Transition Threshold for the 2-D Couette Flow in a Finite Channel

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Abstract

In this paper, we study the transition threshold problem for the 2-D Navier–Stokes equations around the Couette flow \((y, 0)\) at high Reynolds number \(Re\) in a finite channel. We develop a systematic method to establish the resolvent estimates of the linearized operator and space-time estimates of the linearized Navier–Stokes equations. In particular, three kinds of important effects—enhanced dissipation, inviscid damping and a boundary layer—are integrated into the space-time estimates in a sharp form. As an application, we prove that if the initial velocity \(v_0\) satisfies

\[
\|v_0 - (y, 0)\|_{H^2} \leq c Re^{-\frac{1}{2}}
\]

for some small \(c\) independent of \(Re\), then the solution of the 2-D Navier–Stokes equations remains within \(O(Re^{-\frac{1}{2}})\) of the Couette flow for any time.

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1. Introduction

Since Reynolds’s famous experiment [37], hydrodynamic stability at high Reynolds number has become an important field in fluid mechanics [39, 49], which is mainly concerned with how the laminar flows become unstable and transition to turbulence. On one hand, theoretical analysis shows that some laminar flows such as plane Couette flow and pipe Poiseuille flow are linearly stable for any Reynolds number [19, 38]; on the other hand, the experiments show that they could be unstable and transition to turbulence for small but finite perturbations at high Reynolds number [18, 41]. In addition, some laminar flows such as plane Poiseuille flow become turbulent at a much lower Reynolds number than the critical Reynolds number of linear instability. The resolution of these paradoxes is a long-standing problem in fluid mechanics.

There have been many attempts to understand these paradoxes (see [15] and references therein). One resolution going back to Kelvin [25] is that the basin of attraction of the laminar flow shrinks as $Re \to \infty$ so that the flow could become nonlinearly unstable for small but finite perturbations. Thus, an important question firstly proposed by Trefethen et al. [40] is to study the transition threshold. The following mathematical version was formulated by Bedrossian, Germain and Masmoudi [8]:

*Given a norm $\| \cdot \|_X$, determine a $\beta = \beta(X)$ so that*

$\|u_0\|_X \leq Re^{-\beta} \implies \text{stability},$

$\|u_0\|_X \gg Re^{-\beta} \implies \text{instability}.$

The exponent $\beta$ is referred to as the transition threshold in the applied literature.

The goal of this paper is to study the transition threshold for the 2-D incompressible Navier–Stokes equations in a finite channel $\Omega = \{(x, y) : x \in \mathbb{T}, y \in I = (-1, 1)\}$.
\[
\begin{aligned}
\frac{\partial u}{\partial t} - \nu \Delta u + y \frac{\partial u}{\partial x} + (u_2, 0) + u \cdot \nabla u + \nabla P &= 0, \\
\nabla \cdot u &= 0, \\
\nabla \cdot v &= 0, \\
v(0, x, y) &= v_0(x, y),
\end{aligned}
\]

where \( v \sim Re^{-1} \) is the viscosity coefficient, \( v(t, x, y) = (v^1, v^2) \) is the velocity, \( P(t, x, y) \) is the pressure.

For the 2-D fluid, the vorticity \( \omega = \partial_y v^1 - \partial_x v^2 \) has the following beautiful structure:

\[
\frac{\partial \omega}{\partial t} - \nu \Delta \omega + v \cdot \nabla \omega = 0,
\]

so that nonlinear effect is weak and the lift-up effect is absence. Hence, it seems to mean that the 2-D Navier–Stokes equations are not a suitable model describing the transition to turbulence. However, the boundary effect is still strong in 2-D at high Reynolds number regime when the non-slip boundary condition is imposed on the velocity. This effect could give rise to the instability of the flows. Therefore, the stability of the 2-D fluid should be an important step toward understanding the stability of the 3-D fluid in the presence of the physical boundary.

In this paper, we will study the stability of the Couette flow \( U = (y, 0) \), which may be the simplest solution of (1.1) and linearly stable for any Reynolds number. Let \( u = v - U \) be the perturbation of the velocity, which satisfies

\[
\begin{aligned}
\frac{\partial u}{\partial t} - \nu \Delta u + y \frac{\partial u}{\partial x} + (u_2, 0) + u \cdot \nabla u + \nabla P &= 0, \\
\nabla \cdot u &= 0, \\
u(t, x, \pm 1) &= 0, \\
u(0, x, y) &= u_0(x, y).
\end{aligned}
\]

Here we impose non-slip boundary condition on the perturbation \( u \) in order to include the boundary effect.

There are a lot of works \([20, 32, 36, 49]\) in applied mathematics and physics devoted to estimating transition threshold for various flows such as Couette flow and Poiseuille flow. Recently, Bedrossian, Germain, Masmoudi et al. made an important progress on the stability threshold problem for the Couette flow in a series of works \([5–7, 10, 11]\). Roughly speaking, their results could be summarized as follows:

When \( \Omega = \mathbb{T} \times \mathbb{R} \times \mathbb{T} \),

- if the perturbation is in Gevrey class, then \( \beta = 1 \) \([5, 6]\);
- if the perturbation is in Sobolev space, then \( \beta \leq \frac{3}{2} \) \([7]\).

When \( \Omega = \mathbb{T} \times \mathbb{R} \), the threshold is much smaller due to the absence of lift-up effect:

- if the perturbation is in Gevrey class, then \( \beta = 0 \) \([10]\);
- if the perturbation is in Sobolev space, then \( \beta \leq \frac{1}{2} \) \([11]\).

In a recent work \([45]\), Wei and Zhang proved that the transition threshold \( \beta \leq 1 \) still holds in Sobolev regularity for the Couette flow when \( \Omega = \mathbb{T} \times \mathbb{R} \times \mathbb{T} \). This result confirms the transition threshold conjecture proposed in \([15, 40]\).
Previous results show that three kinds of linear effects (including enhanced dissipation, inviscid damping, 3-D lift-up) and nonlinear structure play a key role in determining the transition threshold in the absence of the boundary. In this paper, we would like to understand how various effects, especially the boundary effect, influence the transition threshold in the presence of the boundary. There are some mathematical papers [12,26,29,38] devoting to nonlinear stability of the Couette flow in a channel, where they gave a rough bound of $\beta$, for example, $\beta \leq 3$ in 2-D and $\beta \leq 4$ in 3-D.

To study nonlinear stability, the key step is to establish the space-time estimates for the linearized Navier–Stokes equations around the Couette flow:

\[
\begin{align*}
\partial_t u - \nu \Delta u + y \partial_y u + (u_2, 0) + \nabla P &= 0, \\
\nabla \cdot u &= 0.
\end{align*}
\]

In terms of the vorticity $w = \partial_y u^1 - \partial_x u^2$, it takes the form

\[
\partial_t w - \nu \Delta w + y \partial_y w = 0.
\]

When $\Omega = \mathbb{T} \times \mathbb{R}$, the space-time estimates can be established by using the Fourier transform in $(x, y)$ (see [45]). When $\Omega = \mathbb{T} \times I$, we need to use the resolvent estimates of the linearized operator. Let us emphasize that the spectrum of the operator $A$ is not enough to determine the behaviour of semigroup $e^{tA}$ when $A$ is a non normal operator [42]. In [2,13,14,26,30,40], the authors established some resolvent estimates in some regimes of parameters $Re$ and wave number by using the rigorous analysis combined with numerical computations. However, based on these estimates, one can only establish a rough bound of transition threshold.

In this paper, we first develop a systematic method to establish some sharp resolvent estimates for the linearized operator under the Navier-slip boundary condition and non-slip boundary condition respectively. In particular, for non-slip boundary condition, we use many deep properties of the Airy function to give precise $L^p$ bounds on the solutions of homogeneous Orr–Sommerfeld (OS) equation. Moreover, our resolvent estimates show that the linearized operator has a much wider spectrum gap $O(\nu^{\frac{1}{3}} k^{\frac{2}{3}})$ (usually $O(\nu)$) when the wave number $k \neq 0$, which is related to the enhanced dissipation induced by mixing due to the Couette flow. See [3,22,28,31,48] for the enhanced dissipation induced by the Kolmogorov flow and [17] for more general situation.

With the resolvent estimates at hand, a standard method for semigroup estimates is to use the Dunford integral by choosing a suitable contour to obtain the semigroup estimate. However, the semigroup estimate obtained in this way is not enough to obtain a sharp threshold. In Sect. 5, we develop a complete new method to establish the space-time estimates of the solution of the linearized Navier–Stokes equations. Our space-time estimates include $\|u\|_{L^2_t L^2_y}$ due to inviscid damping, enhanced dissipation estimates, and $L^\infty$ estimate of the velocity in the spirit of maximum principle. Recently, the inviscid damping as an analogue of Landau damping has been well understood at least at the linear level [4,9,23,24,46–48,50].
We believe that the method we develop to establish the resolvent estimates and space-time estimates could be used to other related problems such as the transition threshold for general flows and the stability analysis of boundary layer.

As an application of the space-time estimates, we prove the nonlinear stability of the Couette flow. To state our result, we define

\[ P_0 f = \overline{f}(t, y) = \int_T f(t, x, y) dx, \quad f_{\neq} = f - \overline{f} = \sum_{k \neq 0} f_k(t, y)e^{ikx}. \]

Our result is stated as follows:

**Theorem 1.1.** Assume that \( u_0 \in H^1_0(\Omega) \cap H^2(\Omega) \) with \( \text{div} u_0 = 0 \). There exist constants \( v_0 \) and \( C > 0 \) independent of \( v \) so that if \( \|u_0\|_{H^2} \leq cv^2 \), \( 0 < v \leq v_0 \), then the solution \( u \) of the system (1.3) is global in time and satisfies the following stability estimate:

\[ \sum_{k \in \mathbb{Z}} E_k \leq Cc \nu_0, \]

where \( E_0 = \|\overline{w}\|_{L^\infty L^2} \) and for \( k \neq 0 \),

\[ E_k = \|(1 - |y|)^{\frac{1}{2}} w_k\|_{L^\infty L^2} + |k|\|u_k\|_{L^2 L^2}^2 + |k|^{\frac{3}{2}} \|u_k\|_{L^\infty L^\infty}^2 + (v^2)^{\frac{1}{4}} \|w_k\|_{L^2 L^2}. \]

Let us give some remarks on our result.

- The threshold is the same as one obtained by [11] for the 2-D Navier–Stokes equation in \( \Omega = \mathbb{T} \times \mathbb{R} \). This means that the boundary effect does not give rise to strong nonlinear instability of the Couette flow due to weak nonlinear effect in 2-D fluid. However, the threshold has been improved to \( \beta = \frac{1}{3} \) in a very recent work [33] when \( \Omega = \mathbb{T} \times \mathbb{R} \). It remains unknown whether our result is optimal in a finite channel even for smooth data. Another very interesting issue is to prove the inviscid limit in the Gevrey class as in [10].
- The estimate \( |k|\|u_k\|_{L^2 L^2} \) is due to the inviscid damping. The point-wise type damping estimate is a highly nontrivial problem even for the linearized equation at high Reynolds number regime.
- Stability norm \( \sum_k E_k \) includes the part of \( \sum_k |k|^{\frac{3}{2}} \|u_k\|_{L^\infty L^\infty} \), which means that the 2-D Couette flow is absolutely stable.
- In 3-D case, nonlinear structure of the system is more complex so that it is very hard to analyze how stabilizing mechanism (enhanced dissipation and inviscid damping) and destabilizing mechanism (boundary layer and lift-up) influence different modes of the solution and different components of the velocity, and complex interactions among them. We will leave the transition threshold problem in 3-D to our future work [16].
2. The Linearized Equation and Key Ideas of the Proof

2.1. The linearized equation

The linearized Navier–Stokes equations around the Couette flow take the form

$$\begin{align*}
\partial_t u - \nu \Delta u + y \partial_x u + (u_2, 0) + \nabla P &= 0, \\
\nabla \cdot u &= 0.
\end{align*}$$

(2.1)

Thanks to the unknown pressure, it is not easy to handle the velocity equation directly. Thus, we introduce two formulations in terms of the vorticity $w$ and the stream function $\Phi$ respectively, which are defined by

$$w = \partial_y u^1 - \partial_x u^2, \quad u = \nabla \perp \Phi = (\partial_y \Phi, -\partial_x \Phi).$$

Then the vorticity formulation of (2.1) takes the form

$$\partial_t w - \nu \Delta w + y \partial_x w = 0.$$  

(2.2)

Thanks to $\Delta \Phi = w$, it holds that

$$\partial_t \Delta \Phi - \nu \Delta^2 \Phi + y \partial_x \Delta \Phi = 0.$$  

(2.3)

Due to the beautiful structure (2.2), the 2-D Navier–Stokes equations have no lift-up effect.

Taking the Fourier transform in $x \in \mathbb{T}$, we get

$$w(t, x, y) = \sum_{k \in \mathbb{Z}} \hat{w}_k(t, y) e^{ikx} = \sum_{k \in \mathbb{Z}} e^{ikx} (\partial_y^2 - k^2) \hat{\Phi}_k(t, y).$$

Then we have

$$\partial_t \hat{w}_k(t, y) + L_k \hat{w}_k(t, y) = 0,$$  

(2.4)

where $L_k = \nu (k^2 - \partial_y^2) + i ky$.

For the linearized equation, we will consider two kinds of boundary conditions. The first one is the non-slip boundary condition

$$u(t, x, \pm 1) = 0.$$  

(2.5)

In this case, the stream function $\hat{\Phi}_k(t, y), k \neq 0$ takes on the boundary

$$\hat{\Phi}_k(t, \pm 1) = \hat{\Phi}'_k(t, \pm 1) = 0.$$  

(2.6)

The second one is the Navier-slip boundary condition

$$u^2(t, x, \pm 1) = 0, \quad \partial_y u^1(t, x, \pm 1) = 0.$$  

(2.7)

Thus, $w(t, x, \pm 1) = 0$, and for $k \neq 0$,

$$\hat{\Phi}_k(t, \pm 1) = \hat{\Phi}'_k(t, \pm 1) = 0.$$  

(2.8)
Standard method for studying stability is to make the eigenvalue analysis for the linearized equation. That is, we seek the solution of the form

\[ \hat{w}_k(t, y) = \omega_k(y)e^{-ikt\lambda}, \quad \hat{\phi}_k(t, y) = \varphi_k(y)e^{-ik\lambda t}. \]

Then \( w_k(y) \) and \( \varphi_k(y) \) satisfy the following Orr–Sommerfeld (OS) equation

\[
-v(\partial_y^2 - k^2)\omega_k + ik(y - \lambda)\omega_k = 0,
\]

(2.9)

\[
-v(\partial_y^2 - k^2)^2\varphi_k + ik(y - \lambda)(\partial_y^2 - k^2)\varphi_k = 0.
\]

(2.10)

If there exists a nontrivial solution of (2.10) with the boundary condition (2.6) or (2.8) for \( \lambda \in \mathbb{C}, k > 0 \) with \( \text{Im}\lambda > 0 \), we say that the flow is linearly unstable.

For Navier-slip boundary condition, it is easy to see that the Couette flow is linearly stable for any \( \nu > 0 \). Indeed, it follows from (2.9) and \( \omega_k(\pm 1) = 0 \) that

\[
\int_{-1}^{1} \nu(|\partial_y\omega_k|^2 + k^2|\omega_k|^2) + k\text{Im}\lambda|\omega_k|^2dy = 0,
\]

which implies that \( \omega_k = 0 \) if \( k\text{Im}\lambda > 0 \).

For non-slip boundary condition, Romanov [38] proved that the Couette flow is also linearly stable for any \( \nu > 0 \). In fact, Romanov studied the Navier–Stokes equations in an infinite channel \( \mathbb{R} \times I \), and proved that the eigenvalues of the linearized operator \( L_k \) must lie in a region with \( \text{Re}\lambda < cv \) for some \( c > 0 \). In a finite channel \( \mathbb{T} \times I \), we proved that the eigenvalues must lie in a region with \( \text{Re}\lambda < ck^2\nu^{1/2} \) for some \( c > 0 \) if the wave number \( |k| \geq 1 \), which is related to the enhanced dissipation induced by the Couette flow. Thus, it is very interesting to investigate the long wave effect on nonlinear stability, which is one of main mechanisms leading to the instability of many plane shear flows. For example, the unstable wave numbers \( k \) lie in a band \( O(Re^{-1/7}) \leq k \leq O(Re^{-1/11}) \) for the plane Poiseuille flow [19], which is, however, linearly stable for any \( \nu > 0 \) in a finite channel.

### 2.2. Key ideas and structure of the paper

In Sect. 3, we study the resolvent estimates of the linearized operator under the Navier-slip boundary condition

\[
-v(\partial_y^2 - k^2)w + ik(y - \lambda)w = F, \quad w(\pm 1) = 0.
\]

The proof used the multiplier idea introduced in [27, 28].

In Sect. 4, we study the resolvent estimates of the linearized operator under the non-slip boundary condition

\[
\begin{aligned}
- v(\partial_y^2 - k^2)^2\varphi + ik(y - \lambda)(\partial_y^2 - k^2)\varphi &= F, \\
\varphi(\pm 1) &= 0, \quad \varphi'(\pm 1) = 0.
\end{aligned}
\]
First of all, we decompose $w = (\partial_y^2 - k^2)\varphi$ into the solution $w_{Na}$ of the inhomogeneous OS equation with the Navier-slip boundary condition and the solutions $w_1, w_2$ of the homogeneous OS equation, i.e.,

$$w = w_{Na} + c_1 w_1 + c_2 w_2.$$ 

There are two key ingredients:

- We give a very precise estimates for the coefficients $c_1, c_2$, which are based on the resolvent estimates of the linearized operator under the Navier-slip boundary condition, especially, a weak type resolvent estimate.
- We derive the sharp $L^p$ bounds and weighted $L^2$ bounds on $w_1, w_2$ in Sect. 5, which are based on many deep estimates of the Airy function derived in Sect. 8. In fact, some estimates have been implied in Romanov’s beautiful paper [38].

In Sect. 6, we derive the space-time estimates of the linearized Navier–Stokes equations. To apply them to nonlinear problem, we consider the inhomogeneous problem

$$\partial_t \omega + L_k \omega = -ik f_1 - \partial_y f_2, \quad \omega|_{t=0} = \omega_0(y),$$

where $\omega = (\partial_y^2 - k^2)\varphi$ with $\varphi(\pm 1) = \varphi'(\pm 1) = 0$.

Usually, the space-time estimates can be established by using the Dunford integral and resolvent estimates. Indeed, for Navier-slip boundary condition, we can obtain the sharp bound of semigroup by using Gearhart-Prüss theorem, since the linearized operator $L_k$ is accretive in this case. However, for non-slip boundary condition, one can only obtain a rough bound of semigroup in this way. Using this rough bound, we can also improve previous bounds on the transition threshold. However, it is far from the bound obtained in Theorem 1.1.

Our key idea is that we do not use the resolvent estimates directly, and instead use the ideas of establishing the resolvent estimates in order to know how the boundary layer influences the space-time estimates more precisely. To this end, we first decompose the problem into inhomogeneous problem and homogeneous problem. After taking the Laplace transform for the inhomogeneous problem, we use the ideas of establishing the resolvent estimates to prove $L_2^2(L^2)$ estimate, and then use energy method combined with the precise estimates for $c_1, c_2$ and $w_1, w_2$ to obtain $L_2^\infty(L^2)$ estimate. For the solution $\omega_H$ of the homogeneous problem, we split things into three parts:

$$\omega_H = \omega_H^{(1)} + \omega_H^{(2)} + \omega_H^{(3)},$$

where $\omega_H^{(1)}(t, k, y) = e^{-(vk^2)^{1/3}t - itky} \omega_0(k, y)$, and $\omega_H^{(2)}$ solves

$$(\partial_t - v(\partial_y^2 - k^2) + iky)\omega_H^{(2)} = -(vk^2 - (vk^2)^{1/3})\omega_H^{(1)} + v\partial_y^2 \omega_H^{(1)},$$

$$\omega_H^{(2)}|_{t=0} = 0, \quad \langle \omega_H^{(2)}, e^{\pm ky} \rangle = 0,$$

and $\omega_H^{(3)}$ solves

$$(\partial_t - v(\partial_y^2 - k^2) + iky)\omega_H^{(3)} = 0, \quad \omega_H^{(3)}|_{t=0} = 0, \quad \langle \omega_H^{(3)}(t) + \omega_H^{(1)}(t), e^{\pm ky} \rangle = 0.$$
Here the orthogonality condition for $\omega_{H}^{(2)}$ is equivalent to the non-slip boundary condition: $\omega_{H}^{(2)} = (\partial_{y}^{2} - k^{2})\varphi_{H}^{(2)}$, $\varphi_{H}^{(2)}(\pm 1) = \partial_{y}\varphi_{H}^{(2)}(\pm 1) = 0$. The decomposition makes sense. Indeed, $\omega_{H}^{(1)}$ is given explicitly, and the existence for the solution $\omega_{H}^{(3)}$ is due to the local well-posedness of the inhomogeneous problem under non-slip boundary condition (by first solving $\varphi_{H}^{(2)}$), and then $\omega_{H}^{(3)}$ is defined as $\omega_{H}^{(3)} = \omega_{H} - \omega_{H}^{(1)} - \omega_{H}^{(2)}$. The choice of $\omega_{H}^{(1)}$ takes into consideration the effect of enhanced dissipation and the linearized Euler equation. With $\omega_{H}^{(1)}$, the choices of $\omega_{H}^{(2)}$ and $\omega_{H}^{(3)}$ are to meet the equation and match the boundary condition.

Now the estimates for $\omega_{H}^{(1)}$ are direct. For $\omega_{H}^{(2)}$, we can use the space-time estimates obtained for the inhomogeneous problem, since we know much information about $\omega_{H}^{(1)}$. For $\omega_{H}^{(3)}$, we again need to use the $L^{p}$ estimates of $w_{1}$, $w_{2}$ and new weighted $L^{2}$ estimates for $c_{1}$, $c_{2}$. In summary, the space-time estimates are as follows:

\[
|k||u||_{L^{\infty}L^{\infty}}^{2} + k^{2}||u||_{L^{2}L^{2}}^{2} + (vk^{2})^{\frac{1}{2}}||\omega||_{L^{2}L^{2}}^{2} + ||(1 - |y|)^{\frac{1}{2}}\omega||_{L^{\infty}L^{2}}^{2} \leq C \left( ||\omega(0)||_{L^{2}}^{2} + k^{-2}||\partial_{y}\omega(0)||_{L^{2}}^{2} \right) + C \left( v^{-\frac{1}{2}}|k||f_{1}||_{L^{2}L^{2}}^{2} + v^{-1}||f_{2}||_{L^{2}L^{2}}^{2} \right).
\]

Let us give some explanations on our estimates.

- The estimate $||u||_{L^{2}L^{2}}$ is due to the inviscid damping and plays an important role in this work (also in [45]). The proof is relatively easier than the polynomial decay estimates established in [47].
- The $L^{\infty}$ estimate of the velocity is very surprising and takes the following form in the homogeneous case ($f_{1} = f_{2} = 0$):

\[
|k||u||_{L^{\infty}L^{\infty}}^{2} \leq C \left( ||\omega(0)||_{L^{2}}^{2} + k^{-2}||\partial_{y}\omega(0)||_{L^{2}}^{2} \right)
\]

In some sense, this result means that maximum principle still holds for the linearized Navier–Stokes equation (2.1). This is similar to Abe and Giga’s breakthrough work on the analyticity of the Stokes semigroup in spaces of bounded functions [1].
- The weight $(1 - |y|)^{\frac{1}{2}}$ in the norm $||(1 - |y|)^{\frac{1}{2}}\omega||_{L^{\infty}L^{2}}^{2}$ is due to the boundary effect.
- The estimate $(vk^{2})^{\frac{1}{2}}||\omega||_{L^{2}L^{2}}^{2}$ is due to the enhanced dissipation.

In Sect. 7, we prove nonlinear stability by using the vorticity formulation and the space-time estimates.

Throughout this paper, we always assume $v \in (0, 1]$ and $|k| \geq 1$, and denote by $C$ a constant independent of $v$, $k$, $\lambda$, which may be different from line to line.

### 3. Resolvent Estimates with Navier-Slip Boundary Condition

In this section, we study the resolvent estimates of the linearized operator under the Navier-slip boundary condition. More precisely, we consider the vorticity equation

\[
-v(\partial_{y}^{2} - k^{2})w + ik(y - \lambda)w = F, \quad w(\pm 1) = 0,
\]
and the stream function equation
\[
\begin{aligned}
- \nu (\partial_y^2 - k^2) \varphi + ik(y - \lambda)(\partial_y^2 - k^2) \varphi &= F, \\
\varphi(\pm 1) &= 0, \quad \varphi''(\pm 1) = 0.
\end{aligned}
\quad (3.2)
\]

## 3.1. Resolvent estimate from \(L^2\) to \(H^2\)

First of all, we consider the case of \(\lambda \in \mathbb{R}\).

**Proposition 3.1.** Let \(w \in H^2(I)\) be a solution of (3.1) with \(\lambda \in \mathbb{R}\) and \(F \in L^2(I)\). Then it holds that
\[
\nu^{\frac{1}{6}} |k|^{\frac{4}{3}} \|u\|_{L^2} + \nu^{\frac{1}{6}} |k|^{\frac{5}{3}} \|w\|_{L^1}
\]
\[
+ \nu^{\frac{2}{3}} |k|^{\frac{1}{3}} \|w'\|_{L^2} + (\nu k^2)^{\frac{1}{3}} \|w\|_{L^2} + |k| \|\varphi - (\partial_y^2 - k^2) \varphi\|_{L^2}
\]
\[
\leq C \|F\|_{L^2},
\]
where \(u = (\partial_y \varphi, -ik \varphi)\) and \((\partial_y^2 - k^2) \varphi = 0\) with \(\varphi(\pm 1) = 0\).

