Research Article

On Some Approximation Theorems for Power $q$-Bounded Operators on Locally Convex Vector Spaces

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This paper deals with the study of some operator inequalities involving the power $q$-bounded operators along with the most known properties and results, in the more general framework of locally convex vector spaces.

1. Introduction

Let $X$ be a Hausdorff locally convex vector space over the complex field $C$. By calibration for the locally convex space $X$ we understand a family $P$ of seminorms generating the topology $\tau_P$ of $X$, in the sense that this topology is the coarsest with respect to the fact that all the seminorms in $P$ are continuous. Such a family of seminorms was used by the author and Wu [1] and many others in different contexts (see [2–5]).

It is well known that calibration $P$ is characterized by the property that the set

$$E(p, e) = \{x \in X : p(x) < e\}, \quad e > 0, \quad p \in P$$  \hspace{1cm} (1)

is a neighborhood subbase at $0$. Denote by $(X, P)$ the locally convex space $X$ endowed with calibration $P$.

Recall that a locally convex algebra is an algebra with a locally convex topology in which the multiplication is separately continuous. Such an algebra is said to be locally $m$-convex (l.m.c.) if it has a neighborhood base $\mathcal{U}$ at $0$ such that each $U \in \mathcal{U}$ is convex and balanced (i.e., $\lambda U \subseteq U$ for $|\lambda| \leq 1$) and satisfies the property $U^2 \subseteq U$.

Any algebra with identity will be called unital. It is well known that unital locally $m$-convex algebra $\mathcal{A}$ is characterized by the existence of calibration $P$ such that each $p \in P$ is submultiplicative (i.e., $p(xy) \leq p(x) \cdot p(y)$, for all $x, y \in \mathcal{A}$) and satisfies $p(e) = 1$, where $e$ is the unit element.

An element $a$ of locally convex algebra $\mathcal{A}$ is said to be bounded in $\mathcal{A}$ if there exists $\alpha \in C$ such that the set $\{a(x)\}_{x \in \mathcal{A}}$ is bounded in $\mathcal{A}$ (see [6]).

The set of all bounded elements in $\mathcal{A}$ will be denoted by $\mathcal{A}_0$.

Let $C_{\infty} := C \cup \{\infty\}$ be the Alexandroff one-point compactification of $C$. Following Waelbroeck [7, 8], we introduce the following.

Definition 1. We call resolvent set in the Waelbroeck sense of an element $x$ from a locally convex unital algebra $(X, \mathcal{P})$ the set of all elements $\lambda_0 \in C_{\infty}$ for which there exists $V \in \mathcal{V}_{\lambda_0}$ such that the following conditions hold:

(a) the element $\lambda e - x$ is invertible in $X$, for any $\lambda \in V \setminus \{\infty\}$;

(b) the set $\{(\lambda e - x)^{-1} : \lambda \in V \setminus \{\infty\}\}$ is bounded in $(X, \mathcal{P})$.

The resolvent set in Waelbroeck sense of an element $x$ will be denoted by $\rho_W(x)$. The Waelbroeck spectrum of $x$ will be defined as

$$\sigma_W(x) := C_{\infty} \setminus \rho_W(x).$$ \hspace{1cm} (2)
2. q-Bounded Operators

Following Michael [9] (see also [2, 10]), we introduce the following.

**Definition 2.** We say that a linear operator $T : X \rightarrow X$ is $q$-bounded (quotient-bounded) with respect to $\mathcal{P}$ if for any $p \in \mathcal{P}$ there exists $c_p > 0$ such that

$$p(Tx) \leq c_p p(x), \quad \forall x \in X. \quad (3)$$

Denote by $Q_{p\mathcal{P}}(X)$ the set which consists of all $q$-bounded operators with respect to calibration $\mathcal{P}$.

For a seminorm $p \in \mathcal{P}$, the application $\hat{p} : Q_{p\mathcal{P}}(X) \rightarrow \mathbb{R}$ defined as

$$\hat{p}(T) = \inf \{ r > 0 : p(Tx) \leq rp(x), \forall x \in X \} \quad (4)$$

is also a seminorm. Note that

$$\hat{p}(T_1T_2) \leq \hat{p}(T_1)\hat{p}(T_2), \quad T_1, T_2 \in Q_{p\mathcal{P}}(X), \quad p \in \mathcal{P}. \quad (5)$$

We denote by $\hat{\mathcal{P}}$ the family of seminorms $\{\hat{p} : p \in \mathcal{P}\}$. The space $Q_{p\mathcal{P}}(X)$ will be endowed with a topology $\tau_{\hat{\mathcal{P}}}$ generated by $\hat{\mathcal{P}}$. Remark that [9, Proposition 2.4(j)] implies that under this topology $Q_{p\mathcal{P}}(X)$ becomes a Hausdorff locally $m$-convex topological algebra (in the sense of [9, Definition 2.1]).

