ON ESTIMATION OF INTERNAL STATE BY AN OPTIMAL CONTROL APPROACH FOR ELASTOPLASTIC MATERIAL

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ABSTRACT. After a general formulation of the evolution of an elastoplastic body using duality based on the constitutive behaviour, some classes of inverse problems (estimation of the internal state, determination of an unknown history, ...) for such materials are investigated. A general formulation based on optimal control is proposed, the control variables are related to the internal state. In each class of inverse problem, the solution is obtained by introducing an adjoint state and a suitable cost function.

1. Introduction. Many years ago Bui H. D. proposes in his book [2] some examples based on control optimal theory to solve inverse problems especially for an elastic body, the goal were to determine unknown boundary conditions on a part of the boundary by the knowledge of the displacement and loading on the complementary part of the body boundary. This problem is not well posed in the sense of Hadamard, and can be solved by introducing classical problems and a cost function which is minimum at the solution. Such an approach has been proposed in [8] with extension to elasto-visco-plasticity.

An elasto (visco) plastic body is considered. For given boundary conditions which follows a prescribed history, the internal variables evolves and obeys some requirements. The constitutive law is described by two potentials, the free energy and the pseudo potential of dissipation. For the given history, the solution of the evolution problem satisfies a variational inequality based on a quadratic functional defined in terms of the displacement velocity and rate of internal variables.

The rate boundary value problem is reformulated in a variational form introducing an adjoint state. Then the stationarity of the functional with respect to primal and adjoint variables is equivalent to the system of the local equations which govern the evolution of the system. In this article such a formulation is used to solve particular inverse problems in elasto (visco) plasticity.

In many situations the boundary conditions imposed on a body are not well known on one part Γ_i of the boundary ∂Ω. The first class of inverse problem is to determine these conditions by the knowledge both the fields displacement and traction on the complementary part Γ_o of the boundary. The second class proposes to determine the internal state resulting from an unknown loading history, by the knowledge of both the initial and the final shape of the body, some techniques have
been proposed in [1] to determine residual stresses. The solution proposed next is based on optimal control theory as briefly discussed in [8].

We are interested by estimating the internal state, or by the domain where the plasticity has occurred, and if needed for further investigation by knowing the history of the loading when the constitutive material, the initial shape and the residual shape of the body are given.

In a first section the problem of evolution of an elasto(visco)plastic body is presented to introduce a variational formulation of the rate boundary value problem. Then the inverse problem is presented in a version which is the most general one. The inverse problem is solved among a set of given loading history, for each prescribed history a suitable direct problem is solved. This is a tedious task, because the direct problem is solved many times and in elastoplasticity this resolution is made step by step due to the loading path dependency. For practical applications, we propose another way to obtain an estimation of the internal state. This point of view is developed and analytical solutions are given for particular cases of geometry and loading.

2. The problem of evolution: The primal problem. Let $\Omega$ a domain with external boundary $\partial \Omega$. The body has an elasto(visco)plastic behaviour. Under external loading, the body is deformed. The displacement of the point $x$ is $u(x,t)$. The state of the body is defined by the value of the strain $\varepsilon(u)$ and the internal parameters $\alpha(x,t)$. The local behaviour is defined by a free energy $w(\varepsilon, \alpha)$ which is related to the reversible part of the behaviour

$$w(\varepsilon, \alpha) = \frac{1}{2}(\varepsilon - \alpha) : C(\varepsilon - \alpha) + W(\alpha), \quad \varepsilon(u) = \frac{1}{2}(\nabla^t u + \nabla u)$$

The state equations are given by

$$\sigma = \frac{\partial w}{\partial \varepsilon} = C : (\varepsilon - \alpha), \quad A = -\frac{\partial w}{\partial \alpha}$$

Then the local dissipation of the system is

$$D_m = \sigma : \dot{\varepsilon} - \dot{w} = A.\dot{\alpha}$$

which must be positive. The introduction of a pseudo potential of dissipation $\Phi$ convex function of $A$ ensures this positivity if the normality rule defines the evolution of the internal state.

$$\dot{\alpha} = \frac{\partial \Phi}{\partial A}$$

Introducing the dual potential $d$ associated to $\Phi$, we have

$$d(\dot{\alpha}) + A.(\dot{\alpha} - \beta) \leq d(\beta)$$

then $A \in \partial d(\dot{\alpha})$.

