Sharp optimal recovery in the Two Component Gaussian Mixture Model

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Abstract
In this paper, we study the problem of clustering in the Two component Gaussian mixture model when the centers are separated by some $\Delta > 0$. We present a non-asymptotic lower bound for the corresponding minimax Hamming risk improving on existing results. We also propose an optimal, efficient and adaptive procedure that is minimax rate optimal. The rate optimality is moreover sharp in the asymptotics when the sample size goes to infinity. Our procedure is based on a variant of Lloyd’s iterations initialized by a spectral method. As a consequence of non-asymptotic results, we find a sharp phase transition for the problem of exact recovery in the Gaussian mixture model. We prove that the phase transition occurs around the critical threshold $\bar{\Delta}$ given by

$$\bar{\Delta}^2 = \sigma^2 \left( 1 + \sqrt{1 + \frac{2p}{n \log n}} \right) \log n.$$ 

Keywords: Gaussian Mixture Model, unsupervised clustering, sharp phase transition, spectral methods, Lloyd’s algorithm, non-asymptotic minimax risk.

1. Introduction
The problems of supervised or unsupervised clustering have gained huge interest in the machine learning literature. In particular, many clustering algorithms are known to achieve good empirical results. A very useful model to study and compare these algorithms is the Gaussian mixture model. In this model, we assume that the data are attributed to different centers and that we only have access to observations corrupted by Gaussian noise. For this specific model, one can consider the problem of estimation of the centers, see, e.g., Klusowski and Brinda (2016), Mixon et al. (2016) or the problem of detecting the communities, see, e.g., Lu and Zhou (2016), Giraud and Verzelen (2018), Royer (2017). This paper focuses on community detection.

1.1 The Gaussian Mixture Model
We observe $n$ independent random vectors $Y_1, \ldots, Y_n \in \mathbb{R}^p$. We assume that there exist two unknown vectors $\theta \in \mathbb{R}^p$ and $\eta \in \{-1,1\}^n$, such that, for all $i = 1, \ldots, n$,

$$Y_i = \theta \eta_i + \sigma \xi_i,$$

where $\sigma > 0$, $\xi_1, \ldots, \xi_n$ are standard Gaussian random vectors and $\eta_i$ is the $i$th component of $\eta$. We denote by $Y$ (respectively, $W$) the matrix with columns $Y_1, \ldots, Y_n$ (respectively,
σξ₁, . . . , σξₙ). Model (1) can be written in matrix form

\[ Y = \theta \eta^\top + W. \]

We denote by \( P(\theta, \eta) \) the distribution of \( Y \) in model (1) and by \( E(\theta, \eta) \) the corresponding expectation. We assume that \( (\theta, \eta) \) belongs to the set

\[ \Omega_\Delta = \{ \theta \in \mathbb{R}^p : \| \theta \| \geq \Delta \} \times \{-1,1\}^n, \]

where \( \Delta > 0 \) is a given constant. The value \( \Delta \) characterizes the separation between the clusters and equivalently the strength of the signal.

In this paper, we study the problem of recovering the communities, that is, of estimating the vector \( \eta \). As estimators of \( \eta \), we consider any measurable functions \( \hat{\eta} = \hat{\eta}(Y₁, . . . , Yₙ) \) of \( (Y₁, . . . , Yₙ) \) taking values in \( \{-1,1\}^n \). We characterize the loss of a given \( \hat{\eta} \) by the Hamming distance between \( \hat{\eta} \) and \( \eta \), that is, by the number of positions at which \( \hat{\eta} \) and \( \eta \) differ:

\[ |\hat{\eta} - \eta| := \sum_{j=1}^{n} |\hat{\eta}_j - \eta_j| = 2 \sum_{j=1}^{n} 1(\hat{\eta}_j \neq \eta_j). \]

Here, \( \hat{\eta}_j \) and \( \eta_j \) are the \( j \)th components of \( \hat{\eta} \) and \( \eta \), respectively. Since for community detection it is enough to determine \( \eta \) up to a sign change, one can also consider the loss defined by

\[ r(\hat{\eta}, \eta) := \min_{\nu \in \{-1,1\}} |\hat{\eta} - \nu \eta|. \]

In what follows, we use this loss. The expected loss of \( \hat{\eta} \) is defined as \( E(\theta, \eta) r(\hat{\eta}, \eta) \).

In the rest of the paper, we will always denote by \( \eta \) the vector to estimate, while \( \hat{\eta} \) will denote the corresponding estimator. We consider the following minimax risk

\[ \Psi_\Delta := \inf_{\hat{\eta}} \sup_{(\theta, \eta) \in \Omega_\Delta} \frac{1}{n} E(\theta, \eta) r(\hat{\eta}, \eta), \quad (2) \]

where \( \inf \) denotes the infimum over all estimators \( \hat{\eta} \) in \( \{-1,1\}^n \). A simple lower bound for the risk \( \Psi_\Delta \) is given by (cf. Proposition 3 below):

\[ \Psi_\Delta \geq \frac{c}{1 + \Delta/\sigma} e^{-\Delta^2/2\sigma^2} \quad (3) \]

for some \( c > 0 \). Inspecting the proof one may also notice that this bound is attained at the oracle \( \eta^* \) given by

\[ \eta^*_i = \text{sign} \left( \frac{Y_i^\top \theta}{\| \theta \|} \right). \]

This oracle assumes a prior knowledge of \( \theta \). It turns out that for \( p \geq n \), there exists a regime where the lower bound (3) is not optimal, as pointed by Giraud and Verzelen (2018). The intuitive explanation is that for \( p \) larger than \( n \), the vector \( \theta \) is hard to estimate. To the best of our knowledge, there are no lower bounds for \( \Psi_\Delta \) that capture the issue of estimating \( \theta \). This is one of the main questions addressed in the present paper.

**Notation.** In the rest of this paper we use the following notation. For given sequences \( a_n \) and \( b_n \), we write that \( a_n = O(b_n) \) (respectively, \( a_n = \Omega(b_n) \)) when \( a_n \leq c b_n \) (respectively,
\( a_n \geq c b_n \) for some absolute constant \( c > 0 \). We write \( a_n \asymp b_n \) when \( a_n = \mathcal{O}(b_n) \) and \( a_n = \Omega(b_n) \). For \( x, y \in \mathbb{R}^p \), we denote by \( x^\top y \) the Euclidean scalar product, by \( \|x\| \) the corresponding norm of \( x \) and by \( \text{sign}(x) \) the vector of signs of the components of \( x \). For \( x, y \in \mathbb{R} \), we denote by \( x \lor y \) (respectively, \( x \land y \)) the maximum (respectively, minimum) value between \( x \) and \( y \). To any matrix \( M \in \mathbb{R}^{n \times p} \), we denote by \( \|M\|_{op} \) its operator norm with respect to the \( L^2 \) norm, by \( M^\top \) its transpose and by \( \text{Tr}(M) \) its trace in case \( p = n \). Further, \( I_n \) denotes the identity matrix of dimension \( n \) and \( \mathbf{1}(.) \) denotes the indicator function. We denote by \( \Phi \) the complementary cumulative distribution function of the standard Gaussian random variable i.e., \( \forall t \in \mathbb{R}, \Phi(t) = \mathbb{P}(z > t) \). We denote by \( c \) and \( C \) positive constants that may vary from line to line.

We assume that \( p, \sigma \) and \( \Delta \) depend on \( n \) and the asymptotic results correspond to the limit as \( n \to \infty \). All proofs are deferred to the Appendix.

1.2 Related literature

The present work can be related to two parallel lines of work.

1. Community detection in the sub-Gaussian mixture model:

Lu and Zhou (2016) were probably the first to present statistical guarantees for community detection in the sub-Gaussian mixture model using the well-known Lloyd’s algorithm, cf. Lloyd (1982). The results of Lu and Zhou (2016) require a better initialization than a random guess in addition to the condition

\[ \Delta^2 = \Omega(\sigma^2 \left(1 \lor \frac{p}{n}\right)) \],

(4)

in order to achieve almost full recovery recovery and

\[ \Delta^2 = \Omega(\sigma^2 \log n \left(1 \lor \frac{p}{n}\right)) \],

(5)

in order to achieve exact recovery. The notions of almost full and exact recovery are defined in Section 5 and Appendix A. More recently, Rover (2017) and Giraud and Verzelen (2018) have shown that conditions (4) and (5) are not optimal in high dimension i.e. for \( n = o(p) \). In particular, Giraud and Verzelen (2018) study an SDP relaxation of the K-means criterion that achieves almost full recovery under a milder condition

\[ \Delta^2 = \Omega\left(\sigma^2 \left(1 \lor \sqrt{\frac{p}{n}}\right)\right) \],

(6)

and exact recovery under the condition

\[ \Delta^2 = \Omega\left(\sigma^2 \left(\log n \lor \sqrt{\frac{p \log n}{n}}\right)\right) \].

(7)

To the best of our knowledge, conditions (6) and (7) are the mildest in the literature, but no matching necessary conditions are known so far. Giraud and Verzelen (2018) provide insightful heuristics about optimality of these conditions. In the supervised
setting, where all labels are known similar conditions seem necessary to achieve either almost full or exact recovery. It is still not clear whether optimal conditions in supervised mixture learning are also optimal in the unsupervised setting.

Another difference between the previous papers is in computational aspects. While, in Giraud and Verzelen (2018), an SDP relaxation is proposed, a faster algorithm based on Lloyd’s iterations is developed in Lu and Zhou (2016). It remains not clear whether we can achieve almost full (respectively, exact) recovery under condition (6) (respectively, (7)) through faster methods than SDP relaxations, for instance, through Lloyd’s iterations.

Lu and Zhou (2016) suggest to initialize Lloyd’s algorithm using a spectral method. It would be interesting to investigate whether Lloyd’s algorithm initialized by a spectral method, in the same spirit as in Vempala and Wang (2004), can achieve optimal performance in the more general setting where \( p \) is allowed to be larger than \( n \).

In this paper, we shed some light on these issues. Specifically, we address the following questions.

- Are conditions (6) and (7) necessary for both almost full and exact recovery?
- Are optimal requirements similar in both supervised and unsupervised settings?
- Can we achieve results similar to Giraud and Verzelen (2018) using a faster algorithm?
- In case the answer to previous questions is positive, can we achieve the same results adaptively to all parameters?