If $T \in Q_{p\mathcal{P}}(X)$, the $\mathcal{P}$-spectral radius, denoted by $r_{p\mathcal{P}}(T)$, is considered as the boundedness radius in the sense of Allan [6] (see also [11-13]),

$$r_{p\mathcal{P}}(T) = \inf \{ \lambda > 0 : \text{the sequence } (\lambda^{-1}T)^n \text{ is bounded in } Q_{p\mathcal{P}}(X) \}, \quad (6)$$

where, by common consent, $\inf \emptyset := +\infty$.

The set of all bounded elements in $Q_{p\mathcal{P}}(X)$ will be denoted by $(Q_{p\mathcal{P}}(X))_0$ (see [12]). It easily follows from [6, Proposition 2.14(ii)] that

$$(Q_{p\mathcal{P}}(X))_0 = \{ T \in Q_{p\mathcal{P}}(X) : r_{p\mathcal{P}}(T) < \infty \}. \quad (7)$$

For $T \in (Q_{p\mathcal{P}}(X))_0$ we denote by $\rho_{p\mathcal{P}}(T)$ the Waelbroeck resolvent set of $T$ and by $\sigma_{p\mathcal{P}}(T)$ the Waelbroeck spectrum of $T$. The function

$$\rho_{p\mathcal{P}}(T) \ni \lambda \mapsto R(\lambda, T) := (\lambda I - T)^{-1} \in (Q_{p\mathcal{P}}(X))_0 \quad (8)$$

is called the resolvent function of $T$. It is well known that

$$R(\lambda, T) = \sum_{n=0}^{\infty} T^n / \lambda^{n+1}. \quad (9)$$

In this paper we evaluate the behaviour of the power of a $q$-bounded operator from the algebra $(Q_{p\mathcal{P}}(X))_0$ by some type of approximations. The main results have been announced in [14].

3. The Main Results

We continue to employ the notations from the previous sections and we will introduce two types of operatorial approximations for operators from the algebra $(Q_{p\mathcal{P}}(X))_0$ which approximate a given operator $T$ on a convergent power bounded series. The power boundedness problem for operators acting on Banach spaces was largely developed in various frameworks by many authors (see [15-17]).

In the following, using the functional calculus from the $(Q_{p\mathcal{P}}(X))_0$ algebra (see [7, 8]), some important boundedness properties are obtained. Denote $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$. First we have the following.

**Theorem 3.** If $T \in (Q_{p\mathcal{P}}(X))_0$ satisfies

$$\sup_{p \in \mathcal{P}} \hat{p}(T^k) \leq C, \quad (10)$$

for $k \in \mathbb{N}^*$, then

$$\sup_{p \in \mathcal{P}} \hat{p}(R(\lambda, T)^k) \leq \frac{C}{|\lambda| - 1}^k, \quad (11)$$

for $k \in \mathbb{N}^*$ and for all $\lambda \in \mathbb{C}$ with $|\lambda| > 1$.

**Proof.** Assume that $\sup_{p \in \mathcal{P}} \hat{p}(T^k) \leq C$ for $k \in \mathbb{N}^*$. Since

$$R(\lambda, T) = \sum_{j=0}^{\infty} \frac{T^j}{\lambda^j}, \quad (12)$$

for $|\lambda| > 1$, then, by using the generalized binomial formula, we get

$$R(\lambda, T)^k = \lambda^{-k}(I - \frac{T}{\lambda})^k = \frac{1}{\lambda^k} \sum_{j=0}^{\infty} \binom{j+k-1}{j} \frac{T^j}{\lambda^j}. \quad (13)$$

from where we deduce

$$\hat{p}(R(\lambda, T)^k) \leq \frac{C}{|\lambda|^k} \sum_{j=0}^{\infty} \binom{j+k-1}{j} \left(\frac{1}{|\lambda|}\right)^j \leq \frac{C}{|\lambda| - 1}^k, \quad (14)$$

for any $k \in \mathbb{N}^*$ and any $\hat{p} \in \hat{\mathcal{P}}$. Therefore, the conclusion is verified. 

