Uniqueness of Gibbs States of a Quantum System on Graphs

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Abstract

Gibbs states of an infinite system of interacting quantum particles are considered. Each particle moves on a compact Riemannian manifold and is attached to a vertex of a graph (one particle per vertex). Two kinds of graphs are studied: (a) a general graph with locally finite degree; (b) a graph with globally bounded degree. In case (a), the uniqueness of Gibbs states is shown under the condition that the interaction potentials are uniformly bounded by a sufficiently small constant. In case (b), the interaction potentials are random. In this case, under a certain condition imposed on the probability distribution of these potentials the almost sure uniqueness of Gibbs states has been shown.

1 Introduction

Let \((M, \sigma)\) be a compact Riemannian manifold with the corresponding normalized Riemannian volume measure \(\sigma\). Let also \(G = (L, E)\) be a (countable) graph with no loops, isolated vertices, and multiple edges. The model we consider in this article describes a system of interacting quantum particles,

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each of which is attached to a vertex $\ell \in L$ (one particle per vertex), its position is described by $q_\ell \in M$. The formal Hamiltonian of the model is

$$H = -\frac{\hbar^2}{2m} \sum_{\ell \in L} \Delta_\ell - \sum_{(\ell, \ell') \in E} v_{\ell \ell'}(q_\ell, q_{\ell'})$$

(1)

where $m$ is the particle mass, $\Delta_\ell$ is the Laplace-Beltrami operator with respect to $q_\ell$, and $v_{\ell \ell'}$ is a symmetric continuous function on $M \times M$. By $(\ell, \ell')$ we denote the edge of the graph which connects the named vertices. The Hamiltonian of the free particle

$$H_\ell = -\frac{\hbar^2}{2m} \Delta_\ell$$

(2)

is a self-adjoint operator in the physical Hilbert space $H_\ell = L^2(M, \sigma)$. It is such that

$$\text{trace } \exp(-\tau H_\ell) < \infty$$

for all $\tau > 0$. If one sets the periodic conditions at the endpoints of the interval $[0, \beta]$, $\beta > 0$ being the inverse temperature, then the semi-group $\exp(-\tau H_\ell)$, $\tau \in [0, \beta]$ defines a $\beta$-periodic Markov process, see [6], called a Brownian bridge, described by the Wiener bridge measure. More on the properties of such operators and measures can be found e.g. in [4].

A complete description of the equilibrium thermodynamic properties of an infinite particle system may be made by constructing its Gibbs states. In the quantum case, such states are defined as positive normal functionals on the algebras of observables satisfying the Kubo-Martin-Schwinger (KMS) conditions, which reflect the consistency between the dynamic and thermodynamic properties of the system proper to the equilibrium. Often, also in the case considered here, the KMS conditions cannot be formulated explicitly, which makes the problem of constructing the Gibbs states as KMS states unsolvable. Since 1975, an approach based on the properties of the semigroups generated by the local Hamiltonians is developed. In this approach, in [1, 2] the Gibbs states of the model (1), with $G$ being a simple cubic lattice $\mathbb{Z}^d$ with the edges connecting nearest neighbors, were constructed in terms of probability measures on path spaces – the so called Euclidean Gibbs measures. So far, this is the only possible way to define Gibbs states of such models and hence to give a mathematical description of their thermodynamic properties. The existence of Euclidean Gibbs measures in a simple manner follows from the compactness of the manifold $M$. In the small mass limit, which can also be treated as the strong quantum
limit, the uniqueness of Euclidean Gibbs measures at all $\beta$ was proven in [2]. The existence and uniqueness of the ground state, i.e., of the Euclidean Gibbs measure at $\beta = +\infty$, was proven in [1], also for small $m$. In both cases $\beta < +\infty$ and $\beta = +\infty$, the results were obtained by means of cluster expansions. The same method yields the proof of the uniqueness for small $\beta$ (high temperature uniqueness), which is equivalent to the case of small

$$\sup_{\ell, \ell'} \sup_{q, q'} \sup_{v_{\ell, \ell'}} |v_{\ell, \ell'}(q_{\ell}, q_{\ell'})|$$

In the cluster expansion method, the properties of the lattice $\mathbb{Z}^d$ are crucial. Thus, it would be of interest to extend the latter result to the case of more general graphs, especially those of globally unbounded degree. Another possible extension would be considering random potentials.