2. Community detection in the Stochastic Block Model (SBM):

The Stochastic Block Model, cf. Holland et al. (1983), is probably the most popular framework for node clustering. This model with two communities can be seen as a particular case of model (1) when both the signal and the noise are symmetric matrices. A non-symmetric variant of SBM is the Bipartite SBM, cf. Feldman et al. (2015). Unlike the case of sub-Gaussian mixtures where most results in the literature are non-asymptotic, results on almost full or exact recovery for the SBM and its variants are mostly asymptotic and focus on sharp phase transitions. Abbe (2017) poses an open question on whether it is possible to characterize sharp phase transitions in other related problems, for instance, in the Gaussian mixture model.

The first polynomial method achieving exact recovery in the SBM with two communities is due to Abbe et al. (2014). The algorithm splits the initial sample into two independent samples. A black-box algorithm is used on the first sample for almost full recovery, then a local improvement is applied on the second sample. As stated in Abbe et al. (2014), it is not clear whether algorithms achieving almost full recovery can be used to achieve exact recovery. It remains interesting to understand whether similar results can be achieved through direct algorithms ideally without the splitting step.

For the Bipartite SBM, sufficient computational conditions for exact recovery are presented in Feldman et al. (2015), Florescu and Perkins (2016). While the sharp
phase transition for the problem of detection is fully answered in Florescu and Perkins (2016), it is still not clear whether the condition they require, for exact recovery, is optimal. More interestingly, the sufficient condition for exact recovery is different for $p$ of the same order as $n$ and for $p$ larger than $n^2$ for instance. This shows a kind of phase transition with respect to $p$, where for some critical dimension $p^*$ the hardness of the problem changes.

We resume potential connections between our work and these recent developments in the following questions.

- Is it possible to characterize a sharp phase transition for exact recovery in model (1)?
- Are algorithms achieving almost full recovery useful in order to achieve exact recovery in the Gaussian mixture model?
- Is there a critical dimension $p^*$ that separates different regimes of hardness in the problem of exact recovery?

1.3 Main contribution

In this work, we provide a sharp analysis of almost full and exact recovery in the two component Gaussian mixture model. Moreover, we give non-asymptotic lower bounds for the risk $\Psi_\Delta$ and matching upper bounds through a variant of Lloyd’s iterations initialized by a spectral method. To do so, we define a key quantity $r_n$ that turns out to be the right signal-to-noise ratio (SNR) of the problem:

$$ r_n = \frac{\Delta^2 / \sigma^2}{\sqrt{\Delta^2 / \sigma^2 + p/n}}. \quad (8) $$

This SNR is strictly smaller than the "naive" one $\Delta/\sigma$, cf. (3). In particular, it states that the hardness of the problem depends on the dimension $p$. Among other results, we prove that for some $c_1, c_2, C_1, C_2 > 0$, we have

$$ C_1 e^{-c_1 r_n^2} \leq \Psi_\Delta \leq C_2 e^{-c_2 r_n^2}. $$

Moreover, we give a sharp characterization of the constants in this relation.

Inspecting the proofs of the lower bounds in Section 2, one may learn that, in a setting where no prior information on $\theta$ is given, the supervised learning estimator is optimal. Interestingly, supervised and unsupervised risks are almost equal, and the problem of community detection in the Gaussian mixture model is almost transparent to any supervised information on labels as long as the centers are unknown.

As for the upper bound, we introduce and analyze a fully adaptive rate optimal and computationally simple procedure. In order to achieve optimal decay of the risk, it turns out that it is enough to consider $H(Y^\top Y)$ where for any squared matrix $M$, $H(M) = M - \text{diag}(M)$ and $\text{diag}(M)$ is the diagonal of $M$. We set the initializer $\eta^0$ such that $\eta^0 = \text{sign}(\hat{v})$ and $\hat{v}$ is the eigenvector corresponding to the top eigenvalue of $H(Y^\top Y)$. The risk of $\eta^0$ is studied in Section 3. In particular, we observe that $\eta^0$ can achieve almost full recovery but cannot show it is rate optimal. The lack of rate optimality is probably due to
the fact that spectral methods do not benefit from the structure of binary vectors. As an improvement, we consider in Section 4 the iterative sequence of estimators \((\eta^k)_{k \geq 1}\) defined as

\[
\forall k \geq 0, \quad \eta^{k+1} = \text{sign} \left( H(Y^\top Y)\eta^k \right).
\]

In comparison to Lu and Zhou (2016), we get better results, in particular for large \(p\). In their approach, a spectral initialization on \(\theta\) is considered and estimation of \(\theta\) is handled at each iteration. The main difference compared to our procedure lies in the fact that we get around the step of estimating \(\theta\). We only need the matrix \(H(Y^\top Y)\) that is almost blind to the direction of \(\theta\). Giraud and Verzelen (2018) present a rate optimal procedure without capturing the sharp optimality. Our procedure differs in two ways from Giraud and Verzelen (2018). First, it is not an SDP relaxation method and hence is faster. Second, by using the operator \(H\), we do not need to de-bias the Gram matrix, as this operator handles the task.

In Section 5, we show the existence of a sharp phase transition for exact recovery in the Gaussian mixture model, around the threshold \(\Delta = \bar{\Delta}_n\) such that

\[
\bar{\Delta}_n^2 = \sigma^2 \left(1 + \sqrt{1 + 2\frac{p}{n \log n}}\right) \log n.
\]

In particular, this phase transition gives rise to two different regimes around a critical dimension \(p^* = n \log n\), showing that the hardness of exact recovery depends on whether \(p\) is larger or smaller than \(p^*\).

2. Non-asymptotic fundamental limits in the Gaussian mixture model

In this section, we derive a sharp optimal lower bound for the risk \(\Psi_\Delta\). As stated in the Introduction, a simple lower bound is given by (3). The next proposition provides a sharper statement.

**Proposition 1.** For any \(\Delta > 0\), we have

\[
\Psi_\Delta \geq c \Phi^c(\Delta/\sigma),
\]

for some \(c > 0\).

Following the same lines as in Ndaoud (2018), we obtain two different lower bounds for the minimax risk. Proposition 1 gives a bound responsible for the hardness of recovering communities due to the lack of information on the labels. It still benefits from the knowledge of \(\theta\). In Giraud and Verzelen (2018), it becomes clear that for large \(p\), the hardness of the problem results from the hardness of estimating \(\theta\). Hence, in order to capture this phenomenon, one may try to hide the information about the direction of \(\theta\) in order to make its estimation difficult.

More precisely, in order to bound the risk \(\Psi_\Delta\) from below, we place a prior on both \(\eta\) and \(\theta\). Ideally, we would choose a Gaussian prior for \(\theta\) in order to make its estimation harder, but one should keep in mind that \(\theta\) is constrained to the set \(\Omega_\Delta\). To derive lower bounds on constrained sets, we act as in Butucea et al. (2018). Let \(\pi = \pi_\theta \times \pi_\eta\) be a product probability measure on \(\mathbb{R}^p \times \{-1, 1\}^n\) (a prior on \((\theta, \eta)\)). We denote by \(\mathbb{E}_\pi\) the expectation with respect to \(\pi\).
Theorem 2. Let $\Delta > 0$ and $\pi = \pi_\theta \times \pi_\eta$ a product probability measure on $\mathbb{R}^p \times \{-1,1\}^n$. Then,

$$\Psi_\Delta \geq c \left( \frac{1}{[n/2]} \sum_{i=1}^{[n/2]} \inf_{T_i \in [-1,1]} \mathbb{E}_\pi \mathbb{E}_{(\theta,\eta)} |T_i - \eta_i| - \pi_\theta (\|\theta\| < \Delta) \right),$$

where $\inf_{T_i \in [-1,1]}$ is the infimum over all estimators $\hat{T}_i(Y)$ with values in $[-1,1]$ and $c > 0$.

Theorem 2 is useful to derive non-asymptotic lower bounds for constrained minimax risks. For the corresponding lower bound to be optimal, we need the remainder term $\pi_\theta (\|\theta\| < \Delta)$ to be negligible. In other words, the prior on $\theta$ must ensure that $\|\theta\|$ is greater than $\Delta$ with high probability. This would make the problem of recovery easier. Hence, it is clear that there exists some trade-off concerning the choice of $\pi_\theta$.

Let $\pi^\alpha = \pi_\theta^\alpha \times \pi_\eta$ be a product prior on $\mathbb{R}^p \times \{-1,1\}^n$, such that $\pi_\theta^\alpha$ is the distribution of the Gaussian random vector with i.i.d. centered entries of variance $\alpha^2$, $\pi_\eta$ is the distribution of the vector with i.i.d. Rademacher entries, and $\theta$ is independent of $\eta$. For this specific choice of prior we get the following result.

Proposition 3. For any $\alpha > 0$, we have for all $i = 1, \ldots, n$,

$$\inf_{T_i \in [-1,1]^n} \frac{1}{n} \mathbb{E}_{\pi^\alpha} \mathbb{E}_{(\theta,\eta)} |T_i - \eta_i| \geq \frac{1}{n} \mathbb{E}_{\pi^\alpha} \mathbb{E}_{(\theta,\eta)} |\eta_i^* - \eta_i|,$$

where $\eta_i^*$ is a supervised learning oracle given by

$$\forall i = 1, \ldots, n, \quad \eta_i^* = \text{sign} \left( Y_i^\top \left( \sum_{j \neq i} \eta_j Y_j \right) \right).$$

It is interesting to notice that each entry of the supervised learning oracle $\eta_i^*$ only depends on $\theta$ through its best estimator under the Gaussian prior when the labels for other entries are known. The lower bound of Proposition 3 confirms the intuition that the supervised learning oracle is optimal in a minimax sense. For $\sigma > 0$, define $G_\sigma$ by the relation:

$$\forall t \in \mathbb{R}, \quad G_\sigma(t, \theta) = \mathbb{P} \left( (\theta + \sigma \xi_1) \top \left( \theta + \frac{\sigma}{n-1} \sum_{j=2}^{n} \xi_j \right) \leq \|\theta\|^2 t \right), \quad (9)$$

where $\xi_1, \ldots, \xi_n$ are i.i.d. standard Gaussian random vectors. Combining Theorem 2 and Proposition 3 and using the fact that all entries of the prior $\pi^\alpha$ are i.i.d. we obtain the next proposition.

