HARMONIC SUPERSPACE: SOME NEW TRENDS

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Abstract
We give a brief account of two recent applications of the harmonic superspace method: (i) an off-shell description of torsionful (4, 4) supersymmetric 2D sigma models in the framework of $SU(2) \times SU(2)$ harmonic superspace and (ii) the harmonic superspace formulation of “small” $N = 4$ $SU(2)$ superconformal algebra and the related super KdV hierarchy.
1. **Introduction.** Harmonic superspace has been proposed in 1984 in Dubna as an efficient tool for treating theories with extended supersymmetry [1]. This concept allowed to solve the long-standing problem of constructing off-shell superfield formulations of all \( N = 2, 4 \)D supersymmetric theories: theory of \( N = 2 \) matter (sigma models), \( N = 2 \) super Yang-Mills and supergravity theories [1 - 4] as well as of \( N = 3 \) super Yang-Mills theory [5]. Later on, the same method was applied to purely bosonic manifolds to get a new formulation of the Ward construction for self-dual Yang-Mills fields [6] and to find unconstrained potentials for hyper-Kähler and quaternionic geometries [7, 8].

The essence of the harmonic (super)space approach consists in extending the original (super)manifold by some extra variables which parametrize the automorphism group of the given (super)manifold. The basic advantage of considering such extended manifolds is the possibility to single out in them a submanifold of lower dimension, the so-called “analytic subspace”. In most examples the unconstrained functions on this subspace, analytic (super)fields, turn out to be the fundamental entities of the given theory. In particular, the potentials underlying the hyper-Kähler and quaternionic geometries live as unconstrained objects on the relevant analytic subspaces.

Since its invention, the harmonic superspace approach has been further advanced and developed along several directions. In the present report we review two of its recent developments.

Firstly, we sketch an off-shell description of torsionful \((4,4)\) supersymmetric sigma models in the framework of the \( SU(2) \times SU(2) \) harmonic superspace [9, 10, 11].

Secondly, we present the harmonic superspace formulation of “small” \( N = 4 \) superconformal algebra and the associated \( N = 4 \) KdV hierarchy [12, 13].

2. **SU(2)xSU(2) harmonic superspace.** The \( SU(2) \times SU(2) \) harmonic superspace is an extension of the standard real \((4,4)\) 2D superspace by two independent sets of harmonic variables \( u^{\pm 1} \) and \( v^{\pm 1}_a \) \((u^1 v^{-1}_1 = v^1 a u^{-1}_a = 1)\) associated with the automorphism groups \( SU(2)_L \) and \( SU(2)_R \) of the left and right sectors of \((4,4)\) supersymmetry [3] (see also [14]). The most relevant analytic harmonic subspace is spanned by the following set of coordinates \([\zeta, u, v] = (x^{++}, x^{--}, \theta^{1,0}_1, \theta^{0,1}_a, u^{\pm 1}_i, v^{\pm 1}_a)\),

\[ (\zeta, u, v) = (x^{++}, x^{--}, \theta^{1,0}_1, \theta^{0,1}_a, u^{\pm 1}_i, v^{\pm 1}_a) , \]

where we omitted the light-cone indices of odd coordinates. The superscript “\( n, m \)” stands for two independent harmonic \( U(1) \) charges, left (\( n \)) and right (\( m \)) ones. The additional doublet indices of odd coordinates, \( i \) and \( a \), refer to two extra automorphism groups \( SU(2)^L \) and \( SU(2)^R \) which together with \( SU(2)_L \) and \( SU(2)_R \) form the full \((4,4)\) supersymmetry automorphism group \( SO(4)_L \times SO(4)_R \).

The usefulness of \( SU(2) \times SU(2) \) harmonic superspace consists in that it allows for a natural off-shell superfield description for the important class of \((4,4)\) supersymmetric sigma models, those with torsion on the bosonic manifold. This type of sigma models is interesting mainly because they can provide non-trivial backgrounds for superstrings (see, e.g., [15]).

