Simple Analytic Formula Relating the Mass and Spin of Accreting Compact Objects to Their Rapid X-Ray Variability

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Abstract
Following the previous research on epicyclic oscillations of accretion disks around black holes (BHs) and neutron stars (NSs), a new model of high-frequency quasi-periodic oscillations (QPOs) has been proposed, so-called cusp torus (CT) model, which deals with oscillations of fluid in marginally overflowing accretion tori (i.e., tori terminated by cusps). According to preliminary investigations, the model provides better fits of the NS QPO data compared to the relativistic precession (RP) model. It also implies a significantly higher upper limit on the Galactic microquasar BH spins. A short analytic formula has been noticed to well reproduce the model’s predictions on the QPO frequencies in Schwarzschild spacetimes. Here we derive an extended version of this formula that applies to rotating compact objects. We start with the consideration of Kerr spacetimes and derive a formula that is not restricted to a particular specific angular momentum distribution of the inner accretion flow, such as a Keplerian or constant one. Finally, we consider Hartle–Thorne spacetimes and include corrections implied by the NS oblateness. For a particular choice of a single parameter, our relation provides frequencies predicted by the CT model. For another value, it provides frequencies predicted by the RP model. We conclude that the formula is well applicable to rotating oblate NSs and both models. We briefly illustrate the application of our simple formula on several NS sources and confirm the expectation that the CT model is compatible with realistic values of the NS mass and provides better fits of data than the RP model.

1. Introduction

Accreting compact sources such as low-mass X-ray binaries (LMXBs) and active galactic nuclei provide a unique opportunity to explore the effects associated with strong gravity in black hole (BH) and neutron star (NS) systems where they may also serve as a good tool for the exploration of supradense matter (van der Klis 2006; Lewin & van der Klis 2010). There is a common aim within the large astrophysical community to relate the mass and spin of compact objects to their spectral and timing behavior. In this paper, we focus on the rapid X-ray variability and its models.

The high-frequency part of the power density spectra (PDS) of many sources reveals more or less sharp peaks that are called the high-frequency quasi-periodic oscillations (HF QPOs). Commonly, the HF QPOs seem to have frequencies close to those of the orbital motion in the innermost part of a given accreting system. Detections of elusive HF QPO peaks in BH LMXB sources have been reported at rather constant frequencies, which tend to appear in ratios of small natural numbers (often including a 3:2 ratio; Abramowicz & Kluźniak 2001; Remillard et al. 2002; McClintock & Remillard 2006). The observed HF QPOs are however very weak and the overall picture can be more complex (Belloni et al. 2012; Belloni & Altamirano 2013; Varniere & Rodríguez 2018). In NS sources, HF QPOs are commonly referred to as twin-peak QPOs because they often appear in pairs observed simultaneously at the upper and lower QPO frequency, \( \nu_U > \nu_L \). Notably, robust correlations are observed between the frequencies of twin-peak QPOs. Each source reveals its specific frequency correlation, \( \nu_U = \nu_0(\nu_L) \), although the sources roughly follow a common pattern (Psaltis et al. 1999a; Abramowicz et al. 2005a, 2005b; Zhang et al. 2006). This is illustrated in Figure 1(a) where we show the frequencies of QPOs in the 3:2 frequency ratio observed in Galactic microquasars along with the HF QPO correlations in a group of 14 NS sources. The data used in the figure come from the works of Barret et al. (2005b, 2005c), Boirin et al. (2000), Altamirano et al. (2010), Linares et al. (2005), van der Klis et al. (1997), Boutoulokos et al. (2006), Homan et al. (2002), and Jonker et al. (2000, 2002) for NSs, and Strohmayer (2001), Remillard et al. (2002), Homan et al. (2003), and Remillard et al. (2003) for BHs.

