Robertson-Walker fluid sources endowed with rotation characterised by quadratic terms in angular velocity parameter.

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Abstract

Einstein’s equations for a Robertson-Walker fluid source endowed with rotation are presented upto and including quadratic terms in angular velocity parameter. A family of analytic solutions are obtained for the case in which the source angular velocity is purely time-dependent. A subclass of solutions is presented which merge smoothly to homogeneous rotating and non-rotating central sources. The particular solution for dust endowed with rotation is presented. In all cases explicit expressions, depending sinusoidally on polar angle, are given for the density and internal supporting pressure of the rotating source. In addition to the non-zero axial velocity of the fluid particles it is shown that there is also a radial component of velocity which vanishes only at the poles. The velocity four-vector has a zero component between poles.

1 Introduction

Perturbation techniques, so important in the General Theory of Relativity, have frequently been applied successfully in the description of slowly rotating compact perfect fluid sources. Of considerable importance has been the analysis of Hartle [1] who presented the equations for the equilibrium configurations of cold stars up to and including the first order of angular velocity parameter. This work has formed the basis of an extended first order analysis for example, Kojima [2] and Beyer & Kokkotas [3], Abramowicz et al [4], to the address the problem of the non-radial quasi-periodic oscillations of rotating compact sources and the resulting r-mode spectrum of relativistic stars.
In examples of non-compact rotating sources Kegeles [6] and Wiltshire [7] successfully applied the perturbation method to a slowly rotating non-equilibrium configuration. In this case a Robertson-Walker dust source endowed with rotation was successfully matched to the Kerr exterior solution of Einstein’s equations to the first order in angular velocity parameter.

Recent second order perturbation analyses have largely been confined to non-rotating cases. For example, Salopek et al. [8], Russ et al [9] use the method to discuss the gravitational instabilities of an expanding inhomogeneous universe. However, there is seemingly an absence of literature on the use of second order techniques to describe rotating bodies. This is perhaps surprising, since such analyses can reveal the relativistic effects of the spatial distribution of a rotating fluid as characterised by internal density and pressure. Such effects are not revealed in first order approximations where fluid density and pressure are shown to be same as for a non-rotating source. Moreover, the second order effects result in calculated deviations of spherical symmetry in the fluid boundary of a compact or extended body which may be used in the context of the matching problem with a Kerr vacuum.

It is the aim here to present an example of a second order perturbation analysis applied to an extended rotating source. In particular a Robertson-Walker source will be endowed with rotation. As will be seen the approach naturally brings about solutions of Einstein’s equations which exhibit the non-homogeneities in internal density and supporting pressure due to rotation.

In the following a non-rotating source will be described using the Robertson-Walker metric in the form:

\[
dσ^2_{RW} = dη^2 - R^2(η) \left( \frac{dξ^2}{1 - kξ^2} + ξ^2 dθ^2 + ξ^2 \sin^2(θ) dφ^2 \right)
\]

(1)

where \( R = R(η) \), \( k = -1, 0, 1 \) and where the speed of light \( c = 1 \), and gravitational constant \( G = 1 \). The homogeneous density and supporting pressure will be denoted by \( ρ_{RW} \) and \( p_{RW} \) respectively. The fluid source will be endowed with rotation which will be characterised in terms of an angular speed parameter, denoted by \( q \), and the mathematical analysis to follow will be accurate up to and including second order terms in \( q \).

The rotating fluid source will be described in terms of an extended form of the Robertson-Walker metric which is taken to be:

\[
dσ^2 = \left(1 + \tilde{Q}q^2\right)dη^2 - \frac{R^2(η)}{1 - kξ^2} \left(1 + \tilde{U}q^2\right) dξ^2 - 2\tilde{J}ξ^2 R^2q^2 dξ dη - ξ^2 R^2 \left(1 + \tilde{V}q^2\right) dθ^2 - ξ^2 R^2 \sin^2(θ) \left(1 + \tilde{W}q^2\right) dφ^2 - 2ξ^2 \sin^2(θ) R^2q \left(Y dξ dφ + X dφ dη\right) + O(q^3)
\]

(2)

in which each of the functions, \( \tilde{J}, \tilde{Q}, \tilde{U}, \tilde{V}, \tilde{W} \) depend on \( ξ, θ \) and \( η \) whilst \( X, Y \) depend on \( ξ \) and \( η \) alone. Note the this form of metric is an extension of the linearly perturbed form of the Robertson-Walker metric used by Kegeles.
The components of the fundamental tensor from (2) are determined using Einstein’s equations for a perfect fluid written here in the form:

\[ G_{ab} = -8\pi T_{ab}, \quad T_{ab} = (\rho + p) u^a u_b - \delta_{ab}^0 p, \]  

where \( \rho \), \( p \) are the respective rotating source density and supporting internal pressure and \( u^a \) are the components of the velocity four-vector with the property that \( u^a u_a = 1 \).

The use of seven as yet unknown functions in (2) will naturally give rise to ambiguity in the solution of Einstein’s equations. This ambiguity can be removed by further specification of the gauge in which the solutions are to be determined. Although convenient gauge choices are much discussed in the literature, including recently, Bruni et al [10] it is convenient at this stage to continue the solution process without further specification of the gauge other than that which is explicit from the metric choice (2).

For brevity the term \( O(q^3) \), included in (2) will be omitted from all future expressions. The fact that all terms including \( q^n, n \geq 3 \) are taken as negligibly small will of course be implied.

In summary it is the aim in the following to determine solutions of Einstein’s equations in the form (3) for rotating sources described by the metric (2).

## 2 Solution Approach

Direct calculation of the components of the Einstein tensor for (2) show that the components \( G^3_1 \) and \( G^2_3 \) are identically zero and so for a rotating perfect fluid it follows that the velocity four vector component \( u^2 = 0 \) and one form component \( u_2 = 0 \). It thus follows that the conditions \( T_1^2 = 0 = T_2^1 \) and \( T_1^3 = 0 = T_2^3 \) must hold up to and including terms in \( q^2 \). Moreover \( T_2^2 + p = 0 \).

