A General Approach To
Photon Radiation Off Fermions

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Abstract:
Soft or collinear photon emission potentially poses numerical problems in the phase-space integration of radiative processes. In this paper, a general subtraction formalism is presented that removes such singularities from the integrand of the numerical integration and adds back the analytically integrated contributions that have been subtracted. The method is a generalization of the dipole formalism of Catani and Seymour, which was formulated for NLO QCD processes with massless unpolarized particles. The presented formalism allows for arbitrary mass and helicity configurations in processes with charged fermions and any other neutral particles. Particular attention is paid to the limit of small fermion masses, in which collinear singularities cause potentially large corrections. The actual application and the efficiency of the formalism are demonstrated by the discussion of photonic corrections to the processes $\gamma \gamma \to t\bar{t}(\gamma)$, $e^-\gamma \to e^-\gamma(\gamma)$, and $\mu^+\mu^- \to \nu\bar{\nu}\gamma(\gamma)$.

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1 Introduction

Precision experiments with $e^\pm$ beams, such as at LEP, at the SLC, or at future linear colliders, allow for an investigation of electroweak processes with a typical accuracy of some per cent down to some fractions of a per cent. An adequate description of such reactions—and a theoretical understanding of them that goes beyond a qualitative level—forces us to control higher-order corrections in perturbative predictions. An important source of such radiative corrections is due to the virtual exchange and the real emission of photons, or of gluons if quarks are involved. Although photonic corrections are formally of $\mathcal{O}(\alpha)$ relative to the lowest order, leading to the naive expectation of $\sim 1\%$ as the typical size, the actual effects very often amount to $\sim 10\%$ or more. Apart from large kinematical effects caused by real photon radiation in particular processes, this enhancement mainly originates from collinear photon emission off highly relativistic particles, such as $e^\pm$ at the GeV scale, and from the corresponding virtual photon exchange. For initial-state radiation off electrons, this kind of correction is proportional to $\alpha \ln(m_e/Q)$, where $Q \gg m_e$ is a typical energy scale of the process. The remaining $\mathcal{O}(\alpha)$ corrections amount to one to a few per cent and have to be included in precision calculations as well. For per-cent accuracy even the leading $\mathcal{O}(\alpha^2)$ corrections, or higher, can be relevant.

In this paper we focus on the calculation of the full $\mathcal{O}(\alpha)$ correction that is induced by real photon radiation. Such calculations will be performed for practically all realistic observables numerically, owing to the complexity of the squared amplitudes of the most interesting processes and the necessity of phase-space cuts. Usually the integration over the multidimensional phase space is performed by Monte Carlo integration. Thus, a linear increase in accuracy is roughly accompanied by a quadratic increase in the CPU time needed for the evaluation. In this context, the singularities of a squared amplitude cause problems. For example, the integrand of a bremsstrahlung process blows up if the photon energy becomes small, leading to the well-known logarithmic IR singularity in the phase-space integral. Following a frequently applied standard procedure, known as phase-space slicing, one introduces a small cutoff energy $\Delta E$ and integrates over the photon energy only down to $\Delta E$ numerically. The soft-photon part, $E_\gamma < \Delta E$, is known to factorize from the Born cross section, and the corresponding correction factor, which contains the IR singularity, can be calculated analytically. Since the results obtained this way are correct up to $\mathcal{O}(\Delta E/Q)$, precise predictions require rather small values of $\Delta E$. For $\Delta E \to 0$ the numerical integration result grows like $\alpha \ln(\Delta E/Q)$. Consequently, more and more CPU time is wasted in the precise calculation of this known singular term that cancels in the final result anyhow. Therefore, procedures that avoid such singular numerical integrations are desirable.

Similar problems arise from collinear photon emission off a charged particle with mass $m \ll Q$. Integrating over small emission angles $\theta$ results in mass-singular corrections proportional to $\alpha \ln(m/Q)$. Applying phase-space slicing, the collinearity region is excluded by a small cutoff angle $\Delta \theta$ so that the singularity appears as a $\alpha \ln(\Delta \theta)$ contribution to the numerical integration result. The missing contribution from the region $\theta < \Delta \theta$ is related to the lowest-order cross section and can be obtained without singular numerical integration, similar to the IR case. Concerning the precision of the integration procedure,
$\Delta \theta$ plays a similar role as $\Delta E$ above, and a procedure that avoids the singular integration is preferable. Singular numerical integrations are absent in so-called subtraction methods. The idea of such methods is to subtract and to add a simple auxiliary function from the singular integrand. This auxiliary function has to be chosen in such a way that it cancels all singularities of the original integrand so that the phase-space integration of the difference can be performed numerically, even over the singular regions of the original integrand. In this difference the original matrix element can be evaluated without regulators for IR or collinear singularities, i.e. it is possible to apply powerful spinor techniques (see e.g. Ref. [1, 2, 3] and references therein) that have been developed for four space-time dimensions. The auxiliary function has to be simple enough so that it can be integrated over the singular regions analytically, when the subtracted contribution is added again. This part contains the singular contributions and requires regulators. In general, the statistical uncertainty of the finally obtained correction is smaller than the one of the corresponding result of phase-space slicing, because the absolute value of the numerical integral is usually much smaller for the subtraction method, owing to the absence of singular contributions. Unfortunately, the above requirements set highly non-trivial conditions on the subtraction functions, rendering the construction of a general subtraction procedure difficult. Although various subtraction formalisms have been described for NLO corrections in massless QCD [4, 5], to the best of our knowledge, up to now no general subtraction method has been presented that is able to deal with massive particles in any given process. For the special case of heavy-quark correlations in hadron–hadron collisions, a subtraction procedure has been described in Ref. [6].

In the following we work out a rather general subtraction method for the treatment of photon radiation for any given process involving massive or massless, polarized or unpolarized fermions and any kind of neutral bosons. The inclusion of charged bosons is completely straightforward. Our method follows the guideline provided by the dipole formalism, which has been presented by Catani and Seymour [5] for QCD with massless, unpolarized partons. Since the colour flow in QCD processes is more involved than the charge flow in electroweak processes, our presentation is simpler than the one in Ref. [5] in this respect. However, the generalization of the dipole formalism to arbitrary masses turns out to be highly non-trivial. Even in the limit of small fermion masses, which is of particular interest, there is an important difference between our approach and the subtraction procedures of Refs. [1, 5] for massless QCD partons. We consistently regularize IR and collinear singularities with finite masses, as it is commonly applied to photon radiation in electroweak processes, whereas the above-mentioned QCD studies are carried out in dimensional regularization.

Although we treat only photon radiation explicitly, one should realize that the presented results can also be used for gluon radiation in processes that involve only massive quarks as QCD partons; in this case the colour flow has to be handled as described in Ref. [5]. Our work also represents a first step towards the generalization of the dipole formalism in QCD to include massive partons.

The paper is organized as follows: in the next section we review the general structure of IR and collinear singularities, and describe the strategy of the subtraction procedure. In Section 3 we anticipate our results on the subtraction function and its integrated coun-
terpart for the special case of light fermions, in order to illustrate the structure of the formalism. The general results for arbitrary fermion masses are given in Section 4, where the details of the derivation are described, too. Section 5 contains the numerical examples, including discussions of the photonic $O(\alpha)$ corrections to the processes $\gamma \gamma \rightarrow t\bar{t}(\gamma)$, $e^-\gamma \rightarrow e^-\gamma(\gamma)$, and $\mu^+\mu^- \rightarrow \nu_e\bar{\nu}_e(\gamma)$. In Section 6 we discuss salient features of subtraction formalisms and of the dipole approach. The discussion, in particular, includes comments on the implementation of phase-space cuts, some practical advice, and remarks on the partial generalization to QCD. Our summary is presented in Section 7. In the appendix we provide important special cases, further details of the calculation, and the virtual photonic corrections to $\mu^+\mu^- \rightarrow \nu_e\bar{\nu}_e$.

2 General strategy

2.1 Preliminary remarks and conventions

We consider photon emission in processes that involve arbitrary fermions and any massive neutral bosons. The initial state may also contain photons. The presented method remains applicable to reactions with more than one photon in the final state if only a single photon can become soft or collinear with a light fermion in phase space. Note that situations with more than one photon being soft or collinear correspond to corrections of $O(\alpha^2)$, or higher, relative to the lowest-order process without photon emission. In other words, the method to be described covers all kinds of real-photonic $O(\alpha)$ corrections to processes involving charged fermions and any neutral particles.

The relative charge and the mass of a fermion $f$ are denoted by $Q_f$ and $m_f$, the momentum and the helicity of $f$ are assigned to $p_f$ and $\kappa_f$, respectively. Instead of the general indices $f, f'$ for any fermions, we use the indices $a, b$ only for initial-state fermions and $i, j$ only for final-state fermions. Moreover, we define the sign factors $\sigma_f = \pm 1$ for the charge flow related to the fermion $f$; specifically, we set $\sigma_f = +1$ for incoming fermions and outgoing anti-fermions, and $\sigma_f = -1$ for outgoing fermions and incoming anti-fermions. Consequently, charge conservation of the whole reaction implies

$$\sum_f Q_f \sigma_f = 0. \quad (2.1)$$

Since IR and collinear divergences are regularized by particle masses, we consistently work within four space-time dimensions. The invariant phase-space measure is abbreviated by

$$d\phi(k_1, \ldots, k_n; K) = \left[ \prod_{l=1}^n \frac{d^4k_l}{(2\pi)^3} \Theta(k_l^0) \delta(k_l^2 - m_l^2) \right] (2\pi)^4 \delta^{(4)}(K - \sum_{l=1}^n k_l). \quad (2.2)$$

In the following, $M_1$ is the transition matrix element of the considered process that involves an outgoing photon with momentum $k$. The matrix element of the corresponding process without photon emission is denoted by $M_0$. For brevity, we explicitly write down only those momenta and helicities as arguments of $M_1$ and $M_0$ that are important in the considered equation. The collections of all momenta of the corresponding reactions are abbreviated by $\Phi_1$ and $\Phi_0$, and the respective phase-space measures by $d\Phi_1$ and $d\Phi_0$.
2.2 IR and collinear singularities

If the momentum \( k \) of the radiated photon becomes soft \( (k \to 0) \), the squared matrix element \( \sum_{\lambda_\gamma} |M_1|^2 \), summed over all photon polarizations \( \lambda_\gamma \), becomes IR-singular and asymptotically proportional to \( |M_0|^2 \) in the well-known form (see e.g. Ref. [7])

\[
\sum_{\lambda_\gamma} |M_1|^2 \sim k \to 0 - \sum_{f,f'} Q_f \sigma_f \sigma_{f'} e^2 \frac{p_f p_{f'}}{(p_f k)(p_{f'} k)} |M_0|^2,
\]  

(2.3)

where the sums on the r.h.s. run over all charged fermions of the reaction, and \( e \) is the positron charge. Eq. (2.3) is valid for all polarization configurations separately. The phase-space integral of (2.3) over the soft-photon region is logarithmically divergent. We choose an infinitesimal photon mass \( m_\gamma \) as regulator, yielding singular contributions proportional to \( \alpha \ln(m_\gamma) \) to the real \( \mathcal{O}(\alpha) \) corrections. According to the Bloch–Nordsieck theorem [8], these singular contributions cancel against IR-singular counterparts in the virtual corrections.

Another type of singularity occurs in the limit of a vanishing fermion mass, \( m_f \to 0 \), if the region of collinear photon emission from \( f \) is included in the phase-space integration. The squared amplitude \( |M_1|^2 \) develops poles in \( (p_f k) \to 0 \), leading to logarithmic singularities in the phase-space integral. The asymptotic form of \( \sum_{\lambda_\gamma} |M_1|^2 \) in the collinearity regions is related to the squared amplitude \( |M_0|^2 \) and well known [2,9]. Distinguishing between photon emission from outgoing and incoming fermions, we have

\[
\sum_{\lambda_\gamma} |M_1(p_i; \kappa_i)|^2 \sim p_i k \to 0 - \sum_{p_f} Q_f^2 e^2 g_{i,\tau}^{(out)}(p_i, k)|M_0(p_i + k; \tau \kappa_i)|^2,
\]

\[
\sum_{\lambda_\gamma} |M_1(p_a; \kappa_a)|^2 \sim p_a k \to 0 - \sum_{p_f} Q_f^2 e^2 g_{a,\tau}^{(in)}(p_a, k)|M_0(x_a p_a; \tau \kappa_a)|^2,
\]

(2.4)

where the signs \( \tau = \pm \) account for a possible spin flip of the considered fermion. Whenever \( \tau \) appears more than once in products, we assume summation over \( \tau = \pm \). Note that we take helicity eigenstates as polarization basis throughout. The functions \( g_{f,\tau}^{(out/in)} \) are given by

\[
g_{i,\tau}^{(out)}(p_i, k) = \frac{1}{p_i k} \left[ P_{ff}(z_i) - \frac{m_i^2}{p_i k} \right] - g_{i,-}^{(out)}(p_i, k),
\]

\[
g_{i,-}^{(out)}(p_i, k) = \frac{m_i^2}{2(p_i k)^2} \frac{(1 - z_i)^2}{z_i},
\]

\[
g_{a,\tau}^{(in)}(p_a, k) = \frac{1}{x_a p_a k} \left[ P_{ff}(x_a) - \frac{x_a m_a^2}{p_a k} \right] - g_{a,-}^{(in)}(p_a, k),
\]

\[
g_{a,-}^{(in)}(p_a, k) = \frac{m_a^2}{2(p_a k)^2} \frac{(1 - x_a)^2}{x_a},
\]

(2.5)

where \( P_{ff}(y) \) is the usual splitting function,

\[
P_{ff}(y) = \frac{1 + y^2}{1 - y}.
\]

(2.6)
If one is only interested in unpolarized fermions, the summation of \( g^{(\text{out}/\text{in})}_{\tau}\mathcal{M}_0(\tau\kappa_f) \)^2 over \( \tau \) in (2.4) reduces to \((g^{(\text{out}/\text{in})}_{\tau+} + g^{(\text{out}/\text{in})}_{\tau-})\mathcal{M}_0(\kappa_f)\)^2. In the following, all formulas are written in a form that facilitates this replacement. The variables \( z_i \) and \( x_a \) are the fractions of the fermion energies that are kept by the fermions after photon emission,

\[
z_i = \frac{p^{0}_i}{p^{0}_i + k^0}, \quad x_a = \frac{p^0_a - k^0}{p^0_a}. \tag{2.7}
\]

While final-state radiation does not change any momentum other than \( p_i \) in the hard scattering process, initial-state radiation scales the fermion momentum \( p_a \) down to \( x_a p_a \sim p_a - k \), thereby reducing the centre-of-mass (CM) energy of the hard scattering process. Integrating the squared amplitudes over a collinearity region yields contributions proportional to \( \alpha \ln(m_f) \) to the real \( \mathcal{O}(\alpha) \) corrections. According to the Kinoshita–Lee–Nauenberg theorem [10], the mass-singular corrections \( \alpha \ln(m_i) \), which originate from final-state radiation, cancel against their counterparts in the virtual corrections in the total cross section. Mass singularities from final-state fermions can only survive in specific distributions, such as distributions of invariant-masses that are built of fermion momenta only, i.e. without taking into account photon recombination. For fermions in the initial state the sum of real and virtual corrections remains mass-singular, and the \( \alpha \ln(m_a) \) terms are a potential source of large corrections.

The asymptotic relations (2.3) and (2.4) both relate the full squared matrix element \( |\mathcal{M}_1(\Phi_1)|^2 \) for the radiative process to \( |\mathcal{M}_0(\Phi_0)|^2 \), which corresponds to the process without photon emission. Note that the phase spaces on which these functions are defined are different. In order to guarantee that both sides of (2.3) and (2.4) are defined on the phase space spanned by the momenta \( \Phi_1 \), one has to specify an appropriate mapping from \( \Phi_1 \) to \( \Phi_0 \) that respects all mass-shell relations. The definition of such mappings is of central importance in the construction of a subtraction function \( |\mathcal{M}_{\text{sub}}|^2 \) that is parametrized by \( \Phi_1 \) and has the same asymptotic behaviour as \( \sum_{\lambda\gamma} |\mathcal{M}_1|^2 \) in the singular limits.

### 2.3 The dipole subtraction formalism

Our final aim is to perform the phase-space integral of \( \sum_{\lambda\gamma} |\mathcal{M}_1|^2 \), which involves IR and collinear singularities, without carrying out singular numerical integrations. The basic idea in a subtraction method is to subtract and to add the integral of an appropriate subtraction function \( |\mathcal{M}_{\text{sub}}|^2 \),

\[
\int d\Phi_1 \sum_{\lambda\gamma} |\mathcal{M}_1|^2 = \int d\Phi_1 \left( \sum_{\lambda\gamma} |\mathcal{M}_1|^2 - |\mathcal{M}_{\text{sub}}|^2 \right) + \int d\Phi_1 |\mathcal{M}_{\text{sub}}|^2, \tag{2.8}
\]

where \( |\mathcal{M}_{\text{sub}}|^2 \) possesses the same asymptotic behaviour as \( \sum_{\lambda\gamma} |\mathcal{M}_1|^2 \) in the singular limits. Specifically, we demand

\[
|\mathcal{M}_{\text{sub}}|^2 \sim \sum_{\lambda\gamma} |\mathcal{M}_1|^2 \quad \text{for } k \to 0 \text{ or } p_i k \to 0 \text{ or } p_a k \to 0, \tag{2.9}
\]

\(^1\text{There are also corrections of the forms } \alpha \ln(m_f) \ln(m_{\gamma}) \text{ and } \alpha \ln^2(m_f), \text{ which originate from soft photons. These corrections, however, always cancel against virtual corrections.}\)
where $i$ and $a$ label all outgoing and incoming light fermions. Owing to (2.9), the phase-space integration of the difference $\sum_\gamma |M_1|^2 - |M_{\text{sub}}|^2$ in (2.8) is non-singular, i.e. it can be performed numerically without regulators. The singular contributions of the original integral $\int d\Phi_1 \sum_\gamma |M_1|^2$ are completely contained in $\int d\Phi_1 |M_{\text{sub}}|^2$. If $|M_{\text{sub}}|^2$ is chosen appropriately the singular integrations can be carried out analytically. To this end, the phase-space integral $\int d\Phi_1$ is factorized into a part $\int d\Phi_0$ connected to the non-radiative process and a part $\int [dk]$ connected to the photon phase space,

$$\int d\Phi_1 = \int d\Phi_0 \otimes \int [dk].$$

(2.10)

The sign “$\otimes$” indicates that this factorization is not an ordinary product, but may contain also summations and convolutions. Since $\sum_\gamma |M_1(\Phi_1)|^2$ is related to $|M_0(\Phi_0)|^2$ of the non-radiative process in the singular limits, the subtraction function $|M_{\text{sub}}(\Phi_1)|^2$ can be defined in such a way that it depends on the momenta of $\Phi_0$ only via $|M_0(\Phi_0)|^2$. The integration variables of $\int [dk]$ occur only in the remaining terms of the subtraction function. Since those terms are process-independent, the singular integration of $|M_{\text{sub}}(\Phi_1)|^2$ over $[dk]$ can be performed analytically once and for all. Finally, the integral $\int d\Phi_1 \sum_\gamma |M_1|^2$ takes the schematic form

$$\int d\Phi_1 \sum_\gamma |M_1|^2 = \int d\Phi_1 \left( \sum_\gamma |M_1|^2 - |M_{\text{sub}}|^2 \right) + \int d\Phi_0 \otimes \left( \int [dk] |M_{\text{sub}}|^2 \right).$$

(2.11)

The integrations over $d\Phi_1$ and $d\Phi_0$ on the r.h.s. are free of singularities, and thus are well-suited for numerical evaluations. Since the singularities in $\int [dk] |M_{\text{sub}}|^2$ are controlled analytically, they can be easily combined with their counterparts in the virtual corrections.

We have seen that the subtraction function $|M_{\text{sub}}|^2$ has to obey two non-trivial conditions. It must possess the asymptotic behaviour given in (2.9), and it must still be simple enough so that it can be integrated over the singular regions analytically. Note that all the collinearity regions of phase space overlap and have the IR part ($k \to 0$) in common. Therefore, the naive sum of all collinear singularities, which are proportional to $Q_f^2$, leads to an overcounting of the IR singularity, and thus cannot be used in the subtraction function. In the following we show how this overcounting is avoided and how the subtraction function is constructed within the dipole formalism. In contrast to Ref. [4], where this formalism is described for massless, unpolarized partons in QCD, we have to take care of fermion masses and polarizations.

The subtraction function $|M_{\text{sub}}|^2$ is constructed from auxiliary functions $g_{ff',\tau}^{(\text{sub})}$, which are labelled by pairs of different fermions $f \neq f'$:

$$|M_{\text{sub}}(\Phi_1)|^2 = - \sum_{f \neq f'} Q_f \sigma_f Q_{f'} \sigma_{f'} e^{2g_{ff',\tau}^{(\text{sub})}(p_f, p_{f'}, k)} |M_0 \left( \Phi_{0,ff';\tau}k \right)|^2.$$  

(2.12)

Since only the kinematics of fermion $f$ gives rise to singular contributions in the subtraction function, $f$ is called emitter, whereas $f'$ is called spectator. The summation over $\tau$ accounts for the helicity flip of the emitter $f$. The auxiliary functions $g_{ff',\tau}^{(\text{sub})}$ have to possess an appropriate asymptotic behaviour. In the IR limit one globally demands

$$g_{ff',\tau}^{(\text{sub})}(p_f, p_{f'}, k) \sim \frac{1}{p_f k} \left[ \frac{2(p_f p_{f'})}{p_f k + p_{f'} k} - \frac{m_f^2}{p_f k} \right],$$

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The sign “$\otimes$” indicates that this factorization is not an ordinary product, but may contain also summations and convolutions. Since $\sum_\gamma |M_1(\Phi_1)|^2$ is related to $|M_0(\Phi_0)|^2$ of the non-radiative process in the singular limits, the subtraction function $|M_{\text{sub}}(\Phi_1)|^2$ can be defined in such a way that it depends on the momenta of $\Phi_0$ only via $|M_0(\Phi_0)|^2$. The integration variables of $\int [dk]$ occur only in the remaining terms of the subtraction function. Since those terms are process-independent, the singular integration of $|M_{\text{sub}}(\Phi_1)|^2$ over $[dk]$ can be performed analytically once and for all. Finally, the integral $\int d\Phi_1 \sum_\gamma |M_1|^2$ takes the schematic form

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$$|M_{\text{sub}}(\Phi_1)|^2 = - \sum_{f \neq f'} Q_f \sigma_f Q_{f'} \sigma_{f'} e^{2g_{ff',\tau}^{(\text{sub})}(p_f, p_{f'}, k)} |M_0 \left( \Phi_{0,ff';\tau}k \right)|^2.$$  

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Since only the kinematics of fermion $f$ gives rise to singular contributions in the subtraction function, $f$ is called emitter, whereas $f'$ is called spectator. The summation over $\tau$ accounts for the helicity flip of the emitter $f$. The auxiliary functions $g_{ff',\tau}^{(\text{sub})}$ have to possess an appropriate asymptotic behaviour. In the IR limit one globally demands

$$g_{ff',\tau}^{(\text{sub})}(p_f, p_{f'}, k) \sim \frac{1}{p_f k} \left[ \frac{2(p_f p_{f'})}{p_f k + p_{f'} k} - \frac{m_f^2}{p_f k} \right].$$
\begin{align}
  g_{ij}^{(\text{sub})} & : \\
  & \begin{array}{c}
  \hspace{1cm} p_i \\
  \gamma \hspace{1cm} \gamma \hspace{1cm} \gamma \\
  k \hspace{1cm} j \\
  \end{array} \\
  g_{ia}^{(\text{sub})} & : \\
  & \begin{array}{c}
  \hspace{1cm} p_i \\
  \gamma \hspace{1cm} \gamma \\
  k \hspace{1cm} a \\
  \end{array} \\
  g_{ai}^{(\text{sub})} & : \\
  & \begin{array}{c}
  \hspace{1cm} a \\
  \gamma \hspace{1cm} \gamma \\
  k \hspace{1cm} p_a \\
  \end{array} \\
  g_{ab}^{(\text{sub})} & : \\
  & \begin{array}{c}
  \hspace{1cm} a \\
  \gamma \hspace{1cm} \gamma \\
  k \hspace{1cm} b \\
  \end{array}
\end{align}

Figure 1: Effective diagrams for the different emitter/spectator cases.

\begin{equation}
  g_{ff',\tau}(p_f, p_{f'}, k) \xrightarrow{k \to 0} O(1). \tag{2.13}
\end{equation}

In the collinear limits one demands separate conditions for final- and initial-state fermions \(f = i, a\):

\begin{align}
  g_{i'\tau}(p_i, p_{i'}, k) & \xrightarrow{p_{i,k} \to 0} g_{i,\tau}(p_i, k), \\
  g_{a'\tau}(p_a, p_{a'}, k) & \xrightarrow{p_{a,k} \to 0} g_{a,\tau}(p_a, k), \tag{2.14}
\end{align}

where \(f'\) can be outgoing or incoming. The analytical form of \(g_{ff',\tau}^{(\text{sub})}\) is, of course, not uniquely determined by the asymptotic conditions. A convenient choice for these auxiliary functions, which are graphically represented by the effective diagrams of Fig. 1, is given in the next sections.

