Second-order Relativistic Hydrodynamic Equations for Viscous Systems; how does the dissipation affect the internal energy?

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We derive the second-order dissipative relativistic hydrodynamic equations in a generic frame with a continuous parameter from the relativistic Boltzmann equation. We present explicitly the relaxation terms in the energy and particle frames. Our results show that the viscosities are frame-independent but the relaxation times are generically frame-dependent. We confirm that the dissipative part of the energy-momentum tensor in the particle frame satisfies $\delta T_{\mu}^{\nu} = 0$ obtained for the first-order equation before, in contrast to the Eckart choice $u_{\mu} \delta T_{\mu}^{\nu} u_{\nu} = 0$ adopted as a matching condition in the literature. We emphasize that the new constraint $\delta T_{\mu}^{\nu} = 0$ can be compatible with the phenomenological derivation of hydrodynamics based on the second law of thermodynamics.

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I. INTRODUCTION

After the discovery that perfect hydrodynamics can be valid for describing the phenomenology of Relativistic Heavy Ion Collider (RHIC) at Brookhaven National Laboratory \cite{1,2}, people are now interested in relativistic hydrodynamics for dissipative systems; see the recent excellent review articles \cite{3,4}.

Recently, Tsumura, Kunihiro (the present authors) and Ohnishi (abbreviated as TKO) \cite{6} derived generic covariant hydrodynamic equations for a viscous fluid from the relativistic Boltzmann equation in a systematic manner with no heuristic arguments on the basis of the so-called renormalization group (RG) method \cite{7,8,9,10}. Although the hydrodynamic equations they derived are the so-called first-order ones, the equations have remarkable aspects: The generic equation derived by TKO can produce a relativistic dissipative hydrodynamic equation in any frame with an appropriate choice of a macroscopic flow vector $a^{\mu}$ ($\mu = 0, 1, 2, 3$), which defines the coarse-grained space and time; the resulting equation in the energy frame coincides with that of Landau and Lifshitz \cite{11}, while that in the particle frame is similar to, but slightly different from, the Eckart equation \cite{12}.

Let $\delta T^{\mu\nu}$ and $\delta N^{\mu}$ be the dissipative term of the symmetric energy-momentum tensor and the particle-number vector, respectively. Owing to the ambiguity in the separation of the energy and the mass inherent in relativistic theories, one must choose the local rest frame (LRF) where the flow velocity $u^{\mu}$ with $u^{\mu} u_{\mu} = 1$ is defined: One of the typical frame is the energy (Landau) frame in which $\delta T^{\mu\nu} u_{\mu} \Delta_{\nu\rho} = 0$ with $\Delta^{\mu\nu} = g^{\mu\nu} - u^{\mu} u^{\nu}$ and $g^{\mu\nu} = \text{diag}(+1, -1, -1, -1)$, i.e. there is no dissipative energy flow. On the other hand, another typical frame is the particle (Eckart) frame in which $\delta N^{\mu} \Delta_{\mu\nu} = 0$, i.e. there is no dissipative particle flow. Both in the energy and particle frames, the dissipative terms of the energy-momentum tensor and the particle-number vector are usually assumed to satisfy the constraints,

$$u_{\mu} \delta T^{\mu\nu} u_{\nu} = 0,$$

(I.1)

and $u_{\mu} \delta N^{\mu} = 0$. These phenomenological ansatz have been employed as the matching conditions even in the subsequent “derivations” of the so-called second-order equations \cite{13,14,15,16}; note that even in the Grad’s moment method \cite{17,18}, some ansatz are needed to $\delta T^{\mu\nu}$ and $\delta N^{\mu}$ as the matching conditions, for which different proposals exist \cite{19,20}.

Here we emphasize that the matching conditions touch on the fundamental but not yet fully understood problem how to define the LRF in the relativistic fluid dynamics for a viscous system. The way how to define the LRF or equivalently to fix the matching condition is unsolved yet, and remains a nontrivial and fundamental problem in the field of nonequilibrium relativistic dynamics, although there have been no serious consideration on this difficult problem in the literature. Actually, we shall argue that these phenomenological ansatz, especially Eq. (I.1), can be false and actually is not compatible with the underlying kinetic equation.

