Godeaux surfaces I

Frank-Olaf Schreyer and Isabel Stenger

Abstract

In this paper we describe an 8-dimensional locally complete family of simply connected numerical Godeaux surfaces, building on an homological algebra approach to their construction. We also describe how the families of Reid and Miyaoka with torsion \( \mathbb{Z}/3\mathbb{Z} \) and \( \mathbb{Z}/5\mathbb{Z} \) arise in our homological setting.

Contents

1 Preliminaries .............................................. 3
2 A structure theorem for Godeaux rings .................. 5
3 Normal form and an intersection of four quadrics in \( \mathbb{P}^{11} \) ........ 10
4 The dominant component ................................. 18
   4.1 Barlow surfaces ...................................... 22
5 Special bicanonical fibers .................................. 24
   5.1 The hyperelliptic locus ............................... 27
6 Torsion surfaces ............................................ 32
   6.1 \( \mathbb{Z}/5\mathbb{Z} \)-surfaces ........................... 32
   6.2 \( \mathbb{Z}/3\mathbb{Z} \)-surfaces ........................... 34
7 Special surfaces and ghost components .................. 37
   7.1 Lines meeting one special locus ..................... 37
   7.2 Lines meeting two special loci ..................... 40

Introduction

The minimal surfaces of general type with the smallest possible numerical invariants are the numerical Godeaux surfaces. They have always been of a particular interest in the classification of algebraic surfaces. Miyaoka ([Miy76]) showed that the torsion group of such surfaces is cyclic of order \( m \leq 5 \). Whereas numerical Godeaux surfaces for
Let $X$ be a numerical Godeaux surface, and let $x_0, x_1$ (respectively $y_0, \ldots, y_3$) denote a basis of $H^0(X, \mathcal{O}_X(2K_X))$ (respectively of $H^0(X, \mathcal{O}_X(3K_X))$). We consider the weighted polynomial ring $S = \mathbb{k}[x_0, x_2, y_0, \ldots, y_3]$. The canonical ring $R(X)$ is a finitely generated $S$-module. Using the structure result from [Ste18], $R(X)$ as an $S$-module admits a self-dual minimal free resolution $F$ of length three with a skew-symmetric middle map. There are only finitely many isomorphism classes of the complex $F/(x_0, x_1)$ possible.

In this article, we study the case where the bicanonical system $|2K_X|$ has no fixed part and four distinct base points, which we can assume to be mapped to the coordinate points of $\mathbb{P}^3$ under the tricanonical map. The main idea of the construction is to recover the minimal free resolution $F$ of $R(X)$ as a deformation of the complex $F/(x_0, x_1)$. The essential technique is to consider an unfolding of $F/(x_0, x_1)$ and to interpret the flatness as equations with respect to the unfolding parameters. Removing unfolding parameters which have to be zero by the equations, the remaining parameters which are linear functions in $x_0, x_1$ satisfy a quadratic system of equations. This system consists of four quadrics defining a complete intersection variety $Q$ in $\mathbb{P}^{11}$. The first step in our construction method for numerical Godeaux surfaces is to choose a line $\ell$ in $Q$. After this choice, the remaining relations define a linear system of equations which can be solved by a syzygy computation. For a general line in $Q$ we get a 4-dimensional linear solution space. To obtain a complete classification of all numerical Godeaux surfaces, we have to determine the loci of all lines in $Q$ at which the linear solution space in the second step is so big that we obtain another component.

We succeeded to construct an 8-dimensional locally complete family of simply connected numerical Godeaux surfaces. Moreover, we reconstructed the torsion $\mathbb{Z}/3\mathbb{Z}$- and $\mathbb{Z}/5\mathbb{Z}$-components of Reid ([Rei72]) and the Barlow surfaces ([Bar85]) with our method. Furthermore, we give a complete characterization for the existence of hyperelliptic bicanonical fibers and non-trivial torsion groups in terms of our homological setting. In a subsequent work, we will present our construction of a locally complete 8-dimensional unirational family of numerical Godeaux surfaces with torsion group $\mathbb{Z}/2\mathbb{Z}$.

For the time being, we failed to give a complete characterization of all numerical Godeaux surfaces due to the existence of ghost components. These ghost components arise from other components of the space of all solutions of our deformation problem. We were not able to determine the general stratification of the Fano scheme of lines $F_1(Q)$ with regard to the dimension of the solution space in the second step.

Acknowledgments. This work is a contribution to the Project 1.7 of the SFB-TRR 195 ‘Symbolic Tools in Mathematics and their Application’ of the German Research Foundation (DFG). We thank Wolfram Decker and Miles Reid for inspiring conversations. Our work makes essential use of Macaulay2 ([GS]). We thank Dan Grayson and Mike Stillman for their program. Our work would not have been possible without computer algebra.
1 Preliminaries

Throughout this paper, we use the following notation.

- $X$ denotes a numerical Godeaux surface;
- $\pi : X \rightarrow X_{\text{can}} = \text{Proj}(R(X))$ denotes the morphism to the canonical model;
- $K_X$ and $K_{X_{\text{can}}}$ denote canonical divisors;
- Tors $X$ denotes the torsion subgroup of the Picard group;
- $k$ denotes the ground field.

We are mainly interested in the case $k = \mathbb{C}$ but for computations we also use $k = \mathbb{Q}$ or number fields. In our experiments, $k$ can also be a finite field which we often may regard as a specialization of a number field.

In our construction we use some classical results on the bi- and the tricanonical system of a numerical Godeaux surface $X$ over $\mathbb{C}$ which we will briefly recall here. Let us start with the bicanonical system. We write

$$|2K_X| = |M| + F,$$

where $M$ denotes a generic member of the moving part and $F$ the fixed part of $|2K_X|$.

**Proposition 1.1** ([Miy76], Lemma 6). If $M$ is generically chosen, $M$ is reduced and irreducible. Moreover, $M$ and $F$ satisfy one of the following conditions

(i) $F = 0$,
(ii) $K_X F = 0, F^2 = -2, M^2 = 2, MF = 2$,
(iii) $K_X F = 0, F^2 = -4, M^2 = 0, MF = 4$.

**Remark 1.2.** The statement shows that the fixed part of $|2K_X|$, if non-empty, is supported on the $(-2)$-curves of $X$. Hence, $|2K_{X_{\text{can}}}|$ is free from fixed components and its generic member is irreducible.

Next we summarize some results on the tricanonical system:

**Theorem 1.3** ([Miy76], Theorem 3, Proposition 2 and Proposition 3). The tricanonical map $\phi_{|3K_X|}$ is birational onto its image. The linear system $|3K_X|$ has no fixed part. The generic element $M$ of the moving part of $|2K_X|$ contains no base points of $|3K_X|$.

**Remark 1.4.** Combining the previous statements, we conclude that no base point of $|3K_X|$ (respectively of $|3K_{X_{\text{can}}}|$) is a base point of $|2K_X|$ (respectively of $|2K_{X_{\text{can}}}|$). Hence, for a base point $P$ of $|3K_X|$ there exists a unique divisor $D \in |2K_X|$ which contains $P$. Furthermore, Miyaoka showed that a point $\hat{P}$ is a base point of $|3K_{X_{\text{can}}}|$ if and only if $\hat{P} = \hat{D}_1 + \hat{D}_2$, where $\hat{D}_1, \hat{D}_2$ are two distinct effective curves which are numerically equivalent to $K_{X_{\text{can}}}$ with $\hat{D}_1 + \hat{D}_2 \in |2K_{X_{\text{can}}}|$. The last fact gives indeed a very precise description of the number of base points of $|3K_X|$.
**Theorem 1.5** ([Miy76], Theorem 2). *Every base point of the tricanonical system \( |3K_X| \) is simple, and the number \( b \) of base points is given as follows:*

\[
b = \frac{\# \{ t \in H^2(X, \mathbb{Z})_{\text{tors}} \mid t \neq -t \}}{2}.
\]

Note that for a numerical Godeaux surface \( X \), \( H^2(X, \mathbb{Z})_{\text{tors}} = \text{Tors } X = H_1(X, \mathbb{Z}) \).

Bombieri showed that the order of the torsion group of a numerical Godeaux surface is \( \leq 6 \) (see [Bom73], Theorem 11.14). Miyaoka refined this result in the following way:

**Lemma 1.6** ([Miy76], Lemma 11). *The torsion group of \( X \) is cyclic of order \( \leq 5 \).*

Combining these two statements we obtain the following important result:

**Theorem 1.7** ([Miy76], Theorem 2’ and subsequent Remark). *As above, let \( b \) denote the number of base points of \( |3K_X| \). Then*

\[
b = \begin{cases} 
0 & \text{if } \text{Tors } X \cong 0 \text{ or } \mathbb{Z}/2\mathbb{Z}, \\
1 & \text{if } \text{Tors } X \cong \mathbb{Z}/3\mathbb{Z} \text{ or } \mathbb{Z}/4\mathbb{Z}, \\
2 & \text{if } \text{Tors } X \cong \mathbb{Z}/5\mathbb{Z}.
\end{cases}
\]

Later we will use this characterization and Lemma 1.9 below to determine the torsion group of our constructed surfaces.

**Remark 1.8.** Note that for a non-trivial torsion element \( \tau \in \text{Tors } X \) we have

\[
h^0(X, K_X + \tau) = 1, \ h^1(X, K_X + \tau) = 0.
\]

Indeed, by the Riemann-Roch theorem we have \( h^0(X, K_X + \tau) - h^1(X, K_X + \tau) = 1 \). Now \( h^1(X, K_X + \tau) > 0 \) implies that for the finite étale covering \( f: Y \to X \) corresponding to \( \tau \) we get \( h^1(Y, \mathcal{O}_Y) > 0 \). Hence, \( Y \) has finite cyclic coverings of any large order (see [Bom73], Lemma 10.14), and so does \( X \), which is not possible.

**Lemma 1.9.** *Assume that \( |2K_X| \) has no fixed part and \( (2K_X)^2 = 4 \) distinct (simple) base points. Then the order of \( \text{Tors } X \) is odd. In particular, for the number \( b \) of base points of \( |3K_X| \) we find that*

- \( b = 0 \) if and only if \( \text{Tors } X \cong 0 \),
- \( b = 1 \) if and only if \( \text{Tors } X \cong \mathbb{Z}/3\mathbb{Z} \),
- \( b = 2 \) if and only if \( \text{Tors } X \cong \mathbb{Z}/5\mathbb{Z} \).

**Proof.** Suppose to the contrary that \( \text{Tors } X \cong \mathbb{Z}/2\mathbb{Z} \) or \( \text{Tors } X \cong \mathbb{Z}/4\mathbb{Z} \). Let \( \tau \in \text{Tors } X \) be a non-trivial torsion element of order 2. By Remark 1.8 there exists an effective divisor \( D \in |K_X + \tau| \). But then \( |2K_X| \) contains the double curve \( 2D \) and thus, cannot have 4 distinct base points. The second part is an immediate consequence of Theorem 1.7. \( \square \)
2 A structure theorem for Godeaux rings

In this section we present a structure theorem for the canonical ring

\[ R(X) = \bigoplus_n H^0(X, \mathcal{O}_X(nK_X)) = \bigoplus_n H^0(X, nK_X) \]

of a numerical Godeaux surface \( X \) in which we describe the minimal free resolution of \( R(X) \) as a module over a weighted polynomial ring \( S \).

We first determine a minimal set of generators of \( R(X) \) as an \( \mathbb{k} \)-algebra. Using the Riemann-Roch theorem we see that the plurigenera of \( X \) are

\[ P_n = h^0(X, nK_X) = \begin{cases} 
1 & \text{for } n = 0, \\
0 & \text{for } n = 1, \\
\binom{n}{2} + 1 & \text{for } n \geq 2.
\end{cases} \]

Let \( x_0, x_1 \) be a basis of \( H^0(X, 2K_X) \), and let \( y_0, y_1, y_2, y_3 \) be a basis of \( H^0(X, 3K_X) \). Now \( R(X) \) being an integral domain implies that the elements \( x_0^2, x_0x_1, x_1^2 \) are linearly independent. Thus, as \( H^0(X, 4K_X) \) is 7-dimensional, we can choose \( z_0, \ldots, z_3 \in H^0(X, 4K_X) \) extending these elements to a basis. To give a basis for the vector space \( H^0(X, 5K_X) \), we use the following:

**Lemma 2.1.** The multiplication map \( \mu: H^0(X, 2K_X) \otimes H^0(X, 3K_X) \rightarrow H^0(X, 5K_X) \) is injective.

**Proof.** As \( R(X) \cong R(X_{\text{can}}) \) it is sufficient to show that

\[ \tilde{\mu}: H^0(X_{\text{can}}, 2K_{X_{\text{can}}}) \otimes H^0(X_{\text{can}}, 3K_{X_{\text{can}}}) \rightarrow H^0(X_{\text{can}}, 5K_{X_{\text{can}}}) \]

is injective. Let \( x_0, x_1 \) be a basis of \( H^0(X, 2K_X) \cong H^0(X_{\text{can}}, 2K_{X_{\text{can}}}) \). By Remark 1.2 we know that the bicanonical system has no fixed part on the canonical model. Hence, the following sequence is exact

\[
0 \rightarrow \mathcal{O}_{X_{\text{can}}}(-4K_{X_{\text{can}}}) \xrightarrow{\left(\begin{array}{c} x_1 \\ -x_0 \end{array}\right)} \mathcal{O}_{X_{\text{can}}}(-2K_{X_{\text{can}}}) \oplus \mathcal{O}_{X_{\text{can}}}(-2K_{X_{\text{can}}}) \xrightarrow{(x_0, x_1)} \mathcal{O}_{X_{\text{can}}} \rightarrow \mathcal{O}_Z \rightarrow 0,
\]

where \( Z = \text{div}(x_0) \cap \text{div}(x_1) \). Now tensoring with \( \mathcal{O}_{X_{\text{can}}}(5K_{X_{\text{can}}}) \) and taking global sections, the statement follows since \( h^0(X_{\text{can}}, 5K_{X_{\text{can}}}) = h^1(X_{\text{can}}, \mathcal{O}_{X_{\text{can}}}) = 0. \)

The lemma shows that the global sections \( x_i y_j \) for \( i = 0, 1 \) and \( j = 0, \ldots, 3 \) define an 8-dimensional subspace of \( H^0(X, 5K_X) \). Now as \( h^0(X, 5K_X) = 11 \), we can choose sections \( w_0, w_1, w_2 \in H^0(X, 5K_X) \) extending these elements to a basis. Since we will use the same notation for the generators in the following, we summarize the previous results in one table:
| \(n\) | \(h^0(X, nK_X)\) | basis of \(H^0(X, nK_X)\) |
|------|-----------------|------------------|
| 2    | 2               | \(x_0, x_1\)    |
| 3    | 4               | \(y_0, \ldots, y_3\) |
| 4    | 7               | \(x_0^2, x_0x_1, x_1^2, z_0, \ldots, z_3\) |
| 5    | 11              | \(x_0y_0, \ldots, x_1y_3, w_0, w_1, w_2\) |

The entries marked in blue give a minimal generating set of \(R(X)\) (as a \(k\)-algebra) up to degree 5. Ciliberto showed that the canonical ring of any surface of general type is generated in degree \(\leq 6\) (see [Cil83], Theorem 3.5). Using this result, we get the following refinement for numerical Godeaux surfaces:

**Lemma 2.2.** As a \(k\)-algebra, \(R(X)\) is generated in degree \(\leq 5\).

**Proof.** First recall that no base point of the tricanonical system of \(X\) is a base point of the bicanonical system. Hence, there exists a global section \(y \in H^0(X, 3K_X)\) such that \(\text{Proj}(R(X)/(x_0, x_1, y)) = \emptyset\). Then, as \(R(X)\) is Cohen-Macaulay, \(R(X)\) is a free module over \(A = k[x_0, x_1, y]\). Using the Hilbert series \(\Psi(t)\) of \(R(X)\), the degrees of the free generators are given by

\[
(1 - t^2)(1 - t^3)\Psi(t) = 1 + 3t^3 + 4t^4 + 3t^5 + t^8.
\]

This implies that any homogeneous element of \(R(X)\) of degree 6 is an \(A\)-linear combination of elements of \(R(X)\) of degree \(\leq 5\) which shows the claim. \(\square\)

So, if \(\hat{S} := k[x_0, x_1, y_0, \ldots, y_3, z_0, \ldots, z_3, w_0, w_1, w_2]\) is the weighted polynomial ring with degrees as assigned before, there is a closed embedding

\[
(1) \quad X_{\text{can}} = \text{Proj}(R(X)) \hookrightarrow \text{Proj}(\hat{S}) = \mathbb{P}(2^2, 3^4, 4^4, 5^3).
\]

Thus, we can consider the canonical model of \(X\) as a subvariety of a weighted projective space of dimension 12. Studying this embedding is difficult because we have no structure theorem for Gorenstein ideals of such high codimension. Furthermore, from a computational point of view, codimension 10 is not promising for irreducibility or non-singularity tests. The original construction idea in [Sch05] addresses these problems:

**Basic idea:** We do not consider \(R(X)\) as an algebra but as a finitely generated \(S\)-module, where \(S \subset \hat{S}\) is a subring chosen appropriately. Geometrically, we study the image of \(X_{\text{can}}\) under the projection into the (smaller) projective space \(\text{Proj}(S)\).

So let \(S = k[x_0, x_1, y_0, y_1, y_2, y_3]\) be the graded polynomial ring, where the \(x_i\) and \(y_j\) are as before with \(\deg(x_i) = 2\) and \(\deg(y_j) = 3\). The natural homomorphism

\[f : S \rightarrow R(X)\]

gives \(R(X)\) the structure of a graded \(S\)-algebra. In the following, we consider \(R(X)\) as a graded \(S\)-module via the homomorphism \(f\).

