I. INTRODUCTION

Systems with glassy dynamics typically exhibit non-trivial behavior when they undergo temperature shifts within the glassy phase. These systems, being in an out-of-equilibrium condition, have properties which are expected to depend on their history. This is the ‘memory’ of glassy systems. One memory effect that shows up in a one-time observable is the so called “Kovacs effect”, which manifests itself under a specific experimental protocol. This effect has been the subject of a variety of recent investigations. The characteristic non-monotonic evolution of the observable under examination (the volume in the original Kovacs’ experiment), with the other thermodynamic variables held constant, shows clearly that a non-equilibrium state of the system cannot be fully characterized only by the (time-dependent) values of thermodynamic variables, but that further inner variables are needed to give a full description of the non-equilibrium state of the system. The memory in this case consists in these internal variables keeping track of the history of the system.

The purpose of this paper is to use a specific model for fragile glass to implement the protocol in order to get some insight into the Kovacs effect. We show that in spite of its simplicity, this model captures the phenomenology of the Kovacs effect, makes possible to implement the Kovacs protocol not only with temperature shifts but with magnetic field shifts as well, and allows in specific regimes to obtain analytical expressions for the evolution of the variable of interest. Furthermore the possibility of affording a thermodynamical-like picture through the introduction of effective parameters can be investigated.

This paper is organized as follows: in Section II we review the experimental protocol generating the effect, in Sections III and IV we introduce our model and use it to implement the protocol, in Section V we draw out of this model some analytical results, in Section VI an interpretation of the effect in terms of effective parameters is illustrated with final conclusions. An appendix collects all terms and coefficients employed in the main text.

II. KOVACS PROTOCOL

The experimental protocol, as originally designed by A. J. Kovacs in the ’60s [1], consists of three main steps:

1st step (1) The system is equilibrated at a given high temperature \( T_i \).

2nd step (2) At time \( t = 0 \) the system is quenched to a lower temperature \( T_l \), close to or below the glass transition temperature, and it is let to evolve a period \( t_a \). One then follows the evolution of the proper thermodynamic variable (in the original Kovacs experiment this was the volume \( V(t) \) of a sample of polyvinyl acetate, in our model it will be the “magnetization” \( m_1(t) \)).

3rd step (3) After the time \( t_a \), the volume, or other corresponding observable, has reached a value equal, by definition of \( t_a \), to the equilibrium value corresponding to an intermediate temperature \( T_f \) (\( T_i < T_f < T_l \)), i.e. such that \( V(t_a) = V_{eq}^{T_f} \). At this time, the bath temperature is switched to \( T_f \).

FIG. 1: Kovacs protocol. Starting from an equilibrium condition at \( T = T_i \) (step (1)) at time \( t = 0 \), the system is quenched to \( T = T_l \) and let to evolve (step (2)). In step (3) the temperature is switched to \( T_f \). This is done at the time \( t_a \) for which: \( V(t_a) \equiv V_{eq}^{T_f} \). In the frame, the typical evolution of the volume \( V(t) \) at \( T = T_f \), after the temperature switch, is illustrated.
The pressure (or corresponding variable) is kept constant throughout the whole experiment.

Naively one would expect the observable under consideration, after the third step, to remain constant, since it already has (at time \( t = t_0^+ \)) its equilibrium value. But the system has not equilibrated yet and so the observable goes through a non monotonic evolution before relaxing back to its equilibrium value, showing a characteristic hump whose maximum increases with the magnitude of the final jump of temperature \( T_f - T_i \) and occurs at a time which decreases with increasing \( T_f - T_i \).

We want to implement this protocol on a model for both strong and fragile glass first introduced in [5] the Harmonic Oscillators-Spherical Spins model (HOSS). This model is based on interacting fast and slow modes, this property turns out to be necessary for the memory effect, object of this paper, to occur.