In fact, it is found that the TKO equation in the particle frame derived from the relativistic Boltzmann equation satisfies

$$\delta T^{\mu}_{\mu} = 0,$$

(I.2)

but does not satisfy Eq. (I.1). One should here note that the derived condition (I.2) is identical to a matching condition postulated by Marle \cite{19} and advocated by Stewart \cite{20} in the derivation of the relativistic hydrodynamics from the
relativistic Boltzmann equation with use of the Grad’s moment method. In their paper \cite{8}, TKO proved that the Eckart constraint \cite{11} in the particle frame cannot be compatible with the underlying relativistic Boltzmann equation for the first-order hydrodynamic equation. In spite of the first-order one, the TKO equation in the particle frame is free from the pathological properties \cite{21} in contrast to the original Eckart equation with which the thermal equilibrium for the first-order hydrodynamic equation. In spite of the first-order one, the TKO equation in the particle frame is free from the pathological properties \cite{21} in contrast to the original Eckart equation with which the thermal equilibrium for the first-order hydrodynamic equation. In spite of the first-order one, the TKO equation in the particle frame is free from the pathological properties \cite{21} in contrast to the original Eckart equation with which the thermal equilibrium for the first-order hydrodynamic equation.

One may naturally ask if the Eckart constraint \cite{11} should be replaced with \cite{12} even for the so-called second-order equation like Israel-Stewart one. And are any modifications needed to the constraints in the Landau frame? A purpose of this Letter \cite{23} is to answer these questions both by phenomenological and microscopic analyses. We shall see that the Eckart constraint should be replaced with the new one even in the second-order equation, while no modification is necessary for the constraints in the energy frame. We shall derive the second-order dissipative relativistic hydrodynamic equations in a generic frame with a continuous parameter $\theta$ from the relativistic Boltzmann equation. We shall derive the relaxation terms for a generic frame with the new constraint, and present explicitly those in the energy and particle frames. We shall show that the viscosities are frame-independent but the relaxation times are generically frame-dependent in accordance with the observation by Betz et al. \cite{17}, although the constraint to $\delta T^{\mu\nu}$ is quite different.

II. A GENERAL PHENOMENOLOGICAL DERIVATION OF RELATIVISTIC DISSIPATIVE HYDRODYNAMIC EQUATIONS; EXISTENCE OF POSSIBLE EXTRA TERMS IN THE DISSIPATIVE TERMS

Let $T^{\mu\nu}$ and $N^\mu$ be the symmetric energy-momentum tensor and the particle-number vector of the system we consider, respectively. The total number of independent variables is fourteen, and the dynamical evolution of these variables are governed by the respective balance equations;

\begin{align}
\partial_\mu T^{\mu\nu} &= 0, \\
\partial_\mu N^\mu &= 0. 
\end{align}

(II.1)

(II.2)

With use of an arbitrary four vector $u^\mu$ with $u^\mu u_\mu = 1$, $T^{\mu\nu}$ and $N^\mu$ can be cast into the tensor-decomposed forms, \begin{align}
T^{\mu\nu} &= (e + \delta e) u^\mu u^\nu - (p + \delta p) \Delta^{\mu\nu} + q^\mu u^\nu + q^\nu u^\mu + \pi^{\mu\nu}, \\
N^\mu &= (n + \delta n) u^\mu + \nu^\mu, 
\end{align}

(II.3)

(II.4)

respectively. Here, $e + \delta e, p + \delta p$, and $n + \delta n$ are the internal energy, pressure, and particle-number density in the dissipative system; $e + \delta e \equiv T_{ab} u^a u^b$, $p + \delta p \equiv -1/3 T_{ab} \Delta^{ab}$, and $n + \delta n \equiv N_a u^a$, with $e = e(T, \mu)$, $p = p(T, \mu)$, and $n = n(T, \mu)$ being the corresponding quantities in the local equilibrium state characterized by the temperature $T$ and the chemical potential $\mu$. Note that we have made it explicit by $\delta e$, $\delta p$, and $\delta n$ that the dissipations may cause corrections to all these quantities, although only the correction to the pressure has been considered in the literature; $\delta p$ is identified with the bulk pressure $\Pi$. We emphasize that there is no persuading reasoning that only the pressure acquires corrections due to the dissipative process. The dissipative parts of the energy-momentum tensor and particle-number vector are identified as $\delta T^{\mu\nu} = \delta e u^\mu u^\nu + \delta p \Delta^{\mu\nu} + q^\mu u^\nu + q^\nu u^\mu + \pi^{\mu\nu}$ and $\delta N^\mu = \delta n u^\mu + \nu^\mu$, respectively. The energy flow relative to $u^\mu$ is denoted by $q^\mu$, $\nu^\mu$ is the flow of particle number relative to $u^\mu$, and finally $\pi^{\mu\nu}$ is the shear stress tensor; $q^\mu \equiv T_{ab} u^a \Delta^{b\mu}$, $\nu^\mu \equiv N_a \Delta^{a\mu}$, and $\pi^{\mu\nu} \equiv T_{ab} \Delta^{ab\mu\nu}$. Here the space-like, symmetric and traceless tensor $\Delta^{\mu\nu\rho\sigma} \equiv 1/2 (\Delta^{\mu\nu} \Delta^{\rho\sigma} + \Delta^{\mu\rho} \Delta^{\nu\sigma} - 2/3 \Delta^{\mu\rho} \Delta^{\nu\sigma})$ is introduced. One can easily confirm that $q^\mu u_\mu = 0$, $\nu^\mu u_\mu = 0$, $\pi^{\mu\nu} = \pi^{\nu\mu}$, and $u_\mu \pi^{\mu\nu} = \pi^{\mu\mu} = 0$. This implies that the total number of independent components of $q^\mu$, $\nu^\mu$, and $\pi^{\mu\nu}$ is eleven. Since $T^{\mu\nu}$ and $N^\mu$ have the fourteen components in total, $\delta e$, $\delta p$, and $\delta n$ have only one independent component other than $T$ and $\mu$. We take $\delta p = \Pi$ as the independent component as a natural choice, then $\delta e$ and $\delta n$ can be expressed as $\delta e = f_e \Pi$ and $\delta n = f_n \Pi$, where $f_e$ and $f_n$ are functions of $T$ and $\mu$; $f_e = f_e(T, \mu)$ and $f_n = f_n(T, \mu)$. Here we have assumed that the dissipative order of $\delta e$ and $\delta n$ are the same as that of $\delta p$ at most. We remark that although $f_e$ and $f_n$ may take finite values generically, the functional forms of $f_e$ and $f_n$ cannot be determined by the phenomenological theory; as those of $e$, $p$, and $n$ can not, either. All the previous analyses assumed $f_e = f_n = 0$, which has not been recognized so far.