**Proof.** By integration by parts, we get
\[
\langle F, w \rangle = \nu \|w'\|_{L^2}^2 + \nu k^2 \|w\|_{L^2} + ik \int_{-1}^{1} (y - \lambda) |w|^2 dy.
\]
Taking the real part, we get
\[
\nu \|w'\|_{L^2}^2 \leq \|F\|_{L^2} \|w\|_{L^2}.
\]
We also get by integration by parts that
\[
\langle F, (y - \lambda) w \rangle = -\nu \int_{-1}^{1} w''(y - \lambda) \bar{w} dy + \nu k^2 \int_{-1}^{1} (y - \lambda) |w|^2 dy
\]
\[
+ ik \|w - (\partial_y^2 - k^2) w\|_{L^2}^2
\]
\[
= \nu \int_{-1}^{1} w' \bar{w} dy + \nu \int_{-1}^{1} (y - \lambda) |w'|^2 dy + \nu k^2 \int_{-1}^{1} (y - \lambda) |w|^2 dy
\]
\[
+ ik \|w - (\partial_y^2 - k^2) w\|_{L^2}^2.
\]
Taking the imaginary part, we get
\[
|k| \|(y - \lambda) w\|_{L^2}^2 \leq \|F\|_{L^2} \|w\|_{L^2} + \nu \|w'\|_{L^2} \|w\|_{L^2},
\]
\[
|k| \|(y - \lambda) w\|_{L^2} \leq |k|^{-2} \nu \|F\|_{L^2}^2 + 2 |k|^{-1} \nu \|w'\|_{L^2} \|w\|_{L^2}.
\]
Let \(\delta = \nu^{\frac{1}{3}} |k|^{-\frac{1}{2}}, E = (-1, 1) \cap (\lambda - \delta, \lambda + \delta), E^c = (-1, 1) \setminus (\lambda - \delta, \lambda + \delta).\) Then we have
\[
\|w\|_{L^2}^2 = \|w\|_{L^2(E^c)}^2 + \|w\|_{L^2(E)}^2 \leq \delta^{-2} \|(y - \lambda) w\|_{L^2}^2 + 2 \delta \|w\|_{L^\infty}^2
\]
\[
\leq \delta^{-2} |k|^{-2} \|F\|_{L^2}^2 + 2 \delta^{-2} |k|^{-1} \nu \|w'\|_{L^2} \|w\|_{L^2} + 2 \delta \|w'\|_{L^2} \|w\|_{L^2}
which implies
\[
\|w\|_{L^2}^2 \leq C((vk^2)^{-\frac{2}{3}} + \delta^4 v^{-2}) \|F\|_{L^2}^2 \leq C(vk^2)^{-\frac{2}{3}} \|F\|_{L^2}^2,
\]
and thus,
\[
v \|w'\|_{L^2}^2 \leq \|F\|_{L^2}^2 \|w\|_{L^2}^2 \lesssim (vk^2)^{-\frac{1}{3}} \|F\|_{L^2}^2.
\]
For \(\|(y - \lambda)w\|_{L^2}\), we have
\[
\|(y - \lambda)w\|_{L^2}^2 \leq |k|^{-1} \|F\|_{L^2}^2 + 2|k|^{-1} v \|w'\|_{L^2} \|w\|_{L^2}
\leq |k|^{-2} \|F\|_{L^2}^2 + C|k|^{-1} v(v^{-\frac{2}{3}}k^{-\frac{2}{3}} \|F\|_{L^2})(vk^2)^{-\frac{1}{3}} \|F\|_{L^2}
\leq C|k|^{-2} \|F\|_{L^2}^2.
\]
Notice that
\[
\|w\|_{L^1} = \int_E |w| \, dy + \int_{(-1,1) \setminus E} |w| \, dy
\leq \delta^2 \|w\|_{L^2} + \left(\int_{(-1,1) \setminus E} \frac{1}{(y - \lambda)^2} \, dy\right)^\frac{1}{2} \|(y - \lambda)w\|_{L^2},
\]
which gives
\[
\|w\|_{L^1} \lesssim \delta^\frac{1}{2} (vk^2)^{-\frac{1}{3}} \|F\|_{L^2} + \delta^{-\frac{1}{2}} |k|^{-1} \|F\|_{L^2} \lesssim v^{-\frac{1}{6}} |k|^{-\frac{5}{6}} \|F\|_{L^2},
\]
from which and Lemma 9.3, we infer that
\[
\|u\|_{L^2}^2 \lesssim \|\varphi'\|_{L^2}^2 + k^2 \|\varphi\|_{L^2}^2 \lesssim |k|^{-1} \|w\|_{L^1}^2.
\]
Summing up, we conclude the proof. □

Proposition 3.1 implies that the resolvent set of the linearized operator \(L_k\) contains the region
\[
\Omega = \{\lambda = \lambda_r + i\lambda_i : \lambda_r \leq \epsilon v^{\frac{1}{2}} |k|^{\frac{2}{3}}, \lambda_i \in \mathbb{R}\}
\]
for all small \(\epsilon \leq C^{-1}\) with \(C\) given by Proposition 3.1. As a corollary of Proposition 3.1, we can deduce the following resolvent estimate for \(\lambda \in \Gamma_\epsilon = \{-i \epsilon v^{\frac{1}{2}} k^{-\frac{1}{3}} + \lambda_r : \lambda_r \in \mathbb{R}\}\).

**Corollary 3.2.** Let \(w \in H^2(I)\) be a solution of (3.1) with \(\lambda \in \Gamma_\epsilon, F \in L^2(I)\), and \(0 \leq \epsilon \leq C^{-1}\). Then it holds that
\[
v^{-\frac{1}{6}} |k|^{\frac{2}{3}} \|u\|_{L^2} + v^{\frac{1}{6}} |k|^{\frac{5}{6}} \|w\|_{L^1} + v^{\frac{2}{3}} |k|^{\frac{2}{3}} \|w'\|_{L^2}
+ (vk^2)^{\frac{1}{3}} \|w\|_{L^2} + |k|\|(y - \lambda)w\|_{L^2} \leq C \|F\|_{L^2}.
\]
Proof. Let \( \widetilde{F} = F + \epsilon v^{\frac{1}{3}} |k|^{\frac{2}{3}} w \). Then by Proposition 3.1, we get
\[
\|F\|_{L^2} \geq \|\widetilde{F}\|_{L^2} - \epsilon v^{\frac{1}{3}} |k|^{\frac{2}{3}} \|w\|_{L^2} \geq \|\widetilde{F}\|_{L^2} - C\epsilon \|\widetilde{F}\|_{L^2} \\
\geq (v^{\frac{1}{3}} |k|^{\frac{2}{3}} \|u\|_{L^2} + v^{\frac{1}{3}} |k|^{\frac{2}{3}} \|w\|_{L^1} + v^{\frac{2}{3}} k^{\frac{1}{3}} \|w\|_{L^2} + (vk^2)^{\frac{1}{3}} \|w\|_{L^2} + |k||(y-\lambda)w\|_{L^2}),
\]
if we take \( \epsilon \) so that \( C\epsilon \leq \frac{1}{2} \). \( \square \)

3.2. Resolvent estimate from \( H^{-1} \) to \( H^1 \)

For this, we need to use the stream function formulation as follows:
\[
\begin{aligned}
- v(\partial_y^2 - k^2)\varphi + ik(y-\lambda)(\partial_y^2 - k^2)\varphi - \epsilon v^{\frac{1}{3}} |k|^{\frac{2}{3}} (\partial_y^2 - k^2)\varphi &= F, \\
\varphi(\pm 1) = 0, \quad \varphi''(\pm 1) = 0,
\end{aligned}
\]
with \( \lambda \in \mathbb{R} \).

Proposition 3.3. Let \( \varphi \in H^3(I) \) be a solution of (3.3) with \( F \in H^{-1}(I) \). Then it holds that
\[
(v |k|^{\frac{2}{3}}) \|u\|_{L^2} + v \|w'\|_{L^2} + v^{\frac{2}{3}} |k|^{\frac{1}{3}} \|w\|_{L^2} \leq C \|F\|_{H^{-1}}.
\]
Here \( u = (\partial_y \varphi, -ik\varphi) \) and \( w = (\partial_y^2 - k^2)\varphi \).

The proposition is a direct consequence of Lemmas 3.4 and 3.5.

Lemma 3.4. Let \( w \) be as in Proposition 3.3. It holds that
\[
v \|w'\|_{L^2} + v^{\frac{2}{3}} |k|^{\frac{1}{3}} \|w\|_{L^2} \leq C \|F\|_{H^{-1}}.
\]

Proof. Let \( \delta = v^{\frac{1}{3}} |k|^{-\frac{1}{3}} \). By integration by parts, we have
\[
\|F\|_{H^{-1}} \|w\|_{H^1} \geq |\langle F, w \rangle| = |(-v(\partial_y^2 - k^2)w + ik(y-\lambda)w - \epsilon v^{\frac{1}{3}} |k|^{\frac{2}{3}} w, w)| \\
\geq v \|w'\|_{L^2}^2 + v k^2 \|w\|_{L^2}^2 - \epsilon v^{\frac{1}{3}} |k|^{\frac{2}{3}} \|w\|_{L^2}^2 \\
\geq v \|w\|_{H^1}^2 - \epsilon v^{\frac{1}{3}} |k|^{\frac{2}{3}} \|w\|_{L^2}^2,
\]
which gives
\[
v \|w\|_{H^1}^2 \leq \|F\|_{H^{-1}} \|w\|_{H^1} + \epsilon v^{\frac{1}{3}} |k|^{\frac{2}{3}} \|w\|_{L^2}^2 \\
\leq \frac{v^{-1}}{2} \|F\|_{H^{-1}}^2 + \frac{v}{2} \|w\|_{H^1}^2 + \epsilon v^{\frac{1}{3}} |k|^{\frac{2}{3}} \|w\|_{L^2}^2.
\]
This shows that
\[
\|w\|_{H^1} \leq v^{-1} \|F\|_{H^{-1}} + \sqrt{2\epsilon} \delta^{-1} \|w\|_{L^2}.
\]
Let us introduce a cutoff function $\rho(y)$ as follows:

$$
\rho(y) = \begin{cases} 
-1 & y \in (-1, 1) \cap (-1, \lambda - \delta), \\
\sin\left(\frac{\pi(y - \lambda)}{2\delta}\right) & y \in (-1, 1) \cap (-\delta, \lambda + \delta), \\
1 & y \in (-1, 1) \cap (\lambda + \delta, 1).
\end{cases} 
$$

(3.6)

We get by integration by parts that

$$
\left|\text{Im}(ik(y - \lambda)w - v(\partial_y^2 - k^2)w - \epsilon\sqrt{3}|k|^2w, \rho w)\right|
\geq \delta|k|\int_{(-1,1)\setminus(-\delta,\lambda+\delta)} |w|^2 dy - v \left|\text{Im}\int_{-1}^1 \partial_y w \bar{w} \rho dy\right|
\geq \delta|k|\int_{(-1,1)\setminus(-\delta,\lambda+\delta)} |w|^2 dy - v \left|\int_{-1}^1 w \bar{w} \rho dy\right|
\geq \delta|k|\int_{(-1,1)\setminus(-\delta,\lambda+\delta)} |w|^2 dy - v \|w\|_{L^\infty}\|w\|_{L^2}\|\rho\|_{L^2},
$$

which implies

$$
\delta|k|\int_{(-1,1)\setminus(-\delta,\lambda+\delta)} |w|^2 dy \leq \|F\|_{H^{-1}}\|\rho w\|_{H^1} + v\delta^{-\frac{3}{2}}\|w\|_{L^2}\|w\|_{L^\infty}.
$$

This along with $\|\rho w\|_{H^1} \lesssim \|w\|_{H^1} + \delta^{-1}\|w\|_{L^2}$ shows

$$
\|w\|_{L^2((-1,1)\setminus(-\delta,\lambda+\delta))} \lesssim \delta^{-1}|k|^{-1} \left(\|F\|_{H^{-1}}\|w\|_{H^1} + \delta^{-1}\|F\|_{H^{-1}}\|w\|_{L^2} + \delta^{-\frac{1}{2}}v\|w\|_{L^2}\|w\|_{L^\infty}\right),
$$

which implies

$$
\|w\|_{L^2}^2 \lesssim \delta\|w\|_{L^\infty}^2 + \frac{\|F\|_{H^{-1}}\|w\|_{H^1}}{|k|\delta} + \frac{\|F\|_{H^{-1}}\|w\|_{L^2}}{|k|\delta^\frac{3}{2}} + \frac{v\|w\|_{L^2}\|w\|_{L^\infty}}{|k|\delta^\frac{3}{2}}.
$$

(3.7)

Thanks to $\|w\|_{L^\infty}^2 \lesssim \|w\|_{L^2}\|w\|_{L^2}$ and (3.5), we get

$$
\|w\|_{L^2}^2 \lesssim \delta\|w\|_{H^1}\|w\|_{L^2} + \frac{\|F\|_{H^{-1}}\|w\|_{H^1}}{|k|\delta} + \frac{\|F\|_{H^{-1}}\|w\|_{L^2}}{|k|\delta^\frac{3}{2}} + \frac{v\|w\|_{H^1}^\frac{3}{2}\|w\|_{L^2}^\frac{1}{2}}{|k|\delta^\frac{3}{2}}
\lesssim \frac{\delta}{v}\|F\|_{H^{-1}}\|w\|_{L^2} + \frac{\|F\|_{H^{-1}}^2}{|k|\delta^v} + \frac{(1 + \sqrt{\epsilon})\|F\|_{H^{-1}}\|w\|_{L^2}}{|k|\delta^2} + (\epsilon^\frac{3}{2} + \sqrt{\epsilon})\|w\|_{L^2}^2.
$$

Due to $\delta = v^\frac{1}{2}|\lambda|^{-\frac{1}{2}}$ and $\epsilon$ small, we get, by Young’s inequality, that

$$
\|w\|_{L^2}^2 \leq Cv^{-\frac{3}{4}}|\lambda|^{-\frac{3}{2}}\|F\|_{H^{-1}}^2.
$$

This along with (3.5) shows that $v\|w\|_{H^1} \lesssim \|F\|_{H^{-1}}$. □
Lemma 3.5. Let $u$ be as in Proposition 3.3. It holds that

$$(v|k|^2)^{1/2} \|u\|_{L^2} \leq C \|F\|_{H^{-1}}.$$  

Proof. Let $\delta = v^{1/3} |k|^{-1/3}$ and introduce a cut-off function $\chi(y)$ as follows:

$$\chi(y) = \begin{cases} 
\frac{1}{y - \lambda} & y \in (-1, 1) \cap (-1, \lambda - \delta), \\
\frac{2 y - \lambda}{\delta^2} - \frac{(y - \lambda)^3}{\delta^4} & y \in (-1, 1) \cap (\lambda - \delta, \lambda + \delta), \\
\frac{1}{y - \lambda} & y \in (-1, 1) \cap (\lambda + \delta, 1),
\end{cases}$$

which satisfies

$$\|\chi\|_{L^2} \lesssim \delta^{-1/2}, \quad \|\chi\|_{L^\infty} \lesssim \delta^{-1}, \quad \|\chi\|_{L^2} \lesssim \delta^{-3/2}.$$  

We get by integration by parts that

$$\langle F, \chi \varphi \rangle = v \int_{-1}^{1} w' (\chi \varphi') dy + v k^2 \int_{-1}^{1} w \varphi \chi dy + ik \int_{(-1, 1) \setminus (\lambda - \delta, \lambda + \delta)} w \varphi dy$$

$$+ ik \int_{(-1, 1) \cap (\lambda - \delta, \lambda + \delta)} ((y - \lambda) \chi) w \varphi dy - \epsilon v^{1/3} |k|^{3/2} \int_{-1}^{1} w \chi dy.$$  

Using the facts that $\|w\|_{L^\infty}^2 \leq \|w'\|_{L^2} \|w\|_{L^2}$ and $\|\varphi\|_{L^\infty}^2 \leq \|\varphi'\|_{L^2} \|\varphi\|_{L^2}$, we get

$$\|u\|_{L^2}^2 = \int_{(-1, 1) \setminus (\lambda - \delta, \lambda + \delta)} w \varphi dy + \int_{(-1, 1) \setminus (\lambda - \delta, \lambda + \delta)} w \varphi dy$$

$$\lesssim \delta \|w\|_{L^\infty} \|\varphi\|_{L^\infty} + |k|^{-1} |\{F, \chi \varphi\}|$$

$$+ v |k|^{-1} \|w'\|_{L^2} \|\chi \varphi'\|_{L^2} + v |k| \|w\|_{L^2} \|\chi \varphi\|_{L^2} \|\chi \varphi\|_{L^\infty}$$

$$+ \|w\|_{L^2} \|\varphi\|_{L^\infty} \|\chi \varphi\|_{L^\infty}$$

$$+ \epsilon v^{1/3} |k|^{-3} \|w\|_{L^2} \|\chi \varphi\|_{L^2} \|\chi \varphi\|_{L^\infty}$$

$$\lesssim \delta \|w'\|_{L^2} \|w\|_{L^2} \|\varphi'\|_{L^2} \|\varphi\|_{L^2}^2 + |k|^{-1} \|F\|_{H^{-1}} \|\chi \varphi\|_{H^1}$$

$$+ v |k|^{-1} \|w'\|_{L^2} \|\chi \varphi'\|_{L^2}$$

$$+ v |k| \|w\|_{L^2} \|\varphi'\|_{L^2} \|\varphi\|_{L^2}^2 + \delta^{3/2} \|w\|_{L^2} \|\varphi'\|_{L^2} \|\varphi\|_{L^2}^2$$

$$+ \epsilon v^{1/3} |k|^{-3} \delta^{-3/2} \|w\|_{L^2} \|\varphi'\|_{L^2} \|\varphi\|_{L^2}^2.$$  

Thanks to $\|(\chi \varphi')(\varphi')_{L^2} \lesssim \delta^{-1} \|(\varphi')(\varphi')_{L^2} + \delta^{-3/2} \|(\varphi')(\varphi')_{L^\infty}$, we get by Lemma 3.4 that

$$\|u\|_{L^2}^2 \lesssim \delta^{1/2} |k|^{-1} \|F\|_{H^{-1}} \|\varphi'\|_{L^2} \|\varphi\|_{L^2}^2 + |k|^{-1} \|F\|_{H^{-1}} (\delta^{-1} \|(\varphi')(\varphi')_{L^2} + \delta^{-3/2} \|(\varphi')(\varphi')_{L^\infty})$$

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As \( \|\phi\|_{L^2} \gtrsim \|\phi^j\|_{L^2} + |\phi| |\phi|_{L^2} \), \( \|\phi^j\|_{L^2} + \|\phi\|_{L^2} \leq \|\phi\|_{L^2} \), we have

\[
\|u\|_{L^2}^2 \lesssim (\delta v^{-\frac{5}{6}}|k|^{-\frac{2}{3}} + \delta^{-\frac{1}{2}}|\phi|^{-\frac{2}{3}} + \delta^{-\frac{1}{2}}v^{-\frac{1}{3}}|k|^{-1} + \delta^{-\frac{1}{2}}v^{-\frac{2}{3}}|k|^{-\frac{5}{6}} + \epsilon v^{-\frac{1}{2}}|k|^{-\frac{2}{3}} + v^{-\frac{5}{6}}|k|^{-\frac{1}{3}})\|F\|_{H^{-1}} \|u\|_{L^2} \lesssim (v^{-\frac{1}{2}}|k|^{-1} + v^{-\frac{1}{3}}|k|^{-\frac{2}{3}} + v^{-\frac{5}{6}}|k|^{-\frac{1}{3}})\|F\|_{H^{-1}} \|u\|_{L^2}. 
\]

When \( vk^2 \leq 1 \), we deduce that \( v^{-\frac{1}{2}}|k|^{-1} \|u\|_{L^2} \lesssim \|F\|_{H^{-1}} \). When \( vk^2 \gtrsim 1 \), we get by Lemma 3.4 that

\[
\|u\|_{L^2} \lesssim |k|^{-1} \|w\|_{L^2} \lesssim v^{-\frac{1}{2}}|k|^{-1} (v^{-\frac{1}{3}}|k|^{-\frac{2}{3}}) \|F\|_{H^{-1}} \lesssim v^{-\frac{1}{2}}|k|^{-1} \|F\|_{H^{-1}}. 
\]

\[ \square \]

3.3. Weak type resolvent estimate

The following lemma will be used to estimate the coefficients of the boundary correction when \( F \in H^{-1} \):

Lemma 3.6. Let \( (\phi, w) \) be as in Proposition 3.3. Assume that \( vk^2 \leq 1 \) and \( f \in H^1(-1, 1), \ j \in \{\pm 1\} \) and \( f(-j) = 0 \). Then it holds that

\[
|\langle w, f \rangle| \leq C |k|^{-1} \|F\|_{H^{-1}} \left( \delta^{-\frac{1}{2}} \|f\|_{L^\infty((-1,1) \cap (\lambda-\delta, \lambda+\delta))} + |f(j)|(|j-\lambda| + \delta)^{-\frac{1}{2}} \delta^{-\frac{1}{2}} + \|f \chi\|_{H^1} + \delta^{-1} \|f \chi\|_{L^2} \right). 
\]

Here \( \delta = v^{-\frac{1}{2}}|k|^{-\frac{1}{3}} \) and \( \chi \) is a cut-off function defined by (3.8).

Proof. For \( \phi \in H^1_0(-1,1) \), we get by integration by parts that

\[
\|F\|_{H^{-1}} \|\phi\|_{H^1} \gtrsim |\langle F, \phi \rangle| = \left| -v (\delta_y^2 - k^2) w + i k (y - \lambda) w - \epsilon v^{\frac{1}{3}}|k|^{\frac{2}{3}} w, \phi \right| \gtrsim -v \|w\|_{L^2} |\phi\|_{L^2} + |vk^2 - v^{\frac{1}{3}}|k|^{\frac{2}{3}} \|w\|_{L^2} |\phi\|_{L^2} + |(k(y-\lambda) w, \phi)|. 
\]

As \( vk^2 \leq 1 \), we have \( |vk^2 - v^{\frac{1}{3}}|k|^{\frac{2}{3}}| \lesssim v^{\frac{1}{3}}|k|^{\frac{2}{3}} \). Then by Lemma 3.4, we get

\[
|\langle k \langle w, (y-\lambda) \phi \rangle \rangle| \lesssim \|F\|_{H^{-1}} \|\phi\|_{H^1} + v \|w\|_{L^2} |\phi\|_{L^2} + v^{\frac{1}{3}}|k|^{\frac{2}{3}} \|w\|_{L^2} |\phi\|_{L^2} \lesssim C \|F\|_{H^{-1}} \|\phi\|_{H^1} + CV^{\frac{1}{3}}|k|^{\frac{1}{3}} \|F\|_{H^{-1}} \|\phi\|_{L^2} \leq C \|F\|_{H^{-1}} \|\phi\|_{H^1} + \delta^{-1} \|\phi\|_{L^2}. 
\]
\( \phi_1(y) = \phi(y) - \phi(1)\chi_1(y) \) for \( y \in [-1, 1] \). Then we have \( \chi_1 \in H^1(-1, 1), \phi_1 \in H_0^1(-1, 1), \supp \chi_1 = [1 - \delta_*, 1] \) and

\[
\| \chi_1 \|_{L^\infty} = 1, \quad \| \chi_1 \|_{L^2} \leq \delta_*^{\frac{1}{2}}, \quad \| \chi_1' \|_{L^2} \leq \delta_*^{-\frac{1}{2}}, \\
\| (y - \lambda) \chi_1 \|_{L^\infty} \leq \| y - \lambda \|_{L^\infty((1 - \delta_*, 1))} \| \chi_1 \|_{L^\infty} \leq \| 1 - \lambda \| + \delta_*.
\]

As \( w(1) = 0 \), we have \( |w(y)| = |\int_1^y w'(z)dz| \leq |1 - y|^{\frac{1}{2}} \|w'\|_{L^2} \leq \delta_*^{\frac{1}{2}} \|w'\|_{L^2} \) for \( y \in [1 - \delta_*, 1] \) and \( \| w \|_{L^1((1 - \delta_*, 1))} \leq \delta_*^{\frac{1}{2}} \|w'\|_{L^2} \). By Lemma 3.4 and (3.9), we get

\[
|(w, (y - \lambda)\phi)| \leq |(\phi(1)(w, (y - \lambda)\chi_1)) + |(w, (y - \lambda)\phi_1)| \\
\leq |(\phi(1))| \|w\|_{L^1((1 - \delta_*, 1))} \| (y - \lambda) \chi_1 \|_{L^\infty} + C|k|^{-1} \| F \|_{H^{-1}}((\| \phi \|_{H^1} + \delta^{-1} \| \phi_1 \|_{L^2})) \\
\leq |(\phi(1))| \delta_*^{\frac{3}{2}} \|w'\|_{L^2} ((1 - \lambda) + \delta) + C|k|^{-1} \| \phi(1) \| \| F \|_{H^{-1}} ((\| \chi_1 \|_{H^1} + \delta^{-1} \| \chi_1 \|_{L^2}) \\
+ C|k|^{-1} \| F \|_{H^{-1}} ((\| \phi \|_{H^1} + \delta^{-1} \| \phi \|_{L^2})) \\
\leq C|k|^{-1} \| \phi(1) \| \| F \|_{H^{-1}} ((\| \delta_*^{\frac{3}{2}} ((1 - \lambda) + \delta) + \delta^{-3} + \delta_*^{\frac{1}{2}}) \\
+ C|k|^{-1} \| F \|_{H^{-1}} ((\| \phi \|_{H^1} + \delta^{-1} \| \phi \|_{L^2})).
\]

Here we used the fact that \( \nu^{-1}|k| = \delta^{-3} \). Taking \( \delta_* = ((1 - \lambda) + \delta)^{-\frac{1}{2}} \delta_*^{\frac{3}{2}} \), we get

\[
|(w, (y - \lambda)\phi)| \leq C|k|^{-1} \| \phi(1) \| \| F \|_{H^{-1}} ((1 - \lambda) + \delta)^{\frac{1}{2}} \delta^{-\frac{3}{2}} \\
+ C|k|^{-1} \| F \|_{H^{-1}} ((\| \phi \|_{H^1} + \delta^{-1} \| \phi \|_{L^2})).
\]

for \( \phi \in H^1(-1, 1) \) with \( \phi(-1) = 0 \).