In addition the perfect fluid (3) must also satisfy the following consistency relationships

\[
\begin{align*}
(T_1^1 + p) (T_3^3 + p) - T_1^1 T_3^3 &= 0 \\
(T_1^1 + p) (T_4^4 + p) - T_1^1 T_4^4 &= 0 \\
(T_3^3 + p) (T_4^4 + p) - T_3^3 T_4^4 &= 0
\end{align*}
\]  

(4)

However by direct calculation \( G^3_1, G^1_3, G^1_2, G^2_3 \) and \( G^3_3 \) depend on linear terms in \( q \), whilst \( G^2_1, G^2_1, G^1_1, G^1_4, G^2_4 \) and \( G^1_2 \) depend on quadratic terms in \( q \). Also when \( q = 0 \) each of these components is zero. It follows that for the Robertson-Walker source endowed with rotation that solutions of Einstein’s equations must satisfy the following perturbation equations:

\[
\begin{align*}
T_1^1 + p &= 0 \\
T_2^2 + p &= 0 \\
(T_3^3 + p) (T_4^4 + p) - T_3^3 T_4^4 &= 0 \\
T_3^1 &= 0 \\
T_1^3 &\neq 0
\end{align*}
\]
\[
\begin{align*}
T_1^2 &= 0 = T_2^1 \\
T_4^2 &= 0 = T_2^4
\end{align*}
\tag{5}
\]

The first or second of these equations may be used to calculate the internal pressure \( p \) whilst the density \( \rho \) is calculated using
\[
\rho = T_3^3 + 3p
\tag{6}
\]
where the repeated index indicates summation. It is perhaps worthy of note that although the velocity four vector \( u_{1RW}^1 = 0 \) for the standard Robertson-Walker case, the system of equations (5) do not imply that this condition is retained for the rotating source.

The angular velocity of the source will be denoted by \( L(\xi, \eta) \) where:
\[
L(\xi, \eta) = \frac{u_3}{u_4} = \frac{T_4^3}{T_4^4 + p}
\tag{7}
\]
Since the fourth of conditions (5) \( T_3^3 = 0 \) may be solved immediately to give:
\[
Y_\eta = X_\xi + \frac{h(\xi)}{(1 - k\xi^2)^{\frac{1}{2}}}\xi^4 R^3
\tag{8}
\]
where \( h(\xi) \) is an arbitrary function of \( \xi \), it follows from (7) that the angular velocity of the source is given by:
\[
L(\xi, \eta) = -\frac{q\sqrt{1 - k\xi^2}h_\xi}{16\pi^4 R^5 (\rho_{RW} + p_{RW})} - qX
\tag{9}
\]
where the suffix denotes a partial derivative. The density \( \rho_{RW} \) and pressure \( p_{RW} \) for the standard Robertson-Walker metric (1) are such that:
\[
8\pi (p_{RW} + \rho_{RW}) = -\frac{2R_{\eta\eta}}{R} + \frac{2R_{\eta}^2}{R^2} + \frac{2k}{R^2}
\tag{10}
\]
Moreover, a particle moving in the field of (2) will have zero angular momentum whenever \( u_3 = 0 \), so that the quantity:
\[
\frac{u_3}{u_4} = \frac{q\sin^2 \theta \sqrt{1 - k\xi^2}h_\xi}{16\pi^4 R^5 (\rho_{RW} + p_{RW})}
\tag{11}
\]
will also be zero for such a particle. It follows that the induced angular velocity \( \Omega_f(\xi, \eta) \) of the inertial frame is given by:
\[
\Omega_f(\xi, \eta) = -qX\quad ,
\tag{12}
\]
and that the angular velocity of a particle moving in the field of (2) is:
\[
\Omega_p(\xi, \eta) = -\frac{q\sqrt{1 - k\xi^2}h_\xi}{16\pi^4 R^5 (\rho_{RW} + p_{RW})}
\tag{13}
\]
Clearly therefore it is the nature of \( h (\xi) \) which determines the actual angular velocity of the system and that \( h = 0 \) defines a non-rotating source and that the solution of (8) is then:

\[
X = \Phi_\eta \quad Y = \Phi_\xi
\]

for some \( \Phi = \Phi (\xi, \eta) \).

### 3 Simplification of the Perturbation equations

Using the second equation \( p = -T_2^2 \) the first and third of (5) with (3) become:

\[
G_1^1 - G_2^2 = 0 \quad (15)
\]

\[
(G_3^1 - G_2^2) (G_4^1 - G_2^2) - G_3^1 G_4^2 = 0 \quad (16)
\]

In addition the fifth and sixth of (5) namely, \( T_1^2 = 0 = T_2^1 \) and \( T_2^2 = 0 = T_4^4 \) will be satisfied by taking:

\[
G_1^1 = 0 \quad (17)
\]

\[
G_2^2 = 0 \quad (18)
\]

These equations may be simplified somewhat by firstly, defining a new time variable \( \tau (\eta) \) through:

\[
\tau (\eta) = \int \frac{d\eta}{R^3} \quad (19)
\]

and, secondly by introducing the functions \( S (\tau) \) and \( T (\tau) \) expressed in terms of the density \( \rho_{RW} \) and pressure \( p_{RW} \) for the standard Robertson-Walker metric as follows:

\[
S (\tau) \equiv \frac{1}{R^6 T} = 8\pi (\rho_{RW} + p_{RW}) = - \frac{2R T_\tau}{R^7} + \frac{8R^2}{R^8} + \frac{2k}{R^2} \quad (20)
\]

