Graph theory and qubit information systems of extremal black branes

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Abstract
Using graph theory based on Adinkras, we reconsider the study of extremal black branes in the framework of quantum information. More precisely, we propose a one-to-one correspondence between qubit systems, Adinkras and certain extremal black branes obtained from a type IIA superstring compactified on $T^n$. We accordingly interpret the real Hodge diagram of $T^n$ as the geometry of a class of Adinkras formed by $2^n$ bosonic nodes representing $n$ qubits. In this graphic representation, each node encodes information on the qubit quantum states and the charges of the extremal black branes built on $T^n$. The correspondence is generalized to $n$ superqubits associated with odd and even geometries on the real supermanifold $T^{n|n}$. Using a combinatorial computation, general expressions describing the number of the bosonic and the fermionic states are obtained.

Keywords: string theory, black branes, qubit systems, Adinkras, supermanifolds

(Some figures may appear in colour only in the online journal)

1. Introduction

Recently, extremal black branes in arbitrary dimensions have been investigated using different methods in the framework of string theory and related topics including M-theory compactified on manifolds having special holonomy groups [1–4]. These black objects have
been studied using the so-called attractor mechanism [5–7]. In this way, the scalar fields can be fixed in terms of the black brane charges. This analysis can be done in terms of an effective potential depending on the charges and the stringy moduli obtained from the compactification of higher-dimensional theories. Extremizing the potential with respect to such moduli, the minimum generates the scalar fixed values. Moreover, the corresponding entropy functions have been computed using the U-duality symmetry acting on the invariant black brane charges of the compactified theories. In this regard, the Calabi–Yau compactifications have been explored to produce several interesting results dealing with the black objects in higher-dimensional theories including string theory [6–18].

More recently, a connection with quantum information has been proposed using the qubit formalism [19–24]. It is recalled that the qubit is the building piece of quantum information theory. In fact, a possible interplay between STU black holes having eight charges and three qubits have been given in [25, 26]. The analysis has been extended to include superqubits using a supersymmetric approach based on the theory of Lie superalgebras. In particular, the osp(2|1) Lie superalgebra has been explored to deal with the physics of the superqubits [27–30].

In this paper, we reconsider the study of the extremal black branes in the framework of quantum information using graph theory based on the so-called Adinkras [32, 33]. More precisely, we establish a one-to-one correspondence between qubit systems, Adinkras and extremal black branes embedded in maximally supersymmetric supergravity obtained from a low energy limit of a type IIA superstring compactified on $T^n$. We accordingly interpret the real Hodge diagram of $T^n$ as the geometry of a class of Adinkras involving $2^n$ bosonic nodes representing $n$ qubits. In this graphic representation, each node encodes information on the qubit quantum states and the charges of the extremal black branes constructed from the compactification of a type IIA superstring on $T^n$. The correspondence is generalized to $n$ superqubits associated with odd and even geometries on the real supermanifold $T^{n|n}$. Using combinatorial computation, general expressions describing the number of bosonic and fermionic states are obtained. Then, illustrated models are given. It is worth noting that another nice connection between cohomology of extra dimensions and qubit systems has been also given in [24, 31].

The organization of the paper is as follows. In section 2, we reconsider the study of the extremal black branes in a type IIA superstring compactified on $T^n$. Section 3 concerns an elaboration of a correspondence between qubit systems, Adinkras and the real Hodge diagrams of $T^n$. In section 4, we give a graphic representation of the extremal black branes using qubit systems based on Adinkras. The generalization to superqubit systems is given in section 5. The analysis is done in terms of the real supermanifold $T^{n|n}$. The last section is devoted to the conclusion and open questions.

2. Extremal black branes in string theory

In this section, we reconsider the discussion of the extremal black branes in string theory compactified on $n$-dimensional compact manifolds $X^n$. It has been shown that the black $p$-objects can be produced by a system of $(p + k)$-branes wrapping the $k$-cycles of $X^n$. In such a compactification, it has been shown that the near-horizon geometries of the extremal black $p$-brane are defined by the product of AdS spaces and spheres

$$\text{AdS}_{p+2} \times S^{8-n-p}. \quad (2.1)$$
It is obvious that the integers \( n \) and \( p \) should satisfy

\[
1 \leq n, \quad 2 \leq 8 - n - p. \tag{2.2}
\]

The generalization of such geometries to investigate intersecting attractors has been also given in \([18]\).

