Selberg’s zeta functions for congruence subgroups of modular groups in $SL_2(\mathbb{R})$ and $SL_2(\mathbb{C})$

Yasufumi Hashimoto *

Abstract

It is known that the Selberg zeta function for the modular group has an expression in terms of the class numbers and the fundamental units of the indefinite binary quadratic forms. In the present paper, we generalize such a expression to any congruence subgroup of the modular groups in $SL_2(\mathbb{R})$ and $SL_2(\mathbb{C})$.

1 Introduction

Let $\mathbb{H}$ be the upper half plane with the hyperbolic metric and $\Gamma$ a discrete subgroup of $SL_2(\mathbb{R})$ such that the volume of $\Gamma \backslash \mathbb{H}$ is finite. We denote by $\text{Prim}(\Gamma)$ the set of primitive hyperbolic conjugacy classes of $\Gamma$ and $N(\gamma)$ the square of the larger eigenvalue of $\gamma \in \text{Prim}(\Gamma)$. The Selberg zeta functions for $\Gamma$ are defined as follows.

$$Z_\Gamma(s) := \prod_{\gamma \in \text{Prim}(\Gamma)} \prod_{n=0}^{\infty} (1 - N(\gamma)^{-s-n}) \text{ Res } s > 1.$$ 

It is known that $Z_\Gamma(s)$ is analytically continued to the whole complex plane as a meromorphic function of order 2. By virtue of the analytic properties of $Z_\Gamma(s)$, we can obtain the following asymptotic formula called by the prime geodesic theorem (see, e.g., [He]).

$$\pi_\Gamma(x) := \# \{ \gamma \in \text{Prim}(\Gamma) \mid N(\gamma) < x \} = \text{li}(x) + O(x^\delta) \text{ as } x \to \infty,$$

where $\text{li}(x) := \int_2^x (\log t)^{-1} dt$ and the constant $\delta$ ($0 < \delta < 1$) depends on $\Gamma$.

When $\Gamma = SL_2(\mathbb{Z})$, since there is a one-to-one correspondence between elements of $\text{Prim}(\Gamma)$ and equivalence classes of primitive indefinite binary quadratic forms, and $N(\gamma)$ for $\gamma \in \text{Prim}(\Gamma)$ coincides the square of the fundamental unit of the quadratic form corresponding to $\gamma$ (see e.g. [G]), we can express the Selberg zeta function and the prime

*Partially supported by JSPS Grant-in-Aid for Young Scientists (B) no. 20740027.
MSC: primary: 11M36; secondary:11E41
geodesic theorem for \( \Gamma = \text{SL}_2(\mathbb{Z}) \) as follows (see [Sa1]).

\[
Z_{\text{SL}_2(\mathbb{Z})}(s) = \prod_{D \in \mathcal{D}} \prod_{n=0}^{\infty} (1 - \epsilon(D)^{-2(s+n)}h(D)) \quad \text{Res} > 1, \tag{1.1}
\]

\[
\pi_{\text{SL}_2(\mathbb{Z})}(x) = \sum_{D \in \mathcal{D}, \epsilon(D) < x} h(D) \sim \text{li}(x^2) \quad \text{as} \quad x \to \infty, \tag{1.2}
\]

where \( \mathcal{D} \) is the set of integers \( D > 0 \) such that \( D \) is not square and \( D \equiv 1, 0 \mod 4 \), \( h(D) \) and \( \epsilon(D) \) are respectively the class number and the fundamental unit of the discriminant \( D \) in the narrow sense.

For a three dimensional hyperbolic manifolds whose fundamental group is \( \text{SL}_2(\mathcal{O}) \) where \( \mathcal{O} \) is the integer ring of the imaginary quadratic field, similar expressions of the Selberg zeta function and the prime geodesic theorem were obtained in [Sa2].

Such arithmetic expressions of the Selberg zeta functions essentially give those of the length spectra on the corresponding manifolds. Though it is not easy in general to study the distributions of the length spectra and its multiplicities in detail, by virtue of the expressions by \( h(D) \) and \( \epsilon(D) \), the distributions of them for the cases of the modular group and the congruence subgroups could be studied by the classical tools in the analytic number theory (see [BLS], [Pe] and [H2]). Moreover, by writing down the prime geodesic theorem for the modular group with \( h(D) \) and \( \epsilon(D) \), one can get the density formula of the sum of the class numbers with a different form to that conjectured by Gauss and proven by Siegel [Si] (see [Sa1], [H1] and [H3]).

The aim of the present paper is to generalize such arithmetic expressions of the Selberg zeta functions for congruence subgroups of \( \text{SL}_2(\mathbb{Z}) \) and \( \text{SL}_2(\mathcal{O}) \). For the groups \( \Gamma_0(N) \), \( \Gamma_1(N) \) and \( \Gamma(N) \) of \( \text{SL}_2(\mathbb{Z}) \), the author [H1] already gave explicit expressions of the logarithmic derivatives of the Selberg zeta functions in terms of the class numbers and the fundamental units. In the present paper, we show that the Selberg zeta functions for any congruence subgroup of \( \text{SL}_2(\mathbb{Z}) \) and \( \text{SL}_2(\mathcal{O}_K) \) are expressed in terms of the class numbers and the fundamental units such like (1.1).

\[2 \quad \text{Quadratic forms and modular groups}\]

In this section, we give notations of the quadratic forms, explain the relations between the quadratic forms and elements in the modular group and study the difference between the conjugation in \( \text{SL}_2(\mathcal{O}) \) and that in \( \text{SL}_2(\mathcal{O}/n) \) for an integer ring \( \mathcal{O} \) of an algebraic number fields and an ideal \( n \) in \( \mathcal{O} \).

Let \( K \) be an algebraic number field over \( \mathbb{Q} \) such that \( [K : \mathbb{Q}] < \infty \) and \( \mathcal{O} \) the ring of integers of \( K \). We denote by

\[
Q(x, y) = [a, b, c] := ax^2 + bxy + cy^2
\]

a binary quadratic form over \( \mathcal{O} \) where \( a, b, c \in \mathcal{O} \). We write \( Q \sim Q' \) if there exists \( g \in \text{SL}_2(\mathcal{O}) \) such that \( Q(x, y) = Q'(g.(x, y)) \). Let \( D := b^2 - 4ac \) be the discriminant of
Selberg’s zeta functions for congruence subgroups

$[a, b, c]$ and $P(D)$ the set of solutions $(t, u) \in \mathcal{O}^2$ of the Pell equation $t^2 - Du^2 = 4$. The set $P(D)$ is an abelian group with the identity $(2, 0)$ and the product

$$(t_1, u_1) \ast (t_2, u_2) = \left( \frac{t_1 t_2 + u_1 u_2 D}{2}, \frac{t_1 u_2 + t_2 u_1}{2} \right).$$