**Lemma 2.3.** \(R(X)\) is a finitely generated \(S\)-module.
Proof. By Proposition 1.3 and Remark 1.4 the elements \( x_i \) and \( y_j \) have an empty vanishing locus in \( X_{\text{can}} \), hence \( R(X) \) is finite over \( S \). 

Using the closed embedding in (1), we will from now on identify \( X_{\text{can}} \) with its image in \( \mathbb{P}(2^2, 3^4, 4^4, 5^3) \). Now, since \( R(X) \) is finitely generated as an \( S \)-module, the homomorphism \( f: S \to R(X) \) induces a finite morphism of projective schemes

\[
\varphi: X_{\text{can}} \to \mathbb{P}(2^2, 3^4)
\]

with image \( Y = \text{Proj}(S_Y) \subset \mathbb{P}(2^2, 3^4) \), where \( S_Y = S/\text{ann}_S(R(X)) \).

Lemma 2.4. \((X_{\text{can}}, \varphi)\) is the normalization of \( Y \).

Proof. We already know that \( \varphi: X_{\text{can}} \to Y \) is a finite morphism. Furthermore, \( \varphi \) is birational because the tricanonical map \( \phi_3: X_{\text{can}} \dashrightarrow \mathbb{P}^3 \) is birational onto its image and factors over \( Y \). Hence, as \( X_{\text{can}} \) is normal, \((X_{\text{can}}, \varphi)\) is the normalization of \( Y \). \( \square \)

As a next step we will describe the minimal free resolution of \( R(X) \) as an \( S \)-module. First we note that \( R(X) \) is a Cohen-Macaulay graded \( S \)-module, hence by the Auslander-Buchsbaum formula \( R(X) \) has projective dimension 3. The fact that \( R(X) \) is a Gorenstein ring implies the following symmetry condition:

**Proposition 2.5.** Let

\[
0 \leftarrow R(X) \leftarrow F_0 \leftarrow \ldots \leftarrow F_3 \leftarrow 0
\]

be a minimal free resolution of \( R(X) \) as an \( S \)-module, where \( F_i = \bigoplus_{j \geq 0} S(-j)^{\beta_{i,j}} \). Then

\[
\beta_{i,j} = \beta_{3-i, 17-j} \quad \text{for} \quad 0 \leq i \leq 3, \ 0 \leq j \leq 17,
\]

and the Betti numbers are of the following form:

\[
\begin{array}{cccc}
0 & 1 & 2 & 3 \\
\text{total} : & 8 & 26 & 26 & 8 \\
0 : & 1 & . & . & . \\
1 : & . & . & . & . \\
2 : & . & . & . & . \\
3 : & . & . & . & . \\
4 : & 4 & . & . & . \\
5 : & 3 & 6 & . & . \\
6 : & . & 12 & . & . \\
7 : & . & 8 & 8 & . \\
8 : & . & . & 12 & . \\
9 : & . & . & 6 & 3 \\
10 : & . & . & . & 4 \\
11 : & . & . & . & . \\
12 : & . & . & . & . \\
13 : & . & . & . & . \\
14 : & . & . & . & 1 \\
\end{array}
\]
Proof. Let \( \omega_S \cong S(-16) \) denote the canonical module of \( S \). Applying \( \text{Hom}_S(-, \omega_S) \) to a minimal free resolution of \( R(X) \) yields a minimal free resolution of \( \text{Ext}^3_S(R(X), \omega_S) \cong \omega_R(X) \)

\[
0 \leftarrow \omega_R(X) \leftarrow \text{Hom}_S(F_3, \omega_S) \leftarrow \text{Hom}_S(F_2, \omega_S) \leftarrow \text{Hom}_S(F_1, \omega_S) \leftarrow \text{Hom}_S(F_0, \omega_S) \leftarrow 0.
\]

On the other hand, as \( \omega_R(X) \cong R(X)(1) \), we obtain

\[
0 \leftarrow R(X) \leftarrow \text{Hom}_S(F_3, S(-17)) \leftarrow \text{Hom}_S(F_2, S(-17)) \leftarrow \text{Hom}_S(F_1, S(-17)) \leftarrow \text{Hom}_S(F_0, S(-17)) \leftarrow 0,
\]

which is another minimal free resolution of \( R(X) \) and shows the first claim. To determine the exact Betti numbers we consider the \( R(X) \)-regular sequence \( x_0, x_1, y \) as in the proof of Proposition 2.2. We know that \( M_a = R(X)/(x_0, x_1, y) \) as an \( S/(x_0, x_1, y) \)-module has the same Betti numbers as \( R(X) \) as an \( S \)-module. Modulo \( x_0, x_1, y \) the Artinian module \( M_a \) decomposes into a direct sum of three modules \( M_a^{(i)} \) with Hilbert series

\[
h_0 = 1 + 3t^3,
\]
\[
h_1 = 4t^4,
\]
\[
h_2 = 3t^5 + t^8
\]

and Hilbert numerators

\[
h_0 \cdot (1 - t^3)^3 = 1 - 6t^6 + 8t^9 - 3t^{12},
\]
\[
h_1 \cdot (1 - t^3)^3 = 4t^4 - 12t^7 + 12t^{10} - 4t^{13},
\]
\[
h_2 \cdot (1 - t^3)^3 = 3t^5 - 8t^8 + 6t^{11} - t^{17}.
\]

Thus, \( M_a \) and \( R(X) \) have the Betti numbers as claimed. \( \square \)

Thus \( R(X) \) has a self-dual resolution of length 3, but we cannot apply the famous structure theorem of Buchsbaum-Eisenbud ([BE77]), since \( R(X) \) is not a cyclic \( S \)-module. However, we can apply the structure result for Gorenstein algebras of [Ste19]:

**Theorem 2.6** ([Ste19], Theorem 1.5 and Corollary 5.6). There exists a minimal free resolution of \( R(X) \) as an \( S \)-module of type

\[
0 \leftarrow R(X) \leftarrow F_0 \xleftarrow{d_2} F_1 \xleftarrow{d_2} F_1^*(-17) \xleftarrow{d_1} F_0^*(-17) \leftarrow 0,
\]

where \( d_2 \) is alternating.

Using this statement we can translate the question of constructing or classifying numerical Godeaux surfaces from a problem in algebraic geometry into a problem in homological algebra. Thus, the main focus of our approach is to construct and describe \( S \)-modules \( R \) having a minimal free resolution as described above.

We end this section by determining a minimal set of generating relations of \( X_{can} \subset \mathbb{P}(2^2, 3^4, 4^4, 5^3) \). Using the proof of the structure theorem in [Ste19], these relations can
be computed explicitly from a minimal free resolution $F$ of $R(X)$ since the ring structure of $R(X)$ induces a multiplicative structure on the complex $F \otimes F$ and a morphism of complexes $F \otimes F \to F$. Using this morphism, we compute a minimal generating set of $R(X)$ with the procedure `canonicalRing` from our package `NumericalGodeaux`.

From (1) we know that there exists a surjective ring homomorphism

$$f: \hat{S} \to R(X),$$

where $\hat{S} = \mathbb{k}[x_0, x_1, y_0, \ldots, y_3, z_0, \ldots, z_3, w_0, w_1, w_2]$ is the graded polynomial ring as defined before. Let $r_0 = 1, r_1 = z_0, \ldots, r_4 = z_3, r_5 = w_0, r_6 = w_1, r_7 = w_2$ which generate $R(X)$ as an $S$-module. Proposition 2.5 shows that there are 26 $S$-linear relations between these module generators:

$$0 = \sum_{k=0}^{7} g_{m,k}^r r_k.$$  

Furthermore, for the 28 elements $r_i r_j \in R(X), 1 \leq i \leq j \leq 7$, there exist elements $s_{i,j,k} \in S$ such that

$$r_i r_j = \sum_{k=0}^{7} s_{i,j,k} r_k.$$  

These relations are linearly independent and are uniquely determined modulo the relations in (2). Let $I_X \subset \hat{S}$ be the ideal generated by the relations in (2) and (3). Then:

**Lemma 2.7.** $R(X) \cong \hat{S}/I_X$

**Proof.** Since all generators of $I_X$ define relations in $R(X)$, $f$ factors through a surjective homomorphism $\hat{S}/I_X \to R(X)$. On the other hand, as an $S$-module, $\hat{S}/I_X$ is also generated by $r_0, \ldots, r_7$, and every relation in (2) is also an $S$-linear relation between the module generators of $\hat{S}/I_X$. Hence, there exists a surjective $S$-linear homomorphism $R(X) \to \hat{S}/I_X$ which shows the claim.

We conclude that the homogeneous ideal of $X_{can} \subset \mathbb{P}(2^2, 3^4, 4^4, 5^3)$ is minimally generated by 54 equations with the following degrees:

| Degree | Total |
|--------|-------|
| 0      | 54    |
| 1      | 1     |
| 2      | 6     |
| 3      | 12    |
| 4      | 18    |
| 5      | 6     |
| 6      | 12    |
| 7      | 6     |

9
Remark 2.8. In Theorem 2.6 we have seen that the canonical ring of any numerical Godeaux surface, considered as an $S$-module, admits a minimal free resolution with an alternating middle map. Conversely, given an exact sequence

$$0 \leftarrow R(X) \leftarrow F_0 \xleftarrow{d_1} F_1 \xleftarrow{d_2} F_1(-17) \xleftarrow{d_1^*} F_0^*(-17) \leftarrow 0,$$

where $F_0$ and $F_1$ are defined as above and $d_2$ is alternating, the question is whether the $S$-module $R$ admits a ring structure and, if so, whether $R$ defines the canonical ring of a numerical Godeaux surface. In [Ste18], Theorem 5.0.2, we give a sufficient condition for $R$ carrying a ring structure depending only on the first syzygy matrix $d_1$: let $d_1'$ be the matrix obtained from $d_1$ by erasing the first row, and let $I'$ denote the ideal generated by the maximal minors of $d_1'$. Then $R$ admits the structure of a Gorenstein ring if

(R.C.)  $\text{depth}(I', S) \geq 5$.

If this condition is satisfied, then $\text{Proj} R$ defines a surface embedded in a weighted projective space. Moreover, under the condition that $\text{Proj} R$ has only Du Val singularities, $\text{Proj} R$ defines indeed the canonical model of a numerical Godeaux surface.

3 Normal form and an intersection of four quadrics in $\mathbb{P}^{11}$

In this section we will introduce our unfolding techniques for the minimal free resolution of $R(X)$ as an $S$-module. We set up the first two syzygy matrices with unfolding parameters and study the system of relations which arises by setting the product of these matrices to zero. Using these relations, we introduce a normal form for the matrices in a minimal free resolution of $R(X)$ as an $S$-module.

From the last section we know that there exists a minimal free resolution of $R(X)$ of the form

$$0 \leftarrow R(X) \leftarrow F_0 \xleftarrow{d_1} F_1 \xleftarrow{d_2} F_1^* \xleftarrow{d_1^*} F_0^* \leftarrow 0$$

with $d_2' = -d_2$, $F_0 = S \oplus S(-4)^4 \oplus S(-5)^3$, $F_1 = S(-6)^6 \oplus S(-7)^{12} \oplus S(-8)^8$ and $F_i^* = \text{Hom}(F_i, S(-17))$.

Notation 3.1 (The general set-up). We write

$$d_1 = \begin{bmatrix} 
S & 6S(-6) & 12S(-7) & 8S(-8) \\
4S(-4) & b_0(y) + * & * & * \\
3S(-5) & a & b_1(y) & c \\
\end{bmatrix}$$

$$d_2 = \begin{bmatrix} 
6S(-6) & 6S(-11) & 12S(-10) & 8S(-9) \\
12S(-7) & o & n & b_3(y) \\
8S(-8) & -n^t & b_3(y) & p \\
\end{bmatrix}$$
The matrices \( o \) and \( b_4 \) are both skew-symmetric. Since there are no elements of degree 1 in \( S \), the maps \( S(-5)^3 \leftarrow S(-6)^6 \) and \( S(-8)^8 \leftarrow S(-9)^8 \) are both zero. The red matrices are the submatrices of \( d_1 \) and \( d_2 \) which depend only on the variables \( y_0, \ldots, y_3 \). More precisely, for each \( i \), all entries of \( b_i(y) \) are linear combinations of \( y_0, \ldots, y_3 \) with coefficients in \( \mathbb{k} \). By \( d_1' \) we denote the matrix obtained from \( d_1 \) by erasing the first row.

We do not assign names to the matrices indicated by \( * \) since they won’t play a role in the following. For the matrices marked in blue we obtain the characterization:

| degree | entries                      |
|--------|-----------------------------|
| 2      | linear combinations of \( x_0, x_1 \)                      |
| 4      | linear combinations of \( x_0^2, x_0x_1, x_1^2 \)          |
| 5      | linear combinations of \( x_iy_j, i = 0, 1, \) \( j = 0, \ldots, 3 \) |

We start by describing the minimal free resolution modulo the regular sequence \( x_0, x_1 \). Let \( \bar{R} := R(X)/(x_0, x_1)R[X] \) and \( T := S/(x_0, x_1) \cong \mathbb{k}[y_0, \ldots, y_3] \) with \( \deg(y_i) = 3 \). As a \( T \)-module \( \bar{R} \) splits into a direct sum

\[
\bar{R} = \bigoplus_{k=0}^2 \bar{R}^{(k)} \text{ with } \bar{R}^{(k)} := \bigoplus_{j \equiv k \pmod{3}} \bar{R}_j.
\]

Moreover, for any minimal free resolution \( F \) of \( R(X) \) the complex \( \bar{F} = F/(x_0, x_1) \) decomposes into a direct sum of three \( T \)-complexes which are minimal free resolutions of \( \bar{R}^{(k)} \) as \( T \)-modules.

**Lemma 3.2.** \( \text{Proj}(\bar{R}^{(0)}) \) is a finite scheme of length 4 in \( \mathbb{P}^3 \).

**Proof.** We know that \( \bar{R}^{(0)} = \bigoplus_{k \geq 0} \bar{R}_{3k} \) is a graded ring. Furthermore, the minimal free resolution of \( \bar{R}^{(0)} \) as a \( T \)-module is of the form

\[
0 \leftarrow \bar{R}^{(0)} \leftarrow T \leftarrow T(-2)^6 \leftarrow T(-3)^8 \leftarrow T(-4)^3 \leftarrow 0,
\]

where we consider the variables \( y_j \) with degree 1 now. The Hilbert polynomial of \( \bar{R}^{(0)} \) is the constant polynomial 4, thus \( \text{Proj}(\bar{R}^{(0)}) \subset \mathbb{P}^3 \) is a finite scheme of length 4. \( \square \)

**Remark 3.3.** The Hilbert scheme of length 4 subschemes of \( \mathbb{P}^3 \) which span \( \mathbb{P}^3 \) is known to be irreducible with finitely many orbits under the \( \text{PGL}(4) \)-action. In this paper we focus on the dense orbit which consists of collections of 4 distinct points in general position. The fact that there are only finitely many orbits give us hope that a complete classification of numerical Godeaux surfaces along the lines of our approach might be possible with further work.

**Proposition 3.4.** Let \( X \) be a numerical Godeaux surface. The following are equivalent

(a) \( |2K_X| \) has no fixed part and four distinct base points.
(b) \( \text{Proj}(\bar{R}^{(0)}) \) consists of four distinct points.

**Proof.** The scheme \( \text{Proj}(\bar{R}^{(0)}) \) is the image of the base points of \( |2K_X| \) under the tricanonical map from \( X_{\text{can}} \) to \( \mathbb{P}^3 \). The fixed part \( F \) in case (ii) and (iii) of Proposition 1.1 gets contracted to rational double points leading to a non-reduced scheme structure of \( \text{Proj}(\bar{R}^{(0)}) \). So these cases are excluded under the assumption (b). To prove \((a) \Rightarrow (b)\), we note that a general member of \( |M| = |2K_X| \) is a smooth non-hyperelliptic curve which passes through the 4 base points and is canonically embedded by \( |3K_X| \). Thus the points stay distinct. \(\square\)

Note that the image points span \( \mathbb{P}^3 \) since the homogeneous ideal \( J \) of \( \text{Proj}(\bar{R}^{(0)}) \) in \( T \) contains no linear forms.

**Definition 3.5.** Let \( X \) be a numerical Godeaux surface satisfying the equivalent conditions of Proposition 3.4. A marking on \( X \) is an enumeration \( p_0, \ldots, p_3 \) of the base points of \( |2K_X| \). A marked numerical Godeaux surface is a numerical Godeaux surface together with a marking.

A marked numerical Godeaux surface can only have the torsion groups \( 0, \mathbb{Z}/3\mathbb{Z} \) or \( \mathbb{Z}/5\mathbb{Z} \) because a divisor \( D \in |K + \tau| \) with \( 2\tau = 0 \) leads to a fiber \( 2D \in |2K_X| \) and hence to an everywhere non-reduced base locus. The moduli space of marked numerical Godeaux surfaces is an \( S_4 \) cover of an open part of the moduli space of numerical Godeaux surfaces. Introducing a marking allows us to change coordinates such that \( |3K_X| \) maps the base points to the coordinate points of of \( \mathbb{P}^3 \):

\[
p_0 \mapsto (1 : 0 : 0 : 0), \quad p_1 \mapsto (0 : 1 : 0 : 0), \quad p_2 \mapsto (0 : 0 : 1 : 0) \text{ and } p_3 \mapsto (0 : 0 : 0 : 1).
\]

The stabilizer \( G \leq \text{Aut}(\mathbb{P}^3) = \text{PGL}(4, \mathbb{k}) \) of the coordinate points as a set is \( G \cong (\mathbb{k}^*)^3 \rtimes S_4 \).