Proposition 4. Let $\Delta > 0$ and let $G_\sigma$ be the function defined in (9). For any $\alpha > 0$, we have

$$\Psi_\Delta \geq c \mathbb{E}_{\pi^\alpha} G_\sigma(0, \theta) - c \mathbb{P} \left( \sum_{j=1}^{p} \varepsilon_j^2 \leq \frac{\Delta^2}{\alpha^2} \right),$$

where $\varepsilon_j$ are i.i.d. standard Gaussian random variables and $c > 0$. 7
We are now ready to state the main result of this section. As explained in Giraud and Verzelen (2018), the main limitation of the analysis in Lu and Zhou (2016) is partially due to the choice of the signal-to-noise ratio (SNR) as $\Delta / \sigma$. We use here the SNR $r_n$ given in (8). It is of the same order as the SNR presented in Giraud and Verzelen (2018).

**Theorem 5.** Let $\Delta > 0$. For $n$ large enough, there exists a sequence $\epsilon_n$ such that $\epsilon_n = o(1)$ and

$$
\Psi_\Delta \geq c \Phi^c ((r_n (1 + \epsilon_n)) ,
$$

for some $c > 0$.

It is worth saying that the result of Theorem 5 holds without any assumption on $p$ and can be interpreted in a non-asymptotic sense by replacing $\epsilon_n$ by some small $c > 0$. Moreover, since $r_n < \Delta / \sigma$, it improves upon the lower bound in Proposition 1. This improvement is most dramatic in the regime $\Delta^2 / \sigma^2 = o(p/n)$ that can be called the hard estimation regime.

3. Spectral initialization

In this section, we analyze the non-asymptotic minimax risk of the spectral initializer $\eta_0$. As it is the case in SDP relaxations of the problem, the matrix of interest is the Gram matrix $Y^\top Y$. It is well known that it suffers from a bias that grows with $p$. In Royer (2017), a de-biasing procedure is proposed using an estimator of the covariance of the noise. This step is important to obtain a procedure adaptive to the noise level. Our approach is different but is still adaptive and consists in removing the diagonal entries of the Gram matrix. We give here some intuition about this procedure. Define the linear operator $H: \mathbb{R}^{n \times n} \to \mathbb{R}^{n \times n}$ as follows:

$$
\forall M \in \mathbb{R}^{n \times n}, \quad H(M) = M - \text{diag}(M),
$$

where $\text{diag}(M)$ is a diagonal matrix with the same diagonal as $M$. Going back to Proposition 3, we may observe that the oracle $\eta^{**}$ can be written as

$$
\eta^{**} = \text{sign} \left( H \left( Y^\top Y \right) \eta \right),
$$

where the sign is applied entry-wise. This suggests that the matrix $H(Y^\top Y)$ appears in a natural way. We can decompose it as follows:

$$
H(Y^\top Y) = \|\theta\|^2 \eta^{\top} + H(W^\top W) + H(W^\top \theta \eta^{\top} + \eta \theta^\top W) - \|\theta\|^2 I_n.
$$

Apart from the scalar factor $\|\theta\|^2$, this expression is similar to SBM or symmetric spiked model, with the noise having a more complex structure. It turns out that the main driver of the noise is $H(W^\top W)$. A simple lemma (cf. Appendix C) shows that our approach is probably an alternative to de-biasing the Gram matrix. Specifically, Lemma 17 gives that

$$
\|H(W^\top W)\|_{op} \leq 2 \left\| W^\top W - E \left( W^\top W \right) \right\|_{op}
$$

for any random matrix $W$ with independent columns. Hence, the noise term can be controlled as if its covariance were known. Nevertheless, the operator $H(.)$ may affect dramatically the signal since it also removes its diagonal entries. Fortunately, the signal term is
almost insensitive to this operation since it is a rank-one matrix where the spike energy is spread all over the spike. For instance, we have
\[ \| H(\eta\eta^\top) \|_{op} = \left( 1 - \frac{1}{n} \right) \| \eta \eta^\top \|_{op}. \]

Hence as \( n \) grows the signal does not get affected by removing the diagonal terms while we get rid of the bias in the noise. It is worth noticing that our approach succeeds thanks to the specific form of \( \eta \) and cannot be generalized to any spiked model. For the general case, a more consistent approach is proposed in Zhang et al. (2018), where the diagonal entries can be used to achieve optimal estimation accuracy. Motivated by (11), the spectral estimator \( \eta^0 \) is defined by
\[ \eta^0 = \text{sign}(\hat{v}), \] (12)
where \( \hat{v} \) is the eigenvector corresponding to the top eigenvalue of \( H(Y^\top Y) \).

**Theorem 6.** Let \( \Delta > 0 \) and let \( \eta^0 \) be the estimator given by (12). Under the condition \( r_n \geq C \), for some absolute constant \( C > 0 \), we have
\[ \sup_{(\theta, \eta) \in \Omega_{\Delta}} \frac{1}{n} E_{\theta, \eta}(\eta^0, \eta) \leq \frac{C'}{r_n^2} + \frac{32}{n^2}, \]
and
\[ \sup_{(\theta, \eta) \in \Omega_{\Delta}} P_{\theta, \eta} \left( \frac{1}{n} | \eta^\top \eta^0 | \leq 1 - \frac{\log n}{n} - \frac{C'}{r_n^2} \right) \leq \epsilon_n \Phi(c(r_n)), \]
for some sequence \( \epsilon_n \) such that \( \epsilon_n = o(1) \) and \( C' > 0 \).

As we may expect the appropriate Hamming distance risk is decreasing with respect to \( r_n \). The residual term \( \frac{32}{n^2} \) is due to removing the diagonal and can be seen as the price to pay for adaptation. It is obvious that as \( r_n \) gets larger than \( n \), removing the diagonal terms is sub-optimal.

As \( n, r_n \rightarrow \infty \), \( \eta^0 \) achieves almost full recovery (cf. Definition 13). We show later that this condition is optimal but cannot show that \( \eta^0 \) is rate optimal. In particular, it is not clear whether \( \eta^0 \) can achieve exact recovery. To bring some evidence that \( \eta^0 \) cannot achieve exact recovery, we rely on asymptotic random matrix theory. In Benach-Georges and Nadakuditi (2012), it is shown that, in the asymptotics when \( p/n \rightarrow c \in (0, 1) \) and when the noise is Gaussian, detection is possible only for \( \Delta^2 \geq \sqrt{c\sigma^2} \). Moreover, the asymptotic correlation between \( \eta \) and its spectral approximation is given by \( \sqrt{1 - \frac{c\sigma^2 + \Delta^2}{\Delta^2(1 + \Delta^2/\sigma^2)}} \). When \( r_n = \Omega(1) \), we observe that \( \frac{c\sigma^2 + \Delta^2}{\Delta^2(1 + \Delta^2/\sigma^2)} \approx \frac{1}{r_n^2} \). Hence, the decay in Theorem 6 is expected for general spiked models, but not necessarily rate optimal in our specific setting. The condition \( r_n = \Omega(1) \) is very natural, since it is necessary even for detection as shown in Banks et al. (2018).
4. A rate optimal practical algorithm

In this section, we present an algorithm that is minimax optimal, adaptive to $\Delta$ and $\sigma$ and faster than SDP relaxation. In the same spirit as in Lu and Zhou (2016), we are tempted by using Lloyd’s iterations. If properly initialized, Lloyd’s algorithm may achieve the optimal rate under mild conditions after only a logarithmic number of steps. We present here a variant of Lloyd’s iterations. Motivated by (10), and given an estimator $\hat{\eta}^0$, we define a sequence of estimators $(\hat{\eta}^k)_{k \geq 0}$ such that

$$\forall k \geq 0, \quad \hat{\eta}^{k+1} = \text{sign}\left( H(Y^\top Y) \hat{\eta}^k \right).$$ (13)

Notice that Lloyd’s iterations correspond to the procedure (13), where $H(Y^\top Y)$ is replaced by $Y^\top Y$. If the initialization is good in a sense that we describe below, then at each iteration $\hat{\eta}^k$ gets closer to $\eta$ and achieves the minimax optimal rate after a logarithmic number of steps. The logarithmic number of steps is crucial computationally as it is the case in many other iterative procedures.

**Theorem 7.** Let $\Delta > 0$ and let $\hat{\eta}^0$ be an estimator satisfying

$$\frac{1}{n} \eta^\top \hat{\eta}^0 \geq 1 - \frac{C'}{r_n^2} - \nu_n$$

for some $C' > 0$ and $\nu_n = o(1)$. Let $(\hat{\eta}^k)_{k \geq 0}$ be the corresponding iterative sequence (13). If $r_n \geq C$ for some $C > 0$, then after $k = \lfloor 3 \log n \rfloor$ steps, we have

$$\sup_{(\theta, \eta) \in \Omega_\Delta} \mathbb{E}_{(\theta, \eta)} r(\hat{\eta}^k, \eta) \leq C' r_n^2 \sup_{\|\theta\| \geq \Delta} G_\sigma \left( \epsilon_n + \frac{C'}{r_n} \theta \right) + \epsilon_n \Phi^c(r_n),$$

for some sequence $\epsilon_n$ such that $\epsilon_n = o(1)$ and $C' > 0$.

Recall that $G(t, \theta)$ is close to $G(0, \theta)$ for small $t$. Theorem 7 can be interpreted as follows. Given a good initialization, the iterative procedure (13) achieves an error close to the supervised learning risk within a logarithmic number of steps. Observing that under the condition $r_n \geq C$ for some $C > 0$, the spectral estimator $\eta^0$ is a good initializer, we state a general result showing that our variant of Lloyd’s iterations initialized with a spectral estimator is minimax optimal.

**Theorem 8.** Let $\Delta > 0$. Let $\eta^0$ be the spectral estimator defined in (12) and let $(\hat{\eta}^k)_{k \geq 0}$ be the iterative sequence (13). Assume that $r_n > C$ for some $C > 0$. Then, after $k = \lfloor 3 \log n \rfloor$ steps we have

$$\sup_{(\theta, \eta) \in \Omega_\Delta} \mathbb{E}_{(\theta, \eta)} r(\hat{\eta}^k, \eta) \leq C' \Phi^c \left( r_n \left( 1 - \epsilon_n - \frac{C' \log r_n}{r_n} \right) \right),$$

for some sequence $\epsilon_n$ such that $\epsilon_n = o(1)$ and $C' > 0$.

Notice that the upper bound in Theorem 8 is almost optimal, and gets closer to the optimal minimax rate as $n, r_n \to \infty$. Hence, under mild conditions, we get a matching upper
bound to the lower bound in Theorem 5. Moreover, we figure out that a good initialization combined with smart iterations is almost equivalent to the supervised learning oracle. In fact, the rate in Theorem 8 is almost the same as the rate of the supervised oracle $\eta^{**}$. We conclude that unsupervised learning is asymptotically as easy as supervised learning in the Gaussian mixture model. The next proposition gives a full picture of the minimax risk $\Psi_\Delta$.