In higher-dimensional theories, one may classify the black \( p \)-brane solutions using the extended electric/magnetic duality connecting a \( p \)-dimensional electrical black brane to a \( q \)-dimensional magnetic one via the following relation

\[
p + q = 6 - n. \tag{2.3}
\]

The solution of this equation provides three different black objects organized in terms of the values of the charge couple \((p, q)\). Indeed, they are given by

- \((p, q) = (0, 6 - n)\), associated with the electrical charged black holes,
- \((p, q) = (3 - \frac{n}{2}, 3 - \frac{n}{2})\) \((n \text{ even})\), corresponding to the dyonic black branes, having both electric and magnetic charges,
- \((p, q) \neq (0, 6 - n)\), where \((p, q) \neq (3 - \frac{n}{2}, 3 - \frac{n}{2})\), describing the magnetic black branes.

It is noted that \( p = 0 \) describes electric charged black holes in \( -n \)-dimensions. Their dual magnetic are black \((6 - n)\)-branes. These black objects carry charges corresponding to the gauge-invariant field strengths \( (F = dA)\) of the corresponding maximally supersymmetric supergravity theory. It is worth recalling that even-dimensional space–times produce dyonic black branes. The charges of these objects can fix the value of the dilaton using the attractor mechanism as reported in \([17]\).

Before examining a particular supergravity theory solution, we should recall that the black object charges depend on the choice of the internal space. It turns out that each compactification involves a Hodge diagram carrying not only geometric information but also physical data on the black brane solutions.

In the following sections, we will be concerned with the compactification of type IIA superstring theory on the \( n \)-dimensional tori \( T^n \). This compactification may produce the black \( p \)-brane configurations in \((10 - n)\)-dimensional maximally supersymmetric supergravity coupled to Abelian gauge symmetries associated with the NS-NS and R-R bosonic fields of various ranks. In this way, the black objects can be constructed using the following brane configurations

- D0-branes, F-strings, D2-branes, D4-branes, NS5-branes, D6-branes.

As in the case of the Calabi–Yau manifold, the toroidal compactification involves a real Hodge diagram playing a primordial role in the elaboration of the type IIA superstring charge spectrum in 10 − \( n \) dimensions. Roughly speaking, \( T^n \) is a flat compact space which can be defined in different ways. One of them is to use the trivial fibration of \( n \) circles modeled by the following orbital relations

\[
x_i \equiv x_i + 1, \quad i = 1, \ldots, n. \tag{2.4}
\]

To build the real Hodge diagrams, it is convenient to introduce a binary number notation \( h^{e_1 \cdots e_n} \) playing a similar role as the Hodge numbers appearing in the cohomology of complex Calabi–Yau geometries. More precisely, this number is associated with the following differential form
where \( e_ℓ \) is a binary number taking either 0 or 1, and where \( \overline{e}_ℓ \) is its conjugate. It is obvious that \( \prod_{ℓ=1}^n (\overline{e}_ℓ + e_ℓ dx_ℓ) \) is a real differential form of degree \( k \) which can be written as follows:

\[
\prod_{ℓ=1}^n (\overline{e}_ℓ + e_ℓ dx_ℓ) = \prod_{ℓ=1}^n e_ℓ = k = \sum_{ℓ=1}^n e_ℓ.
\]  

By using the Poincaré duality, this \( k \)-form is dual to a \( k \)-cycle embedded in \( T^n \) on which type IIA branes could wrap to produce black objects in \( 10 - n \) dimensions. By inverting the order of the Hodge diagram corresponding to the Calabi–Yau manifolds, we can give a graphic representation of the cohomology space of \( T^n \) in terms of the real real Hodge diagrams. For simplicity, we illustrate the \( n = 2 \) and \( n = 3 \) cases:

\[
\begin{array}{c|c|c|c}
 n = 2 & h^{1,1} & h^{1,0} & h^{0,1} & h^{0,0} \\
 & 1 & 1 & 1 & 1 \\
\end{array}
\]

\[
\begin{array}{c|c|c|c|c|c}
 n = 3 & h^{1,1,1} & h^{1,0,1} & h^{0,1,1} & h^{0,0,1} \\
 & 1 & 1 & 1 & 1 \\
\end{array}
\]

It is noted that the general real Hodge diagram associated with \( T^n \) can be constructed by using the above-mentioned notation and terminology. It encodes all possible non-trivial cycles of \( T^n \) describing its geometric data including the size and the shape parameters.

