Connections and dynamical trajectories
in generalised Newton-Cartan gravity I.

An intrinsic view

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Abstract

The “metric” structure of nonrelativistic spacetimes consists of a one-form (the absolute clock) whose kernel is endowed with a positive-definite metric. Contrarily to the relativistic case, the metric structure and the torsion do not determine a unique Galilean (i.e. compatible) connection. This subtlety is intimately related to the fact that the timelike part of the torsion is proportional to the exterior derivative of the absolute clock. When the latter is not closed, metric-compatibility and torsionfreeness are thus mutually exclusive. We explore the two corresponding alternative roads. Firstly, we define a torsionfree connection conformally related to a Newtonian connection when the absolute spaces foliate the spacetime. The corresponding geodesics are interpreted as dynamical trajectories and we argue that the corresponding Lagrangian can be thought as the natural extension of the above metric structure. Secondly, we propose a notion of torsional Newtonian connection which include the torsional Newton-Cartan geometry, recently investigated in the literature, as a particular case. In both cases, we identify the necessary data which allows to uniquely fix these connections. In a companion paper, the relativistic ambient origin of both notions will be presented.
# Contents

1 Introduction 1

2 Nonrelativistic metric structures 7
   2.1 Relativistic structures .......................... 7
   2.2 Nonrelativistic structures .......................... 9
   2.3 Observers ........................................ 10
   2.4 Absolute time and spaces .......................... 12
   2.5 Milne boosts .................................... 15

3 Torsionfree connections with a closed absolute clock 20
   3.1 Galilean manifolds ................................ 20
   3.2 Newtonian manifolds ................................. 28
   3.3 Variational approach ................................ 29
   3.4 Towards the ambient formalism ..................... 35

4 Torsionfree connections with a twistless absolute clock 38
   4.1 Conformally Galilean manifolds ..................... 38
   4.2 Variational approach ................................ 43

5 Torsional Galilean connections 47

6 Conclusion 55

A Curvature of a torsionfree Galilean manifold 57

B Technical proofs 62
1 Introduction

As advocated by Élie Cartan after the birth of Einstein’s theory, the geometrisation of gravity induced by the equivalence principle is by no means restricted to General Relativity [1] (cf. also [2]). In this light, Einstein’s and Newton’s theories of gravity both admit geometrical formulations which are, in particular, diffeomorphism invariant. Since the sixties, the corresponding Newton-Cartan geometry has known a revival of interest among relativists and geometers (cf. e.g. [3, 4, 5, 6, 7, 8, 9] for early contributions) but it is only recently that Newton-Cartan geometry – possibly with torsion – has been intensively applied to condensed matter problems\(^1\) (such as the quantum and thermal Hall effects [12, 13] as well as superfluids [14]) for which it proved a very efficient tool to construct effective field theories or for computing Ward identities.

As celebrated in the famous quote\(^2\) of Wheeler, there are two facets of the interaction between the geometry of spacetime and the motion of matter. We will focus on the “kinematical” facet, \textit{i.e.} the motion of test particles in a fixed gravitational background and will ignore the “dynamical” facet, \textit{i.e.} gravitational field equations. In this restricted case, the equivalence and relativity principles strongly prescribe the geometric structures the spacetime is endowed with. On the one hand, the equivalence principle imply that dynamical trajectories of free falling observers are geodesics of a suitable connection, the latter providing a notion of parallelism on the spacetime manifold. Furthermore, such unparameterised geodesics define a \textit{projective structure} on spacetime. On the other hand, the relativity principle\(^3\) further dictates the underlying structure group (Lorentzian \textit{vs} Galilean\(^4\)) of the reduced frame bundle. The corresponding invariant tensor(s) define a \textit{metric structure} on spacetime. An important issue is the interplay between these two structures. More precisely, one should answer the following question: What are the ingredients (\textit{e.g.}\n
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\(^1\)Among the early applications of Newton-Cartan geometry to condensed matter systems is the pioneering work [10] on superfluid dynamics. More recently, the related concept of “nonrelativistic general covariance” was applied to the unitary Fermi gas [11].

\(^2\)“Space[time] tells matter how to move. Matter tells space[time] how to curve.”

\(^3\)As emphasised by many authors (\textit{e.g.} [15]), the so-called “nonrelativistic” theories also embody the principle of relativity, the only actual (but decisive) difference between Special and Galilean relativity being the expression of (Lorentz \textit{vs} Galilei) boosts. Although the terminology “nonrelativistic” is rather unfortunate, we will use it following common practice.

\(^4\)An exhaustive enumeration of homogeneous kinematical groups [16] must also include the (homogeneous) Carroll group (\textit{cf.} [15, 17]).
the torsion) one must add to the metric structure in order to fix uniquely the connection? Providing precise answers to this question (sometimes referred to as the “equivalence problem” in the mathematics literature) for some generalisations of Newton-Cartan geometry is the main subject of this paper.

In (pseudo)-Riemannian geometry, the answer is well known and provides a clear relation between the various elements constituting the kinematical content of general relativity which can be summarised in the following diagram:

Let us briefly make some comments in order to present the logic that will be generalised in the less familiar nonrelativistic case. On top of the triangle sits the metric structure of general relativity: a Lorentzian metric, i.e. a field of nondegenerate bilinear forms on the spacetime manifold. This metric structure uniquely determines a compatible torsionfree connection known as the Levi-Civita connection (Arrow 1). This connection provides the spacetime manifold with a notion of parallelism, thus allowing the definition of a distinguished class of curves: the geodesics (Arrow 2). A geodesic is thus defined as an autoparallel curve with respect to Levi-Civita’s parallelism, i.e. the tangent vector stays parallel to itself along a geodesic. Alternatively, the geodesics can be characterised as curves extremising locally the Lorentzian distance. As a result, the geodesic equation can be obtained as the equation of motion derived from a Lagrangian density built in terms of the metric structure (Arrow 3).

The relations between these different structures can be abstractly summed up in a commutative diagram which will be our leitmotiv:

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Interestingly, the kinematical content of Newton-Cartan gravity can be equally described via a similar diagram, with the important difference that the basic non-relativistic analogue of the metric structure consists in a degenerate contravariant metric $h^\mu\nu$ (a collection of absolute rulers) whose radical is spanned by a nowhere vanishing 1-form $\psi_\mu \neq 0$ (an absolute clock): $h^\mu\nu \psi_\nu = 0$. Connections compatible with such a structure are called Galilean. Two features of nonrelativistic compatible connections $\Gamma^\lambda_{\mu\nu}$ are notably distinct from the relativistic case.

Firstly, the torsion of a Galilean connection obeys a compatibility condition: its timelike part is proportional to the exterior derivative of the absolute clock ($\nabla_\mu \psi_\nu = 0 \Rightarrow \Gamma^\lambda_{[\mu\nu]} \psi_\lambda = \partial_{[\mu} \psi_{\nu]}$). In particular, torsionfree ($\Gamma^\lambda_{[\mu\nu]} = 0$) Galilean connections are only defined for closed absolute clocks ($\partial_{[\mu} \psi_{\nu]} = 0$). Such absolute clocks are synchronised in the sense that they define a notion of absolute time $t$ (locally, $\psi_\mu = \partial_\mu t$). The simultaneity leaves ($t = \text{constant}$) foliate spacetime.

Secondly, the uniqueness of the torsionfree compatible connection is lost. This arbitrariness has a natural physical interpretation: the above “metric” structure is too weak to determine the motion of particles. Indeed, motions can be measured via absolute clocks and rulers, but are not constrained by them. In Newtonian mechanics, the spacetime is a mere container and one should prescribe force fields to determine motion.

The diagram 2 suggests to define a richer metric structure (dubbed here Lagrangian structure) allowing to restore the uniqueness of the torsionfree compatible connection (Arrow 1). In Newton-Cartan gravity, the absolute clock and rulers are in fact supplemented by the gift of a gravitational potential. This Lagrangian structure defines a unique Newtonian connection:

![Figure 3: Kinematical content of Newton-Cartan theory](image)

A Newtonian connection endows the spacetime with a notion of parallelism (different from the Levi-Civita one) allowing in turn the definition of self-parallel curves, similarly to the relativistic case. Such curves acquire the interpretation of dynami-
cal trajectories (Arrow 2) for Lagrangians which are of degree two in the velocities\(^5\): they can be derived from an action principle built in terms of the Lagrangian structure (Arrow 3). In a sense, the nonrelativistic analogue of the Lorentzian distance between two events is in fact the value of the action \(\int L \, dt\), which is a sort of “Lagrangian distance”.

When the absolute clock is not closed, metric-compatibility and torsionfreeness are mutually exclusive. Therefore two alternatives open up: consider either non-Galilean or torsional connections. We start by investigating the first option when the absolute clock is twistless, \(i.e.\) obeys to the Frobenius integrability condition \(\psi_{\mu} \partial_{\nu} \psi_{\rho} = 0\). In such case, the time units vary for each clock and the measured time \(\tau\) will depend on the observers. Nevertheless, spacetime is still foliated by simultaneity slices and a notion of absolute time can be defined. We will show that one can also define a Lagrangian structure in the case of a twistless absolute clock associated with the action principle \(\int L \, d\tau\) making use of the measured time \(\tau\) instead of the absolute one \(t\). Furthermore, the latter Lagrangian structure is conformally related to one for a closed absolute clock. Correspondingly, we generalise the diagram 3 by defining a torsionfree (non Galilean) connection which is uniquely determined by the Lagrangian structure (Arrow 1) and conformally related to a Newtonian connection (in a sense made precise below) whose geodesics describe dynamical trajectories (Arrow 2) extremising the corresponding action principle (Arrow 3).

Secondly, we explore the alternative route by considering generalisations of Newton-Cartan gravity characterised by torsional connections which have known a recent surge of interest regarding applications in the geometrisation of the fractional quantum Hall effect [13] as well as in the context of Lifshitz holography [19]. In such approaches, the torsion is tuned in order to ensure compatibility with the absolute clock. Of particular mathematical interest for us are the works [20, 21] which exhibit a torsional connection compatible with the metric structure while remaining invariant under local Galilean boosts (called Milne boosts). We extend these tor-

\(^5\)This class is natural in Newtonian mechanical systems with holonomic constraints. Recall that a dynamical system with Euclidean coordinates \(x^1, \ldots, x^{d+n}\) is said holonomic if its constraints can be put in the form \(f^\alpha (x^1, \ldots, x^{d+n}, t) = 0\), with \(\alpha \in \{1, \ldots, n\}\) and \(n\) the number of independent constraints. The constraints of an holonomic system whose kinetic energy takes the standard form \(T = \frac{1}{2} \delta_{ab} \dot{x}^a \dot{x}^b\) (with \(a, b = 1, \ldots, d+n\)) can always be solved. Such a system is therefore equivalent to an unconstrained system with Lagrangian of the form \(L = \frac{1}{2} \gamma_{ij}(q) \dot{q}^i \dot{q}^j + A_i(q) \dot{q}^i - U(q)\) (with \(i, j = 1, \ldots, d\)). The corresponding class of Hamiltonians (of degree two in the momenta) are nowadays called “natural Hamiltonians”, cf. [18].
sional Newton-Cartan geometries and make use of Lagrangian structures\(^6\) in order to identify the necessary data which allows to uniquely fix these connections.

**Outline**

The plan of the paper is as follows:

In Section 2, we review various geometric structures of nonrelativistic spacetimes. After a brief reminder of standard definitions and properties regarding relativistic structures, we switch to the investigation of nonrelativistic ones by emphasising their points of divergence with their relativistic counterparts. We focus on a nonrelativistic metric structure (called *Leibnizian* structure) defined as a manifold endowed with a degenerate contravariant metric whose radical is spanned by the absolute clock. The role played by fields of observers in nonrelativistic physics is discussed at length as well as related objects. We then discuss two restrictions that can be imposed on the absolute clock, namely closure (*Augustinian* structure) or the Frobenius criterion (*Aristotelian* structure).

In Section 3, we discuss the possibility of endowing nonrelativistic metric structures with a notion of parallelism, in the guise of a connection. We first focus on torsionfree connections compatible with the underlying metric structure, thus restricting the scope of the analysis to Augustinian structures. We thus review the notions of torsionfree *Galilean* and *Newtonian* connections, with particular attention given to the equivalence problem (*i.e.* the search for structures that uniquely determine a given compatible connection). Apart from the standard characterisation of Newtonian connections in terms of equivalence classes of field of observers and gauge 1-forms, this motivation will lead us to review the less standard solution of the equivalence problem making use of a *Lagrangian* structure. The latter can be thought of as the proper nonrelativistic analogue of the (pseudo)-Riemannian metric structure in that it determines uniquely the compatible torsionfree connection.

In Section 4, we propose a generalisation of torsionfree Galilean (resp. Newtonian) connections to the case of Aristotelian structures, inspired by the commutative diagram 3, and call them conformally Galilean (resp. Newtonian) connections. They

\(^6\)Note however that the action principle for the geodesic equation becomes unclear whenever torsion is involved, so that we will not consider the third arrow of diagram 2 in this case.
are torsionfree, hence they are compatible with the underlying metric structure only when the absolute clock is closed in which case they reduce to usual torsionfree Galilean (resp. Newtonian) connections. More generally, we show that they are conformally related (in a sense that we precise) to the latter connections and define the same projective structure.

In Section 5, we discuss an extension of torsional Newton-Cartan geometry and identify the necessary data which allows to uniquely fix these connections. This section can be read independently from Section 4, but the prerequisites from Section 3 are assumed.

The Section 6 is our conclusion where we briefly summarise our main results and announce some future ones. In a forthcoming paper, we will show how the generalisations of Newton-Cartan geometry we have discussed can be obtained as null dimensional reductions of suitable Lorentzian geometries.

Two appendices close the paper. A detailed discussion of the equivalences between the Trautman and Duval-Künzle conditions is provided in Appendix A while several technical proofs have been relegated to Appendix B.

**Notations**

Let $V$ be a vector space and $v, w \in V$ two vectors. We will denote by $v \vee w = \frac{1}{2} (v \otimes w + w \otimes v)$ (respectively $v \wedge w = \frac{1}{2} (v \otimes w - w \otimes v)$) the (anti)symmetric product, and similarly for higher products. The (anti)symmetrisation of indices is performed with weight one and is denoted by round (respectively, square) brackets, e.g. $\Phi_{(\mu \nu)} \equiv \frac{1}{2} (\Phi_{\mu \nu} + \Phi_{\nu \mu})$ and $\Phi_{[\mu \nu]} \equiv \frac{1}{2} (\Phi_{\mu \nu} - \Phi_{\nu \mu})$.

The spacetime manifold will be written $\mathcal{M}$ and is of dimension $d + 1$. Let $\mathcal{V}$ be a vector bundle over $\mathcal{M}$ with typical fibre the vector space $V$. By $\Gamma(\mathcal{V})$, we will denote the space of its sections, i.e. globally defined $V$-valued fields on $\mathcal{M}$. For instance, $\Gamma(\wedge^p T^* \mathcal{M}) = \Omega^p(\mathcal{M})$ is the space of $p$-forms on $\mathcal{M}$. 
2 Nonrelativistic metric structures

We start by reviewing some standard material about relativistic structures in order to draw comparison with nonrelativistic ones and fix some terminology.

2.1 Relativistic structures

Definition 2.1 (Riemannian structure). A Riemannian structure designates a manifold endowed with a positive-definite metric.

Although this definition restricts to the case of signature $(+, \ldots, +)$, a similar one can be given in the (pseudo)-Riemannian case:

Definition 2.2 (Lorentzian structure). A Lorentzian structure consists in a manifold endowed with a nondegenerate metric of signature $(-, +, \ldots, +)$.

These structures are therefore characterised by a metric structure but, as such, are not endowed with a notion of parallel transport. This supplementary notion of parallelism can be implemented under the features of a Koszul connection\(^7\) compatible with the metric structure. We are thus led to define:

Definition 2.3 (Riemannian/Lorentzian manifold). A Riemannian (Lorentzian) manifold consists in a Riemannian (Lorentzian) structure supplemented with a metric-compatible Koszul connection on the tangent bundle.

We will retain this terminology in the sequel and use the word “structure” in order to designate a manifold endowed with a metric-like structure while keeping the term “manifold” for cases where a Koszul connection is added. However, in the present case the distinction drawn here is only relevant when the Koszul connection has torsion due to the following well-known theorem:

Theorem 2.4 (Space of metric compatible connections). Let $g$ be a (pseudo)-Riemannian metric on a manifold $\mathcal{M}$. The set of Koszul connections compatible with $g$ forms a group isomorphic to the space of sections of the tangent bundle $\Gamma(T\mathcal{M})$.

\(^7\)We will prefer the denomination “Koszul connection” to the more widespread designations of “affine connection” or “covariant derivative” in order to avoid confusion with the slightly different meanings of these terms in some of the mathematical literature. For the sake of completeness, let us remind that a Koszul connection on a vector bundle $E$ over $\mathcal{M}$ is a $C^\infty(\mathcal{M})$-linear map $\nabla : \Gamma(T\mathcal{M}) \to \text{End}(\Gamma(E))$ such that, for any vector field $X \in \Gamma(T\mathcal{M})$, the endomorphism $\nabla_X$ on the space $\Gamma(E)$ of sections obeys to the Leibniz rule: $\nabla_X (f \sigma) = X[f] \sigma + f \nabla_X \sigma$ for any function $f \in C^\infty(\mathcal{M})$ and section $\sigma \in \Gamma(E)$. If the vector bundle is unspecified, it will be implicitly assumed to be the tangent bundle: $E = T\mathcal{M}$.  

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with the metric $g$ forms a vector space isomorphic to the vector space of torsion tensors $T \in \Gamma (\wedge^2 T^*M \otimes T.M)$.

Recall that, given a Koszul connection $\nabla$, the associated torsion tensor $T \in \Gamma (\wedge^2 T^*M \otimes T.M)$ is defined by its action on vector fields $X, Y \in \Gamma (T.M)$ as

$$T (X, Y) = \nabla_X Y - \nabla_Y X - [X,Y].$$

(2.1)

Theorem 2.4 is a consequence of Cartan’s lemma and can be restated in our terminology by saying that a given Riemannian/Lorentzian structure supplemented with a vector-field-valued 2-form (the torsion) defines uniquely a Riemannian/Lorentzian manifold. Furthermore, given a particular metric structure, there is no restriction on the possible torsion which can span the whole vector space of vector-field-valued 2-forms. As we will see, these features are characteristic of the relativistic structures and will be one of the main points of discrepancy with nonrelativistic structures.

An immediate corollary of Theorem 2.4 is the standard result:

**Corollary 2.5** (Fundamental Theorem of (pseudo)-Riemannian geometry). There is a unique torsionfree Koszul connection compatible with a given (pseudo)-Riemannian metric.

