Unitarity, Lorentz invariance and causality in Lee-Wick theories:
An asymptotically safe completion of QED

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Abstract

We revisit the previously unsolved problems of ensuring Lorentz invariance and non-perturbative unitarity in Lee-Wick theories. We base our discussion on an ultraviolet completion of QED by Lee-Wick ghost fields, which is argued to be asymptotically safe. We argue that as long as the state space is based upon a suitable choice of distributions of a type invented by Gel’fand and Shilov, the Lee-Wick ghosts can be eliminated while preserving Lorentz invariance to produce a unitary theory. The method for eliminating ghosts is in principle non-perturbatively well-defined, in contrast with some previous proposals. We also point out a second, independent mechanism for producing a unitary theory, based on a covariant constraint on the maximum four-momentum, which would imply an amusing connection, based on naturalness, between the coupling constant and the hierarchy of scales in the theory. We further emphasize that the resulting theory is causal, and point out some analogies between between the behaviour of Lee-Wick ghost degrees of freedom and black holes.
1 Introduction

We argue that there is justification for reconsidering quantum field theories of Lee-Wick type as potentially realistic fundamental or effective theories of the world. The original justification for these theories, which remains as valid as ever, was their good ultraviolet behaviour. We believe that the methods expounded in this paper can be applied to solve the problems related to unitarity, non-perturbative definition, and Lorentz invariance identified by previous authors.

In the sixties, T.D. Lee and G.C. Wick initiated a program to extract ultraviolet complete unitary quantum field theories embedded in certain larger indefinite inner product field theories [1, 2, 3]. These authors proposed extending certain quantum field theories by adding ghost degrees of freedom, similar to the familiar Pauli-Villars regulator fields [4], to make the theory finite by cancelling divergences. Unlike Pauli-Villars ghosts, however, the Lee-Wick ghost masses were to be kept finite. Lee and Wick argued that, as long as all ghost degrees of freedom in the interacting theory could be shown to have complex energies, one could obtain a unitary theory by constraining the a physical subspace to be exactly those states that have real energy.

However, this program ran into two serious obstacles that were never satisfactorily resolved. We propose to resolve both of these problems in the present article.

First, it was quickly realized that in quantum field theories with the required complex mass ghosts, one would expect multi-ghost states that have real energy [3]. It seemed that these states could not be eliminated by the Lee-Wick real energy constraint, and other arguments would be needed to get rid of them. Ad hoc prescriptions to do so order by order in perturbation theory were proposed by Lee [3], as well as by Cutkosky et al. [5], but no unambiguous all-order prescription was ever found and, as was argued among others by Boulware and Gross [6] and by Nakanishi [7, 8], it is questionable whether these prescriptions have any non-perturbative meaning.

Second, as was shown by Nakanishi and Gleeson et al. [7, 9], a Hamiltonian-based non-perturbatively well-defined approach to eliminating multi-ghost states seemed doomed to fail if we insist on Lorentz-invariance. In particular, there did not seem to be a Lorentz-invariant way of applying the real-energy constraint in a Hamiltonian formulation of the theory.

We will show first that the problem with Lorentz invariance can be solved in a non-perturbative setting by carefully constructing the multi-ghost state
spaces so that they are Lorentz-invariant. The problem with Lorentz invariance identified by Nakanishi and Gleeson et al. can be traced to their use of state spaces consisting of degrees of freedom with complex energies but real momenta. Such a description is obviously not Lorentz invariant. A Lorentz-invariant construction has to address the problem of complex momenta. Fortunately, a framework that can be used for such states exists in the form of a class of spaces of generalized functions based on analytic test functions that was invented by Gel’fand and Shilov \([10, 11]\). The construction of the required states is quite subtle, but we argue that it resolves this part of the problem.

The resulting sets of multi-ghost states will be Lorentz-invariant. However, at first glance, their energies would seem to be real, so that naively they would not seem to be eliminated by the Lee-Wick real energy constraint. However, we will show that once we take into account interactions in a careful non-perturbatively well-defined approach, these energies will generally become complex, and as long as we apply the Lee-Wick real energy constraint and the infinite-volume limit in the correct order, these states will disappear from the physical spectrum, and the resulting theory can be expected to be unitary.

We illustrate these methods with a simple extension of QED with high-mass ghost particles that we argue to be asymptotically safe \([12]\). By this we mean that the fine structure constant remains small at all energies and approaches a fixed point at infinite energy. Our proposal differs from Lee’s in that the theory is only finite after mass renormalizations. Still, unlike ordinary QED, which is not believed to exist up to arbitrarily high energies, our asymptotically safe extension should be an ultraviolet complete theory. In particular, in a non-perturbative construction relying on some discretization, asymptotic safety implies that the theory should have a continuum limit, which is a prerequisite for exact Lorentz invariance. We argue that all ghosts can be eliminated in a Lorentz-invariant and non-perturbatively well-defined way.

We also discuss a second (poor man’s) method for getting rid of ghost states. A universe with finite mass-energy could perfectly well be described by an ultraviolet-complete theory containing ghosts, as long as the mass-energy of the universe is smaller than that of any real-energy ghost states. In other words, as long as the ghost masses are large enough, the subspace of the state space available to the universe, or any subsystem of the universe, is positive-definite, and the description of the world will be unitary. Taking
this idea a little further, we point out an interesting connection, based on a naturalness argument, between the large mass-energy of the universe and the smallness of the fine structure constant.

We also discuss the somewhat misleadingly named “acausality” of these theories, and emphasize, as did previous authors [3, 6, 13, 25], that no inconsistencies can arise. We discuss, by way of example, how a particular form of the grandfather paradox is avoided. We propose that it may be less confusing to think of these theories as non-local, rather than acausal, at sufficiently high energy scales. Initial states typically contain precursors (as they were called by Lee [3]) beyond the limits of experimental precision, that get exponentially magnified by the time evolution and may only become observable at later times. For the reader uncomfortable with this aspect of these theories, we point out that these precursors are morally not so different from the precursors invoked to explain aspects of the bulk-boundary correspondence in String Theory holography [14, 15]. We further point out that Horowitz and Maldacena recently proposed resolving the black hole information paradox via a final state boundary condition [16]. Again, this is not so alien in the context of Lee-Wick theories, since the Lee-Wick real energy constraint can be reinterpreted as incorporating a final state boundary condition, namely, no blow-up at future infinity [6]. In the work of Horowitz and Maldacena, the black hole final state boundary condition ensured that everything falling into the black hole annihilated completely, leaving nothing behind. But as we shall see, this is analogous to what happens in Lee-Wick theories in typical scattering processes mediated by ghosts, where the incoming particles annihilate to form a null state that decays exponentially in the future.

It is worth noting that, while our approach is an adaptation of Lee and Wick’s original proposal for eliminating ghost states, recent work of Bender et al. [17] propose an alternative interpretation of these theories based on PT-symmetry. Their approach differs from ours, which does not rely on PT-symmetry.

Given our proposed resolutions to some prior foundational difficulties encountered in Lee-Wick theories, we believe that these theories deserve another look in the search for descriptions of nature. Indeed, Lee-Wick theories have recently enjoyed a revival in the form of the Lee-Wick Standard Model [18, 19, 20, 21, 22, 23, 24, 25], proposed to address, among other things, the hierarchy problem and the stability of the Higgs mass. Hopefully the results of this paper can be generalized to provide solutions to the problems of non-perturbative unitarity and Lorentz invariance in these theories.
In the light of our results, it may also be productive to revisit renormalizable higher-derivative modifications of gravity, which contain Lee-Wick degrees of freedom \[26\].

## 2 Ghost quantum electrodynamics

We will show that quantum electrodynamics may be completed by the addition of a ghost field to obtain a theory that will be argued to be asymptotically safe. The construction is in principle non-perturbatively well-defined, and will be argued in subsequent sections to satisfy the properties of Lorentz covariance and unitarity necessary for a consistent physical interpretation, provided certain constraints are imposed on the state space to effectively eliminate the ghosts.

Consider the Lagrangian

\[
\mathcal{L} = \sum_{i=0}^{1} \bar{\psi}_i \left( i \left( \partial - iA \right) - m_i \right) \psi_i - \frac{1}{4e^2} F_{\mu\nu}F^{\mu\nu} - \frac{\lambda}{2e^2} \left( \partial_\mu A^\mu \right)^2 + \cdots
\]

\[
= \sum_{i=0}^{1} Z_{i\Lambda} \bar{\psi}_i \left( i \left( \partial - iA \right) - m_{i\Lambda} \right) \psi_i - \frac{1}{4e^2_\Lambda} F_{\mu\nu}F^{\mu\nu} - \frac{\lambda_\Lambda}{2e^2_\Lambda} \left( \partial_\mu A^\mu \right)^2. \tag{1}
\]

Here \(\psi_0\) is an ordinary fermion, while \(\psi_1\) is a Dirac boson – a particle with Dirac action that is quantized using bosonic statistics \[27\]. A theory containing such a field would ordinarily violate unitarity, be unstable, or both. These problems will be avoided later by constraining the state space.

The first line shows the physical fields, masses, and renormalized charge, defined as usual in terms of the positions and residues of poles of the full propagators. The second line shows the full bare action, whose form is constrained by gauge invariance, in terms of the bare masses, field renormalizations and charge at regularization scale \(\Lambda\). The dots on the first line stand for the counter-terms that make up the difference. In electrodynamics, the gauge fixing term does not get renormalized, so \(\lambda/e^2 = \lambda_\Lambda/e^2_\Lambda\).

In contrast to the situation in QED, we shall argue that, under certain conditions, the theory is asymptotically safe and there is no obstruction to taking the limit \(\Lambda \to \infty\).

This type of action is not entirely new. Bosonic fields of Dirac type have been used for many years as Pauli-Villars regulator fields. However, whereas Pauli-Villars regulator masses are taken to infinity as part of renormalization,
we keep all masses finite. A similar, though not identical, proposal was made by Lee with the aim of making QED finite [2]. To achieve finiteness, Lee also included a massive counterpart to the photon. Since we do not include a massive vector particle, our theory will be finite only once we perform mass renormalizations. In addition, whereas Lee’s extra Dirac fields were given complex masses, all masses appearing in our action will be assumed real.

3 Dirac bosons

Dirac bosons are discussed in [27]. A Dirac boson can be quantized in a way that satisfies micro-causality (locality) using either a positive-definite or an indefinite state space inner product.

Choosing the positive-definite representation would lead directly to an unstable theory in which the energy is unbounded below. We will therefore consider instead the indefinite representation, in which the unperturbed energies are nonnegative.

\[
\psi(x) = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_s \left( a^s_p \eta^s(p) e^{-ip \cdot x} + b^s_p \eta^s(p) e^{ip \cdot x} \right),
\]

where

\[
[a^s_p, a^{s'}_{q}] = (2\pi)^3 \delta^3(p - q) \delta^{rs},
\]

\[
[b^s_p, b^{s'}_{q}] = -(2\pi)^3 \delta^3(p - q) \delta^{rs},
\]

\[a^s_p |0\rangle = 0,\]

\[b^s_p |0\rangle = 0.\]

In this representation, any state containing an odd number of the bosonic \(b\)-anti-particles has negative metric. The propagator is the same as for a Dirac fermion, and is given by

\[
\frac{i}{p^0 - m + i0}.
\]

However, because the field is bosonic, loops do not come with a factor \(-1\). As a result, the presence of Dirac bosons in ghost QED will cause high-energy loop diagram cancellations that will make the theory asymptotically safe.
Note that the $i0$-prescription does not follow from a conventional convergence factor argument. It can, however, be derived from the operator representation and also from a properly defined non-perturbative path integral [27].

Since the action is real-valued, the associated Hamiltonian, which acts on an indefinite inner product space, will be pseudo-hermitian. Pseudo-hermitian operators may have, in addition to real eigenvalues, complex eigenvalues occurring in conjugate pairs and associated to null eigenmodes. Since such modes grow or decay exponentially with time, their production would signal an instability in the indefinite inner product theory, and we will have to find a way of avoiding production of such states.

A more serious problem with indefinite representations is the issue of unitarity. These representations contain states whose inner product with themselves is negative, also sometimes called negative-metric states. In our case, any state containing an odd number of Dirac boson anti-particles is a negative metric state. The presence of these states would be a disaster for the probability interpretation unless there is a mechanism for eliminating these states or preventing their production.

We will later impose constraints on the state space under which the interacting theory remains stable and unitarity will be satisfied.

4 Asymptotic safety

We first argue that the theory defined by (1) is asymptotically safe [12] for a large region of the parameter space. In other words, the fine structure constant $\alpha_\Lambda = e^2/4\pi$ will remain a finite small parameter, approaching a fixed point, as $\Lambda \to \infty$. We will motivate this by a calculation to second order, and conclude with a general argument that the theory should remain asymptotically safe to all orders.

