CONSTRUCTING HYPERBOLIC MANIFOLDS

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Abstract. The Coxeter simplex with symbol \( \square\square\square\square\) is a compact hyperbolic 4-simplex and the related Coxeter group \( \Gamma \) is a discrete subgroup of \( \text{Isom}(\mathbb{H}^4) \). The Coxeter simplex with symbol \( \square\square\) is a spherical 3-simplex, and the related Coxeter group \( G \) is the group of symmetries of the regular 120-cell. Using the geometry of the regular 120-cell, Davis constructed an epimorphism \( \Gamma \rightarrow G \) whose kernel \( K \) was torsion-free, thus obtaining a small volume compact hyperbolic 4-manifold \( \mathbb{H}^4/K \).

In this paper we show how to obtain representations \( \Gamma \rightarrow G \) of Coxeter groups \( \Gamma \) acting on \( \mathbb{H}^n \) to certain classical groups \( G \). We determine when the kernel \( K \) of such a homomorphism is torsion-free and thus \( \mathbb{H}^n/K \) is a hyperbolic \( n \)-manifold. As an example, this is applied to the two groups described above, with \( G \) suitably interpreted as a classical group. Using this, further information on the quotient manifold is obtained.

1. Introduction

Let \( M^n \) be an \( n \)-dimensional hyperbolic manifold, that is, an \( n \)-dimensional Riemannian manifold of constant sectional curvature \(-1\). Thus \( M^n \) is isometric to a quotient space \( \mathbb{H}^n/K \) of \( \mathbb{H}^n \) by the free action of a discrete group \( K \cong \pi_1(M^n) \) of hyperbolic isometries.

This paper presents a method of constructing such groups \( K \) as the kernels of representations \( \Gamma \rightarrow G \) of hyperbolic Coxeter groups \( \Gamma \) into finite classical groups. The homomorphisms arise by first representing \( \Gamma \) as a subgroup of the orthogonal group of a quadratic space over a number field \( k \) which preserves a lattice. Then reducing modulo a prime ideal in the ring of integers in the number field yields a representation into a finite classical group given as an orthogonal group of a quadratic space over a finite field.

Such kernels \( K \) will act freely if and only if they are torsion free. The volume of the resulting manifold \( M^n = \mathbb{H}^n/K \) will be \( N \times \text{Vol}(P) \), where \( N \) is the order of the image in the finite classical group and \( P \) is the polyhedron defining the Coxeter group \( \Gamma \). Starting from a suitable Coxeter group \( \Gamma \), the method yields infinitely many examples of manifolds. There has been some interest lately in constructing small volume examples when \( n \geq 4 \) (see [10, 11]). In dimension 4, the compact Davis manifold \( \mathbb{H}^4 \) is constructed by a geometric technique using the existence of a regular compact 120-cell in \( \mathbb{H}^4 \), which has volume \( 26 \times 4\pi^2/3 \). As an application of our method, we construct a compact 4-manifold \( M_0 \) of the same volume which turns out to be isometric to the Davis manifold. With the help of computational techniques, our method gives additional information, producing a presentation for the fundamental group from which we obtain that \( H_1(M_0) = \mathbb{Z}^24 \).

2. Finite representations of hyperbolic Coxeter groups

Consider an \((n+1)\)-dimensional real space \( V \) equipped with a quadratic form \( q \) of signature \((n,1)\). Thus with respect to an orthogonal basis,

\[
q(x) = -x_{n+1}^2 + \sum_{i=1}^{n} x_i^2.
\]  

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The quadratic form $q$ determines a symmetric bilinear form
\[(2) \quad B(x, y) := q(x + y) - q(x) - q(y),\]
with $B(x, x) = 2q(x)$. The Lobachevski (or hyperboloid) model of $\mathbb{H}^n$ is the positive sheet of the sphere of unit imaginary radius in $V$ (see [12]). Equivalently, we take the projection of the open cone $C = \{x \in V \mid q(x) < 0 \text{ and } x_{n+1} > 0\}$ with the induced form.

The isometries of the model are the positive $(n+1) \times (n+1)$ Lorentz matrices, that is, the orthogonal maps of $V$, with respect to $q$, that map $C$ to itself.

