A STACKELBERG GAME OF BACKWARD STOCHASTIC DIFFERENTIAL EQUATIONS WITH PARTIAL INFORMATION

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Abstract. This paper is concerned with a Stackelberg game of backward stochastic differential equations (BSDEs) with partial information, where the information of the follower is a sub-σ-algebra of that of the leader. Necessary and sufficient conditions of the optimality for the follower and the leader are first given for the general problem, by the partial information stochastic maximum principles of BSDEs and forward-backward stochastic differential equations (FBSDEs), respectively. Then a linear-quadratic (LQ) Stackelberg game of BSDEs with partial information is investigated. The state estimate feedback representation for the optimal control of the follower is first given via two Riccati equations. Then the leader’s problem is formulated as an optimal control problem of FBSDE. Four high-dimensional Riccati equations are introduced to represent the state estimate feedback for the optimal control of the leader. Theoretic results are applied to a pension fund management problem of two players in the financial market.

1. Introduction. The Stackelberg game is also known as the leader-follower game, which can be traced back to the early work by Stackelberg [37], when he defined a concept of a hierarchical solution for markets where some firms have power of domination over others. The solutions, in the context of the differential game, is called the corresponding Stackelberg equilibrium points in which there are two players with asymmetric roles, one leader and one follower. For obtaining the Stackelberg solutions, it is usual to divide the game problem into two parts. In the first part, which is also known as the follower’s problem, firstly the leader announces his strategy, then the follower will make an instantaneous response, and choose an optimal strategy corresponding to the given leader’s strategy to minimize (or maximize) his cost functional. In the second part, knowing the follower would take such an optimal strategy, the leader will choose an optimal strategy to minimize (or maximize) his cost functional. Overall, the decisions must be made by two player and one of
them is subordinated to the other because of the asymmetric roles, therefore one player must make a decision after the other player’s decision is made.

The Stackelberg game has wide practical financial and economical backgrounds, and has attracted more and more research attentions with applications. Simann and Cruz [36] made an early study on the properties of the Stackelberg solution in static and dynamic non-zero sum two-player games. Bagchi and Başar [2] investigated an LQ stochastic Stackelberg differential game, where the state and control variables do not enter the diffusion coefficient in the state equation. Yong [46] studied an LQ leader-follower differential game in a more general framework where the coefficients of the system and the cost functionals are random, the diffusion of the state equation contains the control variables, and the weight matrices for the controls in cost functionals are not necessarily positive definite. Øksendal et al. [25] proved a maximum principle for a Stackelberg differential game with jump-diffusion, and applied the result to a continuous time newsvendor problem. Bensoussan et al. [3] introduced several solution concepts in terms of the players’ information sets, and studied LQ Stackelberg games under both adapted open-loop and closed-loop memoryless information structures, whereas the control variables do not enter the diffusion coefficient of the state equation. Meanwhile, the Stackelberg games have been investigated in the mean-field, time-delay, partial information and other fields. Recently, Xu and Zhang [45] studied the discrete-time leader-follower game with time delay and the new co-states which capture the future information of the control and the new state which contains the past effects are introduced to overcome the noncausality of strategy design caused by the delay, then the same technique is used to deal with the continuous-time system. Then Xu et al. [44] studied the leader-follower differential game with time delay appearing in the leader’s control, the open-loop solution is given in the form of the conditional expectation with respect to several symmetric Riccati equations by mainly establishing the nonhomogeneous relationship between the forward and the backward variables. Moon and Başar [24] investigated an LQ mean field Stackelberg differential game with the adapted open-loop information structure of the leader where there are only one leader but arbitrarily large number of followers. Lin et al. [22] studied the open-loop LQ Stackelberg game of the mean-field stochastic systems in finite horizon, and a sufficient condition for the existence and uniqueness of the stackelberg strategy was given in terms of the solvability of some Riccati equations and a convexity condition by introducing new state and costate variables. Shi et al. [33] introduced a new explanation for the asymmetric information feature that the information available to the follower is based on the some sub-$\sigma$-algebra of that available to the leader for the Stackelberg differential game. Then an LQ stochastic Stackelberg differential game with noisy observation was solved via some measure transformation, filtering technique, linear FBSDE and mean-field FBSDE decoupling technique, where not all the diffusion coefficients contain the control variables. Shi et al. [34] studied an LQ stochastic Stackelberg differential game with asymmetric information, where the control variables enter both diffusion coefficients of the state equation. Shi et al. [35] investigated a kind of stochastic LQ Stackelberg differential game with overlapping information which means that the follower’s and the leader’s information have some joint part, while they have no inclusion relation. Li and Yu [19] proved the solvability of a kind of coupled FBSDEs with a multilevel self-similar domination-monotonicity structure, then it is used to characterize the unique equilibrium of an LQ generalized Stackelberg game with hierarchy in a closed form.
Different from forward SDEs where a prescribed initial condition \( x(0) = x_0 \) is given, the BSDEs is short for a kind of backward SDEs with a given terminal condition \( y(T) = \xi \). And BSDE admits a pair of adapted solution \((y(\cdot), z(\cdot))\) under some conditions, where the additional term \( z(\cdot) \) may be interpreted as a risk-adjustment factor and is required for the equation to have adapted solution. The linear version of this type of equation was first introduced by Bismut [4] as the adjoint equation in the stochastic maximum principle. General nonlinear BSDEs, introduced independently by Pardoux and Peng [26] and Duffie and Epstein [11], have received considerable research attention in recent years due to their nice structure and wide applicability in a number of different areas, especially in mathematical finance, optimal control and differential games. El Karoui et al. [12] discussed different properties of BSDEs and their application to finance. Two recent monographs about BSDEs can be seen in Pardoux and Răşcanu [27] and Zhang [50].

The optimal control problem of BSDEs was first studied by Peng [28, 29] and El Karoui et al. [12], when solving the recursive utility maximization problems. Dokuchaev and Zhou [8] studied a stochastic control problem where the system dynamics is a controlled nonlinear BSDE. Kohlmann and Zhou [18] explored the relationship between BSDEs and stochastic controls by interpreting BSDEs as some stochastic optimal control problems. Chen and Zhou [6] investigated an optimization model of stochastic LQ regulators with indefinite control cost weighting matrices, involving a backward LQ problem. Lim and Zhou [21] studied an optimal control of linear BSDEs with a quadratic cost criteria, and the solution is obtained by using the completion-of-squares technique and new kinds of Riccati equations. Huang et al. [14] studied a partial information control problem of backward stochastic systems, and obtained a new stochastic maximum principle. Shi [30] investigated an optimal control problem for systems described by BSDEs with time delayed generators, and proved a sufficient maximum principle. The mean-field BSDE was firstly introduced by Buckdahn et al. [5]. Ma and Liu [23] investigated an optimal control of an infinite horizon system governed by mean-field BSDE with delay and partial information, and establish the existence and uniqueness results for a mean-field BSDE with average delay. Li et al. [20] studied the LQ optimal control problem for mean-field BSDEs.

When it comes to the differential game problem of BSDEs, Hamadene and Leppelier [13] discussed a stochastic zero-sum differential games of the results on BSDEs, and obtained the existence of a saddle point in the bounded case under the Isaacs’ condition. Yu and Ji [49] studied an existence and uniqueness result for an initial coupled FBSDE under some monotone conditions, which was applied to backward LQ non-zero sum stochastic differential game problem. Wang and Yu [40] established a necessary condition and a sufficient condition in the form of maximum principle for open-loop equilibrium point of the game systems described by the BSDEs. Wang and Yu [41] continued to establish a necessary condition in the form of maximum principle for open-loop Nash equilibrium point of this type of partial information game, and then gave a verification theorem which is a sufficient condition for Nash equilibrium point. Shi and Wang [32] investigated a non-zero sum differential game, where the state dynamics follows a BSDE with time-delayed generator, and an Arrow’s sufficient condition for open-loop equilibrium point is proved. Huang et al. [15] studied a backward mean-field linear-quadratic-Gaussian games of weakly coupled stochastic large-population system, and two classes of foregoing games are discussed and their decentralized strategies are derived through the
consistency condition. Huang and Wang [16] discussed a kind of non-zero sum differential game of mean-field BSDE. Wang et al. [39] studied a kind of LQ non-zero sum differential game driven by BSDE with asymmetric information. Aurell [1] studied a mean-field type games between two players with backward stochastic dynamics, and made up a class of non-zero sum, non-cooperating, differential games where the players’ state dynamics solve a BSDE that depends on the marginal distributions of player states. Zheng and Shi [51] researched the Stackelberg game of BSDEs with complete information. Du et al. [9] studied the mean-field game of \( N \) weakly-coupled linear BSDE system. Du and Wu [10] investigated a kind of Stackelberg game of mean-field BSDEs. Huang et al. [17] focused on a kind of non-zero sum differential game driven by mean-field BSDE with asymmetric information.

Inspired by the above literatures, in this paper we study the Stackelberg game of BSDEs with partial information, where the coefficients of the backward game system and cost functionals are deterministic, and the control domain is convex.

The novelty of the formulation and the contribution in this paper is the following.

1. A new kind of general Stackelberg game of BSDEs with partial information is introduced, where a terminal condition \( \xi \) is given in advance. In our framework, the information filtration available to the leader and the follower, are both based on the sub-\( \sigma \)-algebra of the complete information filtration naturally generated by the random noise source. And the information filtration available to the follower is included in that available to the leader.

   For the follower’s problem, the partial information maximum principle and verification theorem are given, which are direct from Theorem 2.1 and Theorem 2.3 of Wang and Yu [41]. For the leader’s problem, inspired by Shi et al. [33], the partial information maximum principle could be derived, where the state equation satisfying a conditional mean-field FBSDE with the initial coupling, and the partial information verification theorem is derived, by the Clarke generalized gradient.

2. For the LQ case, it consists of a stochastic optimal control problem of BSDE with partial information for the follower, and followed by a stochastic optimal control problem of coupled conditional mean-field FBSDEs with complete information for the leader, which are different from that in the (forward) Stackelberg differential game studied in Shi et al. [34].

3. For giving the state feedback representations for the optimal control of the follower, two Riccati equations, a linear backward stochastic differential filtering equation (BSDFE), and a linear stochastic differential filtering equation (SDFE) are introduced. See Theorem 4.1. Then, four high-dimensional Riccati equations, a linear BSDFE, and a linear SDFE are introduced to represent the optimal control of the leader as the state estimate feedback form. See Theorem 4.2.

4. A pension fund problem of two-players Stackelberg game with asymmetric information in the financial market is studied, the Stackelberg equilibrium point is represented and the optimal initial wealth reserve is obtained explicitly.

The rest of this paper is organized as follows. In Section 2, the general Stackelberg game of BSDEs with partial information is formulated. Then this general problem is studied in Section 3. The follower’s problem of the BSDE with partial information is considered first in Subsection 3.1, while the leader’s problem of the conditional mean-field FBSDE is studied in Subsection 3.2. By the partial information maximum principle approach, necessary and sufficient conditions for the optimal controls of the follower and the leader’s are given, respectively. Then the
LQ Stackelberg game problem with partial information is investigated in Section 4. Specially, Subsection 4.1 is devoted to the solution of an LQ stochastic optimal control problem of BSDE with partial information of the follower, via two Riccati equations, a BSDFE and a SDFE, the optimal control of the follower is given in the state estimate feedback form. Subsection 4.2 is devoted to the solution of an LQ stochastic optimal control problem of coupled conditional mean-field FBSDE with complete information of the leader, the optimal control of the leader is represented as the state feedback form by the solutions to four new high-dimensional Riccati equations, a BSDFE and a SDFE. In Section 5, the theoretic results in the previous sections are applied to a pension fund management problem of two players with asymmetric information in the financial market. Finally, Section 6 gives some concluding remarks.

2. Problem formulation. In this paper, we use $\mathbb{R}^n$ to denote the Euclidean space of $n$-dimensional vectors, $\mathbb{R}^{n \times d}$ to denote the space of $n \times d$ matrices, and $\mathcal{S}^n$ to denote the space of $n \times n$ symmetric matrices. $\langle \cdot, \cdot \rangle$ and $| \cdot |$ are used to denote the scalar product and norm in the Euclidean space, respectively. A $\top$ appearing in the superscript of a matrix, denotes its transpose. $f_x, f_{xx}$ denote the first/second-order partial derivatives with respect to $x$ for a differentiable function $f$, respectively.

Let $T > 0$ be fixed. We consider a complete filtered probability space $(\Omega, \mathcal{F}, (\mathbb{F}, \mathbb{P})$, on which two standard $m/\tilde{m}$-dimensional independent Brownian motions $W(\cdot)/\tilde{W}(\cdot)$ are defined, with $W(0) = \tilde{W}(0) = 0$. The filtration $\{\mathcal{F}_t\} = \sigma\{W(r), \tilde{W}(r) : 0 \leq r \leq t\}$ is augmented by all the $\mathbb{P}$-null sets in $\mathcal{F}$, $t \in [0, T]$. $\mathbb{F} = \{\mathcal{F}_t\}_{0 \leq t \leq T}$. $L^2_{\mathbb{F}}(\Omega, \mathbb{R}^n)$ denotes the set of $\mathbb{R}^n$-valued, $\mathcal{F}_t$-measurable, square-integrable random vectors, $L^2_{\mathbb{F}}(0, T; \mathbb{R}^n)$ denotes the set of $\mathbb{R}^n$-valued, $\mathcal{F}_t$-measurable, square-integrable random processes on $[0, T]$, $L^2_{\mathbb{F}}(0, T; \mathbb{R}^{n \times d})$ denotes the set of $n \times d$-matrix-valued, $\mathcal{F}_t$-adapted, square integrable processes on $[0, T]$, and $L^\infty(0, T; \mathbb{R}^{n \times d})$ denotes the set of $n \times d$-matrix-valued, bounded functions on $[0, T]$.

Let us consider the following controlled BSDE:

$$
\begin{cases}
-dg^{v_1,v_2}(t) = f(t, g^{v_1,v_2}(t), z_1^{v_1,v_2}(t), z_2^{v_1,v_2}(t), v_1(t), v_2(t))dt \\
\quad - z_1^{v_1,v_2}(t)dW(t) - z_2^{v_1,v_2}(t)d\tilde{W}(t), \quad t \in [0, T],
\end{cases}
$$

(1)

where $f : [0, T] \times \mathbb{R}^n \times \mathbb{R}^{n \times m} \times \mathbb{R}^{n \times \tilde{m}} \times \mathbb{R}^k_1 \times \mathbb{R}^k_2 \rightarrow \mathbb{R}^n$ is a given continuous function in $(y, z_1, z_2, v_1, v_2)$ and $\xi \in L^2_{\mathbb{F}_T}(\Omega, \mathbb{R}^n)$ is given. $v_1(\cdot) \in U_1$ is the control process of the follower, and $v_2(\cdot) \in U_2$ is the control process of the leader, where $U_i$ is a nonempty convex subset of $\mathbb{R}^k_i$, $i = 1, 2$. In the backward game system (1), the two players work together to achieve a common goal $\xi$ at the terminal time $T$.

Let $\mathcal{G}_i^1 \subseteq \mathcal{F}_t$ be a given sub-filtration which represents the information available to the follower and the leader at time $t \in [0, T]$, $i = 1, 2$, respectively, and $\mathcal{G}_i^2 \subseteq \mathcal{G}_i^1 \subseteq \mathcal{F}_t$. We define the admissible control sets by

$$
U_i[0, T] = \{v_i(\cdot) \in L^2_{\mathcal{G}_i^1}(0, T; \mathbb{R}^k_i) : v_i(t) \in U_i, \ a.e. \ t \in [0, T], \ a.s. \}, \quad i = 1, 2,
$$

(2)

respectively.

We define the cost functionals of the follower and the leader as

$$
J_i(v_1(\cdot), v_2(\cdot); \xi) = \mathbb{E}\left[\int_0^T L_i(t, g^{v_1,v_2}(t), z_1^{v_1,v_2}(t), z_2^{v_1,v_2}(t), v_1(t), v_2(t))dt + h_i(g^{v_1,v_2}(0))\right],
$$

(3)
for \( i = 1, 2 \), respectively. Here \( L_i : [0, T] \times \mathbb{R}^n \times \mathbb{R}^{n \times m} \times \mathbb{R}^{n \times \tilde{m}} \times \mathbb{R}^{k_1} \times \mathbb{R}^{k_2} \to \mathbb{R} \) are given continuous functions in \((y, z_1, z_2, v_1, v_2)\) and \( h_i : \mathbb{R}^n \to \mathbb{R} \) are given continuous functions, for \( i = 1, 2 \). We remark that the cost functional (3) describe that the players have their own benefits besides the terminal common goal \( \xi \).

Now, we give some assumptions that will be in force through this paper.