In this article, we present two statements establishing the uniqueness of Gibbs states of the model (1), which occurs due to weak interaction. In the first one (Theorem 2.2), the graph $G$ obeys a condition, by which vertices of large degree should be at large distance of each other. This is the only condition imposed on the graph – we do not suppose that it has globally bounded degree. In the second statement – Theorem 2.3 – we suppose such a boundedness, however the functions $v_{\ell, \ell'}$, $\langle \ell, \ell' \rangle \in E$, now are random. We claim that there exists a family of the probability distributions on the space of these functions such that, for every member of this family, the Gibbs state of the model is almost surely unique. The proof of these statements is based on an extension of the method of [3], where the single-spin spaces were finite. Here we give its brief sketch. A detailed presentation of this proof, as well as a more detailed explanation of the connection between the Gibbs states and Euclidean Gibbs measures of the model (1), will be given in a separate publication.

2 Setup and Results

Since in our study the role of the particle mass and the temperature will be trivial, for notational convenience we set $m = \beta = \hbar = 1$. Throughout the paper, for a topological space $Y$, by $\mathcal{B}(Y)$ we denote the corresponding Borel $\sigma$-field. By saying that $\mu$ is a measure on $Y$, we mean that $\mu$ is a measure on the measure space $(Y, \mathcal{B}(Y))$. For a measurable function $f : Y \to \mathbb{R}$, which is $\mu$-integrable, we write

$$\mu(f) = \int_Y f \, d\mu.$$
Given $\ell \in L$, we denote

$$X_\ell = \{ x_\ell \in C([0,1] \to M) \mid x_\ell(0) = x_\ell(1) \},$$

which is a metric space with the metric

$$\rho(x,y) = \sup_{\tau \in [0,1]} d(x(\tau),y(\tau)),$$

$d$ being the Riemannian metric. This is our single-spin space. The Wiener bridge measure $\chi_\ell$ is defined on $X_\ell$ by its values on the cylinder sets

$$B_{\tau_1, \ldots, \tau_n} = \{ x_\ell \in X_\ell \mid x_\ell(\tau_1) \in B_1, \ldots, x_\ell(\tau_n) \in B_n \},$$

where $B_i \in B(M)$, $i = 1, \ldots, n$, and $\tau_i \in [0,1]$ are such that $\tau_1 < \tau_2 \cdots < \tau_n$. For such a set,

$$\chi_\ell(B_{\tau_1, \ldots, \tau_n}) = \int_{B_1 \times \cdots \times B_n} p_{\tau_2-\tau_1}(\xi_1,\xi_2) \times \cdots \times p_{\tau_n-\tau_{n-1}}(\xi_{n-1},\xi_n)p_{1+\tau_1-\tau_n}(\xi_n,\xi_1)\sigma(d\xi_1) \cdots \sigma(d\xi_n),$$

where $p_\tau(\xi,\eta)$, $\tau > 0$ is the integral kernel of the operator $\exp(-\tau H_\ell)$.

Let $L^\text{fin}$ be the family of all finite non-void subsets of $L$. As usual, for $\Delta \subset L$, we use the notation $\Delta^c = L \setminus \Delta$. For $\Delta \subset L$, the Cartesian product

$$X_\Delta = \prod_{\ell \in \Delta} X_\ell,$$

is equipped with the product topology. Its elements are denoted by $x_\Delta$; we write $X = X_L$ and $x = x_L$. Given $\langle \ell, \ell' \rangle \in E$, let $\Omega_{\ell\ell'}$ be a copy of the Banach space $C(M \times M \to \mathbb{R})$ of continuous symmetric functions equipped with the supremum norm. Then we set

$$\Omega = \prod_{\langle \ell, \ell' \rangle \in E} \Omega_{\ell\ell'},$$

and equip this space with the product topology. Elements of $\Omega$ are denoted by $v$. For $v_{\ell\ell'} \in \Omega_{\ell\ell'}$ and given $x_\ell, x_{\ell'}$, we set

$$V_{\ell\ell'}(x_\ell, x_{\ell'}) = \int_0^1 v_{\ell\ell'}(x_\ell(\tau), x_{\ell'}(\tau))d\tau.$$

Then $V_{\ell\ell'}$ is a bounded continuous symmetric function on $X_\ell \times X_{\ell'}$. Obviously,

$$\|V_{\ell\ell'}\| \leq \|v_{\ell\ell'}\|,$$
where \( \| \cdot \| \) stands for the supremum norm in both cases.

