ADAPTIVE FINITE ELEMENT METHOD FOR AN ELLIPTIC OPTIMAL
CONTROL PROBLEM WITH INTEGRAL STATE CONSTRAINTS

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Abstract. In this article, we develop a posteriori error analysis of a nonconforming finite element method for a linear quadratic elliptic distributed optimal control problem with two different set of constraints, namely (i) integral state constraint and integral control constraint (ii) integral state constraint and pointwise control constraints. In the analysis, we have taken the approach of reducing the state-control constrained minimization problem into a state minimization problem obtained by eliminating the control variable. The reliability and efficiency of a posteriori error estimator are discussed. Numerical results are reported to illustrate the behavior of the error estimator.

Key words. Elliptic optimal control problem, Fourth order variational inequality, Integral state constraints, Adaptive finite element method

1. Introduction

Optimal control problems (OCPs) play an important role in various applications in physics, mechanics and other engineering sciences. For the theoretical and numerical development of the OCPs, we refer to [55, 41, 44, 31]. The finite element method is a popular and widely used numerical method to approximate OCPs. The finite element approximation of the elliptic optimal control problems started with articles of Falk [23] and Geveci [25]. In these papers, piecewise constant approximation of the control is considered and optimal order error estimates are obtained for the optimal variables. The authors of [3] have established the optimality conditions and introduce the Ritz-Galerkin discretization for elliptic optimal control problem and obtained error estimates for the control and state variables. The authors of [30] have introduced the variational discretization method, therein the error estimates are obtained by exploiting the relationship between the state and adjoint state. The numerical approximation of the elliptic optimal control problems with control variable from measure spaces can be found in [16, 17].

There have been abundant research on the adaptive finite method for OCPs governed by differential equations in last few decades. The use of adaptive techniques based on a posteriori error estimation is well accepted in the context of finite element discretization of partial differential equations [2, 56]. In this direction, the pioneer work has been made by Liu and Yan [43] for residual based a posteriori error estimates, and Becker et al. [4] for dual-weighted goal oriented adaptivity for optimal control problems. In [36] the authors have proved that the sequence of adaptively generated discrete solutions converge to the true solutions of OCPs. Recently, Gong and Yan [27] have presented a rigorous proof for convergence and quasi-optimality of adaptive finite element method for an OCP with pointwise control constraints by means of variational discretization technique. In [57], Wolkmayr has derived functional type a posteriori error estimates for elliptic optimal control problems with control constraints. The authors of [54] have studied the finite element approximation of OCPs governed by elliptic equations with measure data, therein they have derived both a priori and a posteriori error bounds for the
state and control variables. We refer to the reference section for other notable works on the 
adaptive finite element methods for OCPs with control constraints.

In recent years, numerical analysis of OCPs with state constraints has been an active area of 
research. The articles [14, 15, 18, 19, 22, 49, 48] are devoted to the control problems 
with pointwise state constraints. These articles are concentrated on the existence, uniqueness, 
regularity results of the optimal variables and also analyze asymptotic convergence of the errors 
in optimal variables. The authors of [53] have considered a posteriori error estimates for a state-constrained OCP with 
A priori analysis of OCPs with integral state constraint is discussed in [46, 59, 60, 52]. The 
of pure state constraints OCP and derived residual type a posteriori error estimates. To avoid the intrinsic difficulties arising from measure-valued Lagrange multipliers in the case 
authors have used mixed-control state constraints as a relaxation of originally state constrained 
with state and control constraints, and derived reliable a posteriori error estimator. In [32], 
in optimal variables. The authors of [53] have considered the elliptic optimal control problem 
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with pointwise state constraints. These articles are concentrated on the existence, uniqueness, 
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Let \( \Omega \subset \mathbb{R}^2 \) be a convex polygonal domain with smooth boundary \( \partial \Omega \). For any \( 1 \leq p \leq \infty \) and \( D \subset \Omega \), we denote \( L^p(D) \) norm by \( \| \cdot \|_{L^p(D)} \). We adopt the standard notations \( W^{m,p} (\Omega) \) and \( W^{m,p}_0 (\Omega) \) for Sobolev spaces for \( p \in [1, \infty] \) and \( m \geq 0 \) equipped with norm \( \| \cdot \|_{W^{m,p}(\Omega)} \) and seminorm \( | \cdot |_{W^{m,p}(\Omega)} \). When \( p = 2 \), we denote \( W^{m,p}(\Omega) \) by \( H^m(\Omega) \) and \( W^{m,p}_0 (\Omega) \) by \( H^m_0 (\Omega) \) and corresponding norm and seminorm are denoted by \( \| \cdot \|_{H^m(\Omega)} \) and \( | \cdot |_{H^m(\Omega)} \), respectively. We consider the following state and control constrained optimal control problem: to find \( (\hat{u}, \hat{y}) \in L^2(\Omega) \times H^1_0(\Omega) \) such that 

\[
J(\hat{u}, \hat{y}) = \min_{(u, y) \in L^2(\Omega) \times H^1_0(\Omega)} J(u, y)
\]

subject to

\[
\begin{cases}
\int_{\Omega} \nabla y \cdot \nabla w \, dx = \int_{\Omega} w z \, dx & \forall w \in H^1_0(\Omega) \\
\int_{\Omega} u \, dx \geq \delta_1, \\
\int_{\Omega} y \, dx \geq \delta_2,
\end{cases}
\]

where \( J(u, y) = \| y - y_d \|^2_{L^2(\Omega)} + \frac{\beta}{2} \| u \|^2_{L^2(\Omega)} \) with \( y_d \) as the given desired state, \( \delta_1, \delta_2 \in \mathbb{R} \) and \( \beta > 0 \) is a given constant.
The rest of the article is organized as follows. In Section 2, we obtain the characterization of the solution of the optimization problem (1.1)-(1.2) by the solution of a fourth order variational inequality and discuss the optimality conditions of the underlying OCP with integral state as well as integral control constraints. In Section 3, we introduce notations and preliminary results required in the subsequent sections. Therein, we also discuss the finite element discretization of the continuous problem by a bubble enriched Morley finite element method and present the optimality conditions associated to the discrete problem. A posteriori error estimator of the underlying finite element method is introduced in Section 4, followed by that reliability and efficiency estimates are established. In Section 5, we discuss a posteriori error bounds for an OCP with integral state and pointwise control constraints using the proposed finite element method. Finally, in Section 6, we present numerical results to illustrate the performance of derived a posteriori error estimators.

2. CONTINUOUS VARIATIONAL INEQUALITY AND OPTIMALITY CONDITIONS

This section is devoted to characterize the solution of (1.1)-(1.2) by the solution of a variational inequality and discuss the associated optimality conditions.

For \( u \in L^2(\Omega) \), Lax-Milgram lemma [21] ensures the existence of a unique solution \( y \in H^1_0(\Omega) \) satisfying the variational formulation

\[
\int_{\Omega} \nabla y \cdot \nabla w \, dx = \int_{\Omega} uw \, dx \quad \forall \, w \in H^1_0(\Omega).
\]

(2.1)

Moreover, from elliptic regularity theory (cf. [1, 28]) we obtain \( y \in H^2(\Omega) \). Set \( W = H^2(\Omega) \cap H^1_0(\Omega) \). Using \( u = -\Delta y \), we can rewrite the optimization problem (1.1)-(1.2) as follows: to find \( y^* \in K \) such that

\[
y^* = \arg \min_{y \in K} \left[ \frac{1}{2} \int_{\Omega} (y - y_d)^2 \, dx + \frac{\beta}{2} \int_{\Omega} (-\Delta y)^2 \, dx \right]
\]

(2.2)

where \( K \) is defined by

\[
K = \{ w \in W : \int_{\Omega} w \, dx \geq \delta_2 \quad \text{and} \quad \int_{\Omega} (-\Delta w) \, dx \geq \delta_1 \}.
\]

(2.3)

The minimizer of (2.2) can further be characterized by the minimizer of the following optimization problem: find \( y^* \in K \) such that

\[
y^* = \arg \min_{y \in K} \left[ \frac{1}{2} A(y, y) - (y_d, y) \right]
\]

(2.4)

where

\[
A(v, w) = \beta \int_{\Omega} D^2 v : D^2 w \, dx + \int_{\Omega} vw \, dx, \quad v, w \in W,
\]

(2.5)

with \( D^2 v : D^2 w = \sum_{i,j=1}^{2} \left( \frac{\partial^2 v}{\partial x_i \partial x_j} \right) \left( \frac{\partial^2 w}{\partial x_i \partial x_j} \right) \) where \( D^2 \) denotes the Hessian matrix.

We assume the following Slater condition holds [55]: there exists \( y \in W \) satisfying \( \int_{\Omega} y \, dx > \delta_2 \) and \( \int_{\Omega} (-\Delta y) \, dx \geq \delta_1 \). This ensures that the set \( K \) is nonempty, together with closed and convex. Since the bilinear form \( A(\cdot, \cdot) \) is bounded, coercive and symmetric on \( W \), by the standard theory (cf. [24, 35]) there exists a unique solution \( y^* \in K \) of (2.4) satisfying the following variational inequality

\[
A(y^*, w - y^*) \geq \int_{\Omega} y_d(w - y^*) \, dx \quad \forall \, w \in K.
\]

(2.6)
Using the Lagrange multiplier approach, we obtain the following Karush-Kuhn-Tucker conditions (cf. [34, 47]) together with complementarity conditions (2.8)-(2.11): there exist \( \lambda \in \mathbb{R} \) and \( \mu \in \mathbb{R} \) such that

\[
A(y^*, w) = \int_{\Omega} y_d w \, dx - \int_{\Omega} \lambda(\Delta w) \, dx + \int_{\Omega} \mu w \, dx \quad \forall w \in W,
\]

with

\[
\lambda \geq 0, \quad \text{if} \quad \int_{\Omega} (-\Delta y^*) \, dx = \delta_1,
\]

\[
\lambda = 0, \quad \text{if} \quad \int_{\Omega} (-\Delta y^*) \, dx > \delta_1,
\]

\[
\mu \geq 0, \quad \text{if} \quad \int_{\Omega} y^* \, dx = \delta_2,
\]

\[
\mu = 0, \quad \text{if} \quad \int_{\Omega} y^* \, dx > \delta_2.
\]

Note that, the adjoint state \( p \in H^1_0(\Omega) \) satisfy

\[
\int_{\Omega} \nabla p \cdot \nabla w \, dx = \int_{\Omega} (y^* - y_d) w \, dx - \int_{\Omega} \mu w \, dx \quad \forall w \in H^1_0(\Omega).
\]

3. Notations and Finite Element Discretization

In this section, we introduce the discrete control problem and present some useful tools required for subsequent analysis. Let \( \mathcal{T}_h \) be a regular triangulation of the domain \( \Omega \). The following notations will be used throughout this article.

