Ternary $q$-Virasoro–Witt Hom–Nambu–Lie algebras

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Abstract
In this paper we construct ternary $q$-Virasoro–Witt algebras which $q$-deform the ternary Virasoro–Witt algebras constructed by Curtright, Fairlie and Zachos using $su(1,1)$ enveloping algebra techniques. The ternary Virasoro–Witt algebras constructed by Curtright, Fairlie and Zachos depend on a parameter and are not Nambu–Lie algebras for all but finitely many values of this parameter. For the parameter values for which the ternary Virasoro–Witt algebras are Nambu–Lie, the corresponding ternary $q$-Virasoro–Witt algebras constructed in this paper are also Hom–Nambu–Lie because they are obtained from the ternary Nambu–Lie algebras using the composition method. For other parameter values this composition method does not yield a Hom–Nambu–Lie algebra structure for $q$-Virasoro–Witt algebras. We show however, using a different construction, that the ternary Virasoro–Witt algebras of Curtright, Fairlie and Zachos, as well as the general ternary $q$-Virasoro–Witt algebras we construct, carry a structure of the ternary Hom–Nambu–Lie algebra for all values of the involved parameters.

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Introduction
Lie algebras and Poisson algebras have played an extremely important role in physics for a long time. Their generalizations, known as $n$-Lie algebras and ‘Nambu algebras’ [20, 47, 48], also arise naturally in physics and have recently been studied in the context of ‘M-branes’ [8, 28]. It turns out that in the dynamic study of strings and M-branes appears naturally an algebra with ternary multiplication called the Bagger–Lambert algebra. It was used in [8]...
as one of the main ingredients in the construction of a new type of supersymmetric gauge theory that is consistent with all the symmetries expected of a multiple M2-brane theory: 16 supersymmetries, conformal invariance and an $SO(8)$ $R$-symmetry that acts on the eight transverse scalars. Hundreds of papers are dedicated to the Bagger–Lambert algebra by now. Other applications of Nambu algebras to M-branes, quantization of Nambu mechanics, volume preserving diffeomorphisms, integrable systems and related generalization of the Lax equation have been considered in [28].

A long-standing problem related to Nambu algebras is their quantization. For Poisson algebras, the problem of finding an operator algebra where the commutator Lie algebra corresponds to the Poisson algebra is a well-studied problem. For higher order algebras much less is known and the corresponding problem seems to be hard. The Nambu–Lie algebra is defined in general by $n$-ary multi-linear multiplication which is skew symmetric (see (2.4) for $n = 3$) and satisfies an identity extending the Jacobi identity for the Lie algebras. For $n = 3$ this identity is

$$[[x_1, x_2, [x_3, x_4, x_5]], [x_3, [x_1, x_2, x_4]], x_5] + [[x_3, x_4, [x_1, x_2, x_5]], x_5] = [x_1, x_2, [x_3, x_4, x_5]].$$

In Nambu–Lie algebras, the additional freedom in comparison with Lie algebras is mainly limited to extra arguments in the multi-linear multiplication. The identities of Nambu–Lie algebras are also closely resembling the identities for Lie algebras. As a result, there is a close similarity between Lie algebras and Nambu–Lie algebras in their appearances in connection to other algebraic and analytic structures and in the extent of their applicability. Thus it is not surprising that it becomes unclear how to associate, in meaningful ways, ordinary Nambu–Lie algebras with the generalizations important in physics and quantum deformations of Lie algebras when typically the ordinary skew-symmetry and Jacobi identities of Lie algebras are violated. However, if the class of Nambu–Lie algebras is extended with enough extra structure beyond just adding more arguments in multi-linear multiplication, the natural ways of association of such multi-linear algebraic structures with generalizations and quantum deformations of Lie algebras may become feasible. Hom–Nambu–Lie algebras are defined by a similar but more general identity than that of Nambu–Lie algebras involving some additional linear maps (see definition 2.4). These linear maps twisting or deforming the main identities introduce substantial new freedom in the structure allowing to consider Hom–Nambu–Lie algebras as deformations of Nambu–Lie algebras ($n$-Lie algebras). The extra freedom built into the structure of Hom–Nambu–Lie algebras may provide a path to quantization beyond what is possible for ordinary Nambu–Lie algebras. All this also gives important motivation for investigation of mathematical concepts and structures such as Leibniz $n$-ary algebras [12, 20] and their modifications and extensions, as well as Hom-algebra extensions of Poisson algebras [46]. For discussion of physical applications of these and related algebraic structures to models for elementary particles, and unification problems for interactions see [1, 9, 31–34]. For a more general theory of $n$-ary algebras see also [4, 11, 23, 24, 30].

