Kernel Mode Decomposition
and programmable/interpretable regression networks

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July 22, 2019

Abstract

Mode decomposition is a prototypical pattern recognition problem that can be addressed from the (a priori distinct) perspectives of numerical approximation, statistical inference and deep learning. Could its analysis through these combined perspectives be used as a Rosetta stone for deciphering mechanisms at play in deep learning? Motivated by this question we introduce programmable and interpretable regression networks for pattern recognition and address mode decomposition as a prototypical problem. The programming of these networks is achieved by assembling elementary modules decomposing and recomposing kernels and data. These elementary steps are repeated across levels of abstraction and interpreted from the equivalent perspectives of optimal recovery, game theory and Gaussian process regression (GPR). The prototypical mode/kernel decomposition module produces an approximation \( \hat{p}_{w_1,w_2,\ldots,w_m} \) of a product of Hilbert subspaces \( p_{V_1,\ldots,V_m} \) of a common Hilbert space from the observation of the sum \( v := v_1 + \cdots + v_m \in V_1 + \cdots + V_m \). This approximation is minmax optimal with respect to the relative error in the product norm \( \| \cdot \|_{V_1} \) and obtained as \( w_i = Q_i(\sum_j Q_j)^{-1} v = \mathbb{E}[\xi_i \sum_j \xi_j = v] \) where \( Q_i \) and \( \xi_i \sim \mathcal{N}(0,Q_i) \) are the covariance operator and the Gaussian process defined by the norm \( \| \cdot \|_{V_i} \). The prototypical mode/kernel recomposition module performs partial sums of the recovered modes \( w_i \) and covariance operators \( Q_i \) based on the alignment between each recovered mode \( w_i \) and the data \( v \) with respect to the inner product defined by \( S^{-1} \) with \( S := \sum_i Q_i \) (which has a natural interpretation as model/data alignment \( \langle w_i,v \rangle_{S^{-1}} = \mathbb{E}[\xi_i \sum_j \xi_j = v]^2 \) and variance decomposition in the GPR setting). We illustrate the proposed framework by programming regression networks approximating the modes \( v_i = a_i(t)y_i(\theta_i(t)) \) of a (possibly noisy) signal \( \sum_i v_i \) when the amplitudes \( a_i \), instantaneous phases \( \theta_i \) and periodic waveforms \( y_i \) may all be unknown and show near machine precision recovery under regularity and separation assumptions on the instantaneous amplitudes \( a_i \) and frequencies \( \theta_i \). The structure of some of these networks share intriguing similarities with convolutional neural networks while being interpretable, programmable and amenable to theoretical analysis.

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1 Introduction

The purpose of the Empirical Mode Decomposition (EMD) algorithm [16] can be loosely expressed as solving a (usually noiseless) version of the following problem, illustrated in Figure 1.

**Problem 1.** For \( m \in \mathbb{N}^* \), let \( a_1, \ldots, a_m \) be piecewise smooth functions on \([0, 1]\) and let \( \theta_1, \ldots, \theta_m \) be strictly increasing functions on \([0, 1]\). Assume that \( m \) and the \( a_i, \theta_i \) are unknown. Given the (possibly noisy) observation of \( v(t) = \sum_{i=1}^{m} a_i(t) \cos(\theta_i(t)), t \in [0, 1] \), recover the modes \( v_i := a_i(t) \cos(\theta_i(t)) \).

**Figure 1:** A prototypical mode decomposition problem: given \( v = v_1 + v_2 + v_3 \) recover \( v_1, v_2, v_3 \).

In practical applications, generally the instantaneous frequencies \( \omega_i = \frac{d\theta_i}{dt} \) are assumed to be smooth and well separated. Furthermore the \( \omega_i \) and the instantaneous amplitudes are assumed to be varying at a slower rate than the instantaneous phases \( \theta_i \) so that near \( \tau \in [0, 1] \) the intrinsic mode function \( v_i \) can be approximated by a trigonometric function, i.e.

\[
v_i(t) \approx a_i(\tau) \cos(\omega_i(\tau)(t - \tau) + \theta_i(\tau)) \quad \text{for} \quad t \approx \tau .
\] (1.1)

The difficulty of analyzing and generalizing the EMD approach and its popularity in practical applications [15] have stimulated the design of alternative methods aimed at solving Problem 1. Methods that are amenable to a greater degree of analysis include synchrosqueezing [3, 20], variational-mode decomposition [4] and non-linear \( L_1 \) minimization with sparse time-frequency representations [13, 14].

**A Rosetta stone for deep learning?** Since Problem 1 can be seen as prototypical pattern recognition problem that can be addressed from the perspectives of numerical approximation, statistical inference and machine learning, one may wonder if its analysis, from the combined approaches of numerical approximation and statistical inference, could be used as a Rosetta stone for deciphering deep learning.

Indeed, although successful industrial applications [19] have consolidated the recognition of artificial neural networks (ANNs) as powerful pattern recognition tools, their
utilization has recently been compared to “operating on an alien technology” [17] due to the challenges brought by a lag in theoretical understanding: (1) because ANNs are not easily interpretable the resulting models may not be interpretable (and identifying causes of success or failure may be challenging) (2) because ANNs rely on the resolution of non-convex (possibly stochastic) optimization problems, they are not easily amenable to a complete uncertainty quantification analysis (3) because the architecture design of ANNs essentially relies on trial and error, the design of architectures with good generalization properties may involve a significant amount of experimentation.

Since elementary operations performed by ANNs can be interpreted [29] as stacking Gaussian process regression steps with nonlinear thresholding and pooling operations across levels of abstractions, it is natural to wonder whether interpretable Gaussian process regression (GPR) based networks could be conceived for mode decomposition/pattern recognition. Could such networks (1) be programmable based on rational and modular (object oriented) design? (2) be amenable to analysis and convergence results? (3) help our understanding of fundamental mechanisms that might be at play in pattern recognition and thereby help elaborate a rigorous theory for Deep Learning? This paper is an attempt to address these questions, while using mode decomposition [16] as a prototypical pattern recognition problem. As an application of the programmable and interpretable regression networks introduced in this paper, we will also address the following generalization of Problem 1, where the periodic waveforms may all be non-trigonometric, distinct, and unknown and present an algorithm producing near machine precision ($10^{-7}$ to $10^{-4}$) recoveries of the modes.

**Problem 2.** For $m \in \mathbb{N}^*$, let $a_1, \ldots, a_m$ be piecewise smooth functions on $[-1, 1]$, let $\theta_1, \ldots, \theta_m$ be piecewise smooth functions on $[-1, 1]$ such that the instantaneous frequencies $\dot{\theta}_i$ are strictly positive and well separated, and let $y_1, \ldots, y_m$ be square-integrable $2\pi$-periodic functions. Assume that $m$ and the $a_i, \theta_i, y_i$ are all unknown. Given the observation $v(t) = \sum_{i=1}^{m} a_i(t) y_i(\theta_i(t))$ (for $t \in [-1, 1]$) recover the modes $v_i := a_i(t) y_i(\theta_i(t))$.

One fundamental idea is that although Problems 1 and 2 are nonlinear, they can be, to some degree, linearized by recovering the modes $v_i$ as aggregates of sufficiently fine modes living in linear spaces (which, as suggested by the approximation (1.1), can be chosen as linear spans of functions $t \mapsto \cos(\omega(t - \tau) + \theta)$ windowed around $\tau$, i.e. Gabor wavelets). The first part of the resulting network recovers those finer modes through a linear optimal recovery operation. Its second part recovers the modes $v_i$ through a hierarchy of (linear) aggregation steps sandwiched between (nonlinear) ancestor/descendant identification steps. These identification steps are obtained by composing the alignments between $v$ and the aggregates of the fine modes with simple and interpretable nonlinearities (such as thresholding, graph-cuts, etc...), as presented in Section 3.

### 2 The mode decomposition problem

To begin the general (abstract) formulation of the mode decomposition problem, let $V$ be a separable Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and corresponding norm $\| \cdot \|$. Also
let $I$ be a finite set of indices and let $(V_i)_{i \in I}$ be linear subspaces $V_i \subset V$ such that
\[ V = \sum_{i \in I} V_i. \]  

The mode decomposition problem can be informally formulated as follows

**Problem 3.** Given $v \in V$ recover $v_i \in V_i$, $i \in I$, such that $v = \sum_{i \in I} v_i$.

Our solution to Problem 3 will use the interface between numerical approximation, inference and learning (as presented in [26, 27]), which although traditionally seen as entirely separate subjects, are intimately connected through the common purpose of making estimations with partial information [27]. Since the study of this interface has been shown to help automate the process of discovery in numerical analysis and the design of fast solvers [25, 26, 32], this paper is also motivated by the idea it might, in a similar manner and to some degree, also help the process of discovery in machine learning. Here, these interplays will be exploited to address the general formulation of the mode recovery problem from the three perspectives of optimal recovery, game theory and Gaussian process regression. The corresponding minmax recovery framework (illustrated in Figure 2 and presented below) will then be used as a building block for the proposed programmable networks.

### 2.1 Optimal recovery setting

Problem 3 is ill-posed if the subspaces $(V_i)_{i \in I}$ are not linearly independent, in the sense that such a recovery will not be unique. Nevertheless, optimal solutions can be defined in the optimal recovery setting of Micchelli and Rivlin [24]. To this end, let $\| \cdot \|_B$ be a quadratic norm on the product space
\[ B = \prod_{i \in I} V_i, \]  

making $B$ a Hilbert space, and let
\[ \Phi : B \to V \]
be the information map defined by

\[ \Phi(u) := \sum_{i \in I} u_i, \quad u = (u_i)_{i \in I} \in \mathcal{B}. \]  

(2.3)

An optimal recovery solution mapping

\[ \Psi : V \to \mathcal{B} \]

for the mode decomposition problem is defined as follows: for given \( v \in V \), we define \( \Psi(v) \) to be the minimizer of

\[ \min_{w \in \mathcal{B} : \Phi(w) = v} \max_{u \in \mathcal{B} : \Phi(u) = v} \| u - w \|_{\mathcal{B}}. \]

(2.4)

Lemma 2.1. Let \( \Phi : \mathcal{B} \to V \) be surjective. For \( v \in V \), the solution \( w \) of the convex optimization problem

\[
\begin{align*}
\text{Minimize} & \quad \| w \|_{\mathcal{B}} \\
\text{Subject to} & \quad w \in \mathcal{B} \text{ and } \Phi(w) = v.
\end{align*}
\]

(2.5)

determines the unique optimal minmax solution \( w = \Psi(v) \) to (2.4). Moreover,

\[ \Psi(v) = \Phi^{-}v, \]

where the Moore-Penrose inverse \( \Phi^{-} : V \to \mathcal{B} \) of \( \Phi \) is defined by

\[ \Phi^{-} := \Phi^T (\Phi \Phi^T)^{-1}. \]

Now let us be more specific about the structure of \( \mathcal{B} \) that we will assume. Indeed, let the subspaces \( (V_i)_{i \in I} \) be equipped with quadratic norms \( \| \cdot \|_{V_i} \) making each

\[ (V_i, \| \cdot \|_{V_i}) \]

a Hilbert space, and equip their product \( \mathcal{B} = \prod_{i \in I} V_i \) with the product norm

\[ \| u \|_{\mathcal{B}}^2 := \sum_{i \in I} \| u_i \|_{V_i}^2, \quad u = (u_i)_{i \in I} \in \mathcal{B}. \]

(2.6)

We use the notation \([\cdot, \cdot]\) for the duality product between \( V^* \) on the left and \( V \) on the right, and also for the duality product between \( V_i^* \) and \( V_i \) for all \( i \). The norm \( \| \cdot \|_{V_i} \) makes \( V_i \) into a Hilbert space if and only if

\[ \| u_i \|_{V_i}^2 = [Q_i^{-1} u_i, v_i], \quad u_i \in V_i, \]

(2.7)

for some positive symmetric linear bijection

\[ Q_i : V_i^* \to V_i, \]
where by positive and symmetric we mean $[\phi, Q_i \phi] \geq 0$ and $[\phi, Q_i \phi] = [\phi, Q_i \phi]$ for all $\phi, \phi \in V_i^*$. For each $i \in \mathcal{I}$, the dual space $V_i^*$ to $(V_i, \| \cdot \|_{V_i})$ is also a Hilbert space with norm
\[
\| \phi_i \|_{V_i^*}^2 := [\phi_i, Q_i \phi_i], \quad \phi_i \in V_i^*,
\] (2.8)
and therefore the dual space $B^*$ of $B$ can be identified with the product of the dual spaces
\[
B^* = \prod_{i \in \mathcal{I}} V_i^*
\] (2.9)
with (product) duality product
\[
[\phi, u] = \sum_{i \in \mathcal{I}} [\phi_i, u_i], \quad \phi = (\phi_i)_{i \in \mathcal{I}} \in B^*, \quad u = (u_i)_{i \in \mathcal{I}} \in B,
\] (2.10)
and that the symmetric positive linear bijection
\[
Q : B^* \to B
\] (2.11)
defining the quadratic norm $\| \cdot \|_B$ is the block-diagonal operator
\[
Q := \text{diag}(Q_i)_{i \in \mathcal{I}}
\]
defined by its action $Q\phi = (Q_i \phi_i)_{i \in \mathcal{I}}, \phi \in B^*$.

Let
\[
e_i : V_i \to V
\]
be the subset inclusion and let its adjoint
\[
e_i^* : V^* \to V_i^*
\]
be defined through $[e_i^* \phi, v_i] = [\phi, e_i v_i]$ for $\phi \in V^*, v_i \in V_i$. These operations naturally transform the family of operators
\[
Q_i : V_i^* \to V_i, \quad i \in \mathcal{I},
\]
into a family of operators
\[
e_i Q_i e_i^* : V^* \to V, \quad i \in \mathcal{I},
\]
all defined on the same space, so that we can define their sum $S : V^* \to V$ by
\[
S = \sum_{i \in \mathcal{I}} e_i Q_i e_i^*.
\] (2.12)

The following proposition demonstrates that $S$ is invertible and that $S^{-1}$ and $S$ naturally generate dual Hilbert space norms on $V$ and $V^*$ respectively.
Lemma 2.2. The operator $S : V^* \rightarrow V$, defined in (2.12), is invertible. Moreover,
\[ \|v\|_{S^{-1}}^2 := [S^{-1}v, v], \quad v \in V, \]  
(2.13)
defines a Hilbert space norm on $V$ and
\[ \|\phi\|_{S}^2 := [\phi, S\phi] = \sum_{i \in I} \|e_i^*\phi\|_{V_i^*}^2, \quad \phi \in V^*. \]  
(2.14)
defines a Hilbert space norm on $V^*$ which is dual to that on $V$.

The following theorem determines the optimal recovery map $\Psi$.

Theorem 2.3. For $v \in V$, the minimizer of (2.5) and therefore the minmax solution of (2.4) is
\[ \Psi(v) = (Q_ie_i^*S^{-1}v)_{i \in I}. \]  
(2.15)

Furthermore
\[ \Phi(\Psi(v)) = v, \quad v \in V, \]
and
\[ \Psi : (V, \| \cdot \|_{S^{-1}}) \rightarrow (\mathcal{B}, \| \cdot \|_{\mathcal{B}}) \]
and
\[ \Phi^* : (V^*, \| \cdot \|_{\mathcal{B}^*}) \rightarrow (\mathcal{B}^*, \| \cdot \|_{\mathcal{B}^*}) \]
are isometries. In particular, writing $\Psi_i(v) := Q_ie_i^*S^{-1}v$, we have
\[ \| v \|_{S^{-1}}^2 = \| \Psi(v) \|_{\mathcal{B}}^2 = \sum_{i \in I} \| \Psi_i(v) \|_{V_i}^2, \quad v \in V. \]  
(2.16)

Observe that the adjoint
\[ \Phi^* : V^* \rightarrow B^* \]
of $\Phi : B \rightarrow V$, defined by $[\varphi, \Phi(u)] = [\Phi^*(\varphi), u]$ for $\varphi \in V^*$ and $u \in B$, is computed to be
\[ \Phi^*(\varphi) = (e_i^*\varphi)_{i \in I}, \quad \varphi \in V^*. \]  
(2.17)
The following theorem presents optimality results in terms of $\Phi^*$.

Theorem 2.4. We have
\[ \| u - \Psi(\Phi(u)) \|_{\mathcal{B}}^2 = \inf_{\varphi \in V^*} \| u - Q\Phi^*(\varphi) \|_{\mathcal{B}}^2 = \inf_{\varphi \in V^*} \sum_{i \in I} \| u_i - Q_ie_i^*\varphi \|_{V_i}^2. \]  
(2.18)
2.2 Game/decision theoretic setting

Optimal solutions to Problem 3 can also be defined in the setting of the game/decision theoretic approach to numerical approximation presented in [26]. In this setting the minmax problem (2.4) is interpreted as an adversarial zero sum game (illustrated in Figure 2) between two players and lifted to mixed strategies to identify a saddle point. Let $\mathcal{P}_2(\mathcal{B})$ be the set of Borel probability measures $\mu$ on $\mathcal{B}$ such that $E_{u \sim \mu} \left[ \|u\|_B^2 \right] < \infty$, and let $L(V, \mathcal{B})$ be the set of Borel measurable functions $\psi : V \to \mathcal{B}$. Let $\mathcal{E} : \mathcal{P}_2(\mathcal{B}) \times L(V, \mathcal{B}) \to \mathbb{R}$ be the loss function defined by

$$
\mathcal{E}(\mu, \psi) = \frac{E_{u \sim \mu} \left[ \|u - \psi(\Phi(u))\|_B^2 \right]}{E_{u \sim \mu} \left[ \|u\|_B^2 \right]}, \quad \mu \in \mathcal{P}_2(\mathcal{B}), \psi \in L(V, \mathcal{B}). \tag{2.19}
$$

Let us also recall the more general notion of a Gaussian field as described in [26, Chap. 17]. To that end, a Gaussian space $\mathbf{H}$ is a linear subspace $\mathbf{H} \subset L^2(\Omega, \Sigma, \mathbb{P})$ of the $L^2$ space of a probability space consisting of centered Gaussian random variables. A centered Gaussian field $\xi$ on $\mathcal{B}$ with covariance operator $Q : \mathbf{B}^* \to \mathcal{B}$, written $\xi \sim \mathcal{N}(0, Q)$, is an isometry

$$
\xi : \mathbf{B}^* \to \mathbf{H}
$$

from $\mathbf{B}^*$ to a Gaussian space $\mathbf{H}$, in that

$$
[\phi, \xi] \sim \mathcal{N}(0, [\phi, Q\phi]), \quad \phi \in \mathbf{B}^*,
$$

where we use the notation $[\phi, \xi]$ to denote the action $\xi(\phi)$ of $\xi$ on the element $\phi \in \mathbf{B}^*$, thus indicating that $\xi$ is a weak $\mathcal{B}$-valued Gaussian random variable. As discussed in [26, Chap. 17], there is a one to one correspondence between Gaussian cylinder measures and Gaussian fields\footnote{The cylinder sets of $\mathcal{B}$ consists of all sets of the form $F^{-1}(B)$ where $B \in \mathbb{R}^n$ is a Borel set and $F : \mathcal{B} \to \mathbb{R}^n$ is a continuous linear map, over all integers $n$. A cylinder measure $\mu$, see also [26, Chap. 17], on $\mathcal{B}$, is a collection of measures $\mu_F$ indexed by $F : \mathcal{B} \to \mathbb{R}^n$ over all $n$ such that each $\mu_F$ is a Borel measure on $\mathbb{R}^n$ and such that for $F_1 : \mathcal{B} \to \mathbb{R}^{n_1}$ and $F_2 : \mathcal{B} \to \mathbb{R}^{n_2}$ and $G : \mathbb{R}^{n_1} \to \mathbb{R}^{n_2}$ linear and continuous with $F_2 = GF_1$, we have $G_*\mu_{F_1} = \mu_{F_2}$, where $G_*$ is the pushforward operator on measures corresponding to the map $G$, defined by $(G_*\nu)(B) := \nu(G^{-1}B)$. When each measure $\mu_F$ is Gaussian, the cylinder measure is said to be a Gaussian cylinder measure. A sequence $\mu_n$ of cylinder measures such that the sequence $(\mu_n)_F$ converges in the weak topology for each $F$, is said to converge in the weak cylinder measure topology.}. Let $\xi$ denote the Gaussian field $\xi \sim \mathcal{N}(0, Q)$ on $\mathcal{B}$ where $Q : \mathbf{B}^* \to \mathcal{B}$ is the block diagonal operator $Q := \text{diag}(Q_i)_{i \in \mathcal{I}}$. Theorem 2.5 shows that the optimal strategy of Player I is the Gaussian field $\xi - E[\xi | \Phi(\xi)]$ and the optimal strategy of Player II is the conditional expectation

$$
\Psi(v) = E[\xi | \Phi(\xi) = v], \tag{2.20}
$$

that is also equal to (2.15). Write $\mu^1$ for the cylinder measure defined by the Gaussian field $\xi - E[\xi | \Phi(\xi)]$, or the corresponding Gaussian measure in finite dimensions.
We say that a tuple \((\mu', \psi')\) is a saddle point of the loss function \(\mathcal{E} : \mathcal{P}_2(\mathcal{B}) \times \mathcal{L}(V, \mathcal{B}) \rightarrow \mathbb{R}\) if
\[
\mathcal{E}(\mu, \psi) \leq \mathcal{E}(\mu', \psi') \leq \mathcal{E}(\mu', \psi), \quad \mu \in \mathcal{P}_2(\mathcal{B}), \ \psi \in \mathcal{L}(V, \mathcal{B}).
\]

**Theorem 2.5.** Let \(\mathcal{E}\) be defined as in (2.19). It holds true that
\[
\max_{\mu \in \mathcal{P}_2(\mathcal{B})} \min_{\psi \in \mathcal{L}(V, \mathcal{B})} \mathcal{E}(\mu, \psi) = \min_{\psi \in \mathcal{L}(V, \mathcal{B})} \max_{\mu \in \mathcal{P}_2(\mathcal{B})} \mathcal{E}(\mu, \psi). \tag{2.21}
\]
Furthermore,

- If \(\dim(V) < \infty\) then \((\mu^\dagger, \Psi)\) is a saddle point for the loss (2.19), where \(\Psi\) is as in (2.15) and (2.20).
- If \(\dim(V) = \infty\), then the loss (2.19) admits a sequence of saddle points \((\mu_n, \Psi) \in \mathcal{P}_2(\mathcal{B}) \times \mathcal{L}(V, \mathcal{B})\) where \(\Psi\) is as in (2.15) and (2.20), and the \(\mu_n\) are Gaussian measures, with finite dimensional support, converging towards \(\mu^\dagger\) in the weak cylinder measure topology.

**Proof.** The proof is similar to that of [26, Thm. 18.2] \(\square\)

![Figure 3: The minmax solution of the mode decomposition problem.](image)

**2.3 Gaussian process regression setting**

It follows from Theorem 2.5 that the minmax optimal solution to Problem 3 with loss measured as the relative error in the norm (2.6) can be obtained via Gaussian process regression as follows. For \(i \in \mathcal{I}\), let \(\xi_i \sim \mathcal{N}(0, Q_i)\) be independent \(V_i\)-valued Gaussian fields defined by the norms \(\| \cdot \|_{V_i}\). Recall that \(Q_i\) is defined in (2.7) and that \(\xi_i\) is an isometry from \((V_i^*, \| \cdot \|_{V_i^*})\) onto a Gaussian space mapping \(\phi \in V_i^*\) to \([\phi, \xi_i] \sim \mathcal{N}(0, [\xi_i, Q_i])\). Theorem 2.5 asserts that the minmax estimator is (2.20), which, written componentwise, makes the optimal reconstruction of each mode \(v_j\) of \(v = \sum_{i \in \mathcal{I}} v_i\) to be
\[
\mathbb{E}\left[\xi_j \mid \sum_{i \in \mathcal{I}} \xi_i = v\right] = Q_j (\sum_{i \in \mathcal{I}} Q_i)^{-1} v. \tag{2.22}
\]
where the right hand side of (2.22) is obtained from (2.15) and \( \sum_{i \in I} Q_i \) is a shorthand notation for \( \sum_i e_i Q_i e_i^* \) obtained by dropping the indications of the injections \( e_i \) and their adjoint projections \( e_i^* \). From now on, we will use such simplified notations whenever there is no risk of confusion. In summary, the minmax the minmax solution of the abstract mode decomposition problem is obtained as illustrated in Figure 3 based on the specification of the operators \( Q_i : V_i^* \to V_i \) which can be interpreted as quadratic norm defining operators or as covariance operators. Table 1 illustrates the three equivalent interpretations -optimal recovery/operator kernel/Gaussian process regression of our methodology.

| Norm                     | Operator/Kernel | GP                                      |
|--------------------------|-----------------|-----------------------------------------|
| \( \|v_i\|_{V_i}^2 := \langle Q_i^{-1} v_i, v_i \rangle \) | \( Q_i : V_i^* \to V_i \) | \( \xi_i \sim \mathcal{N}(0, Q_i) \) |
| \( \arg \min \left\{ \text{minimize } \sum_i \|w_i\|_{V_i}^2 \right\} \) \( Q_i(\sum_j Q_j)^{-1} v \) | \( \xi_i \sim \mathcal{N}(0, Q_i) \) | \( \mathbb{E}[\xi_i | \sum_j \xi_j = v] \) |

Table 1: Three equivalent interpretations-optimal recovery/operator kernel/Gaussian process regression of our methodology.

