GEOMETRIC DIVISORS IN NORMAL LOCAL DOMAINS

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Abstract. Let $A$ be the local ring at a point of a normal complex variety with completion $R = \hat{A}$. Srinivas has asked about the possible images of the induced map $\text{Cl}_A \hookrightarrow \text{Cl}_R$ over all geometric normal domains $A$ with fixed completion $R$. We use Noether-Lefschetz theory to prove that all finitely generated subgroups are possible in some familiar cases. As a byproduct we show that every finitely generated abelian group appears as the class group of the local ring at the vertex of a cone over some smooth complex variety of each positive dimension.

1. Introduction

In his survey on geometric methods in commutative algebra [27], Srinivas poses several interesting questions about geometric local rings, the local rings $A = \mathcal{O}_{X,x}$ of a point $x$ on a complex algebraic variety $X$, or equivalently local domains of essentially finite type over $\mathbb{C}$. For $A$ normal with completion $R = \hat{A}$, there is a natural map

$$\iota : \text{Cl}_A \hookrightarrow \text{Cl}_R$$

whose injectivity is attributed to Mori [25, Thm. 6.5], so we call $\iota$ the Mori map. Many non-isomorphic geometric rings can have isomorphic completions. For example, the local rings at smooth points of $d$-dimensional complex varieties have isomorphic completions by Cohen’s theorem [18, Thm. 60, Cor. 2], but unless the varieties are birational they do not share the same fraction field. Srinivas asks about the images of the inclusion (1) as $A$ varies with fixed completion $R = \hat{A}$, there is a natural map

$$\iota : \text{Cl}_A \hookrightarrow \text{Cl}_R$$

Question 1.1. Which subgroups of $\text{Cl}_R$ arise as images $\text{Cl}_A \hookrightarrow \text{Cl}_R$ where $R \cong \hat{A}$?

Example 1.2. Srinivas notes that not all images are possible, because the complete local ring $R = \mathbb{C}[[x, y, z]]/(x^2 + y^3 + z^5)$ has class group $\text{Cl}_R \cong \mathbb{C}$, but for every geometric local ring $A$ with $\hat{A} \cong R$, the image $\text{Cl}_A \hookrightarrow \text{Cl}_R$ must be finitely generated [27, Ex. 3.9]. He reasons that if $A = \mathcal{O}_{X,x}$ for a surface $X$ and $Y \to X$ is a resolution of singularities, then the induced map $\text{Pic}^0 Y \to \text{Cl}_A$ is surjective. Since $\text{Pic}^0 Y$ is projective, it has trivial image in the affine group $\mathbb{G}_a = \mathbb{C}$, therefore $\text{Cl}_A \to \mathbb{C}$ factors through the finitely generated Neron-Severi group.

Remark 1.3. Example [1.2] shows that in fact $\mathbb{C}$ is never the class group of any geometric local ring. More generally, the argument applies to any infinitely generated group whose connected component of the identity is an affine group.

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In view of Example 1.2, Srinivas concludes that “probably the only reasonable general question” one can ask is for the smallest possible images. If $A$ is the codimension $r$ quotient of a regular local ring $B$, the dualizing module $\omega_A = \text{Ext}^r_{\mathcal{O}_A}(B, A) \in \text{Cl} R$ is trivial if and only if $R$ is Gorenstein \[21\]. This class is independent of $B$ and the image in $\text{Cl} R$ is independent of $A$, because $\widetilde{\omega}_A \cong \omega_{\hat{A}}$ \[7, Thm. 3.3.5(c)\]. These considerations led Srinivas to ask the following \[27, Ques. 3.5 and 3.7\]:

**Question 1.4.** Let $R$ be the completion of a normal geometric local ring.

(a) If $R$ is Gorenstein, is $R$ the completion of a UFD?

(b) More generally, does there exist $B$ for which $\text{Cl} B = \langle \omega_R \rangle \subset \text{Cl} R$?

Here there has been progress, starting with Grothendieck’s solution \[12, XI, Cor. 3.14\] to Samuel’s conjecture stating that a local complete intersection ring that is factorial in codimension $\leq 3$ is a UFD. Hartshorne and Ogus proved that if $R$ has an isolated singularity, depth $\geq 3$ and embedding dimension at most $2 \dim R - 3$, then $R$ is a UFD \[15\]. Expanding on Srinivas’ observation that local rings of rational double point surface singularities are completions of UFDs \[26\], Parameswaran and Srinivas \[22\] used methods along the lines of Lefschetz’ proof of Noether’s theorem to construct a UFD defining a singularity analytically isomorphic to isolated complete intersection singularity of dimension 2 or 3. We gave a positive answer for hypersurface singularities of dimension $\geq 2$ \[5\] and arbitrary local complete intersection singularities of dimension $\geq 3$ \[6\]. The only result we’ve seen in the non-Gorenstein case is due to van Straten and Parameswaran \[23\], who showed that Question 1.4 (a) has a positive answer if $\dim R = 2$. Heitmann has characterized completions of UFDs \[16\], but his constructions are rarely geometric.

While Question 1.4 has received much attention, Question 1.1 remains wide open. In an effort to better understand it, we call an element $\alpha \in \text{Cl} R$ a geometric divisor if it is in the image of the inclusion \[1\] for some geometric local ring $A$ with $R = \hat{A}$. In view of Example 1.2 we pose the following:

**Question 1.5.** Let $R$ be the completion of a normal geometric local ring.

(a) Given any finitely generated group $G \subset \text{Cl} R$, is there a geometric local ring $B$ with $\hat{B} = R$ and $G = \text{Cl} B$?

(b) Given $\alpha_1, \ldots, \alpha_r \in \text{Cl} R$, is there a geometric local ring $B$ with $\hat{B} = R$ and $\alpha_i \in \text{Cl} B$ for each $1 \leq i \leq r$?

(c) Is every $\alpha \in \text{Cl} R$ a geometric divisor?

(d) Do the geometric divisors form a subgroup of $\text{Cl} R$?

Note the implications $(a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (d)$. Our ignorance about the nature of geometric divisors is revealed in part (c): could there be transcendental divisors that cannot be accessed geometrically? The methods of \[5, 6\] inspire the following possibility:

**Conjecture 1.6.** Let $R$ be the completion of a normal geometric complete intersection ring. Then for any finitely generated group $G \subset \text{Cl} R$, there is a geometric local ring $B$ with $\hat{B} = R$ and $G = \text{Cl} B \subset \text{Cl} R$.

In the best-understood case that $R$ is a completed rational double point surface singularity, we used our Noether-Lefschetz theorem \[2\] with carefully chosen base loci to show...
that every subgroup of Cl $R$ is the image of Cl $A$ for the local ring $A = \mathcal{O}_{S,p}$ of a surface $S \subset \mathbb{P}^3$ \cite{5}, thereby proving Conjecture \ref{conj} for two dimensional rational double point singularities.

**Remark 1.7.** This result contrasts sharply with case in which the function field is rational, where Mohan Kumar \cite{20} shows that for most $A_n$ and $E_n$ singularities there is in fact only one possible isomorphism class for the local ring (and thus the class group). The three exceptions, with two possibilities each, are the $E_8$, $A_7$, and $A_8$; for all other $E_n$ and $A_n$ singularities, the Mori map is an isomorphism. Since an $E_8$ is a UFD under completion, any $E_8$ is a UFD. By following Mohan Kumar’s constructions of the $A_7$ and $A_8$ carefully, one sees that the image of the Mori map for the $A_7$ is either the full completed class group $\mathbb{Z}/8\mathbb{Z}$ or the subgroup of order 4, while in the $A_8$ case the Mori map is either surjective onto $\mathbb{Z}/9\mathbb{Z}$ or its image is of order 3.

