Conjectures on the ring of commuting matrices

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Abstract

Let $X = (x_{ij})$ and $Y = (y_{ij})$ be generic $n$ by $n$ matrices and $Z = XY - YX$. Let $S = k[x_{11}, \ldots, x_{nn}, y_{11}, \ldots, y_{nn}]$, where $k$ is a field, let $I$ be the ideal generated by the entries of $Z$ and let $R = S/I$. We give a conjecture on the first syzygies of $I$, show how these can be used to give a conjecture on the canonical module of $R$. Using this and the Hilbert series of $I$ we give a conjecture on the Betti numbers of $I$ in the $4 \times 4$ case. We also give some guesses on the structure of the resolution in general.

1 Introduction

Throughout this article we let $R$ be the ring defined in the abstract. It was shown by Motzkin and Taussky [11] that the variety of commuting matrices in $M_n(k)$ is irreducible of dimension $n^2 + n$. Gerstenhaber [5] also showed that the variety is irreducible. From this it follows that $\text{Rad}(I)$ is prime and that the dimension of $R$ is $n^2 + n$.

It has been conjectured that $R$ is Cohen-Macaulay and this has been shown for $n = 3$ in [2] and for $n = 4$ in [7]. It has also been conjectured that $R$ is a domain which follows from the ring being CM (see [12]).

Recently, Knutson [10] proved that the off-diagonal elements in $XY - YX$ form a regular sequence. For the cases $n = 2, 3, 4$ the computer programs MACAULAY [1] and MACAULAY 2 [6] can be used to compute a Gröbner basis (and thus the Hilbert series) of the ideal $I$. The resolution can be computed for $n = 2, 3$ and partially for several $n \geq 4$. For the ideal generated by the off-diagonal elements, which we call $J$, a Gröbner basis can easily be computed for the cases $n = 2, 3$ but we have not yet found a term order that works for $n = 4$. In this article we use MACAULAY and MACAULAY 2 to verify many conjectures.

By exploring simple facts concerning the trace of a matrix we can give "many" first syzygies of the ideal $I$ and in section 2 we give a conjecture on the first Betti numbers. In sections 3 and 4 we use our syzygy conjecture to give a conjecture on the generators of the ideal $(J : I)$ and the canonical module of $S/I$. We then compute a partial resolution of the (conjectured) canonical module in the case $n = 4$ and splice this together with a partial resolution of $I$ to give a conjecture of the Betti numbers of $I$ in that case.

In section 5 we give some guesses (mostly based on computer calculations) on the resolution in general and in section 6 we comment on Knutson’s conjecture concerning the prime ideals of $J$.

2 First syzygies

We restate here a conjecture on the first syzygies that was given in [8].

Write $I = (f_1, \ldots, f_{n^2})$, with $f_1 = Z_{11}$, $f_2 = Z_{21}$, \ldots, $f_{n^2} = Z_{nn}$, where $Z = XY - YX$. A syzygy on $I$ is an $n^2$-tuple $(a_1, \ldots, a_{n^2})$ such that

$$f_1 a_1 + f_2 a_2 + \cdots + f_{n^2} a_{n^2} = 0. \quad (1)$$
This can be rewritten as

\[
\text{tr}\left( \begin{bmatrix}
a_1 & \cdots & a_n \\
a_{n+1} & \cdots & \vdots \\
\vdots & \cdots & \vdots \\
a_{n^2+n-1} & \cdots & a_{n^2} 
\end{bmatrix} \begin{bmatrix}
f_1 & \cdots & f_{n^2-n+1} \\
f_2 & \cdots & \vdots \\
\vdots & \cdots & \vdots \\
f_n & \cdots & f_{n^2} 
\end{bmatrix} \right) = 0
\]

i.e. as

\[
\text{tr}(A(XY - YX)) = 0.
\]

So solving (3) for \(A\) is equivalent to solving (1) for \((a_1, \ldots, a_n)\). We claim the following:

**Conjecture:** the module of first syzygies on \(I\) is generated by the Koszul relations of \(I\) and matrices \(A\) that are polynomials in \(X\) and \(Y\). The highest degree of a first syzygy is \(n - 1\) and in degree \(h\) we get generators of bidegrees \((h, 0), (h - 1, 1), \ldots, (0, h)\) (considering the bidegree \((x\text{-deg}, y\text{-deg})\)).

