Barycentric Lagrange interpolation for solving Volterra integral equations of the second kind

E S Shoukralla\(^1\), H Elgohary\(^1\) and B M Ahmed\(^2\*)

\(^1\) Department of Engineering Mathematics and Physics, Faculty of Electronic Engineering, Menoufia University, El Mostashfa Street, Menouf, Egypt.
\(^2\) Department of Engineering Mathematics and Physics, Faculty of Engineering and Technology, Future University in Egypt, End of 90th Street, Fifth Settlement, New Cairo, Cairo, Egypt.

Corresponding author*: Basma.magdy@fue.edu.eg

Abstract. An improved version of Barycentric Lagrange interpolation with uniformly spaced interpolation nodes is established and applied to solve Volterra integral equations of the second kind. The given data function and the unknown functions are transformed into two separate interpolants of the same degree, while the kernel is interpolated twice. The presented technique provides the possibility to reduce the solution of the Volterra equation into an equivalent algebraic linear system in matrix form without any need to apply collocation points. Convergence in the mean of the solution is proved and the error norm estimation is found to be equal to zero. Moreover, the improved Barycentric numerical solutions converge to the exact ones, which ensures the accuracy, efficiency, and authenticity of the presented method.

Keywords: Volterra integral equations, interpolation, Barycentric Lagrange interpolation, Numerical methods.

1. Introduction

The solutions of the initial or boundary value problems of mathematical physics, such as diffraction problems, and scattering in quantum mechanics particularly for singular unknown functions or kernels have been successfully found via integral equations [1-3]. Many methods are used for solving integral equations of different kinds and types [4-8]. Obviously, most of these methods are applicable for solving Volterra equations of the second kind, but the goal of this paper is focused on establishing a new one that be applied easily through a few steps that gives exact solutions and saves time. The given technique begins by redefining the Barycentric Lagrange interpolation [9,10] in a matrix form to get an improved version in such a manner that the round-off error is remarkably minimized, particularly near the endpoints of the integration domain.

The advantages of this method are not only simplifying the calculations of Lagrange interpolations, but also gaining access to an equivalent system of equations without any need to apply collocation points. Moreover, a worthy formula was established for easy computations of the complicated integral entries of the system coefficients matrix. By solving the obtained equivalent algebraic system, the unknown coefficients matrix can be computed and thereby the interpolate unknown function can be found. Two examples are solved by the proposed method, and the solutions are found to be converging to the exact solutions.
2. Methodology

2.1. Barycentric Interpolation method

Consider Volterra integral equation of the second kind
\[ u(x) = f(x) + \int_0^x k(x,t)u(t)\,dt; 0 \leq x \leq b < \infty. \] (1)

where \( f(x), k(x,t) \) are known functions, \( u(x) \) is the unknown function and the Volterra operator
\[ K u = \int_0^x k(x,t)u(t)\,dt \]
acting in \( L_2[0,x]; 0 \leq x \leq b \) and the kernel \( k(x,t) \) is defined on the square \( \{(x,t); 0 \leq x, t \leq b\} \) and vanishing in the triangle \( 0 \leq x < t \leq b \). Let \( \tilde{u}(x) \) be the Lagrange interpolating polynomial of degree \( n \) that interpolates \( u(x) \) at the \( n+1 \) equally spaced distinct nodes \( \{x_i\}_{i=0}^n \subset [0,b] \). By choosing a step size \( h > 0 \) such that \( h = \frac{b}{n} \) we get the \( n+1 \) equidistant interpolation nodes \( x_j = jh; j = 0, n \). Using matrix algebra, we can redefine the Barycentric Lagrange interpolation in a new form and use it to interpolate the unknown function \( u(x) \) to get its interpolant polynomial \( \tilde{u}(x) \) as follows

\[
[\tilde{u}(x)] = \left[ \frac{\zeta_0(x)}{\phi(x)} \quad \frac{\zeta_1(x)}{\phi(x)} \quad \ldots \quad \frac{\zeta_n(x)}{\phi(x)} \right] \text{diag} [\gamma_0, \gamma_1, \ldots, \gamma_n] \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_n \end{bmatrix}. \] (2)

where
\[
\tilde{\phi}(x) = \sum_{j=0}^n \gamma_j \epsilon_j(x); \gamma_j = (-1)^j \binom{n}{j}; \quad j = 0, n; \quad \epsilon_j(x) = \frac{1}{x-x_j}; \quad j = 0, n. \] (3)

