A Renormalization Group Approach to Relativistic Cosmology

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Abstract

We discuss the averaging hypothesis tacitly assumed in standard cosmology. Our approach is implemented in a “3+1” formalism and invokes the coarse graining arguments, provided and supported by the real-space Renormalization Group (RG) methods, in parallel with lattice models of Statistical Mechanics. Block variables are introduced and the recursion relations written down explicitly enabling us to characterize the corresponding RG flow. To leading order, the RG flow is provided by the Ricci-Hamilton equations studied in connection with the geometry of three-manifolds. The possible relevance of the Ricci-Hamilton flow in implementing the averaging in cosmology has been previously advocated, but the physical motivations behind this suggestion were not clear. The RG interpretation provides us with such physical motivations. The properties of the Ricci-Hamilton flow make it possible to study a critical behaviour of cosmological models. This criticality is discussed and it is argued that it may be related to the formation of sheet-like structures in the universe. We provide an explicit expression for the renormalized Hubble constant and for the scale dependence of the matter distribution. It is shown that the Hubble constant is affected by non-trivial scale dependent shear terms, while the spatial anisotropy of the metric influences significantly the scale-dependence of the matter distribution.

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1 Introduction

A successful physical theory usually enables us to isolate some limited range of length scales, or select a not too big set of variables, to render the problem tractable and at the same time preserve its essence. Fortunately, in many circumstances it is not necessary to resolve the details associated with each scale, since generally phenomena of each size can be treated independently. For example, in hydrodynamics there is no need to specify the motion of each water molecule and yet waves can be described as a disturbance of a continuous fluid.

There are also problems that have many various scales of length, that is to say, phenomena or processes where each length-scale’s contribution is of equal importance. To handle them one has to take into account the entire spectrum of length scales, dealing with fluctuations of almost any wavelength and consequently many coupled degrees of freedom. For example, critical phenomena, turbulent flow, the internal structure of elementary particles and confinement in QCD belong to the above class of problems [50].

We argue here that also in gravitational physics we encounter a problem of a similar nature, namely, the so-called averaging problem in relativistic cosmology.

Relativistic cosmologies rely on some form of Cosmological Principle. The latter is usually a smoothing-out hypothesis imposed a priori on the distribution of matter in the universe. A well known example is provided by the Friedmann-Lemaître-Robertson-Walker (FLRW) metric which is assumed to describe the real universe.

There are at least two reasons why one should be very careful in doing so.

Firstly, observations of the distribution of matter have shown that the scale at which the background homogeneity is reached is probably of the order of hundreds of Mpc, meaning that locally, i.e. at least up to this scale, the universe is quite lumpy. Consequently, the local geometry is very complex and its nature not very illuminating from the point of view of cosmology. Seen this way, the approach usually taken is to average out all the matter, i.e. to redistribute them in the form of a homogeneous perfect fluid and use continuous functions (e.g. matter density, pressure) in modeling the universe, assuming that they represent “volume averages” of the corresponding fine scale quantities. In doing so one tacitly assumes that such a smoothed-out universe and the real locally inhomogeneous one, behave identically under their own gravitation. However, it has been stressed in [14] that the above assumption, though usually taken for granted, is by no means justified. First of all, Einstein’s equations are highly non-linear which is why any averaging process on them is far from trivial in general.

Secondly, and this is a remark of a philosophical flavour, most of the observational data are theory dependent, i.e. their meaning can be interpreted only by assuming a particular theoretical explanation.

Thus it is of great importance that the very foundations of a cosmological model should be as sound as possible, in particular, they should be free from assumptions which may not be warranted.

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1If this scale has been reached at all is still a matter of controversy.
2Moreover, additional care is required since a volume average of a tensor is not a covariant quantity, unlike scalars.
2 Coarse-graining in Cosmology

A possible solution to the averaging problem would be to explicitly construct a procedure for carrying out the smoothing process in the full theory. Almost all existing attempts were concerned with the linearized theory, with a possible exception of [9] (see also [24]), for a review see [31].

In [9] a covariant smoothing-out procedure was put forward for the space-times associated with gravitational configurations which may be considered near to the standard ones, generating closed FLRW universes. The procedure makes use of Hamilton’s theorem about smooth deformations of three-metrics and is adapted for smoothing-out an initial data set for cosmological solutions to the Einstein equations. While interesting in its own right, this approach seemed rather ad hoc and not yet capable of resolving the issues of actual limits of validity of the FLRW models in cosmology.

Now, our hope is that there is a smart and simpler way to the heart of the problem, borrowing from the known theories and methods of statistical mechanics, based on the real-space Renormalization Group (RG) approach to study critical phenomena in lattice models [27, 48]. Although the usual renormalization transformations, invoking averaging over square blocks, are designed mostly having ferromagnetic systems in mind, there are many more problems suitable for RG methods. These are difficult problems where the reductionist approach fails and where the effective degrees of freedom of a physical system are scale dependent. The difficulty of this kind of issues can be traced to a multiplicity of scales and, moreover, there can presumably be a gross mismatch between the largest and smallest scales in the problem. The averaging problem in cosmology can be looked at and studied as belonging to precisely this kind of problems. This is our main objective in this paper.

Some form of RG is active on any system where there are fluctuations present (they by no means do need to be quantum). This is so, since one can integrate the fluctuations out of the physical quantities of interest, e.g. the partition function, and depending on the “scale” up to which one is integrating the same emerging quantities are different. The functional relations between them provide recursion relations between the physical parameters, the coupling constants, which characterize the physics at each scale, and this is precisely what RG is all about.

Often a major step consists in finding a way of looking at things. Therefore we stress that the problem we face with the averaged description in cosmology is effectively a question of how a system behaves under changes of “scales”. As such it is most naturally addressed using RG approach, understood here rather as a general strategy to handle problems of multiple length scales enabling us to extract the long distance behaviour of the system by making the scale successively coarser. In cosmology, we have the curvature inhomogeneities and to consistently tackle this problem, we will have to consider a procedure operating on the metric, not only on or apart from the matter present.

To provide even more support in favour of the presented above idea, that RG arguments might be applicable to the problem of the inhomogeneous universe, let us notice that one can also be guided by scaling ideas. Scaling is exhibited (approximately or exactly) by many natural

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3 Although the renormalization procedure might seem purely formal there are important physical ideas behind it, namely, that of scaling and universality.
phenomena and mathematical models. The universe, namely, the distribution of matter in it does exhibit certain scaling properties \cite{7, 26}, of which the power law behaviour of the two-point correlation function for galaxies, clusters and quasars, is a fair example. Scaling, on its own right, is deeply understood within the underlying mathematical scaffold which is RG. These are hints therefore that one can regard the universe as a gravitational dynamical system not far from criticality (understood intuitively by analogy with e.g. a ferromagnet). Later one can also try to qualify the precise nature of the critical behaviour within the phase transition context, but this will not be of our concern in the present paper (apart from a simplified example in section 4.3).

In passing, let us note that interesting results were obtained in \cite{12} on spherical scalar field collapse and in \cite{1} on axisymmetric gravitational waves collapse, which show a surprising scaling and critical behaviour reminiscent of that found in many (second-order) phase transition phenomena in condensed matter. They could be described, we believe, using RG approach again, treating the Einstein equations as generating a RG flow on the space of initial data.

The real space renormalization techniques are mostly applicable to discretized models, based on a lattice. Therefore we now turn to describing a suitably discretized manifold model we are going to work with.

### 2.1 Discretized manifold model

The approach taken is that of a (3+1) formulation of General Relativity (GR) \cite{2}. Let us suppose we have a differentiable, compact three-manifold (without a boundary) $M$. Generally, in this case we will always assume that these manifolds possess certain natural constraints on their diameter and a suitably defined notion of curvature. The point of this requirement is that the manifolds, or more precisely the riemannian structures, can then be classified according to how they can be covered by small metric balls (to be defined later). Moreover the space of riemannian structures has some remarkable compactness properties. This is a classical result obtained by M. Gromov \cite{20}. On a set of riemannian structures it is possible to introduce a distance function, the Gromov distance, which roughly speaking enables one to say something about how close particular manifolds are to each other. For the riemannian manifolds which can be considered close to each other (in the sense of Gromov distance) it is possible to cover them with the balls arranged in similar packing configurations \cite{10}.

In order to define such coverings \cite{21}, let us parameterize the geodesics by arc-length, and for any point $p \in M$ let $d_M(x,p)$ denotes a distance function of the generic point $x$ from the chosen one $p$. Then for any given $\epsilon > 0$ it is always possible to find an ordered set of points \{p$_1$, ..., p$_N$\} in $M$, so that \cite{21}

\[ \begin{align*}
\text{i) } \text{the open metric balls (the geodesic balls) } B_M(p_i, \epsilon) & = \{ x \in M | d_M(x, p_i) < \epsilon \}, \quad i = 1, \ldots, N, \\
\text{cover } M; \text{ in other words the collection } & \{ p_1, \ldots, p_N \} \text{ is an } \epsilon\text{-net in } M. \\
\text{ii) the open balls } B_M(p_i, \epsilon/2), \quad i = 1, \ldots, N \text{ are disjoint, i.e., } & \{ p_1, \ldots, p_N \} \text{ is a minimal } \epsilon\text{-net in } M.
\end{align*} \]

It is fair to say that as a consequence of the compactness properties of the set of riemannian structures that we consider, for each “length scale $\epsilon$” there exists a finite number of “model”
geometries which describe, with an $\epsilon-$approximation, any given riemannian geometry. Namely, given a ball of a certain radius $> \epsilon$ in any riemannian manifold (with suitable restrictions on their volume, diameter and sectional curvature, as we have said earlier) there exists a ball metrically similar (up to an $\epsilon$ scale) in one of the model geometries which does not retain the details of the original manifold on scales smaller than $\epsilon$. Roughly speaking, $\epsilon$ is a measure of the typical curvature inhomogeneity with respect to the model background. Let us stress that this is a highly non-trivial result, in the sense that the metrical properties of the manifolds from an infinite dimensional set are, up to an $\epsilon$ scale, described by the metrical properties of just a finite number of model riemannian manifolds.

The $\epsilon-$nets underlying the balls-coverings precisely provide the discretized manifold model. This coarse-graining of a manifold according to Gromov is the most natural coarse-graining one can think of, pertinent for manifolds with a lower bound to the sectional curvature. This assumption does not limit the generality of our analysis which is basically motivated by a concrete physical problem, whose nature allows us to deal from the beginning with manifolds that are already in a certain sense quasi-homogeneous (Cf. comments on the solvability of the Ricci-Hamilton flow, later on).

In what follows, when speaking of balls we will always mean geodesic balls here.

### 2.2 An empirical averaging procedure

We assume that we have chosen a particular space-like hypersurface $\Sigma$ of the four-dimensional manifold, on which the average of a scalar function $f: \Sigma \to \mathbb{R}$ is given as

$$< f >_{\Sigma(g)} = \frac{\int_{\Sigma} f d\mu_g}{vol(\Sigma, g)},$$

where $vol(\Sigma, g) = \int_{\Sigma} d\mu_g$ and $\mu_g$ is the riemannian measure associated with the three-metric $g$ of $\Sigma$. Since at this stage we simply wish to put forward a few elementary geometrical considerations, we do not specify yet the choice of the hypersurface $\Sigma$ and we do not attribute any particular physical meaning to the function $f$.

If the geometry of $\Sigma$ is not known on a large scale, we cannot take (1) as an operational way of defining the average of $f$. From a more pragmatic point of view, supposing that we can only experience geometry in sufficiently small neighborhoods of a finite set of instantaneous observers, it makes much more sense to replace (1) with a suitable average based on the geometrical information available on the length scale of such observers.

For simplicity, given a finite set of instantaneous observers, located at the points $x_1, \ldots, x_N \in \Sigma$, we may assume that these susceptible to observation regions are suitably small geodesic balls of radius $\epsilon$, scattered over the hypersurface $\Sigma$ so as to cover it. In other words, we assume that $\{x_1, \ldots, x_N\}$ is a minimal $\epsilon-$net in $\Sigma$. Further, we denote by $U_\epsilon$ the corresponding set of geodesic balls $\{B_{\Sigma}(x_i, \epsilon)\}, i = 1, \ldots, N$. Then we can bound (1) as

$$< f >_{\epsilon/2} \frac{\sum h vol(B_h, \epsilon/2)}{vol(\Sigma, g)} \leq < f >_{\Sigma(g)} \leq < f >_{\epsilon} \frac{\sum h vol(B_h, \epsilon)}{vol(\Sigma, g)},$$

where
\[ <f> \equiv \frac{\sum_i \int_{(B_i, \epsilon)} f \, d\mu_g}{\sum_i \text{vol}(B_i, \epsilon)}, \tag{3} \]

and

\[ <f>_{\epsilon/2} \equiv \frac{\sum_i \int_{(B_i, \epsilon/2)} f \, d\mu_g}{\sum_i \text{vol}(B_i, \epsilon/2)}, \tag{4} \]

and where we explicitly indicated the dependence of the averages on a particular “covering”. This suggests to consider \(<f>_{\epsilon}\) as a suitable scale dependent approximation to \(<f>\).

There are certain problems lurking that we have to clear up. Obviously, there are “unwanted” details affecting the average function over the discretized manifold, as given by its partition with a collection of geodesic balls, the immediate one being the underlying discretization. The important question to ask is what happens to the average when we change the length scale. Depending on whether we are actually increasing or decreasing it, respectively less or more details of the underlying geometry, will be felt by the average values. The natural philosophy is that over scales big enough no details should be discerned since the homogeneity and isotropy prevails. This is the reason why on constant curvature spaces averaging is well defined since there one can move the balls freely and deform them, but by so doing no new geometric details that measure the inhomogeneities will be felt in the averaged values of quantities we are interested in.

A natural question to ask now is then how the geometry, specifically curvature inhomogeneities, should depend on scale so that the average over the balls is scale independent, or equivalently, how do we have to deform the geometry in order to achieve the scaling limit when size of the balls matters no more?

First, we give some details on how we calculate the averages according to (3). In order to do this, we employ a preferred system of coordinates on the set of balls \(\{B_i\}\) given by the local diffeomorphism

\[ \exp_x : T_x \Sigma \to \Sigma \]  

i.e., we make use of the exponential mapping

\[ \varphi_i \equiv \exp \big| \exp^{-1} B_i \equiv D_i : D_i \to B_i, \tag{6} \]

where \(D_i = D(x_i, \epsilon)\) is the ball in \(T_{x_i} \Sigma\).

On \(D_i\) we use polar coordinates and pull-back the riemannian measure accordingly, namely,

\[ \varphi_i^* (\mu_g) = \theta(t, x_i) dt \otimes dx_i, \tag{7} \]

4The transition from \((\Sigma, g)\), at a given moment of time \(t_o\), to a tangent space parallels the prescription in riemannian geometry for measuring [43]. Let \(U\) be a given neighborhood of an instantaneous observer in \(\Sigma\) and suppose it is so small that there is a neighborhood \(A\) of \(0 \in T_x \Sigma\) such that \(\exp_x : A(\subset T_x \Sigma) \to U(\subset \Sigma)\) is a diffeomorphism. One can then replace the considerations in \((U, g)\) by those in \(A\) (with the riemannian measure pulled back) via \(\exp^{-1}_x\). Namely, we can say that an instantaneous observer in \((\Sigma, g)\) observes the universe with the help of the exponential mapping, which just means projecting structures from an open neighborhood \(U \subset \Sigma\) of \(x\) by \(\exp^{-1}_x\) and treating them as structures on \(T_x \Sigma\).
where $dx_i$ denotes the canonical measure (euclidean volume form) on the unit sphere $D(x_i, 1) = S^2_1 \subset T_x \Sigma$ and where $dt$ is the Lebesgue measure on $\mathbb{R}$ ($t \geq 0$).

For $t$ small enough one can prove Puiseux’ formula
\[
\theta(t, x) = t^{n-1}(1 - \frac{1}{3} r(x)t^2 + O(t^2)),
\]
where $n = \text{dim} \Sigma$ and $r(x)$ is the Ricci curvature $Ric(g)$ (at the point $x$).

Using this result we have
\[
\int_{B(x_i,t)} f d\mu_g = \int_{S_1^2} f \theta(t, x) dx_i dt.
\]

Let us consider the asymptotic expansion with respect to $t$
\[
\int_{B(x_i,t)} f d\mu_g = \omega_n t^n [f(x_i) + \frac{t^2}{2(n+2)}(\Delta f(x_i) - \frac{R(x_i)}{3} f(x_i)) + O(t^2)],
\]
where $\omega_n$ is the volume of the unit ball of $\mathbb{R}^n$, $R$ is the scalar curvature at the center of the ball and $\Delta$ the Laplacian operator relative to the manifold.

Substituting $f = 1$ in the above formula we get the asymptotic expansion of the volume of a geodesic ball:
\[
\text{vol}(B(x_i, t)) = \omega_n t^n (1 - \frac{R(x_i)}{6(n+2)} t^2 + O(t^2)).
\]

These standard formulae are what we need in order to calculate how the average value behaves when we change the scale.

Since we are interested in discussing how $< f >_\epsilon$ behaves upon changing the radius of the balls $\{B(x_i, \epsilon)\}$, let us consider the average $< f >_{\epsilon_0 + \eta}$, with $\eta$ a positive number with $\eta/\epsilon_o \ll 1$. According to the formulae recalled above, we can write
\[
< f >_{\epsilon_0 + \eta} = \sum_i [f_i + (\frac{\Delta f_i - R_i f_i}{2(n+2)})(\epsilon_o + \eta)^2] + O((\epsilon_o + \eta)^4),
\]
where we have introduced somewhat simplified but otherwise obvious notation. Upon expanding this expression in $\eta$, we get to leading order ($O(\epsilon_o^2)$ in $\epsilon_o$, and $O([\frac{\eta}{\epsilon_o}]^2)$ in $\frac{\eta}{\epsilon_o}$),
\[
< f >_{\epsilon_0 + \eta} \approx < f >_{\epsilon_0} + \frac{1}{n+2} < \Delta f >_{\epsilon_0} \frac{2\eta}{\epsilon_o} + \frac{1}{3(n+2)} [< R >_{\epsilon_0} < f >_{\epsilon_0} - < Rf >_{\epsilon_0}] \epsilon_o \frac{2\eta}{\epsilon_o},
\]
where $< f >_{\epsilon_0}$ is the average of the function $f$ over the set of $N$ instantaneous observers $U_{\epsilon_0}$, (with similar expressions for $< R >_{\epsilon_0}$, $< Rf >_{\epsilon_0}$ and $< \Delta f >_{\epsilon_0}$). Thus, under a change of the cutoff we can write
\[
\epsilon_o \frac{d}{d\eta} < f >_{\epsilon_0 + \eta} |_{\eta/\epsilon_o = 0} = \frac{1}{n+2} < \Delta f >_{\epsilon_0} + \frac{1}{3(n+2)} [< R >_{\epsilon_0} < f >_{\epsilon_0} - < Rf >_{\epsilon_0}]
\]
to leading order. In the next sections we will discuss the consequences of these formulae and the connection of our averaging procedure with the Ricci-Hamilton flow.
3 The Renormalization Group view

3.1 Block variables and recursion relations

The real space RG technique is based on the recursive introduction of block variables. A method of “blocking” is trivial to introduce on regular lattices. In particular, in the case of the Ising model (2−dim) it consists of a subdivision of the spin system into cells, which supposedly interact in a similar way as the original spins. This can be done by introducing the block-spin variables via the majority rule or decimation procedure [6]. The behaviour of blocked lattice on large scales is equivalent to the behaviour of the original lattice corresponding to a different temperature. The slopes of the parameters’ surface close to the critical fixed point determine the macroscopic characteristics of the model (see e.g. [6]).

