Fan-Crossing Free Graphs

Franz J. Brandenburg
94030 Passau, Germany
brandenb@informatik.uni-passau.de

Abstract. A graph is fan-crossing free if it admits a drawing in the plane so that each edge can be crossed by independent edges. Then the crossing edges have distinct vertices. In complement, a graph is fan-crossing if each edge can be crossed by edges of a fan. Then the crossing edges are incident to a common vertex. Graphs are $k$-planar if each edge is crossed by at most $k$ edges, and $k$-gap-planar if each crossing is assigned to an edge involved in the crossing, so that at most $k$ crossings are assigned to each edge.

We use the $s$-subdivision, path-addition, and node-to-circle expansion operations to show that there are fan-crossing free graphs that are not fan-crossing, $k$-planar, and $k$-gap-planar for $k \geq 1$, respectively. A path-addition adds a long path between any two vertices to a graph. An $s$-subdivision replaces an edge by a path of length $s$, and a node-to-circle expansion substitutes a vertex by a 3-regular circle, so that each vertex of the circle inherits an edge incident to the original vertex.

We introduce universality for an operation and a graph class, so the every graph has an image in the graph class. In particular, we show the fan-crossing free graphs are universal for 2-subdivision and for node-to-circle expansion.

Finally, we show that some graphs have a unique fan-crossing free embedding, that there are maximal fan-crossing free graphs with less edges than the density, and that the recognition problem for fan-crossing free graphs is NP-complete.

1 Introduction

We consider graphs that are simple both in a graph theoretic and a topological sense, so that there are no multi-edges or loops, and in a drawing, adjacent edges do not cross and two edges cross at most once. Graphs are often defined by particular properties of a drawing. The planar graphs, in which edge crossings are excluded, are the most prominent example. There has been recent interest in the study of beyond-planar graphs [31], which are defined by drawings with specific restrictions on crossings. If a graph admits a drawing with a specific property, then it is named accordingly, and there is a respective graphs class. For example, there are planar drawings and planar graphs.

A $(k, \ell)$-grid [1] consists of two (or more) sets of edges of a graph, so that, in a drawing, each edge from the first set of $k$ edges crosses each edge from the second set of $\ell$ edges. Special cases are radial grids or fans, in which the edges from the
first set are incident to a single vertex, and natural grids or independent edges, in which the edges from the first set have distinct vertices, see Fig. 3. A graph is planar if it admits a drawing without (1,1)-grids and k-planar if (k + 1, 1)-grids are avoided. Then each edge is crossed by at most k edges. Graphs are fan-crossing free if they do not admit radial (2, 1)-grids and fan-crossing if there are no natural (2, 1)-grids in a drawing. Then an edge is crossed at most once or by two or more edges that are independent and form a fan, respectively. A drawing of a graph is k-quasi-planar if k edges do not mutually cross. 3-quasi-planar graphs are called quasi-planar and can be considered as (1, 1, 1)-grid free graphs. The aforementioned graphs can also be defined by first order logic formulas [20]. A drawing is k-gap-planar [13] if each crossing is assigned to an edge involved in the crossing and at most k crossings are assigned to each edge. Note that these properties are topological. They hold for embeddings, which are equivalence classes of topologically equivalent drawings. There are some special cases, such as outer and layered drawings [31]. If all vertices are in the outer face, then there is an outer drawing, which is outer planar if there are no crossings. Outer fan-crossing (fan-crossing free, k-planar, k-gap-planar, quasi-planar) drawings are defined accordingly. Right-angle crossing is a geometric property. A drawing is right angle crossing (RAC) if the edges are drawn straight line and may cross at a right angle [30].

The classes of k-planar, fan-crossing free, fan-crossing, quasi-planar, k-gap-planar, and right-angle crossing graphs are denoted by kPLANAR, FCF, FAN, QUASI, kGAP, and RAC, respectively. Then 0PLANAR = 0GAP is the class of planar graphs. These and other classes have been studied with different intensity and depth. In particular, the density [31], which is the maximal number of edges of n-vertex graphs, and the size of the largest complete (bipartite) graph [9] is well explored. It has been shown that the recognition problem is NP-complete for 1-planar, fan-crossing (fan-planar), 1-gap-planar, and RAC graphs, whereas a proof for the NP-completeness is still missing for k-planar, k-gap-planar, fan-crossing free and quasi-planar graphs if k ≥ 2.

Consider inclusion relations. In drawings, fan-crossing and fan-crossing free are complementary on edges that are crossed at least twice [20]. Both admit edges that are uncrossed or are crossed once. Hence, every 1-planar graph is both fan-crossing and fan-crossing free, but not conversely, since there are graphs that are fan-crossing and fan-crossing free, but are not 1-planar [21]. There are fan-crossing graphs that are not fan-crossing free, since the density of fan-crossing

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**Fig. 1.** (a) A fan crossing or a radial grid, (b) crossing independent edges or a natural grid, (c) a 2-planar crossing and (d) a quasi-planar crossing
free graphs is $4n - 8$, whereas there are $n$-vertex fan-crossing graphs with $5n - 10$ edges. Particular examples of fan-crossing and non-fan-crossing free graphs are $K_7$ and $K_{p, q}$ with $p = 3, 4$ and $q \geq 7$ [9]. We prove the opposite direction and therefore use path-additions and node-to-circle expansions.

Many more classes of beyond planar graphs have been studied, see [31, 44], some of which are relevant for this paper. A 1-planar drawing is IC-planar [7, 12, 24] if each vertex is incident to at most one crossed edges. It is fan-planar [14, 16, 42] if it is fan-crossing and excludes crossings of an edge from both sides, called configuration II in [42]. This is a restriction, since there are fan-crossing graphs that are not fan-planar [19]. However, it has no impact on the density and the results on fan-planar graphs proved in [9, 14, 16].

Fan-crossing free graphs were introduced by Cheong et al. [28]. They focus on the density of $(k)$-fan-crossing free graphs. Complete and complete bipartite graphs were studied by Angelini et al. [9]. The state of the art is as follows, see also [31].

1. Every $n$-vertex fan-crossing free graph has at most $4n - 8$ edges and the bound is tight.
2. Every fan-crossing free drawing of a graph with $4n - 8$ edges is 1-planar.
3. Every fan-crossing free graph with a drawing with straight line edges has at most $4n - 9$ edges and the bound is tight.
4. 1-planar graphs and right angle crossing (RAC) graphs are fan-crossing free.
5. $K_6$ is fan-crossing free, whereas $K_7$ is not. $K_{p, q}$ is fan-crossing free if and only if $p \leq 2$ or $p \leq 4$ and $q \leq 6$.

For fan-planar graphs, the following has been proved [9, 14, 16, 19, 42]. The shown results also hold for fan-crossing graphs, since the restriction from configuration II is not used in the proofs.