For elastoplastic material. This equation is replaced by the definition of a convex function $f$ associated to the convex domain of reversibility $C$

$$C = \{A^*/f(A^*) \leq 0\}$$

and the evolution is governed by the normality rule which takes now the form

$$\dot{\alpha} = \lambda \frac{\partial f}{\partial A} = \lambda N, \quad f(A) \leq 0, \quad \lambda \geq 0, \quad \lambda f(A) = 0.$$
2.1. **The problem of evolution.** Initially the state is natural i.e

\[ u(x,0) = 0, \quad \alpha(x,0) = 0, \quad \forall x \in \Omega \]

On \( \Gamma_u \) the displacement is prescribed \( u(x,t) = 0 \), on the complementary part \( \Gamma_T \) of the boundary the traction \( T_o(x,t) \) is applied. The solution of the boundary value problem must satisfy the set of local equations

**Compatibility:**

\[ 2\varepsilon(u) = \nabla^T u + \nabla u, \quad \text{over} \ \Omega, \quad u = 0, \quad \text{along} \ \Gamma_u \]

**Equilibrium:**

\[ \text{div} \ \sigma = 0, \quad \text{over} \ \Omega, \quad u, \sigma = T_o \quad \text{along} \ \Gamma_T \]

**Constitutive law:**

\[ \sigma = \frac{\partial w}{\partial \varepsilon} = C : (\varepsilon(u) - \alpha), \quad A = -\frac{\partial w}{\partial \alpha}, \quad \dot{\alpha} = \frac{\partial \Phi}{\partial A} \]

For this behaviour, the displacement \( u \), the local state \( \varepsilon, \alpha \), the stresses \( \sigma \) are functions of position and time.

The evolution is loading path dependent and the solution is obtained step by step during the history of the loading.

2.2. **A variational approach- introduction of an adjoint state.** But for the duration \( t_f \) the primal problem can be formulated in a variational form by introducing the functional \( L \) \[8\]

\[
L = -\int_{t_0}^{t_f} \int_{\Omega} \left( \dot{\varepsilon} : \dot{\sigma} + \dot{\alpha} B^* + \dot{\varepsilon}^* \sigma + \dot{\alpha}^* B \right) \mathrm{d}\Omega \mathrm{d}t + \int_{t_0}^{t_f} \int_{\Gamma_T} T_o u^* \mathrm{d}s \mathrm{d}t + \int_{t_0}^{t_f} \int_{\Omega} \left( A^*(-\dot{\alpha} + \frac{\partial \Phi}{\partial A}) - \alpha^* A \right) \mathrm{d}\Omega \mathrm{d}t
\]

where \( W \) is related to the second derivative of the free energy

\[
\begin{bmatrix}
\dot{\sigma} \\
\dot{\alpha}
\end{bmatrix} = \begin{bmatrix}
\frac{\partial^2 w}{\partial \varepsilon^2} & \frac{\partial^2 w}{\partial \varepsilon \partial \alpha} \\
\frac{\partial^2 w}{\partial \alpha^2} & \frac{\partial^2 w}{\partial \alpha \partial \alpha}
\end{bmatrix} \begin{bmatrix}
\dot{\varepsilon} \\
\dot{\alpha}
\end{bmatrix} = W(\varepsilon, \alpha). \begin{bmatrix}
\dot{\varepsilon} \\
\dot{\alpha}
\end{bmatrix}
\]

\[
(8)
\]

**Theorem 2.1.** The solution \( u(x,t), \alpha(x,t) \) of the problem of evolution satisfies the stationarity of the functional \( L \).

Let us introduce the notations

\[
\begin{bmatrix}
\dot{\sigma} \\
\dot{\alpha}
\end{bmatrix} = W(\varepsilon, \alpha). \begin{bmatrix}
\dot{\varepsilon} \\
\dot{\alpha}
\end{bmatrix} = \begin{bmatrix}
\sigma^* \\
\alpha^*
\end{bmatrix}
\]

\[
(9)
\]

The variations of \( L \) are

\[
\delta L = -\int_{t_0}^{t_f} \int_{\Omega} \left( \delta \dot{\varepsilon} : \sigma^* + \delta \dot{\alpha} B^* + \delta \varepsilon^* \dot{\sigma} + \delta \alpha^* \dot{B} \right) \mathrm{d}\Omega \mathrm{d}t + \int_{t_0}^{t_f} \int_{\Gamma_T} (\delta T_o u^* + \dot{T}_o \delta u^*) \mathrm{d}s \mathrm{d}t + \int_{t_0}^{t_f} \int_{\Omega} \left( \delta A^*(-\dot{\alpha} + \frac{\partial \Phi}{\partial A}) + A^*(-\delta \dot{\alpha} + \frac{\partial^2 \Phi}{\partial A \partial A} \delta A) - \delta \varepsilon^* \dot{A} - \alpha^* \delta \dot{A} \right) \mathrm{d}\Omega \mathrm{d}t
\]
Due to the integration with respect to time $t \in [0,t_f]$, the variations of the functional give the evolution of the mechanical fields and suitable boundary conditions at final time $t_f$.