In the context of the problem we are considering, the application of RG method forces us to invent some analogue of Kadanoff’s blocking (block-spin transformation) applied to the geometry itself. This appears to be a difficult problem since in a general case when the geometry is curved the “lattice” itself takes on a dynamical rôle. Moreover, since the manifold Σ we are dealing with is compact, (closed, without boundary), we must implement a Renormalization Group strategy in a finite geometry, and thus the relevant phenomena are here related to finite size scaling. Roughly speaking, the size of our manifold is characterized by a length scale, say L, (actually the volume, the diameter and possibly bounds on curvatures), which is large in terms of the microscopic scale (the radius of the typical geodesic ball coverings we shall use in the blocking procedure). A continuous theory, describing the (universal) properties of the field f on Σ, arises when the correlation length associated with the distribution of f is large (of the order of L, or bigger). This being the underlying rationale, let us proceed and be guided by the formula (14).

Let us consider our system as nearly infinite, i.e. the manifold Σ divided by the collection of the geodesic balls of radius ϵm ≡ (m + 1)ϵo for m = 0, 1, 2, . . . and where ϵo, the chosen cutoff, is much smaller than the typical length scale associated with Σ, (this length scale can be identified with the injectivity radius of the manifold). Each ball will be labelled by k in the sequel and the original geodesic packing-covering is for m = 0. We can introduce this way a convenient notation for the integral of the generic function f over the ball B(xk, ϵm) as

\[ \psi_m(k; f) \equiv \int_{B(x_k, \epsilon_m)} f \, d\mu_g, \tag{15} \]

which can be seen as the block variable since it allows us to eliminate \( f \) from the distribution of the field \( f \) all fluctuations on scales smaller than the cutoff distance \( \epsilon_m \). We wish to emphasize that if the geometry of the ball \( B(x_k, \epsilon_m) \) is not flat, then the definition of \( \psi_m(k; f) \) can be interpreted as that of a weighted sum over a flat ball, namely,

\[ \psi_m(k; f) \equiv \int_{\exp^{-1}B(\epsilon_m)} f(t, x) \, dt \otimes dx, \tag{16} \]

where the weight \( \theta(t, x) \) is provided by Puiseux’ formula (8).

If we consider the covering of \( B(x_k, \epsilon_{m+1}) \) induced by the geodesic balls \( \{B(x_j, \epsilon_m)\} \), i.e.,
the collection of $N_k(m, m + 1)$ open sets $\{B(x_k, \epsilon_{m+1})\} \cap \{B(x_j, \epsilon_m)\}$, then the above definition of block variables can be written recursively in terms of the values the function $f$ takes correspondingly to balls of larger and larger radii. To this end, let us consider a partition of unity $\{\xi_h\}_{h=1, \ldots, N_k(0, m+1)}$, subordinated to the covering of the generic enlarged ball $B(x_k, \epsilon_{m+1})$, induced by the geodesic balls $\{B(x_h, \epsilon_o)\}$. Namely, a set of smooth functions such that: $0 \leq \xi_h \leq 1$, for each $h$; the support of each $\xi_h$ is contained in the corresponding $B(x_h, \epsilon_o)$; and $\sum_h \xi_h(p) = 1$, for all $p \in B(x_k, \epsilon_{m+1})$.

Under such assumptions, the block variables $\psi_m(k; f)$ can be written recursively as

$$
\psi_0(k; f) = \int_{B(x_k, \epsilon_o)} \xi_k f d\mu_g,
$$

$$
\psi_{m+1}(h; f) = \sum_k \psi_m(k; f).
$$

(17)

Indeed, we have

$$
\psi_1(h; f) = \int_{B(y_h, \epsilon_1)} f d\mu_g = \sum_k \int_{B(x_k, \epsilon_o)} \xi_k f d\mu_g
$$

$$
= \sum_k \psi_o(k; f),
$$

(18)

and

$$
\psi_2(j; f) = \int_{B(z_j, \epsilon_2)} f d\mu_g = \sum_h \int_{B(y_h, \epsilon_1)} \xi_h f d\mu_g
$$

$$
= \sum_h \sum_k \int_{B(x_k, \epsilon_o)} \xi_k \xi_h f d\mu_g
$$

$$
= \sum_h \psi_1(h; f),
$$

(19)

where we have exploited the fact that the functions $\{\xi_k \xi_i\}$ have $\xi_k \xi_i(p) = 0$, except for a finite number of indices $(k, i)$, and $\sum_k \xi_k(p) = 1$, for all $p \in \Sigma$. The above expressions readily generalize for every $m$.

More explicitly, we can write these block variables as
\[
\psi_{m+1}(k; f) = \int_{B(x_k, \epsilon_{m+1})} f \, d\mu_g = \omega_n \epsilon_m^N \sum_h \left[ f(h)\xi_h + \left( \frac{\Delta(f\xi_h)(h) - R(h)f(h)\xi_h/3}{2(n+2)} \right) \epsilon_m^2 + O(\epsilon_m^4) \right].
\] (20)

Notice that in terms of the block variables \( \psi_m(k; f) \) we can rewrite the empirical averaging (3) as

\[
\langle f \rangle_{\epsilon_m}(\psi_m) = \frac{\sum_k N(\epsilon_m) \psi_m(k; f)}{\sum_k N(\epsilon_m) \text{vol}[B(x_k, \epsilon_m)]},
\] (21)

where \( N(\epsilon_m) \) denotes the number of distinct \( B(x_k, \epsilon_m) \) balls providing a minimal \( \epsilon_m \)-covering of the manifold \( \Sigma \). Thus, when \( m \) is sufficiently large, the variation in \( \langle f \rangle_{\epsilon_m}(\psi_m) \) under a block transformation \( \psi_m(k; f) \to \psi_{m+1}(h; f) \) is given (to leading order) by

\[
\langle f \rangle_{\epsilon_{m+1}}(\psi_{m+1}) - \langle f \rangle_{\epsilon_m}(\psi_m) \simeq \frac{1}{n+2} < \Delta(f\xi_h) >_{\epsilon_m} \epsilon_m^2 \frac{1}{m+1} + \frac{1}{3(n+2)} [\langle R >_{\epsilon_m} < f\xi_h >_{\epsilon_m} - < Rf\xi_h >_{\epsilon_m}] \epsilon_m^2 \frac{1}{m+1}.
\] (22)

The above choice of block variables brings out the coupling between averaging a scalar field over a manifold and the presence of fluctuations in the curvature of the underlying geometry.

In order to be more precise, let us assume that the variables \( f(k) \) are randomly distributed according to some probability law \( P(\{f(k)\}) \) (later on we shall come back to this point with a definite prescription). Upon blocking the system and thus renormalizing the variables \( f(k) \) by increasing the scale size, the probability distribution \( P(\{f(k)\}) \) induces a corresponding probability distribution on the variables \( \psi_m \), viz., \( P(\{\psi_m(k; f)\}) \).

From equation (17) it is clear that if the geometrical properties of any two balls, \( B(x_i, \epsilon_m) \) and \( B(x_j, \epsilon_m) \) (with \( B(x_i, \epsilon_m) \cap B(x_j, \epsilon_m) = \emptyset \)), are not correlated then the corresponding block variables, \( \psi_{\bar{m}}(i; f) \) and \( \psi_{\bar{m}}(j; f) \), are uncorrelated. Such length scale \( L \equiv \epsilon_{\bar{m}} \) characterizes the correlation (or persistence) length of the manifold \( (\Sigma, g) \). It is a measure of the typical linear dimension of the largest ball exhibiting a correlated spatial structures. This correlation length can be seen in close analogy with the usual correlation length (usually denoted by \( \xi \)) in condensed matter systems. It depends there upon the coupling constants in particular upon temperature, and diverges to infinity at the phase transition point.

Since \( g \) plays here the rôle of a running coupling, or if you prefer, of “temperature”, the existence of a finite correlation length corresponds to a rather “irregular”, crumpled geometry (as seen on scales of the order of \( L \)), or, equivalently, a high temperature phase of our system.
According to the central limit theorem it follows, for \( m \) large enough, \((\epsilon_m \gg L)\), that the block variables \( \psi_m(k; f) \), being the sum of uncorrelated random variables, are normally distributed, (let us say around zero, for simplicity), with a variance

\[
E_P(\psi_m^2(i; f)) = N(\epsilon_o, m) \bar{\chi},
\]

where \( E_P(\ldots) \) denotes the expectation according to the probability law \( P(\{\psi_m\}) \), \( \bar{\chi} \) is related to the variance of the variables \( f(k) \), \( R(k) \), and \( N(\epsilon_o, m) \) denotes the number of \( \epsilon_o \)-balls in the \( m \)-ball \( B(x_i, \epsilon_m) \). 

Thus, irrespective of the details of the local distribution of the random variables \( f(k) \) and \( R(k) \), we can write for the distribution of \( \{\psi_m(k; f)\} \)

\[
dP(\{\psi_m\}) = \prod_k \left[ d\psi_m(k; f) \left( 2\pi N(\epsilon_o, m) \bar{\chi} \right)^{-1/2} \exp \left[ -\psi_m^2(k; f) / 2N(\epsilon_o, m) \bar{\chi} \right] \right].
\]

This shows that by rescaling the block variables \( \psi_m(k; f) \) according to

\[
\phi_m(k; f) \equiv \left[ N(\epsilon_o, m) \right]^{-1/2} \psi_m(k; f),
\]

we get new block variables with a finite variance as \( m \to \infty \), and for random metrics \( (\Sigma, g) \) we can write

\[
dP(\{\phi_m\}) = \prod_k \left[ d\phi_m(k; f) \left( 2\pi \bar{\chi} \right)^{-1/2} \exp \left[ -\phi_m^2(k; f) / 2\bar{\chi} \right] \right].
\]

The above remarks, paradigmatic of the real space Renormalization Group philosophy, show that the definition of a sensible blocking procedure, in our geometrical setting, consists of a transformation increasing the scale size, realized by passing from the variables \( f(k) \) to the variables \( \psi_m(k; f) \), (namely, by taking the average over all values of \( f \) in a larger and larger ball), followed by a rescaling obtained by dividing \( \psi_m(k; f) \) by a suitable power of the number, \( N(\epsilon_o, m) \), of elementary \( \epsilon_o \)-balls contained in the \( \epsilon_m \)-ball considered, (for random geometries this power is \( 1/2 \)). 

Following standard usage, and in order to arrive at an interesting geometrical notion of blocking, we assume that for a generic metric \( g \) this rescaling follows by dividing \( \psi_m(k; f) \) by \( \left[ N(\epsilon_o, m) \right]^{\omega_m} \), where \( \omega_m \) will in general depend on \( m \). Thus, the rescaled block variables of relevance are

\[
\phi_m(k; f) = \left[ N(\epsilon_o, m) \right]^{-\omega_m} \psi_m(k; f).
\]

The value of \( \omega_m \) will be fixed by the requirement that, as \( m \to \infty \), and for some (critical) metric \( g_{crit} \), (in general for an open set of such metrics), such normalized large scale block
variables have a limiting probability distribution with a finite variance. Namely,

$$\lim_{m \to \infty} P_m(\{\phi_m(k; f)\}) = P_\infty(\{\phi_\infty(f)\}),$$

$$E P_\infty(\phi_\infty^2(f)) = 1.$$  \hspace{1cm} (28)

Notice that if $(\Sigma, g)$ is a nice manifold, e.g. a constant curvature simply connected three-manifold, we are obviously expecting that the corresponding $\omega_m$ is

$$\omega_m(\Sigma, g) = 1.$$ \hspace{1cm} (29)

In general, we can assume that there is a set of critical metrics and a corresponding $\omega$, such that the above requirements are satisfied. Such critical metrics are not necessarily constant curvature metrics and the corresponding $\omega$ is not necessarily 1, (one may conjecture that $1/2 \leq \omega \leq 1$, as happens in the Renormalization Group analysis of many magnetic systems). At this stage, this is only a tentative assumption in order to arrive at an interesting concept of geometric Renormalization Group in our setting. Later on, we shall see that such assumptions are justified by exhibiting examples of such non-trivial metrics.

### 4 Averaging matter and geometry

Until now, our discussion has addressed mainly geometrical issues and the function $f$ entering the cut-off dependent averaging $< f >_\epsilon$ was not specified. Now we wish to apply the obtained results to the averaging of the matter sources, namely, the matter density $\rho$, the spatial stress tensor $s_{ab}$, and the momentum density $J_a$, entering in the phenomenological description of the matter energy-momentum tensor with respect to the instantaneous observers comoving with $\Sigma$, viz.

$$T_{ab} = \rho u_a u_b + J_a u_b + J_b u_a + s_{ab},$$ \hspace{1cm} (30)

where $u$ is a unit, future directed normal to the slice $\Sigma$, $(u^a u_a = -1)$.

In fact, since in the present epoch the universe is mainly matter dominated, i.e. the pressure can be safely neglected, it would in principle be justified to start our analysis with

$$T_{ab} = \rho u_a u_b + J_a u_b + J_b u_a,$$ \hspace{1cm} (31)

for the stress-energy tensor. Even the terms involving the momentum density can be eliminated if we pick up a sensible slicing $\Sigma_t$ (the comoving frame) provided that the cosmological matter fluid is an irrotational fluid in equilibrium. In any case, wishing to maintain the following discussion to a sufficient degree of generality, without particular restrictive assumptions on the matter sources, we assume in our analysis that the matter energy-momentum tensor has the perfect fluid form

$$T_{ab} = \rho u_a u_b + J_a u_b + J_b u_a + g_{ab}p,$$ \hspace{1cm} (32)
with a pressure $p$ which is a priori not-vanishing, and where $g_{ab}$ is the three-metric of $\Sigma$, namely, $g_{ab} = g^{(4)}_{ab} + n_a n_b$. $g^{(4)}_{ab}$ denoting the space-time metric.

As argued in the previous section, smoothing-out the matter sources as described by a set of instantaneous observers (represented by the three-dimensional hypersurface $\Sigma$), means eliminating from the distribution of such sources on $\Sigma$ all fluctuations on scales smaller than the cutoff distance $\epsilon$, leaving an effective probability distribution of fluctuations for the remaining degrees of freedom. The underlying philosophy being that this effective distribution has the same properties as the original one at distances much larger than $\epsilon$ (i.e. for fluctuations with wavelengths much larger than $\epsilon$).

In order to implement this idea along the geometrical lines discussed in the last section, we first need to better specify what we mean by the assignment of a collection of instantaneous observers (endowed with clocks) on $\Sigma$. Since we are adopting a Hamiltonian point of view, such observers are specified by the assignment, on the (abstract) three-manifold $\Sigma$, of the lapse function $\alpha$ and the shift vector field $\alpha^i$. The former provides the local rate of the coordinate clocks of such observers, while the latter is the three-velocity vector of the observers with respect to the set of instantaneous observers at rest on $\Sigma$.

The macroscopic variables of interest characterizing the matter sources are in the present framework, the matter density $\rho$, and the momentum density $J_i$. Actually, it would make more sense to consider the cosmological fluid phenomenologically, described by a pressure $p$, a baryon number density $n_b$, energy density $\rho$, specific entropy $s$, and temperature $T$; such variables being related by

$$dp = n_b dh - n_b T ds$$

and

$$h = \frac{p + \rho}{n_b},$$

where $h$ is the specific enthalpy. However, for simplicity, we shall in the sequel consider a barotropic fluid.

Thus, the field $f$ characterizing the matter sources, as described by the instantaneous observers on $\Sigma$, is given by

$$f \equiv \alpha \rho + \alpha^i J_i.$$  \hspace{1cm} (35)

Notice that $2\alpha p(\rho)$ is the Hamiltonian density of the fluid in the Hamiltonian formulation of Taub’s variational principle for relativistic perfect fluids \[39\]; we also assume that the dominant energy condition holds.

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5The fact that we are adopting a stress tensor with a non-vanishing energy flux can be traced back to the observation that even in an exact $k = 0$ or $k = \pm 1$ FLRW universe there can be a non-zero particle flux tilted with respect to the frame of the fundamental observers, even though $T_{ab}$ has the perfect fluid form. This is possible if the macroscopic quantities (e.g. a particle flux) are derived from kinetic theory with anisotropic distribution function $f(x^i, p^\mu)$ \[40\].
The averaging of the matter sources along the lines described in the previous section would then require considering a finite set of instantaneous observers, \( \{x_1, \ldots, x_N\} \), located on \( \Sigma \) and setting a standard for the cutoff distance \( \epsilon_o \) over which the (experimental) distribution of matter sources, (i.e. the probability that the matter variables, \( \rho(i) \) and \( J(i) \), conform to a given distribution \( \rho(i)dp, J(i)dJ \)), is determined. Then we proceed with the blocking prescription for eliminating unwanted degrees of freedom and consider the behaviour as the averaging regions become larger and larger.

We may decide to treat the riemannian geometry of \( \Sigma \) as uncoupled with the matter sources if we are simply interested in smoothing-out the sources, or if we wish to consider fluctuations of the matter as essentially uncoupled to the fluctuations in geometry. Namely, the curvature fluctuations appearing in the definitions of the block variables, \( \psi_m(k; f) \) and \( \phi_m(k; f) \), can be thought of, under appropriate circumstances, as independent random variables with a given distribution.

In general, however, the sources are coupled to the gravitational field, and we ought to treat the full dynamical system, the cosmological fluid plus the geometry of \( \Sigma \), in the procedure of blocking. Moreover, we should bear in mind that without taking explicitly into account the backreaction of the geometry one cannot really provide a reasonable averaging procedure for the sources.

In order to do so, we consider for a given cutoff distance \( \epsilon_o \), the variables \( \rho(k) \) and \( J(k) \) associated with a minimal geodesic ball covering \( \{B(x_k, \epsilon_o)\}_{k=1}^N \). The change of the cutoff is naturally realized by considering balls \( \{B(x_l, \epsilon_m = (m+1)\epsilon_o)\} \) with \( m = 0, 1, \ldots \). The block variables, \( \psi_o(k; \rho, J) \) and \( \psi_m(k; \rho, J) \), are defined according to (17) with \( f \) given by (35). This transformation can be seen as thinning-out the degrees of freedom which is at the heart of any coarse-graining.

The original cutoff \( \epsilon_o \) is chosen to set the scale over which General Relativity is experimentally verified and it can be taken as, say, the scale of planetary systems. We can thus safely assume that the Einstein field equations hold on that scale. It is however rather impossible to provide a mathematical model of the distribution of matter in the universe going down to such fine scales; besides this task would be impractical. What one does instead is to use continuous functions assuming that they represent appropriately “volume averages”. The results of such an averaging in an inhomogeneous medium obviously depend on the scale. The point is that if the Einstein equations hold on the scale where they have been verified (here taken to be that of the planetary scale), then they do not seem to hold \textit{a priori} on larger, cosmological, scales that require averaging. To see this, notice that the Einstein tensor \( \tilde{G}_{\mu\nu} \), calculated from an “averaged” metric \( \tilde{g}_{\mu\nu} \), cannot be equal to the Einstein tensor \( \bar{G}_{\mu\nu} \) which was first calculated \( \dot{\psi} \) from the fine-scale metric \( g_{\mu\nu} \) and then averaged. This is so due to the non-commutativity of “averaging the metric” with calculating the Einstein tensor being strongly non-linear in the metric components.

Below we shall assume a Hamiltonian point of view. Then the probability \( P(\{\rho(k), J(k)\}) \) that the matter variables, according to the records of the instantaneous observers \( \{B(x_k, \epsilon); \alpha(x_k), \alpha^i(x_k)\} \) in \( \Sigma \), take on some particular set of values, \( \{\bar{\rho}(k), \bar{J}(k)\} \), is given by the equation

\[
P(\{\psi_o(k; \rho, J)\}_{1,\ldots,N}) = \frac{1}{Z} \exp[-H(\{\rho(k), J(k)\})],
\]

(36)
where $Z$ is a normalization factor, and $H(\{\rho(k), J(k)\})$ is the Hamiltonian associated with the matter variables $\rho$ and $J$. Namely,

$$P(\{\psi_o(k; \rho, J)\}_{1, \ldots, N}) = \exp\left[{-\alpha \rho + \alpha^h J_h} \right] \sum_{\rho, \ldots} \exp\left[{-\alpha \rho + \alpha^h J_h} \right],$$

(37)

where $\sum_{\rho, \ldots}$ denotes summation over all possible configurations of the matter variables $\rho$ and $J$, that can be experienced by the instantaneous observers $\{B(x_k, \epsilon); \alpha(x_k), \alpha^i(x_k)\}$ in $\Sigma$.