For \( \Lambda \subset L \), we set
\[
\chi_{\Lambda} = \bigotimes_{\ell \in \Lambda} \chi_{\ell},
\]
which is a probability measure on \( X_{\Lambda} \). For such a \( \Lambda \), we also set
\[
E_{\Lambda} = \{ (\ell, \ell') \in E | \ell, \ell' \in \Lambda \},
\]
\[
\partial E_{\Lambda} = \{ (\ell, \ell') \in E | \ell \in \Lambda, \ell' \in \Lambda^c \},
\]
\[
\partial_{\Lambda} \Lambda = \{ \ell' \in \Lambda^c | \exists \ell \in \Lambda : (\ell, \ell') \in E \}.
\]
The latter sets are called the edge and the vertex boundary of \( \Lambda \) respectively.

For \( \Delta \subset \Delta' \), each element of \( X_{\Delta'} \) can be decomposed
\[
x_{\Delta'} = x_{\Delta} \times x_{\Delta' \setminus \Delta}.
\]
In particular, one has \( x = x_{\Delta} \times x_{\Delta^c} \). Given \( \Lambda \in \mathcal{L}_{\text{fin}}, y \in X \), and \( B \in \mathcal{B}(X) \), we set
\[
\pi_{\Lambda}(B|y) = \frac{1}{Z_{\Lambda}(y)} \int_{X_{\Lambda}} \mathbb{I}_B(x_{\Lambda} \times y_{\Lambda^c}) \exp \left[ V_{\Lambda}(x_{\Lambda}|y) \right] \chi_{\Lambda}(dx_{\Lambda}),
\]
where \( \mathbb{I}_B \) is the indicator of \( B \),
\[
V_{\Lambda}(x_{\Lambda}|y) = \sum_{(\ell, \ell') \in E_{\Lambda}} V_{\ell\ell'}(x_{\ell}, x_{\ell'}) + \sum_{(\ell, \ell') \in \partial_{\Lambda} \Lambda} V_{\ell\ell'}(x_{\ell}, y_{\ell'}),
\]
and
\[
Z_{\Lambda}(y) = \int_{X_{\Lambda}} \exp \left[ V_{\Lambda}(x_{\Lambda}|y) \right] \chi_{\Lambda}(dx_{\Lambda}).
\]
Thus, for every \( v \in \Omega \), \( \pi_{\Lambda} \) is a probability kernel from \( (X, \mathcal{B}(X)) \) into itself. The family \( \{ \pi_{\Lambda} \}_{\Lambda \in \mathcal{L}_{\text{fin}}} \) is the local Gibbs specification of the model considered.

**Definition 2.1** A probability measure \( \mu \) on \( X \) is called a Euclidean Gibbs measure of the model (1) if for every \( \Lambda \in \mathcal{L}_{\text{fin}} \) and any \( B \in \mathcal{B}(X) \),
\[
\mu(B) = \int_X \pi_{\Lambda}(B|x)\mu(dx).
\]
The set of all such Euclidean Gibbs measures will be denoted by \( \mathcal{G} \). If necessary, we write \( \mathcal{G}(v) \) to indicate the dependence on the choice of the interaction potentials.