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| \( n = 2 \) | \( h^{1,1} \) | \( h^{1,0} \) | \( h^{0,1} \) | \( h^{0,0} \) |
|-----------|------------|------------|------------|------------|
|           | 1          | 1          | 1          | 1          |

3. Qubit systems, Adinkras and real Hodge diagrams

In this section, we would like to elaborate a link between qubits, Adinkras and the real Hodge diagrams of \( T^n \). This correspondence will be explored to engineer graphically the extremal black brane geometries using qubit systems. It is recalled that the qubit is a primordial piece in quantum information theory which has been extensively investigated using different physical and mathematical approaches [34–36]. It is a two-configuration system which can be associated, for instance, with the electron in the hydrogen atom. The general state of a single qubit is usually given by the Dirac notation as follows

\[
|ψ⟩ = c_0 |0⟩ + c_1 |1⟩.
\]
where $c_i$ are complex coefficients satisfying the normalization condition

$$|c_0|^2 + |c_i|^2 = 1. \quad (3.2)$$

This equation can be interpreted geometrically in terms of the so-called Bloch sphere. The two quits are four-configuration systems. In this case, the general state takes the following form

$$|\psi\rangle = c_{00} |00\rangle + c_{10} |10\rangle + c_{01} |01\rangle + c_{11} |11\rangle, \quad (3.3)$$

where $c_{ij}$ are complex numbers satisfying the normalization condition

$$|c_{00}|^2 + |c_{10}|^2 + |c_{01}|^2 + |c_{11}|^2 = 1, \quad (3.4)$$

describing a three-dimensional complex projective space $\mathbb{C}P^3$ generalizing the Bloch sphere. This analysis can be extended to $n$ qubits associated with $2^n$ configuration states using the above binary notation.

It is observed that the qubit systems can be represented by diagrams having a strong resemblance with a particular class of Adinkras. Henceforth, we refer to them as bosonic graphs. The Adinkras have been introduced in the study of supersymmetric representation theory [32, 33, 37–41]. In the usual diagram representation, Adinkras are formed by nodes and lines. It has been shown that there are various classes which have been explored in the classification of supersymmetric theories. These diagrams contain bosonic and fermionic nodes like Dynkin diagrams of Lie superalgebras. A particular graph is called regular, formed by $2^n$ nodes connected with $n$ colored lines. In fact, this graph has been explored to represent $n$-qubits. More specifically, to each node, one associates a state of the qubit. To be precise, the correspondence can be formulated as follows

$$\text{node} \longrightarrow \text{state} \quad (3.5)$$

$$\text{number of colors} \longrightarrow \text{number of qubits}. \quad (3.6)$$

To illustrate, we present two bosonic graphs associated with $n = 2$ and $n = 3$. The first example concerns $n = 2$ qubits and it is given by

![Diagram](image-url)
The second example describes \( n = 3 \) qubits and it is represented by

![Diagram of a 3-qubit Adinkra graph]

An inspection shows that the real Hodge diagrams of \( T^n \) can be interpreted as a regular Andikra. To each node, we associate then a single cycle in \( T^n \). As we will see shortly in an example, this link is as follows:

\[
thone \rightarrow \ldots \rightarrow \text{node} = (e_1, \ldots, e_n).
\]  

(3.7)

The number of nodes in the \( k \)-level is exactly the combinatorial number

\[
\text{nbr}(k\text{-level nodes}) = C_n^k
\]  

(3.8)

indicating also the number of \( k \)-cycles in \( T^n \). The total number of cycles is identified with the total number of nodes. This number is given by

\[
\text{nbr (cycles)} = \sum_{k=0}^{n} C_n^k = 2^n.
\]  