A hyperplane in $\mathbb{H}^n$ is the image of the intersection with $C$ of a Euclidean hyperplane in $V$. Each hyperplane is the projective image of the orthogonal complement $e^+ = \{x \in C \mid B(x, e) = 0\}$ of a vector $e$ with $q(e) > 0$. Such a vector is said to be space-like, and it is convenient to normalise so that $q(e) = 1$. The map $r_e : V \to V$ defined by
\[r_e(x) = x - B(x, e)e,\]
when restricted to $\mathbb{H}^n$, is the reflection in the hyperplane corresponding to $e$.

A polyhedron $P$ in $\mathbb{H}^n$ is the intersection of a finite collection of half-spaces, that is, the image of
\[\Lambda = \{x \in C \mid B(x, e_i) \leq 0, i = 1, 2, \ldots, m\},\]
for some space-like vectors $e_i$. The intersections of the hyperplanes $e_i^+$ with $P$ are the faces of the polyhedron. The dihedral angle $\theta_{ij}$ subtended by two intersecting faces of $P$ is determined by $-2\cos \theta_{ij} = B(e_i, e_j)$. On the other hand, non-intersecting faces of $P$ have a common perpendicular geodesic of length $\eta_{ij}$, where $-2\cosh \eta_{ij} = B(e_i, e_j)$.

All of this information is encoded in the Gram matrix $G(P)$ of $P$, an $m \times m$ matrix with $(i, j)$-th entry $a_{ij} = B(e_i, e_j)$. Let $\Gamma$ be the group generated by the reflections $r_i := r_{e_i}$ in the faces of $P$ so that $\Gamma$ is a subgroup of the isometry group $\text{Isom}(\mathbb{H}^n)$. Moreover, $\Gamma$ is discrete exactly when all the dihedral angles $\theta_{ij}$ of $P$ are integer multiples $\pi/n_{ij}$ of $\pi$ [12], and in this case, $\Gamma$ is a hyperbolic Coxeter group. The polyhedron $P$ is depicted by means of its Coxeter symbol, with a node for each face, two nodes joined by $n-2$ edges when the corresponding faces subtend a dihedral angle of $\pi/n$ and other pairs of nodes joined by an edge labelled with the geodesic length between the faces. We use the Coxeter symbol to denote both $P$ and the group $\Gamma$ arising from it.

Using the Lobachevski model, such a hyperbolic Coxeter group $\Gamma$ is a subgroup of $O(V, q)$. Using this, [13] gave necessary and sufficient conditions for such a group to be arithmetic. We adopt Vinberg’s method to conveniently describe the groups $\Gamma$, although they need not be arithmetic.

We first give this method and some general notation which we will use throughout. Let $M$ be a finite-dimensional space over a field $F$. Equipped with a quadratic form $f$ which induces a symmetric bilinear form (as for instance in (1) and (2)), $M$ is a quadratic space over $F$. The group $O(M, f)$ of orthogonal maps consists of linear transformations $\sigma : M \to M$ such that $f(\sigma(m)) = f(m)$ for all $m \in M$.

Consider the Gram matrix $G(P) = [a_{ij}]$, and for any $\{i_1, i_2, \ldots, i_r\} \subseteq \{1, 2, \ldots, m\}$, define
\[b_{i_1 i_2 \cdots i_r} = a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_r i_1},\]
and let $k = \mathbb{Q}(\{b_{i_1 i_2 \cdots i_r}\})$. Take the space-like vectors in $V$ defined by
\[v_{i_1 i_2 \cdots i_r} = a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_r i_1} e_{i_r}.\]
Let $M$ be the $k$-subspace of $V$ spanned by the $v_{i_1 i_2 \cdots i_r}$. A simple calculation gives
\[(3) \quad r_i(v_{i_1 i_2 \cdots i_r}) = v_{i_1 i_2 \cdots i_r} - v_{i_1 i_2 \cdots i_r},\]
and
\[ B(v_{i_1i_2\cdots i_s}, v_{j_1j_2\cdots j_r}) = b_{i_1\cdots i_s, j_1\cdots j_r}. \]
Thus, $M$ is a quadratic space over $k$ under the restriction of $q$, and from (3) and (4)
\[ B(r_i(v_{i_1i_2\cdots i_s}), r_j(v_{j_1j_2\cdots j_r})) = B(v_{i_1i_2\cdots i_s}, v_{j_1j_2\cdots j_r}). \]
It follows that $\Gamma \to O(M, q)$.

