Symmetry protected topological orders of 1D spin systems with $D_2 + T$ symmetry

Zheng-Xin Liu,1,2 Xie Chen,2 and Xiao-Gang Wen2,1

1Institute for Advanced Study, Tsinghua University, Beijing, 100084, P. R. China
2Department of Physics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139, USA

In [Z.-X. Liu, M. Liu, X.-G. Wen, arXiv:1101.5680], we studied 8 gapped symmetric quantum phases in $S = 1$ spin chains which respect a discrete spin rotation $D_2 \subset SO(3)$ and time reversal $T$ symmetries. In this paper, using a generalized approach, we study all the 16 possible gapped symmetric quantum phases of 1D integer spin systems with only $D_2 + T$ symmetry. Those phases are beyond Landau symmetry breaking theory and cannot be characterized by local order parameters, since they do not break any symmetry. They correspond to 16 symmetry protected topological (SPT) orders. We show that all the 16 SPT orders can be fully characterized by the physical properties of the symmetry protected degenerate boundary states (end ‘spins’) at the ends of a chain segment. So we can measure and distinguish all the 16 SPT orders experimentally. We also show that all these SPT orders can be realized in $S = 1$ spin ladder models. The gapped symmetric phases protected by subgroups of $D_2 + T$ are also studied. Again, all these phases can be distinguished by physically measuring their end ‘spins’.

PACS numbers: 75.10.Pq, 64.70.Tg

I. INTRODUCTION

In recent years, topological order1,2 and symmetry protected topological (SPT) order3,4 for gapped quantum ground states has attracted much interest. Here ‘topological’ means that this new kind of orders is different from the symmetry breaking orders.5–7 The new orders include fractional quantum Hall states8–9, 1D Haldane phase10, chiral spin liquids,11–12 $Z_2$ spin liquids,13–15 non-Abelian fractional quantum Hall states,16–19 quantum orders characterized by projective symmetry group (PSG),14,20 topological insulators,21–26 etc.

Recent studies indicate that the patterns of entanglements provide a systematic and comprehensive point of view to understand topological orders and SPT orders.27–30 The phases with long-ranged entanglement have intrinsic topological orders, while symmetric short-range entangled nontrivial phases are said to have SPT orders. With a definition of phase and phase transition using local unitary transformations, one can get a complete classification for all 1D gapped quantum phases,31–33 and partial classifications for some gapped quantum phases in higher dimensions.29,34,35

In contradiction to the suggestion from the symmetry breaking theory, even when the ground states of two Hamiltonians have the same symmetry, sometimes, they still cannot be smoothly connected by deforming the Hamiltonian without closing the energy gap and causing a phase transition, as long as the deformed Hamiltonians all respect the symmetry. So those two states with the same symmetry can belong to two different phases. Those kind of phases, if gapped, are called SPT phases. The Haldane phase of spin-1 chain10 is the first example of SPT phase, which is known to be protected by the $D_2 = \{ E, R_x = e^{i \pi S^y}, R_y = e^{i \pi S^y}, R_z = e^{i \pi S^z} \}$ symmetry.36 Interestingly, when additional time reversal symmetry is present, more SPT phases emerges.33,37

Topological insulators,21–26 is another examples of SPT phases which has attracted much interest in literature. Compared to the topological insulators formed by free electrons, most SPT phases (including the ones discussed in this paper) are strongly correlated. A particular kind of strongly correlated SPT phases protected by time reversal symmetry is called the fractionalized topological insulators by some people.

An interesting and important question is how to classify different 1D SPT phases even in presence of strong correlations/interactions. For the Haldane phase in spin chains, it was thought that the degenerate end states and non-local string order can be used to describe the hidden topological order. However, if we remove the spin rotation symmetry but keep the parity symmetry, the Haldane phase is still different from the trivial phase, despite that the degenerate end states and non-local string order are destroyed by the absence of spin rotation symmetry.4,36,38

Recently, it was argued in Ref. 39 that the entanglement spectrum degeneracy (ESD) can be considered as the criteria to tell whether a phase is topologically ordered or not. However, it is known that all 1D gapped states are short range entangled and have no intrinsic topological orders from entanglement point of view.31,40 On the other hand, many gapped 1D phases have nontrivial ESD. So ESD cannot correspond to the intrinsic topological orders. Then, one may try to use ESD to characterize non-trivial SPT orders as suggested in Ref. 41. ESD does appear to describe non-translation invariant SPT phases protected by on-site symmetry. In particular, the ESD reveal an important connection to the projective representation of the on-site symmetry group.41

It turns out that a clear picture and a systematic classification of all 1D SPT phases can be obtained after realizing the deep connection between local unitary transformation and gapped (symmetry protected) topological phases.31–33 In particular, for 1D systems, all gapped
phase that do not break the symmetry are classified by the 1D representations and projective representations of the symmetry group $G$ (i.e., by the group cohomology classes $H^1(G, U_T(1))$ and $H^2(G, U_T(1))$, see appendix A).$^{31-33}$

In our previous paper, we have calculated the eight classes of unitary projective representations of the point group $D_{2h} = D_2 + T$, based on which we predicted eight SPT phases in integer spin models that respect the $D_{2h}$ symmetry. We realized four interesting SPT phases in $S = 1$ spin chains, and showed that these phases can be distinguished experimentally by their different responses of the end states to magnetic field. In this paper we will show that the group $D_2 + T$ has totally 16 projective representations, and the representation of $T$ is anti-unitary. We then study the properties of the corresponding 16 SPT phases, such as the dimension of their degenerate end states and their response to perturbations. Interestingly, we find that all these SPT phases can be distinguished by their different responses of the end states to various physical perturbations. We also show that all these SPT phases can be realized in spin ladders. Finally we discuss the situations when the symmetry reduces to the subgroups of $D_2 + T$.

This paper is organized as following. In section II we show that there are 16 SPT phases that respect $D_2 + T$ symmetry, and all these phases can be distinguished experimentally. The realization of the 16 SPT phases in $S = 1$ spin chains and spin ladders are given in section III. In section IV, we discuss the projective representations and SPT phases of two subgroups of $D_2 + T$. Section V is the conclusion and discussion. Some details about the derivations, together with a brief introduction to projective representations (and group cohomology) and general classification of SPT phases, are given in the appendices.

\section{Distinguishing 16 SPT Phases with $D_2 + T$ Symmetry}

All the linear representations of the group anti-unitary $D_2 + T$ are 1-dimensional (1-D). The number of linear representations of depends on the representation space. When acting on Hilbert space, the linear representations are classified by $H^1(D_2 + T, U_T(1)) = (Z_2)^2$, which contains four elements. When acting on Hermitian operators, the linear representations are classified by $H^1(D_2 + T, (Z_2)_T) = (Z_2)^3$, which contains eight elements. More details about linear representations and the first group cohomology are given in appendix C. The 8 linear representations (with Hermitian operators as the representation space) are shown in Table IV. These 8 representations collapse into 4 if the representation space is a Hilbert space, because the bases $|1, x\rangle$ and $i|1, x\rangle$ (similarly, $|1, y\rangle$ and $i|1, y\rangle$, $|1, z\rangle$ and $i|1, z\rangle$, $|0, 0\rangle$ and $i|0, 0\rangle$) are not independent. In the following discussion, if there is no further clarification, we will assume the linear representations are defined on a Hermitian operator space. Some of these Hermitian operators, called active operators which will be defined later, are very important to distinguish different SPT phases.

The projective representations are classified by the group cohomology $H^2(D_2 + T, U_T(1))$. There are totally 16 different classes of projective representations for $D_2 + T$, as shown in Table I. More discussions about group cohomology and projective representation are given in appendices A, B, D and E. The 16 classes of projective representations correspond to 16 SPT phases. Our result agrees with the classification in Ref. 33, and the correspondence is illustrated by the indices $(\omega(D_2), \beta(T), \gamma(D_2))$.

In all these 16 SPT phases, the bulk is gapped and we can only distinguish them by their different edge states which are described by the projective representations. We stress that all the properties of each SPT phase are determined by the edge states and can be detected experimentally. The idea is to add various perturbations that break the $D_2 + T$ symmetry, and to see how those perturbations split the degeneracy of the edge states.

