Embedding rationally independent languages into maximal ones

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Abstract. We consider the embedding problem in coding theory: given an independence (a code-related property) and an independent language $L$, find a maximal independent language containing $L$. We consider the case where the code-related property is defined via a rational binary relation that is decreasing with respect to any fixed total order on the set of words. Our method works by iterating a max-min operator that has been used before for the embedding problem for properties defined by length-increasing-and-transitive binary relations. By going to order-decreasing rational relations, represented by input-decreasing transducers, we are able to include many known properties from both the noiseless and noisy domains of coding theory, as well as any combination of such properties. Moreover, in many cases the desired maximal embedding is effectively computable.

Keywords. codes, embedding, error control codes, independence, languages, maximal, transducers, variable-length codes

1 Introduction

The embedding problem for a language $L$ satisfying a property $P$ is to find a language $L'$ that contains $L$ and is maximal satisfying $P$. This problem is meaningful when the property $P$ is an independence. In particular, many natural code-related properties are independences with respect to binary relations on words. In this setting, a binary relation $\rho$ defines the property that consists of all languages in which no two different words are related via $\rho$. Such languages are called $\rho$-independent. The embedding problem has been addressed well for properties defined by length-increasing-and-transitive relations [24], as well as for several fixed properties like the bifix code property [28], the solid code property [14], and the bounded deciphering delay property [3]. In [6], the authors consider properties where the relation $\rho$ is rational and, therefore, described by a finite transducer $t$. In this setting, assuming the given language $L$ is regular, one can decide whether $L$ is a maximal $t$-independent language. The contributions of the present paper are as follows.

• We introduce the concept of input-decreasing transducer, which realizes order-decreasing relations, as a tool for defining many natural code-related properties, including variable-length code properties and error-detection
properties, as well as any combinations of those. Assuming a fixed, but arbitrary total order on words, an input-decreasing transducer $\mathbf{t}$ is such that, for any input word $w$, all output words of $\mathbf{t}$ have a (strictly) smaller order than $w$.

- We show that starting with any $\mathbf{t}$-independent language $L$, we can embed $L$ into a maximal $\mathbf{t}$-independent language $\mu^*_t L$, by iterating the max-min operator $\mu_t$ on $L$. The non-iterated operator $\mu_t$ is considered in [24] where it is shown that if $\mathbf{t}$ is length-decreasing and transitive, then any $\mathbf{t}$-independent language $L$ is embedded into $\mu_t L$ which is maximal $\mathbf{t}$-independent. In many cases, $\mu^*_t$ converges after finitely many steps. We also show a natural example of a $\mathbf{t}$ where $\mu^*_t$ does not converge after finitely many steps.

- Our embedding results hold for any fixed, but arbitrary, language $M$ relative to which maximality is considered, that is, we embed any $L \subseteq M$ into a maximal $\mathbf{t}$-independent subset of $M$—this idea of relative maximality has been considered before, e.g., in [4, 6, 20]. When $M$ is finite, $\mu_t$ always converges after $i$ iterations, for some $i$, to $\mu^*_t L$. When both $M$ and $L$ are regular and $\mu_t$ converges after finitely many operations, then $\mu^*_t L$ is computable. With our approach we provide a solution to the embedding problem for many classical cases of both variable-length codes (for $M = \text{all possible words}$) and error-detecting codes for substitution and for synchronization types of errors (for $M = \text{all words of a certain length}$).

The paper is organized as follows. The next section contains information about the basic notation and terminology used in the paper, and Section 3 provides some background information on independent languages and maximal embeddings, and introduces the iterated max-min operator. Section 4 contains a few technical results and the weak condition of a transducer being smooth, which guarantees that when the max-min operator converges in finitely many iterations, then it produces a maximal embedding. Section 5 focuses on input-decreasing transducers, which are always smooth and guarantee that the iterated max-min operator produces a maximal embedding. Section 6 demonstrates with several examples that the concept of input-decreasing transducer can be used to define many known properties from both the noiseless and noisy domains of coding theory. In that section we also show an example of an input-decreasing transducer for which the iterated operator does not converge finitely. Finally, the last section contains a few concluding remarks and directions for future research.

2 Basic notions and notation

In this section we present our notation and terminology about words, languages, transducers and word operators.
We write \( \mathbb{N}, \mathbb{N}_0 \) for the sets of natural numbers (not including 0) and nonnegative integers, respectively. If \( S \) is a set, then \(|S|\) denotes the cardinality of \( S \), and \( 2^S \) denotes the set of all subsets of \( S \). An **alphabet** is a finite nonempty set of symbols. In this paper, we write \( \Sigma \) for any arbitrary alphabet. The set of all words, or strings, over \( \Sigma \) is written as \( \Sigma^* \) and includes the empty word \( \lambda \). A **language** (over \( \Sigma \)) is any set of words. In the rest of this paragraph, we use the following arbitrary object names: \( i, j \) for nonnegative integers, \( K, L \) for languages and \( u, v, w, x, y \) for words. If \( w \in L \) then we say that \( w \) is an **\( L \)-word**.

When there is no risk of confusion, we write a singleton language \( \{ \} \) simply as \( w \). For example, \( L \cup w \) and \( v \cup w \) mean \( L \cup \{w\} \) and \( \{v\} \cup \{w\} \), respectively. We use standard operations and notation on words and languages [18, 21, 25]. For example, \([w], uv, w^i, KL, L^i, L^*, L^+ \) denote respectively, the length of \( w \), the concatenation of \( u, v \), \( w \), \( L \), \( L^i \), \( L^* \), \( L^+ \) denote respectively, the length of \( w \), the word consisting of \( i \) copies of \( w \), the concatenation of \( K \) and \( L \), the language consisting of all words obtained by concatenating any \( i \) \( L \)-words, the Kleene star of \( L \), and \( L^+ = L^* \setminus \lambda \). If \( w \) is of the form \( uv \) then \( u \) is a **prefix** and \( v \) is a **suffix** of \( w \). If \( w \) is of the form \( uwx \) then \( x \) is an **infix** of \( w \). If \( u \neq w \) then \( u \) is called a **proper prefix** of \( w \)—the definitions of proper suffix and proper infix are similar.