In this paper we give more positive evidence for Conjecture \ref{conj}. Our main tool is the following theorem, which proves $(b) \Rightarrow (a)$ in Question \ref{quest} for most complete intersection domains.

**Theorem 1.8.** Let $p \in V \subset \mathbb{P}^n$ be a normal complete intersection point on a complex variety, where $n = 3$ if $\dim V = 2$. Then for any finitely generated subgroup $G \subset \text{Cl} \mathcal{O}_{V,p}$, there is a geometric local ring $B$ with $\hat{B} = \hat{\mathcal{O}}_{V,p}$ and $\text{Cl} B = G \subset \text{Cl} \mathcal{O}_{V,p} \subset \text{Cl} \hat{\mathcal{O}}_{V,p}$.

**Remark 1.9.** The restriction $n = 3$ if $\dim V = 2$ shows the reliance of our argument on the Noether-Lefschetz theorems \cite{2, 6}. We are currently working to remove this restriction.

One application of Theorem 1.8 is a proof of Conjecture \ref{conj} for the completed local ring at the vertex of a cone over certain projective varieties, even though the class groups of these local rings tend to be infinitely generated.

**Corollary 1.10.** Conjecture \ref{conj} holds for the completed local ring at the vertex $p$ of the cone $V$ over a smooth variety $X \subset \mathbb{P}^n$ satisfying $H^1(\mathcal{O}_X(k)) = H^2(\mathcal{O}_X(k)) = 0$ for $k > 0$.

As in Remark 1.9, we must take $n = 2$ when $X$ is a curve. Thus examples of $X$ in Corollary 1.10 include plane curves of degree $d \leq 3$, surfaces $X \subset \mathbb{P}^3$ of degree $d \leq 4$ and Arithmetically Cohen-Macaulay subvarieties $X \subset \mathbb{P}^n$ of dimension at least 2.

Claborn showed in the 1960s that every abelian group is the class group of a Dedekind domain \cite{8}. A recent refinement of Clark shows that the Dedekind domain may be taken as the integral closure of a PID in a quadratic extension field \cite{9}. Results like these are impossible for geometric Dedekind domains because they have trivial class group, but we are led to ask:

**Question 1.11.** Which abstract groups are class groups of normal geometric rings?

Not all infinitely generated groups are possible by Remark 1.3, but Example 1.2 suggests that finite extensions of quotients of Abelian varieties by finite subgroups are candidates. For example, the quotient of the Picard group of a smooth non-rational plane curve $C$ by $\langle \mathcal{O}_C(1) \rangle$ arises in this way (see Proposition 3.1 (a)). We apply Theorem 1.8 to show that every finitely generated abelian group is possible:
Theorem 1.12. Let $G$ be any finitely generated abelian group. Then there is a point $p$ on a normal surface $S \subset \mathbb{P}^3$ for which $G \cong \text{Cl}\mathcal{O}_{S,p} \subset \hat{\text{Cl}}\mathcal{O}_{S,p}$.

One of our goals here is to publicize these interesting questions. The main result is Theorem 1.12 and the applications above. In the long run we hope that this theorem will pave the way for a proof of Conjecture 1.6. In section 2 we prove the theorem and apply it to rational double point singularities. Section 3 addresses vertex singularities on cones. In Section 4 we compute the completed class group of the singularity at $p$ of a general surface containing $r$ general lines passing through $p$.

2. FINITELY GENERATED SUBGROUPS OF LOCAL CLASS GROUPS

In this section we prove Theorem 1.12 from the introduction, which follows immediately from Lemma 2.1 and Theorem 2.2 below. We note that Conjecture 2.5 would remove the one technical assumption in Theorem 2.2 and recover in Corollary 2.6 the main theorem of [5] with a very short proof.

Lemma 2.1. Let $p \in V \subset \mathbb{P}^n$ be a normal singularity, and let $g \in \text{Cl}\mathcal{O}_{V,p}$. Then there exists an integral divisor on $V$ whose class in $\mathcal{O}_{V,p}$ is $g$.

Proof. The element $g$ lifts to a Weil divisor $C \in \text{Cl}V$ because height one primes in the local ring corresponds to codimension one subvarieties of $V$. Replacing $C$ with $C+nH$ we may assume that $C$ is effective and we claim that $C$ may be taken integral. To see this, choose $n > 0$ so that $I_C(n)$ is generated by global sections. Since $I_C$ is generically a principal ideal along $C$, the general section $s \in H^0(I_C(n))$ generates $I_C$ off a set of codimension two. Therefore the hypersurface $s = 0$ on $V$ consists of $C$ along with another divisor $E$ meeting $C$ properly and $C+E = nH$ on $V$. Choosing $m > n$ so that $I_E(m)$ is generated by global sections, the linear system $H^0(I_E(m+1))$ separates points and tangent vectors away from the base locus $E$ and the general member consists of a proper union $E \cup D$. The restricted linear system separates points and tangent vectors away from $E$, hence gives a map $V - E \to \mathbb{P}^N$ whose image has dimension $\dim V \geq 2$ and we may apply Bertini’s theorem [17] Thm. 6.3] to see that $D$ is integral. Finally

$$D \cong mH - E \cong mH - (nH - C) \cong (m-n)H + C$$

so $D$ is linearly equivalent to $C$ modulo the hyperplane class $H$ (which is trivial in $\text{Cl}\mathcal{O}_{V,p}$), hence equivalent to $g$ in $\text{Cl}\mathcal{O}_{V,p}$. □

Theorem 2.2. Let $V \subset \mathbb{P}^n$ be a variety with normal local complete intersection point $p \in V$, where $n = 3$ if $\dim V = 2$. Given prime divisors $C_i \subset V$, there is a closed subscheme $B \subset \mathbb{P}^n$ containing the $C_i$ such that the general complete intersection $W$ of dimension $\dim V$ containing $B$ is analytically isomorphic to $V$ at $p$, the isomorphism identifying $\text{Cl}\mathcal{O}_{W,p} \subset \hat{\text{Cl}}\mathcal{O}_{W,p}$ with the subgroup generated by the $C_i$.

Remark 2.3. The base locus $B$ in Theorem 2.2 can be taken to be the union of the $C_i$ and a multiplicity structure supported on Sing $V$. The existence of the analytic isomorphism $\hat{\psi} : \hat{\mathcal{O}}_{V,p} \to \hat{\mathcal{O}}_{W,p}$ is given by our extension of Ruiz’ lemma [6] Prop. 4.1] and the classes $C_i$ generate $\text{Cl}\mathcal{O}_{W,p}$ by [6] Cor. 1.5]. The hard part is showing that the abstractly constructed isomorphism $\hat{\psi}$ takes the classes $C_i$ to their expected destinations in $\text{Cl}\mathcal{O}_{W,p}$. 
Proof. By Lemma 2.1 each $g_i \in G$ is represented by an integral divisor $C_i \subset V$ corresponding to a height one prime $\mathfrak{p}_i \subset \mathcal{O}_{V,p}$ via the ideal sheaf $\mathcal{I}_{C,V}$ and hence to a prime $Q_i \subset \mathcal{O}_{V,p}$ of height $n - d + 1$, where $d = \dim V$. Denote by $\widehat{\mathfrak{p}}_i$ and $\widehat{Q}_i$ the ideals generated in their respective completions.

