Decay properties of solutions toward a multiwave pattern to the Cauchy problem for the scalar conservation law with degenerate flux and viscosity

Natsumi Yoshida

BKC Research Organization of Social Sciences, Ritsumeikan University, Kusatsu, Shiga 525-8577, Japan /Osaka City University Advanced Mathematical Institute, Sumiyoshi, Osaka 558-8585, Japan.

Abstract

In this paper, we study the precise decay rate in time to solutions of the Cauchy problem for the one-dimensional conservation law with a nonlinearly degenerate viscosity \( \partial_t u + \partial_x (f(u)) = \mu \partial_x \left( | \partial_x u |^{p-1} \partial_x u \right) \) where the far field states are prescribed. Especially, we deal with the case when the flux function is convex or concave but linearly degenerate on some interval. As the corresponding Riemann problem admits a Riemann solution as a multiwave pattern which consists of the rarefaction waves and the contact discontinuity, it has already been proved by Yoshida that the solution to the Cauchy problem tends toward the linear combination of the rarefaction waves and contact wave for \( p \)-Laplacian type viscosity as the time goes to infinity. We investigate that the decay rate in time of the corresponding solutions toward the multiwave pattern. Furthermore, we also investigate that the decay rate in time of the solution for the higher order derivative. The proof is given by \( L^1, L^2 \)-energy and time-weighted \( L^q \)-energy methods under the use of the precise asymptotic properties of the interactions between the nonlinear waves.

Keywords: viscous conservation law, decay estimates, asymptotic behavior, nonlinearly degenerate viscosity, rarefaction wave and viscous contact wave

1. Introduction and main theorems

In this paper, we are concerned with the asymptotic behavior and the time-decay of solutions for the one-dimensional scalar conservation law with a non-
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linearly degenerate viscosity ($p$-Laplacian type viscosity with $p > 1$)

\[
\begin{aligned}
\frac{\partial_t u + \partial_x f(u)}{\mu} &= \partial^2_x \left( |\partial_x u|^{p-1} \partial_x u \right) & (t > 0, x \in \mathbb{R}), \\
u(0, x) &= u_0(x) & (x \in \mathbb{R}).
\end{aligned}
\] (1.1)

Here, $u = u(t, x)$ denotes the unknown function of $t > 0$ and $x \in \mathbb{R}$, the so-called conserved quantity, $f = f(u)$ is the flux function depending only on $u$, $\mu$ is the viscosity coefficient, $u_0$ is the given initial data and $u$ is assumed to be asymptotically far field constant states $u_\pm$ as $x \to \pm \infty$, that is,

\[\lim_{x \to \pm \infty} u(t, x) = u_\pm \in \mathbb{R} \quad (t \geq 0).\]

We suppose the given flux $f = f(u)$ is a $C^1$-function satisfying $f(0) = f'(0) = 0$, $\mu$ is a positive constant and far field states $u_\pm$ satisfy $u_- < u_+$ without loss of generality.

We are interested in the asymptotic behavior and its precise estimates in time of the global solution to the Cauchy problem (1.1). It can be expected that the large-time behavior is closely related to the weak solution ("Riemann solution") of the corresponding Riemann problem (cf. [13], [28]) for the non-viscous hyperbolic part of (1.1):

\[
\begin{aligned}
\frac{\partial_t u + \partial_x f(u)}{\mu} &= 0 & (t > 0, x \in \mathbb{R}), \\
u(0, x) &= u_0^R(x) & (x \in \mathbb{R}),
\end{aligned}
\] (1.2)

where $u_0^R$ is the Riemann data defined by

\[u_0^R(x) = u_0^R(x; u_-, u_+) := \begin{cases} 
  u_- & (x < 0), \\
  u_+ & (x > 0).
\end{cases}\]

In fact, for $p = 1$ in (1.1), the usual linear viscosity case:

\[
\begin{aligned}
\frac{\partial_t u + \partial_x f(u)}{\mu} &= \partial^2_x u & (t > 0, x \in \mathbb{R}), \\
u(0, x) &= u_0(x) & (x \in \mathbb{R}), \\
\lim_{x \to \pm \infty} u(t, x) &= u_\pm & (t \geq 0),
\end{aligned}
\] (1.3)

when the smooth flux function $f$ is genuinely nonlinear on the whole space $\mathbb{R}$, i.e., $f''(u) \neq 0$ ($u \in \mathbb{R}$), Il'in-Oleinik [10] showed the following: if $f''(u) > 0$ ($u \in \mathbb{R}$), that is, the Riemann solution consists of a single rarefaction wave solution, the global solution in time of the Cauchy problem (1.3) tends toward the rarefaction wave; if $f''(u) < 0$ ($u \in \mathbb{R}$), that is, the Riemann solution consists of a single shock wave solution, the global solution of the Cauchy problem (1.3) does the corresponding smooth traveling wave solution ("viscous shock wave") of (1.3) with a spatial shift (cf. [9]). Hattori-Nishihara [7] also proved that the asymptotic decay rate in time, of the solution toward the single rarefaction wave(see also [6], [23]). More generally, in the case of the flux functions which
are not uniformly genuinely nonlinear, when the Riemann solution consists of a single shock wave satisfying Oleĭnik’s shock condition, Matsumura-Nishihara [19] showed the asymptotic stability of the corresponding viscous shock wave. Moreover, Matsumura-Yoshida [20] considered the circumstances where the Riemann solution generically forms a complex pattern of multiple nonlinear waves which consists of rarefaction waves and waves of contact discontinuity (refer to [14]), and investigated that the case where the flux function \( f \) is smooth and genuinely nonlinear (that is, \( f \) is convex function or concave function) on the whole \( \mathbb{R} \) except a finite interval \( I := (a, b) \subset \mathbb{R} \), and linearly degenerate on \( I \), that is,

\[
\begin{align*}
\text{for } u \in (-\infty, a] \cup \mathbb{R}, \quad f''(u) > 0 & \quad \\text{(1.4)} \\
\text{for } u \in (a, b) \quad f''(u) = 0
\end{align*}
\]

Under the conditions (1.4), the corresponding Riemann solution does form multiwave pattern which consists of the contact discontinuity with the jump from \( u = a \) to \( u = b \) and the rarefaction waves, depending on the choice of \( a, b, u_- \) and \( u_+ \). Thanks to that the cases in which the interval \( (a, b) \) is disjoint from the interval \( (u_-, u_+) \) are similar as in the case the flux function \( f \) is genuinely nonlinear on the whole space \( \mathbb{R} \), and the case \( u_- < a < u_+ < b \) is the same as that for \( a < u_- < b < u_+ \), we may only consider the typical cases

\[
a < u_- < b < u_+ \quad \text{or} \quad u_- < a < b < u_+. \quad \text{(1.5)}
\]

Under the conditions (1.4) and (1.5), they have shown the unique global solution in time to the Cauchy problem (1.3) tends uniformly in space toward the multiwave pattern of the combination of the viscous contact wave and the rarefaction waves as the time goes to infinity. It should be noted that the rarefaction wave which connects the far field states \( u_- \) and \( u_+ \) \((u_\pm \in (-\infty, a] \text{ or } u_\pm \in [b, \infty))\) is explicitly given by

\[
u = \nu^r \left( \frac{x}{t}; u_-, u_+ \right) := \begin{cases} 
    u_- & \quad \text{for } x \leq \lambda(u_-) \ t, \\
    (\lambda)^{-1} \left( \frac{x}{t} \right) \quad \text{for } \lambda(u_-) \ t < x \leq \lambda(u_+) \ t, \\
    u_+ & \quad \text{for } x \geq \lambda(u_+) \ t,
\end{cases}
\]

where \( \lambda(u) := f'(u) \), and the viscous contact wave which connects \( u_- \) and \( u_+ \) \((u_\pm \in [-a, b])\) is given by an exact solution of the linear convective heat equation

\[
\partial_t u + \tilde{\lambda} \partial_x u = \mu \partial_x^2 u \quad \left( \tilde{\lambda} := \frac{f(b) - f(a)}{b - a}, \ t > 0, x \in \mathbb{R} \right) \quad \text{(1.7)}
\]

which has the form

\[
u = U \left( \frac{x - \tilde{\lambda} t}{\sqrt{t}}; u_-, u_+ \right)
\]

where \( U \left( \frac{x}{\sqrt{t}}; u_-, u_+ \right) \) is explicitly defined by

\[
U \left( \frac{x}{\sqrt{t}}; u_-, u_+ \right) := u_- + \frac{u_+ - u_-}{\sqrt{\pi}} \int_{-\infty}^{\sqrt{4\mu t}} e^{-\xi^2} \ d\xi \quad \text{for } t > 0, x \in \mathbb{R}. \quad \text{(1.8)}
\]
Yoshida [29] also obtained the almost optimal decay properties for the asymptotics toward the multiwave pattern. In fact, owing to [29], the decay rate in time is \((1 + t)^{-\frac{1}{2}}(\frac{1}{2} - \frac{1}{p})\) in the \(L^p\)-norm \((2 \leq p < +\infty)\) and \((1 + t)^{-\frac{1}{2} + \epsilon}\) for any \(\epsilon > 0\) in the \(L^\infty\)-norm if the initial perturbation from the corresponding asymptotics satisfies \(H^1\). Furthermore, if the perturbation satisfies \(H^1 \cap L^1\), the decay rate in time is \((1 + t)^{-\frac{1}{2} + \epsilon}\) for any \(\epsilon > 0\) in the \(L^p\)-norm \((1 \leq p < +\infty)\) and \((1 + t)^{-\frac{1}{2} + \epsilon}\) for any \(\epsilon > 0\) in the \(L^\infty\)-norm. For \(p > 1\), there are few results for the asymptotic behavior for the problem (1.1) (the related problems are studied in [4], [21], [22] and so on). In the case where the flux function is genuinely nonlinear on the whole space \(\mathbb{R}\), Matsumura-Nishihara [19] proved that if the far field states satisfy \(u_- = u_+ =: \bar{u}\), then the solution tends toward the constant state \(\bar{u}\), and if the far field states \(u_- < u_+\), then the solution tends toward a single rarefaction wave. Furthermore, Yoshida [31] recently showed their precise decay estimates. In the case where the flux function is given as (1.4) and the far field states as (1.5), Yoshida [30] also showed that the similar asymptotics as the one in [20] which tends toward the multiwave pattern of the combination of the rarefaction waves which connect the far field states \(u_-\) and \(u_+\) \((u_\pm \in (-\infty, a] \text{ or } u_\pm \in [b, \infty))\), and the viscous contact wave which connects \(u_-\) and \(u_+\) \((u_\pm \in [a, b])\). In more detail, the viscous contact wave is said to be “contact wave for \(p\)-Laplacian type viscosity” and explicitly given by

\[
U \left( \frac{x - \bar{t}}{t^{1 + 1/p}}; u_-, u_+ \right) = u_- + \int_{-\infty}^{\frac{x - \bar{t}}{t^{1 + 1/p}}} \left( (A - B\xi^2) \vee 0 \right)^{\frac{1}{p-1}} \, d\xi, \quad (1.9)
\]

\[
\begin{cases}
A = A_{p,p,\pm} := \left( \frac{(p-1)(u_+ - u_-)}{8 \mu p(p+1) \left( \int_0^{\frac{\pi}{2}} (\sin \theta)^{\frac{p+1}{p-1}} \, d\theta \right)^2} \right)^{\frac{p+1}{p-1}}, \\
B = B_{p,p} := \frac{p-1}{2 \mu p(p+1)}, \\
2A^{\frac{p+1}{p-1}}B^{-\frac{1}{2}} \int_0^{\frac{\pi}{2}} (\sin \theta)^{\frac{p+1}{p-1}} \, d\theta = u_+ - u_-.
\end{cases}
\]

It should be noted that the wave (1.9) is constructed by the Barenblatt-Kompanecev-Zel’dovič solution (see also [2], [8], [11])

\[
v(t, x) := \frac{1}{(1 + t)^{1/p+1}} \left( \left( A - B \left( \frac{x}{(1 + t)^{1/p+1}} \right)^2 \right) \vee 0 \right)^{\frac{1}{p-1}}, \quad (1.10)
\]
of the following Cauchy problem of the porous medium equation
\[
\begin{aligned}
\begin{cases}
\partial_tv = \mu \partial_x^2 \left( |v|^{p-1} v \right) & (t > -1, x \in \mathbb{R}), \\
v(-1, x) = (u_+ - u_-) \delta(x) & (x \in \mathbb{R}; u_- < u_+), \\
\lim_{x \to \pm\infty} v(t, x) = 0 & (t \geq -1),
\end{cases}
\end{aligned}
\]  
(1.11)

where \( \delta(x) \) is the Dirac \( \delta \)-distribution.

The aim of the present paper is to obtain the precise time-decay estimates for the asymptotics of the previous study in [30].

**Stability Theorem** (Yoshida [30]). Let the flux function \( f \) satisfy (1.4) and the far field states \( u_{\pm} \) (1.5). Assume that the initial data satisfies \( u_0 - u_0^R \in L^2 \) and \( \partial_x u_0 \in L^{p+1} \). Then the Cauchy problem (1.1) with \( p > 1 \) has a unique global solution in time \( u = u(t, x) \) satisfying
\[
\begin{aligned}
\begin{cases}
u - u_0^R \in C^0 \left( [0, \infty) ; L^2 \right) \cap L^\infty \left( \mathbb{R}^+ ; L^2 \right), \\
\partial_x u \in L^\infty \left( \mathbb{R}^+ ; L^{p+1} \right), \\
\partial_t u \in L^\infty \left( \mathbb{R}^+ ; L^{p+1} \right), \\
\partial_x \left( |\partial_x u|^{p-1} \partial_x u \right) \in L^2 \left( \mathbb{R}^+_t \times \mathbb{R}_x \right),
\end{cases}
\end{aligned}
\]
and the asymptotic behavior
\[
\lim_{t \to \infty} \sup_{x \in \mathbb{R}} |u(t, x) - U_{\text{multi}}(t, x ; u_-, u_+) | = 0,
\]
where \( U_{\text{multi}}(t, x) = U_{\text{multi}}(t, x ; u_-, u_+) \) is defined as follows: in the case \( a < u_- < b < u_+ \),
\[
U_{\text{multi}}(t, x) := U \left( \frac{x - \lambda t}{t^{\frac{1}{p+1}}} ; u_-, b \right) + u^r \left( \frac{x}{t} ; b, u_+ \right) - b
\]
and, in the case \( u_- < a < b < u_+ \),
\[
U_{\text{multi}}(t, x) := u^r \left( \frac{x}{t} ; u_-, a \right) - a + U \left( \frac{x - \lambda t}{t^{\frac{1}{p+1}}} ; a, b \right) + u^r \left( \frac{x}{t} ; b, u_+ \right) - b.
\]

Now we are ready to state our main results.

**Theorem 1.1** (Main Theorem 1). Under the same assumptions in Stability Theorem, the unique global solution in time \( u \) of the Cauchy problem (1.1) satisfying
\[
\begin{aligned}
\begin{cases}
u - u_0^R \in C^0 \left( [0, \infty) ; L^2 \right) \cap L^\infty \left( \mathbb{R}^+ ; L^2 \right), \\
\partial_x u \in L^\infty \left( \mathbb{R}^+ ; L^{p+1} \right), \\
\partial_t u \in L^\infty \left( \mathbb{R}^+ ; L^{p+1} \right), \\
\partial_x \left( |\partial_x u|^{p-1} \partial_x u \right) \in L^2 \left( \mathbb{R}^+_t \times \mathbb{R}_x \right),
\end{cases}
\end{aligned}
\]
satisfies the following time-decay estimates

\[
\begin{align*}
\| u(t) - U_{\text{multi}}(t, \cdot ; u_-, u_+) \|_{L^q} & \leq C(p, q, u_0)(1 + t)^{-\frac{3}{2}p(1 - \frac{2}{q})}, \\
\| u(t) - U_{\text{multi}}(t, \cdot ; u_-, u_+) \|_{L^\infty} & \leq C(\epsilon, p, q, u_0, \partial_x u_0)(1 + t)^{-\frac{1}{2}p + \epsilon},
\end{align*}
\]

for \( q \in [2, \infty) \) and any \( \epsilon > 0 \).