**Example 2.6.** Consider the problem of recovering the modes \( v_1, v_2, v_3, v_4 \) from the observation of the signal \( v = v_1 + v_2 + v_3 + v_4 \) illustrated in Fig. 4. In this example all modes are defined on the interval \([0, 1]\), \( v_1(t) = (1 + 2t^2) \cos(\theta_1(t)) - 0.5t \sin(\theta_1(t)) \), \( v_2(t) = 2(1 - t^2) \cos(\theta_2(t)) + (-t + 0.5t^2) \sin(\theta_2(t)) \), \( v_3(t) = 2 + t - 0.2t^2 \), and \( v_4 \) is white-noise (the instantiation of a centered GP with covariance function \( \delta(s - t) \)). \( \theta_1(t) = \int_0^t \omega_1(s) \, ds \) and \( \theta_2(t) = \int_0^t \omega_2(s) \, ds \) are defined by the instantaneous frequencies \( \omega_1(t) = 16t(1 + t) \) and \( \omega_2(t) = 30\pi(1 + t^2/2) \). In this recovery problem \( \omega_1(t) \) and \( \omega_2(t) \) are known, \( v_3 \) and the amplitudes of the oscillations of \( v_1, v_2 \) are unknown smooth functions of time, only the distribution of \( v_4 \) is known. To define optimal recovery solutions one can either define the normed subspaces \( \langle V_i, \| \cdot \|_{V_i} \rangle \) or (equivalently via (2.7)) define the covariance functions/operators of the Gaussian Processes \( \xi_i \). In this example it is simpler to use the latter. To define the covariance function of the GP \( \xi_1 \) we assume that \( \xi_1(t) = \zeta_{1,c}(t) \cos(\theta_1(t)) + \zeta_{1,s}(t) \sin(\theta_1(t)) \), where \( \zeta_{1,c} \) and \( \zeta_{1,s} \) are independent identically distributed centered Gaussian processes with covariance function

\[
\mathbb{E}[\zeta_{1,c}(s)\zeta_{1,c}(t)] = \mathbb{E}[\zeta_{1,s}(s)\zeta_{1,s}(t)] = e^{-\frac{(s-t)^2}{2}} \quad \text{(chosen with } \gamma = 0.2 \text{ as a prior regularity assumption).}
\]

Note that under this choice \( \xi_1 \) is a centered GP with covariance function

\[
K_1(s, t) = e^{-\frac{(s-t)^2}{2}} \left( \cos(\theta_1(s)) \cos(\theta_1(t)) + \sin(\theta_1(s)) \sin(\theta_1(t)) \right) .
\]

Note that the trigonometric identities \( \cos(a + b) = \cos a \cos b - \sin a \sin b \) and \( \sin(a + b) = \sin a \cos b + \cos a \sin b \) imply that translating \( \theta_1 \) by an arbitrary phase \( b \) leaves \( K_1 \) invariant (knowing \( \theta_1 \) up to a phase shift is sufficient to construct that kernel). Similarly we select the covariance func-
Figure 4: (1) The signal \( v = v_1 + v_2 + v_3 + v_4 \) (2) The modes \( v_1, v_2, v_3, v_4 \) (3) \( v_1 \) and its approximation \( w_1 \) (4) \( v_2 \) and its approximation \( w_2 \) (5) \( v_3 \) and its approximation \( w_3 \) (6) \( v_4 \) and its approximation \( w_4 \).

	

The recovery approach of Example 2.6 is based on the design of an appropriate additive regression model. Additive regression models are not new. They were introduced in [34] for approximating multivariate functions with sums of uni-
Table 2: A summary of the approach of Example 2.6, illustrating the connection between the assumed mode structure and corresponding Gaussian process structure and its corresponding reproducing kernel structure. Note that, for clarity of presentation, this summary does not exactly match that of Example 2.6.

| Mode | GP | Kernel |
|------|----|--------|
| \(v_1(t) = a_1(t) \cos(\theta_1(t))\) | \(\xi_1(t) = \xi_1(t) \cos(\theta_1(t))\) | \(K_1(s, t) = e^{-\frac{|s-t|^2}{\gamma^2}} \cos(\theta_1(s)) \cos(\theta_1(t))\) |
| \(\theta_1\) known | \(\mathbb{E}[\xi_1(s)\xi_1(t)] = e^{-\frac{|s-t|^2}{\gamma^2}}\) | |
| \(a_1\) unknown smooth | | |
| \(v_2(t) = a_2(t) \cos(\theta_2(t))\) | \(\xi_2(t) = \xi_2(t) \cos(\theta_2(t))\) | \(K_2(s, t) = e^{-\frac{|s-t|^2}{\gamma^2}} \cos(\theta_2(s)) \cos(\theta_2(t))\) |
| \(\theta_2\) known | \(\mathbb{E}[\xi_2(s)\xi_2(t)] = e^{-\frac{|s-t|^2}{\gamma^2}}\) | |
| \(a_1\) unknown smooth | | |
| \(v_3\) unknown smooth | \(\mathbb{E}[\xi_3(s)\xi_3(t)] = e^{-\frac{|s-t|^2}{\gamma^2}}\) | \(K_3(s, t) = e^{-\frac{|s-t|^2}{\gamma^2}}\) |
| \(v_4\) unknown white noise | \(\mathbb{E}[\xi_4(s)\xi_4(t)] = \sigma^2 \delta(s-t)\) | \(K_4(s, t) = \sigma^2 \delta(s-t)\) |
| \(v = v_1 + v_2 + v_3 + v_4\) | \(\xi = \xi_1 + \xi_2 + \xi_3 + \xi_4\) | \(K = K_1 + K_2 + K_3 + K_4\) |

variante functions. Generalized additive models (GAMs) [11] replace a linear regression model \(\sum_i \alpha_i X_i\) with an additive regression model \(\sum_i f_i(X_i)\) where the \(f_i\) are unspecified (smooth) functions estimated from the data. Since their inception GAMs have become increasingly popular because they are both easy to interpret and easy to fit [30]. This popularity has motivated the introduction of additive Gaussian processes [8, 6] defined as Gaussian processes whose high dimensional covariance kernels are obtained from sums of low dimensional ones (such kernels are expected to to overcome the curse of dimensionality by exploiting additive non-local effects when such effects are present [8]). Of course, performing regression or mode decomposition with Gaussian processes (GPs) obtained as sums of independent GPs (i.e. performing kriging with kernels obtained as sums of simpler kernels) is much older since Tikhonov regularization (for signal/noise separation) has a natural interpretation as a conditional expectation \(\mathbb{E}[\xi_s|\xi_s + \xi_o]\) where \(\xi_s\) is a GP with a smooth prior (for the signal) and \(\xi_o\) is a white noise GP independent from \(\xi_s\). More recent applications include classification [22], source separation [28, 21], and the detection of the periodic part of a function from partial point evaluations [7, 1]. For that latter application, the approach of [7] is to (1) consider the RKHS \((H, \langle \cdot, \cdot \rangle_H)\) defined by a Matérn kernel \(K\) (2) interpolate the data with the kernel \(K\) and (3) recover the periodic
part by projecting the interpolator (using a projection that is orthogonal with respect to the RKHS scalar product $\langle \cdot, \cdot \rangle_H$) onto $H_p := \text{span}\{\cos(2\pi kt/\lambda), \sin(2\pi kt/\lambda) \mid 1 \leq k \leq q\}$ (the parameters of the Matérn kernel and the period $\lambda$ are obtained via maximum likelihood estimation). Defining $K_p$ and $K_{np}$ as the kernels induced on $H_p$ and its orthogonal complement in $H$, we have $K = K_p + K_{np}$ and the recovery (after MLE estimation of the parameters) can also be identified as the conditional expectation of the GP induced by $K_p$ conditioned on the GP induced by $K_p + K_{np}$.

Figure 5: Left: Problem 1 is hard as a mode decomposition problem because the modes $v_j = a_j(t)\cos(\theta_j(t))$ live in non-linear functional spaces. Right: One fundamental idea is to recover those modes as aggregates of finer modes $v_i$ living in linear spaces.

3 Kernel Mode Decomposition Networks

The recovery approach described in Example 2.6 is based on the prior knowledge of (1) the number of quasi-periodic modes (2) their instantaneous frequencies and (3) their base periodic waveform (which need not be a cosine function). In most applications (1) and (2) are not available and the base waveform may not be trigonometric and may not be known. Even when the periodic waveforms are known and trigonometric (as in Problem 1), the recovery of the modes is still significantly harder because, as illustrated in Figure 5, the functional spaces defined by the modes $a_j(t)\cos(\theta_j(t))$ (under regularity assumptions on the $a_j$ and $\theta_j$) are no longer linear spaces and the simple calculus of Section 2 requires the spaces $V_j$ to be linear. One fundamental idea is to recover those modes $v_j$ as aggregates of finer modes $v_i$ living in linear spaces $V_i$ (see Figure 5). For Problem 1 we will identify $i$ with time-frequency-phase triples $(\tau, \omega, \theta)$ and the spaces $V_i$ with one dimensional spaces spanned by functions that are maximally localized in
the time-frequency-phase domain (i.e. by Gabor wavelets as suggested by the approximation (1.1)) and recover the modes $a_j(t) \cos(\theta_j(t))$ by aggregating the finer recovered modes. The implementation of this idea will therefore transform the nonlinear mode

\[ v^{(1)}_i \]

\[ + \]

\[ v \]

\[ v^{(2)}_j \]

Identification of $i \sim j$: nonlinear

\[ \rightarrow \]

Linear

\[ w^{(1)}_i \]

\[ + \]

\[ w^{(2)}_j \]

Linear

Figure 6: Mode decomposition/recomposition problem.

decomposition problem illustrated on the left hand side of Figure 5 into the mode decomposition/recomposition problem illustrated in Figure 6 and transfer its nonlinearity to the identification of ancestor/descendant relationships $i \sim j$.

To identify these ancestor/descendant relations we will compute the energy $E(i) := \| w_i \|_2^2$ for each recovered mode $w_i$, which as illustrated in Figure 7 and discussed in Section 3.1, can also be identify as the alignment $\langle w_i, v \rangle_{S-1}$ between recovered mode $w_i$ and the signal $v$ or as the alignment $\mathbb{E}[\text{Var}(\xi_i, v)_{S-1}]$ between the model $\xi_i$ and the data $v$. Furthermore $E$ satisfy an energy preservation identity $\sum_i E(i) = \| v \|_{S-1}^2$, which leads to its variance decomposition interpretation. Although alignment calculations are linear, the calculations of the resulting child-ancestor relations may involve a nonlinearity (such as thresholding, graph-cut, computation of a maximizer) and the resulting network can be seen as a sequence of sandwiched linear operations and simple non-linear steps having striking similarities with artificial neural networks.

Of course this strategy can be repeated across levels of abstractions and its complete deployment will also require the generalization of the setting of Section 2 (illustrated in Figure 3) to a hierarchical setting (illustrated in Figure 10 and described in Section 3.3).
3.1 Model/data alignment and energy/variance decomposition

Using the setting and notations of Section 2 and fixing the observed data \( v \in V \), let \( E : \mathcal{I} \to \mathbb{R}_+ \) be the function defined by

\[
E(i) := \|\psi_i(v)\|_{V_i}^2, \quad i \in \mathcal{I}.
\]

We will refer to \( E(i) \) as the energy of the mode \( i \) in reference to its numerical analysis interpretation (motivated by \( E(i) = [Q_i^{-1}\psi_i(v), \psi_i(v)] \) and the interpretation of \( Q_i^{-1} \) as an elliptic operator) and our general approach will be based on using its local and/or global maximizers to decompose/recompose kernels.

Writing \( E_{\text{tot}} := \|v\|^2_{S^{-1}} \), note that (2.16) implies that

\[
E(i) = \langle \psi_i(v), v \rangle_{\mathcal{S}^{-1}} = \text{Var}([\phi, \xi]) = \text{Var}(\langle \xi_i, v \rangle_{\mathcal{S}^{-1}}).
\]

Observe that \( E(i) = \text{Var}(\langle \xi_i, v \rangle_{\mathcal{S}^{-1}}) \) implies that \( E(i) \) is a measure of the alignment between the Gaussian process (GP) model \( \xi_i \) and the data \( v \) in \( V \) and (3.2) corresponds to the variance decomposition

\[
\text{Var}(\langle \sum_{i \in \mathcal{I}} \xi_i, v \rangle_{\mathcal{S}^{-1}}) = \sum_{i \in \mathcal{I}} \text{Var}(\langle \xi_i, v \rangle_{\mathcal{S}^{-1}}).
\]
Therefore, the stronger this alignment $E(i)$ is, the better the model $\xi_i$ is at explaining/representing the data. Consequently, we refer to the energy $E(i)$ as the alignment energy. Observe also that the identity $E(i) = \langle w_i, v \rangle_{S^{-1}}$ with $w_i = \Psi_i(v)$ implies that $E(i)$ is also a measure of the alignment between the optimal approximation $w_i$ of $v_i$ and the signal $v$. Table 3 illustrates the relations between the conservation of alignment energies and the variance decomposition derived from Theorem 2.3 and Proposition 3.1.

| $E(i)$ | $\|\Psi_i(v)\|_{S^{-1}}^2 = \langle \Psi_i(v), v \rangle_{S^{-1}}$ | $[S^{-1}v, Q_i S^{-1}v]$ | $\text{Var}(\langle \xi_i, v \rangle_{S^{-1}})$ | $[S^{-1}v, v]$ | $\text{Var}(\langle \sum_i \xi_i, v \rangle_{S^{-1}})$ |
|--------|------------------------------------------------|-----------------|-----------------|-----------------|-----------------|
| $\sum_i E(i)$ | $\|v\|_{S^{-1}}^2$ | $[S^{-1}v, v]$ | $\text{Var}(\langle \sum_i \xi_i, v \rangle_{S^{-1}})$ |

Table 3: Identities for $E(i)$ and $\sum_i E(i)$

![Figure 8](image.png)

Figure 8: Elementary programming modules for Kernel Mode Decomposition.

3.2 Programming modules and feedforward network

We will now combine the alignment energies of Section 3.1 with the mode decomposition approach of Section 2 to design elementary programming modules (illustrated in Figure 8) for kernel mode decomposition networks (KMDNets). These will be introduced in this section and developed in the following ones. Per Section 2 and Theorem 2.3, the optimal recoveries of the modes $(v_i)_{i \in \mathcal{I}}$ given the covariance operators $(Q_i)_{i \in \mathcal{I}}$ and the observation of $\sum_{i \in \mathcal{I}} v_i$ are the elements $Q_i(\sum_{j} Q_j)^{-1}v$ in $V_i$. This operation is illustrated in module (1) of Figure 8. An important quantity derived from this recovery is the energy function $E: \mathcal{I} \to \mathbb{R}_+$, defined in (3.1) by $E(i) := [Q_i^{-1}w_i, w_i]$ with $w_i := \Psi_i(v)$, and illustrated in module (2). Since, per (3.2), $E_{\text{tot}} = \sum_{i \in \mathcal{I}} E(i)$, where $E_{\text{tot}} := \|v\|_{S^{-1}}^2$ is the total energy (3.1), the function $E$ can be interpreted as performing a decomposition of the total energy over the set of labels $\mathcal{I}$. When $\mathcal{I}$ can be identified with the set of vertices of a graph, the values of the $E(i)$ can be used to cut that graph into subgraphs indexed by labels $j \in \mathcal{J}$ and define a relation $i \sim j$ mapping $i \in \mathcal{I}$ to its subgraph $j$. This graph-cut operation is illustrated in module (3). Since, per Section 3.1, $E(i)$ is also the mean squared
alignment between the model $\xi_i$ and the data $v$, and \((3.4)\) is a variance decomposition, this clustering operation combines variance/model alignment information (as done with PCA) with the geometric information (as done with mixture models \([23]\)) provided by the graph to assign a class $j \in \mathcal{J}$ to each element $i \in \mathcal{I}$. However, the $i \rightsquigarrow j$ relation may also be obtained through a projection step, possibly ignoring the values of $E(p_i)$, as illustrated in module (4) (e.g. when $i$ is an $r$-tuple $(i_1, i_2, \ldots, i_r)$ then the truncation/projection map naturally defines a $\rightsquigarrow$ relation via $(i_1, \ldots, i_r) \rightsquigarrow (i_1, \ldots, i_{r-1})$). As illustrated in module (5), combining the $\rightsquigarrow$ relation with a sum $\sum_{i \rightsquigarrow j}$ produces aggregated covariance operators $Q_j := \sum_{i \rightsquigarrow j} Q_i$, modes $w_j := \sum_{i \rightsquigarrow j} w_i$ and energies $E(j) := \sum_{i \rightsquigarrow j} E(i)$ such that for $V_j := \sum_{i \rightsquigarrow j} V_i$, the modes $(w_i)_{i \rightsquigarrow j}$ are optimal recovery modes in $\prod_{i \rightsquigarrow j} V_i$ given the covariance operators $(Q_i)_{i \rightsquigarrow j}$ and the observation of $w_j = \sum_{i \rightsquigarrow j} w_i$ in $V_j$. Furthermore, we have $E(j) = [Q_j^{-1} w_j, w_j]$. Naturally, combining these elementary modules leads to more complex secondary modules (illustrated in Figure 9) whose nesting produces a network aggregating the fine modes $w_i$ into increasingly coarse modes with the last node corresponding to $v$.

Figure 9: Programming modules derived from the elementary modules of Figure 8.

3.3 Hierarchical mode decomposition

The hierarchy of mode decomposition/recomposition steps discussed in Section 3.2 naturally produces a hierarchy of labels, covariance operators, subspaces and recoveries (illustrated in Figure 10) whose geometries and relationships will now be described. This description will lead to the meta-algorithm 1, presented in Section 3.4, aimed at the production of a KMDNet such as the one illustrated in Figure 9. Section 3.5 will present a practical application to Problem 1.

Our first step is to generalize the recovery approach of Section 2 to the case where $V$ is the sum of a hierarchy of linear nested subspaces labeled by a hierarchy of indices, as defined below.

**Definition 3.2.** For $q \in \mathbb{N}^*$, let $\mathcal{I}^{(1)}, \ldots, \mathcal{I}^{(q)}$ be finite sets of indices such that $\mathcal{I}^{(q)} = \{1\}$ has only one element. Let $\bigcup_{i=1}^{q} \mathcal{I}^{(i)}$ be endowed with a relation $\rightsquigarrow$ that is (1) transitive, i.e., $i \rightsquigarrow j$ and $j \rightsquigarrow k$ implies $i \rightsquigarrow k$ (2) directed, i.e., $i \in \mathcal{I}^{(s)}$ and $j \in \mathcal{I}^{(r)}$ with $r \leq s$
implies \( i \Rightarrow j \) (that is, \( i \) does not lead to \( j \)) and (3) such that for \( r > 1 \) any element \( j \in I^{(r)} \) has at least one \( i \in I^{(r-1)} \) such that \( i \leadsto j \). For \( 1 \leq k < r \leq q \) and an element \( i \in I^{(r)} \), write \( i^{(k)} := \{ j \in I^{(k)} : j \leadsto i \} \) for the level \( k \) ancestors of \( i \).

Let \( V_i^{(k)}, i \in I^{(k)}, k \in \{1, \ldots, q\} \) be a hierarchy of nested linear subspaces of a separable Hilbert space \( V \) such that
\[
V_1^{(q)} = V
\]
and, for each level in the hierarchy \( k \in \{1, \ldots, q - 1\} \),
\[
V_i^{(k+1)} = \sum_{j \in i^{(k)}} V_j^{(k)}, \quad i \in I^{(k+1)}.
\]
(3.5)

Let \( B^{(q)} = V \) and for \( k \in \{1, \ldots, q - 1\} \), let \( B^{(k)} \) be the product space
\[
B^{(k)} := \prod_{i \in I^{(k)}} V_i^{(k)}.
\]
(3.6)

For \( j \in I^{(r)} \) with \( r > k \), let
\[
B_j^{(k)} := \prod_{i \in j^{(k)}} V_i^{(k)},
\]
(3.7)
and let
\[
\Phi_j^{(r,k)} : B_j^{(k)} \to V_j^{(r)}
\]

Figure 10: The generalization of abstract mode decomposition problem of Figure 3 to a hierarchy as described in Section 3.3.
be defined by
\[ \Phi_{j}^{(r,k)}(u) := \sum_{i \in j^{(k)}} u_i, \quad u \in B_{j}^{(k)}. \tag{3.8} \]

Putting these components together as \( \Phi^{(r,k)} = (\Phi_{j}^{(r,k)})_{j \in \mathcal{I}^{(r)}} \), we obtain the multi-linear map
\[ \Phi^{(r,k)} : B^{(k)} \rightarrow B^{(r)}, \quad 1 \leq k < r \leq q, \]
defined by
\[ \Phi^{(r,k)}(u) := \left( \sum_{i \in j^{(k)}} u_i \right)_{j \in \mathcal{I}^{(r)}}, \quad u = (u_i)_{i \in \mathcal{I}^{(k)}} \in B^{(k)}. \tag{3.9} \]

To put hierarchical metric structure on these spaces, for \( k \in \{1, \ldots, q\} \) and \( i \in \mathcal{I}^{(k)} \), let
\[ Q_{i}^{(k)} : V_{i}^{(k),*} \rightarrow V_{i}^{(k)} \]
be positive symmetric linear bijections determining the quadratic norms
\[ \|v\|_{V_{i}^{(k)}}^2 = [Q_{i}^{(k)},-1] v, v \in V_{i}^{(k)}, \tag{3.10} \]
on the \( V_{i}^{(k)} \). For \( k \in \{1, \ldots, q\} \), let \( B^{(k)} \) be endowed with the quadratic norm defined by
\[ \|u\|_{B^{(k)}}^2 = \sum_{i \in \mathcal{I}^{(k)}} \|u_i\|_{V_{i}^{(k)}}^2, \quad u \in B^{(k)}, \tag{3.11} \]
and for \( k < r \leq q \) and \( j \in \mathcal{I}^{(r)} \), let \( B_{j}^{(k)} := \prod_{i \in j^{(k)}} V_{i}^{(k)} \) be endowed with the quadratic norm defined by
\[ \|u\|_{B_{j}^{(k)}}^2 = \sum_{i \in j^{(k)}} \|u_i\|_{V_{i}^{(k)}}^2, \quad u \in B_{j}^{(k)}. \]

For \( 1 \leq k < r \leq q \), the nesting relations (3.5) imply that
\[ V_{i}^{(k)} \subset V_{j}^{(r)}, \quad i \in j^{(k)}, \quad j \in \mathcal{I}^{(r)}, \]
so that the subset injection
\[ e_{j,i}^{(r,k)} : V_{i}^{(k)} \rightarrow V_{j}^{(r)} \tag{3.12} \]
is well defined for all \( i \in j^{(k)}, \quad j \in \mathcal{I}^{(r)}, \) and since all spaces are complete, they have well-defined adjoints, which we write
\[ e_{i,j}^{(k,r)} : V_{j}^{(r),*} \rightarrow V_{i}^{(k),*}. \tag{3.13} \]

For \( 1 \leq k < r \leq q \), \( i \in \mathcal{I}^{(k)} \) and \( j \in \mathcal{I}^{(r)} \), let \( \Psi_{i,j}^{(k,r)} : V_{j}^{(r)} \rightarrow V_{i}^{(k)} \) be defined by
\[ \Psi_{i,j}^{(k,r)}(v_j) = Q_{i}^{(k)} e_{i,j}^{(k,r)} Q_{j}^{(r),-1} v_j, \quad v_j \in V_{j}^{(r)}. \tag{3.14} \]
that, when putting the components together as
\[ \Psi_{j}^{(k,r)} := (\Psi_{i,j}^{(k,r)})_{i \in j^{(k)}}, \]  
(3.15)
determines the multi-linear map
\[ \Psi_{j}^{(k,r)} : V_{j}^{(r)} \to B_{j}^{(k)}. \]

Further collecting components simultaneously over the range and domain as
\[ \Psi^{(k,r)} = (\Psi_{j}^{(k,r)})_{j \in I^{(r)}} \]
we obtain the multi-linear map
\[ \Psi^{(k,r)} : B^{(r)} \to \prod_{j \in I^{(r)}} B_{j}^{(k)} \]
defined by
\[ \Psi^{(k,r)}(v) = (Q_{i}^{(k)} c_{i,j}^{(k,r)} Q_{j}^{(r)} v_{j})_{i \in j^{(k)}}, \quad v = (v_{j})_{j \in I^{(r)}} \in B^{(r)}. \]  
(3.16)

The following condition assumes that the relation \( \rightsquigarrow \) determines a mapping \( \rightsquigarrow : I^{(k)} \to I^{(k+1)} \) for all \( k = 1, \ldots, q - 1 \).

**Condition 3.3.** For \( k \in \{1, \ldots, q - 1\} \), every \( i \in I^{(k)} \) has a unique descendant in \( I^{(k+1)} \). That is, there exists a \( j \in I^{(k+1)} \) with \( i \rightsquigarrow j \) and there is no other \( j' \in I^{(k+1)} \) such that \( i \rightsquigarrow j' \).

Condition 3.3 simplifies the previous results as follows: the subsets \( \{\{i \in j^{(k)}\}\}_{j \in I^{(k+1)}} \) form a partition of \( I^{(k)} \), so that we obtain the simultaneous product structure
\[ B^{(k)} = \prod_{j \in I^{(r)}} B_{j}^{(k)} \]
\[ B^{(r)} = \prod_{j \in I^{(r)}} V_{i}^{(r)} \]  
(3.17)
so that both
\[ \Phi^{(k,r)} : B^{(k)} \to B^{(r)} \]
and
\[ \Psi^{(k,r)} : B^{(r)} \to B^{(k)} \]
are diagonal multi-linear maps with components
\[ \Phi_{j}^{(r,k)} : B_{j}^{(k)} \to V_{j}^{(r)} \]
and
\[ \Psi_{j}^{(k,r)} : V_{j}^{(r)} \to B_{j}^{(k)} \]
respectively. Moreover, both maps are linear under the isomorphism between products and external direct sums of vector spaces. For \( r > k \), we have the following connections between \( B(k), B(r), V_i^{(k)} \) and \( V_j^{(r)} \).

The following theorem is a consequence of Theorem 2.3.

**Theorem 3.4.** Assume that Condition 3.3 holds and that the \( Q_i^{(k)} : V_i^{(k),*} \to V_i^{(k)} \) satisfy the nesting relations

\[
Q_j^{(k+1)} = \sum_{i \in I} e_{j,i}^{(k+1,k)} Q_i^{(k)} e_{i,j}^{(k,k+1)}, \quad j \in I^{(k+1)},
\]

for \( k \in \{1, \ldots, q - 1\} \). Then for \( 1 \leq k < r \leq q \),

- \( \Psi^{(r,k)} \circ \Phi^{(r,k)}(u) \) is the minmax recovery of \( u \in B(k) \) given the observation of \( \Phi^{(r,k)}(u) \in B(r) \) using the relative error in \( \| \cdot \|_{B(k)} \) norm as a loss.

- \( \Phi^{(r,k)} \circ \Psi^{(k,r)} \) is the identity map on \( B(r) \).

- \( \Psi^{(k,r)} : (B(r), \| \cdot \|_{B(r)}) \to (B(k), \| \cdot \|_{B(k)}) \) is an isometry.

- \( \Phi^{(k,r),*} : (B(r), \| \cdot \|_{B(r),*}) \to (B(k), \| \cdot \|_{B(k),*}) \) is an isometry.

Moreover we have the following semigroup properties for \( 1 \leq k < r < s \leq q \):

- \( \Phi^{(s,k)} = \Phi^{(s,r)} \circ \Phi^{(r,k)} \)

- \( \Psi^{(k,s)} = \Psi^{(k,r)} \circ \Psi^{(r,s)} \)

- \( \Psi^{(r,s)} = \Phi^{(r,k)} \circ \Psi^{(k,s)} \)

**Remark 3.5.** The proof of Theorem 3.4 also demonstrates that, under its assumptions, for \( 1 \leq k < r \leq q \) and \( j \in I^{(r)} \), \( \Psi_j^{(r,k)} \circ \Phi_j^{(r,k)}(u) \) is the minmax recovery of \( u \in B_j^{(k)} \) given the observation of \( \Phi_j^{(r,k)}(u) \in V_j^{(r)} \) using the relative error in \( \| \cdot \|_{B_j^{(k)}} \) norm as a loss.