Now for \( f \in H^1(-1, 1), f(-1) = 0 \), let \( \phi = f \chi_1 \). Then we have \( \phi \in H^1(-1, 1), \phi(-1) = 0 \). Thus, we have

\[
|(w, f)| \leq \|w\|_{L^2} \| f - (y - \lambda)\phi \|_{L^2} + |(w, (y - \lambda)\phi)| \\
\leq C\nu^{-\frac{3}{2}} |k|^{-\frac{1}{2}} \| F \|_{H^{-1}} \| f - (y - \lambda)\phi \|_{L^2} + C|k|^{-1} \| \phi(1) \| \| F \|_{H^{-1}} ((1 - \lambda) + \delta)^{\frac{1}{2}} \delta^{-\frac{3}{2}} \\
+ C|k|^{-1} \| F \|_{H^{-1}} ((\| \phi \|_{H^1} + \delta^{-1} \| \phi \|_{L^2})) \\
= C|k|^{-1} \| F \|_{H^{-1}} (\delta^{-2} \| f - (y - \lambda)\phi \|_{L^2} \\
+ \| \phi(1) \| ((1 - \lambda) + \delta)^{\frac{1}{2}} \delta^{-\frac{3}{2}} + \| \phi \|_{H^1} + \delta^{-1} \| \phi \|_{L^2}).
\]

As \( 0 \leq 1 - (y - \lambda)\chi_1 \leq 1 \) for \( y \in [-1, 1] \) and \( 1 - (y - \lambda)\chi_1 = 0 \) for \( y \not\in (\lambda - \delta, \lambda + \delta) \), we have

\[
\| f - (y - \lambda)\phi \|_{L^2} = \| f (1 - (y - \lambda)\chi_1) \|_{L^2} \leq (2\delta)^{\frac{1}{2}} \| f \|_{L^\infty((-1, 1) \cap (\lambda - \delta, \lambda + \delta))}.
\]
Thanks to $|\chi(y)| \leq C(|y - \lambda| + \delta)^{-1}$ for $y \in [-1, 1]$, we get

$$|\phi(1)| = |f(1)||\chi(1)| \leq C|f(1)|(|1 - \lambda| + \delta)^{-1}.$$ 

Thus, we conclude that

$$|\langle w, f \rangle| \leq C|k|^{-1}\|F\|_{H^{-1}}\left(\delta^{-\frac{3}{2}}\|f\|_{L^\infty((-1,1) \cap (\lambda - \delta, \lambda + \delta))} + |f(1)|(|1 - \lambda| + \delta)^{-\frac{3}{4}}\delta^{-\frac{3}{4}} + |f\chi|_{H^1} + \delta^{-1}\|f\chi\|_{L^2}\right).$$ 

The case of $f(1) = 0$ can be proved similarly.  \qed

4. Resolvent Estimates with Non-slip Boundary Condition

In this section, we study the resolvent estimates of the linearized operator under the non-slip boundary condition. To this end, we will use the stream function formulation

$$\begin{cases}
- \nu(\partial_y^2 - k^2)^2 \varphi + ik(y - \lambda)(\partial_y^2 - k^2)\varphi - \epsilon \nu^{\frac{1}{2}}|k|^\frac{3}{2}(\partial_y^2 - k^2)\varphi = F, \\
\varphi(\pm 1) = 0, \quad \varphi'(\pm 1) = 0,
\end{cases} \quad (4.1)$$

where $\lambda \in \mathbb{R}$ and $\epsilon \geq 0$ small enough independent of $\nu, k, \lambda$. We introduce

$$w = (\partial_y^2 - k^2)\varphi, \quad u = (-\partial_y \varphi, ik \varphi).$$

4.1. Reformulation of the problem

We introduce the following decomposition:

$$\varphi = \varphi_{Na} + c_1 \varphi_1 + c_2 \varphi_2, \quad (4.2)$$

where $\varphi_{Na}$ solves the OS equation with the Navier-slip boundary condition

$$\begin{cases}
- \nu(\partial_y^2 - k^2)^2 \varphi_{Na} + ik(y - \lambda)(\partial_y^2 - k^2)\varphi_{Na} - \epsilon \nu^{\frac{1}{2}}|k|^\frac{3}{2}(\partial_y^2 - k^2)\varphi_{Na} = F, \\
\varphi_{Na}(\pm 1) = 0, \quad \varphi'_{Na}(\pm 1) = 0,
\end{cases} \quad (4.3)$$

and $\varphi_1, \varphi_2$ solve the following homogeneous OS equations

$$\begin{cases}
- \nu(\partial_y^2 - k^2)^2 \varphi_1 + ik(y - \lambda)(\partial_y^2 - k^2)\varphi_1 - \epsilon \nu^{\frac{1}{2}}|k|^\frac{3}{2}(\partial_y^2 - k^2)\varphi_1 = 0, \\
\varphi_1(\pm 1) = 0, \quad \varphi'_1(1) = 1, \quad \varphi'_1(-1) = 0,
\end{cases} \quad (4.4)$$

and

$$\begin{cases}
- \nu(\partial_y^2 - k^2)^2 \varphi_2 + ik(y - \lambda)(\partial_y^2 - k^2)\varphi_2 - \epsilon \nu^{\frac{1}{2}}|k|^\frac{3}{2}(\partial_y^2 - k^2)\varphi_2 = 0, \\
\varphi_2(\pm 1) = 0, \quad \varphi'_2(-1) = 1, \quad \varphi'_2(1) = 0.
\end{cases} \quad (4.5)$$
Let \( w_{Na} = (\partial_y^2 - k^2)\varphi_{Na} \) and \( w_i = (\partial_y^2 - k^2)\varphi_i \). Then we have
\[
w = w_{Na} + c_1 w_1 + c_2 w_2 = (\partial_y^2 - k^2)\varphi.
\] (4.6)

Next we determine the coefficients \( c_1, c_2 \). The boundary condition \( \varphi(\pm 1) = \varphi'(\pm 1) = 0 \) implies that
\[
\begin{align*}
\int_{-1}^{1} e^{\pm ky} w(y) dy &= \int_{-1}^{1} e^{\pm ky} (\partial_y^2 - k^2)\varphi(y) dy = 0, \\
\int_{-1}^{1} e^{\pm ky} w_1(y) dy &= \int_{-1}^{1} e^{\pm ky} (\partial_y^2 - k^2)\varphi_1(y) dy = e^{\pm k}, \\
\int_{-1}^{1} e^{\pm ky} w_2(y) dy &= \int_{-1}^{1} e^{\pm ky} (\partial_y^2 - k^2)\varphi_2(y) dy = -e^{\mp k}.
\end{align*}
\]

Then we infer that
\[
\begin{align*}
0 &= \int_{-1}^{1} e^{ky} w(y) dy = \int_{-1}^{1} e^{ky} w_{Na}(y) dy + c_1 \int_{-1}^{1} e^{ky} w_1(y) dy \\
&\quad + c_2 \int_{-1}^{1} e^{ky} w_2(y) dy \\
&= \int_{-1}^{1} e^{ky} w_{Na}(y) dy + e^k c_1 - e^{-k} c_2,
\end{align*}
\]
and
\[
\begin{align*}
0 &= \int_{-1}^{1} e^{-ky} w(y) dy = \int_{-1}^{1} e^{-ky} w_{Na}(y) dy + c_1 \int_{-1}^{1} e^{-ky} w_1(y) dy \\
&\quad + c_2 \int_{-1}^{1} e^{-ky} w_2(y) dy \\
&= \int_{-1}^{1} e^{-ky} w_{Na}(y) dy + e^{-k} c_1 - e^k c_2.
\end{align*}
\]

That is,
\[
\begin{align*}
\begin{cases}
e^k c_1 - e^{-k} c_2 = -\int_{-1}^{1} e^{ky} w_{Na}(y) dy, \\
e^{-k} c_1 - e^k c_2 = -\int_{-1}^{1} e^{-ky} w_{Na}(y) dy.
\end{cases}
\end{align*}
\]

Hence, we obtain
\[
c_1(\lambda) = -\frac{1}{e^{2k} - e^{-2k}} \left( \int_{-1}^{1} e^{k(y+1)} w_{Na}(y) dy - \int_{-1}^{1} e^{-k(y+1)} w_{Na}(y) dy \right) \\
= -\int_{-1}^{1} \frac{\sinh k(y + 1)}{\sinh 2k} w_{Na}(y) dy, \tag{4.7}
\]
\[c_2(\lambda) = \frac{1}{e^{2k} - e^{-2k}} \left( \int_{-1}^{1} e^{ky} w_{Na}(y) \, dy - \int_{-1}^{1} e^{-ky} w_{Na}(y) \, dy \right)\]
\[= \int_{-1}^{1} \frac{\sinh k(1 - y)}{\sinh 2k} w_{Na}(y) \, dy. \quad (4.8)\]

4.2. Bounds on \(c_1\) and \(c_2\)

We assume that \(vk^2 \leq 1\).

**Lemma 4.1.** If \(F \in L^2(I)\), then we have

\[(1 + |k(\lambda - 1)|)c_1| + (1 + |k(\lambda + 1)|)c_2| \leq C v^{-\frac{1}{6}} |k|^{-\frac{5}{6}} \|F\|_{L^2}. \quad (4.9)\]

**Proof.** For the case of \(|\lambda - 1| \leq |k|^{-1}\), we get by (4.7) and Corollary 3.2 that

\[|c_1| \leq C \|w_{Na}\|_{L^1} \leq C v^{-\frac{1}{6}} |k|^{-\frac{5}{6}} \|F\|_{L^2}. \]

For the case of \(|\lambda - 1| \geq |k|^{-1}\), we have \(\lambda - y \geq \lambda - 1 > 0\) for \(y \in (-1, 1)\). Then we get by Corollary 3.2 and Lemma 9.1 that

\[|c_1| = \left| \int_{-1}^{1} \frac{\sinh k(1 + y)}{\sinh 2k} w_{Na} \, dy \right| \]
\[\leq \left( \int_{-1}^{1} \left( \frac{\sinh k(1 + y)}{(y - \lambda) \sinh 2k} \right)^2 \, dy \right)^{\frac{1}{2}} \|y - \lambda\|_{L^2} w_{Na} \]
\[\lesssim |k|^{-\frac{1}{2}} \|F\|_{L^2} \leq v^{-\frac{1}{6}} |k|^{-\frac{5}{6}} \|k(\lambda - 1)^{-1}\| F \|_{L^2}. \]

For the case of \(1 - \lambda \geq |k|^{-1}\), let \(E_1 = (-1, 1) \cap (-\infty, (\lambda + 1)/2)\). Then \(|\lambda - y| \geq |\lambda - 1|/2 > 0\) for \(y \in (-1, 1) \setminus E_1\). By Corollary 3.2, Lemmas 9.1 and 9.2, we get

\[|c_1| = \left| \int_{-1}^{1} \frac{\sinh k(1 + y)}{\sinh 2k} w_{Na} \, dy \right| \]
\[\leq \left( \int_{(-1,1) \setminus E_1} \frac{\sinh k(1 + y)}{(y - \lambda) \sinh 2k} \, dy \right)^{\frac{1}{2}} \|y - \lambda\|_{L^2} w_{Na} \]
\[+ \left\| \frac{\sinh k(1 + y)}{\sinh 2k} \right\|_{L^\infty(E_1)} \|w_{Na}\|_{L^1} \]
\[\lesssim |k|^{-\frac{1}{2}} |\lambda - 1|^{-1} \|y - \lambda\|_{L^2} w_{Na} + e^{-|k|(|1 - \lambda|/2)} \|w_{Na}\|_{L^1} \]
\[\lesssim |k|^{-\frac{1}{2}} |\lambda - 1|^{-1} \|F\|_{L^2} + e^{-|k|(|1 - \lambda|/2)} v^{-\frac{1}{6}} |k|^{-\frac{5}{6}} \|F\|_{L^2} \]
\[\lesssim |k(\lambda - 1)|^{-1} v^{-\frac{1}{6}} |k|^{-\frac{5}{6}} \|F\|_{L^2}. \]

Summing up, we conclude that

\[(1 + |k(\lambda - 1)|) |c_1| \lesssim v^{-\frac{1}{6}} |k|^{-\frac{5}{6}} \|F\|_{L^2}. \]

The estimate of \(c_2\) is similar. \(\Box\)
Lemma 4.2. If $F \in H^{-1}(I)$, then we have

$$(1 + |k(\lambda - 1)|)^{\frac{3}{2}} |c_1| + (1 + |k(\lambda + 1)|)^{\frac{3}{2}} |c_2| \leq C \nu^{-\frac{1}{2}} |k|^{-\frac{1}{2}} \|F\|_{H^{-1}}. \quad (4.10)$$

Proof. Let $\delta = \frac{1}{2} |k|^{-\frac{1}{2}} \leq |k|^{-1}$. We split the proof into three cases.

Case 1. $\lambda \geq 1 + 2|k|^{-1}$.
As $(-1, 1) \cap (\lambda - \delta, \lambda + \delta) = \emptyset$, $\chi(y) = \frac{1}{\lambda - y}$, where $\chi(y)$ is defined in (3.8). Applying Lemma 3.6 with $f(y) = \frac{\sinh(k(1+y))}{\sinh(k)}$ and $j = 1$, we deduce that

$$|c_1| = \left| \langle w_{Na}, f \rangle \right| \leq C|k|^{-1} \|F\|_{H^{-1}} \left((|1 - \lambda| + \delta)^{-\frac{3}{2}} \delta^{-\frac{3}{4}} + \|f(y) - \lambda\|_{H^1} + \delta^{-1} \|f(y)\|_{L^2} \right).$$

Thanks to Lemma 9.1, we have

$$\|f(y)\|_{H^1} \leq \|f\|_{L^2} \frac{1}{|y - \lambda|} \|L_{L^\infty} + 2\|f\|_{L^2} \frac{1}{(y - \lambda)^2} \|L_{L^\infty} + \|f\|_{L^2} \frac{1}{|y - \lambda|} \|L_{L^\infty}. \leq |k|^{\frac{3}{2}} |1 - \lambda|^{-1} + |k|^{\frac{3}{2}} |1 - \lambda|^{-2} + |k|^{\frac{3}{2}} |1 - \lambda|^{-1} \leq |k|^{\frac{3}{2}} (1 + |k(\lambda - 1)|)^{-\frac{3}{4}},$$

$$\|f(y)\|_{L^2} \leq \|f\|_{L^2} \frac{1}{|y - \lambda|} \|L_{L^\infty} \leq |k|^{-\frac{1}{2}} |1 - \lambda|^{-1} \leq |k|^{\frac{1}{2}} (1 + |k(\lambda - 1)|)^{-\frac{3}{4}}.$$  

Due to $\delta^{-1} \geq |k|$, $(|1 - \lambda| + \delta)^{-\frac{3}{2}} \delta^{-\frac{3}{4}} \leq \delta^{-\frac{3}{2}} (1 + |k(\lambda - 1)|)^{-\frac{3}{4}}$. Thus, we obtain

$$|c_1| \leq |k|^{-1} \|F\|_{H^{-1}} \delta^{-\frac{3}{2}} (1 + |k(\lambda - 1)|)^{-\frac{3}{4}} = v^{-\frac{1}{2}} |k|^{-\frac{1}{2}} (1 + |k(\lambda - 1)|)^{-\frac{3}{4}} \|F\|_{H^{-1}}.$$  

Case 2. $|\lambda - 1| \leq 2|k|^{-1}$.
Applying Lemma 3.6 with $f(y) = \frac{\sinh(k(1+y))}{\sinh(k)}$ and $j = 1$, we get

$$|c_1| = \left| \langle w_{Na}, f \rangle \right| \leq C|k|^{-1} \|F\|_{H^{-1}} \left((|1 - \lambda| + \delta)^{-\frac{3}{2}} \delta^{-\frac{3}{4}} + \|f\|_{H^1} + \delta^{-1} \|f\|_{L^2} \right). \quad (4.11)$$

where $E = (-1, 1) \cap (\lambda - \delta, \lambda + \delta)$. Using the facts that

$$\|f\|_{H^1} \leq \|f\|_{L^2} \|F\|_{L^\infty} + \|f\|_{L^\infty} \|\chi\|_{L^2} + \|f\|_{L^\infty} \|\chi\|_{L^2}, \leq \delta^{-\frac{3}{2}} |k|^{\frac{1}{2}} + \delta^{-\frac{3}{2}} + \delta^{-\frac{3}{4}} \leq \delta^{-\frac{3}{2}} (1 + |k(\lambda - 1)|)^{-\frac{3}{4}},$$

$$\|f\|_{L^2} \leq \|f\|_{L^\infty} \|\chi\|_{L^2} \leq \delta^{-\frac{3}{2}} \leq \delta^{-\frac{1}{2}} (1 + |k(\lambda - 1)|)^{-\frac{3}{4}},$$

we deduce that

$$|c_1| \leq |k|^{-1} \|F\|_{H^{-1}} \delta^{-\frac{3}{2}} (1 + |k(\lambda - 1)|)^{-\frac{3}{4}} = v^{-\frac{1}{2}} |k|^{-\frac{1}{2}} (1 + |k(\lambda - 1)|)^{-\frac{3}{4}} \|F\|_{H^{-1}}.$$
Case 3. \( \lambda \leq 1 - 2|k|^{-1} \).

Let \( E_1 = (-1, 1) \cap (-\infty, (\lambda + 1)/2) \) and \( E_1^c = (-1, 1) \setminus (-\infty, (\lambda + 1)/2) \).

Due to \( \frac{\lambda + 1}{2} \geq \lambda + \delta \), we have \( \chi \big|_{E_1^c} = \frac{1}{\chi} \) and \( E \subset E_1 \). It is easy to see that

\[
\| f \chi \|_{L^2} \leq \| f \|_{L^\infty(E_1)} \| \chi \|_{L^2(E_1)} + \| f \|_{L^2(E_1^c)} \| \chi \|_{L^\infty(E_1^c)} \\
\leq \| f \|_{L^\infty(E_1)} \| \chi \|_{L^2(-1,1)} + \| f \|_{L^2(-1,1)} \| 1/(y - \lambda) \|_{L^\infty(E_1^c)},
\]

\[
\| f' \chi \|_{L^2} \leq \| f' \|_{L^\infty(E_1)} \| \chi \|_{L^2(E_1)} + \| f' \|_{L^2(E_1^c)} \| \chi \|_{L^\infty(E_1^c)} \\
\leq \| f' \|_{L^\infty(E_1)} \| \chi \|_{L^2(-1,1)} + \| f' \|_{L^2(-1,1)} \| 1/(y - \lambda) \|_{L^\infty(E_1^c)},
\]

\[
\| f' \chi' \|_{L^2} \leq \| f \|_{L^\infty(E_1)} \| \chi' \|_{L^2(E_1)} + \| f \|_{L^2(E_1^c)} \| \chi' \|_{L^\infty(E_1^c)} \\
\leq \| f \|_{L^\infty(E_1)} \| \chi' \|_{L^2(-1,1)} + \| f \|_{L^2(-1,1)} \| 2/(y - \lambda)^2 \|_{L^\infty(E_1^c)}.
\]

By Lemmas 9.1, 9.2 and \( \| 1/(y - \lambda) \|_{L^\infty(E_1^c)} \leq 2/(1 - \lambda) \), we infer that

\[
\| f \chi \|_{L^2} \lesssim e^{-|k|(1-\lambda)/2} \delta^{-\frac{3}{2}} + |k|^{-\frac{1}{2}} (1 - \lambda)^{-1} \lesssim \delta^{-\frac{3}{2}} (1 + |k(1 - \lambda)|)^{-\frac{3}{2}},
\]

\[
\| f' \chi \|_{L^2} \lesssim |k| e^{-|k|(1-\lambda)/2} \delta^{-\frac{3}{2}} + |k|^{\frac{1}{2}} (1 - \lambda)^{-1} \lesssim \delta^{-\frac{3}{2}} (1 + |k(1 - \lambda)|)^{-\frac{3}{2}},
\]

\[
\| f' \chi' \|_{L^2} \lesssim e^{-|k|(1-\lambda)/2} \delta^{-\frac{3}{2}} + |k|^{-\frac{1}{2}} (1 - \lambda)^{-2} \lesssim \delta^{-\frac{3}{2}} (1 + |k(1 - \lambda)|)^{-\frac{3}{2}}.
\]

This shows that

\[
\| f \chi \|_{H^1} \leq \| f' \chi \|_{L^2} + \| f' \chi' \|_{L^2} + \| f \chi \|_{L^2} \lesssim \delta^{-\frac{3}{2}} (1 + |k(\lambda - 1)|)^{-\frac{3}{2}}.
\]

On the other hand, we have

\[
\delta^{-\frac{3}{2}} \| f \|_{L^\infty(E)} \leq \delta^{-\frac{3}{2}} \| f \|_{L^\infty(E_1)} \leq \delta^{-\frac{3}{2}} e^{-|k|(1-\lambda)/2} \lesssim \delta^{-\frac{3}{2}} (1 + |k(1 - \lambda)|)^{-\frac{3}{2}},
\]

\[
(|1 - \lambda| + \delta)^{-\frac{3}{2}} \delta^{-\frac{3}{4}} \lesssim \epsilon^{-\frac{3}{2}} (1 + |k(\lambda - 1)|)^{-\frac{3}{4}}.
\]

Plugging these inequalities above into (4.12), we get

\[
|c_1| \lesssim \nu^{-\frac{1}{2}} |k|^{-\frac{1}{2}} (1 + |k(\lambda - 1)|)^{-\frac{3}{2}} \| F \|_{H^{-1}}.
\]

Combining three cases, we get

\[
(1 + |k(\lambda - 1)|)^{\frac{3}{2}} |c_1| \lesssim \nu^{-\frac{1}{2}} |k|^{-\frac{1}{2}} \| F \|_{H^{-1}}.
\]

In a similar way, we can deduce the estimate of \( c_2 \). \( \square \)

### 4.3. Bounds on \( w_1 \) and \( w_2 \)

For the solutions \( w_1, w_2 \) of the homogeneous OS equation, we have the following uniform bounds. These bounds cover the case of \( \nu k^2 \leq 1 \) for \( \nu \) small enough, and will be used in Sect. 4.5.

In the sequel, we will introduce \( L = (\frac{\epsilon}{\nu})^{\frac{1}{3}} \) for convenience, which in fact equal \( \delta^{-1} \).
Proposition 4.3. There exists $k_0$ and $\delta_0$ independent of $\nu$ such that if $L \geq 6k$ or $L \geq k \geq k_0$ and $\epsilon \in [0, \delta_0)$, there holds
\[
\|w_1\|_{L^\infty} \leq C \nu^{-\frac{1}{2}} (1 + |k||\lambda - 1|)^{\frac{1}{2}}, \\
\|w_2\|_{L^\infty} \leq C \nu^{-\frac{1}{2}} (1 + |k||\lambda + 1|)^{\frac{1}{2}}, \\
\|w_1\|_{L^1} + \|w_2\|_{L^1} \leq C.
\]

Let $\rho_k$ be a weight function defined by
\[
\rho_k(y) = \begin{cases} 
  L(y + 1) & y \in [-1, -1 + L^{-1}], \\
  1 & y \in [-1 + L^{-1}, 1 - L^{-1}], \\
  L(1 - y) & y \in [1 - L^{-1}, 1]. 
\end{cases}
\]

We also need the following weighted version:

Proposition 4.4. There exists $k_0$ and $\delta_0$ independent of $\nu$ so that if $L \geq 6k$ or $L \geq k \geq k_0$ and $\epsilon \in [0, \delta_0)$, it holds that
\[
\|\rho_k^{-\frac{1}{4}} w_1\|_{L^2} \leq C \nu^{-\frac{2}{3}} |k|^{-\frac{1}{2}} (1 + |k(\lambda - 1)|)^{\frac{3}{8}}, \\
\|\rho_k^{-\frac{1}{4}} w_2\|_{L^2} \leq C \nu^{-\frac{2}{3}} |k|^{-\frac{1}{2}} (1 + |k(\lambda + 1)|)^{\frac{3}{8}}, \\
\|\rho_k^{\frac{1}{4}} w_1\|_{L^2} + \|\rho_k^{\frac{1}{4}} w_2\|_{L^2} \leq C L^{\frac{1}{2}}.
\]

4.4. Resolvent estimates when $vk^2 \geq 1$

This case can be proved directly by using integration by parts.

Proposition 4.5. Let $\varphi$ be a solution of (4.1). If $F \in L^2(I)$, then we have
\[
\nu^{\frac{5}{12}} |k|^{\frac{5}{6}} \|w\|_{L^2} \leq vk^2 \|w\|_{L^2} \leq C \|F\|_{L^2}.
\]

If $F \in H^{-1}(I)$, then we have
\[
(vk^2)^{\frac{1}{2}} \|u\|_{L^2} \leq vk^2 \|u\|_{L^2} \leq C \|F\|_{H^{-1}}, \\
\nu^{\frac{3}{4}} |k|^{\frac{1}{2}} \|w\|_{L^2} \leq \nu |k| \|w\|_{L^2} \leq C \|F\|_{H^{-1}}.
\]
Proof. By integration by parts, we get
\[
\langle -F, \varphi \rangle = \langle \nu (\partial_y^2 - k^2) \varphi - ik (y - \lambda) (\partial_y^2 - k^2) \varphi \rangle + \epsilon \nu \frac{1}{3} |k|^{\frac{5}{3}} (\partial_y^2 - k^2) \varphi, \varphi \rangle
\]
\[
= \nu (\|\varphi''\|_{L^2}^2 + 2k^2 \|\varphi'\|_{L^2}^2 + k^4 \|\varphi\|_{L^2}^2)
\]
\[
- \epsilon \nu \frac{1}{3} |k|^{\frac{5}{3}} (\|\varphi'\|_{L^2}^2 + k^2 \|\varphi\|_{L^2}^2) + ik \int_{-1}^{1} \overline{\varphi}' \varphi \, dy
\]
\[
+ ik \int_{-1}^{1} (y - \lambda) |\varphi|^2 \, dy + ik \int_{-1}^{1} (y - \lambda) |\varphi'|^2 \, dy,
\]
which implies
\[
|\langle F, \varphi \rangle| \geq \nu (\|\varphi''\|_{L^2}^2 + 2k^2 \|\varphi'\|_{L^2}^2 + k^4 \|\varphi\|_{L^2}^2)
\]
\[
- \epsilon \nu \frac{1}{3} |k|^{\frac{5}{3}} (\|\varphi'\|_{L^2}^2 + k^2 \|\varphi\|_{L^2}^2) - |k| \int_{-1}^{1} |\varphi'| \, dy
\]
\[
\geq \nu (\|\varphi''\|_{L^2}^2 + 2k^2 \|\varphi'\|_{L^2}^2 + k^4 \|\varphi\|_{L^2}^2) - \epsilon \nu \frac{1}{3} |k|^{\frac{5}{3}} (\|\varphi'\|_{L^2}^2 + k^2 \|\varphi\|_{L^2}^2)
\]
\[
- \frac{1}{4} (\nu \|\varphi''\|_{L^2}^2 + v k^4 \|\varphi\|_{L^2}^2)
\]
\[
\geq \frac{1}{4} \nu (\|\varphi''\|_{L^2}^2 + 2k^2 \|\varphi'\|_{L^2}^2 + k^4 \|\varphi\|_{L^2}^2) = \frac{1}{4} \nu \|w\|_{L^2}^2 \geq \frac{1}{4} v k^2 \|w\|_{L^2} \|\varphi\|_{L^2},
\]
by using the facts that \(v k^2 \geq 1\), \(\|w\|_{L^2} \geq k^2 \|\varphi\|_{L^2}\), and taking \(\epsilon \leq \frac{1}{4}\). This implies the first inequality of the lemma.