**Theorem 1.2** (Main Theorem II). Under the same assumptions in Theorem 1.1, if the initial data further satisfies \( u_0 - u_0^R \in L^1 \), then it holds that the unique global solution in time \( u \) of the Cauchy problem (1.1) satisfies the following time-decay estimates

\[
\begin{align*}
\| u(t) - U_{\text{multi}}(t, \cdot ; u_-, u_+) \|_{L^q} & \leq C(p, q, u_0)(1 + t)^{-\frac{3}{2}p(1 - \frac{2}{q})}, \\
\| u(t) - U_{\text{multi}}(t, \cdot ; u_-, u_+) \|_{L^\infty} & \leq C(\epsilon, p, q, u_0, \partial_x u_0)(1 + t)^{-\frac{1}{2}p + \epsilon},
\end{align*}
\]

for \( q \in [1, \infty) \) and any \( \epsilon > 0 \). Furthermore, the solution satisfies the following time-decay estimates for the higher order derivative

\[
\| \partial_x u(t) \|_{L^{p+1}}, \| \partial_x u(t) - \partial_x U_{\text{multi}}(t, \cdot ; u_-, u_+) \|_{L^{p+1}} \leq \begin{cases} 
C(p, u_0, \partial_x u_0)(1 + t)^{-\frac{3}{2}p(1 - \frac{2}{q})} & \left( 1 < p < \frac{7 + \sqrt{73}}{12} \right), \\
C(\epsilon, p, u_0, \partial_x u_0)(1 + t)^{-\frac{3}{2}p(1 - \frac{2}{q}) + \epsilon} & \left( \frac{7 + \sqrt{73}}{12} \leq p \right),
\end{cases}
\]

for any \( \epsilon > 0 \).

**Theorem 1.3** (Main Theorem III). Under the same assumptions in Theorem 1.2, if the initial data further satisfies \( \partial_x u_0 \in L^{r+1} (r > p) \), then it holds that the unique global solution in time \( u \) of the Cauchy problem (1.1) satisfies the
following time-decay estimates for the higher order derivative

\[ \| \partial_x u(t) \|_{L^{r+1}}, \quad \| \partial_x u(t) - \partial_x U_{\text{multi}}(t; \cdot; u_-, u_+) \|_{L^{r+1}} \leq \begin{cases} 
C(p, r, u_0, \partial_x u_0) (1 + t)^{-\frac{4p(r-p)+7p+3}{6p(r+1)(r+1)}} & \left( 1 < p < \frac{7+\sqrt{73}}{12}, \quad r > \frac{-4p^2 + 7p + 3}{2p} > p \right), \\
C(\epsilon, p, r, u_0, \partial_x u_0) (1 + t)^{-\frac{p+2r}{2r(p-2)(r+1)}} & \left( 1 < p < \frac{7+\sqrt{73}}{12}, \quad p < r \leq \frac{-4p^2 + 7p + 3}{2p} \right), \\
C(\epsilon, p, r, u_0, \partial_x u_0) (1 + t)^{-\frac{p+2r}{2r(p-2)(r+1)}} & \left( \frac{7+\sqrt{73}}{12} \leq p \right), 
\end{cases} \]

for any \( \epsilon > 0 \).

This paper is organized as follows. In Section 2, we shall prepare the basic properties of the rarefaction wave and the contact wave for \( p \)-Laplacian type viscosity. In Section 3, we reformulate the problem in terms of the deviation from the asymptotic state (similarly in [20], [29], [30]). Following the arguments in [18], we also prepare some uniform boundedness and energy estimates of the deviation as the solution to the reformulated problem. We further introduce the precise properties of the interactions between the nonlinear waves. In order to obtain the time-decay estimates (Theorem 1.1 and Theorem 1.2), in Section 4 and Section 5, we establish the uniform energy estimates in time by using a very technical time-weighted energy method. Finally, in Section 6, we prove the time-decay \( L^{r+1} \)-estimate for the higher order derivative, Theorem 1.3.

**Some Notation.** We denote by \( C \) generic positive constants unless they need to be distinguished. In particular, use \( C(\alpha, \beta, \cdots) \) or \( C_{\alpha, \beta, \cdots} \) when we emphasize the dependency on \( \alpha, \beta, \cdots \). Use \( \mathbb{R}^+ \) as \( \mathbb{R}^+ := (0, \infty) \), and the symbol "\( \vee \)" as

\[ a \vee b := \max\{a, b\}. \]

We also use the Friedrichs mollifier \( \rho_\delta^* \), where, \( \rho_\delta(x) := \frac{\delta}{\pi} \rho \left( \frac{x}{\delta} \right) \) with

\[ \rho \in C_0^\infty(\mathbb{R}), \quad \rho(x) \geq 0 (x \in \mathbb{R}), \]

\[ \text{supp}\{\rho\} \subset \{x \in \mathbb{R} \mid |x| \leq 1\}, \quad \int_{-\infty}^{\infty} \rho(x) \, dx = 1, \]

and \( \rho_\delta^* f \) denote the convolution. For function spaces, \( L^p = L^p(\mathbb{R}) \) and \( H^k = H^k(\mathbb{R}) \) denote the usual Lebesgue space and \( k \)-th order Sobolev space on the
whole space $\mathbb{R}$ with norms $\| \cdot \|_{L^p}$ and $\| \cdot \|_{H^k}$, respectively. We also define the bounded $C^m$-class $B^m$ as follows

$$f \in B^m(\Omega) \iff f \in C^m(\Omega), \sup_{\Omega} \sum_{k=0}^{m} |D^k f| < \infty$$

for $m < \infty$ and

$$f \in B^\infty(\Omega) \iff \forall n \in \mathbb{N}, f \in C^n(\Omega), \sup_{\Omega} \sum_{k=0}^{n} |D^k f| < \infty$$

where $\Omega \subset \mathbb{R}^d$ and $D^k$ denote the all of $k$-th order derivatives.

2. Preliminaries

In this section, we shall arrange the several lemmas concerned with the basic properties of the rarefaction wave for accomplishing the proof of our main theorems. Since the rarefaction wave $u^r$ is not smooth enough, we need some smooth approximated one as in the previous results in [6], [16], [17], [20], [29], [30], [31]. We start with the well-known arguments on $u^r$ and the method of constructing its smooth approximation. We first consider the rarefaction wave solution $w^r$ to the Riemann problem for the non-viscous Burgers equation

$$\begin{align*}
\left\{ \begin{array}{ll}
\partial_t w + \partial_x \left( \frac{1}{2} w^2 \right) &= 0 \quad (t > 0, x \in \mathbb{R}), \\
w(0, x) &= w_0^R(x; w^-, w^+) := \begin{cases} \\
    w_+ & (x > 0), \\
    w_- & (x < 0), \\
\end{cases}
\end{array} \right.
\end{align*} \quad (2.1)$$

where $w_\pm \in \mathbb{R}$ are the prescribed far field states. If the far field states satisfy $w_- < w_+$, then the Riemann problem (2.1) has a unique global weak solution $w = w^r \left( \frac{x}{t}; w^-, w^+ \right)$ given explicitly by

$$w^r \left( \frac{x}{t}; w^-, w^+ \right) := \begin{cases} \\
    w_- & (x \leq w_- t), \\
    \frac{x}{t} & (w_- t \leq x \leq w_+ t), \\
    w_+ & (x \geq w_+ t), \\
\end{cases} \quad (2.2)$$

Next, under the condition $f''(u) > 0$ ($u \in \mathbb{R}$) and $u_- < u_+$, the rarefaction wave solution $u = u^r \left( \frac{x}{t}; u^-, u^+ \right)$ of the Riemann problem (1.2) for hyperbolic conservation law is exactly given by

$$u^r \left( \frac{x}{t}; u^-, u^+ \right) = (\lambda)^{-1} \left( u^r \left( \frac{x}{t}; \lambda^-, \lambda^+ \right) \right) \quad (2.3)$$

which is nothing but (1.6), where $\lambda_\pm := \lambda(u_\pm) = f'(u_\pm)$. We also consider the Cauchy problem for the following non-viscous Burgers equation

$$\begin{align*}
\left\{ \begin{array}{ll}
\partial_t w + \partial_x \left( \frac{1}{2} w^2 \right) &= 0 \quad (t > 0, x \in \mathbb{R}), \\
w(0, x) &= w_0(x) := \frac{w_- + w_+}{2} + \frac{w_+ - w_-}{2} \tanh x \quad (x \in \mathbb{R}).
\end{array} \right.
\end{align*} \quad (2.4)$$
By using the method of characteristics, we easily see that the Cauchy problem (2.4) has a unique classical solution

\[ w = w(t, x; w_-, w_+) \in \mathcal{B}_\infty ([0, \infty) \times \mathbb{R}) \]

satisfying the following formula

\[
\begin{cases}
  w(t, x) = w_0(x_0(t, x)) = \frac{\lambda_- + \lambda_+}{2} + \frac{\lambda_+ - \lambda_-}{2} \tanh(x_0(t, x)), \\
  x = x_0(t, x) + w_0(x_0(t, x)) t.
\end{cases}
\] (2.5)

We define a smooth approximation of \( w^r \left( \frac{x}{t}; w_-, w_+ \right) \) by the solution \( w^r \). We also note the assumption of the flux function \( f \) to be

\[ \lambda' \left( u \right) \left( \frac{d^2 f}{du^2} \left( u \right) \right) > 0. \]

Now we summarize the results for the smooth approximation \( w(t, x; w_-, w_+) \) in the next lemma. Since the proof is given by the direct calculation as in [17], we omit it.

**Lemma 2.1.** Assume that the far field states satisfy \( w_- < w_+ \). Then the classical solution \( w(t, x) = w(t, x; w_-, w_+) \) given by (2.4) satisfies the following properties:

1. \( w_- < w(t, x) < w_+ \) and \( \partial_x w(t, x) > 0 \) \( (t > 0, x \in \mathbb{R}) \).
2. For any \( 1 \leq q \leq \infty \), there exists a positive constant \( C_q \) such that

\[
\| \partial_x w(t) \|_{L^q} \leq C_q (1 + t)^{-1 + \frac{1}{q}} \quad (t \geq 0),
\]

\[
\| \partial_x^2 w(t) \|_{L^q} \leq C_q (1 + t)^{-1} \quad (t \geq 0).
\]

3. \( \lim_{t \to \infty} \sup_{x \in \mathbb{R}} \left| w(t, x) - w^r \left( \frac{x}{t} \right) \right| = 0. \)

We define the approximation for the rarefaction wave \( u^r \left( \frac{x}{t}; u_-, u_+ \right) \) by

\[
U^r (t, x; u_-, u_+) := \left( \lambda \right)^{-1} \left( w(t, x; \lambda_-, \lambda_+) \right).
\] (2.6)

Then we have the next lemma as in the previous works (cf. [6], [10], [17], [20], [29], [30], [31]).

**Lemma 2.2.** Assume that the far field states satisfy \( u_- < u_+ \), and the flux function \( f \in C^3(\mathbb{R}) \), \( f''(u) > 0 \) \( (u \in [u_-, u_+]) \). Then we have the following properties:

1. \( U^r(t, x) \) defined by (2.6) is the unique \( C^2 \)-global solution in space-time of the Cauchy problem

\[
\begin{cases}
  \partial_t U^r + \partial_x \left( f(U^r) \right) = 0 \quad (t > 0, x \in \mathbb{R}), \\
  U^r(0, x) = \left( \lambda \right)^{-1} \left( \frac{\lambda_- + \lambda_+}{2} + \frac{\lambda_+ - \lambda_-}{2} \tanh x \right) \quad (x \in \mathbb{R}), \\
  \lim_{x \to \pm \infty} U^r(t, x) = u_\pm \quad (t \geq 0).
\end{cases}
\]
(2) $u_- < U^r(t, x) < u_+$ and $\partial_x U^r(t, x) > 0$ \hspace{1em} (t > 0, x \in \mathbb{R}).$

(3) For any $1 \leq q \leq \infty$, there exists a positive constant $C_q$ such that

$$\| \partial_x U^r(t) \|_{L^q} \leq C_q (1 + t)^{-1 + \frac{1}{q}} \quad (t \geq 0),$$

$$\| \partial^2_x U^r(t) \|_{L^q} \leq C_q (1 + t)^{-1} \quad (t \geq 0).$$

(4) \hspace{1em} \lim_{t \to \infty} \sup_{x \in \mathbb{R}} \left| U^r(t, x) - u^r \left( \frac{x}{t} \right) \right| = 0.

(5) For any $\epsilon \in (0, 1)$, there exists a positive constant $C_\epsilon$ such that

$$|U^r(t, x) - u_+| \leq C_\epsilon (1 + t)^{-1 + \epsilon} e^{-\epsilon|x - \lambda_+ t|} \quad (t \geq 0, x \geq \lambda_+ t).$$

(6) For any $\epsilon \in (0, 1)$, there exists a positive constant $C_\epsilon$ such that

$$|U^r(t, x) - u_-| \leq C_\epsilon (1 + t)^{-1 + \epsilon} e^{-\epsilon|x - \lambda_- t|} \quad (t \geq 0, x \leq \lambda_- t).$$

(7) For any $\epsilon \in (0, 1)$, there exists a positive constant $C_\epsilon$ such that

$$\left| U^r(t, x) - u^r \left( \frac{x}{t} \right) \right| \leq C_\epsilon (1 + t)^{-1 + \epsilon} \quad (t \geq 1, \lambda_- t \leq x \leq \lambda_+ t).$$

(8) For any $(\epsilon, q) \in (0, 1) \times [1, \infty]$, there exists a positive constant $C_{\epsilon, q}$ such that

$$\left\| U^r(t, \cdot) - u^r \left( \frac{\cdot}{t} \right) \right\|_{L^q} \leq C_{\epsilon, q} (1 + t)^{-1 + \frac{1}{q} + \epsilon} \quad (t \geq 0).$$

Because the proofs of (1) to (4) are given in [17], (5) to (7) are in [20] and (8) is in [29], we omit the proofs here.

We also prepare the next lemma for the properties of the contact wave for $p$-Laplacian type viscosity $U \left( \frac{x}{t^{1/p}} ; u_-, u_+ \right)$ defined by (1.11). In the following, we abbreviate “contact wave for $p$-Laplacian type viscosity” to “viscous contact wave”. We rewrite the viscous contact wave as

$$U(t, x) = U \left( \frac{x}{t^{1/p}} ; u_-, u_+ \right)$$

$$= u_+ - \int_x^{\infty} \frac{1}{t^{1/p}} \left( \left( A - B \left( \frac{y}{t^{1/p}} \right)^2 \right) \vee 0 \right)^{\frac{2}{p+1}} dy,$$

where

$$A = A_{p, \mu, u_\pm} := \left( \frac{\mu(p - 1)}{8 \mu p(p + 1)} \int_0^\pi \left( \sin \theta \right)^{\frac{p+1}{p+1}} d\theta \right)^{\frac{p-1}{p+1}},$$

$$B = B_{p, \mu} := \frac{p-1}{2 \mu p(p + 1)},$$

$$2A^{\frac{p+1}{p(p-1)}} B^{-\frac{1}{2}} \int_0^\pi \left( \sin \theta \right)^{\frac{p+1}{p+1}} d\theta = u_+ - u_-.$$
Then, the following properties hold.

