RIESZ TRANSFORMS AND SOBOLEV SPACES ASSOCIATED TO THE PARTIAL HARMONIC OSCILLATOR

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Abstract. In this paper, our goal is to establish the Sobolev space associated to the partial harmonic oscillator. Based on its heat kernel estimate, we firstly give the definition of the fractional powers of the partial harmonic oscillator

\[ H_{\text{par}} = -\partial^2_\rho - \Delta_x + |x|^2, \]

and show that its negative powers are well defined on \( L^p(\mathbb{R}^{d+1}) \) for \( p \in [1, \infty] \). We then define associated Riesz transforms and show that they are bounded on classical Sobolev spaces by the calculus of symbols.

Secondly, by a factorization of the operator \( H_{\text{par}} \), we define two families of Sobolev spaces with positive integer indices, and show the equivalence between them by the boundedness of Riesz transforms. Moreover, the adapted symbolic calculus also implies the boundedness of Riesz transforms on the Sobolev spaces associated to the partial harmonic oscillator \( H_{\text{par}} \).

Lastly, as applications of our results, we obtain the revised Hardy–Littlewood–Sobolev inequality, the Gagliardo–Nirenberg–Sobolev inequality, and Hardy’s inequality in the potential space \( L^{\alpha,p}_{H_{\text{par}}} \).

1. Introduction

In this paper, we consider the following Schrödinger operator in \( \mathbb{R}^{d+1} \),

\[ H_{\text{par}} = -\partial^2_\rho - \partial^2_{x_1} - \cdots - \partial^2_{x_d} + |x|^2, \]

which we will call a partial harmonic oscillator. The anisotropy of the operator is used in [2, 9] to model magnetic traps in the Bose–Einstein condensation. It is well known in [2] that \( H_{\text{par}} \) is a self-adjoint operator, \( \sigma(H_{\text{par}}) = \sigma_{ac}(H_{\text{par}}) = [d, \infty) \) with embedded generalized eigenvalues \( 2k + d \) for \( k \geq 1 \), and that there is no singular spectrum.

In this paper, we investigate the Riesz transforms and Sobolev spaces adapted to the operator \( H_{\text{par}} \). The fractional integrals and the Riesz transform associated to the classical Laplacian \( -\Delta_{\mathbb{R}^{d+1}} \) are fundamental in harmonic analysis and have significant applications in the study of partial differential equations, see [8, 18, 20] for example.

Our work is motivated by the series of papers [4, 10, 11, 12, 13, 14, 15, 17, 24, 25, 27]. Especially in [10], R. Killip, etc, studied the Sobolev spaces adapted to the Schrödinger equation with the inverse-square potential by using the heat kernel estimates. These results were crucially used in [11, 17] to obtain the scattering result of the solution for non-linear Schrödinger and wave equations with the inverse-square potential. Riesz transforms
outside a convex obstacle were studied in [13]. Other works can be found in [12, 14, 15]. The Riesz transforms for the Hermite operators were also investigated in [25, 27], some techniques in this paper are also inspired by that in [4] where the Sobolev spaces adapted to the Hermite operator are discussed.

Firstly, based on the Fourier–Hermite expansion and the heat kernel estimate for the operator $H_{\text{par}}$, the fractional powers $H^\alpha_{\text{par}}$ can be well-defined on $C^\infty_0(\mathbb{R}^{d+1})$ for $\alpha \in \mathbb{R}$. For the operator $H^\alpha_{\text{par}}$ with the negative powers ($\alpha < 0$), we can obtain that its integral kernel $K_\alpha(z, z')$ is controlled by an integrable function of $z - z'$, and therefore $H^\alpha_{\text{par}}$ can be extended to an operator on $L^p(\mathbb{R}^{d+1})$ for $p \in [1, \infty]$.

After the study of fractional powers, we can consider the potential spaces $L^{\alpha,p}_{H_{\text{par}}} = H^{-\alpha/2}_{\text{par}}(L^p(\mathbb{R}^{d+1}))$, and define the Sobolev spaces $W^{k,p}_{H_{\text{par}}}$, $k \in \mathbb{N}$ by using a factorization of the operator $H_{\text{par}}$. It turns out that they are equivalent for $\alpha = k \in \mathbb{N}$, due to the $L^p$-boundedness of Riesz transforms in the classical Sobolev spaces. Moreover, we can also show that the Riesz transforms are bounded on $L^{\alpha,p}_{H_{\text{par}}}$ by the symbolic calculus in class $G^m$. We will also compare three Sobolev spaces associated to $-\Delta_{\mathbb{R}^{d+1}}$, the Hermite operator $H = -\Delta_{\mathbb{R}^{d+1}} + \rho^2 + |x|^2$, and the operator $H_{\text{par}}$ respectively. Finally, we use the above results to show the revised Hardy–Littlewood–Sobolev inequality, Gagliardo–Nirenberg–Sobolev inequality and Hardy’s inequality in the potential space $L^{\alpha,p}_{H_{\text{par}}}$.

We remark that the operator $H_{\text{par}}$ is a polynomial perturbation of the Laplacian and some results have been established in [6] by using the singular integral operators on nilpotent Lie groups. Instead, our results closely rely on the heat kernel estimate for the operator $H_{\text{par}}$ and Mehler’s formula. By the way, the authors will establish a Mikhlin–Hörmander multiplier theorem for the operator $H_{\text{par}}$ and obtain its applications in the Littlewood–Paley square function estimate together with the Khintchine inequality in the subsequent paper [21].

The results in this paper can be adapted to obtain analogous results for general partial harmonic oscillators $L = -\Delta_x - \Delta_y + |x|^2$, $(x, y) \in \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$. There are some scattering results for nonlinear Schrödinger equations with partial harmonic potentials, see [1, 5], and some regularity problem of the fundamental solutions for the Schrödinger flows with partial harmonic oscillator has been discussed by Zelditch in [29].

Lastly, this paper is organized as follows: some preliminary results and adapted symbol class are introduced in Section 2. In Section 3, by the symbolic calculus, we discuss the fractional powers and the Riesz transforms associated to the operator $H_{\text{par}}$. In Section 4, we define the Sobolev spaces and discuss some inclusion properties. In Section 5, we show the revised Hardy–Littlewood–Sobolev inequality, Gagliardo–Nirenberg–Sobolev inequality, and Hardy’s inequality in the space $L^{\alpha,p}_{H_{\text{par}}}$.

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2. Preliminaries

2.1. Notation. Let $D := \partial/i$. We write $z = (\rho, x)$ or $z' = (\rho', x)$ to denote elements in $\mathbb{R}^{d+1}$ with $\rho, \rho' \in \mathbb{R}$ and $x, x' \in \mathbb{R}^d$. We write $A \lesssim B$ if $A \leq CB$ for some constant
Then we have which form a complete orthonormal basis for \( L^2(\mathbb{R}^{d+1}) \), i.e.
\[
H = -\partial^2_\rho - \Delta + \rho^2 + |x|^2, \quad -\Delta_{\mathbb{R}^{d+1}} = -\partial^2_\rho - \Delta.
\]

For any \( \alpha > 0 \), we define the classical Sobolev space \( W^{\alpha,p}(\mathbb{R}^{d+1}) \) and the Hermite–Sobolev space \( L^{\alpha,p}_H(\mathbb{R}^{d+1}) \) as those in \([4, 8, 18, 20]\) as follows:
\[
W^{\alpha,p}(\mathbb{R}^{d+1}) = \{ f \in L^p(\mathbb{R}^{d+1}) : (1 - \Delta_{\mathbb{R}^{d+1}})^{-\alpha/2} f \in L^p(\mathbb{R}^{d+1}) \},
\]
\[
L^{\alpha,p}_H(\mathbb{R}^{d+1}) = \{ f \in L^p(\mathbb{R}^{d+1}) : H^{-\alpha/2} f \in L^p(\mathbb{R}^{d+1}) \}.
\]

The Fourier transform and the inverse Fourier transform of a Schwartz function \( f \in \mathcal{S}(\mathbb{R}) \) are defined by
\[
\mathcal{F}_\rho \pm f(\tau) = \frac{1}{(2\pi)^{1/2}} \int_\mathbb{R} e^{\pm i\rho \tau} f(\rho) \, d\rho,
\]
and they are similar in the higher dimensional cases.

2.2. Hermite functions. We recall some basic results about Hermite functions from \([3, 7, 26]\). For \( k = 0, 1, \ldots \), the Hermite functions \( h_k(x) \) on \( \mathbb{R} \) are defined by
\[
h_k(x) = (2^k k! \sqrt{\pi})^{-1/2} (\frac{1}{2} x^2 e^{-x^2/2})^k,
\]
which form a complete orthonormal basis for \( L^2(\mathbb{R}) \). For \( \mu \in \mathbb{N}^d \), the Hermite functions \( \Phi_\mu(x) \) on \( \mathbb{R}^d \) are defined by taking the product of the 1-dimensional Hermite functions \( h_{\mu_j} \),
\[
\Phi_\mu(x) = \prod_{j=1}^d h_{\mu_j}(x_j).
\]
It is well known that they are eigenfunctions of the operator \( -\Delta + |x|^2 \) with eigenvalues \( 2|\mu| + d \),
\[
(-\Delta + |x|^2) \Phi_\mu(x) = (2|\mu| + d) \Phi_\mu(x).
\]
Let us define the following first order differential operators for \( 1 \leq j \leq d \) as
\[
A_0 = -\partial_\rho, \quad A_0^* = \partial_\rho, \quad A_j = -\frac{\partial}{\partial x_j} + x_j, \quad A_{-j} = A_j^* = \frac{\partial}{\partial x_j} + x_j.
\]
Then we have
\[
A_j \Phi_\mu = \sqrt{2(\mu_j + 1)} \Phi_{\mu + e_j}, \quad \text{and} \quad A_{-j} \Phi_\mu = \sqrt{2\mu_j} \Phi_{\mu - e_j},
\]
where \( e_j \) is the \( j \)th unit vector in \( \mathbb{N}^d \). We use \( A_j \) and its adjoint operator \( A_j^* \) to factorize the operator \( H_{\text{par}} \) as follows,
\[
H_{\text{par}} = \frac{1}{2} \sum_{j=0}^d (A_j A_j^* + A_j^* A_j).
\]
Denote by $P_k$ the spectral projection to the $k$th eigenspace of $-\Delta + |x|^2$, 
\begin{equation}
P_k f(x) = \int_{\mathbb{R}^d} \sum_{|\mu|=k} \Phi_\mu(x) \Phi_\mu(x') f(x') \, dx'.
\end{equation}

These projections are integral operators with kernels 
\[ \Phi_k(x, x') = \sum_{|\mu|=k} \Phi_\mu(x) \Phi_\mu(x'). \]

The useful Mehler formula for $\Phi_k(x, x')$ is 
\begin{equation}
\sum_{k=0}^{\infty} \frac{r^k}{k!} \Phi_k(x, x') = \pi^{-d/2} (1 - r^2)^{-d/2} e^{-\frac{1}{2} \sum_{i=1}^{d} (|x|^2 + |x'|^2) + \frac{2rxr'}{1 - r^2}},
\end{equation}
for $0 < r < 1$, please refer to [3, 26].

