Quasi-Local Energy in Loop Quantum Gravity

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Abstract

Although there is no known meaningful notion of the energy density of the gravitational field in general relativity, a few notions of quasi-local energy of gravity associated to extended but finite domains have been proposed. In this paper, the notions of quasi-local energy are studied in the framework of loop quantum gravity, in order to see whether these notions can be carried out at quantum level. Two basic quasi-local geometric quantities are quantized, which lead to well-defined operators in the kinematical Hilbert space of loop quantum gravity. We then use them as basic building blocks to construct different versions of quasi-local energy operators. The operators corresponding to Brown-York energy, Liu-Yau energy, Hawking energy, and Geroch energy are obtained respectively. The virtue of the Geroch energy operator is beneficial for us to derive a rather general entropy-area relation and thus a holographic principle from loop quantum gravity.

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1 Introduction

It is well known that there are inherent difficulties in defining energy in general relativity (GR), essentially owing to its non-localizability. By now there is no known meaningful notion of the energy density of gravitational field in GR. Globally, for spacetimes which are asymptotically flat, there are well-defined notions for the total energy, given by the Bondi and ADM expressions integrated over spheres at null infinity and spatial infinity. These global notions are directly related to quantities that can be measured physically by distant observers. However, finding an appropriate notion of energy-momentum would be important from the point of view of applications as well. For example, the correct, ultimate formulation of black hole thermodynamics should probably be based on quasi-local defined internal energy, entropy, angular momentum etc. So far, considerable efforts have been put in to formulate a satisfactory definition of quasi-local energy (QLE) (see [1] for a review). In this paper, a few expressions of quasi-local energy are quantized in the framework of loop quantum gravity (LQG) (see [2–5] for reviews). Our purpose is in two folds. Firstly, we want to check whether the quasi-local notions of gravitational energy can be carried out at quantum level. Secondly, we wish to use these notions of quantum gravitational energy to study the relation between quantum gravity and gravitational thermodynamics. A similar

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effect was made in [6] to quantize the Hamiltonian surface term of a bounded spatial region. The candidates of QLE that we are considering include the Brown-York energy [7], Liu-Yau energy [8], Hawking energy [9], and Geroch energy [10]. All of these energy expressions are constituted by two basic quasi-local quantities representing the extrinsic curvatures of a two-sphere in a spatial slice and a timelike slice respectively. Thus the key task is to quantize these two basic building blocks.

In Section 2, the kinematical framework of LQG is briefly introduced. We then construct in section 3 two basic operators in the kinematical Hilbert space of LQG, which corresponding to the two basic building blocks of quasi-local energies. In section 4, the two basic operators are employed to construct different versions of QLE operators. In section 5, the Geroch energy operator is used to derive a rather general entropy-area relation and thus a holographic principle from LQG.

2 Elements of LQG

The Hamiltonian formalism of GR is formulated on a 4-dimensional manifold $M = \mathbb{R} \times \Sigma$, where $\Sigma$ represents a 3-dimensional manifold with arbitrary topology. Introducing Ashtekar-Barbero variables [11, 12], GR can be casted into a $SU(2)$ connection dynamical theory. The phase space consists of canonical pairs $(A_a^i, E_a^i)$ of fields on $\Sigma$, where $A_a^i$ is a connection 1-form which takes values in the Lie algebra $su(2)$, and $E_a^i$ is a vector density of weight 1 which takes value in the dual of $su(2)$. Here $a, b, c...$ are abstract spatial indices and $i, j, k... = 1, 2, 3$ are internal $su(2)$-indices. The density-weighted triad $E_a^i$ is related to the co-triad $e_j^a$ by the relation $E_a^i = \frac{1}{2} \epsilon^{abc} \epsilon_{ijk} e_j^b e_k^c \text{sgn}(\det(e_a^i))$, where $\epsilon^{abc}$ is the naturally defined levi-civita density and $\text{sgn}(\det(e_a^i))$ denotes the sign of $\det(e_a^i)$. The 3-metric on $\Sigma$ is related to the co-triad by $q_{ab} = e_j^a e_j^b \delta_{ij}$. The only non-trivial Poisson bracket is given by

$$\{A_a^i(x), E_b^j(y)\} = \kappa \beta \delta_a^b \delta_{ij} \delta(x, y),$$ (2.1)

where $\kappa = 8\pi G$ ($G$ denotes Newton’s constant) and $\beta$ is the Barbero-Immirzi parameter. There are three first-class constraints in this Hamiltonian formalism of gravity:

$$G_i = \mathcal{D}_a E_a^i = \partial_a E_a^i + \epsilon_{ijk} A_a^j E_a^k,$$
$$V_a = F_{ab}^i E_i^b,$$
$$H = \frac{E_{ab} E^{ab}}{2\kappa \sqrt{\text{det}(q)}} \left[ \epsilon_{ijk} F_{ab}^i - (1 + \beta^2) 2K_{[a}^j K_{b]}^i \right],$$ (2.2)

where $\mathcal{D}_a$ denotes the covariant derivative defined by the connection $A_a^i$, $F_{ab}^i := \partial_a A_b^i - \partial_b A_a^i + \epsilon_{ijk} A_a^j A_b^k$ is the curvature of $A_a^i$, and $K_a^i$ is the extrinsic curvature of $\Sigma$.

One element of LQG is the notion of graphs embedded in $\Sigma$. By $\gamma$ we denote a closed, piecewise analytic graph. The set of edges of $\gamma$ is denoted by $e_{\gamma}$ and the set of vertices of $\gamma$ by $V(\gamma)$. For an oriented edge $e$ of $\gamma$, its beginning point is denoted by $b(e)$ and its final point by $f(e)$. To construct quantum kinematics, one has to extend the configuration
space $\mathcal{A}$ of smooth connections to the space $\tilde{\mathcal{A}}$ of distributional connections. Through projective techniques, $\tilde{\mathcal{A}}$ is equipped with a natural, faithful, ‘induced’ measure $\mu_o$, called Ashtekar-Isham-Lewandowski measure [13, 14]. In certain sense, this measure is the unique diffeomorphism-invariant measure on $\tilde{\mathcal{A}}$ [15]. The kinematical Hilbert space then reads $\mathcal{H}_{\text{kin}} = L^2(\tilde{\mathcal{A}}, d\mu_o)$. The so-called spin-network basis $T_{\gamma,j,m,n}$ provide an orthonormal basis for $\mathcal{H}_{\text{kin}}$ [2].

One of successes of LQG is the rigorous construction of spatial geometrical operators, such as the area, the volume and the length operators in $\mathcal{H}_{\text{kin}}$ [16–18]. Moreover it turns out that these geometrical operators have a discrete spectrum. The same conclusion are also tenable in the internal gauge invariant Hilbert space $\mathcal{H}_o = L^2(\mathcal{A}/\mathcal{G}, d\mu_o)$.

3 Two basic operators for QLE

Most of the quasi-local energy expressions appeared so far involve two quasi-local quantities defined by the integrals of two extrinsic scalar curvatures of some spatial 2-surface [1]. In this section, we will construct two well-defined basic operators corresponding to the two quasi-local quantities in the kinematic Hilbert space $\mathcal{H}_{\text{kin}}$ of LQG, which will be used as basic building blocks to construct different versions of quasi-local energy operators in the next section. To this aim, we have to first re-express the two quasi-local quantities in terms of real connection variable or its conjugate. Then we regulate the classical expressions in order to get quantities with quantum analogues. It turns out that in the regularization procedure, as the regularization of the Hamiltonian constraint, we need to triangulate the 3-d spatial manifold $\Sigma$ in adaption to a graph, which comes from the cylindrical function in $\mathcal{H}_{\text{kin}}$ that is going to be acted by the constructed operators.

3.1 QLE-like operator

Let $S$ be a 2-d surface with two-sphere topology in the 3-d spatial manifold $\Sigma$ and $\sigma_{ab}$ be the induced metric on $S$ of metric $q_{ab}$ on $\Sigma$. For simplicity, we choose adapted coordinates $\{x^1, x^2, x^3\}$ in $\Sigma$ such that $S$ is given by $x^3 = 0$, and $x^1, x^2$ parameterize $S$. The QLE-like observable is defined as

$$E_{Q,k}(S) := -\frac{1}{\kappa} \int_S d^2x \sqrt{\det(\sigma)} \, k,$$

where $k$ is the extrinsic scalar curvature of $k_{ab}$ of $S$ corresponding to the unit normal $n^a$ in $\Sigma$. In order to quantize the expression, we need first to express it in terms of the real Ashtekar variables. The extrinsic curvature tensor $k_{ab}$ of $S$ corresponding to $n^a$ reads

$$k_{ab} = \sigma_{ac}D_c n_b,$$

where $D_a$ is the derivative operator on $\Sigma$ compatible with $q_{ab}$, i.e., $D_a q_{bc} = 0$. So the extrinsic scalar curvature $k$ of $S$ is

$$k = D_a n^a.$$
From (3.3), we have the following identity in the adapted coordinates.

\[ k = D_a \left( \frac{E_i^a E_i^b}{\det(q) n_b} \right) = \frac{1}{\sqrt{\det(q)}} \partial_a \left( \frac{E_i^a E_i^b}{\sqrt{\det(q)}} \right) = \frac{1}{\sqrt{\det(q)}} \partial_a \left( \frac{E_i^a}{\sqrt{\det(\sigma)}} \right). \quad (3.4) \]

To regularize \( E_{Q,k}(S) \), let \( \epsilon \) be a small number and \( \chi^3_{\epsilon}(x, y) \) be the (smoothed out) characteristic function such that \( \lim_{\epsilon \to 0} \chi^3_{\epsilon}(x, y)/\epsilon^3 = \delta^3(x, y) \). The volume of the cube as measured by \( q_{ab} \) is given by \( V(x, \epsilon) := \int d^3y \chi^3_{\epsilon}(x, y) \sqrt{\det(q)}(y) \) such that \( \lim_{\epsilon \to 0} \frac{V(x, \epsilon)}{\epsilon^3} = \sqrt{\det(q)}(x) \).

We choose again adapted coordinates \( \{y^1, y^2, y^3\} \) in \( \Sigma \) such that \( S \) is given by \( y^3 = 0 \) and each 2-d surface \( y_{\epsilon^3} \) of the family given by \( y^3 = \text{constant} \) is parameterized by \( y^1, y^2 \). We denote also the induced metric of \( q_{ab} \) in \( y_{\epsilon^3} \) by \( \sigma_{ab} \). We then have the following identity by Eqs. (3.1) and (3.4)

\[ E_{Q,k}(S) = -\frac{1}{\kappa} \int_S d^2x \sqrt{\det(\sigma)} \partial_a \left( \frac{E_i^3 E_i^a}{\sqrt{\det(\sigma)}} \right) \]

\[ = -\lim_{\epsilon \to 0} \frac{1}{\kappa} \int_S d^2x \sqrt{\det(\sigma)}(x) \int dy^3 \int_{y_{\epsilon^3}} d^1y^2 \frac{\chi^3_{\epsilon}(x, y)}{\epsilon^3} \partial_a \left( \frac{E_i^3 E_i^a}{\sqrt{\det(\sigma)}}(y) \right) \]

\[ \times \int \Sigma d^3u \frac{\chi^3_{\epsilon}(x, u)}{\epsilon^3} \frac{1}{\sqrt{\det(q)}(u)} \]

\[ = -\lim_{\epsilon \to 0} \frac{1}{\kappa} \int_S d^2x \sqrt{\det(\sigma)}(x) \int dy^3 \int_{y_{\epsilon^3}} d^1y^2 \frac{\chi^3_{\epsilon}(x, y)}{\epsilon^3} \partial_a \left( \frac{E_i^3 E_i^a}{\sqrt{\det(\sigma)}}(y) \right) \]

\[ \times \int \Sigma d^3u \frac{\chi^3_{\epsilon}(x, u)}{\epsilon^3} \frac{[\det(e^i_j)](u)}{[\sqrt{V(u, \epsilon)}]^3} \int \Sigma d^3w \frac{\chi^3_{\epsilon}(x, w)}{[\sqrt{V(w, \epsilon)}]^3} \]