Generalizations of $n$-ary algebras of Lie type and associative type by twisting the identities using linear maps were introduced in [6]. These generalizations include $n$-ary Hom-algebra structures generalizing the $n$-ary algebras of Lie type such as $n$-ary Nambu algebras, $n$-ary Nambu–Lie algebras and $n$-ary Lie algebras, and $n$-ary algebras of associative type such as $n$-ary totally associative and $n$-ary partially associative algebras. The general Hom-algebra structures arose first in connection to quasi-deformation and discretizations of Lie algebras of vector fields. These quasi-deformations lead to quasi-Lie algebras, a generalized Lie algebra structure in which the skew-symmetry and Jacobi conditions are twisted. The first examples were concerned with $q$-deformations of the Witt and Virasoro algebras (see for example [2, 13–17, 19, 27, 29, 40–42]). Motivated by these and new examples arising as applications...
of the general quasi-deformation construction in [26, 35, 36] on the one hand, and the desire to be able to treat within the same framework such well-known generalizations of Lie algebras as the color and Lie superalgebras on the other hand, quasi-Lie algebras and their subclasses of quasi-Hom–Lie algebras and Hom–Lie algebras were introduced in [26, 35–37]. In Hom–Lie algebras skew-symmetry is untwisted, whereas the Jacobi identity is twisted by a linear map and contains three terms as for Lie algebras, reducing to ordinary Lie algebras when the twisting map is the identity map. Hom-associative algebras, replacing associative algebras in the context of Hom–Lie algebras, and also more general classes of Hom–Lie admissible algebras, \(G\)-Hom-associative algebras, were introduced in [35]. Universal enveloping algebras for Hom–Lie algebras were considered in [49]. Formal deformations and elements of (co-)homology for Hom–Lie algebras were studied in [46, 50], whereas dual structures such as Hom-coalgebras, Hom-bialgebras and Hom–Hopf algebras appeared first in [44, 45] and further investigated in [10, 51]. Also there are some connections of Hom–Lie algebras to number theory, arithmetic geometry, elliptic curves and Galois extensions [38].

The aim of this paper is to construct ternary \(q\)-Virasoro–Witt algebras which \(q\)-deform the ternary Virasoro–Witt algebras introduced by Curtright, Fairlie and Zachos [18] in such a way that they carry a structure of ternary Hom–Nambu–Lie algebras for all values of involved parameters. The ternary Virasoro–Witt algebras constructed by Curtright, Fairlie and Zachos are ternary Nambu–Lie algebras only for two values of the parameter \(z = \pm 2i\). We show in this paper that for these values the corresponding ternary \(q\)-Virasoro–Witt algebras are ternary Hom–Nambu–Lie algebras as they are obtained from those ternary Nambu–Lie algebras using the composition with a homomorphism according to a general method described in [6]. For the other values of the parameter \(z\) this does not automatically yield the ternary Hom–Nambu–Lie structure since the algebras one starts with are not Nambu–Lie. We prove however that these ternary algebras of Curtright, Fairlie and Zachos, as well as the general ternary \(q\)-Virasoro–Witt algebras we construct, carry a structure of a ternary Hom–Nambu–Lie algebra in other ways, for any values of the parameters.

The paper is organized as follows. In section 1, we summarize the definition of ternary Virasoro–Witt algebras constructed using \(su(1, 1)\) enveloping algebra techniques by Curtright, Fairlie and Zachos [18]. In section 2, we recall the definitions and properties of ternary Nambu–Lie algebras and ternary Hom–Nambu–Lie algebras including also the composition method of construction of ternary Hom–Nambu–Lie algebras from ternary Nambu–Lie algebras. In section 3, we describe a family of homomorphisms of the ternary Nambu–Lie algebras of Curtright, Fairlie and Zachos and use this and the composition method to define \(q\)-Virasoro–Witt algebras. Section 4 is devoted to the structure of ternary Virasoro–Witt algebras. We show that for any value of the parameter they carry a Hom–Nambu–Lie structure. Also we describe all the twisting maps of a specific natural type. Finally, in section 5 we discuss in more general situation the problem of ternary Hom–Nambu–Lie algebras induced by ternary Nambu–Lie algebras and make some observations about ternary \(q\)-Virasoro–Witt algebras.

By \(K\) we denote throughout an algebraically closed field of characteristic zero.

1. Ternary Virasoro–Witt algebras

In this section, we define ternary Virasoro–Witt algebras constructed using \(su(1, 1)\) enveloping algebra techniques by Curtright, Fairlie and Zachos [18].
Definition 1.1. A ternary Virasoro–Witt algebra is the linear space $W$ spanned by $\{Q_n, R_n\}_{n \in \mathbb{Z}}$ with skewsymmetric trilinear multiplication given by the ternary brackets

\[
[Q_k, Q_m, Q_n] = (k - m)(m - n)(k - n)R_{k+m+n}
\]  
(1.1)

\[
[Q_k, Q_m, R_n] = (k - m)(Q_{k+m+n} + z n R_{k+m+n})
\]  
(1.2)

\[
[Q_k, R_m, R_n] = (n - m)R_{k+m+n}
\]  
(1.3)

\[
[R_k, R_m, R_n] = 0.
\]  
(1.4)

Remark 1.2. Actually the previous ternary algebra is a ternary Nambu–Lie algebra only in the cases $z = \pm 2i$. In section 3, we show that they carry for any $z$ a structure of ternary Hom–Nambu–Lie algebras.

Let us recall the connection to the ternary algebras considered by Larsson in [39]. He considered the operators

\[
E_m = e^{imx}, \quad L_m = e^{imx} \left(-i\frac{d}{dx} + \lambda m\right).
\]  
(1.5)

Here $E_m$ should be understood as an operator of multiplication by $e^{imx}$ on a suitable space of functions.

These operators satisfy the following commutation relations:

\[
[L_m, L_n] = (n - m)L_{m+n}, \quad [E_m, E_n] = 0, \quad [L_m, E_n] = nE_{m+n},
\]

where $[A, B] = AB - BA$ is the usual commutator of linear operators.

