ON THE GRAVITATIONALLY INDUCED SCHWINGER MECHANISM

GUGLIELMO FUCCI
Department of Mathematics, Baylor University, Waco, TX 76798, USA
E-mail: Guglielmo_Fucci@Baylor.edu

IVAN G A VRAMIDI
Department of Mathematics, New Mexico Institute of Mining and Technology Socorro, NM 87801, USA
E-mail: iavramid@nmt.edu

In this paper we will present very recent results obtained in the ambit of quantum electrodynamics in curved spacetime. We utilize a newly developed non-perturbative heat kernel asymptotic expansion on homogeneous Abelian bundles over Riemannian manifolds in order to compute the one-loop effective action for scalar and spinor fields in curved spacetime under the influence of a strong covariantly constant electromagnetic field. In this framework we derived, in particular, the gravitational corrections, up to linear terms in Riemannian curvature, to Schwinger’s result for the creation of particles in a strong electric field.

Keywords: Heat kernel expansion; Effective action; Schwinger mechanism.

1. Introduction

It is generally recognized that the effective action is a tool of fundamental importance in quantum field theory and quantum gravity. In fact, its knowledge allows one to compute the full propagator and the full vertex functions of the quantum theory under consideration and, in turn, the $S$-matrix. Amongst other methods, the effective action can be computed by utilizing the heat kernel approach which was first developed by Schwinger and later generalized to include curved spacetime by De-Witt. In particular Schwinger computed the effective action for constant electromagnetic fields in Minkowski spacetime. He noticed that in presence of an electric field the effective action acquires an imaginary part.
interpreted as creation of pairs induced by the electric field. This effect has been since then known as Schwinger mechanism. From a formal point of view the presence of an imaginary part of the effective action can be understood as follows: the effective action is given in terms of a particular integral, properly regularized, over $t$ of the heat kernel diagonal.\textsuperscript{2,3} In the presence of an electric field the heat kernel diagonal becomes a meromorphic function with isolated single poles on the real axis. These poles are, then, avoided by deforming the contour of integration which leads to an imaginary part given by the sum of the residues of all the poles.\textsuperscript{4}

In this paper we will utilize a newly developed non-perturbative heat kernel asymptotic expansion for homogeneous Abelian bundles in order to obtain the gravitational correction (up to the first order in the Riemannian curvature) to the Schwinger mechanism for a covariantly constant electric field.

2. Non-perturbative Heat Kernel Asymptotic Expansion

Let $(M, g)$ be a $n$-dimensional Riemannian manifold without boundary and $\mathcal{S}$ be a complex vector bundle over $M$ realizing a representation of the group $G \otimes U(1)$, where $G$ is a compact semisimple Lie group. Let $\nabla$ be the total connection on the vector bundle $\mathcal{S}$, $\mathcal{R}_{\mu\nu}$ be the curvature of the $G$-connection and $F_{\mu\nu}$ be the curvature of the $U(1)$-connection (which will be called the electromagnetic field).

Let $U(t; x, x')$ be the heat kernel of the Laplacian $L = -g^{\mu\nu}\nabla_\mu \nabla_\nu$ and $\Theta(t) = U(t; x, x)$ be the heat kernel diagonal. As $t \to 0$ the heat kernel diagonal has a well known asymptotic expansion

$$\Theta(t) \sim (4\pi t)^{-n/2} \sum_{k=0}^{\infty} t^k a_k ,$$  \hspace{1cm} (1)

where $a_k$ are the well known local heat kernel coefficients, which are polynomials in the curvatures (both $R$ and $F$) and their derivatives.

In Ref. 8 we have considered the case in which

$$R \ll F , \quad \nabla \nabla R \ll F^2 , \quad \nabla F = 0 .$$  \hspace{1cm} (2)

