ON THE $f$-VECTORS OF $r$-MULTICHAIN SUBDIVISIONS

SHAHEEN NAZIR

Abstract. For a poset $P$ and an integer $r \geq 1$, let $P_r$ be a collection of all $r$-multichains in $P$. Corresponding to each strictly increasing map $i : [r] \to [2r]$, there is an order $\preceq_i$ on $P_r$. Let $\Delta(G_i(P_r))$ be the clique complex of the graph $G_i$ associated to $P_r$ and $i$. In a recent paper [NW21], it is shown that $\Delta(G_i(P_r))$ is a subdivision of $P$ for a class of strictly increasing maps. In this paper, we show that all these subdivisions have the same $f$-vector. We give an explicit description of the transformation matrices from the $f$- and $h$-vectors of $\Delta$ to the $f$- and $h$-vectors of these subdivisions when $P$ is a poset of faces of $\Delta$. We study two important subdivisions Cheeger-Müller-Schrader’s subdivision and the $r$-colored barycentric subdivision which fall in our class of $r$-multichain subdivisions.

1. Introduction

Stanley laid a foundation for the enumerative theory of subdivisions of simplicial complexes in [Sta92]. His goal was to understand the behavior of the $h$-polynomial under iterated subdivisions. In recent years, a lot of studies has been done continuing the Stanley’s program for important classes of subdivisions, e.g., barycentric subdivisions in [BW08], edgewise subdivisions in [Joc18], interval subdivisions in [AN20a], antiprism subdivisions in [ABJK22], and uniform subdivisions in [Ath20]. All this enumerative study began with the work of Brenti and Welker [BW08] on barycentric subdivisions. They studied the transformation matrix of the $h$-vector of a simplicial complex under the barycentric subdivision. They proved that the $h$-polynomial of the barycentric subdivision of a simplicial complex with non-negative $h$-vector is real-rooted. Recently, Athanasiadis in [Ath20] investigated the entries of the transformation matrix of the $h$-vector of a simplicial complex under the $r$-colored barycentric subdivision. He described them in terms of $r$-colored Eulerian numbers. He also showed that the $h$-polynomial of the $r$-colored barycentric subdivision of a simplicial complex with non-negative $h$-vector is real-rooted.

Let $P$ be a poset with order relation $\leq$. For a non-negative integer $r$, an $r$-multichain $p : p_1 \leq \cdots \leq p_r$ in $P$ is a monotonicly increasing sequence of elements in $P$ of length $r$. We consider the set $P_r$ of all $r$-multichains in $P$. If $r = 1$ then $P_r = P$ and the order complex $\Delta(P)$ of all linearly ordered subsets of $P$ together with its geometric realization are well studied geometric and topological objects. They have been shown to encode crucial information about $P$ and have important applications in combinatorics and many other fields in mathematics (see e.g. [Wac06]). For every strictly monotone map $i : [r] \to [2r]$, define a binary relation $\preceq_i$ on $P_r$. For $p : p_1 \leq \cdots \leq p_r$ and $q : q_1 \leq \cdots \leq q_r$ we set:

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\[
p \preceq q : \iff \begin{cases} p_t \geq q_s, & \text{for } s \leq \iota(t) - t; \\ p_t \leq q_s, & \text{for } s > \iota(t) - t. \end{cases}
\]
for \(p, q \in P_r\). Here for a natural number \(n\) we write \([n]\) for \(\{1, \ldots, n\}\). Through the undirected graph \(G_\iota(P_r) = (P_r, E)\) with edge set
\[E = \{\{p, q\} \subseteq P_r : p \preceq q \text{ and } p \neq q\}\]
we associate to \(P_r\) and \(\iota\) the clique complex \(\Delta(G_\iota(P_r))\) of \(G_\iota(P_r)\); that is the simplicial complex of all subsets \(A \subseteq P_r\) which form a clique in \(G_\iota(P_r)\).

\textbf{Theorem 1.1.} \,[NW21, Theorem 1.1] \textit{For } \(r \geq 2\), the following are equivalent.

- The relation \(\preceq\) is reflexive;
- The map \(\iota\) satisfies the condition that \(\iota(t) \in \{2t - 1, 2t\}\) for all \(1 \leq t \leq r\).
- The complex \(\Delta(G_\iota(P_r))\) is a subdivision of \(\Delta(P)\).

It is also shown in \,[NW21\] that all subdivisions mentioned in Theorem 1.1 are non-isomorphic. It arises a natural question whether these subdivisions have the same face enumeration or not. We answer this question affirmatively in Theorem 1.2.

\textbf{Theorem 1.2.} Let \(\mathcal{I}\) be the collection of all strictly increasing maps \(\iota : [r] \to [2r]\) such that \(\iota(1) = 1\) and \(\preceq_\iota\) is reflexive. Then the \(f\)-vector of the clique complex \(\Delta(G_\iota(P_r))\) is the same for all \(\iota \in \mathcal{I}\).

We give explicit formulae for the transformation matrix of the \(f\)-vector under these multichain subdivisions of a simplicial complex. It is shown that the entries of the transformation matrix of the \(h\)-vector of the \(r\)-multichain subdivisions are given in terms of the descent numbers of the \(r\)-colored permutations. On the way, we formulate some interesting recurrence relations between the \(r\)-colored Eulerian polynomials. Using these relations and \,[SV15, Theorem 2.3\], we derive the real-rootedness of the \(h\)-polynomial of these chain subdivisions(also given in \,[Ath20, Proposition 7.5\]).

We also investigate two special cases of \(r\)-multichain subdivisions. We call the clique complex \(\Delta(G_\iota(P_r))\) an \(r\)-multichain subdivision of type I of \(\Delta(P)\) and denote it by \(\Delta(G_\iota(P_r))\) if \(\iota\) is defined as \(\iota(t) = 2t - 1\) for all \(1 \leq t \leq r\). For this \(\iota\), the relation \(\preceq_\iota\) is denoted as \(\preceq_{\iota I}\). We call the clique complex \(\Delta(G_\iota(P_r))\) an \(r\)-multichain subdivision of type II of \(\Delta(P)\) and denote it by \(\Delta(G_{\iota II}(P_r))\) when \(\iota\) is defined as \(\iota(t) = 2t\), for \(t\) even; \(\iota(t) = 2t - 1\), for \(t\) odd. In this case, the relation \(\preceq_\iota\) is denoted as \(\preceq_{\iota II}\).

The main motivation to study these two chain subdivisions is that it leads us two important geometric subdivisions. One of them is a generalization of the interval subdivision introduced by Walker \,[Wal88\]. In fact, the interval subdivision is a special case of a subdivision described by Cheeger-Müller-Schrader in \,[CMS84\] for \(N = 1\). The other subdivision is the \(r\)-colored barycentric subdivision(the \(r\)-edgewise subdivision of the barycentric subdivision). We give a combinatorial equivalence of these subdivisions (CMS and \(r\)-colored barycentric) in terms of the \(r\)-multichain subdivisions. These connections also lead us to answer a couple of questions posed by Mohammadi and Welker in \,[BGScC17\].
The paper is organized as follows. In the second section, we provide some background about simplicial complexes and related key words. We recall some important subdivisions, e.g., barycentric, \(r\)-edgewise, \(r\)-colored barycentric, CMS’s subdivisions. In Section 3, the \(r\)-colored Eulerian polynomials are defined along with underlined recurrence relations. Furthermore, it is shown that these polynomials are real-rooted. We give some combinatorial description of the \(\gamma\)-coefficients of the symmetric \(r\)-colored Eulerian polynomials. In Section 4, we prove the main theorem that the \(f\)-vector of the clique complex \(\Delta(G_i(P_r))\) of \(G_i(P_r)\) does not depend on \(i\) when the relation \(\preceq_i\) is reflexive. We also describe the transformation of the \(f\)- and \(h\)-vectors under these chain subdivisions of a simplicial complex and show that every \(r\)-multichain subdivision of a Cohen-Macaulay simplicial complex has the real-rooted \(h\)-vector. In the last section, we discuss the connection between the \(r\)-multichain subdivisions with other well-known subdivisions. In Proposition 5.1, we show that for even values of \(r\), the \(r\)-multichain subdivision of type I (defined in Section 1) gives a combinatorial description of the CMS subdivision. In Proposition 5.2, we show that the \(r\)-multichain subdivision of type II (defined in Section 1) is isomorphic to the \(r\)-colored barycentric subdivision.