(3.9)

On the basis of the above link, the cycles in \( T^n \) should be associated with the states defining the \( n \)-qubit systems. In this way, a quantum state is interpreted as the Poincaré dual of the real homology cycle on which branes can wrap to produce a black object in a \((10 - n)\)-dimensional type IIA superstring. We expect that the Adinkras should encode information on the black brane physics in higher-dimensional supersymmetric supergravity theories. This may offer a new take on the graphical representation of such a physics using techniques of graph theory. The following sections concern graphic representations of the extremal black branes using qubit and Adinkra notions.

### 4. Extremal black branes and Adinkra representation of qubit systems

In this section, we examine the correspondence between the extremal black \( p \)-branes in maximally supersymmetric supergravity and qubit systems using a graph theory based on bosonic Adinkras. To understand how such a surprising connection could be true, we first consider the case of the elliptic curve. Then, we give a general picture which may appear in lower-dimensional theories.

#### 4.1. The elliptic curve compactification

As mentioned, we start by discussing the eight-dimensional extremal black branes. In particular, we will give two explicit models corresponding to \( p = 0 \) and \( p = 4 \). These models can be obtained from the compactification of a type IIA superstring on \( \mathbb{T}^2 \), considered as a trivial fibration of two circles: \( S^1 \times S^1 \). The compactification of the massless bosonic ten-dimensional type IIA superstring fields
\[ \phi = N S, S N, S G, B R, A C, M N K \] 

This spectrum, which can be alternatively obtained from the compactification of M-theory on \( T^3 \) with the \( SL(R) \times SL(R) \) U-duality group [42], contains a graviton, three 2-form gauge fields, six vector gauge fields, one self-dual 3-form gauge field and seven scalar fields.

The associated scalar manifold reads as

\[ \frac{SL(3, R)}{SO(3)} \times \frac{SL(2, R)}{SO(2)}. \]  

Motivated by the study of the attractors on the K3 surface in a type IIA superstring, another factorization of such a moduli space has been given in [16]. In type IIA six dimensions obtained from the compactification of the K3 surface, the moduli space contains two factors associated with two possible brane charges given by black strings and holes. According to [16], the separation of the extremal black brane charges in eight dimensions provides a possible factorization of the above scalar manifold given by

\[ \frac{SO(2, 2)}{SO(2) \times SO(2)} \times \frac{SO(2, 1)}{SO(2)} \times SO(1, 1), \]  

associated with three possible black brane charges. In this eight-dimensional \( N = 2 \) supergravity, the extremal near-horizon geometries of the black \( p \)-branes take the following form [18]:

\[ \text{AdS}_{p+2} \times S^{6-p}. \]  

Indeed, they are classified into three categories:

1. \( p = 0 \) is associated with \( \text{AdS}_2 \times S^6 \) which describes the near-horizon geometry of eight-dimensional electric charged black holes. Their dual magnetic objects are black 4-branes having the \( \text{AdS}_4 \times S^2 \) near-horizon geometry. They are charged under six gauge field strengths \( F_i = dA_i (i = 1, \ldots, 6) \) corresponding to the following gauge symmetry:

\[ G = U(1)^2 \times U(1)^2. \]  

2. \( p = 1 \) corresponds to the extremal black strings with the \( \text{AdS}_3 \times S^5 \) near-horizon geometry. The magnetic dual horizon geometry is \( \text{AdS}_3 \times S^3 \) associated with the black 3-branes.

3. \( p = 2 \) describes a dyonic black 2-brane having \( \text{AdS}_4 \times S^4 \) near-horizon geometry. This black object can carry both electric and magnetic charges.