We do not have a proof of this conjecture but we show below that a number of solutions to (3) exist and compare this with computer calculations.

In general we have \(\text{tr}(BC) = \text{tr}(CB)\) for matrices \(B\) and \(C\) so we get that \(\text{tr}(A(XY - YX)) = 0\) whenever the matrix \(A\) commutes with \(X\) or \(Y\). This gives that any polynomial in \(X\) and any polynomial in \(Y\) is a solution to equation (3). We note the following:

If \(M\) is a monomial in \(X\) and \(Y\) then \(\text{tr}(M(XY - YX)) = 0\) if \(MXY\) can be cyclically permuted into \(MYX\).

and

Let \(M_1\) and \(M_2\) be monomials in \(X\) and \(Y\) such that \(\text{tr}(M_1(XY - YX)) \neq 0\) and \(\text{tr}(M_2(XY - YX)) \neq 0\). If \(B = M_1 + M_2\), then \(\text{tr}(B(XY - YX)) = 0\) if \(M_1XY\) can be cyclically permuted into \(M_2XY\) and \(M_2XY\) can be cyclically permuted into \(M_1XY\).

If \(B = M_1 - M_2\) then \(\text{tr}(B(XY - YX)) = 0\) if \(M_1XY\) can be cyclically permuted into \(M_2XY\) and \(M_1XY\) can be cyclically permuted into \(M_2XY\).

Using the above we can guess a number of solutions:

**Degree 0:** Here we only have one syzygy \(A = E\) (the identity matrix), i.e. the ideal is minimally generated by \(n^2 - 1\) elements.

**Degree 1:** We have \(\text{tr}(X(XY - YX)) = \text{tr}(X^2Y) - \text{tr}(XYX) = 0\) so \(A = X\) is a solution and similarly we get that \(A = Y\) is a solution. The two syzygies we get are obviously independent over \(k\) as they have the bidegrees \((1, 0)\) and \((0, 1)\). In [9] we proved that these are the only ones of degree 1.

**Degree 2:** We see that \(A = X^2\) and \(A = Y^2\) are solutions. The only other monomials in \(X\) and \(Y\) are \(XY\) and \(YX\) and neither of those is a solution. We have

\[
\text{tr}((XY + YX)(XY - YX)) = \text{tr}(XYXY) - \text{tr}(XYXY) + \text{tr}(YXXY) - \text{tr}(YXYX) = \text{tr}(XYXY) - \text{tr}(X^2Y^2) + \text{tr}(X^2Y^2) - \text{tr}(XYXY) = 0
\]

so \(A = XY + YX\) gives a syzygy. We thus have syzygies of bidegrees \((2, 0), (1, 1), (0, 2)\).

**Degree 3:** Here we get at least the monomial solutions \(X^3, Y^3, XYX, YXY\) and the binomial solutions \(X^2Y + YX^2, XY^2 + Y^2X\). Macaulay calculations indicate that it is enough to take one syzygy of each bidegree i.e. \(X^3, Y^3, XYX, YXY\) will do.
Degree 4: $X^4, Y^4, X^3Y + YX^3, Y^3X + XY^3, X^2YX + XYX^2, Y^2XY + YXY^2$ and $XY^2X - YX^2Y$.

Degree 5: $X^5, Y^5, X^2YX^2, Y^2XY^2, X^4Y + YX^4, XYX^2Y + YX^2XY, YXY^2X + XY^2XY$.

The syzygies given above work for any $n$. For $n = 2, 3, 4$ (and partially for $n = 5, 6, 7$) we can compare this with Macaulay calculations.