\( \mathcal{T}_e \): set of elements in \( \mathcal{T}_h \) that share the common edge \( e \),

\( h_T \): the diameter of the triangle \( T \), \( h = \max_{T \in \mathcal{T}_h} h_T \)

\( V_h \): set of all vertices of \( \mathcal{T}_h \),

\( V_T \): set of three vertices of \( T \),

\( \mathcal{E}_h = \mathcal{E}_h^i \cup \mathcal{E}_h^b \): the set of the edges of the triangle in \( \mathcal{T}_h \), where \( \mathcal{E}_h^i \) (resp., \( \mathcal{E}_h^b \)) is the subset of \( \mathcal{E}_h \) consisting of edges interior to \( \Omega \) (resp., along \( \partial \Omega \)),

\( h_e \): length of an edge \( e \in \mathcal{E}_h \)

\( \Delta_h \): piecwise (element-wise) Laplacian operator

\( \mathbb{P}_k(T) \): space of polynomials defined on \( T \) of degree less than or equal to \( k \), \( k \geq 0 \) integer,

\( X \lesssim Y \): there exists a positive constant \( C \) (independent of mesh parameter) such that \( X \leq CY \),

\( X \approx Y \): there exists positive constants \( C_1 \) and \( C_2 \) such that \( C_1 Y \leq X \leq C_2 Y \).

Throughout this article, the constant \( C \) will denote a positive generic constant.

We denote by \( H^k(\Omega, \mathcal{T}_h) \) the broken Sobolev space

\[
H^k(\Omega, \mathcal{T}_h) := \{ w \in L^2(\Omega) : w_T = w|_T \in H^k(T) \quad \forall T \in \mathcal{T}_h \}.
\]

Let \( e \in \mathcal{E}_h^i \) be the common side shared by elements \( T_+ \) and \( T_- \). Further, suppose \( n_+ \) is the unit normal of \( e \) pointing from \( T_+ \) to \( T_- \), and \( n_- = -n_+ \). For any scalar valued function \( w \in H^2(\Omega, \mathcal{T}_h) \), we define the jumps \([ \cdot ]\), and averages \( \{ \{ \cdot \} \} \) across the edge \( e \) as follows:

\[
\left[ \frac{\partial w}{\partial n} \right] = \frac{\partial w_+}{\partial n} \bigg|_e - \frac{\partial w_-}{\partial n} \bigg|_e \quad \text{and} \quad \left\{ \left\{ \frac{\partial w}{\partial n} \right\} \right\} = \frac{1}{2} \left( \frac{\partial w_+}{\partial n} \bigg|_e + \frac{\partial w_-}{\partial n} \bigg|_e \right).
\]
For any $w \in H^3(\Omega, \mathcal{T}_h)$, we define

$$
\left[ \frac{\partial^2 w}{\partial n^2} \right]_e = \frac{\partial^2 w_+}{\partial n^2} - \frac{\partial^2 w_-}{\partial n^2} \\
\left( \frac{\partial^2 w}{\partial n^2} \right)_e = \frac{1}{2} \left( \frac{\partial^2 w_+}{\partial n^2} + \frac{\partial^2 w_-}{\partial n^2} \right),
$$

where $w_\pm = w|_{T_\pm}$. For $e \in \mathcal{E}_h$, we choose $n$ be the unit outward normal of $e$ and let $T \in \mathcal{T}_h$ be such that $e = \partial T \cap \partial \Omega$. Set

$$
\left[ \frac{\partial w}{\partial n} \right]_e = \frac{\partial w|_T}{\partial n} \text{ for any } w \in H^2(\Omega, \mathcal{T}_h).
$$

Before introducing the finite element spaces, we define for each triangle $T \in \mathcal{T}_h$ a cubic bubble function $b_T \in \mathbb{P}_3(T)$ by

$$
b_T = 60\lambda^T_1 \lambda^T_2 \lambda^T_3,
$$

where $\lambda^T_i$, $i = 1, 2, 3$ are the barycentric coordinates of $T$ associated with the vertices $p_i \in V_T$.

**Discrete Spaces:** Let $V_M$ denote the Morley finite element space [50] defined by

$$
V_M = \{ w_h \in L^2(\Omega) : w_h|_T \in \mathbb{P}_2(T) \; \forall \; T \in \mathcal{T}_h, \; w_h \text{ is continuous at the vertices of } \mathcal{T}_h, \; \text{and the normal derivative of } w_h \text{ is continuous at the midpoint of the edges of } \mathcal{T}_h, \; \text{and } w_h \text{ vanish on } \partial \Omega \},
$$

and define the space $V_h$ as

$$
V_h = \{ w_h \in L^2(\Omega) : w_h|_T \in \text{span}(b_T) \; \forall \; T \in \mathcal{T}_h \}.
$$

The finite element space $W_h$ is defined as

$$
W_h = V_M \oplus V_h.
$$

The discrete norm $\| \cdot \|_h$ on $W_h$ is defined by

$$
\|w_h\|^2_h = \beta \sum_{T \in \mathcal{T}_h} |w_h|^2_{L^2(T)} + \|w_h\|^2_{L^2(\Omega)} \; \text{ for } w_h \in W_h.
$$

The discrete approximation of the convex set $\mathcal{K}$ is then given by

$$
\mathcal{K}_h = \{ w_h \in W_h : \int_{\Omega} w_h \, dx \geq \delta_2 \ \text{ and } \int_{\Omega} (-\Delta w_h) \, dx \geq \delta_1 \}.
$$

Next, we define the projection, interpolation and enriching operators and tabulate their approximation properties required in further analysis.

**Discrete Operators:** For any $T \in \mathcal{T}_h$ and $w \in L^1(T)$, define

$$
Q_T(w) = \frac{1}{|T|} \int_T w \, dx.
$$

Let $W_{pc,h} := \{ w \in L^1(\Omega) : w|_T \in \mathbb{P}_0(T) \; \forall \; T \in \mathcal{T}_h \}$. Then, $Q_h : L^1(\Omega) \to W_{pc,h}$ is defined by setting $Q_h(w)|_T = Q_T(w)$ for all $w \in L^1(\Omega)$, $T \in \mathcal{T}_h$.

Define interpolation operator $I_h : W \to W_h$ as: for $\xi \in W$,

$$
(I_h \xi)(p) = \xi(p) \; \forall p \in V_h,
$$

$$
\int_e \frac{\partial (I_h \xi)}{\partial n} \, ds = \int_e \frac{\partial \xi}{\partial n} \, ds, \; \forall e \in \mathcal{E}_h,
$$

$$
Q_T(I_h \xi) = Q_T(\xi), \; \forall T \in \mathcal{T}_h.
$$

The interpolation operator is well-defined and $(I_h w)|_T = w$ for any $w \in \mathbb{P}_2(T)$. 


For any $\xi \in W$, using (3.5) we find
\begin{equation}
\int_{\Omega} I_h \xi \, dx = \int_{\Omega} \xi \, dx.
\end{equation}

Further, a use of integration by parts and (3.4) yields
\begin{equation}
\int_{T} \Delta(I_h \xi) \, dx = \sum_{e \in \partial T} \int_{e} \frac{\partial (I_h \xi)}{\partial n} \, ds = \int_{T} (\Delta \xi) \, dx \quad \forall T \in \mathcal{T}_h,
\end{equation}
which implies that
\begin{equation}
Q_h(\Delta_h(I_h \xi)) = Q_h(\Delta \xi), \quad \forall \xi \in W.
\end{equation}
In view of (2.3), (3.1), (3.6) and (3.8), we have
\begin{equation}
E_h \in K_{\mathcal{T}_h},
\end{equation}

This relation also depicts that the discrete set $K_{\mathcal{T}_h}$ is non-empty. We would like to remark here that enriching the Morley finite element space $V_M$ by the bubble function space $V_h$ plays a crucial role in obtaining (3.9).

Below, we state the stability and approximation properties of $I_h$, whose proof follows by using Bramble Hilbert lemma and scaling arguments; see [8, 21] for details.

**Lemma 3.1.** Let $T \in \mathcal{T}_h$ and $s$ be an integer such that $0 \leq s \leq 2$ and $\psi \in H^s(T)$. Then,
\begin{align}
|I_h \psi|_{H^s(T)} & \lesssim |\psi|_{H^s(T)}, & 0 \leq s \leq 2 \\
\sum_{k=0}^{s} & h_T^{k-s} |\psi - I_h \psi|_{H^k(T)} \lesssim |\psi|_{H^s(T)}, & 0 \leq s \leq 2.
\end{align}

Now we define an important tool for the analysis, the enriching operator $E_h : W_h \to (M_h \oplus V_h) \cap W$, where $M_h$ is the Hsieh-Clough-Tocher macro element space [21] associated with $\mathcal{T}_h$.

The operator $E_h$ can be constructed by averaging techniques (cf. [13, 12, 52]) satisfying
\begin{equation}
\int_{e} \frac{\partial (E_h w_h)}{\partial n} \, ds = \int_{e} \frac{\partial w_h}{\partial n} \, ds \quad \forall e \in \mathcal{E}_h,
\end{equation}
\begin{equation}
\text{and} \quad \int_{T} E_h w_h \, dx = \int_{T} w_h \, dx \quad \forall T \in \mathcal{T}_h.
\end{equation}

An application of integration by parts and (3.12) leads to
\begin{equation}
Q_h(\Delta E_h w_h) = Q_h(\Delta_h w_h).
\end{equation}
Moreover, the enriching operator satisfies the following approximation properties (cf. [12]).

**Lemma 3.2.** For any $w_h \in W_h$, we have
\begin{align}
\sum_{T \in \mathcal{T}_h} \left( h_T^{-4} ||w_h - E_h w_h||_{L^2(T)}^2 + h_T^{-2} |w_h - E_h w_h|_{H^1(T)}^2 + |w_h - E_h w_h|_{H^2(T)}^2 \right) & \lesssim ||w_h||_{H^1}^2, \\
\sum_{T \in \mathcal{T}_h} |w_h - E_h w_h|_{H^2(T)}^2 & \lesssim \sum_{e \in \mathcal{E}_h} \frac{1}{h_T} \left[ \left\| \frac{\partial w_h}{\partial n} \right\|_{L^2(e)} \right]^2 \forall w_h \in W_h.
\end{align}

We recall the following inverse and trace inequalities which will be useful in later analysis [21].