The validity of General Relativity at this scale implies that this Hamiltonian is a part of $H_{\text{ADM}}(\{\rho(k), J(k)\})$, the Arnowitt-Deser-Misner Hamiltonian, associated with the data of the three-geometry $g_{ab}$ of $\Sigma$, its conjugate momentum $\pi_{ab}$ and of the matter variables $\rho$ and $J$, evaluated in correspondence with the $\{B(x_k, \epsilon)\}$-approximation associated with the net of points $\{x_k\}$. Namely,

$$H_{\text{ADM}}(\{\rho(k), J(k)\})\big|_{\epsilon} = \alpha H(g, \pi, G, \rho) + \alpha^h H_h(g, \pi, G, J) \big|_{\epsilon} \sum_i \text{vol}(B(x_i, \epsilon)),$$

(38)

where

$$H(g, \pi, G, \rho) = (\det(g))^{-1/2} [\pi_{ab} \pi_{ab} - \frac{1}{2} (\pi^a_a)^2] - (\det(g))^{1/2} R(g) + 8\pi (\det(g))^{1/2} G\rho,$$

(39)

and

$$H_a(g, \pi, G, J) = -2\pi_{a,b}^h + 16 \pi (\det(g))^{1/2} G J_a,$$

(40)

and where the momentum conjugate to the three-metric $g_{ab}$ is given in terms of the second fundamental form $K_{ab}$ of the embedding of $\Sigma$ in the resulting space-time, as

$$\pi^{ab} = (\det(g))^{1/2} (K^{ab} - K^c_c g^{ab}).$$

(41)

We wish to recall that the lapse and the shift appear in the Hamiltonian as arbitrary Lagrange multipliers (their evolution is not specified by the equations of motion), and as such they enforce the constraints

$$H(g, \pi, G, \rho) = (\det(g))^{-1/2} [\pi_{ab} \pi_{ab} - \frac{1}{2} (\pi^a_a)^2] - (\det(g))^{1/2} R(g) + 8\pi (\det(g))^{1/2} G\rho = 0$$

(42)

and

$$H_a(g, \pi, G, J) = -2\pi_{a,b}^h + 16 \pi (\det(g))^{1/2} G J_a = 0.$$  

(43)

As is well known, these constraints are related to the invariance of the theory under the (four-dimensional) diffeomorphism group of the space-time resulting from the evolution of the initial data satisfying them. The momentum constraint, $H_a(g, \pi, G, J) = 0$, generates the (spatial) diffeomorphisms into $\Sigma$, while the Hamiltonian constraint, $H(g, \pi, G, \rho) = 0$, generates
the deformation of the manifold \( \Sigma \) in the resulting space-time, i.e. the dynamics. (Notice that such deformations can be interpreted as four-dimensional diffeomorphisms only after that space-time has been actually constructed).

The Hamiltonian \( \mathcal{H}_{ADM}(\{\rho(k), J(k)\}) \), apart from the lapse and the shift, depends on the three-metric \( g_{ab} \) on \( \Sigma \) and its conjugate momentum \( \pi_{ab} \), the gravitational coupling \( G \), and on the configuration which the matter variables, \( \rho(k) \) and \( J(k) \), take on the set of instantaneous observers, \( \{B(x_k, \epsilon); \alpha(x_k), \alpha'(x_k)\} \), chosen to describe the distribution of matter at the given length scale \( \epsilon \).

The basic question to understand is how the block transformations,

\[
\{\phi_m(k; \rho, J)\} \rightarrow \{\phi_{m+1}(k; \rho, J)\},
\]

followed by rescaling, affect the Hamiltonian associated with the matter variables \( \rho \) and \( J \), and then discuss how this in turn affects the full Hamiltonian \( \mathcal{H}_{ADM} \).

Starting from the probability distribution \( P(\{\psi_0(k; \rho, J)\}_{1,...,N}) \) we can inductively define the probability distribution \( P^{(m+1)}(\{\psi_{m+1}(k; \rho, J)\}) \) of the block variables \( \{\psi_{m+1}(k; \rho, J)\} \) (and of the corresponding rescaled variables \( \{\phi_{m+1}(k; \rho, J)\} \)). Since the block variables \( \{\psi_{m+1}(k; \rho, J)\} \) are recursively obtained from the knowledge of the block variables at the \( m^{th} \) stage, \( \{\psi_m(k; \rho, J)\} \), the distribution \( P^{(m+1)}(\{\psi_{m+1}(k; \rho, J)\}) \) only depends on the knowledge of \( P^{(m)}(\{\psi_{m}(k; \rho, J)\}) \). We can formally write

\[
P^{(m+1)}(\{\psi_{m+1}(k; \rho, J)\}) = \sum_{\{\psi_{m}(k; \rho, J)\}} P^{(m)}(\{\psi_{m}(k; \rho, J)\}),
\]

where the sum is over the probabilities of all the configurations \( \{\psi_{m}(k; \rho, J)\} \) consistent with the configuration \( \{\psi_{m+1}(k; \rho, J)\} \) of the block variables.

As usual, this allows us to define the effective Hamiltonian for the matter variables after \( m \) iterations of the block transformation, according to

\[
P^{(m)}(\{\psi_{m}(k; \rho, J)\}) \equiv \frac{1}{Z^{(m)}} \exp[-H^{(m)}(\{\psi_{m}(k; \rho, J)\})],
\]

where

\[
Z^{(m)} \equiv \sum_{\{\psi_{m}(k; \rho, J)\}} \exp[-H^{(m)}(\{\psi_{m}(k; \rho, J)\})].
\]

Such \( H^{(m)}(\{\psi_{m}(k; \rho, J)\}) \) are defined up to an additive constant term (e.g. \( \mathbb{1} \)). If we also stipulate, as is standard usage in the Renormalization Group approach, that the effective Hamiltonian \( H^{(m+1)}(\{\psi_{m+1}(k; \rho, J)\}) \) takes the same functional form as \( H^{(m)}(\{\psi_{m}(k; \rho, J)\}) \), i.e., if

\[
P^{m+1}(\{\psi_{m+1}(k; \rho, J)\}_{1,...,N}) = \frac{\exp[-<\alpha \rho^{(m+1)} + \alpha^h J_h^{(m+1)}>_{\epsilon_{m+1}}]}{\sum_{\rho, J} \exp[-<\alpha \rho^{(m+1)} + \alpha^h J_h^{(m+1)}>_{\epsilon_{m+1}}]},
\]
then this indeterminacy can be transferred to the effective matter variables $\rho^{(m+1)}$ and $\mathbf{J}^{(m+1)}$, in terms of which $H^{(m+1)}(\{\psi_{m+1}(k; \rho, \mathbf{J})\})$ is defined.

It is immediately checked that such effective matter variables are defined by (46) up to the transformations
\[
\begin{align*}
\rho^{(m+1)} &\to \rho^{(m+1)} + h, \\
\mathbf{J}^{(m+1)} &\to \mathbf{J}^{(m+1)} + \mathbf{v},
\end{align*}
\]
where $h$ and $\mathbf{v}$ are, respectively, a scalar function (sufficiently regular) and a vector field defined on $\Sigma$.

Among the various normalization conditions that we may adopt, in order to avoid the indeterminacy connected with (49), the natural one comes about by requiring the divergence and the Hamiltonian constraints to hold at each stage of the renormalization. This being the case, upon coupling matter to the geometry, the lapse $\alpha$ and the shift $\alpha^i$ maintain their role of Lagrange multipliers enforcing the (four-dimensional) diffeomorphism invariance of the theory.

In this way, the renormalization is set up so that the constraints, if initially satisfied, will remain satisfied at every renormalization step. The three-metric $g$, (and the second fundamental form), in its role of running coupling describing the interaction between matter and geometry, is then governed through (Coulomb like) effects that are expressed by these constraint equations. In this sense, the Coulomb like part of the gravitational interaction is driving the renormalization mechanism, and such requirements imply that the full effective Hamiltonian, (matter plus geometry), takes on the standard ADM form pertaining to gravity interacting with a barotropic fluid at every stage of the renormalization.

According to the remarks above, we can consider as an independent parameter in the (effective) fluid Hamiltonian $H^{(m)}(\{\psi_{m}(k; \rho, \mathbf{J})\})$ the three-metric $g_{ab}^{(m)}$ of the three-manifold $\Sigma$, whereas the matter density $\rho^{(m)}$ and the current density $\mathbf{J}^{(m)}$ are at each stage connected to $g_{ab}^{(m)}$ and $K_{ab}^{(m)}$ by the Hamiltonian and divergence constraints that hold at each stage. Then the effect of the renormalization induced by the blocking procedure $\{\psi_{m}(k; \rho, \mathbf{J})\} \to \{\psi_{m+1}(k; \rho, \mathbf{J})\}$ and the corresponding rescaling, can be symbolized as a non-linear operation acting on the metric $(g_{ab}^{(m)})$ so as to produce the metric $(g_{ab}^{(m+1)})$, i.e.
\[
(g_{ab}^{(m+1)}) = \mathcal{R}(g_{ab}^{(m)}),
\]
whereas the renormalization of the second fundamental form $K_{ab}$ is generated by the linearization of (50), as will be discussed in section 5.

This way, this renormalization transformation $\mathcal{R}$ defines a trajectory in the space of riemannian metrics of $\Sigma$. One moves on such trajectory by discrete jumps. However, in what follows we shall replace such discrete dynamical system by a smooth dynamical system, describing renormalizations of the parameters $(g_{ab})$ and $(K_{ab})$.

The deformation of the initial data set for the Einstein field equations, symbolically denoted by $\mathcal{R}$ above in (50), realizes in fact a formal, (at least at this stage), mapping between the initial
data sets for the field equations. As we have seen above, this renormalization acts in such a way that at each its step the deformed data satisfy the constraints. The time evolution, in turn, of any such data set generates a one-parameter family of solutions to the field equations which interpolates between the initial inhomogeneous space-time to be averaged-out and its renormalized (deformed) counterpart.

4.1 The Ricci–Hamilton flow

In order to replace the discrete operator \( R \), describing the effect of renormalization of \( g_{ab} \), with a continuous flow we can start by discussing some geometrical implications of equation (14). According to the Renormalization Group analysis of the previous section, they follow by considering the average \( < f >_\epsilon \) as a functional of the metric and thinking of the metric \( g_{ab} \) as a running coupling constant, depending on the cut-off. In this connection, it can be verified that we can equivalently interpret (14) as obtained by considering the variation of \( < f >_\epsilon \) under a suitable smooth deformation of the background metric \( g_{ab} \), rather than by deforming the (euclidean) radius of the balls \( \{ B(x_i, \epsilon) \} \).

As a matter of fact, we can equivalently rewrite the second term on the right hand side of (14) as

\[
<R >_\epsilon < f >_\epsilon - < R f >_\epsilon = -D < f >_\epsilon \frac{\partial g_{ab}}{\partial \eta},
\]

where \( D < f >_\epsilon \frac{\partial g_{ab}}{\partial \eta} \) denotes the formal linearization of the functional \( < f >_\epsilon \) in the direction of the symmetric 2–tensor \( \frac{\partial g_{ab}}{\partial \eta} \), and where

\[
\frac{\partial g_{ab}(\eta)}{\partial \eta} = \frac{2}{3} < R(\eta) > g_{ab}(\eta) - 2R_{ab}(\eta),
\]

\( R_{ab}(\eta) \) being the components of the Ricci tensor \( Ric(g(\eta)) \), and \( < R(\eta) > \) is the average scalar curvature given by

\[
< R(\eta) > = \frac{1}{vol(\Sigma)} \int_\Sigma R(\eta) d\mu_{\eta}.
\]

Indeed, the linearization of \( < f >_\epsilon \) in the direction of the generic 2–tensor \( \frac{\partial g_{ab}}{\partial \eta} \) is provided by

\[
D < f >_\epsilon \frac{\partial g_{ab}}{\partial \eta} = \frac{1}{2} < fg_{ab} \frac{\partial}{\partial \eta} g_{ab} >_\epsilon - \frac{1}{2} < f >_\epsilon < g_{ab} \frac{\partial}{\partial \eta} g_{ab} >_\epsilon,
\]

so (51) follows, given the expression (52) for \( \frac{\partial g_{ab}}{\partial \eta} \).

According to what has been said in the previous paragraphs, the effective distribution of matter sources, according to a set of instantaneous observers \( \{ B(x_i, \epsilon) \} \), is characterized by the underlying three-geometry thought of as an effective parameter depending on the cutoff \( \epsilon \). Since the Hamiltonian and the divergence constraints hold at each stage of the renormalization procedure, (they fix the effective Hamiltonians which otherwise are undetermined up to a constant factor), the renormalization of the matter fields is intrinsically tied with the renormalization of the three-metric.
The invariance of the long distance properties of the matter distribution, under simultaneous change of the cutoff $\epsilon$ and the parameter $g_{ab}$, can be expressed as a differential equation for the effective Hamiltonian $H(\rho, J)$, (actually for the partition function associated with this effective Hamiltonian), namely,

$$[-\epsilon \frac{\partial}{\partial \epsilon} + \beta_{ab}(g) \frac{\partial}{\partial g_{ab}}] \sum_{\rho, J} \exp[-H(\rho, J)] = 0. \quad (55)$$

Recall that $H(\rho, J)$ is explicitly provided by (in the given approximation)

$$H(\rho, J) = \langle \alpha p(\rho) + \alpha^i J_i > \epsilon \sum_h \text{vol}(B(x_h, \epsilon)), \quad (56)$$

where the average $\langle \alpha p(\rho) + \alpha^i J_i > \epsilon$ is a functional of the three-metric $g_{ab}$, here thought of as the running coupling constant.

Thus, in order that the equation (55) is satisfied, it is sufficient that

$$[-\epsilon \frac{\partial}{\partial \epsilon} + \beta_{ab}(g) \frac{\partial}{\partial g_{ab}}] \langle \alpha p(\rho) + \alpha^i J_i > \epsilon = 0. \quad (57)$$

The Renormalization Group equation (57) states that increasing the cutoff length (i.e., the radius of the averaging balls) from $\epsilon$ to $\epsilon^\gamma \epsilon$, while deforming the metric $g_{ab}$ by flowing along the beta function $\beta_{ab}(g)$ for a parameter-time $\gamma$, has no net effect on the long distance properties of the considered system.

According to the above remarks, and equation (21), it can be verified that the beta function yielding for (57), defined by

$$\epsilon \frac{\partial}{\partial \epsilon} g_{ab} = \beta_{ab}(g), \quad (58)$$

is exactly provided by (54), namely,

$$\beta_{ab}(g) = \frac{\partial g_{ab}(\eta)}{\partial \eta} = \frac{2}{3} \langle R(\eta) > g_{ab}(\eta) - 2R_{ab}(\eta), \quad (59)$$

where the parameter $\eta$ is the logarithmic change of the cutoff length $\epsilon$.

Since the manifold $\Sigma$ is compact, (52) has to be interpreted as a Renormalization Group equation in a finite geometry, and thus the relevant phenomena are here related to finite size scaling (see e.g. [8]). A continuous theory, describing the (universal) properties of the cosmological sources and of the corresponding geometry, may arise when the correlation length associated with the distribution of cosmological matter is of the order of the size of the underlying manifold.

The metric flow (52) is known as the Ricci-Hamilton flow [22], studied in connection with the quasi-parabolic flows on manifolds; quite independently it has been discussed in investigating the Renormalization Group flow for general $\sigma$-models (see e.g. [34] and references quoted
therein). The Ricci-Hamilton flow is always solvable \[22\] for sufficiently small $\eta$ and has a number of useful properties, (apart from being volume preserving which is simply a consequence of the normalization chosen), namely, any symmetries of $g_{ab}(\eta_0)$ are preserved along the $g_{ab}(\eta)$ flow for all $\eta > \eta_0$, and the limiting metric (if attained) $\overline{g}_{ab} = \lim_{\eta \to \infty} g_{ab}(\eta)$ has constant sectional curvature. Thus equation (52), with the initial condition $g_{ab}(0) = g_{ab}$, defines (when globally solvable) a smooth family of deformations of the initial three-manifold, deforming it into a three-space of constant curvature.

4.2 Fixed points and basins of attraction

Thus the point of the above discussion is that in order to arrive at a fixed point of the RG equation (14), the geometry has to be deformed according to the Ricci-Hamilton flow (52). In this setting of the problem, the Ricci-Hamilton equations appear naturally and in fact the approach proposed, enables us to attach a physical meaning to them within the coarse-graining picture. This element was lacking in \[9\] where Hamilton’s theorem appeared rather \textit{ad hoc}. On the other hand, our approach demonstrates that the smoothing issue is deeply connected with the geometry and exhibits how this relationship works. First some general remarks.

A fixed point is a point in the coupling constants space that satisfies

$$g^*_{ab} = R(g_{ab}),$$

i.e. it is mapped onto itself by RG transformation.

Under RG transformation length scales are reduced by a factor $m + 1$. Namely, for the block variables, the correlation length measured in units appropriate for them, $L_m(\epsilon_m)$, is smaller than the correlation length $L_o$ of the initial system measured in units of $\epsilon_o$. The actual physical value of the correlation length $L$ is of course unchanged by the process of blocking, thus $L = L_m(\epsilon_m) = L_o\epsilon_o$, so

$$L_m = \frac{L_o}{m + 1}.$$  \hfill (61)

Since $L_m < L_o$, the system with Hamiltonian $\mathcal{H}^{(m)}$ must be further from criticality than the original system. Thus, we conclude that the system is at a new effective reduced “temperature” $g_{ab}^{(m)}$. At a fixed point (60), $L^*$ can be only zero or infinity since then $L^* = L^*/(m + 1)$.

As is standard in RG analysis, we will refer to a fixed point with $L = \infty$ as a critical fixed point, when $L = 0$ we will call it trivial. Each fixed point has its domain or basin of attraction, namely, the points in the coupling constants space in such a basin necessarily flow towards and end up at the fixed point, after an infinite number of iterations of RG transformation.

Let us, for the purpose of clarity, employ for a moment the ferromagnetic analogy. In this case, for a system exhibiting a phase transition, there are two attractive fixed points. One is the high-temperature fixed point which attracts each point with $T > T_{\text{crit}}$ in the coupling constants space, and it corresponds to the effective Hamiltonian for the system as $T \to \infty$. In this phase the variables assume random values and are uncorrelated. Upon a sensible blocking of such a system the probability distribution of the block variables remains unchanged.

The second fixed point is the low-temperature fixed point which is the effective Hamiltonian for the system when $T \to 0$. This corresponds to a system in a complete spins alinement and
the block variables are ordered then. Every point corresponding to \( T < T_{\text{crit}} \) is eventually attracted to this fixed point.

Across the Hamiltonian space there should exist then a surface, the so-called critical surface, which separates the effective Hamiltonians flowing to the high-temperature fixed point from those flowing to the low-temperature fixed point. Notice that the word surface here has rather a heuristic meaning. Indeed, the set of metrics, separating those flowing towards a “low-temperature” fixed points from those flowing towards a “high-temperature” fixed point, has quite a complex structure whose understanding is deeply connected with some, yet unsolved, conjectures in 3D-manifold topology, (see below for more details).

If we now choose to start with a point on the critical surface, then upon RG transformation it will stay within the critical surface. There is a possibility that, as the number of iterations of RG goes to infinity, the Hamiltonian will tend to a finite limit \( \mathcal{H}^* \). This point is the critical fixed point and within the critical surface it is attractive, (this is roughly speaking the basic mechanism for universality), along the direction out of the critical surface it is repulsive. This fixed point is related to the singular critical behaviour of the system due to the fact that all points in its basin of attraction have infinite correlation length. The simplest case is when the fixed points are isolated points, but it is also possible to have lines or surfaces of fixed points.

With these preliminary remarks along the way, let us discuss how some of the above general characteristic of the Renormalization Group flows find their proper counterpart in the particular situation we are analysing.

