BPS saturation of the $N = 4$ monopole
by infinite composite-operator renormalization

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ABSTRACT

Quantum corrections to the magnetic central charge of the monopole in $N = 4$ supersymmetric Yang-Mills theory are free from the anomalous contributions that are crucial for BPS saturation of the two-dimensional supersymmetric kink and the $N = 2$ monopole. However, these quantum corrections are nontrivial and they require infinite renormalization of the supersymmetry current, central charges, and energy-momentum tensor, in contrast to $N = 2$ and even though the $N = 4$ theory is finite. Their composite-operator renormalization leads to counterterms which form a multiplet of improvement terms. Using on-shell renormalization conditions the quantum corrections to the mass and the central charge then vanish both, thus verifying quantum BPS saturation.

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Solitons in supersymmetric (SUSY) quantum field theories \[1\ 2\] (and further references in \[3\]) continue to produce surprises. A new anomalous contribution to the central charge of the 1+1-dimensional $N = 1$ kink was needed for BPS saturation \[4\ 5\ 6\ 7\]; using supersymmetry-preserving dimensional regularization, suitably adapted to take solitons into account \[8\], this anomaly was due to a kink-induced left-right asymmetry of the momenta of the fermion in the extra dimension \[9\]. For the 2+1-dimensional $N = 2$ vortex no such anomalies appeared, but a finite quantum correction to the central charge, induced by the winding of the background fields, was essential for BPS saturation \[10\]. Recently, we studied quantum corrections to the monopole in 3+1-dimensional $N = 2$ SUSY Yang-Mills theory. While the infinite renormalization of the coupling constant cancelled a logarithmic divergence in the mass \[15\] and the central charge \[16\], we obtained unexpected finite quantum corrections to both the mass and the central charge \[17\] which are completely analogous to the anomalous contributions of the $N = 1$ kink of Ref. \[9\], and in fact essential for consistency with the $N = 2$ low-energy effective action of Seiberg and Witten \[18\ 19\ 20\].

In this Letter we consider the monopole in the $N = 4$ SUSY Yang-Mills model in 3+1 dimensions. Since this is a finite theory, one expects no contributions from anomalies, nor infinite counterterms from the renormalization of physical parameters.\[2\] Thus the sum over zero point energies should give a finite, perhaps vanishing, one-loop quantum correction to the mass, and this is indeed the case as shown in Ref. \[16\]. Then also the central charge ought to be finite, and equal to the mass to comply with BPS saturation \[2\]. But comparison with the renormalization of infinities in the $N = 2$ case as worked out in Ref. \[16\] produces a perplexing puzzle.\[3\] In the $N = 2$ model the quantum corrections to the central charge in the monopole background have an ultraviolet divergence which is cancelled by the infinite renormalization of the coupling constant. However, in the $N = 4$ model the extra four scalar fields and the additional fermions do not contribute to the central charge, while the coupling constant no longer requires a counterterm. Hence,\[1\] For a superfield treatment of the SUSY kink and the vortex see Refs. \[11\ 12\ 13\] and \[14\], respectively.

\[2\] However there are divergent wave-function renormalizations in Wess-Zumino gauge \[21\], infrared-divergences in general supergauges \[21\], and logarithmic divergences in Green functions of gauge-invariant composite operators corresponding to unprotected long multiplets \[22\].

\[3\] We thank S. Mukhi for drawing our attention to it.
the one-loop central charge seems divergent, while the one-loop contribution to the mass is zero, so that the $N = 4$ model in the sector with solitons, in contrast to $N = 2$, is not finite, in apparent conflict with BPS saturation.

Confronted with this situation, one may consider various possibilities for a solution:

(i) that fermionic terms in the central charge could contribute. Fermionic terms of course contribute to the mass, but common lore has it that they do not contribute to central charges and other integrals of total derivatives. However, while the supersymmetry algebra involves numerous fermionic terms (which are in fact responsible for the anomalous contribution to the central charge of the $N = 2$ theory [17]), they disappear from central charges in strictly four dimensions and thus should not be able to give rise to infinities;

(ii) that one should add improvement terms to the currents. Since in the spontaneously broken phase one has a mass scale (the vev of the Higgs scalar), it is doubtful that improved currents should play a role. Indeed, a SUSY variation of the improvement term to the SUSY current produces an extra bosonic total derivative in the central charge current and in the energy density that would modify mass and central charge already at the classical level. If the SUSY algebra is to represent the monopole with the standard value of mass and central charge, the relevant SUSY current has to be unimproved, in the form that is determined by dimensional reduction of the 10-dimensional theory;

(iii) that one should simply renormalize the central charge. This would solve the problem of the finiteness of the central charge, but supersymmetry would then require that one also renormalize the SUSY current and the stress tensor. This would seem to clash with the calculated finiteness of the quantum mass of the monopole.