1. Every $n$-vertex fan-crossing graph has at most $5n - 10$ edges and the bound is tight.
2. Every fan-crossing graph is quasi-planar.
3. The fan-crossing graphs and the $k$-planar graphs for $k \geq 2$ are incomparable.
4. The recognition problem for fan-crossing graphs is NP-complete.
5. $K_7$ is fan-crossing, whereas $K_8$ is not. The complete bipartite graph $K_{p, q}$ is fan-crossing if and only if $\min\{p, q\} \leq 4$.

Similar facts are known for 1-planar graphs [44], RAC graphs [30, 34], 1-gap-planar graphs [13], and quasi-planar graphs [2, 3, 8]. For a survey see [31].

In this work, we add some more facts on fan-crossing free graphs, which demonstrate the power of these graphs. Some of the results come as expected, for example the NP-hardness, which is stated as an open problem in [31]. As our main contribution, we study three operations on graphs, which are used to distinguish fan-crossing free graphs from other classes of graphs including fan-crossing graphs, and we establish properties of fan-crossing free graphs that are known for 1-planar and fan-crossing graphs. In particular, we prove the following:
The closure properties of some classes of beyond-planar graphs are summarized in Table 1, and inclusion relations and the containment of complete (bipartite) graphs are displayed in Fig. 2, where graphs $A$, $B$ and $C$ are contributed by this work.

The rest of the paper is structured as follows: In Section 2, we study three graph operations and introduce the universality of graph classes for an operation. In Section 3, we establish incomparabilities between some classes of beyond-planar graphs that were unknown so far. Finally, in Section 4, we study properties of fan-crossing free graphs and show that some graphs have a unique fan-crossing free embedding, that there are sparse maximal fan-crossing free graphs, and that the recognition problem is NP-complete.

2 Graph Operations and Universality

The linear density is a common feature of some well-known classes of beyond-planar graphs. Also planar subgraphs and planarizations play an important role for their study, in particular for the density and drawings. These properties may indicate that planar and beyond-planar graphs are close to each other,
and motivate the term “nearly planar” [36]. A major distinction comes from subdivisions, which substitute edges by paths. Every graph $G$ has a subdivision $\sigma(G)$ so that $\sigma(G)$ is 1-planar (or in some other beyond-planar graph class). On the other hand, a graph is planar if and only if its subdivision is planar. The subdivision operation is important for taking minors [32]. An $s$-subdivision replaces an edge by a path of length $s$.

**Definition 1.** A class of graphs $\mathcal{G}$ is universal for a graph operation $f$, or simply $f$-universal, if for every graph $G$ there is a graph $G' \in \mathcal{G}$ such that $G' = f(G)$.

Clearly, the set of discrete graphs is universal for edge insertion, and the set of complete graphs is universal for edge removal. A graph class $\mathcal{G}'$ is universal for $f'$ if $\mathcal{G} \subseteq \mathcal{G}'$, $f'$ extends $f$, and $\mathcal{G}$ is universal for $f$.

Universality provides a new perspective on the set of (all) graphs. By operation $f$, the class of all graphs is projected to the special class $\mathcal{G}$. In other words, all graphs look like graphs from $\mathcal{G}$ if graphs are seen through the lens of the operation. Hence, $\text{GRAPHS} = f^{-1}(\mathcal{G})$, where $\text{GRAPHS}$ denotes the set of all simple graphs.

### 2.1 Subdivision

As of today, the IC-planar graphs are the least class of beyond-planar graphs in the sense of [31] extending the planar graphs. A graph is IC-planar [7] if it admits a 1-planar drawing so that each vertex is incident to at most one crossed edge. Brandenburg et al. [24] have shown that IC-planar graphs admit a straight-line drawing with right angle crossings, so that they are RAC graphs. Further
structural properties have been studied in [12]. RAC graphs are generalized by $k$-bend RAC graphs, which admit drawings with straight-line segments and right angle crossings, so that each edge has at most $k$-bends [30].

Next we consider the universality of some graph classes under subdivision.

**Theorem 1.** (i) The IC-planar graphs are universal for $O(n^2)$-subdivision.
(ii) The $k$-planar and $k$-gap-planar graphs ($k \geq 1$) are universal for $f(n)$-subdivision if $f(n) \in \Theta(n^2)$.
(iii) The RAC graphs are universal for 3-subdivision and not for 2-subdivision.
(iv) The fan-crossing free graphs are universal for 2-subdivision.
(v) The 1-bend RAC graphs are universal for 1-subdivision.

**Proof.** Consider a drawing of a graph $G$ and subdivide an edge into segments, so that each segment is crossed at most once and each subdivision point is incident to at most one crossed segment. A segment is a piece of an edge between two points, which are a vertex or a subdivision point. The so obtained graph $\sigma(G)$ is 1-planar, and is even IC-planar and $O(n^2)$ subdivisions suffice, since each edge of $G$ is crossed at most by all other edges.

From the aforesaid, $O(n^2)$ subdivision suffice for $k$-planar and $k$-gap-planar graphs with $k \geq 1$, and $O(n^2)$ subdivisions are necessary, since the complete graph $K_n$ has $\Omega(n^4)$ many crossings [43]. Since $k$-planar and $k$-gap-planar graphs admit $O(m)$ crossings, where $m$ is the number of edges, some edges have $\Omega(n^2)$ crossings and need $\Omega(n^2)$ many subdivisions for $k$-(gap)-planarity.

For (iii) and (v), Didimo et al. [30] showed that every graph has a 3-bend RAC drawing, whereas $n$-vertex graphs with a 2-bend RAC drawing have $O(n^{7/4})$ edges.

Finally, for (iv), every graph can be drawn such that each edge consists of three segments, the first and last of which are uncrossed and only the middle segments cross, so that the drawing is fan-crossing free. \qed

Note that there are more graph classes that are universal for subdivision, e.g., 3-stack, 2-queue and mixed 1-stack 1-queue graphs [33]. On the other hand, subdivisions do not change the genus of a graph, which is the minimal integer $g$ so that a graph can be embedded on a sphere with $g$ handles [39]. The genus of the $n$-clique $K_n$ is $\left\lceil \frac{(n-3)(n-4)}{12} \right\rceil$ [47]. Hence, graphs of bounded genus are not subdivision universal.