To see how the interplay works, we discuss the case of the black hole solution where the Kaluza–Klein states associated with the symmetry \( U(1)^{10}_{\text{bh}} \) are turned off. More precisely, we
construct a black hole from a system of D0-branes, F-strings, D2-branes wrapping appropriate cycles of the elliptic curve $T^2$. In this way, the dictionary between these cycles, four node Andinkra and two qubits may offer a new way to investigate eight-dimensional black holes. Indeed, the $SO(2, 2)$ appearing in (4.3) should correspond to a black hole with four electric charges which are exactly the entries appearing in the real Hodge diagram of the elliptic curve $T^2$. Its brane representation can be organized in the following brane diagram:

$$\begin{array}{c}
D2 \\
1 \\
1 = \text{F-string} = \text{F-string}
\end{array}$$

Shearing similarities with the real Hodge diagram of $T^2$. Using equation (2.3) for $n = 2$, we can give a graphic representation for the magnetic black 4-brane. It is given by the following brane diagram:

$$\begin{array}{c}
D6 \\
1 \\
1 = \text{NS5-brane} = \text{NS5-brane}
\end{array}$$

(4.7)

Using the previous notations, we propose the correspondence given in Table 1.

The same analysis can be done for $p = 1$ and $p = 2$ but with some differences required by the $T$-duality between D2-branes and D3-branes which has been extensively studied in connection with string theory compactification [43, 44]. The corresponding brane diagrams should involve D3-branes.

### 4.2. Lower-dimensional cases

Here, we would like to extend the above results to higher orders of Adinkras. Particularity, we can do something similar for the more general case associated with ($n > 2$). The corresponding compactification has been extensively studied to generate lower-dimensional type IIA superstring. In this way, the Adinkras will be based on the polyvalent geometry in which the nodes are connected with more than two other ones as shown in the previous section associated with the trivalent geometry. The latter generalizes the bivalent one appearing in the case of the elliptic curve $T^2$. Based on this observation, we believe that the physics of the extremal black branes in a type IIA superstring on $T^n$ shares similarities with $n$-qubits. Before giving a general picture, we illustrate the case of $T^3$. The corresponding compactification

| $T^3$ | Adinkra | Qubit system | Black hole system |
|-------|----------|-------------|------------------|
| $dx_1$ | (000) | $|000\rangle$ | D0-brane |
| $dx_2$ | (001) | $|001\rangle$ | F-string |
| $dx_3$ | (010) | $|010\rangle$ | F-string |
| $dx_4$ | (011) | $|011\rangle$ | D2-brane |

| $dx_1dx_2$ | (110) | $|110\rangle$ | D2-brane |
| $dx_2dx_3$ | (101) | $|101\rangle$ | D2-brane |
| $dx_3dx_4$ | (111) | $|111\rangle$ | D3-brane (T-dual of D2-brane) |

Table 2. The correspondence between eight-dimensional black hole, Adinkra and qubit systems.
produces seven-dimensional extremal black branes. Their extremal near-horizon geometries are given by

$$\times + - S_{A dS}.$$  (4.8)

These asymptotically flat, static and spherical solutions are classified by the following fundamental solutions

1. seven-dimensional black holes dual to black 3-branes,
2. seven-dimensional black strings dual to black 2-branes.

The seven-dimensional correspondence, in the presence of the D3-brane, is illustrated in table 2.

The same analysis can be done for the extremal black hole in a type IIA superstring on $T^n$. In this case, the near-horizon geometry of the extremal black holes reduces to

$$\times - S_{A dS}.$$  (4.9)

Using the fact that the real Hodge diagram of $T^n$ can encode the information on the black brane charges and $n$-qubit systems, we can give a possible brane configuration producing extremal black branes in $10 - n$ dimensions, obtained from the compactification of a type IIA superstring on $T^n$. A similar discussion can be held for the extremal black branes. For illustration, the general picture for black holes is summarized in table 3.

### Table 3

| $T^n$ | Adinkra | Qubit system | Black hole system |
|-------|---------|--------------|------------------|
| $\prod_{\ell=1}^n (\bar{T}_\ell + \epsilon_{\ell}(d\tau_\ell))$ | $(e_{\ell_1}, \ldots, e_{\ell_n})$ | $|e_{\ell_1}, \ldots, e_{\ell_n}|$ | $k$-brane |

5. **Odd and even geometries on $T^{n+n}$ and superqubits**

Having discussed the bosonic qubits, it is very natural to consider superqubit systems and supersymmetric Adinkras in the above correspondence. In fact, superqubits have been investigated in connection with Lie supersymmetries [22, 23, 29]. Roughly speaking, a superqubit can take three values: 0 or 1 or $\bullet$. In fact, 0 and 1 are bosonic while $\bullet$ is fermionic.