**Lemma 2.3.** For any \( p > 1 \) and \( u_\pm \in \mathbb{R} \), we have the following:

(i) \( U \) defined by (1.11) satisfies

\[
U \in \mathcal{B}^1((0, \infty) \times \mathbb{R}) \setminus \mathcal{C}^2 \left( \left\{ (t, x) \in \mathbb{R}^+ \times \mathbb{R} \mid x = \pm \sqrt{\frac{A}{B} t^{\frac{1}{p+1}}} \right\} \right),
\]

and is a self-similar type strong solution of the Cauchy problem

\[
\begin{aligned}
\partial_t U - \mu \partial_x \left( \left| \partial_x U \right|^{p-1} \partial_x U \right) &= 0 & (t > 0, x \in \mathbb{R}), \\
U(0, x) &= u_0^R(x; u_-, u_+) = \begin{cases} u_- & (x < 0), \\
& u_+ & (x > 0), \\
\lim_{x \to \pm \infty} U(t, x) &= u_\pm & (t \geq 0).
\end{cases}
\end{aligned}
\]

(ii) For \( t > 0 \) and \( x \in \mathbb{R} \),

\[
\begin{aligned}
U(t, x) &= u_-, & (x \leq -\sqrt{\frac{A}{B} t^{\frac{1}{p+1}}}), \\
u_- < U(t, x) < u_+, \partial_x U(t, x) > 0 & (-\sqrt{\frac{A}{B} t^{\frac{1}{p+1}}} < x < \sqrt{\frac{A}{B} t^{\frac{1}{p+1}}}), \\
U(t, x) &= u_+, & (x \geq \sqrt{\frac{A}{B} t^{\frac{1}{p+1}}}).
\end{aligned}
\]

(iii) It holds that for any \( 1 \leq q < \infty \),

\[
\left\| \partial_x U(t) \right\|_{L^q} = C_1(A, B; p, q) t^{-\frac{2q-1}{4q+1}}, \quad (t > 0)
\]

where

\[
C_1(A, B; p, q) := \left( 2A \frac{p+2q-1}{2q-1} B^{-\frac{q}{2}} \int_0^{\frac{\pi}{2}} (\sin \theta)^{\frac{2q+1}{2}} \, d\theta \right)^{\frac{1}{q}}.
\]

If \( q = \infty \), we have

\[
\left\| \partial_x U(t) \right\|_{L^\infty} = (2A) \frac{p-1}{p-2} t^{-\frac{1}{p-1}}, \quad (t > 0).
\]

(iv) It holds that for any \( 1 \leq q < \frac{p-1}{p-2} \) with \( p > 2 \), or any \( 1 \leq q < \infty \) with \( 1 < p \leq 2 \),

\[
\left\| \partial_x^2 U(t) \right\|_{L^q} = C_2(A, B; p, q) t^{-\frac{2q-1}{4q+1}}, \quad (t > 0)
\]

where

\[
C_2(A, B; p, q) := \left( 2 \left( \frac{2A \frac{p-2}{p-1} B}{p-1} \right)^q \left( \frac{B}{A} \right)^{-\frac{p+1}{2q+1}} \int_0^{\frac{\pi}{2}} (\sin \theta)^{-\frac{2q+1}{p-2} \frac{p-1}{q} + 1} (\cos \theta)^q \, d\theta \right)^{\frac{1}{q}}.
\]
If $1 < p \leq 2$, for $q = \infty$, we have

$$\| \partial_x^2 U(t) \|_{L^\infty} = \frac{2A^{\frac{p-2}{p-1}}B}{p-1} \left( \frac{B}{A} \right)^{-\frac{1}{2}} t^{-\frac{q}{p+1}} (t > 0).$$

(v) It holds that

$$\left\| \partial_x \left( |\partial_x U|^{p-1} \partial_x U \right)(t) \right\|_{L^2} = C_3(A, B ; p) t^{-\frac{2p+1}{2(p+1)}} (t > 0)$$

where

$$C_3(A, B ; p) := \left( 2 \left( \frac{2B^p}{p-1} \right)^2 \left( \frac{B}{A} \right)^{-\frac{3p-7}{2(p+1)}} \int_0^{\frac{\pi}{2}} (\sin \theta)^{\frac{p+3}{p+1}} (\cos \theta)^2 d\theta \right)^{\frac{1}{2}}.$$

(vi) \( \lim_{t \to \infty} \sup_{x \in \mathbb{R}} |U(1 + t, x) - U(t, x)| = 0. \)

(vii) For any $1 \leq q \leq \infty$ with $p > 1$, there exists a positive constant $C_{p,q}$ such that

$$\| U(1 + t, \cdot) - U(t, \cdot) \|_{L^q} \leq C_{p,q} t^{-1+\frac{1}{p+1}} (t > 0).$$

3. Reformulation of the problem

In this section, we reduce our Cauchy problem (1.1) to a simpler case and reformulate the problem in terms of the deviation from the asymptotic state (the same as in [20], [29], [30]). At first, without loss of generality, we shall consider the case where $a < 0$, $b = 0$ and the flux function $f(u)$ satisfies

$$\begin{cases}
  f''(u) > 0 & (u \in (-\infty, a] \cup [0, +\infty)), \\
  f(u) = 0 & (u \in (a, 0)),
\end{cases} \tag{3.1}$$

under changing the variables and constant as $x - \tilde{\lambda}t \mapsto x$, $u - b \mapsto u$, $f(u + b) - f'(b) u - f(a) \mapsto f(u)$ and $a - b \mapsto a$ in this order. For the far field states $u_{\pm} \in \mathbb{R}$, we only deal with the typical case $a < u_- < 0 < u_+$ for simplicity, since the case $u_- < a < 0 < u_+$ can be treated technically in the same way of the proof as $a < u_- < 0 < u_+$. Indeed, in the case $u_- < a < 0 < u_+$, as we shall see in Section 4 and Section 5, there appears the extra nonlinear interaction terms between two rarefaction waves $u'(\frac{\lambda}{2} ; u_- , a)$ and $u'(\frac{\lambda}{2} ; 0 , u_+)$ with $\lambda(a) = \lambda(0) = 0$ in the remainder term of the viscous conservation law for the asymptotics $U_{multi}$ (see the right-hand side of (3.4)). These terms can be handled in much easier way by Lemma 2.2 than that for other essential nonlinear interaction terms between the rarefaction and the viscous contact waves. Furthermore, we should point out that the problem under the assumptions for the flux function (3.1) and the far field states $a < u_- < 0 < u_+$ is essentially the same as that for $a = -\infty$, because obtaining the a priori and the uniform energy estimates for the former
one can be given in almost the same way as the latter one. Therefore, it is quite natural for us to treat only a simple case
\[
\begin{aligned}
& f''(u) > 0 \quad (u \in [0, \infty)), \\
& f(u) = 0 \quad (u \in (-\infty, 0)),
\end{aligned}
\] (3.2)
and assume \( u_- < 0 < u_+ \). The corresponding stability theorem and our main theorems are the following.

Under the situation, we reformulate the problem in terms of the deviation from the asymptotic state. We first should note by Lemma 2.2 and Lemma 2.3, the asymptotic state \( u^r \left( \frac{x}{t}; u_-, u_+ \right) \) can be replaced by a following approximated multiwave pattern
\[
\tilde{U}(t, x) := U(1 + t, x) + U^r(t, x),
\]
where
\[
U(1 + t, x) = U \left( \frac{x}{(1 + t)^{\frac{1}{p+1}}}; u_-, 0 \right), \quad U^r(t, x) = U^r(t, x; 0, u_+).
\]

In fact, from Lemma 2.1 (especially (8)) and Lemma 2.3 (especially (vii)), it follows that for any \( \varepsilon > 0 \)
\[
\left\| \tilde{U}(t, \cdot) - U_{\text{multi}}(t, \cdot) \right\|_{L^q} \leq \left\| U(1 + t, \cdot) - U(t, \cdot) \right\|_{L^q} + \left\| U^r(t, \cdot) - u^r \left( \frac{\cdot}{t} \right) \right\|_{L^q}.
\]
\[
\leq C_{\varepsilon, q}(1 + t)^{-\left(1 - \frac{1}{q}\right)+\varepsilon} \quad (t \geq 0; 1 \leq q \leq \infty).
\]

Then it is noted that \( \tilde{U} \) is monotonically increasing and approximately satisfies the equation of (1.1) as
\[
\partial_t \tilde{U} + \partial_x \left( f(\tilde{U}) \right) - \mu \partial_x \left( |\partial_x \tilde{U}|^{p-1} \partial_x \tilde{U} \right) = -F_p(U, U^r), \quad (3.3)
\]
where the remainder term \( F_p(U, U^r) \) is explicitly given by
\[
F_p(U, U^r) := \tilde{F}_p(U, U^r)
\]
\[
+ \mu \partial_x \left( |\partial_x U + \partial_x U^r|^{p-1} \left( \partial_x U + \partial_x U^r \right) - |\partial_x U|^{p-1} \partial_x U \right)
\]
\[
= - \left( f'(U + U^r) - f'(U^r) \right) \partial_x U^r - f'(U + U^r) \partial_x U
\]
\[
+ \mu \partial_x \left( |\partial_x U + \partial_x U^r|^{p-1} \left( \partial_x U + \partial_x U^r \right) - |\partial_x U|^{p-1} \partial_x U \right)
\] (3.4)
which consists of the interaction terms of the viscous contact wave \( U \) and the approximation of the rarefaction wave \( U^r \), and the approximation error of \( U^r \) as solution to the conservation law for the \( p \)-Laplacian type viscosity. Here we should note that \( U \) is monotonically nondecreasing and \( U^r \) is monotonically
increasing, that is, \( \partial_x \hat{U}(t, x) > 0 \) \((t \geq 0, x \in \mathbb{R})\) which is frequently used hereinafter. Now putting
\[
u(t, x) = \hat{U}(t, x) + \phi(t, x)
\]
and using (3.5), we can reformulate the problem (1.1) in terms of the deviation \( \phi \) from \( \hat{U} \) as
\[
\begin{align*}
\partial_t \phi + \partial_x \left( f(\hat{U} + \phi) - f(\hat{U}) \right) &- \mu \partial_x \left( |\partial_x \hat{U} + \partial_x \phi|^{p-1}(\partial_x \hat{U} + \partial_x \phi) - |\partial_x \hat{U}|^{p-1}\partial_x \hat{U} \right) = F_p(U, U') \quad (t > 0, x \in \mathbb{R}), \\
\phi(0, x) &= \phi_0(x) := u_0(x) - \hat{U}(0, x) \quad (x \in \mathbb{R}), \\
\lim_{x \to \pm \infty} \phi(t, x) &= 0 \quad (t \geq 0).
\end{align*}
\]
Then we look for the global solution in time
\[
\phi \in C^0([0, \infty); L^2) \cap L^\infty(\mathbb{R}^+; L^2)
\]
with
\[
\partial_x \phi \in L^\infty(\mathbb{R}^+; L^{p+1}) \cap L^{p+1}(\mathbb{R}^+_t \times \mathbb{R}_x).
\]
Here we note the fact \( \phi_0 \in L^2 \) and \( \partial_x \phi_0 \in L^{p+1} \) by the assumptions on \( u_0 \) and the fact
\[
\partial_x \hat{U}(0, \cdot) = \partial_x U(0, \cdot) + \partial_x U'(0, \cdot) \in L^{p+1}.
\]
In the following, we always assume that the flux function \( f \in C^1(\mathbb{R}) \cap C^3([0, \infty)) \) satisfies (3.2), and the far field states satisfy \( u_- < 0 < u_+ \). Then the corresponding our main theorems for \( \phi \) we should prove are as follows.

**Theorem 3.1.** Assume that the flux function \( f \in C^1(\mathbb{R}) \cap C^3([0, \infty)) \) satisfies (3.2), the far field states satisfy \( u_- < 0 < u_+ \), and the initial data \( \phi_0 \in L^2 \) and \( \partial_x \phi_0 \in L^{p+1} \). Then, the unique global solution in time \( \phi \) of the Cauchy problem (3.6) satisfying
\[
\begin{align*}
\phi &\in C^0([0, \infty); L^2) \cap L^\infty(\mathbb{R}^+; L^2), \\
\partial_x \phi &\in L^\infty(\mathbb{R}^+; L^{p+1}) \cap L^{p+1}(\mathbb{R}^+_t \times \mathbb{R}_x), \\
\partial_t (\hat{U} + \phi) &\in L^\infty(\mathbb{R}^+; L^{p+2}) \cap L^{p+2}(\mathbb{R}^+_t \times \{x \in \mathbb{R} \mid u > 0\}), \\
\partial_t (\hat{U} + \phi) &\in L^\infty(\mathbb{R}^+; L^{p+1}), \\
\partial_x \left( |\partial_x (\hat{U} + \phi)|^{p-1}\partial_x (\hat{U} + \phi) \right) &\in L^2(\mathbb{R}^+_t \times \mathbb{R}_x),
\end{align*}
\]
and
\[
\lim_{t \to \infty} \sup_{x \in \mathbb{R}} |\phi(t, x)| = 0.
\]
satisfies the following time-decay estimates

\[
\begin{align*}
\| \phi(t) \|_{L^q} &\leq C(p, q, \phi_0) (1 + t)^{-\frac{1}{2} \left(1 - \frac{2}{q}\right)}, \\
\| \phi(t) \|_{L^\infty} &\leq C(\epsilon, p, q, \phi_0, \partial_x u_0) (1 + t)^{-\frac{1}{2} + \epsilon},
\end{align*}
\]

for \( q \in [2, \infty) \) and any \( \epsilon > 0 \).

**Theorem 3.2.** Under the same assumptions in Theorem 3.1, if the initial data further satisfies \( \phi_0 \in L^1 \), then it holds that the unique global solution in time \( \phi \) of the Cauchy problem (3.6) satisfies the following time-decay estimates

\[
\begin{align*}
\| \phi(t) \|_{L^q} &\leq C(p, q, \phi_0) (1 + t)^{-\frac{1}{2} \left(1 - \frac{2}{q}\right)}, \\
\| \phi(t) \|_{L^\infty} &\leq C(\epsilon, p, q, \phi_0, \partial_x u_0) (1 + t)^{-\frac{1}{2} + \epsilon},
\end{align*}
\]

for \( q \in [1, \infty) \) and any \( \epsilon > 0 \). Furthermore, the solution satisfies the following time-decay estimates for the higher order derivative

\[
\begin{align*}
\| \partial_x u(t) \|_{L^{p+1}}, \| \partial_x \phi(t) \|_{L^{p+1}} &\leq \\
C(\epsilon, p, \phi_0, \partial_x u_0) (1 + t)^{-\frac{p}{p+1} \left(1 - \frac{2}{q}\right)} \left(1 < p \leq \frac{7}{12} + \sqrt{\frac{73}{144} - \frac{(p + 1)^2(3p - 2)}{3}} \epsilon \right), \\
C(\epsilon, p, \phi_0, \partial_x u_0) (1 + t)^{-\frac{p}{p+1} \left(1 - \frac{2}{q}\right) + \epsilon} \left(\frac{7}{12} + \sqrt{\frac{73}{144} - \frac{(p + 1)^2(3p - 2)}{3}} \epsilon < p\right),
\end{align*}
\]

for any \( 0 < \epsilon \ll 1 \).

