Some optimal links between generations of correlation averages

GIOVANNI COPPOLA 1 - MAURIZIO LAPORTA

Abstract. For a real-valued and essentially bounded arithmetic function \( f \), i.e., \( f(n) \ll_{\varepsilon} n^\varepsilon, \forall \varepsilon > 0 \), we give some optimal links between non-trivial bounds for the sums \( \sum_{h \leq H} \sum_{N<n \leq 2N} f(n) f(n-h) \), \( \sum_{N<x \leq 2N} \left| \sum_{x<n \leq x+H} f(n) \right|^2 \) and \( \sum_{N<n \leq 2N} \left| \sum_{0 \leq |n-x| \leq H} (1 - \frac{|n-x|}{H}) f(n) \right|^2 \), with \( H = o(N) \) as \( N \to \infty \).

1. Introduction.

The correlation of a complex-valued arithmetic function \( f \) is a shifted convolution sum of the form

\[
\mathcal{C}_f(h) \overset{\text{def}}{=} \sum_{n \geq N} f(n) \overline{f(n-h)},
\]

where \( n \sim N \) means that \( n \in (N,2N] \cap \mathbb{N} \), while \( N \) and the shift \( h \) are integers such that \( |h| \leq N \). In particular, this allows restricting \( f \) to \( 1 \leq n \leq 3N \) when dealing with \( \mathcal{C}_f(h) \). Further, note that

\[
\mathcal{C}_f(h) = \sum_{n \sim N} \sum_{m \sim N} f(n) \overline{f(m)} + O(\|f\|_\infty \|h\|), \quad \text{with} \quad \|f\|_\infty \overset{\text{def}}{=} \max_{1 \leq n \leq 3N} |f(n)|.
\]

In [CL1] and [CL3] we have investigated the connection between the correlations of \( f \) and its Selberg integral

\[
J_f(N,H) \overset{\text{def}}{=} \sum_{x \sim N} \left| \sum_{n \sim N \leq x+H} f(n) - M_f(x,H) \right|^2,
\]

where \( M_f(x,H) \) is the expected mean value of \( f \) in short intervals, with \( H = o(N) \) as \( N \to \infty \) (to avoid trivialities, hereafter we assume that \( H \to \infty \)). More in general, we extended such an investigation to weighted versions of \( J_f(N,H) \), that include, also, the so-called modified Selberg integral

\[
\tilde{J}_f(N,H) \overset{\text{def}}{=} \sum_{x \sim N} \left| \sum_n C_H(n-x) f(n) - M_f(x,H) \right|^2,
\]

where the Cesàro weight \( C_H(t) \overset{\text{def}}{=} \max(1 - |t|/H,0) \) allows taking the same mean value that appears in \( J_f(N,H) \). Indeed, in [CL1] (see Lemma 7 there), by means of an elementary Dispersion Method, it is shown that weighted Selberg integrals, for a wide class of arithmetic functions, are actually linked to averages of their correlations (see also the proof of Lemma 1 below). Mainly inspired by the prototype of the divisor function \( d_k \), we stuck to the case of a real-valued and essentially bounded \( f \), i.e. \( f(n) \ll_{\varepsilon} n^{\varepsilon} \ (\forall \varepsilon > 0) \). Here we recall that \( \ll \) is Vinogradov’s notation, synonymous to Landau’s O-notation. In particular, \( \ll_{\varepsilon} \) means that the implicit constant might depend on an arbitrarily small \( \varepsilon > 0 \), which might change at each occurrence. Moreover, we abbreviate \( A(N,H) \ll B(N,H) \) whenever \( A(N,H) \ll_{\varepsilon} N^\varepsilon B(N,H), \forall \varepsilon > 0 \).