Finally, we have to specify the conditions on the momenta to be inserted in \(|M_0|^2\) in (2.12). As explained above, it is necessary to define a mapping \(\tilde{\Phi}_{f,f'}\) from the momenta of \(\Phi_1\) to the ones of \(\Phi_0\) that respects all mass-shell relations. The symbol \(\tilde{\Phi}_{f,f'}\) indicates that different mappings are used for different pairs \(ff'\). Denoting the momenta of \(f\) and \(f'\) in \(\tilde{\Phi}_{f,f'}\) by \(\tilde{p}_f\) and \(\tilde{p}_{f'}\), and writing \(k_n\) and \(\tilde{k}_n\) for the remaining momenta in \(\Phi_1\) and \(\tilde{\Phi}_{f,f'}\), respectively, we require

\begin{align}
  \tilde{p}_f & \xrightarrow{k \to 0} p_f, \quad \tilde{p}_{f'} \xrightarrow{k \to 0} p_{f'}, \\
  \tilde{k}_n & \xrightarrow{k \to 0} k_n \tag{2.15}
\end{align}

for the IR limit and

\begin{align}
  \tilde{p}_i & \xrightarrow{p_i \to 0} p_i + k, \quad \tilde{p}_a \xrightarrow{p_a \to 0} x_a p_a, \quad \tilde{p}_{f'} \xrightarrow{p_{f'} \to 0} p_{f'}, \\
  \tilde{k}_n & \xrightarrow{p_{f'} \to 0} k_n \tag{2.16}
\end{align}

for the collinear limits. The mass-shell conditions \(\tilde{p}_f^2 = p_f^2 = m_f^2, \quad \tilde{p}_{f'}^2 = p_{f'}^2 = m_{f'}^2, \quad \tilde{k}_n^2 = k_n^2 = n_n^2\) have to be fulfilled for arbitrary photon momentum \(k\).
Using (2.13) and (2.14), it is rather easy to check that the subtraction function (2.12) possesses the asymptotic behaviour required in (2.9). For the IR limit, the asymptotic relation is verified upon inserting (2.13) into (2.12) and rearranging the terms in ∑f≠f':

\[
\sum_{f≠f'} Q_f \sigma_f Q_{f'} \sigma_{f'} \frac{1}{p_{f'k}} \left[ \frac{2(p_{f'p_f})}{p_{f'k} + p_{f'k}} - \frac{m_f^2}{p_{f'k}} \right] = \sum_{f≠f'} Q_f \sigma_f Q_{f'} \sigma_{f'} \frac{p_{f'p_f}}{(p_{f'k})(p_{f'k})} + \sum_{f} Q_f^2 \frac{m_f^2}{(p_{f'k})^2} = \sum_{f, f'} Q_f \sigma_f Q_{f'} \sigma_{f'} \frac{p_{f'p_f}}{(p_{f'k})(p_{f'k})},
\]

(2.17)

Note that charge conservation (2.4) was used in the form

\[
\sum_{f'(f'≠f)} Q_f \sigma_f = -Q_f \sigma_f
\]

(2.18)
in the term proportional to m_f^2. The arguments of |M_0|^2 in |M_{sub}|^2 behave in the desired way owing to (2.15). The asymptotic relation (2.9) for the collinear limits follows after inserting the conditions (2.14) into (2.12) and using again charge conservation (2.18). The correct behaviour of the momenta of |M_0|^2 in |M_{sub}|^2 is guaranteed by (2.16).

3 Subtraction functions and integrated counterparts—special case of light fermions

Before we turn to the treatment of the general case of massive fermions in the next section, we first describe the dipole subtraction formalism for light fermions, i.e. we neglect fermion masses in this section whenever possible. In this way, the structure of the formalism becomes clear without being obscured by all kind of complications that are related to particle masses. Moreover, this section provides a condensed instruction to the formalism for light fermions, since details of the method that are only relevant for its derivation are also postponed to the next section.

3.1 Final-state emitter and final-state spectator

We define the auxiliary functions g_{ij,τ}^{(sub)}, which correspond to a final-state emitter i and a final-state spectator j, by

\[
g_{ij,+}^{(sub)}(p_i, p_j, k) = \frac{1}{(p_i k)(1 - y_{ij})} \left[ \frac{2}{1 - z_{ij}(1 - y_{ij})} - 1 - z_{ij} \right],
\]

\[
g_{ij,-}^{(sub)}(p_i, p_j, k) = 0,
\]

(3.1)

where the variables y_{ij} and z_{ij} are given by

\[
y_{ij} = \frac{p_i}{p_i p_j + p_i k + p_j k}, \quad z_{ij} = \frac{p_i p_j}{p_i p_j + p_j k}.
\]

(3.2)

\footnote{The dipole subtraction formalism for this important special case has been worked out independently by M. Roth [1]. Comparing both approaches, we find full consistency.}
Since we assume \( m_{i,j} \rightarrow 0 \), the explicit mass terms in \( g_{i,j}^{(\text{out})} \) and in \( g_{i,j}^{(\text{sub})} \) are negligible in the difference \( |M_{\text{sub}}|^2 - \sum_\lambda |M_\lambda|^2 \). In particular, this implies that \( g_{i,j}^{(\text{sub})} \) vanishes. Note, however, that those mass terms are relevant in the integration of \( g_{i,j}^{(\text{sub})} \) over the photonic part of phase space (see next section). It is straightforward to check that the functions \( g_{i,j}^{(\text{sub})} \) of (3.1) obey the asymptotic conditions (2.13) and (2.14) in the IR and collinear limits, in which we get

\[
y_{ij} \xrightarrow{k \to 0} 0, \quad z_{ij} \xrightarrow{k \to 0} 1, \quad y_{ij} \xrightarrow{p_i \to 0} 0, \quad z_{ij} \xrightarrow{p_i \to 0} z_i, \quad (3.3)
\]

with \( z_i \) from (2.7). For the evaluation of \( |M_0(\Phi_{0,ij})|^2 \) we have to define the mapping \( \Phi_{0,ij} \) from \( \Phi_1 \) to \( \Phi_0 \). Of course, it is desirable to leave as many momenta unchanged as possible. Therefore, we redefine only the momenta of \( f \) and \( f' \), and leave all other momenta \( k_n \) unaffected, \( k_n = k_n \). The momenta \( \tilde{p}_i \) and \( \tilde{p}_j \) are chosen as

\[
\tilde{p}_i^\mu = p_i^\mu + k^\mu - \frac{y_{ij}}{1 - y_{ij}} p_j^\mu, \quad \tilde{p}_j^\mu = \frac{1}{1 - y_{ij}} p_j^\mu. \quad (3.4)
\]

The on-shell relations \( \tilde{p}_i^2 = \tilde{p}_j^2 = 0 \) and the validity of the required asymptotic behaviour (2.13) and (2.16) can be checked easily. Moreover, momentum conservation,

\[
P_{ij} = p_i + p_j + k = \tilde{p}_i + \tilde{p}_j, \quad (3.5)
\]

is trivially fulfilled, and \( P_{ij}^2 \geq 0 \) holds for all phase-space points. The above definitions comprise all ingredients for the evaluation of the \( ij \) contribution to the difference \( (\sum_\lambda |M_1|^2 - |M_{\text{sub}}|^2) \), which is integrated over the full phase space numerically. We recall that this integration can be performed with vanishing photon and fermion masses.

The construction of the above contribution to the subtraction function entirely follows the pattern of Ref. [4] for massless QCD partons. The same applies to the other emitter/spectator cases. Note that we have added a factor of \( 1/(1 - y_{ij}) \) in \( g_{i,j}^{(\text{sub})} \) that is not included in the approach of Ref. [4]. This factor, which is introduced for convenience (see massive case in Section 4.4), affects only non-singular contributions.

The differences between Ref. [3] and our approach for light fermions become apparent in the analytical integration of the subtraction function over the photonic part of phase space, where both IR and collinear regulators have to be taken into account. The photonic part of phase space is defined by extracting the phase-space measure \( d\tilde{\Phi}_{0,ij} \), which is spanned by the momenta \( \tilde{p}_i, \tilde{p}_j, \) and \( k_n \), from the full phase-space measure \( d\Phi_1 \), which is spanned by the momenta \( p_i, p_j, k, \) and \( k_n \). The details of this splitting and of the integration over the photonic part can be found in Section 4.4 and in App. 4. Denoting the integral of \( g_{i,j}^{(\text{sub})} \) over the photonic phase space by \( G_{ij,\tau}^{(\text{sub})} \), and including an appropriate normalization factor, the \( ij \) contribution \( |M_{\text{sub},ij}(\Phi_1)|^2 \) to the phase-space integral of the subtraction function reads

\[
\int d\Phi_1 |M_{\text{sub},ij}(\Phi_1)|^2 = -\frac{\alpha}{2\pi} Q_i \sigma_i Q_j \sigma_j \int d\tilde{\Phi}_{0,ij} G_{ij,\tau}^{(\text{sub})} (P_{ij}^2) |M_0(\tilde{p}_i, \tilde{p}_j; \tau \kappa_i)|^2, \quad (3.6)
\]

where \( \alpha = e^2/(4\pi) \) is the fine-structure constant. The functions \( G_{ij,\tau}^{(\text{sub})} \) are explicitly given by

\[
G_{ij,\tau}^{(\text{sub})} (P_{ij}^2) = \mathcal{L}(P_{ij}^2, m_i^2) - \frac{\pi^2}{3} + 1, \quad G_{ij,\tau}^{(\text{sub})} (P_{ij}^2) = \frac{1}{2}, \quad (3.7)
\]
where the singular terms are contained in the function

\[ L(P^2, m^2) = \ln \left( \frac{m_\gamma^2}{P^2} \right) \ln \left( \frac{m^2}{P^2} \right) + \ln \left( \frac{m_\gamma^2}{P^2} \right) - \frac{1}{2} \ln^2 \left( \frac{m^2}{P^2} \right) + \frac{1}{2} \ln \left( \frac{m_\gamma^2}{P^2} \right). \]  

(3.8)

As required, only the emitter mass \( m_i \) gives rise to logarithmic singularities, whereas the spectator mass \( m_j \) can be set to zero exactly. The spin-flip part \( G_{ij,-}^{(\text{sub})} \) is non-vanishing and entirely induced by photons that are emitted collinearly to the emitter \( i \). Since all singular terms are factorized into \( G_{ij,+}^{(\text{sub})} \), the masses \( m_\gamma, m_i, \) and \( m_j \) can be set to zero everywhere in (3.6) apart from \( G_{ij,+}^{(\text{sub})} \).

### 3.2 Final-state emitter and initial-state spectator, and vice versa

Emitter/spectator pairs from the final/initial state and vice versa, i.e. the cases \( i a \) and \( a i \), always occur in pairs for a given process. Since the kinematics is identical in both cases, we treat them in one go. The corresponding auxiliary functions \( g_{j f,\tau}^{(\text{sub})} \) are given by

\[
\begin{align*}
g_{ia,+}^{(\text{sub})} (p_i, p_a, k) &= \frac{1}{(p_i k) x_{ia}} \left[ \frac{2}{2 - x_{ia} - z_{ia}} - 1 - z_{ia} \right], \\
g_{ai,+}^{(\text{sub})} (p_a, p_i, k) &= \frac{1}{(p_a k) x_{ia}} \left[ \frac{2}{2 - x_{ia} - z_{ia}} - 1 - x_{ia} \right], \\
g_{ia,-}^{(\text{sub})} (p_i, p_a, k) &= g_{ai,-}^{(\text{sub})} (p_a, p_i, k) = 0,
\end{align*}
\]  

(3.9)

with the variables

\[
\begin{align*}
x_{ia} &= \frac{p_a p_i + p_a k - p_i k}{p_a p_i + p_a k}, \\
z_{ia} &= \frac{p_a p_i}{p_a p_i + p_a k}.
\end{align*}
\]  

(3.10)

The desired asymptotic behaviour (2.13) and (2.14) in the singular limits, which imply

\[
\begin{align*}
x_{ia} \underset{k \to 0}{\longrightarrow} 1, & \quad z_{ia} \underset{k \to 0}{\longrightarrow} 1, & \quad x_{ia} \underset{p_i k \to 0}{\longrightarrow} 1, & \quad z_{ia} \underset{p_i k \to 0}{\longrightarrow} z_i, \\
x_{ia} \underset{p_a k \to 0}{\longrightarrow} x_a, & \quad z_{ia} \underset{p_a k \to 0}{\longrightarrow} 1,
\end{align*}
\]  

(3.11)

can be verified easily. The modified momenta \( \tilde{p}_i \) and \( \tilde{p}_a \) of the sets \( \Phi_{0,ia} = \Phi_{0,ai} \) are chosen as

\[
\tilde{p}_i^\mu = p_i^\mu + k^\mu - (1 - x_{ia}) p_a^\mu, \quad \tilde{p}_a^\mu = x_{ia} p_a^\mu,
\]  

(3.12)

and the remaining momenta \( \tilde{k}_n \) coincide with the corresponding momenta \( k_n \) of \( \Phi_1 \). The on-shell relations \( \tilde{p}_i^2 = \tilde{p}_a^2 = 0 \), the required asymptotic behaviour (2.15) and (2.16), as well as momentum conservation,

\[
P_{ia} = p_i + k - p_a = \tilde{p}_i - \tilde{p}_a,
\]  

(3.13)

can be checked easily. For massless fermions \( i \) and \( a \) we always have \( P_{ia}^2 \leq 0 \). This completes the definitions of all quantities for the evaluation of the \( ia \) and \( ai \) parts of
the difference \((\sum_{\lambda} |M_{1}|^2 - |M_{\text{sub}}|^2)\), which can be performed numerically with vanishing photon and fermion masses.

The analytical integration of the \(ia\) and \(ai\) parts of the subtraction function is more involved than in the \(ij\) case, since the modified momenta \(\tilde{p}_i\) and \(\tilde{p}_a\) correspond to a new initial state. In the following, we consider a scattering reaction with the two incoming light-like momenta \(p_a\) and \(p_b\). Owing to \(\tilde{p}_a = x_{ia}p_a\), the CM frames of \(p_a + p_b\) and \(\tilde{p}_a + p_b\) are related by a boost along the beam axis. The strength of this boost is determined by \(x_{ia}\), which is the ratio of the corresponding squared CM energies \(s\) and \(\tilde{s}\),

\[
\tilde{s} = 2(\tilde{p}_a p_b) = 2x_{ia}(p_a p_b) = x_{ia}s. \tag{3.14}
\]

The photonic part of the phase space, which results from the extraction of the phase-space measure \(d\tilde{\Phi}_{0,ia}\) from the full measure \(d\Phi_1\), involves an integration over \(x_{ia}\). This integration over \(x_{ia}\) cannot be carried out analytically, since the complete phase space spanned by the new momenta \(\tilde{\Phi}_{0,ia}\) implicitly depends on \(x_{ia}\) via the CM energy \(\sqrt{s}\). Thus, the integral over the photonic part of phase space is written in terms of a convolution over \(x_{ia}\). Including an appropriate normalization, the contributions to the phase-space integral of the subtraction function read

\[
\int d\Phi_1 |M_{\text{sub},ff'}(\Phi_1)|^2 = -\frac{\alpha}{2\pi}Q_a \sigma_a Q_i \sigma_i \\
\times \int_0^1 dx \int d\tilde{\Phi}_{0,ia}(x) \frac{1}{x} G^{(\text{sub})}_{ff',\tau}(P_{ia}^2, x) \left| M_0(xp_a + P_{ia}, xp_a; \tau \kappa_f) \right|^2, \tag{3.15}
\]

with \(ff' = ia, ai\) for the two different cases. In this convolution \(x\) plays the role of \(x_{ia}\), and the argument of the phase-space measure \(d\tilde{\Phi}_{0,ia}(x)\) indicates that each value of \(x\) determines a different phase space. The momenta to be inserted in \(|M_0|^2\) are \(\tilde{p}_i = xp_a + P_{ia}\) and \(\tilde{p}_a = xp_a\), where \(p_a\) is fixed, and \(P_{ia}\) varies with the phase-space point in \(\tilde{\Phi}_{0,ia}(x)\).

Since the distributions \(G^{(\text{sub})}_{ff',\tau}(P_{ia}^2, x)\) become IR-singular at the point \(x \rightarrow 1\), the convolution is not yet suited for a numerical evaluation. A possible way out is provided by the application of the \([...]_+\) prescription to this distribution,

\[
\int_0^1 dx \left[ f(x) \right]_+ g(x) = \int_0^1 dx f(x)[g(x) - g(1)]. \tag{3.16}
\]

Using this trick, the IR singularities in the endpoint contributions

\[
G^{(\text{sub})}_{ff',\tau}(P_{ia}^2, x) = \int_0^1 dx G^{(\text{sub})}_{ff',\tau}(P_{ia}^2, x) \tag{3.17}
\]

are separated from \(G^{(\text{sub})}_{ff',\tau}(P_{ia}^2, x)\), and the convolution reads

\[
\int d\Phi_1 |M_{\text{sub},ff'}(\Phi_1)|^2 = -\frac{\alpha}{2\pi}Q_a \sigma_a Q_i \sigma_i \\
\times \left\{ \int_0^1 dx \left[ \int d\tilde{\Phi}_{0,ia}(P_{ia}^2) \frac{1}{x} G^{(\text{sub})}_{ff',\tau}(P_{ia}^2, x) \left| M_0(xp_a + P_{ia}, xp_a; \tau \kappa_f) \right|^2 \\
- \int d\tilde{\Phi}_{0,ia}(P_{ia}^2, 1) G^{(\text{sub})}_{ff',\tau}(P_{ia}^2, x) \left| M_0(p_a + P_{ia}, p_a; \tau \kappa_f) \right|^2 \right] \\
+ \int d\tilde{\Phi}_{0,ia}(P_{ia}^2, 1) G^{(\text{sub})}_{ff',\tau}(P_{ia}^2) \left| M_0(p_a + P_{ia}, p_a; \tau \kappa_f) \right|^2 \right\}. \tag{3.18}
\]
Note that we have included $P^2_{ia}$ as additional argument in the phase-space measure $d\Phi_{0,ia}(P^2_{ia}, x)$, in order to signalize that we have kept $P^2_{ia}$ fixed during the integration over $x$ in the calculation of the endpoint contributions $G^{(\text{sub})}_{ff',\tau}(P^2_{ia})$. In (3.18) all singular contributions are factorized into $G^{(\text{sub})}_{ff',\tau}$ or $G^{(\text{sub})}_{ff',\tau}$ so that this equation is well-suited for numerical evaluations, and the photon and fermion masses can be set to zero everywhere apart from $G^{(\text{sub})}_{ff',\tau}$ and $G^{(\text{sub})}_{ff',\tau}$.

Finally, we give the explicit form of the distributions $G^{(\text{sub})}_{ff',\tau}(P^2_{ia}, x)$,

$$G^{(\text{sub})}_{ia,+}(P^2_{ia}, x) = \frac{1}{1 - x} \left[ 2 \ln \left( \frac{2 - x}{1 - x} \right) - \frac{3}{2} \right],$$

$$G^{(\text{sub})}_{ai,+}(P^2_{ia}, x) = P_{ff}(x) \left[ \ln \left( \frac{|P^2_{ia}|}{m_a x} \right) - 1 \right] - \frac{2}{1 - x} \ln(2 - x) + (1 + x) \ln(1 - x),$$

$$G^{(\text{sub})}_{ia,-}(P^2_{ia}, x) = 0,$$

$$G^{(\text{sub})}_{ai,-}(P^2_{ia}, x) = 1 - x,$$

and the corresponding endpoint parts $G^{(\text{sub})}_{ff',\tau}(P^2_{ia})$,

$$G^{(\text{sub})}_{ia,+}(P^2_{ia}) = \mathcal{L}(|P^2_{ia}|, m_i^2) - \frac{\pi^2}{2} + 1,$$

$$G^{(\text{sub})}_{ai,+}(P^2_{ia}) = \mathcal{L}(|P^2_{ia}|, m_a^2) + \frac{\pi^2}{6} - \frac{3}{2},$$

$$G^{(\text{sub})}_{ia,-}(P^2_{ia}) = G^{(\text{sub})}_{ai,-}(P^2_{ia}) = \frac{1}{2},$$

(3.20)

where $\mathcal{L}$ is the function defined in (3.8), which contains the logarithmic singularities. Only the emitter masses lead to mass singularities, as it should be. From the general discussion of mass singularities in Section 2.2 it is also clear that the singularity of final-state emitter appears only in the endpoint contribution $G^{(\text{sub})}_{ia,+}$. For an initial-state emitter also the distribution $G^{(\text{sub})}_{ai,+}$ contains a mass-singular part, which is proportional to the splitting function $P_{ff}(x)$. The spin-flip contributions are regular, and the one for a final-state emitter is completely contained in the endpoint part.