\[ = \lim_{\epsilon \to 0} \frac{1}{\kappa} \int_S d^2x \sqrt{\det(\sigma)}(x) \int dy^3 \int_{y_{\epsilon^3}} d^1y^2 [\partial_a \chi^3_{\epsilon}(x, y)] \left( \frac{E_i^3 E_i^a}{\sqrt{\det(\sigma)}}(y) \right) \]

\[ \times \int \Sigma d^3u \frac{\chi^3_{\epsilon}(x, u)}{\epsilon^3} \frac{[\det(e^i_j)](u)}{[\sqrt{V(u, \epsilon)}]^3} \int \Sigma d^3w \frac{\chi^3_{\epsilon}(x, w)}{[\sqrt{V(w, \epsilon)}]^3}, \quad (3.5) \]

where we have inserted the identity \( 1 = [\det(e^i_j)]^2/[\sqrt{\det(q)} ]^2 \) in the third step, and performed an integration by parts in the last step. Let \( \chi^2_{\epsilon}(y, z) \) is the 2-d characteristic function of a coordinate box with center \( y \) and coordinate area \( \epsilon^2 \) and \( Ar(z, \epsilon') := \int_{y_{\epsilon^3}} d^3w \chi^2_{\epsilon}(z, w) \sqrt{\det(\sigma)}(w) \), satisfying \( \lim_{\epsilon' \to 0} \frac{Ar(z, \epsilon')}{\epsilon'^2} = \sqrt{\det(\sigma)}(z) \), is the area of the box as measured by \( \sigma_{ab} \). Then we have

\[ \frac{E_i^3}{\sqrt{\det(\sigma)}}(y^1, y^2, y^3) = \lim_{\epsilon' \to 0} \int_{y_{\epsilon^3}} d^2z \frac{\chi^2_{\epsilon}(y^1, y^2; z^1, z^2)}{\epsilon^2} \frac{E_i^3}{\sqrt{\det(\sigma)}}(z^1, z^2, y^3) \]

\[ = \lim_{\epsilon' \to 0} \int_{y_{\epsilon^3}} d^2z \frac{\chi^2_{\epsilon}(y^1, y^2; z^1, z^2) E_i^3(z^1, z^2, y^3)}{Ar(z^1, z^2, y^3; \epsilon')} \].
Thus we can rewrite Eq. (3.5) as

\[
E_{Q,k}(S) = \lim_{\epsilon, \epsilon' \to -K} \frac{1}{\kappa} \int_S d^2 x \sqrt{\det(\sigma)}(x) \\
\times \int \sum d^3 y \left[ \int_{S_{y}} d^2 z \chi^2_i(y^1, y^2; z^1, z^2) \frac{E^3_i(z^1, z^2, y^3)}{\operatorname{Ar}(z^1, z^2, y^3; \epsilon')} \right] \left[ \partial_a \chi^3_i(x, y) \right] E^a_i(y) \\
\times \int \sum d^3 u \chi^3_i(x, u) \left[ \det(e_i^1) \right] (u) \int \sum d^3 w \chi^3_i(x, w) \left[ \det(e_i^1) \right] (w), \tag{3.6}
\]

Recall the following classical identities

\[
\int \sum d^3 u \left[ \det(e_i^1) \right] (u) = \frac{1}{3!} \int \sum \epsilon_{ijk} e^i \wedge e^j \wedge e^k = \frac{-4}{3!} \int \sum \operatorname{Tr}(e \wedge e \wedge e), \tag{3.7}
\]

and

\[
\epsilon_i^a(u) = \frac{2}{\kappa} \{ A_i^a(u), V(u, \epsilon) \}, \tag{3.8}
\]

where \(\epsilon_{ijk} = -\frac{1}{2} \operatorname{tr}(\tau_i \tau_j \tau_k)\), \(e = e^i \tau_i / 2\), here \(\tau_i = -i \sigma_i\) (\(\sigma_i\) is the Pauli matrix) is the generator of \(su(2)\) obeying \([\tau_i, \tau_j] = 2 \epsilon_{ijk} \tau_k\). We can rewrite Eq. (3.6) as

\[
E_{Q,k}(S) = \frac{2^{14}}{9 \kappa^2} \lim_{\epsilon, \epsilon' \to 0} \int_S d^2 x \sqrt{\det(\sigma)}(x) \\
\times \int \sum d^3 y \left[ \int_{S_{y}} d^2 z \chi^2_i(y^1, y^2; z^1, z^2) \frac{E^3_i(z^1, z^2, y^3)}{\operatorname{Ar}(z^1, z^2, y^3; \epsilon')} \right] \left[ \partial_a \chi^3_i(x, y) \right] E^a_i(y) \\
\times \int \sum \chi^3_i(x, u) \operatorname{Tr} \left( \frac{\{ A(u), V(u, \epsilon) \} \wedge \{ A(u), V(u, \epsilon) \} \wedge \{ A(u), V(u, \epsilon) \}}{\sqrt{V(u, \epsilon)}} \right) \\
\times \int \sum \chi^3_i(x, w) \operatorname{Tr} \left( \frac{\{ A(w), V(w, \epsilon) \} \wedge \{ A(w), V(w, \epsilon) \} \wedge \{ A(w), V(w, \epsilon) \}}{\sqrt{V(w, \epsilon)}} \right) \\
\times \int \sum \chi^3_i(x, u) \operatorname{Tr} \left( \frac{\{ A(u), \sqrt{V(u, \epsilon)} \} \wedge \{ A(u), \sqrt{V(u, \epsilon)} \} \wedge \{ A(u), \sqrt{V(u, \epsilon)} \}}{\sqrt{V(u, \epsilon)}} \right) \\
\times \int \sum \chi^3_i(x, w) \operatorname{Tr} \left( \frac{\{ A(w), \sqrt{V(w, \epsilon)} \} \wedge \{ A(w), \sqrt{V(w, \epsilon)} \} \wedge \{ A(w), \sqrt{V(w, \epsilon)} \}}{\sqrt{V(w, \epsilon)}} \right) \\
= \lim_{\epsilon, \epsilon' \to 0} E^{\epsilon, \epsilon'}_{Q,k}(S), \tag{3.9}
\]
where we have used \( \{ \cdot, \sqrt{V(u, \epsilon)} \} = \{ \cdot, V(u, \epsilon) \}/(2\sqrt{V(u, \epsilon)}) \). Since a typical state \( f_\gamma \in H_{kin} \) is some cylindrical function over a graph \( \gamma \) in \( \Sigma \), as in the construction of the Hamiltonian constraint operator [2, 19], we triangulate \( \Sigma \) in adaption to \( \gamma \) as follows. At every vertex \( v \in V(\gamma) \) we choose a triple \((e_I, e_J, e_K)\) of edges of \( \gamma \) and a tetrahedron \( \Delta^\epsilon_{\gamma,v,e_I,e_J,e_K} \) based at \( v \) which is spanned by segments \( s_I, s_J, s_K \) of the triple. Each segment \( s_I \) is given by the part with the curve parameter \( t^I \in [0, \epsilon] \) of the corresponding edge \( e_I(t^I) \). The holonomy of the connection along a segment \( s_I \) reads

\[
h_{s_I}(A) = \frac{1}{2} + \epsilon \dot{s}_I^a(0) A^a_u(v) \tau_i/2 + O(\epsilon^2),
\]

and for one segment \( s_I \), we have

\[
\int_{s_I} \{ A(u), \sqrt{V(u, \epsilon)} \} \approx \epsilon \dot{s}_I^a(0) \{ A_u(v), \sqrt{V(u, \epsilon)} \}
\]

up to \( O(\epsilon^2) \). Hence for each \( \Delta^\epsilon_{\gamma,v,e_I,e_J,e_K} \), we have

\[
\int_{\Delta^\epsilon_{\gamma,v,e_I,e_J,e_K}} \text{Tr} \left( \left\{ A(u), \sqrt{V(u, \epsilon)} \right\} \wedge \left\{ A(u), \sqrt{V(u, \epsilon)} \right\} \wedge \left\{ A(u), \sqrt{V(u, \epsilon)} \right\} \right) \\
\approx -\frac{1}{6} \epsilon(s_I s_J s_K) \epsilon^{IJK} \text{Tr} \left( h_{s_I}(\Delta) \left\{ h^{-1}_{s_I}(\Delta), \sqrt{V(v(\Delta), \epsilon)} \right\} h_{s_J}(\Delta) \left\{ h^{-1}_{s_J}(\Delta), \sqrt{V(v(\Delta), \epsilon)} \right\} \\
x h_{s_K}(\Delta) \left\{ h^{-1}_{s_K}(\Delta), \sqrt{V(v(\Delta), \epsilon)} \right\} \right),
\]

where \( \epsilon(s_I s_J s_K) := \text{sgn}(\text{det}(\dot{s}_I \dot{s}_J \dot{s}_K(v))) \) takes the values \(+1, -1, 0\) if the tangents of the three segments \( s_I, s_J, s_K \) at \( v \) (in that sequence) form a matrix of positive, negative or vanishing determinant. Then the integration over \( \Sigma \) can be split as follows [2, 19]:

\[
\int_{\Sigma} \approx \int_{\tilde{U}_\gamma} + \sum_{v \in V(\gamma)} \int_{\tilde{U}_{\gamma,v}} = \int_{\tilde{U}_\gamma} + \sum_{v \in V(\gamma)} \frac{1}{E(v)} \sum_{b(e_I) \cap b(e_J) \cap b(e_K) = v} \left[ \int_{\tilde{U}_{\gamma,v,e_I,e_J,e_K}} \int_{\tilde{U}_{\gamma,v,e_I,e_J,e_K}} \right] \\
\approx \int_{\tilde{U}_\gamma} + \sum_{v \in V(\gamma)} \frac{1}{E(v)} \sum_{b(e_I) \cap b(e_J) \cap b(e_K) = v} \left[ 8 \cdot \int_{\Delta^\epsilon_{\gamma,v,e_I,e_J,e_K}} + \int_{\tilde{U}_{\gamma,v,e_I,e_J,e_K}} \right].
\]

Here we have first decomposed \( \Sigma \) into a region \( \tilde{U}_\gamma \) not containing the vertices of \( \gamma \) and the regions \( U_{\gamma,v} \) around the vertices. Then choose a triple \((e_I, e_J, e_K)\) of edges outgoing from \( v \) and decompose \( U_{\gamma,v} \) into the region \( U_{\gamma,v,e_I,e_J,e_K} \) covered by the tetrahedron \( \Delta^\epsilon_{\gamma,v,e_I,e_J,e_K} \) spanned by \( e_I, e_J, e_K \) and its 8 mirror images and the rest \( \tilde{U}_{\gamma,v,e_I,e_J,e_K} \) not containing \( v \). Note that the integral over \( U_{\gamma,v,e_I,e_J,e_K} \) classically converges to \( 8 \) times the integral over the original single tetrahedron \( \Delta^\epsilon_{\gamma,v,e_I,e_J,e_K} \) as we shrink the tetrahedron to zero. We average over all such triples \((e_I, e_J, e_K)\) and divide by the number of possible choices of triples for a vertex \( v \) with \( n(v) \) edges, \( E(v) = \binom{n(v)}{3} \). We can now decompose the \( u \) and \( w \) integration over \( \Sigma \) in Eq. (3.9) according to Eq. (3.12). To quantize the two integration, we replace
Poisson brackets by commutators times $1/(i\hbar)$, holonomies by multiplication and $V$ by the volume operator $\hat{V}$, which acts on a function cylindrical over a graph $\gamma$ as follows [17]:

$$\hat{V}(R)f_{\gamma} = (\hbar\kappa\beta)^{3/2} \sum_{e\in V(\gamma)\cap R} \left[ \frac{i}{3!} \cdot \frac{8}{36} \sum_{e''n'\cap e'' = v} \epsilon(e, e', e'') \epsilon_{ijk} X_i^e X_j^e X_k^e \right]. \quad (3.13)$$

