THE ORDERED $K_0$-GROUP OF A GRAPH $C^*$-ALGEBRA

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Abstract. We calculate the ordered $K_0$-group of a graph $C^*$-algebra and mention applications of this result to AF-algebras, states on the $K_0$-group of a graph algebra, and tracial states of graph algebras.

1. Preliminaries

We provide some basic facts about graph algebras and refer the reader to [8], [2], and [3] for more details. A (directed) graph $E = (E^0, E^1, r, s)$ consists of a countable set $E^0$ of vertices, a countable set $E^1$ of edges, and maps $r, s : E^1 \to E^0$ identifying the range and source of each edge. A vertex $v \in E^0$ is called a sink if $|s^{-1}(v)| = 0$, and $v$ is called an infinite emitter if $|s^{-1}(v)| = \infty$. If $v$ is either a sink or an infinite emitter, we call $v$ a singular vertex. A graph $E$ is said to be row-finite if it has no infinite emitters. The vertex matrix of $E$ is the square matrix $A$ indexed by the vertices of $E$ with $A(v, w)$ equal to the number of edges from $v$ to $w$.

If $E$ is a graph we define a Cuntz-Krieger $E$-family to be a set of mutually orthogonal projections $\{p_v : v \in E^0\}$ and a set of partial isometries $\{s_e : e \in E^1\}$ with orthogonal ranges which satisfy the Cuntz-Krieger relations:

1. $s_e^* s_e = p_{r(e)}$ for every $e \in E^1$;
2. $s_e s_e^* \leq p_{s(e)}$ for every $e \in E^1$;
3. $p_v = \sum_{s(e) = v} s_e s_e^*$ for every $v \in E^0$ that is not a singular vertex.

The graph algebra $C^*(E)$ is defined to be the $C^*$-algebra generated by a universal Cuntz-Krieger $E$-family.

The graph algebra $C^*(E)$ is unital if and only if $E$ has a finite number of vertices, cf. [3] Proposition 1.4, and in this case $1_{C^*(E)} = \sum_{v \in E^0} p_v$. If $E$ has an infinite number of vertices, and we list the vertices of $E$ as $E^0 = \{v_1, v_2, \ldots\}$ and define $p_n := \sum_{i=1}^n p_{v_i}$, then $\{p_n\}_{n=1}^\infty$ will be an approximate unit for $C^*(E)$.

2. The ordered $K_0$-group

If $A$ is a $C^*$-algebra let $P(A)$ denote the set of projections in $A$. It is a fact that if $A$ is unital (or more generally, if $A \otimes \mathcal{K}$ admits an approximate unit consisting of projections), then $K_0(A) = \{[p]_0 - [q]_0 : p, q \in P(A \otimes \mathcal{K})\}$. In addition, the positive cone $K_0(A)^+ = \{[p]_0 : p \in P(A \otimes \mathcal{K})\}$ makes $K_0(A)$ a pre-ordered abelian group. If $A$ is also stably finite, then $(K_0(A), K_0(A)^+)$ will be an ordered abelian group.

Here we compute the positive cone of the $K_0$-group of a graph $C^*$-algebra. Throughout this section we let $\mathbb{Z}^K$ and $\mathbb{N}^K$ denote $\bigoplus_K \mathbb{Z}$ and $\bigoplus_K \mathbb{N}$, respectively.

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Lemma 2.1. Let \( E = (E^0, E^1, r, s) \) be a row-finite graph. Also let \( W \) denote the set of sinks of \( E \) and let \( V := E^0 \setminus W \). Then with respect to the decomposition \( E^0 = V \cup W \) the vertex matrix of \( E \) will have the form

\[
A_E = \begin{pmatrix}
B & C \\
0 & 0
\end{pmatrix}.
\]

For \( v \in E^0 \), let \( \delta_v \) denote the element of \( \mathbb{Z}^V \oplus \mathbb{Z}^W \) with a 1 in the \( v \)th entry and 0’s elsewhere.

