Mass-23 nuclei in astrophysics

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Abstract. The formation of mass-23 nuclei by radiative capture is of great interest in astrophysics. A topical problem associated with these isobars is the so-called \(^{22}\)Na puzzle of ONe white dwarf novae, where the abundance of \(^{22}\)Na observed is not as is predicted by current stellar models, indicating there is more to learn about how the distribution of elements in the universe occurred. Another concerns unexplained variations in elements abundance on the surface of aging red giant stars. One method for theoretically studying nuclear scattering is the Multi-Channel Algebraic Scattering (MCAS) formalism. Studies to date have used a simple collective-rotor prescription to model the target states which couple to projectile nucleons. While, in general, the target states considered all belong to the ground state rotor band, for some systems it is necessary to include coupling to states outside of this band. Herein we discuss an extension of MCAS to allow coupling of different strengths between such states and the ground state band. This consideration is essential when studying the scattering of neutrons from \(^{22}\)Ne, a necessary step in studying the mass-23 nuclei mentioned above.

1. Introduction

Capture of nucleons by mass-22 nuclei is central to several important problems in astrophysics. These include element abundances in red giant stars; according to current stellar models, as stars ascend the red giant branch of the Hertzsprung-Russell diagram of observed stellar luminosity versus temperature, surface abundances of elements should remain constant. However, it is observed that anti-correlations exist between sodium and oxygen. It has been proposed that high-temperature, non-convective mixing may allow leakage to the NeNa cycle from the CNO cycles, producing sodium and depleting oxygen. Such might result from rotation. The reaction that generates sodium in the NeNa cycle is \(^{22}\text{Ne}(p, \gamma)^{23}\text{Na}\), a process that is not yet well understood [1] and is therefore currently being investigated at LUNA, Gran Sasso, with early results published [2].

Another problem at this mass relates to lack of \(^{22}\)Na in ONe white dwarf novae [3, 4, 5]. In these events, the runaway reaction chain

\[
^{20}\text{Ne}(p, \gamma)^{21}\text{Na}(p, \gamma)^{22}\text{Mg}(\beta^+, \nu)^{22}\text{Na}
\]
was expected to stall due to the long half-life of \(^{22}\)Na to \(\beta\)-decay. However, the \(\gamma\)-ray from the resultant excited \(^{22}\)Ne is not observed. This indicates that some other mechanism reduces the abundance of \(^{22}\)Na. Proposed competitors are the reaction chains:

\[ ^{20}\text{Ne}(p, \gamma)^{21}\text{Na}(p, \gamma)^{22}\text{Mg}(p, \gamma)^{23}\text{Al} \]

and

\[ ^{20}\text{Ne}(p, \gamma)^{21}\text{Na}(p, \gamma)^{22}\text{Mg}(\beta^+, \nu)^{22}\text{Na}(p, \gamma)^{23}\text{Mg}. \]

Direct measurement of the reaction \(^{22}\text{Mg}(p, \gamma)^{23}\text{Al}\) is not yet possible, but indirect methods have suggested that this reaction does not account for the lack of \(^{22}\)Na \([3, 4]\). Nonetheless, it will be instructive to investigate this theoretically.

Figure 1 shows the nucleon-emission thresholds for the mass-23 compound states of these reactions, as well as their mirrors.

![Figure 1](image_url)

**Figure 1.** The well-studied mass-23 isobars, and the nucleon-emission thresholds for processes described in Section 1. Data taken from Ref. [6].

### 2. The Multi-channel algebraic scattering formalism

To study these systems, we employ a novel approach to solving the Lippmann-Schwinger scattering formalism. In the following, channels are defined

\[
c : \left[ \left( (\mathcal{L} \otimes \frac{1}{2}) \otimes I \right) \otimes J^- \right],
\]

with \(\mathcal{L}\) being projectile orbital angular momentum, \(J\) projectile total angular momentum, \(I\) target angular momentum, and \(J^-\) the compound system’s spin-parity.
The multichannel algebraic scattering formalism (MCAS) [7] uses nuclear interaction input potentials in the Lippmann-Schwinger equations to realistically model the dynamics of the system. The basic potential used is:

$$V_{cc'}(r) = f(r) \left\{ V_0 \delta_{cc'} + V_\ell [\ell \cdot \ell]_{cc'} + V_{ss}[s \cdot I]_{cc'} \right\} + g(r) V_{ls}[\ell \cdot s]_{cc'},$$

(2)

with deformed Woods-Saxon form factors:

$$f(r) = \left[ 1 + e^{\left(\frac{r-R_a}{a}\right)} \right]^{-1}; \quad g(r) = \frac{1}{r} \frac{df(r)}{dr}. \quad (3)$$