For a topological space \( Y \), by \( \mathcal{P}(Y) \) we denote the set of all probability measures on \( Y \). We equip it with the usual weak topology defined by all bounded continuous functions \( f : Y \to \mathbb{R} \), the set of which will be denoted.
by \(C_b(Y)\). One can show that for every \(f \in C_b(X)\) and any \(\Lambda \in \mathcal{L}_{\text{fin}}\), \(\pi_\Lambda(f|\cdot) \in C_b(X)\); thus, the local Gibbs specification has the Feller property. By the latter property and by the compactness of \(M\), one can show that for every \(x \in X\), the family \(\{\pi_\Lambda(\cdot|x)\}_{\Lambda \in \mathcal{L}_{\text{fin}}} \subset \mathcal{P}(X)\) is relatively compact in the weak topology and that its accumulation points are elements of \(\mathcal{G}\). Hence, the latter set is non-void. For \(v = 0\), the family \(\{\pi_\Lambda\}_{\Lambda \in \mathcal{L}_{\text{fin}}}\) is consistent in the Kolmogorov sense. Then by the Kolmogorov extension theorem, \(\mathcal{G}(0)\) is a singleton. We are going to prove that this property persists if \(v\) is nonzero but \(\|v\|\) is small. Our results cover the following two cases. Given \(\ell \in L\), by \(m_\ell\) we denote the degree of \(\ell\), that is, \(m_\ell = \#\partial E\{\ell\}\). For \(\ell, \ell' \in L\), by \(\rho(\ell, \ell')\) we denote the distance between these vertices – the length of the shortest path connecting \(\ell\) with \(\ell'\). In the first case, the only condition imposed on the graph \(G\) is that for any \(\ell, \ell' \in L\),

\[
\rho(\ell, \ell') \geq \phi(\min\{m_\ell, m_{\ell'}\}),
\]

where \(\phi : \mathbb{N} \to [1, +\infty)\) is a nondecreasing function, such that

\[
\sum_{n=1}^{+\infty} \frac{n}{\phi(n)} < \infty.
\]

Set

\[
\kappa(v) = \sup_{\langle \ell, \ell' \rangle \in E} 16 \left[\exp(4\|v_\ell\ell'\|) - 1\right],
\]

and, for \(\delta > 0\),

\[
\Omega_\delta = \{v \in \Omega \mid \kappa(v) \leq \delta\}.
\]

**Theorem 2.2** There exists \(\delta \in (0, 1)\), depending on the choice of \(\phi\), such that for every \(v \in \Omega_\delta\), the set \(\mathcal{G}(v)\) is a singleton.

Our second result describes the case where the potential \(v \in \Omega\) is random. Let \(Q\) be the family of all probability measures on \(\Omega\) having the product form

\[
Q = \bigotimes_{\langle \ell, \ell' \rangle \in E} Q_{\ell\ell'}, \quad Q_{\ell\ell'} \in \mathcal{P}(\Omega_{\ell\ell'}).
\]

Given \(\lambda > 0\) and \(\theta \in (0, 1)\), we set

\[
Q(\lambda, \theta) = \{Q \in Q \mid \forall \langle \ell, \ell' \rangle \in E : Q_{\ell\ell'}(\|v_\ell\ell'\| > \lambda) < \theta\}.
\]
**Theorem 2.3** Let the graph $G$ be such that

$$\sup_{\ell \in L} m_\ell < \infty.$$  \hfill (21)

Then there exist $\lambda > 0$ and $\theta \in (0, 1)$ such that, for every $Q \in Q(\lambda, \theta)$, the set $G(v)$ is a singleton for $Q$-almost all $v$.

### 3 Comments on the Proof

To prove the above results we will use the method developed in [3], where similar statements were proven for a model of classical spin $s$ with finite single-spin spaces. Thus, the only generalization we need is to extend this method to the case of the single-spin space $X_\ell$ as introduced above.

Given $\Delta \in \mathcal{L}_{\text{fin}}$, we set

$$\nu_{\Delta}(dx_\Delta) = \frac{1}{Z_\Delta} \exp \left[ \sum_{(\ell, \ell') \in E_\Delta} V_{\ell \ell'}(x_\ell, x_{\ell'}) \right] \chi_\Delta(dx_\Delta),$$  \hfill (22)

where $1/Z_\Delta$ is a normalization Gibbs factor. Thus, $\nu_\Delta$ is a probability measure on $X_\Delta$ – the local Euclidean Gibbs measure. For $\Lambda \subset \Delta \subset L$, we set

$$E_\Delta(\Lambda) = \partial \Lambda \cap E_\Delta,$$  \hfill (23)