2.3. Symbol class. First, we recall the definition of the standard symbol class. We denote the spatial variable by $z = (\rho, x)$, and the frequency variable by $\omega = (\tau, \xi)$.

Definition 2.1 ([22, 23]). Let $0 \leq \epsilon, \delta \leq 1$. We say a function $\sigma(z, \omega) \in C^\infty(\mathbb{R}^{d+1} \times \mathbb{R}^{d+1})$ is a symbol of order $m$ with type $(\epsilon, \delta)$, written $\sigma \in S_{\epsilon, \delta}^m$ if for all multi-indices $\beta, \gamma$, there is a constant $C_{\beta, \gamma}$ such that 
\[ |D_z^\beta \sigma(z, \omega)| \leq C_{\beta, \gamma} |\omega|^{m-|\gamma|+\delta|\beta|}. \]

Definition 2.2 ([19]). We say a function $\sigma(z, \omega) \in C^\infty(\mathbb{R}^{d+1} \times \mathbb{R}^{d+1})$ belongs to $\Gamma^m$ if for all multi-indices $\beta, \gamma$, there is constant $C_{\beta, \gamma}$ such that 
\[ |D_z^\beta \sigma(z, \omega)| \leq C_{\beta, \gamma} (|z| + |\omega|^m)^{m-|\beta|-|\gamma|}. \]

Proposition 2.3 ([27]). Let $\alpha < 0$. The symbols of the negative fractional powers $H^\alpha$ of the Hermite operators belong to $\Gamma^{2\alpha}$.

For later use, we need introduce a new symbol class, adapted to the operator $H_{\text{par}}$, in which the upper bounds also depend on partial spatial variable $x$.

Definition 2.4. We say that a function $\sigma(\rho, x, \tau, \xi) \in C^\infty(\mathbb{R}^{d+1} \times \mathbb{R}^{d+1})$ belongs to $G^m$ if for all multi-indices $\beta, \gamma$, there is constant $C_{\beta, \gamma}$ such that 
\[ |D_z^\beta \sigma(\rho, x, \tau, \xi)| \leq C_{\beta, \gamma} (|x| + |\omega|^m)^{m-|\beta|-|\gamma|}. \]

It is easy to see that $\Gamma^m \subset G^m \subset S_{1,0}^m$ for $m \leq 0$.

We quantize a symbol in $\sigma \in G^m$ by defining 
\[ T_\sigma f(z) = \frac{1}{(2\pi)^{d+1}} \int_{\mathbb{R}^{d+1} \times \mathbb{R}^{d+1}} e^{i(z-z')\omega} \sigma(z, \omega) f(z') \, dz' \, d\omega. \]

Using integration by parts, we know that $T_\sigma$ is bounded from $S(\mathbb{R}^{d+1})$ to $S(\mathbb{R}^{d+1})$ if $\sigma \in G^m$. For $p \in G^m_1$, $q \in G^{m_2}$, define $p\# q(z, \omega)$ by the oscillatory integral 
\[ p\# q(z, \omega) := \frac{1}{(2\pi)^{d+1}} \int_{\mathbb{R}^{d+1} \times \mathbb{R}^{d+1}} e^{-i(z+\omega) p(z, \omega + \omega') q(z + z', \omega') \, dz' \, d\omega}. \]

Proposition 2.5. If $p \in G^m$, $q \in G^{m'}$ then $p\# q \in G^{m+m'}$. In particular, if $m + m' \leq 0$, then $p\# q \in S_{1,0}^0$. 

Proof. It suffices to prove that \(|p\#q| \lesssim \langle |x| + |\omega| \rangle^{m+m'}\). In fact, the derivative estimates of \(p\#q\) will follow from the fact that \(D^\alpha p \in G^{m-|\alpha|}\) and \(D^\beta q \in G^{m'-|\beta|}\). By definition, we have for any sufficiently large \(k\) that,

\[
p\#q(z, \omega) = \int_{\mathbb{R}^{d+1} \times \mathbb{R}^{d+1}} F(z') \, dz' \, d\omega' = \frac{1}{(2\pi)^{d+1}} \int_{\mathbb{R}^{d+1} \times \mathbb{R}^{d+1}} e^{-iz' \omega} \langle \nabla z' \rangle^{2k} q(z' + \omega') \langle \nabla \omega' \rangle^{2k} F(z') \langle \nabla \omega' \rangle^{2k} \, dz' \, d\omega'.
\]

The integrand \(F\) is controlled by

\[
p(z, \omega) q(z + \omega', \omega) \lesssim \langle |x| + |\omega + \omega'| \rangle \langle |x' + |\omega| \rangle^{m'} \langle |x + x'| + |\omega| \rangle^{m'} \langle \omega' \rangle^{2k} \langle \omega \rangle^{2k}
\]

Case 1: \(m \geq 0, m' \geq 0\). Peetre's inequality immediately gives for \(k \gg 1\)

\[
|p\#q| \lesssim \langle |x| + |\omega| \rangle^{m+m'} \int_{\mathbb{R}^{d+1} \times \mathbb{R}^{d+1}} \langle z' \rangle^{m'-2k} \langle \omega' \rangle^{-2k} \, dz' \, d\omega' \lesssim \langle |x| + |\omega| \rangle^{m+m'}
\]

Case 2: \(m \geq 0, m' < 0\). We partition \(\mathbb{R}^{d+1} = A_\leq \cup A_\geq\), where

\[
A_\leq = \left\{ \left| z' \right| < \frac{|x| + |\omega|}{2} \right\}, \quad A_\geq = \left\{ \left| z' \right| \geq \frac{|x| + |\omega|}{2} \right\}
\]

**Estimate On** \(A_\leq\): note that \(|x + x'| + |\omega| \geq |x| - |x'| + |\omega| \geq \frac{|x| + |\omega|}{2}\), we have \(\langle |x + x'| + |\omega| \rangle^{m'} \lesssim \langle |x| + |\omega| \rangle^{m'}\) and

\[
\left| \int_{A_\leq \times \mathbb{R}^{d+1}} F(z') \, dz' \, d\omega' \right| \lesssim \langle |x| + |\omega| \rangle^{m+m'} \int_{\mathbb{R}^{d+1} \times \mathbb{R}^{d+1}} \langle z' \rangle^{-2k} \langle \omega' \rangle^{-2k+m} \, dz' \, d\omega'.
\]

**Estimate On** \(A_\geq\): we have \(\langle |x + x'| + |\omega| \rangle^{m'} \leq 1\) and use the denominator to control the \(\langle |x| + |\omega| \rangle^{m}\) term:

\[
\left| \int_{A_\geq \times \mathbb{R}^{d+1}} F(z') \, dz' \, d\omega' \right| \lesssim \langle |x| + |\omega| \rangle^{m+m'} \int_{\mathbb{R}^{d+1} \times \mathbb{R}^{d+1}} \langle z' \rangle^{-2k-m'} \langle \omega' \rangle^{-2k} \, dz' \, d\omega'.
\]

Both bounds are controlled by \(\langle |x| + |\omega| \rangle^{m+m'}\) if \(k\) is large enough.

Case 3: \(m < 0, m' \geq 0\). We decompose \(\mathbb{R}^{d+1} = B_\leq \cup B_\geq\), where

\[
B_\leq = \left\{ \left| \omega' \right| < \frac{|x| + |\omega|}{2} \right\}, \quad B_\geq = \left\{ \left| \omega' \right| \geq \frac{|x| + |\omega|}{2} \right\}
\]

On the set \(B_\leq\), we have \(\langle |x| + |\omega + \omega'| \rangle^m \lesssim \langle |x| + |\omega| \rangle^m\), and on the set \(B_\geq\) we have \(\langle \omega' \rangle^m \lesssim \langle |x| + |\omega| \rangle^m\), so we can obtain the result by similar argument in Case 2 (despite the fact that the required upper bounds depend on \(\tau\), but not on \(\rho\)).

Case 4: \(m < 0, m' < 0\). We use the decompositions as follows

\[
\mathbb{R}^{d+1} = (A_\leq \times B_\leq) \cup (A_\leq \times B_\geq) \cup (A_\geq \times B_\leq) \cup (A_\geq \times B_\geq).
\]
For the above decomposition, we have the following estimates:

\[ A_\subset : \langle |x + x'| + |\omega| \rangle^{m'} \lesssim \langle |x| + |\omega| \rangle^{m'}, \]
\[ A_\supset : \langle |x + x'| + |\omega| \rangle^{m'} \lesssim 1, \]
\[ B_\subset : \langle |x| + |\omega + \omega'| \rangle^{m'} \lesssim \langle |x| + |\omega| \rangle^{m}, \]
\[ B_\supset : \langle |x| + |\omega + \omega'| \rangle^{m} \lesssim 1. \]

By combining the above estimates on the respective sets, we have

\[ \left| \int F d\omega \right| \lesssim \langle |x| + |\omega| \rangle^{m + m'} \int_{\mathbb{R}^{d+1} \times \mathbb{R}^{d+1}} \langle z' \rangle^{-2k} \langle \omega' \rangle^{-2k} d\omega \]
\[ \left| \int_{A_\subset \times B_\supset} F d\omega \right| \lesssim \langle |x| + |\omega| \rangle^{m + m'} \int_{\mathbb{R}^{d+1} \times \mathbb{R}^{d+1}} \langle z' \rangle^{-2k - m} \langle \omega' \rangle^{-2k - m} d\omega. \]

The integrals over \( A_\subset \times B_\supset \) and \( A_\supset \times B_\subset \) are similar, and both of them are bounded by \( \langle |x| + |\omega| \rangle^{m + m'} \) if \( k \) is sufficiently large.

This completes the proof. \( \square \)

3. Fractional Powers of the Operator \( H_{\text{par}} \)

3.1. Functional Calculus for the operator \( H_{\text{par}} \). Fix \( z = (\rho, x) \in \mathbb{R}^{d+1} \). By the continuous Fourier transform in \( \rho \in \mathbb{R} \) and the discrete Hermite expansion in \( x \in \mathbb{R}^{d} \) of the operator \( H_{\text{par}} \), we can write \( H_{\text{par}}f \) for a function \( f \in C^\infty_0(\mathbb{R}^{d+1}) \) as

\[
H_{\text{par}}f(\rho, x) = \sum_{\mu} \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{i\rho \tau} (\tau^2 + 2|\mu| + d)(\mathcal{F}_\rho f(\tau, \cdot), \Phi_\mu(\cdot)) \Phi_\mu(x) d\tau
\]
\[ = \sum_{k=0}^{\infty} \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{i\rho \tau} (\tau^2 + 2k + d)P_k \mathcal{F}_\rho f(\tau, x) d\tau, \]

where \( \mathcal{F}_\rho f \) is the Fourier transform with respect to \( \rho \), and \( P_k \) is the projection operator in (2.2). Thus, for a Borel measurable function \( F \) defined on \( \mathbb{R}_+ \), we can define the operator \( F(H_{\text{par}}) \) by the spectral theory as

\[
F(H_{\text{par}})f(\rho, x) = \sum_{k=0}^{\infty} \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{i\rho \tau} F(\tau^2 + 2k + d)P_k \mathcal{F}_\rho f(\tau, x) d\tau,
\]
so long as the right hand side makes sense.

