A NOTE ON GENERAL QUADRATIC GROUPS

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Abstract: We deduce an analogue of Quillen–Suslin’s local-global principle for the transvection subgroups of the general quadratic (Bak’s unitary) groups. As an application we revisit the result of Bak–Petrov–Tang on injective stabilization for the $K_1$-functor of the general quadratic groups.

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1. Introduction

In this article we are going to discuss three major problems in algebraic K-theory studied rigorously during 1950’s to 1970’s, viz. stabilization problems for the $K_1$-functor, Quillen–Suslin’s local-global principle, and Bak’s unitary groups over form rings.

Let us begin with the stabilization problem for the $K_1$-functor of the general linear groups initiated by Bass–Milnor–Serre during mid 60’s. For details, see [6]. For a commutative ring $R$ with identity, let $GL_n(R)$ be the general linear group, and $E_n(R)$ the subgroup generated by elementary matrices. They had studied the following sequence of maps

$$
\frac{GL_n(R)}{E_n(R)} \rightarrow \frac{GL_{n+1}(R)}{E_{n+1}(R)} \rightarrow \frac{GL_{n+2}(R)}{E_{n+2}(R)} \rightarrow \cdots = K_1(R)
$$

and posed the problem that when does the natural map stabilize? That time it was not known that the elementary subgroup is a normal subgroup in the general linear group for size $n \geq 3$. They showed that the above map is surjective for $n \geq d + 1$, and injective for $n \geq d + 3$, where $d$ is the Krull dimension of $R$, and conjectured that it must be injective for $n \geq d + 2$. In 1969, L.N. Vaserstein proved the conjecture for an associative ring with identity, where $d$ is the stable rank of the ring (cf. [25]). In 1974, he generalized his result for projective modules. After that, in [26], he had studied the orthogonal and the unitary $K_1$-functors, and deduced analogue results for those groups, and showed that for the non-linear cases above mentioned natural map is surjective for $n \geq 2d + 2$ and injectivity for $n \geq 2d + 4$.

It was a period when people were working on Serre’s problem on projective modules, which states that the finitely generated projective modules over polynomial rings over a field are free. If the number of variables $n$ in the polynomial ring is 1, then the affirmative answer follows from a classical result in commutative
algebra. The first non-trivial case \( n = 2 \) was proved affirmatively by C.S. Seshadri in 1958. For details see [19]. After 16 years, in 1974, Murthy–Swan–Towber (then Roitman, Vaserstein et al., for details cf. [19]) proved the result for algebraically closed fields. The general case was proved by Daniel Quillen and Andrei Suslin independently in 1976. Suslin proved that for an associative ring \( A \), which is finite over its center, the elementary subgroup \( \text{E}_n(A) \) is a normal subgroup of \( \text{GL}_n(A) \) for \( n \geq 3 \). It was first appeared in a paper by M.S. Tulenbaev, for details see [24]. Quillen, in his proof, introduced a localization technique which was one of the main ingredient for the proof of Serre’s problem (now widely known as Quillen–Suslin Theorem). Shortly after the original proof, motivated by Quillen’s idea, Suslin introduced the following matrix theoretic version of the local-global principle. We are stating the theorem for a commutative ring with identity, where \( \text{Max}(R) \) is the maximum spectrum of the ring \( R \). The statement is true for almost commutative rings, i.e. rings which are finite over its center.

**Suslin’s Local-Global Principle:** Let \( R \) be a commutative ring with identity and \( \alpha(X) \in \text{GL}_n(R[X]) \) with \( \alpha(0) = I_n \). If \( \alpha_m(X) \in \text{E}_n(R_m[X]) \) for every maximal ideal \( m \in \text{Max}(R) \), then \( \alpha(X) \in \text{E}_n(R[X]) \).

In [12], we have deduced an analogue of the above statement for the transvection subgroups of the full automorphism groups of global rank at least 1 in the linear case, and at least 2 for the symplectic and orthogonal cases. In this article we generalize the statement for the general quadratic (Bak’s unitary) groups. The main aim of this article is to show that using this result one can generalize theorems on free modules to the classical modules. We shall discuss such technique for the stabilization problems for the \( K_1 \)-functor. In similar manner one can also generalize results for unstable \( K_1 \)-groups (cf. [5], [12]), local-global principle for the commutator subgroups (cf. [10]), and may be for the results on congruence subgroup problems, etc.