### 3.3 Initial-state emitter and initial-state spectator

For an emitter $a$ and a spectator $b$ from the initial state we introduce the variables

$$x_{ab} = \frac{p_a p_b - p_a k - p_b k}{p_a p_b}, \quad y_{ab} = \frac{p_a k}{p_a p_b},$$

(3.21)

and define the auxiliary functions $g^{(\text{sub})}_{ab,\tau}$ by

$$g^{(\text{sub})}_{ab,+}(p_a, p_b, k) = \frac{1}{(p_a k) x_{ab} \left[ \frac{2}{1 - x_{ab}} - 1 - x_{ab} \right]},$$

$$g^{(\text{sub})}_{ab,-}(p_a, p_b, k) = 0.$$

(3.22)
They possess the required asymptotic behaviour in the singular limits, which are characterized by
\[ x_{ab} \xrightarrow{k \to 0} 1, \quad y_{ab} \xrightarrow{k \to 0} 0, \quad x_{ab} \xrightarrow{p_a \to 0} x_a, \quad y_{ab} \xrightarrow{p_a \to 0} 0. \] (3.23)
Following the guideline of Ref. [3], the construction of the modified momenta \( \tilde{\Phi}_{0,ab} \), which are used to evaluate \( |\mathcal{M}_0 (\tilde{\Phi}_{0,ab})|^2 \) in (2.12), differs from the previous cases. Instead of changing only the emitter and spectator momenta, we now keep the spectator momentum \( p_b \) fixed and change all outgoing momenta \( k_j \) other than \( k \). Note that \( k_j \) also includes the momenta of neutral outgoing particles, i.e. we have
\[ P_{ab} = p_a + p_b - k = \sum_j k_j. \] (3.24)
The new momenta
\[ \tilde{p}_a^\mu = x_{ab}p_a^\mu, \quad \tilde{p}_{ab}^\mu = x_{ab}p_a^\mu + p_b^\mu \] (3.25)
are chosen in such a way that \( \tilde{p}_a^2 = 0 \) and \( \tilde{p}_{ab}^2 = P_{ab}^2 \). In the IR limit they obviously tend to \( p_a \) and \( P_{ab} \), respectively; in the collinear limit \( \tilde{p}_a^\mu \) approaches \( x_a p_a \) with \( x_a \), and
\[ \tilde{k}_j^\mu = \Lambda^\mu_\nu k_j^\nu \] (3.26)
with
\[ \Lambda^\mu_\nu = g^\mu_\nu - \frac{(P_{ab} + \tilde{P}_{ab})^\mu (P_{ab} + \tilde{P}_{ab})_\nu}{P_{ab}^2 + \tilde{P}_{ab}P_{ab}} + \frac{2\tilde{P}_{ab}^\mu P_{ab,\nu}}{P_{ab}^2}, \] (3.27)
so that the mass-shell relations \( \tilde{k}_j^2 = k_j^2 \) are retained. The necessary condition \( \Lambda^\mu_\nu \Lambda^\nu_\rho = g^\rho_\nu \) and the relation \( \sum_j k_j = \tilde{P}_{ab} \) are easily checked by direct calculation using \( \tilde{P}_{ab}^2 = P_{ab}^2 \).
The above relations comprise the necessary input for the construction of the \( ab \) part \( |\mathcal{M}_{sub,ab}|^2 \) of the subtraction function.

Concerning the analytical integration of the \( ab \) contribution to the subtraction function over the photonic part of phase space, the situation is similar to the previous section. The incoming momenta \( p_a \) and \( p_b \) are modified in an analogous way, namely \( p_a \) is scaled down to \( \tilde{p}_a \) by the variable \( x_{ab} \), while \( p_b \) is kept fixed. The CM frames of \( p_a + p_b \) and \( \tilde{p}_a + p_b \) are related by a boost along the beam axis, and the squared CM energies \( s \) and \( \tilde{s} \) are related by
\[ \tilde{s} = 2(\tilde{p}_a p_b) = 2x_{ab}(p_ap_b) = x_{ab}s. \] (3.28)
Note also that \( P_{ab}^2 = \tilde{s} = x_{ab}s \), owing to definition (3.21). The separation of the photonic part of phase space again leads to a convolution over \( x = x_{ab} \) for the integrated \( ab \) contribution \( |\mathcal{M}_{sub,ab}|^2 \) to the subtraction function,
\[ \int d\Phi_1 |\mathcal{M}_{sub,ab}(\Phi_1)|^2 = -\frac{\alpha}{2\pi} Q_a \sigma_a Q_b \sigma_b \int_0^1 dx \frac{1}{x} G_{ab,\tau}^{(sub)}(s, x) d\Phi_0(x) |\mathcal{M}_0(xp_a, \tilde{k}_n(x); \tau \kappa_a)|^2. \] (3.29)
The argument of the modified momenta \( \tilde{k}_n(x) \) indicates that the new phase space implicitly depends on \( x \). For the numerical evaluation of the convolution, it is appropriate to separate the IR-singular endpoint part
\[ G_{ab,\tau}^{(sub)}(s) = \int_0^1 dx G_{ab,\tau}^{(sub)}(s, x) \] (3.30)
from the distribution $G_{ab,\tau}^{(\text{sub})}$ with the help of the $[\ldots]_+$ prescription. The numerically accessible form of the convolution is

$$
\int d\Phi_1 |M_{\text{sub},ab}(\Phi_1)|^2 = -\frac{\alpha}{2\pi} Q_a \sigma_a Q_b \sigma_b 
\times \left\{ \int_0^1 dx G_{ab,\tau}^{(\text{sub})}(s, x) \left[ \frac{1}{x} \int d\Phi_{0,ab}(s, x) |M_0(x p_a, \tilde{k}_n(x); \tau \kappa_a)|^2 
- \int d\Phi_{0,ab}(s, 1) |M_0(p_a, \tilde{k}_n(1); \tau \kappa_a)|^2 \right] 
+ G_{ab,\tau}^{(\text{sub})}(s) \int d\Phi_{0,ab}(s, 1) |M_0(p_a, \tilde{k}_n(1); \tau \kappa_a)|^2 \right\},
$$

(3.31)

where the distributions are given by

$$
G_{ab,\tau}^{(\text{sub})}(s, x) = P_{ff}(x) \left[ \ln \left( \frac{s}{m_a^2} \right) - 1 \right], \quad G_{ab,\tau}^{(\text{sub})}(s, x) = 1 - x,
$$

(3.32)

and the endpoint parts read

$$
G_{ab,+}^{(\text{sub})}(s) = L(s, m_a^2) - \frac{\pi^2}{3} + \frac{3}{2}, \quad G_{ab,-}^{(\text{sub})}(s) = \frac{1}{2}.
$$

(3.33)

Note that the original squared CM energy $s$ is kept fixed in the integration over $x$ that defines the endpoint contributions in (3.30). This is also indicated in the phase-space measure $d\Phi_{0,ab}(s, x)$, which is to be parametrized for fixed $s$ and $x$. The mass singularities of the emitter $a$ are completely factorized into $G_{ab,+}^{(\text{sub})}$ and $G_{ab,+}^{(\text{sub})}$ so that the convolution over $x$ can be carried out numerically for vanishing photon and fermion masses. Of course, the spectator mass is set to zero everywhere.

4 Subtraction functions and integrated counterparts—general case

In this section we turn to the case of arbitrary finite fermion masses. Here we include also details of the derivation, which have been omitted in the previous section for brevity. The anticipated results for light fermions can be obtained from the general ones of this section by carefully expanding the corresponding formulas for small fermion masses.

Moreover, it is phenomenologically important to consider the case of light fermions only in the initial state, which is of particular interest for $e^+e^-$ collisions at high energies, as observed at LEP or the SLC. In this case, the dipole formalism is also considerably simpler than for general fermion masses. The results of the corresponding expansion are listed in App. [A.1].

4.1 Final-state emitter and final-state spectator

In order to construct the contribution to $|M_{\text{sub}}|^2$ corresponding to an emitter $i$ and a spectator $j$ from the final state, we have to define the auxiliary functions $g_{ij,\tau}^{(\text{sub})}(p_i, p_j, k)$ and the embedding of the momenta $\tilde{p}_i$ and $\tilde{p}_j$ into the phase space spanned by $p_i, p_j,$ and $k$. This embedding has to respect the mass-shell conditions $\tilde{p}_i^2 = p_i^2 = m_i^2, \tilde{p}_j^2 = p_j^2 = m_j^2,$ and
\( k^2 = m_\gamma^2 \), where the photon mass \( m_\gamma \) is taken to be infinitesimal in the final result. In the following we make use of the variables \( y_{ij} \) and \( z_{ij} \) of \((3.2)\) and introduce the abbreviations

\[
P_{ij} = p_i + p_j + k, \quad \bar{P}_{ij}^2 = P_{ij}^2 - m_i^2 - m_j^2 - m_\gamma^2, \quad \lambda_{ij} = \lambda(P_{ij}^2, m_i^2, m_j^2),
\]

where

\[
\lambda(x, y, z) = x^2 + y^2 + z^2 - 2xy - 2xz - 2yz.
\]

In the physical phase space we always have \( P_{ij}^2, \bar{P}_{ij}^2, \lambda_{ij} > 0 \). For later convenience, we define the auxiliary functions

\[
R_{ij}(y) = \frac{\sqrt{(2m_i^2 + \bar{P}_{ij}^2 y)^2 - 4\bar{P}_{ij}^2 m_i^2}}{\sqrt{\lambda_{ij}}},
\]

\[
r_{ij}(y) = 1 - \frac{2m_i^2(2m_i^2 + \bar{P}_{ij}^2)}{\lambda_{ij}} y \left( \frac{1 - y}{1 - y} \right).
\]

Their actual form is convention except for their behaviour near \( y = 0 \), where they are regular with \( R_{ij}(0) = r_{ij}(0) = 1 \) for \( m_\gamma = 0 \).

Using the above quantities, we define the functions \( g_{ij,r}^{(\text{sub})} \) as follows:

\[
g_{ij,+}^{(\text{sub})}(p_i, p_j, k) = \frac{1}{(p_i k) R_{ij}(y_{ij})} \left[ \frac{2}{1 - z_{ij}(1 - y_{ij})} - 1 - z_{ij} - \frac{m_i^2}{p_i k} \right] - g_{ij,-}^{(\text{sub})}(p_i, p_j, k),
\]

\[
g_{ij,-}^{(\text{sub})}(p_i, p_j, k) = \frac{m_i^2}{2(p_i k)^2} \frac{(1 - z_{ij})^2}{z_{ij}} r_{ij}(y_{ij}) R_{ij}(y_{ij}).
\]

It is straightforward to check that these functions obey the asymptotic conditions \((2.13)\) and \((2.14)\) in the IR and collinear limits, in which \( y_{ij} \) and \( z_{ij} \) behave as given in \((3.3)\). Note that the limits \((3.3)\) implicitly assume \( m_\gamma \to 0 \); the collinear limit additionally requires \( m_i \to 0 \). In the mapping \( \Phi_{0ij} \) from \( \Phi_1 \) to \( \Phi_0 \) we leave all momenta \( k_n \) other than \( p_i, p_j, \) and \( k \) unaffected, as in the case of light fermions. The momenta \( \tilde{p}_i \) and \( \tilde{p}_j \) are chosen as

\[
\tilde{p}_j^\mu = \frac{\sqrt{\lambda_{ij}}}{\sqrt{\lambda((p_i + k)^2, P_{ij}^2, m_j^2)}} \left( p_j^\mu - \frac{P_{ij} p_j^\mu}{P_{ij}^2} \right) + \frac{P_{ij}^2 + m_j^2 - m_i^2}{2P_{ij}^2} P_{ij}^\mu,
\]

\[
\tilde{p}_i^\mu = P_{ij}^\mu - \tilde{p}_j^\mu.
\]

The on-shell relations \( \tilde{p}_j^2 = m_j^2 \) and \( \tilde{p}_i^2 = m_i^2 \) directly follow by expanding the squared momenta, and momentum conservation \( P_{ij} = p_i + p_j + k = \tilde{p}_i + \tilde{p}_j \) is fulfilled by definition. Moreover, the validity of the required asymptotic behaviour \((2.13)\) and \((2.14)\) is obvious.

The relations given above include all ingredients that are necessary to calculate the \( ij \) contribution to the subtraction function \( |M_{\text{sub}}(\Phi_1)|^2 \). The phase-space integral of the difference \( \sum \lambda_{ij} |M_1|^2 - |M_{\text{sub}}|^2 \) is non-singular in the IR limit, and also in the collinear limit, which occurs for \( m_i \to 0 \). Therefore, this integral can be evaluated with \( m_\gamma = 0 \) everywhere, and with \( m_f = 0 \) for light fermions. However, we need the dependence on
Our aim is to perform the integration of the \(ij\) one that is spanned by \(k\) remaining phase-space variables contained in the measure \(\mathrm{d}m\). The form of this measure, which is derived in App. B, is given by

\[
\int [\mathrm{d}k(P^2_{ij}, y_{ij}, z_{ij})] = \frac{1}{4(2\pi)^3} \frac{P^4_{ij}}{\lambda_{ij}} \int_0^{2\pi} \mathrm{d}\varphi_{ij} \int_{y_1}^{y_2} \mathrm{d}y_{ij} (1 - y_{ij}) \int_{z_1(y_{ij})}^{z_2(y_{ij})} \mathrm{d}z_{ij}.
\] (4.7)

The angle \(\varphi_{ij}\) is the azimuthal angle of \(p_i\) with respect to the \(p_j\) axis in the CM frame of \(P_{ij}\). The integration boundary for the variables \(y_{ij}\) and \(z_{ij}\) is given by

\[
y_1 = \frac{2m_i m_j}{P^2_{ij}}, \quad y_2 = 1 - \frac{2m_j \left(\sqrt{P^2_{ij} - m_j}\right)}{P^2_{ij}},
\]

\[
z_{1,2}(y_{ij}) = \frac{A_{i,ij} + A_{j,ij} (1 - y_{ij}) + \sqrt{y_1 y_2 - A_{i,ij}^2}}{2(1 - y_2)(m_i^2 + m_j^2 + P^2_{ij} y_{ij})}. \quad (4.8)
\]

Since the integrand \(g^{(\text{sub})}_{ij,\tau}(p_i, p_j, k) \left| \mathcal{M}_0 \left(\tilde{c}_{ij} \tau \kappa_i\right)\right|^2\) of the phase-space integral does not depend on the angle \(\varphi_{ij}\), the integral over \(\varphi_{ij}\) simply yields a trivial factor of 2\(\pi\). Moreover, \(\left| \mathcal{M}_0 \left(\tilde{c}_{ij}\right)\right|^2\) is independent of \(y_{ij}\) and \(z_{ij}\) so that the integrations over \(y_{ij}\) and \(z_{ij}\) only concern the auxiliary functions \(g^{(\text{sub})}_{ij,\tau}\), and we define

\[
G^{(\text{sub})}_{ij,\tau}(P^2_{ij}) = \frac{P^4_{ij}}{2\sqrt{\lambda_{ij}}} \int_{y_1}^{y_2} \mathrm{d}y_{ij} (1 - y_{ij}) \int_{z_1(y_{ij})}^{z_2(y_{ij})} \mathrm{d}z_{ij} g^{(\text{sub})}_{ij,\tau}(p_i, p_j, k). \quad (4.9)
\]

While the integration over \(z_{ij}\) is very simple, the one over \(y_{ij}\) is non-trivial, but can be performed analytically. Details of the calculation can be found in App. C. We obtain

\[
G^{(\text{sub})}_{ij,+}(P^2_{ij}) = \ln \left(\frac{m_i^2 m_j^2}{m_i^2 a_3^2}\right) - 2 \ln(1 - a_3^2) + \frac{3}{2} + \frac{P^2_{ij}}{2\sqrt{\lambda_{ij}}} \left[ \ln(a_1) \ln \left(\frac{m_j^2 m_i^2}{\lambda_{ij} a_2}\right) \right.
\]

\[
+ 2 \text{Li}_2(a_1) + 4 \text{Li}_2 \left(1 + \frac{a_2}{a_1}\right) - 4 \text{Li}_2 \left(-\sqrt{a_1 a_2}\right) + \frac{1}{2} \ln^2(a_1) - \frac{\pi^2}{3}
\]

\[
 - G^{(\text{sub})}_{ij,-}(P^2_{ij}),
\]

\[
G^{(\text{sub})}_{ij,-}(P^2_{ij}) = \frac{P^4_{ij}}{\lambda_{ij}} \left\{ \begin{array}{l}
2m_i^2 m_j^2 \ln(a_3) + m_j^2 m_i^2 \ln \left(1 + a_3^2\right) + m_j^2 m_i^2 \left(\frac{P^2_{ij} + m_i^2}{P^2_{ij} + m_j^2}\right) \ln \left(\frac{2m_j m_i a_3}{P^2_{ij}}\right) \\
+m_j^2 m_i^2 \left(\frac{P^2_{ij} + m_i^2}{P^2_{ij}}\right) \left[ 4\arctan(a_3) - \pi \right] \\
+ (1 - a_3^2) \left\{ \frac{1}{2} + \frac{m_i^2 P^2_{ij}}{P^2_{ij}(P^2_{ij} + m_i^2) + 2 m_i^2 m_j} \right\}
\end{array} \right\}, \quad (4.10)
\]

\(m_i^3\), and light fermion masses \(m_f\) in the integral of \(|\mathcal{M}_{\text{sub}}|^2\) over the photon phase space, which is calculated next.
with the shorthands
\[
a_1 = \frac{\bar{P}^2_{ij} + 2m_i^2 - \sqrt{\lambda_{ij}}}{\bar{P}^2_{ij} + 2m_i^2 + \sqrt{\lambda_{ij}}}, \quad a_2 = \frac{\bar{P}^2_{ij} - \sqrt{\lambda_{ij}}}{\bar{P}^2_{ij} + \sqrt{\lambda_{ij}}}, \quad a_3 = \frac{m_i}{\sqrt{\bar{P}^2_{ij} - m_j}},
\]
(4.11)
and the usual dilogarithm \( \text{Li}_2(x) = -\int_0^1 dt \ln(1 - xt)/t \). The contribution of the \( ij \) part \( |M_{\text{sub},ij}(\Phi_1)|^2 \) to the phase-space integral of the subtraction function is formally the same as given in (3.6) for light fermions, where we have to insert the functions \( G_{ij,\tau}^{(\text{sub})} \) of (4.10) for finite fermion masses. Expanding these functions for \( m_{i,j} \to 0 \), we obtain the results of (3.7).

4.2 Final-state emitter and initial-state spectator

For a final-state emitter \( i \) and an initial-state spectator \( a \) we keep the definitions (3.10) of the variables \( x_{ia}, z_{ia} \) and introduce the abbreviations
\[
P_{ia} = p_i + k - p_a, \quad \bar{P}^2_{ia} = P^2_{ia} - m_a^2 - m_i^2 - m^2, \quad \lambda_{ia} = \lambda(P^2_{ia}, m_a^2, m_i^2).
\]
(4.12)
In the following, we only consider fixed momenta \( P_{ia} \) that obey
\[
P^2_{ia} < (m_a - m_i)^2,
\]
(4.13)
because other values of \( P^2_{ia} \) do not admit the limits \( k \to 0 \) or \( p_i k \to 0 \), as can be checked easily. In particular, (4.13) ensures that \( P^2_{ia} < 0 \). Moreover, we introduce an auxiliary parameter \( x_0 \) with \( 0 \leq x_0 < 1 \) that specifies the lower limit on \( x_{ia} \) for which the subtraction function will be applied. We are forced to deviate from the simple choice \( x_0 = 0 \), because \( x_{ia} \to 0 \) is not allowed for all configurations of \( P^2_{ia}, m_a, \) and \( m_i \). More precisely, one has to require
\[
x_0 > \hat{x} = \frac{-\bar{P}^2_{ia}}{2m_a \left(m_a - \sqrt{P^2_{ia}}\right)} \quad \text{if} \quad 0 < \sqrt{P^2_{ia}} < m_a - m_i,
\]
(4.14)
which is only possible for \( P^2_{ia} > 0 \) and \( m_a > m_i \). Otherwise we can take any value for \( x_0 \) with \( 0 \leq x_0 < 1 \). For instance, it is possible to set \( x_0 = 0 \) for vanishing fermion masses, as done in Section 3.2. For \( P^2_{ia} > (m_a - m_i)^2 \) or \( x_{ia} < x_0 \) the subtraction functions \( g_{ia,\tau}^{(\text{sub})} \) are set to zero consistently. Note that both IR and collinear singularities appear at \( x_{ia} \to 1 \), i.e. applying the subtraction function for \( x_0 < x_{ia} < 1 \) correctly cancels these singularities. The final results on observables must not depend on \( x_0 \), which will not be further specified in the following. Checking the \( x_0 \) independence of observables is a non-trivial check on the complete subtraction procedure.

It is convenient to introduce the auxiliary functions
\[
R_{ia}(x) = \sqrt{(\bar{P}^2_{ia} + 2m_a x)^2 - 4m_a^2 P^2_{ia} x^2},
\]
\[
r_{ia}(x) = \frac{\bar{P}^2_{ia} (\bar{P}^2_{ia} + 2m_a^2) \frac{1}{x}}{\lambda_{ia} 1 - x},
\]
(4.15)
with \( R_{ia}(1) = r_{ia}(1) = 1 \) for vanishing photon mass \( m_\gamma \). The subtraction functions for finite fermion masses read

\[
g^{(\text{sub})}_{ia,+}(p_i, p_a, k) = \frac{1}{(p_ik)_{xia}} \left[ \frac{2}{2 - x_a - z_{ia}} - 1 - z_{ia} - \frac{m_i^2}{p_ik} \right] - g^{(\text{sub})}_{ia,-}(p_i, p_a, k),
\]

\[
g^{(\text{sub})}_{ia,-}(p_i, p_a, k) = \frac{m_i^2}{2(p_ik)^2} \frac{(1 - z_{ia})^2 r_{ia}(x_{ia})}{z_{ia} x_{ia}}.
\] (4.16)

They possess the desired asymptotic behaviour (2.13) and (2.14) in the singular limits. The behaviour of \( x_{ia} \) and \( z_{ia} \) for \( k \to 0 \) or \( p_ik \to 0 \) is given in (3.11). The auxiliary momenta \( \tilde{\Phi}_{0,ia} \), which are needed to evaluate \( |M_0(\tilde{\Phi}_{0,ia})|^2 \), are constructed in a similar way as for light fermions. The emitter and spectator momenta are given by

\[
\tilde{p}_{ia}^\mu = \left( \frac{\sqrt{\lambda_{ia}}}{\sqrt{\lambda((p_i + k)^2, P_{ia}^2, m_a^2)}} \right) (P_{ia}^\mu - \frac{p_{ia} p_a}{P_{ia}^2} P_{ia}^\mu) + \frac{P_{ia}^2 - m_a^2 + m_i^2}{2P_{ia}^2} P_{ia}^\mu,
\]

\[
\tilde{p}_{a}^\mu = \tilde{p}_{ia}^\mu - P_{ia}^\mu,
\] (4.17)

and the remaining momenta \( \tilde{k}_a \) coincide with the corresponding momenta \( k_n \) of \( \tilde{\Phi}_1 \). The on-shell relations \( p_{ia}^2 = P_{ia}^2 = m_a^2 \) and \( \tilde{p}_{a}^2 = \tilde{p}_{ia}^2 = m_i^2 \) can be verified easily. Momentum conservation \( P_{ia} = p_i + k - p_a = \tilde{p}_i - \tilde{p}_{ia} \) and the validity of the required asymptotics (2.13) and (2.14) are obvious.

Using the above relations, the \( ia \) part of \( |M_{\text{sub}}(\Phi_1)|^2 \) can be evaluated. Concerning the role of the masses \( m_\gamma \) and \( m_f \) in the difference \( (\sum_{\lambda_n} |M_1|^2 - |M_{\text{sub}}|^2) \), the remarks of the previous section apply as well.