Notice that
\[
\nu (\|\varphi''\|_{L^2}^2 + 2k^2 \|\varphi'\|_{L^2}^2 + k^4 \|\varphi\|_{L^2}^2) \geq v k^2 \|u\|_{L^2}^2.
\]
Then we have
\[
\frac{1}{4} v k^2 \|u\|_{L^2}^2 \leq |\langle F, \varphi \rangle| \leq \|F\|_{H^{-1}} \|\varphi\|_{H^1} \leq \|F\|_{H^{-1}} \|u\|_{L^2},
\]
which gives the second inequality. On the other hand,
\[
\frac{1}{4} \nu \|w\|_{L^2}^2 \leq |\langle F, \varphi \rangle| \leq \|F\|_{H^{-1}} \|u\|_{L^2} \leq C (v k^2)^{-1} \|F\|_{H^{-1}}^2,
\]
which gives the third inequality. \(\square\)

4.5. Resolvent estimates when \(v k^2 \leq 1\) and \(v \leq \epsilon_0\)

**Proposition 4.6.** Let \(\varphi\) be a solution of (4.1). If \(F \in L^2(I)\), then we have
\[
\nu \frac{1}{5} |k|^{\frac{5}{3}} \|w\|_{L^1} + \nu \frac{5}{3} |k|^{\frac{5}{3}} \|w\|_{L^2} \leq C \|F\|_{L^2}.
\]
If \(F \in H^{-1}(I)\), then we have
\[
\nu \frac{1}{3} |k|^{\frac{2}{3}} \|w\|_{L^2} + \nu \frac{2}{3} |k|^{\frac{1}{3}} \|\rho_k \frac{1}{3} \|w\|_{L^2} + (v k^2)^{-1} \frac{1}{2} \|u\|_{L^2} \leq C \|F\|_{H^{-1}}.
\]
**Proof.** First of all, we consider the case of $F \in L^2(I)$. By Corollary 3.2, Proposition 4.3 and Lemma 4.1, we get

$$\|w\|_{L^1} \leq \|w_{Na}\|_{L^1} + |c_1| \|w_1\|_{L^1} + |c_2| \|w_2\|_{L^1}$$

$$\leq \|w_{Na}\|_{L^1} + C|c_1| + C|c_2|$$

$$\lesssim \|w_{Na}\|_{L^1} \lesssim v^{-\frac{1}{6}}|k|^{-\frac{5}{6}} \|F\|_{L^2}.$$ 

By Proposition 4.3 and Lemma 4.1, we have

$$|c_1| \|w_1\|_{L^2} \leq |c_1| \|w_1\|_{L^1} \|w_1\|_{L^\infty}$$

$$\leq C|c_1|v^{-\frac{1}{2}}(1 + |k|\lambda - 1)^\frac{1}{2} \leq C\nu^{-\frac{5}{12}}|k|^{-\frac{5}{6}} \|F\|_{L^2}.$$ 

Similarly, we have

$$|c_2| \|w_2\|_{L^2} \leq C\nu^{-\frac{5}{12}}|k|^{-\frac{5}{6}} \|F\|_{L^2}.$$ 

Then by Corollary 3.2, we get

$$\|w\|_{L^2} \leq \|w_{Na}\|_{L^2} + |c_1| \|w_1\|_{L^2} + |c_2| \|w_2\|_{L^2}$$

$$\lesssim (vk^2)^{-\frac{1}{2}} \|F\|_{L^2} + v^{-\frac{5}{12}}|k|^{-\frac{5}{6}} \|F\|_{L^2} \lesssim v^{-\frac{5}{12}}|k|^{-\frac{5}{6}} \|F\|_{L^2}.$$ 

Next we consider the case of $F \in H^{-1}(I)$. By Lemma 4.2 and Proposition 4.3, we have

$$|c_1| \|w_1\|_{L^2} + |c_2| \|w_2\|_{L^2} \leq C\nu^{-\frac{3}{2}}|k|^{-\frac{3}{2}} \|F\|_{H^{-1}},$$

which along with Proposition 3.3 gives

$$\|w\|_{L^2} \leq \|w_{Na}\|_{L^2} + |c_1| \|w_1\|_{L^2} + |c_2| \|w_2\|_{L^2}$$

$$\lesssim v^{-\frac{2}{3}}|k|^{-\frac{1}{3}} \|F\|_{H^{-1}} + v^{-\frac{2}{3}}|k|^{-\frac{1}{3}} \|F\|_{H^{-1}} \lesssim v^{-\frac{2}{3}}|k|^{-\frac{1}{3}} \|F\|_{H^{-1}}.$$ 

By Lemma 4.2 and Proposition 4.4, we have

$$|c_1| \|\rho_k^\frac{1}{2}w_1\|_{L^2} + |c_2| \|\rho_k^\frac{1}{2}w_2\|_{L^2} \leq C\nu^{-\frac{3}{2}}|k|^{-\frac{3}{2}} \|F\|_{H^{-1}},$$

which along with Proposition 3.3 gives

$$\|\rho_k^\frac{1}{2}w\|_{L^2} \leq \|\rho_k^\frac{1}{2}w_{Na}\|_{L^2} + |c_1| \|\rho_k^\frac{1}{2}w_1\|_{L^2} + |c_2| \|\rho_k^\frac{1}{2}w_2\|_{L^2}$$

$$\lesssim \|w_{Na}\|_{L^2} + v^{-\frac{2}{3}}|k|^{-\frac{1}{3}} \|F\|_{H^{-1}} \lesssim v^{-\frac{2}{3}}|k|^{-\frac{1}{3}} \|F\|_{H^{-1}}.$$ 

By Lemma 4.2 and Proposition 4.3, we have

$$|c_1| \|w_1\|_{L^1} + |c_2| \|w_2\|_{L^1} \leq C\nu^{-\frac{1}{2}}|k|^{-\frac{1}{2}} \|F\|_{H^{-1}}.$$ 

For the velocity, we have

$$u = u_{Na} + c_1u_1 + c_2u_2,$$

where $u_i = (-\partial_y \varphi_i, ik \varphi_i)$ and $(\partial_y^2 - k^2)\varphi_i = w_i$ with $\varphi_i(\pm 1) = 0, i = 1, 2$. Then by Proposition 3.3 and Lemma 9.3, we get

$$\|u\|_{L^2} \lesssim |u_{Na}|_{L^2} + |c_1| \|u_1\|_{L^2} + |c_2| \|u_2\|_{L^2} \lesssim v^{-\frac{1}{2}}|k|^{-\frac{1}{2}} \|F\|_{H^{-1}}.$$ 

This completes the proof. □
5. $L^p$ bounds on $w_1$ and $w_2$

Recall that $w_i = (\partial_y^2 - k^2)\varphi_i (i = 1, 2)$, where $\varphi_1, \varphi_2$ solve

$$
\begin{cases}
-\nu(\partial_y^2 - k^2)\varphi_1 + i k (y - \lambda)(\partial_y^2 - k^2)\varphi_1 - \epsilon \nu^\frac{1}{3}|k|^\frac{2}{3}(\partial_y^2 - k^2)\varphi_1 = 0, \\
\varphi_1(\pm1) = 0, \quad \varphi'_1(1) = 1, \quad \varphi'_1(-1) = 0,
\end{cases}
$$

and

$$
\begin{cases}
-\nu(\partial_y^2 - k^2)\varphi_2 + i k (y - \lambda)(\partial_y^2 - k^2)\varphi_2 - \epsilon \nu^\frac{1}{3}|k|^\frac{2}{3}(\partial_y^2 - k^2)\varphi_2 = 0, \\
\varphi_2(\pm1) = 0, \quad \varphi'_2(-1) = 1, \quad \varphi'_2(1) = 0.
\end{cases}
$$

In this section, we use the Airy function to solve $w_i$ and give the $L^p$ bounds on $w_i$. Let $L = \left(\frac{k}{\nu}\right)^\frac{1}{3}$. We always assume that $L \geq 6k$ or $L \geq k \geq k_0$ for some big $k_0$.

5.1. Airy function and the OS equation

Let $Ai(y)$ be the Airy function, which is a nontrivial solution of $f'' - yf = 0$.

Let

$$f_1(y) = Ai(e^{\frac{y}{6}}y), \quad f_2(y) = Ai(e^{\frac{5y}{6}}y).$$

Then $f_1$ and $f_2$ are two linearly independent solutions of $f'' - iyf = 0$. Hence,

$$W_1(y) = Ai(e^{\frac{y}{6}}(L(y - \lambda - ik\nu) + i\epsilon)), \quad W_2(y) = Ai(e^{\frac{5y}{6}}(L(y - \lambda - ik\nu) + i\epsilon))$$

are two linearly independent solutions of the homogeneous OS equation

$$-\nu(w'' - k^2w) + i k (y - \lambda)w - \epsilon \nu^\frac{1}{3}|k|^\frac{2}{3}w = 0.$$

Thus, $w_1$ and $w_2$ can be expressed as

$$w_1 = C_{11}W_1(y) + C_{12}W_2(y), \quad w_2 = C_{21}W_1(y) + C_{22}W_2(y), \quad (5.1)$$

where $C_{ij}, i, j = 1, 2$ are constants. Thanks to the facts that

$$\int_{-1}^{1} e^{ky}w_1(y)dy = e^k, \quad \int_{-1}^{1} e^{-ky}w_1(y)dy = e^{-k},$$

we get

$$\begin{cases}
e^k = C_{11}\int_{-1}^{1} e^{ky}W_1(y)dy + C_{12}\int_{-1}^{1} e^{ky}W_2(y)dy, \\
e^{-k} = C_{11}\int_{-1}^{1} e^{-ky}W_1(y)dy + C_{12}\int_{-1}^{1} e^{-ky}W_2(y)dy.
\end{cases}$$
Define the matrix $J$ as

$$J = \begin{pmatrix} A_1 & B_2 \\ B_1 & A_2 \end{pmatrix},$$

where

$$A_1 = \int_{-1}^{1} e^{ky} W_1(y)dy, \quad A_2 = \int_{-1}^{1} e^{-ky} W_2(y)dy,$$

$$B_1 = \int_{-1}^{1} e^{-ky} W_1(y)dy, \quad B_2 = \int_{-1}^{1} e^{ky} W_2(y)dy.$$ 

If $A_1 A_2 - B_1 B_2 \neq 0$, then we have

$$\begin{pmatrix} C_{11} \\ C_{12} \end{pmatrix} = \frac{\begin{pmatrix} A_2 e^k - B_2 e^{-k} \\ -B_1 e^k + A_1 e^{-k} \end{pmatrix}}{A_1 A_2 - B_1 B_2}. $$

Similarly, we have

$$\begin{pmatrix} C_{21} \\ C_{22} \end{pmatrix} = \frac{\begin{pmatrix} -A_2 e^{-k} + B_2 e^k \\ B_1 e^{-k} - A_1 e^k \end{pmatrix}}{A_1 A_2 - B_1 B_2}. $$

### 5.2. Estimates of $C_{ij}$ and $W_i$

We introduce some notations

$$A_0(z) = \int_{e^{\pi/6} z}^{\infty} Ai(t)dt = e^{i\pi/6} \int_{z}^{\infty} Ai(e^{i\pi/6} t)dt, $$

$$d = -1 - \lambda - ik\nu, \quad \tilde{d} = -1 + \lambda - ik\nu. $$

**Lemma 5.1.** It holds that

$$|C_{11}| \leq \frac{CLE^{-2k}}{|A_0(Ld + i\epsilon)|}, \quad |C_{12}| \leq \frac{CL}{|A_0(Ld + i\epsilon)|},$$

$$|C_{21}| \leq \frac{CL}{|A_0(Ld + i\epsilon)|}, \quad |C_{22}| \leq \frac{CLE^{-2k}}{|A_0(L\tilde{d} + i\epsilon)|}. $$

**Lemma 5.2.** It holds that

$$\frac{L}{|A_0(Ld + i\epsilon)|} \|W_1\|_{L^\infty} \leq CV^{-\frac{1}{2}} \left(1 + |k(\lambda + 1)|\right)^{\frac{1}{2}},$$

$$\frac{L}{|A_0(L\tilde{d} + i\epsilon)|} \|W_2\|_{L^\infty} \leq CV^{-\frac{1}{2}} \left(1 + |k(\lambda - 1)|\right)^{\frac{1}{2}},$$

$$\frac{L}{|A_0(Ld + i\epsilon)|} \|W_1\|_{L^1} + \frac{L}{|A_0(L\tilde{d} + i\epsilon)|} \|W_2\|_{L^1} \leq C,$$

and

$$\frac{L}{|A_0(Ld + i\epsilon)|} \|\rho_k W_1\|_{L^2} + \frac{L}{|A_0(L\tilde{d} + i\epsilon)|} \|\rho_k W_2\|_{L^2} \leq C L^{\frac{1}{2}}.$$ 

In order to prove Lemmas 5.1 and 5.2, we need to use many deep estimates on the Airy function. To lighten the reader’s burden, a complete proof will be presented in Sect. 8.
5.3. Proof of Propositions 4.3 and 4.4

Proposition 4.3 is a direct consequence of Lemmas 5.1 and 5.2. Next we prove Proposition 4.4.

Proof. First of all, (4.16) is a direct consequence of Lemmas 5.1 and 5.2.

Let $\delta_s = \nu^{\frac{1}{2}} (1 + |k(\lambda + 1)|)^{-\frac{1}{2}}$. Due to (4.13), we know that $\rho_k(y) = 1 \geq L\delta_s$ for $|y| \leq 1 - L^{-1}$, $\rho_k(y) = L (1 - |y|) \geq L\delta_s$ for $1 - L^{-1} \leq |y| \leq 1 - \delta_s$, and $\rho_k(y) = L (1 - |y|)$ for $1 - \delta_s \leq |y| \leq 1$. Then we deduce from Proposition 4.3 that

$$
\|w_1\|_{L^2} \leq \|w_1\|_{L^1}^\frac{1}{2} \|w_1\|_{L^{\infty}}^\frac{1}{2} \leq C (\nu^{-\frac{1}{2}} (1 + |k|\lambda - 1))^\frac{1}{2}
$$

and that

$$
\|\rho_k^{-\frac{1}{2}} w_1\|_{L^2} \leq \|\rho_k^{-\frac{1}{2}} w_1\|_{L^2((-1+\delta,1-\delta))} + \|\rho_k^{-\frac{1}{2}} w_1\|_{L^2((-1,1+\delta)\cup(1-\delta,1))}
$$

$$
\leq (L\delta_s) - \nu^{-\frac{1}{2}} \|w_1\|_{L^2} + \|(L (1 - |y|))\|^{-\frac{3}{2}} \|w_1\|_{L^2((-1,1+\delta)\cup(1-\delta,1))} \|w_1\|_{L^{\infty}}
$$

$$
\leq C (L\delta_s)^{-\nu^{-\frac{1}{2}} ((1 + |k|\lambda - 1))^\frac{1}{2} + CL - \nu^{-\frac{1}{2}} \lambda (1 + |k|\lambda - 1)^\frac{1}{2}
$$

$$
\leq C L - \nu^{-\frac{1}{2}} ((1 + |k|\lambda - 1))^\frac{1}{2} = C \nu\lambda (1 + |k|\lambda - 1)^\frac{1}{2}
$$

The estimate of $\|\rho_k^{-\frac{1}{2}} w_2\|_{L^2}$ is similar. This proves (4.14) and (4.16). □

6. Space-Time Estimates of the Linearized NS Equations

In this section, we establish the space-time estimates of the linearized 2-D Navier–Stokes equation in the vorticity formulation:

$$
\partial_t \omega + L_k \omega = -ikf_1 - \partial_y f_2, \quad \omega|_{t=0} = \omega_0(k, y),
$$

(6.1)

where $\omega = (\partial_y^2 - k^2)\phi$ with $\phi(\pm 1) = \phi'(\pm 1) = 0$ and

$$
L_k \omega = \nu(\partial_y^2 - k^2)\omega + iky\omega.
$$

We introduce the following norms:

$$
\| f \|_{L^p H^s} = \| f(t) H^s(I) \|_{L^p(\mathbb{R}^+)}^s, \quad \| f \|_{L^p L^q} = \| f(t) L^q(I) \|_{L^p(\mathbb{R}^+)}^q.
$$

The main result of this section are the following space-time estimates:

Proposition 6.1. Let $0 < \nu \leq \varepsilon_0$ and $\omega$ be a solution of (6.1) with $\omega_0 \in H^1(I)$ and $f_1, f_2 \in L^2 L^2$, where $\omega_0$ satisfies $\langle \omega_0, e^{\pm ky} \rangle = 0$. Then there exists a constant $C > 0$ independent of $\nu, k$ so that

$$
|k| \|u\|_{L^\infty L^\infty} + k^2 \|u\|_{L^2 L^2}^2 + (\nu k^2)^\frac{1}{2} \|\omega\|_{L^2 L^2}^2 + \|(1 - |y|)^\frac{1}{2} \omega\|_{L^\infty L^2}^2
$$

$$
\leq C (\|\omega_0\|_{L^2}^2 + k^{-2} \|\partial_y \omega_0\|_{L^2}^2) + C (\nu^{-\frac{1}{2}} |k| \|f_1\|_{L^2 L^2} + \nu^{-1} \|f_2\|_{L^2 L^2}^2).
$$

Here $u = (\partial_y \phi, -ik\phi)$.

We remark that the condition $\langle \omega_0, e^{\pm ky} \rangle = 0$ is equivalent to $\partial_y \phi_0(\pm 1) = 0$ or $u_0 \in H_0^1(I)$. 
6.1. Semigroup bounds

First of all, we consider the linearized equation with the Navier-slip boundary condition:
\[ \partial_t \omega_{Na} + L_k \omega_{Na} = 0, \quad \omega_{Na}(t, k, \pm 1) = 0, \quad \omega_{Na}|_{t=0} = \omega_0(k, y), \quad \lambda < 0, \]
where \( L_k = v(k^2 - \Delta_y^2) + ik \) with \( D(L_k) = H^2 \cap H_0^1(-1, 1) = \{ f \in H^2(-1, 1) : f(\pm 1) = 0 \} \).

Thanks to the fact that for \( f \in D(L_k) \)
\[ \text{Re}(L_k f, f) = v k^2 \| f \|_{L^2}^2 + v \| f' \|_{L^2}^2, \quad \] (6.3)
\( L_k \) is an accretive operator for any \( k \in \mathbb{Z} \). Let us recall that an operator \( A \) in a Hilbert space \( H \) is accretive if \( \text{Re}(A f, f) \geq 0 \) for all \( f \in D(A) \), or equivalently \( \| (\lambda + A) f \| \geq \| \lambda \| \| f \| \) for all \( f \in D(A) \) and all \( \lambda > 0 \). The operator \( A \) is called \( m \)-accretive if in addition any \( \lambda < 0 \) belongs to the resolvent set of \( A \). We define
\[ \Psi(A) = \inf \{ \| (A - i\lambda) f \| : f \in D(A), \lambda \in \mathbb{R}, \| f \| = 1 \}. \]

We need the following Gearhart-Prüss type lemma with sharp bound [44]:

**Lemma 6.2.** Let \( A \) be a \( m \)-accretive operator in a Hilbert space \( H \). Then \( \| e^{-tA} \| \leq e^{-t\Psi + \pi/2} \) for any \( t \geq 0 \).

**Lemma 6.3.** Let \( \omega_{Na} \) be a solution of (6.2) with \( \omega_0 \in L^2(I) \). Then for any \( k \in \mathbb{Z} \), there exist constants \( C, c > 0 \) independent with \( v, k \) such that
\[ \| e^{-tL_k} \omega_0 \|_{L^2} \leq C e^{-c k^2 \sqrt{v} t} \| \omega_0 \|_{L^2}, \]
Moreover, for any \( |k| \geq 1 \),
\[ (vk^2)^\frac{1}{2} \| e^{-tL_k} \omega_0 \|_{L^2}^2 \leq C \| \omega_0 \|_{L^2}^2. \]

**Proof.** Thanks to Proposition 3.1 and (6.3), there exists \( c > 0 \) so that for any \( k \in \mathbb{Z} \),
\[ \Psi(L_k) \geq c (vk^2)^\frac{1}{2} + v, \]
which along with Lemma 6.2 gives the first inequality. The second inequality is a direct consequence of the first one. \( \Box \)

Next we consider the linearized equation (6.1) with non-slip boundary condition. In this case, \( L_k \) is not an accretive operator. By [38] and the argument in Sects. 5 and 8, we know that the eigenvalues of \( L_k \) must lie in a region with \( \text{Im} \lambda < -ck^2 \sqrt{v} \frac{1}{2} \) for some \( c > 0 \), \( vk^2 \leq 1 \). This implies a rough bound
\[ \| e^{-tL_k} \omega_0 \|_{L^2} \leq C(v, k) e^{-c k^2 \sqrt{v} t} \| \omega_0 \|_{L^2}. \]
This bound ensures that we can take the Fourier transform in \( t \). In fact, we can give a more precise bound of \( e^{-tL_k} \) via the Dunford integral and the resolvent estimates:
\[ \| e^{-tL_k} \omega_0 \|_{L^2} \leq \begin{cases} C e^{-ct} \| \omega_0 \|_{L^2}, & vk^2 \geq 1, \\ C(t^{-1} + \sqrt{\frac{v}{\pi}} |k| \frac{\sqrt{2}}{12} e^{-c k^2 \sqrt{v} t}) \| \omega_0 \|_{L^2}, & vk^2 \leq 1. \end{cases} \]
We will not use this improved bound, which is still far from one obtained in Theorem 1.1.
6.2. Space-time estimates when $vk^2 \geq 1$

**Proposition 6.4.** Let $vk^2 \geq 1$ and $\omega$ be a solution of (6.1) with $\omega_0 \in L^2(I)$ and $f_1, f_2 \in L^2 L^2$. Then there exists a constant $C > 0$ independent of $v, k$ so that

$$
k^2 \|u\|^2_{L^\infty L^2} + k^2 \|u\|^2_{L^2 L^2} + vk^2 \|\omega\|^2_{L^2 L^2} + \|\omega\|^2_{L^\infty L^2} \leq C v^{-1} (\|f_1\|^2_{L^2 L^2} + \|f_2\|^2_{L^2 L^2}) + \|\omega_0\|^2_{L^2}.
$$

**Proof.** Taking $L^2$ inner product between (6.1) and $\varphi$, we get

$$
((\partial_t - v(\partial_y^2 - k^2) + iky)\omega, -\varphi) = \langle -ik f_1 - \partial_y f_2, -\varphi \rangle,
$$

which gives

$$
\langle \partial_t u, u \rangle + v \|\omega\|^2_{L^2} + ik \int_{-1}^{1} \varphi \varphi' dy + ik \int_{-1}^{1} y|\varphi|^2 dy + ik^3 \int_{-1}^{1} y|\varphi'|^2 dy
$$

$$
= \langle -ik f_1 - \partial_y f_2, -\varphi \rangle = \langle ik f_1, \varphi \rangle - \langle f_2, \partial_y \varphi \rangle.
$$

Taking the real part of the above equality, we get

$$
\frac{1}{2} \frac{d}{dt} \|u\|^2_{L^2} + v \|\omega\|^2_{L^2}
$$

$$
\leq |k| \int_{-1}^{1} |\varphi' \varphi| dy + (vk^2)^{-1} \|f_1\|^2_{L^2}
$$

$$
+ \frac{1}{4} v k^4 \|\varphi'\|^2_{L^2} + (vk^2)^{-1} \|f_2\|^2_{L^2} + \frac{1}{4} v k^2 \|\varphi\|^2_{L^2}
$$

$$
\leq \frac{1}{2} \left( \frac{1}{vk^2} \|\varphi'\|^2_{L^2} + vk^4 \|\varphi\|^2_{L^2} \right) + \frac{1}{4} v k^2 \|\varphi'\|^2_{L^2}
$$

$$
+ k^4 \|\varphi\|^2_{L^2} + (vk^2)^{-1} (\|f_1\|^2_{L^2} + \|f_2\|^2_{L^2})
$$

$$
\leq \frac{3v}{4} (k^2 \|\varphi'\|^2_{L^2} + k^4 \|\varphi\|^2_{L^2}) + (vk^2)^{-1} (\|f_1\|^2_{L^2} + \|f_2\|^2_{L^2})
$$

$$
\leq \frac{3v}{4} \|\omega\|^2_{L^2} + (vk^2)^{-1} (\|f_1\|^2_{L^2} + \|f_2\|^2_{L^2}).
$$

This shows that

$$
\frac{d}{dt} \|u\|^2_{L^2} + \frac{1}{2} v \|\omega\|^2_{L^2} \leq (vk^2)^{-1} (\|f_1\|^2_{L^2} + \|f_2\|^2_{L^2}),
$$

which gives

$$
\|u(t)\|^2_{L^2} + \int_{0}^{t} \|\omega(s)\|^2_{L^2} ds \leq (vk^2)^{-1} (\|f_1\|^2_{L^2} + \|f_2\|^2_{L^2}) + \|u_0\|^2_{L^2},
$$

that is,

$$
k^2 \|u\|^2_{L^\infty L^2} + vk^2 \|\omega\|^2_{L^2 L^2} \leq v^{-1} (\|f_1\|^2_{L^2 L^2} + \|f_2\|^2_{L^2 L^2}) + k^2 \|u_0\|^2_{L^2}.
$$
Thanks to $k^2\|u\|_{L^2}^2 \leq \|\omega\|_{L^2}^2 \leq vk^2\|\omega\|_{L^2}^2$, we get
\[
k^2\|u\|_{L^\infty L^2} + k^2\|u\|_{L^2 L^2} + vk^2\|\omega\|_{L^2 L^2} \lesssim v^{-1}(\|f_1\|_{L^2 L^2} + \|f_2\|_{L^2 L^2}) + \|\omega_0\|_{L^2}.
\]