In this situation the electromagnetic field cannot be treated as a perturbation and instead of (1) we obtained a new, non-perturbative expansion of the heat kernel diagonal\textsuperscript{8}

$$\Theta(t) \sim (4\pi t)^{-n/2} \sum_{k=0}^{\infty} t^k b_k(t) .$$  \hspace{1cm} (3)
The coefficients $b_k(t)$ are polynomials in the Riemann curvature and its derivatives with the coefficients that are some universal dimensionless tensor-valued analytic functions that depend on $F$ only in the dimensionless combination $t F$. These new non-perturbative coefficients of the heat kernel asymptotic expansion have been computed in Ref. 8 by utilizing a promising algebraic approach developed in Refs. 9–12. The form of the coefficients of the asymptotic expansion for the heat kernel diagonal can be expressed as follows

$$
\begin{align}
  b_2(t) &= \frac{1}{6} R + \Psi^\mu_\nu(t) R^\mu_\nu, \\
  b_3(t) &= 0, \\
  b_4(t) &= -\frac{1}{72} R^2 + \frac{1}{6} R b_2(t) + \Phi^\mu_\nu \Phi^\alpha_\beta \Phi^\gamma_\delta \Phi^\tau_\sigma
  + \Phi^\mu_\nu \Phi^\alpha_\beta \Phi^\gamma_\delta \Phi^\tau_\sigma, \\
\end{align}
$$

where $\Psi(t), \Phi_1(t)$ and $\Phi_2(t)$ are quite involved analytic tensorial functions of $F$ which have been explicitly obtained in Ref. 8.

### 3. One-Loop Effective Action and its Imaginary Part

The heat kernel asymptotic expansion described in the previous section has been used in Ref. 13 in order to evaluate the one-loop effective action, up to linear orders in the Riemannian curvature, for scalar and spinor fields under the influence of a strong covariantly constant electromagnetic field in curved spacetime. In the framework of $\zeta$-function regularization, one can write the one-loop effective Lagrangian in terms of the heat kernel diagonal as follows

$$
\mathcal{L} = -\sigma \int_{\epsilon/\mu^2}^{\infty} \frac{dt}{t} e^{-tm^2} \Theta(t),
$$

where $\sigma = +1$ for bosons and $\sigma = -1$ for fermions, $m$ is the mass of the field and $\epsilon$ and $\mu$ are, respectively, the regularization and renormalization parameters.

As pointed out before, in presence of an electric field the heat kernel diagonal $\Theta(t)$ yields poles on the positive real axis which contribute to the imaginary part of the effective Lagrangian through their residues as

$$
\text{Im } \mathcal{L} = -\sigma \pi \sum_{k=1}^{\infty} \text{Res} \left\{ t^{-1} e^{-tm^2} \Theta(t); t_k \right\}.
$$

By utilizing the spectral decomposition of the electromagnetic 2-form $F$, it was shown in Ref. 13 that the imaginary part of the effective Lagrangian...
for scalar and spinor fields under the influence of solely an electric field $E$ has the form

$$\text{Im } \mathcal{L} = \pi(4\pi)^{-\frac{1}{2}} E \mathcal{P} G_0(y) + \pi(4\pi)^{-\frac{1}{2}} E \mathcal{P}^{-1} \left[ G_1(y) R + G_2(y) \Pi^{\mu\nu} R_{\mu\nu} \\
+ G_4(y) \Pi^{\mu\nu} \Pi^{\alpha\beta} R_{\mu\alpha\nu\beta} + G_5(y) E^{\mu\nu} E^{\rho\beta} R_{\mu\rho\alpha\nu\beta} \right],$$  \hspace{1cm} (8)

where $y = m^2/E$, $E_k$ are antisymmetric matrices satisfying

$$E_{k\mu\nu} = -E_{k\nu\mu}, \quad E^{k}_{[\mu} E^{k}_{\nu]} = 0, \quad E_k E_m = 0, \quad (\text{for } k \neq m) \hspace{1cm} (9)$$

and the projectors $\Pi_k$ are defined by

$$\Pi_k = -E_k^2. \hspace{1cm} (10)$$

One can easily recognize that $G_0$ in the expression (8) represents the original term computed by Schwinger,\(^1\) while the additional functions proportional to the Riemannian curvature have been explicitly evaluated in arbitrary dimensions in Ref. 13 and represent the new gravitational contribution to the Schwinger mechanism.