This torsionfree Koszul connection is called the Levi-Civita connection and plays the role of the zero of the vector space corresponding to all possible Koszul connections compatible with the metric structure.\(^8\) In local coordinates, if one writes $\nabla_\mu Y^\lambda = \partial_\mu Y^\lambda + \Gamma^\lambda_{\mu \nu} Y^\nu$, then the components $\Gamma^\lambda_{\mu \nu}$ defining the Levi-Civita connection are equal to the usual Christoffel symbols:

$$\Gamma^\lambda_{\mu \nu} = \frac{1}{2} g^{\lambda \rho} \left( \partial_\mu g_{\rho \nu} + \partial_\nu g_{\rho \mu} - \partial_\rho g_{\mu \nu} \right).$$

(2.2)

We stress that Corollary 2.5 involves no restriction on the metric structure, so that any Riemannian/Lorentzian structure induces a unique torsionfree Koszul connection. As we will see, this property is lost when one deals with degenerate metric

---

\(^8\)This should be distinguished from the case where no metric structure is involved: in that case, the space of all Koszul connections on a manifold possesses a structure of affine space, with associated vector space the space of 2-covariant, 1-contravariant tensor fields $\Gamma (\wedge^2 T^*M \otimes T.M)$. This translates the well-known fact that the difference between two Koszul connections on the same manifold is a tensor (an element of the vector space $\Gamma (\wedge^2 T^*M \otimes T.M)$) although a Koszul connection is not.
structures. We conclude this brief review of relativistic structures by mentioning a special class of bases of the tangent space:

**Definition 2.6 (Lorentzian basis).** Let $(\mathcal{M}, g)$ be a $(d + 1)$-dimensional Lorentzian structure with nondegenerate covariant metric $g$. A Lorentzian basis of the tangent space $T_x\mathcal{M}$ at a point $x \in \mathcal{M}$ is an ordered basis $B_x = \{e_0|_x, \ldots, e_d|_x\}$ which is orthonormal with respect to $g_x$.

The basis vectors $e_a|_x \in T_x\mathcal{M}$, with $a \in \{0, \ldots, d\}$ thus satisfy the condition $g_x(e_a|_x, e_b|_x) = \eta_{ab}$, with $\eta_{ab}$ the Minkowski metric. The denomination Lorentzian is justified by the fact that at each point $x \in \mathcal{M}$, the group of endomorphisms of $T_x\mathcal{M}$ mapping each Lorentzian basis into another one is isomorphic to the Lorentz group $O(d, 1)$.

### 2.2 Nonrelativistic structures

As mentioned in the introduction, a distinguishing feature of nonrelativistic space-times is the existence of a degenerate metric $^9$ structure $[1, 2]$, in the guise of a contravariant degenerate metric (absolute rulers) whose radical is spanned by a given 1-form (absolute clock), which must be separately specified. More precisely, one defines:

**Definition 2.7 (Absolute clock $[5, 6]$).** An absolute clock $\psi$ on a manifold $\mathcal{M}$ is a nowhere vanishing 1-form $\psi \in \Omega^1(\mathcal{M})$.

An absolute clock allows to distinguish between *timelike* tangent vectors $X_x \in T_x\mathcal{M}$ for which $\psi_x(X_x) \neq 0$ from *spacelike* tangent vectors $Y_x \in T_x\mathcal{M}$ satisfying $\psi_x(Y_x) = 0$. The distribution $\text{Ker } \psi$ is the vector subbundle of $T\mathcal{M}$ spanned by spacelike vectors.

**Definition 2.8 (Absolute rulers $[5, 6]$).** A collection of absolute rulers on a manifold $\mathcal{M}$ endowed with an absolute clock $\psi$ is a positive semi-definite contravariant metric $h \in \Gamma(\vee^2 T\mathcal{M})$ on $\mathcal{M}$ whose radical is spanned by the absolute clock i.e.

$$\text{Rad } h = \text{Span } \psi.$$  \hfill (2.3)

---

$^9$Throughout this work, the term “metric” will be used in a slightly broader sense than the customary one in the physics literature. Namely, we will employ the term to designate a covariant or contravariant symmetric bilinear form being either degenerate or nondegenerate.
Alternatively, a collection of absolute rulers can be defined as a field $\gamma \in \Gamma (\sqrt{2} \left( \text{Ker } \psi \right)^*)$ on $\mathcal{M}$ of positive-definite quadratic forms acting on spacelike vectors.

These two definitions can be shown to be equivalent. In components, the condition (2.3) reads $h^{\mu\nu} \psi_\nu = 0$. Armed with these notions of clocks and rulers, we can now define the nonrelativistic analogue of a Riemannian structure as:

**Definition 2.9** (Leibnizian structure [5, 6, 22]). A Leibnizian structure consists of a triplet composed by the following elements:

- a manifold $\mathcal{M}$
- an absolute clock $\psi$
- a collection of absolute rulers $h$ (or, equivalently, $\gamma$)

Such a Leibnizian structure will be interchangeably denoted $\mathcal{L}(\mathcal{M}, \psi, h)$ or $\mathcal{L}(\mathcal{M}, \psi, \gamma)$.

As mentioned previously, Leibnizian structures are purely “metric” structures and as such, do not involve a notion of parallelism. Before addressing nonrelativistic connections, we must digress a little on the role played by observers in nonrelativistic physics. This discussion will justify the introduction of two refinements of Leibnizian structures, namely Aristotelian and Augustinian structures.

### 2.3 Observers

A map $\lambda : I \subseteq \mathbb{R} \to \mathcal{N}$ from a subset $I \subseteq \mathbb{R}$ of the real line into a manifold $\mathcal{N}$ will be called a parameterised curve on $\mathcal{N}$, while a 1-dimensional submanifold $\mathcal{C}$ of $\mathcal{N}$ will be called an unparameterised curve on $\mathcal{N}$. A parameterised curve $\lambda : I \to \mathcal{C}$ on an unparameterised curve $\mathcal{C}$ will be called a parameterisation of $\mathcal{C}$ when $\lambda$ is invertible. If the unparameterised curve $\mathcal{C}$ on $\mathcal{N}$ is defined by the embedding\(^{10}\) $i : \mathcal{C} \hookrightarrow \mathcal{N}$ then $n \equiv i \circ \lambda$ is called the corresponding parameterised curve on $\mathcal{N}$.

In the following, we let $\mathcal{L}(\mathcal{M}, \psi, \gamma)$ be a Leibnizian structure. We start by defining the notion of (nonrelativistic) observer and its vector field generalisation:

\(^{10}\)In this paper, an embedding will be defined in the weak sense: an injective immersion. Therefore, strictly speaking a submanifold is here an immersed submanifold.
Definition 2.10 (Observer [6]). A (nonrelativistic) observer is a timelike parameterised curve \( n : I \subseteq \mathbb{R} \rightarrow \mathcal{M} \) : \( s \mapsto n(s) \) normalised such that the tangent vector \( N_{n(s)} \in T_{n(s)}\mathcal{M} \) (defined\(^{11}\) as \( N_{n(s)} \equiv n_s D_s \)) satisfies:

\[
\psi_{n(s)}(N_{n(s)}) = 1, \quad \forall s \in I. \quad (2.4)
\]

The parameter \( s \) will soon acquire the interpretation of (nonrelativistic) proper time of the observer \( n \) (cf. Proposition 2.13). In local coordinates, the observer \( n \) is a timelike curve \( x^\mu(s) \) with parameterisation chosen such that \( \psi_\mu \frac{dx^\mu}{ds} = 1 \). This notion can be generalised to define vector fields whose integral curves are observers:

Definition 2.11 (Field of observers [6]). A field of (nonrelativistic) observers is a vector field \( N \in \Gamma(T\mathcal{M}) \) such that \( \psi(N) = 1 \). The space of all fields of observers on \( \mathcal{M} \) is denoted \( \text{FO} (\mathcal{M}) \).

Definition 2.12 (Proper time [22]). Let \( \mathcal{C} \) be a timelike unparameterised curve on \( \mathcal{M} \) defined by the embedding \( i : \mathcal{C} \hookrightarrow \mathcal{M} \). We will call (nonrelativistic) proper time any function \( \tau \in C^\infty (\mathcal{C}) \) satisfying \( d\tau = i^*\psi \).

The fact that the submanifold \( \mathcal{C} \) is of dimension 1 ensures that the 1-form \( i^*\psi \) is closed, so that locally there always exists a function \( \tau \) such that \( d\tau = i^*\psi \). Obviously, this condition only defines the proper time up to a constant. The parameter \( s \) in Definition 2.10 is closely related to the proper time \( \tau \) of the unparameterised curve associated with an observer:

Proposition 2.13. Let \( \mathcal{C} \) be an unparameterised curve on \( \mathcal{M} \) defined by the embedding \( i : \mathcal{C} \hookrightarrow \mathcal{M} \). Let \( \tau \in C^\infty (\mathcal{C}) \) be a proper time on \( \mathcal{C} \).

The parameterised curve \( n = i \circ \lambda \) defined by the parameterisation \( \lambda : I \subseteq \mathbb{R} \rightarrow \mathcal{C} : s \mapsto \lambda(s) \) is an observer if and only if

\[
\tau \circ \lambda(s) = s + a, \quad \forall s \in I \quad (2.5)
\]

with \( a \in \mathbb{R} \) a constant.

The proof is a straightforward application of the previous definitions.

\(^{11}\)The vector \( D_s \in T\mathbb{R}_s \) is defined by its action on functions \( f \in C^\infty (\mathbb{R}) \) as \( D_s[f] = \frac{df}{dt}\bigg|_{t=s} \).
The proper time on an unparameterised curve is defined up to a constant thus, without loss of generality one may assume \( a = 0 \). In such case, the parameterisation \( \lambda \) is the inverse function of the proper time \( \tau \), so that it is natural to identify the parameter \( s \) with the value \( \tau \) of the proper time at the corresponding point on the curve.

**Definition 2.14** (Spacelike projection of vector fields [5]). Let \( N \in FO(\mathcal{M}) \) be a field of observers. The field of endomorphisms \( P^N : \Gamma (T\mathcal{M}) \to \text{Ker} \psi \) defined as

\[
P^N(X) = X - \psi(X)N
\]

where \( X \) is any vector field, is called a spacelike projector of vector fields.

The transpose of a spacelike projector can be defined as the field of linear maps\(^{12}\) \( \bar{P}^N : \Omega^1(\mathcal{M}) \to \text{Ann} N \) defined as \( \bar{P}^N(\alpha) = \alpha - \alpha(N)\psi \), with \( \alpha \in \Omega^1(\mathcal{M}) \). In components, these two spacelike projectors read as:

\[
P_{\mu\nu} = \delta_{\mu\nu} - N_{\mu}\psi_{\nu} = \bar{P}_{\nu\mu}.
\]

### 2.4 Absolute time and spaces

As such, a Leibnizian structure does not allow generically a global definition of absolute time and space since it only provides a set of local clocks and rulers. This drawback can be circumvented by restricting the class of absolute clocks. The suitable restriction comes in two versions, a weak one and a strong one. Denoting \( \mathcal{D} \) the distribution of spacelike hyperplanes \( \mathcal{D}_x \equiv \text{Ker} \psi_x \) (\( \forall x \in \mathcal{M} \)), the weak version consists in imposing that the distribution \( \mathcal{D} \) is involutive. One is then led to define what we called an Aristotelian structure\(^{13}\) as:

**Definition 2.15** (Aristotelian structure [23]). An Aristotelian structure is a Leibnizian structure whose absolute clock induces an involutive distribution, i.e. satisfies the Frobenius integrability condition: \( \psi \wedge d\psi = 0 \).

This supplementary condition ensures, by Frobenius Theorem, that the kernel of \( \psi \) defines a foliation of \( \mathcal{M} \) by a family of hypersurfaces of codimension one called absolute spaces. The absolute time has a fixed value on each absolute space. Therefore,

\(^{12}\)At each point \( x \in \mathcal{M} \), \( \text{Ann} N_x \) stands for the annihilator of \( \text{Span} N_x \) in \( T^*_x\mathcal{M} \) and \( \text{Ann} N \)

\(^{13}\)In the terminology of [22], it would be called a Leibnizian structure with locally synchronizable absolute clock.
absolute spaces can be identified with simultaneity slices $t = \text{const}$. These are the maximal integral submanifolds of $\mathcal{D}$, so that the tangent space $T_x \mathcal{M}$ at each point $x$ of the simultaneity slice is isomorphic to $\text{Ker } \psi_x$. Locally, the 1-form $\psi$ can be written as $\psi = \Omega \, dt$ where $\Omega \in C^\infty (\mathcal{M})$ is a positive function called \textit{time unit} and the function $t \in C^\infty (\mathcal{M})$ will be referred to as the \textit{absolute time}. In contradistinction with $\mathcal{M}$, absolute spaces are Riemannian manifolds since they are endowed with the positive-definite metric $\gamma$.

$$\text{Ker } \psi$$

\begin{figure}[h]
\centering
\includegraphics{foliation.png}
\caption{Foliation of an Aristotelian structure by absolute spaces.}
\end{figure}

Now, let $\mathcal{C}$ be an unparameterised curve on $\mathcal{M}$ defined by the embedding $i : \mathcal{C} \hookrightarrow \mathcal{M}$. The local condition $\psi = \Omega \, dt$ allows to write $d\tau = i^* \psi = (i^* \Omega) \, d(i^* t)$, where $\tau \in C^\infty (\mathcal{M})$ is a proper time on $\mathcal{C}$ while $i^* \Omega = \Omega \circ i$ and $i^* t = t \circ i$ are the pullbacks on $\mathcal{C}$ of the time unit and absolute time, respectively. Integrating the pullback of the absolute clock on a curve joining the events $A$ and $B$, one finds the proper time interval $\tau_{A \rightarrow B} = \int_A^B i^* \psi$. Any observer on an Aristotelian structure can make use of the time unit $\Omega$ in order to compare or “synchronise” its proper time $\tau$ with the absolute time $t$.

The situation regarding synchronisation is even clearer when considering the more restrictive case in which the absolute clock is a closed 1-form. We thus define an \textit{Augustinian structure} as:

\begin{definition}[Augustinian structure] An Augustinian structure is a Leibnizian structure whose absolute clock $\psi$ is closed.
\end{definition}

\footnote{We chose to refer to Augustine of Hippo (also known as “Saint Augustine”) in order to pay tribute to the role he played regarding the philosophy of time, \textit{cf.} Book X of his \textit{Confessions.}}
This stronger condition allows locally to write $\psi = dt$, so that any observer of an Augustinian structure is automatically synchronised\(^\text{15}\) with the absolute time ($\tau = i^*t = t \circ i$). Consequently, if the spacetime is simply connected then two observers sharing the same endpoints $A, B \in \mathcal{M}$ will agree when comparing the proper time passed when going from $A$ to $B$, since the integral

$$\tau_{A \rightarrow B} = \int_A^B i^*\psi = \int_A^B d(t \circ i) = t(i(B)) - t(i(A))$$

does not depend on the path followed.

**Example 2.17** (Aristotle spacetime). The most simple example of a Leibnizian structure is given by a $(d + 1)$-dimensional Aristotle spacetime characterised by a closed absolute clock and flat absolute spaces:

$$\begin{cases}
\psi = dt \\
\gamma = \delta_{ij} dx^i \lor dx^j
\end{cases}$$

where $i, j \in \{1, \ldots, d\}$ and $\delta_{ij}$ the Kronecker delta. Equivalently, one may consider the following contravariant metric: $h = \delta^{ij} \frac{\partial}{\partial x^i} \lor \frac{\partial}{\partial x^j}$ (cf. [24]).

In the Aristotle spacetime, time is absolute and space is Euclidean. Obviously, this spacetime was the only arena where physical events were conceived to take place before the breakthroughs of non-Euclidean geometry in the 19th century and special relativity in the 20th century.

The hierarchy

$$\text{Augustinian} \subset \text{Aristotelian} \subset \text{Leibnizian}$$

of the three types of nonrelativistic “metric” structures introduced so far is summarised in Table 1.

\(^{15}\)Indeed, in the terminology of [22] it would be called a Leibnizian structure with *proper time locally synchronizable* absolute clock.
| Nonrelativistic structure | Absolute clock |
|---------------------------|----------------|
| Leibnizian                | Arbitrary $\psi$|
| Aristotelian              | Frobenius $\psi \wedge d\psi = 0$|
| Augustinian               | Closed $d\psi = 0$|

Table 1: Absolute clocks of nonrelativistic structures

2.5 Milne boosts

Consider an Augustinian structure (locally, $\psi = dt$). One may introduce an *adapted* coordinate system $x^\mu = (t, x^i)$ where the first coordinate is the absolute time and $x^i$ are coordinates on the absolute spaces. In this coordinate system, a field of observers decomposes as $N = \frac{\partial}{\partial t} + v^i \frac{\partial}{\partial x^i}$. The integral curves of $N$ are such that $v^i = \frac{dx^i}{dt}$. By analogy with the proper velocity spacetime vector, a field of observers is then sometimes called a "velocity vector" (e.g. [12]). For an Aristotelian structure ($\psi = \Omega \, dt$), the analogous expression reads $N = \frac{1}{\Omega} \left( \frac{\partial}{\partial t} + v^i \frac{\partial}{\partial x^i} \right) = \frac{1}{\Omega} \frac{\partial}{\partial t} + \tilde{v}^i \frac{\partial}{\partial x^i}$ and its integral curves are such that $\tilde{v}^i = \frac{1}{\Omega} v^i = \frac{dx^i}{d\tau}$ where $\tau$ is the proper time.

Let us turn back to the general case of a Leibnizian structure. Given two fields of observers $N'$ and $N$, their difference $v = N' - N$ is a spatial vector field, i.e. it belongs to the kernel of the absolute clock, $\psi(v) = 0$. Therefore, the difference $v \in \Gamma(\text{Ker } \psi)$ is not a field of observer (rather, it can be thought as the relative spacelike velocity between two fields of observers). This observation prevents the space $\text{FO}(\mathcal{M})$ of all fields of observers $\text{FO}(\mathcal{M})$ from being a vector space. However, $\text{FO}(\mathcal{M})$ possesses a natural structure of affine space [22] with associated vector space $\Gamma(\text{Ker } \psi)$.

**Definition 2.18** (Milne boost [10, 25, 26]). *Given two fields of observers $N$ and $N' \in \text{FO}(\mathcal{M})$, there exists a spacelike vector field $v \in \Gamma(\text{Ker } \psi)$ such that $N' = N + v$. The fields of observers $N$ and $N'$ are said to be related by a Milne boost parameterised by the spacelike vector field $v$.***

15
Since any element $X \in \text{Ker } \psi$ can be expressed using the contravariant metric $h$ as $X \equiv h(\chi)$ for some 1-form $\chi \in \Omega^1(\mathcal{M})$, the next Proposition follows straightforwardly:

**Proposition 2.19** (Milne boost [10, 25, 26]). Let $N$ and $N' \in \text{FO}(\mathcal{M})$ be two fields of observers on $\mathcal{M}$. Then there exists a 1-form $\chi \in \Omega^1(\mathcal{M})$ such that $N' = N + h(\chi)$. The fields of observers $N$ and $N'$ are said to be related by a Milne boost parameterised by the 1-form $\chi$.

In components, the previous Proposition can be restated, assuming $\psi^\mu N^\mu = 1$ as

$$\psi^\mu N'^\mu = 1 \iff \exists \chi^\nu \text{ such that } N'^\mu = N^\mu + h^{\mu\nu} \chi^\nu.$$  

In the case of some adapted coordinates for an Aristotelian structure, one has $N' - N = v^i \frac{\partial}{\partial x^i}$ where $v^i = h^{ij} \chi_j$. The 1-form $\chi$ only appears through the combination $h(\chi)$ and thus can be taken spacelike, i.e. $\chi$ may be everywhere replaced by $\bar{P}_N(\chi)$ for some arbitrary $N$. Milne boosts are sometimes referred to as “local Galilean boosts”, denomination that will be justified in Proposition 2.23.