In this Letter we present a solution which is as surprising as it is simple: if one considers the multiplet associated with the SUSY current [23, 24, 25, 26, 27, 28], the multiplet of unimproved currents of the $N = 4$ model requires additive infinite renormalization involving a multiplet of improvement currents as counterterms. The $N = 2$ model differs by the absence of a 5-form charge in the higher-dimensional SUSY algebra, which permits cancellations not possible in $N = 4$, and both the unimproved and the improved central charge density turn out to be finite by themselves.

To demonstrate these features explicitly, we describe the $N = 4$ model in more detail and set up our notations. The $N = 4$ super Yang-Mills theory is most easily obtained from trivial dimensional reduction from 9+1
dimensions \cite{29, 30}. We shall write $x^M$ for the coordinates and $A_M^a$ for the ten-dimensional Yang-Mills field with $M$ running over $0, \ldots, 3, 5, \ldots, 10$ and $a$ the index for colour (for simplicity we shall take SU(2)). The action reads

$$\mathcal{L} = -\frac{1}{4} (F_{MN}^a)^2 - \frac{1}{2} \bar{\lambda}^a \Gamma^M (D_M \lambda)^a$$

(1)

with a Majorana-Weyl spinor $\lambda$ and is invariant under $\delta A_M = -\epsilon \Gamma_M \lambda$ and $\delta \lambda = \frac{1}{2} F_{MN} \Gamma^{MN} \epsilon$.

The SUSY current is $j^M = \frac{1}{2} F_{NP} \Gamma^{NP} \Gamma^M \lambda$ and varies under SUSY into

$$\frac{1}{2} \delta j^M (x) = (F^{MP} F_{NP} - \frac{1}{4} \delta_N^M F^{RS} F_{RS} + \frac{1}{2} \bar{\lambda} \Gamma^M D_N \lambda) \Gamma^N \epsilon$$

$$+ \frac{1}{16} (\bar{\lambda} \Gamma^M \Gamma^{PQ} D^R \lambda) \Gamma_{PQR} \epsilon$$

$$+ \frac{1}{8} F_{NP} F_{QR} \Gamma^{NP} \Gamma^{QR} \epsilon$$

(2)

The term with $\Gamma_{PQR}$ vanishes after integration over $x$ since $\{Q, Q\}$ is symmetric in the two spinor indices.

For the purpose of dimensional reduction, the 16-component Majorana-Weyl spinor $\lambda^a$ is written as $(\lambda^\alpha I^a, 0)$, where $\alpha = 1, \ldots, 4$ is the 4-dimensional spinor index, $I = 1, \ldots, 4$ is the rigid SU(4) index, and $a$ is the adjoint SU(2) colour index. For the gamma matrices we use the representation\footnote{We use $\{\gamma_\mu, \gamma_\nu\} = 2 \eta_{\mu\nu}$ with metric signature $-+++$, and $\mu, \nu = 0, 1, 2, 3$. Further, $\gamma_5 = i \gamma_0 \gamma_1 \gamma_2 \gamma_3$, so $\{\gamma_5, \gamma_\mu\} = 0$ and $\gamma_5^2 = 1$. Majorana spinors satisfy $\lambda^T C_{11} = \lambda^0 \Gamma^0$ in 9+1 dimensions and $(\lambda')^T C_4 = (\lambda')^0 i \gamma_5$ in 3+1 dimensions.}

$$\Gamma_\mu = \gamma_\mu \otimes 1 \otimes \sigma_2, \quad \mu = 0, 1, 2, 3,$$

$$\Gamma_{4+j} = 1 \otimes \alpha_j \otimes \sigma_1,$$

$$\Gamma_{7+j} = \gamma_5 \otimes \beta_j \otimes \sigma_2, \quad j = 1, 2, 3,$$

$$\Gamma_{11} = 1 \otimes 1 \otimes \sigma_3, \quad C_{11} = -i C_4 \otimes 1 \otimes \sigma_1.$$

(3)

The $\alpha_j$ and $\beta_j$ are the six generators of SO(3)$\times$SO(3) in the representation of purely imaginary antisymmetric $4 \times 4$ matrices \cite{31, 32}, self-dual and anti-self-dual, respectively, and satisfying $\{\alpha_i, \alpha_j\} = \{\beta_i, \beta_j\} = 2 \delta_{ij}$, $[\alpha_i, \alpha_j] = 2 i \epsilon_{ijk} \alpha_k$, $[\beta_i, \beta_j] = 2 i \epsilon_{ijk} \beta_k$, and $[\alpha_j, \beta_k] = 0$.