Thirdly the equations (15) to (17) may be rendered independent of \( \tilde{W} \) with the aid of the following substitutions:

\[
\begin{align*}
\tilde{J} & = XY \sin^2 \theta + \frac{J}{R^3 \xi \sqrt{1 - k \xi^2}} \\
\tilde{U} & = \xi^2 (1 - k \xi^2) Y^2 \sin^2 \theta + U + W \\
\tilde{V} & = V + W \\
\tilde{W} & = W \\
\tilde{Q} & = -\xi^2 R^2 X^2 \sin^2 \theta + Q - W
\end{align*}
\]

where each of \( U, V, W, Q \) and \( J \) are again functions of \( \xi, \theta \) and \( \tau \).
It follows from the transformations (21) and (19) that the metric (2) may now be written in the form:

\[
d\sigma^2 = \xi^2 R^2 \left\{ \frac{(1 + (Q - W) q^2)}{\xi^2} d\tau^2 - \frac{(1 + (U + W) q^2)}{\xi^2 (1 - k\xi^2)} d\xi^2 - \frac{2J q^2}{\xi^2 \sqrt{1 - k\xi^2}} d\tau d\tau \right. \\
- \left. (1 + (V + W) q^2) d\theta^2 - W q^2 \sin^2(\theta) d\phi^2 - \sin^2(\theta) (d\phi + qY d\xi + qR^3 X d\tau)^2 \right\}
\]

(22)

Note that the entity \(d\phi + qY d\xi + qR^3 X d\tau\) is itself an exact differential only in the non-rotating case when equation (14) applies.

In this way the first of equations (15) becomes:

\[
\left\{ -\xi (1 - k\xi^2) (U_\xi + V_\xi) - \frac{\xi^2 (U_{\tau\tau} - V_{\tau\tau})}{R^4} + 2(U - V) + 2\xi^3 \sqrt{1 - k\xi^2} J_{\tau\xi} + \xi^2 (1 - k\xi^2) Q_{\xi\xi} - \xi Q_\xi - Q_{\theta\theta} \right\} \\
\left\{ + \frac{(V_\theta + U_\theta) \cos \theta}{\sin \theta} - \frac{h^2 \sin^2 \theta}{\xi^2 R^4} = 0 \right. \\
(23)
\]

In addition the second of equations (16) has the form:

\[
\xi^2 (1 - k\xi^2) V_{\xi\xi} - 3\xi^2 V_\xi V_\xi \right\} \\
\left\{ -\frac{(U_\theta + Q_\theta) \cos \theta}{\sin \theta} - \sin^2 \theta \left( \frac{h_\xi^2 (1 - k\xi^2) T}{2\xi^2} + \frac{h_\xi^2}{\xi^2 R^4} \right) = 0 \right. \\
\]

(24)

The condition (17) that \(G_{12}^2 = 0\) becomes:

\[
-\frac{U_\theta + Q_\theta}{\xi} + \frac{\xi J_{\tau\theta}}{R^4 \sqrt{1 - k\xi^2}} + Q_{\theta\xi} - \frac{V_\xi \cos \theta}{\sin \theta} = 0 \right. \\
(25)
\]

Only the remaining equation (18) explicitly contains \(W\) as follows:

\[
\frac{2R_\tau (W_\theta - Q_\theta)}{R} + 2W_{\tau\theta} + U + W \sqrt{1 - k\xi^2} (J_{\xi\theta} + J_\theta) - \frac{V_\tau \cos \theta}{\sin \theta} = 0 \right. \\
(26)
\]

Equations (23) to (26) are the final forms of the perturbation equations which determine \(U, V, W, Q\) and \(J\) for given \(h(\xi)\) for the metric (22).

The internal supporting pressure, calculated using the second of (14) is:

\[
8\pi p \right\} \left\{ \sqrt{1 - k\xi^2} \right\} R^6 \right\} (\xi J_{\tau\xi} + 2J_\tau) + (W - Q) \left( \frac{1}{R^6 T} - \frac{3R^2}{R^2} \right)
\]

6
whilst the internal density, calculated using (6) is:

\[
8\pi \rho = q^2 \left\{ \left( \frac{1 - k\xi^2}{R^2} \right) \left( \frac{Q\xi}{2} - W\xi - V\xi + \frac{U\xi}{2\xi} \right) - \frac{(W_{\theta\theta} + U_{\theta\theta} + Q_{\theta\theta})}{\xi^2 R^2} \right.
\]

\[
+ \left. \frac{(V_{\tau\tau} - \frac{U_{\tau\tau}}{R^2})}{R^2} + \frac{R_{\tau}}{R^6} (3W_{\tau} + V_{\tau} + U_{\tau}) + \frac{3R_{\tau}^2 (W - Q)}{R^8} - \frac{3k(U + W)}{R^2} \right.
\]

\[
+ \left. \frac{2(U - V)}{\xi^2 R^2} + \frac{1}{R^2} \left( 3k\xi W_{\tau} - \frac{2W_{\tau}}{\xi} + 4k\xi V_{\tau} - \frac{3V_{\tau}}{\xi} - \frac{Q_{\tau}}{2\xi} \right) \right.
\]

\[
+ \left. \frac{\sqrt{1 - k\xi^2}}{R^6} \left( -2\xi J_{\tau\tau} R_{\tau} - \frac{6JR_{\tau} R_{\tau}}{R} + \xi J_{\tau\tau} \right) \right\}
\]

\[
+ \frac{q^2 (Q_{\theta} + U_{\theta} + 2V_{\theta} - 2W_{\theta}) \cos \theta}{2\xi^2 R^2 \sin \theta} - \frac{h^2 q^2 \sin^2 \theta}{4\xi^6 R^6} + 8\pi \rho_{RW} \tag{28}
\]

Further note that the application of the transformations (21) and (19) has enabled each of the equations (23) to (28) to be written in a form which is independent of both \(X(\xi, \tau)\) and \(Y(\xi, \tau)\) but explicitly contains terms in \(h(\xi)\). This is expected from (12) since one would not expect internal pressure and density to be dependent on the frame dragging term \(X\) but rather on that function defined in (9) and (11) which directly determines the true angular velocity of the source, namely \(h\). It follows that further analysis may continue without further detailed specification of \(X\) and \(Y\), only \(h\) needs consideration.

