CONVEX FUNCTION APPROXIMATIONS FOR MARKOV DECISION PROCESSES

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ABSTRACT. This paper studies function approximation for finite horizon discrete time Markov decision processes under certain convexity assumptions. Uniform convergence of these approximations on compact sets is proved under several sampling schemes for the driving random variables. Under some conditions, these approximations form a monotone sequence of lower or upper bounding functions. Numerical experiments involving piecewise linear functions demonstrate that very tight bounding functions for the fair price of a Bermudan put option can be obtained with excellent speed (fractions of a cpu second). Results in this paper can be easily adapted to minimization problems involving concave Bellman functions.

Keywords. Convexity, Dynamic programming, Function approximation, Markov decision processes

1. Introduction

Sequential decision making under uncertainty can often be framed using Markov decision processes (see [10, 7, 17] and the references within). However, due to the tedious nature of deriving analytical solutions, some authors have suggested the use of approximate solutions instead [16] and this view has been readily adopted due to the advent of cheap and powerful computers. Typical numerical methods either use a finite discretization of the state space [15, 6] or using a finite dimensional approximation of the target functions [20]. This paper will focus on the latter approach for Markov decision processes containing only convex functions and a finite number of actions. Convexity assumptions are often used because it affords many theoretical benefits and the literature is well developed [18]. When there are a finite number of actions, there are two main issues facing function approximation methods in practice. The first involves estimating the conditional expectation in the Bellman recursion. The second deals with representing the reward functions and the expected value functions using tractable objects. This paper approaches the expectation operator using either an appropriate discretization of the random variables or via Monte Carlo sampling. The resulting Bellman functions are then approximated using more tractable convex functions. While many approaches have appeared in the literature [2, 16], this paper differs from the usual in the following manner. Typical approaches assume a countable state space or bounded rewards/costs. However, many problems in practice do not satisfy this and so this paper will not take this approach. This paper also directly exploits convexity to extract desirable convergence properties. Given that decisions are often made at selected points in time for many realistic problems, this paper assumes a discrete time setting and this avoids the many technical details associated with continuous time.

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Regression based methods \[4, 19, 14\] have become a popular tool in the function approximation approach. These methods represent the conditional expected value functions as a linear combination of basis functions. However, the choice of an appropriate regression basis is often difficult and there are often unwanted oscillations in the approximations. This paper is related closest to the work done by \[9\] and \[11\]. The authors in \[9\] considered monotone increasing and decreasing bounding function approximations for discrete time stochastic control problems using an appropriate discretization of the random variable. Like this paper, they assumed the functions in the Bellman recursion satisfy certain convexity conditions. However, their functions are assumed to be bounded from below unlike here. In their model, an action is chosen to minimise the value as opposed to maximise in our setting. In addition, this paper considers an extra layer of function approximation to represent the resulting functions in the Bellman recursion. Nonetheless, this paper adapts some of their brilliant insights. In \[11\], the author exploits convexity to approximate the value functions by using convex piecewise linear functions formed using operations on the tangents from the reward functions. The scheme in \[11\] results in uniform convergence on compact sets of the approximations and has been successfully applied to many real world applications \[13, 12\]. Unlike \[11\], this paper will not impose global Lipschitz continuity on the functions in the Bellman recursion and does not assume linear state dynamics. A much more general class of function approximation is also studied here. Moreover, this paper considers an extra layer of function approximation to represent the resulting functions in the Bellman recursion. Nonetheless, this paper adapts some of their brilliant insights. In \[11\], the author exploits convexity to approximate the value functions by using convex piecewise linear functions formed using operations on the tangents from the reward functions. The scheme in \[11\] results in uniform convergence on compact sets of the approximations and has been successfully applied to many real world applications \[13, 12\]. Unlike \[11\], this paper will not impose global Lipschitz continuity on the functions in the Bellman recursion and does not assume linear state dynamics. A much more general class of function approximation is also studied here. Moreover, this paper proves the same type of convergence under random Monte Carlo sampling of the state disturbances as well as deriving bounding functions. In this sense, this paper significantly generalizes the remarkable work done by \[11\].

This paper is organized as follows. The next section introduces the finite horizon discrete time Markov decision process setting and the convexity assumptions used. The state is assumed to consist of a discrete component and a continuous component. This is a natural assumption since it covers many practical applications. Convexity of the target and approximating functions is assumed on the continuous component of the state. A general convex function approximation scheme is then presented in Section 3. In Section 4, uniform convergence of these approximations to the true value functions on compact sets is proved using straightforward arguments. This convergence holds under various sampling schemes for the random variables driving the state evolution. Under conditions presented in Section 5 and Section 6, these approximations form a non-decreasing sequence of lower bounding or a non-increasing sequence of upper bounding functions for the true value functions, respectively. This approach is then demonstrated in Section 7 using piecewise linear function approximations for a Bermudan put option. The numerical performance is impressive, both in terms of the quality of the results and the computational times. Section 8 concludes this paper. Note that in this paper, global Lipschitz continuity is referred to as simply Lipschitz continuity for shorthand.

2. Markov decision process

Let \( (\Omega, \mathcal{F}, \mathbb{P}) \) represent the probability space and denote time by \( t = 0, \ldots, T \). The state is given by \( X_t := (P_t, Z_t) \) consisting of a discrete component \( P_t \) taking values in some finite set \( \mathbf{P} \) and a continuous component \( Z_t \) taking values in an open convex set \( \mathbf{Z} \subseteq \mathbb{R}^d \). This paper will refer to \( \mathbf{X} := \mathbf{P} \times \mathbf{Z} \) as the state space. Now at each \( t = 0, \ldots, T - 1 \), an action \( a \in \mathbf{A} \) is chosen by the agent from a finite set \( \mathbf{A} \). Suppose the starting state is given by \( X_0 = x_0 \) with probability one. Assume that \( (P_t)_{t=0}^T \) evolves as a controlled Markov chain with transition probabilities \( \alpha_{t+1}^{a,p,p'} \) for \( a \in \mathbf{A}, p,p' \in \mathbf{P} \), and \( t = 0, \ldots, T - 1 \). Here \( \alpha_{t+1}^{a,p,p'} \) is the probability from moving from \( P_t = p \) to \( P_{t+1} = p' \) after applying action \( a \) at time \( t \). Assume the Markov
process \((Z_t)_{t=1}^T\) is governed by action \(a\) via
\[
Z_{t+1} = f_{t+1}(W_{t+1}^a, Z_t)
\]
for some random variable \(W_{t+1}^a : \Omega \to \mathbf{W} \subseteq \mathbb{R}^d\) and measurable transition function \(f_{t+1} : \mathbf{W} \times \mathbf{Z} \to \mathbf{Z}\). At time \(t = 0, \ldots, T-1\), \(W_{t+1}^a\) is the random disturbance driving the state evolution after action \(a\) is chosen. The random variables \(W_{t+1}^a\) and \(f_{t+1}(W_{t+1}^a, z)\) are assumed to be integrable for \(a \in \mathbf{A}, z \in \mathbf{Z}\), and \(t = 0, \ldots, T-1\).

The decision rule \(\pi_t\) gives a mapping \(\pi_t : \mathbf{X} \to \mathbf{A}\) which prescribes an action \(\pi_t(x) \in \mathbf{A}\) for a given state \(x \in \mathbf{X}\). A sequence \(\pi = (\pi_t)_{t=0}^{T-1}\) of decision rules is called a policy. For each starting state \(x_0 \in \mathbf{X}\) and each policy \(\pi\), there exists a probability measure such that \(\mathbb{P}^{x_0,\pi}(X_0 = x_0) = 1\) and
\[
\mathbb{P}^{x_0,\pi}(X_{t+1} \in \mathbf{B} \mid X_0, \ldots, X_t) = K_t^{\pi_t(X_t)}(X_t, \mathbf{B})
\]
for each measurable \(\mathbf{B} \subset \mathbf{X}\) at \(t = 0, \ldots, T - 1\) where \(K_t^\pi\) denotes our Markov transition kernel after applying action \(a\) at time \(t\). The reward at time \(t = 0, \ldots, T-1\) is given by \(r_t : \mathbf{X} \times \mathbf{A} \to \mathbb{R}\). A scrap value \(r_T : \mathbf{X} \to \mathbb{R}\) is collected at terminal time \(t = T\). Given starting \(x_0\), the controller’s goal is to maximize the expectation
\[
v_0^\pi(x_0) = \mathbb{E}^{x_0,\pi} \left[ \sum_{t=0}^{T-1} r_t(X_t, \pi_t(X_t)) + r_T(X_T) \right]
\]
over all possible policies \(\pi\). That is, to find an optimal policy \(\pi^* = (\pi_t^*)_{t=0}^{T-1}\) satisfying \(v_0^{\pi^*}(x_0) \geq v_0^\pi(x_0)\) for any policy \(\pi\). There are only a finite number of possible policies in our setting. Note that this paper focuses only on Markov decision policies since history dependent policies do not improve the above total expected rewards [10, Theorem 18.4].

Let \(K_t^a\) represent the one-step transition operator associated with the transition kernel. For each action \(a \in \mathbf{A}\), the operator \(K_t^a\) acts on functions \(v : \mathbf{X} \to \mathbb{R}\) by
\[
(K_t^a v)(x) = \int_{\mathbf{X}} v(x') K_t^a(x, dx') \quad \text{or}
\]
\[
(K_t^a v)(p, z) = \sum_{p' \in \mathbf{P}} a_{t+1}(a, p', p') \mathbb{E}[v(p', f_{t+1}(W_{t+1}^a, z))].
\]
If an optimal policy exists, it may be found using the following dynamic programming principle. Introduce the Bellman operator
\[
\mathcal{T}_t v(p, z) = \max_{a \in \mathbf{A}} \{r_t(p, z, a) + K_t^a v(p, z)\}
\]
for \(p \in \mathbf{P}, z \in \mathbf{Z}\), and \(t = 0, \ldots, T-1\). The resulting Bellman recursion is given by
\[
v_T^* = r_T, \quad v_t^* = \mathcal{T}_t v_{t+1}^*
\]
for \(t = T - 1, \ldots, 0\). If it exists, the solution \((v_t^*)_{t=0}^T\) gives the value functions and determines an optimal policy \(\pi_t^*\) by
\[
\pi_t^*(p, z) = \arg \max_{a \in \mathbf{A}} \{r_t(p, z, a) + K_t^a v_t^*(p, z)\}
\]
for \(p \in \mathbf{P}, z \in \mathbf{Z}\), and \(t = 0, \ldots, T-1\).
2.1. Convex value functions. Since $A$ is finite, the existence of the value functions is guaranteed if the functions in the Bellman recursion are well defined. This occurs, for example, when the reward, scrap, and transition operator satisfies the following Lipschitz continuity. Let $\| \cdot \|$ represent some norm below.