### 2.2 Path-Addition

A path-addition adds an internally vertex-disjoint path $P$ between any two vertices of a graph. Such paths are also known as ears. They are used in ear decompositions of 2-connected graphs [40] and in subdivisions. We wish to use path-additions so that they preserve a given class of graphs. Therefore, the added paths are long and have length at least $cn$ for some $c > 0$. This property distinguishes our path-addition operation from ear decompositions [40]. A path-addition adds a path between any two vertices, whereas a subdivision can do so if there is an edge. A more general version of path-addition was introduced...
in [27] and studied in [26]. For a graph $G$ and a path $P$, let $G \oplus P$ denote the graph obtained by adding the vertices and edges of $P$ to $G$, where the internal vertices of $P$ are new and have degree two. A class of graphs $\mathcal{G}$ is closed under path-addition if there is some function $D_\mathcal{G}(n)$ so that $G \oplus P$ is in $\mathcal{G}$ if $G$ is an $n$-vertex graph in $\mathcal{G}$ and $P$ has length at least $D_\mathcal{G}(n)$.

It has been shown [26] that the graph classes $\text{RAC}$, $\text{FCF}$, $\text{GAP}$, and $\text{QUASI}$ are closed under path-addition if $D_\mathcal{G}(n)$ is a linear function. The idea of the proof is to route path $P$ along a path $S$ in $G$. Path $P$ makes a bend and creates a further subdivision if it crosses an edge incident to an internal vertex of $S$, so that the crossing introduces a violation without the subdivision. Each bend can be charged to an edge or a vertex of the given graph, which are each charged at most once. If a graph is disconnected and vertices $s$ and $t$ are in different components, then there are exits, which are vertices or crossing points in the outer face of a component, so that $P$ can be routed along such exists. On the other hand, the classes of $k$-planar and fan-crossing graphs are not closed under path-addition. That is, there are $k$-planar (fan-crossing) graphs $G$ and vertices $s$ and $t$, so that for any path $P$ from $s$ to $t$ of any length, graph $G \oplus P$ is not $k$-planar (fan-crossing).

We summarize these results:

**Theorem 2.** The graph classes $\text{RAC}$, $\text{FCF}$, $k\text{GAP}$ ($k \geq 1$), and $\text{QUASI}$ are closed under path-addition, whereas the graph classes $k\text{PLANAR}$ ($k \geq 0$) and $\text{FAN}$ are not closed under path-addition.

Hence, the path-addition operation distinguishes the graph classes $\text{FAN}$ and $k\text{PLANAR}$ from $\text{RAC}$, $\text{FCF}$, $p\text{GAP}$ ($p \geq 1$), and $\text{QUASI}$.

### 2.3 Node-to-Circle Expansion

A node-to-circle expansion substitutes each vertex $v$ of a graph by a 3-regular circle $C$ of length $k$ if $v$ has degree $k$, so that each edge incident to $v$ is inherited by a vertex of the circle. If there is a rotation system with the cyclic ordering of the edges incident to $v$, then this ordering is preserved by the node-to-circle expansion. Edges between consecutive vertices of $C$ are called inner edges, and edges between vertices of different circles are called binding edges. They are one-to-one related to the original edges of $G$. A node-to-circle expansion of a graph $G$, denoted by $\eta(G)$, is a cubic graph with $2m$ vertices and $3m$ edges if $G$ has $m$ edges.

A node-to-circle expansion is a special split operation, in which each vertex is replaced by a subgraph $H$, so that the vertices of $H$ inherit all adjacencies from the vertex. In our case, $H$ is a circle. Graph $H$ is a discrete graph in the $k$-split operation from [35]. Here the objective is to transform a graph into a planar one by as few $k$-split operations as possible.

Node-to-circle expansions have been used in VLSI theory to transform a hypercube into a cube-connected cycle [46]. Thereby, each corner of a $d$-dimensional hypercube is beveled so that there is a slope with $d$ corners of degree three. This
Fig. 3. A 1-planar drawing of the node-to-circle expansion of (a) $K_{4,n}$ and (b) $K_7$. In (b) the topmost circle intersects two other circles, drawn as hexagons.

increases the number of vertices from $2^d$ to $d2^d$. Hypercubes and cube-connected cycles have similar properties, such as diameter and separation width.

The node-to-circle expansion operation preserves more properties, such as 3-connectivity and the genus, and it may simplify graphs. For example, complete bipartite graphs $K_{4,n}$ are fan-crossing but are not $k$-planar for $k = 0, 1, 2, 3, 4$ and $n = 3, 5, 7, 10, 19$, respectively, as shown by Angelini et al. [9]. Also $K_{4,7}$ is not fan-crossing free. However, the node-to-circle expansion of $K_{4,n}$ is 1-planar, as Fig. 3(a) illustrates.

Lemma 1. The node-to-circle expansion $\eta(G)$ of a graph $G$ is planar if and only if $G$ is planar. Similarly, $\eta(G)$ is $k$-planar (fan-crossing free, $k$-gap-planar, quasi-planar, RAC) if so is $G$.

Proof. In a drawing of $G$, expand each vertex $v$ to an $\epsilon$-ball, so that two $\epsilon$-balls do not intersect and an $\epsilon$-ball does not intersect a non-incident edge. For each edge $e$ incident to $v$ place a new vertex $v_e$ at the last intersection of $e$ and the ball. For each edge $e = \{u, v\}$ let $e'$ be the segment of $e$ between the $\epsilon$-balls around $u$ and $v$. The boundary of each $\epsilon$-ball is used for the inner edges of the circles. If two edges of $G$ cross, then they cross outside $\epsilon$-balls. Thereby, properties of crossings of $e$ are transferred to $e'$, for example $k$-crossings, independent crossings or right-angle crossings.

Clearly, $G$ is planar if $\eta(G)$ is planar, since the circle from the expansion of a node can be contracted to a single point. \hfill $\square$

Corollary 1. The graph classes $k$PLANAR, FCF, $k$GAP, QUASI, and RAC are closed under node-to-circle expansion ($k \geq 0$).

The proof of Lemma 1 shows that node-to-circle expansions do not increase the number of crossings, so that $\eta(G)$ has crossing number at most $k$ if $G$ has crossing number $k$. In fact, the number of crossings may decrease by a node-to-circle expansion if an edge is routed through the expanded circle of vertex $v$. 

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Fig. 4. Quasi-planar drawings of (a) $K_{10}$ and (b) $\eta(K_{10})$, where edges from $K_{10}$, drawn dashed and red and blue, are drawn dotted and red and blue in $\eta(K_{10})$. These edges are crossed 6-times in $K_{10}$, whereas the transformed edges are crossed 4-times in $\eta(K_{10})$ including two crossings on the boundary of the traversed circle, drawn as a big node.

and not around it. As an example consider a quasi-planar drawings of $K_{10}$ and $\eta(K_{10})$, respectively, as shown in Fig. 4. Note that $\eta(K_{10})$ is 4-planar, whereas $K_{10}$ is not [9].

Lemma 2. Graph $\eta(G)$ is 2-planar and fan-crossing if $\eta(G)$ is fan-crossing.