**Proposition 9.** Let $\Delta > 0$. For some $c_1, c_2, C_1, C_2 > 0$ and $n$ large enough, we have

$$C_1 e^{-c_1 r_n^2} \leq \Psi_\Delta \leq C_2 e^{-c_2 r_n^2}.$$

Notice that the procedure we present here has a different rate of decay compared to the spectral procedure (12), that may be non-asymptotically sub-optimal. Recent papers by Xia and Zhou (2017) and Abbe et al. (2017) show that a simple spectral algorithm can achieve exact recovery using refined sup-norm perturbation techniques. Although their results are striking, they match the optimal conditions for exact recovery in the Gaussian mixture model only in the zone $r_n \asymp \Delta/\sigma$.

5. Asymptotic analysis. Phase transitions

This section deals with asymptotic analysis of the problem of community detection in the two component Gaussian mixture model. The results are derived as corollaries of the minimax bounds of previous sections. We will assume that $n \to \infty$ and that parameters $p, \sigma$ and $\Delta$ depend on $n$. For the sake of readability we do not equip some parameters with the index $n$.

The two asymptotic properties we study here are **exact recovery** and **almost full recovery**. The complete characterization of the sharp phase transition for almost full recovery is deferred to Appendix A. We use the terminology following Butucea et al. (2018) that we recall here.

**Definition 10.** Let $(\Omega_{\Delta_n})_{n \geq 2}$ be a sequence of classes corresponding to $(\Delta_n)_{n \geq 2}$:

- We say that **exact recovery** is possible for $(\Omega_{\Delta_n})_{n \geq 2}$ if there exists an estimator $\hat{\eta}$ such that

$$\lim_{n \to \infty} \sup_{(\theta, \eta) \in \Omega_{\Delta_n}} E_{(\theta, \eta)} r(\hat{\eta}, \eta) = 0. \quad (14)$$

In this case, we say that $\hat{\eta}$ achieves exact recovery.

- We say that exact recovery is impossible for $(\Omega_{\Delta_n})_{n \geq 2}$ if

$$\liminf_{n \to \infty} \inf_{\hat{\eta}} \sup_{(\theta, \eta) \in \Omega_{\Delta_n}} E_{(\theta, \eta)} r(\hat{\eta}, \eta) > 0, \quad (15)$$

where $\inf_{\hat{\eta}}$ denotes the infimum over all estimators in $\{-1, 1\}^n$.

Informally, we would like to get a “phase transition” value $\Delta_n$ such that exact recovery is possible for $\Delta_n$ greater than $\Delta_n$ and is impossible for $\Delta_n$ smaller than $\Delta_n$. Our aim now
is to find such “phase transition” values. For the problem of exact recovery, the “phase transition” is described in the next theorem. Let \( \bar{\Delta}_n > 0 \) be defined by

\[
\bar{\Delta}^2_n = \sigma^2 \left( 1 + \sqrt{1 + \frac{2p}{n \log n}} \right) \log n. \tag{16}
\]

The next theorem is a direct consequence of Theorem 14, cf. Appendix A.

**Theorem 11.** (i) If \( \Delta_n \geq \bar{\Delta}_n (1 + \epsilon) \) for some \( \epsilon > 0 \). Then, the estimator \( \eta^k \) defined in (12)-(13), with \( k = \lfloor 3 \log n \rfloor \), achieves exact recovery.

(ii) If the complementary condition holds, i.e., \( \Delta_n \leq \bar{\Delta}_n (1 - \epsilon) \) for some \( \epsilon > 0 \), then exact recovery is impossible.

Some remarks are in order here. First of all, Theorem 11 shows that the “phase transition” for exact recovery occurs at \( \bar{\Delta}_n \) given in (16). It is worth noticing that this sharp threshold for exact recovery holds for all values of \( p \). In particular, there exists a critical dimension \( p^* = n \log n \). If \( p = o(p^*) \), then \( \bar{\Delta}_n = (1 + o(1)) \sigma \sqrt{2 \log n} \). In this case, the phase transition threshold for exact recovery is the same as if \( \theta \) were known. While if \( p^* = o(p) \), then \( \bar{\Delta}_n = (1 + o(1)) \sigma \left( \frac{2p \log n}{n} \right)^{1/4} \). This new condition reflects the hardness of estimation, and \( p^* \) can be interpreted as a phase transition with respect to the cluster dimension \( p \).

6. Discussion and open problems

A key objective of this paper was to establish sharp phase transition for exact recovery in the two component Gaussian mixture model. All upper bounds remain valid in the case of sub-Gaussian noise. It would be interesting to generalize the methodology used to derive both lower and upper bounds to the case of multiple communities and general covariance structure of the noise. We also expect the procedure (12)-(13) to achieve exact recovery in asymptotically sharp way in other problems, for instance in the Bipartite Stochastic Block Model.

We conclude this paper with an open question. Let \( p^* = n \log n \). In the regime \( p^* = o(p) \), we proved that for any \( \epsilon > 0 \), the condition

\[
\Delta^2 \geq (1 - \epsilon) \sigma^2 \left( \frac{2p}{p^*} \right)^{1/2} \log n
\]

is necessary to achieve exact recovery. This is a consequence of considering a Gaussian prior on \( \theta \) which makes recovering its direction the hardest. We give here a heuristics that this should hold independently on the choice of prior as long as \( \theta \) is uniformly well-spread (i.e., not sparse). Suppose that we put a Rademacher prior on \( \theta \) such that \( \theta = \frac{\Delta}{\sqrt{p}} \zeta \), where \( \zeta \) is a random vector with i.i.d. Rademacher entries. Following the same argument as in Proposition 1, it is clear that a necessary condition to get non-trivial correlation with \( \zeta \) is given by

\[
\Delta^2 \geq c \sigma^2 \frac{p}{n},
\]
for some $c > 0$. Observing that, in the hard estimation regime, we have
\[
\left( \frac{p}{p^*} \right)^{1/2} \log n = o \left( \frac{p}{n} \right),
\]
it comes that, while exact recovery of $\eta$ is possible, non-trivial correlation with $\zeta$ is impossible. Consequently, there is no hope achieving exact recovery through non-trivial correlation with $\theta$ in the hard estimation regime.

**Conjecture 12.** Let $\Delta > 0$. Assume that $Y$ follows model (1). Let $\eta$ be a random vector with i.i.d. Rademacher random entries, and $\theta = \frac{\Delta}{\sqrt{p}} \zeta$ where $\zeta$ is a random vector with i.i.d. Rademacher entries and independent of $\eta$. Assume that $n \log n = o(p)$. Prove or disprove that, for any $\epsilon > 0$,
\[
\Delta^2 \geq (1 - \epsilon) \sigma^2 \sqrt{\frac{2p \log n}{n}}
\]
is necessary to achieve exact recovery.

In particular, a positive answer to the previous question will be very useful to derive optimal conditions for exact recovery in bipartite graph models among other problems.

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Appendix A. Asymptotic analysis: almost full recovery

In this section, we conduct the asymptotic analysis of the problem of almost full recovery in the two component Gaussian mixture model. We first recall the terminology used in Butucea et al. (2018) that we adopt for the problem of almost full recovery.

Definition 13. Let \((\Omega_{\Delta_n})_{n \geq 2}\) be a sequence of classes corresponding to \((\Delta_n)_{n \geq 2}\):

- We say that almost full recovery is possible for \((\Omega_{\Delta_n})_{n \geq 2}\) if there exists an estimator \(\hat{\eta}\) such that
  \[
  \lim_{n \to \infty} \sup_{(\theta, \eta) \in \Omega_{\Delta_n}} \frac{1}{n} E_{(\theta, \eta)} r(\hat{\eta}, \eta) = 0. \tag{17}
  \]
  In this case, we say that \(\hat{\eta}\) achieves almost full recovery.

- We say that almost full recovery is impossible for \((\Omega_{\Delta_n})_{n \geq 2}\) if
  \[
  \liminf_{n \to \infty} \inf_{\hat{\eta}} \sup_{(\theta, \eta) \in \Omega_{\Delta_n}} \frac{1}{n} E_{(\theta, \eta)} r(\hat{\eta}, \eta) > 0, \tag{18}
  \]
  where \(\inf_{\hat{\eta}}\) denotes the infimum over all estimators in \([-1, 1]^n\).

The following general characterization theorem is a straightforward corollary of the results of previous sections.

Theorem 14. (i) Almost full recovery is possible for \((\Omega_{\Delta_n})_{n \geq 2}\) if and only if
  \[
  \Phi^c(r_n) \to 0 \quad \text{as } n \to \infty. \tag{19}
  \]
  In this case, the estimator \(\eta^k\) defined in (12)-(13), with \(k = \lfloor 3 \log n \rfloor\), achieves almost full recovery.

(ii) Exact recovery is impossible for \((\Omega_{\Delta_n})_{n \geq 2}\) if for some \(\epsilon > 0\)
  \[
  \liminf_{n \to \infty} n \Phi^c(r_n(1 + \epsilon)) > 0 \quad \text{as } n \to \infty, \tag{20}
  \]
  and possible if for some \(\epsilon > 0\)
  \[
  n \Phi^c(r_n(1 - \epsilon)) \to 0 \quad \text{as } n \to \infty, \tag{21}
  \]
  In this case, the estimator \(\eta^k\) defined in (12)-(13), with \(k = \lfloor 3 \log n \rfloor\), achieves exact recovery.

Although this theorem gives a complete solution to the problem of almost full and exact recovery, conditions (19), (20) and (21) are not quite explicit. The next theorem is a consequence of Theorem 14. It describes a “phase transition” for \(\Delta_n\) in the problem of almost full recovery.

Theorem 15. (i) If \(\sigma^2 \left(1 + \sqrt{p/n} \right) = o(\Delta_n^2)\). Then, the estimator \(\eta^k\) defined in (12)-(13), with \(k = \lfloor 3 \log n \rfloor\), achieves almost full recovery.

(ii) Moreover, if \(\Delta_n^2 = O \left(\sigma^2(1 + \sqrt{p/n}) \right)\). Then, almost full recovery is impossible.