In any associative algebra with multiplication ‘·’, define the ternary multi-linear multiplication (bracket) by

\[
[x, y, z] = x \cdot [y, z] + y \cdot [z, x] + z \cdot [x, y]
\]

\[
= x \cdot (y \cdot z) - x \cdot (z \cdot y) + y \cdot (z \cdot x) - y \cdot (x \cdot z) + z \cdot (x \cdot y) - z \cdot (y \cdot x)
\]

where $[., .]$ denotes the binary commutator bracket for its corresponding Lie algebra $([a, b] = a \cdot b - b \cdot a)$. Using (1.6), one gets

\[
[L_k, L_m, L_n] = (\lambda - \lambda^2)(k - m)(m - n)(n - k)E_{k+m+n}
\]  
(1.7)

\[
[L_k, L_m, E_n] = (m - k)(L_{k+m+n} + (1 - 2\lambda)nE_{k+m+n})
\]  
(1.8)

\[
[L_k, E_m, E_n] = (m - n)E_{k+m+n}
\]  
(1.9)

\[
[E_k, E_m, E_n] = 0.
\]  
(1.10)

The brackets (1.1)–(1.4) are obtained by taking, for $\lambda \neq 0$ and $\lambda \neq 1$, $L_m = -(\lambda - \lambda^2)^{-1/4}Q_m$, $E_m = (\lambda - \lambda^2)^{-1/4}R_m$, $z = \frac{1 - 2\lambda}{(\lambda - \lambda^2)^{1/2}}$.

However, exactly the ternary Nambu–Lie algebras of Curtright, Fairlie and Zachos ($z = \pm 2i$) cannot be obtained from (1.7)–(1.10) in this way because $z = \frac{1 - 2\lambda}{(\lambda - \lambda^2)^{1/2}}$ leads to $(4 + z^2)\lambda^2 - (4 + z^2)\lambda + 1 = 0$ which has no solution for $z = \pm 2i$.

For $z \neq \pm 2i$, using that change of generators, we obtain a realization of ternary Virasoro–Witt algebras of Curtright, Fairlie and Zachos by the following operators:

\[
Q_m = -(\lambda - \lambda^2)^{-1/4}me^{imx}
\]

\[
R_m = e^{imx} \left(-i(\lambda - \lambda^2)^{1/4}\frac{d}{dx} + \lambda(\lambda - \lambda^2)^{1/4}m\right).
\]
Note that these operators also satisfy the following commutation relations:
\[
[Q_m, Q_n] = -(\lambda - \lambda^2)^{-1/4} (n - m) Q_{m+n}, \quad [R_m, R_n] = 0,
\]
which also lead to (1.1)–(1.4) using (1.6). In the presentation of Larsson appeared also second-order differential operators \( S_m = e^{i\lambda m} \left( -\frac{i}{\lambda} + \lambda m \right)^3 \). These operators can be expressed in several ways using operators \( E_m \) and \( L_m \). For example \( S_m = L_m E_{-m} L_m \). They are not needed to recover the ternary Virasoro–Witt algebras.

**Remark 1.3.** One may note that the ternary bracket (1.6) does not lead automatically to a ternary Nambu–Lie algebra when starting from an associative algebra and the corresponding Lie algebra given by the binary commutators. Another way to construct ternary Nambu–Lie algebras starting from a Lie algebra and a trace function could be found in [7]. This construction has been extended to ternary Hom–Nambu–Lie algebras in [3].

### 2. Nambu–Lie ternary algebras and Hom–Nambu–Lie ternary algebras

In this section, we recall the definitions of ternary Hom–Nambu algebras and ternary Hom–Nambu–Lie algebras introduced in [6] generalizing the usual ternary Nambu algebras and ternary Nambu–Lie algebras (called also Filippov ternary algebras).

**Definition 2.1.** A ternary Hom–Nambu algebra is a triple \((V, [, , ], \alpha)\), consisting of a vector space \(V\), a trilinear map \([, , ]: V \times V \times V \to V\) and a pair \(\alpha = (\alpha_1, \alpha_2)\) of linear maps \(\alpha_i : V \to V, i = 1, 2\), satisfying
\[
[\alpha_1(x_1), \alpha_2(x_2), [x_3, x_4, x_5]] = [[x_1, x_2, x_3], \alpha_1(x_4), \alpha_2(x_5)]
\]
\[
+ [\alpha_1(x_3), [x_1, x_2, x_4], \alpha_2(x_5)] + [\alpha_1(x_1), \alpha_2(x_4), [x_1, x_2, x_5]].
\]

(2.1)

The identity (2.1) is called the Hom–Nambu identity.

**Remark 2.2.** When the maps \((\alpha_i)_{i=1,2}\) are all identity maps, one recovers the classical ternary Nambu algebras and the identity (2.1) is called the Nambu identity. The Nambu identity is also known as the fundamental identity or Filippov identity [20, 47, 48].

**Remark 2.3.** Let \((V, [, , ], \alpha)\) be a ternary Hom–Nambu algebra where \(\alpha = (\alpha_1, \alpha_2)\). Let \(x = (x_1, x_2) \in V \times V\), \(\alpha(x) = (\alpha_1(x_1), \alpha_2(x_2)) \in V \times V\) and \(y \in V\). Let \(L_y\) be a linear map on \(V\), given by \(L_y(x) = [x_1, x_2, y]\). Then the Hom–Nambu identity is
\[
L_{\alpha(x)}([x_3, x_4, x_5]) = [L_y(x_3), \alpha_1(x_4), \alpha_2(x_5)]
\]
\[
+ [\alpha_1(x_3), L_y(x_4), \alpha_2(x_5)] + [\alpha_1(x_3), \alpha_2(x_4), L_y(x_5)].
\]

