THE DISTRIBUTIVITY SPECTRUM OF BAKER’S VARIETY

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(Received 7 September 2018; accepted 20 May 2020; first published online 26 October 2020)

Communicated by James East

Abstract

For every $n$, we evaluate the smallest $k$ such that the congruence inclusion $\alpha(\beta \circ_n \gamma) \subseteq \alpha \beta \circ_k \alpha \gamma$ holds in a variety of reducts of lattices introduced by K. Baker. We also study varieties with a near-unanimity term and discuss identities dealing with reflexive and admissible relations.

2020 Mathematics subject classification: primary 08B10; secondary 06B75.

Keywords and phrases: Baker’s variety, Jónsson distributivity spectrum, congruence identity, relation identity, congruence distributive variety, near-unanimity term, edge term.

1. Introduction

1.1. Levels of congruence distributive varieties. A variety is a class of algebraic structures, algebras for short, closed under taking subalgebras, products and homomorphic images. A variety is congruence distributive if the congruence lattice of each of its algebras is a distributive lattice. Examples of congruence distributive varieties are (1) any trivial variety with only one-element algebras, (2) the variety of lattices or, more generally, every nontrivial variety with a majority term operation, and (3) the variety of implication algebras, that is, the variety generated by reducts of Boolean algebras in which the term corresponding to implication is the only basic operation.

Hence examples of congruence distributive varieties abound. On the other hand, congruence distributivity is a quite strong and restrictive property; for example, the varieties of groups and of rings are not congruence distributive. The class of congruence distributive varieties has been one of the first classes of varieties intensively studied by general algebraists. A fundamental result by Jónsson [15] states, in an equivalent formulation, that a variety $\mathcal{V}$ is congruence distributive if and only if...
there is some natural number \( k \) such that the congruence inclusion
\[
\alpha(\beta \circ \gamma) \subseteq \alpha\beta \circ \alpha\gamma \circ \alpha\beta \circ \cdots
\] (1-1)
holds in every algebra in \( \mathcal{V} \), with \( k - 1 \) occurrences of \( \circ \) on the right-hand side and where juxtaposition denotes intersection.

Thus it is natural to classify congruence distributive varieties according to their \textit{Jónsson level} [10], that is, the minimal \( k \) for which the above condition holds. The examples (1)–(3) above have been chosen in such a way that the tagging numbers correspond exactly to Jónsson levels. An example of a variety with Jónsson level 4 has been introduced by Baker [1] and this example will be one of the main objects of study in the present work.

While the Jónsson level is surely a natural tool to classify congruence distributive varieties, it is not the only one. In the footnote on [9, page 5] Freese suggests exchanging the role of \( \alpha\beta \) and \( \alpha\gamma \) in the inclusion (1-1), motivated by McKenzie \textit{et al.} [28]. In this way one gets a genuinely new invariant, see also Freese and Valeriote [10].

As another approach, since \( \alpha(\beta \circ \gamma \circ \beta \circ \cdots) \subseteq \alpha(\beta + \gamma) \), for an arbitrary number of factors on the left-hand side and where \( + \) denotes \textit{join}, we can consider inclusions similar to (1-1) in which the left-hand side \( \alpha(\beta \circ \gamma) \) is replaced by \( \alpha(\beta \circ \gamma \circ \beta) \), \( \alpha(\beta \circ \gamma \circ \beta \circ \gamma) \) and so on. It follows either from Jónsson’s argument or from general considerations involving the so-called Maltsev conditions (see, for example, the proof of [24, Theorem 2.1(A) \iff (B)]) that, for every congruence distributive variety \( \mathcal{V} \) and for every fixed number \( n \) of factors on the left, there is some \( k \) such that the analogue of (1-1) holds, namely, the inclusion
\[
\alpha(\beta \circ \gamma \circ \beta \circ \cdots) \subseteq \alpha\beta \circ \alpha\gamma \circ \alpha\beta \circ \cdots
\] (1-2)
holds, with \( n - 1 \) occurrences of \( \circ \) on the left and \( k - 1 \) occurrences of \( \circ \) on the right. In \[24\] we have called the set of such ‘higher levels’ the \textit{Jónsson distributivity spectrum} of \( \mathcal{V} \), namely, to every congruence distributive variety \( \mathcal{V} \) we have associated the function \( J_{\mathcal{V}} \) defined by \( J_{\mathcal{V}}(n - 1) = k - 1 \), where \( k \) is the least natural number such that (1-2) holds, again, with \( n - 1 \) occurrences of \( \circ \) on the left and \( k - 1 \) occurrences of \( \circ \) on the right. The shift by 1 is convenient for notational purposes.

In \[24\] we have presented various examples of functions which can be realized as \( J_{\mathcal{V}} \), for some congruence distributive variety \( \mathcal{V} \); moreover, we have shown that the set of functions representable as \( J_{\mathcal{V}} \) is closed under taking pointwise maximum and that nontrivial constraints are imposed by the value of \( J_{\mathcal{V}}(\ell) \) on the values of \( J_{\mathcal{V}}(\ell') \), for \( \ell' > \ell \). The possible spectra of varieties with Jónsson level 2 or 3 are almost completely characterized in \[24\]; the only problem left is whether \( J_{\mathcal{V}}(2) \leq 2 \) holds for every variety \( \mathcal{V} \) of Jónsson level 3.

1.2. Varieties of Jónsson level 4 and the relational spectrum. In this work we present, among others, the explicit distributivity spectrum of the variety of Jónsson level 4 introduced by Baker [1]. This variety will be called \( \mathcal{B} \) and its definition will
be given in the next subsection. We show that $J_\mathcal{B}(\ell) = 2\ell + 1$ for $\ell$ odd and $J_\mathcal{B}(\ell) = 2\ell$ for $\ell$ even. The theoretical bound for $J_\mathcal{V}$ in the case of varieties of Jónsson level 4 is $J_\mathcal{V}(\ell) \leq 3\ell$, by taking $m = 1$ in [24, Corollary 2.2]. On the other hand, varieties of level less than 4 satisfy $J_\mathcal{V}(\ell) \leq \ell$, for $\ell \geq 3$ [24, Corollary 2.4], and hence $\mathcal{B}$ has an intermediate behavior.

We shall consider many other examples of varieties $\mathcal{V}$ with Jónsson level 4; these varieties will all turn to have exactly the same distributivity spectrum $J_\mathcal{V}$ of $\mathcal{B}$. Among these examples there is a variety with a quaternary near-unanimity term; we shall write $\mathcal{N}_4$ for this variety. Since Baker’s variety $\mathcal{B}$ has no near-unanimity term, the Jónsson distributivity spectrum might appear a quite inadequate classifying tool, since it fails to distinguish between varieties with rather different properties.

Luckily, this is only half the story. It follows from Kazda et al. [18] that in the identities (1-1) and (1-2) we can consider reflexive and admissible relations $S$ and $T$ in place of the congruences $\beta$ and $\gamma$. In principle, the value of $k$ might be different in this case, and thus we get another version of the spectrum. In [24] we have written $J'_\mathcal{V}$ for this modified ‘relational’ spectrum. As we mentioned, $\mathcal{B}$ and $\mathcal{N}_4$ are two varieties of Jónsson level 4 with exactly the same Jónsson distributivity spectrum $J_\mathcal{V}$. However, we show that $J'_\mathcal{B}$ and $J'_\mathcal{N}_4$ are different functions. Thus the relational spectrum $J'_\mathcal{V}$ actually distinguishes between these two structurally different varieties; in particular, $J'_\mathcal{V}$ and $J_\mathcal{V}$ are not always the same function. From a more general point of view, the above results confirm the validity of an idea already hinted at in Jónsson [16, page 370], namely, that relation identities might be useful for classifying varieties. We believe that this is the main contribution of the present work to the study of varieties, in particular, congruence distributive ones.

As a marginal comment, we must confess that our original aim has been the classification of congruence modular varieties. In the case of congruence modular varieties the situation is much more involved, since there are at least three kinds of characterizations, involving three different sets of identities, equivalently, terms, discovered by, respectively, Day [8], Gumm [12] and Tschantz [30]. We naively thought that the classification of congruence distributive varieties—which, of course, are a very special case of congruence modular varieties—was already sufficiently tame; on the other hand, during our investigation of congruence modular varieties [21, 23, 26] we discovered subtle issues which already involve congruence distributive varieties of Jónsson level 3 and 4. Here we deal with level 4; quite unexpected results about varieties of Jónsson level 3 appear in [24, Corollary 2.4] and [27].

1.3. Preliminaries and further details. Baker [1] considered the variety generated by polynomial reducts of lattices in which the only basic operation is $b$ defined by $b(a, c, d) = a(c + d)$, where juxtaposition denotes meet and $+$ denotes join. In a few cases, for clarity, the meet of $a$ and $c$ will be denoted by $a \cdot c$. We shall denote the above variety by $\mathcal{B}$ and we shall call it the Baker’s variety, but let us mention that [1] contains a more general study of varieties which arise as reducts of lattices; see, in particular, [1, Theorem 2]. Notice that, in every algebra in $\mathcal{B}$, the term $x \cdot y = b(x, y, y)$ provides a
semilattice operation; in particular, we can consider any algebra in \( \mathcal{B} \) as an ordered set in a natural way. A related variety is obtained by taking polynomial reducts of lattices in which the only basic operation is \( u \) defined by \( u(a_1, a_2, a_3, a_4) = \prod_{i<j} (a_i + a_j) \), where the indices on the product vary on the set \( \{1, 2, 3, 4\} \). We shall denote this variety by \( \mathcal{N}_4 \). Notice that \( u \) is a near-unanimity term in \( \mathcal{N}_4 \) and that \( b(a, c, d) = u(a, a, c, d) \) provides an interpretation of \( \mathcal{B} \) in \( \mathcal{N}_4 \).

Baker showed that \( \mathcal{B} \) is 4-distributive but not 3-distributive. A variety \( \mathcal{V} \) is \( m \)-distributive, or \( \Delta_m \), if \( \mathcal{V} \) satisfies the congruence identity \( \alpha(\beta \circ \gamma) \subseteq \alpha \beta \circ_m \alpha \gamma \). Thus the smallest \( m \) for which \( \mathcal{V} \) is \( m \)-distributive is the Jónsson level of \( \mathcal{V} \), as introduced in the preceding subsections. We use \( \alpha, \beta, \ldots \) as variables for congruences of some algebra in \( \mathcal{V} \); juxtaposition denotes intersection and we have used the shorthand \( \beta \circ_m \gamma \) for \( \beta \circ \gamma \circ \beta \ldots \) with \( m \) factors, that is, with \( m - 1 \) occurrences of \( \circ \). If, say, \( m \) is odd, we sometimes write \( \beta \circ \gamma \circ \beta \cdots \circ \beta \) in place of \( \beta \circ_m \gamma \) in order to make clear that \( \beta \) is the last factor. Conventionally, \( \beta \circ_0 \gamma = 0 \), the minimal congruence of the algebra under consideration; otherwise the reader might always suppose that \( m \geq 1 \). Notice that an inclusion \( A \subseteq B \) can be equivalently written as \( A = AB \), and hence we are free to use the expression identity in place of inclusion. We refer to Baker [1], Jónsson [16] or Lipparini [24] for other unexplained notions and notation. General textbooks about universal algebra are, among many others, Bergman [3], Burris and Sankappanavar [5], Grätzer [11], Ježek [14] and McKenzie et al. [28].

The original definition of \( m \)-distributivity involves the existence of a certain number of terms introduced by Jónsson [15]; Jónsson terms are exactly the terms arising from the Maltsev condition associated to \( \alpha(\beta \circ \gamma) \subseteq \alpha \beta \circ_m \alpha \gamma \). Here it will be more convenient to express results by means of congruence identities rather than terms. See [24] for a more detailed discussion and further references. Jónsson proved that a variety is congruence distributive if and only if it is \( m \)-distributive, for some \( m \). As already mentioned, it follows from Jónsson’s proof that, for every \( n \) and \( m \), there is some \( k \) such that every \( m \)-distributive variety satisfies the congruence identity \( \alpha(\beta \circ_n \gamma) \subseteq \alpha \beta \circ_k \alpha \gamma \). We initiated the study of the related Jónsson distributivity spectra in [24]. Here we evaluate exactly the distributivity spectra of \( \mathcal{N}_4 \) and of Baker’s variety \( \mathcal{B} \). We also show that we get exactly the same spectra if we consider the corresponding reducts of distributive lattices; call such reducts \( \mathcal{N}^d_4 \) and \( \mathcal{B}^d \).