**Transducers and (word) relations** [1, 22, 26]. A (word) **relation** over \( \Sigma \) is a subset of \( \Sigma^* \times \Sigma^* \), that is, a set of pairs \((x, y)\) of words over the alphabet. The **inverse** of a relation \( \rho \), denoted by \( \rho^{-1} \), is the relation \(\{(y, x) \mid (x, y) \in \rho\}\). The relation is **transitive** if \( (x, y), (y, z) \in \rho \) implies \( (x, z) \in \rho \), for all words \( x, y, z \); that is, \( \rho \circ \rho \subseteq \rho \), where ‘\( \circ \)’ denotes composition. Following [24], the relation is called **length-increasing** (resp. length-decreasing) if \((x, y) \in \rho \) implies \( |x| < |y| \) (resp. \( |x| > |y| \)).

A (finite) **transducer** is a quintuple \( t = (Q, \Sigma, T, I, F) \) such that \( Q \) is the set of states, \( I, F \subseteq Q \) are the sets of initial and final states, respectively, \( \Sigma \) is the alphabet and \( T \subseteq Q \times \Sigma^* \times \Sigma^* \times Q \) is the finite set of transitions. Note that, in general transducers, one considers an input and an output alphabet, but in this paper the input and output alphabets are the same. The **relation realized** by the transducer \( t \), denoted by \( R(t) \), is the set of labels in all the accepting paths of \( t \). We write \( t(x) \) for the set of **possible outputs of** \( t \) on input \( x \), that is, \( y \in t(x) \) iff \((x, y) \in R(t)\). This notation is extended naturally to any language \( X \):

\[
t(X) = \bigcup_{x \in X} t(x).
\]

The **inverse** of a transducer \( t \), denoted by \( t^{-1} \), is the transducer that results from \( t \) by simply switching the input and output parts of the labels in the transitions of \( t \). It follows that \( t^{-1} \) realizes the inverse of the relation realized by \( t \). If \( t \) and \( s \) are transducers, then there are (effectively) a transducer \( (t \lor s) \) realizing \( R(t) \cup R(s) \) and a transducer \( (t \circ s) \) realizing \( R(t) \circ R(s) \). By composing a transducer with itself \( i \) times, for \( i \in \mathbb{N} \), we obtain a transducer which we denote by \( t^i \). We define

\[
t^{-i} \triangleq (t^{-1})^i.
\]
Remark 1. For all \(i \in \mathbb{N}\), we have that
\[
R(t^{-i}) = (R(t^i))^{-1}.
\]
Indeed, note that \((y, x) \in R(t^{-i}) \iff \text{“there are words } x_1, \ldots, x_{i-1} \text{ such that } (y, x_1) \in R(t), \ldots, (x_2, x_1) \in R(t), (x_1, y) \in R(t)\” \iff (x, y) \in R(t^i) \iff (y, x) \in (R(t^i))^{-1}\).

A transducer \(t\) is transitive if \(R(t)\) is transitive, that is, \(R(t^2) \subseteq R(t)\).

For any regular language \(L\), the relations \(R(t) \cap (\Sigma^* \times L)\) and \(R(t) \cap (L \times \Sigma^*)\) are regular. The details of a transducer realizing \(R(t) \cap (\Sigma^* \times L)\), denoted by \(t \uparrow L\), and of a transducer realizing \(R(t) \cap (L \times \Sigma^*)\), denoted by \(t \downarrow L\), are shown in [11]; thus,
\[
u \in (t \uparrow L)(w) \quad \text{if and only if} \quad u \in t(w) \quad \text{and} \quad u \in L.
\]

Language operators. A language operator is a function \(\text{Op} : 2^{\Sigma^*} \rightarrow 2^{\Sigma^*}\).

If \(\text{Op}\) is any language operator, \(X\) is any language and \(i\) is any nonnegative integer, then we can define the following language operators.
\[
\begin{align*}
\text{Op}^0(X) &= X \quad \text{and} \quad \text{Op}^{i+1}(X) = \text{Op}(\text{Op}^i(X)) \\
\text{Op}^{\leq i}(X) &= X \cup \text{Op}(X) \cup \cdots \cup \text{Op}^i(X) \\
\text{Op}^{\geq i}(X) &= \text{Op}^i(X) \cup \text{Op}^{i+1}(X) \cup \cdots \\
\text{Op}^+(X) &= \bigcup_{i=0}^{\infty} \text{Op}^i(X), \quad \text{Op}^+(X) = \bigcup_{i=1}^{\infty} \text{Op}^i(X) \\
\text{Op}^i(X) &= \bigcap_{i=1}^{\infty} \text{Op}^i(X)
\end{align*}
\]

If \(\text{Op}_1\) is also a language operator then we write
\[
\text{Op} \subseteq \text{Op}_1
\]
to indicate that \(\text{Op}(X) \subseteq \text{Op}_1(X)\) for all languages \(X\).

We view a transducer \(t\) as a language operator, so the expressions \(t^*\) and \(t^\uparrow\), for instance, are legitimate in this paper. With this convention we can say that a transducer is transitive if and only if
\[
\text{t}^2 \subseteq \text{t}.
\]

For transducer operators we also have that \(t(\bigcup_i X_i) = \bigcup_i t(X_i)\), for all language families \((X_i)_i\). Using the above notation for language operators and Remark 1 we have the following.