For the rest of the paper, we will use the following handy notation for the difference between the left-hand side and the right-hand side of the Nambu identity for the ternary Nambu algebras:

\[
FI(x_1, x_2, x_3, x_4, x_5) = [x_1, x_2, [x_3, x_4, x_5]] - [[x_1, x_2, x_3], x_4, x_5]
\]
\[
- [x_3, [x_1, x_2, x_4], x_5] - [x_3, x_4, [x_1, x_2, x_5]],
\]
(2.2)

and analogous notation for the Hom–Nambu identity of Hom–Nambu algebras

\[
HFI(x_1, x_2, x_3, x_4, x_5) = [\alpha_1(x_1), \alpha_2(x_2), [x_3, x_4, x_5]]
\]
\[
- [[x_1, x_2, x_3], \alpha_1(x_4), \alpha_2(x_5)] - [\alpha_1(x_3), [x_1, x_2, x_4], \alpha_2(x_5)]
\]
\[
- [\alpha_1(x_3), \alpha_2(x_4), [x_1, x_2, x_5]].
\]
(2.3)
**Definition 2.4.** A ternary Hom–Nambu algebra \((V, [\cdot, \cdot, \cdot], \alpha)\), where \(\alpha = (\alpha_1, \alpha_2)\), is called ternary Hom–Nambu–Lie algebra if the bracket is skew-symmetric that is
\[
[x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}] = Sgn(\sigma)[x_1, x_2, x_3], \quad \forall \sigma \in S_3 \quad \text{and} \quad \forall x_1, x_2, x_3 \in V,
\]
where \(S_3\) stands for the permutation group on three elements.

In particular, the skewsymmetric ternary Nambu algebras are called ternary Nambu–Lie algebras.

The morphisms of ternary Hom–Nambu–Lie algebras are defined in the natural way. It should be pointed out however that the morphisms should intertwine not only the ternary products but also the twisting linear maps.

**Definition 2.5.** Let \((V, [\cdot, \cdot, \cdot], \alpha)\) and \((V', [\cdot, \cdot, \cdot], \alpha')\) be two \(n\)-ary Hom–Nambu algebras (resp. \(n\)-ary Hom–Nambu–Lie algebras) where \(\alpha = (\alpha_1, \alpha_2)\) and \(\alpha' = (\alpha'_1, \alpha'_2)\). A linear map \(\rho : V \rightarrow V'\) is a ternary Hom–Nambu algebras morphism (resp. ternary Hom–Nambu–Lie algebras morphism) if it satisfies
\[
\rho([x_1, x_2, x_3]) = [\rho(x_1), \rho(x_2), \rho(x_3)]' \quad \rho \circ \alpha_i = \alpha'_i \circ \rho \quad \forall i = 1, 2.
\]

The following theorem, given in [6] for \(n\)-ary algebras of Lie type, provides a way to construct a ternary Hom–Nambu–Lie algebra (resp. ternary Hom–Nambu–Lie algebra) starting from a ternary Nambu algebras (resp. ternary Nambu–Lie algebra) and a ternary algebra’s endomorphism.

**Theorem 2.6 ([6]).** Let \((V, [\cdot, \cdot, \cdot])\) be a ternary Nambu algebra (resp. ternary Nambu–Lie algebra) and let \(\rho : V \rightarrow V\) be a ternary Nambu (resp. ternary Nambu–Lie) algebra endomorphism. Set \([\cdot, \cdot, \cdot]_{\rho} = \rho \circ [\cdot, \cdot, \cdot]\) and \(\tilde{\rho} = (\rho, \rho).

Then \((V, [\cdot, \cdot, \cdot]_{\rho}, \tilde{\rho})\) is a ternary Hom–Nambu algebra (resp. ternary Hom–Nambu–Lie algebra). Moreover, suppose that \((V', [\cdot, \cdot, \cdot]_{\rho'})\) is another ternary Nambu algebra (resp. ternary Nambu–Lie algebra) and \(\rho' : V' \rightarrow V'\) is a ternary Nambu (resp. ternary Nambu–Lie) algebra endomorphism. If \(f : V \rightarrow V'\) is a ternary Nambu algebras morphism (resp. ternary Nambu–Lie algebras morphism) obeying \(f \circ \rho = \rho' \circ f\), then \(f : (V, [\cdot, \cdot, \cdot], \tilde{\rho}) \rightarrow (V', [\cdot, \cdot, \cdot], \tilde{\rho'})\) is a ternary Hom–Nambu–Lie algebra’s morphism (resp. ternary Hom–Nambu–Lie algebras morphism).

**Example 2.7.** An algebra \(V\) consisting of polynomials or possibly of other differentiable functions in three variables \(x_1, x_2, x_3\), equipped with well-defined bracket multiplication given by the functional jacobian \(J(f) = (\frac{\partial f_i}{\partial x_j})_{1 \leq i, j \leq 3}:\)
\[
[f_1, f_2, f_3] = \det \begin{pmatrix}
\frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \frac{\partial f_1}{\partial x_3} \\
\frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \frac{\partial f_2}{\partial x_3} \\
\frac{\partial f_3}{\partial x_1} & \frac{\partial f_3}{\partial x_2} & \frac{\partial f_3}{\partial x_3}
\end{pmatrix},
\]
is a ternary Nambu–Lie algebra. By considering a ternary Nambu–Lie algebra endomorphism of such algebra, we construct a ternary Hom–Nambu–Lie algebra. Let \(\gamma(x_1, x_2, x_3)\) be a polynomial or more general differentiable transformation of three variables mapping elements of \(V\) to elements of \(V\) and such that \(\det J(\gamma) = 1\). Let \(\rho_\gamma : V \mapsto V\) be the composition transformation defined by \(f \mapsto f \circ \gamma\) for any \(f \in V\). By the general chain rule for composition of transformations of several variables
\[
J(\rho_\gamma(f)) = J(f) \circ \gamma = (J(f) \circ \gamma) = \rho_\gamma(J(f))J(\gamma),
\]
\[
det J(\rho_\gamma(f)) = \det (J(f) \circ \gamma) \det J(\gamma) = \det \rho_\gamma(J(f)) \det J(\gamma).
\]
Thus, for any $\gamma$ with $\det J(\gamma) = 1$, the map $\rho_\gamma$ yields an endomorphism of the ternary Nambu–Lie algebra with product (2.5). Thus $(V, [\cdot, \cdot, \cdot])_\gamma = \rho_\gamma \circ [\cdot, \cdot, \cdot]$ is a ternary Hom–Nambu–Lie algebra for any such $\gamma$ by theorem 2.6.