**Proof.** It is obvious that the variations with respect to $u^*$ ensure that $\dot{\sigma}$ is statically admissible with the prescribed history loading on $\Gamma_T$.

\[
\text{div} \dot{\sigma} = 0, \quad n.\dot{\sigma} = T_o \tag{10}
\]

The variations with respect to $A^*$ imply the evolution of the internal state $\alpha$:

\[
\dot{\alpha} = \frac{\partial \Phi}{\partial A} \tag{11}
\]

The variation with respect to $\alpha^*$ imply the constitutive law:

\[
A^* + B^* = 0, \quad \dot{A}^* = -\frac{\partial^2 \Phi}{\partial A \partial A} A^*, \quad \alpha^*(t_f) = 0. \tag{14}
\]

Conversely the variations with respect to $u, \alpha$ lead to the adjoin problem

**Equilibrium:**

\[
\text{div} \dot{\sigma}^* = 0, \quad n.\dot{\sigma}^* = 0, \text{on } \Gamma_T \tag{12}
\]

**Boundary condition:**

\[
v^*(x,t) = 0, \quad u^*(x,t_f) = 0, \text{along } \Gamma_u \tag{13}
\]

**Constitutive law:**

\[
A^* + B^* = 0, \quad \dot{A}^* = -\frac{\partial^2 \Phi}{\partial A \partial A} A^*, \quad \alpha^*(t_f) = 0. \tag{14}
\]

\[
\begin{bmatrix}
\sigma^* \\
B^*
\end{bmatrix} = \mathcal{W}(\epsilon, \alpha). \begin{bmatrix}
\epsilon^* \\
\alpha^*
\end{bmatrix} \tag{15}
\]

\[
\sigma^*(x,t_f) = C : \epsilon^*(x,t_f), \text{ in } \Omega \tag{16}
\]

To ensure the existence of solution, the potentials $w, \Phi$ must have regular second derivative.

The direct problem is a problem of viscoplasticity and the adjoin problem is viscoelastic.

This formulation is powerful to describe globally the evolution of the system. The stationarity of $\mathcal{L}$ is equivalent to the local equations of the evolution problem.

2.3. **The particular case of an elastoplastic material.** The solution of the rate boundary value problem satisfies a variational inequality based on the functional $(v = \bar{u})$

\[
\mathcal{F} = \int_{\Omega} \frac{1}{2} (\epsilon(v), \lambda N)^t \mathcal{W} \epsilon(v, \lambda N) \, d\Omega + \int_{\Gamma_T} T_o \cdot v \, ds \tag{17}
\]

defined among the set $\mathcal{K}$ of admissible fields $(v, \lambda)$. The plasticity can evolve only on $\Omega_p = \{ x \in \Omega / f(A) = 0 \}$

\[
\mathcal{K} = \left\{ v, \lambda / \begin{array}{c}
v = 0 \text{ over } \Gamma_o \\
\lambda = 0, \text{ over } \Omega/\Omega_p \\
\lambda \geq 0, \text{ over } \Omega_p
\end{array} \right\} \tag{18}
\]

**Theorem 2.2.** The solution $(v, \lambda)$ of the problem of evolution satisfies the variational inequality

\[
\frac{\partial \mathcal{F}}{\partial v} (v - v^*) + \frac{\partial \mathcal{F}}{\partial \lambda} (\lambda - \lambda^*) \leq 0 \tag{19}
\]

among the set of admissible fields $(v^*, \lambda^*) \in \mathcal{K}$.
Due to the condition $\lambda f = 0$ we can conclude that $\dot{f} \leq 0$ and $\lambda \geq 0$ if $\dot{f} = 0$. Then for a state such that $\dot{f} = 0$ the condition on $\lambda$ is given by

$$\forall \beta \geq 0 \text{ over } \Omega_p, \quad (\lambda - \beta) \dot{f} \geq 0$$

the condition is rewritten as

$$(\lambda - \beta)(N : \frac{\partial^2 w}{\partial \alpha \partial \varepsilon} : \dot{\varepsilon} + \frac{\partial^2 w}{\partial \alpha \partial \alpha} \lambda N) \leq 0$$

that is exactly the second term of the variational inequality.

2.4. Application of the functional $L$. The functional $L$ have been used to determine the asymptotic answer of viscoplastic structure under cyclic loading [6, 7]. For cyclic loading and for standard generalized materials, it is known that the local fields $\sigma(x,t), A(x,t)$ tends to be periodic in time [3]. Introducing the gap on periodicity on the generalized stress space $A = (\sigma, A)$ as a quadratic functional $J$

$$J = \int_{\Omega} \frac{1}{2}(A(T) - A(0)).H.(A(T) - A(0)) \, d\Omega$$

The operator $H$ can be given by the complementary energy associated to the free energy $w$. For example, consider a free energy defined by

$$w(\varepsilon, \alpha, \beta) = \frac{1}{2}(\varepsilon - \alpha).C.(\varepsilon - \alpha) + \frac{1}{2} \beta.Z.\beta$$

In this case

$$\sigma = \frac{\partial w}{\partial \varepsilon} = -\frac{\partial w}{\partial \alpha}, \quad A = \frac{\partial w}{\partial \beta}$$

and

$$w^* = \frac{1}{2} A : H : A = \frac{1}{2} \sigma : C^{-1} : \sigma + \frac{1}{2} A.Z^{-1}.A$$

The asymptotic response in time is obtained by a resolution of the problem of minimization based on $J + L$ and the uniqueness is proved [6, 7].

Changing the nature of the cost function permits to develop new optimisation problems and then to solve other class of problem. The next section presents family of cost functions adapted to inverse problems.

3. A class of inverse problems. Consider a body $\Omega$ for which the local behaviour is known. The initial shape of the body is given but after an unknown history of loading the body takes a residual shape.