In contrast with the case of ordinary quantum electrodynamics, ghost loops will cancel the infinities that would otherwise appear in the vacuum polarization, as a result of which the bare charge does not need an infinite renormalization. Therefore, provided the bare charge remains small under renormalization, we are not required to separate an $F_{\mu\nu}F^{\mu\nu}$ counter-term and are free to organize our perturbation expansion in terms of the bare charge.
\( e_\Lambda \), writing the action as

\[
\mathcal{L} = \sum_{i=0}^{1} \bar{\psi}_i \left( i(\partial - i\mathcal{A}) - m_i \right) \psi_i - \frac{1}{4e_\Lambda^2} F_{\mu\nu} F^{\mu\nu} - \frac{\lambda_\Lambda}{2e_\Lambda^2} (\partial_{\mu} A^\mu)^2 \tag{2}
\]

\[
+ \sum_{i=0}^{1} (\bar{\psi}_i (i(\partial - i\mathcal{A}) - m_i) \psi_i - Z_i \Lambda (m_i - m_i) \bar{\psi}_i \psi_i. \tag{3}
\]

The counter-terms in the second line may be expressed as functions of \( \Lambda \) and \( e_\Lambda \), for which an asymptotic expansion in \( e_\Lambda \) may be derived, as usual, order by order in perturbation theory. As we shall discuss, for a large range of parameters, no large logarithmic corrections will spoil perturbation theory, and all mass terms will be real. Mass renormalization will be discussed in the next section.

Consider the one-loop correction to the vacuum polarization \( \Pi_{\mu\nu}(q) \), given by the diagrams

\[ \text{(4)} \]

that are superficially quadratically divergent. In fact, the correction is finite, as a consequence of gauge invariance and the ghost contribution.

Gauge invariance requires the current conservation Ward identity

\[ q^\mu \Pi_{\mu\nu}(q) = 0, \tag{5} \]

which in turn implies that no quadratically divergent contribution can occur in a gauge-invariant regularization.\(^1\) The remaining logarithmic divergence

\[ \frac{e_\Lambda^2}{2\pi^2} (q_\mu q_\nu - g_{\mu\nu} q^2) \int_0^1 d\beta \beta (1 - \beta) \sum_i c_i \ln \frac{\Lambda_0^2}{m_i^2 - \beta (1 - \beta) q^2} - \frac{e_\Lambda^2}{8\pi^2} g_{\mu\nu} \sum_i c_i m_i^2, \]

where the sums include physical, ghost and regulator fields, and the \( c_i = \pm 1 \) denote the statistics. The first term is independent of the cutoff \( \Lambda_0 \) and can be written in the form (7), where \( \Lambda \) is a function of the regulator masses, for large values of the latter. The second term is only consistent with the Ward identity (5) if the Pauli-Villars mass condition \( \sum_i c_i m_i^2 = 0 \) is imposed. Thus, gauge invariance prohibits the quadratic divergence.
is canceled by the ghost contribution, as is clear from the fact that
\[
\Pi_{\mu\nu}(q) = (q_\mu q_\nu - g_{\mu\nu} q^2) \Pi(q^2),
\]
where
\[
\Pi(q^2) = \frac{e_\Lambda^2}{2\pi^2} \int_0^1 d\beta \beta(1 - \beta) \sum_i c_i \ln \frac{\Lambda^2}{m_i^2 - \beta(1 - \beta) q^2 - i0},
\]
valid as long as \(|q^2| \ll \Lambda^2\). The logarithmic term is independent of the regulator scale \(\Lambda\) due to our choice of statistics
\[
\sum_i c_i = 0,
\]
where \(c_i = 1\) for a fermionic field and \(c_i = -1\) for a bosonic field. We obtain
\[
\Pi(q^2) = \frac{e_\Lambda^2}{2\pi^2} \int_0^1 d\beta \beta(1 - \beta) \ln \frac{M^2 - \beta(1 - \beta) q^2 - i0}{m^2 - \beta(1 - \beta) q^2 - i0}, \quad |q^2| \ll \Lambda^2.
\]
Here \(m = m_0\) denotes the physical lepton mass and \(M = m_1\) denotes the ghost mass.

No infinite renormalization of \(e_\Lambda\) is needed to make this well-defined as \(\Lambda \to \infty\), and we therefore take the bare charge to be a finite value independent of \(\Lambda\), i.e.,
\[
e_\Lambda = \text{constant} \equiv e_\infty,
\]
and take \(\Lambda \to \infty\) to obtain
\[
\Pi(q^2) = \frac{e_\infty^2}{2\pi^2} \int_0^1 d\beta \beta(1 - \beta) \ln \frac{M^2 - \beta(1 - \beta) q^2 - i0}{m^2 - \beta(1 - \beta) q^2 - i0},
\]
now valid for all \(q^2\).

The effective coupling \(\alpha_\mu = e_\mu^2 / 4\pi\) at space-like momentum transfer
\[
q^2 = -\mu^2
\]
is then given by a geometric series over 1PI diagrams, which gives to this order,
\[
\alpha_\mu = \frac{\alpha_\infty}{1 + \Pi(-\mu^2)} = \frac{\alpha_\infty}{1 + \frac{2\alpha_\infty}{\pi} \int_0^1 d\beta \beta(1 - \beta) \ln \frac{M^2 + \beta(1 - \beta) \mu^2}{m^2 + \beta(1 - \beta) \mu^2}}.
\]
The geometric series converges as long as \( \Pi(-\mu^2) < 1 \), which will be the case as long as the bare coupling is small enough that

\[
\frac{a_\infty}{3\pi} \ln \frac{M^2}{m^2} < 1
\]

since the logarithmic term is a monotonic decreasing function of \( \mu \) if \( M > m \). We will discuss the physical implications of this bound below.

As in ordinary QED, the effective coupling grows with increasing energy. However, in contrast to ordinary QED, the effective coupling is well-defined at all energy scales, and asymptotically approaches the finite bare coupling \( a_\infty \) at infinite energy or \( \mu \rightarrow \infty \).\(^2\) The effective coupling is always smaller than the bare coupling, so if the bare coupling is chosen sufficiently small, perturbation theory can be trusted for all scales. In other words, to this order in perturbation theory the theory appears to be asymptotically safe.

To this order, then, the theory will be free of the Landau pole, the unphysical tachyonic singularity at large space-like momentum transfer (short space-like distances) appearing in perturbative QED. While such perturbative arguments are not conclusive, the existence of this divergence in ordinary QED is usually regarded as strong evidence against its existence as a non-trivial theory, and its absence in our theory is therefore good news.

Inverting (10), let us express the bare coupling in terms of the effective coupling \( a = a_0 \) at zero momentum transfer \( \mu = 0 \) to this order as

\[
a_\infty = \frac{a}{1 - \frac{a}{3\pi} \ln \frac{M^2}{m^2}}.
\]

(11)

To investigate the plausibility of having a small bare coupling \( a_\Lambda \) in a model motivated by the QED sector of our world, we take

\[
a \sim \frac{1}{137}
\]

and \( m \sim 0.511 \text{ MeV} \) to be of the order of the electron mass. Even if the mass \( M \) of the ghost is a large as the mass-energy of the visible part of our universe, for which we use, for the sake of the argument, the estimate

\[
M \sim 1.25 \times 10^{82} \text{ MeV},
\]

\(^2\)For time-like momentum transfer \( q^2 > m^2 \) the vacuum amplitude has an imaginary part. But this vanishes as \( q^2 = -\mu^2 \rightarrow \infty \), so in this limit we find the same effective coupling \( a_\Lambda \).
we still find a bare coupling

\[ \alpha_\infty \sim \frac{1}{100}, \]

which is indeed small. In addition, we find that

\[ \frac{\alpha_\infty}{3\pi} \ln \frac{M^2}{m^2} \sim 0.4161 < 1, \]  
(12)

which justifies our prior performing of the geometric sum to obtain (10). The smallness of the first-order \( \alpha_\infty \) leaves plenty of room for possible higher-order contributions. For example, the second order contribution can be calculated and gives

\[ \alpha_\infty = \frac{\alpha}{1 - \frac{\alpha}{3\pi} \ln \frac{M^2}{m^2} - \frac{\alpha^2}{4\pi^2} \ln \frac{M^2}{m^2}}, \]  
(13)

so that

\[ \frac{\alpha_\infty}{3\pi} \ln \frac{M^2}{m^2} \sim 0.4164, \]

which is only a 7/10000 correction to the first order result.

The evidence, to this order, is therefore that the theory has an ultraviolet fixed point at \( \alpha_\infty \sim 1/100 \), and that the effective coupling constant at any finite energy is smaller than this value. We can therefore trust the accuracy of the perturbative calculation that we performed to produce this evidence. This makes it feasible that the theory is indeed asymptotically safe. Further support for this assertion will be given by an all-order argument below.

Note that, since we could directly sum the geometric series, we did not need to rely on a renormalization group equation to arrive at the result for \( \alpha_\mu \). Our result does reproduce the familiar first-order solution to the QED renormalization group equation in the range

\[ m^2 \ll \mu^2, \mu'^2 \ll M^2 \]

by using (10) to write \( \alpha_\mu \) and \( \alpha_\mu' \) in terms of \( \alpha_\infty \) and eliminating the latter to find the expected

\[ \alpha_\mu' = \frac{\alpha_\mu}{1 - \frac{\alpha_\mu}{3\pi} \ln \frac{\mu'^2}{\mu^2} + o(\alpha_\mu^2)}. \]  
(14)
In this range, the \( \beta \)-function coincides with that of ordinary electrodynamics. However, it is clear from (10) that the \( \beta \)-function gets modified when \( \mu \) becomes comparable to the ghost mass \( M \), where \( \beta \) becomes explicitly dependent on \( \mu \). However, we can state that, to this order,

\[
\beta > 0
\]
on the whole range of positive \( \mu \), and that

\[
\beta \to 0
\]
asymptotically as \( \mu \to \infty \), as one would expect in an asymptotically safe theory.

Let us consider whether higher-order corrections might spoil the argument for asymptotic safety. A plausibility argument that the theory should remain asymptotically safe to all orders in perturbation theory can be based on the observation that all perturbative contributions to the vacuum polarization are finite, just like the first order (7). This follows from the known fact that the powers of

\[
\ln \Lambda
\]

appearing in the vacuum polarization \( \Pi(q^2) \) come from internal Dirac loops only. This is a consequence of gauge invariance.\(^3\) Since each Dirac fermion loop can be individually replaced by a Dirac boson ghost loop of opposite sign, the \( \Lambda \)-dependent terms in a 1PI correction of arbitrary order are proportional to

\[
\left( \sum_i c_i (\ln \Lambda) \right)^n = 0,
\]

so that the result is independent of the cutoff \( \Lambda \).

Since \( \Pi(q^2) \) is independent of \( \Lambda \), the bare parameter \( \alpha_\Lambda \) still does not need any infinite perturbative renormalization as \( \Lambda \to \infty \) and can, as before, be chosen as a constant independent of \( \Lambda \). We conclude that perturbation theory, at least to the extent that it remains accurate given that it is an asymptotic series, should not spoil asymptotic safety.

\(^3\)For example, to second order there is a single Dirac loop, and indeed double logarithms from self-energy and vertex sub-divergences cancel because, as a consequence of gauge invariance, the corresponding counterterms occur with the same coefficient \( (Z_\Lambda - 1) \) in the bare action (3).
5 Mass and field renormalizations

In the previous section we saw that corrections to the vacuum polarization are finite due to cancellations between ordinary fermion and ghost loops. On the other hand, self-energy diagrams are not finite without $\Lambda$-dependent mass and field renormalization counter-terms.

Since the action is real-valued, the associated Hamiltonian, which acts on an indefinite inner product space, is pseudo-hermitian. Pseudo-hermitian operators may have, in addition to real eigenvalues, complex eigenvalues occurring in conjugate pairs and associated to null eigenmodes [28, 29]. In other words, as the coupling constant increases from 0, a single field of real mass may split into a pair of degrees of freedom of complex conjugate masses. It should be clear that the original field can only supply one local counter-term to renormalize the common real part of these masses, so that their imaginary part, if any, is not an independent parameter.

We shall assume that real-valued mass counter-terms have been added so that the parameters $m = m_0$ and $M = m_1$ in the action are the real parts of the positions of the physical one-particle poles of the full propagator. This can always be done while maintaining reality of the action, and therefore pseudo-hermiticity.

Complex energies typically occur in cases where a positive-metric and a negative-metric state in the free theory are close compared to the scale of their mixing term in the Hamiltonian. This can be understood in the elementary example of a two-state Hamiltonian of the form

$$H = \begin{pmatrix} \omega_1 & \gamma \\ -\gamma & \omega_2 \end{pmatrix},$$

which is pseudo-hermitian with respect to the inner product $\eta = \text{diag}(1, -1)$, and whose eigenvalues become complex when

$$|\omega_1 - \omega_2| < 2\gamma.$$

In perturbation theory, this effect is a result of summation over virtual negative-metric intermediate states that displaces the energy poles into the complex plane and requires a modification of the usual integration contours in the energy plane. To avoid creating the impression that the contours are ad-hoc, let us derive the correct contour from first principles by applying perturbation theory to the above example, for simplicity taking $\omega_1 = \omega_2 \equiv \omega$. 