Now we shall show that the just usual phenomenological derivation of the hydrodynamic equations in which the second law of thermodynamics is utilized allows the existence of $\delta e$ and $\delta n$, i.e., finite values of $f_e$ and $f_n$, in the relativistic dissipative hydrodynamic equations. It is found that the essential point of the proof is the same for the first- and second-order equations where the entropy current $S^0$ is at most linear and bilinear with respect to $\Pi$, $q^\mu$, $\nu^\mu$, and $\pi^{\mu\nu}$, respectively, although the resulting mathematical expressions are much more complicated in the second-order one \cite{24}. Thus we here take the first-order equation, for the sake of simplicity. The second-order equations with...
finite $f_e$ and $f_n$ will be derived microscopically later in this article. So the entropy current is given by

$$T^S u^\mu = p u^\mu + u^\mu T_{\mu\nu} - \mu N^\mu.$$  \hspace{1cm} (II.5)

The second law of thermodynamics reads $\partial_\mu S^\mu \geq 0$.

The divergence of $S^\mu$ is found to take the form

$$\partial_\mu S^\mu = \Pi \left[ f_e D \frac{1}{T} - \frac{1}{T} \nabla^\mu u_{\mu} - f_n D \frac{\mu}{T} \right] + q^\mu \left[ \frac{1}{T} D u_{\mu} + \nabla^\mu \frac{1}{T} \right] - \nu^\mu \nabla^\mu \frac{\mu}{T} + \pi^{\mu\nu} \frac{1}{T} \nabla_\mu u_\nu, \hspace{1cm} (II.6)$$

where $D \equiv u^a \partial_a$ and $\nabla^\mu \equiv \Delta^\mu a \partial_a$. Here, we have used the conservation laws, Eq.’s (II.1) and (II.2), and the first law of thermodynamics, $D(p/T) + e D(1/T) - n D(\mu/T) = 0$.

The frames define the flow velocity $\nu^\mu$ of the fluid: The flow velocity in the particle frame and the energy frame are defined by setting $u^\mu = N^\mu/\sqrt{N^\nu N_\nu}$ and $u^\mu = T^{\mu\alpha} u_\alpha/\sqrt{T^{\nu\beta} u_\nu T_{\nu\beta} u^\beta}$, respectively. By these settings, a closed system of the relativistic dissipative hydrodynamic equations is obtained. Note that $u^\mu = N^\mu/\sqrt{N^\nu N_\nu}$ is equivalent to $\nu^\mu = 0$ ($q^\mu = 0$).