At present, it is commonly accepted that QPOs originate in the innermost parts of accreting systems, mostly likely being related to orbital motion. This view is supported by spectroscopic arguments and the similarity between the frequencies of QPOs and those of the orbital motion. There is, however, no commonly accepted QPO model. In fact, it has not even yet been resolved whether the generic mechanism is the same for both classes of the sources (e.g., van der Klis 2006; Méndez & Belloni 2021).

The kinematics of the orbital motion allows for a straightforward consideration of variability that is connected to inhomogeneities propagating in the accretion flow. Even before the era of the Rossi X-ray Timing Explorer QPO
observations (van der Klis 1999), it was proposed that inhomogeneities moving from the disk to the NS surface can produce imprints in the observed variability, which can be used to determine the NS properties (Kluzniak et al. 1990). Later, several “kinematic” models of QPOs were proposed, the most often-quoted one being the “relativistic precession” model (hereafter the RP model) introduced in a series of papers by Stella & Vietri (1998a, 1999, 2002) and Morsink & Stella (1999) (see also Abramowicz et al. 1992). A similar concept is represented by models that include interactions between the disk and the surface or magnetosphere of rotating NSs (Strohmayer et al. 1996; Miller et al. 1998; Psaltis et al. 1999b; Huang et al. 2016; Wang et al. 2018, 2020). Another example of this type of model was discussed by Čadež et al. (2008), Kostić et al. (2009), and

Figure 1. (a) The data of several sources and examples of the expected frequency relations that are drawn for $M = 1.7 M_\odot$. The expected frequency relations are drawn for a nonrotating NS. (b) Sketch of the trajectory of a test particle on a slightly eccentric orbit that plays a crucial role in the hot-spot interpretation of QPOs. (c) Topology of equipotential surfaces that determine the spatial distribution of fluid in thick disks. The orange (along with the yellow) region corresponds to a torus with a cusp. The cusp is situated between the ISCO (the marginally stable circular orbit) and the marginally bound circular orbit (denoted as $r_{\text{mb}}$). (d) The data of the atoll source 4U 1636-53 (Barret et al. 2005b, 2005c) and their best fits for a nonrotating NS. For the sake of clarity, the data set that corresponds to the individual continuous observations is compared to the data set associated with the common processing of all observations. The right panel indicates individual contributions of each data point to the total chi-square value of a given fit (see Török et al. 2016a, for details). (e) The equipressure contours seen within the general relativistic three-dimensional global radiative magnetohydrodynamic simulation of Lančová et al. (2019) who have reported on a new class of realistic solutions of black hole accretion flows—the so-called puffy accretion disks. (f) Equidensity profiles corresponding to panel (e).
Germanà et al. (2009). They introduced a scenario in which the QPOs are generated by a “tidal disruption” (TD) of large accreted inhomogeneities.