Relying heavily on Kazda et al. [18], we observed in [24] that congruence distributive varieties satisfy also identities of the form \( \alpha(R \circ_n T) \subseteq \alpha R \circ_k \alpha T \), where \( R \) and \( T \) denote reflexive and admissible relations. In Sections 3 and 4 we find the best bounds for relation identities of this kind in \( \mathcal{B} \) and \( \mathcal{N}_4 \); moreover, we show that in the case of \( \mathcal{B} \) and \( \mathcal{N}_4 \) it is possible to take \( \alpha \), too, as a reflexive and admissible relation. As far as relation identities are concerned, \( \mathcal{B} \) and \( \mathcal{N}_4 \) exhibit a subtly different behavior. As mentioned, this confirms the suggestion implicit in [16, page 370] and explicitly advanced in [25] that the study of relation identities provides a finer classification of varieties (in particular, congruence distributive varieties), in comparison with the study of congruence identities alone.
1.4. Solutions to two problems. The relation identities found in Sections 3 and 4 solve also some earlier problems. In [20] we have showed that, under a fairly general assumption, a congruence identity is equivalent to the same identity when considered for representable tolerances, instead. In Remark 3.4 we show that the assumption of representability of tolerances is necessary in the above equivalence.

It is known [7, 21, 26] that the identities $\alpha(\Theta \circ \Theta) \subseteq \alpha \Theta \circ_k \alpha \Theta$ and $\alpha(R \circ R) \subseteq \alpha R \circ_k \alpha R$, for some $k, k'$, both characterize congruence modularity, where $\Theta$ denotes a tolerance. Remarks 4.4 and 4.5 below show that, for a variety $\mathcal{V}$, the best values of $k$ or $k'$ in the above identities are not determined by the Day modularity level of $\mathcal{V}$. It is an open problem to find the example of a variety for which the best values for $k$ and $k'$ above are distinct.

1.5. Further results. Section 5 contains a few remarks about relation identities satisfied by varieties with a near-unanimity term and by varieties with an edge term. Here we are dealing with the general case, not with specific examples such as $\mathcal{N}_4$. Further remarks are contained in Section 6. Among other, following the lines of [1], we consider identities satisfied by arbitrary polynomial reducts of lattices. We also consider polynomial reducts of Boolean algebras.

2. The distributivity spectra of $\mathcal{B}$ and $\mathcal{N}_4$

Recall that $\mathcal{B}$ is the variety generated by polynomial reducts of lattices in which the only basic operation is $b(a, c, d) = a(c + d)$ and that $\mathcal{N}_4$ is defined similarly with respect to the operation $u(a_1, a_2, a_3, a_4) = \prod_{i<j}(a_i + a_j)$. Here we maintain the terminology from [1], though probably the expression ‘term-reduct’ is nowadays more frequent. The varieties obtained by considering only reducts of distributive lattices are denoted, correspondingly, by $\mathcal{B}_d$ and $\mathcal{N}_4^d$.

**Theorem 2.1.** Suppose that $n \geq 2$ and $\mathcal{V}$ is either $\mathcal{B}$, $\mathcal{B}_d$, $\mathcal{N}_4$ or $\mathcal{N}_4^d$. Then $\mathcal{V}$ satisfies the following congruence identities:

\begin{align*}
\alpha(\beta \circ_n \gamma) & \subseteq \alpha \beta \circ_{2n} \alpha \gamma \quad \text{for } n \text{ even, and} \quad (2-1) \\
\alpha(\beta \circ_n \gamma) & \subseteq \alpha \beta \circ_{2n-1} \alpha \gamma \quad \text{for } n \text{ odd,} \quad (2-2)
\end{align*}

and the subscripts on the right-hand sides are best possible; actually, $\mathcal{V}$ does not even satisfy

\begin{align*}
\alpha(\beta \circ (\alpha \gamma \circ \alpha \beta \circ_{n-2} \alpha \beta) \circ \gamma) & \subseteq \alpha \beta \circ_{2n-1} \alpha \gamma \quad \text{for } n \text{ even, and} \quad (2-3) \\
\alpha(\beta \circ (\alpha \gamma \circ \alpha \beta \circ_{n-2} \alpha \gamma) \circ \beta) & \subseteq \alpha \beta \circ_{2n-2} \alpha \gamma \quad \text{for } n \text{ odd.} \quad (2-4)
\end{align*}

**Proof.** The positive result that Equations (2-1) and (2-2) hold in $\mathcal{B}$ is an observation in [24, Section 3], but inserted there in a quite abstract and general context. In the special case of $\mathcal{B}$ the proof is direct and is an almost immediate generalization of Baker’s argument. Indeed, if $n$ is even and $(a, d) \in \alpha(\beta \circ_n \gamma)$, then $a \not\equiv d$ and there are
elements $c_i$ such that $a = c_0 \beta c_1 \gamma c_2 \beta \cdots c_{n-1} \gamma c_n = d$. Then the elements

$$
\begin{align*}
  a &= b(a, c_0, d), \quad b(a, c_1, d), \quad b(a, c_2, d), \quad \ldots, \quad b(a, c_{n-1}, d), \\
  b(a, c_n, d) &= b(d, a, a) = b(d, c_0, a), \\
  b(d, c_1, a) &= b(d, c_2, a), \quad \ldots, \quad b(d, c_{n-1}, a), \\
  b(d, c_n, a) &= b(d, d, d) = d
\end{align*}
$$

(2-5)

witness $(a, d) \in \alpha \beta \circ_{2n} \alpha \gamma$. Notice that, say, $b(a, c_j, d) \alpha b(a, c_j, a) = a = b(a, c_{j+1}, a) \alpha b(a, c_{j+1}, d)$, for every $j$. The same chain of elements works in the case $n$ odd, but in this case $c_{n-1} \beta c_n = d$, and hence $b(a, c_{n-1}, d) \alpha \beta b(a, d, d) = a \cdot d = b(d, a, a) \alpha \beta b(d, c_1, a)$ and, in particular, $b(a, c_{n-1}, d) \alpha \beta b(d, c_1, a)$; thus one passage might be skipped and we get $(a, d) \in \alpha \beta \circ_{2n-1} \alpha \gamma$. Since $B$ is interpretable in $N_4$, then (2-1) and (2-2) hold in $N_4$, too. The result for $N_4$ can be obtained also directly from the case $m = 2$ of Equations (5-1) and (5-2) in Proposition 5.1 below. Clearly, if some congruence identity holds in $B$, respectively, $N_4$, then it holds in $B^d$, respectively, $N_4^d$.

Now we show that Equations (2-3) and (2-4) fail, and hence the bounds in (2-1) and (2-2) are optimal. We present the argument for $N_4$ and $N_4^d$. This is enough, since, say, $B$ is interpretable in $N_4$. In any case, the same argument works for $B$ and $B^d$, too, with no essential modification.

For $h \geq 1$, let $C_{h+1}$ denote the $h + 1$-elements chain with underlying set $C_{h+1} = \{0, 1, \ldots, h\}$ and with the standard ordering, inducing the standard lattice operations of min and max. Let $L$ be the lattice $C_{n+1} \times C_{n+1} \times C_2$. Since in what follows the ‘last’ $C_2$ will play a different role with respect to the first two $C_{n+1}$, we usually denote the larger element of $C_2$ by $\uparrow$ and the smaller element of $C_2$ by $\downarrow$. Consider the following elements of $L$:

$$
\begin{align*}
  a &= c_0 = (n, 0, \uparrow), \\
  d &= c_n = (0, n, \uparrow), \quad \text{and} \\
  c_i &= (n-i, i, \downarrow), \quad \text{for } i = 1, \ldots, n-1.
\end{align*}
$$

Recall that we have set $u(x_1, x_2, x_3, x_4) = \prod_{j<k}(x_j + x_k)$ and $b(x_1, x_3, x_4) = u(x_1, x_1, x_3, x_4) = x_1(x_3 + x_4)$. Let

$$
B = \{a \in L \mid a \leq c_i, \text{ for some } i \leq n\}.
$$

We show that $B$ is closed under $u$, and hence $B = (B, u)$ is an algebra in $N_4$, actually, in $N_4^d$, since $L$ is a distributive lattice. Indeed, suppose that $a_1 \leq c_{i_1}, \ldots, a_4 \leq c_{i_4}$. Since $u$ is invariant under any permutation of its arguments, it is no loss of generality to assume that $i_1 \leq i_2 \leq i_3 \leq i_4$. If $u(c_{i_1}, c_{i_2}, c_{i_3}, c_{i_4})$ has a third $\uparrow$ component, then at least three among $c_{i_1}, c_{i_2}, c_{i_3}, c_{i_4}$ have a third $\uparrow$ component, and hence at least two among $c_{i_1}, c_{i_2}, c_{i_3}, c_{i_4}$ are either $c_0$ or $c_n$. Say, $c_{i_1} = c_{i_2} = c_0$, and hence, since $u$ is a monotone operation, $u(a_1, a_2, a_3, a_4) \leq u(c_0, c_0, c_{i_3}, c_{i_4}) = c_0(c_{i_3} + c_{i_4}) \leq c_0$. Otherwise $u(c_{i_1}, c_{i_2}, c_{i_3}, c_{i_4})$ has a third $\downarrow$ component and $u(a_1, a_2, a_3, a_4) \leq u(c_{i_1}, c_{i_2}, c_{i_3}, c_{i_4}) = (n-i_3, i_2, \downarrow) = c_{i_2}c_{i_3} \leq c_{i_4}$, since we have assumed $i_1 \leq i_2 \leq i_3 \leq i_4$. In any case, $u(a_1, a_2, a_3, a_4) \in B$.

Hence $B = (B, u)$ is an algebra in $N_4$; in particular, $(B, b)$ is an algebra in $B$. Let $B^\uparrow$ denote the set of all the elements of $B$ with a last $\uparrow$. By the definition of $B$, the elements
Similarly, \(i\) appropriate values of \(i\) turn out to be kernels of appropriate projections. See \(\beta\) the distributivity spectrum of Baker’s variety \(\alpha\) is a congruence on \(B\).

Let \(\beta\) be the congruence on \(B\) defined in such a way that two elements \((i_1, i_2, i_3)\) and \((j_1, j_2, j_3)\) of \(B\) are \(\beta\)-related if and only if, for every \(\ell = 1, 2\), their components \(i_\ell\) and \(j_\ell\) differ at most by 1, and:

\[
\begin{align*}
(a) & \quad \text{if } i_1 \neq j_1, \text{ then max } \{i_1, j_1\} \text{ has the same parity as } n; \text{ and} \\
(b) & \quad \text{if } i_2 \neq j_2, \text{ then max } \{i_2, j_2\} \text{ is odd.}
\end{align*}
\]

It can be checked directly that \(\beta\) is a congruence; otherwise, argue as follows. Let \(\beta'_2\) be the congruence on \(C_{n+1}\) whose blocks are \(\{n, n-1\}, \{n-2, n-3\}, \ldots\). If \(\beta''_1\) is the counterimage in \(L\) of \(\beta'_1\) through the first projection, then \(\beta''_1\) is a congruence on \(L\), and hence a congruence on \((L, u)\). Thus the restriction \(\beta_1\) of \(\beta''_1\) to \(B\) is a congruence on \((B, u)\). Similarly, define \(\beta_2\) using the counterimage through the second projection of the congruence on \(C_{n+1}\) whose blocks are \(\{0, 1\}, \{2, 3\}, \ldots\). Then \(\beta = \beta_1 \beta_2\), and hence \(\beta\) is a congruence, being the meet of two congruences.

The congruence \(\gamma\) on \(B\) is defined in a similar way; two elements \((i_1, i_2, i_3)\) and \((j_1, j_2, j_3)\) of \(B\) are \(\gamma\)-related if and only if, for every \(\ell = 1, 2\), their components \(i_\ell\) and \(j_\ell\) differ at most by 1, and:

\[
\begin{align*}
(a) & \quad \text{if } i_1 \neq j_1, \text{ then max } \{i_1, j_1\} \text{ has not the same parity as } n; \text{ and} \\
(b) & \quad \text{if } i_2 \neq j_2, \text{ then max } \{i_2, j_2\} \text{ is even.}
\end{align*}
\]

In passing, let us mention that there is an alternative construction of \(B\) which makes the definitions of \(\beta\) and \(\gamma\) simpler; actually, in this alternative construction \(\beta\) and \(\gamma\) turn out to be kernels of appropriate projections. See [22]. However, all the remaining arguments are much more involved in [22]; moreover, the current presentation has the advantage of being more compact.

