LOCAL INDICATORS FOR
PLURISUBHARMONIC FUNCTIONS

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ABSTRACT. - The notion of index, classical in number theory and extended in [18] to plurisubharmonic functions, allows to define an indicator which is applied to the study of the Monge-Ampère operator and a pluricomplex Green function.

1 Introduction

We recall some local notions which are often used in various investigations.

a). The index $I(F, x^0, a)$ of a zero $x^0 \in \Omega$ of a holomorphic function $F \in Hol(\Omega)$, where $\Omega$ is a domain of $\mathbb{C}^n$, is used in important results of number theory [12]; $I(F, x^0, a)$ is defined by means of the set $\omega \in \mathbb{N}^n$ of $n$-tuples $(i) = (i_1, \ldots, i_n)$ such that $D^{(i)}F(x^0) \neq 0$. Given a direction $(a) = (a_k > 0; 1 \leq k \leq n) \in \mathbb{R}^n$,

$$I(F, x^0, a) = \inf_{(i)} (a, i) \quad \text{for} \quad (a, i) = \sum_k a_k i_k \geq 0 \quad \text{and} \quad (i) \in \omega. \quad (1)$$

b). In fact, the index $I(F, x^0, a)$ is a property (function of $a$ and $x^0$) of the current of integration $[W] = dd^c \log |F|$ over the analytic set $W = \{x \in \Omega : F(x) = 0\}$, where $d = \partial + \bar{\partial}$ and $d^c = (2\pi i)^{-1}(\partial - \bar{\partial})$. We denote by $PSH(\Omega)$ the class of plurisubharmonic functions in a domain $\Omega$ of $\mathbb{C}^n$ and by $\Theta_p(\Omega)$ the class of positive closed currents represented by homogeneous forms of $dx_k, d\bar{x}_k$, $1 \leq k \leq n$, of bidegree $(n - p, n - p)$. The Lelong number $\nu(T, x^0)$ at $x^0 \in \Omega$ for $T \in \Theta_p(\Omega)$ is related to the trace measure of $T$,

$$\sigma_T = T \wedge \beta_p, \quad (2)$$

where $\beta_p = (p!)^{-1}\beta_1^p$ is the volume element of $\mathbb{C}^p$. By the definition,

$$\nu(T, x^0) = \lim_{r \to 0} (\tau_2^p r^{2p})^{-1} \sigma_T [B^{2n}(x^0, r)] \quad (3)$$

where $\tau_2^p$ is the volume of the unit ball $B^{2p}(0, 1)$ of $\mathbb{C}^p$. In (3), the trace measure $\sigma_T$ belongs to the remarkable class $P_p(\Omega)$ of positive measures characterized by the property that the quotient in the right hand side of (3) is an increasing function of
Another definition of the number \( \nu(T, x^0) \) is derived from (2) and (3) by setting \( \varphi_1(x) = \log |x - x^0| \):

\[
\nu(T, x^0) = \lim_{t \to -\infty} \int_{\{\varphi_1 < t\}} T \wedge (dd^c \varphi_1)^p.
\] (4)

c). By replacing in (4) \( \varphi_1 \) with a function \( \varphi \in \text{PSH}(\Omega) \) such that \( \exp \varphi \) is continuous and the set \( \{ \varphi(x) = -\infty \} \) is relatively compact, J.-P. Demailly (see [4] and [5]) has found new important applications of (4). The choice of the function

\[
\varphi_a(x) = \sup_k a_k^{-1} \log |x_k - x_k^0|
\] (5)

which is circled with the center \( x^0 \) and such that \( \{ \varphi_a(x) = -\infty \} \) is reduced to \( x^0 \), allows us to put into this framework the notion of index \( I(F, x^0, a) \) of a zero of a holomorphic function \( F \) at \( x^0 \). As was proved in [18],

\[
I(F, x^0, a) = (a_1 \cdot a_2 \ldots a_n) \nu[T, x^0, \varphi_a(x - x^0)]
\] (6)

for \( T = dd^c \log |F| \in \Theta_1(\Omega) \). It can be extended to arbitrary plurisubharmonic functions by setting for \( f \in \text{PSH}(\Omega) \) and \( x^0 \in \Omega \),

\[
n(f, x^0, a) = (a_1 \cdot a_2 \ldots a_n) \nu[dd^c f, x^0, \varphi_a(x - x^0)],
\] (7)

so that

\[
I(F, x^0, a) = n(f, x^0, a) \quad \text{for} \quad f = \log |F|.
\] (8)

d). In the recent paper [18], \( n(f, x^0, a) \) has appeared in a simpler form given for \( x^0 = 0 \) by the relation

\[
n(f, 0, a) = \lim_{w \to 0} (\log w)^{-1} f(w, x, a), \quad 0 \leq w \leq 1, \ x \in \Omega \setminus A,
\] (9)

where \( f(w, x, a) = f(w^{a_1} x_1, \ldots, w^{a_n} x_n) \) and \( A \) is an algebraic set for \( f = \log |F| \). In the general case, for \( f \in \text{PSH}(\Omega) \), \( A \) is of zero measure and

\[
n(f, 0, a) = \liminf_{w \to 0} (\log w)^{-1} f(w, x, a)
\]

outside a set \( A' \subset A \) which is pluripolar in \( \Omega \).