Another, qualitatively rather different, possibility relies on the consideration of a collective motion of the accreted matter, in particular some fluid accretion disk oscillatory modes. Such a concept has been explored for both thin (diskoseismology) and thick disks. Within the former case, a possible QPO mechanism has been suggested based on the early studies of Kato & Fukue (1980), Kato (1990), Nowak & Wagoner (1991), Kato & Honma (1991), Nowak & Wagoner (1992), and others. Reviews of this pioneering research can be found in Wagoner (1999) and Kato (2001), while several subsequent papers were also published (Wagoner et al. 2001; Wagoner 2012; Kato & Machida 2020; Smith et al. 2021). As for the latter, a large collection of papers has been published suggesting that the QPOs are related to oscillations of accretion tori (Abramowicz & Kluźniak 2001; Rezzolla et al. 2003; Bursa et al. 2004; Bursa 2005; Abramowicz et al. 2006; Blaes et al. 2006; Ingram & Done 2010; Fragile et al. 2016; de Avellar et al. 2017; Mishra et al. 2017). In the framework of fluid flow oscillations, based on the evidence for the appearance of ratios of small natural numbers, Kluźniak & Abramowicz (2001) have introduced the idea of a nonlinear resonant coupling between different pairs of disk oscillation modes. This idea has been further discussed extensively (Abramowicz & Kluźniak 2001; Abramowicz et al. 2003a, 2003b; Kluźniak et al. 2004; Lee et al. 2004; Abramowicz et al. 2005b; Püttig 2005; Török et al. 2005; Horák & Karas 2006; Horák et al. 2009). Among these often-mentioned models, there are several alternative explanations, such as QPOs arising from shocks in advective accretion flows (Chakrabarti & Titarchuk 1995; Chakrabarti 1997; Le et al. 2016), the Rossby wave instability (Li et al. 2000; Vincent et al. 2013), or magnetic reconnection between the boundaries of the disk (Zhao et al. 2009; Huang et al. 2013). While we do not intend to provide a full review of the QPO models here, we note that numerous modifications of models discussed in this paper exist along with other different concepts (e.g., Rodríguez et al. 2002; Titarchuk & Wood 2002; Zhang 2004; Wang et al. 2008; Mukhopadhyay 2009; Stuchlík & Kotrllová 2009; Bachetti et al. 2010; Dönmez et al. 2011; Stuchlík et al. 2013; Stuchlík & Kolos 2014; Kolos et al. 2015; Germanà 2017; Fragile 2020; Stuchlík et al. 2020; Wang & Zhang 2020).

Having briefly illustrated the large collection of ideas proposed to explain the QPO phenomenon, in this work we focus solely on two particular QPO models. Following our previous study (Török et al. 2016a, 2016b, 2018, 2019), we derive a simple analytic formula relating the QPO frequencies to parameters of rotating compact objects within the framework of these models. Finally, we apply the formula to data of several NS sources and compare the predictions of the models.

2. QPO Models under Consideration

We focus on two particular QPO models that deal with orbital motion of the accreted fluid. First is the above-mentioned RP model, which in its usual form incorporates the assumption that the observed rapid X-ray variability originates in the local orbital motion of hot inhomogeneities orbiting in the innermost parts of the accretion disk, such as blobs or vortices (see Kluźniak et al. 1990; Abramowicz et al. 1992; Stella & Vietri 1998b, 1999).

The RP model has been used and quoted in numerous studies. It frequently serves as a rough tool for the estimation of a compact object’s mass based on its variability (e.g., Barret et al. 2006; Boutloukos et al. 2006, 2007, 2008; Barret et al. 2008; Török et al. 2010; Lin et al. 2011; Motta et al. 2014; du Buisson et al. 2019; Maselli et al. 2020, and references therein). The relation between QPO frequencies postulated within the model has been shown to roughly match the NS sources data (Stella & Vietri 1998b, 1999; Morris & Stella 1999b; Belloni et al. 2007; Lin et al. 2011; Török et al. 2012, 2016b; see Figure 1(a)). It is, however, highly questionable whether the local motion of hot spots can be responsible for the observed QPOs’ high amplitudes and coherence times (Barret et al. 2005a, 2005b).

The other QPO model under consideration, which was proposed recently by Török et al. (2016a), assumes marginally overflowing accretion tori (see the works of Abramowicz & Kluźniak 2001, Kluźniak & Abramowicz 2001, Rezzolla et al. 2003, Blaes et al. 2006, and de Avellar et al. 2018, for a broader context). This concept, to which we hereafter refer as the cusp torus (CT) model, was suggested as a disk oscillation–based alternative to the RP model. It utilizes the expectation that toroidal structures and cusps are likely to appear in real accretion flows, in which case the overall accretion rate through the inner edge of the disk could be strongly modulated by the torus oscillations (Kozłowski et al. 1978; Abramowicz et al. 1978; Paczynski & Abramowicz 1982; Horák 2005; Abramowicz et al. 2007; Parthasarathy et al. 2017). A sketch of the marginally overflowing accretion torus geometry is shown in Figure 1(c).

