New limit theorems related to free multiplicative convolution

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Abstract

Let ⊞, ⊠ and ⊎ be the free additive, free multiplicative, and boolean additive convolutions, respectively. For a probability measure µ on [0, ∞) with finite second moment, we find the scaling limit of (µ ⊠ N) ⊞ N as N goes to infinity. The R–transform of the limit distribution can be represented by the Lambert’s W function. We also find similar limit theorem by replacing the free additive convolution with the boolean convolution.

1 Introduction

In free probability theory, some limit theorems are known as in classical probability theory. The most famous limit theorem is the free central limit theorem, which was found by Voiculescu. If \{X_i\}_{i∈N} is a sequence of freely independent identically distributed (for short free iid) random variables with finite second moment, then the normalized sum \( \frac{X_1 + \cdots + X_N}{\sqrt{N}} \) converges to the Wigner’s semi-circle law in distribution as N goes to infinity. In addition, we know the Poisson limit theorem, the stable limit theorem and so on, for details, see [9], [3], [3]. Recently other new limit theorems with respect to the free convolutions [6], [16] and [14] have been investigated.

In this paper, we consider not only addition but also multiplication of free iid random variables. Especially, we introduce a new normalized sum of multiplications of freely independent random variables, that is, for the double sequence of free iid random variables
\[ \{X_i^{(j)}\}_{i \in \mathbb{N}} \text{ having distribution } \mu \text{ on } [0, \infty) \text{ with finite second moment we consider a new normalization } Y_N \]
\[ Y_N = \frac{\sqrt{X_N^{(j)}} \cdots \sqrt{X_2^{(j)}X_1^{(j)}} \sqrt{X_2^{(j)}} \cdots \sqrt{X_N^{(j)}}}{m_1^N N}, \]

where \( m_1 \) is the mean of distribution \( \mu \). We shall see that the limit distribution depends only on the first and second moments. In its proof, we shall investigate the Taylor type expansion of the \( S \)-transform. A similar limit theorem can be found under boolean independence.

In Section 2, we shall gather the tools for free and boolean probability. Especially, we recall \( \mathcal{R}, S \) and \( \Sigma \)-transforms and infinite divisibility. In Section 3, we shall prove the Taylor type expansions for \( S \) and \( \Sigma \)-transforms and some limit theorems. Finally, in Section 4, we discuss the limit distributions with focusing on infinite divisibility and moments.

2 Preliminaries

Let \( \mathbb{R}_+ \) be \([0, +\infty)\) and \( \mathbb{C}^+ \) be \( \{z = x + iy \in \mathbb{C}; y > 0\} \). We fix notation that \( \mathcal{P} \) and \( \mathcal{P}_+ \) mean the set of all Borel probability measures on \( \mathbb{R} \) and \( \mathbb{R}_+ \), respectively. We denote the free additive, free multiplicative and boolean additive convolutions by \( \boxplus, \boxtimes \) and \( \oplus \), respectively, see for details of convolutions, \( [13], [15] \) and \( [11] \). Afterword, \( \delta_0 \) stands the Dirac probability measure concentrated on 0.

2.1 Analytic tools

Here, we will gather the analytic tools for free and boolean probability and mention some their important facts.

We denote the Cauchy transform of a probability measure \( \mu \) on \( \mathbb{R} \) by
\[ G_\mu(z) = \int_{\mathbb{R}} \frac{1}{z-x} \mu(dx), \quad z \in \mathbb{C}^+, \]
and
\[ \Psi_\rho(z) = \int_{\mathbb{R}} \frac{xz}{1-xz} \rho(dx), \quad z \in \mathbb{C} \backslash \mathbb{R} \]
denotes the moment generating function of \( \rho \) on \( \mathbb{R}_+ \). Then the Speicher’s \( R \) and Voiculescu’s \( \mathcal{R} \)-transforms of \( \mu \) are defined as follow:
\[ R_\mu(z) = zR_\mu(z) = zG_\mu^{-1}(z) - 1, \quad 1/z \in \Gamma_{\alpha,\beta}, \]
where \( G_\mu^{-1}(z) \) is the inverse of \( G_\mu(z) \) with respect to composition and \( \Gamma_{\alpha,\beta} = \{z = x + iy \in \mathbb{C}^+; y > \beta, |y| > \alpha x\} \). Note that we use both \( R \) and \( \mathcal{R} \)-transforms for convenience. The \( S \) and \( \Sigma \)-transforms of \( \rho \) are defined by
\[ S_\rho(z) = \frac{(z + 1)\Psi_\rho^{-1}(z)}{z}, \quad z \in \Psi_\rho(i\mathbb{C}^+), \]
and
\[ \Sigma_\rho(z) := S_\rho\left(\frac{z}{1-z}\right), \quad z/1-z \in \Psi_\rho(i\mathbb{C}^+), \]
respectively, where \( \Psi_\rho^{-1}(z) \) is the inverse of \( \Psi_\rho(z) \) with respect to composition. Now, we summarize the relations between the transforms and convolutions, see for proofs [5] and [1].