In the first part of the present paper we will use (3) for a study of singularities of plurisubharmonic functions \( f \), supposing \( f \in \text{PSH}(\Omega) \) and \( D \subset \subset \Omega \), where \( D \) is the unit polydisk \( \{ x \in \mathbb{C}^n : \sup |x_k| < 1 \} \); we denote by \( \text{PSH}_-(D) \) the class of \( f \) satisfying \( \sup \{ f(x) : x \in D \} \leq 0 \) and \( f \not\equiv -\infty \). The domain \( D \) as well as the weights \( \varphi_a \) in (3) being circled, it is natural to work with a circled image \( f_c \) of \( f \) and then with its convex image on the space \( \mathbb{R}^n_- \) of \( u_k = \log |x_k|, \ 1 \leq k \leq n \).
Developing the results of J.-P. Demailly and C.O. Kiselman [10] we get the value $n(f, 0, a)$ which produces, via $a_k = -\log |y_k|$, a function $\Psi_{f,0}(y)$, the local indicator of $f$ at 0, which is plurisubharmonic in $D(y)$, the unit polydisk in the space $C^n_y$. It satisfies the Monge-Ampère equation

\[(ddc\Psi_{f,0})^n = \tau_f(0) \delta(0), \quad \tau_f(0) > 0,\]  

(10)

where $\delta(0)$ is the Dirac measure at the origin of $C^n_y$, and

\[(ddc f)^n \geq \tau_f(x^0) \delta(x^0)\]  

(11)

for $f \in PSH(\Omega)$ such that $(ddc f)^n$ is well defined, and $x^0 \in \Omega$. Moreover, there is the relation $\tau_f(x^0) \geq [\nu(T, x^0)]^n$ for $T = ddc f \in \Theta_{n-1}(\Omega)$.

Then we consider a class of plurisubharmonic functions on $\Omega$ with singularities on a finite set $\{x^1, \ldots, x^N\}$, controlled by given indicators $\Psi_m$, $1 \leq m \leq N$. We construct a plurisubharmonic function $G$ vanishing on $\partial \Omega$ and such that $\Psi_{G,x^m} = \Psi_m$, $1 \leq m \leq N$, and $(ddcG)^n = \sum_m \tau_m \delta(x^m)$ with $\tau_m$ the mass of $(ddc\Psi_m)^n$. We prove that $G$ is the unique plurisubharmonic function with these properties. For the case $\Psi_m(x) = \mu_m \log |x - x^m|$, it coincides with the pluricomplex Green function with weighted poles at $x^1, \ldots, x^N$ (10). We prove a variant of comparison theorem for plurisubharmonic functions with controlled singularities and study a Dirichlet problem for this class of functions.

A part of the results of Section 3 are close to those from [19] where a weighted pluricomplex Green function with infinite singular set was introduced and the corresponding Dirichlet problem was studied.

## 2 Circled functions and convex projections

We will consider here the case $x^0 = 0$ and $f \in PSH(\Omega)$ supposing $0 \in D \subset \subset \Omega$, where $D$ is the unit polydisk $\{x \in C^n : \sup |x_k| < 1\}$, and $f \in PSH_{-}(D)$, that is $f(x) \leq 0$ for $x \in \overline{D}$ and $f \not\equiv -\infty$.

A set $A \subset C^n$ is called 0-circled (or just circled) if $x = (x_k) \in A$ implies $x' = (x_k e^{i\theta_k}) \in A$ for $0 \leq \theta_k \leq 2\pi$, $1 \leq k \leq n$. We will say that a function $f(x)$ defined on $A$, is circled if it is invariant with respect to the rotations $x_k \mapsto x_k e^{i\theta_k}$, $1 \leq k \leq n$.

Given a function $f \in PSH(\Omega)$, $\Omega$ being a circled domain, we consider a circled function $f_c \in PSH(\Omega)$ equal to the mean value of $f(x_k e^{i\theta_k})$ with respect to $0 \leq \theta_k \leq 2\pi$, $1 \leq k \leq n$. In what follows, we will also use another circled function $f'_c \geq f_c$, equal to the maximum of $f(x_k e^{i\theta_k})$ for $0 \leq \theta_k \leq 2\pi$, $1 \leq k \leq n$. Note that the differential operators, namely $\partial, \bar{\partial}$, $d = \partial + \bar{\partial}$ and $dc = (2\pi i)^{-1}(\partial - \bar{\partial})$, commute with the mapping $f \mapsto f_c$, so $(\partial f)_c = \partial f_c$, however it is not the case for $f \mapsto f'_c$.

To a Radon measure $\sigma$ on a circled domain $\Omega$, we relate a circled measure $\sigma_c$ defined by $\sigma_c(f) = \sigma(f_c)$ for continuous functions $f$. In the same way, to a current $T \in \Theta_p(\Omega)$ we associate a circled current $T_c$ which is defined on homogeneous
forms $\lambda$ of bidegree $(p, p)$ by $T_c(\lambda) = T(\lambda_c)$, where $\lambda_c$ are obtained by replacing the coefficients of $\lambda$ with their mean values with respect to $\theta_k$. In particular, if $T = dd^c f$, $f \in PSH(\Omega)$ and $\Omega$ circled, $T_c = dd^c f_c$. It gives us a specific property of the value $n(f, 0, a)$ defined by (4) and (5), and of the index $I(F, 0, a)$.

**Proposition 1** Let $\Omega$ be a 0-circled domain and $T \in \Theta_p(\Omega)$. For every 0-circled weight $\varphi$, $\nu(T, \varphi) = \nu(T_c, \varphi)$. In particular, since the weight $\varphi_a$ defined by (5) for $x^0 = 0$, is circled, the number $n(f, 0, a)$ in (4) for $f \in PSH(\Omega)$ at the origin can be calculated by replacing $f$ with $f_c$: $n(f, 0, a) = n(f_c, 0, a)$, and the number $n(f, 0, a)$ is calculated on the convex image $g(u)$ of $f_c$, $g(u) = f[\exp(u_k + i\theta_k)]$:

$$n(f, 0, a) = \lim_{v \to -\infty} v^{-1} g(u_k + a_k v).$$

For $f \in PSH_-(D)$, as was shown in [18],

$$n(f, 0, a) = \lim_{w \to 0} (\log w)^{-1} f(w^{a_1} x_1, \ldots, w^{a_n} x_n)$$

for almost all $x \in D$. The limit exists for all $x \in D$ when replacing $f(x)$ by $f_c(x)$ or by $f_c'(x)$. Indeed, for $R > 1$

$$f_c(x) \leq f_c'(x) \leq \gamma_R f_c(Rx) \leq 0$$

with $\gamma_R = (R - 1)^n (R + 1)^{-n}$ which satisfies $1 - \epsilon \leq \gamma_R \leq 1$ for $R > R_0(\epsilon)$.