Theorem 15 shows that almost full recovery occurs if and only if
  \[
  \sigma^2 \left(1 + \sqrt{p/n} \right) = o(\Delta_n^2). \tag{22}
  \]
Appendix B. Main proofs

In all the proofs of lower bounds, we follow the same argument as in Theorem 2 in Gao et al. (2018) in order to substitute the minimax risk of $r(\tilde{\eta}, \eta)$ by a Hamming minimax risk. Let $z^*$ be a vector of labels in $\{-1, 1\}^n$ and let $T$ be a subset of $\{1, \ldots, n\}$ of size $\lfloor n/2 \rfloor + 1$. A lower bound of the minimax risk is given on the subset of labels $Z$, such that for all $i \in T$, we have $\eta_i = z_i^*$. Observe that in that case

$$r(\eta^1, \eta^2) = |\eta^1 - \eta^2|,$$

for any $\eta^1, \eta^2 \in Z$. The argument in Gao et al. (2018), leads to

$$\Psi_\Delta \geq \frac{c}{|T^c|} \sum_{i \in T^c} \inf_{\tilde{\eta}_i} \mathbb{E}_{\pi_{(\theta, \eta)}} |\tilde{\eta}_i - \eta_i|,$$

for some $c > 0$ and for any prior $\pi_\eta$ such that $\pi_\eta$ is invariant by a sign change. That is typically the case under Rademacher prior on labels. As a consequence, a lower bound of $\Psi_\Delta$ is given by a lower bound of the R.H.S minimax Hamming risk.

B.1 Proof of Proposition 1

Let $\tilde{\theta}$ be a vector in $\mathbb{R}^p$ such that $||\tilde{\theta}|| = \Delta$. Placing an independent Rademacher prior $\pi$ on $\eta$, and fixing $\theta$, it follows that

$$\inf_{\tilde{\eta}_j} \mathbb{E}_{\pi_{(\tilde{\theta}, \eta)}} |\tilde{\eta}_j - \eta_j| \geq \inf_{\tilde{\eta}_j} \mathbb{E}_{\pi_{(\tilde{\theta}, \eta)}} |\tilde{\eta}_j(Y_j) - \eta_j|, \tag{23}$$

where $\tilde{\eta}_j \in [-1, 1]$. The last inequality holds because of independence between the priors.

We define, for $\epsilon \in \{-1, 1\}$, $\tilde{f}_\epsilon(.)$ the density of the observation $Y_j$ conditionally on the value of $\eta_j = \epsilon$. Now, using Neyman-Pearson lemma and the explicit form of $\tilde{f}_\epsilon$, we get that the selector $\eta^*_j$ given by

$$\eta^*_j = \text{sign}\left(\tilde{\theta}^\top Y_j\right), \quad \forall j = 1, \ldots, n$$

is the optimal selector that achieves the minimum of the RHS of (23). Plugging this value in (23), we get further that

$$\inf_{\tilde{\eta}_j} \mathbb{E}_{\pi_{\eta}} |\tilde{\eta}_j(Y_j) - \eta_j| = 2\Phi\left(\frac{\Delta}{\sigma}\right).$$

B.2 Proof of Theorem 2

Throughout the proof, we write for brevity $A = \Omega_\Delta$. Set $\eta^A = \eta 1((\theta, \eta) \in A)$ and denote by $\tilde{\pi}_A$ the probability measure $\pi$ conditioned by the event $\{(\theta, \eta) \in A\}$, that is, for any $C \subseteq \mathbb{R}^p \times \{-1, 1\}^n$

$$\tilde{\pi}_A(C) = \frac{\pi((\{\theta, \eta\} \in C) \cap \{(\theta, \eta) \in A\})}{\pi((\theta, \eta) \in A)}.$$  

The measure $\tilde{\pi}_A$ is supported on $A$ and we have

$$\inf_{\tilde{\eta}_j} \mathbb{E}_{\tilde{\pi}_A_{(\theta, \eta)}} |\tilde{\eta}_j - \eta_j| \geq \inf_{\tilde{\eta}_j} \mathbb{E}_{\tilde{\pi}_A_{(\theta, \eta)}} |\tilde{\eta}_j - \eta^A_j| \geq \inf_{\tilde{T}_j} \mathbb{E}_{\tilde{\pi}_A_{(\theta, \eta)}} |\tilde{T}_j - \eta^A_j|.$$
where $\inf_{\hat{T}_j}$ is the infimum over all estimators $\hat{T}_j = \hat{T}_j(Y)$ with values in $\mathbb{R}$. According to Theorem 1.1 and Corollary 1.2 on page 228 in Lehmann and Casella (2006), there exists a Bayes estimator $B_j^A = B_j^A(Y)$ such that

$$\inf_{\hat{T}_j} \mathbb{E}_{\pi_A} E_{(\theta,\eta)} | \hat{T}_j - \eta_j^A | = \mathbb{E}_{\pi_A} E_{(\theta,\eta)} | B_j^A - \eta_j^A |,$$

and this estimator is a conditional median of $\eta_j^A$ given $Y$. Therefore,

$$\Psi_\Delta \geq c \left( \frac{1}{\lfloor n/2 \rfloor} \sum_{j=1}^{\lfloor n/2 \rfloor} \mathbb{E}_{\pi_A} E_{(\theta,\eta)} | B_j^A - \eta_j^A | \right). \quad (24)$$

Note that $B_j^A \in [-1,1]$ since $\eta_j^A$ takes its values in $[-1,1]$. Using this, we obtain

$$\inf_{\hat{T}_j \in [-1,1]} \mathbb{E}_{\pi} E_{(\theta,\eta)} | \hat{T}_j - \eta_j | \leq \mathbb{E}_{\pi} E_{(\theta,\eta)} | B_j^A - \eta_j |$$

$$= \mathbb{E}_{\pi} E_{(\theta,\eta)} \left( | B_j^A - \eta_j | 1((\theta,\eta) \in A) \right) + \mathbb{E}_{\pi} E_{(\theta,\eta)} \left( | B_j^A - \eta_j | 1((\theta,\eta) \in A^c) \right)$$

$$\leq \mathbb{E}_{\pi_A} E_{(\theta,\eta)} | B_j^A - \eta_j^A | + \mathbb{E}_{\pi_A} E_{(\theta,\eta)} \left( | B_j^A - \eta_j | 1((\theta,\eta) \in A^c) \right)$$

$$\leq \mathbb{E}_{\pi_A} E_{(\theta,\eta)} | B_j^A - \eta_j^A | + 2P((\theta,\eta) \not\in A). \quad (25)$$

The result follows combining (24) and (25).

**B.3 Proof of Proposition 3**

We start by using the fact that

$$\mathbb{E}_{\pi^\alpha} E_{(\theta,\eta)} | \tilde{\eta}_i - \eta_i | = \mathbb{E}_{p_{-i}} \mathbb{E}_{p_i} E_{(\theta,\eta)} (| \tilde{\eta}_i - \eta_i || (\eta_j)_{j \neq i} ),$$

where $p_i$ is the marginal of $\pi^\alpha$ on $(\theta,\eta_i)$, while $p_{-i}$ is the marginal of $\pi^\alpha$ on $(\eta_j)_{j \neq i}$. Using the independence between different priors, one may observe that $\pi^\alpha = p_i \times p_{-i}$. We define, for $\epsilon \in \{-1,1\}$, $\tilde{f}_i^\epsilon$ the density of the observation $Y$ given $(\eta_j)_{j \neq i}$ and given $\eta_i = \epsilon$. Using Neyman-Pearson lemma, we get that

$$\eta_i^{**} = \begin{cases} 1 & \text{if } \tilde{f}_i^1(Y) \geq \tilde{f}_i^{-1}(Y), \\ -1 & \text{else}, \end{cases}$$

minimizes $\mathbb{E}_{p_i} E_{(\theta,\eta)} (| \tilde{\eta}_i - \eta_i || (\eta_j)_{j \neq i} )$ over all functions of $(\eta_j)_{j \neq i}$ and of $Y$ with values in $[-1,1]$. Using the independence of the rows of $Y$, we have

$$\tilde{f}_i^\epsilon(Y) = \prod_{j=1}^p e^{-\frac{1}{2}L_j^\epsilon \Sigma^{-1}_\epsilon L_j} (2\pi)^{p/2} | \Sigma_\epsilon |,$$

where $L_j$ is the $j$-th row of $Y$ and $\Sigma_\epsilon = I_n + \alpha^2 \eta_\epsilon \eta_\epsilon^\top$. We denote by $\eta_\epsilon$ the binary vector such that $\eta_\epsilon = \epsilon$ and the other components are known. It is easy to check that $| \Sigma_\epsilon | = 1 + \alpha^2 n$, hence it does not depend on $\epsilon$. A simple calculation leads to

$$\Sigma^{-1}_\epsilon = I_n - \frac{\alpha^2}{1 + \alpha^2 n} \eta_\epsilon \eta_\epsilon^\top.$$
Hence
\[
\frac{\tilde{f}_i(Y)}{f_{i-1}(Y)} = \prod_{j=1}^p e^{-\frac{1}{2} L_j^T (\Sigma^{-1}_i - \Sigma^{-1}) L_j}
= \prod_{j=1}^p e^{-\frac{1}{2} \sum_{k \neq i} L_j \eta_k}
= e^{\frac{1}{2} \sum_{k \neq i} \eta_k \sum_{j=1}^p L_j \eta_k}
= e^{\alpha^2 (Y, \sum_{k \neq i} \eta_k Y_k)}.
\]

It is now immediate that
\[
\eta_i^{**} = \text{sign} \left( Y_i^\top \left( \sum_{k \neq i} \eta_k Y_k \right) \right).
\]

**B.4 Proof of Proposition 4**

Combining Theorem 2 and Proposition 3, we get that
\[
\Psi_{\Delta} \geq c \left( \frac{1}{|n/2|} \sum_{i=1}^{[n/2]} \mathbb{E}_{\pi \in \pi^n} \mathbb{E}_{(\theta, \eta)} |\eta_i^{**} - \eta_i| - \pi_{\theta}^0 (\|\theta\| \leq \Delta) \right).
\]

Recall that here \(\theta\) has i.i.d. centered Gaussian entries with variance \(\alpha^2\). This yields the second term on the R.H.S of the inequality of Proposition 4. While, for the first term, one may notice that the vectors \(\eta_i Y_i\) for \(i = 1, \ldots, n\) are i.i.d. and that
\[
|\eta_i^{**} - \eta_i| = 2 \left( \eta_i Y_i^\top \left( \sum_{j \neq i} \eta_j Y_j \right) \right) \leq 0.
\]

Then, we use the definition of \(G_{\sigma} (9)\) in order to conclude.

**B.5 Proof of Theorem 5**

We prove the result by considering separately the following three cases.