It is known that $P(d) \simeq (\mathbb{Z}/l\mathbb{Z}) \times \mathbb{Z}$ where $l$ is the number of roots of 1 in $\mathcal{O}_{K(\sqrt{D})}$, $r_1$ is the number of embeddings $K(\sqrt{D})$ into $\mathbb{R}$ and $2r_2$ is the number of embeddings $K(\sqrt{D})$ into $\mathbb{C}$ with $\text{Im}(K(\sqrt{D})) \not\subset \mathbb{R}$. Note that $r_1 + 2r_2 = [K(\sqrt{D}) : \mathbb{Q}]$. For a quadratic form $Q = [a, b, c]$ and a solution $(t, u) \in P(D)$, we put

$$\gamma(Q, (t, u)) := \left( \frac{t + bu}{2a}, \frac{-cu}{t - bu} \right) \in \text{SL}_2(\mathcal{O}).$$

Conversely, for $\gamma = (\gamma_{i,j})_{1 \leq i, j \leq 2} \in \text{SL}_2(\mathcal{O})$, we put

$$t_\gamma := \gamma_{11} + \gamma_{22}, \quad u_\gamma := \gcd(\gamma_{21}, \gamma_{11} - \gamma_{22}, -\gamma_{12}),$$

$$a_\gamma := \gamma_{21}/u_\gamma, \quad b_\gamma := (\gamma_{11} - \gamma_{22})/u_\gamma, \quad c_\gamma := -\gamma_{12}/u_\gamma,$$

$$Q_\gamma := [a_\gamma, b_\gamma, c_\gamma], \quad D_\gamma := \frac{t_\gamma^2 - 4}{u_\gamma^2} = b_\gamma^2 - 4a_\gamma c_\gamma.$$ 

We note the following elementary facts without proofs.

**Fact 2.1.** Suppose that $\gamma, \gamma_1, \gamma_2 \in \text{SL}_2(\mathcal{O})$ are not in the center of $\text{SL}_2(\mathcal{O})$. Then we have

1. $\gamma(Q_g, (t_g, u_g)) = g$ for any $g \in \text{SL}_2(\mathcal{O})$.
2. $D(Q) = D_\gamma(Q, (t, u))$ for any $(t, u) \in P(D)$ and $D_\gamma = D(Q_\gamma)$.
3. $(t_\gamma(Q, (t, u)), u_\gamma(Q, (t, u))) = (t, u)$ and $(t_\gamma, u_\gamma) \in P(d_\gamma)$.
4. If $Q_{\gamma_1} = Q_{\gamma_2}$ then $Q_{\gamma_1 \gamma_2} = Q_{\gamma_1} = Q_{\gamma_2}$ and $(t_{\gamma_1 \gamma_2}, u_{\gamma_1 \gamma_2}) = (t_{\gamma_1}, u_{\gamma_1}) \ast (t_{\gamma_2}, u_{\gamma_2})$.
5. $Q_{g^{-1}g}(x, y) = Q_{\gamma}(g, (x, y))$ for any $g \in \text{SL}_2(\mathcal{O})$.

For a fixed discriminant $D \in \mathcal{O}$, let $I(D)$ be the set of equivalence classes of quadratic forms $[a, b, c]$ with $b^2 - 4ac = D$. It is known that $h(D) := \# I(D)$ is finite. According to Fact 2.1, we see that

$$h(d) = \# \{ \gamma \in \text{Conj}(\text{SL}_2(\mathcal{O})) \mid D_\gamma = D \}/P(D)$$

$$= \# \{ \gamma \in \text{Conj}(\text{SL}_2(\mathcal{O})) \mid t_\gamma = t, u_\gamma = u \},$$

where $(t, u)$ is a non-trivial solution of $t^2 - Du^2 = 4$.

For quadratic forms $[a_1, b_1, c_1], [a_2, b_2, c_2] \in I(d)$, we give the following product.

$$[a_1, b_1, c_1] \ast [a_2, b_2, c_2] = \left[ \frac{a_1 a_2}{\beta^2}, \frac{a_1}{\beta} b_2 + v_1 \frac{a_2}{\beta} b_1 + w \frac{b_1 b_2 + D}{2\beta^2}, C \right],$$

where $v_1, w$ are integers.
Lemma 2.1. Let \( a \in \mathcal{O} \) be an ideal in \( \mathcal{O} \). For \( a, b, c \in \mathcal{O}/a, d = b^2 - 4ac \) and \( t, u \in \mathcal{O}/a \) satisfying \( t^2 - Du^2 = 4 \) without \( (t, u) = (\pm 2, 0) \), we denote by

\[
\gamma := \begin{pmatrix} \frac{1}{2}(t + bu) & -cu \\ au & \frac{1}{2}(t - bu) \end{pmatrix}, \quad \gamma_{\nu} := \begin{pmatrix} \frac{t + \delta u}{2} & \frac{D - \delta^2}{\nu u} -1u \\ \frac{4}{\nu} & t - \delta u \end{pmatrix},
\]

where \( \nu \in (\mathcal{O}/a)^* \) and \( \delta \) is taken as \( 2 \mid b + \delta \) when \( \gcd(a, 2) \neq 1 \) and is taken by 0 when \( \gcd(a, 2) = 1 \). Then we have \( \gamma \sim \gamma_{\nu} \) in \( \text{PSL}_2(\mathcal{O}/a) \) for some \( \nu \in \mathcal{O}_a^{(2)} \).

Proof. When \( \gcd(a, a) \neq 1 \), it is easy to see that there exists \( g \in \text{PSL}_2(\mathcal{O}/a) \) such that \( \gcd((g^{-1}\gamma g)_{21}/u, a) = 1 \). Then it is sufficiently to consider only the case for \( \gcd(a, a) = 1 \).

When \( a = \nu a^2 \) for \( \nu \in \mathcal{O}_a^{(2)} \) and \( \alpha \in \mathcal{O}/a \), we have \( g'^{-1}\gamma g' = \gamma_{\nu} \) where

\[
g' := \begin{pmatrix} \alpha^{-1} & (\nu\alpha)^{-1}(b + \delta)/2 \\ 0 & \alpha \end{pmatrix}.
\]

This completes the proof. \( \square \)

Lemma 2.2. Let \( a \) be an ideal in \( \mathcal{O} \). For \( (t, u) \in \mathcal{O}^2 - \{(\pm 2, 0)\} \), we denote by \( \mathcal{I}(t, u) := \{ \gamma \in \text{Conj}(\text{PSL}_2(\mathcal{O})) \mid t_{\gamma} = t, u_{\gamma} = u \} \) and \( \mathcal{J}_\nu(t, u) := \{ \gamma \in \mathcal{I}(t, u) \mid \gamma \sim \gamma_{\nu} \text{ in } \text{PSL}_2(\mathcal{O}/a) \} \).