**Inverse Inequalities:** For any $w_h \in W_h$ and $1 \leq p, q < \infty$,
\begin{align}
||w_h||_{W^{m,p}(T)} & \lesssim h_T^{m - \frac{2}{p}} ||w_h||_{W^{m,q}(T)} \quad \forall T \in \mathcal{T}_h, \\
||\nabla w_h||_{L^p(T)} & \lesssim h_T^{-1} ||w_h||_{L^p(T)} \quad \forall T \in \mathcal{T}_h.
\end{align}
Discrete trace inequality: Let \( \psi \in W^{1,p}(T) \), \( T \in T_h \) and let \( e \in E_h \) be an edge of \( T \). Then for any \( 1 \leq p < \infty \), it holds that

\[
\| \psi \|_{L^p(e)}^p \lesssim h_e^{-1}(\| \psi \|_{L^p(T)}^p + h_e^p \| \nabla \psi \|_{L^p(T)}^p).
\]

Discrete Problem: The discrete form of the minimization problem (2.4) is defined as follows: Find \( y^*_h \in K_h \) such that

\[
y^*_h = \arg\min_{y_h \in K_h} \left( \frac{1}{2} A_h(y_h, y_h) - (y_d, y_h) \right),
\]

where

\[
A_h(w_h, v_h) = \beta \sum_{T \in T_h} \int_T D^2 w_h : D^2 v_h \, dx + \int_\Omega w_h v_h \, dx, \quad w_h, v_h \in W_h.
\]

Since \( K_h \) is non-empty, closed, convex and the bilinear form \( A_h(\cdot, \cdot) \) is symmetric and positive definite on \( W_h \), the discrete problem (3.18) is well-posed and its solution is characterized by the solution of the discrete variational inequality

\[
A_h(y^*_h, w_h - y^*_h) \geq (y_d, w_h - y^*_h) \quad \forall w_h \in K_h.
\]

As in the case of the continuous problem, we have the following optimality conditions associated with the discrete problem:

Lemma 3.3. Let \( y^*_h \in K_h \) be the optimal solution of the discrete problem, then there exists Lagrange multipliers \( \lambda_h \in \mathbb{R} \) and \( \mu_h \in \mathbb{R} \) such that the following conditions hold:

\[
A_h(y^*_h, w_h) - \int_\Omega y_d w_h \, dx = \int_\Omega \mu_h w_h \, dx - \int_\Omega \lambda_h (\Delta_h w_h) \, dx, \quad \forall w_h \in W_h,
\]

together with

\[
\mu_h \geq 0, \quad \lambda_h \geq 0,
\]

\[
\mu_h (\delta_2 - \int_\Omega y^*_h \, dx) = 0,
\]

\[
\lambda_h (\delta_1 + \int_\Omega \Delta_h y^*_h \, dx) = 0.
\]

4. A Posteriori Error Analysis

In this section we introduce a posteriori error estimator and present the first main result of the paper, namely, the reliability analysis of the error estimator. Followed by that, we also discuss the efficiency estimates of a posteriori error estimator. The contributions of error
estimator are defined by

\[ \eta_1^2 = \beta^{-1} \sum_{T \in T_h} h_T^4 \| y_d + \mu_h - y_h^* \|_{L^2(T)}^2, \]
\[ \eta_2^2 = \beta \sum_{e \in E_h} h_e \left\| \frac{\partial y_h^*}{\partial n} \right\|_{L^2(e)}^2, \]
\[ \eta_3^2 = \beta \sum_{e \in E_h} h_e \left\| \frac{\partial^2 y_h^*}{\partial n^2} \right\|_{L^2(e)}^2, \]
\[ \eta_4^2 = \beta \sum_{e \in E_h} h_e \left\| \frac{\partial (\Delta y_h^*)}{\partial n} \right\|_{L^2(e)}^2, \]
\[ \eta_5^2 = \beta^{-1} \sum_{T \in T_h} h_T^2 |\lambda_h|^2. \]

The full error estimator \( \eta_h \) is given by

\[ (4.1) \quad \eta_h^2 = \eta_1^2 + \eta_2^2 + \eta_3^2 + \eta_4^2 + \eta_5^2. \]

4.1. Reliability of Error Estimator. Below, we establish the reliability estimates of a posteriori error estimator \( \eta_h \).

**Theorem 4.1.** Let \( y^* \) and \( y_h^* \) be solutions of variational inequalities \((2.6)\) and \((3.20)\), respectively. Then, it holds that,

\[ \| y^* - y_h^* \|_h \lesssim \eta_h. \]

**Proof.** We set \( \phi = y^* - E_h y_h^* \in W \) and let \( \phi_h \in W_h \). Using the coercive property of the bilinear form \( A(\cdot, \cdot) \), \((2.7)\) and \((3.21)\) we obtain

\[ \| y^* - E_h y_h^* \|_h^2 \lesssim A(y^* - E_h y_h^*, \phi) = A(y^*, \phi) - A_h(y_h^*, \phi) + A_h(y_h^* - E_h y_h^*, \phi) \]
\[ \lesssim \int_{\Omega} y_d \phi \, dx - \int_{\Omega} \lambda(\Delta \phi) \, dx + \int_{\Omega} \mu \phi \, dx - A_h(y_h^*, \phi) + \lambda_h(y_h^* - E_h y_h^*, \phi) \]
\[ \lesssim \int_{\Omega} y_d(\phi - \phi_h) \, dx - \int_{\Omega} \lambda(\Delta \phi) \, dx + \int_{\Omega} \mu \phi \, dx + \int_{\Omega} \lambda_h(\Delta h \phi_h) \, dx - \int_{\Omega} \mu \phi_h \, dx \]
\[ - A_h(y_h^*, \phi - \phi_h) + \lambda_h(y_h^* - E_h y_h^*, \phi) \]
\[ \lesssim \int_{\Omega} y_d(\phi - \phi_h) \, dx - \int_{\Omega} \mu h(\phi_h - \phi) \, dx - A_h(y_h^*, \phi - \phi_h) - \int_{\Omega} (\lambda - \lambda_h)(\Delta \phi) \, dx \]
\[ + \int_{\Omega} (\mu - \mu_h) \phi \, dx + \int_{\Omega} \lambda h \Delta h (\phi_h - \phi) \, dx + \lambda h(y_h^* - E_h y_h^*, \phi). \]

(4.2)

Now, we bound the terms of the right hand side of the last estimate. The estimation of first three terms is discussed towards the end. We first handle the rest terms other than the first three terms. For the fourth term in \((4.2)\), using the discrete and continuous complementarity
conditions we get

\[
\int_{\Omega} (\lambda - \lambda_h)(-\Delta \phi) \, dx = \int_{\Omega} (\lambda - \lambda_h)(-\Delta (y^* - E_h y^*_h)) \, dx
\]

\[
= \int_{\Omega} \lambda(-\Delta y^* + \Delta E_h y^*_h) \, dx - \int_{\Omega} \lambda_h(-\Delta y^* + \Delta E_h y^*_h) \, dx
\]

\[
= \lambda \left( \int_{\Omega} (-\Delta y^*) \, dx - \delta_1 \right) + \lambda \left( \delta_1 + \int_{\Omega} \Delta E_h y^*_h \, dx \right)
\]

\[
- \lambda_h \left( \int_{\Omega} (-\Delta y^*) \, dx - \delta_1 \right) - \lambda_h \left( \delta_1 + \int_{\Omega} \Delta E_h y^*_h \, dx \right)
\]

\[
\leq \lambda \left( \delta_1 + \int_{\Omega} \Delta h y^*_h \, dx \right) - \lambda_h \left( \delta_1 + \int_{\Omega} \Delta h y^*_h \, dx \right)
\]

\[
\leq 0,
\]

where in obtaining second last estimate we have used that \(\int_{\Omega} \Delta E_h y^*_h \, dx = \int_{\Omega} \Delta y^*_h \, dx\),

\[
\lambda_h \left( \int_{\Omega} (-\Delta y^*) \, dx - \delta_1 \right) \geq 0
\]

and \(\lambda \left( \delta_1 + \int_{\Omega} \Delta h y^*_h \, dx \right) \leq 0\).

Next, we handle the fifth term of right hand side of (4.2). A use of (2.10), (2.11), (3.13) together with \(\mu_h \geq 0\), \(\int_{\Omega} y^* \, dx \geq \delta_2\) and \(\mu \left( \delta_2 - \int_{\Omega} y^*_h \, dx \right) \leq 0\), yields

\[
\int_{\Omega} (\mu - \mu_h) \phi \, dx = \int_{\Omega} \mu(y^* - E_h y^*_h) \, dx - \int_{\Omega} \mu_h(y^* - E_h y^*_h) \, dx
\]

\[
= \mu \left( \int_{\Omega} y^* \, dx - \delta_2 \right) + \mu \left( \delta_2 - \int_{\Omega} E_h y^*_h \, dx \right)
\]

\[
- \mu_h \left( \int_{\Omega} y^* \, dx - \delta_2 \right) - \mu_h \left( \delta_2 - \int_{\Omega} E_h y^*_h \, dx \right)
\]

\[
\leq \mu \left( \delta_2 - \int_{\Omega} E_h y^*_h \, dx \right) - \mu_h \left( \delta_2 - \int_{\Omega} E_h y^*_h \, dx \right)
\]

\[
\leq \mu \left( \delta_2 - \int_{\Omega} y^*_h \, dx \right) - \mu_h \left( \delta_2 - \int_{\Omega} y^*_h \, dx \right)
\]

\[
\leq 0.
\]

The estimate on the last two terms of (4.2) can be realized with an application of Lemmas 3.2 and 3.1 as,

\[
A_h(y^*_h - E_h y^*_h, \phi) \leq \|y^*_h - E_h y^*_h\|_h \|\phi\|_h \leq \eta_2 \|\phi\|_h,
\]

and

\[
\int_{\Omega} \lambda_h \Delta h (\phi_h - \phi) \, dx = \sum_{T \in T_h} |\lambda_h| \int_T |\Delta (\phi_h - \phi)| \, dx \leq \sum_{T \in T_h} h_T |\lambda_h| \|\Delta (\phi_h - \phi)\|_{L^2(T)}
\]

\[
\leq \sum_{T \in T_h} h_T |\lambda_h| \|\phi\|_{H^2(T)} \leq \eta_5 \|\phi\|_h.
\]
We now proceed to handle the first three terms of (4.2). Performing integration by parts twice yields

\[ A_h(y_h, \phi - \phi_h) = \beta \sum_{T \in \mathcal{T}_h} \int_T D^2 y_h^* : D^2 (\phi - \phi_h) \, dx + \int_{\Omega} y_h (\phi - \phi_h) \, dx \]

\[ = \beta \sum_{T \in \mathcal{T}_h} \int_T \Delta^2 y_h^* (\phi - \phi_h) \, dx + \beta \sum_{T \in \mathcal{T}_h} \int_{\partial T} \left( \frac{\partial^2 y_h^*}{\partial n^2} \right) \left( \frac{\partial (\phi - \phi_h)}{\partial n} \right) \, ds \]

\[ + \beta \sum_{T \in \mathcal{T}_h} \int_{\partial T} \left( \frac{\partial^2 y_h^*}{\partial n_1} \right) \left( \frac{\partial (\phi - \phi_h)}{\partial n} \right) \, ds - \beta \sum_{T \in \mathcal{T}_h} \int_{\partial T} \left( \frac{\partial^2 y_h^*}{\partial n_2} \right) (\phi - \phi_h) \, ds \]

\[ + \int_{\Omega} y_h (\phi - \phi_h) \, dx. \]

(4.7)

Thus,

\[ \int_{\Omega} y_d (\phi - \phi_h) \, dx - \int_{\Omega} \mu_h (\phi_h - \phi) \, dx - A_h(y_h^*, \phi - \phi_h) = \int_{\Omega} (y_d + \mu_h - y_h^*) (\phi - \phi_h) \, dx \]

\[ - \beta \sum_{e \in \mathcal{E}_h} \int_e \left[ \frac{\partial \Delta y_h^*}{\partial n} \right] (\phi - \phi_h) \, ds + \beta \sum_{e \in \mathcal{E}_h} \int_e \left[ \frac{\partial^2 y_h^*}{\partial n^2} \right] \left\{ \frac{\partial (\phi - \phi_h)}{\partial n} \right\} \, ds \]

\[ + \beta \sum_{e \in \mathcal{E}_h} \int_e \left\{ \frac{\partial^2 y_h^*}{\partial n_1} \right\} \left[ \frac{\partial (\phi - \phi_h)}{\partial n} \right] \, ds + \beta \sum_{e \in \mathcal{E}_h} \int_e \left[ \frac{\partial^2 y_h^*}{\partial n_2} \right] \frac{\partial (\phi - \phi_h)}{\partial t} \, ds. \]