**Theorem 1.** If $r_t(p, z, a)$, $r_T(p, z)$, and $K^p_t\|z\|$ are Lipschitz continuous in $z$ for $a \in A$, $p \in P$, and $t = 0, \ldots, T - 1$, then the value functions exists.

**Proof.** If the functions in the Bellman recursion have an upper bounding function [1, Definition 2.4.1], there will be no integrability issues [1, Proposition 2.4.2]. For $t = 0, \ldots, T - 1$, $a \in A$ and $p \in P$, Lipschitz continuity yields

$$ |r_t(p, z, a)| \leq |r_t(p, z', a)| + c\|z - z'| \quad \text{and} \quad |r_T(p, z)| \leq |r_T(p, z')| + c\|z - z'|$$

for some constant $c$ and for $z, z' \in Z$. Therefore, $|r_t(p, z, a)| \leq c'(1 + \|z\|)$ and $|r_T(p, z)| \leq c'(1 + \|z\|)$ for some constant $c'$ for $p \in P$ and $a \in A$. By assumption on $K^p_t\|z\|$, for $a \in A$, $p \in P$, and $t = 0, \ldots, T - 1$,

$$ \left| \int_X 1 + \|z''\|K^p_t((p, z), d(p', z')) \right| \leq c''(1 + \|z\|)$$

for some constant $c''$. Therefore, an upper bounding function for our Markov decision process is given by $1 + \|z\|$ with constant coefficient given by $\max\{c', c''\}$. \qed

Note that on a finite dimensional vector space ($d < \infty$), all norms are equivalent and so the choice of $\| \cdot \|$ is not particularly important. The existence of the value functions and an optimal policy is problem dependent and so the following simplifying assumption is made to avoid any further technical diversions.

**Assumption 1.** Bellman functions $r_T(p, z)$, $r_t(p, z, a)$, and $K^p_tv_{t+1}^\pi(p, z)$ are well defined and real-valued for all $p \in P$, $z \in Z$, $a \in A$, and $t = 0, \ldots, T - 1$.

Under the above assumption, value functions $(v_t^\pi)^{T-1}_{t=0}$ and an optimal policy $\pi^*$ exists. Further, they can be found via the Bellman recursion presented earlier. Since convex functions plays a central role in this paper and to avoid any potential misunderstanding regarding this, the following definition is provided.

**Definition 1.** A function $h : Z \to \mathbb{R}$ is convex in $z$ if

$$ h(\alpha z' + (1 - \alpha) z'') \leq \alpha h(z') + (1 - \alpha) h(z'')$$

for $z', z'' \in Z$ and $0 \leq \alpha \leq 1$.

Note that since $Z$ is open, convexity in $z$ automatically implies continuity in $z$. Now impose the following continuity and convexity assumptions.

**Assumption 2.** Let $f_t(w, z)$ be continuous in $w$ for $z \in Z$ and $t = 1, \ldots, T$.

**Assumption 3.** Let $r_t(p, z, a)$ and $r_T(p, z)$ be convex in $z$ for $a \in A$, $p \in P$ and $t = 0, \ldots, T - 1$. If $h(z)$ is convex in $z$, then $K^p_t h(z)$ is also convex in $z$ for $a \in A$.

Assumption 2 will be used to apply the continuous mapping theorem in the proofs and Assumption 3 guarantees that all the value functions $(v_t^\pi(p, z))^{T-1}_{t=0}$ are convex in $z$ for $t = 0, \ldots, T$ and $p \in P$. Convexity can be preserved by the transition operator (as stated in Assumption 3) due to the interaction between the value function and the transition function. Let us explore just a few examples for $d = 1$:
• If \( v(p, z) \) is non-decreasing convex in \( z \) and \( f_{t+1}(w, z) \) is convex in \( z \), the composition \( v(p, f_{t+1}(w, z)) \) is convex in \( z \).

• Similarly, \( v(p, f_{t+1}(w, z)) \) is convex in \( z \) if \( v(p, z) \) is convex non-increasing in \( z \) and \( f_{t+1}(w, z) \) is concave in \( z \).

• If \( f_{t+1}(w, z) \) is affine linear in \( z \), \( K^a_{t+1} v(p, z) \) is also convex in \( z \) if \( v(p, z) \) is convex in \( z \). Note this is a consequence of the first two cases since an affine linear function is both convex and concave.

• If the space \( Z \) can be partitioned so that in each component any of the above cases hold, the function composition \( v(p, f_{t+1}(w, z)) \) is also convex in \( z \).

3. Approximation

Denote \( G^{(m)} \subset Z \) to be a \( m \)-point grid. This paper adopts the convention that \( G^{(m)} \subset G^{(m+1)} \) and \( \bigcup_{m=1}^\infty G^{(m)} \) is dense in \( Z \). Suppose \( h : Z \to \mathbb{R} \) is a continuous function and introduce some function approximation operator \( S_{G^{(m)}} \) dependent on the grid (this will be clarified shortly) which approximates \( h \) using another more tractable continuous function. Some examples involving piecewise linear approximations are depicted in Figure 1.

![Figure 1. Approximating the smooth target functions using piecewise linear functions.](image)

The first step in dealing with the Bellman recursion is to approximate the transition operator (1). For each time \( t = 0, \ldots, T-1 \) and action \( a \in A \), choose a suitable \( n \)-point disturbance sampling \( (W_{t+1}^a(k))_{k=1}^n \) with weights \( (p_{t+1}^{a,n}(k))_{k=1}^n \). Define the modified transition operator by

\[
K_{t}^{a,n}(p, z) = \sum_{p' \in P} \alpha_{t+1}^{a,p,p'} \sum_{k=1}^n \rho_{t+1}^{a,n}(k) v(p', f_{t+1}(W_{t+1}^a(k), z))
\]

and the modified Bellman operator by

\[
T_{t}^{(m,n)}(p, z) = \max_{a \in A} \left( S_{G^{(m)}} r_t(p, z, a) + S_{G^{(m)}} K_{t}^{a,n}(p, z) \right)
\]

where \( v(p, z) \) is a function continuous in \( z \) for \( p \in P \). In the above, the approximation scheme \( S_{G^{(m)}} \) is applied to the functions for each \( p \in P \) and \( a \in A \). The resulting backward induction

\[
v_T^a = r_T, \quad v_{T-1}^{(m,n)} = T_{T-1}^{(m,n)} S_{G^{(m)}} v_T^a, \quad v_t^{(m,n)} = T_{t}^{(m,n)} v_{t+1}^{(m,n)}
\]
for \( t = T - 2, \ldots, 0 \) gives the modified value functions \( (v_t^{(m,n)})_{t=0}^{T-1} \). From Assumption 2 and the continuity of the modified value functions in \( z \) for \( p \in P \), it is clear that \( v_t^{(m,n)}(p, f_t(w, z)) \) is continuous in \( w \) for \( p \in P \), \( z \in Z \), and \( t = 1, \ldots, T - 1 \) since we have the composition of continuous functions.

The central idea behind this paper is to approximate the original problem in (2) with a more tractable problem given by (6). Therefore, one would like the functions in modified Bellman recursion to resemble their true counterparts. Since Assumption 3 imposes convexity on the functions in the original Bellman recursion and \( S_{G(m)} \) is used to approximate these functions, the following assumption on convexity and pointwise convergence on the dense grid is only natural.

**Assumption 4.** For all convex functions \( h : Z \to \mathbb{R} \), suppose that \( S_{G(m)}h(z) \) is convex in \( z \) for \( m \in \mathbb{N} \) and that \( \lim_{m \to \infty} S_{G(m)}h(z) = h(z) \) for \( z \in \bigcup_{m=1}^{\infty} G^{(m)} \).

The following assumption is now made to hold throughout this paper to guarantee the convexity of the modified value functions.

**Assumption 5.** Assume \( K_{T-1}^{(n)} S_{G(m)} v_T^{(p,z)} \) and \( K_t^{(n)} v_{t+1}^{(m,n)}(p,z) \) are convex in \( z \) for \( m, n \in \mathbb{N} \), \( a \in A \), \( p \in P \) and \( t = T - 2, \ldots, 0 \).

**Theorem 2.** The functions in the modified Bellman recursion (6) are convex in \( z \) for \( p \in P \) under Assumptions 3, 4 and 5.

**Proof.** By assumption, \( K_{T-1}^{(n)} S_{G(m)} v_T^{(p,z)} \) is convex in \( z \) for \( p \in P \), \( a \in A \), and \( m, n \in \mathbb{N} \). The reward functions are convex in \( z \) (by Assumption 3) for \( p \in P \) and \( a \in A \). Therefore, \( S_{G(m)}k_t^{(n)} v_T^{(a,p,z)} \) is convex in \( z \) by Assumption 4. The sum and pointwise maximum of convex functions is convex. Therefore, \( v_{T-1}^{(m,n)}(p,z) \) is convex in \( z \) for \( p \in P \) and \( m, n \in \mathbb{N} \) due to application of \( S_{G(m)} \). Proceeding inductively for \( t = T - 2, \ldots, 0 \) gives the desired result. □

Please note that the grid \( G^{(m)} \) can be easily made time dependent without affecting the convergence results in the next section. But for notational simplicity, this dependence is omitted. Also note that the modified Bellman operator (5) is not necessarily monotone since the operator \( S_{G(m)} \) is not necessarily monotone. It turns out that the convergence results presented in the next section does not require this property. However, to obtain lower and upper bounding functions in Section 5 and Section 6, this paper will impose Assumption 6 in Section 5 to induce monotonicity in the modified Bellman operator.

### 4. Convergence

This section proves convergence of the modified value functions. There are two natural choices for the disturbance sampling and weights:

- Use Monte Carlo to sample the disturbances randomly and the realizations are given equal weight or;
- Partition \( W \) and use some derived value (e.g. the conditional averages) on each of the components for the sampling. The sampling weights are determined by the probability measure of each component.

While the first choice is easier to use and more practical in high dimensional settings, the second selection confers many desirable properties which will be examined later on. First introduce the following useful concepts which will be used extensively.
Lemma 1. Let \((h^{(n)})_{n \in \mathbb{N}}\) be a sequence of real-valued convex functions on \(Z\) i.e. \(h^{(n)} : Z \to \mathbb{R}\) for \(n \in \mathbb{N}\). If the sequence converges pointwise to \(h\) on a dense subset of \(Z\), then the sequence \((h^{(n)})_{n \in \mathbb{N}}\) converges uniformly to \(h\) on all compact subsets of \(Z\).

Proof. See [18, Theorem 10.8]. \(\square\)

Definition 2. A sequence of convex real-valued functions \((h^{(n)})_{n \in \mathbb{N}}\) on \(Z\) is called a CCC (convex compactly converging) sequence in \(z\) if \((h^{(n)})_{n \in \mathbb{N}}\) converges uniformly on all compact subsets of \(Z\).