Furthermore, \( \Phi_j^{(r,k)} \circ \Psi_j^{(k,r)} \) is the identity map on \( V_j^{(r)} \) and \( \Psi_j^{(k,r),*} : (V_j^{(r),*}, \| \cdot \|_{V_j^{(r),*}}) \to (B_j^{(k)}, \| \cdot \|_{B_j^{(k),*}}) \) are isometries.
Gaussian process regression interpretation  As in the setting of Section 3.3, for $k \in \{1, \ldots, q\}$, let $Q^{(k)} : \mathcal{B}_{(k),*} \to \mathcal{B}_{(k)}$ be the block-diagonal operator $Q^{(k)} := \text{diag}(Q_{i,i}^{(k)})_{i \in \mathcal{I}_{(k)}}$ defined by its action $Q^{(k)} \phi := (Q_{i,i}^{(k)} \phi_i)_{i \in \mathcal{I}_{(k)}}, \ \phi \in \mathcal{B}_{(k),*}$, and, as discussed in Section 2.2, write $\xi^{(k)} := N(0, Q^{(k)})$ for the centered Gaussian field on $\mathcal{B}_{(k)}$ with covariance operator $Q^{(k)}$.

**Theorem 3.6.** Under the assumptions of Theorem 3.4, for $1 \leq k \leq q$, the distribution of $\xi^{(k)}$ is that of $\Phi^{(k)}(\xi^{(1)})$. Furthermore $\xi^{(1)}$ conditioned on $\Phi^{(k)}(\xi^{(k)})$ is a time reverse martingale in $k$ and, for $1 \leq k < r \leq q$, we have

$$
\Psi^{(k,r)}(v) = \mathbb{E}[\xi^{(k)} | \Phi^{(r,k)}(\xi^{(k)}) = v], \quad v \in \mathcal{B}^{(r)}. \quad (3.20)
$$

### 3.4 Mode decomposition through partitioning and integration

In the setting of Section 3.3, recall that $\mathcal{I}^{(q)} = \{1\}$ and $V_1^{(q)} = V$ so that the index $j$ in $\Psi_{i,j}^{(k,q)}$ defined in (3.14) only has one value $j = 1$ and $1^{(k)} = \mathcal{I}^{(k)}$, and therefore

$$
\Psi_{i,1}^{(k,q)}(v) := Q_{i,1}^{(k,q)} e_{1,1}^{(k,q)} Q_{1,1}^{(q)} v^{-1}, \quad v \in V, i \in \mathcal{I}^{(k)}. \quad (3.21)
$$

Fix a $v \in V$ and for $k \in \{1, \ldots, q\}$, let

$$
E^{(k)} : \mathcal{I}^{(k)} \to \mathbb{R},
$$

defined by

$$
E^{(k)}(i) := \|\Psi_{i,1}^{(k,q)}(v)\|_{V_1^{(k)}}^2, \quad i \in \mathcal{I}^{(k)}, \quad (3.21)
$$

be the alignment energy of the mode $i \in \mathcal{I}^{(k)}$. Under the nesting relations (3.19), the definition (3.10) of the norms and the semigroup properties of the subspace embeddings imply that

$$
E^{(k+1)}(i) = \sum_{i' \in \mathcal{I}^{(k)}} E^{(k)}(i'), \quad i \in \mathcal{I}^{(k+1)}, k \in \{1, \ldots, q - 1\}. \quad (3.22)
$$

We will now consider applications where the space $(V, \|\cdot\|_V)$ is known, and the spaces $(V_i^{(1)}, \|\cdot\|_{V_i^{(1)}})$, including their index set $\mathcal{I}^{(1)}$, are known, but the spaces $(V_j^{(k)}, \|\cdot\|_{V_j^{(k)}})$ and their indices $\mathcal{I}^{(k)}$, are unknown for $1 < k < q$, as is any relation $\rightsquigarrow$ connecting
them. Instead, they will be constructed by induction from model/data alignments as illustrated in Figures 7 and 11. In these applications

\[
(V, \| \cdot \|_V) = (V^{(q)}_1, \| \cdot \|_{V^{(q)}_1}),
\]

\[
V = \sum_{i \in I^{(1)}} V_i^{(1)}
\]

and the operator \( Q_{1}^{(q)} : \mathcal{V}^{*} \to \mathcal{V} \) associated with the norm \( \| \cdot \|_{V^{(q)}_1} \) is the sum

\[
Q_{1}^{(q)} = \sum_{i \in I^{(1)}} e_{1,i}^{(q,1)} Q_{i}^{(1,1)} e_{i,1}^{(1,q)}.
\]  

(3.23)

In this construction we assume that the set of indices \( I^{(1)} \) are vertices of a graph \( G^{(1)} \), whose edges provide neighbor relations among the indices. The following meta-algorithm, Algorithm 1, forms a general algorithmic framework for the adaptive determination of the intermediate spaces \( (V_j^{(k)}, \| \cdot \|_{V_j^{(k)}}) \), their indices \( I^{(k)} \), and a relation \( \leadsto \), in such a way that Theorem 3.4 applies. Observe that this meta-algorithm is obtained by combining the elementary programming modules illustrated in Figures 8 and 9 and
Algorithm 1 Mode decomposition through partitioning and integration.

1: for \( k = 1 \) to \( q - 2 \) do
2: Compute the function \( E^{(k)} : \mathcal{I}^{(k)} \rightarrow \mathbb{R}_+ \) defined by (3.21).
3: Use the function \( E^{(k)} \) to segment/partition the graph \( G^{(k)} \) into subgraphs \( (G_j^{(k+1)})_{j \in \mathcal{I}^{(k+1)}} \), thereby determining the indices \( \mathcal{I}^{(k+1)} \). Define the ancestors \( j^{(k)} \) of \( j \in \mathcal{I}^{(k+1)} \) as the vertices \( i \in \mathcal{I}^{(k)} \) of the sub-graph \( G_j^{(k+1)} \).
4: Identify the subspaces \( V_j^{(k+1)} \) and the operators \( Q_j^{(k+1)} \) through (3.5) and (3.19).
5: end for
6: Recover the modes \( (\Psi_{(q-1,q)}(v))_{i \in \mathcal{I}(q-1)} \) of \( v \).

discussed in Section 3.2. In the following Section 3.5, it is demonstrated on a problem in time-frequency mode decomposition.

3.5 Application to time-frequency decomposition

We will now propose a solution to Problem 1 (note that for that problem the observed data is the whole function \( v = u + v_\sigma \)) based on the hierarchical segmentation approach described in Section 3.4. We will employ the GPR interpretation of Section 2.3 and assume that \( v \) is the realization of a Gaussian process \( \xi \) obtained by integrating Gabor wavelets [9] against white noise. To that end, for \( \tau, \theta \in \mathbb{R} \) and \( \omega, \alpha > 0 \), let

\[
\chi_{\tau,\omega,\theta}(t) := \left( \frac{2}{\pi} \right)^{\frac{1}{2}} \sqrt{\frac{\omega}{\alpha}} \cos(\omega(t - \tau) + \theta) e^{-\frac{\omega^2(t-\tau)^2}{\alpha^2}}, \quad t \in \mathbb{R}, \tag{3.24}
\]

be the shifted/scaled Gabor wavelet, whose scaling is motivated by the normalization \( \int_{-\pi}^{\pi} \int_{\mathbb{R}} \chi_{\tau,\omega,\theta}^2(t) \, dt \, d\theta = 1 \). See Figure 12 for an illustration of the Gabor wavelets. Recall [9] that each \( \chi \) is minimally localized in the time-frequency domain (it minimizes the product of standard deviations in the time and frequency domains) and the parameter \( \alpha \) is proportional to the ratio between localization in frequency and localization in space.

Figure 12: Gabor wavelets \( \chi_{\tau,\omega,\theta} (3.24) \) for various parameter values.
Let $\zeta(\tau, \omega, \theta)$ be a white noise process on $\mathbb{R}^3$ (a centered GP with covariance function $E[\zeta(\tau, \omega, \theta)\zeta(\tau', \omega', \theta')] = \delta(\tau - \tau')\delta(\omega - \omega')\delta(\theta - \theta')$) and let

$$
\xi_u(t) := \int_{-\pi}^\pi \int_{\omega_{\text{min}}}^{\omega_{\text{max}}} \int_0^1 \zeta(\tau, \omega, \theta)\chi_{\tau, \omega, \theta}(t) d\tau d\omega d\theta, \quad t \in \mathbb{R}.
$$

(3.25)

Letting, for each $\tau, \omega$ and $\theta$,

$$
K_{\tau, \omega, \theta}(s, t) := \chi_{\tau, \omega, \theta}(s)\chi_{\tau, \omega, \theta}(t), \quad s, t \in \mathbb{R},
$$

(3.26)

be the reproducing kernel associated with the wavelet $\chi_{\tau, \omega, \theta}$, it follows that $\xi_u$ is a centered GP with covariance function

$$
K_u(s, t) = \int_{-\pi}^\pi \int_{\omega_{\text{min}}}^{\omega_{\text{max}}} \int_0^1 K_{\tau, \omega, \theta}(s, t) d\tau d\omega d\theta, \quad s, t \in \mathbb{R}.
$$

(3.27)

Given $\sigma > 0$, let $\xi_\sigma(t)$ be a white noise process on $\mathbb{R}$ (independent from $\zeta$) of variance $\sigma^2$ (a centered GP with covariance function $E[\xi_\sigma(s)\xi_\sigma(t)] = \sigma^2\delta(s - t)$) and let $\xi$ be the GP defined by

$$
\xi := \xi_u + \xi_\sigma.
$$

(3.28)

$\xi$ is a centered GP with covariance function defined by the kernel

$$
K := K_u + K_\sigma
$$

(3.29)

with

$$
K_\sigma(s, t) = \sigma^2\delta(s - t).
$$

(3.30)

Hence, compared to the setting of Section 2, and apart from the mode corresponding to the noise $\xi_\sigma$, the finite number of modes indexed by $\mathcal{I}$ has been turned into a continuum of modes indexed by

$$
\mathcal{I} := \{ (\tau, \omega, \theta) \in [0, 1] \times [\omega_{\text{min}}, \omega_{\text{max}}] \times (-\pi, \pi) \}
$$

with corresponding one dimensional subspaces

$$
V^{(1)}_{(\tau, \omega, \theta)} = \text{span}\{\chi_{\tau, \omega, \theta}\},
$$

positive operators $Q_{\tau, \omega, \theta}$ defined by the kernels $K_{\tau, \omega, \theta}(s, t)$ and the integral

$$
K_u(s, t) = \int_{-\pi}^\pi \int_{\omega_{\text{min}}}^{\omega_{\text{max}}} \int_0^1 K_{\tau, \omega, \theta}(s, t) d\tau d\omega d\theta, \quad s, t \in \mathbb{R},
$$

of these kernels (3.27) to obtain a master kernel $K_u$ instead of a sum

$$
S = \sum_{i \in \mathcal{I}} e_i Q_i e_i^* 
$$

as in (2.12). Table 4 illustrates the time-frequency version of Table 2 we have just developed and the following remark explains the connection between kernels and operators in more detail.
Table 4: The time-frequency version of Table 2

| Mode | GP | Kernel |
|------|----------------|---------|
| $v_{\tau,\omega,\theta}(t) = a_{\tau,\omega,\theta}(t)\chi_{\tau,\omega,\theta}(t)$ | $\xi_{\tau,\omega,\theta}(t) = \zeta(\tau,\omega,\theta)\chi_{\tau,\omega,\theta}(t)$ | $K_{\tau,\omega,\theta}(s,t) = \chi_{\tau,\omega,\theta}(s)\chi_{\tau,\omega,\theta}(t)$ |
| $a_{\tau,\omega,\theta}$ unknown and $L^2$ | $E[\zeta(\tau,\omega,\theta)\zeta(\tau',\omega',\theta')]=\delta(\tau-\tau')\delta(\omega-\omega')\delta(\theta-\theta')$ | |
| $v_{\tau,\omega} = \int_{-\pi}^{\pi} v_{\tau,\omega,\theta}d\theta$ | $\xi_{\tau,\omega}(t) = \int_{-\pi}^{\pi} \xi_{\tau,\omega,\theta}(t)d\theta$ | $K_{\tau,\omega}(s,t) = \int_{-\pi}^{\pi} K_{\tau,\omega,\theta}(s,t)d\theta$ |
| $v_{\omega} = \int\int\int v_{\tau,\omega,\theta}d\tau d\omega d\theta$ | $\xi_{\omega}(t) = \int\int\int \xi_{\tau,\omega,\theta}(t)d\tau d\omega d\theta$ | $K_{\omega}(s,t) = \int\int\int K_{\tau,\omega,\theta}(s,t)d\tau d\omega d\theta$ |
| $v_{\sigma}$ unknown white noise | $E[\xi_{\sigma}(s)\xi_{\sigma}(t)] = \sigma^2\delta(s-t)$ | $K_{\sigma}(s,t) = \sigma^2\delta(s-t)$ |
| $v = v_{\omega} + v_{\sigma}$ | $\xi = \xi_{\omega} + \xi_{\sigma}$ | $K = K_{\omega} + K_{\sigma}$ |
| $v_{\omega} = \int\int v_{\tau,\omega,\theta}d\tau d\omega$ | $\xi_{\omega}(t) = \int\int \xi_{\tau,\omega,\theta}(t)d\tau d\omega$ | $K_{\omega}(s,t) = \int\int K_{\tau,\omega,\theta}(s,t)d\tau d\omega d\theta$ |

Remark 3.7 (Kernels, operators, and discretizations). This kernel mode decomposition framework constructs reproducing kernels $K$ through the integration of elementary reproducing kernels, but the recovery formula of Theorem 2.3 requires the application of operators, and their inverses, corresponding to these kernels. In general, there is no canonical connection between kernels and operators, but here we consider restricting to the unit interval $[0,1] \subset \mathbb{R}$ in the time variable $t$. Then, each kernel $K$ under consideration other than $K_{\sigma}$ corresponds to the symmetric positive integral operator

$$K : L^2[0,1] \to L^2[0,1]$$

defined by

$$(Kf)(s) := \int_0^1 K(s,t)f(t)dt, \quad s \in [0,1], \ f \in L^2[0,1].$$

Moreover, these kernels all have sufficient regularity that $K$ is compact and therefore not invertible, see e.g. Steinwart and Christmann [33, Thm. 4.27]. On the other hand, the operator

$$K_{\sigma} : L^2[0,1] \to L^2[0,1]$$

corresponding to the white noise kernel $K_{\sigma}$ (3.30) is

$$K_{\sigma} = \sigma^2 I$$

where

$$I : L^2[0,1] \to L^2[0,1]$$

26
is the identity map. Since $K = K_u + K_\sigma$ (3.29), the operator $\bar{K} = \bar{K}_u + \bar{K}_\sigma$ is a symmetric positive compact operator plus a positive multiple of the identity and therefore it is Fredholm and invertible. Consequently, we can apply Theorem 2.3 for the optimal recovery.

In addition, in numerical applications, $\tau$ and $\omega$ are discretized (using $N + 1$ discretization steps) and the integrals in (3.34) are replaced by sums over $\tau_k := k/N$ and $\omega_k := \omega_{\min} + \frac{k}{N}(\omega_{\max} - \omega_{\min})$ ($k \in \{0,1,\ldots, N\}$). Moreover, as in Example 2.6, the time interval $[0,1]$ is discretized into $M$ points and the corresponding operators on $\mathbb{R}^M$ are $\sigma^2 I$, where $I : \mathbb{R}^M \to \mathbb{R}^M$ is the identity, plus the kernel matrix $(K_u(t_i, t_j))_{i,j=1}^M$ corresponding to the sample points $t_i, i = 1,\ldots, M$. Nevertheless, for simplicity and conciseness, we will keep describing the proposed approach in the continuous setting. Moreover, we will overload notation and not use the $\bar{K}$ notation, but instead use the same symbol for a kernel and its corresponding operator.

Figure 13: Mode decomposition through partitioning and integration. $q = 4$, $w^{(3)} := \psi^{(3,4)} v$, $w^{(2)} := \psi^{(2,4)} v$, and $\sigma$ corresponds to the noise component.

We now describe the hierarchical approach of Section 3.4 to this time-frequency setting and illustrate it in Figure 13. To that end, we identify $\mathcal{I}$ with $\mathcal{I}^{(1)}$ so that

$$\mathcal{I}^{(1)} = \{(\tau, \omega, \theta) \in [0,1] \times [\omega_{\min}, \omega_{\max}] \times (-\pi, \pi)\} \cup \{\sigma\},$$

where the noise mode has been illustrated in Figure 13 by adding an isolated point with label $\sigma$ to each set $\mathcal{I}^{(k)}$ with $k < q = 4$.

Although Line 3 of Algorithm 1 uses the energy $E^{(1)}$ at level $k = 1$ to partition the index set $\mathcal{I}^{(1)}$, the algorithm is flexible in how we use it or if we use it. In this particular application we first ignore the computation of $E^{(1)}$ and straightforward partition $\mathcal{I}^{(1)}$ into a family of subsets

$$\mathcal{I}^{(1)}_{\tau, \omega} := \{(\tau, \omega, \theta) : \theta \in (-\pi, \pi)\} \cup \{\sigma\}, \quad (\tau, \omega) \in [0,1] \times [\omega_{\min}, \omega_{\max}],$$
indexed by \( \tau \) and \( \omega \), so that the corresponding index set at level \( k = 2 \) is
\[
\mathcal{I}^{(2)} = \{ (\tau, \omega) \in [0, 1] \times [\omega_{\min}, \omega_{\max}] \} \cup \{ \sigma \},
\]
and the ancestors of \((\tau, \omega, \sigma)\) are
\[
(\tau, \omega, \sigma)^{(2)} = \{ (\tau, \omega, \theta) : \theta \in (-\pi, \pi] \} \cup \{ \sigma \}.
\]
The subspace corresponding to the label \((\tau, \omega)\) is then
\[
V_{(\tau, \omega)}^{(2)} = \text{span}\{ \chi_{\tau, \omega, \theta} | \theta \in (-\pi, \pi] \}
\]
and, as in (3.19), its associated positive operator is characterized by the kernel
\[
K_{\tau, \omega} := \int_{-\pi}^{\pi} K_{\tau, \omega, \theta} d\theta. \tag{3.31}
\]
We can evaluate \( K_{\tau, \omega} \) using (3.26) and (3.24) by defining
\[
\chi_{\tau, \omega, c} := \left( \frac{2}{\pi} \right)^{\frac{1}{4}} \sqrt{\frac{\omega}{\alpha}} \cos(\omega(\cdot - \tau)) e^{-\frac{\omega^2(\cdot - \tau)^2}{\alpha^2}},
\]
\[
\chi_{\tau, \omega, s} := \left( \frac{2}{\pi} \right)^{\frac{1}{4}} \sqrt{\frac{\omega}{\alpha}} \sin(\omega(\cdot - \tau)) e^{-\frac{\omega^2(\cdot - \tau)^2}{\alpha^2}}, \tag{3.32}
\]
and using the trigonometric identity \( \cos(a + \theta) = \cos a \cos \theta - \sin a \sin \theta \), to obtain
\[
K_{\tau, \omega}(s, t) := \chi_{\tau, \omega, c}(s)\chi_{\tau, \omega, c}(t) + \chi_{\tau, \omega, s}(s)\chi_{\tau, \omega, s}(t). \tag{3.33}
\]
Therefore \( V_{(\tau, \omega)}^{(2)} = \text{span}\{ \chi_{\tau, \omega, c}, \chi_{\tau, \omega, s} \} \) and (3.27) reduces to
\[
K_u(s, t) = \int_{\omega_{\min}}^{\omega_{\max}} \int_{0}^{1} K_{\tau, \omega}(s, t) d\tau d\omega. \tag{3.34}
\]
Using \( K := K_u + K_\sigma \) (3.29), let \( f \) be the solution of the linear system \( \int_{0}^{1} K(s, t) f(t) dt = v(s) \), i.e.
\[
K f = v, \tag{3.35}
\]
and let \( E(\tau, \omega) \) be the energy of the recovered mode indexed by \((\tau, \omega)\), i.e.
\[
E(\tau, \omega) = \int_{0}^{1} \int_{0}^{1} f(s) K_{\tau, \omega}(s, t) f(t) ds dt, \quad (\tau, \omega) \in [0, 1] \times [\omega_{\min}, \omega_{\max}]. \tag{3.36}
\]
Since \( K f = v \) implies that
\[
v^T K^{-1} v = f^T K f,
\]
it follows that
\[
v^T K^{-1} v = \int_{\omega_{\min}}^{\omega_{\max}} \int_{0}^{1} E(\tau, \omega) d\tau d\omega + f^T K_\sigma f. \tag{3.37}
\]
For the recovery of $n$ modes using Algorithm 1, at the second level $k = 2$ we use $E(\tau, \omega)$ to partition the time-frequency domain of $(\tau, \omega)$ into $n$ disjoint subsets $A(1), A(2), \ldots, A(n)$ as illustrated in Figure 13 for $n = 3$ (in particular, $n$ is not assumed to be known beforehand and recovered from $E(\tau, \omega)$), and then define $\mathcal{I}^{(3)}$ as $\{1, 2, \ldots, n, \sigma\}$, the subspace corresponding to the mode $i \neq \sigma$ as $V_i^{(3)} = \text{span}\{\chi_{\tau, \omega, c}, \chi_{\tau, \omega, s} \mid (\tau, \omega) \in A(i)\}$ and the kernel associated with the mode $i \neq \sigma$ as

$$K_i(s, t) = \int_{(\tau, \omega) \in A(i)} K_{\tau, \omega}(s, t) d\tau d\omega, \quad s, t \in \mathbb{R},$$

(3.38)

as displayed in the bottom row in Table 4, so that

$$K_u = \sum_{i=1}^{n} K_i.$$ (3.39)

We then apply the optimal recovery formula of Theorem 2.3 to approximate the modes of $v_1, \ldots, v_n$ of $u$ from the noisy observation of $v = u + v_\sigma$ (where $v_\sigma$ is a realization of $\xi_\sigma$) with the elements $w_1, \ldots, w_n$ obtained via the integration $w_i = K_i K^{-1} v = K_i f$.

Figure 14 illustrates a three mode $n = 3$ noisy signal and the recovery of its modes. Sub-figure (1) displays the total observed signal $v = u + v_\sigma$ and the three modes $v_1, v_2, v_3$ constituting $u = v_1 + v_2 + v_3$ are displayed in sub-figures (5), (6), and (7) along with their recoveries $w_1, w_2$ and $w_3$. Sub-figure 8 of Figure 14 also shows approximations of the instantaneous frequencies obtained as

$$\omega_{i, E}(t) := \arg\max_{(\omega, t) \in A(i)} E(t, \omega).$$ (3.40)

4 Additional programming modules and squeezing

In the approach described in Section 3.4, $\mathcal{I}^{(k)}$ was partitioned into subsets $(j^{(k)})_{j \in \mathcal{I}^{(k+1)}}$ and the $Q_i^{(k)}$ were integrated (that is, summed over or average-pooled) using (3.5) and (3.19) in Line 4 of Algorithm 1, over each subset to obtain the $Q_j^{(k+1)}$. This partitioning approach can naturally be generalized to a domain decomposition approach by letting the subsets be non-disjoint and such that $\cup_{j \in \mathcal{I}^{(k+1)}} j^{(k)}$ forms a strict subset\(^2\) of $\mathcal{I}^{(k)}$ (i.e. some $i \in \mathcal{I}^{(k)}$ may not have descendants). We will now generalize the relation $\rightarrow$ so as to (1) not satisfy Condition 3.3 (2) be non directed, that is, not satisfy Definition 3.2

\(^2\)Although the results of Theorem 3.4 do not hold true under this general domain-decomposition, those of Theorem 2.3 remain true between levels $k$ and $q$ (at each level $k$ the $v_i^{(k)}$ are optimal recovered modes given the $Q_i^{(k)}$ and the observation $v$).
Figure 14: (1) The signal $v = u + v_\sigma$ where $u = v_1 + v_2 + v_3$, $v_\sigma \sim \mathcal{N}(0, \sigma^2 \delta(t-s))$ and $\sigma = 0.01$ (2) $(\tau, \omega) \rightarrow E(\tau, \omega)$ defined by (3.36) (one can identify three stripes) (3) $\omega \rightarrow E(0.6, \omega)$ (4) Partitioning $[0,1] \times [\omega_{\text{min}}, \omega_{\text{max}}] = \mathcal{P}_{i=1} E(i)$ of the time frequency domain into three disjoint subsets identified from $E$ (5) $v_1$ and its approximation $w_1$ (6) $v_2$ and its approximation $w_2$ (7) $v_3$ and its approximation $w_3$ (8) $\omega_1, \omega_2, \omega_3$ and their approximations $\omega_1, E, \omega_2, E, \omega_3, E$.

(some $j \in \mathcal{I}^{(k+1)}$ may have descendants in $\mathcal{I}^{(k)}$ (3) to not define a map (a label $i$ may have multiple descendants) and (4) enable loops.

With this generalization the proposed framework is closer (in spirit) to an object oriented programming language than to a meta-algorithm. We will therefore describe it as such via the introduction of additional elementary programming modules and illustrate the proposed language by programming increasingly efficient networks for mode decomposition.