In matrix form, \( \tilde{u}(x) \) can be rewritten in the form

\[
[\tilde{u}(x)] = \Psi(x) W U; \quad \Psi(x) = \left[ \psi_j(x) \right]_{j=0}^n; \quad \psi_j(x) = \frac{\zeta_j(x)}{\phi(x)}; \quad j = 0, n. \] (4)

Here \( W = \text{diag} [\gamma_0, \gamma_1, \ldots, \gamma_n] \) is a square diagonal matrix and \( U = [u_0, u_1, \ldots, u_n] \) is the unknown Barycentric coefficients column matrix. Similarly, we can find the interpolant given function \( \tilde{f}(x) \) in the matrix form

\[
[\tilde{f}(x)] = \Psi(x) W F; \quad F = [f_j]_{j=0}^n; \quad f_j = f(x_j); \quad j = 0, n. \] (5)

The kernel \( k(x,t) \) is now interpolated twice; the first interpolation with respect to the argument \( t \), while the second interpolation with respect to the argument \( x \). Thus, we get

\[
[\tilde{k}(x,t)] = \Psi(x) K \Psi^T \left( t \right). \] (6)

where the square matrix \( K \) is defined by
By virtue of Eqs. (4) and (6), we have

\[ k(x,t)u(t) = \Psi(x)K\Psi^T(t)\Psi(t)WU = \Psi(x)K\tilde{\Psi}(t)WU. \]  

where

\[ \tilde{\Psi}(t) = \Psi^T(t)\Psi(t) = [\psi_{ij}(t)]_{i,j=0}^{n}; \psi_{ij}(t) = \psi(t_j)\psi(t_i); i,j=0,n. \]  

By Substituting Eq. (8) and Eq. (9) into Eq. (1) we get

\[ \Psi(x)K\Phi(x)WU = \tilde{\Psi}(x)K\Phi(x)WF; \Phi(x) = \frac{x}{0} \Psi_{ij}(t)dt. \]  

Once again, replacing \( u(x) \), and \( k(x,t) \) of Eq. (1) with the new interpolated forms \( \tilde{u}(x) \), and \( \tilde{k}(x,t) \) to get

\[ \Psi(x)K\Phi(x)WU = \tilde{\Psi}(x)K\Phi(x)WF; \Phi(x) = \frac{x}{0} \tilde{\Psi}(t)K\Phi(t)dt. \]  

Consequently, and by virtue of Eq. (5) we get the following linear system of algebraic equations

\[ \Psi(x)K\Phi(x)WU - \Psi(x)K\tilde{\Phi}(x)WU = \Psi(x)K\Phi(x)WF; \tilde{\Phi}(x) = \frac{x}{0} \tilde{\Psi}(t)K\Phi(t)dt. \]  

Simplifying the system given by Eq. (12) we get

\[ U = \left( \tilde{\Phi}(x) - \Phi(x) \right)^{-1} \Phi(x)WF. \]  

To minimize the complexities of the integral entries of \( \Phi(x) \), let

\[ \Phi(t) = [\phi_{ij}(t)]_{i,j=0}^{n}; \phi_{ij}(t) = \frac{t}{0} \psi_{ij}(x)dx; \psi_{ij}(x) = \psi(x_j)\psi(x_i); i,j=0,n. \]  

Now, if we define a matrix \( C \) such that

\[ C = k_{ij}_{i,j=0} = \Psi(t)K; c_{ij} = \sum_{p=0}^{n} k_{pj}\psi_{ip}(t); K = [k_{ij}]_{i,j=0}^{n}; \tilde{\Psi}(t) = [\psi_{ij}(t)]_{i,j=0}^{n}. \]  