**Proposition 3.6.** Let \( X \) be a marked Godeaux surface. Then after a change of coordinates of \( \mathbb{P}(2^2, 3^4, 4^4, 3^5) \) and a change of basis of the free resolution \( F \) we may assume that the summands of

\[
\bar{F} = F/(x_0, x_1) = \bar{F}^{(0)} \oplus \bar{F}^{(1)} \oplus \bar{F}^{(2)}
\]

as \( T = \mathbb{k}[y_0, \ldots, y_3] = S/(x_0, x_1) \) are the following complexes

1. \( \bar{F}^{(0)} \) is the Macaulay2 resolution

\[
T \leftarrow 6T(-2 \cdot 3) \leftarrow 8T(-3 \cdot 3) \leftarrow 3T(-4 \cdot 3) \leftarrow 0,
\]

of \( T/J \), where

\[
J = (y_i y_j \mid 0 \leq i < j \leq 3) = \bigcap_{i=0}^{3} J_i \quad \text{and} \quad J_i = (y_j \mid j \neq i).
\]
(1) $\bar{F}^{(1)}$ is up to a twist the sum of 4 Koszul complexes

$$T(-4) \leftarrow 3T(-4 - 3) \leftarrow 3T(-4 - 2 \cdot 3) \leftarrow T(-4 - 3 \cdot 3) \leftarrow 0$$

resolving $T/J_i$.

(2) $\bar{F}^{(2)} = (\bar{F}^{(0)})^\vee = \text{Hom}(\bar{F}^{(0)}, T(-17))[-3]$ is up to twist and shift the dual of $\bar{F}^{(0)}$.

Proof. Since $x_0, x_1$ are a regular sequence on $R(X)$ and on $S$, the complex

$$0 \leftarrow \bar{R}(X) \leftarrow \bar{F}_0 \xleftarrow{\partial_1} \bar{F}_1 \xleftarrow{\partial_2} \bar{F}_1^\vee \xleftarrow{\partial_1^\vee} \bar{F}_0^\vee \leftarrow 0$$

is still exact. We choose coordinates on $\mathbb{P}(3^4)$ such that

$$\bar{R}^{(0)} = T/J$$

with $J = (y_i y_j \mid 0 \leq i < j \leq 3)$ holds. Then we can choose a basis of $F$ such that $\bar{F}^{(0)}$ has the desired shape and $\bar{F}^{(2)} = (\bar{F}^{(0)})^\vee$ holds. It remains to normalize $\bar{F}^{(1)}$. We start to prove that the sheaf associated to $\bar{R}^{(1)}$ on $\mathbb{P}(3^4) = \mathbb{P}^3$ has support in the coordinate points.

Lemma 3.7. $\text{ann}_T(\bar{R}^{(1)}) = J$.

Proof. We know that

$$\text{ann}_T(\bar{R}) = \bigcap_{i=0}^{2} \text{ann}_T(\bar{R}^{(i)}) = \text{ann}_T(\bar{R}^{(0)}) = J$$

since $\bar{R}$ is a $T$-algebra, in particular $J \subset \text{ann}_T(\bar{R}^{(1)})$. Thus $V(\text{ann}_T(\bar{R}^{(1)})) \subset V(J)$ is clear, and it is enough to prove equality of the vanishing loci, since $J$ is a radical ideal.

Suppose that there is a coordinate point $p_i$ which is not contained in $V(\text{ann}_T(\bar{R}^{(1)}))$. Then there exists an integer $n_i \geq 1$ such that $y_i^{n_i} \in \text{ann}_T(\bar{R}^{(1)})$. Hence for each $z \in R_4 = H^0(X, 4K_X)$ there is a relation of the form

$$r_0 x_0 + r_1 x_1 + y_i^{n_i} z = 0$$

in $R(X)$, where $r_0, r_1 \in R(X)$. Since $y_i^{n_i}(p_i) \neq 0$, all forms $z \in H^0(X, 4K_X)$ must vanish at the point $p_i \in X_{can}$. But $|4K_X|$ is base point free by [Bom73], Theorem 5.2, a contradiction.

Now let $N = \bar{R}^{(1)}(4)$ and consider the associated sheaf $\tilde{N}$ on $\mathbb{P}(3^4)$. As a module over $T = \mathbb{k}[y_0, \ldots, y_3]$ with the standard grading $\deg y_i = 1$ the module $N$ has a linear resolution

$$0 \leftarrow N \leftarrow T^4 \leftarrow T(-1)^{12} \leftarrow T(-2)^{12} \leftarrow T(-3)^4 \leftarrow 0.$$ 

Thus in this grading $N$ is a 0-regular Cohen-Macaulay module. Hence

$$N = \oplus_{d \geq 0} H^0(\mathbb{P}^3, \tilde{N}(d)).$$
Since \( J \) and \( J_i \) generate the same ideal in \( T_{y_i} \), the restriction of \( \hat{N} \) to the affine chart \( U_i = \{ y_i \neq 0 \} \) is \( \mathcal{O}_{P_i}^{r_i} \) for some \( r_i \geq 1 \) and

\[
\hat{N} \cong \bigoplus_{i=0}^{3} \mathcal{O}_{P_i}^{r_i}.
\]

Since \( H^0(\mathbb{P}^3, \hat{N}(d)) = N_d \) is 4-dimensional for each \( d \geq 1 \) we conclude \( r_i = 1 \) for all \( i \) and

\[
N = \bigoplus_{i=0}^{3} T/J_i.
\]

Thus \( \check{R}^{(1)} = \bigoplus_{i=0}^{3} T/J_i(-4) \) holds with respect to the grading of \( S \). Hence for suitable chosen generators for \( H^0(X, 4K_X) \) and suitable basis of \( F \) the complex \( \check{R}^{(1)} \) has the desired shape.

\[ \square \]

**Remark 3.8.** The Macaulay2 choice of the resolution of \( J_i = \{ y_j \mid j \neq i \} \) has not a skew-symmetric middle matrix. We take the following resolution of \( T/J_0(-4) \)

\[
T(-4) \xleftarrow{(y_1, y_2, y_3)} T(-7)^3 \xleftarrow{\begin{pmatrix} 0 & y_3 & -y_2 \\ -y_3 & 0 & y_1 \\ y_2 & -y_1 & 0 \end{pmatrix}} T(-10)^3 \xleftarrow{(y_1, y_2, y_3)} T(-13)^3 \xleftarrow{0}.
\]

and similarly for the other \( T/J_i(-4) \).

The Macaulay2 choice of the resolution of \( T/J \) has the following differentials

\[ (4) \]

\[
T \xleftarrow{\begin{pmatrix} y_0y_1 & y_0y_2 & y_1y_2 & y_0y_3 & y_1y_3 & y_2y_3 \end{pmatrix}} T(-6)^6
\]

\[
\begin{pmatrix} -y_2 & 0 & -y_3 & 0 & 0 & 0 & 0 & 0 \\ y_1 & -y_1 & 0 & 0 & -y_3 & 0 & 0 & 0 \\ 0 & y_0 & 0 & 0 & 0 & 0 & -y_3 & 0 \\ 0 & 0 & y_1 & -y_1 & y_2 & -y_2 & 0 & 0 \\ 0 & 0 & 0 & y_0 & 0 & 0 & y_2 & -y_2 \\ 0 & 0 & 0 & 0 & y_0 & 0 & y_1 & 0 \end{pmatrix}
\]

\[ (5) \]

\[
T(-6)^6 \xleftarrow{\begin{pmatrix} -y_3 & 0 & y_2 & 0 & -y_1 & 0 & 0 & 0 \\ -y_3 & -y_3 & y_2 & y_2 & 0 & 0 & -y_0 & 0 \\ 0 & 0 & 0 & -y_2 & 0 & y_1 & 0 & -y_0 \end{pmatrix}} T(-9)^8
\]

\[ (6) \]

\[
T(-9)^8 \xleftarrow{\begin{pmatrix} -y_3 & 0 & y_2 & 0 & -y_1 & 0 & 0 & 0 \\ -y_3 & -y_3 & y_2 & y_2 & 0 & 0 & -y_0 & 0 \\ 0 & 0 & 0 & -y_2 & 0 & y_1 & 0 & -y_0 \end{pmatrix}} T(-12)^3
\]

**Definition 3.9.** We call any minimal free resolution of the type

\[ (7) \]

\[
0 \xleftarrow{} R(X) \xleftarrow{d_1} F_0 \xleftarrow{d_2} F_1 \xleftarrow{F_1^\vee} d_1 \xleftarrow{} F_0^\vee \xleftarrow{} 0
\]

with a skew-symmetric middle matrix \( d_2 \) and which restricts with \( x_0 = x_1 = 0 \) to the complex described above a standard resolution of \( R(X) \).
Over an algebraically closed field any numerical Godeaux surface $X$ such that the base locus of $|2K_X|$ consists of 4 distinct base points $R(X)$ admits a standard resolution. Note that such a standard resolution is in general not unique.

In a standard resolution the red parts of 3.1 are completely determined. We now plan to determine the remaining blue parts of the resolution by using (unfolding) parameters for their entries and analyzing their relations which are imposed by the condition $d_1d_2 = 0$. We start analyzing the relations imposed by $d_1^\prime d_2 = 0$, where $d_1^\prime$ is the matrix obtained from $d_1$ by erasing the first row. The first row of $d_1$ will be added at the last step in our construction, since we can recover it from $d_2$ by a syzygy computation.

$$0 = d_1^\prime d_2 = \begin{pmatrix} ao - b_1(y)n^t - cb_3(y)^t & an - cp^t & ab_3(y) + b_1(y)p \\ -en^t & eb_1(y) - b_2(y)p^t & ep \end{pmatrix}.$$  

Using the order of $p_0, \ldots, p_3$, we introduce a natural labeling $a_{i,j}^{(k)}$ for the 24 entries of the matrix $a$ as indicated below in the $(1 + 4) \times (6 + 12)$ submatrix of $d_1$:  

$$\begin{pmatrix} y_0y_1 + * & y_0y_2 + * & y_1y_2 + * & y_0y_3 + * & y_1y_3 + * & y_2y_3 + * & * \\ a_{0,1}^{(0)} & a_{0,2}^{(0)} & a_{1,2}^{(0)} & a_{0,3}^{(0)} & a_{1,3}^{(0)} & a_{2,3}^{(0)} & D_0 \\ a_{0,1}^{(1)} & a_{0,2}^{(1)} & a_{1,2}^{(1)} & a_{0,3}^{(1)} & a_{1,3}^{(1)} & a_{2,3}^{(1)} & 0 \\ a_{0,1}^{(2)} & a_{0,2}^{(2)} & a_{1,2}^{(2)} & a_{0,3}^{(2)} & a_{1,3}^{(2)} & a_{2,3}^{(2)} & 0 \\ a_{0,1}^{(3)} & a_{0,2}^{(3)} & a_{1,2}^{(3)} & a_{0,3}^{(3)} & a_{1,3}^{(3)} & a_{2,3}^{(3)} & 0 \end{pmatrix} = \begin{pmatrix} y_1 & y_2 & y_3 \end{pmatrix}$$  

with $D_0 = (y_1 \ y_2 \ y_3), D_1 = (y_0 \ y_2 \ y_3), D_2 = (y_0 \ y_1 \ y_3)$ and $D_3 = (y_0 \ y_1 \ y_2)$. Imposing the condition $d_1^\prime d_2 = 0$ on the general setup for the matrices, we see that 12 out of the 24 $a$-variables are a priori zero:

**Proposition 3.10.** Let $d_1^\prime$ and $d_2$ be as in Notation 3.1 in our standard form satisfying $d_1^\prime d_2 = 0$. If $k \notin \{i, j\}$, then $a_{i,j}^{(k)} = 0$. Furthermore, every non-zero entry of the matrix $p$ and $e$ can be expressed up to a sign by one of the 12 remaining $a$-variables.

**Proof.** The proof of this statement can be deduced either theoretically or computationally by evaluating the relations $eb_4(y) - b_2(y)p^t = 0$ and $ab_3(y) + b_1(y)p = 0$. For a theoretical treatment we refer to [Ste18], Section 7.1. A Macaulay2 computation is given by the procedure `getRelationsAndNormalForm` from our Macaulay2 package NumericalGodeaux.

Using this statement, we assume from now on that the $a$-matrix is of the form

$$a = \begin{pmatrix} a_{0,1}^{(0)} & a_{0,2}^{(0)} & 0 & a_{0,3}^{(0)} & 0 & 0 \\ a_{0,1}^{(1)} & 0 & a_{1,2}^{(1)} & 0 & a_{1,3}^{(1)} & 0 \\ 0 & a_{0,2}^{(2)} & a_{1,2}^{(2)} & 0 & 0 & a_{2,3}^{(2)} \\ 0 & 0 & 0 & a_{0,3}^{(3)} & a_{1,3}^{(3)} & a_{2,3}^{(3)} \end{pmatrix}.$$  

Using this statement, we assume from now on that the $a$-matrix is of the form

Using this statement, we assume from now on that the $a$-matrix is of the form
Furthermore, expressing any \( e \)- and \( p \)-variable by one of the \( a \)-variables, we obtain a new matrix of relations:

\[
d_1^t d_2 = \begin{pmatrix}
a o - b_1(y) n^{tr} - c b_3(y)^{tr} & a n - c p^{tr} & 0 \\
- e n^{tr} & 0 & e p
\end{pmatrix}.
\]

In particular, we see from the new matrix of relations that if the entries of the matrix \( a \) are known and satisfy the equation \( e p = 0 \), all the remaining relations are linear in the unknown \( n \), \( c \) and \( o \)-variables.

Now let us recall that a possible entry of the matrix \( a \) is a linear combination of \( x_0 \), \( x_1 \) with coefficients in \( k \). We will think of these coefficients as Stiefel coordinates, hence as the entries of \( 2 \times 12 \)-matrices having at least one non-vanishing maximal minor.

**Notation 3.11.** For a matrix \( \hat{\ell} \in \text{St}(2, 12) \), we denote by \( \ell \) the line in \( \mathbb{P}^{11} \) spanned by the rows of \( \hat{\ell} \).

An assignment to the 12 remaining \( a \)-variables gives a matrix \( \hat{\ell} \in \text{St}(2, 12) \), and hence a line \( \ell \subset \mathbb{P}^{11} \). We have to choose lines in \( \mathbb{P}^{11} \) such that the quadratic relations coming from \( e p = 0 \) are satisfied. Using again our procedure `getRelationsAndNormalForm`, we see that there are exactly 4 different forms

\[
q_0 = a_{1,2}^{(1)} a_{1,3}^{(1)} - a_{1,2}^{(2)} a_{2,3}^{(2)} + a_{1,3}^{(3)} a_{2,3}^{(3)},
q_1 = a_{0,2}^{(0)} a_{0,3}^{(0)} - a_{0,3}^{(3)} a_{2,3}^{(2)} + a_{0,2}^{(2)} a_{2,3}^{(2)},
q_2 = a_{0,1}^{(1)} a_{1,3}^{(1)} - a_{0,1}^{(0)} a_{0,3}^{(0)} + a_{0,3}^{(3)} a_{1,3}^{(3)},
q_3 = a_{0,1}^{(0)} a_{0,2}^{(0)} - a_{0,1}^{(1)} a_{1,2}^{(1)} + a_{0,2}^{(2)} a_{1,2}^{(2)}
\]

which the assignment of the remaining 12 \( a \)-variables have to satisfy. The \( q_i \) are quadrics of rank 6. Hence, there are skew-symmetric matrices \( M_0, \ldots, M_3 \) of size 4 so that \( q_0, \ldots, q_3 \) are the Pfaffians of these matrices. One possible choice for these skew-symmetric matrices is:

\[
M_0 = \begin{pmatrix}
0 & a_{1,2}^{(1)} & a_{1,3}^{(1)} & a_{1,3}^{(3)} \\
0 & a_{2,3}^{(3)} & a_{2,3}^{(2)} & a_{1,3}^{(1)} \\
0 & 0 & a_{1,3}^{(1)} & 0
\end{pmatrix},
M_1 = \begin{pmatrix}
0 & a_{0,2}^{(0)} & a_{0,3}^{(3)} & a_{0,3}^{(2)} \\
0 & a_{2,3}^{(2)} & a_{2,3}^{(3)} & a_{0,3}^{(0)} \\
0 & 0 & a_{0,3}^{(0)} & 0
\end{pmatrix},
\]

\[
M_2 = \begin{pmatrix}
0 & a_{0,1}^{(1)} & a_{0,1}^{(0)} & a_{0,3}^{(3)} \\
0 & a_{1,3}^{(3)} & a_{0,3}^{(0)} & a_{1,3}^{(1)} \\
0 & 0 & a_{1,3}^{(1)} & 0
\end{pmatrix},
M_3 = \begin{pmatrix}
0 & a_{0,1}^{(0)} & a_{0,1}^{(1)} & a_{0,2}^{(2)} \\
0 & a_{1,2}^{(2)} & a_{1,2}^{(1)} & a_{0,2}^{(0)} \\
0 & 0 & a_{0,2}^{(0)} & 0
\end{pmatrix}.
\]
The corresponding variety $Q = V(q_0, \ldots, q_3) \subset \mathbb{P}^{11}$ is an irreducible complete intersection. $Q$ is irreducible because its singular loci has codimension $8 > 5$ in $\mathbb{P}^{11}$. The first step of our construction method for a marked numerical Godeaux surface $X$ is to choose a line $\ell \subset Q$.