In the following two subsections, let \(p \in P\), \(a \in A\), and \(t = 0, \ldots, T - 1\) be arbitrary chosen. The sequence \((v^{(n)}_{t+1}(p, z))_{n \in \mathbb{N}}\) will be used to demonstrate the behaviour of the modified value functions under the modified transition operator. Assume \((v^{(n)}_{t+1}(p', z))_{n \in \mathbb{N}}\) forms a CCC sequence in \(z\) converging to value functions \(v^*_{t+1}(p', z)\) for all \(p' \in P\). Note that by Assumption 2, \(v^{(n)}_{t+1}(p', f_{t+1}(w, z))\) is continuous in \(w\) since we have a composition of continuous functions.

4.1. Monte Carlo sampling. The below establishes uniform convergence on compact sets under Monte Carlo sampling.

Theorem 3. Let \((W^{a,(n)}_{t+1}(k))_{k=1}^n\) be independently and identically distributed copies of \(W^a_{t+1}\) and \(\rho^{a,(n)}_{t+1}(k) = \frac{1}{n}\) for \(k = 1, \ldots, n\). Assume these random variables reside on the same probability space as \(W^a_{t+1}\). If \(W\) is compact, it holds that

\[
\lim_{n \to \infty} K_{t}^{a,(n)} v^{(n)}_{t+1}(p, z) = K_t v^*_{t+1}(p, z), \quad z \in Z.
\]

If \(K_t^{a,(n)} v^{(n)}_{t+1}(p, z)\) is also convex in \(z\) for \(n \in \mathbb{N}\), then \((K_t^{a,(n)} v^{(n)}_{t+1}(p, z))_{n \in \mathbb{N}}\) forms a CCC sequence in \(z\).

Proof. From the strong law of large numbers,

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^n v^*_{t+1}(p', f_{t+1}(W^{a,(n)}_{t+1}(k), z)) = \mathbb{E}[v^*_{t+1}(p', f_{t+1}(W^a_{t+1}, z))]
\]

holds with probability one. The summands can be expressed as

\[
v^{(n)}_{t+1}(p', f_{t+1}(W^{a,(n)}_{t+1}(k), z)) + v^*_{t+1}(p', f_{t+1}(W^a_{t+1}(k), z)) - v^{(n)}_{t+1}(p', f_{t+1}(W^a_{t+1}(k), z)).
\]

Define \(M_n := \sup_{w \in W} |v^*_{t+1}(p', f_{t+1}(w, z)) - v^{(n)}_{t+1}(p', f_{t+1}(w, z))|\). The continuity of \(f_{t+1}(w, z)\) in \(w\), the uniform convergence on compact sets of \(v^{(n)}_{t+1}\) to \(v^*_{t+1}\), and the compactness of \(W\) gives \(\lim_{n \to \infty} M_n = 0\). From Cesaro means,

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^n |v^*_{t+1}(p', f_{t+1}(W^{a,(n)}_{t+1}(k), z)) - v^{(n)}_{t+1}(p', f_{t+1}(W^a_{t+1}(k), z))| \leq \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^n M_k = 0
\]

with probability one and so

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^n v^{(n)}_{t+1}(p', f_{t+1}(W^{a,(n)}_{t+1}(k), z)) = \mathbb{E}[v^*_{t+1}(p', f_{t+1}(W^a_{t+1}, z))]
\]

with probability one. Therefore,

\[
\lim_{n \to \infty} \sum_{p' \in P} \alpha^{a,p',p}_{t+1} \frac{1}{n} \sum_{k=1}^n v^{(n)}_{t+1}(p', f_{t+1}(W^{a,(n)}_{t+1}(k), z)) = \sum_{p' \in P} \alpha^{a,p',p}_{t+1} \mathbb{E}[v^*_{t+1}(p', f_{t+1}(W^a_{t+1}, z))]
\]
almost surely and so the first part of the statement then follows. Now observe that the almost sure convergence in the first part of the statement holds for any choice of \( z \in \mathbb{Z} \). There are a countable number of \( z \in \bigcup_{m \in \mathbb{N}} \mathcal{G}^{(m)} \). A countable intersection of almost sure events is also almost sure. Therefore, \( \mathcal{K}_t^{a,(n)} v_{t+1}^{(n)}(p, \cdot) \) converges to \( \mathcal{K}_t^{a} v_{t+1}^{*}(p, \cdot) \) pointwise on a dense subset \( \bigcup_{m \in \mathbb{N}} \mathcal{G}^{(m)} \) of \( \mathbb{Z} \) with probability one. The second part of the statement then results from Lemma 1.

The above assumes that \( \mathbf{W} \) is compact. While this compactness assumption may seem problematic for unbounded \( \mathbf{W} \) cases, one can often find a compact subset \( \overline{\mathbf{W}} \subset \mathbf{W} \) so obscenely large that it contains the vast majority of the probability mass. Therefore, from at least a numerical work perspective, the drawback from this compactness is not that practically significant especially considering computers typically have a limit on the size of the numbers they can generate and because of machine epsilon. For example, if \( \mathbf{W} = \mathbb{R}_+ \), one can set \( \overline{\mathbf{W}} \) where \( \max \overline{\mathbf{W}} \) is orders of magnitudes greater than this size limit and \( \min \overline{\mathbf{W}} \) is drastically smaller than the machine epsilon. With this, one can then use Monte Carlo sampling in practice as normal without restriction. The above convergence when \( \mathbf{W} \) is not compact will be addressed in future research.

### 4.2. Disturbance space partition

This subsection proves the same convergence under partitioning of the disturbance space \( \mathbf{W} \). Introduce partition \( \Pi^{(n)} = \{ \Pi^{(n)}(k) \subset \mathbf{W} : k = 1, \ldots, n \} \) and define the diameter of the partition by

\[
\delta^{(n)} := \max_{k=1, \ldots, n} \sup \{ \| w' - w'' \| : w', w'' \in \Pi^{(n)}(k) \}
\]

if it exists. The case where \( \mathbf{W} \) is compact is considered first.

**Theorem 4.** Suppose \( \mathbf{W} \) is compact and let \( \lim_{n \to \infty} \delta^{(n)} = 0 \). Choose sampling \( (W_{t+1}^{a,(n)}(k))_{k=1}^{n} \) where \( W_{t+1}^{a,(n)}(k) \in \Pi^{(n)}(k) \) and \( \rho_{t+1}^{a,(n)}(k) = \mathbb{P}(W_{t+1}^{a} \in \Pi^{(n)}(k)) \) for \( k = 1, \ldots, n \). It holds that

\[
\lim_{n \to \infty} \mathcal{K}_t^{a,(n)} v_{t+1}^{(n)}(p, z) = \mathcal{K}_t^{a} v_{t+1}^{*}(p, z), \quad z \in \mathbb{Z}.
\]

If \( \mathcal{K}_t^{a,(n)} v_{t+1}^{(n)}(p, z) \) is also convex in \( z \) for \( n \in \mathbb{N} \), then \( (\mathcal{K}_t^{a,(n)} v_{t+1}^{(n)}(p, z))_{n \in \mathbb{N}} \) forms a CCC sequence in \( z \).

**Proof.** Denote \( n \)-point random variable

\[
W_{t+1}^{a,(n)} = \sum_{k=1}^{n} W_{t+1}^{a,(n)}(k) \mathbb{1}(W_{t+1}^{a} \in \Pi^{(n)}(k))
\]

where \( \mathbb{1}(\mathbf{B}) \) denotes the indicator function of the set \( \mathbf{B} \). Now

\[
\lim_{n \to \infty} \mathbb{E}[\| W_{t+1}^{a,(n)} - W_{t+1}^{a} \|] \leq \lim_{n \to \infty} \delta^{(n)} = 0
\]

and so \( W_{t+1}^{a,(n)} \) converges to \( W_{t+1}^{a} \) in distribution as \( n \to \infty \). Using this convergence, the fact that \( \mathbf{W} \) is compact, the fact that \( v_{t+1}^{(n)}(p', f_{t+1}(w, z)) \) and \( v_{t+1}^{*}(p', f_{t+1}(w, z)) \) are continuous in \( w \), and the fact that \( v_{t+1}^{(n)} \) converges to \( v_{t+1}^{*} \) uniformly on compact sets, it can be seen that

\[
\lim_{n \to \infty} \mathbb{E}[\| v_{t+1}^{(n)}(p', f_{t+1}(W_{t+1}^{a,(n)}), z) \|] = \mathbb{E}[\| v_{t+1}^{*}(p', f_{t+1}(W_{t+1}^{a}), z) \|]
\]

for \( p' \in \mathbf{P} \) and \( z \in \mathbb{Z} \). This first part of the statement then follows easily. The second part of the theorem follows from Lemma 1. \( \square \)
The next theorem examines the case when \( W \) is not necessarily compact. In addition, conditional averages are used for the disturbance sampling and this is perhaps a more sensible choice given that it minimizes the mean square error from the discretization of \( W_{t+1}^a \). In the following, \( \mathbb{E}[W_{t+1}^a | W_{t+1}^a \in \Pi(n)(k)] \) refers to the expectation of \( W_{t+1}^a \) conditioned on the event \( \{W_{t+1}^a \in \Pi(n)(k)\} \). This paper sometimes refers to this as the local average.

**Theorem 5.** Suppose generated sigma-algebras \( \sigma_{a,t+1}^{(n)} = \sigma(\{W_{t+1}^a \in \Pi(n)(k)\}, k = 1, \ldots, n) \) satisfy \( \sigma(W_{t+1}^a) = \sigma(\cup_{n \in \mathbb{N}} \sigma_{a,t+1}^{(n)}) \). Select sampling \( (W_{t+1}^{a,n}(k))_{k=1}^{n} \) such that

\[
W_{t+1}^{a,n}(k) = \mathbb{E}[W_{t+1}^a | W_{t+1}^a \in \Pi(n)(k)]
\]

with \( \rho_{t+1}^{a,n}(k) = \mathbb{P}(W_{t+1}^a \in \Pi(n)(k)) \) for \( k = 1, \ldots, n \). If \( (v_{t+1}^{(n)}(p', f_{t+1}(W_{t+1}^{a,n}(z))))_{n \in \mathbb{N}} \) is uniformly integrable for \( p' \in \mathbb{P} \) and \( z \in \mathbb{Z} \), then:

\[
\lim_{n \to \infty} v_{t+1}^{(n)}(p, z) = K_t v_{t+1}^*(p, z), \quad z \in \mathbb{Z}.
\]

If \( K_t v_{t+1}^{(n)}(p, z) \) is also convex in \( z \) for all \( n \in \mathbb{N} \), then \( (K_t v_{t+1}^{(n)}(p, z))_{n \in \mathbb{N}} \) forms a CCC sequence in \( z \).

