SUM-FREE SETS WHICH ARE CLOSED UNDER MULTIPLICATIVE INVERSES

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Abstract. Let $A$ be a subset of a finite field $F$. When $F$ has prime order, we show that there is an absolute constant $c > 0$ such that, if $A$ is both sum-free and equal to the set of its multiplicative inverses, then $|A| < (0.25 - c)|F| + o(|F|)$ as $|F| \to \infty$. We contrast this with the result that such sets exist with size at least $0.25|F| - o(|F|)$ when $F$ has characteristic 2.

1. Introduction

Let $A$ be a subset of a finite field $F$. We say $A$ is sum-free if $A \cap (A + A) = \emptyset$, where

$$A + A := \{a + b : a, b \in A\}.$$ 

We say $A$ is closed under (multiplicative) inverses if $0 \not\in A$ and $A = A^{-1}$, where

$$A^{-1} := \{a^{-1} : a \in A\}.$$ 

In this paper, we study sets which are both sum-free and closed under inverses.

When $F$ has prime order, a simple application of the Cauchy-Davenport inequality (see e.g. [TV06, Theorem 5.4]) shows that $|A| \leq (|F| + 1)/3$ when $A$ is sum-free.

Lev showed in [Lev06] that when $|A|$ is close to $|F|/3$, $A$ is similar in structure to an arithmetic progression, and therefore unlikely to be closed under inverses. So, we might expect $|A|$ to be smaller than $|F|/3$ if $A$ is also closed under inverses.

In this direction, Bienvenu et al. showed in [BHS19, Corollary 5.1] that $|A| < 0.3051|F| + o(|F|)$ as $|F| \to \infty$. We offer the following improvement on this:

Theorem 1.1. There is an absolute constant $c > 0$ so that if $F$ is a field of prime order and $A \subseteq F^*$ is sum-free and closed under inverses then $|A| < (0.25 - c)|F| + o(|F|)$ as $|F| \to \infty$.

This is in contrast to fields of characteristic 2, where we show:

Proposition 1.2. If $F$ is a field of characteristic 2 then there exists $A \subseteq F^*$ which is both sum-free and closed under inverses, such that $|A| = 0.25|F| + o(|F|)$ as $|F| \to \infty$.

Write $\mu(F)$ for the density $|A|/|F|$ of the largest $A \subseteq F$ which is both sum-free and closed under inverses. Theorem 1.1 says that $\mu(F_p) \leq 0.25 - c + o(1)$, whereas Proposition 1.2 says that $\mu(F_{2^n}) \geq 0.25 - o(1)$. So we can deduce that:

Corollary 1.3. The limit $\lim_{|F| \to \infty} \mu(F)$ does not exist.

The rest of the paper is structured as follows. In Section 2 we recall some basic definitions of Fourier analysis, and establish some notation. In Section 3 we consider fields of prime order. We establish some Fourier analytic results and use them to prove Theorem 1.1. Then, in Section 4 we consider fields of even characteristic, and prove Proposition 1.2. In Section 5 we make some final remarks.
2. Notation and definitions from Fourier analysis

Let $F$ be a finite field. We recall some basic definitions from Fourier analysis (see e.g. [TV06, Section 4] or [Wol15, Section 1.1]).

If $X \subseteq F$ is non-empty and $f : X \rightarrow \mathbb{C}$ is any function, we define the mean

$$E_{x \in X}[f(x)] := \frac{1}{|X|} \sum_{x \in X} f(x).$$

We will also write $E[f] = E_{x \in F}[f(x)] = E_{x \in F}[f]$ when it is unambiguous to do so. We denote by $1_X$ the indicator function

$$1_X(x) := \begin{cases} 1 & \text{if } x \in X, \\ 0 & \text{otherwise}. \end{cases}$$

When $F$ has prime order $p$ we can view the set of functions $F \rightarrow \mathbb{C}$ as a Hilbert space by equipping it with the inner product

$$\langle f, g \rangle := E[f \cdot g].$$

Write $e^\theta = \exp(i \theta)$ for the exponential map $\mathbb{R} \rightarrow \mathbb{C}$. For each $r \in F$, define the character

$$\chi_r : F \rightarrow \mathbb{C} \text{ by } \chi_r(x) := e(2\pi rx/p).$$

The characters enjoy the following orthogonality property:

$$\langle \chi_r, \chi_s \rangle = \begin{cases} 1 & \text{if } r = s, \\ 0 & \text{otherwise}. \end{cases}$$

This motivates the definition of the Fourier coefficient of $f$ at $r$ as

$$\hat{f}(r) := \langle f, \chi_r \rangle.$$

Parseval’s identity is then

$$E[|f|^2] = \sum_{r \in F} |\hat{f}(r)|^2.$$

3. Fields of prime order

The goal of this section is to prove Theorem 1.1. Let $F = F_p$ be a field of prime order $p > 2$. Let $A$ be a subset of $F^*$, not necessarily sum-free or closed under inverses, with density $\alpha = |A|/p$. We fix some $0 < \alpha_0 < 0.25$ and assume $\alpha \geq \alpha_0$, since otherwise Theorem 1.1 is immediate.

