The quantum and Klein-Gordon oscillators in a non-commutative complex space and the thermodynamic functions

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Abstract

In this work we study the quantum and Klein-Gordon oscillators in non-commutative complex space. We show that the quantum and Klein-Gordon oscillators in non-commutative complex space are obey an particle similar to the an electron with spin in a commutative space in an external uniform magnetic field. Therefore the wave function $\psi(z, \bar{z})$ takes values in $\mathbb{C}^4$, spin up, spin down, particle, antiparticle, a result which is obtained by the Dirac theory. The energy levels could be obtained by exact solution. We also derived the thermodynamic functions associated to the partition function, and we show that the non-commutativity effects are manifested in energy at the high temperature limit.

1 Introduction

In recent years many arguments have been suggested to motivate a deviation from the flat-space concept at very short distances [1, 2] so that we have a new concept of quantum spaces [3, 4, 5, 6, 7]. Quantum spaces depend on parameters such that for a particular value of these parameters they become the usual flat space. We consider the simplest version of quantum spaces as the natural extension of the usual quantum mechanical commutation relations between position and momentum, by imposing further commutation relations between position coordinates themselves. The non-commutative space can be obtained by the coordinate operators where we replace the ordinary product by the star product:

$$[x^\mu, x^\nu] = i\theta^{\mu\nu}, \quad (1)$$

or using the new non-commutative coordinate operators $\hat{x}$ satisfying the commutation relations:

$$[\hat{x}^\mu, \hat{x}^\nu] = i\theta^{\mu\nu}, \quad (2)$$

where

$$\hat{x}^\mu = x^\mu - \frac{\theta^{\mu\nu}}{2}p^\nu. \quad (3)$$

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For the non-commutative canonical-type space, the parameter $\theta^{\mu\nu}$ is an anti-symmetric real matrix of dimension length-square. The star-product $(\star)$ defined between two function is given by:

$$\psi (x) \star \varphi (x) = \psi (x) \exp \left\{ -i\theta_{\mu\nu} \left\langle \frac{\partial}{\partial \mu} - i \frac{\partial}{\partial \nu} \right\rangle \right\} \varphi (y) \mid_{x=y}.$$  \hspace{1cm} (4)

It is easy to verify the relation (1) by simply replacing $\psi (x) = x^\mu, \varphi (x) = x^\nu$ in (4). This idea is similar to the physical meaning of the Plank constant in the relation $[x,p] = i\hbar$, which as is known is the smallest phase-space in quantum mechanics. There is a lot of interest in recent years in the study of non-commutative canonical-type quantum mechanics, quantum field theory and string theory [8,9,10]. On other hand the solutions of classical dynamical problems of physical systems obtained in terms of complex space variables are well-known. There are also interests in the complex quantum mechanical systems (in two dimensions) [11,12], in which we consider a quantum harmonic oscillator in non-commutative complex space (so coordinate and momentum operators of this space are written $\hat{z} = x + iy$ and $\hat{p}_z = (p_x - ip_y)/2$).

In this paper we first present a harmonic oscillator in the non-commutative complex space. The system is described by the wave function $\psi (z, \bar{z})$ which takes values in $C^4$, spin up, spin down, particle, antiparticle, which is obtained by the Dirac formalism. Secondly we shall also study relativistic quantum mechanics particularly we show that the Klein-Gordon oscillator in the non-commutative complex space is similar to the equation of motion for a fermion with spin $1/2$ [13].

This paper is organized as follows. In section 2 we present the quantum oscillator in a non-commutative complex space. We shall see the effect of non-commutativity of space on the thermodynamics function associated to the harmonic oscillator with a non-commutative complex space, and we show how our findings differ from the results obtained in ref [14,15]. In section 3 we derive the deformed Klein-Gordon equation for the harmonic oscillator with non-commutative complex space. We solve this equation and obtain the non-commutative modification of the energy levels. Finally, in section 5, we draw our conclusions.