In the previous paragraph we showed that it was possible to replace the discrete operation of increasing the scale size of the observational averaging region by a “transformation”, smoothly deforming the background metric \( g_{ab} \), which turned out to be the Ricci-Hamilton flow. In this setting, following the example above, we would like to adopt the fundamental hypothesis linking RG to the critical phenomena, namely, the existence of a “critical” metric (on the critical surface) \( g_{ab}^{\text{crit}} \), and of a “fixed point” metric \( g_{ab}^\star \), such that

\[
\lim_{m \to \infty} \mathcal{R}^{(m)}(g_{ab}^{\text{crit}}) = g_{ab}^\star.
\]  

(62)

In (62) \( g_{ab}^\star \) is a mathematical object invariant under RG and we assume that \( g_{ab}^{\text{crit}} \) represents the physics (in a sense to be clarified further) of curvature fluctuations of a manifold at its critical point (“critical manifold”). Since the Ricci-Hamilton flow can be interpreted as a dynamical system on a set of closed riemannian manifolds, we can adopt the following interpretation of (62). Suppose, that we look at our manifold through a “microscope” and are able to discern the curvature fluctuations down to a size \( \epsilon_m \). Then \( \mathcal{R}^{(m)} \) represents the operation of decreasing the magnification factor, by \( m \) say, i.e. the sample seen appears to shrink by this factor. We have to assume that the system is sufficiently large so that the edges of the sample will not appear in the view. The hypothesis (62) states that if we decrease the magnification by a sufficiently large amount, we shall not see any change if we decrease it even further.

We are going to describe to what extent such hypothesis holds.
We already said that the Ricci-Hamilton flow (52), while always solvable for sufficiently small \( \eta \), may not yield for a non-singular solution as \( \eta \) increases. Hamilton noticed that there are patterns in the kind of singularities that may develop. Typically, the curvature blows up but in a very regular way (e.g. for \( S^1 \times S^2 \) with the standard symmetric metric). This has led him to a research program which, roughly speaking, amounts to saying that any three-manifold can be decomposed into pieces on which the Ricci-Hamilton flow is global and thereby, each of these pieces can be smoothly deformed into a locally homogeneous three-manifold. Singularities may develop in the regions connecting the smooth-able pieces, but such singularities should be of a finite number of types and all of a rather symmetric nature (namely, if they are blown up, they should be associated with symmetric manifolds as e.g. \( S^1 \times S^2 \) [11, 25]).

It may be said that Hamilton’s program is an analytic approach to prove Thurston’s conjecture, which claims that any closed three-manifold can be cut into pieces, such that each of them admits a locally homogeneous geometry. The rationale, underlying this analytical program towards Thurston’s geometrization conjecture, lies in the above nice structural properties of the Ricci-Hamilton flow. Several steps are involved in this program. Let us briefly recall them since, even if some of them are yet unproven, they shed light on the assumption (62) and on (52) when interpreted as a Renormalization Group flow.

The first step is the assignment of an arbitrary metric \( g \) on the three-manifold \( \Sigma \). In the Renormalization Group approach, this corresponds to picking up a vastly inhomogeneous and anisotropic geometry for the physical space, (here equivalent to the high-temperature phase). Such a choice may not conform to the actual quasi-homogeneous three-geometry of the physical space as is experienced now. However this quasi-homogeneity, in our opinion, may be related to the possibility that the actual universe is near criticality, a circumstance that we want to discuss rather than assume from the outset.

The second step is to deform this metric \( g \) via the Ricci-Hamilton flow (52). In general, this flow develops local singularities which should be related to the manifold decomposition in Thurston’s conjecture. Away from each of the local singularities it is conjectured (but not yet proved) that the Ricci-Hamilton flow approaches that of a locally homogeneous geometry in each disconnected piece.

This picture may be consistent with our blocking procedure yielding for (52) as the Renormalization Group flow. According to the analysis, carried out in section 3.1, a highly inhomogeneous and anisotropic geometry \((\Sigma, g)\) can be characterized by minimal geodesic ball coverings \( \{B(x_i, \epsilon_0)\} \), whose balls are to a good approximation largely uncorrelated when seen on a suitable scale. This means that the values of the scalar curvatures, \( R(i) \), evaluated at the centers of the balls of the covering, are random variables. Upon enlarging the balls and rescaling, certain regions of the manifold might be such that correlations develop among the corresponding \( R(i) \), whereas in other regions, no matter how we enlarge the balls and rescale, the \( R(i) \) will remain independently distributed random variables. The former regions, exhibiting a persistence length, are those that should approach a locally homogeneous geometry under blocking and rescaling (i.e., under the flow (52)). The latter regions should rather develop singularities under (52), since no matter how much we block and rescale, the curvature will maintain its white noise character.
The third step is to study the behavior of (52) for the locally homogeneous geometries, for this accounts for the structure of the critical set of (52). This has been accomplished [25], and one finds that, depending on the initial locally homogeneous geometry, the Ricci-Hamilton flows either, (i) converges to a constant curvature metric, (ii) asymptotically approaches (as $\eta \to \infty$) a flat degenerate geometry, of either two or one dimensions (pancake or cigar degeneracy), with the curvature decaying at the rate $1/\eta$, or, (iii) hits a curvature singularity in finite time, with this singularity being that of the Ricci-Hamilton flow for the standard metric on $S^2 \times S^1$. Note that constant curvature geometries always occur whenever the manifold can support them, (in dimension three, constant curvature manifolds and Einstein manifolds are synonymous). It is also quite interesting to note that the Ricci-Hamilton flow of homogeneous metrics usually (with a few exceptions) tends to approach or converge to the maximally symmetric homogeneous metric in the class considered (see [25] for details).

Interpreting (52) as the Renormalization Group flow, it follows that locally homogeneous geometries evolving under (52) towards an isolated constant curvature manifold, are sinks describing a stable phase of the corresponding cosmological model. For instance a FLRW model (with closed spatial sections) characterizes such a phase. Non-isolated constant curvature manifolds (e.g. flat three-tori) provide less trivial examples of limiting behaviour of (52), (see e.g. [23, 11]). The locally homogeneous manifolds non-admitting Einstein manifolds (i.e. there are no left-invariant Einstein metrics on the group SL(2, IR)), provide even more interesting behavior. In this case, a metric renormalized under the action of (52) develops degeneracies and one gets, in our setting, an effective cosmological model with spatial sections of lower dimensionality.

All these limit points of (52), either fixed or not, have their own basins of attraction. Rigorously speaking, they are the sets of three-metrics flowing under (52) to the respective limit points, briefly discussed above. There are nine such basins of attraction, corresponding to the nine classes of homogeneous geometries that can be used to model (by passing to the universal cover) the local inhomogeneous geometries on closed three-manifolds. By labelling these classes according to the minimal isometry group of the geometries considered we distinguish the following basins (here we follow closely the exposition in [25]):

(i): The $\mathbb{R}^3$-Basin. It contains all three-metrics flowing towards the homogeneous flat $\mathbb{R}^3$ metrics. This basin is eventually attracted by flat space, (flat tori, when reverting to the original manifold rather than to its universal cover).

(ii): The $SU(2)$-Basin. It contains all three-metrics flowing towards the three-parameter family of homogeneous $SU(2)$ metrics. This class admits Einstein metrics, in particular the round metrics on the three-sphere. This basin is exponentially attracted to the round three-sphere, (modulo identifications). It is the basin of attraction yielding for closed FLRW cosmological models.

(iii): The $SL(2, \mathbb{R})$-Basin. It contains all three-metrics flowing towards the three-parameter family of homogeneous $SL(2, \mathbb{R})$ metrics. This class does not admit Einstein metrics. This basin goes degenerate, yielding for a pancake degeneracy whereby a two-dimensional geometry
survives: two of the components of the metric increase without bound while the other shrinks to zero.

(iv): The Heisenberg-Basin. It contains all three-metrics flowing towards the three-parameter family of homogeneous Nil-metrics. Again, this class does not contain any Einstein metrics. This basin too undergoes a pancake degeneracy.

(v): The $E(1, 1)$-Basin, where $E(1, 1)$ is the group of isometries of the plane with flat Lorentz metric. It contains all three-metrics flowing towards the three-parameter family of homogeneous Solv-metrics. Also this basin fails to contain Einstein metrics. This basin eventually exhibits a cigar degeneracy: the curvature dies away, and while one diameter expands without bound, the other two diameters shrink to zero.

(vi): The $E(2)$-Basin, where $E(2)$ is the group of isometries of the euclidean plane. It contains all three-metrics flowing towards the three-parameters family of homogeneous Solv-metrics containing the flat geometry. This basin is eventually attracted by flat metrics.

(vii): The $H(3)$-Basin, where $H(3)$ is the group of isometries of hyperbolic three-space. It contains all three-metrics flowing towards the one-parameter family of homogeneous metrics constant multiples of the standard hyperbolic metric. This basin is attracted to hyperbolic space.

(viii): The $SO(3) \times \mathbb{R}^1$-Basin. It contains all three-metrics flowing towards the two-parameter family of homogeneous metrics obtained by rescaling the standard product metric on $S^2 \times \mathbb{R}^1$. It does not contain Einstein metrics. This is a singular basin, it is attracted towards a curvature singularity: the round two-sphere shrinks while the scale on $\mathbb{R}^1$, (or if you prefer, the $S^1$ factor in the original manifold), expands.

(ix): The $H^2 \times \mathbb{R}^1$-Basin, where $H(2)$ is the group of isometries of the hyperbolic plane. It contains all three-metrics flowing towards the two-parameter family of homogeneous metrics obtained by rescaling the product metric on the product manifold, $\mathbb{R}^1 \times H^2$. Again, this basin does not contain Einstein manifolds, and it is attracted towards a pancake degeneracy.

The basins of attraction just described, and in particular those yielding for fixed points (Einstein manifolds), are relatively uninteresting in connection with the Renormalization Group interpretation of (52). Such fixed points, e.g. the round three-sphere, or the flat three-tori, are all totally attractive. As already recalled they can be thought of as distinct stable phases yielding for distinct cosmological models, characterized by the nature of the metric (and of its infinitesimal deformations) at the fixed point. For instance, the $SU(2)$-Basin characterizes FLRW models with closed spatial sections, and the related homogeneous anisotropic models, (see section 6 for more details).
4.3 Critical fixed points: an example

The existence of critical fixed points, (critical in the sense of \(62\)), characterizing the (universal) properties of (continuous) phase transitions between two different cosmological regimes, cannot be immediately read off from the above analysis. It is rather the consequence of the (conjectured) existence of the decomposition of a manifold into pieces that is associated, according to Hamilton, to the local singularities of the flow \(52\), (see the second step of Hamilton-Thurston geometrization program). The origin of this connection between critical behavior of \(52\), in the sense of Renormalization Group, and Hamilton’s program can be seen by considering the following detailed example.

Let us assume that topologically \(\Sigma\) is a three-sphere, \(\Sigma \simeq S^3\). We shall consider on \(\Sigma\) a metric \(g_1\) obtained by glueing through a smooth connected sum two copies of a round three-sphere

\[
\Sigma \simeq S^3_{(1)} \sharp S^3_{(2)} ,
\]

and endowing each \(S^3_{(i)}\), \(i = 1, 2\), factor with a round metric of volume \(v_1 = 1\), and the joining tube

\[
S^2 \times ([0,1] \subset \mathbb{R}^1) ,
\]

with the standard product metric of volume \(\text{const.} e^{-3\tau}\), (\(\tau\) is a suitable parameter, see below), for \(\tau \gg 1\).

To explicitly construct this latter metric we can proceed as follows:

Let \(y_1\) and \(y_2\) respectively denote two chosen points in both factor copies, \(S^3_{(1)}\) and \(S^3_{(2)}\). Let \(h(i): \mathbb{R}^3 \rightarrow S^3_{(i)}\), \(i = 1, 2\), be two imbeddings given by the exponential mappings

\[
\exp_{y_1} : T_{y_1}S^3_{(1)} \simeq \mathbb{R}^3 \rightarrow S^3_{(1)} ,
\]

\[
\exp_{y_2} : T_{y_2}S^3_{(2)} \simeq \mathbb{R}^3 \rightarrow S^3_{(2)} .
\]

We assume that \(h(1)\) preserves the orientation while \(h(2)\) reverses it.

Let \(\alpha: (0, \infty) \rightarrow (0, \infty)\) denote an orientation reversing diffeomorphism, and define \(\alpha_3: \mathbb{R}^3/\{0\} \rightarrow \mathbb{R}^3/\{0\}\) by the map \(\alpha_3(v) = \alpha(|v|) \frac{v}{|v|}\) for every vector \(v \in \mathbb{R}^3/\{0\}\). For every point \(x_1 = h_1(v)\) in the geodesic ball \(B(y_1, r)/\{0\} \subset S^3_{(1)}\), with \(0 < r \leq \frac{\pi}{2}\), (with \(\pi/2\) the injectivity radius of the unit three-sphere \(S^3_{(1)}\)), we identify \(h_1(v)\) with \(h_2(\alpha_3(v)) \in B(y_2, r)/\{0\} \subset S^3_{(2)}\).

The space obtained in this way

\[
S^3_{(1)} \sharp S^3_{(2)} = (S^3_{(1)} - \{y_1\}) \cup_{h(2)\alpha_3h^{-1}(1)} (S^3_{(2)} - \{y_2\})
\]

is a particular realization of the connected sum \(30\) of two copies of unit three-spheres.

In order to give to the neck, joining the two \(S^3_{(i)}\), a cylindrical shape we blow up \(17\) the metrics of the three-spheres in the neighborhoods of the points \(y_1\) and \(y_2\). Consider, for
simplicity, only the $S^3_{(1)}$ factor since the argument goes in an analogous way also for the $S^3_{(2)}$ factor.

Exploiting the exponential mapping the round metric of $S^3_{(1)}$, (actually any sufficiently smooth metric), can be written in a neighborhood of $y_1$ in geodesic polar coordinates as
\[ g(x) = dr^2 + r^2 h_{ij} d\theta^i d\theta^j + \mathcal{O}(r^4), \] (67)

where $r = \text{dist}(x, y_1)$ is the distance between $y_1$ and the point considered, $h_{ij} d\theta^i d\theta^j$ is the metric on the two-dimensional unit sphere $S^2$, and as usual, the higher order correction terms involve the curvature.

Now, if we blow up this metric by rescaling it through $r^2$ we get, up to curvature corrections, a distorted cylindrical metric
\[ \tilde{g}(x) = \frac{g(x)}{r^2} = \frac{dr^2}{r^2} + h_{ij} d\theta^i d\theta^j + \mathcal{O}(r^2). \] (68)

In order to eliminate the axial distortion due to $\frac{dr^2}{r^2}$, we substitute for the radial variable $r$, introducing a new coordinate $\tau$ defined as
\[ r = \exp[-\tau]. \] (69)

When expressed in terms of $\tau$ we get that the blown up metric $\tilde{g}(x)$ reduces to
\[ \tilde{g}(x) = d\tau^2 + h_{ij} d\theta^i d\theta^j + \mathcal{O}(e^{-2\tau}). \] (70)

Thus, the blown up metric approaches the cylindrical metric exponentially fast as $\tau \to \infty$. In order to introduce smoothly such a metric on the neck of $S^3_{(1)} \# S^3_{(2)}$ we can proceed as follows.

Choose smooth functions $\delta_i(r), i = 1, 2,$ satisfying
\[ \delta_i(r_i) = \begin{cases} \frac{1}{r_i^2}, & 0 < r_i \leq \frac{\pi}{4}, \\ C^\infty, & \text{decreasing, } \frac{\pi}{4} \leq r_i \leq \frac{\pi}{2}, \\ 1, & r_i \geq \frac{\pi}{2}, \end{cases} \] (71)

where $r_i = \text{dist}(x, y_{(i)})$. Also, let us introduce the following inclusion maps
\[ \chi(1) : S^3_{(1)} - \{y_1\} \hookrightarrow (S^3_{(1)} - \{y_1\}) \cup_{h_{(2)\alpha h_{(1)}^{-1}}} (S^3_{(2)} - \{y_2\}), \]
\[ \chi(2) : S^3_{(2)} - \{y_2\} \hookrightarrow (S^3_{(2)} - \{y_1\}) \cup_{h_{(2)\alpha h_{(1)}^{-1}}} (S^3_{(2)} - \{y_2\}), \] (72)
\[ \chi(3) : h_1(T_{y_1} S^3_{(1)} - \{0\}) \cup h_2(T_{y_2} S^3_{(2)} - \{0\}) \hookrightarrow (S^3_{(1)} - \{y_1\}) \cup_{h_{(2)\alpha h_{(1)}^{-1}}} (S^3_{(2)} - \{y_2\}), \]
(\chi(1) \text{ and } \chi(2) \text{ denote the inclusions of the spheres, minus the points } y_{(i)}, \text{ into the connected sum; } \chi(3) \text{ is the inclusion of the neck}).

If \{\xi_\alpha\} is a partition of unity associated with the covering corresponding to the above inclusions \chi(\alpha), with \alpha = 1, 2, 3, we can then define the following metric on \(S^3_1 \sharp S^3_2\),

\[
\tilde{g}(x) = \sum_\alpha \xi_\alpha \chi(\alpha)^* [g(x) \cdot \delta_{(i)}(\text{dist}(x, y_{(i)}))].
\] (73)

For \(x\) not in the geodesic balls (of radius \(\pi/2\)), centered on \(y_1\) and \(y_2\), this is the standard round metric on the unit three-sphere; for \(x\) in the geodesic balls of radius \(\pi/4\), centered on \(y_1\) and \(y_2\), this is, up to curvature corrections, the cylindrical metric introduced above. For \(x\) in the annuli between such balls \(\tilde{g}\) is a smooth interpolating metric joining the spheres to the cylindrical neck.

By expressing (73) in terms of the variable \(\tau\), we get the metric on \(S^3_1 \sharp S^3_2\) which is the round metric on each \(S^3_{(i)} / B(y_i, e^{-\tau})\) factor, and these factors are connected, for \(\tau\) large enough (i.e., nearby \(y_i\)), by a thin flat cylinder.

The Ricci-Hamilton evolution of \((S^3_1 \sharp S^3_2, \tilde{g})\) can be explicitly constructed as follows.

According to [25] let us write the metric on \(S^2 \times \mathbb{R}^1\) as

\[
\tilde{g}|_{\text{neck of } S^3_1 \sharp S^3_2} = Dg_{\mathbb{R}^1} + Eg_{S^2},
\] (74)

where \(g_{\mathbb{R}^1}\) is the metric on \(\mathbb{R}^1\), \(g_{S^2}\) is the round metric on the unit two-sphere, and \(D\) and \(E\) are constants. The Ricci-Hamilton flow (25) preserves the structure of this metric and reduces to the coupled system of ordinary differential equations

\[
\frac{d}{d\eta} E = -\frac{2}{3}, \quad \frac{d}{d\eta} D = \frac{4}{3} \frac{D}{E},
\] (75)

which immediately integrate to

\[
E = E_0 - \frac{2}{3}\eta, \quad D = \frac{D_0 E_0^2}{(E_0 - (2/3)\eta)^2},
\] (76)

where \(E_0^2 = E^2(\eta = 0)\) is the initial radius of the round two-sphere, whereas \(D_0 = D(\eta = 0)\) is the scale on \(\mathbb{R}^1\).
Given this solution of (52), we can construct the Ricci-Hamilton evolution of \((S^3_{(1)} \times S^3_{(2)}, \tilde{g})\) by looking for a solution of (52) in the form
\[
\tilde{g}(x; \eta) = \sum_{\alpha} \xi_\alpha \chi^*_\alpha(\eta)[g(x) \cdot \delta_{(i)}(\text{dist}_\eta(x, y_{(i)}))],
\]
(77)
where the inclusion maps \(\chi_{(\alpha)}\) depend now on the deformation parameter \(\eta\). We assume that the metric \(g(x)\), (the round metric on the spheres \(S^3_{(i)}\) and the standard product metric on the neck \(\simeq S^2 \times \mathbb{R}^1\)), being locally homogeneous, is preserved by (52) since the Ricci-Hamilton flow preserves isometries. As the plumbing between the spheres and the neck shrinks as \(\eta\) increases, (as the \(S^2\) factor in the neck), the inclusions \(\chi_{(\alpha)}\) are necessarily \(\eta\)-dependent. The dynamics of \(\chi_{(\alpha)}\) can be obtained as follows.