If we consider \( \begin{pmatrix} B^t - I \\ C^t \end{pmatrix} : \mathbb{Z}^V \to \mathbb{Z}^V \oplus \mathbb{Z}^W \), then \( K_0(C^*(E)) \cong \mathrm{coker} \begin{pmatrix} B^t - I \\ C^t \end{pmatrix} \)
via an isomorphism which takes \([p_v])_0 \) to \([\delta_v] \) for each \( v \in E^0 \). Furthermore, this isomorphism takes \( (K_0(C^*(E)))^+ \) to \( \{ [x] : x \in \mathbb{N}^V \oplus \mathbb{N}^W \} \), where \([x] \) denotes the class of \( x \) in \( \mathrm{coker} \begin{pmatrix} B^t - I \\ C^t \end{pmatrix} \).

Proof. The fact that \( K_0(C^*(E)) \cong \mathrm{coker} \begin{pmatrix} B^t - I \\ C^t \end{pmatrix} \) is shown for row-finite graphs in \([6\), Theorem 3.1\]. Thus all that remains to be verified in our claim is that this isomorphism identifies \( (K_0(C^*(E)))^+ \) with \( \{ [x] : x \in \mathbb{N}^V \oplus \mathbb{N}^W \} \). To do this, we will simply examine the proof of \([6\), Theorem 3.1\] to determine how the isomorphism acts. We will assume that the reader is familiar with this proof, and use the notation established in it without comment.

If \( E \times_1 [m, n] \) is the graph defined in \([6\), Theorem 3.1\], then we see that \( E \times_1 [m, n] \) is a row-finite graph with no loops and in which every path has length at most \( n - m \). Therefore we can use the arguments in the proofs of \([5\), Proposition 2.1\], \([8\), Corollary 2.2\], and \([8\), Corollary 2.3\) to conclude that \( C^*(E \times_1 [m, n]) \) is the direct sum of copies of the compact operators (on spaces of varying dimensions), indexed by the sinks of \( E \times_1 [m, n] \) and that each summand contains precisely one projection \( p(v, k) \) associated to a sink as a minimal projection. Thus

\[
K_0(C^*(E \times_1 [m, n])) \cong \bigoplus_{v \in V} \mathbb{Z}[p(v, n)]_0 \oplus \bigoplus_{k=0}^{n-m} \mathbb{Z}[p(v, n-k)]_0
\]

and \( K_0(C^*(E \times_1 [m, n]))^+ \) is identified with

\[
\bigoplus_{v \in V} \mathbb{N}[p(v, n)]_0 \oplus \bigoplus_{k=0}^{n-m} \mathbb{N}[p(v, n-k)]_0.
\]

By the continuity of \( K \)-theory, one can let \( m \) tend to \(-\infty\) and deduce that

\[
K_0(C^*(E \times_1 [-\infty, n])) \cong \bigoplus_{v \in V} \mathbb{Z}[p(v, n)]_0 \oplus \bigoplus_{k=0}^{\infty} \mathbb{Z}[p(v, n-k)]_0
\]

\[
\cong \mathbb{Z}^V \oplus \mathbb{Z}^W \oplus \mathbb{Z}^W \oplus \ldots.
\]

Furthermore, it follows from \([10\), Theorem 6.3.2(ii)\] that this isomorphism identifies \( K_0(C^*(E \times_1 [-\infty, n]))^+ \) with \( \mathbb{N}^V \oplus \mathbb{N}^W \oplus \mathbb{N}^W \oplus \ldots \).

This computation is used later in the proof of \([5\), Theorem 3.1\], where the \( K_0 \) functor is applied to a commutative diagram to obtain Figure (3.5) of \([5\], which we
reproduce here:

\[
\begin{array}{rll}
Z^V \oplus Z^W \oplus Z^W \oplus \ldots \xrightarrow{D} Z^V \oplus Z^W \oplus Z^W \oplus \ldots \xrightarrow{\iota_{n+1}} K_0(C^*(E \times_1 \mathbb{Z})) \\
1-D \hspace{2cm} 1-D \hspace{2cm} 1-\beta^{-1}
\end{array}
\]

Now it is shown in [8, Lemma 3.3] that the homomorphism \(\iota^1\) induces an isomorphism \(\mathfrak{T}^{\iota_1}\) of \(\text{coker}(1-D)\) onto \(K_0(C^*(E))\). We shall show that \(\mathfrak{T}^{\iota_1}(N^V \oplus N^W \oplus N^W \oplus \ldots) = K_0(C^*(E))^+\). To begin, note that it follows from [10, Theorem 6.3.2(ii)] that

\[
K_0(C^*(E \times_1 \mathbb{Z}))^+ = \bigcup_{n=1}^{\infty} \iota^n_0(N^V \oplus N^W \oplus N^W \oplus \ldots).
\]