To model nuclear targets with a rotational character, we consider the quantum radius of a rigid drop of nuclear matter, with axial, permanent deformation to be

$$R(\theta, \phi) = R_0 \left[ 1 + \sum_{L(\geq 2)} \sqrt{\frac{4\pi}{2L+1}} \beta_L \left[ Y_L(\hat{r}) \cdot Y_L(\hat{\gamma}) \right] \right] = R_0 \left[ 1 + \epsilon \right].$$

Next, $f(r)$ and $g(r)$ are expanded to second order in $\epsilon$, which gives the potential the desired rotor character. The full potential is quite detailed, but in a short-hand notation can be written as:

$$V_{cc'}(r) = V_{cc'}^{(0)}(r) + V_{cc'}^{(1)}(r) + V_{cc'}^{(2)}(r)$$

\[
= \left\{ v^{(0)}(r) \right\}_{cc'} + \left\{ v^{(1)}(r) \sum_{L(\geq 2)} \beta_L \sqrt{\frac{4\pi}{2L+1}} \left[ Y_L \cdot Y_L \right]_{cc'} \right\} + \left\{ v^{(2)}(r) \sum_{L,L'}(\geq 2) \beta_L \beta_{L'} \sqrt{(2L+1)(2L'+1)} \sum_{\ell} \frac{1}{2\ell+1} |(L0\ell'0\ell)|^2 \left[ Y_{\ell} \cdot Y_{\ell} \right] \right\}_{cc'}. \quad (4)
\]

We focus on the $\beta_L$, with the functions $v^{(0)}(r), v^{(1)}(r)$ and $v^{(2)}(r)$ subsuming all terms independent of $L$, being derivatives of the Woods-Saxon form factors and potential variables. Full expressions can be found in Ref. [7, 8].

In the MCAS method, these interactions are separated into an ‘optimal’ set of functions (sturmians [9], $\chi$) for algebraic treatment of the integral equations. To summarise, with full details to be found in Ref. [7], the assumption that the $T$-matrix can be split, viz.

$$T_{cc'} = -\sum_p |\chi_{cp}\rangle \frac{1}{[1 - \eta_p]} \eta_p \langle \chi_{c'p}|,$$

(5)

where $|\chi_{cp}\rangle$, called sturmians, are appropriately-defined eigenfunctions of specifically-cast Schrödinger equations, and $\eta_p$ are their eigenvalues, we are led to a separable potential

$$V_{cc'} = -\sum_p |\chi_{cp}\rangle \frac{1}{\eta_p} \langle \chi_{c'p}|.$$

(6)

This is known as the Hilbert-Schmid expansion of amplitudes. Using this procedure in reverse, solutions for the $T$-matrix are defined automatically from the potential, without the need to explicitly solve the integral equations.
In terms of the sturmians, the scattering matrices are:

$$S_{cc'} = \delta_{cc'} - i e^{-i k_c} \frac{1}{\pi \mu} \sum_{n,n'=1}^{N} \sqrt{k_c} \chi_{cn}(k_c) \left( \frac{\gamma - G_0}{\mu} \right)_{nn'}^{cc'} \chi_{cn'}(k_c') \sqrt{k_c'}$$  \hspace{1cm} (7)

where

$$[G_0]_{nn'} = \mu \left[ \sum_{c=1}^{\text{open}} \int_{0}^{\infty} \chi_{cn}(x) \frac{x^2}{k_c^2} - x^2 \delta(x) \chi_{cn'}(x) dx - \sum_{c=1}^{\text{closed}} \int_{0}^{\infty} \chi_{cn}(x) \frac{x^2}{\hbar^2 + x^2} \chi_{cn'}(x) dx \right].$$  \hspace{1cm} (8)

Other features of this method include the location of all resonance centroids and widths of the \( A+1 \) coupled system; the determination of all subthreshold bound states from the use of negative projectile energies; a method of accounting for target states that are particle-unstable resonances [10, 11]; and a mechanism to incorporate the Pauli principle, even with collective models of the target. The latter is performed with the addition of orthogonalising pseudo potentials, to the interaction potentials, \( V_{cc'}(r) \), to define new potentials:

$$V_{cc'}(r, r') = V_{cc'}(r) \delta(r - r') + \lambda A_c(r) A_{c'}(r') \delta_{c,c'},$$  \hspace{1cm} (9)

where \( A_c(r) \) are the radial wave functions of occupied orbitals and \( \lambda \) are parameters that determine the strength of the blocking [12].

To date, the only accommodated projectiles are nucleons, but an expansion to allowing \( \alpha \)-particles will be published within the year. Likewise, only (Tamura) rotational modes for the target are currently accommodated, but a treatment of vibrational modes will also be published this year, with shell model interactions being under development. And, while only elastic scattering and single-channel fusion have been accounted for, the formalism for multi-channel fusion, stripping and pick-up has been published [13], and will be implemented in the future.