13
We write
\[
\theta(t) \langle 1 | e^{-iHt} | 1 \rangle \equiv \int_{\Gamma} \frac{dE}{2\pi} e^{-iEt} f(E).
\]

where \( \Gamma \) indicates a deformation of the real line to be determined. As described in reference [27], both \( \Gamma \) and the function \( f(E) \) may be computed by performing perturbation theory in \( t \)-space to obtain the convergent series (here \( V_{12} = -V_{21} = \gamma \))
\[
\int_{-\infty}^{\infty} \frac{dE}{2\pi} e^{-iEt} \frac{i}{E - \omega + i0} + (-i)^2 \int_{-\infty}^{\infty} \frac{dE}{2\pi} e^{-iEt} V_{12}V_{21} \left( \frac{i}{E - \omega + i0} \right)^3 \\
+ (-i)^4 \int_{-\infty}^{\infty} \frac{dE}{2\pi} e^{-iEt} V_{12}V_{21}V_{12}V_{21} \left( \frac{i}{E - \omega + i0} \right)^5 + \cdots \\
= \theta(t) e^{-i\omega t} \left( 1 + \frac{V_{12}V_{21}(it)^2}{2!} + \frac{(V_{12}V_{21})^2 (it)^4}{4!} + \cdots \right) \\
= \theta(t) e^{-i\omega t} \cosh \gamma t \\
= \theta(t) \frac{1}{2} \left( e^{-i(\omega+i\gamma)t} + e^{-i(\omega-i\gamma)t} \right) \\
= \int_{\Gamma} \frac{dE}{2\pi} e^{-iEt} \frac{i}{E - \omega - V_{12}V_{21}/(E - \omega)}, \tag{15}
\]

where \( \Gamma \) is obtained by a deformation of the real line into the complex plane to run above both poles of
\[
f(E) \equiv \frac{i}{E - \omega + \gamma^2/(E - \omega)}
\]

including when the poles are away from the real axis in the complex plane. It is important to note that we have to first perform the \( E \)-integrals in the first line and then sum the series [27]. Note that the order of integration and summation cannot be interchanged, since the momentum-space geometric series under the integral does not converge for all \( s \). Thus, the perturbative series can be summed in position space where it converges, but we cannot expect to get the correct answers by formally summing the divergent Fourier transformed terms in momentum space. For further examples where such
integration contours are unambiguously determined in a non-perturbative framework, see reference [27].

Notice, though, that the functional form of \( f(E) \), though not the contour \( \Gamma \), could have been obtained directly from a formal geometric sum in momentum space.\(^4\)

For comparison with field theory, notice that the poles of \( f(E) \), which are the energy eigenvalues of \( H \), acquire imaginary parts as a consequence of the fact that the imaginary part of the contribution to the denominator of \( f(E) \),

\[
\text{Im} \left( \frac{-V_{12}V_{21}}{E - \omega} \right)
\]

is negative above the real axis due to \( V_{12} = -V_{21}^* \), which is the case for matrix elements between states of opposite metric. As a result, this term will cancel the imaginary part of the other contribution \( E - \omega \) to the denominator of \( f(E) \) somewhere in the upper half plane, and we get a complex pole as long as the real parts of these two terms also cancel which, in the non-degenerate case where \( \omega_1 \neq \omega_2 \), will happen if \( \omega_1 \) is close to \( \omega_2 \) compared to the scale set by \( \gamma \).

This generalizes to field theory as follows. Consider doing perturbation theory based on a Dirac particle of rest mass \( m \). The propagator is

\[
\frac{i}{\not{p} - m - \Sigma(\not{p})},
\]

where \(-i\Sigma\) denotes a sum of diagrams that are irreducible with respect to the particle whose self-energy we are determining. Again, a full specification of the distribution consists not only of this function but also the associated integration contour, which can be obtained through a careful calculation in the spirit of (15). The irreducible contribution to the forward scattering amplitude for the Dirac particle is

\[
\mathcal{M}(p, s \rightarrow p, s) = -Z \bar{u}_p^s \Sigma u_p^s.
\]

\(^4\)It is important to note, however, that the propagator is a distribution \( \int \! dE \, f(E) (\cdots) \), which is specified not only by the function \( f(E) \) but also by the contour \( \Gamma \). Momentum-space perturbation theory formally gives the function \( f(E) \), but not the contour, and thus does not fully specify the distribution. Knowing the contour becomes essential when we use \( f(E) \) as an internal line in a higher order diagram, so that an integration over \( E \) is required. See reference [1] for an example where using the wrong distribution for an internal line gives incorrect results. Reference [27] contains a discussion on how to calculate such diagrams unambiguously in a non-perturbatively well-defined approach.
This is independent of the spin $s$ by rotational invariance, and is the complex conjugate of the corresponding anti-particle matrix element. With respect to the conjugation defined by $\Sigma \equiv \gamma^0 \Sigma^\dagger \gamma^0$, the optical theorem may be written as

$$\text{Im} (\Sigma) = \frac{1}{2mZ} \text{Im} \mathcal{M}(p, s \rightarrow p, s) \cdot 1$$

$$= \frac{1}{2mZ} \left( \frac{1}{2} \right) \int d\mu_f \mathcal{M}(p, s \rightarrow f) \mathcal{M}(f \rightarrow p, s) \delta^4(p_f - p),$$

where $d\mu_f$ is the appropriate Lorentz-invariant measure on the space of all possible final states that may be obtained by cutting irreducible diagrams.

When $\Sigma(p)$ has a multi-particle cut starting at $p^2 < m^2$, the particle is unstable and can decay to the multi-particle states. Consider the propagator

$$i \frac{1}{p - m - \Sigma(p)}.$$ 

We see that if $\text{Im} (-\Sigma)$ has a cut starting at some $p^2 < m^2$, and is positive above and negative below the cut, the propagator will have no pole on the physical sheet. It will have a pole with nonzero negative imaginary part on the unphysical second sheet. We no longer have a particle but instead a resonance.

In ordinary hermitian quantum field theories, $\text{Im} (-\Sigma)$ will always be positive, and the resonance poles will not be on the physical sheet. In pseudo-hermitian theories, however, we have

$$\mathcal{M}(p, s \rightarrow f) \mathcal{M}(f \rightarrow p, s) = -|\mathcal{M}(p, s \rightarrow f)|^2 < 0,$$

whenever the initial and final states are non-null states of opposite metric. As a result, $\text{Im} (-\Sigma)$ will become negative if this contribution is larger than the total contribution of decays to states of the same metric. We find that in this case, the poles of

$$i \frac{1}{p - m - \Sigma(p)}$$

will be on the physical sheet, at complex conjugate positions with nonzero imaginary parts [1]. Such poles are not resonances. Instead, they correspond to a pair of null energy eigenstates with complex conjugate rest masses. Such
conjugate null states are dual, i.e., the inner product on the subspace spanned by these states can be brought to the form

\[
\begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}.
\]

In other words, whenever a particle is unstable with respect to decay to a multi-particle state of opposite metric, it may split into a pair of degrees of freedom of complex conjugate masses.

Let us consider whether the masses of the two types of Dirac particles in our theory may become complex. *We shall assume, for the rest of this paper, that both \( m \) and \( M \) are nonzero, and that \( M \gg m \).*

First we consider the ghost. The lowest order negative contribution to \( \text{Im}(\Sigma) \) in the ghost forward amplitude is given by the indicated vertical cut in the diagram.

\[=\]

since states containing even an odd numbers of ghost anti-particles have opposite sign metric.

This cut starts at \( p^2 = (3M)^2 \), which is well-separated from \( M^2 \). Assuming \( \text{Re}\Sigma(p = M) = 0 \), which can always be ensured by including an
appropriate real counter-term in the bare action, the mass pole remains at the real position \( p^2 = M^2 \). Indeed, the indicated decay is prevented by kinematics, and so does not lead to an instability and an associated complex mass.

The same goes for the lowest order negative contribution to \( \text{Im} (−\Sigma) \) in the fermion forward scattering amplitude

\[
\begin{align*}
\text{Diagram 1} & \quad \text{Diagram 2} \\
\text{Diagram 3} & \quad \text{Diagram 4}
\end{align*}
\]

where the indicated decay is, once again, prevented by kinematics due the the assumption \( M \gg m \). The fermion, like the Dirac boson, is stable and its mass remains real.

We conclude that the masses of the fundamental Dirac fermion and Dirac bosonic ghost remain real under perturbative corrections.

There are, however, composite degrees of freedom whose masses we do expect to become complex. The lightest of these is the ghost analogue of the singlet ground state of positronium. The singlet state of ordinary positronium is an unstable state consisting of a fermionic particle and anti-particle in the singlet configuration. This state is not part of the spectrum but corresponds to a resonance. The ghost analogue consists of a ghost particle and ghost anti-particle in the singlet configuration. In this case, however, the instability of the state under decay to states of opposite metric will split it into a pair of null degrees of freedom of complex conjugate masses. The corresponding
poles will be on the physical sheet, and will appear, for example, in the full vacuum polarization

Since the singlet is a scalar, we expect a contribution to the vacuum polarization of the approximate form

\[
\frac{iC}{p^2 - \mu^2 - \Sigma(p)},
\]

where \(C\) is a constant and where \(\Sigma\) is again the irreducible forward scattering amplitude for the effective ghost positronium degree of freedom. As in the Dirac case, cuts corresponding to intermediate states of opposite metric will give negative contributions to \(\text{Im}(-\Sigma)\), which may then similarly cause the pole to split and move to complex conjugate positions away from the real axis on the physical sheet.\(^5\)

The parameter \(\mu\) above is approximately the real part of the rest mass of the bound state. As in the case of ordinary positronium, this mass can be approximated by subtracting the non-relativistic binding energy to give, for ghost positronium, the value

\[
\mu \equiv 2M - \alpha^2 M/4.
\] (16)

Then \(\text{Im}(-\Sigma)\) is determined by the various decay processes that may affect the composite degree of freedom. A first approximation may be obtained by considering the constituent particles at relative rest, which translates to calculating certain simple Feynman diagrams near the two-particle threshold \(p^2 = (2M)^2\) as described, for example, in [30].

For ghost positronium, the first relevant contribution to \(\text{Im}(-\Sigma)\) is given

---

\(^5\)Notice that singlet ghost positronium is a \textit{bosonic} Klein-Gordon degree of freedom with \textit{negative} metric. If such a field had appeared in the original Lagrangian, it would have violated micro-causality [38]. For describing a composite object, it is acceptable.
by the vertical cut in the diagram

\[
\begin{array}{c}
\begin{array}{c}
\text{Diagram 1} \\
\text{Diagram 2}
\end{array}
\end{array}
\]

calculated at the threshold \( p^2 = (2M)^2 \). This cut corresponds to the decay of the composite state to a pair of photons. The indicated cut starts at \( p^2 = 0 \) and contributes negatively to \( \text{Im}(-\Sigma) \) along the real axis. If negative contributions are dominant at the energy \( \mu \) (16), this contribution will make the pole at \( \mu^2 \) split into a pair of complex conjugate poles away from the real axis. To the same order, the following diagrams also contribute to \( \text{Im}(-\Sigma) \).

\[
\begin{array}{c}
\text{Diagram 3} \\
\text{Diagram 4}
\end{array}
\]

(17)

Due to the assumption, \( m \ll M \), the threshold \( p^2 = (2m)^2 \) for creating two fermions is lower than \( \mu^2 \), so the indicated cut in the first of these diagrams also corresponds to an instability that contributes negatively to \( \text{Im}(-\Sigma) \) in the vicinity of \( \mu^2 \). The cut in the second diagram starts to contribute positively to \( \text{Im}(-\Sigma) \), but only above the threshold \( p^2 = (2M)^2 \). Since \( \mu < 2M \), the decay is kinematically ruled out and does not affect our conclusion that we obtain complex conjugate physical poles.

In addition to these unstable states, we also expect stable bound states consisting of ghost-lepton atoms. One consists of an electron and an anti-ghost and has negative metric. Its anti-atom has positive metric. These have approximate rest mass

\[ M + m - \frac{\alpha^2}{2} \left( \frac{mM}{M + m} \right). \]

The analytic structure of diagrams involving multi-particle intermediate states is much more problematic. For example, it has been argued that
diagrams involving intermediate states of complex conjugate rest masses, calculated in a Hamiltonian approach, have non-covariant singularities that break Lorentz invariance \[7, 9\]. We will discuss this issue in section 8, where we propose a solution based on carefully constructed Lorentz invariant multiparticle state spaces that are different from the spaces used by prior authors. The resulting theory will be Lorentz-invariant by construction, and so will the analytic structure of Feynman diagrams.

As an example, in our approach we would expect a cut starting on the real axis at

\[
p^2 = \{2(2M - \alpha^2 M/4)\}^2,
\]

corresponding to pairs of conjugate ghost positronium states. However, as we will discuss in the section on Lorentz invariance, when considered properly in a specific non-perturbatively well-defined framework, interactions are expected to cause this cut to split into a pair of conjugate cuts “infinitesimally” above and below the complex plane.

Note that we will also have cuts starting at complex branch points. For example, two of these cuts, corresponding to two-particle states consisting of an electron and a ghost positronium, will start approximately at

\[
\{m + 2M - \alpha^2 M/4 \pm i\gamma/2\}^2,
\]

where \(2M + \alpha^2 M/4 \pm i\gamma/2\) are the complex ghost positronium masses. For each cut, its complex conjugate mirror image also appears as a consequence of the pseudo-hermiticity of the Hamiltonian.

Since mass and field renormalizations are \(\Lambda\)-dependent, unlike the charge renormalization, it is perhaps not obvious that they do not suffer from Landau-pole type obstructions similar to those that appear in ordinary quantum electrodynamics. We verify that such an obstruction is absent for the field renormalization constant \(Z_{i\Lambda}\). To second order, we have

\[
Z_{i\Lambda} = \frac{1}{1 + \frac{\alpha_\Lambda}{4\pi} \left( \ln \frac{\Lambda^2}{m_i^2} - 3 \ln 3 + \frac{9}{4} \right)}
\]

in a gauge where infrared divergences vanish \[30\]. Unlike the case of the charge, this has an explicit dependence on the regularization \(\Lambda\). In ordinary electrodynamics, we are prevented from taking the limit \(\Lambda \to \infty\), at least in perturbation theory, due to the fact that \(\alpha_\Lambda \to \infty\) for some finite value of
Λ. In the present case, however, we have arranged for $\alpha_\Lambda$ to remain small as $\Lambda \to \infty$, so $Z_\Lambda$ remains nonzero for all $\Lambda$. As a result, there is no obstruction to removing the regularization.