2. Preliminaries

2.1. Simplicial Complexes and Face Vectors: An abstract simplicial complex \(\Delta\) on a finite vertex set \(V\) is a collection of subsets of \(V\), such that \(\{v\} \in \Delta\) for all \(v \in V\), and if \(F \in \Delta\) and \(E \subseteq F\), then \(E \in \Delta\). The members of \(\Delta\) are known as faces. The dimension \(\dim(F)\) of a face \(F\) is \(|F| - 1\). Let \(d = \max\{|F| : F \in \Delta\}\) and define the dimension of \(\Delta\) to be \(\dim \Delta = d - 1\). For each \(F \in \Delta\), we denote \(2^F\) as the simplex with vertex set \(F\). One can associate to an abstract simplicial complex \(\Delta\) a topological space \(|\Delta|\) known as geometric realization of \(\Delta\) by taking the convex hull \(\operatorname{conv}(F)\) in some Euclidean space \(\mathbb{R}^m\).
for every face \( F \) in \( \Delta \). For more details, see [TOG17, Chapter 16].

The \( f \)-polynomial of a \((d - 1)\)-dimensional simplicial complex \( \Delta \) is defined as:

\[
f_{\Delta}(t) = \sum_{F \in \Delta} t^{\dim(F)+1} = \sum_{i=0}^{d} f_{i-1} t^i,
\]

where \( f_{i} \) is the number of faces of dimension \( i \). Note that \( \dim \emptyset = -1 \), therefore \( f_{-1} = 1 \).

The sequence \( f(\Delta) = (f_{-1}, f_0, \ldots, f_{d-1}) \) is called the \( f \)-vector of \( \Delta \). Define the \( h \)-vector \( h(\Delta) = (h_0, h_1, \ldots, h_d) \) of \( \Delta \) by the \( h \)-polynomial:

\[
h_{\Delta}(t) := (1-t)^d f_{\Delta}(t/(1-t)) = \sum_{i=0}^{d} h_i t^i.
\]

We say that two simplicial complexes \( \Delta \) and \( \Gamma \) on the vertex sets \( V \) and \( W \) are isomorphic if there is a bijection \( \theta : V \to W \) such that \( F \in \Delta \) iff \( \theta(F) \in \Gamma \).

### 2.2. Subdivisions

A **topological subdivision** of a simplicial complex \( \Delta \) is a (geometric) simplicial complex \( \Delta' \) with a map \( \theta : \Delta' \to \Delta \) such that, for any face \( F \in \Delta \), the following holds: (a) \( \Delta'_F := \theta^{-1}(2^F) \) is a subcomplex of \( \Delta' \) which is homeomorphic to a ball of dimension \( \dim(F) \); (b) the interior of \( \Delta'_F \) is equal to \( \theta^{-1}(F) \). The face \( \theta(G) \in \Delta \) is called the carrier of \( G \in \Delta' \). The subdivision \( \Delta' \) is called **quasi-geometric** if no face of \( \Delta' \) has the carriers of its vertices contained in a face of \( \Delta \) of smaller dimension. Moreover, \( \Delta' \) is called **geometric** if there exists a geometric realization of \( \Delta' \) which geometrically subdivides a geometric realization of \( \Delta \), in the way prescribed by \( \theta \).

Clearly, all geometric subdivisions (such as the barycentric, edgewise and chain subdivisions considered in this paper) are quasi-geometric. For more detail, we refer to [Sta92] and a survey by Athanasiadis [Ath16]. Moving forward, we recall some well-known subdivisions.

#### 2.2.1. The barycentric subdivision

Let \( \{v_1, \ldots, v_n\} \) be an affinely independent set of vectors in \( \mathbb{R}^d \). For \( \emptyset \neq A \subseteq \{v_1, \ldots, v_n\} \), let

\[
b_A := \frac{1}{|A|} \sum_{v \in A} v
\]

be the barycenter of the simplex \( \text{conv}(A) \). Then for any chain \( \emptyset \neq A_0 \subset A_1 \subset \cdots \subset A_k \) of subsets of \( \{v_1, \ldots, v_n\} \), let \( b_{A_0 \subset A_1 \subset \cdots \subset A_k} := \text{conv}(b_{A_0}, \ldots, b_{A_k}) \) be the convex hull.

Let \( \Delta_{d-1} \) be a geometric \( d-1 \)-simplex with the vertex set \( V = \{e_1, \ldots, e_d\} \) of the unit vectors in \( \mathbb{R}^d \). Then the set of simplices \( b_{A_0 \subset A_1 \subset \cdots \subset A_k} \) for chains \( \emptyset \neq A_0 \subset A_1 \subset \cdots \subset A_k \) of subsets in \( V \) defines a subdivision of \( \Delta_{d-1} \) which is called the barycentric subdivision, denoted by \( \text{sd}(\Delta_{d-1}) \), of \( \Delta_{d-1} \). In general, the barycentric subdivision \( \text{sd}(\Delta) \) is obtained from a simplicial complex \( \Delta \) by applying it to every simplex in \( \Delta \).

#### 2.2.2. The \( r \)th edgewise subdivision

Let \( \Delta \) be a simplicial complex with the vertex set \( V_1 = \{e_1, e_2, \ldots, e_m\} \) of the unit vectors in \( \mathbb{R}^m \). For \( u = (u_1, \ldots, u_m) \in \mathbb{Z}^m \), let \( \text{Supp}(u) := \{e_i : u_i \neq 0\} \), and \( \iota(u) := (u_1, u_1+u_2, \ldots, u_1+u_2+\cdots+u_m) \). The \( r \)th edgewise subdivision of \( \Delta \) is the simplicial complex \( (\Delta)^{<r>} \) consisting of subsets \( G \subseteq \Gamma_r = \{(u_1, \ldots, u_m) : \emptyset \neq \)}
\[ \sum_{i=1}^m u_i = r, \ u_i \geq 0 \] \] with \( \cup_{u \in G} \text{Supp}(u) \in \Delta \) and either \( \iota(u) - \iota(v) \in \{0, 1\}^m \) or \( \iota(u) - \iota(v) \in \{0, -1\}^m \) for all \( u, v \in G \). For more details, see in [BR05, Definition 6.1] and [EG00].

2.2.3. The \( r \)-colored barycentric subdivision: The \( r \)-colored barycentric subdivision, denoted by \( \text{sd}_r(\Delta) \) of a simplicial complex \( \Delta \) is the \( r \)th edgewise subdivision of the barycentric subdivision of \( \Delta \).

2.2.4. The Cheeger-Müller-Schrader’s subdivision ([CMS84]): Let \( \Delta_{d-1} \) be the standard simplex of dimension \( d-1 \) in \( \mathbb{R}^d \) with the unit vectors \( e_j \) as vertices, then

\[ \Delta_{d-1} := \{(t_1, \ldots, t_d) \in \mathbb{R}^d : \sum_{i=1}^d t_i = 1 \text{ and } t_i \geq 0 \text{ for } i = 1, 2, \ldots, d \}. \]

For each vertex \( e_j \), define a hypercube \( C_j \) as:

\[ C_j := \{(t_1, \ldots, t_d) \in \Delta_{d-1} : t_j \geq t_i \text{ for all } i \}. \]

For \( i \neq j \), the opposing faces of \( C_j \) are given by the pair of hyperplanes

\[ H_{j,0}^i = \{(t_1, \ldots, t_d) \in \Delta_{d-1} : t_i = 0 \} \]

and

\[ H_{j,1}^i = \{(t_1, \ldots, t_d) \in \Delta_{d-1} : t_i = t_j \}. \]

Figure 3. CMS subdivision of the 2-simplex

For a non-negative integer \( N \), the hypercube \( C_j \)'s are further subdivided by hyperplanes \( H_{j,k/N}^i = \{(t_1, \ldots, t_d) \in \Delta_{d-1} : t_i = \frac{k}{N} t_j \}, 0 \leq k \leq N \) into \( N^{d-1} \) regions, each of which is a parallelepiped \( P \). Now, take the barycentric subdivision of each parallelepiped \( P \). The resulting simplicial complex is in fact a subdivision, call it Cheeger-Müller-Schrader’s Subdivision, denoted as \( \text{CMS}(\Delta_{d-1}) \) of the simplex \( \Delta_{d-1} \). The CMS subdivision \( \text{CMS}(\Delta) \) of a simplicial complex \( \Delta \) is obtained by applying it to every simplex in \( \Delta \).
3. The $r$-colored Permutation Group $\mathbb{Z}_r \wr \Omega_d$

Let $d \geq 1$ and $r \geq 0$ be fixed integers. We present here some notations and statistics for the $r$-colored permutation group $\mathbb{Z}_r \wr \Omega_d$, where $\mathbb{Z}_r = \{0, 1, \ldots, r - 1\}$ is the cyclic group of order $r$ and $\Omega_d$ is the group of usual permutations on $[d]$. It is the group consisting of all the bijections $\sigma$ of the set

$$S := \{1^{(0)}, \ldots, d^{(0)}, 1^{(1)}, \ldots, d^{(1)}, \ldots, 1^{(r-1)}, \ldots, d^{(r-1)}\}$$

onto itself with the condition that if $\sigma(i^{(s)}) = j^{(t)}$, then $\sigma(i^{(s+1)}) = j^{(t+1)}$, where the exponents are taken modulo $r$. By the above condition, it is clear that $\sigma \in \mathbb{Z}_r \wr \Omega_d$ can be fully determined by the first $d$ elements of the set $S$. Therefore, we may write $\sigma$ as $(\sigma_1^1, \ldots, \sigma_d^{t_d})$. The exponent $\epsilon_i$ can be viewed as the color assigned to $\sigma_i$.

