 16 from

\[
\begin{bmatrix}
\Phi_{xx} \\
\Phi_{ux} \\
\Phi_{uy}
\end{bmatrix}
\begin{bmatrix}
zI - A & -B \\
O & I - K \\
-C & -D & I
\end{bmatrix}
\]

\[
= \begin{bmatrix}
zI - A & -B \\
O & I - K^{-1} \Delta_{SLS}^{SLS} & O \\
-C & -D & I
\end{bmatrix}.
\]

Also, by Lemma 1, the stability matrix for the realization in Fig. 6 is

\[
\begin{bmatrix}
\Phi_{xx}(\Delta_{SLS}) & S_{xx}(\Delta_{SLS}) & \Phi_{xy}(\Delta_{SLS}) & S_{yx}(\Delta_{SLS}) \\
\Phi_{ux}(\Delta_{SLS}) & S_{uu}(\Delta_{SLS}) & \Phi_{uy}(\Delta_{SLS}) & S_{yu}(\Delta_{SLS}) \\
\Phi_{xy}(\Delta_{SLS}) & S_{yx}(\Delta_{SLS}) & \Phi_{uy}(\Delta_{SLS}) & S_{yy}(\Delta_{SLS})
\end{bmatrix}
\]

\[
= \begin{bmatrix}
(I + \Delta_{SLS}^{SLS}) O & O & O \\
\Delta_{2}^{SLS} I & O & O \\
O & O & I
\end{bmatrix}
\begin{bmatrix}
\Phi_{xx} & S_{xx} & \Phi_{xy} \\
\Phi_{ux} & S_{uu} & \Phi_{uy} \\
S_{yx} & S_{yu} & S_{yy}
\end{bmatrix}
\]

\[
= \begin{bmatrix}
(I + \Delta_{SLS}^{SLS})^{-1} O & O & O \\
\Delta_{2}^{SLS}(I + \Delta_{SLS}^{SLS})^{-1} I & O & O \\
O & O & I
\end{bmatrix}
\begin{bmatrix}
\Phi_{xx} & S_{xx} & \Phi_{xy} \\
\Phi_{ux} & S_{uu} & \Phi_{uy} \\
S_{yx} & S_{yu} & S_{yy}
\end{bmatrix},
\]

which derives the system response. Since

\[
[zI - A & -B] \begin{bmatrix}
\Phi_{xx} \\
\Phi_{ux} \\
\Phi_{uy}
\end{bmatrix} [zI - A & -C]
\]

\[
= [zI - A & -B] \begin{bmatrix}
I & \Delta_{SLS}^{SLS} \\
\Delta_{2}^{SLS} & I
\end{bmatrix}
\]

\[
= [I & 0] [zI - A & -C],
\]

we have \((zI - A)\Delta_{1}^{SLS} = I\Delta_{2}^{SLS}\). Therefore,

\[
[zI - A & -B] \begin{bmatrix}
\Phi_{xx}(\Delta_{SLS}) \\
\Phi_{ux}(\Delta_{SLS}) \\
\Phi_{xy}(\Delta_{SLS}) \\
\Phi_{uy}(\Delta_{SLS})
\end{bmatrix}
\]

\[
= [zI - A & -B] \begin{bmatrix}
I & \Delta_{SLS}^{SLS} \\
\Delta_{2}^{SLS} O & I
\end{bmatrix}^{-1} \begin{bmatrix}
\Phi_{xx} \\
\Phi_{ux} \\
\Phi_{xy} \\
\Phi_{uy}
\end{bmatrix}
\]

\[
= [zI - A & -B] \begin{bmatrix}
\Phi_{xx} \\
\Phi_{ux} \\
\Phi_{xy} \\
\Phi_{uy}
\end{bmatrix}
\]

Also, (20) and \(z\Phi_{ux}, \Phi_{uy} \in \mathcal{R}\mathcal{H}_{\infty}\) imply

\[\hat{\Phi}_{ux}(zI - A) - \hat{\Phi}_{uy}C = \Delta_{2}^{SLS},\]

and hence \(\Delta_{2}^{SLS} \in \mathcal{R}\mathcal{H}_{\infty}\). Along with \((I + \Delta_{SLS}^{SLS})^{-1} \in \mathcal{R}\mathcal{H}_{\infty}\), we know that \(\{\Phi_{xx}(\Delta_{SLS}), \Phi_{ux}(\Delta_{SLS}), \Phi_{xy}(\Delta_{SLS}), \Phi_{uy}(\Delta_{SLS})\}\) satisfies Corollary 4 through the same proof in [23].