The Ricci-Hamilton flow (52) for \(\tilde{g}(x; \eta)\) is given by
\[
\begin{align*}
\frac{\partial \tilde{g}_{ab}(\eta)}{\partial \eta} &= \frac{2}{3} \langle \tilde{R}(\eta) \rangle - \tilde{g}_{ab}(\eta) - 2\tilde{R}_{ab}(\eta) \\
\tilde{g}_{ab}(\eta = 0) &= \tilde{g}_{ab},
\end{align*}
\]
and, in terms of the pulled-back metric it is
\[
\begin{align*}
\frac{\partial}{\partial \eta} [\chi^*_\alpha(\eta)g(x) \cdot \delta_{(i)}]_{ab}(x) &= \frac{2}{3} \langle \tilde{R}(\eta) \rangle - [\chi^*_\alpha(\eta)g(x) \cdot \delta_{(i)}]_{ab} \\
&\quad - 2\tilde{R}(\chi^*_\alpha(\eta)g(x) \cdot \delta_{(i)})_{ab}.
\end{align*}
\]
(78)

The left hand side of (78) can be evaluated according to a suggestion first exploited by D. DeTurck, viz.,
\[
\begin{align*}
\frac{\partial}{\partial \eta} [\chi^*_\alpha(\eta)g \cdot \delta_{(i)}]_{ab}(x) &= \chi^*_\alpha(\eta) \left[ \frac{\partial}{\partial \eta} [g \cdot \delta_{(i)}]_{ab}(\chi(\eta, x)) \right] \\
&\quad + \chi^*_\alpha(\eta) \left[ L_w(\eta) [g \cdot \delta_{(i)}]_{ab}(\chi(\eta, x)) \right],
\end{align*}
\]
(79)
where the Lie derivative \(L_w(\eta)[g \cdot \delta_{(i)}]_{ab}(\chi(\eta, x))\) is along the vector field \(w(\eta; \alpha)\) which generates the \(\eta\)-evolution of the inclusions \(\chi_{(\alpha)}\), viz.,
\[
\begin{align*}
\frac{\partial \chi_{(\alpha)}(\eta)}{\partial \eta} &= w(\eta; \chi_{(\alpha)}(\eta)), \\
\chi_{(\alpha)}(\eta = 0) &= \chi_{(\alpha)}.
\end{align*}
\]
(80)

Let us denote by \(< R >_{(\alpha)}\) the average of the scalar curvature \(R(\chi_{(\alpha)}(x))\) over the images of the inclusions \(\chi_{(\alpha)}\), (viz., for \(\alpha = 1, 2, < R >_{(\alpha)}\) is the average over \(S^3_{(i)}\), while for \(\alpha = 3\), the average is over the neck). In terms of these averages, the Ricci-Hamilton flow (78) can be written as

\[28\]
\[
\frac{\partial}{\partial \eta} [\delta^{(i)} \cdot g_{ab}(\chi_\alpha(x))] = \frac{2}{3} \left< R(\eta) >_{(\alpha)} \delta^{(i)} \cdot g_{ab}(\chi_\alpha(x)) - 2R_{ab}(\chi_\alpha(x)) \right.
+ \frac{2}{3} \left[ \delta^{(i)} \cdot g_{ab}(\chi_\alpha(x)) \left< R(\eta) > - \left< R(\eta) >_{(\alpha)} \right. \right]
- L_{\alpha(\eta)} [g \cdot \delta^{(i)}]_{ab}(\chi(\eta, x)).
\] (81)

As can be checked, the Ricci-Hamilton flow preserves the local homogeneous geometries over the spheres \(S^3_{(i)}\), and on the neck, if and only if the vector field \(w(\eta; \alpha)\) satisfies
\[
\frac{2}{3} \delta^{(i)} \cdot g_{ab}(\chi_\alpha(x)) \left< R(\eta) > - \left< R(\eta) >_{(\alpha)} \right. \right] = L_{\alpha(\eta)} [g \cdot \delta^{(i)}]_{ab}(\chi(\eta, x)).
\] (82)

By taking the trace of (82) with respect to \(g \cdot \delta^{(i)}\), we get
\[
< R > - < R >_{(\alpha)} = \nabla^k w_k(\eta; \alpha),
\] (83)
where \(\nabla\) denotes the riemannian connection with respect to \(g \cdot \delta^{(i)}\).

For \(\alpha = 1, 2\), i.e. for the punctured spheres \(S^3_{(i)} \setminus \{ y_i \}\), the above relation yields upon integration over \(S^3_{(i)} \setminus B(y_i, r)\)
\[
\left< R > - < R >_{(\alpha)} \right> = \frac{\text{vol} \left( S^3_{(i)} \setminus B(y_i, r) \right) \eta(x) \sigma_k}{\int_{S^3_{(i)} \setminus B(y_i, r)} w_k(\eta; \alpha) d\sigma^k},
\] (84)
where \(S^2(\eta)\) is the \(\eta\)-dependent boundary of \(S^3_{(i)} \setminus B(y_i, r)\). Since \(< R > - < R >_{(\alpha)}\), for \(\alpha = 1, 2\), is proportional to the average scalar curvature of the neck, the term on the left hand side of (84) blows up as \((E_0 - \frac{2}{\pi} \eta)^{-1}\) as \(\eta\) increases. On the other hand, by the \(S^2(\eta)\)-rotational symmetry, the term \(w_k(\eta; \alpha) d\sigma^k\) is spatially constant on the two-sphere boundary \(S^2(\eta)\) of the punctured three-spheres. Thus, (84) implies that as \(\eta\) increases, the surface area of \(S^2(\eta)\) shrinks to zero, as \((E_0 - \frac{2}{\pi} \eta)^{-1}\), by moving along the outer normal direction of \(S^2(\eta) \subset (S^3_{(i)} \setminus B(y_i, r(\eta))).\)

For \(\alpha = 3\), i.e., for the neck, (84) yields
\[
\left[ < R > - < R >_{(\alpha=3)} \right> = \frac{\text{vol} \left( S^2(\eta) \times \mathbb{R}^1 \right)}{\int_{S^2(\eta)} w_k(\eta; \alpha) d\sigma^k - \int_{S^2(\eta)} w_k(\eta; \alpha) d\sigma^k}, \] (85)
where \(S^2(\eta) - S^2(\eta)\) is the oriented boundary of the cylinder \(S^2(\eta) \times \mathbb{R}^1\). Since \(< R > - < R >_{(\alpha=3)}\) is, up to small correction terms (coming from the collars joining the spheres to the neck), the average curvature over the spheres \(S^3_{(i)}\), \(i = 1, 2\), (84) simply tells us that as the \(S^2(\eta)\) factor in the neck shrinks, the scale-length of the \(\mathbb{R}^1\) factor grows, so that the volume \(\text{vol} \left( S^2(\eta) \times \mathbb{R}^1 \right)\) remains constant during the Ricci-Hamilton evolution, namely,
\[
\text{vol} \left( S^2(\eta) \times \mathbb{R}^1 \right) = \frac{\int_{S^2(\eta)} w_k(\eta; \alpha) \, d\sigma^k - \int_{S^2(\eta)} w_k(\eta; \alpha) \, d\sigma^k}{<R> - <R>_{(\alpha=3)}}.
\] (86)

Notice that by introducing, as above, a new variable
\[
\text{dist}_\eta(x, y_{(i)}) = \exp[-\tau(\eta)],
\] (87)
with
\[
\tau(\eta) = \frac{\tau}{1 - \frac{2}{3} \eta},
\] (88)
and by rescaling the unit two-sphere metric \( h_{ij} \) according to
\[
h_{ij}(\eta) = \left[1 - \frac{2}{3} \eta\right] h_{ij},
\] (89)
the above analysis of the Ricci-Hamilton flow (78) provides on the neck the metric
\[
\tilde{g}(x; \eta) = d\tau^2(\eta) + h_{ij}(\eta) d\theta^i d\theta^j + \mathcal{O}(e^{-2\tau(\eta)}),
\] (90)
and on \( S^3_{(i)} \) the standard round metric.

Notice also that (90) goes cylindrical exponentially fast in \( \eta \). As \( \eta \) increases \( 0 \leq \eta < \frac{3}{2} \), the neck becomes longer and longer while getting thinner and thinner. In the limit, \( \eta \to \frac{3}{2} \), we get
\[
(S^3_{(1)} \# S^3_{(2)}, \tilde{g}) \to S^3_{(1)} \coprod S^3_{(2)},
\] (91)
where the three-spheres \( S^3_{(i)} \) carry the round metric of volume 1, while the smooth joining regions shrink exponentially fast around the points \( y_{(i)}, i = 1, 2 \).

4.4 Critical surfaces and topological crossover
Let \( g_o \) be a metric on \( \Sigma \) with positive Ricci curvature,
\[
\text{Ric}(g_o) > 0
\] (92)
and with volume \( \text{vol}(\Sigma, g_o) = 2 \).

By means of \( g_o \) and the metric \( \tilde{g} \) introduced in the previous section, we can construct on \( \Sigma \) a smooth one-parameter, \( (0 \leq t \leq 1) \), family of metrics \( g_t \), with \( g_{t=0} = g_o \) and \( g_{t=1} = \tilde{g} \) by setting
\[
g_t \equiv (1 - t) g_o + t \tilde{g},
\]
\[0 \leq t \leq 1.\] (93)
According to Hamilton’s theorem \cite{22}, there is a right-open neighborhood of \( t = 0 \) such that all metrics \( g_t \) in this neighborhood are in \( SU(2) \)-Basins and are attracted, under the action of the Ricci-Hamilton flow \eqref{52}, towards the round metrics on \( S^3 \) with volumes \( v_t \), (since the volume of \( g_t \) changes with \( t \), we have a family of fixed points, namely, round three-spheres parameterized by the corresponding volumes \( v_t \) which are kept constant under the Ricci-Hamilton flow; thus \( v_{t=0} = 2 \)).

On the other hand, according to the remarks above, we have an open neighborhood of \( t = 1 \) such that all metrics \( g_t \) in this neighborhood go singular under the Ricci-Hamilton flow. Indeed, for \( t = 1 \), the Ricci-Hamilton flow of \( \tilde{g} \) fixes the round \( S^3 \) factors while the joining tube, \( S^2 \times \mathbb{R}^1 \), is driven towards a curvature singularity. By continuity, this behavior extends to a left-open neighborhood of \( t = 1 \), and a three-sphere in this neighborhood splits apart into two round spheres.

It follows that there is a neighborhood of the \( \tilde{g} \) metric such that some of the metrics in this neighborhood are driven towards attractive \( SU(2) \)-Basins, while others are driven towards a singular \( (S^3_{(1)} \# S^3_{(2)}) \)-Basin.

The set of metrics driven towards this basin defines a critical surface. Below we shall define a critical fixed point. Since the Ricci-Hamilton flow preserves the metric \( \tilde{g} \), up to a trivial rescaling of the neck in \( S^3_{(1)} \# S^3_{(2)} \), (rigorously speaking this is true only up to exponentially small correction terms which arise in the regions joining the three-spheres with the \( S^2 \times \mathbb{R}^1 \) neck), we can characterize the critical fixed point for \eqref{52}, in the example considered, as the metric on \( S^3_{(1)} \# S^3_{(2)} \) given by

\[
G_{\text{crit}} \equiv \{ \tilde{g}(x; \eta) | \eta \to \frac{3}{2} \}, \tag{94}
\]

where \( \tilde{g}(x; \eta) \) is the one parameter \( \eta \)-family of metric solution of the Ricci-Hamilton initial value problem \eqref{78}. This is a consistent characterization of a critical fixed point for \eqref{52}, since \eqref{52} fixes \( (S^3_{(1)} \# S^3_{(2)}, G_{\text{crit}}) \), and as required for \eqref{62}, the correlation length associated with \( (S^3_{(1)} \# S^3_{(2)}, G_{\text{crit}}) \) is \( L = \infty \). This is so since, roughly speaking, in order to describe \( (S^3_{(1)} \# S^3_{(2)}, G_{\text{crit}}) \) we fix any representative \( (S^3_{(1)} \# S^3_{(2)}) \) which is characterized by a correlation length \( L_o \simeq \text{the length of the neck} \), and then rescale with the Ricci-Hamilton flow \eqref{78} from \( L_o \) to \( L = \infty \).

Notice however that in this particular case, the critical point is not to be related to a phase transition. As mentioned before, we are here in presence of significant finite-size effects which are concerned with a dimensional crossover and they show up, as usual, as an effective reduction of dimensionality \cite{8}. In this case, the three-dimensional neck of \( (S^3_{(1)} \# S^3_{(2)}) \) goes one-dimensional. This crossover to (quasi-)one-dimensional behavior is not accompanied by a singular behavior in thermodynamical quantities, such as correlation functions, but nonetheless an anomalous behavior is present. Indeed, geometrically speaking, the critical fixed point \( (S^3_{(1)} \# S^3_{(2)}, G_{\text{crit}}) \), and the corresponding critical surface, separates two stable phases under the renormalization generated by \eqref{52}. One is given by the manifolds eventually evolving towards
the round three-sphere of volume 2. The other is generated by those manifolds which pinch off and eventually evolve towards two round three-spheres, each of volume 1. The *pinching off* through thinner and thinner necking is necessary for such a topological crossover.

From a physical point of view, and as we shall see in section 6, the fixed point described above separates two possible different closed FLRW regimes. One with closed spatial sections which, at some particular instant, are a three-sphere of volume 2, while in the other regime we have two distinct closed FLRW universes having spatial sections (at a given instant) of volume 1. We may have also many different necks corresponding to a regime whereby the spatial section \( \Sigma \) yields for many closed FLRW universes.

It is clear that the above explicit construction of a critical fixed point for (52) can in principle be generalized to more general situations. The strategy is to take two or more trivial fixed points, such as those associated with the \( SU(2) \) and \( H(3) \)-Basins, and connect them through the degenerate basins (such as \( SO(3) \times \mathbb{R}^1 \), as in the above example, or through the \( H(2) \times \mathbb{R}^1 \)-Basin, etc.). In this connection notice that the connected sum mechanism yielding for the \( (S^3_1 \sharp S^3_2) \)-critical fixed point, can be generalized to an operation of joining two (or more) manifolds (corresponding to stable attractors) along tubular neighborhoods of surfaces (rather than points, as in the case for the standard connected sum).

A particularly interesting connecting geometry would be \( H(2) \times \mathbb{R}^1 \), (by compactifying the hyperbolic plane in a closed surface). In this latter case the scale of the hyperbolic geometry goes to infinity under the Ricci-Hamilton flow (*pancake degeneracy* [25]). Finite size effects are again at work, but this time the effective reduction of dimensionality is more interesting than the one in the \( (S^3_1 \sharp S^3_2, \mathcal{G}_{\text{crit}}) \) case. Indeed, two out of three dimensions are infinite, and there will be a crossover to a critical behavior with critical exponents characteristic of a two-dimensional system.

One can consider the analysis presented above as a physically non-trivial application of Hamilton-Thurston’s geometrization program. In some of its aspects it is rather conjectural and speculative, but in our opinion, it is quite intriguing that motivations coming from geometry and a physical problem, like the one addressed here, namely, the construction of cosmological models out of a local gravitational theory, go hand in hand in such a way.

### 5 Linearized RG flow

The relative slopes of \( \langle f \rangle_{\epsilon_m}(g_1) \) and \( \langle f \rangle_{\epsilon_m}(g_2) \), with \( f \) given by equation (35), as \( m \to \infty \), and for \( g_1 \) in a neighborhood of \( g_2 \), are of some relevance to our discussion. In a standard RG analysis such relative slopes are related to critical exponents. Given the blocking procedure for \( \langle f \rangle_{\epsilon_m}(g_1) \) as \( m \to \infty \), yielding for a \( g_1 \) “renormalized” according to the Ricci-Hamilton flow (32), one can sensibly ask what happens if \( g_1 \) is slightly perturbed, namely, if we replace it by

\[
g_{ab} \to g_{ab} + \delta K_{ab},
\]
where $K_{ab}$ is a symmetric bilinear form (a choice of the symbol is quite intentional, since later the above consideration will be applied to the second fundamental form). It can be shown that if $g_1$ is scaled, according to the Ricci-Hamilton flow (52), then $K_{ab}$ gets renormalized according to the linearized Ricci-Hamilton flow, namely ($\eta$ in the brackets suppressed),

$$
\frac{\partial}{\partial \eta} K_{ab} = \frac{2}{3} < R > K_{ab} + \frac{2}{3} g_{ab} (\frac{1}{2} < R g^{ab} K_{ab} > - \frac{1}{2} < R > g^{ab} K_{ab} > - < R^{ab} K_{ab} >) - \Delta_L K_{ab} + 2 [\text{div}^*(\text{div}(K - \frac{1}{2}(\text{Tr} K) g))]_{ab}, \tag{96}
$$

with the initial data $K_{ab}(\eta = 0) = K_{ab}$, where $K \in S^2 \Sigma$ is a given symmetric bilinear form, $\Delta_L$ is the Lichnerowicz-DeRham Laplacian on bilinear forms

$$
\Delta_L K_{ab} \equiv -\nabla^s \nabla_s K_{ab} + R^{asb}_{\phantom{asb}bc} K^c_a - 2 R^{abst} K_{st}, \tag{97}
$$

and the operators $\Delta_L$, $\text{div}^*$, $\text{div}$ and $\text{Tr}$ are considered with respect to the flow of metric $(g, \eta) \to g(\eta)$, solution of (52). The $\text{div}$ (here, minus the usual divergence) is the divergence operator on $S^2 \Sigma$, $\text{div}^*$ is the $L^2$-adjoint of $\text{div}$, acting from the space of vector fields on $\Sigma$ to $S^2 \Sigma$ (it can be identified with $\frac{1}{2}[\text{Lie derivative}]$ of the metric tensor along a vector field).

Note that $K(\eta)$ solution of the linear (weakly) parabolic initial value problem (96) always exists and is unique $\text{[22]}$, and represents an infinitesimal deformation of metrics connecting the two neighbouring flows of metrics, $g(\eta)$ and $g'(\eta)$, (obtained as solutions of problem (52) with initial data $g(\eta = 0) = g$ and $g'(\eta = 0) = g(\eta = 0) + \epsilon K(\eta = 0) + O(\epsilon^2)$, respectively). For what concerns the structure of this solution, one can verify that corresponding to the “trivial” initial datum $K(\eta = 0) = L_X g$ (where $X : \Sigma \to T \Sigma$ is a smooth vector field on $\Sigma$), the solution of (94) is provided by

$$
K_{ab}(\eta) = L_X g_{ab}(\eta). \tag{98}
$$

This property expresses the $\text{Diff}(\Sigma)$ equivariance of the Ricci-Hamilton flow. (Notice that $X$ is $\eta$-independent).

The above fact follows by noticing that along the trajectories of the flow $(\eta, g) \to g(\eta)$, solution of (52), we have

$$
\frac{\partial}{\partial \eta} L_X g_{ab}(\eta) = L_X [\frac{\partial}{\partial \eta} g_{ab}(\eta)] = \frac{2}{3} < R(\eta) >_\eta L_X g_{ab}(\eta) - 2 L_X R_{ab}(\eta). \tag{99}
$$

But the $\text{Diff}(\Sigma)$-equivariance of the Ricci tensor, i.e. the fact that $\text{Ric}(\varphi^* g) = \varphi^* \text{Ric}(g)$ for any smooth diffeomorphism $\varphi : \Sigma \to \Sigma$, implies that

$$
L_X R_{ab} = D \text{Ric}(g) \cdot L_X g_{ab}, \tag{100}
$$

where $D \text{Ric}(g) K$ is the formal linearization of $\text{Ric}(g)$, around $g$, in the direction $K$:

$$
D \text{Ric}(g) \cdot K \equiv \frac{d}{dt} [\text{Ric}(g + tK)]_{t=0} = \frac{1}{2} \Delta_L K - \text{div}^* [\text{div}(K - \frac{1}{2}(\text{Tr} K) g)]. \tag{101}
$$
Upon introducing (100) in (99) we get
\[ \frac{\partial}{\partial \eta} L_X g_{ab}(\eta) = \frac{2}{3} < R(\eta) >_g L_X g_{ab}(\eta) - 2D \text{Ric}(g(\eta)) \cdot L_X g_{ab}(\eta). \] (102)

One can check that the right hand side of the above expression coincides with the right hand side of (96), when this latter is evaluated for \( K_{ab}(\eta) = L_X g_{ab}(\eta) \). Hence \( L_X g_{ab}(\eta) \) solves the partial differential equation (96) and, since for \( \eta = 0 \), \( K_{ab} = L_X g_{ab} \), the uniqueness of any solution of the initial value problem (96) implies that \( K_{ab}(\eta) = L_X g_{ab}(\eta) \), whenever \( K_{ab}(\eta = 0) = L_X g_{ab} \), as stated.