In [CL1] we also searched for links between non-trivial bounds for \( J_f(N,H), \tilde{J}_f(N,H) \) and the so-called deviation of \( f \), that is defined as

\[
\mathbb{D}_f(N,H) \overset{\text{def}}{=} \sum_{h \leq H} \mathcal{C}_f(h) - \sum_{x \sim N} \frac{M_f(x,H)^2}{H}.
\]

\footnote{titolare di un Assegno “Ing.Giorgio Schirillo” dell’Istituto Nazionale di Alta Matematica (Fellow “Ing.Giorgio Schirillo” of the Istituto Nazionale di Alta Matematica).}

Mathematics Subject Classification (2010) : 11N37.
In this paper we focus on the special case of a real-valued and essentially bounded function \( f \) which is also balanced, that is \( M_f(x, H) \) vanishes identically. Therefore, this yields

\[
\mathbb{D}_f(N, H) = \sum_{h \leq H} C_f(h), \\
J_f(N, H) = \sum_{x \sim N} \left| \sum_{x < n \leq x + H} f(n) \right|^2, \\
\tilde{J}_f(N, H) = \sum_{x \sim N} \left| \sum_n C_H(n - x) f(n) \right|^2.
\]

In this regard, we have the following result that improves on the bounds given in [CL1].

**Theorem.** Let \( H, N \in \mathbb{N} \) with \( H = o(N) \) as \( N \to \infty \), and let \( f : \mathbb{N} \to \mathbb{R} \) be essentially bounded and balanced.

i) If there is \( A \in [-1, 1) \) such that \( J_f(N, H) \ll NH^{1+A} \) and \( J_f(N, H_1) \ll NH_1^{1+A} \) for \( H_1 = \left[H^{1-\frac{4(1-A)}{5-A}}\right] \), then

\[
\mathbb{D}_f(N, H) \ll (N + H^{2-A})H^{1-\frac{4(1-A)}{5-A}}.
\]

ii) If there is \( A \in [-3, 1) \) such that \( \tilde{J}_f(N, H) \ll NH^{1+A} \) and \( \tilde{J}_f(N, H_2) \ll NH_2^{1+A} \) for \( H_2 = \left[H^{1-\frac{4(1-A)}{5-A}}\right], \)

\[
J_f(N, H) \ll (N + H^{2-A})H^{2-\frac{4(1-A)}{5-A}}.
\]

Hereafter, \( [t] \) denotes the integer part of \( t \in \mathbb{R} \).

**Remarks.**

1. Note that

\[
A \in [-1,1) \Leftrightarrow 1 + A \in [0,2) \Leftrightarrow 1 - \frac{2(1-A)}{3-A} \in [0,1) \Leftrightarrow (2-A)^{-1} \in [1/3,1),
\]

\[
A \in [-3,1) \Leftrightarrow 1 + A \in [-2,2) \Leftrightarrow 1 - \frac{2(1-A)}{5-A} \in [0,1) \Leftrightarrow (2-A)^{-1} \in [1/5,1).
\]

Finally, the minimal values \(-1 \) and \(-3 \) for \( A \), resp., in i) and ii), are sharp. To see this, it is enough to consider the function \( f(n) = (-1)^{n+1} \), which is essentially bounded and balanced (its Dirichlet series is known as *Dirichlet’s eta function*). Since its correlation is \( C_f(h) = (-1)^h N \), then

\[
\mathbb{D}_f(N, H) = N \sum_{h \leq H} (-1)^h = \begin{cases} -N & \text{if } H \text{ is odd} \\ 0 & \text{otherwise}, \end{cases} \\
J_f(N, H) = \sum_{x \sim N} \left| \sum_{x < n \leq x + H} (-1)^n \right|^2 = \begin{cases} N & \text{if } H \text{ is odd} \\ 0 & \text{otherwise}, \end{cases} \\
\tilde{J}_f(N, H) = \frac{1}{H^2} \sum_{x \sim N} \left| \sum_{h \leq H} \sum_{0 \leq |n-x| < h} (-1)^n \right|^2
\]

\[
= \frac{1}{H^2} \sum_{x \sim N} \left| \sum_{h \leq H} \sum_{0 \leq |n-x| < h} 1 \right|^2 = \begin{cases} N/H^2 & \text{if } H \text{ is odd} \\ 0 & \text{otherwise}, \end{cases}
\]

where we have used the Cesaro identity

\[
\sum_{0 \leq |n-x| \leq H} \left( 1 - \frac{|n-x|}{H} \right) f(n) = \frac{1}{H} \sum_{h \leq H} \sum_{0 \leq |n-x| < h} f(n) .
\]

2. Through an application of the so-called *length inertia* (see [CL3] and [CL4]), it could be shown that the Theorem hypothesis on \( J_f(N, H) \) and \( \tilde{J}_f(N, H) \) might be dropped without affecting the result.
2. Lemmata.