Now, we estimate the terms of right hand side as follows: a use of the Cauchy-Schwarz inequality and Lemma 3.1 yields

\[ \left| \int_{\Omega} (y_d + \mu_h - y_h^*) (\phi - \phi_h) \, dx \right| \leq \sum_{T \in \mathcal{T}_h} h_T^2 \| y_d + \mu_h - y_h^* \|_{L^2(T)} h_T^{-2} \| \phi - \phi_h \|_{L^2(T)} \]

\[ \leq \sum_{T \in \mathcal{T}_h} h_T^2 \| y_d + \mu_h - y_h^* \|_{L^2(T)} \| \phi \|_{H^2(T)} \]

(4.8)

Using Cauchy-Schwarz inequality, discrete trace inequality and Lemma 3.1 we find

\[ \left| \beta \sum_{e \in \mathcal{E}_h} \int_e \left[ \frac{\partial^2 y_h^*}{\partial n^2} \right] \left\{ \frac{\partial (\phi - \phi_h)}{\partial n} \right\} \, ds \right| \leq \]

\[ \beta \left( \sum_{e \in \mathcal{E}_h} \| \frac{\partial^2 y_h^*}{\partial n^2} \|_{L^2(e)}^2 \right)^{1/2} \left( \sum_{e \in \mathcal{E}_h} h_e^{-1} \| \frac{\partial (\phi - \phi_h)}{\partial n} \|_{L^2(e)}^2 \right)^{1/2} \]

\[ \leq \eta_3 \| \phi \|_h. \]

(4.9)
Invoking Lemma 3.1 together with inverse inequality and discrete trace inequality (3.17), we obtain

$$\left| \beta \sum_{e \in E_h^i} \int_e \left[ \frac{\partial^2 y_h^*}{\partial n \partial t} \right] \frac{\partial (\phi - \phi_h)}{\partial t} \, ds \right| \leq$$

$$\beta \left( \sum_{e \in E_h^i} h_e \left\| \left[ \frac{\partial^2 y_h^*}{\partial n \partial t} \right] \right\|_{L^2(e)} \right)^{\frac{1}{2}} \left( \sum_{e \in E_h^i} h_e^{-1} \left\| \frac{\partial (\phi - \phi_h)}{\partial t} \right\|_{L^2(e)} \right)^{\frac{1}{2}}$$

(4.10)

$$\lesssim \eta_2 \| \phi \|_h$$

and,

$$\left| \beta \sum_{e \in E_h^i} \int_e \left[ \frac{\partial (\Delta y_h^*)}{\partial n} \right] (\phi - \phi_h) \, ds \right| \leq$$

$$\beta \left( \sum_{e \in E_h^i} h_e^3 \left\| \left[ \frac{\partial (\Delta y_h^*)}{\partial n} \right] \right\|_{L^2(e)} \right)^{\frac{1}{2}} \left( \sum_{e \in E_h^i} h_e^{-3} \left\| \phi - \phi_h \right\|^2_{L^2(e)} \right)^{\frac{1}{2}}$$

(4.11)

$$\lesssim \eta_4 \| \phi \|_h.$$

Finally, combining the estimates (4.3)-(4.11) together with (4.2), we get the desired result. ∎

In order to obtain the reliability estimates for Lagrange multiplier errors $|\mu - \mu_h|$ and $|\lambda - \lambda_h|$, we introduce the auxiliary variables $z_h \in W$ and $\hat{p}_h \in H^1_0(\Omega)$ satisfying the following equations.

(4.12) \[ \mathcal{A}(z_h, w) = \int_\Omega y_d w \, dx + \int_\Omega \lambda_h (-\Delta w) \, dx + \int_\Omega \mu_h w \, dx \quad \forall w \in W, \]

and

(4.13) \[ \int_\Omega \nabla \hat{p}_h \cdot \nabla w \, dx = \int_\Omega (z_h - y_d) w \, dx - \int_\Omega \mu_h w \, dx \quad \forall w \in H^1_0(\Omega). \]

The well-posedness of these auxiliary problems 4.12 and 4.13 follows from Lax-Milgram lemma [21]. These auxiliary problems help in estimating the errors in Lagrange multipliers. In the next lemma, we estimate the error $\|z_h - y^*\|_h$.

**Lemma 4.2.** There exists a positive constant $C$, depending only on the shape regularity of $T_h$, such that

$$\|z_h - y^*\|_h \leq C \eta_h.$$

**Proof.** We have,

(4.14) \[ \|z_h - y^*\|_h \lesssim \left( \|z_h - y_h^*\|_h + \|y_h^* - y^*\|_h \right). \]

The estimation of $\|z_h - y_h^*\|_h$ follows in similar steps as in Theorem 4.1. For completeness, we briefly discuss the proof. A use of triangle inequality gives

(4.15) \[ \|z_h - y_h^*\|_h \leq \|z_h - E_h y_h^*\|_h + \|E_h y_h^* - y_h^*\|_h. \]
For $\phi = z_h - E_h \psi_h \in W$ and $\phi_h = I_h \phi \in W_h$, a use of coercive property of $A(\cdot, \cdot)$, (3.21) and (4.12) leads to

$$\|z_h - E_h \psi_h\|_h^2 = \|\phi\|_h^2 \lesssim A(z_h - E_h \psi_h, \phi) \lesssim A(z_h, \phi) - A_h(y_h, \phi) + A_h(y_h - E_h \psi_h, \phi) \lesssim \int_\Omega y_d(\phi - \phi_h) \, dx$$

(since $\phi_h$ is the solution of $A_h(y_h, \phi_h) = 0$).

Using the fact

$$\|\phi\|_h \lesssim \|\phi\|_h$$

(4.19)

Next, we show that the error in Lagrange multipliers can be estimated in terms of $\|z_h - y^*_h\|_h$.

**Lemma 4.3.** There exists a positive constant $C$ depending only on the shape regularity of $T_h$, such that

$$|\mu - \mu_h| \leq C\|y^* - z_h\|_{L^2(\Omega)},$$

(4.17)

$$|\lambda - \lambda_h| \leq C\|y^* - z_h\|_h.$$  

(4.18)

**Proof.** Upon subtracting (2.12) and (4.13), we find

$$\int_\Omega \nabla(p - \hat{p}_h) \cdot \nabla w \, dx = \int_\Omega (y^* - z_h) w \, dx - \int_\Omega (\mu - \mu_h) w \, dx \quad \forall w \in H^1_0(\Omega).$$

(4.19)

We choose the cut-off function $\psi \in C_0^\infty(\Omega)$ with $\frac{1}{|\Omega|} \int_\Omega \psi \, dx = 1$ and $\|\psi\|_{H^1(\Omega)} \leq C$. Let $\hat{C} = \frac{1}{|\Omega|} \int_\Omega (p - \hat{p}_h) \, dx$, we observe that $\hat{C} \psi \in C_0^\infty(\Omega) \subset H^1_0(\Omega)$. Take $w = p - \hat{p}_h - \hat{C} \psi \in H^1_0(\Omega)$ in (4.19) to obtain

$$\int_\Omega \nabla(p - \hat{p}_h) \cdot \nabla(p - \hat{p}_h - \hat{C} \psi) \, dx = \int_\Omega (y^* - z_h)(p - \hat{p}_h - \hat{C} \psi) \, dx - \int_\Omega (\mu - \mu_h)(p - \hat{p}_h - \hat{C} \psi) \, dx.$$  

Using the fact $\mu - \mu_h \in \mathbb{R}$ and $\int_\Omega (p - \hat{p}_h - \hat{C} \psi) \, dx = 0$, we obtain

$$\|\nabla(p - \hat{p}_h)\|^2 = \int_\Omega \nabla(p - \hat{p}_h) \hat{C} \psi \, dx + \int_\Omega (y^* - z_h)(p - \hat{p}_h - \hat{C} \psi) \, dx$$

$$\lesssim \|\nabla(p - \hat{p}_h)\|_{L^2(\Omega)} \|\hat{C} \psi\|_{L^2(\Omega)} + \|y^* - z_h\|_{L^2(\Omega)} \|p - \hat{p}_h - \hat{C} \psi\|_{L^2(\Omega)}$$

$$\lesssim \|\nabla(p - \hat{p}_h)\|_{L^2(\Omega)} \|\hat{C} \psi\|_{L^2(\Omega)} + \|y^* - z_h\|_{L^2(\Omega)} \|p - \hat{p}_h\|_{L^2(\Omega)}$$

(4.20)

$$+ \|p - \hat{p}_h\|_{L^2(\Omega)} \|\hat{C} \psi\|_{L^2(\Omega)}.$$
Upon summing up for all \( T \), the following integration by parts formula estimates and skipped the standard details. For \( z \) function techniques \[5(6)\]. We have discussed the main ideas involved in proving these efficiency estimates. Finally, in view of lemma 4.2 and 4.3, we have the estimation of error in Lagrange multipliers by the error estimator \( \eta_h \).

### 4.2. Local Efficiency Estimates

In this section, we derive the local efficiency estimates of a posteriori error estimator \( \eta_h \) obtained in last subsection. \( \text{Th} \), we use the standard bubble function techniques \[5(6)\]. We have discussed the main ideas involved in proving these efficiency estimates and skipped the standard details. For \( z \in H^1(\Omega, T_h) \) and \( v \in H^2(\Omega, T_h) \), we have the following integration by parts formula.

\[
\int_T (\Delta z)v \, dx = \int_T \Delta z : D^2v \, dx + \int_{\partial T} \frac{\partial \Delta z}{\partial n} v \, ds - \int_{\partial T} \frac{\partial^2 z}{\partial n \partial t} \frac{\partial v}{\partial t} \, ds - \int_{\partial T} \frac{\partial^2 z}{\partial n^2} \frac{\partial v}{\partial n} \, ds.
\]

Upon summing up for all \( T \in T_h \), we obtain

\[
\sum_{T \in T_h} \int_T \Delta^2 z v \, dx = \sum_{T \in T_h} \int_T \Delta z : D^2v \, dx + \sum_{e \in \mathcal{E}_h} \int_e \left\{ \frac{\partial \Delta z}{\partial n} \right\} [v] \, ds
\]

\[
+ \sum_{e \in \mathcal{E}_h} \int_e \left\{ \frac{\partial^2 z}{\partial n^2} \right\} \left\{ \frac{\partial v}{\partial n} \right\} ds + \sum_{e \in \mathcal{E}_h} \int_e \left\{ \frac{\partial^2 z}{\partial n \partial t} \right\} \left\{ \frac{\partial v}{\partial t} \right\} ds
\]

\[
+ \sum_{e \in \mathcal{E}_h} \int_e \left\{ \frac{\partial^2 z}{\partial n \partial t} \right\} \left\{ \frac{\partial v}{\partial t} \right\} ds.
\]

**Theorem 4.4.** It holds that,

\[
\int_T (\Delta - \lambda)(\Delta w) \, dx = A(z_h - y^*, w) + \int_T (\mu - \mu_h) w \, dx \quad \forall w \in W.
\]

Then, the estimate (4.18) can be realized from (4.22) and (4.17).