In the particle frame where $\nu^\mu$ vanishes, Eq. (II.6) is reduced to

$$\partial_\mu S^\mu = \Pi \left[ f_e D \frac{1}{T} - \frac{1}{T} \nabla^\mu u_{\mu} - f_n D \frac{\mu}{T} \right] + q^\mu \left[ \frac{1}{T} D u_{\mu} + \nabla^\mu \frac{1}{T} \right] + \pi^{\mu\nu} \frac{1}{T} \nabla_\mu u_\nu. \hspace{1cm} (II.7)$$

It is found that the following constitutive equations,

$$\Pi = \zeta T \left[ f_e D \frac{1}{T} - \frac{1}{T} \nabla^\mu u_{\mu} - f_n D \frac{\mu}{T} \right], \hspace{1cm} (II.8)$$

$$q^\mu = -\lambda T^2 \left[ \frac{1}{T} D u_{\mu} + \nabla^\mu \frac{1}{T} \right], \hspace{1cm} (II.9)$$

$$\pi^{\mu\nu} = 2 \eta \Delta^{\mu\rho\sigma} \nabla^\rho u_\sigma, \hspace{1cm} (II.10)$$
guarantees the second law of thermodynamics, $\partial_\mu S^\mu \geq 0$, with $\zeta$, $\lambda$, and $\eta$ being the bulk viscosity, heat conductivity, and shear viscosity, respectively. This is because the divergence $\partial_\mu S^\mu$ now becomes positive semi-definite;

$$\partial_\mu S^\mu = \frac{\Pi^2}{\zeta T} \left[ q^\mu \frac{\mu}{T^2} + \pi^{\mu\nu} \pi_{\mu\nu} 2\eta T \right] \geq 0. \hspace{1cm} (II.11)$$

Thus we realize that there is nothing wrong with the resultant relativistic dissipative hydrodynamic equations with finite $f_e$ and $f_n$, or equivalently finite $\delta e$ and $\delta n$, which is compatible with the second law of thermodynamics. Eq.’s (II.8) and (II.10) with a restricted condition $f_e = f_n = 0$ are identical to the constitutive equations proposed by Eckart that are commonly used.

In the energy frame where $\nu^\mu = 0$, we can obtain the constitutive equations in the same way as the particle-frame case with $f_e$ and $f_n$ being kept finite. The resultant equations are given by Eq.’s (II.8), (II.10), and

$$\nu^\mu = \lambda \hat{h}^{-1} \nabla^\mu \frac{\mu}{T}, \hspace{1cm} (II.12)$$

with $\hat{h} \equiv (e + p)/n T$ being the enthalpy. It is noted that these equations are reduced to the constitutive equations by Landau if we can set $f_e = f_n = 0$.

By applying the above argument to the entropy current at most bilinear with respect to $\Pi$, $q^\mu$, $\nu^\mu$, and $\pi^{\mu\nu}$, we can obtain the relaxation equations with $f_e$ and $f_n$ being finite, which make up the so-called second-order relativistic dissipative hydrodynamic equations together with the conservation laws in Eq.’s (II.1) and (II.2).

Now the dissipative part of the energy-momentum tensor satisfies $u_{\mu} \delta T_{\mu\nu} u_\nu = \delta \Pi$ and $\delta T_{\mu} = \delta e - 3 \delta p = (f_e - 3) \Pi$. As emphasized before, the values of $f_e$ and $f_n$ can be determined only from a microscopic theory. The phenomenological theory cannot proceed further because no such logic to determine them is implemented in the theory. In the following section, we shall show that the microscopic theory gives $f_e = 3$ together with $f_n = 0$ in the particle frame while $f_e = f_n = 0$ in the energy frame, and hence $\delta T_{\mu} = 0$ but $u_{\mu} \delta T_{\mu\nu} u_\nu = 3 \Pi \neq 0$ in the particle frame. This fact tells us that the usual constraint employed for the particle frame must be abandoned, and all the analyses based on this constraint should be redone.
The argument so far is in the stage of thermodynamics where the argument is robust but the parameters such as $f_c$ and $f_n$ as well as the equations of state $c$, $p$ and $n$ appearing in the theory remain undetermined. The problem which we encounter is how to reduce a dynamical equation to a slower one described with fewer dynamical variables. For this purpose, we will investigate the infrared limit of the relativistic Boltzmann equation with use of a powerful reduction method, the “RG method” [11].

The RG method is a systematic reduction theory of the dynamics leading to the coarse-graining of temporal and spatial scales. The full presentation of the reduction of the relativistic Boltzmann equation to the second-order hydrodynamic equation is technical and involved. So we here only present main results with key several equations, leaving the detailed account to another publication [24], although the derivation of a wide class of the first-order equations is presented in Ref. [12].

We start with the simple relativistic Boltzmann equation,

$$ p^\mu \partial_\tau f_p(x) = C[f_p](x), \quad (\text{III.1}) $$

where $f_p(x)$ denotes the one-particle distribution function defined in the phase space $(x^\mu, p^\mu)$ with $p^\mu$ being the four momentum of the on-shell particle. The right-hand side of Eq. (III.1) is the collision integral, $C[f_p](x) = \frac{1}{2} \sum_{a=1}^3 \sum_{b=1}^3 \sum_{c=1}^3 \sum_{d=1}^3 \omega(p_a p_b p_c p_d) (f_{p_c}(x) f_{p_d}(x) - f_p(x) f_{p_b}(x))$, where $\omega(p_a p_b p_c p_d)$ denotes the transition probability owing to the microscopic two-particle interaction.