For the knowledge of the residual geometry, the inverse problem proposes to determine or to give estimate of the internal state and of the plastic strain distribution. We can be simultaneously interested by the determination of a possible loading history which transforms the initial shape to the residual one in a loading - unloading process.

The resolution of such an inverse problem in elastoplasticity is based on some general ideas.

We consider different classes of loading history $T_o(x,t)$ applied on a part of the boundary $\Gamma_o$. For each history, the primal problem can be solved step by step or by the stationarity of the functional $L$. Then among the set of classes of loading history, the optimal history $T_{op}^*(x,t)$ is chosen such that the residual shape issued
from the optimal solution \( u^{op}(x, t_f) \) is close to the measured residual one \( u_o(x) \) over \( \Gamma_o \) : this condition is explained as a minimisation of a cost function \( J \)

\[
J(T_o, \alpha, t_f) = h \int_{\Gamma_o} \frac{1}{2} ||u(t_f) - u_o||^2 \, ds
\]  

(22)

This process gives an estimation of the loading history simultaneously with the determination of the internal state for prescribed condition \( u_o \).

In this problem the control variables are the loading \( T_o(x, t) \) and the internal variables \( \alpha(x, t) \).

If we consider visco-plastic material, the inverse problem can be solved by minimisation of a functional. Find the best history of loading \( T_o(x, t) \) applied on \( \Gamma_o \) such that:

\[
J_f(T_o, \alpha, \alpha^*, u, u^*, A, A^*) = J + L + \int_0^{t_f} \int_{\Gamma_o} \frac{1}{2} HT_o \cdot \dot{T}_o \, ds \, dt
\]  

(23)

The variations with respect to the arguments define the local behaviour for the direct problem and the adjoin state.

The constant \( h \) and \( H \) are positive. The \( H \) terms ensure regularity on \( T_o \).

The conditions of optimality is defined by the relation with respect to \( u(t_f) \) and \( T_o \) on the boundary \( \Gamma_o \). These conditions are defined on \( \Gamma_o \) by

\[
0 = v^* + HT_o
\]

\[
0 = n \cdot \sigma^*(x, t_f) - h(u(x, t_f) - u_o)
\]

The first condition is the boundary condition applied during the time for the adjoin displacement as a function of the traction applied at optimality; the second one is the boundary condition on the traction associated to the adjoin state.

Due to the dependency with respect to time in viscoplasticity, the solution depends on the duration \( t_f \) then a minimization on this parameter is also performed.

To determine the optimal loading, the problem of evolution must be solved for each choice of \( T_o(x, t) \), this is a tedious task. It is the reason why another way is now proposed.

4. Estimation of the internal state in elastoplasticity. Consider a body \( \Omega \).

The body is submitted to an increasing loading, the history of which is not known. Assume that the final shape \( u_o(x) \) is known simultaneously with the final loading \( T_o \) on the boundary \( \Gamma_o \).

The internal variables \( \alpha \) are now the plastic strain \( \varepsilon_p \) for sake of simplicity. At the final state, the plastic strain is \( \varepsilon_p \), the stress satisfies the constitutive law:

\[
\sigma = C : (\varepsilon - \varepsilon_p), \quad f(\sigma, \varepsilon_p) \leq 0, \quad \text{Tr} \varepsilon_p = 0
\]  

(24)

where the function \( f(\sigma, \varepsilon_p) = f(A) \) is convex. In general, \( f \) is only function of the deviator of \( A \), then the plastic strain is isochoric \( \text{Tr} \varepsilon_p = 0 \).

The stresses are in equilibrium with the given boundary condition at final state

\[
\text{div} \sigma = 0, \quad \text{over} \quad \Omega_p, \quad n \cdot \sigma = T_o \quad \text{along} \quad \Gamma_o
\]  

(25)

To estimate the internal state from the knowing of the residual shape, we can solve a problem of linear elasticity controlled by \( \varepsilon_p \) : find the displacement \( u \) such
that

\[ 2\epsilon(u) = \nabla u + \nabla' u \]
\[ \sigma = C : (\epsilon - \epsilon_p), \quad f(\sigma, \epsilon_p) \leq 0 \]
\[ n.\sigma = T_n, \quad \text{along } \Gamma_o \]
\[ u = 0, \quad \text{along } \Gamma_u \]

The best estimate of \( \epsilon_p \) is such that the displacement solution \( u_{sol}(T_o, \epsilon_p) \) is close to the known displacement \( u_o \) along \( \Gamma_o \).

To obtain a solution close to a problem of elastoplasticity, we decompose the body into two domains, one where \( \epsilon_p = 0 \) and a plastic zone \( \Omega_p \) where the stresses are in a domain close to the domain of reversibility. These constraints are incorporated into the functional to minimize

\[
J(\epsilon_p) = \frac{1}{2} \int_{\Gamma_o} h||u - u_o||^2 d\Omega + \frac{\alpha}{2} \int_{\Gamma_p} (\text{Tr } \epsilon_p)^2 d\Omega + \frac{\beta}{2} \int_{\Gamma_p} (f(\sigma, \epsilon_p))^2 d\Omega
\]

where the stresses \( \sigma \) are solutions of the problem of equilibrium satisfying the constitutive law. The second term relaxes the condition of traceless of \( \epsilon_p \). It can be omitted if we consider isochoric plasticity.