4.1 Elementary programming modules

We will now introduce new elementary programming modules in addition to the five illustrated in Figure 8 and discussed in Section 3.2. These new modules, beginning with module (6), are illustrated in Figure 15. Here they will be discussed abstractly but forward reference to specific examples.. The first module (module (6)) of Figure 15 replaces the average-pooling operation to define the energy $E$ by a max-pool operation. More precisely module (6) combines an $i \rightarrow j$ relation with an energy $E$ to produce a max-pool energy via

$$S(j) = \max_{i \rightarrow j} E(i),$$

(4.1)
where $i \leadsto j$ here is over $i$ from the previous level to that of $j$. In what follows we will adhere to this semantic convention. As shown in module (7), this combination can also be performed starting with a max-pool energy, i.e. module (7) combines a $i \leadsto j$ relation with a max-pool energy $S$ to produce a max-pool energy via

$$S(j) = \max_{i \leadsto j} S(i). \quad (4.2)$$

Maximizers can naturally be derived from this max-pooling operation and modules (8) and (9) define $i(j)$ as the maximizer (or the set of maximizers if non-unique) of the energy or the max-pool energy. More precisely module (8) combines an $i \leadsto j$ relation with an energy function $E(i)$ to produce

$$i(j) = \arg\max_{i \leadsto j} E(i), \quad (4.3)$$

and module (9) combines an $i \leadsto j$ relation with a max-pool energy function $S(i)$ to produce

$$i(j) = \arg\max_{i \leadsto j} S(i). \quad (4.4)$$

Similarly to module (3) of Figure 8, module (10) of Figure 15 combines the max-pool energy $S$ with a graph operation to produce the ancestor-descendant relation $i \leadsto j$. We will show that module (10) leads to a more robust domain decomposition than module (3) due to its insensitivity to domain discretization. Module (11) uses the functional dependence $j(i)$ to define the relation $i \leadsto j$. Module (12) expresses the transitivity of function dependence, i.e. it combines $j(i)$ and $k(j)$ to produce $k(i)$. Similarly, module (13) expresses the transitivity of the $\leadsto$ relation, i.e. $i \leadsto j$ and $j \leadsto k$ can be combined to produce $i \leadsto k$. Module (14) (analogously to module (4)) uses an injection step to define a functional dependence $i(j)$ (e.g. for the time-frequency application in Figure 19, if $\mathcal{J}$ is the set of $(\tau, \omega')$ and $\mathcal{I}$ is that of $(\tau, \omega)$ the injection $\iota : \mathcal{I} \cap \mathcal{J} \to \mathcal{I}$ defines a functional dependence $i(j)$). Module (15) uses a functional dependence $i(j)$ to produce another functional dependence $k(j)$ (e.g. for the time-frequency-phase application in Figures 21 and 22, we can define the functional dependence $(\tau, \omega')(\tau, \omega)$ from the functional dependence $(\tau, \omega, \theta)(\tau, \omega)$ via $\omega'(\tau, \omega) = \hat{c}_\tau \theta(\tau, \omega)$). Module (16) utilizes the functional dependence $k(j)$ to produce another functional dependence $l(j)$ (e.g. for the time-frequency-phase-application in Figure 23, we can define the functional dependence $(\tau, \omega, \phi)(\tau, \omega)$ from the functional dependence $(\tau, \omega, \theta)(\tau, \omega)$ via $\phi(\tau, \omega) = \hat{c}_\tau \theta(\tau, \omega)$).
dependence \(i(j)\) to produce a pullback covariance operator \(Q_j := Q_{i(j)}\) (\(:= \sum_{i \in i(j)} Q_i\) if \(i(j)\) is a set-valued rather than a single-valued mapping). Module (17) combines a functional dependence \(i(j)\) with a relation \(i \rightarrow i\) to produce a pullback covariance operator \(Q\).

if \(i(p)j(q)\) is a set-valued rather than a single-valued mapping). Module (17) combines a functional dependence \(i(p)j(q)\) with a relation \(j \rightarrow k\) to produce a pullback covariance operator \(Q_k\).

4.2 Programming the network

Programming of the network is achieved by assembling the modules of Figures 8 and 15 in a manner that (1) \(v\) is one of the inputs of the network and (if the network is used for mode decomposition/pattern recognition) (2) the modes \(v_m\) are one of the outputs of the network. As with any interpretable programming language avoiding inefficient coding and bugs remains important. We will now use this language to program KMDNets.

4.3 Squeezing

We will now present an interpretation and a variant (illustrated in Figure 17) of the synchrosqueezing transform due to Daubechies, Lu and Wu \([3]\) in the setting of KMDNets, and thereby initiate its GP regression version. We will demonstrate that this version generalizes to the case where the basic waveform is non-periodic and/or unknown. We use the setting and notations of Section 3.5.

Let \(f\) be the solution of \(Kf = v\) (3.35) and let

\[
E(\tau, \omega, \theta) := \int_0^1 \int_0^1 f(s)K_{\tau,\omega,\theta}(s,t)f(t)\,ds\,dt
\]

be the energy of the mode indexed by \((\tau, \omega, \theta)\). For \((\tau, \omega) \in [0,1] \times [\omega_{\min}, \omega_{\max}]\), write

\[
\theta_e(\tau, \omega) := \arg\max_{\theta \in [\omega_{\min}, \omega_{\max}]} E(\tau, \omega, \theta).
\]

Since the definitions (3.24) of \(\chi_{\tau,\omega,\theta}\) and \(\chi_{\tau,\omega,c}\) and \(\chi_{\tau,\omega,s}\) in (3.32), together with the identity \(\cos(a + \theta) = \cos a \cos \theta - \sin a \sin \theta\) imply that

\[
\chi_{\tau,\omega,\theta}(t) = \frac{1}{\sqrt{\pi}}(\chi_{\tau,\omega,c}(t) \cos(\theta) - \chi_{\tau,\omega,s}(t) \sin(\theta)), \quad t \in \mathbb{R},
\]

it follows that, if we define

\[
W_c(\tau, \omega) := \int_0^1 \chi_{\tau,\omega,c}(t)f(t)\,dt
\]

\[
W_s(\tau, \omega) := \int_0^1 \chi_{\tau,\omega,s}(t)f(t)\,dt,
\]

32
we obtain
\[
\int_0^1 \chi_{\tau,\omega}\phi(t)f(t)\,dt = \frac{1}{\sqrt{\pi}} \left(\cos(\theta)W_c(\tau,\omega) - \sin(\theta)W_s(\tau,\omega)\right).
\] \hspace{1cm} (4.8)

Consequently, we deduce from (4.5) and (3.26) that
\[
E(\tau,\omega,\theta) = \frac{1}{\pi} \left(\cos(\theta)W_c(\tau,\omega) - \sin(\theta)W_s(\tau,\omega)\right)^2.
\] \hspace{1cm} (4.9)

It follows that, when either \(W_c(\tau,\omega) \neq 0\) or \(W_s(\tau,\omega) \neq 0\), that
\[
\theta_c(\tau,\omega) = \text{phase} \left(W_c(\tau,\omega) - iW_s(\tau,\omega)\right),
\] \hspace{1cm} (4.10)

where, for a complex number \(z\), \(\text{phase}(z) := \theta \in (-\pi, \pi] : z = re^{i\theta}, r > 0\). Moreover, it follows from (4.9) that with \(E(\tau,\omega)\) defined as in (3.36), that
\[
E(\tau,\omega) = W_c^2(\tau,\omega) + W_s^2(\tau,\omega).
\] \hspace{1cm} (4.11)

Now consider the mode decomposition problem with observation \(v \sum v_i\), and let \(t \to \theta_i(t)\) be the phase of the mode
\[
v_i(t) = a_i(t) \cos(\theta_i(t)) + b_i(t) \sin(\theta_i(t))
\]
where \(a_i\) and \(b_i\) are slowly varying compared to \(\theta_i\). Since near the time \(\tau\), \(v_i(t) \approx a_i(\tau) \cos((t - \tau)\theta_i(\tau) + \theta_i(\tau)) + b_i(\tau) \sin((t - \tau)\theta_i(\tau) + \theta_i(\tau))\). It follows from the representation (3.32) of \(\chi_{\tau,\omega,c}\) and \(\chi_{\tau,\omega,s}\) that for \(\omega \approx \theta_i(\tau), \theta_i(\tau,\omega)\) is an approximation of \(\theta_i(\tau)\) and
\[
\omega_i(\tau,\omega) = \frac{\partial \theta_i}{\partial \tau}(\tau,\omega), \hspace{1cm} (4.12)
\]
is an approximation of the instantaneous frequency \(\dot{\theta_i}(\tau)\).

**Remark 4.1.** In the discrete setting, to minimize errors, (4.12) is approximated by the solution of
\[
e^{i\omega_c(\tau_k,\omega)(\tau_k+1-\tau_k)}e^{i\text{phase}(W_c(\tau_k,\omega)-iW_s(\tau_k,\omega))} = e^{i\text{phase}(W_c(\tau_{k+1},\omega)-iW_s(\tau_{k+1},\omega))},
\]
i.e. (writing \(\text{atan2}\) for Matlab’s four-quadrant inverse tangent)
\[
\omega_c(\tau_k,\omega) = \frac{1}{\tau_{k+1} - \tau_k} \text{atan2} \left(\frac{W_c(\tau_{k+1},\omega)W_s(\tau_k,\omega) - W_s(\tau_{k+1},\omega)W_c(\tau_k,\omega)}{W_c(\tau_{k+1},\omega)W_c(\tau_k,\omega) + W_s(\tau_{k+1},\omega)W_s(\tau_k,\omega)}\right) \hspace{1cm} (4.13)
\]

In preparation for illustrating the application of the programming of KMDNets, as a synchrosqueezing algorithm, to the decomposition problem when \(v\) and its modes are as in Figure 14, Figure 16 illustrates the basic quantities we have just been developing. In particular,

- The functions \(W_c\) and \(W_s\) are shown in sub-figures (1) and (2) of Figure 16.
- The function \(\tau \to (W_c(\tau,300),-W_s(\tau,300))\) is shown in sub-figure (3) of Figure 16. The functions \(\theta_c(\tau,300), E(\tau,300)\) and \(\omega_c(\tau,300)\) are the phase, square modulus and angular velocity of this function.
Figure 16: (1) $W_c(\tau, \omega)$ (2) $W_s(\tau, \omega)$ (3) $\tau \rightarrow (W_c(\tau, 300), W_s(\tau, 300), \tau)$ (4) $(\tau, \omega) \rightarrow \theta_c(\tau, \omega)$ (5) $(\tau, \omega) \rightarrow \omega_c(\tau, \omega)$ (6) $\omega \rightarrow \omega_c(0.6, \omega)$ and $\omega \rightarrow E(0.6, \omega)$ (7) $\omega \rightarrow S(0.6, \omega)$ (8) $\omega \rightarrow S_E(0.6, \omega)$ (9) $(\tau, \omega) \rightarrow S(\tau, \omega)$ (10) $t \rightarrow \omega_i(t)$ and $t \rightarrow \omega_{i,E}(t)$ for $i \in \{1, 2, 3\}$ (11) $t \rightarrow \cos(\theta_1(t))$ and $t \rightarrow \cos(\theta_{1,E}(t))$ (12) $t \rightarrow \sin(\theta_1(t))$ and $t \rightarrow \sin(\theta_{1,E}(t))$.

- The functions $(\tau, \omega) \rightarrow \theta_c(\tau, \omega)$, $\tau \rightarrow \theta_c(\tau, \omega_{i,E}(\tau))$ (with $\omega_{i,E}$ defined in (3.40)) and $t \rightarrow \theta_i(t)$ are shown in sub-figures (4), (11) and (12) of Figure 16. Observe that $\tau \rightarrow \theta_c(\tau, \omega_{i,E}(\tau))$ is an approximation of $\tau \rightarrow \theta_i(\tau)$.

- The functions $(\tau, \omega) \rightarrow \omega_c(\tau, \omega)$, $\omega \rightarrow \omega_c(0.6, \omega)$ and $\tau \rightarrow \omega_c(\tau, \omega_{i,E}(\tau))$ are shown in sub-figures (5), (6) and (10) of Figure 16. Observe that $\tau \rightarrow \omega_c(\tau, \omega_{i,E}(\tau))$ is an approximation of the instantaneous frequency $\tau \rightarrow \omega_i(\tau) = \theta_i(\tau)$ of the mode $v_i$.

To describe the remaining components of Figure 16 and simultaneously complete the application of the programming of KMDNets as a synchrosqueezing algorithm and introduce a max-pool version of synchrosqueezing, we now introduce the synchrosqueezed energy $S_E(\tau, \omega)$ and the max-pool energy $S(\tau, \omega)$: The synchrosqueezed energy $S_E(\tau, \omega)$ is defined by

$$
\int_{\omega_{\text{min}}}^{\omega_{\text{max}}} \varphi(\omega) S_E(\tau, \omega) \, d\omega = \int_{\omega_{\text{min}}}^{\omega_{\text{max}}} \varphi(\omega_c(\tau, \omega')) E(\tau, \omega') \, d\omega'
$$
for all regular test function $\varphi$, i.e.

$$S_E(\tau, \omega) = \lim_{\delta \to 0} \frac{1}{\delta} \int_{\omega' : \omega - \delta \leq \omega' \leq \omega + \delta} E(\tau, \omega') \, d\omega',$$

where numerically we use the approximation

$$S_E(\tau, \omega) = \frac{1}{\delta} \int_{\omega' : \omega - \delta \leq \omega' \leq \omega + \delta} E(\tau, \omega') \, d\omega'$$

(4.14)

for $\delta$ small. The function $S_E(\tau, \omega)$ is closely related to the synchrosqueezed transform introduced in [3] in that the value of $S_E(\tau, \omega)$ is obtained by transporting the energy $E(\tau, \omega)$ via the map $(\tau, \omega) \to (\tau, \omega_\tau(\tau, \omega))$.

Returning to the application, the transport of the energy $E(\tau, \omega)$ via the map $(\tau, \omega) \to (\tau, \omega_\tau(\tau, \omega))$ is illustrated for $\tau = 0.6$ by comparing the plots of the functions $\omega \to \omega_\tau(0.6, \omega)$ and $\omega \to E(0.6, \omega)$ in sub-figure (6) with the function $\omega \to S_E(0.6, \omega)$ shown in sub-figure (8) of Figure 16. As in [3] the value of $S_E(\tau, \omega)$ (and thereby the height of the peaks in sub-figure (8)) depends on the discretization and the measure $d\omega$ used in the integration (4.14). For example, using a logarithmic discretization or replacing the Lebesgue measure $d\omega$ by $\omega d\omega$ in (4.14) will impact the height of those peaks. To avoid this dependence on the choice of measure, we define the max-pool energy

$$S(\tau, \omega) = \max_{\omega' : \omega_\tau(\tau, \omega') = \omega} E(\tau, \omega'),$$

(4.15)

illustrated in sub-figure (9) of Figure 16. Comparing sub-figures (6), (7) and (8) of Figure 16, observe that, although both synchrosqueezing and max-pooling decrease the width of the peaks of the energy plot $\omega \to E(0.6, \omega)$, only max-squeezing preserves their heights (as noted in [3, Sec. 2] a discretization dependent weighting of $d\omega$ would have to be introduced to avoid this dependence).

Figure 17: Synchro-squeezed (left) and max-pool (right) energies.

Figure 17 provides an interpretation of the synchrosqueezed and max-pool energies $S_E(\tau, \omega)$ and $S(\tau, \omega)$ in the setting of KMDNet programming. The left (synchrosqueezed)
and right (max-pool) sub-figures are identical except for the highlighted portions near their top center. In that interpretation $I^{(1)}$ and $I^{(2)}$ are, as in Section 3.5 and modulo the noise mode $\sigma$, respectively, the set of time-frequency-phase labels $(\tau, \omega, \theta) \in [0, 1] \times [\omega_{\text{min}}, \omega_{\text{max}}] \times (-\pi, \pi)$ and the set of time-frequency labels $(\tau, \omega) \in [0, 1] \times [\omega_{\text{min}}, \omega_{\text{max}}]$. Modulo the noise label $\sigma$, $I^{(3)}$ is the range of $(\tau, \omega) \to (\tau, \omega_{e}(\tau, \omega))$ and the ancestors of $(\tau, \omega') \in I^{(3)}$ are the $(\tau, \omega)$ such that $\omega' = \omega_{e}(\tau, \omega)$. Then, in that interpretation, the synchrosqueezed energy is simply the level 3 energy $E^{(3)}$, whereas $S^{(3)}$. Note that the proposed approach naturally generalizes to the case where the periodic waveform $y$ is known and non-trigonometric by simply replacing the cosine function in (3.24) by $y$.

Figure 18: (1) The signal $v = v_1 + v_2 + v_3 + v_\sigma$ where $v_\sigma \sim \mathcal{N}(0, \sigma^2 \delta(s-t))$ and $\sigma = 0.01$ (2) instantaneous frequencies $t \to \omega_i(t)$ of the modes $i = 1, 2, 3$ (3) $(\tau, \omega) \to S(\tau, \omega)$ (4) Sub-domains $A(1), A(2)$ and $A(3)$ of the time-frequency domain (5) approximated instantaneous frequencies $t \to \omega_{i,e}(t)$ of the modes $i = 1, 2, 3$ (6, 7, 8) $v_1, v_2, v_3$ and their approximations $w_1, w_2, w_3$ obtained from the network shown in Figure 19 (9) phase $\theta_1$ and its approximation $\theta_{1,e}$ (10, 11, 12) $v_1, v_2, v_3$ and their approximations $w_1, w_2, w_3$ obtained from the network shown in Figure 21.
Let us now demonstrate the effectiveness of the max-pooling technique in its ability to perform mode recovery when the instantaneous frequencies of the modes cross. Consider the noisy signal \( v \) illustrated in sub-figure (1) of Figure 18. This signal is composed of 4 modes, \( v = v_1 + v_2 + v_3 + v_\sigma \), where \( v_\sigma \sim N(0, \sigma^2 \delta(s - t)) \) is a white noise realization with \( \sigma = 0.01 \). The modes \( v_1, v_2, v_3 \) are shown in sub-figures (6), (7), (8) and their instantaneous frequencies \( \omega_1, \omega_2, \omega_3 \) are shown in sub-figure (2). Note that \( \omega_1 \) and \( \omega_2 \) cross each other around \( t \approx 0.6 \) and \( v_3 \) vanishes around \( t \approx 0.3 \). We now program two KMDNets and describe their accuracy in recovering those modes.

The first network, illustrated in Figures 19 and 20 recovers approximations to \( v_1, v_2, v_3 \) by identifying three subsets \( A(1), A(2), A(3) \) of the time-frequency domain \((\tau, \omega)\) and integrating the kernel \( K_{\tau, \omega} \) (defined as in (3.31)) over those subsets (as in (3.38)). For this example, the subsets \( A_1, A(2), A(3) \) are shown in sub-figure (4) of Figure 18 and identified as narrow sausages defined by the peaks of the max-pool energy \( S^{(3)}(\tau, \omega) \) (computed as in (4.15)) shown in sub-figure (3). The corresponding approximations \( w_1, w_2, w_3 \) (obtained as in (3.39)) of the modes \( v_1, v_2, v_3 \) are shown in sub-figures (6), (7) and (8) of Figure 18. Note the increased approximation error around \( t \approx 0.6 \) corresponding to the crossing point between \( \omega_1 \) and \( \omega_2 \) and \( A(1) \) and \( A(2) \). The estimated instantaneous frequencies \( \omega_{i,e}(\tau) = \omega_e(\tau, \arg\max_{\omega_c(\tau, \omega) \in A(i)} S^{(3)}(\tau, \omega)) \) illustrated in sub-
figure (5) of Figure 18 also show an increased estimation error around that crossing point.

![Diagram](image)

Figure 20: The KMDNet program corresponding to Figure 19. Upper left provides the symbolic connections between the indices \( i, j, k \) and the time-frequency parameters along with the functional dependencies \( i \xrightarrow{p} j \) and \( k \xrightarrow{p} j \). Beginning with the input \( v \) in the lower left, the operators \( Q_i \) corresponding to the baby kernels \( K_{\tau,\omega,\theta} \) are used to produce optimal recovery estimates \( w_i \) and the corresponding alignment energies \( E_p(i) \). The projection function \( j(i) \) taking \( (\tau, \omega) \) to \( (\tau, \omega) \) is the relation \( i \xrightarrow{p} j \) which determines the integration operation \( \int d\theta \) indicated as \( \Sigma_{i \rightarrow j} \) which then determines summed energies \( E(j) := \sum_{i \rightarrow j} E(i) \) and covariances \( Q_j := \sum_{i \rightarrow j} Q_i \). Moreover, the projection \( i \xrightarrow{p} j \) also determines a max operation \( \text{arg max}_\theta \) which we denote by \( \text{arg max}_{\theta} \). This function is then differentiated to obtain the functional relation \( k(j) = (\tau, \omega, \theta) \) where \( \omega_e(\tau, \omega) := \frac{\partial}{\partial \omega} \theta_e(\tau, \omega) \). This determines the relation \( j \xrightarrow{p} k \) which determines the maximization operation \( \text{max}_{j \rightarrow k} \) that, when applied to the alignment energies \( E(j) \), produces the max-pooled energies \( S(k) \). These energies are then used to determine a graph cut establishing a relation \( k \xrightarrow{p} m \) where \( m \) is a mode index. Combining this relation with the injection \( j \xrightarrow{p} k \) determines the relation \( j \xrightarrow{p} m \), that then determines the summation \( \sum_{j \rightarrow m} \) over the preimages of the relation, thus determining operators \( Q_m \) indexed by the mode \( m \) by \( Q_m := \sum_{j \rightarrow m} Q_j \). Optimal recovery is then applied to obtain the estimates \( w_m := Q_m(\sum_{m} Q_{m'})^{-1} \).

The second network, illustrated in Figures 21 and 22, proposes a more robust ap-
Figure 21: Recovery from instantaneous phases approximations. The left-hand side of the figure is that of the right-hand side (corresponding to max-pooling) of Figure 17 and therefore also that of Figure 19, and proceeding to the right as in Figure 19, the three subsets $A_1, A_2, A_3$ of the time-frequency domain $(\tau, \omega)$ and integrating the kernel $K_{\tau,\omega}$ (defined as in (3.31)) over those subsets (as in (3.38)). However, to define the kernels $K_m$ for the final optimal recovery, we define $\omega_{m,e}(\tau) := \arg \max_{\omega' \in A(i)} S(3)(\tau, \omega')$ to produce the $\theta$ function for each mode $m$ through $\theta_{m,e}(\tau) = \theta_e(\tau, \omega_{m,e}(\tau))$. These functions are inserted into (4.17) to produce $K_m$ and their associated operators $Q_m$ which are then used in the finally recovery $w_m = Q_m(\sum_{m'} Q_{m'})^{-1} v$.

Approach based on the estimates $\theta_{i,e}$ of instantaneous phases $\theta_i$ obtained as

$$\theta_{i,e}(\tau) = \theta_e(\tau, \arg \max_{\omega \in (\tau, \omega) \in A(i)} S(3)(\tau, \omega)),$$  

where the $A(i)$ are obtained as in the first network, illustrated in Figure 19, and $\theta_e(\tau, \omega)$, used in the definition (4.16) of $\theta_{e,i}(\tau)$, is identified as in (4.10). To recover the modes $v_i$, the proposed network proceeds as in Example 2.6 by introducing the kernels

$$K_i(s,t) = e^{-\frac{(s-t)^2}{\gamma^2}} \left( \cos(\theta_{i,e}(t)) \cos(\theta_{i,e}(s)) + \sin(\theta_{i,e}(t)) \sin(\theta_{i,e}(s)) \right),$$

with $\gamma = 0.2$. Defining $K_{\sigma}$ as in (3.30), the approximations $w_1, w_2, w_3$ of the modes $v_1, v_2, v_3$, shown in sub-figures (10), (11) and (12) of Figure 18, are obtained as in (3.39) with $f$ defined as the solution of $(K_1 + K_2 + K_3 + K_{\sigma}) f = v$. Note that the network illustrated in Figure 21 can be interpreted as the concatenation of 2 networks. One aimed at estimating the instantaneous phases and the other aimed at recovering the modes based on those phases. This principle of network concatenation is evidently generic.


Figure 22: The KMDNet program corresponding to Figure 21. Upper left provides the symbolic connections between the indices $i, j, k, l$ and the time-frequency parameters along with the functional dependencies $i \mapsto j$ and $k \mapsto l$ and the definition of $\theta_{m,e}$. Beginning with the input $v$ in the lower left, ignoring the bottom two rows for the moment, we begin very much as in Figure 20 moving to the right until the determination of the energies $S(k)$, the determination of a graph cut and its resulting $k \mapsto l$, and the resulting arg max relation $k(l) := \arg \max_{i \mapsto j} S(k)$ which amounts to $k(l) = (\tau, \omega_{m,e}(\tau))$. Returning to the second row from the bottom, we compose the functional relations of the injection $j(k)$ and the arg max function $i(l)$ determined by the relation $i \mapsto j$ and the energy $E(i)$, to obtain $i(k)$ and then compose this with the argmax function $k(l)$ to produce the functional dependence $i(l)$ defined by $i(l) = (\tau, \omega_{m,e}(\tau), \theta_{m,e}(\tau))$. Using the projection $l \mapsto m$, this determines the function $\theta_{m,e}(\cdot)$ corresponding to the mode label $m$. These functions are inserted into (4.17) to produce $K_m$ and their associated operators $Q_m$ which are then used in the finally recovery $w_m = Q_m(\sum_m Q_m^{-1})^{-1}v$.

5 Alignments calculated in $L^2$

The calculation of the energies for our prototypical application was done with respect to the inner product defined by the inverse of the operator associated with $K$ defined in (3.29), i.e. the energy of the mode $(\tau, \omega, \theta)$ was defined as $E(\tau, \omega, \theta) = v^T K^{-1} K_{\tau,\omega,\theta} K^{-1} v$ with $K_{\tau,\omega,\theta}$ defined in (3.26). The computational complexity of the method can be accelerated by (1) using the $L^2$ inner product instead of the one defined by $K^{-1}$ (i.e. defining the energy of the mode $(\tau, \omega, \theta)$ by $E_2(\tau, \omega, \theta) = v^T K_{\tau,\omega,\theta} v$ (2) localizing this calculation in a time-window centered around $\tau$ and of width proportional to $1/\omega$.

Our experiments shows that simplification lowers the computational complexity of the proposed approach without impacting its accuracy. Three points justify this observation: (1) Replacing $E$ by $E_2$ is equivalent to calculating mean-squared alignments with respect to the $L^2$-scalar product instead of the one induced by the inverse of the operator
defined by $K$ (2) In the limit where $\sigma \to \infty$ we have $E \approx \sigma^{-4}E_2$, therefore $E$ and $E_2$ are proportional to each other in the high noise regime (3) If $\omega_{\min} = 0$ and $\omega_{\max} = \infty$ then $K_u$ defined by (3.27) is the identity operator on $L^2$. We will now rigorously show that point (3) holds true when the periodic waveform is trigonometric and show in Section 6 that this results holds true independently of the periodic waveform being used.

Let us recall the Schwartz class of test functions

$$S := \{ f \in C^\infty(\mathbb{R}) : \sup_{x \in \mathbb{R}} |x^{m_1} D^{m_2} f(x)| < \infty, m_1, m_2 \in \mathbb{N} \}$$

and the confluent hypergeometric function $1F_1$, defined by

$$1F_1(\alpha, \gamma; z) = 1 + \frac{\alpha z}{\gamma 1!} + \frac{\alpha(\alpha + 1) z^2}{\gamma(\gamma + 1) 2!} + \frac{\alpha(\alpha + 1)(\alpha + 2) z^3}{\gamma(\gamma + 1)(\gamma + 2) 3!} + \ldots,$$

see e.g. see Gradshteyn and Ryzhik [10, Sec. 9.21].