III. THE HARMONIC OSCILLATOR-SPHERICAL SPIN MODEL

The HOSS model contains a set of \( N \) spins \( S_i \) locally coupled to a set of \( N \) harmonic oscillator \( x_i \) according to the following hamiltonian:

\[
\mathcal{H} = \sum_{i=1}^{N} \left( \frac{K}{2} x_i^2 - H x_i - J x_i S_i - LS_i \right) \quad (1)
\]

The spins have no fixed length but satisfy the spherical constraint: \( \sum_{i=1}^{N} S_i^2 = N \). The spin variables are assumed to relax on a much shorter time scale than the harmonic oscillator variables, so the oscillator variables are the slow modes and on their dynamical evolution the fast spin modes act just as noise. One can then integrate out the spin variables to obtain the following effective Hamiltonian for the oscillators (for details see [5], explicit expressions of undefined terms appearing in all equations hereafter are reported in the Appendix):

\[
\frac{\mathcal{H}_{\text{eff}}(\{x_i\})}{N} = \frac{K}{2} m_2 - H m_1 - w_T(m_1, m_2) + T \frac{1}{2} \log \left( \frac{w_T(m_1, m_2) + \frac{T}{2}}{\frac{T}{2}} \right) \quad (2)
\]

which depends on the temperature and on the first and second moment of the oscillator variables, namely:

\[
m_1 = \frac{1}{N} \sum_{i=1}^{N} x_i \quad , \quad m_2 = \frac{1}{N} \sum_{i=1}^{N} x_i^2 \quad \quad (3)
\]

These variables encode the dynamics of the system which is implemented through a Monte Carlo parallel update of the oscillator variables:

\[
x_i \rightarrow x_i + r_i / \sqrt{N} \quad \quad (4)
\]

The variables \( r_i \) are normally distributed with zero mean value and variance \( \sigma^2 \). The update is accepted according to the Metropolis acceptance rule applied to the variation \( \Delta \) of the energy of the oscillator variables, which is determined by \( \mathcal{H}_{\text{eff}} \) and, in the limit of large \( N \), is given by:

\[
\frac{\Delta \epsilon}{N} = \frac{K_T(m_1, m_2)}{2} \Delta m_2 - H_T(m_1, m_2) \Delta m_1. \quad (5)
\]

This simple model turns out to have a slow dynamics and can be solved analytically.

Following [5] one can derive the dynamical equations for \( m_1 \) and \( m_2 \)

\[
\begin{align*}
\dot{m}_1 &= \frac{H_T(m_1, m_2) - m_1}{K_T(m_1, m_2)} f_T(m_1, m_2) \quad (6) \\
\dot{m}_2 &= \frac{2}{K_T(m_1, m_2)} [I_T(m_1, m_2) + H_T(m_1, m_2)\dot{m}_1]
\end{align*}
\]

The stationary solutions of these equations coincide with the saddle point of the partition function of the whole system at equilibrium at temperature \( T \) and are given by:

\[
\begin{align*}
\bar{m}_1 &= \frac{H_T(\bar{m}_1, \bar{m}_2)}{K_T(\bar{m}_1, \bar{m}_2)} = \frac{\bar{H}_T}{\bar{K}_T} \quad \quad (7) \\
\bar{m}_2 - \bar{m}_1^2 &= \frac{T}{K_T(\bar{m}_1, \bar{m}_2)} = \frac{T}{\bar{K}_T}
\end{align*}
\]

with barred variables from now on indicating their equilibrium values.

A. Strong and Fragile Glasses with the HOSS model

In spite of its simplicity, the HOSS model allows to describe both strong and fragile glasses, characterized respectively by an Arrhenius or a Vogel-Fulcher law in the relaxation time. The following constraint on the configurations space is applied:

\[
\mu_2 = m_2 - m_1^2 - m_0 \geq 0 \quad \quad (8)
\]

When \( m_0 = 0 \) there exists a single global minimum in the configurations space of the oscillators, therefore the role of the constraint with \( m_0 > 0 \) is to avoid the existence of a “crystalline state” and to introduce a finite transition temperature. The stationary solutions for the dynamics with this constraint are given by:

\[
\begin{align*}
\bar{m}_1 &= \frac{H_T(\bar{m}_1, \bar{m}_2)}{K_T(\bar{m}_1, \bar{m}_2)} = \frac{\bar{H}_T}{\bar{K}_T} \quad (9) \\
\bar{m}_2 - \bar{m}_1^2 &= \begin{cases} 
\frac{T}{K_T(\bar{m}_1, \bar{m}_2)} & \text{for } T > T_k \\
m_0 & \text{for } T \leq T_k
\end{cases} 
\end{align*}
\]