Let us briefly discuss the historical aspects of the general quadratic group defined with the concept of *form ring* introduced by Antony Bak in his Ph.D. thesis (cf. [1]) around 1967. For details see [1] and [15].

We know that a quadratic form on an \( R \)-module \( M \) is a map \( q : M \to R \) such that

1. \( q(ax) = a^2 q(x), \quad a \in R, \quad x \in M, \)
2. \( B_q : M \times M \to A \) defined by \( B_q(x, y) = q(x + y) - q(x) - q(y) \) is bilinear (symmetric).

The map \( B_q \) is called the *bilinear form associated to* \( q \). If \( 2 \in R^* \), then for all \( x \in M \) we get \( q(x) = \frac{1}{2} B_q(x, x) \). The pair \( (M, q) \) is called the quadratic \( R \)-module.

Suppose \( B : V \times V \to K \) is a bilinear form on a vector space \( V \) over a field \( K \). We say \( B \) is

1. symmetric if \( B(u, v) = B(v, u) \),
2. anti-symmetric if \( B(u, v) = -B(v, u) \), and
3. symplectic (alternating) if \( B(u, u) = 0 \) for all \( u \in V \).
Now, symplectic ⇒ anti-symmetric, and if char($K$) ≠ 2, then symplectic ⇔ anti-symmetric. On the other hand, if char($K$) = 2, then anti-symmetric ⇔ symmetric. Also, if char($K$) ≠ 2, then $q$ is a quadratic form if and only if it is symmetric bilinear form $B(u,v) = q(u + v) - q(u) - q(v)$, as we get $\frac{1}{2} B(u,u) = q(u)$. The study of Dickson–Dieudonne shows that the cases char($K$) = 2 and char($K$) ≠ 2 differs even at the level of definition. During 1950’s–1970’s it was a problem that how to generalize classical groups by constructing theory which does not depend on the invertibility of 2. In 1967, A. Bak came up with the following:

First of all, a classical group should be considered as preserving a pair of forms $(B, q)$. Secondly, a quadratic form $q$ should take its value not in the ring $R$, but in its factor group $R/\Lambda$, where $\Lambda$ is certain additive subgroup of $R$, the so called form parameter.

In his seminal work “K-Theory of Forms”, Bak generalized many results for the general quadratic groups which were already known for the general linear, symplectic and orthogonal groups. In 2003, Bak–Petrov–Tang proved that the injective stabilization bound for the $K_1$-functor of the general quadratic groups over form rings is also $2d + 4$, like traditional classical groups of even size, (cf. [4]). Later Petrov in his Ph.D Thesis (cf. [21]), and Tang (unpublished) independently studied the result for the classical modules. But none of those works are published. In this article we revisit their results for the general quadratic modules, as an application of the local-global principle for the transvection subgroups. But, even though the local-global principle holds for the module finite rings, our method is applicable only for the commutative rings. On the other hand, it is possible to apply this method to deduce analogue results for any other kind of classical groups, in particular, for the Bak–Petrov’s groups (cf. [20]), where we have analogue local-global principle. In this connection, we mention that for the structure of unstable $K_1$-groups of the general quadratic groups, we refer [12] and [14]. In both the papers it has been shown that the unstable $K_1$-groups are nilpotent-by-abelian for $n \geq 6$, which generalizes the result of Bak for the general linear groups in [2]. Following our method one can generalize that result for the module case. i.e. it can be shown that the unstable $K_1$-groups for the (extended) general quadratic modules with rank $\geq 6$ are nilpotent-by-abelian.

2. Preliminaries

In this section we shall recall necessary definitions.

Form Rings: Let $- : R \to R$, defined by $a \mapsto \overline{a}$, be an involution on $R$, and $\lambda \in C(R) =$ center of $R$ such that $\lambda \overline{\lambda} = 1$. We define two additive subgroups of $R$

$$
\Lambda_{\text{max}} = \{ a \in R | a = -\lambda \overline{a} \} \quad \& \quad \Lambda_{\text{min}} = \{ a - \lambda \overline{a} | a \in R \}.
$$

One checks that $\Lambda_{\text{max}}$ and $\Lambda_{\text{min}}$ are closed under the conjugation operation $a \mapsto \overline{\lambda} x$ for any $x \in R$. A $\lambda$-form parameter on $R$ is an additive subgroup $\Lambda$ of $R$ such that $\Lambda_{\text{min}} \subseteq \Lambda \subseteq \Lambda_{\text{max}}$, and $\overline{\lambda} x \subseteq \Lambda$ for all $x \in R$. A pair $(R, \Lambda)$ is called a form ring.
General Quadratic Groups:

Let $V$ be a right $R$-module. By $\text{GL}(V)$ we denote the group of all $R$-linear automorphisms of $V$. Throughout the paper we shall consider $V$ as a projective $R$-module. To define the general quadratic module we need following definitions:

**Definition 2.1.** A sesquilinear form is a map $f : V \times V \to R$ such that $f(ua, vb) = \overline{\lambda}f(u, v)b$ for all $u, v \in V$ and $a, b \in R$.