We start with an off-shell formulation of the twisted \((4,4)\) multiplet which until now was the basic object in constructing actions for such sigma models (actually, a subclass of them with mutually commuting left and right quaternionic structures [16, 17]). It is described by a real analytic superfield \( q^{1,1}(\zeta, u, v) \) subjected to the harmonic constraints [6]

\[ D^{2,0} q^{1,1} = D^{0,2} q^{1,1} = 0 . \]

where

\[ D^{2,0} = \partial^{2,0} + i \theta^{1,0}_1 \frac{\partial^{1,0}_1}{2} \partial^{--}, \quad D^{0,2} = \partial^{0,2} + i \theta^{0,1}_a \frac{\partial^{0,1}_a}{2} \partial^{++} \]
are the left and right mutually commuting analyticity-preserving harmonic derivatives. These constraints leave in $q^{1,1} 8 + 8$ independent components, that is just the irreducible off-shell component content of $(4,4)$ twisted multiplet [8,16].

The general off-shell action of $n$ superfields $q^{1,1 M} (M = 1,2,...n)$ can be written as the following integral over the analytic superspace (4)

$$ S_q = \int \mu^{-2,-2} h^{2,2}(q^{1,1 M},u^{\pm 1},v^{\pm 1}), $$

$\mu^{-2,-2}$ being the relevant integration measure. The analytic superfield lagrangian $h^{2,2}$ is an arbitrary function of its arguments (the only restriction on its dependence on the harmonics $u$ and $v$ is the consistency with the external $U(1)$ charges 2, 2).

As a non-trivial example of the $q^{1,1}$ action with four-dimensional bosonic manifold we quote the action of $(4,4)$ extension of the $SU(2) \times U(1)$ WZNW sigma model

$$ S_{wzw} = {1 \over 4\kappa^2} \mu^{-2,-2} q^{1,1 q^{(1,1)}} \frac{\ln(1 + X)}{X^2} - \frac{1}{(1 + X)X} \right). $$

Here

$$ q^{1,1} = q^{1,1} - c^{1,1}, \ X = c^{1,1} - 1 q^{1,1}, \ c^{\pm 1,\pm 1} = c^{ia} a^{i \pm 1} v^{\pm 1}, \ c^{ia} c^{ia} = 2. $$

Despite the presence of an extra quartet constant $\delta^a$ in the analytic superfield lagrangian, the action (5) actually does not depend on $\delta^a$ as it is invariant under arbitrary rigid rescalings and $SU(2) \times SU(2)$ rotations of this constant.

The $SU(2) \times SU(2)$ harmonic superspace description of $(4,4)$ twisted multiplet suggests a new off-shell formulation of the latter via unconstrained analytic superfields. After implementing the constraints (2) in the action with superfield lagrange multipliers we arrive at the following new action (6)

$$ S_{q,\omega} = \mu^{-2,-2} \left\{ q^{1,1 M} (D^{2,0} \omega^{-1,1 M} + D^{0,2} \omega^{-1,1 M}) + h^{2,2}(q^{1,1},u,v) \right\}. $$

In (6) all the involved superfields are unconstrained analytic, so from the beginning the action (7) contains an infinite number of auxiliary fields coming from the double harmonic expansions with respect to the harmonics $u^{\pm 1,i}, v^{\pm 1,a}$. Varying with respect to the Lagrange multipliers $\omega^{-1,1 M}, \omega^{-1,1 M}$ takes one back to the action (4) and constraints (2). On the other hand, varying with respect to $q^{1,1 M}$ yields an algebraic equation for eliminating this superfield, which enables one to get a new dual off-shell representation of the twisted multiplet action through unconstrained analytic superfields $\omega^{-1,1 M}, \omega^{-1,1 M}$.