i.e., $E_\Delta(\Lambda)$ is the smallest subset of $E_\Delta$ along which one can cut out $\Lambda$ from $\Delta$. Then, for $\Lambda \subset \Delta \in \mathcal{L}_{\text{fin}}$, we have

$$\nu_\Delta(dx_\Delta) = \nu_\Delta(dx_{\Delta \setminus \Lambda} | x_\Lambda) \nu_\Delta^\Lambda(dx_\Lambda),$$  \hfill (24)

where $\nu_\Delta^\Lambda$ is the projection of $\nu_\Delta$ onto $B(X_\Lambda)$ and

$$\nu_\Delta(dx_{\Delta \setminus \Lambda} | x_\Lambda) = \frac{1}{N_\Lambda(x_\Lambda)} \exp \left[ \sum_{(\ell, \ell') \in E_{\Delta \setminus \Lambda} \cup \mu_\Lambda} V_{\ell \ell'}(x_\ell, x_{\ell'}) \right] \chi_{\Delta \setminus \Lambda}(dx_{\Delta \setminus \Lambda}),$$  \hfill (25)

$$N_\Lambda(x_\Lambda) = \int_{X_{\Delta \setminus \Lambda}} \exp \left[ \sum_{(\ell, \ell') \in E_{\Delta \setminus \Lambda} \cup \mu_\Lambda} V_{\ell \ell'}(x_\ell, x_{\ell'}) \right] \chi_{\Delta \setminus \Lambda}(dx_{\Delta \setminus \Lambda}).$$

Given $\ell, \ell' \in L$, by $\vartheta(\ell, \ell')$ we denote a path, which connects these vertices. That is, $\vartheta(\ell, \ell')$ is the sequence of vertices $\ell_0, \ldots, \ell_n$, $n \in \mathbb{N}$, such that: (a)
\ell_0 = \ell \text { and } \ell_n = \ell'; (b) for all } j = 0, \ldots, n - 1, \langle \ell_j, \ell_{j+1} \rangle \in E. \text { The path } \vartheta(\ell, \ell') \text { will be called admissible if: (c) } m_{\ell_k} \geq 2 \text { for all } k = 1, \ldots, n - 1; \text { (d) if } \ell_i = \ell_j \text { for certain } i < j, \text { then there exists a } k, i < k < j, \text { such that } m_{\ell_k} \geq m_{\ell_i} = m_{\ell_j}. \text { The length of a path } |\vartheta(\ell, \ell')| \text { is the number of vertices in it, i.e., } |\vartheta(\ell, \ell')| = n.

For } \langle \ell, \ell' \rangle \in E, \text { we set, c.f., (17), }
\zeta_{\ell \ell'} = 16 \left[ \exp \left( 4 \|v_{\ell \ell'}\| \right) - 1 \right]. \tag{26}

For a path } \vartheta(\ell, \ell'), \text { we define
\[ R[\vartheta(\ell, \ell')] = 4^{\varsigma(\ell) + \varsigma(\ell')} \prod_{i=0}^{n-1} \zeta_{\ell \ell'}, \tag{27} \]
where } \varsigma(\ell) = -1 \text { if } m_{\ell} = 1 \text { and } \varsigma(\ell) = 0 \text { if } m_{\ell} \geq 2. \text { Finally, for } \Delta \subset L, \text { we set
}\[ S_\Delta(\ell, \ell') = \sum R[\vartheta(\ell, \ell')], \tag{28} \]
where the summation is performed over all admissible paths connecting } \ell \text { with } \ell'. \text { The main element of the proof of both our theorems is the following, lemma which is an adaptation of Theorem 1 of \[3\] to the model considered here.