The temperature \( T_k \) is determined by:

\[
T_k = m_0 K_{T_k}(\bar{m}_1^{T_k}, \bar{m}_2^{T_k}) = m_0 K_{T_k}. \quad (10)
\]
This is the highest temperature at which the constraint is fulfilled, for smaller temperatures the system relaxes to equilibrium configurations which fulfill the constraint. For \( T > T_k \) therefore the dynamics is not affected by the constraint. For \( T \leq T_k \) the system eventually reaches a configuration which fulfills the constraint, when this happens it gets trapped for ever in such a configuration. This is equivalent to have a “Kauzmann-like” transition, occurring at \( T = T_k \) with vanishing configuration entropy, meaning the system gets stuck forever in one single configuration fulfilling the constraint (see also: [8]).

When there is no constraint, i.e. when \( m_0 = 0 \), then \( T_k = 0 \), if the Monte Carlo updates are done with Gaussian variables with constant variance \( \sigma^2 \), this model is characterized by an Arrhenius relaxation law:

\[
\tau_{eq} \sim e^{\frac{\Delta}{kT}}
\]  

in so resembling the relaxation properties of strong glasses.

The HOSS model with constraint strictly positive \((m_0 > 0)\) can easily be extended to describe fragile glasses by further introducing in the variance of the Monte Carlo update, the following dependence on the dynamics:

\[
\sigma^2 = 8(m_2 - m_0^2)(m_2 - m_1^2 - m_0)^{-\gamma}
\]

In this case the relaxation time turns out to follow the generalized Vogel-Fulcher law:

\[
\tau_{eq} \sim e^{\frac{\Delta}{kT}}(T - T_k)^{\gamma}
\]

The parameter \( \gamma \) is introduced to make the best Vogel-Fulcher type fit for the relaxation time in experiments, making this model valid for a wide range of fragile glasses. When the temperature approaches the value \( T_k \) defined by \( \bar{m} \), from above, the system relaxes towards configurations close to the ones fulfilling the constraint. The variance \( \sigma^2 \) then tends to diverge, the updates become large and so unfavorable, meaning that almost every update of the oscillator variables is refused. This produces the diverging relaxation time following the Vogel-Fulcher law of Eq. (13).

### IV. KOVACS EFFECT IN THE HOSS MODEL

We implement the Kovacs protocol in the model above introduced for a fragile glass. The system is prepared at a temperature \( T_i \) and quenched to a region of temperature close to the \( T_k \), i.e. \( T_i > T_k \). Solving numerically Eqs. (9) we determine the evolution of the system in both step 2 and 3 of the protocol. In step 2 the time \( t_a \), at which \( m_1^T(t_a) = \bar{m}_1^T \), is calculated so that:

\[
m_1^T(t_a) = \bar{m}_1^T
\]

\[
m_2^T(t_a) = m_2^T(t_a)
\]

The evolution of the fractional ”magnetization”:

\[
\delta m_1(t) = \frac{m_1(t) - \bar{m}_1^T}{\bar{m}_1^T}
\]

after step 3 \((t > t_a)\) for different values of \( T_i \) is reported in Figs. 2 and 3 respectively for \( \gamma = 1 \) and \( \gamma = 2 \). In all the figures, the values for the parameters of the model are: \( J = K = 1, L = H = 0.1, m_0 = 5 \). For such parameter values, the Kauzmann temperature turns out to be \( T_k = 4.00248 \).

**FIG. 2:** Fragile glass with \( \gamma = 1 \). The Kovacs protocol is implemented with a quench from temperature \( T_i = 10 \) to \( T_f \), and final jump (at \( t = t_a^+ \)) to the intermediate temperature \( T_f = 4.3 \). The curves, starting from the lowest, refer to \( T_i = 4.005, 4.05, 4.15, 4.25 \), the dashed line refers to condition \( T_i = T_f \) (simple aging with no final temperature shift).

**FIG. 3:** Fragile glass with \( \gamma = 2 \). The Kovacs protocol implemented with a quench from temperature \( T_i = 10 \) to \( T_f \), and final jump (at \( t = t_a^+ \)) to the intermediate temperature \( T_f = 4.3 \). The curves, starting from the lowest, refer to \( T_i = 4.005, 4.05, 4.15, 4.25 \), the dashed line refers to condition \( T_i = T_f \) (simple aging with no final temperature shift).