Because the non-vanishing contributions of $\hat{V}$ acting on a cylindrical function $f_{\gamma}$ come from the vertices $v \in V(\gamma)$, only the integration over the tetrahedra $\Delta_{e',v,e',1,e,e,k}$ needs to be considered. Hence the resulted operator corresponding the last two integrals in Eq. (3.9) acts on a cylindrical function as

$$\frac{1}{36i\hbar^6} \sum_{v'\in V(\gamma)} \chi_9^3(x, v') \frac{8}{E(v')} \sum_{v(\Delta') = v'} \epsilon(s_J s_J s_K) \epsilon^{JK} \text{Tr}(h_{s_J(\Delta)}^{-1} \hat{V}(v', \epsilon) h_{s_K(\Delta)}^{-1} \hat{V}(v', \epsilon)) \times h_{s_J(\Delta)}^{-1} \hat{V}(v', \epsilon) h_{s_K(\Delta)}^{-1} \hat{V}(v', \epsilon)) \times \sum_{v''\in V(\gamma)} \chi_9^3(x, v'') \frac{8}{E(v'')} \sum_{v(\Delta'') = v''} \epsilon(s_L s_M s_N) \epsilon^{LMN} \text{Tr}(h_{s_L(\Delta')}^{-1} \hat{V}(v'', \epsilon) h_{s_N(\Delta')}^{-1} \hat{V}(v'', \epsilon)) \cdot f_{\gamma}. \quad (3.14)$$

We now come to the quantization of the second integral in Eq. (3.9). Given a graph $\gamma$ and a 2-surface $S$, we can change the orientations of some edges of $\gamma$ and subdivide edges of $\gamma$ into two halves at an interior point if necessary, and obtain a graph $\gamma_S$ adapted to $S$ such that the edges of $\gamma_S$ belong to the following four types [2]: (i) $e$ is the up type edge if $e \cap S = b(e)$ and $e^a(0)n_a(e(0)) > 0$ where $n_a$ is the co-vector field normal to $S$; (ii) $e$ is the down type edge if $e \cap S = b(e)$ and $e^a(0)n_a(e(0)) < 0$; (iii) $e$ is the inside type edge if $e \cap S = e$; (iv) $e$ is the outside type edge if $e \cap S = \emptyset$. In the following, we only use the graphs adapted to some 2-surfaces. For convenience, we will abbreviate the coordinate $(z^1, z^2, y^3)$ of a point in 2-d surface $S_{y^3}$ as $(z, y^3)$. In a suitable operator-ordering, that integral can be quantized as an operator acting on a cylindrical function as follows:

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\delta \sum_{S_{y^3}} \left[ \int_{S_{y^3}} d^2z \frac{\chi_9^3(y^1, y^2; z^1, z^2)}{Ar(z, y^3; e')} \hat{E}_i^3(z^1, z^2, y^3) \right] \left[ \partial_a \chi_9^3(x, y) \right] \hat{E}_i^a(y) \cdot f_{\gamma}$$

$$= \frac{(-i\hbar\kappa\beta)^2}{8} \sum_{e \in E(\gamma)} \int_{\Sigma} d^3y \sum_{e' \in E(\gamma), e'(0) \in S_{y^3}} g(e', S_{y^3}) \chi_9^3(y^1, y^2; e'(0)) \frac{\partial_a \chi_9^3(x, y)}{Ar(e'(0), e')} X_i^e(0)$$

$$\times \left[ \partial_a \chi_9^3(x, y) \right] \int_0^1 dt \hat{e}^a(t) \delta^3(y, (e(t))) X_i^e(t) \cdot f_{\gamma}$$

$$= \frac{(-i\hbar\kappa\beta)^2}{8} \sum_{e \in E(\gamma)} \lim_{n \to \infty} \sum_{k=1}^n \left[ \chi_9^3(x, e(t_k)) - \chi_9^3(x, e(t_{k-1})) \right]$$

$$\sum_{e \in E(\gamma)} \lim_{n \to \infty} \sum_{k=1}^n \left[ \chi_9^3(x, e(t_k)) - \chi_9^3(x, e(t_{k-1})) \right]$$
where \( 0 = t_0 < t_1 < \ldots < t_n = 1 \) is an arbitrary partition of the interval \([0, 1]\), \(X^i_e(t) := [h_e(0, t) \tau_t h_e(t, 1)]_{AB} \partial/\partial [h_e(0, 1)]_{AB} \) (we denote \(X^i_e := X^i_e(0)\) in the following) and

\[
\varrho(e, S) = \begin{cases} 
+1, & \text{if } e \text{ is of the up type with respect to } S; \\
-1, & \text{if } e \text{ is of the down type with respect to } S; \\
0, & \text{if } e \text{ is of the inside or outside type with respect to } S.
\end{cases}
\]

Let us introduce the set of isolated intersection points of \(\gamma\) and \(S\)

\[
P(\gamma, S) := \{e \cap S | \varrho(e, S) \neq 0, e \in E(\gamma)\}.
\]

The first integral in Eq. (3.9) can be quantized straightforwardly, since \(\sqrt{\det(\sigma)}(x)\) is given by [4,16]

\[
\sqrt{\det(\sigma)}(x) \cdot f_\gamma = \frac{1}{4} \hbar \kappa \beta \sum_{v \in P(\gamma, S)} \delta^2(x, v) \sqrt{-\sum_{e,e' \in E(\gamma); b(e) = b(e') = v} \varrho(e, e') X^i_e X^j_e \cdot f_\gamma} =: \sum_{v \in P(\gamma, S)} \delta^2(x, v) \hat{A}r_v \cdot f_\gamma,
\]

where \(\varrho(e, e') := \varrho(e, S) \varrho(e', S)\). Putting Eqs. (3.14), (3.15) and (3.17) together, we finally obtain the regularized operator corresponding to Eq. (3.9), acting on a cylindrical function as

\[
\hat{E}_{Q,k}(S) \cdot f_\gamma = \frac{2^n \beta^2}{81 \hbar^4 \kappa^5} \sum_{v \in P(\gamma, S)} \hat{A}r_v \sum_{e \in E(\gamma)} \sum_{k=1}^n \left[ \chi^3_e(v, e(t_k)) - \chi^3_e(v, e(t_{k-1})) \right] \\
\times \sum_{e' \in E(\gamma), e'(0) = S_e^3(t_{k-1})} \varrho(e', S_e^3(t_{k-1})) \frac{X^i_e(e_1(t_{k-1}), e_1(t_{k-1})); e'(0))}{\hat{A}r(e'(0), e')} X^i_e(0) X^i_e(0) \\
\times \sum_{v' \in V(\gamma)} \frac{8}{E(v')} \sum_{v(\Delta) = v'} \epsilon(s_{IJ} s_{JK}) \epsilon^{JK} \text{Tr} \left[ h_{sJK}(\Delta) \left[ h_{sJK}^{-1}(\Delta), \sqrt{\hat{V}'(v', \epsilon)} \right] \right] \\
\times h_{sJK}(\Delta) \left[ h_{sJK}^{-1}(\Delta), \sqrt{\hat{V}'(v', \epsilon)} \right] h_{sJK}(\Delta) \left[ h_{sJK}^{-1}(\Delta), \sqrt{\hat{V}'(v', \epsilon)} \right] \\
\times \sum_{v'' \in V(\gamma)} \frac{8}{E(v'')} \sum_{v(\Delta) = v''} \epsilon(s_{LM} s_{SN}) \epsilon^{LM} \text{Tr} \left[ h_{sLM}(\Delta) \left[ h_{sLM}^{-1}(\Delta), \sqrt{\hat{V}'(v'', \epsilon)} \right] \right]
\]

(3.15)
Now we perform the limit $n \to \infty$, $\epsilon' \to 0$ and $\epsilon \to 0$ in reversed order. Keeping $n$ fixed, for small enough $\epsilon$, only the term with $k = 1$ in the sum survives provided that $b(\epsilon) = v$, and only terms with $v = v' = v''$ contribute. So for small enough $\epsilon$, the above operator reduces to

$$
\hat{E}_{Q,k}^n(S) \cdot f_\gamma = -\frac{2^9 \beta^2}{81 \hbar^4 k^5} \sum_{v \in P(\gamma,S)} \hat{A} r_v
\times \sum_{e' \in E(\gamma), e'(0) \in S_{\mathcal{E}(t_0) = S}} \varrho(e', S_{e'(0)}) \frac{\chi^2_e(v; e'(0))}{\hat{A} r(e'(0), e')} \left( X_i^e(0) \sum_{b(e) = v} X_i^e(0) \right)
\times \frac{8}{E(v)} \sum_{v(\Delta) = v} \epsilon(s I s J s K) \epsilon^{IJK} \text{Tr} \left( h_{s I(\Delta)} \left[ h_{s J(\Delta)}^{-1} \sqrt{\hat{V}(v, \epsilon)} \right] h_{s K(\Delta)} \left[ h_{s K(\Delta)}^{-1} \sqrt{\hat{V}(v, \epsilon)} \right] \right)
\times h_{s M(\Delta')} \left[ h_{s M(\Delta')}^{-1} \sqrt{\hat{V}(v, \epsilon)} \right] h_{s N(\Delta')} \left[ h_{s N(\Delta')}^{-1} \sqrt{\hat{V}(v, \epsilon)} \right] \cdot f_\gamma.
$$

For small enough $\epsilon'$, the function $\chi^2_e(v; e'(0))$ vanishes unless $v = e'(0)$. Hence the above regularized operator reduces to

$$
\hat{E}_{Q,k}^{n,\epsilon'}(S) \cdot f_\gamma = -\frac{2^{11} \beta^2}{81 \hbar^4 k^5} \sum_{v \in P(\gamma,S)} \hat{A} r_v \frac{1}{\hat{A} r(v, \epsilon')} \sum_{b(e') = v} \varrho(e', S) X_i^e \sum_{b(e) = v} X_i^e
\times \frac{8}{E(v)} \sum_{v(\Delta) = v} \epsilon(s I s J s K) \epsilon^{IJK} \text{Tr} \left( h_{s I(\Delta)} \left[ h_{s J(\Delta)}^{-1} \sqrt{\hat{V}(v, \epsilon)} \right] h_{s K(\Delta)} \left[ h_{s K(\Delta)}^{-1} \sqrt{\hat{V}(v, \epsilon)} \right] \right)
\times h_{s M(\Delta')} \left[ h_{s M(\Delta')}^{-1} \sqrt{\hat{V}(v, \epsilon)} \right] h_{s N(\Delta')} \left[ h_{s N(\Delta')}^{-1} \sqrt{\hat{V}(v, \epsilon)} \right] \cdot f_\gamma.
$$

Notice that $\varrho(e', S)$ implies that the edges inside $S$ have no contribution to the operation, while the action of area operator $\hat{A} r(v, \epsilon')$ on the edges transversal to $S$ is non-vanishing. Hence $1/\hat{A} r(v, \epsilon')$ is well defined. Thus one can take the limits and obtain an operator as

$$
\hat{E}_{Q,k}(S) \cdot f_\gamma = -\frac{2^{11} \beta^2}{81 \hbar^4 k^5} \sum_{v \in P(\gamma,S)} \sum_{b(e') = v} \varrho(e', S) X_i^e \sum_{b(e) = v} X_i^e
$$
\[ \times \frac{8}{E(v)} \sum_{v(\Delta)=v} \epsilon(s_{ISJSK}) \epsilon^{iJK} \text{Tr} \left( h_{sJ(\Delta)} \left[ h_{sJ(\Delta)}^{-1}, \sqrt{V_v} \right] \right) \]
\[ \times h_{sJ(\Delta)} \left[ h_{sJ(\Delta)}^{-1}, \sqrt{V_v} \right] h_{sK(\Delta)} \left[ h_{sK(\Delta)}^{-1}, \sqrt{V_v} \right] \]
\[ \times \frac{8}{E(v)} \sum_{v(\Delta)=v} \epsilon(s_{LMNS}) \epsilon^{LMN} \text{Tr} \left( h_{sL(\Delta')} \left[ h_{sL(\Delta')}^{-1}, \sqrt{V_{v'}} \right] \right) \]
\[ \times h_{sM(\Delta')} \left[ h_{sM(\Delta')}^{-1}, \sqrt{V_{v'}} \right] h_{sN(\Delta')} \left[ h_{sN(\Delta')}^{-1}, \sqrt{V_{v'}} \right] \]
\[ =: \sum_{v \in P(\gamma,S)} \hat{E}_{Q,K,v} \cdot f_{\gamma}. \quad (3.19) \]