Since \(\text{coker}(1-\beta^{-1}) = K_0(C^*(E))\), this implies that

\[
K_0(C^*(E))^+ = \bigcup_{n=1}^{\infty} \{[\iota^n_0(y)] : y \in N^V \oplus N^W \oplus N^W \oplus \ldots\}
\]

where \([\iota^n_0(y)]\) denotes the equivalence class of \(\iota^n_0(y)\) in \(\text{coker}(1-\beta^{-1})\). We shall show that the right hand side of this equation is equal to \(\{[\iota^n_0(y)] : y \in N^V \oplus N^W \oplus N^W \oplus \ldots\}\). Let \([\iota^n_0(y)]\) be a typical element in the right hand side. Then from the commutativity of the above diagram \(\iota^n_0(y) - \iota^n_0(Dy) = \iota^n_0((1-D)y) = (1-\beta^{-1})(\iota^n_0(y))\) which is 0 in \(\text{coker}(1-\beta^{-1})\). But then \(\iota^n_0(y) = \iota^n_0(D^{n+1}y) = \iota^n_0(y)\) in \(\text{coker}(1-\beta^{-1})\). Hence

\[
(1.1) \quad K_0(C^*(E))^+ = \{[\iota^n_0(y)] : y \in N^V \oplus N^W \oplus N^W \oplus \ldots\}.
\]

Next, recall that [3, Lemma 3.4] shows that the inclusion \(j : Z^V \oplus Z^W \hookrightarrow Z^V \oplus Z^W \oplus Z^W \oplus \ldots\) induces an isomorphism \(j\) of \(\text{coker}K\) onto \(\text{coker}(1-D)\). We wish to show that

\[
(2.2) \quad j([x] : x \in N^V \oplus N^W) = ([y] : y \in N^V \oplus N^W \oplus N^W \oplus \ldots).
\]

It suffices to show that any element \((n, m_1, m_2, m_3, \ldots) \in N^V \oplus N^W \oplus N^W \oplus \ldots\) is equal to an element of the form \((a, b, 0, 0, 0, \ldots)\) in \(\text{coker}(1-D)\). But given \((n, m_1, m_2, m_3, \ldots)\) we see that since this element is in the direct sum, there exists a positive integer \(k\) for which \(i > k\) implies \(m_i = 0\). Thus

\[
\begin{pmatrix}
1 & 0 & 0 & \ldots & 0 \\
-B & 1 & 0 & \ldots & 0 \\
0 & -1 & 1 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & \ldots & 0
\end{pmatrix}
\begin{pmatrix}
m_1 \\
m_2 \\
m_3 \\
m_k \\
0
\end{pmatrix}
= \begin{pmatrix}
0 \\
m_2 + \ldots + m_k \\
m_3 + \ldots + m_k \\
m_4 + \ldots + m_k \\
\vdots
\end{pmatrix}
\]

and so \((n, m_1, m_2, \ldots)\) equals \((n, m_1 + \ldots + m_k, 0, 0, \ldots)\) in \(\text{coker}(1-D)\), and \(2.2\) holds.
Finally, the isomorphism between \( \operatorname{coker} \left( \frac{B^t - I}{C^t} \right) \) and \( K_0(C^*(E)) \) is defined to be \( \sigma_l \circ \rho \). But \( \frac{B^t}{C^t} \) and \( \frac{B^t}{C^t} \) show that this isomorphism takes \( \{ [x] : x \in N^V \oplus N^W \} \) onto \( (K_0(C^*(E)))^+ \).

**Theorem 2.2.** Let \( E = (E^0, E^1, r, s) \) be a graph. Also let \( W \) denote the set of singular vertices of \( E \) and let \( V : = E^0 \setminus W \). Then with respect to the decomposition \( E^0 = V \cup W \) the vertex matrix of \( E \) will have the form

\[
A_E = \begin{pmatrix} B & C \\ * & * \end{pmatrix}
\]

where \( B \) and \( C \) have entries in \( \mathbb{Z} \) and the *'s have entries in \( \mathbb{Z} \cup \{ \infty \} \). Also for \( v \in E^0 \), let \( \delta_v \) denote the element of \( \mathbb{Z}^V \oplus \mathbb{Z}^W \) with a 1 in the \( v \)th entry and 0’s elsewhere.