1. **Case \(\Delta \leq \frac{\log^2(n)}{\sqrt{n}}\).** In this case we use Proposition 1.

   Since \(0 \leq \frac{\Delta^2}{\Delta^2 + p/n} \leq \Delta\), we have \(\Delta - \frac{\Delta^2}{\sqrt{\Delta^2 + p/n}} \leq \frac{\log^2(n)}{\sqrt{n}}\). Hence

   \[
   \left| \Phi^c(\Delta) - \Phi^c \left( \frac{\Delta^2}{\sqrt{\Delta^2 + p/n}} \right) \right| \leq c \frac{\log^2(n)}{\sqrt{n}} \Phi^c \left( \frac{\Delta^2}{\sqrt{\Delta^2 + p/n}} \right),
   \]

   for some \(c > 0\). Hence we get the result with \(\epsilon_n = c \frac{\log^2(n)}{\sqrt{n}}\).

2. **Case \(\Delta \geq \sqrt{\frac{p \log n}{n}}\).** In this case, we have \(\sqrt{1 + \frac{p}{n \Delta^2}} \leq \Delta\). It is easy to check that

   \[
   \left| \sqrt{1 + \frac{p}{n \Delta^2}} - 1 \right| \leq \frac{1}{\log n}.
   \]
Hence
\[
\Delta \leq \Delta^2 \sqrt{\Delta^2 + p/n}(1 + \epsilon_n),
\]
for \(\epsilon_n = \frac{1}{\log n}\). We conclude using Proposition 1.

3. Case \(\frac{\log^2(n)}{\sqrt{n}} < \Delta < \frac{p\log n}{n}\). Notice that \(p \geq \log^3(n)\) in this regime. We will use Proposition 4. Set \(\alpha^2\) such that
\[
\alpha^2 = \frac{\Delta^2}{p(1 - \nu_n)} \quad \text{and} \quad \nu_n = \sqrt{\frac{n\Delta^2}{p\log^2(n)}}.
\]
It is easy to check that \(0 < \nu_n^2 \leq 1/\log n\), Hence
\[
P\left(\sum_{j=1}^{p} \epsilon_j^2 \leq \frac{\Delta^2}{\alpha^2}\right) = P\left(\frac{1}{p} \sum_{j=1}^{p} (\epsilon_j^2 - 1) \leq -\nu_n\right) \leq e^{-c\frac{n}{\log^2(n)}\Delta^2},
\]
for some \(c > 0\). Hence, for any \(\epsilon_n \to 0\) we have
\[
P\left(\sum_{j=1}^{p} \epsilon_j^2 \leq \frac{\Delta^2}{\alpha^2}\right) \leq e^{-c'\log n}\Phi\left(\Delta(1 + \epsilon_n)\right) \leq e^{-c'\log n}\Phi\left(\frac{\Delta^2}{\sqrt{\Delta^2 + p/n}}(1 + \epsilon_n)\right),
\]
for some \(c' > 0\). Since \(e^{-c'\log n} \to 0\), then in order to conclude, we just need to prove that
\[
E_{\pi^\alpha G_\sigma(0, \theta)} \geq (1 - \epsilon_n)\Phi\left(\frac{\Delta^2}{\sqrt{\Delta^2 + p/n}}(1 + \epsilon_n)\right),
\]
for some sequence \(\epsilon_n \to 0\).

We recall that
\[
E_{\pi^\alpha G_\sigma(0, \theta)} = P\left(\theta + \xi_1 \geq 0\right),
\]
where \(\xi_1, \xi_2\) are two independent random vectors with i.i.d. standard Gaussian entries and \(\theta\) is an independent Gaussian prior. Moreover, using independence, we have
\[
P\left(\theta + \xi_1 \geq 0\right) = P\left(\theta + \xi_2 \geq 0\right) = P\left(\epsilon \sqrt{\|\theta\|^2 + \frac{\|\xi_2\|^2}{n-1}} + \frac{2}{\sqrt{n-1}}\theta^\top \xi_2 \geq \|\theta\|^2 + \frac{1}{\sqrt{n-1}}\theta^\top \xi_2\right),
\]
where \(\epsilon\) is a standard Gaussian random variable. Fix \(\theta\) and define the random event
\[
\mathcal{A} = \left\{\frac{\|\xi_2\|^2}{n-1} \geq \frac{p}{n-1}(1 - \zeta_n)\right\} \cap \{\|\theta\|^2 \leq \sqrt{n-1}\beta_n\|\theta\|^2\},
\]
where \(\beta_n > 0\) and \(\zeta_n \in (0, 1)\). It is easy to check that
\[
P\left(\mathcal{A}^c\right) \leq e^{-c\log^3(n)}\zeta_n^2 + e^{-c\beta_n^2 n\|\theta\|^2},
\]
(26)
for some $c > 0$. Hence conditioning on $\theta$, we have
\[
P\left( (\theta + \xi_1)^\top (\theta + \frac{\xi_2}{\sqrt{n-1}}) \leq 0 \right) \geq \mathbb{E} \left[ \Phi^c \left( \frac{||\theta||^2 (1 + \beta_n)}{\sqrt{||\theta||^2 (1 - 2\beta_n) + \frac{p}{n-1} (1 - \zeta_n)}} \right) \right] P(\mathcal{A}) .
\]
where the last expectation is over $\theta$. Define now the random event $\mathcal{B} = \{ ||\theta||^2 - \Delta^2 \leq \Delta^2 \gamma_n \}$ where $\gamma_n \in (0, 1)$. Then, using (26), we get
\[
P\left( (\theta + \xi_1)^\top (\theta + \frac{\xi_2}{\sqrt{n-1}}) \leq 0 \right) \geq \Phi^c \left( U_n \left( 1 - e^{-c \log^3 (n) \zeta_n^2} - e^{-c \beta_n^2 (1 - \gamma_n) \log^4 (n)} \right) \right) P(\mathcal{B}),
\]
where $U_n := \frac{\Delta^2 (1 + \beta_n) (1 + \gamma_n)}{\sqrt{\Delta^2 (1 - 2\beta_n) (1 - \gamma_n) + \frac{p}{n-1} (1 - \zeta_n)}}$. Now we may check that
\[
P(\mathcal{B}^c) = P \left( \left| \sum_{j=1}^p \varepsilon_j^2 - \frac{\Delta^2}{\alpha^2} \right| \geq \frac{\Delta^2}{\alpha^2} \gamma_n \right).
\]
Hence
\[
P(\mathcal{B}^c) \leq P \left( \left| \sum_{j=1}^p \varepsilon_j^2 - p \right| \geq \frac{\Delta^2}{\alpha^2} \gamma_n - p - \frac{\Delta^2}{\alpha^2} \right).
\]
Using the definition of $\alpha^2$ we get
\[
P(\mathcal{B}^c) \leq P \left( \left| \sum_{j=1}^p \varepsilon_j^2 - p \right| \geq p(1 - \nu_n) \gamma_n - \nu_n \right) \leq 2 e^{-c \log^3 (n) \gamma_n^2}, \tag{28}
\]
for some $c > 0$ whenever $4 \nu_n \leq \gamma_n \leq 1$. Using the inequality $\nu_n^2 \leq 1/ \log n$, and choosing $\beta_n^2 = 1/ \log n$, $\gamma_n^2 = 16/ \log n$ and $\zeta_n^2 = 1/ \log n$, we get the desired result by combining (27) and (28).

### B.6 Proof of Theorem 6

We begin by writing that
\[
\frac{1}{n} Y^\top Y = \frac{||\theta||^2}{n} \eta \eta^\top + Z_1,
\]
where
\[
Z_1 = \frac{1}{n} \eta \theta^\top W + \frac{1}{n} W^\top \theta \eta^\top + \frac{1}{n} W^\top W.
\]
Next observe that
\[
H \left( \frac{1}{n} Y^\top Y \right) = \frac{||\theta||^2}{n} \eta \eta^\top + Z_2,
\]
where $Z_2$ is given by
\[
Z_2 = H (Z_1) - \frac{||\theta||^2}{n} I_n.
\]
Based on Lemma 17, we have
\[ \|Z_2\|_{op} \leq 4 \left\| \frac{1}{n} \eta \theta^\top W \right\|_{op} + 2 \left\| \frac{1}{n} W^\top W - E \left( \frac{1}{n} W^\top W \right) \right\|_{op} + \frac{\|\theta\|^2}{n}. \] (29)

Using the Davis-Kahan sin \( \theta \) Theorem cf. Theorem 4.5.5 in Vershynin (2018), we obtain
\[ \min_{\nu \in \{-1,1\}} \left\| \hat{v} - \frac{1}{\sqrt{n}} \nu \eta \right\|^2 \leq 8 \|Z_2\|_{op}^2 \|\theta\|^4. \] (30)

Hence, using Lemma 20, we get
\[ \frac{1}{n} r(\eta^0, \eta) \leq 16 \|Z_2\|_{op}^2 \left\|\theta\right\|^4 \leq 512 \left( \left\| \frac{1}{n} \eta \theta^\top W \right\|_{op}^2 + \left\| \frac{1}{n} W^\top W - E \left( \frac{1}{n} W^\top W \right) \right\|_{op}^2 \right) + \frac{32}{n^2}. \] (31)

Since \( r_n \geq C \), for some \( C \) large enough. We may assume that \( \|\theta\|^2 \geq 1 \) so that \( 1 + \frac{p}{n} \leq \|\theta\|^2 + p/n \). The inequality in expectation is a consequence of Lemma 18 and Lemma 19.

