Zero forcing parameters and minimum rank problems

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Abstract
The zero forcing number \( Z(G) \), which is the minimum number of vertices in a zero forcing set of a graph \( G \), is used to study the maximum nullity/minimum rank of the family of symmetric matrices described by \( G \). It is shown that for a connected graph of order at least two, no vertex is in every zero forcing set. The positive semidefinite zero forcing number \( Z_+(G) \) is introduced, and shown to be equal to \( |G| - \text{OS}(G) \), where \( \text{OS}(G) \) is the recently defined ordered set number that is a lower bound for minimum positive semidefinite rank. The positive semidefinite zero forcing number is applied to the computation of positive semidefinite minimum rank of certain graphs. An example of a graph for which the real positive symmetric semidefinite minimum rank is greater than the complex Hermitian positive semidefinite minimum rank is presented.

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1 Introduction

The minimum rank problem for a (simple) graph asks for the determination of the minimum rank among all real symmetric matrices with the zero-nonzero pattern of off-diagonal entries described by a given graph (the diagonal of the matrix is free); the maximum nullity of the graph is the maximum nullity over the same set of matrices. This problem arose from the study of possible eigenvalues of real symmetric matrices described by a graph and has received considerable attention over the last ten years (see [7] and references therein). There has also been considerable interest in the related positive semidefinite minimum rank problem, where the minimum rank is taken over (real or complex Hermitian) positive semidefinite matrices described by a graph (see, for example, [4, 6, 10, 12, 13, 15]).

Zero forcing sets and the zero forcing number were introduced in [1]. The zero forcing number is a useful tool for determining the minimum rank of structured families of graphs and small graphs, and is motivated by simple observations about null vectors of matrices. The zero forcing process is the same as graph infection used by physicists to study control of quantum systems [5], and the zero forcing number is becoming a graph parameter of interest in its own right.

A graph \( G = (V_G, E_G) \) means a simple undirected graph (no loops, no multiple edges) with a finite nonempty set of vertices \( V_G \) and edge set \( E_G \) (an edge is a two-element subset of vertices). All matrices discussed are Hermitian; the set of real symmetric \( n \times n \) matrices is denoted by \( S_n \) and the set of (possibly complex) Hermitian \( n \times n \) matrices is denoted by \( H_n \). For \( A \in H_n \), the graph of \( A \), denoted by \( \mathcal{G}(A) \), is the graph with vertices \( \{1, \ldots, n\} \) and edges \( \{(i, j) : a_{ij} \neq 0, 1 \leq i < j \leq n\} \). Note that the diagonal of \( A \) is ignored in determining \( \mathcal{G}(A) \). The study of minimum rank has focused on real symmetric matrices (or in some cases, symmetric matrices over a field other than the real numbers), whereas much of the work on positive semidefinite minimum rank involves (possibly complex) Hermitian matrices. Whereas it is well known that using complex Hermitian matrices can result in a lower minimum rank than using real symmetric matrices, one of the issues in the study of minimum positive semidefinite rank has been whether or not using only real matrices or allowing complex matrices matters to minimum positive semidefinite rank. Example [7] below shows that complex Hermitian positive semidefinite minimum rank can be strictly lower than real symmetric positive semidefinite minimum rank.

Let \( G \) be a graph. The set of real symmetric matrices described by \( G \) is

\[ S(G) = \{ A \in S_n : \mathcal{G}(A) = G \}. \]

The minimum rank of \( G \) is

\[ \text{mr}(G) = \min \{ \text{rank} A : A \in S(G) \} \]

and the maximum nullity of \( G \) is

\[ \text{M}(G) = \max \{ \text{null} A : A \in S(G) \}. \]
Clearly $\text{mr}(G) + \text{M}(G) = |G|$, where the order $|G|$ is the number of vertices of $G$. The set of real positive semidefinite matrices described by $G$ and the set of Hermitian positive semidefinite matrices described by $G$ are, respectively,

$$S_+(G) = \{ A \in S_n : \mathcal{G}(A) = G \text{ and } A \text{ is positive semidefinite} \}$$

$$\mathcal{H}_+(G) = \{ A \in H_n : \mathcal{G}(A) = G \text{ and } A \text{ is positive semidefinite} \}.$$ 

The minimum positive semidefinite rank of $G$ and minimum Hermitian positive semidefinite rank of $G$ are, respectively,

$$\text{mr}^R_+(G) = \min \{ \text{rank} A : A \in S_+(G) \} \quad \text{and} \quad \text{mr}^C_+(G) = \min \{ \text{rank} A : A \in \mathcal{H}_+(G) \}.$$ 

The maximum positive semidefinite nullity of $G$ and the maximum Hermitian positive semidefinite nullity of $G$ are, respectively,

$$\text{M}^R_+(G) = \max \{ \text{null} A : A \in S_+(G) \} \quad \text{and} \quad \text{M}^C_+(G) = \max \{ \text{null} A : A \in \mathcal{H}_+(G) \}.$$ 

Clearly $\text{mr}^R_+(G) + \text{M}^R_+(G) = |G|$ and $\text{mr}^C_+(G) + \text{M}^C_+(G) = |G|$. There are a variety of symbols in the literature (see, for example, [14, 15]) for these parameters, including $\text{msr}(G)$ and $\text{hmr}_+(G)$ for what we denote by $\text{mr}^C_+(G)$. Clearly $\text{M}^R_+(G) \leq \text{M}(G)$ and $\text{mr}(G) \leq \text{mr}^R_+(G)$ for every graph $G$, and it is well known that these inequalities can be strict (for example, any tree $T$ that is not a path has $\text{mr}(T) < \text{mr}^R_+(T)$).