$$Q_i \subset \mathcal{O}_{V,p} \quad \downarrow \quad \widehat{Q}_i \subset \widehat{\mathcal{O}}_{V,p}$$

$$\mathfrak{p}_i \subset \mathcal{O}_{V,p} \quad \downarrow \quad \widehat{\mathfrak{p}}_i \subset \widehat{\mathcal{O}}_{V,p}$$

Recall that a ring $A$ is excellent [18, § 13] if it is Noetherian, universally catenary, a G-ring (the completion map $A_p \to \widehat{A}_p$ is regular for all $p \in \text{Spec } A$) and the regular locus of any finitely generated $A$-algebra $B$ is open in $\text{Spec } B$. This class of rings includes complete Noetherian local rings and localizations of quotients of regular local rings. Thus the quotients $\mathcal{O}_{V,p}/Q_i$ are excellent and hence their completions $\mathcal{O}_{V,p}/\mathfrak{q}_i \cong \widehat{\mathcal{O}}_{V,p}/\widehat{Q}_i$ retain the property of being reduced [19, Thm. 32.2], therefore $\widehat{Q}_i$ can be written as an intersection of prime ideals $\bigcap \widehat{Q}_{i,j}$. Furthermore, normality of $\mathcal{O}_{V,p}$ implies that of $\widehat{\mathcal{O}}_{V,p}$ [19, Thm. 32.2], so both rings are equidimensional and catenary as well, since both are excellent. It follows [19, Thm. 31.5] that $\widehat{\mathcal{O}}_{V,p}/\widehat{Q}_i$ is equidimensional so that each $\widehat{Q}_{i,j}$ has height $n - d + 1$ (like $Q_i$ itself). Likewise for the quotient rings $\mathcal{O}_{V,p}$ and $\widehat{\mathcal{O}}_{V,p}$ we may write $\widehat{\mathfrak{p}}_i = \bigcap \widehat{\mathfrak{p}}_{i,j}$ with $\widehat{\mathfrak{p}}_{i,j}$ (the images of the $\widehat{Q}_{i,j}$) of height one. By the uniqueness of this irredundant primary decomposition [11, Thm. 18.24], one sees that $\widehat{\mathfrak{p}}_i$ contains elements in $\widehat{\mathfrak{p}}_{i,j} \setminus \widehat{\mathfrak{p}}_{j,i}$, so the valuation $v_{i,j}$ corresponding to $\widehat{\mathfrak{p}}_{i,j}$ satisfies $v_{i,j}(\widehat{\mathfrak{p}}_i) = 1$. Therefore in Samuel’s formula [25] for the injection $j_V : \text{Cl } \mathcal{O}_{V,p} \to \text{Cl } \widehat{\mathcal{O}}_{V,p}$, we have $j_V(\widehat{\mathfrak{p}}_i) = \sum_j \widehat{\mathfrak{p}}_{i,j}$, and $j_V(G) = \langle \sum_j \widehat{\mathfrak{p}}_{1,j}, \sum_j \widehat{\mathfrak{p}}_{2,j}, \ldots, \sum_j \widehat{\mathfrak{p}}_{r,j} \rangle$.

Since $p \in V$ is a normal local complete intersection, $V$ is locally defined by an ideal $I_V = (F_1, F_2, \ldots, F_n)$ and the singular locus is defined by $I_V + J$, where $J$ is the Jacobian ideal generated by partial derivatives of the $F_i$, and $(F_i) + J$ has height $\geq n - d + 2$ by normality. Now let $Z$ be the subscheme of $\mathbb{P}^d$ defined by the ideal $(\bigcap I_{C_i} \cdot J_F^2) + I_V$. Scheme-theoretically, $Z$ the union of the $C_i$ of dimension $d - 1$ along with a multiplicity structure on the singular locus of dimension $\leq d - 2$.

The general complete intersection $W$ of dimension $d$ containing $Z$ is locally defined by $F'_i = F_i + T$, where $T \in J^2 I_C$. By [6, Prop. 4.1], $W$ is analytically isomorphic to $V$ at $p$; that is, there is an automorphism $\alpha$ of $\mathbb{C}[[x_1, x_2, \ldots, x_n]]$ given by the change of variables $x_i \mapsto x_i + h_i$, $h_i \in J I_C \mathbb{C}[[x_1, x_2, \ldots, x_n]]$, $1 \leq i \leq n$, such that $\alpha(F_i) = F'_i$, that is $F'_i(x_1, x_2, \ldots, x_n) = F_i(x_1 + h_1, x_2 + h_2, \ldots, x_n + h_n)$. Moreover, for $f \in \widehat{Q}_{i,j}$, the difference

$$f(x_1 + h_1, x_2 + h_2, \ldots, x_n + h_n) - f(x_1, x_2, \ldots, x_n)$$

clearly lies in $(h_1, h_2, \ldots, h_n) \subset I_C \mathbb{C}[[x_1, x_2, \ldots, x_n]] \subset \widehat{Q}_{i,j}$, so $\alpha(\widehat{Q}_{i,j}) \subset \widehat{Q}_{i,j}$ and therefore $\alpha(\widehat{Q}_{i,j}) = \widehat{Q}_{i,j}$ since each $\widehat{Q}_{i,j}$ is prime.

From the preceding paragraph we see that $\alpha$ induces an isomorphism

$$\bar{\alpha} : \widehat{\mathcal{O}}_{V,p} = \mathbb{C}[[x_1, x_2, x_3]]/(F_i) \to \mathbb{C}[[x_1, x_2, x_3]]/(F'_i) = \widehat{\mathcal{O}}_{W,p}$$
satisfying \( \overline{\alpha}(\hat{P}_{i,j}) = \hat{P}_{W,i,j} \), where the subscript \( W \) denotes passing to the quotient in \( \mathcal{O}_{W,p} \) (or \( \hat{\mathcal{O}}_{W,p} \) in the sequel).

By our Noether-Lefschetz theorem [6, Cor. 1.6] (2 Thm. 1.1 if \( d = 2 \) and \( n = 3 \)), the class group \( \text{Cl} \mathcal{W} \) is generated by the \( C_i \), and therefore the class group \( \text{Cl} \mathcal{O}_{W,p} \) is generated by the images of the primes \( \hat{P}_i \). Above we showed that these have image \( \sum \hat{P}_{i,j} \) in the completed local ring and that \( \overline{\alpha}(\hat{P}_{i,j}) = \hat{P}_{W,i,j} \), and therefore \( \overline{\alpha}(\sum \hat{P}_{i,j}) = \sum_k \hat{P}_{W,i,j} \). Putting this all together we have

\[
\overline{\alpha} \circ j_W(G) = \left( \sum_j \hat{P}_{W,1,j}, \sum_j \hat{P}_{W,2,j}, \ldots, \sum_j \hat{P}_{W,r,j} \right) = j_W(\text{Cl} \mathcal{O}_{W,p}),
\]

where \( j_W \) is the natural injection \( \text{Cl} \mathcal{O}_{W,p} \to \text{Cl} \hat{\mathcal{O}}_{W,p} \), so we have constructed the local ring \( \mathcal{O}_{W,p} \) having the desired property. \( \square \)

**Remark 2.4.** Although stated for local rings at normal points on local complete intersection varieties, Theorem 2.2 has a purely algebraic formulation: Let \( A \) be the localization of a complete intersection ring of finite type over \( \mathbb{C} \) at a maximal ideal with completion \( \hat{R} = \hat{A} \) and let \( G \subset \text{Cl} B \) be any finitely generated subgroup. Then there is a ring \( B \) with \( \hat{B} \cong \hat{R} \) such that the subgroup \( \text{Cl} B \hookrightarrow \text{Cl} R \) is exactly the subgroup \( G \subset \text{Cl} A \subset \text{Cl} R \).