Finally, the two equations (23) to (26) contain the four unknowns \(J, Q, U, \) and \(V\) and may be used to determine families of rotating extended sources for a range gauges and or physical conditions. As a particular example of a solution procedure it should be noted that in cases when \(V\) is known explicitly then equation (24) may be integrated directly to determine \(U + Q\). Thus if either \(U\) or \(Q\) is known then the equation (25) may be integrated immediately to determine \(J\).

Finally, the velocity four-vector component \(u^1/u^4\) may be calculated through

\[
u^1\nu^4 = \frac{T^1_4}{T^4_4 + p} \tag{29}
\]

and so up to and including quadratic terms in \(q^2\) it follows that:

\[
u^1\nu^4 = -\frac{G^1_4}{8\pi (\rho_{RW} + \rho_{RW})} \tag{30}
\]
where $G_4^1$ is given by:

\[
- \frac{R^5 G_4^1}{\sqrt{1 - k\xi^2}} = \left\{ \left( \frac{W_{\tau\xi} + V_{\tau\xi}}{2} \right) + \frac{1}{\xi} \left( \frac{V_{\tau}}{2} - U_{\tau} \right) + \frac{R_{\tau}}{R} \left( W_{\xi} - Q_{\xi} \right) \right\} \sqrt{1 - k\xi^2}
\]

\[
+ \frac{J_{\theta\theta}}{2\xi} - \frac{\xi J}{R^4 T} + 2k\xi J + \frac{J_{\theta} \cos \theta}{2\xi \sin \theta}
\]

Note also that in a similar way $u_1/u_4$ may be found using:

\[
\frac{u_1}{u_4} = \frac{T_4^4}{T_4^3 + p} = - \frac{G_4^1}{8\pi (\rho_{RW} + p_{RW})}
\]

where $G_4^1$ is

\[
G_4^1 = -h_\xi q^2 \sin^2 \theta Y \sqrt{1 - k\xi^2} - q^2 \left\{ \left( -\frac{2W_{\tau\xi} + V_{\tau\xi}}{2R^3} + \frac{2U_{\tau} - V_{\tau}}{2R^3} \right) \right\}
\]

\[
+ \frac{R_{\tau} \left( Q_{\xi} - W_{\xi} \right)}{R^4} + \frac{2J \sqrt{1 - k\xi^2}}{\xi R^3} - \left( \frac{J_{\theta\theta} + 4J}{2\xi \sqrt{1 - k\xi^2} R^3} \right)
\]

\[
+ \frac{q^2 \cos \theta J_{\theta}}{2\sin \theta \xi \sqrt{1 - k\xi^2} R^3}
\]

Thus only \[83\] depends explicitly on $Y(\xi, \tau)$ and is a further manifestation of the frame dragging effect expressed in \[8\]. This may be removed by choosing $Y = 0$.

4 Characterisation of angular velocity leading to analytic solutions

Consider first equations the three equations (23) to (25) and note that particular solutions may be found in principle by setting:

\[
U(\xi, \theta, \tau) = \frac{u_1}{\xi^5} \sin^2 \theta + u_2
\]

\[
V(\xi, \theta, \tau) = \frac{u_3}{\xi^6} \sin^2 \theta + u_4
\]

\[
J(\xi, \theta, \tau) = \sqrt{1 - k\xi^2} \left( u_5 \sin^2 \theta + u_6 \right)
\]

\[
Q(\xi, \theta, \tau) = u_7 \sin^2 \theta + u_8
\]

where $u_i$ are functions of $\xi$ and $\tau$ alone. It is straightforward to show that equation [24] may be used to determine $u_7$ and $u_4$ since:

\[
\xi^6 u_7 = -\frac{\xi^2 \left( h^2 + u_{3\xi\xi} \right)}{2R^4} + \frac{\xi^2 (1 - k\xi^2)}{2} \left( u_{3\xi\xi} \frac{k\xi^2 T}{2} \right)
\]

\[
+ \frac{9k\xi^3 u_3}{2} - 12k\xi^2 u_4 - 5\xi u_3 + 15u_3 - u_1
\]
and \( u_4 = 0 \). Moreover direct substitution of these relationships into (25) shows that this equation is satisfied provided that \( u_5 \) is such that:

\[
\frac{\partial u_5}{\partial \tau} = \Psi_1 R^4 T + \Psi_2 R^4 + \Psi_3
\]  

(39)

where \( \Psi_1, \Psi_2 \) and \( \Psi_3 \) are fully defined in terms of \( h, u_1 \) and \( u_3 \) and are presented in the appendix. The equation (38) and (39) together with \( u_4 = 0 \) may then be substituted directly into equation (23) to produce a very lengthy relationship, reproduced in the appendix, having the general form:

\[
\Psi_4 \sin^2 \theta + \Psi_5 = 0
\]  

(40)

where \( \Psi_4 \) and \( \Psi_5 \) are again fully defined in terms of \( h, u_1 \) and \( u_3 \) and their partial derivatives with respect to \( \xi \) and \( \tau \).