If we consider \( \left( \frac{B^t - I}{C^t} \right) : \mathbb{Z}^V \to \mathbb{Z}^V \oplus \mathbb{Z}^W \), then \( K_0(C^*(E)) \cong \operatorname{coker} \left( \frac{B^t - I}{C^t} \right) \) via an isomorphism which takes \( [p_v]_0 \) to \( [\delta_v] \) for each \( v \in E^0 \). Furthermore, this isomorphism takes \( (K_0(C^*(E)))^+ \) onto the semigroup generated by \( \{ [\delta_v] : v \in E^0 \} \cup \{ [\delta_v] - \sum_{e \in S}[\delta_{r(e)}] : v \text{ is an infinite emitter and } S \text{ is a finite subset of } s^{-1}(v) \} \).

**Proof.** The fact that \( K_0(C^*(E)) \cong \operatorname{coker} \left( \frac{B^t - I}{C^t} \right) \) was established in [3, Theorem 3.1] using the isomorphisms constructed in [4, Lemma 2.3]. We shall examine the proof of [4, Theorem 3.1] to determine where the positive cone of \( K_0(C^*(E)) \) is sent. Again, we shall assume that the reader is familiar with the proof, and use the notation established in it without comment.

We begin by letting \( F \) denote a desingularization of \( E \) (see [3, §2]). Then [4, Theorem 2.11] shows that there exists a homomorphism \( \phi : C^*(E) \to C^*(F) \) which embeds \( C^*(E) \) onto a full corner of \( C^*(F) \) and takes each \( p_v \) to the projection in \( C^*(F) \) corresponding to \( v \). Since \( \phi \) is an embedding onto a full corner, it induces an isomorphism \( \phi_v : K_0(C^*(E)) \to K_0(C^*(F)) \) which takes the class of \( p_v \) in \( K_0(C^*(E)) \) to the class of the corresponding projection in \( K_0(C^*(F)) \). By Theorem 2.1 if \( A_F \) denotes the vertex matrix of \( F \), then \( K_0(C^*(E)) \cong \operatorname{coker}(A_F - I) \) and \( K_0(C^*(E))^+ \) is identified with \( \{ [x] : x \in \bigoplus_V \mathbb{Z} \oplus \bigoplus_W \mathbb{Z} \oplus \bigoplus_Q \mathbb{Z} \} \) where \( Q : = \bigoplus_Q \mathbb{Z} \). Now it is shown in the proof of [4, Lemma 2.3] that the inclusion map \( \rho : \bigoplus_V \mathbb{Z} \oplus \bigoplus_W \mathbb{Z} \oplus \bigoplus_W \mathbb{Z} \oplus \bigoplus_W \mathbb{Z} \oplus \bigoplus_W \mathbb{Z} \) induces an isomorphism \( \overline{\rho} : \operatorname{coker} \left( \frac{B^t - I}{C^t} \right) \to \operatorname{coker}(A_F - I) \). Since this isomorphism identifies the class of \( \delta_v \in \bigoplus_V \mathbb{Z} \oplus \bigoplus_W \mathbb{Z} \) with the class of \( \left( \begin{array}{c} \delta_v \\ 0 \end{array} \right) \in \bigoplus_V \mathbb{Z} \oplus \bigoplus_W \mathbb{Z} \oplus \bigoplus_Q \mathbb{Z} \), it follows that \( [p_v]_0 \in K_0(C^*(E)) \) is identified with \( [\delta_v] \in \operatorname{coker} \left( \frac{B^t - I}{C^t} \right) \).