As already explained in Section 3.2, the photonic part of phase space, which is obtained by separating \( d\tilde{\Phi}_{0,ia} \) from \( d\Phi_1 \), involves an integration over the variable \( x_{ia} \). This integration over \( x_{ia} \) turns into a convolution over \( x = x_{ia} \) in the integration of the subtraction function, where \( x \) determines the CM energy of the reduced phase space spanned by the momenta \( \tilde{\Phi}_{0,ia} \). For the explicit splitting of phase space, we decompose the momentum \( P_{ia} = p_b - K_{ia} \) into the second incoming momentum \( p_b \) and the total momentum \( K_{ia} \) of all outgoing particles other than \( i \) and the inspected photon. The phase-space separation is defined by

\[
\int d\phi(p_i, k, K_{ia}; p_a + p_b) \theta(x_{ia} - x_0)
= \int_{x_0}^{x_1} dx \int d\phi(\tilde{p}_i(x), K_{ia}; \tilde{p}_a(x) + p_b) \int [dk(P_{ia}^2, x, z_{ia})],
\] (4.18)

with \( x_1 \) given below. The \( x \)-dependent momenta

\[
\tilde{p}_{ia}^\mu(x) = \frac{1}{R_{ia}(x)} \left( x P_{ia}^\mu + \tilde{p}_{ia}^2 + 2m_a^2 x P_{ia}^\mu \right) - \frac{P_{ia}^2 + m_a^2 - m_i^2}{2P_{ia}^2} P_{ia}^\mu,
\]

\[
\tilde{p}_i^\mu(x) = \tilde{p}_{ia}^\mu(x) + P_{ia}^\mu
\] (4.19)

result from \( \tilde{p}_a \) and \( \tilde{p}_i \) upon substituting \( (p_a P_{ia}) \to -m_a^2 - \tilde{p}_{ia}^2/(2x) \), which eliminates \( (p_a P_{ia}) \) in favour of \( x \) and \( P_{ia}^2 \). Note that the substitution also concerns \( (p_i + k)^2 = \)
\((p_a + P_{ia})^2\) in \([1,17]\). If we now try to reconstruct \(\vec{p}_{\gamma}(x)\) from \(x\) and \(p_a\), as it was possible for \(m_a = 0\) in Section 3.2, we find that the knowledge of \(x\) and \(p_a\) is not sufficient for \(m_a \neq 0\). This complication is due to the fact that the boost relating the CM frames of \(p_a + p_b\) and \(\vec{p}_a + p_b\) does not simply go along the beam axis for finite \(m_a\). We will come back to this problem and to its solution at the end of this section and proceed by performing the integral of the contribution to the subtraction function over \([dk(P_{ia}^2, x, z_{ia})]\). In App. 3 the explicit form of this measure is derived. The result is

\[
\int [dk(P_{ia}^2, x, z_{ia})] = \frac{1}{4(2\pi)^3} \frac{P_{ia}^4}{\sqrt{\lambda_{ia} R_{ia}(x)}} \frac{\rho_{ia}(\bar{s})}{x^2} \int_{z_1(x)}^{z_2(x)} d\bar{z}_{ia} \int_0^{2\pi} d\varphi_\gamma, \tag{4.20}
\]

where \(\varphi_\gamma\) is the azimuthal angle of the photon in the CM frame of \(p_i + k\). The function \(\rho_{ia}(\bar{s})\) reads

\[
\rho_{ia}(\bar{s}) = \left[ \frac{\lambda(\bar{s}, m_{\gamma}, m_\gamma^2)}{\lambda(s, m_{\gamma}, m_\gamma^2)} \right]. \tag{4.21}
\]

where \(s\) and \(\bar{s}\) denote the squared CM energies of \(p_a + p_b\) and \(\vec{p}_a + p_b\), respectively. The integration boundary for \(z_{ia}\) is given by

\[
z_{1,2}(x) = \frac{P_{ia}^2[\bar{P}_{ia}^2 - x(P_{ia}^2 + 2m_\gamma^2)] + \sqrt{P_{ia}^4(1-x)^2 - 4m_\gamma^2m_{\gamma}^2x^2} \sqrt{\lambda_{ia} R_{ia}(x)}}{2P_{ia}^2[\bar{P}_{ia}^2 - x(P_{ia}^2 - m_\gamma^2)]}. \tag{4.22}
\]

From this relation we read off that the maximal value of \(x\) is given by

\[
x_1 = \frac{\bar{P}_{ia}^2}{P_{ia}^2 - 2m_\gamma m_{\gamma}} = 1 - \frac{2m_\gamma m_{\gamma}}{[P_{ia}^2]} + \mathcal{O}(m_{\gamma}^2). \tag{4.23}
\]

While the integration of \(g_{ia,\tau}^{(sub)}(p_i, p_a, k) |\mathcal{M}_0 (\Phi_{0,ia}; \tau R_i)|^2\) over \(\varphi_\gamma\) yields a trivial factor of \(2\pi\), the integration over \(z_{ia}\) depends on the actual form of \(g_{ia,\tau}^{(sub)}\). Defining

\[
G_{ff',\tau}^{(sub)}(P_{ia}^2, x_{ia}) = \frac{\bar{P}_{ia}^4}{2\sqrt{\lambda_{ia} R_{ia}(x_{ia})}} \int_{z_1(x_{ia})}^{z_2(x_{ia})} d\bar{z}_{ia} g_{ff',\tau}^{(sub)}(p_f, p_{f'}, k) \tag{4.24}
\]

for \(ff' = ia\), we obtain

\[
G_{ia,+}^{(sub)}(P_{ia}^2, x) = - \frac{\bar{P}_{ia}^2}{\sqrt{\lambda_{ia} R_{ia}(x)(1-x)}} \left\{ 2\ln \left[ \frac{2 - x - z_1(x)}{2 - x - z_2(x)} \right] + \frac{z_1(x) - z_2(x)}{2} \right\} - G_{ia,-}^{(sub)}(P_{ia}^2, x),
\]

\[
G_{ia,-}^{(sub)}(P_{ia}^2, x) = \frac{m_i^2}{\sqrt{\lambda_{ia} (1-x)^2 R_{ia}(x)}} \left\{ \ln \left[ \frac{z_2(x)}{z_1(x)} \right] + \frac{z_1(x) - z_2(x)}{2} \right\}. \tag{4.25}
\]

after a simple integration. The function \(G_{ia,+}^{(sub)}\) is singular in the limit \(x \to 1\) for \(m_{\gamma} = 0\). Since our aim is to perform the convolution over \(x\) numerically with \(m_{\gamma} = 0\), we separate
the singularity at $x \to 1$ by introducing the $[\ldots]_+$ prescription. Considering that our lower integration limit $x_0$ can be different from zero, we write

$$G^{(\text{sub})}_{ff',\tau}(P^2_{ia},x) = \left[ G^{(\text{sub})}_{ff',\tau}(P^2_{ia},x)\delta(x-x_0) \right]_+ + G^{(\text{sub})}_{ff',\tau}(P^2_{ia},x_0)\delta(1-x)$$

(4.26)

with

$$G^{(\text{sub})}_{ff',\tau}(P^2_{ia},x_0) = \int_{x_0}^{x_1} dx \, G^{(\text{sub})}_{ff',\tau}(P^2_{ia},x),$$

(4.27)

where $ff' = ia$. In writing (4.26), we have already used that the photon mass $m_\gamma$ can be set to zero, and thus $x_1$ set to 1, everywhere apart from $G^{(\text{sub})}_{ia,\tau}$, which contains the IR singularity. The integration over the distribution $G^{(\text{sub})}_{ia,\tau}(P^2_{ia},x)\delta(x-x_0)$ can be performed with $m_\gamma = 0$ and, if desired, with $m_f = 0$ for the fermion masses. The endpoint contributions $G^{(\text{sub})}_{ia,\tau}(P^2_{ia},x_0)$ are obtained after performing the non-trivial integration over $x$. Details of this integration can be found in App. [A]. The results are

$$G^{(\text{sub})}_{ia,\tau}(P^2_{ia},x_0) = 2 \ln \left( \frac{m_\gamma m_i}{P^2_{ia}(1-x_0)} \right) + \frac{P^4_{ia}}{2(P^2_{ia} + m_i^2)^2} \ln \left[ \frac{m_\gamma x_0 - P^2_{ia}(1-x_0)}{m_i^2} \right]$$

$$- \frac{P^4_{ia}(1-x_0)}{2(P^2_{ia} + m_i^2)[P^2_{ia}(1-x_0) - m_i^2x_0]} + 2$$

$$+ \frac{P^2_{ia}}{\lambda_{ia}} \left\{ 2 \ln(b_1) \ln \left[ \frac{b_0 \sqrt{\lambda_{ia}(m_i^2 + m_i^2 - P^2_{ia})}}{m_\gamma m_i^2} \right] - \frac{1}{2} \ln^2(b_1) + \frac{\pi^2}{3} $$

$$+ 2 \sum_{k=1}^{5} (-1)^k \text{Li}_2(b_k) \right\} - G^{(\text{sub})}_{ia,-}(P^2_{ia},x_0),$$

$$G^{(\text{sub})}_{ia,-}(P^2_{ia},x_0) = \frac{2m_\gamma m_i(P^2_{ia} + 2m_i^2)}{\lambda_{ia}} \arctan \left[ \frac{m_i}{m_\gamma (1-x_0)} \right] - \frac{m_i}{\sqrt{\lambda_{ia}}} R_{ia}(x_0) \ln(b_0)$$

$$+ \frac{m_i^2 P^2_{ia}}{\lambda_{ia}} \left\{ \ln(x_0) - \left[ 1 + \frac{m_i^2 P^2_{ia}}{(P^2_{ia} + m_i^2)^2} \right] \ln \left[ x_0 - \frac{P^2_{ia}}{m_i^2}(1-x_0) \right] $$

$$+ \frac{2m_i^2}{P^2_{ia}} \ln \left[ 1 + \frac{m_i^2}{m_i^2}(1-x_0)^2 \right] - \frac{x_0 [P^2_{ia} - 2m_i^2(1-x_0)]}{2[P^2_{ia}(1-x_0) - m_i^2x_0]}$$

$$- \frac{4m_i^2}{P^2_{ia}} + \frac{m_i^2(1-x_0)}{P^2_{ia} + m_i^2} \right\} + \frac{P^4_{ia}}{2\lambda_{ia}},$$

(4.28)

with the abbreviations

$$b_0 = \frac{-4m_a^2 P^2_{ia}(1-x_0)}{\lambda_{ia}[1 + R_{ia}(x_0)]^2 + 4m_a^2(P^2_{ia} + m_i^2)(1-x_0)^2}, \quad b_1 = \frac{2m_i^2 - P^2_{ia} - \sqrt{\lambda_{ia}}}{2m_i^2 - P^2_{ia} + \sqrt{\lambda_{ia}}},$$

$$b_{2,3} = \frac{2m_i^2 + P^2_{ia} \pm \sqrt{\lambda_{ia}}}{-P^2_{ia} \pm \sqrt{\lambda_{ia}}} b_0, \quad b_{4,5} = \frac{2m_i^2 - m_a^2 \mp \sqrt{\lambda_{ia}}}{2m_i^2 - P^2_{ia} \pm \sqrt{\lambda_{ia}}} b_0, $$

$$b_6 = \frac{2m_i^2 x_0 - P^2_{ia}(1-x_0) + \sqrt{\lambda_{ia}} R_{ia}(x_0)(1-x_0)}{2m_i^2 x_0 - P^2_{ia}(1-x_0) - \sqrt{\lambda_{ia}} R_{ia}(x_0)(1-x_0)}.$$  

(4.29)

We have checked that all arguments of the logarithms and dilogarithms in (4.28) lie on the first Riemann sheet of the corresponding function for the allowed regions of the various
parameters. Although the spin-flip contribution \( G_{ia,-}^{(sub)}(P_{ia}^2, x) \) is not IR-singular at \( x \to 1 \), we have nevertheless introduced the \([\ldots]_+\) distribution. This is advantageous for a large momentum transfer, since in the limit \( m_{a,i} \to 0 \) the complete contribution of \( G_{ia,-}^{(sub)}(P_{ia}^2, x) \) is contained in the endpoint part \( G_{ia,-}^{(sub)}(P_{ia}^2, x_0) \). If the limit \( x_0 \) can be set to zero, many terms in the endpoint contributions \( G_{ia,-}^{(sub)}(P_{ia}^2, x_0) \) simplify. In order to facilitate the application of our results, we list the corresponding results \( G_{ia,\tau}^{(sub)}(P_{ia}^2) = G_{ia,\tau}^{(sub)}(P_{ia}^2, 0) \) in App. A.2 explicitly.

The final result for the \( ia \) contribution to the phase-space integral of the subtraction function reads

\[
\int d\Phi_1 \left| M_{\text{sub},ff'}(\Phi_1) \right|^2 = -\frac{\alpha}{2\pi} Q_a \sigma_a Q_i \sigma_i \\
\times \left\{ \int_{x_0}^1 dx \left[ \int d\tilde{\Phi}_{0,ia}(K_{ia}^2, P_{ia}^2, x) \frac{P_{ia}(\tilde{s})}{x^2} G_{ff',\tau}^{(sub)}(P_{ia}^2, x) \left| M_0(\tilde{p}_a(x), \tilde{p}_i(x); \tau \kappa_f) \right|^2 \right. \\
- \int d\tilde{\Phi}_{0,ia}(K_{ia}^2, P_{ia}^2, 1) G_{ff',\tau}^{(sub)}(P_{ia}^2, x) \left| M_0(\tilde{p}_a(1), \tilde{p}_i(1); \tau \kappa_f) \right|^2 \right. \\
+ \int d\tilde{\Phi}_{0,ia}(K_{ia}^2, P_{ia}^2, 1) G_{ff',\tau}^{(sub)}(P_{ia}^2, x_0) \left| M_0(\tilde{p}_a(1), \tilde{p}_i(1); \tau \kappa_f) \right|^2 \right\}, \tag{4.30}
\]

with \( ff' = ia \). As already mentioned above, the phase-space integration over the \( x \)-dependent momenta has to be performed carefully. First, one has to determine the squared CM energy \( \tilde{s} = (\tilde{p}_a(x) + p_b)^2 \) of the new initial state. The needed scalar product \( p_a(x)p_b \) is obtained upon contracting the first equation of (4.19) with \( p_{b,\mu} \). The product \( p_{a,\mu}p_b \), which appears on the r.h.s., can be replaced by \( P_{ia}p_b = (m_b^2 + P_{ia}^2 - K_{ia}^2)/2 \), according to the definition of the outgoing momentum \( K_{ia} \). In summary, we obtain \( \tilde{s} \) as a function of \( x, P_{ia}^2 \), and \( K_{ia}^2 \):

\[
\tilde{s} = m_a^2 + m_b^2 + \frac{1}{R_{ia}(x)} \left[ x(s - m_a^2 - m_b^2) \right. \\
\left. + (\tilde{P}_{ia} + 2m_a^2)(\tilde{P}_{ia} - K_{ia}^2) \right] \\
- \frac{(P_{ia}^2 + m_a^2 - m_b^2)(m_b^2 + P_{ia}^2 - K_{ia}^2)}{2P_{ia}^2}. \tag{4.31}
\]

Thus, it is necessary first to fix \( x, P_{ia}^2 \), and \( K_{ia}^2 \) in the phase-space integration over \( d\Phi_{0,ia}(K_{ia}^2, P_{ia}^2, x) \), before the other phase-space-variables can be parametrized, which is indicated by the arguments of \( d\Phi_{0,ia} \).

### 4.3 Initial-state emitter and final-state spectator

The case of an initial-state emitter \( a \) and a final-state spectator \( i \) is kinematically identical with the previous one, where the roles played by \( a \) and \( i \) are interchanged. Therefore, we can make use of the variables and auxiliary quantities \( x_{ia}, z_{ia}, P_{ia}, \) etc. of the previous section and adopt the same restrictions on \( P_{ia}^2 \) and \( x_{ia} \). For finite fermion masses the auxiliary functions \( g_{ai,\tau}^{(sub)} \) read

\[
g_{ai,+}^{(sub)}(p_a, p_i, k) = \frac{1}{(p_a k) x_{ia}} \left[ \frac{2}{2 - x_{ia} - z_{ia}} - R_{ia}(x_{ia})(1 + x_{ia}) - \frac{x_{ia}m_a^2}{p_a k} \right] - g_{ai,-}^{(sub)}(p_a, p_i, k),
\]
\begin{align}
g_{a_i,-}(p_a, p_i, k) &= \frac{m_a^2}{2(p_a k)^2} \frac{(1 - x_{ia})^2}{x_{ia}}. \tag{4.32}
\end{align}

In the IR and collinear limits the functions \((4.32)\) behave as required in \((2.13)\) and \((2.14)\). The behaviour of \(x_{ia}\) and \(z_{ia}\) in those limits is given in \((3.11)\). The auxiliary momenta \(p_a\) and \(p_i\) are constructed as specified in \((4.17)\), completing the construction prescription for the subtraction contribution \(|\mathcal{M}_{\text{sub}, ai}|^2\).

The separation of the photon phase space also proceeds along the same lines as in the previous section, leading to the same kind of convolution over \(x\). In contrast to the previous case, in which the singularities appeared for \(x \to 1\), the collinear singularity \((p_a k \to 0)\) is not restricted to a single point in \(x\), as can be seen in \((3.11)\). Therefore, it is necessary to choose the lower limit \(x_0\) for \(x\) small enough so that the complete range of \(x = x_{ia}\) is covered for small \(m_a\); otherwise collinear singularities in \((\sum_\gamma |\mathcal{M}_1|^2 - |\mathcal{M}_{\text{sub}}|^2)\) remain uncancellable. Since negative values of \(x_{ia}\) (if they occur at all) can never lead to collinear singularities, our initial restriction \(x_0 \geq 0\) is consistent. If effects of \(O(m_a)\) are consistently neglected, one can simply take \(x_0 = 0\), as already done in Section 3.

The analytical integration of the subtraction function over the photon phase space is performed as in the previous section. Hence, we define \(G_{ai,\tau}^{(\text{sub})}\) according to \((4.24)\) with \(ff' = ai\) and carry out the simple integration over \(z_{ia}\), yielding

\begin{align}
G_{ai, \tau}^{(\text{sub})}(P_{ia}^2, x) &= -\frac{P_{ia}^2}{\sqrt{\lambda_{ia} R_{ia}(x)}} \left\{ \frac{2}{1 - x} \ln \left[ \frac{1 - z_1(x)}{1 - z_2(x)} \right] \right. \\
&+ R_{ia}(x)(1 + x) \ln \left[ \frac{1 - z_2(x)}{1 - z_1(x)} \right] + \frac{2m_a^2}{P_{ia}^2} \frac{x^2}{1 - z_2(x)} - \frac{1}{1 - z_1(x)} \left. \right\}
\end{align}

\begin{align}
G_{ai, -}^{(\text{sub})}(P_{ia}^2, x) &= 1 - x. \tag{4.33}
\end{align}

For \(m_\gamma = 0\), the function \(G_{ai, \tau}^{(\text{sub})}(P_{ia}^2, x)\) becomes singular at \(x \to 1\). This singularity is split off by introducing the \([\ldots]_+\) distribution as specified in \((4.26)\) and \((4.27)\) with \(ff' = ai\), thereby defining the endpoint contributions \(G_{ai,\tau}^{(\text{sub})}(P_{ia}^2, x_0)\). Actually, this splitting is not needed for \(G_{ai, -}^{(\text{sub})}(P_{ia}^2, x)\), which is a simple regular function; we proceed this way in order to keep the generic description of the polarized and unpolarized cases. The integration over \(x\) can be performed analytically, yielding

\begin{align}
G_{ai, +}^{(\text{sub})}(P_{ia}^2, x_0) &= 2 \ln \left[ \frac{m_\gamma m_i}{P_{ia}^2(1 - x_0)} \right] + \frac{P_{ia}^4 - P_{ia}(1 - R_{ia}(x_0))}{4m_a^2(P_{ia}^2 + m_i^2)} + 2(1 - x_0) \\
&+ \frac{P_{ia}^4 (3P_{ia}^2 + 2m_i^2)}{2\sqrt{\lambda_{ia}(P_{ia}^2 + m_i^2)^2}} \left\{ \frac{1}{\gamma} \ln \left[ \frac{P_{ia}^2 \gamma^2 + 2m_i^2 + \gamma \sqrt{\lambda_{ia}}}{P_{ia}^2 \gamma^2 + 2m_i^2 x_0 + \gamma \sqrt{\lambda_{ia}} R_{ia}(x_0)} \right] \right. \\
&+ \ln \left[ x_0 - \frac{P_{ia}^2}{m_i^2}(1 - x_0) \right] + \ln \left[ \frac{P_{ia}^2(1 - \gamma^2) - 2m_i^2}{P_{ia}^2(1 - \gamma^2) - 2m_i^2 x_0 + \sqrt{\lambda_{ia}} R_{ia}(x_0)} \right] \right. \\
&+ \frac{P_{ia}^2}{\sqrt{\lambda_{ia}}} \left\{ 2 \ln \left( \frac{m_\gamma m_i}{b_0 \sqrt{\lambda_{ia}}} \right) \ln \left( \frac{c_1}{c_0} \right) - \ln \left( \frac{m_i^2 + \gamma^2 P_{ia}^2}{m_i^2} \right) \ln(c_1) + \frac{1}{2} \ln(c_0 c_1) \ln \left( \frac{c_1}{c_0} \right) \right. \\
&+ \frac{x_0}{2} (2 + x_0) \ln \left( \frac{P_{ia}^2 + \sqrt{\lambda_{ia}} R_{ia}(x_0)}{P_{ia}^2 - \sqrt{\lambda_{ia}} R_{ia}(x_0)} \right) - \frac{1}{2} \ln(c_0) - 2 \sum_{k=0}^{5} (-1)^k \text{Li}_2(c_k) \right\}
\end{align}
\[ G_{ai,-}^{(\text{sub})}(P_{ia}^2, x_0) = \frac{1}{2}(1 - x_0)^2, \]

with the shorthands
\[
c_0 = \frac{\bar{P}_{ia}^2 + \sqrt{\lambda_{ia}}}{\bar{P}_{ia}^2 - \sqrt{\lambda_{ia}}} \quad c_1 = b_1, \quad c_{2,3} = -\frac{2m_a^2 + \bar{P}_{ia}^2 \mp \sqrt{\lambda_{ia}}}{2m_a^2} b_0, \quad c_{4,5} = b_{4,5}, \]

\[ \gamma = \frac{m_a}{\sqrt{-\bar{P}_{ia}^2 - m_i^2 + i\epsilon}}. \]

The variables \( b_{0,1,4,5} \) are defined in (4.29). Although the variable \( \gamma \) can become imaginary for some values of \( P_{ia}^2 \), the result for \( G_{ai,+}^{(\text{sub})} \) is always real and does not depend on the sign of the infinitesimal imaginary part \( i\epsilon \). The endpoint contributions \( G_{ai,\tau}^{(\text{sub})}(P_{ia}^2, x_0) = G_{ai,\tau}^{(\text{sub})}(P_{ia}^2, 0) \) for the simpler value \( x_0 = 0 \) are explicitly listed in App. A.2.