Order the elements $r_1, \ldots, r_{(p-1)/2}$ of the interval $\{1, \ldots, (p-1)/2\} \subseteq F$ so that $\delta_1 \geq \cdots \geq \delta_{(p-1)/2}$, where $|\Gamma_A(r_i)| = \delta_i \alpha$. Note that

$$F^* = \{r_1, \ldots, r_{(p-1)/2}\} \cup \{-r_1, \ldots, -r_{(p-1)/2}\}$$

and that $\Gamma_A(-r_i) = \overline{\Gamma_A(r_i)}$ for each $i$. We will also write $\theta_i \in [0, 2\pi)$ for the argument of $\Gamma_A(r_i)$, so that $\Gamma_A(r_1) = (\delta_1 \alpha) e(\theta_1)$ and $\Gamma_A(r_1) + \Gamma_A(-r_1) = 2\delta_1 \alpha \cos \theta_1$.

3.1. Properties of sum-free sets. We begin by recalling a standard identity, which can be derived by considering the convolution $1_A * 1_A$ (see e.g. [TV06, p. 153]).

**Proposition 3.1.** If $A$ is sum-free then

$$\alpha^3 + \sum_{r \neq 0} |\Gamma_A(r)|^2 \overline{\Gamma_A(r)} = 0.$$

In fact, this sum is dominated by its largest terms.

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1 We follow the notation of [TV06]. It is also common to write $e_p(x) = e(2\pi x/p)$. 
Lemma 3.2. Let $k$ be a positive integer. For any $p$ such that $k < (p-1)/2$, if $A \subseteq \mathbb{F}_p$ then
\[
\sum_{i \geq k} \delta_i^3 \to 0
\]
as $k \to \infty$, uniformly in $A$ provided $\alpha \geq \alpha_0$.

Proof. From Parseval’s identity we know
\[
\alpha^2 + 2\alpha^2 \sum_{i \geq 1} \delta_i^2 \leq \alpha,
\]
whence, looking at the first $k$ terms of the sum,
\[
\delta_k^2 \leq \frac{1 - \alpha^2}{2k \alpha}.
\]
So
\[
\sum_{i > k} \delta_i^3 \leq \delta_k \sum_{i > k} \delta_i^2 \leq k^{-1/2} \left( \frac{1 - \alpha^2}{2\alpha} \right)^{3/2} \leq k^{-1/2} \left( \frac{1 - \alpha_0^2}{2\alpha_0} \right)^{3/2} \to 0.
\]

\[\square\]

Corollary 3.3. If $A$ is sum-free then
\[
\sum_{i=1}^k \delta_i^3 \geq \delta_1^3 \left| \cos \theta_1 \right| + \sum_{i=2}^k \delta_i^3 \geq \frac{1}{2} - o_{k \to \infty}(1),
\]
where the error is uniform in $A$ provided $\alpha \geq \alpha_0$.

Proof. The first inequality is immediate. For the second, we begin with Proposition 3.1 and make two applications of the triangle inequality.

\[
\alpha^3 = \left| \sum_{r \neq 0} \hat{1}_A(r) \hat{1}_A(r) \right| = \left| \sum_{i=1}^{(p-1)/2} \delta_i^2 \alpha^2 \left( \hat{1}_A(r_i) + \hat{1}_A(-r_i) \right) \right| \leq \sum_{i=1}^{(p-1)/2} \delta_i^2 \alpha^2 \left| \hat{1}_A(r_i) + \hat{1}_A(-r_i) \right| \leq \delta_1^2 \alpha^2 \left| 2\delta_1 \alpha \cos \theta_1 \right| + \sum_{i=2}^{(p-1)/2} \delta_i^2 \alpha^2 \left( \left| \hat{1}_A(r_i) \right| + \left| \hat{1}_A(-r_i) \right| \right) = 2\delta_1^3 \alpha^3 \cos \theta_1 + \sum_{i=2}^{(p-1)/2} 2\delta_i^3 \alpha^3.
\]
Now divide through by $2\alpha^3$ and apply Lemma 3.2.

Another corollary of Proposition 3.1 gives bounds on $\alpha$ in terms of the sizes of the largest two Fourier coefficients. The first, which considers only $\delta_1$, is standard (c.f. [Lev06, p. 226]). The second is stronger when $\delta_2$ is small compared to $\delta_1$.

Corollary 3.4. If $A$ is sum-free then
\[
\alpha \leq \frac{\delta_1}{1 + \delta_1}.
\]
Moreover, if $1 + \delta_2 + 2\delta_1^2 \delta_2 - 2\delta_1^3 > 0$ then
\[
\alpha \leq \frac{\delta_2}{1 + \delta_2 + 2\delta_1^2 \delta_2 - 2\delta_1^3}.
\]
Proof. We prove the second bound. The first is proved similarly. We begin with Proposition 3.1 by Weil’s bound. Also, using the fact that the characters are orthonormal, we have

\[
\psi(x) = \prod_{\alpha \neq 0} |\frac{\alpha}{p}| \sum_{r \neq 0} |\overline{\Lambda}_A(r)|^2 |\overline{\Lambda}_A(r)| \leq 2\delta_1^3 \alpha^3 + \left| \sum_{r \neq 0, \pm 1} |\overline{\Lambda}_A(r)|^2 |\overline{\Lambda}_A(r)| \right|
\]

\[
\leq 2\delta_1^3 \alpha^3 + \delta_2 \alpha \sum_{r \neq 0, \pm 1} |\overline{\Lambda}_A(r)|^2 = 2\delta_1^3 \alpha^3 + \delta_2 \alpha (\alpha^2 - 2\delta_1^2 \alpha^2).
\]