We hope to remove the restriction that \( n = 3 \) when \( d = 2 \) of Theorem 2.2 in the future. For this it suffices to prove Conjecture 2.5 below [6, Rmk. 1.4]. The conjecture holds for \( X = \mathbb{P}^3 \) [2] and the analogous statement holds when \( \text{dim} \ X \geq 4 \) [6].

**Conjecture 2.5.** Let \( X \subset \mathbb{P}^n \) be a normal threefold containing a closed subscheme \( Z \) of dimension \( \leq 1 \) lying on a normal surface \( S \subset X \) with curve components \( Z_i \). Then the very general \( Y \in |H^0(X, \mathcal{I}_Z(d))| \) for \( d \gg 0 \) is normal and the natural homomorphisms \( \text{Cl} X \to \text{Cl} Y \) and \( \bigoplus \mathbb{Z} \text{Supp}^{Z_i} \to \text{Cl} Y \) induce an isomorphism \( \alpha : \bigoplus \mathbb{Z} \oplus \text{Cl} X \to \text{Cl} Y \).

As an application of Theorem 2.2 we recover our earlier result [5] with a short proof:

**Corollary 2.6.** If \( (R, m) \) is a complete 2-dimensional rational double point singularity and \( G \subset \text{Cl} A \) is a subgroup, then there exists a geometric normal domain \( (B, n) \) such that \( \hat{B} \cong \hat{R} \) and \( H = \text{Cl} B \subset \text{Cl} R \).

**Proof.** For these singularities \( \text{Cl} R \) is a finite group, so it suffices to find a geometric local ring \( (B, n) \) with \( \hat{B} \cong \hat{R} \) and \( \text{Cl} B = \text{Cl} R \) for each singularity type. For the \( A_n \) singularity with equation \( xy - z^{n+1} = 0 \), the line \( L : x = z = 0 \) is a generator for \( \text{Cl} \mathcal{O}_{S,p} \cong \text{Cl} \hat{\mathcal{O}}_{S,p} \cong \mathbb{Z}/(n+1)\mathbb{Z} \) [3, Ex. 2.1] (smaller subgroups of \( \text{Cl} \hat{\mathcal{O}}_{S,p} \) are constructed in [3, Prop. 3.3 and Prop. 3.4]). Similarly [5, Prop. 4.5], the \( D_n \) singularity is obtained by completing the local ring at the origin \( p \) of a surface \( S \) with equation \( x^2 + y^2z - z^{n-1} = 0 \): for \( n = 2k \) the curves \( x = z = 0 \) and \( x = y - z^{k-1} = 0 \) generate \( \text{Cl} \hat{\mathcal{O}}_{S,p} \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \), while for \( n = 2k + 1 \) the curve \( y = x - z^k = 0 \) generates \( \text{Cl} \hat{\mathcal{O}}_{S,p} \cong \mathbb{Z}/4\mathbb{Z} \). For an \( E_6 \) use the equation \( x^2 + y^2 - z^4 \) and the curve having ideal \( (x - z^2, y) \) to generate \( \text{Cl} \hat{\mathcal{O}}_{S,p} \cong \mathbb{Z}/3\mathbb{Z} \); for an \( E_7 \), use the equation \( x^2 + y^2 + z^3 \) and the line \( (x, y) \) to generate \( \text{Cl} \hat{\mathcal{O}}_{S,p} \cong \mathbb{Z}/2\mathbb{Z} \).

A complete \( E_8 \) is a UFD, so \( \text{Cl} \hat{\mathcal{O}}_{S,p} = 0 \). \( \square \)
3. Vertex singularities

Here we study the singularity at the vertex $p$ of a cone $V$ over a smooth variety $X$. Our main results state that (a) every finitely generated abelian group occurs as the class group of a vertex singularity and (b) Conjecture 1.6 holds when $X$ is a smooth arithmetically Cohen-Macaulay variety. Both follow from Theorem 2.2 and Proposition 3.1 which gives a geometric description of the groups $\text{Cl} \mathcal{O}_{V,p} \hookrightarrow \hat{\text{Cl}}_{V,p}$. The first three parts are well-known, but the last part is new to our knowledge.

**Proposition 3.1.** Let $X \subset \mathbb{P}^n$ be a smooth projective variety over an algebraically closed field with projective cone $V \subset \mathbb{P}^{n+1}$ having vertex $p$ and hyperplane section $H$. Then

(a) $\text{Cl} \, V \cong \text{Cl} \, X$.

(b) The natural map $\text{Spec} \mathcal{O}_{V,p} \to (V - H)$ induces an isomorphism on class groups $\text{Cl} \mathcal{O}_{V,p} \cong \text{Cl}(V - H) \cong \text{Pic} X/(\mathcal{O}_X(1))$.

(c) The blow-up $\tilde{V}$ of $V$ at $p$ is a $\mathbb{P}^1$-bundle over $X$ with exceptional divisor $E \cong X$.

(d) The map $\iota: \text{Cl} \mathcal{O}_{V,p} \to \hat{\text{Cl}}_{V,p}$ is a split injection. If $H^1(\mathcal{O}_X(k)) = H^2(\mathcal{O}_X(k)) = 0$ for $k \geq d$, then $\iota$ may be identified with the map $\text{Pic} E_d/\langle E \rangle \to \text{Pic} E/\langle E \rangle$ induced by the restriction, where $E_d$ is the $d$th infinitesimal neighborhood of $E \subset V$.

**Proof.** Parts (a) and (b) are [13, II, Exer. 6.3] and (c) is [13, V, Ex. 2.11.4]. For (d), let $W = \text{Spec} \mathcal{O}_{V,p}$ and $Z = \text{Spec} \hat{\mathcal{O}}_{V,p}$. The morphisms $Z \to W \to (V - X)$ extend to morphisms $\tilde{Z} \to \tilde{W} \to (\tilde{V} - X)$ of the blow-ups at $p$. All three are smooth and share the exceptional divisor $E = \text{Proj} \oplus \mathbb{m}_p^n/\mathbb{m}_p^{n+1}$ because the quotients $\mathbb{m}_p^n/\mathbb{m}_p^{n+1}$ are isomorphic for $n \geq 1$. Likewise, they share the infinitesimal neighborhoods $E_k$ for $k \geq 1$ because the local rings $\mathcal{O}_p/\mathbb{m}_p^n$ are isomorphic for $n \geq 1$. Taking Picard groups gives a diagram with short exact rows in which the map $\alpha$ is the Mori map, geometrically realized as the pull-back map of line bundles:

\[
\begin{array}{cccccc}
\mathbb{Z} & \xrightarrow{E} & \text{Pic}(\tilde{V} - X) & \longrightarrow & \text{Pic}(\tilde{V} - E - X) \\
\downarrow & & \downarrow\beta & & \downarrow\beta \\
\mathbb{Z} & \xrightarrow{E} & \text{Pic} \tilde{W} & \longrightarrow & \text{Pic}(\tilde{W} - E) \cong \text{Cl} \mathcal{O}_{V,p} & \\
\downarrow & & \downarrow\gamma & & \downarrow\alpha \\
\mathbb{Z} & \xrightarrow{E} & \text{Pic} \tilde{Z} & \longrightarrow & \text{Pic}(\tilde{Z} - E) \cong \hat{\text{Cl}}_{V,p} \\
\end{array}
\]

The map $W \to V - X$ induces an isomorphism $\text{Cl}(V - X) \to \text{Cl} \mathcal{O}_{V,p}$ by part (b). Blowing up the vertex $p$ we obtain $\tilde{\beta}: \text{Pic}(\tilde{V} - E - X) \cong \text{Pic}(\tilde{W} - E)$, so $\beta$ is also an isomorphism by the Snake lemma. Since $\tilde{V} - X \to E$ is an $\mathbb{A}^1$-bundle, the pullback $\text{Pic} E \to \text{Pic}(\tilde{V} - X)$ is an isomorphism whose inverse coincides with the restriction $\text{Pic}(\tilde{V} - X) \to \text{Pic} E$, hence the restriction $\text{Pic} \tilde{W} \to \text{Pic} E$ is also an isomorphism.