Since the equilibrium value of \( m_1 \) decreases with in-
creasing temperature (as opposed to what happens for instance with the volume) we observe a reversed 'Kovacs hump'. The curves keep the same properties typical of the Kovacs effect, the minima occur at a time which decreases and have a depth that increases with increasing magnitude of the final switch of temperature. As expected, since increasing $\gamma$ corresponds to further slowing the dynamics, the effect shows on a longer time scale in the case $\gamma = 2$ as compared to $\gamma = 1$.

Actually, since in the last step of the protocol: $m_1(t = t_a) = \tilde{m}_1^{T_f}$ and $f_T(m_1, m_2)$ is always positive, from the first of Eqs. (6) one soon realizes that the hump for this model can be either positive or negative, depending on the sign of the term:

$$\frac{H_{T_f}(\tilde{m}_1^{T_f}, m_2)}{K_{T_f}(\tilde{m}_1^{T_f}, m_2)} - \tilde{m}_1^{T_f}$$

at $t = t_a^+$. This term is zero when $m_1 = \tilde{m}_1^{T_f}$, $m_2 = \tilde{m}_2^{T_f}$, so one would expect $m_2(t = t_a^+) = \tilde{m}_2^{T_f}$ to be the border value determining the positivity or negativity of the hump. Since $H_{T_f}(\tilde{m}_1^{T_f}, m_2)$ decreases with increasing $m_2$ while $K_{T_f}(\tilde{m}_1^{T_f}, m_2)$ increases, it follows that the condition for a positive hump is:

$$m_2(t = t_a^+) < \tilde{m}_2^{T_f}$$

For shifts of temperature in a wide range close to the transition temperature $T_f$, where the dynamics is slower and the effect is expected to show up significantly on a long time scale, the condition $m_2(t = t_a) > \tilde{m}_2^{T_f}$ is always fulfilled and therefore a negative hump is expected.

### A. Kovacs protocol at constant temperature with magnetic field shifts

Interchanging the roles of $T$ and $H$, the Kovacs protocol can be implemented at constant temperature, by changing the magnetic field $H$ instead. From Eqs. (9) and (11) one can see that the value of the transition temperature $T_k$ depends on $H$ as well. Different values of $H$ determine different values of $T_k$. Therefore the protocol must be implemented in the following way. The temperature is kept fixed at $T_i$. At this temperature there is a "Kauzmann" transition for a specific value of the field $H = H_k$. The temperature $T_k$ decreases with decreasing $H$. So if we work at $T = T_i$ with magnetic fields $H < H_k$ we are sure to implement every step of the protocol keeping the system always above the "Kauzmann" transition corresponding to the value of $H$ applied. We start with system equilibrated at $T = T_i$ and $H = H_i < H_k$, and at time $t = 0$ we shift instantaneously the field to a larger value $H_f$, such that $H_i < H_f < H_k$. Then we let the system age for a time $t_a$ such that $m_1^{H_f}(t_a) = \tilde{m}_1^{H_f}$. At this time the field is shifted to $H_f$ (with $H_f < H_i < H_k$).

The subsequent evolution of the fraction magnetization $\delta m_1(t)$ is shown in Fig. 4. Again the curves show all the typical properties of the Kovacs hump, with a very slow relaxation back to equilibrium due to the fact that $H_f$ has been chosen very close to $H_k$.

![FIG. 4: Fragile glass with $\gamma = 1$. The Kovacs protocol is implemented at constant temperature $T_i = 4.2$ with a sequence of magnetic field shifts. $H_k = 2.24787$ is the value of the field such that $T_k = T_i = 4.2$. Starting from $H_i = 0.1$ (step (1)), the field is switched to $H_f$ and the system is let to evolve a time $t_a$ (step (2)). At $t = t_a$, i.e. when $m_1^{H_f}(t_a) = \tilde{m}_1^{H_f}$, the field is switched to $H_f = 2.17$ (step (3)). The curves, starting from the lowest, refer to $H_l = 2.22$, 2.20, 2.18, the dashed line refers to condition $H_l = H_f$. (The values of the other parameters of the model are $J = K = 1$ and $L = 0.1$)](image)

### V. ANALYTICAL SOLUTION IN THE LONG-TIME REGIME

In the previous Section we have shown, through a numerical solution of the dynamics, that the HOSS model reproduces the phenomenology of the Kovacs effect, showing the same qualitative properties of the Kovacs hump as obtained in experiments (see for ex. [1][1]) or with other models [2][2][2][2][2][2][2].