After reduction to 3+1 dimensions, the energy operator $T_{00}$ contains the sum of the stress tensors for the gauge fields $A^a_\mu$, the four Majorana spinors $\lambda^{\alpha I a}$, the three scalars $S_j = A_{4+j}$, and the three pseudoscalars $P_j = A_{7+j}$. 


The 12 real central charges appear as\(^5\)

\[
\frac{1}{2}\{Q^\alpha I, Q^\beta J\} = \delta^{IJ}(\gamma^\mu C^{-1})^{\alpha\beta} P_\mu \\
+ i(\gamma_5 C^{-1})^{\alpha\beta}(\alpha^I)^{IJ} \int d^3 x U_j - (C^{-1})^{\alpha\beta}(\beta^I)^{IJ} \int d^3 x V_j \\
+ (C^{-1})^{\alpha\beta}(\beta^I)^{IJ} \int d^3 x \tilde{U}_j + i(\gamma_5 C^{-1})^{\alpha\beta}(\beta^I)^{IJ} \int d^3 x \tilde{V}_j, \quad (4)
\]

where \(U_j\) and \(V_j\) are due to the five-gamma term in (2), and \(\tilde{U}_j\) and \(\tilde{V}_j\) due to the one-gamma terms. The indices \(I\) and \(J\) are lowered and raised by the charge conjugation matrix in this space, which is \(\delta^{IJ}\), see (3).

The \(N = 2\) monopole (and dyon) can be embedded into the \(N = 4\) model by selecting \(j = 1\) and one then has \((S = S_1, P = P_1, U = U_1 \text{ etc.})\)^6

\[
U = \partial_i(S^a \frac{1}{2} \epsilon^{ijk} F_{jk}^a), \quad \tilde{U} = \partial_i(P^a F_{0i}^a) \\
V = \partial_i(P^a \frac{1}{2} \epsilon^{ijk} F_{jk}^a), \quad \tilde{V} = \partial_i(S^a F_{0i}^a). \quad (5)
\]

The classical contribution to the central charge of a magnetic monopole is due to \(U\), and it is determined by the asymptotic values

\[
A_i^a \rightarrow \epsilon_{aij} \frac{1}{g} \hat{x}^j r, \quad F_{ij}^a \rightarrow -\epsilon_{ijk} \frac{1}{g} \hat{x}^a \hat{x}^k r^2, \\
S_1^a \rightarrow -\delta^a_b \hat{x}^i (v - \frac{1}{gr}), \quad \hat{x}^i \equiv x^i / r, \quad (6)
\]

yielding \(\int d^3 x U = 4\pi v / g\).

For the above realization of central charges it is crucial that the SUSY current \(j\) is exactly as determined by the 9+1-dimensional theory and does

\(^5\)The complex matrix \(Z^{IJ} = -Z^{JI}\) of central charges contains 6 complex (12 real) elements, as shown in \(\text{[4]}\). The magnetic charge \(U_j\) and the electric charge \(V_j\) only appear in the combination \(Z^{IJ} = (iU_1 + \tilde{V}_1)(\alpha^I)^{IJ} + \ldots\). By a unitary transformation one can block-diagonalize \(Z^{IJ}\), with two real antisymmetric \(2 \times 2\) matrices along the diagonal. The \(N = 4\) action has a rigid \(R\) symmetry group \(\text{SU}(4)\) (not \(\text{U}(4)\) \(\text{[33]}\)). To exhibit this \(\text{SU}(4)\), the spin zero fields are combined into \(M^{IJ} = (\alpha^I)^{IJ} S_j + i(\beta^I)^{IJ} P_j\), but if \(S_1^a = v \delta^a_3\), there is a central charge and \(R\) is broken down to the manifest stability subgroup of \(M^{23} = M^{14} = S_1\), which is \(\text{USp}(4)\).