(A1) \textbf{The function } \textbf{f} \textbf{ is continuously differentiable in } (y, z_1, z_2, v_1, v_2). \textbf{Moreover, the partial derivatives } f_y, f_{z_1}, f_{z_2}, f_{v_1}, \textbf{and } f_{v_2} \textbf{ with respect to } y, z_1, z_2, v_1 \textbf{ and } v_2 \textbf{ are uniformly bounded.}

From Pardoux and Peng [26], it is easy to see that if both \( v_1(\cdot) \in \mathcal{U}_1[0, T], v_2(\cdot) \in \mathcal{U}_2[0, T] \), and (A1) holds, then BSDE (1) admits a unique solution triple \((y^{v_1, v_2}(\cdot), z_1^{v_1, v_2}(\cdot), z_2^{v_1, v_2}(\cdot)) \in L_2^0(0, T; \mathbb{R}^n) \times L_2^0(0, T; \mathbb{R}^{n \times m}) \times L_2^0(0, T; \mathbb{R}^{n \times \tilde{m}})\) which we called the state trajectory.

(A2) \( L_i \) \textbf{ is continuously differentiable with respect to } (y, z_1, z_2, v_1, v_2) \textbf{ and } h_i \textbf{ is continuously differentiable in } y, i = 1, 2. \textbf{Moreover, there exists a constant } C \textbf{ such that } L_{iy}, L_{iz_1}, L_{iz_2}, L_{iv_1}, L_{iv_2} \textbf{ are bounded by } C(1 + |y| + |z_1| + |z_2| + |v_1| + |v_2|), \textbf{ and } h_{iy} \textbf{ is bounded by } C(1 + |y|), \textbf{i} = 1, 2.

The problem studied in this paper is proposed in the following definition.

\textbf{Definition 2.1.} \textbf{The pair } \((\bar{v}_1(\cdot), \bar{v}_2(\cdot)) \in \mathcal{U}_1[0, T] \times \mathcal{U}_2[0, T] \textbf{ is called an optimal solution to the Stackelberg game of BSDEs with partial information, if it satisfies the following condition:}

(i) \textbf{For given } \xi \in L_2^0(\Omega, \mathbb{R}^n) \textbf{ and any } v_2(\cdot) \in \mathcal{U}_2[0, T], \textbf{there exists a map } \Gamma : \mathcal{U}_2[0, T] \times L_2^0(\Omega, \mathbb{R}^n) \to \mathcal{U}_1[0, T] \textbf{ such that}

\[
J_1(\Gamma(v_2(\cdot), \xi), v_2(\cdot); \xi) = \min_{v_1(\cdot) \in \mathcal{U}_1[0, T]} J_1(v_1(\cdot), v_2(\cdot); \xi). \tag{4}
\]

(ii) \textbf{There exists a unique } \bar{v}_2(\cdot) \in \mathcal{U}_2[0, T] \textbf{ such that}

\[
J_2(\Gamma(\bar{v}_2(\cdot), \xi), \bar{v}_2(\cdot); \xi) = \min_{v_2(\cdot) \in \mathcal{U}_2[0, T]} J_2(\Gamma(v_2(\cdot), \xi), v_2(\cdot); \xi). \tag{5}
\]

(iii) \textbf{The optimal strategy of the follower is } \bar{v}_1(\cdot) = \Gamma(\bar{v}_2(\cdot), \xi).

We call the above \((\bar{v}_1(\cdot), \bar{v}_2(\cdot))\) \textbf{ a Stackelberg equilibrium point.}

3. The general problem.

3.1. \textbf{Optimization for the follower.} \textbf{In this subsection, we seek the necessary and sufficient conditions of the partial information optimal control for the follower.}

\textbf{Let } \xi \in L_2^0(\Omega, \mathbb{R}^n) \textbf{ and the leader’s strategy } v_2(\cdot) \in \mathcal{U}_2[0, T] \textbf{ be given. Let } \bar{v}_1(\cdot) \textbf{ be an optimal control of the follower, and } (y^{\bar{v}_1, v_2}(\cdot), z_1^{\bar{v}_1, v_2}(\cdot), z_2^{\bar{v}_1, v_2}(\cdot)) \in L_2^0(0, T; \mathbb{R}^n) \times L_2^0(0, T; \mathbb{R}^{n \times m}) \times L_2^0(0, T; \mathbb{R}^{n \times \tilde{m}}) \textbf{ be the corresponding state trajectory. Let the process } x(\cdot) \in L_2^0(0, T; \mathbb{R}^n) \textbf{ satisfy the following adjoint equation:}

\[
\begin{cases}
  dx(t) = -H_{1y}dt - H_{1z_1}dW(t) - H_{1z_2}d\tilde{W}(t), \ t \in [0, T], \\
  x(0) = h_{1y}(y^{\bar{v}_1, v_2}(0)),
\end{cases}
\tag{6}
\]

where \( H_{1\phi} \equiv H_{1\phi}(t, y^{\bar{v}_1, v_2}, z_1^{\bar{v}_1, v_2}, z_2^{\bar{v}_1, v_2}, v_1, v_2, x) \) \textbf{ denotes the partial derivatives of } \( H_1 \) \textbf{ with respect to } \phi = y, z_1, z_2, \textbf{ respectively, and the Hamiltonian function } H_1 : [0, T] \times \mathbb{R}^n \times \mathbb{R}^{n \times m} \times \mathbb{R}^{n \times \tilde{m}} \times \mathbb{R}^{k_1} \times \mathbb{R}^{k_2} \to \mathbb{R} \textbf{ is defined as}

\[
H_1(t, y, z_1, z_2, v_1, v_2) = -L_1(t, y, z_1, z_2, v_1, v_2) - (f(t, y, z_1, z_2, v_1, v_2), x). \tag{7}
\]
In the above, we have omitted some time variables $t$ for simplicity, which will be frequently done in this paper.

The following two results are direct from Theorems 2.1, 2.3 of Wang and Yu [41].

**Theorem 3.1. (Partial information maximum principle for the follower)** Let (A1), (A2) hold and $ξ$ be given. Giving the leader’s strategy $v_2(\cdot) ∈ U_2[0,T]$. Let $v_1(\cdot) ∈ U_1[0,T]$ be an optimal control of the follower, $(y^{1,v_2}(\cdot), z_1^{1,v_2}(\cdot), z_2^{1,v_2}(\cdot))$ be the corresponding state trajectory. Then we have

\[
E\left[\left<H_{1,v_1}(t,y^{1,v_2},z_1^{1,v_2},z_2^{1,v_2},v_1,v_2,x), v_1 - v_1(t)\right|G_t^1\right]\leq 0,
\]

for a.e. $t ∈ [0,T]$, a.s., for any $v_1 ∈ U_1$, where $x(\cdot)$ is the solution to the adjoint equation (6).

**Theorem 3.2. (Partial information verification theorem for the follower)** Let (A1), (A2) hold and $ξ$ be given. Giving the leader’s strategy $v_2(\cdot) ∈ U_2[0,T]$. Assume that $L_1$ is continuously differentiable in $v_1$, for a.e. $t ∈ [0,T]$. Let $\tilde{v}_1(\cdot) ∈ U_1[0,T]$ be given such that $L_{v_1}(t,y^{1,v_2},z_1^{1,v_2},z_2^{1,v_2},v_1,v_2) ∈ L^2_h(0,T; \mathbb{R})$, for $Φ = y_1, z_1, z_2, v_1, v_2$. Suppose that the adjoint equation (6) admits a solution $x(\cdot)$, and

\[
E[H_1(t,y^{1,v_2},z_1^{1,v_2},z_2^{1,v_2},v_1,v_2,x)|G_t^1] = \max_{v_1 ∈ U_1} E[H_1(t,y^{1,v_2},z_1^{1,v_2},z_2^{1,v_2},v_1,v_2,x)|G_t^1],
\]

for a.e. $t ∈ [0,T]$, a.s. Moreover, suppose $E[H_{1,v_1}(t,y^{1,v_2},z_1^{1,v_2},z_2^{1,v_2},v_1,v_2,x)|G_t^1]$ is continuous at $v_1 = \tilde{v}_1(t)$ for a.e. $t ∈ [0,T]$. For each $t ∈ [0,T]$, suppose $H_1$ is concave in $(y_1, z_1, z_2, v_1)$, and $h_1$ is convex in $y$. Then $\tilde{v}_1(\cdot)$ is an optimal control of the follower.

3.2. **Optimization for the leader.** In this subsection, we firstly restate the partial information stochastic optimal control problem of the leader in detail. For any $v_2(\cdot) ∈ U_2[0,T]$, by the maximum condition (8), we assume that a functional $\bar{v}_1(t) = \tilde{v}_1(t, y^{0,v_2},z_1^{0,v_2},z_2^{0,v_2},\hat{x},\hat{x})$ is uniquely defined, where

\[
\begin{align*}
\hat{y}^{0,v_2}(t) & = E[y^{1,v_2}(t)|G_t^1], \\
\hat{z}_1^{0,v_2}(t) & = E[z_1^{1,v_2}(t)|G_t^1], \\
\hat{z}_2^{0,v_2}(t) & = E[z_2^{1,v_2}(t)|G_t^1], \\
\hat{v}_2(t) & = E[v_2(t)|G_t^1], \\
\hat{x}(t) & = E[x(t)|G_t^1], \quad t ∈ [0,T].
\end{align*}
\]

(10)

Denote $φ^{v_2}(\cdot) := φ^{v_1,v_2}(\cdot)$ for $φ = y, z_1, z_2$, respectively. And let $Φ^L : [0,T] × \mathbb{R}^n × \mathbb{R}^n × U_1 × U_2 → \mathbb{R}$ as

\[
Φ^L = Φ^L(t, y^{0,v_2},z_1^{0,v_2},z_2^{0,v_2},v_2) = Φ(t, y^{0,v_2},z_1^{0,v_2},z_2^{0,v_2},\bar{v}_1,v_2)
\]

\[
= Φ(t, y^{0,v_2},z_1^{0,v_2},z_2^{0,v_2},\bar{v}_1(t, y^{0,v_2},z_1^{0,v_2},z_2^{0,v_2},\hat{x},\hat{x}), v_2)
\]

for $Φ = f, L_1, L_2$ and their derivatives with respect to $φ = y, z_1, z_2$, respectively. Then, substituting $\bar{v}_1(\cdot)$ into (1), and combining it with (6), we derive the following FBSDE

\[
\begin{align*}
\begin{cases}
\begin{align*}
dx(t) & = \left[(f_y)^{\top}x + (L_{1y}^1)\right]dt + \left[(f_{z_1}^L)^{\top}x + (L_{1z_1}^L)\right]dW(t) \\
& + \left[(f_{z_2}^L)^{\top}x + (L_{1z_2}^L)\right]d\tilde{W}(t), \\
-dy^{v_2}(t) & = f^tx dt - z_1^{v_2}(t)dW(t) - z_2^{v_2}(t)d\tilde{W}(t), \quad t ∈ [0,T], \\
x(0) & = h_1(y^{v_2}(0)), \quad y^{v_2}(T) = ξ.
\end{align*}
\end{cases}
\end{align*}
\]

(11)
Then we redefine

\[
\tilde{J}_2(v_2(\cdot); \xi) := J_2(\tilde{v}_1(\cdot), v_2(\cdot); \xi) = \mathbb{E} \left[ \int_0^T L_2^v(t) dt + h_2(y^{v_2}(0)) \right]
\]

\[
:= \mathbb{E} \left[ \int_0^T L_2^v(t, y^{v_2}(t), z_1^{v_2}(t), z_2^{v_2}(t), v_2(t)) dt + h_2(y^{v_2}(0)) \right].
\]

(12)

The target of the leader is to find an optimal control \(\tilde{v}_2(\cdot) \in U_2[0, T]\).

Suppose that there exists an optimal control \(\tilde{v}_2(\cdot) \in U_2[0, T]\) for the leader, and the corresponding “state trajectory” \((y^{\tilde{v}_1, \tilde{v}_2}(\cdot), z_1^{\tilde{v}_2}(\cdot), z_2^{\tilde{v}_2}(\cdot), \tilde{x}(\cdot))\) is the solution to (11). Define the Hamiltonian function of the leader

\[
H(\cdot) := \mathbb{E} \left[ \int_0^T L_2^v(t, y^{v_2}(t), z_1^{v_2}(t), z_2^{v_2}(t), v_2(t)) dt + h_2(y^{v_2}(0)) \right].
\]

Indeed, the conditional mean-field term, compared to [52] where the state equation contains conditional mean-field FBSDE can be regarded as a non-standard equation with coupling appears in the terminal time. And the second feature is that the special feature different from the existing literatures is that we derive the special partial information maximum principle for the leader

\[
\text{Theorem 3.3. (Partial information maximum principle for the leader)} \quad \text{Suppose (A1), (A2) holds, let } \tilde{v}_2(\cdot) \in U_2[0, T] \text{ be an optimal control of the leader and (y}^{v_2}(\cdot), z_1^{v_2}(\cdot), z_2^{v_2}(\cdot), \tilde{x}(\cdot)) \text{ be the corresponding optimal state trajectory. Let } (p(\cdot), q_1(\cdot), q_2(\cdot), Q(\cdot)) \text{ be the solution to (14), then}
\]

\[
\mathbb{E} \langle H_{2v_2}(t, y^{v_2}, z_1^{v_2}, z_2^{v_2}, \tilde{v}_2, \tilde{x}, p, q_1, q_2, Q), v_2 - \tilde{v}_2(t) \rangle
\]

\[
+ \mathbb{E} \langle H_{2x}(t, y^{v_2}, z_1^{v_2}, z_2^{v_2}, \tilde{v}_2, \tilde{x}, p, q_1, q_2, Q | G_t^1), \tilde{v}_2 - \tilde{v}_2(t) | G_t^2 \rangle \geq 0,
\]

(15)

for a.e. \(t \in [0, T]\), a.s., for any \(v_2 \in U_2\).
Proof. The maximum condition (15) is similar to that in Theorem 2.3 of Shi et al. [33], which can be obtained applying the convex variation and adjoint technique. We omit the detail for the space limit. See also Zuo and Min [52], Wang et al. [38] for optimal control problems of mean-field FBSDEs with partial information.

Theorem 3.4. (Partial information verification theorem for the leader) Let (A1), (A2) hold, $\tilde{\nu}_2(\cdot) \in U_2$ and $(y^\nu_2(\cdot), z^\nu_2(\cdot), \bar{x}(\cdot))$ be the corresponding state trajectory. Suppose $h_{1,y}(y) = \tilde{h}_1y$ with $h_1 \in S^n$. Let $(p(\cdot), q_1(\cdot), q_2(\cdot), Q(\cdot))$ be the solution to (14). For each $t \in [0, T]$, suppose that $H_2$ is convex in $(y, z_1, z_2, v_2, x)$ and $h_2$ is convex in $y$, and

$$
E[H_2(t, y^\nu_2, z^\nu_2, \nu_2, \bar{x}, p, q_1, q_2, Q) + E[H_2(t, y^\nu_2, z^\nu_2, \nu_2, \bar{x}, p, q_1, q_2, Q)|G^1_t]|G^2_t] = \min_{v_2 \in U_2} E[H_2(t, y^\nu_2, z^\nu_2, v_2, \bar{x}, p, q_1, q_2, Q) + E[H_2(t, y^\nu_2, z^\nu_2, v_2, \bar{x}, p, q_1, q_2, Q)|G^1_t]|G^2_t],
$$

holds for a.e. $t \in [0, T]$, a.s., then $\tilde{\nu}_2(\cdot)$ is an optimal control of the leader.

Proof. Inspired by Yong and Zhou [48], Shi [31], we can prove it by using the Clarke generalized gradient. We omit the details and let it to the interested readers.

4. The linear quadratic problem. In this section, we study an LQ case with $m = \bar{m} = 1$ and to give some explicit forms of the previous results. On account of the abstract concept of the sub-$\sigma$-algebra $G_1$ and $G_2$ which is not generated by known Brownian motion or observation process, we have difficulty in obtaining the specific filtering equations under the general information filtration. Same phenomenon can also be seen in the reference such as [17], [34] and [35]. Therefore, we only consider the special case when the follower’s information filtration is

$$
G^1_t = \sigma\{W(r) : 0 \leq r \leq t\}, \ t \in [0, T],
$$

and the leader’s information filtration is

$$
G^2_t = \mathcal{F}_t = \sigma\{W(r), \bar{W}(r) : 0 \leq r \leq t\}, \ t \in [0, T].
$$

We will consider some general cases in the future.