**Proof.** Denote random variable \( W_{t+1}^{a,n} = \mathbb{E}[W_{t+1}^a | \sigma_{a,t+1}^{(n)}] \) which takes values in the set of local averages \( \{\mathbb{E}[W_{t+1}^a | W_{t+1}^a \in \Pi(n)(k)] : k = 1, \ldots, n\} \). On the set of paths where the almost sure convergence \( W_{t+1}^{a,n} \to W_{t+1}^a \) holds by Levy’s upward theorem [21, Section 14.2], the set \( \{W_{t+1}^{a,n} : n \in \mathbb{N}\} \) is bounded on each sample path since a convergent sequence is bounded and so

\[
\lim_{n \to \infty} v_{t+1}^{(n)}(p', f_{t+1}(W_{t+1}^{a,n}(z))) = v_{t+1}^*(p', f_{t+1}(W_{t+1}^a, z))
\]

with probability one since there is uniform convergence on bounded sets. Using the Vitali convergence theorem [21, Section 13.7],

\[
\lim_{n \to \infty} \mathbb{E}[v_{t+1}^{(n)}(p', f_{t+1}(W_{t+1}^{a,n}(z)))] = \mathbb{E}[v_{t+1}^*(p', f_{t+1}(W_{t+1}^a, z))]
\]

for \( p' \in \mathbb{P} \). This proves the first part of the statement. The second part of the statement stems from Lemma 1. \( \square \)

The above assumes that \( (v_{t+1}^{(n)}(p', f_{t+1}(W_{t+1}^{a,n}(z))))_{n \in \mathbb{N}} \) is uniformly integrable. This is satisfied when \( W \) is compact. For \( W \) not compact, Theorem 6 may be useful.

**Lemma 2.** Let random variable \( Y \) be integrable on \( (\Omega, \mathcal{F}, \mathbb{P}) \). The class of random variables \( \{\mathbb{E}[Y | \mathcal{G}] : \mathcal{G} \text{ is a sub-sigma-algebra of } \mathcal{F}\} \) is uniformly integrable.

**Proof.** See [21, Section 13.4]. \( \square \)

In the following, the function \( f_{t+1}(w, z) \) is said to be convex in \( w \) component-wise if each component of \( f_{t+1}(w, z) \) is convex in \( w \).

**Theorem 6.** Let \( W_{t+1}^{a,n} = \mathbb{E}[W_{t+1}^a | \sigma_{a,t+1}^{(n)}] \) and \( v_{t+1}^{(n)}(p', z) \) be Lipschitz continuous in \( z \) with Lipschitz constant \( c_n \). If \( \sup_{n \in \mathbb{N}} c_n < \infty \) and for \( z \in \mathbb{Z} \) either:

- \( f_{t+1}(w, z) \) is Lipschitz continuous in \( w \), or
- \( \|f_{t+1}(w, z)\| \) is convex in \( w \), or
- \( f_{t+1}(w, z) \) is convex in \( w \) component-wise,

holds, then \( (v_{t+1}^{(n)}(p', f_{t+1}(W_{t+1}^{a,n}(z)))_{n \in \mathbb{N}} \) is uniformly integrable for \( z \in \mathbb{Z} \).
Proof. From Lipschitz continuity,
\[
|v_{t+1}^{(n)}(p', f_{t+1}(W_{t+1}^{a,(n)}(z)))| \leq |v_{t+1}^{(n)}(p, z')| + c_n \|f_{t+1}(W_{t+1}^{a,(n)}(z)) - z'\|
\]
holds for \(z' \in \mathbb{Z}\). Since \(v_{t+1}^{(n)}\) converges to \(v_{t+1}^*\), it is enough to verify that \((c_n \|f_{t+1}(W_{t+1}^{a,(n)}(z)) - z'\|)_{n \in \mathbb{N}}\) is uniformly integrable to prove the above statement. Now if \(f_{t+1}(w, z)\) is Lipschitz continuous in \(w\),
\[
\|f_{t+1}(W_{t+1}^{a,(n)}(z))\| \leq \|f_{t+1}(w', z)\| + c \|E[W_{t+1}^a | \sigma_{a,t+1}] - w'\|.
\]
for some constant \(c\) and \(w' \in \mathbb{W}\). Now suppose instead that \(\|f_{t+1}(w, z)\|\) is convex in \(w\). We know from Jensen’s inequality that
\[
\|f_{t+1}(W_{t+1}^{a,(n)}(z))\| \leq E[\|f_{t+1}(W_{t+1}^a, z)\| | \sigma_{a,t+1}].
\]
Finally, if \(f_{t+1}(w, z)\) is convex in \(w\) component-wise, Jensen’s gives
\[
f_{t+1}(W_{t+1}^{a,(n)}(z)) \leq E[f_{t+1}(W_{t+1}^a, z) | \sigma_{a,t+1}]
\]
holding component-wise. From the above inequality and the fact that convex functions are bounded below by an affine linear function (e.g. tangents), the following holds component-wise for some constants \(b, c'\):
\[
|f_{t+1}(W_{t+1}^{a,(n)}(z))| \leq \|E[f_{t+1}(W_{t+1}^a, z) | \sigma_{a,t+1}]| + b + c'E[W_{t+1}^a | \sigma_{a,t+1}].
\]
Using Lemma 2 and sup\(n \in \mathbb{N}\) \(c_n < \infty\), Equations (7), (8) or (9) reveals that \((c_n \|f_{t+1}(W_{t+1}^{a,(n)}(z)) - z'\|)_{n \in \mathbb{N}}\) is uniformly integrable for \(z' \in \mathbb{Z}\) because it is dominated by a family of uniformly integrable random variables.

The above generalizes the condition used by [11] to ensure uniform integrability his approximation scheme. Before proceeding, note that \(\Pi^{(n)}\) can be made time dependent without affecting the convergence above.

4.3. Modified value functions. The following establishes the uniform convergence on compact sets of the resulting modified value functions under each of the disturbance sampling methods. Let \((m_n)_{n \in \mathbb{N}}\) and \((n_m)_{m \in \mathbb{N}}\) be sequences of natural numbers increasing in \(n\) and \(m\), respectively.

Lemma 3. Suppose \((h_1^{(n)})_{n \in \mathbb{N}}\) and \((h_2^{(n)})_{n \in \mathbb{N}}\) are CCC sequences on \(\mathbb{Z}\) converging to \(h_1\) and \(h_2\), respectively. Define \(h_3^{(n)}(z) := \max(h_1^{(n)}(z), h_2^{(n)}(z))\) and \(h_3(z) := \max(h_1(z), h_2(z))\). Then:
- \((h_1^{(n)} + h_2^{(n)})_{n \in \mathbb{N}}\) is a CCC sequence on \(\mathbb{Z}\) converging to \(h_1 + h_2\),
- \((S_{G_m} h_1^{(n)})_{n \in \mathbb{N}}\) is a CCC sequence on \(\mathbb{Z}\) converging to \(h_1\), and
- \((h_3^{(n)})_{n \in \mathbb{N}}\) is a CCC sequence on \(\mathbb{Z}\) converging to \(h_3\).

Proof. They can be proved easily using the definition of uniform convergence on compact sets, Assumption 4, and Lemma 1.

Theorem 7. The sampling in Theorem 3, Theorem 4 or Theorem 5 gives
\[
\lim_{n \to \infty} v_t^{(m,n)}(p, z) = \lim_{m \to \infty} v_t^{(m,n)}(p, z) = v_t^*(p, z)
\]
for \(p \in \mathbb{P}, \ z \in \mathbb{Z}\), and \(t = T - 1, \ldots, 0\). Also, \((v_t^{(m,n)}(p, z))_{n \in \mathbb{N}}\) and \((v_t^{(m,n)}(p, z))_{m \in \mathbb{N}}\) both form CCC sequences in \(z\) for all \(p \in \mathbb{P}\) and \(t = T - 1, \ldots, 0\).
Proof. Let us consider the limit as $n \to \infty$ first and prove this via backward induction. At $t = T - 1$, Lemma 3 reveals that $(\mathcal{S}_G(m_n) v_{T-1}^{a}(p, z))_{n \in \mathbb{N}}$ forms a CCC sequence in $z$ for $p \in P$ and converges to $v_{T-1}^{a}(p, z)$. Now from Assumption 5, $K_{T-1}^{a}(n) \mathcal{S}_G(m_n) v_{T}^{a}(p, z)$ is convex in $z$. Therefore, using Assumption 5 and either Theorem 3, Theorem 4 or Theorem 5 reveals that $(K_{T-1}^{a}(n) \mathcal{S}_G(m_n) v_{T}^{a}(p, z))_{n \in \mathbb{N}}$ forms a CCC sequence in $z$ converging to $K_{T-1}^{a}(n) v_{T}^{a}(p, z)$. From the above and Lemma 3, we know that $(\mathcal{S}_G(m_n) r_{T-1}(p, z, a) + \mathcal{S}_G(m_n) K_{T-1}^{a}(n) \mathcal{S}_G(m_n) v_{T}^{a}(p, z))_{n \in \mathbb{N}}$ forms a CCC sequence in $z$ and converges to $r_{T-1}(p, z, a) + K_{T-1}^{a}(n) v_{T}^{a}(p, z)$. Since $A$ is finite, Lemma 3 implies that $(v^{a,m}(m_n)(p, z))_{n \in \mathbb{N}}$ forms a CCC sequence in $z$ and converges to $v_{T-1}^{a}(p, z)$ for $p \in P$. At $t = T - 2$, it can be shown using the same logic above for $p \in P$ and $a \in A$ that $(K_{T-2}^{a}(n) v_{T-1}^{a}(p, z))_{n \in \mathbb{N}}$ forms a CCC sequence in $z$ and converges to $K_{T-2}^{a}(n) v_{T-1}^{a}(p, z)$. Following the same lines of argument above eventually leads to $(v_{T-2}^{a,m}(m_n)(p, z))_{n \in \mathbb{N}}$ forming a CCC sequence in $z$ and that it converges to $v_{T-2}^{a}(p, z)$ for $p \in P$. Proceeding inductively for $t = T - 3, \ldots, 0$ gives the desired result. The proof for the $m \to \infty$ case follows the same lines as above. □

5. LOWER BOUNDERS

Observe that the convergence results presented so far does require the modified Bellman operator (5) to be a monotone operator. However, this is needed to obtain lower and upper bounding functions.

Assumption 6. For all convex functions $h^l, h^u: \mathbb{R} \to \mathbb{R}$ where $h^l(z) \leq h^u(z)$ for $z \in \mathbb{R}$, assume $\mathcal{S}_G(m) h^l(z) \leq \mathcal{S}_G(m) h^u(z)$ for $z \in \mathbb{Z}$ and $m \in \mathbb{N}$.