Remark 2. If the transducer \(t\) is transitive then \(t^+ = t\) and \(t^{-1}\) is also transitive.
3 Codes and the max-min operator

Here we provide background information on code-related properties (independence properties) and introduce the iterated max-min operator that is used to embed a given independent language to a maximal one. A property (over Σ) is any set P of languages. If L is in P then we say that L satisfies P. A code property, or independence, [10], is a property P for which there is n ∈ N∪{ℵ₀} such that

\[ L \in P \text{ if and only if } L' \in P, \text{ for all } L' \subseteq L \text{ with } 0 < |L'| < n, \]

that is, L satisfies the property exactly when all nonempty subsets of L with less than n elements satisfy the property. In the rest of the paper we only consider properties P that are independences. A language L ∈ P is called P-maximal, or a maximal P code, if L ∪ w /∈ P for any word w /∈ L. From [10] we have that every L satisfying P is included in a maximal P code. To our knowledge, with possibly very few exceptions, all known code related properties in the literature [2, 5, 6, 10, 17, 19, 23, 27] are code properties as defined above. In this work we focus on input-altering transducer properties. A transducer t is called input-altering if w /∈ t(w), for all words w. A language L is called t-independent if

\[ t(L) \cap L = \emptyset. \] (1)

The independence \( P_t \) described by t is the set of all t-independent languages. It is easy to verify that the above equation is equivalent to

\[ t^{-1}(L) \cap L = \emptyset \] (2)

and also to

\[ (t^{-1} \lor t)(L) \cap L = \emptyset \] (3)

Thus, any of t, \( t^{-1} \), \( t \lor t^{-1} \) can be used to describe the same code property.

Remark 3. Let t be an input-altering transducer. Every singleton language \( \{w\} \) is t-independent.

Remark 4. The approach of input-altering transducers constitutes a realization in algorithmic terms of independences defined via binary relations and includes many known properties such as prefix codes, bifix codes, outfix codes, and many error-detecting languages, as well as all the intersections of any two such properties. In particular, for any binary relation \( \rho \), a language L is \( \rho \)-independent if

\[ u, v \in L \text{ and } (u, v) \in \rho \text{ implies } u = v. \] (4)

The above statement implies that \( \rho \)-independence is the same as \( \rho^{-1} \) independence. Let \( \rho_\neq = \{(x, y) \in \rho \mid x \neq y\} \). If \( \rho_\neq \) is rational then there is an input-altering transducer t realizing it, and condition (4) is equivalent to any of (1)—(3) above. The representation of code properties by transducers (or other formal objects such as trajectories [5]) has lead to the implementation of a package for manipulating objects representing code properties [7], as well as to an online tool for answering questions about code properties [15].
In the rest of the paper we consider a fixed, but arbitrary, input-altering transducer $t$, and a fixed, but arbitrary, language $M$. Let $X$ be any language. We define the following language operators.

$$ I_t(X) = M - (t(X) \cup t^{-1}(X)) \quad \text{and} \quad \mu_t(X) = I_t(X) - t^{-1}(I_t(X)) $$

When the transducer $t$ is understood, we omit above the subscript $t$. Also, as the operator $\mu$ is used heavily, we usually omit parentheses when applying $\mu$ on a language $X$. So the two operators are also written, respectively, as

$$ I(X) = M - (t(X) \cup t^{-1}(X)) \quad \text{and} \quad \mu X = \mu(X) = I(X) - t^{-1}(I(X)) $$

The above operators are essentially translated to our transducer notation from the corresponding ones in [24]. The operator $I(\cdot)$ is the set of all possible words that are either in $X$ or $t$-independent from $X$, so in some sense it is the maximum set in which $X$ can be embedded. However, two words in $I(X) - X$ might be $t$-dependent. The operator mapping any $Y$ to $Y - t^{-1}(Y)$ is the ‘$t$-minimize’ operator which returns all $Y$-elements that cannot produce another $Y$-element via $t$. The term ‘minimize’ makes sense in our context of input-decreasing transducers further below.

**Definition 5.** The operator $\mu_t$, or simply $\mu$ when $t$ is understood, shown above is called the **max-min operator**. The operator $\mu^*$ is called the **iterated max-min operator**. We say that it converges finitely on a language $L$, if there is $i \in \mathbb{N}_0$ such that $\mu^* L = \mu^i L$.

In the case of codes defined by length-increasing-and-transitive relations (equivalently, length-decreasing-and-transitive relations), already the language $\mu L$ is maximal and constitutes a solution to the embedding problem, where $L$ is the given language satisfying the code property. As stated in [24], however, this does not work for other codes like bifix codes, and also for error-detecting codes. A main observation in this paper is that for any $t$-independent language $L$, the language $\mu^* L$ is an embedding of $L$, provided that $t$ satisfies a reasonable condition—see Section 5.

### 4 Smooth Transducer Operators

In this section we obtain several technical results about the max-min operator $\mu$ and we demonstrate Theorem 11, which states that when $t$ is smooth and $\mu^*$ converges finitely on some initial $t$-independent language $L$, then the resulting language is a $t$-independent maximal embedding of $L$. The concept of a smooth transducer is rather technical and is intended to keep the results general. All input-decreasing transducers of the next section are smooth.

The second statement of the next lemma is the analogue of a statement in [5] concerning codes defined via trajectories.

**Lemma 6.** Let $X, Y$ be any languages and let $L$ be a language satisfying the property $P_t$. The following statements hold true.
1. If \( X \subseteq Y \) then \( I(Y) \subseteq I(X) \).
2. \( X - t^{-1}(X) \) satisfies \( \mathcal{P}_t \).
3. \( \mu^i L \) satisfies \( \mathcal{P}_t \) and \( L \subseteq \mu^i L \subseteq \mu^{i+1} L \subseteq I(L) \), for all \( i \in \mathbb{N} \).