All that remains is to determine where this isomorphism sends the positive cone of \( K_0(C^*(E)) \). Let \( \Gamma \) denote the semigroup of elements that \( \overline{\rho} \) sends to \( \{ [x] : x \in \bigoplus_V \mathbb{Z} \oplus \bigoplus_W \mathbb{Z} \oplus \bigoplus_W \mathbb{Z} \} \). Now certainly \( \{ [\delta_v] : v \in E^0 \} \) is in \( \Gamma \). Furthermore, for any infinite emitter \( v \) and finite subset \( S \subseteq s^{-1}(v) \) we have that

\[
[p_v]_0 - \sum_{e \in S}[p_{r(e)}]_0 = [p_v]_0 - \sum_{e \in S}[s_s s_e]_0 = [p_v]_0 - \sum_{e \in S}[s_s s_e]_0 = [p_v]_0 - \sum_{e \in S}[s_s s_e]_0
\]
and this element belongs to $K_0(C^*(E))^+$. Since $K_0(C^*(E))^+$ is identified with $\{[x] : x \in \bigoplus_V \mathbb{Z} \oplus \bigoplus_W \mathbb{Z} \oplus \bigoplus_W Q\}$ this implies that the class of $\begin{pmatrix} \delta_v \\ 0 \end{pmatrix}$ is in $\{[x] : x \in \bigoplus_V \mathbb{Z} \oplus \bigoplus_W \mathbb{Z} \oplus \bigoplus_W Q\}$ and thus $[\rho_v] - \sum_{v \in S} [\rho_{r(e)}]$ is in $\Gamma$. On the other hand, we know that $\Gamma$ is generated by the elements that $\overline{\rho}$ sends to the classes of the generators of $\bigoplus_V \mathbb{Z} \oplus \bigoplus_W \mathbb{Z} \oplus \bigoplus_W Q$. Now certainly the inverse image under $\overline{\rho}$ of the class of $\begin{pmatrix} \delta_v \\ 0 \end{pmatrix}$ for $v \in V \cup W$ will be $[\delta_v]$. In addition, if $v_i$ is a vertex on the tail added to an infinite emitter $v$, then we see that the inverse image under $\overline{\rho}$ of the element $\begin{pmatrix} 0 \\ \delta_{v_i} \end{pmatrix}$ will be $\begin{pmatrix} u \\ v \end{pmatrix}$ where $u$ and $v$ are as defined in the final paragraph of [4, Lemma 2.3]. However, one can verify from how $u$ and $v$ are defined that $\begin{pmatrix} u \\ v \end{pmatrix}$ will have the form $\begin{pmatrix} \delta_v - \sum_{v \in S} \delta_{r(e)} \end{pmatrix}$ for some finite $S \subseteq s^{-1}(v)$. Thus $\Gamma$ is generated by the elements $[\delta_v]$ and $[\delta_v] - \sum_{v \in S} [\delta_{r(e)}]$.  

3. Applications

3.1. AF-algebras. The graph algebra $C^*(E)$ is an AF-algebra if and only if $E$ has no loops [8, Theorem 2.4]. By Elliott’s Theorem AF-algebras are classified by their ordered $K_0$-groups. Hence for two graphs containing no loops, Theorem 2.2 can be used to determine if their associated $C^*$-algebras are isomorphic (as well as stably isomorphic).

3.2. States on $K_0(C^*(E))$. If $A$ is a $C^*$-algebra containing a countable approximate unit $\{p_n\}_{n=1}^\infty$ consisting of projections, then a state $\rho$ on $K_0(A)$ is a homomorphism $\rho : K_0(A) \to \mathbb{R}$ such that $\rho(K_0(A)^+) \subseteq \mathbb{R}^+$ and $\lim_{n \to \infty} \rho([p_n]) = 1$. The set of all states on $K_0(A)$ is denoted $S(K_0(A))$ and we make it into a topological space by giving it the weak-* topology.

Definition 3.1. If $E$ is a graph, then a graph trace on $E$ is a function $g : E^0 \to \mathbb{R}^+$ with the following two properties:

1. For any nonsingular vertex $v \in E^0$ we have $g(v) = \sum_{e \in E^1, s(e) = v} g(r(e))$.
2. For any infinite emitter $v \in E^0$ and any finite set of edges $e_1, \ldots, e_n \in s^{-1}(v)$ we have $g(v) \geq \sum_{i=1}^n g(r(e_i))$.

We define the norm of $g$ to be the (possibly infinite) value $\|g\| := \sum_{v \in E^0} g(v)$, and we shall use $T(E)$ to denote the set of all graph traces on $E$ with norm 1.