**Remark 2.8.** One may expect that a ternary generalization of the binary Witt algebra defined on the generators $Q_n$ by the bracket $[Q_n, Q_m] = (m - n)Q_{n+m}$ could be given by a ternary bracket defined on the generators $Q_n$ by
\[
[Q_k, Q_m, Q_n] = (m - k)(n - m)(k - n)Q_{k+m+n}.
\]
It turns out that in this case the Nambu identity is not satisfied:
\[
FI(Q_u, Q_v, Q_k, Q_m, Q_n) = -2(k - m)(k - n)(m - n)(-u + v) \times (-nu + u^2 + m(n - u - v) + k(m + n - u - v) - nv + uv + v^2)Q_{k+m+n+v},
\]
Moreover, it is not a ternary Hom–Nambu–Lie algebra for any nontrivial linear maps of the form $\alpha_1(Q_n) = a_n Q_n, \alpha_2(Q_n) = b_n Q_n$.

3. **Ternary $q$-Virasoro–Witt algebras**

In this section, we describe a family of homomorphisms of the ternary Nambu–Lie algebras of Curtright, Fairlie and Zachos and use this and the composition method to define $q$-Virasoro–Witt algebras. We also use this and theorem 2.6 to construct Hom–Nambu–Lie algebras associated with those two ternary Virasoro–Witt algebras defined by (1.1)–(1.4) which are ternary Nambu–Lie algebras ($z = \pm 2i$).

Let us provide first a class of algebra homomorphisms of ternary Virasoro–Witt algebras. We consider the linear maps $f$ acting on the generators by $f(Q_n) = a_n Q_n, f(R_n) = b_n Q_n$ and satisfying for any $X, Y, Z$ of the ternary Virasoro–Witt algebra $f([X, Y, Z]) = [f(X), f(Y), f(Z)]$. The previous identity applied to the triples $(Q_k, Q_m, Q_n), (Q_k, Q_m, R_n), (Q_k, R_m, R_n)$ and $(R_k, R_m, R_n)$, for any $k, m, n \in \mathbb{Z}$, leads to the system of equations
\[
\begin{align*}
    b_{k+m+n} &= a_k a_m a_n & \text{for} & & k \neq m \neq n \neq k & (3.1) \\
    a_{k+m+n} &= a_k a_m b_n & \text{for} & & k \neq m \neq n & (3.2) \\
    b_{k+m+n} &= a_k a_m b_n & \text{for} & & k \neq m \neq n \neq 0 & (3.3) \\
    b_{k+m+n} &= a_k b_m b_n & \text{for} & & m \neq n & (3.4)
\end{align*}
\]
which has as solution $a_n = b_n = q^n$, with $q \in \mathbb{K}$.

**Proposition 3.1.** The linear map defined on the ternary Virasoro–Witt algebra by
\[
f(Q_n) = q^n Q_n, \quad f(R_n) = q^n Q_n, \quad \text{for} \quad q \in \mathbb{K},
\]
is a ternary algebra endomorphism.

Theorem 2.6 and the ternary Virasoro–Witt algebra endomorphism described in proposition 3.1 lead to the following result.

**Theorem 3.2.** Let $W$ be the linear space generated by $(Q_n, R_n)_{n \in \mathbb{Z}}$. Let skewsymmetric trilinear bracket multiplication $[\cdot, \cdot, \cdot] : W^3 \to W$ be defined by
\[
[Q_k, Q_m, Q_n]_q = q^{k+m+n}(k - m)(m - n)(k - n)R_{k+m+n}
\]

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Remark 3.3. In theorem 3.2, for $q \in \mathbb{K}$, assume that the supports of $q \neq 0$. Curtright, Fairlie and Zachos. In what follows we assume that $q \neq 0$, since otherwise we have a zero ternary algebra.

Remark 3.4. The ternary Hom–Nambu–Lie algebras are actually not ternary Nambu–Lie algebras for $q \neq 0$. Indeed, we recover the ternary Nambu–Lie algebras obtained by Curtright, Fairlie and Zachos which are Nambu–Lie for $z = \pm 2i$.

\begin{align}
[Q_k, Q_m, R_n]_q &= q^{k+m+n}(k-m)(Q_{k+m-n} + znR_{k+m-n}) \\
[Q_k, R_m, R_n]_q &= q^{k+m+n}(n-m)R_{k+m+n} \\
[R_k, R_m, R_n]_q &= 0
\end{align}

for $q \in \mathbb{K}$. Let $\alpha_1$ and $\alpha_2$ be the linear maps on $W$ given by $\alpha_1(Q_n) = q^nQ_n$, $\alpha_1(R_n) = q^nR_n$, $\alpha_2(Q_n) = q^nQ_n$, $\alpha_2(R_n) = q^nR_n$. Then $(W, [\cdot, \cdot, \cdot]_q, (\alpha_1, \alpha_2))$ are the ternary Hom–Nambu–Lie algebras if $z = \pm 2i$.

We call $(W, [\cdot, \cdot, \cdot]_q)$ for any values of $z$ a ternary $q$-Virasoro–Witt algebra.