Finally, in view of lemma 4.2 and 4.3, we have the estimation of error in Lagrange multipliers by the error estimator \( \eta_h \).
where \( \|w\|_{2,T} := \beta |w|_{H^2(T)} + h_T^2 \|w\|_{L^2(T)} \) for any \( w \in H^2(\Omega, T_h) \), \( \text{Osc}(y_d; T) := h_T^2 \|y_d - \bar{y}_d\|_{L^2(T)}^2 \) with \( \bar{y}_d := \frac{1}{T} \int_T y_d \, dx \) and \( T_e \) denotes the union of elements sharing the edge \( e \).

**Proof.** (i) **(Local bound for \( \eta_1 \))** Let \( \tilde{b}_T \) be a polynomial bubble function vanishing up to the first order on \( \partial T \), i.e., \( \tilde{b}_T \) and \( \nabla \tilde{b}_T \) vanish on \( \partial T \), and set \( \phi_T = (\bar{y}_d + \mu_h - y_h^*) \tilde{b}_T \). Let \( \tilde{\phi} \) be the extension of \( \phi_T \) to \( \Omega \) by zero, clearly \( \tilde{\phi} \in W \). We further have,

\[
\|(\phi_T)_T\|_{L^2(T)} \lesssim \|y_d + \mu_h - y_h^*\|_{L^2(T)},
\]

and

\[
\|y_d + \mu_h - y_h^*\|^2_{L^2(T)} \approx \int_T (y_d + \mu_h - y_h^*) \phi_T \, dx
\]

\[
= \int_T (y_d + \mu_h - y_h^*) \phi_T \, dx + \int_T (\bar{y}_d - y_d) \phi_T \, dx.
\]

Using equation (2.7) and the fact that, \( \beta \int \Omega D^2 \bar{y}_h : D^2 \bar{\phi} \, dx = \int \Omega \lambda_h (\Delta \bar{\phi}) \, dx \), we have

\[
\int_T (y_d + \mu_h - y_h^*) \phi_T \, dx = \beta \int \Omega D^2 y^* : D^2 \bar{\phi} \, dx + \int \Omega y^* \bar{\phi} \, dx + \int \Omega \lambda (\Delta \bar{\phi}) \, dx
\]

\[
- \int \Omega \mu \bar{\phi} \, dx + \int \Omega \mu_h \bar{\phi} \, dx - \int \Omega y_h^* \bar{\phi} \, dx
\]

\[
= \beta \int \Omega D^2 (y^* - y_h^*) : D^2 \bar{\phi} \, dx + \int \Omega (y^* - y_h^*) \bar{\phi} \, dx
\]

\[
+ \int \Omega (\mu_h - \mu) \bar{\phi} \, dx + \int \Omega (\lambda - \lambda_h) \Delta \bar{\phi} \, dx
\]

\[
\lesssim (\beta \|y^* - y_h^*\|_{H^2(T)} \|\phi_T\|_{H^2(T)} + \|y^* - y_h^*\|_{L^2(T)} \|\phi_T\|_{L^2(T)}
\]

\[
+ \|\mu - \mu_h\|_{L^2(T)} \|\phi_T\|_{L^2(T)} + \|\lambda - \lambda_h\|_{L^2(T)} \|\Delta \phi_T\|_{L^2(T)}
\]

\[
\lesssim (\beta h_T^{-2} \|y^* - y_h^*\|_{H^2(T)} + \|y^* - y_h^*\|_{L^2(T)} + \|\mu - \mu_h\|_{L^2(T)}
\]

\[
+ h_T^{-2} \|\lambda - \lambda_h\|_{L^2(T)} \|\phi_T\|_{L^2(T)}
\]

where in the last step, we used the inverse estimate \( \|\phi_T\|_{H^2(T)} \leq C h_T^{-2} \|\phi_T\|_{L^2(T)} \). Thus, from equation (4.31) and (4.32), we obtain

\[
h_T^{-2} \|y_d + \mu_h - y_h^*\|_{L^2(T)} \lesssim \|y^* - y_h^*\|_{2,T} + h_T^2 \|\mu - \mu_h\|_{L^2(T)} + \|\lambda - \lambda_h\|_{L^2(T)} + h_T^2 \|y_d - \bar{y}_d\|_{L^2(T)},
\]

thus we get the desired estimate.

(ii) **(Local bound for \( \eta_2 \))** We skip the proof of (4.26) which follows using standard bubble function techniques together with the realization that

\[
h_e^{-1/2} \left\| \left( \frac{\partial y_h^*}{\partial n} \right) \right\|_{L^2(e)} = h_e^{-1/2} \left\| \left( \frac{\partial (y^* - y_h^*)}{\partial n} \right) \right\|_{L^2(e)} \quad \forall e \in E_e^i.
\]

(iii) **(Local bound for \( \eta_3 \))** Let \( \Theta = \beta \left[ \frac{\partial y_h^*}{\partial n} \right] \) along \( e \) and define \( \theta_1 \in \mathbb{P}_1(T_e) \) by

\[
\theta_1 = 0 \quad \text{and} \quad \frac{\partial \theta_1}{\partial n} = \Theta \quad \text{on the edge} \ e.
\]

It is easy to verify that \( \|\theta_1\|_{1,T_e} \approx h_e |\Theta| \) and \( |\theta_1|_{\infty,T_e} \approx h_e |\Theta| \). Next, define \( \theta_2 \in \mathbb{P}_0(T_e) \) satisfying the following properties:
(a) $\theta_2$ is positive on the edge $e$ and takes unit value at the midpoint of the edge.
(b) $\theta_2$ vanishes up to first order on $(\partial T_+ \cup \partial T_-) \setminus e$.
It follows from the scaling that

\[
(4.35) \quad \|\theta_2\|_{1,T_\pm} \approx 1 \approx \|\theta_2\|_{\infty,T_\pm}.
\]

Using integration by parts formula (4.24), Poincaré inequality, inverse inequality and equations (2.7), (3.21), we find

\[
\beta^2 \left\| \frac{\partial^2 y_h^*}{\partial n^2} \right\|_{L^2(e)}^2 = \int_e \beta^2 \left[ \frac{\partial^2 y_h^*}{\partial n^2} \right]^2 ds = \int_e \beta \left[ \frac{\partial^2 y_h^*}{\partial n^2} \right] \Theta ds \\
\lesssim \beta \int_e \left[ \frac{\partial^2 y_h^*}{\partial n^2} \right] \frac{\partial \theta_1}{\partial n} \theta_2 ds = \beta \int_e \left[ \frac{\partial^2 y_h^*}{\partial n^2} \right] \frac{\partial (\theta_1 \theta_2)}{\partial n} ds \\
= -\beta \sum_{T \in T_e} \int_T D^2 y_h^* : D^2 (\theta_1 \theta_2) dx \\
= \beta \sum_{T \in T_e} \int_T D^2 (y^* - y_h^*) : D^2 (\theta_1 \theta_2) dx - \beta \sum_{T \in T_e} \int_T D^2 y^* : D^2 (\theta_1 \theta_2) dx \\
\lesssim \sum_{T \in T_e} \left\{ \beta \int_T D^2 (y^* - y_h^*) : D^2 (\theta_1 \theta_2) dx - \int_T (y_d + \mu_h - y_h^*) \theta_1 \theta_2 dx \\
+ \int_T (\mu_h - \mu) \theta_1 \theta_2 dx + \int_T (y^* - y_h^*) \theta_1 \theta_2 dx + \int_T (\lambda - \lambda_h) \Delta_h (\theta_1 \theta_2) dx \right\} \\
\lesssim \sum_{T \in T_e} \left\{ \|y^* - y_h^*\|_{H^2(T)} + \|y^* - y_h^*\|_{L^2(T)} + \|y_d + \mu_h - y_h^*\|_{L^2(T)} \\
+ \|\mu - \mu_h\|_{L^2(T)} + \|\lambda - \lambda_h\|_{L^2(T)} \right\} |\theta_1 \theta_2|_{H^1(T_e)} \\
+ \|\lambda - \lambda_h\|_{L^2(T)} h_e^{-1} |\theta_1 \theta_2|_{H^1(T_e)}. \\
\right.
\]

therein, using (4.34) and (4.35), we have

\[
(4.36) \quad |\theta_1 \theta_2|_{H^1(T_e)} \leq |\theta_1|_{\infty,T_\pm} |\theta_2|_{1,T_\pm} + |\theta_1|_{1,T_\pm} |\theta_2|_{\infty,T_\pm} \lesssim (h_e \beta)^2 \left\| \frac{\partial^2 y_h^*}{\partial n^2} \right\|_{L^2(e)}^2. \\
\]

Finally, the estimate (4.27) can be realized by using (4.25).

(iv)(Local bound for $\eta_4$) Let $e \in E_1^i$ be an interior edge sharing the elements $T_+$ and $T_-$ and $T_e = T_+ \cup T_-$. Define $\theta_3 \in \mathbb{P}_0(T_e)$, by assigning $\theta_3 = \beta \left[ \frac{\partial (\Delta y_h^*)}{\partial n} \right]$ on $e$ and $\theta_3$ satisfies:

\[
(4.37) \quad \|\theta_3\|_{L^2(T_\pm)} \lesssim \frac{1}{h_e} \left\| \left[ \frac{\partial (\Delta y_h^*)}{\partial n} \right] \right\|_{L^2(e)}. \\
\]

Further, let $\theta_4 \in \mathbb{P}_8(T_e)$ satisfies the following:

(a) $\theta_4$ is positive on the edge $e$ and takes unit value at the midpoint of the edge.
(b) $\theta_4$ vanishes up to first order on $(\partial T_+ \cup \partial T_-) \setminus e$.