With the above definitions of \(\alpha\), \(\beta\) and \(\gamma\), we have \(c_0 \alpha c_n\) and \(c_j \alpha c_{j+1}\), for \(j = 1, \ldots, n-2\). Moreover, \(c_{2i} = (n-2i, 2i, -) \beta (n-2i-1, 2i+1, -) = c_{2i+1}\), for all the appropriate values of \(i\) and where the value of the third component is not relevant. Similarly, \(c_{2i+1} \gamma c_{2i+2}\), for every appropriate \(i\). So \((c_0, c_n) \in \alpha (\beta \circ (\alpha \gamma \circ n-2 \alpha \beta) \circ \gamma), \quad \text{for } n \text{ even, and } (c_0, c_n) \in \alpha (\beta \circ (\alpha \gamma \circ n-2 \alpha \beta) \circ \gamma), \quad \text{for } n \text{ odd.}

On the other hand, in view of the above description of \(B^\dagger\), the only elements \(\alpha \beta\)-connected to \(c_0 = e_0 = (n, 0, \uparrow)\) are \(c_0\) itself and \(e_1 = (n-1, 1, \uparrow)\). No other element of \(B^\dagger\) is \(\alpha \gamma\)-connected to \(c_0\), and hence there is no advantage in ‘staying at \(c_0\).’ The only other element \(\alpha \gamma\)-connected to \(e_1 = (n-1, 1, \uparrow)\) is \(e_2 = (n-2, 0, \uparrow)\) and, so on, the only other element \(\alpha \beta\)-connected to \(e_{2i}\) is \(e_{2i+1}\) and the only other element \(\alpha \gamma\)-connected to \(e_{2i+1}\) is \(e_{2i+2}\), until we reach \(e_{n-1}\), where the situation splits into two cases.
If \( n \) is even, then \((1, 0, \uparrow) = e_{n-1} \) \( \alpha \gamma e_n = (0, 0, \uparrow) = f_0 \) \( \alpha \beta f_1 = (0, 1, \uparrow) \) and no other nontrivial \( \alpha \beta \)- or \( \alpha \gamma \)-relation holds among these elements. Symmetrical considerations hold for the \( f_i \) and, since \( f_n = c_n \), we get that any chain from \( c_0 \) to \( c_n \) in which each pair of elements is either \( \alpha \beta \) or \( \alpha \gamma \)-connected must involve all the \( 2n + 1 \) elements of \( B^\uparrow \), hence any chain as above is of length at least \( 2n \), and thus (2-3) fails in \( B \).

On the other hand, if \( n \) is odd, then \( e_{n-1} = (1, 0, \uparrow) \) \( \alpha \beta (0, 1, \uparrow) = f_1 \); thus \( e_{n-1} \) \( \alpha \beta f_1 \) and we do not need all the elements of \( B^\uparrow \) to get an \( \alpha \beta \)-or- \( \alpha \gamma \)-chain, and we can skip \( e_n = f_0 \). However, all the rest is the same and we need \( 2n - 1 \) steps from \( c_0 \) to \( c_n \), and hence (2-4) fails.

The case \( n = 2 \) for \( B \) in Theorem 2.1 gives a proof of Baker’s result that \( B \) is 4-distributive but not 3-distributive. Our proof of 4-distributivity is the same as Baker’s; the counterexample to 3-distributivity is only slightly different.

**Remarks 2.2.** (a) There is a short and simple syntactical folklore proof that Baker’s variety is not 2-distributive, that is, that \( B \) has no majority term. Actually, the proof shows that \( B \) has no near-unanimity term. If \( t \) is a term of \( B \), define the relevant variable of \( t \) inductively as follows. If \( t \) is a variable \( x_j \), then \( x_j \) is the relevant variable of \( t \). Otherwise, \( t = b(t_1, t_2, t_3) \) and we define the relevant variable of \( t \) to be the relevant variable of \( t_1 \). If \( B \in B \), \( B \) has a minimal element 0 and we substitute 0 for the relevant variable of some term \( t \), then \( t \) is evaluated as 0, no matter what we substitute for the other variables. Thus \( B \) has no near-unanimity term, in particular, no majority term.

More generally, the argument shows that, for every \( k \)-ary term \( t \), there is some ‘place’ \( i \leq k \) such that \( B \) satisfies no equation of the form \( t(\ldots, y, \ldots) = x \), where \( y \) is put in place \( i \), \( x \) is a variable distinct from \( y \) and the other arguments of \( t \) are arbitrary variables. This shows that \( B \) has no cube term, as introduced by Berman et al. [4].

(b) Using a different method, Mitschke [29] proved that the variety \( I \) of implication algebras has no near-unanimity term. Since Baker’s variety \( B \) is interpretable in \( I \), then Mitschke’s result furnishes another proof that \( B \) has no near-unanimity term. The method in (a) can be applied also to \( I \), providing a shorter proof of the mentioned result by Mitschke. Simply argue as in (a) above, defining the relevant variable of \( t_1 \rightarrow t_2 \) to be the relevant variable of \( t_2 \) and dealing with some maximal element 1 rather than with 0. Thus we also get that \( I \) has no cube term. Essentially, this is the argument hinted at on [4, page 1470]. In particular, 3-distributive 3-permutable varieties do not necessarily have a cube term.

(c) The argument in (a) can be extended in order to give still another proof that Baker’s variety is not 3-distributive. Indeed, 3-distributivity is equivalent to the existence of ternary terms \( j_1 \) and \( j_2 \) satisfying \( x = j_1(x, x, y) = j_1(x, y, x) \), \( j_1(x, y, y) = j_2(x, y, y) \) and \( j_2(x, x, y) = j_2(y, x, y) = y \) [15]. With the same assumptions and definitions as in (a) above, the first equations imply that the relevant variable of \( j_1(x, y, z) \) is \( x \), and hence \( 0 = j_1(0, b, b) = j_2(0, b, b) \), for every \( b \in B \). Under the order induced by the semilattice operation, we have that every term operation is monotone (this applies to \( B \) but not to the variety of implication algebras!), and hence \( 0 = j_2(0, b, b) \geq \ldots \)
Suppose that \( n = 2 \), which is impossible if \( B \) is taken to be of cardinality \( \geq 2 \). Thus \( B \) is not 3-distributive.

In fact, in the above argument we have not used the equation \( j_2(y, x, y) = y \). This shows that \( B \) does not even satisfy \( \alpha(y \circ \beta) \subseteq \alpha y \circ \beta \circ \gamma \), equivalently, taking converses, \( B \) does not satisfy \( \alpha(\beta \circ \gamma) \subseteq \gamma \circ \beta \circ \alpha \). This negative result will be improved in the following proposition. Compare Equation (2-8) below.

In the terminology from [24], Theorem 2.1 implies that \( J_2(n - 1) = 2n - 1 \), for \( n \) even and that \( J_2(n - 1) = 2n - 2 \), for \( n \) odd. In [24] we have also considered ‘reversed’ Jónsson spectra, given by identities like \( \alpha(\beta \circ_n \gamma) \subseteq \alpha y \circ_k \alpha \beta \). We are going to see that the proof of Theorem 2.1 gives exact bounds for identities of the above kind both in \( B \) and \( J_0 \), as well as in their distributive counterparts.

Moreover, it follows from results by Tschandt [30] that, for every congruence modular variety \( V \) and every \( n \), there is some \( k \) such that \( V \) satisfies \( \alpha(\beta \circ_n \gamma) \subseteq \alpha(y \circ \beta) \circ (\alpha \circ_{k-2} \alpha \beta) \). See, for example, [24, Section 4] for details. Of course, in a congruence distributive variety we already know that \( \alpha(\beta \circ_n \gamma) \subseteq \alpha \gamma \circ_k \alpha \beta \), for some \( k' \). However, in principle, it might happen that Tschandt-like formulae provide a value of \( k \) much smaller than \( k' \). This is not the case for \( B \) and \( J_0 \).

**Proposition 2.3.** Suppose that \( n \geq 2 \) and \( V \) is either \( B \), \( B^d \), \( N_4 \) or \( N^d_4 \). Then \( V \) satisfies the following congruence identities:

\[
\alpha(\beta \circ_n \gamma) \subseteq \alpha y \circ_{2n+1} \alpha \beta \quad \text{for } n \text{ even,} \tag{2-6}
\]

\[
\alpha(\beta \circ_n \gamma) \subseteq \alpha y \circ_{2n} \alpha \beta \quad \text{for } n \text{ odd,} \tag{2-7}
\]

\[
\alpha(\beta \circ_n \gamma) \subseteq \alpha(y \circ \beta) \circ (\alpha \circ_{2n-1} \alpha \beta) \quad \text{for } n \text{ even,} \tag{2-8}
\]

\[
\alpha(\beta \circ_n \gamma) \subseteq \alpha(y \circ \beta) \circ (\alpha \circ_{2n-2} \alpha \beta) \quad \text{for } n \text{ odd,} \tag{2-9}
\]

and the values of the indices on the right-hand sides give the best possible bounds. Actually, \( V \) fails to satisfy

\[
\alpha(\beta \circ (\alpha \circ_{n-2} \alpha \beta) \circ \gamma) \subseteq \alpha(y \circ \beta) \circ (\alpha \circ_{2n-4} \alpha \beta) \circ \alpha(y \circ \beta) \quad \text{for } n \text{ even,} \tag{2-10}
\]

\[
\alpha(\beta \circ (\alpha \circ_{n-2} \alpha \beta) \circ \gamma) \subseteq \alpha(y \circ \beta) \circ (\alpha \circ_{2n-5} \alpha \beta) \circ \alpha(\beta \circ \gamma) \quad \text{for } n \text{ odd.}
\]

**Proof.** Equations (2-6)–(2-9) are immediate from (2-1) and (2-2), since, say, \( \alpha \beta \circ_{2n} \alpha y \circ_{n+1} \alpha \beta \) and \( \alpha \beta \subseteq \alpha(y \circ \beta) \).

The proof of Theorem 2.1 shows that the bounds on the right-hand sides of (2-6) and (2-7) are optimal. Indeed, in the proof that (2-3) and (2-4) fail we have observed that \( c_0 \) is \( \alpha y \)-connected to no other element of \( B^1 \), and hence we ‘lose one turn’ if we want the chain to start with \( \alpha y \). Actually, we have that, say, for \( n \) even, already \( \alpha(\beta \circ (\alpha \circ_{n-2} \alpha \beta) \circ \gamma) \subseteq \alpha y \circ_{2n} \alpha \beta \) fails in \( V \).

In order to show that the indices in (2-8) and (2-9) are the best possible, we shall modify the construction in the proof of Theorem 2.1. With the definitions and notation in the mentioned proof, let \( B^- = B \setminus \{(n, 0, \downarrow), (0, n, \downarrow)\} \). We claim that \( B^- \) is (the base
set for) an algebra in, say, \( N_4 \). We shall show that if \( a_1, a_2, a_3, a_4 \in B^- \), then it is not the case that \( u(a_1, a_2, a_3, a_4) = (n, 0, \bot) \). Indeed, since \( c_0 \) is the only element of \( B^- \) with first component \( n \), if the first component of \( u(a_1, a_2, a_3, a_4) \) is \( n \), then at least three arguments of \( u \) have \( n \) as the first component, hence at least three arguments of \( u \) are equal to \( c_0 \), and thus \( u(a_1, a_2, a_3, a_4) \) is itself \( c_0 \). Notice that, by construction, \( n \) is the maximum possible value for the first component. Similarly, if \( a_1, a_2, a_3, a_4 \in B^- \), then \( u(a_1, a_2, a_3, a_4) \) is not \((0, n, \bot)\), and hence \( B^- \) is an algebra in \( N_4 \).

We have that \( c_0 \) is \( \gamma \)-connected to no other element of \( B^- \), because of clause (a_\( \gamma \)) in the definition of \( \gamma \). Thus if \( c_0 \alpha(\gamma \circ \beta) f \) in \( B^- \), for some \( f \), then \( c_0 \gamma e \beta f \), for some \( e \), and hence necessarily \( c_0 = e \) and \( c_0 \alpha \beta f \). Thus if, say, \( n \) is even and we suppose by contradiction that \( (c_0, c_n) \in \alpha(\gamma \circ \beta) \circ (\alpha \gamma \circ \alpha \beta \circ \alpha \gamma) \), then we would have \( (c_0, c_n) \in \alpha \beta \circ \alpha \gamma \). Thus if, say, \( n \) is even and we suppose by contradiction that \( (c_0, c_n) \in \alpha(\gamma \circ \beta) \circ (\alpha \gamma \circ \alpha \beta \circ \alpha \gamma) \), then we would have \( (c_0, c_n) \in \alpha \beta \circ \alpha \gamma \). But this is impossible because of the example constructed in the proof of Theorem 2.1. Hence we cannot get better bounds in (2-8) or (2-9). Performing also the symmetric argument, we have that \( \mathcal{V} \) fails to satisfy the last equations in the statement. □

Recall that a variety \( \mathcal{V} \) is \( n \)-modular if \( \mathcal{V} \) satisfies the identity \( \alpha(\beta \circ \alpha \gamma \circ \beta) \subseteq \alpha \beta \circ_n \alpha \gamma \). See Day [8]. Equations (2-2) and (2-4) in Theorem 2.1 in the case \( n = 3 \) provide the following corollary.