Conversely, we have the following.

**Theorem 4.** If $T \in (Q_{p\mathcal{P}}(X))_0$ and

$$\sup_{p \in \mathcal{P}} \hat{p}(R(\lambda, T)) \leq \frac{C}{|\lambda| - 1}, \quad (15)$$

for all $\lambda \in \mathbb{C}$ with $|\lambda| > 1$, then

$$\sup_{p \in \mathcal{P}} \hat{p}(T^k) \leq Ce(k+1), \quad (16)$$

for $k \in \mathbb{N}^*$. 

Proof. Let us suppose condition \( \tilde{p}(R(\lambda, T)) \leq C/(|\lambda| - 1) \) is true for all \( \tilde{p} \in \tilde{\mathcal{D}} \), for any \( k \in \mathbb{N}^* \) and \( |\lambda| > 1 \). For \( k \in \mathbb{N}^* \) fixed, by choosing the integration path \( \Gamma : |\lambda| = 1 + 1/k \), with the aid of the functional calculus from the algebra \((Q,\rho(X))_0\), we obtain
\[
T^k = \frac{1}{2\pi i} \int_{\Gamma} \lambda^k R(\lambda, T) \, d\lambda.
\]
(17)
Thus, for all \( \tilde{p} \in \tilde{\mathcal{D}} \), we have
\[
\tilde{p}(T^k) \leq \frac{1}{2\pi} \int_{\Gamma} |\lambda|^k \tilde{p}(R(\lambda, T)) \, d\lambda
\leq \frac{1}{2\pi} \max_{|\lambda| \leq 1} |\lambda|^{k} \cdot \max_{|\lambda| > 1} \frac{C}{|\lambda| - 1} \cdot \int_{\Gamma} d\lambda
\leq \frac{1}{2\pi} \left(1 + \frac{1}{k}\right) C k \cdot 2\pi \left(1 + \frac{1}{k}\right) \leq Ce(k + 1)
\]
which implies the desired result.

Moreover, we can formulate the following.

Theorem 5. If \( T \in (Q,\rho(X))_0 \) and
\[
\sup_{\tilde{p} \in \tilde{\mathcal{D}}} \tilde{p}(R(\lambda, T)T^k) \leq \frac{C}{(|\lambda| - 1)^k},
\]
(18)
for \( k \in \mathbb{N}^* \) and for all \( \lambda \in \mathbb{C} \) with \( |\lambda| > 1 \), then
\[
\sup_{\tilde{p} \in \tilde{\mathcal{D}}} \tilde{p}(T^k) \leq C \frac{k!}{k^k} \leq C \sqrt{2\pi}(k + 1), \quad k \in \mathbb{N}^*.
\]
(19)
Proof. Integrating (17) by parts \( j - 1 \) times, for \( j > 2 \), we obtain
\[
T^k = \frac{(-1)^{j-1}}{2\pi i} \int_{\Gamma} \frac{(j + 1)!|\lambda|^{k+j-1}}{(k + 1) \cdots (j + 1)} R(\lambda, T) \, d\lambda.
\]
(20)
Now choosing \( \Gamma \) the circle of radius \( 1 + j/k \) and by using the hypothesis, for \( j \to \infty \), we get
\[
\sup_{\tilde{p} \in \tilde{\mathcal{D}}} \tilde{p}(T^k) \leq C \frac{k!}{k^k} \leq C \sqrt{2\pi}(k + 1).
\]
(21)
The last inequality was obtained by using Stirling’s approximation.

Now, for \( T \in (Q,\rho(X))_0 \) we introduce (see [18]) the following.

Definition 6. The Yosida approximation \( Y(\lambda, T) \) of \( T \), for \( \lambda \in \rho_W(T) \cap \mathbb{C} \), is defined as
\[
Y(\lambda, T) = \lambda TR(\lambda, T).
\]
(22)
Next theorem shows how an operator from the \((Q,\rho(X))_0\) algebra is related to its Yosida approximation.