Proof. Graph $\eta(G)$ is 3-regular, so that an edge can be crossed by at most three edges from a fan. If edge $e$ is crossed by three edges incident to vertex $v$, then move $v$ to the other side of $e$ and reroute the edges incident to $v$ so that they do not cross $e$. $\square$

Lemma 3. The node-to-circle expansion of complete graphs $K_n$ for $n \geq 540$ is not fan-crossing.

Proof. Suppose there is a fan-crossing drawing of $\eta(K_n)$ so that it has the least number of crossings of inner edges. For every vertex $v$ there is a closed curve $C$ of length $n - 1$ for the expansion $v$. Circle $C$ may self-intersect. Graph $\eta(G)$ is 2-planar by Lemma 2.

Four inner edges are involved in each crossing with two inner edges. If edges from $C_i$ and $C_j$ cross, then either (at least) two edges of each of $C_i$ and $C_j$ cross or there is an edge $e$ of $C_i$ that is crossed by two edges incident to a vertex $v$ of $\eta(C_j)$. Then we assume that the binding edge $\{v, w\}$ incident to $v$ for some vertex $w$ crosses some circle $C_q$. Here $w$ is not in $C_i$, since otherwise vertex $w$ can be placed on $e$, so that the drawing of $\eta(K_n)$ is not minimal for the number of crossings. If the binding edge $\{v, w\}$ is uncrossed, then we can change the embedding so that it crosses $e$ instead of the two other edges incident to $v$, so
that $C_i$ and $C_j$ no longer cross. For every circle $C_i$ let $d_i$ be the number of binding edges that cross $C_i$, so that each binding edge is incident to a vertex $v$ whose inner edges cross a single edge of another circle. Then at most $n - 1 - d_i$ edges of $C_i$ can be crossed by edges from other circles. Since there are $n(n - 1)$ inner edges and four inner edges are involved in each crossing, there are at most $n(n - 1)/4$ crossings of circles.

Consider the complete graph $K_n$ with the circles $C_i$ as vertices and a 2-coloring of the edges so that edge $\{C_i, C_j\}$ is colored red if $C_i$ and $C_j$ cross, and blue otherwise. According to [49], the Ramsey number $r(7, 7)$ is bounded by 540, so that cliques of size 540 have a subclique of size seven with red or with blue edges. Since there are no more blue than red edges, there is a $K_7$ with blue edges.

Let $O_1, \ldots, O_p$ be a maximal set of circles that do not cross each other, that is the edge between $O_i$ and $O_j$ is colored blue. Consider the subgraph $H$ of $\eta(K_n)$ induced by $O_1, \ldots, O_p$. Then the inner edges of $H$ do not cross each other. Each binding edge of $H$ crosses at most one other edge, so that the drawing of $H$, which is a restriction of a fan-crossing drawing of $\eta(K_n)$, is 1-planar.

If a binding edge $\{v_i, v_j\}$ between vertices $v_i$ in $O_i$ and $v_j$ in $O_j$ crosses an inner edge of circle $O_k$, then $O_i$ and $O_j$ are on opposite sides of $O_k$, since inner edges from circles do not cross. Suppose that $O_j$ is inside $O_r$ and $O_i$ is outside. Then no vertex of another circle $O$ is inside $O_k$. If $O$ is inside $O_k$ (or vice versa), then the edge between vertices of $O$ and $O_j$ crosses two independent edges. Otherwise, at least $2(p - 3)$ edges must cross $O_k$ so that at least one of its edges is crossed by at least two independent edges if $p \geq 7$. By the same reasoning, there are at most two concentric circles.

Suppose that $O_j$ is inside $O_k$. Then all other circles $O_i$ are outside $O_k$ and the edges between vertices of $O_j$ and $O_i$ cross an edge of $O_k$, so that they are not crossed by another edge. Then there is a $(p - 2)$-star with center $O_k$ and $p - 2$ satellites $O_i$. For $p \geq 7$, some binding edges are crossed at least twice, since the $(p - 2)$-clique from the satellites includes a pentagram drawing of $K_5$, as shown in Fig. 6(b), with all vertices in one face. Since binding edges are independent, there is a contradiction. Otherwise, graph $H$ is homeomorphic to $K_p$. Each binding edge of $H$ between vertices from different circles can cross at most one circle in a fan-crossing embedding. However, graph $K_p$ for $p \geq 7$ is not 1-planar, so that at least one of its edges is crossed at least twice. This is a contradiction against the fan-crossing drawing of $\eta(K_n)$, since binding edges are independent.

In consequence, the subgraph of $\eta(K_n)$ induced by the set of circles that do not mutually cross is not 1-planar and is not fan-crossing, since each crossing in this subgraph is a 1-planar crossing. Hence, $\eta(K_n)$ is not fan-crossing for $n \geq 540$.

Note that both $K_7$ and $\eta(K_7)$ are fan-crossing, as Figs. 7(c) and 3(b) show. Lemma 3 does not show that the fan-crossing graphs are not closed under node-to-circle expansions. Nevertheless, it helps to distinguish the fan-crossing and the fan-crossing free graphs.
Theorem 3. The fan-crossing free graphs are universal for node-to-circle expansion, that is, for every graph $G$ there is a fan-crossing free graph $H$ so that $\eta(G) = H$.

Proof. As in the proof of Lemma 1, consider a drawing of $G$. Expand each vertex $v$ into an $\epsilon$-ball with $d$ points if $v$ has degree $d$ and 3-subdivide each edge using the intersection points of the $\epsilon$-balls. Then all edges between intersection points on the balls are independent. Hence, the drawing of $\eta(G)$ is fan-crossing free. □

From the density or maximal complete (bipartite) graphs it was known that there are fan-crossing graphs that are not fan-crossing free. The opposite direction was open [31]. We claim that also the classes of $k$-planar, $k$-gap-planar and quasi-planar graphs are not universal for node-to-circle expansion. Observe that the Crossing Lemma [4, 45] does not help to prove this claim for $k$-planar and $k$-gap-planar graphs, since graphs $\eta(G)$ are sparse.

Corollary 2. The graph classes $FAN$ and $FCF$ are incomparable. In particular, $K_7$ or $K_{4,n}$ with $n \geq 7$, are fan-crossing and not fan-crossing free, whereas $\eta(K_n)$ for $n \geq 540$ is fan-crossing free and not fan-crossing.

For graphs that are defined by a drawing, there is the special case in which all vertices are in the outer face. We call these outer graphs, for example outer-planar (outer-fcf) graphs if the drawing is planar (fan-crossing free).