To satisfy the problem of equilibrium, the adjoin displacement \( u^* \) is introduced taking account of the local behaviour

\[
\int_{\Omega} \nabla u^* : C : (\nabla u - \epsilon_p) d\Omega - \int_{\Gamma_o} u^*.T_o ds = 0, \forall u^* \text{ with } u^* = 0 \text{ over } \Gamma_u
\]

The yielding function \( f(\sigma, \epsilon_p) \) is rewritten in terms of the local strains

\[ f(\sigma(\epsilon(u), \epsilon_p), \epsilon_p) = Y(\epsilon(u), \epsilon_p) \]

The displacement \( u \) and \( u^* \) are continuous along the boundary of the plastic zone.

\[ ||u|| = 0, \quad ||u^*|| = 0, \quad \text{over } \Gamma_p \]

where \( ||f|| = f^+ - f^- \) is the jump of the function \( f \) across \( \Gamma_p \) oriented from \( \Omega_p \), and we note \( 2\bar{f} = f^+ + f^- \).

Finally the functional to optimize among a set of admissible fields \( u, u^*, \epsilon_p \) is reduced to

\[
\mathcal{J}(u, u^*, \epsilon_p, \Omega_p) = \frac{h}{2} \int_{\Gamma_o} ||u - u_o||^2 d\Omega + \frac{\alpha}{2} \int_{\Omega_p} (\text{Tr } \epsilon_p)^2 d\Omega + \frac{\beta}{2} \int_{\Omega_p} (Y(\epsilon(u), \epsilon_p))^2 d\Omega
\]

\[ - \int_{\Omega} \nabla u^* : C : (\nabla u - \epsilon_p) d\Omega + \int_{\Gamma_o} u^*.T_o ds
\]

The optimality conditions are given by the variations of \( \mathcal{J} \). It can be noticed that the plastic domain \( \Omega_p \) is not known a priori.

The displacements \( u \) and \( u^* \) being continuous on the boundary \( \Gamma_p \) of \( \Omega_p \), then their variations satisfy Hadamard compatibility relations. When the position of \( \Gamma_p \) is moved with a normal velocity \( \delta\phi(s) \), the variations of the displacements satisfy the jump condition

\[
||[\delta u] + \delta\phi||[\nabla u]|.n = 0, \quad ||[\delta u^*] + \delta\phi||[\nabla u^*]|.n = 0
\]
4.1. The variations of $J$.

- The variations with respect to $\mathbf{u}^*$ provide equations of local behaviour

\[
\sigma = \mathcal{C} : (\mathbf{\varepsilon}(\mathbf{u}) - \mathbf{\varepsilon}_p),
\]
\[
\text{div} \sigma = 0,
\]
\[
\mathbf{n} \cdot \sigma = \mathbf{T}_o, \text{ over } \Gamma_o
\]
\[
\mathbf{n} \cdot |\sigma| = 0, \text{ over } \Gamma_p
\]

- The variations with respect to $\mathbf{u}$ determine the equations for the adjoin state

\[
\sigma^*_- = \mathcal{C} : \mathbf{\varepsilon}(\mathbf{u}^*)_-, \text{ over } \Omega_p
\]
\[
\sigma^*_+ = \mathcal{C} : \mathbf{\varepsilon}(\mathbf{u}^*)_+, \text{ over } \Omega_0
\]
\[
\text{div} \sigma^* = 0
\]
\[
\mathbf{n} \cdot \sigma^* = h(\mathbf{u} - \mathbf{u}_o), \text{ over } \Gamma_o
\]
\[
\mathbf{n} \cdot |\sigma^*| = 0, \quad ||\mathbf{u}^*|| = 0, \text{ over } \Gamma_p
\]

- The condition of optimality with respect to $\varepsilon_p$ determines the plastic strain

\[
\mathcal{C} : \nabla \mathbf{u}^* + \alpha \text{Tr} \varepsilon_p \mathbf{I} + \beta Y \frac{\partial Y}{\partial \varepsilon_p} = 0 \quad (29)
\]

where $\mathbf{I}$ is the identity.

- The variations with respect to the motion of $\Gamma_p$ represents a conservation of energy

\[
- ||\mathbf{\varepsilon}(\mathbf{u}^*) : \sigma|| + \sigma : ||\nabla \mathbf{u}^*|| + \sigma^*_- : ||\nabla \mathbf{u}|| + \frac{\beta}{2} Y^2 = 0 \quad (30)
\]