**Corollary 2.4.** The varieties \( \mathcal{B}, \mathcal{B}^d, N_4 \) and \( N_4^d \) are 4-distributive, 5-modular and not 4-modular.

### 3. Some relation identities

We shall use \( R, S \) and \( T \) as variables for reflexive and admissible binary relations and \( \Theta \) as a variable for tolerances. All the relations considered in the present note are assumed to be reflexive. In [24, Proposition 3.1] we noticed that congruence distributive varieties satisfy also relation identities of the form \( \Theta(R \circ_n S) \subseteq \Theta R \circ_k \Theta S \). This is a consequence of results by Kazda et al. [18]. See [24] for further details and references.

We do not know whether every congruence distributive variety satisfies \( T(R \circ S) \subseteq TR \circ_k TS \), for some \( k \). However, we showed in [25] that the above relation holds in Baker’s variety with \( k = 4 \). In the present section we provide exact bounds for identities of the above kind, both in the case of \( \mathcal{B} \) and in the case of \( N_4 \). We exhibit a subtle difference between the two varieties, when relation identities are concerned. Compare Theorems 3.1, 4.1 and Proposition 3.2 below. On the other hand, as we showed in the previous section, \( \mathcal{B} \) and \( N_4 \) behave in the same way as far as many congruence identities are concerned.

Each result in the present and in the following section holds for \( \mathcal{B} \) if and only if it holds for \( \mathcal{B}^d \). In fact, we never use lattice distributivity in the proofs; on the other hand, all the counterexamples we deal with are based on the construction in the proof of Theorem 2.1, and this construction is the polynomial reduct of a distributive lattice. Similarly, each result holds for \( N_4 \) if and only if it holds for \( N_4^d \). For the sake
of simplicity, we do not mention $B^d$ and $N_4^d$ explicitly in the following statements. However, by the above comments, the reader might always consider $B^d$ in place of $B$ and $N_4^d$ in place of $N_4$ in what follows.

Recall that, if $n$ is odd, we sometimes write $R \circ_n S \circ \cdots \circ R$ in place of $R \circ_n S$, when we want to make clear that the last factor is $R$.

**Theorem 3.1.** If $n \geq 2$, then the following identities are satisfied:

\[ T(R \circ_n S) \subseteq TR \circ_{2n} TS \quad \text{by} \quad B, N_4, n \text{ even}; \tag{3-1} \]

\[ T(R \circ_n S) \subseteq (TR \circ TS \circ \cdots \circ TR) \circ (TR \circ TS \circ \cdots \circ TR) \quad \text{by} \quad B, n \text{ odd}; \tag{3-2} \]

\[ T(R \circ_n S) \subseteq TR \circ_{2n-1} TS \quad \text{by} \quad N_4, n \text{ odd}, \tag{3-3} \]

and the values of the indices on the right are best possible. Moreover, $B$ fails to satisfy

\[ \alpha(\Theta \circ_n \gamma) \subseteq \alpha\Theta \circ_{2n} \alpha\gamma, \tag{3-4} \]

for $n$ odd, where $T, R, S$ are reflexive and admissible relations, $\Theta$ is a tolerance and $\alpha$ and $\gamma$ are congruences.

Before proving Theorem 3.1 we prove Proposition 3.2 below, a more general result on the positive side. However, the formulation of 3.2 is more involved. The statement of Theorem 3.1 suggests that, for $n$ even, $B$ and $N_4$ behave in the same way and no essential difference seems to appear in comparison with the previous section. On the other hand, for $n$ odd, Theorem 3.1 shows that $N_4$ satisfies the somewhat stronger identity (3-3). However, in a sense, $N_4$ behaves better than $B$ for every value of the index $n$, as we show in Proposition 3.2. The difference between $B$ and $N_4$ appears in a clearer light in Theorem 4.1 in the next section.

After the proof of Proposition 3.2, furnishing the positive side of Theorem 3.1, we present in 3.3 the example of a tolerance on $(B, b)$, the algebra constructed in the proof of Theorem 2.1. The example is then used in order to show that the bounds in identity (3-2) are optimal.

For a binary relation $R$, let $R^n$ denote the $n$-fold composition of $R$ with itself, that is, $R^n = R \circ_n R$. If $R$ and $S$ are binary relations, let $R \cup S$ denote the smallest reflexive and admissible relation containing both $R$ and $S$.

**Proposition 3.2.** For every $n \geq 2$, the following identities are satisfied:

by $B$:

\[ T(R_1 \circ R_2 \circ \cdots \circ R_n) \subseteq (TR_1 \circ TR_2 \circ \cdots \circ TR_n)^2; \tag{3-5} \]

by $N_4$:

\[ T(R_1 \circ R_2 \circ \cdots \circ R_{n-1} \circ R_n) \subseteq TR_1 \circ TR_2 \circ \cdots \circ TR_{n-1} \circ TR_n; \tag{3-6} \]

**Proof.** Were $T$ a congruence, the proof of (2-1) in Theorem 2.1 would give a proof of (3-5), since in the proof of Theorem 2.1 we have not used the assumption that $\beta$ and $\gamma$ are congruences, we have only used that $\beta$ and $\gamma$ are reflexive and admissible relations.
Dealing with many relations instead of just two relations presents no new difficulty. At certain places in Theorem 2.1 we need transitivity of $\beta$, but this is not necessary here, according to the formulations of Theorem 3.1 and Proposition 3.2. For example, were both $T$ and $R$ transitive in Equation (3-2), then the two adjacent occurrences of $TR$ would absorb into one. This is the reason why the indices in Theorem 2.1 can be improved by 1 in the case $n$ odd, since $\alpha$ and $\beta$ there are congruences, and hence transitive. But we are not assuming transitivity in Theorem 3.1 and in Proposition 3.2; correspondingly, we are not asking for the stronger conclusions.

We have to give a proof of Proposition 3.2 in the case when $T$ is just a reflexive and admissible relation. This involves considering a different sequence of elements in comparison with the sequence described in (2-5). We first give the proof of (3-5) for $B$; then we improve (3-5) to (3-6) for $N_4$ by an additional and somewhat delicate argument.

Suppose that $(a, d) \in T(R_1 \circ R_2 \circ \cdots \circ R_n)$ in some algebra in $B$. This means that $a T d$ and that there are elements $c_j$ such that $a = c_0 R_1 c_1 R_2 c_2 \cdots R_{n-1} c_{n-1} R_n c_n = d$. In what follows, we write, say, $a(d + c_1)$ for $b(a, d, c_1)$, and $a(d + c_1)(d + c_2)$ for $b(b(a, d, c_1), d, c_2)$. Equations like $a(d + c_1)(d + c_1) = a(d + c_1)$ or $a(d + c_1)(a + c_2) = a(d + c_1)$ hold in $B$ since corresponding equations hold in lattices. Consider the following elements:

\[
g_0 = a, \quad g_1 = a(d + c_1), \quad g_2 = a(d + c_1)(d + c_2), \quad \cdots, \quad g_n = h_0 = ad, \quad g_{n-1} = a(d + c_1)(d + c_2) \cdots (d + c_{n-1}),
\]

\[
h_1 = d(a + c_1)(a + c_2) \cdots (a + c_{n-1}), \quad h_2 = d(a + c_2) \cdots (a + c_{n-1}), \quad \cdots, \quad h_{n-1} = d(a + c_{n-1}), \quad h_n = d.
\]

We have $g_i, TR_{i+1} g_{i+1}$ for $i < n$ and similarly for the $h_i$. Indeed, for example,

\[
g_1 = a(d + c_1) = a(d + c_1)(a + c_2) T a(d + c_1)(d + c_2) = g_2, \quad \text{and}
\]

\[
g_1 = a(d + c_1) = a(d + c_1)(d + c_1) R_2 a(d + c_1)(d + c_2) = g_2.
\]

Notice that, in the definition of the $g_i$, when going from $g_{n-1}$ to $g_n$ we follow the preceding pattern; indeed, according to the pattern, $g_n$ would be

\[
a(d + c_1) \cdots (d + c_{n-1})(d + c_n) = a(d + c_1) \cdots (d + c_{n-1})(d + d)
\]

which in fact is equal to $ad$. Thus we have $(a, g_n) \in TR_1 \circ TR_2 \circ \cdots \circ TR_n$ and $(h_0, d) \in TR_1 \circ TR_2 \circ \cdots \circ TR_n$, and hence $(a, d) \in (TR_1 \circ TR_2 \circ \cdots \circ TR_n)^2$, since $g_n = h_0$.

We have proved (3-5).

In passing, let us remark that the above elements $g_0, g_1, \ldots, g_n = h_0, \ldots, h_{n-1}, h_n$ satisfy some additional relation identities. We have that each element in the above chain is $T$-related with every element which follows. For example, $g_{n-1} T h_1$, since

\[
g_{n-1} = a(d + c_1) \cdots (d + c_{n-1}) = a(d + c_1) \cdots (d + c_{n-1})(a + c_1) \cdots (a + c_{n-1}) T
\]

\[
d(d + c_1) \cdots (d + c_{n-1})(a + c_1) \cdots (a + c_{n-1}) = d(a + c_1) \cdots (a + c_{n-1}) = h_1.
\]

All the other relations are proved in a similar way.
The proof of Equation (3-6) involves the same chain of elements, this time working in \( N_4 \). Notice that the above elements are expressible in \( N_4 \), since \( b(x_1, x_3, x_4) = u(x_1, x_1, x_3, x_4) = x_1(x_3 + x_4) \). In the above-displayed formula we have shown \( g_{n-1} T h_1 \). It remains to show that, in addition, \( g_{n-1} \overline{R_n} \cup \overline{R_1} h_1 \). In order to prove this relation, we write \( g_{n-1} = a(d + c_1) \cdots (d + c_{n-2})(d + c_{n-1}) \) as \( u(a, a, d, c_{n-1}) \cdot (d + c_1) \cdots (d + c_{n-2}) \). This formula should be interpreted in the sense that, say, \( u(a, a, d, c_{n-1}) \cdot (d + c_1) \cdots (d + c_{n-2}) \). Thus \( g_{n-1} \overline{R_n} \cup \overline{R_1} h_1 \), and the proof of Proposition 3.2 is complete. □

Recall the definitions of \( B, B^\dagger, \alpha, \beta, \gamma \) from the proof of Theorem 2.1. Let \( B^\dagger = B \setminus B^\dagger \) be the set of those elements of \( B \) with a third \( \downarrow \) component. Let \( E = \{ e_0, \ldots, e_n \} \) and \( F = \{ f_0, \ldots, f_n \} \), where \( e_0, \ldots, f_n \) are the elements in the displayed list (\( B^\dagger \)) in the proof of Theorem 2.1. We now present the example of a tolerance on \( (B, b) \). Recall that \( (B, b) \) is an algebra in \( B \).

**Example 3.3.** Let \( \Lambda \) be the binary relation on \( B \) defined as follows. Two elements \( x \) and \( y \) of \( B \) are \( \Lambda \)-related if and only if either:

(a) both \( x \) and \( y \) belong to \( E \); or
(b) both \( x \) and \( y \) belong to \( F \); or
(c) at least one of \( x \) or \( y \) belongs to \( B^\dagger \).

We show that \( \Lambda \) is a tolerance on \( (B, b) \).