In particular, the heat semigroup \( e^{-tH_{\text{par}}} \) can be defined by

\[
e^{-tH_{\text{par}}} f(\rho, x) = \sum_{\mu} \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{i\rho \tau} e^{-t(\tau^2 + 2|\mu| + d)}((\mathcal{F}_\rho f)(\tau, \cdot), \Phi_\mu(\cdot)) \Phi_\mu(x) d\tau
\]
\[ = \sum_{k=0}^{\infty} \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{i\rho \tau} e^{-t(\tau^2 + 2k + d)}P_k(\mathcal{F}_\rho f)(\tau, x) d\tau, \]

for any \( f \in C^\infty_0(\mathbb{R}^{d+1}) \). By Mehler’s formula (2.3), the integral kernel of the operator \( e^{-tH_{\text{par}}} \) is

\[
E(t, z, z') = 2^{-d+2} \pi^{-\frac{d+1}{2}} t^{-1/2} (\sinh 2t)^{-d/2} e^{-B(t, z, z')},
\]
where
\[ B(t, z, z') = \frac{1}{4} (2 \coth 2t - \tanh t)|x - x'|^2 + \frac{\tanh t}{4} |x + x'|^2 + \frac{(\rho - \rho')^2}{4t}. \]

3.2. **Fractional powers of the operator** \( H_{\text{par}} \) **and the heat semigroup.** On the one hand, for \( \alpha \in \mathbb{R} \), we can define the fractional powers \( H_{\text{par}}^\alpha \) on \( C^\infty_0(\mathbb{R}^{d+1}) \) by
\[
H_{\text{par}}^\alpha f(\rho, x) = \sum_{k=0}^{\infty} \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{i\tau \rho}(\tau^2 + 2k + d)^\alpha P_k(\mathcal{F}_\rho f)(\tau, x) \, d\tau.
\]
Simple calculation shows that the identity
\[
(3.4) \quad H_{\text{par}}^\alpha \cdot H_{\text{par}}^\beta f = H_{\text{par}}^{\alpha + \beta} f,
\]
holds for \( f \in C^\infty_0(\mathbb{R}^{d+1}) \) and \( \alpha, \beta \in \mathbb{R} \).

On the other hand, we can also formulate the powers \( H_{\text{par}}^\alpha \) with \( \alpha \in \mathbb{R} \) by the semigroup \( e^{-tH_{\text{par}}} \) and the Gamma function. Firstly, for any \( \alpha > 0 \), the negative powers \( H_{\text{par}}^{-\alpha} \) can be written as
\[
(3.5) \quad H_{\text{par}}^{-\alpha} f(\rho, x) = \frac{1}{\Gamma(\alpha)} \int_{0}^{\infty} t^{\alpha - 1} e^{-tH_{\text{par}}} f(\rho, x) \, dt.
\]
Notice that the integral kernel of the operator \( H_{\text{par}}^{-\alpha} \) is positive since the integral kernel \( (3.3) \) of \( e^{-tH_{\text{par}}} \) is positive. Similarly for any \( a \in \mathbb{R} \) and \( d > -a \), we have
\[
(3.6) \quad (H_{\text{par}} + a)^{-\alpha} f(\rho, x) = \frac{1}{\Gamma(\alpha)} \int_{0}^{\infty} t^{\alpha - 1} e^{-tH_{\text{par}}} f(\rho, x) \, dt
\]
so long as the integral exists.

We now express the positive fractional powers in terms of derivatives of the semigroup. Let \( N \) be the smallest integer which is larger than \( \alpha \). Using the identity \(-H_{\text{par}} e^{-tH_{\text{par}}} = d^t e^{-tH_{\text{par}}} \) and (3.4), we have for any \( f \in C^\infty_0(\mathbb{R}^{d+1}) \) that
\[
(3.7) \quad H_{\text{par}}^\alpha f(\rho, x) = \frac{(-1)^N}{\Gamma(N - \alpha)} \int_{0}^{\infty} t^{N - \alpha - 1} \frac{d^N}{dt^N} e^{-tH_{\text{par}}} f(\rho, x) \, dt.
\]

We have the following properties between \( H_{\text{par}}^\alpha \) and \( A_{\pm j} \) for \( 1 \leq j \leq d \).

**Lemma 3.1.** For any \( \alpha \in \mathbb{R} \), \( d \geq 3 \), \( 1 \leq j \leq d \), and \( f \in C^\infty_0(\mathbb{R}^{d+1}) \), we have
\[
A_0 H_{\text{par}}^\alpha f = H_{\text{par}}^\alpha A_0 f, \quad A_j H_{\text{par}}^\alpha f = (H_{\text{par}} - 2)^\alpha A_j f, \quad A_{-j} H_{\text{par}}^\alpha f = (H_{\text{par}} + 2)^\alpha A_{-j} f, \quad H_{\text{par}}^\alpha A_j f = A_j (H_{\text{par}} + 2)^\alpha f, \quad H_{\text{par}}^\alpha A_{-j} f = A_{-j} (H_{\text{par}} - 2)^\alpha f.
\]

**Proof.** The first result is trivial. For \( 1 \leq j \leq d \), we only give the details for \( A_j H_{\text{par}}^\alpha f = (H_{\text{par}} - 2)^\alpha A_j f \), as the other cases are dealt with in analogous argument. By the definition of \( H_{\text{par}}^\alpha \), we have
\[
H_{\text{par}}^\alpha f(\rho, x) = \sum_{\mu} \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{i\tau \rho}(\tau^2 + 2|\mu| + d)^\alpha (\mathcal{F}_\rho f(\tau, \cdot), \Phi_{\mu}(\cdot)) \Phi_{\mu}(x) \, d\tau.
\]
On one hand, by (2.1), we have

\[ A_j H_\text{par}^\alpha f(\rho, x) \]

\[ = \sum_\mu \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{i\tau \rho}(\tau^2 + 2|\mu| + d)^\alpha (F_\rho f(\tau, \cdot), \Phi_\mu(\cdot)) A_j \Phi_\mu(x) d\tau \]

\[ = \sum_\mu \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{i\tau \rho}(\tau^2 + 2|\mu| + d)^\alpha (F_\rho f(\tau, \cdot), \Phi_\mu(\cdot)) d\tau \cdot \sqrt{2(\mu_j + 1)} \Phi_{\mu+\epsilon_j}(x). \]

On the other hand, by the fact that \( A_{-j} = A_j^* \), we obtain

\[ A_j f(\rho, x) = \sum_\mu \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{i\tau \rho}(A_j F_\rho f(\tau, \cdot), \Phi_\mu(\cdot)) \Phi_\mu(x) d\tau \]

\[ = \sum_\mu \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{i\tau \rho}(F_\rho f(\tau, \cdot), A_{-j} \Phi_\mu(\cdot)) \Phi_\mu(x) d\tau \]

\[ = \sum_\mu \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{i\tau \rho}(F_\rho f(\tau, \cdot), \Phi_{\mu-\epsilon_j}(\cdot)) d\tau \cdot \sqrt{2\mu_j} \Phi_\mu(x). \]

By (3.1), we have for a function \( g \in C^\infty_0(\mathbb{R}^{d+1}) \) that

\[ (H_\text{par} - 2)^\alpha g = \sum_\mu \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{i\tau \rho}(\tau^2 + 2|\mu| + d - 2)^\alpha (F_\rho g(\tau, \cdot), \Phi_\mu(\cdot)) \Phi_\mu(x) d\tau, \]

which implies by choosing \( g = A_j f \) that

\[ (H_\text{par} - 2)^\alpha A_j f(\rho, x) \]

\[ = \sum_\mu \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{i\tau \rho}(\tau^2 + 2|\mu| + d - 2)^\alpha (F_\rho f(\tau, \cdot), \Phi_{\mu-\epsilon_j}(\cdot)) d\tau \sqrt{2\mu_j} \Phi_\mu(x) \]

\[ = \sum_\mu \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{i\tau \rho}(\tau^2 + 2|\mu| + d)^\alpha (F_\rho f(\tau, \cdot), \Phi_\mu(\cdot)) d\tau \sqrt{2\mu_j + 2\Phi_{\mu+\epsilon_j}(x) \}

\[ = A_j H_\text{par}^\alpha f(\rho, x). \]

We finish the proof. \qed

3.3. Properties of negative fractional powers of \( H_\text{par} \). In this subsection, we explore the properties of the negative powers of the operator \( H_\text{par} \). The first result is:

**Proposition 3.2.** Given \( \alpha > 0 \), the operator \( H_\text{par}^-\alpha \) has the integral representation

\[ H_\text{par}^-\alpha f(z) = \int_{\mathbb{R}^{d+1}} K_\alpha(z, z') f(z') dz' \]

for all \( f \in C^\infty_0(\mathbb{R}^{d+1}) \). Moreover, there exist a functions \( \Psi_\alpha \in L^1(\mathbb{R}^{d+1}) \) and a constant \( C > 0 \) such that

\[ (3.8) \]

\[ K_\alpha(z, z') \leq C \Psi_\alpha(z - z'), \text{ for all } z, z' \in \mathbb{R}^{d+1}. \]

Hence, \( H_\text{par}^-\alpha \) is well defined and bounded on \( L^p(\mathbb{R}^{d+1}) \) for \( p \in [1, +\infty] \).
Proof. By (3.3) and (3.5), we have

\[ K_\alpha(\rho, x, \rho', x') = \frac{C_\alpha}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1/2} (\sinh 2t)^{-d/2} e^{-B(t, z, z')} \, dt. \]

We decompose \( K_\alpha(\rho, x, \rho', x') \) into two parts,

\[ K^1_\alpha(\rho, x, \rho', x') = \frac{1}{\Gamma(\alpha)} \int_0^1 t^{\alpha-1} E(t, \rho, x, \rho', x') \, dt, \]
\[ K^2_\alpha(\rho, x, \rho', x') = \frac{1}{\Gamma(\alpha)} \int_1^\infty t^{\alpha-1} E(t, \rho, x, \rho', x') \, dt. \]

We firstly estimate the term \( K^2_\alpha(\rho, x, \rho', x') \). Together the inequality that \( 2 \coth 2t - \tanh t > \coth t > 1 \), with the fact that \( t \to \infty \), we have

\[
|K^2_\alpha(\rho, x, \rho', x')| \leq C e^{-|x-x'|^2-(\rho-\rho')^2-|x+x'|^2} \int_1^\infty t^{a-3/2} e^{-t\rho} \, dt
\leq C e^{-|x-x'|^2-(\rho-\rho')^2-|x+x'|^2}.
\]

We next estimate \( K^1_\alpha(\rho, x, \rho', x') \). We further split it into two cases.