**Theorem 3.3.** Under the same assumptions in Theorem 3.2, if the initial data further satisfies \( \partial_x u_0 \in L^{r+1} \) \((r > p)\), then it holds that the unique global solution in time \( \phi \) of the Cauchy problem (3.6) satisfies the following time-decay
estimates for the higher order derivative

\[ \left\| \partial_x u(t) \right\|_{L^{r+1}}, \left\| \partial_x \phi(t) \right\|_{L^{r+1}} \leq \begin{cases} C(\epsilon, p, r, \phi_0, \partial_x u_0) (1 + t)^{-\frac{4p(r-p) + 7p + 3}{6p(r+1)(r+1)}} & \\
 & \left( 1 < p \leq \frac{7}{12} + \sqrt{\frac{73}{144} - \frac{p(r+1)(3p-2)(r+1)}{2(r-p+1)}} \right) , \epsilon > \frac{-4p^2 + 7p + 3}{2p} > p, \right. \\
C(\epsilon, p, r, \phi_0, \partial_x u_0) (1 + t)^{-\frac{p+2r}{6p+1}} & \\
& \left( 1 < p \leq \frac{7}{12} + \sqrt{\frac{73}{144} - \frac{p(r+1)(3p-2)(r+1)}{2(r-p+1)}} \right) , \epsilon < \frac{-4p^2 + 7p + 3}{2p} \right) , \\
C(\epsilon, p, r, \phi_0, \partial_x u_0) (1 + t)^{-\frac{4p(r-p) + 7p + 3}{6p(r+1)(r+1)}} & \left. + \epsilon \right) \\
& \left( \frac{7}{12} + \sqrt{\frac{73}{144} - \frac{p(r+1)(3p-2)(r+1)}{2(r-p+1)}} \right) , \epsilon < p, \right) \\
& \left. p < r \leq \frac{-4p^2 + 7p + 3}{2p} \right) , \\
\end{cases} \]

for any \( 0 < \epsilon \ll 1 \).

In order to accomplish the proofs of Theorem 3.1, Theorem 3.2 and Theorem 3.3, we will need some estimates about boundedness of the perturbation \( \phi \) and \( u \). We shall arrange some lemmas for them.

By using the maximum principle (cf. \[9\], \[10\]), we first have the following uniform boundedness of the perturbation \( \phi \) and \( u \), that is,

**Lemma 3.1** (uniform boundedness). It holds that

\[ \sup_{t \in [0, \infty), x \in \mathbb{R}} \left| \phi(t, x) \right| \leq \sup_{x \in \mathbb{R}} \left| u_0(x) \right| + \sup_{x \in \mathbb{R}} \left| U(t, x) \right| + \sup_{t \in [0, \infty), x \in \mathbb{R}} \left| U^r(t, x) \right| \quad (3.7) \]

\[ = \| \phi_0 \|_{L^\infty} + 2 \left( |u_-| + |u_+| \right), \]

\[ \sup_{t \in [0, \infty), x \in \mathbb{R}} \left| u(t, x) \right| \leq \| \phi_0 \|_{L^\infty} + 2 \left( |u_-| + |u_+| \right) + |u_-| \vee |u_+| =: \tilde{C}. \quad (3.8) \]

Secondly, we also have the uniform estimates of \( \phi \) as follows (for the proof of it, see \[30\]).

**Lemma 3.2** (uniform estimates). The unique global solution in time \( \phi \) of the Cauchy problem (3.3) satisfies the following uniform energy inequalities.
(1) There exists a positive constant \( C_p(\phi_0) = C_p(\| \phi_0 \|_{L^2}) \) such that
\[
\| \phi(t) \|_{L^2}^2 + \int_0^\infty G(t) \, dt + \int_0^\infty \int_{-\infty}^{\infty} (\partial_x \phi)^2 \left( |\partial_x \phi|^{p-1} + |\partial_x U|^{p-1} + |\partial_x U^r|^{p-1} \right) \, dx \, dt \leq C_p(\phi_0) \quad (t \geq 0),
\]
where \( G(t) \) is exactly given by
\[
G(t) := \left( \int_{U \geq 0} \phi^2 \partial_x \tilde{U} \, dx \right) (t) + \left( \int_{\tilde{U} + \phi \geq 0, \tilde{U} < 0} (\tilde{U} + \phi)^2 \partial_x \tilde{U} \, dx \right) (t)
+ \left( \int_{\tilde{U} + \phi < 0, \tilde{U} \geq 0} (\tilde{U} + |\phi|)^2 \partial_x \tilde{U} \, dx \right) (t).
\]

(2) There exists a positive constant \( C_p(\phi_0, \partial_x u_0) = C_p(\| \phi_0 \|_{L^2}, \| \partial_x u_0 \|_{L^{p+1}}) \) such that
\[
\| \partial_x u(t) \|_{L^{p+1}}^{p+1} + \int_0^\infty \int_{-\infty}^{\infty} |\partial_x u|^{2(p-1)} (\partial_x^2 u)^2 \, dx \, dt \leq C_p(\phi_0, \partial_x u_0) \quad (t \geq 0).
\]

(3) There exists a positive constant \( C_p(\phi_0, \partial_x u_0) = C_p(\| \phi_0 \|_{L^2}, \| \partial_x u_0 \|_{L^{p+1}}) \) such that
\[
\int_0^\infty \| \partial_x u(t) \|_{L^{p+2}}^{p+2} \, dt \leq C_p(\phi_0, \partial_x u_0) \quad (t \geq 0).
\]

We also prepare the precise properties for the nonlinear interaction terms of the viscous contact wave \( U \) and the approximation of rarefaction wave \( U^r \). Namely, due to Lemma 2.2 and Lemma 2.3, we can easily see the fact that for any \( t \geq 0 \) there uniquely exists \( x = X(t) \in \mathbb{R} \) such that
\[
\tilde{U}(t, X(t)) = U(t, X(t)) + U^r(t, X(t)) = 0 \quad (t \geq 0). \tag{3.9}
\]
that is,
\[
U^r(t, X(t)) = -U(t, X(t))
= \int_{X(t)}^\infty \frac{1}{(1 + t)^{\frac{1}{p+1}}} \left( A - B \left( \frac{y}{(1 + t)^{\frac{1}{3}} + 0} \right)^2 \right) \, dy \quad (t \geq 0).
\]

More precisely, we have the following lemma.

**Lemma 3.3.** The function
\[
X : [0, \infty) \ni t \mapsto X(t) \in \mathbb{R}
\]
defined by (3.9) has following asymptotic properties.

(i) There exists a positive time $T_0$ such that for some $\delta \in \left(0, \sqrt{\frac{A}{B}}\right),$
\[
\left(\sqrt{\frac{A}{B}} - \delta\right) (1 + t)^{\frac{1}{p+1}} < X(t) < \sqrt{\frac{A}{B}} (1 + t)^{\frac{1}{p+1}} \quad (t \geq T_0).
\]

(ii) For any $\epsilon \in (0, 1)$, there exists a positive constant $C_{p, \epsilon}$ such that
\[
\left\| (\lambda - 1)^{-1} \left( \frac{X(t)}{1 + t} \right) - \int_0^t \left( (A - B \xi^2) \vee 0 \right)^{\frac{1}{p+1}} d\xi \right\| \leq C_{p, \epsilon} (1 + t)^{-1+\epsilon} \quad (t \geq T_0).
\]

(iii) There exists a positive constant $C_p$ such that
\[
\left\| \sqrt{\frac{A}{B}} - \frac{X(t)}{(1 + t)^{\frac{1}{p+1}}} \right\| \leq C_p (1 + t)^{-\frac{1}{p+1}} \quad (t \geq T_0).
\]

4. Time-decay estimates with $2 \leq q \leq \infty$

In this section, we show the time-decay estimates with $2 \leq q \leq \infty$ (not assuming $L^1$-integrability to the initial perturbation), that is, Theorem 3.1. To do that, we shall obtain the time-weighted $L^q$-energy estimates to $\phi$ with $2 \leq q < \infty$ (cf. [29]).

**Proposition 4.1.** Suppose the same assumptions in Theorem 3.1. For any $q \in [2, \infty)$, there exist positive constants $\alpha$ and $C_{\alpha, p, q}$, such that the unique global solution in time $\phi$ of the Cauchy problem (3.6) satisfies the following $L^q$-energy estimate
\[
(1 + t)^\alpha \| \phi(t) \|_{L^q}^q + \int_0^t (1 + \tau)^\alpha G_q(\tau) d\tau
\]
\[
+ \int_0^t (1 + \tau)^\alpha \int_{-\infty}^\infty |\phi|^{q-2} |\partial_x \phi|^2 d\kappa d\tau
\]
\[
+ \int_0^t (1 + \tau)^\alpha \int_{-\infty}^\infty |\phi|^{q-2} \left| |\partial_x \phi + \partial_x U + \partial_x U^r|^{p-1} - |\partial_x U + \partial_x U^r|^{p-1}\right| d\kappa d\tau
\]
\[
\leq C_{\alpha, p, q} \| \phi_0 \|_{L^q}^q + C(\alpha, p, q, \phi_0) (1 + t)^{\alpha - \frac{2}{p+1}} \quad (t \geq T_0),
\]
(4.1)
Lemma 4.2. Assume \( G_q = G_q(t) \) is explicitly given by

\[
G_q(t) := \left( \int_{\tilde{U} + \phi < 0, \tilde{U} \geq 0} |\phi|^q \partial_x \tilde{U} \, dx \right)(t)
\]

\[
+ \left( \int_{\tilde{U} + \phi > 0, \tilde{U} \geq 0} |\phi|^{q-1} \tilde{U} \partial_x \tilde{U} \, dx \right)(t)
\]

\[
+ \left( \int_{\tilde{U} + \phi > 0, \tilde{U} < 0} \left( |\phi|^{q-1} (q \tilde{U} + (q - 1) |\phi|) + |\tilde{U}^q| \right) \partial_x \tilde{U} \, dx \right)(t).
\]

The proof of Proposition 4.1 is provided by the following two lemmas.

**Lemma 4.1.** For any \( 2 \leq q < \infty \), there exist positive constants \( \alpha \) and \( C_q \) such that

\[
(1 + t)^{\alpha} \|\phi(t)\|_{L^q}^q + q (q - 1) \int_0^t (1 + \tau)^{\alpha} G_q(\tau) \, d\tau
\]

\[
+ C_q \int_0^t (1 + \tau)^{\alpha} \int_{-\infty}^{\infty} |\phi|^{q-2} \left( \partial_x \phi \right)^2
\]

\[
\times \left( |\partial_x \phi|^{p-1} + |\partial_x U|^{p-1} + |\partial_x U'|^{p-1} \right) \, dx \, d\tau
\]

\[
+ C_q \int_0^t (1 + \tau)^{\alpha} \int_{-\infty}^{\infty} |\phi|^{q-2} \left| \partial_x \phi \partial_x \tilde{U} \right|^{p-1} \, dx \, d\tau
\]

\[
\leq \|\phi_0\|_{L^q}^q + \alpha \int_0^t (1 + \tau)^{\alpha-1} \|\phi(\tau)\|_{L^r}^r \, d\tau
\]

\[
+ q \int_0^t (1 + \tau)^{\alpha} \|\phi(\tau)\|_{L^r}^{p-1} \left\| F_\mu(U, U') (\tau) \right\|_{L^1} \, d\tau
\]

\[
+ \mu q \int_0^t (1 + \tau)^{\alpha} \|\phi(\tau)\|_{L^r}^{p-2} \times \int_{-\infty}^{\infty} |\partial_x \phi| \left( (\partial_x U + \partial_x U')^p - (\partial_x U)^p \right) \, dx \, d\tau (t \geq 0).
\]

(4.2)

**Lemma 4.2.** Assume \( p > 1 \) and \( 2 \leq q < \infty \). We have the following interpolation inequalities.

1. For any \( 2 \leq r < \infty \), there exists a positive constant \( C_{p,q,r} \) such that

\[
\|\phi(t)\|_{L^r} \leq C_{p,q,r} \left( \int_{-\infty}^{\infty} |\phi|^2 \, dx \right)^{\frac{p^r + p + 1}{(p+1)(p+q-1)r}}
\]

\[
\times \left( \int_{-\infty}^{\infty} |\phi|^{q-2} |\partial_x \phi|^{p+1} \, dx \right)^{\frac{r-2}{(p+q-1)r}} (t \geq 0).
\]
There exists a positive constant $C_{p,q}$ such that
\[
\| \phi(t) \|_{L^\infty} \leq C_{p,q} \left( \int_{-\infty}^{\infty} |\phi|^2 \, dx \right)^{\frac{p}{q+p+1}} \times \left( \int_{-\infty}^{\infty} |\phi|^{q-2} |\partial_x \phi|^{p+1} \, dx \right)^{\frac{1}{q+p+1}} (t \geq 0).
\]

In what follows, we first prove Lemma 4.1 and finally give the proof of Proposition 4.1 (the proof of Lemma 4.2 is given in [31], so we omit here).

**Proof of Lemma 4.1.** Multiplying the equation in (3.6) by $|\phi|^{q-2} \phi$ with $2 \leq q < \infty$, we obtain the divergence form
\[
\partial_t \left( \frac{1}{q} |\phi|^q \right) + \partial_x \left( |\phi|^{q-2} \phi \left( f(\tilde{U} + \phi) - f(\tilde{U}) \right) \right)
+ \partial_x \left( -(q-1) \int_0^\phi \left( f(\tilde{U} + \eta) - f(\tilde{U}) \right) |\eta|^{q-2} \, d\eta \right)
+ \partial_x \left( -\mu |\phi|^{q-2} \phi \right.
\times \left( |\partial_x \tilde{U} + \partial_x \phi|^{p-1} (\partial_x \tilde{U} + \partial_x \phi) - |\partial_x \tilde{U}|^{p-1} (\partial_x \tilde{U}) \right) \bigg) \tag{4.3}
+ (q-1) \int_0^\phi \left( \lambda(\tilde{U} + \eta) - \lambda(\tilde{U}) \right) |\eta|^{q-2} \, d\eta \left( \partial_x \tilde{U} \right)
+ \mu (q-1) |\phi|^{q-2} \partial_x \phi
\times \left( |\partial_x \tilde{U} + \partial_x \phi|^{p-1} (\partial_x \tilde{U} + \partial_x \phi) - |\partial_x \tilde{U}|^{p-1} (\partial_x \tilde{U}) \right)
= |\phi|^{q-2} \phi F_p(U, U').
\]
Integrating (4.3) with respect to $x$, we have
\[
\frac{1}{q} \frac{d}{dt} \| \phi(t) \|_{L^q}^q
+ \int_{-\infty}^{\infty} (q-1) \left( \lambda(\tilde{U} + \eta) - \lambda(\tilde{U}) \right) |\eta|^{q-2} \, d\eta \left( \partial_x \tilde{U} \right) \, dx
+ \mu (q-1) \int_{-\infty}^{\infty} |\phi|^{q-2} \partial_x \phi
\times \left( |\partial_x \tilde{U} + \partial_x \phi|^{p-1} (\partial_x \tilde{U} + \partial_x \phi) - |\partial_x \tilde{U}|^{p-1} (\partial_x \tilde{U}) \right) \, dx \tag{4.4}
= \int_{-\infty}^{\infty} |\phi|^{q-2} \phi F_p(U, U') \, dx.
\]
In order to estimate the second term on the left-hand side of (4.4), noting the shape of the flux function $f$, we divide the integral region of $x$ depending on the signs of $U + \phi$, $\tilde{U}$ and $\phi$ as

$$
\int_{-\infty}^{\infty} \left( \int_{0}^{\phi} \left( \lambda(\tilde{U} + \eta) - \lambda(\tilde{U}) \right) |\eta|^{q-2} \, d\eta \right) \left( \partial_x \tilde{U} \right) \, dx
$$

$$
= \int_{\tilde{U} + \phi \geq 0} + \int_{\tilde{U} + \phi \geq 0, \phi \leq 0} + \int_{\tilde{U} + \phi \geq 0, \phi < 0} + \int_{\tilde{U} + \phi < 0, \tilde{U} \geq 0}
$$

where we used the fact that the integral is clearly zero on the domain $\tilde{U} + \phi \leq 0$ and $\tilde{U} \leq 0$. By Lagrange’s mean-value theorem, we easily get as

$$
\left( \int_{-\infty}^{\infty} \left( \int_{0}^{\phi} \left( \lambda(\tilde{U} + \eta) - \lambda(\tilde{U}) \right) |\eta|^{q-2} \, d\eta \right) \left( \partial_x \tilde{U} \right) \, dx \right) (t) \sim G_q(t)
$$

where $G_q = G_q(t)$ is defined in Proposition 4.1 (cf. [20], [29], [30]). Next, by using the uniform boundedness, Lemma 3.1, and the following absolute equality with $p > 1$, for any $a, b \in \mathbb{R},$

$$
\left( | a |^{p-1} a - | b |^{p-1} b \right) (a - b)
$$

$$
= \frac{1}{2} \left( | a |^{p-1} + | b |^{p-1} \right) (a - b)^2 + \frac{1}{2} \left( | a |^{p-1} - | b |^{p-1} \right) (a^2 - b^2)
$$

$$
\geq \frac{1}{4} \left( | a |^{p-1} + | b |^{p-1} + | a - b |^{p-1} \right) (a - b)^2 + \frac{1}{2} \left( | a |^{p-1} - | b |^{p-1} \right) (a^2 - b^2),
$$

we have

$$
\frac{1}{q} \frac{d}{dt} \| \phi(t) \|_{L^q}^q + C_{p,q}^{-1} G_q(t)
$$

$$
+ \frac{\mu (q-1)}{4} \int_{-\infty}^{\infty} \phi |\eta|^{q-2} \left( |\partial_x \phi|^{p-1} + |\partial_x U|^{p-1} + |\partial_x U^r|^{p-1} \right) \, dx
$$

$$
+ \frac{\mu (q-1)}{2} \int_{-\infty}^{\infty} \phi |\eta|^{q-2} \left| |\partial_x \phi + \partial_x U^r|^{p-1} - |\partial_x U|^{p-1} \right|
$$

$$
\times \left( |\partial_x \phi + \partial_x U^r|^2 - (\partial_x U^r)^2 \right) \, dx
$$

$$
\leq \int_{-\infty}^{\infty} |\phi|^{q-2} \phi F_p(U, U^r) \, dx.
$$

(4.5)