**Theorem 5.1.** Consider extending the definition (3.27) of the kernel $K_u$ so that the range of $\omega$ is extended from $[0, 1]$ to $\mathbb{R}_+$ and that of $\tau$ is extended from $[0, 1]$ to $\mathbb{R}$, so that

$$K_\beta(s, t) = \int_{-\pi}^{\pi} \int_{\mathbb{R}_+} K_{\tau, \omega, \theta}(s, t) d\tau d\omega d\theta, \quad s, t \in \mathbb{R},$$

where, as before,

$$K_{\tau, \omega, \theta}(s, t) := \chi_{\tau, \omega, \theta}(s) \chi_{\tau, \omega, \theta}(t), \quad s, t \in \mathbb{R},$$

but where we have introduced a perturbation parameter $0 \leq \beta \leq 1$ defining the Gabor wavelets

$$\chi_{\tau, \omega, \theta}(t) := \left( \frac{2}{\alpha^2 \pi^3} \right)^{\frac{1}{2}} \omega^{\frac{1-\beta}{2}} \cos(\omega(t - \tau) + \theta) e^{-\frac{\omega^2 (\omega - \tau)^2}{\alpha^2}}, \quad t \in \mathbb{R}, \quad (5.1)$$

defining the baby kernels. Defining the scaling constant

$$H(\beta) := 2^{\beta-1} \sqrt{\pi} (\sqrt{2\alpha})^{1-\beta} \Gamma\left(\frac{\beta}{2}\right) e^{-\frac{\omega^2}{\alpha^2}} 1F_1\left(\frac{\beta}{2}, 1; \frac{\alpha^2}{2}\right),$$

let $K_\beta$ denote the integral operator

$$(K_\beta f)(s) := \frac{1}{H(\beta)} \int_{\mathbb{R}} K_\beta(s, t) f(t) dt$$

associated to the kernel $K_\beta$ scaled by $H(\beta)$. Then we have the semigroup property

$$K_{\beta_1} K_{\beta_2} f = K_{\beta_1 + \beta_2} f, \quad f \in S, \quad \beta_1, \beta_2 > 0, \beta_1 + \beta_2 < 1,$$

and

$$\lim_{\beta \to 0} (K_\beta f)(x) = f(x), \quad x \in \mathbb{R}, \quad f \in S$$

where the limit is taken from above.
6 Universality of the aggregated kernel

Let

\[ y(t) := \sum_{n=-N}^{N} c_n e^{int} \]

be the Fourier expansion of a general \(2\pi\) periodic complex-valued waveform and use it to define wavelets

\[ \chi_{\tau,\omega,\theta}(t) := \omega^{\frac{1-\beta}{2}} y(\omega(t - \tau) + \theta) e^{-\frac{\alpha^2}{2}|t - \tau|^2} \]

as in the \(\beta\)-parameterized wavelet versions of (3.24) in Theorem 5.1, using the waveform \(y\) instead of the cosine. The following lemma evaluates the mama kernel

\[ K_\beta(s, t) := \Re \int_{-\pi}^{\pi} \int_{\bb{R}^2} \chi_{\tau,\omega,\theta}(s)\chi^*_{\tau,\omega,\theta}(t) d\tau d\omega d\theta. \quad (6.1) \]

**Lemma 6.1.** Define the norm

\[ \|y\|^2 := \sum_{n=-N}^{N} e^{-\frac{|n\alpha|^2}{2}} |c_n|^2 \quad (6.2) \]

of the base waveform \(y\). We have

\[ K_\beta(s, t) = 2\pi |s - t|^{\beta - 1} \sum_{n=-N}^{N} a_n(s, t)|c_n|^2 \]

where

\[ a_n(s, t) = \frac{\alpha\sqrt{\pi} (\sqrt{2}\alpha)^{1-\beta}}{2\sqrt{2}} \Gamma\left(\frac{1 - \beta}{2}\right) e^{-\frac{|n\alpha|^2}{2}} {}_1F_1\left(\frac{\beta}{2}; \frac{1}{2}; \frac{|n\alpha|^2}{2}\right). \]

In particular, at \(\beta = 0\) we have

\[ K_0(s, t) = \alpha^2 \pi^2 |s - t|^{-1} \|y\|^2. \]

### 6.1 Characterizing the norm \(\sum_{n=-N}^{N} e^{-\frac{|n\alpha|^2}{2}} |c_n|^2\)

The norm (6.2) of the function \(y(t) := \sum_{n=-N}^{N} c_n e^{int}\) is expressed in terms of its Fourier coefficients \(c_n\). The following lemma evaluates it directly in terms of the function \(y\).

**Lemma 6.2.** The norm (6.2) of the function \(y(t) := \sum_{n=-N}^{N} c_n e^{int}\) satisfies

\[ \|y\|^2 = \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} G(t, t')y(t)y^*(t') dt dt' \]

where

\[ G(t, t') = 2\pi \frac{\sinh(\frac{\alpha^2}{2})}{\cosh(\frac{\alpha^2}{2}) - \cos(t - t')}, \quad t, t' \in [-\pi, \pi]. \]
Remark 6.3. The norm (6.2) is clearly insensitive to the size of the high frequency (large $n$) components $c_n e^{int}$ of $y$. On the other hand, the alternative representation of this norm in Lemma 6.2 combined with the fact that the kernel $G$ satisfies

$$\frac{\sinh(\frac{\alpha^2}{2})}{\cosh(\frac{\alpha^2}{2}) + 1} \leq G(t, t') \leq 2\pi \frac{\sinh(\frac{\alpha^2}{2})}{\cosh(\frac{\alpha^2}{2}) - 1}, \quad t, t' \in [-\pi, \pi],$$

which, for $\alpha \geq 10$, implies

$$1 - 10^{-21} \leq G(t, t') \leq 1 + 10^{-21}, \quad t, t' \in [-\pi, \pi],$$

implies that

$$\left| \|y\|^2 - \int_{-\pi}^{\pi} |y(t)|^2 dt \right|^2 \leq 10^{-21} \int_{-\pi}^{\pi} |y(t)|^2 dt^2$$

that is, $\|y\|^2$ is exponentially close to the square of its integral.

7 Non-trigonometric waveform and iterated KMD

We will now consider the mode recovery Problem 1 generalized to the case where the periodic waveform of each mode is the same known, possibly non-trigonometric, square-integrable $2\pi$-periodic function $t \rightarrow y(t)$, which we will refer to as the base waveform. The objective of this problem can be loosely expressed as solving the following generalization of Problem 1 towards the resolution of the more general Problem 2. We now switch the time domain from $[0, 1]$ to $[-1, 1]$.  

Figure 23: (1) Triangle base waveform (2) EKG base waveform.
Problem 4. For \( m \in \mathbb{N}^* \), let \( a_1, \ldots, a_m \) be piecewise smooth functions on \([-1, 1]\), let \( \theta_1, \ldots, \theta_m \) be strictly positive and increasing functions on \([-1, 1]\), and let \( y \) be a square-integrable \( 2\pi \)-periodic function. Assume that \( m \) and the \( a_i, \theta_i \) are unknown and the base waveform \( y \) is known. Given the observation \( v(t) = \sum_{i=1}^{m} a_i(t)y(\theta_i(t)) \) (for \( t \in [-1, 1] \)) recover the modes \( v_i := a_i(t)y(\theta_i(t)) \).

Example 7.1. Figure 23 shows two \( 2\pi \)-periodic base waveforms (triangle and EKG) which we will use in our numerical experiments/illustrations. The EKG (-like) waveform is \( y_{EKG}(t) = (2\pi)^{-1} \int_0^{2\pi} y_{EKG}(s) \, ds / \| y_{EKG} \|_{L^2([0,2\pi])} \) with \( y_{EKG}(t) \) defined on \([0,2\pi]\) as

1. \( 0.3 - |t - \pi| \) for \( |t - \pi| < 0.3 \)
2. \( 0.03 \cos^2(\tfrac{\pi}{0.6}(t - \pi + 1)) \) for \( |t - \pi + 1| < 0.3 \)
3. \( 0.03 \cos^2(\tfrac{\pi}{0.6}(t - \pi - 1)) \) for \( |t - \pi - 1| < 0.3 \)
4. \( 0 \) otherwise.

Our approach, summarized in Algorithm 2 and explained in the following sections, will be to (1) use the max-pool energy \( S \) (4.15) to obtain, using (4.16), an estimate of the phase \( \theta_{\text{low}}(t) \) associated with the lowest instantaneous frequency \( \omega_{\text{low}} = \hat{\theta}_{\text{low}} \) (as described in Section 7.2) (2) iterate a micro-local KMD (presented in Section 7.1) of the signal \( v \) to obtain a highly accurate estimate of the phase/amplitude \( \theta_i, a_i \) of the corresponding mode \( v_i \) (this iteration can achieve near machine-precision accuracies when the instantaneous frequencies are separated) (3) peel off the mode \( v_i \) from \( v \) (4) iterate to obtain all the modes (5) perform a last micro-local KMD of the signal for higher accuracy. To illustrate this approach, in the next two sections we will apply it to the signals \( v \) displayed in Figures 24 and 25, where the modes of Figure 24 are triangular and those of Figure 25 are EKG.
Figure 25: (1) Signal $v$ (2) Instantaneous frequencies $\omega_i := \dot{\theta}_i$ (3) Amplitudes $a_i$ (4, 5, 6) Modes $v_1, v_2, v_3$.

### 7.1 Micro-local KMD

We will now describe the micro-local KMD which takes a time $\tau$, a model (estimate) phase function $t \rightarrow \theta_e(t)$ and a signal $v(t) := a(t)\cos(\theta(t))$ as inputs and produces, as outputs, (1) an estimate $a(\tau, \theta_e, v)$ of the amplitude $a(\tau)$ of the mode $v$ and (2) a correction $\delta\theta(\tau, \theta_e, v)$ determining an estimate $\theta_e(\tau) \sim \theta_e(\tau) + \delta\theta(\tau, \theta_e, v)$ of the mode phase function $\theta_e$.

Indeed, given $\alpha > 0$, $\tau \in [-1, 1]$, a differentiable function $t \rightarrow \theta_e(t)$ on $[-1, 1]$, and $n \in \{0, 1, 2\}$, let $\chi^\tau_{n,c}$ and $\chi^\tau_{n,s}$ be the wavelets defined by

\[
\chi^\tau_{n,c}(t) := \cos(\theta_e(t))(t - \tau)^n e^{-\left(\frac{\theta_e(t)(t-\tau)}{\alpha}\right)^2}, \\
\chi^\tau_{n,s}(t) := \sin(\theta_e(t))(t - \tau)^n e^{-\left(\frac{\theta_e(t)(t-\tau)}{\alpha}\right)^2},
\]

where the derivative $\dot{\theta}_e(\tau)$ of $\theta_e$ at time $\tau$ is the instantaneous frequency, and let $\xi_{\tau,\theta_e}$ be the Gaussian process defined by

\[
\xi_{\tau,\theta_e}(t) := \sum_{n=0}^{2} \left( X_{n,c} \chi^\tau_{n,c}(t) + X_{n,s} \chi^\tau_{n,s}(t) \right),
\]

where $X_{n,c}, X_{n,s}$ are independent $N(0, 1)$ random variables. Let $v_{\tau,\theta_e}$ be the Gaussian windowed signal defined by

\[
v_{\tau,\theta_e}(t) = e^{-\left(\frac{\theta_e(t)(t-\tau)}{\alpha}\right)^2} v(t), \quad t \in [-1, 1],
\]
and, for \((n, j) \in \{0, 1, 2\} \times \{c, s\}\)\(^3\),

\[
Z_{n, j}(\tau, \theta_e, v) := \lim_{\sigma \to 0} \mathbb{E}[X_{n, j} | \xi_{\tau, \theta_e} + \xi_{\sigma} = v_{\tau, \theta_e}], \tag{7.4}
\]

where \(\xi_{\sigma}\) is white noise, independent of \(\xi_{\tau, \theta_e}\), with variance \(\sigma^2\). To compute \(Z_{n, j}\), observe that since both \(\xi_{\tau, \theta_e}\) and \(\xi_{\sigma}\) are Gaussian fields, it follows from (2.22) that

\[
\mathbb{E}[\xi_{\tau, \theta_e} | \xi_{\tau, \theta_e} + \xi_{\sigma}] = A_{\sigma}(\xi_{\tau, \theta_e} + \xi_{\sigma})
\]

for the linear mapping

\[
A_{\sigma} = Q_{\tau, \theta_e}(Q_{\tau, \theta_e} + \sigma^2 I)^{-1},
\]

where \(Q_{\tau, \theta_e} : L^2 \to L^2\) is the covariance operator of the Gaussian field \(\xi_{\tau, \theta_e}\) and \(\sigma^2 I\) is the covariance operator of \(\xi_{\sigma}\). Using the characterization of the limit of Tikhonov regularization as the Moore-Penrose inverse, see e.g. Barata and Hussein [2, Thm. 4.3], along with the orthogonal projections connected with the Moore-Penrose inverse, we conclude that \(\lim_{\sigma \to 0} A_{\sigma} = P_{\chi}\) and therefore

\[
\lim_{\sigma \to 0} \mathbb{E}[\xi_{\tau, \theta_e} | \xi_{\tau, \theta_e} + \xi_{\sigma}] = P_{\chi}(\xi_{\tau, \theta_e} + \xi_{\sigma}), \tag{7.5}
\]

where \(P_{\chi}\) is the \(L^2\) orthogonal projection onto the span \(\chi := \text{span}\{\chi_{n,c}, \chi_{n,s}, n = 0, 1, 2\}\).

Since the definition (7.2) can be written

\[
\xi_{\tau, \theta_e} = \sum_{n, j} X_{n, j} \chi_{n, j}^{\tau, \theta_e},
\]

summing (7.4) and using (7.5), we obtain

\[
\sum_{n, j} Z_{n, j}(\tau, \theta_e, v) \chi_{n, j}^{\tau, \theta_e}(t) = P_{\chi} v_{\tau, \theta_e}(t), \quad t \in [-1, 1]. \tag{7.6}
\]

Consider the vector function \(Z(\tau, \theta_e, v) \in \mathbb{R}^6\) with components \(Z_{n, j}(\tau, \theta_e, v)\), the 6 dimensional Gaussian random vector \(X\) with components \(X_{n, j}\), \((n, j) \in \{0, 1, 2\} \times \{c, s\}\), and the \(6 \times 6\) matrix \(A^{\tau, \theta_e}\) defined by \(A^{\tau, \theta_e}_{(n, j), (n', j')} := \langle \chi_{n, j}^{\tau, \theta_e}, \chi_{n', j'}^{\tau, \theta_e} \rangle_{L^2}\). Straightforward linear algebra along with (7.6) establish that the vector \(Z(\tau, \theta_e, v)\) can be computed as the solution of the linear system

\[
A^{\tau, \theta_e} Z(\tau, \theta_e, v) = b^{\tau, \theta_e}(v), \tag{7.7}
\]

where \(b^{\tau, \theta_e}(v)\) is the \(\mathbb{R}^6\) vector with components \(b^{\tau, \theta_e}_{n, j}(v) := \langle \chi_{n, j}^{\tau, \theta_e}, v_{\tau, \theta_e} \rangle_{L^2}\). See subfigures (1) and (2) of both the top and bottom of Figure 26 for illustrations of the windowed signal \(v_{\tau, \theta_e}(t)\) and of its projection \(\lim_{\sigma \to 0} \mathbb{E}[\xi_{\tau, \theta_e} | \xi_{\tau, \theta_e} + \xi_{\sigma} = v_{\tau, \theta_e}]\) in (7.5) corresponding to the signals \(v\) displayed in Figures 24 and 25.

Now suppose that

\[
v(t) = a(t) \cos(\theta(t)),
\]

\(^3\)Experimental evidence indicates that the extra fitting power obtained using \(n \in \{0, 1, 2\}\) versus \(n = 0\) or \(n \in \{0, 1\}\) provides better results. The quantitative analysis of these options we do not consider here.
Figure 26: Top: $v$ is as in Figure 24 (the periodic waveform is triangular). Bottom: $v$ is as in Figure 25 (the periodic waveform is EKG). Both top and bottom: (1) The windowed signal $v_{\tau,\theta,e}(t)$ (2) $\lim_{\sigma \to 0} \mathbb{E}[\xi_{\tau,\theta_1,e} + \xi_\sigma = v_{\tau,\theta_1,e}]$ (3) $(v - v_{1,e})_{\tau,\theta_2,e}(t)$ (4) $\lim_{\sigma \to 0} \mathbb{E}[\xi_{\tau,\theta_2,e} + \xi_\sigma = (v - v_{1,e})_{\tau,\theta_2,e}]$ (5) $(v - v_{1,e} - v_{2,e})_{\tau,\theta_3,e}(t)$ (6) $\lim_{\sigma \to 0} \mathbb{E}[\xi_{\tau,\theta_3,e} + \xi_\sigma = (v - v_{1,e} - v_{2,e})_{\tau,\theta_3,e}]$.

where the model $\theta_e$ is some approximation of the unknown $\theta$ and $\sigma$ oscillates slower than
Moreover, the smoothness of \( \theta \), so that
\[
v_{\tau, \theta}(t) = e^{-\left( \frac{\theta(t)(t-\tau)}{\alpha} \right)^2} a(t) \cos(\theta(t)),
\]
and consider the modified function
\[
\tilde{v}_{\tau, \theta}(t) = e^{-\left( \frac{\theta(t)(t-\tau)}{\alpha} \right)^2} \left( a(\tau) + \dot{a}(\tau)(t - \tau) + \frac{1}{2} \ddot{a}(\tau)(t - \tau)^2 \right) \cos(\theta(t))
\]
obtained by replacing the function \( a \) with the first three terms of its Taylor series about \( \tau \). In what follows, we will use the expression \( \sim \) to articulate an informal approximation analysis. All these “approximations” will be quantified when we proceed to a quantitative analysis. All these “approximations” will be quantified when we proceed to a quantitative performance analysis of our methods. It is clear that \( \tilde{v}_{\tau, \theta} \in \chi \) and, since \( \alpha \) is small and \( \dot{\theta} \) is positive and bounded away from 0, that \( \langle \chi_{n, j} \rangle v_{\tau, \theta} - \tilde{v}_{\tau, \theta} \rangle_{L^2} \sim 0 \), and therefore \( P_{\chi} v_{\tau, \theta} \sim \tilde{v}_{\tau, \theta} \), which implies that \( (P_{\chi} v_{\tau, \theta})(t) \sim \tilde{v}_{\tau, \theta}(t), t \in [-1, 1] \), and therefore (7.6) implies that
\[
\sum_{n' j'} Z_{n', j'}(\tau, \theta, v) \chi_{n', j'}^{\tau, \theta}(t) \sim e^{-\left( \frac{\theta(t)(t-\tau)}{\alpha} \right)^2} \left( a(\tau) + \dot{a}(\tau)(t - \tau) + \frac{1}{2} \ddot{a}(\tau)(t - \tau)^2 \right) \cos(\theta(t))
\]
for all \( t \in [-1, 1] \). Let \( B_{\epsilon}(\tau) \) be the open ball about \( \tau \) of radius \( \epsilon \) and select \( \epsilon \) small enough so that (7.10) implies that
\[
\sum_{n' j'} Z_{n', j'}(\tau, \theta, v) \chi_{n', j'}^{\tau, \theta}(t) \sim e^{-\left( \frac{\theta(t)(t-\tau)}{\alpha} \right)^2} a(\tau) \cos(\theta(t)), \quad t \in B_{\epsilon}(\tau).
\]
Furthermore, observing that the definition (7.1) of \( \chi_{n, e}^{\tau, \theta} \) and \( \chi_{n, s}^{\tau, \theta} \) implies that we can further reduce the size of \( \epsilon \) so that, for \( n > 0 \), \( \chi_{n, e}^{\tau, \theta} \) and \( \chi_{n, s}^{\tau, \theta} \) are small for \( t \in B_{\epsilon}(\tau) \), so that consequently
\[
\sum_{j'} Z_{0, j'}(\tau, \theta, v) \chi_{0, j'}^{\tau, \theta}(t) \sim e^{-\left( \frac{\theta(t)(t-\tau)}{\alpha} \right)^2} a(\tau) \cos(\theta(t)), \quad t \in B_{\epsilon}(\tau),
\]
which implies that
\[
Z_{0, e}(\tau, \theta, v) \cos(\theta_e(t)) + Z_{0, s}(\tau, \theta, v) \sin(\theta_e(t)) \sim a(\tau) \cos(\theta(t)), \quad t \in B_{\epsilon}(\tau),
\]
for \( \epsilon \) small enough. Setting \( \theta_\delta := \theta - \theta_e \) as the approximation error, using the cosine sum formula \( \cos \theta = \cos(\theta_\delta + \theta_e) = \cos \theta_\delta \cos \theta_e - \sin \theta_\delta \sin \theta_e \), we obtain
\[
Z_{0, e}(\tau, \theta, v) \cos(\theta_e(t)) + Z_{0, s}(\tau, \theta, v) \sin(\theta_e(t)) = a(\tau) \left( \cos(\theta(t)) \cos(\theta_e(t)) - \sin(\theta(t)) \sin(\theta_e(t)) \right).
\]
Moreover, the smoothness of \( \theta_\delta \) implies, further reducing the size of \( \epsilon \), that \( \theta_\delta(t) \sim \theta_\delta(t), t \in B_{\epsilon}(\tau) \), so that, for all \( t \in B_{\epsilon}(\tau) \), we have
\[
Z_{0, e}(\tau, \theta, v) \cos(\theta_e(t)) + Z_{0, s}(\tau, \theta, v) \sin(\theta_e(t)) \sim a(\tau) \left( \cos(\theta(t)) \cos(\theta_e(t)) - \sin(\theta(t)) \sin(\theta_e(t)) \right),
\]
Figure 27: Top: \( v \) is as in Figure 24 (the periodic waveform is triangular). Bottom: \( v \) is as in Figure 25 (the periodic waveform is EKG). Both top and bottom: \( \tau = 0 \).

(1) the amplitude of the first mode \( a_1(t) \) and its local Gaussian regression estimation \( a(\tau, \theta_{1,e}, v)(t) \) (2) the error in estimated phase of the first mode \( \theta_1(t) - \theta_{1,e}(t) \) and its local Gaussian regression \( \delta \theta(\tau, \theta_{1,e}, v)(t) \) (3, 4) are as (1,2) with \( v \) and \( \theta_{1,e} \) replaced by \( v - v_{1,e} \) and \( \theta_{2,e} \) (5,6) are as (1,2) with \( v \) and \( \theta_{1,e} \) replaced by \( v - v_{1,e} - v_{2,e} \) and \( \theta_{3,e} \).
which, since $\dot{\theta}_e(t)$ positive and bounded away from 0, implies that

$$Z_{0,e}(\tau, \theta_e, v) \sim a(\tau) \cos(\theta_\delta(\tau))$$

$$Z_{0,s}(\tau, \theta_e, v) \sim -a(\tau) \sin(\theta_\delta(\tau)) .$$

Consequently, writing

$$a(\tau, \theta_e, v) := \sqrt{Z_{0,e}^2(\tau, \theta_e, v) + Z_{0,s}^2(\tau, \theta_e, v)}$$

$$\delta \theta(\tau, \theta_e, v) := \text{atan2}(-Z_{0,s}(\tau, \theta_e, v), Z_{0,e}(\tau, \theta_e, v)) ,$$

we obtain that $a(\tau, \theta_e, v) \sim a(\tau)$ and $\delta \theta(\tau, \theta_e, v) \sim \theta_\delta(\tau)$. We will therefore use $a(\tau, \theta_e, v)$ to estimate the amplitude $a(\tau)$ of the mode $v$ using the model estimate $\theta_\delta$ and $\delta \theta(\tau, \theta_e, v)$ to estimate the mode phase $\theta$ through $\theta(\tau) = \theta_e(\tau) + \theta_\delta(\tau) + \delta \theta(\tau, \theta_e, v)$. Iterating this refinement process will allow us to achieve near machine-precision accuracies in our phase/amplitude estimates. See sub-figures (1) and (2) of the top and bottom of Figure 27 for illustrations of $a(t, \alpha_e, \omega(t), \theta(t) - \theta_e(t)$ and $\delta \theta(\tau, \theta_e, v)(t)$ corresponding to the first mode $v_1$ of the signals $v$ displayed in sub-figure (4) of Figures 24 and 25.

7.2 The lowest instantaneous frequency

We will use the network illustrated in Figure 18 to design a module taking a signal $v$ as input and producing, as output, the instantaneous phase $\theta_{\text{low}}(v)$ of the mode of $v$ having

![Figure 28: Max-squeezing with the EKG base waveform and derivation of the instantaneous phase estimates $\theta_{\text{low}}$. (1,2) $\tau, \omega \rightarrow S(\tau, \omega, v)$ and identification of $A_{\text{low}}$ (3, 4) $\tau, \omega \rightarrow S(\tau, \omega, v - v_{1,e})$ and identification of its $A_{\text{low}}$ (5,6) $\tau, \omega \rightarrow S(\tau, \omega, v - v_{1,e} - v_{2,e})$ and identification of its $A_{\text{low}}$.](image)
the lowest instantaneous frequency. The main steps of the computation performed by
this module are as follows. Let $S(\tau, \omega, v)$ be the max-pool energy defined as in (4.15),
where now it is useful to indicate its dependence on $v$.

Let $A_{\text{low}}$ be a subset of the time-frequency domain $(\tau, \omega)$ identified (as in sub-figure
(2) of Figure 28) as a narrow sausage around the lowest instantaneous frequency defined
by the local maxima of the $S(\tau, \omega, v)$ (we restrict our presentation to the situation where
the instantaneous frequencies $\omega_i$ do not cross each other). If no modes can be detected
(above a given threshold) in $S(\tau, \omega, v)$ then we set $\theta_{\text{low}}(v) = \emptyset$ otherwise we proceed as
follows.

Let $\omega_{\text{low}}(\tau) = \omega_e(\tau, \arg\max_{\omega: (\tau, \omega) \in A_{\text{low}}} S(\tau, \omega))$ be the estimated instantaneous fre-
quency of the mode having the lowest instantaneous frequency and, with $\theta_e$ defined as
in (4.6), let

$$\theta_{\text{low}}(t) := \theta_e(t, \omega_{\text{low}}(t)),$$

be the corresponding estimated instantaneous phase (obtained as in (4.16)).

7.3 The iterated micro-local KMD algorithm.

Figure 29: Modular representation of Algorithm 2, described in this section. The blue
module represents the estimation of the lowest frequency as illustrated in Figure 28. The
brown module represents the iterative estimation of the mode with lowest instantaneous
frequency of steps 10 through 13. The yellow module represents the iterative refinement
of all the modes in steps 20 through 29. The brown and yellow modules used to refine
phase/amplitude estimates use the same code.

The method of estimating the mode with the lowest instantaneous frequency, de-
scribed in Section 7.2, provides a foundation for the iterated micro-local KMD algo-
Algorithm 2 Iterated micro-local KMD.