To get the final step here we use Parseval’s identity. Now rearrange to find

\[
\alpha (1 + \delta_2 + 2\delta_1^2 \delta_2 - 2\delta_1^2) \leq \delta_2
\]

and apply the hypothesis. \(\Box\)

3.2. Properties of sets which are closed under inverses. To exploit the fact that \(A = A^{-1}\) we will make use of the following result from [Bom71 Proposition 1], which can be thought of as a version of Bessel’s inequality for vectors which are ‘almost orthogonal’.

Lemma 3.5. Let \(H\) be a Hilbert space with inner product \(\langle , \rangle\). Then for any \(f, \phi_1, \ldots, \phi_M \in H\) we have the inequality

\[
\|f\|^2 \geq \sum_{i=1}^{M} \frac{|\langle f, \phi_i \rangle|^2}{\sum_{j=1}^{M} |\langle \phi_i, \phi_j \rangle|}.
\]

We also recall Weil’s estimate for Kloosterman sums [Wei48, p. 207].

Lemma 3.6 (Weil’s estimate). If \(p\) is prime and \(a, b\) are integers with \(ab \neq 0\) then

\[
\left| \sum_{x \in \mathbb{F}_p^*} e_a(x) e_b(x^{-1}) \right| \leq 2\sqrt{p}.
\]

We arrive at a useful bound on the size of a set which is closed under inverses.

Proposition 3.7. Suppose \(A = A^{-1}\) and let \(m \geq 0\). Suppose \(s_1, \ldots, s_m\) are distinct elements of \(\mathbb{F}_p^*\) with \(|\overline{\Lambda}_A(s_i)| = \lambda_i \alpha\). Then

\[
\alpha \leq \frac{1}{1 + 2 \sum_{i=1}^{m} \lambda_i^2} + O\left( m/\sqrt{p} \right).
\]

Moreover, if \(k \geq 0\) then we have the bound

\[
\alpha \leq \frac{1}{1 + 4 \sum_{i=1}^{k} \delta_i^2} + O\left( k/\sqrt{p} \right).
\]

Proof. Define \(s_0 := 0\), and so \(\lambda_0 = 1\). For each \(i\) define \(\phi_i := e_{s_i}\) and, if \(i > 0\), \(\psi_i(x) := \phi_i(x^{-1})\), with the convention that \(0^{-1} = 0\). We aim to apply Lemma 3.5 to \(1_A\) and these ‘almost orthogonal’ functions. For \(i \geq 0\) and \(j > 0\) we have

\[
|\langle \phi_i, \phi_j \rangle| = \frac{1}{p} \left| \sum_{x \in \mathbb{F}_p} e_{s_i}(x) e_{s_j}(x^{-1}) \right| = \frac{1}{p} \left| \sum_{x \in \mathbb{F}_p} e_{s_i}(x) e_{-s_j}(x^{-1}) \right| \leq \frac{1 + 2 \sqrt{p}}{p}
\]

by Weil’s bound. Also, using the fact that the characters are orthonormal, we have

\[
\langle \psi_i, \psi_j \rangle = \mathbb{E}_x \left[ \phi_i(x^{-1}) \overline{\phi_j(x^{-1})} \right] = \mathbb{E}_x \left[ \phi_i(x) \overline{\phi_j(x)} \right] = \langle \phi_i, \phi_j \rangle = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}
\]
Finally,
\[ |\{1_A, \psi_i\}| = \frac{1}{p} \sum_{a \in A} \varphi_i(a^{-1}) = \frac{1}{p} \sum_{a \in A} \varphi_i(a) = |\{1_A, \varphi_i\}| = |\overline{1_A}(s_i)| = \lambda_i \alpha. \]

So, applying Lemma 3.3, we find
\[
\alpha \geq \sum_{i=0}^{m} \frac{\lambda_i^2 \alpha^2}{1 + m (1 + 2 \sqrt{p}) / p} + \sum_{i=1}^{m} \frac{\lambda_i^2 \alpha^2}{1 + (m + 1) (1 + 2 \sqrt{p}) / p},
\]
from which the result follows.

For the moreover part, take \( m = 2k \) and \( s_i = r_i = -s_{m-i} \) for each \( i \leq k \).

3.3. Constructing large coefficients. If \( |\overline{1_A}(r)| = \delta \alpha \) then an observation of Yudin recorded in [Lev01, p. 258] yields the following bound on \( |\overline{1_A}(2r)| \):

\[ |\overline{1_A}(2r)| \geq (2 \delta^2 - 1) \alpha. \]

We strengthen this in two ways. First, we show that, given conditions on \( \delta \) and the argument \( \theta \) of \( \overline{1_A}(r) \), the coefficient \( \overline{1_A}(2r) \) lies in the right-half plane of \( C \). Second, we show that given some lower bound on \( \alpha \), we can obtain a slightly stronger lower bound on \( |\overline{1_A}(2r)| \). We shall prove (1) along the way.