Case 1: \((z, z') \in D_+ := \{(z, z') \in \mathbb{R}^{d+1} \times \mathbb{R}^{d+1}; |z - z'| \geq 1\}\). Using the fact that \( \sinh 2t \sim t \) and \( \coth 2t \sim e^{2t}, \coth 2t \sim e^{2t} \), as \( t \to \infty \), we have

\[
|K^1_\alpha(\rho, x, \rho', x')| \leq \int_0^1 t^{\alpha-1} \frac{e^{-\frac{1}{2t}|x-x'|^2+(\rho-\rho')^2}}{\sqrt{2\pi |x-x'|^2+(\rho-\rho')^2}} \, dt
\leq e^{-\frac{1}{16} |x-x'|^2+(\rho-\rho')^2} \int_0^1 t^{\alpha-1} \frac{e^{-\frac{1}{2t} |x-x'|^2+(\rho-\rho')^2}}{\sqrt{2\pi |x-x'|^2+(\rho-\rho')^2}} \, dt
\leq e^{-\frac{1}{16} |x-x'|^2+(\rho-\rho')^2}.
\]

Case 2: \((z, z') \in D_- := \{(z, z') \in \mathbb{R}^{d+1} \times \mathbb{R}^{d+1}; |z - z'| < 1\}\). If \( \alpha < \frac{d+1}{2} \), we have

\[
|K^1_\alpha(\rho, x, \rho', x')| \leq \int_0^1 t^{\alpha-1} \frac{e^{-\frac{1}{2t} |x-x'|^2+(\rho-\rho')^2}}{\sqrt{2\pi |x-x'|^2+(\rho-\rho')^2}} \, dt
\leq \frac{1}{|x-x'|^2+(\rho-\rho')^2)^{(d+1)/2-a}},
\]

if \( \alpha = \frac{d+1}{2} \), we have

\[
|K^1_\alpha(\rho, x, \rho', x')| \leq \log[|x-x'|^2+(\rho-\rho')^2];
\]

and finally, if \( \alpha > \frac{d+1}{2} \), we have

\[
|K^1_\alpha(\rho, x, \rho', x')| \leq 1.
\]

It follows that (3.8) holds with the integrable function \( \Psi_\alpha \) defined by

\[
(3.9) \quad \Psi_\alpha(z - z') = \begin{cases} 1_{D_-} |z - z'|^{2\alpha - (d+1)} + 1_{D_+} e^{-\frac{1}{4\alpha} |z - z'|^2}, & \alpha < \frac{d+1}{2}, \\ 1_{D_-} \log |z - z'| + 1_{D_+} e^{-\frac{1}{4\alpha} |z - z'|^2}, & \alpha = \frac{d+1}{2}, \\ 1_{D_-} + 1_{D_+} e^{-\frac{1}{16} |z - z'|^2}, & \alpha > \frac{d+1}{2}, \end{cases}
\]
which completes the proof. □

By (3.6), we can obtain similar estimates for the integral kernels of the operators \((H_{\text{par}} + 2)^{-\alpha}\) and \((H_{\text{par}} - 2)^{-\alpha}\):

**Corollary 3.3.** Assume that \(\alpha > 0\). Let \(M_\alpha(\rho, x, \rho', x')\) and \(N_\alpha(\rho, x, \rho', x')\) be the integral kernels of the operators \((H_{\text{par}} + 2)^{-\alpha}\) and \((H_{\text{par}} - 2)^{-\alpha}\) respectively. Then, we have

\[
M_\alpha(\rho, x, \rho', x') \leq K_\alpha(\rho, x, \rho', x') \leq \Psi_\alpha(z - z'),
\]

and

\[
N_\alpha(\rho, x, \rho', x') \leq \Psi_\alpha(z - z'), \quad \text{if } d \geq 3.
\]

**Proof.** By (3.6), the estimate of \(M_\alpha\) is trivial. It suffices to show the estimate of \(N_\alpha\). By definition, we have

\[
N_\alpha(\rho, x, \rho', x') = \frac{1}{\Gamma(\alpha)}\int_0^\infty t^{\alpha-1}e^{2t}E(t, \rho, x, \rho', x') \, dt.
\]

We once again decompose \(N_\alpha(\rho, x, \rho', x')\) into two parts as follows.

\[
N_\alpha^1(\rho, x, \rho', x') = \frac{1}{\Gamma(\alpha)}\int_0^1 t^{\alpha-1}e^{2t}E(t, \rho, x, \rho', x') \, dt,
\]

\[
N_\alpha^2(\rho, x, \rho', x') = \frac{1}{\Gamma(\alpha)}\int_1^\infty t^{\alpha-1}e^{2t}E(t, \rho, x, \rho', x') \, dt.
\]

For the term \(N_\alpha^1(\rho, x, \rho', x')\), we can proceed by the same way as the estimate of \(K_\alpha^1(\rho, x, \rho', x')\), we omit the details here. For \(N_\alpha^1(\rho, x, \rho', x')\) term, we obtain for \(d \geq 3\) that

\[
|N_\alpha^2(\rho, x, \rho', x')| \leq Ce^{-|x - x'|^2 - (\rho - \rho')^2 - |x + x'|^2} \int_1^\infty t^{\alpha-3/2}e^{-td}e^{2t} \, dt
\]

\[
\leq Ce^{-|x - x'|^2 - (\rho - \rho')^2 - |x + x'|^2},
\]

which completes the proof. □

**Remark 3.4.** We make some comments about the condition that \(d > -a = 2\) for the second result of \(N_\alpha^2(\rho, x, \rho', x')\) in the above corollary. Since the integral \(\int_1^\infty t^{\alpha-3/2}e^{2t}e^{-t} \, dt\) is divergent, we cannot obtain similar estimates for \((H_{\text{par}} - 2)^{-\alpha}\) for the case \(d = 1\). It is consistent with the spectral property that \(\sigma(H_{\text{par}} - 2) = [-1, \infty)\) when \(d = 1\). When \(d = 2\), the spectral property is that \(\sigma(H_{\text{par}} - 2) = [0, \infty)\), and we have different behavior depending on the power \(\alpha\). If \(0 < \alpha < 1/2\), the kernel of \((H_{\text{par}} - 2)^{-\alpha}\) has exponential decay as \(|z - z'| \to \infty\) since \(\int_1^\infty t^{\alpha-3/2}e^{2t}e^{-2t} \, dt\) is convergent, and if \(\alpha \geq 1/2\), no useful result can be derived.

In addition, we also have the following boundedness result for the negative fractional powers of the operator \(H_{\text{par}}\).

**Proposition 3.5.** Let \(p \in [1, \infty]\) and \(\alpha > 0\). Then the weighted operator

\[
|x|^{2\alpha}H_{\text{par}}^{-\alpha}
\]

is bounded on \(L^p(\mathbb{R}^{d+1})\).
Proof. By Schur’s lemma [8], we only need to verify that
\begin{equation}
\sup_z |x|^{2\alpha} \int_{\mathbb{R}^{d+1}} |K_\alpha(z, z')| \, dz' \lesssim 1,
\end{equation}
and
\begin{equation}
\sup_{z'} \int_{\mathbb{R}^{d+1}} |x|^{2\alpha} |K_\alpha(z, z')| \, dz \lesssim 1.
\end{equation}

By the boundedness in Proposition 3.2, we may assume $|x| \geq 2$.
Firstly, we prove (3.11). We partition $\mathbb{R}^d = E_x \cup E^c_x$, where
\begin{equation*}
E_x = \{ x' \in \mathbb{R}^d : |x| > 2|x - x'| \}.
\end{equation*}
When $x' \in E^c_x$, i.e., $|x| \leq 2|x - x'|$. By the fact that
\begin{equation*}
2 \coth 2t - \tanh t \geq \frac{1}{2} (2 \coth 2t - \tanh t) + \frac{1}{4},
\end{equation*}
we have
\begin{equation*}
K_\alpha(\rho, x, \rho', x') \lesssim e^{-\frac{1}{4}|x-x'|^2} \int_0^\infty t^{\alpha-1/2} (\sinh 2t)^{-d/2} e^{-\frac{1}{8} (B(t, \rho, \rho', \rho', \rho'))} \, dt \lesssim e^{-\frac{1}{4}|x-x'|^2} K_\alpha \left( \rho, \frac{x}{\sqrt{2}}, \rho', \frac{x'}{\sqrt{2}} \right).
\end{equation*}
So it follows that
\begin{align*}
|x|^{2\alpha} \int_{\mathbb{R}} \int_{E^c_x} |K_\alpha(\rho, x, \rho', x')| \, dz' \, d\rho' &\lesssim \int_{\mathbb{R}^{d+1}} |x - x'|^{2\alpha} e^{-\frac{1}{4}|x-x'|^2} |K_\alpha \left( \rho, \frac{x}{\sqrt{2}}, \rho', \frac{x'}{\sqrt{2}} \right)| \, dz' \, d\rho' \\
&\lesssim 1.
\end{align*}

When $x' \in E_x$, i.e., $|x| > 2|x - x'|$, we once again decompose $K_\alpha = K^1_\alpha + K^2_\alpha$ as in the proof of Proposition 3.2. For $K^2_\alpha(z, z')$ term. Since $|x| > 2|x - x'|$ implies $|x| < |x + x'|$, we have
\begin{align*}
|x|^{2\alpha} \int_{\mathbb{R}} \int_{E_x} |x|^{2\alpha} e^{-\frac{1}{4}|x+x'|^2} e^{-\frac{1}{4}|x-x'|^2} e^{-(\rho-\rho')^2} \, dz' \, d\rho' &\lesssim \int_{\mathbb{R}} \int_{\mathbb{R}} |x|^{2\alpha} e^{-\frac{1}{4}|x|^2} e^{-\frac{1}{4}|x-x'|^2} e^{-(\rho-\rho')^2} \, dz' \, d\rho' \\
&\lesssim 1.
\end{align*}
As for the $K^1_\alpha(z, z')$ term, we have
This completes the proof of (3.11).

To prove (3.12), we similarly partition $\mathbb{R}^d = E_{x'} \cup E^c_{x'}$, with
\[ E_{x'} = \{ x \in \mathbb{R}^d : |x'| \geq 2|x - x'| \}, \]
and write
\[ \int_{\mathbb{R}^{d+1}} |x|^{2\alpha} K_\alpha(z, z') \, dz = \left( \int_{\mathbb{R}} \int_{E_{x'}} + \int_{\mathbb{R}} \int_{E^c_{x'}} \right) |x|^{2\alpha} K_\alpha(\rho, x, \rho', x') \, dx \, d\rho. \]

When $x \in E_{x'}$, i.e., $|x'| \geq 2|x - x'|$, we have $|x| \leq \frac{3}{2}|x'|$. Since the kernel is symmetric in $z$ and $z'$, that is, $K_\alpha(\rho, x, \rho', x') = K_\alpha(\rho', x', \rho, x)$, we have
\[ \int_{\mathbb{R}} \int_{E_{x'}} |x|^{2\alpha} K_\alpha(\rho, x, \rho', x') \, dx \, d\rho \lesssim |x'|^{2\alpha} \int_{\mathbb{R}} \int_{|x'| > 2|x - x'|} |K_\alpha(\rho', x', \rho, x)| \, dx \, d\rho. \]

By (3.11), the right hand side of the above inequality is finite.