Let us firstly consider the case that the space of degenerate end `spin' is 2-dimensional. We have three Pauli matrices $(\sigma_x, \sigma_y, \sigma_z)$ to lift the end `spin' degeneracy. During various perturbations of the system, only those that reduce to the Pauli matrices $(\sigma_x, \sigma_y, \sigma_z)$ can split the degeneracy of the ground states. These perturbations will be called active operators. To identify whether a perturbation is an active operator, one can compare its symmetry transformation properties under $D_2 + T$ with those of the three Pauli matrices $(\sigma_x, \sigma_y, \sigma_z)$. For different SPT phases, the end spin forms different projective representations of the $D_2 + T$ group, and consequently the three Pauli matrices $(\sigma_x, \sigma_y, \sigma_z)$ form different linear representations of $D_2 + T$. So they correspond to different active operators in different SPT phases.

Let $O$ be a perturbation operator, under the symmetry operation $g$ it varies as

$$u(g)^{\dagger}Ou(g) = \eta_g(O)O,$$

where $u(g)$ is the representation of symmetry group $G$ on the physical spin Hilbert space, $\eta_g(O)$ is equal to 1 or -1 and forms a 1-D representation of the symmetry group $D_2 + T$. On the other hand, the three Pauli matrices $(\sigma_x, \sigma_y, \sigma_z)$ also form linear representations of $D_2 + T$. In the end `spin' space, the Pauli matrices transform as $(m = x, y, z)$

$$M(g)^{\dagger}\sigma_m M(g) = \eta_g(\sigma_m)\sigma_m,$$

where $M(g)$ is the projective representation of $g$ (see Table I) on the end `spin' Hilbert space. If the physical operator $O$ and the end `spin' operator $\sigma_m$ form the same linear representation of the symmetry group, namely, $\eta_g(O) = \eta_g(\sigma_m)$, then they should have the same matrix elements (up to a constant factor) in the end spin subspace. In Table I, the sequence of operators $(O_1, O_2, O_3)$ are the active operators corresponding to the end `spin' operators $(\sigma_x, \sigma_y, \sigma_z)$, respectively.
TABLE I. All the projective representations of group $D_{2n} = D_2 + T$. We only give the representation matrices for the three generators $R_x, R_z$ and $T$. $K$ stands for the anti-linear operator. The 16 projective representations corresponds to 16 different SPT phases. This result agrees with the classification of combined symmetry $D_2 + T$ given in Ref. 33. The indexes $(\omega, \beta, \gamma) = (\omega(D_2), \beta(T), \gamma(D_2))$ show this correspondence. Five of these SPT phases can be realized in $S = 1$ spin chain models and others can be realized in $S = 1$ spin ladders or large-spin spin chains. The active operators are those physical perturbations which (partially) split the irreducible end states.

| $R_x$ | $R_z$ | $T$ | $\omega, \beta, \gamma$ | dim. | active operators | spin models ($S = 1$) |
|-------|-------|-----|------------------------|-----|----------------|----------------------|
| $E_0$ | 1     | 1   | $K$                    | 1, 1,A | 1               | $(S_{xyz}, S_{xyz}, S_{xyz})^b$ | chain(trivial phase) |
| $E_1$ | 1     | $i\sigma_z$ | $\sigma_z K$          | 1,-1,B | 2               | $(S_{xyz}, S_{xyz})$ | ladder |
| $E_1'$| 1     | $i\sigma_z$ | $\sigma_z K$          | 1, 1,B | 2               | $(S_{xyz}, S_{xyz})$ | ladder |
| $E_2$ | $i\sigma_z$ | 1   | $i\sigma_z K$         | 1,-1,B | 2               | $(S_{xyz}, S_{xyz})$ | ladder |
| $E_3$ | $i\sigma_z$ | 1   | $i\sigma_z K$         | 1, 1,B | 2               | $(S_{xyz}, S_{xyz})$ | ladder |
| $E_3'$| $i\sigma_z$ | 1   | $i\sigma_z K$         | 1,-1,B | 2               | $(S_{xyz}, S_{xyz})$ | ladder |
| $E_4$ | $i\sigma_z$ | $i\sigma_z$ | $i K$                  | -1, 1,A | 2               | $(S_{xyz}, S_{xyz})$ | chain($T_y$ phase) |
| $E_5$ | $i\sigma_z$ | $i\sigma_z$ | $i K$                  | 1, 1,B | 4               | $(S_{xyz}, S_{xyz})$ | ladder |
| $E_6$ | $i\sigma_z$ | $i\sigma_z$ | $i K$                  | 1, 1,B | 2               | $(S_{xyz}, S_{xyz})$ | ladder |
| $E_6'$| $i\sigma_z$ | $i\sigma_z$ | $i K$                  | 1, 1,B | 4               | $(S_{xyz}, S_{xyz})$ | ladder |
| $E_7$ | $i\sigma_z$ | $i\sigma_z$ | $i K$                  | 1,-1,B | 2               | $(S_{xyz}, S_{xyz})$ | ladder |
| $E_7'$| $i\sigma_z$ | $i\sigma_z$ | $i K$                  | 1, 1,B | 4               | $(S_{xyz}, S_{xyz})$ | ladder |
| $E_8$ | $i\sigma_z$ | $i\sigma_z$ | $i K$                  | 1,-1,B | 2               | $(S_{xyz}, S_{xyz})$ | ladder |
| $E_9$ | $i\sigma_z$ | $i\sigma_z$ | $i K$                  | 1, 1,B | 4               | $(S_{xyz}, S_{xyz})$ | ladder |
| $E_{10}$ | $i\sigma_z$ | $i\sigma_z$ | $i K$                  | 1,-1,B | 2               | $(S_{xyz}, S_{xyz})$ | ladder |
| $E_{11}$ | $i\sigma_z$ | $i\sigma_z$ | $i K$                  | 1, 1,B | 4               | $(S_{xyz}, S_{xyz})$ | ladder |
| $E_{12}$ | $i\sigma_z$ | $i\sigma_z$ | $i K$                  | 1,-1,B | 2               | $(S_{xyz}, S_{xyz})$ | ladder |
| $E_{13}$ | $i\sigma_z$ | $i\sigma_z$ | $i K$                  | 1, 1,B | 4               | $(S_{xyz}, S_{xyz})$ | ladder |

\[\chi_m(T) = \frac{N g_m^2 B y}{3 k_B T},\]

where $N$ is the number of end ‘spins’.

Notice that different projective representations have different active operators. Thus we can distinguish all of the 16 SPT phases experimentally. For instance, the active operators of the $E_1$ and $E_1'$ phases are $(S_x, S_z, S_{xy})$ and $(S_y, S_{xy}, S_{xy})$, respectively. Here $S_m = S_m S_n + S_n S_m$ is a spin quadrupole operator, and $S_{xy}$ is a third order spin operator, such as $S_{xy}, S_{xy,1}, S_{xy,2}$. We will show that the two SPT phases $E_1$ and $E_1'$ can be distinguished by the perturbation (3).