**Definition 2.2.** A $\Lambda$-quadratic form on $V$ is a map $q : V \to R/\Lambda$ such that $q(v) = f(v, v) + \Lambda$.

**Definition 2.3.** An associated $\lambda$-Hermitian form is a map $h : V \times V \to R$ with the property $h(u, v) = f(u, v) + \lambda f(v, u)$.

**Definition 2.4.** A quadratic module over $(R, \Lambda)$ is a triple $(V, h, q)$.

The $\lambda$-Hermitian form $h : V \times V \to R$ induces a map $V \to \text{Hom}_R(V, R)$, given by $v \mapsto h(v, -)$. We say that $V$ is non-singular if $V$ is a projective $R$-module and the Hermitian form $h$ is non-singular.

**Definition 2.5.** A morphism of quadratic modules over $(R, \Lambda)$ is a map $\mu : (V, q, h) \to (V', q', h')$ such that $\mu : V \to V'$ is $R$-linear, $\mu(\lambda) = \lambda'$, and $\mu(\Lambda) = \Lambda'$.

**General Quadratic (Bak’s Unitary) Groups:** Let $(V, q, h)$ be a non-singular quadratic module over $(R, \Lambda)$. We define the general quadratic group as follows:

$$\text{GQ}(V, q, h) = \{ \alpha \in \text{GL}(V) \mid h(\alpha u, \alpha v) = h(u, v), \; q(\alpha u) = q(v) \}.$$  

i.e. the group consisting of all automorphisms which fixes the $\lambda$-Hermitian form and the $\Lambda$-quadratic form. One observes that the traditional classical groups are the special cases of Bak’s unitary groups. The central concept is “form parameter” due to Bak. Earlier version is due to K. McCrimmon which plays an important role in his classification theory of Jordan Algebras. He defined for the wider class of alternative rings (not just associative rings), but for associative rings it is a special case of Bak’s concept. For details, see N. Jacobson, Lectures on Quadratic Jordan Algebras, TIFR, Bombay 1969, (cf. [17]) and [16].

**Free Case:** Let $V$ be a non-singular free right $R$-module of rank $2n$ with ordered basis $e_1, e_2, \ldots, e_{-n}, e_{-n+1}, \ldots, e_{-2}, e_{-1}$. Consider the sesquilinear form $f : V \times V \to R$ defined by $f(u, v) = \overline{\lambda_1} v_{-1} + \cdots + \overline{\lambda_n} v_{-n}$. Let $h$ be the $\lambda$-Hermitian form, and $q$ be the $\Lambda$-quadratic form defined by $f$. We get

$$h(u, v) = \overline{\lambda_1} v_{-1} + \cdots + \overline{\lambda_n} v_{-n} + \lambda v_n + \cdots + \lambda v_1,$$

$$q(u) = \Lambda + \overline{\lambda_1} u_{-1} + \cdots + \overline{\lambda_n} u_{-n}.$$  

Using the above basis we can identify $\text{GQ}(V, h, q)$ with a subgroup of $\text{GL}_{2n}(R, \Lambda)$ of rank $2n$, say $\text{GQ}_{2n}(R, \Lambda)$. Hence

$$\text{GQ}_{2n}(R, \Lambda) = \{ \sigma \in \text{GL}_{2n}(R, \Lambda) \mid \overline{\lambda_n} \psi_n \sigma = \psi_n \},$$

where

$$\psi_n = \begin{pmatrix} 0 & \lambda I_n \\ I_n & 0 \end{pmatrix}.$$
Unitary Transvections $\text{EQ}(V)$: Let $(V,h,q)$ be a quadratic module over $(R,\Lambda)$. Let $u,v \in V$ and $a \in R$ be such that $f(u,u) \in \Lambda$, $h(u,v) = 0$ and $f(v,v) = a$ modulo $\Lambda$. Then we quote the definition of unitary transvection from ([§5.1, 8]). We define $\sigma = \sigma_{u,a,v} : M \to M$ defined by

$$\sigma(x) = x + h(v,x) - v\overline{\lambda}h(u,x) - u\overline{\lambda}ah(u,x).$$