In this section we show that, by carefully choosing the working conditions in which the protocol is implemented, our model provides with an analytical solution for the evolution of the variable of interest.

#### A. Auxiliary variables

In order to ease calculations, as done in [4][8] it is convenient to introduce the following variables:

$$\mu_1 = \frac{H_{T_f}(m_1, m_2)}{K_{T_f}(m_1, m_2)} - m_1$$

$$\mu_2 = m_2 - m_1^2 - m_0$$
for which the dynamical equations read:

\[
\dot{\mu}_1 = -JQ_T(m_1, m_2)I_T(m_1, m_2) - (1 + DQ_T(m_1, m_2))\mu_1 f_T(m_1, m_2)
\]

\[
\dot{\mu}_2 = \frac{2I_T(m_1, m_2)}{K_T(m_1, m_2)} + 2\mu_2^2 f_T(m_1, m_2)
\]

We will choose to implement steps 2 and 3 of the protocol in a range of temperature very close to the Kauzmann temperature \( T_k \). As exhaustively shown in (21), in the long time regime the variable \( \mu_2(t) \) decays logarithmically to its equilibrium value which is small for \( T \sim T_k \).

So, if \( t_a \) is very large, the value of the variable \( \mu_2(t) \), which is continuous at the jump, will be small enough to fulfill the condition for which the following equation is shown to be valid (21):

\[
\frac{d\mu_1}{d(\delta\mu_2)} = \frac{(1 + Q_T(m_1, m_2)D)\left(\frac{\mu_1}{\mu_2}ight)^{-\gamma} - JQ_T(m_1, m_2)T}{2(m_0 + \mu_2)} \tag{20}
\]

where now the variable \( \delta\mu_2(t) = \mu_2(t) - \bar{\mu}_2 \) is used and barred variables always refer to equilibrium condition. Of course choosing \( T_i \) close to \( T_k \) and waiting a long time \( t_a \) so that the system approaches equilibrium, allows only small temperature shifts for the final step of the protocol, meaning that also \( T_f \) will be close to \( T_k \). All the coefficients which appear in equation (20) (see Appendix for complete expressions) in the regime chosen, can be assumed constant and equal to their equilibrium values with a very good approximation. The equation can then be easily integrated to give:

\[
\mu_1(\delta\mu_2) = \exp \left[ -A_Q \frac{2F_1(\gamma, \gamma, \gamma + 1, -\frac{\mu_2}{\delta\mu_2})}{\gamma(\delta\mu_2)^\gamma} \left( \mu_1 + \bar{\mu}_2 \right) \right] + C_Q \int_{\delta\mu_2}^{\delta\mu_2} dz \exp \left[ A_Q \frac{2F_1(\gamma, \gamma, \gamma + 1, -\frac{\mu_2}{\delta\mu_2})}{\gamma z^{\gamma}} \right] \tag{21}
\]

where the superscript + indicates \( t = t_a^+ \) and \( 2F_1 \) the hypergeometric function. This expression simplifies in cases \( \gamma = 1, 3/2, 2 \). All these solutions and relative coefficients are reported in the appendix, here we limit ourselves to the case \( \gamma = 1 \) which corresponds to ordinary Vogel-Fulcher relaxation law. In this case the solution is:

\[
\mu_1(t) = \left( \frac{\delta\mu_2(t)}{\delta\mu_2(t) + \bar{\mu}_2} \right)^{\frac{\gamma}{2}} \left[ \mu_1^+ \left( \frac{\delta\mu_2^+ + \bar{\mu}_2}{\delta\mu_2^+} \right)^{\frac{\gamma}{2}} \right] - C_Q \int_{\delta\mu_2}^{\delta\mu_2} dz \left( \frac{z}{z + \bar{\mu}_2} \right)^{-\frac{\gamma}{2}} \tag{22}
\]

where:

\[
\int_a^b \frac{d\eta}{\eta + \alpha} = \frac{x^{\alpha+1} - \frac{x}{\alpha} - \frac{x}{\alpha} - \frac{x}{\alpha}}{\eta^{\alpha+1}(1+\alpha)} \bigg|_{x=a}^{x=b} = \frac{a^{\alpha+1} - \frac{a}{\alpha} - \frac{a}{\alpha} - \frac{a}{\alpha}}{b^{\alpha+1}(1+\alpha)}
\]