Now consider the map $\gamma$ and let $\tilde{Z}$ be the formal completion of $Z$ along $E$. The category of coherent $\mathcal{O}_Z$-modules is equivalent to the category of coherent $\mathcal{O}_{\tilde{Z}}$-modules [13, II, Thm. 9.7] (see also [11, III, Thm. 5.1.4]), therefore we obtain an isomorphism...
Pic \( \hat{Z} \cong \Pic \hat{Z} \). Furthermore, there is an isomorphism Pic \( \hat{Z} \cong \lim_k \Pic E_k \) where \( E_k \) is the scheme structure on \( E \) given by the ideal \( T^k_E \) \cite{I3}, II, Exer. 9.6. This allows us to express the restriction map Pic \( \hat{Z} \to \Pic E \) as the composite

\[
\text{(3)} \quad \Pic \hat{Z} \cong \Pic \hat{Z} \cong \lim_k \Pic E_k \to \Pic E
\]

where the last map is the projection onto Pic \( E_1 = \Pic E \) from the inverse limit. The long exact cohomology sequences associated to the exact sequences

\[
0 \to \mathcal{I}^k_E/\mathcal{I}^{k+1}_E \to \mathcal{O}^*_{E_{k+1}} \to \mathcal{O}^*_{E_k} \to 0
\]

yield exact fragments

\[
\text{(4)} \quad H^1(\mathcal{I}^k_E/\mathcal{I}^{k+1}_E) \to \Pic E_{k+1} \to \Pic E_k \to H^2(\mathcal{I}^k_E/\mathcal{I}^{k+1}_E).
\]

Making the identification \( E \cong X \subset H \cong \mathbb{P}^n \), the canonical \( \mathcal{O}(1) \) from the Proj construction of blow-up is both the pull-back of \( \mathcal{O}_{\mathbb{P}^n}(1) \) and given by \( -E \), so \( \mathcal{I}_E \cong \mathcal{O}(1) \) and \( \mathcal{I}^k_E/\mathcal{I}^{k+1}_E \cong \mathcal{O}(k) \otimes \mathcal{O}_E \cong \mathcal{O}_X(k) \) via the isomorphism \( X \cong E \). In view of (4) and the vanishing hypotheses we see that Pic \( E_{k+1} \to \Pic E_k \) is an isomorphism for each \( k \geq d \), so the direct limit is isomorphic to Pic \( E_d \). In other words, the restriction map Pic \( \hat{Z} \to \Pic E_d \) is an isomorphism. Restricting Diagram \[2\] to \( E \subset E_d \) gives the commuting diagram

\[
\begin{array}{cccccc}
0 & \longrightarrow & \Pic((\hat{V} - X)) & \longrightarrow & \Pic E_d & \longrightarrow & \Pic E & \longrightarrow & 0 \\
& & \downarrow \gamma_{0,\beta} & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Pic \hat{Z} & \longrightarrow & \Pic E_d & \longrightarrow & \Pic E & \longrightarrow & 0
\end{array}
\]

in which the horizontal maps are restrictions. Since the composite \( \Pic((\hat{V} - X)) \to \Pic E \) is an isomorphism, the vertical map \( \Pic(\hat{V} - E) \to \Pic \hat{Z} \) is a splitting for the restriction map \( \Pic E_d \to \Pic E \). When we quotient out by the subgroup \( \langle E \rangle \) we obtain the diagram

\[
\begin{array}{cccccc}
0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \Pic(\hat{V} - E) & \longrightarrow & \Cl \mathcal{O}_{\hat{V},p} & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \Pic \hat{Z} & \longrightarrow & \Cl \hat{\mathcal{O}}_{\hat{V},p} & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \Pic E & \longrightarrow & \Cl \mathcal{O}_{\hat{V},p} & \longrightarrow & 0
\end{array}
\]

whose rows are short exact sequences. This diagram identifies \( \Cl \mathcal{O}_{\hat{V},p} \cong \Pic E/\langle E|E \rangle \) and \( \Cl \hat{\mathcal{O}}_{\hat{V},p} \cong \Pic E_d/\langle E|E_d \rangle \). Since the composite of the two rightmost vertical maps is the identity, the Mori map \( \Cl \mathcal{O}_{\hat{V},p} \to \Cl \hat{\mathcal{O}}_{\hat{V},p} \) is a split injection. \( \Box \)

**Remark 3.2.** The Mori map \[1\] does not split in general. For example, the \( \mathbb{A}_3 \) singularity ring \( R \) has completed class group \( \mathbb{Z}/4\mathbb{Z} \) and there are geometric rings \( A \) with completion \( R \) for which \( \Cl A = \mathbb{Z}/2\mathbb{Z} \) as a subgroup \[5\].
3.1. Finitely generated groups are class groups at vertex singularities.

**Theorem 3.3.** Let $G$ be any finitely generated abelian group. Then there is a geometric normal domain $B$ of dimension two with $\text{Cl } B \cong G$.

**Proof.** Write $G \cong \mathbb{Z}^r \oplus \bigoplus_{i=1}^s \mathbb{Z}/n_i \mathbb{Z}$ for suitable $r, s, n_i$. Choose a smooth plane curve $C$ of sufficiently high degree that its genus satisfies $g \geq \frac{1}{2}(r + s)$. The vertex $p$ of the cone $S$ over $C$ has class group $\text{Cl}\mathcal{O}_{S,p} \cong \text{Pic}(C)/(\mathcal{O}_C(1))$ by Proposition 3.1. Since the only degree-0 class in $\langle \mathcal{O}_C(1) \rangle$ is 0, the composite map

$$\text{Pic}^0(C) \rightarrow \text{Pic} C \rightarrow \text{Pic} C/(\mathcal{O}_C(1))$$

is injective, where $\text{Pic}^0(C)$ is the subgroup of $\text{Pic} C$ consisting of the degree-0 classes. Since $\text{Pic}^0(C)$ is isomorphic to the Jacobian variety $J(C)$, which for the complex curve $C$ is isomorphic to $\mathbb{C}^g/\Lambda$ with $\Lambda$ a rank-$(2g)$ lattice in $\mathbb{C}^g$, we see that

$$\text{Pic}^0(C) \cong \mathbb{R}^{2g}/\mathbb{Z}^{2g} \cong (\mathbb{R}/\mathbb{Z})^{2g}$$

as an additive group. Since $\mathbb{R}/\mathbb{Z}$ has elements of all positive orders (including $\infty$), we can choose $r$ elements of summands having order $\infty$ and $s$ elements having respective orders $n_i$, which generate a subgroup of $\mathbb{R}^{2g}/\mathbb{Z}^{2g}$ isomorphic to $G$. Now apply Theorem 2.2 to these elements. $\square$