We note the right-hand side of (4.5) can be estimated as

$$
\left| \int_{-\infty}^{\infty} |\phi|^{q-2} \phi F_p(U, U^r) \, dx \right| \leq \left| \int_{-\infty}^{\infty} |\phi|^{q-1} \tilde{F}_p(U, U^r) \, dx \right|
$$

$$
+ \int_{-\infty}^{\infty} |\phi|^{q-2} \left| \partial_x \phi \right| \left( (\partial_x U + \partial_x U^r)^p - (\partial_x U)^p \right) \, dx.
$$

(4.6)
Thus, substituting (4.6) into (4.5), multiplying the inequality by \((1 + t)^\alpha\) with \(\alpha > 0\) and integrating over \((0, t)\) with respect to the time, we complete the proof of Lemma 4.1.

**Proof of Proposition 4.1.** By using Lemma 4.1 and Lemma 4.2, we shall estimate the second term, the third term and the fourth term on the right-hand side of (4.2) as follows: for any \(\epsilon > 0\),

\[
\alpha \int_0^t (1 + \tau)^{\alpha - 1} \| \phi(\tau) \|^q_{L^q} d\tau \\
\leq C_{\alpha,p,q} \int_0^t (1 + \tau)^{\alpha - 1} \left( \int_{-\infty}^{\infty} |\phi|^{q-2} |\partial_x \phi|^{p+1} dx \right)^{\frac{q-2}{2(p+q-1)}} \\
\times \left( \int_{-\infty}^{\infty} |\phi|^2 dx \right)^{\frac{q-2}{2(p+q-1)}} d\tau \\
\leq \epsilon \int_0^t (1 + \tau)^{\alpha - 1} \left( \int_{-\infty}^{\infty} |\phi|^{q-2} |\partial_x \phi|^{p+1} dx \right) d\tau \\
+ C_{\alpha,p,q}(\epsilon) \int_0^t (1 + \tau)^{\alpha - 1} \| \phi(\tau) \|^q_{L^q\tau}\bar{F}_p(U, U^r)(\tau) \|_{L^1} d\tau,
\]

\[
q \int_0^t (1 + \tau)^{\alpha} \| \phi(\tau) \|^q_{L^q\tau} \left\| \bar{F}_p(U, U^r)(\tau) \right\|_{L^1} d\tau \\
\leq C_{p,q} \int_0^t (1 + \tau)^{\alpha} \left( \int_{-\infty}^{\infty} |\phi|^{q-1} |\partial_x \phi|^{p+1} dx \right)^{\frac{q-1}{q(p+q-1)}} \\
\times \left( \int_{-\infty}^{\infty} |\phi|^2 dx \right)^{\frac{q(p-1)}{2(p+q-1)}} \left\| \bar{F}_p(U, U^r)(\tau) \right\|_{L^1} d\tau \\
\leq \epsilon \int_0^t (1 + \tau)^{\alpha} \left( \int_{-\infty}^{\infty} |\phi|^{q-1} |\partial_x \phi|^{p+1} dx \right) d\tau \\
+ C_{p,q}(\epsilon) \int_0^t (1 + \tau)^{\alpha} \| \phi(\tau) \|^q_{L^q\tau} \left\| \bar{F}_p(U, U^r)(\tau) \right\|_{L^1}^{\frac{3}{2}-\frac{3}{p-q-1}} d\tau.
\]
\[
\mu q \int_0^t (1 + \tau)^\alpha \| \phi(\tau) \|_{L^\infty}^{p-2} \times \int_{-\infty}^\infty |\partial_x \phi| \left( (\partial_x U + \partial_x U^r)^p - (\partial_x U)^p \right) d\tau \\
\leq C_{p,q} \int_0^t (1 + \tau)^\alpha \left( \int_{-\infty}^\infty |\phi| q^{-1} |\partial_x \phi|^{p+1} d\tau \right) \\
\times \left( \int_{-\infty}^\infty |\phi|^2 d\tau \right) \\
\times \left( \int_{-\infty}^\infty \left( (\partial_x U + \partial_x U^r)^p - (\partial_x U)^p \right)^{\frac{p+1}{p}} d\tau \right) \\
\leq \epsilon \int_0^t (1 + \tau)^\alpha \left( \int_{-\infty}^\infty |\phi| q^{-1} |\partial_x \phi|^{p+1} d\tau \right) d\tau \\
+ C_{p,q} \epsilon \int_0^t (1 + \tau)^\alpha \| \phi(\tau) \|_{L^\infty}^{2q(\epsilon-2)} \times \left( \int_{-\infty}^\infty \left( (\partial_x U + \partial_x U^r)^p - (\partial_x U)^p \right)^{\frac{p+1}{p}} d\tau \right) \times \left( \int_{-\infty}^\infty \left( (\partial_x U + \partial_x U^r)^p - (\partial_x U)^p \right)^{\frac{p+1}{p}} d\tau \right) \times \left( \int_{-\infty}^\infty \left( (\partial_x U + \partial_x U^r)^p - (\partial_x U)^p \right)^{\frac{p+1}{p}} d\tau \right) \times \left( \int_{-\infty}^\infty \left( (\partial_x U + \partial_x U^r)^p - (\partial_x U)^p \right)^{\frac{p+1}{p}} d\tau \right). \tag{4.9}
\]
Substituting (4.7), (4.8) and (4.9) into (4.2), we have

\[
(1 + t)^\alpha \| \phi(t) \|^p_{L^p} + \int_0^t (1 + \tau)^\alpha G_q(\tau) \, d\tau \\
+ \int_0^t (1 + \tau)^\alpha \int_{-\infty}^{\infty} |\phi|^q - 2(\partial_x \phi)^2 \\
\times \left( |\partial_x \phi|^{p-1} + |\partial_x U|^{p-1} + |\partial_x U^\tau|^{p-1} \right) \, dx \, d\tau \\
+ \int_0^t (1 + \tau)^\alpha \int_{-\infty}^{\infty} |\phi|^q \left( |\partial_x \phi + \xi \tilde{U}|^{p-1} - |\partial_x U|^{p-1} \right) \, dx \, d\tau
\]

\[
\leq C_{\alpha,p,q} \| \phi_0 \|^p_{L^p} + C_{\alpha,p,q}(\epsilon) \int_0^t (1 + \tau)^{\alpha - \frac{2p+1}{p+1}} \| \phi(\tau) \|^q_{L^2} \, d\tau \\
+ C_{p,q}(\epsilon) \int_0^t (1 + \tau)^\alpha \| \phi(\tau) \|^2_{L^2} \left\| \tilde{F}_p(U,U^\tau)(\tau) \right\|_{L^1}^{\frac{2(\alpha - 1)}{p \alpha}} \, d\tau \\
+ C_{p,q}(\epsilon) \int_0^t (1 + \tau)^\alpha \| \phi(\tau) \|^2_{L^2} \left( \int_{-\infty}^{\infty} \left( (\partial_x U + \partial_x U^\tau)^p - (\partial_x U)^p \right)^{\frac{p+1}{p}} \, dx \right)^{\frac{p}{p+1}} \, d\tau
\]

By using the \( L^2 \)-boundedness of \( \phi \), Lemma 3.2, we first get

\[
\| \phi(t) \|^2_{L^2} \leq C_p(\phi_0). \tag{4.11}
\]

By using Lemma 2.2, Lemma 2.3 and Lemma 3.3, we also get

**Lemma 4.3.** For any fixed \( p \in (1, \infty) \), we have the following time-decay estimates.

1. For any \( \delta \in (0, 1) \), there exists positive constants \( C_p, C_\delta \) and \( T_0 \) such that

\[
\left\| \tilde{F}_p(U,U^\tau)(t) \right\|_{L^1} \leq C_p \left( (1 + t)^{-\frac{2p}{p+1}} + C_\delta (1 + t)^{-2(1-\delta)} \right) \quad (t \geq T_0).
\]

2. There exists a positive constant \( C_p \) such that

\[
\left( \int_{-\infty}^{\infty} \left( (\partial_x U + \partial_x U^\tau)^p - (\partial_x U)^p \right)^{\frac{p+1}{p}} \, dx \right)(t) \leq C_p (1 + t)^{-1} \quad (t \geq 0).
\]

We estimate the each terms on the right-hand side of (4.10) as follows:

\[
C_{\alpha,p,q} \int_0^t (1 + \tau)^{\alpha - \frac{2p+1}{p+1}} \| \phi(\tau) \|^q_{L^2} \, d\tau \\
\leq C_{\alpha,p,q}(C_p(\phi_0)) \int_0^t (1 + \tau)^{\alpha - \frac{2p+1}{p+1}} \, d\tau \tag{4.12}
\]

\[
\leq C_{\alpha,p,q}(C_p(\phi_0)) (1 + t)^{\alpha - \frac{2}{p+1}},
\]
We use the following Gagliardo-Nirenberg inequality:

\[ \left\| \frac{\phi}{\| \phi \|_{L^2}} \right\|_{L^p} \leq C \left( 1 + \frac{p}{q} + 1 \right)^{1+1} \left( 1 + \frac{p}{q} + 1 \right)^{-1}, \]

for \( 2 \leq q < \infty \).

**Proof of Theorem 3.1.** We already have proved the decay estimate of \( \| \phi(t) \|_{L^q} \) with \( 2 \leq q < \infty \). Therefore we only show the \( L^\infty \)-estimate. We first note by Lemma 2.2 and Lemma 2.3 that

\[ \left\| \frac{\partial_x \phi(t)}{\| \partial_x \phi(t) \|_{L^{p+1}}} \right\|_{L^{p+1}} \leq \left( \| \partial_x u(t) \|_{L^{p+1}} + \| \partial_x U(t) \|_{L^{p+1}} \right)^{-1}, \]

for any \((q, \theta) \in [1, \infty) \times (0, 1]\) satisfying

\[ \frac{p}{p+1} \theta = (1 - \theta) \frac{1}{q}. \]

Substituting (4.15) and (4.16) into (4.17), we have

\[ \| \phi(t) \|_{L^\infty} \leq C(p, q, \theta, \phi_0, \partial_x u_0) (1 + t)^{-\frac{1}{p+1} + \frac{1}{p+1}(1-\theta)} \]

for \( \theta \in (0, 1] \). Consequently, we do complete the proof of Theorem 3.1.
5. Time-decay estimates with $1 \leq q \leq \infty$

In this section, we show the time-decay estimates with $1 \leq q \leq \infty$ and time-decay estimate for the higher order derivative in the $L^{p+1}$-norm, in the case where $\phi_0 \in L^1 \cap L^2$ with $\partial_x u_0 \in L^{p+1}$, that is, Theorem 3.2. Then, we first establish the $L^1$-estimate to the solution $\phi$ of the reformulated Cauchy problem (3.6). To do that, we use the Friedrichs mollifier $\rho_\delta$, where, $\rho_\delta(\phi) := \frac{1}{\delta} \rho\left(\frac{\phi}{\delta}\right)$ with $\rho \in C^\infty_0(\mathbb{R})$, $\rho(\phi) \geq 0$ ($\phi \in \mathbb{R}$), $\supp\{\rho\} \subset \{\phi \in \mathbb{R} \mid \phi \leq 1\}$, $\int_{-\infty}^{\infty} \rho(\phi) d\phi = 1$.

Some useful properties of the mollifier are as follows.

Lemma 5.1.

(i) $\lim_{\delta \to 0} (\rho_\delta * \text{sgn}) (\phi) = \text{sgn}(\phi)$ ($\phi \in \mathbb{R}$),

(ii) $\lim_{\delta \to 0} \int_0^\phi (\rho_\delta * \text{sgn}) (\eta) d\eta = |\phi|$ ($\phi \in \mathbb{R}$),

(iii) $(\rho_\delta * \text{sgn})\big|_{\phi=0} = 0$,

(iv) $\frac{d}{d\phi} (\rho_\delta * \text{sgn}) (\phi) = 2 \rho_\delta(\phi) \geq 0$ ($\phi \in \mathbb{R}$),

where

$$(\rho_\delta * \text{sgn}) (\phi) := \int_{-\infty}^{\infty} \rho_\delta(\phi - y) \text{sgn}(y) dy$$.  ($\phi \in \mathbb{R}$)

and

$$\text{sgn}(\phi) := \begin{cases} -1 & (\phi < 0), \\ 0 & (\phi = 0), \\ 1 & (\phi > 0). \end{cases}$$

Making use of Lemma 5.1, we obtain the following $L^1$-estimate.

Proposition 5.1. Assume that the same assumptions in Theorem 3.2. For any $p > 1$ and any $\epsilon > 0$, there exist positive constants $C_p$ and $C_\epsilon$ such that the unique global solution in time $\phi$ of the Cauchy problem (3.6) satisfies the following $L^1$-estimate

$$\| \phi(t) \|_{L^1} \leq \| \phi_0 \|_{L^1} + C_p(1 + t) \frac{2}{p+1} + C_\epsilon(1 + t)^{-2(1-\epsilon)} \quad (t \geq T_0) \quad (5.1)$$

for any $\epsilon > 0$.