First, let us introduce some notation and some auxiliary functions. For the unit step weight

\[ u_h(a) \overset{\text{def}}{=} \begin{cases} 1 & \text{if } a \in [1, H] \cap \mathbb{N} \\ 0 & \text{otherwise,} \end{cases} \]

we set

\[ U_h(h) \overset{\text{def}}{=} \sum_{a, b \leq H} u_h(b) u_h(a), \quad U_h(h) \overset{\text{def}}{=} \frac{1}{H^2} \sum_{a, b \leq H} U_h(b) U_h(a), \]

\[ \hat{u}_h(\beta) \overset{\text{def}}{=} \sum_{h \leq H} e(h \beta), \quad \hat{U}_h(\beta) \overset{\text{def}}{=} \sum_{h \leq H} U_h(h) e(h \beta), \quad \forall \beta \in \mathbb{R}. \]

where \( e(\alpha) \overset{\text{def}}{=} e^{2\pi i \alpha} \). Moreover, for the function \( f \) under consideration we denote

\[ \hat{f}(\beta) \overset{\text{def}}{=} \sum_{n \sim N} f(n) e(n \beta). \]

Next Lemma is a consequence of Lemma 7 from [CL1]. Somehow the formulæ (I)–(III) were already implicit between the lines of [CL1], where, however, the underlying assumption that \( f \) has to be also essentially bounded is, in fact, redundant.

**Lemma 1.** For every balanced \( f : \mathbb{N} \to \mathbb{R} \) one has

(I) \[ D_f(N, H) = \int_0^1 |\hat{f}(\beta)|^2 \hat{u}_h(-\beta) d\beta + O(H^2 \|f\|_2^2), \]

(II) \[ J_f(N, H) = \int_0^1 |\hat{f}(\beta)|^2 |\hat{u}_h(\beta)|^2 d\beta + O(H^3 \|f\|_2^3), \]

(III) \[ \bar{J}_f(N, H) = \int_0^1 |\hat{f}(\beta)|^2 \frac{|\hat{u}_h(\beta)|^4}{H^2} d\beta + O(H^3 \|f\|_2^3). \]

**Proof.** Since

\[ D_f(N, H) = \sum_{h \leq H} \mathcal{C}_f(h), \]

then (I) follows immediately because it is plain that

\[ \mathcal{C}_f(h) = \sum_{n \sim N} \sum_{n \sim N} f(n) f(m) + O(\|f\|_2^2 |h|) = \int_0^1 |\hat{f}(\beta)|^2 e(-h \beta) d\beta + O(\|f\|_2^2 |h|). \]

In order to show (II) and (III) let us recall that the Selberg integral and the modified one for any real arithmetic function \( f \) are related to the correlation averages (see [CL1], Lemma 7), respectively as

\[ J_f(N, H) = \sum_h U_h(h) \mathcal{C}_f(h) - \frac{2}{H} \sum_n f(n) \sum_{x \sim N} u_h(n-x) M_f(x, H) + \sum_{x \sim N} M_f(x, H)^2 + O(H^3 \|f\|_2^3), \]

\[ \bar{J}_f(N, H) = \sum_h \bar{U}_h(h) \mathcal{C}_f(h) - \frac{2}{H} \sum_n f(n) \sum_{x \sim N} U_h(n-x) M_f(x, H) + \sum_{x \sim N} M_f(x, H)^2 + O(H^3 \|f\|_2^3). \]