Let \( E_{v,s}(\gamma) = \{ e \in E(\gamma); v = b(e); e = \ast \text{type} \} \) where \( \ast = u, d, i \) for ‘up, down, inside’ with respect to \( S \) respectively, and let \( X^i_{v,s} = \sum_{e \in E_{v,s}} X^i_e \). Then one can check the commutation relation [2]

\[ [X^i_{v,s}, X^j_{v',s'}] = -2\epsilon_{ijk} X^k_{v,s} \delta_{v,v'} \delta_{s,s'}. \quad (3.20) \]

Hence one has

\[ \sum_{b(e')=v} g(e',S) X^i_{e'} \sum_{b(e)=v} X^i_e = (X^i_{v,u} - X^i_{v,d})(X^i_{v,u} + X^i_{v,d} + X^i_{v,i}) \]
\[ = \sum_{b(e)=v} X^i_e \sum_{b(e')=v} g(e',S) X^i_{e'}. \quad (3.21) \]

Thus there is no operator-ordering problem for these two operators. Moreover, the operator
\[ \sum_{b(e')=v} g(e',S) X^i_{e'} \sum_{b(e)=v} X^i_e \] is gauge invariant since

\[ \left[ \sum_{b(e')=v} g(e',S) X^i_{e'} \sum_{b(e)=v} X^i_e \sum_{b(e'')=v} X^j_{e''} \right] \]
\[ = [(X^i_{v,u} - X^i_{v,d})(X^i_{v,u} + X^i_{v,d} + X^j_{v,i}), X^j_{v,u} + X^j_{v,d} + X^j_{v,i}] = 0. \quad (3.22) \]

Therefore our quasi-local energy operator \( \hat{E}_{Q,K}(S) \) in (3.19) is gauge invariant. Moreover, we can also define a symmetric quantum version of \( E_{Q,K}(S) \) as

\[ \hat{E}_{Q,K}^s(S) := \frac{1}{2} \left( \hat{E}_{Q,K}(S) + \hat{E}_{Q,K}^\dagger(S) \right). \quad (3.23) \]

A special property of the \( \hat{E}_{Q,K}(S) \) (or \( \hat{E}_{Q,K}^s(S) \)) is immediately clear. Because \( \sum_{b(e)=v} X^i_e \) generates the internal gauge transformations, \( \hat{E}_{Q,K}(S) \) (or \( \hat{E}_{Q,K}^s(S) \)) vanishes on gauge-invariant states. Note that the quantization of \( E_{Q,K}(S) \) is not unique. An alternative quantization with a partial gauge fixing is given in Appendix A.
3.2 Quasi-local normal momentum operator

The so-called normal-directional momentum of $S$ in adapted coordinates can be expressed as [20]

$$J_{Q,l}(S) := \frac{1}{\kappa} \int_S d^2x \sqrt{\det(\sigma)} \ l,$$

where $l$ is the trace of $l_{ab}$, which is the extrinsic curvature tensor of $S$ with respect to its unit normal $u^a$ orthogonal to $\Sigma$, i.e.,

$$l = \sigma^{ab} l_{ab} = \sigma^{ab} \nabla_a u_b.$$  \hfill (3.25)

Let $Ar(S)$ be the area of a closed 2-surface $S$. The Euclidean Hamiltonian constraint with lapse function $N = 1$ reads

$$H^E(1) = \frac{1}{2\kappa} \int_\Sigma d^3x \ \frac{1}{\sqrt{\det(q)}} \ \epsilon_{ijk} F_{ai}^b E_j^a E_k^b.$$  \hfill (3.26)

Then we have (see Appendix B for a proof)

$$J_{Q,l}(S) = \frac{1}{\kappa} \{H^E(1), Ar(S)\}.$$  \hfill (3.27)

Thus the quasi-local momentum $J_{Q,l}(S)$ of $S$ can be regarded as the “time derivative” of the area of $S$. Since there are densely defined operators corresponding to $H^E(1)$ and $Ar(S)$ in $H_{\text{kin}}$, we may replace Poisson brackets by commutators times $1/(i\hbar)$, functions by operators, and obtain the operator $\hat{J}_{Q,l}(S)$. However, there are two quantum versions of $H^E(1)$ in the literature, the graph-changing one [19,21] and the no-graph-changing one [22]. It turns out that the no-graph-changing version is more convenient in defining the quantum version of $\hat{J}_{Q,l}(S)$.

We now introduce the minimal loop prescription [23]. Given a vertex $v$ of $\gamma$ and two different edges $e_i, e_j$ incident at and outgoing from $v$, a loop $\alpha_{ij}$ within $\gamma$ starting at $v$ along $e_i$ and ending at $v$ along $e_j^{-1}$ is said to be minimal provided that there is no other loop within $\alpha_{ij}$ satisfying the same restrictions with fewer edges traversed. We denote by $L(v, e_i, e_j)$ the set of minimal loops with the data indicated. The non-graph-changing symmetric operator $\hat{H}^E(1)$ acts on a cylindrical function as [22]

$$\hat{H}^E(1) \cdot f_\gamma = \frac{1}{3i\hbar \kappa^2 \beta} \sum_{v \in V(\gamma)} \sum_{e_i, e_j, e_k = v} \epsilon_{ijk} \epsilon(e_i, e_j, e_k) \frac{1}{|L(v, e_i, e_j)|} \times \sum_{\alpha_{ij} \in L(v, e_i, e_j)} \text{Tr} \left( \left\{ h_{\alpha_{ij}}, h_{e_k} [h_{e_k}^{-1}, V_v] \right\} \right) \cdot f_\gamma$$

$$=: \sum_{v \in V(\gamma)} \hat{H}_v^E \cdot f_\gamma,$$  \hfill (3.28)
where \( \{ \cdot, \cdot \} \) denotes the anti-commutator.

Hence it is easy to obtain a well-defined operator corresponding to \( J_{Q,l}(S) \) as

\[
\hat{J}_{Q,l}(S) = \frac{1}{i\hbar\kappa} [\hat{H}^E(1), \hat{A}r(S)].
\] (3.29)

It is easy to show that \( \hat{J}_{Q,l}(S) \) is a gauge-invariant, diffeomorphism-covariant and symmetric operator. For later purposes we write a more explicit expression for \( \hat{J}_{Q,l}(S) \) operating on a cylindrical function as

\[
\hat{J}_{Q,l}(S) \cdot f_\gamma = \frac{1}{i\hbar\kappa} \sum_{v' \in V(\gamma), v \in P(\gamma,S)} [\hat{H}^E_{v'}, \hat{A}r_v] \cdot f_\gamma = \frac{1}{i\hbar\kappa} \sum_{v \in P(\gamma,S)} [\hat{H}^E_v, \hat{A}r_v] \cdot f_\gamma
\]

\[
= : \sum_{v \in P(\gamma,S)} \hat{J}_{Q,l,v} \cdot f_\gamma,
\] (3.30)

where in the first step we used the fact that \( \hat{H}^E(1) \) is a non-graph-changing operator, and in the second step we exploited that \( \hat{A}r_v \) only acts on the edges incident at \( v \in P(\gamma,S) \) so that the commutator with \( \hat{H}^E_{v'} \), which contains only holonomies of edges incident at \( v' \), vanishes if \( v' \neq v \).

4 QLE operators

In this section, we will use the two well-defined operators \( \hat{E}_{Q,k}(S) \) and \( \hat{J}_{Q,l}(S) \) constructed in the last section as building blocks to quantize several types of QLE expressions.

4.1 Brown-York energy operator

The system under consideration is a spatial three-surface \( \Sigma \) bounded by a two-surface \( S \) in a spacetime region that can be decomposed as a product of a spatial three-surface and a real line-interval representing time. Suppose that the 2-metric \( \sigma_{ab} \) induced on \( S \) has positive scalar curvature. Then by the embedding theorem there is a unique isometric embedding of \( (S, \sigma_{ab}) \) into the flat 3-space. Let \( k_o \) be the trace of extrinsic curvature of \( S \) in this embedding, which is completely determined by \( \sigma_{ab} \) and is necessarily positive. The time evolution of the two-surface boundary \( S \) is the timelike three-surface boundary \( ^3B \).

Brown and York defined their QLE by the Hamiltonian-Jacobi method as [7]:

\[
E_{BY}(S) := \frac{1}{\kappa} \int_S d^2x \sqrt{\det(\sigma)}(k_o - k),
\] (4.1)

where \( k \) is the trace of the extrinsic curvature \( k_{ab} \) of \( S \) corresponding to the normal \( n^a \) orthogonal to \( ^3B \) (“orthogonal boundaries assumption”), and the integral of \( k_o \) is a reference term that is used to normalize the energy with respect to a reference spacetime, not necessarily flat. The second integral in Eq. (4.1) have been quantized as Eq. (3.19). The
construction of reference term from a reference space amounts to posing and solving an isomeric embedding problem. One natural choice is to embed $S$ isometrically into Euclidean three space $(\mathbb{R}^3, \delta_{ab})$ in order to obtain an extrinsic curvature tensor $(k_o)_{ab}$ (and hence $k_o$). With the second fundamental form $(k_o)_{ab}$ expressed in terms of the embedding’s coordinate chart, this is the Weyl’s problem, a classic problem of differential geometry for which an extensive literature exists [24]. However, it is very difficult to obtain the solution of $(k_o)_{ab}$ in terms of $\sigma_{ab}$. Nevertheless, since the function of the reference term is to normalize the energy, in the quantum version one may regard it as a c-number $-K_o \equiv \frac{1}{\kappa} \int_S d^2x \sqrt{\det(\sigma)} k_o$. In this sense the Brown-York energy has been quantized as

$$\hat{E}_{BY}(S) = \hat{E}_{Q,k}(S) - K_o. \quad (4.2)$$

Of course, one may also take the other viewpoint that the reference term is dynamical and thus should be quantized. In certain symmetric models, it is indeed possible to solve $k_o$.

### 4.1.1 QLE in spherically symmetric model

We now study the Brown-York QLE in spherically symmetric quantum geometry. Consider a static spherically symmetric space-time with line element

$$ds^2 = -N^2 dt^2 + H^2 dr^2 + R^2 (d\theta^2 + \sin^2 \theta d\phi^2), \quad (4.3)$$

where $N$ and $H$ are functions of $r$ only. Let $\Sigma_t$ be the interior of a $t = constant$ slice with two-boundary $S_{r=const}$ specified by $r = constant$. A straightforward calculation gives the trace $k$ of $k_{ab}$ as

$$k = \frac{2R'}{RH}, \quad (4.4)$$

where the prime denotes the partial differentiation with respect to coordinate $r$. Now consider a round sphere with radius $R$ embedded in $(\mathbb{R}^3, \delta_{ab})$. Such a sphere has an extrinsic curvature $(k_o)_{ab}$ with trace

$$k_o = \frac{2}{R}. \quad (4.5)$$

So the Brown-York quasi-local energy can be written as

$$E_{BY}(S_{r=const}) = \frac{1}{\kappa} \int_0^{2\pi} d\phi \int_0^\pi d\theta R^2 \sin \theta \left( \frac{2}{R} - \frac{2R'}{RH} \right)$$

$$= \frac{8\pi}{\kappa} R(1 - \frac{R'}{H}). \quad (4.6)$$

In the spherically symmetric model, the invariant connections and triads can be written as [25]

$$A_a = A_r(r) \Lambda_3 (dr)_a + [A_1(r) \Lambda_1 + A_2(r) \Lambda_2] (d\theta)_a + [A_1(r) \Lambda_2 - A_2(r) \Lambda_1] \sin \theta (d\phi)_a$$
+ \Lambda_3 \cos \theta (d\varphi)_\alpha. \]

\[
E^a = E^r(r) \Lambda_3 \sin \theta \left( \frac{\partial}{\partial r} \right)^a + [E^1(r) \Lambda_1 + E^2(r) \Lambda_2] \sin \theta \left( \frac{\partial}{\partial \theta} \right)^a \\
+ \left[ E^1(r) \Lambda_2 - E^2(r) \Lambda_1 \right] \left( \frac{\partial}{\partial \varphi} \right)^a,
\]