2.1 $n = 2$

% betti s2
; total:  3  2
; -----------------
; 2:  3  2

Which means that we have 3 generators and the only syzygies we get are the 2 linear ones. The matrices $X$ and $Y$ are $2 \times 2$ matrices so they satisfy a characteristic polynomial of degree 2, i.e. $X^2 - \text{tr}(X)X + \det(X)E = 0$ so the syzygy given by $X^2$ can be written in terms of smaller degree syzygies. Similarly for $Y^2$. For $2 \times 2$ matrices (see e.g. [4]) we have the following identity:

$$YX = (\text{tr}(XY) - \text{tr}(X) \text{tr}(Y))E + \text{tr}(Y)X + \text{tr}(X)Y - XY$$

(4)

So the syzygy that $XY + YX$ gives can be written in terms of lower degree syzygies.

2.2 $n = 3$

We get the following Betti numbers

% betti s3
; total:  8  33
; -----------------
; 2:  8  2
; 3: - 31

As expected we get 2 linear first syzygies. There are 31 first syzygies of degree 2, $\binom{6}{2} = 28$ of those are the trivial syzygies (Koszul relations) and the 3 nontrivial ones correspond to $A = X^2$, $A = Y^2$ and $A = XY + YX$. There are no syzygies of degree 3 so the solutions from before given by $X^3$, $Y^3$, $XYX$, $YXY$, much be linear combinations of the syzygies of smaller degree. The characteristic equation takes care of $X^3$ and $Y^3$ and we get

$$XYX = \frac{1}{2}(x_1 + x_5)(XY + YX) + y_9X^2 + (x_2x_4 - x_1x_5)Y + (x_3y_7 - x_1y_9 + x_6y_8 - x_5y_9)X + cE + \sum aT_a$$

where the $T_a$ are matrices we get from the trivial syzygies. The coefficient of $E$ is

$$c = \det \begin{pmatrix} x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 \\ y_7 & y_8 & y_9 \end{pmatrix}$$
and the coefficients of \(XY + YX, X^2, X\) and \(Y\) are given by traces and determinants of minors of this matrix.

Considering \(n = 4\) and \(n = 5, 6, 7\) (partial computation) we give the conjecture below on the first Betti numbers. We use the notation of Macaulay 2 to display the Betti numbers, i.e. the number in column \(i\) row \(j\) (starting with column 0, row 0) is \(\beta_{i,j}\).

\[
\begin{array}{cccc}
\text{total} & 1 & n^2 - 1 & (\binom{n^2-1}{2}) + (\binom{n+1}{2}) - 1 \\
0 & 1 & \cdot & \cdot \\
1 & \cdot & n^2 - 1 & 2 \\
2 & \cdot & \cdot & (\binom{n^2-1}{2}) + 3 \\
3 & \cdot & \cdot & 4 \\
4 & \cdot & \cdot & 5 \\
\cdot & \cdot & \cdot & \cdot \\
n-1 & \cdot & \cdot & n \\
n & \cdot & \cdot & \cdot \\
\end{array}
\]

Where the \(\binom{n^2-1}{2}\) syzygies of degree 2 are the Koszul relations.

We also conjecture that the \(k\) syzygies of degree \(k-1\) have the following bidegrees (i.e. \((x\text{-deg}, y\text{-deg})\))

\((k-1, 0), (k-2, 1), \ldots, (1, k-2), (0, k-1)\).

### 3 The ideal quotient \((J : I)\)

In the following let \(J\) be the ideal generated by the off-diagonal elements of \(XY - YX\). Knutson [10] has shown that these elements form a regular sequence for any \(n\). It is known [11] that the height of \(I\) is \(n^2 - n\) which is equal to the number of generators of \(J\) so these form a maximal regular sequence in \(I\).