Proposition 3.2. If $E$ is a graph, then the state space $S(K_0(C^*(E)))$ with the weak-* topology is naturally isomorphic to $T(E)$ with the topology generated by the subbasis $\{N_{v, \epsilon}(g) : v \in E^0, \epsilon > 0, \text{ and } g \in T(E)\}$, where $N_{v, \epsilon}(g) := \{h \in T(E) : |h(v) - g(v)| < \epsilon\}$.

Proof. We define a map $\iota : S(K_0(C^*(E))) \to T(E)$ by $\iota(f)(v) := f([p_v])$. We shall show that $\iota$ is an affine homeomorphism. To see that $\iota$ is injective note that if $\iota(f_1) = \iota(f_2)$, then for each $v \in E^0$ we have that $f_1([p_v]) = \iota(f_1)(v) = \iota(f_2)(v) = f_2([p_v])$, and since the $[p_v]$’s generate $K_0(C^*(E))$ it follows that $f_1 = f_2$.

To see that $\iota$ is surjective, let $g : E^0 \to \mathbb{R}^+$ be a graph trace. We shall define a homomorphism $f : \text{coker} \begin{pmatrix} B^t - 1 \\ C^t \end{pmatrix} \to \mathbb{R}$ by setting $f([\delta_v]) := g(v)$. Because $g$ satisfies (3.1) we see that $f$ is well defined. Also, since the values of $g$ are
positive and $g$ satisfies (3) of Definition 3.3 we see that $f(K_0(C^*(E))^+) \subseteq \mathbb{R}^+$. Finally, since $g$ has norm 1 we see that $\lim_{n \to \infty} f(\sum_{i=1}^n p_i) = \lim_{n \to \infty} \sum_{i=1}^n g(v_i) = \|g\| = 1$. So $f$ is a state on $K_0(C^*(E))$ and $\iota(f) = g$.

It is straightforward to verify that $\iota$ is an affine homeomorphism. \hfill $\square$

3.3. Tracial states on $C^*(E)$. A trace on a $C^*$-algebra $A$ is a linear functional $\tau : A \to \mathbb{C}$ with the property that $\tau(ab) = \tau(ba)$ for all $a, b \in A$. We say that $\tau$ is positive if $\tau(a) \geq 0$ for all $a \in A^+$. If $\tau$ is positive and $\|\tau\| = 1$ we call $\tau$ a tracial state. The set of all tracial states is denoted $T(A)$ and when $T(A)$ is nonempty we equip it with the weak-* topology. Let $A$ be a $C^*$-algebra with a countable approximate unit consisting of projections. If $\tau$ is a trace on $A$, then it induces a map $K_0(\tau) : K_0(A) \to \mathbb{R}$ given by $K_0(\tau)([p]_0 - [q]_0) = \tau(p) - \tau(q)$. The map $K_0(\tau)$ will be an element of $S(K_0(A))$ (see [14], §5.2 for more details) and thus there is a continuous affine map $r_A : T(A) \to S(K_0(A))$ defined by $r_A(\tau) := K_0(\tau)$.

It is a fact that any quasi-trace on an exact $C^*$-algebra extends to a trace (this was proven by Haagerup for unital $C^*$-algebras [1] and shown to hold for nonunital $C^*$-algebras by Kirchberg [5]). Furthermore, Blackadar and Rørdam showed in [2] that when $A$ is unital every element in $K_0(A)$ lifts to a quasi-trace. It is straightforward to extend the result of Blackadar and Rørdam to $C^*$-algebras with a countable approximate unit consisting of projections. Thus when $A$ is a graph algebra we see that the map $r_A : T(A) \to S(K_0(A))$ is surjective.

If $A$ has real rank zero, then the span of the projections in $A$ is dense in $A$ and $r_A$ is injective. It was shown in [3] that a graph algebra $C^*(E)$ has real rank zero if and only if the graph $E$ satisfies Condition (K); that is, no vertex in $E$ is the base of exactly one simple loop. Therefore, when $A = C^*(E)$ and $E$ is a graph satisfying Condition (K), the map $r_A$ is a homeomorphism and Proposition 3.2 shows that the tracial states on $C^*(E)$ are identified in a canonical way with $T(E)$.

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