The crucial feature of the action (6) (and its $\omega$ representation) is the abelian gauge invariance

$$ \delta \omega^{1,-1 M} = D^{2,0} \sigma^{1,-1 M}, \ \delta \omega^{-1,1 M} = -D^{0,2} \sigma^{1,-1 M}, $$

where $\sigma^{1,-1 M}$ are unconstrained analytic superfield parameters. This gauge freedom ensures the on-shell equivalence of the $q, \omega$ or $\omega$ formulations of the twisted multiplet action to its original $q$ formulation (4): it neutralizes superfluous physical dimension component fields in the superfields $\omega^{1,-1 M}$ and $\omega^{-1,1 M}$ and thus equalizes the numbers of propagating fields in both formulations. It holds already at the free level, with $h^{2,2}$ quadratic in $q^{1,1 M}$,
so it is natural to expect that any reasonable generalization of the action \((\mathcal{I})\) respects this symmetry or a generalization of it.

It is well known that with making use of \((4,4)\) twisted multiplets one may construct invariant off-shell actions only for those torsionful \((4,4)\) sigma models in which left and right triplets of the covariantly constant complex structures on the bosonic target mutually commute \([16, 17]\). It turns out that the above dual form of the twisted multiplet action is a good starting point for constructing more general actions which admit no inverse duality transformation to the twisted multiplets ones and provide an off-shell description of sigma models with non-commuting left and right complex structures.

These more general actions are obtained by allowing for the dependence on the superfields \(\omega\) as also in \(h^{2,2}\). With making use of freedom of reparametrizations in the target space together with the self-consistency constraints following from the commutativity condition

\[
[D^{2,0}, D^{0,2}] = 0 ,
\]

one may show that the most general action of the analytic superfields triples \(\omega^{1,-1M}, \omega^{-1,1M}, q^{1,1M}\) can be reduced to the form \([\mathcal{I}], [\mathcal{I}]\)

\[
S_{q,\omega} = \int \mu^{-2-2} \left\{ q^{1,1M} D^{0,2} \omega^{1,-1M} + q^{1,1M} D^{2,0} \omega^{-1,1M} + \omega^{1,-1M} h^{1,3M} + \omega^{-1,1M} h^{3,1M} + \omega^{1,-1N} h^{2,2}[M,N] + h^{2,2} \right\} ,
\]

where the involved potentials depend only on \(q^{1,1M}\) and target harmonics and satisfy the following constraints

\[
\nabla^{2,0} h^{1,3N} - \nabla^{0,2} h^{3,1N} + h^{2,2}[N,M] \frac{\partial h^{2,2}}{\partial q^{1,1M}} = 0 \quad (11)
\]

\[
\nabla^{2,0} h^{2,2}[N,M] - \frac{\partial h^{3,1N}}{\partial q^{1,1T}} h^{2,2}[T,M] + \frac{\partial h^{3,1M}}{\partial q^{1,1T}} h^{2,2}[T,N] = 0 \quad (12)
\]

\[
\nabla^{0,2} h^{2,2}[N,M] - \frac{\partial h^{1,3N}}{\partial q^{1,1T}} h^{2,2}[T,M] + \frac{\partial h^{1,3M}}{\partial q^{1,1T}} h^{2,2}[T,N] = 0 \quad (13)
\]

\[
\nabla^{2,0} h^{2,2}[M,L] \frac{\partial h^{2,2}}{\partial q^{1,1T}} + h^{2,2}[L,T] \frac{\partial h^{2,2}}{\partial q^{1,1T}} = 0 \quad (14)
\]

\[
\nabla^{2,0} = \partial^{2,0} + h^{3,1N} \frac{\partial}{\partial q^{1,1N}}, \quad \nabla^{0,2} = \partial^{0,2} + h^{1,3N} \frac{\partial}{\partial q^{1,1N}} . \quad (15)
\]

Here \(\partial^{2,0}, \partial^{0,2}\) act only on the target harmonics.