Proof. The first statement follows from the definition of \( I \). For the second statement, we need to show that Eq. (1) holds for \( L = X - t^{-1}(X) \). For the sake of contradiction assume that there is \( w \in X - t^{-1}(X) \) and \( w \in t(X - t^{-1}(X)) \). Then \( w \in t(u) \) for some \( u \in X - t^{-1}(X) \), which implies \( u \notin t^{-1}(w) \) and, then \( w \notin t(u) \), which is impossible.

For the third statement, we first show that for any language \( K \) satisfying \( \mathcal{P}_t \), we have

\[
K \subseteq \mu K \subseteq I(K) \quad \text{and} \quad \mu K \text{ satisfies } \mathcal{P}_t. \tag{5}
\]

The previous statement of the lemma implies that indeed \( \mu K \) satisfies \( \mathcal{P}_t \). The definition of \( \mu \) implies that \( \mu K \subseteq I(K) \). Now, as \( K \) satisfies both Eq. (1) and (2), we have that \( K \cap (t(K) \cup t^{-1}(K)) = \emptyset \) and, therefore, \( K \subseteq I(K) \). If it were the case that \( K \cap t^{-1}(I(K)) \neq \emptyset \), then also \( t(K) \cap I(K) \neq \emptyset \), which is impossible. Hence, \( K \subseteq \mu K \). Now the statement follows if we use \( L \) or \( \mu^i(L) \) in place of \( K \) in (5), taking also into account the first statement of the lemma.

In going from the length-increasing-and-transitive binary relations of \([24]\) to the input-altering ones of \([6]\), we need to obtain a few somewhat subtle relationships between the operators \( \mu(\cdot) \) and \( I(\cdot) \).

**Definition 7.** Let \( X \) be any language, and consider again our fixed input-altering transducer \( t \). We define the following notation and concepts.

1. \( \sigma_{X,t} = t^{-1} \uparrow I(X) \). When \( t \) is understood we simply write \( \sigma_X \) instead of \( \sigma_{X,t} \).
2. \( t \) is called **exhaustive**, if \( t^0(X) = \emptyset \), for every language \( X \).
3. \( t \) is called **smooth**, if \( \sigma_{X}^{i}(I(X)) \subseteq \sigma_{X}^{i}(\mu X) \), for every language \( X \).

One verifies that exhaustive \( t^{-1} \) implies smooth \( t \).

**Lemma 8.** Let \( X, Y \) be any languages. The following statements hold true.

1. \( \sigma_X(X) = \emptyset = \sigma_X^+(X) \) and \( \sigma_X(Y) \subseteq I(X) \). Also, if \( X \subseteq Y \) then \( I(Y) \cap t^{-1}(X) = \emptyset \) and \( \sigma_Y(A) \subseteq \sigma_X(A) \) for all languages \( A \).
2. \( \mu X = I(X) - \sigma_X(I(X)) \) and \( I(X) = \mu X \cup \sigma_X(I(X)) \).
3. If \( X \subseteq Y \) and \( i \in \mathbb{N} \) then \( \sigma_Y^i(A) \subseteq \sigma_X^i(B) \) for all languages \( A, B \) with \( A \subseteq B \).
4. If \( t \) is transitive then also \( \sigma_X \) is transitive.
5. \( I(X) = \sigma_X^*(\mu X) \cup \sigma_X^*(I(X)) \).
6. If $X = \mu X$ then $I(X) = X \cup \sigma_X^2(I(X))$.

Proof. The first two statements follow from the definitions of the operators $\sigma_X$, $I$, and $\mu$. For example,

$$\sigma_X(X) = t^{-1}(X) \cap I(X) = t^{-1}(X) \cap (M - t(X) - t^{-1}(X)) = \emptyset.$$  

The third statement follows from the second one using induction on $i$. The fourth statement follows when we note that $I(X)$ is smooth. Let $\sigma_X^j$ denote the $j$-th power of $\sigma_X$. The following statements hold true.

1. $I(X) = \sigma_X^x(\mu X)$ and $\mu X \cap \sigma_X^+ (\mu X) = \emptyset$.
2. $\sigma_{\mu^{-1}L}^j(\mu^i L) = \sigma_{\mu^{-1}L}^j(\Delta_i L)$, for all $j \in \mathbb{N}$.
3. $I(\mu^i L) \subseteq \mu^i L \cup \sigma_{\mu^{-1}L}^{2i+2}(\Delta_i L) \subseteq \mu^i L \cup (t^{-1})^{\geq 2}(\Delta_i L)$
4. $\Delta_{i+1} L \subseteq \sigma_{\mu^{-1}L}^{2i+4}(\Delta_i L)$.
5. $\sigma_{\mu^{-1}L}^{i+1}(\mu^i L) \subseteq \sigma_{\mu^{-1}L}^{i+4}(\mu^i L)$
6. $\sigma_{\mu^{-1}L}^{i+1}(\mu^i L) = \sigma_{\mu^{-1}L}^{i+2}(\Delta_i L) \subseteq \sigma_{L}^{\geq 2i}(\Delta_i L)$.
Proof. We use the previous lemma. In particular, as $t$ is smooth, we have
\[ I(X) = \sigma_X^+(\mu X). \]

1. Now, $\mu X = I(X) - \sigma_X(I(X)) = \sigma_X^+(\mu X) - \sigma_X^-(\mu X) = \mu X - \sigma_X^-(\mu X)$.
2. As $\sigma_{\mu_{i-1}^+}^+(\mu^i L) = \emptyset$, we have $\sigma_{\mu_{i-1}^-}^+(\mu^i L) = \sigma_{\mu_{i-1}^-}^+(\Delta_i L \cup \mu^i L) = \sigma_{\mu_{i-1}^-}^+(\Delta_i L) \cup \sigma_{\mu_{i-1}^-}^+(\mu^i L) = \sigma_{\mu_{i-1}^-}^+(\Delta_i L)$. The statement now follows.
3. Using the previous statement we have,
\[ I(\mu^i L) \subseteq I(\mu^{i-1} L) = \mu^i L \cup \sigma_{\mu_{i-1}^-}^+(\mu^i L) = \mu^i L \cup \sigma_{\mu_{i-1}^-}^+(\Delta_i L). \]
4. If $\mu^i L \subseteq I(\mu^i L)$, the statement follows from the previous one.
5. First note that, for all $j \in \mathbb{N}$,
\[ \sigma_{\mu_{i-1}^-}^+(\mu^i L) = \sigma_{\mu_{i-1}^-}^+(\sigma_{\mu_{i-1}^-}^+(\mu^i L)). \]