One may vary the node-to-circle expansion operation and expand a vertex $v$ of degree $d$ into a $d'$-circle with $d' \geq d$ so that $d$ vertices on the circle have degree three and each of them inherits an edge incident to $v$ and the remaining $d' - d$ vertices have degree two, or expand $v$ into a path with vertices of degree at most three. We denote this expansion by $\eta'(G)$. Then Theorem 3 can be sharpened so that all vertices of a fan-crossing free drawing of a graph $\eta'(G)$ are in the outer face. In other words, graph $\eta'(G)$ is outer fan-crossing free. It is readily seen that neither $K_6$ nor its expansion $\eta'(K_6)$ admit an outer quasi-planar drawing.

3 Relationships

The following incomparabilities are known: (i) 1-planar and RAC graphs [34] (ii) 2-planar and fan-planar [16], and (iii) fan-crossing and 1-gap-planar, where the latter, stated as an open problem in [13], follows from $K_8$ and $K_{4,n}$ for $n \geq 9$ [9].

Theorem 4. There are RAC and fan-crossing free graphs that are not fan-crossing.

Proof. A tile $T$ is an internally crossed $5 \times 5$ grid with vertices $t_{p,q}$ for $1 \leq p, q \leq 5$ and edges $\{t_{p,q}, t_{r,s}\}$ if $\max\{|p - r|, |q - s|\} = 1$. It has 25 vertices and 72 edges and diameter four, see Fig. 3.

Since outer fan-planar graphs have a density of $3n - 5$ [16], a tile is not outer fan-planar. Note that every outer fan-crossing graph is outer fan-planar, since an edge cannot be crossed from both sides if all vertices are in the outer face.
Fig. 5. An internally crossed $5 \times 5$ grid with three added paths, drawn dotted and/or dashed in a RAC drawing.

Hence, in every fan-crossing embedding, each tile $T$ has at least one vertex in its interior, called the center of $T$, and at most 24 edges in the boundary of its outer face. There are at least three vertex-disjoint paths between the center and vertices in the boundary, since a tile is 3-connected.

Let graph $G$ consists of at least 650 tiles and at least 60 paths between any two vertices from different tiles. Suppose there is a fan-crossing embedding of $G$. Each tile has a Jordan curve for its boundary, so that the center is in the interior of the Jordan curve. We say that two tiles cross, nest, and are disjoint if their boundaries cross, nest or are disjoint, respectively. They nest if one tile is in the interior of the boundary of the other tile. Define the distance between two tiles by the minimum number of boundaries that must be crossed by a path $P$ between their centers if $P$ is added to $G$ in a fan-crossing embedding of $G \oplus P$. In particular, the distance is at least two if the tiles are disjoint, since an added path must cross the boundary of each of them.

Suppose there are (at least) five tiles $T_i$ with a mutual distance at least two. Then consider $p \geq 60$ paths of length 15 between their centers. Each path must cross at least two boundaries. Since each boundary consists of at most 24 edges, at least one of its edges is crossed twice if there are at least 25 paths. In a fan-crossing embedding, this edge is crossed by the first (or last) edges of the paths, since all other edges of the paths are independent. We call $P$ a red path if the first and last edges cross an edge of a boundary. Suppose there is a red path between the centers of tiles $T_i$ and $T_j$ for $1 \leq i, j \leq 5$. Then there is a subgraph homeomorphic to $K_5$, so that at least two paths must cross. Since the paths have length 15 and their first and last edges are crossed by edges of boundaries, at least one inner edge of a path is crossed twice by independent edges from two paths if there are at least 14 parallel red paths. Then there is a violation of a fan-crossing embedding.

Since the centers of the tiles are unknown, there are $p \geq 60$ paths of length 15 between any two vertices from different tiles. Graph $G$ is not fan-crossing if there are at least five tiles at mutual distance two. Since the boundary of a tile has length 24, at least 60 – 23 paths from the center cross the boundary in fans for size at least two, and at least 60 – 46 paths between two centers are red. Each red path has 13 edges between the boundaries, so that there is an independent crossing if there are at least 14 red paths between the centers of five tiles.
Graph $G$ is RAC and, hence, fan-crossing free. Therefore, draw each tile as a RAC graph as shown in Fig. 3, so that the edges between grid points are parallel to the axis. Place the tiles from left to right so that they are disjoint and their bottom row is on the $x$-axis. There is a bundle of paths between any two vertices from different tiles. Route each added path $P$ from a vertex to the outer face of the tile, so that internally there are right angle crossings. This needs paths of length at most six, so that the sixth edge is parallel to the $y$-axis. The middle edges between the seventh and tenth vertex of each path are horizontal or vertical and are routed outside the tiles, so that edges of two paths do not overlap. If they cross, they cross at a right angle. If there are several paths between two vertices, then they are piecewise in parallel, except for the first and last edges.

It remains to show that there is a graph consisting of at least five tiles at distance at least two. Consider a fan-crossing embedding $\mathcal{E}(G)$ of $k$ tiles, so that only the outer boundary and the center in its interior are taken into account. In other words, there are $k$ Jordan curves and $k$ points, so that a point is in the interior of a particular Jordan curve. The boundary of a tile can be crossed at most 24 times by the boundaries of other tiles, since it has length at most 24. In fact, three or four edges from both boundaries are involved in a crossing. Consider the planar dual of $\mathcal{E}(G)$. If center $c$ of tile $T$ is in face $f$, then assign $T$ to $f$. Face $f$ has at most 25 adjacent faces, that are accounted to $f$, since the boundary of a tile must be crossed for a new face and it can be crossed at most 24 times. In addition, there is the outer face or the face from an nesting tile enclosing $T$. Faces from tiles in its interior are accounted to these tiles. Moreover, at most five tiles can be assigned to a face, since a tile has diameter four and any path from the center to a vertex in the boundary of $T$ can cross at most four boundaries. Suppose there are at least 650 tiles. If several tiles are assigned to a face, then keep only one them. Each face has at most 25 neighbors with an assigned tile at distance one, so that there are 26 candidates. Hence, at least five tiles remain, so that the faces, to which they are assigned, have distance at least two.

We claim that an internally crossed $10 \times 10$ grid with an added path of length 12 from $t_{4,4}$ to $t_{7,7}$ is not fan-crossing, since in any fan-crossing embedding, vertices $t_{4,4}$ and $t_{7,7}$ are separated by a cycle of crossed edges from the grid, so that an internal edge of the path must cross such an edge.

Now we can distinguish fan-crossing free graphs from some other classes of beyond-planar graphs [31].

Clearly, a drawing is 1-planar if it is both fan-crossing and fan-crossing free [20]. On the other hand, there are graphs that are both fan-crossing and fan-crossing free, but are not 1-planar [21]. From the density or complete (bipartite) graphs and graph $G$ from Lemma 4, we obtain non-fan-crossing graphs via additions of many short paths, so that we can conclude.