In the physically relevant case of a four-dimensional spacetime one obtains for scalar fields\(^1\)

$$G_{0,\text{scalar}}(y) = -\frac{1}{\pi^2} \text{Li}_2(-e^{-\pi y}) , \quad G_{1,\text{scalar}}(y) = -\left(\frac{1}{6} - \xi\right) \frac{1}{\pi} \ln(1 + e^{-\pi y}) , \hspace{1cm} (11)$$

$$G_{2,\text{scalar}}(y) = \frac{1}{48\pi^3} \left\{ \frac{2\pi^3 y e^{-\pi y}}{1 + e^{-\pi y}} + 8\pi^2 \ln(1 + e^{-\pi y}) \\
+ 18\pi y \text{Li}_2(-e^{-\pi y}) + 54\text{Li}_3(-e^{-\pi y}) \right\} , \hspace{1cm} (12)$$

$$G_{4,\text{scalar}}(y) = \frac{1}{384\pi^3} \left\{ \frac{16\pi^3 y e^{-\pi y}}{1 + e^{-\pi y}} + 4\pi^2 (17 - 3y^2) \ln(1 + e^{-\pi y}) \\
+ 192\pi y \text{Li}_2(-e^{-\pi y}) + 504\text{Li}_3(-e^{-\pi y}) \right\} , \hspace{1cm} (13)$$

$$G_{5,\text{scalar}}(0, y) = -\frac{1}{256\pi^3} \left\{ 4\pi^2 (1 - 3y^2) \ln(1 + e^{-\pi y}) + 48\pi y \text{Li}_2(-e^{-\pi y}) \\
+ 72\text{Li}_3(-e^{-\pi y}) \right\} , \hspace{1cm} (14)$$

where $\xi$ represents the coupling constant, while for spinor fields one has\(^1\)

$$G_{0,\text{spinor}}(y) = \frac{4}{\pi^2} \text{Li}_2(e^{-\pi y}) , \quad G_{1,\text{spinor}}(y) = \frac{1}{3\pi} \ln(1 - e^{-\pi y}) , \hspace{1cm} (15)$$
\[ G_{\text{spinor}}^{0}(y) = -\frac{1}{12\pi^2} \left\{ \frac{2\pi^3 ye^{-\pi y}}{1 - e^{-\pi y}} - \frac{8\pi^2 \ln(1 - e^{-\pi y})}{-18\pi y \text{Li}_2(e^{-\pi y})} - 54\text{Li}_3(e^{-\pi y}) \right\}, \tag{16} \]

\[ G_{\text{spinor}}^{4}(y) = -\frac{1}{96\pi^3} \left\{ \frac{16\pi^3 ye^{-\pi y}}{1 - e^{-\pi y}} - 4\pi^2(20 - 3y^2) \ln(1 - e^{-\pi y}) - 192\pi y \text{Li}_2(e^{-\pi y}) - 504\text{Li}_3(e^{-\pi y}) \right\}, \tag{17} \]

\[ G_{\text{spinor}}^{5}(y) = \frac{3}{16\pi^3} \left\{ \pi^2(4 + y^2) \ln(1 - e^{-\pi y}) - 4\pi y \text{Li}_2(e^{-\pi y}) - 6\text{Li}_3(e^{-\pi y}) \right\}, \tag{18} \]

where \(\text{Li}_j(z)\) in the above formulas represents the polylogarithmic function.

It is interesting, at this point, to analyze the limit as \(y\) approaches zero, namely the situation in which \(E \gg m^2\). For a scalar field one obtains,\(^{13}\) by taking the limit \(y \to 0\) of the expressions (11)-(14),

\[ G_{\text{scalar}}^{0} = \frac{1}{12}, \quad G_{\text{scalar}}^{1} = -\left(\frac{1}{6} - \xi\right) \frac{1}{\pi} \ln 2, \tag{19} \]

\[ G_{\text{scalar}}^{2} = \frac{1}{6\pi} \ln 2 - \frac{27}{32\pi^3} \zeta(3), \quad G_{\text{scalar}}^{4} = \frac{17}{96\pi} \ln 2 - \frac{63}{64\pi^3} \zeta(3), \tag{20} \]

\[ G_{\text{scalar}}^{5} = -\frac{1}{64\pi} \ln 2 + \frac{27}{128\pi^3} \zeta(3), \tag{21} \]

where \(\zeta(s)\) is the Riemann zeta function. \(G_{i}^{\text{spinor}}(y)\), for \(i \neq 0\), in four dimensions represent a special case since there is an infrared divergence as \(m \to 0\) (or \(y \to 0\)). This means that there is no well-defined value for the small mass limit. Instead, one finds a logarithmic divergence, \(\log(\pi y)\), as follows\(^{13}\)

\[ G_{\text{spinor}}^{0} = \frac{2}{3}, \quad G_{\text{spinor}}^{1}(y) = \frac{1}{3\pi} \log(\pi y) + O(y), \tag{22} \]