The calculation of $n(f, 0, a)$ for $f \in PSH_-(D)$ uses the convex image $g_f(u) = f_c[\exp(u_k + i\theta_k)]$ or $g'_f(u) = f'_c[\exp(u_k + i\theta_k)]$ obtained by setting $x_k = \exp(u_k + i\theta_k)$, the functions $g_f$ and $g'_f$ being defined on $R^n = \{-\infty \leq u_k \leq 0\}$.

**Proposition 2** In order that a function $h(u_1, \ldots, u_n) : R^n \to R_-$ be the image of $f \in PSH_-(D)$ obtained by $x_k = \exp(u_k + i\theta_k)$ and

$$f(x) = f[\exp(u_k + i\theta_k)] = h(u),$$

it is necessary and sufficient that $h$ be convex of $u \in R^n$, increasing in each $u_k$, $-\infty \leq u_k \leq 0$, and $h(u) \not\equiv -\infty$, $h(u) \leq 0$.

The necessity condition results from the classic properties of $f \in PSH_-(D)$. To show the sufficiency, we remark that convexity of $h$ implies its continuity on $R^n$. On the other hand, we have (the derivatives being taken in the sense of distributions), for any $\lambda \in C^n$,

$$4 \sum \frac{\partial^2}{\partial x_k \partial x_j} \lambda_k \lambda_j = \sum \frac{\partial^2}{\partial u_k \partial u_j} \lambda_k \lambda_j$$

(13)
where \( \lambda'_k = x_k^{-1}\lambda_k \). Let \( A \subset D \) be the union of the subspaces \( \{ x_k = 0 \} \) in \( D \). By (L3), \( f \in PSH(D \setminus A) \). The condition \( f(x) \leq 0 \) implies that \( f \) extends by upper semicontinuity to \( A \), so \( f \in PSH_-(D) \) for \( f(x) = h(\log|x_1|, \ldots, \log|x_n|) \).

**Definition.** We denote by \( \text{Conv}(\mathbb{R}^n) \) the class of functions \( h(u_1, \ldots, u_n) \leq 0, -\infty \leq u_k \leq 0 \), satisfying the conditions listed in Proposition 2.

**Proposition 3** Let \( h \in \text{Conv}(\mathbb{R}^n) \) be the image of \( f(x_k) = h(\exp(u_k + i\theta_k)) \in PSH_-(D) \). Then

a) \[ \lim_{v \to -\infty} v^{-1}h(u_1 + v, \ldots, u_n + v) = \lim_{v \to -\infty} \frac{\partial}{\partial v}h(u_1 + v, \ldots, u_n + v) = \nu(f, 0), \]

where \( \nu(f, 0) \) is the Lelong number of \( f \) at \( x = 0 \). More generally,

\[ \lim_{v \to -\infty} v^{-1}h(u_1 + a_1v, \ldots, u_n + a_nv) = n(f, 0, a) \quad (14) \]

is independent of \( u_k \);

b) \( \lim_{v \to -\infty} v^{-1}h(u_1 + v, u_2, \ldots, u_n) = \nu_1(f, 0) \) is independent of \( u_k \) and is the generic Lelong number (cf. [7]) of the current \( T = dd^c f \) along the variety \( D_1 = \{ x \in D : x_1 = 0 \} \). Moreover, for \( x'_1 = (x_2, \ldots, x_n) \), the function

\[ h_1(r_1, x'_1) = (2\pi)^{-1} \int_0^{2\pi} f(r_1e^{i\theta_1}, x'_1) d\theta_1, \]

has the property

\[ \lim_{w \to 0} (\log w)^{-1}h_1(wr_1, x'_1) = \nu_1(f, 0) \]

for \( x'_1 \in D_1 \) with exception of a pluripolar subset of \( D_1 \);

c) \( \sum_k \nu_k(f, 0) \leq \nu(f, 0) \).

**Proof.** Existence and equality of the limits in a) follow from the increasing with respect to \( v, -\infty < v \leq 0 \), and from the condition \( h \leq 0 \). Moreover, if \( l(\rho) \) is the mean value of \( f(x) \) over the sphere \( |x| = \rho \), then

\[ \nu(f, 0) = \lim_{\rho \to 0} \frac{\partial l(\rho)}{\partial \log \rho} = \lim_{\rho \to 0} (\log \rho)^{-1}l(\rho). \quad (15) \]

We compare the mean values with respect to \( \theta_k \) over the circled domains \( B(0, \rho) \) and \( D(\rho) = \{ \sup |x_k| \leq \rho < 1 \} \) for the image \( h(u) \) of the circled function \( f_c(x) \), for \( u_k = \log \rho - \frac{1}{2} \log n \) and \( u'_k = \log \rho, 1 \leq k \leq n \):

\[ h(u) \leq l(\log \rho) \leq h(u'), \]

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since \( D(\rho/\sqrt{n}) \subset B(0, \rho) \subset D(\rho) \). This gives us (14) and (a).

Statement (b) is known (cf. [15]). The limit

\[
- c(x'_1) = \lim_{r \to 0} \left( \log \frac{1}{r} \right)^{-1} h(r, x'_1) \leq 0
\]

for \( r \searrow 0 \) exists and is obtained by increasing negative values, the second term of (16) belonging to \( PSH(D_1) \) for \( r > 0 \). If \( c(\hat{x}'_1) = 0 \) for a point \( \hat{x}'_1 \in D_1 \), then \( c(x'_1) = 0 \) except for a pluripolar subset of \( D \) and the statement is proved. Otherwise, consider the set \( D_1(r) \subset D_1 \) and \( c_0 = \sup c(x'_1) \) for \( x'_1 \in D_1(r) \) and apply the preceding argument to

\[
\lim_{r \to 0} \left[ \left( \log \frac{1}{r} \right)^{-1} h(r, x'_1) + c_0 \right].
\]

The statement for \( h \in \text{Conv}(\mathbb{R}^n) \) follows from this precise property of the plurisubharmonic image.