1: \( i \leftarrow 1 \)
2: \( v^{(1)} \leftarrow v \)
3: \( \textbf{while} \; \text{true} \; \textbf{do} \)
4: \( \textbf{if} \; \theta_{\text{low}}(v^{(i)}) = \emptyset \; \textbf{then} \)
5: \( \text{break loop} \)
6: \( \textbf{else} \)
7: \( \theta_{i,e} \leftarrow \theta_{\text{low}}(v^{(i)}) \)
8: \( \textbf{end if} \)
9: \( a_{i,e}(t) \leftarrow 0 \)
10: \( \textbf{while} \; \sup_{t} |\delta \theta_{i,e}(t)| > \epsilon_1 \; \textbf{do} \)
11: \( v_{\text{res}}^{(i)} \leftarrow v^{(i)} - a_{i,e} \bar{y}(\theta_{i,e}) \)
12: \( a_{i,e}(\tau) \leftarrow a(\tau, \theta_{i,e}, v_{\text{res}}^{(i)}) \)
13: \( \theta_{i,e}(\tau) \leftarrow \theta_{i,e}(\tau) + \frac{1}{2} \delta \theta(\tau, \theta_{i,e}, v_{\text{res}}^{(i)}) \)
14: \( \textbf{end while} \)
15: \( v^{(i+1)} \leftarrow v^{(i)} - a_{i,e} y(\theta_{i,e}) \)
16: \( i \leftarrow i + 1 \)
17: \( \textbf{end while} \)
18: \( m \leftarrow i - 1 \)
19: \( i \leftarrow 1 \)
20: \( \textbf{while} \; \sup_{t} |\delta \theta_{i,e}(t)| > \epsilon_2 \; \textbf{do} \)
21: \( v_{i,\text{res}} \leftarrow v - a_{i,e} \bar{y}(\theta_{i,e}) - \sum_{j \neq i} a_{j,e} y(\theta_{j,e}) \)
22: \( a_{i,e}(\tau) \leftarrow a(\tau, \theta_{i,e}, v_{i,\text{res}}) \)
23: \( \theta_{i,e}(\tau) \leftarrow \theta_{i,e}(\tau) + \frac{1}{2} \delta \theta(\tau, \theta_{i,e}, v_{i,\text{res}}) \)
24: \( \textbf{if} \; i < m \; \textbf{then} \)
25: \( i \leftarrow i + 1 \)
26: \( \textbf{else} \)
27: \( i \leftarrow m \)
28: \( \textbf{end if} \)
29: \( \textbf{end while} \)
30: \( \text{Return the modes } v_{i,e} \leftarrow a_{i,e}(t) y(\theta_{i,e}(t)) \text{ for } i = 1, \ldots, m \)

Algorithm 2. We now describe Algorithm 2 using Figures 26, 27, 28, presenting its modular representation in Figure 29. To that end, let

\[
y(t) = c_1 \cos(t) + \sum_{n=2}^{\infty} c_n \cos(nt + d_n)
\]  

be the Fourier representation of the base waveform \( y \) (which, without loss of generality, has been shifted so that the first sine coefficient is zero) and write

\[
\tilde{y}(t) := y(t) - c_1 \cos(t)
\]

for its overtones.
Let us describe how steps 1 to 17 provide refined estimates for the amplitude and the phase of each mode \( v_i, i \in \{1, \ldots, m\} \) of the signal \( v \). Although the overtones of \( y \) prevent us from simultaneously approximating all the instantaneous frequencies \( \theta_i \) from the max-pool energy of the signal \( v \), since the lowest mode \( v_{\text{low}} = a_{\text{low}} \cos(\theta_{\text{low}}) \) can be decomposed into the sum \( v_{\text{low}} = a_{\text{low}} c_1 \cos(\theta_{\text{low}}) + a_{\text{low}} \bar{y}(\theta_{\text{low}}) \) of a signal \( a_{\text{low}} c_1 \cos(\theta_{\text{low}}) \) with a cosine waveform plus the signal \( a_{\text{low}} \bar{y}(\theta_{\text{low}}) \) containing its higher frequency overtones, the method of Section 7.2 can be applied to obtain an estimate \( a_{\text{low},e} c_1 \cos(\theta_{\text{low},e}) \) of the primary component \( a_{\text{low}} c_1 \cos(\theta_{\text{low}}) \) of the first mode. Since \( c_1 \) is known, this estimate produces the estimate \( a_{\text{low},e} \bar{y}(\theta_{\text{low},e}) \) for the overtones of the lowest mode. To improve the accuracy of this estimate, in steps 12 and 13 the micro local KMD of Section 7.1 is iteratively applied to the residual signal \( v_{\text{res}}^{(2)} \leftarrow v - a_{\text{low},e} \bar{y}(\theta_{\text{low},e}) \), consisting of the signal \( v \) with the estimated overtones of the lowest mode removed. The rate parameter \( 1/2 \) in step 13 is to avoid overcorrecting the phase estimates, while the parameters \( \epsilon_1 \) and \( \epsilon_2 \) in steps 10 and 20 are pre-specified accuracy thresholds. The resulting estimated lowest mode is then removed from the signal to determine the first residual \( v^{(2)} := v - a_{\text{low},e} \bar{y}(\theta_{\text{low},e}) \) in step 15.

Iterating this process, we can peel off an estimate \( a_{i,e} \bar{y}(\theta_{i,e}) \) of the mode corresponding to the lowest instantaneous frequency of the residual \( v^{(i)} := v - \sum_{j=1}^{i-1} a_{j,e} \bar{y}(\theta_{j,e}) \) of the signal \( v \) obtained in step 15, removing the interference of the first \( i-1 \) modes, including their overtones, in our estimate of the instantaneous frequency and phase of the \( i \)-th mode. See Figure 28 for the evolution of the \( A_{\text{low}} \) sausage as these modes are peeled off. See sub-figures (3) and (5) of the top and bottom of Figure 26 for the results of peeling off the first two estimated modes of the signal \( v \) corresponding to both Figures 24 and 25 and sub-figures (4) and (6) for the results of the corresponding projections in (7.5). See sub-figures (3) and (4) of the top and bottom of Figure 27 for amplitude and its estimate of the results of peeling off the first estimated mode and sub-figures (5) and (6) corresponding to peeling of the first two estimated modes of the signal \( v \) corresponding to both Figures 24 and 25.

Once the amplitude/phase estimates \( a_{i,e}, \bar{y}(\theta_{i,e}), i \in \{1, \ldots, m\} \) have been obtained in steps 1 to 17, steps 19 to 29 enable us to achieve even higher accuracies by iterating the micro local KMD of Section 7.1 on the residual signals \( v_{i,\text{res}} \leftarrow v - a_{i,e} \bar{y}(\theta_{i,e}) - \sum_{j \neq i} a_{j,e} \bar{y}(\theta_{j,e}) \), consisting of the signal \( v \) with all the estimated modes \( j \neq i \) and estimated overtones of the mode \( i \) removed.

The proposed algorithm can be further improved by (1) applying a Savitsky-Golay filter to locally smooth (de-noise) the curves corresponding to each estimate \( \theta_{i,e} \) (which corresponds to refining our phase estimates through GPR filtering) (2) starting with a slightly larger \( \alpha \) (to decrease interference from other modes/overtones) and slowly reducing its value in the final steps (to further localize our estimates after other components have been mostly eliminated).
7.4 Numerical experiments

Here we present results for both the triangle wave and EKG wave examples. As discussed in the previous section, these results are visually displayed in 26 and 27.

7.4.1 Triangle wave example

The base waveform is the triangle wave displayed in Figure 23. To avoid boundary artifacts we observe the signal \( v \) on a mesh spanning \([-3, 3]\) spaced at intervals of \( \frac{1}{2000} \) and aim to recover each mode \( v_i \) on \([-1, 1]\). We slowly increased \( \alpha \) from 10 to 15 in the loop corresponding to steps 1 to 17 and slowly decreased it to 6 in the final loop corresponding to steps 19 to 29. The amplitudes and frequencies of each of the modes are shown in Figure 24. The recovery errors of each mode as well as their amplitude and phase functions are displayed in Table 5. They were found to be on the order of \( 10^{-7} \) for the first signal component and approximately \( 10^{-6} \) for the higher two. A plot superimposing \( v_i \) and \( v_{i,e} \) would visually appear to be one curve due to the negligible recovery errors.

Table 5: Signal component recovery errors in the triangle base waveform example.

| Mode | \( \frac{\|v_{i,e} - v_i\|_2}{\|v_i\|_2} \) | \( \frac{\|v_{i,e} - v_i\|_{L\infty}}{\|v_i\|_{L\infty}} \) | \( \frac{\|a_{i,e} - a_i\|_2}{\|a_i\|_2} \) | \( \frac{\|\theta_{i,e} - \theta_i\|_2}{\|\theta_i\|_2} \) |
|------|-----------------|-----------------|-----------------|-----------------|
| \( i = 1 \) | \( 1.94 \times 10^{-7} \) | \( 3.65 \times 10^{-7} \) | \( 9.99 \times 10^{-8} \) | \( 1.50 \times 10^{-7} \) |
| \( i = 2 \) | \( 3.71 \times 10^{-6} \) | \( 2.52 \times 10^{-6} \) | \( 5.68 \times 10^{-7} \) | \( 3.56 \times 10^{-6} \) |
| \( i = 3 \) | \( 2.19 \times 10^{-6} \) | \( 3.89 \times 10^{-6} \) | \( 1.40 \times 10^{-6} \) | \( 1.51 \times 10^{-6} \) |

7.4.2 EKG wave example

The base waveform is the EKG wave displayed in Figure 23. We use the same discrete mesh as in the triangle case. Here, we slowly increased \( \alpha \) from 20 to 30 in the loop corresponding to steps 1 to 17 and slowly decreased it to 10 in the final loop corresponding to steps 19 to 29. The amplitudes and frequencies of each of the modes are shown in Figure 25, while the recovery error of each mode as well as their amplitude and phase functions are shown in Table 6. Amplitude and phase relative errors are found to be on the order of \( 10^{-2} \) to \( 10^{-4} \) in this setting. Again, \( v_{i,e} \) and \( v_i \) are visually indistinguishable due to the small recovery errors.

Table 6: Signal component recovery errors in the EKG base waveform example.

| Mode | \( \frac{\|v_{i,e} - v_i\|_2}{\|v_i\|_2} \) | \( \frac{\|v_{i,e} - v_i\|_{L\infty}}{\|v_i\|_{L\infty}} \) | \( \frac{\|a_{i,e} - a_i\|_2}{\|a_i\|_2} \) | \( \frac{\|\theta_{i,e} - \theta_i\|_2}{\|\theta_i\|_2} \) |
|------|-----------------|-----------------|-----------------|-----------------|
| \( i = 1 \) | \( 2.72 \times 10^{-3} \) | \( 1.04 \times 10^{-2} \) | \( 9.11 \times 10^{-4} \) | \( 3.25 \times 10^{-3} \) |
| \( i = 2 \) | \( 1.72 \times 10^{-3} \) | \( 4.30 \times 10^{-3} \) | \( 3.04 \times 10^{-4} \) | \( 2.31 \times 10^{-4} \) |
| \( i = 3 \) | \( 5.28 \times 10^{-3} \) | \( 2.89 \times 10^{-2} \) | \( 1.02 \times 10^{-3} \) | \( 8.91 \times 10^{-4} \) |
8 Unknown base waveforms

We next consider the extension to the mode recovery problem to the case where the periodic base waveform of each mode may be unknown (and different across modes). In this context, our objective can be summarized as solving Problem 2. To avoid ambiguities caused by overtones when the waveforms \( y_i \) are not only non-trigonometric but also unknown, we will assume that the functions \( (k\hat{\theta}_i)_{t\in[-1,1]} \) and \( (k'\hat{\theta}_i)_{t\in[-1,1]} \) are distinct for \( i \neq i' \) and \( k, k' \in \mathbb{N}^* \) (those functions may be equal for some \( t \) but not for all \( t \)).

To describe the algorithm we write

\[
y_i(t) = \cos(t) + \sum_{k=2}^{k_{\text{max}}} c_{i,(k,c)} \cos(kt) + c_{i,(k,s)} \sin(kt)
\]  

(8.1)

for the Fourier representation of \( y_i \), which, without loss of generality has been scaled, translated and truncated \((k_{\text{max}} \text{ can be arbitrarily large and in a discrete setting it is bounded by the inverse of the resolution of the discretization in time). \)

At the cost of some degree of forward referencing we will use the signal \( v \) and its corresponding modes \( v_1, v_2, v_3 \) displayed in Figure 30 to illustrate our approach. The waveforms \( y_1, y_2 \) and \( y_3 \) composing that signal are shown in Figure 31 and described in Section 8.3.

![Figure 30: (1) Signal \( v \) (the signal is defined over \([-1,1]\) but displayed over \([0,0.4]\) for visibility) (2) Instantaneous frequencies \( \omega_i := \dot{\theta}_i \) (3) Amplitudes \( a_i \) (4, 5, 6) Modes \( v_1, v_2, v_3 \) over \([0,0.4]\) (mode plots have also been zoomed in for visibility).](image)
8.1 Micro-local waveform KMD

\[
\begin{align*}
\mathcal{U} & \rightarrow \theta_{1,e} \rightarrow y_{1,e} \rightarrow \mathcal{V}_{1,e} \rightarrow \mathcal{V} - \mathcal{V}_{1,e}
\end{align*}
\]

Figure 32: High level structure of the algorithm for the case when waveforms are unknown.

We now describe the micro-local waveform KMD, Algorithm 3, which takes as inputs a time \( \tau \), an estimated instantaneous amplitude and phase \( t \rightarrow a(t), \theta(t) \), and a signal \( v \), and outputs an estimate of the waveform \( y(t) \) associated with the phase. The proposed approach is a direct extension of the one presented in Section 7.1 and the shaded part of Figure 32 shows the new block which will be added to Algorithm 2, the algorithm designed for the case when waveforms are non-trigonometric and known. As described below this new block produces an estimator \( y_{i,e} \) of the waveform \( y_i \) from an estimate \( \theta_{i,e} \) of the phase \( \theta_i \).

Given \( \alpha > 0 \), \( \tau \in [-1, 1] \), and differentiable function \( t \rightarrow \theta(t) \), define the following Gaussian process

\[
\xi_{\tau, \theta}^b(t) = e^{-\left(\frac{\phi(t) \beta(t)}{\alpha}\right)^2} \left( X_{1,c}^b \cos \left( \theta(t) \right) + \sum_{k=2}^{k=\max} \left( X_{k,c}^b \cos \left( k \theta(t) \right) + X_{k,s}^b \sin \left( k \theta(t) \right) \right) \right),
\]

(8.2)

where \( X_{1,c}^b, X_{k,c}^b \), and \( X_{k,s}^b \) are independent \( \mathcal{N}(0, 1) \) random variables. Let

\[
v_{\tau, \theta}(t) := e^{-\left(\frac{\phi(t) \beta(t)}{\alpha}\right)^2} v(t), \quad \tau \in [-1, 1],
\]

(8.3)
be the windowed signal, and define

$$Z_{k,j}^y(\tau, \theta, v) := \lim_{\sigma \to 0} \mathbb{E}[X_{k,j}^y \xi_{\tau,\theta} + \xi_\sigma = v, \theta],$$

(8.4)

and for \(k \in \{2, \ldots, k_{\text{max}}\}, j \in \{c, s\}, \)

$$c_{k,j}(\tau, \theta, v) := \frac{Z_{k,j}^y(\tau, \theta, v)}{Z_{1,c}^y(\tau, \theta, v)}.$$

(8.5)

This yields an estimate of each Fourier coefficient of the waveform \(y\) expressed as in Problem 2 at time \(t = \tau\). This waveform recovery is susceptible to error when there is interference in the overtone frequencies (that is for the values of \(\tau\) at which \(j_1 \theta_{i_1} \approx j_2 \theta_{i_2}\) for \(i_1 < i_2\)). To overcome this, we compute \(c_{k,j}(\tau, \theta, v)\) at each time \(\tau\) and take the most common approximate value over all \(\tau\). More specifically, let \(\mathcal{T}\) be the finite set of values of \(\tau\) used in the numerical discretization of the time axis. For an interval \(I \subset \mathbb{R}\) let \(T_I := \{\tau \in T | c_{k,j}(\tau, \theta, v) \in I\}\),

(8.6)

and \(N_I\) for the number of elements of \(T_I\). Let \(I_{\text{max}}\) be a maximizer of \(N_I\) over intervals of fixed width \(L\).

We then define our overall estimate \(c_{k,j}(\theta, v)\) of the Fourier coefficient \(c_{k,j}\) to be the average of the values of \(c_{k,j}(\tau, \theta, v)\) over \(\tau \in I_{\text{max}} \cap T\), i.e.

$$c_{k,j}(\theta, v) := \begin{cases} \frac{1}{N_{\text{max}}} \sum_{\tau \in T_{\text{max}}} c_{k,j}(\tau, \theta, v) & \text{for } \frac{N_{\text{max}}}{N} \geq 0.05 \\ 0 & \text{for } \frac{N_{\text{max}}}{N} < 0.05 \end{cases},$$

(8.7)

where \(N\) is the size of the time mesh, i.e. the number of elements of \(T\). Figure 33 shows a zoomed-in histogram of the values of \(c_{2,c}(\tau, \theta, v)\) associated with the waveform of the first mode and the plots of \(\tau \rightarrow c_{1,(3,c)}, c_{1,(3,s)}\). The interpretation of the selection of the cutoff 0.05 is as follows: if \(\frac{N_{\text{max}}}{N}\) is small then there is interference in the overtones at all time \([-1, 1]\) and no information may be obtained about the corresponding Fourier coefficient.

**On the interval width.** The recovered modes and waveforms show little sensitivity to the specific value of \(L\) which has been set to be 0.002 in our numerical experiments (widths between 0.001 and 0.01 yield similar results). The rationale for the selection of the (rough) value of \(L\) is as follows. Suppose \(v = \cos(\omega t)\) and \(v' = v + \cos(1.5\omega t)\). Define the quantity

$$\max_{\tau} c_{2,c}(\tau, \theta, v') - c_{2,c}(\tau, \theta, v)$$

(8.8)

with the intuition of approximating the maximum corruption by the \(\cos(1.5\omega t)\) term in the estimated first overtone. This quantity is mainly dependent on the selection of \(\alpha\) and marginally on \(\omega\). For our selection of \(\alpha = 10\), we numerically found its value to be approximately 0.002.
Figure 33: (1) A histogram (with cropped outliers) with bin width 0.002 of \( c_{1,3} \) values (the true value of \( c_{1,3} \) is 1/9 since \( y_1 \) is a triangle wave) (2) The analogous histogram of \( c_{1,3,6} \) values (the true coefficient of this overtone is 0) (3) \( \tau \rightarrow c_{1,3} \) (4) \( \tau \rightarrow c_{1,3,6} \).

8.2 Iterated micro-local KMD with unknown waveforms algorithm

Except for the steps discussed in Section 8.1 Algorithm 3 is identical to Algorithm 2. We first identify the lowest frequency of the cosine component of each mode (step 8) to obtain a refined estimate of \( \theta_{i,e} \). Then the base waveform \( y_i \) recovery is estimated in steps 16-17. Finally, once each mode has been identified, we again apply waveform estimation in steps 27-28 (after nearly eliminating other modes and reducing interference in overtones for higher accuracies).

| Mode | \( \frac{|v_{i,e} - v_i|_{L^2}}{|v_i|_{L^2}} \) | \( \frac{|v_{i,e} - v_i|_{L^\infty}}{|v_i|_{L^\infty}} \) | \( \frac{|\theta_{i,e} - \theta_i|_{L^2}}{|\theta_i|_{L^2}} \) | \( \frac{|y_{i,e} - y_i|_{L^2}}{|y_i|_{L^2}} \) |
|------|-----------------|-----------------|-----------------|-----------------|
| \( i = 1 \) | \( 6.59 \times 10^{-3} \) | \( 2.65 \times 10^{-2} \) | \( 1.52 \times 10^{-5} \) | \( 1.75 \times 10^{-5} \) |
| \( i = 2 \) | \( 2.62 \times 10^{-4} \) | \( 5.61 \times 10^{-4} \) | \( 8.12 \times 10^{-5} \) | \( 1.25 \times 10^{-4} \) |
| \( i = 3 \) | \( 6.55 \times 10^{-4} \) | \( 9.76 \times 10^{-4} \) | \( 3.99 \times 10^{-4} \) | \( 3.67 \times 10^{-4} \) |

Table 7: Signal component recovery errors when the base waveforms are unknown

8.3 Numerical experiments

To illustrate this learning of the base waveform of each mode, we take \( v(t) = \sum_{i=1}^{3} a_i(t) y_i(\theta_i(t)) \). We set the (unknown) waveform \( y_1 \) to be triangle as in Figure 23. For \( i = 2, 3 \), \( k \in \{2, \ldots, 7\} \) and \( j \in \{c, s\} \) we set \( y_{i,(k,j)} \) to be zero with probability 1/2 or to be a random sample from \( \mathcal{N}(0, 1/k^4) \) with probability 1/2 (recall our convention that \( y_{i,(0,c)} = 1 \) and \( y_{i,(0,s)} = 0 \)). The waveforms \( y_1, y_2, y_3 \) are illustrated in Figure 31. The modes \( v_1, v_2, v_3 \), their amplitudes and instantaneous frequencies are shown in Figure 30.

We use the same mesh and the same value of \( \alpha \) values as in Section 7.4.1. The main source of error for the recovery of the first mode’s base waveform stems from the fact that a triangle wave has an infinite number of overtones, while in our implementation,

\footnote{unrefined estimates of \( \theta_{i,e} \) in step 7 lead to poor recoveries of the \( y_i \)}
Algorithm 3 Iterated micro-local KMD with unknown waveforms.

1: \( i \leftarrow 1 \)
2: \( v^{(1)} \leftarrow v \)
3: while true do
4:   if \( \theta_{\text{low}}(v^{(i)}) = \emptyset \) then
5:     break loop
6:   else
7:     \( \theta_{i,e} \leftarrow \theta_{\text{low}}(v^{(i)}) \)
8:     \( y_{i,e} \leftarrow \cos(t) \)
9:   end if
10: \( a_{i,e}(t) \leftarrow 0 \)
11: while \( \sup_t |\delta\theta_{i,e}(t)| > \epsilon_1 \) do
12:   \( v_{\text{res}}^{(i)} \leftarrow v^{(i)} - a_{i,e}y_{i,e}(\theta_{i,e}) \)
13:   \( a_{i,e}(\tau) \leftarrow a(\tau, \theta_{i,e}, v_{\text{res}}^{(i)}) \)
14:   \( \theta_{i,e}(\tau) \leftarrow \theta_{i,e}(\tau) + \frac{1}{2} \delta\theta(\tau, \theta_{i,e}, v_{\text{res}}^{(i)}) \)
15: end while
16: \( c_{i,(k,j),e} \leftarrow c_{k,j}(\theta_{i,e}, v_{\text{res}}^{(i)}) \)
17: \( y_{i,e}(t) \leftarrow \cos(t) + \sum_{k=1}^{\text{max}} c_{i,(k,e)} \cos(kt) + c_{i,(k,s),e} \sin(kt) \)
18: \( v^{(i+1)} \leftarrow v^{(i)} - a_{i,e}y_{i,e}(\theta_{i,e}) \)
19: \( i \leftarrow i + 1 \)
20: end while
21: \( m \leftarrow i - 1 \)
22: \( i \leftarrow 1 \)
23: while \( \sup_t |\delta\theta_{i,e}(t)| > \epsilon_2 \) do
24:   \( v_{\text{res}} \leftarrow v - a_{i,e}y_{i,e}(\theta_{i,e}) - \sum_{j \neq i} a_{j,e}y_{j,e}(\theta_{j,e}) \)
25:   \( a_{i,e}(\tau) \leftarrow a(\tau, \theta_{i,e}, v_{\text{res}}) \)
26:   \( \theta_{i,e}(\tau) \leftarrow \theta_{i,e}(\tau) + \frac{1}{2} \delta\theta(\tau, \theta_{i,e}, v_{\text{res}}) \)
27:   \( c_{i,(k,j),e} \leftarrow c_{k,j}(\theta_{i,e}, v - \sum_{j \neq i} a_{j,e}y_{j,e}(\theta_{j,e})) \)
28:   \( y_{i,e}(t) \leftarrow \cos(t) + \sum_{k=1}^{\text{max}} c_{i,(k,e)} \cos(kt) + c_{i,(k,s),e} \sin(kt) \)
29:   if \( i < m \) then
30:     \( i \leftarrow i + 1 \)
31: else
32:   \( i \leftarrow m \)
33: end if
34: end while
35: Return the modes \( y_{i,e}(\theta_{i,e}(t)) \) for \( i = 1, ..., m \)

we estimate only the first 15 overtones. Compared to the triangle wave truncated to the first 15 overtones, the \( L^2 \)-norm recovery error of the waveform is \( 2.09 \times 10^{-3} \). We omitted the plots of the \( y_{i,e} \) as they are visually indistinguishable from those of the \( y_i \). Recovery errors are presented in Table 7.
9 Proofs

9.1 Proof of Lemma 2.1

We first establish that $\Psi(v) = \Phi^v$, where the Moore-Penrose inverse is defined by $\Phi^v := \Phi^T(\Phi\Phi^T)^{-1}$. To that end, let $w^*$ be the solution of (2.5). Since $\Phi : \mathcal{B} \rightarrow V$ is surjective it follows that $\Phi : \text{Ker}^\perp(\Phi) \rightarrow V$ is a bijection and therefore

$$\{ w : \Phi(w) = v \} = w_0 + \text{Ker}(\Phi)$$

for a unique $w_0 \in \text{Ker}^\perp(\Phi)$. Therefore, setting $w' := w - w_0$ we find that $(w^*)' := w^* - w_0$ is a solution of

\[
\begin{cases}
\text{Minimize} & \|w' + w_0\|_B \\
\text{Subject to} & w' \in \mathcal{B} \text{ and } \Phi(w') = 0,
\end{cases}
\]

so that by the projection theorem we have $(w^*)' = P_{\text{Ker}(\Phi)}(-w_0)$ where $P_{\text{Ker}(\Phi)}$ is the orthogonal projection onto $\text{Ker}(\Phi)$. Therefore $w^* = w_0 + (w^*)' = w_0 - P_{\text{Ker}(\Phi)}(w_0) = P_{\text{Ker}^\perp(\Phi)}w_0$, so that we obtain

$$w^* = P_{\text{Ker}^\perp(\Phi)}w_0.$$ 

Since $\Phi$ is surjective and continuous it follows from the closed range theorem, see e.g. Yosida [35, Pg. 208] that $\text{Im}(\Phi^T) = \text{Ker}^\perp(\Phi)$ and $\text{Ker}(\Phi^T) = \emptyset$, which implies that $\Phi\Phi^T : V \rightarrow V$ is invertible, so that the Moore-Penrose inverse $\Phi^\perp : V \rightarrow \mathcal{B}$ of $\Phi$, is well-defined by

$$\Phi^\perp := \Phi^T(\Phi\Phi^T)^{-1}.$$ 

It follows that $P_{\text{Ker}^\perp(\Phi)} = \Phi^\perp\Phi$ and $\Phi\Phi^\perp = I_V$ so that

$$w^* = P_{\text{Ker}^\perp(\Phi)}w_0 = \Phi^\perp\Phi w_0 = \Phi^v,$$

that is, we obtain the assertion $w^* = \Phi^v$.