\(^6\)To obtain the total derivatives in \(\text{[5]}\), one needs to use Bianchi identities in the case of \(U\) and \(V\), and equations of motion in the case of \(\tilde{U}\) and \(\tilde{V}\).
not have improvement terms at the classical level. In 3+1 dimensions one could add a gauge invariant improvement term of the form

$$\Delta j^\mu = c \Gamma^{\mu \nu} \partial_\nu (A_j \Gamma^J \lambda), \quad J = 5, \ldots, 10,$$

(7)

which for \( c = -2/3 \) would provide \( \gamma_\mu (j + \Delta j)^\mu = 0 \). However, this would change \( U \) into \( U + \Delta U \) with

$$\Delta U = \frac{c}{2} \left( U + \frac{i}{8} \partial_i (\epsilon^{ijk} \lambda \gamma^1 \gamma_{jk} \lambda) \right),$$

(8)

and would change the value of the central charge \( \int d^3 x U \) at the classical level.

Similarly, the stress tensor would be modified, resulting in a corresponding shift of mass ascribed to the monopole.

For the calculation of quantum corrections we need to fix the gauge. The most convenient choice turns out to be the background-covariant Feynman-R \( \xi \) gauge (with \( \xi = 1 \)) obtained by dimensional reduction of the gauge fixing term

$$\mathcal{L}_{g.f.} = -\frac{1}{2\xi} (D_M [\hat{A}] a^M)^2$$

(9)

where \( a^M \) are the quantum gauge fields and \( \hat{A}^M \) the background fields.

To determine potential counterterms, it is sufficient to consider the spontaneously broken phase in the trivial background with no monopoles. Choosing \( \langle S_1^a \rangle = v \delta_3^a \), one finds that tadpole diagrams with an external background field \( \hat{S}_1 \) or an external quantum field \( s_1 \) vanish. Coupling constant renormalization can be fixed by wave-function renormalization of the gauge boson background fields due to \( Z_g = Z_A^{-1/2} \). Using the on-shell two-point self-energy of the massless photon with external background fields, one finds after a Wick rotation

$$\frac{1}{g_0^2} = \frac{1}{g^2} + c_g I, \quad I \equiv \int \frac{d^{4+\epsilon} k_E}{(2\pi)^{4+\epsilon}} \frac{1}{(k_E^2 + m^2)^2}, \quad m = g_0 v_0 = g v,$$

(10)

with \( c_g = 4 \) in \( N = 2 \), but \( c_g = 0 \) in the present case of \( N = 4 \). Renormalizing similarly the massless Higgs boson on-shell, one finds \( Z_S = Z_A \) in the gauge \( \xi = 1 \), but

$$Z_S = (1 + 2 g^2 [4 - N + 1 - \xi] I)$$

(11)

for \( \xi \neq 1 \), while \( Z_A \) is \( \xi \)-independent.
The one-loop result for the mass of a monopole is given by

\[ M^{(1)} = \frac{4\pi m}{g_0^2} + \frac{1}{2} \sum (\omega_B - \omega_F). \]  

(12)

In the \( N = 2 \) case the sum over zero-point energies gives rise to an integral over a nontrivial difference of spectral densities which is UV divergent such that the infinities cancel with those of (10), leaving a finite remainder, namely the anomalous contribution discussed in Ref. [17]. In \( N = 4 \), however, there is complete cancellation of the sums over zero-point energies [16]. The requirement of quantum BPS saturation thus demands a nontrivial contribution to the central charge in the case of \( N = 2 \), but none for \( N = 4 \).

In Ref. [17], the anomalous contribution to the central charge for \( N = 2 \) was obtained as a fermionic contribution to the momentum operator in the extra dimension required for supersymmetry-preserving dimensional regularization. This involved a difference of spectral densities for the different components of the fermionic fluctuations. In the case of \( N = 4 \), there are twice as many fermions, and it turns out that the extra fermions lead to a complete cancellation of anomalous contributions.