\[ G_{\text{spinor}}^{2}(y) = \frac{2}{3\pi} \log(\pi y) - \frac{1}{6\pi} + \frac{9}{2\pi^3} \zeta(3) + O(y), \tag{23} \]

\[ G_{\text{spinor}}^{4}(y) = \frac{5}{6\pi} \log(\pi y) - \frac{1}{6\pi} + \frac{21}{4\pi^3} \zeta(3) + O(y), \tag{24} \]

\[ G_{\text{spinor}}^{5}(y) = \frac{3}{4\pi} \log(\pi y) + \frac{9}{8\pi^3} \zeta(3) + O(y). \tag{25} \]
From the last formulas, one can clearly see that there are new infrared divergences in the imaginary part of the effective action for spinor fields in four dimensions when $E \gg m^2$. This means, physically, that the creation of massless spinor particles (or massive particles in supercritical electric field) is magnified substantially by the presence of the gravitational field.\textsuperscript{13} We would like to mention, here, that similar infrared divergences appear in the real part of the effective Lagrangian for massless spinor fields under the influence of a pure magnetic field. This means that the vacuum energy of charged spinors with small mass (or equivalently massive charged spinors for which $m^2 \ll B$) dramatically increases due to the presence of the gravitational field.\textsuperscript{15}

4. Conclusions

In this paper we have briefly described interesting new developments in the ambit of quantum electrodynamics in arbitrarily curved spacetimes. In particular, by using a new non-perturbative heat kernel asymptotic expansion obtained in Ref. 8, we have found, in Ref. 13, the gravitational corrections to the creation of pairs in a strong electric field. The Schwinger mechanism has, in fact, lately gained importance in light of new experiments proposed in order to observe the creation of electron-positron pairs from vacuum.\textsuperscript{16–18} It would certainly be of interest to specialize the gravitational corrections to the Schwinger mechanism to gravitational backgrounds which are relevant in astrophysical settings. This would be important in order to understand in which situations the gravitationally induced Schwinger mechanism might be observable \textsuperscript{a}. This effect could also have important consequences for theories with spontaneous breakdown of symmetry when the mass of charged particles is generated by a Higgs field. These theories would exhibit an enhancement of created particles (in the massless limit an infinite amount) at the phase transition point when the symmetry is restored and the massive charged particles become massless. This seems to be an interesting new physical effect that, for the reasons mentioned before, deserves further investigation.

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References

1. J. S. Schwinger, *Phys. Rev.* **82**, 664 (1951)
2. B. S. DeWitt, *Dynamical Theory of Groups and Fields* (Gordon and Breach Science Publishers, 1965)
3. B. S. DeWitt, *The Global Approach to Quantum Field Theory* (Oxford University Press, Oxford, 2003)
4. J. S. Schwinger, *Phys. Rev.* **93**, 615 (1954)
5. B. S. DeWitt, *Phys. Rev* **162**, 1195 (1967)
6. B. S. DeWitt, *Phys. Rev* **162**, 1239 (1967)
7. B. S. DeWitt, *Phys. Rep.* **19**, No. 6, 295 (1975)
8. I. G. Avramidi and G. Fucci, *Comm. Math. Phys.* **291**, 543 (2009)
9. I. G. Avramidi, *Phys. Lett. B* **305**, 27 (1993)
10. I. G. Avramidi, *J. Math. Phys.* **36**, 1557 (1995)
11. I. G. Avramidi, *J. Math. Phys.* **36**, 5055 (1995)
12. I. G. Avramidi, *J. Math. Phys.* **37**, 374 (1996)
13. I. G. Avramidi and G. Fucci, *J. Math. Phys.* **50**, 102302 (2009)
14. S. W. Hawking, *Comm. Math. Phys.* **55**, 133 (1977)
15. G. Fucci, *J. Math. Phys.* **50**, 102301 (2009)
16. G. V. Dunne, New Strong-Field QED Effects at ELI: Nonperturbative Vacuum Pair Production, Key Lecture at the *ELI Workshop and School on Fundamental Physics with Ultra-High Fields*, (Frauenworth Monastery, Germany)
17. G. V. Dunne, H. Gies and R. Schtzhold, arXiv 0908.0948 [hep-th]
18. F. Hebenstreit, R. Alkofer, G. V. Dunne and H. Gies, *Phys.Rev.Lett.* **102**, 150404 (2009)