Inspection shows that the equation \( \Psi_4 = 0 \) may in general terms only be solved numerically for \( u_1 \) (say) in terms any given \( u_3 (\xi, \tau) \) and \( h (\xi) \). Remarkably, however there is at least one case where considerable simplification is possible, namely when:

\[
h (\xi) = n\xi^5
\]  

(41)

with constant \( n \), for which a basic separation of variables approach reveals that:

\[
\begin{align*}
u_1 (\xi, \tau) &= n^2 \xi^{10} (\psi_1 + \psi_2 \xi^2) \\
u_3 (\xi, \tau) &= n^2 \xi^{10} \psi_3
\end{align*}
\]  

(42)

where \( \psi_1, \psi_2 \) and \( \psi_3 \) are functions of \( \tau \) alone which satisfy:

\[
125k^2 T - \frac{10k}{R^4} (1 + \psi_3 \tau) + \psi_2 \tau - 24k (\psi_2 + 10k \psi_3) = 0
\]  

(43)

\[
-\frac{225kT}{2} + \frac{7\psi_3 \tau - \psi_1 \tau + 5}{R^4} + 4k (49\psi_3 - 2\psi_1) + 14\psi_2 = 0
\]  

(44)

Although these equations need to be solved numerically when \( k = 1, -1 \) the case \( k = 0 \) yields the analytic result that:

\[
7\psi_3 \tau - \psi_1 \tau + 5 = 0 \quad \psi_2 = 0
\]  

(45)

In the following the family of solutions presented will be based upon equation (45). Thus using (41) and (19) substituted in (13) and the angular velocity \( \Omega_p (\xi, \tau) \) of a particle in the field of (22) is:

\[
\Omega_p (\xi, \tau) = -\frac{nj}{16\pi R^2 (\rho_{RW} + p_{RW})}
\]  

(46)

It follows that the angular velocity is purely time dependent. The frame dragging effect (12) is

\[
\Omega_f = -qR^3 X
\]  

(47)

where from (8) and (19) \( X (\xi, \tau) \) is determined through:

\[
R^3 X_\xi = Y_\tau - n\xi
\]  

(48)
In cases when $Y = 0$ so that the frame dragging effect due to rotation is removed from (33) then (46) becomes:

$$\Omega_f = \frac{nq\xi^2}{2}$$  \hspace{1cm} (49)

where for choice $X(0, \tau) = 0$.

5 Development of solutions with $k=0$ and with purely time dependent angular velocity

Thus using the solution procedure outlined in the previous section with (45) it is straightforward to show that the corresponding solutions of (23) to (26) are:

$$U(\xi, \theta, \tau) = n^2\psi_1\xi^4\sin^2\theta + n^2z_2$$  \hspace{1cm} (50)

$$V(\xi, \theta, \tau) = \frac{n^2}{7} \left(\psi_1 - \frac{5\tau^2}{2}\right)\xi^4\sin^2\theta$$  \hspace{1cm} (51)

$$J(\xi, \theta, \tau) = n^2z_5\sin^2\theta + n^2z_6$$  \hspace{1cm} (52)

$$Q(\xi, \theta, \tau) = n^2\left\{\xi^4 \left(\frac{3\psi_1}{7} - \frac{25T^2}{4} - \frac{25\tau^2}{7}\right) - \frac{\xi^6}{14R^4} \left(\psi_{1rr} + 2\right)\right\}\sin^2\theta + n^2z_8$$  \hspace{1cm} (53)

$$W(\xi, \theta, \tau) = \frac{n^2z_9\sin^2\theta}{R} + \frac{n^2z_{10}}{R}$$  \hspace{1cm} (54)

where $z_5(\xi, \tau)$, $z_6(\xi, \tau)$ and $z_9(\xi, \tau)$ are determined in terms of $\psi_1(\tau)$, $z_2(\xi, \tau)$ and $z_8(\xi, \tau)$ through:

$$\frac{\partial z_5}{\partial \tau} = \xi^2R^4 \left(\frac{75T^2}{4} + 10\tau^2\right) + \frac{5\xi^4}{14} \left(\psi_{1rr} + 2\right)$$  \hspace{1cm} (55)

$$\frac{\partial^2 z_6}{\partial \xi \partial \tau} = -R^4 \left\{\xi \left(\frac{25T^2}{4} + \frac{45\tau^2}{14} + \frac{5\psi_1}{7}\right) + \frac{z_{8rr}}{2\xi} - \frac{z_{8r} + z_{2r}}{2\xi^2} + \frac{z_{2}}{\xi^4}\right\}$$

$$\frac{\partial^3 z_9}{\partial \xi \partial \tau} = \frac{\xi^3}{14} \left(\psi_{1rr} + 2\right) + \frac{z_{2rr}}{2\xi}$$  \hspace{1cm} (56)

$$\frac{\partial z_9}{\partial \tau} = \xi^4R_{\tau} \left(\frac{3\psi_1}{7} - \frac{25T^2}{4} - \frac{25\tau^2}{7}\right) - \frac{\xi^6R_{\tau}}{14R^4} \left(\psi_{1rr} + 2\right)$$

$$\frac{\partial^2 z_9}{\partial \xi \partial \tau} = -\frac{\xi^4}{28} \left(5\tau + 13\psi_1\right) + \frac{R}{2} \left(\xi z_{8r} + z_8\right)$$  \hspace{1cm} (57)
The function \( z_{10}(\xi, \tau) \) is arbitrary. The supporting internal pressure is given by:

\[
8\pi p = 8\pi p_{RW} + n^2 q^2 \sin^2 \theta \left\{ \frac{25\xi^2 T}{8R^2} + \frac{\xi^4}{14R^6 T} (25\tau^2 - 3\psi_1) + \frac{z_9}{2R^2 T} \right\} \\
+ \frac{\xi^6}{28R^{10} T} (\psi_{1,\tau} + 2) + \frac{25\xi^4}{8R^6} \left\{ -25T \frac{R^6}{R^2} - \frac{95\xi^2}{14R^2} + \frac{5\psi_1}{7R^2} \right\} - \frac{\xi^4}{7R^6} (\psi_{1,\tau} + 2) \\
+ n^2 q^2 \left\{ -\frac{\xi^2}{7R^2} \left( -\frac{25T}{4} + 5\tau^2 + 6\psi_1 \right) - \frac{\xi^4 R}{4R^2} \left( 5\tau + \psi_{1,\tau} \right) \\
- \frac{\xi^2}{R^2} \left( \frac{\xi^2}{2} + 13\psi_1 \right) + \frac{1}{\xi R^2} \left( \frac{z_2}{\xi^2} \right) - \frac{1}{R^3} \left( \frac{2z_{10}}{\xi} + \frac{4z_9}{\xi^2} + z_{10,\tau} \right) \\
- 3\frac{z_8 R^2}{R^8} + \frac{R_\tau}{R^2} \left( -2\xi z_6 + z_2 - 6z_6 + \frac{3z_{10}}{R^2} \right) \right\}
\]