Fields of observers are bestowed upon a greater importance in nonrelativistic physics in comparison with the relativistic case, since a great deal of structures can only be defined by making use of a particular $N$ (thus in a non-canonical way). Indeed, since the contravariant metric $h$ of a Leibnizian structure is degenerate, there is no natural covariant metric defined on the whole tangent bundle $T\mathcal{M}$ (remember that the absolute rulers $\gamma$ are only defined on $\text{Ker } \psi$). However, the gift of a field of observers $N$ allows to uniquely define a (degenerate) covariant bilinear form $\gamma^N$ transverse to $N$ as:

**Definition 2.20** (Transverse metric). Let $\mathcal{L}(\mathcal{M}, \psi, \gamma)$ be a Leibnizian structure and $N \in \text{FO}(\mathcal{M})$ a field of observers on $\mathcal{M}$. The transverse metric $\gamma^N \in \Gamma(\vee^2 T^*\mathcal{M})$ is defined by its action on vector fields $X, Y \in \Gamma(TM)$ as

$$\gamma^N(X, Y) = \gamma\left(P_N(X), P_N(Y)\right)$$  

where $\gamma \in \Gamma(\vee^2 (\text{Ker } \psi)^*)$ is the collection of absolute rulers and $P_N$ stands for the spacelike projector associated to the field of observers $N$.  

16
The right-hand side of eq.(2.7) is well-defined since the image of a spacelike projector lies in \( \text{Ker} \psi \). The epithet “transverse” is justified by the fact that \( N \in \text{Rad} \gamma \), i.e. \( \frac{\partial}{\partial t} (X, N) = 0 \) \( \forall X \in \Gamma (T\mathcal{M}) \). Furthermore, it is easy to show that the contraction of \( N \) with the contravariant metric \( h \) satisfies the relation: \( h \left( \frac{\partial}{\partial t} (X, N) \right) = P^N (X) \), \( \forall X \in \Gamma (T\mathcal{M}) \). In components, we thus have the two relations:

\[
\begin{align*}
\frac{\partial}{\partial t} N^\mu & = 0 \\
\frac{\partial}{\partial t} h^{\mu \nu} & = \delta^\mu_\nu - N^\mu \psi^\nu.
\end{align*}
\]

(2.8)

In fact, these two conditions completely determine \( \frac{\partial}{\partial t} \gamma \), as expressed by:

**Proposition 2.21** (cf. e.g. [6]). Let \( \mathcal{L} (\mathcal{M}, \psi, \gamma) \) be a Leibnizian structure and \( N \in \text{FO} (\mathcal{M}) \) a field of observers on \( \mathcal{M} \). There is a unique covariant bilinear form \( \frac{\partial}{\partial t} \gamma \in \Gamma (\sqrt{2} T^* \mathcal{M}) \) satisfying the conditions (2.8).

As suggested by the superscript, the covariant metric \( \gamma_x \) depends on the choice of field of observers \( N \). More precisely, it can be shown that under a change of field of observers \( N \rightarrow N' \) via the Milne boost parameterised by the 1-form \( \chi \in \Omega^1 (\mathcal{M}) \), the covariant metric \( \frac{\partial}{\partial t} \gamma \) varies as

\[
\frac{\partial}{\partial t} \gamma_x \rightarrow \frac{\partial}{\partial t} \gamma_x' = \left( 2 \chi (N) + h (\chi, \chi) \right) \psi_x \psi_x - 2 \chi (\mu \nu) \psi_x.
\]

(2.9)

A nonrelativistic avatar of a Lorentzian basis (cf. Definition 2.6) can be formulated:

**Definition 2.22** (Galilean basis [6]). Let \( \mathcal{L} (\mathcal{M}, \psi, \gamma) \) be a Leibnizian structure. A Galilean basis of the tangent space \( T_x \mathcal{M} \) at a point \( x \in \mathcal{M} \) is an ordered basis \( B_x = \{ N_x, e_1|_x, \ldots, e_d|_x \} \) with \( N_x \) the tangent vector of an observer and \( \{ e_1|_x, \ldots, e_d|_x \} \) a basis of \( \text{Ker} \psi_x \) which is orthonormal with respect to \( \gamma_x \).

Explicitly, the basis \( B_x = \{ N_x, e_1|_x, \ldots, e_d|_x \} \) must satisfy the conditions:

1. \( \psi_x (N_x) = 1 \)
2. \( \psi_x (e_i|_x) = 0 \), \( \forall i \in \{1, \ldots, d\} \)
3. \( \gamma_x (e_i|_x, e_j|_x) = \delta_{ij} \), \( \forall i, j \in \{1, \ldots, d\} \).
A basis of $T^*_x\mathcal{M}$ dual to $B_x = \{N_x, e_i|x\}$ is given by $B^*_x \equiv \{\psi_x, \theta^i_x\}$, where the $d$ one-forms $\theta^i_x$ satisfy the requirement: $\theta^i_x(e_j|x) = \delta^i_j$.

The reference to Galilei in Definition 2.22 is justified by the following Proposition:

**Proposition 2.23 (cf. e.g. [22]).** At each point $x \in \mathcal{M}$, the set of endomorphisms of $T_x\mathcal{M}$ mapping each Galilean basis into another one forms a group isomorphic to the homogeneous Galilei group.

We detail the proof since it clarifies the interpretation of Milne boosts as local Galilean boosts.

**Proof:** Let us denote by $T : T_x\mathcal{M} \to T_x\mathcal{M}$ one of the endomorphisms considered. Since $T$ maps bases into bases, it must be a vector space isomorphism so that it can be represented by an element of $GL(T_x\mathcal{M})$ as the invertible matrix

$$T \equiv \begin{pmatrix} a & b \\ c^T & R \end{pmatrix} \quad (2.10)$$

where $a \in \mathbb{R}$, $b, c \in \mathbb{R}^d$ and $R \in GL(\mathbb{R}^d)$. Let $B_x = \{N_x, e_{ix}\}$ be a Galilean basis of $T_x\mathcal{M}$, the basis $T(B_x) = \{N'_x, e'_{ix}\}$ reads (dropping the index $x$ for notational simplicity):

$$T \begin{pmatrix} N \\ e_i \end{pmatrix} = \begin{pmatrix} a & b \\ c^T & R \end{pmatrix} \begin{pmatrix} N \\ e_i \end{pmatrix} = \begin{pmatrix} aN + b'e_i \\ c^T_j N + R^j_i e_i \end{pmatrix}. \quad (2.11)$$

Requiring that $T(B_x)$ is a Galilean basis (Conditions 1-3 following Definition 2.22) imposes that $T$ satisfy:

1. $\psi_x(N'_x) = 1 \Rightarrow a = 1$
2. $\psi_x(e'_{ix}) = 0, \forall i \in \{1, \ldots, d\} \Rightarrow c^T_j = 0$
3. $\gamma_x(e'_{ix}, e'_{jx}) = \delta_{ij}, \forall i, j \in \{1, \ldots, d\} \Rightarrow R \in O(d)$. 

18
The set of matrices representing the set of isomorphisms $T$ is thus of the form

$$T = \begin{pmatrix} 1 & b \\ 0 & R \end{pmatrix}$$  \hspace{1cm} (2.12)$$

with $b \in \mathbb{R}^d$ and $R \in O(d)$. This set of matrices form a subgroup of $GL(T_x\mathcal{M})$ isomorphic to the homogeneous Galilei group Gal$_0$. The homogeneous Galilei group therefore acts regularly on the space of Galilean basis via the group action:

$$\{ N, e_i \} \mapsto \{ N + b^i e_i, R^j_i e_j \}.$$  \hspace{1cm} (2.13)

Proposition 2.23 together with Definition 2.22 can be generalised in a straightforward way from the tangent space at a point of $\mathcal{M}$ to the tangent bundle of $\mathcal{M}$. A Galilean basis of $T\mathcal{M}$ is thus defined as the ordered set of fields $B = \{ N, e_1, \ldots, e_n \}$ with $N$ a field of observers and $\{ e_1, \ldots, e_n \}$ a basis of Ker $\psi$, orthonormal with respect to $\gamma$. Two Galilean bases $\{ N', e'_i \}$ and $\{ N, e_i \}$ are mapped via a local transformation where $R : \mathcal{M} \rightarrow O(d)$ parameterises a local rotation and $b^i : \mathcal{M} \rightarrow \mathbb{R}^d$ a local Galilean boost. Explicitly, one has:

$$\begin{cases} 
N' = N + b^i e_i \\
e'_i = R^j_i e_j 
\end{cases}$$  \hspace{1cm} (2.14)$$

where the first expression is a Milne boost (cf. Proposition 2.19).
3 Torsionfree connections with a closed absolute clock

We now switch to the definition of nonrelativistic manifolds, i.e. Leibnizian structures endowed with a compatible Koszul connection and discuss some of the peculiarities arising, which are absent in the relativistic case.

3.1 Galilean manifolds

It should first be noted that the compatibility condition with the metric-like structure must apply to both the absolute rulers and clock. One then defines:

Definition 3.1 (Galilean manifold [6]). A Leibnizian structure supplemented with a Koszul connection compatible with the absolute clock and rulers is called a Galilean manifold. The Koszul connection is then referred to as a Galilean connection.

If we let \( \mathcal{L}(\mathcal{M}, \psi, h) \) be a Leibnizian structure, the compatibility conditions read

1. \( \nabla \psi = 0 \)
2. \( \nabla h = 0 \).

These two conditions can be more explicitly stated as:

1. \( X[\psi(Y)] = \psi(\nabla_X Y), \forall X, Y \in \Gamma(T\mathcal{M}) \)
2. \( X[h(Y, Z)] = h(\nabla_X Y, Z) + h(Y, \nabla_X Z), \forall X, Y, Z \in \Gamma(T\mathcal{M}) \).

When the absolute rulers are formulated in terms of a field \( \gamma \in \Gamma(\vee^2 (\text{Ker } \psi)^*) \), the second condition can be restated as \( \nabla \gamma = 0 \) or equivalently:

\[ X[\gamma(V, W)] = \gamma(\nabla_X V, W) + \gamma(V, \nabla_X W), \forall X \in \Gamma(T\mathcal{M}) \text{ and } \forall V, W \in \text{Ker } \psi. \]

The right-hand-side of the previous equation is well defined as \( Y \in \text{Ker } \psi \) implies \( \psi(\nabla_X Y) = 0 \) (cf. Condition 1.) which in turn, ensures that \( \nabla_X Y \in \text{Ker } \psi, \forall X \in \Gamma(T\mathcal{M}) \).

In components, these two sets of equivalent conditions read:

\[
\begin{align*}
\nabla_\mu \psi_\nu = 0 & \iff \nabla_\mu \psi_\nu = 0 \\
\nabla_\mu h^{\alpha\beta} = 0 & \iff \nabla_\mu \gamma_{\alpha\beta} = 0
\end{align*}
\]
One should note that the compatibility requirement of a Galilean connection with the absolute clock \((\nabla \psi = 0)\) ensures that \(\psi\) is a conservor, in the following sense:

**Definition 3.2** (Conservor [9]). Let \(\nabla\) be a Koszul derivative on \(\mathcal{M}\) and \(\psi \in \Omega^1 (\mathcal{M})\) be a 1-form field. The 1-form \(\psi\) is said to be a conservor if it satisfies

\[
(\nabla_X \psi)(Y) + (\nabla_Y \psi)(X) = 0, \quad \forall X, Y \in \Gamma (T\mathcal{M}).
\]  

(3.16)

The conservor condition takes the form of a Killing equation in the sense that, in components, it reads as \(\nabla_{(\mu} \psi_{\nu)} = 0\). The term “conservor” follows from the Proposition:

**Proposition 3.3** (cf. [9]). Let \(\mathcal{L} (\mathcal{M}, \psi, \gamma)\) be a Leibnizian structure endowed with the Koszul connection \(\nabla\) and let \(X \in \Gamma (T\mathcal{M})\) be a vector field which is affine geodesic with respect to \(\nabla\), i.e. \(\nabla_X X = 0\). If \(\psi\) is a conservor for \(\nabla\), then the quantity \(\psi (X)\) is conserved along the integral curves of \(X\).

**Proof:** Starting with the equality \(X [\psi (X)] = (\nabla_X \psi) (X) + \psi (\nabla_X X)\), the first term on the right-hand side vanishes due to the conservor condition while the second term is null since \(X\) is affine geodesic. Therefore \(X [\psi (X)] = 0\), so that the quantity \(\psi (X)\) is conserved along the integral curves of \(X\). \(\square\)

**Corollary 3.4** (cf. [9]). Let \(\mathcal{L} (\mathcal{M}, \psi, \gamma)\) be a Leibnizian structure endowed with the Koszul connection \(\nabla\) and assume that the absolute clock \(\psi\) is a conservor for \(\nabla\). Then any geodesic whose tangent vector lies in the absolute space \(\Sigma_t\) at one point stays entirely in the hypersurface \(\Sigma_t\).

**Proof:** Let \(x : I \subseteq \mathbb{R} \rightarrow \mathcal{M} : s \mapsto x(s)\) be a parameterised curve which is integral for the vector field \(X\) and assumed to be affine geodesic with respect to \(\nabla\). Suppose furthermore that there exists a value \(s_0 \in I\) of the curve parameter such that \(\psi_{x(s_0)} (X_{x(s_0)}) = 0\). According to Proposition 3.3, \(\psi (X) = 0\) is conserved along \(X\) so that \(\psi_{x(s)} (X_{x(s)}) = 0\) for all \(s \in I\). Therefore, denoting \(\Sigma_t\) the absolute space of absolute time \(t\), \(x(s_0) \in \Sigma_t \Rightarrow x(s) \in \Sigma_t\), \(\forall s \in I\). \(\square\)
In the case of a torsionfree Galilean manifold, it can be shown that the torsionfree Galilean connection $\nabla$ reduces to the Levi-Civita connection for the spatial metric $\gamma$ on the absolute spaces $\Sigma_t$, so that geodesics whose tangent vector lies in the absolute space $\Sigma_t$ at one point are geodesics of $\Sigma_t$ for the spatial Levi-Civita connection. In this particular sense, the absolute spaces of a torsionfree Galilean manifold can be qualified as totally geodesic.

A first peculiarity of a Galilean manifold, in contradistinction with the relativistic case, is the fact that not all the torsion tensors are compatible with a given Leibnizian structure, as the following Proposition shows:

**Proposition 3.5** (cf. [6, 22]). Let $\mathcal{G} (\mathcal{M}, \psi, \gamma, \nabla)$ be a Galilean manifold and denote $T$ the torsion of the Galilean connection $\nabla$. The following relation holds:

$$\psi (T (X, Y)) = d\psi (X, Y)$$

(3.17)

for all $X, Y \in \Gamma (T \mathcal{M})$.

In components, relation (3.17) reads $\psi_\lambda \Gamma^\lambda_{[\mu\nu]} = \partial_{[\mu} \psi_{\nu]}$, where the torsion $T$ decomposes as $T \equiv \Gamma^\lambda_{[\mu\nu]} dx^\mu \wedge dx^\nu \otimes \partial_\lambda$.

In particular, the previous Proposition implies that only Augustinian structures (i.e. satisfying $d\psi = 0$) admit a torsionfree Koszul connection. This is clearly a distinctive feature of nonrelativistic structures as there exists no such restriction in the relativistic case. Furthermore, while in the relativistic case, Corollary 2.5 ensures that a torsionfree Lorentzian manifold is uniquely determined by the metric structure, in the nonrelativistic case however, the degeneracy of the metric prevents the gift of an Augustinian structure to uniquely fix a compatible torsionfree Koszul connection.

As one will see later, this arbitrariness has a natural physical interpretation: the Leibnizian structure merely encodes the properties of a nonrelativistic spacetime which would be a mere container where particles can be placed and measured. In Newtonian mechanics, their movement will be fixed by prescribed forces, *a priori* independent of the Leibnizian structure (usually taken to be flat, i.e. the Aristotle spacetime of Example 2.17). According to the equivalence principle, these dynamical trajectories acquire the interpretation of spacetime geodesics. In other words, the arbitrariness in the choice of the external forces prescribed on top of the Leibnizian structure corresponds to the arbitrariness in the choice of a Galilean connection.
This freedom is already manifest in the following two paradigmatic examples of Galilean manifolds:

**Example 3.6** (Galilei and Newton-Hooke spacetimes). The Aristotle spacetime (Example 2.17) with absolute clock $\psi = dt$ and rulers $\gamma = \delta_{ij}dx^i \vee dx^j$ can be supplemented with a flat connection $\Gamma_{\mu\nu}^\lambda = 0$ in order to yield the standard Galilei spacetime [24]. Alternatively, one can endow the Aristotle spacetime with the (equally compatible) connection $\Gamma$ whose only nonvanishing components are $\Gamma_{00}^i = -\frac{k}{\tau} x^i$. This Galilean manifold is referred to as the Newton-Hooke spacetime (cf. [27] for a nice introduction to its physical interpretation as a nonrelativistic cosmological model). The constant $k$ can take the values $+1$ (expanding spacetime) or $-1$ (oscillating spacetime). It can even take the value $k = 0$ (Galilei spacetime). The corresponding force field is simply the one of a harmonic oscillator for $k = -1$, i.e. a force linear in the displacement (attractive for $k = -1$, repulsive for $k = +1$).

As illustrated in the previous example, to a given Augustinian structure corresponds a whole class of compatible torsionfree Koszul connections. More precisely, as expressed in the Theorem below, they form an affine space with associated vector space $\Omega^2(\mathcal{M})$, the space of 2-forms on $\mathcal{M}$. A field of observers can be used to select a privileged vector (the zero) endowing this affine space with a structure of vector space.

**Theorem 3.7** (cf. [5, 6]). Given a field of observers $N \in \text{FO}(\mathcal{M})$, the set of torsionfree Galilean manifolds compatible with a given Augustinian structure $\mathcal{I}(\mathcal{M}, \psi, \gamma)$ is in bijective correspondence with the set $\Omega^2(\mathcal{M})$ of 2-forms $F$ on $\mathcal{M}$.

In other words, given an Augustinian structure (the “container”) and a field of observers $N \in \text{FO}(\mathcal{M})$, the arbitrariness of choice in the torsionfree compatible connection $\nabla$ is encoded into a 2-form on $\mathcal{M}$ (which can be physically interpreted as the “force field” which can be freely specified).

**Definition 3.8** (Gravitational fieldstrength measured by a field of observers). Let $\mathcal{G}(\mathcal{M}, \psi, \gamma, \nabla)$ be a Galilean manifold and $N \in \text{FO}(\mathcal{M})$ a field of observers. The gravitational fieldstrength measured by the field of observers $N$ is defined as the 2-form

$$\mathcal{N}_F(X, Y) \equiv \gamma(\nabla_X N, P^N(Y)) - \gamma(\nabla_Y N, P^N(X))$$

(3.18)
where $X,Y \in \Gamma(T\mathcal{M})$ are vector fields on $\mathcal{M}$ and $P^N$ designates the spacelike projector (cf. Definition 2.14).

In components, eq. (3.18) reads [28]

$$\hat{N}_\alpha F^\alpha = -2\gamma_{[\alpha} N^\beta \gamma_{\beta]} N^\lambda.$$

The above terminology pursues the analogy between the geodesic equation and the Lorentz force, in which the gravitational fieldstrength is the exact analogue of the Faraday tensor.

