Four-quark energies in SU(2) lattice Monte Carlo using a tetrahedral geometry

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This contribution – a continuation of the work in refs.\cite{1}-\cite{4} – reports on recent developments in the calculation and understanding of 4-quark energies generated using lattice Monte Carlo techniques.

1. THE CALCULATION OF 4-QUARK ENERGIES IN A TETRAHEDRAL GEOMETRY USING MONTE CARLO SIMULATION

In refs.\cite{3,4} – the binding energies of four quarks in various geometries (e.g. at the corners of a rectangle or on a straight line) were calculated in static-quenched-SU(2) on a lattice $16^3 \times 32$ with $\beta = 2.4$. This showed that the 4-quark binding energies were greatest when two of the three possible partitions of the four quarks – into two colour-singlet pairs – were degenerate in energy e.g. as in a square geometry. The main reasons for studying other geometries were: 1) to investigate further this point and 2) to give a representative set of geometries illustrating the different possibilities for arranging four quarks. The latter would then serve as a very stringent test of models attempting to describe four interacting quarks – provided such models involve approximations analogous to those contained in the corresponding lattice calculation e.g. static quarks and SU(2).

Interest in the tetrahedral geometry arises, since in this case all three partitions are degenerate in energy. For completeness, a series of related geometries, where the four quarks have the coordinates $(0, 0, 0), (r, 0, d), (0, d, d)$ and $(r, d, 0)$, are now studied. Preliminary results are given in table 1.

Table 1

| (d/a, r/a) | (1,0) | (1,1) | (1,2) |
|-----------|-------|-------|-------|
| $E_0$     | -0.0696(2) | -0.016(2) | -0.003(1) |
| $E_1$     | -0.016(2) | 0.265(2) |

| (d/a, r/a) | (2,1) | (2,2) | (2,3) |
|-----------|-------|-------|-------|
| $E_0$     | -0.043(3) | -0.020(1) | -0.008(2) |
| $E_1$     | 0.085(2) | -0.020(1) | 0.147(2) |

| (d/a, r/a) | (3,2) | (3,3) | (3,4) |
|-----------|-------|-------|-------|
| $E_0$     | -0.041(1) | -0.028(1) | -0.010(2) |
| $E_1$     | 0.048(2) | -0.028(1) | 0.105(5) |

| (d/a, r/a) | (4,3) | (4,4) | (4,5) |
|-----------|-------|-------|-------|
| $E_0$     | -0.039(2) | -0.036(1) | -0.012(2) |
| $E_1$     | 0.03(1) | -0.036(1) | 0.089(4) |

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Here two points are noteworthy:

1) For the tetrahedral case (i.e. \( r = d \)), the ground state energy \( (E_0) \) and the first excited state energy \( (E_1) \) are degenerate. As will be seen in section 3 below, this degeneracy could have a natural explanation using a simple model.

2) As \( r \) becomes small, the results match on to the earlier ones in ref.[3] for \( r = 0 \) – the \( d \times d \) square geometry, where only the two degenerate quark partitions were employed to give the binding energies. For example, with \((d,r) = (2,1)\) and \((3,2)\) the values of \( E_0 \) are the same when using either the two degenerate partitions or the complete set of three. However, the results for \( E_1 \) with two partitions get less accurate as \( r \) increases giving 0.086(2) and 0.053(4) for the above values of \((d,r)\). This gives added confidence that, for squares, the earlier use of two partitions was certainly adequate.

2. EXTRACTION OF BINDING ENERGIES USING THE FULL CORRELATION MATRIX

In refs.[1]-[3] the energies were extracted by first solving for the eigenvalues of

\[
W^{T}a_{n}^{T} = \lambda_{i}^{(T)}W^{T-1}a_{n}^{T}.
\]

Here, for euclidean time differences \( T \), the \( W_{mn}^{T} \) are the values of Wilson loops with \( m,n \) = 1, 2, 3 for the 2-quark energies (i.e. 3 fuzzing levels 12,16,20) and \( m,n \) = 1, 2 or 1, 2, 3 for the 4-quark energies depending on the number of partitions used – all with fuzzing level 20. Knowing \( \lambda_{i}^{(T)} \), the 2- and 4-quark energies are given by the plateau as

\[
E_{i} = -\lim_{T\to\infty} \ln \left( \lambda_{i}^{(T)} \right).
\]

In practice, since it is the 4-quark binding energy that is of prime interest, the plateau is extracted directly for the difference of the 4- and 2-quark energies. These differences are quoted in table 1. The errors are then estimated by bootstrapping this procedure. As discussed in ref.[3], this technique does not make full use of the available Monte Carlo data, since it disregards correlations between different values of \( T \). A more complete fit minimizes the following function

\[
\sum_{T,T',P,P'} D(T'T')C^{-1}D(TP),
\]

where \( D(TP) = [F(TP) - W(TP)] \). Here \( W \) is the average of the Wilson loop data and \( F \) is a function to be fitted to this data. Both of these are dependent on \( T \) and also \( P \) – an index denoting the fuzzings or partitions involved i.e. correlations among the data in both \( T \) and in \( P \) are now included. \( C(T'T',P'P) \) is the full correlation matrix among the measured Wilson loop data samples. Since inverting a large matrix with limited data samples is unstable, we need to approximate \( C \) by some model e.g.