When $x \in E^c_{x'}$, and $|x - x'| < 1$, it follows that $|x| \leq |x'| + |x - x'| \leq 4$. Hence,
\[ \int_{\mathbb{R}} \int_{|x'| < 2|x - x'|} |x|^{2\alpha} |K_\alpha(\rho, x, \rho', x')| \, dx \, d\rho \lesssim \int_{\mathbb{R}} \int_{|x'| < 2|x - x'|} |K_\alpha(\rho, x, \rho', x')| \, dx \, d\rho \lesssim 1. \]

When $x \in E^c_{x'}$ and $|x - x'| > 1$, we have
\[ \int_{\mathbb{R}} \int_{|x'| < 2|x - x'|} |x|^{2\alpha} |K_\alpha(\rho, x, \rho', x')| \, dx \, d\rho \lesssim \int_{\mathbb{R}} \int_{|x - x'| \geq 1} |x - x'|^{2\alpha} K_\alpha(\rho, x, \rho', x') \, dx \, d\rho \lesssim \int_{\mathbb{R}^{d+1}} \left| x - x' \right|^{2\alpha} e^{-\frac{1}{4\alpha} |x - x'|^2} K_\alpha(\rho, \frac{x}{\sqrt{2}}, \rho', \frac{x'}{\sqrt{2}}) \, dx' \, d\rho' \lesssim 1. \]

This proves (3.12) and completes the proof. \(\square\)

### 3.4. Symbols for fractional powers $H_{\text{par}}^s$

Using the heat kernel representation of the fractional powers operator $H_{\text{par}}^s$ in Subsection 3.2, we firstly calculate the symbol of the
heat semigroup $e^{-tH_{\text{par}}}$. Together (3.2), the fact that $\hat{\Phi}_\mu = (-i)^|\mu|\Phi_\mu$, with Plancherel’s theorem, we have

$$e^{-tH_{\text{par}}} f(\rho, x) = \sum_\mu \frac{1}{\sqrt{2\pi}} \int_\mathbb{R} e^{ix\rho} e^{-\tau^2 t} e^{-(2|\mu|+d)t} e^{-\frac{\pi i}{4}|\mu|^2} (\mathcal{F}_{\rho,x} f(\tau, \xi), \Phi_\mu(\xi)) \Phi_\mu(x) d\tau$$

(3.13)

$$= \frac{1}{(2\pi)^{(d+1)/2}} \int_{\mathbb{R}^{d+1}} e^{ix\rho} e^{\xi \cdot x} p_t(\rho, x, \tau, \xi) \mathcal{F}_{\rho,x} f(\tau, \xi) d\tau d\xi,$$

where

$$p_t(\rho, x, \tau, \xi) = \sum_\mu e^{-i\xi \cdot x} e^{-\tau^2 t} e^{-(2|\mu|+d)t} e^{-\frac{\pi i}{4}|\mu|^2} \Phi_\mu(\xi) \Phi_\mu(x).$$

In view of Mehler’s formula (2.3), we have

$$p_t(\rho, x, \tau, \xi) = c_d (\cosh 2t)^{-d/2} e^{-b(t,x,\tau,\xi)},$$

where $c_d = (2\pi)^{-d/2}$ and

(3.14)  \[ b(t, x, \tau, \xi) = \frac{1}{2} (|x|^2 + |\xi|^2) \tanh 2t + 2ix \cdot \xi \sech 2t (\sinh t)^2 + \tau^2 \]

Now we have

**Lemma 3.6.** Let $\alpha \in \mathbb{R}$. The symbol $\sigma_\alpha(\rho, x, \tau, \xi)$ of the operator $H_{\text{par}}^\alpha$ belongs to the symbol class $G^{2\alpha}$ defined by Definition 2.4.

**Proof.** From the explicit formula of $p_t$, we know that $\sigma_\alpha$ does not depend on $\rho$. Thus, the estimates for $\sigma_\alpha$ do not depend on $\rho$ and its derivatives on $\rho$ are zero. For brevity, we will write $\sigma_\alpha(x, \tau, \xi)$ instead of $\sigma_\alpha(\rho, x, \tau, \xi)$, and similarly for other terms which are independent of $\rho$.

The case $\alpha = 0$ is obvious.

For the case $\alpha < 0$. Using the equality of (3.5), we have

$$\sigma_\alpha(x, \tau, \xi) = \frac{1}{\Gamma(-\alpha)} \int_0^\infty t^{-\alpha-1} p_t(x, \tau, \xi) dt.$$

We split $\sigma_\alpha(x, \tau, \xi)$ into two parts as follows.

$$\sigma_\alpha^1(x, \tau, \xi) = \frac{1}{\Gamma(-\alpha)} \int_0^1 t^{-\alpha-1} p_t(x, \tau, \xi) dt,$$

$$\sigma_\alpha^2(x, \tau, \xi) = \frac{1}{\Gamma(-\alpha)} \int_1^\infty t^{-\alpha-1} p_t(x, \tau, \xi) dt.$$

To estimate $\sigma_\alpha^1$, on the one hand, by the lower bound estimate for the real part of $b$ as $t \in (0, 1)$,

$$\Re b(t, x, \tau, \xi) \geq ct(|x|^2 + |\xi|^2 + \tau^2),$$

we have

$$|\sigma_\alpha^1(x, \tau, \xi)| \leq \frac{1}{\Gamma(-\alpha)} \int_0^1 t^{-\alpha-1} e^{-c(|x|^2 + |\xi|^2 + \tau^2)t} dt \leq (1 + |x|^2 + |\xi|^2 + \tau^2)^\alpha.$$
On the other hand, by (3.14), the derivatives of $b(t, x, \tau, \xi)$ satisfy
\[
\begin{align*}
\frac{\partial}{\partial x_j} b(t, x, \tau, \xi) &= x_j \tanh 2t + 2i \xi_j \sech 2t (\sinh t)^2 \sim x_j t + \xi_j t^2, \\
\frac{\partial^2}{\partial x_j \partial x_k} b(t, x, \tau, \xi) &= \tanh 2t \sim t, \\
\frac{\partial}{\partial \tau} b(t, x, \tau, \xi) &= \xi_j \tanh 2t + 2ix_j \sech 2t (\sinh t)^2 \sim \xi_j t + x_j t^2, \\
\frac{\partial^2}{\partial \tau \partial \xi_k} b(t, x, \tau, \xi) &= \tanh 2t \sim t, \\
\frac{\partial}{\partial \tau} b(t, x, \tau, \xi) &= 2\tau t, \quad \frac{\partial^2}{\partial t \partial \tau} b(t, x, \tau, \xi) = 2t,
\end{align*}
\]
as $t \to 0$. In conclusion, we obtain with the shorthand $X = (x, \tau, \xi)$ that
\[
|\partial^\beta_X b(t, x, \tau, \xi)| \lesssim \begin{cases}
|X|^{2-|\beta|} t, & |\beta| \leq 2, \\
0, & |\beta| \geq 3.
\end{cases}
\]
By using Faà di Bruno’s formula, we obtain
\[
\begin{align*}
\frac{\partial^\beta}{\partial \sigma^\alpha} \sigma^1_\alpha (x, \tau, \xi) &= \frac{1}{\Gamma(-\alpha)} \int_0^1 t^{-\alpha-1} p(t, x, \tau, \xi) \prod_{1 \leq j \leq |\beta|} (\partial^\beta_X b)^{n_j} dt. \\
\text{Hence, for any } |\beta| \geq 1, \text{ we get}
\end{align*}
\]
\[
|\partial^\beta_X \sigma^1_\alpha (x, \tau, \xi)| \lesssim \int_0^1 t^{-\alpha-1} e^{-c|X|^2} \prod_{j=1}^{\beta} t^{n_j} dt \prod_{1 \leq j \leq |\beta|} |X|^{2n_j-|\beta|n_j} \lesssim (X)^{2\alpha-|\beta|}.
\]
That is, $\sigma^1_\alpha \in G^{2\alpha}$ for all $\alpha < 0$.

To estimate $\sigma^2_\alpha$, since $\cosh t \sim e^t$, and $\tanh t \sim t$ as $t \to \infty$, we have
\[
\text{Re } b(t, x, \tau, \xi) \geq ct|X|^2,
\]
which implies that
\[
|\sigma^2_\alpha (x, \tau, \xi)| \lesssim \int_1^\infty t^{-\alpha-1} e^{-ct|X|^2} dt \lesssim e^{-|X|^2}.
\]
When we take partial derivatives in $X = (x, \tau, \xi)$, we only change the degree of the polynomials in $X$. The dominating term is still $e^{-|X|^2}$. Hence, we have $\sigma^2_\alpha \in G^\alpha$.

For the case $\alpha > 0$. By (3.7), we obtain the symbol of the operator $H^\text{par}_\alpha$,
\[
\sigma_\alpha (x, \tau, \xi) = \frac{(-1)^N}{\Gamma(N-\alpha)} \int_0^\infty t^{N-\alpha-1} \frac{d^N}{dt^N} p(t, x, \tau, \xi) dt.
\]
Arguing as in the case $\alpha < 0$, we split it into two parts
\[
\begin{align*}
\sigma^1_\alpha (x, \tau, \xi) &= \frac{(-1)^N}{\Gamma(N-\alpha)} \int_0^1 t^{N-\alpha-1} \frac{d^N}{dt^N} p(t, x, \tau, \xi) dt, \\
\sigma^2_\alpha (x, \tau, \xi) &= \frac{(-1)^N}{\Gamma(N-\alpha)} \int_1^\infty t^{N-\alpha-1} \frac{d^N}{dt^N} p(t, x, \tau, \xi) dt.
\end{align*}
\]
Note that
\[
\partial_t b(t, x, \tau, \xi) = 2(|x|^2 + |\xi|^2) \operatorname{sech}^2 2t + \tau^2 + 2ix \cdot \xi [-2 \operatorname{sech}^2 2t \sinh 2t \sinh^2 t + \tanh 2t]
\]
\[
= 2(|x|^2 + |\xi|^2) \operatorname{sech}^2 2t + \tau^2 + 2ix \cdot \xi \operatorname{sech} 2t \sinh 2t.
\]

Thus, for \(t \in (0, 1)\) we have
\[
|\partial_t (\cosh 2t)^{-d/2}| \lesssim 1, \quad |\partial_t b(t, x, \tau, \xi)| \lesssim |X|^2,
\]
and
\[
|\partial_t^N (\cosh 2t)^{-d/2}| \lesssim 1, \quad |\partial_t^N b(t, x, \tau, \xi)| \lesssim |X|^{2N}.
\]

To compute the derivative estimates of \(p_t(x, \tau, \xi)\) in the term \(\sigma_1^\alpha(x, \tau, \xi)\), we use the above estimates, Leibniz rule and Faà di Bruno’s formula, and obtain that
\[
|\sigma_1^\alpha(x, \tau, \xi)| \lesssim \int_0^1 t^{N-\alpha-1} e^{-ct} |X|^{2N} dt \lesssim \langle X \rangle^{2\alpha}.
\]

For the derivative estimates, we have
\[
|\partial^\beta_X \sigma_1^\alpha(x, \tau, \xi)| \lesssim \int_0^1 t^{N-\alpha-1} e^{-ct} |X|^2 \prod_{1 \leq j \leq |\beta|} |X|^{2n_j - |\beta_j| n_j} d\tau \lesssim \langle X \rangle^{2\alpha - |\beta|}.
\]

For the term \(\sigma_2^\alpha\), the exponential decay can be obtained as the case \(\alpha < 0\).