4.1. Optimization for the follower. We consider the following controlled linear BSDE

$$
\begin{cases}
-dy^v_1(t) = \left[ A(t)y^{v_1,v_2}(t) + B_1(t)v_1(t) + B_2(t)v_2(t) + C_1(t)z_1^{v_1,v_2}(t) \
+ C_2(t)z_2^{v_1,v_2}(t) \right] dt - z_1^{v_1,v_2}(t)dW(t) - z_2^{v_1,v_2}(t)d\bar{W}(t), \ t \in [0, T],

y^{v_1,v_2}(T) = \xi,
\end{cases}
$$

and the cost functional

$$
J_1(v_1(\cdot), v_2(\cdot); \xi) = \frac{1}{2}E\left[ \int_0^T \left( \langle Q_1(t)y^{v_1,v_2}(t), y^{v_1,v_2}(t) \rangle + \langle R_1(t)v_1(t), v_1(t) \rangle \
+ \langle S_1(t,z_1^{v_1,v_2}(t), z_1^{v_1,v_2}(t) \rangle + \langle S_2(t,z_2^{v_1,v_2}(t), z_2^{v_1,v_2}(t) \rangle \right) dt \
+ \langle G_1 y^{v_1,v_2}(0), y^{v_1,v_2}(0) \rangle \right].
$$

Here, $A(\cdot), B_1(\cdot), C_1(\cdot), Q_1(\cdot), R_1(\cdot), S_i(\cdot), i = 1, 2$ are deterministic matrix-valued functions, and $G_1$ is an $\mathbb{R}^n$-valued vector. We give the following assumptions.
where $\Gamma(\cdot)$ is also bounded and the inverse $R_1^{-1}(\cdot)$ is also bounded.

For given control $v_2(\cdot)$, suppose there exists a $\mathcal{G}_t^1$-adapted optimal control $\bar{v}_1(\cdot)$ of the follower, and the corresponding optimal state is $(y^{\bar{v}_1,v_2}(\cdot), z_1^{\bar{v}_1,v_2}(\cdot), z_2^{\bar{v}_1,v_2}(\cdot))$. According to Theorem 3.1, it is necessary that

$$E[H_{10}(t, y^{\bar{v}_1,v_2}, z_1^{\bar{v}_1,v_2}, z_2^{\bar{v}_1,v_2}, v_1, v_2, x)|\mathcal{G}_t^1] = 0, \quad a.e. \ t \in [0,T], \ a.s.,$$

(19)

where the Hamiltonian function of the follower is

$$H_{11}(t, y, z_1, z_2, v_1, v_2, x) = -\frac{1}{2}Q_{1}(t)y - \frac{1}{2}(R_1(t)v_1 + R_2(t)v_2 + C_1(t)z_1 + C_2(t)z_2),$$

(20)

Then we have

$$\bar{v}_1(t) = -R_1^{-1}(t)B_1(t)^\top \hat{x}(t), \quad a.e. \ t \in [0,T], \ a.s.,$$

(21)

where $(y^{\bar{v}_1,v_2}(\cdot), z_1^{\bar{v}_1,v_2}(\cdot), z_2^{\bar{v}_1,v_2}(\cdot), x(\cdot))$ satisfies

$$-dy^{\bar{v}_1,v_2}(t) = \left( A_y^{\bar{v}_1,v_2} - B_1R_1^{-1}B_1^\top \hat{x} + B_2v_2 + C_1z_1^{\bar{v}_1,v_2} + C_2z_2^{\bar{v}_1,v_2} \right)dt$$

$$-z_1^{\bar{v}_1,v_2}(t)dW(t) - z_2^{\bar{v}_1,v_2}(t)d\hat{W}(t),$$

(22)

$$dx(t) = \left( A^\top x + Qy^{\bar{v}_1,v_2} \right)dt + \left( C_1^\top x + S_1z_1^{\bar{v}_1,v_2} \right)dW(t)$$

$$+ \left( C_2^\top x + S_2z_2^{\bar{v}_1,v_2} \right)d\hat{W}(t), \quad t \in [0,T],$$

$$x(0) = G_1y^{\bar{v}_1,v_2}(0), \quad y^{\bar{v}_1,v_2}(T) = \xi.$$  

We then wish to obtain the state feedback form of $\bar{v}_1(\cdot)$, from (21). Let

$$y^{\bar{v}_1,v_2}(t) = -P_1(t)x(t) - \phi(t),$$

(23)

where $P_1(\cdot)$ is a deterministic and differentiable $\mathbb{R}^{n \times n}$-matrix-valued function with $P_1(T) = 0$, and $\phi(\cdot)$ is an $\mathbb{R}^n$-valued, $\mathbb{F}$-adapted process satisfying the BSDE:

$$d\phi(t) = \Gamma(t)dt + \eta(t)dW(t), \quad t \in [0,T]$$

$$\phi(T) = -\xi.$$  

(24)

In the above equation, $\Gamma(\cdot)$ and $\eta(\cdot)$ are both $\mathbb{R}^n$-valued, $\mathbb{F}$-adapted processes, to be determined later. Applying Itô’s formula to (23), we get

$$dy^{\bar{v}_1,v_2}(t) = \left( -\dot{P}_1x - P_1A^\top x + P_1Q_1P_1x + P_1Q_1\phi - \Gamma \right)dt$$

$$- \left[ P_1(C_1^\top x + S_1z_1^{\bar{v}_1,v_2}) + \eta \right]dW(t) - P_1(C_2^\top x + S_2z_2^{\bar{v}_1,v_2})d\hat{W}(t).$$

(25)

Comparing (25) with the first equation of (22), we derive

$$(-AP_1 - \dot{P}_1 - P_1A^\top + P_1Q_1P_1)x - B_1R_1^{-1}B_1^\top \hat{x} + B_2v_2 + C_1z_1^{\bar{v}_1,v_2} + C_2z_2^{\bar{v}_1,v_2} + (P_1Q_1 - A)\phi = \Gamma.$$  

(26)
By assuming $P_1S_1 + I$ and $P_1S_2 + I$ to be invertible, we have
\[
\begin{aligned}
\dot{x}_1^{\bar{v}_1,v_2} &= -(P_1S_1 + I)^{-1}P_1C_1^T \dot{x} - (P_1S_1 + I)^{-1}\xi, \\
\dot{x}_2^{\bar{v}_1,v_2} &= -(P_1S_2 + I)^{-1}P_1C_2^T \dot{x}.
\end{aligned}
\] (27)

Taking $\mathbb{E}[\cdot|G_t]$ on both sides of (23), (24), (26) and (27), we get
\[
\begin{aligned}
\dot{y}_1^{\bar{v}_1,v_2}(t) &= -P_1(t)\dot{x}(t) - \phi(t), \\
d\hat{\phi}(t) &= \hat{\Gamma}(t)dt + \hat{\eta}(t)dW(t), \ t \in [0,T], \ \hat{\phi}(T) = -\xi, \\
\hat{\Gamma} &= \left( -\hat{P}_1 - AP_1 - P_1A^T + P_1Q_1P_1 - B_1R_1^{-1}B_1^T \right) \dot{x} + B_2\hat{v}_2 \\
&\hspace{1cm} + C_1\dot{z}_1^{\bar{v}_1,v_2} + C_2\dot{z}_2^{\bar{v}_1,v_2} + (P_1Q_1 - A)\hat{\phi},
\end{aligned}
\] (28)
and
\[
\begin{aligned}
\dot{z}_1^{\bar{v}_1,v_2} &= -(P_1S_1 + I)^{-1}P_1C_1^T \dot{x} - (P_1S_1 + I)^{-1}\xi, \\
\dot{z}_2^{\bar{v}_1,v_2} &= -(P_1S_2 + I)^{-1}P_1C_2^T \dot{x}.
\end{aligned}
\] (29)

Inserting (30) into (29), we get
\[
\hat{\Gamma} = \left[ -\hat{P}_1 - AP_1 - P_1A^T + P_1Q_1P_1 - B_1R_1^{-1}B_1^T - C_1(P_1S_1 + I)^{-1}P_1C_1^T \\
&- C_2(P_1S_2 + I)^{-1}P_1C_2^T \right] \dot{x} + B_2\hat{v}_2 - C_1(P_1S_1 + I)^{-1}\hat{\eta} + [P_1Q_1 - A]\hat{\phi}. \tag{31}
\]

If the following Riccati equation
\[
\begin{aligned}
\hat{P}_1 + AP_1 + P_1A^T - P_1Q_1P_1 - B_1R_1^{-1}B_1^T + C_1(P_1S_1 + I)^{-1}P_1C_1^T \\
&+ C_2(P_1S_2 + I)^{-1}P_1C_2^T = 0, \ t \in [0,T], \\
P_1(T) = 0,
\end{aligned}
\] (32)

admits a unique differentiable solution $P_1(\cdot)$, then
\[
\begin{aligned}
-d\phi(t) &= \left[ -(P_1Q_1 - A)\phi + C_1(P_1S_1 + I)^{-1}\hat{\eta} - B_2\hat{v}_2 \right] dt \\
&\quad - \hat{\eta}(t)dW(t), \ t \in [0,T], \\
\phi(T) &= -\xi.
\end{aligned}
\] (33)

Next, we set
\[
x(t) = P_2(t)y^{\bar{v}_1,v_2}(t) + \varphi(t), \tag{34}
\]
where $P_2(\cdot)$ is a deterministic and differentiable $\mathbb{R}^{n \times n}$-matrix-valued function with $P_2(0) = G_1$, and $\varphi(\cdot)$ is an $\mathbb{R}^n$-valued, $\mathbb{F}$-adapted process satisfying the SDE:
\[
\begin{aligned}
d\varphi(t) &= \alpha(t)dt + \beta(t)dW(t), \ t \in [0,T], \\
\varphi(0) &= 0, \tag{35}
\end{aligned}
\]
where $\alpha(\cdot), \beta(\cdot)$ are both $\mathbb{R}^n$-valued, $\mathbb{F}$-adapted processes to be determined later.

Applying Itô’s formula to (34), we get
\[
dx(t) = \left( \hat{P}_2y^{\bar{v}_1,v_2} - P_2A\bar{y}^{\bar{v}_1,v_2} + P_2B_1R_1^{-1}B_1^T \dot{x} - P_2B_2\hat{v}_2 - P_2C_1\dot{z}_1^{\bar{v}_1,v_2} \\
&- P_2C_2\dot{z}_2^{\bar{v}_1,v_2} + \alpha(t)dt + \beta(t)dW(t) + P_2\dot{z}_1^{\bar{v}_1,v_2}dW(t) + P_2\dot{z}_2^{\bar{v}_1,v_2}dW(t) \right). \tag{36}
\]

Comparing (36) with the second equation of (22), we have
\[
\begin{aligned}
\alpha &= \left( -\hat{P}_2 + P_2A + Q_1 \right)y^{\bar{v}_1,v_2} + A^T \dot{x} + P_2C_1\dot{z}_1^{\bar{v}_1,v_2} \\
&\hspace{1cm} + P_2C_2\dot{z}_2^{\bar{v}_1,v_2} - P_2B_1R_1^{-1}B_1^T \dot{x} + P_2B_2\hat{v}_2, \\
C_1^T \dot{x} + S_1\dot{z}_1^{\bar{v}_1,v_2} &= P_2\dot{z}_1^{\bar{v}_1,v_2} + \beta, \quad C_2^T \dot{x} + S_2\dot{z}_2^{\bar{v}_1,v_2} = P_2\dot{z}_2^{\bar{v}_1,v_2}. \tag{37}
\end{aligned}
\]
Taking $\mathbb{E}[\cdot | G_t]$ on both sides of (34), (35) and (37), we have
\begin{align}
\dot{z}(t) &= P_2(t)\dot{y}^{\hat{e}_1, \hat{e}_2}(t) + \hat{\phi}(t), \\
\dot{\phi}(t) &= \hat{\alpha}(t)dt + \hat{\beta}(t)dW(t), \quad t \in [0, T], \quad \hat{\phi}(0) = 0,
\end{align}
(38)
\begin{align}
\hat{\alpha} &= \left[ -\dot{P}_2 + P_2A + A^TP_2 + Q_1 - P_2B_1R_1^{-1}B_1^TP_2 \\
&\quad - P_2C_1(P_1S_1 + I)^{-1}P_1C_1^TP_2 - P_2C_2(P_1S_2 + I)^{-1}P_1C_2^TP_2 \right] \dot{y}^{\hat{e}_1, \hat{e}_2} \\
&\quad + \left[ A^T - P_2B_1R_1^{-1}B_1^T - P_2C_1(P_1S_1 + I)^{-1}P_1C_1^T \\
&\quad - P_2C_2(P_1S_2 + I)^{-1}P_1C_2^T \right] \hat{\phi} - P_2C_1(P_1S_1 + I)^{-1}\hat{\eta} + P_2B_2\hat{v}_2,
\end{align}
(39)
\begin{align}
\hat{\beta} &= \left[ C_1^T - (S_1 - P_2)(P_1S_1 + I)^{-1}P_1C_1^T \right] \dot{x} - (S_1 - P_2)(P_1S_1 + I)^{-1}\hat{\eta}.
\end{align}

If the following Riccati equation
\begin{align}
\begin{cases}
\dot{P}_2 - A^TP_2 - P_2A + P_2B_1R_1^{-1}B_1^TP_2 + P_2C_1(P_1S_1 + I)^{-1}P_1C_1^TP_2 \\
+ P_2C_2(P_1S_2 + I)^{-1}P_1C_2^TP_2 - Q_1 = 0, \quad t \in [0, T], \\
P_2(0) = G_1,
\end{cases}
\end{align}
(40)

admits a unique differentiable solution $P_2(\cdot)$, then

\begin{align}
\dot{\phi}(t) &= \left\{ \begin{array}{l}
\left[ A^T - P_2B_1R_1^{-1}B_1^T - P_2C_1(P_1S_1 + I)^{-1}P_1C_1^T \\
- P_2C_2(P_1S_2 + I)^{-1}P_1C_2^T \right] \dot{\phi} - P_2C_1(P_1S_1 + I)^{-1}\hat{\eta} + P_2B_2\hat{v}_2 \\
+ \left\{ \begin{array}{l}
C_1^T - (S_1 - P_2)(P_1S_1 + I)^{-1}P_1C_1^T \dot{x} \\
- (S_1 - P_2)(P_1S_1 + I)^{-1}\hat{\eta}
\end{array} \right\} dW(t).
\end{array} \right.
\end{align}
(41)

From the above relationship between $\dot{x}(\cdot)$ and $\dot{y}^{\hat{e}_1, \hat{e}_2}(\cdot)$:
\begin{align}
\begin{cases}
\dot{y}^{\hat{e}_1, \hat{e}_2}(t) = -P_1(t)\dot{x}(t) - \hat{\phi}(t), \\
\dot{x}(t) = P_2(t)\dot{y}^{\hat{e}_1, \hat{e}_2}(t) + \hat{\phi}(t),
\end{cases}
\end{align}
(42)

by assuming $I + P_2P_1$ to be invertible, we get
\begin{align}
\dot{x} &= -(I + P_2P_1)^{-1}P_2\hat{\phi} + (I + P_2P_1)^{-1}\hat{\phi}.
\end{align}
(43)

Putting (43) into (41), after dealing with the $dW(t)$ term and using the fact that
\begin{align}
(I + AB)^{-1}A = A(I + BA)^{-1}, \quad \text{for any } A, B \in \mathcal{S}^n,
\end{align}

we derive
\begin{align}
\begin{cases}
\dot{\phi}(t) &= \left\{ \begin{array}{l}
\left[ A^T - P_2B_1R_1^{-1}B_1^T - P_2C_1(P_1S_1 + I)^{-1}P_1C_1^T \\
- P_2C_2(P_1S_2 + I)^{-1}P_1C_2^T \right] \dot{\phi} - P_2C_1(P_1S_1 + I)^{-1}\hat{\eta} + P_2B_2\hat{v}_2 \\
+ \left\{ \begin{array}{l}
- (I + P_2P_1)(S_1P_1 + I)^{-1}C_1^T(1 + P_2P_1)^{-1}P_2\hat{\phi} \\
(1 + P_2P_1)(S_1P_1 + I)^{-1}C_1^T(1 + P_2P_1)^{-1}\hat{\phi} \\
- (S_1 - P_2)(P_1S_1 + I)^{-1}\hat{\eta}
\end{array} \right\} dW(t)
\end{array} \right.
\end{cases}
\end{align}
(44)
Remark 2. We introduce two Riccati equations (32), (40) for \( P_1(\cdot), P_2(\cdot) \), respectively, to build the relation between \( y^{\hat{v}_1, \hat{v}_2}(\cdot) \) and \( x(\cdot) \). Similarly to [21], we can obtain the unique solvability of them. And the unique solvability of (33) and (44), with the solutions \((\hat{\phi}(\cdot), \hat{\eta}(\cdot))\) and \(\hat{\varphi}(\cdot)\) respectively, is evident as they can be regarded as two kinds of linear BSDE and SDE with bounded deterministic coefficients and square integrable nonhomogeneous terms.

We then have the following result.