(4.7)

where \( A_r, A_1, A_2, E^r, E^1 \) and \( E^2 \) are real functions on an one-dimensional, radial manifold \( M \) with coordinate \( r \), and \( \Lambda_I \) are the \( su(2) \)-matrices and identical to \( \tau_i/2 \) or a rigid rotation thereof. For convenience, one introduces variables

\[
A_\varphi(r) := \sqrt{ (A_1(r))^2 + (A_2(r))^2 },
\]

(4.8)

\[
E^\varphi(r) := \sqrt{ (E^1(r))^2 + (E^2(r))^2 },
\]

(4.9)

and \( \alpha(r), \beta(r) \) defined by

\[
\Lambda_1 \cos \beta(r) + \Lambda_2 \sin \beta(r) = (A_1(r) \Lambda_2 - A_2(r) \Lambda_1) / A_\varphi(r),
\]

(4.10)

\[
\Lambda_1 \cos (\alpha(r) + \beta(r)) + \Lambda_2 \sin (\alpha(r) + \beta(r)) = (E^1(r) \Lambda_2 - E^2(r) \Lambda_1) / E^\varphi(r).
\]

(4.11)

In the spherical coordinate system, the components of the spatial 3-metric \( q_{ab} \) on \( \Sigma \) take the form

\[
(q_{ab}) = \text{diag} \left( \frac{(E^\varphi)^2}{E^r}, E^r, E^r \sin^2 \theta \right).
\]

(4.12)

So we have the relation

\[
H = \frac{E^\varphi}{\sqrt{E^r}}, \quad R = \sqrt{E^r}.
\]

(4.13)

The QLE reads

\[
E_{\text{BY}}(S_r=\text{const}) = \frac{8\pi}{\kappa} \sqrt{E^r} \left[ 1 - \frac{(E^r)'^2}{2 E^\varphi} \right].
\]

(4.14)

In Ref. [26], a canonical transformation from \((A_r, A_1, A_2; E^r, E^1, E^2)\) to \((A_r, \beta K_\varphi, \eta; E^r, E^\varphi, P^\eta)\), where

\[
K_\varphi(r) := \sqrt{ (K_1(r))^2 + (K_2(r))^2 }
\]

(4.15)

is the extrinsic curvature component and

\[
\eta(r) := \alpha(r) + \beta(r),
\]

(4.16)

\[
P^\eta(r) := 2 A_\varphi(r) E^\varphi(r) \sin \alpha(r)
\]

(4.17)

has been made. The symplectic structure is then given by

\[
\Omega_B = \frac{4\pi}{\kappa \beta} \int_B d\tau \delta A_r \wedge \delta E^r + 2 \beta \delta K_\varphi \wedge \delta E^\varphi + \delta \eta \wedge \delta P^\eta).
\]

(4.18)
The resulted Hilbert space is spanned by an orthonormal basis of spin network states:

\[
T_{g,k,\mu}(A) = \prod_{e \in g} \exp \left( \frac{i}{2} k_e \int A_e(r) dr \right) \prod_{v \in V(g)} \exp(i \mu_v \beta K_v (v)) \exp(ik_v \eta(v))
\]  

(4.19)

with edge labels \( k_e \in \mathbb{Z} \), vertex labels \( \mu_v \in \mathbb{R} \) and \( k_v \in \mathbb{Z} \) for graphs \( g \) in the 1-dimensional radial manifold \( M \). The flux operators corresponding to the momenta \( E^r \) and \( E^\varphi \) act on the spin network states as:

\[
\hat{E}^r(r) T_{g,k,\mu} = \frac{\beta \hbar \kappa}{8\pi} \frac{k_e^{+}(r) + k_e^{-}(r)}{2} T_{g,k,\mu},
\]  

(4.20)

\[
\int_{I} \hat{E}^\varphi(r) T_{g,k,\mu} = \frac{\beta \hbar \kappa}{4\pi} \sum_{v \in I} \mu_v \delta(r,v) T_{g,k,\mu},
\]  

(4.21)

where \( e^{\pm}(r) \) are the two edges (or two parts of a single edge) meeting at \( r \), \( \delta(r,v) \) is the Dirac delta distribution, and \( I \) is a region of the reduced (radial) manifold \( M \).

We now consider the quantization of the QLE. The spin connection component can be regularized as [26]

\[
-(E^r)’ = -\frac{1}{4} \left( \frac{E^r(v_+) - E^r(v)}{\int_{v_+}^{v} E^\varphi dr} + \frac{E^r(v) - E^r(v_-)}{\int_{v_-}^{v} E^\varphi dr} \right) + O(\epsilon).
\]  

(4.22)

Now the \( E^r \) in Eq. (4.22) can be promoted as well-defined operators, and the \( \frac{1}{\int E^\varphi dr} \) may also become well-defined operator by suitable treatments [27, 28]. Hence the quasi-local energy operator \( \hat{E}_{BY}(S_{r=\text{const}}) \) in the spherically symmetric sector of LQG can be well defined.

### 4.2 Liu-Yau energy operator

Let \( l \) be the trace of the extrinsic curvatures of the 2-surface \( S \) in the physical spacetime corresponding to the future pointing timelike normal. Liu and Yau define a quasi-local energy by [8]

\[
E_{LY}(S) := \frac{1}{\kappa} \int_{S} d^2x \sqrt{\det(\sigma)}(k_o - \sqrt{|k^2 - l^2|}).
\]  

(4.23)

Since the first term in Eq. (4.23) is again a reference term, we will only consider the second term

\[
E_{Q,k,l}(S) \equiv -\frac{1}{\kappa} \int_{S} d^2x \sqrt{\det(\sigma)} \sqrt{|k^2 - l^2|}.
\]  

(4.24)

Let \( g_\epsilon(x,y) \) be a 1-parameter family of fields on \( S \) which tend to \( \delta^2(x, y) \) as \( \epsilon \) tends to zero, i.e., such that

\[
\lim_{\epsilon \to 0} \int_{S} d^2y \ g_\epsilon(x^1, x^2; y^1, y^2) f(y^1, y^2) = f(x^1, x^2)
\]  

(4.25)
for all smooth densities $f$ of weight 1 and of compact support on $S$. (Thus, $g_\epsilon(x,y)$ is a density of weight 1 in $x$ and a function in $y$.) Using $g_\epsilon(x,y)$ as smearing function, one can regularize $E_{Q,k,l}(S)$ as

$$
E_{Q,k,l}^\epsilon(S) = - \int_S d^2x \left[ \frac{1}{\kappa} \int_S d^2y \; g_\epsilon(x,y) \sqrt{\det(\sigma)}(-k + l) \frac{1}{\kappa} \int_S d^2z g_\epsilon(x,z) \sqrt{\det(\sigma)}(-k - l) \right]^{1/2} 
$$

$$
\equiv - \int_S d^2x \left[ \left| \left[ E_{Q,-k+l} \right]_{g_\epsilon}(x) [E_{Q,-k-l}]_{g_\epsilon}(x) \right|^{1/2} \equiv - \int_S d^2x \sqrt{|E_{S}|_{g_\epsilon}(x)}. \tag{4.26}
$$

It is easy to see that $E_{Q,k,l}^\epsilon(S)$ tends to $E_{Q,k,l}(S)$ as $\epsilon$ tends to zero. Let us now turn to the integrand of Eq. (4.26). It can be promoted as an operator acting on a cylindrical function as

$$
[\hat{E}_S]_{g_\epsilon}(x) \cdot f_\gamma = \left| \hat{E}_{Q,-k+l} \right|_{g_\epsilon}(x) \left| \hat{E}_{Q,-k-l} \right|_{g_\epsilon}(x) \cdot f_\gamma \equiv \sum_{v,v' \in P(\gamma)} g_\epsilon(x,v) g_\epsilon(x,v') \left| (\hat{E}_{Q,k,v} + \hat{J}_{Q,l,v})(\hat{E}_{Q,k,v'} - \hat{J}_{Q,l,v'}) \right| \cdot f_\gamma. \tag{4.27}
$$

where the absolute value $\left| (\hat{E}_{Q,k,v} + \hat{J}_{Q,l,v})(\hat{E}_{Q,k,v} - \hat{J}_{Q,l,v}) \right| \equiv \hat{A}$ indicates that one is supposed to take the square root of the operator $\hat{A}^\dagger \hat{A}$. We choose $\epsilon$ sufficiently small so that $g_\epsilon(x,v) g_\epsilon(x,v')$ is zero unless $v = v'$. Then one obtains

$$
[\hat{E}_S]_{g_\epsilon}(x) \cdot f_\gamma = \sum_{v \in P(\gamma)} g_\epsilon(x,v)^2 \left| (\hat{E}_{Q,k,v} + \hat{J}_{Q,l,v})(\hat{E}_{Q,k,v} - \hat{J}_{Q,l,v}) \right| \cdot f_\gamma. \tag{4.28}
$$

Notice that $[\hat{E}_S]_{g_\epsilon}(x)$ is a non-negative self-adjoint operator and hence have a well defined square root, which is also a positive-definite self-adjoint operator. Since we have chosen $\epsilon$ to be sufficiently small, for any given point $x$ in $S$, $g_\epsilon(x,v)$ is non-zero for at most one vertex $v$. We can therefore take the sum over $v$ outside the square root and thus obtain

$$
\sqrt{[\hat{E}_S]_{g_\epsilon}(x)} \cdot f_\gamma = \sum_{v \in P(\gamma)} g_\epsilon(x,v) \sqrt{\left| (\hat{E}_{Q,k,v} + \hat{J}_{Q,l,v})(\hat{E}_{Q,k,v} - \hat{J}_{Q,l,v}) \right|} \cdot f_\gamma. \tag{4.29}
$$

Finally, we can remove the regulator. By integrating both sides of Eq. (4.29) on $S$ and then taking the limit $\epsilon \to 0$, we obtain the desired operator corresponding to Eq. (4.24) as

$$
\hat{E}_{Q,k,l}(S) \cdot f_\gamma = - \sum_{v \in P(\gamma)} \sqrt{\left| (\hat{E}_{Q,k,v} + \hat{J}_{Q,l,v})(\hat{E}_{Q,k,v} - \hat{J}_{Q,l,v}) \right|} \cdot f_\gamma. \tag{4.30}
$$

### 4.3 Hawking energy operator

By studying the perturbation of the dust-filled open Friedmann-Robertson-Walker spacetime, Hawking found that [9]

$$
E_H(S) := \frac{2\sqrt{\pi}}{\kappa} \sqrt{Ar(S)} \left[ 1 - \frac{1}{16\pi} \int_S d^2x \sqrt{\det(\sigma)} (k^2 - l^2) \right] \tag{4.31}
$$
behaves as an appropriate notion of energy surrounded by the space-like topological 2-
sphere \( S \). The virtue of Hawking energy is that it does not need a reference term. We can
regularize it as

\[
E_H(S) = \frac{2\sqrt{\pi}}{\kappa} \sqrt{A_r(S)} - \lim_{\epsilon \to 0} \frac{\sqrt{A_r(S)}}{8\sqrt{\pi} \kappa} \int_S d^2x \sqrt{\det(\sigma)}(-k + l) \times \int_S d^2y \frac{\chi^2(x, y)}{\epsilon^2 \sqrt{\det(\sigma)}} \sqrt{\det(\sigma)}(-k - l) \\
= \frac{2\sqrt{\pi}}{\kappa} \sqrt{A_r(S)} - \lim_{\epsilon \to 0} \frac{\sqrt{A_r(S)}}{8\sqrt{\pi} \kappa} \int_S d^2x \sqrt{\det(\sigma)}(-k + l) \times \int_S d^2y \frac{\chi^2(x, y)}{A_r(x, \epsilon)} \sqrt{\det(\sigma)}(-k - l).
\]