Then the statement follows when we use induction on $i$ to show $\sigma_{\mu_{i-1}^-}^+(\mu^i L) \subseteq \sigma_{\mu_{i-1}^-}^+(\mu L)$.

\[ \square \]

Lemma 10. Let $L$ be a language satisfying the property $P_i$. The following statements hold true.

1. $L$ is $P_i$-maximal if and only if $I(L) \subseteq L$.
2. If $L$ is $P_i$-maximal then $L = \mu^i L$, for all $i \in \mathbb{N}$.

Proof. For the first statement, following [6] we have that $L$ is $P_i$-maximal if and only if,
\[ M - (L \cup t(L) \cup t^{-1}(L)) = \emptyset, \]
if and only if, $I(L) - L = \emptyset$, if and only if, $I(L) \subseteq L$.

For the second statement, let $L$ be $P_i$-maximal. Then $I(L) \subseteq L$, and the statement follows from Lemma 6. \[ \square \]
Theorem 11. Let $L$ be a language satisfying the property $\mathcal{P}_t$. If $t$ is smooth and there is $i \in \mathbb{N}_0$ such that $\mu^{i+1}L = \mu^iL$ then $\mu^iL$ is $\mathcal{P}_t$-maximal and contains $L$.

Proof. Assume $\mu^{i+1}L = \mu^iL$ and let $K = \mu^iL$. As $t$ is smooth, Lemmata 8 and 9 imply

$$I(K) = \sigma_K^\mu(\mu K) = \mu K \cup \sigma_K^\mu(\mu K) = K \cup \sigma_K^\mu(K) = K.$$ 

This implies that $K$ is maximal using Lemma 10.

The next theorem is a slightly stronger version of a result in [24] which states that $L$ is included in the maximal $\mu L$ when $t$ is length-decreasing-and-transitive.

Theorem 12. Let $L$ be a language satisfying the property $\mathcal{P}_t$. If $t$ is transitive and smooth then $\mu L$ is $\mathcal{P}_t$-maximal and contains $L$.

Proof. Assume $t$ is transitive and smooth. Then, $t^2 \subseteq t$. By Lemma 10, it is sufficient to show that $I(\mu L) \subseteq \mu L$. First note that Lemmata 8 and 9 imply $I(L) = \mu L \cup \sigma_L(\mu L)$. Then,

$$I(\mu L) \subseteq I(L) = \mu L \cup \sigma_L(\mu L).$$

By definition of $I(\cdot)$, we have $I(\mu L) \cap t^{-1}(\mu L) = \emptyset$, which implies that $I(\mu L) \subseteq \mu L$, as required.

Example 13. The input-altering transducer $t_1$ in Fig. 1 is transitive and smooth but neither length-decreasing nor length-increasing. It is transitive because of the fact that $t_1^2(x) = \emptyset$ for all $x$. This last fact implies that $t_1^i(x) = \emptyset$ for all $x$ and $i \geq 2$, which implies further that $t_1^{-1}$ is exhaustive and, therefore, $t$ is smooth.

![Figure 1: An example of an input-altering transducer that is smooth and transitive, but neither length-decreasing nor length-increasing.](image)

5 Input-decreasing Transducer Properties

In this section, we consider a fixed, but arbitrary, total order $\prec$ on the set $\Sigma^*$ of all words. Then, every word $w$ has a position $\text{pos}(w)$ with respect to
that order, starting from position 0. Moreover, \( v \prec w \) implies \( v \neq w \), for any \( v, w \in \Sigma^* \). We also consider a fixed, but arbitrary, transducer \( t \) such that
\[
y \in t(x) \text{ implies } y \prec x
\]
for all words \( x, y \). Any transducer satisfying the above condition is called an input-decreasing transducer.

**Definition 14.** An input-decreasing transducer property is a property that is equal to \( P_t \) for some input-decreasing transducer \( t \).

**Remark 15.** Input-decreasing transducer properties are closed under intersection, as \((t \lor s)\) is input-decreasing when both \( t \) and \( s \) are.

**Remark 16.** For any binary relation \( \rho \), let
\[
\rho_{\prec} = \{ (x, y) \in \rho \mid x \prec y \},
\]
and assume that \( \rho_{\prec} \) can be realized by an input-decreasing transducer \( t \). One verifies that a language \( L \) is \( \rho \)-independent if and only if it satisfies \( P_t \) (that is, \( L \) is \( t \)-independent).

**Lemma 17.** Consider the fixed input-decreasing transducer \( t \). The following statements hold true.

1. \( t \) is input-altering
2. \( t(x) \) is finite, for all words \( x \).
3. \( t \) is smooth.

**Proof.** The first two statements follow from the assumption that \( t \) is input-decreasing. For the last statement, we show that \( t^{-1} \) is exhaustive using contradiction. So assume there is a language \( X \) and a word \( w \) such that \( w \in (t^{-1})^\cap(X) \). Let \( p \) be the position of \( w \) with respect to the total order \( \prec \). As \( w \in (t^{-1})^{p+1}(X) \), there are words \( x_0, x_1, \ldots, x_p \in X \) such that
\[
x_1 \in t^{-1}(x_0), \quad x_2 \in t^{-1}(x_1), \ldots, \quad x_p \in t^{-1}(x_{p-1}), \quad w \in t^{-1}(x_p).
\]
Then, \( x_0 \prec x_1 \prec \cdots \prec x_p \prec w \), which implies that the position of \( w \) is greater than \( p \), a contradiction.