with:

\[
8\pi p_{RW} = \frac{1}{R^6 T} - \frac{3R^2}{R^8}
\]

and the internal density is:

\[
8\pi \rho = 8\pi \rho_{RW} + n^2 q^2 \sin^2 \theta \left\{ \frac{\xi^2}{R^2} \left( -\frac{25T}{4} + 5\tau^2 + 6\psi_1 \right) - \frac{\xi^4 R}{4R^2} \left( 5\tau + \psi_{1,\tau} \right) \\
- \frac{\xi^2}{7R^2} \left( 5\tau^2 - \frac{2z_9}{\xi} + 6z_6 - z_{9,\tau} \right) - \frac{\xi^4}{4R^6} \left( 5\tau + \psi_{1,\tau} \right) \\
+ n^2 q^2 \left\{ -\frac{\xi^2}{7R^2} \left( -\frac{25T}{4} + 5\tau^2 + 6\psi_1 \right) - \frac{\xi^4 R}{4R^2} \left( 5\tau + \psi_{1,\tau} \right) \\
- \frac{\xi^2}{R^2} \left( \frac{\xi^2}{2} + 13\psi_1 \right) + \frac{1}{\xi R^2} \left( \frac{z_2}{\xi^2} \right) - \frac{1}{R^3} \left( \frac{2z_{10}}{\xi} + \frac{4z_9}{\xi^2} + z_{10,\tau} \right) \\
- 3\frac{z_8 R^2}{R^8} + \frac{R_\tau}{R^2} \left( -2\xi z_6 + z_2 - 6z_6 + \frac{3z_{10}}{R^2} \right) \right\}
\]

with:

\[
8\pi \rho_{RW} = \frac{3R^2}{R^8}
\]

Also using (30) and (31) it may be shown that:

\[
\frac{1}{RT} \frac{u^1}{u^4} = n^2 q^2 \left\{ -\frac{\xi z_6}{R^2 T} + \frac{z_{10,\tau} - z_6 R_\tau}{\xi} + \frac{2z_5 - z_2}{\xi} \right\} \\
+ n^2 q^2 \sin^2 \theta \left\{ \frac{\xi}{2} \left( z_{9,\tau} - \frac{2z_5}{R^2 T} \right) + \frac{z_5 - \frac{3z_5}{R^2}}{\xi} - \frac{5\xi^3 (\tau + \psi_{1,\tau})}{2} \right\}
\]

Inspection of (62) shows that in general terms \( u^1(\xi, \theta, \tau) \) although it is possible to choose \( z_{10} \) so that \( u^1(\xi, 0, \tau) = 0 \). Thus \( u^1 = 0 \) is possible for a particle...
moving in the field of \( \frac{\partial^2 z_{10}}{\partial \tau \partial \xi} = \frac{z_6 \xi}{R^3 T} + z_8 R_{\tau} - \frac{R}{\xi} (2z_5 - z_2, \) \( \) (63)

6 A subclass of solutions

Consider now the particular case:

\[
\psi_1 = -\tau^2 \quad z_2 = \omega_2 \xi^2 \quad \psi_8 = \omega_8 \xi^2
\] (64)

where \( \omega_2 \) and \( \omega_8 \) are functions of \( \tau \). The equations (55) to (57) become:

\[
z_5 = \psi_5 \xi^2 \quad \frac{d\psi_5}{d\tau} = R^4 \left( \frac{75T}{4} + 10\tau^2 \right)
\] (65)

\[
z_6 = \psi_6 \xi^2 \quad \frac{d\psi_6}{d\tau} = -R^4 \left( \frac{25T}{8} + \frac{5\tau^2}{4} \right) + \frac{\omega_2 \tau}{4}
\] (66)

Moreover

\[
z_9 = \alpha_2 \xi^2 + \alpha_4 \xi^4
\] (67)

\[
\frac{d\alpha_2}{d\tau} = 3\psi_5 R - \frac{\omega_2}{2} \quad \frac{d\alpha_4}{d\tau} = 3\tau R - R_{\tau} \left( \frac{25T}{4} + 4\tau^2 \right)
\] (68)

where \( \psi_5, \psi_6, \alpha_2 \) and \( \alpha_4 \) are functions of \( \tau \) alone. The condition (63) is:

\[
z_{10} = \beta_2 \xi^2 + \beta_4 \xi^4
\] (69)

\[
\frac{d\beta_2}{d\tau} = \omega_8 R_{\tau} + R \left( \frac{\omega_2}{2} - \psi_5 \right) \quad \frac{d\beta_4}{d\tau} = \frac{\psi_6}{4R^3 T}
\] (70)

where \( \beta_2 = \beta_2 (\tau) \) and \( \beta_4 = \beta_4 (\tau) \).