The modified Bellman operator (5) is now monotone i.e. for $m, n \in \mathbb{N}$, $p \in P$, $z \in \mathbb{Z}$, and $t = 0, \ldots, T - 1$, we have $T_t^{a,m}(p, z, a) \leq \mathcal{T}_t^{a,m}(p, z, a)$ if $v(p', z) \leq v(p', z)$ for $p' \in P$. This stems from the monotonicity of (4) and Assumption 6. Under the following conditions, the modified value functions constructed using the disturbance sampling in Theorem 5 leads to a non-decreasing sequence of lower bounding functions. Partition $\Pi^{n+1}$ is said to refine $\Pi^n$ if each component in $\Pi^{n+1}$ is a subset of a component in $\Pi^n$.

Lemma 4. Let $t = 0, \ldots, T - 1$ and $v(p, f_{t+1}(w, z))$ be convex in $w$ for all $p \in P$. If $\Pi^{n+1}$ refines $\Pi^n$, then Theorem 5 gives $K_{t}^{a}(n) v(p, z) \leq K_{t}^{a}(n+1) v(p, z)$.

Proof. Without loss of generality, assume the last two components of $\Pi^{n+1}$ are both subsets of the last two components of $\Pi^n$. With this, $K_{t}^{a}(n) v(p, z) - \sum_{p' \in P} \alpha_{t+1}^{a,p,p'} \rho_{t+1}^{a}(n) v(p', f_{t+1}(W_{t+1}^{a}(n), z))$ $= K_{t}^{a}(n+1) v(p, z) - \sum_{p' \in P} \alpha_{t+1}^{a,p,p'} \sum_{k=n}^{n+1} \rho_{t+1}^{a}(n+1)(k) v(p', f_{t+1}(W_{t+1}^{a}(n+1)(k), z))$. Since $v(p', f_{t+1}(w, z))$ is convex in $w$, it holds that $v(p', f_{t+1}(W_{t+1}^{a}(n), z)) \leq \frac{\rho_{t+1}^{a}(n+1)}{\rho_{t+1}^{a}(n)} v(p', f_{t+1}(W_{t+1}^{a}(n), z)) + \frac{\rho_{t+1}^{a}(n+1)}{\rho_{t+1}^{a}(n)} v(p', f_{t+1}(W_{t+1}^{a}(n+1)(n + 1), z))$
because
\[ W_{t+1}^{a(n)}(n) = \rho_{t+1}^{a(n+1)}(n) W_{t+1}^{a(n)}(n) + \rho_{t+1}^{a(n+1)}(n+1) W_{t+1}^{a(n+1)}(n+1) \]
and \( \rho_{t+1}^{a(n+1)}(n) + \rho_{t+1}^{a(n+1)}(n + 1) = \rho_{t+1}^{a(n)}(n) \). Therefore, for all \( p' \in P \)
\[ \rho_{t+1}^{a(n)}(n) v(p', f_{t+1}(W_{t+1}^{a(n)}(n), z)) \leq \sum_{k=n}^{n+1} \rho_{t+1}^{a(n+1)}(k) v(p', f_{t+1}(W_{t+1}^{a(n+1)}(k), z)) \]
and so \( \mathcal{K}_{t}^{a(n)} v(p, z) \leq \mathcal{K}_{t}^{a(n+1)} v(p, z) \) as claimed. \[ \Box \]

**Theorem 8.** Using Theorem 5 gives for \( p \in P, z \in Z, m, n \in \mathbb{N}, \) and \( t = 0, \ldots, T - 1 \):

- \( v_{t}^{(m,n)}(p, z) \leq v_{t}^{*}(p, z) \) if \( v_{t}^{*}(p', f_{t}(w, z')) \) is convex in \( w \) and if \( S_{G(m')}h(z') \leq h(z') \) for \( p' \in P, z' \in Z, t' = 1, \ldots, T, \) and all convex functions \( h \).
- \( v_{t}^{(m,n)}(p, z) \leq v_{t}^{(m,n+1)}(p, z) \) when \( \Pi^{(n+1)} \) refines \( \Pi^{(n)} \) and if \( S_{G(m')}v_{t}^{*}(p', f_{t}(w, z')) \) and \( v_{t}^{(m',n')} \) are convex in \( w \) for \( m', n' \in \mathbb{N}, p' \in P, z' \in Z, t' = 1, \ldots, T - 1 \).
- \( v_{t}^{(m,n)}(p, z) \leq v_{t}^{(m,n+1)}(p, z) \) if \( S_{G(m')}h(z') \leq S_{G(m'+1)}h(z') \) for \( z' \in Z, m' \in \mathbb{N}, \) and all convex functions \( h \).

**Proof.** The three inequalities are proven separately using backward induction.

1) Recall that \( W_{t}^{a(n)} = \mathbb{E}[W_{t+1}^{a(n)} | \sigma_{a,t+1}] \) for \( a \in A \) and \( t = 0, \ldots, T - 1 \). From the tower property, Jensen’s inequality and the monotonicity of (4):

\[ \mathbb{E}[v_{t}^{*}(p', f_{t}(W_{t}^{a(n)})), z')] \geq \mathbb{E}[v_{t}^{*}(p', f_{t}(W_{t}^{a(n)}(k), z'))] \]

for all \( p' \in P \) and \( z' \in Z \). Therefore, \( \mathcal{K}_{t}^{a(n)} S_{G(m)} v_{t}^{*}(p, z) \leq \mathcal{K}_{t-1}^{a(n)} v_{t}^{*}(p, z) \) which in turn implies that \( v_{t}^{(m,n)}(p, z) \leq v_{t-1}^{(m,n)}(p, z) \) since \( S_{G(m')}r_{T-1}(p, z, a) + S_{G(m')} \mathcal{K}_{t-1}^{a(n)} S_{G(m)} v_{t}^{*}(p, z) \leq r_{T-1}(p, z, a) + \mathcal{K}_{t-1}^{a(n)} v_{t}^{*}(p, z) \). Proceeding inductively for \( t = T - 2, \ldots, 0 \) using a similar argument as above gives the first inequality in the statement.

2) Lemma 4 gives \( \mathcal{K}_{t}^{a(n)} S_{G(m)} v_{t}^{*}(p, z) \leq \mathcal{K}_{t-1}^{a(n+1)} S_{G(m)} v_{t}^{*}(p, z) \) which implies \( v_{t-1}^{(m,n)}(x) \leq v_{t-1}^{(m,n+1)}(p, z) \). Using the monotonicity of (4) and a similar argument as above, one can show

\[ \mathcal{K}_{t-2}^{a(n)} v_{t-2}^{(m,n)}(p, z) \leq \mathcal{K}_{t-2}^{a(n)} v_{t-2}^{(m,n+1)}(p, z) \leq \mathcal{K}_{t-2}^{a(n+1)} v_{t-2}^{(m,n+1)}(p, z) \]

\[ \Rightarrow v_{t-2}^{(m,n)}(p, z) \leq v_{t-2}^{(m,n+1)}(p, z) \]. Proceeding inductively for \( t = T - 3, \ldots, 0 \) proves the desired result.

3) This can be easily proved by backward induction using the monotonicity of (4) and by the fact that \( S_{G(m')}r_{T}(p, z, a) \leq S_{G(m'+1)}r_{T}(p, z, a) \) and \( S_{G(m')}r_{T}(p, z) \leq S_{G(m'+1)}r_{T}(p, z) \) for \( p \in P, z \in Z, a \in A, m \in \mathbb{N}, \) and \( t = 0, \ldots, T - 1 \). \[ \Box \]

It turns out that if the modified value functions are bounded above by the true value functions such as in the first case point of Theorem 8, the uniform integrability assumption in Theorem 5 holds automatically. This is proved in the next subsection.
5.1. Uniform integrability condition. Theorem 9 below differs from Theorem 6 in that Lipschitz continuity in $z$ is not assumed for the approximating functions. Instead, the value functions are assumed to bound the approximating functions from above. In the following, the sequence of functions $(v_{t+1}^{(n)}(p, z))_{n \in \mathbb{N}}$ from the previous section is reused. Recall that this sequence converges uniformly to $v_{t+1}$ on compact sets.

**Lemma 5.** Suppose that $v_{t+1}^{(n)}(p, z) \leq v_{t+1}^{*}(p, z)$ for all $n \in \mathbb{N}$, $p \in \mathcal{P}$, and $z \in \mathcal{Z}$. For fixed $p \in \mathcal{P}$ and all $z' \in \mathcal{Z}$, there exists constants $c_{n,z'} \in \mathbb{R}^d$ such that

$$v_{t+1}^{*}(p, z) \geq v_{t+1}^{(n)}(p, z) \geq v_{t+1}^{(n)}(p, z') + c_{n,z'}^T(z' - z)$$

for $z \in \mathcal{Z}$ and $n \in \mathbb{N}$. It also holds that $\sup_{n \in \mathbb{N}} \|c_{n,z'}\| < \infty$.

**Proof.** The first part follows from the definition of a tangent for a convex function. Now note that if $\sup_{n \in \mathbb{N}} \|c_{n,z'}\|$ is not bounded for $z' \in \mathcal{Z}$, then there exists $\tilde{n}$ and $\tilde{z}$ where

$$v_{t+1}^{*}(p, \tilde{z}) \leq v_{t+1}^{(\tilde{n})}(p, z') + c_{\tilde{n},z'}^T(z' - \tilde{z})$$

since $v_{t+1}^{(n)}$ converges to $v_{t+1}^{*}$. This yields a contradiction. \qed

**Theorem 9.** Suppose $v_{t+1}^{(n)}(p, z) \leq v_{t+1}^{*}(p, z)$ for all $n \in \mathbb{N}$, $p \in \mathcal{P}$, and $z \in \mathcal{Z}$. Assume $v_{t+1}^{*}(p, f_{t+1}(w, z))$ is convex in $w$ and let $W_{t+1}^{a,n} = E[W_{t+1}^{a} \mid \sigma_{a,t+1}^{(n)}]$. If for $z \in \mathcal{Z}$ either:

- $f_{t+1}(w, z)$ is Lipschitz continuous in $w$,
- $\|f_{t+1}(w, z)\|$ is convex in $w$, or
- $f_{t+1}(w, z)$ is convex in $w$ component-wise,

holds, then $(v_{t+1}^{(n)}(p, f_{t+1}(W_{t+1}^{a,n}, z)))_{n \in \mathbb{N}}$ is uniformly integrable.

**Proof.** Jensen’s inequality and $v_{t+1}^{(n)}(p, f_{t+1}(W_{t+1}^{a,n}, z)) \leq v_{t+1}^{*}(p, f_{t+1}(W_{t+1}^{a,n}, z)) \leq E[v_{t+1}^{*}(p, f_{t+1}(W_{t+1}^{a,n}, z)) \mid \sigma_{a,t+1}^{(n)}]$.