In this section we study the ideal \((J : I)\) and use our conjecture on the first syzygies to give a conjecture on its generators. For the cases \(n = 2, 3\) a Gröbner basis of \(J\) can be computed using Macaulay so we can test the conjecture. We demonstrate first the cases \(n = 3, 4\).

#### 3.1 \(n = 3\)

The nontrivial syzygies on \(I\) are given by \(A \in \{E, X, Y, X^2, Y^2, XY + YX\}\).

The ideal \(I\) is generated by \((f_1, \ldots, f_9)\) where \(f_1, f_5\) and \(f_9\) are from the diagonal of \(XY - YX\) and \(J = (f_2, f_3, f_4, f_6, f_7, f_8)\). Pick 3 different syzygies, \(A, B\) and \(C\). Then

\[
\begin{align*}
& a_1f_1 + a_5f_5 + a_9f_9 = a_2f_2 + a_3f_3 + a_4f_4 + a_6f_6 + a_7f_7 + a_8f_8 \\
& b_1f_1 + b_5f_5 + b_9f_9 = b_2f_2 + b_3f_3 + b_4f_4 + b_6f_6 + b_7f_7 + b_8f_8 \\
& c_1f_1 + c_5f_5 + c_9f_9 = c_2f_2 + c_3f_3 + c_4f_4 + c_6f_6 + c_7f_7 + c_8f_8
\end{align*}
\]

so

\[
\begin{vmatrix}
a_1 & b_1 & c_1 \\
a_5 & b_5 & c_5 \\
a_9 & b_9 & c_9
\end{vmatrix} \cdot f_i \in J \quad \text{for } i = 1, 5, 9.
\]

Direct calculations using MACAULAY give that it suffices to take the generators of \(J\) and the elements given by \((A, B, C) \in \{(E, X, Y), (E, X, X^2), (E, X, Y^2), (E, Y, X^2), (E, Y, X^4)\}\) to get all the generators of \((J : I)\). The bidegrees of these additional generators are \((1, 1), (3, 0), (2, 1), (1, 2)\) and \((0, 3)\).
We can also partially see directly that these suffice, denote by $A_d$ the diagonal of the matrix $A$ and by $I_d$ the diagonal of $E$. Consider the $3 \times 4$ matrix $[1_d, A_d, B_d, C_d]$ and add one row by repeating say the first row. We then have a $4 \times 4$ matrix whose determinant is zero and expanding by the first row we get

$$0 = \det[A_d, B_d, C_d] - A_{11} \det[1_d, A_d, C_d] + B_{11} \det[1_d, A_d, B_d] - C_{11} \det[1_d, A_d, B_d]$$

So we only have to consider triples of the form $(E, A, B)$. There are 10 of these, the ones given above and $(E, X, XY + YX), (E, Y, XY + YX), (E, X^2, Y^2), (E, XY, XY + YX)$ and $(E, Y^2, XY + YX)$. We have

$$\det[1_d, X_d, XY + YX_d] - 2 \det[1_d, X_d, Y_d^2] - 2 \text{tr} X \det[1_d, X_d, Y_d] = x_2 f_2 - x_3 f_3 - x_4 f_4 + x_6 f_6 + x_7 f_7 - x_8 f_8$$

so the triple $(E, X, XY + YX)$ (and for symmetry reasons $(E, Y, XY + YX)$) does not give a new generator. It is probably possible to get similar simple equations explaining why $\det[1_d, X_d^2, Y_d^2]$, $\det[1_d, X_d^2, XY + YX_d]$ and $\det[1_d, Y_d^2, XY + YX_d]$ are not needed as minimal generators of $(J : I)$.

It seems that it suffices to use enough syzygies to get one generator of each bidegree.

### 3.2 $n = 4$

The nontrivial syzygies on $I$ are given by $A = X, Y, X^2, Y^2, XY + YX, X^3, XYX, YXY, Y^3$.