When \(C\) is perturbed, we have \(R(\Delta) = \hat{R} + \Delta\) where

\[\Delta = \begin{bmatrix}
O & 0 & 0 \\
O & 0 & 0 \\
O & 0 & 0
\end{bmatrix}
\]

and the result follows from Lemma 3.

Since we have shown \(S(\Delta) \in \mathcal{R}\mathcal{H}_{\infty}\) if and only if \((I + \Delta_{SLS}^{SLS})^{-1}\) exists above, the nominal controller \(K\) stabilizes the system under the same condition, and Corollary 16 follows.

\[\square\]

**VII. Applications and Beyond**

In addition to unifying existing results, Lemma 1 and Lemma 3 also allow us to easily derive robust results by extending the nominal results. Below, we provide some new results to demonstrate the effectiveness of the lemmas. On the other hand, although the conditions derived from Lemma 1 and Lemma 3 in this work can unify a large portion of robust controller synthesis results in the literature, there are still results beyond its scope, which we briefly discuss at the end of this section.

**A. New Robust Results**

As examples, we derive the following robust results for output-feedback SLS that generalize Corollary 15 and a condition for robust IOP.

**Corollary 17.** For output-feedback systems as in Fig. 6 with arbitrary \(D\), let \(\{\Phi_{xx}, \Phi_{ux}, \Phi_{xy}, \Phi_{uy}\}\) satisfy Corollary 4. Consider an additive perturbation \(\Delta\) on the plant such that

\[A(\Delta) = A + \Delta A, \quad B(\Delta) = B + \Delta B, \quad C(\Delta) = C + \Delta C, \quad D(\Delta) = D + \Delta D.\]
where $\Delta_A, \Delta_B, \Delta_C, \Delta_D \in \mathcal{RH}_\infty$. Then the nominal controller

$$\hat{K} = \hat{K}_0 \left( I + D \hat{K}_0 \right)^{-1},$$

where $\hat{K}_0 = \hat{\Phi}_{uy} - \hat{\Phi}_{ux} \hat{\Phi}_{xx}^{-1} \hat{\Phi}_{xy}$, internally stabilizes the perturbed system if and only if $\Psi \in \mathcal{RH}_\infty$ where

$$\Psi = \left( I - \begin{bmatrix} \Delta_A & \Delta_B \\ \Delta_C & \Delta_D \end{bmatrix} \begin{bmatrix} \hat{\Phi}_{xx} & \hat{\Phi}_{xy} \\ \hat{\Phi}_{ux} & \hat{\Phi}_{uy} \end{bmatrix} \right)^{-1}. $$

**Proof.** Since $\{\hat{\Phi}_{xx}, \hat{\Phi}_{ux}, \hat{\Phi}_{xy}, \hat{\Phi}_{uy}\}$ satisfies Corollary 4, we have the nominal stability $\hat{S} \in \mathcal{RH}_\infty$ and the nominal controller $\hat{K}$ is given by [23, Corollary 5].

Given the realization in Fig. 6, the perturbation $\Delta$ can be expressed as

$$\Delta = \begin{bmatrix} \Delta_A & \Delta_B & O \\ O & O & O \\ \Delta_C & \Delta_D & O \end{bmatrix}.$$ 

Since $\hat{S} \in \mathcal{RH}_\infty$, Lemma 3 requires $(I - \Delta \hat{S})^{-1} \in \mathcal{RH}_\infty$, or the inverse of

$$I - \begin{bmatrix} \Delta_A & \Delta_B & O \\ O & O & O \\ \Delta_C & \Delta_D & O \end{bmatrix} = \begin{bmatrix} \hat{\Phi}_{xx} & \hat{\Phi}_{xy} \\ \hat{\Phi}_{ux} & \hat{\Phi}_{uy} \\ \hat{\Phi}_{ux} & \hat{\Phi}_{uy} \end{bmatrix},$$

should be in $\mathcal{RH}_\infty$. Computing the inverse of (21) is equivalent to computing

$$\begin{bmatrix} \Psi & -\Psi \begin{bmatrix} \Delta_A & \Delta_B \\ \Delta_C & \Delta_D \end{bmatrix} \\ O & I \end{bmatrix}. $$

We know $\Delta_A, \Delta_B, \Delta_C, \Delta_D, \hat{S}_{ux}, \hat{S}_{uu} \in \mathcal{RH}_\infty$. Therefore, $\hat{K}$ still internally stabilizes the perturbed plant if and only if $\Psi \in \mathcal{RH}_\infty$, which concludes the proof. $\square$

We can extend Corollary 17 to provide an SLS version of Corollary 13.