For \( \nu, \nu' \in \mathcal{O}_a^{(2)} \), define the equivalence \( \nu \sim \nu' \) such that \( \gamma_{\nu} \) is conjugate to \( \gamma_{\nu'} \) in \( \text{PSL}_2(\mathcal{O}/a) \). Put \( \mathcal{L}_a(t, u) := \{ \nu \in \mathcal{O}_a^{(2)} \mid \sim \} \) and \( \mu_a(t, u) := \# \mathcal{L}_a(t, u) \).

Number the elements of \( \mathcal{L}_a(t, u) \) by \( \nu_1(=1), \nu_2, \ldots, \nu_\mu \). For simplicity, rewrite \( \mathcal{J}_{\nu_1} \) by \( \mathcal{J}_1 \).

Then we have \( \# \mathcal{J}_i = h(D)/\mu \) for any \( 1 \leq i \leq \mu \).

\[
\gcd(\nu_1, \nu_2, \ldots, \nu_\mu) = 1,
\]

and \( \mathcal{I}(t, u) = \mathcal{I}(t', u') \) if and only if \( \nu_1 = \nu_1' \).
Proof. For \(\gamma_1, \gamma_2 \in \mathcal{I}(t, u)\), denote by \(\gamma_1 \ast \gamma_2 := \gamma(Q_{\gamma_1} \ast Q_{\gamma_2}, (t, u))\), where the product \(\ast\) in the right hand side is given by (2.3).

Fix \(\gamma' \in \mathcal{I}_i\). It is easy to see that \(\gamma \ast \gamma' \in \mathcal{I}_i\) for any \(\gamma \in \mathcal{I}_i\). When \(\gamma_1, \gamma_2 \in \mathcal{I}_i\) are not conjugate in \(\text{SL}_2(\mathcal{O})\), it is trivial that \(\gamma_1 \ast \gamma' \not\sim \gamma_2 \ast \gamma'\) in \(\text{PSL}_2(\mathcal{O})\). Then we have \(\mathcal{I}_i \supseteq \gamma' \ast \mathcal{I}_i\). Conversely, \(\gamma \ast \gamma' \in \mathcal{I}_i\) for any \(\gamma \in \mathcal{I}_i\). Then we can get \(\mathcal{I}_1 = \gamma' \ast \mathcal{I}_i\) similarly. This yields that \(\# \mathcal{I}_1 = \# \mathcal{I}_i\) for any \(1 \leq i \leq \mu\). Since \(\sum_{i=1}^{\mu} \# \mathcal{I}_i = h(d)\), we conclude that \(\# \mathcal{I}_i = h(d)/\mu\) for any \(1 \leq i \leq \mu\).

Consider the case of \(\mathcal{O} = \mathbb{Z}\). Note that, when \(\mathcal{O} = \mathbb{Z}\),

\[
\mathbb{Z}_p^{(2)} = \begin{cases} 
\{1\}, & p = 2, r = 1, \\
\{1, 3\}, & p = 2, r = 2, \\
\{1, 3, 5, 7\}, & p = 2, r \geq 3, \\
\{1, \eta\}, & p \geq 3,
\end{cases}
\]

where \(\eta \in (\mathbb{Z}/p\mathbb{Z})^\times\) is a non-quadratic residue of \(p\).

Lemma 2.3. Let \(p \geq 5\) be a prime number. If \(p \nmid \alpha\) then there exists \(l \in \mathbb{Z}/p\mathbb{Z}\) such that \(1 + \alpha l^2\) is a non-quadratic residue.

Proof. When \((\alpha/p) = 1\), it is easy to see that \(\{\alpha l^2 \mid l \in (\mathbb{Z}/p\mathbb{Z})^*\}\) coincides \(\{\beta \in (\mathbb{Z}/p\mathbb{Z})^* \mid (\beta/p) = 1\}\). Since \(#\{\beta \in (\mathbb{Z}/p\mathbb{Z})^* \mid (\beta/p) = 1\} = 1/2#(\mathbb{Z}/p\mathbb{Z})^*\), the claim of the lemma is trivial.

When \((\alpha/p) = -1\), then \(\{\alpha l^2 \mid l \in (\mathbb{Z}/p\mathbb{Z})^*\}\) \(= \{\beta \in (\mathbb{Z}/p\mathbb{Z})^* \mid (\beta/p) = -1\}\). Assume that \(1 + \alpha l^2\) is quadratic residue for any \(l \in (\mathbb{Z}/p\mathbb{Z})^*\), namely \(\{1 + \alpha l^2 \mid l \in (\mathbb{Z}/p\mathbb{Z})^*\} = \{\beta \in (\mathbb{Z}/p\mathbb{Z})^* \mid (\beta/p) = 1\}\). Since \(#\{\beta \in (\mathbb{Z}/p\mathbb{Z})^* \mid (\beta/p) = 1\} = #\{\beta \in (\mathbb{Z}/p\mathbb{Z})^* \mid (\beta/p) = -1\}\), it should hold that \((1/p) = (3/p) = \cdots = 1\), \((2/p) = (4/p) = \cdots = -1\) or \((1/p) = (3/p) = \cdots = -1\), \((2/p) = (4/p) = \cdots = 1\). However \((1/p) = (4/p) = 1\). This contradicts to the assumption. Therefore the claim holds.

Lemma 2.4. (i) When \(p = 2\) we have

if \(d \equiv 1 \mod 4\) then \(\gamma_1 \sim \gamma_\nu\) in \(\text{PSL}_2(\mathbb{Z}/2^r\mathbb{Z})\) for \(\nu = 3, 5, 7\),
if \(d \equiv 3 \mod 4\) then \(\gamma_1 \sim \gamma_5, \gamma_3 \sim \gamma_7, \gamma_3^3 \sim \gamma_3\) in \(\text{PSL}_2(\mathbb{Z}/2^r\mathbb{Z})\),
if \(d \equiv 2 \mod 8\) then \(\gamma_1 \sim \gamma_7, \gamma_3 \sim \gamma_5, \gamma_1^3 \sim \gamma_3\) and \(\gamma_3^3 \sim \gamma_1\) in \(\text{PSL}_2(\mathbb{Z}/2^r\mathbb{Z})\),
if \(d \equiv 6 \mod 8\) then \(\gamma_1 \sim \gamma_7, \gamma_3 \sim \gamma_5, \gamma_1^5 \sim \gamma_5\) and \(\gamma_5^3 \sim \gamma_1\) in \(\text{PSL}_2(\mathbb{Z}/2^r\mathbb{Z})\),
if \(d \equiv 4 \mod 8\) then \(\gamma_1 \sim \gamma_7, \gamma_3 \sim \gamma_5, \gamma_3^3 \sim \gamma_3\) and \(\gamma_3^3 \sim \gamma_1\) in \(\text{PSL}_2(\mathbb{Z}/2^r\mathbb{Z})\),
if \(d \equiv 0 \mod 8\) then \(\gamma_1^\nu \sim \gamma_\nu\) and \(\gamma_\nu^\nu \sim \gamma_1\) in \(\text{PSL}_2(\mathbb{Z}/2^r\mathbb{Z})\) for \(\nu = 3, 5, 7\),

where \(d = D\) if the square free factor of \(D\) is equivalent to 1 modulo 4 and \(d = D/4\) otherwise.