Similarly to the case $n = 3$ we pick $E$ and $3$ more syzygies and get that the determinant of a matrix consisting of the diagonals gives an element in $(J : I)$. Assuming that we get only one element in $(J : I)$ of each bidegree we get the following possibilities (ordered by total degree):

| total degree | triple of syzygies                                      | bidegree of element in $(J : I)$          |
|--------------|--------------------------------------------------------|------------------------------------------|
| 4            | $(X, Y, X^2), (X, Y, XY + YX), (X, Y, Y^2)$             | $(3, 1), (2, 2), (1, 3)$                 |
| 5            | $(X, Y, X^3), (X, Y, XYX), (X, Y, YXY), (X, Y, Y^3)$    | $(4, 1), (3, 2), (2, 3), (1, 4)$         |
| 6            | $(X, X^2, X^3), (Y, X^2, X^3), (X, Y^2, X^3), (X, X^2, Y^3)$ | $(6, 0), (5, 1), (4, 2), (3, 3)$         |
|              | $(Y, X^2, Y^3), (X, Y^2, Y^3), (Y, Y^2, Y^3)$           | $(2, 4), (1, 5), (0, 6)$                 |
| 7            | $(XY + YX, X^2, X^3)$, ...                             | $(6, 1), ...$                           |

We believe that it is enough to take the elements in total degrees 4, 5 and 6. So the highest degree is high enough for us to pick syzygies in the $x$-variables only (and $y$-variables only). This is partially based on comparison of the Hilbert series with a conjectured canonical module (see section 4).

### 3.3 General case

We generalize the idea above for any $n$ and conjecture that $(J : I)$ is generated by the elements of $J$ and elements of the form $u = \det U$ where $U$ is an $n \times n$ matrix whose columns are the diagonals of $E$ and the matrices defining the syzygies. Below we give a conjecture on the degrees of these elements.

We consider the following table of possible bidegrees of syzygies and find the smallest total degree of $u$ we can get from picking $n$ syzygies:

$$
\begin{align*}
(0, 0) \\
(1, 0), (0, 1) \\
(2, 0), (1, 1), (0, 2) \\
(3, 0), (2, 1), (1, 2), (0, 3) \\
\ldots 
\end{align*}
$$
We pick bidegrees from the $k$ first rows where $k = [-\frac{1}{2} + \sqrt{2n + \frac{1}{4}}]$. If $s := n - \frac{k(k+1)}{2} \neq 0$ we pick $s$ bidegrees from row $k+1$ starting from the left. The total bidegree from the first $k$ rows is $(\frac{k}{6}(k^2 - 1), \frac{s}{6}(k^2 - 1)) =: (a, a)$. We have 2 cases:

**Case** $s = 0$: then $n = \frac{k(k+1)}{2}$ and we get exactly $n$ matrices from the first $k$ rows. The smallest possible total degree of $u = \det U$ is $d_{\text{min}} = 2\frac{k}{3}(k^2 - 1) = \frac{k}{3}(k^2 - 1)$ where $k = -\frac{1}{2} + \sqrt{2n + \frac{1}{4}}$. As there is one possibility of picking the $n$ matrices we get one generator of this minimal degree (the values of $n$ where this occurs are for instance $n = 3, 6, 10, \ldots$).

**Case** $s \neq 0$: we pick $s$ bidegrees from row $k+1$ starting from the left. The total bidegree is $(sk - \frac{s(2s-1)}{2}, \frac{s(2s-1)}{2}) := (c - b, b)$. The smallest total degree is then:

$$d_{\text{min}} = \frac{k}{3}(k^2 - 1) + sk$$

Considering the different possibilities of bidegrees of elements of this total degree (picking bidegrees further to the right in the table of bidegrees) we get elements of bidegrees:

$$(a + c - b, a + b), (a + c - b - 1, a + b + 1), \ldots, (a + b, a + c - b)$$

the total number of such elements is $c - 2b + 1 = s(k - s + 1) + 1$.