It is worth emphasising that Theorem 3.7 pinpoints the arbitrariness in a torsionfree Galilean connection for a fixed field $N$ of observers. A natural (and often implicit) assumption in geometric theories of gravity is the absence of a privileged field of observers. Correspondingly, the above Galilean connection $\nabla$ will from now be assumed *Milne-invariant*, in the sense that no privileged field of observers is involved in the set of ingredients characterising uniquely the connection. Under this assumption, the gravitational fieldstrength (3.18) measured by $N$ transforms as $\hat{N} \rightarrow \hat{N} + d\chi$ under a Milne boost $N \rightarrow N + h (\chi)$, where the 1-form $\hat{\Phi} \in \Omega^1(\mathcal{M})$ is defined as

$$\hat{\Phi} = \hat{P}^N (\chi) - \frac{1}{2} h (\chi, \chi) \psi$$

and $\hat{P}^N$ is the transpose of the spacelike projector defined in Definition 2.14. The components of the 1-form $\hat{\Phi}$ expressed in terms of the 1-form $\chi$ read as [28]

$$\hat{\Phi}_\mu = \chi_\mu - \left( \chi_\nu N^\nu + \frac{1}{2} h^{\nu\rho} \chi_\nu \chi_\rho \right) \psi_\mu. \quad (3.20)$$

The corresponding orbit of a couple $\left( N, \hat{N} \right)$ under all Milne boosts defines an equivalence class $\left[ N, \hat{N} \right]$ as follows: two couples $\left( N', \hat{N}' \right)$ and $\left( N, \hat{N} \right)$ are said to be equivalent if there exists a 1-form $\chi \in \Omega^1(\mathcal{M})$ such that

$$\begin{cases}
N' = N + h (\chi) \\
\hat{N}' = \hat{N} + d\hat{\Phi}.
\end{cases} \quad (3.21)$$

24
Definition 3.9 (Gravitational fieldstrength). A Milne orbit $[N, F]$ is dubbed a gravitational fieldstrength.

Using this terminology, Theorem 3.7 implies the following:

Corollary 3.10. Given an Augustinian structure, the Milne-invariant torsionfree Galilean connections compatible are in bijective correspondence with gravitational fieldstrengths.

From now on, all Galilean connections will be implicitly assumed Milne-invariant so this adjective will be dropped. The components of a Galilean connection read [6, 9]:

$$
\Gamma^\lambda_{\mu\nu} = N^\lambda \partial_\mu (N^\nu) + \frac{1}{2} h^\lambda_\rho \bigg[ \partial_\mu N^\rho \gamma^\nu + \partial_\nu N^\rho \gamma^\mu - \partial_\rho N^\mu \gamma^\nu \bigg] + h^\lambda_\rho \psi(\mu, F)_{\rho\nu}.
$$

(3.22)

Notice that this expression, although involving the field of observers $N$, can be checked to be invariant under a Milne boost, as it should. In other words, the connection is indeed independent of the choice of representative in the equivalence class $[N, F]$.

Expressing the 2-form $F$ on the Galilean basis $(N, e_i)$ (with $(\psi, \theta^i)$ the associated dual basis) leads to the following decomposition:

$$
N F = 2 N (N, e_i) \psi \wedge \theta^i + N (e_i, e_j) \theta^i \wedge \theta^j.
$$

(3.23)

The first term defines a spacelike vector field $G^N \in \text{Ker } \psi$ as $G^N = F^N (N, e_i) e^i$ (where $e^i \equiv e_j \delta^{ij}$) called the gravitational force field measured by the field of observers $N$. The second term corresponds to the action of $F$ on spacelike vector fields and will be referred to as the Coriolis 2-form $\omega^N \in \Gamma (\wedge^2 (\text{Ker } \psi)^*)$ measured by the field of observers $N$. It is defined as $\omega^N (V, W) = F^N (V, W)$, with $V, W \in \text{Ker } \psi$. Again, this terminology somehow pursues the analogy between the geodesic equation and the Lorentz force, in which the gravitational force field is the analogue of the electric field, while the Coriolis 2-form plays the role of the (Hodge dual to the) magnetic field.

Using eq.(3.18), these two definitions can be recast in a more geometric way which justifies further the terminology used:
Definition 3.11 (Gravitational force field and Coriolis 2-form [22]). Let \( \mathcal{M} (\mathcal{M}, \psi, \gamma, \nabla) \) be a Galilean manifold and \( N \in FO (\mathcal{M}) \) a field of observers. The gravitational field induced by \( \nabla \) on \( N \) is the spacelike vector field \( N^{\mathcal{G}} \in \text{Ker} \psi \):

\[
N^{\mathcal{G}} \equiv \nabla N.
\] (3.24)

The Coriolis 2-form induced by \( \nabla \) on \( N \) is the 2-form \( N^{\mathcal{\omega}} \in \Gamma (\wedge^2 (\text{Ker} \psi)^*) \), acting on \( V, W \in \text{Ker} \psi \) as\(^{16}\):

\[
N^{\mathcal{\omega}} (V, W) \equiv \gamma (\nabla V N, W) - \gamma (V, \nabla W N).
\] (3.25)

The compatibility condition of the Galilean connection \( \nabla \) with the absolute clock \( \psi \) (cf. Definition 3.1) ensures that \( \psi (\nabla_X N) = X \left[ \psi (N) \right] = 0, \forall X \in \Gamma (T.\mathcal{M}) \). This expression ensures \( \psi (\nabla N N) = 0 \), which in turn guarantees that \( \mathcal{G} \) is spacelike.

As one can see from eq. (3.24), the gravitational force field represents the obstruction of the field of observers \( N \) to be geodesic. In turn, for such a field of observers, free falling objects (i.e. that follow geodesics) appear to experience a gravitational force field. Similarly, the Coriolis 2-form is related to the “Coriolis force” associated to rotations of local observers with respect to each other. According to the decomposition (3.23), the gravitational field \( \mathcal{G} \) and the Coriolis 2-form \( \mathcal{\omega} \) associated to the field of observers \( N \) encode all the information contained in the 2-form \( \mathcal{F} \). Hence, given a field of observers \( N \), torsionfree Galilean manifolds compatible with a given Augustinian structure can equivalently be put in bijective correspondence with couples \( \left( N^{\mathcal{G}}, N^{\mathcal{\omega}} \right), \mathcal{G} \in \text{Ker} \psi, \mathcal{\omega} \in \Gamma (\wedge^2 (\text{Ker} \psi)^*) \) (cf. Corollary 5.28 in [22]).

The Milne-invariant connection (3.22) splits in a non Milne-invariant way as

\[
\Gamma^\lambda_{\mu
u} = \Gamma^\lambda_{\mu
u} + h^\lambda_{\rho\nu} \psi (\mu \mathcal{F}^\rho_{\nu}) \rho
\]

where the second term is a tensor, so that \( \Gamma^\lambda_{\mu
u} \) is a connection sometimes referred to as the special connection (cf. e.g. [6, 29]) associated to the field of observers \( N \). Note that the field of observers \( N \) is both geodesic (\( \mathcal{G} = 0 \)) and Coriolis-free (\( \mathcal{\omega} = 0 \)) with respect to the special connection \( \Gamma^\lambda_{\mu
u} \).

\(^{16}\)Note that our normalisation for the Coriolis 2-form differs by a factor \( \frac{1}{2} \) from the one used in [22].
**Proposition 3.12** (Special connection [6, 29]). Given an Augustinian structure $\mathcal{H}(\mathcal{M}, \psi, \gamma)$ and a field of observers $N$, the special connection is the unique torsion-free Galilean connection $\nabla$ such that $N$ is geodesic and Coriolis-free, i.e. such that the gravitational force field and the Coriolis 2-form both vanish.

For a prescribed Augustinian structure and field of observers, the class of torsionfree Galilean connections forms a vector space whose zero is the special connection.

In physical terms, a Galilean manifold endowed with such a special connection describes a nonrelativistic spacetime where there exists a privileged field (possibly a class) of “inertial” observers, i.e. measuring no gravitational field (in the sense of Proposition 3.12). The simplest example is the Galilei spacetime where $N = \frac{\partial}{\partial t} + v^i \frac{\partial}{\partial x^i}$ with $v^i$ constant. In order to account for the general case, the field of forces experienced by $N$ must be separately specified, hence the term $h^{\lambda \rho} \psi^N_{(\mu} F^N_{\nu)\rho}$ in eq.(3.24).

One way to partially reduce the ambiguity in the definition of the torsionfree Galilean connection is to impose supplementary conditions. The following condition [6, 7] has been proved very useful:

**Definition 3.13** (Duval-Künzle condition [6, 7]). Let $\mathcal{G}(\mathcal{M}, \psi, \gamma, \nabla)$ be a Galilean manifold and denote $R$ the curvature of the Galilean connection $\nabla$. The Duval-Künzle condition then reads:

$$\alpha \left( R(X, h(\beta); Y) \right) = \beta \left( R(Y, h(\alpha); X) \right)$$

(3.26)

$\forall X, Y \in \Gamma (T\mathcal{M})$ and $\forall \alpha, \beta \in \Omega^1(\mathcal{M})$.

This condition on the curvature operator $R$ is written more transparently in components as:

$$R^\mu_{\alpha \beta} = R^\nu_{\beta \alpha}$$

with $R^\mu_{\alpha \beta} \equiv h^{\nu \rho} R^\mu_{\alpha \rho \beta}$. Appendix A is devoted to the study of the curvature tensor for a Galilean manifold, we discuss in particular some useful identities as well as classic constraints encountered in the literature, focusing on the torsionfree case.
3.2 Newtonian manifolds

We now turn our attention to the study of torsionfree Galilean manifolds satisfying the Duval-Künzle condition (cf. Definition 3.13).

**Definition 3.14** (Newtonian manifold [6, 7]). A Newtonian manifold $\mathcal{N}(\mathcal{M}, \psi, \gamma, \nabla)$ is a torsionfree Galilean manifold whose Galilean connection satisfies the Duval-Künzle condition. The Koszul connection $\nabla$ is then referred to as a Newtonian connection.

Theorem 3.7 can then be specialised as:

**Theorem 3.15** (cf. [6, 7]). Given a field of observers $N \in \text{FO}(\mathcal{M})$, the set of Newtonian manifolds compatible with a given Augustinian structure $\mathcal{S}(\mathcal{M}, \psi, \gamma)$ is in bijective correspondence with the set of closed 2-forms $N^F$.

In the light of the previous Theorem, the Duval-Künzle condition can be reinterpreted as a geometric characterisation for the closedness of the 2-forms $N^F$ belonging to the equivalence class $[N, N^F]$ characteristic of the Galilean manifold. Applying Poincaré Lemma, one can locally write a given $N^F$ as an exact form so that there exists a class of 1-forms $\mathcal{A} \in \Omega^1(\mathcal{M})$ satisfying $F = d\mathcal{A}$. Two equivalent 1-forms $\mathcal{A}'$ and $\mathcal{A}$ differ by an exact differential: $\mathcal{A}' = \mathcal{A} + df$, with $f \in C^\infty(\mathcal{M})$. The transformation $\mathcal{A} \rightarrow \mathcal{A} + df$ will be referred to as a Maxwell gauge transformation. On the other hand, the transformation law of a 1-form $\mathcal{A}$ under a Milne boost $N \rightarrow N + h(\chi)$ follows directly from the one for $F$ and is given by $\mathcal{A} \rightarrow \mathcal{A} + \Phi$ with $\Phi$ as in (3.19).

Similarly to the Galilean case, one is led to define an additional structure supplementing the Augustinian one in order to solve the equivalence problem for Newtonian manifolds. The Newtonian analogue of a gravitational fieldstrength is thus defined as:

**Definition 3.16** (Gravitational potential). Let $N$ denote a field of observers and $\mathcal{A}$ a 1-form on $\mathcal{M}$. An orbit $[N, \mathcal{A}]$ under Milne boosts and Maxwell gauge transformations is called a gravitational potential. In other words, a gravitational potential is an equivalence class where two couples $\left(N', \mathcal{A}'\right)$ and $\left(N, \mathcal{A}\right)$ are said to be equivalent.
if there exists a 1-form $\chi \in \Omega^1(M)$ and a function $f \in C^\infty(M)$ such that

$$\begin{cases} N' = N + h(\chi) \\ A' = A + \Phi + df. \end{cases}$$ \hfill (3.27)

In a representative $(N, A)$, the second entry $A$ is called a gravitational gauge 1-form for the field of observers $N$.

The next Corollary follows from Theorem 3.15:

**Corollary 3.17.** Milne-invariant Newtonian connections compatible with a given Augustinian structure are in bijective correspondence with gravitational potentials.

### 3.3 Variational approach

The present section revisits the equivalence problem for Newtonian manifolds (i.e. the search for extensions of a given Augustinian structure determining uniquely a Newtonian connection) by displaying an alternative formulation [9], based on Coriolis-free fields of observers (cf. Definition 3.11). We start by proving the following Proposition:

**Proposition 3.18.** Let $\mathcal{N}(M, \psi, \gamma, \nabla)$ be a Newtonian manifold associated to the gravitational potential $[N, A]$. The field of observers $Z \in FO(M)$ is Coriolis-free if and only if

$$Z = N - h\left(\frac{N}{A}\right) + h(df)$$ \hfill (3.28)

for a function $f \in C^\infty(M)$ and a couple $(N, A)$ in the equivalence class.

The proof of Proposition 3.18 can be found in Appendix B.

In the following, we let $\mathcal{N}(M, \psi, \gamma, \nabla)$ be a Newtonian manifold associated to the gravitational potential $[N, A]$. The quantity $Z = N - h\left(\frac{N}{A}\right)$ is invariant under a Milne boost (this fact is shown in the proof of Proposition 3.18), so that the 1-form $\frac{N}{A}$ can be thought of as a compensator field, used in order to construct Milne-invariant objects (cf. Table 2 below). Moreover, Proposition 3.18 ensures that any Newtonian
manifold admits Coriolis-free fields of observers and even provides an explicit way
to construct them: namely, one can go from any field of observers \( N \in FO(\mathcal{M}) \) to
a Coriolis-free field of observers \( Z \) via a Milne boost parameterised by the 1-form
\( \chi = -A \). Under such a Milne boost, the gravitational gauge 1-form \( A \) for \( N \) gets
mapped to a gravitational gauge 1-form \( \bar{A} \equiv \frac{1}{2} \phi \psi \) which is along the absolute clock
and where the explicit form of the function \( \phi \in C^\infty(\mathcal{M}) \) is given in:

**Definition 3.19** (Gravitational gauge scalar). Consider a gravitational gauge 1-
form \( A \) for the field of observers \( N \). The function

\[
\phi \equiv 2^N A(N) - h \left( \frac{N}{A}, \frac{N}{A} \right)
\]

(3.29)
is called the gravitational gauge scalar corresponding to \( A \).

This denomination is justified by the form taken by the gravitational force field
\( \bar{Z} \equiv \nabla_Z Z = -\frac{1}{2} h (d\phi) \). As one can see, the gravitational force field measured by
a Coriolis-free field of observers derives from a potential (up to a factor, the scalar
potential \( \phi \)). It can be checked that the gravitational gauge scalar \( \phi \) is also a Milne-
invariant object. However, it is not gauge invariant, a point which will be adressed
in details after the following example.

**Example 3.20** (Galilei and Newton-Hooke spacetimes). The Augustinian structure
of these spacetimes is composed of the absolute clock \( \psi = dt \) and rulers \( \gamma = \delta_{ij} dx^i \vee \)
\( dx^j \). The Galilei and Newton-Hooke spacetimes (Example 3.6) are also endowed
with a Newtonian connection, the only nonvanishing components of which are \( \Gamma^i_{00} = -\frac{k}{c^2} x^i \) with \( k = 0 \) for the Galilei spacetime. The field of observers \( Z = \frac{\partial}{\partial t} \) is
Coriolis-free and measures the gravitational force field \( \bar{Z} = -\frac{k}{c^2} x \) which derives
from the gravitational gauge scalar \( \phi = \frac{k}{c^2} x \). The gravitational gauge 1-form
for the Coriolis-free field of observers \( Z \) is thus \( \bar{A} = \frac{k}{2c^2} x dt \). Notice that the
collection of all Coriolis-free field of observers are obtained from \( Z \) by shifting its
spatial part by an irrotational relative velocity field, \( i.e. \) a gradient \( \nu^i = \partial^i f \).

The couple \( (Z, \frac{1}{2} \phi \psi) \) is a distinguished representative of the gravitational potential
\( [N, \bar{A}] \) characterising the Newtonian manifold \( \mathcal{N} \). Conversely, the whole equivalence class \( [N, \bar{A}] \) can be reconstructed from one of its representatives using relations
(3.27). Therefore, one can characterise a Newtonian manifold $\mathcal{N}$ by an Augustinian structure $\mathcal{N} (\mathcal{M}, \psi, \gamma)$ together with a couple $\{(Z, \phi)\}$. In order to make a converse statement, one needs first to acknowledge the fact that a given Newtonian manifold does not define a unique Coriolis-free field of observers but rather a class thereof. Indeed, two Coriolis-free fields of observers $Z$ and $Z' \in FO (\mathcal{M})$ have been seen to be related by a Maxwell transformation $Z' = Z - h(df)$ with gauge function $f \in C^\infty (\mathcal{M})$ (cf. Proposition 3.18). This is a direct consequence of the previously mentioned fact that to a given field of observers $N$ corresponds a class of 1-forms $[N_A]$ differing by $N_{A'} = N_A + df$, for some function $f$ on $\mathcal{M}$. Consequently, the respective gravitational gauge scalars $\phi$ and $\phi' \in C^\infty (\mathcal{M})$ can be checked to be related according to $\phi' = \phi + 2 df (Z) - h (df, df)$. Newtonian manifolds thus can be put in correspondence with the following equivalence classes:

**Definition 3.21** (Gravitational potential). A gravitational potential is an equivalence class $[Z, \phi]$, where two couples $(Z', \phi')$ and $(Z, \phi)$ are said to be equivalent if there exists a function $f \in C^\infty (\mathcal{M})$ such that

\[
\begin{align*}
Z' &= Z - h(df) \\
\phi' &= \phi + 2 df (Z) - h (df, df)
\end{align*}
\]

We sum up the whole discussion in the following Proposition:

**Proposition 3.22** (cf. [9]). Let $\mathcal{N} (\mathcal{M}, \psi, \gamma)$ be an Augustinian structure. There is a bijective correspondence between Newtonian manifolds $\mathcal{N} (\mathcal{M}, \psi, \gamma, \nabla)$ and gravitational potentials $[Z, \phi]$. An obvious benefit of the present formulation is that it allows to gather up the supplementary information needed to define Newtonian manifolds in a Milne-invariant way. Indeed, two representatives of this equivalence class only differ by a Maxwell gauge-transformation. Another interesting feature of this formulation is embodied by the following Proposition. We first define the notion of a Lagrangian metric:

**Definition 3.23** (Lagrangian metric). Let $\mathcal{L} (\mathcal{M}, \psi, \gamma)$ be a Leibnizian structure. A covariant metric $g \in \Gamma (\mathcal{V}^2 T^* \mathcal{M})$ on $\mathcal{M}$ satisfying the condition $g (X, Y) = \gamma (X, Y)$, for any $X, Y \in \Gamma (\ker \psi)$ will be called a Lagrangian metric.
Proposition 3.24. Let $\mathcal{S}(\mathcal{M}, \psi, \gamma)$ be an Augustinian structure. Let $Z \in \text{FO}(\mathcal{M})$ designate a field of observers and $\phi \in C^\infty(\mathcal{M})$ a function on $\mathcal{M}$. There is a bijective correspondence between couples $(Z, \phi)$ and Lagrangian metrics $g \in \Gamma(\vee^2 T^* \mathcal{M})$.

The somewhat lengthy proof is relegated to Appendix B. It rests on the fact that the only Lagrangian metric $g \in \Gamma(\vee^2 T^* \mathcal{M})$ satisfying $g(Z) = \phi \psi$ reads as $g \equiv \tilde{Z} + \phi \psi \vee \psi$, with $\tilde{Z}$ the metric transverse to $Z$. For an arbitrary field of observers $N$, the analogue expression reads $g \equiv \tilde{N} + 2 \psi \vee A$.