**Proposition 1.** For \( \mu_1 \in \mathcal{P}, \mu_2 \in \mathcal{P}, \rho_1 \in \mathcal{P}^+ \) and \( \rho_2 \in \mathcal{P}^+ \), which are not \( \delta_0 \), there exist \( \alpha > 0 \) and \( \beta > 0 \) such that
\[ R_{\mu_1 \boxplus \mu_2}(z) = R_{\mu_1}(z) + R_{\mu_2}(z), \quad 1/z \in \Gamma_{\alpha,\beta}, \]
\[ S_{\rho_1 \boxplus \rho_2}(z) = S_{\rho_1}(z)S_{\rho_2}(z), \quad z \in \Psi_{\rho_1}(i\mathbb{C}^+) \cap \Psi_{\rho_2}(i\mathbb{C}^+), \]
\[ S_{\rho_1 \boxdot t}(z) = \frac{1}{t} S_{\rho_1}\left(\frac{z}{t}\right), \]
\[ \Sigma_{\rho_1 \boxplus \rho_2}(z) = \Sigma_{\rho_1}(z)\Sigma_{\rho_2}(z), \quad z/(z-1) \in \Psi_{\rho_1}(i\mathbb{C}^+) \cap \Psi_{\rho_2}(i\mathbb{C}^+), \]
\[ \Sigma_{\rho_1 \boxdot t}(z) = \frac{1}{t} \Sigma_{\rho_1}\left(\frac{z}{t}\right). \]

For \( c > 0 \), the dilation operator \( D_c \) on \( \mathcal{P} \) is defined by
\[ D_c(\mu)(B) := \mu\left(\frac{1}{c} B\right) \]
for any Borel set \( B \) on \( \mathbb{R}_+ \), where \( \frac{1}{c} B = \{ x \in \mathbb{R} ; \frac{1}{c} x \in B \} \). In the paper [1], they showed
\[ S_{D_c(\mu)}(z) = \frac{1}{c} S_{\mu}(z), \]
and
\[ \Sigma_{D_c(\mu)}(z) = \frac{1}{c} \Sigma_{\mu}(z). \]

A probability measure \( \mu \) is \( \boxplus \)–infinitely divisible if for any \( n \in \mathbb{N} \) there exists \( \mu_n \in \mathcal{P} \) such that
\[ \mu = \mu_n \boxplus \cdots \boxplus \mu_n. \]

We denote the class of all \( \boxplus \)–infinitely divisible distributions by \( \mathcal{I}_{\boxplus} \).

**Remark 2.** We can define other infinite divisibility replacing \( \boxplus \) by \( \boxtimes \) or \( \triangledown \). But for boolean convolution, all probability measures are \( \triangledown \)–infinitely divisible. So we shall not discuss \( \triangledown \)–divisibility any longer.

The next proposition characterizes the \( \boxplus \)–infinitely divisible laws.

**Proposition 3.** The followings are equivalent:
(1) $\mu \in \mathbb{I}^\mathbb{I}$.

(2) $\mathcal{R}_\mu$ has an analytic extension defined on $\mathbb{C}^-$ with value $\mathbb{C}^- \cup \mathbb{R}$.

(3) There exist unique $b_\mu \in \mathbb{R}$ and finite measure $\nu_\mu$ such that

$$\mathcal{R}_\mu(z) = b_\mu + \int_{\mathbb{R}} \frac{1}{1-tz} \nu_\mu(dt), \quad z \in \mathbb{C}^-.$$  

The above expression is called $\mathbb{I}$–Lévy–Khintchine representation, or simply Lévy–Khintchine representation.