6 Unitarity and scattering

The time evolution described by our action is pseudo-unitary. This means that inner products and normalizations are preserved by time evolution, as is the trace of the density matrix, which can be normalized to unity. However, the density matrix is not positive definite, and does not have a conventional probability interpretation.

However, Lee and Wick noted that if all negative-definite states are “unstable” in the sense of acquiring non-real complex energies due to interactions, we will obtain a unitary theory by constraining the state space to real-energy states [1]. This proposal worked well to eliminate single-ghost states, but was never satisfactorily generalized to the elimination of multi-ghost states. Attempts by previous authors to do so were either ad hoc, ambiguous, and probably nonperturbatively ill-defined [2, 5, 6, 8], or broke Lorentz invariance [7, 9].

We will argue that the original real-energy proposal can, with some care, be generalized to the elimination of multi-ghost states in a Lorentz invariant and non-perturbatively well-defined manner, and is therefore sufficient for obtaining a unitary theory in the case of ghost QED.

Specifically, we will argue in the following sections that multi-ghost states can be quantized in a Lorentz invariant way. At this point, their energies may be real, but taking into account interactions, the energies generically become non-real as long we impose a long-distance cutoff. We then apply the Lee-Wick real-energy constraint to eliminate these states first and only then remove the cutoff. The resulting theory should be unitary.

Therefore, in this section and the next, we review how the elimination of complex-energy states affects scattering calculations. We essentially follow Lee and Wick [1].

Ordinary scattering theory assumes a free Hamiltonian $H_0$ and an interacting hermitian Hamiltonian $H$, both hermitian and with coinciding continuous spectra, and constructs operators $W_-$ and $W_+$ that allow us to obtain the generalized eigenstates of the interacting Hamiltonian from those of the
free Hamiltonian as

\[ |E_\alpha, \alpha, \text{in}\rangle_H = W_- |E_\alpha, \alpha\rangle_{H_0}, \quad H \langle E_\alpha, \alpha, \text{out} | = H_0 \langle E_\alpha, \alpha | W_+^\dagger, \]

where \( \alpha \) stands for any additional quantum numbers. Given these two operators, the scattering matrix can be written as

\[ S = W_+^\dagger W_. \]

To obtain a probability interpretation in an indefinite theory, we need to restrict the state space to a physical subspace that is positive definite and invariant under time evolution. Since any complex-energy eigenstates have zero norm, we may follow Lee and Wick and first eliminate these by constraining the physical subspace to lie within the real eigenvalue spectrum of \( H \).

Let us discuss the implications of this constraint for scattering. We assume that the free Hamiltonian \( H_0 \) has real spectrum and we assume, as before, that the parameters in \( H_0 \) have been chosen so that, for a subset of quantum numbers, its spectrum coincides with the real part of the interacting spectrum. The Lippmann-Schwinger equations may be used to formally represent \( W_\pm \) as

\[ W_- |E, \cdots\rangle_{H_0} = \left( 1 + \frac{1}{E - H + i\epsilon} (H - H_0) \right) |E, \cdots\rangle_{H_0}, \]

with the hermitian conjugate equation for \( W_+^\dagger \).

In an indefinite theory, the Lippmann-Schwinger equations remain sensible for the states of real \( E \), but the interpretation of the S-matrix in terms of a scattering process runs into the following hurdle: Ordinarily, the operators \( W_\pm \) may be also obtained as weak limits

\[ W_- = \lim_{t \to -\infty} e^{-iHt} e^{iH_0 t}, \]
\[ W_+^\dagger = \lim_{t \to -\infty} e^{iH_0 t} e^{-iHt}, \]

from which the physical scattering interpretation may be derived. However, if \( H \) has complex eigenvalues with positive imaginary part, these limits will not in general exist. Simply stated, not all free eigenstates of real energy are orthogonal to the complex energy eigenstates of the interacting \( H \), so
that applying \( e^{-iHt} \) to a free eigenstate will give an exponentially diverging contribution as \( t \to \infty \).

The solution proposed by Lee and Wick is to make sure that we project the complex energy null states, which are not in the physical subspace, out of the state \( e^{iH_0 t} |E_\alpha, \alpha\rangle_{H_0} \) at time \(-t\), and instead use as initial state

\[
e^{iH_0 t} |E_\alpha, \alpha\rangle_{H_0} - \int_{\mathbb{R}} d\beta \langle E_\beta, \beta | H \langle E_\beta^*, \beta | e^{iH_0 t} |E_\alpha\rangle_{H_0}.
\]

The projection depends on the solution to the full dynamics of \( H \) and can be calculated order by order in perturbation theory.

Equivalently, we have to apply \( e^{-iHt} e^{iH_0 t} \) not to the free eigenstate \( |E_\alpha, \alpha\rangle_{H_0} \) but instead to the state

\[
|E_\alpha, \alpha\rangle^p_{H_0} = |E_\alpha, \alpha\rangle_{H_0} - \int_{\mathbb{R}} d\beta e^{iE_\alpha t} e^{-iH_0 t} |E_\beta, \beta\rangle \langle E_\beta^*, \beta | E_\alpha\rangle_{H_0} \\
- \int_{\mathbb{R}} d\beta \int_{\mathbb{R}} d\gamma e^{i(E_\alpha - E_\gamma)t} |E_\gamma, \gamma\rangle_{H_0} \langle E_\gamma, \gamma | E_\beta, \beta\rangle H \langle E_\beta^*, \beta | E_\alpha\rangle_{H_0}.
\]

Interestingly, we can argue that the second term vanishes in the limit \( t \to \infty \) that interests us, but only as far as a local observer in the far past is concerned. A careful way of doing this is by considering a smooth wave packet

\[
\int d\alpha f(\alpha) |E_\alpha, \alpha\rangle^p_{H_0} = \int d\alpha f(\alpha) |E_\alpha, \alpha\rangle_{H_0} \\
- \int_{\mathbb{R}} d\beta \int_{\mathbb{R}} d\gamma |E_\gamma, \gamma\rangle_{H_0} \langle E_\gamma, \gamma | E_\beta, \beta\rangle H \times \\
x \int d\alpha f(\alpha) e^{i(E_\alpha - E_\gamma)t} \langle E_\beta^*, \beta | E_\alpha, \alpha\rangle_{H_0}.
\]

Assuming \( H \langle E_\beta^*, \beta | E_\alpha, \alpha\rangle_{H_0} \) is sufficiently smooth – which is expected to be the case since the real \( E_\alpha \) never coincides with the complex \( E_\beta \) – we use
the Riemann-Lebesgue lemma to conclude that the last integral goes to zero polynomially as $e^{-iE_\gamma t}t^a$ when $t \to \infty$. In other words,

$$\int d\alpha f(\alpha) |E_\alpha, \alpha\rangle_{H_0} \to \int d\alpha f(\alpha) |E_\alpha, \alpha\rangle_{H_0}$$

as $t \to \infty$. This equation expresses the interesting property that two states, known to have very different time evolution, are locally indistinguishable to arbitrary precision as $t \to \infty$ in the usual topology [28]. Specifically, the state on the left remains bounded when we apply $e^{-iHt}e^{iH_0t}$ to it, whereas the one on the right depends exponentially on $t$ due to the factor $e^{-iHt}$. The reason for this is that the local distinguishability of the states decreases polynomially in the far past, and a polynomial decrease will always be overcome by the exponential growth in the time evolution.

This means that an experimenter far enough in the past cannot locally distinguish the input states that he or she is preparing from free eigenstates of $H_0$. Although the prepared states unavoidably contain nonlocal correlations encoded in the second term in equation (18), these correlations are not locally detectable to an experimenter located far enough in the past that the wavelength of these correlations is smaller than the experimental resolution. In contrast to ordinary scattering theory where polynomially decaying terms can be discarded in the limit, here the correlations represented by the extra terms are not innocuous and have to be kept, since they are exponentially magnified by the time evolution as the input states evolve towards the scattering region, and their presence cancels out the exponential divergence of the state that would otherwise have occurred.

Since the correlations in the initial state are not locally detectable in the sufficiently far past, their effect unfolding as time goes on may give the appearance of acausality, as has been discussed by previous authors. It should be clear from the formalism that there is no real acausality, since the information, called “precursors” by Lee, is in fact present in the initial state (18). We discuss the issue of causality more fully in section 12.

7 Calculation of the S-matrix

Given the rather complex characterization (18) of the asymptotic states, how does one in fact calculate physical scattering amplitudes perturbatively
in a way that properly takes into account the rather complicated precursor information so as to avoid exponential blow-up of the S-matrix?

As explained by Lee and Wick [1], it is straightforward in principle to calculate amplitudes in the reduced theory obtained by dropping the non-real energy states from the spectrum. One proceeds by calculating amplitudes without this restriction, as one would in ordinary quantum field theory, between states \(|p_\alpha, \alpha\rangle_{H_0}\). One does this on a large but finite time interval \(2t\). As we discussed, the resulting S-matrix elements contain terms that diverge exponentially as \(t \to \infty\). After identifying and discarding these terms, Lee and Wick showed that we obtain exactly the desired S-matrix elements between the states \(|p_\alpha, \alpha\rangle_{H_0}\), expressed in equation (18), of the constrained theory without the non-real energy states.

In practice, the exponentially growing terms may often be discarded via the following trick. If the state depends analytically on the various scattering invariants, scattering amplitudes in the full unconstrained theory can often be expressed as contour integrals over the space of complexified invariants. Discarding exponentially growing terms from scattering amplitudes can then often be implemented in a straightforward way by modifying the contour integrals on the complex plane. For example, for a simple two-particle scattering, this can be done by taking the integration contour in the \(s\)-plane to be along the real line instead of the contour that runs above the non-analyticities in the upper complex plane representing exponentially growing contributions of complex energy states in the full theory.

Useful as such contour tricks are, they are not foundational to the definition of the theory. The Lee-Wick real-energy constraint is non-perturbatively well-defined and does not assume any particular analytic properties of amplitudes.

8 Lorentz invariance

In its Lagrangian formulation, the theory appears manifestly Lorentz-invariant. We shall argue that the theory can be quantized in a way that preserves this Lorentz invariance even after applying the Lee-Wick real energy constraint. Our construction will be in principle non-perturbatively well-defined.

This assertion may seem to be in conflict with prior work by Nakanishi [7] and Gleeson et al. [9]. These authors argued that, if the unconstrained theory has complex mass states, certain amplitudes where such states ap-
pear in loops lose Lorentz invariance after applying the Lee-Wick real energy constraint. They based their calculations on the Hamiltonian approach originally advocated by Lee [3], which is in principle non-perturbatively well-defined. The fact that they ran into problems should therefore be taken very seriously.\(^6\)

We will argue that the problem pointed out by Nakanishi and by Gleeson et al. can be overcome by basing the state space on a class of distributions studied by Gel’fand and Shilov [10, 11]. Our construction will allow us to define the multi-ghost state spaces and the Lee-Wick real energy constraint in a Lorentz-invariant way. Our method is in principle non-perturbatively well-defined.

Let us discuss the problem in more detail by way of an example. Consider a theory that contains ghost scalar particles of complex conjugate masses \(M\) and \(M^\ast\) (for example, the two ghost positronium states considered previously). Next consider a one-loop diagram whose intermediate two-particle state consists of an \(M\)-particle and an \(M^\ast\)-particle. Writing their energies as \(\sqrt{M^2 + \vec{k}^2}\) and \(\sqrt{M^\ast^2 + (\vec{p} - \vec{k})^2}\) and assuming, as these previous authors did, real space-like momenta \(\vec{k}\) and \(\vec{p}\), the description of the two-ghost state space is not Lorentz invariant, and the resulting diagram contains a fish-shaped non-analytic region in the \(s = p^2\) plane \([7, 9]\), whose shape depends on the Lorentz frame.\(^7\)

\(^6\)An alternative approach to obtaining a unitary theory was investigated by Cutkosky et al [5]. These authors proposed modifying diagrams order by order according to heuristic prescriptions based on unitarity considerations. The resulting amplitudes are Lorentz-invariant, but as was noted by these authors themselves, their prescription was at best incomplete due to unresolved ambiguities. As was argued also, amongst others, by Nakanishi [7] and Boulware and Gross [6], the prescription may not have non-perturbative meaning. For example, it would appear that no Hamiltonian can reproduce their prescription.

For these reasons, we shall not discuss the approach of Cutkosky et al. further. Instead, we shall insist on sticking to calculations that are in principle non-perturbatively well-defined, based either on path integral or Hamiltonian arguments.

\(^7\)The non-invariance of the two-ghost state space will nevertheless not be visible in the unconstrained theory as long as the external legs are non-ghost particles of real mass with asymptotic wave packets that are analytic at least on this fish-shaped non-analyticity region in \(s\). In this case, scattering amplitudes in the unconstrained theory between non-ghost states will be Lorentz invariant. This is because the integration path for calculating scattering amplitudes in the unconstrained theory does not run along the real line in the \(s = p^2\) plane, but instead runs above the region of non-analyticity, as we saw in a toy example in section 5. So if the asymptotic wave packets are chosen, for example, to be holomorphic in \(s\), we can move the contour down towards the real line while continuing
The apparent conflict with Lorentz invariance occurs when we apply the Lee-Wick real energy constraint to obtain a unitary theory. Given the real-momentum characterization of the two-ghost states described in the previous paragraph, it is clear that, apart from a set of measure zero, the two-ghost intermediate states have complex energies. As we discussed, Lee and Wick showed that removing complex-energy states translates into discarding exponentially growing terms from the scattering amplitudes \[1\], and that this can be done in simple cases such as this one by taking the integration contour in the \(s\)-plane to be along the real line instead of running above the non-analyticities. Applying the Lee-Wick trick, the modified contour along the real line will bisect the non-invariant region of non-analyticity, and the scattering amplitude will not be Lorentz invariant \([7, 9]\).