**Lemma 1.** $M$ is an $(n + 1)$-dimensional space over $k = \mathbb{Q}((b_{i_1i_2\cdots i_s})).$

**Proof.** If $P$ has finite volume then the vectors $e_i$ span $V$ and the Gram matrix is indecomposable \cite{K}. So for each $i$, there is a $j \neq i$ such that $a_{ij} \neq 0$. Successively choose indices $1 = i_0, i_1, \ldots$ such that the $i_k$-th row contains a non-zero entry in the $(i_{k+1})$-st column, for $k \geq 1$. We can ensure that the $i_k$ are distinct. For, if the only non-zero entries of the $k$-th row are those in the columns with indices $1, i_1, \ldots, i_k$, throw away $i_k$ and go back to the $(i_{k-1})$-st row to rechoose a different column. Eventually, by discarding and moving backwards, we must be able to rechoose, in the $i_j$-th row, an index different from all the $i_{j+1}, \ldots, i_k$ discarded. Otherwise, $\{e_1, \ldots, e_k\}$ are orthogonal to the other basis vectors, contradicting indecomposability. In this way we must arrive at a sequence $1 = i_0, i_1, \ldots, i_m - 1$ of length $m$. Hence, for any $i$, $e_i = e_{i_k}$ for some $i_k$, and $v_{i_1\cdots i_k} = a_{i_1i_2}\cdots a_{i_{k-1}i_k}e_{i_k}$ with coefficient non-zero. Thus, the vectors $v_{i_1i_2\cdots i_k}$ span $V$ over $\mathbb{R}$ and hence $M$ is $(n + 1)$-dimensional over $\mathbb{R}$. Now, if $\{v_1, \ldots, v_{n+1}\}$ is an $\mathbb{R}$-basis for $M$ and $v = \sum x_i v_i \in M$, then the system of equations $B(v, v_j) = \sum x_i B(v_i, v_j)$ has a unique solution, since the matrix with $(i, j)$-th entry $B(v_i, v_j)$ is invertible. But the solutions $x_i \in k$, since $B(u, v) \in k$ for all $u, v \in M$. Thus $\{v_1, \ldots, v_{n+1}\}$ is a $k$-basis for $M$. \hfill \Box

We make a number of simplifying assumptions which hold for many examples. Suppose that $k$ is a number field and let $\mathcal{O}$ denote the ring of integers in $k$. Suppose furthermore that all $b_{i_1\cdots i_s}$ are $\mathcal{O}$-lattices in $M$ spanned by the elements $v_{i_1\cdots i_s}$, and assume that $N$ is a free $\mathcal{O}$-lattice. This will hold, in particular, when $\mathcal{O}$ is a principal ideal domain.

By (3) above, $N$ is invariant under $\Gamma$ so that
\[ \Gamma \subset O(N, q) := \{ \sigma \in O(M, q) \mid \sigma(N) = N \}. \]
With the restriction of $q$, $N$ is a quadratic module over $\mathcal{O}$. If $\mathfrak{P}$ is any prime ideal in $\mathcal{O}$, let $\bar{k} = \mathcal{O}/\mathfrak{P}$. Reducing modulo $\mathfrak{P}$, we obtain a quadratic space $\bar{N}$ over $\bar{k}$ with respect to $\bar{q}$ and an induced map $\Gamma \to O(\bar{N}, \bar{q})$.

The groups $O(\bar{N}, \bar{q})$ are essentially the finite classical groups referred to earlier. However, the quadratic space $(\bar{N}, \bar{q})$ may not be a regular quadratic space, in which case we must factor out the radical to obtain a regular quadratic space (see Section 4 below). This will occur if the discriminant of $\bar{N}$ is zero. Since the discriminant of $\bar{N}$ is the image in $\bar{k}$ of the discriminant of $N$ this will only occur for finitely many prime ideals $\mathfrak{P}$.