**Theorem 4.1.** Under the assumptions (L1) and (L2), for given \( \xi \in L^2_{\mathcal{F}_T}(\Omega; \mathbb{R}^n) \) and \( v_2(\cdot) \in \mathcal{U}_2[0, T] \), the follower’s problem is solvable with the optimal strategy \( \hat{v}_1(\cdot) \) being a state estimate feedback representation

\[
\hat{v}_1(t) = -R_1^{-1}(t)B_1(t)^T \left[ P_2(t)\hat{y}^{\hat{v}_1, \hat{v}_2}(t) + \hat{\varphi}(t) \right], \quad \text{a.e. } t \in [0, T], \text{ a.s.,}
\]

where \( P_2(\cdot) \) and \( \hat{\varphi}(\cdot) \) are the solutions to (40) and (44), respectively. The optimal state trajectory \((\hat{g}^{\hat{v}_1, \hat{v}_2}(\cdot), \hat{z}_1^{\hat{v}_1, \hat{v}_2}(\cdot), \hat{z}_2^{\hat{v}_1, \hat{v}_2}(\cdot))\) is the unique solution to the following FBSDE:

\[
\begin{aligned}
-d\hat{g}^{\hat{v}_1, \hat{v}_2}(t) &= \left( A\hat{y}^{\hat{v}_1, \hat{v}_2} - B_1R_1^{-1}B_1^TP_2\hat{y}^{\hat{v}_1, \hat{v}_2} - B_1R_1^{-1}B_1^T\hat{\varphi} + B_2\hat{v}_2 \\
&\quad + C_1\hat{z}_1^{\hat{v}_1, \hat{v}_2} + C_2\hat{z}_2^{\hat{v}_1, \hat{v}_2} \right) dt - \hat{z}_1^{\hat{v}_1, \hat{v}_2}(t) dW(t), \\
\hat{d}(t) &= \left( A^T \hat{\dot{x}} + Q_1\hat{y}^{\hat{v}_1, \hat{v}_2} \right) dt + \left( C_1^T \hat{\dot{x}} + S_1\hat{z}_1^{\hat{v}_1, \hat{v}_2} \right) dW(t), \quad t \in [0, T], \\
\hat{g}(0) &= G_1\hat{y}^{\hat{v}_1, \hat{v}_2}(0) + \hat{g}^{\hat{v}_1, \hat{v}_2}(T) = \hat{\xi}.
\end{aligned}
\]

**Proof.** For given \( \xi \) and \( v_2(\cdot) \), let \( P_1(\cdot) \) satisfy (32). By the standard BSDE theory we can solve (33) to obtain \((\hat{\phi}(\cdot), \hat{\eta}(\cdot))\). Let \( P_2(\cdot) \) satisfy (40), and by the standard SDE theory we can solve (44) to obtain \( \hat{\varphi}(\cdot) \). Then according to the (30), (43) and (28), we can obtain \((\hat{g}^{\hat{v}_1, \hat{v}_2}(\cdot), \hat{z}_1^{\hat{v}_1, \hat{v}_2}(\cdot), \hat{z}_2^{\hat{v}_1, \hat{v}_2}(\cdot))\), which are the \( \mathcal{G}_t^1 \)-adapted solution to (46). Therefore, the state estimate feedback representation (45) can be obtained. The proof is complete.

4.2. Optimization for the leader. By (17) and (45), we have

\[
\begin{aligned}
dy^{\hat{v}_1, \hat{v}_2}(t) &= \left( A\hat{y}^{\hat{v}_1, \hat{v}_2} - B_1R_1^{-1}B_1^TP_2\hat{y}^{\hat{v}_1, \hat{v}_2} - B_1R_1^{-1}B_1^T\hat{\varphi} \\
&\quad + B_2\hat{v}_2 + C_1\hat{z}_1^{\hat{v}_1, \hat{v}_2} + C_2\hat{z}_2^{\hat{v}_1, \hat{v}_2} \right) dt \\
&\quad - \hat{z}_1^{\hat{v}_1, \hat{v}_2}(t) dW(t) - \hat{z}_2^{\hat{v}_1, \hat{v}_2}(t) dW(t), \quad t \in [0, T], \\
y^{\hat{v}_1, \hat{v}_2}(T) &= \xi,
\end{aligned}
\]

From the relationships (42) and (30), we get

\[
\begin{aligned}
\hat{\phi} &= -(P_1P_2 + I)\hat{y}^{\hat{v}_1, \hat{v}_2} - P_1\hat{\varphi}, \\
\hat{\eta} &= -P_1C_1^TP_2\hat{y}^{\hat{v}_1, \hat{v}_2} - P_1C_1^T\hat{\varphi} - (P_1S_1 + I)\hat{z}_1^{\hat{v}_1, \hat{v}_2}.
\end{aligned}
\]

Substituting (48) into (44), we get

\[
\begin{aligned}
\hat{d}\hat{\varphi}(t) &= \left[ A^T - P_2B_1R_1^{-1}B_1^T - P_2C_2(P_1S_2 + I)^{-1}P_1C_2^T \right] \hat{\varphi} \\
&\quad + P_2C_1(P_1S_1 + I)^{-1}P_1C_1^TP_2\hat{y}^{\hat{v}_1, \hat{v}_2} + P_2C_1\hat{z}_1^{\hat{v}_1, \hat{v}_2} + P_2B_2\hat{v}_2 \right) dt \\
&\quad + \left[ C_1^TP_2\hat{y}^{\hat{v}_1, \hat{v}_2} + C_1^T\hat{\varphi} + (S_1 - P_2)\hat{z}_1^{\hat{v}_1, \hat{v}_2} \right] dW(t), \quad t \in [0, T], \\
\hat{\varphi}(0) &= 0.
\end{aligned}
\]
Combining (47) and (49), we can get the state equation of the leader:

\[
\begin{align*}
    d\hat{\phi}(t) &= \left\{ \left[ A^T - P_2 B_1 R_1^{-1} B_1^T - P_2 C_2 (P_1 S_2 + I)^{-1} P_1 C_2^T \right] \hat{\phi} \\
    &\quad + P_2 C_1 (P_1 S_1 + I)^{-1} P_1 C_1^T P_2 y_1 \bar{v}_1, \bar{v}_2 + P_2 C_1 \bar{z}_1, \bar{v}_2 + P_2 B_2 \bar{v}_2 \right\} dt \\
    &\quad + \left\{ C_1^T P_2 y_1 \bar{v}_2 + C_1^T \dot{\phi} + (S_1 - P_2) \bar{z}_1, \bar{v}_2 \right\} dW(t), \\
    -d\bar{y}_{1,v_2}(t) &= \left( Ay_{1,v_2} - B_1 R_1^{-1} B_1^T P_2 y_1 \bar{v}_2 - B_1 R_1^{-1} B_1^T \hat{\phi} \\
    &\quad + B_2 v_2 + C_1 \bar{z}_1, \bar{v}_2 + C_2 \bar{z}_2, \bar{v}_2 \right) dt - \bar{z}_{1,v_2}(t) dW(t) \\
    y_{1,v_2}(T) &= \xi, \hat{\phi}(0) = 0.
\end{align*}
\]

(50)

For any given \( \xi, v_2(\cdot) \), from the proof above, the solvability for \((y_{1,v_2}(\cdot), z_{1,v_2}(\cdot), z_{2,v_2}(\cdot), \hat{\phi}(\cdot))\) to (50) can be guaranteed, though it is fully coupled.

The leader would like to choose an \( F \)-adapted optimal control \( \bar{v}_2(\cdot) \), such that the cost functional

\[
J_2(v_1(\cdot), v_2(\cdot); \xi) = \frac{1}{2} \mathbb{E} \left[ \int_0^T \left( \langle Q_2 y_{1,v_2}(t), y_{1,v_2}(t) \rangle + \langle R_2 v_2(t), v_2(t) \rangle \right. \\
+ \langle N_1 z_{1,v_2}(t), z_{1,v_2}(t) \rangle + \langle N_2 z_{2,v_2}(t), z_{2,v_2}(t) \rangle \right) dt \\
\left. + \langle G_2 y_{1,v_2}(0), y_{1,v_2}(0) \rangle \right] 
\]

is minimized. We suppose the following holds:

\[
\begin{align*}
    (L3) \quad &\begin{cases}
        Q_2(\cdot), N_1(\cdot), N_2(\cdot) \in L^\infty(0, T; \mathbb{S}^n), \quad Q_2(\cdot), N_1(\cdot), N_2(\cdot) \geq 0, \\
        R_2(\cdot) \in L^\infty(0, T; \mathbb{S}^{k_2}), \quad R_2(\cdot) > 0, \quad G_2 \in \mathbb{S}^n, G_2 \geq 0,
    \end{cases}
    \\
    \text{and the inverse } R_2^{-1}(\cdot) \text{ is also bounded.}
\end{align*}
\]

Define the Hamiltonian function of the leader as

\[
\begin{align*}
    H_2(t, y_{1,v_2}, z_{1,v_2}, z_{2,v_2}, v_2, \hat{\phi}, p, q_1, q_2, Q) &= \langle A^T - P_2 B_1 R_1^{-1} B_1^T - P_2 C_2 (P_1 S_2 + I)^{-1} P_1 C_2^T \rangle \hat{\phi} \\
    &\quad + P_2 C_1 (P_1 S_1 + I)^{-1} P_1 C_1^T P_2 y_{1,v_2} + P_2 C_1 \bar{z}_1, \bar{v}_2 + P_2 B_2 \bar{v}_2, p) \\
    &\quad + \langle C_1^T P_2 y_{1,v_2} + C_1^T \dot{\phi} + (S_1 - P_2) \bar{z}_1, \bar{v}_2, q_1 \rangle + \langle A y_{1,v_2} \\
    &\quad - B_1 R_1^{-1} B_1^T P_2 y_{1,v_2} - B_1 R_1^{-1} B_1^T \hat{\phi} + B_2 v_2 + C_1 \bar{z}_1, \bar{v}_2 + C_2 \bar{z}_2, \bar{v}_2, Q \rangle \\
    &\quad + \frac{1}{2} \left[ \langle Q_2 y_{1,v_2}, v_{1,v_2} \rangle + \langle R_2 v_2, v_2 \rangle + \langle N_1 z_{1,v_2}, z_{1,v_2} \rangle + \langle N_2 z_{2,v_2}, z_{2,v_2} \rangle \right].
\end{align*}
\]

(52)

Noting that the variable \( q_2 \) does not appear explicitly. Suppose that there exists an \( F \)-adapted optimal control \( \bar{v}_2(\cdot) \) of the leader, and the corresponding optimal state trajectory is \((y_{1,v_2}(\cdot), z_{1,v_2}(\cdot), z_{2,v_2}(\cdot), \hat{\phi}(\cdot)) \equiv (y_{1,\bar{v}_2}(\cdot), z_{1,\bar{v}_2}(\cdot), z_{2,\bar{v}_2}(\cdot), \hat{\phi}(\cdot))\), then by Theorem 3.3, we obtain

\[
B_2^T Q(t) + R_2 \bar{v}_2(t) + B_2^T P_2 \hat{\phi}(t) = 0, \quad a.e. \ t \in [0, T], \ a.s.,
\]

(53)
where the $\mathcal{F}$-adapted process triple $(p(\cdot), q_1(\cdot), Q(\cdot))$ satisfies

$$
\begin{aligned}
    dQ(t) &= \left\{ [P_2C_1(P_1S_1 + I)^{-1}P_1C_1^\top P_2] \right. \\
    &\quad \left. + P_2C_1P_1^\top \hat{p} + P_2C_1\hat{q}_1 + A^\top Q \\
    &\quad - P_2B_1R_1^{-1}B_1^\top \hat{Q} + Q_2^\top y_{\tilde{v}_2} \right\} dt + \left[ C_1^\top P_2\hat{p} + (S_1 - P_2)^\top \hat{q}_1 \\
    &\quad + C_1^\top Q + N_1 z_{\tilde{v}_2}^\top \right] dW(t) + \left( C_2^\top Q + N_2 z_{\tilde{v}_2}^\top \right) d\tilde{W}(t), \\
    -dp(t) &= \left\{ [A^\top - P_2B_1R_1^{-1}B_1^\top - P_2C_2(P_1S_2 + I)^{-1}P_1C_1^R] \right. \\
    &\quad \left. p + C_1(q_1 - B_1R_1^{-1}B_1^\top Q) \right\} dt - q_1(t) dW(t), \quad t \in [0, T], \\
    Q(0) &= G_2y_{\tilde{v}_2}(0), \quad p(T) = 0.
\end{aligned}
$$

Then, to make the problem clear, let us put (50) and (54) together:

$$
\begin{aligned}
    d\hat{\varphi}(t) &= \left\{ [A^\top - P_2B_1R_1^{-1}B_1^\top - P_2C_2(P_1S_2 + I)^{-1}P_1C_1^\top] \right. \\
    &\quad \left. \hat{\varphi} + P_2C_1(P_1S_1 + I)^{-1}P_1C_1^\top P_2y_{\tilde{v}_2} + P_2C_1z_{\tilde{v}_2} + P_2B_2\hat{\varphi}_2 \right\} dt \\
    &\quad + \left[ C_1^\top P_2\hat{\varphi} + (S_1 - P_2)^\top \hat{q}_1 + A^\top Q \\
    &\quad - P_2B_1R_1^{-1}B_1^\top \hat{Q} + Q_2^\top y_{\tilde{v}_2} \right\} dt + \left[ C_1^\top P_2\hat{p} + (S_1 - P_2)^\top \hat{q}_1 \\
    &\quad + C_1^\top Q + N_1 z_{\tilde{v}_2}^\top \right] dW(t) + \left( C_2^\top Q + N_2 z_{\tilde{v}_2}^\top \right) d\tilde{W}(t), \\
    dQ(t) &= \left\{ [P_2C_1(P_1S_1 + I)^{-1}P_1C_1^\top P_2] \right. \\
    &\quad \left. + P_2C_1P_1^\top \hat{p} + P_2C_1\hat{q}_1 + A^\top Q \\
    &\quad - P_2B_1R_1^{-1}B_1^\top \hat{Q} + Q_2^\top y_{\tilde{v}_2} \right\} dt + \left[ C_1^\top P_2\hat{p} + (S_1 - P_2)^\top \hat{q}_1 \\
    &\quad + C_1^\top Q + N_1 z_{\tilde{v}_2}^\top \right] dW(t) + \left( C_2^\top Q + N_2 z_{\tilde{v}_2}^\top \right) d\tilde{W}(t), \\
    -dp(t) &= \left\{ [A^\top - P_2B_1R_1^{-1}B_1^\top - P_2C_2(P_1S_2 + I)^{-1}P_1C_1^R] \right. \\
    &\quad \left. p + C_1(q_1 - B_1R_1^{-1}B_1^\top Q) \right\} dt - q_1(t) dW(t), \\
    -dy_{\tilde{v}_2}(t) &= [Ay_{\tilde{v}_2} - B_1R_1^{-1}B_1^\top P_2y_{\tilde{v}_2} - B_1R_1^{-1}B_1^\top \hat{\varphi} + B_2\tilde{v}_2 \\
    &\quad + C_1z_{\tilde{v}_2} + C_2z_{\tilde{v}_2}^\top] dt - z_{\tilde{v}_2}(t) dW(t) - z_{\tilde{v}_2}(t) d\tilde{W}(t), \quad t \in [0, T], \\
    \hat{\varphi}(0) &= 0, \quad Q(0) = G_2y_{\tilde{v}_2}(0), \quad p(T) = 0, \quad y_{\tilde{v}_2}(T) = \xi.
\end{aligned}
$$

We may look at the above equations in a different way. To this end, let us set

$$
X = \left( \begin{array}{c} \hat{\varphi} \\ Q \end{array} \right), \quad Y = \left( \begin{array}{c} p \\ y_{\tilde{v}_2} \end{array} \right), \quad Z_1 = \left( \begin{array}{c} q_1 \\ z_{\tilde{v}_2} \end{array} \right), \quad Z_2 = \left( \begin{array}{c} 0 \\ z_{\tilde{v}_2} \end{array} \right).
$$

$$
(56)
$$
and
\[
\begin{align*}
A_1 &= \begin{pmatrix} A^T - P_2 B_1 R_1^{-1} B_1^T - P_2 C_2 (P_1 S_2 + I)^{-1} P_1 C_2^T & 0 \\ 0 & A^T \end{pmatrix}, \\
\tilde{A}_1 &= \begin{pmatrix} C_1^T & 0 \\ 0 & C_1^T \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 0 \\ 0 & -P_2 B_1 R_1^{-1} B_1^T \end{pmatrix}, \quad \tilde{A}_2 = \begin{pmatrix} 0 & 0 \\ 0 & C_2^T \end{pmatrix}, \\
B_1 &= \begin{pmatrix} 0 & 0 \\ 0 & Q_2 \end{pmatrix}, \quad \tilde{B}_2 = \begin{pmatrix} 0 \\ B_2 \end{pmatrix}, \quad \mathcal{F}_1 = \begin{pmatrix} 0 & 0 \\ -B_1 R_1^{-1} B_1^T & 0 \end{pmatrix}, \\
\tilde{B}_2 &= \begin{pmatrix} P_2 C_1 (P_1 S_1 + I)^{-1} P_1 C_1^T P_2 \end{pmatrix}, \\
\tilde{C}_1 &= \begin{pmatrix} 0 & 0 \\ 0 & N_1 \end{pmatrix}, \quad \tilde{C}_2 = \begin{pmatrix} (S_1 - P_2)^T & S_1 - P_2 \\ 0 & 0 \end{pmatrix}, \quad \tilde{C}_3 = \begin{pmatrix} 0 & 0 \\ 0 & N_2 \end{pmatrix}, \\
\mathcal{C} &= \begin{pmatrix} P_2 C_1 \\ 0 \end{pmatrix}, \quad \mathcal{D} = \begin{pmatrix} P_2 B_2 \\ 0 \end{pmatrix}, \quad \tilde{\xi} = \begin{pmatrix} 0 \\ \xi \end{pmatrix}, \quad \tilde{G} = \begin{pmatrix} 0 & 0 \\ 0 & C_2 \end{pmatrix}.
\end{align*}
\]