The regime in which MCAS works best is that of light mass nuclei at resonant-scattering energies where there are few compound states, but work such as discussed here expands these limitations by considering more channels.

The approach used to study proton scattering is to first define the nuclear potential by scattering of neutrons from the mirror of the target, and then add a Coulomb potential. To date, the Coulomb potential used has considered the target’s protons to be in a similar geometry to the entire drop of nuclear matter, but a more sophisticated Fermi charge distribution is under development. In either case, the Coulomb and nuclear potentials are added and new sturmians generated for the mirror system. (Development proceeds as per Appendix C of Ref. [7], with Riccati-Coulomb instead of Riccati-Bessel functions.)

3. MCAS evaluation of the \( n + {^{22}Ne} \) system

As shown in Figure 1, it is clear that the mirror systems \( n + {^{22}Ne} \) and \( p + {^{22}Mg} \) have the fewest bound states. Thus, within MCAS’ purview as outlined in Section 2, it will be easiest to characterise the nuclear potential over an energy range that will describe well both the bound spectra and, in the case of \( p + {^{22}Mg} \), scattering cross sections.

Elaborations on the model required for this system, related to one aspect of the \( {^{22}Na} \) problem in ONe white dwarf novae, will then act as a good starting point for studying the other mass-23 nuclei of the \( {^{22}Na} \) nova problem and the red giant element abundance problem.

Figure 2 shows the experimental spectra of \( {^{22}Ne} \) and \( {^{22}Mg} \). They are shown on individual energy axes, zeroed at their own ground states, illustrating the strong correlation in sequence and spacing of states. This clearly indicates the validity of the mirror symmetry.
Shown in thick (green) lines are the rotor-like states used to define channels. Though the fourth state in this band, a 6$^+$ state, is observed at 6.3 MeV in both nuclei, it is not used in calculations as its coupling to nucleons has negligible effect on the low-spin states observed at low energies in $^{23}$Ne and $^{23}$Al. Shown in dashed (blue) lines are states used in the calculation which are outside the ground-state rotor band but which are known to couple to said band by γ-emissions, which are shown by (red) arrows.

![Diagram of low-energy experimental spectra of $^{22}$Ne and $^{23}$Mg.](image)

**Figure 2.** The low-energy experimental spectra of $^{22}$Ne and $^{23}$Mg. Details in text. Data taken from Ref. [4, 14].

It was found that the inclusion of states outside the main band was necessary in order to obtain an adequate fit to spectral data. To include, in the first instance, the second 2$^+$ state, consider that the ground-state γ-decay half-lives of states relate to $E2$ transition probabilities by

$$
\tau_2 = \frac{\ln(2)}{W_{(E2)}(E_\gamma)} = \frac{0.693}{W_{(E2)}(E_\gamma)} : \quad (10)
$$

and that transition probabilities are linked to $B(E2)$ values by

$$
W_{(E2)}(E_\gamma) = 1.23 \times 10^9 \ (E_\gamma)^5 \ B(E2), \quad (11)
$$

with $E_\gamma$ being photon energy. To first order, and without considering band quantum numbers, a collective (rotational) model gives $B(E2)$ proportional to $\beta_2^2$.

For $^{22}$Ne, the first 2$^+$ states at 1.275 and 4.456 MeV both decay by $E2$ γ-emission to the ground state, with half lives of 3.63 ps and by 37 fs, respectively. This means that the relevant
Table 1. Parameter values defining the $n^+^{22}$Ne interaction. $\lambda^{OPP}$ are blocking strengths of occupied shells, in MeV.

| Parameter      | Odd parity | Even parity |
|----------------|------------|-------------|
| $V_0$ (MeV)    | -65.20     | -51.30      |
| $V_{II}$ (MeV) | -1.01      | -0.30       |
| $V_{ls}$ (MeV) | 7.00       | 7.00        |
| $V_{ss}$ (MeV) | -0.20      | -1.45       |

| $R_0$ (fm) | $\alpha$ (fm) | $\beta_2$ | $\overline{\beta_2}$ | $\beta_4$ |
|------------|----------------|-----------|----------------------|---------|
| 3.1        | 0.75           | 0.22      | 0.1034               | -0.08   |

| $\lambda^{OPP}$ | $1s_{1/2}$ | $1p_{3/2}$ | $1p_{1/2}$ | $1d_{5/2}$ |
|------------------|------------|------------|------------|------------|
| $0^+_1$          | $10^6$     | $10^6$     | $10^6$     | 0.0        |
| $2^+_1$          | $10^6$     | $10^6$     | $10^6$     | 0.0        |
| $4^+_1$          | $10^6$     | $10^6$     | $10^6$     | 0.0        |
| $2^+_2$          | $10^6$     | $10^6$     | $10^6$     | 0.0        |

$\overline{\beta_2}$ for linking $2^+_2$ to other states; 43% of 0.22. See text.