**Theorem 18.** Assume that \( t \) is input-decreasing. If a language \( L \) satisfies \( P_t \) then the language \( \mu L \) is \( P_t \)-maximal and contains \( L \).

**Proof.** That \( \mu L \) contains \( L \) follows from Lemma 6. Also, using the same lemma it follows that no two words \( u, v \) in \( \mu L \) are related via \( R(t) \) and, therefore, \( \mu L \) satisfies \( P_t \). To show that \( \mu L \) is maximal we pick any word \( w \in I(\mu L) \) and show that \( w \in \mu L \)—see Lemma 10. By the definition of the operator \( I \), we have that
\[
w \in \bigcap_{i \geq 0} \left( \mathcal{M} \setminus (t \lor t^{-1})(\mu L) \right) = \bigcap_{i \geq 0} I(\mu^i L).
\]
Then by Lemma 9, for every nonnegative integer \(i\), we have that
\[
w \in \sigma_{\mu^i L}^* (\mu^{i+1} L) = (\mu^{i+1} L) \cup \sigma_{\mu^i L}(\mu^{i+1} L) \cup \sigma_{\mu^i L}^{\geq 2}(\mu^{i+1} L).
\]
Now let \(i = \lfloor \text{pos}(w)/2 \rfloor\). As \(w \in (M - (t \vee t^{-1})(\mu^{i+1} L))\), we have that \(w \notin \sigma_{\mu^i L}(\mu^{i+1} L)\). Also, by Lemma 9(6), any \(u \in \sigma_{\mu^i L}^{\geq 2}(\mu^{i+1} L)\) must have \(\text{pos}(u) \geq 2i + 2\) and, therefore, \(w \notin \sigma_{\mu^i L}^{\geq 2}(\mu^{i+1} L)\). Hence, \(w \in (\mu^{i+1} L) \subseteq \mu^* L\), as required.

In the above theorem, the premise that \(t\) be input-decreasing is essential. This is shown next with examples.

**Example 19.** Let \(p\) and \(s\) be the input-decreasing transducers describing, respectively, prefix codes and suffix codes. Let \(b = (p \vee s)\) be the input-decreasing transducer describing bifix codes. Then, for any bifix code \(L\), the language \(\mu^* b L\) is a maximal bifix code containing \(L\). On the other hand, if we describe bifix codes using \(\text{any}\) of the three transducers
\[
(p^{-1} \vee s), \ (p \vee s^{-1}), \ (p^{-1} \vee s^{-1}),
\]
then the theorem does not hold—although all three are input-altering, none of them is input-decreasing. For example, with \(t\) being any of those three transducers, and for \(\Sigma = \{0, 1\}\), we have \(\mu(001) = 001\), hence \(\mu^*(001) = 001\); that is on input 001, the iterated max-min operator returns 001 itself, which is not maximal bifix—here we have used FAdo [7] for computations on automata and transducers.

**Remark 20.** If the language \(L\) is regular then also the language \(\mu^i L\) is regular, for all \(i \in \mathbb{N}\). This follows by the definition of \(\mu\) and the standard closure properties of regular languages. In particular, an automaton accepting \(\mu^i L\) can be effectively computed from any automaton accepting \(L\). Thus, if \(t\) is such that \(\mu^* = \mu^i\) for some index \(i\), then a maximal regular embedding of any given regular \(L\) can be effectively computed.
Theorem 21. Assume that $M$ is finite and $t$ is input-decreasing. If a language $L \subseteq M$ satisfies property $P_t$ then there is $i \in \mathbb{N}$ such that the language $\mu^i L$ is regular, $P_t$-maximal and contains $L$.

Proof. Let $M = \{w_1, \ldots, w_n\}$ for some $n \in \mathbb{N}$. As each $t(w_i)$ is finite there is $p_i \in \mathbb{N}$ such that $t^{p_i}(w_i) = \emptyset$. Hence, $t^p(M) = \emptyset$, where $p = \max_i \{p_i\}$. Then we have that $(t^p)^{-1}(M) = \emptyset$, and also $(t^{-p})(M) = \emptyset$, by Remark 1. Then Lemma 9 implies

$$\mu^{1+[p/2]}(L) - \mu^{[p/2]}(L) \subseteq \sigma^{\geq [p/2]}_L(\mu L) \subseteq \sigma^{\geq p}_L(\mu L) \subseteq \emptyset.$$ 

Hence, $\mu^{[p/2]}(L)$ is $P_t$-maximal by Theorem 11. \hfill \Box

As before, the premise that $t$ be input-decreasing is essential. This is shown next with an example.

Example 22. Let $\text{sub}_1$ be the input-altering transducer (shown below) describing 1-substitution-detecting languages. A language $L$ is $k$-substitution detecting if no $L$-word can result into another $L$-word using up to $k$ symbol substitutions (one substitution = one symbol replaced with another one). The transducer is not input-decreasing, as $0 \in \text{sub}_1(1)$ and $1 \in \text{sub}_1(0)$. Moreover for $t = \text{sub}_1$, we have that $\mu(0000) = 0000$ and, hence, $\mu^*(0000) = 0000$, which is not maximal 1-substitution-detecting.

![Figure 3: This transducer describes 1-substitution-detecting languages—see caption of the previous figure for explanations on transducer diagrams. It is input-altering but not input-decreasing.](image-url)

6 Examples and further observations

In this section we use the standard quasi-lexicographic (or radix) total order on all words over $\{0, 1, \ldots, q-1\}$, for some integer $q \geq 2$. Thus, $u \prec v$ means that, either $u$ is shorter, or $u$ and $v$ are of the same length and, for the first position in which they differ, the symbol of $u$ at that position is smaller than that of $v$. All the examples presented below have been confirmed using the well-maintained Python package FAdo [7], which was recently updated to include a module on codes described by input-altering transducers [12].