**Theorem 4.1.** Let $W$ be the linear space generated by $\{Q_n, R_n\}_{n \in \mathbb{Z}}$. Let $\alpha_1$ and $\alpha_2$ be two linear maps on $W$ defined on the generators by

\begin{align}
\alpha_1(Q_n) &= a_nQ_n + b_nR_n, \\
\alpha_1(R_n) &= c_nQ_n + d_nR_n \\
\alpha_2(Q_n) &= a_n^2Q_n + b_n^2R_n, \\
\alpha_2(R_n) &= c_n^2Q_n + d_n^2R_n.
\end{align}

Assume that the supports of $a_n, b_n, c_n, d_n$, $i = 1, 2$, as functions of $n$ are $\emptyset$ or $\mathbb{Z}$.

Then the ternary $q$-Virasoro–Witt algebras on $W$ defined by brackets (3.6)–(3.9) are ternary Hom–Nambu–Lie algebras with twisting maps $\alpha_1$ and $\alpha_2$, for any value of $z$ and $q$, if
and only if

\[ \alpha_1(Q_n) = \beta_1 q^n R_n, \quad \alpha_1(R_n) = 0, \]  
\[ \alpha_2(Q_n) = \beta_2 q^n R_n, \quad \alpha_2(R_n) = 0, \]  

where \( \beta_1, \beta_2 \in \mathbb{K}. \)

**Proof.** The Hom–Nambu identities are equivalent to a large system of equations for the coefficients \( a_n^i, b_n^i, c_n^i, d_n^i, i = 1, 2. \) The exact relation between the Hom–Nambu identities and the final quite special form of \( \alpha_1 \) and \( \alpha_2 \) is a mystery. This proof pinpoints instances of this relation on the computational level which might help to understand this interdependence on a more conceptual level.

The identity \( \text{HFI}(Q_u, Q_v, Q_k, Q_m, Q_n) = 0, \) for any \( u, v, k, m, n \), yields

\[ a_n^1 (q^n a_m^2 - q^m a_n^2) = 0. \]  

Thus, \( a_n^1 = 0 \) or \( q^n a_m^2 - q^m a_n^2 = 0 \) for any \( n \) and \( m \). Hence, \( a_n^2 = \lambda_2 q^n, \lambda_2 \in \mathbb{K}. \)

**Case 1.** \( a_n^1 = 0. \) The identity \( \text{HFI}(Q_u, Q_v, Q_k, Q_m, Q_n) = 0 \) leads to

\[ a_n^2 b_n^1 = 0, \quad (q^n a_m^2 - q^m a_n^2) b_n^1 = 0, \quad a_n^2 (q^n b_m^1 - q^m b_n^1) = 0. \]  

Therefore, we have two subcases to study \( a_n^2 = 0 \) or \( a_n^2 \neq 0 \) and \( b_n^1 = 0 \).

**Case 1.1.** \( a_n^1 = 0, a_n^2 = 0. \) The identity \( \text{HFI}(Q_u, Q_v, Q_k, Q_m, Q_n) = 0 \) is fulfilled in this case, but \( \text{HFI}(Q_u, R_v, Q_k, Q_m, Q_n) = 0 \) for any \( u, v, k, m, n \) is equivalent to

\[ (q^n b_k^1 - q^k b_n^1) b_n^2 = 0, \quad b_n^1 (q^n b_m^2 - q^m b_n^2) = 0, \quad (q^n b_k^1 b_n^2 - q^k b_m^2 b_n^2) = 0, \]  
\[ b_n^1 c_n^1 = 0. \]  

Then, we consider the following subcases.

**Case 1.1.1.** \( a_n^1 = 0, a_n^2 = 0, b_n^1 = 0. \) The identity \( \text{HFI}(Q_u, Q_v, R_k, Q_m, Q_n) = 0 \) is equivalent to the system \( c_n^1 (q^n b_m^2 - q^m b_n^2) = 0, b_n^1 c_n^1 = 0. \) In case \( b_n^1 = 0, \) the identity \( \text{HFI}(Q_u, R_v, Q_k, Q_m, Q_n) = 0 \) is equivalent to the equations \( c_n^1 c_n^2 = 0, c_n^1 d_n^1 = 0, c_n^1 d_n^2 = 0, d_n^1 d_n^2 = 0. \) This case leads to \( \alpha_1 \) or \( \alpha_2 \) identically trivial. In case \( b_n^1 \neq 0 \) we have \( c_n^1 = 0. \) The remaining identities lead to \( \alpha_1 \) identically trivial.

**Case 1.1.2.** \( a_n^1 = 0, a_n^2 = 0, b_n^1 \neq 0. \) According to the second equation of the system (4.6) we should consider \( b_n^1 = \beta_2 q^n, \) where \( \beta_2 \) is any scalar in \( \mathbb{K}. \) The identity \( \text{HFI}(Q_u, R_v, Q_k, Q_m, Q_n) = 0 \) translates to the equations \( q^n b_k^1 - q^k b_n^1 = 0, b_n^1 c_n^1 = 0. \) Thus, \( b_n^1 = \beta_1 q^n, \) where \( \beta_1 \) is any scalar in \( \mathbb{K} \) and \( c_n^1 = 0. \) The remaining identities imply that \( c_n^1 = 0, d_n^1 = 0, d_n^2 = 0. \) Therefore, we get only \( \alpha_1, \alpha_2 \) given by (4.3) and (4.4).

**Case 1.2.** \( a_n^1 = 0, a_n^2 \neq 0. \) According to the system (4.6) we have \( b_n^1 = 0. \) The identity \( \text{HFI}(Q_u, Q_v, R_k, Q_m, Q_n) = 0 \) implies \( a_n^2 c_n^1 = 0, b_n^1 c_n^1 = 0, a_n^2 d_n^1 = 0. \) Since \( a_n^1 \neq 0, \) then \( c_n^1 = 0 \) and \( d_n^1 = 0. \) Thus \( \alpha_1 \) is identically trivial.