2 Quantum oscillator in non-commutative complex space

In two dimensional space, the complex coordinate system $(z, \bar{z})$ and momentum $(p_z, p_{\bar{z}})$ is defined by:

$$z = x + iy, \hspace{0.5cm} \bar{z} = x - iy, \hspace{0.5cm} \text{and} \hspace{0.5cm} p_z = \frac{1}{2} (p_x - ip_y), \hspace{0.5cm} p_{\bar{z}} = - \bar{p}_z = \frac{1}{2} (p_x + ip_y). \hspace{1cm} (5)$$

We are interested in introducing the non-commutative complex operators of
coordinates and momentum in a two-dimensional space:

\[ \hat{\mathbf{z}} = \hat{x} + i \hat{y} = z + i \theta p_z, \quad \hat{\bar{\mathbf{z}}} = \hat{x} - i \hat{y} = \bar{z} - i \theta p_z, \]  

\[ \hat{p}_z = p_z, \quad \hat{\bar{p}}_z = p_{\bar{z}}. \]  

The non-commutative algebra (1) can be rewritten as:

\[ [\hat{\mathbf{z}}, \hat{\bar{\mathbf{z}}}] = 2 \theta, \quad [\hat{\mathbf{z}}, \hat{p}_z] = [\hat{\bar{\mathbf{z}}}, \hat{p}_{\bar{z}}] = 0, \quad [\hat{\bar{\mathbf{z}}}, \hat{p}_z] = 2 \hbar, \quad [p_z, p_{\bar{z}}] = 0. \]  

Now we will discuss the oscillator systems on non-commutative complex quantum space. In this formulation we consider the isotropic oscillator on a two-dimensional space:

\[ H = \frac{1}{2m} (p_x^2 + p_y^2) + \frac{m \omega^2}{2} (x^2 + y^2) \]
\[ = \frac{1}{2m} (p_x - ip_y) (p_x + ip_y) + \frac{m \omega^2}{2} (x + iy) (x - iy). \]  

Using the eqs. (5), eq. (9) takes the form:

\[ H = H_{\mathbf{z}\bar{\mathbf{z}}} = \frac{2}{m} p_z p_{\bar{z}} + \frac{m \omega^2}{2} z \bar{z} = H_{\bar{z}z}. \]  

This Hamiltonian of an oscillator on a complex space is Hermitian. The solution of the corresponding eigenvalue is real. We can introduce the new annihilation operators \( a \) and \( b \) defined as:

\[ a = \frac{2i p_{\bar{z}} + m \omega z}{2 \sqrt{m \omega}}, \quad (11) \]
\[ b = \frac{2i p_z + m \omega \bar{z}}{2 \sqrt{m \omega}}, \quad (12) \]

and the corresponding creation operators \( a^+ \) and \( b^+ \) satisfying the usual commutation relations:

\[ [a, a^+] = [b, b^+] = 1. \]  

All other commutations are zero. We can express \( z, \bar{z} \) and \( p_z, p_{\bar{z}} \) in terms of these operators as:

\[ p_z = \frac{i}{\sqrt{m \omega}} (-a + b^+), \quad (14) \]
\[ \bar{p}_z = \frac{i}{\sqrt{m \omega}} (a^+ - b), \quad (15) \]
\[ z = \frac{1}{\sqrt{m \omega}} (a + b^+), \quad (16) \]
\[ \bar{z} = \frac{1}{\sqrt{m \omega}} (b + a^+). \quad (17) \]

After replacing (14) into (17) the Hamiltonian is written as:

\[ H_{z\bar{z}} = \omega (a^+ a + b^+ b + 1). \]  

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The operators \( a^+ a = N^+ \) and \( b^+ b = N_+ \) satisfy the eigenvalue equations
\[ N^+ |n^+, n_+\rangle = n^+ |n^+, n_+\rangle \]
with \( n^+_+ = 0, 1, 2, \ldots \). The eigenvalues for the Hamiltonian (18) are given by:
\[ E_{n^+ n_+} = \omega (n^+ + n_+ + 1) . \tag{19} \]