The action \([\mathcal{I}], [\mathcal{I}], [\mathcal{I}]\), with taking account of the constraints \([\mathcal{I}], [\mathcal{I}], [\mathcal{I}])\), enjoys the following non-abelian generalization of the gauge freedom \([\mathcal{I}]\)

\[
\delta \omega^{1,-1M} = \left( D^{2,0} \delta^{MN} + \frac{\partial h^{3,1N}}{\partial q^{1,1M}} \right) \sigma^{-1,-1N} - \omega^{1,-1L} \frac{\partial h^{2,2}[L,N]}{\partial q^{1,1M}} \sigma^{-1,-1N},
\]

\[
\delta \omega^{-1,1M} = - \left( D^{0,2} \delta^{MN} + \frac{\partial h^{1,3N}}{\partial q^{1,1M}} \right) \sigma^{-1,-1N} - \omega^{-1,1L} \frac{\partial h^{2,2}[L,N]}{\partial q^{1,1M}} \sigma^{-1,-1N},
\]

\[
\delta q^{1,1M} = \sigma^{-1,-1N} h^{2,2}[N,M].
\]

In general, these gauge transformations close with a field-dependent Lie bracket parameter:

\[
\delta b_r q^{1,1M} = \sigma^{-1,-1N}_r h^{2,2}[N,M], \quad \sigma^{-1,-1N}_b = -\sigma^{-1,-1L}_1 \sigma^{-1,-1T}_2 \frac{\partial h^{2,2}[L,T]}{\partial q^{1,1N}} . \quad (17)
\]
We see that eq. (14) guarantees the nonlinear closure of the algebra of gauge transformations (15) and so it is a group condition similar to the Jacobi identity. Remarkably, these gauge transformations augmented with the group condition (14) are precise bi-harmonic counterparts of the two-dimensional version of basic relations of the so-called Poisson nonlinear gauge theory which recently received some attention [19]. The manifold \((q, u, v)\) can be interpreted as a kind of bi-harmonic extension of some Poisson manifold and the potential \(h^{2,2}_{\{N,M\}}(q, u, v)\) as a tensor field inducing the Poisson structure on this extension.

It should be pointed out that it is the presence of the antisymmetric potential \(h^{2,2}_{\{N,M\}}\) that makes the considered case non-trivial and, in particular, the gauge invariance (16) non-abelian. If \(h^{2,2}_{\{N,M\}}\) is vanishing, (10) proves to be reducible to the dual action of twisted multiplets (7). In the \(n = 1\) case the potential \(h^{2,2}_{\{N,M\}}\) vanishes identically, so (10) for \(n = 1\) is actually equivalent to (7). Thus only for \(n \geq 2\) a new class of torsionful (4, 4) sigma models comes out.

It is easy to see that the action (14) with non-zero \(h^{2,2}_{\{N,M\}}\) does not admit any duality transformation to the form with the superfields \(q^{1,1\,M}\) only. So, the obtained system certainly does not admit in general an equivalent description in terms of twisted multiplets and, for this reason, the left and right complex structures on the target space can be non-commuting. In refs. [10, 11] we have explicitly shown this non-commutativity for a particular class of the models in question.

This subclass corresponds to the choice of the following ansatz

\[
\begin{align*}
\h^{1,3\,N} &= \h^{3,1\,N} = 0 ; \ h^{2,2} = h^{2,2}(t, u, v) , \ t^{2,2} = q^{1,1\,M}q^{1,1\,M} ; \\
\h^{2,2}_{\{N,M\}} &= b^{1,1\,f_{NML}}q^{1,1\,L} , \ b^{1,1} = b^a u_1^a v_1^a , \ b^a = \text{const} ,
\end{align*}
\]

where the real constants \(f_{NML}\) are totally antisymmetric. The constraints (11) - (13) are identically satisfied with this ansatz, while (14) is now none other than the Jacobi identity which tells us that the constants \(f_{NML}\) are structure constants of some real semi-simple Lie algebra. Thus the associated (4, 4) sigma models can be interpreted as a kind of Yang-Mills theories in the harmonic superspace. They provide the direct non-abelian generalization of the twisted multiplet sigma models with the action (7) which are thus analogs of two-dimensional abelian gauge theory. The action (14) specialized to the case (13) is as follows

\[
S^{YM}_{q,\omega} = \int \mu^{-2,2}\{ q^{1,1\,M}( D^{0,2}\omega^{1,1\,M} + D^{2,0}\omega^{-1,1\,M} + b^{1,1} \omega^{-1,1\,L}\omega^{1,1\,N} f_{LNM} ) \\
+ h^{2,2}(q, u, v) \} .
\]

It is a clear analog of the Yang-Mills action in the first order formalism, \(q^{1,1\,N}\) being an analog of the YM field strength and \(b^{1,1}\) of the YM coupling constant.