Indeed, \( \Lambda \) is clearly symmetric and reflexive, since \( B^\dagger = E \cup F \), and hence \( B^\dagger = B \setminus (E \cup F) \). We have to check that \( \Lambda \) is admissible. Suppose that \( x_1 \Lambda y_1, x_2 \Lambda y_2 \) and \( x_3 \Lambda y_3 \), for certain elements \( x_1, y_1, \ldots \) of \( B \). Letting \( x = b(x_1, x_2, x_3) \) and \( y = b(y_1, y_2, y_3) \), we have to show that \( x \Lambda y \). If either \( x \in B^\dagger \) or \( y \in B^\dagger \), there is nothing to prove. If both \( x = b(x_1, x_2, x_3) \) and \( y = b(y_1, y_2, y_3) \) belong to \( B^\dagger \), that is, they have a last \( \uparrow \) component, then both \( x_1 \) and \( y_1 \) have a last \( \uparrow \) component, that is, \( x_1, y_1 \in B^\dagger \). By the definition of \( \Lambda \), either \( x_1, y_1 \in E \) or \( x_1, y_1 \in F \), and, correspondingly, \( x_1 \) and \( y_1 \) either both have a null second component or have a null first component; hence this applies to \( x \) and \( y \), as well. By the description of \( B^\dagger \) in the proof of Theorem 2.1, if \( x \) and \( y \) have a null second component, then they both belong to \( E \), and if \( x \) and \( y \) have a
null first component, then they both belong to $F$. We have shown that $x \Lambda y$, and thus $\Lambda$ is admissible (we had no need to use the assumption that $x_2 \Lambda y_2$ and $x_3 \Lambda y_3$).

On the other hand, $\Lambda$ is not admissible on $(B, u)$; see Remark 4.2.

We now give the proof of Theorem 3.1.

\textbf{Proof.} The positive result that Equations (3-1)–(3-3) hold is an immediate consequence of Proposition 3.2, taking $R_1 = R_3 = \cdots = R$ and $R_2 = R_4 = \cdots = \emptyset$. Notice that if $n$ is odd, then $R_n = R_1 = R$, hence the factor $T(TR_n \circ TR_1)(R_n \cup R_1)$ in (3-6) becomes $TR$, and thus we get (3-3) from (3-6).

The bounds given by Equations (3-1) and (3-3) are optimal even in the case of congruences, as shown by Equations (2-1) and (2-2) in Theorem 2.1. As soon as we show that (3-4) fails, we get that (3-2) is the best possible result; in particular, the two adjacent occurrences of $TR$ in the middle of the right-hand side of (3-2) do not always ‘absorb into one’, even in the case when $T$ and $\emptyset$ are congruences and $R$ is a tolerance.

To show that (3-4) can fail, consider the construction in the proof of Theorem 2.1 in the case $n$ odd, this time taking $B' = (B, b)$ instead of $B = (B, u)$. Let $\Theta = \Lambda \beta$, where $\Lambda$ is the tolerance constructed in Example 3.3. Proceed as in the proof of Theorem 2.1 and notice that $(c_0, c_n) \in (\alpha \Theta \circ 2n)\gamma$, as witnessed, again, by $c_1, c_2, \ldots$. In the case at hand $e_{n-1} = (1, 0, \uparrow)$ and $f_i = (0, 1, \uparrow)$ are not $\Theta$-related, since they are not $\Lambda$-related. Hence here we cannot skip the passage from $e_{n-1} \alpha \gamma$ to $e_n$, as we did in the case $n$ odd in the proof of Theorem 2.1. Of course, we do have $(c_0, c_n) \in (\alpha \Theta \circ (\alpha \gamma \circ \alpha \Theta) \circ \alpha \Theta) \circ \alpha \Theta \circ 2n \alpha \gamma$, as shown by Equation (3-2); however, we lose one more turn if we want that $\alpha \Theta$ and $\alpha \gamma$ strictly alternate on the right-hand side of (3-4), and hence we cannot have $(c_0, c_n) \in (\alpha \Theta \circ 2n \alpha \gamma)$.

$\square$

The above argument shows that also the identity $\alpha (\Theta \circ (\alpha \gamma \circ \alpha \Theta \circ \alpha \gamma) \circ \Theta) \subseteq \alpha \Theta \circ 2n \alpha \gamma$ fails in $\mathcal{B}$, for $n$ odd.

\textbf{Remark 3.4.} In [20] we have shown that, under a fairly general hypothesis, any congruence identity is equivalent to the corresponding tolerance identity, provided that only tolerances representable as $R \circ R^{-1}$ are considered in the latter identity. Here $R^{-1}$ denotes the converse of $R$.

Equations (2-2) in Theorem 2.1 and (3-4) in Theorem 3.1 show that, in general, the assumption of representability is necessary in the results from [20]. A similar counterexample has been presented in [27].

It follows from [20] that the tolerance $\Theta$ used in the proof of Theorem 3.1 is not representable. It can be checked directly that in $(B, b)$ neither $\Theta$ for $n$ odd, nor $\Lambda$ for every $n$ are representable, where $\Lambda$ is the tolerance constructed in Example 3.3. In fact, if $R$ is reflexive and admissible, $c_0 R \circ R^{-1} c_1$ and $c_{n-1} R \circ R^{-1} c_n$, then $e_{n-1} R \circ R^{-1} f_1$. Indeed, if $c_0 R g R^{-1} c_1$ and $c_{n-1} R h R^{-1} c_n$, then $c_n R h$ and $g R^{-1} c_0$, and hence $e_{n-1} = c_0(c_{n-1} + c_n) R g(h + h) = gh = h(g + g) R^{-1} c_n(c_0 + c_1) = f_1$.

4. Identities with just two relations

If $R$ is a congruence, or just a transitive relation, then there is no point in considering identities of the form $T(R \circ R) \subseteq something$, since $R \circ R = R$. In passing, let us
mention that the latter identity $R \circ R = R$ is equivalent to congruence permutability, as noticed independently by Hutchinson [13] and Werner [31]. On the other hand, if we only suppose that $R$ is reflexive and admissible, many more identities of the form $T(R \circ R) \subseteq \text{something}$ become interesting. For example, we showed in [21, 26] that a variety $\mathcal{V}$ is congruence modular if and only if there is some $k$ such that $\mathcal{V}$ satisfies the identity $\Theta(R \circ R) \subseteq (\Theta R)^k$.

In the present section we evaluate the best possible bounds for identities of the above kind both in $\mathcal{B}$ (equivalently, $\mathcal{B}^d$) and $\mathcal{N}_4$ (equivalently $\mathcal{N}_4^d$). In a certain respect, here the situation is simpler than in the preceding sections, since we do not need the division into the two cases $n$ odd and $n$ even; moreover, the bounds for $\mathcal{N}_4$ are always better than the bounds for $\mathcal{B}$. Notice that all the identities considered in the present section are strictly weaker than congruence distributivity, since they might hold in nondistributive congruence modular varieties; actually, all the identities below hold in congruence permutable varieties.

**Theorem 4.1.** For $n \geq 2$, the following identities are satisfied:

by $\mathcal{B}$: \[ TR^n \subseteq (TR)^{2n}, \] (4-1)

by $\mathcal{N}_4$: \[ TR^n \subseteq (TR)^{2n-1} \] (4-2)

and the exponents on the right are best possible; actually,

$\mathcal{B}^d$ fails to satisfy \[ \alpha(\Theta \circ (\alpha \Theta)^{n-2} \circ \Theta) \subseteq (\alpha \Theta)^{2n-1}, \] and \[ N_4^d \text{ fails to satisfy } \alpha(\Theta \circ (\alpha \Theta)^{n-2} \circ \Theta) \subseteq (\alpha \Theta)^{2n-2}, \] (4-3) (4-4)

where $T$, $R$ are reflexive and admissible relations, $\Theta$ is a tolerance and $\alpha$ is a congruence.

**Proof.** Identities (4-1) and (4-2) are immediate from identities (3-1)–(3-3) in Theorem 3.1, taking $S = R$, and from identity (3-6) in Proposition 3.2.

We first show that (4-4) can fail, and hence the bound in (4-2) is best possible. Consider again the counterexample $(B, u)$ constructed in the proof of Theorem 2.1. Let $\Psi$ be the binary relation on $B$ defined in such a way that two elements $x$ and $y$ in $B$ are $\Psi$-related if and only if:

(d) for each $\ell = 1, 2$, the components $x_\ell$ and $y_\ell$ differ at most by 1.

We claim that $\Psi$ is a tolerance on $B$. Indeed, condition (d) defines a tolerance $\Psi_L$ on the lattice $L = C_{n+1} \times C_{n+1} \times C_2$, since $\Psi_L$ is a product of tolerances on the factors. Hence $\Psi_L$ is a tolerance on the polynomial reduct $(L, u)$ and $\Psi$, being the restriction of $\Psi_L$ to $B$, is a tolerance, too.

Now we have $(c_0, c_n) \in \alpha(\Psi \circ (\alpha \Psi)^{n-2} \circ \Psi)$, again, as witnessed by $c_1, c_2, \ldots$. On the other hand, as in the proof of Theorem 2.1, the only other element $\alpha \Psi$-connected to $c_0 = e_0$ is $e_1$, the only further element $\alpha \Psi$-connected to $e_1$ is $e_2$ and so on, until we reach $e_{n-1}$, which is $\alpha \Psi$-connected only to $e_{n-2}$ (but this has no use), to $e_n = f_0$ and...
to \(f_i\). We get the fastest path going directly through \(f_1\); in any case, we need \(2n - 1\) steps, and thus \((c_0, c_n) \in (\alpha \Psi)^{2n-2}\) fails in \(N_4\). Hence \((4-4)\) fails with \(\Psi\) in place of \(\Theta\).

In order to disprove the identity in \((4-3)\), let us work in \((B, b)\) instead. Recall that \((B, b) \in B\). The relation \(\Psi\) defined above is a tolerance on \((B, b)\), being a tolerance on \((B, u)\). Let \(\Theta = \Lambda \Psi\), where \(\Lambda\) is the tolerance defined in Example 3.3. As above, we have \((c_0, c_n) \in (\alpha \Theta) (\alpha \Theta)^{n-2} \Theta\). We have that \((c_0, c_n) \in (\alpha \Theta)^{2n-1}\) fails, since any chain of \(\alpha \Theta\)-related elements from \(c_0\) to \(c_n\) must contain all the elements of \(B^\uparrow\). The difference with the previous case dealing with \(N_4\) is that here \(e_{n-1}\) and \(f_i\) are not \(\Theta\)-related, being not \(\Lambda\)-related, and hence one more step is necessary. \(\square\)

**Remark 4.2.** In Example 3.3 we have seen that \(\Lambda\) is a tolerance on \((B, b)\). It follows from the above proof that \(\Lambda\) is not a tolerance on \((B, u)\). It is easy to see directly that \(\Lambda\) is not even compatible in \((B, u)\); otherwise, we would get \(e_{n-1} = u(c_0, c_0, c_{n-1}, c_n) \Lambda u(c_0, c_1, c_n, c_n) = f_1\), a contradiction.

As a small improvement on some results in this and in the previous section, notice that in the identities in \((3-1)\), \((3-2)\), \((3-5)\) and \((4-1)\) it is enough to assume that \(T, R\) and \(S\) are set-theoretical unions of reflexive and admissible relations. Indeed, in the proofs only one element is moved at a time. Compare [25].

The following lemma provides another argument showing that the exponent on the right in the identity in \((4-2)\) cannot be improved. Recall that if \(T\) is a binary relation on some algebra, \(\overline{T}\) denotes the smallest reflexive and admissible relation containing \(T\). The definition of \(n\)-modularity has been recalled shortly before Corollary 2.4.

**Lemma 4.3.** Let \(\mathcal{V}\) be any variety.

1. If \(\mathcal{V}\) satisfies \(\alpha(R \circ R) \subseteq (\alpha R)^k\), then \(\mathcal{V}\) is \(2k\)-modular.
2. More generally, if \(\mathcal{V}\) satisfies \(\alpha(R \circ_n R) \subseteq (\alpha R)^k\), for some \(n \geq 2\), then \(\mathcal{V}\) satisfies \(\alpha(\beta \circ_{2n} \alpha \gamma) \subseteq \alpha \beta \circ_{2k} \alpha \gamma\).
3. If \(k \geq 2\) and \(\mathcal{V}\) satisfies \(\alpha(R \circ S) \subseteq \alpha R \circ (\alpha(R \cup S))^{k-1}\), then \(\mathcal{V}\) is \((2k - 1)\)-modular.

**Proof.**

1. Taking \(R = \beta \circ \alpha \gamma\), we have \(\alpha(\beta \circ \alpha \gamma) = \alpha \beta \circ \alpha \gamma\), and hence
   \[
   \alpha(\beta \circ \alpha \gamma \circ \beta) \subseteq \alpha(\beta \circ \alpha \gamma \circ \beta \circ \alpha \gamma) = \alpha(R \circ R) \subseteq (\alpha R)^k = (\alpha(\beta \circ \alpha \gamma))^k = (\alpha \beta \circ \alpha \gamma)^k = \alpha \beta \circ_{2k} \alpha \gamma.
   \]

2. The second part is proved in the same way.
3. Take \(R = \beta\), \(S = \alpha \gamma \circ \beta\) and observe that \(R \cup S = \alpha \gamma \circ \beta\) is reflexive and admissible. \(\square\)

Thus \(N_4\) fails to satisfy \(TR^2 \subseteq (TR)^2\), since otherwise \(N_4\) would be \(4\)-modular, by 4.3(1), contradicting Corollary 2.4.