We need some additional graph terminology. The complement of a graph $G = (V, E)$ is the graph $\overline{G} = (V, \overline{E})$, where $\overline{E}$ consists of all two element sets from $V$ that are not in $E$. We denote the complete graph on $n$ vertices by $K_n$; a complete graph is also called a clique. The degree of vertex $v$ in graph $G$ is the number of edges incident with $v$, and the minimum degree of the vertices of $G$ is denoted by $\delta(G)$. A set of subgraphs of $G$, each of which is a clique and such that every edge of $G$ is contained in at least one of these cliques, is called a clique covering of $G$. The clique covering number of $G$, denoted by $\text{cc}(G)$, is the smallest number of cliques in a clique covering of $G$.

**Observation 1.1.** [7] For every graph $G$, $\text{mr}^R_+(G) \leq \text{cc}(G)$, so $|G| - \text{cc}(G) \leq \text{M}^R_+(G)$.

For an $n \times n$ matrix $A$ and $W \subseteq \{1, \ldots, n\}$, the principal submatrix $A[W]$ is the submatrix of $A$ lying in the rows and columns that have indices in $W$. For a graph $G = (V_G, E_G)$ and $W \subseteq V_G$, the induced subgraph $G[W]$ is the graph with vertex set $W$ and edge set $\{ \{v, w\} \in E_G : v, w \in W \}$. The induced subgraph $\mathcal{G}(A)[W]$ of the graph of $A$ is naturally associated with the graph of the principal submatrix for $W$, i.e., $\mathcal{G}(A[W])$. The subgraph induced by $W = V_G \setminus W$ is usually denoted by $G - W$, or in the case $W$ is a singleton $\{v\}$, by $G - v$.

The path cover number $P(G)$ of $G$ is the smallest positive integer $m$ such that there are $m$ vertex-disjoint induced paths $P_1, \ldots, P_m$ in $G$ that cover all the vertices of $G$ (i.e., $V_G = \bigcup_{i=1}^m V_{P_i}$). A graph is planar if it can be drawn
in the plane without crossing edges. A graph is outerplanar if it has such a
drawing with a face that contains all vertices. Given two graphs \( G \) and \( H \), the
Cartesian product of \( G \) and \( H \), denoted \( G \square H \), is the graph whose vertex set is
the Cartesian product of \( V_G \) and \( V_H \), with an edge between two vertices exactly
when they are identical in one coordinate and adjacent in the other.

Let \( G = (V_G, E_G) \) be a graph. A subset \( Z \subseteq V_G \) defines an initial set of black
vertices (with all the vertices not in \( Z \) white), called a coloring. There are no
constraints on permissible colorings; instead there are constraints on how new
colorings can be derived. The color change rule (for the zero forcing number) is
to change the color of a white vertex \( w \) to black if \( w \) is the unique white neighbor
of a black vertex \( u \); in this case we say \( u \) forces \( w \) and write \( u \rightarrow w \). Given a
coloring of \( G \), the derived set is the set of black vertices obtained by applying the
color change rule until no more changes are possible. A zero forcing set for \( G \)
is a subset of vertices \( Z \) such that if initially the vertices in \( Z \) are colored black
and the remaining vertices are colored white, the derived set is \( V_G \). The zero
forcing number \( Z(G) \) is the minimum of \( |Z| \) over all zero forcing sets \( Z \subseteq V_G \).

**Theorem 1.2.** [1, Proposition 2.4] For any graph \( G \), \( M(G) \leq Z(G) \).

Suppose \( S = (v_1, v_2, \ldots, v_m) \) is an ordered subset of vertices from a given
graph \( G \). For each \( k \) with \( 1 \leq k \leq m \), let \( G_k \) be the subgraph of \( G \) induced
by \( \{v_1, v_2, \ldots, v_k\} \), and let \( H_k \) be the connected component of \( G_k \) that contains
\( v_k \). If for each \( k \), there exists a vertex \( w_k \) that satisfies: \( w_k \neq v_l \) for \( l \leq k \),
\( \{w_k, v_k\} \in E \), and \( \{w_k, v_s\} \notin E \), for all \( v_s \) in \( H_k \) with \( s \neq k \), then \( S \) is called an
ordered set of vertices in \( G \), or an OS-set. As defined in [10], the OS number of
a graph \( G \), denoted by \( OS(G) \), is the maximum of \( |S| \) over all OS-sets \( S \) of \( G \).