To establish (c), we observe that for \( u \in \mathbb{R}^n \) and \( h(u) \) the image in \( \text{Conv}(\mathbb{R}^n) \) of \( f \in PSH_-(D) \),

\[
\frac{\partial}{\partial v} h(u_1 + v, \ldots, u_n + v) = \sum_{k=1}^n \frac{\partial h}{\partial u_k}(u_1 + v, \ldots, u_n + v),
\]

the derivatives are positive and decreasing for \( v \searrow -\infty \), and the limit of

\[
\frac{\partial h}{\partial u_k}(u_1 + v, u_2, \ldots, u_n)
\]

is equal to \( \nu_1(f, 0) \), the Lelong number of \( dd^c f \) along \( D_1 \). Therefore

\[
\frac{\partial}{\partial v} h(u_1 + v, \ldots, u_n + v) \geq \sum_{k=1}^n \nu_k(f, 0),
\]

so taking \( v \searrow -\infty \) we get by (a),

\[
\nu(f, 0) \geq \sum_{k=1}^n \nu_k(f, 0).
\]  

(17)

**Remark.** Actually, by the theorem of Y.T. Siu, (17) is a particular case of the following statement: the number \( \nu(f, 0) \) is at least equal to the sum of the generic numbers \( \nu(W_i) \) for \( T = dd^c f \) along analytic varieties \( W_j \) of codimension 1 containing the origin.

In what follows, we will use a special subclass of circled plurisubharmonic functions \( f \in PSH_-(D) \) that have the following "conic" property: the convex image \( g_f(u) \) of \( f \) satisfies the equation

\[
g_f(c u) = c g_f(u) \text{ for every } c > 0.
\]  

(18)
Such a function $f$ will be called an indicator. For example, the weights $\varphi_a$ in (3) are indicators.

**Proposition 4** Let $f \in PSH_-(D)$ be an indicator. Then $(dd^cf)^n = 0$ on $D_0 = \{x \in D : x_1 \ldots x_n \neq 0\}$.

**Proof.** It is sufficient to show that the domain $D_0$ can be foliated by one-dimensional analytic varieties $\gamma_y$ such that the restriction of $f$ to each leaf $\gamma_y$ is harmonic on $\gamma_y$. So, given $y = (|y_k|e^{i\theta_k}) \in D_0$, consider an analytic variety $\gamma_y$, the image of $C$ under the holomorphic mapping $\lambda = (\lambda_1, \ldots, \lambda_n)$ with $\lambda_k(\zeta) = |y_k|e^{i\theta_k}$. Note that $y = \lambda(1) \in \gamma_y$. As $f$ is circled, the function $f_y(\zeta) = f(\lambda(\zeta))$, the restriction of $f$ to $\gamma_y$, depends only on $\text{Re} \zeta$. By (18), $f_y(\zeta)$ satisfies $f_y(c\zeta) = cf_y(\zeta)$ for all $c > 0$. Therefore, it is linear and thus harmonic on $\gamma_y$.

## 3 Indicator of a plurisubharmonic function

Given a function $f \in PSH(\Omega)$ and a point $x^0 \in \Omega$, we will construct a function $\Psi_{f,x^0}(y)$ related to local properties of $f$ at $x^0$. We will have $\Psi_{f,x^0} \in PSH_-(D)$, $D$ being the unit polydisk in the space $C^n_{(y)}$, and $\Psi_{f,x^0}(y) < 0$ in $D$ if and only if the Lelong number of $f$ at $x^0$ is strictly positive, otherwise $\Psi_{f,x^0}(y) \equiv 0$.

**Definition.** The local indicator (or just indicator) $\Psi_{f,0}$ of a function $f \in PSH_-(D)$, $D \subset C^n_{(x)}$, at $x^0 = 0$ is defined for $y \in D \subset C^n_{(y)}$ by

$$\Psi_{f,0}(y) = -n(f, 0, -\text{log} |y_k|).$$

Referring to (3) with $R = -\text{log} w$, $0 < R < +\infty$, we rewrite this as

$$\Psi_{f,0}(y) = \lim_{R \to +\infty} R^{-1} f[\exp(u_k + i\theta_k + R \text{log} |y_k|)].$$

(19)

The limit (19) exists almost everywhere for $x_k = \exp(u_k + i\theta_k)$, however (see Introduction) the value $n(f, 0, a)$ can be calculated as well by replacing $f(x)$ with the circled functions $f_c(x)$ or $f'_c(x)$. One can then substitute them for $f$ in (19) to get $\Psi_{f,0}$. At $x^0 \neq 0$, the function $\Psi_{f,x^0}(y)$ is defined by means of $f[x^0_k + \exp(u_k + i\theta_k + R \text{log} |y_k|)]$.

If $f$ is replaced by $f_c[\exp(u_k + i\theta_k)] = g_f(u_k)$ or by $f'_c[\exp(u_k + i\theta_k)] = g'_f(u_k)$, the limit exists, by Proposition 4, for every $u = (u_k) \in \mathbb{R}^n_-$:

$$\Psi_{f,0}(y) = \lim_{R \to +\infty} R^{-1} g(u_k + R \text{log} |y_k|),$$

(20)

and does not depend on $u$.

**Proposition 5** Let $f \in PSH_-(D)$. Then
a) $\Psi_{f,0}(y) \in PSH_-(D)$ and is 0-circled;

b) the convex image $g_\psi(u)$ in $\mathbb{R}^n$ has the conic property $g_\psi(cu) = cg_\psi(u)$ for every $c > 0$, i.e. $\Psi_{f,0}$ is an indicator;

c) $\Psi_{f,0}(y) \geq f'_c(y) \geq f(y), \forall y \in D$;

d) the mapping $f \mapsto \Psi_{f,0}$ is a projection, $\Psi_{f,0}(y)$ is its own indicator at the origin;

e) the indicator $\Psi_{f,0}$ is the least indicator majorizing $f$ on $D$;

f) if $f_j(x_j)$ is the restriction of $f$ to the complex subspace $\{x_s = 0, \forall s \neq j\}$ and

$$f_j(x_j) \neq -\infty,$$ (22)

then $\Psi_{f,0}(y) \geq \nu_j \log |y_j|, \nu_j$ being the Lelong number of $f_j(x_j)$ at the origin;

g) if (22) holds for each $j$, then the Monge-Ampère operator $(dd^c\Psi_{f,0})^n$ is well defined on the whole polydisk $D$ and

$$(dd^c\Psi_{f,0})^n = 0$$ (23)

on $D \setminus \{0\}$.