Discontinuities have general properties, they satisfy the relations

\[
||f \mathbf{g}|| = ||f|| \mathbf{g} + \mathbf{f} ||\mathbf{g}||
\]

then

\[
||\mathbf{\varepsilon}(\mathbf{u}^*) : \sigma|| = \sigma : ||\nabla \mathbf{u}^*|| + ||\sigma||. ||\nabla \mathbf{u}^*||
\]

as $\mathbf{u}^*$ is continuous then $||\nabla \mathbf{u}^*|| = \mathbf{v}^* \otimes \mathbf{n}$ and due to the continuity of $\sigma \mathbf{n}$ the variation with respect to the motion of $\Gamma_p$ is simplified to

\[
- ||\sigma|| : \nabla \mathbf{u}^* + \sigma^*_- : ||\nabla \mathbf{u}|| + \frac{\beta}{2} Y^2 + \frac{\alpha}{2} (\text{Tr} \varepsilon_p)^2 = 0 \quad (31)
\]

This optimization leads to the determination of the plastic zone with an evaluation of the plastic strain. The quality of the solution depends on the choice of the constant $\alpha, \beta, r$. However, for an increasing loading process, the final state is given by a distribution of $\varepsilon_p$ such that

\[
\varepsilon_p = 0, \quad f = 0 \text{ along the boundary } \Gamma_p
\]

It can be noticed that for the exact solution $Y = 0$, then $\sigma^* = 0, \mathbf{u}^* = 0$. Then the exact solution ensures that $J = 0$.

The ability of the minimisation to rebuild the internal state inside a body is now investigated on analytical solution for elastoplastic materials on a hollow sphere.

An analytical solution gives the opportunity to know the sensibility of the solution to the boundary conditions, when the measurements of these quantities are noised.
5. On the hollow sphere in elastoplasticity.

5.1. Exact solution under radial loading. The solution is well known for an elasto perfectly plastic material. The external radius of the sphere is \( R_e \), the radius of the void is \( R_i \) then the porosity is \( c = R_i^3 / R_e^3 \). For an increasing radial loading, plasticity expands from the radius \( R_i \) to the radius \( R_p \). The plastic zone \( \Omega_p \) is then spherical. The volume fraction of plastic domain \( p = R_p^3 / R_e^3 \). The displacement solution is radial, \( u = u(r)e_r \), and the cauchy stress is searched as \( \sigma = \sigma_r e_r \otimes e_r + \sigma_t (e_\theta \otimes e_\theta + e_\phi \otimes e_\phi) \)

The domain of reversibility is determined by the yield function \( f(\sigma) = \sigma_t - \sigma_r - \sigma_o \leq 0 \) where \( \sigma_o \) is a constant. The plastic strain is isochoric and \( \epsilon_p(r) = \epsilon_p(r)(2e_r \otimes e_r - e_\theta \otimes e_\theta - e_\phi \otimes e_\phi) \)

After some calculations, the plastic strain and the stresses inside the plastic zone are

\[
\begin{align*}
\epsilon_p(r) & = \frac{3\kappa + 4\mu}{18\kappa\mu} \sigma_o (1 - \frac{R_p^3}{r^3}) \\
\sigma_r & = 2\sigma_o \ln \frac{r}{R_i} \\
\sigma_t & = \sigma_r + \sigma_o
\end{align*}
\]

The global response of the elastoplastic sphere is obtained as

\[
\begin{align*}
\Sigma(R_p) & = \sigma_r(R_e) = \frac{2}{3} \sigma_o (1 - p + \ln \frac{P}{c}) \quad (32) \\
E(R_p) & = \frac{u(R_e)}{R_e} = \frac{2}{3} \sigma_o \left( \frac{P}{4\mu} + \frac{1}{3\kappa} (1 + \ln \frac{P}{c}) \right) \quad (33)
\end{align*}
\]

and the residual shape after unloading \( (E - 3K(c)\Sigma) \) is defined by a plastic strain \( E_{ir}(R_p) = \frac{3\kappa + 4\mu}{18\kappa\mu} \sigma_o \left( \frac{1}{1 - c} (p - c - c \ln \frac{P}{c}) \right) \quad (34) \)

the global modulus of elasticity being \( 3K(c) = 12\kappa\mu(1 - c)/(4\mu + 3\kappa) \).

Along the boundary \( \Gamma_p \), the traction and the displacement are given by

\[
\Sigma_p = \sigma_r(R_p) = \frac{2}{3} \sigma_o \ln \frac{P}{c}, \quad E_p = \frac{u(R_p)}{R_p} = E - \frac{\sigma_o}{6\mu} (1 - p)
\]

5.2. The inverse problem. We consider the hollow sphere, at the initial state the sphere has external radius \( R_e \) at the final state the loading is \( \Sigma_m \) and the shape is given by \( E_m = u_m(R_e)/R_e \). The fields depend only upon \( r \) and \( \epsilon_p \) is assumed isochoric. Under these assumptions the functional to minimize is defined by

\[
\mathcal{J}(u, u^*, \epsilon_p, R_p) = \frac{1}{2} hR_e^3 \left( \frac{u(R_e)}{R_e} - E_m \right)^2 + \frac{\beta}{2} \int_{R_i}^{R_p} Y^2 r^2 dr
\]

\[
- \int_{R_i}^{R_p} \nabla u^* : C : (\nabla u - \epsilon_p)^2 dr + R_e^2 \Sigma_m u^*(R_e)
\]

where the plastic strain vanishes \( (\epsilon_p = 0) \) in the domain \( [R_p, R_e] \).