(4.32)

To quantize the expression (4.32), one can replace the \( \sqrt{A_r(S)} \) by \( \sqrt{\hat{A}_r(S)} \) and use the well-
defined operators \( \hat{E}_{Q,k,v} \) and \( \hat{J}_{Q,l,v} \). We then formally get an operator acts on cylindrical
functions as

\[
\hat{E}^\ell_H(S) \cdot f_\gamma = \frac{2\sqrt{\pi}}{\kappa} \sqrt{\hat{A}_r(S)} \cdot f_\gamma - \frac{1}{8\sqrt{\pi} \kappa} \sum_{v \in P(\gamma, S)} \left( \hat{E}_{Q,k,v} + \hat{J}_{Q,l,v} \right) \sqrt{\hat{A}_r(S)} \hat{A}_r_v \times \sum_{v' \in P(\gamma, S)} \chi^2(v, v')(\hat{E}_{Q,k,v} - \hat{J}_{Q,l,v}) \cdot f_\gamma.
\]

(4.33)

For sufficiently small \( \epsilon \), \( \chi^2(v, v') \) is zero unless \( v = v' \). Thus one has

\[
\hat{E}_H(S) \cdot f_\gamma = \frac{2\sqrt{\pi}}{\kappa} \sqrt{\hat{A}_r(S)} \cdot f_\gamma - \frac{1}{8\sqrt{\pi} \kappa} \sum_{v \in P(\gamma, S)} (\hat{E}_{Q,k,v} + \hat{J}_{Q,l,v}) \sqrt{\hat{A}_r(S)} \hat{A}_r_v (\hat{E}_{Q,k,v} - \hat{J}_{Q,l,v}) \cdot f_\gamma.
\]

(4.34)

However, this is not a densely defined operator due to \( \frac{1}{\hat{A}_r_v} \). Fortunately, since \( \hat{E}_{Q,k,v} \) vanishes the internal gauge-invariant states, for a gauge-invariant state \( \Psi_\gamma \in \mathcal{H}_o \) we have

\[
\hat{E}_H(S) \cdot \Psi_\gamma = \frac{2\sqrt{\pi}}{\kappa} \sqrt{\hat{A}_r(S)} \cdot \Psi_\gamma + \frac{1}{8\sqrt{\pi} \kappa} \sum_{v \in P(\gamma, S)} \hat{J}_{Q,l,v} \sqrt{\hat{A}_r(S)} \hat{J}_{Q,l,v} \cdot \Psi_\gamma.
\]

(4.35)

Hence we can obtain a well-defined operator corresponding to Hawking energy in \( \mathcal{H}_o \) as

\[
\hat{E}_H(S) \cdot \Psi_\gamma = \frac{2\sqrt{\pi}}{\kappa} \sqrt{\hat{A}_r(S)} \cdot \Psi_\gamma + \frac{1}{2\sqrt{\pi} \hbar^2 \kappa^3} \sum_{v \in P(\gamma, S)} \left[ \hat{H}^E_v, \sqrt{\hat{A}_r_v} \right]^\dagger \sqrt{\hat{A}_r(S)} \left[ \hat{H}^E_v, \sqrt{\hat{A}_r_v} \right] \cdot \Psi_\gamma.
\]

(4.36)

It is easy to see that this operator is symmetric in \( \mathcal{H}_o \).
4.4 Geroch energy operator

Geroch modified the Hawking energy and gave the other definition for QLE as [10]

\[ E_G(S) := \sqrt{\frac{Ar(S)}{16\pi G^2}} \left(1 - \frac{1}{16\pi} \int_S d^2x \sqrt{\det(\sigma)} k^2 \right). \]  \hspace{1cm} (4.37)

It can be regularized as

\[ E_G(S) = \frac{2\sqrt{\pi}}{\kappa} \sqrt{Ar(S)} - \lim_{\epsilon \to 0} \frac{\sqrt{Ar(S)} \int_S d^2x \sqrt{\det(\sigma)} k \int_S d^2y \chi^2(x,y) \sqrt{\det(\sigma)} k}{8\sqrt{8\pi} \kappa}. \]  \hspace{1cm} (4.38)

The quantum operator corresponding to the Geroch energy then formally reads

\[ \hat{E}_G(S) \cdot f\gamma(A) = \frac{2\sqrt{\pi}}{\kappa} \sqrt{\hat{A}r(S)} \cdot f\gamma(A) - \frac{1}{8\sqrt{8\pi} \kappa} \sqrt{\hat{A}r(S)} \sum_{v \in P(\gamma,S)} \frac{1}{Ar_v} \hat{E}_{Q,k,v}^2 \cdot f\gamma(A). \]  \hspace{1cm} (4.39)

Let \( T_s \) be the gauge-invariant spin network function in \( \mathcal{H}_o \). Then \( \hat{E}_G(S) \) acts on \( T_s \) as

\[ \hat{E}_G(S) \cdot T_s = \frac{2\sqrt{\pi}}{\kappa} \sqrt{\hat{A}r(S)} \cdot T_s \]  \hspace{1cm} (4.40)

Thus \( \hat{E}_G(S) \) is well defined in \( \mathcal{H}_o \). Hence the spectrum of the area operator [16] implies the spectrum of \( \hat{E}_G(S) \) as

\[ \text{Spec}[\hat{E}_G(S)] = \sqrt{\frac{2\pi \hbar \beta}{\kappa}} \left[ \sum_{v \in P(\gamma,S)} \sqrt{2j_v^{(d)}(j_v^{(d)} + 1) + 2j_v^{(u)}(j_v^{(u)} + 1) - j_v^{(d+u)}(j_v^{(d+u)} + 1)} \right]^{\frac{1}{2}}, \]  \hspace{1cm} (4.41)

where \( j_v^{(d)} \), \( j_v^{(u)} \) and \( j_v^{(d+u)} \) are half-integers subject to the usual condition:

\[ j_v^{(d+u)} \in \left\{ |j_v^{(d)} - j_v^{(u)}|, |j_v^{(d)} - j_v^{(u)}| + 1, \ldots, j_v^{(d)} + j_v^{(u)} \right\}. \]  \hspace{1cm} (4.42)

In the case that all edges of the graph \( \gamma \) underlying \( T_s \) puncture \( S \), i.e., \( \gamma \) has no edges tangential to \( S \), the spectrum is reduced to

\[ \text{Spec}[\hat{E}_G(S)] = \sqrt{\frac{4\pi \hbar \beta}{\kappa}} \left[ \sum_{v \in P(\gamma,S)} \sqrt{j_v(j_v + 1)} \right]^{\frac{1}{2}} = m \left[ \sum_{v \in P(\gamma,S)} \sqrt{j_v(j_v + 1)} \right]^{\frac{1}{2}}, \]  \hspace{1cm} (4.43)

where \( j_v \) are half-integers. In summary, \( \hat{E}_G(S) \) is a densely defined, positive semi-definite operator in \( \mathcal{H}_o \), and its spectrum is entirely discrete inherited from the property of area operator. Thus we have proved a quantum positivity QLE theorem. Moreover, \( \hat{E}_G(S) \) is both internal gauge invariant and invariant under the diffeomorphism transformations tangent to \( S \). Furthermore, the Geroch gravitational energy in a quantum state labelled by a graph \( \gamma \) is concentrated at the vertices of \( \gamma \) which live on 2-surface \( S \) and the edges which puncture \( S \) transversely. The discreteness of quantum gravitational energy enable us to estimate the statistical entropy of the region enclosed by \( S \) in next section.
5 Discussions: entropy-area relation in LQG

It was first speculated by Bekenstein that one could associate an entropy $S_{BH}$ to a black hole with horizon area $A$ as [29]

$$S_{BH} = k_B \frac{A}{4\hbar G},$$

(5.1)

where $k_B$ is the Boltzmann constant. The statistically mechanical origin of this entropy has been an outstanding mystery for physicists. Some intuitive arguments and accurate calculations have been done in the framework of LQG to account for Eq. (5.1) [30–33]. Now we have the quantum gravitational energy of any finite region bounded by a closed 2-surface $S$. So, in principle, one may study the thermodynamical properties of an arbitrary bounded gravitational system in LQG by the standard statistical mechanics method. For example, if one considers it as a canonical system, the partition function reads

$$Z(S) = \text{Tr} e^{-\frac{\hat{E}(S)}{k_B T}}$$

(5.2)

for certain QLE operator $\hat{E}(S)$. Then all the thermodynamical quantities including the entropy of the system can be derived in principle. However, since the spectrum of the above QLE operators are either too complicated or unknown yet, it is still difficult to do practical calculations. Thus, further investigations in this strict approach are needed to understand the spectrum properties of the QLE operators.

On the other hand, the so-called holographic principle says that, at the fundamental (quantum) level, one should be able to characterize the state of any physical system located in a bounded spatial domain by degrees of freedom on the surface of the domain. Consequently, the number of physical degrees of freedom in the domain is bounded from above by the area of the boundary of the domain instead of its volume. If the entropy representing the degrees of freedom including gravity in a bounded domain could be calculated, one would be able to check whether the holographic principle is valid or not in LQG. Some QLE operator and the corresponding partition function provide a possible approach to this issue. Again, we need more control on the spectrum of the QLE operators constructed above.

Nevertheless, the virtue of Geroch QLE operator $\hat{E}_G(S)$ in (4.40) is beneficial for us to generalize the entropy-area relation in the framework of LQG. Our discussion is restricted a simple self-gravitating system bounded by a space-like closed 2-dimensional surface $S$. One can count the number of quantum states corresponding to eigenvalue of $\hat{E}_G(S)$. Thus, it is regarded as a microcanonical ensemble where the energy of the system is fixed. The spectrum (4.43) of the Geroch energy operator implies that the energy eigenvalue involves only the number $N$ of punctures and the spins $\vec{j}_v$ of the edges that intersect the surface $S$. Thus the number of the eigenstates of a given eigenvalue of $E_G(S)$ is infinite because different positions of the punctures give different states. This is no longer the case after modeling out the spatial diffeomorphisms tangent to $S$. Following [30, 34], we shall treat the punctures as distinguishable. Our task is to count the number $N(E_G(S))$ of quantum
states corresponding to the classical QLE $E_G(S)$. It is to see that $\mathcal{N}(E_G(S))$ is the same as the number $\mathcal{N}(Ar(S))$ corresponding to the classical area $Ar(S) = \frac{\kappa^2}{4\pi} E_G(S)^2$. Hence the states which we are considering satisfy

$$Ar(S) = \kappa h \beta \sum_j n_j \sqrt{j(j+1)},$$

where $n_j$ is the number of punctures with spin $j$. Following [33], the number of the states is given by

$$\mathcal{N}(Ar(S)) = \frac{(\sum_j n_j)!}{\prod_j n_j!} \prod_j (2j + 1)^{n_j}.$$  

(5.4)

Using Stirling’s formula, one gets the entropy as [33]

$$S = k_B \ln \mathcal{N}(E_G(S)) = k_B \ln \mathcal{N}(Ar(S)) = k_B \frac{Ar(S) \beta_0}{4hG} \beta,$$

(5.5)

where $\beta_0$ is the solution of the equation

$$1 = \sum_j (2j + 1) e^{-2\pi \beta_0 \sqrt{j(j+1)}}.$$  

(5.6)

Moreover, if one assumes that $\mathcal{N}(E_G(S))$ represent the physical degrees of freedom in the domain bounded by $S$, the holographic principle is realized in LQG. Note that for marginally trapped surfaces, the QLE must be the irreducible mass $\sqrt{A(S)}/16\pi G^2$ [1]. Hence our discussion for these cases do not depend on a specific definition of QLE.

We conclude with a few open issues related to the present work. (i) More understanding of the spectrum of the QLE operators is needed in order to do further practical calculations. (ii) Semiclassical analysis on the QLE operators is yet to be done. (iii) Since classically the integral of $k$ is non-zero in general, the vanishing of QLE-like operator $\hat{E}_{Q,k}(S)$ on gauge invariant states is in some sense awkward. One may consider other possible operator orderings in its construction in order to avoid the weakness. (iv) The knowledge on the physical Hilbert space of LQG would be a great help to our scheme.