Theorem 7. The Yosida approximation \( Y(\lambda, T) \) is analytic for \( \lambda \in \rho_W(T) \cap \mathbb{C} \) and the series representation
\[
Y(\lambda, T) = \sum_{j=0}^{\infty} \frac{T^{j+1}}{\lambda^j}
\]
(23)
converges for \( |\lambda| > r_\rho(T) \). Moreover,
\[
(1) \quad Y(\lambda, T) = \lambda^2 R(\lambda, T) - \lambda I;
\]
\[
(2) \quad \text{if there exists } \tilde{p} \in \tilde{\mathcal{D}} \text{ such that } r_\rho(T) < \tilde{p}(T), \text{ then}
\]
\[
\tilde{p}(Y(\lambda, T) - T) \leq \frac{\tilde{p}(T^2)}{|\lambda| - \tilde{p}(T)},
\]
(24)
for \( |\lambda| > \tilde{p}(T) \);
\[
(3) \quad \sigma_W(Y(\lambda, T)) = \{z/(1 - z/\lambda), z \in \sigma_W(T)\}.
\]
Proof. By evaluating \( Y(\lambda, T) \) in terms of the resolvent \( R(\lambda, T) \), for \( |\lambda| > r_\rho(T) \) we obtain
\[
Y(\lambda, T) = \lambda TR(\lambda, T) = \lambda (\lambda I - T)^{-1}
\]
(25)
from where it follows that the assertion of the theorem is true. Moreover,
\[
Y(\lambda, T) = \sum_{j=0}^{\infty} \frac{T^{j+1}}{\alpha^j} - \lambda I = \lambda^2 R(\lambda, T) - \lambda I,
\]
(26)
so (1) is true.

To prove (2) one can observe that, from
\[
Y(\lambda, T) = \sum_{j=0}^{\infty} \frac{T^{j+1}}{\alpha^j},
\]
(27)

it follows that
\[
Y(\lambda, T) - T = \sum_{j=0}^{\infty} \frac{T^2}{\alpha} \left( \frac{T^j}{\alpha^j} \right)
\]
(28)
on a set for which \( |\lambda| > r_\rho(T) \). Moreover,
\[
\tilde{p}(Y(\lambda, T) - T) \leq \sum_{j=0}^{\infty} \frac{\tilde{p}(T^2)}{\alpha} \tilde{p}\left( \frac{T^j}{\alpha^j} \right)
\]
\[
\leq \tilde{p}\left( \frac{T^2}{\alpha} \right)^{\infty} \sum_{j=0}^{\infty} \tilde{p}\left( \frac{T^j}{\alpha^j} \right)
\]
\[
= \tilde{p}\left( \frac{T^2}{\alpha} \right) \frac{1}{1 - \tilde{p}(T/\alpha)} = \frac{\tilde{p}(T^2)}{|\lambda| - \tilde{p}(T)}
\]
(29)
for \( |\lambda| > \tilde{p}(T) > r_\rho(T) \).

A simple reasoning shows that \( R(\lambda, T) \in (Q,\rho(X))_0 \); then it follows \( Y(\lambda, T) \in (Q,\rho(X))_0 \).

From [19, Theorem 3.1.14], for \( |\lambda| > |z| \), we have
\[
\sigma_W(Y(\lambda, T)) = Y(\lambda, \sigma_W(T)),
\]
(30)
for all \( z \in \sigma_W(T) \), and
\[
Y(\lambda, z) = \sum_{j=0}^{\infty} \frac{z^{j+1}}{\lambda^j}
\]
(31)
on \( |\lambda| > |z| \), which could be written as \( Y(\lambda, z) = z/(1 - z/\lambda) \), for any \( z \in \sigma_W(T) \), so (3) is proved.
Below we state an equivalence between a power bounded operator from the \((Q_\varphi(X))_0\) algebra and the power of its Yosida approximation.

**Theorem 8.** Let \(T \in (Q_\varphi(X))_0\) and \(Y(\lambda, T)\) its Yosida approximation. Then the following assertions are equivalent:

(i) \(\sup_{\hat{p} \in \hat{\mathcal{P}}} \hat{p}(T^k) \leq c\), for any \(k \in \mathbb{N}^*\);

(ii) \(\sup_{\hat{p} \in \hat{\mathcal{P}}} \hat{p}(Y(\lambda, T)^k) \leq c/(1 - 1/|\lambda|)^k\), for any \(k \in \mathbb{N}^*\) and for all \(\lambda \in \mathbb{C}\) with \(|\lambda| > 1\).