The function \( Y \) is given by

\[
Y = 2\mu (\frac{u}{r} - \frac{du}{dr}) + 6\mu \epsilon_p(r) - \sigma_o \quad (35)
\]
The variations $\delta J$ are now expressed.

- Variations with respect to $\varepsilon_p$
  \[ 4\mu \left( \frac{d u^*}{dr} - \frac{u^*}{r} \right) - 6\mu \beta Y = 0, \quad \forall R_i \leq r \leq R_p \] (36)

- Variations with respect to $u(r)$
  \[ \sigma^* = C : \varepsilon(u^*) \] (37)
  \[ 0 = \frac{d \sigma^*}{dr} + \frac{2}{r} (\sigma^*_r - \sigma^*_t), \quad r \geq R_p \] (38)
  \[ 0 = -2\mu \beta \left( \frac{dY}{dr} + \frac{3Y}{r} \right) + \frac{d\sigma^*}{dr} + \frac{2}{r} (\sigma^*_r - \sigma^*_t), \quad R_i \leq r \leq R_p \] (39)
  \[ \sigma^*(R_e) = h \left( \frac{u(R_e)}{R_e} - E_m \right) \] (40)
  \[ 0 = \sigma^*_r(R_i) + 2\mu \beta Y(R_i) \] (41)
  \[ 0 = |[\sigma^*_r]| - 2\mu \beta Y(R_p) \] (42)

- Variations with respect to $u^*(r)$
  \[ \sigma = C : (\varepsilon(u) - \varepsilon_p), \quad \varepsilon_p = 0, R_p \leq r \leq R_e \] (43)
  \[ 0 = \frac{d\sigma_r}{dr} + \frac{2}{r} (\sigma_r - \sigma_t), \] (44)
  \[ \Sigma_m = \sigma_r(R_e) \] (45)
  \[ 0 = \sigma_r(R_i) \] (46)
  \[ 0 = |[\sigma_r]|(R_p) \] (47)

- Variations with respect to $R_p$, at point $R_p$ we have
  \[ \frac{\beta}{2} Y^2 + \sigma^* : \nabla u^*_+ - \sigma^- : \nabla u^*_+ + \sigma^*_r(R_p) - |\left[ \frac{du}{dr} \right]| - \sigma_t| \left[ \frac{du^*}{dr} \right] = 0 \]

  which is reduced to
  \[ \frac{\beta}{2} Y^2 - 2\sigma_t |u^*(R_p)| - \sigma^*_r(R_p) - |[\frac{du}{dr}]| = 0. \] (48)

5.3. **Form of the solution.** The expression of the adjoin displacement $u^*$ by elimination of $Y$ between 36,39 is obtained

\[
\begin{cases}
    r \geq R_p, \ u^* = a_e r + \frac{R_p^3}{r^2} b_e \\
    r \leq R_p, \ u^* = a_p r + \frac{R_p^3}{r^2} b_p 
\end{cases}
\]

Then the value of $Y$ is determined as

\[ Y = -\frac{2}{3\beta^3} \frac{d}{dr} \left( \frac{u^*}{r} \right) = \frac{2R_p^3}{r^3} b_p = \sigma_t - \sigma_r - \sigma_o \]

then the equilibrium equation can be integrated and the value of $\sigma_r$ in the plastic domain is

\[ \sigma_r = 2\sigma_o \ln \frac{r}{R_i} - 4b_p \frac{R_p^3}{3} \left( \frac{1}{r^3} - \frac{1}{R_i^3} \right) \] (49)

In the plastic domain the stress is the primal one with an extra contribution which has the form of an elastic contribution.
Therefore $\sigma_t$ is evaluated from $Y$ and as $3\kappa Tr(\varepsilon) = \sigma_r + 2\sigma_t$ the displacement in $\Omega_p$ is determined

$$3\kappa(r^2 u - R_i^2 u_i) = 2\sigma_o r^3 \log \frac{r}{R_i} + \frac{4b_p R_i^3}{3R_i^3} (r^3 - R_i^3)$$

and along $r = R_p$ we obtain:

$$3\kappa(p_u - c u_i) = \frac{2\sigma_o}{3} p \log \frac{p}{c} + (p - c) \frac{4b_p R_p^3}{3R_p^3}$$

In the domain $R_p \leq r \leq R_e$, the displacement $u(r)$ is $ar + bR^3_p/r^2$. Then the problem of optimisation is reduced to the system

$$\sigma_r(R_e) = 3\kappa a - 4\mu b p = \Sigma_m$$
$$\sigma_r^*(R_e) = 3\kappa a - 4\mu b c = hR_e(a + bp - E_m)$$
$$a + b = \frac{u_p}{R_p}$$
$$a_c + b_c = a_p + b_p$$
$$3\kappa a_p - 4\mu b_p p/c = -2\mu Y(R_i) = -\frac{4\mu p}{c} b_p$$
$$3\kappa(a_c - a_p) - 4\mu(b_c - b_p) = 4\mu b_p$$
$$3\kappa a - 4\mu b = 2\frac{\sigma_o}{3} p \log \frac{p}{c} + \frac{4}{3} (1 - \frac{p}{c})$$