Moreover, if \( K(\eta) \) is a solution of (96), with initial datum \( K(\eta = 0) = K \), then the space average of \( Tr K(\eta) \) over \((\Sigma, g(\eta))\) is preserved along the flow \((\eta, g) \rightarrow g(\eta)\), namely,
\[ < Tr K(\eta) >_{\eta} = < Tr K >_{\eta = 0}, \quad 0 \leq \eta < \infty. \] (103)

This property of the solutions of (96) is an immediate consequence of the volume-preserving character of the Ricci-Hamilton flow.

Finally, another relevant property of the initial value problem (96) can be stated as follows. If \((\eta, K_{ab}) \rightarrow K_{ab}(\eta)\) is the flow solution of (96), with initial datum \( K_{ab}(\eta = 0) = K_{ab} \), then it can always be written as
\[ K_{ab}(\eta) = \hat{K}_{ab}(\eta) + L_{v(\eta)} g_{ab}(\eta), \] (104)

where the bilinear form \( \hat{K}_{ab}(\eta) \) and the \( \eta \)-dependent vector field \( v(\eta) \), respectively, are the solutions of the initial value problems:
\[ \frac{\partial}{\partial \eta} \hat{K}_{ab} = \frac{2}{3} < R > \hat{K}_{ab} + \frac{2}{3} g_{ab}[\frac{1}{2} < R g^{ab} \hat{K}_{ab} > - \frac{1}{2} < R > g^{ab} \hat{K}_{ab} > - \frac{1}{2} < R^{ab} \hat{K}_{ab} > - \Delta L \hat{K}_{ab}, \]
\[ \hat{K}_{ab}(\eta = 0) = K_{ab}, \] (105)

and
\[ \frac{\partial}{\partial \eta} v_a(\eta) = -\nabla^c (\hat{K}_{ca} - \frac{1}{2} \hat{K}^{rs} g_{rs} g_{ca}), \quad v(\eta = 0) = 0. \] (106)

To summarize, as \( \eta \rightarrow \infty \), \( K_{ab}(\eta) \) may either approach a Lie derivative term, such as \( L_{v(\eta)} g_{ab}(\eta) \), or a non-trivial deformation \( \hat{K}_{ab}(\eta) \) [35]. The non-trivial deformation is present only if the corresponding Ricci-Hamilton flow for \( g_{ab}(\eta) \) approach an Einstein metric on \( \Sigma \) which is not isolated. In such a case, (e.g. flat tori), there is a finite dimensional set of such Einstein metrics, and the non-trivial \( \hat{K}_{ab} \) simply are the infinitesimal deformations connecting one Einstein metric \( \bar{g}_1 \) in \( \Sigma \) and the infinitesimally non-equivalent one. Also in this case the Lie derivative term may be present. What it represents is a reparameterization of the metric \( \bar{g}_1 \) (under the action of the infinitesimal group of diffeomorphisms generated by \( v \) (see equation (104)), (“gauge artifact”). This latter Lie derivative term is the only surviving term when \( \bar{g} \) is isolated (like e.g. in the case of the round three-sphere. As is known, the round metric \( \bar{g} \) on
the three-sphere $S^3$ is isolated, in the sense that there are not volume-preserving infinitesimal deformations of $\bar{g}$ mapping it to another inequivalent constant curvature metric $\bar{g}'$.

To summarize, the flow (105), (106), tells us how to renormalize the second fundamental form in such a way that the blocking prescription $<f>_{t_m} \rightarrow <f>_{t_{m+1}}$ works both for the initial metric $g$ as well as for the perturbed metric $g + \delta K_{ab}$.

5.1 Scaling and critical exponents

From the characterization of critical fixed points $g_{ab}^*$ for (52), (see section 4.2), we can get information on the critical exponent characterizing critical behavior of the metrics nearby $g_{ab}^*$. This is the content of this section. In particular, we shall discuss the critical exponents related to the critical fixed point $(S^3 \# S^3, G_{\text{crit}})$. Even if this point is not a thermodynamically interesting critical point, it exhibits many of the general features of the more interesting type of singularities.

In order to characterize these critical exponents we can use the linearized Ricci-Hamilton flow associated with the one-parameter family of metrics $(S^3 \# S^3, G_{\text{crit}})$. In the following we shall however proceed more directly and examine the properties of the two-point correlation function associated with $(S^3 \# S^3, G_{\text{crit}})$ and the probability law of relevance to our analysis.

Let $y_i \in S^3_i, i = 1, 2$, be the two points in $(S^3 \# S^3, G_{\text{crit}})$ around which the round metrics of $S^3_i$ have been blown up. Let $f$ denote a non-negative scalar field on $(S^3 \# S^3)$, distributed according to the probability law formally defined by

$$dP \equiv \frac{\exp[-H(f)] \prod_x df(x)}{\int \exp[-H(f)] \prod_x df(x)}.$$  \hspace{1cm} (107)

where the functional integration is over the space of fields $f: (S^3 \# S^3) \rightarrow \mathbb{R}$, equipped with the $L^2$-inner product

$$(f|f') \equiv \int_{(S^3 \# S^3)} f(x)f'(x) d\mu_g(x),$$  \hspace{1cm} (108)

and where

$$H(f) = \int_{(S^3 \# S^3)} f(x) d\mu_g(x).$$  \hspace{1cm} (109)

(We could consider $f$ as related to the matter fields, as in $f = \alpha \rho + \alpha' J_i$, but the following analysis is quite independent from a particular meaning of $f$). Notice also that the measure $d\mu_g(x)$ in the above formulae is the riemannian measure on $(S^3 \# S^3)$ associated with the metric $\tilde{g}(x)$ defined by (73).

Let us now concentrate on the behavior of (107) when the neck of $(S^3 \# S^3)$ gets thinner and longer under the action of (52).

It can then be checked that along the Ricci-Hamilton flow (76) associated with $\tilde{g}|_{\text{neck}}$, both the $L^2$-inner product (108) and the Gibbs factor $\exp[-H(f)]$ corresponding to (109), are
invariant. Thus it follows that the probability measure (107) is invariant under (52), and the correlation function defined by

\[ E_{dP}[f(y_1)f(y_2)], \quad (110) \]

where \( E_{dP} \) denotes the expectation with respect to \( dP \), is well defined over the critical fixed point \( G_{crit} \).

In section 4.1 we have interpreted the Ricci-Hamilton flow (52) as the RG flow in a finite geometry, characterized by the length scale \( L \), which is large compared to the microscopic scale (in particular, it is much larger than the radius of the geodesic ball coverings used to discretize the theory). The correlation function depends on such a dimensional parameter. If

\[ L(y_1, y_2) \equiv \int_{y_1}^{y_2} \sqrt{Dz} \quad (111) \]

denotes the distance between the points \( y_1 \) and \( y_2 \) along the cylindrical neck of \((S^3_1 \# S^3_2)\), and if this distance is large (as compared with the radius of geodesic ball coverings), then the correlation length \( \xi \) associated with the two-point (connected) correlation function can be read off from

\[ \xi \simeq L(y_1, y_2) \frac{-L(y_1, y_2)}{\ln E_{dP}[f(y_1)f(y_2)]_{conn}}. \quad (112) \]

Since the correlation function remains invariant under the Ricci-Hamilton deformation of the cylindrical neck, we get that along (52), \( \xi/L(y_1, y_2) \) remains constant which implies that on \((S^3_1 \# S^3_2), G_{crit}\) the correlation length \( \xi \) behaves as

\[ \xi \simeq \sqrt{D_0}E_0[E_0 - \frac{2}{3}\eta]^{-\frac{1}{4}}. \quad (113) \]

According to standard usage, we can define the critical exponent \( \nu \) associated with the correlation length of a finite size system (with typical size \( L \)) by the condition

\[ \frac{\partial}{\partial p} \xi(L, p)|_{p=p_c} \simeq L^{1+\frac{1}{\nu}}, \quad (114) \]

where \( p \) is a parameter driving the system to criticality, and \( p_c \) is its corresponding critical value. In our case, it is natural to set \( p = \eta \), with \( p_c = \eta_c = \frac{3}{2} \). This immediately yields for the critical exponent \( \nu \) the value

\[ \nu = 1. \quad (115) \]

A similar computation for the critical exponent associated with the correlation length can be carried out when the connecting geometry is \( H^2 \times \mathbb{R}^1 \). This takes place when two riemannian
manifolds $M_1$ and $M_2$, which are supposed to evolve nicely under the Ricci-Hamilton flow, are connected through a tubular neighborhood of a surface $S_h$ of genus $h$, viz., $\sigma = M_1 \natural_{S_h} M_2$.

In this case, the metric on the neck can be written as\(^{25}\)

$$g_{\text{neck}} = Dg_{\mathbb{R}^1} + Eg_{H^2},$$

where $g_{H^2}$ is the metric on the hyperbolic plane. The Ricci-Hamilton flow equations take a form similar to the $S^2 \times \mathbb{R}^1$ case, up to a minus (important!) sign, namely,

$$\frac{d}{d\eta} E = \frac{2}{3},$$
$$\frac{d}{d\eta} D = -\frac{4}{3} \left( \frac{D}{E} \right),$$

which upon integration provide

$$E = E_0 + \frac{2}{3} \eta,$$
$$D = \frac{D_0 E_0^2}{|E_0 + (2/3)\eta|^2},$$

where $E_0^2 = E^2(\eta = 0)$ is the initial scale of the hyperbolic geometry, whereas $D_0 = D(\eta = 0)$ is the scale on $\mathbb{R}^1$.

The scale $E$ of the hyperbolic geometry increases linearly with $\eta$, while the scale factor $D$ on the line $\mathbb{R}^1$ shrinks. It can be checked\(^{25}\) that corresponding to this scale dynamics, the curvature decays according to $||Ric|| = \sqrt{2/(E_0 + \frac{2}{3}\eta)}$, and we get in the limit $\eta \to \infty$ a pancake degeneracy.

The relevant correlation function is now $E_d P(f(y_1)f(y_2))_{\text{conn}}$, with $y_1$ and $y_2$ fixed points in the $H^2$ factor, (i.e., on the surface $S_h$). Again, owing to the symmetries of the geometry involved, it immediately follows that the ratio between the correlation length $\xi$ and the distance $L(y_1, y_2)$ must remain constant under the Ricci-Hamilton flow. This implies that the correlation length behaves as

$$\xi \simeq \sqrt{\frac{E_0}{E_0 + \frac{2}{3}\eta}},$$

which corresponds again a critical exponent $\nu = 1$, (in this case one cannot apply\(^{114}\) since the system is actually going to an infinite size).

A striking feature of these topological crossover phenomena, associated with the renormalization of the cosmological matter distribution, is that their pattern resembles the linear sheet-like (or sponge-like) structure in the distribution of galaxies on large scales (e.g.\(^{18}\)). It
is evident that if, \((\Sigma, g, K, \rho, J)\), the initial data set for the real universe (see the next section), is close to criticality, in the sense discussed in the previous section, then the corresponding averaged model exhibits a tendency to topological crossover in various regions (the ones where the inhomogeneities are larger). Filament-like and sheet-like structures would emerge, and the overall situation would be the one where such structures appear together with regions of high homogeneity and isotropy, in some sort of hierarchy. This situation is akin to that of a ferromagnet nearby its critical temperature, whereby we have islands of spins up and down in some sort of nested pattern. Even if there is a tendency to homogeneity at very large scales, the picture just sketched, of the “hierarchy of structures”, might be qualitatively valid in a good part of the universe; indeed recent observational data seem to suggest the existence of still larger and larger structures (e.g. [13]). Notice that this picture bears some resemblance to the “cascade of fluctuations” in critical phenomena [19]. Droplet fluctuations nucleated at the lattice scale in the critical state can grow to the size of the correlation length where the details of the lattice structure become lost and the scale invariant distributions of the large “droplets” are universal.

A possible (though not yet clarified) connection of this whole picture with the self-organized criticality (SOC) [4] can be envisaged, whereby the problem of structure formation, in terms of growth phenomena, could be tackled in the framework of avalanche activity used in SOC (cf. [46]). It would be particularly interesting to estimate the scales in the universe, where it is necessarily critical and trapped into self-organized (critical) states (similar suggestion was recently posed in [42]).

6 Effective cosmological models

The results of the previous sections have interesting consequences when applied in a cosmological setting.

Let us assume that at the scale over which General Relativity is experimentally verified, a cosmological model of our universe is provided by evolving a set of consistent initial data \((\Sigma, g, K, \rho, J)\), according to the evolutive part of Einstein’s equations.

The data \((\Sigma, g, K, \rho, J)\) describing the interaction between the actual distribution of sources and the inhomogeneous geometry of the physical space, \((\Sigma, g, K)\), are required to satisfy the Hamiltonian and the divergence constraints. Respectively,

\[
8\pi G\rho = R(g) - K^{ab}K_{ab} + k^2, \tag{120}
\]

\[
\nabla^iK_{ih} - \nabla_h k = 16\pi G J_h, \tag{121}
\]

where, \(k \equiv K^a_a\), (see also section 4, where they were written in terms of the three-metric \(g_{ab}\) and its associated conjugate momentum \(\pi_{ab}\)).

According to the blocking and renormalization procedure, discussed at length in this paper, we can implement a coarse-graining transformation on this actual data set by suitably renormalizing the metric and the second fundamental form, while retaining the functional form and the validity of the constraints.
The metric is renormalized according to the Ricci-Hamilton flow (52),
\[
\begin{cases}
\frac{\partial g_{ab}(\eta)}{\partial \eta} = \frac{2}{3} < R(\eta) > g_{ab}(\eta) - 2R_{ab}(\eta) \\
g_{ab}(\eta = 0) = g_{ab}.
\end{cases}
\]

In order to renormalize the second fundamental form, let us rewrite it in terms of the deformation tensor \( H_{ab} \) defined by
\[
H_{ab} = K_{ab} + L_{\alpha}g_{ab}, \tag{122}
\]
where \( \vec{\alpha} \) is the shift vector field providing the three-velocity of chosen instantaneous observers on \( \Sigma \), with respect to which we are implementing the renormalization procedure. We renormalize \( K_{ab} \) by rescaling this deformation tensor \( H_{ab} \) according to (96), viz.,
\[
\frac{\partial}{\partial \eta} H_{ab} = \frac{2}{3} < R > H_{ab} + \frac{2}{3}g_{ab}\frac{1}{2} < Rg_{ab} > H_{ab} - \frac{1}{2} < R > g_{ab}H_{ab} - \Delta LH_{ab} + 2[div^*(div(H - \frac{1}{2}(Tr H)g))]_{ab}, \tag{123}
\]
with the initial data \( H_{ab}(\eta = 0) = (K_{ab} + L_{\alpha}g_{ab})_{\eta=0} \). Notice that we renormalize the deformation tensor \( H_{ab} \), rather than \( K_{ab} \) directly, because in this way we can get rid of the possible \( Diff \)-induced shear which may develop in \( \lim_{\eta \to \infty} K_{ab}(\eta) \). Indeed, according to equation (104), as \( \eta \to \infty \) the solution of this initial value problem, \( K_{ab}(\eta) \), approaches a non-trivial deformation \( \lim_{\eta \to \infty} \tilde{K}_{ab}(\eta) \) plus a Lie-derivative term, \( \lim_{\eta \to \infty} L_{\nu(\eta)}g_{ab}(\eta) \). The former is present only if the Ricci-deformed metric \( \tilde{g}_{ab} = \lim_{\eta \to \infty} g_{ab}(\eta) \) is not isolated.

Since at this stage we are mainly interested in FLRW space-times, let us assume that \( \tilde{g}_{ab} \) is isolated, while in order to take care of the \( Diff \)-induced shear, \( \lim_{\eta \to \infty} L_{\nu(\eta)}g_{ab}(\eta) \), we can choose the shift vector field \( \alpha^i \) in such a way that \( \lim_{\eta \to \infty} L_{\nu(\eta)}g_{ab}(\eta) \) is compensated by \( L_{\alpha}g_{ab} \). Since \( L_{\alpha}g_{ab} \) is a trivial datum for (96), it is thus sufficient to choose
\[
\alpha^i = \lim_{\eta \to \infty} \nu(\eta). \tag{124}
\]
Notice that \( \alpha^i \) is the three-velocity vector of the chosen instantaneous observers on \( \Sigma \), thus (124) provides a map identifying corresponding points between the initial actual manifold \((\Sigma, g, K, \rho, J)\) and its renormalized counterpart \( \lim_{\eta \to \infty}(\Sigma, g, K, \rho, J)(\eta) \).

In this section, and mainly for actual computational purposes, we assume that the original inhomogeneous initial data set is such that the Ricci-Hamilton flow is global. As already mentioned, we are interested in connecting an inhomogeneous cosmological space-time to its corresponding FLRW model. This is the case, in particular, if we assume that the original manifold \((\Sigma, g)\) has a positive Ricci tensor, (this case is obviously quite similar to the analysis in [4], there are however important differences that we are going to emphasize). Or more generally, if we assume that the original manifold \((\Sigma, g)\) is in the \( SU(2) \)-Basin of attraction or
nearby the critical point $\mathpzc{z}_i S^3_{(i)}$, with $i = 1, 2, \ldots$, yielding for a manifold $(\Sigma, g)$, nucleating under the Ricci-Hamilton flow $(78)$, (extended to many connected sums), to disjoint three-spheres $S^3_{(i)}$.

Given this setting, we see that due to the properties of the Ricci-Hamilton flow we have $\lim_{\eta \to \infty} K_{ab}(\eta) = \frac{\bar{k}}{3} \bar{g}_{ab}$. The given $K_{ab}$ is deformed by gradual elimination of its shear $K_{ab} - \frac{1}{3} k \bar{g}_{ab}$ and the original (position dependent) rate of volume expansion $k$ is being replaced with its corresponding average value.

Since the constraints, (120) and (121), are required to hold at each step of the renormalization procedure, we get

$$8\pi G(\eta) \rho(\eta) = R(g(\eta)) - K^{ab}(\eta) K_{ab}(\eta) + k^2(\eta),$$

(125)

$$\nabla(\eta)^i K_{ih}(\eta) - \nabla(\eta)_h k = 16\pi G(\eta) J_h(\eta),$$

(126)

where we have explicitly introduced a possible $\eta$-dependence into the gravitational coupling $G$.

Let us now explore the consequences of (125) and (126). From the stated hypothesis on the Ricci-Hamilton flow it follows immediately that, $K_{ab}(\eta) \to \frac{1}{3} \bar{k} \bar{g}_{ab}$ as $\eta \to \infty$, thus (126) implies that

$$\lim_{\eta \to \infty} J_h(\eta) = 0.$$

(127)

In order to analyze (125), we will make use of a property of the Ricci-Hamilton flow, namely that the flow $K(\eta)$, solution of (96), is such that $\frac{\partial}{\partial \eta} < k(\eta) >_\eta = 0$, i.e. the space average of the trace of the second fundamental form remains constant during the deformation. This allows us to write

$$< k >^2 = \bar{k}^2,$$

(128)

since in the limit, the volume expansion is simply a constant, and where $< \ldots >_\eta$ denotes the full average of the enclosed quantity with respect to the initial metric.

Equation (128) provides the Hubble constant on the FLRW time slice associated with the smoothed data corresponding to $(\Sigma, g, K, \rho, J)$.