Summing up, we have \(\sigma_\alpha \in G^{2\alpha}\) for all \(\alpha \in \mathbb{R}\), and this concludes the proof. \(\square\)

### 3.5. Riesz transforms and symbols.

The \(j\)th Riesz transforms associated with the operator \(H_{\text{par}}\) are defined as
\[
R_j = A_j H_{\text{par}}^{-1/2}, \quad -d \leq j \leq d.
\]

In general, for any \(m \in \mathbb{N}\) and \(j = (j_1, \ldots, j_m), -d \leq j_k \leq d\), the \(j\)th Riesz transform of order \(m\) is the operator
\[
R_j = R_{j_1, \ldots, j_m} = A_{j_1} A_{j_2} \ldots A_{j_m} H_{\text{par}}^{-m/2}
\]
\[
= P_m(\partial_\rho, \partial_x, x) H_{\text{par}}^{-m/2},
\]
where \(P_m\) is a polynomial of degree \(m\).

In this subsection, we will prove that the Riesz transforms defined by (3.15) and (3.16) are bounded on classical Sobolev spaces by verifying that their symbols belong to the symbol class \(S_{1,0}^0\). There are two ways to show this. The most obvious way would be a direct calculation by the heat semigroup. The symbol of the operator \(A_0 e^{-tH_{\text{par}}}\) is \(-i\tau p_t(x, \tau, \xi)\), so the symbol of Riesz transform \(R_0\) is
\[
\sigma_{R_0}(\rho, x, \tau, \xi) = \frac{1}{\sqrt{\pi}} \int_0^\infty t^{-\frac{1}{2}} i\tau p_t(x, \tau, \xi) dt.
\]
For the symbol of the operator \( A_j e^{-t H_{\text{par}}} \), \( 1 \leq j \leq d \), we use the formula (3.13) to get
\[
A_j e^{-t H_{\text{par}}} f(\rho, x) = \int_{\mathbb{R}^{d+1}} e^{i \tau \rho} e^{i \xi \cdot x} p_t(x, \tau, \xi)(-i \xi_j + x_j + \partial_{x_j} b)(\tau, \xi) \, d\tau \, d\xi.
\]
This shows that the symbol of the operator \( A_j e^{-t H_{\text{par}}} \) is
\[
p_t(x, \tau, \xi)(-i \xi_j + x_j + \partial_{x_j} b(t, \rho, x, \tau, \xi)).
\]
So, the symbols for Riesz transforms \( R_j \) for \( 1 \leq j \leq d \) are
\[
\sigma_{R_j}(\rho, x, \tau, \xi) = -\frac{1}{\sqrt{d}} \int_0^\infty t^{\frac{d}{2}} (-i \xi_j + x_j + \partial_{x_j} b) p_t(x, \tau, \xi) \, d\tau.
\]
Similar calculations give the symbols for Riesz transforms \( R_j \) for \( -d \leq j \leq -1 \). It is then possible to use the integral forms (3.17) and (3.18) in a lengthy calculation like in the proof of Lemma 3.6 to show that they belong to the symbol class \( S_{1,0}^0 \).

The second, simpler and shorter way is to take advantage of the symbol calculus for compositions in \( G^m \), which we present below.

**Proposition 3.7.** The symbols \( \sigma_{R_j} \) of Riesz transforms \( R_j \) for \( 0 \leq |j| \leq d \) belong to the symbol class \( S_{1,0}^0 \), hence they are bounded on classical Sobolev spaces \( W^{\alpha, p}(\mathbb{R}^{d+1}) \) for any \( \alpha \in \mathbb{R} \) and \( 1 < p < \infty \). In addition, the same result holds for Riesz transforms \( R_j \) of high order.

**Proof.** The symbols of the operators \( A_j \) are given by either \( i \tau \) or \( \pm i \xi_j + x_j \), which belong to class \( G^1 \). From Proposition 3.6, the symbol of the operator \( H_{\text{par}}^{-1/2} \) belongs to class \( G^{-1} \). By symbolic calculus in class \( G^m \) (Proposition 2.5), we obtain that symbols of Riesz transform \( R_j \) for \( 0 \leq |j| \leq d \) belong to class \( S_{1,0}^0 \), which implies the boundedness of Riesz transform \( R_j \) for \( 0 \leq |j| \leq d \) on \( W^{\alpha, p} \) for all \( 1 < p < \infty \), (see [22, 23]).

For Riesz transform \( R_j \) with high order, By Proposition 2.5 and the fact that the symbols of the operators \( A_{j_1} A_{j_2} \ldots A_{j_m} \) and \( H_{\text{par}}^{-m/2} \) belongs to symbol classes \( G^m \) and \( G^{-m} \) respectively. Hence, the symbol of \( R_j \) belongs to the symbol the symbol class \( S_{1,0}^0 \), which implies the result and completes the proof. \( \square \)

### 4. Sobolev spaces associated to the partial harmonic oscillator

Given any \( p \in [1, \infty) \), and \( \alpha > 0 \), we define the potential spaces associated to \( H_{\text{par}} \) by
\[
L^{\alpha, p}_{H_{\text{par}}}(\mathbb{R}^{d+1}) = H_{\text{par}}^{-\alpha/2}(L^p(\mathbb{R}^{d+1})),
\]
with the norm
\[
\|f\|_{L^{\alpha, p}_{H_{\text{par}}}} = \|g\|_{L^p(\mathbb{R}^{d+1})},
\]
where \( g \in L^p(\mathbb{R}^{d+1}) \) satisfies \( H_{\text{par}}^{-\alpha/2} g = f \).

**Remark 4.1.** The norm is well defined since \( H_{\text{par}}^{-\alpha/2} \) is one-to-one and bounded in \( L^p(\mathbb{R}^{d+1}) \). Also, \( C_0^\infty(\mathbb{R}^{d+1}) \) is dense in \( L^{\alpha, p}_{H_{\text{par}}}(\mathbb{R}^{d+1}) \).
For any nonnegative integer $k \geq 0$, we can also define the Sobolev spaces associated to $H_{\text{par}}$ by the differential operators $A_j$ as follows:

$$W^{k,p}_{H_{\text{par}}} = \left\{ f \in L^p(\mathbb{R}^{d+1}) \left| \begin{array}{c}
A_{j_1}A_{j_2} \cdots A_{j_m} f \in L^p(\mathbb{R}^{d+1}), \\
\text{for any } 1 \leq m \leq k, \ 0 \leq |j_1|, \ldots, |j_m| \leq d
\end{array} \right. \right\},$$

with the norm

$$\|f\|_{W^{k,p}_{H_{\text{par}}}} = \sum_{m=1}^{k} \left( \sum_{d=1}^{d} \cdots \sum_{d=1}^{d} \|A_{j_1}A_{j_2} \cdots A_{j_m} f\|_{L^p} \right) + \|f\|_{L^p}.$$

**Theorem 4.2.** Let $k \in \mathbb{N}$ and $p \in (1, \infty)$. Then we have

$$W^{k,p}_{H_{\text{par}}} (\mathbb{R}^{d+1}) = L^{k,p}_{H_{\text{par}}} (\mathbb{R}^{d+1})$$

with equivalence of norms.

**Proof.** We firstly prove that $L^{k,p}_{H_{\text{par}}} (\mathbb{R}^{d+1}) \subset W^{k,p}_{H_{\text{par}}} (\mathbb{R}^{d+1})$. For any function $f \in L^{k,p}_{H_{\text{par}}} (\mathbb{R}^{d+1})$, there exists a function $g \in L^p(\mathbb{R}^{d+1})$, such that $f = H_{\text{par}}^{-k/2} g$. Hence, by the $L^p$ boundedness of Riesz transforms in Proposition 3.7, we have

$$\|A_{j_1}A_{j_2} \cdots A_{j_m} f\|_p = \|A_{j_1}A_{j_2} \cdots A_{j_m} H_{\text{par}}^{k/2} g\|_p \lesssim \|g\|_p \lesssim \|f\|_{L^{k,p}_{H_{\text{par}}} (\mathbb{R}^{d+1})},$$

which implies that $L^{k,p}_{H_{\text{par}}} (\mathbb{R}^{d+1}) \subset W^{k,p}_{H_{\text{par}}} (\mathbb{R}^{d+1})$.

Next, we show that $W^{k,p}_{H_{\text{par}}} (\mathbb{R}^{d+1}) \subset L^{k,p}_{H_{\text{par}}} (\mathbb{R}^{d+1})$ by induction. First, it is easy to check that for any $f, g \in C_c^\infty(\mathbb{R}^{d+1})$, we have

$$\int_{\mathbb{R}^{d+1}} fg = 2 \int_{\mathbb{R}^{d+1}} \sum_{-d \leq j \leq d} R_j f R_j g.$$

Thus, by duality and the boundedness of Riesz transform, we obtain for any $g \in L^p(\mathbb{R}^d)$ that

$$\|g\|_p \lesssim \sum_{-d \leq j \leq d} \|R_j g\|_p.$$

Hence, we obtain by choosing $g = H_{\text{par}}^{1/2} f$ that

$$\|f\|_{L^{1,p}_{H_{\text{par}}} (\mathbb{R}^{d+1})} = \|H_{\text{par}}^{1/2} f\|_p \lesssim \sum_{-d \leq j \leq d} \|A_j f\|_p \lesssim \|f\|_{W^{1,p}_{H_{\text{par}}} (\mathbb{R}^{d+1})}.$$  \(\text{(4.1)}\)

That is, $W^{k,p}_{H_{\text{par}}} (\mathbb{R}^{d+1}) \subset L^{k,p}_{H_{\text{par}}} (\mathbb{R}^{d+1})$ for $k = 1$.