Then, \( \Phi(x) \) takes the form

\[ \Phi(x) = [\phi_{ij}(x)]_{i,j=0}^{n}; \phi_{ij}(x) = \sum_{q=0}^{n} \sum_{p=0}^{n} k_{pq} \int_{0}^{x} \psi_{ip}(t)\psi_{qj}(x)dt; \forall i,j=0,n. \]
By computing $\Phi(x)$ and $\Phi(x)$, the unknown Barycentric coefficients column matrix $U$ can be computed and thereby the interpolated unknown function $\tilde{u}(x)$ can be found.

2.2. Error Norm Estimation
Rewrite Eq. (1) in the form $u = Tu$, where the operator $T$ is defined by $Tu = f + Ku$, where $Ku = \int_0^x k(x,t)u(t)dt$ is the Volterra operator acting in $L_2[0,x];0 \leq x \leq b$. Sample the total error norm of the interpolation by $E_n(x)$ such that $E_n(x) = \|Tu - T\tilde{u}\|_2$ where $\| \cdot \|_2$ denotes the Euclidean norm in $\mathbb{R}^2$ and $T\tilde{u} = \tilde{f} + \tilde{Ku}$. Thus, we have

$$\|Tu - T\tilde{u}\|_2 = \left\| (f - \tilde{f}) + (Ku - \tilde{Ku}) \right\| \leq \|f - \tilde{f}\|_2 + \|Ku - \tilde{Ku}\|_2. \tag{17}$$

For $\|f - \tilde{f}\|_2$, we have

$$\|f - \tilde{f}\|_2 = \left[ \int_a^b \left( f(x) - \tilde{f}(x) \right)^2 dx \right]^{1/2} \tag{18}.$$  

Assuming that $u(x)$, $f(x)$ belong to $L_2(0,x);0 \leq x \leq b$ with $\max_{x \in [0,b]} f(x) = N$; $N$ is a real number and $\max_{t,x \in [0,b]} |k(x,t)| = M$, and $\int_a^b \int_a^b |k(x,t)|^2 dx dt < \infty$; $M$ is a real number, we find that

$$\int_a^b \int_a^b |f(x)|^2 dx \leq \eta \int_a^b \left| \tilde{f}(x) \right|^2 dx = \int_a^b |\Psi(x)|^2 dx \leq r_2, \tag{19}$$

where $\eta, r_2$ real numbers. Using the following inequality

$$\int_a^b |f(x)| \tilde{f}(x) dx \leq \frac{1}{2} \int_a^b |f(x)|^2 dx + \frac{1}{2} \int_a^b \left| \tilde{f}(x) \right|^2 dx = \frac{1}{2} (\eta + r_2). \tag{20}$$

we find that $\|f - \tilde{f}\|_2 = 0$. In the same context, we have

$$\|Ku - \tilde{Ku}\|_2 = \| \int_0^x k(x,t)u(t)dt - \int_0^x \tilde{k}(x,t)u(t)dt \|_2 = \left[ \int_0^x \left( \int_0^x k(x,t)u(t)dt - \int_0^x \tilde{k}(x,t)u(t)dt \right)^2 dx \right]^{1/2}. \tag{21}$$

By Cauchy–Bunyakowski inequality, we get
\[
\int_0^t \left( \int_0^t k(x,t)u(t) \, dt \right)^2 \, dx \leq \int_0^t \left( \int_0^t \left| k(x,t) \right|^2 \, dt \right) \, dx = \left\| u(t) \right\|_2^2 \left( \int_0^t \left( \int_0^t \left| k(x,t) \right|^2 \, dt \right) \, dx \right) \nabla^2 \left( \frac{(Mt)^2}{2} \left\| u(t) \right\|_2^2 \right) ; 0 < t \leq b. \tag{22}
\]