For the inequality in probability, we first observe, using (31), that
\[ \frac{1}{n} |\eta^\top \eta^0| \geq 1 - 8 \|Z_2\|_{op}^2 \left\|\theta\right\|^4. \]

Next, and since \( r_n \geq C \) for some \( C \) large enough, observe that
\[ P(\theta, \eta) \left( \frac{1}{n} |\eta^\top \eta^0| \leq 1 - \frac{\log n}{n} - \frac{C}{r_n^2} \right) \leq A_1 + A_2, \]
where
\[ A_1 = P(\theta, \eta) \left( \left\| \frac{1}{n} \eta \theta^\top W \right\|_{op} \geq \sqrt{\frac{\log n}{2n} \|\theta\|^2 + 2} \right), \]
and
\[ A_2 = P(\theta, \eta) \left( \left\| \frac{1}{n} W^\top W - E \left( \frac{1}{n} W^\top W \right) \right\|_{op} \geq \sqrt{\frac{\log n}{2n} \|\theta\|^2 + C \left( 1 + \sqrt{\frac{p}{n}} \right)} \right), \]
Using Lemma 18 and Lemma 19, we get
\[ P(\theta, \eta) \left( \frac{1}{n} |\eta^\top \eta^0| \leq 1 - \frac{\log n}{n} - \frac{C}{r_n^2} \right) \leq 2e^{-c\sqrt{n}\log n}\|\theta\|^2 \left( 1 + \frac{\log n}{p} \|\theta\|^2 \right) \leq c\log n r_n^2, \]

Using the tail Gaussian function, we conclude easily that
\[ e^{-c\sqrt{n}\log n} = o(\Phi^c(r_n)). \]
B.7 Proof of Theorem 7

By the definition of \( r(\hat{\eta}, \eta) \), we may assume w.l.o.g that \( \eta^\top \hat{\eta}^0 > 0 \). Define the random events \( A_i \) for \( i = 1, \ldots, n \), \( B \) and \( C \) such that for all \( i = 1, \ldots, n \)

\[
A_i = \left\{ \left( \frac{1}{n} H(Y^\top Y)^\top_i \eta \right) \eta_i \geq \|\theta\|^2 \left( \frac{8C}{r_n} + \frac{C'}{r_n^2} + 8c' \sqrt{\frac{\log n}{n}} + \nu_n \right) \right\},
\]

\[
C = \left\{ \frac{1}{n} \sum_{i=1}^{n} 1_{A_i} \leq \frac{C'}{4r_n^2} \right\}
\]

and

\[
B = \left\{ \|Z_2\|_{op} \leq c' \sqrt{\frac{\log n}{n}} \|\theta\|^2 + C \left( 1 \lor \sqrt{p/n} \right) \right\},
\]

where we use the same notation of the previous proof and \( c' \) a positive constant that we may choose large enough.

We first prove, by induction, that on the event \( B \cap C \), we have

\[
\frac{1}{n} \eta^\top \hat{\eta}^k \geq 1 - \frac{C'}{r_n^2} - \nu_n, \quad \forall k = 0, 1, \ldots
\]

For \( k = 0 \), the result is obvious. Let \( k \geq 1 \). Assume that the result holds for \( k \), and we prove it for \( k + 1 \). Remember that

\[
\frac{1}{n} H(Y^\top Y) = \frac{1}{n} \|\theta\|^2 \eta^\top + Z_2.
\]

A simple calculation leads to

\[
\frac{1}{n} H(Y^\top Y)^\top_i \hat{\eta}^k = (Z_2)^\top_i (\hat{\eta}^k - \eta) + \frac{1}{n} H(Y^\top Y)^\top_i \eta - \|\theta\|^2 \eta_i \frac{n - \eta^\top \hat{\eta}^k}{n}.
\]

Hence if \( \eta_i = -1 \) and if \( A_i \) is true, then using the induction hypothesis we get

\[
\frac{1}{n} H(Y^\top Y)^\top_i \hat{\eta}^k \leq (Z_2)^\top_i (\hat{\eta}^k - \eta) - \|\theta\|^2 \left( 8C \frac{1}{r_n} + 8c' \sqrt{\frac{\log n}{n}} \right).
\]

Hence when \( \eta_i = -1 \) we have

\[
1_{\{ \frac{1}{n} H(Y^\top Y)^\top_i \hat{\eta}^k \geq 0 \}} 1_{A_i} \leq 1_{\{ (Z_2)^\top_i (\hat{\eta}^k - \eta) \geq \|\theta\|^2 \left( \frac{8C}{r_n} + 8c' \sqrt{\frac{\log n}{n}} \right) \}} \leq \left( \frac{(Z_2)^\top_i (\hat{\eta}^k - \eta)}{\|\theta\|^2 \left( \frac{8C}{r_n} + 8c' \sqrt{\frac{\log n}{n}} \right)} \right)^2.
\]

similarly we get for \( \eta_i = 1 \) that

\[
1_{\{ \frac{1}{n} H(Y^\top Y)^\top_i \hat{\eta}^k \leq 0 \}} 1_{A_i} \leq \left( \frac{(Z_2)^\top_i (\hat{\eta}^k - \eta)}{\|\theta\|^2 \left( \frac{8C}{r_n} + 8c' \sqrt{\frac{\log n}{n}} \right)} \right)^2.
\]
It is clear that
\[
\frac{1}{2}|\hat{\eta}^{k+1} - \eta| = \sum_{\eta = -1}^{1} 1\{\frac{1}{n}H(Y^\top Y_i)^\top \hat{\eta}^k \geq 0\} + \sum_{\eta = 1}^{n} 1\{\frac{1}{n}H(Y^\top Y_i)^\top \hat{\eta}^k \leq 0\}.
\]

Hence we get using the events \(A_i\) for \(i = 1, \ldots, n\), that
\[
\frac{1}{2n}|\hat{\eta}^{k+1} - \eta| \leq \frac{\|Z_2\|_{op}^2}{\|\theta\|^4 \left(\frac{8C}{r_n} + 8c' \sqrt{\log \frac{n}{n}}\right)^2} \frac{\|\hat{\eta}^k - \eta\|^2}{n} + \frac{1}{n} \sum_{i=1}^{n} 1_{A_i^c}.
\]

Using the induction hypothesis and the events \(B\) and \(C\), we get
\[
1 - \frac{1}{n} \eta^\top \hat{\eta}^{k+1} \leq 4 \left(\frac{c' \sqrt{\log \frac{n}{n}}}{\|\theta\|^2} + C \left(1 \vee \sqrt{\frac{p}{n}}\right) \right)^2 \left(\frac{C'/r_n^2 + \nu_n}{2}\right).
\]

Since \(r_n > C\) for \(C\) large enough, then \(1 \vee \sqrt{\frac{p}{n}} \leq \|\theta\|^2/r_n\), it comes that
\[
\frac{1}{n} \eta^\top \hat{\eta}^{k+1} \geq 1 - \frac{C'}{r_n^2} - \nu_n.
\]

That concludes that on \(B \cap C\), for all \(k = 0, 1, \ldots\) we get
\[
\frac{1}{n} \eta^\top \hat{\eta}^{k} \geq 1 - \frac{C'}{r_n^2} - \nu_n.
\]

Hence, and using (32), we obtain
\[
\frac{1}{n} |\hat{\eta}^{k+1} - \eta| 1_{B \cap C} \leq \frac{1}{4} n |\hat{\eta}^{k} - \eta| 1_B 1_C + \frac{2}{n} \sum_{i=1}^{n} 1_{A_i^c}.
\]

As a consequence we find that for \(k = 0, 1, \ldots\)
\[
\frac{1}{n} |\hat{\eta}^{k} - \eta| 1_B 1_C \leq 2 \left(\frac{1}{4}\right)^k + \frac{8}{3n} \sum_{i=1}^{n} 1_{A_i^c}.
\]

Observe that for \(k \geq \lfloor 3 \log n \rfloor\), we have \(k \geq 2 \frac{\log n}{\log 4}\) and
\[
\left(\frac{1}{4}\right)^k \leq \frac{1}{n^2}.
\]

Hence for \(k \geq \lfloor 3 \log n \rfloor\),
\[
\frac{1}{n} |\hat{\eta}^{k} - \eta| 1_B 1_C \leq \frac{2}{n^2} + \frac{8}{3n} \sum_{i=1}^{n} 1_{A_i^c}.
\]
Observe that if $\frac{1}{n}\sum_{i=1}^{n} 1_{A_i^c} = 0$ then $\frac{1}{n} |\tilde{\eta}_k - \eta| 1_{B_1C} = 0$. Else, $\frac{1}{n}\sum_{i=1}^{n} 1_{A_i^c} \geq \frac{1}{n}$. This leads to
$$\frac{1}{n} |\tilde{\eta}_k - \eta| 1_{B_1C} \leq \frac{14}{3n} \sum_{i=1}^{n} 1_{A_i^c}.$$  

Finally we get for $k \geq \lfloor 3 \log n \rfloor$,
$$\frac{1}{n} E (|\tilde{\eta}_k - \eta|) \leq \frac{14}{3n} \sum_{i=1}^{n} P(A_i^c) + P(B^c) + P(C^c) \leq \left( \frac{14}{3} + \frac{4r_n^2}{C'} \right) \frac{1}{n} \sum_{i=1}^{n} P(A_i^c) + P(B^c).$$

The term $P(B^c)$ is upper bounded exactly as in the previous proof and we have
$$P(B^c) = o(\Phi^c(r_n)).$$

For the other term observe that
$$P(A_i^c) = G_\sigma \left( \frac{C''}{r_n} + \epsilon_n, \|\theta\|^2 \right),$$
for some $C'' > 0$ and $\epsilon_n = o(1)$. That concludes the proof.

**B.8 Proof of Theorem 8**

Combining Theorem 6 and Theorem 7, it is enough to prove that
$$r_n^2 \sup_{\|\theta\| \geq \Delta} G_\sigma \left( \epsilon_n + \frac{C'}{r_n}, \theta \right) \leq \Phi^c \left( \epsilon_n \left( 1 - \epsilon' - \frac{C'' \log r_n}{r_n} \right) \right) + \epsilon'_n \Phi^c(r_n),$$
for some $\epsilon'_n = o(1)$ and $C'' > 0$. Recall that
$$G_\sigma \left( \epsilon_n + \frac{C'}{r_n}, \theta \right) = P \left( \left( \theta + \xi_1 \right)^\top \left( \theta + \frac{\xi_2}{\sqrt{n-1}} \right) \leq \left( \epsilon_n + \frac{C'}{r_n} \right) \|\theta\|^2 \right),$$
where $\xi_1, \xi_2$ are two independent Gaussian random vector with i.i.d. standard entries and $\theta$ and independent Gaussian prior. Moreover, using independence, we have
$$G_\sigma \left( \epsilon_n + \frac{C'}{r_n}, \theta \right) = P \left( \epsilon \sqrt{\|\theta\|^2 + \frac{\|\xi_2\|^2}{n-1}} + 2 \sqrt{n-1} \theta^\top \xi_2 \geq \|\theta\|^2 \left( 1 - \epsilon_n - \frac{C'}{r_n} \right) + \frac{1}{\sqrt{n-1}} \theta^\top \xi_2 \right),$$
where $\epsilon$ is a standard Gaussian random variable. Set the random event
$$\mathcal{A} = \left\{ \frac{\|\xi_2\|^2}{n-1} \leq \frac{p}{n-1} + \zeta_n \|\theta\|^2 \right\} \cap \left\{ \theta^\top \xi_2 \leq \sqrt{n-1} \beta_n \|\theta\|^2 \right\},$$
where $\zeta_n$ and $\beta_n$ are positive sequences. It is easy to check that
$$P(\mathcal{A}^c) \leq e^{-c\|\theta\|^4 n^2 \xi_2^2 / p} + e^{-c\beta_n^2 n\|\theta\|^2} + e^{-c\zeta_n n\|\theta\|^2},$$
for some $c > 0$. Hence using the event $\mathcal{A}$, we get
$$G_\sigma \left( \epsilon_n + \frac{C'}{r_n}, \theta \right) \leq P \left( \epsilon \sqrt{\|\theta\|^2 (1 + \zeta_n + 2\beta_n) + \frac{p}{n-1}} \geq \|\theta\|^2 \left( 1 - \epsilon_n - \frac{C'}{r_n} - \beta_n \right) \right) + P(\mathcal{A}^c).$$