**Lemma 3.12.** Let $\hat{\ell} \in \text{St}(2, 12)$ be a matrix. Then

$$e((x_0, x_1)\hat{\ell})p((x_0, x_1)\hat{\ell}) = 0 \iff \ell \subset Q \subset \mathbb{P}^{11}.$$ 

**Proof.** Clear from the definition of the variety $Q \subset \mathbb{P}^{11}$. $\square$

After the choice of a parametrized line $\ell \subset Q$, hence the choice of $a$, $e$ and $p$, the second step of our construction consists in solving the remaining equations

$$ao - b_1(y)n^t - cb_3(y)^t = 0,$$

$$an - cp^t = 0,$$

$$-en^t = 0.$$ 

This is a system of linear equations for the $c$-, $o$- and $n$-variables.

We end this section by introducing a normal form for the $o$-matrix which is the $6 \times 6$ skew-symmetric submatrix of $d_2$ whose entries are homogeneous of degree 5. Let

$$0 \leftarrow R(X) \leftarrow F_0 \overset{d_1}{\leftarrow} F_1 \overset{d_2}{\leftarrow} F_1^\vee \overset{d_3}{\leftarrow} F_0^\vee \leftarrow 0$$

be a standard resolution of $R(X)$ and define the maps

$$\alpha_0 = \text{id}_{F_0}, \quad \alpha_1 = \begin{pmatrix} \text{id}_6 & \gamma \\ \text{id}_{12} & \text{id}_8 \end{pmatrix}, \quad \alpha_2 = \begin{pmatrix} \text{id}_6 & \text{id}_{12} \\ -\gamma & \text{id}_8 \end{pmatrix}, \quad \text{and} \quad \alpha_3 = \text{id}_{F_0^\vee},$$

where $\gamma$ is a $6 \times 8$ matrix whose entries are linear forms in $x_0, x_1$. Then, setting $e_1 = \alpha_0d_1\alpha_1^{-1}$ and $e_2 = \alpha_1d_2\alpha_2^{-1}$, we obtain another isomorphic standard resolution of $R(X)$

$$0 \leftarrow R(X) \leftarrow F_0 \overset{e_1}{\leftarrow} F_1 \overset{e_2}{\leftarrow} F_1^\vee \overset{e_3}{\leftarrow} F_0^\vee \leftarrow 0,$$

with a new skew-symmetric matrix $o + b_3\gamma^t - \gamma b_3$. Motivated by this, we can reduce the original $o$-matrix by the $6 \times 8$ matrix $b_3(y)$ and its transpose in a way so that we keep the skew-symmetry and obtain a new matrix which depend only on $12$ $o$-variables instead of $60 = 4 \cdot \binom{6}{2}$ bilinear $o$-variables

$$\begin{pmatrix}
0 & o_{1,0,0}y_0 & o_{2,0,1}y_1 & o_{3,0,0}y_0 & o_{4,0,1}y_1 & 0 \\
0 & o_{2,1,2}y_2 & o_{3,1,0}y_0 & 0 & o_{5,1,2}y_2 \\
0 & 0 & o_{4,2,1}y_1 & o_{5,2,2}y_2 & 0 \\
0 & 0 & o_{4,3,3}y_3 & o_{5,3,3}y_3 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}.$$
For more details we refer again to our procedure \texttt{getRelationsAndNormalForm}. Now using the relations coming from $ao - b_1(y)n^t - cb_3(y)t^r = 0$, we can express any $n$-variable as a linear combination of $a$, $o$ and $c$-variables. The final system of linear equations depend then on 12 $o$-variables and 20 $c$-variables, and a solution for these variables can be computed using syzygies. We will study these linear solution spaces more extensively in the following sections.

Remark 3.13. A different normalization pursued in [Ste18] is to make the $c$-matrix as much zero as possible. In particular, for torsion-free numerical Godeaux surfaces with no hyperelliptic bicanonical fibers, the $c$-matrix can be assumed to be zero in this setting.

4 The dominant component

In Section 3 we have seen that our construction method of numerical Godeaux surfaces consists mainly of two big steps: first, choosing a line in the quadratic complete intersection $Q \subset \mathbb{P}^{11}$ and second, choosing a solution for a linear system of equations. This section concerns the second step. We summarize some of the main results of this section.

Theorem 4.1. For a general line $\ell \subset Q \subset \mathbb{P}^{11}$ the linear system of equations for the remaining $o$- and $c$-variables has a 4-dimensional solution space, and $\ell \in F_1(Q)$ is a smooth point in the Fano variety of lines in $Q$.

Corollary 4.2. There exists a $8+3$-dimensional irreducible family in the unfolding parameter space for Godeaux surfaces, which modulo the $(\mathbb{C}^*)^3$-action gives a 8-dimensional locally complete family of Godeaux surfaces with trivial fundamental group hence torsion group $\text{Tors} = 0$.

We call this family the dominant family, because it dominates the Fano variety of lines $F_1(Q)$ ([Ste18] established the irreducibility of $F_1(Q)$ numerically).

A sufficient condition for a line $\ell$ to lead only to Godeaux surfaces in the dominant family is that $\ell$ does not intersect the homology loci of the complexes $C_1$ and $C_2$ we introduce below. A key point is that the Barlow surfaces are part of the dominant family. Lines leading to torsion $\mathbb{Z}/3\mathbb{Z}$- and $\mathbb{Z}/5\mathbb{Z}$-Godeaux surfaces have to intersect some of the homology loci non-trivially.

Starting with matrices $d_1'$ and $d_2$ in normal form, the remaining unfolding parameters satisfy exactly 46 homogeneous relations coming from $d_1'd_2 = 0$: the four quadratic relations $q_0, \ldots, q_3$ which are Pfaffians and 42 relations which are linear in the unknown $o$-variables and $c$-variables, see \texttt{getRelationsAndNormalForm}.

We start with representing the 42 relations by a matrix. By $c$ we denote the column vector of the 20 unknown $c$-variables and by $o$ the column vector of the remaining 12 $o$-variables. The linear system of equations is of the form

$$
\begin{pmatrix}
0 & l_1 \\
l_2 & q
\end{pmatrix}
\begin{pmatrix}
c \\
o
\end{pmatrix} = 0
$$
where $l_1$ is a $12 \times 12$-matrix and $l_2$ is a $30 \times 20$-matrix, both having entries linear in the $a$-variables, and $q$ is a $30 \times 12$-matrix with quadratic entries. We denote the $42 \times 32$-matrix by $m_a$ and the standardly graded polynomial ring with the 12 remaining $a$-variables by $S_a$.

First let us describe the quadratic matrix $l_1$. Arranging the $a$-variables and the corresponding 12 relations of degree 4 properly, $l_1$ is a skew-symmetric matrix which is the direct sum of the matrices

$$
\left(\begin{array}{ccc}
0 & a_{1,3}^{(3)} & a_{0,3}^{(3)} \\
0 & 0 & a_{2,3}^{(2)} \\
0 & 0 & 0
\end{array}\right),
\left(\begin{array}{ccc}
0 & a_{1,2}^{(2)} & a_{0,2}^{(2)} \\
0 & 0 & a_{2,2}^{(3)} \\
0 & 0 & 0
\end{array}\right),
\left(\begin{array}{ccc}
0 & a_{1,3}^{(1)} & a_{0,1}^{(1)} \\
0 & 0 & a_{2,1}^{(2)} \\
0 & 0 & 0
\end{array}\right),
\left(\begin{array}{ccc}
0 & a_{0,2}^{(0)} & a_{0,1}^{(0)} \\
0 & 0 & a_{0,3}^{(0)}
\end{array}\right).
$$

Note that the entries of such a $3 \times 3$-matrix are exactly the entries of a row of the $a$-matrix. Resolving $l_1$ and $l_1^t = -l_1$ yields a complex $C_1$ which is a direct sum of four Koszul complexes

| 0 | 1 | 2 | 3 |
|---|---|---|---|
| total: | 4 | 12 | 12 | 4 |
| -3: | 4 | 12 | 12 | 4 |

The $30 \times 20$-matrix $l_2$ is not as easy to describe. Arranging the $c$-variables and the 30 relations of degree 6, we get

$$
l_2 = \left(\begin{array}{c}
l_1 \\
0 \\
n_1 \\
n_2
\end{array}\right),
$$

where $n_1$ is $18 \times 12$-matrix and $n_2$ a $18 \times 8$-matrix, both having full rank. Furthermore, the matrix $l_2$ has full rank 20 and hence no non-trivial syzygies. However, over the quotient ring $S_Q = S_a/I(Q)$ we obtain syzygies. Moreover, rather unexpectedly, both modules $\ker(l_2 \otimes S_Q) \cong 2S_Q(-1)$ and $\ker(l_2^t \otimes S_Q) \cong 12S_Q(-4)$ are free over this quotient ring. Putting these together, we get a generically exact complex $C_2$ with Betti numbers

| 0 | 1 | 2 | 3 |
|---|---|---|---|
| total: | 12 | 30 | 20 | 2 |
| -4: | 12 | 30 | 20 | .. |
| -3: | .. | .. | .. | .. |
| -2: | .. | .. | 2 | .. |

Similarly, $\ker(m_a^t \otimes S_Q) \cong 4S_Q(-3) \oplus 12S_Q(-4)$ is free, however $E = \ker(m_a \otimes S_Q)$ is not free.

**Proof of Theorem 4.1.** Since any syzygy of $l_2$ induces a syzygy of $m_a$ and every syzygy of $m_a$ projects to one of $l_1$, we can combine these complexes into a commutative diagram
with three split exact rows. The columns are only generically exact. Following the red arrows in the diagram, we obtain a boundary map $4S_Q \to 12S_Q(4)$. We verify computationally that this boundary map is the zero homomorphism, and hence we obtain a complex

\begin{equation}
0 \to 2S_Q(-1) \to E \to 4S_Q \to 0
\end{equation}

which is exact except at the last position, due to the homology $H_1(C_2) = \ker(g_2) / \im(l_2)$. Below we will prove computationally that the support of the homology groups $H_i(C_2)$ and $H_i(C_1)$ have codimension $\geq 2$ in $Q$. Thus a general line $\ell \subset Q$ does not intersect this locus. If $\ell$ is such a line, then the complex in (8) restricted to $\ell$ is exact and we get an exact sequence of global sections on $\mathbb{P}^1 \cong \ell$

\begin{equation}
0 \to H^0(\mathcal{O}_{\mathbb{P}^1}(-1)^2) \to H^0(E|_{\ell}) \to H^0(\mathcal{O}_{\mathbb{P}^1}^4) \to 0.
\end{equation}

Consequently, we obtain a 4-dimensional solution space for the remaining $o$- and $c$-variables in this case.

We will prove that $F_1(Q)$ is smooth at a general $\ell$ computationally by studying a Barlow line in Section 4.1. Thus with these two further assertions we obtain the proof of Theorem 4.1.

In the following we report our results on the homology loci of the complexes introduced above using Macaulay2. The complex $C_1$ has just homology at the zeroth position, and we compute that the three entries of each row of the $a$-matrix form a regular sequence in $Q$. More precisely, $H_0(C_1) = \coker g_1$ is supported at a 4-dimensional scheme in $Q$ which decomposes into eight irreducible varieties. Setting the entries of one row of the $a$-matrix to zero, the corresponding locus in $Q$ decomposes into a union of two varieties.
with Betti tables

\[
\begin{array}{cccccc}
0 & 1 & 0 & 1 \\
\text{total:} & 1 & 8 & \text{total:} & 1 & 7 \\
0: & 1 & 3 & \text{and} & 0: & 1 & 6. \\
1: & . & 4 & 1: & . & 1 \\
2: & . & 1 \\
\end{array}
\]

For example, restricted to the components corresponding to the entries of the first row, the \( a \)-matrix is of the form

\[
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
a_{1,0,1} & 0 & a_{1,1,2} & a_{1,1,3} & 0 \\
0 & a_{2,0,2} & a_{2,1,2} & 0 & 0 & a_{2,2,3} \\
0 & 0 & 0 & a_{3,0,3} & a_{3,1,3} & a_{3,2,3}
\end{pmatrix}
\]

or

\[
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & a_{1,1,2} & 0 & a_{1,1,3} & 0 \\
0 & 0 & a_{2,1,2} & 0 & 0 & a_{2,2,3} \\
0 & 0 & 0 & a_{3,1,3} & a_{3,2,3} \\
\end{pmatrix}
\]

For the second complex \( C_2 \) we have \( H_2(C_2) = H_3(C_2) = 0 \) by construction. The module \( H_1(C_2) = \ker g_2/\operatorname{im} l_2 \) is supported at a 5-dimensional scheme of degree 72 in \( Q \). There are exactly 24 irreducible components all of codimension 6 in \( \mathbb{P}^{11} \) with only two different Betti tables

\[
\begin{array}{cccccc}
0 & 1 & 0 & 1 \\
\text{total:} & 1 & 6 & \text{and} & 1: & . & 1 \\
0: & 1 & 5 & 1: & . & 2 \\
1: & . & 4 \\
\end{array}
\]

and exactly 12 components for each Betti table. We checked that all components with the same Betti table are equivalent under the \( S_1 \)-operation on the coefficients and present only one example for each class here:

\[
\begin{align*}
(a_{0,0,3}, a_{1,1,3}, a_{2,2,3}, a_{3,1,3}, a_{3,2,3}, a_{2,1,2}a_{2,0,2} - a_{1,1,2}a_{1,0,1} + a_{0,0,2}a_{0,0,1}), \\
(a_{0,0,3}, a_{1,1,2}, a_{2,2,3}, a_{3,2,3}, a_{2,1,2}a_{2,0,2} + a_{0,0,2}a_{0,0,1}, a_{3,1,3}a_{3,0,3} + a_{1,1,3}a_{1,0,1}).
\end{align*}
\]

The module \( H_0(C_2) = \coker g_2 \) is supported at a 5-dimensional scheme of degree 72 in \( Q \). By inspecting this locus, we see that its minimal primes are exactly the minimal primes of the ideal of \( 4 \times 4 \)-minors of the \( a \)-matrix in \( Q \). It decomposes into 5 components with Betti numbers

\[
\begin{array}{cccc}
0 & 1 & 0 & 1 \\
\text{total:} & 1 & 12 & \text{and} & 1: & . & 4 \\
0: & 1 \\
1: & . & 4 \\
2: & . & 4 \\
3: & . & 4 \\
\end{array}
\]

In particular we deduce from these computations that the homology loci of \( C_1 \) and \( C_2 \) have codimension \( \geq 2 \) in \( Q \).

To end this section, we will briefly explain how to construct lines in \( Q \) meeting no or some special loci. The idea is the following: first we choose a point \( p \in Q \). Then, for a
general \( p \in Q \), the variety \( Z = Q \cap T_p Q \) is a cone over a surface. Afterwards we choose a (general) point \( q \in Z \) different from \( p \). Now, as \( Q \) is a complete intersection of quadrics, the line \( \ell = \overline{pq} \) is completely contained in \( Q \). Proceeding like this, the constructed line \( \ell \) does not meet any of the codimension \( \geq 2 \) subloci of \( Q \) introduced above. If we want to construct lines meeting one or two loci, we simply choose the two spanning points in these loci (if possible).

**Remark 4.3.** Most of our Macaulay2 computations are performed over finite fields. Over a finite field \( \mathbb{F} \) we can find a \( \mathbb{F} \)-rational point \( p \in Q \) by intersecting \( Q \) with random hyperplanes down to a zero-dimensional scheme and repeat this until this intersection contains a \( \mathbb{F} \)-rational point. For the second point \( q \) we proceed analogously with the scheme \( Z \). Note that under mild hypotheses we can regard lines over a finite field \( \mathbb{F} \) as a specialization of lines over an algebraic number field. For a construction method of lines defined over a finite field extension of \( \mathbb{Q} \), we refer to [Ste18], Section 7.2. In [Ste18] we constructed an element of the dominant family over an algebraic number field of degree 8.

### 4.1 Barlow surfaces

The Barlow surface ([Bar85]) was the first example of a simply connected numerical Godeaux surface. In this subsection we first sketch the original construction due to Barlow, reconstruct the surface then with our construction and show in the end that the Barlow surfaces are deformation equivalent to the members of our dominant family. In particular, every surface in our dominant family is simply connected.

For the reconstruction of the original Barlow surface we follow the descriptions in [Bar85] and [Lee01]. To start with, we recall that there is an 8-dimensional family of numerical Godeaux surfaces with \( T_{\text{ors}} = \mathbb{Z}/5\mathbb{Z} \) which are given as the quotient of quintics in \( \mathbb{P}^5 \) under a free action of \( \mathbb{Z}/5\mathbb{Z} \). In [Cat83], Catanese showed that there is a 4-dimensional subfamily in which the corresponding quintic is the determinant of a \( 5 \times 5 \) symmetric matrix. Moreover, in this 4-dimensional family there exists a 2-dimensional subfamily in which the group action of \( \mathbb{Z}/5\mathbb{Z} \) can be extended to a group action of the dihedral group \( D_5 \). Using a twist of this action, Barlow realized a simply connected numerical Godeaux surface as a quotient of a double cover of such a quintic. This construction shows the existence of a 2-dimensional family of simply connected numerical Godeaux surfaces.