It remains to estimate $\|\omega\|_{L^\infty L^2}^2$. Let $F_1 = \partial_t \varphi + iky\varphi$, so it holds that
\[
(\partial_y^2 - k^2)F_1 = \partial_t \varphi + iky\omega + 2ik\partial_y \varphi, \quad F_1|_{y=\pm 1} = \partial_y F_1|_{y=\pm 1} = 0.
\]

We get by integration by parts that
\[
\langle ikf_1, F_1 \rangle - \langle f_2, \partial_y F_1 \rangle = \langle -ikf_1 - \partial_y f_2, -F_1 \rangle \\
= \langle (\partial_y - v(\partial_y^2 - k^2) + iky)\omega, -F_1 \rangle \\
= \langle (\partial_y^2 - k^2)F_1 - v(\partial_y^2 - k^2)\omega - 2ik\partial_y \varphi, -F_1 \rangle \\
= \|\partial_y F_1\|_{L^2}^2 + k^2\|F_1\|_{L^2}^2 + v(\omega, (\partial_y^2 - k^2)F_1) + \langle 2ik\partial_y \varphi, F_1 \rangle \\
= \|\partial_y F_1\|_{L^2}^2 + k^2\|F_1\|_{L^2}^2 + v(\omega, \partial_y \varphi + iky\omega + 2ik\partial_y \varphi) + \langle 2ik\partial_y \varphi, F_1 \rangle.
\]

Taking the real part of the above equality, we get
\[
\frac{v}{2} \frac{d}{dt}\|\omega\|_{L^2}^2 + \|\partial_y F_1\|_{L^2}^2 + k^2\|F_1\|_{L^2}^2 \\
\leq 2v|k|\|\omega, \partial_y \varphi\| + 2|k|\|\partial_y F_1\| + \|ikf_1, F_1\| + \|f_2, \partial_y F_1\| \\
\leq v\|\omega\|_{L^2}^2 + vk^2\|\partial_y \varphi\|_{L^2}^2 + 2\|\partial_y \varphi\|_{L^2}^2 + k^2\|F_1\|_{L^2}^2/2 \\
+ \|f_1\|_{L^2}^2/2 + \|f_2\|_{L^2}^2/2 + \|\partial_y F_1\|_{L^2}^2/2 \\
= v\|\omega\|_{L^2}^2 + (vk^2 + 2)\|\partial_y \varphi\|_{L^2}^2 + k^2\|F_1\|_{L^2}^2 + \|f_1\|_{L^2}^2 \\
+ \|f_2\|_{L^2}^2/2 + \|\partial_y F_1\|_{L^2}^2/2,
\]

which gives
\[
\frac{v}{2} \frac{d}{dt}\|\omega\|_{L^2}^2 + \|\partial_y F_1\|_{L^2}^2 \leq \|f_1\|_{L^2}^2 + \|f_2\|_{L^2}^2 + 2v\|\omega\|_{L^2}^2 + 2(vk^2 + 2)\|\partial_y \varphi\|_{L^2}^2.
\]

This, along with the fact that
\[
(vk^2 + 2)\|\partial_y \varphi\|_{L^2}^2 \leq 3vk^2\|\partial_y \varphi\|_{L^2}^2 \leq 3v\|\omega\|_{L^2}^2,
\]

yields that
\[
v\|\omega(t)\|_{L^2}^2 \leq \|f_1\|_{L^2 L^2}^2 + \|f_2\|_{L^2 L^2}^2 + 8v\|\omega\|_{L^2 L^2}^2 + v\|\omega_0\|_{L^2}^2.
\]

Thus, we have
\[
\|\omega\|_{L^\infty L^2}^2 \lesssim v^{-1}(\|f_1\|_{L^2 L^2}^2 + \|f_2\|_{L^2 L^2}^2) + \|\omega\|_{L^2 L^2}^2 + \|\omega_0\|_{L^2}^2 \\
\lesssim v^{-1}(\|f_1\|_{L^2 L^2}^2 + \|f_2\|_{L^2 L^2}^2) + \|\omega_0\|_{L^2}^2.
\]

Summing up, we conclude the proof. □
6.3. Space-time estimates when $vk^2 \leq 1$

We decompose $\omega = \omega_I + \omega_H$, where $\omega_I$ solves

$$(\partial_t - v(\partial_y^2 - k^2) + iky)\omega_I = -ikf_1 - \partial_y f_2, \quad \omega_I|_{t=0} = 0, \quad (6.4)$$

and $\omega_H$ solves

$$(\partial_t - v(\partial_y^2 - k^2) + iky)\omega_H = 0, \quad \omega_H|_{t=0} = \omega_0(k, y), \quad (6.5)$$

together with the boundary conditions

$$\omega_I = (\partial_y^2 - k^2)\varphi_I, \quad \varphi_I(\pm 1) = \varphi'_I(\pm 1) = 0,$$

$$\omega_H = (\partial_y^2 - k^2)\varphi_H, \quad \varphi_H(\pm 1) = \varphi'_H(\pm 1) = 0.$$

6.3.1. Space-time estimates of the inhomogeneous problem

**Proposition 6.5.** Let $vk^2 \leq 1$ and $\omega_I$ be a solution of \((6.4)\). Then there exists a constant $C > 0$ independent of $v, k$ such that

$$(vk^2)^{\frac{1}{2}} \parallel\rho_k^{\frac{1}{2}}\omega_I\parallel^2_{L^2_{L^2}} + k^2 \parallel u_I\parallel^2_{L^2_{L^2}} + (vk^2)^{\frac{1}{2}} \parallel\omega_I\parallel^2_{L^\infty_{L^2}} \leq C (v^{-\frac{3}{2}} |k|^2 \parallel f_1\parallel^2_{L^2_{L^2}} + v^{-1} \parallel f_2\parallel^2_{L^2_{L^2}}),$$

**Proof.** We take the Fourier transform in $t$:

$$w(\lambda, k, y) = \int_0^{+\infty} \omega_I(t, k, y)e^{-it\lambda} dt, \quad F_j(\lambda, k, y) = \int_0^{+\infty} f_j(t, k, y)e^{-it\lambda} dt, \quad j = 1, 2.$$

Thus, we have

$$(i\lambda - v(\partial_y^2 - k^2) + iky)w(\lambda, k, y) = -ikF_1(\lambda, k, y) - \partial_y F_2(\lambda, k, y) \quad (6.6)$$

with $\int_{-1}^{1} w(\lambda, k, y)e^{\pm ky} dy = 0$. Using Plancherel’s theorem, we know that

$$\int_0^{+\infty} \parallel\omega_I(t)\parallel^2_{L^2} dt \sim \int_{\mathbb{R}} \parallel w(\lambda)\parallel^2_{L^2} d\lambda, \quad \int_0^{+\infty} \parallel\rho_k^{\frac{1}{2}}\omega_I(t)\parallel^2_{L^2} dt \sim \int_{\mathbb{R}} \parallel\rho_k^{\frac{1}{2}} w(\lambda)\parallel^2_{L^2} d\lambda,$$

$$\int_0^{+\infty} \parallel u_I(t)\parallel^2_{L^2} dt \sim \int_{\mathbb{R}} \parallel (k, \partial_y)(\partial_y^2 - k^2)^{-1} w(\lambda)\parallel^2_{L^2} d\lambda,$$

$$\int_0^{+\infty} \parallel f_j(t)\parallel^2_{L^2} dt \sim \int_{\mathbb{R}} \parallel F_j(\lambda)\parallel^2_{L^2} d\lambda, \quad j = 1, 2.$$

We further decompose $w$ as follows:

$$w(\lambda, k, y) = w_{Na}^{(1)} + w_{Na}^{(2)} + (c_1^{(1)}(\lambda) + c_1^{(2)}(\lambda))w_1 + (c_2^{(1)}(\lambda) + c_2^{(2)}(\lambda))w_2,$$

where $w_{Na}^{(1)}$ and $w_{Na}^{(2)}$ solve

$$(i\lambda - v(\partial_y^2 - k^2) + iky)w_{Na}^{(1)}(\lambda, k, y) = -ikF_1(\lambda, k, y), \quad w_{Na}^{(1)}|_{y=\pm 1} = 0, \quad (6.7)$$
\[(i\lambda - v(\partial_y^2 - k^2) + iky)w^{(2)}_{N\alpha}(\lambda, k, y) = -\partial_y F_2(\lambda, k, y), \quad w^{(2)}_{N\alpha}|_{y=\pm 1} = 0,\]

\[(6.8)\]

and \(w_i = (\partial_y^2 - k^2)\varphi_i\) with \(\varphi_1, \varphi_2\) solving \((4.4), (4.5)\) with \(\epsilon = 0, \lambda\) replaced by \(\lambda' = -\lambda/k,\) and

\[
c_1^{(j)}(\lambda) = -\int_{-1}^{1} \frac{\sinh k(y + 1)}{\sinh 2k} w^{(j)}_{N\alpha}(\lambda, k, y) dy,
\]

\[
c_2^{(j)}(\lambda) = \int_{-1}^{1} \frac{\sinh k(1 - y)}{\sinh 2k} w^{(j)}_{N\alpha}(\lambda, k, y) dy.
\]

By Corollary 3.2, we have

\[v^{\frac{1}{2}}|k|^\frac{1}{3} \|(k, \partial_y) (\partial_y^2 - k^2)^{-1} w^{(1)}_{N\alpha}(\lambda)\|_{L^2} + (vk^2)^{\frac{1}{2}} \|w^{(1)}_{N\alpha}(\lambda)\|_{L^2} \leq C \|k F_1(\lambda)\|_{L^2},\]

and by Proposition 3.3,

\[v^{\frac{1}{2}}|k|^\frac{1}{3} \|(k, \partial_y) (\partial_y^2 - k^2)^{-1} w^{(2)}_{N\alpha}(\lambda)\|_{L^2} + v^{\frac{1}{2}}|k|^\frac{1}{3} \|w^{(2)}_{N\alpha}(\lambda)\|_{L^2} \leq C \|F_2(\lambda)\|_{L^2}.
\]

By Lemma 4.1, Propositions 4.3 and 4.4, we have

\[|c_1^{(1)}(\lambda)| \|w_1\|_{L^2} + |c_2^{(1)}(\lambda)| \|w_2\|_{L^2} \leq Cv^{-\frac{5}{12}}|k|^{-\frac{5}{6}} \|k F_1(\lambda)\|_{L^2},\]

\[|c_1^{(2)}(\lambda)| \|\rho_k^\frac{1}{2} w_1\|_{L^2} + |c_2^{(1)}(\lambda)| \|\rho_k^\frac{1}{2} w_2\|_{L^2} \leq Cv^{-\frac{1}{2}}|k|^{-\frac{5}{6}} \|k F_1(\lambda)\|_{L^2},\]

\[|c_1^{(1)}(\lambda)| \|w_1\|_{L^1} + |c_2^{(1)}(\lambda)| \|w_2\|_{L^1} \leq Cv^{-\frac{1}{4}}|k|^{-\frac{5}{6}} \|k F_1(\lambda)\|_{L^2},\]

and by Lemma 4.2, Propositions 4.3 and 4.4,

\[|c_1^{(2)}(\lambda)| \|w_1\|_{L^2} + |c_2^{(2)}(\lambda)| \|w_2\|_{L^2} \leq Cv^{-\frac{3}{4}}|k|^{-\frac{1}{2}} \|F_2(\lambda)\|_{L^2},\]

\[|c_1^{(2)}(\lambda)| \|\rho_k^\frac{1}{2} w_1\|_{L^2} + |c_2^{(2)}(\lambda)| \|\rho_k^\frac{1}{2} w_2\|_{L^2} \leq Cv^{-\frac{1}{2}}|k|^{-\frac{1}{2}} \|F_2(\lambda)\|_{L^2},\]

\[|c_1^{(2)}(\lambda)| \|w_1\|_{L^1} + |c_2^{(2)}(\lambda)| \|w_2\|_{L^1} \leq Cv^{-\frac{1}{4}}|k|^{-\frac{1}{2}} \|F_2(\lambda)\|_{L^2}.
\]

This shows that

\[\|w(\lambda)\|_{L^2} \leq \|w^{(1)}_{N\alpha}(\lambda)\|_{L^2} + \|w^{(2)}_{N\alpha}(\lambda)\|_{L^2}
\]

\[+ |c_1^{(1)}(\lambda)| \|w_1\|_{L^2} + |c_2^{(1)}(\lambda)| \|w_2\|_{L^2}
\]

\[+ |c_1^{(2)}(\lambda)| \|w_1\|_{L^2} + |c_2^{(2)}(\lambda)| \|w_2\|_{L^2}
\]

\[\leq C(vk^2)^{-\frac{1}{2}} \|k F_1(\lambda)\|_{L^2} + Cv^{-\frac{5}{12}}|k|^{-\frac{5}{6}} \|F_2(\lambda)\|_{L^2}
\]

\[+ Cv^{-\frac{5}{12}}|k|^{-\frac{5}{6}} \|k F_1(\lambda)\|_{L^2} + Cv^{-\frac{1}{2}}|k|^{-\frac{1}{2}} \|F_2(\lambda)\|_{L^2}
\]

\[\leq C(vk^2)^{-\frac{1}{2}} \|k F_1(\lambda)\|_{L^2} + Cv^{-\frac{3}{4}}|k|^{-\frac{1}{2}} \|F_2(\lambda)\|_{L^2},
\]

and

\[\|\rho_k^\frac{1}{2} w(\lambda)\|_{L^2} \leq C(vk^2)^{-\frac{1}{2}} \|k F_1(\lambda)\|_{L^2} + Cv^{-\frac{3}{4}}|k|^{-\frac{1}{2}} \|F_2(\lambda)\|_{L^2},\]
and by Lemma 9.3,

\[ \| (k, \partial_y) (\partial_y^2 - k^2)^{-1} w(\lambda) \|_{L^2} \]
\[ \leq \| (k, \partial_y) (\partial_y^2 - k^2)^{-1} w^{(1)}_{Na}(\lambda) \|_{L^2} + \| (k, \partial_y) (\partial_y^2 - k^2)^{-1} w^{(2)}_{Na}(\lambda) \|_{L^2} \]
\[ + |c_L^{(1)}(\lambda)| \| (k, \partial_y) (\partial_y^2 - k^2)^{-1} w_1 \|_{L^2} + |c_L^{(1)}(\lambda)| \| (k, \partial_y) (\partial_y^2 - k^2)^{-1} w_2 \|_{L^2} \]
\[ + |c_L^{(2)}(\lambda)| \| (k, \partial_y) (\partial_y^2 - k^2)^{-1} w_1 \|_{L^2} + |c_L^{(2)}(\lambda)| \| (k, \partial_y) (\partial_y^2 - k^2)^{-1} w_2 \|_{L^2} \]
\[ \leq C v^{-\frac{3}{8}} |k|^{-\frac{3}{2}} \| k F_1(\lambda) \|_{L^2} + C (v k^2)^{-\frac{1}{2}} \| F_2(\lambda) \|_{L^2} + |k|^{-\frac{1}{2}} |c_L^{(1)}(\lambda)| \| w_1 \|_{L^1} \]
\[ + |c_L^{(2)}(\lambda)| \| w_2 \|_{L^1} + |c_L^{(1)}(\lambda)| \| w_1 \|_{L^1} + |c_L^{(2)}(\lambda)| \| w_2 \|_{L^1} \]
\[ \leq C v^{-\frac{3}{8}} |k|^{-\frac{3}{2}} \| k F_1(\lambda) \|_{L^2} + C (v k^2)^{-\frac{1}{2}} \| F_2(\lambda) \|_{L^2}. \]

In summary, we conclude that

\[ (v k^2)^{\frac{3}{4}} \| \rho_k^{\frac{1}{2}} \omega_I \|_{L^2 L^2}^2 + k^2 \| u_I \|_{L^2 L^2}^2 + (v k^2)^{\frac{1}{2}} \| \omega_I \|_{L^2 L^2}^2 \]
\[ \sim (v k^2)^{\frac{3}{4}} \| \rho_k^{\frac{1}{2}} \omega_I \|_{L^2 L^2}^2 + C (v k^2)^{\frac{1}{2}} \| k F_1(\lambda) \|_{L^2 L^2}^2 + C (v k^2)^{\frac{1}{2}} \| F_2(\lambda) \|_{L^2 L^2}^2 \]
\[ + C k^2 v^{-\frac{1}{3}} |k|^{-\frac{5}{2}} \| k F_1(\lambda) \|_{L^2 L^2}^2 + C (v k^2)^{\frac{1}{2}} \| F_2(\lambda) \|_{L^2 L^2}^2 \]
\[ + C (v k^2)^{\frac{1}{2}} v^{-\frac{1}{3}} |k|^{-\frac{5}{2}} \| k F_1(\lambda) \|_{L^2 L^2}^2 + C (v k^2)^{\frac{1}{2}} v^{-\frac{1}{3}} |k|^{-\frac{5}{2}} \| F_2(\lambda) \|_{L^2 L^2}^2 \]
\[ \sim v^{-\frac{1}{3}} |k|^{-\frac{1}{2}} \| f_1 \|_{L^2 L^2}^2 + v^{-\frac{1}{3}} \| f_2 \|_{L^2 L^2}^2. \]

It remains to consider \( L^\infty L^2 \) estimate of \( \omega_I \). Let \( \omega_{Na} \) be a solution of

\[ (\partial_t - v(\partial_y^2 - k^2) + i k y) \omega_{Na} = -i k f_1 - \partial_y f_2, \quad \omega_{Na}|_{t=0} = 0, \quad \omega_{Na}|_{y=\pm 1} = 0. \]

Set \( w_{Na}(\lambda, k, y) = \int_0^{+\infty} \omega_{Na}(t, k, y) e^{-it\lambda} dt \), which satisfies

\[ (i \lambda - v(\partial_y^2 - k^2) + i k y) w_{Na}(\lambda, k, y) = -i k F_1(\lambda, k, y) - \partial_y F_2(\lambda, k, y), \quad \omega_{Na}|_{y=\pm 1} = 0. \]

Thus, \( w_{Na} = w_{Na}^{(1)} + w_{Na}^{(2)} \). By Plancherel’s theorem, we have

\[ (v k^2)^{\frac{1}{2}} \| \omega_{Na} \|_{L^2 L^2}^2 \sim (v k^2)^{\frac{1}{2}} \| w_{Na}(\lambda) \|_{L^2}^2 \]
\[ \leq 2 (v k^2)^{\frac{1}{2}} \| w_{Na}^{(1)}(\lambda) \|_{L^2}^2 \| w_{Na}^{(2)}(\lambda) \|_{L^2}^2 \]
\[ \leq C (v k^2)^{\frac{1}{2}} \| (v k^2)^{-\frac{1}{2}} \| k F_1(\lambda) \|_{L^2}^2 \| F_2(\lambda) \|_{L^2}^2 \]
\[ = C v^{-\frac{1}{3}} |k|^{-\frac{1}{2}} \| F_1(\lambda) \|_{L^2}^2 \| F_2(\lambda) \|_{L^2}^2 \]
\[ \sim v^{-\frac{1}{3}} |k|^{-\frac{1}{2}} \| f_1 \|_{L^2 L^2}^2 + v^{-\frac{1}{3}} \| f_2 \|_{L^2 L^2}^2. \]
Moreover, we have

\[
\int_{0}^{+\infty} (\omega_I(t, k, y) - \omega_N(t, k, y))e^{-it\lambda} dt = w(\lambda, k, y) - w_{N}(\lambda, k, y) = (c_1^{(1)}(\lambda) + c_1^{(2)}(\lambda))w_1 + (c_2^{(1)}(\lambda) + c_2^{(2)}(\lambda))w_2.
\]

Thus, we can write

\[
\omega_I = \omega_N + \omega_1^{(1)} + \omega_1^{(2)} + \omega_2^{(1)} + \omega_2^{(2)},
\]

where for \(j = 1, 2\),

\[
\omega_j^{(l)}(t, k, y) = \frac{1}{2\pi} \int_{\mathbb{R}} c_j^{(l)}(\lambda)w_j(\lambda, k, y)e^{it\lambda} d\lambda, \quad t > 0.
\]

Now we estimate \(\|\omega_N\|_{L^{\infty}L^2} \) and \(\|\omega_j^{(k)}\|_{L^{\infty}L^2}\) separately. Notice that

\[
\partial_t \|\omega_N\|_{L^2}^2 + v\|\partial_y \omega_N\|_{L^2}^2 + vk^2\|\omega_N\|_{L^2}^2 = \text{Re}(\partial_t - v(\partial_y^2 - k^2) + iky)\omega_N, \quad \omega_N \\
= \text{Re}\{-ikf_1 - \partial_y f_2, \omega_N\} = \text{Re}\{-ik(f_1, \omega_N) + (f_2, \partial_y \omega_N)\} \leq |k||f_1||\omega_N|_{L^2} + ||f_2||_{L^2}||\partial_y \omega_N||_{L^2},
\]

which gives

\[
\partial_t \|\omega_N\|_{L^2}^2 + v\|\partial_y \omega_N\|_{L^2}^2 + 2vk^2\|\omega_N\|_{L^2}^2 \leq v^{-\frac{1}{2}}|k|^\frac{3}{2} ||f_1||_{L^2}^2 + (vk^2)^{\frac{1}{2}} \omega_N ||_{L^2}^2 + v^{-1}||f_2||_{L^2}^2.
\]

As \(\omega_N |_{t=0} = 0\), this shows that

\[
\|\omega_N(t)||_{L^2}^2 \leq \int_0^t \left(v^{-\frac{1}{2}}|k|^\frac{3}{2} ||f_1(s)||_{L^2}^2 + (vk^2)^{\frac{1}{2}} \omega_N(s)||_{L^2}^2 + v^{-1}||f_2(s)||_{L^2}^2\right) ds \leq v^{-\frac{1}{2}}|k|^\frac{3}{2} ||f_1||_{L^2L^2}^2 + (vk^2)^{\frac{1}{2}} \omega_N ||_{L^2L^2}^2 + v^{-1}||f_2||_{L^2L^2}^2.
\]

Thus, we get

\[
\|\omega_N||_{L^{\infty}L^2}^2 \leq C(v^{-\frac{1}{2}}|k|^\frac{3}{2} ||f_1||_{L^2L^2}^2 + v^{-1}||f_2||_{L^2L^2}^2).
\]

Notice that (6.7) is equivalent to

\[
(-v(\partial_y^2 - k^2) + ik(y - \lambda'))w_N^{(1)}(\lambda, k, y) = -ikF_1(\lambda, k, y), \quad w_N^{(1)}|_{y=\pm 1} = 0,
\]

with \(\lambda' = -\lambda/k\). By Lemma 4.1 and Proposition 4.3, we have

\[
(1 + |k(-\lambda/k - 1)|)\frac{3}{2} |c_1^{(1)}(\lambda)| ||w_1||_{L^2} \leq Cv^{-\frac{5}{12}}|k|^{-\frac{5}{6}}||kF_1(\lambda)||_{L^2}.
\]
Thanks to $|k(-\lambda/k - 1)| = |\lambda + k|$ and $\|(1 + |\lambda + k|)^{-3}||_{L^2(\mathbb{R})} \leq C$, we have

$$\|\omega_1(t)\|_{L^2} \leq \frac{1}{2\pi} \int_{\mathbb{R}} |c_1^{(1)}(\lambda)||w_1(\lambda)||_{L^2} d\lambda \leq C(1 + |\lambda + k|)^{-3} v^{-\frac{5}{12}}|k|^{-\frac{5}{12}} \|F_1(\lambda)||_{L^1(\mathbb{R})} \leq C v^{-\frac{5}{12}}|k|^\frac{1}{8} \|f_1\|_{L^2(\mathbb{R})},$$

which shows that

$$(vk^2)^{\frac{1}{2}} \|\omega_1^{(1)}\|_{L^2(\mathbb{R})} \leq C(vk^2)^{\frac{1}{2}} (v^{-\frac{5}{12}}|k|^\frac{1}{8} \|f_1\|_{L^2(\mathbb{R})}) = C v^{-\frac{7}{24}} |k|^\frac{1}{8} \|f_1\|_{L^2(\mathbb{R})}^2.$$ 

Similarly, we have

$$(vk^2)^{\frac{1}{2}} \|\omega_2^{(1)}\|_{L^2(\mathbb{R})} \leq C v^{-\frac{7}{24}} |k|^\frac{1}{8} \|f_1\|_{L^2(\mathbb{R})}^2.$$ 

By Lemma 4.2 and Proposition 4.4, we have

$$(1 + |k(-\lambda/k - 1)|)^{\frac{3}{2}} |c_1^{(2)}(\lambda)||\rho_k^{-\frac{1}{2}}w_1||_{L^2} \leq C v^{-\frac{7}{8}} |k|^{-\frac{3}{8}} \|F_2(\lambda)||_{L^2},$$

$$(1 + |k(-\lambda/k - 1)|)^{\frac{3}{2}} |c_1^{(2)}(\lambda)||\rho_k^{-\frac{1}{2}}w_1||_{L^2} \leq C v^{-\frac{19}{24}} |k|^{-\frac{7}{24}} \|F_2(\lambda)||_{L^2},$$

which give

$$|\lambda + k|^3 |c_1^{(2)}(\lambda)||\rho_k^{-\frac{1}{2}}w_1||_{L^2}^2 + (vk^2)^{\frac{1}{2}} |\lambda$$

$$+ k|^3 |c_1^{(2)}(\lambda)||\rho_k^{-\frac{1}{2}}w_1||_{L^2}^2 \leq C v^{-\frac{3}{4}} |k|^{-\frac{7}{4}} \|F_2(\lambda)||_{L^2}^2.$$ 