In this section, we investigate the superqubits in the framework of the toroidal compactification with bosonic and fermionic coordinates. In particular, we explore the superform cohomology of a particular real supermanifold $T^{n+n}$ equipped with $n$ bosonic coordinates $x_{\ell}$ and $m$ fermionic coordinates $\theta_{\ell}$. It is recalled that the supermanifolds have been extensively studied in string theory and related topics including mirror geometries [45–47].

In what follows, we consider a special geometry where $n = m$ corresponding to the real supermanifold $T^{n+n}$ and make contact with superqubits. More precisely, we would like to elaborate the real Hodge diagram of $T^{n+n}$. Following some assumptions specified below, we show that a part of this extended real Hodge diagram can be explored to represent $n$ superqubits using the supersymmetric Adinkras involving bosonic and fermionic nodes. In fact, these supersymmetric real Hodge diagrams can be built using the extension of the above binary notation. For simplicity and keeping contact with the previous section, we use the notation $h^{e_{\ell_1}, \ldots, e_{\ell_n}|a_{p_1}, \ldots, a_{p_m}}$ associated with the following differential superform:
Its degree is given by

\[ d = \sum_{\ell=1}^{n} e_{\ell} + \sum_{a=1}^{n} e_{\alpha}. \] (5.2)

To give a differential geometry representation of \( n \) superqubits, we postulate the following constraints on the superform degrees:

- the lowest degree is zero associated with a bosonic state,
- the highest degree is \( n \) associated either with a bosonic or a fermionic state according to the parity of \( n \).

More precisely, we will be interested in the following superforms on the real supermanifold \( T^{n|n} \):

\[ \prod_{\ell=1}^{n} (\bar{\tau}_\ell + e_{\ell}dx_\ell) \prod_{a=1}^{n} (\bar{\tau}_a + e_{\alpha}d\theta_\alpha). \] (5.3)

Based on the above assumption, combinatorial calculation reveals that the total number of the associated cycles is given by

\[ \sum_{\ell=0}^{n} \sum_{k=0}^{n-\ell} C_{\ell}^{n} C_{k}^{n-\ell}. \] (5.4)

It is worth noting that if we put \( \ell = 0 \), we recover the bosonic case as discussed in section 2.

Using the expression of the Taylor series \((1 + x)^n\), we can show that this number is exactly the state number of \( n \) superqubits

\[ \sum_{\ell=0}^{n} \sum_{k=0}^{n-\ell} C_{\ell}^{n} C_{k}^{n-\ell} = 3^n. \] (5.5)

This includes the bosonic and the fermionic states. To get the state number of each sector, we use the splitting

\[ 3^n = \frac{3^n - 1}{2} + \frac{3^n + 1}{2}. \] (5.6)

The calculation shows the relations

\[ \sum_{\ell=\text{even}}^{n} C_{\ell}^{n} \sum_{k=0}^{n-\ell} C_{k}^{n-\ell} = \frac{3^n + 1}{2}, \] (5.7)

\[ \sum_{\ell=\text{odd}}^{n} C_{\ell}^{n} \sum_{k=0}^{n-\ell} C_{k}^{n-\ell} = \frac{3^n - 1}{2}. \] (5.8)

The number of the bosonic and the fermionic states can be identified by exploring the formal supersymmetry structure
where $B$ and $F$ denote the bosonic and the fermionic generators respectively. An inspection shows that the number of the bosonic states is given by

\[
\text{Number of bosonic states} = \frac{3^n + 1}{2}.
\]  \hspace{1cm} (5.9)

Similarly, the number of fermionic states reads as

\[
\text{Number of fermionic states} = \frac{3^n - 1}{2}.
\]  \hspace{1cm} (5.10)