With this, (55) is equivalent to the FBSDE:
\[
\begin{align*}
&\begin{cases}
&\begin{aligned} &dX(t) = \left[ A_1 X + A_2 \hat{X} + B_1 Y + B_2 \hat{Y} + C \hat{Z}_1 + D \hat{\nu}_2 \right] dt \\
&\quad + \left[ \tilde{A}_1 \hat{X} + \tilde{C}^T \hat{Y} + \tilde{C}_1 \hat{Z}_1 + \tilde{C}_2 \hat{Z}_2 \right] dW(t) + \left[ \tilde{A}_2 X + \tilde{C}_3 Z_2 \right] d\tilde{W}(t),
\end{aligned}
\end{cases} \\
&\begin{cases}
&\begin{aligned} &-dY(t) = \left[ \mathcal{F}_1 X + \mathcal{A}_1^T Y + \mathcal{A}_2^T \hat{Y} + \mathcal{A}_1^T Z_1 + \mathcal{A}_2^T Z_2 + \tilde{B}_2 \hat{\nu}_2 \right] dt \\
&\quad - Z_1(t) dW(t) - Z_2(t) d\tilde{W}(t), \quad t \in [0, T],
\end{aligned}
\end{cases}
\end{align*}
\]
\[
X(0) = GY(0) , \quad Y(T) = \tilde{\xi}.
\]

Then from (53), we have
\[
\hat{\nu}_2(t) = -R_2^{-1} \left[ \tilde{B}_2^T X(t) + D^T \hat{Y}(t) \right], \quad a.e. \ t \in [0, T], \quad a.s. \quad (58)
\]

Taking $\mathbb{E}[\cdot | G_t^T]$ on (58), we get
\[
\hat{\nu}_2(t) = -R_2^{-1} \left[ \tilde{B}_2^T \hat{X}(t) + D^T \hat{Y}(t) \right], \quad a.e. \ t \in [0, T], \quad a.s. \quad (59)
\]

Putting (58) and (59) into (57), we get
\[
\begin{align*}
&\begin{cases}
&\begin{aligned} &dX(t) = \left[ A_1 X + (A_2 - DR_2^{-1} \tilde{B}_2^T) \hat{X} + B_1 Y + (B_2 - DR_2^{-1} D^T) \hat{Y} + C \hat{Z}_1 \right] dt \\
&\quad + \left[ \tilde{A}_1 \hat{X} + C^T \hat{Y} + \tilde{C}_1 \hat{Z}_1 + \tilde{C}_2 \hat{Z}_2 \right] dW(t) + \left[ \tilde{A}_2 X + \tilde{C}_3 Z_2 \right] d\tilde{W}(t),
\end{aligned}
\end{cases} \\
&\begin{cases}
&\begin{aligned} &-dY(t) = \left[ (\mathcal{F}_1 - \tilde{B}_2 R_2^{-1} \tilde{B}_2^T) X + A_1^T Y + (A_2^T - \tilde{B}_2 R_2^{-1} D^T) \hat{Y} \\
&\quad + \mathcal{A}_1^T Z_1 + \mathcal{A}_2^T Z_2 \right] dt - Z_1(t) dW(t) - Z_2(t) d\tilde{W}(t), \quad t \in [0, T],
\end{aligned}
\end{cases}
\end{align*}
\]
\[
X(0) = GY(0) , \quad Y(T) = \tilde{\xi}.
\]

Noting that (60) is a coupled conditional mean-field FBSDE, we need to decouple it with the similar method before. We set
\[
Y(t) = \Pi_1(t) X(t) + \Pi_2(t) \hat{X}(t) + \tilde{\phi}(t),
\]

where $\Pi_1(\cdot), \Pi_2(\cdot)$ are $\mathbb{R}^{2n \times 2n}$-matrix-valued deterministic, differentiable functions with $\Pi_1(T) = \Pi_2(T) = 0$, and $(\tilde{\phi}(\cdot), \tilde{\eta}(\cdot))$ are both $\mathbb{R}^{2n}$-valued, $\mathbb{F}$-adapted processes satisfying the BSDE:
\[
\begin{align*}
&\begin{cases}
&\begin{aligned} &-d\tilde{\phi}(t) = \tilde{\alpha}(t) dt - \tilde{\eta}(t) dW(t), \quad t \in [0, T],
\end{aligned}
\end{cases} \\
&\begin{cases}
&\begin{aligned} &\tilde{\phi}(T) = \tilde{\xi},
\end{aligned}
\end{cases}
\end{align*}
\]
Comparing the diffusion coefficients of \( dt \) then comparing the drift coefficients of the \( E \) taking \( I \) where 

\[
\tilde{\alpha} = (\Pi_1 + \Pi_1 A_1) X + \Pi_1 (A_2 - D R_2^{-1} B_2^T) \hat{X} + \Pi_1 B_1 Y + \Pi_1 B_2 - D R_2^{-1} D^T \hat{Y}
\]

\[
+ \Pi_1 C \hat{Z}_1 + \Pi_2 (A_1 + A_2 - D R_2^{-1} B_2^T) \hat{X} + \Pi_2 (B_1 + B_2 - D R_2^{-1} D^T) \hat{Y}
\]

\[
+ \Pi_3 C \hat{Z}_1 + (F_1 - B_2 R_2^{-1} B_2^T) X + A_1^T Y + (A_1^T - B_2 R_2^{-1} D^T) \hat{Y} + \tilde{\alpha} \]  

(64)

Comparing the diffusion coefficients of \( dW(\cdot) \) term and \(dwW(\cdot) \) term on both sides of (64), we derive

\[
Z_1 = \Pi_1 \tilde{\alpha} X + \Pi_1 C^T \hat{Y} + \Pi_1 \tilde{\alpha} Z_1 + \Pi_1 \tilde{\alpha} Z_1
\]

\[
+ \Pi_2 \tilde{\alpha} X + \Pi_2 C^T \hat{Y} + \Pi_2 (\tilde{\alpha} + \tilde{\alpha}) Z_1 + \tilde{\eta}
\]

(65)

and

\[
Z_2 = \Pi_1 \tilde{\alpha} X + \Pi_1 \tilde{\alpha} Z_2.
\]

(66)

Then comparing the drift coefficients of \( dt \) term on both sides of (64), we have

\[
\tilde{\alpha} = (\Pi_1 + \Pi_1 A_1) X + \Pi_1 (A_2 - D R_2^{-1} B_2^T) \hat{X} + \Pi_1 B_1 Y + \Pi_1 B_2 - D R_2^{-1} D^T \hat{Y}
\]

\[
+ \Pi_1 C \hat{Z}_1 + \Pi_2 (A_1 + A_2 - D R_2^{-1} B_2^T) \hat{X} + \Pi_2 (B_1 + B_2 - D R_2^{-1} D^T) \hat{Y}
\]

\[
+ \Pi_3 C \hat{Z}_1 + (F_1 - B_2 R_2^{-1} B_2^T) X + A_1^T Y + (A_1^T - B_2 R_2^{-1} D^T) \hat{Y} + \tilde{\alpha} \]

(67)

Taking \( E[|G_1^1|] \) on (65), and supposing \( \Sigma_1 := [I - (\Pi_1 + \Pi_2) (\tilde{\alpha} + \tilde{\alpha})]^{-1} \) exists, we get

\[
\hat{Z}_1 = \Sigma_1 (\Pi_1 + \Pi_2) \tilde{\alpha} \hat{X} + \Sigma_1 (\Pi_1 + \Pi_2) C^T \hat{Y} + \Sigma_1 \tilde{\eta}.
\]

(68)

Putting (68) into (65), and supposing \( (I - \Pi_1 \tilde{\alpha})^{-1} \) exists, we get

\[
Z_1 = \Sigma_2 \Pi_1 \tilde{\alpha} \hat{X} + \Sigma_3 \Pi_1 \tilde{\alpha} \hat{X} + \Sigma_4 \Pi_2 \tilde{\alpha} \hat{X}
\]

\[
+ \Sigma_1 (\Pi_1 + \Pi_2) C^T \hat{Y} + \Sigma_4 \tilde{\alpha} \hat{Y} + \Sigma_4 \tilde{\eta},
\]

(69)

where

\[
\begin{align*}
\Sigma_2 := (\Pi_1 + \Pi_2) C^T (\Pi_1 + \Pi_2), \\
\Sigma_3 := (\Pi_1 + \Pi_2) \tilde{\alpha} + \Pi_2 \tilde{\alpha}, \quad \Sigma_4 = (I - \Pi_1 \tilde{\alpha})^{-1}.
\end{align*}
\]

From (66), and assuming \( I - \Pi_1 \tilde{\alpha} \) to be invertible, we can get

\[
Z_2 = (I - \Pi_1 \tilde{\alpha})^{-1} \Pi_1 \tilde{\alpha} X.
\]

(70)
Putting (68), (69) and (70) into (67), we drive
\[
\begin{align*}
\hat{\alpha} &= \left[ \Pi_1 + \Pi_1 A_1 + \Pi_1 B_1 \Pi_1 + \Sigma_9 + A_1^\top \Sigma_4 \Pi_1 \hat{A}_1 + \hat{A}_2^\top \Sigma_10 \Pi_1 \hat{A}_2 \right] X \\
&+ \left\{ \Pi_1 \Sigma_5 + \Pi_1 B_1 \Pi_2 + \Pi_1 \Sigma_4 (\Pi_1 + \Pi_2) + (\Pi_1 + \Pi_2) C \Sigma_1 (\Pi_1 + \Pi_2) \hat{A}_1 \right. \\
&+ (\Pi_1 + \Pi_2) C \Sigma_1 A_2 + \Pi_2 \Sigma_7 + \Pi_2 \Sigma_8 (\Pi_1 + \Pi_2) + A_1^\top \Pi_2 \\
&+ \Sigma_5 (\Pi_1 + \Pi_2) + \hat{A}_1^\top \left[ \Sigma_1 \Sigma_2 + \Sigma_4 \Sigma_3 \Pi_1 \hat{A}_1 + \Sigma_1 \Pi_2 \hat{A}_1 \right] \} \dot{X} \\
&+ [\Pi_1 B_1 + A_1^\top] \hat{\phi} + [\Pi_1 \Sigma_6 + (\Pi_1 + \Pi_2) C \Sigma_1 (\Pi_1 + \Pi_2) C^\top + \Pi_2 \Sigma_8 + \Sigma_5^T \\
&+ \hat{A}_1^\top \Sigma_4 (\Pi_1 + \Pi_2) C^\top] \hat{\eta} + \left[ (\Pi_1 + \Pi_2) C \Sigma_1 + \hat{A}_1^\top \Sigma_4 \Sigma_3 \Sigma_1 \right] \hat{\eta},
\end{align*}
\]

(71)

where
\[
\begin{cases}
\Sigma_5 := A_2 - DR_2^{-1} B_2^\top, & \Sigma_6 := B_2 - DR_2^{-1} D^\top, \\
\Sigma_7 := A_1 + A_2 - DR_2^{-1} B_2^\top, & \Sigma_8 := B_1 + B_2 - DR_2^{-1} D^\top, \\
\Sigma_9 := F_1 - B_2 R_2^{-1} B_2^\top, & \Sigma_{10} := (I - \Pi_1 C_3)^{-1}.
\end{cases}
\]

Then, if \(\Pi_1(\cdot)\) and \(\Pi_2(\cdot)\) satisfy the following two Riccati equations, one by one:
\[
\begin{cases}
\Pi_1 + \Pi_1 A_1 + A_1^\top \Pi_1 + \Pi_1 B_1 \Pi_1 + \hat{A}_1^\top \Sigma_4 \Pi_1 \hat{A}_1 \\
+ \hat{A}_2^\top \Sigma_{10} \Pi_1 \hat{A}_2 + \Sigma_9 = 0, & t \in [0, T], \\
\Pi_1(T) = 0,
\end{cases}
\]

(72)

and
\[
\begin{cases}
\Pi_2 + \Pi_1 \Sigma_5 + \Sigma_5^T \Pi_1 + \Pi_2 \Sigma_7 + \Sigma_7^T \Pi_2 + \Pi_1 B_1 \Pi_2 + \Pi_2 B_2 \Pi_1 \\
+ (\Pi_1 + \Pi_2) \Sigma_6 (\Pi_1 + \Pi_2) + \Pi_2 B_1 \Pi_2 + (\Pi_1 + \Pi_2) C \Sigma_1 (\Pi_1 + \Pi_2) \hat{A}_1 \\
+ \Sigma_{11} \Sigma_1 \Sigma_2 + \hat{A}_1^\top \Sigma_4 \Sigma_3 \Sigma_1 \Pi_1 \hat{A}_1 + \hat{A}_1^\top \Sigma_1 \Pi_2 \hat{A}_2 = 0, & t \in [0, T], \\
\Pi_2(T) = 0,
\end{cases}
\]

(73)

where \(\Sigma_{11} := \hat{A}_1^\top + (\Pi_1 + \Pi_2) C\), we have
\[
\begin{align*}
\hat{\alpha} &= \left[ \Pi_1 \Sigma_9 + \Sigma_{11} \Sigma_1 (\Pi_1 + \Pi_2) C^\top + \Pi_2 \Sigma_8 + \Sigma_5^T \right] \hat{\phi} + [\Pi_1 B_1 + A_1^\top] \hat{\phi} \\
&+ \left[ (\Pi_1 + \Pi_2) C \Sigma_1 + \hat{A}_1^\top \Sigma_4 \Sigma_3 \Sigma_1 \right] \hat{\eta} + \hat{A}_1^\top \Sigma_4 \hat{\eta}.
\end{align*}
\]

(74)

Taking \(E[\cdot | G_t^1]\) on both sides of (74), we get
\[
\hat{\alpha} = [\Pi_1 + \Pi_2) C \Sigma_8 + \Sigma_{11} \Sigma_1 (\Pi_1 + \Pi_2) C^\top + \Sigma_5^T] \hat{\phi} + \Sigma_{11} \Sigma_1 \hat{\eta}.
\]

(75)

After taking \(E[\cdot | G_t^1]\) on (62), noting (75), we can derive the equation of \((\hat{\phi}(\cdot), \hat{\eta}(\cdot))\):
\[
\begin{cases}
-d \hat{\phi}(t) = \left\{ \left[ (\Pi_1 + \Pi_2) C \Sigma_8 + \Sigma_{11} \Sigma_1 (\Pi_1 + \Pi_2) C^\top + \Sigma_5^T \right] \hat{\phi} + \Sigma_{11} \Sigma_1 \hat{\eta} \right\} dt \\
- \hat{\eta}(t) dW(t), & t \in [0, T], \\
\hat{\phi}(T) = \hat{\xi}.
\end{cases}
\]

(76)

In the meanwhile, we set
\[
X(t) = \Pi_3(t) Y(t) + \Pi_4(t) \tilde{Y}(t) + \tilde{\varphi}(t),
\]

(77)
where $\Pi_3(\cdot), \Pi_4(\cdot)$ are $\mathbb{R}^{2n \times 2n}$-matrix-valued deterministic, differentiable functions with $\Pi_3(0) = \Gamma, \Pi_4(0) = 0$, and $\tilde{\varphi}(\cdot)$ is an $\mathbb{R}^{2n}$-valued, $\mathbb{F}$-adapted process satisfying the following SDE:
\[
\begin{cases}
  d\tilde{\varphi}(t) = \tilde{\beta}(t)dt + \tilde{\gamma}(t)dW(t), & t \in [0, T], \\
  \tilde{\varphi}(0) = 0,
\end{cases}
\]
(78)

where $\tilde{\beta}(\cdot), \tilde{\gamma}(\cdot)$ are both $\mathbb{R}^{2n}$-valued, $\mathbb{F}$-adapted processes to be determined later.