Thus, we obtain

$$|\omega_1^{(2)}(t, k, y)|^2 \leq \left| \int_{\mathbb{R}} |c_1^{(2)}(\lambda)w_1(\lambda, k, y)| d\lambda \right|^2 \leq \int_{\mathbb{R}} (|\lambda + k|^3 |c_1^{(2)}(\lambda)||\rho_k^{-\frac{1}{2}}w_1(\lambda, k, y)|^2$$

$$+ (vk^2)^{\frac{1}{2}} |\lambda + k|^3 |c_1^{(2)}(\lambda)||\rho_k^{-\frac{1}{2}}w_1(\lambda, k, y)|^2) d\lambda \times \int_{\mathbb{R}} (|\lambda + k|^3 \rho_k(y) + (vk^2)^{\frac{1}{2}} |\lambda + k|^3 \rho_k^{-\frac{1}{2}}(y))^{-1} d\lambda,$$

for $t > 0, \ y \in (-1, 1)$. Notice that

$$\int_{\mathbb{R}} (|\lambda + k|^3 \rho_k(y) + (vk^2)^{\frac{1}{2}} |\lambda + k|^3 \rho_k^{-\frac{1}{2}}(y))^{-1} d\lambda$$

$$z=(\lambda+k)\rho_k(y)(vk^2)^{-1}\left\{ \int_{\mathbb{R}} |z|^3 \rho_k^{-2}(y)(vk^2)^{-1} dz + (vk^2)^{\frac{1}{2}} |z|^3 \rho_k^{-2}(y))^{-1} \rho_k^{-2}(y)(vk^2)^{-1} dz \right\}$$

$$= (vk^2)^{-\frac{1}{2}} \int_{\mathbb{R}} (|z|^3 + |z|^3)^{-1} dz = C (vk^2)^{-\frac{1}{6}} < +\infty.$$
Thus, we have
\[
\|w_1^{(2)}(t)\|_{L^2}^2 = \int_{-1}^{1} |w_1^{(2)}(t, k, y)|^2 dy \leq C(vk^2)^{-\frac{1}{6}} \int_{\mathbb{R}} \left( |\lambda| + k^\frac{3}{2} |c_1^{(2)}(\lambda)|^2 \rho_k(y) |w_1(\lambda, k, y)|^2 + (vk^2)^{\frac{1}{2}} |\lambda| + k^\frac{3}{2} |c_1^{(2)}(\lambda)|^2 \rho_k^{-\frac{1}{2}}(y) |w_1(\lambda, k, y)|^2 \right) dy d\lambda
\]
\[
= C(vk^2)^{-\frac{1}{6}} \int_{\mathbb{R}} (|\lambda| + k^\frac{3}{2} |c_1^{(2)}(\lambda)|^2 \rho_k^{-\frac{1}{2}} w_1^2) d\lambda
\]
\[
\leq C(vk^2)^{-\frac{1}{6}} \int_{\mathbb{R}} v^{-\frac{4}{3}} |k|^{-\frac{2}{3}} \|F_2(\lambda)\|_{L^2}^2 d\lambda \sim v^{-\frac{3}{2}} |k|^{-1} \|f_2\|_{L^2}^2,
\]
which shows that
\[
(vk^2)^{\frac{1}{2}} \|w_1^{(2)}\|_{L^\infty L^2} \leq C(vk^2)^{\frac{1}{2}} v^{-\frac{3}{2}} |k|^{-1} \|f_2\|_{L^2}^2 = Cv^{-1} \|f_2\|_{L^2 L^2}^2.
\]
Similarly, we have
\[
(vk^2)^{\frac{1}{2}} \|w_2^{(2)}\|_{L^\infty L^2} \leq C(vk^2)^{\frac{1}{2}} v^{-\frac{1}{2}} \|f_2\|_{L^2 L^2}^2.
\]
In summary, we conclude that
\[
(vk^2)^{\frac{1}{2}} \|w\|_{L^\infty L^2}^2 \leq C\|w_{Na}\|_{L^\infty L^2}^2 + C \sum_{j, l=1}^2 (vk^2)^{\frac{1}{2}} \|w_j^{(l)}\|_{L^\infty L^2}^2
\]
\[
\leq C\left(v^{-\frac{3}{2}} |k|^{-\frac{2}{3}} \|f_1\|_{L^2 L^2}^2 + v^{-1} \|f_2\|_{L^2 L^2}^2\right).
\]
\[\square\]

### 6.3.2. Space-time estimates of the homogeneous problem

**Proposition 6.6.** Let $vk^2 \leq 1$ and $w_H$ be a solution of (6.5) with $(\omega_0, e^{\pm ky}) = 0$. Then there exists a constant $C > 0$ independent of $v, k$ such that

\[
(vk^2)^{\frac{1}{2}} \|w_H\|_{L^2 L^2}^2 + (vk^2)^{\frac{1}{2}} \|w_H\|_{L^\infty L^2}^2 + (vk^2)^{\frac{1}{2}} \|\rho_k^{\frac{1}{2}} w_H\|_{L^2 L^2}^2 + k^2 \|u_H\|_{L^2 L^2}^2 \leq C \|w_0\|_{L^2}^2 + C v^{\frac{1}{2}} |k|^{-\frac{4}{3}} \|\partial_y \omega_0\|_{L^2}^2.
\]

**Proof.** We introduce
\[
\omega_H^{(0)}(t, k, y) = e^{-itky} \omega_0(0, k, y), \quad t \in \mathbb{R},
\]
\[
\omega_H^{(1)}(t, k, y) = e^{-(vk^2)^{1/3}t} \omega_H^{(0)}(t, k, y), \quad t > 0.
\]
Then we have
\[
(\partial_t + i ky)\omega_H^{(0)} = 0, \quad (\partial_t + i ky + (vk^2)^{1/3})\omega_H^{(1)} = 0,
\]
\( \omega_H^{(0)}(0) = \omega_H^{(1)}(0) = \omega(0), \quad \| \omega_H^{(0)}(t) \|_{L^2} = \| \omega_H(0) \|_{L^2}. \)

\((\partial_t - v(\partial_y^2 - k^2) + i ky)\omega_H^{(1)} = (vk^2 - (vk^2)^{\frac{1}{3}})\omega_H^{(1)} - v\partial_y^2 \omega_H^{(1)}.\)

Thus, we can decompose \( \omega_H \) as follows:

\[ \omega_H = \omega_H^{(1)} + \omega_H^{(2)} + \omega_H^{(3)}, \]

where \( \omega_H^{(2)} \) solves

\((\partial_t - v(\partial_y^2 - k^2) + i ky)\omega_H^{(2)} = -(vk^2 - (vk^2)^{\frac{1}{3}})\omega_H^{(1)} + v\partial_y^2 \omega_H^{(1)}, \)

\(\omega_H^{(2)} \big|_{t=0} = 0, \quad \langle \omega_H^{(2)}, e^{\pm ky} \rangle = 0.\)

and \( \omega_H^{(3)} \) solves

\((\partial_t - v(\partial_y^2 - k^2) + i ky)\omega_H^{(3)} = 0, \quad \omega_H^{(3)} \big|_{t=0} = 0, \quad \langle \omega_H^{(3)}(t) + \omega_H^{(1)}(t), e^{\pm ky} \rangle = 0.\)

We denote

\[ u_H^{(j)} = (\partial_y, -ik)\varphi_H^{(j)}, \quad \varphi_H^{(j)} = (\partial_y^2 - k^2)^{-1} \omega_H^{(j)}, \quad j = 0, 1, 2, 3. \]

**Step 1.** Estimates of \( u_H^{(j)}, j = 1, 2. \)

By Proposition 6.5, we have

\((vk^2)^{\frac{1}{3}} \| \omega_H^{(2)} \|_{L^2}^2 + k^2 \| u_H^{(2)} \|_{L^2}^2 + (vk^2)^{\frac{1}{3}} \| \omega_H^{(1)} \|_{L^2}^2 + v \| \partial_y \omega_H^{(1)} \|_{L^2}^2 \)

\[ \leq C\left(\| (vk^2)^{\frac{1}{3}} \| \omega_H^{(1)} \|_{L^2}^2 + v \| \partial_y \omega_H^{(1)} \|_{L^2}^2 \right). \]

It is easy to see that

\(\| \omega_H^{(1)}(t) \|_{L^2} = \| e^{-(vk^2)^{\frac{1}{3}} t} \omega_H^{(0)}(0) \|_{L^2} = e^{-(vk^2)^{\frac{1}{3}} t} \| \omega_0 \|_{L^2}, \)

\((vk^2)^{\frac{1}{3}} \| \omega_H^{(1)} \|_{L^2}^2 = (vk^2)^{\frac{1}{3}} \| \omega_0 \|_{L^2}^2 = \| \omega_0 \|_{L^2}^2/2, \)

and

\(\| \partial_y \omega_H^{(1)}(t) \|_{L^2} = \| \partial_y (e^{-(vk^2)^{\frac{1}{3}} t - iky} \omega_0(k, y)) \|_{L^2} \)

\[ = e^{-(vk^2)^{\frac{1}{3}} t} \| e^{-iky} (\partial_y - ik) \omega_0(k, y) \|_{L^2} \]

\[ \leq e^{-(vk^2)^{\frac{1}{3}} t} (\| \partial_y \omega_0 \|_{L^2} + |t| \| \omega_0 \|_{L^2}), \]

and

\(v \| \partial_y \omega_H^{(1)} \|_{L^2}^2 \leq 2v \| e^{-(vk^2)^{\frac{1}{3}} t} \|_{L^2}^2 \| \partial_y \omega_0 \|_{L^2}^2 \)

\[ + 2v |t| e^{-(vk^2)^{\frac{1}{3}} t} \| \omega_0 \|_{L^2}^2 \]

\[ = v^{\frac{1}{3}} |k|^{-\frac{2}{3}} \| \partial_y \omega_0 \|_{L^2}^2 + 2 \| \omega_0 \|_{L^2}^2. \]
This shows that
\[
(\nu k^2)^{\frac{1}{2}} \| \rho_k \|_{L^2}^2 + k^2 \| u_H^{(0)} \|_{L^2}^2 + \nu k^2 \| \omega_H^{(2)} \|_{L^2}^2 \\
+ (\nu k^2)^{\frac{1}{2}} \| \omega_H^{(2)} \|_{L^2}^2 + (\nu k^2)^{\frac{1}{2}} \| \omega_H^{(2)} \|_{L^\infty}^2 \\
\leq C \| \omega(0) \|_{L^2}^2 + C \nu^\frac{1}{2} |k|^{-\frac{3}{4}} \| \partial_y \omega(0) \|_{L^2}^2,
\]
(6.9)
and
\[
(\nu k^2)^{\frac{1}{2}} \| \rho_k \|_{L^2}^2 + (\nu k^2)^{\frac{1}{2}} \| \omega_H^{(1)} \|_{L^2}^2 + (\nu k^2)^{\frac{1}{2}} \| \omega_H^{(1)} \|_{L^\infty}^2 \\
\leq 2(\nu k^2)^{\frac{1}{2}} \| \omega_H^{(1)} \|_{L^2}^2 + \| \omega_H^{(1)} \|_{L^\infty}^2 \leq C \| \omega(0) \|_{L^2}^2.
\]
(6.10)

We use the basis in \( L^2([-1, 1]) : \varphi_j(y) = \sin(\pi j(y + 1)/2), j \in \mathbb{Z}_+. \) Then we have
\[
\omega_H^{(0)} = \sum_{j=1}^{+\infty} (\omega_H^{(0)}, \varphi_j) \varphi_j, \quad \| \omega_H^{(0)} \|_{L^2}^2 = \sum_{j=1}^{+\infty} |(\omega_H^{(0)}, \varphi_j)|^2, \quad \varphi_H^{(0)}
\]
\[
= - \sum_{j=1}^{+\infty} \frac{(\omega_H^{(0)}, \varphi_j)}{(\pi j/2)^2 + k^2} \varphi_j,
\]
\[
\| u_H^{(0)}(t) \|_{L^2}^2 = \| \partial_y \varphi_H^{(0)}(t) \|_{L^2}^2 + k^2 \| \varphi_H^{(0)}(t) \|_{L^2}^2 = -\langle \varphi_H^{(0)}(t), \omega_H^{(0)}(t) \rangle
\]
\[
= \sum_{j=1}^{+\infty} \frac{|(\omega_H^{(0)}(t), \varphi_j)|^2}{(\pi j/2)^2 + k^2}.
\]

Thanks to \( \omega_H^{(0)}(t, k, y) = e^{-itky} \omega_0(k, y), \) we have
\[
\langle \omega_H^{(0)}(t), \varphi_j \rangle = \int_{-1}^{1} e^{-itky} \omega_0(k, y) \varphi_j(y) dy,
\]
from which and Plancherel’s formula, we infer that
\[
\int_{\mathbb{R}} |\langle \omega_H^{(0)}(t), \varphi_j \rangle|^2 dt = \frac{2\pi}{|k|} \int_{-1}^{1} |\omega_0(k, y) \varphi_j(y)|^2 dy \leq \frac{2\pi}{|k|} \| \omega_0 \|_{L^2}^2.
\]

Therefore, we have
\[
\int_{\mathbb{R}} \| u_H^{(0)}(t) \|_{L^2}^2 dt = \sum_{j=1}^{+\infty} \int_{\mathbb{R}} \frac{|\langle \omega_H^{(0)}(t), \varphi_j \rangle|^2}{(\pi j/2)^2 + k^2} dt \leq \sum_{j=1}^{+\infty} \frac{2\pi}{|k|} \| \omega_0 \|_{L^2}^2
\]
\[
\leq \int_{0}^{+\infty} \frac{2\pi}{|k|} \frac{\| \omega_0 \|_{L^2}^2}{(\pi z/2)^2 + |k|^2} dz = \frac{2\pi}{k^2} \| \omega_0 \|_{L^2}^2,
\]
from which, along with \( u_H^{(1)} = e^{-(\nu k^2)^{1/3}t} u_H^{(0)}, \) we infer that
\[
k^2 \| u_H^{(1)} \|_{L^2}^2 \leq k^2 \| u_H^{(0)} \|_{L^2}^2 \leq C \| \omega_0 \|_{L^2}^2.
\]
(6.11)
Thus, we have
\[ w(\lambda, k, y) := \int_{-1}^{1} \frac{\sinh k(1+y)}{\sinh 2k} \omega_H^{(1)}(t, k, y) dy = -\int_{-1}^{1} \frac{\sinh k(1+y)}{\sinh 2k} \omega_H^{(3)}(t, k, y) dy, \]
\[ a_1(t) = \int_{-1}^{1} \sinh k(1+y) \omega_H^{(1)}(t, k, y) dy = -\int_{-1}^{1} \frac{\sinh k(1+y)}{\sinh 2k} \omega_H^{(3)}(t, k, y) dy, \]
\[ a_2(t) = \int_{-1}^{1} \sinh k(1-y) \omega_H^{(1)}(t, k, y) dy = -\int_{-1}^{1} \frac{\sinh k(1-y)}{\sinh 2k} \omega_H^{(3)}(t, k, y) dy, \]
\[ a_1^{(0)}(t) = \int_{-1}^{1} \frac{\sinh k(1+y)}{\sinh 2k} \omega_H^{(0)}(t, k, y) dy, \]
\[ a_2^{(0)}(t) = \int_{-1}^{1} \frac{\sinh k(1-y)}{\sinh 2k} \omega_H^{(1)}(t, k, y) dy. \]

Then we have
\[ a_1(t) = e^{-(\nu k^2)^{1/4} t} a_1^{(0)}(t), \]
\[ a_2(t) = e^{-(\nu k^2)^{1/4} t} a_2^{(0)}(t). \]

We take the Fourier transform in \( t \):
\[ w(\lambda, k, y) := \int_{0}^{+\infty} \omega_H^{(3)}(t, k, y) e^{-it\lambda} dt, \quad c_j(\lambda) := \int_{0}^{+\infty} a_j(t) e^{-it\lambda} dt, \quad j = 1, 2. \]

Then we have
\[ (i\lambda - \nu (\partial_y^2 + k^2) + iky) w(\lambda, k, y) = 0, \]
\[ c_1(\lambda) = -\int_{-1}^{1} \frac{\sinh k(1+y)}{\sinh 2k} w(\lambda, k, y) dy, \]
\[ c_2(\lambda) = -\int_{-1}^{1} \frac{\sinh k(1-y)}{\sinh 2k} w(\lambda, k, y) dy. \]

Thus, we have
\[ w = -c_1(\lambda) w_1 - c_2(\lambda) w_2, \]
where \( w_1, w_2 \) are defined as in section 4.1. Let us first claim that
\[ \| (1 + |\lambda + k|) c_1 \|_{L^2(\mathbb{R})}^2 \leq C |k|^{-1} \| \omega_0 \|_{L^2}^2, \quad (6.12) \]
\[ \| (1 + |\lambda - k|) c_2 \|_{L^2(\mathbb{R})}^2 \leq C |k|^{-1} \| \omega_0 \|_{L^2}^2. \quad (6.13) \]

By Proposition 4.3, we know that
\[ \| w_1 \|_{L^2} \leq C \nu^{-\frac{1}{4}} (1 + |k| - \lambda/k - 1)^{\frac{1}{4}} = C \nu^{-\frac{1}{4}} (1 + |\lambda + k|)^{\frac{1}{4}}, \]
\[ \| w_2 \|_{L^2} \leq C \nu^{-\frac{1}{4}} (1 + |k| - \lambda/k + 1)^{\frac{1}{4}} = C \nu^{-\frac{1}{4}} (1 + |\lambda - k|)^{\frac{1}{4}}, \]
from which we infer that
\[ \| w(\lambda) \|_{L^2} = \| c_1(\lambda) w_1 + c_2(\lambda) w_2 \|_{L^2} \leq |c_1(\lambda)| \| w_1 \|_{L^2} + |c_2(\lambda)| \| w_2 \|_{L^2} \]
\[
\leq C \nu^{-\frac{1}{2}} \left([c_1(\lambda)| (1 + |\lambda + k|)^{\frac{1}{2}} + |c_2(\lambda)| (1 + |\lambda - k|)^{\frac{1}{2}}\right).
\]

By Proposition 4.3, we have \(\|w_1\|_{L^1} + \|w_2\|_{L^1} \leq C\), which, along with Lemma 9.3, implies

\[
\| (k, \partial_y) (\partial_y^2 - k^2)^{-1} w(\lambda) \|_{L^2} \\
\leq |c_1(\lambda)| \| (k, \partial_y) (\partial_y^2 - k^2)^{-1} w_1 \|_{L^2} + |c_2(\lambda)| \| (k, \partial_y) (\partial_y^2 - k^2)^{-1} w_2 \|_{L^2} \\
\leq |k|^{-\frac{1}{2}} \left( |c_1(\lambda)| \| w_1 \|_{L^1} + |c_2(\lambda)| \| w_2 \|_{L^1} \right) \leq C |k|^{-\frac{1}{2}} \left( |c_1(\lambda)| + |c_2(\lambda)| \right).
\]

By Proposition 4.4, we have \(\|^{\frac{1}{2}} w \|_{L^2} + \|^{\frac{1}{2}} w \|_{L^2} \leq CL^{\frac{1}{2}}\) with \(L = (k/\nu)^{\frac{1}{2}}\), which gives

\[
\|_{L^2} = |c_1(\lambda)|^{\frac{1}{2}} w_1 + |c_2(\lambda)|^{\frac{1}{2}} w_2 \|_{L^2} \leq |c_1(\lambda)|^{\frac{1}{2}} w_1 \|_{L^2} \\
+ |c_2(\lambda)|^{\frac{1}{2}} w_2 \|_{L^2} \\
\leq C \| (1 + |\lambda + k|)^{\frac{1}{2}} c_1 \|_{L^2} + (1 + |\lambda - k|)^{\frac{1}{2}} c_2 \|_{L^2}.
\]

Summing up, we conclude that

\[
(vk^2)^{\frac{1}{2}} \|^{\frac{1}{2}}\omega_H(\lambda) \|_{L^2}^2 \leq (vk^2)^{\frac{1}{2}} \| w(\lambda) \|_{L^2}^2 \\
\leq C \| (1 + |\lambda + k|)^{\frac{1}{2}} c_1 \|_{L^2} + (1 + |\lambda - k|)^{\frac{1}{2}} c_2 \|_{L^2}.
\]

Thanks to \(\omega_H(t) = \frac{1}{2} \int_{\mathbb{R}} w(\lambda) e^{it\lambda} d\lambda\), we also have

\[
(vk^2)^{\frac{1}{2}} \| \omega_H(\lambda) \|_{L^\infty}^2 \leq (vk^2)^{\frac{1}{2}} \| w(\lambda) \|_{L^1}^2 \\
\leq C \| (1 + |\lambda + k|)^{\frac{1}{2}} c_1 \|_{L^1} + (1 + |\lambda - k|)^{\frac{1}{2}} c_2 \|_{L^1}.
\]

Then it follows from (6.12) and (6.13) that

\[
(vk^2)^{\frac{1}{2}} \|^{\frac{1}{2}}\omega_H(\lambda) \|_{L^2}^2 \leq (vk^2)^{\frac{1}{2}} \| w(\lambda) \|_{L^2}^2 + (vk^2)^{\frac{1}{2}} \| \omega_H(\lambda) \|_{L^2}^2 \\
\leq C |k| \| (1 + |\lambda + k|)^{\frac{1}{2}} c_1 \|_{L^2} + (1 + |\lambda - k|)^{\frac{1}{2}} c_2 \|_{L^2} \\
+ C \| (1 + |\lambda |)^{\frac{1}{2}} c_1 \|_{L^2} + (1 + |\lambda - k|)^{\frac{1}{2}} c_2 \|_{L^2} \\
\leq C \| (1 + |\lambda + k|)^{\frac{1}{2}} c_1 \|_{L^2} + (1 + |\lambda - k|)^{\frac{1}{2}} c_2 \|_{L^2} \\
\leq C \|\omega_0\|_{L^2}^2.
\]
Step 3. Proof of (6.12) and (6.13).

Let us assume that \((\omega_0, e^{\pm iky}) = 0\). Then we have \(a_1(0) = a_2(0) = 0\). Thanks to \(\phi_0(t, k, y) = e^{-iky}\omega(k, y)\), we get

\[
a_1^{(0)}(t) = \int_{-1}^{1} \sinh k(y + 1) \omega_0^{(0)}(t, k, y) \, dy = \int_{-1}^{1} \sinh k(y + 1) e^{-iky} \omega_0(k, y) \, dy,
\]

from which, with Plancherel’s formula, we infer that

\[
\int_{\mathbb{R}} |a_1^{(0)}(t)|^2 \, dt = \frac{2\pi}{|k|} \int_{-1}^{1} \left| \sinh k(y + 1) \omega_0(k, y) \right|^2 \, dy \leq \frac{2\pi}{|k|} \|\omega_0\|^2_{L^2}.
\]

Using the formula

\[
\partial_t a_1^{(0)}(t) + ika_1^{(0)}(t) = \int_{-1}^{1} ik(1 - y) \sinh k(y + 1) e^{-iky} \omega_0(k, y) \, dy,
\]

we get

\[
\int_{\mathbb{R}} \left| \partial_t a_1^{(0)}(t) + ika_1^{(0)}(t) \right|^2 \, dt \leq \frac{2\pi}{|k|} \|\omega_0\|^2_{L^2};
\]

here we used \(\left| ik(1 - y) \sinh k(y + 1) \right| \leq k(1 - y) (2 e^{-k(1-y)}) \leq 1\) for \(y \in [-1, 1]\).

As \(a_1(t) = e^{-(vk^2)^{1/3}} a_1^{(0)}(t)\), we have

\[
\| e^{ikt} a_1(t) \|^2_{L^2(0, +\infty)} = \| e^{ikt - (vk^2)^{1/3}} a_1^{(0)}(t) \|^2_{L^2(0, +\infty)} \leq C|k|^{-1} \|\omega_0\|^2_{L^2},
\]

and (using \(vk^2 \leq 1\)),

\[
\| \partial_t (e^{ikt} a_1(t)) \|^2_{L^2(0, +\infty)} = \| e^{ikt - (vk^2)^{1/3}} (\partial_t a_1^{(0)}(t) + ika_1^{(0)}(t)) - (vk^2)^{1/3} a_1^{(0)}(t) \|^2_{L^2(0, +\infty)} \leq 2 \| \partial_t a_1^{(0)}(t) + ika_1^{(0)}(t) \|^2_{L^2} + 2 \| (vk^2)^{1/3} a_1^{(0)}(t) \|^2_{L^2} \leq C|k|^{-1} \|\omega_0\|^2_{L^2}.
\]

We define \(\tilde{a}_1(t) = e^{ikt} a_1(t)\) for \(t \geq 0\) and \(\tilde{a}_1(t) = 0\) for \(t \leq 0\). Due to \(a_1(0) = 0\), we have \(\tilde{a}_1(t) \in H^1(\mathbb{R})\) and \(\|\tilde{a}_1(t)\|^2_{H^1(\mathbb{R})} \leq C|k|^{-1} \|\omega_0\|^2_{L^2}\). Moreover,

\[
\int_{\mathbb{R}} \tilde{a}_1(t) e^{-ik\lambda} \, dt = \int_{0}^{+\infty} a_1(t) e^{ikt - ik\lambda} \, dt = c_1(\lambda - k),
\]

which gives

\[
\|\tilde{a}_1(t)\|^2_{H^1(\mathbb{R})} \sim \int_{\mathbb{R}} (1 + |\lambda|)^2 c_1(\lambda - k)^2 \, d\lambda = (1 + |\lambda + k|) c_1 \|\omega_0\|^2_{L^2(\mathbb{R})}.
\]

Thus, we obtain

\[
\| (1 + |\lambda + k|) c_1 \|^2_{L^2} \leq C|k|^{-1} \|\omega_0\|^2_{L^2}.
\]

Similarly, we can prove (6.13).