**Lemma 3.8.** Suppose \( r \neq 0 \) and \( \overline{1_A}(r) = (\delta \alpha)e(\theta) \). Then
\[ 2 \Re \overline{1_A}(2r) = \overline{1_A}(2r) + \overline{1_A}(-2r) \geq 2 \alpha (2 \delta^2 \cos^2 \theta - 1). \]

Moreover, if \( \alpha \geq \alpha_0 > 0 \) then
\[ |\overline{1_A}(2r)| \geq (2 \delta^2 - 1 + \varepsilon - o(1)) \alpha \]
as \( p \to \infty \), where the error is uniform in \( A \) and \( \varepsilon > 0 \), which depends only on \( \alpha_0 \), is given by
\[ \varepsilon = \frac{2^9}{3^4 \times 5^5} \alpha^4. \]

**Proof.** For any \( \omega \in S^1 \), it can be seen that
\[ \mathbb{E}_x \left[ 1_A(x) (\overline{\varphi_r(x)} + \omega \overline{e_{-r}(x)})^2 \right] = 2 \alpha + \omega^2 \overline{1_A}(2r) + \overline{\omega}^2 \overline{1_A}(-2r). \]

By applying Cauchy-Schwarz we can compute
\[
\mathbb{E}_x \left[ 1_A(x) \right] \mathbb{E}_x \left[ 1_A(x) (\overline{\varphi_r(x)} + \omega \overline{e_{-r}(x)})^2 \right] \geq \mathbb{E}_x \left[ 1_A(x) \right] \mathbb{E}_x \left[ 1_A(x) (\varphi_r(x) + \omega e_{-r}(x))^2 \right] = \left( \omega \overline{1_A}(r) + \overline{\omega} \overline{1_A}(-r) \right)^2.
\]

Setting \( \omega = 1 \) and substituting in (2) then gives
\[ \alpha \left( 2 \alpha + \overline{1_A}(2r) + \overline{1_A}(-2r) \right) \geq \left( \overline{1_A}(r) + \overline{1_A}(-r) \right)^2 = 4 \delta^2 \alpha^2 \cos^2 \theta, \]
from which the first inequality follows.

If instead we take \( \omega = e(\theta) \) then we find
\[ \alpha \left( 2 \alpha + \omega^2 \overline{1_A}(2r) + \omega \overline{1_A}(-2r) \right) \geq \left( |\overline{1_A}(r)| + |\overline{1_A}(r)| \right)^2 = (2 \delta \alpha)^2, \]
which rearranges with the triangle inequality to give (1).

The Cauchy-Schwarz inequality \( \mathbb{E}[XY]^2 \leq \mathbb{E}[X^2] \mathbb{E}[Y^2] \) is only close to equality when the random variables \( X \) and \( Y \) are close to proportional. However, \( 1_A(x) \) and \( 1_A(x) \cdot (\overline{\varphi_r(x)} + \omega \overline{e_{-r}(x)}) = 1_A(x) \cdot 2 \cos(2 \pi r x / p + \theta) \)
are not approximately proportional, since $A$ is not thin.

Concretely, set $\omega = e(-\theta)$ again. Using the fact that $\mathbb{E}[X^2] = \mathbb{E}[(X - \mathbb{E}[X])^2] + \mathbb{E}[X]^2$ for a random variable $X$, we can compute

$$\mathbb{E}_{x \in \mathbb{F}_p} [1_A(x) (\omega e_{r}(x) + \omega e_{-r}(x))^2] = \alpha \mathbb{E}_{x \in A} [(\omega e_{r}(x) + \omega e_{-r}(x))^2]$$

$$= \alpha \mathbb{E}_{x \in A} [(\omega e_{r}(x) + \omega e_{-r}(x) - 2\delta)^2] + 4\delta^2 \alpha$$

$$= \alpha \mathbb{E}_{x \in A} [(2 \cos(2\pi x / p + \theta) - 2 \cos \varphi)^2] + 4\delta^2 \alpha$$

$$= 16\alpha \mathbb{E}_{x \in A} [\sin^2 (t_1(x)) \sin^2 (t_2(x))] + 4\delta^2 \alpha,$$

where $\varphi := \arccos(\delta) \in [0, \pi/2]$, $t_1(x) := \pi x / p + \theta / 2 + \varphi / 2$ and $t_2(x) := \pi x / p + \theta / 2 - \varphi / 2$.

We should be explicit about the fact that we are dealing with lifts $\tilde{y} \in \mathbb{Z}$ of the elements $y = rx \in \mathbb{F}_p$. We can make any choice of lift we like, so let us fix the lift so that $|\pi x / p + \theta / 2| \leq \pi / 2$. It follows that

$$|t_i(x)| \leq \pi / 2 + \varphi / 2 \leq 3\pi / 4$$

for $i = 1, 2$. Writing

$$m = \frac{2\sqrt{2}}{3\pi},$$

we therefore have that

$$|\sin(t_i(x))| \geq m |t_i(x)|.$$

Now observe that, for any $\gamma$, $|t_1(x)| \leq \gamma$ for at most $1 + \frac{2\gamma}{\pi} p$ values of $x$. Similarly