In the following we briefly recall the description of a symmetric determinantal quintic \( Q_5 \subset \mathbb{P}^3 \) and the definition of the action of \( D_5 \) on \( Q_5 \). Let \( u_1, \ldots, u_4 \) denote the coordinates of \( \mathbb{P}^3 \), and let \( \xi \) be a primitive fifth root of unity. Then the group \( D_5 = \langle \beta, \tau \rangle \) acts on \( \mathbb{P}^3 \) via

\[
\beta: (u_1 : u_2 : u_3 : u_4) \mapsto (\xi u_1 : \xi^2 u_2 : \xi^3 u_3 : \xi^4 u_4), \\
\tau: (u_1 : u_2 : u_3 : u_4) \mapsto (u_4 : u_3 : u_2 : u_1).
\]

A quintic in \( \mathbb{P}^3 \) which is invariant under this action is the determinant of the symmetric
matrix

\[ A = (a_{ij}) = \begin{pmatrix}
0 & a_{1u1} & a_{2u2} & a_{2u3} & a_{1u4} \\
a_{1u1} & a_{3u2} & a_{4u3} & a_{5u4} & 0 \\
a_{2u2} & a_{4u3} & a_{6u4} & 0 & a_{5u1} \\
a_{2u3} & a_{5u4} & 0 & a_{6u1} & a_{4u2} \\
a_{1u4} & 0 & a_{5u1} & a_{4u2} & a_{3u3}
\end{pmatrix}, \]

where \( a_1, \ldots, a_6 \in k \) are parameters (see [Lee01]). A generic surface \( Q_5 = \det A \) has an even set of 20 nodes given by the \( 4 \times 4 \) minors of \( A \). Hence, there exists a double cover \( \Phi: F \to Q_5 \) branched over these nodes. Then \( \Phi \) is the canonical map of \( F \), and the canonical ring \( R = R(F) \) is generated by \( u_1, \ldots, u_4 \in R_1^+, v_1, \ldots, v_5 \in R_2^- \) with the following relations

\[
\sum_j a_{ij}v_j, \quad (5 \text{ relations of degree } 3)
\]

\[
v_jv_k - B_{jk}, \quad (15 \text{ relations of degree } 4)
\]

where \( B_{jk} \) is the entry in row \( j \) and column \( k \) of the adjoint matrix of \( A \) (see [Cat83], Theorem 3.5).

**Theorem 4.4** ([Bar85], Theorem 2.5 and subsequent Corollary). There exists an action of \( D_5 = \langle \beta, \alpha \rangle \) on \( F \) such that \( \beta \) acts freely on \( F \) and \( \alpha \) with a finite fixed locus. The corresponding quotient is a surface \( B \) with four double points whose resolution is a minimal surface of general type with \( K^2 = 1, p_g = 0 \) and \( \pi_1 = \{1\} \).

The element \( \alpha \) acts on \( F \) via an induced action of \( \tau \) on \( F \) twisted by the canonical involution \( \iota: F \to F \). The \( D_5 = \langle \beta, \alpha \rangle \)-action on \( F \) is given by

\[
\beta(u_i) = \xi^i u_i, \quad \beta(v_i) = \xi^{-i} v_i, \\
\alpha(u_i) = u_{-i}, \quad \alpha(v_i) = -v_{-i},
\]

using indices in \( \mathbb{Z}/5\mathbb{Z} \), see [BvBKP12], Remark 4.2.

**Example 4.5.** The special surface constructed in [Bar85] corresponds to a symmetric quintic as in (1) with parameters

\[ a_1 = a_2 = a_4 = a_5 = 1, \quad a_3 = a_6 = -4, \]

see [BvBKP12], Remark 2.1. We take these parameters and construct with the help of Macaulay2 the canonical ring \( R(X) = R(F)^{D_5} \), the canonical model \( X_{can} \), a standard resolution of \( R(X) \) as an \( S \)-module and the corresponding line \( \ell \) in \( Q \).

**Proposition 4.6.** The Barlow surface is deformation equivalent to a general member of our 8-dimensional family of torsion-free numerical Godeaux surfaces. In particular, all members of our dominant family are simply connected.
Proof. We verify computationally that over the Barlow line \( \ell \) the solution space in the second step is a 4-dimensional linear space, whence, as in the case of a line in \( Q \) intersecting no homology loci, we obtain a \( \mathbb{P}^3 \) of solutions. Using our procedure `normalBundleLineInQ`, we determine the normal sheaf \( N_\ell|Q \) which is a line bundle as \( \ell \) does not meet the singular locus of \( Q \) and obtain

\[
N_\ell|Q \cong \mathcal{O}_{\mathbb{P}^1}(1)^2 \oplus \mathcal{O}_1^1.
\]

Thus, \( h^0(N_\ell|Q) = 8 \), \( h^1(N_\ell|Q) = 0 \) and we can move the Barlow line to a line in \( Q \) not meeting any of the homology loci. Thus the Barlow surface lies in the dominant component.

\[ \square \]

5 Special bicanonical fibers

In this section we characterize special bicanonical divisors of marked Godeaux surfaces. Recall the following result of Catanese and Pignatelli:

**Lemma 5.1.** Let \( X \) be a numerical Godeaux surface and let \( C \in |2K_{X,\text{can}}| \). Then one of the following holds:

(i) \( C \) is embedded by \( \omega_C \) and \( \phi_3(C) = \phi_{\omega_C}(C) \) is the complete intersection of a quadric and a cubic.

(ii) \( C \) is honestly hyperelliptic and \( \phi_3(C) = \phi_{\omega_C}(C) \) is a twisted cubic curve.

(iii) \( \pi^*C = D_1 + D_2 \) with \( D_i \in |K_X + \tau_i| \), \( \tau_i \in \text{Tors } X \) nontrivial, \( \tau_1 + \tau_2 = 0 \).

Case (i) is the general one.

Note that a Gorenstein curve \( C \) is called *honestly hyperelliptic* if there exists a finite morphism \( C \rightarrow \mathbb{P}^1 \) of degree 2. This definition does not require that \( C \) is smooth or irreducible.

**Proof.** We follow the proof of [CP00], Lemma 1.10. Suppose \( C \) is not 3-connected. Then \( \pi^*C \in |2K_X| \) is not 3-connected as well by [CFHR99] Lemma 4.2, and we have a decomposition

\[
\pi^*C = D_1 + D_2 \text{ with } D_1D_2 \leq 2 \text{ and } K_XD_i = 1.
\]

Hence \( D_1^2 + D_2^2 = (2K_X)^2 - D_1D_2 \geq 0 \), and we may assume that \( D_1^2 \) is non-negative. \( D_1^2 \) is odd, since \( D_1(D_1+K_X) \) is even. Hence \( D_1^2 > 0 \) and the algebraic Hodge index theorem implies \( D_1^2 = 1 \) and \( D_1 = K_X + \tau_1 \) with \( \tau_1 \in \text{Tors } X \setminus \{0\} \), hence \( D_2 = K_X - \tau_1 \). So if we are not in case (iii), we may assume that \( C \) is 3-connected and the result follows from [CFHR99] Theorem 3.6.

We start characterizing torsion fibers, i.e., fibers of type (iii).

**Theorem 5.2.** Let \( X \) be a marked numerical Godeaux surface with standard resolution \( F \). A point \( q = (q_0 : q_1) \in \mathbb{P}^1 \cong |2K_X| \) corresponds to a torsion fiber if and only if \( \text{rank}(e(q)) \leq 2 \).
Proof. Let $\varphi : X_{\text{can}} \to Y \subset \mathbb{P}(2^2, 3^4)$ be our model in the weighted $\mathbb{P}^5$. Suppose $\text{rank}(e(q)) \leq 2$. Then $\hat{q} = (q_0 : 0 : 0 : 0 : 0) = (q_0 : q_1 : 0 : 0 : 0)$ is a point where the matrix $d_1$ drops rank, because

$$d_1(\hat{q}) = \begin{pmatrix} * & 0 & * \\ a(q) & 0 & c(q) \\ 0 & e(q) & 0 \end{pmatrix}.$$ 

So $\hat{q} \in V(\text{ann coker } d_1) \cap V(y_0, \ldots, y_3) \subset Y$ corresponds to a base point of $|3K_{X_{\text{can}}}|$. By Miyaoka’s results $\hat{q}$ is the intersection point $\hat{D}_1 \cap \hat{D}_2$, where $D_1 + D_2 \in |2K_X|$ is a torsion fiber which coincides with the fiber over $q$. Thus $\text{rank}(e(q)) \leq 2$ is sufficient.

The proof that the condition is also necessary is more subtle. The ideal $I_3(e) + I(Q) \subset S_a$ of $3 \times 3$-minors of $e$ decomposes in 7 components with Betti numbers

0 1 0 1 1 0 1
0 1 6 or 0 1 8
1 . 1

and degree 2 and 1 respectively. We have encountered the 4 components of the first type already in Section 4. The 3 components of the second type lead to torsion surfaces. Below we will show that there are 8-dimensional families of $\mathbb{Z}/3\mathbb{Z}$ and $\mathbb{Z}/5\mathbb{Z}$ marked Godeaux surfaces, where the $e$ matrix drops rank in 1 respectively 2 points of the loci of the second type. Since the family of torsion $\mathbb{Z}/3\mathbb{Z}$ and $\mathbb{Z}/5\mathbb{Z}$ Godeaux surfaces is known to be irreducible by [Rei72] and [Miy76] and the condition of Proposition 3.4 is an open condition, we see that intersecting one or two loci of the second kind is a necessary condition for torsion marked Godeaux surfaces.

As a corollary of the proof we obtain the following:

**Proposition 5.3.** Let $\ell \subset \mathbb{P}^{11}$ be a line which intersects $V(I_3(e)) \cap Q$ in a point of a degree 2 component which is not contained in a degree 1 component. Then $\ell$ does not lead to a numerical Godeaux surface.

**Remark 5.4.** In Section 7 we will see that such lines do lead to surfaces which however are always reducible.

**Theorem 5.5.** Let $X$ be a marked numerical Godeaux surface, and let $q \in \mathbb{P}^1$. Then $\text{rank}(a(q)) = 3$ if and only if the corresponding fiber $C_q \subset |2K_{X_{\text{can}}}|$ is hyperelliptic.

**Proof.** After applying a linear change of coordinates if necessary, we may assume that $q = (0 : 1)$ and

$$C = \text{Proj}(R(X)/(x_0)).$$

Furthermore, as $h^1(X, \mathcal{O}_X(nK_X)) = 0$ for all $n$, we get

$$R(X)/(x_0) \cong \bigoplus_{n \geq 0} H^0(C, \mathcal{O}_C(nK_X|_C)).$$
Using this, we compute that $h^0(C, \mathcal{O}_C(6K_X|_C)) = h^0(C, \mathcal{O}_C(2K_C)) = 9$ and thus, there are 6 relations between the 15 global sections

$$x_1^3, \{x_1z_j\}_{0 \leq j \leq 3}, \{y_iy_j\}_{0 \leq i \leq j \leq 3} \in H^0(C, \mathcal{O}_C(6K_X|_C)).$$

These relations are given by the first 6 columns of $\tilde{d}_1 := d_1 \otimes S/(x_0)$.

If $C_q$ is hyperelliptic, then by Lemma 5.1 there are 3 equations among these relations which are quadrics in the $y_i$'s alone. Thus $\text{rank } a(q) \leq 3$. By computation we see that $I_3(e) + I(Q) \subset I_3(a) + I(Q)$, hence a point with $\text{rank } a(q) \leq 2$ belongs to a torsion fiber.

Conversely, suppose that $\text{rank } a(q) = 3$. Then there exists a three-dimensional space of linear combinations of the first 6 entries

$$y_0y_1 + \alpha_1x_1^3, \ldots, y_2y_3 + \alpha_6x_1^3$$

of the first row of $\tilde{d}_1$ in the ideal of $C_q$ in $R(X)/(x_0)$. We have to show that these equations are quadratic in the $y_i$'s alone and that they define a rational normal curve.

The ideal $I_4(a) + I(Q)$ of $4 \times 4$ minors of $a$ decomposes into 5 components with Betti numbers

|        | 0   | 1   |         | 0   | 1   |
|--------|-----|-----|---------|-----|-----|
| total: | 1   | 7   |         | 1   | 12  |
| 0:     | 1   | 6   |         | 0   | 1   |
| 1:     | .   | 1   |         | 1   | .   |
| 2:     | .   | 4   |         | 2   | .   |
| 3:     | .   | 4   |         | 3   | .   |

The four degree 2 components belong also to $I_3(e) + I(Q)$, hence are excluded, since they do not lead to marked Godeaux surfaces by Proposition 5.3 unless the point $q$ lies also in a linear component of $I_3(e) + I(Q)$. Since $\text{rank } a(q) \leq 2$ for any $q$ contained in one of the linear components of $I_3(e) + I(Q)$, the fiber over $q$ is not a torsion fiber, because $\text{rank } a(q) = 3$. Since there are at least 2 quadrics in the $y_i$'s alone and $C_q$ is not at a torsion fiber, it must be a hyperelliptic fiber, and there are actually precisely 3 quadrics in the $y_i$'s alone by Lemma 5.1.

We denote by $J_{hyp} \subset S_q$ the fifth component of $I_4(a) + I(Q)$. From the proof of Theorem 5.5 we obtain the following:

**Corollary 5.6.** Let $q \in \ell \subset Q \subset \mathbb{P}^{11}$ be a point on a line in $Q$ and let $X_{can}$ be a Godeaux surface constructed from $\ell$. Then $C_q \subset X_{can}$ is a hyperelliptic fiber if and only if $q \in V(J_{hyp})$. □

Next we describe the image of a hyperelliptic fiber under the birational morphism $\varphi: X_{can} \to Y$. We already know that a hyperelliptic curve is mapped 2-to-1 under the rational map $\phi_2 \times \phi_3: X \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^3$.

**Proposition 5.7.** Let $C \in |2K_{X_{can}}|$ be a honestly hyperelliptic curve. Then the restriction of $\varphi: X_{can} \to Y$ to $C$ is a birational morphism onto its image $G$ with $p_a(G) = p_a(C) + 1$. □
Corollary 5.8. If there exists a hyperelliptic fiber, then the morphism \( \varphi : X_{can} \to Y \subset \mathbb{P}(2^2, 3^4) \) is not an isomorphism.

Proof of Proposition. The idea of the proof is to embed \( C \) (respectively \( G \)) into projective spaces and show that the induced morphism is the restriction of a projection from a point on a rational normal scroll.

Let \( K_0 \) be the Cartier divisor corresponding to the \( g_2^1 \) on \( C \). We consider the very ample line bundle \( \mathcal{O}_C(6K_0) = \mathcal{O}_C(K_0) \otimes \mathcal{O}_C(5K_0) \) which is the restriction of \( \mathcal{O}_X(6K_X) \) to \( C \) and set \( V = H^0(C, \mathcal{O}_C(6K_0)) \). Hence \( C \) embedded in \( \mathbb{P}(V) \cong \mathbb{P}^8 \) is contained in a smooth rational normal scroll of type \( S(6, 1) \) (as \( H^0(C, \mathcal{O}_C(5K_0)) = \text{Sym}^5(H^0(C, \mathcal{O}_C(K_0)) + \langle u \rangle \) for some global section \( u \)). The divisor class of \( C \) in \( S(6, 1) \) of type \( 2H - 2L \), where \( L \) denotes the class of a ruling and \( H \) the class of a hyperplane section. Indeed, let \( C \sim aH + bL \) for some \( a, b \). Then, clearly \( a = 2 \) by the definition of the scroll and \( C \) being hyperelliptic, whereas \( b = -2 \) follows from the fact that \( C \) has degree 12 under the morphism induced by \( 6K_X \). In particular, \( C \) does not meet the directrix \( H - 6L \) of the scroll.

Now we consider the image \( G \subset \mathbb{P} = \mathbb{P}(2, 3^4) \) of \( C \) under the birational morphism \( \varphi : X_{can} \to Y \). Denoting by \( R_i \) a rational normal curve of degree \( i \) in \( \mathbb{P}^i \), we have a diagram

\[
\begin{array}{ccc}
C & \xrightarrow{\mathcal{O}_2(6)|_G} & S(6, 1) \subset \mathbb{P}^8 \\
& \searrow & \\
& \quad G & \leftarrow S(6, 0) \subset \mathbb{P}^7 \\
& \nearrow & \\
K_C & \quad & R_3 \rightarrow R_6 \subset \mathbb{P}^6
\end{array}
\]

The image of \( G \) is contained in a cone \( S(6, 0) \) over \( R_6 \subset \mathbb{P}^6 \). The morphism \( C \to G \) is the restriction of the projection from a point \( p \) on the directrix of \( S(6, 1) \) to \( S(6, 0) \). Moreover, the unique line of the ruling of \( S(6, 1) \) through \( p \) is the only line through \( p \) which intersects \( C' \) in two points (counted with multiplicity). Let \( H' \) denote the class of a hyperplane section on \( S(6, 0) \). Since \( G \) does not meet the vertex of the cone \( S(6, 0) \) and \( \deg(G) = 12 \) in \( \mathbb{P}^7 \), we see that \( C' \sim 2H' \). Hence, using adjunction, we obtain \( p_a(G) = 5 \). So \( \varphi|_C : C \to G \) is birational, and \( p_a(G) = p_a(C) + 1 \) holds.

5.1 The hyperelliptic locus

By Corollary 5.6 a point \( p \in \ell \) corresponds to a hyperelliptic curve in \( |2K| \) if and only if \( p \in V_{hyp} = V(J_{hyp}) \). The ideal \( J_{hyp} \) has codimension 6 and its free resolution has Betti table...
The function \texttt{precomputedHyperellipticLocus} records the generators of $J_{hyp}$.