Now the result follows from (6.9), (6.10), (6.11) and (6.15). \(\square\)
6.3.3. Space-time estimates of the full problem

Proposition 6.7. Let \( \nu k^2 \leq 1 \) and \( \omega \) be a solution of (6.1). Then there exists a constant \( C > 0 \) independent of \( \nu, k \) such that

\[
\| \rho_k \|_{L^\infty_L L^2} + (\nu k^2)^{\frac{1}{2}} \| \omega \|_{L^2_L L^2} + (\nu k^2)^{\frac{1}{2}} \| \omega \|_{L^\infty_L L^2} + (\nu k^2)^{\frac{1}{2}} \| \rho_k \|_{L^2_L L^2} + k^2 \| u \|_{L^2_L L^2} \\
\leq C \| \omega_0 \|_{L^2} + C \nu^\frac{1}{3} |k|^{-\frac{2}{3}} \| \partial_y \omega_0 \|_{L^2} + C (\nu^{-\frac{1}{3}} |k|^\frac{4}{3} \| f_1 \|_{L^2_L L^2} + v^{-1} \| f_2 \|_{L^2_L L^2}).
\]

Proof. It follows from Propositions 6.5 and 6.6 that

\[
(\nu k^2)^{\frac{1}{2}} \| \omega \|_{L^2_L L^2} + (\nu k^2)^{\frac{1}{2}} \| \omega \|_{L^\infty_L L^2} + (\nu k^2)^{\frac{1}{2}} \| \rho_k \omega \|_{L^2_L L^2} + k^2 \| u \|_{L^2_L L^2} \\
\leq C \| \omega_0 \|_{L^2} + C \nu^\frac{1}{3} |k|^{-\frac{2}{3}} \| \partial_y \omega_0 \|_{L^2} + C (\nu^{-\frac{1}{3}} |k|^\frac{4}{3} \| f_1 \|_{L^2_L L^2} + v^{-1} \| f_2 \|_{L^2_L L^2}).
\]

(6.16)

It remains to estimate \( \| \rho_k \omega \|_{L^\infty_L L^2} \). For this, we introduce a new weight function \( \tilde{\rho}_k \) defined as follows:

\[
\tilde{\rho}_k(y) = \begin{cases} 
(Ly + L - 1)^3 + 1 & y \in [-1, -1 + L^{-1}], \\
1 & y \in [-1 + L^{-1}, 1 - L^{-1}], \\
(L - Ly - 1)^3 + 1 & y \in [1 - L^{-1}, 1],
\end{cases}
\]

(6.17)

where \( L = (\frac{|k|}{\nu})^{\frac{1}{3}} \). It is easy to see that

\[ \tilde{\rho}_k \in C^2(-1, 1), \quad |\tilde{\rho}_k'| \lesssim L, \quad |\tilde{\rho}_k''| \lesssim L^2, \]

and there exists a constant number \( C \), independent of \( \nu, k \), so that \( C^{-1} \rho_k \lesssim \tilde{\rho}_k \lesssim C \rho_k \). We also have \( \nu L^2 = (\nu k^2)^{\frac{1}{3}} \). Recall that \( \omega \) satisfies (6.1). By integration by parts, we get

\[
\frac{1}{2} \frac{d}{dr} \| \tilde{\rho}_k \omega \|_{L^2}^2 + v(\| \tilde{\rho}_k \omega \|_{L^2}^2 + k^2 \| \tilde{\rho}_k \omega \|_{L^2}^2) = \text{Re} \left( -v \int_{-1}^{1} \omega \tilde{\omega} (\tilde{\rho}_k') dy \\
-ik \{ f_1, \tilde{\rho}_k^3 \omega \} + \{ f_2, \partial_y (\tilde{\rho}_k^3 \omega) \} \right) \\
\leq v \int_{-1}^{1} |\omega|^2 (\tilde{\rho}_k')^2 dy \\
+ |k| \| f_1 \|_{L^2} \| \tilde{\rho}_k^3 \omega \|_{L^2} + \| f_2 \|_{L^2} (3 \| \tilde{\rho}_k^3 \omega \|_{L^\infty} \| \tilde{\rho}_k^2 \omega \|_{L^2} + \| \tilde{\rho}_k^3 \omega' \|_{L^2}) \\
\leq v(6 \| \tilde{\rho}_k \|_{L^\infty}^2 + 3 \| \tilde{\rho}_k'' \|_{L^\infty}) \int_{-1}^{1} |\omega|^2 \tilde{\rho}_k dy + \frac{1}{2} v^{-\frac{1}{3}} |k|^\frac{4}{3} \| f_1 \|_{L^2}^2 \\
+ \frac{1}{2} \| \nu k^2 \|_{L^2}^2 + v^{-1} \| f_2 \|_{L^2}^2 + 3 v \| \tilde{\rho}_k \|_{L^\infty} \| \tilde{\rho}_k^2 \omega \|_{L^2}^2 + v \| \tilde{\rho}_k^3 \omega' \|_{L^2}^2,
\]

which shows that

\[
\frac{d}{dr} \| \tilde{\rho}_k \omega \|_{L^2}^2 \lesssim v L^2 \| \tilde{\rho}_k \omega \|_{L^2}^2 + v^{-1} |k|^{\frac{4}{3}} \| f_1 \|_{L^2}^2 + (\nu k^2)^{\frac{1}{3}} \| \tilde{\rho}_k^3 \omega \|_{L^2}^2 + v^{-1} \| f_2 \|_{L^2}^2.
\]
from which and (6.16), we infer that
\[
\|\rho_k \omega\|^2_{L^\infty L^2} \leq C \|\omega\|^2_{L^2} + C v^\frac{1}{2} |k|^{-\frac{4}{3}} \|\partial_y \omega\|^2_{L^2} + C (v^{-\frac{1}{2}} |k|^\frac{4}{3} \|f_1\|^2_{L^2 L^2} + v^{-1} \|f_2\|^2_{L^2 L^2}).
\]

This completes the proof. □

6.4. Proof of Proposition 6.1

Proof. For the case of \(vk^2 \leq 1\), we have
\[
v^\frac{1}{2} |k|^{-\frac{4}{3}} = k^{-2} (vk^2)^{\frac{1}{2}}, \quad v^{-\frac{1}{2}} |k|^\frac{4}{3} = v^{-\frac{1}{2}} |k|(vk^2)^{\frac{1}{6}} \leq v^{-\frac{1}{2}} |k|,
\]
which, along with Proposition 6.7, implies that
\[
k^2 \|u\|^2_{L^2 L^2} + (vk^2)^{\frac{1}{2}} \|\omega\|^2_{L^2 L^2} + (vk^2)^{\frac{1}{2}} \|\omega\|^2_{L^\infty L^2} + \|\rho_k \omega\|^2_{L^\infty L^2}
\leq C (\|\omega\|^2_{L^2} + k^{-2} \|\partial_y \omega\|^2_{L^2}) + C (v^{-\frac{1}{2}} |k| \|f_1\|^2_{L^2 L^2} + v^{-1} \|f_2\|^2_{L^2 L^2}).
\]

Due to the definition of \(\rho_k\), we know that \(\rho_k^\frac{3}{2}(y) = 1 \geq (1 - |y|)^{\frac{3}{2}}\) for \(|y| \leq 1 - L^{-1}\), \(\rho_k^\frac{3}{2}(y) = L^\frac{3}{2} (1 - |y|)^{\frac{3}{2}} \geq L^\frac{3}{2} v^{\frac{3}{2}} (1 - |y|)^{\frac{3}{2}} = (|k|)^\frac{3}{2} (1 - |y|)^{\frac{3}{2}} \geq (1 - |y|)^{\frac{3}{2}}\) for \(1 - L^{-1} \leq |y| \leq 1 - v^{\frac{3}{2}}\), where \(L = (|k|/v)^{\frac{1}{2}} \leq v^{-\frac{1}{2}}\) and \((vk^2)^{\frac{1}{2}} \geq v^{\frac{1}{2}} \geq 1 - |y|\) for \(1 - v^{\frac{3}{2}} \leq |y| \leq 1\). Thus,
\[
\|(1 - |y|)^{\frac{3}{2}} \omega\|^2_{L^\infty L^2} \leq (vk^2)^{\frac{1}{2}} \|\omega\|^2_{L^\infty L^2} + \|\rho_k \omega\|^2_{L^\infty L^2}.
\]

Since \(0 \leq 1 - \rho_k^\frac{3}{2} \leq 1\) for \(|y| \leq 1\) and \(1 - \rho_k^\frac{3}{2} = 0\) for \(|y| \leq 1 - L^{-1}\), we have
\[
\|(1 - \rho_k^\frac{3}{2}) \omega\|_{L^1} \leq \|\omega\|_{L^1(1 - L^{-1},1)} + \|\omega\|_{L^1((-1,-1+L^{-1})}.
\]

Notice that
\[
\|\rho_k^\frac{3}{2}\|^2_{L^2(1 - L^{-1},1 - v^{\frac{3}{2}})} = \int_{1 - L^{-1}}^{1 - v^{\frac{3}{2}}} L^{-3} (1 - y)^{-3} dy = \frac{1}{2} L^{-3} (1 - y)^{-2} |1 - L^{-1}| \leq \frac{1}{2} L^{-3} v^{-\frac{1}{2}} = \frac{1}{2} |k|^{-1}.
\]

Thus, we have
\[
\|\omega\|_{L^1(1 - L^{-1},1)} = \|\omega\|_{L^1(1 - L^{-1},1 - v^{\frac{3}{2}})} + \|\omega\|_{L^1(1 - v^{\frac{3}{2}},1)} \leq \|\rho_k^\frac{3}{2} \omega\|_{L^2(1 - L^{-1},1 - v^{\frac{3}{2}})} + v^{\frac{1}{2}} \|\omega\|_{L^2}.
\]
Similarly, we have

$$\|u\|_{L^1(-1,-1+L^{-1})} \leq 2|k|^{-\frac{1}{2}}\|\rho_k^2 \omega\|_{L^2} + v^\frac{1}{4}\|\omega\|_{L^2}.$$  

This shows that

$$\|(1 - \rho_k^2)\omega\|_{L^1} \leq \|\omega\|_{L^1(1-1',1')} + \|\omega\|_{L^1(-1',-1+L^{-1})} \leq C\left(|k|^{-\frac{1}{2}}\|\rho_k^2 \omega\|_{L^2} + v^\frac{1}{4}\|\omega\|_{L^2}\right).$$

By Lemma 9.3, we have

$$\|u\|_{L^\infty} \leq \|(\partial_y, k)(\partial_x^2 - k^2)^{-1}\rho_k^2 \omega\|_{L^\infty} + \|(\partial_y, k)(\partial_x^2 - k^2)^{-1}(1 - \rho_k^2)\omega\|_{L^\infty} \leq C\left(|k|^{-\frac{1}{2}}\|\rho_k^2 \omega\|_{L^2} + \|(1 - \rho_k^2)\omega\|_{L^1}\right) \leq C\left(|k|^{-\frac{1}{2}}\|\rho_k^2 \omega\|_{L^2} + v^\frac{1}{4}\|\omega\|_{L^2}\right),$$

which gives

$$|k|\|u\|_{L^2}^2 \leq C\|\rho_k^2 \omega\|_{L^2}^2 + C|k|v^\frac{1}{4}\|\omega\|_{L^2}^2. \quad (6.20)$$

Then the desired result follows from (6.18), (6.19) and (6.20).

For the case of $vk^2 \geq 1$, we have $|k|\|u\|_{L^\infty}^2 \leq C\|\omega\|_{L^\infty}^2$, and then the result follows from Proposition 6.4 and the facts that $(vk^2)^{-\frac{1}{2}} \leq vk^2$, $v^{-1} = v^{-\frac{1}{2}}|k|(vk^2)^{-\frac{1}{2}} \leq v^{-\frac{1}{2}}|k|$.  

7. Nonlinear Stability

In this section, we prove Theorem 1.1. For the 2-D Navier-Stokes equation, the global existence of smooth solution is well-known for the data $u_0 \in H^2(\Omega)$. The main interest of Theorem 1.1 is the stability estimate $\sum_{k \in \mathbb{Z}} E_k \leq C C v^{\frac{1}{2}}$. Let us recall that $E_0 = \|\bar{w}\|_{L^\infty L^2}$, and for $k \neq 0$,

$$E_k = \|(1 - |y|)^{\frac{1}{2}} w_k\|_{L^\infty L^2} + |k|\|u_k\|_{L^2 L^2} + |k|\|u_k\|_{L^\infty L^2} + (vk^2)^{\frac{1}{2}}\|w_k\|_{L^2 L^2}.$$

First of all, we derive the evolution equations of $\bar{\bar{u}}(t, y)$ and $w_k(t, y) = \frac{1}{2\pi} \int_{\mathbb{T}} w(t, x, y)e^{-ik^3} dx$. We denote

$$f^1_k(t, y) = \sum_{l \in \mathbb{Z}} u^1_l(t, y) w_{k-l}(t, y), \quad f^2_k(t, y) = \sum_{l \in \mathbb{Z}} u^2_l(t, y) w_{k-l}(t, y).$$

Thanks to $\text{div} u = 0$, we have $\bar{\bar{u}}^2(t, y) = 0$. Due to $P_0(u^1 \partial_x u^1) = 0$, we find that

$$(\partial_t - \nu \partial_y^2)\bar{\bar{u}}^1(t, y) = -\sum_{l \in \mathbb{Z}\setminus\{0\}} u^2_l(t, y) \partial_x u^1_{-l}(t, y)$$
\[
\sum_{l \in \mathbb{Z} \setminus \{0\}} u_l^2(t, y) w_{-l}(t, y) = -f_0^2(t, y),
\]
and \(w_k(t, y)(k \neq 0)\) satisfies
\[
(\partial_t - \nu(\partial_y^2 - k^2) + iky)w_k(t, y) = -ikf_k^1(t, y) - \partial_y f_k^2(t, y).
\]

Next we estimate \(E_0\). By integration by parts, we get
\[
\left\langle \partial_t - \nu(\partial_y^2 - k^2) \bar{u}^1, -\partial_y \bar{u}^1 \right\rangle = \frac{1}{2} \partial_t \| \partial_y \bar{u}^1(t) \|_{L^2}^2 + \nu \| \partial_y \bar{u}^1(t) \|_{L^2}^2 = \langle \bar{f}^1, \partial_y \bar{u}^1 \rangle,
\]
which gives
\[
\partial_t \| \partial_y \bar{u}^1(t) \|_{L^2}^2 + \nu \| \partial_y \bar{u}^1(t) \|_{L^2}^2 \lesssim v^{-1} \| f_0^2(t, y) \|_{L^2}^2,
\]
from which, along with \(\partial_y \bar{u}^1(t, y) = \bar{w}(t, y)\), we infer that
\[
E_0^2 = \| \bar{w} \|_{L^\infty L^2}^2 \lesssim v^{-1} \| f_0^2 \|_{L^2 L^2}^2 + \| \bar{w} \|_{L^2}^2.
\]

Now we estimate \(E_k\). It follows from Proposition 6.1 that
\[
E_k^2 \lesssim v^{-\frac{1}{2}} |k| \| f_k^1 \|_{L^2 L^2}^2 + v^{-1} \| f_k^2 \|_{L^2 L^2}^2 + \| w_{0,k} \|_{L^2}^2 + |k|^{-2} \| \omega_{0,k} \|_{L^2}^2.
\]

For \(k \neq 0\), we have
\[
\| \frac{u_k^2(t, y)}{(1 - |y|)^{\frac{1}{2}}} \|_{L^2 L^\infty}^2 = \sup_{y \in [-1, 1]} \frac{|u_k^2(t, y)|^2}{1 - |y|} \|_{L^1}
= \max \left\{ \sup_{y \in [0, 1]} \frac{|f_y^1 \partial_y u_k^2(t, z) dz|^2}{1 - |y|}, \sup_{y \in [-1, 0]} \frac{|f_y^1 \partial_y u_k^2(t, z) dz|^2}{1 - |y|} \right\} \|_{L^1}
\leq 4 \| \partial_y u_k^2 \|_{L^2 L^2}^2 = 4 |k|^2 \| u_k^1 \|_{L^2 L^2}^2 \leq 4 E_k^2.
\]
from which we infer that, for \(k \in \mathbb{Z}\),
\[
\| f_k^2 \|_{L^2 L^2} \leq \sum_{l \in \mathbb{Z}} \| \frac{u_l^2(t, y)}{(1 - |y|)^{\frac{1}{2}}} \|_{L^2 L^\infty} \| (1 - |y|)^{\frac{1}{2}} w_{k-l} \|_{L^\infty L^2} \leq 2 \sum_{l \in \mathbb{Z}} E_l E_{k-l},
\]
and
\[
\| f_k^1 \|_{L^2 L^2} \leq \| u_k^1 \|_{L^\infty L^\infty} \| w_k \|_{L^2 L^2} + \| u_k^1 \|_{L^2 L^\infty} \| \bar{w} \|_{L^\infty L^2}
\]
\[
+ \sum_{l \in \mathbb{Z} \setminus \{0,k\}} \| u_l^1 \|_{L^\infty L^\infty} \| w_{k-l} \|_{L^2 L^2}.
\]

Thanks to \(|l||k - l| \geq |k||l| (l \neq 0, k)\), we have
\[
\sum_{l \in \mathbb{Z} \setminus \{0,k\}} \| u_l^1 \|_{L^\infty L^\infty} \| w_{k-l} \|_{L^2 L^2} \lesssim \sum_{l \in \mathbb{Z} \setminus \{0,k\}} |l|^{-\frac{1}{2}} E_l v^{-\frac{1}{2}} |k - l|^{-\frac{1}{2}} E_{k-l}
\]
\[
\lesssim |k|^{-\frac{1}{2}} v^{-\frac{1}{2}} \sum_{l \in \mathbb{Z} \setminus \{0,k\}} E_l E_{k-l},
\]
and
\[ \| \bar{u}^1 \|_{L^\infty L^2} \| w_k \|_{L^2 L^2} + \| u_k^1 \|_{L^2 L^\infty} \| \bar{w} \|_{L^\infty L^2} \lesssim \| \bar{w} \|_{L^\infty L^2} \| w_k \|_{L^2 L^2} \lesssim (vk^2)^{-\frac{1}{4}} E_k E_0. \]

This shows that
\[ \| f_k^1 \|_{L^2 L^2} \lesssim (vk^2)^{-\frac{1}{4}} \sum_{l \in \mathbb{Z}} E_l E_{k-l}. \]  \hspace{1cm} (7.4)

It follows from (7.1)–(7.3) that
\[ E_k \lesssim v^{-\frac{1}{2}} \sum_{l \in \mathbb{Z}} E_l E_{k-l} + \| w_{0,k} \|_{L^2} + \| \partial_y w_{0,k} \|_{L^2}, \]
\[ E_0 \lesssim v^{-\frac{1}{2}} \sum_{l \in \mathbb{Z}\setminus\{0\}} E_l E_{-l} + \| \bar{w}_0 \|_{L^2}, \]

which leads to
\[ \sum_{k \in \mathbb{Z}} E_k \lesssim v^{-\frac{1}{2}} \sum_{k \in \mathbb{Z}} \sum_{l \in \mathbb{Z}} E_l E_{k-l} + \sum_{k \in \mathbb{Z}} \| w_{0,k} \|_{L^2} + \sum_{k \in \mathbb{Z}\setminus\{0\}} |k|^{-\frac{1}{2}} \| \partial_y w_{0,k} \|_{L^2}. \]  \hspace{1cm} (7.5)

Due to \( \| u_0 \|_{H^2} \leq cv^{\frac{1}{2}} \), it is easy to verify that
\[ \sum_{k \in \mathbb{Z}} \| w_{0,k} \|_{L^2} + \sum_{k \in \mathbb{Z}\setminus\{0\}} |k|^{-\frac{1}{2}} \| \partial_y w_{0,k} \|_{L^2} \leq Ccv^{\frac{1}{2}}. \]

If \( c \) is suitably small, then we can deduce from (7.5) and a continuous argument that
\[ \sum_{k \in \mathbb{Z}} E_k \leq Ccv^{\frac{1}{2}}. \]

This completes the proof of Theorem 1.1.

8. Some Key Estimates Related to the Airy Function

8.1. Basic properties of the Airy function

Let \( Ai(z) \) be the classical Airy function which satisfies
\[ \partial_z^2 Ai(z) - z Ai(z) = 0. \]

We have the following asymptotic formula for \( |\arg z| \leq \pi - \varepsilon, \varepsilon > 0 \)(see [38,43]):
\[ Ai(z) = \frac{1}{2\sqrt{\pi}} z^{-\frac{1}{4}} e^{-\frac{2}{3}z^{\frac{3}{2}}} \left( 1 + R(z) \right), \hspace{0.5cm} R(z) = O(z^{-\frac{3}{2}}). \]

Thus, we may define
\[ A_0(z) = \int_{e^{i\pi/6}z}^{+\infty} Ai(t) dt = e^{i\pi/6} \int_{z}^{+\infty} Ai(e^{i\pi/6}t) dt. \]

For \( A_0(z) \), we have the following important properties from [38]:
Lemma 8.1. It holds that

1. There exists $\delta_0 > 0$ so that $A_0(z)$ has no zeros in the half plane $\text{Im} z \leq \delta_0$.
2. Let $a(\delta) = \sup \left\{ \text{Re} \left( \frac{A_0'(z)}{A_0(z)} \right) : \text{Im} z \leq \delta \right\}$. There exists $\delta_0 > 0$ so that $a(\delta) \in C([0, \delta_0])$ and

$$a(0) = -0.4843 \ldots < -1/3.$$ 

3. For $|\text{arg}(ze^{i\pi})| \leq \pi - \varepsilon, \varepsilon > 0$, we have the asymptotic formula

$$\frac{A_0'(z)}{A_0(z)} = -e^{i\pi/6}(ze^{i\pi/6})^{1/2} + O(z^{-1}).$$

Using Lemma 8.1, we can deduce the following important estimates on $A_0(z)$:

Lemma 8.2. Let $\delta_0$ be as in Lemma 8.1. There exists $c > 0$ so that for $\text{Im} z \leq \delta_0$,

$$\left| \frac{A_0'(z)}{A_0(z)} \right| \leq 1 + |z|^{1/2}, \quad \text{Re} \frac{A_0'(z)}{A_0(z)} \leq -c(1 + |z|^{1/2}).$$

Proof. For $|z| \geq R_0 \gg 1$ and $\text{Im} z \leq \delta_0$, we use the asymptotic formula in Lemma 8.1. For $|z| \leq R_0$ and $\text{Im} z \leq \delta_0$, we use the facts that

$$|A_0(z)| \sim 1, \quad \left| \frac{A_0'(z)}{A_0(z)} \right| \leq C, \quad \text{Re} \frac{A_0'(z)}{A_0(z)} \leq -c.$$

We introduce

$$\omega(z, x) = \frac{A_0(z + x)}{A_0(z)} = \exp \left( \int_0^x \frac{A_0'(z + t)}{A_0(z + t)} \, dt \right).$$

Lemma 8.3. There exists $\delta_1 > 0$ so that for $\text{Im} z \leq \delta_1$ and $x \geq 0$,

$$|\omega(z, x)| \leq e^{-\frac{x}{3}}.$$

Proof. Thanks to Lemma 8.1, there $\delta_1 > 0$ so that $a(\varepsilon) \geq \frac{1}{3}$ for $\varepsilon \in [0, \delta_1]$. Thus, for $\text{Im} z \leq \delta_1$ and $x \geq 0$, we have

$$|\omega(z, x)| = \left| \exp \left( \int_0^x \frac{A_0'(z + t)}{A_0(z + t)} \, dt \right) \right| = \left| \exp \left( \text{Re} \int_0^x \frac{A_0'(z + t)}{A_0(z + t)} \, dt \right) \right| \leq e^{-\frac{x}{3}}.$$ 

$\square$
8.2. Estimates of $W_1, W_2$

In this subsection, we prove Lemma 5.2. Let us recall that

$$W_1(y) = Ai(e^{\frac{x}{n}}(L(y - \lambda - ik\nu) + i\epsilon)), \quad W_2(y) = Ai(e^{\frac{x}{n}}(L(y - \lambda - ik\nu) + i\epsilon)),$$

where $L = (\frac{t}{n})^{\frac{1}{2}}$ and $\epsilon > 0$. We need the following technical lemma:

**Lemma 8.4.** Let $0 < \epsilon \leq 1$. It holds that, for any $x \geq 0$,

$$\int_0^{Lx} (1 + |t + Ld + i\epsilon|^2) dt \gtrsim Lx|Ld|^{\frac{1}{2}} + \int_0^{Lx} (1 + (vk^2)^{\frac{3}{2}} + |t - L(\lambda + 1)|^{\frac{1}{2}}) dt,$$

(8.1)

where $d = -1 - \lambda - ik\nu$.

**Proof.** Notice that

$$1 + |t + Ld + i\epsilon|^2 = 1 + |t - L(\lambda + 1) - i(vk^2)^{\frac{3}{2}} + i\epsilon|^2 \gtrsim 1 + |t - L(\lambda + 1)|^{\frac{1}{2}} + (vk^2)^{\frac{3}{2}}$$

and $|Ld| \sim |L(\lambda + 1)| + (vk^2)^{\frac{3}{2}}$. Thus, we have

$$\int_0^{Lx} (1 + |t + Ld + i\epsilon|^2) dt \sim \int_0^{Lx} (1 + (vk^2)^{\frac{3}{2}} + |t - L(\lambda + 1)|^{\frac{1}{2}}) dt.$$

For the case of $Lx \leq L(\lambda + 1)$, we have

$$\int_0^{Lx} (1 + (vk^2)^{\frac{1}{2}} + |t - L(\lambda + 1)|^{\frac{1}{2}}) dt = \int_0^{Lx} (1 + (vk^2)^{\frac{1}{2}}$$

$$+ (L(\lambda + 1) - t)^{\frac{1}{2}}) dt$$

$$= Lx(1 + (vk^2)^{\frac{1}{2}}) + \frac{2}{3}[(L(\lambda + 1))^{\frac{3}{2}} - (L(\lambda + 1) - Lx)^{\frac{3}{2}}]$$

$$\gtrsim Lx((L(\lambda + 1))^{\frac{1}{2}} + (vk^2)^{\frac{1}{2}}) \sim Lx|Ld|^\frac{1}{2} + |Lx|^\frac{3}{2}.$$

For the case of $Lx \geq L(\lambda + 1) \geq 0$, we have

$$\int_0^{Lx} (1 + (vk^2)^{\frac{1}{2}} + |t - L(\lambda + 1)|^{\frac{1}{2}}) dt$$

$$= Lx(1 + (vk^2)^{\frac{1}{2}}) + \int_0^{L(\lambda + 1)} (L(\lambda + 1) - t)^{\frac{1}{2}} dt + \int_{L(\lambda + 1)}^{Lx} (t - L(\lambda + 1))^{\frac{1}{2}} dt$$

$$= Lx(1 + (vk^2)^{\frac{1}{2}}) + \frac{2}{3}[(L(\lambda + 1))^{\frac{3}{2}} + (Lx - L(\lambda + 1))^{\frac{3}{2}}]$$

$$\gtrsim |Lx|^\frac{3}{2} + Lx(vk^2)^{\frac{1}{2}} \sim Lx|Ld|^\frac{1}{2} + |Lx|^\frac{3}{2}.$$
For the case of $L(\lambda + 1) \leq 0$, we have

$$\int_0^{Lx} (1 + (vk^2)^{\frac{1}{3}} + |t - L(\lambda + 1)|^{\frac{1}{3}}) dt \sim (Lx(1 + (vk^2)^{\frac{1}{3}} + |L(\lambda + 1)|^{\frac{1}{3}}) + \int_0^{Lx} t^{\frac{1}{3}} dt$$

$$\geq Lx((L(\lambda + 1))^{\frac{1}{2}} + (vk^2)^{\frac{1}{3}}) + (2/3)|Lx|^{\frac{1}{2}} \sim Lx|Ld|^{\frac{1}{2}} + |Lx|^{\frac{1}{2}}.$$  

Summing up, we conclude the lemma. $\square$

Now we are in a position to prove Lemma 5.2.