Unitary Transvections $\text{EQ}(M)$ in $M = V \perp \mathbb{H}(P)$: Let $P$ be a projective $R$-module of rank at least one, and $\mathbb{H}(P)$ the hyperbolic space. Let $x = (v,p,q) \in M$ for some $v \in V$, $p \in P$, and $q \in P^*$. For any element $p_0 \in P$, $w_0 \in V$ and $a_0 \in A$ such that $a_0 = f(w_0,w_0)$ modulo $\Lambda$, the above conditions hold, and hence we can define $\sigma_{p_0,a_0,w_0}$ as follows:

$$\sigma_{p_0,a_0,w_0}(x) = x + p_0h(w_0,x) - w_0\overline{\lambda}h(p_0,x) - p_0\lambda a_0 h(p_0,x).$$

Elementary Quadratic Matrices: (Free Case)

Let $\rho$ be the permutation, defined by $\rho(i) = n + i$ for $i = 1, \ldots, n$. Let $e_i$ denote the column vector with 1 in the $i$-th position and 0’s elsewhere. Let $e_{ij}$ be the matrix with 1 in the $ij$-th position and 0’s elsewhere. For $a \in R$, and $1 \leq i, j \leq n$, we define

$$q\varepsilon_{ij}(a) = I_{2n} + ae_{ij} - a\overline{\lambda}e_{\rho(j)\rho(i)} \quad \text{for} \ i \neq j,$$

$$q\rho_{ij}(a) = \begin{cases} I_{2n} + ae_{\rho(j)} - \lambda \overline{\lambda} e_{\rho(i)} & \text{for} \ i \neq j, \\ I_{2n} + ae_{\rho(i)} & \text{for} \ i = j, \end{cases}$$

$$q\lambda_{ij}(a) = \begin{cases} I_{2n} + ae_{\rho(i)j} - \lambda e_{\rho(j)i} & \text{for} \ i \neq j, \\ I_{2n} + ae_{\rho(i)j} & \text{for} \ i = j. \end{cases}$$

(Note that for the second and third type of elementary matrices, if $i = j$, then we get $a = -\lambda a$, and hence it forces that $a \in \Lambda_{\max}(R)$. One checks that these above matrices belong to $GQ_{2n}(R,\Lambda)$; cf. [1].)

$n$-th Elementary Quadratic Group $\text{EQ}_{2n}(R,\Lambda)$: The subgroup generated by $q\varepsilon_{ij}(a)$, $q\rho_{ij}(a)$ and $q\lambda_{ij}(a)$, for $a \in R$ and $1 \leq i, j \leq n$.

Stabilization map: There are standard embeddings:

$$GQ_{2n}(R,\Lambda) \longrightarrow GQ_{2n+2}(R,\Lambda)$$

given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a & 0 & 0 & b \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ c & 0 & 0 & d \end{pmatrix} \text{ or } \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a & b & 0 \\ 0 & c & d & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$ 

Hence we have

$$GQ(R,\Lambda) = \lim_{\longrightarrow} GQ_{2n}(R,\Lambda).$$

It is clear that the stabilization map takes generators of $\text{EQ}_{2n}(R,\Lambda)$ to the generators of $\text{EQ}_{2n+2}(R,\Lambda)$.
Commutator Relations: There are standard formulas for the commutators between quadratic elementary matrices. For details, we refer [1] (Lemma 3.16). In later sections there are repeated use of those relations.

Remark: If $M = R^{2n}$, then under the choice of the above basis we get $\text{EQ}(M) = \text{EQ}_{2n}(R, \Lambda)$. (cf. proof of Lemma 2.20 in [5]). Hence we have

$$\text{EQ}(R, \Lambda) = \lim_{\to} \text{EQ}_{2n}(R, \Lambda)$$

Using analogue of the Whitehead Lemma for the general quadratic groups (cf. [1]) due to Bak, one gets

$$[\text{GQ}(R, \Lambda), \text{GQ}(R, \Lambda)] = [\text{EQ}(R, \Lambda), \text{EQ}(R, \Lambda)] = \text{EQ}(R, \Lambda).$$

Hence we define the Whitehead group of the general quadratic group

$$K_1 \text{GQ} = \text{GQ}(R, \Lambda)/\text{EQ}(R, \Lambda).$$

And, the Whitehead group at the level $m$

$$K_{1,m} \text{GQ} = \text{GQ}_m(R, \Lambda)/\text{EQ}_m(R, \Lambda),$$

where $m = 2n$ in the non-linear cases. For classical modules we replace $(R, \Lambda)$ by $M = V \oplus H(P)$.