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Appendix

A The partial gauge fixing version of $\hat{E}_{Q,k}(S)$

Classically, we have

$$k = D_a n^a = D_a \left( \frac{1}{\sqrt{\det(q)}} n^i E^a_i \right) = \frac{1}{\sqrt{\det(q)}} \partial_a (n^i E^a_i), \quad (A.1)$$

where we have introduced the internal vector $n^i = n_a e^a_i$ in the “internal space”. It is convenient to first carry out a partial gauge fixing. Let us fix an internal vector field $n^i$ with $n^i n_i = 1$. We restrict ourselves to flat derivative operators $\partial$ which annihilate $n^i$ in addition to $\delta_{ij}$. We call the partial gauge fixing as $n^i$-gauge fixing. Fixed $\vec{n}$ gives a fixed direction in the internal gauge group $SU(2)$. Thus the structure group is reduced from $SU(2)$ to $U(1)$. Physical states and observables should, of course, be independent of this choice. The $n^i$ gauge transformations on the 2-d surface $S$, which keep $n^i$ invariant, are generated by the following Gauss constraint

$$G(\lambda n^i) = \int d^3x \lambda n^i G_1, \quad (A.2)$$

where $\lambda$ is an arbitrary real number. Hence the corresponding Gauss constraint operator is given by

$$\hat{G}(\lambda n^i) \cdot f_\gamma = -\frac{i\hbar\kappa\beta}{2} \lambda n^i \sum_{e \in E(\gamma)} X^i_e \cdot f_\gamma. \quad (A.3)$$

Let $\psi_\gamma$ be a $n^i$-gauge-invariant cylindrical function corresponding the above Gauss constraint on $\mathcal{A}$. Then at every vertex $v$ of $\gamma$, the following condition must hold:

$$n^i \sum_{v \in V(\gamma)} X^i_v \cdot \psi_\gamma = 0. \quad (A.4)$$

where $X^i_v = \sum_{e \in E(\gamma), b(e) = v} X^i_e$. Under the $n^i$-gauge fixing, the extrinsic scalar curvature $k$ reduces to

$$k = \frac{n^i}{\sqrt{\det(q)}} \partial_a E^a_i. \quad (A.5)$$

Note that now $n^i$ is a non-dynamical constant which need not to be quantized. For simplicity, we choose again adapted coordinates $\{x^1, x^2, x^3\}$ with respect to $S$. Our aim is to quantize the following quantity under the partial gauge fixing

$$E_{Q,k}(S) = -\frac{1}{\kappa} \int_S d^2x \sqrt{\det(\sigma)(x)} \frac{\sqrt{\det(q)}(x)}{\sqrt{\det(q)}(y)} n^i(x) \partial_a E^a_i(y)$$
\[
\int_S d^2 x \left( \frac{\sqrt{\det(\sigma)(x)} V(x, \epsilon)}{\sqrt{\det(q)(x)}} \right) n^i(x) \int_S d^3 y \frac{\chi^3(x, y)}{\epsilon^3} \partial_a E^a_i(y) \\
= -\lim_{\epsilon \to 0} \frac{1}{\epsilon^3} \int_S d^2 x \sqrt{\det(q)(x)} n^i(x) \int_S d^3 y \frac{\chi^3(x, y)}{\epsilon^3} \partial_a E^a_i(y) \\
= \lim_{\epsilon \to 0} \frac{1}{\epsilon^3} \int_S d^2 x \sqrt{\det(\sigma)(x)} (V(x, \epsilon)) n^i(x) \int_S d^3 y \frac{\chi^3(x, y)}{\epsilon^3} \partial_a E^a_i(y) \\
=: \lim_{\epsilon \to 0} E^a_{Q,k}(S). \tag{A.6}
\]

The first integration of Eq. (A.6) can be written as
\[
\int_S d^2 x \left( \frac{\sqrt{\det(\sigma)(x)} V(x, \epsilon)}{\sqrt{\det(q)(x)}} \right) n^i(x) = \int_S d^2 x \left( \frac{\sqrt{\det(q)(x)} V(x, \epsilon)}{\sqrt{\det(q)(x)}} \right) n^i(x) \int_S d^3 y \frac{\chi^3(x, y)}{\epsilon^3} \partial_a E^a_i(y),
\]
where \( \bar{\eta}_a = (dx^3)_a \). Using \( g_{cd}(x, y) \) satisfying Eq. (4.25) as smearing function, we define
\[
\begin{align*}
\left[ \frac{\bar{\eta}_a(x) E^a_i(x)}{V(x, \epsilon)} \right]_{g_{cd}} & := \int_S d^2 u g_{cd}(x, u) \frac{\bar{\eta}_a(u) E^a_i(u)}{V(u, \epsilon)} = \int_S \frac{g_{cd}(x, u) \frac{1}{2} \epsilon_{ijk}e^j(u) \wedge e^k(u)}{V(u, \epsilon)} \\
& = \frac{8}{\kappa^2} \int_S g_{cd}(x, u) \epsilon_{ijk} \left\{ A^i(u), \sqrt{V(u, \epsilon)} \right\} \wedge \left\{ A^k(u), \sqrt{V(u, \epsilon)} \right\}, \tag{A.8}
\end{align*}
\]
where we have used the classical identity
\[
e^i_a(u) = \frac{2}{\kappa} \left\{ A^i_a(u), V(u, \epsilon) \right\}, \tag{A.9}
\]
and absorbed \( 1/V(u, \epsilon) \) into the Poisson brackets. Thus we have
\[
\int_S d^2 x \left( \frac{\sqrt{\det(\sigma)(x)} V(x, \epsilon)}{\sqrt{\det(q)(x)}} \right) n^i(x) = \lim_{\epsilon \to 0} \int_S d^2 x \left( \frac{\sqrt{\det(q)(x)} V(x, \epsilon)}{\sqrt{\det(q)(x)}} \right) n^i(x) \int_S d^3 y \frac{\bar{\eta}_a(x) E^a_i(x)}{V(x, \epsilon)} \left[ \frac{\bar{\eta}_a(x) E^a_i(x)}{V(x, \epsilon)} \right]_{g_{cd}}, \tag{A.10}
\]

We introduce a triangulation of the 2-d surface \( S [35, 36] \). Denote by \( \Delta \) a solid triangle. Single out one of the corners of the triangle and call it \( v(\Delta) \). At \( v(\Delta) \) there are incident two edges \( s_1(\Delta), s_2(\Delta) \) of \( \partial \Delta \) which we equip with outgoing orientation, that is, they start at \( v(\Delta) \). Let us now write the integral over \( S \) as a sum of integrals over \( \Delta \) where \( \Delta \) are triangles of some triangulation \( T \) of \( S \),
\[
\left[ \frac{\bar{\eta}_a(x) E^a_i(x)}{V(x, \epsilon)} \right]_{g_{cd}} = \frac{4}{\kappa^2} \sum_{\Delta \in T} g_{cd}(x, v(\Delta)) \epsilon_{s_1(\Delta) \wedge s_2(\Delta) \wedge v(\Delta)} \epsilon_{ijkl} \epsilon_{ijk} \left\{ h^{s_1(\Delta)}_{s_1(\Delta)}, \sqrt{V(v(\Delta), \epsilon)} \right\} \left\{ h^{s_2(\Delta)}_{s_2(\Delta)}, \sqrt{V(v(\Delta), \epsilon)} \right\}, \tag{A.11}
\]
where \( \epsilon_{s_1s_2} := \text{sgn}(\det(\hat{s}_1 s_2)) \) takes the values \( +1, -1, 0 \) if the tangents of the two segments \( s_1, s_2 \) at \( v \) (in that sequence) form a matrix of positive, negative or vanishing determinant. For convenience, we introduce
\[
e_{s_1}^i(v(\Delta)) := \text{Tr} \left( \tau_j h_{s_1(\Delta)} \{ h^{s_1(\Delta)}_{s_1(\Delta)}, \sqrt{V(v(\Delta), \epsilon)} \} \right). \tag{A.12}
\]
Then the quantity in the square root of the expression in (A.10) can be regularized as

\[
\frac{\tilde{n}_a(x)E_i^a(x)}{V(x,\epsilon)} \tilde{n}_b(x)E_i^b(x) = 16 \kappa^4 \sum_{\Delta,\Delta'\in T} g_{\epsilon}(x, v(\Delta))g_{\epsilon}(x, v(\Delta')) \epsilon(s_{I} s_{J})\epsilon_{ijk}\epsilon_{I}^{J} e_{I}^{J}(v(\Delta)) e_{J}^{k}(v(\Delta))
\]

\[\times \epsilon(s_{K} s_{L})\epsilon_{imn}\epsilon_{K}^{L} e_{K}^{m}(v(\Delta')) e_{L}^{n}(v(\Delta')). \quad (A.13)\]

We introduce a triangulation of the 2-d surface \( S \) in adaption to the graph \( \gamma [35, 36] \) and only consider the terms in (A.13) which sum over triangles \( \Delta \) whose basepoint \( v(\Delta) \) coincides with a vertex \( v \) of the graph.

\[
\frac{16}{\kappa^4} \sum_{v, v' \in V(\gamma) \cap S} g_{\epsilon}(x, v)g_{\epsilon}(x, v')\epsilon_{ijk}\epsilon_{imn} \sum_{s_{j}(\Delta) \cap s_{J}(\Delta') = v} \frac{4}{E(v)} \epsilon(s_{I} s_{J})\epsilon_{I}^{J} e_{I}^{J}(v) e_{J}^{k}(v)
\]

\[\times \sum_{s_{K}(\Delta') \cap s_{L}(\Delta') = v'} \frac{4}{E(v')} \epsilon(s_{K} s_{L})\epsilon_{K}^{L} e_{K}^{m}(v') e_{L}^{n}(v'). \quad (A.14)\]

For sufficiently small \( \epsilon' \), \( g_{\epsilon}(x, v)g_{\epsilon}(x, v') \) is zero unless \( v = v' \). Then the double sum over vertices reduces to a single one, Eq. (A.14) reduces to

\[
\frac{16}{\kappa^4} \sum_{v \in V(\gamma) \cap S} [g_{\epsilon}(x, v)]^2 \epsilon_{ijk}\epsilon_{imn} \frac{16}{E(v)^2} \epsilon(s_{I} s_{J})\epsilon(s_{K} s_{L})\epsilon_{I}^{J} e_{I}^{J}(v) e_{J}^{k}(v) e_{K}^{m}(v) e_{L}^{n}(v)
\]

\[= \frac{32}{\kappa^4} \sum_{v \in V(\gamma) \cap S} [g_{\epsilon}(x, v)]^2 \frac{16}{E(v)^2} \sum_{IJKL} \epsilon(s_{I} s_{J})\epsilon(s_{K} s_{L})\epsilon_{I}^{J} e_{I}^{J}(v) e_{K}^{K}(v) e_{L}^{L}(v). \quad (A.15)\]

where we have abbreviated \( \sum_{s_{l}(\Delta) \cap s_{j}(\Delta) = s_{K}(\Delta) \cap s_{L}(\Delta) = v} \rightarrow \sum_{IJKL} \). We further introduce the manifestly gauge invariant quantities [37]

\[
q_{IK}(v) = e_{I}^{J}(v) e_{K}^{J}(v). \quad (A.16)\]

The (A.14) can be written as

\[
\frac{32}{\kappa^4} \sum_{v \in V(\gamma) \cap S} [g_{\epsilon}(x, v)]^2 \frac{16}{E(v)^2} \sum_{IJKL} \epsilon(s_{I} s_{J})\epsilon(s_{K} s_{L})\epsilon_{I}^{J} e_{I}^{J}(v) q_{IK}(v) q_{JL}(v). \quad (A.17)\]

