Folding back and pseudo-folding back of the student when solving the limit problem

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Abstract. Definition of limits is one of the complex definitions in mathematics. Difficulties occur when students link the solution of limit problem to the definition of the limit. When the students face the difficulties, they need to folding back to recall the previous knowledge to overcome those difficulties. The keyword of folding back is the occurrence of thickness on previous knowledge. The fact often happens that students are no thickening of knowledge so that the students are not able to solve the problem. This fact is called as pseudo-folding back. This study aims to reveal the folding back and the pseudo-folding back of the students when students link the solution of limit problem to the definition of the limit. The approach of this study is case study and one student became the participant of this study. The results of this study showed that the folding back and the pseudo-folding back occurred when the student do the cancellation and link the statement is approaching two but not equal with two with the definition of limit.

1. Introduction
Understanding is fundamental in learning mathematics. Various experts have been studied a lot about understanding. Skemp [1] divides this understanding into two major groups, namely relational understanding and instrumental understanding. This relational and instrumental understanding of Skemp [1] is another viewpoint of Hibert's [2] conceptual knowledge and procedural knowledge. Meanwhile, Pirie and Kieren [3] view understanding as a process of growth that is intact, dynamic, layered but not linear, and never ends. They reject the concept of growth of understanding as a function that rises monotonically.

Understanding growth theory of Pirie and Kieren states that understanding is a dynamic, active, and continuous process but not linear. Understanding is a continuous process that involves levels or layers of different understandings and not a linear system like a ladder. The process of returning to a inner layer of certain layers in Pirie's and Kieren's understanding theory is called folding back. Pirie & Kieren [3] provide a theoretical framework in the form of eight levels of understanding which are also called layers of understanding, namely: primitive knowing, image making, image having, property noticing, formalising, observing, structuring, and inventorying. One important element of Pirie & Kieren's [3] understanding theory is folding back.

Folding back occurs when students are faced with a problem in any outer layer but cannot quickly solve it so that it returns to a deeper layer. The keyword of folding back according to Pirie and Kieren is the occurrence of thickening of knowledge in the inner layer. This will happen if the student's inner
layer is sufficient so as to strengthen his knowledge in this inner layer [4]. The fact is that the inner layer of students is not enough so that there is no thickening in this inner layer [5] so the students cannot move to the outer layer. This event is called a pseudo-folding back. This pseudo-folding back occurs when the students solve the limit problem associated with the definition of limit.

Existing studies have not revealed the link between difficulties with primitive knowing adequacy of students. When the primitive knowing students are not enough, then pseudo-folding back occurs. Therefore, this study aims to uncover folding back and pseudo-folding students when linking the limit definition with the solution of limit problems. This research yields insight into the interesting phenomenon of the occurrence of folding back and pseudo-folding back to limit problems that have never been discussed in the previous literature. When mathematics educators know folding back and pseudo folding back students in solving limit problems, the educator will also know the location of the student's difficulties in solving the limit problem. Thus, the main benefit of this study was that it could help mathematics educators to stimulate the growth of students' mathematical understanding of the concept of limits.

2. Method
This research aims to uncover folding back and pseudo-folding back of the students when solving limit problems. This study uses a case study approach [6, 7]. This research was carried out in the Mathematics Department of FMIPA State University of Malang (UM), Indonesia. Since the topic of limit is a difficult topic for students, then the study participants are the best students of the third semester of one class (from five parallel classes), called Ika. The consideration that this student is the best is best based on the first and the second semester grades. She got an A in the Calculus course. In addition, this consideration is also based on the observations of the researcher in learning during the two semesters that have been running. At first Ika was asked to solve the limit problem

\[ \lim_{x \to 2} \frac{x^2 - 4}{x - 2}. \]

Furthermore, the occurrence of folding back and pseudo-folding back is explored through interviews. Due to page limitations, only two interviews are included in the results. To guarantee the accuracy of the data, the interview is equipped with a video recorder.

3. Results and discussion
This study reveals folding back and pseudo-folding back when students associate the solution of limit with the limit definition. Fig. 1 below is a form of Ika’s solution.

![Image of Ika's solution to the limit problem](image)

**Figure 1.** The solution of limit problem by Ika

Ika stated that her solution is based on the limit theorem. Therefore, she is at the level of understanding structuring.

This article reveals folding back and pseudo-folding back when she associated the solution in the first line with the definition of limit, namely when describing the solution

\[ \lim_{x \to 2} \frac{(x+2)(x-2)}{(x-2)} = \lim_{x \to 2} (x + 2). \]
Ika stated that because in the limit concept when $x$ is approaching 2 means $x$ is only approaching 2 but it is not equal to 2 so that it she can cancel $x - 2$ (Interview 1). Ika can solve mathematical problems and see the connection between the concept of limits and algebraic operations. She can see the connection between a concept with another concept [1, 2, 8]. Ika returned to the level of understanding primitive knowing and can connect her solution with the intuitive definition of limit. So, understanding at the primitive level of knowing experiencing thickening (folding back).