In our examples below we use notation of regular expressions. For instance, $01^*0(0 + 1)$ denotes the language $\{01^*0 \mid i \in \mathbb{N}_0\}\{0, 1\}$.
Example 23. Let $M = \{0,1\}^*$ and $t = b = \text{the input-decreasing transducer describing bifix codes}$. We have that 

\[
\mu(001) = \{001, 000, 10, 11\}, \quad \text{and } \mu^2(001) = 01*0(0 + 1) + 10 + 11
\]

which is maximal. Again, we have 

\[
\mu((0 + 1)^30) = (0 + 1)^4, \quad \text{which is maximal,}
\]

\[
\mu^2((0 + 1)^311) = (0 + 1)^3(0 + 10^*1), \quad \text{which is maximal.}
\]

The last code above is the reverse of a code in [2], which is called there reversible Golomb-Rice code. Finally, note that $\mu^5$ on 11111 generates a maximal bifix code.

For the next examples we use the two transducers shown below over the binary alphabet $\{0,1\}$.

![Transducers](image)

Figure 4: On input $x$, the left transducer outputs any word resulting by substituting exactly one 1 in $x$ with a 0. Note that $(\text{sub}_1^\prec) \lor (\text{sub}_1^\supset)^{-1}$ is equal to the transducer $\text{sub}_1$ and, therefore, $\text{sub}_1^\supset$ describes the 1-substitution-detecting languages over $\{0,1\}$. The right transducer describes the 2-substitution-detecting languages. Both transducers are input-decreasing.

Example 24. Let $M = \{0,1\}^5$ and $t = \text{sub}_1^\supset = \text{the input-decreasing transducer describing 1-substitution-detecting languages}$. We have that 

\[
\mu^3(01111) = \{w \in \{0,1\}^5 \mid w's \text{ count of 1s is even}\}
\]

This code is maximal and known as the even-parity code of length 5, which constitutes a vector space of dimension 4 consisting of $2^4$ codewords.

Example 25. Let $M = \{0,1\}^7$ and $t = \text{sub}_2^\supset = \text{the input-decreasing transducer describing 2-substitution-detecting languages}$. We have that 

\[
\mu^6(111111) = \{0000000, 1001011, 0101010, 1100001, 0011001, 1010010,
0110011, 1111000, 0000111, 1001100, 0101101, 1100110,
0011110, 1010101, 0110100, 1111111\}
\]

This code is the reverse of the Hamming code of length 7 [8]. It is maximal and constitutes a vector space of dimension 4 consisting of $2^4$ codewords. It is also 1-substitution-correcting.
In the next example we use the input-decreasing transducer $\text{id}_2^\prec$ shown in Fig 5, which describes 2-insertion-deletion-detecting languages. A language $L$ is $k$-insertion-deletion detecting if no $L$-word can result into another $L$-word using a total of up to $k$ symbol insertions/deletions. The challenge in designing the transducer is to make sure that two insertion-deletion errors on some input word $x$ do cause the resulting word to be different from $x$ and smaller than $x$. Moreover, the transducer $(\text{id}_2^\prec \lor (\text{id}_2^\prec)^{-1}$ is such that, on any input word $x$, one or two insertion/deletion errors are applied resulting into a word not equal to $x$. The main idea is that $\text{id}_2^\prec$ applies an insertion and a deletion in two ways: (i) A deletion of 1 immediately followed by either a 0 not changed or an inserted 0. This is justified, as deleting a 1 in a run of 1s has the same effect as deleting the last 1 of that run; (ii) An insertion of a 0 immediately followed by either a 1 not changed or a deleted 1. Again, this is justified as inserting a 0 in a run of 0s has the same effect as inserting the 0 at the end of that run.

Figure 5: This is an input-decreasing transducer describing the 2-insertion-deletion-detecting languages over $\{0, 1\}$.

**Example 26.** Let $M = \{0, 1\}^6$ and $t = \text{id}_2^\prec$ = the input-decreasing transducer
describing 2-insertion-deletion-detecting languages. We have that
\[ \mu^5(001011) = \{000000, 001011, 001100, 010001, 011101, \\
101010, 110000, 110011, 111100, 111111\} \]
This code is maximal and consists of 10 codewords. Any 2-insertion-deletion-detecting code of fixed length has a Levenshtein distance greater than 2 and, therefore, it is also 1-insertion-deletion-correcting. We note that the Levenshtein 1-insertion-deletion-correcting code of length 6 in [16] is maximal and consists of 10 codewords as well.

**Example 27.** Let \( M = \{0, 1\} \leq 6 \) and \( t = p \lor \text{sub}_1^\prec \) the input-decreasing transducer describing languages over \( \{0, 1\} \) that are both 1-substitution-detecting and prefix codes. We have that
\[ \mu^5(111) = \{0, 10, 111, 1100, 11010, 110110\} \]
is maximal (relative to \( \{0, 1\} \leq 6 \)).

The next result shows an example of an input-decreasing transducer and language on which the \( \mu^* \) does not converge finitely. First we establish the following lemma.