The characterisation of Newtonian manifolds using Coriolis-free fields of observers thus allows to define a covariant metric $g$. Although we are in a nonrelativistic context, the latter metric can be nondegenerate (when the gravitational gauge scalar $\phi$ is nowhere vanishing). Under a Maxwell-gauge transformation $Z \rightarrow Z - h(df)$, the metric $g$ transforms as

$$g \rightarrow g' = g + 2 \psi \vee df$$  \hspace{1cm} (3.31)

so that we are led to define an equivalence class $[g]$ as:

Definition 3.25 (Lagrangian structure). Let $\mathcal{L}(\mathcal{M}, \psi, \gamma)$ be a Leibnizian structure. We define an equivalence class $[g]$ of Lagrangian metrics $g \in \Gamma(\vee^2 T^* \mathcal{M})$ on $\mathcal{M}$ such that two representatives $g'$ and $g$ are related by $g' = g + 2 \psi \vee df$, for some function $f \in C^\infty(\mathcal{M})$. The triplet $\mathcal{L}(\mathcal{M}, \psi, [g])$ is called a Lagrangian structure.

Now, one can combine Propositions 3.22 and 3.24 in order to show:

Proposition 3.26 (cf. [9]). Let $\mathcal{S}(\mathcal{M}, \psi, \gamma)$ be an Augustinian structure. There is a bijective correspondence between Newtonian manifolds $\mathcal{N}(\mathcal{M}, \psi, \gamma, \nabla)$ and Lagrangian structures $\mathcal{L}(\mathcal{M}, \psi, [g])$.

The following table sums up the Milne-invariant objects introduced in this Section along with their Maxwell-gauge transformation law:
| Type                                                                 | Name                                      | Definition                                                                 | Maxwell-gauge transformation law |
|---------------------------------------------------------------------|-------------------------------------------|----------------------------------------------------------------------------|----------------------------------|
| \( Z \in FO (\mathcal{M}) \)                                      | Coriolis-free field of observer           | \( Z \equiv N - h \binom{N}{A} \)                                        | \( Z \rightarrow Z - h(df) \)    |
| \( \phi \in C^\infty (\mathcal{M}) \)                            | Gravitational gauge scalar                | \( \phi \equiv 2A(N) - h \binom{N}{A} \)                               | \( \phi \rightarrow \phi + 2 df \left(Z\right) - h(df, df) \) |
| \( g \in \Gamma (\sqrt{}^2 T^* \mathcal{M}) \)                  | Lagrangian metric                         | \( g \equiv \gamma + 2 \psi \vee \binom{N}{A} \)                        | \( g \rightarrow g + 2 \psi \vee df \) |

Table 2: Milne-invariant objects

The use of the “Lagrangian” denomination is justified by the fact that a Lagrangian metric \( g \) defines a Lagrangian as \( L \equiv \frac{1}{2} g \left(X, X\right) \) with \( X \in FO(\mathcal{M}) \) the tangent vector field associated to an (arbitrary) observer \( x : I \subseteq \mathbb{R} \rightarrow \mathcal{M} : \tau \mapsto x(\tau) \). In components, the Lagrangian then reads

\[
L = \frac{1}{2} g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau}.
\]  

(3.32)

In order to find the associated equations of motion, it must be kept in mind that the variation of the Lagrangian (3.32) is not performed over the whole space of tangent vectors but is constrained to the space of tangent vectors parameterised by the proper time \( \tau \), i.e. to the space of observers (cf. Proposition 2.13). In the generic case, the constraint \( \psi_\mu \frac{dx^\mu}{d\tau} = 1 \) is linear in the velocities and, in general, non-holonomic (since it is of the form \( f(x^i, \dot{x}^i, t) = 0 \)). However, in the Augustinian case, the absolute clock is closed (\( \psi = dt \)) so that the constraint can be integrated to give a holonomic constraint (i.e. of the form \( f(x^i, t) = 0 \)) which can be resolved by adopting the absolute time \( t \) as parameter:

\[
L = \frac{1}{2} g_{\mu\nu} \frac{dx^\mu}{dt} \frac{dx^\nu}{dt}.
\]  

(3.33)

In an adapted coordinate system \( (t, x^i) \), the Lagrangian reads

\[
L = \frac{1}{2} \gamma_{ij} \dot{x}^i \dot{x}^j + A_i \dot{x}^i - U
\]
where we used the relation \( g \equiv N + 2 \psi \) and \( \dot{N} \equiv \frac{d}{dt} N \equiv -U dt + A_i dx^i \) and \( \gamma_{ij} \) the components of the collection of absolute rulers \( \gamma \). The Lagrangian (3.33) is therefore formally identical to the one describing the motion of a charged particle minimally coupled to an electromagnetic field through the vector potential \( A_i \) and the scalar potential \( U \) and moving on a Riemannian manifold with metric \( \gamma_{ij} \).

**Example 3.27.** In the particular case of the Aristotle spacetime (\( \psi = dt \) and \( \gamma = \delta_{ij} dx^i \vee dx^j \)) endowed with a Coriolis-free field of observer \( Z = \frac{\partial}{\partial t} \), the Lagrangian takes the standard form \( \mathcal{L} = \frac{1}{2} \left( \frac{dx^i}{dt} \frac{dx^j}{dt} + \phi \right) \) since the gravitational gauge 1-form reads \( A = \frac{1}{2} \phi dt \) for this field of observers. Notice that the usual potential would be \( U = -\frac{1}{2} \phi \).

For a generic Augustinian structure, the Euler-Lagrange equations of motion derived from \( \mathcal{L} \) take the form [9]:

\[
\frac{d^2 x^\beta}{dt^2} + \frac{1}{2} \left[ \partial_\mu g_{\nu\alpha} + \partial_\nu g_{\mu\alpha} - \partial_\alpha g_{\mu\nu} \right] \frac{dx^\mu}{dt} \frac{dx^\nu}{dt} = 0.
\]

Contracting with \( h^{\lambda\alpha} \) and using the relation \( g_{\alpha\beta} h^{\lambda\alpha} = \delta_\lambda^\beta - Z^\lambda \psi_\beta \) (as can be deduced from the expression of the Lagrangian metric \( g \)) leads to:

\[
\frac{d^2 x^\lambda}{dt^2} - Z^\lambda \psi_\nu \frac{d^2 x^\nu}{dt^2} + \frac{1}{2} h^{\lambda\alpha} \left[ \partial_\mu g_{\nu\alpha} + \partial_\nu g_{\mu\alpha} - \partial_\alpha g_{\mu\nu} \right] \frac{dx^\mu}{dt} \frac{dx^\nu}{dt} = 0. \tag{3.34}
\]

Now, differentiating the constraint \( \psi_\mu \frac{dx^\mu}{dt} = 1 \), one obtains the relation

\[
\psi_\nu \frac{d^2 x^\nu}{dt^2} = -\partial_\alpha \psi_\beta \frac{dx^\alpha}{dt} \frac{dx^\beta}{dt}
\]

which can be substituted in eq.(3.34) to give

\[
\frac{d^2 x^\lambda}{dt^2} + \Gamma^\lambda_{\mu\nu} \frac{dx^\mu}{dt} \frac{dx^\nu}{dt} = 0
\]

where the components \( \Gamma^\lambda_{\mu\nu} \) read

\[
\Gamma^\lambda_{\mu\nu} = Z^\lambda \partial_\mu \psi_\nu + \frac{1}{2} h^{\lambda\alpha} \left[ \partial_\mu g_{\nu\alpha} + \partial_\nu g_{\mu\alpha} - \partial_\alpha g_{\mu\nu} \right]. \tag{3.35}
\]

Using Table 2, one can check that the expression (3.35) is identical to the one of eq.(3.22) (with \( F = dA \)), so that the Lagrangian \( \mathcal{L} \) describes a free particle in
geodesic motion with respect to a Newtonian connection, hence providing a concrete implementation of Proposition 3.26. Note that, although being explicitly Milne-invariant, the Lagrangian $\mathcal{L}$ is not invariant under a Maxwell-gauge transformation of $g$ as $g_{\mu\nu} \to g_{\mu\nu} + 2\psi(\mu)\partial_{\nu}f$ but transforms by adjonction of a total derivative $\mathcal{L} \to \mathcal{L} + \frac{df}{dt}$ which only contributes to the boundary term, so that the equations of motion (and thus the expression of $\Gamma^\lambda_{\mu\nu}$) are Maxwell-gauge invariant. Finally, notice that, when the gravitational potential vanishes, then eq.(3.35) is just the expression of the special connection $\tilde{\Gamma}^\lambda_{\mu\nu}$ (since $g = \tilde{\gamma}^\lambda$ when $\phi = 0$).

### 3.4 Towards the ambient formalism

Before addressing the question of parallelism for Aristotelian structures, we conclude the present Section by a heuristic discussion regarding the natural emergence of the ambient formalism through the study of Newtonian manifolds, thus paving the way to the more systematic discussion to appear in [30].

Let\(^{17}\) $\tilde{\mathcal{N}}(\tilde{\mathcal{M}}, \tilde{\psi}, \tilde{\gamma}, \tilde{\nabla})$ be a Newtonian manifold where $\tilde{\mathcal{M}}$ is $(d+1)$-dimensional. Pick a field of observers $\tilde{N} \in FO(\tilde{\mathcal{M}})$. The characterisation of a Newtonian manifold $\tilde{\mathcal{N}}$ has been seen to require the introduction of a set of 1-forms $\tilde{N}^A \in \Omega^1(\tilde{\mathcal{M}})$ with Maxwell-like transformation law $\tilde{N}^A \to \tilde{N}^A + df$, where $f \in C^\infty(\tilde{\mathcal{M}})$. To the bundle-minded physicist, this transformation law suggests to reinterpret the 1-forms $\tilde{N}^A$ as gauge connections for a principal $(\mathbb{R}, +)$-bundle of projection $\pi : \tilde{\mathcal{M}} \to \tilde{\mathcal{M}}$, where $\tilde{\mathcal{M}}$ is a $(d+2)$-dimensional manifold. Recall that, if $\tilde{A} \in \Omega^1(\tilde{\mathcal{M}})$ is an $(\mathbb{R}, +)$-Ehresmann connection on $\tilde{\mathcal{M}}$, choosing a section $\sigma : \tilde{\mathcal{U}} \to \tilde{\mathcal{M}}$ (where $\tilde{\mathcal{U}} \subset \tilde{\mathcal{M}}$ is an open subset of $\tilde{\mathcal{M}}$) allows to define a gauge connection $\tilde{A} \in \Omega^1(\tilde{\mathcal{U}})$ as $\tilde{A} \equiv \sigma^*\tilde{A}$. Reciprocally, a collection $\{\tilde{\mathcal{U}}_\alpha, \tilde{A}_\alpha\}$ (where the $\tilde{\mathcal{U}}_\alpha$ form an open cover of $\tilde{\mathcal{M}}$ and the set of $\tilde{A}_\alpha$ differ by Maxwell-like transformation laws) defines an unique Ehresmann connection $\tilde{A}$.

The principal $(\mathbb{R}, +)$-bundle involves a supplementary “internal” direction, the vertical fiber foliation, which is a congruence of integral curves (called rays) for the unique fundamental vector field of the principal bundle $\tilde{\mathcal{M}}$, denoted $\xi \in \Gamma(T\tilde{\mathcal{M}})$ and designated as the wave vector field. Since $\xi$ is the fundamental vector field, it

\(^{17}\)In the present section, we anticipate on the notation to be used in [30] where nonrelativistic objects will be topped with a bar.
satisfies \( A(\xi) = 1 \) (since 1 is the generator of the Abelian Lie algebra \( \mathbb{R} \)). Usually \( e.g. \) in Yang-Mills theories, the fiber of an Ehresmann bundle is interpreted as an auxiliary geometric object allowing to define an internal symmetry. The key to the ambient approach consists in reinterpreting this additional direction as a new spacetime dimension.

Now, we investigate how structures on \( \bar{\mathcal{M}} \) can be lifted up to \( \mathcal{M} \). First, the absolute clock \( \bar{\psi} \in \Omega^1(\bar{\mathcal{M}}) \) defines an unique closed 1-form \( \psi \in \Omega^1(\mathcal{M}) \) as \( \psi \equiv \pi^* \bar{\psi} \), called \textit{wave covector field}. It can be checked that, since \( \pi_* \xi = 0 \), one has \( \psi(\xi) = 0 \), so that \( \xi \in \ker \psi \). The kernel of \( \psi \) defines an involutive distribution \( \psi \) being closed whose integral submanifolds are called \textit{wavefront worldvolumes}. Each wavefront worldvolume can thus be envisaged as the union of an absolute space with the corresponding fibers. A wavefront wordlvolume \( \mathcal{W} \) is therefore endowed with a contravariant metric \( \gamma \in \Gamma(\bigwedge^2 \ker \psi^*) \) defined as the generalised pullback of the collection of absolute rulers \( \bar{\gamma} \in \Gamma(\bigwedge^2 \ker \bar{\psi}^*) \), \( i.e. \ \gamma \equiv \pi^* \bar{\gamma} \). Contrarily to its nonrelativistic counterpart, the metric \( \gamma \) is degenerate since \( \gamma(\xi) = 0 \) (in the language of [17], the triplet \( \mathcal{W}, \gamma, \xi \) is thus a Carroll metric structure). The field of observers \( \bar{N} \in \mathcal{F}O(\bar{\mathcal{M}}) \) can be lifted up to \( \mathcal{M} \) by defining \( N \in \mathcal{F}O(\mathcal{M}) \) as the horizontal lift of \( \bar{N} \) with respect to \( \bar{A} \) (\( i.e. \ \pi_* N = \bar{N} \) and \( \bar{A}(N) = 0 \)) while an ambient covariant metric \( \bar{N} \gamma \in \Gamma(\bigwedge^2 T^* \bar{\mathcal{M}}) \) can be defined as the generalised pullback of the transverse metric \( \bar{N} \gamma \in \Gamma(\bigwedge^2 T^* \bar{\mathcal{M}}) \). It can be checked that \( \text{Span} \{ \xi, N \} \in \text{Rad} \gamma \).

According to Proposition 3.26, a Newtonian manifold defines a class of Lagrangian metrics \( [\bar{g}] \) where each metric \( \bar{g} \in \Gamma(\bigwedge^2 T^* \bar{\mathcal{M}}) \) is given by \( \bar{g} \equiv \bar{\gamma} + 2 \bar{\psi} \wedge \bar{A} \) and transforms under a gauge transformation as \( \bar{g} \rightarrow \bar{g}' = \bar{g} + 2 \bar{\psi} \wedge df \). Similarly to the definition of an Ehresmann connection on \( \mathcal{M} \), it can be shown that the set \( [\bar{g}] \) defines a unique covariant metric \( g \in \Gamma(\bigwedge^2 T^* \mathcal{M}) \) satisfying \( \bar{g} = \sigma^* g \). Explicitly, the metric \( g \) can be expressed as \( g \equiv \gamma + 2 \psi \wedge A \). Furthermore, the metric \( g \) can be shown to be nondegenerate. The expression for \( g \) can be used in order to compute \( g(\xi, \xi) = 0 \) and \( g(N, N) = 0 \) (so that \( \xi \) and \( N \) are null vector fields). Furthermore, \( g(\xi) = \psi \) and \( g(N) = \bar{A} \). This implies \( g(\xi, N) = 1 \) so that \( \xi \) and \( N \) form a lightcone basis (\( cf. \) [23] and Figure 5) and \( \mathcal{M} \) is thus a Lorentzian manifold. Since \( g \) is nondegenerate, it defines a notion of parallelism on \( \mathcal{M} \) in the guise of the Levi-Civita connection \( \nabla \) and it will be shown in [30] (following [31]) how the Levi-Civita connection \( \nabla \)
projects down to the Newtonian connection $\nabla$ on $\mathcal{M}$. The wavevector field can then be shown to be parallel with respect to $\nabla$, so that $\mathcal{M}$ can be characterised as a Bargmann-Eisenhart wave (cf. the terminology used in [23]).

The conclusion emerging from the line of reasoning sketched here is that the usual hierarchy between Bargmann-Eisenhart waves and Newtonian manifolds (where the latter are obtained from the former) can in fact be reversed given that a geometrical understanding of nonrelativistic spacetimes (Newtonian manifolds) leads naturally to the reconstruction of an ambient relativistic spacetime (Bargmann-Eisenhart waves). As always in the process of dimensional reduction, a spacetime symmetry of the ambient manifold becomes interpreted as an internal symmetry on the reduced manifold. A Maxwell gauge symmetry is always found in the reduced theory along one-dimensional orbits independently of the type of curves, whether spacelike (in the usual case à la Kaluza-Klein) or lightlike (here).
4 Torsionfree connections with a twistless absolute clock

Having reviewed how a Leibnizian structure with closed absolute clock (i.e. an Augustinian structure, cf. Definition 2.16) can be endowed with a notion of parallelism, we now turn our attention to the weaker case when the clock is twistless, i.e. when it satisfies the Frobenius criterion (the structure is then Aristotelian, cf. Definition 2.15) and propose a candidate connection, that can be defined on such structures.

As guiding lines for our proposal, we rely on the two following requirements:

- Conformal equivalence
- Projective structure

on which we will draw for the rest of this Section.

4.1 Conformally Galilean manifolds

We start by noting that, since the absolute clock $\psi$ is assumed to be twistless (i.e. $\psi \wedge d\psi = 0$), Frobenius Theorem ensures that locally there exists a closed 1-form $\tilde{\psi}$ and a positive function $\Omega$ (called time unit of $\psi$) such that $\psi = \Omega \tilde{\psi}$. This last relation thus provides a natural notion of conformal equivalence at the level of clocks.

The following definition extend this notion to absolute rulers in order to precisely express in which sense an Augustinian and an Aristotelian structures can be said conformally related.

**Definition 4.1** (Conformally related structures). An Augustinian structure $\mathcal{H}(\mathcal{M}, \tilde{\psi}, \tilde{\gamma})$ and an Aristotelian structure $\mathcal{A}(\mathcal{M}, \psi, \gamma)$ are said to be conformally related if the following relations hold:

$$\psi = \Omega \tilde{\psi}, \quad \gamma = \Omega \tilde{\gamma}$$

(4.36)

with $\Omega \in C^\infty(\mathcal{M})$ the time unit of $\psi$.