The supporting internal pressure is given by:

\[
8\pi p = 8\pi p_{RW} + n^2 q^2 \sin^2 \theta \left\{ \xi^2 \left( \frac{25T}{8R^2} + \frac{\alpha_2}{2R^3 T} \right) + \xi^4 \left( \frac{25}{8R^6} + \frac{2\tau^2}{R^6 T} + \frac{\alpha_4}{2R^3 T} \right) \right\} + n^2 q^2 \xi^2 \left\{ -\frac{75T}{4R^2} - \omega_8 \frac{R_{\tau}}{R^3 T} + \frac{\beta_2}{2R^3 T} + \frac{3\omega_8 R_2^2}{R^6} + \frac{R_{\tau}}{R^3} \left( \omega_8 + \frac{\beta_2}{R} \right) - \frac{10\tau^2}{2R^6} + \omega_{2,\tau} \frac{R_{\tau}}{2R^6} - \frac{\beta_{2,\tau}}{R^3} \right\} + n^2 q^2 \xi^4 \left\{ \frac{\beta_4}{2R^3 T} + \frac{\beta_4 R_{\tau}}{R^6} - \frac{\beta_4}{R^3} \right\} + n^2 q^2 \left\{ 2\omega_8 - \omega_2 \right\}
\] (71)
whilst the density is:

\[
8\pi \rho = 8\pi \rho_{RW} + n^2 q^2 \sin^2 \theta \left\{ \xi^2 \left( -\frac{25T}{4R^2} - \frac{11\psi_5 R_x}{2R^7} - \frac{\tau^2}{R^2} - \frac{14\alpha_4}{R^3} \right) 
- \frac{\xi^4}{4R^6} \left( 1 + \frac{3\tau R_x}{R} \right) 
+ n^2 q^2 \xi^2 \left\{ -\frac{3\omega_8 R_x^2}{R^8} + \frac{(\omega_2 - 10\psi_6) R_x}{R^3} + \frac{3\tau^2}{2R^2} - \frac{4\alpha_4}{R^3} \right\} 
+ n^2 \xi^2 \left\{ \frac{3\omega_2}{R^2} - \frac{4\alpha_2}{R^3} \right\} \right\}
\]  
(72)

Notice that in region of \( \xi = 0 \) that both the expressions for pressure and density are well behaved and in particular when \( \xi = 0 \) the pressure and density of the centre of the rotating source are

\[
8\pi p(0, \tau) = 8\pi p_{RW} + n^2 q^2 \left\{ \frac{2\omega_8}{R^2} - \omega_2 \right\}
\]  
(73)

\[
8\pi \rho(0, \tau) = 8\pi \rho_{RW} + n^2 q^2 \left\{ \frac{3\omega_2}{R^2} - \frac{4\alpha_2}{R^3} \right\}
\]  
(74)

Note that the centre of the source will be non-rotating \( 8\pi p(0, \tau) = 8\pi p_{RW} \) and \( 8\pi \rho(0, \tau) = 8\pi \rho_{RW} \) when:

\[
\omega_2 = \frac{4\alpha_2}{3R} \quad \omega_8 = \frac{2\alpha_2}{3R}
\]  
(75)

For this solution defining \( u^1 \) becomes:

\[
\frac{u^1}{a^1} = n^2 q^2 \xi^3 RT \sin^2 \theta \left( \frac{5\tau}{2} - \frac{\psi_5}{R^4T} \right)
\]  
(76)

7 Robertson-Walker dust endowed with rotation

As a specific example of the above subclass consider the case of dust endowed with rotation so that:

\[
R(\tau) = \kappa \tau^2 \quad T(\tau) = \frac{3\tau^2}{4}
\]  
(77)

where \( \kappa \) is constant and suppose also that \( 8\pi p(0, \tau) = 8\pi p_{RW} \) and \( 8\pi \rho(0, \tau) = 8\pi \rho_{RW} \) in (73) and (74), then (75) to (76) become:

\[
U(\xi, \theta, \tau) = -\frac{207\kappa^4 \tau^3 n^2 \xi^2}{544} - n^2 \tau^2 \xi^4 \sin^2 \theta
\]  
(78)

\[
V(\xi, \theta, \tau) = \frac{n^2 \tau^2 \xi^4 \sin^2 \theta}{2}
\]  
(79)
\[ J(\xi, \theta, \tau) = -\frac{1155\kappa^4\tau^{17}n^2\xi^2}{272}\sin^2 \theta \]  

(80)

\[ Q(\xi, \theta, \tau) = -\frac{139\tau^2n^2\xi^4}{16}\sin^2 \theta - \frac{207\kappa^4\tau^{20}n^2\xi^2}{1088} \]  

(81)

\[ W(\xi, \theta, \tau) = \frac{n^2\kappa^4\tau^6\sin^2 \theta}{64} \left( \frac{945\tau^2\xi^2}{17} - \frac{121\xi^4}{\kappa^4\tau^4} \right) - \frac{837\kappa^4\tau^{20}n^2\xi^2}{1088} \]  

(82)



\[ 8\pi p = \frac{5n^2q^2\sin^2 \theta}{16\kappa^2} \left\{ \frac{159\tau^2\xi^2}{17} + \frac{29\xi^4}{2\tau^4\kappa^4} \right\} + \frac{7715\tau^2n^2q^2\xi^2}{272\kappa^2} \]  

(83)

\[ 8\pi \rho = \frac{4}{3\tau^6\kappa^6} + \frac{3n^2q^2}{4\kappa^2} \sin^2 \theta \left\{ \frac{945\tau^2\xi^2}{136} - \frac{\xi^4}{\tau^4\kappa^4} \right\} + \frac{2915\tau^2n^2q^2\xi^2}{272\kappa^2} \]  

(84)

and

\[ \frac{u^1}{u^4} = -\frac{645\kappa^6\tau^7n^2q^2\xi^3\sin^2 \theta}{272} \]  

(85)

\section{Conclusion}

Einstein’s equations for a Robertson-Walker source endowed with rotation up to and including quadratic terms in angular velocity parameter have been presented. It has been shown that a family of analytic solutions of the equations are possible for the case when \( k = 0 \) and the angular velocity of the fluid is purely time dependent. The corresponding density and supporting internal density are explicitly presented in a form containing perturbations from their respective Robertson-Walker counterparts. A subclass of the solutions merges seamlessly with the Robertson-Walker source at the origin. The work presented here is very much a preliminary investigation and further research is now being conducted since it is possible that further mathematical analysis will reveal new analytic solutions of Einstein’s equations with for example spatially varying fluid angular velocity. Moreover a numerical analysis will be applied to reveal a broader range of properties of the perturbation equations for varying forms of \( h = h(\xi) \) and for \( k = 1, 0, -1 \). Clearly, the determination of rotating sources incorporating gravitational radiation would also be an important development. Further work will also address gauge issues since the analytic solutions presented in this paper have the property that the velocity four vector components have the general property that \( u^2 = 0 \) and \( u^1 \neq 0 \) except at the poles where \( u^1 = 0 \). It would be interesting to consider solutions for which \( u^2 = 0 \) and \( u^1 = 0 \) for all values of \( \xi, \theta \) and \( \tau \).