Example 4. The typical examples of $\mathbb{I}$–infinitely divisible distribution are the Wigner’s semi-circle law, the Dirac’s delta distribution and the free Poisson distribution $\pi_t$ with parameter $t \geq 0$ having density

$$\pi_t(dx) = \max(0,(1-t))\delta_0(dx) + \frac{1}{2\pi x} \sqrt{4t-(x-1-t)^2} \mathbb{I}_{[1-\sqrt{t},1+\sqrt{t}]}(x)dx. \quad (2.1)$$

The Lévy measure $\nu_\mu$ and $b_\mu$ of the semi-circle law are $\delta_0$ and 0, and the free Poisson law $\pi_t$ has $b_\mu = t$ and $\nu_\mu = t\delta_1$. We put $\pi_1$ by $\pi$.

The following functional equation of the $R$ and $S$–transforms can be found in, for instance, [12, Lemma 2]:

Proposition 5. Assume that $\mu \in \mathbb{P}_+$. For some sufficiently small $\varepsilon > 0$, we have a region $D_\varepsilon$ that includes $\{-it; 0 < t < \varepsilon\}$ such that

$$z = R_\mu \left( zS_\mu(z) \right), \quad (2.2)$$

for $z \in D_\varepsilon$.

3 New limit theorems

In this section, we prove a new limit theorem related to both free additive and multiplicative convolutions. We also discuss a similar result with replacing $\mathbb{I}$ by $\mathbb{U}$. It was proved in [10] by Młotkowski that for the free Poisson law $\pi$, we have

$$D_n \left( \left( \pi^\otimes(n-1) \right)^{\otimes n} \right)_{n \to \infty} \nu_0 \quad \text{in dist.},$$

where the $p^{th}$ moment of $\nu_0$ is given by $\frac{p^p}{p!}$. We find that theorem of this type holds for any probability distributions under finite second moment condition.
3.1 Taylor type expansion

We prove the Taylor type expansion for the $S$–transform under the finite moment condition. Concerning with the $R$–transform, it was proved by Benaych-Georges in [2]. For each region $A$ in $\mathbb{C}$, we denote $z \to 0$ with $z \in A$ by $z \xrightarrow{z \in A} 0$.

**Lemma 6.** Let $\rho \in P_{+}$ has the moment of order $p$, that is, for $k = 0, 1, 2, \ldots, p$,

$$m_k(\rho) := \int_{\mathbb{R}_{+}} x^k \rho(dx) < \infty.$$

Then its moment generating function $\Psi_{\rho}(z)$ has a Taylor expansion

$$\Psi_{\rho}(z) = \sum_{k=1}^{p-1} m_k(\rho) z^k + O(z^p), \quad z \xrightarrow{z \in i\mathbb{C}^{+}} 0.$$

**Proof.** We have

$$\Psi_{\rho}(z) - \sum_{k=1}^{p-1} m_k(\rho) z^k = \int_{\mathbb{R}_{+}} \frac{x^p z^p}{1 - xz} \rho(dx).$$

Note that for given $x \in \mathbb{R}_{+}$, the inequality

$$\left| \frac{x^p z^p}{1 - xz} \right| < |xz|^p$$

holds for $z$ with sufficiently small $|z|$. By the Lebesgue dominated convergence theorem,

$$\lim_{z \xrightarrow{z \in i\mathbb{C}^{+}} 0} \frac{1}{z^p} \left( \Psi_{\rho}(z) - \sum_{k=1}^{p-1} m_k(\rho) z^k \right) = m_p(\rho).$$

\[\square\]

**Lemma 7.** Let $\rho \in P_{+}$ has the moment of order $p$ and $\rho \neq \delta_0$. Then we have the followings:

1. $\Psi_{\rho}(z)$ is univalent in $i\mathbb{C}^{+}$.
2. The inverse function $\Psi_{\rho}^{-1} : \Psi_{\rho}(i\mathbb{C}^{+}) \to i\mathbb{C}^{+}$ of $\Psi_{\rho}$ admits the Taylor expansion

$$\Psi_{\rho}^{-1}(z) = \sum_{k=1}^{p} b_k(\rho) z^k + o(z^p), \quad z \xrightarrow{z \in i\mathbb{D}^{+}} 0.$$

In particular, $b_1 = \frac{1}{m_1(\rho)}$ and $b_2 = -\frac{m_2(\rho)}{(m_1(\rho))^2}$. 