Now from Lemma 5, there exists constants $c_{n,z'} \in \mathbb{R}^d$ such that for all $n \in \mathbb{N}$:

$$v_{t+1}^{(n)}(p, f_{t+1}(W_{t+1}^{a,n}, z)) \geq v_{t+1}^{(n)}(p, z') + c_{n,z'}^T(z' - f_{t+1}(W_{t+1}^{a,n}, z))$$

for some $z' \in \mathcal{Z}$ with probability one. Using the above inequalities:

$$|v_{t+1}^{(n)}(p, f_{t+1}(W_{t+1}^{a,n}, z))| \leq |E[v_{t+1}^{*}(p, f_{t+1}(W_{t+1}^{a,n}, z)) \mid \sigma_{a,t+1}^{(n)}]| + |v_{t+1}^{(n)}(p, z') + c_{n,z'}^T(z' - f_{t+1}(W_{t+1}^{a,n}, z))|$$

almost surely. The first term on the right forms a uniformly integrable family of random variables from Lemma 2. Since $v_{t+1}^{(n)}$ converges to $v_{t+1}^{*}$, $\sup_{n \in \mathbb{N}} v_{t+1}^{(n)}(p, z') < \infty$. From Lemma 5, $\sup_{n \in \mathbb{N}} \|c_{n,z'}\| < \infty$. Now $(f_{t+1}(W_{t+1}^{a,n}, z))_{n \in \mathbb{N}}$ can be shown to be uniformly integrable using a similar argument as in the proof of Theorem 6. Therefore, $(v_{t+1}^{(n)}(p, f_{t+1}(W_{t+1}^{a,n}, z)))_{n \in \mathbb{N}}$ is dominated by a family of uniformly integrable random variables and so is also uniformly integrable. \qed
6. Upper bounds

For the case where \( W \) is compact, a sequence of upper bounding function approximations can also be constructed using the following setting from [9]. It is well known that a convex function continuous on a compact convex set attains its maximum at an extreme point of that set. The following performs a type of averaging of these extreme points to obtain an upper bound using Jensen’s inequality (see [3, Section 5]). Let us assume that the closures of each of the components in partition \( \Pi^{(n)} \) are convex and contain a finite number of extreme points. Denote the set of extreme points of the closure \( \tilde{\Pi}^{(n)}(k) \) of each component \( \Pi^{(n)}(k) \) by

\[
E(\tilde{\Pi}^{(n)}(k)) = \{ e_{k,i}^{(n)} : i = 1, \ldots, L_k^{(n)} \}
\]

where \( L_k^{(n)} \) is the number of extreme points in \( \tilde{\Pi}^{(n)}(k) \). Suppose there exists weighting functions \( q_{t+1,k,i}^{a,n} : W \to [0,1] \) satisfying

\[
\sum_{i=1}^{L_k^{(n)}} q_{t+1,k,i}^{a,n}(w) = 1 \quad \text{and} \quad \sum_{i=1}^{L_k^{(n)}} q_{t+1,k,i}^{a,n}(w)e_{k,i}^{(n)} = w
\]

for \( w \in \tilde{\Pi}^{(n)}(k) \) and \( k = 1, \ldots, n \). Suppose \( \rho_{t+1}(k) = \mathbb{P}(W_{t+1}^a \in \Pi^{(n)}(k)) > 0 \) for \( k = 1, \ldots, n \) and define random variables \( \tilde{W}_{t+1,k}^{a,n} \) satisfying

\[
\mathbb{P} \left( \tilde{W}_{t+1,k}^{a,n} = e_{k,i}^{(n)} \right) = \frac{q_{t+1,k,i}^{a,n}}{\rho_{t+1}(k)} \quad \text{where} \quad \tilde{q}_{t+1,k,i}^{a,n} = \int_{\tilde{\Pi}^{(n)}(k)} q_{t+1,k,i}^{a,n}(w)\rho_{t+1}(k)dw
\]

and \( \rho_{t+1}(B) = \mathbb{P}(W_{t+1}^a \in B) \). To grasp the intuition for the upper bound, note that if \( g : W \to \mathbb{R} \) is a convex and continuous function, then

\[
\mathbb{E}[g(W_{t+1}^a)1(W_{t+1}^a \in \Pi^{(n)}(k))] \leq \sum_{i=1}^{L_k^{(n)}} \tilde{q}_{t+1,k,i}^{a,n}g(e_{k,i}^{(n)})
\]

for \( k = 1, \ldots, n \) (see [9, Corollary 7.2]) and so

\[
(10) \quad \mathbb{E}[g(W_{t+1}^a)] \leq \sum_{k=1}^{n} \sum_{i=1}^{L_k^{(n)}} \tilde{q}_{t+1,k,i}^{a,n}g(e_{k,i}^{(n)}).
\]

For the following theorem, define random variable

\[
\tilde{W}_{t+1,k}^{a,n} := \sum_{k=1}^{n} \tilde{W}_{t+1,k}^{a,n}1 \left( W_{t+1}^a \in \Pi^{(n)}(k) \right).
\]

Recall that \( (v_{t+1}(p,z))_{n \in \mathbb{N}} \) is a CCC sequence in \( z \) for \( p \in P \) and that \( v_{t+1}^{(n)} \) converges to \( v_{t+1}^* \). Also, recall \( v_{t+1}^{(n)}(p,f_{t+1}(w,z)) \) is continuous in \( w \).

**Lemma 6.** If the diameter of the partition vanishes i.e. \( \lim_{n \to \infty} \delta^{(n)} = 0 \), then

\[
\lim_{n \to \infty} \mathbb{E}[v_{t+1}^{(n)}(p,f_{t+1}(\tilde{W}_{t+1,k}^{a,n} + z))] = \mathbb{E}[v_{t+1}^*(p,f_{t+1}(W_{t+1}^a + z))], \quad z \in \mathbb{Z}.
\]

If, in addition, \( \mathbb{E}[v_{t+1}^{(n)}(p,f_{t+1}(\tilde{W}_{t+1,k}^{a,n} + z))] \) is convex in \( z \) for \( n \in \mathbb{N} \), then the sequence of functions \( (\mathbb{E}[v_{t+1}^{(n)}(p,f_{t+1}(\tilde{W}_{t+1,k}^{a,n} + z))])_{n \in \mathbb{N}} \) form a CCC sequence in \( z \).
Proof. By construction, $W_{t+1,k}^{n_0}$ takes values in the extreme points of $\Pi^{(n)}(k)$ for $k = 1, \ldots, n$ and so
\[
\lim_{n \to \infty} E[|W_{t+1}^{n_0} - W_{t+1}^{n_0}|] \leq \lim_{n \to \infty} \delta^{(n)} = 0.
\]
Thus, $\overline{W}_{t+1}^{n_0}$ converges to $W_{t+1}^{n_0}$ in distribution as $n \to \infty$. Therefore, the proof for the above statement then follows the same lines as the proof for Theorem 4.

Let us define the following alternative modified transition operator:

\[
k_t^{a_0}(p, z) = \sum_{p' \in \mathcal{P}} \alpha_t^{a,p,p'} E[v(p', f_{t+1}(\overline{W}_{t+1}^{n_0}, z))].
\]

The next theorem establishes the uniform convergence on compact sets of the resulting modified value functions when the new modified transition operator (11) is used in place of the original modified transition operator (4). Recall that $(n_n)_{n \in \mathbb{N}}$ and $(n_m)_{m \in \mathbb{N}}$ are sequences of natural numbers increasing in $n$ and $m$, respectively.

**Theorem 10.** Suppose $\lim_{n \to \infty} \delta^{(n)} = 0$. Using (11) gives

\[
\lim_{m \to \infty} v_t^{(m,n)}(p, z) = \lim_{n \to \infty} v_t^{(m,n)}(p, z) = v_t^*(p, z)
\]

for $p \in \mathcal{P}$, $z \in \mathcal{Z}$, and $t = T - 1, \ldots, 0$. Also, $(v_t^{(m,n)}(p, z))_{m \in \mathbb{N}}$ and $(v_t^{(m,n)}(p, z))_{m \in \mathbb{N}}$ both form CCC sequences in $z$ for all $p \in \mathcal{P}$ and $t = T - 1, \ldots, 0$.

**Proof.** The proof follow the same lines as the proof in Theorem 7 but using Lemma 6 instead. \qed

Under certain conditions, these modified expected value functions also form a non-decreasing sequence of upper bounding functions as shown below. The following gives analogous versions of Lemma 4 and Theorem 8 but for the upper bound case.

**Lemma 7.** Suppose $\Pi^{(n+1)}$ refines $\Pi^{(n)}$, that $v_{t+1}^{(n)}(p, f_{t+1}(w, z))$ is convex in $w$, and that $v_{t+1}^{(n)}(p, z) \geq v_{t+1}^{(n+1)}(p, z) \geq v_{t+1}^*(p, z)$ for $p \in \mathcal{P}$, $z \in \mathcal{Z}$, and $n \in \mathbb{N}$. Then

\[
E[v_{t+1}^{(n)}(p, f_{t+1}(\overline{W}_{t+1}^{n_0}, z))] \geq E[v_{t+1}^{(n)}(p, f_{t+1}(\overline{W}_{t+1}^{n_0}, z))] \geq E[v_{t+1}^{(n)}(p, f_{t+1}(W_{t+1}, z))]
\]

for $a \in \mathcal{A}$, $p \in \mathcal{P}$, $z \in \mathcal{Z}$, and $n \in \mathbb{N}$.

**Proof.** See [9, Theorem 7.8] or [8, Theorem 1.3]. \qed

**Theorem 11.** Using (11) gives for $p \in \mathcal{P}$, $z \in \mathcal{Z}$, $m, n \in \mathbb{N}$, and $t = 0, \ldots, T - 1$:

- $v_t^{(m,n)}(p, z) \geq v_t^*(p, z)$ when $v_t^*(p', f_t'(w, z'))$ is convex in $w$ and if $S_{G(m')}h(z') \geq h(z')$ for $p' \in \mathcal{P}$, $z' \in \mathcal{Z}$, $t' = 1, \ldots, T$, and all convex functions $h$.
- $v_t^{(m,n)}(p, z) \geq v_t^{(m,n+1)}(p, z)$ if $\Pi^{(n+1)}$ refines $\Pi^{(n)}$ and if $S_{G(m')}v_t^*(p', f_t'(w, z'))$ and $v_t^{(m,n)}(p', f_t'(w, z'))$ are convex in $w$ for $m', n' \in \mathbb{N}$, $p' \in \mathcal{P}$, $z' \in \mathcal{Z}$, and $t = 1, \ldots, T - 1$.
- $v_t^{(m,n)}(p, z) \geq v_t^{(m+1,n)}(p, z)$ if $S_{G(m')}h(z') \geq S_{G(m'+1)}h(z')$ for $z' \in \mathcal{Z}$, $m' \in \mathbb{N}$, and all possible convex functions $h$.