Similarly, in the case that the end ‘spin’ is 4-dimensional, there are 15 4 × 4 matrices that can (partially) lift the degeneracy of the end states, namely, $(\sigma_x \otimes \sigma_x, \sigma_y \otimes \sigma_y, \sigma_z \otimes \sigma_z, \sigma_x \otimes \sigma_y, \sigma_y \otimes \sigma_x, \sigma_z \otimes \sigma_y, \sigma_y \otimes \sigma_z, \sigma_x \otimes \sigma_z, I \otimes \sigma_x, I \otimes \sigma_y)$. And the corresponding active operators are given in Table I. Since the active operators are perturbations that split the ground state degeneracy, through linear response theory, they correspond to measurable physical quantities. For example, if the spin $S_m$ is an active operator, it couples to a magnetic field through the interaction

\[H' = \sum_i (g_x \mu B B_x S_x, i + g_y \mu B B_y S_y, i + g_z \mu B B_z S_z, i)(3)\]

The end ‘spins’ may be polarized by above perturbation. In a real spin-chain materials, due to structural defects, there are considerable number of end ‘spins’. They behave as impurity spins (the gapped bulk can be seen as a paramagnetic material). Thus, the polarization of the end ‘spins’ can be observed by measuring the magnetic susceptibility, which obeys the Curie law $(m = x, y, z)$.
need to add perturbations by the spin-quadrupole operators \( S_{xy}, S_{yz}, S_{xz} \) and the third-order spin operators such as \( S_{xy}, S_{x,i+1} \). Actually, these perturbations may be realized experimentally. For instance, the interaction between the spin-quadrupole and a nonuniform magnetic field is reasonable in principle:

\[
H' = g_{xy} \left( \frac{\partial B_x}{\partial y} + \frac{\partial B_y}{\partial x} \right) S_{xy} + ... 
\]

One can measure the corresponding ‘quadrupole susceptibility’ corresponding to above perturbation. Similar to the spin susceptibility, different SPT phases have different coupling constants for the ‘quadrupole susceptibility’. Consequently, from the information of the spin dipole- and quadrupole- susceptibilities (and other information corresponding to the third-order spin operators), all the 16 SPT phases can be distinguished.

### III. REALIZATION OF SPT PHASES IN \( S = 1 \) SPIN CHAINS AND LADDERS

In this section, we will illustrate that all these 16 SPT phases can be realized in \( S = 1 \) spin chains or ladders.

#### A. spin-chains

1. **SPT phases for nontrivial projective representations**

In Ref. 37, we have studied four nontrivial SPT phases \( T_0, T_x, T_y, T_z \) in \( S = 1 \) spin chains. The ground states of these phases are written as a matrix product state (MPS)

\[
|\phi\rangle = \sum_{\{m_i\}} \text{Tr}(A_1^{m_1} A_2^{m_2} ... A_N^{m_N}) |m_1 m_2 ... m_N\rangle.
\]

where \( m_i = x, y, z \). More information about MPS is given in appendix. B.

1) \( T_0 \) phase. The end ‘spins’ of this phase belong to the projective representation \( E_{13} \), and a typical MPS in this phase is

\[
A^x = a \sigma_x, \quad A^y = b \sigma_y, \quad A^z = c \sigma_z,
\]

where \( a, b, c \) are real numbers.\(^{44}\) Table I shows that there is only one active operator \( S_z \) in this phase, so the end spins will response to the magnetic field along all the three directions.

2) \( T_x \) phase. The end ‘spins’ of this phase belong to the projective representation \( E_{11} \), and a typical MPS in this phase is

\[
A^x = a \sigma_x, \quad A^y = b i \sigma_y, \quad A^z = c i \sigma_z,
\]

where \( a, b, c \) are real numbers. Table I shows that there is only one active operator \( S_x \) in this phase, so the end spins will only response to the magnetic field along \( x \) direction.

3) \( T_y \) phase. The end ‘spins’ of this phase belong to the projective representation \( E_5 \), and a typical MPS in this phase is

\[
A^x = i a \sigma_x, A^y = b \sigma_y, A^z = i c \sigma_z,
\]

where \( a, b, c \) are real numbers. Table I shows that there is only one active operator \( S_y \) in this phase, so the end spins will only respond to the magnetic field along \( z \) direction.

2. **SPT phases for trivial projective representations**

Corresponding to the trivial projective IRs, we can also construct trivial phases. Here ‘trivial’ means that the ground state is in some sense like a direct product state. In these phase the matrix \( A^m \) also vary as Eqs. (B3) and (D1), except that \( A^m \) is a 1-D matrix, and \( M(y) \) is a 1-d representation of \( D_2 + T \). Since all the 1-D representation belongs to the same class, there is only one trivial phase.

A simple example of the states in this phase is a direct product state

\[
|\phi\rangle = |m_1\rangle |m_2\rangle ... |m_N\rangle.
\]

This state can be realized by a strong (positive) on-site single-ion anisotropy term \( (S_m)^2 \), \( m = x, y, z \). In this phase, there is no edge state, and no linear response to all perturbations.

#### B. spin ladders

In last section we have realized 5 of the 16 different SPT phases (with only \( D_2 + T \) symmetry) in \( S = 1 \) spin chains. In this section, we will show that all the other phases can be realized in \( S = 1 \) ladders.

1. **General discussion for spin ladders**

For simplicity, we will consider the spin-ladder models without inter-chain interaction.\(^{45}\) In that case, the ground state of the spin ladder is a direct product of the ground states of the independent chains. For example, for a two-leg ladder, the physical Hilbert space at each site is a direct product space \( \mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \) spanned by bases \( |m_1 n_1\rangle = |m_1\rangle |n_1\rangle \), with \( m_1, n_1 = x, y, z \). If the
ground state of the two chains are $|\phi_1\rangle$ and $|\phi_2\rangle$ respectively,
\[
|\phi_1\rangle = \sum_{\{m\}} \text{Tr}(A^{m_1\ldots A^{m_N}})|m_1\ldots m_N\rangle,
\]
\[
|\phi_2\rangle = \sum_{\{n\}} \text{Tr}(B^{n_1\ldots B^{n_N}})|n_1\ldots n_N\rangle,
\]
with
\[
\sum_{m'} u(g)_{mm'}A^{m'} = e^{i\alpha_1(g)}M(g)\dagger A^mM(g),
\]
\[
\sum_{n'} v(g)_{nn'}B^{n'} = e^{i\alpha_2(g)}N(g)\dagger B^nN(g),
\]
for an unitary operator $\hat{g}$ and
\[
\sum_{m'} u(T)_{mm'}(A^{m'})\dagger = M(T)\dagger A^mM(T),
\]
\[
\sum_{n'} v(T)_{nn'}(B^{n'})\dagger = N(T)\dagger B^nN(T),
\]
for the time reversal operator $T$. Then the ground state of the ladder is
\[
|\phi\rangle = |\phi_1\rangle \otimes |\phi_2\rangle
= \sum_{\{m,n\}} \text{Tr}(A^{m_1\ldots A^{m_N}})\text{Tr}(B^{n_1\ldots B^{n_N}})|m_1n_1\ldots m_Nn_N\rangle
\]
\[
= \sum_{\{m,n\}} \text{Tr}[(A^{m_1} \otimes B^{n_1})\ldots(A^{m_N} \otimes B^{n_N})]
\times |m_1n_1\ldots m_Nn_N\rangle
\]
which satisfies
\[
\sum_{m,n,m',n'} [u(g) \otimes v(g)]_{mn,m'n'}(A^{m'} \otimes B^{n'})
= e^{i\alpha(g)}(M \otimes N)\dagger (A^m \otimes B^n)(M \otimes N)
\]
for an unitary $\hat{g}$ (here $\alpha(g) = \alpha_1(g) + \alpha_2(g)$) and
\[
\sum_{m,n,m',n'} [u(T) \otimes v(T)]_{mn,m'n'}(A^{m'} \otimes B^{n'})\dagger
= (M \otimes N)\dagger (A^m \otimes B^n)(M \otimes N)
\]
for the time reversal operator $T$. This shows that the ground state of the ladder is also a MPS which is represented by $A^m \otimes B^n$, and $M \otimes N$ is a projective representation of the symmetry group $G$.

Specially, if $B^n$ is 1-D and $N(g) = 1$ (representing a trivial phase), then we have
\[
\sum_{m,n,m',n'} [u(g) \otimes v(g)]_{mn,m'n'}(A^{m'} \otimes B^{n'})
= e^{i\alpha(g)}M\dagger (A^m \otimes B^n)M.
\]
In general the projective representation $M(g) \otimes N(g)$ is reducible. This means that the end ‘spin’ of the ladder is a direct sum space of several irreducible projective representations (IPRs). These IPRs are degenerate and belong to the same class. However, this degeneracy is accidental, because only irreducible representation protected by symmetry is robust. Notice that we didn’t consider the inter-chain interaction in the ladder. If certain interaction is considered, the degeneracy between the same classes of IPRs can be lifted, and only one IPR remains as the end ‘spin’ in the ground state. This IPR (or more precisely the class it belongs to) determines which phase the spin ladder belongs to.