One can then expand the variable of interest \( m_1(t) \) in terms of \( \mu_1 \) and \( \delta\mu_2 \) and obtain the following expression for the Kovacs curves:

\[
\delta m_1(t) = A_1^i (\bar{m}_1^+, \bar{m}_1^+)(\mu_1(t) - \mu_1^+) + A_2^i (\bar{m}_2^+, \bar{m}_2^+)(\delta\mu_2(t) - \delta\mu_2^+) \tag{23}
\]

where the coefficients are approximately constant in the regime chosen and can be evaluated at equilibrium.

\begin{figure}[h!]
\centering
\includegraphics[width=\textwidth]{fig5}
\caption{Numerical solution (continuous lines) of the Kovacs’ curves compared to approximate analytical solution at short-intermediate (dashed line) and intermediate-long time (dashed line). The protocol is applied between \( T_i = 10 \) and \( T_f = 4.018 \). The curves starting from the lowest refer to \( T_i = 4.005, 4.008 \).}
\end{figure}

**B. Short and intermediate \( t - t_a \)**

For small \( t - t_a \), a linear approximation for the variable \( \delta\mu_2 \), with slope given by the second equation of the set \( 2 \), evaluated at \( t = t_a^+ \), turns out to be very good. Inserting this expression in Eq. \( 22 \) to get \( \mu_1(t) \) and then in Eq. \( 20 \) a good approximation of the first part of the hump for small and intermediate \( t - t_a \) is obtained, as shown in Fig. 5.

**C. Intermediate and long \( t - t_a \)**

When \( t - t_a \) is very large, we can use Eq. \( 22 \) and the pre-asymptotic approximation for \( \mu_2(t) \) (see: \( 7 \))

\[
\mu_2(t) = \left( \log(t/t_0) + \frac{1}{2} \log(\log(t/t_0)) \right)^{-1/\alpha} \tag{24}
\]

Inserting this expression in Eq. \( 22 \) to get \( \mu_1(t) \) and then in Eq. \( 20 \), a good approximation for the hump and the tail of the Kovacs curves is obtained. In Fig. 5 we show the agreement between the analytical expression so obtained and the numerical solution.
VI. EFFECTIVE TEMPERATURE AND EFFECTIVE FIELD

The out-of-equilibrium state of the system can be expressed by a number of effective parameters which is in general equal to the number of independent observables considered. In the HOSS model, given the solution of the dynamics, a quasi-static approach can be followed to generalize the equilibrium thermodynamics (see: [7, 10]) by computing the partition function of all the macroscopic equivalent states at a given time $t$. The measure on which this out of equilibrium partition function is evaluated is not the Gibbs measure. One can assume an effective temperature $T_e$ and an effective field $H_e$, and substitute the equilibrium measure by $\exp(-H_{\text{eff}}(\{x_i\}, T, H_e)/T_e)$, and in this way determine the partition function and then the free energy. The values of the effective parameters at a given time $t$ are those that minimize the free energy so calculated. In this way one then obtains:

$$T_e(t) = K_T(m_1(t), m_2(t))[m_2(t) - m_2^2(t)]$$  \hspace{1cm} (25)