It should be clear that the source of the problem lies in the fact that the set of two-particle null states that are supposed to be eliminated by the Lee-Wick real energy constraint is not Lorentz-invariant. This observation suggests that the problem may be fixable if we can recast the full theory in a manifestly covariant state space. The discussion in the footnote also suggests that, if we base our state space on holomorphic test functions, the unconstrained theory may in fact have a hidden covariance, which can be made explicit by using the appropriate sets of generalized functions as continuum states. We now proceed to do so.

In reference \([27]\), by the present author, a general non-perturbatively well-defined Hamiltonian and path integral approach to indefinite inner product theories was described. The approach was based on a class Gel’fand-Shilov distributions on analytic test spaces that provided a very natural description of the complex-energy states that tend to arise in pseudo-unitary theories.

We will continue that work and formulate the current theory in a Gel’fand-Shilov rigged state space based on a test space of analytic functions. The space will be tailor-made for building covariant descriptions of complex mass degrees of freedom. There are many spaces of analytic functions, but our choice is constrained by the requirement that the test function space be invariant under time evolution. In the relativistic case, the relevant evolution (at least for the free theory) is given by the Klein-Gordon or Dirac equations, and we shall choose as our test space for a particle of mass \(m\) the Gel’fand-

---

the integrand into the original non-analytic region. This analytically continued integrand turns out to be a Lorentz invariant function with a cut along the real axis starting at \((M + M^*)^2\) \([3, 9]\). Thus, provided the asymptotic wave packets are holomorphic, the scattering amplitude in question is in fact Lorentz invariant.
Shilov space of entire functions \([10, 11, 27]\) on the mass shell hyperboloid parameterized by the three-momentum \(\bar{k}\) in some frame.

\[
S_{1/2, A}^{1/2, B} (\bar{k}) \equiv \bigotimes_{i=1}^{3} S_{1/2, A} (k_i), \quad k_0 = \sqrt{m^2 + \bar{k}^2}, \quad \bar{k} = m\lambda, \quad \lambda_i \in \mathbb{R},
\]

The mass \(m\) may be complex, in which case the momenta \(\bar{k}\) are complex, as we discuss further below. Members of \(S_{1/2, A} (z)\) are entire functions with order of growth

\[
|\phi(x + iy)| < Ce^{-a|x|^2 + b|y|^2}, \quad a = \frac{1}{2eA^2}, \quad b = \frac{eB^2}{2}.
\]

Here \(e\) denotes the base of the natural logarithm. The parameters \(A\) and \(B\) will be chosen such that test functions will decay along the possibly complex directions \(\bar{k} \in m\mathbb{R}^3\) for all masses \(m\) in the theory.

The space \(S_{1/2, A}^{1/2, B} (\bar{k})\) is invariant under Klein-Gordon evolution \([11]\), and has Fourier transform

\[
\mathcal{F} \left( S_{1/2, A}^{1/2, B} (\bar{k}) \right) = S_{1/2, A} (\bar{x}),
\]

where \(\bar{x}\) parameterizes some space-like surface. Because the set of four-momenta

\[
\left\{ (k_0, \bar{k}) \mid k_0 = \sqrt{m^2 + \bar{k}^2}, \quad \bar{k} = m\lambda, \quad \lambda_i \in \mathbb{R} \right\}
\]

is Lorentz-invariant for real or complex \(m\), it follows that the test function space \(S_{1/2, A}^{1/2, B} (\bar{k})\) is in fact frame-independent. The full test space on an arbitrary number of particles is generated via the Fock construction from sums of tensor products of single-particle test functions. We shall denote the resulting Fock space by

\[
S_{1/2, A}^{1/2, B}.
\]

So far \(A\) and \(B\) have not been fixed. However, let us make the physical assumption that the energy and each component of the momentum of all states lie within a double wedge \(|y| < \alpha|x|\) in the complex plane. By choosing \(a > \alpha^2b\) in the above formula for the order of growth, or equivalently \(AB < 1/e\alpha\), we can ensure that test functions decay as \(e^{-\beta|z|^2}\) in any direction lying within the wedge.

29
This choice of test space allows us to formulate complex mass-energy theories in a manifestly Lorentz-invariant way. Consider, for example, the two-point function

\[ D(x - y) = \langle 0 | \phi(x) \phi(y) | 0 \rangle \]

of a scalar field with complex mass \( M \). A Lorentz-invariant set of one-particle states can be generated from the state with four-momentum \((M, \bar{0})\) via Lorentz transformations to obtain the invariant hyperboloid

\[ \left\{ (k_0, \bar{k}) \mid k_0 = \sqrt{M^2 + \bar{k}^2}, \bar{k} = M\lambda, \lambda_i \in \mathbb{R} \right\}. \]

The momenta of these states are not real, so that the formal, manifestly Lorentz-invariant expression for the two-point function

\[ D(x - y) = \int_{M\mathbb{R}^3} \frac{d^3\bar{p}}{(2\pi)^3} \frac{e^{ip_\mu(x^\mu - y^\mu)}}{2\sqrt{M^2 + \bar{p}^2}} \]

would naively appear to diverge due to the exponentially growing integrand. However, the two-point function defines a perfectly meaningful distribution with respect to \( S_{1/2, B}^{1/2, A}(\bar{k}) \). Specifically, in momentum space it is defined via its effect on a test function \( \phi \) as

\[ \int_{M\mathbb{R}^3} \frac{d^3\bar{p}}{(2\pi)^3} \frac{e^{i(p_0x^0 - p_\mu y^\mu)}}{2\sqrt{M^2 + \bar{p}^2}} \phi(\bar{p}), \quad p_0 = \sqrt{M^2 + \bar{p}^2}, \]

which converges due to the Gaussian decay of \( \phi \) in the wedge regions containing each component of the complex momentum in \( M\mathbb{R}^3 \). To obtain the position space form of the distribution, note that we can deform the complex contours in the above convergent integral to the real line without encountering singularities to obtain an integral over real momenta, and then carefully
exchange limits as follows:

\[
\int_{MR^3} \frac{d^3 \tilde{p}}{(2\pi)^3} \frac{e^{i(p_0 x^0 - p_\mu y^\mu)}}{2\sqrt{M^2 + \tilde{p}^2}} \phi(\tilde{p})
\]

\[= \int_{R^3} \frac{d^3 \tilde{p}}{(2\pi)^3} \frac{e^{i(p_0 x^0 - p_\mu y^\mu)}}{2\sqrt{M^2 + \tilde{p}^2}} \phi(\tilde{p})
\]

\[= \int_{R^3} \frac{d^3 \tilde{p}}{(2\pi)^3} \frac{e^{-i p_\mu y^\mu}}{2\sqrt{M^2 + \tilde{p}^2}} \int_{R^3} d^3 \tilde{x} \ e^{i p_\mu x^\mu} \phi(x)
\]

\[= \lim_{\epsilon \to 0^+} \int_{R^3} \frac{d^3 \tilde{p}}{(2\pi)^3} \frac{e^{-i p_\mu y^\mu - \epsilon |\tilde{p}|}}{2\sqrt{M^2 + \tilde{p}^2}} \int_{R^3} d^3 \tilde{x} \ e^{-i p_\mu x^\mu} \phi(x)
\]

\[= \int_{R^3} d^3 \tilde{x} \ \phi(x) \left( \lim_{\epsilon \to 0^+} \int_{R^3} \frac{d^3 \tilde{p}}{(2\pi)^3} \frac{e^{i p_\mu (x^\mu - y^\mu) - \epsilon |\tilde{p}|}}{2\sqrt{M^2 + \tilde{p}^2}} \right). \quad (19)
\]

Note that the limit added in the third step is entirely superfluous, but allows us to exchange integrals in the last step. The factor in parentheses can then be evaluated in terms of Hankel functions as in the case of real $M$. The factor $e^{-\epsilon |\tilde{p}|}$ happens to be convenient for evaluating the integral for time-like $x - y$, and could have been replaced by something like $(\tilde{p}^2 + M^2)^{-\epsilon}$ for convenience in evaluating the integral for space-like $x - y$.

It is important to note that the use of an analytic test space does not limit our ability to describe the world. For example, any square-integrable single-particle wave function can be approximated arbitrarily closely by an element of $S^{1/2, B}(\tilde{k})$. Furthermore, because the analytic test space is smaller than the usual space of Schwartz test functions, the dual space of generalized functions is larger than the space of Schwartz distributions. We therefore gain the ability to describe extra interesting generalized states, such as states with complex energy-momentum, that are not Schwartz distributions. Indeed, there is no first principle that forces one to accept Schwartz distributions as canonical, and in fact Schwartz distributions are known to be insufficiently general for describing such useful entities in ordinary scattering theory as the complex energy-momentum Gamow states describing resonances or unstable particles [27, 31, 32, 33, 34, 35, 36, 37].

One important consequence of choosing an analytic test space is that the continuum part of the spectral decomposition of operators is no longer unique. This gives us considerable freedom in choosing sets of continuum states to use in completeness relations. In the present case, this freedom
will allow us to find an equivalent formulation of the full theory based on a Lorentz-invariant state space.

As a simple one-dimensional example of this freedom, consider an element \( \phi \in S_{1/2,A}^{1/2,B}(x) \). We have

\[
\phi(x) = \int \frac{dk}{2\pi} e^{ikx} \tilde{\phi}(k),
\]

where the Fourier transform \( \tilde{\phi} \in S_{1/2,B}^{1/2,A}(k) \) is also an entire function. As a result, we can deform, say, a finite section of the integration path from the real line into the complex plane. Call the deformed path \( \Gamma \). Then

\[
\phi(x) = \int_{\Gamma} \frac{dk}{2\pi} e^{ikx} \tilde{\phi}(k)
= \int_{\Gamma} \frac{dk}{2\pi} e^{ikx} \int_{\mathbb{R}} dy e^{-iky} \phi(y).
\]

The integral over \( y \) converges for any complex \( k \) due to the Gaussian decay of \( \phi \in S_{1/2,A}^{1/2,B}(x) \). But for complex \( k \), we have

\[
e^{-iky} = (e^{ik^*y})^* = \langle y|k^*\rangle^* = \langle k^*|y\rangle.
\]

Therefore

\[
\tilde{\phi}(k) = \langle k^*|\phi\rangle,
\]

and

\[
\phi(x) = \int_{\Gamma} \frac{dk}{2\pi} \langle x|k\rangle \langle k^*|\phi\rangle.
\]

In other words, the set of complex-momentum null generalized states lying along \( \Gamma \) is also complete when acting on test functions. Formally, the identity on \( S_{1/2,A}^{1/2,B}(x) \) has many possible spectral decompositions in terms of momentum eigenstates, such as

\[
\phi(x) = \int_{\mathbb{R}} \frac{dk}{2\pi} \langle k|\phi\rangle
= \int_{\Gamma} \frac{dk}{2\pi} \langle k|k^*\rangle
= \frac{1}{2} \int_{\Gamma} \frac{dk}{2\pi} \langle k|k^*\rangle + \frac{1}{2} \int_{\Gamma^*} \frac{dk}{2\pi} \langle k|k^*\rangle
= \cdots.
\]
All these representations are equivalent for doing physics, given the assumption that we can approximate physical wave packets by test functions. However, notice that the second completeness relation does not extend to the formal generalized states $|k\rangle$ and $|k^*\rangle$ themselves since, for example, $\langle k^*|k^* \rangle = 0$. In other words, we cannot carelessly generalize relations outside the domain of the distributions involved.\textsuperscript{8}

This is analogous to what is happening in our field theory. Consider the previously described scattering of a non-ghost incoming wave packet in $S^{1/2,B}_{1/2,A}$. We could deform the integration contour lying above the fish-shaped non-analyticity region to the real line along the analytic continuation of the integrand in the amplitude. The analytic continuation was Lorentz-invariant and had a cut along the real axis starting at $s = (M + M^*)^2$ [3, 7, 9]. This suggests that the amplitude obtained from the set of intermediate complex $s$ two-ghost states indexed by the points in the non-invariant fish-shaped region is equivalent, modulo the $S^{1/2,B}_{1/2,A}$ test space, to the amplitude obtained from some set of real $s$ intermediate states indexed by the points along the cut on the real axis. As we shall see, the latter set of states can be chosen to be manifestly Lorentz-invariant, and so that their momenta and energies are real.

In the unconstrained theory, it therefore will not make a difference, for the calculation of scattering amplitudes of non-ghost states, whether we use this manifestly Lorentz invariant set of intermediate real-energy ghost states or the Lorentz non-invariant fish-shaped set of complex-energy ghost states. Either set of continuum states can be expressed in terms of the other. With one set, Lorentz invariance is explicit; with the other, it is hidden. However, the choice will make a difference when applying the Lee-Wick real energy constraint, which we will do in a non-perturbative setting where we apply an infrared regularization (which discretizes momenta and causes the continuum states to become actual normalizable states), then apply the Lee-Wick real energy constraint, and then remove the regularization. This procedure will give a Lorentz-invariant result only if we use the manifestly Lorentz-invariant multi-ghost states.