**Remark 3.4.** In Theorem 3.3 above we constructed $B$ as the local ring at the vertex of the cone over a smooth plane curve. To obtain rings $B$ of higher dimension we can adjust the proof as follows. Choose $C$ as above and let $X = C \times \mathbb{P}^l$ for any $l \geq 1$. Then $\text{Pic } X \cong \text{Pic } C \oplus \mathbb{Z}$ [13, II. Exer. 6.1] and $\mathcal{O}_C(1) \otimes \mathcal{O}_{\mathbb{P}^l}(1)$ is very ample on $X$, giving a closed immersion $X \hookrightarrow \mathbb{P}^{N}$. Letting $V \subset \mathbb{P}^{N+1}$ be the projective cone with vertex $p$, we obtain $\text{Cl } \mathcal{O}_{V,p} \cong \text{Pic } C \oplus \mathbb{Z}/\langle \mathcal{O}_C(1) \otimes \mathcal{O}_{\mathbb{P}^l}(1) \rangle$ and again the composite map

$$J(C) = \text{Pic}^0(C) \rightarrow \text{Pic } C \oplus \mathbb{Z} \rightarrow \text{Pic } C \oplus \mathbb{Z}/\langle \mathcal{O}_C(1) \otimes \mathcal{O}_{\mathbb{P}^l}(1) \rangle$$

is injective allowing us to run the same construction.

3.2. Cones over Arithmetically Cohen-Macaulay varieties.

**Theorem 3.5.** Let $X \subset \mathbb{P}^n$ be a smooth complex variety of dimension at least two for which $H^1(\mathcal{O}_X(k)) = H^2(\mathcal{O}_X(k)) = 0$ for all $k > 0$. If $V \subset \mathbb{P}^{n+1}$ is the projective cone with vertex $p$, then Conjecture 1.6 holds for $R = \hat{\mathcal{O}}_{V,p}$.

**Proof.** The vanishing hypothesis allows us to apply Proposition 3.1 with $d = 1$, so we obtain the isomorphism $\text{Cl } \mathcal{O}_{V,p} = \text{Cl } \hat{\mathcal{O}}_{V,p}$. Now apply Theorem 2.2 to $V$. $\square$

**Corollary 3.6.** Let $X \subset \mathbb{P}^n$ be a smooth ACM subvariety of dimension $\geq 2$ with projective cone $V \subset \mathbb{P}^{n+1}$ having vertex $p$. Then Conjecture 1.6 holds for $R = \hat{\mathcal{O}}_{V,p}$.

**Example 3.7.** Corollary 3.6 provides many completed local rings of vertex singularities for which Conjecture 1.6 holds, but to apply Theorem 3.5 we only used the vanishings $H^1(\mathcal{O}_X(k)) = H^2(\mathcal{O}_X(k)) = 0$ for $k > 0$. Here are examples which need not be ACM:

(a) Plane curves $X \subset \mathbb{P}^2$ of degree $d \leq 3$. Note that the cone $V = C(X) \subset \mathbb{P}^3$, so we are staying within the restricted hypothesis in Theorem 2.2.

(b) Surfaces $X \subset \mathbb{P}^3$ of degree $d \leq 4$. 




Lemma 4.1. Let $C$ be mon integral divisors that with the strict transforms $\tilde{E}$ as in the proof, The exact sequence (4) takes the form
\begin{equation}
0 \to \mathbb{C} \to \text{Pic } E_2 \to \text{Pic } E_1 \to 0
\end{equation}
for $k = 1$ since $H^0(E_2, \mathcal{O}_{E_2}^*) \to H^0(E_1, \mathcal{O}_{E_1}^*)$ is surjective (nonzero constants are global units on $E_2$) and hence the Mori map $\text{Cl } \mathcal{O}_{V,p} \to \text{Cl } \hat{\mathcal{O}}_{V,p}$ is a split injection with cokernel isomorphic to $\mathbb{C}$. We can no longer apply Theorem 2.2 with $V$, but there may well be another geometric local ring with completion $\hat{\mathcal{O}}_{V,p}$ giving geometric divisors outside the image above. Are all the divisors in $\text{Cl } \hat{\mathcal{O}}_{V,p}$ geometric?

4. SURFACES CONTAINING GENERAL LINES THROUGH A POINT

In Theorem 2.2, we represented elements of a local class group $\text{Cl } \mathcal{O}_{V,p}$ by integral divisors $C_i$ with Lemma 2.1 and carefully chose a base locus $B$ consisting of the $C_i$ and some infinitesimal data to pick out the group generated by them for general $W$ containing $B$. For $B = \bigcup C_i$ we do not expect the isomorphism $\text{Cl } \hat{\mathcal{O}}_{W,p} \cong \text{Cl } \hat{\mathcal{O}}_{V,p}$, but we can still expect some kind of approximation. In this section we give a method for comparing class groups of singularities having the same tangent cones and apply it to compute the local class group of the general surface containing a set of general lines through a point (Proposition 1.3) and for a general surface containing three double lines (Example 1.6).

Given integral divisors $C_i$ on $V \subset \mathbb{P}^n$ as in Theorem 2.2, what can be said about $\text{Cl } \mathcal{O}_{W,p}$ for the general complete intersection $W$ containing $\bigcup C_i$? If $V$ and $W$ have equal smooth tangent cones $E_V = E_W$ in their respective blow-ups at $p$ in $\widetilde{\mathbb{P}^n}$, we can at least make a first-order estimate in the sense that
$$\text{Cl } \hat{\mathcal{O}}_{W,p} \cong \lim_{\text{Pic } E_W^k} \Rightarrow \text{Cl } \hat{\mathcal{O}}_{V,p} \cong \lim_{\text{Pic } E_V^k}$$
where $E^n$ is the $n$th infinitesimal neighborhood of $E$.

Lemma 4.1. Let $V$ and $W$ be subvarieties of $\mathbb{P}^n$ of the same dimension containing common integral divisors $C_1, C_2, \ldots C_s$ and having an isolated singularity at $p$. Suppose further that $V$ and $W$ have equal tangent cones at $p$ that induce equal smooth exceptional divisors $E_V = E_W$ on the blowups $\tilde{V}, \tilde{W} \subset \mathbb{P}^n$ at $p$. Then the diagram
\begin{equation}
\begin{array}{cccc}
\mathbb{Z}^s & \overset{C_i}{\longrightarrow} & \text{Cl } \mathcal{O}_{V,p} & \subset \text{Cl } \hat{\mathcal{O}}_{W,p} & \longrightarrow & \text{Pic } E_W \\
\bigg\downarrow & & \bigg\downarrow & & \bigg\downarrow & \\
\mathbb{Z}^s & \overset{C_i}{\longrightarrow} & \text{Cl } \mathcal{O}_{W,p} & \subset \text{Cl } \hat{\mathcal{O}}_{V,p} & \longrightarrow & \text{Pic } E_V
\end{array}
\end{equation}
commutes, where the rightmost vertical map is the equality induced by $E_V = E_W \subset \mathbb{P}^{n-1}$.