(ii) When \(p \geq 3\) we have

if \(p \nmid D\) then \(\gamma_1 \sim \gamma_\eta\) in \(\text{PSL}_2(\mathbb{Z}/p^r\mathbb{Z})\),
if \(p \mid D\) then \(\gamma_1^\eta \sim \gamma_\eta\) and \(\gamma_\eta^\mu \sim \gamma_1\) (\(\eta \mu \equiv 1 \mod p^r\)) in \(\text{PSL}_2(\mathbb{Z}/p^r\mathbb{Z})\).
Proof. When \( p = 2 \), let

\[
g_{i} := \begin{pmatrix} 1 & 0 \\ i & 1 \end{pmatrix}.
\]

We have

\[
\begin{aligned}
g_{1}^{-1}g_{1}/u_2 &\equiv \begin{cases} 6 & d \equiv 2, \\ 5 & d \equiv 4, \\ 3 & d \equiv 6, \end{cases} \\
g_{2}^{-1}g_{2}/u_2 &\equiv \begin{cases} 3 & d \equiv 1, \\ 5 & d \equiv 3, \\ 7 & d \equiv 5, \end{cases}
\end{aligned}
\]

\[
\begin{aligned}
g_{4}^{-1}g_{4}/u_2 &\equiv \begin{cases} 3 & d \equiv 5, \\ 5 & d \equiv 1, \end{cases} \\
g_{6}^{-1}g_{6}/u_2 &\equiv \begin{cases} 5 & d \equiv 5, \\ 7 & d \equiv 1. \end{cases}
\end{aligned}
\]

Then the relations \( \gamma_1 \sim \gamma_\nu \) in (i) of the lemma are true.

When \( p = 3 \) and \( 3 \nmid D_\gamma \), we have \((g_{1}^{-1}g_{1})_{21}/u_2 \equiv 2u_\gamma \mod 3^r\), where

\[
g := \begin{cases}
(0, -(D/8)^{-1/2}) & D_\gamma \equiv 1 \mod 3, \\
(2 + D/4)^{1/2} & D_\gamma \equiv 2 \mod 3.
\end{cases}
\]

Then we see that \( \gamma_1 \sim \gamma_2 \) in \( PSL_2(\mathbb{Z}/3^r\mathbb{Z}) \) if \( 3 \nmid D \).

When \( p \geq 5 \), according to Lemma 3.3, we see that if \( p \nmid \alpha \) the equation \( x^2 - \alpha y^2 \equiv \eta \mod p^r \) has a solution such that \( p \nmid x \) or \( p \nmid y \). For \( p \nmid D_\gamma \), let \((l, m) \in (\mathbb{Z}/p^r\mathbb{Z})^2\) be the solution of \( x^2 - 4^{-1}Dy^2 \equiv \eta \mod p^r \) and

\[
g := \begin{pmatrix} l \\ m \end{pmatrix}.
\]

Then we have \((g_{1}^{-1}g_{1})_{21}/u_2 \equiv \eta \mod p^r \). This means that \( \gamma_1 \) is conjugate to \( \gamma_\eta \) in \( PSL_2(\mathbb{Z}/p^r\mathbb{Z}) \).

By virtue of 4. of Fact 2.1, we have

\[
\frac{(\gamma_{1}^{\nu})_{21}}{u} = u^{\nu} \frac{(t + u\sqrt{D})^\nu - (t - u\sqrt{D})^\nu}{2u\sqrt{d}} = \sum_{l=0}^{[\nu/2]} \binom{\nu}{2l+1} t^{\nu-1} \left( \frac{Du^2}{Du^2 + 4} \right)^l.
\]

When \( \nu = 3 \), we have

\[
\frac{(\gamma_{1}^{3})_{21}}{u} = \binom{3}{2} \left( \nu + \frac{Du^2}{Du^2 + 4} \right) \equiv 3 \mod 8
\]

for \( d \equiv 0, 2, 3, 4, 7 \mod 8 \). Then in these cases \( \nu_3^3 \sim \nu_3 \). Similarly we can see that \( \nu_1^5 \sim \nu_5 \) for \( d \equiv 6 \mod 8 \). When \( 8 \mid d \) or \( p \mid d \), we have

\[
\frac{(\gamma_{1}^{\nu})_{21}}{u} = \nu \binom{\nu}{2}^{-1} + \sum_{l=1}^{[\nu/2]} \binom{\nu}{2l+1} t^{\nu-1} \left( \frac{Du^2}{Du^2 + 4} \right)^l \equiv \nu \mod 8 \text{ or } p.
\]

Then \( \gamma_\nu \sim \gamma_\nu \) holds for \( p = 2, \nu = 3, 5, 7 \) and \( p \geq 3, \nu = \eta \). \( \square \)
3 Generalized Venkov-Zograf’s formula

In this section, we give a generalization of Venkov-Zograf’s formula [VZ] used in the proof of the main theorem.

Let $G$ be a connected non-compact semi-simple Lie group with finite center, and $K$ a maximal compact subgroup of $G$. We put $d = \dim (G/K)$. We denote by $\mathfrak{g}$, $\mathfrak{k}$ the Lie algebras of $G$, $K$ respectively and $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ a Cartan decomposition with respect to the Cartan involution $\theta$. Let $a_\mathfrak{p}$ be a maximal abelian subspace of $\mathfrak{p}$. Throughout this paper we assume that $\text{rank}(G) = 1$, namely $\dim a_\mathfrak{p} = 1$. We extend $a_\mathfrak{p}$ to a $\theta$-stable maximal abelian subalgebra $a$ of $\mathfrak{g}$, so that $a = a_\mathfrak{p} + a_\mathfrak{b}$, where $a_\mathfrak{p} = a \cap \mathfrak{p}$ and $a_\mathfrak{b} = a \cap \mathfrak{b}$. We put $A = \exp a$, $A_\mathfrak{p} = \exp a_\mathfrak{p}$ and $A_\mathfrak{b} = \exp a_\mathfrak{b}$.