\textit{We will therefore insist that any non-perturbative approach to applying the Lee-Wick real energy constraint should be based, from the start, on a}

\textsuperscript{8}Although the third line above can be made to work also when applied to generalized momentum states as long as we choose the inner products $\langle k|k^* \rangle$ proportional to the appropriate delta functions with respect to the integrals along $\Gamma$ and $\Gamma^*$.  

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33
manifestly Lorentz-invariant state space.

We now proceed to find a Lorentz-invariant set of two-particle states that can represent a ghost of complex mass $M$ together with the corresponding anti-ghost of mass $M^*$. An obvious, but unworkable, approach would consist of following the Fock construction by making two-particle states from the free tensor product of the invariant mass hyperboloids, discussed above for a single particle, of masses $M$ and $M^*$. This space, convenient as it would have been, is unsuitable for the following reason. It is not difficult to see that a given component of the combined space-like momentum of the two particles in this parameterization can be an arbitrary complex number. As a result, the scattering amplitude with intermediate two-particle states in this space will include an integration over the complex plane, weighed by the incoming wave packet in $S_{1/2}^B$. However, as we have specified the test space, this wave packet has exponential growth in the imaginary direction, and we will not get a finite result.

Therefore, we have to be more careful in specifying an invariant two-ghost state space. We will first write down one particular suitable choice of invariant space and then justify it below by considering the explicit expression for certain Hamiltonian matrix elements.

In (19) we saw that single-particle three-momenta can be deformed from complex to real modulo the analytic test space. Let $b_\bar{k}$ and $\tilde{b}_\bar{k}$ be creation operators for the single ghost states with four-momenta $(\sqrt{M^2+k^2},\bar{k})$ and $(\sqrt{(M^*)^2+k^2},\bar{k})$ respectively, where $\bar{k}$ is real. The particular Lorentz-invariant space of real-energy two-ghost states that we will use is spanned by

$$\left\{ L\left(b_\bar{k}\tilde{b}_{-\bar{k}}\right) | 0 \rangle \mid \bar{k} \text{ real} \right\}.$$  \hspace{1cm} (20)

Here

$$L\left(b_\bar{k}\tilde{b}_{-\bar{k}}\right) \equiv L\left(b_\bar{k}\right)L\left(\tilde{b}_{-\bar{k}}\right),$$

and, taking $L$ to be, for example, a boost in the $x_3$-direction, we have

$$L\left(b_\bar{k}\right) \equiv b_{(k_1,k_2,\sqrt{M^2+k^2}\sinh\beta+k_3\cosh\beta)}$$

and likewise for $\tilde{b}_\bar{k}$ with $M$ replaced by $M^*$. The transformed operators individually have complex momenta, but it is easily checked that the combined four-momentum of the two-particle state is real, and therefore fall inside the wedge where the test functions have exponential decay. This space of states

34
should therefore be suitable for the calculation of scattering amplitudes with incoming wave packets in the space $S^{1/2}_{1/2, A}$. It should be noted that this space of two-particle states is not a free tensor product of single ghost states.

To justify this choice of two-particle space from a non-perturbative point of view, we need to find the matrix elements of the Hamiltonian for creation of the two-ghost states in this invariant set from a single-particle state. (More generally, we need the $m \to n$ particle matrix elements, which can be obtained via a similar procedure.) Assuming for simplicity that all fields are scalars, a typical cubic term in the Hamiltonian will have the form

$$\int_{\mathbb{R}^3} \frac{d^3 \tilde{p}}{(2\pi)^3} \frac{1}{2E_{\tilde{p}}} a_p \int_{\mathbb{R}^3} \frac{d^3 k}{(2\pi)^3} \frac{1}{2E_{\frac{1}{2} \tilde{p}+k}} \frac{1}{2E_{\frac{1}{2} \tilde{p}-k}} b_{\frac{1}{2} \tilde{p}+k}^\dagger \tilde{b}_{\frac{1}{2} \tilde{p}-k}^\dagger$$

with respect to invariantly normalized real-momentum creation-annihilation operators. Here $a$ is the ordinary scalar of real mass $m$ and $b$ and $\tilde{b}$ the ghosts of complex masses $M$ and $M^*$ respectively. Acting on one-particle states, this term will not explicitly create two-ghost states in the manifestly covariant space constructed above, but modulo the analytic test space, we can deform the integral over $\tilde{k}$ to do so. Without loss of generality, take

$$\tilde{p} = (0, 0, p_3) \equiv (0, 0, \sinh \beta \left( \sqrt{k^2 + M^2} + \sqrt{k^2 + M^*^2} \right),$$

which defines $\beta$, and deform the integral over $\tilde{k}$ to

$$\int_{\mathbb{R}} \frac{dk_1'}{(2\pi)} \int_{\mathbb{R}} \frac{dk_2'}{(2\pi)} \int_{\Gamma_{k_1', k_2'}} \frac{dk_3'}{(2\pi)} \frac{1}{2E_{\frac{1}{2} \tilde{p}+k'}} \frac{1}{2E_{\frac{1}{2} \tilde{p}-k'}} b_{\frac{1}{2} \tilde{p}+k'}^\dagger \tilde{b}_{\frac{1}{2} \tilde{p}-k'}^\dagger$$

where the complex curve $\Gamma_{k_1', k_2'}$ in the $k_3'$ plane is defined by the equation

$$k_3' = \frac{1}{2} \sinh \beta \left( \sqrt{k^2 + M^2} - \sqrt{k^2 + M^*^2} \right) + k_3 \cosh \beta,$$

$$k_{1,2} = k_1', k_3 \in \mathbb{R}.$$

This particular deformation is justified when we take the matrix elements of the expression between the one-particle and two-ghost subspaces of the

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9There are other possible choices of Lorentz-invariant two-ghost state spaces. For example, we could have taken the space-like momenta of both particles from $M\mathbb{R}^3$ instead of $\mathbb{R}^3$. We conjecture that all such choices should be equivalent as far as scattering of real-energy non-ghost particles are concerned.

35
analytic test space $S_{1/2,A}^1$, and different deformations will be needed for higher matrix elements. The curve $\Gamma_{k_1,k_2}$ asymptotically approaches the real line as $k_3 \to \infty$, and therefore lies within the region of Gaussian decay of the test functions. Since

$$\frac{1}{2} \bar{p} + \bar{k}' = (k_1, k_2, \sqrt{k_2^2 + M^2} \sinh \beta + k_3 \cosh \beta),$$

$$\frac{1}{2} \bar{p} - \bar{k}' = (-k_1, -k_2, \sqrt{k_2^2 + M^*^2} \sinh \beta - k_3 \cosh \beta),$$

it follows that the states $b_{\frac{1}{2}p+k}^\dagger \tilde{b}_{\frac{1}{2}p-k}^\dagger |0\rangle$ lie in the Lorentz invariant two-ghost state space, and we can read off the matrix elements linking the one-particle states to the manifestly invariant two-ghost space from the deformed Hamiltonian.

Note that this deformation is only valid for the specific $1 \to 2$ matrix elements considered above. For general $m \to n$ matrix elements, Lorentz-invariant state spaces of multi-particle states can be generated in a similar way as for the two-ghost states above. However, different deformations of the cubic term in the Hamiltonian will generally be required to obtain the matrix elements linking these invariant spaces. A fuller analysis is left for future work.

Given the equivalence, modulo the test space, of the original and the deformed Hamiltonian matrix elements, it follows that both should give equivalent amplitudes for scattering of wave packets of non-ghost states. It is the amplitude calculated using the original Hamiltonian matrix elements (21) that has the previously mentioned non-Lorentz-invariant fish-shaped non-analyticity region in $s$-space (corresponding to the complex energies of the intermediate ghost states) with the contour of integration running above this region, and it is the amplitude calculated using the equivalent deformed matrix elements (22) that has the Lorentz-invariant cut in $s$-space along the real line corresponding to the real energies of the intermediate two-ghost states from the Lorentz-invariant set. The equivalence of these amplitudes means that the integrand in the latter should be the continuation of the same analytic function as that in the former from outside the fish-shaped region. This is supported by the observation that the continuation of the former amplitude from outside the fish-shaped non-analyticity region does give an analytic function with a cut starting at $(M + M^*)^2$ along the real axis in the $s$-plane, whose explicit form can be found in [3, 9].
We have succeeded in rewriting the amplitudes in terms of a Lorentz-invariant set of two-ghost states. However, at this point it appears that these states have real energy, and therefore will not be eliminated directly by the Lee-Wick real energy constraint. This seems, at first glance, to be a disaster for unitarity. However, we shall indicate that the problematic negative-definite states do get eliminated after taking into account their interactions in a non-perturbatively well-defined way. We shall further see that, for the diagram we consider, the method will effectively reproduce Lee’s and Cutkosky et al.’s prescription for projecting out these states [3], without relying on their ad hoc assumptions.

9 Nonperturbative unitarity via elimination of multi-ghost states

Our analysis so far would suggest that the theory contains real-energy multi-ghost states. If this were the case, there would be a problem with unitarity, since the theory would have negative-metric states that could not be eliminated by the Lee-Wick real energy constraint. In this section, we will argue that, in a specific non-perturbatively well-defined approach, the energies of these states in fact become non-real due to interactions. As a result, many multi-ghost states will be eliminated by the Lee-Wick real-energy constraint. The rest will be removed by a ghost number constraint on the asymptotic states.\(^{10}\)

The simplest case where the problem might occur are the two-particle states consisting of a ghost and anti-ghost degree of freedom of the original Lagrangian. We have seen that the mass of these particles remains real in the presence of interactions, and so the two-particle states would be expected to have real energies. A slightly more complicated case where this may occur are the two-particle states consisting of a ghost positronium particle and anti-particle. These have complex conjugate masses, but as we saw in the previous section, we can re-express the two-particle states in terms of

\(^{10}\)In the next section, we will describe an alternative, less general but very simple method, that may also, in certain cases, be useful for eliminating these multi-ghost states. The method in that section constrains the mass of the ghosts to be of the same order as the total mass-energy of the modeled universe. The more general method in the current section imposes no such constraint on the ghost mass.
a Lorentz-invariant space of real-energy states.\footnote{In this case, individual particles have complex energy, and are therefore null states. However, the degenerate real-energy (anti-)symmetric combinations of these are positive (negative) definite.} Since in both cases we find negative metric states with real energy, this seems to be a disaster for unitarity, since the Lee-Wick real energy constraint will not eliminate such states.

*However, interactions can be expected quite generally to move the mass-energy of these states away from the real axis into the complex plane, at which point the Lee-Wick real energy constraint can be applied to eliminate them.*

Note that this statement would not be true for ordinary unitary local field theories where, given the exact single-particle energies, we can infer the exact energies of two-particle states to be simple sums of single-particle energies by simply increasing the distance between the two particles to the point where the interactions are sufficiently negligible. For interactions to make a difference to the two-particle energies, we could impose a large-distance (infrared) cutoff, which would prevent us from increasing the separation arbitrarily. However, in ordinary unitary local field theories, the energy shift due to interactions is real and vanishes once we remove the cutoff.

In pseudo-unitary theories, the energies of any two-particle negative-metric states will likewise be modified by interactions when we impose a large-distance cutoff preventing indefinite separation. However, as we will argue, with certain mild conditions, these modified energies will generally become non-real. Therefore, as long as we implement the Lee-Wick real energy constraint before removing the large-distance cutoff, these states will be eliminated.

Technically, what happens is analogous to our previous discussion of the single pole due to a negative-metric state lying on a cut due to positive definite states. Unless a symmetry forbade the interaction, the negative-metric state became unstable with respect to decay into the positive definite states forming the cut, and as a result the pole split into a pair of complex conjugate positions on the physical sheet away from the real axis. The single negative metric state can linearly combine with positive metric states to give precisely two dual null states. All other states had to remain on the real axis.

In general, the two-ghost negative metric cut is degenerate with a cut due to positive-metric states of much lower mass, and to which the two-ghost states can transition. For example, a pair of complex ghost-positronium
atoms can annihilate and transition to a multi-photon or multi-lepton state. We then assume a long-distance regularization that temporarily discretizes the four-momenta. Because the masses of the constituents of the negative metric two-ghost states are by assumption much larger than those of the constituents of the positive-metric transition products, as we increase the large distance cutoff the positive-metric states can, at the high energies where the two-ghost states appear, be represented to an increasingly good approximation by a continuous cut in $s$, while the two-ghost states, which have comparatively much smaller phase volume in the now discretized momentum space, may still be considered discrete. Just as in the single-pole case, the interaction will split each negative-definite state into a pair of dual null states with complex-conjugate mass-energies.

The conditions for this argument to work can be expressed in another way. As we saw by way of the toy example in section 5, interactions between a positive metric state and a negative metric state, both of whose unperturbed energies are real, will generally split the spectrum into a pair of null states with complex conjugate energies as long as the distance between the unperturbed energies is comparable to the scale of the interaction. In the field theory case, as long as the ghost masses are much larger than the masses of the transition products (in this case lepton or photon multi-particle states), each two-ghost state will, as we increase the large-distance cutoff, approach degeneracy with many positive-metric states into which they can transition. Thus, the conditions are met for all these negative-metric states to be removed from the real spectrum by the interaction, combining with suitable positive metric states to form dual null states with complex conjugate energies. Only positive-metric states should remain in the real spectrum.