3. Suslin’s Local-Global Principle for Transvection Subgroups

In this section we prove analogue of Quillen–Suslin’s local-global principle for the transvection subgroups of the general quadratic groups. We start with the following splitting lemma.

**Lemma 3.1.** (cf. pg. 43-44, Lemma 3.16, [1]) Let $q_{ij}$ denote any one of the elementary generator of $q_{\varepsilon_{ij}}$, $q_{ij}$ and $q_{rij}$ in $\text{GQ}_{2n}(R, \Lambda)$. Then, for all $x, y \in R$,

$$q_{ij}(x + y) = q_{ij}(x)q_{ij}(y).$$

We shall need following standard fact.

**Lemma 3.2.** Let $G$ be a group, and $a_1, b_1 \in G$, for $i = 1, \ldots, n$. Then for

$$r_i = \prod_{j=1}^{i} a_j, \text{ we have } \prod_{i=1}^{n} a_i b_i = \prod_{i=1}^{n} r_i b_i r_i^{-1} \prod_{i=1}^{n} a_i.$$

**Notation 3.3.** By $\text{GQ}_{2n}(R[X], \Lambda[X], (X))$ we shall mean the group of all invertible matrices in $\text{GQ}_{2n}(R[X], \Lambda[X])$ which are $I_{2n}$ modulo $(X)$. Let $\Lambda[X]$ denote the $\lambda$-form parameter on $R[X]$ induced from $(R, \Lambda)$, i.e., $\lambda$-form parameter on $R[X]$ generated by $\Lambda$, i.e., the smallest form parameter on $R[X]$ containing $\Lambda$. Let $\Lambda_s$ denote the $\lambda$-form parameter on $R_s$ induced from $(R, \Lambda)$.

**Lemma 3.4.** The group $\text{GQ}_{2n}(R[X], \Lambda[X], (X)) \cap \text{EQ}_{2n}(R[X], \Lambda[X])$ is generated by the elements of the types $\varepsilon q_{ij}(f(X)) \varepsilon^{-1}$, where $\varepsilon \in \text{EQ}_{2n}(R, \Lambda)$, $f(X) \in R[X]$ and $q_{ij}(f(X))$ are congruent to $I_{2n}$ modulo $(X)$. 
Proof. Let $\alpha(X) \in EQ_{2n}(R[X], \Lambda[X])$ be such that $\alpha(X) = I_{2n}$ modulo $(X)$. Then we can write $\alpha(X)$ as a product of elements of the form $q_{ij}(f(X))$, where $f(X)$ is a polynomial in $R[X]$. We write each $f(X)$ as a sum of a constant term and a polynomial which is identity modulo $(X)$. Hence by using the splitting property described in Lemma 3.1, each elementary generator $q_{ij}(f(X))$ can be written as a product of two such elementary generators with the left one defined on $R$ and the right one defined on $R[X]$ which is congruent to $I_{2n}$ modulo $(X)$.

Therefore, we can write $\alpha(X)$ as a product of elementary generators of the form

$$q_{ij}(f(0))q_{ij}(Xg(X))$$

for some $g(X) \in R[X]$ with $g(0) \in R$. Now the result follows by using the identity described in Lemma 3.2. $\square$

By repeated application of commutator formulas stated in ([1], pg. 43-44, Lemma 3.16) one gets the following lemma.

**Lemma 3.5.** Suppose $\vartheta$ is an elementary generator of the general quadratic group $GQ_{2n}(R[X], \Lambda[X])$. Let $\vartheta$ be congruent to identity modulo $(X^{2m})$, for $m > 0$. Then, if we conjugate $\vartheta$ with an elementary generator of the general quadratic group $GQ_{2n}(R, \Lambda)$, we get the final matrix as a product of elementary generators of the general quadratic group $GQ_{2n}(R[X], \Lambda[X])$ each of which is congruent to identity modulo $(X^{m})$.

**Corollary 3.6.** In Lemma 3.5 we can take $\vartheta$ as a product of elementary generators of the general quadratic group $GQ_{2n}(R[X], \Lambda[X])$.

Let us recall following useful and well known fact. We use this fact for the proof of dilatation lemma.