**Lemma 28.** Let \( n \in \mathbb{N}_0 \), let \( \Sigma = \{0, 1\} \), let \( L_n = \{1, 00, 010, \ldots, 01^{n-1}0\} \), and let \( t = p \lor \text{sub}_1^\prec \) the input-decreasing transducer describing the languages that are both 1-substitution-detecting and prefix codes. We have that
\[ I(L_n) = L_n \cup 01^n \Sigma^+ . \]

**Proof.** Recall that \( \text{sub}_1^\prec \) substitutes exactly one 1 with a 0, and hence, \( (\text{sub}_1^\prec)^{-1} \) substitutes exactly one 0 with a 1. Thus,
\[ (\text{sub}_1^\prec)^{-1}(L_n) = (10 + 110 + \cdots + 1^n 0) + (01 + 011 + \cdots + 01^n). \]
We use the notation \( (x)^1/0 \) to denote the set of all words that result by substituting exactly one 1 with a 0 in the word \( x \). Thus,
\[ \text{sub}_1^\prec(L_n) = 0 + 0(1)^{1/0}0 + \cdots + 0(1^{n-1})^{1/0}0. \]
Using the definition of $I(\cdot)$ and the fact $\Sigma^* = \lambda + 0 + 1 + 0\Sigma^+ + 1\Sigma^+$, we have

\[
I(L_n) = \Sigma^* - p(L_n) - L_n\Sigma^+ - \text{sub}_1^*(L_n) - (\text{sub}_1^*)^{-1}(L_n) = 1 + 0\Sigma^+ - 00\Sigma^+ - \cdots - 01^{n-1}0\Sigma^+ - 01 - 01^2 - \cdots - 01^n \\
- 000 - 0(11)^{1/0}0 - \cdots - 0(1^{n-1})^{1/0}0 = (1 + 00) + (01\Sigma^+ - 010\Sigma^+ - \cdots - 01^{n-1}0\Sigma^+) - 01^2 - \cdots - 01^n \\
- 000 - 0(11)^{1/0}0 - \cdots - 0(1^{n-1})^{1/0}0 = (1 + 00 + 010) + (011\Sigma^+ - 0120\Sigma^+ - \cdots - 01^{n-1}00\Sigma^+) - 01^2 - \cdots - 01^n \\
- 000 - 0(111)^{1/0}0 - \cdots - 0(1^{n-1})^{1/0}0 = \cdots = (1 + 00 + 010 + \cdots + 01^{n-2}0) \\
+ (01^{n-1}\Sigma^+ - 01^{n-1}0\Sigma^+ - 01^n - 0(1^{n-1})^{1/0}0 = L_n + (01^n\Sigma^+ - 0(1^{n-1})^{1/0}0) = L_n + 01^n\Sigma^+,
\]
as required.  

\[\square\]

**Theorem 29.** Let $M = \Sigma^*$ and $\Sigma = \{0, 1\}$. There is an input-decreasing transducer $t$ such that $(\mu^*)$ does not converge finitely.

**Proof.** We consider the notation in the above lemma, and we use induction on $n \in \mathbb{N}_0$ to show that

\[
\mu^n1 = \{1, 00, 010, \ldots, 01^{n-1}0\}.
\]

The statement holds for $n = 0$. Assume it holds for some $n$, as displayed above, and consider calculating $(\mu^{n+1}1)$. Using the definition of $I(\cdot)$ we have that

\[
(\mu^{n+1}1) = I(\mu^n1) - t^{-1}(I(\mu^n1)) = I(\mu^n1) - t^{-1}(I(\mu^n1) - \mu^n1),
\]

where we have used the fact $t^{-1}(\mu^n1) \cap \mu^n1 = 0$ as $\mu^n1$ satisfies $P_{t^{-1}}$. Now using the above lemma we have that

\[
\mu^{n+1}1 = (\mu^n1) \cup 01^n\Sigma^+ - t^{-1}(01^n\Sigma^+).
\]

Using the definition of $t$ one verifies that $t(\mu^n1) \cap 01^n\Sigma^+ = 0$ and that the only element of $01^n\Sigma^+$ that does not belong to $t^{-1}(01^n\Sigma^+)$ is $01^n0$. Then it follows that $(\mu^{n+1}1) = (\mu^n1) + 01^n0$, as required.  

\[\square\]

**Example 30.** In [9] the authors consider computing a maximal prefix code that is a subset of a given regular language $L$. We can replace prefix code with $t$-independent language, for any suitable input-decreasing transducer $t$, and approach this generalized problem by first computing $X = L - t^{-1}(L)$, which is always $t$-independent, and then use the iterated max-min operator to embed $X$ into a maximal $t$-independent language relative to $M = L$. The work in [9] provides further results on maximal prefix codes that can possibly be extended to certain $t$-independences—see also the last section for further comments.
7 Conclusion

We have shown that when an independence property is described by an input-decreasing transducer \( t \), then the max-min operator \( \mu_t \) can be iterated on any language to produce a maximal embedding. This approach works for many natural independence properties from both the noiseless and noisy domains of coding theory, as well as for any combinations of such properties. We conclude with a few directions for future research.

- Find out whether, for any given regular bifix code, the max-min operator converges finitely. We believe that the answer here is yes.
- Find out whether any regular maximal \( t \)-independent language is the result of applying the iterated max-min operator on some initial finite language. This might be true for some cases of \( t \). Related results of this type exist in [13, 14].
- Study the behaviour of \( \mu^* \) on various finite languages, in particular on singleton languages \( \{w\} \). In this setting, we can talk about the code generated by \( w \). We note that many substitution-detecting codes (CRC codes in particular) are generated from a single word, which in fact is represented by a polynomial [17].
- Explore the quality of the maximal languages generated by \( \mu^* \). In terms of information theory, quality could be the average word length, or the efficiency of encoding information, for instance. In terms of automaton theory, investigate the state complexity of the regular maximal languages in terms of the state complexity of the initial language, for various cases of \( t \). A study of this type for prefix codes can be found in [9].
- Find out whether the following problem is computable: given any input-altering transducer \( t \), return (if possible) an input-decreasing transducer \( x \) such that \( t \lor t^{-1} = x \lor x^{-1} \) — that is, \( \mathcal{P}_t = \mathcal{P}_x \).

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