Then, $\theta_4$ satisfies

\[
(4.38) \quad h_e^{-1} \|\theta_4\|_{L^2(T_\pm)} + \|\theta_4\|_{L^\infty(T_\pm)} \lesssim 1.
\]
Define $\hat{\phi}_e = \theta_3 \theta_4$ on $T_e$ and let $\hat{\phi} \in W$ be the extension of $\hat{\phi}_e$ by zero outside $T_e$. From (4.24), (2.7), (3.21), discrete trace inequality and inverse inequality, it follows that

\[
\beta^2 \left\| \frac{\partial (\Delta y^*_h)}{\partial n} \right\|^2_{L^2(e)} = \beta^2 \int_{T_e} \left[ \frac{\partial (\Delta y^*_h)}{\partial n} \right]^2 \, ds \leq \int_{T_e} \beta \left[ \frac{\partial (\Delta y^*_h)}{\partial n} \right] \hat{\phi} \, ds
\]

\[
= \beta \left\{ \int_{T_e} \Delta^2 y^*_h \hat{\phi} \, dx - \sum_{T \in T_e} \int_T D^2 y^*_h : D^2 \hat{\phi} \, dx - \int_{T_e} \left[ \frac{\partial^2 y^*_h}{\partial n^2} \right] \frac{\partial \hat{\phi}}{\partial n} \, ds \right\}
\]

\[
= \sum_{T \in T_e} \left\{ \beta \int_T D^2 (y^* - y^*_h) : D^2 \hat{\phi} \, dx - \int_T (y_d + \mu_h - y^*_h) \hat{\phi} \, dx \right\}
\]

\[
+ \int_{T_e} \left( \frac{\partial \hat{\phi}}{\partial n} \right) \, ds - \beta \int_{T_e} \left[ \frac{\partial^2 y^*_h}{\partial n^2} \right] \frac{\partial \hat{\phi}}{\partial n} \, ds
\]

\[
\leq \left( \beta h_e^{-2} \| y^* - y^*_h \|_{H^2(T_e)} + \| y^* - y^*_h \|_{L^2(T_e)} + \| y_d + \mu_h - y^*_h \|_{L^2(T_e)} \right)
\]

\[
+ \| \mu - \mu_h \|_{L^2(T_e)} + h_e^{-2} \| \lambda - \lambda_h \|_{L^2(T_e)} \right\} \hat{\phi} \|_{L^2(T_e)} + \beta \left\| \frac{\partial^2 y^*_h}{\partial n^2} \right\|_{L^2(e)} \left\| \frac{\partial \hat{\phi}}{\partial n} \right\|_{L^2(e)}
\]

\[
\lesssim \left( h_e^{-2} \{ \beta \| y^* - y^*_h \|_{H^2(T_e)} + h_e^{-2} \| y^* - y^*_h \|_{L^2(T_e)} + h_e^2 \| y_d + \mu_h - y^*_h \|_{L^2(T_e)} \right)
\]

\[
+ h_e^2 \| \mu - \mu_h \|_{L^2(T_e)} + \| \lambda - \lambda_h \|_{L^2(T_e)} \right\} + \beta h_e^{-2} \left\| \frac{\partial^2 y^*_h}{\partial n^2} \right\|_{L^2(e)} \left\| \frac{\partial \hat{\phi}}{\partial n} \right\|_{L^2(e)}
\]

\[
\lesssim \left( h_e^{-2} \{ \beta \| y^* - y^*_h \|_{H^2(T_e)} + h_e^{-2} \| y^* - y^*_h \|_{L^2(T_e)} + h_e^2 \| y_d + \mu_h - y^*_h \|_{L^2(T_e)} \right)
\]

\[
+ h_e^2 \| \mu - \mu_h \|_{L^2(T_e)} + \| \lambda - \lambda_h \|_{L^2(T_e)} \right\} + \beta h_e^{-2} \left\| \frac{\partial^2 y^*_h}{\partial n^2} \right\|_{L^2(e)} \beta h_e^{3/2} \left\| \frac{\partial (\Delta y^*_h)}{\partial n} \right\|_{L^2(e)}
\]

(4.39)

where in the last step we have used (4.37) and (4.38). Hence,

\[
\beta h_e^{3/2} \left\| \frac{\partial (\Delta y^*_h)}{\partial n} \right\|_{L^2(e)} \lesssim \sum_{T \in T_e} \left( \| y^* - y^*_h \|_{L^2(T)} + h_e^2 \| y_d + \mu_h - y^*_h \|_{L^2(T)} + h_e^2 \| \mu - \mu_h \|_{L^2(T)} \right)
\]

\[
+ \| \lambda - \lambda_h \|_{L^2(T)} \right\} + h_e^{1/2} \beta \left\| \frac{\partial^2 y^*_h}{\partial n^2} \right\|_{L^2(e)}
\]

(4.40)

Finally, we obtain the bound (4.28) by a use of (4.25) and (4.27).

(v) **Local bound for $\eta_5$** Let $\hat{b}_T$ be a polynomial bubble function vanishing up to the second order on $\partial T$. Let $\Delta \psi_T = \lambda_h \Delta \hat{b}_T$ and $\hat{\psi} \in W$ be the extension of $\psi_T$ by zero to $\Omega$. In view of (4.24), $\int_T D^2 y^*_h : D^2 \hat{\psi} \, dx = 0$. Therefore, using (2.7) we have,
\[ \| \lambda_h \|_{L^2(T)} \lesssim \int_T \lambda_h \Delta \psi_T \, dx = \int_\Omega (\lambda_h - \lambda) \Delta \tilde{\psi} \, dx + \int_\Omega \lambda \Delta \tilde{\psi} \, dx \]
\[ = \int_\Omega (\lambda_h - \lambda) \Delta \tilde{\psi} \, dx + \int_\Omega y_d \tilde{\psi} \, dx + \int_\Omega \mu \tilde{\psi} \, dx - A(y^*, \tilde{\psi}) \]
\[ = \int_T (\lambda_h - \lambda) \Delta \tilde{\psi} \, dx + \int_T (y_d + \mu_h - y_h^*) \tilde{\psi} \, dx + \int_T (\mu - \mu_h) \tilde{\psi} \, dx \]
\[ - \int_T (y^* - y_h^*) \tilde{\psi} \, dx - \beta \int_T D^2(y^* - y_h^*) : D^2 \tilde{\psi} \, dx \]
\[ \lesssim \left( \| \lambda - \lambda_h \|_{L^2(T)} + h_T^2 \| \mu - \mu_h \|_{L^2(T)} + h_T^2 \| y_d + \mu_h - y_h^* \|_{L^2(T)} \right. \]
\[\left. + \| y^* - y_h^* \|_{2,T} \| \psi_T \|_{H^2(T)} \right) \]
\[ \lesssim \left( \| \lambda - \lambda_h \|_{L^2(T)} + h_T^2 \| \mu - \mu_h \|_{L^2(T)} + h_T^2 \| y_d + \mu_h - y_h^* \|_{L^2(T)} \right. \]
\[\left. + \| y^* - y_h^* \|_{2,T} \| \lambda_h \|_{L^2(T)} \right) \]

where in obtaining the second last estimate, we have used Poincaré inequality with scaling arguments. Finally, we get the desired estimate by taking into account (4.25)-(4.28).

\[ \square \]

5. ADAPTIVE FEM FOR OCPs WITH INTEGRAL STATE CONSTRAINT AND POINCEWISE CONTROL CONSTRAINTS

This section is devoted to the a posteriori error analysis of OCPs with integral state constraint and pointwise control constraints. We consider the following minimization problem: find \((y^*, u^*) \in K\), such that
\[
(5.1) \quad (y^*, u^*) = \arg\min_{(y, u) \in K} \left( \frac{1}{2} \| y - y_d \|^2 + \frac{\beta}{2} \| u \|^2 \right)
\]
subject to the constraints
\[
(5.2) \quad \begin{cases} 
\int_\Omega \nabla y \cdot \nabla w \, dx = \int_\Omega uw \, dx, & \forall w \in H_0^1(\Omega) \\
\int_\Omega y \, dx \geq \delta_3, \\
u_a \leq u \leq u_b \text{ a.e. in } \Omega,
\end{cases}
\]
where \(K = H_0^1(\Omega) \times L^2(\Omega)\) and \(\delta_3\) is a constant. The functions \(u_a, u_b\) are assumed to satisfy
(i) \(u_a, u_b \in W^{1, \infty}(\Omega)\), (ii) \(u_a < u_b\) on \(\Omega\).

As discussed in Section 1, we then rewrite this optimization problem into a reduced minimization problem involving only the state variable. Analogously to (2.4), the reduced optimal control problem is to find \(y^* \in \tilde{K}\) such that
\[
(5.3) \quad y^* = \arg\min_{y \in \tilde{K}} \left( \frac{1}{2} A(y, y) - (y_d, y) \right),
\]
where the bilinear form \(A(\cdot, \cdot)\) is same as in (2.5) and set \(\tilde{K}\) is defined as
\[
(5.4) \quad \tilde{K} = \{ w \in W : \int_\Omega w \, dx \geq \delta_3 \text{ and } u_a \leq -\Delta w \leq u_b \text{ a.e. in } \Omega \}.
\]

We assume the following Slater condition: there exists \(y \in W\) such that \(\int_\Omega y \, dx > \delta_3\) and \(u_a \leq -\Delta y \leq u_b\). Thus, the closed convex set \(\tilde{K}\) is nonempty. The minimizer of (5.3) have the
characterization in terms of the solution of the following fourth order variational inequality: find \( y^* \in \tilde{K} \) satisfying

\[
A(y^*, w - y^*) \geq \int_{\Omega} y_d(w - y^*) \, dx \quad \forall w \in \tilde{K}.
\]

The following (generalized) Karush-Kuhn-Tucker conditions hold (see [34, 47]): there exist \( \lambda \in L^2(\Omega) \) and \( \mu \in \mathbb{R} \) such that

\[
\beta \int_{\Omega} (\Delta y^*)(\Delta w) \, dx + \int_{\Omega} y^*w \, dx = \int_{\Omega} y_dw \, dx - \int_{\Omega} \lambda(\Delta w) \, dx + \int_{\Omega} \mu w \, dx
\]

for all \( w \in W \) together with the complementarity conditions

\[
\begin{align*}
\lambda \geq 0 & \quad \text{if} \quad -\Delta y^* = u_a, \\
\lambda \leq 0 & \quad \text{if} \quad -\Delta y^* = u_b, \\
\lambda = 0 & \quad \text{otherwise}, \\
\mu \geq 0 & \quad \text{if} \quad \int_{\Omega} y^* \, dx = \delta_3, \\
\mu = 0 & \quad \text{if} \quad \int_{\Omega} y^* \, dx > \delta_3.
\end{align*}
\]

The adjoint state \( p \in H^1_0(\Omega) \) associated to the problem (5.1) -(5.2) is given by

\[
\int_{\Omega} \nabla p \cdot \nabla w \, dx = \int_{\Omega} (y^* - y_d)w \, dx - \int_{\Omega} \mu w \, dx \quad \forall w \in H^1_0(\Omega).
\]

5.1. **Discrete Problem.** The discretization of (5.3) is to find \( y^*_h \in \tilde{K}_h \) such that

\[
y^*_h = \arg \min_{y_h \in \tilde{K}_h} \left[ \frac{1}{2} A_h(y_h, y_h) - (y_d, y_h) \right],
\]

where

\[
\tilde{K}_h = \{ w_h \in W_h : \int_{\Omega} \, w_h \, dx \geq \delta_3 \text{ and } Q_h u_a \leq Q_h(-\Delta_h w_h) \leq Q_h u_b \}.
\]

Remark 5.1. *Owing to the property (3.6) and (3.4) of \( I_h \), we have \( I_h \tilde{K} \subset \tilde{K}_h \).*

As in the continuous case, the minimizer of (5.13) can be characterized by the solution of the following variational inequality: find \( y^*_h \in \tilde{K}_h \) such that

\[
A_h(y^*_h, w_h - y^*_h) \geq (y_d, w_h - y^*_h) \quad \forall w_h \in \tilde{K}_h.
\]

where the bilinear form \( A_h(\cdot, \cdot) \) is defined in (3.20). It can be easily checked that the discrete problem (5.15) is well-posed.