Proof. Statement a) follows from (20), $g(R \log |y_k|)$ being a convex negative function for $R > 0$, and the limit of the quotient is obtained by increasing negative values. When setting $v_k = \log |y_k| = -a_k$, the image of $\Psi_{f,0}$ belongs to $\text{Conv}(\mathbb{R}^n)$ and $\Psi_{f,0}(y)$ is a 0-circled plurisubharmonic function.

The property b), essential for the indicator $\Psi_{f,0}$, results from the equality

$$n(f,0,ca) = cn(f,0,a)$$ for all $c > 0$.

Relations c) are a consequence of (20) where $g(u)$ is the convex image $g'_f(u)$ of $f'_c(x) = \sup_{h_k} f(x_k e^{ih_k})$. We have $g'_f(\log |y_k|) \geq f(y)$. On the other hand, the quotient $m(R) = R^{-1}g'_f(R \log |y_k|), R > R_0 > 1$, is a convex, negative and increasing function of $R$ for $|y_k| < 1$. Therefore, $\lim_{R \to +\infty} m(R) \geq m(1)$, and by (20),

$$0 \geq \Psi_{f,0}(y) \geq g'_f(\log |y_k|) \geq f(y)$$

for $y \in D$.

Statement d) follows from (20) for $f = \Psi_{f,0}$ and from relation b).

To prove e), consider any indicator $\psi(y) \geq f(y)$ on $D$. Then $\Psi_{\psi,0}(y) \geq \Psi_{f,0}(y)$, and by d), $\Psi_{\psi,0} = \psi$, so $\psi(y) \geq \Psi_{f,0}(y) \forall y \in D$. 

The bound in $f$ results from $c$ and the maximum principle for plurisubharmonic functions, since (for $j = 1$)

$$\Psi_{f,0}(y) \geq \sup_{\theta_k} f(y_k e^{i\theta_k}) \geq \sup_{\theta_i} f(y_1 e^{i\theta_1}, 0, \ldots, 0)$$

and for $|y_1| \searrow 0$ the quotient $(\log |y_1|)^{-1} \sup_{\theta_1} f_1(y_1 e^{i\theta_1})$ for the restriction $f_1$ to the complex subspace $\{x_s = 0, \forall s > 1\}$, decreases to $\nu_1$.

Finally, in the assumptions of $g$, the function $\Psi_{f,0}(y)$ is locally bounded on $D \setminus \{0\}$ by $f$, so the operator $(dd^c \Psi_{f,0})^n$ is well defined on $D$. Equation (23) is valid on the domain $D \setminus \{y : y_1 y_2 \ldots y_n = 0\}$ by Proposition 4 and then on $D \setminus \{0\}$, because the Monge-Ampère measure of a bounded plurisubharmonic function has zero mass on any pluripolar set (see [2]).

Remark. Statement $d$ of Proposition 5 is, in other words, that all the directional numbers $\nu(dd^c \Psi_{f,0}, \varphi_a)$ of $\Psi_{f,0}$ coincide with the directional numbers $\nu(dd^c f, \varphi_a)$ of the original function $f$, $\forall a \in \mathbb{R}^n_+$. The above construction is in fact of local character and Proposition 5 remains valid for the indicator $\Psi_{f,x_0}$ of any function $f(x)$ plurisubharmonic in a neighbourhood $\omega$ of a point $x^0 \in \mathbb{C}^n$, with the the following change in the statement $c)$: $(21)$ should be replaced by

$$\Psi_{f,x_0}(x - x_0^0) \geq f(x) + C \quad \forall x \in D(x_0^0, r), \quad C = C(u, r), \quad (24)$$

where $D(x_0^0, r) = \{x : |x_k - x_0^k| < r, 1 \leq k \leq n\}$ and $r > 0$ is such that the polydisk $D(x_0^0, r) \subset \subset \omega$. And of course the restriction $f_j$ in (22) should be taken to the subspaces $\{x_s = x_0^s, \forall s \neq j\}$.

Let now $f(x) \in PSH(\omega)$ be locally bounded on $\omega \setminus \{x^0\}$. Then its indicator $\Psi_{f,x_0}$ satisfies the equation

$$(dd^c \Psi_{f,x_0})^n = \tau_f(x_0^0) \delta(0) \quad (25)$$

with some number $\tau_f(x_0^0) \geq 0$ and $\delta(0)$ the Dirac measure at 0, and $\tau_f(x_0^0) > 0$ if and only if the Lelong number of the function $f$ at $x_0^0$ is strictly positive. And now we relate this value to $(dd^c f)^n$.

**Theorem 1** Let $f \in PSH(\omega)$ be locally bounded out of a point $0 \in \omega$. Then

$$(dd^c f)^n \geq \tau_f(0) \delta(0). \quad (26)$$
Proof. In view of (24), the function $f$ satisfies
\[
\limsup_{x \to 0} \frac{\Psi_{f,0}(x)}{f(x)} \leq 1. \tag{27}
\]
By the Comparison theorem of Demailly [7], Theorem 5.9, this implies
\[
(dd^c \Psi_{f,0})^n|_{\{0\}} \leq (dd^c f)^n|_{\{0\}}.
\]
On the other hand,
\[
(dd^c \Psi_{f,0})^n|_{\{0\}} = (dd^c \Psi_{f,0})^n = \tau_f(0) \delta(0)
\]
by (25), that gives us (26).

The theorem is proved.

Remark. It is well known that for any plurisubharmonic function $v$ with isolated singularity at 0, there is the relation
\[
(dd^c v)^n \geq \left[\nu(dd^c v, 0)\right]^n \delta(0). \tag{28}
\]
By the remark after the proof of Proposition 4, $\nu(dd^c f, 0)$ is equal to $\nu(dd^c \Psi_{f,0}, 0)$. Applying (28) to $v = \Psi_{f,0}$ we get, in view of Theorem 4
\[
(dd^c f)^n \geq (dd^c \Psi_{f,0})^n \geq \left[\nu(dd^c \Psi_{f,0}, 0)\right]^n = \left[\nu(dd^c f, 0)\right]^n,
\]
so (29) is an improvement of (28).