The result for the \( ai \) contribution to the integrated subtraction function takes the same form as in the \( ia \) case, but now we have to identify \( f f' = ai \) in (4.30). Concerning the phase-space integration over \( d\hat{\Phi}_{0,ia}(K_{ia}^2, P_{ia}^2, x) \), the remarks made at the end of the previous section apply as well. The IR singularity is contained in the endpoint part \( G_{ai,+}^{(\text{sub})}(P_{ia}^2, x_0) \). However, the collinear singularity also appears in the function \( G_{ai,+}^{(\text{sub})}(P_{ia}^2, x_0) \). Since all singular terms are factorized, the convolution over \( x \) itself can be carried out with \( m_\gamma = 0 \), and with \( m_f = 0 \) for light fermions.

### 4.4 Initial-state emitter and initial-state spectator

For an emitter \( a \) and a spectator \( b \) we keep the definition (3.21) of the variables \( x_{ab} \) and \( y_{ab} \). Moreover, we introduce the abbreviations
\[
P_{ab} = p_a + p_b - k, \quad s = (p_a + p_b)^2, \quad \bar{s} = s - m_a^2 - m_b^2, \quad \lambda_{ab} = \lambda(s, m_a^2, m_b^2) \]

and the auxiliary function
\[
R_{ab}(x) = \frac{\sqrt{(\bar{s}x + m_a^2)^2 - 4m_a^2m_b^2}}{\sqrt{\lambda_{ab}}}. \]

The function is regular at \( x \to 1 \) with \( R_{ab}(1) = 1 \) for \( m_\gamma = 0 \). For the contribution to the subtraction function we define
\[
g_{ab,+}^{(\text{sub})}(p_a, p_b, k) = \frac{1}{(p_a k)x_{ab}} \left[ \frac{2}{1 - x_{ab}} - 1 - x_{ab} \frac{m_a^2}{p_a k} \right] - g_{ab,-}^{(\text{sub})}(p_a, p_b, k),
\]
\[
g_{ab,-}^{(\text{sub})}(p_a, p_b, k) = \frac{m_a^2}{2(p_a k)^2} \frac{(1 - x_{ab})^2}{x_{ab}}. \]

These functions possess the required asymptotic behaviour in the singular limits, which are characterized by (3.23). The functions \( g_{ab,\tau}^{(\text{sub})} \) are set to zero for \( x_{ab} < x_0 \), where the kinematical lower bound
\[
x_0 \geq \hat{x} = \frac{2m_a m_b - m_\gamma^2}{\bar{s}}. \]
has to be respected. For $x_{ab}$ smaller than $\hat{x}$, collinear photon emission cannot occur, and the following construction of new momenta would break down. Note that $x_0$ can be set to zero in either limit of $m_a \to 0$ or $m_b \to 0$, in consistency with our treatment of light fermions above.

As already explained in the case of light fermions in Section 3.3, the spectator momentum $p_b$ is kept fixed, but the emitter momentum $p_a$ and the momenta $k_j$ of all outgoing particles other than the photon are changed, resulting in a modification of the total momentum $P_{ab} = \sum_j k_j$. We map the momenta $p_a$ and $P_{ab}$ to

$$\tilde{p}_a^\mu = \frac{\sqrt{\lambda(P_{ab}^2, m_a^2, m_b^2)}}{\sqrt{\lambda_{ab}}} \left( p_a^\mu - \frac{p_a p_b}{m_b^2} p_b^\mu \right) + \frac{P_{ab}^2 - m_a^2 - m_b^2}{2m_b^2} p_b^\mu,$$

$$\tilde{P}_{ab}^\mu = \tilde{p}_a^\mu + p_b^\mu,$$

leading to the mass-shell relations $\tilde{p}_a^2 = m_a^2$ and $\tilde{P}_{ab}^2 = P_{ab}^2$. These relations and the validity of the required asymptotics in the IR and collinear limits can be checked easily. The individual momenta $k_j$ are modified by a Lorentz transformation in the same way as for light fermions, i.e. we have $\tilde{k}_j^\mu = \Lambda^\mu_{\nu} k_j^\nu$ as defined in (3.26). The actual form (3.27) of the transformation $\Lambda^\mu_{\nu}$ remains valid, but the momenta $P_{ab}$ and $\tilde{P}_{ab}$ of this section have to be inserted. This completes the necessary input for the construction of the differential subtraction contribution $|\mathcal{M}_{\text{sub},ab}|^2$.

The separation of the photon phase space is again written in terms of a convolution over an auxiliary parameter $x$,

$$\int d\phi(k, P_{ab}; p_a + p_b) \theta(x - x_0) = \int_{x_0}^{x_1} dx \int d\phi(\tilde{P}_{ab}(x); \tilde{p}_a(x) + p_b) \int [dk(s, x, y_{ab})].$$

(4.41)

The $x$-dependent momenta

$$\tilde{p}_a^\mu(x) = P_{ab}(x) \left( p_a^\mu - \frac{s}{2m_b^2} p_b^\mu \right) + \frac{sx + m_{\gamma}^2}{2m_b^2} p_b^\mu,$$

$$\tilde{P}_{ab}^\mu(x) = \tilde{p}_a^\mu(x) + p_b^\mu$$

(4.42)

are obtained from $\tilde{p}_a$ and $\tilde{P}_{ab}$ upon replacing $P_{ab}$ by $(sx + m_a^2 + m_b^2 + m_{\gamma}^2)$. Thus, they coincide with $\tilde{p}_a$ and $\tilde{P}_{ab}$ at $x = x_{ab}$,

$$\tilde{p}_a^\mu(x_{ab}) = \tilde{p}_a^\mu, \quad \tilde{P}_{ab}^\mu(x_{ab}) = \tilde{P}_{ab}^\mu.$$  

(4.43)

Note that $\tilde{p}_a^2(x) = m_a^2$ even for $x \neq x_{ab}$. The measure $[dk(s, x, y_{ab})]$ for the photon phase space is derived in App. B. The result is

$$\int [dk(s, x, y_{ab})] = \frac{1}{4(2\pi)^3} \frac{s^2}{\sqrt{\lambda_{ab}}} \int_{y_1(x)}^{y_2(x)} dy_{ab} \int_0^{2\pi} \! d\varphi_{\gamma},$$

(4.44)

where $\varphi_{\gamma}$ is the azimuthal angle of the photon in the CM frame. The integration boundary for $y_{ab}$ is given by

$$y_{1,2}(x) = \frac{s + 2m_a^2}{2s} (1 - x) \mp \frac{\sqrt{\lambda_{ab}}}{2s} \sqrt{(1 - x)^2 - \frac{4m_a^2 s}{s^2}}.$$  

(4.45)
This relation provides us also with the maximal value $x_1$ of $x$, for which $x = x_{ab}$ is possible,

$$x_1 = 1 - \frac{2m_\gamma \sqrt{s}}{s}. \quad (4.46)$$

Integrating $g_{ab,\tau}(p_a, p_b, k) \left| \mathcal{M}_0(\tilde{\Phi}_{0,ab}; \tau \kappa_\alpha) \right|^2$ over $\varphi_\gamma$ results in a trivial factor of $2\pi$. The integration of $g_{ab,\tau}^{(\text{sub})}$ over $y_{ab}$ is also performed easily. Defining

$$G_{ab,\tau}^{(\text{sub})}(s, x) = \frac{x s^2}{2\sqrt{\lambda_{ab}}} \int_{y_{1}(x)}^{y_{2}(x)} dy_{ab} g_{ab,\tau}^{(\text{sub})}(p_a, p_b, k), \quad (4.47)$$

we obtain

$$G_{ab,+}^{(\text{sub})}(s, x) = \frac{s}{\sqrt{\lambda_{ab}}} \left\{ \frac{1 + x^2}{1 - x} \ln \left[ \frac{y_2(x)}{y_1(x)} \right] + \frac{2m_a^2 x}{s} \left[ \frac{1}{y_2(x)} - \frac{1}{y_1(x)} \right] \right\} - G_{ab,-}^{(\text{sub})}(s, x), \quad (4.48)$$

$$G_{ab,-}^{(\text{sub})}(s, x) = 1 - x.$$

The singularity at $x \to 1$, which appears for $m_\gamma = 0$, is split off by introducing the $[...]_+$ distribution. Integration over $x$ yields the endpoint contributions

$$G_{ab,+}^{(\text{sub})}(s, x_0) = \ln \left[ \frac{m_a^2 s}{s^2(1 - x_0)^2} \right] + 2(1 - x_0) + \frac{s}{\sqrt{\lambda_{ab}}} \left\{ \ln \left[ \frac{m_a^2 \lambda_{ab}}{m_a^2 s^2 (1 - x_0)^2} \right] \ln(d_1) + \frac{1}{2} \ln(d_1) + 2 \text{Li}_2(d_1) + \frac{1}{2} \ln^2(d_1) - \frac{\pi^2}{3} \right\} - G_{ab,-}^{(\text{sub})}(s, x_0),$$

$$G_{ab,-}^{(\text{sub})}(s, x_0) = \frac{1}{2}(1 - x_0)^2, \quad (4.49)$$

where

$$d_1 = \frac{s + 2m_a^2 - \sqrt{\lambda_{ab}}}{s + 2m_a^2 + \sqrt{\lambda_{ab}}}. \quad (4.50)$$

The integral of the $ab$ part of the subtraction function finally reads

$$\int d\Phi_1 \left| \mathcal{M}_{\text{sub},ab}(\Phi_1) \right|^2 = -\frac{\alpha}{2\pi} Q_a \sigma_a Q_b \sigma_b$$

$$\times \left\{ \int_{x_0}^1 dx \, G_{ab,\tau}^{(\text{sub})}(s, x) \left[ \frac{1}{x} \int d\tilde{\Phi}_{0,ab}(s, x) \left| \mathcal{M}_0(\tilde{p}_a(x), \tilde{k}_j(x); \tau \kappa_\alpha) \right|^2 \right. \right.$$  

$$\left. - \int d\tilde{\Phi}_{0,ab}(s, 1) \left| \mathcal{M}_0(\tilde{p}_a(1), \tilde{k}_j(1); \tau \kappa_\alpha) \right|^2 \right]$$

$$+ G_{ab,\tau}^{(\text{sub})}(s, x_0) \int d\tilde{\Phi}_{0,ab}(s, 1) \left| \mathcal{M}_0(\tilde{p}_a(1), \tilde{k}_j(1); \tau \kappa_\alpha) \right|^2 \right\}, \quad (4.51)$$

where the IR singularity is contained in the function $G_{ab,+}^{(\text{sub})}(s, x_0)$, and the mass singularity, which appears in $G_{ab,+}^{(\text{sub})}(s, x_0)$ as well, is factorized (see Section 3). The convolution over $x$ itself can be carried out with $m_\gamma = 0$, and with $m_f = 0$ for light fermions. The relation
between the momenta $\tilde{\Phi}_{0,ab}(x)$ and the variable $x$ is much simpler than in the case with an emitter or a spectator in the final state. Contracting the first equation of (4.42) with $p_{b,\mu}$ and cancelling some terms, yields

$$\tilde{s} = P_{ab}^2 = \tilde{s}x + m^2_a + m^2_b + O(m^2_\gamma),$$  \hspace{1cm} (4.52)

i.e. the CM energy $\sqrt{\tilde{s}}$ of $\tilde{\Phi}_{0,ab}(s,x)$ is completely determined by the original CM energy $\sqrt{s}$ and $x$. Knowing $\tilde{s}$ from $s$ and $x$, it is straightforward to parametrize $d\tilde{\Phi}_{0,ab}(s,x)$.

5 Applications

In this section, we compare some numerical results on QED corrections obtained by the phase-space slicing method with the ones of the subtraction formalism described in this paper. In this context, we mention that we have adjusted all phase-space parametrizations to the peaking structure of the integrand in the application of the slicing method. For instance, $\ln(E_\gamma)$ is used as integration variable, in order to flatten the IR pole $1/E_\gamma$ for small values of the energy $E_\gamma$ of the outgoing photon. Photon emission angles are treated in a similar way if collinear photon emission from light fermions can take place. These reparametrizations have improved the efficiency of the slicing method considerably, whereas such improvements are not necessary for the subtraction method.

We consider the sample processes $\gamma\gamma \rightarrow ff(\gamma)$, $e^-\gamma \rightarrow e^-\gamma(\gamma)$, and $\mu^+\mu^- \rightarrow \nu_e\bar{\nu}_e(\gamma)$. This choice provides separate applications for the cases $ij$, $ia + ai$, and $ab$ of emitter/spectator pairs $ff'$.

For the numerical evaluations we take the following set of parameters \[12\]:

$$\alpha = 1/137.0359895, \quad M_W = 80.41 \text{ GeV}, \quad M_Z = 91.187 \text{ GeV}, \quad \Gamma_Z = 2.49 \text{ GeV},$$

$$m_e = 0.51099907 \text{ MeV}, \quad m_\mu = 105.658389 \text{ MeV}, \quad m_t = 173.8 \text{ GeV}.$$ \hspace{1cm} (5.1)

The weak mixing angle $\theta_w$ is fixed by

$$\cos \theta_w = c_w = \frac{M_W}{M_Z}, \quad s_w = \sqrt{1 - c_w^2}.$$ \hspace{1cm} (5.2)

The fermionic couplings to the $Z$ boson are expressed in terms of the vector and axial-vector factors

$$v_f = \frac{I_{w,f}^3}{2c_w s_w} - \frac{s_w}{c_w} Q_f, \quad a_f = \frac{I_{w,f}^3}{2c_w s_w},$$ \hspace{1cm} (5.3)

where $I_{w,f}^3 = \pm 1/2$ is the third component of the weak isospin of the fermion $f$.

5.1 The processes $\gamma\gamma \rightarrow ff(\gamma)$

The QED and weak corrections to the production of light fermion–anti-fermion pairs have been discussed recently in Ref. \[13\]. Details about different variants of phase-space slicing and about the dipole formalism presented here can also be found there. In particular, the treatment of angular cuts in the phase-space integral is described for the dipole formalism. Actually, the subtraction functions of Ref. \[13\] and the ones given in this
paper differ by a non-singular factor, leading to a different constant contribution in the integrated counterparts. We have repeated the numerics of Ref. [13] for the functions defined in this paper and found results of the same quality. Using the same number of phase-space points in the Monte Carlo integration, which is performed by Vegas [14], the integration error of the results obtained by phase-space slicing are larger by factors of 10–20.

Here we focus on the case of massive fermions; specifically, we consider the process

$$\gamma(k_1, \lambda_1) + \gamma(k_2, \lambda_2) \longrightarrow t(p, \sigma) + \bar{t}(\bar{p}, \bar{\sigma}) [+\gamma(k, \lambda)],$$

(5.4)

where $k_{1,2}$, $p$, $\bar{p}$ are the particle momenta, and $\lambda_{1,2}$, $\sigma$, $\bar{\sigma}$ are the corresponding helicities. There are two emitter/spectator pairs, $t\bar{t}$ and $\bar{t}t$, both of type $ij$, and the subtraction function (2.12) is given by

$$|M_{\text{sub}}(p, \bar{p}, k; \sigma, \bar{\sigma})|^2 = Q^2te^2g_{ij,\tau}(p, \bar{p}, k) |M_0(\bar{p}1, \bar{p}1; \tau\sigma, \bar{\sigma})|^2$$

$$+ Q^2te^2g_{ij,\tau}(\bar{p}, p, k) |M_0(\bar{p}2, \bar{p}2; \sigma, \tau\bar{\sigma})|^2.$$  

(5.5)

The construction of the auxiliary functions $g_{ij,\tau}$ with $ij = tt, \bar{t}t$ proceeds as described in Section 4.1. In particular, the invariant masses $P_{ij}$ are given by the square of the CM energy $\sqrt{s}$,

$$P^2_{tt} = P^2_{\bar{t}t} = (p + \bar{p} + k)^2 = (k_1 + k_2)^2 = s.$$  

(5.6)

The pairs of auxiliary momenta $(\bar{p}l, \bar{p}l)$ with $l = 1, 2$ are obtained from (4.3) upon setting $p_i = p$, $p_j = \bar{p}$ and $p_i = \bar{p}$, $p_j = p$, respectively. We recall that the spatial parts of the spectator momenta and their corresponding auxiliary momenta have the same direction in the CM frame, i.e. $\mathbf{p}||\bar{\mathbf{P}}_1$ and $\mathbf{p}||\bar{\mathbf{P}}_2$. This fact is useful for the implementation of angular cuts (see Ref. [13]). The integrated counterpart to the differential subtraction function (5.5) reads

$$\int d\phi(p, \bar{p}, k; k_1 + k_2) |M_{\text{sub}}(p, \bar{p}, k; \sigma, \bar{\sigma})|^2$$

$$= \frac{Q^2t\alpha}{2\pi} G_{tt,\tau}^{(\text{sub})}(s) \left[ \int d\phi(\bar{p}l_1, \bar{p}l_1; k_1 + k_2) |M_0(\bar{p}l_1, \bar{p}l_1; \tau\sigma, \bar{\sigma})|^2$$

$$+ \int d\phi(\bar{p}l_2, \bar{p}l_2; k_1 + k_2) |M_0(\bar{p}l_2, \bar{p}l_2; \sigma, \tau\bar{\sigma})|^2 \right],$$  

(5.7)

where we have exploited that the auxiliary functions $G_{tt,\tau}^{(\text{sub})}(s)$ and $G_{tt,\tau}^{(\text{sub})}(s)$ are already fixed by the initial state and coincide.

For the numerical evaluation of the matrix element $M_1$ of the radiative process $\gamma \gamma \rightarrow tt\gamma$, we apply crossing relations to the result on the related reaction $f\bar{f} \rightarrow \gamma\gamma\gamma$, which is listed in Ref. [8]. The phase-space integration is performed by Vegas [14]. Finally, we combine the real-photic corrections with the virtual photonic corrections, the evaluation of which is described in Ref. [15]. The resulting QED correction $\delta_{\text{QED}}$ to the total unpolarized cross section is given in Table 1 for some CM energies $\sqrt{s}$. As expected, the statistical error of the result of phase-space slicing grows roughly proportional to $\ln(\Delta E/E)$, where $E = \sqrt{s}/2$ is the photon beam energy in the CM frame. It is obvious that the value $\Delta E/E = 10^{-2}$ is still not small enough to guarantee reliable results. For smaller values of $\Delta E$ the integration error is again larger by a factor of 10–20 than the corresponding error obtained by the application of the dipole subtraction method.
Table 1: Results on the QED correction $\delta_{\text{QED}}$ to the unpolarized total cross section of $\gamma\gamma \rightarrow t\bar{t}(\gamma)$.

5.2 The process $e^-\gamma \rightarrow e^-\gamma(\gamma)$

(i) Moderate scattering energies

We consider the Compton process

$$e^-(p,\sigma) + \gamma(k_\gamma, \lambda) \rightarrow e^-(p',\sigma') + \gamma(k_1', \lambda_1') [\gamma(k_2', \lambda_2')]$$

(5.8)

where the momenta and helicities are given in parentheses. Owing to the strong polarization dependence of its polarized cross sections, this process is well-suited to determine the degrees of beam polarization of $e^\pm$ beams. For incoming laser photons and $e^\pm$ beams in the energy region of 1 GeV to 1 TeV, the CM energy is in the MeV range, i.e. the CM energy is not large with respect to the electron mass. Details of precision calculations, which include the photonic corrections of $O(\alpha)$, for such Compton polarimeters can be found in Refs. [16, 17]. In the following we make use of the analytical results on the virtual corrections and on the amplitudes for real-photonic bremsstrahlung given in Ref. [14] and evaluate the real corrections with the dipole formalism.

The subtraction function receives contributions of the mixed emitter/spectator types $ia$ and $ai$. Denoting the incoming and outgoing electrons in (5.8) by $e$ and $e'$, respectively, these contributions are labelled by $e'e$ and $e'e'$. Since both outgoing photons can become soft, we have to introduce subtraction functions for each individual final-state photon.
Note the IR regions of the two photons are separated in phase space so that the two subtraction functions can simply be added. Thus, the full subtraction function reads

$$|\mathcal{M}_{\text{sub}}|^2 = \sum_{l=1,2} |\mathcal{M}_{\text{sub}}^{(l)}(p,p',k'_l;\sigma,\sigma')|^2$$  \hspace{1cm} (5.9)

with

$$|\mathcal{M}_{\text{sub}}^{(l)}(p,p',k'_l;\sigma,\sigma')|^2 = e^2 g^{(\text{sub})}_{e'e',\tau}(p',p,k'_l) |\mathcal{M}_0(\tilde{p}_l,\tilde{p}'_l;\sigma,\sigma')|^2 + e^2 g^{(\text{sub})}_{e'e',\tau}(p,p',k'_l) |\mathcal{M}_0(\tilde{p}_l,\tilde{p}'_l;\tau\sigma,\sigma')|^2.$$  \hspace{1cm} (5.10)

The functions $g^{(\text{sub})}_{e'e',\tau}$ and $g^{(\text{sub})}_{e'e',\tau}$ are defined in Sections 4.2 and 4.3, where we have to identify $p_a = p$, $p_i = p'$, and $k = k'_l$. The auxiliary electron momenta $\tilde{p}_l$ and $\tilde{p}'_l$ play the roles of $\tilde{p}_a$ and $\tilde{p}_i$ in (4.17), respectively, where the index $l$ refers to the inserted photon momentum $k = k'_l$. The subtraction function is completely fixed by the above identifications. Because of $m_a = m_i = m_e$, we can take $x_0 = 0$ as the lower limit on $x_{ia}$ [see (4.11)].