We now attend to the matter of when the kernel of a representation of $\Gamma$ is torsion-free. In certain circumstances, this can be decided arithmetically using a small variation of a result of Minkowski (see for example \cite{K} page 176).

**Lemma 2.** Let $k$ be a quadratic number field, whose ring of integers $\mathcal{O}$ is a principal ideal domain. Let $p$ be a rational prime. Let $\alpha \in \mathcal{O}$ be such that $\alpha \not\mid 2$, and, if $3$ is ramified in the extension $k \mid \mathbb{Q}$, then $\alpha \not\mid 3$. If $A \in GL(n, \mathcal{O})$ is such that $A^p = I$ and $A \equiv I(\text{mod } \alpha)$, then $A = I$.

**Proof.** Suppose $A \not\equiv I$ so that $A = I + \alpha E$ where $E \in M_n(\mathcal{O})$ and we can take the g.c.d. of the entries of $E$ to be $1$. From $(I + \alpha E)^p = I$ we have
\[ pE + \frac{p(p-1)}{2}\alpha E^2 \equiv 0 \pmod{\alpha^2}. \]
Lemma 3. Suppose \( \Gamma \) is a hyperbolic Coxeter group generated by reflections in the faces of some polyhedron \( P \) in \( n \)-dimensional Euclidean space \( \mathbb{E}^n \) or \( n \)-dimensional hyperbolic space \( \mathbb{H}^n \). If \( v \in \Gamma \) is a torsion element, then for some vertex \( v \in P \), \( \gamma \) is \( \Gamma \)-conjugate to an element of \( \Gamma_v \).

Notice that in the situation described in the lemma, a Coxeter symbol for \( \Gamma_v \) is obtained in the following way: take the sub-symbol of \( \Gamma \) with nodes (and their mutually incident edges) corresponding to faces of \( P \) containing \( v \). For brevity’s sake, when we say torsion element from now on, we will mean non-trivial torsion element.

Corollary 1. If \( \alpha \not| 2 \) and if \( 3 \) is ramified in \( k | \mathbb{Q} \), \( \alpha \not| 3 \), then the kernel of the mapping on \( GL(n, \mathcal{O}) \) induced by reduction (mod \( \alpha \)) is torsion-free.

More generally, a geometrical argument allows us to determine when \( \alpha \) any representation has torsion-free kernel, albeit by expending a little more effort. Suppose \( v \in P \) is a vertex of the polyhedron \( P \), and \( \Gamma_v \) is the stabiliser in \( \Gamma \) of \( v \). For \( P \) of finite volume, \( v \) is either in \( \mathbb{H}^n \) or on the boundary, and \( v \) is called finite or ideal respectively. We have the following “folk-lore” result,

Lemma 3. Suppose \( \Gamma \) is a discrete group generated by reflections in the faces of some polyhedron \( P \) in \( n \)-dimensional Euclidean space \( \mathbb{E}^n \) or \( n \)-dimensional hyperbolic space \( \mathbb{H}^n \). If \( \gamma \in \Gamma \) is a torsion element, then for some vertex \( v \in P \), \( \gamma \) is \( \Gamma \)-conjugate to an element of \( \Gamma_v \).

Thus \( pE \equiv 0 \pmod{\alpha} \) and so \( p \equiv 0 \pmod{\alpha} \). Since \( \alpha \not| 2 \), \( p \) is odd. Suppose \( p \) is unramified in the extension \( k | \mathbb{Q} \). Then either \( p = \alpha \) or \( p = \alpha \alpha^\prime \) with \( \alpha^\prime \in \mathcal{O} \) and \( (\alpha, \alpha^\prime) = 1 \). So \( pE \equiv 0 \pmod{\alpha^2} \) so \( E \equiv 0 \pmod{\alpha} \). This is a contradiction.

Now suppose that \( p \) is ramified. Then \( p = u\alpha^2 \) where \( u \in \mathcal{O}^\ast \) and by assumption \( p \neq 3 \). Expanding as above, but to three terms, gives

\[
u \alpha^2 E + \frac{p-1}{2} \alpha E^2 + \frac{(p-1)(p-2)}{6} \alpha^2 E^3 \equiv 0 \pmod{\alpha^3}.
\]

This yields the contradiction \( E \equiv 0 \pmod{\alpha} \).