For example, if $f(x) = \log(|x_1|^{k_1} + |x_2|^{k_2})$ with $0 < k_1 < k_2$, then
\[
(dd^c f)^2 = \tau_f(0) \delta(0) = k_1 k_2 \delta(0) > k_1^2 \delta(0) = \left[\nu(dd^c f, 0)\right]^2 \delta(0),
\]
and thus $\tau_f(0) > \left[\nu(dd^c f, 0)\right]^2$.

More generally, if $F$ is a holomorphic mapping to $\mathbb{C}^n$ with an isolated zero at 0 of multiplicity $\mu_0$, and $f = \log |F|$, then
\[
\left[\nu(dd^c f, 0)\right]^n \leq \tau_f(0) \leq \mu_0.
\]

In fact, relation (27) makes it possible to obtain extra bounds for $(dd^c f)^n$ in case of $exp f \in C(\Omega)$. Such a function $f$ can be then considered as a plurisubharmonic weight $\varphi$ for Demailly’s generalized numbers $\nu(T, \varphi)$ of a closed positive current $T$ of bidimension $(p, p)$, $1 \leq p \leq n - 1$ [4]:
\[
\nu(T, \varphi) = \lim_{s \to -\infty} \int_{\{\varphi < s\}} T \wedge (dd^c \varphi)^p = T \wedge (dd^c \varphi)^p|_{\{0\}}.
\]
Moreover, the function $\Psi_{f,0}$ is such a weight, too. By Comparison theorem from [7], Theorem 5.1, relation (27) implies

$$\nu(T, \Psi_{f,0}) \leq \nu(T, f).$$

Take

$$T_k = (dd^c f)^k \wedge (dd^c \Psi_{f,0})^{n-k-1}, \ 1 \leq k \leq n-1.$$ 

These currents are well defined on a neighbourhood of 0 and are of bidimension $(1,1)$. Applying (29) to $T=T_k$ we obtain

$$T_k \wedge dd^c f|_0 \geq T_k \wedge dd^c \Psi_{f,0}|_0,$$

that gives us

**Proposition 6** Let $f \in PSH_-(\Omega)$ be locally bounded out of $\{0\}$ and $\exp f \in C(\Omega)$. Then

$$(dd^c f)^n|_0 \geq (dd^c f)^{n-1} \wedge dd^c \Psi_{f,0}|_0 \geq \ldots \geq (dd^c f)^{n-k} \wedge (dd^c \Psi_{f,0})^k|_0 \geq \ldots \geq (dd^c \Psi_{f,0})^n.$$ 

### 4 Dirichlet problem with local indicators

Let $\Omega$ be a bounded pseudoconvex domain in $\mathbb{C}^n$ and $K$ be a compact subset of $\Omega$. By $PSH(\Omega, K)$ we denote the class of plurisubharmonic functions on $\Omega$ that are locally bounded on $\Omega \setminus K$.

Let $K = \{x^1, \ldots, x^N\} \subset \Omega$ and $\{\Psi_m\}$ be $N$ indicators, i.e. circled functions in $PSH_-(D)$ whose convex images satisfy ([18]). In the sequel we assume that $\Psi_m \in PSH(D, \{0\})$. Then by Proposition [4],

$$(dd^c \Psi_m)^n = \tau_m \delta(0), \ 1 \leq m \leq N.$$ 

Let us fix the system $\Phi = \{(x^1, \Psi_1), \ldots, (x^N, \Psi_N)\}$ and consider a positive measure $T_\Phi$ on $\Omega$, defined as

$$T_\Phi = \sum_{1 \leq m \leq N} \tau_m \delta(x^m).$$ 

Each function $\Psi_m$ can be extended from a neighbourhood of the origin to a function $\tilde{\Psi}_m \in PSH(\mathbb{C}^n, \{0\})$, and the indicators of the functions

$$\tilde{\Psi}(x) = \sum_m \tilde{\Psi}_m(x - x^m) + A,$$

at $x^m$ are equal to $\Psi_m, 1 \leq m \leq N$, for any real number $A$. So the class

$$N_{\Phi, \Omega} = \{v \in PSH_-(\Omega, K) : \Psi_{v,x^m} \leq \Psi_m, 1 \leq m \leq N\}$$ 

is not empty.

Theorem [4] implies
Theorem 2 \((dd^c f)^n \geq T_\Phi \forall f \in N_{\Phi,\Omega}\).

Now we introduce the function
\[ G_{\Phi,\Omega}(x) = \sup\{v(x) : v \in N_{\Phi,\Omega}\}. \] (34)

Theorem 3 Let \(\Omega\) be a hyperconvex domain in \(\mathbb{C}^n\). Then the function \(G = G_{\Phi,\Omega}\) has the following properties:

a) \(G \in PSH^-(\Omega, K)\);

b) \(G(x) \to 0\) as \(x \to \partial \Omega\);

c) \(\Psi_{G,x^m} = \Psi_m, 1 \leq m \leq N\);

d) \((dd^c G)^n = T_\Phi\), the measure \(T_\Phi\) being defined by (31);

e) \(G \in C(\overline{\Omega} \setminus K)\).

Remark. In the case where \(\Psi_m = \nu_m \log |x|\), the function \(G_{\Phi,\Omega}\), the pluricomplex Green function with several weighted poles, was introduced in [16]. A situation with infinite number of poles was considered in [19], where a function \(G_{f,\Omega}\) was introduced as the upper envelope of the class \(\{v \in PSH^-(\Omega, K) : \nu(dd^c v, x) \geq \nu(dd^c f, x), \forall x\}\), \(f\) being a plurisubharmonic function with the following properties: 

e\(f\) is continuous, \(f^{-1}(-\infty)\) is a compact subset of \(\Omega\), and the set \(\{x : \nu(dd^c f, x) > 0\}\) is dense in \(f^{-1}(-\infty)\). Our proof is much the same as of the corresponding statements of [19].