The integrated counterpart to the subtraction function receives contributions from convolutions of the form (5.9). Owing to Bose symmetry with respect to the interchange of the outgoing photons, the two contributions corresponding to the two photons are equal. Therefore, we calculate only the integrated subtraction contribution for the photon with momentum $k'_1$ and weight this contribution with a factor of 2. Let us first consider the phase-space integration in the convolution. The squares of the momenta $P_{ia}$ and $K_{ia}$ are given by

$$P_{ia}^2 = (p' + k'_1 - p)^2 = (k_\gamma - k'_2)^2 = \tilde{t}, \quad K_{ia}^2 = (k_\gamma - P_{ia})^2 = k_2^2 = 0.$$  \hspace{1cm} (5.11)

Inserting these quantities and $m_b = 0$ into (5.9), we obtain

$$\tilde{s} = m_e^2 - \frac{\tilde{t}}{2} + \frac{2xs + \tilde{t} - 2m_e^2}{2R_{e'e}(x)}.$$  \hspace{1cm} (5.12)

for the new squared CM energy used in the convolution over $x$. The phase-space measure $d\bar{\Phi}_{0,ia}(K_{ia}^2, P_{ia}^2, x)$ reads

$$\int d\bar{\Phi}_{0,e'e}(0,\tilde{t},x) = \frac{1}{4(2\pi)^2} \int_{\tilde{t}_{\text{min}}(x)}^{0} d\tilde{t} \int_{0}^{2\pi} d\tilde{\phi}'_2 \frac{1}{\tilde{s} - m_e^2}.$$  \hspace{1cm} (5.13)

The lower limit $\tilde{t}_{\text{min}}(x)$ on $\tilde{t}$ is determined by two kinematical conditions. Firstly, $\tilde{t}$ cannot be lower than $-4E_\gamma^2$, where $E_\gamma$ is the energy of the incoming photon in the original CM frame. This condition corresponds to the “edge” of phase space where $k'_1 \rightarrow 0$. Secondly, the requirement $\tilde{s} > m_e^2$ sets another lower limit on $\tilde{t}$ in the calculation of $\tilde{s}$ from (5.12) for fixed $x$. Hence, $\tilde{t}_{\text{min}}(x)$ is the maximum of these two limits. The integration over the azimuthal angle $\tilde{\phi}'_2$ in the CM frame of $\tilde{p}_l(x) + k_\gamma$ yields a factor of $2\pi$ owing to the rotational invariance of the integrand. In the integrand of the convolution we insert the distributions $g_{e'e',\tau}^{(\text{sub})}$ and $g_{e'e',\tau}^{(\text{sub})}$ defined in Sections 4.2 and 4.3, respectively. The endpoint contributions $G_{e'e',\tau}^{(\text{sub})}$ and $G_{e'e',\tau}^{(\text{sub})}$ for $x_0 = 0$ are taken from App. A.2. The auxiliary function

$$\rho_{ia}(\tilde{s}) = \frac{\tilde{s} - m_e^2}{\tilde{s} - m_e^2}.$$  \hspace{1cm} (5.14)
| $P_e, P_\gamma$ | $\sigma_0$/mb | Method                        | $\Delta E/E_\gamma$ | $\delta_{\text{QED}}$/% |
|---------------|--------------|-------------------------------|----------------------|-------------------------|
| +, +          | 110.946      | Phase-space slicing           | $10^{-2}$            | 0.4094 ± 0.0014         |
|               |              |                               | $10^{-4}$            | 0.4016 ± 0.0030         |
|               |              |                               | $10^{-6}$            | 0.4014 ± 0.0047         |
|               |              | Dipole formalism              | –                    | 0.4013 ± 0.00033        |
| +, -          | 65.2608      | Phase-space slicing           | $10^{-2}$            | 0.4996 ± 0.0016         |
|               |              |                               | $10^{-4}$            | 0.4921 ± 0.0035         |
|               |              |                               | $10^{-6}$            | 0.4898 ± 0.0053         |
|               |              | Dipole formalism              | –                    | 0.49699 ± 0.00092       |

Table 2: Results on the $\mathcal{O}(\alpha)$ QED correction $\delta_{\text{QED}}$ to the total Born cross section $\sigma_0$ of $e^-\gamma \to e^-\gamma(\gamma)$ for $\sqrt{s} = 2.21836\text{ MeV}$ and different degrees of beam polarization $P_e$ and $P_\gamma$.

| $P_e, P_\gamma$ | $\sigma_0$/pb | Method                        | $\Delta E/E$ | $\Delta \theta$/rad | $\delta_{\text{QED}}$/% |
|---------------|--------------|-------------------------------|--------------|---------------------|-------------------------|
| +, +          | 90.4372      | IR slicing and effective mass factor | $10^{-2}$   | –                   | 5.441 ± 0.016           |
|               |              |                               | $10^{-4}$   | –                   | 5.416 ± 0.031           |
|               |              |                               | $10^{-6}$   | –                   | 5.468 ± 0.047           |
|               |              | Phase-space slicing           | $10^{-2}$   | $10^{-2}$           | 5.3783 ± 0.0074         |
|               |              |                               | $10^{-4}$   | $10^{-4}$           | 5.385 ± 0.026           |
|               |              |                               | $10^{-6}$   | $10^{-6}$           | 5.454 ± 0.055           |
|               |              | Dipole formalism              | –           | –                   | 5.3588 ± 0.0041         |
| +, -          | 12.2425      | IR slicing and effective mass factor | $10^{-2}$   | –                   | 15.686 ± 0.015          |
|               |              |                               | $10^{-4}$   | –                   | 15.685 ± 0.026          |
|               |              |                               | $10^{-6}$   | –                   | 15.679 ± 0.037          |
|               |              | Phase-space slicing           | $10^{-2}$   | $10^{-2}$           | 15.655 ± 0.0056         |
|               |              |                               | $10^{-4}$   | $10^{-4}$           | 15.656 ± 0.019          |
|               |              |                               | $10^{-6}$   | $10^{-6}$           | 15.687 ± 0.045          |
|               |              | Dipole formalism              | –           | –                   | 15.649 ± 0.011          |

Table 3: Results on the $\mathcal{O}(\alpha)$ QED correction $\delta_{\text{QED}}$ to the total Born cross section $\sigma_0$ of $e^-\gamma \to e^-\gamma(\gamma)$ for $\sqrt{s} = 100\text{ GeV}$, $20^\circ < \theta_e' < 160^\circ$, and different degrees of beam polarization $P_e$ and $P_\gamma$. 
For the squared Born amplitudes $|M_0(\tilde{p}_a(x), \tilde{p}_i(x))|^2$ the invariants $\tilde{s}$ and $\tilde{t}$ correspond to the Mandelstam variables $s$ and $t$ as defined in Ref. [17], respectively.

In Table 2 we give the total cross sections for polarized incoming particles and unpolarized outgoing particles for a CM energy that is typical for a Compton polarimeter of a future $e^+e^-$ collider; the beam energies are $\tilde{E}_e = 500$ GeV and $\tilde{E}_\gamma = 2.33$ eV. The statistical error of the result of phase-space slicing grows with decreasing $\Delta E/E_\gamma$. For $\Delta E/E_\gamma = 10^{-2}$ the influence of the finite value of $\Delta E$ is still visible. For the smaller values of $\Delta E$ the integration error of the results obtained by the dipole subtraction formalism is smaller than the one of the slicing method by at least an order of magnitude.

(ii) High scattering energies

Compton scattering represents an important reference process in possible future electron–photon colliders with CM energies in the GeV to TeV range. In this case the electron mass $m_e$ is small with respect to the CM energy and can be neglected in predictions whenever mass singularities are avoided. Detailed discussions of the corresponding lowest-order cross sections and the electroweak $\mathcal{O}(\alpha)$ corrections can be found in Refs. [18, 19].

In the following we take over the results on the virtual QED corrections given there and supplement the calculation of the real-photonic corrections of Ref. [19] by the application of the dipole formalism.

The construction of the subtraction function and its integrated counterpart proceeds analogously to the case of finite $m_e$ above. One can either expand the above results for $m_e \to 0$ or make direct use of the general results presented in Section 3 for light fermions. There is, however, a difference to the massive case as far as the kinematics is concerned. For $m_e \to 0$, exact backward Compton scattering has to be excluded by appropriate cuts because of a kinematical $u$-channel pole in the lowest-order cross section, which is only regularized by a finite electron mass. We avoid this singular region by requiring a finite angle $\theta_e'$ of the outgoing electron with the beam axis in the CM frame. To this end, we introduce the step function

$$g_{\text{cut}}(\theta) = \Theta(\theta - \theta_{\text{cut}})\Theta(180^\circ - \theta_{\text{cut}} - \theta)$$

and set $\theta_{\text{cut}} = 20^\circ$ in the numerical evaluation. While the original squared matrix element $|\mathcal{M}_1|^2$ is simply multiplied by $g_{\text{cut}}(\theta'_e)$ in the phase-space integration, the cuts on the subtraction function have to be chosen in such a way that the same cuts can be applied in the integrated counterpart to the subtraction function. At the same time, one has to ensure that the subtraction function still compensates all singularities of $|\mathcal{M}_1|^2g_{\text{cut}}(\theta'_e)$. Applying the cuts to the polar angles $\theta'_e, l$ of the two momenta $\tilde{p}_l$ in the original CM frame fulfills these requirements. Thus, $|\mathcal{M}_{\text{sub}}^{(l)}|^2$ in (5.9) is replaced by $|\mathcal{M}_{\text{sub}}^{(l)}|^2g_{\text{cut}}(\theta'_e, l)$.

In the limit $m_e \to 0$ the integrated counterpart to the subtraction function simplifies drastically. The boost that relates the CM frames of $p + k_\gamma$ and $\tilde{p}_1 + k_\gamma = xp + k_\gamma$ goes along the beam axis, and the squared CM energies are related by $\tilde{s} = xs$. Therefore, the phase-space measure and the auxiliary function given above reduce to

$$\int d\tilde{\Phi}_0(x, t, x) = \frac{1}{8\pi xs} \int_{-xs}^0 d\tilde{t}, \quad \rho_{t\alpha}(\tilde{s}) = x,$$

$$\rho_{a\alpha}(\tilde{s}) = x,$$
where rotational invariance is already exploited to perform the integration over $\varphi'_2$. According to the cutting procedure described above, we have to apply the angular cut on the angle of $\hat{p}_1'$ in the original CM frame, i.e. we have to transform the polar angle $\tilde{\vartheta}_{e,1}'(x)$ of $\hat{p}_1'$ defined in the CM frame of $xp + k_\gamma$ back to the CM frame of $p + k_\gamma$. Denoting the angle in the latter frame by $\tilde{\vartheta}_{e,1}(x)$, the two angles are related by

$$\cos \tilde{\vartheta}_{e,1}(x) = \frac{x - 1 + (1 + x) \cos \tilde{\vartheta}_{e,1}'(x)}{1 + x + (x - 1) \cos \tilde{\vartheta}_{e,1}'(x)}. \quad (5.17)$$

The cuts are consistently introduced in the convolutions over $x$ if all squared matrix elements $|M_0(\hat{p}_a(\xi), \hat{p}_i(\xi))|^2$ get the factor $g_{\text{cut}}(\tilde{\vartheta}_{e,1}'(\xi))$, where $\xi$ is equal to $x$ or 1.

Let us inspect the IR and mass singularities explicitly. From the results of Ref. [19] we deduce that the factor $\delta_{\text{QED}}$ for the virtual corrections can be decomposed into a polarization-independent singular part and a polarization-dependent regular part $\delta_{\text{rem}}(\sigma, \lambda, \sigma', \lambda'_1)$,

$$\delta_{\text{QED}} = -\frac{\alpha}{\pi} \mathcal{L}(-t, m_e^2) + \delta_{\text{rem}}(\sigma, \lambda, \sigma', \lambda'_1), \quad (5.18)$$

with the auxiliary function $\mathcal{L}$ of (3.8). Since the Mandelstam variable $t = (p - p')^2$ of Ref. [19] corresponds to $\bar{t}$ in the convolution over $x$ described above, the IR and mass singularities of the virtual correction exactly cancel against the ones contained in the endpoint parts $G_{\varphi e,+,+}^{(\text{sub})}$ and $G_{\varphi e',+,+}^{(\text{sub})}$. Therefore, the only uncancelled mass-singular corrections are the ones contained in the distribution $G_{\varphi e',+,+}^{(\text{sub})}$, where they are weighted with the splitting function $P_{ff}(x)$ in the convolution over $x$.

Table 3 shows our results on the $\mathcal{O}(\alpha)$ QED corrections to the integrated cross section for $\sqrt{s} = 100$ GeV and different beam polarizations. The table does not only contain the results from the slicing and subtraction methods, but also includes the results obtained by a formalism called “IR slicing and effective mass factor”. In this approach only the IR regions are removed from phase space by cuts, and the collinear poles are regularized by applying appropriate factors that replace the poles by the correct mass-dependent behaviour. More details about the application of this procedure and of the slicing method can be found in Ref. [19]. For both slicing variants, the statistical integration errors increase with decreasing cut parameters $\Delta E/E$ and $\Delta \vartheta$. Here $E$ is the beam energy in the CM frame, and the cut angle $\Delta \vartheta$ defines cones around the electron directions that are excluded from phase space. For a cut size of $10^{-2}$, the integration errors of the different methods are of the same order of magnitude, but at least for the approach with effective mass factors the finiteness of the cut is still visible. Therefore, smaller cuts are advisable. In this case the superiority of the subtraction formalism becomes obvious. For the inspected cuts, there is an improvement of a factor of 2 or more in the integration error.\footnote{We expect that the superiority of the subtraction method is more enhanced if more realistic cuts are applied. Cutting the electron angle directly, without taking into account a recombination with soft or collinear photons in the detector, is a strong idealization. Technically this leads to regions in phase space where $g_{\text{cut}}(\tilde{\vartheta}_{e}) = 1$ and $g_{\text{cut}}(\tilde{\vartheta}_{e,1}) = 0$ or vice versa. For collinear photons these regions shrink to zero, but nevertheless induce strong peaks in the integrand. Realistic cuts should avoid such pathologies, leading to an improvement in the numerical integration.}

3
5.3 The process $\mu^+\mu^- \to \nu_\ell \bar{\nu}_\ell (\gamma)\n
(i) Moderate scattering energies

As a final example, we consider the process

$$\mu^- (p, \sigma) + \mu^+ (p', \sigma') \to \nu_\ell (q, -) + \bar{\nu}_\ell (q', +) [+ \gamma (k, \lambda)], \tag{5.19}$$

which is phenomenologically less important, but—owing to its simplicity—it is well suited for demonstrating the application of the dipole formalism in situations with two charged fermions in the initial state. At lowest order, there is only an $s$-channel diagram with $Z$-boson exchange, and the Born amplitude reads

$$\mathcal{M}_0 (p, p', q, q'; \sigma, \sigma') = \frac{e^2 (\nu_\ell + a_{\nu_\ell})}{2 (s - M_Z^2 + i M_Z \Gamma_Z)} \left[ \bar{v}_{\mu^+} (p'; \sigma') \gamma^\rho (v_\mu - a_\mu \gamma_5) u_{\mu^-} (p; \sigma) \right] \times \left[ \bar{u}_{\nu_\ell} (q; -) \gamma^\rho (1 - \gamma_5) v_\nu (q'; +) \right]. \tag{5.20}$$

For our purposes, it is sufficient to describe the $Z$ resonance with the constant experimental width $\Gamma_Z$ given above. The virtual photonic corrections consist of a correction to the $Z\mu\mu$ vertex and the muon wave-function correction. The derivation of the Born cross section and the virtual correction is standard and has been performed using the techniques described in Ref. [3]. The results are listed in App. D for a finite muon mass $m_\mu$. The bremsstrahlung corrections involve photon emission from the muons in the initial state only. The amplitudes for these real corrections have been obtained from the general results for $\mu^+ \mu^- \to f \bar{f} \gamma$ given in Ref. [3].

The subtraction function receives the two contributions $g^{(\text{sub})}_{\mu^- \mu^+ \tau}$ and $g^{(\text{sub})}_{\mu^+ \mu^- \tau}$, which are both of type $ab$, and reads

$$|\mathcal{M}_{\text{sub}} (p, p', k, q, q'; \sigma, \sigma')|^2 = e^2 g^{(\text{sub})}_{\mu^- \mu^+ \tau} (p, p', k) |\mathcal{M}_0 (\bar{p}, \bar{p}', \bar{q}_1, \bar{q}_1'; \tau \sigma, \sigma')|^2 + e^2 g^{(\text{sub})}_{\mu^+ \mu^- \tau} (p', p, k) |\mathcal{M}_0 (p, p', \bar{q}_2, \bar{q}_2'; \tau \sigma')|^2. \tag{5.21}$$

In the following, we only describe the construction of the auxiliary momenta $\bar{p}$, $\bar{q}_1$, and $\bar{q}_1'$ for the contribution of $g^{(\text{sub})}_{\mu^- \mu^+ \tau}$. The case of $g^{(\text{sub})}_{\mu^+ \mu^- \tau}$ can be treated analogously. The auxiliary variables $x = x_{ab}$ and $y = y_{ab}$ for the $g^{(\text{sub})}_{\mu^- \mu^+ \tau}$ contribution read

$$x = \frac{pp' - pk - p k}{pp'} = \frac{s - 2 m_\mu^2}{s - 2 m_\mu^2}, \quad y = \frac{pk}{pp'}, \tag{5.22}$$

where we have included the relation between $x$ and the two squared CM energies $s = (p + p')^2$ and $\tilde{s} = (p + p' - k)^2 = (\bar{p} + \bar{p}')^2$. The subtraction function is consistently set to zero for $x < x_0$ with

$$x_0 = \frac{2 m_\mu^2}{s - 2 m_\mu^2}. \tag{5.23}$$

Inserting $p_a = p$, $p_b = p'$, and $P = P_{ab} = p + p' - k$ into (4.40), we get the new momenta

$$\bar{p}' = \sqrt{\frac{x}{x_0}} \frac{4 m_\mu^2 (pp')(1 - x^2)}{\sqrt{x} \sqrt{x_0} + \sqrt{x_0}} p', \quad \bar{p} = \bar{p}' + p'. \tag{5.24}$$
where we have rearranged some terms in order to reveal the behaviour of $\bar{p}$ in the limit $m_\mu \to 0$. The abbreviations $\lambda_s$ and $\lambda_\tilde{s}$ are given by

$$
\lambda_s = \lambda(s, m_\mu^2, m_\mu^2) = \sqrt{s(4m_\mu^2)}, \\
\lambda_\tilde{s} = \lambda(\tilde{s}, m_\mu^2, m_\mu^2) = \sqrt{\tilde{s}(4m_\mu^2)}.
$$

(5.25)

For the evaluation of the Born matrix element $M_0(p, p', q_1)$ we need a scalar product between an initial-state and a final-state momentum, such as $p'q_1 = p'_\mu \Lambda^{\mu}_\nu q_1^{\nu}$, in addition to the already known quantity $\tilde{s}$. At this point, the explicit form (3.27) of the Lorentz transformation $\Lambda_{\mu\nu}$ enters. The calculation of the desired scalar product simply requires some contractions among the original momenta and $\tilde{p}$.

The evaluation of the integrated counterpart to the subtraction function leads to convolutions of the form given in (5.21). Since the variable $x$ enters the phase space $\tilde{\Phi}_{0,ab}(s, x)$ only by the CM energy $\sqrt{s}$, the phase-space integration over the squared amplitudes $|M_0|^2$ is the same as for the lowest-order cross section $\sigma_0(s)$ at the CM energy $\sqrt{s}$. Consequently, the convolution can be formulated in terms of lowest-order cross sections, and the integrated counterpart to the $g_{-\mu^+,\tau}$ contribution in (5.21) reads

$$
\Delta \sigma_{-\mu^+,\tau}(s, P_-, P_+) = \frac{\alpha}{2\pi} \int_{x_0}^1 dx \frac{G_{-\mu^+,\tau}(s, x)}{x^2} \left[ \frac{\sqrt{\lambda_s}}{x \sqrt{\lambda_\tilde{s}}} \sigma_0(\tilde{s}, \tau P_-, P_+) - \sigma_0(s, \tau P_-, P_+) \right] \right.

\left. + \frac{\alpha}{2\pi} G_{-\mu^+,\tau}(s, x_0) \sigma_0(s, \tau P_-, P_+),
$$

(5.26)

where $P_\pm$ are the degrees of beam polarization of the $\mu^\pm$ beams. The factor $\sqrt{\lambda_s}/\sqrt{\lambda_\tilde{s}}$ stems from the flux factors in the transition from squared matrix elements to cross sections.

Table 3 shows some results on the photonic $O(\alpha)$ corrections to the lowest-order cross sections $\sigma_0$ for different $\mu^-$-beam polarizations. The $\mu^+$ beam is assumed to be unpolarized. The considered CM energy of $\sqrt{s} = 500$ MeV is too small for a neglect of the muon mass $m_\mu$ in the non-singular contributions. Therefore, the $m_\mu$ dependence is treated exactly. The results of the dipole subtraction formalism are compared to the ones obtained by phase-space slicing, where $\Delta E$ is the cut energy on the outgoing photon, and $E = \sqrt{s}/2$ denotes the beam energy in the CM frame. Similar to the examples inspected previously, for $\Delta E/E = 10^{-2}$ the influence of the finiteness of $\Delta E$ is still visible at the chosen level of accuracy. On the other hand, using the same integration parameters for the subtraction method, the improvement in the integration error is between one and two orders of magnitude.

(ii) High scattering energies

Now we turn to high scattering energies and neglect the muon mass whenever possible, i.e. we apply the results of Section 3.3. In this limit, the virtual correction reduces to the simple polarization-independent factor $\delta_{QED}^{virt}$ to the Born cross section $\sigma_0$.

$$
\delta_{QED}^{virt} = -\frac{\alpha}{\pi} \left[ \mathcal{L}(s, m_\mu^2) - \frac{2\pi^2}{3} + 2 \right],
$$

(5.27)

in agreement with the result given in Ref. [20] on initial-state radiation in $e^+e^- \to Z^* \to f\bar{f}$. Concerning the real correction, the subtraction procedure described above becomes
Table 4: Results on the $O(\alpha)$ QED correction $\delta_{\text{QED}}$ to the total Born cross section $\sigma_0$ of $\mu^+\mu^- \rightarrow \nu_e\bar{\nu}_e(\gamma)$ for $\sqrt{s} = 500$ MeV and different degrees of $\mu^-$-beam polarization $P_-$ and unpolarized $\mu^+$. 

| $P_-$ | $\sigma_0/10^{-5}\,\text{pb}$ | Method | $\Delta E/E$ | $\Delta \theta/\text{rad}$ | $\delta_{\text{QED}}/%$ |
|-------|-------------------------------|--------|--------------|----------------|-----------------|
| +     | 5.74003                       | Phase-space slicing | $10^{-2}$  | 0.3827 ± 0.0048 |
|       |                               |        | $10^{-4}$  | 0.352 ± 0.011  |
|       |                               |        | $10^{-6}$  | 0.341 ± 0.018  |
|       | Dipole formalism              | –      |             | 0.36637 ± 0.00022 |
| −     | 9.35981                       | Phase-space slicing | $10^{-2}$  | 0.3275 ± 0.0045 |
|       |                               |        | $10^{-4}$  | 0.299 ± 0.011  |
|       |                               |        | $10^{-6}$  | 0.292 ± 0.017  |
|       | Dipole formalism              | –      |             | 0.31238 ± 0.00020 |

Table 5: Results on the $O(\alpha)$ QED correction $\delta_{\text{QED}}$ to the total Born cross section $\sigma_0$ of $\mu^+\mu^- \rightarrow \nu_e\bar{\nu}_e(\gamma)$ for $\sqrt{s} = 50$ GeV and different degrees of $\mu^-$-beam polarization $P_-$ and unpolarized $\mu^+$. 

| $P_-$ | $\sigma_0/\text{pb}$ | Method | $\Delta E/E$ | $\Delta \theta/\text{rad}$ | $\delta_{\text{QED}}/%$ |
|-------|----------------------|--------|--------------|----------------|-----------------|
| +     | 1.32547              | IR slicing and effective mass factor | $10^{-2}$  | –              | −4.157 ± 0.021  |
|       |                      |        | $10^{-4}$  | –              | −4.331 ± 0.055  |
|       |                      |        | $10^{-6}$  | –              | −4.353 ± 0.089  |
|       | Phase-space slicing  | $10^{-2}$  | $10^{-2}$  | −4.162 ± 0.018 |
|       |                      | $10^{-4}$  | $10^{-4}$  | −4.321 ± 0.090 |
|       |                      | $10^{-6}$  | $10^{-6}$  | −4.36 ± 0.22   |
|       | Dipole formalism     | –      | –            | −4.29135 ± 0.00022 |
| −     | 2.06497              | IR slicing and effective mass factor | $10^{-2}$  | –              | −4.168 ± 0.021  |
|       |                      |        | $10^{-4}$  | –              | −4.335 ± 0.054  |
|       |                      |        | $10^{-6}$  | –              | −4.356 ± 0.087  |
|       | Phase-space slicing  | $10^{-2}$  | $10^{-2}$  | −4.151 ± 0.018 |
|       |                      | $10^{-4}$  | $10^{-4}$  | −4.257 ± 0.091 |
|       |                      | $10^{-6}$  | $10^{-6}$  | −4.24 ± 0.22   |
|       | Dipole formalism     | –      | –            | −4.30390 ± 0.00020 |
technically simpler, since $m_\mu$ can be neglected in the kinematics everywhere. Owing to the simplicity of the total Born cross section for $m_\mu = 0$,

$$\sigma_0(s) = (1 - \tau P_+)(1 + \tau P_-) \frac{\pi \alpha^2}{3} (v_{ve} + a_{ve})^2 (v_\mu - \tau a_\mu)^2 \frac{s}{s - M_Z^2 + iM_Z\Gamma_Z}$$

the convolution (5.26) over $x$ can be easily performed analytically. Note that the IR and mass-singular part of the virtual correction $\delta_{\text{QED}}^{\text{virt}}$ is again exactly cancelled by the singular terms in the endpoint contributions $G^{(\text{sub})}_{\mu^-\mu^+,+}$ and $G^{(\text{sub})}_{\mu^+\mu^-,+}$. The remaining mass-singular contributions are contained in $G^{(\text{sub})}_{\mu^-\mu^+,+}$ and $G^{(\text{sub})}_{\mu^+\mu^-,+}$ and enter the convolution (5.26) over $x$ weighted by the splitting function $P_{ff}(x)$.