2. $S = 1$ spin ladders in different SPT phases

In appendix E, we show how to obtain all the other IPRs by reducing the direct product representations of $E_{13}, E_{11}, E_5, E_0$. We start with these four IPRs because the corresponding SPT phases $T_0, T_x, T_y, T_z$ have been realized in spin chains. Actually, the reduction procedure provides a method to construct spin ladders from spin chains and to realize all the SPT phases.

By putting two different spin chains (belonging to the $T_0, T_x, T_y, T_z$ phases) into a ladder, we obtain 6 new phases corresponding to $E_1, E'_1, E_3, E'_3, E_7, E'_7$, respectively. If we put one more spin chain into the ladder, then we obtain 5 more new phases corresponding to $E'_0, E'_5, E'_6, E'_{11}, E'_{13}, E'_{17}$, respectively. Therefore, together with $T_0, T_x, T_y, T_z$ and the trivial phase in spin chains, we have realized all the 16 SPT phases listed in Table I. Furthermore, if we have translational symmetry, then from section III A 1 and Eq. (14), we have totally $16 \times 4 = 64$ different SPT phases in spin ladders, in accordance with the result of Ref. 31.

IV. SPT PHASES FOR SUBGROUPS OF $D_2 + T$

From the projective representations of group $D_2 + T$, we can easily obtain the projective representations of its subgroups. According to Table I, the representation matrices for the subgroups also form a projective representation, but usually it is reducible. By reducing these matrices, we can obtain all the IPRs of the subgroup.

A. $D_2 = \{E, R_x T, R_y T, R_y\}$

This group is also a $D_2$ group except that half of its elements are anti-unitary. Notice that $T$ itself is not a group element. This group has four 1-D linear representations. In Table V in appendix C, we list the representation matrix elements, representational bases of physical spin and spin operators (for $S = 1$) according to each linear representation.

The projective representations of the subgroup $D_2$ are shown in Table II. By reducing the representation matrix of $D_2 + T$, we obtained 8 projective representations.
TABLE II. Projective representations of group $\bar{D}_2 = \{E, R_z, R_xT, R_y\}$. There are 4 classes of projective representations, meaning that the second group cohomology contains 4 elements.

| class | $E$ | $R_z$ | $R_xT$ | $R_y$ | dimension | effective/active operators | spin models ($S = 1$) |
|-------|-----|-------|-------|-------|-----------|-----------------------------|---------------------|
| 1     | 1   | 1     | $K$   | $K$   | 1         | $\sigma_x \sim S_x, S_y, S_{xy}$; $\sigma_y \sim S_y$; $\sigma_z \sim S_z, S_{xy}$ | chain (trivial phase) |
| 2     | 1   | 1     | $\sigma_y$ | $\sigma_y$ | 2         | $\sigma_x, \sigma_y, \sigma_z \sim S_{xx}$ | ladder |
|       | $\otimes 1$ | $\otimes 1$ | $\sigma_y \otimes \sigma_y$ | $\sigma_y \otimes \sigma_y$ | 4         | $\sigma_x, \sigma_y \sim S_x, S_y$; $\sigma_z \sim S_z$ | ladder |
| 3     | 1   | $i\sigma_z$ | $\sigma_y$ | $\sigma_y$ | 2         | $\sigma_x, \sigma_y \sim S_z, S_{xy}$; $\sigma_z \sim S_{xx}$ | chain |
|       | $\otimes 1$ | $\otimes 1$ | $i\sigma_y$ | $i\sigma_y$ | 4         | $\sigma_x, \sigma_y \sim S_z, S_{xy}$; $\sigma_z \sim S_{xx}$ | chain |
| 4     | 1   | $i\sigma_z$ | $\sigma_y$ | $i\sigma_y$ | 2         | $\sigma_x, \sigma_z \sim S_z, S_{xy}$; $\sigma_y \sim S_x$ | chain |
|       | $\otimes 1$ | $\otimes 1$ | $i\sigma_y$ | $i\sigma_y$ | 4         | $\sigma_x, \sigma_z \sim S_z, S_{xy}$; $\sigma_y \sim S_x$ | chain |

They are classified into 4 classes. This can be shown by calculating the corresponding 2-cocycles of these projective representations. Two projective representations belonging to the same class means that the corresponding 2-cocycle differ by a 2-coboundary (see appendices A, B and D).

As shown in Table II, the 2-dimensional representation in class-1 is trivial (or linear), it belongs to the same class as the 1-D representation. This means that the edge states in this phase is not protected by symmetry, the ground state degeneracy can be smoothly lifted without phase transition. The class-3 and class-4 nontrivial SPT phases can be realized in spin chains. These two phases can be distinguished by magnetic fields. The phase corresponding to the class-3 projective representation only respond to the magnetic field along $z$ direction, and the phase corresponding to class-4 projective representation only respond to the magnetic field along $x$ direction. The remaining two nontrivial SPT phases of class 2 can be realized by spin ladders.

B. $Z_2 + T = \{E, R_z, T, R_xT\}$

This subgroup is also a direct product group. The linear representations and projective representations are given in Table. VI (see appendix C) and III, respectively. This group is isomorphic to $\bar{D}_2 = \{E, R_zT, R_xT, R_y\}$, so its projective representations and SPT phases are one to one corresponding to those in II. However, the corresponding SPT phases in III and II are not the same, because they have different response to external perturbations.

Notice that, this simple symmetry is very realistic for materials. For example, the quasi-1D anti-ferromagnets CaRuO$_3$ and NaRO$_3$ respect this $Z_2 + T$ symmetry due to spin-orbital coupling. Their ground state, if non-symmetry breaking, should belong to one of the four SPT phases listed in Table III.

V. CONCLUSION AND DISCUSSION

In summary, through the projective representations, we studied all the 16 different SPT phases for integer spin systems that respect only $D_{2h} = D_2 + T$ on-site symmetry. We provided a method to measure all the SPT orders. We showed that in different SPT phase the end ‘spins’ respond to various perturbations differently. These perturbations include spin dipole- (coupling to uniform magnetic fields) and quadrupole- (coupling to nonuniform magnetic fields). We illustrated that the SPT orders in different SPT phases can be observed by experimental measurements, such as the temperature dependence of the magnetic susceptibility and asymmetric $g$-factors. We illustrated that all the 16 SPT phases can be realized in $S = 1$ spin chains or ladders. Finally we studied the SPT phases for two subgroups of $D_2 + T$, one of the subgroup is the symmetry group of some interesting materials. Certainly, our method of studying SPT orders can be generalized to other symmetry groups.
VI. ACKNOWLEDGEMENTS

We thank Ying Ran for helpful discussions. This research is supported by NSF Grant No. DMR-1005541 and NSFC 11074140.

Appendix A: Group cohomology

We consider a finite group $G = \{g_1, g_2, \ldots\}$ with its module space $U_T(1)$. The group elements of $G$ are operators on the module space. A $n$-cochain $\omega_n(g_1, g_2, \ldots, g_n)$ is a function on the group space which maps $\otimes^n G \rightarrow U(1)$. The cochains can be classified with the coboundary operator.

Suppose the cochain $\omega_n(g_1, g_2, \ldots, g_n) \in U(1)$, then the coboundary operator is defined as

$$
(d\omega_n)(g_1, g_2, \ldots, g_{n+1}) = g_1 \cdot \omega_n(g_2, g_3, \ldots, g_{n+1})
$$

for $n \geq 1$, and

$$
(d\omega_0)(g_1) = \frac{g_1 \cdot \omega_0}{\omega_0},
$$

(A1)

for $n = 0$. Here $g \cdot \omega_n$ is a group action on the module space $U(1)$. If $g$ is an unitary operator, it acts on $U(1)$ trivially $g \cdot \omega_n = \omega_n$. If $g$ is anti-unitary (such as the time reversal operator $T$), then the action is given as $g \cdot \omega_n = \omega_n^{-1}$. We will use $U_T(1)$ to denote such a module space. We note that, if $G$ contain no time reversal transformation, then $U_T(1) = U(1)$.