Proof of Theorem 3. Since \(N_{\Phi,\Omega} \neq \emptyset\), the function \(G = G_{\Phi,\Omega}\) is well defined and
\[ G^* = \limsup_{y \to x} G(y) \in PSH^-(\Omega, K). \]

The function \(\tilde{\Psi}\) in (32) can be modified in a standard way to \(\tilde{\Psi}' \in PSH^-(\Omega, K)\) such that \(\tilde{\Psi}'(x) = \alpha \rho(x)\) in a neighbourhood of \(\partial \Omega\), \(\alpha\) being a positive number and \(\rho\) a bounded exhaustion function on \(\Omega\), and \(\tilde{\Psi}'(x) = \tilde{\Psi}(x) - \beta\) on a neighbourhood of \(K\). It shows us that
\[ G^* \geq \tilde{\Psi}'. \] (35)

It implies, in particular, that
\[ \Psi_{G^*,x^m} \geq \Psi_{\tilde{\Psi}',x^m} = \Psi_m, 1 \leq m \leq N. \] (36)
Since $\Psi_{\sup\{v,w\},x} \leq \sup\{\Psi_{v,x}, \Psi_{w,x}\}$ for any plurisubharmonic functions $v$ and $w$, there exists an increasing sequence of functions $v_j \in N_{\Phi,\Omega}$ such that $\lim_{j \to \infty} v_j = v \leq G$ and $v^* = G^*$.

The indicator of $v_j$ at $x^m$ is the limit of $R^{-1}g_{v_j,x^m}(R \log |y_k|)$ for $R \to +\infty$, the function $g_{v_j,x^m}(u)$ being the convex image of the mean value of $v_j(x^m_k + e^{u_k+i\theta_k})$ with respect to $\theta_k$ for $u_k < \log \text{dist}(x^m, \partial \Omega)$, $1 \leq k \leq n$, and the limit is obtained by the increasing values. It gives us

$$R^{-1}g_{v_j,x^m}(R \log |y_k|) \leq \Psi_{v_j,x^m}(y) \leq \Psi_m(y).$$

(37)

The functions $v_j$ increase to $G^*$ out of a pluripolar set $X = \{x \in \Omega : v(x) < v^*(x)\}$. Since the restriction of $X$ to the distinguished boundary of any polydisk is of zero Lebesgue measure \cite{14}, (37) implies that

$$R^{-1}g_{G^*,x^m}(R \log |y_k|) \leq \Psi_m(y).$$

and thus, taking $R \to +\infty$,

$$\Psi_{G^*,x^m} \leq \Psi_m, \quad 1 \leq m \leq N.$$  

(38)

As $G^* \in PSH_-(\Omega)$, the function $G^*$ belongs to the class $N_{\Phi,\Omega}$ and so

$$G^* \equiv G.$$  

(39)

By (38) and (39), $\Psi_{G^*,x^m} = \Psi_m$. It proves statements a) and c); statement b) follows from inequality (38).

Continuity of $G$ can be proved as in \cite{13}, Theorem 2.6, with the following modification. Instead of Demailly’s approximation theorem \cite{8} we use the similar fact: for any function $u \in PSH(\Omega)$ there exists a sequence of continuous plurisubharmonic functions $u_m$ satisfying

$$u(x) - \frac{c_1}{m} \leq u_m(x) \leq \sup\{u(x+y) : |y_k - x_k| \leq r_k, \ 1 \leq k \leq n\} + \frac{1}{m} \log \frac{c_2}{r_1 \ldots r_n}$$

and

$$\Psi_{u,x}(y) \leq \Psi_{u_m,x}(y) \leq \Psi_{u,x}(y) - \frac{1}{m} \log |y_1 \ldots y_n|, \quad \forall x \in \Omega, \ \forall y \in D.$$

To prove d), observe that in view of (38) and (39), $\tilde{\Psi}^\prime \leq G$. By the comparison theorem of Demailly (\cite{7}, Theorem 5.9), this implies

$$(dd^c G)^n |_{\{x^m\}} \leq (dd^c \tilde{\Psi}^\prime)^n |_{\{x^m\}} \leq [dd^c \Psi_m(x - x^m)]^n, \quad 1 \leq m \leq N,$$

and therefore

$$(dd^c G)^n |_{K} \leq T_\Phi.$$  

(40)
On the other hand, by Theorem 2,
\[(dd^c G)^n \geq T_\Phi.\]
Being comparing to (40) this provides
\[(dd^c G)^n |_K = T_\Phi.\]
Finally, the equality \((dd^c G)^n = 0\) on \(\Omega \setminus K\) can be proved in a standard way by showing it is maximal on \(\Omega \setminus K\) (see [1] , [3], that proves d).
The theorem is proved.

As a consequence, we get an "indicator" variant of the Schwarz type lemma (see [16], [19]):

**Theorem 4** Let the indicator of a function \(g \in PSH(\Omega)\) at \(x^m\) does not exceed \(\Psi_m, 1 \leq m \leq N\), and let \(g(x) \leq M\) on \(\Omega\). Then \(g(x) \leq M + G_{\Phi,\Omega}(x), \forall x \in \Omega\).

Now we are going to show that the function \(G_{\Phi,\Omega}\) is the unique plurisubharmonic function with the properties a) – d) of Theorem 3. It is known that for unbounded plurisubharmonic functions \(u\), the Dirichlet problem
\[
\begin{cases}
(dd^c u)^n = \mu \geq 0 & \text{on } \Omega \\
v = h & \text{on } \partial \Omega
\end{cases}
\] (41)
need not have a unique solution even in a simple case \(\mu = \delta(0), \ h \equiv 0\). However, a solution is unique under some regularity assumptions on the functions \(v\). For example, as was established in [19], (41) has a unique solution for
\[
\mu = \sum [\nu(dd^c f, x^m)]^n \delta(x^m) \tag{42}
\]
with \(f(x)\) specified in the remark after the statement of Theorem 3, if the functions \(v(x) \in PSH_-(\Omega, K)\) have to satisfy
\[
\nu(dd^c v, x^m) = \nu(dd^c f, x^m), \ 1 \leq m \leq N. \tag{43}
\]
These additional relations mean that
\[
v(x) \sim \nu(dd^c v, x^m) \log |x - x^m| \text{ near } x^m \tag{44}
\]
\((v\) has regular densities at its poles, in the terminology of [10]).