According to our introductory arguments, as we remove the long-distance cutoff, the energies of these null states will become real. For this reason, we propose imposing the Lee-Wick real energy constraint before removing the cutoff. In particular, the procedure is as follows:

**Impose a large-distance cutoff and show that the interaction moves negative-definite states away from the real line. Then apply the Lee-Wick real energy constraint before removing the cutoff. In symbols, the state space is obtained as follows.**

$$\lim_{L \to \infty} P_{RL}^{H_L} \mathcal{H},$$

where $\mathcal{H}$ denotes the full indefinite inner product state space and $P_{RL}^{H_L}$ is the projection onto the real-energy eigenspace of the regularized Hamiltonian.
$H_L$, where $L$ is the long-distance cutoff.

It is interesting that this method will, in a simple example, produce a result very much like Lee’s ad hoc prescription (and that of Cutkosky et al.) for eliminating these continuum states. Our method provides a justification for their prescription. Unlike their approach, ours is non-perturbatively well-defined in principle, and we rely on higher-order interaction effects.

As an illustration, let us reconsider a scattering process in which the photon appears as intermediate state in the $s$-channel. The transverse part of the photon propagator is

$$D(q^2) = -\frac{ie^2}{q^2} \frac{1}{1 + e^2 \Pi(q^2)}, \quad (23)$$

where the vacuum polarization contribution was calculated in section 4 to be

$$\Pi(q^2) = \frac{e^2}{2\pi^2} \int_0^1 d\beta \beta(1 - \beta) \ln \frac{M^2 - \beta(1 - \beta)q^2 - i0}{m^2 - \beta(1 - \beta)q^2 - i0}. \quad (24)$$

The scattering amplitude is then proportional to

$$D(s) = -ie^2 \left( \frac{1}{s} + \int_{4m^2}^{\infty} ds' \frac{\rho_m(s')}{s - s' + i0} + \int_{4M^2}^{\infty} ds' \frac{\rho_M(s')}{s - s' + i0} \right). \quad (25)$$

Here $\rho_m$ represents the unitarity sum over intermediate pairs of ordinary fermions, and $\rho_M$ the unitarity sum over intermediate pairs of ghosts.

This expression is valid as it stands below the $2M$-threshold. However, above the threshold, the off-shell amplitude will be modified when we apply the non-perturbatively well-defined projection described above. This will be fortunate, because the cut starting at the $2M$ threshold is incompatible with unitarity, corresponding as it does to the production of negative-definite states.

We now follow our suggested procedure and impose a large-distance cut-off, which discretizes the momenta. The cut starting at $4M^2$ will become a series of poles. The ghost states represented by these poles can annihilate to give a virtual photon that decays to ordinary electron-positron states of much lower mass. According to the general argument above, this interaction will make these two-ghost poles move away from the real axis to give a pair of complex conjugate poles corresponding to null states. Only positive definite states remain on the real line.
The Lee-Wick real energy constraint, which eliminates the null states, is equivalent to changing the integration contour in the $s$-plane, which in the full theory runs above all these complex poles, to run along the real axis. Our non-perturbatively well-defined prescription consists of doing this change of contour before removing the regularization. Therefore, half of the null state poles will unambiguously lie at complex positions above the integration contour and half below. The result when we do finally take the continuum limit is therefore equivalent to calculating the amplitude with the discrete two-ghost poles becoming a pair of cuts, one running infinitesimally above the real axis and the other running below it, and with the integration contour running along the real axis between them. The cuts meet at the threshold $2M$, so that the values of the amplitude below and above the threshold belong to separate sheets of an analytic function.

Interestingly, the numerical value of the amplitude above the threshold can in fact be read off from the original continuum calculation of the amplitude. First we note that the photon propagator was obtained from a geometric series that already included interactions between the negative and positive definite states that should be responsible for moving the ghost pairs into the complex plane. Indeed, the propagator summed sequences of bubbles to all orders, including diagrams in which ghost pairs and ordinary fermion pairs alternate. Therefore, in the no-cutoff limit, the result (25) already takes into account the contribution of the conjugate pair of cuts described in the previous paragraph. As far as the integration contour of the full unconstrained theory, which runs above all singularities in the $s$-plane, is concerned, these cuts can be deformed to the real line and the result is indistinguishable from the single cut in the expression (25). However, according to the reasoning of the previous paragraph, the integration path in the projected theory must go between the conjugate cuts infinitesimally above and below the real line. The one function that will give the correct result with respect to both the unconstrained theory’s contour above the singularities and the constrained
theory's real-line contour can therefore only be

\[
D(s) = -ie^2 \left( \frac{1}{s} + \int_{4m^2}^{\infty} ds' \frac{\rho_m(s')}{s - s' + i0} \right)
\]

\[
+ \frac{1}{2} \int_{4M^2}^{\infty} ds' \rho_M(s') \left\{ \frac{1}{s - s' + i0} + \frac{1}{s - s' - i0} \right\}
\]

\[
= -ie^2 \left( \frac{1}{s} + \int_{4m^2}^{\infty} ds' \frac{\rho_m(s')}{s - s' + i0} + P \int_{4M^2}^{\infty} ds' \frac{\rho_M(s')}{s - s'} \right). \tag{28}
\]

This result is now consistent with unitarity, since the last term in parentheses does not contribute a imaginary part for real \(s\).

Similar considerations will apply to states of higher ghost number. The above construction (20) of a Lorentz-invariant space of two-ghost states can be generalized to obtain Lorentz-invariant state spaces containing an arbitrary number of ghost excitations. For example, take all individual space-like momenta real in the center of mass frame, and then generate an invariant set of states by applying the Lorentz group. Other Lorentz-invariant constructions are possible (as we saw before, there is no unique continuum basis dual to spaces of holomorphic test functions), but this non-uniqueness should not matter for calculating amplitudes of non-ghost particles, which should be unique. Again, these states are not free tensor products of single-particle states. However, their matrix elements in the Hamiltonian can again be obtained via a similar procedure as was used in the previous section to obtain matrix elements involving two-ghost states. A more exhaustive analysis is deferred to future work.

Again, this construction will in general give negative-metric ghost states of real energy, which would be disastrous for unitarity unless we can eliminate them by applying the Lee-Wick real energy constraint as we did to eliminate the two-ghost states above. The construction will also give positive-metric states containing ghosts (for example, any state containing any number of ghosts or an even number of anti-ghosts). Of these, our non-perturbatively well-defined limiting procedure should eliminate all multi-particle states containing at least one ghost and anti-ghost (irrespective of whether the overall metric of the state is positive or negative) since these can annihilate pairwise and transition into multi-photon or multi-lepton states consisting of particles of much lower mass. The metric of the resulting state has sign opposite to the original. As before, for large enough distance cutoff, these ghost states appear discrete compared to the effective continua of ordinary physical par-
articles into which they can transition. Thus, the conditions are satisfied for the state to split into a pair of complex-conjugate energy states. Again, applying the Lee-Wick real energy constraint and then removing the cutoff will eliminate these states.

Our method so far has eliminated all states containing at least one ghost and one antighost. We now discuss the elimination of the remaining states that contain ghosts.

We now argue that we will obtain a unitary ghost-free theory by further restricting the space of asymptotic scattering states (in other words, the eigenstates of the free Hamiltonian $H_0$ used to set up the scattering problem in section 6) to the zero ghost number sector. Ordinarily, the interaction would allow ghosts and anti-ghosts to be created in pairs from initial states with zero ghost number, but since all states containing such pairs are already removed from the asymptotic spectrum by the Lee-Wick real-energy constraint, no ghosts will be created by the S-matrix of the constrained theory.\textsuperscript{12}\textsuperscript{13}

To summarize, we have argued that we can obtain a unitary S-matrix applying the Lee-Wick real energy constraint and large-distance regularization in proper order, and by restricting the asymptotic state space to the ghost number zero sector.

10 A second mechanism for unitarity

In the previous section, we described a general method for eliminating real-energy multi-ghost states in a non-perturbatively well-defined framework that consists of applying the usual Lee-Wick real energy constraint before removing the long-distance cutoff.

In this section, we will describe a simpler but less general alternative construction that may be useful in specific cases. However, the approach of this section requires the ghost mass to be of the order of the mass-energy of the modeled universe.

\textsuperscript{12}However, note that the exact states that are obtained from zero-ghost asymptotic states via the mappings $W_\pm$ in section 6 are still in general nontrivial superpositions containing ghost terms. The zero-ghost constraint is imposed on the eigenstates of $H_0$ labeling the asymptotic states and make sense only in the context of the S-matrix.

\textsuperscript{13}The zero ghost number constraint also eliminates otherwise stable states states containing only ghosts or only anti-ghosts, which have real mass-energy even in the presence of interactions, as we saw in section 5.
Our approach here is based on the idea that a theory describing a universe of finite mass-energy only needs to be unitary below this scale. Negative-metric states with mass-energy larger than the total mass of the universe are never inhabited, so a theory that has such states can still be unitary for all physically realizable states. For the sake of this argument, we assume a flat Minkowski universe, and proceed as follows:

- Assume that the mass-energy of the universe is bounded by a finite number. More precisely, assume that the state of the universe is expressible as a superposition of eigenstates of the Hamiltonian, the energies of which have an upper bound. Any separable subsystem of the universe will necessarily have a mass-energy lower than this number.

- A quantum theory of the universe must be able to predict a unitary S-matrix for the universe and for any separable subsystem of the universe. A quantum theory can be a complete description of a universe of bounded mass-energy without needing to model states of mass-energy above this bound unitarily, or at all.

- We can retain exact Lorentz invariance, as well as locality and cluster decomposition to within the observational limits implied by these assumptions, by using pseudo-unitary quantum field theories that are unitary at mass-energies lower than the universe bound.

This approach immediately considerably widens the range of possible quantum field theories that may be considered as complete theories describing universes of bounded mass-energy. In particular:

Consider an indefinite metric quantum field theory that is asymptotically safe, so that it is ultraviolet compete and has exact Lorentz invariance. Suppose that the center of mass energy of the lowest negative definite eigenstate with real energy of the Hamiltonian $H$ is $M$. Then the subspace of real-energy eigenstates of $H$ satisfying the constraint $P_\mu P^\mu < M^2$ is positive definite. By restricting to this subspace, we obtain a unitary and Lorentz-invariant quantum theory.

Note the following:
• The requirement of asymptotic safety is very important. It is a necessary condition for theory to have a well-defined continuum limit, without which Lorentz-invariance would not hold, and the constraint on the state space would not be invariant.

• The resulting theory has exact Lorentz covariance. Its space of physical states is Lorentz-invariant.

• Because the physical sector is embedded in a larger quantum field theory, and we expect cluster decomposition to hold in the latter due to the local nature of its formulation, independently of the signature of inner product, we also expect to see cluster decomposition in the physical sector to within the observational limits implied by the constraint on physical states.

• It is important to note that virtual momenta in Feynman loops are not cut off. Loops are already finite in the unconstrained theory through asymptotic cancellations as in section 4, or through renormalization as in section 5, before imposing the constraint. The constraint \( P_\mu P^\mu < M^2 \) selects a subspace of physical states post hoc from a larger indefinite metric theory that is well-defined at arbitrarily high energy, but does not affect virtual loops.

• Since the theory is ultraviolet complete, the constraint is not a regularization, and it is not a measure of our imperfect knowledge of higher-energy physics.

The constraint on \( P_\mu P^\mu \) above implies the Lee-Wick real-energy constraint. As far as S-matrix calculations are concerned, we have to use the Lee-Wick methods as before to eliminate the exponentially growing contributions of any complex-energy eigenstates the theory may have. Imposing the \( P_\mu P^\mu \) constraint on the free asymptotic states in the far past will ensure, as in ordinary field theories, that no states with real mass-energy above the bound will be produced. However, as we saw before, any non-zero overlap of free states with complex-energy eigenstates of the full Hamiltonian leads to exponential terms that cannot be ignored.

Stated another way, applying the constraint \( P_\mu P^\mu \) on the asymptotic scattering states is complicated by the fact that, as we have seen, we are not entitled to regard scattering states in the far past or future as approaching
eigenstates of the free Hamiltonian, even though to a local observer they become locally indistinguishable from free eigenstates to arbitrary accuracy. Small discrepancies due to overlaps with complex-energy states grow exponentially under time evolution. Constraining the spectrum to real energies can be done by identifying and discarding these exponential contributions, just as we discussed before.

The feasibility of a constraint of this type in modeling somewhat realistic universes is supported by the calculation in section 4. In particular, taking the fine structure constant

$$\alpha \sim \frac{1}{137}$$

in the theory of ghost QED, and $m \sim 0.511 \text{ MeV}$ and, as using, for the sake of applying the constraint $P_\mu P^\mu < M^2$, a ghost mass $M$ as large as the mass-energy of the visible part of our universe, estimated as

$$M \sim 1.25 \times 10^{82} \text{ MeV},$$

we find a small bare coupling

$$\alpha_\infty \sim \frac{1}{100},$$

and, more importantly, an effective perturbative expansion parameter to first order

$$\frac{\alpha_\infty}{3\pi} \ln \frac{M^2}{m^2} \sim 0.42 < 1,$$

(29)

despite the enormous value of $M/m$. In other words, perturbation theory remains valid for all energies, and the theory could have been a complete theory if the universe consisted of QED only. In the approach of this section based on the $P_\mu P^\mu$ constraint, and given the measured values of $\alpha$ and $m$, ghost QED can only be a complete theory if the mass-energy of the modeled universe is finite and less than $M$, where $M$ is constrained by the requirement that (29) be sufficiently smaller than unity.