\[
C(T'T',P'P) = A(P'P)\exp(-a|T' - T|)
\]

would be one particularly simple choice. Usually \( F \) has a form such as \( \sum a_{i}(P)\exp(-m_{i}T) \), where the \( m_{i} \) are then interpreted as the energies \( (E_{i}) \) of the system. Such an analysis is now being carried out. Preliminary results in table 2 indicate that the earlier energies and their error estimates were indeed correct.

Table 2

Comparison between the energies \( E_{i}(C) \) extracted using the full correlation matrix and the earlier results \( E_{i}(P) \) with the plateau method. These results are for quarks on the corners of rectangles with sides \((d,r)\).

| (d,r)     | (2,2) | (2,3) | (2,4) |
|-----------|-------|-------|-------|
| \( E_{0}(C) \) | -0.0590(3) | -0.005(1) | 0.0006(7) |
| \( E_{0}(P) \) | -0.0585(4) | -0.006(1) | -0.0004(4) |
| \( E_{1}(C) \) | 0.141(1) | 0.325(2) | 0.514(1) |
| \( E_{1}(P) \) | 0.143(1) | 0.324(1) | 0.512(1) |

3. A MODEL FOR UNDERSTANDING 4-QUARK ENERGIES

In refs.[3]-[4] a model for attempting to understand the above lattice results was introduced. This amounts to comparing the lattice energies with the eigenvalues \( [E_{i}(f)] \) from

\[
[V(f) - E_{i}(f)N(f)] \Psi_{i} = 0,
\]

where
with
\[ N(f) = \begin{pmatrix} 1 & f/2 \\ f/2 & 1 \end{pmatrix}, \]
\[ V(f) = \begin{pmatrix} v_{13} + v_{24} & fV_{AB} \\ fV_{BA} & v_{14} + v_{23} \end{pmatrix}. \] (4)

Here \( V_{AB} = V_{BA} = \frac{1}{2} (v_{13} + v_{24} + v_{14} + v_{23} - v_{12} - v_{34}) \) (5)
as expected from a \( \tau_i \tau_j \) colour-vector potential.

For squares of a given size \( d \times d \), one value of \( f(d) \) gives a reasonable fit to both \( E_0(d) \) and \( E_1(d) \). However, for the tetrahedral geometry, since all the \( v_{ij} \) are equal, the two eigenvalues of eq.(3) are degenerate at zero binding energy. To push this degeneracy to some other energy could be explained by including a colour-scalar interquark potential in addition to the colour-vector potential used to derive eq.(5).

Further support for the above model comes from ref.\[6\], where it is shown explicitly that perturbation theory up to fourth order in the quark-gluon coupling does indeed correspond to the \( f = 1 \) limit.

Of course, the above replacement of the \( E_{0,1} \) by \( f \) is not particularly useful unless \( f \) – a function of all the four quark coordinates – can be parametrized in some convenient and simple manner. So far, the most successful appear to be the forms
\[ f = f_c \exp[-\alpha b_0 A - \gamma \sqrt{b_0 P}], \] (6)
where \( b_0 \) is the string tension dictated by the 2-quark potential, \( A \) is the minimal area of the surface bounded by the straight lines connecting the quarks, \( P \) is a perimeter connecting the four quarks and \( \alpha, \gamma \) are phenomenological constants adjusted to fit the lattice data. In this way, the original rapid variations of \( E_{0,1} \) with geometry can be first described in terms of a smoothly varying function \( f \), which can then be expressed in terms of the constants \( \alpha, \gamma \). In practice, two specific parametrizations have been attempted. The first \[f_c \] assumes \( f_c = 1 \) – consistent with the expectation that \( f \to 1 \) as \( A \) and \( P \) become small i.e. in the perturbative limit. In addition, guided by arguments based on the lattice cubic symmetry, \( P \) is dependent on the underlying lattice – not just the quark positions – and so, being a lattice artefact, must be dropped in the continuum limit. In the second parametrization, \( P \) is defined simply in terms of the quark coordinates by the straight lines joining the quarks. However, \( f_c \) is taken to be \( \beta \) dependent. A preliminary fit gives \( f_c(\beta = 2.4) = 0.88(2) \) and \( f_c(\beta = 2.5) = 0.94(2) \) i.e. apparently \( f_c \to 1 \) as \( \beta \) increases – again as expected in the perturbative limit. The continuum form of \( f \) then is taken to have \( f_c = 1 \), with \( A \) and \( P \) defined purely by the four quark positions. At present, it is not clear which of these two parametrizations is to be preferred.

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