We are interested in the hydrodynamical regime where the time- and space-dependence of the physical quantities are small. In another word, the time and space entering the hydrodynamic equation are the ones coarse-grained from the zeroth-order approximate solution we construct is a stationary solution, which is identical to a local equilibrium, leaving the detailed account to another publication [24], although the derivation of a wide class of the first-order equations is presented in Ref. [11].

We note that the small quantity $\varepsilon$ has been introduced to tag that the space derivatives are small for the system we are interested in. $\varepsilon$ may be identified with the ratio of the average particle distance over the mean free path, i.e., the Knudsen number.

In this coordinate system, Eq. (III.1) can be cast into

$$ \frac{\partial}{\partial \tau} f_p(\tau, \sigma) = \frac{1}{p \cdot a_p(\tau, \sigma)} C[f_p](\tau, \sigma) = \frac{1}{p \cdot a_p(\tau, \sigma)} \nabla f_p(\tau, \sigma), \quad (\text{III.2}) $$

where $a_p(\tau, \sigma) \equiv a_p^0(x)$, $\Delta_{\mu\nu}(\tau, \sigma) \equiv \Delta_{\mu\nu}^p(\tau, \sigma)$, and $f_p(\tau, \sigma) \equiv f_p(x)$. Since $\varepsilon$ appears in front of $\nabla f_p$, $\nabla f_p$ can be applied. In the perturbative expansion, we shall take the coordinate system where $a_p^0(\tau, \sigma)$ has no $\tau$ dependence, i.e., $a_p^0(\tau, \sigma) = a_p^0(\sigma)$.

The zeroth-order approximate solution we construct is a stationary solution, which is identical to a local equilibrium distribution function given by the Juetner function $f_{eq} \equiv (2\pi)^{-3} \exp[(\mu - p \cdot u)/T]$. Note that this solution contains five would-be integration constants, $T$, $\mu$, and $u^\mu$ with $u^\mu u_\mu = 1$, which can be identified with the temperature, the chemical potential, and the fluid velocity, respectively.

The collision integral is expanded around the zeroth-order solution and is reduced to the linear operator $A_{pq} \equiv (p \cdot a_p)^{-1} \frac{\partial}{\partial \sigma} C[f_{eq}^p]_q$. Furthermore, it is found to be convenient to convert $A_{pq}$ to $L_{pq} f_{eq}^{-1} A_{pq} f_{eq}^q = [f_{eq}^{-1} A f_{eq}]_{pq}$, with the diagonal matrix $f_{pq}^eq \equiv f_{pq}^eq \delta_{pq}$. We also define the inner product between arbitrary vectors $\varphi_p$ and $\psi_p$ by

$$ \langle \varphi, \psi \rangle \equiv \sum_p \frac{1}{p_0} (p \cdot a_p) f_{pq}^eq \varphi_p \psi_p. \quad (\text{III.3}) $$

With this inner product, we can define a normed linear space.

Now the first-order solution is given in terms of the five zero modes of $L$, $\varphi_0^{pq} = (1, p^\mu)$. The corresponding variables are just $T$, $\mu$, and $u^\mu$ with $u_\mu u^\mu = 1$. The zero modes span a linear space $P_0$, which is an invariant manifold for the asymptotic dynamics of the relativistic Boltzmann equation in the terminology in the dynamical systems [8, 23].

Then the second-order solution is given by incorporating the next slow modes, which span a linear space $P_1$. We naturally require $P_1$ is orthogonal to $P_0$, that is, $P_0 \perp P_1$. We find that $P_1$ is expanded by the bilinear forms of momenta; $\varphi_1^{pq} \equiv \{Q_0 \varphi_0^{pq} \}$, where $\varphi_0^{pq} \equiv p^\mu p^\nu$, and $Q_0$ is the projection to complement to $P_0$. By definition, $\langle \varphi_1^p, \varphi_0^q \rangle = 0$ is satisfied. Note that the dimension of $\varphi_1^{pq}$ is nine, which correspond to the number of the new would-be integration constants, $J$ with $J^\mu u_\mu = 0$, and $\pi^{\mu\nu}$ with $\pi^{\mu\nu} = \pi^\nu_{\mu\nu}$ and $\pi^{\mu\nu} u_\nu = \pi^\mu_{\mu\nu} = 0$. 