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(3) $\mathcal{D}_\rho := \Psi_\rho(i\mathbb{C}^+)$ is a region contained in the circle with diameter $(\rho(\{0\}) - 1, 0)$. In addition, $\Psi_\rho(i\mathbb{C}^+) \cap \mathbb{R} = (\rho(\{0\}) - 1, 0)$,

$$\lim_{t \uparrow 0} \Psi_\rho^{-1}(t) = 0$$

and

$$\lim_{t \downarrow \rho(\{0\})-1} \Psi_\rho^{-1}(t) = \infty.$$  

**Proof.** (1) and (3) are proved in Bercovici and Voiculescu \[5, Proposition 6.2\].

(2) **Step 1** We shall first prove for $p = 1$. Take any continuous path $\{z(t)\}_{t \in (0, 1]}$ in $\mathcal{D}_\rho$ such that $\lim_{t \downarrow 0} z(t) = 0$. By (1), we can choose a unique continuous path $\{\omega(t)\}_{t \in [0, 1]}$ on $t \in (0, 1]$ such that $\lim_{t \downarrow 0} \omega(t) = 0$ and $\Psi_\rho(\omega(t)) = z(t)$.

$$\lim_{t \downarrow 0} \frac{\Psi_\rho^{-1}(z(t))}{z(t)} = \lim_{t \downarrow 0} \frac{\omega(t)}{\Psi_\rho(\omega(t))} = \lim_{t \downarrow 0} \frac{1}{\Psi_\rho(\omega(t))/\omega(t)} = \frac{1}{m_1}.$$  

By arbitrariness of the paths, it follows that

$$\Psi_\rho^{-1}(z) = b_1(\rho)z + o(z), \quad z \xrightarrow{z \in \mathcal{D}_\rho} 0.$$  

**Step 2** Assume that the statement of (2) holds for $l < p$. From Lemma \[3\] we have

$$\Psi'_\rho(z) = m_1(\rho) + \sum_{k=1}^l (k + 1)m_{k+1}(\rho)z^k + o(z^l), \quad z \xrightarrow{z \in \mathbb{C}^+} 0.$$  

By the assumption, we have

$$\Psi_\rho^{-1}(z) = \sum_{k=1}^l b_k(\rho)z^k + o(z^l), \quad z \xrightarrow{z \in \mathcal{D}_\rho} 0.$$  

Thus,

$$\Psi'_\rho \circ \Psi_\rho^{-1}(z) = m_1(\rho) + \sum_{k=1}^l \tilde{m}_{k+1}(\rho)z^k + o(z^l), \quad z \xrightarrow{z \in \mathcal{D}_\rho} 0,$$

where $\tilde{m}_2(\rho) = 2m_2(\rho)(b_1(\rho))^2$. Since the derivation of $\Psi^{-1}_\rho(z)$ is given by

$$(\Psi^{-1}_\rho)'(z) = \frac{1}{\Psi'_\rho \circ \Psi^{-1}_\rho(z)},$$  

we obtain

$$(\Psi^{-1}_\rho)'(z) = \frac{1}{m_1(\rho)} + \sum_{k=1}^l (k + 1)b_{k+1}(\rho)z^k + o(z^l), \quad z \xrightarrow{z \in \mathcal{D}_\rho} 0.$$  

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which implies that
\[ \Psi^{-1}_\rho(z) = \sum_{k=1}^{l+1} b_k(\rho) z^k + o(z^{l+1}), \quad z \xrightarrow{z \in \mathcal{D}_\rho} 0. \]

Comparing the first two coefficients, it follows that
\[ b_1(\rho) = \frac{1}{m_1(\rho)} \]
and
\[ 2b_2(\rho) = -\frac{\tilde{m}_2(\rho)}{m_1(\rho)} = -2 \frac{m_2(\rho)}{(m_1(\rho))^3}, \]
hence,
\[ b_2(\rho) = -\frac{m_2(\rho)}{(m_1(\rho))^3} < 0. \]

As a consequence, we can derive the following theorem:

**Theorem 8.** Let \( \rho \in \mathcal{P}_+ \) has the moment of order \( p \) and \( \rho \neq \delta_0 \). We denote the variance of \( \rho \) by \( \text{Var}(\rho) \).