The non-anomalous quantum corrections to the central charge of a monopole can be calculated along the lines of Ref. [16]. At one-loop order they are given by

\[ U^{(1)}_{\text{non-ano}} = \frac{4\pi m}{g_0^2} + \frac{1}{2} \int d^3x \partial_i \left( \hat{S}^a_{ijk} \langle F_{jk}^a[\hat{A} + a] - F_{jk}^a[\hat{A}] \rangle \right). \]  

(13)

Expanding in terms of \( a \), the term quadratic in \( a \) produces a loop formed by a propagator for the \( a^a_\mu \) field in the monopole background, whose spatial components in background-covariant Feynman-\( R_\xi \) gauge read [16]

\[ \langle a_j^b(x)a_k^c(y) \rangle = i[(D^2)^{-1b} \delta_{jk} + 2(D^2)^{-1bb'}ge^{b'c'e}F_{jk}^e (D^2)^{-1c'c} + \ldots] \delta^4(\xi)(x - y) \]  

(14)

where \((\tilde{D}^2)^{-1}\) is the dimensionally reduced version of \((D_M[\hat{A}]D^M[\hat{A}])^{-1}\) with asymptotic behaviour

\[ (\tilde{D}^2)^{-1ab} \rightarrow (\Box - m^2 + i\epsilon)^{-1}(\delta^{ab} - \hat{x}^a\hat{x}^b) + (\Box + i\epsilon)^{-1}\hat{x}^a\hat{x}^b. \]  

(15)

Using that \( \hat{S}^a \rightarrow -v \hat{x}^a \) and that to leading order in \( 1/r \) the vector \( \hat{x}^a \) can be treated as constant, one readily finds [16]

\[ U^{(1)}_{\text{non-ano}} = \frac{4\pi m}{g_0^2} + 16\pi im \lim_{x \rightarrow y} \frac{1}{(\Box - m^2 + i\epsilon)^2} \delta^4(x - y). \]  

(16)
In $N = 2$, where $c_g = 4$ in (10), this leads to $U^{(1)}_{\text{nom--ano}} = U^{(1)}_{cl} = 4\pi m/g^2$. However, in $N = 4$ the coupling constant does not renormalize, $g_0 = g$, and we are left with an uncanceled ultraviolet divergence in $U^{(1)}$.

As we already discussed, the solution to this problem does not come from possible fermionic contributions to the central charge density $U$ in $N = 4$, as there are none. Using the improved SUSY current and the corresponding central charge would in fact change the situation, but then mass and central charge would not agree with the standard value at the classical level. Keeping the unimproved SUSY current as obtained by dimensional reduction, the uncanceled divergence in (10) has instead to be absorbed in a composite-operator renormalization of $U$.

This perhaps surprising result can already be seen in the renormalization of the $N = 4$ theory in the trivial sector (i.e. without monopoles). To discuss composite-operator renormalization we consider the partition function with an external source $K$ for the composite operator $U$ defined in (3). $\mathcal{L} \to \mathcal{L} + K(x)U(x)$. All additional ultraviolet divergences that appear in matrix elements with elementary fields (or further insertions of $U$) have to be absorbed by counterterms linear in $K$ (or of higher order in $K$). For evaluating $\langle U \rangle$, which will of course give something nontrivial only in the topologically nontrivial sector, we only need counterterms linear in $K$. These are determined by the 32 proper diagrams in Fig. 1 which list the potential divergent matrix elements with elementary fields. Of particular interest to us are those involving external scalars $S_1^a$ and gauge fields. The superficially quadratically divergent diagrams (a) and (b) vanish identically. The diagrams (c)–(g) are superficially linearly divergent, but if the external scalar is the massless Higgs field $S_1^3$ and the external gauge field the massless photon $A_3^\mu$, diagram (c) vanishes in background-covariant Feynman-$R$ gauge. Diagrams (d)–(f) turn out to be logarithmically divergent individually, but finite when added together. Diagrams (t)–(e) vanish for external $S_1^a$ and $A_3^\mu$ fields. The only remaining divergent diagram with these external fields is diagram (g), which gives the logarithmically divergent contribution

$$4ig^2 q^i \epsilon^{ijk} p^j \delta^k_{\mu} \int \frac{d^4+\epsilon k}{(2\pi)^4} \frac{1}{(k^2 + m^2 - i\epsilon)(k^2 + m^2 - i\epsilon)}$$

(17)

where $q$ is the incoming momentum carried by $U$ and $p$ the outgoing momen-
Figure 1: One-particle-irreducible divergent one-loop diagrams in matrix elements involving the central charge density $U$. Wavy lines indicate gauge bosons, dashed lines scalar bosons, and full lines fermions.