**Proof.** Property (i) implies \(r_{\mathcal{P}}(T) \leq 1\) so that the argumentation given in the proof of Theorem 7 implies that any \(\lambda \in \mathbb{C}\) with \(|\lambda| > 1\) belongs to the resolvent set of \(T\). Hence, using the generalized binomial formula, we get

\[
Y(\lambda, T)^k = \sum_{j=0}^{\infty} \binom{k+j-1}{j} \frac{T^j \lambda^j}{1 - 1/|\lambda|^k}. \tag{32}
\]

Now, by applying (i) again we obtain

\[
\hat{p}(Y(\lambda, T)^k) \leq c \sum_{j=0}^{\infty} \binom{k+j-1}{j} \frac{(1 - 1/|\lambda|)^j}{(1 - 1/|\lambda|)^k} \tag{33}
\]

for any \(\hat{p} \in \hat{\mathcal{P}}\), whence by passing to supremum, the inequality (ii) holds.

Conversely, (i) is a direct consequence of (ii).

**4. Application**

For \(L > 0\) let \(X := \mathscr{C}[0, L]\) be the space of continuous functions on \([0, L]\) endowed with the norm \(|u|_L := \max_{t \in [0, L]} |u(t)|\).

Consider \(T : X \rightarrow X\), given by

\[
Tu(t) = \int_0^t u(s) \, ds. \tag{41}
\]

Following [19], we see that the resolvent of \(T\) is given by

\[
R(\lambda, T) u(t) = \frac{1}{\lambda} u(t) + \frac{1}{\lambda^2} \int_0^t e^{(t-s)/\lambda} u(s) \, ds, \tag{42}
\]

the Yosida approximation of \(T\) is

\[
Y(\lambda, T) u(t) = \int_0^t e^{(t-s)/\lambda} u(s) \, ds, \tag{43}
\]

and the Möbius approximation of \(T\) is

\[
A(\lambda, T) u(t) = \left(1 - \frac{1}{\lambda}\right)^k Y(\lambda, T)^k, \tag{40}
\]

for \(k \in \mathbb{N}^*\).

A similar result as in Theorem 8 is given below.

**Theorem 11.** Let \(T \in (Q_\varphi(X))_0\) and \(A(\lambda, T)\) its approximation as above. Then the following assertions are equivalent:

(i) \(\sup_{\hat{p} \in \hat{\mathcal{P}}} \hat{p}(T^k) \leq C\), for any \(k \in \mathbb{N}^*\);

(ii) \(\sup_{\hat{p} \in \hat{\mathcal{P}}} \hat{p}(A(\lambda, T)^k) \leq C\), for any \(k \in \mathbb{N}^*\) and for every \(\lambda \in \mathbb{C}\) with \(|\lambda| > 1\).

**Proof.** From Theorem 8, for \(T \in (Q_\varphi(X))_0\),

\[
\sup_{\hat{p} \in \hat{\mathcal{P}}} \hat{p}(T^k) \leq C \tag{38}
\]

is equivalent to

\[
\sup_{\hat{p} \in \hat{\mathcal{P}}} \hat{p}(Y(\lambda, T)^k) \leq \frac{C}{(1 - 1/|\lambda|)^k}. \tag{39}
\]

The conclusion follows taking into account that

\[
A(\lambda, T) = \frac{1 - 1/|\lambda|}{\lambda} Y(\lambda, T)^k, \tag{40}
\]

for \(k \in \mathbb{N}^*\).\hfill \qed
If $L > 1$, then we can introduce for each $\varepsilon > 0$ the following norm on $C[0, L]$:

$$\|u\|_\varepsilon := \max_{t \in [0, L]} e^{t/\varepsilon} |u(t)|, \quad u \in C[0, L].$$

(46)

Then a simple computation gives that

$$\|Tu\|_\varepsilon < \varepsilon \|u\|_\varepsilon, \quad u \in C[0, L].$$

(47)

On the other hand,

$$\|u\|_\varepsilon \leq |u|_L \leq e^{L/\varepsilon} \|u\|_\varepsilon.$$  

(48)

Remark that, by Theorem 11, for all $\lambda > 1$, we get

$$|A(\lambda, T)|_L = (\lambda - 1) \left(e^{T/\lambda} - 1\right) \leq 1$$

(49)

if and only if $|T|_L \leq 1$.

It is clear that for estimating the powers of $T$ it seems to be better to use the Yosida approximation or M"obius approximation than the resolvent approximation.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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