Proof. The images of the classes of $C_i$ in $\text{Pic } E_W$ and $\text{Pic } E_V$ are given by the intersection with the strict transforms $\tilde{C_i} \subset \widetilde{\mathbb{P}^n}$, hence are equal by construction. \qed
Now consider \( r \) lines containing a fixed point \( p \in \mathbb{P}^3 \). When \( r = 2 \), the general surface \( S \) containing the lines is smooth and \( \text{Pic} \, S = \mathbb{Z}^3 \) is freely generated by \( \mathcal{O}(1) \) and the two lines [2, Cor. 1.3]. In this section we give the answer when \( r > 2 \) and the lines are in general position, determining the nature of the singularity on \( S \) where the lines meet and which classes represent Cartier divisors. It turns out that the singularity is analytically isomorphic to that of a cone over a plane curve. The following lemma tells us that the classes of are independent in the class group of such a cone.

**Lemma 4.2.** Fix \( r > 2 \) and \( d > 1 \) satisfying

\[
(7) \quad \left( \frac{d + 1}{2} \right) \leq r < \left( \frac{d + 2}{2} \right).
\]

Then \( r \) very general points \( \{p_1, p_2, \ldots, p_r\} \subset \mathbb{P}^2 \) lie on a smooth curve \( D \) of degree \( d \) and no curve of degree \( d - 1 \). If \( r > 5 \), then for the general such \( D \) containing the points, the natural map \( \mathbb{Z}^r \xrightarrow{p_i} \text{Pic} \, D/\langle \mathcal{O}_D(1) \rangle \) is injective.

**Proof.** The first statement is clear because \( r \) points \( p_i \in \mathbb{P}^2 \) in general position impose independent conditions on curves of fixed degree and lie on a smooth curve of degree \( d \). For the second statement let \( U_d \subset \mathbb{P}^r(H^0(\mathcal{O}_{\mathbb{P}^2}(d))) \) be the smooth curve locus and let

\[
W = \{(p_1, p_2, \ldots, p_r, D) \in (\mathbb{P}^2)^r \times U_d : p_i \in D \}
\]

which comes equipped with the family of curves

\[
\begin{array}{c}
\mathcal{D} \subset (\mathbb{P}^2 \times W) \\
\pi \downarrow \downarrow \\
W = W
\end{array}
\]

obtained by pulling back the universal curve over \( U_d \) along the projection \( W \to U_d \). Let \( \mathcal{H} \) be the pullback of \( H = \mathcal{O}_{\mathbb{P}^2}(1) \) on \( \mathcal{D} \) and for each \( 1 \leq i \leq r \) the assignment \( (p_1, \ldots, p_r, D) \to p_i \in D \) gives a section to \( \pi \) whose image is an effective Cartier divisor \( \mathcal{P}_i \subset \mathcal{D} \). For fixed \( (n_i, m) = (n_1, \ldots, n_r, m) \in \mathbb{Z}^{r+1} \), let \( T(n_i, m) \subset W \) defined by \( \mathcal{O}_D(\sum n_ip_i + mH) = 0 \in \text{Pic} \, D \). If \( \sum n_i + md \neq 0 \), then \( T(n_i, m) \) is empty by reason of degree. If \( \sum n_i + md = 0 \), the line bundle \( \mathcal{M} = \mathcal{O}_D(\sum n_i \mathcal{P}_i + md\mathcal{H}) \) restricts to \( \mathcal{O}_D(\sum n_ip_i + mdH) \) on the fibers and has degree zero, hence is trivial if and only if \( H^0(D, \mathcal{M}_D) \neq 0 \), a Zariski closed condition by semicontinuity (\( \mathcal{M} \) is flat over \( W \) by constancy of Hilbert polynomial). Thus each \( T(n_i, m) \subset W \) is a proper closed subset of \( W \), hence a very general set of \( r \) points is contained in a smooth degree \( d \) curve \( D \) with no non-trivial relations in \( \text{Pic} \, D/\langle \mathcal{O}_D(1) \rangle \). \( \square \)

**Proposition 4.3.** Let \( Z = \bigcup_{i=1}^r L_i \) be a union of \( r \) general lines passing through \( p \in \mathbb{P}^3 \).

For a very general surface \( S \in |H^0(\mathcal{I}_Z(d))| \) with \( d \gg 0 \), define \( \theta : \mathbb{Z}^r \xrightarrow{L_i} \text{Cl} \, \mathcal{O}_{S,p} \).

(a) The surface \( S \) has the same tangent cone at \( p \) as the vertex of a cone \( X \) over a smooth plane curve of minimal degree containing a hyperplane section of the lines.

(b) If \( r \leq 2 \), then \( \text{Cl} \, \mathcal{O}_{S,p} = 0 \).

(c) If \( 3 \leq r \leq 5 \), then \( \text{Cl} \, \mathcal{O}_{S,p} \cong \mathbb{Z}/2\mathbb{Z} \) and \( \theta \) takes each \( L_i \) to a generator.

(d) If \( r > 5 \), then \( \theta \) is injective.
Proof. The surface $S$ is smooth at $p$ for $r \leq 2$, hence $\text{Cl} \mathcal{O}_{S,p} = 0$ and $\theta = 0$, proving (b). For $r > 2$, let $Z \subset \mathbb{A}^3$ be the cone over $r$ general points in $\mathbb{P}^2$ with vertex $p = (0,0,0)$. The ideal $I_Z$ is the homogeneous ideal of the $r$ points in $\mathbb{P}^2$, so the general surface $S$ containing $Z$ has equation $F + G$, where $\deg F = d$ and $G$ consists of higher order terms. By Lemma 4.2, $F$ is the equation of the cone $X$ over a smooth degree $d$ plane curve $V$ for $S$ general, so $S$ and $X$ have equal tangent cones at $p$ which resolve by one blowup with equal exceptional curves $E_S = E_X$ isomorphic to the plane curve defined by $F$. This gives (a).

For $3 \leq r \leq 5$, $d = 2$, $X$ has an $A_1$ singularity, and thus so does $S$ [10, Char. A3]. This, the fact that a complete $A_1$ has class group $\mathbb{Z}/2\mathbb{Z}$, and the fact that the $L_i$ are non-Cartier on $S$ immediately gives (c).

For $r > 5$, by Lemma 4.1 we obtain the diagram

$$
\begin{array}{c}
\mathbb{Z}^r & \xrightarrow{L_i} & \text{Cl} \mathcal{O}_{S,p} \subset \text{Cl} \hat{\mathcal{O}}_{S,p} \longrightarrow \text{Pic} E_S \\
\end{array}
$$

$$
\begin{array}{c}
\mathbb{Z}^r & \xrightarrow{L_i} & \text{Cl} \mathcal{O}_{V,p} \subset \text{Cl} \hat{\mathcal{O}}_{V,p} \longrightarrow \text{Pic} E_V \\
\end{array}
$$

Since the lines are general, their classes are independent in $\text{Pic} E_X$ by Lemma 4.2 so the composition along the bottom row is injective, and therefore so is that along the top, which gives (d). \hfill \square

Corollary 4.4. Let $S$ and $r$ be as in Proposition 4.3. Then

(a) $r \leq 2 \Rightarrow \text{Pic} S = \text{Cl} S$,
(b) $3 \leq r \leq 5 \Rightarrow \text{Pic} S = \{ \sum a_i L_i + b H \in \text{Cl} S : 2| \sum a_i \}$,
(c) $r > 5 \Rightarrow \text{Pic} S = \{ b H \in \text{Cl} S : b \in \mathbb{Z} \}$.