We denote by $\mathfrak{g}^c$, $a^c$ the complexification of $\mathfrak{g}$, $a$ respectively. Let $\Phi$ be the set of roots of $(\mathfrak{g}^c, a^c)$, $\Phi^+$ the set of positive roots in $\Phi$, $P_+ = \{ \alpha \in \Phi^+ | \alpha \neq 0 \text{ on } a_\mathfrak{p} \}$, and $P_- = \Phi^+ - P_+$. We put $\rho = 1/2 \sum_{\alpha \in P_+} \alpha$. For $h \in A$ and linear form $\lambda$ on $a$, we denote by $\xi_{\lambda}$ the character of $a$ given by $\xi_{\lambda}(h) = \exp \lambda(\log h)$. Let $\Sigma$ be the set of restrictions to $a_\mathfrak{p}$ of the elements of $P_+$. Then the set $\Sigma$ is either of the form $\{ \beta \}$ or $\{ \beta, 2\beta \}$ with some element $\beta \in \Sigma$. We fix an element $H_0 \in a_\mathfrak{p}$ such that $\beta(H_0) = 1$, and put $\rho_0 = \rho(H_0)$.

Note that

$$2\rho_0 = \begin{cases} m - 1, & (G = SO(m, 1), m \geq 2), \\ 2m, & (G = SU(m, 1), m \geq 2), \\ 4m + 2, & (G = SP(m, 1), m \geq 1), \\ 16, & (G = F_4). \end{cases}$$

Let $\Gamma$ be a discrete subgroup of $G$ such that the volume of $X_\Gamma = \Gamma \setminus G/K$ is finite. We denote by $\text{Prim}(\Gamma)$ the set of primitive hyperbolic conjugacy classes of $\Gamma$, and $Z(\Gamma)$ the center of $\Gamma$. For $\gamma \in \text{Prim}(\Gamma)$, we denote by $h(\gamma)$ an element of $A$ which is conjugate to $\gamma$, and $h_\mathfrak{p}(\gamma)$, $h_\mathfrak{b}(\gamma)$ the elements of $A_\mathfrak{p}$, $A_\mathfrak{b}$ respectively such that $h(\gamma) = h_\mathfrak{p}(\gamma)h_\mathfrak{b}(\gamma)$. Let $N(\gamma)$ be the norm of $\gamma$ given by $N(\gamma) = \exp (\beta(\log (h_\mathfrak{p}(\gamma))))$.

For a finite dimensional representation $\chi$ of $\Gamma$, we define the Selberg zeta function as follows.

$$Z_\Gamma(s, \chi) := \prod_{\gamma \in \text{Prim}(\Gamma)} \prod_{\lambda \in \mathfrak{L}} \det(\text{Id} - \chi(\gamma)\xi_{\lambda}(h(\gamma))^{-1}N(\gamma)^{-s})^{m_\lambda} \quad \text{Res } > 2\rho_0,$$

where $L$ is the semi-lattice of linear forms on $a$ given by $L := \{ \sum_{i=1}^l m_i \alpha_i | \alpha_i \in P_+, m_i \in \mathbb{Z}_{\geq 0} \}$ and $m_\lambda$ is the number of distinct $(m_1, \cdots, m_l) \in (\mathbb{Z}_{\geq 0})^l$ such that $\lambda = \sum_{i=1}^l m_i \alpha_i \in L$. We note that the logarithmic derivative of the above is written as follows.

$$\frac{Z'_\Gamma(s, \chi)}{Z_\Gamma(s, \chi)} = \sum_{\gamma \in \text{Prim}(\Gamma), k \geq 1} \text{tr} \chi(\gamma^k) \log N(\gamma)D(\gamma^k)^{-1}N(\gamma)^{-ks} \quad \text{Res } > 2\rho_0, \quad (3.1)$$

where $D(\gamma) := \prod_{\alpha \in P_+} [1 - \xi_\alpha(h(\gamma))^{-1}]$.

We now generalize the Venkov-Zograf formula [VZ] for two dimensional case as follows.
**Lemma 3.1.** Let $\Gamma$ be a discrete subgroup of $G$ such that the volume of $\Gamma \backslash G/K$ is finite and $\Gamma'$ a subgroup of $\Gamma$ with the finite index. Then, for a finite dimensional representation $\chi$ of $\Gamma'$, we have

$$Z_{\Gamma'}(s, \chi) = Z_{\Gamma}(s, \text{Ind}_{\Gamma'}^{\Gamma} \chi),$$

where $\text{Ind}_{\Gamma'}^{\Gamma} \chi$ is a representation of $\Gamma$ induced by $\chi$.

**Proof.** It is easy to see that, for $\gamma \in \Gamma$, there exist positive integers $m_1, \ldots, m_k$ and a complete system of representatives $\{a_j^{(i)}\}$ of $\Gamma/\Gamma'$ such that $m_1 \geq \cdots \geq m_k$, $\sum_{i=1}^k m_i = [\Gamma : \Gamma']$ and $(a_j^{(i)})^{-1} \gamma a_{j+1}^{(i)}$ for $m_i \geq 2$, $1 \leq j \leq m_i - 1$ and $(a_m^{(i)})^{-1} \gamma a_1^{(i)}$ are in $\Gamma'$. Then by the definition of the induced representation, we see that

$$\text{Ind}_{\Gamma'}^{\Gamma} \chi(\gamma) \sim \begin{pmatrix} S_{m_1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & S_{m_k} \end{pmatrix} 	ag{3.2}$$

where $S_{m_i}$ is given by

$$S_{m_i} = \left\{ \begin{array}{ll}
\chi(a_1^{(i)} \gamma a_1^{(i)}) & (m_i = 1), \\
0 & \\
\chi(a_1^{(i)} \gamma a_2^{(i)}) & \\
\vdots & \\
0 & \\
\chi(a_m^{(i)} \gamma a_1^{(i)}) & (m_i \geq 2).
\end{array} \right.$$  

Then we have

$$Z_{\Gamma}(s, \text{Ind}_{\Gamma'}^{\Gamma} \chi) = \prod_{\gamma \in \text{Prim}(\Gamma)} \prod_{\lambda \in L} \det(\text{Id} - \text{Ind}_{\Gamma'}^{\Gamma} \chi(\gamma) \xi_\lambda(h(\gamma))^{-1} N(\gamma)^{-s})^{m_\lambda}$$

$$= \prod_{\gamma \in \text{Prim}(\Gamma)} \prod_{\lambda \in L} \prod_{i=1}^k \det(\text{Id} - \chi(a_1^{(i)} \gamma \gamma^{m_i} a_1^{(i)}) \xi_\lambda(h(\gamma^{m_i}))^{-1} N(\gamma^{m_i})^{-s})^{m_\lambda}.$$