To calculate the wave functions \( \psi_{n^+ n_+} (z, \bar{z}) \), we simply apply the operators \( a^+ \) and \( b^+ \) on the ground state \( \psi_{00} (z, \bar{z}) \) which is given by:
\[ \psi_{00} (z, \bar{z}) = \sqrt{\frac{m \omega}{\pi}} \exp \left( -\frac{m \omega}{2} z \bar{z} \right) . \tag{20} \]
We see from eq. (19) that the eigenvalues of \( H_{zz} \) take the form:
\[ E_n = \omega (n + 1) , \tag{21} \]
where
\[ n = n^+ + n_+ . \tag{22} \]
For each value of the energy (labeled by \( n \)), the wave functions of the system is described by two wave functions \([16, 12]\):
\[ \psi_{n0} (z, \bar{z}) = C z^n \exp \left( -\frac{m \omega}{2} z \bar{z} \right) , \quad \psi_{0n} (z, \bar{z}) = C \bar{z}^n \exp \left( -\frac{m \omega}{2} z \bar{z} \right) \tag{23} \]
where \( C = \sqrt{\frac{(m \omega)^{n+1}}{\pi n!}} \).

In the non-commutative complex space we notice that \( \hat{z} \hat{\bar{z}} \neq \hat{\bar{z}} \hat{z} \). Then the Hamiltonian of equation (10) can be written as:
\[ \hat{H} = \begin{pmatrix} \hat{H}_{zz} & 0 \\ 0 & \hat{H}_{\bar{z}\bar{z}} \end{pmatrix} , \tag{24} \]
where
\[ \hat{H}_{zz} = \frac{2}{m} p_z p_{\bar{z}} + \frac{m \omega^2}{2} \hat{z} \hat{\bar{z}}, \tag{25} \]
and
\[ \hat{H}_{\bar{z}\bar{z}} = \frac{2}{m} p_{\bar{z}} p_z + \frac{m \omega^2}{2} \hat{\bar{z}} \hat{z} . \tag{26} \]

The Hamiltonian of equation (25) takes the form:
\[ \hat{H}_{zz} = \frac{2}{m} p_z p_{\bar{z}} + \frac{1}{2} m \omega^2 z \bar{z} - \frac{m \omega^2}{2} \theta (L_z - 1) , \tag{27} \]
where \( \frac{1}{m} = \frac{1}{m} + \frac{m \omega^2 \theta^2}{2} \), \( \bar{\omega} = \omega \sqrt{1 + \frac{m^2 \omega^2 \theta^2}{2} } \) and \( L_z = i (zp_z - \bar{z} p_{\bar{z}}) \). The last term can be written as:
\[ -\frac{m \omega^2}{2} \theta (L_z - 1) = -\frac{m \omega^2}{2} \theta (L_z + 2 s_z) , \quad \text{where} \quad s_z = -1/2 , \tag{28} \]
which is the same as the total magnetic moment of particle with spin 1/2 where the non-commutativity parameter plays the role of a magnetic field.

Then the Hamiltonian of eq. (27) takes the form:

$$\hat{H} = \frac{2}{m}p_z p_z + \frac{1}{2} m\tilde{\omega}^2 z \bar{z} - \frac{m\omega^2}{2} \theta (L_z + 2s_z) = \hat{H}_z, \text{ where } s_z = -1/2. \quad (29)$$

This Hamiltonian is Hermitian and with represents the oscillation of a particle with spin 1/2 in a uniform external magnetic field where the direction of spin is opposite to that of the magnetic field.

Furthermore the Hamiltonian of equation (26) takes the form:

$$\hat{H} = \frac{2}{m}p_z p_z + \frac{1}{2} m\tilde{\omega}^2 z \bar{z} - \frac{m\omega^2}{2} \theta (L_z + 2s_z) = \hat{H}_t, \text{ where } s_z = 1/2. \quad (30)$$

This Hamiltonian is Hermitian and represents a particle with spin 1/2 in a uniform external magnetic field where the projection of spin is along the direction of the magnetic field.