For the first assertion, suppose that $\text{Ker}\Phi = \emptyset$. Since it is surjective, it follows that $\Phi$ is a bijection. Then, the unique solution to the minmax problem is the only feasible one $w^* = \Phi^{-1}v = \Phi^v$. When $\text{Ker}\Phi \neq \emptyset$, observe that since all $u$ which satisfy $\Phi(u) = v$ have the representation $u = w_0 + u'$ for fixed $w_0 \in \text{Ker}^\perp(\Phi)$ and some $u' \in \text{Ker}(\Phi)$, it follows that the inner maximum satisfies

$$\max_{u \in \mathcal{B}|\Phi(u) = v} \frac{\|u - w\|_B}{\|u\|_B} = \max_{u' \in \text{Ker}(\Phi)} \frac{\|u' + w_0 - w\|_B}{\|u' + w_0\|_B} = \max_{u' \in \text{Ker}(\Phi)} \max_{t \in \mathbb{R}} \frac{\|tu' + w_0 - w\|_B}{\|tu' + w_0\|_B} \geq 1.$$
On the other hand, for \( w := \Phi^{-} v \), we have

\[
\max_{u \in \mathcal{B}|\Phi(u) = v} \frac{\|u - w\|_\mathcal{B}}{\|u\|_\mathcal{B}} = \max_{u \in \mathcal{B}|\Phi(u) = v} \frac{\|u - \Phi^{-} v\|_\mathcal{B}}{\|u\|_\mathcal{B}} = \max_{u \in \mathcal{B}|\Phi(u) = v} \frac{\|u - \Phi^{-} \Phi(u)\|_\mathcal{B}}{\|u\|_\mathcal{B}} = \max_{u \in \mathcal{B}|\Phi(u) = v} \frac{\|u - P_{\text{Ker} \Phi} u\|_\mathcal{B}}{\|u\|_\mathcal{B}} \leq 1,
\]

which implies that \( w := \Phi^{-} v \) is a minmax solution. To see that it is the unique optimal solution, observe that we have just established that

\[
\max_{u \in \mathcal{B}|\Phi(u) = v} \frac{\|u - \Psi(v)\|_\mathcal{B}}{\|u\|_\mathcal{B}} = 1 \tag{9.2}
\]

for any optimal \( \Psi : V \to \mathcal{B} \). It then follows that

\[
\max_{u \in \mathcal{B}} \frac{\|u - \Psi(\Phi(u))\|_\mathcal{B}}{\|u\|_\mathcal{B}} = 1
\]

which implies that the map \( I - \Psi \circ \Phi : \mathcal{B} \to \mathcal{B} \) is a contraction. Moreover, by selecting \( u \in \text{Ker}(\Phi) \) tending to 0, it follows from (9.2) that \( \Psi(0) = 0 \). Since, by definition, \( \Phi \circ \Psi = I_V \), we have\( (I - \Psi \circ \Phi)^2(u) = (I - \Psi \circ \Phi)(u - \Psi \circ \Phi(u)) = u - \Psi \circ \Phi(u) - \Phi \circ \Psi(\Phi(u)) = u - \Psi \circ \Phi(u) - \Psi(\Phi(u) - \Phi \circ \Psi(\Phi(u))) = u - \Psi \circ \Phi(u) - \Phi(0) = u - \Phi \circ \Phi(u) \)

so that the map \( I - \Psi \circ \Phi \) is a projection. Since \( \Phi(\Phi(u) - \Psi \circ \Phi(u)) = 0 \) it follows that \( \text{Im}(I - \Psi \circ \Phi) \subset \text{Ker}(\Phi) \), but since for \( b \in \text{Ker}(\Phi) \), we have \( (I - \Psi \circ \Phi)(b) = b - \Psi \circ \Phi(b) = b \), we obtain the equality \( \text{Im}(I - \Psi \circ \Phi) = \text{Ker}(\Phi) \).

To show that a projection of this form is necessarily linear, let us demonstrate that \( \text{Im}(\Psi \circ \Phi) = \text{Ker}^\perp(\Phi) \). To that end, use the decomposition \( \mathcal{B} = \text{Ker}(\Phi) \oplus \text{Ker}^\perp(\Phi) \) to write \( u = u' + u'' \) with \( u' \in \text{Ker}(\Phi) \) and \( u'' \in \text{Ker}^\perp(\Phi) \) and write the contractive condition \( \|u - \Psi \circ \Phi(u)\|^2 \leq \|u\|^2 \) as

\[
\|u' + u'' - \Psi \circ \Phi(u' + u'')\|^2 \leq \|u' + u''\|^2,
\]

which using the linearity of \( \Phi \) and \( u' \in \text{Ker}(\Phi) \) we obtain

\[
\|u' + u'' - \Psi \circ \Phi(u'')\|^2 \leq \|u' + u''\|^2,
\]

61
Suppose that $\Psi \circ \Phi(u'') = v' + v''$ with $v' \in \text{Ker}(\Phi)$ nontrivial. Then, selecting $u' = tv'$, with $t \in \mathbb{R}$, we obtain
\[
\|(t-1)v' + u'' - v''\|^2 \leq \|tv' + u''\|^2
\]
which amounts to
\[
(t-1)^2\|v'\|^2 + \|u'' - v''\|^2 \leq t^2\|v'\|^2 + \|u''\|^2
\]
which amounts to
\[
(1-2t)\|v'\|^2 + \|u'' - v''\|^2 \leq \|u''\|^2
\]
which provides a contradiction for $t$ large enough negative. Consequently, $v' = 0$ and $\text{Im}(\Psi \circ \Phi) \subset \text{Ker}^\perp(\Phi)$. Since $I = \Psi \circ \Phi + (I - \Psi \circ \Phi)$ with $\text{Im}(\Psi \circ \Phi) \subset \text{Ker}^\perp(\Phi)$ and $\text{Im}(I - \Psi \circ \Phi) \subset \text{Ker}(\Phi)$ it follows that $\text{Im}(\Psi \circ \Phi) = \text{Ker}^\perp(\Phi)$. Since $\Psi \circ \Phi$ is a projection it follows that
\[
\Psi \circ \Phi(u'') = u'', \quad u'' \in \text{Ker}^\perp(\Phi).
\]
Consequently, for two elements $u_1 = u'_1 + u''_1$ and $u_2 = u'_2 + u''_2$ with $u'_1 \in \text{Ker}(\Phi)$ and $u''_i \in \text{Ker}^\perp(\Phi)$ for $i = 1, 2$ we have
\[
(I - \Psi \circ \Phi)(u_1 + u_2) = u_1 + u_2 - \Psi \circ \Phi(u_1 + u_2) = u'_1 + u'_2 + u''_1 + u''_2 - \Psi \circ \Phi(u''_1 + u''_2) = u'_1 + u'_2 + u''_1 - \Psi \circ \Phi(u'_1) + u''_2 - \Psi \circ \Phi(u''_2) = (I - \Psi \circ \Phi)(u_1) + (I - \Psi \circ \Phi)(u_2),
\]
and similarly, for $t \in \mathbb{R}$,
\[
(I - \Psi \circ \Phi)(tu_1) = t(I - \Psi \circ \Phi)(u_1),
\]
so we conclude that $I - \Psi \circ \Phi$ is linear.

Since according to Rao [31, Rem. 9, pg. 51], a contractive linear projection on a Hilbert space is an orthogonal projection, it follows that the map $I - \Psi \circ \Phi$ is an orthogonal projection, and therefore $\Psi \circ \Phi = P_{\text{Ker}^\perp(\Phi)}$. Since $\Phi^-$ is the Moore-Penrose inverse, it follows that $P_{\text{Ker}^\perp(\Phi)} = \Phi^- \Phi$ so that $\Psi \circ \Phi = \Phi^- \Phi$, and therefore the assertion $\Psi = \Phi^-$ follows by right multiplication by $\Psi$ using the identity $\Phi \circ \Psi = I_{V}$.

### 9.2 Proof of Lemma 2.2

Let us write $\Phi : \mathcal{B} \to V$ as
\[
\Phi(u) = \sum_{i \in I} e_i u_i, \quad u = (u_i \in V_i)_{i \in I},
\]
where we now include the subspace injections $e_i : V_i \to V$ in its description. Let $\tilde{e}_i : V_i \to \mathcal{B}$ denote the component injection $\tilde{e}_i v_i := (0, \ldots, 0, v_i, 0, \ldots, 0)$ and let $\tilde{e}_i^T : \mathcal{B} \to V_i$ denote the component projection. Using this notation, the norm (2.6) on $\mathcal{B}$ becomes
\[
\|u\|^2_{\mathcal{B}} := \sum_{i \in I} \|\tilde{e}_i^T u\|^2_{V_i}, \quad u \in \mathcal{B},
\]
(9.3)
with inner product
\[ \langle u_1, u_2 \rangle_B := \sum_{i \in I} \langle e_i^T u_1, e_i^T u_2 \rangle_{V_i}, \quad u_1, u_2 \in B. \]

Clearly, \( e_j^T e_i = 0, i \neq j \) and \( e_i^T e_i = I_{V_i} \), so that
\[
\langle e_i^T u, v_i \rangle_{V_i} = \sum_{j \in I} \langle e_j^T u, e_j^T e_i v_i \rangle_{V_i} = \langle u, e_i v_i \rangle_B,
\]
implies that \( e_i^T \) is indeed the adjoint of \( e_i \). Consequently we obtain
\[
\Phi = \sum_{i \in I} e_i e_i^T
\]
and therefore its Hilbert space adjoint \( \Phi^T : V \to B \) is
\[
\Phi^T = \sum_{i \in I} e_i e_i^T,
\]
where \( e_i^T : V \to V_i \) is the Hilbert space adjoint of \( e_i \). To compute it, use the Riesz isomorphism
\[
\iota : V \to V^*
\]
and the usual duality relationships to obtain
\[
e_i^T = Q_i e_i^* \iota,
\]
where \( e_i^* : V^* \to V_i^* \) is the dual adjoint projection. Consequently we obtain
\[
\Phi \Phi^T = \sum_{j \in I} \sum_{i \in I} e_j e_j^T e_i e_i^T
\]
\[
= \sum_{i,j \in I} e_j e_j^T e_i e_i^T
\]
\[
= \sum_{i \in I} e_i e_i^T
t
\]
\[
= \sum_{i \in I} e_i Q_i e_i^* \iota,
\]
and therefore defining
\[
S := \sum_{i \in I} e_i Q_i e_i^*
\]
it follows that
\[
\Phi \Phi^T = S \iota.
\]
Since \( \Phi \Phi^T \) and \( \iota \) are invertible, \( S \) is invertible. The invertibility of \( S \) implies both assertions regarding norms and their duality follows in a straightforward way from the definition of the dual norm. For the Hilbert space version see, e.g., [26, Prop. 11.4].
9.3 Proof of Theorem 2.3

We use the notations and results in the proof of Lemma 2.2. The assumption \( V = \Sigma_i V_i \) implies that the information map \( \Phi : B \to V \) defined by

\[
\Phi(u) = \sum_{i \in I} u_i, \quad u = (u_i \in V_i)_{i \in I},
\]

is surjective. Consequently, Lemma 2.1 asserts that the minimizer of (2.5) is \( w^* = \Psi(v) := \Phi^*v \), where the Moore-Penrose inverse \( \Phi^* := \Phi^T(\Phi \Phi^T)^{-1} \) of \( \Phi \) is well defined, with \( \Phi^T : V \to B \) being the Hilbert space adjoint to \( \Phi : B \to V \). The proof of Lemma 2.2 obtained \( \Phi \Phi^T = S_\iota \) where \( S := \sum_{i \in I} e_i Q_i e_i^* \) and \( \iota : V \to V^* \) is the Riesz isomorphism, \( e_i^T = Q_i e_i^* \iota \), where \( e_i^T : V \to V_i \) is the Hilbert space adjoint of \( e_i \) and \( e_i^* : V^* \to V_i^* \) is its dual space adjoint, and \( \Phi^T = \sum_{i \in I} e_i e_i^T \), where \( e_i : V_i \to B \) denotes the component injection \( e_i v_i := (0, \ldots, 0, v_i, 0, \ldots, 0) \).

Therefore, since \( (\Phi \Phi^T)^{-1} = \iota^{-1} S^{-1} \), we obtain \( \Phi^- = \sum_{i \in I} e_i Q_i e_i^* \iota^{-1} S^{-1} \), which amounts to

\[
\Phi^- = \sum_{i \in I} e_i Q_i e_i^* S^{-1}, \tag{9.4}
\]

or in coordinates

\[
(\Phi^- v)_i = Q_i e_i^* S^{-1} v, \quad i \in I,
\]

establishing the first assertion. The second follows from the general property \( \Phi \Phi^- = \Phi \Phi^T(\Phi \Phi^T)^{-1} = I \) of the Moore-Penrose inverse. The first isometry assertion follows from

\[
\|\Phi^- v\|_B^2 = \sum_{i \in I} \| (\Phi^- v)_i \|_{V_i}^2 = \sum_{i \in I} \| Q_i e_i^* S^{-1} v \|_{V_i}^2 = \sum_{i \in I} [Q_i^{-1} Q_i e_i^* S^{-1} v, Q_i e_i^* S^{-1} v] = \sum_{i \in I} [e_i^* S^{-1} v, Q_i e_i^* S^{-1} v] = \sum_{i \in I} [S^{-1} v, e_i Q_i e_i^* S^{-1} v] = [S^{-1} v, S S^{-1} v] = [S^{-1} v, v] = \| v \|_{S^{-1}}^2
\]

for \( v \in V \).
For the second, write \( \Phi = \sum_{i \in \mathcal{I}} e_i e_i^T \) and consider its dual space adjoint \( \Phi : V^* \to B^* \) defined by

\[
\Phi^* = \sum_{i \in \mathcal{I}} e_i^T e_i^*.\]

A straightforward calculation shows that \( e_i^T e_i^* : \mathcal{I} \to B^* \) is the component injection into the product \( B^* = \prod_{i \in \mathcal{I}} V_i^* \). Consequently, we obtain

\[
e_i^T Q e_j^* = \delta_{i,j} Q, \quad i, j \in \mathcal{I},
\]

so that

\[
\Phi Q \Phi^* = \sum_{i \in \mathcal{I}} e_i e_i^T Q \sum_{j \in \mathcal{I}} e_j^T e_j^* = \sum_{i, j \in \mathcal{I}} e_i e_i^T Q e_j^* e_j^* = \sum_{i \in \mathcal{I}} e_i Q_i e_i^* = S,
\]

and since, for \( \phi \in V^* \),

\[
\|\Phi^* \phi\|_B^2 = \langle \Phi^* \phi, Q \Phi^* \phi \rangle_{B^*} = [\Phi^* \phi, Q \Phi^* \phi] = [\phi, \Phi Q \Phi^* \phi] = [\phi, S \phi] = \|\phi\|_S^2
\]

it follows that \( \Phi^* \) is an isometry.

### 9.4 Proof of Theorem 2.4

Use the Riesz isomorphism between \( V \) and \( V^* \) to represent the dual space adjoint \( \Phi^* : V^* \to B^* \) of \( \Phi : B \to V \) as \( \Phi^* : V \to B^* \). It follows from the definition of the Hilbert space adjoint \( \Phi^T : V \to B \) that

\[
[\Phi^* v, b] = \langle v, \Phi b \rangle = \langle \Phi^T v, b \rangle_B.
\]

Since \( Q : B^* \to B \) (2.11) defines the \( B \) inner product through

\[
\langle b_1, b_2 \rangle_B = [Q^{-1} b_1, b_2], \quad b_1, b_2 \in B
\]

it follows that \( [\Phi^* v, b] = \langle Q \Phi^* v, b \rangle_B \) and therefore \( \langle Q \Phi^* v, b \rangle_B = \langle \Phi^T v, b \rangle_B, v \in V, b \in B \), so we conclude that

\[
\Phi^T = Q \Phi^*.
\]

Since Theorem 2.3 demonstrated that \( \Psi \) is the Moore-Penrose inverse \( \Phi^\dagger \) which implies that \( \Psi \circ \Phi \) is the orthogonal projection onto \( \text{Im}(\Phi^T) \) it follows that \( \Psi \circ \Phi(u) \in \text{Im}(\Phi^T) \). However, the identity \( \Phi^T = Q \Phi^* \) implies that \( \text{Im}(\Phi^T) = Q \text{Im}(\Phi^*) \) so that we obtain the first part

\[
\|u - \Psi(\Phi(u))\|_B = \inf_{\phi \in V^*} \|u - Q \Phi^* (\phi)\|_B
\]

of the assertion. The second half follows from the definition (2.6) of \( \| \cdot \|_B \).
9.5 Proof of Proposition 3.1

Restating the assertion using the injections $e_i : V_i \to V$, our objective is to establish that

$$E(i) = \text{Var}( [\phi, e_i \xi_i]) = \text{Var}(\langle e_i \xi_i, v \rangle_{S^{-1}}).$$

Since $[\phi, e_i \xi_i] = [e_i^* \phi, \xi_i]$ it follows that $[\phi, e_i \xi_i] \sim \mathcal{N}(0, [e_i^* \phi, Q_i e_i^* \phi])$ so that $\text{Var}( [\phi, e_i \xi_i]) = [e_i^* \phi, Q_i e_i^* \phi]$, which using $\phi = S^{-1}v$ becomes

$$\text{Var}( [\phi, e_i \xi_i]) = [S^{-1}v, e_i Q_i e_i^* S^{-1}v].$$

On the other hand, the definitions (3.1) of $E(i)$, (2.7) of $\|v\|_V$, and Theorem 2.3 imply that

$$E(i) := \|\Psi_i(v)\|_{V_i}^2 = [Q_i^{-1}\Psi_i(v), \Psi_i(v)] = [Q_i^{-1} Q_i e_i^* S^{-1}v, Q_i e_i^* S^{-1}v] = [e_i^* S^{-1}v, Q_i e_i^* S^{-1}v] = [S^{-1}v, e_i Q_i e_i^* S^{-1}v],$$

so that we conclude the first part $E(i) = \text{Var}( [\phi, e_i \xi_i])$ of the assertion. Since $[\phi, e_i \xi_i] = [S^{-1}v, e_i \xi_i] = \langle v, e_i \xi_i \rangle_{S^{-1}}$ we obtain the second.

9.6 Proof of Theorem 3.4

Fix $1 \leq k < r \leq q$. To apply Theorem 2.3, we select $\mathcal{B} := \mathcal{B}^{(k)}$ and $V := \mathcal{B}^{(r)}$ and endow them with the external direct sum vector space structure of products of vector spaces. Since the information operator $\Phi^{(r,k)} : \mathcal{B}^{(k)} \to \mathcal{B}^{(r)}$ defined in (3.9) is diagonal with components $\Phi_j^{(r,k)} : B_j^{(k)} \to V_j^{(r)}$, $j \in J^{(r)}$ and the norm on $\mathcal{B}^{(k)} = \prod_{j \in J^{(r)}} B_j^{(k)}$ is the product norm $\|u\|_\mathcal{B}^{(k)} = \prod_{j \in J^{(r)}} \|u_j\|_{B_j^{(k)}}$, $u = (u_i)_{i \in J^{(r)}}$, it follows from the variational characterization of Lemma 2.1, the diagonal nature of the information map $\Phi^{(r,k)}$ and the product metric structure on $\mathcal{B}^{(k)}$ that the optimal recovery solution $\Psi^{(k,r)}$ is the diagonal operator with components the optimal solution operators corresponding to the component information maps $\Phi_j^{(r,k)} : B_j^{(k)} \to V_j^{(r)}$, $j \in J^{(r)}$. Since each component (3.8) of the observation operator is

$$\Phi_j^{(r,k)}(u) := \sum_{i \in j^{(k)}} u_i, \quad u \in B_j^{(k)}$$

it follows that the appropriate subspaces of $V_j^{(r)}$ are

$$V_j^{(k)} \subset V_j^{(r)}, \quad i \in j^{(k)}.$$
Moreover, Condition 3.3 and the semigroup nature of the hierarchy of subspace embeddings implies that

\[ e_{j,i}^{(k+2,k)} = \sum_{l \in j^{(k+1)}} e_{j,l}^{(k+2,k+1)} e_{l,i}^{(k+1,k)}, \quad i \in j^{(k)}, \]

where the sum, despite its appearance, is over one term, and by induction we can establish that assumption (3.19) implies that

\[ Q_j^{(r)} = \sum_{i \in j^{(k)}} e_{j,i}^{(r,k)} Q_i^{(k)} e_{i,j}^{(k,r)}, \quad j \in \mathcal{I}^{(r)}. \tag{9.5} \]

Utilizing the adjoint \( e_{i,j}^{(k,r)} : V_j^{(r),*} \to V_i^{(k),*} \) (3.13) to the subspace embedding \( e_{i,j}^{(r,k)} : V_i^{(k)} \to V_j^{(r)} \), it now follows from Theorem 2.3 and (9.5) that these component optimal solution maps \( \Psi_j^{(k,r)} : V_j^{(r)} \to \mathcal{B}_j^{(k)} \) are those assumed in the theorem in (3.14) and (3.15) as

\[ \Psi_j^{(k,r)}(v_j) := (e_{i,j}^{(k,r)} Q_j^{(r),-1} v_j)_{i \in j^{(k)}}, \quad v_j \in V_j^{(r)}. \tag{9.6} \]

The first three assertions for each component \( j \) then follow from Theorem 2.3, thus establishing the first three assertions in full.

For the semigroup assertions, Condition 3.3 implies that, for \( k < r < s \) and \( l \in I^{(s)} \), there is a one to one relationship between \( \{ j \in I^{(r)}, i \in j^{(k)} \} \) and \( \{ i \in j^{(k)} \} \). Consequently, the definition (3.9) of \( \Phi^{(r,k)} \) implies

\[ \Phi^{(s,r)} \circ \Phi^{(r,k)}(u) = \left( \sum_{j \in I^{(s)}} \left( \sum_{i \in j^{(k)}} u_i \right) \right)_{j \in I^{(r)}} = \left( \sum_{i \in j^{(k)}} u_i \right)_{i \in j^{(k)}} = \Phi^{(s,k)}(u), \]

establishing the fourth assertion \( \Phi^{(s,k)} = \Phi^{(s,r)} \circ \Phi^{(r,k)} \).

For the fifth, the definition (3.16) of \( \Psi^{(r,s)} \) implies that

\[ \Psi^{(k,r)} \circ \Psi^{(r,s)}(v) = \left( Q_j^{(s)} e_{i,j}^{(r,s)} Q_j^{(r),-1} \Psi_j^{(s)}(v) \right)_{i \in j^{(k)}} = \left( Q_j^{(s)} e_{i,j}^{(r,s)} Q_j^{(r),-1} Q_j^{(r),-1} Q_j^{(s),-1} v_i \right)_{i \in j^{(k)}} = \left( Q_j^{(s)} e_{i,j}^{(s)} Q_i^{(s),-1} v_i \right)_{i \in j^{(k)}} = \left( Q_j^{(s)} e_{i,j}^{(s)} Q_i^{(s),-1} v_i \right)_{i \in j^{(k)}} \]

establishing \( \Psi^{(k,s)} = \Psi^{(k,r)} \circ \Psi^{(r,s)}. \)

The last assertion follows directly from the second and the fifth.

### 9.7 Proof of Theorem 3.6

Since \( \xi^{(k)} : \mathcal{B}^{(k),*} \to H \) is an isometry to a Gaussian space of real variables we can abuse notation and write \( \xi^{(k)}(b^*) = [b^*, \xi^{(k)}] \) which emphasizes the interpretation of \( \xi^{(k)} \) as a weak \( \mathcal{B}^{(k)} \)-valued random variable. Since, by Theorem 3.4

\[ \Phi^{(k,1),*} : \mathcal{B}^{(k),*} \to \mathcal{B}^{(1),*} \]

is an isometry (9.7)
and $\xi^{(1)} : B^{(1),*} \to H$ is an isometry, it follows that

$$\Phi^{(k,1)} \xi^{(1)} := \xi^{(1)} \circ \Phi^{(k,1),*} : B^{(k),*} \to H$$

is an isometry, and therefore a Gaussian field on $B^{(k)}$. Since Gaussian fields transform like Gaussian measures with respect to continuous linear transformations, we obtain that

$$\xi^{(1)} \sim \mathcal{N}(0, Q^{(1)})$$

for some $E$. Thus establishing the assertion that

$$\Phi^{(k,1)} \xi^{(1)} \sim \mathcal{N}(0, \Phi^{(k,1),*} Q^{(1)} \Phi^{(k,1),*})$$

but the isometric nature (9.7) of $\Phi^{(k,1),*}$ implies that

$$\Phi^{(k,1)} Q^{(1)} \Phi^{(k,1),*} = Q^{(k)}$$

so we conclude that

$$\Phi^{(k,1)} \xi^{(1)} \sim \mathcal{N}(0, Q^{k})$$

thus establishing the assertion that $\xi^{(k)}$ is distributed as $\Phi^{(k,1)} \xi^{(1)}$.