The application of the slicing method additionally requires the analytic treatment of photons that are emitted nearly collinearly from the muon beams, i.e. which have emission angles $\theta$ within the ranges $0^\circ < \theta < \Delta \theta$ or $0^\circ < 180^\circ - \theta < \Delta \theta$ with $\Delta \theta \ll 1$. These effects are calculated in the same way as described in Ref. [19] for initial-state radiation in Compton scattering. Details about the variant with effective mass factors can also be found there.

In Table 5 we show the photonic corrections for $\sqrt{s} = 50 \text{ GeV}$, $P_- = \pm 1$, and $P_+ = 0$, obtained in the small-mass limit $m_\mu \to 0$. The numbers again underline the superiority of the subtraction formalism. The integration error is reduced by one to two orders of magnitude, without the necessity to look for a plateau in auxiliary parameters, such as $\Delta E$ and $\Delta \theta$.

6 Discussion and outlook

6.1 Features of subtraction methods and the dipole formalism

As already explained in the introduction, the basic motivation for the development of subtraction methods is to avoid singular numerical integrations in the calculation of real-photonic (or real-gluonic) corrections. In the previous section, we have compared the results for various radiative processes obtained by applying the dipole subtraction formalism of this paper with the ones obtained by phase-space slicing. We have found that the application of the subtraction formalism typically reduces the integration error by an order of magnitude with respect to the results of phase-space slicing, when all integrations are performed with the same statistics. As mentioned at the beginning of Section 5, the efficiency of the slicing method has been improved by introducing appropriate parametrizations of phase space, whereas such improvements are not needed for the subtraction method.

Moreover, a successful application of the slicing method requires a careful investigation of the dependence on the soft-photon cut $\Delta E$ and, if relevant, on the angular cut $\Delta \theta$. It is necessary to optimize the choice of the cut parameters for all considered observables. The integration error roughly grows proportional to the logarithm of a cut parameter if the cut becomes small. The optimal choice of cut parameters loosens the cuts as much as possible, but still suppresses remnant effects of their finiteness. The optimal set of cuts varies with the desired accuracy and, in most cases, also with input parameters, such as the scattering energy. In practice, one often tends to choose rather small cuts
One of the great advantages of the dipole formalism is certainly its process independence, which distinguishes this approach from most of the other subtraction procedures. In this paper, the dipole formalism is worked out for photon radiation in processes involving charged fermions and any other neutral particles. We stress that all different configurations of particle masses and helicities are supported. The subtraction function, which removes IR and possible collinear singularities from the differential cross section, is constructed in such a way that the transition to the region of small fermion masses proceeds smoothly. In other words, there is one subtraction function that interpolates the regions of large and small masses.

Finally, one has to admit that the actual application of subtraction methods, in general, is more involved than the use of phase-space slicing for complicated electroweak processes. The presentation in this paper certainly shows that the application of a subtraction procedure can be quite involved for processes with massive particles. The implementation of phase-space cuts within subtraction methods is straightforward, but nevertheless can be laborious (see also next section). On the other hand, once the procedure is applied to a process, such complications are completely overcome, and the advantages described above become apparent.

6.2 Phase-space cuts and distributions

In the above formulation of the dipole formalism, we mainly concentrated on the calculation of total cross sections, but we did not pay particular attention to phase-space cuts or to the calculation of distributions. We recall that the difference of the differential cross section and the subtraction function is integrated over the full phase space $\Phi_{1}$ of $n + 1$ particles numerically, but the integrated counterpart to the subtraction function implicitly contains the integration over the photonic part of phase space, which is carried out analytically. The cuts that are applied to the subtraction function have to be identical with the ones that are applied and to the integrated counterpart of the subtraction function. Otherwise these two contributions will not compensate each other, leading to wrong results. In practice, this means that we have to distinguish two types of cuts. Firstly, we have the original cuts that are applied to the original differential cross section; these cuts concern the full phase space $\Phi_{1}$ of $n + 1$ particles. Secondly, we have auxiliary cuts that are applied to the subtraction function and to its integrated counterpart; they are defined in the reduced phase spaces $\tilde{\Phi}_{0,ff'}$ of $n$ particles. Simple examples for the implementation of angular cuts have been described in Section 5.2 for Compton scattering at high energies and in Ref. [13] for the production of light fermion–anti-fermion pairs in photon–photon collisions.

The calculation of distributions is similar to the application of cuts, since a histogram of a distribution is nothing but a series of cuts. Hence, the histogram routine that generates the desired distribution during the Monte Carlo integration has to handle each column of the histogram in the same way as a cutted contribution to the integrated cross
section. Note that this procedure implies that the original differential cross section and the subtraction function may contribute to different columns of the histogram for one and the same event. The final result for each column is nevertheless finite, because such events are in general far away from the singular regions.

6.3 Practical advice

Subtraction methods offer a number of checks, which are very useful in practice. The basic principle of subtraction methods is that all contributions originating from the subtraction function add up to zero in the final result. In the following we describe some possible checks for the dipole formalism that are mainly based on this principle. The described checks have been successfully carried out in the applications discussed in Section 5.

The auxiliary functions $g^{(\text{sub})}_{\ell f, \tau}, \bar{G}^{(\text{sub})}_{\ell f, \tau}$, and $G^{(\text{sub})}_{\ell f', \tau}$ can be checked for consistency without application to a specific process. To this end, one should carry out all integrations numerically that have been performed analytically for the derivation of $\bar{G}^{(\text{sub})}_{\ell f', \tau}$ and $G^{(\text{sub})}_{\ell f', \tau}$. Since some of these integrations involve IR singularities, a small photon mass $m_\gamma$ has to be consistently used in the numerics.

For the treatment of specific processes, the construction of the phase spaces $\tilde{\Phi}_{0, \ell f'}$ deserves particular care. It can be very useful to compare the corresponding phase-space volumes entering the integrations over the phase spaces of $n + 1$ and $n$ particles. The two volumes are obtained as follows:

(a) in the original integration over $d\Phi_1$ we set $M_1 \rightarrow 0$, $M_0 \rightarrow 1$, and $g^{(\text{sub})}_{\ell f, \tau} \rightarrow 1$;

(b) in the integrations of the counterparts over $d\tilde{\Phi}_{0, \ell f'}$ we set $M_0 \rightarrow 1$ and use the expressions for $\bar{G}^{(\text{sub})}_{\ell f', \tau}$ and $G^{(\text{sub})}_{\ell f', \tau}$ that correspond to $g^{(\text{sub})}_{\ell f', \tau} \rightarrow 1$. Those expressions can be derived easily, using $m_\gamma = 0$.

Note that this phase-space comparison, in particular, represents a non-trivial check on the convolutions (4.30) in the $ia$ and $ai$ cases, which can be quite complicated for massive initial-state fermions.

In many cases, the phase-space check can be extended by including the full form of the functions $g^{(\text{sub})}_{\ell f', \tau}, \bar{G}^{(\text{sub})}_{\ell f', \tau}$, and $G^{(\text{sub})}_{\ell f', \tau}$, i.e. the only substitutions are $M_1 \rightarrow 0$ and $M_0 \rightarrow 1$ in the phase-space integrations. Owing to the IR and collinear singularities in $g^{(\text{sub})}_{\ell f', \tau}$, this kind of check is not always possible in a simple way. The check is, for instance, useful in the $ia$ and $ai$ cases with $m_\alpha \neq 0$. In these cases, all singularities appear for $x \rightarrow 1$ and can be removed by applying the additional cut $x_{ia} < 1 - \Delta x$ with any small $\Delta x > 0$ in the integration over $d\Phi_1$. This additional cut has to be incorporated in the convolution of $\bar{G}^{(\text{sub})}_{\ell f', \tau}$ over $x$, too. The simplest possibility to achieve this is to omit the introduction of the $[\ldots]_+$ prescription and to perform the convolution in the range $x_0 < x < 1 - \Delta x$.

Of course, many other variants of such consistency checks may be useful in actual applications.

4At the edges of the histogram columns this can also occur for “singular events”. The finiteness of such contributions is guaranteed by the suppression of phase space for those events.
6.4 Generalization to QCD

In this paper, we have focussed on photon radiation off fermions only. The presented formalism can, however, be carried over to gluon radiation for a certain class of processes. Consider, for instance, a process that involves a heavy quark–anti-quark pair $q\bar{q}$, but no other QCD partons. In this case, the gluonic corrections can be obtained from the photonic corrections by the replacement $Q_q^2\alpha_s^0 \rightarrow 4\alpha_s/3$, and the infinitesimal photon mass $m_\gamma$ turns into an infinitesimal gluon mass $m_g$. Since the IR singularity is abelian, the transition to dimensional regularization is performed by the well-known substitution

$$\ln(m_g^2) \rightarrow \frac{(4\pi\mu^2)^\epsilon \Gamma(1+\epsilon)}{\epsilon} + \mathcal{O}(\epsilon), \quad (6.1)$$

where $D = 4 - 2\epsilon$ is the dimension and $\mu$ the reference mass of dimensional regularization.

The results of this paper can also be used to deal with processes involving more than one heavy quark–anti-quark pair if the colour flow is treated properly. The colour algebra is identical to the corresponding process with massless quarks and can be taken over from Ref. [5].

The presented results do not cover the cases of gluon radiation in which collinear singularities are treated within dimensional regularization. This includes real-gluonic corrections to all processes involving massless partons in the initial state. However, the presented results can serve as a starting point for a full generalization of the dipole formalism to QCD with heavy quarks.

7 Summary

Following the guideline of Ref. [5], where the dipole subtraction formalism is presented for NLO QCD corrections involving massless unpolarized partons, we have formulated this method for photon radiation off massive fermions. The dipole formalism represents a process-independent subtraction procedure that removes all IR and collinear singularities from differential cross sections of bremsstrahlung processes. The subtracted singular structures are calculated separately, where the integration over the singular regions is performed analytically. Consequently, no singular numerical integrations are needed for the final result. This advantage distinguishes subtraction formalisms from methods that employ phase-space slicing. Slicing methods require a careful optimization of small cuts that exclude the singular regions from the numerical phase-space integration.

Since the consistent inclusion of finite fermion masses turned out to be highly non-trivial, we have presented the derivation of the method in a rather detailed way. Our formulation, which allows for fermions with definite helicity eigenstates, is applicable to processes involving charged fermions and any type of neutral particles. The generalization to charged bosons is straightforward. In the limit of small fermion masses, which is of particular importance phenomenologically, the dipole formalism simplifies considerably and is easy to use.

In order to illustrate the use of the method in practice, we have applied the dipole subtraction method to the processes $\gamma\gamma \rightarrow t\bar{t}(\gamma)$, $e^-\gamma \rightarrow e^-\gamma(\gamma)$, and $\mu^+\mu^- \rightarrow \nu_\mu\bar{\nu}_e(\gamma)$. Comparing the corresponding results to the ones obtained by slicing methods, we find
improvements in the integration errors of typically an order of magnitude, where Monte Carlo integrations are performed with the same statistics in both approaches.

Finally, we conclude that the dipole subtraction formalism is superior to methods that are based on phase-space slicing. Moreover, the presented procedure for photon radiation off massive fermions is a first step towards the full generalization of the dipole formalism to QCD with heavy quarks.

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Appendix

A Special cases

A.1 Light incoming particles

The case of light fermions in the initial state is of particular importance. For instance, it is relevant for $e^+e^-$ collisions at high energies, as observed at LEP or the SLC. Since the $ab$ case with $m_{a,b} \to 0$ is already covered by Section 3.3, here we concentrate on the mixed cases $ia$ and $ai$ with $m_a \to 0$.

Using the auxiliary parameters $x_{ia}$ and $z_{ia}$ of (3.10), the subtraction functions are given by

$$
g_{ia,+}(p_i, p_a, k) = \frac{1}{(p_i k) x_{ia}} \left[ \frac{2}{2 - x_{ia} - z_{ia}} - 1 - z_{ia} - \frac{m_i^2}{p_i k} \right] - g_{ia,-}(p_i, p_a, k),
$$

$$
g_{ia,-}(p_i, p_a, k) = \frac{m_i^2}{2 (p_i k)^2 x_{ia}^2} \left[ 1 - z_{ia} \right]^2 x_{ia},
$$

$$
g_{ai,+}(p_a, p_i, k) = \frac{1}{(p_a k) x_{ia}} \left[ \frac{2}{2 - x_{ia} - z_{ia}} - 1 - x_{ia} \right],
$$

$$
g_{ai,-}(p_a, p_i, k) = 0. \tag{A.1}
$$

The lower limit $x_0$ on $x_{ia}$ can be set to zero consistently. The construction of the auxiliary momenta $\tilde{p}_a$ and $\tilde{p}_i$, which is given in (4.17) for finite fermion masses, becomes particularly simple for $m_a \to 0$. The result is formally identical with (3.12) for the fully massless case, but one should note that $\tilde{p}_i^2 = m_i^2$ still holds. Using the above relations, the $ai$ and $ia$ contributions to the subtraction function $|M_{\text{sub}}|^2$ of (2.12) can be constructed easily.

The calculation of the integrated counterparts to $|M_{\text{sub}}|^2$ also considerably simplifies in the limit $m_a \to 0$. Since the construction of the auxiliary momenta $\tilde{p}_i$ and $\tilde{p}_a$ proceeds
as in the fully massless case described in Section 3, the convolution over \( x \) takes the simple form (3.18) even for finite \( m_i \). The distributions for these convolutions read

\[
\mathcal{G}^{(\text{sub})}_{ia,+}(P^2_{ia}, x) = \frac{1}{1-x} \left\{ 2 \ln \left[ \frac{2 - x - z_1(x)}{1-x} \right] \\
+ \frac{1}{2} \left[ z_1(x) - 1 \right] \left[ 3 + z_1(x) - \frac{4m_i^2 x}{(P^2_{ia} - m_i^2)(1-x)} \right] \right\} - \mathcal{G}^{(\text{sub})}_{ia,-}(P^2_{ia}, x),
\]

\[
\mathcal{G}^{(\text{sub})}_{ia,-}(P^2_{ia}, x) = \frac{m_i^2}{P^2_{ia} - m_i^2} \frac{1}{(1-x)^2} \left\{ \ln \left[ z_1(x) \right] + \frac{1}{2} (1 - z_1(x)) [3 - z_1(x)] \right\},
\]

\[
\mathcal{G}^{(\text{sub})}_{a,+(P^2_{ia}, x) = P_{ff}(x) \left\{ \ln \left( \frac{m_i^2 - P^2_{ia}}{m^2_{\text{sub}} x} \right) + \ln [1 - z_1(x)] - 1 \right\} \\
- \frac{2}{1-x} \ln [2 - x - z_1(x)] + (1 + x) \ln (1 - x),
\]

\[
\mathcal{G}^{(\text{sub})}_{ai,-}(P^2_{ia}, x) = 1 - x,
\]

where

\[
z_1(x) = \frac{m_i^2 x}{m_i^2 - P^2_{ia}(1-x)}
\]

is the lower limit on \( z_{ia} \) for \( m_\gamma = m_a = 0 \). The endpoint contributions are given by

\[
G^{(\text{sub})}_{ia,+}(P^2_{ia}) = \ln \left( \frac{m_i^2}{m_\gamma^2} \right) \ln \left( \frac{2 - P^2_{ia}}{m_i^2} \right) + 2 \ln \left( \frac{m_\gamma m_i}{m_i^2 - P^2_{ia}} \right) - 2 \text{Li}_2 \left( \frac{P^2_{ia}}{P^2_{ia} - 2m_i^2} \right) \\
+ \frac{1}{2} \ln^2 \left( \frac{m_i^2}{m_\gamma^2} \right) + \frac{(P^2_{ia} - m_i^2)^2}{2P^2_{ia}} \ln \left( 1 - \frac{P^2_{ia}}{m_i^2} \right) - \frac{\pi^2}{6} + 1 + \frac{m_i^2}{2P^2_{ia}},
\]

\[
G^{(\text{sub})}_{ai,+}(P^2_{ia}) = \ln \left( \frac{m_i^2}{m_a^2} \right) \ln \left( \frac{m_i^2 (2m_i^2 - P^2_{ia})}{m_i^2 - P^2_{ia}} \right) + \ln \left( \frac{m_i^2}{m_a^2} \right) + 2 \text{Li}_2 \left( \frac{P^2_{ia}}{2m_i^2 - P^2_{ia}} \right) \\
- 2 \text{Li}_2 \left( \frac{m_i^2}{2m_i^2 - P^2_{ia}} \right) + 2 \ln \left[ \frac{m_i^2}{(m_i^2 - P^2_{ia})(2m_i^2 - P^2_{ia})} \right] \ln \left( \frac{2m_i^2 - P^2_{ia}}{m_i^2 - P^2_{ia}} \right) \\
+ \frac{1}{2} \ln^2 \left( \frac{m_i^2}{m_a^2} \right) + \frac{3}{2} \ln \left( \frac{m_i^2}{m_a^2} + \frac{m_i^2 (m_i^2 - 4P^2_{ia})}{2P^4_{ia}} \right) \ln \left( \frac{P^2_{ia}}{m_i^2} \right) \\
+ \frac{\pi^2}{3} - \frac{3}{2} + \frac{m_i^2}{2P^2_{ia}},
\]

\[
G^{(\text{sub})}_{ia,-}(P^2_{ia}) = G^{(\text{sub})}_{ai,-}(P^2_{ia}) = \frac{1}{2}.
\]

The fully massless limit, in which we additionally have \( m_i \to 0 \), can be read off from the above results easily, and we get back the corresponding results of Section 3.

### A.2 Endpoint contributions for \( x_0 = 0 \)

In Section 4 we have given the endpoint contributions \( G^{(\text{sub})}_{ff',r}(P^2_{ff'}, x_0) \) with \( ff' = ia, ai, ab \) for an arbitrary lower limit \( x_0 \geq 0 \) and finite fermion masses. In many applica-
tions it is possible to set $x_0$ to zero, which simplifies the formulas for these contributions. For convenience, we list the results on $G^{(\text{sub})}_{\mathbf{f}', \mathbf{f}'}(P_{\mathbf{f}'}^2) = G^{(\text{sub})}_{\mathbf{f}', \mathbf{f}'}(P_{\mathbf{f}'}^2, 0)$ explicitly,

$$G^{(\text{sub})}_{ia, +}(P_{ia}^2) = 2 \ln \left( \frac{m_ia}{P_{ia}^2} \right) + \frac{2 \lambda_{ia}^2}{\lambda_{ia}} \ln \left( \frac{m_a^2}{m_i^2} \right) - \frac{\lambda_{ia}^2}{\lambda_{ia}} \ln \left( \frac{m_a^2}{m_i^2} \right) + 2$$

$$G^{(\text{sub})}_{ia, -}(P_{ia}^2) = \frac{2b}{\lambda_{ia}} \left[ \ln \left( \frac{m_a^2}{m_i^2} \right) + \frac{m_i^2}{m_a^2} \right] - 4$$

$$G^{(\text{sub})}_{ai, +}(P_{ia}^2) = 2 \ln \left( \frac{m_i^2}{P_{ia}^2} \right) + \frac{P_{ia}^4(3P_{ia}^2 + 2m_a^2 + \gamma \lambda_{ia})}{4m_a^2(P_{ia}^2 + m_i^2)^2} \ln \left( \frac{m_a^2}{m_i^2} \right) + 2 \ln \left( \frac{m_a^2}{m_i^2} \right)$$

$$G^{(\text{sub})}_{ai, -}(P_{ia}^2) = \frac{1}{2},$$

$$G^{(\text{sub})}_{ab, +}(s) = \ln \left( \frac{m_a^2 s^2}{s^2} \right) + \frac{3}{2} + \frac{s}{\lambda_{ab}} \left\{ \ln \left( \frac{m_a^2 \lambda_{ab}}{m_a^2 s^2} \right) \ln(d_1) + \frac{1}{2} - \frac{2m_a^2}{s} \right\} \ln(d_1)$$

$$G^{(\text{sub})}_{ab, -}(s) = \frac{1}{2},$$

(A.5)

where the abbreviations $b_i, c_i, d_i, \gamma$ are defined as in Section [ ]. In particular, we have

$$b_0 = -\frac{2m_a^2}{2m_a^2 + P_{ia}^2 - \sqrt{\lambda_{ia}}}.$$  

(A.6)

**B  Derivation of phase-space splittings**

In this appendix, we outline the derivation of the photonic parts of phase space needed for the analytical integration of the subtraction function. The emitter/spectator cases $ij$, $ia + ai$, and $ab$ are kinematically different and are treated separately.
so that (4.6) and (B.1) directly lead to
\[
\int d\phi(p_i, p_j, k; P_{ij}) = \frac{1}{8(2\pi)^6} \int d\Omega_j \int d\varphi_{ij} \int dp_i^0 \int dp_j^0,
\]

in the CM frame of $P_{ij}$. In (B.1) we have exploited the fact that the momenta $p_j$ and $\tilde{p}_j$ possess the same solid angle $\Omega_j$ in this frame, as a consequence of definition (4.3). The phase-space variables of $d\phi(p_i, p_j, k; P_{ij})$ are illustrated in Fig. 2. The angles $\theta_{ij}$ and $\varphi_{ij}$ assign the polar and azimuthal angles of $p_i$ with respect to the $p_j$ axis, respectively. The particle energies $p_i^0$ and $p_j^0$ can be expressed in terms of the variables $y_{ij}$ and $z_{ij}$,

\[
p_i^0 = \frac{2m_i^2 + P_{ij}^2(y_{ij} + z_{ij} - y_{ij}z_{ij})}{2\sqrt{P_{ij}^2}}, \quad p_j^0 = \frac{2m_j^2 + P_{ij}^2(1 - y_{ij})}{2\sqrt{P_{ij}^2}},
\]

so that (4.6) and (B.1) directly lead to $[dk(P_{ij}^2, y_{ij}, z_{ij})]$ as given in (4.7). The integration boundary for the particle energies, which is determined by

\[
1 \geq |\cos \theta_{ij}| = \frac{|P_{ij}^2 - 2\sqrt{P_{ij}^2(p_i^0 + p_j^0) + 2p_i^0p_j^0 + m_i^2 + m_j^2 - m_{ij}^2}|}{2|p_i||p_j|}
\]

translates into the boundary (4.8) of the variables $y_{ij}$ and $z_{ij}$.
B.2 Final-state emitter and initial-state spectator, and vice versa