A cochain $\omega_n$ satisfying $d\omega_n = 1$ is called a $n$-cocycle. If $\omega_n$ satisfies $\omega_n = d\omega_{n-1}$, then it is called a $n$-coboundary. Since $d^2 = 1$, a coboundary is always a cocycle. The following are two examples of cocycle equations. 1-cocycle equation:

$$
\frac{g_1 \cdot \omega_2(g_2)\omega(g_1)}{\omega_2(g_1g_2)} = 1.
$$

(A2)

2-cocycle equation:

$$
\frac{g_1 \cdot \omega_3(g_2, g_3)\omega_2(g_1, g_2, g_3)}{\omega_2(g_1g_2g_3)\omega_2(g_1, g_2, g_3)} = 1.
$$

(A3)

The group cohomology is defined as $H^n(G, U_T(1)) = Z^n/B^n$. Here $Z^n$ is the set of $n$-cocycles and $B^n$ is the set of $n$-coboundaries. If two $n$-cocycles $\omega_n$ and $\omega'_n$ differ by a $n$-coboundary $\tilde{\omega}n$, namely, $\omega'_n = \omega_n + \tilde{\omega}n$, then they are considered to be equivalent. The set of equivalent $n$-cocycles is called a equivalent class. Thus, the $n$-cocycles are classified with different equivalent classes, these classes form the (Abelian) cohomology group $H^n(G, U_T(1)) = Z^n/B^n$.

As an example, we see the cohomology of $Z_2 = \{E, \sigma\}$, where $E$ is the identity element and $\sigma^2 = E$. Since this group $Z_2$ is unitary, it acts on the module space trivially and $U_T(1) = U(1)$: $g \cdot \omega_n = \omega_n$. From (A2) the first cohomology is the 1-D representations.

$$
H^1(Z_2, U(1)) = Z_2,
$$

The second cohomology classifies the projective representations (see appendix B). It can be shown that all the solutions of (A3) are 2-coboundaries $\omega = d\omega_1$. So all the 2-cocycles belong to the same class, consequently,

$$
H^2(Z_2, U(1)) = 0.
$$

Let us see another example, the time reversal group $Z_2^T = \{E, T\}$. Notice that the time reversal operator $T$ is antiunitary, it acts on $U_T(1)$ nontrivially: $T \cdot \omega_n = \omega_n^{-1}$. As a result, the cohomology of $Z_2^T$ is different from that of $Z_2$:

$$
H^1(Z_2^T, U_T(1)) = 0,
$$

$$
H^2(Z_2^T, U_T(1)) = Z_2.
$$

The group $Z_2^T$ have two orthogonal 1-D representations (see appendix C), but above result shows that these two 1-D representations belong to the same class. Further more, the nontrivial second group cohomology shows that $Z_2^T$ has a nontrivial projective representation, which is well known: $M(E) = I, M(T) = i\sigma_yK$.

Appendix B: Brief review of the classification of 1D SPT orders

A key trick to use local unitary transformation to study/classify 1D gapped SPT phases is the matrix product state (MPS) representation of the ground states. The simplest example is the $S = 1$ AKLT wave function$^{42}$ in the Haldane phase which can be written as a $2 \times 2$ MPS. Later it was shown that in 1D all gapped many-body spin wave functions (it was generalized to fermion systems) can be well approximated by a MPS as long as the dimension $D$ of the matrix is large enough$^{43}$

$$
|\phi\rangle = \sum_{\{m_i\}} \text{Tr}(A_1^{m_1}A_2^{m_2}\ldots A_N^{m_N})|m_1m_2\ldots m_N\rangle.
$$

(B1)

Here $m$ is the index of the $d$-component physical spin, and $A_i^{m_i}$ is a $D \times D$ matrix. Provided that the system is translationally invariant, then one set all the matrices $A^m$ as the same over all sites.

In the MPS picture, it is natural to understand that projective representations can be used as a label of different SPT phase. Suppose that a system has an on-site unitary symmetry group $G$ which keep the ground state $|\phi\rangle$ invariant

$$
\hat{g}|\phi\rangle = u(g) \otimes u(g) \otimes \ldots \otimes u(g)|\phi\rangle = (e^{i\alpha_g})^N|\phi\rangle.
$$

(B2)
where \( g \in G \) is a group element of \( G \), \( u(g) \) is its \( d \)-dimensional (maybe reducible) representation and \( e^{i\alpha(g)} \) is its 1-D representation. We only consider the case that \( u(g) \) is a linear representation of \( G \). The case that \( u(g) \) forms a projective representation of \( G \) (such as half-integer spin chains) has been studied in Ref. 31 and 33. Eqs. (B1) and (B2) require that the matrix \( A^m \) should vary in the following way\(^{31,41}\)

\[
\sum_m u(g)_{mm'} A^{m'} = e^{i\alpha(g)} M(g)^\dagger A^m M(g), \quad (B3)
\]

where \( M(g) \) is an invertible matrix and is essential for the classification of different SPT phases. Notice that if \( M(g) \) satisfies Eq. (B3), so does \( M(g) e^{i\varphi(g)} \). Since \( u(g_1 g_2) = u(g_1) u(g_2) \) and \( e^{i\alpha(g_1 g_2)} = e^{i\alpha(g_1)} e^{i\alpha(g_2)} \), we obtain

\[
M(g_1 g_2) = M(g_1) M(g_2) e^{i\theta(g_1 g_2)}. \quad (B4)
\]

Above equation shows that up to a phase \( e^{i\theta(g_1, g_2)} \), \( M(g) \) satisfies the multiplication rule of the group. Further, \( M(g) \) satisfies the associativity condition \( M(g_1) M(g_2) M(g_3) = M(g_1 g_2 g_3) \), or equivalently

\[
e^{i\theta(g_2 g_3)} e^{i\theta(g_1, g_2 g_3)} = e^{i\theta(g_1, g_2)} e^{i\theta(g_1, g_3)}.
\]

Above equation coincide with the cocycle equation (A3) when \( G \) is unitary. The matrices \( M(g) \) that satisfies above conditions are called projective representation of the symmetry group \( G \). Above we also shows the relation between projective representations and 2-cocycle.

For a projective representation, the two-element function \( e^{i\theta(g_1, g_2)} \) has redundant degrees of freedom. Suppose that we introduce a phase transformation, \( M(g_1) y = e^{i\varphi(g_1)} M(g_1) \) and \( M(g_2) y = e^{i\varphi(g_2)} M(g_2) \), then the function \( e^{i\theta(g_1, g_2)} \) becomes

\[
e^{i\varphi(g_1 g_2)} e^{i\varphi(g_1) - i\varphi(g_2)} e^{i\theta(g_1, g_2)}. \quad (B5)
\]

Notice that \( e^{i\theta(g_1, g_2)} \) and \( e^{i\theta(g_1, g_3)} \) differs by a 2-coboundary, so they belong to the same class. Thus, the projective representations are classified by the second group cohomology \( H^2(G, U(1)) \). If \( M(g) \) and \( M(g) \) belong to different (classes of) projective representations, then they cannot be smoothly transformed into each other, therefore the corresponding quantum states \( A^m \) and \( A^m \) fall in different phases. In other words, the projective representation \( \omega_2 \in H^2(G, U(1)) \) provides a label of a SPT phase. If the system is translationally invariant, then \( e^{i\alpha(g)} \in H^1(G, U(1)) \) is also a label of a SPT phase. In this case, the complete label of a SPT phase is \( (\omega_1, \alpha) \). If translational symmetry is absent, we can regroup the matrix \( A^m \) such that \( e^{i\alpha(g)} = 1 \), then each SPT phase is uniquely labeled by \( \omega_2 \).

Appendix C: Linear representations for \( D_2 + T \) and its subgroups

Generally, the 1-D linear representations of a group \( G \) are classified by its first group cohomology \( H^1(G) \). However, there is a subtlety to choose the coefficient of \( H^1(G) \). We will show that if the representation space is a Hilbert space, the 1-D representations are characterized by \( H^1(G, U(1)) \) (or \( H^1(G, U_T(1)) \) if \( G \) contains anti-unitary elements); while if the representation space is a Hermitian operator space, then the 1-D representations are characterized by \( H^1(G, Z_2) \) (notice that \( H^1(G, (Z_2)_T) = H^1(G, Z_2) \), there is no difference whether \( G \) contains anti-unitary elements or not).