Proof of Proposition 5.1. Multiplying the equation in the problem (3.6) by $(\rho_\delta * \text{sgn}) (\phi)$, we obtain the divergence form
\[
\begin{align*}
\partial_t \left( \int_0^\phi (\rho_\delta \ast \text{sgn})(\eta) \, d\eta \right) \\
+ \partial_x \left( (\rho_\delta \ast \text{sgn})(\phi) \left( f(\bar{U} + \phi) - f(\bar{U}) \right) \right) \\
+ \partial_x \left( -\int_0^\phi (f(\bar{U} + \eta) - f(\bar{U})) \frac{d(\rho_\delta \ast \text{sgn})}{d\phi}(\eta) \, d\eta \right) \\
+ \partial_x \left( -\mu (\rho_\delta \ast \text{sgn})(\phi) \times \left( |\partial_x \bar{U} + \partial_x \phi|^{p-1}(\partial_x \bar{U} + \partial_x \phi) - |\partial_x \bar{U} |^{p-1}(\partial_x \bar{U}) \right) \right) \\
+ \int_0^\phi \left( \lambda(\bar{U} + \eta) - \lambda(\bar{U}) \right) \frac{d(\rho_\delta \ast \text{sgn})}{d\phi}(\eta) \, d\eta \left( \partial_x \bar{U} \right) \\
+ \mu \frac{d(\rho_\delta \ast \text{sgn})}{d\phi}(\phi) \partial_x \phi \\
\times \left( |\partial_x \bar{U} + \partial_x \phi|^{p-1}(\partial_x \bar{U} + \partial_x \phi) - |\partial_x \bar{U} |^{p-1}(\partial_x \bar{U}) \right)
\end{align*}
\]

(5.2)

Integrating (5.2) with respect to \( x \) and \( t \), we have

\[
\begin{align*}
&\int_{-\infty}^\infty \int_{0}^{\phi(t)} (\rho_\delta \ast \text{sgn})(\eta) \, d\eta \, dx \\
+ \int_{0}^{t} \int_{-\infty}^\infty \int_{0}^{\phi} \left( \lambda(\bar{U} + \eta) - \lambda(\bar{U}) \right) \frac{d(\rho_\delta \ast \text{sgn})}{d\phi}(\eta) \, d\eta \left( \partial_x \bar{U} \right) \, dx \, dr \\
+ \frac{\mu}{2} \int_{0}^{t} \int_{-\infty}^\infty \frac{d(\rho_\delta \ast \text{sgn})}{d\phi}(\phi) \left[ |\partial_x \phi + \partial_x \bar{U}|^{p-1} - |\partial_x \bar{U}|^{p-1} \right] \\
\times \left( \partial_x \phi + \partial_x \bar{U} \right)^2 \, dx \, dr \\
= \int_{-\infty}^\infty \int_{0}^{\phi(t)} (\rho_\delta \ast \text{sgn})(\eta) \, d\phi \, dx \\
+ \int_{0}^{t} \int_{-\infty}^\infty (\rho_\delta \ast \text{sgn})(\phi) F_p(U, U') \, dx \, dr.
\end{align*}
\]

(5.3)

By using Lemma 5.1, we first note that for \( t \in [0, \infty) \),

\[
\left| \int_{0}^{\phi(t)} (\rho_\delta \ast \text{sgn})(\eta) \, d\eta \right| \leq (\rho_\delta \ast \text{sgn})(|\phi(t)|) |\phi(t)| \leq |\phi(t)|,
\]

(5.4)

\[
\lim_{\delta \to 0} \int_{-\infty}^\infty \int_{0}^{\phi(t)} (\rho_\delta \ast \text{sgn})(\eta) \, d\eta = \| \phi(t) \|_{L^1}.
\]

(5.5)
We also note the following (the proof is similar to the one in [29]).

**Lemma 5.2.** It holds that

\[
\int_{-\infty}^{\infty} \int_{0}^{\phi(t)} \left( \lambda(\tilde{U} + \eta) - \lambda(\tilde{U}) \right) \frac{d(\rho_\delta \ast \text{sgn})}{d\phi}(\eta) \, d\eta \left( \partial_x \tilde{U} \right) \, dx \\
\geq C^{-1} \left( \int_{\tilde{U} + \phi \geq 0, \tilde{U} \geq 0} |\phi| \rho_\delta(\eta) \, d\eta \left( \partial_x \tilde{U} \right) \right) (t) \\
+ C^{-1} \left( \int_{\tilde{U} + \phi \geq 0, \tilde{U} < 0} |\phi| (\tilde{U} + \eta) \rho_\delta(\eta) \, d\eta \left( \partial_x \tilde{U} \right) \right) (t) \\
+ C^{-1} \left( \int_{\tilde{U} + \phi < 0, \tilde{U} \geq 0} |\phi| \tilde{U} \rho_\delta(\eta) \, d\eta \left( \partial_x \tilde{U} \right) \right) (t) \geq 0 \quad (t \geq 0).
\]

(5.6)

So we can get

\[
\| \phi(t) \|_{L^1} \leq \| \phi_0 \|_{L^1} + \lim_{\delta \to 0} \int_{0}^{t} \left| \int_{-\infty}^{\infty} (\rho_\delta \ast \text{sgn}) (\phi) \tilde{F}_p(U, U') \, dx \right| \, d\tau \\
\leq \| \phi_0 \|_{L^1} + \lim_{\delta \to 0} \int_{0}^{t} \left| \int_{-\infty}^{\infty} (\rho_\delta \ast \text{sgn}) (\phi) \tilde{F}_p(U, U') \, dx \right| \, d\tau \\
+ \mu \lim_{\delta \to 0} \int_{0}^{t} \left| \int_{-\infty}^{\infty} (\rho_\delta \ast \text{sgn}) (\phi) \right| \times \partial_x \left( |\partial_x U + \partial_x U' - 1| (\partial_x U + \partial_x U') - |\partial_x U| (\partial_x U) \right) \, dx \, d\tau
\]

(5.7)

Noting by the asymptotic properties of $X(t)$ ($t \geq T_0$), Lemma 3.3, that

\[
\left( \int_{-\infty}^{\infty} + \int_{X(t)}^{\infty} \right) \left| \lambda(U + U') - \lambda(U') \right| \partial_x U' \, dx \\
\leq C_p(1 + t)^{-\frac{\alpha - 1}{\alpha}} + C_\epsilon(1 + t)^{-1 - (1 - 2\epsilon)} \quad (\epsilon \in (0, 1))
\]

and

\[
\int_{-\infty}^{\infty} \lambda(U + U') \partial_x U \, dx \leq C_p(1 + t)^{-\frac{\alpha - 1}{\alpha}} + C_\epsilon(1 + t)^{-1 - (1 - 2\epsilon)} \quad (\epsilon \in (0, 1)),
\]

we immediately get

\[
\lim_{\delta \to 0} \left| \int_{-\infty}^{\infty} (\rho_\delta \ast \text{sgn}) (\phi) \tilde{F}_p(U, U') \, dx \right| (t) \\
\leq \left( \int_{-\infty}^{\infty} |\text{sgn}(\phi)| \, \left| \tilde{F}_p(U, U') \right| \, dx \right) (t) \\
\leq C_p(1 + t)^{-\frac{\alpha}{\alpha + 1}} + C_\epsilon(1 + t)^{-2(1 - \epsilon)} \quad (t \geq T_0).
\]

(5.8)
Next, we estimate

\[
\left| \int_{-\infty}^{\infty} (\rho_\delta \ast \text{sgn}) (\phi) \partial_x \left( | \partial_x U + \partial_x U^r |^{p-1} (\partial_x U + \partial_x U^r) - | \partial_x U |^{p-1} (\partial_x U) \right) \, dx \right|
\]

\[
\leq p \left| \int_{-\infty}^{\infty} (\rho_\delta \ast \text{sgn}) (\phi) \left( (\partial_x U + \partial_x U^r)^{p-1} - (\partial_x U)^{p-1} \right) \partial_x^2 U \, dx \right|
\]

\[
+ p \left| \int_{-\infty}^{\infty} (\rho_\delta \ast \text{sgn}) (\phi) (\partial_x U + \partial_x U^r)^{p-1} \partial_x^2 U^r \, dx \right|
\]

By using Lagrange’s mean-value theorem, we have

\[
\left| \int_{-\infty}^{\infty} (\rho_\delta \ast \text{sgn}) (\phi) \left( (\partial_x U + \partial_x U^r)^{p-1} - (\partial_x U)^{p-1} \right) \partial_x^2 U \, dx \right|
\]

\[
\leq \begin{cases} 
(p - 1) \left| \int_{-\infty}^{\infty} (\rho_\delta \ast \text{sgn}) (\phi) (\partial_x U)^{p-2} \partial_x U^r \partial_x^2 U \, dx \right| & (1 < p < 2), \\
(p - 1) \left| \int_{-\infty}^{\infty} (\rho_\delta \ast \text{sgn}) (\phi) (\partial_x U + \partial_x U^r)^{p-2} \partial_x U^r \partial_x^2 U \, dx \right| & (p \geq 2).
\end{cases}
\]

By using Lemma 2.2 and Lemma 2.3, we have for \(1 < p < 2\),

\[
\lim_{\delta \to 0} \left| \int_{-\infty}^{\infty} (\rho_\delta \ast \text{sgn}) (\phi) (\partial_x U)^{p-2} \partial_x U^r \partial_x^2 U \, dx \right| (t)
\]

\[
\leq \int_{-\infty}^{\sqrt{\frac{\pi}{\theta (1+t)^{1+\alpha}}}} (\partial_x U)^{p-2} \partial_x U^r \partial_x^2 U \, dx
\]

\[
\leq C_p (1 + t)^{-(1 + \frac{\alpha}{2p})} \int \frac{x}{(1 + t)^{1+\alpha}} \, dx
\]

\[
\leq C_p (1 + t)^{-\frac{2p}{2p+\alpha}} \quad (t \geq T_0),
\]

and for \(p \geq 2\),

\[
\lim_{\delta \to 0} \left| \int_{-\infty}^{\infty} (\rho_\delta \ast \text{sgn}) (\phi) (\partial_x U + \partial_x U^r)^{p-2} \partial_x U^r \partial_x^2 U \, dx \right| (t)
\]

\[
\leq C_p \int_{-\infty}^{\infty} (\partial_x U)^{p-2} \partial_x U^r \partial_x^2 U \, dx + C_p \int_{-\infty}^{\infty} (\partial_x U^r)^{p-1} \partial_x^2 U \, dx
\]

\[
\leq C_p (1 + t)^{-\frac{2p}{2p+\alpha}} + C_p (1 + t)^{-(p-1 + \frac{2\alpha}{2p})}
\]

\[
\times \int_{-\sqrt{\frac{\pi}{\theta (1+t)^{1+\alpha}}}}^{\sqrt{\frac{\pi}{\theta (1+t)^{1+\alpha}}}} \left( A - B \left( \frac{x}{(1 + t)^{1+\alpha}} \right)^2 \right)^{-\frac{\alpha}{2p}} |x| \, dx
\]

\[
\leq C_p (1 + t)^{-\frac{2p}{2p+\alpha}} + C_p (1 + t)^{-\frac{2\alpha}{2p+\alpha}} \quad (t \geq T_0).
\]
Similarly, we have
\[
\lim_{\delta \to 0} \int_{-\infty}^{\infty} (\rho \ast \text{sgn}) (\phi) \left( \partial_x U + \partial_x U^r \right)^{p-1} \partial_x^2 U^r \, dx \left( t \right) = C_p (1 + t)^{-\frac{2p}{p-1}} \quad (t \geq T_0).
\]
(5.11)

Then, substituting (5.8), (5.9), (5.10) and (5.11) into (5.7), we have the desired \(L^1\)-estimate (5.1).

Next, we show the time-weighted \(L^q\)-energy estimates to \(\phi\).

**Proposition 5.2.** Suppose the same assumptions in Theorem 3.2. For any \(q \in [1, \infty)\), there exist positive constants \(\alpha\) and \(C_{\alpha,p,q}\), such that the unique global solution in time \(\phi\) of the Cauchy problem (3.6) satisfies the following \(L^q\)-energy estimate
\[
(1 + t)^\alpha \| \phi(t) \|_{L^q}^2 + \int_0^t (1 + \tau)^\alpha G_q(\tau) \, d\tau
\]
\[
+ \int_0^t (1 + \tau)^\alpha \int_{-\infty}^{\infty} |\phi|^{q-2} \left( \partial_x \phi \right)^2 \times \left( |\partial_x \phi|^{p-1} + |\partial_x U|^{p-1} + |\partial_x U^r|^{p-1} \right) \, dx d\tau
\]
\[
+ \int_0^t (1 + \tau)^\alpha \int_{-\infty}^{\infty} |\phi|^{q-2} \left| \partial_x \phi + \partial_x U + \partial_x U^r \right|^{p-1} - \left| \partial_x U + \partial_x U^r \right|^{p-1} \right| \, dx d\tau
\]
\[
\leq C_{\alpha,p,q} \| \phi_0 \|_{L^q}^2 + C(\alpha, p, q, \phi_0) (1 + t)^{\alpha - \frac{q-1}{p}} \quad (t \geq T_0).
\]
(5.12)

The proof of Proposition 5.2 is given by the following two lemmas.

**Lemma 5.3.** For any \(1 \leq q < \infty\), there exist positive constants \(\alpha\) and \(C_q\) such
where

\[
(1 + t)^\alpha \| \phi(t) \|_{L^p} + q (q - 1) \int_0^t (1 + \tau)^\alpha G_q(\tau) \, d\tau \\
+ C_q \int_0^t (1 + \tau)^\alpha \int_{-\infty}^{\infty} |\phi|^{q-2} (\partial_x \phi)^2 \times \left| \left| \partial_x \phi \right|^{p-1} + \left| \partial_x U \right|^{p-1} + \left| \partial_x U' \right|^{p-1} \right| \, dx \, d\tau \\
+ C_q \int_0^t (1 + \tau)^\alpha \int_{-\infty}^{\infty} |\phi|^{q-2} \left| (\partial_x \phi + \partial_x \tilde{U}) \right|^{p-1} - \left| \partial_x \tilde{U} \right|^{p-1} \times \left( (\partial_x \phi + \partial_x \tilde{U})^2 - (\partial_x \tilde{U})^2 \right) \, dx \, d\tau
\]

\[
\leq \| \phi_0 \|_{L^q}^q + \alpha \int_0^t (1 + \tau)^{\alpha - 1} \| \phi(\tau) \|_{L^p}^p \, d\tau \\
+ q \int_0^t (1 + \tau)^\alpha \| \phi(\tau) \|_{L^\infty}^{q-1} \left\| \widetilde{F_p}(U, U') (\tau) \right\|_{L^1} \, d\tau \\
+ C_q \int_0^t (1 + \tau)^\alpha \widetilde{F_p}(\phi, U, U') (\tau) \, d\tau \quad (t \geq T_0),
\]

(5.13)

where

\[
\widetilde{F_p}(\phi, U, U')(t) := \begin{cases} 
\| \phi(t) \|_{L^\infty}^{q-1} \int_{-\infty}^{\infty} \partial_x \left( (\partial_x U + \partial_x U')^p - (\partial_x U)^p \right) \, dx \big| (t) \quad (1 < q < 2), \\
\| \phi(t) \|_{L^\infty}^{q-2} \int_{-\infty}^{\infty} \left( (\partial_x U + \partial_x U')^p - (\partial_x U)^p \right)^{q-1} \, dx \big| (t) \quad (q \geq 2).
\end{cases}
\]

**Lemma 5.4.** Assume \( p > 1 \) and \( 1 \leq q < \infty \). We have the following interpolation inequalities.

1. For any \( 1 \leq r < \infty \), there exists a positive constant \( C_{p,q,r} \) such that

\[
\| \phi(t) \|_{L^r} \leq C_{p,q,r} \left( \int_{-\infty}^{\infty} |\phi|^2 \, dx \right)^{p+r-1 \over (2p+q-1)r} \times \left( \int_{-\infty}^{\infty} |\phi|^{q-2} |\partial_x \phi|^p \, dx \right)^{(r-1)/(2p+q-1)} \quad (t \geq 0).
\]

2. There exists a positive constant \( C_{p,q} \) such that

\[
\| \phi(t) \|_{L^\infty} \leq C_{p,q} \left( \int_{-\infty}^{\infty} |\phi|^2 \, dx \right)^{p \over p+r-1} \times \left( \int_{-\infty}^{\infty} |\phi|^{q-2} |\partial_x \phi|^p \, dx \right)^{1 \over p+r-1} \quad (t \geq 0).
\]
The proofs of Lemma 5.3, Lemma 5.4 and Proposition 5.2 are given in the quite similar way as those of Lemma 4.1, Lemma 4.2 and Proposition 4.1, so we omit them. We particularly note that we have by Proposition 5.2
\[ \| \phi(t) \|_{L^q} \leq C(p, q, \phi_0) (1 + t)^{-\frac{1}{2p}(1 - \frac{1}{q})} \] (5.14)
for \( 1 \leq q < \infty \).