**Lemma 3.7.** (cf., [3], Lemma 5.1) Let $A$ be Noetherian ring and $s \in A$. Let $s \in A$ and $s \neq 0$. Then there exists a natural number $k$ such that the homomorphism $G(A, s^{k}A, s^{k}\Lambda) \to G(A_{s}, \Lambda_{s})$ (induced by the localization homomorphism $A \to A_{s}$) is injective.

We recall that one has any module finite ring $R$ as a direct limit of its finitely generated subrings. Also, $G(R, \Lambda) = \lim_{\rightarrow} G(R_{i}, \Lambda_{i})$, where the limit is taken over all finitely generated subring of $R$. Thus, one may assume that $C(R)$ is Noetherian. For the rest of this section we shall consider module finite rings $(R, \Lambda)$ with identity.

In [12], we have given a proof of the dilation lemma for the general Hermitian groups. For the general quadratic modules, the proof is similar, in fact easier. We are giving the sketch of the proof, as it is not clearly written any available literature. It is almost done in [21] for the odd unitary groups which contains the general quadratic groups. For the hyperbolic unitary groups it is done in [3].

**Lemma 3.8. (Dilation Lemma: Free Case)** Let $\alpha(X) \in GQ_{2n}(R[X], \Lambda[X])$, with $\alpha(0) = I_{2n}$. If $\alpha_{s}(X) \in EQ_{2n}(R_{s}[X], \Lambda_{s}[X])$ for some non-nilpotent $s \in R$, then $\alpha(bX) \in EQ_{2n}(R[X], \Lambda[X])$ for $b \in (s^{l})C(R)$, $l \gg 0$.

(Actually, we mean there exists some $\beta(X) \in EQ_{2n}(R[X], \Lambda[X])$ such that $\beta(0) = I_{2n}$ and $\beta_{s}(X) = \alpha_{s}(bX)$.)
Proof. Given that \( \alpha_s(X) \in \text{EQ}_2(R_s[X], \Lambda_s[X]) \). Since \( \alpha(0) = I_{2n} \), using Lemma 3.9 we can write \( \alpha_s(X) \) as a product of the matrices of the form \( \varepsilon \varrho_{ij}(h(X))^{-1} \), where \( \varepsilon \in \text{EQ}_2(R_s, \Lambda_s) \), \( h(X) \in R_s[X] \) with \( \varrho_{ij}(h(X)) \) congruent to \( I_{2n} \) modulo \( (X) \). Applying the homomorphism \( X \mapsto XT^d \), for \( d \gg 0 \), from the polynomial ring \( R[X] \) to the polynomial ring \( R[X, T] \), we look on \( \alpha(XT^d) \).

Note that \( R_s[X, T] \cong (R_s[X])[T] \). As \( C(R) \) is Noetherian, it follows from Lemma 3.7 and Corollary 3.6 that over the ring \( (R_s[X])[T] \) we can write \( \alpha_s(XT^d) \) as a product of elementary generators of the general quadratic group such that each of those elementary generator is congruent to identity modulo \( (T) \). Let \( l \) be the maximum of the powers occurring in the denominators of those elementary generators. Again, as \( C(R) \) is Noetherian, by applying the homomorphism \( T \mapsto s^mT \) for \( m \geq l \) it follows from Lemma 3.6 that over the ring \( R[X, T] \) we can write \( \alpha_s(XT^d) \) as a product of elementary generators of general quadratic group such that each of those elementary generator is congruent to identity modulo \( (T) \). i.e. there exists some \( \beta(X, T) \in \text{EQ}_2(R[X, T], \Lambda[X, T]) \) such that \( \beta(0, 0) = I_{2n} \) and \( \beta_s(X, T) = \alpha_s(bXT^d) \), for some \( b \in (s^l)C(R) \). Finally, the result follows by putting \( T = 1 \).

Lemma 3.9. (Dilation Lemma: Module Case) Assume \( M = V \oplus \mathbb{H}(R) \), where \( V \) is a right \( R \)-module, and \( \mathbb{H}(R) \) is the usual hyperbolic space. Let rank of \( M \) is \( 2n \). Let us denote \( M[X] = (V \perp \mathbb{H}(R))[X] \). Let \( s \in R \) be such that \( V_s \) is free. Let \( \alpha(X) \in \text{GQ}(M[X]) \) with \( \alpha(0) = \text{Id} \). Suppose \( \alpha_s(X) \in \text{EQ}_2(R_s[X], \Lambda_s[X]) \). Then there exists \( \tilde{\alpha}(X) \in \text{EQ}(M[X]) \) and \( l > 0 \) such that \( \tilde{\alpha}(X) \) localizes to \( \alpha(bX) \) for some \( b \in (s^l) \) and \( \tilde{\alpha}(0) = \text{Id} \).