Since the discusses for unitary group and anti-unitary group are very similar, we will only consider a group \( G \) which contains anti-unitary elements. Firstly, we consider the 1-D linear representations on a Hilbert space \( \mathcal{H} \). Suppose \( \phi \in \mathcal{H} \) is a basis, and \( g \in G \) is an anti-unitary element, then

\[
\hat{g} |\phi\rangle = \eta(g) K |\phi\rangle, \quad (C1)
\]

where the number \( \eta(g) \) is the representation of \( g \). Notice that \( g \) is anti-linear, which may change the phase of \( |\phi\rangle \). To see that, we suppose \( K |\phi\rangle = |\phi\rangle \), and introduce a phase transformation for the basis \( |\phi\rangle \), namely, \( |\phi'\rangle = e^{i\theta} |\phi\rangle \). Now we choose \( |\phi'\rangle \) as the basis, then

\[
\hat{g} |\phi'\rangle = \eta(g) e^{i\theta} K |\phi'\rangle, \quad (C2)
\]

so the representation \( \eta(g) e^{i\theta} \) of \( g \) changes accordingly. This means that the 1-D representation of the group \( G \) is \( U(1) \)-valued, and is characterized by the first cohomology group \( H^1(G, U(1)) \). In the case of \( D_2 + T \), we have

\[
H^1(D_2 + T, U_T(1)) = (Z_2)^2,
\]

so \( D_2 + T \) has 4 different 1-D linear representations on Hilbert space, which can be labeled as \( A, B_1, B_2, B_3 \) respectively.

Now we consider the 1-D representations on a Hermitian operator space. Suppose \( O_1, O_2, ..., O_n \) are orthonormal Hermitian operators satisfying \( \text{Tr}(O_n O_m) = \delta_{mn} \), an anti-unitary element \( g \in G \) act on these operators as

\[
\hat{g} O_m = K M(g) O_m M(g) K = \sum_n \zeta(g)_{mn} O_n, \quad (C3)
\]

Here \( M(g) K \) is either a linear or a projective representation of \( g \), while \( \zeta(g) \) is always a linear representation. Since \( [K M(g) O_m M(g) K] = K M(g) O_m M(g) K \), we have

\[
|\sum_n \zeta(g)_{mn} O_n| = |\sum_n \zeta(g)_{mn} O_n| = \sum_n \zeta(g)_{mn} O_n = \sum_n \zeta(g)_{mn} O_n,
\]

which gives

\[
\zeta(g)^* = \zeta(g).
\]

The same result can be obtained if \( G \) is unitary. So we conclude that, all the linear representations defined on
Hermitian operator space are real. Now we focus on 1-D linear representations. Since \( g \) is either unitary or anti-unitary, we have \( \zeta(g) = 1 \). On the other hand, \( \zeta(g) \) must be real, so \( \zeta(g) = \pm 1 \). As a result, all the 1-D linear representations on Hermitian operator space are \( \mathbb{Z}_2 \), which are characterized by the first group cohomology \( H^1(G, (\mathbb{Z}_2)_T) \). For the group \( D_2 + T \),

\[
H^1(D_2 + T, (\mathbb{Z}_2)_T) = (\mathbb{Z}_2)^3,
\]

so there are 8 different 1-D linear representation, corresponding to 8 classes of Hermitian operators as shown in Table IV. Since all the linear representations of \( D_2 + T \) are 1-dimensional, this 8 1-D representations are all of its linear representations.

Above discussion is also valid for the subgroups of \( D_2 + T \). In Tables V and VI, we give the linear representations of its two subgroups (the number of 1-D linear representations on Hilbert space is half of that on Hermitian operator space).

We have shown that for 1-D linear representations defined on Hermitian operator space, there is no difference whether a group element is unitary or anti-unitary. This conclusion is also valid for higher dimensional linear representations (however, if the representation space is a Hilbert space, unitary or anti-unitary group elements will be quite different). The linear representations on Hermitian operator space are used to define the active operators.

For a general group \( G \), if it has a nontrivial projective representation, which correspond to a SPT phase, then the active operators are defined in the following way: for a set of Hermitian operators \( O_1^{\text{ph}}, ..., O_n^{\text{ph}} \) acting on the physical spin Hilbert space, if we can find a set of Hermitian operators \( O_1^{\text{lin}}, ..., O_n^{\text{lin}} \) acting on the internal-spin Hilbert space (or the projective representation space), such that \( O_i^{\text{ph}} \) and \( O_i^{\text{lin}} \) form the same \( n \)-dimensional real linear representation of \( G \), then the operators \( O_i^{\text{ph}} \) are called active operators. Different SPT phases have different set of active operators, so we can use these active operators to distinguish different SPT phases.

### Appendix D: 16 projective representations of \( D_2 + T \) group

We have shown in appendices A and B that the projective representations are classified by the second group cohomology \( H^2(G, U_1(1)) \). However, usually it is not easy to calculate the group cohomology. So we choose to calculate the projective representations directly. In the following we give the method through which we obtain all the 16 projective representations of \( D_2 + T \) in Table I.

The main trouble comes from the anti-unitarity of some symmetry operators, such as the time reversal operator \( T \). Under anti-unitary operators (such as \( T \)), the matrix \( A^m \) varies as

\[
\sum_{m'} u(T)(m'm')^* = M(T)^T A^m M(T).
\]

Notice that \( e^{i\alpha(T)} \) is absent because we can always set it to be 1 by choosing proper phase of \( A^m \). To see more difference between the unitary operator and anti-unitary operators, we introduce an unitary transformation to the bases of the virtual ‘spin’ such that \( A^m \) becomes \( A^m = U^T A^m U \). Then for an unitary symmetry operation \( g \), Eq. (B3) becomes

\[
\sum_{m'} u(g)(m'm') A^m = e^{i\alpha(g)} \bar{M}(g)^T A^m \bar{M}(g),
\]

where \( \bar{M}(g) = U^T M(g) U \). However, for the anti-unitary operator \( T \), \( A^m \) varies as

\[
\sum_{m'} u(T)(m'm')^* = \bar{M}(T)^T A^m \bar{M}(T),
\]

where \( \bar{M}(T) = U^T M(T) U^* = U^T [M(T)K] U \). Therefore, we can see that \( M(T)K \) as a whole is the anti-unitary projective representation of \( T \) when acting on the virtual ‘spin’ space.

The question is how to obtain the matrix \( M(T) \). In Ref. 37, we firstly treated \( T \) as an unitary operator, and

---

**Table IV.** Linear representations of \( D_{2h} = D_2 + T \)

| \( g \) | \( E \) | \( R_x \) | \( R_y \) | \( R_T \) | \( R_{T'} \) | \( R_T \) |
|---|---|---|---|---|---|---|
| \( A_g \) | 1 | 1 | 1 | 1 | 1 | 1 | \( (0, 0) \) |
| \( B_{yz} \) | 1 | -1 | 1 | 1 | -1 | -1 | \( i(1, z) \) |
| \( B_{xy} \) | 1 | -1 | 1 | 1 | -1 | -1 | \( i(1, y) \) |
| \( B_{yx} \) | 1 | -1 | 1 | 1 | -1 | -1 | \( i(1, x) \) |
| \( A_x \) | 1 | 1 | 1 | 1 | -1 | -1 | \( (0, 0)(S_{xz}, S_{y}, x, i)^{+1} \) |
| \( B_{zx} \) | 1 | -1 | 1 | 1 | 1 | 1 | \( (1, z) \) |
| \( B_{xz} \) | 1 | 1 | -1 | 1 | 1 | 1 | \( (1, y) \) |
| \( B_{z} \) | 1 | -1 | -1 | 1 | -1 | -1 | \( (1, x) \) |