and
\[
\int_0^t \left( \int_0^t \tilde{k}(x,t)\tilde{u}(t) \, dt \right)^2 \, dx \leq \int_0^t \left( \int_0^t \left| \tilde{k}(x,t) \right|^2 \, dt \right) \, dx = \left\| \tilde{u}(t) \right\|_2^2 \left( \int_0^t \left( \int_0^t \left| \tilde{k}(x,t) \right|^2 \, dt \right) \, dx \right) \nabla^2 \left( F(t) \left\| \tilde{u}(t) \right\|_2^2 \right) ; 0 < t \leq b. \tag{23}
\]

where
\[
\int_0^x \tilde{k}(x,t)^2 \, dt = \int_0^x \Psi(x) K \tilde{\Psi}(t) f(t) \, dt = \sum_{s=0}^n \sum_{j=0}^n \sum_{i=0}^n \left( \sum_{i=0}^n k_{ij} \tilde{\psi}_{ji}(x) \right) \tilde{\psi}_{vs}(x) ; v = 0, n
\]

\[
\tilde{\psi}_{ij}(x) = \int_0^x \psi_{ij}(t) \, dt ; F(t) = \int_0^x \left( \sum_{s=0}^n \sum_{j=0}^n \sum_{i=0}^n \left( \sum_{i=0}^n k_{ij} \tilde{\psi}_{ji}(x) \right) \right) \tilde{\psi}_{vs}(x) \, dx. \tag{24}
\]

Since,
\[
\int_0^t \int_0^x \tilde{k}(x,t)^2 \, dtdx \leq \frac{1}{2} \int_0^t \int_0^x \tilde{k}(x,t)^2 \, dtdx + \frac{1}{2} \int_0^t \int_0^x \tilde{k}(x,t)^2 \, dtdx. \tag{25}
\]

Then
\[
-\frac{1}{2} \int_0^t \int_0^x \tilde{k}(x,t)u(t) \, dt \times \int_0^x \tilde{k}(x,t)\tilde{u}(t) \, dt \, dx \leq -\int_0^t \left\| \tilde{k}(x,t)u(t) \right\| \, dx - \int_0^t \left\| \tilde{k}(x,t)\tilde{u}(t) \right\| \, dx
\]

\[
= -\left( \frac{(Mt)^2}{2} \left\| u(t) \right\|_2^2 - F(t) \left\| \tilde{u}(t) \right\|_2^2 \right). \tag{26}
\]

Therefore, and after substituting from Eqs. (22), (23), and (26), we find that \( \left\| Ky - \tilde{K} \tilde{y} \right\| = 0 \). Thus, we have proved that
\[
E_n(x) = \left\| Tu - T\tilde{u} \right\|_2 = 0. \tag{27}
\]

For the convergence in the main of the proposed interpolant solution \( \tilde{u}(x) \), we have
\[ \int_{a}^{b} \tilde{u}(x)u(x)dx = \int_{0}^{n} \sum_{j=0}^{\infty} \psi_j(x) \omega_j \sum_{i=0}^{\infty} \alpha_i \psi_i(x) dx = \int_{0}^{b} \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} \alpha_i \beta_{ji}(x) \omega_j dx = \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} \alpha_i \beta_{ji}(x) \omega_j. \]  \quad (28)

and

\[ \int_{a}^{b} |\tilde{u}(x)|^2 dx = \int_{0}^{n} \left( \sum_{j=0}^{n} \sum_{i=0}^{n} \alpha_i \beta_{ji} \right) \omega_j \int_{0}^{b} |u(x)|^2 dx = \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} \alpha_i \beta_{ji} \omega_j. \]  \quad (29)

where

\[ \omega_k = u_k \gamma_k; \quad k = 0, n; m_{ij} = \int_{0}^{b} \psi_i(x) \psi_j(x) dx = \beta_{ij} \quad \forall i, j = 0, n. \]  \quad (30)