The scalings appearing in Definition 4.1 can be interpreted as generated by the dilatation part of the Lifshitz algebra with dynamical exponent $z = 2$ (cf. [32]), for which time scales twice as much as space.
The previous definition provides a nonrelativistic analogue of the notion of conformal equivalence of two relativistic metrics \( g = \Omega \bar{g} \), where \( \Omega \in C^\infty(\mathcal{M}) \) is the conformal factor. Recall that, in the relativistic case, the notion of conformal equivalence straightforwardly extends from metrics to connections, using the Levi-Civita prescription. The Levi-Civita connections \( \nabla \) and \( \bar{\nabla} \) respectively associated to the metrics \( g \) and \( \bar{g} \) can be shown to satisfy:

\[
\nabla_X Y = \bar{\nabla}_X Y + \Upsilon(X) Y + \Upsilon(Y) X - g(X, Y) g^{-1}(\Upsilon) \quad (4.37)
\]

where \( X \) and \( Y \) are arbitrary vector fields on \( \mathcal{M} \) and the 1-form \( \Upsilon \in \Omega^1(\mathcal{M}) \) is exact and defined as \( \Upsilon \equiv \frac{1}{2} d(ln \Omega) \). Denoting \( \Gamma^\lambda_{\mu\nu} \) and \( \bar{\Gamma}^\lambda_{\mu\nu} \) the components of \( \nabla \) and \( \bar{\nabla} \) respectively allows to reexpress the previous expression (4.37) as:

\[
\Gamma^\lambda_{\mu\nu} = \bar{\Gamma}^\lambda_{\mu\nu} + \Upsilon_\mu \delta^\lambda_\nu + \Upsilon_\nu \delta^\lambda_\mu - g_{\mu\nu} g^{\lambda\rho} \Upsilon_\rho. \quad (4.38)
\]

The following commutative diagram sums up the construction in the relativistic case:

\[
\begin{array}{ccc}
\mathcal{M}, \bar{g} & \longrightarrow & \mathcal{M}, g \\
2 & \downarrow & 2 \\
\nabla & \longrightarrow & \nabla
\end{array}
\]

where the different arrows stand for:

1. Metric Conformal equivalence \( i.e. \ g = \Omega \bar{g} \)

2. Levi-Civita prescription

3. Connection Conformal equivalence given by eq.(4.37) with \( \Upsilon \equiv \frac{1}{2} d(ln \Omega) \).

Note that in the relativistic case, we dispose of a canonical notion of torsionfree connection for all metric structures (the Levi-Civita prescription), which in turn allows to define a notion of conformal equivalence for connections (\( cf. \) eq.(4.37)). Now, in the nonrelativistic case, we do not yet dispose of a notion of parallelism on
Aristotelian structures but aim to deduce one from a notion of conformal equivalence for connections mimicked on the relativistic case. More explicitly, one can display a nonrelativistic avatar of the previous diagram

\[ \mathcal{S}(\mathcal{M}, \bar{\psi}, \bar{\gamma}) \xrightarrow[1]{\text{1}} \mathcal{A}(\mathcal{M}, \psi, \gamma) \xrightarrow[2]{\text{2}} \xrightarrow[3]{\text{3}} \xrightarrow[4]{\text{4}} \]

The first arrow stands for the metric conformal equivalence formulated in Definition 4.1 while the second arrow designates the torsionfree Galilean prescription, which is non-canonical (cf. Theorem 3.7). Our strategy consists in deducing the third arrow from the fourth one which will be chosen by analogy with its relativistic counterpart. Explicitly, given a torsionfree Galilean connection \( \bar{\nabla} \) on an Augustinian structure \( \mathcal{S}(\mathcal{M}, \bar{\psi}, \bar{\gamma}) \), one seeks to define a connection \( \nabla \) on an Aristotelian structure \( \mathcal{A}(\mathcal{M}, \psi, \gamma) \) by adapting relation (4.37) between two conformal connections to the nonrelativistic case. However, the presence of the covariant metric \( g_{\mu\nu} \) in the last term of (4.37) makes this task not entirely straightforward, due to the lack of a canonical nonrelativistic covariant metric. Relying on our second guiding line allows us to select a notion of conformal equivalence between connections, that will be further justified using a Lagrangian approach.

The former can be precisely articulated as follows: the connections \( \nabla \) and \( \bar{\nabla} \) will be assumed to belong to the same projective structure, understood in the sense of the following classic theorem:

**Theorem 4.2** (cf. e.g. [33, 34]). Two torsionfree Koszul connections \( \nabla \) and \( \bar{\nabla} \) share the same geodesics as unparameterised curves if and only if there exists a 1-form \( \Upsilon \in \Omega^1(\mathcal{M}) \) such that, for any pair of vector fields \( X \) and \( Y \in \Gamma(T\mathcal{M}) \), the following relation holds:

\[ \nabla_X Y = \bar{\nabla}_X Y + \Upsilon(X) Y + \Upsilon(Y) X. \quad (4.39) \]

Two Koszul connections \( \nabla \) and \( \bar{\nabla} \) satisfying eq.(4.39) for some 1-form \( \Upsilon \) are said
to belong to the same projective structure.

In holonomic components, relation (4.39) can be stated as

$$\Gamma^\lambda_{\mu\nu} = \tilde{\Gamma}^\lambda_{\mu\nu} + \Upsilon^\mu_{\delta} \delta^\lambda_{\nu} + \Upsilon^\nu_{\delta} \delta^\lambda_{\mu}. \quad (4.40)$$

Merging the projective structure requirement with the insights gained from the relativistic case, we are led to define a nonrelativistic notion of conformal equivalence between connections $\tilde{\nabla}$ and $\nabla$ (for two conformally related Augustinian and Aristotelian structures, respectively) as satisfying relation (4.39) with $\Upsilon \equiv \frac{1}{2}d(\ln \Omega)$ and $\Omega$ the time unit of the absolute clock $\psi$ for the underlying Aristotelian structure. Armed with our guiding lines, we thus have been able to single out a prescription for a connection in the case of an Aristotelian structure, as embodied in the following definition:

**Definition 4.3** (Conformally Galilean/Newtonian manifold). Let $\mathcal{A} (\mathcal{M}, \psi, \gamma)$ be an Aristotelian structure conformally related to the Augustinian structure $\mathcal{S} (\mathcal{M}, \tilde{\psi}, \tilde{\gamma})$. Let $\tilde{\nabla}$ be a torsionfree Galilean (resp. Newtonian) connection preserving $\mathcal{S}$. There is a canonical connection conformally related to $\tilde{\nabla}$, in the sense that

$$\nabla_X Y = \tilde{\nabla}_X Y + \Upsilon (X) Y + \Upsilon (Y) X \quad (4.41)$$

for any pair of vector fields $X$ and $Y \in \Gamma (T\mathcal{M})$, where $\Upsilon \equiv \frac{1}{2}d(\ln \Omega)$ with $\Omega \in C^\infty (\mathcal{M})$ the time unit of $\psi$. Such a connection $\nabla$ is called a conformally Galilean (resp. Newtonian) connection and the quadruple $\mathcal{P} (\mathcal{M}, \psi, \gamma, \nabla)$ will be referred to as a conformally Galilean (resp. Newtonian) manifold.

A solution to the equivalence problem for such connections is straightforwardly obtained from the previous definition together with Corollaries 3.10 and 3.17 as:

**Proposition 4.4.** Conformally Galilean (resp. Newtonian) connections on a given Aristotelian structure are in bijective correspondence with gravitational field strengths (resp. potentials).

Note that the 1-form $\Upsilon$ is closed, so that $\nabla$ and $\tilde{\nabla}$ belong to the same special projective structure (cf. [34]). The following Proposition takes advantage of this fact:
Proposition 4.5. Let $\mathcal{A}(\mathcal{M}, \psi, \gamma)$ be an Aristotelian structure conformally related to the Augustinian structure $\mathcal{S}(\mathcal{M}, \bar{\psi}, \bar{\gamma})$ and let $\nabla$ be a Koszul connection on $\mathcal{M}$. The Koszul connection $\nabla$ is a conformally Galilean (resp. Newtonian) connection for $\mathcal{A}$ if and only if there exists a torsionfree Galilean connection (resp. Newtonian) $\bar{\nabla}$ for $\mathcal{S}$ such that

$$\bar{X} = \Omega X \Rightarrow \bar{\nabla}_X \bar{X} = \Omega^2 \nabla_X X, \quad \forall X \in \Gamma(T\mathcal{M}) \quad (4.42)$$

with $\Omega \in C^\infty(\mathcal{M})$ the time unit of the absolute clock i.e. $\psi = \Omega \bar{\psi}$.

Proof: Both Definition 4.3 and condition (4.42) imply that $\nabla$ and the torsionfree Galilean connection (resp. Newtonian) $\bar{\nabla}$ belong to the same projective structure, in the sense of Theorem 4.2. More precisely, there exists a 1-form $\Upsilon \in \Omega^1(\mathcal{M})$ such that relation (4.39) holds for any pair of vector fields $X$ and $Y \in \Gamma(T\mathcal{M})$. Now, let $X, \bar{X} \in \Gamma(T\mathcal{M})$ be two vector fields satisfying $\bar{X} = \Omega X$, with $\Omega \in C^\infty(\mathcal{M})$ the time unit of $\psi$. Equation (4.39) allows to write

$$\nabla_X X = \bar{\nabla}_X X + 2\Upsilon(X) X$$

$$= \Omega^{-2} \bar{\nabla}_X \bar{X} - \Omega^{-2} \bar{X} [\ln \Omega] \bar{X} + 2 \Omega^{-2} \Upsilon(X) \bar{X}$$

The equality $\nabla_X X = \Omega^{-2} \bar{\nabla}_X \bar{X}$ is met if and only if $2\Upsilon(X) = \bar{X} [\ln \Omega]$ for any $\bar{X} \in \Gamma(T\mathcal{M})$, that is iff $\Upsilon \equiv \frac{1}{2} d[\ln \Omega]$. \hfill \Box

Condition (4.42) is reminiscent of the equation relating the affine parameterisations of a null geodesic vector field in two conformally related Riemannian manifolds (cf. e.g. Appendix D of [35]). Indeed, in [30], we will reinterpret this condition along these lines using hindsight provided by the ambient approach.

An explicit expression for the components of a conformally Galilean connection can be obtained, starting from relation (4.40) and plugging the components of a torsionfree Galilean connection $\bar{\nabla}$ given by eq.(3.22):

$$\Gamma^\lambda_{\mu\nu} = \bar{N}^\lambda \partial_{(\mu} \bar{\psi}_{\nu)} + \frac{1}{2} \bar{h}^\lambda_{\mu\nu} \left[ \partial_{(\mu} \bar{\gamma}_{\nu)} + \partial_{(\nu} \bar{\gamma}_{\mu)} - \partial_{(\rho} \bar{\gamma}_{\mu\nu)} \right] + \bar{h}^\lambda_{\rho\nu} \bar{\psi}_{(\mu} F^\rho_{\nu)}$$

$$+ \frac{1}{2} \left[ \delta^\lambda_{\mu} \partial_{\nu} \ln \Omega + \delta^\lambda_{\nu} \partial_{\mu} \ln \Omega \right]$$

42
where $\tilde{F} \in \Omega^2 (\mathcal{M})$ is closed in the conformally Newtonian case. Substituting along

$$\psi = \Omega \tilde{\psi}, \ h = \Omega^{-1} \tilde{h}, \ N = \Omega^{-1} \tilde{N}, \ \gamma = \tilde{\gamma}, \ F = \tilde{F}$$

leads to the component expression of a conformally Galilean connection in terms of its own Aristotelian structure, field of observers and gravitational fieldstrengths (unbarred quantities):

$$\Gamma^\lambda_{\mu \nu} = N^\lambda \partial_{(\mu} \psi_{\nu)} + \frac{1}{2} h^{\lambda \rho} \left[ \partial_{\mu} \gamma_{\rho \nu} + \partial_{\nu} \gamma_{\rho \mu} - \partial_{\rho} \gamma_{\mu \nu} \right] + h^{\lambda \rho} \psi_{(\mu} F_{\nu)\rho} + \frac{N}{\gamma_{\mu \nu}} h^{\lambda \rho} \partial_{\rho} \psi_{\sigma]} N^\sigma.$$  

(4.44)

Comparison with the formal expressions of the coefficients (3.22) and (4.44) reveals the presence of the extra term $\frac{N}{\gamma_{\mu \nu}} h^{\lambda \rho} \partial_{\rho} \psi_{\sigma]} N^\sigma$. This extra term has the nice feature of making the coefficients (4.44) invariant under a Milne boost, when the absolute clock $\psi$ satisfies the Frobenius Criterion. Thus no field of observers acquires a privileged status, in accordance with the Milne-invariance requirement.

### 4.2 Variational approach

We now provide additional clues regarding the naturality of conformally Newtonian connections via a derivation of expression (4.44) (with closed $\tilde{F}$) by adapting the “Lagrangian” setup for Newtonian connections (cf. Section 3.3) to the Aristotelian case. Our starting point is thus an Aristotelian structure $\mathcal{A} (\mathcal{M}, \psi, \gamma)$ endowed with a Lagrangian metric $g \in \Gamma (\sqrt{2} T^* \mathcal{M})$. We write the Lagrangian:

$$\mathcal{L} = \frac{1}{2} g_{\mu \nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau}$$

(4.45)

where $x : I \subseteq \mathbb{R} \to \mathcal{M} : \tau \mapsto x(\tau)$ is an observer.

Note that, in contradistinction with the Augustinian case, the absolute clock $\psi$ is not closed and the normalisation condition $\psi_\mu \frac{dx^\mu}{d\tau} = 1$ is therefore a non-holonomic constraint. Taking this constraint into account while varying the action

$$\mathcal{S} = \int \frac{1}{2} g_{\mu \nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} d\tau$$

(4.46)
we find the following Euler-Lagrange equations of motion:

\[ \frac{d^2 x^\lambda}{d\tau^2} + \Gamma^\lambda_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = 0 \]  

(4.47)

where the coefficients \( \Gamma^\lambda_{\mu\nu} \) now read:

\[ \Gamma^\lambda_{\mu\nu} = Z^\lambda \partial_{(\mu} \psi_{\nu)} + \frac{1}{2} h^{\lambda \rho} \left[ \partial_\mu g_{\rho\nu} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu} \right] + g_{\mu\nu} h^{\lambda \rho} \partial_{(\rho} \psi_{\sigma)} Z^\sigma. \]  

(4.48)

Comparing with the same derivation in the Augustinian case, we note the appearance of an extra term \( g_{\mu\nu} h^{\lambda \rho} \partial_{(\rho} \psi_{\nu)} Z^\sigma \) whose presence is due to the non-holonomicity of the constraint \( \psi_\mu \frac{dx^\mu}{d\tau} = 1 \). It can be checked that the expression (4.48) is invariant (when the absolute clock satisfies the Frobenius Criterion) under a Maxwell-gauge transformation \((\text{cf. Table 2})\). Similarly to the Newtonian case, one can make use of the relations of Table 2 in order to recast the previous expression in the form (4.44), with closed \( \bar{F} \).

We sum up the previous discussion by displaying an avatar of Propositions 3.22 and 3.26, thus providing a solution to the equivalence problem for conformally Newtonian connections in terms of Lagrangian variables:

**Proposition 4.6.** Let \( \mathcal{A}(\mathcal{M}, \psi, \gamma) \) be a given Aristotelian structure. There is a bijective correspondence between:

- conformally Newtonian manifolds \( \mathcal{N}(\mathcal{M}, \psi, \gamma, \nabla) \)
- gravitational potentials \([Z, \phi]\)
- Lagrangian structures \([g]\).

Before addressing torsional connections, we conclude the present section by proposing an axiomatic formulation of conformally Galilean connections which mimicks Definition 3.1. In order to do so we first compute the “compatibility” conditions of \( \nabla \) with \( \psi \) and \( h \) from the expression (4.44) for the coefficients of a conformally Galilean connection:

\[ \nabla_\mu \psi_\nu = \partial_\mu (\ln \Omega) \psi_\nu \]  

(4.49)

\[ \nabla_\mu h^{\alpha\beta} = \delta^{(\alpha}_{\mu} h^{\beta)}_\rho \partial_\rho (\ln \Omega). \]  

(4.50)
As can be foreseen in Proposition 3.5, the conformally Galilean connection, being torsionfree, is not compatible with the underlying Aristotelian structure. In particular, eq.(4.50) prevents the conformally Galilean connection from reducing to the Levi-Civita connection for the spatial metric $\gamma$ on the absolute spaces $\Sigma_t$. Rather, the last term in (4.44) furnishes a contribution so that

$$\Gamma^i_{jk} = \frac{1}{2} \gamma^{im} \left[ \partial_j \gamma_{km} + \partial_k \gamma_{jm} - \partial_m \gamma_{jk} + \gamma_{jk} \partial_m (\ln \Omega) \right].$$

(4.51)

Concretely, this means that the parallel transport of spacelike objects on an absolute space defined by a proper (i.e. $\Omega \neq$ constant) conformally Galilean connection does not perfectly preserve lengths and angles. This somewhat odd feature (a sort of non-metricity) is in contradistinction with the case of Galilean manifolds. However, from the discussion in the beginning of Subsection 4.1 and the explicit expression (4.51) one can show that $\Gamma^i_{jk}$ and the Levi-Civita associated to $\bar{\gamma}_{ij}$ belong to the same special projective structure and thus define the same unparameterised geodesics on each absolute space.

Adopting a bottom-top approach, we can formulate a set of axioms allowing to recover the “compatibility” conditions (4.49)-(4.50) as follows:

**Proposition 4.7.** Let $\mathcal{A}(\mathcal{M}, \psi, \gamma)$ be an Aristotelian structure and $\nabla$ a torsionfree Koszul connection on $\mathcal{M}$. The compatibility conditions (4.49)-(4.50) are satisfied if and only if $\nabla$ satisfies the two axioms:

1. The absolute clock is a conservor for $\nabla$.

2. There exists a vector field $V$ on $\mathcal{M}$ such that

$$(\nabla_X h)(\alpha, \beta) = \alpha(X)\beta(V) + \alpha(V)\beta(X)$$

for any vector field $X$ and for any 1-forms $\alpha$ and $\beta$ on $\mathcal{M}$.

**Proof:** First, we recall that since $\psi$ is assumed to induce an involutive distribution, being part of an Aristotelian structure (cf. Definition 2.15), Frobenius theorem ensures that locally, there always exists a function $\Omega \in C^\infty(\mathcal{M})$ such that $d\psi = d(\ln \Omega) \wedge \psi$. Equation (4.49) follows straightforwardly from axiom 1.
assuming that $\psi$ is a conservor (i.e. $\nabla_\mu \psi_\nu = 0$), so that $\nabla_\mu \psi_\nu = \nabla_{[\mu} \psi_{\nu]} = \partial_{[\mu} \psi_{\nu]} = \partial_{[\mu} (\ln \Omega) \psi_{\nu]}$.

Before establishing eq.(4.50), let us show how the non-preservation of $\psi$ affects the consistency requirement $\nabla_\mu (h^{\alpha\beta} \psi_\beta) = 0$. Indeed, $(\nabla_\mu h^{\alpha\beta}) \psi_\beta = -h^{\alpha\beta} (\nabla_\mu \psi_\beta) = -h^{\alpha\beta} \partial_\mu (\ln \Omega) \psi_\beta] = \frac{1}{2} h^{\alpha\beta} \partial_\beta (\ln \Omega) \psi_\mu$. Now axiom 2 reads in components as $\nabla_\mu h^{\alpha\beta} = 2 \delta^{(\alpha}_\mu V^{\beta)}$. Together with the former consistency requirement $(\nabla_\mu h^{\alpha\beta}) \psi_\beta = \frac{1}{2} h^{\alpha\beta} \partial_\beta (\ln \Omega) \psi_\mu$, one first deduce that $(\nabla_\mu h^{\alpha\beta}) \psi_\alpha \psi_\beta = 2 \psi_\mu \psi_\alpha V^\alpha = 0$ so that $\psi_\alpha V^\alpha = 0$ which in turn implies that $(\nabla_\mu h^{\alpha\beta}) \psi_\beta = \psi_\mu V^\alpha = \frac{1}{2} h^{\alpha\beta} \partial_\beta (\ln \Omega) \psi_\mu$. The last equality gives the expression for $V$ as $V^\alpha = \frac{1}{2} h^{\alpha\beta} \partial_\beta (\ln \Omega)$, so that $\nabla_\mu h^{\alpha\beta} = \delta^{(\alpha}_\mu h^{\beta)}_\rho \partial_\rho (\ln \Omega)$.
5 Torsional Galilean connections

So far, we chose to restrict the scope of our analysis to nonrelativistic structures endowed with torsionfree connections. Such a restriction is quite natural when one is dealing with nonrelativistic metric structures whose absolute clock is closed (Augustinian structures) since, in this case, there exist torsionfree connections which are furthermore compatible with the metric structure (cf. Theorem 3.7), similarly to the relativistic case. Nonetheless, the introduction of torsional connections acquires increased relevance when considering nonrelativistic metric structures with non-closed absolute clock, since then the torsionfree condition and metric compatibility become mutually exclusive (cf. Proposition 3.5). In the last section, when considering parallelism for Aristotelian structures, we were brought to favour the torsionfree condition at the expense of the metric compatibility, by considering connections “conformally equivalent” to Galilean/Newtonian ones. However, the alternative route is equally worth of exploration, as proven by the recent surge of interest in generalisations of Newton-Cartan geometry characterised by torsional connections (e.g. [13, 14, 19]). Such approaches focus on a Leibnizian structure endowed with a Koszul connection whose torsion is tuned in order to ensure compatibility with the absolute clock and rulers. In particular, the works [20, 21] exhibit a torsional connection compatible with the metric structure while remaining invariant under Milne boosts. This nice feature is achieved by making use of the manifestly Milne-invariant “Lagrangian” variables (i.e. $Z, \phi, g$ in Section 3.3). We start this section by reviewing some standard results regarding relativistic torsional connections before studying their nonrelativistic analogues, namely torsional Galilean connections. We then propose a generalisation of the notion of Newtonian connection to the torsional case, allowing the use of the “Lagrangian” variables. The Koszul connection exhibited in [20, 21] will thus be shown to belong to this special class. The present discussion is intended as prolegomena in order to pave the way to the description of the embedding of torsional Galilean manifolds inside relativistic spacetimes in [30].