In particular, by setting \( M_f(x, H) = 0 \) in these formulæ and by using the properties

\[ U_h(h) = \sum_{a \leq H-|h|} 1, \quad \bar{U}_h(\beta) = |\hat{u}_h(\beta)|^2, \]

\[ \hat{u}_h(\beta) \overset{\text{def}}{=} \sum_{h \leq H} e(h \beta), \quad \hat{U}_h(\beta) \overset{\text{def}}{=} \sum_{h \leq H} U_h(h) e(h \beta), \quad \forall \beta \in \mathbb{R}. \]
we get (II) and (III), because it is easily seen that
\[
\sum_{h} U_{n}(h) C_{f}(h) = \sum_{h \leq H} \sum_{0 \leq |a| < h} C_{f}(a) = \int_{0}^{1} |\hat{f}(\beta)|^{2} \tilde{U}_{n}(\beta) d\beta + O(H^{3}\|f\|_{\infty}^{2}),
\]

\[
\sum_{h} \tilde{U}_{n}(h) C_{f}(h) = \sum_{h} \sum_{b-a=h} \frac{U_{n}(b) U_{n}(a)}{H^{2}} C_{f}(h) = \frac{1}{H^{2}} \int_{0}^{1} |\hat{f}(\beta)|^{2} |\tilde{U}_{n}(\beta)|^{2} d\beta + O(H^{3}\|f\|_{\infty}^{2}).
\]

The Lemma is completely proved.

Remark. Consistently with the terminology introduced in [CL1], we refer to (I), (II) and (III) as a first, second and third generation formula, respectively. As transpires also from the above proof, such formulae correspond to iterations of correlations’ averages.

Next Lemma gives two versions of a Gallagher’s inequality (see [Ga], Lemma 1), that have been discussed in [CL2] and [CL4].

Lemma 2. Let \( N, h \in \mathbb{N} \) be such that \( h \to \infty \) and \( h = o(N) \) as \( N \to \infty \). If \( f : \mathbb{N} \to \mathbb{C} \) is essentially bounded and balanced, then

1) \[
h^{2} \int_{-\frac{1}{2}}^{\frac{1}{2}} |\hat{f}(\alpha)|^{2} d\alpha \ll J_{f}(N, h) + h^{3},
\]

2) \[
h^{2} \int_{-\frac{1}{2}}^{\frac{1}{2}} |\hat{f}(\alpha)|^{2} d\alpha \ll \tilde{J}_{f}(N, h) + h^{3}.
\]

3. Proof of the Theorem.

In what follows, we will appeal to the well-known property

\[
|\hat{u}_{n}(\alpha)| = \frac{|\sin(\pi H\alpha)|}{\sin(\pi \alpha)} \leq \frac{1}{\sin(\pi \alpha)} < \frac{1}{2\alpha}, \quad \text{for } 0 < \alpha < \frac{1}{2}.
\]

In particular, this yields the implication

\[(*) \quad |\hat{u}_{n}(\alpha)| > h \quad \Rightarrow \quad |\alpha| < \frac{1}{2h}.
\]

Proof of i). Since \( f \) is essentially bounded and balanced, then from (I) of Lemma 1 we infer

\[
\mathbb{D}_{f}(N, H) \ll \int_{-\frac{1}{2}}^{1/2} \left| \hat{f}(\alpha) \right|^{2} |\hat{u}_{n}(\alpha)| d\alpha + H^{2}
\]

\[
\ll H^{1-\delta} \int_{|\hat{u}_{n}(\alpha)| \leq [H^{1-\delta}]} \left| \hat{f}(\alpha) \right|^{2} d\alpha + H^{1-\gamma} \int_{[H^{1-\delta}] < |\hat{u}_{n}(\alpha)| \leq H^{1-\gamma}} \left| \hat{f}(\alpha) \right|^{2} d\alpha
\]

\[
+ H^{\gamma-1} \int_{|\hat{u}_{n}(\alpha)| > H^{1-\gamma}} \left| \hat{f}(\alpha) \right|^{2} |\hat{u}_{n}(\alpha)|^{2} d\alpha + H^{2},
\]
where $\gamma, \delta$ are real numbers to be determined later, so that $0 < \gamma \leq \delta$. Thus, by applying ($*$), Parseval’s identity and (II) of Lemma 1, we get

$$
\mathbb{D}_f(N, H) \ll NH^{1-\delta} + H^{1-\gamma} \int_{|\alpha| \leq \frac{1}{2^{1/2}}} |\hat{f}(\alpha)|^2 \, d\alpha + J_f(N, H) H^{\gamma-1} + H^{2+\gamma},
$$