Corollary 2. The kernel of a representation \( \Gamma \to G \) is torsion-free exactly when every torsion element of every vertex stabiliser \( \Gamma_v \) has the same order as its image in \( G \).

At this point the situation bifurcates into two cases: if \( v \in P \) is a finite vertex, then \( \Gamma_v \) is isomorphic to a discrete group acting on the \((n-1)\)-sphere \( S^{n-1} \) centered on \( v \), hence is finite. Thus, the conditions of the corollary are satisfied exactly when \( \Gamma_v \) and its image in \( G \) have the same order.

If \( v \) is ideal, then consider a horosphere \( \Sigma \) based at \( v \), and restrict the action of \( \Gamma_v \) to \( \Sigma \). Then \( \Sigma \) is isometric to an \((n-1)\)-dimensional Euclidean space \( \mathbb{E}^{n-1} \), and \( \Gamma_v \) acts on it discretely with fundamental region \( P^\prime \), the intersection with \( \Sigma \) of \( P \). Any torsion element of \( \Gamma_v \) is then \( \Gamma_v \)-conjugate by Lemma 3 to the stabiliser in \( \Gamma_v \) of a vertex \( v^\prime \) of \( P^\prime \). Write \( \Gamma_{v,v^\prime} \) for this stabiliser, and observe that it is isomorphic to a discrete group acting on the \((n-2)\)-sphere \( S^{n-2} \) in \( \mathbb{E}^{n-1} \), centered on \( v^\prime \), and hence is also finite. The conditions of the corollary are satisfied exactly when for each \( v^\prime \in P^\prime \), the group \( \Gamma_{v,v^\prime} \) and its image in \( G \) have the same order.

Summarising,

Proposition 1. Suppose \( \Gamma \) is a hyperbolic Coxeter group generated by reflections in the faces of a polyhedron \( P \) as above. For each finite vertex \( v \) of \( P \), take the stabiliser \( \Gamma_v \). For each ideal vertex, take the stabilisers \( \Gamma_{v,v^\prime} \) for each vertex \( v^\prime \) of the Euclidean polyhedron \( P^\prime \). Then kernel(\( \Gamma \to G \)) is torsion-free if and only if each such \( \Gamma_v \) and \( \Gamma_{v,v^\prime} \) has the same order as its image in \( G \).

It is an elementary process to verify the conditions of the proposition. For, each vertex stabiliser is a finite spherical reflection group of some lower dimension, hence from the well-known list (see [3], Section 2.11 for their orders). To find the orders of their images in \( G \), the computational algebra package Magma is enlisted.
3. Polyhedra in $\mathbb{H}^n$

Let $P$ be a polyhedron in $\mathbb{H}^n$, thus the image of 

$$\Lambda = \{x \in C \mid B(x, e_i) \leq 0, i = 1, 2, \ldots, m\},$$

for some space-like vectors $e_i$. On occasion, a connected union of several copies of $P$ will yield another polyhedron of interest. In particular, we may want to glue copies of $P$ onto its faces using some of the reflections $r_i$ as gluing maps.

**Lemma 4.** If $r_i$ is a reflection in a face of $P$, then 

$$\Lambda \cup r_i(\Lambda) = \Lambda' := \{x \in C \mid B(x, e_i) \text{ and } B(x, r_i(e_j)) \leq 0, \text{ for all } j \neq i\}.$$ 

**Proof.** If $x \in \Lambda'$, then either $B(x, e_i) \leq 0$, in which case $x \in \Lambda$, or $B(x, e_i) > 0$, in which case $B(x, r_i(e_i)) \leq 0$, hence $x \in r_i(\Lambda)$. Conversely, if $x \in \Lambda$, then $B(x, e_i) \leq 0$ for all $j$. If $j \neq i$, then 

$$B(x, r_i(e_j)) = B(x, e_j) - B(x, e_i)B(e_i, e_j) \leq 0,$$

since all three terms are $\leq 0$. A similar argument deals with the $x \in r_i(\Lambda)$. \qed