For $\sigma \in \mathbb{Z}_r \wr \Omega_d$, the descent set is defined as

$$\text{Des}(\sigma) := \{1 \leq i \leq d : \text{ either } \epsilon_i > \epsilon_{i+1} \text{ or } \epsilon_i = \epsilon_{i+1} \text{ and } \sigma_i > \sigma_{i+1}\}$$

with the assumption that $\sigma_{d+1} := d + 1$ and $\epsilon_{d+1} := 0$. In particular, $d$ is a descent of $\sigma$ if and only if $\sigma_d$ has nonzero color. The descent number of $\sigma$ is defined as $\text{des}(\sigma) := |\text{Des}(\sigma)|$.

Set $A_d := \{\sigma \in \mathbb{Z}_r \wr \Omega_d : \epsilon_1 = 0\}$ and $A_{d,j} := \{\sigma \in A_d : \sigma_d = d + 1 - j\}$. For $s \in \{0, 1, 2, \ldots, r - 1\}$, set $A_{d,j}^{(s)} := \{\sigma \in A_{d,j} : \epsilon_d = s\}$, $A_d^{(s)} := \{\sigma \in A_d : \epsilon_d = s\}$ and $A_d^{(\neq 0)} := \{\sigma \in A_d : \epsilon_d \neq 0\}$. The $r$-colored Eulerian polynomials are defined as follows:

$$A_{d,j}^{(s)}(t) := \sum_{\sigma \in A_{d,j}^{(s)}} t^{\text{des}(\sigma)} = \sum_{m=0}^{d} A^{(s)}(d, j, m) t^m, \quad (1)$$

and

$$A_d^{(s)}(t) := \sum_{\sigma \in A_d^{(s)}} t^{\text{des}(\sigma)} = \sum_{j=1}^{d} \sum_{m=0}^{d} A^{(s)}(d, j, m) t^m, \quad (2)$$

where $A^{(s)}(d, j, m)$ be the number of elements in $A_{d,j}^{(s)}$ with exactly $m$ descents.

Since $A_{d,j} = \bigcup_{s=0}^{r-1} A_{d,j}^{(s)}$ and $A_d = \bigcup_{j=1}^{d} A_{d,j}$ so we have:

$$A_{d,j}(t) = \sum_{s=0}^{r-1} A_{d,j}^{(s)}(t) \quad \text{and} \quad A_d(t) = \sum_{j=1}^{d} A_{d,j}(t). \quad (3)$$

Some interesting elementary properties and recurrence relations of $A^{(s)}(d + 1, k, 1, m)$ are given in the following lemma:

**Lemma 3.1.** For $0 \leq s \leq r - 1$ and $0 \leq k \leq d$, let

$$S_d^{(s)}(k) := (A^{(s)}(d + 1, k + 1, 0), A^{(s)}(d + 1, k + 1, 1), \ldots, A^{(s)}(d + 1, k + 1, d)).$$

Then we have the following relations:
(1) \( A^{(0)}(d + 1, k + 1, m) = A^{(0)}(d + 1, d + 1 - k, d - m) \) and thus
\[
\mathcal{S}^{(0)}_d(k) = \mathcal{S}^{(0)}_d(d - k),
\]
where \((a_0, a_1, \ldots, a_d, a_0) = (a_d, a_{d-1}, \ldots, a_1, a_0)\).

(2) For \( s \neq 0 \), \( A^{(s)}(d + 1, k + 1, m) = A^{(r-s)}(d + 1, d + 1 - k, d + 1 - m) \) and thus
\[
(\mathcal{S}^{(s)}_d(k), 0) = (\mathcal{S}^{(r-s)}_d(d - k), 0).
\]

(3)
\[
A^{(0)}(d + 1, k + 1, m) = \sum_{j=k}^{d-1} A^{(0)}(d, j + 1, m) + \sum_{s=1}^{r-1} \sum_{j=0}^{d-1} A^{(s)}(d, j + 1, m)
+ \sum_{j=0}^{k-1} A^{(0)}(d, j + 1, m - 1).
\]
Thus, we have:
\[
\mathcal{S}^{(0)}_d(k) = \sum_{j=k}^{d-1} (\mathcal{S}^{(0)}_{d-1}(j), 0) + \sum_{s=1}^{r-1} \sum_{j=0}^{d-1} (\mathcal{S}^{(s)}_{d-1}(j), 0) + \sum_{j=0}^{k-1} (0, \mathcal{S}^{(0)}_{d-1}(j)),
\]
with \( \mathcal{S}^{(0)}_0(0) = (1) \) and \( \mathcal{S}^{(s)}_0(0) = (0) \).

(4) For \( s \neq 0 \),
\[
A^{(s)}(d + 1, k + 1, m) = \sum_{j=k}^{d-1} A^{(s)}(d, j + 1, m) + \sum_{l=1}^{s-1} \sum_{j=0}^{d-1} A^{(l)}(d, j + 1, m)
+ \sum_{j=0}^{d-1} A^{(0)}(d, j + 1, m - 1) + \sum_{j=0}^{k-1} A^{(s)}(d, j + 1, m - 1)
+ \sum_{l=s+1}^{r-1} \sum_{j=0}^{d-1} A^{(l)}(d, j + 1, m - 1).
\]
Thus, we have
\[
\mathcal{S}^{(s)}_d(k) = \sum_{j=k}^{d-1} (\mathcal{S}^{(s)}_{d-1}(j), 0) + \sum_{l=1}^{s-1} \sum_{j=0}^{d-1} (\mathcal{S}^{(l)}_{d-1}(j), 0) + \sum_{j=0}^{d-1} (0, \mathcal{S}^{(0)}_{d-1}(j))
+ \sum_{j=0}^{k-1} (0, \mathcal{S}^{(s)}_{d-1}(j)) + \sum_{l=s+1}^{r-1} \sum_{j=0}^{d-1} (0, \mathcal{S}^{(l)}_{d-1}(j)).
\]

Proof. There is a bijection \( \sigma = (\sigma_1^d, \ldots, \sigma_{d+1}^d) \mapsto \bar{\sigma} = (\bar{\sigma}_1^d, \ldots, \bar{\sigma}_{d+1}^d) \) between the set enumerated by the given two numbers, where \( \bar{\sigma}_i := d + 1 - \sigma_i \) and
\[
\bar{\epsilon}_i := \begin{cases} \epsilon_i, & \epsilon_i = 0; \\ r - \epsilon_i, & \epsilon_i \neq 0. \end{cases}
\]
For \( 1 \leq i \leq d + 1 \), we have the following four possible cases:
- \( \epsilon_i > \epsilon_{i+1} = 0 \)
Corollary 3.2. For some $\gamma$

$\epsilon_i > \epsilon_{i+1} > 0$

$\epsilon_i = \epsilon_{i+1} = 0$ and $\sigma_i > \sigma_{i+1}$

$\epsilon_i = \epsilon_{i+1} \neq 0$ and $\sigma_i > \sigma_{i+1}$

In the first case, $i \in \text{Des}(\sigma)$ if and only if $i \in \text{Des}(\bar{\sigma})$ and in other three cases, we have $i \in \text{Des}(\sigma)$ if and only if $i \notin \text{Des}(\bar{\sigma})$.

1. In this case, it is clear that $d+1$ is not a descent of $\sigma$ and $\bar{\sigma}$. Thus, $\text{des}(\sigma) + \text{des}(\bar{\sigma}) = d$ gives the required assertion.

2. In this case, $d+1$ is always a descent of $\sigma$ and $\bar{\sigma}$. Therefore, the required assertion follows from the relation $\text{des}(\sigma) + \text{des}(\bar{\sigma}) = d + 1$.

3. The recursion formula follows from the effect of removing $\sigma_{d+1} = d + 1 - k$ from the colored permutation $\sigma$ in $A_{d+1,k+1}$ with $\text{des}(\sigma) = r$.