A generic choice of the macroscopic frame vector is $a^p_\mu = ((p \cdot u) \cos \theta + m \sin \theta)/(p \cdot u) u^\mu$, where $\theta$ is a parameter defining the frame. For example, $\theta = 0$ ($\theta = \pi/2$) gives the energy (particle) frame.

The resultant generic relaxation equations of the second-order hydrodynamic equation with $\theta$ being kept are

$$
\Pi = X_{\Pi} - \tau_{\Pi} D\Pi - \ell_{\Pi j} \nabla^a J_a + X_{\Pi a} J_a + X_{\Pi b} \pi_{ab},
$$

(III.4)

$$
J^\mu = X^\mu_j - \tau_j \Delta^\mu a D J_a - \ell_{j k} \nabla^\mu J_{k} - \ell_{j \pi} \Delta^{abc} \nabla_a \pi_{b c} + X^\mu_{j a} J_a + X^\mu_{j a} \pi_{ab},
$$

(III.5)

$$
\pi^{\mu \nu} = X^{\mu \nu} - \tau_{\pi} \Delta^{\mu \nu} D \pi_{a b} - \ell_{\pi j} \Delta^{\mu \nu \ab} \nabla_a J_b + X^{\mu \nu}_{\pi a} J_a + X^{\mu \nu}_{\pi a} \pi_{a b}. 
$$

(III.6)

Here, $X_{\Pi}$, $X^\mu_j$, and $X^{\mu \nu}_\pi$ are the thermodynamic forces; their simple forms retaining only $X_{\Pi}$, $X^\mu_j$, and $X^{\mu \nu}_\pi$ are the usual constitutive equations. The relaxation equations of $\Pi$, $J^\mu$, and $\pi^{\mu \nu}$ are characterized by the relaxation times $\tau_{\Pi}$, $\tau_j$, and $\tau_\pi$, while $\ell_{\Pi j}$, $\ell_{j \Pi}$, $\ell_{j \pi}$, and $\ell_{\pi j}$ mean the relaxation lengths. The correction to the thermodynamic forces $X_{\Pi}$, $X^\mu_j$, and $X^{\mu \nu}_\pi$ are given by $X_{\pi a} J_a$, $X^{\mu \nu}_{\pi a} J_a$, and $X^{\mu \nu}_{\pi a} \pi_{a b}$.

The continuity equations of the second-order equation in the energy frame is found to be given by setting $\theta = 0$ as was anticipated.

The energy-momentum tensor and particle-number vector in the particle frame with $\theta = \pi/2$ read

$$
T^{\mu \nu} = e u^\mu u^\nu - (p + \Pi) \Delta^{\mu \nu} + \pi^{\mu \nu},
$$

(III.7)

$$
N^\mu = n u^\mu + J^\mu.
$$

(III.8)

The thermodynamic forces are $X_{\Pi} = -\zeta \nabla^a u_a$, $X^\mu_j = \lambda \hat{h}^{-2} \nabla^\mu (\mu / T)$, and $X^{\mu \nu}_\pi = 2 \eta \Delta^{\mu \nu \ab} \nabla_a u_b$, which clearly show that $f_n = 0$ and $f_e = 0$ as was anticipated.

The energy-momentum tensor and particle-number vector in the particle frame with $\theta = \pi/2$ read

$$
T^{\mu \nu} = (e + 3 \Pi) u^\mu u^\nu - (p + \Pi) \Delta^{\mu \nu} + u^\mu J^\nu + u^\nu J^\mu + \pi^{\mu \nu},
$$

(III.9)

$$
N^\mu = n u^\mu,
$$

(III.10)

and $X_{\Pi} = -\zeta (3\gamma - 4)^{-2} (\nabla^a u_a - 3 T D T^{-1})$, $X^\mu_j = -\lambda T^2 (\nabla^\mu D T^{-1} + D u^\mu)$, and $X^{\mu \nu}_\pi = 2 \eta \Delta^{\mu \nu \ab} \nabla_a u_b$, where $\gamma \equiv 1 + (z^2 - \hat{h}^2 + 5 \hat{h} - 1)^{-1}$ is the ratio of the specific heats. Thus we find that $f_e = 3$ with $f_n = 0$, as we announced.