In the relativistic case, as expressed in Theorem 2.4, the set of Koszul connections compatible with a given Lorentzian structure $(\mathcal{M}, g)$ forms a vector space isomorphic to the vector space of torsion tensors $T \in \Gamma (\wedge^2 T^\ast \mathcal{M} \otimes T\mathcal{M})$, so that any Koszul connection $\nabla$ compatible with $g$ is uniquely determined by the gift of a tensor $T$. 

47
This result can be made manifest by displaying the so-called “Koszul formula” which allows to express the action of $\nabla$ on vector fields solely in terms of the metric and torsion tensors:

$$
2 g (\nabla_X Y, Z) = X [g (Y, Z)] + Y [g (X, Z)] - Z [g (X, Y)]
+ g ([X, Y], Z) - g ([Y, Z], X) - g ([X, Z], Y)
+ g (T (X, Y), Z) - g (T (Y, Z), X) - g (T (X, Z), Y)
$$

(5.52)

with $X, Y, Z \in \Gamma (T \mathcal{M})$.

The previous expression can be recast in components as:

$$
\Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\rho} [\partial_\mu g_{\rho\nu} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu}] + \Gamma^\lambda_{[\mu\nu]} + \Gamma^\rho_{[\sigma\mu]} g^\sigma\lambda g_{\rho\nu} + \Gamma^\rho_{[\sigma\nu]} g^\sigma\lambda g_{\rho\mu}
$$

where $\Gamma^\lambda_{[\alpha\beta]} = \frac{1}{2} T^\lambda_{a\beta}$.

We now switch to the study of nonrelativistic torsional manifolds. The following Theorem, generalising Theorem 3.7, can be seen as a nonrelativistic avatar of Theorem 2.4:

**Theorem 5.1 (cf. [22])**. Given a field of observers $N \in FO (\mathcal{M})$, the space of Koszul connections compatible with a given Leibnizian structure $\mathcal{L} (\mathcal{M}, \psi, \gamma)$ is isomorphic to the vector space $\Omega^2 (\mathcal{M}) \times \Gamma (\wedge^2 T^* \mathcal{M} \otimes Ker \psi)$.

Explicitly, for a given field of observers $N$, the arbitrariness in the compatible Koszul connection is encoded in a couple $\left( F^N, P^N (T) \right)$ with

- $F^N \in \Omega^2 (\mathcal{M})$ a 2-form on $\mathcal{M}$,
- $P^N (T) \in \Gamma (\wedge^2 T^* \mathcal{M} \otimes Ker \psi)$ the spacelike projection (cf. Definition 2.14) of a torsion tensor $T \in \Gamma (\wedge^2 T^* \mathcal{M} \otimes T \mathcal{M})$ whose timelike part is constrained to satisfy eq.(3.17).

As noted in [22], the fibers of the vector bundles $\Gamma (\wedge^2 T^* \mathcal{M} \otimes T \mathcal{M})$ and $\Omega^2 (\mathcal{M}) \times \Gamma (\wedge^2 T^* \mathcal{M} \otimes Ker \psi)$ both have dimension $\frac{d(d+1)^2}{2}$ (for $(d + 1)$-dimensional space-times), so that the amount of freedom in the choice of a (potentially torsional) compatible Koszul connection is the same in the relativistic and nonrelativistic cases. This important observation softens the sharp distinction between relativistic and
nonrelativistic cases that hold in the torsionfree case. An ambient perspective of this fact will be provided in [30].

A nonrelativistic equivalent of the Koszul formula can be formulated as

\[ 2 \gamma (\nabla_X Y, V) = X \left[ \gamma (Y, V) \right] + Y \left[ \gamma (X, V) \right] - V \left[ \gamma (X, Y) \right] + \gamma ([X, Y] , V) - \gamma ([Y, V] , X) - \gamma ([X, V] , Y) \]

\[ + \gamma (T (X, Y) , V) - \gamma (T (Y, V) , X) - \gamma (T (X, V), Y) \]

\[ + \psi (X) F (Y, V) + \psi (Y) F (X, V) \]

with \( X, Y \in \Gamma (T \mathcal{M}) \) and \( V \in \text{Ker} \psi \). In components, this expression reads:

\[ \Gamma^\lambda_{\mu \nu} = N^\lambda \partial_{(\mu} \psi_{\nu)} + \frac{1}{2} h^{\lambda \rho} \left[ \partial_{\mu} N^\rho_{\nu} + \partial_{\nu} N^\rho_{\mu} - \partial_{\rho} N^\rho_{\mu \nu} \right] + h^{\lambda \rho} \psi (N^F)_{\rho \nu} \]

\[ + \Gamma^\lambda_{[\mu \nu]} + \Gamma^p_{[\sigma \mu]} h^{\sigma \lambda \nu} + \Gamma^p_{[\sigma \nu]} h^{\sigma \lambda \mu} \]

(5.54)

Similarly to the torsionfree case, the 2-form \( \bar{F} \in \Omega^2 (\mathcal{M}) \) stands for the gravitational fieldstrength measured by the field of observers \( N \) (cf. Definition 3.8) whose explicit expression is given by (3.18). Once again, we make the assumption that the Galilean connection \( \nabla \) is Milne-invariant (in particular, the torsion tensor is assumed Milne-invariant) and derive the transformation relations of \( \bar{F} \) under a Milne boost (cf. Proposition 2.19). One finds that, under a Milne boost parameterised by the 1-form \( \chi \in \Omega^1 (\mathcal{M}) \), the measured gravitational fieldstrength \( F \) transforms as

\[ F^\alpha_{\beta} \rightarrow F^\alpha_{\beta} + 2 \nabla [\alpha \bar{F}^\gamma_{\beta}] \]

(5.55)

with the 1-form \( \bar{\Phi} \in \Omega^1 (\mathcal{M}) \) as in eq.(3.19).

**Definition 5.2** (Torsional gravitational fieldstrength). Let \( N \) denote a field of observers, \( F \) a 2-form on \( \mathcal{M} \) and \( P^N (T) \) a spacelike vector-field-valued 2-form on \( \mathcal{M} \). An orbit \( [N, F, P^N (T)] \) under Milne boosts is called a torsional gravitational fieldstrength. In other words, a torsional gravitational fieldstrength is an equivalence class where two triplets \( [N', F', P^{N'} (T)] \) and \( [N, F, P^N (T)] \) with \( N', N \in FO (\mathcal{M}) \), \( N', F', F \in \Omega^2 (\mathcal{M}) \) and \( P^N (T), P^{N'} (T) \in \Gamma (\lambda^2 T^* \mathcal{M} \otimes \text{Ker} \psi) \) are said equivalent if

---

18 A Koszul formula for Galilean connections was first obtained in [22]. Our expression (5.53) presents the advantage of being closer to its relativistic avatar (5.52).
there exists a 1-form $\chi \in \Omega^1 (\mathcal{M})$ such that:

$$
\begin{align*}
N^\mu &= N^\mu + h^{\mu\nu} \chi_\nu \\
N'_{\alpha\beta} &= F_{\alpha\beta} + 2\nabla_{[\alpha} \hat{\Phi}_{\beta]} \\
^{P^N'} (T) &= P^N (T) - h (\chi) d\psi
\end{align*}
\tag{5.56}
$$

where the expression of the 1-form $\hat{\Phi} \in \Omega^1 (\mathcal{M})$ is given by eq. (3.19).

**Corollary 5.3.** Milne-invariant torsional Galilean connections compatible with a given Leibnizian structure are in bijective correspondence with torsional gravitational fieldstrengths.

Having reviewed the characterisation of torsional Galilean manifolds, we are now in a position to look for a torsional generalisation of the notion of Newtonian connection. Firstly, in view of expression (5.55), one deduces that, in contradistinction to the torsionfree case, a Milne boost does not preserve the condition that the 2-form $F$ is closed ($dF = 0$), since $\nabla_{[\alpha} \hat{\Phi}_{\beta]}$ is not generically exact. Consequently, since we are considering Milne-invariant connections, we are led to discard the condition $dF = 0$ as a potential candidate aiming at generalising the notion of Newtonian connection since it is inconsistent whenever torsion is involved. However, relying on the form of eq. (5.55), a natural condition consists in imposing that $F$ is covariantly exact, in the following sense:

**Definition 5.4** (Covariantly exact differential form). A differential $p$-form $\alpha \in \Omega^p (\mathcal{M})$ is said to be covariantly exact for the Koszul connection $\nabla$ if there exists a $(p - 1)$-form $\beta \in \Omega^{p-1} (\mathcal{M})$ such that $\alpha_{\mu_1...\mu_p} = \nabla_{[\mu_1} \beta_{\mu_2...\mu_p]}$.

For the case when $\nabla$ is torsionfree, the notion of a covariantly exact form identifies with the one of an exact form. Based on this notion, one can now articulate a generalised definition of Newtonian connection as:

**Definition 5.5** (Torsional Newtonian connection). Let $\mathcal{G} (\mathcal{M}, \psi, \gamma, \nabla)$ be a Galilean manifold whose Koszul connection $\nabla$ is characterised by the torsional gravitational fieldstrength $\left[ N, F, P^N (T) \right]$. The Koszul connection $\nabla$ is said to be a torsional...
Newtonian connection if the 2-form \( N^F \in \Omega^2(\mathcal{M}) \) associated to a given \( N \in FO(\mathcal{M}) \) is covariantly exact.

The transformation law given by eq.\((5.55)\) ensures the consistency of the previous definition, since the covariant exactness of a 2-form \( N^F \) is preserved by a Milne boost.

Given a field of observers \( N \in FO(\mathcal{M}) \), we denote \( A \in \Omega^1(\mathcal{M}) \) the 1-form defined as \( F_{[\alpha\beta]} \equiv \nabla_{[\alpha}A_{\beta]} \).

**Definition 5.6** (Torsional gravitational potentials). Let \( N \) denote a field of observers, \( \tilde{A} \) a 1-form on \( \mathcal{M} \) and \( P^N(T) \) a spacelike vector-field-valued 2-form on \( \mathcal{M} \). An orbit \( \left[ N, \tilde{A}, P^N(T) \right] \) under Milne boosts is called a torsional gravitational fieldstrength. In other words, a torsional gravitational fieldstrength is an equivalence class where two triplets \( \left( N', \tilde{A}, P^{N'}(T) \right) \) and \( \left( N, \tilde{A}, P^N(T) \right) \) with \( N', N \in FO(\mathcal{M}), \ N', \tilde{A}, A \in \Omega^1(\mathcal{M}) \) and \( P^N(T), P^{N'}(T) \in \Gamma(\wedge^2 T^* \mathcal{M} \otimes \text{Ker} \psi) \) are said equivalent if there exists a 1-form \( \chi \in \Omega^1(\mathcal{M}) \) such that:

\[
\begin{aligned}
N'^\mu &= N^\mu + h^{\mu\nu}\chi^\nu \\
N'^A_\alpha &= N_A^\alpha + \tilde{\Phi}_A^\alpha \\
P^{N'}(T) &= P^N(T) - h(\chi) d\psi
\end{aligned}
\]  

where the expression of the 1-form \( \tilde{\Phi} \in \Omega^1(\mathcal{M}) \) is given by eq.\((3.19)\).

**Corollary 5.7.** Milne-invariant torsional Newtonian connections compatible with a given Leibnizian structure are in bijective correspondence with torsional gravitational potentials.

It is worth stressing that, contrarily to the case of a torsionfree Newtonian connection, there is no additional Maxwell gauge symmetry at hand, as can be observed by comparison between the transformation relations \((5.56)\) and \((3.27)\). In the case of vanishing torsion, the origin of the supplementary gauge invariance can be traced back to the closed condition \( dF = 0 \) which allows to locally write the 2-form \( F^N \) as the gravitational fieldstrength for the gravitational potential \( A \in \Omega^1(\mathcal{M}) \) measured by \( N \) i.e. \( F = dA \), so that \( F \) is invariant under a gauge transformation of the form \( A \rightarrow A + df \), with \( f \in C^\infty(\mathcal{M}) \). However, when \( F \) is covariantly exact,
i.e. \( F_{\alpha\beta} = \nabla_{[\alpha} A_{\beta]} \), the torsion term if non-vanishing breaks the invariance. This important distinction motivates the following terminology: a torsionful Newtonian connection is a torsional Newtonian connection with non-vanishing torsion. Accordingly, a Newtonian connection is a torsional Newtonian connection with vanishing torsion.

Similarly to the torsionfree case, an alternative description of torsional Newtonian connections can be given by making use of the “Lagrangian” variables \( Z, \phi \) and \( g \) (cf. Table 2). Starting from the component expression (5.54) with \( F_{[\alpha\beta]} = \nabla_{[\alpha} A_{\beta]} \), this is achieved by performing a Milne boost parameterised by the 1-form \( \chi \equiv -A \), under which the torsional Newtonian connection takes the manifestly Milne-invariant form:

\[
\Gamma^\lambda_{\mu\nu} = Z^\lambda \partial_{(\mu} \psi_{\nu)} + \frac{1}{2} h^{\lambda\rho} [\partial_{\rho} g_{\mu\nu} + \partial_{\nu} g_{\rho\mu} - \partial_{\rho} g_{\mu\nu}]
\]

\[
+ \Gamma^\lambda_{[\mu\nu]} + \Gamma^\rho_{[\sigma\mu]} h^{\sigma\lambda} g_{\rho\nu} + \Gamma^\rho_{[\sigma\nu]} h^{\sigma\lambda} g_{\rho\mu}.
\] (5.58)

By construction, the field of observers \( Z \) is the unique vector field which is Coriolis-free with respect to \( \nabla \) (as follows by repeating the steps in the proof of Proposition 3.18, cf. Appendix A). This reformulation, along with Proposition 3.24, allows to articulate two additional characterisations of torsionful Newtonian manifolds, as formulated in the following Proposition:

**Proposition 5.8.** Let \( \mathcal{L}(\mathcal{M}, \psi, \gamma) \) be a Leibnizian structure. There is a bijective correspondence between torsionful Newtonian manifolds and

1. triplets \((Z, \phi, P^Z(T))\), where \( Z \in FO(\mathcal{M}) \), \( \phi \in C^\infty(\mathcal{M}) \) and \( P^Z(T) \in \Gamma(\wedge^2 T^*\mathcal{M} \otimes \text{Ker} \psi) \).

2. couples \((g, T)\) where \( g \in \Gamma(\sqrt{2} T^*\mathcal{M}) \) is a Lagrangian metric and \( T \in \Gamma(\wedge^2 T^*\mathcal{M} \otimes T\mathcal{M}) \) is a non-vanishing torsion tensor whose timelike part is constrained to satisfy \( \psi(T(X,Y)) = d\psi(X,Y) \) for all \( X,Y \in \Gamma(T\mathcal{M}) \).

The two items in the previous proposition complement the content of Propositions 3.22 and 3.26, respectively, to account for the torsionful case. As commented earlier, the torsionfree case is special in that it involves an additional gauge invariance.

In the torsionful case where no such symmetry is present\(^{19}\), torsionful Newtonian

\(^{19}\)In other words, the coefficients (5.58) are not invariant under the transformations recorded in Table 2.
connections are characterised by individual objects rather than classes thereof. This justifies the formal separation between the torsionfree and torsionful cases in two sets of Propositions.

According to the first item of Proposition 5.8 if one is given a Leibnizian structure and a field of observers $Z$, the space of torsional Newtonian connections possesses a structure of vector space isomorphic to $C^\infty(M) \times \Gamma(\wedge^2 T^*M \otimes \text{Ker} \psi)$. The role of the zero of this vector space is played by the torsional Newtonian connection characterised by $\phi = 0$ and $P^Z(T) = 0$ generalising the special connection to the torsionful case. This last condition allows to write the torsion tensor $T \in \Gamma(\wedge^2 T^*M \otimes T.M)$ as

$$T = Z \, d\psi. \tag{5.59}$$

In components, the previous expression reads $\Gamma^\lambda_{\mu\nu} = Z^\lambda \partial_{[\mu} \psi_{\nu]}$. Plugging $\phi = 0$, $g = \gamma$ and the expression of $T$ inside (5.58) leads to:

$$\Gamma^\lambda_{\mu\nu} = Z^\lambda \partial_{\mu} \psi_{\nu} + \frac{1}{2} \delta^\lambda_{\mu\rho} \left[ \partial_{\mu} Z_{\rho\nu} + \partial_{\nu} Z_{\rho\mu} - \partial_{\rho} Z_{\mu\nu} \right] \tag{5.60}$$

These are the components of the torsional connection introduced in the works [20, 21]. Note that, besides being Coriolis-free, the observers defined as integral curves of $Z$ are also geodesic, since the gravitational gauge scalar $\phi$ has been chosen to vanish.

The different structures necessary to uniquely determine a given manifold are summarised in the following table, both in the relativistic and nonrelativistic cases.
| Metric structure | Supplementary structure | Manifold |
|------------------|-------------------------|----------|
| Lorentzian       | ×                       | Lorentzian |
| Augustinian      | Gravitational fieldstrength $\left[ N, F \right]$ | Galilean |
|                  | Gravitational potential $\left[ N, A^N, \left[ Z, \phi \right], \left[ g \right] \right]$ | Newtonian |
| Aristotelian     | Gravitational fieldstrength $\left[ N, F \right]$ | Conformally Galilean |
|                  | Gravitational potential $\left[ N, A^N, \left[ Z, \phi \right], \left[ g \right] \right]$ | Conformally Newtonian |
| Leibnizian       | Torsional gravitational fieldstrength $\left[ N, F, P^N (T) \right]$ | Torsional Galilean |
|                  | Torsional gravitational potential $\left[ N, A, P^N (T) \right], \left( Z, \phi, P^Z (T) \right), \left( g, T \right)$ | Torsional Newtonian |

Table 3: Solutions to the equivalence problem
6 Conclusion

In this work, we investigated two novel generalisations of Newtonian connections for metric structures with non-closed absolute clock. We went down the two following crossing roads: Firstly, we presented a connection living on the most general nonrelativistic metric structure allowing a notion of absolute time (Aristotelian structure). This torsionfree connection has the nice feature of being both Milne and Maxwell gauge-invariant and such that its geodesic equation follows from a variational principle, similarly to its Newtonian counterpart to which it can be said conformally equivalent. Furthermore, they belong to the same projective structures, hence they define the same unparameterised geodesics. Secondly, we reviewed torsional Galilean connections and isolated a Milne-invariant subclass (dubbed torsional Newtonian connections) characterised by a covariantly exact 2-form. Similarly to their torsionfree counterparts, torsional Newtonian connections can be expressed in terms of (manifestly Milne-invariant) Lagrangian variables which we used in order to make contact with the recently introduced torsional Newton-Cartan geometry [20, 21]. We further discussed the geometric origin behind the lack of Maxwell gauge-invariance of the latter.