An interesting specific feature of this “harmonic Yang-Mills theory” is that the “coupling constant” \(b^{1,1} = b^a u_1^a v_1^a\) is doubly charged (this is necessary for the balance of harmonic \(U(1)\) charges). When \(b^a \to 0\), the non-abelian structure contracts into the abelian one and we reproduce the twisted multiplet action (7). It also turns out that \(b^a\) measures the “strength” of non-commutativity of the left and right complex structures on the bosonic target.

In the simplest case of \(h^{2,2} = q^{1,1\,M}q^{1,1\,M}\) we have calculated [10, 11], in the first non-vanishing order in fields, the bosonic metric and torsion potential, as well as the relevant triplets of complex structures. All these proved to be represented by non-trivial expressions, the commutator of the left and right complex structures being non-vanishing and proportional to \(b^a\).
Thus in the present case in the bosonic sector we encounter a more general geometry compared to the one associated with twisted \((4,4)\) multiplets. The basic characteristic feature of this geometry is the non-commutativity of the left and right complex structures. It is related in a puzzling way to the non-abelian structure of the analytic superspace actions \(\mathcal{H}_{11}, \mathcal{H}_{10}\): the coupling constant \(b_{1,1}\) (or the Poisson potential \(h_{2,2}^{[M,N]}\) in the general case) measures the strength of the non-commutativity of complex structures.

In summary, within the \(SU(2) \times SU(2)\) harmonic superspace approach it becomes possible to solve the long-standing problem of manifestly supersymmetric off-shell description of the torsionful \((4,4)\) supersymmetric sigma models with non-commuting left and right complex structures.

3. Harmonic superspace formulation of \(N=4\) \(SU(2)\) super KdV hierarchy. The supercurrent generating “small” \(N = 4\) \(SU(2)\) superconformal algebra (SCA) admits a concise formulation in the 1D \(SU(2)\) analytic harmonic superspace

\[(\zeta, u^+, u^-) = (z, \theta^+, \bar{\theta}^+, u^+, u^-) ,\]

where it is represented by a doubly charged superfield \(V^{++}(\zeta, u)\) subjected to the constraint

\[D^{++}V^{++} = 0 .\]  

with

\[D^{++} = \partial^{++} - i\theta^+\bar{\theta}^+\partial_z , \quad \partial_{++} \equiv u^{+i}\frac{\partial}{\partial u^{-i}}.\]

Solving \(21\), one gets

\[V^{++}(\zeta) = w^{ij}u^+_iu^+_j - \theta^+\xi^k u^+_k + \bar{\theta}^+\bar{\xi}^k u^-_k + \theta^+\bar{\theta}^+ \left(i\partial w^{ik}u^+_iu^-_k + T\right) .\]  

Here, up to scaling factors, the component fields coincide with the currents of \(N = 4\) \(SU(2)\) SCA: the \(SU(2)\) triplet of spin 1 currents generating \(SU(2)\) affine Kac-Moody subalgebra, a complex doublet of fermionic spin 3/2 currents and the spin 2 conformal stress-tensor, respectively \[21\].

The standard commutation relations for these currents follow from the following superfield Poisson bracket \[12\]

\[\{V^{++}(1), V^{++}(2)\} = \mathcal{D}^{++|++}\Delta(1-2)\]

\[\mathcal{D}^{++|++} \equiv (D_1^+)^2(D_2^-)^2 \left(\left[\left(\frac{u^+_1 u^-_2}{u^+_2 u^-_1}\right) - \frac{1}{2}D_2^-\right]V^{++}(2) - \frac{k}{4}\partial_2\right) ,\]

where \(\Delta(1-2) = \delta(x_1 - x_2) (\theta^1 - \theta^2)^4\) is the ordinary 1D \(N = 4\) superspace delta function and

\[(D^+)^2 \equiv D^+\bar{D}^+\]

are operators ensuring the harmonic analyticity with respect to both superspace arguments, \(D^+, \bar{D}^+\) being the relevant \(u\) projections of 1D, \(N = 4\) spinor derivatives. Note that the harmonic singularity in the r.h.s. of \(23\) is fake: it is cancelled after decomposing the harmonics \(u^\pm_i\) over \(u^\pm_i\) with making use of their completeness relation.