**Theorem 1.3.** [10, Proposition 3.3] For any graph \( G \), \( OS(G) \leq mr_C^+(G) \).

In Section 2 we establish several properties of the zero forcing number, in-
cluding the nonuniqueness of zero forcing sets. In Section 3 we introduce the
positive semidefinite zero forcing number as an upper bound for maximum pos-
itive semidefinite nullity, show that the sum of the positive semidefinite zero
forcing number and the OS number is the order of the graph, and apply the
positive semidefinite zero forcing number to the computation of positive semide-
finite minimum rank. Section 4 provides the first example showing that \( mr_R^+(G) \) and \( mr_C^+(G) \) need not be the same (described as unknown in [7]).

## 2 Properties of the zero forcing number

In this section, we establish several properties of the zero forcing number, in-
cluding the non-uniqueness of zero forcing sets and its relationship to path cover
number. We need some additional definitions related to the zero forcing number.

**Definition 2.1.** A minimum zero forcing set is a zero forcing set \( Z \) such that
\( |Z| = Z(G) \).
Zero forcing chains of digraphs were defined in [2]. We give an analogous definition for graphs.

**Definition 2.2.** Let $Z$ be a zero forcing set of a graph $G$.

- Construct the derived set, recording the forces in the order in which they are performed. This is the **chronological list of forces**.

- A **forcing chain** (for this particular chronological list of forces) is a sequence of vertices $(v_1, v_2, \ldots, v_k)$ such that for $i = 1, \ldots, k - 1$, $v_i \rightarrow v_{i+1}$.

- A **maximal forcing chain** is a forcing chain that is not a proper subsequence of another zero forcing chain.

Note that a zero forcing chain can consist of a single vertex $(v_1)$, and such a chain is maximal if $v_1 \in Z$ and $v_1$ does not perform a force.

As noted in [1], the derived set of a given set of black vertices is unique; however, a chronological list of forces (of one particular zero forcing set) usually is not. At Rocky Mountain Discrete Mathematics Days held Sept. 12 – 13, 2008 at the University of Wyoming, the following questions were raised.

**Question 2.3.** Is there a graph that has a unique minimum zero forcing set?

**Question 2.4.** Is there a graph $G$ and a vertex $v \in V_G$ such that $v$ is in every minimum zero forcing set?

We show the answers to both these questions are negative for nontrivial connected graphs.

**Definition 2.5.** Let $Z$ be a zero forcing set of a graph $G$. A **reversal** of $Z$ is the set of last vertices of the maximal zero forcing chains of a chronological list of forces.

Each vertex can force at most one other vertex and can be forced by at most one other vertex, so the maximal forcing chains are disjoint, and the elements of $Z$ are the initial vertices of the maximal forcing chains. Thus the cardinality of a reversal of $Z$ is the same as the cardinality of $Z$.

**Theorem 2.6.** If $Z$ is a zero forcing set of $G$ then so is any reversal of $Z$.

**Proof.** Write the chronological list of forces in reverse order, reversing each force (call this the reverse chronological list of forces) and let the reversal of $Z$ for this list be denoted $W$. We show the reverse chronological list of forces is a valid list of forces for $W$. Consider the first “force” $u \rightarrow v$ on the reverse chronological list. All neighbors of $u$ except $v$ must be in $W$, since when the last force $v \rightarrow u$ of $Z$ was done, each of them had the white neighbor $u$ and thus did not force any vertex previously (in the original chronological list of forces). Thus $u \rightarrow v$ is a valid force for $W$. Continue in this manner or use induction on $|G|$. \qed
Corollary 2.7. No connected graph of order greater than one has a unique minimum zero forcing set.

Lemma 2.8. Let $G$ be a connected graph of order greater than one and let $Z$ be a minimum zero forcing set. Every $z \in Z$ has a neighbor $w \notin Z$.

Proof. Suppose not. Then there is a vertex $z \in Z$ such that every neighbor of $z$ is in $Z$ (and $z$ does have at least one neighbor $v$). Since $z$ cannot perform a force, $z$ is in the reversal $W$ of $Z$. Using the reversed maximal forcing chains, no neighbor of $z$ performs a force. So $W \setminus \{z\}$ is a zero forcing set of smaller cardinality, because after every vertex except $z$ is black, $v$ can force $z$. □

Theorem 2.9. If $G$ is a connected graph of order greater than one, then

$$\bigcap_{Z \in ZFS(G)} Z = \emptyset,$$

where $ZFS(G)$ is the set of all minimum zero forcing sets of $G$.

Proof. Suppose not. Then there exists $v \in \bigcap_{Z \in ZFS(G)} Z$. In particular, for each $Z$ and each reversal $W$ of $Z$, $v$ is in both $Z$ and $W$. This means that there is a maximal forcing chain consisting of only $v$, or in other words $v$ does not force any other vertex.

Let $Z$ be a zero forcing set. If there is no chronological list of forces in which a neighbor of $v$ performs a force, then replace $Z$ by its reversal (since, by Lemma 2.8, $v$ originally had a white neighbor $u$, in the reversal $u$ performs a force). Let $u \to w$ be the first force in which the forcing vertex $u$ is a neighbor of $v$. We claim that $Z \setminus \{v\} \cup \{w\}$ is a zero forcing set for $G$. The forces can proceed until $u$ is encountered as a forcing vertex. At that time, replace $u \to w$ by $u \to v$, and then continue as in the original chronological list of forces. □

Next we show that for any graph the zero forcing number is an upper bound for the path cover number.