Whilst perturbation analyses are of considerable importance in the General Theory of Relativity there is no doubt that the major goal for future research must be in the determination of exact solutions of Einstein’s equations for rotating sources with physically realistic properties. According to Bradley et al [11]
there is currently an ‘embarrassing hiatus’ in the availability of such solutions. It is the hope that the perturbation analysis presented above may provide a signpost which lead to the possible discovery of such solutions assuming that they exist.

9 Acknowledgments

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[11] Bradley, M., Fodor, G., Marklund M. & Perjes, Z. 2000. Class. Quant. Grav., 17, 351.
A Expressions terms used in section 4

In the following:

\[ P = \sqrt{1 - k\xi^2} \]  

(A0)

so that:

\[ \Psi_1(\xi, \tau) = \frac{h_\xi h_\xi \xi^2 P^2}{2 \xi^7} - \frac{3 h_\xi^2 P^2}{4 \xi^8} - \frac{h_\xi^2}{2 \xi^8} \]  

(A1)

\[ \Psi_2(\xi, \tau) = -\frac{u_{3\xi\xi} P^2}{2 \xi^4} + \frac{6 u_{3\xi} P^2}{\xi^6} - \frac{30 u_3 P^2}{\xi^9} + \frac{60 u_3 P^2}{\xi^{10}} + \frac{3 u_{3\xi\xi}}{2 \xi^4} + \frac{29 u_{3\xi}}{2 \xi^6} + \frac{u_{1\xi}}{\xi^9} + \frac{42 u_3}{\xi^{10}} - \frac{6 u_1}{\xi^{10}} \]  

(A2)

\[ \Psi_3(\xi, \tau) = \frac{h_\xi}{\xi^3} + \frac{u_{3\xi\xi} P^2}{2 \xi^5} - \frac{5 h^2}{2 \xi^8} - \frac{5 u_{3\xi\xi}}{2 \xi^8} \]  

(A3)

\[ \Psi_4(\xi, \tau) = \frac{h_\xi h_\xi \xi^2 P^4 T}{2 \xi^7} + \frac{h_\xi^2 P^4 T}{2 \xi^4} - \frac{3 h_\xi h_\xi P^4 T}{\xi^9} + \frac{3 h_\xi^2 P^4 T}{\xi^4} - \frac{5 h_\xi h_\xi \xi^2 P^2 T}{2 \xi^5} + \frac{9 h_\xi^2 P^2 T}{2 \xi^4} + \frac{h_\xi h_\xi P^2}{2 \xi^9} + \frac{h_\xi^2 P^2}{\xi^2 R^4} + \frac{u_{3\xi\xi\xi} P^2}{2 \xi^4 R^4} - \frac{10 h_\xi h_\xi P^2}{\xi^3 R^4} - \frac{5 u_{3\xi\xi\xi} P^2}{\xi^3 R^4} + \frac{15 h^2 P^2}{\xi^3 R^4} - \frac{15 u_{3\xi\xi\xi} P^4}{2 \xi^3} + \frac{54 u_{3\xi} P^4}{\xi^4 R^4} + \frac{210 u_{3\xi}}{2 \xi^9} - \frac{130 u_3 P^2}{\xi^6} + \frac{350 u_3 P^2}{\xi^6} + \frac{3 u_{3\xi\xi\xi} P^4}{2 \xi^3} - \frac{79 u_{3\xi\xi\xi} P^2}{2 \xi^4} + \frac{u_{3\xi\xi\xi} P^2}{\xi^4} + \frac{247 u_{3\xi\xi\xi} P^2}{2 \xi^6} + \frac{13 u_{1\xi}}{\xi^6} + \frac{456 u_3 P^2}{\xi^6} + \frac{48 u_1 P^2}{\xi^6} - \frac{3 u_{3\xi\xi\xi}}{2 \xi^4} + \frac{25 u_{3\xi\xi}}{2 \xi^5} + \frac{u_{1\xi}}{\xi^6} - \frac{34 u_3}{\xi^6} + \frac{2 u_1}{\xi^6} \]  

(A4)

\[ \Psi_5(\xi, \tau) = \frac{h_\xi^2 P^2 T}{2 \xi^4} + \frac{2 u_{6\xi\xi} \xi^3 P^2}{R^4} + \frac{2 u_{6\xi} \xi^2 P^2}{R^4} - \frac{2 u_{6\xi} \xi^2}{R^4} - \frac{u_{2\xi\xi} \xi^2}{R^4} + \frac{h_\xi^2}{\xi^4 R^4} + \frac{u_{3\xi\xi} \xi^2 P^2}{\xi^9} - \frac{u_{2\xi}}{\xi^4} - \frac{9 u_{3\xi} P^2}{\xi^4} + \frac{9 u_{3\xi} P^2}{\xi^8} + \frac{24 u_3 P^2}{\xi^8} - \frac{u_{8\xi}}{\xi^8} + \frac{u_{3\xi}}{\xi^8} + \frac{4 u_3}{\xi^8} + \frac{4 u_3}{\xi^8} + 2 u_2 \]  

(A5)