**Table V.** Linear representations of \( D_2 = \{ E, R_x T, R_z T, R_y \} \)

| \( g \) | \( E \) | \( R_x T \) | \( R_z T \) | \( R_y \) | \( \text{bases operators} \) |
|---|---|---|---|---|---|
| \( A_g \) | 1 | 1 | 1 | 1 | \( (0, 0) \) |
| \( B_{yz} \) | 1 | 1 | -1 | -1 | \( i(1, z) \) |
| \( B_{xy} \) | 1 | -1 | -1 | -1 | \( i(1, y) \) |
| \( B_{yx} \) | 1 | -1 | -1 | -1 | \( i(1, x) \) |

**Table VI.** Linear representations of \( Z_2 + T = \{ E, R_x T, R_z T \} \)

| \( g \) | \( E \) | \( R_x T \) | \( R_z T \) | \( \text{bases operators} \) |
|---|---|---|---|---|
| \( A_g \) | 1 | 1 | 1 | \( (0, 0) \) |
| \( B_{yz} \) | 1 | 1 | -1 | \( i(1, z) \) |
| \( B_{xy} \) | 1 | -1 | -1 | \( i(1, y) \) |
| \( B_{yx} \) | 1 | -1 | -1 | \( i(1, x) \) |
we got 8 classes of unitary projective representations for the group $D_{2h}$ (see Table VII). By replacing $M(T)$ by $M(T)K$, we obtained 8 different classes of anti-unitary projective representations. However, not all the projective representations can be obtained this way. Notice that $[M(T)K]^2 = 1$ and $[M(T)K]^2 = -1$ belong to two different projective representations, the anti-unitary projective representations are twice as many as the unitary projective representations. Fortunately, all the remaining (anti-unitary) projective representations can be obtained from the known ones. Notice that the direct product of any two projective representations is still a projective representation of the group, which can be reduced to a direct sum of several projective representations. There may be new ones in the reduced representations that are different from the 8 known classes. Repeating this procedure (until it closes), we finally obtain 16 different classes of projective representations (see appendix E). Notice that the Clebsch-Gordan coefficients which reduce the product representation should be real, otherwise it does not commute with $K$ and will not block diagonalize the product representation matrix of $T$ (and other anti-unitary symmetry operators). Because of this restriction, we obtain four $4$-dimensional irreducible projective representations (IPRs) which are absent in the unitary projective representations.

### Appendix E: Realization of SPT phases in $S = 1$ spin ladders

From the knowledge of section IIIA, together with Eqs. (12) and (14), we can construct different SPT phases with spin ladders. From the discussion in section IIIB, the projective representation $M(g) \otimes N(g)$ is usually reducible. It can be reduced to several representations of the same class. This class of projective representation determines which phase the ladder belongs to. Thus, the decomposition of direct products of different projective representations is important. Since the SPT phases corresponding to $E_{13}, E_{11}, E_5, E_9$ (and $E_{13}, E_{17}, E_{13}^*$, separately) have already been realized in spin chains, we will first study the decompositions of the direct product of two of them.

$E_5 \otimes E_9 = (\sigma_z, I, i\sigma_x) \oplus (\sigma_z, -I, i\sigma_x) = E_4 \oplus E_4'$.$E_5 \otimes E_{11} = \{I, i\sigma_z, \sigma_x\} \oplus \{-I, i\sigma_z, -\sigma_x\} = E_4 \oplus E_4'$.$E_5 \otimes E_{13} = (\sigma_z, i\sigma_z, i\sigma_y) \oplus (\sigma_z, -i\sigma_z, -i\sigma_y) = E_4 \oplus E_4'$.$E_5 \otimes E_{11} = \{I, i\sigma_z, \sigma_x\} \oplus \{-I, i\sigma_z, -\sigma_x\} = E_4 \oplus E_4'$.$E_5 \otimes E_{13} = (\sigma_z, i\sigma_z, i\sigma_y) \oplus (\sigma_z, -I, i\sigma_y) = E_4 \oplus E_4'$.

In above decomposition, all the CG coefficients are real. The three matrices in each bracket are the representation matrices for the three generators $R_x, R_y, T$, separately. We omitted the anti-unitary operator $K$ for the representation matrix of $T$. Further, $E_1$ and $E_2$ ($E_3$ and $E_4$, so on and so forth) belong to the same class of projective representation, and differs only by a phase transformation. So with spin ladders, we realize 6 SPT phases corresponding to the projective representations $E_1, E_1', E_3, E_3', E_7, E_7'$.

Using these projective representations $E_1, E_1', E_3, E_3', E_7, E_7'$, together with $E_{13}, E_{11}, E_5, E_9$, we can repeat above procedure and obtain more projective representations and their corresponding SPT phases. The result is shown below:

$E_1 \otimes E_3 = (\sigma_z, -i\sigma_z, i\sigma_x) \oplus (\sigma_z, i\sigma_z, -i\sigma_x) = E_7 \oplus E_9$.$E_1 \otimes E_5 = (-I \otimes i\sigma_z, I \otimes i\sigma_z, -\sigma_y \otimes \sigma_z) = E_{11}'$.$E_1 \otimes E_7 = (\sigma_z, I, i\sigma_y) \oplus (-\sigma_z, I, i\sigma_y) = E_3 \oplus E_4$.$E_1 \otimes E_9 = (-i\sigma_z, -i\sigma_z, -i\sigma_y) \oplus (-i\sigma_z, i\sigma_z, -i\sigma_y) = E_{13} \oplus E_{14}'$.$E_1 \otimes E_{11} = (-I \otimes i\sigma_z, -I \otimes i\sigma_z, \sigma_y \otimes I) = E_5'$.$E_1 \otimes E_{13} = (-i\sigma_z, I, -i\sigma_x) \oplus (-i\sigma_z, -I, -i\sigma_x) = E_3 \oplus E_{10}$.

$E_2 \otimes E_3 = (-\sigma_z, -i\sigma_z, -i\sigma_y) \oplus (\sigma_z, i\sigma_z, i\sigma_y) = E_4 \oplus E_4'$.$E_2 \otimes E_5 = (-i\sigma_z, i\sigma_z, -\sigma_z) \oplus (i\sigma_z, -i\sigma_z, \sigma_z) = E_{11} \oplus E_{12}'$.$E_2 \otimes E_7 = (-\sigma_z, I, \sigma_y) \oplus (-\sigma_z, -I, i\sigma_y) = E_3' \oplus E_4'$.$E_2 \otimes E_9 = (-I \otimes i\sigma_z, I \otimes i\sigma_z, -\sigma_y \otimes \sigma_z) = E_{13}'$.$E_2 \otimes E_{11} = (-i\sigma_z, -\sigma_z, I) \oplus (-i\sigma_z, -\sigma_z, -I) = E_3 \oplus E_6$.$E_2 \otimes E_{13} = (-I \otimes i\sigma_z, -I \otimes i\sigma_z, -\sigma_y \otimes \sigma_z) = E_{14}'$.

$E_3 \otimes E_5 = (-I \otimes i\sigma_z, I \otimes i\sigma_z, \sigma_y \otimes \sigma_z) = E_6'$.$E_3 \otimes E_7 = (-I, i\sigma_x, i\sigma_y) \oplus (I, i\sigma_x, -i\sigma_y) = E_{1} \oplus E_3$.$E_3 \otimes E_9 = (-I \otimes i\sigma_z, I \otimes i\sigma_z, -\sigma_y \otimes I) = E_5'$.$E_3 \otimes E_{11} = (-i\sigma_z, i\sigma_x, i\sigma_y) \oplus (-i\sigma_z, i\sigma_x, -i\sigma_y) = E_{13}'$.
\[ E_{13} \otimes E_{14}; \]
\[ E_{9} \otimes E_{13} = (-i\sigma_z, i\sigma_x, -\sigma_z) \oplus (-i\sigma_z, -i\sigma_x, \sigma_z) = E_{11} \oplus E_{12}; \]
\[ E_{9}' \otimes E_{9}' = (-i\sigma_z, \sigma_x, -i\sigma_x) \oplus (-i\sigma_z, -\sigma_x, i\sigma_x) = E_3 \oplus E_6; \]
\[ E_9 \otimes E_6 = (-i\sigma_z, \sigma_x, -i\sigma_x) \oplus (-i\sigma_z, -\sigma_x, i\sigma_x) = E_6 \oplus E_6; \]
\[ E_3 \otimes E_3 = (-i\otimes i\sigma_y) \oplus (I, -i\sigma_y) = E_0' \oplus E_0'; \]
\[ E_3 \otimes E_9 = (I, I, i\sigma_y) \oplus (I, -i\sigma_y) = E_0' \oplus E_0'; \]
\[ E_7 \otimes E_7 = (I, I, i\sigma_y) \oplus (I, -i\sigma_y) = E_9 \oplus E_9; \]
\[ E_7 \otimes E_9 = (I, I, i\sigma_y) \oplus (I, -i\sigma_y) = E_9 \oplus E_9; \]
\[ E_7 \otimes E_7 = (I, I, i\sigma_y) \oplus (I, -i\sigma_y) = E_9 \oplus E_9; \]

Above we get four SPT phases corresponding to \( E_{13}, E'_{13}, E_{11}, E_{13}' \), all of them have 4-dimensional end 'spins'. We also get a SPT phase corresponding to \( E_0 \), which has 2-dimensional end 'spins'.