More explicitly, let us write the FLRW metric in the standard form (units $c = 1$)

$$ds^2 = -dt^2 + S^2(t)d\sigma^2,$$

(129)

where $d\sigma^2$ is the metric of a three-space of constant curvature and it is time independent. As we are interested in the three-sphere case, the metric $d\sigma^2$ can be written as

$$d\sigma^2 = d\chi^2 + \sin^2 \chi (d\theta^2 + \sin^2 \theta d\phi^2).$$

(130)

Since the volume $\text{vol}(\Sigma, g)$ of the original inhomogeneous manifold is preserved by the Ricci-Hamilton flow, we can relate the factor $S^2(t)$, providing the inverse (sectional) curvature of the FLRW slice $t = t_o$, to $\text{vol}(\Sigma, g)$

$$S^2(t_o) = \left( \frac{\text{vol}(\Sigma, g)}{2\pi^2} \right)^{\frac{2}{3}}.$$

(131)
Notice in particular that the scalar curvature, towards which \( R(g(\eta)) \) evolves under the Ricci-Hamilton flow, is given by

\[
\bar{R} \equiv \lim_{\eta \to \infty} R(g(\eta)) = \frac{3}{S^2} = 3 \left( \frac{\text{vol}(\Sigma, g)}{2\pi^2} \right)^{-\frac{2}{3}}.
\] (132)

Having said this, the equation (125) becomes in the limit, after extracting a trace free part of \( K_{ab} \),

\[
8\pi \bar{G} \bar{\rho} = \bar{R} + \frac{2}{3} \bar{k}^2,
\] (133)
since no residual shear survives, and where we have introduced the renormalized gravitational coupling

\[
\bar{G} \equiv \lim_{\eta \to \infty} G(\eta).
\] (134)

On the other hand (125) gives

\[
\frac{2}{3} \bar{k}^2(\eta) = 8\pi G \rho(\eta) - R(g(\eta)) + \tilde{K}^{ab}(\eta) \tilde{K}_{ab}(\eta),
\] (135)
where the shear \( \tilde{K}_{ab} \equiv K_{ab} - \frac{1}{3} kg_{ab} \) has been explicitly introduced.

We can rewrite the last equation upon taking the average with respect to the initial metric, as

\[
\frac{2}{3} < k^2 >_o = 8\pi G < \rho >_o - < R >_o + < \tilde{K}^{ab} \tilde{K}_{ab} >_o.
\] (136)

By exploiting (128) and (136) we can get the Hubble constant of the FLRW model we want to use to describe the time evolution of the renormalized initial data set \( \lim_{\eta \to \infty} (\Sigma, g, K, \rho, J)(\eta) \).

### 6.1 The renormalized Hubble constant

Taking into account (128), and noticing that the Hubble constant on the FLRW slice corresponding, via the Ricci-Hamilton flow, to \( (\Sigma, g, K) \) is

\[
H^2_o = \left( \frac{1}{S} \frac{dS}{dt} \right)^2 = \frac{2}{9} \bar{k}^2,
\] (137)
we get

\[
H^2_o = \frac{8\pi G}{3} < \rho >_o - \frac{1}{3} < R >_o + \frac{1}{3} < \tilde{K}^{ab} \tilde{K}_{ab} >_o - \frac{2}{9} (< k^2 >_o - < k >_o^2).
\] (138)

This is a rather tautological rewriting of (137), justified by the fact that in this form \( H^2_o \) clearly shows a contribution from the spatial average of the shear. In particular, if originally (i.e., for \( \eta = 0 \)) \( k \) is spatially constant, (138) reduces to

\[
H^2_o = \frac{8\pi G}{3} < \rho >_o - \frac{1}{3} < R >_o + \frac{1}{3} < H^{ab} \tilde{H}_{ab} >_o + \frac{1}{3} < \tilde{L}_\alpha g_{ab} \tilde{L}_\alpha g_{cd} g^{ac} g^{bd} >_o.
\] (139)
where we have explicitly used the decomposition $K_{ab} = H_{ab} - L_{\alpha} g_{ab}$, (see equation (122)), and
\begin{equation}
\tilde{L}_{\alpha} g_{ab} \equiv \nabla_a \alpha_b + \nabla_b \alpha_a - \frac{2}{3} g_{ab} \nabla_c \alpha^c
\end{equation}
is the conformal Lie derivative of the metric $g_{ab}$ along the vector field $\alpha$.

The above expressions for $H_o$ correspond, through (133), to the matter density distribution
\begin{equation}
8\pi \bar{G} \bar{\rho} = 8\pi G <\rho>_o + \bar{R} - <R>_o + <\tilde{K}_{ab} \tilde{K}_{ab}>_o - \frac{2}{3} (<k^2>_o - <k>_o^2).
\end{equation}

We have to consider this expression for $\bar{G} \bar{\rho}$ as the renormalized effective sources entering into the Friedmann equation, if we want to describe the real locally inhomogeneous universe through a corresponding idealized FLRW model. As expected, the renormalized matter density shows contributions of geometric origin, either coming from the shear anisotropies or from the local fluctuations in curvature and volume expansion.

At this stage, a few comments are in order concerning, in particular, the effective Hubble constant $H_o$ (138). First, we wish to emphasize that this is the theoretical expression for the Hubble constant if one wishes to model the real locally anisotropic and inhomogeneous universe with a corresponding FLRW one. Expression (138) clearly shows that apart from the expected contributions to the expansion rate coming from matter and curvature, there are two further contributions:

(i) a negative contribution coming from the local fluctuations in the expansion rate. Apparently, this is not a very significant term, since we may choose, as a convenient initial data set describing the real locally inhomogeneous universe, a data set supported on a three-manifold $\Sigma$ of constant extrinsic time, (i.e., $k$ spatially constant on $\Sigma$). Actually, the optimal choice of the data set with respect to which smooth-out the lumpiness in matter and geometry is a deep and interesting question on which we comment later on, in section 6.3.

(ii) a positive contribution coming from the shear term $\frac{1}{3} <\tilde{H}_{ab} \tilde{H}_{ab}>_o$, and from the observer dependent term $\frac{1}{3} <\tilde{L}_{\alpha} g_{ab} \tilde{L}_{\alpha} g_{cd} g^{ac} g^{bd}>_o$ which depends on the three-velocity $\vec{\alpha}$ of the observers with respect to which the renormalization is carried out.

These latter terms are the most important non-standard contributions to $H_o^2$, and have their origin both in the presence of the gravitational radiation
\begin{equation}
\frac{1}{3} <\tilde{H}_{ab} \tilde{H}_{ab}>_o,
\end{equation}
where $\tilde{H}_\perp$ is the divergence-free part of the deformation tensor $\tilde{H}_{ab}$, and in the anisotropies generated by the motion of matter
\begin{equation}
\frac{1}{3} <\tilde{H}_{ab} \tilde{H}_{ab}>_o,
\end{equation}

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where $\tilde{H}_{\parallel}$ is the longitudinal part of the deformation tensor, obtained as a solution to the equation

$$\nabla^i (\tilde{K}_{\parallel})_{ih} - \frac{2}{3} \nabla_h k = 16\pi G J_h + \nabla^i (\tilde{L}_{\alpha} g_{ih}).$$  \hfill (144)  

Recall that the shift vector field $\vec{\alpha}$ is connected to the current density $\vec{J}$ by the requirement that it is chosen in such a way as to eliminate the longitudinal shear in $\lim_{\eta \to \infty} K_{ab}(\eta)$. Also, notice that

$$< \tilde{H}^{ab} \tilde{H}_{ab} >_o = < \tilde{H}^{ab}_{\perp} \tilde{H}_{\perp ab} >_o + < \tilde{H}^{ab}_{\parallel} \tilde{H}_{\parallel ab} >_o.$$  \hfill (145)  

In practice, the deformation energy $< \tilde{H}^{ab} \tilde{H}_{ab} >_o$ yields a contribution to the Hubble constant $H_o$ which can be roughly estimated by exploiting the anisotropy measurements in the cosmic microwave background (CMB) radiation, as long as the frame used in averaging (i.e., the lapse $\alpha$ and the shift $\alpha^i$) is, on the average, comoving with the cosmological fluid. We also require that the original locally inhomogeneous manifold $(\Sigma, g, K)$ does not differ too much from a standard FLRW $t = const.$ slice. In such case one can apply the analysis of [36] to conclude that, at the present epoch, the ratio between the deformation shear $\tilde{H}_{ab}$ and expansion is of the order

$$\left( \frac{\tilde{H}_{ab}}{H} \right)_o < 4\epsilon,$$  \hfill (146)  

where $\epsilon \equiv \max(\epsilon_1, \epsilon_2, \epsilon_3)$ denotes the upper limit of currently observed anisotropy in the CMB radiation temperature variation, and where $\epsilon_1$, $\epsilon_2$, $\epsilon_3$, respectively, denote the dipole, quadrupole, and octopole temperature anisotropies. On choosing $\epsilon \simeq 10^{-4}$, as indicated by the recent CMB radiation anisotropy measurements, one gets that the shear deformation is at most about $10^{-3}$ of the expansion $[36]$. 

Thus, as long as we assume that the original locally inhomogeneous manifold $(\Sigma, g, K)$ does not differ too much from a standard FLRW $t = const.$ slice, the contribution to $H_o$ from $< \tilde{H}^{ab} \tilde{H}_{ab} >_o$ is certainly quite small, and we can reliably write

$$H_o^2 \simeq \frac{8\pi G}{3} < \rho >_o - \frac{1}{3} < R >_o + \frac{1}{3} < \tilde{L}_{\alpha} g_{ab} \tilde{L}_{\alpha} g_{cd} g^{ac} g^{bd} >_o.$$  \hfill (147)  

The last term involving the shift vector $\vec{\alpha}$ is a velocity effect term which is by no means small, at least a priori. Thus the renormalized Hubble constant needed, to describe by a model-FLRW a locally inhomogeneous and anisotropic universe, is not provided by a naive average even if the actual universe, to be smoothed-out in FLRW-modelling, is not too far from homogeneity and isotropy. Clearly, we are expecting that in such a case the actual contribution of the observer-velocity term is rather small, but this is a question which is difficult to handle. Technically speaking, we would need estimates on the size of the $Diff$-induced shear generated
upon smoothing by the Ricci-Hamilton flow. This is an issue under current investigation. We wish also to stress that in this velocity term there is hidden a non-trivial scale dependence. Indeed, assuming that we wish to model the actual universe by a FLRW one only up to a certain scale, then the shift vector \( \vec{\alpha} \) needed to cure the Diff-induced shear, (which is tantamount to saying the proper selection of a frame with respect to which we are carrying out the partial smoothing), depends on such a scale, and the larger the scale the larger the contribution.

There is another intriguing possible explanation for a larger than expected Hubble constant. Indeed, if we take seriously the possibility that the real universe may be close to the critical phase, as argued in the previous sections, then the contribution from the shear is not just of a conceptual value. For example, the original data set \((\Sigma, g, K)\) may be near the critical surface associated with the \(^\sharp \text{S}^3\)\(^{(1)}\)\(\sharp\text{S}^3\)\(^{(2)}\)\(\cdot\cdot\cdot\sharp\text{S}^3\)\(^{(n)}\)), this being the case if the original metric \(g\) exhibits regions of large inhomogeneities. Assuming, for simplicity, that the rate of volume expansion is spatially constant, we get that each \(\text{S}^3\)\(^{(i)}\) factor, \(i = 1, \ldots, n\), inherits a Hubble constant provided by

\[
H^2_o(i) = \frac{8\pi G}{3} < \rho>_o - \frac{1}{3} < R>_o + \frac{1}{3} < \tilde{K}_{ab}\tilde{K}_{ab}>_o .
\] (148)

These Hubble constants can be quite dominated by the large anisotropies \(< \tilde{K}_{ab}\tilde{K}_{ab}>_o\) of the original manifold. Indeed, the previous estimate on the smallness of the shear term is, strictly speaking, valid only in the observable domain defined by our past light-cone from last scattering to the present day. Thus, if the \(\text{S}^3\)\(^{(i)}\) factors resulting from the critical behavior of \((\Sigma, g, K)\) are not smaller than the spatial sections intercepted by the interior of this light-cone, we would not notice the contribution from the local shear, (since we would have been looking at a rather homogeneous and isotropic island), the large contribution would come from the regions of large inhomogeneities and anisotropies which, under the Ricci-Hamilton renormalization, undergo the topological crossover. Thus, a value of the Hubble constant may quite well depend on a possible large shear outside our observable domain.

The behavior just described also shows that the RG procedure developed preserves the size of the regions where spatial homogeneity and isotropy is observed, even if large inhomogeneity may be globally present. This tells us what are the averaging scales on which a FLRW model is a good approximation. More precisely, in order to determine the averaging scale associated with a FLRW model we want to use, to describe \((\Sigma, g, K)\), we need to understand more in detail the (conjectured) Hamilton-Thurston decomposition of \((\Sigma, g, K)\). The specific example discussed in section 4.3 may serve as an indication.

### 6.2 The scale dependence of the matter distribution

In the above analysis, we also introduced a renormalized gravitational coupling. In a sense this is superfluous since the three-dimensional metric \(g\) of \(\Sigma\) is acting as the running coupling constant, and we can always reabsorb \(G\) into a definition of \(g\). Nevertheless, the use of the renormalized coupling \(G\) may be helpful if one wishes to use the standard average of matter
\(< \rho >_o \) in the Friedmann equation, rather than the effective matter distribution \( \bar{\rho} \). The explicit expression for \( \bar{G} \) can be easily obtained by setting \( \bar{\rho} = < \rho >_o \) in (141):

\[
8\pi \bar{G} = 8\pi G + \frac{\bar{R} - < R >_o + < \bar{K}^{ab} \bar{K}_{ab} >_o - \frac{2}{3} (< k^2 >_o - \bar{k}^2)}{< \rho >_o}.
\] (149)

Notice however, that it is \( G(\eta)\rho(\eta) \) which is inferred from measurements for different scales, and thus the use of \( \bar{G} \) is not particularly remarkable.

In this connection, it is more important to discuss the dependence of \( G(\eta)\rho(\eta) \) as the local scale is varied, namely as \( \eta \) increases, (recall that \( \eta \) is the logarithmic change of the cutoff length associated with the geodesic ball coverings). For simplicity, we do this only for the case in which no shear is present \( (\bar{K}_{ab} = 0) \), and the rate of volume expansion is spatially constant \((k = \text{const.})\). Under such hypothesis, we get for the scale-dependence of the average \( < G\rho > \)

\[
\frac{\partial}{\partial \eta} [8\pi < G(\eta)\rho(\eta)>] = \frac{\partial}{\partial \eta} < R(g(\eta)) > = 2 < \bar{R}^{ik} \bar{R}_{ik} > + \frac{1}{3} \left( < R >^2 - < R^2 > \right),
\] (150)

where \( \bar{R}^{ik} = R^{ik} - \frac{1}{4} g^{ik} R \) is the trace-free part of the Ricci tensor. From this expression we see that, not only shear anistropies but, also metric anisotropies favour an increasing in \( G(\eta)\rho(\eta) \) with the scale. To give an explicit example, let us consider as an initial metric to be smoothed-out a locally homogeneous and anisotropic \( SU(2) \)-metric \( g \). Following the notation and the analysis in the paper of Isenberg and Jackson [25], we can write such a metric and its Ricci-Hamilton evolution, in terms of a left-invariant one-form basis \{\( \theta^a \), \( a = 1, 2, 3 \), on \( SU(2) \) as

\[
g = A(\eta)(\theta^1)^2 + B(\eta)(\theta^2)^2 + C(\eta)(\theta^3)^2,
\] (151)

where \( A, B, C \) are scale \((\eta)\)-dependent variables. With respect to this parameterization, the scalar curvature is given by

\[
R(\eta) = \frac{1}{2} \left[ [A^2 - (B - C)^2] + [B^2 - (A - C)^2] + [C^2 - (A - B)^2] \right].
\] (152)

While the squared trace-free part of the Ricci tensor is given by

\[
||\bar{\text{Ric}}||^2 = \frac{1}{6} \left[ [A^2 - (B - C)^2]^2 + [B^2 - (A - C)^2]^2 + [C^2 - (A - B)^2]^2 \right]
- \frac{1}{6} \left[ [A^2 - (B - C)^2][B^2 - (A - C)^2] + [A^2 - (B - C)^2][C^2 - (A - B)^2] \right]
+ \frac{1}{6} [B^2 - (A - C)^2][C^2 - (A - B)^2].
\] (153)

The Ricci-Hamilton flow for this metric \( g \) exponentially converges to the fixed point \( A = B = C = 1 \), with the normalization \( ABC = 1 \), and with \( A(\eta) \geq B(\eta) \geq C(\eta) \) for all \( \eta \) [25].
From the above expression for \( R(\eta) \) it follows that \( R(\eta) \) monotonically increases from its initial value \( R(\eta = 0) \) and exponentially approaches
\[
\lim_{\eta \to \infty} R(\eta) = \frac{3}{2}.
\] (154)

This increase is generated by the exponential damping of the anisotropic part of the Ricci tensor (154) which is smoothed by the Ricci-Hamilton flow and, roughly speaking, is redistributed uniformly in the form of scalar curvature.

Notice that if the anisotropy is large, the actual increase \( \Delta R \) from \( R(\eta = 0) \) to \( \lim_{\eta \to \infty} R(\eta) = \bar{R} \) may be quite significant. For instance, if for \( \eta = 0 \), we have \( A(\eta = 0) = 1 \), \( B(\eta = 0) = 2 \), \( C(\eta = 0) = 1/2 \), we get \( R(\eta = 0) = 7/8 \), while \( \bar{R} = 3/2 \). The renormalized scalar curvature will have increased by nearly as much as 70% of its original value.

Thus, if we smooth-out the initial data set \( (\Sigma \simeq S^3, g, K, \rho, J) \), with \( g \) the above \( SU(2) \)-metric, \( K_{ab} = \frac{1}{3} g_{ab} k \), with \( k = \text{const.} \), \( J = 0 \), and with the matter density \( \rho \) such that the Hamiltonian constraint holds, we get that \( G(\eta)\rho(\eta) \) monotonically increases as \( \eta \to \infty \), exponentially approaching a fixed value.

Under the above simplifying assumptions concerning the shear and the rate of volume expansion, we get
\[
8\pi G \bar{\rho} = 8\pi G < \rho >_o + \bar{R} - < R >_o
\] (155)

and, as remarked above, the correction term \( \bar{R} - < R >_o \) may be quite large, of the same order of magnitude as the naive average \( 8\pi G < \rho >_o \).

More generally, we can easily get an expression for the variation of \( G(\eta) < \rho(\eta) > \) as \( \eta \) is increased from a given scale, say from \( \eta_o \) to a larger scale \( \eta = \eta_o + \delta \eta \).

From equation (150) we immediately get
\[
8\pi G(\eta) < \rho(\eta) > = 8\pi G(\eta_o) < \rho(\eta_o) > + 2 < \tilde{R}^{ik} \tilde{R}_{ik} >_{(\eta_o)} \delta \eta
\] (156)
\[
+ \frac{1}{3} \left( < R >^2 - < R^2 > \right)_{(\eta_o)} \delta \eta + \mathcal{O}(\delta \eta^2).
\]

With our simplifying assumptions, that there is no shear and that the rate of volume expansion is spatially constant, we can exploit the Hamiltonian constraint and rewrite (157) as

\[
G(\eta) < \rho(\eta) > = G(\eta_o) < \rho(\eta_o) > + \frac{1}{4\pi} < \tilde{R}^{ik} \tilde{R}_{ik} >_{(\eta_o)} \delta \eta
\] (157)
\[
- \frac{8}{3} \pi G^2(\eta_o) < \rho(\eta_o) >^2 \left( \frac{< \rho^2(\eta_o) > - < \rho(\eta_o) >^2}{< \rho(\eta_o) >^2} \right) \delta \eta + \mathcal{O}(\delta \eta^2).
\]

Thus, in leading order, a large density contrast \( \frac{< \rho^2(\eta_o) > - < \rho(\eta_o) >^2}{< \rho(\eta_o) >^2} \), at a given observational scale \( \eta_o \), tends to reduce the value of \( G(\eta) < \rho(\eta) > \) as the scale of observation is increased.
This reduction effect should dominate on local scales where large inhomogeneities are present. On sufficiently large scales \( \eta_0 \), cosmological data suggest that the density contrast tends to decrease, and in such a case, as the scale of observation is further increased, equation (158) shows that \( G(\eta) < \rho(\eta) > \) now tends to become larger, with deviations from the original value driven by anisotropies in the curvature.