Hence, it is proved that

\[ \lim_{n \to \infty} \|\tilde{u}(x) - \tilde{u}(x)\|_{2} = \lim_{n \to \infty} \left\{ \sum_{j=0}^{\infty} \sum_{i=0}^{n} \alpha_i \beta_{ji}(x) \omega_j + \sum_{j=0}^{n} \sum_{i=0}^{\infty} \alpha_i \beta_{ij}(x) \omega_j - 2 \sum_{i=0}^{n} \sum_{j=0}^{\infty} \alpha_i \beta_{ji}(x) \omega_i \right\} = 0. \]  \quad (31)

3. Computational results

The computations were made using MATLAB Version 2018. For example 1, where the kernel and the given data functions are analytic, the parameter \( b \) is cancelled during the computation and the improved Barycentric solution is found in an explicit formula equal to the exact solution for \( n=5 \). For example 2, where the kernel and the given data functions are transcendental functions, the improved Barycentric numerical solutions for \( b=1, n=7 \) are substantial of higher accuracy and strongly converge to the exact solution for any value of \( x \).

Example (1) [4]

Consider Volterra integral equation

\[ u(x) = x + \frac{7}{12}x^5 - \int_{0}^{x} (x^2 + x^2)u(t)dt. \]  \quad (32)

whose exact solution is \( u(x) = x \). For \( n=5 \) we find that

\[ U = \begin{bmatrix} 0 & \frac{b}{5} & \frac{2b}{5} & \frac{3b}{5} & \frac{4b}{5} & b \end{bmatrix}^T. \]

By substituting into equation (4), the parameter \( b \) was cancelled and the obtained improved barycentric numerical solution is found to be \( u(x) = x \) which equals the exact one.

Example (2) [4]

Consider Volterra integral equation

\[ u(x) = x + \frac{7}{12}x^5 - \int_{0}^{x} (x^2 + x^2)u(t)dt. \]  \quad (32)
\[
 u(x) = e^{-x^2} + \frac{x(1-e^{-x^2})}{2} - \int_0^x t \, dt. 
\]

(33)

whose exact solution is \( u(x) = e^{-x^2} \). It is found that \( \tilde{u}_n(x_i) \) for \( n = 7 \) and \( b = 1 \) are strongly convergent to the exact one. The improved barycentric numerical solutions \( \tilde{u}_n(x_i) \) for \( b=1 \) and \( n=17 \) respectively are shown in table 1. The absolute errors \( E_n(x_i) \) are given in table 2. In figure 1, plotted are the graphs of the exact solution \( u(x_i) \) and \( \tilde{u}_n(x_i) \).

**Table 1.** A comparison between the exact solution \( u(x_i) \) and the solutions \( \tilde{u}_n(x_i) \).

| \( x_i \) | \( u(x_i) \) | \( \tilde{u}_1(x_i) \) | \( \tilde{u}_2(x_i) \) | \( \tilde{u}_3(x_i) \) |
|---|---|---|---|---|
| 0 | 0.990049833749168 | 0.968050347315980 | 0.965096305667694 | 0.988178473925050 |
| 0.1 | 0.960784939152323 | 0.934092482039410 | 0.928118477389353 | 0.958942454111887 |
| 0.2 | 0.913931185271228 | 0.896281955156874 | 0.886613585572443 | 0.913385370621379 |
| 0.3 | 0.852143789662611 | 0.853003818700363 | 0.838475311415924 | 0.853111223991993 |
| 0.4 | 0.778800783071405 | 0.80295912157551 | 0.7820344681299 | 0.780453430352598 |
| 0.5 | 0.697676326070131 | 0.745256764842716 | 0.716213659327260 | 0.687945797406495 |
| 0.6 | 0.612626934184416 | 0.679574846335409 | 0.64069675834725 | 0.612149015887457 |
| 0.7 | 0.527292424043049 | 0.606227852169218 | 0.55611209302782 | 0.526519620088494 |
| 0.8 | 0.448580662229241 | 0.52653313836181 | 0.464227140240692 | 0.448841291864112 |
| 0.9 | 0.36789441171442 | 0.44266968086149 | 0.368174832505544 | 0.367967202106808 |