Using the above notations, we can reformulate the equation of $Y(\cdot)$ in (60) as
\[
\begin{cases}
  -dY(t) = (\Sigma_9 X + \mathcal{A}_1^T Y + \Sigma_5^T \tilde{Y} + \tilde{\mathcal{A}}_1^T Z_1 + \tilde{\mathcal{A}}_2^T Z_2)dt \\
  - Z_1(t)dW(t) - Z_2(t)d\tilde{W}(t), & t \in [0, T],
\end{cases}
\]
(79)

By applying Lemma 5.4 in [42], we get
\[
\begin{cases}
  -d\hat{Y}(t) = (\Sigma_9 \hat{X} + \Sigma_5^T \hat{Y} + \tilde{\mathcal{A}}_1^T \hat{Z}_1 + \tilde{\mathcal{A}}_2^T \hat{Z}_2)dt - \hat{Z}_1(t)dW(t), & t \in [0, T], \\
  \hat{Y}(T) = \hat{\xi}.
\end{cases}
\]
(80)

Applying Itô’s formula to (77), we have
\[
dX(t) = (\Pi_3 \tilde{Y} - \Pi_3 \Sigma_9 X - \Pi_3 \mathcal{A}_1^T Y - \Pi_3 \Sigma_5^T \tilde{Y} - \Pi_3 \tilde{\mathcal{A}}_1^T Z_1 - \Pi_3 \tilde{\mathcal{A}}_2^T Z_2 \\
+ \Pi_4 \tilde{Y} - \Pi_4 \Sigma_9 \hat{X} - \Pi_4 \Sigma_5^T \hat{Y} - \Pi_4 \tilde{\mathcal{A}}_1^T \hat{Z}_1 - \Pi_4 \tilde{\mathcal{A}}_2^T \hat{Z}_2 + \beta)dt \\
+ (\Pi_3 Z_1 + \Pi_4 \hat{Z}_1 + \gamma)dw(t) + \Pi_3 Z_2 d\tilde{W}(t)
\]
(81)

Comparing the drift terms of $dW(\cdot)$ and $d\tilde{W}(\cdot)$ on both sides of (81), we obtain
\[
\begin{cases}
  \Pi_3 Z_1 + \Pi_4 \hat{Z}_1 + \gamma = \tilde{\mathcal{A}}_1 X + C^T \hat{Y} + \tilde{\mathcal{C}}_1 Z_1 + \tilde{\mathcal{C}}_2 \hat{Z}_1, \\
  \Pi_3 Z_2 = \tilde{\mathcal{A}}_2 X + \tilde{\mathcal{C}}_3 Z_2.
\end{cases}
\]
(82)

Comparing the drift term of $dW(\cdot)$ on both sides of (81), we derive
\[
\begin{aligned}
\tilde{\beta} &= (\Pi_3 \Sigma_9 + \mathcal{A}_1)X - (\Pi_3 - \Pi_3 \mathcal{A}_1^T + \mathcal{B}_1)Y + \Pi_3 \tilde{\mathcal{A}}_1^T Z_1 + \Pi_3 \tilde{\mathcal{A}}_2^T Z_2 \\
+ (\Pi_4 \Sigma_9 + \Sigma_5)\tilde{X} - (\Pi_4 - \Pi_4 \Sigma_5^T - \Pi_4 \tilde{\mathcal{A}}_1^T - \Sigma_6)\tilde{Y} + (\Pi_4 \tilde{\mathcal{A}}_1^T + C)\tilde{Z}_1 + \Pi_4 \tilde{\mathcal{A}}_2^T \tilde{Z}_2.
\end{aligned}
\]
(83)

Putting (68) and (69) into (82), we get
\[
\begin{aligned}
\tilde{\gamma} &= \left\{ \tilde{\mathcal{A}}_1 \Pi_3 + (\bar{\mathcal{C}}_1 - \Pi_3) \Sigma_4 \Pi_1 \mathcal{A}_3 \Pi_3 \right\} Y + \left\{ \tilde{\mathcal{A}}_1 \Pi_4 + C^T \right. \\
&+ (\bar{\mathcal{C}}_1 - \Pi_3) \Sigma_4 \Pi_1 \tilde{\mathcal{A}}_1 \Pi_4 + (\bar{\mathcal{C}}_1 - \Pi_3) \left\{ \Sigma_1 \Sigma_2 + \Sigma_4 \Sigma_3 \Pi_1 \mathcal{A}_1 \\
&+ \Sigma_1 \Pi_2 \tilde{\mathcal{A}}_1 \right\} (\Pi_3 + \Pi_4) + (\bar{\mathcal{C}}_1 - \Pi_3) \Pi_3 \Sigma_1 + \Pi_2 \Sigma_1 \right\} \tilde{Y} \\
&+ \left[ \tilde{\mathcal{A}}_1 + (\bar{\mathcal{C}}_1 - \Pi_3) \Sigma_4 \Pi_1 \tilde{\mathcal{A}}_1 \right] \tilde{\varphi} + (\bar{\mathcal{C}}_1 - \Pi_3) \Sigma_4 \tilde{\eta} \\
&+ (\bar{\mathcal{C}}_1 - \Pi_3) \Sigma_1 (\Pi_1 + \Pi_2)C^T \tilde{\phi} + \left\{ (\bar{\mathcal{C}}_1 - \Pi_3) \left\{ \Sigma_1 \Sigma_2 + \Sigma_4 \Sigma_3 \Pi_1 \tilde{\mathcal{A}}_1 \\
+ \Sigma_1 \Pi_2 \tilde{\mathcal{A}}_1 \right\} + (\bar{\mathcal{C}}_2 - \Pi_4) \Sigma_1 (\Pi_1 + \Pi_2) \tilde{\mathcal{A}}_1 \right\} \tilde{\varphi} \\
&+ \left[ (\bar{\mathcal{C}}_1 - \Pi_3) \Sigma_4 \Sigma_3 \Pi_1 + (\bar{\mathcal{C}}_2 - \Pi_4) \Sigma_4 \right] \tilde{\eta},
\end{aligned}
\]
(84)
\[ \tilde{Z}_2 = \Sigma_{10} \Pi_1 \tilde{A}_2 [(\Pi_3 + \Pi_4)\hat{Y} + \hat{\varphi}] . \]  

Substituting (68), (69), (70) and (85) into (83), and if \( \Pi \) satisfies the following two Riccati equations, one by one:

\[
\begin{align*}
\dot{\Pi}_3 &= -A_1 \Pi_3 - \Pi_3 A_1^\top - \Pi_3 \Sigma_9 \Pi_3 - \Pi_3 \tilde{A}_1^\top \Sigma_4 \Pi_2 \tilde{A}_1 \Pi_3 \\
- \Pi_3 \tilde{A}_1^\top \Sigma_9 \Pi_2 \tilde{A}_2 \Pi_3 - B_1 &= 0, \quad t \in [0, T], \\
\Pi_3(0) &= \tilde{G},
\end{align*}
\]

and

\[
\begin{align*}
\dot{\Pi}_4 &= -\Pi_3 \Sigma_5^\top - \Sigma_5 \Pi_3 - \Pi_4 \Sigma_7^\top - \Sigma_7 \Pi_4 - \Pi_3 \Sigma_9 \Pi_4 - \Pi_4 \Sigma_6 \Pi_3 \\
- \Pi_4 \Sigma_9 \Pi_4 + \Pi_3 \tilde{A}_1^\top \Sigma_4 \Pi_2 \tilde{A}_1 - \Pi_4 \tilde{A}_1^\top \Sigma_1 \Sigma_2 \Pi_3 + \Pi_4 \\
- \Pi_3 \tilde{A}_1^\top \Sigma_10 \Pi_1 \tilde{A}_2 \Pi_4 - \Pi_3 \Pi_4 + \Pi_4 \tilde{A}_1^\top \Sigma_1 (\Pi_1 + \Pi_2) \tilde{A}_1 (\Pi_3 + \Pi_4) \\
- \Pi_4 \tilde{A}_1^\top \Sigma_1 (\Pi_1 + \Pi_2) \Sigma_10 \Pi_1 \tilde{A}_2 \Pi_3 - \Pi_4 \tilde{A}_1^\top \Sigma_10 \Pi_1 \tilde{A}_2 \Pi_4 \\
- C \Sigma_1 (\Pi_1 + \Pi_2) \Sigma_{12} - \Sigma_6 &= 0, \quad t \in [0, T], \\
\Pi_4(0) &= 0,
\end{align*}
\]

then we get

\[
\begin{align*}
\tilde{\beta} &= \left\{ \Pi_3 \tilde{A}_1^\top \left[ \Sigma_1 \Sigma_2 + \Sigma_4 \Sigma_5 \Sigma_1 \Pi_1 \tilde{A}_1 + \Sigma_1 \Pi_2 \tilde{A}_1 \right] + \Pi_4 \Sigma_9 \\
&\quad - \Pi_4 \tilde{A}_1^\top \Sigma_1 (\Pi_1 + \Pi_2) \tilde{A}_1 + \Pi_4 \tilde{A}_1^\top \Sigma_10 \Pi_1 \tilde{A}_2 + \Sigma_5 + C \Sigma_1 (\Pi_1 + \Pi_2) \tilde{A}_1 \right\} \hat{\varphi} \\
&\quad + \left\{ \Pi_3 \Sigma_9 + \Pi_3 \tilde{A}_1^\top \Sigma_4 \Pi_1 \tilde{A}_1 + \Pi_3 \tilde{A}_1^\top \Sigma_10 \Pi_1 \tilde{A}_2 + \Sigma_5 \right\} \tilde{\phi} + \Pi_3 \tilde{A}_1^\top \Sigma_10 \Pi_1 \tilde{A}_2 \tilde{\eta} \\
&\quad + \Pi_3 \tilde{A}_1^\top \Sigma_1 (\Pi_1 + \Pi_2) C^\top \hat{\phi} + \left\{ \Pi_3 \tilde{A}_1^\top \Sigma_4 \Sigma_1 + \Pi_4 \tilde{A}_1^\top \Sigma_4 + \Sigma \Sigma_1 \right\} \tilde{\eta}.
\end{align*}
\]

Taking \( \mathbb{E}[\cdot | \mathcal{G}_t^1] \) on (84), we get

\[
\hat{\gamma} = \left\{ \tilde{A}_1 \Pi_3 + (\tilde{C}_1 - \Pi_3) \Sigma_4 \Pi_1 \tilde{A}_1 \Pi_3 + \tilde{A}_1 \Pi_4 + C^\top + (\tilde{C}_1 - \Pi_3) \Sigma_4 \Pi_1 \tilde{A}_1 \Pi_4 \\
+ (\tilde{C}_1 - \Pi_3) \left[ \Sigma_1 \Sigma_2 + \Sigma_4 \Sigma_3 \Pi_1 \tilde{A}_1 + \Sigma_1 \Pi_2 \tilde{A}_1 \right] (\Pi_3 + \Pi_4) \\
+ \left( \tilde{C}_2 - \Pi_4 \Sigma_1 (\Pi_1 + \Pi_2) \Sigma_{12} \right) \hat{Y} + \left\{ (\tilde{C}_1 - \Pi_3) \Sigma_1 \Sigma_2 + \Sigma_4 \Sigma_3 \Sigma_1 \Pi_1 \tilde{A}_1 \right\} \hat{\varphi} \\
+ \Sigma_1 \Pi_2 \tilde{A}_1 \right\} + \left( \tilde{C}_2 - \Pi_4 \Sigma_1 (\Pi_1 + \Pi_2) \tilde{A}_1 + \tilde{A}_1 + (\tilde{C}_1 - \Pi_3) \Sigma_4 \Pi_1 \tilde{A}_1 \right) \tilde{\phi} \\
+ (\tilde{C}_1 - \Pi_3) \Sigma_1 (\Pi_1 + \Pi_2) C^\top \hat{\phi} + (\tilde{C}_1 + \tilde{C}_2 - \Pi_3 - \Pi_4) \Sigma_1 \tilde{\eta}.
\]

Then taking \( \mathbb{E}[\cdot | \mathcal{G}_t^1] \) on (61) and (77), we get

\[
\begin{align*}
\dot{X} &= (\Pi_3 + \Pi_4) \dot{Y} + \hat{\varphi}, \\
\dot{Y} &= (\Pi_1 + \Pi_2) \dot{X} + \hat{\phi}.
\end{align*}
\]

Assuming \( I - (\Pi_1 + \Pi_2)(\Pi_3 + \Pi_4) \) to be invertible, then

\[
Y = \Sigma_{13} (\Pi_1 + \Pi_2) \hat{\phi} + \Sigma_{13} \hat{\phi},
\]
where, we set $\Sigma_{13} := [I - (\Pi_1 + \Pi_2)(\Pi_3 + \Pi_4)]^{-1}$. Therefore, we can rewrite the equation (89) as

$$\hat{\gamma} = \left\{ (\tilde{A}_1(\Pi_3 + \Pi_4) + (\tilde{c}_1 - \Pi_3)\Sigma_4\Pi_1\tilde{A}_1(\Pi_3 + \Pi_4) + C^T \\
+ (\tilde{c}_1 - \Pi_3) [\Sigma_1\Sigma_2 + \Sigma_4\Sigma_3\Pi_1\tilde{A}_1 + \Sigma_1\Pi_2\tilde{A}_1] (\Pi_3 + \Pi_4) \\
+ (\tilde{c}_2 - \Pi_4) [\Sigma_1\Sigma_2 + \Sigma_4\Sigma_3\Pi_1\tilde{A}_1 + \Sigma_1\Pi_2(t)\tilde{A}_1] \\
+ (\tilde{c}_2 - \Pi_4) [\Sigma_1(\Pi_1 + \Pi_2)\tilde{A}_1 + \tilde{A}_1 + (\tilde{c}_1 - \Pi_3)\Sigma_4\Pi_1\tilde{A}_1] \right\} \hat{\phi}$$

(92)

Thus, we can derive the equation of $\hat{\phi}(\cdot)$:

$$\begin{cases}
    d\hat{\phi}(t) = \hat{\beta}(t)dt + \hat{\gamma}(t)dW(t), & t \in [0, T], \\
    \hat{\phi}(0) = 0,
\end{cases}$$

(94)

where $\hat{\beta}(\cdot)$ and $\hat{\gamma}(\cdot)$ satisfy (93) and (92), respectively.

Noticing that Riccati equations (72) and (86) are similar with that in the existing literatures [10] and [51]. However, the Riccati equations (73) and (87) are new and introduced firstly due to the appearance of $\hat{X}$ and $\hat{Y}$.

**Theorem 4.2.** Under assumptions (L1), (L2) and (L3), suppose the Riccati equations (72), (73), (86) and (87) admit differentiable solutions $\Pi_1(\cdot), \Pi_2(\cdot), \Pi_3(\cdot)$ and $\Pi_4(\cdot)$, respectively. Then the leader’s problem is solvable with the optimal strategy $\tilde{v}_2(\cdot)$ being of a state estimate feedback representation

$$\tilde{v}_2(t) = -R_{21}^{-1}(t)\tilde{B}_2^T\Pi_3(t)Y(t) - R_{21}^{-1}(t)[\tilde{B}_2^T\Pi_4(t) + D^T]Y(t)$$

$$- R_{21}^{-1}(t)\tilde{B}_2^T\tilde{\varphi}(t), \text{ a.e. } t \in [0, T], \text{ a.s.},$$

(95)

where $Y(\cdot), \hat{Y}(\cdot)$ satisfy the following BSDEs, respectively:

$$\begin{cases}
    -dY(t) = [\Sigma_3\Pi_3 + A_1^TY + (\Sigma_3^T + \Sigma_5^T)\hat{Y} + \Sigma_2\tilde{\varphi} + \tilde{A}_1\tilde{Z}_1 + \tilde{A}_2^T\tilde{Z}_2]dt \\
    - Z_1(t)dW(t) - Z_2(t)d\hat{W}(t), & t \in [0, T], \\
    Y(T) = \xi,
\end{cases}$$

(96)
\[
\begin{cases}
-d\tilde{Y}(t) = \left[ (\Sigma_0(\Pi_3 + \Pi_4) + \Sigma_2^T)\tilde{Y} + \Sigma_0\tilde{\varphi} + \tilde{A}_1^T\tilde{Z}_1 + \tilde{A}_2^T\tilde{Z}_2 \right] dt \\
\quad - \tilde{Z}_1(t) dW(t), \; t \in [0, T], \\
\tilde{Y}(T) = \hat{\xi},
\end{cases}
\]
and \(\tilde{\varphi}(\cdot)\) satisfies the SDE (78). Meanwhile, by (61) and (77), we can get
\[
(I - \Pi_1\Pi_4)Y - \left[ \Pi_1\Pi_4 + \Pi_4(\Pi_3 + \Pi_4) \right] \tilde{Y} = \Pi_2\tilde{\varphi} + \tilde{\phi} + \Pi_1\tilde{\varphi}.
\]  

**Proof.** For given \(\xi\), let \(\Pi_1(\cdot)\) and \(\Pi_4(\cdot)\) satisfy (72) and (73), respectively. By the standard BSDE theory, we can solve (76) to obtain \((\tilde{\phi}(\cdot), \tilde{\eta}(\cdot))\), and due to (74), we can solve (62) to obtain \((\hat{\phi}(\cdot), \hat{\eta}(\cdot))\). Let \(\Pi_3(\cdot)\) and \(\Pi_4(\cdot)\) satisfy (86) and (87), respectively. By the standard SDE theory, we can solve (94) to obtain \(\hat{\varphi}(\cdot)\), then \(\hat{Y}(\cdot)\) can be solved by (91). Due to (96) and (98), we can get \(Y(\cdot)\) and \(\tilde{\varphi}(\cdot)\), thus the state estimate feedback representation (95) can be obtained. The proof is complete. \(\square\)

**Remark 3.** Inspired by [10] and [33], in order to obtain the state estimate feedback representations for the optimal control of the follower and leader, respectively, in addition to the standard Riccati equations, some new Riccati equations and the forward-backward stochastic differential filtering equations (FBSDFEs) are firstly introduced, which is a new feature compared to the existing literatures. To be precise, in the follower’s problem, comparing to [51], two similar Riccati equations are given, in the meanwhile, a BSDFE and a SDFE are introduced in the partial information framework, which is different from the results in [34]. In the leader’s problem, comparing to [51], due to the appearance of \(\hat{X}\) and \(\hat{Y}\) in the leader’s state equation, besides two standard high-dimensional Riccati equations, another two new and complex high-dimensional Riccati equations are introduced, a linear BSDFE is derived from the terminal value \(Y(T)\) in (60), then a linear SDFE is needed for representing the solution to the BSDFE, which is more complicated than that in [34].