The Karush-Kuhn-Tucker conditions for the discrete problem [34, 47] is given as follows: there exist \( \lambda_h \in \mathcal{P}_0(T_h) \) and \( \mu_h \in \mathbb{R} \) such that

\[
A_h(y^*_h, w_h) = \int_{\Omega} y_d w_h \, dx - \int_{\Omega} \lambda_h(\Delta_h w_h) \, dx + \int_{\Omega} \mu_h w_h \, dx, \quad \forall w_h \in W_h
\]
together with the complementary conditions
\begin{align}
(5.17) \quad & \lambda_h \geq 0 \quad \text{on } T \in \mathcal{T}_h \text{ such that } Q_T(-\Delta y^*_h) = Q_T(u_a), \\
(5.18) \quad & \lambda_h \leq 0 \quad \text{on } T \in \mathcal{T}_h \text{ such that } Q_T(-\Delta y^*_h) = Q_T(u_b), \\
(5.19) \quad & \lambda_h = 0 \quad \text{otherwise}, \\
(5.20) \quad & \mu_h \geq 0 \quad \text{if } \int_{\Omega} y^*_h \, dx = \delta_3, \\
(5.21) \quad & \mu_h = 0 \quad \text{if } \int_{\Omega} y^*_h \, dx > \delta_3.
\end{align}

In the following, we derive the reliability estimates of the estimator $\eta_h$ for the error $\|y^* - y^*_h\|_h$. For this we introduce the following auxiliary problem: let $\tilde{z}_h \in W$ be the solution of
\begin{equation}
(5.22) \quad A(\tilde{z}_h, w) = \int_{\Omega} y_d w \, dx + \int_{\Omega} \lambda_h (-\Delta_h w) \, dx + \int_{\Omega} \mu_h w \, dx, \quad \forall w \in W.
\end{equation}

The well-posedness of (5.22) is ensured by Lax-Milgram lemma [21]. Now, we proceed to establish the reliability of the error estimator for the error in solution $y^*$.

**Theorem 5.2.** Let $y^*_h$ and $\tilde{z}_h$ be the solutions of (5.16) and (5.22), respectively. Then,
\begin{equation}
(5.23) \quad \|y^*_h - \tilde{z}_h\|_h \lesssim \eta_h,
\end{equation}
where $\eta_h$ is defined in (4.1).

**Proof.** Let $\phi = \tilde{z}_h - E_h y^*_h \in W$ and $\phi_h = I_h \phi \in W_h$, a use of coercive property of the bilinear form $A(\cdot, \cdot)$, (5.22) and (5.16) leads to
\begin{equation}
(5.24) \quad \|\phi\|^2_h \lesssim A(\tilde{z}_h, \phi) - A_h(E_h y^*_h, \phi)
\begin{align*}
\lesssim & \int_{\Omega} y_d \phi \, dx + \int_{\Omega} \mu_h \phi \, dx - A_h(y^*_h, \phi) + \int_{\Omega} \lambda_h (-\Delta_h \phi) \, dx \\
& + A_h(y^*_h - E_h y^*_h, \phi) \\
\lesssim & \int_{\Omega} y_d (\phi - \phi_h) \, dx - A_h(y^*_h, \phi - \phi_h) \, dx + \int_{\Omega} \mu_h (\phi - \phi_h) \, dx \\
& - \int_{\Omega} \lambda_h (\Delta_h (\phi - \phi_h)) + A_h(y^*_h - E_h y^*_h, \phi) \\
= & I_1 + I_2 + I_3 + I_4 + I_5.
\end{align*}
\end{equation}

Note that, $I_1 + I_2 + I_3$ and $I_5$ can be estimated following same arguments as in the Theorem 4.1. Now, it remains to estimate $I_4$. An application of the Cauchy Schwarz inequality and Lemma 3.1 yields
\begin{equation}
|I_4| \lesssim \sum_{T \in \mathcal{T}_h} | \int_{\Omega} \lambda_h \Delta (\phi_h - \phi) \, dx | \lesssim \sum_{T \in \mathcal{T}_h} h_T | \lambda_h | \| \Delta (\phi_h - \phi) \|_{L^2(T)} \\
\lesssim \sum_{T \in \mathcal{T}_h} h_T | \lambda_h | \| \phi \|_{H^2(T)} \lesssim \eta_h \| \phi \|_h.
\end{equation}

Combining all the estimates together with (5.24), we get
\begin{equation}
\| \phi \|_h = \| \tilde{z}_h - E_h y^*_h \|_h \lesssim \eta_h.
\end{equation}

An use of the triangle inequality $\| \tilde{z}_h - y^*_h \|_h \leq \| \tilde{z}_h - E_h y^*_h \|_h + \| E_h y^*_h - y^*_h \|_h$, in view of Lemma 3.2 leads to the desired estimate. \qed
Theorem 5.3. Let \( y^* \) and \( y^*_h \) be solutions of variational inequalities (5.5) and (5.15), respectively. Then, it holds that

\[
\|y^* - y^*_h\|_h \lesssim \left( \eta_h + \sum_{T \in \Omega_1 \cup \Omega_2} \|\lambda\|_{L^2(T)} \left( \sum_{e \in E_T} \frac{1}{h_e} \left\| \frac{\partial y^*_h}{\partial n} \right\|_{L^2(e)}^2 \right) \right)^{\frac{1}{2}} + \left( \int_{\Omega_1} \lambda(\Delta y^*_h + u_a) \, dx \right)^{\frac{1}{2}} + \left( \int_{\Omega_2} \lambda(\Delta_h y^*_h + u_b) \, dx \right)^{\frac{1}{2}}.
\]

Proof. Set \( \phi = y^* - E_h y^*_h \in W \) and let \( \phi_h \in W_h \). As in Theorem 4.1, using coercivity of the bilinear form \( A(\cdot, \cdot) \), we get

\[
\|y^* - E_h y^*_h\|_h \lesssim A(y^* - E_h y^*_h, \phi) = A(\tilde{z}_h - E_h y^*_h, \phi) + A(y^*, \phi) - A(\tilde{z}_h, \phi).
\]

In view of (5.6) and (5.16), the term \( A(y^*, \phi) - A(\tilde{z}_h, \phi) \) satisfies

\[
A(y^*, \phi) - A(\tilde{z}_h, \phi) = \int_{\Omega} (\lambda_h - \lambda_h)(-\Delta \phi) \, dx + \int_{\Omega} (\mu_h - \mu_h) \phi \, dx.
\]

Following the same arguments as in Theorem 4.1 we obtain

\[
\int_{\Omega} (\mu - \mu_h) \phi \, dx \leq 0.
\]

Now, to estimate the term \( \int_{\Omega} (\lambda_h - \lambda_h)(-\Delta \phi) \, dx \), we split it as

\[
\int_{\Omega} (\lambda - \lambda_h)(-\Delta \phi) \, dx = \int_{\Omega_1} (\lambda - \lambda_h)(-\Delta \phi) \, dx + \int_{\Omega_2} (\lambda - \lambda_h)(-\Delta \phi) \, dx,
\]

where \( \Omega_1 \) and \( \Omega_2 \) are discrete control contact sets defined by

\[
\Omega_1 = \{ T \in \mathcal{T}_h : Q_T(\Delta y^*_h) = Q_T(u_a) \}, \quad \Omega_2 = \{ T \in \mathcal{T}_h : Q_T(\Delta y^*_h) = Q_T(u_b) \}.
\]

The first term of the right hand side of equation (5.28) can be estimates as follows.

\[
\int_{\Omega_1} (\lambda - \lambda_h)(-\Delta \phi) \, dx = \int_{\Omega_1} \lambda_h(\Delta \phi) \, dx - \int_{\Omega_2} \lambda(\Delta \phi) \, dx
\]

\[
= \int_{\Omega_1} \lambda_h(\Delta y^* - \Delta E_h y^*_h) \, dx - \int_{\Omega_1} \lambda(\Delta y^* - \Delta E_h y^*_h) \, dx.
\]

Using \( \lambda_h \geq 0, u_a \leq -\Delta y^* \), relation (3.14) and \( \sum_{T \in \Omega_1} \lambda_h \int_T (u_a + \Delta_h y^*_h) \, dx = 0 \), we find

\[
\int_{\Omega_1} \lambda_h(\Delta y^* - \Delta E_h y^*_h) \, dx = \int_{\Omega_1} \lambda_h(\Delta y^* + u_a) \, dx - \int_{\Omega_1} \lambda_h(u_a + \Delta E_h y^*_h) \, dx
\]

\[
= \int_{\Omega_1} \lambda_h(\Delta y^* + u_a) \, dx - \int_{\Omega_1} \lambda_h(u_a + \Delta y^*_h) \, dx
\]

\[
- \int_{\Omega_1} \lambda_h(\Delta_h y^*_h - \Delta E_h y^*_h) \, dx \leq 0.
\]
In view of (5.7) and Lemma 3.2, we have

\[- \int_{\Omega_1} \lambda(\Delta y^* - \Delta E_h y_h^*) \, dx = \int_{\Omega_1} \lambda(\Delta E_h y_h^* - \Delta y^*) \, dx\]
\[= \int_{\Omega_1} \lambda(\Delta E_h y_h^* - \Delta_h y_h^*) \, dx + \int_{\Omega_1} \lambda(\Delta_h y_h^* + u_a) \, dx\]
\[- \int_{\Omega_1} \lambda(\Delta_h y^* + u_a) \, dx\]
\[(5.31)\]
\[
\lesssim \sum_{T \in \Omega_1} \|\lambda\|_{L^2(T)} \sum_{e \in E_T} \frac{1}{h_e} \left\| \left[ \frac{\partial y_h^*}{\partial n} \right] \right\|^2_{L^2(e)} + \int_{\Omega_1} \lambda(\Delta_h y_h^* + u_a) \, dx.
\]

Combining (5.29), (5.30) and (5.31), we get
\[(5.32)\]
\[\int_{\Omega_1} (\lambda - \lambda_h)(-\Delta \phi) \, dx \lesssim \sum_{T \in \Omega_1} \|\lambda\|_{L^2(T)} \sum_{e \in E_T} \frac{1}{h_e} \left\| \left[ \frac{\partial y_h^*}{\partial n} \right] \right\|^2_{L^2(e)} + \int_{\Omega_1} \lambda(\Delta_h y_h^* + u_a) \, dx.
\]

Repeating the similar arguments, we estimate the second term of the right hand side of equation (5.28)
\[(5.33)\]
\[\int_{\Omega_2} (\lambda - \lambda_h)(-\Delta \phi) \, dx \lesssim \sum_{T \in \Omega_1 \cup \Omega_2} \|\lambda\|_{L^2(T)} \sum_{e \in E_T} \frac{1}{h_e} \left\| \left[ \frac{\partial y_h^*}{\partial n} \right] \right\|^2_{L^2(e)} + \int_{\Omega_2} \lambda(\Delta_h y_h^* + u_b) \, dx.
\]

A use of (5.32) and (5.33) in (5.28) yields
\[(5.34)\]
\[\int_{\Omega} (\lambda - \lambda_h)(-\Delta \phi) \, dx \lesssim \sum_{T \in \Omega_1 \cup \Omega_2} \|\lambda\|_{L^2(T)} \sum_{e \in E_T} \frac{1}{h_e} \left\| \left[ \frac{\partial y_h^*}{\partial n} \right] \right\|^2_{L^2(e)} + \int_{\Omega_1} \lambda(\Delta_h y_h^* + u_a) \, dx
\]
\[+ \int_{\Omega_2} \lambda(\Delta_h y_h^* + u_b) \, dx.
\]