The conditional expectation $\mathbb{E}[\xi^{(k)} | \Phi^{(r,k)}(\xi^{(k)})]$ is uniquely characterized by its field of conditional expectations $\mathbb{E}[\xi^{(k)} \mid \Phi^{(r,k)}(\xi^{(k)})]$, $b^* \in B^{(k),*}$ which because of the linearity of conditional expectation of Gaussian random variables appears as

$$\mathbb{E}[\xi^{(k)} \mid \Phi^{(r,k)}(\xi^{(k)})] = [A_{b^*}, \Phi^{(r,k)}(\xi^{(k)})]$$

for some $A_{b^*} \in V^*$. Furthermore, the Gaussian conditioning also implies that the dependence of $A_{b^*}$ on $b^*$ is linear so we write $A_{b^*} = A b^*$ for some $A : B^* \to V^*$, thereby obtaining

$$\mathbb{E}[\xi^{(k)} \mid \Phi^{(r,k)}(\xi^{(k)})] = [A b^*, \Phi^{(r,k)}(\xi^{(k)})], \quad b^* \in B^{(k),*}.$$ (9.8)

Using the well-known fact, see e.g. Dudley [5, Thm. 10.2.9], that the conditional expectation of a square integrable random variable on a probability space $(\Omega, \Sigma', P)$ with respect to a sub-$\sigma$-algebra $\Sigma' \subset \Sigma$ is the orthogonal projection onto the closed subspace $L^2(\Omega, \Sigma', P) \subset L^2(\Omega, \Sigma, P)$, it follows that the conditional expectation satisfies

$$\mathbb{E}[\xi^{(k)} | \Phi^{(r,k)}(\xi^{(k)})] = v^*, \quad b^* \in B^{(k),*}, v^* \in V^{(k),*}.$$ (9.7)

Rewriting this as

$$\mathbb{E}[[b^*, \xi^{(k)}] - [A b^*, \Phi^{(r,k)}(\xi^{(k)}) v^*, \Phi^{(r,k)}(\xi^{(k))}] = 0, \quad b^* \in B^{(k),*}, v^* \in V^{(k),*}.$$ (9.6)

we obtain

$$[b^*, Q^{(k)} \Phi^{(r,k),*} v^*] = [A b^*, Q^{(k)} \Phi^{(r,k),*} v^*] = [b^*, A^* \Phi^{(r,k)} Q^{(k)} \Phi^{(r,k),*} v^*]$$

for all $b^* \in B^{(k),*}$ and $v^* \in V^{(k),*}$, and so conclude that

$$A^* \Phi^{(r,k)} Q^{(k)} \Phi^{(r,k),*} v^* = Q^{(k)} \Phi^{(r,k),*} v^*, \quad b^* \in B^{(k),*}, v^* \in V^{(k),*}$$

68
which implies that
\[ A^* \Phi^{(r,k)} b = b, \quad b \in \text{Im}(Q^{(k)} \Phi^{(r,k),*}). \] (9.9)

Since
\[
\langle \Phi^{(r,k), T} b^{(r)}, b^{(k)} \rangle_{B^{(k)}} = \langle b^{(r)}, \Phi^{(r,k)} b^{(k)} \rangle_{B^{(k)}} \\
= [Q^{(k)} , -1 b^{(r)}, \Phi^{(r,k)} b^{(k)}] \\
= [\Phi^{(r,k),*} Q^{(k)} , -1 b^{(r)}, b^{(k)}] \\
= [Q^{(r), -1} Q^{(r)} \Phi^{(r,k),*}, Q^{(k)}, -1 b^{(r)}, b^{(k)}] \\
= \langle Q \Phi^{(r,k),*} Q^{(k)}, -1 b^{(r)}, b^{(k)} \rangle_{B^{(r)}},
\]
we conclude that
\[ \Phi^{(r,k), T} = Q^{(r)} \Phi^{(r,k),*} Q^{(k)}, -1 \]
and therefore
\[ \text{Im}(Q^{(r)} \Phi^{(r,k),*}) = \text{Im}(\Phi^{(r,k), T}). \]
Consequently, (9.9) now reads
\[ A^* \Phi^{(r,k)} b = b, \quad b \in \text{Im}(\Phi^{(r,k), T}). \] (9.10)

Since clearly
\[ A^* \Phi^{(r,k)} b = 0, \quad b \in \text{Ker}(\Phi^{(r,k)}) \]
it follows that
\[ A^* \Phi^{(r,k)} = P_{\text{Im}(\Phi^{(r,k), T})} \]
Since \( P_{\text{Im}(\Phi^{(r,k), T})} = (\Phi^{(r,k)})^{-1} \Phi^{(r,k)} \), the identity \( \Phi^{(r,k)} (\Phi^{(r,k)})^{-1} = I \) establishes that
\[ A^* = (\Phi^{(r,k)})^{-1} \]

Since (9.8) implies that
\[ \mathbb{E}[b^*, \xi^{(k)}] = [\xi^{(k)}] \]
which in turn implies that
\[ \mathbb{E}[\xi^{(k)} | \Phi^{(r,k)} (\xi^{(k)})] = A^* \Phi^{(r,k)} (\xi^{(k)}) \]
we obtain
\[ \mathbb{E}[\xi^{(k)} | \Phi^{(r,k)} (\xi^{(k)})] = (\Phi^{(r,k)})^{-1} \Phi^{(r,k)} (\xi^{(k)}) \]
Since Theorem 2.3 established that the optimal solution operator \( \Psi^{(k,r)} \) corresponding to the information map \( \Phi^{(r,k)} \) was the Moore-Penrose inverse \( \Psi^{(k,r)} = (\Phi^{(r,k)})^{-} \) we obtain
\[ \mathbb{E}[\xi^{(k)} | \Phi^{(r,k)} (\xi^{(k)})] = \Psi^{(k,r)} \circ \Phi^{(r,k)} (\xi^{(k)}) \]
so that
\[ \mathbb{E}[\xi^{(k)} | \Phi^{(r,k)} (\xi^{(k)}) = v] = \Psi^{(k,r)} (v), \] (9.11)
thus establishing the final assertion. To establish the martingale property, let us define \( \tilde{\xi}^{(1)} := \xi^{(1)} \) and

\[
\dot{\tilde{\xi}}^{(k)} := \mathbb{E}[\xi^{(1)} \mid \Phi^{(k,1)}(\xi^{(1)})], \quad k = 2, \ldots .
\]

as a sequence of Gaussian fields all on the same space \( \mathcal{B}^{(1)} \). (9.11) implies that

\[
\dot{\tilde{\xi}}^{(k)} = \Psi^{(1,k)} \circ \Phi^{(k,1)}(\tilde{\xi}^{(1)}), \quad (9.12)
\]

so that, the identities \( \Phi^{(r,1)} = \Phi^{(r,k)} \circ \Phi^{(k,1)} \) and \( \Phi^{(k,1)} \circ \Psi^{(1,k)} = I_{\mathcal{B}^{(1)}} \) from Theorem 3.4

\[
\mathbb{E}[\dot{\tilde{\xi}}^{(k)}|\Phi^{(r,1)}(\tilde{\xi}^{(1)})] = \mathbb{E}[\Psi^{(1,k)} \circ \Phi^{(k,1)}(\tilde{\xi}^{(1)})|\Phi^{(r,1)}(\tilde{\xi}^{(1)})] \circ \Phi^{(k,1)}(\tilde{\xi}^{(1)})] = \mathbb{E}[\Psi^{(1,k)} \circ \Phi^{(k,1)}(\tilde{\xi}^{(1)})|\tilde{\xi}^{(1)}] \circ \Psi^{(k,1)}(\tilde{\xi}^{(1)})]
\]

that is \( \dot{\tilde{\xi}}^{(k)} \) is a reverse martingale.

### 9.8 Proof of Theorem 5.1

Let us simplify for the moment and define a scaled wavelet

\[
\tilde{\chi}_{\tau,\omega,\theta}(t) := \omega^{\frac{1-\beta}2} \cos(\omega(t - \tau) + \theta) e^{-\frac{\omega^2(t-\tau)^2}{\alpha^2}}, \quad t \in \mathbb{R}, \quad (9.13)
\]

so that at \( \beta = 0 \) we have

\[
\chi_{\tau,\omega,\theta} = \left(\frac{2}{\pi^3 \alpha^2}\right)^{\frac{1}{4}} \tilde{\chi}_{\tau,\omega,\theta}. \quad (9.14)
\]

Since

\[
K(s, t) := \int_{-\pi}^{\pi} \int_{\mathbb{R}^+} \int_{\mathbb{R}} \tilde{\chi}_{\tau,\omega,\theta}(s) \tilde{\chi}_{\tau,\omega,\theta}(t) d\tau d\omega d\theta
\]

\[
= \int_{-\pi}^{\pi} \int_{\mathbb{R}^+} \int_{\mathbb{R}} \cos(\omega(s - \tau) + \theta) e^{-\frac{\omega^2(s-\tau)^2}{\alpha^2}} \cos(\omega(t - \tau) + \theta) e^{-\frac{\omega^2(t-\tau)^2}{\alpha^2}} d\tau d\omega d\theta
\]

\[
= \int_{-\pi}^{\pi} \int_{\mathbb{R}^+} \int_{\mathbb{R}} \cos(\omega(s - \tau) + \theta) \cos(\omega(t - \tau) + \theta) e^{-\frac{\omega^2(s-\tau)^2}{\alpha^2}} e^{-\frac{\omega^2(t-\tau)^2}{\alpha^2}} d\tau d\omega d\theta
\]

70
the trigonometric identity
\[
\cos(\omega(s - \tau) + \theta) \cos(\omega(t - \tau) + \theta) = \left(\cos(\omega(s - \tau)) \cos \theta - \sin(\omega(s - \tau)) \sin \theta\right) \left(\cos(\omega(t - \tau)) \cos \theta - \sin(\omega(t - \tau)) \sin \theta\right)
\]

and the integral identities \(\int_{-\pi}^{\pi} \cos^2 \theta d\theta = \int_{-\pi}^{\pi} \sin^2 \theta d\theta = \pi\) and \(\int_{-\pi}^{\pi} \cos \theta \sin \theta d\theta = 0\) imply that

\[
K(s, t) = \pi \int_{\mathbb{R}^+} \int_{\mathbb{R}} \left(\cos(\omega(s - \tau)) \cos(\omega(t - \tau)) + \sin(\omega(s - \tau)) \sin(\omega(t - \tau))\right) e^{-\frac{\omega^2(s-\tau)^2}{\alpha^2}} e^{-\frac{\omega^2(t-\tau)^2}{\alpha^2}} d\tau \omega^{1-\beta} d\omega
\]

so that the trigonometric identity \(\cos(a - b) = \cos a \cos b + \sin a \sin b\) implies

\[
K(s, t) = \pi \int_{\mathbb{R}^+} \int_{\mathbb{R}} \cos(\omega(s - t)) e^{-\frac{\omega^2(s-t)^2}{\alpha^2}} e^{-\frac{\omega^2(t-\tau)^2}{\alpha^2}} d\tau \omega^{1-\beta} d\omega
\]

which amounts to

\[
K(s, t) = \pi \Re \int_{\mathbb{R}^+} \int_{\mathbb{R}} e^{i\omega(s-t)} e^{-\frac{\omega^2(s-t)^2}{\alpha^2}} e^{-\frac{\omega^2(t-\tau)^2}{\alpha^2}} d\tau \omega^{1-\beta} d\omega.
\] (9.15)

Using the identity

\[
e^{-\frac{\omega^2(s-t)^2}{\alpha^2}} e^{-\frac{\omega^2(t-\tau)^2}{\alpha^2}} = e^{-\frac{\omega^2}{\alpha^2} (2\tau^2 - 2(s+t)\tau)} e^{-\frac{\omega^2}{\alpha^2} (s^2 + t^2)}
\]

and the integral identity

\[
\int e^{-a\tau^2 - 2br\tau} d\tau = \sqrt{\frac{\pi}{a}} e^{\frac{r^2}{a}}, \quad a > 0, b \in \mathbb{C},
\] (9.16)

with the choice \(a := \frac{2\alpha^2}{\omega^2}\) and \(b := -\frac{\omega^2}{\alpha^2} (s + t)\), so that \(b^2/a = \frac{\omega^2}{2\alpha^2} (s + t)^2\), we can evaluate the integral

\[
\int e^{-\frac{\omega^2}{\alpha^2} (2\tau^2 - 2(s+t)\tau)} d\tau = \frac{\alpha}{\omega} \sqrt{\frac{\pi}{2}} e^{\frac{\omega^2}{2\alpha^2} (s+t)^2}.
\]

Consequently,

\[
K(s, t) = \pi \Re \int_{\mathbb{R}^+} \int_{\mathbb{R}} e^{i\omega(s-t)} e^{-\frac{\omega^2(s-t)^2}{\alpha^2}} e^{-\frac{\omega^2(t-\tau)^2}{\alpha^2}} d\tau \omega^{1-\beta} d\omega
\]

\[
= \pi \Re \int_{\mathbb{R}^+} \int_{\mathbb{R}} e^{i\omega(s-t)} e^{-\frac{\omega^2}{\alpha^2} (s^2 + t^2)} e^{\frac{\omega^2}{\alpha^2} (2\tau^2 - 2(s+t)\tau)} d\tau \omega^{1-\beta} d\omega
\]

\[
= \pi \Re \int_{\mathbb{R}^+} \int_{\mathbb{R}} e^{i\omega(s-t)} e^{-\frac{\omega^2}{\alpha^2} (s^2 + t^2)} \left(\int e^{\frac{-\omega^2}{\alpha^2} (2\tau^2 - 2(s+t)\tau)} d\tau\right) \omega^{1-\beta} d\omega
\]

\[
= \alpha \sqrt{\frac{\pi}{2}} \Re \int_{\mathbb{R}^+} \int_{\mathbb{R}} e^{i\omega(s-t)} e^{-\frac{\omega^2}{\alpha^2} (s^2 + t^2)} e^{\frac{\omega^2}{2\alpha^2} (s+t)^2} \omega^{-\beta} d\omega
\]

\[
= \alpha \sqrt{\frac{\pi}{2}} \Re \int_{\mathbb{R}^+} \int_{\mathbb{R}} e^{i\omega(s-t)} e^{-\frac{\omega^2}{2\alpha^2} (s-t)^2} \omega^{-\beta} d\omega
\]

\[
= \alpha \sqrt{\frac{\pi}{2}} \Re \int_{\mathbb{R}^+} \cos(\omega(s - t)) e^{-\frac{\omega^2}{2\alpha^2} (s-t)^2} \omega^{-\beta} d\omega,
\]

71
that is,
\[
K(p, s, t) = \alpha \sqrt{\frac{\pi^3}{2}} \int \cos(\omega(s-t))e^{-\frac{\omega^2}{2\alpha^2}(s-t)\omega-\beta}d\omega. \tag{9.17}
\]

Utilizing the integral identity
\[
\int_0^\infty x^{\mu-1}e^{-px^2} \cos(ax)dx = \frac{1}{2}p^{-\mu}\Gamma\left(\frac{\mu}{2}\right)e^{-\frac{a^2}{4p^2}}I_1\left(-\frac{\mu}{2} + \frac{1}{2}, \frac{1}{2}, \frac{a^2}{4p^2}\right), \quad a > 0, \mu > 0,
\]
from Gradshteyn and Ryzhik [10, 3.952:8], with
\[
p^2 = \frac{|s-t|^2}{2\alpha^2}, \quad a := |s-t| \quad \text{and} \quad \mu := 1 - \beta,
\]
we obtain
\[
K(p, s, t) = \alpha \sqrt{\frac{\pi^3}{2}} (\sqrt{2\alpha})^{1-\beta}|s-t|^{\beta-1}\Gamma\left(\frac{1-\beta}{2}\right)e^{-\frac{a^2}{2}}I_1\left(\frac{1}{2}, \frac{1}{2}, \frac{\alpha^2}{2}\right).
\]

Consequently, reintroducing the scaling (9.14) obtains \(K_\beta(p, s, t) = \left(\frac{2}{\pi\alpha^2}\right)^{\frac{1}{2}}K(p, s, t)\) when \(\beta = 0\). To indicate the dependence on \(\beta\), we define
\[
K_\beta(p, s, t) = \frac{1}{2}(\sqrt{2\alpha})^{1-\beta}|s-t|^{\beta-1}\Gamma\left(\frac{1-\beta}{2}\right)e^{-\frac{a^2}{2}}I_1\left(\frac{1}{2}, \frac{1}{2}, \frac{\alpha^2}{2}\right), \tag{9.19}
\]
so that \(K_\beta = K_0\). For fixed \(\alpha\), at the limit \(\beta = 0\), we have, recalling that \(\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}\),
\[
K_0(p, s, t) = \frac{\sqrt{2\pi}}{2}\alpha|s-t|^{-1}e^{-\frac{a^2}{2}}I_1\left(0, \frac{1}{2}, \frac{\alpha^2}{2}\right)
\]
and since
\[
I_1\left(0, \frac{1}{2}, \frac{\alpha^2}{2}\right) = 1
\]
we obtain
\[
K_0(p, s, t) = \frac{\sqrt{2\pi}}{2}\alpha|s-t|^{-1}e^{-\frac{a^2}{2}}.
\]

The scaling constant \(H(\beta)\) defined in the theorem satisfies
\[
H(\beta) := \frac{1}{2}(\sqrt{2\alpha})^{1-\beta}\Gamma(1-\frac{\beta}{2})e^{-\frac{a^2}{2}}I_1\left(\frac{1}{2}, \frac{1}{2}, \frac{\alpha^2}{2}\right)\tilde{H}(\beta)
\]
with
\[
\tilde{H}(\beta) := 2^{\beta} \sqrt{\pi} \frac{\Gamma\left(\frac{\beta}{2}\right)}{\Gamma(1-\frac{\beta}{2})}, \tag{9.20}
\]

72
so that, by (9.19) we have
\[
\frac{1}{H(\beta)} K_\beta(s, t) = \frac{|s - t|^{\beta - 1}}{H(\beta)}.
\]
Therefore, if we let \( K_\beta \) denote the integral operator
\[
(K_\beta f)(s) := \frac{1}{H(\beta)} \int_{\mathbb{R}} K_\beta(s, t)f(t)dt
\]
associated to the kernel \( K_\beta \) scaled by \( H(\beta) \), it follows that
\[
(K_\beta f)(s) = \frac{1}{H(\beta)} \int_{\mathbb{R}} |s - t|^{\beta - 1} f(t)dt,
\]
namely that it is a scaled version of the integral operator \( f \mapsto \int_{\mathbb{R}} |s - t|^{\beta - 1} f(t)dt \) corresponding to the Riesz potential \( |s - t|^{\beta - 1} \). Consequently, according to Helgason [12, Lem. 5.4 & Prop. 5.5], this scaling of the Riesz potential by \( H(\beta) \) implies the assertions of the theorem.

9.9 Proof of Lemma 6.1
The outer most integral in the definition (6.1) of \( K_\beta \) is
\[
\int_{-\pi}^{\pi} y(\omega(s - \tau) + \theta)y^*(\omega(t - \tau) + \theta) d\theta = \int_{-\pi}^{\pi} \sum_{n=-N}^{N} c_n e^{im(\omega(s - \tau) + \theta)} \sum_{-N}^{N} c_m^* e^{-im(\omega(t - \tau) + \theta)} d\theta
\]
\[
= \sum_{n=-N}^{N} \sum_{m=-N}^{N} e^{im\omega(s - \tau)} e^{-im\omega(t - \tau)} c_n c_m^* \int_{-\pi}^{\pi} e^{i(n-m)\theta} d\theta
\]
\[
= 2\pi \sum_{n=-N}^{N} e^{i\omega(s - \tau)} e^{-i\omega(t - \tau)} |c_n|^2
\]
\[
= 2\pi \sum_{n=-N}^{N} e^{i\omega(s - t)} |c_n|^2
\]
so that
\[
K_\beta(s, t) = 2\pi \sum_{n=-N}^{N} K_n(s, t)|c_n|^2
\]
where

\[
K_n(s, t) = \Re \int e^{i\omega(s-t)} e^{-\frac{|s-t|^2}{2\alpha^2}} e^{-\frac{|s-t|^2}{2\alpha^2}} d\tau \omega^{1-\beta} d\omega
\]

\[
= \Re \int e^{i\omega(s-t)} e^{-\frac{|s-t|^2}{2\alpha^2}} e^{-\frac{|s-t|^2}{2\alpha^2}} \left( \int e^{\frac{\omega^2}{2\alpha^2}} (2\tau^2 - 2(s+t)\tau) d\tau \right) \omega^{1-\beta} d\omega
\]

\[
= \alpha \sqrt{\frac{\pi}{2}} \Re \int e^{i\omega(s-t)} e^{-\frac{|s-t|^2}{2\alpha^2}} e^{\frac{\omega^2}{2\alpha^2}} (s+t)^2 \omega^{-\beta} d\omega
\]

\[
= \alpha \sqrt{\frac{\pi}{2}} \Re \int e^{i\omega(s-t)} e^{-\frac{|s-t|^2}{2\alpha^2}} \omega^{-\beta} d\omega
\]

\[
= \alpha \sqrt{\frac{\pi}{2}} \int \cos(n\omega(s-t)) e^{-\frac{|s-t|^2}{2\alpha^2}} \omega^{-\beta} d\omega.
\]

Consequently, using the integral identity (9.18) with \(a = |n|s-t|, \mu = 1-\beta, p^2 = \frac{|s-t|^2}{2\alpha^2}\), and therefore \(\frac{a^2}{4p^2} = \frac{|n|\alpha^2}{2}\) and \(p = \frac{|s-t|}{\sqrt{2\alpha}}\) we conclude that

\[
K_n(s, t) = \alpha \sqrt{\frac{\pi}{2}} (\sqrt{2\alpha})^{1-\beta} |s-t|^{\beta-1} \Gamma(\frac{1-\beta}{2}) e^{-\frac{|n|\alpha^2}{2}} \frac{1}{2} \frac{1}{2} \frac{1}{2} F_1 \left( \frac{\beta}{2}, \frac{1}{2}, \frac{|n|\alpha^2}{2} \right),
\]

which does not appear to have a nice dependency on \(n\), except for \(\beta = 0\), where

\[
\frac{1}{2} F_1 \left( 0; \frac{1}{2}; \frac{|n|\alpha^2}{2} \right) = 1
\]

and \(\Gamma(\frac{1}{2}) = \sqrt{\pi}\), so that

\[
K_n(s, t) = \frac{\alpha}{2} \pi e^{-\frac{|n|\alpha^2}{2}} |s-t|^{-1}
\]

and therefore

\[
K_0(s, t) = \alpha^2 \pi^2 \|y\|^2 |s-t|^{-1},
\]

when written in terms of the norm \(\|y\|^2 := \sum_{n=-N}^{N} e^{-\frac{|n|^2}{2}} |c_n|^2\).

### 9.10 Proof of Lemma 6.2

For \(\gamma > 0\), let us evaluate the function

\[
\phi(s) := \sum_{n=-\infty}^{\infty} e^{-|n|\gamma} e^{ins}
\]

(9.21)
with Fourier coefficients $\hat{\phi}(n) = e^{-|n|\gamma}$. Since

$$\phi(s) = \sum_{n=-\infty}^{\infty} e^{-|n|\gamma} e^{ins}$$

$$= \sum_{n=1}^{\infty} e^{-n\gamma} e^{ins} + 1 + \sum_{n=-\infty}^{-1} e^{n\gamma} e^{ins}$$

$$= \sum_{n=1}^{\infty} e^{-n\gamma} e^{ins} + 1 + \sum_{n=1}^{\infty} e^{-n\gamma} e^{-ins}$$

$$= 1 + 2 \sum_{n=1}^{\infty} e^{-n\gamma} \cos(ns),$$

the identity

$$1 + 2 \sum_{n=1}^{\infty} e^{-n\gamma} \cos ns = \frac{\sinh(\gamma)}{\cosh(\gamma) - \cos(s)}$$  \hspace{1cm} (9.22)$$

of Gradshteyn and Ryzhik [10, 1.461:2] implies that

$$\phi(s) = \frac{\sinh(\gamma)}{\cosh(\gamma) - \cos(s)}.$$  \hspace{1cm} (9.23)$$

Consequently, with the choice $\gamma := \frac{a^2}{\pi}$ in (9.21), that is for

$$\hat{\phi}(s) := \sum_{n=-\infty}^{\infty} e^{-|n|\frac{a^2}{\pi}} e^{ins}$$

we find that

$$\hat{\phi}(s) = \frac{\sinh(\frac{a^2}{\pi})}{\cosh(\frac{a^2}{\pi}) - \cos(s)}.$$  \hspace{1cm} (9.23)$$

We will need two basic facts about the Fourier transform of $2\pi$-periodic functions, see e.g. Katznelson [18, Sec. I]. If we denote the Fourier transform by $\hat{f}(n) := \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-int}dt$, $\forall n$, the convolution theorem states that for periodic functions $f, g \in L^1[-\pi, \pi]$ that the convolution $(f \ast g)(s) := \frac{1}{2\pi} \int_{-\pi}^{\pi} f(s-t)g(t)dt$ is a well defined periodic function in $L^1[-\pi, \pi]$ and that $(f \ast g)(n) = \hat{f}(n)\hat{g}(n), \forall n$. Moreover, for square integrable $2\pi$-periodic functions in $L^2[-\pi, \pi]$, the Parseval identity is $\sum_{n=-\infty}^{\infty} |\hat{f}(n)|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(s)|^2$. Consequently, observing that $c_n = 0, n < -N, n > N$, the Parseval identity and the
convolution formula imply that

\[
\|y\|^2 = \sum_{n=-N}^{N} e^{-\frac{|n|^2}{4}} |c_n|^2
\]

\[
= \left\| (e^{-\frac{|n|^2}{4}} c_n) \right\|_{\ell^2}^2
\]

\[
= \left\| (\hat{\phi} y) \right\|_{\ell^2}^2
\]

\[
= \left\| (\hat{\phi} \ast y) \right\|_{\ell^2}^2
\]

\[
= \|\phi \ast y\|^2_{L^2[-\pi, \pi]}
\]

\[
= \int |\phi \ast y|^2
\]

\[
= \int \left| \int \phi(s-t)y(t)dt \right|^2 ds
\]

\[
= \int \left( \int \phi(s-t)y(t)dt \int \phi(s-t')y^*(t')dt' \right) ds
\]

\[
= \int \int \phi(s-t)y(t)\phi(s-t')y^*(t')dt dt' ds
\]

\[
= \int G(t, t') y(t)y^*(t') dt dt',
\]

that is,

\[
\|y\|^2 = \int G(t, t') y(t)y^*(t') dt dt'
\]

where

\[
G(t, t') := \int \phi(s-t)\phi(s-t') ds \tag{9.24}
\]

with

\[
\phi(s) = \frac{\sinh(\frac{\alpha^2}{4})}{\cosh(\frac{\alpha^2}{4}) - \cos(s)}. \tag{9.25}
\]

We can evaluate \(G\) using the identity (9.22) as follows: Since

\[
G(t, t') = \int \phi(s-t)\phi(s-t') ds
\]

\[
= \int \left( 1 + 2 \sum_{n=1}^{\infty} e^{-n^2\frac{\alpha^2}{4}} \cos n(s-t) \right) \left( 1 + 2 \sum_{n'=1}^{\infty} e^{-n'^2\frac{\alpha^2}{4}} \cos n'(s-t') \right) ds,
\]

and, for each product, we have

\[
\int \cos n(s-t) \cos n'(s-t') ds
\]

\[
= \int (\cos ns \cos nt - \sin ns \sin nt)(\cos n's \cos n't' - \sin n's \sin n't') ds
\]

\[
= \delta_{n,n'} \int (\cos ns \cos nt - \sin ns \sin nt)(\cos ns \cos n't' - \sin ns \sin n't') ds,
\]

76
using the $L^2$-orthogonality of the cosines and the sines and the identities $\int \cos^2 ns = \pi$ and $\int \sin^2 ns = \pi$, we conclude that

$$\int (\cos ns \cos nt - \sin ns \sin nt)(\cos ns \cos nt' - \sin ns \sin nt') \, ds = \pi(\cos nt \cos nt' + \sin nt \sin nt')$$

and therefore

$$\int \cos n(s-t) \cos n'(s-t') \, ds = \pi \delta_{n,n'} \cos n(t-t'). \quad (9.26)$$

Consequently, we obtain

$$G(t,t') = \int \left(1 + 2 \sum_{n=1}^{\infty} e^{-n^{2}2} \cos n(s-t) \right) \left(1 + 2 \sum_{n'=1}^{\infty} e^{-n'^{2}2} \cos n'(s-t') \right) \, ds$$

$$= \int \left(1 + 4 \sum_{n=1}^{\infty} e^{-n^{2}2} \cos n(s-t) \cos n(s-t') \right) \, ds$$

$$= 2\pi + 4\pi \sum_{n=1}^{\infty} e^{-n^{2}2} \cos n(t-t')$$

and therefore, using the identity (9.22) again, we conclude

$$G(t,t') = 2\pi \frac{\sinh(\frac{\alpha^2}{2})}{\cosh(\frac{\alpha^2}{2}) - \cos(t-t')}.$$

Acknowledgments The authors gratefully acknowledge support by the Air Force Office of Scientific Research under award number FA9550-18-1-0271 (Games for Computation and Learning).

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