According to Lemma 2.3 of [HW], we see that $(\text{Ind}_{\Gamma'}^{\Gamma} \chi(\gamma))$ is expressed as (3.2) if and only if $a_1^{(i)} \gamma m_i a_1^{(i)}$ $(1 \leq i \leq k)$ are primitive and are not conjugate to each other in $\Gamma'$. This means that

$$\{a_1^{(i)} \gamma m_i a_1^{(i)} | \gamma \in \text{Prim}(\Gamma), 1 \leq i \leq k\} = \text{Prim}(\Gamma').$$

Then we have

$$Z_{\Gamma}(s, \text{Ind}_{\Gamma'}^{\Gamma} \chi) = \prod_{\gamma \in \text{Prim}(\Gamma')} \prod_{\lambda \in L} \det(\text{Id} - \chi(\gamma') \xi_\lambda(h(\gamma'))^{-1} N(\gamma')^{-s})^{m_\lambda} = Z_{\Gamma'}(s, \chi).$$

This completes the proof. \qed
Selberg’s zeta functions for congruence subgroups

We furthermore prepare the following two lemmas.

Lemma 3.2. (see [HW]) Let \( \Gamma, \Gamma_1, \Gamma_2 \) be discrete subgroups of \( G \) such that \( \text{vol}(\Gamma \backslash G/K) < \infty \), \( \Gamma_1, \Gamma_2 \subset \Gamma \), \( \Gamma_1 \Gamma_2 = \Gamma \) and \( [\Gamma : \Gamma_1 \cap \Gamma_2] < \infty \). Then, for any \( \gamma \in \Gamma \), we have
\[
(\text{Ind}_{\Gamma_1}^{\Gamma_2} 1)(\gamma) = (\text{Ind}_{\Gamma_1}^{\Gamma_2} 1)(\gamma) \otimes (\text{Ind}_{\Gamma_1}^{\Gamma_2} 1)(\gamma).
\]

Lemma 3.3. Let \( \Gamma \) be a congruence subgroup of level \( N = \prod p^r \). Then we have \( \Gamma = \cap p \Gamma_p \) for some \( \Gamma_p \supset \hat{\Gamma}(p^r) \).

Proof. Put \( \Gamma_p = \hat{\Gamma}(p^r) \). It is easy to check that \( \cap p \Gamma_p = \Gamma \). \qed

4 Main theorem and its proof

In the case of \( G = \text{SO}(2,1) \cong \text{SL}_2(\mathbb{R}) \) and \( G = \text{SO}(3,1) \cong \text{SL}_2(\mathbb{C}) \), the Selberg zeta function is written as follows.

For \( \gamma \in \text{Prim}(\Gamma) \), let \( |a(\gamma)| > 1 \) and \( m(\gamma) \) be the order of the torsion of the centralizer of \( \gamma \). Note that \( |a(\gamma)|^2 = N(\gamma) \).

It is known that \( \Gamma = \text{SL}_2(\mathbb{Z}) \) and \( \text{SL}_2(\mathcal{O}) \) where \( \mathcal{O} \) is the ring of integers of \( \mathbb{Q}(\sqrt{-M}) \) and \( M > 0 \) is not square are discrete subgroups of \( G = \text{SL}_2(\mathbb{R}) \) and \( \text{SL}_2(\mathbb{C}) \) respectively such that the volume of \( \Gamma \backslash G/K \) is finite.

Let
\[
\mathfrak{D}_K = \mathfrak{D} := \begin{cases}
\{ D \in \mathbb{Z}_{>0} \mid D \text{ is not a square and } D \equiv 0, 1 \mod 4 \} & K = \mathbb{Q}, \\
\{ D \in \mathcal{O} \mid D \text{ is not a square and } D \equiv 3a^2 \mod 4 \} & K = \mathbb{Q}(\sqrt{-M}).
\end{cases}
\]

For \( D \in \mathfrak{D}_K \), we see that \( P(D) \cong (\mathbb{Z}/|D|) \times \mathbb{Z} \). We call that \( \epsilon(D) := (t_1 + u_1 \sqrt{D})/2 \) by the fundamental unit of \( D \). Then the primitive element \( \gamma \) of \( \text{SL}_2(\mathcal{O}) \) is written as \([2,1]\) for \( (t, u) = (t_1, u_1) \). For such a \( \gamma \), we see that \( a(\gamma) = \epsilon(D) \) and \( N(\gamma) = |\epsilon(D)|^2 \). Since \( m(\gamma) = l \), by using \( (t, u) \in P(D) \) with \( 0 \leq \arg \epsilon(D) \leq \pi/2 \), we see that the Selberg zeta function for \( \Gamma = \text{SL}_2(\mathcal{O}) \) as follows (see [Sa1] and [Sa2]).

\[
Z_{\text{SL}_2(\mathbb{Z})}(s) = \prod_{d \in \mathfrak{D}_K} \prod_{n=0}^{\infty} (1 - \epsilon(D)^{-2(s+n)})^{h(D)} \quad \text{Res} > 1,
\]

\[
Z_{\text{SL}_2(\mathcal{O})}(s) = \prod_{D \in \mathfrak{D}_K} \prod_{m,n=0}^{\infty} (1 - \epsilon(D)^{-2m})^{h(D)} \quad \text{Res} > 2.
\]

Let \( \mathfrak{a} \) be an ideal in \( \mathcal{O} \). We call that \( \Gamma \subset \text{SL}_2(\mathcal{O}) \) is a congruence subgroup of \( \text{SL}_2(\mathcal{O}) \) with the level \( \mathfrak{a} \) if \( \Gamma \supset \hat{\Gamma}(\mathfrak{a}) \) and \( \Gamma \not\supset \hat{\Gamma}(\mathfrak{a}') \) for any \( \mathfrak{a}' \mid \mathfrak{a} \). The main result of the present paper is as follows.
Theorem 4.1. Let $\mathcal{O}$ be the integer ring of $\mathbb{Q}$ or its imaginary quadratic extension, $a$ an ideal in $\mathcal{O}$ and $\Gamma$ a congruence subgroup of $\text{SL}_2(\mathcal{O})$ of level $a$. Then the following formulas hold:

(i) The case of $\mathcal{O} = \mathbb{Z}$,

$$Z_\Gamma(s) = \prod_{D \in \mathfrak{D}} \prod_{n=0}^{\infty} \det(I - \chi_\Gamma(\gamma_1)\epsilon(D)^{-2(s+n)}h(D)),$$

$$\frac{Z_\Gamma'(s)}{Z_\Gamma(s)} = \sum_{D \in \mathfrak{D}} \sum_{j \geq 1} \text{tr} \chi_\Gamma(\gamma_1)h(D) \frac{2\log \epsilon(D)}{1 - \epsilon(D)^{-2j}} \epsilon(D)^{-2js}.$$