4. The proof is similar as of the assertion (3). \qed

Corollary 3.2. For $0 \leq s \leq r - 1$ and $0 \leq m \leq d$, we have the following relations:

1. The polynomial $A_{d,k}^{(0)}(t)$ is symmetric.

2. The polynomial $A_{d,k}^{(s)}(t)$ is symmetric.

3. For $d \geq 1$ and $0 \leq k \leq d$, we have

$$A_{d,k}^{(0)}(t) = t \sum_{j=0}^{k-1} A_{d-1,j}^{(0)}(t) + \sum_{j=k}^{d-1} A_{d-1,j}^{(0)}(t) + \sum_{s=1}^{r-1} \sum_{j=0}^{d-1} A_{d-1,j}^{(s)}(t).$$

4. For $s \geq 1$

$$A_{d,k}^{(s)}(t) = t \sum_{j=0}^{d-1} A_{d-1,j}^{(0)}(t) + t \sum_{i=1}^{s+1} \sum_{j=0}^{d-1} A_{d-1,j}^{(l)}(t) + \sum_{j=k}^{d-1} A_{d-1,j}^{(s)}(t) + \sum_{j=k}^{d-1} A_{d-1,j}^{(s)}(t).$$

Remark 3.3. The polynomials $A_{d,k}^{(0)}(t)$ and $A_{d,k}^{(s)}(t)$ from Corollary 3.2 (3) and (4) satisfy the same recurrence relation given in [SV15, Theorem 2.3]. Thus,

$$(A_{d,0}^{(0)}(t), \ldots, A_{d,d}^{(0)}(t), A_{d,0}^{(r-1)}(t), \ldots, A_{d,d}^{(r-1)}(t), A_{d,0}^{(1)}(t), \ldots, A_{d,d}^{(1)}(t))$$

is an interlacing sequence of polynomials. This also shows that the polynomials $A_{d}(t), A_{d}^{(0)}(t)$ and $A_{d}^{(s)}(t)$ are real-rooted.

The $\gamma$-vector. The $\gamma$-vector is also an important enumerative invariant of a flag homology sphere. Gal [Gal05] conjectured that the $\gamma$-vector is non-negative for a flag homological sphere. The non-negativity of the $\gamma$-vector implies the Charnay-Davis conjecture.

It is well-known that a symmetric polynomial $p(x)$ of degree $n$ can be uniquely written in the form

$$p(x) = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \gamma_i x^i (1 + x)^{n-2i},$$

for some $\gamma_i$. The polynomial $p(x)$ is called $\gamma$-nonnegative if $\gamma_i \geq 0$ for all $i$ and $\gamma = (\gamma_1, \ldots, \gamma_{\lfloor \frac{n}{2} \rfloor})$ is known as $\gamma$-vector of polynomial $p(x)$. In this subsection, we aim to provide a combinatorial description of $\gamma$-vectors of symmetric polynomials $A_{d}^{(0)}(t)$ and
Let us first recall the definition of slide. Let \( \sigma^i \) and consider \( \sigma^i_1 \cdots \sigma^i_d \) and \( \sigma^d_0 \sigma^1_1 \cdots \sigma^n_d \sigma^{d+1}_d \), where \( \sigma_0 = \infty, \epsilon_0 = 0, \sigma_{d+1} = d + 1 \) and \( \epsilon_{d+1} = 0 \). Put asterisks at each end and also between \( \sigma^i_1 \) and \( \sigma^i_{i+1} \) whenever \( \sigma^i_i < \sigma^i_{i+1} \) or if \( \epsilon_i = \epsilon_{i+1} \), then \( \sigma_i < \sigma_{i+1} \).

A slide is any segment between asterisks of length at least 2. In other words, a slide of \( \sigma \) is any decreasing run of \( \sigma^0_0 \sigma^1_1 \cdots \sigma^d_d \sigma^{d+1}_d \) of length at least 2. For example, for the permutation \((3^2)^5(1^1)(0^2)(2^1)4(1)\), \( \infty(0) \ast 3(2)^5(1^1)1(0^2) \ast 2(2)^4(1)6(0) \ast \) there are two slides, namely, \( (3(2)^5(1^1)1(0^2), 2(2)^4(1)6(0) \).

The following theorem is a generalization of \[\text{AN20b, Theorem 5.3}].

**Theorem 3.4.** The polynomials \( A_d^{(0)}(t) \) and \( A_d^{(\neq 0)}(t) := \sum_{s=1}^{r-1} A_d(s)(t) \) are symmetric of degree \( d - 1 \), so these can be expressed as:

\[
A_d^{(0)}(t) = \sum_{i=0}^{\lfloor \frac{d-1}{2} \rfloor} a^{(0)}(d, i, i)t^i(1 + t)^{d-1-2i},
\]

and

\[
A_d^{(\neq 0)}(t) = \sum_{i=0}^{\lfloor \frac{d-1}{2} \rfloor} a^{(\neq 0)}(d, i, i)t^i(1 + t)^{d-2i},
\]

where \( a^{(0)}(d, i, i) \) is the number of \( r \)-colored permutation \( \sigma \in A_d^{(0)} \) with \( i \) descents and \( i + 1 \) slides; and \( a^{(\neq 0)}(d, i, i) \) is the number of \( r \)-colored permutation \( \sigma \in A_d^{(\neq 0)} \) with \( i \) descents and \( i + 1 \) slides.

In particular, the polynomials \( A_d^{(0)}(t) \) and \( A_d^{(\neq 0)}(t) \) are \( \gamma \)-nonnegative.

To prove the above theorem, we need to define some notations. Let \( A_d^{(0)}(d, k) \) and \( A_d^{(\neq 0)}(d, k) \) represent the number of all \( r \)-colored permutations of descent \( k \) in \( A_d^{(0)} \) and \( A_d^{(\neq 0)} \) respectively. Let \( a^{(0)}(d, k, s) \) and \( a^{(\neq 0)}(d, k, s) \) denote the number of \( r \)-colored permutations with \( d \) descent and \( s + 1 \) slides in \( A_d^{(0)} \) and \( A_d^{(\neq 0)} \) respectively. It can be observed that every element in \( A_d^{(0)} \) has at least 1 slide while an element in \( A_d^{(\neq 0)} \) has at least 2 slides.

**Lemma 3.5.** We have the following relations:

\[
a^{(0)}(d, k, s) = \binom{d - 1 - 2s}{k - s} a^{(0)}(d, s, s) \quad \text{and} \quad a^{(\neq 0)}(d, k, s) = \binom{d - 1 - 2s}{k - s} a^{(\neq 0)}(d, s, s).
\]

Therefore,

\[
A_d^{(0)}(d, k) = \sum_{s=0}^{k} \binom{d - 1 - 2s}{k - s} a^{(0)}(d, s, s) \quad \text{and} \quad A_d^{(\neq 0)}(d, k) = \sum_{s=0}^{k} \binom{d - 1 - 2s}{k - s} a^{(\neq 0)}(d, s, s).
\]

**Proof.** Let us prove the relation for \( A_d^{(0)} \). Let \( \sigma \in A_d^{(0)} \) with \( s \) descent number and \( s + 1 \) slides. Counting \( \sigma_0 = \infty^{(0)} \), there are \( d + 1 \) symbols and \( d + 1 - 2(s + 1) = d - 1 - 2s \) that are not included in the slides. Choose \( k - s \) of these \( n - 1 - 2s \) elements, move each chosen element \( \sigma^i_i \) to the left if \( \epsilon_i = 0 \) (to right if \( \epsilon_i \neq 0 \), respectively) into the nearest slide.
the proof follows on similar lines. □

A holds. The second assertion follows upon summing \( a^{(0)}(n, k, s) \) over \( 0 \leq s \leq k \). For \( A_d^{(r)} \), the proof follows on similar lines.

The proof of Theorem 3.4 follows from the above lemma and the relations \( A^{(0)}(d, k) = A^{(0)}(d, d - 1 - k) \) and \( A^{(k)}(d, k) = A^{(r-s)}(d, d - 1 - k) \) derived from Lemma 3.1(1) and (2).

4. The \( f \)-vector of \( r \)-multichain Subdivisions

In this section, we will prove one of the main results of this paper. Let \( \mathcal{I} \) be the collection of all strictly increasing maps \( \iota : [r] \to [2r] \) such that \( \iota(1) = 1 \) and \( \leq_{\iota} \) is reflexive, i.e. \( \iota(t) \in \{2t, 2t - 1\} \) for all \( t > 1 \). Let us recall that \( \Delta(G_f(P_r)) \) is the \( r \)-multichain subdivision of type I when \( \iota(t) = 2t - 1 \) for all \( t \) and \( \leq_{\iota} \) is the order relation in \( P_r \) in this case. We will prove that \( f(\Delta(G_f(P_r))) = f(\Delta(G_f(P_r))) \) for all \( \iota \in \mathcal{I} \).