The present analysis restricted to an intrinsic point of view on nonrelativistic connections, with particular emphasis placed on the equivalence problem. In a forthcoming companion paper [30], we will discuss an ambient approach to these classes of connections by generalising the Bargmann framework of Duval et al. (cf. [31, 36]) where nonrelativistic Newtonian manifolds were obtained as null dimensional reduction of a specific class of relativistic ones. As hinted in Subsection 3.4, this approach is indeed quite natural for Newtonian manifolds but can be extended to embed more general nonrelativistic structures. The conformally Newtonian connection will hence be shown to arise as a projection of the Levi-Civita connection for a class of relativistic spacetimes admitting a null and hypersurface-orthogonal Killing vector field. These spacetimes were studied in [37] and dubbed Platonic waves in [23] where they were shown to be conformally related to the class of [31, 36]. This ambient construction will notably allow us to formulate at the level of connections the Eisenhart-Lichnerowicz lift [38] of dynamical trajectories to relativistic geodesics. Furthermore, an ambient account of (torsional) Galilean connections will
be provided by considering a class of relativistic spacetimes endowed with a (possibly torsional) connection parallelising a null Killing vector field. In particular, this setup will allow us to embed torsionfree Galilean manifolds into torsional relativistic spacetimes, thus shedding new light on the torsional origin of the gravitational field-strength. Finally, we will identify the restriction on the relativistic torsion suitable for embedding torsional Newtonian connections.

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A Curvature of a torsionfree Galilean manifold

Let us recall the definition of the curvature of a Koszul connection for the vector bundle $E$ on $\mathcal{M}$:

$$R(X,Y;f) = \nabla_X \nabla_Y f - \nabla_Y \nabla_X f - \nabla_{[X,Y]} f$$

with $X,Y \in \Gamma (T\mathcal{M})$ and $f \in \Gamma (E)$. The components of the Koszul curvature for the tangent bundle $E = T\mathcal{M}$ read:

$$dx^\lambda \left[ R(\partial_\mu, \partial_\nu; \partial_\rho) \right] \equiv R^\lambda_{\rho\mu\nu}.$$

Compatibility conditions (3.15) for the Galilei connection $\nabla$ impose the following constraints on the Koszul curvature:

$$\begin{align*}
\nabla_\mu \psi_\nu &= 0 \Rightarrow \psi_\lambda R^\lambda_{\rho\mu\nu} = 0 \\
\nabla_\mu h^{\alpha\beta} &= 0 \Rightarrow h^\rho_{\beta} R^\alpha_{\mu\rho\nu} + h^\alpha_{\rho} R^\beta_{\rho\mu\nu} = 0
\end{align*}$$

Notation A.1. In the following we will use a Galilean basis $B \equiv \{N, e_i\}$ together with its dual $B^* \equiv \{\psi, \theta^i\}$. Now, let $T^\mu_\nu$ be the holonomic components of a tensor $T \in \Gamma (T\mathcal{M} \times T^*\mathcal{M})$. The following notation will prove to be handy:

$$\begin{align*}
T^0_\nu &\equiv \psi_\mu T^\mu_\nu \\
T^i_\nu &\equiv \theta^i_\mu T^\mu_\nu \\
T^\mu_0 &\equiv N^\mu T^\mu_\nu \\
T^\mu_i &\equiv e^\mu_i T^\mu_\nu
\end{align*}$$

The previously stated constraints on the Koszul curvature can thus be reexpressed as:

$$R^0_{\rho\mu\nu} = 0 = R^{(ij)}_{\mu\nu}. \quad (A.61)$$

Taking these constraints into account, the components of the curvature 2-form $R^\lambda_\rho \in \Omega^2 (\mathcal{M})$ can be expanded as:

$$R^\lambda_\rho = R^i_\rho \theta^j_\rho e^\lambda_i + R^i_0 \psi_\rho e^\lambda_i. \quad (A.62)$$

Proposition A.2 (Symmetries of the Galilean curvature). The curvature tensor of
a torsionfree Galilean connection satisfies the following identities:

\[
\begin{cases}
R^i_{\rho(\mu\nu)} = 0 \\
R^i_{[\rho\mu\nu]} = 0 \\
R^i_{j\,k\,l} = R^j_{i\,k\,l}.
\end{cases}
\]  (A.63)

**Proof:** These equalities follow respectively from the well-known identities

\[
\begin{align*}
R(X,Y;Z,W) &= -R(Y,X;Z,W), \\
R(X,Y;Z) + R(Y,Z;X) + R(Z,X;Y) &= 0, \\
R(X,Y;Z,W) &= R(Z,W;X,Y),
\end{align*}
\]

where \( R(X,Y;Z,W) \equiv \gamma(R(X,Y;Z),W) \). \(\square\)

The second identity of the previous Proposition, known as the first Bianchi identity, can be decomposed further:

**Proposition A.3.** The first Bianchi identity for the Galilei curvature leads to the following set of equations:

\[
\begin{cases}
R^l_{[ij]0} + \frac{1}{2}R^l_{0ij} = 0 \\
R^l_{[ijk]} = 0.
\end{cases}
\]  (A.64)

**Corollary A.4 (cf. [37]).** The curvature tensor of a torsionfree Galilean manifold satisfies the relation:

\[
R^{(i\,j)}_{[\mu\nu]} = 0
\]

**Proof:** The relation is equivalent to the set:

\[
\begin{cases}
R^{(i\,j)}_{[k\,l]} = 0 \\
R^{(i\,j)}_{[0\,k]} = 0.
\end{cases}
\]
The first relation follows straightforwardly from the all-spacelike first Bianchi identity and compatibility relations (A.61). The second identity is obtained by taking the symmetric part in \((k \leftrightarrow i)\) of the temporal/spacelike Bianchi identity:

\[
R_{(k\i)j0} - R_{(k|j|i)0} + R_{(k|0|i)j} = 0. \tag{A.65}
\]

The first term vanishes, leaving \(R^{(k \i)}_{[0 \i]} = 0\). \(\square\)

We now focus on the Duval-Künzle condition (cf. Definition 3.13) which, in components reads:

\[
R^{\mu \nu \alpha \beta} = R^{\nu \mu \beta \alpha}.
\]

Decomposing on a Galilean basis leads to the set of equations:

\[
\begin{cases}
R^{i j k l} = R^{j i l k} \\
R^{[i j]0 0} = 0 \\
R^{j i 0 k} = R^{i j k 0}
\end{cases} \tag{A.66}
\]

The first equation is already implied by the first Bianchi identity. However the two remaining are non-trivial constraints that reduce the number of independent components from \(\frac{1}{12}d^2(d+1)(d+5)\) to \(\frac{1}{12}(d+1)^2 \left[(d+1)^2 - 1\right]\), i.e. to the same number as in a \((d+1)\)-dimensional (pseudo)-Riemannian manifold (cf. e.g. [6]).

**Proposition A.5.** The Duval-Künzle condition can be alternatively written as

\[
R^i_0 \wedge \theta_i = 0. \tag{A.67}
\]

**Proof:** The alternative formulation \(R^i_0 \wedge \theta_i = 0\) of the Duval-Künzle condition imposes the following constraints on the components of \(R^i_0 = R^{i j 0}_0 \theta_j \wedge \psi + \)
\[ \frac{1}{2} R^{i \ j \ k \ 0} \theta_j \wedge \theta_k : \]

\[
\begin{aligned}
R^{[i \ j \ 0]}_0 &= 0 \\
R^{[i \ k \ l \ 0]} &= 0.
\end{aligned}
\]

The first equality matches the second one from (A.66) so what remains to be proved is the following equivalence:

\[ R^j_0 \ i \ k = R^i_0 \ j \ k \Leftrightarrow R^{[i \ j \ k \ 0]} = 0. \quad (A.68) \]

We start by totally antisymmetrising the first of the identities of the Bianchi set (A.64):

\[ R^{[i \ j \ k \ 0]} + \frac{1}{2} R^{[i \ j \ k]} = 0. \quad (A.69) \]

Expanding the first term leads to:

\[ \frac{1}{3} \left( 2 R^{i \ [k \ j \ 0]} - R^{i \ k \ j \ 0} \right) + \frac{1}{2} R^{[i \ j \ k \ 0]} = 0. \quad (A.70) \]

Using again the first Bianchi identity allows to transform the first term on the left-hand side:

\[ \frac{1}{3} \left( R^{j \ 0 \ i \ k} - R^{i \ k \ j \ 0} \right) + \frac{1}{2} R^{[i \ j \ k \ 0]} = 0. \quad (A.71) \]

so that \( R^j_0 \ i \ k = R^i_0 \ j \ k \Leftrightarrow R^{[i \ j \ k \ 0]} = 0. \]

Along the Duval-Künzle condition, another constraint on the curvature, dubbed the Trautman condition\(^{20}\) is frequently encountered in the literature:

**Definition A.6** (Trautman condition, cf. e.g. [39]). Let \( J \) be the Jacobi curvature operator defined as

\[ J (X, Y; Z) \equiv \frac{1}{2} (R (Z, X; Y) + R (Z, Y; X)) \quad (A.72) \]

\(^{20}\)Although the denominations Duval-Künzle and Trautman conditions seem customary in the literature, it is amusing to note that in the respective works commonly cited when these conditions are discussed, Trautman wrote what is usually referred to as the Duval-Künzle condition (cf. eq.(IV) of [3]) while Künzle wrote the Trautman condition (cf. eq.(4.14) of [6]).
where $X$, $Y$ and $Z$ are three vector fields. The Trautman condition states that the Jacobi operator must be self-adjoint when acting on spacelike vectors, i.e.

\[ \gamma(J(X,Y;V),W) = \gamma(J(X,Y;W),V) \]  
(A.73)

$\forall X,Y \in \Gamma(T\mathcal{M})$ and $\forall V,W \in \text{Ker } \psi$.

In components, the Jacobi operator reads $J^\lambda_{\rho\mu\nu} = R^\lambda_{\rho(\mu|\nu)} = -R^\lambda_{(\mu|\rho)\nu}$ while the Trautman condition imposes:

\[ R^{|i|j}_{(\mu|\nu)} = 0. \]  
(A.74)

**Proposition A.7** (cf. [37]). The Duval-Künzle and Trautman conditions are equivalent for a torsionfree Galilean manifold.

**Proof:** Decomposing the Duval-Künzle operator as:

\[
R^{i\ j}_{\mu\nu} - R^{j\ i}_{\nu\mu} \quad \text{(DK)} = \frac{1}{2} \left( R^{(i\ j)}_{\mu\nu} + R^{[i\ j]}_{\mu\nu} - R^{(j\ i)}_{\nu\mu} - R^{[i\ j]}_{\nu\mu} \right)
\]

\[
= \frac{1}{2} \left( R^{(i\ j)}_{\mu\nu} + R^{[i\ j]}_{\mu\nu} - R^{(j\ i)}_{\nu\mu} + R^{[i\ j]}_{\nu\mu} \right)
= R^{|i|j}_{(\mu|\nu)} + R^{[i\ j]}_{(\mu|\nu)} \quad \text{(C)}
\]

\[
= R^{(i\ j)}_{\mu\nu} + R^{[i\ j]}_{(\mu|\nu)} \quad \text{(T)}
\]

(A.75)

one recognises the operator (C) obtained in Corollary A.4 as well as the Trautman operator (T). Provided Corollary A.4, the Duval-Künzle and Trautman conditions are therefore equivalent. □
Proof of Proposition 3.18  Let us first check that the previous definition for $Z$ is well-defined under a change of representative (cf. eq.(3.27)). This is easily seen as:

$$Z' = N' - h \left( \frac{N'}{A} \right) + h (df')$$

$$= N + h (\chi) - h \left( \frac{N}{A + \Phi} + df \right) + h (df')$$

$$= N - h \left( \frac{N}{A} \right) + h (df' - f)$$

since $\Phi$ is given by (3.19).

Now, let us compute the Coriolis 2-form of a field of observers $Z = N + h (\chi)$, with $\chi \in \Omega^1 (\mathcal{M})$:

$$Z \omega (V, W) = \gamma (\nabla_V Z, W) - \gamma (V, \nabla_W Z)$$

$$= N \omega (V, W) + \gamma (\nabla_V h (\chi), W) - \gamma (\nabla_W h (\chi), V)$$

with $V, W \in \text{Ker } \psi$. Note that the second and third terms make sense, since $\psi(\nabla_V h (\chi)) = V [\psi(h(\chi))] = 0$. Using $\nabla \gamma = 0$ allows to reformulate the first term in brackets as $\gamma (\nabla_V h (\chi), W) = V [\gamma (h (\chi), W)] - \gamma (h (\chi), \nabla_V W)$. Proceeding similarly with the second term in brackets leads to:

$$Z \omega (V, W) = N \omega (V, W) + \left( V [\gamma (h (\chi), W)] - \gamma (h (\chi), \nabla_V W) - (V \leftrightarrow W) \right)$$

$$= N \omega (V, W) + V [\chi (W)] - W [\chi (V)] - \chi (\nabla_V W - \nabla_W V)$$

$$= N \omega (V, W) + V [\chi (W)] - W [\chi (V)] - \chi ([V, W])$$

$$= N \omega (V, W) + d\chi (V, W)$$

where one used respectively: in the first step, the equality $\gamma (h (\alpha), X) = \alpha (X)$, with $\alpha \in \Omega^1 (\mathcal{M})$ and $X \in \text{Ker } \psi$; in the second step, the fact that the Newtonian connection is torsionfree; in the third step, the definition of the exterior derivative of a 1-form.

Imposing that $Z \omega$ vanishes and using the local expression of $N \omega$ as $N \omega (V, W) = \omega (V, W)$
\[ d \left( N + \chi \right) (V,W) = 0 \quad \forall V,W \in \text{Ker} \psi. \]  \hspace{1cm} (B.76)

Using the involutivity of the distribution induced by Ker \( \psi \), one can show that the condition (B.76) implies, locally, that

\[ \exists f \in C^\infty (\mathcal{M}) \quad \text{with} \quad \chi (V) = - N (V) + df (V) \quad \forall V \in \text{Ker} \psi. \]

Therefore, there exists a function \( f \) on \( \mathcal{M} \) such that \( Z = N - h \left( N - h \left( N + \chi \right) \right) + h (df) \).

**Proof of Lemma B.1** We start by proving the implication \( (Z, \phi) \Rightarrow g \):

**Lemma B.1.** Let \( \mathcal{S} (\mathcal{M}, \psi, \gamma) \) be an Augustinian structure, \( Z \in FO (\mathcal{M}) \) a field of observers and \( \phi \in C^\infty (\mathcal{M}) \) a function on \( \mathcal{M} \). The metric \( g \in \Gamma (\bigwedge^2 T^* \mathcal{M}) \) defined as:

\[ g \equiv \gamma + \phi \psi \bigwedge \psi \]

with \( \gamma \) the metric transverse to \( Z \), is the only Lagrangian metric satisfying

\[ g (Z) = \phi \psi. \]  \hspace{1cm} (B.77)

**Proof:** Let \( g \in \Gamma (\bigwedge^2 T^* \mathcal{M}) \) be an arbitrary covariant metric on \( \mathcal{M} \). The decomposition of \( g \) on the Galilean basis \( \{ Z, e_i \} \) (with dual basis \( \{ \psi, \theta^i \} \)) reads:

\[ g = g (Z,Z) \psi \bigwedge \psi + 2g (Z, e_i) \psi \bigwedge \theta^i + g (e_i, e_j) \theta^i \bigwedge \theta^j. \]

Requiring that the Lagrangian metric \( g \) satisfies the condition B.77 reduces its expression to:

\[ g = \phi \psi \bigwedge \psi + \gamma (e_i, e_j) \theta^i \bigwedge \theta^j \]

where the second term is nothing but \( \gamma \). \( \square \)

63
A statement converse to Lemma B.1, i.e. the implication $g \Rightarrow (Z, \phi)$, can be formulated as follows:

**Lemma B.2.** Let $\mathcal{S} (\mathcal{M}, \psi, \gamma)$ be an Augustinian structure and $g \in \Gamma (\bigwedge^2 T^* \mathcal{M})$ be a Lagrangian metric on $\mathcal{M}$. There is a unique couple $(Z, \phi)$, with $Z \in FO (\mathcal{M})$ a field of observers and $\phi \in C^\infty (\mathcal{M})$ a function such that:

$$g (Z) = \phi \psi. \quad (B.78)$$

**Proof:** We start by proving that the condition $g (X, Y) = \gamma (X, Y), \forall X, Y \in \Gamma (\text{Ker} \psi)$ implies that $\text{Rad} \ g \cap \text{Ker} \psi = \{0\}$. Suppose there exists a vector field $v \in \Gamma (T \mathcal{M})$ such that $g (v) = \psi (v) = 0$. Since $\psi (v) = 0$, $g (v, w) = \gamma (v, w) = 0, \forall w \in \Gamma (\text{Ker} \psi)$, which leads to a contradiction since $\gamma$ is positive definite. In conclusion, such a vector field $v$ does not exist and $\text{Rad} \ g \cap \text{Ker} \psi = \{0\}$.

The positive definiteness of $\gamma$ implies also that the dimension of $\text{Rad} \ g$ is either 0 or 1, so that we will distinguish these two cases:

| Dim (Rad g) = 1 | Dim (Rad g) = 0 |
|-----------------|-----------------|

Let $v \in \Gamma (T \mathcal{M})$ such that $\text{Rad} \ g = \text{Span} \ v$. The defining relation for $Z$ and $\phi$ then implies $g (Z, v) = \phi \psi (v) = 0$, which in turn ensures $\phi = 0$, since $\psi (v) \neq 0$ in virtue of the precedent discussion. Then, one obtains $g (Z) = 0$ so that $Z \in \text{Rad} \ g$, i.e. $Z \sim v$. The normalization condition $\psi (Z) = 1$ fixes $Z$ uniquely.

Since the metric $g$ is now assumed to be nondegenerate, one can introduce its inverse $g^{-1} \in \Gamma (\bigwedge^2 T^* \mathcal{M})$. Acting on each side of the defining equation for $Z$ and $\phi$ with $g^{-1}$, one gets $Z = \phi \ g^{-1} (\psi)$. Acting now with $\psi$ on each side leads to $\phi \ g^{-1} (\psi, \psi) = 1$, so that $\phi = \frac{1}{g^{-1} (\psi, \psi)}$. Plugging back into the expression for $Z$ leads to $Z = \frac{g^{-1} (\psi)}{g^{-1} (\psi, \psi)}$. We summarise our results in the following table:
Definition of $\phi$

$$\phi = \frac{1}{g^{-1}(\psi, \psi)}$$

Definition of $Z$

$$Z = \frac{g^{-1}(\psi)}{g^{-1}(\psi, \psi)}$$

Table 4: Lagrangian variables

Note that $\phi = 0$ if and only if $\dim (\text{Rad } g) = 1$. □
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