The concept of the CT model stems from the outputs of previous studies devoted to oscillations and stability properties of fluid tori. These were initiated by Papaloizou & Pringle (1984), who investigated the global linear stability of fluid tori with respect to nonaxisymmetric perturbations and, considering small linear perturbations to the torus equilibrium, derived a single partial differential equation governing the linear dynamics of oscillations of a Newtonian constant specific angular momentum torus. Later, a general relativistic form of the Papaloizou–Pringle (PP) equation was introduced by Abramowicz et al. (2006) and Blaes et al. (2006). Since this equation cannot be fully solved analytically, Straub & Sramkova (2009) and Fragile et al. (2016) used a perturbation method to derive fully general relativistic formulæ determining frequencies of the axisymmetric and nonaxisymmetric radial and vertical epicyclic modes in a slightly nonslender constant specific angular momentum torus within a second-order accuracy in the torus thickness. Their outputs were subsequently applied in Török et al. (2016a).

The CT model provides generally better fits of the NS data than the RP model. This is depicted in Figure 1(d). This finding is rather independent of the NS spin (Török et al. 2016a, 2016b). The model also likely predicts a lower NS mass compared to the RP model, which, in some cases, implies a mass estimate that is too high (Török et al. 2016a, 2018, 2019). Moreover, the upper limit on the Galactic microquasar spins given by this model is significantly higher than in the RP model’s case, namely \( j \sim 0.75 \) versus \( j \sim 0.55 \). This is in better agreement with the spectral spin estimates (Kotrllová et al. 2020).

An overview of the physical assumptions of the model, as well as all appropriated references, can be found in the studies
of Török et al. (2016a) and Kotroţovă et al. (2020). Similarly to the RP model, the currently applied concept of an inner torus displaying a cusp is very simplified compared to real accretion flows. Note that, at their present stage, both considered models do not explain why the assumed modes of the accreted matter motion are well observed while other, similar modes are not. Nevertheless, the CT model assumes a global motion of disk fluid as opposed to the local test particle motion considered in the RP model. Moreover, the structure of the inner accretion flow observed in general relativistic radiation magnetohydrodynamic (GRMHD) simulations often resembles that assumed within the model. This is illustrated in Figures 1(e) and (f). Although there still is a rather long way to go in the development of the GRMHD simulations in order to become able to study the possible QPO mechanism (see, however, Musoke et al. 2022), the similarity between the structures supports the expectation that the adopted approximation of the accretion flow posed by marginally overflowing tori allows for the determination of oscillatory frequencies close to those of accretion flows in real systems.

3. Frequencies of QPOs within the RP Model

Within the RP model, the frequencies of the two observed QPOs are given by the Keplerian frequency $\nu_K$ and the relativistic precession frequency $\nu_p$ of a slightly perturbed circular geodesic motion occurring at an arbitrary QPO excitation orbital radius $r_0$ (see Figure 1(b)):

$$\nu_U(r_0) = \nu_K(r_0), \quad \nu_L(r_0) = \nu_p(r_0).$$

(1)

The precession frequency is equal to the difference between the Keplerian and the radial epicyclic frequency:

$$\nu_p(r_0) = \nu_K(r_0) - \nu_1(r_0).$$

(2)

For a nonrotating relativistic compact star with the external spacetime given by the Schwarzschild geometry, the relation between the QPO frequencies implied by the RP model, i.e., by Equation (1), can be written as (Stella & Vietri 1999)

$$\nu_L = \nu_U(1 - B \sqrt{1 - (\nu_U/\nu_0)^{3/2}}),$$

(3)

where $B = 1$ and $\nu_0$ are equal to the Keplerian frequency at the innermost stable circular orbit (ISCO). The ISCO frequency is given solely by the gravitational mass $M$:

$$\nu_0 = \mathcal{F} \frac{1}{6^{3/2}}, \quad \mathcal{F} \equiv c^3 / (2\pi GM).$$