The Hamiltonian in eqs. (29) and (30) is Hermitian and can written in terms of creation/annihilation operators as:

$$\hat{H} = \begin{pmatrix} \tilde{\omega} (a^+ a + b^+ b + 1) + \frac{m\omega^2}{2} \theta (a^+ a - b^+ b + 1) & 0 \\ 0 & \tilde{\omega} (a^+ a + b^+ b + 1) + \frac{m\omega^2}{2} \theta (a^+ a - b^+ b - 1) \end{pmatrix}. \quad (31)$$

Here the annihilation operators $a$ and $b$ are

$$a = \frac{2p_z - i m\tilde{\omega} z}{2\sqrt{m\omega}}, \quad (32)$$

$$b = \frac{2p_z + i m\tilde{\omega} \bar{z}}{2\sqrt{m\omega}}, \quad (33)$$

and we have replaced the angular momentum $L_z$ in terms of $a, a^+$ and $b, b^+$ through:

$$L_z = -(a^+ a - b^+ b). \quad (34)$$

The eigenvalues for the Hamiltonian (31) are:

$$E_{\pm} = \tilde{\omega} (n^+ + n_+ + 1) + \frac{m\omega^2}{2} \theta (n^+ - n_+ \pm 1). \quad (35)$$

We see from eq. (35) that the eigenvalues of $\hat{H}$ take the form:

$$E_n = \tilde{\omega} (n + 1) + \frac{m\omega^2}{2} \theta (m_l \pm 1), \quad (36)$$

where $n = n^+ + n_+$ and $m_l = n^+ - n_+$. For a given value of the energy labeled by $n$, the values of $m_l$ are bound by $(n)$ from above and by $(-n)$ from below.
The wave function of the system is described by four wave functions:

\[ \psi_{n0}^+ (z, \bar{z}) = C z^n \exp \left( -\frac{m\omega}{2} z \bar{z} \right) |\uparrow\rangle, \psi_{n0}^- (z, \bar{z}) = C z^n \exp \left( -\frac{m\omega}{2} z \bar{z} \right) |\downarrow\rangle \]  

\[ \psi_{0n}^+ (z, \bar{z}) = C \bar{z}^n \exp \left( -\frac{m\omega}{2} z \bar{z} \right) |\uparrow\rangle, \psi_{0n}^- (z, \bar{z}) = C \bar{z}^n \exp \left( -\frac{m\omega}{2} z \bar{z} \right) |\downarrow\rangle \]  

which correspond to the limiting values of \( L_z \) being maximal \((n)\) and minimal \((-n)\). Thus a single oscillator state may be split into two pairs each having the same energy leading to twofold degeneracy of the energy levels. This oscillator is described by two double component spinor:

\[ \psi_{n0} = \left( \begin{array}{c} \psi_{n0}^+ \\ \psi_{n0}^- \end{array} \right), \quad \text{and} \quad \psi_{0n} = \left( \begin{array}{c} \psi_{0n}^+ \\ \psi_{0n}^- \end{array} \right), \]  

where the sign \((\pm)\) signifies spin up or down. The oscillator is positioned in the four equivalent points \((z, \bar{z}, z, \bar{z}) \leftrightarrow (z, \bar{z}, -z, -\bar{z})\). Therefore the wave function \(\psi (z, \bar{z})\) takes values in \(C^4\), spin up, spin down, particle, antiparticle.

This result is obtained by the Dirac theory.

For the special case \(n^+ = n_+ = n\) the eigenvalues are:

\[ E_{\pm} = \tilde{\omega} (2n + 1) \pm \frac{m\omega^2}{2} \theta. \]  

They correspond to the energy values for the charged oscillator with spin \(1/2\) in a magnetic field, where the non-commutativity plays the role of magnetic field which creates the total magnetic moment of a particle with spin \(1/2\), which in turn shifts the spectrum of energy resulting in the degeneracy being removed. Such effects are similar to the Zeeman splitting in a commutative space.