The photonic phase-space measure \([dk(P_{ia}^2, x, z_{ia})]\) of the \(ia\) and \(ai\) cases is defined in (4.18). We derive the form of this measure by comparing appropriate parametrizations of both sides of (4.18). On the l.h.s. we factorize the phase space into two two-particle phase spaces,
\[
\int d\phi(p_i, k, K_{ia}; p_a + p_b) = \int \frac{d(p_i + k)^2}{2\pi} \int d\phi(p_i + k, K_{ia}; p_a + p_b) \int d\phi(p_i, k; p_i + k), \tag{B.4}
\]
and insert the parametrizations
\[
\int d\phi(p_i + k, K_{ia}; p_a + p_b) = \frac{1}{4(2\pi)^2} \frac{1}{\sqrt{\lambda(s, m_a^2, m_b^2)}} \int dP_{ia}^2 \int d\varphi_K,
\]
\[
\int d\phi(p_i, k; p_i + k) = \frac{1}{4(2\pi)^2} \frac{1}{\sqrt{\lambda((p_i + k)^2, P_{ia}^2, m_a^2)}} \int (p_i - p_a)^2 \int d\varphi_\gamma, \tag{B.5}
\]
where \(\varphi_K\) is the azimuthal angle of \(K_{ia}\) in the CM frame of \(p_a + p_b\), and \(\varphi_\gamma\) is the one of the photon in the CM frame of \(p_i + k\). Using the relations
\[
x_{ia} = \frac{-P_{ia}^2}{(p_i + k)^2 - P_{ia}^2 + m_a^2}, \quad z_{ia} = \frac{m_a^2 + m_b^2 - (p_a - p_i)^2}{(p_i + k)^2 - P_{ia}^2 + m_a^2}, \tag{B.6}
\]
the integrations over the invariants \((p_i + k)^2\) and \((p_i - p_a)^2\) can be replaced by integrations over \(x_{ia}\) and \(z_{ia}\). The integration limits \(z_{1,2}(x)\), which are given in (4.22), follow from the limits on \((p_a - p_i)\) for fixed \((p_i + k)^2 = (P_{ia} + p_a)^2 = P_{ia}^2 - m_a^2 - P_{ia}^2/x\). They can be easily derived in the CM frame of \(p_i + k\). Finally, we use
\[
\sqrt{\lambda((p_i + k)^2, P_{ia}^2, m_a^2)} = \sqrt{\lambda_{ia} R_{ia}(x_{ia})} \tag{B.7}
\]
on the l.h.s. of (4.18). On the r.h.s. we make use of the parametrization
\[
\int d\phi(\tilde{p}_a(x), K_{ia}; \tilde{p}_a(x) + p_b) = \frac{1}{4(2\pi)^2} \int dP_{ia}^2 \frac{1}{\sqrt{\lambda(s, m_a^2, m_b^2)}} \int d\tilde{\varphi}_K, \tag{B.8}
\]
where \(\tilde{\varphi}_K\) denotes the azimuthal angle of \(K_{ia}\) in the CM frame of \(\tilde{p}_a(x) + p_b\). Note that \(\tilde{s}\), which is the squared CM energy of \(\tilde{p}_a(x) + p_b\), depends on \(s, x, P_{ia}^2\), and \(K_{ia}^2\).

Since we are not dealing with transverse polarizations, but with helicity eigenstates or unpolarized configurations, we can assume rotational invariance of \(|M_0(\tilde{\Phi}_{0,ia})|^2\) around the beam axis in the corresponding CM frame of \(\tilde{p}_a(x) + p_b\). This implies that the integration over the azimuthal angle \(\varphi_K\) yields a trivial factor of \(2\pi\). The integration of \(|M_0(\tilde{\Phi}_{0,ia})|^2\) over \(\varphi_K\) yields a factor of \(2\pi\) as well because of the one-to-one correspondence of \(\varphi_K\) and \(\tilde{\varphi}_K\).\footnote{The angles are related by \(\tan \varphi_K = \tan \tilde{\varphi}_K\), if \(\varphi_K = \tilde{\varphi}_K = 0\) is defined in the plane spanned by \(p_a, p_b\), and \(\tilde{p}_a\). This follows from the fact that components of the direction orthogonal to this plane are not affected by the Lorentz transformation that relates the CM frames of \(p_a + p_b\) and \(\tilde{p}_a + p_b\). The integrals over \(\varphi_K\) and \(\tilde{\varphi}_K\) remain unchanged by the transformation: \(\int_{0}^{2\pi} d\varphi_K = \int_{0}^{2\pi} d\tilde{\varphi}_K = 2\pi\).}

Inserting the above relations into (4.18), the photonic phase space \([dk(P_{ia}^2, x, z_{ia})]\) can be identified for fixed values of \(x = x_{ia}\) and \(P_{ia}^2\). The result is given in (4.20).
B.3 Initial-state emitter and initial-state spectator

The measure $[dk(s, x, y_{ab})]$ for the photon phase space is derived by considering explicit representations of the two phase-space volumes $d\phi(\ldots)$ in (4.41). The full phase space of $P_{ab}$ and $k$ is parametrized in the CM frame of $p_a + p_b$ by

$$\int d\phi(k, P_{ab}; p_a + p_b) = \frac{1}{8(2\pi)^2} \frac{\sqrt{\lambda(s, P_{ab}^2, m^2)}}{s} \int d\Omega_\gamma,$$

where $\Omega_\gamma$ is the solid angle of the photon. The one-particle phase space of $\tilde{P}_{ab}(x)$ reads

$$\int d\phi(\tilde{P}_{ab}(x); \tilde{p}_a(x) + p_b) = (2\pi) \delta(\tilde{P}_{ab}^2(x) - P_{ab}^2) = \frac{2\pi}{s} \delta(x - x_{ab}).$$

Putting everything together and expressing the polar angle $\theta_\gamma$ of the photon in terms of $y_{ab}$, we get the result (4.44) for $[dk(s, x, y_{ab})]$. The integration boundary (4.45) on $y_{ab}$ is determined by $|\cos \theta_\gamma| < 1$.

C Sketch of the calculation of the non-trivial integrals

In Section 4 we have seen that the analytical integration of the subtraction function $|M_{\text{sub}}|^2$ over the photonic parts $[dk(\ldots)]$ of phase space leads to integrals of a non-trivial structure. Therefore, we sketch the calculation of those integrals in this appendix.

C.1 Final-state emitter and final-state spectator

We first consider the integral for $G^{(\text{sub})}_{ij,+}$, as defined in (4.9), for an emitter $i$ and a spectator $j$ in the final state. The integration over the variable $z_{ij}$ is simple and yields

$$G^{(\text{sub})}_{ij,+}(P_{ij}^2) = \int_{y_1}^{y_2} dy \frac{\bar{P}_{ij}^2}{\sqrt{\lambda_{ij} R_{ij}(y)}} \left\{ 2 \ln \left[ \frac{1 - (1 - y)z_1(y)}{1 - (1 - y)z_2(y)} \right] - \frac{1 - y}{2y} \left[ 2 + z_1(y) + z_2(y)[z_2(y) - z_1(y)] - \frac{2m^2}{P_{ij}^2} \frac{1 - y}{y^2} \left[ z_2(y) - z_1(y) \right] \right] \right\} - G^{(\text{sub})}_{ij,-}(P_{ij}^2),$$

where we have renamed $y_{ij}$ to $y$. The contribution of $G^{(\text{sub})}_{ij,-}(P_{ij}^2)$ will be calculated below. The explicit integral over $y$ involves two types of square roots of quadratic forms in $y$, entering via the limits $z_{1,2}(y)$ given in (4.8). The limits $y_{1,2}$ are also defined there. Either of those roots can be removed by splitting the $y$ range into two pieces:

(a) $y_1 < y < \Delta y \ll 1$,

(b) $\Delta y < y < y_2$.

The IR singularity is contained in part (a) so that part (b) can be evaluated with $m_\gamma = 0$, replacing the root $\sqrt{y^2 - y_1^2}$ by $y$. The integration over part (a) is simplified by choosing the auxiliary parameter $\Delta y$ small so that in $O(\Delta y)$ the parameter $y$ can be set to zero in
the non-singular factors of the integrand. Thus, for \( m_\gamma \to 0 \) we can replace \( R_{ij}(y) \) by 1 in this part. Explicitly we get

\[
G_{ij,+}^{\text{(sub)}}(P_{ij}^2) \bigg|_{(a)} = \int_{y_1}^{\Delta y} dy \frac{2 \bar{P}_{ij}^2}{y \sqrt{\lambda_{ij}}} \ln \left[ \frac{(P_{ij}^2 + 2m_{ij}^2)y + \sqrt{\lambda_{ij} y^2 - y_1^2}}{(P_{ij}^2 + 2m_{ij}^2)y - \sqrt{\lambda_{ij} y^2 - y_1^2}} \right] \\
- \int_{y_1}^{\Delta y} dy \frac{2 \bar{P}_{ij}^2}{y^2 \sqrt{y^2 - y_1^2}},
\]

\[
G_{ij,+}^{\text{(sub)}}(P_{ij}^2) \bigg|_{(b)} = \int_{\Delta y}^{y_2} dy \frac{2 \bar{P}_{ij}^2 \ln \left[ \frac{2m_{ij}^2 + P_{ij}^2(1 + y) + \sqrt{(2m_{ij}^2 + P_{ij}^2 - P_{ij}^2 y)^2 - 4P_{ij}^2 m_{ij}^2}}{2m_{ij}^2 + P_{ij}^2(1 + y) - \sqrt{(2m_{ij}^2 + P_{ij}^2 - P_{ij}^2 y)^2 - 4P_{ij}^2 m_{ij}^2}} \right]}{y \sqrt{(2m_{ij}^2 + P_{ij}^2 - P_{ij}^2 y)^2 - 4P_{ij}^2 m_{ij}^2}} \\
- \int_{\Delta y}^{y_2} dy \left\{ \frac{2}{y} - \frac{P_{ij}^4 y}{2(m_{ij}^2 + y P_{ij}^2)^2} \right\} - G_{ij,-}^{\text{(sub)}}(P_{ij}^2). 
\]

The second integrals in both parts are elementary. In the first integrals we remove the square roots by the substitutions

(a) \( y_1 \sqrt{1 + x} = y + \sqrt{y^2 - y_1^2} \),

(b) \( x = y_2 - y + \sqrt{(y_2 - y) \left( 2 - y - y_2 + 4m_{ij}^2 \right)} \).

The resulting integrals are of the form

\[
\int_{x_1}^{x_2} dx \ f(x) \ln[g(x)],
\]

where \( f(x) \) and \( g(x) \) are algebraic functions. Upon decomposing \( f(x) \) into partial fractions and factorizing \( g(x) \), such integrals yield subintegrals that can be expressed in terms of logarithms and dilogarithms. A convenient way to obtain compact results is to transform the limits \( x_{1,2} \) into 0 and \( \infty \) in a first step. This is achieved by the substitution \( \xi = (x - x_1)/(x_2 - x) \). The subintegrals that lead to dilogarithms can then be calculated by using the standard integral

\[
\int_0^\infty d\xi \left( \frac{1}{\xi - \alpha_0} - \frac{1}{\xi - \alpha_1} \right) \ln(1 + \beta \xi) = \sum_{i=0,1} (-1)^i [\text{Li}_2(1 + \beta \alpha_i) + \eta(-\alpha_i, \beta) \ln(1 + \beta \alpha_i)].
\]

Although these steps are straightforward, they nevertheless involve a lot of algebra. Therefore, we omit the details. Instead we comment on the IR singularity and the role of the parameter \( \Delta y \). In part (a) the upper limit of the integration over \( x \) tends to infinity like \( \Delta y^2 \bar{P}_{ij}^4/(m_{ij}^2 m_{ij}^2) \) for fixed \( \Delta y \), since the photon mass \( m_\gamma \) is infinitesimal. This induces terms proportional to \( \ln(\Delta y/m_\gamma) \) in part (a). On the other hand, part (b) is logarithmically divergent for \( \Delta y \to 0 \). The artificial \( \ln(\Delta y) \) terms, of course, cancel in the sum

\footnote{The contributions of the function \( \eta(x, y) = \ln(xy) - \ln(x) - \ln(y) \) are necessary to put the arguments of the dilogarithms onto the first Riemann sheet for complex constants \( \alpha_{0,1} \) and \( \beta \).}
of parts (a) and (b). Finally, we note that we had to exploit some identities for the
dilogarithms in order to obtain the compact form of the final result (4.10) for \(G_{ij,+}^{(\text{sub})}\).

The calculation of \(G_{ij,-}^{(\text{sub})}\) proceeds in a different way. Since this function is IR-finite,
we can set \(m_\gamma\) to zero from the beginning. The defining integral (4.9) explicitly reads
\[
G_{ij,-}^{(\text{sub})}(P_{ij}^2) = \frac{m_\gamma^2}{\sqrt{\lambda_{ij}}} \int_0^{y_2} dy \frac{1 - y}{y^2} \frac{r_{ij}(y)}{R_{ij}(y)} \int_{z_1(y)}^{z_2(y)} dz \frac{(1 - z)^2}{z}.
\]

(C.5)

Note that only the behaviour of \(g_{ij,-}\) at \(y \to 0\) is relevant in the IR and collinear limits [see (3.3)]. Therefore, we have chosen a form of the auxiliary function \(r_{ij}(y)\) that simplifies
the integration. We have defined \(r_{ij}(y)\) in such a way that
\[
\frac{1 - y}{y^2} \frac{r_{ij}(y)}{R_{ij}(y)} = -\frac{d}{dy} \left[ \frac{R_{ij}(y)}{y} \right] + \mathcal{O}(m_\gamma^2).
\]

(C.6)

This choice allows us to perform the integration in (C.5) over \(z\) implicitly upon applying
integration by parts in the integration over \(y\). The boundary terms of this integration by
parts vanish, and the result is
\[
G_{ij,-}^{(\text{sub})}(P_{ij}^2) = \frac{m_\gamma^2}{\sqrt{\lambda_{ij}}} \int_0^{y_2} dy \frac{R_{ij}(y)}{y} \left\{ \frac{[1 - z_2(y)]^2}{z_2(y)} z'_2(y) - \frac{[1 - z_1(y)]^2}{z_1(y)} z'_1(y) \right\},
\]

(C.7)

where \(z'_{1,2}(y) = dz_{1,2}(y)/dy\). The final integration over \(y\) is elementary. Note that the
above trick avoids terms such as \(\ln[z_{1,2}(y)]\) after the integration over \(z\); such terms would
lead to dilogarithms in the final result.

C.2 Final-state emitter and initial-state spectator, and vice versa

The integrals for the endpoint contributions defined in (4.27) for the mixed cases
\(ff' = ia, ai\) are calculated in a similar way. Therefore, we outline only the basic steps.

Inspecting the explicit form of the distributions \(G_{ff',+}^{(\text{sub})}\), we find that the integrals (4.27)
for \(G_{ff',+}^{(\text{sub})}\) again contain two different square roots of quadratic forms in \(x\). Analogously
to the \(ij\) case, we first separate these roots by splitting the range of the \(x\) integration as follows:

(a) \(x_1 > x > 1 - \Delta x\), with \(\Delta x \ll 1\),

(b) \(1 - \Delta x > x > x_0\).

Part (a) contains the IR singularity and involves only values of \(x\) in the vicinity of 1. Thus, in \(\mathcal{O}(\Delta x)\) we can set \(x\) to 1 in all non-singular terms of the integral. In particular,
this replaces the function \(R_{ia}(x)\) by \(1 + \mathcal{O}(m_\gamma^2)\) and removes the root implicitly contained
in \(R_{ia}(x)\). In \(\mathcal{O}(m_\gamma)\) we can replace the explicit root \(\sqrt{P_{ia}^4(1 - x)^2 - 4m_i^2m_\gamma^2x^2}\), which
appears in \(z_{1,2}(x)\) given in (1.22), by \(\sqrt{P_{ia}^4(1 - x)^2 - 4m_i^2m_\gamma^2} \). This root is removed by the substitution

(a) \(\frac{2m_im_\gamma}{P_{ia}^2}\sqrt{1 + y} = 1 - x + \sqrt{(1 - x)^2 - \frac{4m_i^2m_\gamma^2}{P_{ia}^4}}\).
The resulting integral is of the form (C.3) and can be reduced to logarithms and dilogarithms, as described above. The IR singularity appears in contributions proportional to \( \ln(\Delta x/m_\gamma) \). In part (b) we can set \( m_\gamma \) to zero, since the IR singularity is avoided by the finite value of \( \Delta x \). This eliminates the explicit root in \( z_{1,2}(x) \) given in (4.22). The root in \( R_{ia}(x) \) is removed by the substitution

\[
(b) \quad y = -\frac{2m_\gamma A}{P_{ia}^2} x - \frac{\sqrt{\lambda_{ia}}}{P_{ia}^2} R_{ia}(x), \quad \text{with} \quad A = \sqrt{-P_{ia}^2 - m_i^2} > 0.
\]

Note that this substitution is only allowed for \( A > 0 \). This is, e.g., fulfilled if \( P_{ia}^2 < 0 \), but in general not for all \( P_{ia}^2 \). We evaluate the integrals for the allowed range with \( A > 0 \) and cover the full parameter space in the final result by analytical continuation in \( P_{ia}^2 \). The reduction of the obtained integral, which is again of the general form (C.3), to logarithms and dilogarithms proceeds as above. However, particular care is needed in the arguments of those multivalued functions. As required, the singular \( \ln(\Delta x) \) terms cancel in the sum of parts (a) and (b).

The calculation of \( G_{(sub)}^{ab} \) is simplified by an appropriate choice of the auxiliary function \( r_{ia}(x) \). Similar to the \( ij \) case, we have defined this function in such a way that

\[
\frac{x}{(1-x)^2} \frac{r_{ia}(x)}{R_{ia}(x)} = \frac{d}{dx} \left[ \frac{R_{ia}(x)}{1-x} \right] + O(m_\gamma^2).
\]

Hence, integration by parts can be applied as above, and the resulting integral over \( x \) is elementary.

Finally, the calculation of \( G_{(sub)}^{ai,-} \) is trivial.

### C.3 Initial-state emitter and initial-state spectator

In view of the analytical integrations, the \( ab \) case turns out to be the simplest one. The integrals of the endpoint parts are given by

\[
G_{(sub)}^{ab,\tau}(s, x_0) = \int_{x_0}^{x_1} dx \ G_{(sub)}^{ab,\tau}(s, x) \tag{C.9}
\]

with the distributions \( G_{(sub)}^{ab,\tau} \) of (4.48). The calculation of \( G_{(sub)}^{(a,b)} \) is trivial.

The integral for \( G_{(sub)}^{ab,+} \) involves only the square root of the quadratic form in \( x \) that is contained in the limits \( y_{1,2}(x) \) given in (4.43). Note that the auxiliary function \( R_{ab}(x) \) does not occur in the integral. As above, we first split the range of the integration over \( x \) as follows:

(a) \( x_1 > x > 1 - \Delta x \), \quad with \( \Delta x \ll 1 \),

(b) \( 1 - \Delta x > x > x_0 \).

In part (a) we can set \( x \) to 1 in all non-singular terms, and the root is removed by the substitution

\[
(a) \quad \frac{2m_\gamma \sqrt{s}}{s} \sqrt{1 + y} = 1 - x + \sqrt{(1-x)^2 - \frac{4sm_\gamma^2}{s^2}}.
\]

This leads to an integral of the form (C.3), which is evaluated as described above. In part (b) we can set \( m_\gamma \) to zero, directly resulting in an elementary integral, which is expressed in terms of logarithms.
D Photonic corrections to $\mu^+\mu^- \to \nu_e\bar{\nu}_e(\gamma)$

Using the methods described in Ref. [7], we have calculated the virtual photonic corrections of $\mathcal{O}(\alpha)$ for arbitrary muon mass. The one-loop amplitude reads

$$\mathcal{M}_{\text{virt}} = \frac{\alpha}{\pi} \left\{ \frac{s - 2m^2_{\mu}}{4\beta s} \left[ \pi^2 - 4 \ln \left( \frac{m_{\gamma}}{m_{\mu}} \right) \ln(x_s) - \ln^2(x_s) - 4 \text{Li}_2(1 + x_s) + 4\pi i \ln(1 + x_s) \right] \right. \right.$$  

$$+ \ln \left( \frac{m_{\mu}}{m_{\gamma}} \right) - \frac{3\beta}{4} \ln(x_s) - 1 \} \mathcal{M}_0$$ 

$$+ \frac{e(v_{\nu_e} + a_{\nu_e})}{2(s - M_Z^2 + iM_Z\Gamma_Z)} \frac{\alpha m_{\mu}}{\pi \beta s} \ln(x_s) \left\{ 2m_{\mu} a_{\mu} \left[ \bar{v}_{\mu^+} \gamma^\rho \gamma_5 u_{\mu^-} \right] \left[ \bar{u}_{\nu_e} \gamma_\rho (1 - \gamma_5) v_{\nu_e} \right] \right. \right.$$  

$$- v_{\mu} \left[ \bar{v}_{\mu^+} u_{\mu^-} \right] \left[ \bar{u}_{\nu_e} (1 - \gamma_5) v_{\nu_e} \right] \}$$,  

(D.1)

where $\beta$ denotes the muon velocity in the CM frame, and $x_s$ is an auxiliary variable,

$$\beta = \sqrt{1 - \frac{4m^2_{\mu}}{s}}, \quad x_s = \frac{\beta - 1}{\beta + 1} + i\epsilon.$$  

(D.2)

The fermion spinors in (D.1) carry the same arguments as indicated in the Born amplitude $\mathcal{M}_0$ given in (5.20). The spinor chains have been evaluated by applying the Weyl–van der Waerden spinor technique, following the formulation of Ref. [3]. The amplitudes $\mathcal{M}_1$ for the radiative process $\mu^+\mu^- \to \nu_e\bar{\nu}_e\gamma$ are contained in Ref. [3] explicitly. The application of the slicing method to the real corrections requires the separate calculation of the soft-photonic corrections. They are contained in the factor correction $\delta_{\text{soft}}$ to the Born cross section $\sigma_0$,

$$\delta_{\text{soft}} = -\frac{\alpha}{\pi} \left\{ \frac{s - 2m^2_{\mu}}{2\beta s} \left[ 4 \ln \left( \frac{2\Delta E}{m_{\gamma}} \right) \ln(-x_s) + 4 \text{Li}_2(1 + x_s) + \ln^2(-x_s) \right] \right. \right.$$  

$$+ 2 \ln \left( \frac{2\Delta E}{m_{\gamma}} \right) + \ln(-x_s) \}$$,  

(D.3)

which has been deduced from the general results given in Ref. [7].

The above results can be easily expanded in the limit $m_{\mu} \to 0$, which can be used for high energies.

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