Proof. Arguing as in the proof of Proposition 3.1 in [15] we can deduce the proof. One observes the repeated use of the commutator formulas stated in pd. 43. [15]

Lemma 3.10. (Local Global Principle for Translation Subgroups) Let \( M = V \perp \mathbb{H}(R) \), where \( V \) is as above. Let \( M[X] = (V \perp \mathbb{H}(R))[X] \). Let \( \alpha(X) \in \text{GQ}(M[X]) \) with \( \alpha(0) = \text{Id} \). Suppose \( \alpha_m(X) \in \text{EQ}_2(R_m[X], \Lambda_m[X]) \) for every maximal ideal \( m \) in \( R \). Then \( \alpha(X) \in \text{EQ}(M[X]) \).

Proof. Since \( \alpha_m(X) \in \text{EQ}_2(R_m[X], \Lambda_m[X]) \) for all \( m \in \text{Max}(C(R)) \), for each \( m \) there exists \( s \in C(R) - m \) such that \( \alpha_s(X) \in \text{EQ}_2(R_s[X], \Lambda_s[X]) \). We consider a finite cover of \( C(R) \), say \( s_1 + \cdots + s_r = 1 \). Following Suslin’s trick, let \( \theta(X, T) = \alpha_s(X + T)\alpha_s(T)^{-1} \).

Then \( \theta(X, T) \in \text{EQ}_2((R_s[T])[X], (\Lambda_s[T])[X]) \) and \( \theta(0, T) = I_{2n} \).

Since for \( l \gg 0 \), \( (s_1^l, \ldots, s_r^l) = R \), we chose \( b_1, b_2, \ldots, b_r \in C(R) \), with \( b_i \in (s_i^l)C(R) \), \( l \gg 0 \) such that \( (A) \) holds and \( b_1 + \cdots + b_r = 1 \). Then by dilation lemma (Lemma 3.9), applied with base ring \( R[T] \), there exists \( \beta(X) \in \text{EQ}(M[X], \Lambda[X]) \) (considering \( M \) as an \( R[T] \) module) such that \( \beta_s(X) = \theta(b_iX, T) \). Therefore, \( \prod_{i=1}^{r} \beta(X) \in \text{EQ}(M[X], \Lambda[X]) \). But,

\[
\alpha_{s_1 \cdots s_r}(X) = \left( \prod_{i=1}^{r-1} \theta_{s_1 \cdots s_i}(b_iX, T) \right) \theta_{s_1 \cdots s_{r-1}}(b_rX, 0).
\]
Observe that as a consequence of the Lemma 3.7 it follows that the map
\[ \text{EQ}(R, s^k R, s^k \Lambda) \rightarrow \text{E}(R, s^k \Lambda) \]
for \( k \in \mathbb{N} \), is injective for each \( s = s_i \). As \( \alpha(0) = I_{2n} \), we conclude \( \alpha(X) \in \text{EQ}(M[X]) \). □

4. STABILIZATION OF \( K_1 \text{GQ} \)

We are recalling following result of Bak–Petrov–Tang in \(^4\) for free modules of even size. We are stating the theorem for commutative rings, and for this section we shall work for commutative rings with trivial involution. We consider the underline \( R \)-module \( M = V \oplus \mathbb{H}(R) \), where \( V \) is a right \( R \)-module and \( \mathbb{H}(R) \) is the hyperbolic space with usual inner product.

**Theorem 4.1.** (Bak–Petrov–Tang) Let \( (R, \Lambda) \) be a commutative form ring with Krull dimension \( d \). If \( 2n \geq \max(6, 2d + 2) \), then \( K_{1,2n} \text{GQ}(R, \Lambda) \) is a group, the stabilization maps \( K_{1,2n-2} \text{GQ}(R, \Lambda) \rightarrow K_{1,2n} \text{GQ}(R, \Lambda) \) is surjective, and the maps
\[ K_{1,2n} \text{GQ}(R, \Lambda) \rightarrow K_{1,2n+2} \text{GQ}(R, \Lambda) \]
are isomorphism of groups.