In our situation,
\[
\mu = T_f = \sum \tau_m \delta(x^m), \tag{45}
\]
where \(\tau_m\) are defined by (30) with \(\Psi_m = \Psi_{f,x^m}\), and we are going to replace condition (13) by \(\Psi_{v,x^m} = \Psi_{f,x^m}, \ 1 \leq m \leq N\).

To prove the uniqueness, we need a variant of the comparison theorem for unbounded plurisubharmonic functions (see [1] - [4], [3], [11], [19] for different classes of plurisubharmonic functions).
Theorem 5 Let \( f \in PSH(\Omega, K) \), \( K = \{x^1, \ldots, x^m\} \), and
\[
(dd^c f)^n|_K = T_f,
\]
the measure \( T_f \) being given by (44). Let \( v \in PSH(\Omega, K) \) satisfy the conditions
1) \( \lim \inf_{x \to \partial \Omega} (f(x) - v(x)) \geq 0 \);
2) \( (dd^c v)^n \geq (dd^c f)^n \) on \( \Omega \setminus K \);
3) \( \Psi_{v,x^m} \leq \Psi_{f,x^m}, \ 1 \leq m \leq N \).
Then \( v \leq f \) on \( \Omega \).

The proof is just as of Theorem 3.3 of [19], and we omit it here.

Corollary 1 Under the conditions of Theorem 3, the function \( G_{\Phi, \Omega} \) is the unique plurisubharmonic function with the properties a) – d) of that theorem.

Remarks.
1. Condition (46) is essential. Indeed, let \( f(x) = \frac{1}{2} \log(|x_1|^4 + |x_1 + x_2|^2), \ v(x) = \frac{1}{2} \log(|x_1|^2 + |x_2|^4) + m \) with \( m = \inf \{f(x) : |x| = 1\} > -\infty \). Then \( v(x) \leq f(x) \) for \( |x| = 1 \), \((dd^c f)^2 = (dd^c v)^2 = 0\) on \( \{0 < |x| < 1\} \), and \( \Psi_{v,0}(x) = \Psi_{f,0}(x) = \log \max \{|x_1|, |x_2|^2\} \).
However, for \( x_2 = t \in (0, e^m) \), \( x_1 = -x_2^2 \), \( f(x) = 2 \log t < \log t + m < v(x) \). The reason here is that \((dd^c f)^2 = 4\delta(0) > 2\delta(0) = T_f\).

2. By Comparison theorem of Demailly [7], relation (46) is true when \( f(x) \sim \Psi_{f,x^m}(x - x^m) \) near \( x^m \), a weaker than (44) but still controlled regularity.

3. By Proposition 5, an indicator \( \Psi \) possesses the properties a) – d) of \( G_{\Phi, \Omega} \) from Theorem 3 with \( \Omega = D \), the unit polydisk, and \( \Phi = (\{0\}, \Psi) \). Therefore, \( \Psi = G_{D, \Phi} \).

Theorems 3 and 5 allow us also to state the following result.

Theorem 6 Let \( \Omega \) be a bounded strictly pseudoconvex domain, \( K = \{x^1, \ldots, x^m\} \), and let a function \( f \in PSH(\Omega, K) \) satisfy
\[
(dd^c f)^n = T_f.
\]
Then the Dirichlet problem
\[
\begin{aligned}
(d^2 v)^n &= T_f \quad \text{on } \Omega \\
\Psi_{v,x^m} &= \Psi_{f,x^m} \quad \text{for } 1 \leq m \leq N \\
v &= h \quad \text{on } \partial \Omega
\end{aligned}
\]
has a unique solution in the class $PSH(\Omega, K)$ for each function $h \in C(\partial \Omega)$. This solution is continuous on $\Omega \setminus K$.

Proof. Let $\Phi = \{(x^m, \Psi_m)\}$ with $\Psi_m = \Psi_{f,x^m}$. Consider the class
\[
N_{f,h} = \{ v \in PSH(\Omega, K) : \Psi_{v,x^m} \leq \Psi_{f,x^m} \forall m, \lim_{x \to y} v(x) = h(y) \forall y \in \partial \Omega \}.
\]
Let $u_0(x)$ be the unique solution of the corresponding homogeneous problem
\[
\begin{aligned}
(d^2 u)^n &= 0 \quad \text{on } \Omega \\
u &= h \quad \text{on } \partial \Omega.
\end{aligned}
\]
Then $u_0 + G_{\Phi,\Omega} \in N_{f,h}$, so $N_{f,h} \neq \emptyset$.

The desired solution $v_0$ is given as
\[
v_0(x) = \sup \{v(x) : v \in N_{f,h}\}.
\]
Just as in the proof of Theorem 3, one can show that $v_0$ does solve the problem and is continuous on $\overline{\Omega} \setminus K$. The uniqueness follows from Theorem 5.

Theorem 6 can be related to the following question which was one of the motivations of the present study. Let $F : \overline{\Omega} \to \mathbb{C}^n$ be a holomorphic mapping with isolated zeros $\{x^m\} \subset \Omega$ of multiplicities $\mu_m$. Then the function $f(x) = \log |F(x)|$ solves the Dirichlet problem
\[
\begin{aligned}
(d^2 v)^n &= \sum \mu_m \delta(x^m) \quad \text{on } \Omega \\
v &= f \quad \text{on } \partial \Omega.
\end{aligned}
\]
Under what extra conditions on $v$, the function $f$ is the unique solution of the problem? By Theorem 3, if $f$ has regular behaviour at $x^m$ with respect to its indicators, i.e. if
\[
(d^2 \Psi_{f,x^m})^n = \mu_m \delta(x^m), \quad 1 \leq m \leq N,
\]
it gives the unique solution to the problem
\[
\begin{aligned}
(d^2 v)^n &= \sum \mu_m \delta(x^m) \quad \text{on } \Omega \\
\Psi_{v,x^m} &= \Psi_{f,x^m} \quad 1 \leq m \leq N \\
v &= f \quad \text{on } \partial \Omega.
\end{aligned}
\]

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