To deduce the super KdV equation with the second hamiltonian structure given by the \(N = 4\) \(SU(2)\) SCA in the form \(23\) we need to construct the relevant hamiltonian of the
dimension 3. The most general dimension 3 $N = 4$ supersymmetric hamiltonian $H_3$ one may construct out of $V^{++}$ reads

$$H_3 = \int [dZ] V^{++}(D^{--})^2 V^{++} - i \int [d\zeta^2] c^{-4}(u) (V^{++})^3. \quad (24)$$

Here $[dZ]$ and $[d\zeta^2]$ are the integration measure of the full harmonic superspace and its analytic subspace, $D^{--}$ is the second harmonic derivative (not preserving the analyticity) such that

$$[ D^{++}, D^{--} ] = D^0,$$

$D^0$ being the operator counting the harmonic $U(1)$ charge, e.g., $D^0 V^{++} = 2V^{++}$. The $U(1)$ invariance of the integral over analytic subspace requires the inclusion of the harmonic monomial $c^{-4}(u) = c^{ijkl} u^i u^j u^k u^l$ which explicitly breaks $SU(2)$ symmetry. The coefficients $c^{ijkl}$ belong to the dimension 5 spinor representation of $SU(2)$, i.e. form a symmetric traceless rank 2 tensor, and completely break the $SU(2)$ symmetry, unless $c^{-4}$ is of the special form

$$c^{-4}(u) = (a^{-2}(u))^2, \quad a^{-2}(u) = a^{ij} u^i u^j. \quad (25)$$

After taking off the harmonics this condition becomes

$$c^{ijkl} = \frac{1}{3} \left( a^{ij} a^{kl} + a^{ik} a^{jl} + a^{il} a^{jk} \right). \quad (26)$$

In this case, the symmetry breaking parameter belongs to the dimension 3 (vector) representation of $SU(2)$, and thus has $U(1)$ as a little group.

Using the hamiltonian $(24)$, we construct the relevant evolution equation:

$$V_t^{++} = \{ H_3, V^{++} \}. \quad (27)$$

In the explicit form it reads

$$V_t^{++} = i \left( D^+ \right)^2 \left\{ \frac{k}{2} D^{--} V^{++} - \left[ V^{++} (D^{--})^2 V^{++} - \frac{1}{2} (D^{--} V^{++})^2 \right]_x - \frac{3}{20} k A^{-4} (V^{++})^2 x + \frac{1}{2} A^{-6} (V^{++})^3 \right\}. \quad (28)$$

Here $A^{-4}$ and $A^{-6}$ are differential operators on the 2-sphere $\sim SU(2)/U(1)$

$$A^{-4} = \sum_{N=1}^{4} \frac{(-1)^N}{N!} c^{2N-4} \left( \frac{1}{N!} (D^{--})^N, \quad A^{-6} = \frac{1}{5} \sum_{N=0}^{4} \frac{(-1)^N}{N!} c^{2N-4} \frac{(5 - N)}{(N + 1)!} (D^{--})^{N+1}. \quad (29)$$

We have used the notation:

$$c^{2N-4} = \frac{(4 - N)!}{4!} (D^{++})^N c^{-4}, \quad N \in \{ 0 \cdots 4 \}. \quad (30)$$

Equation $(28)$ is the $N = 4$ $SU(2)$ super KdV equation. It is easy to check that its r.h.s is analytic and satisfies the same constraint $(21)$ as the l.h.s.
As is known, the $N = 2$ super KdV equation is integrable only for $a = 4, -2, 1, a$ being a free parameter in the corresponding hamiltonian [20 - 24]. Since the $SU(2)$ breaking tensor $c^{ijkl}$ is a direct analog of this $N = 2$ KdV parameter (and is reduced to it upon the reduction $N = 4 \rightarrow N = 2$), one may expect that the $N = 4$ super KdV equation is integrable only when certain restrictions are imposed on this tensor. To see which kind of restrictions arises, in [12, 13] we examined the issue of the existence of non-trivial conserved charges for (28) which are in involution with the hamiltonian (24). Let us sketch the results of this analysis.