We note that the Hamiltonian of harmonic oscillator in non-commutative ordinary coordinates reads:

\[ \hat{H} = \frac{1}{2} \left( \hat{H}_\downarrow + \hat{H}_\uparrow \right) = \frac{1}{2m} \left( p_x^2 + p_y^2 \right) + \frac{m\omega^2}{2} \left( \hat{x}^2 + \hat{y}^2 \right). \]  

The eigenvalues of this Hamiltonian can be written as:

\[ E = \tilde{\omega} (n^+ + n_+ + 1) + \frac{m\omega^2}{2} \theta \left( n^+ - n_+ \right) \]  

For the special case \(n^+ = n_+ = n\), the energy \(E = \tilde{\omega} (2n + 1)\) is not shifted, contrary to non-commutative complex space where the energy levels are shifted. Thus the system without spin in non-commutative coordinate space has an added advantage that the spin effect is automatically manifested.

The thermodynamic functions associated with the non-commutative complex oscillator are also of interest. First we calculate the partition function \(Z (\beta, \theta) \) [14, 15]:

\[ Z (\beta, \theta) = \text{tr} G (x, \dot{x}), \]  

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where $G(x, \hat{x})$ is the Green function given by:

$$ G(x, \hat{x}; T) = \langle x | \exp \left\{ -i \left( \hat{H} \right) (t - \hat{t}) \right\} | \hat{x} \rangle. $$

The Hamiltonian $\hat{H}$ is diagonal in the basis $|n^+ n_+\rangle$ and the energy eigenvalue given in the relationship (35) can be expressed by:

$$ E_{\pm} = \hbar \tilde{\omega} \left( \left( 1 + \frac{m \omega^2}{2} \right) n^+ + \left( 1 - \frac{m \omega^2}{2} \right) n_+ + \left( 1 \pm \frac{m \omega^2}{2} \right) \right). $$

Then the Green function in eq. (44) is readily found to be:

$$ G(x, \hat{x}; T) = \sum_{n^+ n_+} \psi_{n^+ n_+}^+(x) \psi_{n^+ n_+}^-(\hat{x}) \times \exp \left\{ -i \left( \hbar \tilde{\omega} \left( \left( 1 + \frac{m \omega^2}{2} \right) n^+ + \left( 1 - \frac{m \omega^2}{2} \right) n_+ + \left( 1 \pm \frac{m \omega^2}{2} \right) \right) \right\} T \right\}. $$

where $T = t - \hat{t}$ and $\psi_{n^+ n_+}^+(x)$ are the eigenfunctions of the harmonic oscillator in in commutative space. Using the Euclidean rotation $i (t - \hat{t}) \rightarrow \beta$, one can see that the partition function in equation (43) can be written by as:

$$ Z_{\pm} (\beta, \theta) = \exp \left\{ -\beta \hbar \tilde{\omega} \left( 1 \pm \frac{m \omega}{2} \theta \right) \right\} \sum_{n^+} \exp \left\{ -\beta \hbar \tilde{\omega} \left( 1 + \frac{m \omega}{2} \theta \right) n^+ \right\} \times \sum_{n_+} \exp \left\{ -\beta \hbar \tilde{\omega} \left( 1 - \frac{m \omega}{2} \theta \right) n_+ \right\} \exp \left\{ \mp \beta \hbar \frac{m \omega^2}{2} \theta \right\} \frac{1}{4 \sinh \beta \hbar \tilde{\omega} \left( 1 + \frac{m \omega}{2} \theta \right) \sinh \beta \hbar \tilde{\omega} \left( 1 - \frac{m \omega}{2} \theta \right)} = \exp \left\{ \mp \beta \hbar \frac{m \omega^2}{2} \theta \right\} Z_{HO} (\beta, \theta), $$

where

$$ Z_{HO} (\beta, \theta) = \frac{1}{4 \sinh \beta \hbar \tilde{\omega} \left( 1 + \frac{m \omega}{2} \theta \right) \sinh \beta \hbar \tilde{\omega} \left( 1 - \frac{m \omega}{2} \theta \right)} $$

is the partition function of the harmonic oscillator in non-commutative space [14, 15].