**Table 2.** The absolute errors \( E_n(x_i) \).

| \( x_i \) | \( E_1(x_i) \) | \( E_2(x_i) \) | \( E_3(x_i) \) | \( E_4(x_i) \) |
|---|---|---|---|---|
| 0 | 2.4953528014714 e-02 | 1.87135982411791 e-03 | 8.89442156778082 e-04 |
| 0.1 | 3.26709667623886 e-02 | 1.84895040366222 e-03 | 3.1822510561707 e-04 |
| 0.2 | 3.751999709687852 e-02 | 5.45814649849568 e-03 | 2.2209934174315 e-04 |
| 0.3 | 4.36717655208276 e-02 | 9.67434325782091 e-03 | 2.5937596978242 e-04 |
| 0.4 | 5.155129861465 e-02 | 1.65264728119260 e-03 | 1.2205600458215 e-04 |
| 0.5 | 6.47508047716854 e-02 | 9.888631334563481 e-03 | 7.7295892714702 e-04 |
| 0.6 | 7.69484521509934 e-02 | 2.80708516503093 e-03 | 4.773872965905 e-04 |
| 0.7 | 8.8904511736635 e-02 | 2.2817149897337 e-03 | 4.72809354554977 e-04 |
| 0.8 | 10.0252476132401 e-02 | 1.93600741077513 e-03 | 3.98322564117067 e-03 |
| 0.9 | 1.74905269147062 e-02 | 2.95391369114104 e-03 | 1.98177609536552 e-02 | 2.87459131006709 e-02 |
| $x_i$ | $E_5(x_i)$ | $E_6(x_i)$ | $E_7(x_i)$ |
|------|------------|------------|------------|
| 0    | 0          | 0          | 0          |
| 0.1  | 1.14255980232939 e-04 | 1.22722199727976e-05 | 2.15768410327666e-06 |
| 0.2  | 2.46477348871828 e-07 | 3.1011868984229e-06 | 6.42534446463464e-07 |
| 0.3  | 4.16236941462016 e-05 | 1.6418206349826e-06 | 1.63614737203055e-07 |
| 0.4  | 4.38619152585965 e-06 | 1.69405680972012e-06 | 2.13045502350795e-07 |
| 0.5  | 5.30286494858278 e-05 | 1.28849410141463e-07 | 3.2529630933645e-07 |
| 0.6  | 7.26361548668321 e-05 | 1.53389776424984e-06 | 2.66403742954502e-07 |
| 0.7  | 1.40672551786098 e-04 | 8.87201076382738e-07 | 3.1807496203730e-07 |
| 0.8  | 3.45134102223133 e-04 | 2.00217564438888e-06 | 5.15117003385690e-07 |
| 0.9  | 4.79130858825239 e-04 | 8.05029276740310e-07 | 3.30532527303973e-06 |
| 1    | 5.13259275067295 e-04 | 1.10032359226797e-05 | 1.57360096902925e-06 |

Figure 1. The exact solution $u(x)$ and the obtained improved barycentric numerical solutions for $b=1$, $n=1,7$.

4. Conclusion

The Barycentric Lagrange interpolation formula with uniformly spaced interpolation nodes was modified in matrix form and was used to solve Volterra integral equations of the second kind. The given data function and the unknown function are replaced by two different modified interpolating polynomials of the same degree, while the kernel is interpolated twice with respect to both its variables.

To avoid the application of the collocation method, the unknown function is substituted twice into the integral equation, so that the solution is equivalent to the solution of an algebraic linear system. The presented method produces the exact solutions for any $0 \leq x \leq b < \infty$, where $b$ is often cancelled during the computations for analytic kernel and given functions regardless of the choice of the step-size of the interpolation nodes.

If the kernel and the given data functions are not analytic algebraic functions, then the improved Barycentric numerical solutions are strongly converge to the exact solution for $b=1$. For $b > 1$ it is
necessary to increase the number of interpolating nodes, which yields increasing the degree of the interpolant polynomials.

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