Proof of Theorem 4.2. Let \( F_k(\Delta) \) denote the collection of all \( k \)-dimensional faces of \( \Delta \). It is clear that \( F_0(\Delta(G_f(P_r))) = F_0(\Delta(G_f(P_r))) \) for all \( \iota \in \mathcal{I} \). For \( k \geq 1 \), let \( p_1 \prec_1 \cdots \prec_r p_{k+1} \) be a \( k \)-dimensional face in \( \Delta(G_f(P_r)) \), where \( p_j : p_{j,1} \leq p_{j,2} \leq \cdots \leq p_{j,r} \) is an \( r \)-multichain in \( P_r \) for \( j = 1, \ldots, k + 1 \). One may represent a \( k \)-dimensional face \( p_1 \prec_1 \cdots \prec_r p_{k+1} \) as a matrix

\[
M = \begin{pmatrix}
p_{1,1} & p_{2,1} & \cdots & p_{k+1,1} \\
p_{1,2} & p_{2,2} & \cdots & p_{k+1,2} \\
\vdots & \vdots & \ddots & \vdots \\
p_{1,r} & p_{2,r} & \cdots & p_{k+1,r}
\end{pmatrix}
\]

of order \( r \times (k + 1) \) with monotonically increasing columns and monotonically increasing \( t \)-th row when \( \iota(t) = 2t - 1 \); monotonically decreasing \( t \)-th row when \( \iota(t) = 2t \). One can see that \( j \)-th column of \( M \) represents the \( r \)-multichain \( p_j \).

For \( \iota(t) = 2t - 1 \), define \( \overline{p}_{j,t} := p_{j,t} \). For \( \iota(t) = 2t \), let \( (x_1, x_2, \ldots, x_m) \) be the arrangement of distinct elements of \( t \)-th row \( p_{1,t}, p_{2,t}, \ldots, p_{k+1,t} \) in strictly decreasing order. Define \( \overline{p}_{j,t} := x_{m-b+1} \) when \( p_{j,t} = x_b \) for some \( 1 \leq b \leq m \). For instance, the monotonically decreasing row \( \overline{p}_t : 3 \leq 2 \leq 1 \leq 1 \) will be changed to the monotonically increasing row \( \overline{p}_t : 1 \leq 2 \leq 3 \leq 3 \).

Consider the matrix

\[
\overline{M} = \begin{pmatrix}
\overline{p}_{1,1} & \overline{p}_{2,1} & \cdots & \overline{p}_{k+1,1} \\
\overline{p}_{1,2} & \overline{p}_{2,2} & \cdots & \overline{p}_{k+1,2} \\
\vdots & \vdots & \ddots & \vdots \\
\overline{p}_{1,r} & \overline{p}_{2,r} & \cdots & \overline{p}_{k+1,r}
\end{pmatrix}
\]

of order \( r \times (k + 1) \). By definition, each row is monotonically increasing and each column is also monotonically increasing. Moreover, columns of \( \overline{M} \) are distinct because the matrix \( P \) has distinct columns.
Let $\overline{p}_j : \overline{p}_{j,1} \leq \overline{p}_{j,2} \leq \cdots \leq \overline{p}_{j,r}$ for $j = 1, 2, \ldots, k+1$. Thus, the above matrix $M$ gives us a $k$-dimensional face $\overline{p}_{1} \prec \cdots \prec \overline{p}_{k+1}$ in $\Delta(G_1(P_r))$ by definition of $\prec$.

For $r \in \mathcal{I}$ and $k \geq 1$, define a map $\mathcal{F}_r : F_k(\Delta(G_1(P_r))) \to F_k(\Delta(G_1(P_r)))$ as

$$p_1 \prec \cdots \prec p_{k+1} \mapsto \overline{p}_{1} \prec \cdots \prec \overline{p}_{k+1}.$$

We claim that $\mathcal{F}_r$ is bijection. Let $p : p_1 \prec \cdots \prec p_{k+1}$ be a $k$-dimensional face in $\Delta(G_1(P_r))$. Define $\overline{p} : \overline{p}_{1} \prec \cdots \prec \overline{p}_{k+1}$, where $\overline{p}_{j,t} = p_{j,t}$ if $x(t) = 2t - 1$. For $x(t) = 2t$, define $\overline{p}_{j,t} = x_{m-b+1}$ when $p_{j,t} = x_\ell$ where $(x_1, \ldots, x_m)$ be the arrangement of distinct $p_{t,1}, \ldots, p_{t,k+1}$ in the decreasing order. It is clear by definition that $\overline{p}$ is the unique $k$-dimensional face in $\Delta(G_1(P_r))$ such that $\mathcal{F}_r(\overline{p}_{1} \prec \cdots \prec \overline{p}_{k+1}) = p_1 \prec \cdots \prec p_{k+1}$. Thus, it shows that $\mathcal{F}_r$ is bijective.

4.1. The $f$-vector of $r$-multichain subdivision of type I. In this subsection, we consider $P$ the poset of all faces of a simplicial complex $\Delta$ of dimension $d - 1$. We aim to give an explicit formula for the transformation matrix of the $f$-vector of $\Delta(G_1(P_r))$ when $r$ is reflexive. By Theorem 1.2 it is enough to study the $f$-vector of one of the subdivisions $\Delta(G_1(P_r))$ of $P$. Set $C^I_r(\Delta) := \Delta(G_1(P_r))$ and $[A_1, \ldots, A_r] := A_1 \subseteq \cdots \subseteq A_r$ where $A_t$ is a face in $\Delta$ for all $1 \leq t \leq r$.

By the definition of $C^I_r(\Delta)$, a $k$-dimensional face in $C^I_r(\Delta)$ is a chain

$$[A_{01}, \ldots, A_{0r}] \prec_I [A_{11}, \ldots, A_{1r}] \prec_I \cdots \prec_I [A_{k1}, \ldots, A_{kr}]$$

of $r$-multichains of faces in $\Delta$ of length $k+1$. The $f_0(C^I_r(\Delta))$ is the number of $r$-multichains $[A_1, \ldots, A_r]$ where $A_1 \subseteq \cdots \subseteq A_r$ for $A_1, \ldots, A_r \in \Delta \setminus \{\emptyset\}$. For a fixed $A \in \Delta$, the number of all possible $r$-multichains of the form $[A_1, \ldots, A_{r-1}, A_r = A]$ is

$$\sum_{l_r = 1}^{l} \sum_{l_{r-1} = 1}^{l-1} \cdots \sum_{l_2 = 1}^{l_3} \binom{l_2}{l_1} \cdots \binom{l_{r-2}}{l_{r-3}} \binom{l}{l_{r-1}},$$

where $l = |A|$ and $l_i = |A_i|$ for $1 \leq i \leq r - 1$. By applying binomial theorem successively, we obtain that the expression (4) is equal to $r^l - (r-1)^l$.

Since there are $f_{l-1}(\Delta)$ choices for $A$ with $|A| = l$, the number of all possible $r$-multichains in $C^I_r(\Delta)$ will be

$$f_0(C^I_r(\Delta)) = \sum_{l=0}^{d} (r^l - (r-1)^l) f_{l-1}(\Delta).$$

To compute $f_k(C^I_r(\Delta))$, for $k \geq 0$, let us introduce some notations.

Let $P^0_{\alpha_1, \ldots, \alpha_r}$ denote the number of chains of $r$-multichains of length $k+1$ terminating at some fixed $r$-multichain $[A_1, A_2, \ldots, A_r] = [A_1, A_1 \cup A'_2, \ldots, A_{r-1} \cup A'_r]$, where $A'_i = A_i \setminus A_{i-1}$ and $\alpha_i = |A'_i|$ for all $2 \leq i \leq r$ and $\alpha_1 = |A_1|$. By definition, $P^0_{\alpha_1, \ldots, \alpha_r}$ = 1 and $P^0_{\alpha_1, \ldots, \alpha_r} = 0$ for all $\alpha_i$.

There are $\binom{\alpha_1}{k_1} \cdots \binom{\alpha_{r-1}}{k_{r-1}} \binom{\alpha_r}{k_r}$ choices of $r$-multichains of the form $[B_1, A_1 \cup B_2, \ldots, A_{r-1} \cup B_r]$ with $|B_i| = k_i$ for all $i = 1, \ldots, r$ such that $[B_1, A_1 \cup B_2, \ldots, A_{r-1} \cup B_r] \preceq_I [A_1, A_2, \ldots, A_r]$. The number of $r$-multichains of the form $[B_1, A_1 \cup B_2, \ldots, A_{r-1} \cup B_r]$ with $|B_i| = k_i$ for all $i = 1, \ldots, r$ such that $[B_1, A_1 \cup B_2, \ldots, A_{r-1} \cup B_r] \preceq_I [A_1, A_2, \ldots, A_r]$.
i.e., $B_1 \subseteq A_1 \subseteq A_1 \cup B_2 \subseteq \cdots \subseteq A_{r-1} \cup B_r \subseteq A_r$ and the number of all chains of length $k$ terminating at $[B_1, A_1 \cup B_2, \ldots, A_{r-1} \cup B_r]$ is $P_{k-1}^{k_1, k_2, \ldots, k_r}$.