Thus, the behavior of the product \( G(\eta) < \rho(\eta) > \) under a variation of the observational scale is related to a competition between anisotropy and density contrast, and it may be of significance in the correct interpretation of recent cosmological data. In particular, a sufficiently large spatial anisotropy (even maintaining local homogeneity) may increase \( G(\eta) < \rho(\eta) > \) in a significant way as \( \eta \) is increased.

6.3 The choice of the initial data set

The general picture arising from the above analysis is that we pick up an appropriate initial data set which, when evolved, gives rise to the real space-time. The description of this data set and of the resulting space-time is too detailed for being of relevance to cosmology. Intuitively, one would like to eliminate somehow all the unwanted (coupled) fluctuations of matter and space-time geometry on small scales, and thus extract the effective dynamics capturing the global dynamics of the original space-time. The possibility of actually implementing such an approach is strongly limited by the fact that we do not know a priori the structure of the space-time we are dealing with. But we may alternatively decide to handle the unwanted fluctuations at the level of data sets, since the time evolution of the initial data set for the Einstein equations is actually determined by the very constraints which that data set has to satisfy. As we have seen above, this can be done quite effectively in the framework of the Renormalization Group which is naturally well suited to this purpose. However, and here we come to the point we wish to make clear, different initial data sets giving rise to the same inhomogeneous and anisotropic space-time, may yield smoothed data set giving rise to different FLRW space-times. In other words, the renormalization procedure and the dynamics do not commute. The dynamics gives rise to the crossover between different FLRW space-times, or more generally, between different renormalized models of the same original irregular space-time.

This situation is in fact not so paradoxical as it may seem. From a thermodynamical point of view, we have seen that one of the members of the initial data set, namely the three-metric \( g \), plays the rôle of a temperature. Thus, by varying the metric, one can move through the possible pure phases of the thermodynamical system considered. In this sense, a real, locally inhomogeneous universe, is to be considered as akin to a generic complex thermodynamical system. The possible locally homogeneous cosmological models, arising from it by suitable choices of initial data set to smooth-out, correspond to its distinct pure phases. The resulting dynamics yields, in an analogy with common statistical system, a dynamical crossover between different pure phases. This makes accessible in cosmology too, the whole subject of critical phenomena with a plethora of interesting consequences. Critical phenomena are always manifested macroscopically, as phase transitions are collective phenomena in their nature. This aspect may turn out to be of importance for the study of structure formation and clustering in the universe.

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The issue that now needs consideration concerns the existence of natural initial data sets, in the real lumpy universe, with respect to which the above Renormalization Group smoothing can be implemented. As remarked previously, the formalism is particularly well suited to a data set supported on slices of constant extrinsic time, \((\text{viz.}, \text{ spatially constant rate of volume expansion, } k)\). This is so simply because the Ricci-Hamilton flow, characterizing the scale-dependence in the fluctuations of geometry, is most conveniently normalized to preserve volume. However, depending on the particular geometry we wish to smooth, different normalizations can be envisaged too \([25]\), and the relevance of constant extrinsic time initial data set can be simply traced back to their importance in the standard analysis of the initial value problem in relativistic cosmology.

A suitable slice of a frame comoving with matter is another \textit{natural} choice that almost immediately comes about. However, in general, we cannot choose surfaces orthogonal to the cosmic fluid flow lines, since if rotation is present, they do not exist. A more proper choice would be to select \textit{the surface of constant matter density} as the surfaces of constant time, (see \([15]\) and references quoted therein). It has been argued recently that this choice is rather optimal. In such a slicing \([15]\), and in the \textit{observable domain}, the spacetime metric takes the form

\[
g^{(4)} = -A^2(x^\alpha)dt \otimes dt + S^2(t,x^h)f_{ij}(t,x^h)dx^i \otimes dx^j,
\]

(158)

where \(\alpha = 1,2,3,4,\ \text{and } i,j = 1,2,3\). From an observative point of view, the function \(A^2(x^\alpha)\) is nearly constant since the acceleration of the corresponding timelike congruence is small; \(S^2(t,x^h)\) is nearly independent of time since the expansion of the congruence is almost spatially constant, and finally also \(f_{ij}(t,x^h)\) is nearly independent of time, because the shear is nearly zero. These remarks suggest that the slices of constant time characterized in this way are, at least in our observative domain, nearly spaces of constant curvature, so that they are the most suitable for implementing the above smoothing procedure.

In this connection notice that, for an initial data set associated with such a surface of constant matter density, some of the formulae of the previous sections are simplified. We refer in particular to the expression \([158]\) providing the scale dependence of \(G(\eta) < \rho(\eta) >\) which now reduces to

\[
G(\eta) < \rho(\eta) > = G(\eta_0) < \rho(\eta_0) > + \frac{1}{4\pi} < \tilde{R}^{ik}\tilde{R}_{ik} >_{(\eta_0)} \delta \eta + O(\delta \eta^2).
\]

(159)

Thus, for an initial data set associated with a surface of constant matter density, the product \(G(\eta)\rho(\eta)\) increases with the averaging scale as we average-out the local anisotropies in the geometry. And, as remarked in the previous section, \(\sqrt{G(\eta)} - \sqrt{G\rho}\) can be quite large, of the same order of magnitude of \(G\rho\), if \(f(x^h,t)\) is sufficiently spatially anisotropic.

Finally, one may wish to choose initial data set corresponding to a frame minimizing the anisotropies in the CMB radiation. This is characterized by that unique four-velocity which eliminates the CMB dipole. This corresponds to a family of observers moving with the background radiation. This choice has an advantage that it can be accurately determined by local
7 Concluding remarks

According to the contents of this paper the key idea on which our whole analysis rests is that of RG, namely, the involved physics is that of the running (scale dependence, be it energy or momentum scales) of the couplings and the relevant quantities, accordingly. This philosophy is recognized and established in particle physics, e.g. it is well known that the fine structure constant $\alpha$ measured at low energies is different from the one measured at the LEP energy scale. In each case it is the presence of “fluctuations” (of any kind) that requires a scale dependent redefinition (“dressing”) of the physical parameters which can, in turn, modify them, as well as the very structure of the theory, in a non-trivial way. Applications of the RG to particle physics have usually been in the ultra-violet limit (e.g. in QED, QCD, GUT) whereas in condensed matter physics they have been in the infra-red limit, in the study of critical phenomena and phase transitions.

We have taken this infra-red direction in cosmology. The application of the concept of running of the physical quantities, motivated by RG, appears to be a new important feature in a cosmological setting, providing (at least) partial explanation of some controversies of standard cosmology, which we are going to discuss below. Let us first point out that generally running, also, of cosmological quantities is as such motivated by the asymptotically free higher derivative quantum gravity, according to which the gravitational constant is asymptotically free \[3\]. Taking into account this fact, of running $G$, one can explore its consequences in the standard FLRW cosmology (cf. \[19, 5\]).

A word of caution is in order - since we do not have an ultimate theory of quantum gravity (QG), such approaches to cosmology are not on a rigorous basis from a theoretical point of view and should rather be taken as phenomenological. In principle, once we have a valid QG theory will one be able to directly derive a RG equations for various cosmological quantities. Although we have taken here a more standard view in which a split exists between the background (associated with infra-red effects) and renormalization of fluctuations, there may well be scales where such a split is not sensible at all. However, a RG capable of interpolating between the qualitatively different degrees of freedom in a parameter space of gravity, remains to be developed which may, after all, as well be possible after QG theory is within our reach.

One of the major issues in modern cosmology is concerned with the value of the Hubble constant and the apparent conflict between the observed age of the universe and the predicted one, in the standard FLRW model, based on the recent measurements of the Hubble constant. Namely, the recent measurements of the Hubble constant using the Virgo cluster (distance $\geq 15$ Mpc) strongly support that the Hubble constant $H_o$ has somewhat larger value $h = 0.87 \pm 0.07$\[10\]. At the same time, other distance indicators yield a systematically smaller value of $H_o$, e.g. around $0.55 \pm 0.08$ using NGC 5253 at a distance of 4 Mpc \[15\], while an analysis of the gravitational lensing of QSO 0957+561 indicates $h = 0.50 \pm 0.17$ \[11\]. When one calculates now the age of the universe, using the larger value of $H_o$, in the FLRW model one runs into a

$h$ is $H_o$ measured in units of 100 km/sec/Mpc.
serious problem, as the predicted age turns out to be too small to accommodate the measured ages (∼14-18 Gyr) of the globular clusters in our galaxy [44, 32].

Moreover, typical inferred values of the density parameter \( \Omega_o = \rho_o/\rho_{o}^{\text{crit}} \) (\( \rho_o \) is the present value of the total energy density of the universe and \( \rho_{o}^{\text{crit}} \) its present critical energy density, defined as \( \rho_{o}^{\text{crit}} = 3H_o^2/(8\pi G) \)) increase correspondingly with the increasing size of various structures (e.g. [29]). These measurements can at most account for a fraction of \( \Omega_o \) which, according to the inflationary paradigm, should be equal to 1. This in turn is one of the reasons for postulating the existence of non-baryonic Dark Matter (DM) which is also required to explain the structure formation.

Various people have since then looked at possible theoretical alternatives to these DM scenarios, such as e.g. introducing a cosmological constant in the Einstein equations or \textit{ad hoc} modifications of the usual theory of gravity. The important point in this respect may as well be the one addressed in this paper. It is usually taken for granted that, on large scales, the universe is described by the FLRW solution. There is no alternative really since we do not know any solution of Einstein’s equations capable of describing a clumpy universe. Nevertheless, even in the absence of explicit fine-grained models, we would like to know how in principle, and when, one could extract a background model from an inhomogeneous one, such that: (i) they both obey, “approximately”, the Einstein equations despite the averaging or smoothing involved, and (ii) observational determinations of cosmological parameters (\( H_o, \Omega_o, \ldots \)) correspond in a sensible way to that mathematical averaging procedure. Thus an issue of importance for cosmology [27], is the question on what scale is the FLRW model supposed to describe the universe? Likewise, what averaging scale are we referring to when we give the value of \( \Omega_o \), whose definition necessarily refers to an idealised, i.e. smoothed background model.

There has been recently an increased effort in this direction with some interesting results, as e.g. that the coarse-graining effects could be non-negligible in the context of affecting the age of the universe. For example, [17] considered a model, of locally open (underdense) universe embedded in the spatially flat universe, in which the expansion rate in our local universe is larger than the global average. Similar model was considered in [38] where a local void in the global FLRW model was studied and the inhomogeneity described by the Lemaître-Tolman-Bondi solution. The results indicate that if we happened to live in such a void, but insisted on interpreting the observations by the FLRW model, the Hubble constant measurements could give results depending on the separation of the source and the observer, providing a possible explanation for the wide range of their reported values and capable of resolving the age-of-universe problem.

On the other hand [19] studied the QG effects at cosmological scales (the phenomenon in question is exactly that of quantum coherence, known from a laboratory to happen on macroscopic scales of the order of cm), assuming asymptotic freedom of the gravitational constant and incorporating running \( G \), according to the appropriate RG equations, into FLRW model. Such \( G \) takes the value of Newton’s constant \( G_N \) at short distances but then slowly rises as distance increases. However, as mentioned in [19], the RG equations used there might not be applicable in the infra-red regime studied, but these concerns were put aside having in mind the absence of any other available beta functions for QG in the infra-red regime.

One can also approach the averaging problem, modifying the FLRW metric (or equivalently Cosmological Principle) and the Einstein equations, by an introduction of a generalized scale
factor which depends both on $t$ and the scale $r$ \[28\]. This introduces the scale dependence of $(G\rho)$ and running of other cosmological quantities such as $H_0$, $\Omega_0$, and the age of the universe $t_o$, as functions of distance scales.

Let us notice that this picture of running of cosmological quantities comes about naturally in our approach which is physically motivated by RG, namely, due to increasing of the gravitational constant with scale, (and possibly increasing amount of DM), as discussed in the previous section, $\Omega_0$ has effectively increased too. Moreover, since the scale factor is governed by the scale dependent $(G\rho)$, it now seems to depend on the scale $r$ as well, i.e. it increases at any fixed time with the increasing distance. Consequently, the value of $H_0$ is not the same everywhere in the observable universe and depends on a scale where it is measured. Moreover, since $H_0 \sim 1/t_0$ the universe becomes older when its age is estimated on a smaller scale. This is not to be taken as implying that the age depends on where one calculates (every observer using the same scale $r$ at some time $t_1$ will obtain the same age). The key point to emphasize is that, having in mind the RG arguments and interpretation, a direct comparison of cosmological quantities makes sense only when they are measured (or calculated) with respect to the same scale, since the same quantity can take different values at different scales.

Notice that independently, also Quantum Cosmology advocates, though in a different context of bubble universes, a possibility that we may live in a universe in which the value of Hubble constant and the measured density are different in different places and in our local neighborhood $\Omega_o$ may well be less than $1$ \[34\].

This makes our proposal even more promising.

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We wish to thank Juan Pérez-Mercader and Giancarlo Jug for interesting discussions on the Renormalization Group.

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Figure Captions

Fig. 1 A portion of a minimal geodesic balls covering. The dotted disks are the \( \frac{\epsilon}{2} \)-balls which pack a given region. The larger, undotted disks, represent the \( \epsilon \)-balls which provide the covering.

Fig. 2 We can approximate the average over \( \Sigma \) by averaging over euclidean balls whose Lebesgue measure has been locally weighted through Puiseux’ formula. In this drawing, the \( \frac{\epsilon}{2} \)-geodesic balls are represented by curved disks on \( \Sigma \), while the corresponding euclidean balls are correctly depicted as three-dimensional.

Fig. 3 The local average of a function \( f \) over a geodesic ball \( B(x, \epsilon_0) \) feels the underlying curvature of the manifold through Puiseux’ formula. In particular, in passing from \( B(x, \epsilon_0) \) to the larger ball \( B(x, \epsilon_0 + \eta) \) we get correction terms which depend on the fluctuations in curvature.

Fig. 4 The intricate but symmetrical intersection pattern of an array of \( \frac{\epsilon}{2} \)-balls in the 2D-plane. Here, the \( \frac{\epsilon}{2} \)-balls are dotted while the overlapping \( \epsilon \)-balls, providing the covering, are the dashed circles. The solid circles and the arcs of circles describe the intersection pattern of a \( \frac{3}{2} \epsilon \)-ball. On a curved manifold of bounded geometry, the pattern is more complicated and not symmetrical at all. Nonetheless, one may easily obtain a recursive definition of blocking by exploiting a partition of unity argument, explained in the text.

Fig. 5 The intersection pattern of a \( 2\epsilon \)-ball \( B(x_j, 2\epsilon) \) with five \( \epsilon \)-balls \( B(x_h, \epsilon) \). The \( 2\epsilon \)-ball is represented by a dotted disk, while the \( \epsilon \)-balls are represented by solid circles. The white disks represent the packing of \( B(x_j, 2\epsilon) \) with \( \frac{\epsilon}{2} \) balls. The standard representation of the integral over \( B(x_j, 2\epsilon) \), through the partition of unity subordinated to the \( B(x_h, \epsilon) \)-covering, yields the recursive relation \( \psi_{m+1}(j) = \sum_h \psi_m(h) \).

Fig. 6 Even if the function \( f \) is constant over \( \Sigma \), its average values over the geodesic balls, \( B(x_1, \epsilon) \) and \( B(x_2, \epsilon) \), are generally uncorrelated owing to the random fluctuations in the geometry of the balls.

Fig. 7 The matter distribution \( \rho \), \( \vec{J} \), the lapse function \( \alpha \), and the shift vector field \( \vec{\alpha} \) associated with the instantaneous observers on \( \Sigma \). Here, \( \vec{n} \) denotes the four-dimensional unit, future-pointing, normal to \( \Sigma \). The vector field \( \alpha \vec{n} + \vec{\alpha} \) connects the corresponding points in the two infinitesimally close slices \( \Sigma \) and \( \Sigma_{\delta t} \).
Given a probability law, according to which matter is distributed at a given length scale (say, the planetary scale), we can get the corresponding probability distribution obtained by averaging the matter variables over regions of ever increasing scales. The resulting effective Hamiltonians are defined up to additive constants which affect the renormalized mass density and the renormalized momentum density. Such indeterminacy can be naturally removed by enforcing the Hamiltonian and divergence constraints at each step of the renormalization procedure.

Upon enlarging the ball, from $\epsilon$ to $e^\gamma \epsilon$, we alter the average value of the matter variables $\rho$ and $\vec{J}$, (even if $\rho$ and $\vec{J}$ do not vary appreciably), since there can be strong curvature fluctuations in the annulus $B(x, e^\gamma \epsilon) \setminus B(x, \epsilon)$.

If we enlarge the ball from $\epsilon$ to $e^\eta \epsilon$, while deforming the metric according to the Ricci-Hamilton flow (52), (for a parameter time $\eta = \gamma$), then the average value remains as constant as it can be since (52) smoothes the curvature fluctuations in the annulus $B(x, e^\eta \epsilon) \setminus B(x, \epsilon)$.

If we blow up the metric of two three-spheres $S_3^{(1)}$ and $S_3^{(2)}$, (here represented as two-spheres), around the points $y_1$ and $y_2$ and then join the resulting manifolds, we obtain the connected sum $S_3^{(1)} \# S_3^{(2)}$. According to this procedure, the neck $S^2 \times \mathbb{R}$ inherits a cylindrical metric up to exponentially small correction terms.

The Ricci-Hamilton flow of $S_3^{(1)} \# S_3^{(2)}$ can be expressed in terms of the $\eta$-evolution of the inclusion maps $\chi_{(\alpha)}$, $\alpha = 1, 2, 3$, and of the known evolution of the locally homogeneous metrics on $S_3^{(i)} \setminus B(y_i)$ and $S^2 \times \mathbb{R}$.

Different deformations of the initial three-manifold $\Sigma$ may have quite different fates. If the waist of $\Sigma$ is increased enough by the deformation, (i.e., if it is rounded), the Ricci-Hamilton flow will evolve $\Sigma$ toward the round three-sphere. If the waist is shrunk enough by the initial deformation, the Ricci-Hamilton flow will evolve $\Sigma$ toward $S_3^{(1)} \# S_3^{(2)}$. The three-manifolds attracted towards $S_3^{(1)} \# S_3^{(2)}$ define a critical surface.

Two neighboring three-manifolds, $(\Sigma, g)$ and $(\Sigma, g + \delta K)$, with the same volume, evolving under the Ricci-Hamilton flow toward isometric round three-spheres $(S^3, g_{can})$ and $(S^3, g_{can}^*)$. In general, $g_{can}$ and $g_{can}^*$ differ by an infinitesimal diffeomorphism $\phi$ generated by a vector field $v$, i.e. $g_{can}^* = g_{can} + \delta \left( L_v g_{can} - \frac{1}{2} g_{can} \nabla_i v^i \right)$. By exploiting the freedom in choosing the shift vector field $\tilde{\alpha}$, we can get rid of $\phi$. 

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Fig. 15 The time evolution of an inhomogeneous and anisotropic initial data set and its Ricci-Hamilton renormalized counterpart. Notice, that \((\bar{M}^4, \bar{g}^{(4)})\) depends in a *sensible way* on the particular initial data set \((\Sigma, g, K, \rho, \bar{J})\) which is renormalized according to the Ricci-Hamilton flow. Different initial data sets, corresponding to the same inhomogeneous and anisotropic space-time \((M^4, g^{(4)})\), may generate *distinct* renormalized model space-times \((\bar{M}^4, \bar{g}^{(4)})\). This seemingly paradoxical situation has a natural explanation in terms of the RG approach.
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