**Proof.** The three inequalities are proven separately using backward induction.

1. The monotonicity of (11) and Lemma 7 $\implies K_{T-1}^{a_0(n)}S_{G(m)}v_t^*(p, z) \geq K_{T-1}^{a_0(n)}v_{T-1}^*(p, z) \implies S_{G(m)}r_{T-1}(p, z, a) + S_{G(m')}K_{T-1}^{a_0(n)}S_{G(m')}v_t^*(p, z) \geq v_{T-1}^*(p, z, a) \implies v_{T-1}^{(m,n)}(p, z) \geq v_{T-1}^*(p, z)$. Proceeding inductively for $t = T - 2, \ldots, 0$ using a similar argument as above gives the desired result.
2) Lemma 7 \implies k_{T-1}^{a(n)}S_{G^{(m)}}v_{T}^{z}(p, z) \geq k_{T-1}^{a(n+1)}S_{G^{(m)}}v_{T}^{z}(p, z). Therefore, \( v_{T-1}^{(m,n)}(p, z) \geq v_{T-1}^{(m,n+1)}(p, z) \). Using the monotonicity of (11) and Lemma 7, it holds that

\[
k_{T-2}^{a(n)}v_{T-2}^{(m,n)}(p, z) \geq k_{T-2}^{a(n)}v_{T-2}^{(m,n+1)}(p, z) \geq k_{T-2}^{a(n+1)}v_{T-2}^{(m,n+1)}(p, z)
\]

\[
\implies v_{T-2}^{(m,n)}(p, z) \geq v_{T-2}^{(m,n+1)}(p, z).
\]

Proceeding inductively for \( t = T-3, \ldots , 0 \) proves the second part of the statement.

3) This can be proved via backward induction using the monotonicity of (11) and by the fact that \( S_{G^{(m)}}r_{t+1}(p, z, a) \geq S_{G^{(m+1)}}r_{t}(p, z, a) \) and \( S_{G^{(m)}}r_{T}(p, z) \geq S_{G^{(m+1)}}r_{T}(p, z) \) for \( p \in P, a \in A, m \in \mathbb{N}, \) and \( t = 0, \ldots , T - 1. \)

\[
\Box
\]

7. Numerical demonstration

A natural choice for \( S_{G^{(m)}} \) is to use piecewise linear functions. A piecewise linear function can be represented by a matrix where each row or column captures the relevant information for each linear functional. This is attractive given the availability of fast linear algebra software.

7.1. Bermudan put option. Markov decision processes are common in financial markets. For example, a Bermudan put option represents the right but not the obligation to sell the underlying asset for a predetermined strike price \( K \) at prespecified time points. This problem is characterized by \( P = \{ \text{exercised, unexercised} \} \) and \( A = \{ \text{exercise, don’t exercise} \}. \) At \( P_{t} = \{ \text{unexercised} \}, \) applying \( a = \{ \text{exercise} \} \) and \( a = \{ \text{don’t exercise} \} \) leads to \( P_{t+1} = \{ \text{exercised} \} \) and \( P_{t+1} = \{ \text{unexercised} \}, \) respectively with probability one. If \( P_{t} = \{ \text{exercised} \}, \) then \( P_{t+1} = \{ \text{exercised} \} \) almost surely regardless of action \( a. \) Let \( \Delta \) be the time step and represent the interest rate per annum by \( \kappa \) and underlying asset price by \( z. \) Defining \( (z)^{+} = \max(z, 0), \) the reward and scrap for the option are given by

\[
r_{z}(\text{unexercised, } z, \text{exercise}) = e^{-\kappa \Delta t}(K - z)^{+}
\]

\[
r_{z}(\text{unexercised, } z) = e^{-\kappa \Delta T}(K - z)^{+}
\]

for all \( z \in \mathbb{R}_{+} \) and zero for other \( p \in P \) and \( a \in A. \) The fair price of the option is

\[
v_{0}^{z}(\text{unexercised, } z_{0}) = \max \{ E[e^{-\kappa \Delta T}(K - Z_{\tau})^{+}] : \tau = 0, 1, \ldots , T \}.
\]

The option is assumed to reside in the Black-Scholes world where the asset price process \((Z_{t})_{t=0}^{T}\) follows geometric Brownian motion i.e.

\[
Z_{t+1} = W_{t+1}Z_{t} = e^{(\kappa - \frac{\sigma^{2}}{2})\Delta + \sigma \sqrt{\Delta} N_{t+1}}Z_{t}
\]

where \((N_{t})_{t=1}^{T}\) are independent standard normal random variables and \( \sigma \) is the volatility of stock returns. Note that the disturbance is not controlled by action \( a \) and so the superscript is removed from \( W_{t+1}^{a} \) for notational simplicity in the following subsections. The reward and scrap functions are convex and Lipschitz continuous in \( z. \) It is not hard to see that under the linear state dynamic for \((Z_{t})_{t=0}^{T}\), the resulting expected value functions and value functions are also convex, Lipschitz continuous, and decreasing in \( z. \) In the following two subsections, two different \( S_{G^{(m)}} \) schemes are used to approximate these functions.
7.2. Approximation using tangents. It is well known that a convex real valued function is differentiable almost everywhere [18, Theorem 25.5] and so can be approximated accurately on any compact set using a sufficient number of its tangents. Suppose that convex function \( h : \mathbb{Z} \to \mathbb{R} \) holds tangents on each point in \( G^{(m)} \) given by \( \{h'_1(z), \ldots, h'_m(z)\} \) and that the approximation scheme \( S_{G^{(m)}} \) takes the maximising tangent to form a convex piecewise linear approximation of \( h \) i.e.

\[
S_{G^{(m)}}(z) = \max \{h'_1(z), \ldots, h'_m(z)\}.
\]

It is not hard to see that the resulting approximation \( S_{G^{(m)}}(z) \) is convex, piecewise linear, and converges to \( h \) uniformly on compact sets as \( m \to \infty \). It is also clear that \( S_{G^{(m)}}(z) \leq S_{G^{(m+1)}}(z) \leq h(z) \) for all \( z \in \mathbb{Z} \). Note that while the choice of a tangent may not be unique at some set of points in \( \mathbb{Z} \), the uniform convergence on compact sets of this scheme is not affected. Assumption 5 holds for any choice of grid and disturbance sampling under the linear state dynamics in Section 7.1. To see this, note that if \( v(p, z) \) is any function convex in \( z \) then \( v(p, wz) \) is also convex in \( z \) for any \( w \in \mathbb{W} \).

As a demonstration, a Bermudan put option is considered with strike price 40 that expires in 1 year. The put option is exercisable at 51 evenly spaced time points in the year, which includes the start and end of the year. The interest rate and the volatility is set at 0.18958 and use disturbance space partitions consisting of \( n \) convex components of equal probability mass i.e. \( P(W_t \in \mathbb{W}) = 0.999999999 \) for all \( t = 1, \ldots, T \). To this end, introduce the truncated distribution \( \mathbb{P} \) defined by \( \mathbb{P}(W_t \in \mathbb{B}) = \alpha P(W_t \in \mathbb{B}) \) for all \( \mathbb{B} \subseteq \mathbb{W} \) where \( \alpha = 1/P(W_t \in \mathbb{W}) \) is the normalizing constant. In the following results, set \( \mathbb{W} = [0.841979, 1.18958] \) and use disturbance space partitions consisting of \( n \) convex components of equal probability mass. Suppose that the extreme points are ordered \( e^{(n)}_{k,1} < e^{(n)}_{k,2} \) for all \( k = 1, \ldots, n \). For \( k = 1, \ldots, n \), define points \( e^{(n)}_k = \left[e^{(n)}_{(k+1)/2]}, [k+2-2(k+1)/2]\right] \) and \( e^{(n)}_{n+1} = e^{(n)}_{n,2} \) where \( [\ ] \) denotes the integer part. Defining partial expectations \( E_t(a, b) = E[1 \{W_t \in [a, b]\}] \)
and using
\[ q_{t,k,1}^{(n)}(w) = \frac{e_{j,2} - w}{e_{j,2} - e_{j,1}} \quad \text{and} \quad q_{t,k,2}^{(n)}(w) = \frac{w - e_{j,1}}{e_{j,2} - e_{j,1}} \]
onone can determine
\[
\begin{align*}
\mathbb{P}\left(W_t^{(n)} = e_1^{(n)} \right) &= \frac{e_2^{(n)} / n - \Lambda_t(e_1^{(n)}, e_2^{(n)})}{e_2^{(n)} - e_1^{(n)}}, \\
\mathbb{P}\left(W_t^{(n)} = e_{n+1}^{(n)} \right) &= \frac{\Lambda_t(e_n^{(n)}, e_{n+1}^{(n)}) - e_n^{(n)}/n}{e_{n+1}^{(n)} - e_n^{(n)}}, \\
\mathbb{P}\left(W_t^{(n)} = e_j^{(n)} \right) &= \frac{e_j^{(n)} + 1 / n - \Lambda_t(e_j^{(n)}, e_{j+1}^{(n)})}{e_{j+1}^{(n)} - e_j^{(n)}} + \frac{\Lambda_t(e_{j-1}^{(n)}, e_j^{(n)}) - e_{j-1}^{(n)}/n}{e_j^{(n)} - e_{j-1}^{(n)}}
\end{align*}
\]
for \( j = 2, \ldots, n \).
Recall that the reward, scrap, and the true value functions are Lipschitz continuous, convex, and non-increasing in \( z \). Therefore, there exists a \( z' \) such that these functions are linear in \( z \) when
z < z′ for some z′. In fact, it is well known that \( v_t^*(\text{unexercised}, z) = r_t(\text{unexercised}, z, \text{exercise}) \) when z < z′ for \( t = 0, \ldots, T \) and some z′. Let us define our approximation scheme in the following manner. Suppose \( h: Z \rightarrow \mathbb{R} \) is a convex function non-increasing in z and \( h(z) = h'(z) \) when z < z′. For \( G^m = \{g(1), \ldots, g(m)\} \) where \( g(1) \leq g(2) < \cdots < g(m) \), set

\[
S_{G(m)} h(z) = \begin{cases} 
    h'(z) & \text{if } z \leq g(1); \\
    d_i (z - g(i)) + h(g(i)) & \text{if } g(i) < z \leq g(i+1); \\
    h(g(m)) & \text{if } z > g(m),
\end{cases}
\]

where \( d_i = \frac{h(g(i+1)) - h(g(i))}{g(i+1) - g(i)} \) for \( i = 2, \ldots, m-1 \). For \( g^{(1)} \leq z \leq g^{(m)} \), \( S_{G(m)} \) forms an approximation of h via linear interpolation. It is not hard to see that \( S_{G(m)} h(z) \geq S_{G(m+1)} h(z) \geq h(z) \) for all \( z \in Z \). It is also clear that Assumption 5 will hold for any grid and disturbance sampling due to the linear dynamics of \( Z_t \).