Proposition 2.10. For any graph $G$, $P(G) \leq Z(G)$.

Proof. Let $Z$ be a zero forcing set. The vertices in a forcing chain induce a path in $G$ because the forces in a forcing chain occur chronologically in the order of the chain (since only a black vertex can force). The maximal forcing chains are disjoint, contain all the vertices of $G$, and the elements of the set $Z$ are the initial vertices of the maximal forcing chains. Thus $P(G) \leq |Z|$. By choosing a minimum zero forcing set $Z$, $P(G) \leq Z(G)$. □

In [14] it was shown that for a tree $T$, $P(T) = M(T)$, and in [1] it was shown that for a tree, $P(T) = Z(T)$ (and thus $M(T) = Z(T)$). In [3] it was shown that for graphs in general, $P(G)$ and $M(G)$ are not comparable. However, Sinkovic has established the following relationship for outerplanar graphs: If $G$ is an
outerplanar graph, then $M(G) \leq P(G)$ \[16\]. The next example shows that neither outerplanar graphs nor 2-trees require $M(G) = Z(G)$ or $P(G) = Z(G)$ (a 2-tree is constructed inductively by starting with a $K_3$ and connecting each new vertex to 2 adjacent existing vertices).

**Example 2.11.** Let $G_{12}$ be the graph shown in Figure 1, called the pinwheel on 12 vertices. Note that $G_{12}$ is an outerplanar 2-tree. The set $\{1, 2, 6, 10\}$ is a zero forcing set for $G_{12}$, so $Z(G_{12}) \leq 4$. We show that $Z(G_{12}) \geq 4$, which implies $Z(G_{12}) = 4$. Suppose to the contrary that $Z$ is a zero forcing set for $G_{12}$ and $|Z| = 3$. To start the forcing, at least two of the vertices must be in one of the sets $\{1, 2, 3\}, \{7, 8, 9\}, \{10, 11, 12\}$; without loss of generality, assume that two or three black vertices are in $\{1, 2, 3\}$. Then after several forces the vertices $\{1, 2, 3, 4, 5\}$ are black, and at most one additional vertex $v \notin \{1, 2, 3, 4, 5\}$ is in $Z$. To perform another force with only one more black vertex $v$, either 6 or 7 must be black, and 5 can force the other, but then no additional forces can be performed, so $Z$ was not a zero forcing set for $G_{12}$. Clearly $G_{12}$ can be covered by 9 triangles, so $cc(G_{12}) \leq 9$ and $M^R_{+}(G_{12}) \geq 3$, by Observation 1.1. It is easy to find a path covering of 3 paths, so $M^R_{+}(G_{12}) = M(G_{12}) = P(G_{12}) = 3$ and $mr^R_{+}(G_{12}) = mr(G_{12}) = cc(G_{12}) = 9$. Since $G_{12}$ is chordal, $mr^C_{+}(G_{12}) = cc(G_{12})$ \[14\], and thus $M^C_{+}(G_{12}) = 3$.

### 3 The positive semidefinite zero forcing number

In this section, we introduce the positive definite zero forcing number, relate it to maximum positive semidefinite nullity and to the OS number, and apply it to compute maximum positive semidefinite nullity of several families of graphs. The definitions and terminology for zero forcing (coloring, derived set, etc.) are the same as for the zero forcing number $Z(G)$, but the color change rule is different.

**Definition 3.1.**

- The positive semidefinite color change rule is:

  Let $B$ be the set consisting of all the black vertices. Let $W_1, \ldots, W_k$ be
the sets of vertices of the $k$ components of $G - B$ (note that it is possible that $k = 1$). Let $w \in W_i$. If $u \in B$ and $w$ is the only white neighbor of $u$ in $G[W_i \cup B]$, then change the color of $w$ to black.

- The positive semidefinite zero forcing number of a graph $G$, denoted by $Z_+(G)$, is the minimum of $|X|$ over all positive semidefinite zero forcing sets $X \subseteq V_G$ (using the positive semidefinite color change rule).

Forcing using the positive semidefinite color change rule can be thought of as decomposing the graph into a union of certain induced subgraphs and using ordinary zero forcing on each of these induced subgraphs. The application of the positive semidefinite color change rule is illustrated in the next example.

**Example 3.2.** Let $T$ be a tree. Then $Z_+(T) = 1$, because any one vertex $v$ is a positive semidefinite zero forcing set. Formally, this can be established by induction on $|T|$: If $v$ is a leaf, it forces its neighbor; if not a decomposition takes place. In either case smaller tree(s) are obtained. It has been known for a long time (see, for example, [7]) that $M^C_+(T) = 1$, but the use of $Z_+$ provides an easy proof of this result, because $M^C_+(T) = 1$ is an immediate consequence of $Z_+(T) = 1$ by Theorem 3.5 below.