**Corollary 18.** For output-feedback systems $G(\Delta)$ as in Fig. 6 perturbed by structured $\Delta$ as in Corollary 17, let $K_\epsilon$ be the set of robustly stabilizing controllers defined by

$$K_\epsilon = \{ K : K \text{ internally stabilizes } G(\Delta), \forall \Delta \in \mathcal{D}_\epsilon \}$$

where

$$\mathcal{D}_\epsilon = \{ \Delta \text{ structured as in Corollary 17} : \| \Delta \|_\infty < \epsilon \}.$$ 

Then $K_\epsilon$ is parameterized by $\{ \hat{\Phi}_{xx}, \hat{\Phi}_{ux}, \hat{\Phi}_{xy}, \hat{\Phi}_{uy} \}$ that satisfies Corollary 4 and

$$\begin{bmatrix} \hat{\Phi}_{xx} & \hat{\Phi}_{xy} \\ \hat{\Phi}_{ux} & \hat{\Phi}_{uy} \end{bmatrix} \leq \epsilon^{-1}.$$ 

**Proof.** Similar to the proof of Corollary 13, we know $\Delta \in \mathcal{RH}_\infty$ as its norm is bounded. By the small gain theorem and Corollary 17, $\Psi \in \mathcal{RH}_\infty$ for all $\Delta \in \mathcal{D}_\epsilon$ if and only if

$$\begin{bmatrix} \hat{\Phi}_{xx} & \hat{\Phi}_{xy} \\ \hat{\Phi}_{ux} & \hat{\Phi}_{uy} \end{bmatrix} \leq \epsilon^{-1},$$

which concludes the proof. $\square$

Similar to Corollary 14, 16, and 17, we can derive the following condition for the nominal IOP controller that still stabilizes a perturbed plant using Lemma 3. The proof is mostly the same as the proof of Corollary 13 and omitted.

**Corollary 19.** For the realization in Fig. 4, let $\{\hat{Y}, \hat{W}, \hat{U}, \hat{Z}\}$ satisfy Corollary 2. Consider an additive perturbation $\Delta_G \in \mathcal{RH}_\infty$ on the plant such that $G(\Delta_G) = G + \Delta_G$. Then the nominal controller

$$\hat{K} = \hat{U} \hat{Y}^{-1}$$

internally stabilizes the perturbed system if and only if

$$\hat{I} - \Delta_G \hat{U}^{-1} \in \mathcal{RH}_\infty.$$ 

**B. Discussion**

There are still some robust results beyond the scope of the conditions from Lemma 1 and Lemma 3. In particular, the conditions rely on some other procedure to ensure the perturbed stability matrix is still in $\mathcal{RH}_\infty$ for the whole uncertainty set $\mathcal{D}$, or condition (5). Different $\mathcal{D}$ may invoke different theorems. For example, while the small gain theorem works for ball-like uncertainty set, there is a line of research on bounded perturbation of transfer function coefficients, which builds upon the Kharitonov’s Theorem [29]. Kharitonov’s Theorem suggests that robust stability over the whole uncertainty set $\mathcal{D}$ can be achieved by the stability of 4 elements within $\mathcal{D}$. It is leveraged by [30][31][32] to synthesize robust controllers and further generalized by [33], [34] for larger classes of coefficient-perturbed transfer functions.

In contrast to the realization-centric perspective adopted in this work, there are also alternative approaches for system analysis using the integral quadratic constraints [35] or interval analysis [36]. Those methods also derive robust results, while we argue that the realization-centric perspective is more straightforward and the proofs are much simpler.

There is still a plenty of robust results that could be derived or generalized from Lemma 1 and Lemma 3 but not included here, such as [37, Proposition 1], [38, Lemma 2], [39, Theorem III.5]. We encourage the reader to explore those diverse results and unify them under this paper’s approach.

**VIII. Conclusion**

In this paper, we derived the realization-stability lemma and the corresponding robust stability conditions. The realization-stability lemma is built upon our realization-centric abstraction. Unlike traditional approaches that differentiate the plant from the controller, realization abstraction examines the system as a whole, which leads to the concept of equivalent systems and provides a unified approach to synthesize and analyze controllers.

Via the realization-stability lemma, not only did we formulate the general controller synthesis problem and its robust version, but we also showed that all existing controller synthesis, realization, and some robust results are all special cases of the lemma merely concerning different realizations/system structures. In addition, we leveraged the lemmas to derive new
robust results and discussed some other robust results beyond our current analysis approach.

Through these case studies, we demonstrate a unified procedure/analysis approach based on the realization-stability lemma to perform controller synthesis and realization derivation. Meanwhile, unintentionally but perhaps usefully, the paper serves as a comprehensive survey of contemporary (robust) controller synthesis results.

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