**Remark 4.** We notice an interesting and thoughtful relation between the different forms of maximum principle and the solvability of related high-dimensional Riccati equations, derived from the leader’s control problem. It is acknowledged that we usually apply the variation and adjoint technique to derive the maximum principle, but if we give the different adjoint equations with different initial or terminal value, where the differences depend on the positive/negative signs behind the initial or terminal value. Then the feature would lead to different Hamiltonian functions with different positive/negative signs and, to some degree, different kinds of reasonable maximum principles. But due to the technique (56), we put the previous \(n\)-dimensional state variables into the \(2n\)-dimensional new variables. Therefore, \(\Pi_1(\cdot), \Pi_2(\cdot), \Pi_3(\cdot)\) and \(\Pi_4(\cdot)\) are \(\mathbb{R}^{2n \times 2n}\)-matrix-valued functions. This feature has no influence on the study of standard stochastic control problems and differential games because we do not need the technique for augmentation. However, it indeed has impact on the study of the solvability of the Riccati equations in the leader-follower problem. Especially, such as the \(\Sigma_0\) in (72) and \(\Sigma_4\) in (87) which are matrix-valued functions, the opposite sign appearing in the component of matrix caused by the feature might cause the asymmetry of the \(\Pi_1(\cdot)\) and \(\Pi_4(\cdot)\). Therefore, we must choose the approximate initial or terminal value of the adjoint variable to ensure the symmetry of the Riccati equations at least.
In the following, we pay attention to discuss the solvability of the four Riccati equations (72), (73), (86) and (87) for $\Pi_1(\cdot), \Pi_2(\cdot), \Pi_3(\cdot)$ and $\Pi_4(\cdot)$. However, the solvability of coupled linear conditional mean-field FBSDE (60) where the drift terms contain processes $Z_1(\cdot)$ and $Z_2(\cdot)$, is still a challenging issue. Therefore, we will consider the case of $C_1(\cdot) = C_2(\cdot) \equiv 0$. For simplicity, we will just consider the constant coefficient case.

Firstly, we discuss the solvability of (72) and (73) for $\Pi_1(\cdot)$ and $\Pi_2(\cdot)$, due to $C_1 = C_2 = 0$, then we have $\tilde{A}_1 = \tilde{A}_2 = B_2 = C = 0$. Therefore, (72) and (73) can be rewritten as:

$$\begin{aligned}
\Pi_1 + \Pi_1A_1 + A_1^T\Pi_1 + \Pi_1B_1\Pi_1 + \Sigma_\sigma = 0, & \quad t \in [0, T], \\
\Pi_1(T) = 0,
\end{aligned}$$

(99)

and

$$\begin{aligned}
\Pi_2 + \Pi_2(\Sigma_7 + \Sigma_\sigma\Pi_1) + (\Sigma_7^T + \Pi_1\Sigma_8)\Pi_2 + \Pi_2\Sigma_8\Pi_2 \\
+ \Pi_1\Sigma_6\Pi_1 + \Pi_1\Sigma_5 + \Sigma_7^T\Pi_1 = 0, & \quad t \in [0, T], \\
\Pi_2(T) = 0,
\end{aligned}$$

(100)

According to the Theorem 5.3 of Yong [47], we have the following lemmas.

**Lemma 4.3.** Let (L1), (L2) hold, $C_1 = C_2 = 0$, and $\det \left\{ (0, \ I) e^{A_1t} \left[ \begin{array}{c} 0 \\ I \end{array} \right] \right\} > 0, \ t \in [0, T]$ hold. Then, (99) admits a unique solution $\Pi_1(\cdot)$ which has the following representation

$$\Pi_1(t) = -\left[ (0, \ I) e^{A_1(t-t)} \left[ \begin{array}{c} 0 \\ I \end{array} \right] \right]^{-1} (0, \ I) e^{A_1(T-t)} \left[ \begin{array}{c} 0 \\ I \end{array} \right],$$

(101)

where

$$A = \left( \begin{array}{cc} A_1 & B_1 \\ -\Sigma_\sigma & -A_1^T \end{array} \right).$$

(102)

**Lemma 4.4.** Let (L1), (L2) hold, $C_1 = C_2 = 0$, and $\det \left\{ (0, \ I) e^{A_2t} \left[ \begin{array}{c} 0 \\ I \end{array} \right] \right\} > 0, \ t \in [0, T]$ hold, and (99) hold. Then, (100) admits a unique solution $\Pi_2(\cdot)$. Then, (100) admits a unique solution $\Pi_2(\cdot)$ which has the following representation

$$\Pi_2(t) = -\left[ (0, \ I) e^{\tilde{A}_1(t-t)} \left[ \begin{array}{c} 0 \\ I \end{array} \right] \right]^{-1} (0, \ I) e^{\tilde{A}_1(T-t)} \left[ \begin{array}{c} 0 \\ I \end{array} \right],$$

(103)

where

$$\tilde{A} = \left( \begin{array}{cc} \Sigma_7 + \Sigma_\sigma\Pi_1 & \Sigma_8 \\ -(\Pi_1\Sigma_6\Pi_1 + \Pi_1\Sigma_5 + \Sigma_7^T\Pi_1) & -(\Sigma_7^T + \Pi_1\Sigma_8) \end{array} \right).$$

(104)

Secondly, we discuss the solvability of two Riccati equation (86) and (87) for $\Pi_3(\cdot)$ and $\Pi_4(\cdot)$, by making the time-reversing transformation:

$$\tau = T - t, \ t \in [0, T].$$

Since $C_1 = C_2 = 0$, the equivalent form of (86) and (87) can be obtained, respectively,

$$\begin{aligned}
\hat{\Pi}_3 + \mathcal{A}_1\Pi_3 + \Pi_3\mathcal{A}_1^T + \Pi_3\Sigma_\sigma\Pi_3 + B_1 = 0, & \quad t \in [0, T], \\
\Pi_3(T) = \tilde{G},
\end{aligned}$$

(105)
and
\[
\begin{aligned}
\dot{\Pi}_4 + \Pi_4 (\Sigma_7^T + \Sigma_9 \Pi_3) + (\Sigma_7 + \Pi_3 \Sigma_9) \Pi_4 \\
+ \Pi_3 \Sigma_9 \Pi_4 + \Pi_3 \Sigma_6^T + \Sigma_5 \Pi_3 + \Sigma_6 = 0, \ t \in [0, T],
\end{aligned}
\]
(106)

We introduce the following Riccati equation:
\[
\begin{aligned}
\dot{\Pi}_{3,1}(t) + (A_1 + \bar{G} \Sigma_9) \Pi_{3,1}(t) + \Pi_{3,1}(t)(A_1^T + \Sigma_9 \bar{G}) + \Pi_{3,1}(t) \Sigma_9 \Pi_{3,1}(t) \\
+ A_1 \bar{G} + \bar{G} A_1^T + \bar{G} \Sigma_9 \bar{G} + B_1 = 0, \ t \in [0, T],
\end{aligned}
\]
(107)

Therefore, using the same method as before, we get the sufficient condition to ensure the solvability of Riccati equation of \( \Pi_3(\cdot) \) and \( \Pi_4(\cdot) \).

Lemma 4.5. Let (L1), (L2), (L3) hold, \( C_1 = C_2 = 0 \), and \( \det \left\{ \begin{pmatrix} 0 & I \end{pmatrix} e^{Bi} \begin{pmatrix} 0 \\ I \end{pmatrix} \right\} > 0, \ t \in [0, T] \) hold. Then, (107) admits a unique solution \( \Pi_{3,1}(\cdot) \) which has the following representation
\[
\Pi_{3,1}(t) = -\left[ \begin{pmatrix} 0 & I \end{pmatrix} e^{B(T-t)} \begin{pmatrix} 0 \\ I \end{pmatrix} \right]^{-1} \begin{pmatrix} 0 & I \end{pmatrix} e^{B(T-t)} \begin{pmatrix} 0 \\ I \end{pmatrix},
\]
(109)

where
\[
B = \begin{pmatrix} -A_1^T & \Sigma_9 \bar{G} \\ \bar{G} A_1^T + \Sigma_9 \bar{G} + B_1 & -A_1 + \Sigma_9 \bar{G} \end{pmatrix}.
\]
(110)

Moreover, (108) gives the solution \( \Pi_3(\cdot) \) to the Riccati equation (105).

Lemma 4.6. Let (L1), (L2), (L3) hold, \( C_1 = C_2 = 0 \), and \( \det \left\{ \begin{pmatrix} 0 & I \end{pmatrix} e^{Bi} \begin{pmatrix} 0 \\ I \end{pmatrix} \right\} > 0, \ t \in [0, T] \) hold, and (105) admits a unique solution \( \Pi_3(\cdot) \). Then, (106) admits a unique solution \( \Pi_4(\cdot) \) which has the following representation
\[
\Pi_4(t) = -\left[ \begin{pmatrix} 0 & I \end{pmatrix} e^{\tilde{B}(T-t)} \begin{pmatrix} 0 \\ I \end{pmatrix} \right]^{-1} \begin{pmatrix} 0 & I \end{pmatrix} e^{\tilde{B}(T-t)} \begin{pmatrix} 0 \\ I \end{pmatrix},
\]
(111)

where
\[
\tilde{B} = \begin{pmatrix} -\Sigma_7^T + \Sigma_9 \Pi_3 \\ -\Pi_3 \Sigma_6^T + \Sigma_5 \Pi_3 + \Sigma_6 \end{pmatrix}.
\]
(112)

In the above, by considering the special case, some sufficient conditions (Lemma 4.3 - Lemma 4.6) have been given to ensure the solvability of Riccati equation for \( \Pi_1(\cdot), \Pi_2(\cdot), \Pi_3(\cdot) \) and \( \Pi_4(\cdot) \).

Before the end of this section, from (45), the optimal control \( \bar{v}_1(\cdot) \) of the follower can also be represented in a nonanticipating way:
\[
\bar{v}_1(t) = -R_1^{-1}(t) B_1(t)^T \left\{ \left[ \begin{pmatrix} 0 & P_2(t) \end{pmatrix} + (1 \ 0) (\Pi_3(t) + \Pi_4(t)) \right] \check{y}(t) \\
+ (1 \ 0) \ \check{\varphi}(t) \right\}, \ a.e. \ t \in [0, T], \ a.s.
\]
(113)

Thus, the Stackelberg equilibrium point \((\bar{v}_1(\cdot), \bar{v}_2(\cdot))\) is obtained, by (113) and (95).
5. Application to pension fund management problem. In this section, we are denoted to study a defined benefit (DB) pension fund management problem arising from financial markets, which naturally motivate the above theoretical research of the LQ Stackelberg game for BSDE with partial information.

It is well known that a pension fund can be classified into two main categories: Defined benefit (DB) pension scheme and defined contribution (DC) pension scheme. In a DB scheme, the benefits are fixed in advance by the sponsor, and the contributions are designed to assure the future payments to claim holders in their retirement period. There are two corresponding representative members who makes contributions continuously over time to the pension fund in $[0,T]$. One of the members is the leader with the regular premium proportion $v_2$ as his contribution, who is usually regarded as the supervisory, government or company. And the other one is the follower with the regular premium proportion $v_1$ as his contribution, who is usually regarded as the individual producer or retail investor. Premiums are a proportion of salary or income which are continuously deposited into the pension fund plan member’s account as the contributions.

We consider a continuous-time setup, and the dynamics of pension fund plan member’s account is given by

$$dF(t) = F(t)d\Delta(t) + (v_1(t) + v_2(t) - DB)dt,$$  \hfill (114)

where $F(t)$ is the value process of pension fund plan member’s account at time $t$, $d\Delta(t)$ is the instantaneous return during the time interval $(t,t+dt)$, $v_1(\cdot)$ and $v_2(\cdot)$ are the premium proportions of follower and leader which acts as our control variables, respectively. $DB$ is the pension scheme benefit outgo which is assumed to be a constant for sake of simplicity.

Suppose that the pension fund is invested in a risk-free asset (bond) and two risky assets (stocks). The price $S_0(t)$ of the bond at time $t$ is given by

$$\begin{cases}
    dS_0(t) = r(t)S_0(t)dt, & t \geq 0, \\
    S_0(0) = 1,
\end{cases}$$  \hfill (115)

where $r(t) > 0$ is the instantaneous rate of return at time $t$.

The prices $S_1(t)$ and $S_2(t)$ of the two stocks at time $t$ are given by

$$\begin{cases}
    dS_1(t) = S_1(t)[\mu_1(t)dt + \sigma(t)dW(t)], & t \geq 0, \\
    S_1(0) = S^1_0,
\end{cases}$$  \hfill (116)

$$\begin{cases}
    dS_2(t) = S_2(t)[\mu_2(t)dt + \tilde{\sigma}(t)d\tilde{W}(t)], & t \geq 0, \\
    S_2(0) = S^2_0,
\end{cases}$$  \hfill (117)

respectively, where $W(\cdot)$ and $\tilde{W}(\cdot)$ are two independent one-dimension Brownian motion. Here $\mu_i(t) > r(t), i = 1, 2$ are the instantaneous rates of expected return and $\sigma(t), \tilde{\sigma}(t) > 0$ are the instantaneous rates of volatility, at time $t$. We assume that $\mu_1(\cdot), \mu_2(\cdot), r(\cdot), \sigma(\cdot)$ and $\tilde{\sigma}(\cdot)$ are deterministic bounded functions, and $\sigma^{-1}(\cdot)$ and $\tilde{\sigma}^{-1}(\cdot)$ are also bounded.

In the real financial market, it is reasonable for the investors to make decisions based on the historical price of the risky asset $S_1(\cdot)$ and $S_2(\cdot)$. Therefore, the observable filtration at time $t$ can be set as $\mathcal{F}_t = \sigma\{S_1(s), S_2(s)|0 \leq s \leq t\}$ and it is clear that $\mathcal{F}_t = \sigma\{W(s), \tilde{W}(s)|0 \leq s \leq t\}$. However, in our Stackelberg game background, there exists two different asymmetric information for two players, to some degree, because of some practical phenomenon such as insider trading or the
information asymmetry. So we assume that the one who plays a leader’s role knows the full information from the financial market including the price of the risky assets $S_1(\cdot)$ and $S_2(\cdot)$, which is called $\mathcal{F}_t^1 = \sigma\{W(s), \tilde{W}(s) | 0 \leq s \leq t\}$, but the other one who plays a follower’s role only knows the partial information about the price $S_1(\cdot)$ coming from $\mathcal{G}_t^1 = \sigma\{W(s) | 0 \leq s \leq t\}$. Obviously, $\mathcal{G}_t^1 \subset \mathcal{F}_t$. We define $\mathcal{F} = \{\mathcal{F}_t\}_{0 \leq t \leq T}$.