**Observation 3.3.** Since any zero forcing set is a positive definite zero forcing set,

$$Z_+(G) \leq Z(G).$$

**Example 3.4.** The pinwheel $G_{12}$ shown in Figure 1 has $Z_+(G_{12}) = 3 = M^R_+(G_{12})$ because $X = \{4, 5, 6\}$ is a positive semidefinite zero forcing set ($G_{12} - X$ is disconnected, and $X$ is a zero forcing set for $G[\{1, 2, 3, 4, 5, 6\}]$, etc.).

For any graph $G$ that is the disjoint union of connected components $G_i$, $i = 1, 2, \ldots, k$, $Z_+(G) = \sum_{i=1}^{k} Z_+(G_i)$ (the analogous results for $M, M^R_+, M^C_+$ and $Z$ are all well known).

**Theorem 3.5.** For any graph $G$, $M^C_+(G) \leq Z_+(G)$.

**Proof.** Let $A \in H_+(G)$ with null $A = M^C_+(G)$. Let $x = [x_i]$ be a nonzero vector in ker $A$. Define $B$ to be the set of indices $u$ such that $x_u = 0$ and let $W_1, \ldots, W_k$ be the sets of vertices of the $k$ components of $G - B$. We claim that in $G[B \cup W_1]$, $w \in W_1$ cannot be the unique neighbor of any vertex $u \in B$. Once the claim is established, if $X$ is a positive semidefinite zero forcing set for $G$, then the only vector in ker $A$ with zeros in positions indexed by $X$ is the zero vector, and thus $M^C_+(G) \leq Z_+(G)$.

To establish the claim, renumber the vertices so that the vertices of $B$ are last, the vertices of $W_1$ are first, followed by the vertices of $W_2$, etc. Then $A$
has the block form

\[
A = \begin{bmatrix}
A_1 & 0 & \ldots & 0 & C_1^* \\
0 & A_2 & \ldots & 0 & C_2^* \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \ldots & A_k & C_k^* \\
C_1 & C_2 & \ldots & C_k & D
\end{bmatrix}.
\]

Partition \( x \) conformally as \( x = [x_1^T, \ldots, x_t^T, 0]^T \), and note that all entries of \( x_i \) are nonzero, \( i = 1, \ldots, k \). Then \( Ax = 0 \) implies \( A_i x_i = 0 \), \( i = 1, \ldots, k \). Since \( A \) is positive semidefinite, each column in \( C_i^* \) is in the span of the columns of \( A_i \) by the column inclusion property of Hermitian positive semidefinite matrices \([15]\). That is, for \( i = 1, \ldots, k \), there exists \( Y_i \) such that \( C_i^* = A_i Y_i \). Thus \( C_i x_i = Y_i^* A_i x_i = 0 \), and \( w \in W_i \) cannot be the unique neighbor in \( W_i \) of any vertex \( u \in B \). \( \square \)

Theorem 3.5 is also a consequence of Theorem 3.6 below and Theorem 1.3 above, but using that as a justification obscures the motivation for the definition and the connection between zero forcing and null vectors that is given in the short direct proof.

In \([15]\), Theorem 2.10 it is shown that \( |G| - Z(G) \leq OS(G) \). A similar method can be used to show an a more precise relationship between \( Z_+ \) and the OS number.

**Theorem 3.6.** For any graph \( G = (V, E) \) and any ordered set \( S, V \setminus S \) is a positive semidefinite forcing set for \( G \), and for any positive semidefinite forcing set \( X \) for \( G \), there is an order that makes \( V \setminus X \) an ordered set for \( G \). Thus \( Z_+(G) + OS(G) = |G| \).

**Proof.** Let \( X \) be a positive semidefinite zero forcing set for \( G \) such that \( |X| = Z_+(G) \). Let \( v_i \) be the vertex colored black by the \( i \)th application of the positive semidefinite color change rule. We show that \( S = \{ v_1, v_{t-1}, \ldots, v_t \} \) is an OS set for \( G \), where \( t = |G| - Z_+(G) \). Further define \( X_0 = X \), and \( X_{i+1} = X_i \cup \{ v_{i+1} \} \), for \( i = 0, 1, \ldots, t - 1 \). For each \( v_i \), since it was initially white and then colored black on the \( i \)th application of the positive semidefinite color change rule, there exists a vertex \( w_i \in X_i \) (the current black vertices) such that \( v_i \) is the only neighbor in the subgraph of \( G \) induced by \( X_i \cup H_1 \), where the subgraph \( G \setminus X_i \) has components \( H_1, H_2, \ldots, H_p \) with \( v_i \in H_1 \). Since \( X \) is a positive semidefinite zero forcing set, no other vertex from the set \( \{ v_{i+1}, v_{i+2}, \ldots, v_t \} \) (the remaining white vertices) can be in \( H_1 \) and be a neighbor of \( w_i \). Hence the set \( \{ v_t, v_{t-1}, \ldots, v_1 \} \) is an OS-set. Therefore \( t \leq OS(G) \). Thus

\[
|G| - Z_+(G) \leq OS(G).
\]