Suppose that for any $f \in C_c^\infty(\mathbb{R}^{d+1})$ and any $1 \leq m < k$, we have

$$\|f\|_{L^{m,p}_{H_{\text{par}}} (\mathbb{R}^{d+1})} \leq \|f\|_{W^{m,p}_{H_{\text{par}}} (\mathbb{R}^{d+1})}.$$  \(\text{(4.3)}\)

It follows by duality that

$$\|f\|_{L^{k,p}_{H_{\text{par}}} (\mathbb{R}^{d+1})} = \|H_{\text{par}}^{k/2} f\|_{L^p(\mathbb{R}^{d+1})} = \sup_{g \in C_c^\infty(\mathbb{R}^d), \|g\|_p = 1} \int_{\mathbb{R}^{d+1}} H_{\text{par}}^{k/2} f g \, dz = \sup_{g \in C_c^\infty(\mathbb{R}^d), \|g\|_p = 1} \int_{\mathbb{R}^{d+1}} H_{\text{par}}^{k} f H_{\text{par}}^{-k/2} g \, dz.$$
Since there exist constants $c_1, c_2, \ldots, c_{k-1}$ such that
\begin{equation*}
\sum_{0 \leq |j_1|, \ldots, |j_k| \leq d} A_{j_k}^* \cdots A_{j_1}^* A_{j_1} \cdots A_{j_k} = 2^k H_{\text{par}}^k + \sum_{m=1}^{k-1} c_m H_{\text{par}}^m,
\end{equation*}
we obtain that
\begin{equation*}
2^k \int_{\mathbb{R}^{d+1}} H_{\text{par}}^k f H_{\text{par}}^{-k/2} g \, dz
= \int_{\mathbb{R}^{d+1}} \sum_{0 \leq |j_1|, \ldots, |j_k| \leq d} \left( A_{j_k}^* \cdots A_{j_1}^* A_{j_1} \cdots A_{j_k} - \sum_{m=1}^{k-1} c_m H_{\text{par}}^m \right) f H_{\text{par}}^{-k/2} g
= \sum_{0 \leq |j_1|, \ldots, |j_k| \leq d} \int_{\mathbb{R}^{d+1}} A_{j_k} \cdots A_{j_1} f R_{j_1} \cdots R_{j_k} g - \sum_{m=1}^{k-1} c_m \int_{\mathbb{R}^{d+1}} H_{\text{par}}^{m/2} f H_{\text{par}}^{-k-m} g
\leq \sum_{0 \leq |j_1|, \ldots, |j_k| \leq d} \left( \|A_{j_1} \cdots A_{j_k} f\|_p \|R_{j_1} \cdots R_{j_k} g\|_{p'} + \sum_{m=1}^{k-1} |c_m| \|H_{\text{par}}^{m/2} f\|_p \|H_{\text{par}}^{-k-m} g\|_{p'} \right)
\leq C \|f\|_{W_{\text{par}}^{\alpha,p}(\mathbb{R}^{d+1})},
\end{equation*}
where we used (4.3) and the boundedness of Riesz transforms in the last inequality. This completes the proof that $W_{\text{par}}^{\alpha,p}(\mathbb{R}^{d+1}) \subset L_{\text{par}}^{\alpha,p}(\mathbb{R}^{d+1})$ for $k \geq 1$.

**Proposition 4.3.** Let $p \in (1, \infty)$. The Riesz transforms $R_j$ ($-d \leq j \leq d$) are bounded on the spaces $L_{\text{par}}^{\alpha,p}(\mathbb{R}^{d+1})$.

**Proof.** By the definition of $L_{\text{par}}^{\alpha,p}(\mathbb{R}^{d+1})$, it suffices to show for $0 \leq |j| \leq d$ that the operators
\begin{equation*}
T_j = H_{\text{par}}^{-\alpha/2} A_j H_{\text{par}}^{-1/2} H_{\text{par}}^{\alpha/2}
\end{equation*}
are bounded on $L^p(\mathbb{R}^{d+1})$. By Proposition 2.5, Lemma 3.6 and the fact that the symbol of the operators $A_j$ belongs to the symbol class $G^1$, the symbols of the operator $T_j$ belong to the symbol class $S_{1,0}^0$. Hence, the operators $T_j$ are bounded on $L^p(\mathbb{R}^{d+1})$ for $1 < p < \infty$ (see [22, 23]), which proves the result. □

A direct consequence of Proposition 3.5 is that any function in $L_{\text{par}}^{\alpha,p}(\mathbb{R}^{d+1})$ enjoys some decay in the $x$ direction.

**Corollary 4.4.** If $p \in [1, \infty)$, $\alpha > 0$ and $f \in L_{\text{par}}^{\alpha,p}(\mathbb{R}^{d+1})$, then $|x|^\alpha f$ belongs to $L^p(\mathbb{R}^{d+1})$.

Next, we show the relations between space $L_{\text{par}}^{\alpha,p}(\mathbb{R}^{d+1})$ and spaces $W^{\alpha,p}(\mathbb{R}^{d+1})$, $L_{\text{par}}^{\alpha,p}(\mathbb{R}^{d+1})$ adapted to the Laplacian and Hermite operators, respectively.

**Theorem 4.5.** Let $\alpha > 0$ and $p \in (1, \infty)$, then
\begin{enumerate}
\item $L_{\text{par}}^{\alpha,p}(\mathbb{R}^{d+1}) \subseteq L_{\text{par}}^{\alpha,p}(\mathbb{R}^{d+1}) \subseteq W^{\alpha,p}(\mathbb{R}^{d+1})$.
\item If $f \in W^{\alpha,p}(\mathbb{R}^{d+1})$ and has compact support, then $f \in L_{\text{par}}^{\alpha,p}(\mathbb{R}^{d+1})$.
\end{enumerate}

**Proof.** We firstly show the inclusion in (1). It suffices to verify that the symbols of
\begin{equation*}
(1 - \Delta_{\mathbb{R}^{d+1}})^{\alpha/2} H_{\text{par}}^{-\alpha/2} \text{ and } H_{\text{par}}^{\alpha/2} H_{\text{par}}^{-\alpha/2}
\end{equation*}
belong to the symbol class \( S_{1,0}^0 \) and so they define bounded operators on \( L^p(\mathbb{R}^{d+1}) \), (see [22, 23]).

For the former: since the symbol of the operator \((1 - \Delta_{\mathbb{R}^{d+1}})^{\alpha/2}\) belongs to class \( S_{1,0}^{0} \) and the symbol of the operator \( H_{\text{par}}^{-\alpha/2} \) belongs to class \( G^{-\alpha} \) due to Lemma 3.6 and class \( S_{1,0}^{-\alpha} \), we obtain that the symbol of the operator \((1 - \Delta_{\mathbb{R}^{d+1}})^{\alpha/2}H_{\text{par}}^{-\alpha/2}\) belongs to \( S_{1,0}^{0} \). Therefore, the operator \((1 - \Delta_{\mathbb{R}^{d+1}})^{\alpha/2}H_{\text{par}}^{-\alpha/2}\) is bounded on \( L^p(\mathbb{R}^{d+1}) \).

For the later: it is known for the symbol \( q_{\alpha}(\rho, x, \tau, \xi) \) of the operator \( H^{-\alpha/2} \) that \( q_{\alpha} \in G^{-\alpha} \) in [26] since

\[
|D_{\tau}^{\beta}D_{\xi}^{\gamma}q_{\alpha}(\rho, x, \tau, \xi)| \lesssim (1 + |\tau| + |\xi| + |\rho| + |x|)^{-\alpha-|\beta|-|\gamma|-|\delta|} \\
\lesssim (1 + |\tau| + |\xi| + |x|)^{-\alpha-|\beta|-|\gamma|-|\delta|}.
\]

As the symbol of the operator \( H_{\text{par}}^{\alpha/2} H^{-\alpha/2} \) belongs to \( S_{1,0}^{0} \) by Proposition 2.5, which implies that the operator \( H_{\text{par}}^{\alpha/2} H^{-\alpha/2} \) is bounded on \( L^p(\mathbb{R}^{d+1}) \).

Next, we show the nonequivalence between them. Define

\[
g_1(\rho, x) = \frac{1}{(1 + \rho)^{\frac{1}{2}+\alpha}} \frac{1}{(1 + |x|)^{\frac{1}{2}+\alpha}}, \quad f_1(\rho, x) = (I - \Delta_{\mathbb{R}^{d+1}})^{-\alpha/2} g_1(\rho, x), \\
g_2(\rho, x) = \frac{1}{(1 + \rho)^{\frac{1}{2}+\alpha}} \Phi_{\mu}(x), \quad f_2(\rho, x) = H^{-\alpha/2} g_2(\rho, x).
\]

We claim that

\[ f_1 \in W^{\alpha,p}(\mathbb{R}^{d+1}) \setminus L^{\alpha,p}_{H_{\text{par}}}(\mathbb{R}^{d+1}), \quad f_2 \in L^{\alpha,p}_{H_{\text{par}}}(\mathbb{R}^{d+1}) \setminus L^{\alpha,p}_H(\mathbb{R}^{d+1}). \]

Let \( G_{\alpha}(z) \) be the kernel of the oerpator \((I - \Delta_{\mathbb{R}^{d+1}})^{-\alpha/2} \), that is,

\[
G_{\alpha}(z) = \frac{1}{(4\pi)^{d/2} \Gamma(d/2)} \int_{0}^{\infty} e^{-\frac{|z|^2}{4t}} t^{d/2-1/2} t^{-\alpha/(d+1)/2} dt.
\]

On the one hand, it is easy to see that \( g_1 \in L^p(\mathbb{R}^{d+1}) \), hence \( f_1 \in W^{\alpha,p}(\mathbb{R}^{d+1}) \). On the other hand, since \( G_{\alpha}(z) \) is positive,

\[
f_1(\rho, x) = \int_{\mathbb{R}^{d+1}} G_{\alpha}(z')g_1(z - z') dz' \geq (2 + |\rho|)^{-1/p-\alpha}(2 + |x|)^{-1/p-\alpha} \int_{|z|<1} G_{\alpha}(z') dz',
\]

which implies that \( |x|^\alpha f_1 \notin L^p(\mathbb{R}^{d+1}) \), and therefore we have \( f_1 \notin L^{\alpha,p}_{H_{\text{par}}}(\mathbb{R}^{d+1}) \) by Corollary 4.4.

Similarly, using the fact that \( g_2 \in L^p(\mathbb{R}^{d+1}) \) holds, we have \( f_2 \in L^{\alpha,p}_{H_{\text{par}}}(\mathbb{R}^{d+1}) \). At the same time, we have

\[
f_2(\rho, x) \geq (2 + |\rho|)^{-1/p-\alpha} \int_{|z|<1} \Psi_{\alpha}(z') d\rho' dx',
\]

which implies that \( |\rho|^\alpha f_2 \notin L^p(\mathbb{R}^{d+1}) \), and therefore we have \( f_2 \notin L^{\alpha,p}_H(\mathbb{R}^{d+1}) \).

Lastly, the statement (2) follows from (1) and [4, Theorem 3 (iii)].
5. Some Integral Inequalities Adapted to the Operator $H_{\text{par}}$

In this section, we obtain some integral inequalities associated to the partial harmonic oscillator $H_{\text{par}}$.

5.1. Hardy–Littlewood–Sobolev Inequality. Let $0 < \alpha < d + 1$. By (3.9), we have

$$H_{\text{par}}^{-\alpha/2} f(z) \leq C \int_{\mathbb{R}^{d+1}} \frac{|f(z')|}{|z - z'|^{d+1-\alpha}} dz',$$

then we have

**Proposition 5.1.** Let $p, q > 1$ and $0 < \alpha < d + 1$ with $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{d+1}$, then the operator $H_{\text{par}}^{-\alpha/2}$ is bounded from $L^p(\mathbb{R}^{d+1})$ to $L^q(\mathbb{R}^{d+1})$.

**Proof.** This is obvious from the rough estimate (5.1) and the classical Hardy–Littlewood–Sobolev inequality in [16]. □

In fact, by (3.9), the integral kernel of the operator $H_{\text{par}}^{-\alpha/2} f$ is controlled by an integral function which has exponential decay away from the zero, which implies the following refined estimates.