The diagrams $(h)$–$(s)$ are also logarithmically divergent and evidently needed for making the counterterm gauge invariant, namely proportional to $K(x)\hat{A}^3(x)\hat{S}^3(x))$. 

The diagrams $(h)$–$(s)$ are also logarithmically divergent and evidently needed for making the counterterm gauge invariant, namely proportional to $K(x)\hat{U}(x)$.

In the $N = 4$ theory with background-covariant Feynman-$R_\xi$ gauge, there are no divergent counterterms from bosonic wave function or coupling renormalization, so $U(x)$ needs to be renormalized explicitly. With the on-shell renormalization condition that diagram $(g)$ is subtracted completely at $p = 0$, the counterterm is such that the second term in (16) is cancelled completely, as is required for BPS saturation at the quantum level.

However, the central charge density $U(x)$ is not just multiplicatively renormalized. Consider diagram $(\zeta)$; it yields

$$4ig^2\int \frac{d^{4+\epsilon}k}{(2\pi)^4} \frac{\eta^{ijk}k^j k^s \alpha^1 \gamma_{ks}}{(k^2 + m^2 - i\epsilon)((p + k)^2 + m^2 - i\epsilon)((k + p + q)^2 + m^2 - i\epsilon)}.$$  

(19)

Renormalizing at vanishing external momenta, one can replace $k^j k^s$ by $\eta^{js}k^2/4$.

Taking into account that the fermionic counterterm has to include a factor $1/2$ because of the Majorana property, the complete counterterm required by
the composite-operator renormalization of the central charge density turns out to be proportional to the improvement term $\Delta U$ and explicitly reads

$$4g^2IK(x) \left( \hat{U} + \frac{i}{8} \partial_i (e^{ijk} \bar{\lambda} \alpha^1 \gamma_{jk} \lambda) \right)(x)$$

(20)

where $I$ is the divergent integral appearing in (10). This result is in fact independent of the gauge parameter $\xi$ in (9). While the gauge-parameter independence of diagram (c) is fairly easy to check, the purely bosonic diagrams (c)–(g) are gauge dependent such that it compensates for the gauge dependence of $Z_S$ displayed in (11).\footnote{Explicitly, for $\xi \neq 1$ the factor 4 in front of (17) gets replaced by $5 - \xi$ which arises as $0 + 4 + (-2\xi) + (-2) + (3 + \xi)$ from the five diagrams (c)–(g), respectively. The factor 4 in the final counterterm (21) is recovered by adding to minus the contributions from these diagrams the contribution from $Z_{S/2} = (1 + g^2(1 - \xi)I)$ in $N = 4$.}

The additive renormalization of $U$ can be understood as follows. Writing the unimproved SUSY current as $j = (j + \Delta j) - \Delta j \equiv j_{\text{improved}} - \Delta j$, with $\Delta j$ given by (17) and $c = -2/3$ therein, it is the superconformal current \footnote{\textcopyright 2023 American Physical Society.} $j_{\text{improved}}$ which is protected from renormalization \footnote{\textcopyright 2023 American Physical Society.}, but the multiplet of improvement terms associated to $\Delta j$ is not (compare with the renormalization of an unimproved energy-momentum tensor \footnote{\textcopyright 2023 American Physical Society.} \cite{bib:40, bib:41, bib:42, bib:43, bib:44}). In the presence of monopoles, the relevant SUSY current is the unimproved one, hence the appearance of infinite renormalizations.

To verify that the improved SUSY current and its superpartners are not renormalized while the improvement terms are multiplicatively renormalized, one in fact needs to take into account the wave function renormalization of the fermions. Regardless of background covariance the latter is not related to the wave function renormalization of the gauge bosons, since the gauge fixing term (9) is not SUSY invariant. A straightforward calculation of the one-loop fermion self-energy, renormalized such that it vanishes on-shell for the massless fermions, leads to

$$\lambda_0 = \sqrt{Z_\lambda \lambda}, \quad Z_\lambda = 1 - 2(N + \xi - 1)g^2I$$

(21)

in $N$-extended super-Yang-Mills theory.

Considering now the renormalization of the fermionic composite operator in $\Delta U$, eq. (8), this is determined by the diagrams of Fig. 2. Evaluating diagrams (η) and (θ), which can give rise to an extra fermionic counterterm, one finds that these contribute proportional to $(\xi - 1)$ such that the fermionic
operator in $\Delta U$ is renormalized by a gauge-choice independent counterterm when subtracted on-shell.