for $t_2$. We therefore have that $t_1(x)^2 t_2(x)^2 \leq \gamma^4$ for at most $2 + \frac{2\gamma}{\pi} p$ values of $x$. Thus

$$\mathbb{E}_{x \in A} [\sin^2 (t_1(x)) \sin^2 (t_2(x))] \geq m^4 \mathbb{E}_{x \in A} [t_1(x)^2 t_2(x)^2]$$

$$\geq m^4 \left(1 - \frac{4\gamma}{\alpha_0 \pi} - \frac{2}{\alpha_0 p}\right) \gamma^4$$

$$= m^4 \left(1 - \frac{4\gamma}{\alpha_0 \pi}\right) \gamma^4 - o(1).$$

Taking $\gamma = \frac{\pi}{6} \times \alpha_0$ makes $\left(1 - \frac{4\gamma}{\alpha_0 \pi}\right) \gamma^4 = \alpha_0^4 \times \frac{x^4}{p^4}$.

Starting from (2) we can now compute

$$\omega^2 \tilde{\Gamma}_A(2r) + \omega^2 \tilde{\Gamma}_A(-2r) = \mathbb{E}_{x \in \mathbb{F}_p} [1_A(x) (\omega e_{r}(x) + \omega e_{-r}(x))^2] - 2\alpha$$

$$\geq 16\alpha \mathbb{E}_{x \in A} [\sin^2 (t_1(x)) \sin^2 (t_2(x))] + 4\delta^2 \alpha - 2\alpha$$

$$\geq 2 (2\delta^2 - 1 + 8m^4 \pi^4 \alpha_0^4 / 5^5 - o(1)) \alpha,$$

from which the triangle inequality gives the result with

$$\varepsilon = \frac{8m^4 \pi^4}{5^5} \alpha_0^4 = \frac{2\theta}{3^4 \times 5^5} \alpha_0^4.$$

\[ \square \]

Remarks. If a lower bound on $\delta$ is assumed then $\varepsilon$ can be made slightly larger, by strengthening the bound in (3).

We also have as a corollary that

$$|\tilde{\gamma}_A(r)| \leq (1 - \Omega (\alpha_0^4) + o_{p \to \infty}(1)) \alpha$$

\[2\] This bound can be derived by considering the concavity of $\sin t$ in the region $0 \leq t \leq 3\pi / 4$. 

for any $r 
eq 0$. A consequence of [Lev01, Theorem 5], is the stronger result that

$$|I_A(r)| \leq (1 - \Omega(\alpha_0^2) + o_{p \to \infty}(1)) \alpha$$

for any $r \neq 0$. This suggests that the factor of $\alpha_0^2$ in $\epsilon$ could be replaced with a factor of $\alpha_0^2$ with some more work.

3.4. Proof of Theorem 1.1. The proof of Theorem 1.1 is a case analysis on the values of $I_A(r_i)$. If $\delta_1$ and $\delta_2$ are both small, then Corollary 3.4 is strong enough. Otherwise, we use Proposition 3.7. The question then becomes: given $n$ that $\hat{\epsilon}$ values of $3.4$.

Proof of Theorem 1.1. We can assume that $\alpha \geq 0.24$, since otherwise we are done. We shall reason based on the value of $\delta_1$. First, we make an observation common to several of the cases. If we can show that there is an $h > 0$ so that

$$\sum_{i=1}^k \delta_i^2 \geq 0.75 + h - o_{k \to \infty}(1),$$

where the error is uniform in $A$, then applying Proposition 3.7 will yield

$$\alpha \leq \frac{1}{1 + 4 \times (0.75 + h - o_{k \to \infty}(1)) + O(k/\sqrt{p})} + O(k/\sqrt{p}) < 0.25 - c_h + o_{k \to \infty}(1) + O(k/\sqrt{p})$$

for some $c_h > 0$ depending only on $h$. Now, begin by choosing $k$ large enough that the $o_{k \to \infty}(1)$ in (1) is less than $c_h/3$. Then, choose $p$ large enough that the $O(k/\sqrt{p})$ in (1) is also less than $c_h/3$. Then $\alpha < 0.25 - c_h/3$ as required.

Case 1: $\delta_1 \leq 0.33$. Recall the first bound from Corollary 3.4

$$\alpha \leq \frac{\delta_1}{1 + \delta_1}.$$  

Note that as long as $\delta_1 < 1/3$, this is enough to bound $\alpha < 0.25$. In particular, here we have

$$\alpha \leq \frac{\delta_1}{1 + \delta_1} \leq 0.33 \cdot \frac{1}{1.33} < 0.2482.$$

Case 2: $0.33 \leq \delta_1 \leq 0.45$. Now the first conclusion of Corollary 3.4 is not enough, but we can argue based on the value of $\delta_2$. If $\delta_2$ is small, then the second conclusion of Corollary 3.4 will suffice. Otherwise, we can force $\sum_{i=1}^k \delta_i^2$ to be large and apply (1). So, write $\delta_2 = a\delta_1$ where $a \in (0, 1]$.

Case 2.1: $a \leq 0.7$. Apply the second conclusion of Corollary 3.4 noting that the hypothesis on $\delta_1$ and $\delta_2$ is met, to get

$$\alpha \leq \frac{a\delta_1}{1 + a\delta_1 + 2a\delta_1^2 - 2\delta_1} \leq \frac{\max_{x,y} xy}{1 + xy + 2x^2y - 2x^3},$$

where the maximum is taken over the range $0.33 \leq x \leq 0.45, 0 \leq y \leq 0.7$.