**Proof. Step 1.** $L^\infty$ estimate

Thanks to the definition of $W_1$, we find that

$$\frac{L|W_1(y)|}{|A_0(Ld + i\epsilon)|} = \frac{L[Ai(e^{iLx}(L(y + 1 + d) + i\epsilon))]}{|A_0(Ld + i\epsilon)|}.$$  

Thanks to $|A_0'(z)| = |Ai(e^{i\frac{Lx}{2}} z)|$, we get

$$\frac{L|W_1(x - 1)|}{|A_0(Ld + i\epsilon)|} = \frac{L[A'_0(Lx + Ld + i\epsilon)]}{|A_0'(Lx + Ld + i\epsilon)|} \frac{|A_0(Lx + Ld + i\epsilon)|}{|A_0(Ld + i\epsilon)|}.$$  

By Lemma 8.2, we have

$$\frac{|A'_0(Lx + Ld + i\epsilon)|}{|A_0(Lx + Ld + i\epsilon)|} \lesssim 1 + |Lx + Ld + i\epsilon|^{\frac{1}{2}} \lesssim 1 + |Lx + Ld|^{\frac{1}{2}},$$  

from which, along with Lemma 8.3, we infer that for any $x \in [0, 2]$,

$$\frac{L|W_1(x - 1)|}{|A_0(Ld + i\epsilon)|} \lesssim L(1 + |Lx + Ld|^{\frac{1}{2}}) e^{-Lx/3} \lesssim L(1 + |Ld|^{\frac{1}{2}}) \lesssim L(1 + |L(1 + \lambda)|^{\frac{1}{2}}) = L + |k(1 + \lambda)/v|^{\frac{1}{2}}.$$  

As $L = (|k|/v)^{\frac{1}{2}} = v^{-\frac{1}{2}} (vk^2)^{\frac{1}{6}} \leq v^{-\frac{1}{2}}$, we have

$$\frac{L}{|A_0(Ld + i\epsilon)|} \|W_1\|_{L^\infty} \lesssim C(L + |k(1 + \lambda)/v|^{\frac{1}{2}}) \leq Cv^{-\frac{1}{2}} (1 + |k(\lambda + 1)|)^{\frac{1}{2}}.$$  

The proof of $\frac{L}{|A_0(Ld + i\epsilon)|} \|W_2\|_{L^\infty}$ is similar.

**Step 2.** $L^1$ estimate

Thanks to the definition of $W_1$, we have

$$\frac{L}{|A_0(Ld + i\epsilon)|} \|W_1\|_{L^1} = \frac{L}{|A_0(Ld + i\epsilon)|} \int_{-1}^1 |Ai(e^{i\frac{Lx}{2}} (L(y - \lambda - ikv) + i\epsilon))| dy$$

$$= \frac{L}{|A_0(Ld + i\epsilon)|} \int_0^2 |Ai(e^{i\frac{Lx}{2}} (Lx + Ld + i\epsilon))| dx$$

$$= \frac{L}{|A_0(Ld + i\epsilon)|} \int_0^2 |Ai(e^{i\frac{Lx}{2}} (Lx + Ld + i\epsilon))| dx$$
\[ L \int_0^2 \frac{|A'_0(Lx + Ld + i\epsilon)|}{|A_0(Lx + Ld + i\epsilon)|} \frac{|A_0(Lx + Ld + i\epsilon)|}{|A_0(Ld + i\epsilon)|} \, dx. \]

For the case of \(|Ld| \leq 1\), we get by Lemmas 8.2 and 8.3 that
\[ \frac{L}{|A_0(Ld + i\epsilon)|} \|W_1\|_{L^1} \lesssim L \int_0^2 (1 + |Lx + Ld|^{1/2}) e^{-|Lx|/3} \, dx \lesssim L \int_0^2 (1 + |Lx|^{1/2}) e^{-|Lx|/3} \, dx \lesssim 1. \]

For the case of \(|Ld| \geq 1\), by Lemma 8.1 and the proof of Lemmas 8.3, and 8.4, we infer that
\[ \frac{L}{|A_0(Ld + i\epsilon)|} \|W_1\|_{L^1} \lesssim L \int_0^2 (1 + |Lx + Ld|^{1/2}) e^{-c|Lx|/d(Lx)} \, dx \lesssim L \int_0^2 (1 + |Lx + Ld|^{1/2}) e^{-c|Lx|/d(Lx)} \, dx \lesssim 1, \]

here we used
\[ L \int_0^2 e^{-c|Lx|/d(Lx)} \, dx = \int_0^2 e^{-c|Lx|/d(Lx)} \, d(Lx) \lesssim 1, \]
\[ L \int_0^2 |Lx|^{1/2} e^{-c|Lx|/d(Lx)} \, dx = \int_0^2 |Lx|^{1/2} e^{-c|Lx|/d(Lx)} \, d(Lx) \lesssim 1, \]
\[ L \int_0^2 |Ld|^{1/2} e^{-c|Lx|/d(Lx)} \, dx = \int_0^2 e^{-c|Lx|/d(Lx)} \, d(Lx) \lesssim 1. \]

The proof of \( \frac{L}{|A_0(Ld + i\epsilon)|} \|W_2\|_{L^1} \lesssim 1 \) is similar.

**Step 3.** Weighted \( L^2 \) estimate
We have
\[ \frac{L}{|A_0(Ld + i\epsilon)|} \|\rho_k W_1\|_{L^2} \]
\[ = \frac{L}{|A_0(Ld + i\epsilon)|} \left( \int_{-\infty}^{\infty} \rho_k(y) |A_i(L(y - \lambda - ik\nu) + i\epsilon)|^2 dy \right)^{1/2} \]
\[ = \frac{L}{|A_0(Ld + i\epsilon)|} \left( \int_0^2 \rho_k(x - 1) |A_i(L(x + d) + i\epsilon)|^2 dx \right)^{1/2} \]
\[ = L \left( \int_0^2 \rho_k(x - 1) \frac{|A'_0(Lx + Ld + i\epsilon)|^2}{|A_0(Lx + Ld + i\epsilon)|^2} \frac{|A_0(Lx + Ld + i\epsilon)|}{|A_0(Ld + i\epsilon)|} \, dx \right)^{1/2}. \]

For the case of \(|Ld| \leq 1\), we have
\[ \frac{L}{|A_0(Ld + i\epsilon)|} \|\rho_k W_1\|_{L^2} \lesssim L \left( \int_0^2 (1 + |Lx + Ld|) e^{-|Lx|/3} \, dx \right)^{1/2} \lesssim L^{1/2}. \]
For the case of $|Ld| \geq 1$, we have
\[
\frac{L}{|A_0(Ld + i\epsilon)|} \|\rho_k W_1\|_{L^2} \lesssim L \left( \int_0^2 \rho_k(x - 1)(1 + |Lx + Ld|) e^{-2c'|Lx||Ld|^\frac{1}{2}} dx \right)^{\frac{1}{2}} \\
\lesssim L \left( \int_0^2 (1 + Lx + |Ld|\rho_k(x - 1)) e^{-2c'|Lx||Ld|^\frac{1}{2}} dx \right)^{\frac{1}{2}} \\
\lesssim L \left( \int_0^2 (1 + Lx + Lx|Ld|) e^{-2c'|Lx||Ld|^\frac{1}{2}} dx \right)^{\frac{1}{2}} \lesssim L^\frac{1}{2}.
\]
The proof of $\frac{L}{|A_0(Ld + i\epsilon)|} \|\rho_k^2 W_2\|_{L^2} \lesssim L^\frac{1}{2}$ is similar. □

8.3. Estimates of $C_{ij}$

In this subsection, we prove Lemma 5.1. We need the following lemmas.

Lemma 8.5. Let $\delta_0$ be as in Lemma 8.2. Then it holds that for $\text{Im} z \leq \delta_0$ and $x \geq 0$,
\[
|\omega(z, x)| \leq e^{-cx^{3/2}}.
\]

Proof. By Lemma 8.2, we get
\[
|\omega(z, x)| \leq \left| \exp \left( \text{Re} \int_0^x \frac{A_0'(z + t)}{A_0(z + t)} dt \right) \right| \leq \exp \left( -c \int_0^x (1 + |z + t|^\frac{1}{2}) dt \right) \leq e^{-cx^{3/2}},
\]
where we used Lemma 8.4 so that
\[
\int_0^x (1 + |z + t|^\frac{1}{2}) dt = \int_0^{Lx'} (1 + |t + Ld + i\epsilon|^\frac{1}{2}) dt \gtrsim |Lx'|^{\frac{3}{2}} = x^{\frac{3}{2}},
\]
by writing $z = Ld + i\epsilon, x = Lx'$. □

Lemma 8.6. Let $\delta_1$ be as in Lemma 8.3. There exists $k_0 > 1$ so that if $L \geq 6k$ or $L \geq k \geq k_0$, then we have
\[
e^{2k} \left| 1 - e^{-2k}\omega(z, 2L) - \frac{k}{L} \int_0^{2L} e^{-\frac{kt}{2}} \omega(z, t) dt \right| \\
\geq \sqrt{2} \left| 1 - e^{-2k}\omega(z, 2L) + \frac{k}{L} \int_0^{2L} e^{\frac{kt}{2}} \omega(z, t) dt \right| \text{ for } \text{Im} z \leq \delta_1.
\]

Proof. We first consider the case of $L \geq 6k, k \geq 1$. It follows from Lemma 8.3 that for $\text{Im} z \leq \delta_1$,
\[
\left| 1 - e^{-2k}\omega(z, 2L) - \frac{k}{L} \int_0^{2L} e^{-\frac{kt}{2}} \omega(z, t) dt \right| \\
\geq 1 - e^{-2k}|\omega(z, 2L)| - \frac{k}{L} \int_0^{2L} e^{-\frac{kt}{2}} |\omega(z, t)| dt \\
\geq 1 - e^{-2k}e^{-2L/3} - \frac{k}{L} \int_0^{2L} e^{-\frac{kt}{2} - \frac{L}{3}} dt
\]
\[
\geq 1 - e^{-2k} - \frac{k}{L} / \left( \frac{k}{L} + \frac{1}{3} \right) \geq 1 - \frac{1}{6} - \frac{1}{6} / \left( \frac{1}{6} + \frac{1}{3} \right) = \frac{1}{2},
\]

and
\[
\left| 1 - e^{2k} \omega(z, 2L) + \frac{k}{L} \int_0^{2L} e^{k t} \omega(z, t) dt \right| \\
\leq 1 + e^{2k} |\omega(z, 2L)| + \frac{k}{L} \int_0^{2L} e^{k t} |\omega(z, t)| dt \\
\leq 1 + e^{2k} e^{-2L/3} + \frac{k}{L} \int_0^{2L} e^{k t} dt \\
\leq 1 + e^{2k} + \frac{k}{L} \left( \frac{1}{3} - \frac{k}{L} \right) \leq 1 + \frac{1}{6} + \frac{1}{6} / \left( \frac{1}{3} - \frac{1}{6} \right) \leq 7/3.
\]

This shows that
\[
\sqrt{3} \left| 1 - e^{2k} \omega(z, 2L) + \frac{k}{L} \int_0^{2L} e^{k t} \omega(z, t) dt \right| \\
\leq \frac{7\sqrt{3}}{3} < 10/3 < 7/2 < e^2/2 \leq e^{2k}/2 \\
\leq e^{2k} \left| 1 - e^{-2k} \omega(z, 2L) - \frac{k}{L} \int_0^{2L} e^{-k t} \omega(z, t) dt \right|.
\]

For the case of \( L \geq k \geq k_0 \), on one hand, we have
\[
\left| 1 - e^{-2k} \omega(z, 2L) - \frac{k}{L} \int_0^{2L} e^{-k t} \omega(z, t) dt \right| \geq 1 - e^{-2k} e^{-2L/3} - \frac{k}{L} \int_0^{2L} e^{-k t} dt \\
\geq 1 - e^{-8k/3} - \frac{k}{L} / \left( \frac{k}{L} + \frac{1}{3} \right) \geq \frac{1}{8},
\]
on the other hand, by Lemma 8.5, we have
\[
\left| 1 - e^{2k} \omega(z, 2L) + \frac{k}{L} \int_0^{2L} e^{k t} \omega(z, t) dt \right| \leq 1 + e^{2k} |\omega(z, 2L)| + \int_0^{2L} e^{t} |\omega(z, t)| dt \\
\leq 1 + e^{2k} e^{c(2L)^{3/2}} + \int_0^{2L} e^{t - cr^{3/2}} dt \\
\leq 1 + e^{2k - c(2k)^{3/2}} + C \leq C_0,
\]

here \( C_0 > 3 \) is an absolute constant. Choose \( k_0 > 1 \) so that \( e^{2k_0} > 8\sqrt{2}C_0 \). Then we have
\[
e^{2k} \left| 1 - e^{-2k} \omega(z, 2L) - \frac{k}{L} \int_0^{2L} e^{-k t} \omega(z, t) dt \right| \geq e^{2k_0} / 8 \geq \sqrt{2}C_0 \\
\geq \sqrt{2} \left| 1 - e^{2k} \omega(z, 2L) + \frac{k}{L} \int_0^{2L} e^{k t} \omega(z, t) dt \right|.
\]

This completes the proof of the lemma. \( \square \)
Now we are in a position to prove Lemma 5.1. Let us recall that

\[
\begin{pmatrix}
(C_{11}) \\
(C_{12})
\end{pmatrix} = \frac{(A_2e^k - B_2e^{-k})}{A_1A_2 - B_1B_2},
\begin{pmatrix}
(C_{21}) \\
(C_{22})
\end{pmatrix} = \frac{(-A_2e^{-k} + B_2e^k)}{A_1A_2 - B_1B_2},
\tag{8.2}
\]

where

\[
A_1 = \int_{-1}^{1} e^{ky} W_1(y) dy, \quad A_2 = \int_{-1}^{1} e^{-ky} W_2(y) dy,
\]

\[
B_1 = \int_{-1}^{1} e^{-ky} W_1(y) dy, \quad B_2 = \int_{-1}^{1} e^{ky} W_2(y) dy.
\]

**Proof.** Let \(y + 1 = x = \frac{t}{L} \). Due to \(A'_0(z) = -e^{i\pi/6} Ai(e^{i\pi/6} z)\), we have

\[
B_1 = \int_{0}^{2} e^{-k(x-1)} Ai(e^{i\pi/6}(L(x + d) + i\epsilon)) dx
\]

\[
= \frac{e^k}{L} \int_{0}^{2L} e^{-\frac{k}{L}t} Ai(e^{i\pi/6}((t + Ld) + i\epsilon)) dt
\]

\[
= -\frac{e^{k-i\pi/6}}{L} \int_{0}^{2L} e^{-\frac{k}{L}t} A'_0(t + Ld + i\epsilon) dt
\]

\[
= -\frac{e^{k-i\pi/6}}{L} \left[ e^{-2k} A_0(2L + Ld + i\epsilon) - A_0(Ld + i\epsilon) 
+ \frac{k}{L} \int_{0}^{2L} e^{-\frac{k}{L}t} A_0(t + Ld + i\epsilon) dt \right]
\]

\[
= -A_0(Ld + i\epsilon) \left[ e^{-2k} \omega(Ld + i\epsilon, 2L) - 1 + \frac{k}{L} \int_{0}^{2L} e^{-\frac{k}{L}t} \omega(Ld + i\epsilon, t) dt \right].
\tag{8.3}
\]

Similarly, we have

\[
A_1 = -A_0(Ld + i\epsilon) \left[ e^{2k} \omega(Ld + i\epsilon, 2L) - 1 - \frac{k}{L} \int_{0}^{2L} e^{\frac{k}{L}t} \omega(Ld + i\epsilon, t) dt \right].
\tag{8.4}
\]

Then we infer from Lemma 8.6 that

\[
\left| \frac{A_1}{B_1} \right| = e^{-2k} \left| \frac{1 - e^{2k} \omega(Ld + i\epsilon, 2L) + \frac{k}{L} \int_{0}^{2L} e^{\frac{k}{L}t} \omega(Ld + i\epsilon, t) dt}{1 - e^{-2k} \omega(Ld + i\epsilon, 2L) - \frac{k}{L} \int_{0}^{2L} e^{-\frac{k}{L}t} \omega(Ld + i\epsilon, t) dt} \right| \leq \frac{\sqrt{2}}{2}.
\tag{8.5}
\]

Thanks to \(Ai(z) = \frac{Ai(z)}{z}\), we have

\[
\overline{B_2} = \int_{-1}^{1} e^{ky} Ai(e^{-i\pi/6}(L(y - \lambda + ikv) - i\epsilon)) dy
\]
\[
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\]
\[
\int_{-1}^{1} e^{-ky} Ai(e^{i\pi/6} (L(y + \lambda - ik\nu) + i\epsilon))dy
\]
\[
= -A_0(L\tilde{a} + i\epsilon) \frac{e^{k-i\pi/6}}{L} \left[ e^{-2k\omega(L\tilde{a} + i\epsilon, 2L) - 1}
+ \frac{k}{L} \int_{0}^{2L} e^{-k\omega(L\tilde{a} + i\epsilon, t)}dt \right].
\] (8.6)

Similarly, we have
\[
\overline{A_2} = -A_0(L\tilde{a} + i\epsilon) \frac{e^{k-i\pi/6}}{L} \left[ e^{2k\omega(L\tilde{a} + i\epsilon, 2L) - 1}
- \frac{k}{L} \int_{0}^{2L} e^{k\omega(L\tilde{a} + i\epsilon, t)}dt \right].
\] (8.7)

Thus, by Lemma 8.6, we get
\[
\left| \frac{A_2}{B_2} \right| \leq \frac{\sqrt{2}}{2}. \tag{8.8}
\]

Now it follows from (8.5) and (8.8) that
\[
|A_1A_2 - B_1B_2| \gtrsim |B_1B_2|.
\]

From the proof of Lemma 8.6 and (8.4), we know that
\[
|B_1| \geq \frac{e^k}{L} |A_0(Ld + i\epsilon)| \left( 1 - e^{-2k|\omega(Ld + i\epsilon, 2L)| - \frac{k}{L} \int_{0}^{2L} e^{-k\omega(Ld + i\epsilon, t)}dt \right)
\geq \frac{1}{8} \frac{e^k}{L} |A_0(Ld + i\epsilon)|.
\]

Similarly, \( |B_2| \geq \frac{1}{8} \frac{e^k}{L} |A_0(L\tilde{a} + i\epsilon)|. \) Thus,
\[
|A_1A_2 - B_1B_2| \gtrsim |A_0(Ld + i\epsilon)||A_0(L\tilde{a} + i\epsilon)| \frac{e^{2k}}{L^2}.
\]

Furthermore, we also have
\[
|B_1| \leq 2 \frac{e^k}{L} |A_0(Ld + i\epsilon)|, \quad |B_2| \leq 2 \frac{e^k}{L} |A_0(L\tilde{a} + i\epsilon)|,
\]
\[
|A_1| \leq C_0 \frac{e^{-k}}{L} |A_0(Ld + i\epsilon)|, \quad |A_2| \leq C_0 \frac{e^{-k}}{L} |A_0(L\tilde{a} + i\epsilon)|.
\]

Summing up, we can conclude the estimates of \( C_{ij}. \) \( \square \)

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9. Appendix

Lemma 9.1. It holds that for any $|k| \geq 1$,
\[
\int_{-1}^{1} \left| \frac{\sinh k(1+y)}{\sinh 2k} \right|^2 dy = \int_{-1}^{1} \left| \frac{\sinh k(1-y)}{\sinh 2k} \right|^2 dy \leq 2|k|^{-1},
\]
\[
\int_{-1}^{1} \left| \frac{\cosh k(1+y)}{\sinh 2k} \right|^2 dy = \int_{-1}^{1} \left| \frac{\cosh k(1-y)}{\sinh 2k} \right|^2 dy \leq 8|k|^{-1}.
\]

Proof. Using the facts that
\[
\frac{\sinh k(1+y)}{\sinh 2k} \leq 2e^{-|k|(1-y)}, \quad \left| \frac{\cosh k(1+y)}{\sinh 2k} \right| \leq 4e^{-|k|(1-y)},
\]
we infer that
\[
\int_{-1}^{1} \left| \frac{\sinh k(1+y)}{\sinh 2k} \right|^2 dy \leq \int_{-1}^{1} 4e^{-2|k|(1-y)} dy \leq 2|k|^{-1},
\]
\[
\int_{-1}^{1} \left| \frac{\cosh k(1+y)}{\sinh 2k} \right|^2 dy \leq \int_{-1}^{1} 16e^{-2|k|(1-y)} dy \leq 8|k|^{-1}.
\]

Lemma 9.2. Let $|k| \geq 1$. If $1 - \lambda \geq |k|^{-1}$, then we have
\[
\left\| \frac{\sinh k(1+y)}{\sinh 2k} \right\|_{L^\infty(E_1)} \leq 2e^{-|k|(1-\lambda)/2},
\]
\[
\left\| \frac{\cosh k(1+y)}{\sinh 2k} \right\|_{L^\infty(E_1)} \leq 4e^{-|k|(1-\lambda)/2},
\]
where $E_1 = (-1, 1) \cap (-\infty, (\lambda + 1)/2)$. If $\lambda + 1 \geq |k|^{-1}$, then we have
\[
\left\| \frac{\sinh k(1-y)}{\sinh 2k} \right\|_{L^\infty(E_2)} \leq 2e^{-|k|(1+\lambda)/2},
\]
\[
\left\| \frac{\cosh k(1-y)}{\sinh 2k} \right\|_{L^\infty(E_1)} \leq 4e^{-|k|(1+\lambda)/2},
\]
where $E_2 = (-1, 1) \cap ((\lambda - 1)/2, +\infty)$.

Proof. If $1 - \lambda \geq |k|^{-1}$, then we have for $y \in E_1$,
\[
\left| \frac{\sinh k(1+y)}{\sinh 2k} \right| \leq 2e^{-|k|(1-y)} \leq 2e^{-|k|(1-\lambda)/2},
\]
\[
\left| \frac{\cosh k(1+y)}{\sinh 2k} \right| \leq 4e^{-|k|(1-y)} \leq 4e^{-|k|(1-\lambda)/2}.
\]
This gives the first inequality. The proof of the second inequality is similar. □
Lemma 9.3. If \((\partial_y^2 - k^2)\varphi = w, \varphi(\pm 1) = 0, |k| \geq 1\), then we have
\[
\|\varphi'\|_{L^2}^2 + k^2 \|\varphi\|_{L^2}^2 = \langle -w, \varphi \rangle \lesssim |k|^{-1} \|w\|_{L^1}^2,
\]
\[
\|\varphi'\|_{L^\infty} + |k| \|\varphi\|_{L^\infty} \lesssim \|w\|_{L^1},
\]
\[
\|\varphi'\|_{L^\infty} + |k| \|\varphi\|_{L^\infty} \lesssim |k|^{-\frac{1}{2}} \|w\|_{L^2}.
\]

Proof. The first inequality follows from the following:
\[
\|\varphi'\|_{L^2}^2 + k^2 \|\varphi\|_{L^2}^2 = \langle -w, \varphi \rangle \leq \|w\|_{L^1} \|\varphi\|_{L^\infty} \lesssim \|w\|_{L^1} \|\varphi'\|_{L^2} \|\varphi\|_{L^2}^\frac{1}{2} \lesssim \|w\|_{L^1} (|k|^{-1} \|\varphi'\|_{L^2}^2 + |k| \|\varphi\|_{L^2}^2)^\frac{1}{2}.
\]

Using the first inequality, we infer that
\[
|k| \|\varphi\|_{L^\infty} \lesssim |k| \|\varphi'\|_{L^2} \|\varphi\|_{L^2}^\frac{1}{2} \lesssim (|k| \|\varphi'\|_{L^2}^2 + k^2 \|\varphi\|_{L^2}^2)^\frac{1}{2} \lesssim \|w\|_{L^1}.
\]

For \(y \in [0, 1]\), we choose \(y_1 \in (y - 1/k, y)\) so that \(|\varphi'(y_1)|^2 \leq |k| \|\varphi'\|_{L^2}^2\). Then we have
\[
|\varphi'(y)| \leq |\varphi'(y_1)| + \int_{y_1}^{y} |\varphi''(z)| dz \leq (|k| \|\varphi'\|_{L^2}^2)^\frac{1}{2} + \int_{y_1}^{y} |k^2 \varphi(z) + w(z)| dz \leq C \|w\|_{L^1} + |y - y_1| k^2 \|\varphi\|_{L^\infty} + \|w\|_{L^1} \leq C \|w\|_{L^1} + |k| \|\varphi\|_{L^\infty} \leq C \|w\|_{L^1}.
\]

Similarly, \(|\varphi'(y)| \leq C \|w\|_{L^1}\) for \(y \in [-1, 0]\). This proves the second inequality.

Thanks to \(\|w\|_{L^2}^2 = \|\varphi''\|_{L^2}^2 = \|\varphi''\|_{L^2} \|\varphi'\|_{L^2} \|\varphi\|_{L^2}^\frac{1}{2} = \|\varphi''\|_{L^2} \|\varphi'\|_{L^2} \|\varphi\|_{L^2} \|\varphi\|_{L^2}^\frac{1}{2} \leq \|\varphi''\|_{L^2} \|\varphi'\|_{L^2} + |k| \|\varphi\|_{L^2} \|\varphi\|_{L^2} + |k^2 \|\varphi\|_{L^2}^2 \lesssim |k|^{-\frac{1}{2}} \|w\|_{L^2},\)
which gives the third inequality. \(\Box\)

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