Although we have obtained the relaxation equations for the dissipative forces $\Pi$, $J^\mu$, and $\pi^{\mu \nu}$ for arbitrary $\theta$ [24], we shall only write down them for two typical frames, i.e., the energy ($\theta = 0$) and the particle ($\theta = \pi/2$) frames for the sake of the space.
(A) In the energy frame ($\theta = 0$):

$$\Pi = -\zeta \nabla^a u_a - \tau_\Pi D\Pi - \ell_{\Pi J} \nabla^a J_a$$

$$-\frac{1}{2} \Pi \left\{ \kappa_{\Pi} \nabla^a u_a + \frac{\zeta T}{\tau_\Pi} \partial_a \left( \frac{\tau_\Pi}{\zeta T} u^a \right) \right\}$$

$$-\frac{1}{2} \ell_{\Pi J} \left\{ \kappa^{(0)}_{\Pi} \nabla^a u_a - \kappa^{(1)}_{\Pi} D u^a + \frac{\zeta T}{\ell_{\Pi J}} \partial_b \left( \frac{\ell_{\Pi J}}{\zeta T} \Delta_b \right) \Delta_c^a \right\} J_a$$

$$-\frac{1}{2} \ell_{\Pi} \left\{ - \kappa_{\Pi} \Delta^{abcd} \nabla_{c} u_{d} \right\} \pi_{ab},$$

$$J^\mu = \lambda \frac{h^{-2}}{T} \nabla^\mu - \ell J J^\nu \Pi - \tau J \Delta^{\mu a} D J_a - \ell J \pi \Delta^{\mu abc} \nabla_a \pi_{bc}$$

$$-\frac{1}{2} \ell_{J J} \left\{ \kappa^{(0)}_{J J} \nabla^\mu u_b - \kappa^{(1)}_{J J} D u^a + \frac{\lambda}{\ell_{J J}} \Delta^{\mu a} \nabla_b u_c - 2 \omega^{\mu a} \right\} \Pi$$

$$-\frac{1}{2} \ell_{J J} \left\{ - \kappa_{\Pi} \Delta^{\mu abc} \nabla_a \pi_{bc} \right\} \pi_{ab},$$

$$\pi^{\mu \nu} = 2 \eta \Delta^{\mu abc} \nabla_a u_b - \ell_{J J} \Delta^{\mu abc} \nabla_a J_b - \tau \Delta^{\mu abc} D \pi_{ab}$$

$$-\frac{1}{2} \ell_{\pi J} \left\{ - \kappa_{\Pi} \Delta^{\mu abc} \nabla_a u_b \right\} \Pi$$

$$-\frac{1}{2} \ell_{\pi J} \left\{ \Delta^{\mu abc} \left( \kappa^{(0)}_{J J} \nabla_b \mu - \kappa^{(1)}_{J J} D u_b \right) + \frac{\eta T}{\ell_{J J}} \Delta^{\mu bc} \partial_d \left( \frac{\ell_{J J}}{\eta T} \Delta^{cdef} \right) \Delta_e^a \right\} J_a$$

$$-\frac{1}{2} \tau J \left\{ \Delta^{\mu abc} \left[ \kappa^{(0)}_{J J} \nabla^c u_c + \frac{\eta T}{\tau J} \partial_c \left( \frac{\tau J}{\eta T} u^c \right) \right] - 4 \kappa^{(1)}_{J J} \Delta^{\mu cde} \Delta_e^a \Delta_{cd} D \pi_{fg} \nabla_f u_g - 4 \Delta^{\mu cde} \Delta_e^a \omega_{cd} \right\} \pi_{ab},$$

where $\omega^{\mu \nu} \equiv (\nabla^\mu u^\nu - \nabla^\nu u^\mu)/2$ is the vorticity.

(B) In the particle frame ($\theta = \pi/2$):

$$\Pi = -\zeta (3 \gamma - 4)^{-2} \left( \nabla^a u_a - 3 T D^1 \right) - \tau_\Pi D\Pi - \ell_{\Pi J} \nabla^a J_a$$

$$-\frac{1}{2} \Pi \left\{ \kappa_{\Pi} \nabla^a u_a + \frac{\zeta (3 \gamma - 4)^{-2} T}{\tau_\Pi} \partial_a \left( \frac{\tau_\Pi}{\zeta (3 \gamma - 4)^{-2} T} u^a \right) \right\}$$

$$-\frac{1}{2} \ell_{\Pi J} \left\{ \kappa^{(0)}_{\Pi J} \nabla^a u_a - \kappa^{(1)}_{\Pi J} D u^a + \frac{\zeta (3 \gamma - 4)^{-2} T}{\ell_{\Pi J}} \partial_b \left( \frac{\ell_{\Pi J}}{\zeta (3 \gamma - 4)^{-2} T} \Delta_b \right) \Delta_c^a \right\} J_a$$

$$-\frac{1}{2} \ell_{\Pi} \left\{ - \kappa_{\Pi} \Delta^{abcd} \nabla_{c} u_{d} \right\} \pi_{ab},$$