**Theorem 5.9.** $V_{hyp} \subset \mathbb{P}^{11}$ is birational to a product of a Hirzebruch surface $F = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1} (2))$ with 3 copies of $\mathbb{P}^1$. In terms of homogeneous coordinates $v_0, \ldots, v_1$ on the product, where $v_0$ corresponds to the section of $H^0(F, \mathcal{O}_F (1)) \cong H^0(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1} (2))$ obtained from the first summand and $v_1$ corresponds to the section obtained from the second summand, $w_0, w_1$ denote pull backs of homogeneous coordinates of the base of $F$ and similarly $x_0, \ldots, z_1$ homogeneous coordinates on the three $\mathbb{P}^1$ factors, we obtain the following: The birational map

$$
\varphi : \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1} (2)) \times \mathbb{P}^1 \times \mathbb{P}^1 \longrightarrow V_{hyp} \subset \mathbb{P}^{11}
$$

is given by the transpose of the $12 \times 1$ matrix

$$
\begin{pmatrix}
    a_{2,3}^{(3)} \\
    a_{1,3}^{(3)} \\
    a_{0,3}^{(3)} \\
    a_{2,3}^{(2)} \\
    a_{1,2}^{(2)} \\
    a_{0,2}^{(2)} \\
    a_{1,3}^{(1)} \\
    a_{1,2}^{(1)} \\
    a_{0,1}^{(1)} \\
    a_{0,3}^{(0)} \\
    a_{0,2}^{(0)} \\
    a_{0,1}^{(0)} \\
\end{pmatrix}
= 
\begin{pmatrix}
    v_0^2(x_1 w_1 - x_0 w_0) x_1 (x_1 + x_0) y_0 y_1 z_1^2 \\
    v_0^2(x_1 w_1 - x_0 w_0) x_0 (x_1 + x_0) y_0 y_1 z_0^2 \\
    -v_1^2 w_0^2 w_1^2 (w_1 + w_0) x_0 x_1 (x_1 + x_0) y_0 y_1 z_1^2 \\
    -v_0^2 (x_1 w_1 - x_0 w_0) x_0 x_1 y_0^2 z_1^2 \\
    -v_0^2 (x_1 w_1 - x_0 w_0) x_0 (x_1 + x_0) y_1^2 z_0^2 \\
    v_1^2 w_0^2 w_1 (w_1 + w_0) x_0 x_1 (x_1 + x_0) y_1^2 z_1^2 \\
    -v_0^2 (x_1 w_1 - x_0 w_0) x_0 x_1 y_0^2 z_0^2 \\
    v_0^2 (x_1 w_1 - x_0 w_0) x_1 (x_1 + x_0) y_1^2 z_1^2 \\
    -v_1^2 w_0^2 w_1 (w_1 + w_0) x_0 x_1 (x_1 + x_0) y_1^2 z_1^2 \\
    -v_0^2 v_1 w_0 w_1 (x_1 w_1 - x_0 w_0) x_0 x_1 y_0^2 z_1^2 \\
    v_0 v_1 w_0 (w_1 + w_0) (x_1 w_1 - x_0 w_0) x_1 (x_1 + x_0) y_1^2 z_1^2 \\
    -v_0 v_1 w_0 (w_1 + w_0) (x_1 w_1 - x_0 w_0) x_0 (x_1 + x_0) y_1^2 z_1^2
\end{pmatrix}.
$$

**Proof.** To check that $\varphi (\mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1} (2)) \times \mathbb{P}^1 \times \mathbb{P}^1) \subset V(J_{hyp})$ is an easy \texttt{Macaulay2} computation. To check that $\varphi$ is dominant it remains to check that the rank of the Jacobian matrix of $\varphi$ coincides with the dimension of the affine cone over $V_{hyp}$, which is 6. To prove that the map is birational it suffices to verify that the preimage of the image of a general point in the product, coincides with the point up to parts contained in the base loci, another easy \texttt{Macaulay2} computation which we documented in the file \texttt{verifyThmHypLocus}. \qed

Perhaps more interesting than these computations is the way we found the parametrization. We describe our approach next. The \texttt{Macaulay2} details can be obtained with our function \texttt{computeParametrizationOfHypLocus}.
From the length of the resolution we see that $J_{\text{hyp}}$ is (arithmetically) Cohen-Macaulay. The dualizing sheaf of $V_{\text{hyp}}$ has a linear presentation matrix

$$0 \leftarrow \omega_{\text{hyp}} \leftarrow \mathcal{O}_{\mathbb{P}^{11}}(-2) \leftarrow \mathcal{O}_{\mathbb{P}^{11}}(-3)$$

which is up to a twist the transpose $\psi$ of the last matrix in the free resolution of $J_{\text{hyp}}$. However, $\omega_{\text{hyp}}$ is not an invertible sheaf, since $V_{\text{hyp}}$ has non-Gorenstein singularities. The first step towards the computation of the parametrization is to compute the image under the rational map

$$V_{\text{hyp}} \rightarrow \mathbb{P}^{19}$$

defined by $|\omega_{\text{hyp}}(2)|$. The graph of this map is contained in the scheme defined by

$$(b_0 \ldots b_{19}) \psi$$

and the differentiation with respect to the $a_{ij}$'s gives a $12 \times 92$-presentation matrix

$$0 \leftarrow \mathcal{L} \leftarrow \mathcal{O}_{\mathbb{P}^{19}}^{12} \leftarrow \mathcal{O}_{\mathbb{P}^{19}}^{92}(-1).$$

The annihilator of $\mathcal{L}$ is an ideal $J_1$ of degree 180 and codimension 14 with Betti table

\[
\begin{array}{cccccc}
0 & 1 & 247 & 0 & 1 & 81 \\
\text{total:} & 1 & 247 & \text{total:} & 1 & 81 \\
0: & 1 & . & . & 0: & 1 & . \\
1: & . & 19 & 1: & . & 37 \\
2: & . & 228 & 2: & . & 44 \\
\end{array}
\]

is the Betti table of the ideal $J_{1s}$ of degree 168 and codimension 14 obtained by saturating $J_1$ with respect to $\prod_{j=0}^{19} b_j$.

**Remark 5.10.** On first glance, we were surprised that $J_1$ was not prime. The discovery of the other component is in principal possible via primary decomposition. We discovered them by saturating in $b_0$ by good luck. The residual part $J_{\text{residual}} = J_1 : J_{1s}$ decomposes into 6 components of degree 2. The explanation of the additional components is that these are contributions from the non-Gorenstein loci of $J_{\text{hyp}}$.

The next lucky discovery was that the linear strand of the resolution $J_{1s}$ has Betti table

\[
\begin{array}{cccccc}
0 & 1 & 2 & 3 & 4 & 5 \\
\text{total:} & 1 & 37 & 84 & 54 & 24 & 5 \\
0: & 1 & . & . & . & . & . \\
1: & . & 37 & 84 & 54 & 24 & 5 \\
\end{array}
\]

The cokernel of the last differential in the linear strand transposed and twisted, $\mathcal{O}_{\mathbb{P}^{19}}^5 \leftarrow \mathcal{O}_{\mathbb{P}^{19}}^{24}(-1)$, is supported on a rational normal scroll of degree 6. Indeed, $V_1 = V(J_{1s})$ is contained in a rational normal scroll which is the cone over $\mathbb{P}^5 \times \mathbb{P}^1 \subset \mathbb{P}^{11}$ with vertex $a \mathbb{P}^7 \subset \mathbb{P}^{19}$. Using a scrollar syzygy [vB07], we can compute the $2 \times 6$-matrix of linear forms defining the scroll in $\mathbb{P}^{11}$, and hence an isomorphism of the scroll with $\mathbb{P}^5 \times \mathbb{P}^1$. 

29
The projection of $V(J_{1s})$ from the vertex into $\mathbb{P}^{11}$ is defined by an ideal $J_2$ whose resolution has Betti numbers

\[
\begin{array}{cccccccc}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
\text{total:} & 1 & 19 & 55 & 97 & 99 & 56 & 20 & 3 \\
0: & 1 & . & . & . & . & . & . & . \\
1: & . & 19 & 52 & 45 & 24 & 5 & . & . \\
2: & . & . & 3 & 52 & 75 & 36 & 8 & . \\
3: & . & . & . & . & 15 & 12 & 3 & 1
\end{array}
\]

Using Cox coordinates $\mathbb{Q}[c_0, \ldots, c_5, w_0, w_1]$ we obtain that $V_2 = V(J_2)$ is a complete intersection of two quadric bundles of class $2H - R$ on $\mathbb{P}^5 \times \mathbb{P}^1$, where $H$ and $R$ denote the hyperplane class and the ruling in $\text{Pic}(\mathbb{P}^5 \times \mathbb{P}^1)$. Following [Sch86], the resolution of $J_2$ can be obtained from the exact sequence

\[
0 \leftarrow \mathcal{O}_{V(J_2)} \leftarrow \mathcal{O}_{\mathbb{P}^5 \times \mathbb{P}^1} \leftarrow \mathcal{O}_{\mathbb{P}^5 \times \mathbb{P}^1}(-2H + R) \leftarrow \mathcal{O}_{\mathbb{P}^5 \times \mathbb{P}^1}(-4H + 2R) \leftarrow 0
\]

via an iterated mapping cone

\[
[[\mathcal{C}^0 \leftarrow \mathcal{C}^1(-2) \oplus \mathcal{C}^1(-2)] \leftarrow \mathcal{C}^2(-4)]
\]

over Buchsbaum-Eisenbud complexes $\mathcal{C}^i$ associated to the $2 \times 6$-matrix defining the scroll with Betti tables

\[
\begin{array}{cccccccc}
0 & 1 & 2 & 3 & 4 & 5 & 1 & 2 & 3 & 4 & 5 & 6 & 2 & 3 & 4 & 5 & 6 & 7 \\
0: & 1 & . & . & . & . & 0: & . & . & . & . & . & 0: & . & . & . & . & . & . \\
1: & . & 15 & 40 & 45 & 24 & 5 + 2 & 1: & 2 & 6 & . & . & . & + 1: & . & . & . & . , \\
2: & . & . & . & . & . & 2: & . & 20 & 30 & 18 & 4 & 2: & 3 & 12 & 15 & . & . \\
3: & . & . & . & . & . & 3: & . & . & . & . & . & 3: & . & . & . & 15 & 12 & 3
\end{array}
\]

which gives the table above.

**Proposition 5.11.** $V_2$ is birational to $V'_2 = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(2)) \times \mathbb{P}^1 \times \mathbb{P}^1$.

**Proof.** By inspection we find that in terms of the Cox ring $\mathbb{Q}[c_0, \ldots, c_5, w_0, w_1]$ of $\mathbb{P}^5 \times \mathbb{P}^1$ the variety $V_2$ is defined by two relative rank 4 quadrics

\[
\det \begin{pmatrix}
    c_0 & c_2 \\
    -c_4 w_1 & c_5(w_0 - w_1)
\end{pmatrix}
\quad \text{and} \quad
\det \begin{pmatrix}
    c_1 & c_2 \\
    -c_4 w_1 & c_3 w_0
\end{pmatrix}.
\]

Regarding

\[
\begin{pmatrix}
    c_0 \\
    -c_4 w_1
\end{pmatrix}
\begin{pmatrix}
    c_2 \\
    c_5(w_0 - w_1)
\end{pmatrix}
= \begin{pmatrix}
    y_0 \\
    y_1
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
    z_0 \\
    z_1
\end{pmatrix} = 0
\]

as a linear system for $c_0, \ldots, c_5$ with coefficients in Cox coordinates $(w_0, w_1, y_0, y_1, z_0, z_1)$ of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ we find a $4 \times 6$-matrix $m$ with a rank 2 kernel

\[
\mathcal{O}(0, 1, 1)^2 \oplus \mathcal{O}(1, 1, 1)^2 \xleftarrow{m} \mathcal{O}^6 \xleftarrow{n} \mathcal{O}(0, -1, -1) \oplus \mathcal{O}(-2, -1, -1).
\]
Thus
\[ V'_2 = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(2)) \times \mathbb{P}^1 \times \mathbb{P}^1 \to V_2 \subset \mathbb{P}^5 \times \mathbb{P}^1 \subset \mathbb{P}^{11} \]
defined by
\[(v_0, v_1, w_0, w_1, y_0, y_1, z_0, z_1) \mapsto (v_0, v_1) n^t \otimes (w_0, w_1) \]
gives a rational parametrization of \( V_2 \). Note that the rational map is defined by forms of multidegree \((1, 3, 1, 1)\), where \(\deg v_0 = (1, 2, 0, 0)\), \(\deg v_1 = (1, 0, 0, 0)\) and \(\deg w_0 = \deg w_1 = (0, 1, 0, 0)\).

**Proposition 5.12.** \( V_1 \) is birational to a degree 4 rational normal curve fibration \( \tilde{V}'_1 \to V'_2 \) in a \( \mathbb{P}^4 \)-bundle \( \mathbb{P}(\mathcal{T}) \) over \( V'_2 = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(2)) \times \mathbb{P}^1 \times \mathbb{P}^1 \). The varieties \( V'_1 \) and \( V_{hyp} \) are birational to \( \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(2)) \times \mathbb{P}^1 \times \mathbb{P}^1 \).

**Proof.** Let \( \frac{\mathbb{P}^{19}}{\mathbb{P}_{11}} = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)) \to \mathbb{P}^{19} \to \mathbb{P}^{11} \) denote the blow-up of the projection center of \( \mathbb{P}^{19} \to \mathbb{P}^{11} \). Taking the pullback of this bundle along the rational map \( V'_2 = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(2)) \times \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^{11} \) from Proposition 5.11 we obtain the bundle \( \mathbb{P}(\mathcal{U}) \to V'_2 \) with
\[
\mathcal{U} = \mathcal{O}_{V_2} \oplus 8\mathcal{O}_{V_2}(1, 3, 1, 1).
\]
Let
\[
\psi' : \mathbb{Q}[b_0, \ldots, b_{19}] \to \mathbb{Q}[u_0, \ldots, u_8, v_0, v_1, w_0, w_1, y_0, y_1, z_0, z_1]
\]
be the corresponding ring homomorphism. We will solve the equations defined by \( J_3 = \psi'(J_{18}) \) in two steps. Saturating \( J_3 \) with respect to \( u_0 \) yields 12 equations which are linear in the \( u \)'s. Their solution space defines a \( \mathbb{P}^4 \)-bundle \( \mathbb{P}(\mathcal{T}) \to V'_2 \).

The saturation of the image \( J_4 \) of \( J_3 \) leads to an ideal defined by 6 relative quadrics, which we identify with the minors of a homogeneous \( 2 \times 4 \)-matrix. We compute the \( 2 \times 4 \)-matrix by using a relative scrollar syzygy [vb07]. Luckily, the Gröbner basis computation in Macaulay2 uses a relative scrollar syzygy as one of the basis elements of the third syzygy module. Introducing another \( \mathbb{P}^1 \) factor with coordinates \((x_0, x_1)\) leads to a parametrization
\[
V'_1 = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(2)) \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \to \tilde{V}_1 \subset \tilde{\mathbb{P}}^{19}.
\]
Finally, substituting this parametrization into the \( 12 \times 92 \) presentation matrix of \( \mathcal{L} \) yields a rank 11 matrix over the Cox ring of \( V'_1 \) and the syzygy of the transposed matrix yields the final parametrization of \( V_{hyp} \) from Theorem 5.9.
6 Torsion surfaces

Let $X$ be a numerical Godeaux surface with $\text{Tors } X = \mathbb{Z}/5\mathbb{Z}$ or $\text{Tors } X = \mathbb{Z}/3\mathbb{Z}$, and let $\tau_i, \tau_j \in \text{Pic}(X)$ be two different torsion elements with $\tau_i = -\tau_j$. As $h^0(K_X + \tau) = 1$ for any non-trivial torsion element $\tau$ in Pic$(X)$, we can choose effective divisors $D_i \in |K_X + \tau_i|$ and $D_j \in |K_X + \tau_j|$. Then,

$$C_{i,j} = D_i + D_j \in |2K_X|$$

and $D_i$ and $D_j$ intersect in exactly one point which is a base point of $|3K_X|$. Thus, for $T = \mathbb{Z}/3\mathbb{Z}$ we obtain a special reducible bicanonical fiber $D_1 + D_2$ with $\tau_2 = -\tau_1$, whereas for $T = \mathbb{Z}/5\mathbb{Z}$ we obtain two special reducible bicanonical fibers $D_1 + D_4$ and $D_2 + D_3$ with $\tau_4 = -\tau_1$ and $\tau_3 = -\tau_2$.

In the last case, Reid showed that for any $i \neq j$, $D_i$ and $D_j$ intersect in a unique point and that for three different $i, j, k$, any two points of intersection are distinct (cf. [Rei72], Lemma 0.1 and 0.2). Hence, denoting by $P_{i,j}$ the intersection points of $D_i$ and $D_j$, then the two bicanonical divisors $C_{1,4}$ and $C_{2,3}$ intersect in four distinct points $\{P_{1,2}, P_{1,3}, P_{2,4}, P_{3,4}\}$ which are exactly the base points of $|2K_X|$. In particular, $|2K_X|$ has no fixed components and four different base points.

Next, we study the restriction of the tricanonical map $\phi_3 : X \to \mathbb{P}^3$ to a divisor $C_{i,j} = D_i + D_j$. Working on the canonical model $X_{\text{can}}$ instead, we may assume that any $D_i$ is an irreducible curve with $p_a(D_i) = 2$. We have $|3K_X|_{D_i} = |K_{D_i} + D_j|_{D_i} = |K_{D_i} + Q|$ where $Q$ is a base point of $|3K_X|$. Thus, under the birational map $\phi_3$, the curve $D_i$ is mapped $2 : 1$ to a line.