(4)

4. Frequencies of QPOs within the CT Model

For the CT model, the relation between the QPO frequencies in the Schwarzschild spacetimes also depends purely on $M$ and can be written in the following implicit form (Török et al. 2016a):

$$\nu_U(r_0) = \nu_K(r_0), \quad \nu_L(r_0) = \nu_r^{m=-1},$$

(5)

where $r_0$ denotes the torus center where the density of fluid peaks, $\nu_K$ determines the torus center corotation frequency, and $\nu_r^{m=-1}$ is equal to the frequency of the first nonaxisymmetric radial epicyclic mode calculated for the marginally overflowing torus (i.e., the torus that forms a cusp). Contrary to the RP model, which considers a geodesic test particle precession, this frequency corresponds to the precession of the whole fluid flow, which can strongly modulate accretion rate through the boundary layer. We note that, in contrast to the $m = -1$ mode, the modulation mechanism behind the imprints of the Keplerian frequency has not yet been fully resolved within the model’s framework. We expect that unstable corotation oscillatory modes along with the dynamics of inhomogeneities formed in the flow may contribute to the accretion rate modulation. The timescale of disk oscillations is more than 5 orders of magnitude shorter than the typical integration time required to well identify the two peaks in the PDS. Consequently, an overall emphasis on frequencies that are close or equal to the Keplerian frequency at the torus center can arise in the X-ray PDS, at least when small tori close to the ISCO are considered. Within the model framework, small tori likely correspond to the case of high QPO frequencies observed in the atoll sources. Our expectation is also in agreement with the results of numerical simulations of the (PP) instability. It has been shown that nonlinear evolution of the instability with azimuthal wavenumber $m$ leads to fragmentation of the initial torus configuration into a configuration with $m$ nearly disconnected “planets” (see the early studies of Goldreich et al. 1986; Goodman et al. 1987; Narayan et al. 1987). Since the instabilities can be suppressed by accretion through the cusp (Blaes 1987), the upper and lower QPOs can alternate with each other. Nevertheless, more investigation is needed to obtain a comprehensive understanding of the physical mechanism behind the upper QPO.

There is no explicit analytical evaluation of the $\nu_L(\nu_L)$ function and the Equation (5) must be solved numerically. However, it has been noticed by Török et al. (2018) that there is a solid analytic approximation—the numerical solution nearly coincides with Equation (3) when $B = 0.8$. This is illustrated in Figure 1(b).

5. Rotating Compact Stars

It has been noticed by Török et al. (2010) that in Kerr spacetimes characterized by the $j = cJ/(GM^2)$ rotational parameter, the relation between the QPO frequencies implied by the RP model can be expressed as

$$\nu_L = \nu_U \left\{ 1 - B \left[ 1 + \frac{8j\nu_U}{\mathcal{F} - j\nu_U} - 6 \left( \frac{\nu_U}{\mathcal{F} - j\nu_U} \right)^{2/3} - 3j^2 \left( \frac{\nu_U}{\mathcal{F} - j\nu_U} \right)^{4/3} \right]^{1/2} \right\},$$

(6)

when one sets $B = 1$.

Based on the analogy to nonrotating stars, Equation (6) for a particular choice of $B(j)$ can be expected to reproduce the numerically calculated frequency relation given by the CT model. Having this intention in mind, we presume a simple linear prescription,

$$B(j) = kj + 0.8,$$

(7)

which results in $B = 0.8$ for the $j = 0$ limit.

We made a comparison between the predictions of Equation (6) and the CT model predictions. Following the approach of Török et al. (2016a), we use the results of analytical calculations of the perturbative solution of the PP equation and compute the numerical solution of the dependence between the
expected QPO frequencies. As noticed by Horáček et al. (2017), the applied analytic perturbative solution of the PP equation providing the frequency of the radial epicyclic modes is in good agreement with the appropriate numerical solution. The whole set of formulae necessary for the numerical calculations is given by fairly long expressions; we therefore provide their explicit form in a Wolfram Mathematica notebook (see Kotrlová et al. 2020). A good match between the analytical prescription and the numerically calculated predictions is found for \( k = -0.2 \) and illustrated in Figure 2.