where we have set $H_1 = [H^{1-\delta}]$. Thus, by 1) of Lemma 2 and, then, assuming that $J_f(N, H_1) \ll NH_1^{1+A}$ and $J_f(N, H) \ll NH^{1+A}$, we write

$$
\mathbb{D}_f(N, H) \ll NH^{1-\delta} + H^{1-\gamma} H^{-2} J_f(N, H_1) + H^{1-\gamma} H_1 + J_f(N, H) H^{\gamma-1} + H^{2+\gamma}
$$

$$
\ll NH(H^{-\delta} + H^{-\gamma - (1-\delta)(1-A)} + H^{\gamma + A - 1}) + H^{2+\gamma}.
$$

Now, observe that $\delta = \gamma + (1 - \delta)(1 - A) = 1 - A - \gamma$ is satisfied by $\delta = \frac{2(1-A)}{3-A}$ and $\gamma = \frac{(1-A)^2}{2(3-A)}$, which obey the condition $0 < \gamma \leq \delta$ whenever $A \in [-1, 1)$. This yields the inequality for $\mathbb{D}_f(N, H)$ stated in i).

**Proof of ii.** Since we closely follow the proof of i), we skip some details. By using (II) and (III) of Lemma 1 and applying 2) of Lemma 2, as before we can write

$$
J_f(N, H) \ll NH^{2-2\delta} + H^{2-2\gamma} \int_{|\alpha| \leq \frac{1}{2^{1/2}}} |\hat{f}(\alpha)|^2 \, d\alpha + H^{2\gamma} \int_{-1/2}^{1/2} |\hat{f}(\alpha)|^2 \left|\frac{\hat{g}_n(\alpha)}{H^2}\right|^2 d\alpha + H^3
$$

$$
\ll NH^{2-2\delta} + H^{2-2\gamma} H^{-2} J_f(N, H_2) + H^{2-2\gamma} H_2 + H^{2\gamma} J_f(N, H) + H^{3+2\gamma},
$$

where $H_2 = [H^{1-\delta}]$. Thus, from $J_f(N, H) \ll NH^{1+A}$ and $J_f(N, H_2) \ll NH_2^{1+A}$, it follows

$$
J_f(N, H) \ll NH^2 (H^{-2\delta} + H^{-2\gamma - (1-A)(1-\delta)} + H^{A-1+2\gamma}) + H^{3+2\gamma}.
$$

The conclusion follows by taking $\delta = \frac{2(1-A)}{5-A}$ and $\gamma = \frac{(1-A)^2}{2(3-A)}$, noticing that $0 < \gamma \leq \delta$ whenever $A \in [-3, 1)$. The Theorem is completely proved.

**References**

[CL1] G. Coppola, M. Laporta, *Generations of correlation averages*, Journal of Numbers, Vol. 2014 (2014), Article ID 140840, 13 pages, http://dx.doi.org/10.1155/2014/140840 (draft at arxiv:1205.1706v3)

[CL2] G. Coppola, M. Laporta, *A modified Gallagher’s Lemma*, preprint at arxiv.org/abs/1301.0008v1

[CL3] G. Coppola, M. Laporta, *Symmetry and short interval mean-squares*, (submitted), preprint available at arXiv:1312.5701v1.

[CL4] G. Coppola, M. Laporta, *A generalization of Gallagher’s Lemma for exponential sums*, to appear on Siauliai Mathematical Seminar, (draft at arxiv.org/abs/1411.1739v1)

[Ga] P. X. Gallagher, *A large sieve density estimate near $\sigma = 1$*, Invent. Math. 11 (1970), 329–339. MR 43#4775

Giovanni Coppola
Università degli Studi di Napoli
Home address: Via Partenio 12 -
- 83100, Avellino(AV), ITALY
e-page: www.giovanicoppola.name
e-mail: giovanni.coppola@unina.it

Maurizio Laporta
Università degli Studi di Napoli
Dipartimento di Matematica e Appl.
Compl.Monte S.Angelo
Via Cinthia - 80126, Napoli, ITALY
e-mail: mlaporta@unina.it