$$H_e(t) = H - K_T(m_1(t), m_2(t))\mu_1(t)$$

We plot $H_e$ as a function of $T_e$ in a Kovacs’ setup, in Fig. 6. We see that in step 2 of the protocol (lower continuous line), equivalent to a simple aging experiment, the effective magnetic field relaxes monotonically to the value $H$. In step 3 (upper continuous line CD), after the final jump of the bath temperature, represented in the figure by the jump from point $B$ to point $C$, the effective magnetic field goes through a non-monotonic evolution before relaxing to the equilibrium value $H$. This is when the “Kovacs” effect occurs. A conclusion that can be drawn is that a thermodynamic-like picture in terms of only the effective temperature is not possible in the Kovacs setup if not at cost of neglecting effects of the order of magnitude of the “Kovacs effect” itself. So, while in an aging experiment in the long time regime $H_e(t) - H$ is very small compared to $T_e(t) - T$ (so that one can consider $H_e = H$ and use only $T_e$ as effective parameter), in the Kovacs protocol, it is in the non-monotonic evolution of the effective field that the memory effect manifests itself. An additional effective field is then needed to recover a complete thermodynamic-like picture of the system inclusive of the Kovacs’ effect. The dashed line in the figure represents a simple aging experiment at $T = T_f$, in this case a thermodynamic-like picture with only an effective temperature is possible, assuming $H_e = H$. One can argue from the figure that such a picture would be possible also in step 3 of the protocol (curve CD) since the two curves, for $T_e$ close enough to $T_f$, coincide. But this happens when the system is close to equilibrium and the signature of the memory effect, the non-monotonic evolution, is lost. These conclusions confirm the results obtained in Ref. [6], where the impossibility of a thermodynamic-like picture with only the effective temperature was based on a potential energy landscape (PEL) analysis. The molecular liquid studied in [6], in the last step of the Kovacs protocol, explores regions of the PEL never explored in equilibrium, and so a simple mapping to an equilibrium condition at a different temperature (the effective temperature by definition) is not possible.

VII. CONCLUSION

We have shown that a simple mode with constrained dynamics like the HOSS model, is rich enough to reproduce the Kovacs memory effect, even allowing to obtain analytical expression for the Kovacs hump in a long time regime. The Kovacs effect is observed in many experiments and models, showing common qualitative properties which we have found to be shared also by the model analyzed in this paper. The quantitative properties depend on the particular system or model analyzed.

As far as it concerns the HOSS model, it turns out that for the slow modes, i.e. the oscillator variables, fixing the overall average value, the magnetization $m_1$, does not prevent the existence of memory encoded in the variable $m_2$, which keeps track of the history of the system. The equilibrium value of $m_2$ increases with temperature while the equilibrium value of $m_1$ decreases with increasing temperature. Therefore after the final switch of temperature, since $m_2(t_0) > \bar{m}_2^{T_f}$, the variable $m_2$ has a value corresponding to an equilibrium condition at a higher temperature (memory of the initial state at temperature $T_f$) so driving the system towards a condition corresponding to a higher temperature, i.e. smaller values of $m_1$, determining the hump.
It is important to stress that a fundamental ingredient in the HOSS model is the interaction between slow and fast modes. Due to this interaction the equilibrium configurations of the oscillator variables at a given temperature are determined by both $m_2$ and $m_1$, the first and second moment of their distribution, whose dynamical evolution is interdependent. When such interaction is turned off (by setting $J = 0$) essentially only one variable is sufficient to describe the equilibrium configurations and the dynamics of the system, and the memory effect is lost. In this respect this model constitutes an improvement to the so-called oscillator model [12] within which such memory effect cannot be reproduced. Another important conclusion, confirming previous results [10], which can be drawn from this model is that a complete thermodynamic-like picture inclusive of the Kovacs’ effect, with only an effective temperature is not possible and that also an effective field in this case is needed. In the present model one can also study temperature cycle experiments of the type carried out in spin glasses (see [13]), leaving room for further research.

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**APPENDIX A**

In this Appendix we report all the explicit expressions for terms appearing in the text. In Eqs. 2 and 6 we have:

- \( w_T(m_1, m_2) = \sqrt{J^2 m_2 + 2JLm_1 + L^2 + T^2/4} \)
- \( K_T(m_1, m_2) = K - \frac{J^2}{w_T(m_1, m_2) + T/2} \)
- \( H_T(m_1, m_2) = H + \frac{JL}{w_T(m_1, m_2) + T/2} \)
- \( f_T(m_1, m_2) = \frac{\sigma^2 K_T(m_1, m_2)}{2T} \text{Erfc}[\tilde{\alpha}_T(m_1, m_2)] \cdot \exp\left[\frac{\sigma^2}{8w_T(m_1, m_2)} - \alpha_T^2(m_1, m_2)\right] \)
- \( I_T(m_1, m_2) = \frac{\sigma^2 K_T(m_1, m_2)}{4} \text{Erfc}[\alpha_T(m_1, m_2)] \)

where:

- \( \tilde{\alpha}_T(m_1, m_2) = m_2 - m_1^2 + \left(\frac{H_T(m_1, m_2)}{K_T(m_1, m_2)} - m_1\right)^2 \)
- \( \alpha_T(m_1, m_2) = \sqrt{\frac{\sigma^2}{8w_T(m_1, m_2)}} \)
- \( \alpha_T(m_1, m_2) = \frac{2K_T(m_1, m_2)\tilde{w}_T(m_1, m_2)}{T} - 1 \)

In Eqs. 18, 19, 20, 21, 22 and 23:

\[
A_s = \frac{\sigma^2 K_T}{8}
\]
\[
D = JH + LK = JH_T + LK_T
\]
\[
Q_T(m_1, m_2) = \frac{K_T^2 w_T(w_T + T/2)^2}{J^2 D}
\]
\[
P_T(m_1, m_2) = \frac{1}{2K_T w_T(w_T + T/2)^2}
\]
\[
A_k = \frac{K_T^2 (K - \tilde{K}_T)(1 + D\tilde{Q}_T - \tilde{P}_T)}{(K - \tilde{K}_T)(1 + D\tilde{Q}_T - \tilde{K}_T \tilde{Q}_T)}
\]
\[
2F_1(a, b, c, z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{n=0}^{\infty} \frac{\Gamma(a+n)\Gamma(b+n)\ z^n}{\Gamma(c+n)\ n!}
\]
\[
A_Q = 1 + Q_T(\tilde{m}_1^T, \tilde{m}_2^T)D = 1 + Q_T D
\]
\[
B^2_Q = \exp[A_Q - \frac{2}{\gamma(\delta_m^2)^\gamma}]
\]
\[
C_Q = \frac{2Q_T T_f}{(m_0 + \mu^2)} = \frac{1}{\gamma(1 + D\tilde{Q}_T + \tilde{P}_T)}
\]
\[
A_T^1(m_1, m_2) = \frac{m_1(Jm_1 + L + (w_T + T/2)K_T)}{2(m_0 + \mu^2)}
\]
\[
A_T^2(m_1, m_2) = 2m_1 A_T^1(m_1, m_2)
\]
\[
t_0 = \frac{\sqrt{\pi}}{8\gamma} 1 + D\tilde{Q}_T + \tilde{P}_T
\]

1. Solutions of Eq. (20) for \( \gamma = \frac{3}{2} \)

\[
\mu_1(\delta \mu_2) = \left(1 - \frac{\sqrt{1 + \frac{\delta \mu_2}{\mu^2}}}{1 + \sqrt{1 + \frac{\delta \mu_2}{\mu^2}}}\right)^{\frac{\nu_2}{\nu_2 + \nu_2}} e^{\frac{\nu_2}{\nu_2 + \nu_2} + \frac{\nu_2}{\nu_2 + \nu_2} \nu_2 + \nu_2 + \nu_2} \left(\mu_1 B_Q^2\right)
\]

\[
-C_Q \int_{\delta \mu_2}^{\nu_2} dz \left(1 + \sqrt{1 + \frac{\nu_2}{\nu_2}} \right)^{\frac{\nu_2}{\nu_2 + \nu_2}} e^{\frac{\nu_2}{\nu_2 + \nu_2} + \frac{\nu_2}{\nu_2 + \nu_2} \nu_2 + \nu_2 + \nu_2} \left(\mu_1 B_Q^2\right)
\]

2. Solutions of Eq. (20) for \( \gamma = 2 \)

\[
\mu_1(\delta \mu_2) = \left(\frac{\delta \mu_2}{\delta \mu_2 + \mu^2}\right)^{\frac{\nu_2}{\nu_2}} e^{\frac{\nu_2}{\nu_2 + \nu_2} + \frac{\nu_2}{\nu_2 + \nu_2} \nu_2 + \nu_2 + \nu_2} \left(\mu_1 B_Q^2\right)
\]

\[
-C_Q \int_{\delta \mu_2}^{\nu_2} dz \left(1 + \sqrt{1 + \frac{\nu_2}{\nu_2}} \right)^{\frac{\nu_2}{\nu_2 + \nu_2}} e^{\frac{\nu_2}{\nu_2 + \nu_2} + \frac{\nu_2}{\nu_2 + \nu_2} \nu_2 + \nu_2 + \nu_2} \left(\mu_1 B_Q^2\right)
\]
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