Now we are in position to compute several interesting quantities. The Free energy of a systems at finite temperature is:

$$ F_{\pm} (\beta, \theta) = -\frac{1}{\beta} \ln Z_{\pm} (\beta, \theta) = \pm \frac{\hbar m \omega^2}{2} \theta - \frac{1}{\beta} \ln Z_{HO} (\beta, \theta), $$

which at low temperature limit tends to the ground state energy of system. In this case we have:

$$ \lim_{\beta \rightarrow \infty} F (\beta, \theta) = \hbar \tilde{\omega} \left( 1 \pm \frac{m \omega}{2} \theta \right), $$

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which corresponds to ground state energy of the system, $E_{00} = \hbar \tilde{\omega} (1 \pm \frac{m\omega}{\theta})$.

The mean energy of a systems is:

$$\langle E^{\pm}(\theta,\beta) \rangle = -\frac{\partial}{\partial \beta} \ln Z^{\pm}(\beta,\theta)$$

$$= \mp \frac{\hbar m\omega^2}{2} \theta + \hbar \tilde{\omega} \left[ \left( 1 + \frac{m\omega}{\theta} \right) \cosh \beta h\tilde{\omega} \left( 1 + \frac{m\omega}{\theta} \right) \\
+ \left( 1 - \frac{m\omega}{\theta} \right) \cosh \beta h\tilde{\omega} \left( 1 - \frac{m\omega}{\theta} \right) \right]. \quad (51)$$

Let us now investigate the infinite temperature limit ($\beta \to 0$) where $\theta$ and $\omega$ are fixed. At this limit Eq. (51) collapses into:

$$\lim_{\beta \to 0} \langle E^{\pm}(\theta,\beta) \rangle = \pm \frac{\hbar m\omega^2}{2} \theta + 2\hbar \omega \frac{kT}{\hbar \tilde{\omega}} = \hbar \tilde{\omega} \left( 2\frac{kT}{\hbar \tilde{\omega}} \pm \frac{m\omega}{\theta} \right). \quad (52)$$

This means that the non-commutativity effects are manifested in energy at the high temperature limit, contrary to what was found in the reference [15].

### 3 Klein-Gordon Oscillator in Non-commutative complex space

The Klein-Gordon equation in complex quantum space and in constant magnetic field has the following form:

$$\left( 2p_z - ie\frac{B}{2} \hat{z} \right) \left( 2p_z + ie\frac{B}{2} \hat{z} \right) \psi = (E^2 - m^2) \psi. \quad (53)$$

which can be written in commutative space as:

$$\left( p_x^2 + p_y^2 + \frac{e^2 B^2}{4} (x^2 + y^2) - eB L_z \right) \psi = (E^2 - m^2 + eB) \psi. \quad (54)$$

This equation is same as the Klein-Gordon equation in real space and in magnetic field with the extra constant ($eB$). In a non-commutative complex quantum space, it is described by the following equation:

$$\begin{pmatrix}
2p_z + ie\frac{B}{2} \hat{z} & 0 \\
0 & 2p_z - ie\frac{B}{2} \hat{z}
\end{pmatrix} \begin{pmatrix}
2p_z + ie\frac{B}{2} \hat{z} \\
0
\end{pmatrix} \psi = (E^2 - m^2) \psi. \quad (55)$$

Using the definition of the non-commutative coordinates, we can rewrite this equation in a commutative quantum space as:

$$\begin{pmatrix}
1 + \frac{eB\theta}{4} & p_x^2 + p_y^2 + \frac{e^2 B^2}{4} (x^2 + y^2) \\
-eB L_z - \frac{eB\theta}{4} \left( L_z \pm 2 \left( \frac{1}{2} \right) \right)
\end{pmatrix} \psi = (E^2 - m^2 + eB) \psi, \quad (56)$$
so that the critical point is obtained when the coefficient of the extra non-commutative constant equals to zero. In this case the non-commutativity parameter $\theta = \frac{A}{27}$. This value is the same one obtained in the case of the equivalence between non-commutative quantum mechanics and the Landau problem [14]. At this point it is clear that the equation (56) is similar to the Klein-Gordon one in a commutative space with an external magnetic field.