Notice that the number of classes of unitary projective representations of \( D_{2h} \) is 8, but considering that \( T \) is anti-unitary such that \( T^2 \) can be either 1 or -1, we obtain 16 classes of projective representations for \( D_2 + T \).

---

1. X.-G. Wen, Phys. Rev. B 40, 7387 (1989).
2. X.-G. Wen, Int. J. Mod. Phys. B 4, 239 (1990).
3. X.-G. Wen, Phys. Rev. B 68, 201311 (2003), cond-mat/0107071.
4. Z.-C. Gu and X.-G. Wen, Phys. Rev. B 80, 155131 (2009), arXiv:0903.1069.
5. L. D. Landau, Phys. Z. Sowjetunion 11, 26 (1937).
6. V. L. Ginzburg and L. D. Landau, Zh. Ekaper. Teoret. Fiz. 20, 1064 (1950).
7. L. D. Landau and E. M. Lifschitz, *Statistical Physics - Course of Theoretical Physics Vol 5* (Pergamon, London, 1958).
8. D. C. Tsui, H. L. Stormer, and A. C. Gossard, Phys. Rev. Lett. 48, 1559 (1982).
9. R. B. Laughlin, Phys. Rev. Lett. 50, 1395 (1983).
10. F. D. M. Haldane, Physics Letters A 93, 464 (1983).
11. V. Kaluhrner and R. B. Laughlin, Phys. Rev. Lett. 59, 2095 (1987).
12. X.-G. Wen, F. Wilczek, and A. Zee, Phys. Rev. B 39, 11413 (1989).
13. N. Read and S. Sachdev, Phys. Rev. Lett. 66, 1773 (1991).
14. X.-G. Wen, Phys. Rev. B 44, 2664 (1991).
15. R. Moessner and S. L. Sondhi, Phys. Rev. Lett. 86, 1881 (2001).
16. G. Moore and N. Read, Nucl. Phys. B 360, 362 (1991).
17. X.-G. Wen, Phys. Rev. Lett. 66, 802 (1991).
18. R. Willett, J. P. Eisenstein, H. L. Strörmer, D. C. Tsui, A. C. Gossard, and J. H. English, Phys. Rev. Lett. 59, 1776 (1987).
19. I. P. Rudn, J. B. Miller, C. M. Marcus, M. A. Kastner, L. N. Pfeiffer, and K. W. West, Science 320, 899 (2008).
20. X.-G. Wen, Phys. Rev. D 68, 065003 (2003), hep-th/0302201.
21. C. L. Kane and E. J. Mele, Phys. Rev. Lett. 95, 226801 (2005), cond-mat/0411737.
22. C. L. Kane and E. J. Mele, Phys. Rev. Lett. 95, 146802 (2005), cond-mat/0506581.
23. B. A. Bernevig and S.-C. Zhang, Phys. Rev. Lett. 96, 106802 (2006).
24. J. E. Moore and L. Balents, Phys. Rev. B 75, 121306 (2007), cond-mat/0607314.
25. L. Fu, C. L. Kane, and E. J. Mele, Phys. Rev. Lett. 98, 106803 (2007), cond-mat/0607699.
26. X.-L. Qi, T. Hughes, and S.-C. Zhang, Phys. Rev. B 78, 195424 (2008), arXiv:0802.3537.
27. M. Levin and X.-G. Wen, Phys. Rev. Lett. 96, 110405 (2006), cond-mat/0510613.
28. A. Kitaev and J. Preskill, Phys. Rev. Lett. 96, 110404 (2006).
29. X. Chen, Z.-C. Gu, and X.-G. Wen, Phys. Rev. B 82, 155138 (2010), arXiv:1004.3835.
30. X.-G. Wen, Physics Letters A 300, 175 (2002), cond-mat/0110397.
31. X. Chen, Z.-C. Gu, and X.-G. Wen, Phys. Rev. B 83, 035107 (2011), arXiv:1008.3745.
32. N. Schuch, D. Perez-Garcia, and I. Cirac(2011), arXiv:1010.3732.
33. X. Chen, Z.-C. Gu, and X.-G. Wen(2011), arXiv:1103.3323.
34. M. Levin and X.-G. Wen, Phys. Rev. B 71, 045110 (2005), cond-mat/0404617.
35. Z.-C. Gu, Z. Wang, and X.-G. Wen(2010), arXiv:1010.1517.
36. F. Pollmann, E. Berg, A. M. Turner, and M. Oshikawa(2009), arXiv:0909.4059.
37. Z.-X. Liu, M. Liu, and X.-G. Wen(2011), arXiv:1101.1662.
38. E. Berg, G. D. Torre, T. Giamarchi, and E. Altman, Phys. Rev. B 77, 245119 (2008).
39. H. Li and F. D. M. Haldane, Phys. Rev. Lett. 101, 010504 (2008).
40. F. Verstraete, J. I. Cirac, J. I. Latorre, E. Rico, and M. M. Wolf, Phys. Rev. Lett. 94, 140601 (2005).
41. F. Pollmann, E. Berg, A. M. Turner, and M. Oshikawa, Phys. Rev. B 81, 064439 (2010), arXiv:0910.1811.
42. A. Iylbleek, T. Kennedy, E. H. Lieb, and H. Tasaki, Comm. Math. Phys. 115, 477 (1988).
43. Guifré Vidal, Phys. Rev. Lett. 91, 147902 (2003).
44. When \( a = b = c = 1 \), this state is invariant under \( SO(3) + T \), where \( SO(3) \) is generated by \( S_x, S_y, S_z \) and \( T = e^{i\pi S_y} K \). From Ref. 33, systems with \( SO(3) + T \) symmetry have 4 SPT phases. It seems strange that its subgroup \( D_2 + T \) contains more SPT phases. Actually, there
are four distinct $SO(3) + T$ groups which contain $D_2 + T$ as a subgroup. In these four groups, $T$ is always defined as $T = e^{i\pi S_y} K$, but the $SO(3)$ parts are different. Except for the one mentioned above, we have additional three choices: $-S_x, S_{xz}, S_{xy}$ or $S_{yx}, S_{xy}, S_{xz}, -S_z$. Each of the four groups contains 4 SPT phases, so their common subgroup $D_2 + T$ contains $4 \times 4 = 16$ SPT phases.

Actually, provided that the the symmetry group of the Hamiltonian of the ladder is $D_2 + T$, inter-chain interactions must be considered (otherwise the symmetry group should be $(D_2 + T) \otimes (D_2 + T)$). Here we take the limit that the strength of inter-chain interaction tends to zero.

45 Y. Shirako, H. Satsukawa, X. X. Wang, J. J. Li, Y. F. Guo, M. Arai, K. Yamaura, M. Yoshida, H. Kojitani, T. Katsumata, Y. Inaguma, K. Hiraki, T. Takahashi, M. Akaogi, arXiv:1104.1461.

46 M. Bremholma, S.E. Duttona, P.W. Stephensb and R.J. Cavaa, arXiv:1011.5125.