For fixed $k$ and $\alpha_1, \ldots, \alpha_r$, the number $P_k^{\alpha_1, \ldots, \alpha_r}$ satisfies the following recurrence relation:

$$P_k^{\alpha_1, \ldots, \alpha_r} = \sum_{k_r=0}^{\alpha_r} \sum_{k_{r-1}=0}^{\alpha_{r-1}} \cdots \sum_{k_1=1}^{\alpha_1} \binom{\alpha_1}{k_1} \cdots \binom{\alpha_{r-1}}{k_{r-1}} \binom{\alpha_r}{k_r} P_{k-1}^{k_1, k_2, \ldots, k_r} - P_{k-1}^{\alpha_1, \ldots, \alpha_r}. \quad (6)$$

In the next lemma, we have derived an explicit formula for $P_k^{\alpha_1, \ldots, \alpha_r}$ by induction and binomial theorem.

Lemma 4.1. For given $\alpha_i$ and $k \geq 0$, the number $P_k^{\alpha_1, \ldots, \alpha_r}$ is given as:

$$P_k^{\alpha_1, \ldots, \alpha_r} = \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} [(i+1)^{\alpha_2 + \cdots + \alpha_r} ((2i)^{\alpha_1} - (2i-1)^{\alpha_1})]. \quad (7)$$

Proof. For $k = 0$, $P_0^{\alpha_1, \ldots, \alpha_r} = 1$ and for $k = 1$, we have $P_1^{\alpha_1, \ldots, \alpha_r} = (2^{\alpha_1} - 1)2^{\alpha_2 + \cdots + \alpha_r} - 1$. Thus one can easily see that (7) holds for $k = 0, 1$. Now, suppose that (7) is true for $k - 1$. Substitute the formula of $P_{k-1}^{\alpha_1, \ldots, \alpha_r}$ in the recurrence relation (6), we have

$$P_k^{\alpha_1, \ldots, \alpha_r} = \sum_{k_r=0}^{\alpha_r} \sum_{k_{r-1}=0}^{\alpha_{r-1}} \cdots \sum_{k_1=1}^{\alpha_1} \binom{\alpha_1}{k_1} \cdots \binom{\alpha_{r-1}}{k_{r-1}} \binom{\alpha_r}{k_r} \sum_{i=0}^{k-1} (-1)^{k-i} \binom{k-1}{i} [(i+1)^{\alpha_2 + \cdots + \alpha_r} ((2i)^{\alpha_1} - (2i-1)^{\alpha_1})].$$

Using the binomial formula $r$ times (summing over $k_1, k_2, \ldots, k_r$), we have

$$P_k^{\alpha_1, \ldots, \alpha_r} = \sum_{i=0}^{k-1} (-1)^{k-1-i} \binom{k-1}{i} [(i+1)^{\alpha_2 + \cdots + \alpha_r} ((2i+1)^{\alpha_1} - (2i)^{\alpha_1})].$$

Now, using the identity $\binom{k-1}{i} + \binom{k-1}{i} = \binom{k}{i}$ we get the required identity. \qed

There are $f_{l-1}(\Delta)$ choices for $A$ with $|A| = l$ and for a fixed $A$ we have $(l_2^{l_2}) \cdots (l_r^{l_r})$ $r$-multichain $A_1 \subseteq \cdots \subseteq A_r$, where $A_r = A$ with $|A_i| = l_i$ for $i = 1, \ldots, r$. Hence, we have

$$f_k(C^l_r(\Delta)) = \sum_{l=0}^{d} \left( \sum_{l_r-1}^{l_r} \cdots \sum_{l_1}^{l_2} \binom{l_2}{l_1} \cdots \binom{l_r}{l_{r-1}} P_k^{l_1, l_2-1, \ldots, l_r-l_{r-1}} \right) f_{l-1}(\Delta) \quad (8)$$

Using Lemma 4.1 and the application of binomial theorem, we have the $f$-vector transformation as follows:
Theorem 4.2. Let $\Delta$ be a $(d - 1)$-dimensional simplicial complex. Then
\[
f_k(\mathcal{C}_r^l(\Delta)) = \sum_{l=0}^{d} \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} [(r + ri)^i - (r + ri - 1)^i] f_{l-1}(\Delta).
\]
for $0 \leq k \leq d - 1$ and $f_{-1}(\mathcal{C}_r^l(\Delta)) = f_{-1}(\Delta) = 1$.

The transformation of the $f$-vector of $\Delta$ to the $f$-vector of $r$-multichain subdivision $\mathcal{C}_r^l(\Delta)$ (also for $\mathcal{C}_2^{l,1}(\Delta)$) is given by the matrix:
\[
F_d = [f_{l,m}]_{0 \leq l, m \leq d},
\]
where
\[
f_{0,m} = \begin{cases} 1, & m = 0; \\ 0, & m > 0. \end{cases}
\]
and for $1 \leq l \leq d$, we have
\[
f_{l,m} = \sum_{i=0}^{l-1} (-1)^{l-1-i} \binom{l-1}{i} [(ri + r)^m - (ri + r - 1)^m]
\]
(10)

In the following lemma, we give a recurrence relation for $f_{l,m}$:

Lemma 4.3. For $1 \leq l \leq d - 1$ and $1 \leq m \leq d$,
\[
\sum_{j=1}^{m} r^j \binom{m}{j} f_{l,m-j} = f_{l+1,m}.
\]

Proof. Using (10), we have
\[
\sum_{j=1}^{m} r^j \binom{m}{j} f_{l,m-j} = \sum_{j=1}^{m} r^j \binom{m}{j} \sum_{i=0}^{l-1} (-1)^{l-1-i} \binom{l-1}{i} [(ri + r)^m - (ri + r - 1)^m]
\]
\[
= \sum_{i=0}^{l-1} (-1)^{l-1-i} \binom{l-1}{i} [(ri + 2r)^m - (ri + 2r - 1)^m - (ri + r)^m + (ri + r - 1)^m]
\]
The last assertion follows by taking sum over $j$. Now, after re-summing and using the identity \( \binom{k-1}{i} + \binom{k-1}{i-1} = \binom{k}{i} \), we get the required identity. \( \square \)

In the next lemma, we show how the numbers $f_{l,m}$ are related to the $r$-colored Eulerian numbers.

Lemma 4.4. Let $T_{t,j}$ be the collection of all partition $T = T_1 | \cdots | T_t | T_{t+1}$ of rank $t$ of $d + 1$ elements ranging from $S$ for which every element $1, 2, \ldots, d + 1$ with exactly one color appears in $T$; min $T_1$ of color $(0)$ and max $T_{t+1} = d + 1 - j$. Then
\[
|T_{t,j}| = \sum_{m=0}^{d} \binom{d-j}{d-m} f_{l,m}.
\]
To form such a partition, we first choose $d - m$ elements among $\{1, \ldots, d - j\}$ to put in $T_{i+1}$ along with $d + 1 - j$. This can be done in \(\binom{d-j}{d-m}\) ways. For $t > 0$, to form $T_1 \| \ldots \| T_t$ we need to create a set partition from the remaining $m$ elements, and this can be done in $f_{k,m}$ ways. We proceed with proving this claim by using induction on $t$. For $t = 1$, it is trivial. For $t = 2$, to form $T_1$, we need to put $m$ elements from $\{1, \ldots, d + 1\} \setminus T_2$ such that $\min T_1$ of color $(0)$. This gives $r^m - (r - 1)^m$ choices, which is the same as $f_{1,m}$. Suppose that the number of such set partitions $T_1 \| \ldots \| T_t$ of $m$ elements from $\{1, \ldots, d + 1\}$ (with $\min T_1$ of color $(0)$) is $f_{t,m}$. Now, to form such set partition $T_1 \| T_2 \| \ldots \| T_{t+1}$ of $m$ elements, we first choose $i$ elements from $m$ remaining elements, where $i > 0$. This can be done in $m^i \binom{m}{i}$ ways; and the set partition $T_1 \| T_2 \| \ldots \| T_t$ from remaining $m - i$ elements can be done in $f_{k,m-i}$ ways (by induction hypothesis). Thus we have $\sum_{i=1}^{m} r^i \binom{m}{i} f_{t,m-i} = f_{t+1,m}$

which completes the proof. \(\square\)

4.2. The $h$-vector Transformation: In this subsection, we express the $h$-vector of an $r$-multichain subdivision of simplicial complex $\Delta$ in terms of the $h$-vector of the simplicial complex $\Delta$. It is known that the entries of the transformation matrix of the $h$-vector of $C^I_2(\Delta)$ are given in terms of 2-colored Eulerian numbers, see [AN20a, Theorem 3.1]. The following theorem generalizes that the entries of the transformation matrix of the $h$-vector of $C^I_r(\Delta)$ are given in terms of $r$-colored Eulerian numbers.