$$J^\mu = -\lambda T^2 \left( \nabla^\mu \frac{1}{T} + \frac{1}{T} D u^a \right) - \ell_{J J} \nabla^\mu \Pi - \tau J \Delta^{\mu a} D J_a - \ell J \pi \Delta^{\mu abc} \nabla_a \pi_{bc}$$

$$-\frac{1}{2} \ell_{J J} \left\{ \kappa^{(0)}_{J J} \nabla^a u_b - \kappa^{(1)}_{J J} D u^a + \frac{\lambda T^2}{\ell_{J J}} \Delta^{\mu a} \nabla_b u_c - 2 \omega^{\mu a} \right\} \Pi$$

$$-\frac{1}{2} \ell_{J J} \left\{ - \kappa_{\Pi} \Delta^{\mu abc} \nabla_a \pi_{bc} \right\} \pi_{ab},$$

$$\pi^{\mu \nu} = 2 \eta \Delta^{\mu abc} \nabla_a u_b - \ell_{J J} \Delta^{\mu abc} \nabla_a J_b - \tau \Delta^{\mu abc} D \pi_{ab}$$

$$-\frac{1}{2} \ell_{\pi J} \left\{ - \kappa_{\Pi} \Delta^{\mu abc} \nabla_a u_b \right\} \Pi$$

$$-\frac{1}{2} \ell_{\pi J} \left\{ \Delta^{\mu abc} \left( \kappa^{(0)}_{J J} \nabla_b \mu - \kappa^{(1)}_{J J} D u_b \right) + \frac{\eta T}{\ell_{J J}} \Delta^{\mu bc} \partial_d \left( \frac{\ell_{J J}}{\eta T} \Delta^{cdef} \right) \Delta_e^a \right\} J_a$$

$$-\frac{1}{2} \tau J \left\{ \Delta^{\mu abc} \left[ \kappa^{(0)}_{J J} \nabla^c u_c + \frac{\eta T}{\tau J} \partial_c \left( \frac{\tau J}{\eta T} u^c \right) \right] - 4 \kappa^{(1)}_{J J} \Delta^{\mu cde} \Delta_e^a \Delta_{cd} D \pi_{fg} \nabla_f u_g - 4 \Delta^{\mu cde} \Delta_e^a \omega_{cd} \right\} \pi_{ab},$$

$$-\frac{1}{2} \tau J \left\{ - \kappa_{\Pi} \Delta^{\mu abc} \nabla_{c} u_{d} \right\} \pi_{ab}.$$
FIG. 1: The $\theta$ dependence of $\tau_{\Pi}$ and $\tau_{J}$ at $m/T = 0.5$ and $\mu/T = 0.0$. We normalized the relaxation times by the corresponding transport coefficients. The energy and particle frames correspond to $\theta = 0$ and $\pi/2$.

and Eq. (III.13). Note that the effective bulk viscosity $\zeta_{\text{eff}} \equiv \zeta (3 \gamma - 4)^{-2}$ [21] appears in Eq. (III.14).

In summary, we have derived the second-order dissipative relativistic hydrodynamic equations in a generic frame with a continuous parameter $\theta$; the generic frame is reduced to the energy and particle frame with the parameter choice $\theta = 0$ and $\pi/2$, respectively. A notable point of our result is that the dissipative part of the symmetric energy-momentum tensor $\delta T^{\mu\nu}$ in the particle frame satisfies the equality $\delta T^{\mu}_{\mu} = 0$, in contrast to the usual choice $u_{\mu} \delta T^{\mu\nu} u_{\nu} = 0$, while $\delta T^{\mu\nu}$ of our derived equation in the energy frame satisfies the usual constraint $u_{\mu} \delta T^{\mu\nu} u_{\nu} = 0$. We emphasize that this novel equality in the particle frame is a consequence of the derivation based on the renormalization-group method, a powerful method for the reduction of dynamical systems. We note that the same constraints were also derived for the first-order dissipative relativistic hydrodynamic equation [6, 21]. We have also shown that the phenomenological derivation based on the second law of thermodynamics allows that $u_{\mu} \delta T^{\mu\nu} u_{\nu}$ can be proportional to the bulk pressure $\Pi$ and non-vanishing in the particle frame. Indeed, our microscopic derivation shows that $u_{\mu} \delta T^{\mu\nu} u_{\nu} = 3 \Pi$. We have presented the relaxation equations in the energy and particle frames, explicitly as typical examples, although we have obtained the microscopic expressions for them in a more generic frame [24]. We have shown that the viscosities are frame-independent but the relaxation times are generically frame-dependent, as depicted in FIG. 1. The detailed derivation of the equations and discussions on the phenomenological consequences of the hydrodynamic equations thus obtained will be discussed in forthcoming papers [24].

IV. BRIEF SUMMARY

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