We shall finally obtain the time-decay estimates for the higher order derivatives, that is, \( \partial_x \phi \) and \( \partial_x u \), and also get the \( L^\infty \)-estimate for \( \phi \).

**Proposition 5.3.** Suppose the same assumptions in Theorem 3.2. There exist positive constants \( \alpha \) and \( C_{\alpha, p} \), such that the unique global solution in time \( \phi \) of the Cauchy problem (3.6) satisfies the following \( L^{p+1} \)-energy estimate
\[
(1 + t)^\alpha \| \partial_x u(t) \|_{L^{p+1}}^{p+1} + \int_0^t (1 + \tau)^\alpha \int_{-\infty}^{\infty} |\partial_x u(t)|^{2(p-1)} (\partial_x^2 u)^2 \, dx \, d\tau \\
+ \int_0^t (1 + \tau)^\alpha \| \partial_x u(\tau) \|_{L^{p+2}}^{p+2} \, d\tau \\
\leq C_{\alpha, p} \| \partial_x u_0 \|_{L^{p+1}}^{p+1} + C(\alpha, p, \phi_0, \partial_x u_0) (1 + t)^{\alpha - \frac{1}{2p}} \quad (t \geq T_0).
\] (5.15)

To obtain Proposition 5.3, we first show the following.

**Lemma 5.5.** It follows that
\[
(1 + t)^\alpha \| \partial_x u(t) \|_{L^{p+1}}^{p+1} \\
+ \mu p^2 (p+1) \int_0^t (1 + \tau)^\alpha \int_{-\infty}^{\infty} |\partial_x u(t)|^{2(p-1)} (\partial_x^2 u)^2 \, dx \, d\tau \\
+ p \int_0^t (1 + \tau)^\alpha \int_{\partial_x u \geq 0} f''(u) \, |\partial_x u|^{p+2} \, dx \, d\tau \\
= \| \partial_x u_0 \|_{L^{p+1}}^{p+1} + \alpha \int_0^t (1 + \tau)^{\alpha - 1} \| \partial_x u(\tau) \|_{L^{p+1}}^{p+1} \, d\tau \\
+ p \int_0^t (1 + \tau)^\alpha \int_{\partial_x u < 0} f''(u) \, |\partial_x u|^{p+2} \, dx \, d\tau \quad (t \geq T_0).
\] (5.16)

**Proof of Lemma 5.5.** Multiplying the equation in the problem (1.1), that is,
\[
\partial_t u + \partial_x (f(u)) = \mu \partial_x \left( |\partial_x u|^{p-1} \partial_x u \right)
\]
by
\[
-\partial_x \left( |\partial_x u|^{p-1} \partial_x u \right),
\]
we obtain the divergence form

\[
\partial_t \left( \frac{1}{p+1} |\partial_x u|^{p+1} \right) + \partial_x \left( - |\partial_x u|^{p-1} \partial_x u \cdot \partial_t u \right) + \partial_x \left( - \frac{p}{p+1} f'(u) |\partial_x u|^{p+1} \right) + \frac{p}{p+1} f''(u) |\partial_x u|^{p+1} \partial_x u + \mu p |\partial_x u|^{2(p-1)} \left( \partial_x^2 u \right)^2 = 0.
\]

Integrating the divergence form (5.17) with respect to \( x \), we have

\[
\frac{1}{p+1} \frac{d}{dt} \| \partial_x u(t) \|_{L^{p+1}}^{p+1} + \mu p^2 \int_{-\infty}^{\infty} |\partial_x u|^{2(p-1)} \left( \partial_x^2 u \right)^2 \, dx + \frac{p}{p+1} \int_{-\infty}^{\infty} f''(u) |\partial_x u|^{p+1} \partial_x u \, dx = 0.
\]

We separate the integral region to the third term on the left-hand side of (5.18) as

\[
\int_{-\infty}^{\infty} f''(u) |\partial_x u|^{p+1} \partial_x u \, dx
\]

\[
= \int_{\partial_x u \geq 0} + \int_{\partial_x u < 0}
\]

\[
= \int_{\partial_x u \geq 0} f''(u) |\partial_x u|^{p+2} \, dx - \int_{\partial_x u < 0} f''(u) |\partial_x u|^{p+2} \, dx.
\]

Substituting (5.19) into (5.18), we get the following equality

\[
\frac{1}{p+1} \frac{d}{dt} \| \partial_x u(t) \|_{L^{p+1}}^{p+1} + \mu p^2 \int_{-\infty}^{\infty} |\partial_x u|^{2(p-1)} \left( \partial_x^2 u \right)^2 \, dx + \frac{p}{p+1} \int_{\partial_x u \geq 0} f''(u) |\partial_x u|^{p+2} \, dx = \frac{p}{p+1} \int_{\partial_x u < 0} f''(u) |\partial_x u|^{p+2} \, dx.
\]

Multiplying (5.20) by \((1+t)^\alpha\) with \(\alpha > 0\) and integrating over \((0,t)\) with respect to the time, we complete the proof of Lemma 5.5.

**Proof of Proposition 5.3.** We use the following important results (cf. [30]).

**Lemma 5.6.** For any \( s \geq 0 \), there exists a positive constant \( C_s \) such that

\[
\int_{\partial_x u < 0} f''(u) |\partial_x u|^s \, dx \leq C_s \int_{\partial_x u < 0} |\partial_x \phi|^s \, dx.
\]

In fact, taking care of the relation by using Lemma 2.2 and Lemma 2.3

\[
\partial_x u = \partial_x \tilde{U} + \partial_x \phi < 0 \iff \partial_x \phi < 0, \partial_x \tilde{U} < |\partial_x \phi|,
\]

\[
\partial_x u = \partial_x \tilde{U} + \partial_x \phi < 0 \iff \partial_x \phi < 0, \partial_x \tilde{U} < |\partial_x \phi|,
\]
we immediately have
\[
\int_{\partial_x u < 0} f''(u) |\partial_x u|^s \, dx 
\leq 2^s \left( \max_{|u| \leq C} f''(u) \right) \int_{\partial_x \phi < 0, \partial_x \theta < |\partial_x \phi|} |\partial_x \phi|^s \, dx. \tag{5.23}
\]

Since \( \partial_x u \) is absolutely continuous, we first note that for any \( x \in \{ x \in \mathbb{R} | \partial_x u < 0 \} \), there exists \( x_k \in \mathbb{R} \cup \{-\infty\} \) such that
\[
\partial_x u(x_k) = 0, \quad \partial_x u(y) < 0 \quad (y \in (x_k, x)).
\]

Therefore, it follows that for such \( x \) and \( x_k \) with \( q \geq p (> 1) \),
\[
|\partial_x u|^q = (-\partial_x u)^q = q \int_{x_k}^{x} (-\partial_x u)^{q-1} (-\partial_x^2 u) \, dy \tag{5.24}
\]

By using the Cauchy-Schwarz inequality, we have

**Lemma 5.7.** It holds that
\[
\int_{\partial_x u < 0} |\partial_x u|^{p+2} \, dx 
\leq C_p \left( \int_{\partial_x u < 0} |\partial_x u|^{2(p-1)} (\partial_x^2 u)^2 \, dx \right)^{\frac{1}{3p+1}} \left( \int_{\partial_x u < 0} |\partial_x u|^{p+1} \, dx \right)^{\frac{2p+2}{3p+1}}. \tag{5.25}
\]

By using Young’s inequality to (5.25), we also have

**Lemma 5.8.** It follows that for any \( \epsilon > 0 \), there exists a positive constant \( C_p(\epsilon) \) such that,
\[
\int_{\partial_x u < 0} |\partial_x u|^{p+2} \, dx 
\leq \epsilon \int_{\partial_x u < 0} |\partial_x u|^{2(p-1)} (\partial_x^2 u)^2 \, dx + C_p(\epsilon) \left( \int_{\partial_x u < 0} |\partial_x u|^{p+1} \, dx \right)^{\frac{2p+2}{3p+1}}. \tag{5.26}
\]
By using Lemma 5.6, Lemma 5.7 and Lemma 5.8 with \( \epsilon = \frac{\mu p^2 (p + 1)}{2} \), we have

\[
(1 + t)^\alpha \| \partial_x u(t) \|_{L_{p+1}^{p+1}}^p + \mu p^2 (p + 1) \int_0^t (1 + \tau)^\alpha \int_{-\infty}^\infty |\partial_x u|^2 (\partial_x^2 u)^2 \, dx \, d\tau \\
+ \frac{p}{4} \int_0^t (1 + \tau)^\alpha \int_{\partial_x u \geq 0} f''(u) |\partial_x u|^{p+1} \, dx \, d\tau \leq \| \partial_x u_0 \|_{L_{p+1}^{p+1}}^p \\
+ C_p \int_0^t (1 + \tau)^\alpha \left( \int_{\partial_x u < 0} |\partial_x u|^{p+1} \, dx \right)^{\frac{p+1}{p}} \, d\tau.
\]

(5.27)

By using Proposition 5.2, we get the following time-decay estimates.

**Lemma 5.9.** There exist positive constants \( \alpha \gg 1 \) and \( C_{\alpha, p, q} \), such that

\[
\int_0^t (1 + \tau)^\alpha \int_{-\infty}^\infty \phi |\partial_x \phi(\tau)|^{p+1} \, dx \, d\tau \leq C (\alpha, p, \phi_0) (1 + t)^{\alpha - \frac{2(p+1)}{p}} \quad (t \geq t_0).
\]

(5.28)

By using Lemma 5.9 with \( \alpha \mapsto \alpha - 1 \gg 1 \) and \( q = 2 \), we have

\[
\alpha \int_0^t (1 + \tau)^{\alpha - 1} \| \partial_x \phi(\tau) \|_{L_{p+1}^{p+1}} \, d\tau \leq C (\alpha, p, \phi_0) (1 + t)^{\alpha - \frac{2(p+1)}{p}}.
\]

(5.29)

We can also estimate by using Lemma 2.2 and Lemma 2.3 as

\[
\alpha \int_0^t (1 + \tau)^{\alpha - 1} \| \partial_x \varphi(\tau) \|_{L_{p+1}^{p+1}} \, d\tau \leq C (\alpha, p) (1 + t)^{\alpha - \frac{p}{p+1}}.
\]

(5.30)

By using the uniform boundedness in Lemma 3.2, that is,

\[
\| \partial_x u(t) \|_{L_{p+1}^{p+1}} \leq C_p (\| \phi_0 \|_{L^2}, \| \partial_x u_0 \|_{L_{p+1}^{p+1}})
\]

and Lemma 5.9 with \( q = 2 \), we can estimate as

\[
C_p \int_0^t (1 + \tau)^\alpha \left( \int_{\partial_x u < 0} |\partial_x u|^{p+1} \, dx \right)^{\frac{p+1}{p}} \, d\tau \\
\leq C_p \int_0^t (1 + \tau)^\alpha \int_{\partial_x u < 0} |\partial_x \phi|^{p+1} \, dx \cdot \| \partial_x u(\tau) \|_{L_{p+1}^{p+1}}^2 \, d\tau \\
\leq C (p, \phi_0, \partial_x u_0) \int_0^t (1 + \tau)^\alpha \int_{-\infty}^\infty |\partial_x \phi|^{p+1} \, dx \, d\tau \\
\leq C (\alpha, p, \phi_0, \partial_x u_0) (1 + t)^{\alpha - \frac{p}{p+1}}.
\]

(5.31)
Substituting (5.29), (5.30) and (5.31) into (5.27), we complete the proof of Proposition 5.3. In particular, we have

\[ \| \partial_x u(t) \|_{L^{p+1}}^{p+1} \, d\tau \leq C(p, \phi_0, \partial_x u_0) (1 + t)^{-\frac{1}{p}}, \quad (5.32) \]

and

\[ \| \partial_x \phi(t) \|_{L^{p+1}}^{p+1} \, d\tau \leq \| \partial_x u(t) \|_{L^{p+1}}^{p+1} + \| \partial_x U^r(t) \|_{L^{p+1}}^{p+1} \leq C(p, \phi_0, \partial_x u_0) (1 + t)^{-\frac{1}{p}} \quad (5.33) \]

for \( 1 \leq q < \infty \).

**Proof of Theorem 3.2.** We already have proved the decay estimate of \( \| \phi(t) \|_{L^q} \) with \( 1 \leq q < \infty \). Therefore we only show the following time-decay estimate for the higher order derivative

\[ \| \partial_x u(t) \|_{L^{p+1}} \quad \| \partial_x \phi(t) \|_{L^{p+1}} \]

\[ \leq \begin{cases} 
C(\alpha, p, \phi_0, \partial_x u_0) (1 + t)^{-\frac{1}{(p+1)^2}} \\
C(p, \phi_0, \partial_x u_0) (1 + t)^{-\frac{1}{(p+1)^2} + \epsilon} \\
\left( \frac{7}{12} + \sqrt{\frac{73}{144} - \frac{(p+1)^2(3p-2)}{3}} \epsilon \right) \end{cases} \quad (5.34) \]

for any \( 0 < \epsilon \ll 1 \), and the \( L^\infty \)-estimate for \( \phi \).

We first prove (5.34). Substituting (5.32) into (5.31), we have

\[ C_p \int_0^t (1 + \tau)^{\alpha} \int_{\partial_x u < 0} |\partial_x \phi|^{p+1} \, dx \cdot \| \partial_x u(\tau) \|_{L^{p+1}}^{\frac{2(p+1)}{3}} \, d\tau \leq C(p, \phi_0, \partial_x u_0) \int_0^t (1 + \tau)^{\alpha - \frac{2p}{3}} \frac{2p}{3} \int_{-\infty}^{\infty} |\partial_x \phi|^{p+1} \, dx \, d\tau. \quad (5.35) \]

By using Lemma 5.9 with \( \alpha \mapsto \alpha - \frac{2p}{3} \gg 1 \) and \( q = 2 \), we also have

\[ \alpha \int_0^t (1 + \tau)^{\alpha - 1} \| \partial_x \phi(\tau) \|_{L^{p+1}}^{p+1} \, d\tau \leq C(\alpha, p, \phi_0) (1 + t)^{\alpha - \frac{2p}{3} - \frac{2p}{3}}. \quad (5.36) \]
Substituting (5.36) into (5.27), we have
\[ \| \partial_x u(t) \|_{L^{p+1}}^{p+1}, \| \partial_x \phi(t) \|_{L^{p+1}}^{p+1} \]
\[ \leq C(p, \phi_0, \partial_x u_0) \righttimes \left( (1 + t)^{-\frac{2p+1}{2p}} + (1 + t)^{-\frac{p}{p+1}} + (1 + t)^{-\frac{1}{p} + \frac{1}{p+1}} \right) \]
\[ \leq C(p, \phi_0, \partial_x u_0) \left( (1 + t)^{-\frac{p}{p+1}} + (1 + t)^{-\frac{1}{p} + \frac{1}{p+1}} \right) \]
\[ (5.37) \]

Iterating “\( \infty \)”-times the above process, we will get
\[ \| \partial_x u(t) \|_{L^{p+1}}^{p+1}, \| \partial_x \phi(t) \|_{L^{p+1}}^{p+1} \]
\[ \leq C(\epsilon, p, \phi_0, \partial_x u_0) \left( (1 + t)^{-\frac{p}{p+1}} + (1 + t)^{-\frac{1}{p} + \frac{1}{p+1}} + \epsilon \right) \]
\[ \leq C(\epsilon, p, \phi_0, \partial_x u_0) \left( (1 + t)^{-\frac{p}{p+1}} + (1 + t)^{-\frac{1}{p} + \frac{1}{p+1}} + \epsilon \right) \]
\[ (5.38) \]

\[ \begin{cases} 
C(\epsilon, p, \phi_0, \partial_x u_0) (1 + t)^{-\frac{p}{p+1}} \\
\left( 1 < p \leq \frac{7}{12} + \sqrt{\frac{73}{144} - \frac{(p + 1)(3p - 2)}{3} \epsilon} \right), \\
C(\epsilon, p, \phi_0, \partial_x u_0) (1 + t)^{-\frac{1}{2(3p - 2)}} + \epsilon \\
\left( \frac{7}{12} + \sqrt{\frac{73}{144} - \frac{(p + 1)(3p - 2)}{3} \epsilon} < p \right) 
\end{cases} \]

for any \( 0 < \epsilon \ll 1 \).