**Interview 1** Ika’s Explanation about \( \lim_{x \to 2} \frac{(x-2)(x+2)}{(x-2)} = \lim(x + 2) \)

Furthermore, Ika was asked about statements $x$ is approaching two but $x$ is not equal to two. Ika stated that since $\delta$ is positive then $|x - c| < \delta$ has the same meaning as $x$ is not equal to two. She was wrong in understanding the meaning of the inequality involving absolute value $|x - c| < \delta$. She knew the solution of that inequality, which is the value of $x$ that satisfies the inequality $c - \delta < x < c + \delta$ but she has difficulty in interpreting the solution the inequality. This finding is supported by the result of [9] which states that most of the students cannot make geometrical interpretations of absolute value. Ika still works in the level of primitive knowing on inequality of absolute value. Her understanding at this level is thickening (folding back) but she cannot to move to the next level. This event is referred to as ineffective folding back [4]. Ika stated that it is impossible to find $f(2)$ since 2 is not in the domain. Thus, we approached 2 from left (1.99) and from the right (2.01) to find $\frac{x^2-4}{x-2}$ (Interview 2). Ika still works at the level of understanding primitive knowing. The Ika’s solution at this step is not initially related to the concept of the limit, that is, when she declared that it is impossible to find $f(2)$. Her understanding is not thickening (pseudo-folding back). This happens because prior knowledge is not enough to solve the problem at hand [10]. Furthermore, Ika associated the rational function $f(x) = \frac{x^2-4}{x-2}$ with the left limit and the right limit, but she cannot associate it with the definition of limit. She has difficulty to understand the definition of limit. This is supported by the results of the research by [11, 12, 13] which states that the concept of limit is one of the mathematics concept that is difficult to understand by students.

**Researcher:** Yes, really, earlier Ika said that $x$ is not equal to two in the limit $x$ is close to two. How do you get the statement $x$ is not equal to two?

**Ika:** Oh, for example like this, (while pointing $\frac{x^2-4}{x-2}$) two is not in the domain. So it is impossible to find $f(2)$, because the domain does not exist. Therefore, we use the approaching from the right or from the left, for example 1.99 is approaching from the left while 2.01 is approaching from the right, for example to find this, (while pointing $\frac{x^2-4}{x-2}$).

**Interview 2** The relationship between $x$ is approaching two but $x \neq 2$ with rational Function $f(x) = \frac{x^2-4}{x-2}$.

Ika was asked again about statements $x$ is approaching two but $x$ is not equal to two. She again pointed to the definition of limit. She still works at the level of understanding primitive knowing about the intuitive definition of limit. She cannot connect the intuitive definition of limit with the definition
of limit. She cannot explain the link between statements $x$ is approaching two but $x$ is not equal to two with the definition of limit. Her understanding in this level is not thickening (pseudo-folding back). Furthermore, Ika stated that her reason to say that $x$ is not the same as two is the statement $0 < |x - 2|$. Since this inequality does not involve equality sign. Thus, if $x = 2$ is substituted into the inequality $0 < |x - 2|$, then it yields $0 < 0$. She is still working at the level of understanding primitive knowing and she can link the statement $x$ is not equal to two with absolute inequality $0 < |x - 2|$. Therefore, there is a thickening at the level of understanding primitive knowing (folding back). She can associate the statement $x$ is not equal to two with the definition of limit.

Next, Ika was asked about the statement that $x$ is approaching two. Ika said that the statement $|x - 2|$ means $x - 2$ if $x - 2 \geq 0$ and $2 - x$ if $x - 2 < 0$. Furthermore, she stated that there were two statements, namely $x - 2 < 0$ and $x - 2 \geq 0$. She is working on the definition of limit but it is not in accordance with the problem she faces. Her behaviour is called the pseudo-conceptual behavior by [14]. Ika goes on to say that $x$ can approach two from left and right, namely $x < 2$ means $x$ is approaching two from the right and $x \geq 2$ means $x$ is approaching two from the right. Her reason about $x$ approaching two is not correct. Thus, her level of understanding about this is not thickening (pseudo-folding back). She also stated that since $\delta$ is a small positive number that depends on $\varepsilon$ which is also the smallest positive number, then the statement $|x - 2| < \delta$ implies that the difference between $x$ and 2 is small. She said that because of the absolute value it can be concluded that $x$ is approaching two from the right and from the left. She is working at the level of understanding primitive knowing and can associate statements $x$ is approaching two with the inequality $|x - 2| < \delta$. Thus, her understanding at this level is thickening (folding back). Ika can associate the statement $x$ is approaching two with the definition of limit. She can see the connection between a concept and another concept (see [1, 2, 8]).

4. Conclusion
Folding back and Pseudo-folding back occurs when she was explaining $\lim_{x \to 2} \frac{(x-2)(x+2)}{(x-2)} = \lim_{x \to 2} (x + 2)$, explaining the statement $x$ is not the equal to two, and explaining the statement $x$ is approaching two. Folding back occurs when student can connect the solution with the concept of limits so that there is thickening at the level of knowing primitive understanding [3, 4, 15]. Conversely, a pseudo-folding back occurs when student cannot link her solution with the limit concept because primitive knowing is insufficient. There are three main points that cause the pseudo-folding back. Firstly, student only performs algebraic operations without linking them to the concept of limits [16]. Secondly, student seems to do conceptual and analytical behavior, but actually what is done is pseudo-conceptual and pseudo-analytical behavior [14]. Thirdly, student can determine the solution of the inequality of absolute values but she cannot interpret the solution [9].

At the level of primitive knowing, student cannot associate statement $x$ is approaching two but $x$ is not equal to two with absolute value inequality $0 < |x - 2| < \delta$. Suggestions for teachers are to give an in-depth understanding of absolute value inequality analytically and geometrically to their students.

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