(ii) The case of $\mathcal{O} \subset \mathbb{Q}(\sqrt{-M})$,

$$Z_\Gamma(s) = \prod_{D \in \mathfrak{D}} \prod_{n=0}^{\infty} \prod_{\nu \in \mathcal{L}_a(t_1(D), u_1(D))} \det(I - \chi_\Gamma(\gamma_\nu)\epsilon(D)^{-2m\epsilon(D)^{-n}}\epsilon(D)^{-2s}h(D)/\mu_a(D),$$

$$\frac{Z_\Gamma'(s)}{Z_\Gamma(s)} = \sum_{D \in \mathfrak{D}} \sum_{j \geq 1} \sum_{\nu \in \mathcal{L}_a(t_1(D), u_1(D))} \text{tr} \chi_\Gamma(\gamma_\nu)h(D) \frac{2\log \epsilon(D)}{\mu_a(D) |1 - \epsilon(D)^{-2j}|^2} \epsilon(D)^{-2js}.$$ 

where $\gamma_\nu$ and $\mathcal{L}_a(t, u)$ are respectively defined in Lemma 2.7 and 2.2.

Proof. Since the expressions of the logarithmic derivatives of $Z_\Gamma(s)$ can be obtained easily from those of $Z_\Gamma(s)$, we treat only $Z_\Gamma(s)$.

According to Lemma 3.1, we have

$$Z_\Gamma(s) = Z_{\text{SL}_2(\mathcal{O})}(s, \chi_\Gamma)$$

$$= \prod_{\gamma \in \text{prim} (\text{SL}_2(\mathcal{O}))} \prod_{\lambda \in \mathcal{L}} \det(I - \chi_\Gamma(\gamma)\xi_\lambda(h(\gamma))^{-1}N(\gamma)^{-s})^{\mu_a(D)}$$

$$= \prod_{D \in \mathfrak{D}} \prod_{\lambda \in \mathcal{L}} \prod_{\gamma \in \mathcal{I}(t_1(D), u_1(D))} \det(I - \chi_\Gamma(\gamma)\xi_\lambda(h(\gamma))^{-1}N(\gamma)^{-s})^{\mu_a(D)}$$

where $\chi_\Gamma := \text{Ind}_{\text{SL}_2(\mathbb{Z})}\chi$. Since $\gamma \sim \gamma_\nu$ in $\text{PSL}_2(\mathcal{O}/a)$ and $\chi_\Gamma \sim \text{Ind}_{\Gamma/\Gamma(a)}1$, we see that $\chi_\Gamma(\gamma) \sim \chi_\Gamma(\gamma_\nu)$. Then, from Lemma 2.2, we have

$$Z_\Gamma(s) = \begin{cases} 
\prod_{D \in \mathfrak{D}} \prod_{n=0}^{\infty} \prod_{i=1}^{\mu_a(D)} \det(I - \chi(\gamma_\nu_i)\epsilon(D)^{-2(s+n)}h(D)/\mu_a(D), & \mathcal{O} = \mathbb{Z}, \\
\prod_{D \in \mathfrak{D}} \prod_{m,n=0}^{\infty} \prod_{i=1}^{\mu_a(D)} \det(I - \chi(\gamma_\nu_i)\epsilon(D)^{-m\epsilon(D)^{-n}}\epsilon(D)^{-2(s+n)}h(D)/\mu_a(D), & \mathcal{O} \subset \mathbb{Q}(\sqrt{-M}). 
\end{cases}$$
This completes the proof for imaginary quadratic \( O \).

Consider the case of \( O = \mathbb{Z} \). When \( \Gamma \supset \hat{\Gamma}(p^r) \), by virtue of Lemma 2.4 we see that \( \gamma_1^{l_i} \sim \gamma_{\nu_i} \) and \( \gamma_{\nu_i}^{m_i} \sim \gamma_1 \) in \( \text{PSL}_2(\mathbb{Z}/p^r\mathbb{Z}) \) for some integers \( l_i, m_i \geq 1 \). Then we can get \( \text{tr}\chi_\Gamma(\gamma_1) = \text{tr}\chi_\Gamma(\gamma_{\nu_i}) \). Similarly we have \( \text{tr}\chi_\Gamma(\gamma_1^{k}) = \text{tr}\chi_\Gamma(\gamma_{\nu_i}^{k}) \) for any \( k \). This means that \( \chi_\Gamma(\gamma_1) \) and \( \chi_\Gamma(\gamma_{\nu_i}) \) give the same type of permutations. Then we have \( \chi_\Gamma(\gamma_1) \sim \chi_\Gamma(\gamma_{\nu_i}) \).

Furthermore when \( \Gamma \supset \hat{\Gamma}(N) \) with \( N = \prod_{p|N} p^r \), we see that \( \chi_\Gamma(\gamma) \sim \otimes_{p|N}\chi_\Gamma(p^r) \) from Lemma 3.2 and 3.3. Then \( \chi_\Gamma(\gamma_1) \sim \chi_\Gamma(\gamma_{\nu_i}) \) holds for any congruence subgroup of \( \text{SL}_2(\mathbb{Z}) \). Therefore we get

\[
Z_\Gamma(s) = \prod_{D \in \mathcal{D}} \prod_{n=0}^{\infty} \det(I - \chi_\Gamma(\gamma_1)\epsilon(D)^{-2(s+n)})^{h(D)}.
\]

This completes the proof of the main theorem for \( Z_\Gamma(s) \). Getting the expressions for \( Z'_\Gamma(s)/Z_\Gamma(s) \) is not difficult. \( \square \)

5 Examples

In this section, we give two examples. For convenience, we put

\[
H(x; (m_1^{n_1}, \ldots, m_l^{n_l})) := \prod_{i=1}^{l} (1 - x^{m_i})^{n_i},
\]

where \( m_1, \ldots, m_l \) and \( n_1, \ldots, n_l \) are positive integers. Then Theorem 4.1 and some elementary calculations give the following expressions of the Selberg zeta functions.