For given $R_p$, this is a system of seven equations for seven unknown. At the solution for given $R_p$, the cost function at the solution is reduced to

$$J = \frac{1}{2} hR_e^2 (a + bp - E_m)^2 + \int_{R_i}^{R_e} Y^2 r^2 dr$$

After some calculations, we obtain $a_p = 0$ and with $\frac{4}{3}(1 - \frac{p}{c}) = K$

$$\Sigma_p = \frac{2}{3} \sigma_o \ln \frac{p}{c}$$
$$3\kappa(1 - p) = \Sigma_m - p\Sigma_p + Kpb_p$$
$$4\mu b(1 - p) = \Sigma_m - \Sigma_p + Kbp$$
$$a + bp = \frac{\Sigma_m}{3\kappa(1 + \frac{3\kappa}{4\mu})} - p\Sigma_p \frac{3\kappa + 4\mu}{12\mu \kappa} + Kbp \left( \frac{1}{3\kappa} + \frac{1}{4\mu} \right)$$

As $3\kappa a_c = 4\mu b_c$ the boundary condition for the adjoin stress is rewritten as

$$4\mu b_c(1 - p) = hR_e(a + bp - E_m), b_p = a_c + b_c = a_c \frac{3\kappa + 4\mu}{4\mu}$$

Using the values

$$E_m = E(R_p) + \hat{E}, \quad \Sigma_m = \Sigma(R_p) + \hat{\Sigma}, L = \frac{1}{3\kappa} + \frac{1}{4\mu}$$

$b_p$ is obtained

$$hR_e \left( \frac{\Sigma_0}{3\kappa} + \frac{p}{4\mu} \right) + (p - 1) \hat{E} = b_p \left( \frac{1}{L} - hR_e p K \right)$$

This ensures uniqueness when $R_p$ is known. If $\hat{E} = 0$ and $\hat{\Sigma} = 0$, the value $R_p$ determined by $\Sigma(R_p)$ is unique, and the cost function vanishes $J = 0$.

The variations with respect to $R_p$ is also satisfied.
Then we conclude that if \((E_m, \Sigma_m)\) are related to some \(R_p\), which is equivalent to a couple \((E(R_p), \Sigma(R_p))\) \(^{32,33}\); the solution of the direct problem is recovered, otherwise a value of \(R_p\) differs from an exact value.

The condition obtained by the variation with respect to \(R_p\) is an equation on \(R_p\) or on \(p\) which is complex. Multiple solutions can be obtained depending on the noise \(\hat{E}, \hat{\Sigma}\). A valuable solution must satisfy that \(Y \leq 0\) in the domain \(R_p \leq r \leq R_e\). It is noticed that the solution is unique for the exact value when the noise vanishes.

5.4. Other boundary conditions. In the last example, the state of traction and the shape at the final time will be given at the maximum loading.

Assume now that the residual shape \(E_{ir}\) is given, at this state the traction vanishes. The goal is to determine the internal state for this situation.

The idea is to determine a loading unloading path such that the residual shape at the final state is close to the residual one. Denoting \(\Sigma_m\) the maximum of the traction, for which the strain is \(E_m\), the residual strain is given by \(E_{ir} = E(\Sigma_m) - 3K(c)\Sigma_m\) and the cost function becomes

\[
\mathcal{J}(u, u^*, \epsilon_p, R_p, \Sigma_m) = \frac{1}{2} hR_e^3 \left( \frac{u(R_e)}{R_e} - 3K(c)\Sigma_m - E_{ir} \right)^2 + \frac{\beta}{2} \int_{R_i}^{R_e} Y^2 r^2 dr \\
- \int_{R_i}^{R_e} \nabla u^* : \mathbb{C} : (\nabla u - \epsilon_p) r^2 dr + R_e^2 \Sigma_m u^*(R_e)
\]

The variations with respect to \(\Sigma_m\) determine the maximum of traction to apply in order to obtain the residual shape \(E_{ir}\). Due to preceding section, it is clear that for \(E_{ir}(R_p)\) and \(\Sigma_m = \Sigma(R_p)\) the cost function vanishes and the internal state is the exact one. The uniqueness is obvious due to the property of the condition of variation with respect to \(R_p\).

6. Conclusion. We have presented a general method for solving inverse problem in viscoplasticity. The method is based on a variational formulation of the evolution of the system using an adjoint state and a cost function which is related to gap between displacement solution at a given load and the corresponding measurement. Then the solution is obtained as a problem of optimisation in the framework of control theory, the internal state and the loading history playing the role of control variables.

For estimation of the internal state a simplified model is proposed, the domain \(\Omega_p\) where the plasticity occurs becomes a new control variable. The yielding function is approximate on this domain. The applicability has been shown on an analytical case especially to recover the residual shape after an unknown loading.

In the two last examples, we have taken into account the same form of \(\epsilon_p\) than the solution of the direct problem of elastoplasticity. Assuming that for example \(\epsilon_p\) is uniform gives approximation of the domain of plasticity. This should be investigate in another paper.

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