**Theorem 4.5.** The $h$-vector of $C^I_r(\Delta)$ can be represented as:

$$h(C^I_r(\Delta)) = R_d h(\Delta),$$

where the entries of the matrix $R_d$ are given as:

$$R_d = [A^{(0)}(d + 1, s + 1, t)]_{0 \leq s, t \leq d}.$$

Thus, the $h$-vector of $C^I_r(\Delta)$ will be

$$h(C^I_r(\Delta)) = [A^{(0)}(d + 1, k + 1, m)]_{0 \leq k, m \leq d} h(\Delta) = \sum_{k=0}^{d} h_k s_d^{(0)}(k), \quad (11)$$

where

$$s_d^{(s)}(k) := (A^{(s)}(d + 1, k + 1, 0), A^{(s)}(d + 1, k + 1, 1), \ldots, A^{(s)}(d + 1, k + 1, d))$$

**Proof.** Since each set partition $T = T_1 \| \ldots \| T_{t+1}$ can be mapped to a permutation $\sigma = \sigma(T)$ by removing bars and writing each block in increasing order such that $\sigma_{d+1} = d + 1 - j$, and $\sigma_1$ of color $(0)$. That is, $\sigma \in A_{d+1+j}$ with $\text{Des}(\sigma) \subseteq D$, where $D = D(A) = \{|A_0|, |A_0| + |A_1|, \ldots, |A_0| + |A_1| + \ldots + |A_{r-1}|\}$. Thus, the claim follows from Lemma 4.3 and $h(C^I_r(\Delta)) = H_d F_d H^{-1}_d h(\Delta)$, where $H_d$ is the transformation matrix from the $f$-vector to the $h$-vector. \(\square\)

Using [SV15, Theorem 2.3] and Theorem 4.5, we have the following result.
Corollary 4.6. Let $\Delta$ be a $(d - 1)$-dimensional simplicial complex with non-negative $h$-vector. Then the $h$-vector of $C_r^1(\Delta)$ is real-rooted.

5. Combinatorial equivalences of the CMS and $r$-colored barycentric subdivisions

In this section, it is shown that the $r$-multichain subdivisions of type I and II are the same as the $r$-colored barycentric subdivision and the CMS subdivision described in [CMSv4] for $r = 2N$ respectively.

5.1. The $r$-colored barycentric subdivision: Assume that $\Delta$ is the $d-1$ simplex on the vertex set $[d]$. By definition, $\text{sd}_r(\Delta)$ is the $r$-th edgewise subdivision of the simplicial complex $\text{sd}(\Delta)$. Since the edgewise subdivision depends on the linear ordering on the vertex set $V(\text{sd}(\Delta)) := \{F : \emptyset \neq F \subseteq [d]\}$, therefore we need to fix an ordering on $V(\text{sd}(\Delta))$. Define an ordering $\preceq$ on $V(\text{sd}(\Delta))$ as: $F \preceq G$ if $|F| < |G|$ or $(|F| = |G|$ and $F \leq_{\text{lex}} G)$, where $\leq_{\text{lex}}$ is a lexicographic ordering on finite sets.

Let $U_r$ be the vertex set of $\text{sd}_r(\Delta)$, i.e., a collection of all ordered (given by $\preceq$) $m$-tuples $u = (u_F : F \subseteq V(\text{sd}(\Delta)))$ in $\mathbb{Z}_\geq^m$ such that $\sum_{F \in V(\text{sd}(\Delta))} u_F = r$ and $\text{Supp}(u) \in \text{sd}(\Delta)$; $m = |V(\text{sd}(\Delta))|$. If $u \in U_r$ with Supp($u$) = $\{G_1, \ldots, G_k\}$, then by definition of barycentric subdivision, we have $G_1 \subset \cdots \subset G_k \subseteq [d]$.

Proposition 5.1. Let $\Delta$ be a $d$-1-dimensional simplex. Then the $r$-multichain subdivision $C_r^1(\Delta)$ is isomorphic to the $r$-colored barycentric subdivision $\text{sd}_r(\Delta)$.

Proof. First, we will show that there is a bijection between the vertex sets $U_r$ and $C_r(\Delta)$. Let $u = (u_F : \emptyset \neq F \subseteq [d]) \in U_r$ with $\text{Supp}(u) = \{G_1, \ldots, G_k\}$. Define a map $\theta : U_r \rightarrow C_r(\Delta)$ as:

$$\theta(u) = [A_1, \ldots, A_r],$$

where

$$A_i = \begin{cases} G_1, & 1 \leq i \leq u_{G_1}; \\ G_2, & u_{G_1} + 1 \leq i \leq u_{G_1} + u_{G_2}; \\ \vdots & \vdots \\ G_k, & \sum_{j=1}^{k-1} u_{G_j} + 1 \leq i \leq \sum_{j=1}^{k} u_{G_j} = r. \end{cases}$$

For $A = [A_1, \ldots, A_r] \in C_r(\Delta)$, set $u_F := |\{i : F = A_i\}|$ for $F \in \{A_1, \ldots, A_r\}$ and $u_F := 0$ for $F \notin \{A_1, \ldots, A_r\}$. Since $\sum_{F \in V(\text{sd}(\Delta))} u_F = r$, there is a unique $u = (u_F : F \in V(\text{sd}(\Delta))) \in U_r$ such that $\theta(u) = A$. This shows that $\theta$ is a bijection.

Since both simplicial complexes $\text{sd}_r(\Delta)$ and $C_r^1(\Delta)$ are flag so it is enough to show that $F \in \text{sd}_r(\Delta)$ if and only if $\theta(F) \in C_r^1(\Delta)$ for any 1-dimensional face $F$.

Let $u, v \in U_r$ such that $\{u, v\}$ is a 1-dimensional face in $\text{sd}_r(\Delta)$ with $i(u) - i(v) \in \{0, 1\}^m$. Let $\text{Supp}(u) = \{G_1, \ldots, G_k\}$ and $\text{Supp}(v) = \{H_1, \ldots, H_l\}$. Then

$$i(u)_F = \begin{cases} 0, & F \leq H_1; \\ u_{H_1} + \cdots + u_{H_j}, & H_j \leq F < H_{j+1}; \\ r, & F \geq H_k. \end{cases}$$
and
\[ t(v)_F = \begin{cases} 0, & F \leq G_1; \\ v_{G_1} + \cdots + v_{G_j}, & G_j \leq F < G_{j+1}; \\ r, & F \geq G_1. \end{cases} \]

Since \( \text{Supp}(u) \cup \text{Supp}(v) \) is a face (a chain of \( H \)'s and \( G \)'s) in \( \text{sd}(\Delta) \), therefore we must have \( H_1 \subseteq G_1 \) by the assumption that \( (\iota(u) - \iota(v))_{H_1} = 0 \) or 1. If \( H_2 \subseteq G_1 \), then \( \iota(u)_{H_2} = u_{H_1} + u_{H_2} > 1 \) and \( \iota(v)_{H_2} = 0 \) which contradicts to the supposition that \( (\iota(u) - \iota(v))_{H_2} = 0 \) or 1. Therefore, we must have \( G_1 \subseteq H_2 \). Continuing with this argument, we get consequently that \( H_1 \subseteq G_1 \subseteq H_2 \subseteq \cdots \). This shows that \( \theta(u) \prec_{I} \theta(v) \), i.e., \( \{ \theta(u), \theta(v) \} \) is 1-dimensional face in \( C^I_1(\Delta) \).

Now, let \( A = [A_1, \ldots, A_r] \) and \( B = [B_1, \ldots, B_s] \) in \( C_r(\Delta) \) such that \( A \prec_I B \). Let \( u = \theta^{-1}(A) \) and \( v = \theta^{-1}(B) \). It implies that \( \text{Supp}(u) = \{A_{i_1}, \ldots, A_{i_k}\} \) and \( \text{Supp}(v) = \{B_{j_1}, \ldots, B_{j_l}\} \) and \( A_{i_k} \subseteq B_{j_l} \subseteq \cdots \). Therefore, by definition of \( u \)'s and \( v \)'s, we have \( (\iota(u) - \iota(v))_F = 0 \) or 1 for all \( F \in V(\text{sd}(\Delta)) \). Thus, \( \{u, v\} \) is a 1-dimensional face in \( \text{sd}_r(\Delta) \).

5.2. The CMS subdivision: We begin with fixing a labeling of CMS subdivided simplicial complex through its simplicies constructively. Continuing the description in Subsection 2.2.4, we assert that the vertices appearing in \( C_j \) after choosing hyperplanes are resultant of the intersection of hyperplanes \( \cap_{i \neq j} H_{j, k_i}^{i, k_i}, 0 \leq k_i \leq N \). Therefore, the coordinates of these vertices are:

\[ x_i = \begin{cases} \frac{N}{M}, & i = j; \\ \frac{k_i}{M}, & i \neq j. \end{cases} \]

where \( M = N + \sum_{i \neq j} k_i \).

Let us label these vertices by the \( d \)-tuple \( (k_1, \ldots, k_{j-1}, N, k_{j+1}, \ldots, k_d) \) for \( 0 \leq k_i \leq N \).