Thus, we get (5.34).

We finally show the \( L^\infty \)-estimate for \( \phi \) by using the Gagliardo-Nirenberg
inequality. Substituting (5.14) and (5.34) into (4.17), we get

\[ \| \phi(t) \|_{L^\infty} \leq \begin{cases} C(\epsilon, p, \theta, \phi_0, \partial_x u_0) (1 + t)^{-\frac{1}{p} + \left( \frac{2p+1}{2(p+1)} - \frac{p}{p+1} \right) \theta} & \text{for } \theta \in (0, 1] \text{ and any } 0 < \epsilon \ll 1. \end{cases} \]

for \( \theta \in (0, 1] \) and any \( 0 < \epsilon \ll 1 \). Consequently, we do complete the proof of Theorem 3.2.

6. \( L^{r+1} \)-estimate for the higher order derivative with \( r > p \)

In this section, we show the time-decay estimates for the higher order derivative in the \( L^{r+1} \)-norm with \( r > p \), in the case where \( \phi_0 \in L^1 \cap L^2 \) with \( \partial_x u_0 \in L^{p+1} \cap L^{r+1} \), that is, Theorem 3.3.

**Proposition 6.1.** Suppose the same assumptions in Theorem 3.3. For any \( r > p \), there exist positive constants \( \alpha \) and \( C_{\alpha, p, r} \), such that the unique global solution in time \( \phi \) of the Cauchy problem (3.6) satisfies the following \( L^{r+1} \).
energy estimate

\[(1 + t)^\alpha \| \partial_x u(t) \|_{L_{t+1}}^{r+1} + \int_0^t (1 + \tau)^\alpha \int_{-\infty}^{+\infty} |\partial_x u|^{p+r-2} (\partial_x^2 u)^2 \, dx \, d\tau \]
\[+ \int_0^t (1 + \tau)^\alpha \| \partial_x u(\tau) \|_{L_{t+1}}^{\alpha^2} \, d\tau \]
\[\leq C_{\alpha,p,r} \| \partial_x u_0 \|_{L_{t+1}}^{\alpha+1} \]
\[
\begin{cases}
C(\alpha, \epsilon, p, r, \phi_0, \partial_x u_0) (1 + t)^\alpha - \frac{4n(r-p) + 7p + 3}{6p(p+1)} \\
\left(1 + \frac{\epsilon}{p+1} + \frac{1}{p+1} \right)
\end{cases}
\]

\[
\begin{cases}
C(\alpha, \epsilon, p, r, \phi_0, \partial_x u_0) (1 + t)^\alpha - \frac{4n(r-p) + 7p + 3}{6p(p+1)} \\
\left(1 + \frac{\epsilon}{p+1} + \frac{1}{p+1} \right)
\end{cases}
\]

\[
\begin{cases}
C(\alpha, \epsilon, p, r, \phi_0, \partial_x u_0) (1 + t)^\alpha - \frac{4n(r-p) + 7p + 3}{6p(p+1)} \\
\left(1 + \frac{\epsilon}{p+1} + \frac{1}{p+1} \right)
\end{cases}
\]

\[
\begin{cases}
C(\alpha, \epsilon, p, r, \phi_0, \partial_x u_0) (1 + t)^\alpha - \frac{4n(r-p) + 7p + 3}{6p(p+1)} \\
\left(1 + \frac{\epsilon}{p+1} + \frac{1}{p+1} \right)
\end{cases}
\]

\[
\begin{cases}
C(\alpha, \epsilon, p, r, \phi_0, \partial_x u_0) (1 + t)^\alpha - \frac{4n(r-p) + 7p + 3}{6p(p+1)} \\
\left(1 + \frac{\epsilon}{p+1} + \frac{1}{p+1} \right)
\end{cases}
\]

for \( t \geq T_0 \) and any \( 0 < \epsilon \ll 1 \).

The proof of Proposition 6.1 is given by the following three lemmas. Because the proofs of them are similar to those of Lemma 5.5, Lemma 5.6, Lemma 5.7 and Lemma 5.8, we state only here.

**Lemma 6.1.** There exist positive constants \( C_{p,r} \) and \( C_{\alpha,p,r} \) such that

\[
(1 + t)^\alpha \| \partial_x u(t) \|_{L_{t+1}}^{r+1} + \mu p r (r + 1) \int_0^t (1 + \tau)^\alpha \int_{-\infty}^{+\infty} |\partial_x u|^{p+r-2} (\partial_x^2 u)^2 \, dx \, d\tau \\
+ r \int_0^t (1 + \tau)^\alpha \int_{\partial_x u \geq 0} f''(u) |\partial_x u|^{r+2} \, dx \, d\tau \\
\leq \| \partial_x u_0 \|_{L_{t+1}} \\
+ C_{\alpha,p,r} \int_0^t (1 + \tau)^\alpha - \frac{2p+1}{p+1} \left( \int_{-\infty}^{+\infty} |\partial_x u|^{p+1} \, dx \right)^{\frac{p+2+1}{p+1}} \, d\tau \\
+ C_{p,r} \int_0^t (1 + \tau)^\alpha \left( \int_{\partial_x u < 0} |\partial_x u|^{p+1} \, dx \right)^{\frac{p+2+1}{p+1}} \, d\tau \quad (t \geq T_0).
\]
Lemma 6.2. Assume $p > 1$ and $r > p$. We have the following interpolation inequalities.

1. There exists a positive constant $C_{p,r}$ such that

$$
\| \partial_x u(t) \|_{L^{r+1}} \leq C_{p,r} \left( \int_{-\infty}^{\infty} |\partial_x u|^{2(p-1)} (\partial_x^2 u)^2 \ dx \right)^{\frac{r-p}{2(p+r+1)(r+1)}} \\
\times \left( \int_{\partial_x u < 0} |\partial_x u|^{p+1} \ dx \right)^{\frac{p+2r+1}{2(p+r+1)(r+1)}} \ (t \geq 0).
$$

2. There exists a positive constant $C_{p,r}$ such that

$$
\| \partial_x u(t) \|_{L^{\infty}} \leq C_{p,r} \left( \int_{-\infty}^{\infty} |\partial_x u|^{2(p-1)} (\partial_x^2 u)^2 \ dx \right)^{\frac{1}{2p+r+1}} \\
\times \left( \int_{\partial_x u < 0} |\partial_x u|^{p+1} \ dx \right)^{\frac{1}{p+2r+1}} \ (t \geq 0).
$$

Lemma 6.3. Assume $p > 1$ and $r > p$. We have the following interpolation inequalities.

1. There exists a positive constant $C_{p,r}$ such that

$$
\| \partial_x u(t) \|_{L^{r+1}(\{\partial_x u < 0\})} \leq C_{p,r} \left( \int_{\partial_x u < 0} |\partial_x u|^{2(p-1)} (\partial_x^2 u)^2 \ dx \right)^{\frac{r-p}{2(p+r+1)(r+1)}} \\
\times \left( \int_{\partial_x u < 0} |\partial_x u|^{p+1} \ dx \right)^{\frac{p+2r+1}{2(p+r+1)(r+1)}} \ (t \geq 0).
$$

2. There exists a positive constant $C_{p,r}$ such that

$$
\| \partial_x u(t) \|_{L^{\infty}(\{\partial_x u < 0\})} \leq C_{p,r} \left( \int_{\partial_x u < 0} |\partial_x u|^{2(p-1)} (\partial_x^2 u)^2 \ dx \right)^{\frac{1}{2p+r+1}} \\
\times \left( \int_{\partial_x u < 0} |\partial_x u|^{p+1} \ dx \right)^{\frac{1}{p+2r+1}} \ (t \geq 0).
$$

Proof of Proposition 6.1. By using (5.34), we estimate the each terms on...
the right-hand side of (6.2) as

$$C_{\alpha, p, r} \int_0^t (1 + \tau)^{\alpha - \frac{2p + r + 1}{3p + 1}} \left( \int_{-\infty}^{\infty} |\partial_x u|^{p + 1} \, dx \right)^{\frac{p + 2r + 1}{3p + 1}} \, d\tau$$

$$\leq \begin{cases} C(\alpha, \epsilon, p, r, \phi_0, \partial_x u_0) \int_0^t (1 + \tau)^{\alpha - \frac{2p + r + 1}{3p + 1} - \frac{p(2 + r + 1)}{3(3p + 1)(3p + 2)}} \, d\tau & (1 < p \leq \frac{7}{12} + \sqrt{\frac{73}{144} - \frac{(p + 1)(3p - 2)}{3} \epsilon}) \\ C(\alpha, \epsilon, p, r, \phi_0, \partial_x u_0) \int_0^t (1 + \tau)^{\alpha - \frac{2p + r + 1}{3p + 1} - \frac{3(p + 2r + 1)}{2(3p + 1)(3p + 2)}} \, d\tau & \left( \frac{7}{12} + \sqrt{\frac{73}{144} - \frac{(p + 1)(3p - 2)}{3} \epsilon} < p \right) \end{cases}$$

$$\leq \begin{cases} C(\alpha, \epsilon, p, r, \phi_0, \partial_x u_0) (1 + t)^{\alpha - \frac{p + r + 1}{3p + 1}} & (1 < p \leq \frac{7}{12} + \sqrt{\frac{73}{144} - \frac{(p + 1)(3p - 2)}{3} \epsilon}) \\ C(\alpha, \epsilon, p, r, \phi_0, \partial_x u_0) (1 + t)^{\alpha - \frac{6p + 2r + 3}{2(3p + 1)(3p + 2)}} \epsilon^{\frac{p + 2r + 1}{3p + 1}} & \left( \frac{7}{12} + \sqrt{\frac{73}{144} - \frac{(p + 1)(3p - 2)}{3} \epsilon} < p \right) \end{cases}$$

(6.3)
\[
C_{p,r} \int_0^t (1 + \tau)\alpha \left( \int_{\partial_x u < 0} |\partial_x u|^{p+1} \, dx \right)^{\frac{p+2r+2}{2p}} \, d\tau \\
\leq C_{p,r} \int_0^t (1 + \tau)^\alpha \left( \int_{-\infty}^{\infty} |\partial_x \phi|^{p+1} \, dx \right) \|\partial_x u(\tau)\|_{L^{p+1}}^{2(p+1)(p+1)} \, d\tau
\]

\[
\begin{cases}
C(\alpha, \epsilon, p, r, \phi_0, \partial_x u_0) \int_0^t (1 + \tau)^\alpha \left( \int_{-\infty}^{\infty} |\partial_x \phi|^{p+1} \, dx \right) \|\partial_x \phi(\tau)\|_{L^{p+1}}^{p+1} \, d\tau \\
\left( 1 < p \leq \frac{7}{12} + \sqrt{\frac{73}{144} - \frac{(p+1)(3p-2)}{3} \epsilon} \right)
\end{cases}
\]

for any \(0 < \epsilon \ll 1\).

By using Lemma 5.9 with

\[
\alpha \mapsto \left\{ \begin{array}{ll}
\alpha - \frac{2(r-p+1)}{3(p+1)} & (1 < p \leq \frac{7}{12} + \sqrt{\frac{73}{144} - \frac{(p+1)(3p-2)}{3} \epsilon}) \\
\alpha - \left( \frac{r-p+1}{p(3p-2)} - \frac{2(r-p+1)}{3p} \epsilon \right) & \left( \frac{7}{12} + \sqrt{\frac{73}{144} - \frac{(p+1)(3p-2)}{3} \epsilon < p} \right)
\end{array} \right.
\]

and \(q = 2\), we get

\[
C_{p,r} \int_0^t (1 + \tau)^\alpha \left( \int_{\partial_x u < 0} |\partial_x u|^{p+1} \, dx \right)^{\frac{p+2r+2}{2p}} \, d\tau \\
\leq \begin{cases}
C(\alpha, \epsilon, p, r, \phi_0, \partial_x u_0) (1 + t)^{-\frac{4p(r-p)+7p+1}{3p}} \\
\left( 1 < p \leq \frac{7}{12} + \sqrt{\frac{73}{144} - \frac{(p+1)(3p-2)}{3} \epsilon} \right)
\end{cases}
\]

for any \(0 < \epsilon \ll 1\).
Substituting (6.3) and (6.5) into (6.2), we have

\[
(1 + t)^\alpha \| \partial_x u(t) \|_{L^{r+1}}^{r+1} + \int_0^t (1 + \tau)^\alpha \int_{-\infty}^{\infty} |\partial_x u|^{p+r-2} \left( \partial_x^2 u \right)^2 \, dx \, d\tau \\
+ \int_0^t (1 + \tau)^\alpha \int_{\partial_x u \geq 0} f''(u) |\partial_x u|^{r+2} \, dx \, d\tau \\
\leq C_{\alpha, p, r} \| \partial_x u_0 \|_{L^{p+1}}^{p+1}
\]

\[
\begin{cases}
C(\alpha, \epsilon, p, r, \phi_0, \partial_x u_0)(1 + t)^\alpha \\
\times \left( (1 + t)^{-\frac{4p(r-p)+7p+3}{6p(p+1)}} + (1 + t)^{-\frac{p+2r-1}{3p(p-2)} + \frac{2(r-p+1)}{3p}} \right)
\end{cases}
\]

\[
\begin{cases}
1 < p \leq \frac{7}{12} + \sqrt{\frac{73}{144} - \frac{(p+1)(3p-2)}{3} \epsilon} \\
C(\alpha, \epsilon, p, r, \phi_0, \partial_x u_0)(1 + t)^\alpha \\
\times \left( (1 + t)^{-\frac{6p(r-p)+7p+3}{6p(p+1)}} + (1 + t)^{-\frac{p+2r-1}{3p(p-2)} + \frac{2(r-p+1)}{3p}} \right)
\end{cases}
\]

\[
\begin{cases}
\left( \frac{7}{12} + \sqrt{\frac{73}{144} - \frac{(p+1)(3p-2)}{3} \epsilon} \right) < p
\end{cases}
\]

(6.6)

for any \( 0 < \epsilon \ll 1 \).

Here, we note the following: if \( 1 < p \leq \frac{7}{12} + \sqrt{\frac{73}{144} - \frac{(p+1)(3p-2)}{3} \epsilon} < \frac{7+\sqrt{73}}{12} \), then

\[
p < \frac{-4p^2 + 7p + 3}{2p}.
\]

Therefore, it follows that

\[
(1 + t)^{-\frac{4p(r-p)+7p+3}{6p(p+1)}} \leq (1 + t)^{-\frac{3(p-r)(p-1)(3p+1)}{12(p-p-1)(3p-2)}} \quad \left( r > \frac{-4p^2 + 7p + 3}{2p} > p \right)
\]

and

\[
(1 + t)^{-\frac{6p(r-p)+7p+3}{6p(p+1)}} \leq (1 + t)^{-\frac{3(p-r)(p-1)(3p+1)}{12(p-p-1)(3p-2)}} \quad \left( p < r \leq \frac{-4p^2 + 7p + 3}{2p} \right).
\]

If \( \epsilon < \frac{3(r-p)(p-1)(3p+1)}{12(p-p-1)(3p-2)} \), then, for any \( p > 1 \) with \( r > p \),

\[
(1 + t)^{-\frac{6p(r-p)+7p+3}{6p(p+1)}} \leq (1 + t)^{-\frac{3(r-p)(p-1)(3p+1)}{12(p-p-1)(3p-2)}}.
\]

Thus, we do complete the proof of Proposition 6.1.
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