**Corollary 3.** The fan-crossing graphs and the RAC and fan-crossing free graphs, respectively, are incomparable.
Bae et al. [13] state the relationship between 1-gap-planar and fan-planar (fan-crossing) graphs as an open problem and do not ask for the relationship between 1-gap-planar and fan-crossing free. We solve the latter.

**Theorem 5.** There are fan-crossing free and even RAC graphs that are not \( k \)-gap-planar, and conversely.

**Proof.** The 3-subdivision \( \sigma_3(G) \) of any graph \( G \) is a RAC graph \( G \) [30], and the 2-subdivision \( \sigma_2(G) \) is fan-crossing free by Theorem 1. Graph \( \sigma_3(G) \) has \( n + 2m \) vertices. It has at most \( 5n + 10m - 10 \) edges if it were 1-gap-planar and \( O(\sqrt{k}(n + m)) \) for \( k \geq 2 \) [13], so that a \( k \)-gap-planar drawing has at most that many crossings The complete graph \( K_n \) has \( \Omega(n^4) \) crossings [43]. The number of crossings of a graph is unchanged by subdivision. Hence, there are graphs whose 3-subdivision has too many crossings for \( k \)-gap-planarity. The case of 2-subdivisions is similar. For the converse direction, consider 1-gap-planar graphs containing \( K_7 \) or \( K_9 \) as a subgraph or 1-gap-planar graphs with more than \( 4n - 8 \) edges.

\[ \Box \]

The incomparabilities are summarized in Fig. 2. It remains to show that there are fan-crossing free graphs that are not quasi-planar. Moreover, there are classes of graphs with drawings that simultaneously satisfy two properties, such as fan-crossing and fan-crossing free, and the intersection of two graph classes, such as \( \text{FAN} \cap \text{FCF} \). A study of such graph classes is asked for by Didimo et al. [31].

### 4 Properties of Fan-Crossing Free Graphs

It is well-known that the 4-clique \( K_4 \) admits two embeddings as a planar tetrahedron and 1-planar with a pair of crossing edges. The 5-clique \( K_5 \) has five embeddings [41], as shown in Fig. 6. Only the so-called T-embedding in Fig. 6(a) is fan-crossing free and even 1-planar. The embedding in Fig. 6(e) has an edge which is crossed by the edges of triangle. It is 1-gap-planar and quasi-planar and not fan-crossing and not 2-planar. In fan-crossing embeddings it can be transformed into the Q-embedding of Fig. 6(c) [19].

#### 4.1 Unique Embeddings

Consider a fan-crossing free embedding of \( K_6 \), which is obtained by placing the next vertex into one of the faces of the T-embedding of \( K_5 \). There are two possibilities up to symmetry: in a face with or without a crossing point. Only the latter results in a fan-crossing free embedding. The obtained embedding is unique, since the edges must be routed as shown in Fig. 7. Otherwise, an edge is crossed by at least two edges of a fan. The embedding can be drawn with two or three vertices in the outer face. The 7-clique is not fan-crossing free, since it has too many edges, but fan-crossing (fan-planar) [16], see Fig. 7(c). We summarize:
Fig. 6. All non-isomorphic embeddings of $K_5$ [41], each with two drawings. Only the T-embedding (a) is 1-planar and fan-crossing free.

Fig. 7. The only fan-crossing free embedding of $K_6$ drawn (a) with a triangle and (b) with only two vertices in the outer face. (c) A 2-planar fan-crossing embedding of $K_7$. Edges are colored black, blue, and red if they are uncrossed and crossed once and twice, respectively.

Lemma 4. The cliques $K_5$ and $K_6$ are fan-crossing free and each has a unique fan-crossing free embedding.

Cheong et al. [28] have shown that every fan-crossing free embedding of an $n$-vertex graph with $4n - 8$ edges is 1-planar. Such graphs are called extreme or optimal 1-planar [17]. The embedding consists of a 3-connected planar subgraph, in which faces are quadrangles with a pair of crossing edges. Such graphs exist for $n = 8$ and for all $n \geq 10$ [17]. They have a unique 1-planar embedding, except for the extended wheel graphs $XW_{2k}$ for $k \geq 3$, which have two embeddings, where the poles are exchanged [48]. An extended wheel graph $XW_{2k}$ consists of two poles $p$ and $q$ and a circle of length $2k$. There is an edge between each pole and each vertex of the circle, whereas there is no edge $\{p, q\}$. In addition, there is an edge between a vertex of the circle and the vertex after next (in cyclic order). Each of these edges is crossed by an edge incident to a pole. Moreover, optimal 1-planar graphs can be recognized in linear time [22].

Corollary 4. Every $n$-vertex fan-crossing free graph with $4n - 8$ edges has a unique embedding, except for the extended wheel graphs $XW_{2k}$, which have two embeddings.
Fig. 8. A fan-crossing free embedding of a nested triangle graph, augmented by “hermits”, drawn by pink squares, for the proof of Theorem 6. The drawing is not quasi-planar.

The crossed nested triangle graph $\Delta_k$ consists of $k$ nested triangles $T_1, \ldots, T_k$ with vertices $a_i, b_i, c_i$ for $i = 1, \ldots, k$ so that each quadrangle between two successive sides of the triangles has a pair of crossing edges and is $K_4$, as shown in Fig. 8. Triangle $T_i$ is at level $i$, where $T_1$ is in the outer face and $T_k$ in the inner face. The subgraph induced by two consecutive triangles is $K_6$, which is an inner $K_6$ if it is induced by $T_i, T_{i+1}$ for $i = 2, \ldots, k - 2$. Graph $\Delta_k$ has $3k$ vertices and $12k - 9$ edges. It is 1-planar and admits a straight-line drawing, but it is not a RAC graph, since it has $K_6$ subgraphs.

**Lemma 5.** For every $k \geq 1$, the crossed nested triangle graph $\Delta_k$ has a unique fan-crossing free embedding.

**Proof.** For $k = 1$ there is a triangle, and $K_6$ for $k = 2$ has a unique fan-crossing free embedding by Lemma 4. For $k \geq 3$, the restriction of $\Delta_k$ to two consecutive triangles is $K_6$. Each such $K_6$ has a unique fan-crossing free embedding, and each such $K_6$ must be drawn with a triangle in its outer face, since a W-configuration, as shown in Fig. 7(b), enforces a crossing of an edge by at least two edges of a fan. Hence, all triangles are drawn as nested triangles. Clearly, one can choose the outer face of the drawing with three vertices or with two vertices and a crossing point. \qed

Note that the crossed nested triangle graph $\Delta_k$ admits many fan-crossing drawings, since a $K_6$ has many fan-crossing embeddings with different embeddings of its $K_5$ subgraphs.