In connection with graph theory, the corresponding Adinkras should be formed by $\frac{3^n + 1}{2}$ bosonic nodes and $\frac{3^n - 1}{2}$ fermionic nodes associated with even and odd geometries on $T^{n\text{lin}}$ respectively. In this way, the link can be put as follows:

\begin{align*}
\text{Bosonic nodes} & \longrightarrow \text{Bosonic forms} \\
\text{Fermionic nodes} & \longrightarrow \text{Fermionic forms}.
\end{align*}

To illustrate this analysis, we examine the real supermanifold $T^{2|2}$ associated with two superqubits. In particular, it is obvious to see that the bosonic states correspond to the following differential form

\[ 1, \ dx_1, \ dx_2, \ dx_1dx_2, \ d\theta_1d\theta_2. \]  \hspace{1cm} (5.13)

The number of the corresponding bosonic states is $\frac{3^2 + 1}{2} = 5$. Similarly, we can get the fermionic states associated with

\[ d\theta_1, \ d\theta_2, \ dx_1d\theta_2, \ d\theta_1dx_2. \]  \hspace{1cm} (5.14)

The number is $\frac{3^2 - 1}{2} = 4$. Motivated by the existence of the supermanifolds and brane geometries, the corresponding Adinkras and brane charges are represented in table 4. The table includes superbranes living in supermanifolds.

We can construct many additional examples by considering D3-branes and their supersymmetric versions.
6. Conclusion and discussions

In this paper, we have reconsidered the study of extremal black branes in the framework of quantum information using graph theory based on the so-called Adinkras. In particular, we have elaborated a one-to-one correspondence between qubit systems, Adinkras and extremal black branes embedded in maximally supergravity obtained from a low-energy limit of a type IIA superstring compactified on $T^n$. It has been observed that the physical states of $n$-qubit systems can be represented graphically using Adinkras. These graphs are based on polyvalent geometries appearing in the case of Dynkin diagrams of Lie algebras. In this representation, the $n$-qubits are associated with the $n$-valent geometry in which each node is connected with $n$ colored lines. Based on this observation, the $n$-qubit is represented by a graph of $2^n$ bosonic nodes connected by the colored $n$ polyvalent geometry. In this graphic representation, each node encodes information on the qubit quantum states and the charges of the extremal black branes embedded in a type IIA superstring on $T^n$. Then, we have proposed a possible generalization to $n$ superqubits. More precisely, we have shown that these systems can be associated with odd and even geometries on the real supermanifold $T^{n|n}$. More precisely, the number of the corresponding bosonic and fermionic states are obtained using a combinatorial calculation.

Our paper comes up with many open questions related to quantum information theory. In fact, many concepts have been developed in such a theory including gates, circuits and entanglement states. It should of be of interest to investigate such concepts using graph theory and the physics dealing with black objects. Moreover, there have been many works trying to connect black holes with quantum information [48]. It should be interesting to make contact with such activities. This will be addressed elsewhere.

On other hand, the analysis presented here might be extended in the case of the Calabi–Yau manifolds. To speculate on this extension, let us consider the compactification of a type IIA superstring on the K3 surface [49, 50]. It is recalled that the corresponding complex Hodge diagram reads as

\[
\begin{array}{cccc}
h^{0,0} & h^{1,0} & h^{0,1} & 1 \\
h^{2,0} & h^{1,1} & h^{0,2} = 1 & 0 \\
h^{2,1} & h^{1,2} & 0 & 0 \\
h^{2,2} & & 1 &\
\end{array}
\]

It has been shown that the moduli space of such a compactification is given by the factorization

\[
\frac{SO(4, 20)}{SO(4) \times SO(20)} \times SO(1, 1). \tag{6.1}
\]

It is observed that this factorization is linked to two possible black object solutions in six dimensions which are given by

- $p = 0$: corresponding to black holes with a near-horizon geometry: $AdS_2 \times S^4$
- $p = 1$: associated with a black string having a near-horizon geometry: $AdS_3 \times S^3$.

In fact, the factor $\frac{SO(4, 20)}{SO(4) \times SO(20)}$ is associated with 24 black hole charges identified with entries appearing in the above complex Hodge diagram. The corresponding brane realization can be given by
At first sight, the brane representation could be related to the qudit systems [51]. However, we believe that this connection deserves a deeper study.

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