(1) The \( S \)-transform of \( \rho \) admits the Taylor expansion of the form
\[ S_\rho(z) = \sum_{k=0}^{p-1} s_k(\rho) z^{k+1} + o(z^p), \quad z \xrightarrow{z \in \mathcal{D}_\rho} 0. \]

Furthermore, \( s_0(\rho) = 1/m_1(\rho) \) and
\[ s_1(\rho) = \frac{1}{m_1(\rho)} - \frac{m_2(\rho)}{(m_1(\rho))^3} = -\frac{\text{Var}(\rho)}{m_1(\rho)^3}. \]

(2) The \( \Sigma \)-transform of \( \rho \) admits the Taylor expansion of the form
\[ \Sigma_\rho(z) = \sum_{k=0}^{p-1} \sigma_k(\rho) z^{k+1} + o(z^p), \quad z \xrightarrow{z \in \mathcal{D}_\rho} 0. \]

Furthermore, \( \sigma_0(\rho) = 1/m_1(\rho) \) and
\[ \sigma_1(\rho) = \frac{1}{m_1(\rho)} - \frac{m_2(\rho)}{(m_1(\rho))^3} = -\frac{\text{Var}(\rho)}{m_1(\rho)^3}. \]

### 3.2 Limit theorems

Here we state the main theorem.

**Theorem 9.** We assume that \( \rho \in \mathcal{P}_+ \) has the second moment and put \( \alpha = \frac{\text{Var}(\rho)}{(m_1(\rho))^2}. \)

(1) There exist \( s_0 > 0 \) and \( s_1 < 0 \) such that the \( S \)-transform of \( \rho \) is given by
\[ S_\rho(z) = s_0 + s_1 z + o(z), \quad z \xrightarrow{z \in \mathcal{D}_\rho} 0, \]
and there exists a probability measure \( \eta_\alpha \in \mathcal{P}_+ \) such that

\[
D_{s_0^n/n} \left( \left( \rho^\otimes n \right)^\otimes n \right) \rightarrow \eta_\alpha \quad \text{in dist.}
\]

In addition, the \( S \)-transform of the limit distribution \( \eta_\alpha \) is

\[
S_{\eta_\alpha}(z) = \exp(-\alpha z).
\]

(2) There exist \( \sigma_0 > 0 \) and \( \sigma_1 < 0 \) such that the \( \Sigma \)-transform of \( \rho \) is given by

\[
\Sigma_\rho(z) = \sigma_0 + \sigma_1 z + o(z), \quad \text{as } z \xrightarrow{\mathcal{D}_\rho} 0,
\]

and there exists a probability measure \( s_\alpha \in \mathcal{P}_+ \) such that

\[
D_{s_0^n/n} \left( \left( \rho^\otimes n - 1 \right)^\otimes n \right) \rightarrow s_\alpha \quad \text{in dist.}
\]

In addition, the \( \Sigma \)-transform of the limit distribution \( s_\alpha \) is

\[
\Sigma_{s_\alpha}(z) = \exp(-\alpha z).
\]

Proof. By Proposition 1, we obtain

\[
S_{D_{s_0^n/n}} \left( \left( \rho^\otimes n \right)^\otimes n \right)(z) = \frac{1}{s_0^n} \left( s_0^n + \frac{s_1^n z}{n} \right)^n + o(1) \quad \text{(for sufficiently large } n)\]

\[
\xrightarrow{n \to \infty} \exp \left( \frac{s_1^n z}{s_0^n} \right).
\]

From [4] and [5, Theorem 6.13 (i)], there exists a free multiplicative infinitely divisible measure \( \eta_\alpha \) such that \( S_{\eta_\alpha}(z) = \exp \left( \frac{s_1^n z}{s_0^n} \right) \). The proof for (2) is the same as for the free additive case. \( \square \)

We can exchange the order of free multiplicative and free additive (or boolean additive) convolutions. The difference is in the scaling speed.