(1)

For the converse, we use the fact that if \( S = \{ v_1, v_2, \ldots, v_m \} \) is an OS set, then the set \( S \setminus \{ v_m \} \) is also an OS set. Suppose \( S = \{ v_1, v_2, \ldots, v_m \} \) is an OS set with \( |S| = OS(G) \). Then we claim that \( V \setminus S \) is a positive semidefinite zero forcing set. So color the vertices \( V \setminus S \) black, and suppose the subgraph \( G_m \) induced
by the vertices of \( \{v_1, \ldots, v_m\} \) has components induced by \( U_1, U_2, \ldots, U_\ell \). Let \( v_m \in U_1 \). Since \( S \) is an OS-set there exists a vertex \( w_m \in V \setminus S \) such that \( w_m v_m \in E \) and \( w_m v_s \not\in E \) for all other \( v_s \in U_1 \). This implies that \( v_m \) can be colored black under the positive semidefinite color change rule. Since \( S \setminus \{v_m\} \) is also an OS-set for \( G \), we may continue this argument and deduce that \( V \setminus S \) is a positive semidefinite zero forcing set. Hence

\[
|G| - OS(G) = |V \setminus S| \geq Z_+(G),
\]

as the positive semidefinite zero forcing number is defined as a minimum over all such zero forcing sets. From (1) and (2), \( Z_+(G) + OS(G) = |G| \).

**Corollary 3.7.** For every graph \( G \),

\[
\delta(G) \leq Z_+(G).
\]

**Proof.** By [15, Corollary 2.19], \( OS(G) \leq |G| - \delta(G) \). Combining this with Theorem 3.6 gives the result. \( \square \)

Another consequence of Theorem 3.6 is that there are examples of graphs for which \( Z_+ \) may not be equal to \( M^C_+ \). For example, in [15] it was shown that the Möbius Ladder on 8 vertices, sometimes denoted by \( ML_8 \) or \( V_8 \), satisfies \( OS(ML_8) = 4 \) and \( M^C_+(ML_8) = 5 \). In this case, by Theorem 3.6 it follows that \( Z_+(ML_8) = 4 \), and hence \( Z_+(ML_8) > 3 = M^C_+(ML_8) \).

In [11], the zero forcing number was used to establish the minimum rank/maximum nullity of numerous families of graphs. The positive semidefinite zero forcing number is equally effective. Here we apply it to two families of graphs. The set of vertices associated with (the same) positive semidefinite zero forcing set in each copy of \( G \) is a positive semidefinite zero forcing set for \( G \oplus H \).

**Proposition 3.8.** For all graphs \( G \) and \( H \), \( Z_+(G \oplus H) \leq \min\{Z_+(G)|H|, Z_+(H)|G|\} \).

**Corollary 3.9.** If \( T \) is a tree and \( G \) is a graph, then \( Z_+(T \oplus G) \leq |G| \).

**Theorem 3.10.** If \( T \) is a tree of order at least two, then \( M^C_+(T \circ K_r) = M^C_+(T \oplus K_r) = Z_+(T \circ K_r) = Z_+(T \oplus K_r) = r \).

**Proof.** Let \( T \) be a tree of order \( n \geq 2 \). By Corollary 3.9 \( Z_+(T \circ K_r) \leq r \). We show \( r \leq M^C_+(T \circ K_r) \) by constructing a matrix \( A \in S_+(T \circ K_r) \) of rank at most \( (n-1)r \), and the result then follows from Theorem 3.3. The construction is by induction on \( n \). Let \( P_2 \) denote the path on 2 vertices. To show that \( M^C_+(P_2 \circ K_r) = r \), choose a nonsingular matrix \( M \in S_+(K_r) \) such that \( M^{-1} \in S_+(K_r) \) (for example, \( M = I + J \), where \( I \) is the identity matrix and \( J \) is the all 1s matrix). Then \( B = \begin{bmatrix} M & I \\ I & M^{-1} \end{bmatrix} \in S_+(P_2 \circ K_r) \) and rank \( B = \text{rank} M = r \).

Without loss of generality, in \( T \) vertex \( n \) is adjacent only to vertex \( n - 1 \). We
order the vertices \((i, j)\) of \(T \sqcap K_r\) lexicographically. By the induction hypothesis, there is a matrix \(C \in S_+((T - n) \sqcap K_r)\) such that \(\text{rank} C = (n - 2)r\); let \(C' = C \oplus 0_{r \times r}\). Using \(B \in S_+(P_2 \sqcap K_r)\) already constructed with rank \(r\), let \(B' = 0_{(n-2)r \times (n-2)r} \oplus B\). Then for \(\alpha \in \mathbb{R}\) chosen to avoid cancellation, \(A = C' + \alpha B' \in S_+(T \sqcap K_r)\) and \(\text{rank} A \leq (n - 2)r + r = (n - 1)r\).