**Lemma 3.1** Given } \Lambda \in \mathcal{L}_{\text{fin}}, \text { let } f : X_\Lambda \to \mathbb{R} \text { be a bounded continuous function. Then for every } \Delta \subset \mathcal{L}_{\text{fin}}, \text { such that } \Lambda \subset \Delta, \text { and arbitrary } \ell' \in \Delta \setminus \Lambda, \ x_{\ell'}, x_{\ell'}' \in X_{\ell'}, \text { it follows that
}\[ \left| \frac{\nu(\Delta | x_{\ell'})}{\nu(\Delta | x_{\ell'}')} - 1 \right| \leq \sum_{\ell \in \Lambda} S_\Delta(\ell, \ell'). \tag{29} \]

**Scetch of the proof:** The proof of the lemma is based on the following estimates. Let } (Y, B(Y), P) \text { be a probability space and } a, b, c \text { be positive measurable real valued functions on } Y. \text { Then
}\[ \inf_{y \in Y} \left( \frac{b(y)}{c(y)} \right) \leq \frac{\int a(y) b(y) P(dy)}{\int a(y) c(y) P(dy)} \leq \sup_{y \in Y} \left( \frac{b(y)}{c(y)} \right), \tag{30} \]
\[ \left| \frac{\int a(y) b(y) P(dy)}{\int a(y) c(y) P(dy)} - 1 \right| \leq \sup_{y \in Y} \left( \frac{b(y)}{c(y)} \right) - 1. \tag{31} \]

Now let } a, a' : Y \to (0, + \infty) \text { and this } a \text { obey the conditions } P(a) = P(a') = 1. \text { Set
}\[ S(b, c) = \max \{ \sup_{y \in Y} b(y); \sup_{y \in Y} c(y) \}, \quad I(b, c) = \min \{ \inf_{y \in Y} b(y); \inf_{y \in Y} c(y) \}, \]
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and suppose that $b$ and $c$ are such that $I(b, c) > 0$. Then

$$\left| \frac{\int a(y)b(y)P(dy)}{\int a'(y)c(y)P(dy)} - 1 \right| \leq \frac{S(b, c)}{T(b, c)} - 1. \tag{32}$$

Suppose now that $c$ is such that there exists $y_0 \in Y$ for which $c(y_0) = \inf_{y \in Y} c(y)$. Let positive $\gamma, \epsilon, \delta$ be such that

$$\left| \frac{a(y)}{a'(y)} - 1 \right| \leq \gamma, \quad \left| \frac{c(y)}{c(y_0)} - 1 \right| \leq \epsilon, \quad \left| \frac{b(y)}{c(y)} - 1 \right| \leq \delta.$$

Then

$$\frac{\int a(y)b(y)P(dy)}{\int a'(y)c(y)P(dy)} \leq \delta + \epsilon \gamma + \delta \epsilon \gamma. \tag{33}$$

By means of the estimates (30) – (33) the proof of the lemma follows by the same inductive scheme which was used in the proof of Theorem 1 in \[3\]. \[\Box\]

The proof of both theorems stated above follows from Lemma 3.1 by means of similar arguments the proof of Theorems 2, 3, and 4 in \[3\] was based on.

References

[1] S. Albeverio, Yu. G. Kondratiev, R. A. Minlos, G. V. Shchepan’uk: Ground State Euclidean Gibbs Measures for Quantum Lattice Systems on Compact Manifolds, *Rep. Math. Phys.* 45, 419–429 (2000).

[2] S. Albeverio, Yu. G. Kondratiev, R. A. Minlos, G. V. Shchepan’uk: Uniqueness Problem for Quantum Lattice Systems with Compact Spins, *Lett. Math. Phys.* 52, 185–195 (2000).

[3] L. A. Bassalygo, R. L. Dobrushin: Uniqueness of a Gibbs Field with a Random Potential—an Elementary Approach, (Russian) *Teor. Verojatnost. i Primenen.* 31, 651–670 (1986); English translation: *Theory Probab. Appl.* 31, 572–589 (1987).

[4] B. K. Driver, Heat Kernels Measures and Infinite Dimensional Analysis, In P. Auscher, T. Coulhon, and A. Grigor’yan, editors, *Heat Kernels and Analysis on Manifolds, Graphs, and Metric Spaces*, pages 1–104, Contemp. Math., 338, Amer. Math. Soc., Providence, RI, 2003.

[5] H.-O. Georgii: *Gibbs Measures and Phase Transitions*, Studies in Mathematics, 9, Walter de Gruyter, Berlin New York 1988.
[6] A. Klein and L. Landau, Periodic Gaussian Osterwalder-Schrader Positive Processes and the Two-Sided Markov Property on the Circle. *Pacific J. Math.* **94**, 341–367 (1981).