**Corollary 10.** Under the same setting as in Theorem 9, we have

\[
D_{s_0^n/n^n} \left( \left( \rho^\otimes n \right)^\otimes n \right) \rightarrow \eta_\alpha,
\]

\[
D_{s_0^n/n^n} \left( \left( \rho^\otimes n - 1 \right)^\otimes n \right) \rightarrow s_\alpha,
\]

as \( n \to \infty \).

Proof. As we have done in the proof of Theorem 9, it can be proved by using Proposition 1 and Theorem 8. \( \square \)
4 Lambert $W$ function and infinite divisibility of the limit distribution

4.1 On the limit distribution of free case

When we calculate the $R$–transform or the moment generating function, the Lambert’s $W$–function plays an important role. Let $W_0(z)$ be the principal branch of the Lambert $W$–function, that is, it satisfies the functional equation

$$z = W_0(z) \exp(W_0(z)).$$

For more details of the Lambert $W$ function, see, for instance, [7].

**Theorem 11.**  (1) The $R$ and $R$–transforms of probability measure $\eta_\alpha$ are given as follows:

$$R_{\eta_\alpha}(z) = -\frac{W_0(-\alpha z)}{\alpha z},$$

$$R_{\eta_\alpha}(z) = -\frac{1}{\alpha} W_0(-\alpha z).$$

(2) $\eta_\alpha$ is both $\boxplus$–infinitely divisible and $\boxminus$–infinitely divisible.

(3) The free cumulant sequence of $\eta_\alpha$ is

$$\left\{ \frac{\alpha^n}{n!} \right\}_{n\in\mathbb{N}}.$$

(4) The Lévy measure $\nu_{\eta_\alpha}$ of $\eta_\alpha$ is given by

$$\nu_{\eta_\alpha}(ds) = \frac{1}{\alpha \pi} f^{-1}(\alpha / s) f(\alpha / s) 1_{[0,\infty)}(s) ds,$$

where $f(u) = u \csc u \exp(-u \cot u)$. In case of $\alpha = 1$, for the shape of the density of $\nu_{\eta_1}$, see the graph below.

![Graph showing the density of $\nu_{\eta_1}$](image-url)

(5) It holds the following formulas:

$$\eta_{\alpha t}^{\boxplus t} = D_1(\eta_{\alpha 1}^{\boxplus 1}),$$

$$\eta_{\alpha t}^{\boxminus t} = D_1(\eta_{\alpha 1}^{\boxminus 1}).$$
The proof of this theorem is helped by the following well-known integral representation of the Lambert’s W–function:

**Proposition 12.** For any $z \in \mathbb{C}^+$, we have an integral representation:

$$\frac{W_0(z)}{z} = \frac{1}{\pi} \int_0^\pi \frac{(1 - u \cot u)^2 + u^2}{z + u \csc u \exp(-u \cot u)} \, du.$$  

**Proof of Theorem 11.** (1) The $\boxplus$–infinitely divisibility is trivial from the form of the $S$–transform and the facts in [\text{I}]. By Proposition 5, we have

$$R_{\eta_\alpha}(ze^{-\alpha z}) = z.$$  

Then the $R$–transform is given by using the Lambert’s W–function as follows:

$$R_{\eta_\alpha}(z) = -\frac{1}{\alpha} W_0(-\alpha z),$$

and hence,

$$R_{\eta_\alpha}(z) = \frac{W_0(-\alpha z)}{-\alpha z}.$$  

(2) By the property of $W_0(z)$, $R_{\eta_\alpha}$ has an analytic extension defined on $\mathbb{C}^-$ with value $\mathbb{C}^- \cup \mathbb{R}$, which means that $\eta_\alpha$ is $\boxplus$–infinitely divisible.

(3) The Taylor type expansion of $-W_0(-z)$ at the origin is obtained from Equation (3.1) of [\text{II}, pp. 339].