A book with \(m \geq 2\) pages, denoted \(B_m\) [9, p. 14], is \(m\) copies of a 4-cycle with one edge in common, or equivalently, \(B_m = K_{1,m} \sqcap P_2\), where \(K_{1,m}\) is the complete bipartite graph with partite sets of 1 and \(m\) vertices. For \(m \geq 2, \ t \geq 3\), we call \(m\) copies of a \(t\)-cycle with one edge in common a generalized book, denoted by \(B_t^m\) (obviously, \(B_4^m = B_4^m\)).

**Proposition 3.11.** If \(B_t^m\) is a generalized book, then \(M^R_+(B_t^m) = M^C_+(B_t^m) = Z_+(B_t^m) = 2\).

**Proof.** The two vertices in the common edge are a positive semidefinite zero forcing set, so \(Z_+(B_t^m) \leq 2\). Thus by Theorem 3.5, \(M^C_+(B_t^m) \leq 2\). Since \(B_t^m\) is not a tree, \(M^R_+(B_t^m) \geq 2\) [12].

## 4 Real versus complex minimum positive semidefinite rank

Clearly \(m^C_+(G) \leq m^R_+(G)\) for every graph \(G\). Previously it was not known whether \(m^R_+(G)\) could differ from \(m^R_+(G)\) [7, p. 578]. In this final section we provide an example of a graph for which these parameters are not identical.

**Example 4.1.** The “\(k\)-wheel with 4 hubs” (for \(k\) at least 3) is the graph on \(4k + 4\) vertices such that the outer cycle has \(4k\) vertices, and each of the 4 hubs is attached to every 4th vertex of the cycle, and no others; this graph is denoted \(H_4(k)\), and \(H_4(3)\) is shown in Figure 2. This family arose in Hall’s investigation of graphs having minimum rank 3 [11]. We show \(m^R_+(H_4(3)) = 3\) and \(m^R_+(H_4(3)) = 4\). As numbered in Figure 2, \(H_4(3)\) is bipartite with partite sets consisting of the odd vertices and the even vertices. By [2, Theorem 3.1], \(m^R_+(H_4(3)) = m^R(Y_{H_4(3)})\) where \(Y_{H_4(3)}\) is the biadjacency zero-nonzero pattern of \(H_4(3)\) and \(m^R(Y_{H_4(3)})\) is the asymmetric minimum rank over the real numbers (Theorem 3.1 applies to \(H_4(3)\) because \(H_4(3)\) is a bipartite graph). The same method used to prove Theorem 3.1 also shows that \(m^C_+(H_4(3)) = m^C(Y_{H_4(3)})\) where \(m^C(Y_{H_4(3)})\) is the asymmetric minimum rank over the complex numbers (in [2, Remark 3.2] it is noted that the method in Theorem 3.1 is valid for constructing a symmetric matrix over an infinite field, and the same reasoning applies to constructing a Hermian matrix over \(\mathbb{C}\) by using Hermitian adjoints in place of transposes). After scaling rows and columns,
the principal submatrix in the first three rows and columns is nonsingular, necessary: through 3. Computations show that the following assignments of variables are necessary: 

\[
A = \begin{bmatrix}
0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 \\
0 & 0 & 1 & a_{3,8} & a_{3,10} & a_{3,12} & a_{3,14} & 0 \\
1 & 0 & 0 & a_{5,8} & a_{5,10} & a_{5,12} & 0 & a_{5,16} \\
1 & a_{7,4} & 0 & 0 & a_{7,10} & a_{7,12} & a_{7,14} & 0 \\
1 & a_{9,4} & a_{9,6} & 0 & 0 & a_{9,12} & 0 & a_{9,16} \\
1 & a_{11,4} & a_{11,6} & a_{11,8} & 0 & 0 & a_{11,14} & 0 \\
0 & 1 & 0 & a_{13,8} & 0 & a_{13,12} & a_{13,14} & a_{13,16} \\
1 & 0 & a_{15,6} & 0 & a_{15,10} & 0 & a_{15,14} & a_{15,16} \\
\end{bmatrix},
\]

where the displayed entries \( a_{ij} \) are nonzero (real or complex) numbers. Since the principal submatrix in the first three rows and columns is nonsingular, rank \( A = 3 \) implies that rows 4 through 8 are linear combinations of rows 1 through 3. Computations show that the following assignments of variables are necessary:

\[
\begin{align*}
a_{5,8} &= (a_{3,8} - 1)a_{7,4}, \quad a_{5,10} = (a_{3,10} - 1)a_{7,10}, \quad a_{5,12} = a_{3,12}a_{7,4} + a_{7,12}, \quad a_{5,16} = -a_{7,4}, \\
a_{7,14} &= -a_{3,14}a_{7,4}, \quad a_{9,16} = a_{9,4} - a_{7,4}, \quad a_{9,6} = a_{9,4}, \quad a_{9,12} = a_{3,12}a_{7,4} + a_{7,12} - a_{3,12}a_{9,4} + a_{3,12}a_{9,6}, \\
a_{7,10} &= a_{7,4} - a_{3,10}a_{7,4} - a_{9,4}, \quad a_{9,4} = (1 - a_{3,8})a_{7,4}, \quad a_{11,4} = a_{7,4}, \quad a_{11,14} = a_{3,14}(a_{11,6} - a_{11,4}), \\
a_{7,12} &= -a_{3,12}a_{11,6}, \quad a_{11,8} = a_{3,8}a_{11,6}, \quad a_{3,8} = a_{3,10}(a_{7,4} - a_{11,6})/a_{7,4}, \quad a_{13,16} = 1, \quad a_{13,14} = -a_{3,14}, \\
a_{13,12} &= -a_{3,12}, \quad a_{3,10} = 1, \quad a_{13,8} = a_{11,6}/a_{7,4}, \quad a_{15,16} = -a_{7,4}, \quad a_{15,14} = a_{3,14}a_{15,6}, \\
a_{15,10} &= -a_{11,6} + a_{15,6}, \quad a_{11,6} = a_{7,4} + a_{15,6}.
\end{align*}
\]