In the mechanism for obtaining unitarity discussed in the previous sections, the same upper limit on $M$ is valid. However, in that case, $M$ did not limit the mass-energy of the universe, and could in fact be much smaller than the mass-energy of the universe. In that case, $M$ is the scale at which non-local or unusual causal effects become observable, as we shall see below in section 12.
11 Hierarchy and naturalness

In our second approach to unitarity, based on the constraint \( P_\mu P^\mu < M^2 \) discussed in the previous section, there is an amusing naturalness relationship between the fine-structure constant and the matter-ghost hierarchy constrained by the the mass-energy of the QED universe.

Specifically, the value of the fine structure constant may be related to the upper bound on the mass-energy of the universe via a naturalness argument.

As we saw, for the theory to be well-defined, the leading loop expansion parameter

\[
\frac{\alpha_{\infty}}{3\pi} \ln \frac{M^2}{m^2},
\]

which is the largest perturbative parameter in the theory, must be smaller than one. If we consider it natural that this parameter be of order unity, we may conclude that the fine-structure constant must be small if the mass-energy of the universe is large compared to the mass of the electron. In other words, an observation of the total mass-energy of the QED universe can be used to infer, via naturalness, an upper bound for the value of the fine structure constant.

As we saw in the previous section, the value of 0.42 for the loop expansion parameter, which is very much of order unity and presumably “natural”, is remarkably compatible with realistic values of \( \alpha \), electron mass, and (visible) universe mass energy in our own universe. Thus, the large mass of the universe may be invoked to “explain” the small observed value of the fine structure constant.

The argument so far assumed our second approach to unitarity based on the \( P_\mu P^\mu \) constraint. In the first approach, the ghost mass \( M \) did not constrain the mass-energy of the modeled universe. In this case, we can still make a slightly weaker statement based on naturalness. Specifically, the argument remains true that it is natural for a larger hierarchy \( M/m \) to be associated with a smaller fine structure constant \( \alpha \), with the logarithmic relationship implied by (30). The difference in this case is that now \( M/m \) is unrelated to universe size.
12 Causality

An interesting feature of the unitary theories obtained by the Lee-Wick real energy constraint is an unusual causal behaviour. Aspects of this were described by Lee and Wick and further discussed by Coleman [1, 2, 3, 13]. These authors used the term ‘acingual’ in describing the behaviour. We find this choice of terminology unfortunate, since it may encourage the incorrect conclusion that such theories might allow, for example, for past events to be influenced by future actions. Such effects are not possible.

As we will discuss in this section, it may be preferable to think of theories of this type not as acausal but rather as non-local. Using Lee’s original terminology, states may contain initially undetectable ‘precursors’ that are exponentially amplified by time evolution and become relevant at later times. To emphasize that causality is satisfied, we will perform a typical thought experiment in a simple toy model and illustrate how acausal inconsistencies are avoided.

For the reader uncomfortable with this aspect of these theories, we point out that locally undetectable precursor degrees of freedom have been invoked to explain aspects of the bulk-boundary correspondence in String Theory holography [14, 15]. We further point out that Horowitz and Maldacena recently proposed resolving the black hole information paradox via a final state boundary condition [16]. Again, this is not so alien in the context of Lee-Wick theories, since the Lee-Wick real energy constraint can be reinterpreted as incorporating a final state boundary condition, namely, no blow-up at future infinity [6].

We will discuss the issues in a simple toy model that captures the essential features of the field theory scattering discussed in section 6. Consider a model that has three states, denoted

\[ \{ |\text{incoming}\rangle, |\text{outgoing}\rangle, |\text{ghost}\rangle \}, \]

with inner product

\[ \eta \equiv \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \]

We declare the physical state at time \( t = 0 \) to be

\[ |\text{incoming}\rangle + |\text{outgoing}\rangle + |\text{ghost}\rangle \]
and introduce a scattering matrix between times $-T$ and $T$ as

$$S \equiv W^{-1}_+ W_-$$

where

$$W_- \equiv \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} \left( \frac{1}{\epsilon} + \epsilon \right) & \frac{1}{2} \left( \frac{1}{\epsilon} - \epsilon \right) \\ 0 & \frac{1}{2} \left( \frac{1}{\epsilon} - \epsilon \right) & \frac{1}{2} \left( \frac{1}{\epsilon} + \epsilon \right) \end{pmatrix}$$

and

$$W^{-1}_+ \equiv \begin{pmatrix} \frac{1}{2} \left( \frac{1}{\epsilon} + \epsilon \right) & 0 & -\frac{1}{2} \left( \frac{1}{\epsilon} - \epsilon \right) \\ 0 & 1 & 0 \\ -\frac{1}{2} \left( \frac{1}{\epsilon} - \epsilon \right) & 0 & \frac{1}{2} \left( \frac{1}{\epsilon} + \epsilon \right) \end{pmatrix},$$

where

$$\epsilon \equiv e^{-\gamma T}, \quad \gamma > 0.$$

Applying $W_-^{-1}$ and $W_+$ respectively to the physical state at time $t = 0$, the physical state at time $t = -T$ is obtained as

$$|\text{incoming}\rangle + \epsilon (|\text{outgoing}\rangle + |\text{ghost}\rangle)$$

and the physical state at time $t = T$ is obtained as

$$|\text{outgoing}\rangle + \epsilon (|\text{incoming}\rangle + |\text{ghost}\rangle).$$

As in the field theory discussed in section 6, the in and out states contain an $\epsilon$-contribution that vanishes asymptotically and cannot be locally distinguished by observers in the sufficiently far past and future. As we discussed, these corrections grow exponentially with time and cannot be omitted, unlike in the case of a positive-definite theory.

The above scattering matrix is meant to capture the essentials of what happens in a local field theory with the set of processes indicated in the following set of diagrams:

\[\text{Diagram}(31)\]
Two particles can annihilate to form a ghost, and a ghost can decay into a pair of particles. For example, the first diagram represents the transition

$$|\text{incoming}\rangle \rightarrow |\text{ghost}\rangle$$

via the action of $W_{-1}^{-1}$ at some time $t > 0$. Components of the state that decay in the far past or future are indicated by dotted lines.

Here the vertical axis is meant to connote time flow. Note in particular that the outgoing particles are created at time $t < 0$ before the incoming particles annihilate at time $t > 0$. This seems, at first glance, to be a signal of acausality. However, essential to the resolution of this issue is the observation that the outgoing particles are not created from the incoming particles. In particular, incoming and outgoing particles are always in different branches of the superposition. Each individual branch is causal.

In particular, the interpretation of the scattering process is not as in the following picture, suggested by Coleman [13], who imagined an unstable intermediate particle moving into the past.

This diagram would suggest that there exists an intermediate state consisting of a tensor product of five particles, which would be incorrect. If one were to take such an interpretation seriously, one would have to explain what happens if an experimenter were to stop the incoming particles after the outgoing particles have already been emitted and observed. Coleman suggested that the theory might incorporate a novel uncertainty relation that would make such experiments impossible [13].

Our analysis shows that such an uncertainty relation is unnecessary. In fact, it should be almost immediately clear that the version of the grandfather paradox described above is not possible, since at time $t = 0$ the incoming and outgoing scattered particles are in separate terms of a superposition. Even if we could observe the outgoing scattering products, we would not be able to then observe (and stop) the incoming particles, since these are in separate branches.

In fact, it is impossible to observationally distinguish incoming from outgoing particles in the intermediate region. The problem is that these individual terms are not physical states; indeed, the single physical state at $t = 0$
is given by the combination \( |\text{incoming}\rangle + |\text{outgoing}\rangle + |\text{null}\rangle \). To see what goes wrong if we were to try to distinguish the individual terms, let us try to couple the system to an observer who measures, for example, the outgoing particles at \( t = 0 \). To see that such a measurement is unphysical, note that after the interaction, the combined state would have to be

\[
|\text{incoming}\rangle \otimes |1\rangle + |\text{outgoing}\rangle \otimes |0\rangle + |\text{null}\rangle \otimes |0\rangle ,
\]

where the second factor indicates the state of the observer. However, this combined state cannot be physical, since it diverges exponentially under the subsequent evolution given by \( W^{-1} \otimes 1 \) as \( T \to \infty \). Unsurprisingly, consistency requires us to limit observations to physical states. As a result, we cannot locally measure outgoing and incoming states in the scattering region.

The question of locality should be addressed further. Clearly, the physical states are subject to constraints that may make them nonlocal, and they cannot therefore be prepared and measured by entirely local experimenters. For example, the \( \epsilon \) term of the initial state

\[
|\text{incoming}\rangle + \epsilon (|\text{outgoing}\rangle + |\text{ghost}\rangle)
\]

cannot be discarded without violating the physicality constraint, leading to a divergent evolution in the far future, but it is unclear how such a state could be prepared by a pair of independent local agents, given that the ghost is located at the center of mass of the two incoming particles. However, we should remember that experimenters are subject to the same physicality constraints. Thus, the state of any two separated agents will itself contain a nonlocal term proportional to \( \epsilon \) that emits the intermediate ghost. This does not mean that these agents are not independent. Indeed, the \( \epsilon \)-term is uniquely determined by the \( \epsilon \)-independent piece, and we could therefore label states in the far past by omitting the former. The resulting labeling will coincide with that of an ordinary unconstrained field theory not containing ghosts, in which independent agents are usually assumed to be unproblematic. Given such a labeling, however, the effect of the ghosts show up in the effectively nonlocal interaction.

### 13 Towards quantum black holes?

The diagram that we have discussed has an interesting feature suggestive of the expected behaviour of a microscopic black hole. Of course, the theory
does not contain gravity, and the ghost particles do not have the correct quantum numbers or the ability to accrue matter as a black hole would. However, the causal structure of the scattering diagram (31)

\[
\begin{align*}
\mathcal{A} &+ \mathcal{B} + \cdots
\end{align*}
\]

which is representative of, for example, the scattering of two high-energy leptons via their interaction with the complex-mass ghost positronium poles, is reminiscent of that of a black hole being created and then evaporating. In particular, the particles seen by the future observer are emitted from a point causally prior to the collision of the incoming matter, which is reminiscent of the way Hawking radiation originates causally prior to the singularity that absorbs the incoming matter in a black hole. Our model even incorporates an analogue of an evaporating singularity, in the form of the ghost particle created by the collision of the incoming matter. Indeed, we have seen that the amplitude of the ghost term vanishes in the far future, which is what one would expect from the singularity of an evaporating black hole.

Yet, the theory is by construction unitary with no loss of information or quantum coherence. It is therefore perhaps possible to understand hypothetical features of quantum black holes with the help of relatively simple toy models based on these ideas.

Our observation shows some similarities with the work of Horowitz and Maldacena [16], on the black hole final state boundary condition. Their final state boundary condition ensured that everything falling into the black hole annihilated completely, leaving nothing behind. This is indeed analogous to what happens in the above diagrams, where the incoming particles annihilate into a ghost whose amplitude decays exponentially in the future.

14 Conclusion

Given the results of this paper, we believe that there is justification for reconsidering quantum field theories of Lee-Wick type as potentially realistic fundamental or effective theories of the world. We believe that the ideas described in this paper can be used to solve the previously open problem
of quantizing and constraining these theories non-perturbatively to obtain unitary theories that are also covariant.

We illustrated the ideas by way of a Lee-Wick extension of QED, which we argued to be asymptotically safe. Basing the state space on certain Gel’fand-Shilov distributions, we showed explicitly that specific sets of ghost states can be covariantly eliminated from the theory in a non-perturbatively well-defined way, and provided a plausible general argument, though not yet a rigorous proof, that all ghosts can indeed be eliminated this way, and that a unitary theory is obtained.

We also described a second, orthogonal method for getting rid of ghost states, based on a covariant maximum four-momentum constraint. In this approach, we pointed out an interesting connection, based on a naturalness argument, between the largeness of the universe and the smallness of the fine structure constant.

We then discussed the so-called ‘acausality’ of these theories, emphasizing that they are not really acausal, since no inconsistencies can arise. We propose that it may be less confusing to think of these theories as non-local at sufficiently high energy scales, with initial states containing precursors that may only become observable at late times. We pointed out some analogies between Lee-Wick precursors and the precursors invoked to explain aspects of the bulk-boundary correspondence in String Theory holography, and between scattering processes in Lee-Wick theories mediated by ghosts and the behaviour of the black holes in Horowitz’s and Maldacena’s recently proposed resolution of the black hole unitarity problem using final state boundary conditions. As we pointed out, the causal structure of these kinds of Lee-Wick diagrams have aspects in common with the Penrose diagrams of black holes. Hopefully these comparisons will make the reader less likely to reject Lee-Wick theories solely based on their unusual causal behaviour.

Given our proposed resolutions to some prior foundational difficulties encountered in Lee-Wick theories, we believe that these theories deserve another look in the search for descriptions of nature. Indeed, Lee-Wick theories have recently enjoyed a revival in the form of the Lee-Wick Standard Model [18, 19, 20, 21, 22, 23, 24, 25], proposed to address, among other things, the hierarchy problem and the stability of the Higgs mass. Hopefully the results of this paper can be generalized to provide solutions to the problems of non-perturbative unitarity and Lorentz invariance in these theories.

In the light of our results, it may also be productive to revisit renormalizable higher-derivative modifications of gravity, which contain Lee-Wick
degrees of freedom [26].

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