Conservation of the dimension 1 charge:

$$ H_1 = \int [d\zeta^-] V^{++} $$

(31)

imposes no condition on the parameters of the hamiltonian.

A charge with dimension 2 exists only provided the condition (25) holds. It reads:

$$ H_2 = i \int [d\zeta^-] a^{-2} (V^{++})^2. $$

(32)

The conservation of this charge implies a stringent constraint on $a^{ij}$, namely

$$ a^{i+2}a^{-2} - (a^0)^2 = \frac{1}{2} a^{ij}a_{ij} = -\frac{10}{k}, $$

(33)

where

$$ a^{i+2} = D^{++}a^0 = \frac{1}{2} (D^{++})^2 a^{-2} = a^{ij}u_i^+u_j^+. $$

Assuming that the central charge $k$ is integer (if we restrict ourselves to unitary representations of the $SU(2)$ Kac-Moody algebra [27]), eq. (33) means that $a^{ij}$ parametrizes some sphere $S^2 \sim SU(2)/U(1)$, such that the reciprocal of its radius is quantized.

The next conserved charge is a dimension 4 one $H_4$ (the dimension 3 conserved charge is the $N = 4$ KdV hamiltonian itself). It exists under the same restrictions (24), (33) on $c^{ijkl}$ and reads:

$$ H_4 = \int [dZ] a^{-2}V^{++}(D^{--}V^{++})^2 + \frac{i}{6} \int [d\zeta^-] \left[ \frac{7}{6} (a^{-2})^3 (V^{++})^4 - ka^{-2}(V^{++})^2 \right]. $$

(34)

It is curious that it yields the same $N = 4$ KdV equation (28) via the first hamiltonian structure associated with the Poisson bracket

$$ \{V^{++}(1), V^{++}(2)\}_{(1)} = i \left( a^0(1) - a^{+2}(1) \frac{u_i^-}{u_i^+} \right) (D_1^+)^2(D_2^+)^2\Delta(1 - 2). $$

(35)

This bracket is related to the original one (23) via the shift

$$ V^{++} \rightarrow V^{++} + ia^{+2}(u). $$

(36)

Taking as a new hamiltonian

$$ H_{(1)} = -\frac{9k}{4} H_4, $$

(37)

we can reproduce (28) as the hamiltonian flow:

$$ V^{++}_t = \{H_{(1)}, V^{++}\}_{(1)}. $$

(38)
This comes about in a very non-trivial way, an essential use of the constraint (33) has to be made in the process.

Thus the conditions (25) and (33) are necessary not only for the existence of the first non-trivial conservation laws for eq. (28), but also for it to be bi-hamiltonian. This property persists for the evolution equations associated with other conserved charges. For instance, the equation associated with $H_2$ via the structure (23), with respect to (35) has $H_3$ as the hamiltonian.

The last bosonic conserved charges we have constructed until now [13] are $H_5$ and $H_6$. Once again, they exist iff the conditions (25) and (33) are satisfied. They possess rather complicated structure. For instance, $H_5$ is given by the following expression (without loss of generality, we have chosen $k=2$)

$$H_5 = \frac{1}{2} \int [dZ] \left[ \frac{1}{4} (D^{-}V^{++})^4 + i (D^{-}V^{++})^2 (D^{-})^2 V^{++} ight.$$  

$$+ \frac{15}{4} (a^{-})^2 (D^{-}V^{++})^2 (V^{++})^2 - \frac{1}{2} (D^{-}V^{++})^2 \right]$$  

$$+ i \frac{1}{4} \int [d\zeta^{-2}] \left[ \frac{63}{100} (a^{-})^4 (V^{++})^5 - 5 (a^{-})^2 (V^{++})^2 V^{++} \right].$$  

Both the bi-hamiltonian structure and the existence of non-trivial conserved charges are indications that $N=4$ KdV equation (28) with the restrictions (25), (33) is integrable, i.e. gives rise to a whole $N=4$ SU(2) super KdV hierarchy which is unique. Clearly, for the rigorous proof one should, before all, either find the relevant Lax pair or show the existence of an infinite number of conserved charges of the type given above (e.g., by employing recursion relations implied by the bi-hamiltonian property [28, 26]). These problems still remain to be solved.