25
By choosing $\beta_n = \zeta_n = \sqrt{\log n / n}$, we get that
\[ P(A^c) \leq e^{-c\sqrt{\log n}}. \]
The last fact is due to the condition $r_n \geq C$ for some $C > 0$. Hence
\[ P(A^c) = o(\Phi_c(r_n)). \]
Moreover and since $\zeta_n$ and $\beta_n$ are vanishing sequences as $n \to \infty$, we get that
\[ P\left(\epsilon \sqrt{\|\theta\|^2(1 + \zeta_n + 2\beta_n) + \frac{p}{n-1}} \geq \|\theta\|^2 \left(1 - \epsilon_n - \frac{C'}{r_n} - \beta_n\right)\right) = \Phi_c\left(\frac{\|\theta\|^2}{\sqrt{\|\theta\|^2 + \frac{p}{n}}} \left(1 - \frac{C'}{r_n} - \epsilon_n\right)\right), \]
for some $\epsilon'_n = o(1)$. We conclude using the fact that $x \to x \sqrt{x + \epsilon}$ is non-decreasing on $\mathbb{R}^+$ and the fact that for $C < x < y$, we have $x^2 \Phi_c(y) \leq c_1 \Phi_c(y - c_2 \log x)$, for some $c_1, c_2 > 0$.

**B.9 Proof of Proposition 9**

Set $n$ large enough. According to Theorem 5, we have
\[ \Psi_\Delta \geq \frac{1}{2} \Phi_c(2r_n). \] (33)
For the upper bound. If $r_n$ is larger than $2C$, then using Theorem 8, we get
\[ \Psi_\Delta \leq C' \Phi_c\left(\frac{r_n}{4}\right), \] (34)
for some $C' > 0$. Observe that for $r_n \leq 2C$, we have
\[ c_1 \leq \Phi_c(r_n), \]
for some $c_1 > 0$. Hence, for $r_n \leq 2C$, we get
\[ \Psi_\Delta \leq \frac{\Phi_c(r_n)}{c_1}. \] (35)
We conclude combining (34), (33) and (35).

**B.10 Proof of Theorem 14**

- **Necessary conditions:**
  According to Theorem 5, we have
  \[ \Psi_\Delta \geq (1 - \epsilon_n)\Phi_c(r_n(1 + \epsilon_n)), \]
  for some $\epsilon_n = o(1)$. If for some $\epsilon > 0$,
  \[ \liminf_{n \to \infty} n\Phi_c(r_n(1 + \epsilon)) > 0, \]
  for some $\epsilon_n = o(1)$. If for some $\epsilon > 0$,
then using the monotonicity of $\Phi^c$, we conclude that exact recovery is impossible. For almost full recovery, assume that $\Phi^c(r_n)$ does not converge to 0, and that almost full recovery is possible. Then using continuity and monotonicity of $\Phi^c$, we get that $r_n(1 + \epsilon_n) \to \infty$. Hence $r_n \to \infty$ and $\Phi^c(r_n) \to 0$ which is absurd. That proves that the condition $\Phi^c(r_n) \to 0$ is necessary to achieve almost full recovery.

• Sufficient conditions:
According to Theorem 8, we have that, under the condition $r_n > C$ for some $C > 0$, the estimator $\hat{\eta}_k$ defined in the Theorem satisfies

$$\sup_{(\theta, \eta) \in \Omega_n} \frac{1}{n} \mathbb{E}_{(\theta, \eta)} r(\hat{\eta}_k, \eta) \leq C' \Phi^c\left(r_n \left(1 - \epsilon_n - \frac{C' \log r_n}{r_n}\right)\right),$$

for some sequence $\epsilon_n$ such that $\epsilon_n = o(1)$. If $\Phi^c(r_n) \to 0$, then $r_n \to \infty$. Hence for any $\epsilon > 0$, $r_n(1 - \epsilon) \to \infty$. It follows that $r_n \left(1 - \epsilon_n - \frac{C' \log r_n}{r_n}\right) \to \infty$. We conclude that almost full recovery is possible under the condition $\Phi^c(r_n) \to 0$, and $\hat{\eta}_k$ achieves almost full recovery in that case.

For exact recovery, observe that, if $n \Phi^c(r_n(1 - \epsilon)) \to 0$, for some $\epsilon > 0$, then $r_n \to \infty$. It follows that for $n$ large enough

$$r_n \left(1 - \epsilon_n - \frac{C' \log r_n}{r_n}\right) \geq r_n(1 - \epsilon).$$

We conclude by taking the limit that $\hat{\eta}_k$ achieves exact recovery in that case, and that exact recovery is possible.

**B.11 Proof of Theorem 15 and 11**

By inverting the function $x \to \frac{x}{\sqrt{x + \frac{p}{nA}}}$, we observe that for any $A > 0$,

$$r_n^2 \geq A \iff \Delta_n^2 \geq A \frac{1 + \sqrt{1 + \frac{4p}{nA}}}{2}.$$ 

Using Theorem 14 and the Gaussian tail function, we get immediately the results for both almost full recovery and exact recovery.
Appendix C. Technical Lemmas

Lemma 16. Let $A$ be a matrix in $\mathbb{R}^{n \times n}$. Then
\[ \|H(A)\|_{op} \leq 2\|A\|_{op}. \]

Proof. From the linearity of $H$, we have that
\[ \|H(A)\|_{op} \leq \|A\|_{op} + \|\text{diag}(A)\|_{op}, \]
where
\[ \|\text{diag}(A)\|_{op} = \max_i |A_{ii}| \leq \|A\|_{op}. \]

Lemma 17. For any random matrix $W$ with independent columns, we have
\[ \|H(W^\top W)\|_{op} \leq 2\left\| W^\top W - \mathbb{E}(W^\top W) \right\|_{op}. \]

Proof. Since $\mathbb{E}(W^\top W)$ is a diagonal matrix, it follows that
\[ H(W^\top W) = H \left( W^\top W - \mathbb{E}(W^\top W) \right). \]
The result follows from Lemma 16.

Lemma 18. Let $u \in \mathbb{S}^{p-1}$ and $v \in \mathbb{S}^{n-1}$, and $W \in \mathbb{R}^{p \times n}$ a matrix with i.i.d. centered Gaussian entries of variance at most $\sigma^2$. Then, for some $c, C > 0$
\[ \forall t \geq 2\sigma, \quad \mathbb{P} \left( \left\| \frac{1}{\sqrt{n}} W^\top uv^\top \right\|_{op} \geq t \right) \leq e^{-cnt/\sigma^2}, \]
and
\[ \mathbb{E} \left( \left\| \frac{1}{\sqrt{n}} W^\top uv^\top \right\|_{op}^2 \right) \leq C\sigma^2. \]

Proof. We can easily check that
\[ \left\| \frac{1}{\sqrt{n}} W^\top uv^\top \right\|_{op} \leq \frac{1}{\sqrt{n}} \|W^\top u\|_2. \]
Since $\|u\|_2 = 1$, we have that $W^\top u$ is Gaussian with mean 0 and covariance matrix $\sigma^2 I_n$. We conclude using a tail inequality for quadratic forms of sub-Gaussian random variables using the fact that $t \geq 2\sigma$, see, e.g., Hsu et al. (2012). The inequality in expectation is immediate by integration of the tail function.

Lemma 19. Let $W \in \mathbb{R}^{p \times n}$ be a matrix with i.i.d. centered Gaussian entries of variance at most $\sigma^2$. For some $c, C, C' > 0$ we have
\[ \forall t \geq C\sigma^2 \left( 1 + \sqrt{\frac{p}{n}} \right), \quad \mathbb{P} \left( \frac{1}{n} \|H(W^\top W)\|_{op} \geq t \right) \leq e^{-cnt/\sigma^2 \left( 1 + \frac{p}{n} \right)}, \]
and
\[ \mathbb{E} \left( \frac{1}{n} \|H(W^\top W)\|_{op}^2 \right) \leq C'\sigma^4 (1 + p/n). \]
Proof. Using Lemma 17, we get
\[ \mathbf{P}\left(\frac{1}{n}\|H(W^\top W)\|_{op} \geq t\right) \leq \mathbf{P}\left(\frac{1}{n}\|W^\top W - \mathbf{E}(W^\top W)\|_{op} \geq t/2\right). \]

Now based on Theorem 4.6.1 in Vershynin (2018), we get moreover that
\[ \mathbf{P}\left(\frac{1}{n}\|H(W^\top W)\|_{op} \geq t\sigma^2\right) \leq 9^n2e^{-cnt(1\wedge tn/p)}, \]
for some \( c > 0 \). For \( t \geq C(1 \vee \sqrt{p/n})\sigma^2 \) with \( C \) large enough, we get \( ct(1 \wedge tn/p\sigma^2) \geq 4\sigma^2\log 9 \), hence
\[ \mathbf{P}\left(\frac{1}{n}\|H(W^\top W)\|_{op} \geq t\right) \leq e^{-c'tn/\sigma^2(1\wedge tn/p\sigma^2)}, \]
for some \( c' > 0 \). The result in expectation is immediate by integration.

Lemma 20. For any \( x \in \{-1,1\}^n \) and \( y \in \mathbb{R}^n \), we have
\[ \frac{1}{n}|x - \text{sign}(y)| \leq 2\left\| \frac{x}{\sqrt{n}} - y \right\|^2. \]

Proof. It is enough to observe that if \( x_i \in \{-1,1\} \), then
\[ |x_i - \text{sign}(y_i)| = 21(x_i \neq \text{sign}(y_i)) \leq 2|x_i - \sqrt{n}y_i|^2. \]