We illustrate the lemma by considering the situation in four dimensions. In particular, if $P$ is a compact simplex it has Coxeter symbol one of the five depicted in Figure 1 (see [5], Section 6.9). In fact, and this explains the idiosyncratic numbering, $\text{Vol}(\Delta_i) < \text{Vol}(\Delta_j)$ if and only if $i < j$ (see [5]). Suppose the nodes of $\Delta_2$, read from left to right, correspond to hyperplanes $e_i^\perp$ for $i = 1, \ldots, 5$. If $r_5 = r_{e_5}$, we have 

$$\Delta_2 \cup r_5(\Delta_2) = \{x \in C \mid B(x, e_i) \leq 0, i = 1, \ldots, 4, \text{ and } B(x, r_5(e_4)) \leq 0\},$$

since $r_5(e_i) = e_i$ for $i = 1, 2, 3$. Now, $B(e_3, r_5(e_4)) = -2 \cos \pi/3$ and $B(e_4, r_5(e_4)) = -2 \cos \pi/2$, so $\Delta_2 \cup r_5(\Delta_2)$ is a simplex with Coxeter symbol $\Delta_4$. Thus, if $\Gamma_i$ is the group generated by the reflections in the faces of $\Delta_i$, we have that $\Gamma_4$ has index two in $\Gamma_2$. By comparing the volumes of the simplices using the results of [5], the only other possible inclusions are $\Gamma_4$ and $\Gamma_3$ as subgroups of indices 17 and 26 respectively in $\Gamma_1$. But a low index subgroups procedure in MAGMA shows that $\Gamma_1$ has no subgroups of these indices. Thus Figure 1 is a complete picture of the possible inclusions.

4. An example

In this section, we apply our method in dimension 4 starting with the Coxeter simplex $\Delta_3$ and related group $\Gamma_3$ described above. If $P$ is a finite volume Coxeter polyhedron in $\mathbb{H}^4$, then $\text{vol}(P) = \chi(P)4\pi^2/3$ where $\chi(P)$ is the Euler characteristic of $P$ (see [5]), which coincides with the Euler characteristic of the associated group. This is readily computed from the Coxeter symbol [1], page 250], [2]. For $\Delta_3$, the Euler characteristic is 26/14400. The vertex stabilisers are $\langle \circ \rangle \times \langle \circ \rangle \times \langle \circ \rangle$, $\mathbb{Z}_2 \times \langle \circ \rangle \times \langle \circ \rangle$, $\langle \circ \rangle \times \langle \circ \rangle \times \langle \circ \rangle$, $\langle \circ \rangle \times \langle \circ \rangle \times \langle \circ \rangle \times \mathbb{Z}_2$ and $\langle \circ \rangle \times \langle \circ \rangle \times \langle \circ \rangle$, having orders 14400, 240, 100, 240 and 14400 (see [5], Section 2.11). Thus the minimum index any torsion free
subgroup of $\Gamma_3$ can have is 14400, and we show that there is a normal torsion free subgroup of precisely this index. The corresponding manifold then has Euler characteristic 26, making it the same volume as the Davis manifold $[\mathbb{3}]$. Indeed, it has been shown in $[\mathbb{3}]$ that $\Gamma_3$ has a unique torsion-free normal subgroup of index 14400. It follows that this manifold is the Davis manifold.

We use the same letters $\{e_1, e_2, \ldots, e_5\}$ for the basis of $\tilde{N}$. The images of the generating reflections of $\Gamma_3$ are then $5 \times 5$ matrices with entries in $\mathbb{F}_5$. The computational system MAGMA then shows that the group they generate has order 14400 so that the kernel has index 14400 in $\Gamma_3$ as required.

The index 14400 is too large to allow MAGMA to implement the Reidemeister-Schreier process to obtain a presentation for $K$. However, closer examination of the image group allows this process to be implemented by splitting into two steps. We will deal with this now.