Combining (5.26), (5.27) and (5.34), we have
\[
A(y^*, \phi) - A(\breve{\phi}_h, \phi) \lesssim \sum_{T \in \Omega_1 \cup \Omega_2} \|\lambda\|_{L^2(T)} \sum_{e \in E_T} \frac{1}{h_e} \left\| \left[ \frac{\partial y_h^*}{\partial n} \right] \right\|^2_{L^2(e)} + \int_{\Omega_1} \lambda(\Delta_h y_h^* + u_a) \, dx
\]
\[+ \int_{\Omega_2} \lambda(\Delta_h y_h^* + u_b) \, dx.
\]

Using the continuity of bilinear form and the Young’s inequality in (5.25) leads to
\[(5.35)\]
\[
\|y^* - E_h y_h^*\|_h^2 \lesssim \left( \|\tilde{z}_h - E_h y_h^*\|_h^2 + \sum_{T \in \Omega_1 \cup \Omega_2} \|\lambda\|_{L^2(T)} \sum_{e \in E_T} \frac{1}{h_e} \left\| \left[ \frac{\partial y_h^*}{\partial n} \right] \right\|^2_{L^2(e)} \right) + \int_{\Omega_1} \lambda(\Delta_h y_h^* + u_a) \, dx + \int_{\Omega_2} \lambda(\Delta_h y_h^* + u_b) \, dx.
\]

Finally, a use of triangle inequality, (5.35), Theorem 5.2 and Lemma 3.2 gives
\[
\|y^* - y_h^*\|_h^2 \lesssim \left( \eta_h^2 + \sum_{T \in \Omega_1 \cup \Omega_2} \|\lambda\|_{L^2(T)} \sum_{e \in E_T} \frac{1}{h_e} \left\| \left[ \frac{\partial y_h^*}{\partial n} \right] \right\|^2_{L^2(e)} \right) + \int_{\Omega_1} \lambda(\Delta_h y_h^* + u_a) \, dx + \int_{\Omega_2} \lambda(\Delta_h y_h^* + u_b) \, dx.
\]

This completes the proof. □
We would like to remark here that, in Theorem 5.3 the estimate is not a genuine a posteriori error estimate because of the presence of $\lambda$ in the right hand side, but it is useful in realizing the asymptotic convergence of the adaptive algorithm. Now, following the idea of Lemma 4.3 and Theorem 4.2, we can estimate the error in Lagrange multipliers, hence we state the result omitting details of the proof.

**Lemma 5.4.** It holds that,

$$|\mu - \mu_h| + \|\lambda - \lambda_h\|_{L^2(\Omega)} \lesssim \eta_h.$$  

The following local efficiency estimates can be proved using bubble function techniques as in Theorem 4.4.

**Theorem 5.5.** There exists a positive constant $C > 0$ depending on the shape regularity of $\mathcal{T}_h$ such that

$$h_T^2\|y_d - y_h\|_{L^2(T)} \lesssim \|y^* - y_h^*\|_{L^2(T)} + h_T^2\|\mu - \mu_h\|_{L^2(T)} + \lambda - \lambda_h\|_{L^2(T)} + Osc(y_d; T) \forall T \in \mathcal{T}_h,$$

$$\beta h_e^{1/2} \left\| \frac{\partial y_h}{\partial n} \right\|_{L^2(e)} \lesssim \sum_{T \in \mathcal{T}_e} \left( \|y^* - y_h^*\|_{L^2(T)} + h_T^2\|\mu - \mu_h\|_{L^2(T)} + \lambda - \lambda_h\|_{L^2(T)} \right) + Osc(y_d; T_e) \forall e \in \mathcal{E}^i_h,$$

$$\beta h_e^{1/2} \left\| \frac{\partial^2 y_h}{\partial n^2} \right\|_{L^2(e)} \lesssim \sum_{T \in \mathcal{T}_e} \left( \|y^* - y_h^*\|_{L^2(T)} + h_T^2\|\mu - \mu_h\|_{L^2(T)} + \lambda - \lambda_h\|_{L^2(T)} \right) + Osc(y_d; T_e) \forall e \in \mathcal{E}^i_h,$$

$$\beta h_e^{3/2} \left\| \frac{\partial (\Delta y_h^*)}{\partial n} \right\|_{L^2(e)} \lesssim \sum_{T \in \mathcal{T}_e} \left( \|y^* - y_h^*\|_{L^2(T)} + h_T^2\|\mu - \mu_h\|_{L^2(T)} + \lambda - \lambda_h\|_{L^2(T)} \right) + Osc(y_d; T_e) \forall e \in \mathcal{E}^i_h,$$

where $\|w\|_{L^2(T)} := \beta|w|_{H^2(T)} + h_T^{1/2}\|w\|_{L^2(T)}$ for any $w \in H^2(\Omega, T_h)$, $Osc(y_d; T) := h_T^2\|y_d - \bar{y}_d\|_{L^2(T)}$ with $\bar{y}_d := \frac{1}{|T|}\int_T y_d \, dx$ and $\mathcal{T}_e$ denotes the union of elements sharing the edge $e$.

6. Numerical Assessments

In this section, we perform numerical experiments to illustrate the performance of the error estimators derived in Section 4 and Section 5. For this, we have considered four examples. The data of first example is for the purely integral state constraints, the second one is based on the purely integral control constraint, the third example consists of the integral state and integral control constraints and the last example concerns the integral state and pointwise control constraints. The discrete problem is solved using the primal-dual active set method. For the adaptive refinement, we use the following paradigm

**SOLVE** $\rightarrow$ **ESTIMATE** $\rightarrow$ **MARK** $\rightarrow$ **REFINE**

We compute the discrete state using the primal-dual active set algorithm in step 'SOLVE'. Thereafter in step 'ESTIMATE', we compute the error estimator on each element $T \in \mathcal{T}_h$ and use Dörfler marking strategy with parameter $\theta = 0.3$ to mark the elements for refinement. Finally, a new adaptive mesh is obtained by performing refinement using the newest vertex bisection algorithm. Below, we consider various test examples.
Example 6.1. This example consists of the integral state constraints as active constraints [58]. Here, we solve the following problem on $\Omega = (0, 1)^2$ with $\beta = 1$.

$$
\begin{align*}
\min_{y \in K} & \left\{ \frac{1}{2} \| y - y_d \|_{L^2(\Omega)}^2 + \frac{\beta}{2} \| u \|_{L^2(\Omega)}^2 \right\} \\
\text{s.t.} & \\
-\Delta y = f + u & \text{in } \Omega, \\
y = 0 & \text{on } \partial \Omega, \\
\int_\Omega y \, dx \geq \delta_2 & \text{and } \int_\Omega u \, dx \geq \delta_1,
\end{align*}
$$

(6.1)

with the exact solution and the data as

$$
p = \sin(2\pi x_1)\sin(2\pi x_2) + \frac{3}{8} \sin(2\pi x_1)\sin(4\pi x_2),
$$

$$
y = p,
$$

$$
y_d = y + \Delta p - 0.4,
$$

$$
f = -\Delta y - u,
$$

$$
\delta_2 = -0.4,
$$

$$
\delta_1 = 0,
$$

$$
u = \max\{\tilde{p} + \beta \tilde{\delta}_1, 0\} - p,
$$

where $\tilde{p} = \frac{\int_\Omega p \, dx}{\int_\Omega 1 \, dx}$ and $\tilde{\delta}_1 = \frac{\delta_1}{\int_\Omega 1 \, dx}$.

Figure 1a depicts convergence behavior of the error and the estimator with respect to the increasing number of degrees of freedom (DoFs). From this figure, it is evident that both error and error estimator converge with optimal rate ($1/\sqrt{\text{DoFs}}$). Figure 1a also ensures the reliability of the error estimator. Figure 1b shows the efficiency indices, ensuring that the error estimator is efficient.

(A) Error and Estimator

(B) Efficiency Index

**FIGURE 1.** Error, estimator and efficiency index for Example 6.1
Example 6.2. In this example [24], we consider the optimal control problem (6.1) with purely integral control constraints given on the domain $\Omega = (0, 1) \times (0, 1)$ with $\beta = 1$ as follows

\[
p = \sin(\pi x_1)\sin(\pi x_2),
\]
\[
y = 2\pi^2 p + y_d
\]
\[
y_d = 0,
\]
\[
f = 4\pi^4 p + p - \frac{4}{\pi^2},
\]
\[
\delta_2 = 100,
\]
\[
\delta_1 = 0,
\]
\[
u = \max\{\tilde{p} + \beta\tilde{\delta}_1, 0\} - p,
\]

where $\tilde{p} = \frac{\int_\Omega p \, dx}{\int_\Omega 1 \, dx}$ and $\tilde{\delta}_1 = \frac{\delta_1}{\int_\Omega 1 \, dx}$.

For this example, the convergence behavior of the error and estimator is shown in Figure 2a, which confirms that both error and estimator converges optimally and also that the estimator is reliable. The efficiency of the estimator is ensured by efficiency index depicted in Figure 2b.

![Error and Estimator](A)

![Efficiency Index](B)

**Figure 2.** Error, estimator and efficiency index for Example 6.2
Example 6.3. In this example, we consider the OCP (6.1) with integral state and integral control constraints on the domain \( \Omega = (-1,1) \times (-1,1) \) with the following data:

\[
\begin{align*}
    y &= \frac{-1}{2\pi^2} sin(\pi x_1) sin(\pi x_2), \\
    p &= sin(\pi x_1) sin(\pi x_2), \\
    y_d &= -(2\pi^2 + \frac{1}{2\pi^2}) sin(\pi x_1) sin(\pi x_2) - 0.6, \\
    f &= 0, \\
    \delta_2 &= 0, \\
    \delta_1 &= 0, \\
    \beta &= 1, \\
    u &= -p + \max\{\bar{p} + \beta \bar{\delta}_1, 0\}, \\
\end{align*}
\]

where \( \bar{p} = \int_{\Omega} p \, dx / \int_{\Omega} 1 \, dx \) and \( \bar{\delta}_1 = \delta_1 / \int_{\Omega} 1 \, dx \).

We plot the convergences histories for the error and the error estimator in Figure 3a and the efficiency index in Figure 3b. These figures validates the reliability and efficiency of the error estimator together with the optimal convergence.

Example 6.4. In this example, we consider the problem (5.1)-(5.2) with integral state and pointwise control constraints. The idea of this example is taken from [38]. Therein, the domain \( \Omega = (0,1) \times (0,1) \) and the exact solution is not known.

\[
\begin{align*}
    y_d &= 10(sin(\pi x_1) + sin(\pi x_2)), \\
    \beta &= 0.01, \\
    \delta_3 &= 0, \\
    u_a &= 0 \text{ and } u_b = 30. \\
\end{align*}
\]
The behavior of error estimator is illustrated in Figure 4a confirming the optimal convergence and reliability of the error estimator. The adaptive mesh at a certain refinement level is depicted in Figure 4b.

![Estimator and Adaptive Mesh](image)

**Figure 4.** Estimator and adaptive mesh for Example 6.4

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