The Klein-Gordon oscillator in complex space can be defined by the following equation:

$$(2p_z + im \omega \hat{z}) (2p_z - im \omega \hat{z}) \psi = (E^2 - m^2) \psi. \quad (57)$$

which can be rewritten in commutative space as:

$$(p_x^2 + p_y^2 + m^2 \omega^2 (x^2 + y^2) + 2m \omega L_z) \psi = (E^2 - m^2 + 2m \omega) \psi. \quad (58)$$

It is clear that the K-G oscillator in a complex space is similar to the K-G equation for a particle of positive charge in an external magnetic field. The eigenvalues for the Hamiltonian in equation (58) are:

$$E^2 = 2m \omega (n_x + n_y + 1) + 2m \omega (m_l - 1) + m^2$$

$$= 2m \omega (n_x + n_y + m_l) + m^2. \quad (59)$$

In the non-commutative complex space the K-G oscillator is described by the following equation:

$$
\begin{pmatrix}
(2p_z + im \omega \hat{z}) & 0 & (2p_z - im \omega \hat{z}) \\
0 & (2p_z - im \omega \hat{z}) & (2p_z + im \omega \hat{z})
\end{pmatrix} \psi = (E^2 - m^2) \psi. \quad (60)
$$

Using the relations (6) and (7) we can rewrite the equation (60) in commutative space as:

$$
\begin{pmatrix}
1 - m \omega \theta & \frac{m^2 \omega^2 \theta^2}{4} & 0 \\
0 & (p_x^2 + p_y^2) & m^2 \omega^2 (x^2 + y^2) + 2m \omega L_z - m^2 \omega^2 \theta (L_z \pm 1)
\end{pmatrix} \psi = (E^2 - m^2 + 2m \omega) \psi. \quad (61)
$$

This equation is similar to the equation of motion for a fermion of spin $\frac{1}{2}$ in a constant magnetic field. Then the equation (61) takes the following form:

$$
\begin{pmatrix}
1 + \frac{m \omega \theta}{2} & \frac{m^2 \omega^2 \theta^2}{4} & 0 \\
0 & (p_x^2 + p_y^2) + m^2 \omega^2 (x^2 + y^2) - 2m \omega L_z \\
0 & m^2 \omega^2 \theta (L_z + 2s_z)
\end{pmatrix} \psi = (E^2 - m^2 - 2m \omega) \psi, \quad s_z = \pm \frac{1}{2}. \quad (62)
$$

where the energy eigenvalues for eq. (62) are given by:

$$E^2 = 2m \omega \theta (n_x + n_y + 1) + 2m \omega \theta (m_l \pm 1) + m^2, \quad (63)$$
where $\omega_\theta = \omega (1 - \frac{m \omega \theta}{2})$. We have found that the non-commutativity plays the role of a magnetic field interacting automatically with the spin of a particle, thereby the system with spin in a magnetic field will have a resonance [17]. Then the critical values of $\theta = \frac{2}{m \omega}$ can be considered as a resonance point.

4 Conclusion

In this work we started from quantum charged oscillator in a canonical non-commutative complex space. By using the Moyal product up to first order in the non-commutativity parameter $\theta$, we derived the deformed Hamiltonian and Klein-Gordon equation. By solving it exactly we found that the energy is shifted up to the first order in $\theta$ by two levels, hence we can say that the particle in non-commutative complex space is equivalent to a particle with spin $1/2$ in magnetic field, where the non-commutativity plays the role of magnetic field which creates the total magnetic moment of particle with spin $1/2$, which in turn shifts the spectrum of energy. Such effects are similar to the Zeeman splitting in a commutative space.

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