This expression is increasing in $y$ since $x^3 \leq 1/2$, so

$$\alpha \leq \max_x \frac{0.7x}{1 + 0.7x - 0.6x^3} \leq \max_x \frac{0.7x}{1 + 0.7x - 0.6 \times 0.45^3}.$$
The expression on the right hand side increases with $x$, so plugging in $x = 0.45$ gives $α < 0.24994$.

**Case 2:** $α ≥ 0.7$. Applying Corollary 3.3 gives

\[
\sum_{i=1}^{k} δ_i^3 ≥ \frac{1}{2} - δ_i^3 - o_{k→∞}(1) = \frac{1}{2} - (1 + a^3) δ_i^3 - o_{k→∞}(1)
\]

whence, by (4),

\[
\sum_{i=1}^{k} δ_i^5 ≥ (1 + a^2) δ_i^5 + \left(\frac{1}{2} - (1 + a^3) δ_i^3\right)^{2/3} - o_{k→∞}(1)
\]

\[≥ \min_{x,y} \left((1 + y^2) x^2 + \left(\frac{1}{2} - (1 + y^3)x^3\right)^{2/3}\right) - o_{k→∞}(1),\]

where the minimum is over the range $0.33 ≤ x ≤ 0.45, 0.7 ≤ y ≤ 1$. One can check that the expression being minimised in (5) is increasing with $y$. Hence

\[
\sum_{i=1}^{k} δ_i^2 ≥ \min_x \left(1.49x^2 + \left(0.5 - 1.343x^3\right)^{2/3}\right) - o_{k→∞}(1).
\]

This new expression increases with $x$ (see Figure 1). So, we can compute

\[
\sum_{i=1}^{k} δ_i^2 ≥ 1.49 \times 0.33^2 + \left(\frac{1}{2} - 1.343 \times 0.33^3\right)^{2/3} > 0.7510 - o_{k→∞}(1).
\]

**Case 3:** $0.45 ≤ δ_1 ≤ 0.7455$. Here $δ_1$ is quite large, but $δ_1^3 < 1/2$, so $δ_2$ will have to be quite large also. This will allow us to use (4). In detail, Corollary 3.3 gives

\[
\sum_{i=1}^{k} δ_i^3 ≥ \frac{1}{2} - δ_i^3 - o_{k→∞}(1).
\]

If $k$ is large enough then the right hand side is positive. So from (4) we have

\[
\sum_{i=1}^{k} δ_i^2 ≥ δ_i^2 + \left(\frac{1}{2} - δ_i^3\right)^{2/3} - o_{k→∞}(1)
\]

\[≥ \min_x \left(x^2 + \left(\frac{1}{2} - x^3\right)^{2/3}\right) - o_{k→∞}(1),\]

where the minimum is taken over the range $0.45 ≤ x ≤ 0.7455$. This expression is smallest when $x = 0.7455$ (see Figure 1). So we have

\[
\sum_{i=1}^{k} δ_i^2 ≥ 0.7455^2 + \left(\frac{1}{2} - 0.7455^3\right)^{2/3} - o_{k→∞}(1) > 0.7501 - o_{k→∞}(1).
\]

**Case 4:** $0.7455 ≤ δ_1 ≤ 0.809016$. If $θ_1$ is close to $0$ or $π$ then Lemma 3.5 will give us a large coefficient in the right half-plane. Otherwise, the contribution of $r_1$ to Corollary 3.3 is negligible. In either case, we end up being able to use (4).

Assume $p > 3$ and let $t$ be such that $2r_1 = ±r_t$. Note that $t ≠ 1$, as otherwise either $2r_1 = r_t$ or $3r_1 = 0$, which both imply $r_k = 0$ since $p > 3$. If we write $Δ(δ, θ) = 2δ^2 \cos^2 θ - 1$ for any $δ, θ$, then Lemma 3.5 says that

\[\operatorname{Re} \hat{\Lambda}(r_t) ≥ Δ(δ_1, θ_t)α.\]

---

3 The choice of boundary may seem odd here. The argument in this case gives $α ≤ 0.25 + o(1)$ exactly for $δ_1 = \sqrt{(3 + \sqrt{5})/8} ≈ 0.809017$, so to get below that bound with this argument we consider a region slightly to the left of this critical point.
We also know from (1) that $\delta_1 \geq 2\delta^2_1 - 1$.