Next we consider a traversal of $K_5$ or $K_6$ by a path in a fan-crossing free embedding, which causes a delay by at least four units, and is used in the NP-hardness proof of Theorem 7.
Lemma 6. In every fan-crossing free embedding, if a path $P$ traverses $K_5$ ($K_6$), where $P$ and the clique have disjoint sets of vertices, so that there are different edges of $K_5$ ($K_6$) crossed first and last by $P$, then at least four (five) edges of $P$ cross edges of $K_5$ ($K_6$).

Proof. Let $P = (u_0, \ldots, u_t)$ be a path such that $u_0$ and $u_t$ are in the outer face and $u_1, \ldots, u_{t-1}$ are in an inner triangle of the $T$-embedding of $K_5$. Let the first edge $(u_0, u_1)$ of $P$ cross edge $(v_1, v_2)$ of $K_5$ and let the last edge $(u_{t-1}, u_t)$ of $P$ cross $(v_1, v_3)$ in a fan-crossing free embedding of a graph $G$ that includes $P$ and $K_5$. Since $K_5$ has a unique fan-crossing free embedding, $P$ must cross two more edges of $K_5$ and a simultaneous crossing of two or more edges induces a fan-crossing. Hence, $P$ needs at least four edges for a traversal of $K_5$. The case of $K_6$ is similar.

4.2 Extremal Graphs

A graph $G$ is maximal for a class of graphs if no edge can be added without violation, so that $G + e$ is not in the class. The density is the maximum number of edges of all maximal $n$-vertex graphs in a class. The sparsity is the minimum. Density and sparsity coincide for planar graphs, whereas they differ for some classes of beyond-planar graphs. Brandenburg et al. [25] have shown that the sparsity of 1-planar graphs is at most $\frac{17}{15}n$ and for IC, NIC and outer 1-planar graphs [10,12].

Theorem 6. There are maximal fan-crossing free graphs with $m = 7/2n - 17/2$ edges for every $n = 6k + 1$ with $k \geq 3$.

Proof. Consider a crossed nested triangle graph $\Delta_k$ for $k \geq 3$. Attach three hermits $h_i, h'_i$ and $h''_i$ along the edges $\{a_i, a_{i+1}\}, \{b_i, b_{i+1}\}$ and $\{c_i, c_{i+1}\}$ of each triangle $T_i = (a_i, b_i, c_i)$ for $i = 1, \ldots, k-1$ and connect each hermit with the vertices of the edge to which it is attached. Also link the hermits by edges $\{h_i, h'_i\}, \{h'_i, h''_i\}$ and $\{h_i, h''_i\}$. Add a hermit in the outer face and link it to $a_1, b_1, c_1$, and similarly for the inner face, see Fig. 8.

Graph $H_k$ is fan-crossing free, as shown Fig. 8, where the embedding is not quasi-planar. It is also maximal fan-crossing free. Therefore, observe that $\Delta_k$ has a unique fan-crossing free embedding, which is maximal fan-crossing free.

Consider hermit $h_i$ that is attached to edge $\{a_i, a_{i+1}\}$. Then $h_i$ can be placed into the faces $f$ and $f'$ to either side of $\{a_i, a_{i+1}\}$. It cannot be placed into another face without creating a fan-crossing by two edges. Also $h_i$ cannot be linked to another vertex of $\Delta_k$ without creating a crossing by a fan of at least two edges. The case of $h'_i$ and $h''_i$ is similar. Hermit $h_i$ is linked to $h'_i$ and $h''_i$ by an edge, but not to any other hermit if the embedding is fan-crossing free. Then at least two edges incident to a vertex of $\Delta_k$ must be crossed. If three hermits $h_i, h'_i, h''_i$ are simultaneously placed into another face that is next to the one in which they were placed, then the edges between them create a crossing by a fan.
of two edges, as illustrated in Fig. 9(a). Similarly, the hermit in the inner face can be placed in a neighboring face, as illustrated in Fig. 9(b), but it cannot be linked to another hermit or another vertex without violation.

Graph \( \Delta_k \) has \( 3k \) vertices and \( 12k - 9 \) edges. We have added \( 3(k - 1) + 2 \) hermits and \( 9(k - 1) + 6 \) edges incident to the hermits. Hence, graph \( H_k \) has \( 6k - 1 \) vertices and \( 21k - 12 \) edges, so that \( m = \frac{7}{2}n - \frac{17}{2} \).

\[ \square \]

4.3 The Complexity of Fan-Crossing Free Recognition

The recognition of beyond-planar graphs is mostly NP-hard, see [31]. It has been expected that recognizing fan-crossing free graphs is NP-complete, too, which is proved next.

**Theorem 7.** The recognition of fan-crossing free graphs is NP-complete.

**Proof.** Clearly, the problem is in NP.

For the NP-hardness we adapt the construction given by Grigoriev and Bodlaender [38] for 1-planar graphs. The proof is by reduction from 3-PARTITION, which is a strongly NP-hard problem [37]: An instance \( I \) of 3-PARTITION consists of a multiset \( A \) of \( 3m \) positive integers with \( B/4 < a < B/2 \) and \( \sum_{a \in A} = mB \) for some integer \( B \) and each \( a \in A \). The 3-PARTITION problem asks whether \( A \) can be partitioned into \( m \) subsets \( A_1, \ldots, A_m \), each of size three, such that the sum of the numbers in each subset equals \( B \).

From \( I \) we construct a graph \( G_I \) such that \( I \) has a solution if and only if \( G_I \) admits a fan-crossing free embedding, and \( G_I \) can be constructed in polynomial time. The components of the reduction are a transmitter, a collector, \( m \) splitter and “fat” edges. A fat edge \( \{u, v\} \) is a \( K_5 \) with vertices \( u \) and \( v \) and three more vertices, where \( u \) and \( v \) are in the outer face if there is a drawing. For an illustration see Fig. 10.

The transmitter is a double-wheel with a center \( c_T \) and two circuits of length \( 3m \). Let \( C = (u_1, \ldots, u_{3m}) \) be the outer and \( C' = (u'_1, \ldots, u'_{3m}) \) the inner circuit. Let \( \{c_T, u'_i\} \) and \( \{u'_i, u_i\} \) be fat edges for each \( i \). Then the center \( c_T \) is a vertex of each of the \( K_5 \) graphs of the fat edges that are incident to the center. Each sector
with boundary $u_i, u'_i, c_i, T_i, u'_{i+1}, u_{i+1}$ is partitioned into three triangles by (normal) edges $\{u'_i, u'_{i+1}\}, \{u_i, u_{i+1}\}$ and a diagonal $\{u_i, u'_{i+1}\}$. The triangle $(u_i, u'_i, u'_{i+1})$ is called a sector-triangle, and $(u_i, u_{i+1}, u'_{i+1})$ is called an outer triangle, see the enlargement in Fig. 10(b). The collector is defined accordingly, with a center $c_C$ and $Bm$ sectors with an outer boundary $(v_1, \ldots, v_{Bm})$. The boundary of the transmitter is partitioned into $m$ segments, each of width three, and accordingly, the collector has $m$ segments, each of width $B$. The ends of the segments are connected by a fat edge in circular order. Hence, there is fat edge $(u_{3i}, v_{Bi})$ for $i = 1, \ldots, m$.