We note that the applicability of our result to rapidly rotating BHs is limited. The so far performed numerical calculations of frequencies given by the CT model utilize a perturbative approach valid within the second-order accuracy in torus thickness. Within this approach, the calculations are for high spins very sensitive to small changes in the torus thickness and the rotational parameter. Full numerical investigation of the PP equation, which determines the epicyclic mode frequencies, will be needed for rapidly rotating BHs.

6. QPO Frequencies and Accretion Flow Angular Momentum Distribution

Following the previous studies of Straub & Srámková (2009), Török et al. (2016a), Fragile et al. (2016), and Kotrlová et al. (2020), our simple formula relating the QPO frequencies and the compact object mass and spin was derived under the specific consideration of tori with constant distribution of angular momentum \( \ell \) of the accreted fluid. Nevertheless, it can be easily shown that the formula is of more general importance since its validity is not limited to this particular \( \ell \) prescription.

6.1. Range of QPO Frequencies

If we replace the \( B \) factor in Equation (6) by unity, we obtain exactly the prediction based on the test particle motion. This case also describes the scenario in which the angular momentum distribution of the accreted fluid is Keplerian. It is a limited case, in which the cross section of the oscillating torus is infinitely small (a parameter determining the torus thickness, \( \beta \), goes to zero).\(^2\)

When the \( B \) factor is taken into account, we come up with a scenario in which the torus has its maximal possible size (\( \beta = \beta_{\text{cusp}} \)). As shown by Straub & Srámková (2009), the \( m = -1 \) radial mode frequency evolves as a monotonic function of the torus thickness (see Figure 3(a) for illustration). Accordingly, the black curve (marked as the RP model) and the colored curves (the CT model) in Figure 2 describe the two extreme predictions of the QPO frequencies given by Equation (6). The area between these curves covers the whole range of QPO frequencies determined under the consideration of constant \( \ell \) and any \( \beta \). When we put

\[
B = 1 - 0.2(1 + j) \frac{\beta}{\beta_{\text{cusp}}},
\]

we obtain a continuous set of curves that cover the area between the limiting curves given by \( \beta = 0 \) and \( \beta = \beta_{\text{cusp}} \).

6.2. Generalization for Nonconstant Angular Momentum Distributions

The above consideration can be extended to a more general picture. In panel (a) of Figure 3, we show the behavior of the \( m = -1 \) radial frequency (the expected lower QPO frequency) for the particular case of \( \ell = \ell_{\text{const}} \). One should note the increasing monotonic behavior of the curves. When \( \beta \) increases, the \( m = 0 \) mode frequency decreases and the \( m = -1 \) mode frequency increases getting closer to the Keplerian frequency. The trend of the \( m = -1 \) mode frequency rising with the growing size of the oscillating structure persists for less-simplified situations as well. In panels (b) and (c) of Figure 3, we show a sketch of a possible parameterization of linear angular momentum distributions and its projection in the plane of the expected QPO frequencies. Clearly, taking into account the presumption of a monotonic behavior of the \( \nu_{\ell=-1}(\beta) \) function, any linear prescription for the angular momentum distribution, and all possible torus thicknesses from the \( \beta \in [0, \beta_{\text{cusp}}] \) range, the expected QPO frequencies should fall into the range denoted by the shaded area. An analogical consideration also applies to nonlinear distributions.

Overall, the narrow range between the two extremal curves given by Equation (11), which is indicated by the shaded area in Figure 3(b), represents a rather general limit on the QPO frequencies valid for a variety of plausible angular momentum distributions.