Example 1. In the case of \( \Gamma = \Gamma_0(p) := \{ \gamma \in \text{SL}_2(\mathbb{Z}) \mid p \mid \gamma_{21} \} \) with prime \( p \geq 3 \), we have

\[
Z_{\Gamma_0(p)}(s) = \prod_{D \in \mathcal{D}} \prod_{p|D} \prod_{n=0}^{\infty} H\left(\epsilon(D)^{-2(s+n)}; (1^{p+1})\right)^{h(D)}
\]

\[
\times \prod_{(1<l_1)(p-1)/2} \prod_{D \in \mathcal{D}} \prod_{p|D} \prod_{n=0}^{\infty} H\left(\epsilon(D)^{-2(s+n)}; (1^2, l_1^{(p-1)/l_1})\right)^{h(D)}
\]

\[
\times \prod_{D \in \mathcal{D}} \prod_{p|D} \prod_{n=0}^{\infty} H\left(\epsilon(D)^{-2(s+n)}; (1, p)\right)^{h(D)}
\]

\[
\times \prod_{(1<l_2)(p+1)/2} \prod_{D \in \mathcal{D}} \prod_{p|D} \prod_{n=0}^{\infty} H\left(\epsilon(D)^{-2(s+n)}; (l_2^{(p+1)/l_2})\right)^{h(D)}.
\]
\[
\frac{Z'_{\Gamma_0(p)}(s)}{Z_{\Gamma_0(p)}(s)} = (p + 1) \sum_{D \in \mathcal{D}, j \geq 1} \frac{1}{p|u_j} h(D) \frac{2 \log \varepsilon(D)}{1 - \varepsilon(D)^{-2j}} \varepsilon(D)^{-2js}
\]

\[
+ 2 \sum_{D \in \mathcal{D}, j \geq 1} \frac{1}{p|u_j, (D/p)=1} h(D) \frac{2 \log \varepsilon(D)}{1 - \varepsilon(D)^{-2j}} \varepsilon(D)^{-2js}.
\]

**Example 2.** In the case of \( \Gamma = \Gamma_0(p) := \{ \gamma \in \text{SL}_2(\mathcal{O}) \mid \gamma_{21} \equiv 0 \mod p \} \) with prime ideal \( p \) such that \( p \) does not divide 2 and the discriminant of \( \mathcal{O} \), we have

\[
Z_{\Gamma_0(p)}(s) = \prod_{D \in \mathcal{D}} \prod_{n,m=0}^{\infty} H\left(\varepsilon(D)^{-2m} \varepsilon(D)^{-2n} \mid \varepsilon(D) \mid^{-2s}; \left(1, p^2 + 1\right)\right)^{h(D)}
\]

\[
\times \prod_{(1)<l_1|(p^2-1)/2} \prod_{D \in \mathcal{D}} \prod_{p|u_1, p|u_1, (D/p)=1}^{\infty} H\left(\varepsilon(D)^{-2m} \varepsilon(D)^{-2n} \mid \varepsilon(D) \mid^{-2s}; \left(1, p^{l_1}\right)\right)^{h(D)}
\]

\[
\times \prod_{(1)<l_2|(p^2+1)/2} \prod_{D \in \mathcal{D}} \prod_{p|u_1, p|u_2, (D/p)=-1}^{\infty} H\left(\varepsilon(D)^{-2m} \varepsilon(D)^{-2n} \mid \varepsilon(D) \mid^{-2s}; \left(1, p^{l_2}\right)\right)^{h(D)},
\]

\[
\frac{Z'_{\Gamma_0(p)}(s)}{Z_{\Gamma_0(p)}(s)} = (p^2 + 1) \sum_{D \in \mathcal{D}, j \geq 1} \frac{1}{p|u_j} h(D) \frac{2 \log \varepsilon(D)}{|1 - \varepsilon(D)^{-2j}|^2} \varepsilon(D)^{-2js},
\]

\[
+ 2 \sum_{D \in \mathcal{D}, j \geq 1} \frac{1}{p|u_j, (D/p)=1} h(D) \frac{2 \log \varepsilon(D)}{|1 - \varepsilon(D)^{-2j}|^2} \varepsilon(D)^{-2js},
\]

\[
+ \sum_{D \in \mathcal{D}, j \geq 1} h(D) \frac{2 \log \varepsilon(D)}{|1 - \varepsilon(D)^{-2j}|^2} \varepsilon(D)^{-2js}.
\]

**References**

[BLS] E. Bogomolny, F. Leyvraz and C. Schmit, *Distribution of eigenvalues for the modular group*, Commun. Math. Phys. **176** (1996), 575–617.

[Ga] R. Gangolli, *Zeta functions of Selberg’s type for compact space forms of symmetric spaces of rank one*, Illinois J. Math. **21** (1977), 1–41.
Selberg’s zeta functions for congruence subgroups

[GW] R. Gangolli and G. Warner, Zeta functions of Selberg’s type for some noncompact quotients of symmetric spaces of rank one, Nagoya Math. J. 78 (1980), 1-44.

[G] C. F. Gauss, Disquisitiones arithmeticae, Fleischer, Leipzig, (1801).

[H1] Y. Hashimoto, Arithmetic expressions of Selberg’s zeta functions for congruence subgroups, J. Number Theory, 122 (2007), 324–335.

[H2] Y. Hashimoto, Distributions of length multiplicities for negatively curved locally symmetric Riemannian manifolds, arXiv.math.SP/0701239.

[H3] Y. Hashimoto, Asymptotic formulas for partial sums of class numbers of indefinite binary quadratic forms, arXiv.math/0807.0056.

[HW] Y. Hashimoto and M. Wakayama, Splitting density for lifting about discrete groups, Tohoku Math. J, 59 (2007), 527–545.

[He] D. Hejhal, The Selberg trace formula of PSL(2, R) I, II, Lec. Notes in Math. 548, 1001 Springer-Verlag, (1976, 1983).

[Pe] M. Peter, The correlation between multiplicities of closed geodesics on the modular surface, Commun. Math. Phys. 225 (2002), 171–189.

[Sa1] P. Sarnak, Class numbers of indefinite binary quadratic forms I, II, J. Number Theory, 15 (1982), 229-247 and 21 (1985), 333–346.

[Sa2] P. Sarnak, The arithmetic and geometry of some hyperbolic three-manifolds, Acta Math. 151 (1983), 253–295.

[Si] C. L. Siegel, The average measure of quadratic forms with given determinant and signature, Ann. of Math. II, 45 (1944), 667-685.

[VZ] A. B. Venkov and P. G. Zograf, Analogues of Artin’s factorization formulas in the spectral theory of automorphic functions associated with induced representations of Fuchsian groups, Izv. Akad. Nauk SSSR Ser. Mat. 46 (1982), 1150–1158, 1343 (Russian), Math. USSR-Izv. 21 (1983), 435–443 (English translation).

HASHIMOTO, Yasufumi
Institute of Systems and Information Technologies/KYUSHU,
7F 2-1-22, Momochihama, Fukuoka 814-0001, JAPAN
e-mail:hasimoto@isit.or.jp