As an application of our local-global principle for the transvection subgroup (Theorem 3.10) we generalize the above stabilization result for the general quadratic modules. Let us first recall the following key lemma of Vaserstein (cf. Corollary 5.4, \(^{27}\)). A proof for the absolute case is nicely written in an unpublished paper by Maria Saliani (cf. Theorem 6.1, \(^{22}\)). The relative case follows by using the double ring concept, as it is done in \(^{13}\) (Theorem 4.1).

**Lemma 4.2.** Let \( (R, \Lambda) \) be an associative ring with identity with Krull dimension \( d \), and \( I \) be an ideal of \( R \). Let \( M = V \oplus \mathbb{H}(R) \) and \( (M, h, q) \) a general quadratic module of rank \( 2n \geq \max(6, 2d + 2) \). Then the group of elementary transvection \( \text{EQ}(M \perp \mathbb{H}(R), I) \) acts transitively on the set \( \text{Um}(M \perp \mathbb{H}(R), I) \) of unimodular elements which are congruent to \((0, \ldots, 1, \ldots, 0)\) modulo \( I \). In other words,
\[ \text{GQ}(M \perp \mathbb{H}(R), I) = \text{EQ}(M \perp \mathbb{H}(R), I) \text{GQ}(V, I). \]

Also, we have the standard fact.

**Proposition 4.3.** Let \( M \) and \( V \) be as above, and \( \Delta \in \text{GQ}(M \perp \mathbb{H}(R)) \) and \( n \geq 2 \). If \( \Delta e_{2n} = e_{2n} \), then \( \Delta \in \text{EQ}(M \perp \mathbb{H}(R)) \text{GQ}(M) \).

**Proof.** Proof goes as in Lemma 3.6 in \(^{11}\). □

We deduce the following stabilization result for the general quadratic modules:

**Theorem 4.4.** Let \( (R, \Lambda) \) be an associative ring with identity with Krull dimension \( d \). Let \( M = V \oplus \mathbb{H}(R) \) and \( (M, h, q) \) a general quadratic module of rank \( 2n \geq \max(6, 2d + 2) \). Then, the stabilization map
\[ K_{1,2n} \text{GQ}(M) \rightarrow K_{1,2n+2} \text{GQ}(M \perp \mathbb{H}(R)) \]
is isomorphism of groups.
Proof. It follows from the above result of Bak–Petrov–Tang (Theorem 3.11) that for every localization at maximal ideals of $R$ the map is surjective at the level $2d + 2$. Hence the surjectivity follows by local-global principle, and by the surjectivity result of Bak–Petrov–Tang.

In view of their result, it is enough to prove the injectivity for $2n = 2d + 2$. Let $n_1 = 2n + 2$. Suppose $\gamma \in GQ(M)$ is such that $\gamma = \gamma \oplus \Id$ lies in $\text{EQ}(M \oplus \HH(R))$. Let $\phi(X)$ be the isotopy between $\gamma$ and identity. Viewing $\phi(X)$ as a matrix it follows that $v(X) = \phi(X)v_n$ is in $\text{Um}(M \oplus \HH(R))[X]$, and $v(X) = e_n$ modulo $(X^2 - X)$. It follows from Lemma 4.2 that over $R[X]$ we get $\sigma(X) \in \text{EQ}(M \oplus \HH(R))[X]$ such that $\sigma(X)v(X) = e_n$ and $\sigma(X) = \Id$ modulo $(X^2 - X)$. Therefore, $\sigma(X)\phi(X)e_n = e_n$. Then by Lemma 4.3 over $R[X]$ we can write $\sigma(X)^t\phi(X) = \psi(X)\phi(X)$ for some $\psi(X) \in \text{EQ}(M \oplus \HH(R))[X]$ and $\phi(X) \in GQ(M[X])$.

Since $\sigma(X) = \Id$ modulo $(X^2 - X)$, $\phi(X)$ is an isotopy between $\gamma$ and identity. Therefore, after localization at a maximal ideal $m$, the image $\phi_m(X)$ is stably elementary for every maximal ideal $m$ in $\text{Max}(R)$. Hence by the stability theorem of Bak–Petrov–Tang (Theorem 4.1) for the free modules, it follows that $\phi_m(X) \in \text{EQ}(2d + 2, R_m[X])$. Since $\phi(0) = \Id$, we get $\phi(0) = \Id$. Hence by the above local-global principle for the transvection subgroups (Theorem 3.10) it follows that $\phi(X) \in \text{EQ}(M[X])$. Hence $\gamma = \phi(1) \in \text{EQ}(M)$. This proves that the map at the level $2d + 4$ is injective.

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