(4) We put $g(u) = (1 - u \cot(u))^2 + u^2$. Noting that

$$g(u) = \frac{uf'(u)}{f(u)},$$

we obtain

$$R_{\eta_\alpha}(z) = \frac{1}{\alpha \pi} \int_0^\pi \frac{g(u)}{-\alpha z + f(u)} \, du = \frac{1}{\alpha \pi} \int_0^\pi \frac{g(u)/f(u)}{1 - \alpha z/f(u)} \, du = \frac{1}{\alpha \pi} \int_0^\infty \frac{f^{-1}(\alpha/s)}{1 - sz} \, ds,$$

where we have changed the variables as $s = \alpha/f(u)$.

$$R_{\eta_\alpha}(z) = \frac{1}{\alpha \pi} \int_0^\infty \frac{f^{-1}(\alpha/s)}{1 - sz} \, ds$$

$$= \frac{1}{\alpha \pi} \int_0^\infty \left( \frac{sz}{1 - sz} + 1 \right) f^{-1}(\alpha/s) \, ds$$

$$= \frac{1}{\alpha} + \frac{1}{\alpha \pi} \int_0^\infty \frac{z}{1 - sz} f^{-1}(\alpha/s) \, ds.$$  

Therefore we obtain the Lévy measure $\nu_{\eta_\alpha}(ds) = sf^{-1}(\alpha/s)/(\alpha \pi) \, ds$ of $\eta_\alpha$.

(5) It is direct consequence of Proposition 11. 

\[\square\]
Remark 13. Here we consider the limit distribution with parameter $\alpha = 1$. For example, if $\rho$ is the free Poisson distribution with parameter 1, this is the case. Simply we write $\eta$ instead of $\eta_1$. There exists a probability measure $\rho$ such that
\[
R_{\rho}(z) = \frac{R_\eta(z) - 1}{z}. \tag{4.2}
\]
Indeed, if we consider the shifted free cumulant sequence \( \{k_n(\rho)\}_{n\in\mathbb{N}} = \{(n+1)^n\}_{n\in\mathbb{N}} \), which is a sequence of coefficients of Taylor expansion of $R_{\rho}$ at 0, then it becomes a moment sequence of a probability measure. This means that the measure $\rho$ is a free compound Poisson distribution with a compound measure $\sigma$, the moments of which are $m_n(\sigma) = \frac{(n+1)^n}{(n+1)!}$. From (4.2), we have
\[
z R_\eta(z M_{\rho}(z)) = z M_{\rho}(z). \tag{4.3}
\]
By putting $P(z) = z M_{\rho}(z)$ and using the Lagrange inversion formula, (4.3) implies that
the $n$th coefficient of $\{P(z)\} = \frac{1}{n} \times (n-1)^{th}$ coefficient of $R_{\rho}(z))$.
Hence we obtain the moments of $\rho$ as $m_n(\rho) = \frac{(2n+1)^{n-1}}{n!}$.

4.2 On the limit distribution in boolean case

Let $s := s_1$ denote a probability measure with the moment sequence $\{\frac{n^n}{n!}\}_{n\geq 0}$, the positivity of which is ensured by [10]. Then its moment generating function $M_s(z)$ can be given by
\[
M_s(z) = \sum_{n=0}^{\infty} \frac{n^n}{n!} z^n = \frac{1}{1 - \eta(z)} \tag{1}
\]
where the function $\eta(z)$ is defined by
\[
\eta(z) = -W_0(-z), \quad z \in \mathbb{C} \setminus \left[ \frac{1}{e}, \infty \right).
\]

Remark 14. The following useful facts on the function $\eta$ can be found in [8, Sect.2]: The map
\[
\theta \mapsto \frac{\sin \theta}{\theta} \exp \left( \theta \cot \theta \right)
\]
is a bijection of $(0, \pi)$ onto $(0, e)$, and if we define the functions $\eta^+, \eta^- : \left[ \frac{1}{e}, \infty \right) \to \mathbb{C}$ by
\[
\eta^\pm \left( \theta \frac{\exp \left( - \theta \cot \theta \right)}{\sin \theta} \right) = \theta \cot \theta \pm i \theta, \quad 0 \leq \theta < \pi,
\]
then
\[
\eta^\pm(x) = \lim_{y \downarrow 0} \eta(x + iy), \quad x \in \left[ \frac{1}{e}, \infty \right).
\]