To write the regulated operator corresponding to (A.13), we replace Poisson brackets by commutators times \( 1/(i\hbar) \), holonomies by multiplication and \( V \) by \( \hat{V} \). As we evaluate the operator corresponding to Eq. (A.13), we find out that only those triangles \( \Delta \) contribute whose basepoint \( v(\Delta) \) coincides with a vertex \( v \) of the graph due to the presence of the
volume operators. Hence we obtain the regulated operator corresponding Eq. \(A.13\) acting on a cylindrical function as

\[
\left[ \frac{\tilde{\hat{n}}_a(x)\tilde{E}^a_i(x)}{V(x, \epsilon)} \right]_{g_{\epsilon'}} \left[ \frac{\tilde{\hat{n}}_b(x)\tilde{E}^b_i(x)}{V(x, \epsilon)} \right]_{g_{\epsilon'}} \cdot f_\gamma
\]

\[= \frac{32}{\kappa^4} \frac{1}{(i\hbar)^4} \sum_{v \in V(\gamma) \cap S} [g_{\epsilon'}(x, v)]^2 16 \sum_{IJKL} \epsilon(s ISJ)\epsilon(s KS_L)\epsilon^{IJ} \epsilon^{KL} \hat{q}_{IK}(v)\hat{q}_{KL}(v) \cdot f_\gamma
\]

\[=: \sum_{v \in V(\gamma) \cap S} [g_{\epsilon'}(x, v)]^2 \hat{q}_v \cdot f_\gamma, \quad (A.18)
\]

where

\[\hat{q}_{IK}(v) = \hat{e}^i_J(v)\hat{e}^j_K(v) \quad (A.19)\]

with

\[\hat{e}^i_J(v) = \text{Tr} \left( \tau_i \hat{h}_{s_t}(\Delta) \left[ \hat{h}^{-1}_{s_t}(\Delta) \sqrt{\hat{V}_v} \right] \right)_{\gamma \in V(\gamma)}. \quad (A.20)\]

It is easy to see that the operator in \(A.18\) is gauge invariant. Notice that the self-adjointness of \(i\hat{e}^i_J(v)\) implies that \(\hat{q}_v\) is a non-negative self-adjoint operator and hence has a well defined square root. Since we have chosen \(\epsilon'\) to be sufficiently small, for any given point \(x\) in \(S\), \(g_{\epsilon'}(x, v)\) is non-zero for at most one vertex \(v\). We can therefore take the sum over \(v\) outside the square root and obtain

\[
\left( \left[ \frac{\tilde{\hat{n}}_a(x)\tilde{E}^a_i(x)}{V(x, \epsilon)} \right]_{g_{\epsilon'}} \left[ \frac{\tilde{\hat{n}}_b(x)\tilde{E}^b_i(x)}{V(x, \epsilon)} \right]_{g_{\epsilon'}} \right)^{1/2} \cdot f_\gamma = \sum_{v \in V(\gamma) \cap S} g_{\epsilon'}(x, v) \sqrt{\hat{q}_v} \cdot f_\gamma. \quad (A.21)
\]

Finally, we can remove the regulator, i.e., take the limit as \(\epsilon'\) tends to zero. Then the following equality holds in the distributional sense.

\[
\left[ \frac{\sqrt{\text{det}(\sigma)(x)}}{V(x, \epsilon)} \right] \cdot f_\gamma = \sum_{v \in V(\gamma) \cap S} \delta^2(x, v)[\hat{q}_v]^{1/2} \cdot f_\gamma. \quad (A.22)
\]

The second integration of Eq. \(A.6\) can be similarly quantized as

\[
\int_{\Sigma} d^3 y [\partial_a \chi^3_x(x, y)] \tilde{E}^a_i(y) \cdot f_\gamma
\]

\[= - \frac{i\hbar\kappa\beta}{2} \sum_{e \in E(\gamma)} \lim_{n \to \infty} \sum_{k=1}^{n} \left( \chi^3_x(x, e(t_k)) - \chi^3_x(x, e(t_{k-1})) \right) X^i_e(t_{k-1}) \cdot f_\gamma. \quad (A.23)
\]
Using Eqs. (A.22) and (A.23), we obtain the regularized operator corresponding to $E_{Q,k}(S)$ as

$$
\hat{E}^{\epsilon,n}_{Q,k}(S) \cdot f_{\gamma} = \frac{-i\hbar \kappa}{2} \int_{S} d^{2}x n^{i}(x) \sum_{v \in V(\gamma) \cap S} \delta^{2}(x, v)[\hat{q}_{v}]^{1/2} \\
\times \sum_{e \in E(\gamma)} \sum_{k=1}^{n} \left[ \chi^{3}_{e}(x, e(t_{k})) - \chi^{3}_{e}(x, e(t_{k-1})) \right] X^{i}_{e}(t_{k-1}) \cdot f_{\gamma} \\
= \frac{-i\hbar \kappa}{2} \sum_{v \in V(\gamma) \cap S} [\hat{q}_{v}]^{1/2} n^{i}(v) \\
\times \sum_{e \in E(\gamma)} \sum_{k=1}^{n} \left[ \chi^{3}_{e}(v, e(t_{k})) - \chi^{3}_{e}(v, e(t_{k-1})) \right] X^{i}_{e}(t_{k-1}) \cdot f_{\gamma}. \tag{A.24}
$$

Now we perform the limit $n \to \infty$ and $\epsilon \to 0$ in reversed order. Keeping $n$ fixed, for small enough $\epsilon$ only the term with $k = 1$ in the sum survives provided that $b(e) = v$. We then obtain the operator

$$
\hat{E}_{Q,k}(S) \cdot f_{\gamma} = \frac{-i\hbar \kappa}{2} \sum_{v \in V(\gamma) \cap S} [\hat{q}_{v}]^{1/2} n^{i}(v) \sum_{b(e) = v} X^{i}_{e} \cdot f_{\gamma} \\
= -i\hbar \kappa \sum_{v \in V(\gamma) \cap S} [\hat{q}_{v}]^{1/2} n^{i}(v) \sum_{b(e) = v} Y^{i}_{e} \cdot f_{\gamma}, \tag{A.25}
$$

where $Y^{i}_{e} \equiv -\frac{i}{2} X^{i}_{e}$ is the self-adjoint right-invariant vector field. It is clear that the $\hat{E}_{Q,k}(S)$ in (A.25) is densely defined in $H_{kin}$, and it vanishes the $n^{i}$-gauge-invariant states.

## B Proof of an equality

We first give a proof for Eq. (3.27) in 3.2. By choosing adapted coordinates $\{x^{1}, x^{2}, x^{3}\}$ with respect to $S$, the normal-directional momentum of $S$ is given by

$$
J_{Q,l}(S) := \frac{1}{\kappa} \int_{S} d^{2}x \sqrt{\det(\sigma)} l \\
= \frac{1}{\kappa} \int_{S} d^{2}x \sqrt{\tilde{n} e \tilde{n} c \det(q)} (K - n^{a} n^{b} K_{ab}) \\
\equiv J_{1}(S) + J_{2}(S), \tag{B.1}
$$

where

$$
l := \sigma^{ab} l_{ab} = \sigma^{cd} \nabla_{c} u_{d} = \nabla_{a} u^{a} - n^{c} n^{d} \nabla_{c} u_{d}.
$$

The first term in Eq. (B.1) can be written as

$$
J_{1}(S) = \frac{1}{\kappa} \int_{S} d^{2}x \sqrt{\tilde{n} e \tilde{n} b} E^{a}_{i} K^{a}_{i}, \tag{B.2}
$$
and the second term as
\[
J_2(S) = -\frac{1}{\kappa} \int_S d^2 x \sqrt{\bar{n}_c \bar{n}_c} n^a n_b K^i_a E^b_i. \quad (B.3)
\]
Thus we obtain
\[
J_{Q,l}(S) = \frac{1}{\kappa} \left( \int_S d^2 x \sqrt{\bar{n}_b \bar{n}_b} K^i_a E^a_i - \int_S d^2 x \sqrt{\bar{n}_c \bar{n}_c} n^a n_b K^i_a E^b_i \right). \quad (B.4)
\]
On the other hand,
\[
\{ H^E(1), Ar(S) \} = \kappa \int_{\Sigma} d^3 y \frac{\delta H^E(1) \delta Ar(S)}{\delta A^i_d(y) \delta E^d_i(y)}. \quad (B.5)
\]
Note that the area of $S$ can be written as
\[
Ar(S) = \int_S d^2 x \sqrt{\bar{n}_a \bar{n}_b E^a_i E^b_j \delta^{ij}} = \int_{\Sigma} d^3 x \sqrt{\bar{n}_a \bar{n}_b E^a_i E^b_j \delta^{ij} \delta(x^3, 0)}. \quad (B.6)
\]
One gets
\[
\frac{\delta Ar(S)}{\delta E^d_i(y)} = \frac{\bar{n}_b E^d_j \delta^{ij} \bar{n}_a \delta^d_1 \delta^{1d}}{\sqrt{\bar{n}_e \bar{n}_f E^e_m E^f_m \delta^{mn}}} \delta(y^3, 0) = \frac{\bar{n}_b e^{bl} \bar{n}_d}{\sqrt{\bar{n}_a \bar{n}_a}} \delta(y^3, 0), \quad (B.7)
\]
\[
\frac{\delta H^E(1)}{\delta A^i_d(y)} = \frac{1}{\kappa} \left( \epsilon^{abd} \partial_a e_{bl} + \epsilon^{abd} \epsilon_{ijl} A^i_a e^j_b \right). \quad (B.8)
\]
Plugging Eqs. (B.7) and (B.8) into (B.3), we obtain
\[
\{ H^E(1), Ar(S) \} = \int_{\Sigma} d^3 y \frac{1}{\sqrt{\bar{n}_e \bar{n}_e}} \left( \epsilon^{abd} \partial_a e_{bl} + \epsilon^{abd} \epsilon_{ijl} A^i_a e^j_b \right) \bar{n}^e_c \bar{n}_d \delta(y^3, 0). \quad (B.9)
\]
The two terms in (B.9) can be reduced respectively to
\[
\epsilon^{abd} \partial_a e_{bl} \bar{n}^e_c \bar{n}_d = \epsilon^{[abc]}(-\Gamma^f_{ab} e_{fl} - \epsilon_{imn} \Gamma^m_a e^n_b) \bar{n}^e_c \bar{n}_d = -\Gamma^m_a \bar{n}^c \bar{n}_d \epsilon^{abd} \epsilon_{jbc} e^f_m
\]
\[
= -\Gamma^m_a E^a_m \bar{n}^d \bar{n}_d + \Gamma^m_a E^d_m \bar{n}^a_d,
\]
\[
\epsilon^{abc} \epsilon_{ijl} A^i_a e^j_b \bar{n}^e_c \bar{n}_d = A^i_a \bar{n}^c \bar{n}_d \epsilon^{abc} \epsilon_{jbc} e^f_i = A^m_a E^a_m \bar{n}^d \bar{n}_d - A^d_a E^d_a \bar{n}^a_d. \quad (B.10)
\]
Hence one has
\[
\{ H^E(1), Ar(S) \} = \int_{\Sigma} d^3 y \frac{1}{\sqrt{\bar{n}_e \bar{n}_e}} (K^m_a E^a_m \bar{n}^d \bar{n}_d - K^m_a E^d_m \bar{n}^a_d) \delta(y^3, 0)
\]
\[
= \int_S d^2 y K^m_a E^a_m \sqrt{\bar{n}^d \bar{n}_d} - \int_S d^2 y K^m_a E^d_m \bar{n}^a_d. \quad (B.11)
\]
The second integrand in (B.11) reduces to
\[
\frac{K^m_a E^d_m \bar{n}^a_d}{\sqrt{\bar{n}_c \bar{n}_c}} = \sqrt{\bar{n}_c \bar{n}_c} n^a n_b K^i_a E^b_i.
\]
We thus obtain
\[
\{ H^E(1), Ar(S) \} = \int_S d^2 y \sqrt{\bar{n}^b \bar{n}_b} K^i_a E^a_i - \int_S d^2 y \sqrt{\bar{n}_c \bar{n}_c} n^a n_b K^i_a E^b_i. \quad (B.12)
\]
Comparing (B.4) to (B.12) we complete the proof.
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