transition probabilities are

$$W(E2)(1.275) = \frac{0.693}{3.63} \cdot 10^{12}$$

$$= 1.23 \cdot 10^{0} \cdot (1.275)^{5} \cdot B(E2, 1.275)$$

$$W(E2)(4.456) = \frac{0.693}{37} \cdot 10^{15}$$

$$= 1.23 \cdot 10^{0} \cdot (4.456)^{5} \cdot B(E2, 4.456),$$

from which $B(E2, 1.275) = 46.06 \, e^2 \, \text{fm}$ and $B(E2, 4.456) = 8.67 \, e^2 \, \text{fm}$. Taking the ratio and assuming that the $B(E2)$ scale as $\beta_2^2$, the deformation length for the 4.456 MeV decay would be $\sim 0.43$ times that for the 1.275 MeV decay. Thus, with $\beta_2$ defining the deformation of the ground state band, and $\overline{\beta_2}$ being the deformation of the $2^+_2$ state, we use the value $\overline{\beta_2} = 0.43\beta_2$ to couple the $2^+_2$ state to the states of the ground state band. (The effect of varying $\overline{\beta_2}$ from 0 to $\beta_2$ is explored in Ref. [8].)

The parameter set which gives the best agreement with the low-energy spectrum of $^{23}$Ne, when using the target state set $0^+_1$, $2^+_1$, $4^+_1$, and $2^+_2$ (with the overline denoting the reduced coupling), is shown in Table 1, with results shown in the middle panel of Fig. 3.

This calculation obtains a match, within 0.5 MeV, for the first nine states whose spin-parities are known or postulated. This makes a good result for a range of up to 3.5 MeV.

It does, however, present two states in this range not observed in experiment. The unobserved $9^+_2$ state found by MCAS could possibly exist but is as yet unobserved. In support of this, there exist low-energy $9^+_2$ states in both $^{23}$Na, whose low-lying spectrum has many similarities with $^{23}$Ne, at 2.703 MeV above the ground state, and possibly in its mirror, $^{23}$Mg, which has a state uncertainly designated $9^+_2$, $5^+_2$ at 2.714 MeV above its ground state.

Another calculation was performed, using the same parameter set as in Table 1 with the addition of the $4^+_2$ state, with, for convenience, coupling between this and all other states given 0.43 the strength as between states in the ground state rotor band. Results, shown in the right panel of Fig. 3, show that the addition of this state brings the least-well created low-lying states of the previous calculation into better agreement with experiment, supporting the recent
Figure 3. The experimental $^{23}$Ne spectrum [6] (left) and that calculated from MCAS evaluation of the $n+^{22}$Ne, with target state sets $0^+_1, 2^+_1, 4^+_1, 5/2^+_1$ (middle) and $0^+_2, 2^+_1, 4^+_1, 5/2^+_1, 7/2^+_1$ (right), relative to the neutron emission threshold. The bar denotes the use of reduced coupling for channels involving this state. Data from Ref. [6].

experimental finding [4] that this state contributes to the ground state of the mirror of $^{23}$Ne, $^{23}$Al.
4. Future work
The immediate extension to this work is to calculate elastic cross sections for the mirror system, \( p^+{^{22}}\text{Mg} \). This will be performed with a Fermi charge distribution model.

Secondly, better elaboration of inter- and intra-band coupling strengths must be made. This depends upon how the second band is classified. As is, with a shape co-existence being assumed, it appears as a second \( K = 0 \) \( \beta \)-band, though no \( 0^+ \) state for the band head is known experimentally. Alternatively, as the \( 2^+_2 \) and \( 4^+_2 \) states of \( {^{22}}\text{Ne} \) could be considered the first and third states of a \( K = 2 \) \( \gamma \)-band. There is an uncertainly-assigned state in the mirror, \( {^{22}}\text{Mg} \), that may be the second \( \gamma \)-band state, a \( 3^+ \) [15, 16], in which case all states should possess the same deformation.

With these details clarified, the remaining systems mass-23 systems discussed in Section 1, as well as their mirrors, will be studied. With these systems characterised, a forthcoming extension to MCAS will be used to calculate their capture cross sections.

5. Conclusions
As mass-23 nuclei are of interest in several problems of astrophysical import, relating to ONe white dwarf novae and red giant element abundances. To study these, the MCAS method of (thus far) nucleon-nucleus scattering is being expanded to include states outside the main rotational band. The simplest mass-23 system for the method, \( n^+{^{22}}\text{Ne} \), has been studied, with its mirror, \( p^+{^{22}}\text{Mg} \), currently under consideration. The target states of importance have been found to be \( 0^+_1 \), \( 2^+_1 \), \( 4^+_1 \), \( 2^+_2 \), and \( 4^+_2 \). The latter confirms the recent experimental findings of Banu et al. [4]

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