6.1 $\mathbb{Z}/5\mathbb{Z}$-surfaces

These surfaces are completely classified:

**Theorem 6.1** (see [Miy76] or [Rei72]). *Numerical Godeaux surfaces with torsion group $\mathbb{Z}/5\mathbb{Z}$ form a unirational irreducible family of dimension 8.*

Their proof shows that any $G = \mathbb{Z}/5\mathbb{Z}$ Godeaux surface arises as a quotient $Q_5/G$ of a quintic as follows: Let $\xi$ be a primitive fifth root of unity. The group $G$ acts on $\mathbb{P}^3$ via

$$\beta : (u_1 : u_2 : u_3 : u_4) \mapsto (\xi u_1, \xi^2 u_2, \xi^3 u_3, \xi^4 u_4).$$

Consider the $G$-invariant family of quintic forms

$$f = f_c = u_1^5 + u_2^5 + u_3^5 + u_4^5 + c_0 u_1^3 u_3 u_4 + c_1 u_1 u_2^3 u_3 + c_2 u_2 u_3^3 u_4 + c_3 u_1 u_2 u_4^3 + c_4 u_1^2 u_2^2 u_4 + c_5 u_2^2 u_3^2 u_4 + c_6 u_2 u_3 u_4^2 + c_7 u_1 u_3^2 u_4^2$$

with $c = (c_0, \ldots, c_7) \in \mathbb{A}^8$. The coefficients of $u_i^5$ are chosen to be 1 to guarantee that $Q_5 = V(f)$ does not meet the four fixed points of the action. The quotient $Q_5/G$ is a Godeaux surface if the parameter $c \in \mathbb{A}^8$ is chosen such that $Q_5$ has at most rational double points as singularities.
In [Ste18], Section 9.1, a minimal set of algebra generators for $R(X) = R(Q_5)^G$ with $Q_5$ and $G$ as above is presented. Moreover, using this set of generators, we determined a choice for the first syzygy matrix $d_1$ of $R(X)$ depending on $c \in \mathbb{A}^8$.

In this section, we reconstruct the family of $\mathbb{Z}/5\mathbb{Z}$ surfaces using our approach. We deduce computationally that the locus of possible lines in $Q$ for marked numerical Godeaux surfaces with $Tors = \mathbb{Z}/5\mathbb{Z}$ is isomorphic to a $S_3$-orbit of six $\mathbb{P} \times \mathbb{P}$'s.

First, in Theorem 5.2 and Proposition 5.3 we have seen that for constructing a $\mathbb{Z}/5\mathbb{Z}$-Godeaux surface we have to choose a line in $Q$ intersecting two $\mathbb{P}^3$'s in $V(I_3(e)) \cap Q$ in exactly one point. So we choose two different $\mathbb{P}^3$'s in this locus and evaluate the condition that a line through two general points is completely contained in the variety $Q$ (see our procedure lineConditionsTorsZ5). The resulting zero loci $W \subset \mathbb{P} \times \mathbb{P}$ decomposes into a union of several surfaces of type $\mathbb{P} \times \mathbb{P} \subset \mathbb{P} \times \mathbb{P}$ and $\mathbb{P} \times \mathbb{P} \subset \mathbb{P} \times \mathbb{P}$ or $\mathbb{P} \times \mathbb{P} \subset \mathbb{P} \times \mathbb{P}$. We verify computationally that there are two components of $W$, both isomorphic to a $\mathbb{P} \times \mathbb{P}$, which give lines leading generically to numerical Godeaux surfaces with torsion group $\mathbb{Z}/5\mathbb{Z}$. We display the results in a table, using the following notation:

- $n$ = number of components of $W$ of a given type,
- $f$ = dimension of the family of lines,
- $s$ = projective dimension of the linear solution space in the second step,
- $t$ = total dimension of the constructed family of varieties,
- $J$ = generating ideal of the model in $\mathbb{P} \times \mathbb{P}$,
- $R.C.$ = the ring condition introduced in Remark 2.8.

| $n$ | family of lines | $f$ | $s$ | $t$ | $J$ of a given bidegree | comments |
|-----|----------------|-----|-----|-----|--------------------------|----------|
| 2   | $\mathbb{P} \times \mathbb{P}$ | 2   | 9   | 11  | $\{0, 7\} \Rightarrow 1$ | a general $\mathbb{Z}/5\mathbb{Z}$-Godeaux surface |
|     |                | true|      |     | $\{1, 2\} \Rightarrow 1$ |          |
|     |                |     |      |     | $\{1, 5\} \Rightarrow 1$ |          |
|     |                |     |      |     | $\{2, 3\} \Rightarrow 1$ |          |
| 4   | $\mathbb{P} \times \mathbb{P}$ | 2   | 16  | 18  | $\{0, 2\} \Rightarrow 3$ | $J$ defines a surface in $\mathbb{P} \times \mathbb{P}^3$ which is the union of three $\mathbb{P} \times \mathbb{P}$'s |
| 4   | $\mathbb{P} \times \mathbb{P}$ | 2   | 16  | 18  | $\{0, 2\} \Rightarrow 3$ | $J$ defines a surface in $\mathbb{P} \times \mathbb{P}^3$ which is the union of three $\mathbb{P} \times \mathbb{P}$'s |
| 4   | $\mathbb{P} \times \mathbb{P}$ | 2   | 16  | 18  | $\{0, 2\} \Rightarrow 3$ | $J$ defines a surface in $\mathbb{P} \times \mathbb{P}^3$ which is the union of three $\mathbb{P} \times \mathbb{P}$'s |

Figure 1: $\mathbb{Z}/5\mathbb{Z}$-surfaces
Remark 6.2. Note that for a Godeaux surface \( X \) with \( \text{Tors} X = \mathbb{Z}/5\mathbb{Z} \), the image under the tricanonical map is a hypersurface of degree 7 in \( \mathbb{P}^3 \). Hence the ideal of the image of \( X \) under the product of the bi- and tricanonical map to \( \mathbb{P}^1 \times \mathbb{P}^3 \) must contain a form of bidegree \((0, 7)\).

Example 6.3. For example, lines in \( Q \subset \mathbb{P}^{11} \) given in Stiefel coordinates

\[
\begin{pmatrix}
0 & 0 & 0 & p_0 & 0 & 0 & 0 & 0 & p_1 & 0 & 0 & 0 \\
0 & q_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & q_1 & 0 \\
\end{pmatrix},
\]

where \((p_0, p_1), (q_0, q_1)\) \(\in\) \( \mathbb{P}^1 \times \mathbb{P}^1 \), lead to \( \mathbb{Z}/5\mathbb{Z} \)-Godeaux surfaces whose two reducible fibers \( C_{1,4} \) and \( C_{2,3} \) are mapped to the union of lines \( V(y_0, y_1) \cup V(y_2, y_3) \) and \( V(y_0, y_2) \cup V(y_1, y_3) \).

Corollary 6.4. There is subscheme of the Fano variety \( F_1(Q) \) of lines in \( Q \) isomorphic to the union of \( 6 = 2 \binom{3}{2} \) surfaces of type \( \mathbb{P}^1 \times \mathbb{P}^1 \) whose elements lead generically to numerical Godeaux surface with \( \text{Tors} = \mathbb{Z}/5\mathbb{Z} \). Moreover, there exists a \( 2+9 \)-dimensional unirational family in the unfolding parameter space for marked Godeaux surfaces, which modulo the \((\mathbb{C}^*)^3\)-action gives a \( 8 \)-dimensional family of Godeaux surfaces with \( \text{Tors} = \mathbb{Z}/5\mathbb{Z} \).

6.2 \( \mathbb{Z}/3\mathbb{Z} \)-surfaces

As in the case of \( \text{Tors} = \mathbb{Z}/5\mathbb{Z} \), numerical Godeaux surfaces with \( \text{Tors} = \mathbb{Z}/3\mathbb{Z} \) are completely classified:

Theorem 6.5 (see [Rei72]). Numerical Godeaux surfaces with \( \text{Tors} = \mathbb{Z}/3\mathbb{Z} \) form an irreducible, unirational \( 8 \)-dimensional component of the moduli space of Godeaux surfaces.

As in the previous section, the idea of the construction is to start with a covering \( Y \rightarrow X \), where \( Y \) is a surface of general type with \( K^2 = 3, p_g = 2 \) and \( q = 0 \) on which the group \( G = \mathbb{Z}/3\mathbb{Z} \) acts freely. Reid completely describes those covering surfaces in [Rei72] and gives a refined description for the equations of \( Y \) using unprojection in [Rei13].

In [Ste18], using the unprojection method from [Rei13], we constructed a general cover surface \( Y \) and the canonical ring \( R(X) = R(Y)^G \) and verified that there are four distinct base points of \( |2K_X| \) and that the canonical model \( X_{\text{can}} \) is smooth at these base points. Hence, we may assume that a general \( \mathbb{Z}/3\mathbb{Z} \)-Godeaux surface admits a marking.

As in the previous section, we want to parametrize lines in \( Q \) leading to \( \mathbb{Z}/3\mathbb{Z} \)-Godeaux surfaces. Recall that the tricanonical system of a Godeaux surface \( X \) with \( \text{Tors} = \mathbb{Z}/3\mathbb{Z} \) has a single base point which is contained in a unique bicanonical fiber \( C_{1,2} = D_1 + D_2 \). Thus, by Theorem 5.2 and Proposition 5.3, we have to construct lines in \( Q \) meeting a unique \( \mathbb{P}^3 \subset V(I_3(e)) \cap Q \). We first choose a general point \( p \) in such a \( \mathbb{P}^3 \) and then a second point in the cone

\[ Z = T_p Q \cap Q \subset \mathbb{P}^{11}. \]
In each irreducible component $Z_i$ of $Z$ we choose a point $q$ and examine the surface constructed from the line $\ell = \overline{pq}$. Again we present the computational results in a table, using

$$Z_i = \text{irreducible component of } Z = T_p(Q) \cap Q \subset \mathbb{P}^{11}$$

and $f, s, t$ and $J$ are defined as in Table (1).

| $\text{betti } Z_i$ | $f$ | $s$ | $t$ | $R.C.$ | $\# \text{ of gen. of } J$ of a given bidegree | comments |
|---------------------|-----|-----|-----|--------|---------------------------------|----------|
| $0$ | $1$ | $6$ | $5$ | $11$ | true | $\{0, 8\} \Rightarrow 1$ | a general $\mathbb{Z}/3\mathbb{Z}$ Godeaux surface |
| $0$ | $1$ | $3$ | $6$ | $11$ | true | $\{1, 3\} \Rightarrow 1$ | $\{1, 5\} \Rightarrow 1$ | $\{2, 2\} \Rightarrow 1$ |
| $0$ | $1$ | $8$ | $5$ | $11$ | true | $\{0, 2\} \Rightarrow 1$ | $\{3, 4\} \Rightarrow 2$ | $\{4, 1\} \Rightarrow 1$ |
| $0$ | $1$ | $8$ | $12$ | $17$ | false | $\{0, 2\} \Rightarrow 2$ | $\{1, 2\} \Rightarrow 1$ |

Figure 2: $\mathbb{Z}/3\mathbb{Z}$-surfaces

Remark 6.6. Only the first example leads to a numerical Godeaux surface $X$ with $\text{Tors } X = \mathbb{Z}/3\mathbb{Z}$. Note that the image of $X$ under the tricanonical map is a hypersurface of degree 8 in $\mathbb{P}^3$. Hence the ideal of the image of $X$ under the product of the bi- and tricanonical map to $\mathbb{P}^1 \times \mathbb{P}^3$ must contain a form of bidegree $(0, 8)$.

Analyzing the component $Z_1 \subset T_p(Q)$ from the case 1.1) we see that it is cone with vertex $p$ over a singular surface in $\mathbb{P}^4$ with minimal free resolution

$$
\begin{array}{ccc}
0 & 1 & 2 \\
\text{total}: & 1 & 3 & 2 \\
0: & 1 & . & . \\
1: & . & 3 & 2 \\
\end{array}
$$

Using this observation, we are able to rationally parametrize lines for the case 1.1.):
Theorem 6.7. There exists a unirational 5-dimensional subscheme of $F_1(Q)$ whose elements lead generically to numerical Godeaux surfaces with $\text{Tors} X = \mathbb{Z}/3\mathbb{Z}$ parameterized by

$$\varphi: \mathbb{P}(\mathcal{O}_{\mathbb{P}^3 \times \mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^3 \times \mathbb{P}^1}(2, 3)) \longrightarrow F_1(Q)$$

\[
\begin{pmatrix}
  u_0 \\
  0 \\
  0 \\
  u_1 \\
  0 \\
  0 \\
  u_2 \\
  u_3
\end{pmatrix}
\leftrightarrow
\begin{pmatrix}
  -u_0^2 w_1 u_1^3 z_1 \\
  -u_1 u_2 w_0^2 w_1 z_1 \\
  u_1 u_2 w_0^2 w_1 z_1 \\
  0 \\
  -u_0 u_3^2 w_0^2 w_1 z_1 \\
  u_0 u_3^2 w_0^2 w_1 z_1 \\
  u_3^2 u_0 w_0^2 w_1 z_1 \\
  -u_3^2 u_0 w_0^2 w_1 z_1 \\
  -u_2 z_0 + u_2^2 u_3 w_0^2 z_1 \\
  u_1^2 u_2 w_0^2 w_1 z_1 \\
  0 \\
  -u_0^2 u_2 w_0^2 w_1 z_1 \\
  u_0^2 u_2 w_0^2 w_1 z_1 \\
  -u_3 z_0
\end{pmatrix}
\]

Moreover, there exists a $5 + 6$-dimensional unirational family in the unfolding parameter space for marked Godeaux surfaces, which modulo the $(\mathbb{C}^*)^3$-action gives a 8-dimensional family of Godeaux surfaces with Torsion group $T = \mathbb{Z}/3\mathbb{Z}$.

**Proof.** First, we choose one of the component in $V(I_3(e)) \cap Q$ isomorphic to a $\mathbb{P}^3$, for example

$$V = V(a_{1,3}^{(3)}, a_{0,3}^{(3)}, a_{1,2}^{(2)}, a_{0,2}^{(2)}, a_{1,3}^{(1)}, a_{1,2}^{(1)}, a_{0,3}^{(0)}, a_{0,2}^{(0)}) \subset Q.$$ 

Let $u_0, \ldots, u_3$ be new homogeneous coordinates, then a general point $p$ in $V$ is given by

$$p = \left( u_0, 0, 0, u_1, 0, 0, 0, 0, u_2, 0, 0, u_3 \right)^t$$

and the tangent space is isomorphic to a $\mathbb{P}^7$. Computing the equations, we obtain a $\mathbb{P}^7$-bundle

$$\mathbb{P}(\mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3}(1)^4)$$

with homogeneous coordinates $v_0, \ldots, v_7$. Next, we consider the restriction of $Q$ to this $\mathbb{P}^7$-bundle and choose the irreducible component $Z_1$ of $Q$ specializing to the component $Z_1$ described above. Among the defining equations $Z_1$, there are two determinantal equations given by

$$\begin{pmatrix}
  u_0^2 & v_7 \\
  u_1^2 & -v_6
\end{pmatrix} \text{ and } \begin{pmatrix}
  u_2^2 & v_5 \\
  u_3^2 & -v_4
\end{pmatrix}.$$

Using the substitution

$$v_7 = v_{6,7} u_0^2, \quad v_6 = -v_{6,7} u_1^2, \quad v_5 = v_{4,5} u_2^2, \quad v_6 = -v_{4,5} u_3^2$$

with new coordinates in $v_{0,7}$ and $v_{4,5}$, we obtain a new variety $Z_1'$ in a $\mathbb{P}^1$-bundle over $\mathbb{P}(\mathcal{O}_{\mathbb{P}^3}^4)$ defined by 3 equations whose minimal free resolution is given by the Hilbert-Burch matrix

$$m = \begin{pmatrix}
  -u_1 v_0 + u_0 v_1 & -u_0^2 u_1^2 v_0^2 \\
  u_0^2 u_1^2 v_4^2 & u_2^2 u_3^2 v_{45} \\
  v_6^7 & u_3 v_2 - u_2 v_3
\end{pmatrix}.$$
Now regarding
\[ m \left( \begin{array}{c} w_0 \\ w_1 \end{array} \right) = 0 \]
as a linear system for \( v_0, \ldots, v_3, v_{4,5}, v_{6,7} \), with coefficients depending on the in the coordinates \((u_0, \ldots, u_3, w_0, w_1)\) of \( \mathbb{P}^3 \times \mathbb{P}^1 \) we obtain a \( 3 \times 6 \) matrix \( m \) with a rank 3 kernel
\[ \mathcal{O}(2, 1)^2 \oplus \mathcal{O}(-2, 1) \leftarrow m \mathcal{O}(1, 0)^4 \oplus \mathcal{O}(-2, 0)^2 \leftarrow n \mathcal{O}^2 \oplus \mathcal{O}(-2, -3) \]
and
\[ n = \begin{pmatrix} u_0 & u_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & u_2 & u_3 & 0 & 0 \\ 0 & u_0 u_1^2 w_1^3 & 0 & -u_2 u_3^2 w_0^3 & w_0 w_1^2 & -w_0^2 w_1 \end{pmatrix}^t. \]
As the sum of the first and second line gives the point \( p \) (in the corresponding coordinates) and we are looking for a second point \( q \in T_pQ \) with \( q \neq p \), we obtain only 2-dimensional solution spaces for \( v_0, \ldots, v_3, v_{4,5}, v_{6,7} \). We choose the space spanned by the last two columns of \( n \) which has the right dimension for general points \((u, w) \in \mathbb{P}^3 \times \mathbb{P}^1\). Thus, we obtain a birational map
\[ \mathbb{P}(\mathcal{O}_{\mathbb{P}^3 \times \mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^3 \times \mathbb{P}^1}(2, 3)) \twoheadrightarrow Z'_1 \]
\[ (z_0, z_1, u_0, \ldots, u_3, w_0, w_1) \mapsto (z_0, z_1)n^t. \]
Reversing the single substitution steps, we obtain the desired parametrization for the second point \( q \) and hence for a line \( \ell = \overline{pq} \subset Q \). Moreover, we verify that for a general point \((u, w, z)\) the line specializes to one chosen in the case 1.1) above which leads to a numerical Godeaux surfaces with \( \text{Tors} = \mathbb{Z}/3\mathbb{Z}. \)

\[ \square \]

7 Special surfaces and ghost components

Recall that any line in \( Q \) meeting none of the special loci described in Section 4 leads to a member of our constructed dominant family, that means to a torsion-free marked numerical Godeaux surface with no hyperelliptic fiber. Marked numerical Godeaux surfaces with non-trivial torsion group form an irreducible family and their associated lines in \( Q \) were described in Section 6.1 and Section 6.2. Thus, the question remains whether there exists a different family of torsion-free marked numerical Godeaux surfaces. The associated lines of such a family must necessarily meet one of the special loci.