Suppose that the proportion $\pi_1(t)$ and $\pi_2(t)$ of the pension fund is to be allocated in the two stock, respectively, while $1 - \pi_1(t) - \pi_2(t)$ is to be allocated in the bond, at time $t$. Thus the instantaneous return becomes

$$d\Delta(t) = \left[ \dot{r}(t) + (\mu_1(t) - r(t))\pi_1(t) + (\mu_2(t) - r(t))\pi_2(t) \right] dt + \sigma(t)\pi_1(t)dW(t) + \tilde{\sigma}(t)\pi_2(t)d\tilde{W}(t). \tag{118}$$

Therefore, the pension fund dynamics can be written as the following form:

$$dF(t) = \left[ \dot{r}(t)F(t) + (\mu_1(t) - r(t))\pi_1(t)F(t) + (\mu_2(t) - r(t))\pi_2(t)F(t) + v_1(t) + v_2(t) - DB \right] dt + \sigma(t)\pi_1(t)F(t)dW(t) + \tilde{\sigma}(t)\pi_2(t)F(t)d\tilde{W}(t). \tag{119}$$

On the one hand, if the pension fund manager wants to achieve the wealth level $\xi$ at the terminal time $T$ to fulfill his/her obligations, then the dynamics of pension fund plan member’s account is

$$\begin{aligned}
\begin{cases}
  dF(t) = \left[ \dot{r}(t)F(t) + (\mu_1(t) - r(t))\pi_1(t)F(t) + (\mu_2(t) - r(t))\pi_2(t)F(t) \\
  + v_1(t) + v_2(t) - DB \right] dt + \sigma(t)\pi_1(t)F(t)dW(t) \\
  + \tilde{\sigma}(t)\pi_2(t)F(t)d\tilde{W}(t), \ t \in [0, T],
\end{cases}
\end{aligned} \tag{120}$$

$$F(T) = \xi.$$ 

On the other hand, if we set $\sigma(\cdot)\pi_1(\cdot)F(\cdot) = Z_1(\cdot)$ and $\tilde{\sigma}(\cdot)\pi_2(\cdot)F(\cdot) = Z_2(\cdot)$, then the above equation is equivalent to the BSDE

$$\begin{aligned}
\begin{cases}
  -dF(t) = -\left[ \dot{r}(t)F(t) + \frac{\mu_1(t) - r(t)}{\sigma(t)}Z_1(t) + \frac{\mu_2(t) - r(t)}{\tilde{\sigma}(t)}Z_2(t) + v_1(t) \\
  + v_2(t) - DB \right] dt - Z_1(t)dW(t) - Z_2(t)d\tilde{W}(t), \ t \in [0, T],
\end{cases}
\end{aligned} \tag{121}$$

where the control processes $v_1(\cdot)$ and $v_2(\cdot)$ are adapted to the information filtration $\mathcal{G}_t^1$ and $\mathcal{F}_t$, respectively.

Let $\mathcal{U}_1[0, T] = \{v_1(\cdot) \in L^2_{\mathcal{G}_t^1}(0, T; \mathbb{R}) | v_1(t) \in \mathbb{R}, t \in [0, T] \}$ and $\mathcal{U}_2[0, T] = \{v_2(\cdot) \in L^2_{\mathcal{F}_t}(0, T; \mathbb{R}) | v_2(t) \in \mathbb{R}, t \in [0, T] \}$ denote the admissible control sets for the follower and leader, respectively. For any $(v_1(\cdot), v_2(\cdot)) \in \mathcal{U}_1 \times \mathcal{U}_2$, the BSDE (121) admits a unique solution triple $(F(\cdot), Z_1(\cdot), Z_2(\cdot))$ in $L^2_{\mathcal{G}_t^1}(0, T; \mathbb{R}) \times L^2_{\mathcal{F}_t}(0, T; \mathbb{R}) \times L^2_{\mathcal{F}_t}(0, T; \mathbb{R})$.

Let us introduce the cost functionals

$$J_i(v_1(\cdot), v_2(\cdot); \xi) = \mathbb{E} \left[ \int_0^T \frac{1}{2} e^{-\beta t} (v_1(t) - NC)^2 dt + F^2(0) \right], \quad i = 1, 2, \tag{122}$$

where $\beta$ is a discount factor and $NC$ is a preset target, say, the normal cost. The aim of the members is to minimize the cost functional $J_i(v_1(\cdot), v_2(\cdot); \xi)$ over $\mathcal{U}_i$, $i = 1, 2$. Recall that the first term of $J_i(u_1(\cdot), u_2(\cdot); \xi)$ is the running cost due to the deviation of the contribution from the preset target level. This term is introduced here to measure the stability of the DB pension scheme. The second term $F(0)$ is just the initial reserve to operate the scheme.
Let us now explain the leader-follower feature of the game. At time $t$, first, the big company (leader) announces his/her contribution (premium proportion) $v_2(t)$. Second, with the help of the part of informations the retail investor (follower) knows, he/she would like to set his/her contribution (premium proportion) over $v_1(\cdot) \in U_1$. Knowing the follower would take such an optimal control $\hat{v}_1(\cdot)$ (supposing it exists, which depends on the choice $v_2(\cdot)$ of the leader and the initial state $\xi$, in general), and having the advantages over the follower in case of possessing more information, the big company (leader) would like to choose some $\hat{v}_2(\cdot) \in U_2$ to minimize $J_2(\hat{v}_1(\cdot), v_2(\cdot); \xi)$ over $v_2(\cdot) \in U_2$.

We aim to find the Stackelberg equilibrium point $(\hat{v}_1(\cdot), \hat{v}_2(\cdot)) \in U_1 \times U_2$, which is the optimal control pairs of the Stackelberg game of BSDE with partial information.

There is much literature to study the pension fund management problem by stochastic control approach, such as Huang et al. [14], Di Giacinto et al. [7], etc. However, our problems are essentially different in that we study the pension fund problem in the framework of Stackelberg game of BSDE with partial information. For more details about financial applications for partial information differential games, please refer to Wang and Yu [41], Shi and Wang [32], Huang et al. [17], Xiong et al. [43], etc.

It is obvious that this problems can be regarded as a special case of that in Section 4. So we can use the results to solve it. For the simplicity of the calculations in this example, we set $DB = NC = 0$. Comparing to (17), (18) and (51), we know in this section $A(t) = -r(t), B_1(t) = B_2(t) = -1, C_1(t) = -\frac{\mu_1(\cdot) - r(t)}{\sigma(t)}, C_2(t) = -\frac{\mu_2(\cdot) - r(t)}{\sigma(t)}$, $Q_1(t) = Q_2(t) = 0, R_1(t) = R_2(t) = e^{-\beta t}, S_1(t) = S_2(t) = 0, N_1(t) = N_2(t) = 0$ and $G_1(t) = G_2(t) = 2$ for any $t \in [0, T]$.

Firstly, we solve the follower’s problem. For any given $\xi \in L^2_F(\Omega; \mathbb{R})$ and $v_2(\cdot) \in U_2[0, T]$, using Theorem 4.1, we can get

$$\hat{v}_1(t) = e^{\beta t} \left[ P_2(t) \hat{y}^{\hat{v}_1, \hat{v}_2} + \hat{\varphi}(t) \right], \quad (123)$$

where $(\hat{y}^{\hat{v}_1, \hat{v}_2}(\cdot), \hat{z}^{\hat{v}_1, \hat{v}_2}(\cdot), \hat{z}^{\hat{v}_1, \hat{v}_2}(\cdot))$ satisfy the following FBSDE

$$\begin{aligned}
-d\hat{y}^{\hat{v}_1, \hat{v}_2}(t) &= \left[ -(e^{\beta t} P_2(t) + r(t))\hat{y}^{\hat{v}_1, \hat{v}_2}(t) - e^{\beta t}\hat{\varphi}(t) - \frac{\mu_1(t) - r(t)}{\sigma(t)} \hat{z}^{\hat{v}_1, \hat{v}_2}(t) \\
&\quad - \frac{\mu_2(t) - r(t)}{\sigma(t)} \hat{z}^{\hat{v}_1, \hat{v}_2}(t) - \hat{v}_2(t) \right] dt - \hat{z}^{\hat{v}_1, \hat{v}_2}(t) dW(t), \\
\hat{z}^{\hat{v}_2}(t) &= \frac{\mu_2(t) - r(t)}{\sigma(t)} P_1(t) \hat{x}(t), \quad t \in [0, T], \\
\hat{y}^{\hat{v}_1, \hat{v}_2}(T) &= \hat{\xi}, \quad \hat{x}(0) = 2\hat{y}^{\hat{v}_1, \hat{v}_2}(0),
\end{aligned}$$

(124)

$P_1(\cdot)$ and $P_2(\cdot)$ satisfy the following two Riccati equations

$$\begin{aligned}
&P_1(t) + \left[ \frac{(\mu_1(t) - r(t))^2}{\sigma(t)} + \left( \frac{\mu_2(t) - r(t)}{\sigma(t)} \right)^2 - 2r(t) \right] P_1(t) \\
&\quad + e^{\beta t} = 0, \quad t \in [0, T], \quad P_1(T) = 0,
\end{aligned}$$

(125)
\[
\begin{aligned}
\dot{P}_2(t) + \left[ \left( \frac{\mu_1(t) - r(t)}{\sigma(t)} \right)^2 + \left( \frac{\mu_2(t) - r(t)}{\sigma(t)} \right)^2 \right] P_2^2(t) P_1(t) + e^{\beta t} P_2^2(t) \\
+ 2r(t)P_2(t) = 0, \ t \in [0, T], \ P_2(0) = 2, 
\end{aligned}
\]  

(126)

respectively, and \((\hat{\phi}(\cdot), \hat{\phi}(\cdot), \eta(\cdot))\) satisfy the following FBSDE

\[
\begin{aligned}
d\hat{\phi}(t) &= \left\{ - \left[ \left( \frac{\mu_1(t) - r(t)}{\sigma(t)} \right)^2 + \left( \frac{\mu_2(t) - r(t)}{\sigma(t)} \right)^2 \right] P_1(t) P_2(t) - e^{\beta t} P_2(t) \right. \\
& \left. - \frac{\mu_1(t) - r(t)}{\sigma(t)} P_2(t) \eta(t) - P_2(t)\hat{\eta}(t) \right}\ dt \\
& + \left\{ \frac{\mu_1(t) - r(t)}{\sigma(t)} P_2(t)\hat{\phi}(t) - \frac{\mu_2(t) - r(t)}{\sigma(t)} \hat{\phi}(t) + P_2(t)\hat{\eta}(t) \right\} dW(t), \\
-d\hat{\phi}(t) &= \left\{ - r(t) \hat{\phi}(t) + \hat{\eta}(t) - P_2(t) \right. \\
& \left. - \frac{\mu_1(t) - r(t)}{\sigma(t)} \hat{\eta}(t) \right\} dt - \hat{\eta}(t) dW(t), \ t \in [0, T], \\
\hat{\phi}(T) &= -\xi, \ \hat{\phi}(0) = 0. 
\end{aligned}
\]

(127)

Next, we solve the leader’s problem, noting (56) and putting

\[
\begin{aligned}
A_1 &= \begin{pmatrix} -r(t) - e^{\beta t} P_2(t) - \left( \frac{\mu_2(t) - r(t)}{\sigma(t)} \right)^2 P_1(t) P_2(t) & 0 \\
0 & -r(t) \end{pmatrix}, \\
\tilde{A}_1 &= \begin{pmatrix} -\frac{\mu_1(t) - r(t)}{\sigma(t)} & 0 \\
0 & -\frac{\mu_1(t) - r(t)}{\sigma(t)} \end{pmatrix}, \ \bar{A}_2 = \begin{pmatrix} 0 & 0 \\
0 & -e^{\beta t} P_2(t) \end{pmatrix}, \\
B_2 &= \begin{pmatrix} 0 & 0 \\
0 & \left( \frac{\mu_1(t) - r(t)}{\sigma(t)} \right)^2 P_2^2(t) P_1(t) \end{pmatrix}, \\
C &= \begin{pmatrix} 0 & -\frac{\mu_1(t) - r(t)}{\sigma(t)} P_2(t) \\
-\frac{\mu_1(t) - r(t)}{\sigma(t)} P_2(t) & 0 \end{pmatrix}, \ \bar{C}_1 = \begin{pmatrix} 0 & 0 \\
0 & 0 \end{pmatrix}, \\
\bar{C}_2 &= \begin{pmatrix} 0 & -P_2(t) \\
-P_2(t) & 0 \end{pmatrix}, \ \bar{C}_3 = \begin{pmatrix} 0 & 0 \\
0 & 0 \end{pmatrix}, \ \bar{D} = \begin{pmatrix} -P_2(t) \\
0 \end{pmatrix}, \\
\bar{F}_1 &= \begin{pmatrix} 0 & -e^{\beta t} \\
-e^{\beta t} & 0 \end{pmatrix}, \ \bar{\xi} = \begin{pmatrix} 0 \\
\xi \end{pmatrix}, \ \bar{G} = \begin{pmatrix} 0 & 0 \\
0 & 2 \end{pmatrix}. 
\end{aligned}
\]

By Theorem 4.2, we can get

\[
\hat{\eta}(t) = -e^{\beta t} \bar{E}_2^T \Pi_3(t) Y(t) - e^{\beta t} \bar{E}_2^T \Pi_4(t) + \bar{D}^T \dot{Y}(t) - e^{\beta t} \bar{E}_2^T \bar{\phi}(t),
\]

(128)
where \((Y(\cdot), Z(\cdot), \tilde{Z}(\cdot))\) satisfy the following 2-dimensional BSDE

\[
-dY(t) = \left\{ \begin{array}{l}
\left[ \begin{array}{cc}
-\rho(t) - \rho(t) & -e^{\beta t} P_2(t) - \left( \frac{\mu_2(t) - \rho(t)}{\sigma(t)} \right)^2 P_1(t) P_2(t) \\
0 & -r(t)
\end{array} \right] Y(t) \\
+ \left[ \begin{array}{cc}
0 & -e^{\beta t} \\
0 & -e^{\beta t}
\end{array} \right] \Pi_3(t) \\
+ \left[ \begin{array}{cc}
0 & -e^{\beta t} P_2(t) \\
0 & -e^{\beta t} P_2(t)
\end{array} \right] \Pi_4(t) + \left[ \begin{array}{cc}
0 & 0 \\
0 & 0
\end{array} \right] \tilde{Y}(t) \\
+ \left( \begin{array}{cc}
0 & 0 \\
-\mu_2(t) - \rho(t)
\end{array} \right) \tilde{Y}(t) + \left( \begin{array}{cc}
0 & 0 \\
-\mu_2(t) - \rho(t)
\end{array} \right) \tilde{Z}_1(t) \\
+ \left( \begin{array}{cc}
0 & 0 \\
-\mu_2(t) - \rho(t)
\end{array} \right) \tilde{Z}_2(t)
\end{array} \right\} dt - Z_1(t) d\tilde{W}(t) - Z_2(t) d\tilde{W}(t), \ t \in [0, T],
\]

\(Y(T) = \xi,\)

(129)

\(\tilde{Y}(\cdot)\) satisfies the 2-dimensional SDE (78), and \(\Pi_3(\cdot)\) and \(\Pi_4(\cdot)\) satisfy (86) and (87), respectively. By a dual technique, we have

\[
Y(t) = E \left[ \Gamma_t(T) \xi + \int_t^T \left\{ \left[ \begin{array}{cc}
0 & -e^{\beta s} \\
-\rho(s) & -e^{\beta s}
\end{array} \right] \Pi_4(s) \\
+ \left( \begin{array}{cc}
-e^{\beta s} P_2(s) & 0 \\
0 & -e^{\beta s} P_2(s)
\end{array} \right) \tilde{Y}(s) + \left( \begin{array}{cc}
0 & -e^{\beta s} \\
0 & -e^{\beta s}
\end{array} \right) \tilde{\varphi}(s)
\end{array} \right\} \Gamma_t(s) ds \bigg| \mathcal{F}_t \right],
\]

(130)

where for \(t \in [0, T], \Gamma_t(\cdot)\) is the unique solution to

\[
d\Gamma_t(s) = \left\{ \begin{array}{l}
\left[ \begin{array}{cc}
0 & -e^{\beta s} \\
-\rho(s) & -e^{\beta s}
\end{array} \right] \Pi_3(s) \\
+ \left( \begin{array}{cc}
-e^{\beta s} P_2(s) & 0 \\
0 & -e^{\beta s} P_2(s)
\end{array} \right) \tilde{Y}(s) + \left( \begin{array}{cc}
0 & 0 \\
0 & 0
\end{array} \right) \tilde{\varphi}(s)
\end{array} \right\} \Gamma_t(s) ds \\
+ \left( \begin{array}{cc}
0 & 0 \\
0 & 0
\end{array} \right) \tilde{Z}_1(s) d\tilde{W}(s), \ s \in [t, T],
\]

(131)

Thus, \((\bar{v}_1(\cdot), \bar{v}_2(\cdot))\) determined by (123) and (128) is a Stackelberg equilibrium point of our Stackelberg game of BSDEs with partial information.

Finally, the optimal initial wealth reserve \(y_{\bar{v}_1, \bar{v}_2}(0)\) is the second component of the following 2-dimensional vector

\[
Y(0) = E \left[ \Gamma_0(T) \xi + \int_0^T \left\{ \left[ \begin{array}{cc}
0 & -e^{\beta t} \\
-\rho(t) & -e^{\beta t}
\end{array} \right] \Pi_4(t) \\
+ \left( \begin{array}{cc}
0 & 0 \\
-e^{\beta t} P_2(t) & -e^{\beta t} P_2(t)
\end{array} \right) \tilde{Y}(t) + \left( \begin{array}{cc}
0 & -e^{\beta t} \\
0 & -e^{\beta t}
\end{array} \right) \tilde{\varphi}(t)
\end{array} \right\} \Gamma_0(t) dt \right].
\]

(132)
6. **Concluding remarks.** In this paper, we have discussed the Stackelberg game of BSDEs with partial information. The general problem is studied first and then the LQ special case is researched in some state estimate representations for the Stackelberg equilibrium point, for the follower and the leader, respectively. Theoretical results are applied to the pension fund management problem.

Possible extensions to the Stackelberg game with noisy observations are desired to be researched, and the solvability of the related Riccati equations are very challenging and difficult research topics. We will consider these problems in the future.

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