To get the number of elements in the next smallest degree we count the possibilities of picking $n$ bidegrees and skipping $(1, 0)$ or $(0, 1)$ etc. In each total degree except the largest we get elements of bidegrees of the following form:

$$(r, t), (r - 1, t + 1), \ldots, (r - (r - t), t + (r - t))$$

where $r \neq 0$ and $t \neq 0$.

We believe that the largest total degree we need for a generator of $(J : I)$ is the one where we pick $n$ matrices of bidegrees

$$(0, 0), (1, 0), \ldots, (n - 1, 0)$$

giving us the total degree

$$d_{\text{max}} = 0 + 1 + \cdots + (n - 1) = \frac{n(n - 1)}{2}.$$ 

We get $\frac{n(n - 1)}{2}$ + 1 elements of this total degree, one for each possible bidegree.

In section 5 we will see that this agrees with some guesses we have on the Betti numbers of $I$.

### 4 Canonical module

If $R$ is Cohen-Macaulay (which is known for the cases $n = 2, 3, 4$) then its canonical module is defined as

$$\omega_R := \text{Ext}_R^d(S/I, S)$$

where $d = n^2 - n$ is the height of $I$. Let $J = j_1, \ldots, j_{n^2 - n}$ be the subideal of $I$ consisting of the off-diagonal elements in $XY - YX$. Then we have

$$\text{Ext}_R^d(S/I, S) \cong \text{Ext}_{S/J_1}^d(S/I, S/J_1) \cong \cdots \cdots \cong \text{Hom}_{S/J}(S/I, S/J) \cong (J : I)/J$$

In the previous section we gave a conjecture on the generators of $(J : I)$. For $n = 4$ we compute $(J : I)$ using the conjecture and partially resolve $(J : I)/J$ using MACAULAY. We get the following Betti numbers:
This gives us the Betti numbers of the tail of the resolution of \( I \) (see e.g. cor. 3.3.9 in [3]). So we can compare this with the Hilbert series of \( S/I \):

\[
h_{S/I}(t) = (1 - 15t^2 + 2t^3 + 108t^4 - 26t^5 - 562t^6 + 466t^7 + 1613t^8 - 2742t^9 - 1078t^{10} + 5994t^{11} - 4367t^{12} - 2262t^{13} + 5630t^{14} - 3650t^{15} + 818t^{16} + 166t^{17} - 103t^{18} + 4t^{19} + 3t^{20})/(1 - t)^{32}
\]

We see that our conjecture fits with the (last 6) coefficients of the polynomial in the numerator. Partly computing the resolution of \( I \) we get the Betti numbers:

\[
o_{18} = \text{total: } 1 \ 16 \ 115 \ 595 \ 2127 \ 2791 \ 848 \ 60 \ 5
\]

Splicing together these 2 Betti tables and using the Hilbert series we get we get the following conjecture on the Betti numbers:

\[
\begin{align*}
total & : 1 \ 15 \ 115 \ 595 \ 2127 \ 4713 \ 6902+ \ 4432+ \ 5710+ \ 3821 \ 1170 \ 200 \ 14 \\
0: & 1 \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \ . \\
1: & 15 \ 2 \ . \ . \ . \ . \ . \ . \ . \ . \\
2: & . \ . \ . \ . \ . \\
3: & 4 \ . \ . \ . \\
4: & . \ . \ 1658 \ . \ . \\
5: & . \ . \ 1922 \ . \ . \ . \\
6: & . \ . \ . \ . \ . \\
7: & . \ . \ . \ . \ . \\
8: & . \ . \ . \ . \ . \\
\end{align*}
\]

where \(-d+c=-2262\) (from the Hilbert series). The boldfaced numbers are the 2 earlier macaulay computations and the others are based on the Hilbert series.