More generally, Equation \((4-2)\) cannot be improved to \(TR^n \subseteq (TR)^{2n-2}\), since otherwise Lemma 4.3(2) would give
\[
\alpha(\beta \circ (\alpha \gamma \circ \alpha \beta \circ 2^{n-3} \circ \alpha \gamma \circ \beta)) \subseteq \alpha(\beta \circ 2^n \alpha \gamma) \subseteq 4.3(2) \alpha \beta \circ 4n-4 \alpha \gamma, \tag{4-5}
\]
where the superscript in \(\subseteq 4.3\) means that we are applying item (2) in Lemma 4.3. Taking \(m = 2n - 1\) in (4-5), we get \(2n - 3 = m - 2\) and \(4n - 4 = 2m - 2\), and thus Equation (4-5) becomes Equation (2-4) in Theorem 2.1 with \(m\) in place of \(n\). Theorem 2.1 shows that this equation fails in \(\mathcal{N}_4\).

Remark 4.4. Recall that \(R^-\) denotes the converse of the relation \(R\). It is not difficult to show that a variety \(\mathcal{V}\) is \((k + 1)\)-modular if and only if \(\mathcal{V}\) satisfies the identity

\[
\alpha(R \circ R^-) \subseteq \alpha R \circ_k a R^-.
\tag{4-6}
\]

See [21], where it is also shown that \(\alpha\) can be equivalently taken to vary among tolerances. If we let \(R = \Theta\) be a tolerance in (4-6), we get

\[
\alpha(\Theta \circ \Theta) \subseteq (\alpha \Theta)^k.
\tag{4-7}
\]

Clearly, in turn, (4-7) implies back congruence modularity; actually, (4-7) implies \((2k + 2)\)-modularity (perhaps the bound \(2k + 2\) can be improved). Indeed, taking \(\Theta = \alpha \gamma \circ \beta \circ \alpha \gamma\) in (4-7) we get

\[
\alpha(\beta \circ \alpha \gamma \circ \alpha \gamma) \subseteq \alpha(\alpha \gamma \circ \beta \circ \alpha \gamma \circ \alpha \gamma) = \alpha(\Theta \circ \Theta) \subseteq (\alpha \Theta)^k = (\alpha \gamma \circ \alpha \beta \circ \alpha \gamma)^k = \alpha \gamma \circ 2k+1 \alpha \beta \subseteq \alpha \beta \circ 2k+2 \alpha \gamma.
\]

Theorem 4.1 shows that the best value of \(k\) in (4-7) for a congruence modular variety \(\mathcal{V}\) is not determined by the Day modularity level of \(\mathcal{V}\) (and it is not determined by the Jónsson distributivity level, either, for congruence distributive varieties). Indeed, both \(\mathcal{B}\) and \(\mathcal{N}_4\) are 5-modular, not 4-modular, 4-distributive and not 3-distributive, but the best value for \(k\) in (4-7) is 4 for \(\mathcal{B}\) and 3 for \(\mathcal{N}_4\): just take \(n = 2\) in Theorem 4.1. In particular, the variety \(\mathcal{N}_4\) shows that (4-7) for some given \(k\) does not imply \((k + 1)\)-modularity.

Remark 4.5. Since the identity

\[
\alpha(R \circ R) \subseteq (\alpha R)^k
\tag{4-8}
\]
implies (4-7), then, by the above remark, the identity (4-8) implies congruence modularity. This follows also directly from Lemma 4.3. As we mentioned in the introduction, it is shown in [21, 26] that the converse holds, that is, every congruence modular variety satisfies (4-8), for some \(k\). Again, this argument from [21, 26] relies heavily on [18].

As in Remark 4.4, we get that Theorem 4.1 shows that the best value of \(k\) in (4-8) for a variety \(\mathcal{V}\) is not determined by the Day modularity level of \(\mathcal{V}\). It is likely that there is some variety \(\mathcal{V}\) such that the best value of \(k\) in (4-7) is strictly smaller than the best value of \(k\) in (4-8). The varieties considered here do not furnish a (counter-)example for this possible inequality.
5. Near-unanimity and edge terms

Of course, it is an interesting problem to evaluate the distributivity spectra of other congruence distributive varieties. Varieties with a near-unanimity term appear to be a significant test case. Recall that $R, S, T, \ldots$ denote reflexive and admissible relations, $\Theta$ denotes a tolerance and $\bar{R} \cup \bar{S}$ denotes the smallest admissible relation containing both $R$ and $S$.

**Proposition 5.1.** Suppose that $m \geq 1$ and that the variety $\mathcal{V}$ has an $(m + 2)$-ary near-unanimity term. Then, for every $n \geq 2$, the variety $\mathcal{V}$ satisfies

\[ \alpha(\beta \circ_n \gamma) \subseteq \alpha \beta \circ_{mn} \alpha \gamma \quad \text{for } n \text{ even}, \quad (5-1) \]

\[ \alpha(\beta \circ_n \gamma) \subseteq \alpha \beta \circ_{1 + m(n - 1)} \alpha \gamma \quad \text{for } n \text{ odd}, \quad (5-2) \]

\[ \Theta(R \circ_n R) \subseteq (\Theta R)^{1 + m(n - 1)} \quad \text{for every } n, \quad (5-3) \]

where $R$ is a reflexive and admissible relation, $\Theta$ is a tolerance and $\alpha, \beta$ and $\gamma$ are congruences.

Taking $n = 2$ in Equation $(5-1)$ we get that $\mathcal{V}$ is $2m$-distributive [29]. Taking $n = 3$ in Equation $(5-2)$ with $\alpha \gamma$ in place of $\gamma$, we get that $\mathcal{V}$ is $(2m + 1)$-modular.

In fact, we prove a more general result.

**Proposition 5.2.** Under the assumptions in Proposition 5.1, $\mathcal{V}$ satisfies

\[ \Theta(R_1 \circ R_2 \circ \cdots \circ R_n) \subseteq \Theta R_1 \circ \Theta R_2 \circ \cdots \circ \Theta R_{n-1} \circ \Theta(\bar{R}_n \cup \bar{R}_1) \circ \Theta R_2 \circ \cdots \circ \Theta R_{n-1} \circ \Theta(\bar{R}_n \cup \bar{R}_1) \circ \cdots \]

\[ \Theta R_2 \circ \cdots \circ \Theta R_{n-1} \circ \Theta R_n, \quad (5-4) \]

with $m$ lines, that is, with a total of $1 + m(n - 1)$ factors (a total of $m(n - 1)$ occurrences of $\circ$) on the right-hand side.

Equation $(5-4)$ should read $\Theta(R_1 \circ \cdots \circ R_n) \subseteq \Theta R_1 \circ \Theta R_2 \circ \cdots \circ \Theta R_{n-1} \circ \Theta R_n$ when $m = 1$ and $\Theta(R_1 \circ \cdots \circ R_n) \subseteq \Theta R_1 \circ \cdots \circ \Theta R_{n-1} \circ \Theta(\bar{R}_n \cup \bar{R}_1) \circ \Theta R_2 \circ \cdots \circ \Theta R_n$ when $m = 2$.

**Proof.** Suppose that $u$ is an $(m + 2)$-ary near-unanimity term, $a \Theta d$ and $a R_1 c_1 R_2 c_2 \cdots c_{n-2} R_{n-1} c_{n-1} R_n d$. Then

\[ a = u(a, \ldots, a, a, a, d) R_1 u(a, \ldots, a, a, c_1, d) R_2 u(a, \ldots, a, a, c_2, d) \ldots \]

\[ R_{n-1} u(a, \ldots, a, a, c_{n-1}, d) \bar{R}_n \cup \bar{R}_1 u(a, \ldots, a, c_1, d) R_2 u(a, \ldots, a, c_2, d) \ldots \]

\[ R_{n-1} u(a, a, c_{n-1}, d, \ldots, d) \bar{R}_n \cup \bar{R}_1 u(a, c_1, d, \ldots, d) R_2 u(a, c_2, d, \ldots, d) \ldots R_{n-1} u(a, c_{n-1}, d, \ldots, d) R_n u(a, d, d, \ldots, d) = d. \]
In the above chain of relations we have only used a minimal part of the assumption that \( u \) is a near-unanimity term: we have used only the two special cases in which the ‘dissenter’ is either the first or the last element. The full assumption is used in order to show that all the above elements are \( \Theta \)-related. Were \( \Theta = \alpha \) a congruence, this would be trivial, since
\[
\begin{align*}
\mathbf{u}(a_1, \ldots, a, c_j, d, \ldots, d) & \preceq \mathbf{u}(a_1, \ldots, a, c_k, a, \ldots, a) = a = \\
\mathbf{u}(a_1, \ldots, a, c_k, a, \ldots, a) & \preceq \mathbf{u}(a_1, \ldots, a, c_k, d, \ldots, d),
\end{align*}
\]
for all pairs of indices \( j \) and \( k \) and where the \( c_j \) and the \( c_k \) can occur in any pair of possibly distinct positions.

Notice that the case in which \( \Theta \) is a congruence is enough in order to prove (5-1) and (5-2) in Proposition 5.1. Formally, (5-1) does not follow from (5-4); however, in (5-4) we can replace each occurrence of \( \Theta(R_n \cup R_1) \) by \( \Theta R_n \circ \Theta R_1 \). Indeed, say,
\[
\mathbf{u}(a_1, \ldots, a, c_{n-1}, d) \Theta R_n \mathbf{u}(a_1, \ldots, a, d, d, \ldots, d) \Theta R_1 \mathbf{u}(a_1, \ldots, a, c_1, d, d, \ldots, d).
\]

It remains to consider the case in which \( \Theta \) is only supposed to be a tolerance. The argument resembles a proof in Czédli and Horváth [6]. As above, we show that any two elements in the above chain (disregarding their ordering, that is, slightly more than required) are \( \Theta \)-related. Indeed,
\[
\begin{align*}
\mathbf{u}(a_1, \ldots, a, c_j, d, \ldots, d) & = \\
\mathbf{u}(\mathbf{u}(\ldots, a, c_k, a \ldots), \ldots, \mathbf{u}(\ldots, a, c_k, a \ldots), \ldots, \mathbf{u}(\ldots, d, c_k, d \ldots), \ldots, \mathbf{u}(\ldots, d, c_k, d \ldots)) \Theta \\
\mathbf{u}(\mathbf{u}(\ldots, a, c_k, d \ldots), \ldots, \mathbf{u}(\ldots, a, c_k, d \ldots), \ldots, \mathbf{u}(\ldots, a, c_k, d \ldots)) & = \\
\mathbf{u}(a_1, \ldots, a, c_k, d, \ldots, d),
\end{align*}
\]
again, for all pairs of indices \( j \) and \( k \) and where the \( c_j \) and the \( c_k \) can occur in any pair of possibly distinct positions. In the above-displayed formula the vertical bars link distinct \( \Theta \)-related elements and, in order to keep the formula within a reasonable length, we have written, say, \( \mathbf{u}(\ldots, a, c_k, d \ldots) \) in place of \( \mathbf{u}(a_1, \ldots, a, a, c_k, d, d, \ldots, d, d) \). In conclusion, we get that \((a, d)\) belongs to the right-hand side of (5-4). \( \square \)

Equations (5-1) and (5-2) show that a variety with a near-unanimity term is congruence distributive, a result originally due to Mitschke [29]. The above proof seems simpler than the one from [29] and uses folklore ideas. See, for example, Kaarli and Pixley [17, Lemma 1.2.12], whose proof is credited to E. Fried. Notice that here and in [17, Lemma 1.2.12], as well, it is not necessary to use Jónsson’s characterization [15] of congruence distributive varieties.

From a more recent point of view, (5-4) might be seen as a combination of two observations. First, the fact that a near-unanimity term easily yields a set of directed Jónsson terms; see, for example, Barto and Kozik [2, Section 5.3.1]. Second, the observation in [24] that directed Jónsson terms not only imply congruence distributivity, but also imply certain similar relation identities. The technical idea of merging \( R_n \) and \( R_1 \), so as to obtain a smaller number of factors in (5-3) and (5-4) seems new, at least in the present context. We do not know whether we can replace the
tolerance \( \Theta \) by a reflexive and admissible relation \( T \) in (5-3) and (5-4) (with or without the same number of factors on the right).