**Case 2.** \( a_n^1 \neq 0. \) According to (4.5) we have \( a_n^2 = \lambda_2 q^n \) and \( b_n^2 = \beta_2 q^n. \) The identity \( \text{HFI}(Q_u, R_v, Q_k, Q_m, Q_n) = 0 \) implies that \( a_n^1 c_n^2 = 0. \) Since \( a_n^1 \neq 0, \) then \( c_n^2 = 0. \) We consider in the following two subcases \( \lambda_2 = 0 \) and \( \lambda_2 \neq 0. \)

**Case 2.1.** \( a_n^1 \neq 0, a_n^2 = 0. \) The identities \( \text{HFI}(Q_u, Q_v, Q_k, Q_m, Q_n) = 0 \) and \( \text{HFI}(Q_u, R_v, Q_k, Q_m, Q_n) = 0 \) imply \( b_n^1 = 0, d_n^2 = 0. \) Therefore \( \alpha_2 = 0. \)

**Case 2.2.** \( a_n^1 \neq 0, a_n^2 \neq 0. \) The identity \( \text{HFI}(Q_u, Q_v, R_k, Q_m, Q_n) = 0 \) leads to \( c_n^1 = 0 \) and \( a_n^2 = \lambda_1 q^n \) with \( \lambda_1 \neq 0. \) The identity \( \text{HFI}(Q_u, Q_v, Q_k, Q_m, Q_n) = 0 \) leads to \( b_n^1 = \beta_1 q^n. \)
and \( \beta_2 = \frac{1}{z^3} \). Using HFI\((Q_u, R_v, Q_k, Q_{m}, Q_{n}) = 0 \) and HFI\((Q_u, Q_v, R_k, Q_{m}, Q_{n}) = 0 \) we obtain \( d^1_\alpha = \lambda_2 t^{\alpha}, d^2_\alpha = \lambda_1 t^{\alpha} \) and \( \beta_1 = 0 \). Thus \( \beta_2 = 0 \). In this case, the Hom–Nambu–Lie identities yield \( \lambda_1 \lambda_2 (z^2 + 4) = 0 \), which is impossible for any \( z \neq \pm 2i \), since \( \lambda_1 \lambda_2 \neq 0 \). In fact, this case corresponds to theorem 3.2 where \( z^2 + 4 = 0 \) and \( \alpha_1(Q_n) = \lambda_1 q^n Q_n \), \( \alpha_1(R_n) = \lambda_1 q^n R_n \), \( \alpha_2(Q_n) = \lambda_2 q^n Q_n \), \( \alpha_2(R_n) = \lambda_2 q^n R_n \).

In the case \( q = 1 \), we obtain the following corollary.

**Proposition 4.2.** Let \( \alpha_1 \) and \( \alpha_2 \) be two linear maps defined on the generators of ternary Virasoro–Witt algebras by

\[
\alpha_1(Q_n) = a_1^1 Q_n + b_1^1 R_n, \quad \alpha_1(R_n) = c_1^1 Q_n + d_1^1 R_n
\]

\[
\alpha_2(Q_n) = a_2^1 Q_n + b_2^1 R_n, \quad \alpha_2(R_n) = c_2^1 Q_n + d_2^1 R_n.
\]

Assume further that supports of \( a_i^j, b_i^j, c_i^j, d_i^j, i = 1, 2 \), as functions of \( n \) are \( \emptyset \) or \( \mathbb{Z} \).

Then the ternary Virasoro–Witt algebras on \( W \) defined by the brackets (1.1)–(1.4) are ternary Hom–Nambu–Lie algebras with twisting maps \( \alpha_1 \) and \( \alpha_2 \), for any value of \( z \), if and only if

\[
\alpha_1(Q_n) = \beta_1 R_n, \quad \alpha_1(R_n) = 0,
\]

\[
\alpha_2(Q_n) = \beta_2 R_n, \quad \alpha_2(R_n) = 0.
\]

**Remark 4.3.** We have previously considered only linear maps with a global support but of course it could be possible to get other solutions if one considers different supports for the maps.

An interesting open problem is the complete description and classification of all possible linear maps \( \alpha_1 \) and \( \alpha_2 \) yielding Hom–Nambu or Hom–Nambu–Lie structures on the ternary \( q \)-Virasoro–Witt algebras. Theorem 4.1 and proposition 4.2 are steps in this direction.

5. Ternary Hom–Nambu–Lie algebras induced by ternary Nambu–Lie algebras

In this section, we address for ternary algebras the following question discussed for binary Hom-associative algebras by Fréjigier and Gohr in [21, 25] (see also [22]): when is a ternary Hom–Nambu–Lie algebra \((V, [\cdot, \cdot, \cdot], (\alpha, \alpha))\) induced by a ternary Nambu–Lie algebra \((V, [\cdot, \cdot, \cdot])\) by the composition method according to theorem 2.6? That is, when does a ternary algebra endomorphism \( \rho : V \to V \) exist with respect to \([\cdot, \cdot, \cdot]\) such that \([\cdot, \cdot, \cdot] = \rho \circ [\cdot, \cdot, \cdot]\) and \( \alpha = \rho \)?