![Figure 4](image-url) Figure 4. Price estimate as disturbance sampling (left plot) or grid density (right plot) is increased.

Using the same put option parameters as before and a grid with points equally spaced between \( z = 30 \) and 60, Figure 4 examines the behaviour of the upper bound for the option with \( Z_0 = 36 \) as we increase the size of the partition \( n \) or the grid density \( m \). In the left plot, the size of the partition \( n \) is increased for fixed \( m = 301 \). Similarly, the grid density is increased in the right plot for fixed \( n = 10000 \). Observe that the bounds is decreasing in both \( m \) and \( n \) as proved by Theorem 10.

### 7.4. Accuracy and speed

Table 1 gives points on the lower and upper bounding functions for the fair price of the option at different starting asset prices and expiry dates. Columns 2 to 4 give an option with the 1 year expiry. A grid of 301 equally spaced points from \( z = 30 \) to \( z = 60 \) is used. Columns 5 to 7 gives an option expiring in 2 years and is exerciseable at 101 equally spaced time points including the start. A wider grid of 401 equally spaced points from \( z = 30 \) to \( z = 70 \) is used to account for the longer time horizon. For both, a disturbance partition of size \( n = 1000 \) is used. The other option parameters (e.g. interest rate) remain the same as before. Tighter bounding functions can be obtained by increasing the grid density \( m \) or size of the disturbance sampling \( n \). This is illustrated by columns 8 – 10 where we revisit the 1 year expiry option with 4001 equally spaced grid points from \( z = 30 \) to \( z = 70 \) and a disturbance partition of \( n = 20000 \) is used.
Table 1. Bermuda put valuation with $\text{vol} = 0.2$.

| $Z_0$ | Expiry 1 year | Expiry 2 years | Dense, Expiry 1 year |
|-------|---------------|---------------|---------------------|
|       | Lower | Upper | Gap    | Lower | Upper | Gap    | Lower | Upper | Gap |
| 32    | 8.00000 | 8.00000 | 0.00000 | 8.00000 | 8.00000 | 0.00000 | 8.00000 | 8.00000 | 0e+00 |
| 34    | 6.05155 | 6.05318 | 0.00163 | 6.22898 | 6.23254 | 0.00356 | 6.05198 | 6.05201 | 2e-05 |
| 36    | 4.47689 | 4.48038 | 0.00348 | 4.83885 | 4.84435 | 0.00550 | 4.47780 | 4.47785 | 5e-05 |
| 38    | 3.24898 | 3.25347 | 0.00450 | 3.74319 | 3.74964 | 0.00645 | 3.25011 | 3.25018 | 7e-05 |
| 40    | 2.31287 | 2.31766 | 0.00479 | 2.88294 | 2.88965 | 0.00670 | 2.31405 | 2.31413 | 8e-05 |
| 42    | 1.61582 | 1.62047 | 0.00465 | 2.21077 | 2.21735 | 0.00658 | 1.61696 | 1.61704 | 8e-05 |
| 44    | 1.10874 | 1.11311 | 0.00437 | 1.68826 | 1.69456 | 0.00630 | 1.10985 | 1.10993 | 8e-05 |
| 46    | 0.74795 | 0.75217 | 0.00423 | 1.28419 | 1.29023 | 0.00604 | 0.74915 | 0.74922 | 7e-05 |

The results in Table 1 were computed using the R script provided in the appendix. On a Linux Ubuntu 16.04 machine with Intel i5-5300U CPU @2.30GHz and 16GB of RAM, it takes roughly 0.15 cpu seconds to generate each of the bounding functions represented by columns 2 and 3. For expiry 2 years, it takes around 0.20 cpu seconds to generate each bounding function. For columns 8 or 9, it takes around 40 cpu seconds. Note that the numerical methods used are highly parallelizable. The code uses some multi-threaded code and the times can be reduced to between 0.03 - 0.10 real world seconds to generate columns 2, 3, 5, and 6 each on four CPU cores. For columns 8 and 9, the times are reduced to around 10 real world seconds. Please note that faster computational times can be attained by a more strategic placement of the grid points. The reader is invited to replicate these results using the R script provided in the appendix. Now given the excellent quality of the value function approximations, the optimal policy can then be obtained via (3). This is demonstrated by Figure 5 where the right plot gives the optimal exercise boundary. If at any time the price of the underlying asset is below the cutoff, it is optimal to exercise the option.

![Figure 5](image-url)

**Figure 5.** Left plot gives the lower and upper bounding functions for option price with $\text{vol} = 0.2$ and 1 year expiry. The dashed lines indicate the upper bound while the unbroken curves give the lower bounds. The optimal exercise boundary is given in the right plot.

For the 1 year expiry option, suppose the volatility is doubled to $\text{vol} = 0.4$. Using the same grid and disturbance sampling from columns 2 and 3 leads to poor bounding functions...
as illustrated by the left plot of Figure 6. However, spreading the same number of grid points evenly between \( z = 20 \) and \( z = 80 \) leads to better bounds as shown by the right plot. The computational times remain the same as before. Therefore, the tightness of the bounds can be achieved by the simple modification of either the grid or disturbance sampling and their adjustments can be done independently of each other as shown in Theorem 7 and Theorem 10. This is in contrast to the popular least squares Monte Carlo method where the size of the regression basis should not grow independently of the number of simulated paths [5].

8. Conclusion

This paper studies the use of a general class of convex function approximation to estimate the value functions in Markov decision processes. The key idea is that the original problem is approximated with a more tractable problem and under certain conditions, the solutions to this modified problem converge to their original counterparts. More specifically, this paper has shown that these approximations may converge uniformly to their true unknown counterparts on compact sets under different sampling schemes for the driving random variables. Exploiting further conditions leads to approximations that form either a non-decreasing sequence of lower bounding functions or a non-increasing sequence of upper bounding functions. Numerical results then demonstrate the speed and accuracy of a proposed approach involving piecewise linear functions. While the focus of this paper has been numerical work, one can in principle replace the original problem with a more analytically tractable problem and obtain the original solution by considering the limits.

The starting state \( X_0 = x_0 \) for the decision problem was assumed to be known with certainty. Suppose this is not true and \( X_0 \) is distributed with distribution \( P_{X_0} \). Then, the value function for this case can be obtained simply via

\[
\int_X v_0^*(x') P_{X_0}(dx')
\]

where \( v_0^* \) is obtained assuming the starting state is known. Now note that the insights presented in this paper can be adapted to problems where the functions in the Bellman recursion are not convex in \( z \). For example, it is not hard to see that they can be easily modified for minimization.
problems involving concave functions. That is, problems of the form

$$T_t v(x) = \min_{a \in A} (r_t(x, a) + K^a_t v(x))$$

where the scrap and reward functions are concave in $z$ and the transition operator preserves concavity. To see this, note that the sum of concave functions is concave and the pointwise minimum of concave functions is also concave. Finally, the methods shown in this paper has been adapted to infinite time horizon contracting Markov decision processes in [22]. Extensions to partially observable Markov decision processes will be considered in future research.

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APPENDIX A. R Script for Table 1

The script (along with the R package) used to generate columns 2, 3, and 4 in Table 1 can be found at https://github.com/YeeJeremy/ConvexPaper. To generate the others, simply modify the values on Line 6, Line 10, Line 11, and/or Line 14.

REFERENCES

[1] N. Bauerle and U. Rieder, Markov decision processes with applications to finance, Springer, Heidelberg, 2011.
[2] J. Birge, State-of-the-art-survey–stochastic programming: Computation and applications, INFORMS Journal on Computing 9 (1997), no. 2, 111–133.
[3] J. Birge and R. Wets, Designing approximation schemes for stochastic optimization problems, in particular for stochastic programs with recourse, pp. 54–102, Springer Berlin Heidelberg, 1986.
[4] J. Carriere, Valuation of the early-exercise price for options using simulations and nonparametric regression, Insurance: Mathematics and Economics 19 (1996), 19–30.
[5] P. Glasserman and B. Yu, Number of paths versus number of basis functions in american option pricing, Annals of Applied Probability 14 (2004), 2090–2119.
[6] R. Gray and D. Neuhoff, Quantization, IEEE Trans. Inf. Theor. 44 (2006), no. 6, 2325–2383.
[7] O. Hernandez-Lerma and J. Lasserre, Discrete-time markov control processes : Basic optimality criteria, Springer, New York, 1996.
[8] O. Hernandez-Lerma, C. Piovesan, and W. Runggaldier, Numerical aspects of monotone approximations in convex stochastic control problems, Annals of Operations Research 56 (1995), no. 1, 135–156.
[9] O. Hernandez-Lerma and W. Runggaldier, Monotone approximations for convex stochastic control problems, Journal of Mathematical Systems, Estimation, and Control 4 (1994), no. 1, 99–140.
[10] K. Hinderer, Foundations of non-stationary dynamic programming with discrete time parameter, Springer-Verlag, Berlin, 1970.
[11] J. Hinz, Optimal stochastic switching under convexity assumptions, SIAM Journal on Control and Optimization 52 (2014), no. 1, 164–188.
[12] J. Hinz and J. Yee, Optimal forward trading and battery control under renewable electricity generation, Journal of Banking & Finance InPress (2017).
[13] Stochastic switching for partially observable dynamics and optimal asset allocation, International Journal of Control 90 (2017), no. 3, 553–565.
[14] F. Longstaff and E. Schwartz, Valuing American options by simulation: a simple least-squares approach, Review of Financial Studies 14 (2001), no. 1, 113–147.
[15] G. Pages, H. Pham, and J. Printems, Optimal quantization methods and applications to numerical problems in finance, pp. 253–297, Birkhuser Boston, 2004.
[16] W. Powell, Approximate dynamic programming: Solving the curses of dimensionality, Wiley, Hoboken, New Jersey, 2007.
[17] M. Puterman, Markov decision processes: Discrete stochastic dynamic programming, Wiley, New York, 1994.
[18] R. Rockafellar, *Convex analysis*, Princeton landmarks in mathematics and physics, Princeton University Press, 1970.

[19] J. Tsitsiklis and B. Van Roy, *Optimal stopping of Markov processes: Hilbert space, theory, approximation algorithms, and an application to pricing high-dimensional financial derivatives*, IEEE Transactions on Automatic Control **44** (1999), no. 10, 1840–1851.

[20] ———, *Regression methods for pricing complex American-style options*, IEEE Transactions on Neural Networks **12** (2001), no. 4, 694–703.

[21] D. Williams, *Probability with martingales*, Cambridge mathematical textbooks, Cambridge University Press, 1991.

[22] J. Yee, *Approximate value iteration for dynamic programming under convexity*, Preprint (Preprint).