The remaining diagrams (i)–(λ) are trivially gauge-parameter independent and lead to mixing with the bosonic operator $U$ precisely such that $\Delta U$ is multiplicatively renormalized as claimed. The counterterm is given explicitly by

$$Z_{\Delta U} = 1 + 12g^2 I$$

in $N = 4$.

In the $N = 2$ theory, the situation is subtly different. Because of the absence of a 5-form charge in the higher-dimensional SUSY algebra, the central charge of the $N = 2$ theory is given by the combination

$$U_{N=2} = \frac{1}{2} \epsilon^{ijk} \partial_i (S^a F^a_{jk}) + \partial_i (P^a F^a_{i0}) = U - \tilde{U}.$$  

Both $U$ and $\tilde{U}$ give rise to fermionic counterterms through diagram (ς) of Fig. 1, but they turn out to be equal and cancel when combined.

Correspondingly, one finds that the improvement term $\Delta j^\mu$, which is given by (I) with $J$ running only over $J = 5, 6$, now does not give rise to a fermionic term in $\Delta U_{N=2}$. Instead, the latter has the same structure as $U_{N=2}$ itself:

$$\Delta U_{N=2} = \frac{c}{2} (U - \tilde{U}).$$

Moreover, in the $N = 2$ case, the bosonic counterterms needed to make $U_{N=2}$ finite are already contributed by the elementary wave-function renormalizations of the bosonic fields, and this equally holds for $\Delta U_{N=2}$. Hence, $Z_{\Delta U} = 1$ in $N = 2$.

To summarize, while in the case of $N = 2$ super-Yang-Mills theory both the improved and the unimproved central charge density are finite operators, we have shown that in the $N = 4$ case composite-operator renormalization of
the central charge density is essential for obtaining a finite result for the central charge at the quantum level. Using on-shell renormalization conditions this then matches with the null result obtained from calculating the quantum mass of the $N = 4$ monopole through the sum over zero-point energies. It would be interesting to also calculate the mass of the monopole from the expectation value of $T_{00}$, which by supersymmetry receives also infinite additive renormalization by an improvement term. The latter is given by the total divergence $2g^2 \partial_i \partial_i (A^J A^J)$, which in fact does contribute to the mass of the soliton, since in the monopole background $A^J A^J = S^2 \sim v^2(1 - 2/(mr))$ asymptotically. Because the sum over zero-point energies can be obtained from the integral over $T_{00}$ by partial integrations, which also give rise to total derivatives of the form of the improvement terms, we expect that the composite-operator renormalization of $T_{00}$ is required to compensate the effect of the former.

We conclude that the resolution of the UV-divergence puzzle in the BPS saturation of the $N = 4$ monopole has brought to the surface a new feature, which did not play a role in the models studied before. The multiplet of $N = 4$ currents for the theory with monopoles contains unimproved currents which are not multiplicatively renormalized, but one must add divergent counterterms which form a multiplet of $N = 4$ improvement terms. We have found that these counterterms are not only gauge invariant, but also independent of the choice of gauge fixing parameter\footnote{It is still conceivable that a supersymmetric gauge choice could eliminate these divergences. However, explicit calculations of quantum corrections in the presence of a monopole have been possible so far only in ordinary background-covariant Feynman-$R_\xi$ gauge, and no superfield version of that is known.}, which could imply that they have some wider importance.

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Note added

In the meantime we have performed a direct calculation of the quantum mass of the $N = 2$ and $N = 4$ monopole from the integral over the expectation value of $T_{00}$. We confirmed that the total derivative terms, which result if one rewrite the integral as a sum over zero point energies, indeed give rise to ultraviolet divergent contributions in the case of the $N = 4$ monopole (but not in $N = 2$) which are cancelled by an improvement counterterm with coefficient determined by (22). The details of this calculation will be presented elsewhere [45].

We note again that our results depend on having chosen the unimproved SUSY current multiplet that is singled out by the possibility of formulating the $N = 2$ and $N = 4$ models as dimensional reductions of $N = 1$ super Yang-Mills theory in $D = 6$ and 10, respectively. In the $N = 4$ model the monopole mass would be different already at the classical level, but free from the necessity of composite operator renormalization of the current multiplet, if one started from an improved current multiplet. We believe that both formulations are theoretically consistent, hence only experiment could decide which one Nature chooses (if any).

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