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\(^2\) The beta parameter describes the torus thickness and its relative radial extent such that, at a given \( r_0 \), a higher beta corresponds to a higher torus extent. It is given as \( \beta = \sqrt{2n} \frac{cs_n}{nu_0^2} \), where \( n \) is the polytropic index, \( cs_n \) the polytropic sound speed, \( u_0^2 \) denotes the contravariant time component of the four-velocity, and \( \Omega_0 \) indicates the angular velocity of the flow. All these quantities are defined at the center of the torus, \( r = r_0 \).
7. NSs and Their Oblateness

Equation (6) is valid for Kerr spacetimes relevant to BH sources. Within a reasonable accuracy, it can be applied to NS sources as well provided that the NS mass is high. Considering the restrictions on the NS quadrupole moment \( q = Q / M^3 \) given by present NS equations of state and consequent implications on the orbital frequencies (Urbanec et al. 2013; Urbancová et al. 2019), we can estimate the uncertainty in our formula valid for most of the available NS data. For high NS masses (typically \( M \gtrsim 2 M_\odot \)) and spins corresponding up to \( j \sim 0.3 \), the uncertainty in NS mass induced within our formula by the quadrupole moment should not be higher than 3%. On
the other hand, for low NS masses \((M \lesssim 1.4 \, M_\odot)\), this uncertainty may exceed the value of 10%.

A modification of the formula that would be sufficiently valid for such less-compact NS sources can be obtained assuming the Hartle–Thorne geometry (Hartle 1967; Hartle & Thorne 1968), which applies to slowly rotating NSs. One may expect that the impact of the quadrupole moment consideration on the relation between the QPO frequencies can be roughly included substituting the \((rotational) \, 3^2j^2\) term in Equation \((6)\) by a simple dependency on \(q\), \(Q = Q(j, q)\). In the limit of \(q = j^2\), the Hartle–Thorne formula should coincide with those expressed in the Kerr spacetime (e.g., Urbancová et al. 2019) and there is

\[
Q(j, q) = Q(j, j^2) = 3j^2.
\]

In analogy to Section 5, we use the results of analytical calculations of the perturbative solution of the PP equation and compute the numerical solution of the dependence between the expected QPO frequencies. Then we attempt to find the best evaluation of the \(Q\) term, performing numerical calculations of the CT model frequencies in Hartle–Thorne spacetimes. We utilize the results of Fragile et al. (2016) and Kotrlová et al. (2020) and their extension to Hartle–Thorne spacetimes.

We find that the particular term

\[
Q = \frac{1}{3}(17q - 8j^2)
\]

well matches the numerical calculations. This is illustrated in Figure 4. It is also clear from the figure that for \(B = 1\) we obtain the frequencies predicted by the RP model. We note that for high QPO frequencies, corresponding to radii close to the ISCO, when \(\nu_L\) approaches \(\nu_U\), there are discrepancies between the examined relations. These follow from the limitations of the Hartle–Thorne approach, which is accurate up to the second-order terms in \(q\). The inaccuracies, however, grow only when the difference between the two QPO frequencies, \(\Delta \nu = (\nu_U - \nu_L)/\nu_L\), is smaller than 10%.

8. Discussion and Conclusions

For practical purposes, taking into account the ISCO frequency term for nonrotating stars, \(\nu_L = \nu_U = \nu_0\), Equation \((6)\) can be further rewritten into a final compact form,

\[
\nu_L = \nu_0[1 - B\sqrt{1 + 8j\nu_0 - 6\nu_0^{2/3} - 3Q\nu_0^{2/3}}],
\]

where

\[
B = 0.8 - 0.2j, \quad \nu_0 = \frac{\nu_U/\nu_0}{6^{2/3} - j\nu_U/\nu_0}, \quad \nu_0 = 2198 \frac{M_\odot}{M}, \quad Q = \frac{1}{3}(17q - 8j^2).
\]