We now turn to edge terms, an important generalization of near-unanimity terms. Berman, Idziak, Marković, McKenzie, Valeriote and Willard have introduced edge terms in [4], providing equivalent characterizations for their existence. Further characterizations have been found by Kearnes and Szendrei in [19].

If \( k \geq 2 \), a \((k+1)\)-ary term \( t \) is a \( k \)-edge term for some variety \( \mathcal{V} \) if the equations \( x = t(y, y, x, x, \ldots, x) = t(x, y, y, x, \ldots, x) \) hold and, moreover, all the equations of the form \( x = t(x, x, x, \ldots, y, \ldots) \) hold in \( \mathcal{V} \), where a single occurrence of \( y \) appears in any place after the third place, surrounded by \( x \)s elsewhere. We have used here the formulation from [19] in which the first two places are exchanged. Notice that a \( k \)-ary near-unanimity term becomes a \( k \)-edge term by adding a dummy variable at the second place. The following proposition provides, among other, still another proof that varieties with an edge term are congruence modular.

**Proposition 5.3.** If \( k \geq 3 \) and \( \mathcal{V} \) has a \( k \)-edge term, then \( \mathcal{V} \) satisfies

\[
\Theta(R \circ R) \subseteq (\Theta R)^{k-1} \quad \text{and, more generally,} \quad \Theta(R \circ S) \subseteq \Theta R \circ (\Theta (R \cup S))^{k-2},
\]

where \( R, S \) are reflexive and admissible relations and \( \Theta \) is a tolerance.

*Proof.* The proof is similar to the proof of Proposition 5.1, with a variation ‘near the edge’. Equation (5-5) is the particular case \( R = S \) of Equation (5-6), and hence it is enough to prove the latter. If \( a \Theta d \) and \( a R c S d \), then

\[
\begin{align*}
a &= t(a, a, a, a, a, a, \ldots, a, a, a, a, d) \quad R \\
t(a, a, a, a, a, a, a, \ldots, a, a, c, d) &= R \cup S \\
t(a, a, a, a, a, a, a, \ldots, a, c, d, d) &= R \cup S \\
&\ldots \\
t(a, a, a, a, c, d, d, d, d, d, d, d, d) &= R \cup S \\
t(a, a, a, a, c, d, d, d, d, d, d, d, d, d) &= R \cup S \\
t(c, c, d, d, d, d, d, d, d, d, d, d, d) &= d
\end{align*}
\]

The proof that all the above elements are \( \Theta \)-related is similar to the corresponding proof in 5.1. For example,

\[
\begin{align*}
t(a, c, c, d, \ldots) &= t(t(a, a, a, c, a, \ldots), c, c, t(d, d, d, c, d, \ldots) \ldots) \Theta \\
t(t(a, a, a, c, d, \ldots), c, c, t(a, a, a, a, c, d, \ldots) \ldots) &= t(a, a, a, c, d, \ldots) \quad \Box
\end{align*}
\]

**Remark 5.4.** Merging the proofs of Propositions 5.2 and 5.3 we get that if \( k \geq 4 \) and \( \mathcal{V} \) has a \( k \)-edge term, then, for every \( n \geq 2 \), the variety \( \mathcal{V} \) satisfies
\[ \Theta(R \circ S)(R_1 \circ \cdots \circ R_n) \subseteq \Theta R_1 \circ \Theta R_2 \circ \cdots \circ \Theta R_{n-1} \circ \Theta(R_n \cup R_1) \circ \Theta R_2 \circ \cdots \circ \Theta R_{n-1} \circ \Theta(R_n \cup R_1) \circ \cdots \Theta R_2 \circ \cdots \circ \Theta R_{n-1} \circ \Theta(R_n \cup R) \circ \Theta(R \cup S), \]

with \( k - 3 \) lines.

### 6. Polynomial reducts of lattices and of Boolean algebras

Recall that \( \alpha, \beta, \gamma, \ldots \) denote congruences, \( \Theta \) denotes a tolerance and \( R, S, T, \ldots \) denote reflexive and admissible relations.

**Remark 6.1.** It is standard and easy to show that varieties with a majority term satisfy \( T(R \circ S) \subseteq TR \circ TS \). See, for example, \([24, 25]\). Since the composition of two reflexive and admissible relations is still reflexive and admissible, then, by substitution and an easy induction, we get that if \( 'V \) has a majority term, then, for every \( n \geq 2 \), the variety \( 'V \) satisfies \( T(R_1 \circ R_2 \circ \cdots \circ R_n) \subseteq TR_1 \circ TR_2 \circ \cdots \circ TR_n \).

Moreover, by taking \( T = (R_3 \circ R_2)(R_1 \circ R_3) \), we get that a variety with a majority term satisfies \( (R_1 \circ R_2)(R_3 \circ R_2)(R_1 \circ R_3) \subseteq R_1R_3 \circ R_1R_2 \circ R_1R_2 \circ R_1R_3 \).

**Remark 6.2.** In passing, we notice the curious fact that while, of course, the identity \( \alpha(\beta \circ \gamma) = \alpha \beta \circ a\gamma \) for congruences is equivalent to the existence of a majority term, on the other hand, the identity

\[ (\alpha \circ \delta)(\beta \circ \gamma) = \alpha \beta \circ \alpha \gamma \circ \delta \beta \circ \delta \gamma \] (6-1)

is equivalent to arithmeticity. Notice that we are assuming equality, not just inclusion.

To prove the claim, assume Equation (6-1) and expand the product in two ways, getting \( \alpha \beta \circ \alpha \gamma \circ \delta \beta \circ \delta \gamma = \alpha \beta \circ \delta \beta \circ \alpha \gamma \circ \delta \gamma \). Then taking \( \gamma = \alpha \) and \( \delta = \beta \) we get \( \alpha \circ \beta = \alpha \beta \circ \alpha \circ \beta \circ \alpha = \alpha \beta \circ \beta \circ \alpha \circ \alpha \beta = \beta \circ \alpha \), that is, congruence permutability. On the other hand, by taking \( \beta = \gamma \) in (6-1), we get 2-distributivity, that is, a majority term. It is well known that arithmeticity is equivalent to congruence permutability together with the existence of a majority term, and hence our claim follows.

**Remark 6.3.** The simplest way to check whether a variety \( 'V \) is congruence distributive is to work on the third power of \( F_{'V}(2) \). See \([10]\). However, in the present remark we are concerned with the classical method of working in \( F_{'V}(3) \). It follows from \([15]\) and is by now standard that, for every \( m \), some variety \( 'V \) satisfies the congruence identity

\[ \alpha(\beta \circ \gamma) \subseteq \alpha \beta \circ_m \alpha \gamma \] (6-2)

if and only if (6-2) holds in \( F_{'V}(3) \), the free algebra in \( 'V \) generated by three elements. Henceforth, a possible way to check whether some variety \( 'V \) satisfies (6-2), or even many related identities, is to check them in \( F_{'V}(3) \). This is classical, by now.

Let us compute, for example, \( F_{3' V}(3) \). Since \( F_{3' V}(3) \) is naturally embedded into \( F_{\mathcal{D}}(3) \), where \( \mathcal{D} \) is the variety of distributive lattices, it is easy to see that the elements
of \( F_{\mathcal{V}}(3) \) are
\[
x, \quad x(y + z), \quad xy, \quad xyz,
\]
together with the elements arising from all the possible permutations of \( x, y \) and \( z \). See [1]. The elements in the above list are exactly also the elements of \( F_{\mathcal{N}^d}(3) \), since \( F_{\mathcal{V}}(3) \) is closed under \( u \). Indeed, if \( a \in F_{\mathcal{V}}(3) \), then, by the above description, either \( a \leq x \) or \( a \leq y \), or \( a \leq z \). Hence if \( a_1, a_2, a_3, a_4 \in F_{\mathcal{V}}(3) \), then at least two elements are less than or equal to some generator, say, \( a_1, a_2 \leq x \), thus \( u(a_1, a_2, a_3, a_4) \leq x \), and hence \( u(a_1, a_2, a_3, a_4) \in F_{\mathcal{V}}(3) \). Of course, the above argument would fail, were we considering \( F_{\mathcal{V}}(4) \) in place of \( F_{\mathcal{V}}(3) \). In a sense, computing \( F_{\mathcal{V}}(3) \) is a way to see that \( \mathcal{N}^d_4 \) and \( \mathcal{B}^d \) satisfy exactly the same identities of the form \( a(\beta \circ \gamma) \subseteq \text{something} \), for congruences and for many possible variations on the expression on the right.

Let us now consider the relation identity
\[
T(R \circ R) \subseteq TR \circ_m TR. \quad (6-3)
\]
It is still true that some variety \( \mathcal{V} \) satisfies (6-3) if and only if (6-3) holds in \( F_{\mathcal{V}}(3) \). The argument is standard but not very usual. To prove the nontrivial implication, let \( F_{\mathcal{V}}(3) \) be generated by \( x, y \) and \( z \), let \( T \) be the smallest reflexive and admissible relation containing the pair \((x, z)\) and let \( R \) be the smallest reflexive and admissible relation containing the pairs \((x, y)\) and \((y, z)\). Since \((x, z) \in T(R \circ R)\), then, if (6-3) holds, there are elements \( t_0, \ldots, t_m \in F_{\mathcal{V}}(3) \) witnessing \((x, z) \in TR \circ_m TR \). Since we are working in \( F_{\mathcal{V}}(3) \), the \( t_i \) correspond to ternary terms of \( \mathcal{V} \). What does it mean, say, that \( t_i R t_{i+1} ? \) It is easy to see that \( R = \{(w(x, y, z, x, y), w(x, y, z, y, z)) \mid w \text{ a quinary term of } \mathcal{V} \} \). Hence \( t_i(x, y, z) = w_i(x, y, z, x, y) \) and \( w_i(x, y, z, y, z) = t_{i+1}(x, y, z) \), for some quinary term \( w_i \). Similarly, the \( T \)-relations are witnessed by certain quaternary terms. Once we have found appropriate terms and appropriate equations, it is standard to see that they witness that (6-3) holds in \( \mathcal{V} \). See, e.g., [25, 27] for similar arguments.

We now see an essential difference between the identities (6-2) and (6-3). While \( F_{\mathcal{N}^d}(3) = F_{\mathcal{B}}(3) \), we have seen in Theorem 4.1 that \( \mathcal{N}^d_4 \) and \( \mathcal{B}^d \) do not satisfy the same identities of the form (6-3), in the sense that the best possible indices on the right are not the same. At first, this might generate some perplexity, but in the end the explanation is easy. The point is that, when considering congruence identities of the form (6-2), only ternary terms are relevant; in other words, only the elements of \( F_{\mathcal{V}}(3) \) are relevant. On the other hand, though the validity of (6-3) can be checked in \( F_{\mathcal{V}}(3) \), the relevant terms, in this case, are the quaternary and quinary ones.

Exactly as in [1], some results provable for \( \mathcal{B} \) extend to every variety which is a congruence distributive polynomial reduct of some variety of lattices. This is the content of the next theorem, where we also show that there are limitations to the counterexamples which can be furnished by polynomial reducts of Boolean algebras.

If \( P \) is a set of lattice terms and \( L \) is a lattice, we denote by \( L_P \) the algebra with base set \( L \) and with, as basic operations, those induced by the terms of \( P \). If \( \mathcal{V} \) is a variety of lattices, we let \( \mathcal{V}_P \) be the variety generated by all algebras \( L_P \), with \( L \) varying
in \(V\). If \(W = V_p\), for some \(P\), we say that \(W\) is a polynomial reduct of \(V\). Polynomial reducts of Boolean algebras are defined in a similar fashion.

**Theorem 6.4.** (1) If the variety \(W\) is a congruence distributive polynomial reduct of some variety of lattices, then \(W\) satisfies

\[
\Theta(R \circ_2 S) \subseteq \Theta R \circ_2 \Theta S \quad \text{for } n \text{ even, and}
\]

\[
\Theta(R \circ_2 S) \subseteq (\Theta R \circ_2 \Theta S \circ_2 \cdots \circ_2 \Theta R) \circ_2 (\Theta R \circ_2 \Theta S \circ_2 \cdots \circ_2 \Theta R) \quad \text{for } n \text{ odd,}
\]

where \(R, S\) are reflexive and admissible relations and \(\Theta\) is a tolerance.

(2) If the variety \(W\) is a congruence distributive polynomial reduct of the variety of distributive lattices, then \(W\) satisfies the Equations (3-1) and (3-2) in Theorem 3.1. In other words, we may allow \(\Theta\) to be a reflexive and admissible relation in the identities in (1) above.