Figure 2: The the 3-wheel on 4 hubs, \( H_4(3) \), for Example 4.1
After making these assignments, rows 4 - 7 are linear combinations of rows 1, 2, and 3, and in order for row 8 to be a linear combinations of rows 1, 2, and 3, it is necessary and sufficient that

\[1 + \frac{a_{7,4}}{a_{15,6}} + \left(\frac{a_{7,4}}{a_{15,6}}\right)^2 = 0. \tag{3}\]

Clearly (3) has a solution if and only if the field contains a primitive third root of unity. Thus \(\text{mr}_C(H_4(3)) = 3\) whereas \(\text{mr}_R(H_4(3)) = 4\), giving

\[\text{mr}_C(H_4(3)) = 3 < 4 = \text{mr}_R(H_4(3)).\]

References

[1] AIM Minimum Rank – Special Graphs Work Group (F. Barioli, W. Barrett, S. Butler, S. M. Cioabă, D. Cvetković, S. M. Fallat, C. Godsil, W. Haemers, L. Hogben, R. Mikkelson, S. Narayan, O. Pryporova, I. Sciriha, W. So, D. Stevanović, H. van der Holst, K. Vander Meulen, and A. Wangsness). Zero forcing sets and the minimum rank of graphs. *Linear Algebra and its Applications*, 428/7: 1628–1648, 2008.

[2] F. Barioli, S. M. Fallat, D. Hershkowitz, H. T. Hall, L. Hogben, H. van der Holst, and B. Shader. On the minimum rank of not necessarily symmetric matrices: a preliminary study. *Electronic Journal of Linear Algebra*, 18: 126–145, 2009.

[3] F. Barioli, S. M. Fallat, and L. Hogben. Computation of minimal rank and path cover number for graphs. *Linear Algebra and its Applications*, 392: 289–303, 2004.

[4] M. Booth, P. Hackney, B. Harris, C. R. Johnson, M. Lay, L. H. Mitchell, S. K. Narayan, A. Pascoe, K. Steinmetz, B. D. Sutton, W. Wang, On the minimum rank among positive semidefinite matrices with a given graph. *SIAM Journal of Matrix Analysis and Applications*, 30: 731–740, 2008.

[5] D. Burgarth and V. Giovannetti. Full control by locally induced relaxation. *Physical Review Letters* PRL 99, 100501 (2007).

[6] Y. Colin de Verdière. Multiplicities of eigenvalues and tree-width of graphs. *J. Combin. Theory Ser. B* 74: 121–146, 1998.

[7] S. Fallat and L. Hogben. The minimum rank of symmetric matrices described by a graph: a survey. *Linear Algebra and its Applications*, 426: 558–582, 2007.

[8] S. Fallat and C. R. Johnson. Olga, matrix theory, and the Taussky unification problem. *Linear Algebra and Its Applications*, 280: 39–49, 1998.
[9] J. A. Gallian, A Dynamic Survey of Graph Labeling. *Electronic Journal of Combinatorics*, DS6 (219 pp.), http://www.combinatorics.org/Surveys/ds6.pdf, January 31, 2009.

[10] P. Hackney, B. Harris, M. Lay, L. H. Mitchell, S. K. Narayan, A. Pascoe, Linearly independent vertices and minimum semidefinite rank, *Linear Algebra and Its Applications*, 431: 1105–1115, 2009.

[11] H. T. Hall. Minimum rank 3 is difficult to determine. Preprint.

[12] H. van der Holst. Graphs whose positive semi-definite matrices have nullity at most two. *Linear Algebra and Its Applications*, 375: 1–11, 2003.

[13] H. van der Holst. On the maximum positive semi-definite nullity and the cycle matroid of graphs. *Electronic Journal of Linear Algebra*, 18: 192–201, 2009.

[14] C. R. Johnson and A. Leal Duarte. The maximum multiplicity of an eigenvalue in a matrix whose graph is a tree. *Linear and Multilinear Algebra* 46: 139–144, 1999.

[15] L. Mitchell, S. Narayan, and A. Zimmer, Lower bounds in minimum rank problems, *Linear Algebra and Its Applications*, 432: 430-440, 2010.

[16] J. Sinkovic. Maximum nullity of outerplanar graphs and the path cover number. To appear in *Linear Algebra and Its Applications*.