For the above choice of \(B\), our relation provides frequencies predicted by the CT model with high accuracy. Choosing a constant \(B\), \(B = 1\), it (almost exactly) provides frequencies

![Figure 5](http://example.com/fig5.png)

**Figure 5.** (a) Best fits of the data of the 4U 1636-53 atoll source found for the RP and CT models and a particular choice of NS spin and oblateness. For the other choices within the considered range of parameters, \(j \in [0, 0.4]\) and \(q/j^2 \in [1, 10]\), the resulting fits are similar. (b) The best-fitting mass corresponding to the CT model as it depends on \(q/j^2\). (c) The same as in panel (a) but for the 4U 1735-44 atoll source. (c) The same as in panel (b) but for the 4U 1735-44 atoll source.
predicted by the RP model. We therefore conclude it is applicable for both models in the case of rotating oblate NSs. For \( Q = 3j^2 \), the relation reduces to the case of Kerr spacetimes describing rotating BHs.

8.1. Application to the Atoll Source 4U 1636-53 and Other NSs

Following Török et al. (2016a), we apply Equation (11) to the data of the atoll source 4U 1636-53. The main outputs of our investigation are illustrated in Figure 5. Figure 5(a) includes examples of the best fits given by the CT model. Fits given by the RP model are shown as well for the sake of comparison. Figure 5(b) depicts how the best-fitting \( M \) depends on \( j \) and \( q/j^2 \). It shows that for very compact stars with \( q/j^2 \sim 1 \) the best-fitting \( M \) increases with increasing \( j \), reaching values of \( M \in [2, 2.2] M_\odot \) for \( j \in [0.2, 0.4] M_\odot \). This is in agreement with the investigation of Török et al. (2016a) limited to the case of Kerr spacetimes. On the other hand, for stars of high oblateness, \( q/j^2 > 4 \), the best-fitting \( M \) decreases with increasing \( j \). For stars of moderate oblateness, \( q/j^2 \sim 3 \), there is only a very weak dependency on \( j \) and the estimated mass is around \( M = 1.75 M_\odot \).

The same investigation was performed for the atoll source 4U 1735-44. The results are illustrated in Figure 5 showing a picture very similar to the 4U 1636-53 case. In analogy to the 4U 1636-53 case, we obtain fits better than those of the RP model and the similar quadrupole moment dependence. For very compact stars, \( q/j^2 \sim 1 \), the best-fitting \( M \) increases with increasing \( j \), reaching values of \( M \in [1.9, 2.2] M_\odot \) for \( j \in [0.2, 0.4] M_\odot \), while for stars of high oblateness, \( q/j^2 > 4 \), the best-fitting \( M \) decreases with increasing \( j \). For stars of moderate oblateness, \( q/j^2 \sim 3 \), there is only a very weak dependency on \( j \) and the estimated mass is around \( M = 1.9 M_\odot \). In the same way, we investigated another four atoll sources with a high amount of available data (Barret et al. 2005b, 2005c; Török et al. 2012). Overall, we find that for stars of moderate oblateness, \( q/j^2 \sim 3 \), the mass should be within the interval of \( M \in [1.6, 1.9] M_\odot \).

These findings further confirm the expectation that the CT model not only fits the data better than the RP model, but is also compatible with realistic values of the NS mass.

8.2. Caveats

Our findings on NS mass needs to be expanded to a larger set of sources, namely to a full confrontation of the parameters implied by the model and particular NS equations of state. It should be sufficient if this confrontation is carried out within the framework of the Hartle–Thorne spacetime for most sources and data except for very rapidly rotating sources and data with \( v_u/v_i < 1.2 \). It is questionable whether the present relation can be applied to sources with very strong magnetic fields such as X-ray pulsars. The applicability of our results to rapidly rotating BHs has yet to be explored as well using the full numerical solution of the PP equation.

Despite these caveats, we conclude that the simple Equation (11) can be useful for a brief estimation of the mass and spin of accreting BHs and NSs.

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