**Case 4.1:** $\Delta(\delta_1, \theta_1) > 0$. In this case, Re $\hat{\Gamma}_A(r)$ > 0. From Proposition 3.1 and the triangle inequality we have

$$
\delta_1^3 |\cos \theta_1| + \sum_{i \neq 1, t} \delta_i^3 \geq \frac{1}{2} + \frac{\delta_1^2}{\alpha} \text{Re } \hat{\Gamma}_A(r) \geq \frac{1}{2} + (2\delta_1^2 - 1)^2 \Delta(\delta_1, \theta_1).
$$

By replacing $\theta_1$ with $\pi - \theta_1$ if necessary, we can assume $\theta_1 \in [\pi/2, 3\pi/2]$. Then

$$
\sum_{i \neq 1, t} \delta_i^3 \geq \frac{1}{2} + (2\delta_1^2 - 1)^2 \Delta(\delta_1, \theta_1) + \delta_1^3 \cos \theta_1
$$

$$
\geq \min_{t} \left(\frac{1}{2} + (2\delta_1^2 - 1)^2 \Delta(\delta_1, t) + \delta_1^3 \cos t\right),
$$

where the minimum is taken over the range $\pi/2 \leq t \leq 3\pi/2$. It can be checked that this minimum is attained when $t = \pi$. So

$$
\sum_{i \neq 1, t} \delta_i^3 \geq \frac{1}{2} + (2\delta_1^2 - 1)^3 - \delta_1^3.
$$

Then by Lemma 3.2, since we’ve fixed $\alpha \geq 0.24$, this becomes

$$
\sum_{2 \leq i \leq k, i \neq 1} \delta_i^3 \geq \frac{1}{2} + (2\delta_1^2 - 1)^3 - \delta_1^3 - o_{k \to \infty}(1).
$$

We can lower bound $\frac{1}{2} + (2\delta_1^2 - 1)^3 - \delta_1^3 > 0.000001$ here. Therefore, by taking $k$ large enough we can ensure that the right hand side of [4] is positive. It follows from [4] that

$$
\sum_{i = 1}^{k} \delta_i^2 \geq \delta_1^2 + (2\delta_1^2 - 1)^2 + \left(\frac{1}{2} + (2\delta_1^2 - 1)^3 - \delta_1^3\right)^{2/3} - o_{k \to \infty}(1)
$$

$$
\geq \min_x \left(x^2 + (2x^2 - 1)^2 + \left(\frac{1}{2} + (2x^2 - 1)^3 - x^3\right)^{2/3}\right) - o_{k \to \infty}(1),
$$

where the minimum is taken in the range $0.7455 \leq x \leq 0.809016$. Now, it can be verified that this attains its minimum when $x = 0.809016$ (see Figure 1), so we can calculate

$$
\sum_{i = 1}^{k} \delta_i^2 > 0.750001 - o_{k \to \infty}(1).
$$

---

4 Intuitively, this sum will be smallest when all of the mass is concentrated in $\delta_1$ and $\delta_2$, i.e. when $\delta_1^3 - (2\delta_1^2 - 1)^3$ is close to 1/2, which is when $\delta_1$ is close to $\sqrt{(3 + \sqrt{5})/8} \approx 0.809017$. 

---

Figure 1. The function of $x$ which is minimised to produce a lower bound on $\sum_{i=1}^{k} \delta_i^2$ in different cases, along with the region on which $x$ is minimised in each case (dashed lines) and the constant 0.75 (red). Left: Case 2.2 given by (9). Centre: Case 3 given by (7). Right: Cases 4.1 given by (10) (black) and 4.2 given by (11) (blue).
Case 4.2: $\Delta(\delta_1, \theta_1) \leq 0$. We shall apply Corollary 3.3 which says
\[
\sum_{i=2}^{k} \delta_i^3 \geq \frac{1}{2} - \delta_1^4 |\cos \theta_1| - o_{k \to \infty}(1).
\]
From the assumption that $\Delta(\delta_1, \theta_1) \leq 0$ we know that $\delta_1 |\cos \theta_1| \leq \sqrt{2}/2$. So
\[
\sum_{i=2}^{k} \delta_i^3 \geq \frac{1}{2} - \frac{\sqrt{2}}{2} \delta_1^2 - o_{k \to \infty}(1).
\]
Now, $1 - \delta_1^2 \sqrt{2} \geq 1 - 0.809016^2 \times \sqrt{2} > 0$ here. So after taking $k$ large enough the right hand side above is positive. Then applying (1) gives
\[
\sum_{i=1}^{k} \delta_i^2 \geq 0.7659 - o_{k \to \infty}(1),
\]
where the minimum is taken over the range $0.7455 \leq x \leq 0.809016$. This minimum is attained when $x = 0.809016$ (see Figure 1). So we can calculate
\[
\sum_{i=1}^{k} \delta_i^2 > 0.7659 - o_{k \to \infty}(1).
\]

Case 5: $\delta_1 \geq 0.809016$. Here, Lemma 3.8 will allow us to force $\delta_2^2 + \delta_2^2 > 0.750001$ and use Proposition 3.7. Note that we really do need the improvement over (1), as otherwise we get $\delta_2^2 + \delta_2^2 > 0.75$ when $\delta_1 = ((3 + \sqrt{5})/8)^{1/2}$. First, take $p$ large enough that the error in Lemma 3.8 is less than $0.000001$, given $\alpha_0 \geq 0.24$.