Proof. This follows immediately from the Theorem in view of the exact sequence $0 \to \text{Pic} S \to \text{Cl} S \to \text{Cl} \mathcal{O}_{S,p}$ [14, Prop. 2.15] and the identification $\text{Cl} S \cong \mathbb{Z}^{r+1}$ generated by $L_i$ and $H = \mathcal{O}(1)$. \hfill \square

In Proposition 4.3 it is essential that the lines be in general position. For example, if $C$ consists of $r$ planar lines passing through $p$, then the general surface $S$ containing $C$ is smooth at $p$ and $\text{Cl} S = \text{Pic} S$ is freely generated by $H$ and the lines $L_i$. A more interesting configuration is the pinwheel featured in the following proposition.

Proposition 4.5. For $r \geq 2$, let $C \subset \mathbb{P}^3$ be the union of $r$ general lines $L_1, L_2, \ldots, L_r$ contained in a plane $H$ passing through $p$ and a line $L_0 \not\subset H$ through $p$. Then there are local coordinates $x, y, z$ and $a_i \in k$ for which

$$I_C = (xz, yz, \Pi_{j=1}^r (x + a_j y)).$$

The very general surface $S$ containing $C$ has an $A_{r-1}$ singularity at $p$ and the restriction map $\theta : \text{Cl} S \to \text{Cl} \mathcal{O}_{S,p} \cong \mathbb{Z}/r\mathbb{Z}$ sends the lines $L_1, \ldots, L_r$ to $1$ and $L_0$ to $-1$.

Proof. If $z = 0$ is the plane containing the $r$ general lines, it is clear that the ideal (8) cuts out $C$ set-theoretically for some $a_j$. To see that this is the total ideal for $C$, observe that the closure $\overline{C}$ in $\mathbb{P}^3$ is contained in the complete intersection $X : \Pi(x + a_j y) = zx = 0,$
which links $C$ to the planar multiplicity $(r - 1)$-line $Z$ given by $x = y^{r-1} = 0$. Since $Z$ is planar, we can write down the total ideal for $\overline{C}$ using the cone construction from liaison theory.

The general surface $S$ containing $C$ has local equation $0 = xz - Ayz + B\Pi(x + a_j y)$ for general $A, B$. Setting $x' = x - Ay$ this becomes $0 = x'z + B\Pi(x' + (A + a_j)y)$. The product on the right can be written $x'm + B\Pi(A + a_j)y'$, so if we set $z' = z + m$ the equation for $S$ takes the form $x'z' + uyz' = 0$, where $u = B\Pi(A + a_j)$ is a unit, exhibiting the equation of an $A_{r-1}$ singularity, which has local Picard group $\text{Cl}\, \mathcal{O}_{S,p} \cong \mathbb{Z}/r\mathbb{Z}$.

To find the images of the lines $L_j$ in $\text{Cl}\, \mathcal{O}_{S,p}$, we blow up $S$ and check the intersections of the strict transforms $\tilde{L}_i$ with the exceptional divisors $E_i$, starting with $L_0$, for which the ideal is $(x, y) = (x', y)$. Adding projective coordinates $X, Y, Z$ for the blow-up and looking on the patch $Z = 1$, we have $(x', y) = (zX, zY) = z(X, Y)$ so $\tilde{L}_0$ is the line $X = Y = 0$, which meets the component $x' = y = z' = X = 0$ of the exceptional divisor at the origin, a smooth point of $\tilde{S}$. Since this is the only intersection of $\tilde{L}_0$ and $E$, $\tilde{L}_0 = (1, 0, ...0)$ in $\text{Cl}\, \mathcal{O}_{S,p}$ and is thus a generator.

For $1 \leq j \leq r$, the ideal for $L_j$ is

$$(z, x + a_j y) = (z, x' + (A + a_j)y) = (z' - m, x' + (A + a_j)y)$$

and $\tilde{L}_j$ meets the other component of $E$ in a single point away from the singularity of $\tilde{S}$.

For example, on the patch $x = 0$ we have $(x', y) = (zX, zY) = z(X, Y)$ so $\tilde{L}_0$ is the line $X = Y = 0$, which meets the component $x' = y = z' = X = 0$ of the exceptional divisor at the origin, a smooth point of $\tilde{S}$. Since this is the only intersection of $\tilde{L}_0$ and $E$, $\tilde{L}_0 = (1, 0, ...0)$ in $\text{Cl}\, \mathcal{O}_{S,p}$ and is thus a generator.

For $1 \leq j \leq r$, the ideal for $L_j$ is

$$(z, x + a_j y) = (z, x' + (A + a_j)y) = (z' - m, x' + (A + a_j)y)$$

and $\tilde{L}_j$ meets the other component of $E$ in a single point away from the singularity of $\tilde{S}$.

Example 4.6. Consider the subscheme $Z \subset \mathbb{P}^3$ consisting of the union of three double lines defined by ideals $(x, y^2), (x^2, z)$, and $(y, z^2)$ with respective supports $L_1, L_2$ and $L_3$. There is an obvious containment $I = (xyz, x^2y, y^2z, z^2x) \subset (x, y^2) \cap (x^2, z) \cap (y, z^2)$ and in fact they are equal because $I$ has minimal graded resolution

$$0 \to S(-4)^3 \to S(-3)^4 \to I \to O$$

and sheafifying gives the resolution for an ACM curve $W$ of degree 6 (it’s linked to a degenerate twisted cubic curve by two cubics). Thus $Z \subset W$ and both have degree 6 so they are equal. The general surface $S$ of high degree containing $Z$ has equation of the form $Axyz + Bx^2y + Cxz^2 + Dy^2z$, where $A, B, C, D$ do not vanish at the point $p = (0, 0, 0, 1)$ of intersection of the lines, therefore the singularity of $S$ at $p$ has the same tangent cone as the vertex of the cone $V$ over the plane curve $C_0$ defined by $a_{xyz} + bx^2y + cxz^2 + dy^2z$ where $a, b, c, d$ are the respective constant terms, which is smooth for general $A, B, C, D$. The singularities resolve in one blow-up with exceptional divisors $E_S = E_V \cong C_0$. Thus we can apply Lemma 4.4 to obtain a commutative diagram

$$
\begin{array}{ccc}
\mathbb{P}^3 \xrightarrow{L_i} \text{Cl}\, \mathcal{O}_{S,p} \subset \text{Cl}\, \hat{\mathcal{O}}_{S,p} \longrightarrow \text{Pic}\, E_S \\
\mathbb{P}^3 \xrightarrow{L_i} \text{Cl}\, \mathcal{O}_{V,p} \subset \text{Cl}\, \hat{\mathcal{O}}_{V,p} \longrightarrow \text{Pic}\, E_V
\end{array}
$$
By the main theorem of [2], the class group $\text{Cl}_S$ is freely generated by the lines $L_i$ and $O_S(1)$; The images under the surjective map $\text{Cl}_S \to \text{Cl}_{O,\mathcal{P}} \cong \text{Pic} C_0/\langle O_{C_0}(1) \rangle$ correspond via the construction of blowing up and completing along the exceptional curve to the classes in corresponding to the points $P_1 = (0, 0, 1), P_2 = (0, 1, 0),$ and $P_3 = (1, 0, 0)$ respectively. The line $x = 0$ intersects $C_0$ in a double point at $P_1$ and a reduced point at $P_2$, hence $2P_1 + P_2 \in \mathcal{O}_{C_0}(1)$ and thus $P_3 = -2P_1$ in $Cl_{O,\mathcal{P}}$. Likewise $P_3 = -2P_2 = 4P_1$ and $P_1 = -2P_3 = -8P_1$, so $\text{Cl}_{O,\mathcal{P}} \cong \mathbb{Z}/9\mathbb{Z}$.

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