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From (1), the Cauchy transform of the measure $s$ is given by

$$G_s(\zeta) = \frac{1}{\zeta} \frac{1}{1 - \eta\left(\frac{1}{\zeta}\right)}, \quad \text{for } \zeta \in \mathbb{C} \setminus [0, e].$$

Now we apply the Stieltjes inversion formula to obtain the density function $\varphi_s(t)$ of the measure $s$, that is, for $t \in [0, e]$,

$$\varphi_s(t) = \frac{1}{\pi} \lim_{\varepsilon \downarrow 0} \Im \left( G_s(t + i\varepsilon) \right)$$

$$= \frac{1}{\pi} \lim_{\varepsilon \downarrow 0} \Im \left( \frac{1}{t + i\varepsilon} \frac{1}{1 - \eta\left(\frac{1}{t + i\varepsilon}\right)} \right)$$

$$= \frac{1}{\pi} \lim_{\varepsilon \downarrow 0} \Im \left( \frac{t - i\varepsilon}{t^2 + \varepsilon^2} \frac{1}{1 - \eta\left(\frac{t - i\varepsilon}{t^2 + \varepsilon^2}\right)} \right)$$

$$= \frac{1}{\pi} \Im \left( \frac{1}{1 - \eta\left(\frac{1}{t}\right)} \right),$$

where the function $\eta^-$ is defined as in Remark above. Here we change the variables

$$\frac{1}{t} = \frac{\theta}{\sin \theta} \exp\left(-\theta \cot \theta\right),$$

then it follows that

$$\varphi_s(t) = \frac{1}{\pi} \Im \left( \frac{1}{t} \frac{\theta}{1 - \left(\theta \cot \theta - i\theta\right)} \right)$$

$$= \frac{1}{\pi} \frac{\theta}{t \left(1 - \theta \cot \theta\right)^2 + \theta^2}$$

$$= \frac{1}{\pi} \frac{\theta}{\sin \theta \exp\left(-\theta \cot \theta\right)} \left(\frac{\theta}{\left(1 - \theta \cot \theta\right)^2 + \theta^2}\right)$$

$$= \frac{1}{\pi} \frac{\theta^2 \exp\left(-\theta \cot \theta\right)}{\sin \theta \left(\left(1 - \theta \cot \theta\right)^2 + \theta^2\right)}.$$

Thus we obtain the following proposition:

**Proposition 15.** The probability density function $\varphi_s$ of the measure $s$ can be given by the implicit (parametric) form as

$$\varphi_s\left(\frac{\sin v}{v} \exp\left(v \cot v\right)\right) = \frac{1}{\pi} \frac{v^2 \exp\left(-v \cot v\right)}{\sin v \left(\left(1 - v \cot v\right)^2 + v^2\right)}, \quad 0 < v < \pi.$$
Remark 16. (1) The shape of the density function of $\varphi_s$ is as the graph below, especially non-unimodal.

\[\begin{array}{c}
\begin{array}{c}
0.0 \\
0.5 \\
1.0 \\
1.5 \\
2.0 \\
2.5 \\
0.0 \\
0.5 \\
1.0 \\
1.5 \\
2.0 \\
2.5 \\
0.0 \\
0.5 \\
1.0 \\
1.5 \\
2.0 \\
2.5 \\
0.0 \\
0.5 \\
1.0 \\
1.5 \\
2.0 \\
2.5
\end{array}
\end{array}\]

(2) The function \((1 - v \cot v)^2 + v^2\) also appears in the integral representation of \(W_0(z)/z\) as we mentioned Proposition [12]

\[\frac{W_0(z)}{z} = \frac{1}{\pi} \int_0^{\pi} \frac{(1 - v \cot v)^2 + v^2}{z + \frac{v}{\sin v} \exp(-v \cot v)} dv.\]

Thus using \(f(v) = v \csc v \exp(-v \cot v)\) and (4.1) again, the parametric form of the density function can be rewritten as

\[\varphi_s\left(\frac{1}{f(v)}\right) = \frac{1}{\pi} \left(\frac{f(v)}{f'(v)}\right)^2.\]

Remark 17. We also know that there exists the similar moment sequence. In the paper by Dykema and Haagerup [8], they find a limit distribution of DT-operator DT(0, $\delta_0$). The moment of their one is \(m_n = \frac{n^n}{(n+1)!}.\)

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