Let \((V, [\cdot, \cdot, \cdot], (\alpha, \alpha))\) be a ternary Hom–Nambu–Lie algebra and \( \alpha \) be a ternary algebra endomorphism. The Hom–Nambu identity for any \( x_1, \ldots, x_5 \in V \) is

\[
\text{HFI}(x_1, x_2, x_3, x_4, x_5) = [\alpha(x_1), \alpha(x_2), [x_3, x_4, x_5]] - [[x_1, x_2, x_3], \alpha(x_4), \alpha(x_5)]
\]

\[
- [\alpha(x_3), [x_1, x_2, x_4], \alpha(x_5)] - [\alpha(x_3), \alpha(x_4), [x_1, x_2, x_3]] = 0.
\]

If \( \alpha \) is invertible, then it is equivalent to

\[
[x_1, x_2, [\alpha^{-1}(x_3), \alpha^{-1}(x_4), \alpha^{-1}(x_5)]] - [[\alpha^{-1}(x_1), \alpha^{-1}(x_2), \alpha^{-1}(x_3)], x_4, x_5]
\]

\[
- [x_3, [\alpha^{-1}(x_1), \alpha^{-1}(x_2), \alpha^{-1}(x_4)], x_5]
\]

\[
- [x_3, x_4, [\alpha^{-1}(x_1), \alpha^{-1}(x_2), \alpha^{-1}(x_3)]] = 0.
\]

(5.1)

If \( \alpha \) is a ternary algebra endomorphism, then \( \alpha^{-1} \) is also a ternary algebra endomorphism. Indeed, for any \( x_1, x_2, x_3 \in V \),
formal deformation theory and (co-)homology theory for usual Nambu, Nambu–Lie and Hom–Nambu–Lie algebras is yet to be developed. When this is done it should extend the formal deformation theory and corresponding (co-)homology theory for Hom–Nambu and Nambu–Lie algebras.

It is shown in theorem 4.1 that these ternary algebras carry structures of algebras equipped by linear maps twisting the fundamental identity and generalizing Nambu–Lie algebras. It is shown in theorem 4.1 that these ternary algebras carry structures of ternary Hom–Nambu–Lie algebras for all values of the involved parameters, thus allowing to consider these algebras within the same general framework.

Therefore, identity (5.1) can be written as

\[ [x_1, x_2, \alpha^{-1}([x_3, x_4, x_5])] - [\alpha^{-1}([x_1, x_2, x_3]), x_4, x_5] \]

Assume now that the bracket \([\cdot, \cdot, \cdot]\) is induced by a ternary Nambu–Lie algebra \((V, [\cdot, \cdot, \cdot])\) and a ternary algebra endomorphism \(\rho\), that is \([\cdot, \cdot, \cdot] = \rho \circ [\cdot, \cdot, \cdot]'\). It follows that identity (5.2) is

\[ \rho([x_1, x_2, \alpha^{-1} \circ \rho([x_3, x_4, x_5])]) - [\alpha^{-1} \circ \rho([x_1, x_2, x_3]), x_4, x_5] \]

The previous identity corresponds to a Nambu identity if \(\alpha^{-1} \circ \rho\) is the identity map. Hence, we get the following proposition.

**Proposition 5.1.** Let \((V, [\cdot, \cdot, \cdot], (a, \alpha))\) be a ternary Hom–Nambu–Lie algebra and assume that \(\alpha\) is a ternary algebra automorphism. Then the ternary Hom–Nambu–Lie algebra \((V, [\cdot, \cdot, \cdot], (\alpha, a))\) is induced by a ternary Hom–Nambu–Lie algebra \((V, [\cdot, \cdot, \cdot]')\) where \([\cdot, \cdot, \cdot]' = \alpha^{-1}[\cdot, \cdot, \cdot]\).

**Remark 5.2.** We observe that ternary Hom–Nambu–Lie \(q\)-Virasoro–Witt algebras in theorem 3.2 can be untwisted by the linear map \(v(Q_n) = q^{-n} Q_n, v(R_n) = q^{-n} R_n\), since \(a_1 = a_2\) and \(a_1^{-1} = a_2^{-1} = v\). The ternary Hom–Nambu–Lie \(q\)-Virasoro–Witt algebras constructed in theorem 4.1 could not be untwisted in such a way (for \(\beta_1 = \beta_2\)), as the linear maps are nilpotent and thus are non-invertible. It would be of interest to know whether for \(\beta_1 = \beta_2\) it is possible to untwist these algebras in some other subtle ways.

**Remark 5.3.** In the same way as ternary Nambu–Lie algebra structures can be viewed as ternary generalizations of Poisson structures, the ternary Hom–Nambu–Lie algebras can be viewed as ternary generalizations of Hom–Poisson structures from formal deformation theory and deformation quantization for Hom–Lie and Hom-associative algebras [46]. General formal deformation theory and corresponding (co-)homology theory for Hom–Nambu and Hom–Nambu–Lie algebras is yet to be developed. When this is done it should extend the formal deformation theory and (co-)homology theory for usual Nambu, Nambu–Lie and associative-type ternary algebras [5]. The extra freedom provided in ternary Hom–Nambu–Lie algebras via the linear maps twisting the fundamental identity may provide a new path to quantization within the general framework of ternary Hom–Nambu–Lie algebras, bringing also further insight into Hom–Poisson structures, an extension of Poisson structures to realm of Hom–Lie and quasi-Lie algebras [36, 46].

### 6. Conclusion

The ternary algebras constructed in theorem 3.2 may be viewed as a deformation of the ternary Virasoro–Witt algebras of Curttright, Fairlie and Zachos within Hom–Nambu–Lie algebras, a class of algebras equipped by linear maps twisting the fundamental identity and generalizing Nambu–Lie algebras. It is shown in theorem 4.1 that these ternary algebras carry structures of ternary Hom–Nambu–Lie algebras for all values of the involved parameters, thus allowing to consider these algebras within the same general framework.
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