Under this labeling, every \( m \)-dimensional face \( F \) of some parallelepiped \( P \) in \( C_j \) is determined by \( 2^m \) vertices

\[ \{(l_1, \ldots, l_{j-1}, N, l_{j+1}, \ldots, l_d) : l_i = k_i \text{ or } k_i + 1 \text{ with } |\{i : l_i \neq k_i\}| \leq m\} \]

where \( k_i = \min\{v_i : v = (v_1, \ldots, v_d) \text{ is a vertex of the face } F\} \). For example, two vertices \((k_1, \ldots, k_{j-1}, N, k_{j+1}, \ldots, k_d) \) and \((l_1, \ldots, l_{j-1}, N, l_{j+1}, \ldots, l_d) \) in \( C_j \) form an edge of a face \( F \) of some parallelepiped \( P \) in \( C_j \) if and only if \( |k_{i_0} - l_{i_0}| = 1 \) for some unique \( i_0 \neq j \) and \( |k_i - l_i| = 0 \) for all \( i \neq i_0 \).

The barycenter \( b_F \) of an \( m \)-dimensional face \( F \) of some parallelepiped \( P \) in \( C_j \) can be labeled by \((l_1, \ldots, l_{j-1}, N, l_{j+1}, \ldots, l_d) \), where

\[ l_i = \begin{cases} k_i, & \text{ith coordinate remains fixed for all vertices in } F; \\ k_i + \frac{1}{2}, & \text{otherwise}. \end{cases} \]

where \( k_i = \min\{v_i : v = (v_1, \ldots, v_d) \text{ is a vertex of the face } F\} \). It can be observed that the number of non-integers in the coordinate of the vertex \( b_F \) is the same as the dimension of \( F \). Thus, the vertex set \( V(\text{CMS}(\Delta)) \) of the CMS subdivision can be labelled as

\[ V(\text{CMS}(\Delta)) = \{(k_1/2, \ldots, k_d/2) : \text{there exists } j \text{ such that } k_j = 2N \text{ and } 0 \leq k_i \leq 2N \text{ for all } i\}. \]
Here, we include a figure 5.2 (when $N = 1$ and $d = 3$) to demonstrate the above labelling.

\[
\begin{align*}
(0, 1, 0) & \quad (1, 0, 0) \\
(\frac{1}{2}, 1, 0) & \quad (1, 1, 0) \\
(1, \frac{1}{2}, 0) & \quad (0, \frac{1}{2}, 0) \\
(0, 1, \frac{1}{2}) & \quad (0, 0, 1) \\
(0, 1) & \quad (0, 1) \\
(1, 0) & \quad (1, 0) \\
(\frac{1}{2}, \frac{1}{2}) & \quad (1, \frac{1}{2}) \\
1 & \quad \frac{1}{2}
\end{align*}
\]

**Figure 4.** CMS subdivision of the 2-simplex when $N = 1$

Let $b_{F_0}, \ldots, b_{F_m}$ be an $m$-dimensional simplex in CMS($\Delta$), where $F_0 \subset F_1 \subset \cdots \subset F_m$ is an increasing sequence of faces of some parallelepiped $P$ in $C_j$. Then it is determined by the set of $m + 1$ vertices $\{b_{F_0}, \ldots, b_{F_m}\}$ which satisfies $b_{F_i} = b_{F_0} + (i - 1)\frac{d}{2}$ for all $1 \leq i \leq d$. Since the number of non-integral coordinates in $F$ is the same as the dimension of $F$, therefore the number of non-integral coordinates in $F_i$ is less or equal to the number of non-integral coordinates in $F_j$ and the number of integral coordinates in $F_i$ is greater or equal to the number of integral coordinates in $F_j$ for all $1 \leq i < j \leq m$.

**Proposition 5.2.** Let $\Delta$ be a simplex of dimension $d - 1$. Then for $r = 2N$, the chain subdivision $C^{II}_r(\Delta)$ is isomorphic to the CMS subdivision.

**Proof.** Here, we denote $[A_r, \ldots, A_1]$ by an $r$-multichain $A_r \subset \cdots \subset A_1$. Assume that $\Delta$ is a $d - 1$-simplex on the vertex set $[d]$. Define a bijection $\varphi$ between the vertex sets $C^{2N}(\Delta)$ and $V(\text{CMS}(\Delta))$ as:

\[
v = (\frac{k_1}{2}, \ldots, \frac{k_d}{2}) \mapsto \varphi(v) = [A_{2N}, A_{2N-1}, \ldots, A_1],
\]

where $A_{2N} = \{i \mid k_i = 2N\}$ and for $1 \leq l < 2N$, $A_l = \{i \mid k_i = l\} \cup A_{l+1}$. Since for each vertex $v \in V(\text{CMS}(\Delta))$, there is some $j$ such that $v_j = 2N$, therefore $j \in A_{2N}$, hence $A_{2N}$ is non-empty. Moreover, $A_{2N} \subset \cdots \subset A_1 \subset [d]$. Thus, $[A_{2N}, A_{2N-1}, \ldots, A_1]$ is the unique element of $C^{2N}(\Delta)$ associated to a given vertex $v$ in CMS($\Delta$). Therefore, $\varphi$ is well-defined.
To show the subjectivity of \( \varphi \), let \( \{A_2N, A_{2N-1}, \ldots, A_1\} \) be a vertex in \( C_{2N}(\Delta) \), where

\[
0 \neq A_{2N} \subseteq A_{2N-1} \subseteq \cdots \subseteq A_1
\]

is a chain of subsets of \( [d] \). For each \( l \in [d] \), let \( v_l = \{i : l \in A_i\} \), then \( 0 \leq k_l \leq 2N \). Since \( A_{2N} \) is non-empty therefore, there is an index \( j \in [d] \) such that \( k_j = 2N \). Thus, this gives us a unique vertex

\[
v = (\frac{k_1}{2}, \ldots, \frac{k_d}{2}) \in V(\text{CMS}(\Delta))
\]

and \( \varphi(v) = [A_{2N}, A_{2N-1}, \ldots, A_1] \), since \(|\{i : v_l \geq 1\}| = |\{i : i \in A_l\}| = v_l \) for \( 1 \leq l \leq d \). This shows that \( \varphi \) is a bijection.

Since both simplicial complexes CMS(\( \Delta \)) and \( C_{2N}^{II}(\Delta) \) are flag so it is enough to show that \( \sigma \in \text{sd}_r(\Delta) \) iff \( \theta(\sigma) \in C_{2N}^{II}(\Delta) \) for any 1-dimensional simplex \( \sigma \). Let \( \sigma \) be a 1-dimensional simplex in CMS(\( \Delta \)) with vertices \( \{b_{F_0}, b_{F_1}\} \), where \( F_0 \subset F_1 \) is a strictly increasing sequence of faces of some parallelepiped \( P \) in \( C_j \) and \( b_{F_1} \) is the barycenter of the face \( F_1 \). It can be noted that

\[
\{i : \text{the } i\text{th coordinate remains fixed for all vertices in } F_1\}
\]

\( \subseteq \{i : \text{the } i\text{th coordinate remains fixed for all vertices in } F_0\} \).

Therefore, by definition of \( \varphi \) and \( b_{F_1} \), it follows that

\[
\varphi(b_{F_1})_{2N} \subseteq \varphi(b_{F_0})_{2N} \subseteq \varphi(b_{F_0})_{2N-1} \cdots \subseteq \varphi(b_{F_0})_2 \subseteq \varphi(b_{F_0})_1 \subseteq \varphi(b_{F_1})_1.
\]

Consequently, we have

\[
\varphi(b_{F_1}) \prec_{II} \varphi(b_{F_0})
\]

which gives a chain of length 2 in \( C_{2N}^{II}(\Delta) \).

Now, let \( \{A_{2N}^0, \ldots, A_1^0\} \prec_{II} \{A_{2N}^1, \ldots, A_1^1\} \) be a 2-chain in \( C_{2N}(\Delta) \). This gives 2 vectors

\[
b_{F_0} = (\frac{k_0^0}{2}, \ldots, \frac{k_d^0}{2}) \quad \text{and} \quad b_{F_1} = (\frac{k_0^1}{2}, \ldots, \frac{k_d^1}{2})
\]

for some faces \( F_0, F_1 \). Since \( k_l^0 = |\{i : l \in A_l^0\}| \), then by ordering of \( A_l^0 \), we get \( k_l^0 = k_l^1 \) or \( k_l^0 = k_l^1 + \frac{1}{2} \). Therefore, we must have \( F_1 \subseteq F_0 \). Thus, these vectors give rise an edge in CMS(\( \Delta \)).

\[\square\]

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DEPARTMENT OF MATHEMATICS, LAHORE UNIVERSITY OF MANAGEMENT SCIENCES, LAHORE, PAKISTAN

Email address: shaheen.nazir@lums.edu.pk