7.1 Lines meeting one special locus

In the following, we represent our computational experiments with Macaulay2 for lines in \( Q \) meeting one of the special loci. We work over the finite field \( \mathbb{F}_{32233} \) and proceed as follows. First we choose for each type of the the special loci a representative and compute a general point \( p \) in this component. Afterwards, we determine the intersection
\[ Z = T_p(Q) \cap Q \subset T_p(Q) \subset \mathbb{P}^{11}, \]

37
where $Z$ is a cone with vertex $p$. Except for points in the hyperelliptic locus, we can decompose $Z$ into its irreducible components. For general points in the hyperelliptic locus we believe that $Z$ is an irreducible variety. Next we choose a general point $q$ in each of the irreducible components of $Z$ and obtain a line $\ell = \overline{pq} \subset Q$. Finally, we compute the solution space and determine the dimension of the constructed family of varieties. In the end, we verify whether the resulting module $R = \text{coker} \ d_1$ satisfies the ring condition ($R.C.$) from Remark 2.8 and compute the bihomogeneous model of the variety $V(\text{ann} \ coker \ d_1)$ in $\mathbb{P}^1 \times \mathbb{P}^3$.

Recall that if $R$ is the canonical ring of a numerical Godeaux surface $X$, then the ideal of the model in $\mathbb{P}^1 \times \mathbb{P}^3$ must contain a form of bidegree $(0, 9 - \# \text{ of base points of } 3K_X)$, hence of bidegree $(0, 9), (0, 8)$ or $(0, 7)$.

We represent the results in a table denoting by $W$ the chosen special locus and using the notation $f, s, t, J$ and $Z_i$ as in Table (1). Note that if $Z$ has several irreducible components leading to families with the same values for $f, s$ and $t$ and the same type of irreducible components of the model in $\mathbb{P}^1 \times \mathbb{P}^3$, we state the number of those components and display only the result for one such component.

| betti $W$ | occurrence of $W$ | betti $Z_i$ | $f$ | $s$ | $t$ | $R.C.$ | # of gen. of $J$ of a given bidegree | comments |
|---------|-----------------|-----------|----|----|----|--------|--------------------------------------|----------|
| 1)      | 0: 1 6          | 0: 1 6    | 7  | 7  | 14 | false  | $\{0, 2\} \Rightarrow 2$            | $J$ defines a surface in $\mathbb{P}^1 \times \mathbb{P}^3$ which decomposes into a $\mathbb{P}^1 \times \mathbb{P}^1$ and a $(4, 4)$-hypersurface in $\mathbb{P}^1 \times \mathbb{P}^2$. |
|         | 1: . 1          | 1: . 1    | 3  |    |    |        |                                       | contained in the closure of the dominant component |
| 2.1)    | 0: 1 5          | 0: 1 4    | 7  | 3  | 10 | true   | $\{0, 9\} \Rightarrow 1$            | as in 1) |
|         | 1: . 1          | 1: . 7    |    |    |    |        |                                       |          |
|         | 5               |           |    |    |    |        |                                       |          |
| 2.2)    | 0: 1 6          | 0: 1 6    | 7  | 7  | 14 | false  | $\{0, 2\} \Rightarrow 2$            | as in 1) |
|         | 1: . 1          | 1: . 1    |    |    |    |        |                                       |          |
| 2.1 | 0 1 0: 1 4 1: . 2 support of $H_1(C_2)$ | 0 1 0: 1 4 1: . 5 1 | 7 true | \{0, 9\} \Rightarrow 1 \{1, 6\} \Rightarrow 2 \{1, 6\} \Rightarrow 2 \{7, 2\} \Rightarrow 1 | a torsion-free numerical Godeaux surface with no hyp. fibers |
|-----|---------------------------------|------------------|--------|-------------------------------------------------|
| 2.2 | 0 1 0: 1 8 1 2 6 4 true | \{0, 2\} \Rightarrow 1 \{3, 4\} \Rightarrow 2 \{5, 3\} \Rightarrow 1 | \{0, 2\} \Rightarrow 1 \{3, 4\} \Rightarrow 2 \{5, 3\} \Rightarrow 1 | surface in $\mathbb{P}^1 \times \mathbb{P}^3$ decomposes into a union of two surfaces which are \((3, 3)\) hypersurfaces in $\mathbb{P}^1 \times \mathbb{P}^2$ |
| 2.3 | 0 1 0: 1 8 2 6 4 true | \{0, 2\} \Rightarrow 1 \{3, 4\} \Rightarrow 2 \{5, 3\} \Rightarrow 1 | \{0, 2\} \Rightarrow 1 \{3, 4\} \Rightarrow 2 \{5, 3\} \Rightarrow 1 | as in 2.2 |
| 2.4 | 0 1 0: 1 8 2 6 3 9 true | \{0, 2\} \Rightarrow 1 \{3, 4\} \Rightarrow 2 \{5, 3\} \Rightarrow 1 | \{0, 2\} \Rightarrow 1 \{3, 4\} \Rightarrow 2 \{5, 3\} \Rightarrow 1 | as in 2.2 |
| 4) | 0 1 0: 1 3 1: . 4 2: . 1 support of $H_0(C_1)$, contained in the hyperelliptic locus | 0 1 0: 1 4 1: . 4 1 6 3 9 true | \{0, 9\} \Rightarrow 1 \{1, 6\} \Rightarrow 5 \{1, 6\} \Rightarrow 5 \{5, 2\} \Rightarrow 1 | torsion-free numerical Godeaux surface with one hyperelliptic fiber |
| 5) | 0 1 0: 1 1: . 4 2: . 4 5: . 4 support of $H_0(C_2)$, the hyperelliptic locus | 0 1 0: 1 4 1: . 4 1 7 true | \{0, 9\} \Rightarrow 1 \{1, 6\} \Rightarrow 5 \{1, 6\} \Rightarrow 5 \{5, 2\} \Rightarrow 1 | torsion-free numerical Godeaux surface with one hyperelliptic fiber |

Remark 7.1. Note that lines meeting the components in the vanishing locus of the $3 \times 3$ minors of $e$ with Betti table

```
0 1
total: 1 8
0: 1 8
```

have already been studied in Section 6.2.
7.2 Lines meeting two special loci

In this section, we present our computational experiments for lines meeting at least two special loci. We work over the finite field $\mathbb{F}_{32233}$. Lines meeting components of the vanishing locus of the $3 \times 3$ minors of the $e$-matrix have already been studied in the previous sections. They either lead to marked numerical Godeaux with a non-trivial torsion group or to surfaces which are no numerical Godeaux surfaces (Theorem 5.2 and Proposition 5.3). Thus, we exclude lines meeting any of these components from our study.

Note that meeting two special loci in $\mathbb{Q}$ induces at least a codimension 1 condition. Thus, being interested in further components or ghost components of our construction space, we will only display results, where the projective dimension of the linear solution space in the second step is at least 4. Let $W_1$ and $W_2$ denote the two chosen special loci in $\mathbb{Q}$. Then we choose first a general point $p \in W_1 \subset \mathbb{Q}$. Afterwards we compute (if possible) the irreducible components $Z_i$ of $W_2 \cap T_p(\mathbb{Q}) \subset \mathbb{Q} \subset \mathbb{P}^{11}$ and choose in each component $Z_i$ a general point $q$. The line $\ell = pq$ intersects the two chosen loci in at least one point each. Note that in several cases the resulting line $\ell$ intersects also other loci non-trivially or may also be contained completely in some of the loci. Thus, even for the same choice of pair $(W_1, W_2)$ we can obtain linear solution spaces of different dimensions depending on the choice of the component $Z_i$.

| betti $W_1$ | betti $W_2$ | $s$ | R.C. | # of gen. of $J$ of a given bidegree | comments |
|------------|-------------|-----|------|----------------------------------|----------|
| 1.)        |             |     |      |                                  |          |
| 0 1        | 0 1         | 4   | false| $\{0, 2\} \Rightarrow 1$        |          |
| 0: 1 3     | 0: 1 4      |     |      | $\{1, 2\} \Rightarrow 1$        |          |
| 1: . 4     | 1: . 2      |     |      |                                  |          |
| 2: . 1     |             |     |      |                                  |          |

$J$ defines a surface in $\mathbb{P}^1 \times \mathbb{P}^3$ which decomposes into a $\mathbb{P}^1 \times \mathbb{P}^1$ and a non-complete intersection surface in $\mathbb{P}^1 \times \mathbb{P}^3$.

| 2.1)       |             |     |      |                                  |          |
| 0 1        | 0 1         | 4   | true | $\{0, 2\} \Rightarrow 1$        |          |
| 0: 1 4     | 0: 1 4      |     |      | $\{3, 4\} \Rightarrow 2$        |          |
| 1: . 2     | 1: . 2      |     |      | $\{5, 3\} \Rightarrow 1$        |          |
| 2.2)       |             |     |      |                                  |          |
| 0 1        | 0 1         | 4   | true | as in 2.1)                       | as in 2.1) |
| 0: 1 4     | 0: 1 5      |     |      |                                  |          |
| 1: . 2     | 1: . 1      |     |      |                                  |          |
| 2.3)       |             |     |      |                                  |          |
| 0 1        | 0 1         | 4   | true | $\{0, 2\} \Rightarrow 1$        |          |
| 0: 1 4     | 0: 1 4      |     |      | $\{3, 4\} \Rightarrow 2$        |          |
| 1: . 2     | 1: . 2      |     |      | $\{4, 3\} \Rightarrow 1$        |          |

40
Depending on the dimension of the family of lines (and the group operation of $\mathbb{C}^3$), the cases presented above may lead to new ghost components.

For the sake of completeness, we present the results for lines meeting the hyperelliptic locus in two different points. Recall that the hyperelliptic locus $V_{\text{hyp}}$ is a 5-dimensional subscheme of $\mathcal{Q}$. For a point $p \in V_{\text{hyp}}$, the intersection $V_{\text{hyp}} \cap T_p(\mathcal{Q})$ is a curve through $p$ of degree 72 in $T_p(\mathcal{Q}) \cong \mathbb{P}^7$ (a general point in the hyperelliptic locus is a smooth point of $\mathcal{Q}$). Hence, we have 1-dimensional choice for the second point $q$, and thus in total a 6-dimensional family of lines. Furthermore, we verified that $\mathbb{C}^3$ operates with a trivial stabilizer on the general line in this family. We obtain a 6-dimensional family of torsion-free numerical Godeaux surfaces with two hyperelliptic fibers with the following numerical data:

\[
\begin{array}{c|c|c|c|c|c|c|c}
3.) & 0 & 1 & 0 & 1 & \{0, 9\} \Rightarrow 1 & \{1, 4\} \Rightarrow 1 & \text{a numerical Godeaux surface with trivial torsion group and two hyperelliptic fibers} \\
& 0: & 1 & 0: & 1 & & & \\
1: & . & 4 & 1: & . & 4 & 3 & \text{true} \\
2: & . & 4 & 2: & . & 4 & & \\
5: & . & 4 & 5: & . & 4 & & \\
\end{array}
\]

**Remark 7.2.** It is known that the Barlow surfaces have two hyperelliptic bicanonical fibers. Thus the 2-dimensional locus of Barlow lines is a sublocus of our 6-dimensional family.

**Remark 7.3.** Note that in the examples of torsion-free marked numerical Godeaux surfaces presented in this section, the ideal $J$ of the bi-tri-canonical model in $\mathbb{P}^1 \times \mathbb{P}^3$ contains a form of bidegree $(7 - 2h, 2)$, where $h$ is the number of hyperelliptic bicanonical fibers. Using a different approach, this computational observation is proven in [CP00], Theorem 2.5.

**Summary**

In this paper we found three 8-dimensional families of numerical Godeaux surfaces whose bicanonical system on the canonical model has 4 distinct base points. Whether there are further surfaces of this kind stays open. If such surfaces exist, then they have trivial torsion group. The corresponding lines $\ell \subset \mathcal{Q} \subset \mathbb{P}^{11}$ have to intersect some of the special loci of Section 4. Experimentally over finite fields we investigated surfaces which arise from a general point on one or two of these loci. Several ghost components of such surfaces were found. However, all of the surfaces which we discovered, and which do not lie in the closure of the dominant component, are reducible and mapped to a quadric in $\mathbb{P}^3$. If this is always true then there are no further numerical Godeaux surfaces whose bicanonical system has no fixed part and four distinct base points.

The results on Godeaux surfaces where the base locus of $|2K_{\mathcal{X}_{\text{can}}}|$ is non-reduced will be published in a further paper. In particular, that paper will contain a description of an 8-dimensional unirational locally complete family of Godeaux surfaces with $\text{Tors} = \mathbb{Z}/2\mathbb{Z}$.

Whether our approach will lead to a complete classification of (marked) numerical Godeaux surfaces depends on whether we will be able to exclude the existence of (even more) special lines, which lead to smooth surfaces. In those cases the number of choices...
in the second step has to be even larger than in the corresponding case observed so far experimentally in Section 7.

References

[Bar85] R. Barlow, A simply connected surface of general type with \( p_g = 0 \). *Inventiones mathematicae*, 79:293–301, 1985.

[BvBK12] C. Böhning, H.-C. Graf von Bothmer, L. Katzarkov, and P. Sosna, Determinantal Barlow surfaces and phantom categories. *J. Eur. Math. Soc.*, 17:1569–1592, 2015.

[Bom73] E. Bombieri, Canonical models of surfaces of general type. *Publications Mathématiques de l’IHÉS*, 42:171–219, 1973.

[vB07] H.-C. Graf von Bothmer, Scrollar syzygies of general canonical curves with genus \( \leq 6 \). *Transactions of the AMS* 359, 465–488, 2007.

[BE77] D. A. Buchsbaum and D. Eisenbud, Algebra structures for finite free resolutions and some structure theorems for ideals of codimension 3. *Amer. J. Math.*, 3:447–485, 1977.

[CFHR99] F. Catanese, M. Franciosi, K. Hulek, and M. Reid, Embeddings of curves and surfaces. *Nagoya Mathematical Journal*, 154:185–220, 1999.

[Cil83] C. Ciliberto, Sul grado dei generatori dell’anello canonico di una superficie di tipo generale. *Rend. Sem. Mat. Univ. Pol. Torino*, 41:83–111, 1983.

[Cat81] F. Catanese, Babbage’s conjecture, contact of surfaces, symmetric determinantal varieties and applications. *Inventiones mathematicae*, 63:433–465, 1981.

[CP00] F. Catanese and R. Pignatelli, On simply connected Godeaux surfaces. *Complex Analysis and Algebraic Geometry*, de Gruyter, Berlin, 117–153, 2000.

[Eis05] D. Eisenbud, *The Geometry of Syzygies*. Springer-Verlag, New York, 2005.

[GS] D. R. Grayson, M. E. Stillman, Macaulay2, a software system for research in algebraic geometry. Available at [http://www.math.uiuc.edu/Macaulay2/](http://www.math.uiuc.edu/Macaulay2/)

[Lee01] Y. Lee, Bicanonical pencil of a determinantal Barlow surface. *Transactions of the American Mathematical Society*, 353:893–905, 2001.

[Miy76] Y. Miyaoka, Tricanonical Maps of Numerical Godeaux Surfaces. *Inventiones mathematicae*, 34:99–112, 1976.

[Rei72] M. Reid, Surfaces with \( p_g = 0, K^2 = 1 \). *Sci. Univ. Tokyo Sect. IA* 25, 75–92, 1978.
[Rei13] M. Reid, Parallel unprojection equations for $\mathbb{Z}/3$-Godeaux surfaces. Available from http://homepages.warwick.ac.uk/masda/codim4/God3.pdf, 2013

[Sch86] F.-O. Schreyer, Syzygies of canonical curves and special linear series. Math. Ann. 275: 105–137, 1986.

[Sch05] F.-O. Schreyer, An Experimental Approach to Numerical Godeaux Surfaces. Oberwolfach Report "Komplexe Algebraische Geometrie", 7:434–436, 2005.

[SS20] F.-O. Schreyer, I. Stenger, NumericalGodeaux, Macaulay2-package for the construction of numerical Godeaux surfaces. Available at www.math.uni-sb.de/ag/schreyer/index.php/computeralgebra, 2020.

[Ste18] I. Stenger, A Homological Approach to Numerical Godeaux Surfaces, PhD thesis, 2018, Technische Universität Kaiserslautern.

[Ste19] I. Stenger, A structure result for Gorenstein algebras of odd codimension, arXiv.1910.00516, 2019.

Author Addresses:
Frank-Olaf Schreyer
Mathematik und Informatik, Universität des Saarlandes, Campus E2 4, D-66123 Saarbrücken, Germany.
schreyer@math.uni-sb.de

Isabel Stenger
Mathematik und Informatik, Universität des Saarlandes, Campus E2 4, D-66123 Saarbrücken, Germany.
stenger@math.uni-sb.de