Then by Lemma 3.8 we know that $\delta_2 \geq 2 \delta_1^2 - 1 + \varepsilon - 0.000001$ where
\[
\varepsilon = \frac{2}{3} \times \frac{0.24^4}{3^4} > 0.0000061,
\]
which implies
\[
\delta_2^2 + \delta_2^2 \geq \delta_1^2 + (2 \delta_1^2 - 0.999994)^2 \geq \min_x \left( x^2 + (2x^2 - 0.999994)^2 \right),
\]
where the minimum is taken over the range $0.809016 \leq x \leq 1$. This is increasing since $x \geq 0.809016$ implies $2x^2 > 0.999994$, so
\[
\delta_2^2 + \delta_2^2 \geq 0.809016^2 + (2 \times 0.809016^2 - 0.999994)^2 > 0.7500001.
\]
Now applying Proposition 3.7 with $k = 2$ gives
\[
\alpha \leq \frac{1}{1 + 4 (\delta_1^2 + \delta_2^2)} + O(1/\sqrt{p}) \leq 0.249999975 + o(1).
\]

\[\square\]

4. Fields of characteristic 2

Now suppose that $\mathbb{F}$ is a field of order $q = 2^n$, and let $A$ be a subset of $\mathbb{F}^*$. Define the \textit{trace} $\text{Tr} : \mathbb{F} \to \mathbb{F}_2$ by
\[
\text{Tr}(x) := \sum_{i=0}^{n-1} x^{2^i}.
\]
Note that $\text{Tr}(x) + \text{Tr}(y) = \text{Tr}(x + y)$. We shall make use of the following bound on Kloosterman sums over fields of characteristic 2 (see Con02).
Lemma 4.1. If $a \in F^*$ then
\[ \left| \sum_{x \in F^*} (-1)^{\text{Tr}(x+ax^{-1})} \right| \leq 2\sqrt{q}. \]

Proof of Proposition 1.2. Let $\gamma : F \to \mathbb{C}$ be the additive character on $F$ given by
\[ \gamma(x) = (-1)^{\text{Tr}(x)}. \]

Define $X := F \setminus \ker \gamma$ and, noting that $0 \notin X$ since $0 \in \ker \gamma$, $A := X \cap X^{-1}$. Then $X$ is sum-free, and $A$ is both sum-free and closed under inverses.

Note $1_X = \frac{1}{2}(1 - \gamma)$. So, with the convention that $0^{-1} = 0$, we have
\[
\alpha = \mathbb{E}_x [1_X(x)1_{X^{-1}}(x)] = \mathbb{E}_x [1_X(x)1_{X}(x^{-1})] = \frac{1}{4} \mathbb{E} [(1 - \gamma(x))(1 - \gamma(x^{-1}))] = \frac{1}{4} + \frac{1}{4} \mathbb{E} [\gamma(x)\gamma(x^{-1})].
\]

Since $\text{Tr}(x) + \text{Tr}(x^{-1}) = \text{Tr}(x + x^{-1})$, we have $\gamma(x)\gamma(x^{-1}) = \gamma(x + x^{-1})$. Then
\[
\left| \mathbb{E}_x [\gamma(x) + \gamma(x^{-1})] \right| = \left| \mathbb{E}_x [\gamma(x + x^{-1})] \right| \leq \frac{2\sqrt{q}}{q} = o(1)
\]
by Lemma 4.1 which gives our result. \qed

5. Final remarks

5.1. Write $\sigma(F)$ for the density $|A|/|F|$ of the largest sum-free subset $A$ of $F$. This quantity was studied in the more general context of finite Abelian groups by Diananda and Yap in [DY69]. Recall from Section 1 that we define $\mu(F)$ to be the density of the largest subset of $F$ which is both sum-free and closed under inverses.

When $F$ has characteristic 2 it can be seen that $\sigma(F) = 1/2$, as the set $X$ in the proof of Proposition 1.2 demonstrates. Moreover, Proposition 1.2 itself shows $\mu(F) \geq 1/4 - o(1)$.

When $F$ has prime order $p > 2$, the interval $I = \{x \in F : p/3 < x < 2p/3\}$ has density $1/3 + o(1)$, and this is the best possible by the Cauchy-Davenport inequality. As described in [BHS91, p. 8], the set $I \cap I^{-1}$ is then sum-free and closed under inverses, and has density $1/9 - o(1)$. So $\mu(F) \geq 1/9 - o(1)$.

It is reasonable to suspect that the events ‘$A$ is sum-free’ and ‘$A^{-1}$ is sum-free’ are independent. So, we conjecture that the lower bounds above are in fact tight:

Conjecture 5.1. Let $F$ be a finite field. Then $\mu(F) = \sigma(F)^2 + o(1)$ as $|F| \to \infty$.

5.2. For a set $A \subseteq F^*$ we can use the quantity
\[ I(A) := \frac{|A \cap A^{-1}|}{|A|} \]
to measure ‘how much’ $A$ is closed under inverses. So we have studied sum-free sets $A$ with $I(A) = 1$. When $F$ has prime order $p$ and $A$ is sum-free with $I(A)$ large, we might still expect to do better than the bound of $|A| < (p + 1)/3$ given by the Cauchy-Davenport inequality. Indeed, since $A \cap A^{-1}$ is itself sum-free and closed under inverses we have
\[ \alpha = |A|/p = \frac{|A \cap A^{-1}|}{I(A) \times p} \leq \frac{\mu(F)}{I(A)}. \]

So when $I(A) \geq 0.75$ we can use Theorem 1.1 to deduce
\[ \alpha \leq \frac{\mu(F)}{0.75} \leq \frac{0.25 - c + o(1)}{0.75} \leq (1 - 4c)/3 + o(1). \]
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