**Theorem 5.2.** Let $0 < \alpha < d + 1$. Then the following holds:

1. there exists a constant $C > 0$ such that

$$\|H_{\text{par}}^{-\alpha/2} f\|_q \leq C \|f\|_1,$$

for all $f \in L^1(\mathbb{R}^{d+1})$ if and only if $1 \leq q < \frac{d+1}{d+1-\alpha}$.

2. there exists a constant $C > 0$ such that

$$\|H_{\text{par}}^{-\alpha/2} f\|_\infty \leq C \|f\|_p$$

for all $f \in L^p(\mathbb{R}^{d+1})$ if and only if $p > \frac{d+1}{\alpha}$.

3. If $1 < p < \infty$, $1 < q < \infty$ and $\frac{1}{p} - \frac{\alpha}{d+1} \leq \frac{1}{q} < \frac{1}{p}$, then there exists a constant $C > 0$ such that

$$\|H_{\text{par}}^{-\alpha/2} f\|_q \leq C \|f\|_p$$

for all $f \in L^p(\mathbb{R}^{d+1})$.

**Proof.** For the case (1). By generalized Minkowski's inequality, we obtain for function $f \in L^1(\mathbb{R}^{d+1})$ that

$$\int_{\mathbb{R}^{d+1}} (H_{\text{par}}^{-\alpha/2} f)^q(z) dz \leq \left( \int_{\mathbb{R}^{d+1}} \left( \int_{\mathbb{R}^{d+1}} K_{\alpha/2}(z, z')^q dz' \right)^{1/q} |f(z')| dz' \right)^q$$

By (3.8) and (3.9), we have

$$\int_{\mathbb{R}^{d+1}} K_{\alpha/2}(z, z')^q dz \lesssim \int_{D_-} \frac{dz}{|z - z'|^{q(d+1-\alpha)}} + \int_{D_+} e^{-|z|/2} dz,'
where $D_+ = \{|z - z'| \geq 1\}$ and $D_+ = \{|z - z'| < 1\}$. The right hand side in the above estimate is finite if $1 \leq q < \frac{d+1}{d+1-\alpha}$. For the converse, note that as $|z - z'| < 1$, we have

$$K_{\alpha/2}(z, z') \geq C_\alpha e^{-|x+x'|^2} \int_0^1 t^{\alpha-1-\frac{d+1}{2}} e^{-|z-z'|^2/t} \, dt \geq C_\alpha \frac{e^{-|x+x'|^2}}{|z - z'|^{d+1-\alpha}}.$$  

(5.2)

Let $f_n$ be an approximation of identity. Then we have

$$\int_{\mathbb{R}^{d+1}} \left( \int_{\mathbb{R}^{d+1}} \frac{e^{-|x+x'|^2}}{|z - z'|^{d-\alpha}} f_n(z') \, dz' \right)^q \, dz \to \int_{|z| \leq 1} \frac{e^{-q|x|^2}}{|z|^{q(d+1-\alpha)}} \, dz,$$

which is $\infty$ for $q(d+1-\alpha) \geq d$, and completes the proof of the necessity that $1 \leq q \leq d/(d+1-\alpha)$.

For the case (2). By Hölder’s inequality, we get

$$\left| \int_{\mathbb{R}^{d+1}} K_{\alpha/2}(z, z') f(z') \, dz' \right| \leq \|f\|_p \left( \int_{\mathbb{R}^{d+1}} K_{\alpha/2}(z, z')^{p'} \, dz' \right)^{1/p'}.$$

Using a similar argument as in the proof of the case (1), we know that the right hand side is finite when $p > \frac{d+1}{\alpha}$.

Conversely, by choosing

$$f(z) = \begin{cases} |z|^{-\alpha}(\log |z|)^{-\frac{\alpha}{d+1}(1+\varepsilon)}, & \text{if } |z| \leq 1/2, \\ 0, & \text{if } |z| \geq 1/2, \end{cases}$$

then we have $f \in L^p(\mathbb{R}^{d+1})$ for all $p \leq (d+1)/\alpha$. However, the function $H^{-\alpha/2}_{\text{par}} f$ is essentially unbounded since we have by (5.2) that

$$H^{-\alpha/2}_{\text{par}} f(0) \geq C \int_{|z'| \leq 1/2} |z|^{-(d+1)} \left( \log \frac{1}{|z'|} \right)^{-\frac{\alpha}{d+1}(1+\varepsilon)} \, dz' = \infty,$$

where $\varepsilon$ is small.

The case (3) now follows from Proposition 5.1, the inequality (1), the inequality (2) and the Riesz–Thorin interpolation theorem.

\[ \square \]

**Remark 5.3.** By Corollary 3.3 and the similar argument as in the above proof, we have the following result: Let $p, q > 1$ and $0 < \alpha < d+1$ with $\frac{1}{p} - \frac{\alpha}{d+1} \leq \frac{1}{q} < \frac{1}{p}$, then there exists a constant $C$ such that for all $f \in L^p(\mathbb{R}^{d+1})$, we have

$$\|(H_{\text{par}} + 2)^{-\alpha/2} f\|_q \leq C\|f\|_p,$$

$$\|(H_{\text{par}} - 2)^{-\alpha/2} f\|_q \leq C\|f\|_p, \quad d \geq 3.$$

### 5.2. Gagliardo–Nirenberg–Sobolev inequality

We define the gradient operator associated to the operator $H_{\text{par}}$ as follows:

$$\nabla H_{\text{par}} f := (A_0 f, A_1 f, \ldots, A_d f, A_{-1} f, \ldots, A_{-d} f).$$

**Theorem 5.4.** Let $d \geq 3$ and $1 < p, q < \infty$ satisfy $\frac{1}{p} - \frac{1}{d+1} \leq \frac{1}{q} < \frac{1}{p}$. Then for any $f \in L^1_{\text{par}}(\mathbb{R}^{d+1})$, we have

$$\|f\|_q \leq C\|\nabla H_{\text{par}} f\|_p.$$
Proof. By duality and (4.1), we have
\[ \|f\|_q \leq \sum_{-d \leq j \leq d} \|R_j f\|_q. \]
By Lemma 3.1, we obtain \( R_0 f = H_{par}^{-1/2} A_0 f \) and
\[ R_j f = (H_{par} + 2 \text{sgn} \ j)^{-1/2} A_j f, \quad 1 \leq |j| \leq d. \]
Hence, by Remark 5.3 and Theorem 5.2, we obtain for \( \frac{1}{p} - \frac{1}{d+1} \leq \frac{1}{q} < \frac{1}{p} \)
that
\[ \|f\|_q \leq \sum_{-d \leq j \leq d} \|R_j f\|_q \]
\[ \leq \sum_{1 \leq |j| \leq d} \|((H_{par} + 2 \text{sgn} \ j)^{-1/2} A_j f\|_q + \|H_{par}^{-1/2} A_0 f\|_q \]
\[ \leq \sum_{-d \leq j \leq d} \|A_j f\|_p = \|\nabla H_{par} f\|_p. \]
This completes the proof. \qed

Remark 5.5. The classical Gagliardo–Nirenberg–Sobolev inequality in \( \mathbb{R}^{d+1} \) holds with
\( \frac{1}{q} = \frac{1}{p} - \frac{1}{d+1} \). The result here holds for a larger range of \((p, q)\) due to the extra decay
property of functions \( f \in L^1_{H_{par}}(\mathbb{R}^{d+1}) \).

5.3. Hardy’s inequality. Recall that the Hardy’s inequality is:
\[ ||z|^{-\alpha} f(z)||_{L^p(\mathbb{R}^{d+1})} \lesssim \|(-\Delta_{\mathbb{R}^{d+1}})^{\alpha/2} f\|_{L^p(\mathbb{R}^{d+1})}, \quad 0 < \alpha < \frac{d+1}{p}, \]
see [28]. We extend classical Hardy’s inequality for the Laplacian operator \(-\Delta_{\mathbb{R}^{d+1}}\) to the
operator \( H_{par} \) as follows:

**Theorem 5.6.** Let \( 1 < p < \infty \). Then, for any \( f \in C_0^\infty(\mathbb{R}^{d+1}) \), we have
\[ |||z|^{-\alpha} f(z)||_{L^p(\mathbb{R}^{d+1})} \lesssim \|H_{par}^{\alpha/2} f\|_{L^p(\mathbb{R}^{d+1})}, \quad 0 < \alpha < \frac{d+1}{p}. \]
In particular, the inequality
\[ ||z|^{-1} f(z)||_{L^p(\mathbb{R}^{d+1})} \lesssim \|\nabla H_{par} f\|_{L^p(\mathbb{R}^{d+1})} \]
holds when \( 1 < p < d + 1 \).

**Proof.** Note that the symbols of \( (1 - \Delta_{\mathbb{R}^{d+1}})^{\alpha/2} \) and \( H_{par}^{-\alpha/2} \) belong to \( S_{1,0}^0 \) and \( S_{1,0}^{-\alpha} \) respectively. The composition law gives that the symbol of \( (1 - \Delta_{\mathbb{R}^{d+1}})^{\alpha/2} H_{par}^{-\alpha/2} \) belongs to \( S_{1,0}^{\alpha} \),
which implies that \( (1 - \Delta_{\mathbb{R}^{d+1}})^{\alpha/2} H_{par}^{-\alpha/2} \) are bounded in \( L^p(\mathbb{R}^{d+1}) \) for \( 1 < p < \infty \), (see [22, 23]). In addition, \( (-\Delta_{\mathbb{R}^{d+1}})^{\alpha/2} (-1 - \Delta_{\mathbb{R}^{d+1}})^{-\alpha/2} \) are bounded in \( L^p(\mathbb{R}^{d+1}) \) for \( 1 < p < \infty \)
from [20]. Hence, by the inequality (5.3) we have
\[ |||z|^{-\alpha} f(z)||_{L^p(\mathbb{R}^{d+1})} \lesssim \|(-\Delta_{\mathbb{R}^{d+1}})^{\alpha/2} f\|_{L^p(\mathbb{R}^{d+1})} \]
\[ \lesssim \|(-\Delta_{\mathbb{R}^{d+1}})^{\alpha/2} (1 - \Delta_{\mathbb{R}^{d+1}})^{-\alpha/2} (1 - \Delta_{\mathbb{R}^{d+1}})^{\alpha/2} H_{par}^{-\alpha/2} f\|_{L^p(\mathbb{R}^{d+1})} \]
\[ \lesssim \|H_{par}^{\alpha/2} f\|_{L^p(\mathbb{R}^{d+1})}. \]
For $\alpha = 1$, using the first inequality in (4.2), we have
\[
\| z^{-1} f(z) \|_{L^p(\mathbb{R}^{d+1})} \lesssim \| H^{1/2}_{\text{par}} f \|_{L^p(\mathbb{R}^{d+1})} \lesssim \sum_{0 < |j| \leq d} \| A_j f \|_{L^p(\mathbb{R}^{d+1})} \lesssim \| \nabla H_{\text{par}} f \|_{L^p(\mathbb{R}^{d+1})}.
\]
Hence, we completes the proof. \qed

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