5 Resolution

Computer calculations indicate that there might be a (non-trivial) multiplicative structure on the resolution. The first \( n-1 \) lines in the Betti table can be interpreted as products of the generators and the first syzygies. There is also an interesting "multiplicative pattern" on the top "staircase" of the Betti numbers, i.e. these seem to be products of the 2 linear first syzygies and the generators of the ideal. Consider below the Betti numbers for \( n = 3 \) and partial the Betti numbers for \( n = 4, 5, 6 \):
We seem to get $2(n^2 - 1)$ quadratic second syzygies, $3(n^2 - 1)$ quadratic fourth syzygies etc. We get 2 linear first syzygies, 3 linear third syzygies, 4 linear fifth syzygies etc. In the fourth line of the table for $n = 5$ we have $2096 = (24_n^1) + 3 \cdot 24$ and $558 = 2 \cdot 279$. We interpret the first 4 lines of this table as products, let $f_1, \ldots, f_{24}$ be generators of the ideal, $g_1, g_2$ the first linear syzygies, $h_1, h_2, h_3$ the first syzygies (that are not the Koszul relations) of degree 2 and $k_1, k_2, k_3, k_4$ the first syzygies of degree 3, then the partial Betti table may be interpreted as follows:

So up to a certain row (probably row $n - 1$) the generators and the first syzygies seem to generate everything (and the “multiplication” is nonzero). Our conjecture is that we have the following Betti numbers for a general $n$:

$$
\begin{array}{cccccccc}
\text{hd 1} & \text{hd 2} & \text{hd 3} & \text{hd 4} & \text{hd 5} & \text{hd 6} & \text{hd 7} & \text{hd 8} & \ldots & n^2 - n - 1 & n^2 - n \\
\hline
1: & n^4 - 1 & 2 & - & - & - & - & - & \ldots & - & - \\
2: & - & (n^2 - 1) + 3 & 2(n^2 - 1) & 3 & - & - & - & \ldots & - & - \\
3: & - & 4 & p & p & 3(n^2 - 1) & 4 & - & \ldots & - & - \\
4: & - & 5 & p & p & p & p & p & \ldots & - & - \\
\vdots & - & - & - & - & - & - & - & \ldots & - & - \\
n - 2 & - & n - 1 & p & p & p & p & p & \ldots & - & - \\
n - 1 & - & n & ? & ? & ? & ? & ? & \ldots & - & - \\
\vdots & - & - & - & - & - & - & - & \ldots & - & - \\
\frac{n(n-1)}{2} & - & - & \ldots & \ldots & \ldots & \ldots & \ldots & \frac{n(n-1)}{2} & (n^2 - 1) & \frac{n(n-1)}{2} + 1 \\
M: & - & - & \ldots & \ldots & \ldots & \ldots & \ldots & - & - & \ldots & \ldots \\
M + 1: & - & - & \ldots & \ldots & \ldots & \ldots & \ldots & - & - & \ldots & \ldots \\
\end{array}
$$

where $p$ means products of earlier entries and $M$ is determined by $d_{\min}$ from the conjecture on the canonical module i.e. $M = n(n - 1) - d_{\min}$. The numbers $s$ and $k$ are defined in the section on $(J : I)$. 

8
6 Minimal primes of $J$

It has been conjectured that $I$ is a prime ideal and this can be deduced from $I$ being Cohen-Macaulay \[12\] which we know for $n = 2$, $n = 3$ and $n = 4$ so at least in those cases $I$ is a minimal prime of $J$. Knutson \[10\] gives a conjecture that $J$ has one other minimal prime which is generated by determinants of matrices whose columns are the diagonals of powers of $X$ and $Y$. These determinants are elements of $(J : I)$ but, if our conjecture is true, do not generate it for all $n$. Our conjecture is that the other prime ideal is given by $(J : I)$ which coincides with Knutson’s equations for $n = 3$. For $n = 4$ we get by picking the syzygies $(E, X, Y, XY + YX)$ an element of bidegree $(2, 2)$ that is a zero-divisor on $J$ and that can not, for bidegree reasons, be created by determinants coming from the diagonals of the powers of $X$ and $Y$.

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