The bilinear form $\tilde{B}$ on $\tilde{N}$ is degenerate and there is a one-dimensional radical $\tilde{N}^\perp$ spanned by $v_0 = e_1 - e_2 + e_4 - e_5$. Thus $\tilde{N} = W \oplus \tilde{N}^\perp$. If $w \in W$ and $\sigma \in O(\tilde{N}, \tilde{q})$, then $\sigma(w) = w' + t v_0$ where $w' \in W$ and $t \in \mathbb{F}_5$. The induced mapping $\bar{\sigma}$ defined by $\bar{\sigma}(w) = w'$ is easily seen to be an orthogonal map on $W$ and we obtain a representation $\Gamma_3 \to O(W, \tilde{q})$. We now identify $O(W, \tilde{q})$ as one of the classical finite groups using the notation in $[\mathbb{4}]$. Let $g_i, h_i \in W$, for $i = 1, 2$ be defined by $g_i = e_1 - e_2, h_1 = -e_1 + e_2 + e_3, g_2 = e_1 + 2e_5$ and $h_2 = -e_1 + 2e_5$. Then $\bar{q}(g_i) = \bar{q}(h_i) = 0$ and $B(g_i, h_j) = \delta_{ij}$. Thus $O(W, \tilde{q}) \cong O_4^+(5)$. There is a chain of subgroups

$$1 \subset Z \subset \Omega_4^+(5) \subset SO_4^+(5) \subset O_4^+(5),$$

where $Z$ is the largest normal soluble subgroup of $O_4^+(5)$ and

$$\Omega_4^+(5) \cong \frac{SL(2,5) \times SL(2,5)}{\langle (I, -I) \rangle}.$$ 

The image of $\Gamma_3$ is isomorphic to a subgroup of index 2 in $O_4^+(5)$, different from $SO_4^+(5)$ and the orientation-preserving subgroup $\Gamma_3^+$ maps onto $\Omega_4^+(5)$. The target group has a normal subgroup of index 60 with quotient isomorphic to $PSL(2,5)$ and hence so does $\Gamma_3^+$. Using MAGMA we find a presentation for this subgroup $K_1$ with three generators and nine relations. The group $K$ is then the kernel of the induced map from $K_1$ onto $SL(2,5)$. Again using MAGMA, we obtain a presentation for $K$ on 24 generators and several pages of relations. The abelianisation of $K$ is $\mathbb{Z}^{24}$. This agrees with the homology calculations in $[\mathbb{4}]$.

This calculation is readily carried out once the images of the generators of $\Gamma_3^+$ are identified with pairs of matrices. We sketch the method of obtaining this description.
Let $V$ be a two dimensional space over $\mathbb{F}_5$ with symplectic form $f$ defined with respect to a basis $n_1, n_2$ by

$$f\left(\sum x_i n_i, \sum y_i n_i\right) = x_1 y_2 - x_2 y_1.$$ 

Let $U = V \otimes V$ and define $g$ on $U$ by

$$g(v_1 \otimes v_2, w_1 \otimes w_2) = f(v_1, w_1)f(v_2, w_2).$$ 

Then $g$ is a symmetric bilinear form on $U$ and $O(U, g) \cong O_4^+(5)$. Note that $SL(2,5) \times SL(2,5)$ acts on $U$ by

$$(\sigma, \tau)(v \otimes w) = \sigma(v) \otimes \tau(w),$$ 

and this action preserves $g$ with $(-I, -I)$ acting trivially. This describes the group $\Omega_4^+(5)$.

Additionally, the mapping $\rho : U \to U$ given by $\rho(v \otimes w) = w \otimes v$ also lies in $O(U, g)$ and has determinant $-1$. Let $H$ be the subgroup generated by $\Omega_4^+(5)$ and $\rho$.

We identify the image of $\Gamma_3$ with $H$, by first identifying $U$ and $W$ by the linear isometry induced by

$$n_1 \otimes n_1 \mapsto g_1, n_1 \otimes n_2 \mapsto g_2, n_2 \otimes n_1 \mapsto -h_2, n_2 \otimes n_2 \mapsto h_1.$$ 

It is now easy to check that the image of $r_5$ is $\rho$. Recall that $\Gamma_3^+$ is generated by $x = r_5r_4, y = r_5r_3, z = r_5r_2, w = r_5r_1$. Now determine the images of $x, y, z, w$ as pairs of matrices in $SL(2,5) \times SL(2,5)$.

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