For each element of $A$ there is a splitter $P_a$ with center $c_a$ and $a + 1$ satellites at distance two from the center, i.e., each satellite has a connector on the path to $c_a$. In [38], the satellites are at distance one from the center. We connect one satellite of each splitter with the transmitter center $c_T$ and the remaining satellites with the collector center $c_C$. Hence, there is a path of length six from $c_T$ via each $c_a$ to $c_C$.

Since 3-PARTITION is strongly NP-hard, all numbers $a \in A$ can be given in unary encoding, such that $G_I$ has polynomial size and can be constructed in polynomial time.

For the correctness of the reduction we follow the arguments given in [38]. However, we use $K_5$ instead of $K_6$ as a fat edge, and sector triangles and satellites at distance two from the center of each splitter.

If the instance of 3-PARTITION has a solution, then the transmitter, collector and regions are drawn as illustrated in Fig. 10. The splitters of a 3-set with $a_1 + a_2 + a_3 = B$ appear in one region and between the outer boundaries of the transmitter and the collector. Since each sector of the collector has $B$ sectors, we can put a single satellite in each sector triangle and obtain a 1-planar drawing.

Conversely, for the “only if” part, suppose there is a fan-crossing free drawing of $G_I$, where the drawing is on the sphere. As in [38], we place $c_T$ at the North Pole and $c_C$ at the South Pole. A meridian path $M_i$ is a path of five fat edges between the centers of the transmitter and the collector through the vertices $u_{3i}$ and $v_{Bi}$ on the boundaries, for $i = 1, \ldots, m$. Each meridian path also contains a unique path of length five from $c_T$ to $c_C$. A splitter path $SP_a = (c_T, a_T, c_a, a_C, c_C)$ is a path of length six between the centers of the transmitter and the collector via the center of a splitter $P_a$, such that $a_T$ and $a_C$ are satellites of $P_a$. Note that there is a splitter path from $c_C$ through $a_C$ and $c_a$ for $a$ vertices and there is a single path from $c_T$ through $a_T$ to $c_a$.

From now on assume that we are given a fan-crossing free embedding of $G_I$. First, observe, that two meridian paths $P$ and $P'$ do not cross, i.e., there is no crossing of two edges $e$ and $e'$, where $e$ is in the subgraph induced by $P$ and $e'$ is in the subgraph induced by $P'$. Edges $e$ and $e'$ belong to two distinct $K_5$, and a fan-crossing is unavoidable by the uniqueness of a fan-crossing free embedding of $K_5$ from Lemma 4 if $P$ and $P'$ cross.

Second, a meridian path $P$ and a splitter path $SP_a$ do not cross. Towards a contradiction, suppose that edge $e$ of $SP_a$ crosses an edge $f$ of some fat edge of $P$. Each splitter path must cross the boundaries of the transmitter and the
collector, which needs a path of length at least four. If, in addition, an edge of a $K_5$ is crossed, then at least four more edge is needed. However, splitter paths are tight and have length five.

In consequence, we can follow the arguments given by Grigoriev and Bodlaender. Two successive meridian paths $M_i$ and $M_{i+1}$ define a face and there is a cyclic ordering of faces according to the circuits of the boundaries of the transmitter and the collector. In consequence, there is a unique way to draw the transmitter around the North Pole, and similarly, there is a unique way to draw the collector around the South Pole.

We now consider the drawing of splitters. Each splitter path has length five. It takes a fan-crossing free path of length at least two from the center of the transmitter to cross a diagonal between the inner and outer circuit, and similarly for the collector. Hence, the center $c_a$ of a splitter an be placed in an outer triangle of the transmitter, in an outer triangle of the collector, or in the exterior of the transmitter and the collector.

The center $c_a$ is a fan of $a + 1$ edges, and at most one of them can cross the diagonal towards the sector triangle in a fan-crossing free way. If $c_a$ is placed in an outer triangle of the transmitter (or the collector), then also the connectors of the remaining $a$ satellites must be placed in the triangle; otherwise, there is a fan-crossing. However, then the length of the paths does not suffice, and fan-crossings are unavoidable. Hence, the centers of each splitter must be placed between the outer boundaries of the transmitter and the collector.

In consequence, each satellite is placed in a sector triangle and each connector in an outer triangle. Then there is at most one satellite in each sector triangle; otherwise there is a fan crossing, since the edges to the satellites are incident to the centers $c_T$ and $c_C$ of the transmitter and collector, respectively.

Hence, for each region, we have exactly three splitters, which each have one satellite in a sector triangle, and there is at most one satellite in each of the $B$ sector triangles of the region (face) of the collector. Since we have $Bm$ such paths, each face must contain exactly three splitters with exactly $B$ paths between the splitter centers and the South Pole, which implies that the instance $I$ has a solution. \hfill $\square$

The recognition problem for fan-crossing free graphs with a fixed rotation system is also NP-complete. Here, a graph is given with a rotation system describing the cyclic ordering of the edges incident to each vertex. Therefore, we can modify the proof of [11] for the NP-completeness of recognizing 1-planar graphs with a rotation system in a similar way as in the proof of Theorem 7.

5 Conclusion

In this work, we have shown that the fan-crossing free graphs are incomparable with the fan-crossing, 2-planar, and 1-gap-planar graphs, respectively, and thereby filled a gap in [31]. It remains to show an incomparability between fan-crossing free and quasi-planar graphs. Our tools are the $s$-subdivision, path-addition, and node-to-circle operations, where $2$-subdivision and node-to-circle
expansion can be used to represent any graph by a fan-crossing free graph. The potential of these operations has not yet been exploited. In particular, it is open whether the fan-crossing free graphs are 1-subdivision universal and whether the quasi-planar graphs are universal for 2-subdivision and node-to-circle expansion. Moreover, we have added fan-crossing free graphs to the list of beyond-planar graph classes with an NP-complete recognition problem, which was open so far [31].

How shall we draw fan-crossing free graphs? What is their book-thickness [15]? How are they related to classes of graphs that are defined by visibility representations [29,36]? There are partial answers for 1-planar graphs [5,6,18,23] for these problems, which need a generalization to other classes of beyond-planar graphs.

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