(3) If \(W\) is a congruence modular polynomial reduct of the variety of Boolean algebras, then either \(W\) has a majority term, or \(W\) has a Mal'tsev term, or \(W\) interprets \(B\). If in addition \(W\) is congruence distributive, then \(W\) satisfies the Equations (3-1) and (3-2) in Theorem 3.1.

**Proof.** (1) First, notice that in the proof that Baker’s variety \(B\) satisfies the identities (2-1) and (2-2) in Theorem 2.1 we have only used the equations

\[
x = b(x, x, y) = b(x, y, x) \quad \text{and} \quad b(x, y, y) = b(y, x, x), \quad (6-4)
\]

and that, as we mentioned, the proof works also when \(\beta\) and \(\gamma\) are reflexive and admissible relations. Using the idea from Czédli and Horváth [6], an idea we have already used above, we can replace \(\alpha\) in (2-1) and (2-2) by a tolerance \(\Theta\). Indeed, if \(\Theta\) is a tolerance and \(a \Theta d\), then from the equation \(x = b(x, y, x)\) we get

\[
b(a, c_j, d) = b(b(a, c_k, a), c_j, b(d, c_k, d)) \Theta
\]

\[
b(b(a, c_k, d), c_j, b(a, c_k, d)) = b(a, c_k, d),
\]

for all pairs of indices \(j\) and \(k\). Notice that we have shown a little more than requested, namely, that all the elements from the list in Equation (2-5) in the proof of 2.1 are \(\Theta\)-related.

Hence if a variety has a term satisfying (6-4), then the conclusion in (1) holds. We show that a variety satisfying the assumptions in (1) either has a majority term, or a term satisfying (6-4). The argument goes exactly as in the proof of [1, Theorem 2]. Since the Equations (6-4) depend only on two variables, then, for some given term \(t\) (in place of \(b\)), the Equations (6-4) hold in \(W\) if and only if they hold in the free algebra \(F_w(2)\) in \(W\) generated by two elements. Suppose that \(W = V_p\). The result is obvious if \(V\) is a trivial variety. Otherwise, \(V\) contains the two-element lattice \(C_2\). Since the free lattice generated by two elements is \(C_2 \times C_2\), then \(F_w(2)\) is a (possibly improper) subalgebra of \((C_2)_p \times (C_2)_p\). Hence the Equations (6-4) hold in \(W\) for some term if and only if they hold in \((C_2)_p\), if and only if they hold in the variety \(W''\) generated by \((C_2)_p\). If \(W\) is congruence distributive, then \(W''\)
is congruence distributive, too. The free algebra $\mathbf{F}_{\mathcal{W}}(3)$ in $\mathcal{W}$ generated by three elements $x, y$ and $z$ can be seen as a subalgebra of the free distributive lattice generated by $x, y$ and $z$. Baker [1, proof of Theorem 2] shows that $\mathbf{F}_{\mathcal{W}}(3)$, being congruence distributive, must contain either the median $xy + xz + yz$, or the Baker element $x(y + z)$ or its dual. These elements are given by a ternary term $t$ of $\mathcal{W}$ and the above arguments show that in the former case $t$ is a majority term for $\mathcal{W}$, while in the latter case $t$ satisfies the Equations (6-4). Notice that it is not necessarily the case that $t$ is interpreted as $xy + xz + yz$ or $x(y + z)$ throughout $\mathcal{W}$, we only get that $t$ is either a majority term or satisfies (6-4). However this is enough, by Fact 6.1 in the former case, and by the comment in the first paragraph of the present proof in the latter case. Hence (1) is proved.

(2) This is a particular case of the last statement of (3), however a direct proof along the lines of (1) is easy. Under the additional assumption, we can argue directly in $\mathcal{W}$, rather than in $\mathcal{W}'$, and hence in the present case $t$ can be actually interpreted as $xy + xz + yz$ or $x(y + z)$ or the dual throughout $\mathcal{W}$. In the former case Fact 6.1 is enough and in the latter case the arguments in the proof of (3-5) in Proposition 3.2 carry over.

(3) Let us prove the first statement. If $\mathcal{W}$ is congruence modular, then $\mathcal{W}$ has ternary directed Gumm terms $d_1, \ldots, d_n, q$, as introduced in Kazda et al. [18, page 205]. See [18, Theorem 1.1, Clause 3]. We recall the equations that directed Gumm terms satisfy as soon as needed. Obviously, a ternary term of $\mathcal{W}$ corresponds to a ternary Boolean term $t$; hence it is no loss of generality to assume that $t(x, y, z) = a_1xyz + a_2xyz' + a_3xy'z + a_4xy'z' + a_5x'yz + \ldots$, where $'$ denotes complement and each $a_1, a_2, \ldots$ is either 0 or 1. The first term $d_1$ in the set of directed Gumm terms satisfies the equations $d_1(x, x, y) = x = d_1(x, y, x)$. Represent $d_1$ by a Boolean expression as above. By the first equation, the coefficients of $xyz$ and $xyz'$ must be 1 and the coefficients of $xy'z$ and $xy'z'$ must be 0. By the second equation, the coefficients of $xyz$ and $xy'z$ must be 1 and the coefficients of $x'yz'$ and $x'y'z'$ must be 0. Considering all the possibilities, one easily sees that $d_1$ is either the majority term $xy + xz + yz$, or the Baker term, or the dual of the Baker term, or the first projection. In all but the last case we are done. If $d_1$ is the first projection, then the equations for directed Gumm terms are $d_2(x, x, y) = d_1(x, y, y) = d_1(x, y, x)$, but since $d_1(x, y, y) = x$, we get $d_2(x, x, y) = x$, and hence we can repeat the above argument for $d_2$. Going on, if we either get a majority term, or a Baker term, or its dual, we are done as above. Otherwise, all the $d_j$ are first projections. Then the remaining term $q$ in the set of directed Gumm terms satisfies $q(x, x, y) = y$ and $q(x, y, y) = d_n(x, y, y) = x$, and hence $q$ is a Maltsev term for permutability.

To prove the last statement, if $\mathcal{W}$ is congruence distributive, then we have directed Jónsson terms [18], to the effect that $q$ as above is the third projection (or, simply, discard $q$ and ask for $d_n(x, y, y) = x$). Arguing as above, we get that some $d_j$ satisfies all the equations satisfied in distributive lattices by either the majority term or by the Baker term, and hence, again, either Fact 6.1 or the proof of (3-5) applies. □
We expect that 6.4(2) might fail if $\mathcal{W} = \mathcal{V}_P$ when $\mathcal{V}$ is not the variety of distributive lattices. In other words, we expect that (at least, without affecting the indices) 6.4(1) cannot be improved in such a way that reflexive and admissible relations are taken into account everywhere. In this respect, we notice that in the proof of Theorem 2.1 all the lattices we have considered are indeed distributive. Hence, in view of 6.4(2), in order to provide a counterexample that 6.4(1) cannot be improved in the above sense, one should start with a different and more complicated example, namely, it is not enough to consider different lattice term operations on the same set $B$ considered in the proof of 2.1.

Acknowledgement

We thank an anonymous referee for useful comments.

References

[1] K. A. Baker, ‘Congruence-distributive polynomial reducts of lattices’, Algebra Universalis 9 (1979), 142–145.
[2] L. Barto and M. Kozik, ‘Absorption in universal algebra and CSP’, in: The Constraint Satisfaction Problem: Complexity and Approximability, Dagstuhl Follow-Ups, 7 (Schloss Dagstuhl–Leibniz Zentrum für Informatik, Wadern, 2017), 45–77.
[3] C. Bergman, Universal Algebra: Fundamentals and Selected Topics (CRC Press, Boca Raton, FL, 2012).
[4] J. Berman, P. Idziak, P. Marković, R. McKenzie, M. Valeriote and R. Willard, ‘Varieties with few subalgebras of powers’, Trans. Amer. Math. Soc. 362 (2010), 1445–1473.
[5] B. Burris and H. P. Sankappanavar, A Course in Universal Algebra (Springer, New York, 1981).
[6] G. Czédli and E. K. Horváth, ‘Congruence distributivity and modularity permit tolerances, Acta Univ. Palack. Olomuc. Fac. Rerum Natur. Math. 41 (2002), 39–42.
[7] G. Czédli, E. K. Horváth and P. Lipparini, ‘Optimal Mal’tsev conditions for congruence modular varieties’, Algebra Universalis 53 (2005), 267–279.
[8] A. Day, ‘A characterization of modularity for congruence lattices of algebras’, Canad. Math. Bull. 12 (1969), 167–173.
[9] R. Freese, ‘Alan Day’s early work: congruence identities’, Algebra Universalis 34 (1995), 4–23.
[10] R. Freese and M. A. Valeriote, ‘On the complexity of some Maltsev conditions’, Int. J. Algebra Comput. 19 (2009), 41–77.
[11] G. Grätzer, Universal Algebra, 2nd expanded edn (Springer, New York, 1979).
[12] H. P. Gumm, ‘Geometrical methods in congruence modular algebras’, Mem. Amer. Math. Soc. 45(286) (1983) viii+79.
[13] G. Hutchinson, ‘Relation categories and coproduct congruence categories in universal algebra’, Algebra Universalis, 32 (1994), 609–647.
[14] J. Ježek, Universal Algebra (2008), available at http://ka.karlin.mff.cuni.cz/jezek/ua.pdf.
[15] B. Jónsson, ‘Algebras whose congruence lattices are distributive’, Math. Scand. 21 (1967), 110–121.
[16] B. Jónsson, ‘Congruence varieties’, Algebra Universalis 10 (1980), 355–394.
[17] K. Kaarli and A. F. Pixley, Polynomial Completeness in Algebraic Systems (Chapman & Hall/CRC, Boca Raton, FL, 2001).
[18] A. Kazda, M. Kozik, R. McKenzie and M. Moore, ‘Absorption and directed Jónsson terms’, in: Don Pigozzi on Abstract Algebraic Logic, Universal Algebra, and Computer Science, Outstanding Contributions to Logic, 16 (ed. J. Czelakowski) (Springer, Cham, 2018), 203–220.
[19] K. A. Kearnes and Á. Szendrei, ‘Clones of algebras with parallelogram terms’, *Internat. J. Algebra Comput.* **22**(1250005) (2012), 30.
[20] P. Lipparini, ‘From congruence identities to tolerance identities’, *Acta Sci. Math. (Szeged)* **73** (2007), 31–51.
[21] Lipparini, P. Relation identities equivalent to congruence modularity, Preprint, 2017, arXiv:1704.05274.
[22] Lipparini, P. The distributivity spectrum of Baker’s variety, Preprint, 2017, arXiv:1709.05721v2.
[23] Lipparini, P. On the number of terms witnessing congruence modularity, Preprint, 2017, arXiv:1709.06023.
[24] P. Lipparini, ‘The Jónsson distributivity spectrum’, *Algebra Universalis* **79**(23) (2018), 1–16.
[25] P. Lipparini, ‘Unions of admissible relations and congruence distributivity’, *Acta Math. Univ. Comenian. (N.S.)* **87**(2) (2018), 251–266.
[26] P. Lipparini, ‘A variety \( \mathcal{V} \) is congruence modular if and only if \( \mathcal{V} \) satisfies \( \Theta(R \circ R) \subseteq (\Theta R)^h \), for some \( h \)’, in: *Algebras and Lattice in Hawai‘i A Conference in Honor of Ralph Freese, William Lampe, and J.B. Nation* (eds. K. Adaricheva, W. DeMeo and J. Hyndman) (Lulu.com, Morrisville, NC, 2018), 61–65, available at https://universalalgebra.github.io/ALH-2018/assets/ALH-2018-proceedings-6x9.pdf.
[27] P. Lipparini, ‘Relation identities in 3-distributive varieties’, *Algebra Universalis* **80**(55) (2019), 1–20.
[28] R. N. McKenzie, G. F. McNulty and W. F. Taylor, *Algebras, Lattices, Varieties*, Vol. 1 (Wadsworth & Brooks/Cole Advanced Books & Software, Monterey, CA, 1987).
[29] A. Mitschke, ‘Near unanimity identities and congruence distributivity in equational classes’, *Algebra Universalis* **8** (1978), 29–32.
[30] S. T. Tschantz, ‘More conditions equivalent to congruence modularity’, in: *Universal Algebra and Lattice Theory*, Lecture Notes in Mathematics, 1149 (Springer, Berlin, 1985), 270–282.
[31] H. Werner, ‘A Mal’cev condition for admissible relations’, *Algebra Universalis* **3** (1973), 263.

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