After reduction $N=4 \rightarrow N=2$ at the superfield level [12, 13], $N=4$ SU(2) SCA breaks down to the $N=2$ SCA and one is left with the standard $N=2$ KdV, the free parameter $a$ of the latter being identified with

$$a = \frac{6}{5} e^{1212} = \frac{2}{5} \left( a^{11} a^{22} + 2 a^{12} a^{12} \right), \quad a^{ij} a_{ij} = -2 \left( a^{12} a^{12} - a^{11} a^{22} \right) = -10.$$  

Actually, only for three independent fixings of the SU(2) breaking parameter $a^{ik}$ (corresponding to the three independent choices of the SU(2) frame), the consistent reduction to $N=2$ KdV turns out to be possible [13]. For these three choices the constraint (33) gives rise, through the relations (40), to the following values of the $N=2$ KdV parameter

$$(a) \ a = 4; \quad (b) \ a = -2; \quad (c) \ a = -2.$$  

In other words, we get two $N=2$ super KdV hierarchies, with $a = 4$ and $a = -2$, as two inequivalent reductions of the single $N=4$ SU(2) hierarchy (the cases (b) and (c) yield the same system).

To summarize, the 1D $N=4$ harmonic superspace approach allows us to construct, in a manifestly supersymmetric way, the super KdV equation associated with $N=4$ SU(2) SCA, to reveal the existence of higher-order conservation laws and bi-hamiltonian structure for this system, and to show that it embodies as two different subsystems two of the three inequivalent $N=2$ super KdV hierarchies.

4. Perspectives. As regards the new class of (4,4) sigma models described in Sect. 2, the obvious problems for further study are to compute the relevant metrics and torsions in
a closed form and to try to utilize the corresponding manifolds as backgrounds for some superstrings. An interesting question is as to whether the constraints (11) - (14) admit solutions corresponding to the (4, 4) supersymmetric group manifold WZNW sigma models [29].

It still remains to examine whether the action (10) indeed describes most general (4, 4) models with torsion. The constrained superfield $q^{1,1M}$ the dual action of which was a starting point of our construction, actually represents only one type of (4, 4) twisted multiplet [18]. There exist other types which differ in the $SU(2) \times SU(2)$ assignment of their component fields [16, 30, 31]. At present it is unclear how to simultaneously describe all of them in the framework of the same $SU(2) \times SU(2)$ analytic harmonic superspace. Perhaps, their actions are related to those of $q^{1,1}$ by a kind of duality transformation. It may happen, however, that for their self-consistent description one will need a more general type of (4, 4) harmonic superspace, with the whole $SO(4) \times SO(4)$ automorphism group of (4, 4) supersymmetry harmonized. The related actions will be certainly more general than those presented here.

Among the problems related to the $N = 4$ super KdV hierarchy, let us mention, besides the construction of a Lax pair representation for the $N = 4$ SU(2) KdV, a generalization to the case of the “large” $N = 4$ SCA with the affine subalgebra $so(4) \times u(1)$ [32, 30]. The corresponding $N = 4$ super KdV hierarchy is expected to embrace both the $N = 4$ $SU(2)$ and $N = 3$ KdV ones as particular cases. Since the large $N = 4$ SCA can be viewed as a closure of two “small” $N = 4$ SCAs, each entering with its own affine $su(2)$ subalgebra, one may conjecture that a 1D version of $SU(2) \times SU(2)$ harmonic superspace (or its further extension) can be relevant to this problem.

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