Streaming Quantiles Algorithms with Small Space and Update Time

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ABSTRACT
Approximating quantiles and distributions over streaming data has been studied for roughly two decades now. Recently, Karnin, Lang, and Liberty proposed the first asymptotically optimal algorithm for doing so. This manuscript complements their theoretical result by providing a practical variants of their algorithm with improved constants. For a given sketch size, our techniques provably reduce the upper bound on the sketch error by a factor of two. These improvements are verified experimentally. Our modified quantile sketch improves the latency as well by reducing the worst-case update time from $O(\frac{n}{\delta})$ down to $O(\log\frac{1}{\delta})$. We also suggest two algorithms for weighted item streams which offer improved asymptotic update times compared to naive extensions. Finally, we provide a specialized data structure for these sketches which reduces both their memory footprints and update times.

CCS CONCEPTS
• Computing methodologies → Vector/streaming algorithms; Distributed computing methodologies; • Theory of computation → Sketching and sampling;

KEYWORDS
quantiles, kll, sketching, load balancing, streaming algorithm

1 INTRODUCTION
Estimating the underlying distribution of data is crucial for many applications. It is common to approximate an entire Cumulative Distribution Function (CDF) or specific quantiles. The median (0.5 quantile) and 95-th and 99-th percentiles are widely used in financial metrics, statistical tests, and system monitoring. Quantiles summary found applications in databases [21, 23], sensor networks [14], logging systems [20], distributed systems [7], and decision trees [5]. While computing quantiles is conceptually very simple, doing so naively becomes infeasible for very large data.

Formally the quantiles problem can be defined as follows. Let $S$ be a multiset of items $S = \{s_i\}_{i=1}^n$. The items in $S$ exhibit a full-ordering and the corresponding smaller-than comparator is known. The rank of a query $q$ (w.r.t. $S$) is the number of items in $S$ which are smaller than $q$. An algorithm should process $S$ such that it can compute the rank of any query item. Answering rank queries exactly for every query is trivially possible by storing the multiset $S$. Storing $S$ in its entirety is also necessary for this task.

Throughout this manuscript we assume that each item in the stream requires $O(1)$ space to store.
Felber and Ostrovsky [8] suggested non-trivial techniques of feeding sampled items into sketches from [9] and improved the space complexity to $O(\frac{1}{\epsilon} \log \frac{1}{\delta})$. Recently Karnin et al. in [13] presented an asymptotically optimal but non-mergeable data structure with space usage of $O(\frac{1}{\epsilon} \log \frac{1}{\delta})$ and a matching lower bound. They also presented a fully mergeable algorithm whose space complexity is $O(\frac{1}{\epsilon} \log^2 \frac{1}{\delta})$.

In the current paper, we suggest several further improvements to the algorithms introduced in [13]. These improvements do not affect the asymptotic guarantees of [13] but reduce the upper bounds by constant terms, both in theory and practice. The suggested techniques also improve the worst-case update time. Additionally, we suggest two algorithms for the extended version of the problem where updates have weights. All the algorithms presented operate in the comparison model. They can only store (and discard) items from the stream and compare between them. For more background on quantile algorithms in the streaming model see [11, 25].

2 A UNIFIED VIEW OF PREVIOUS RANDOMIZED SOLUTIONS

To introduce further improvements to the streaming quantiles algorithms we will first re-explain the previous work using simplified concepts of one pair compression and a compactor. Consider a simple problem in which your data set contains only two items $a$ and $b$, while your data structure can only store one item. We focus on the comparison based framework where we can only compare items and cannot compute new items via operations such as averaging. In this framework, the only option for the data structure is to pick one of them and store it explicitly. The stored item $x$ is assigned weight 2. Given a rank query $q$ the data structure will report 0 for $q < x$, and 2 for $q > x$. For $q \notin \{a, b\}$ the output of the data structure will be correct, however, for $q \in \{a, b\}$ the correct rank is 1 and the data structure will output with 0 or 2. It, therefore, introduces a +1/-1 error depending on which item was retained. From this point on, $q$ is an inner query with respect to the pair $(a, b)$ if $q \in \{a, b\}$ and an outer query otherwise. This lets us distinguish those queries for which an error is introduced from those that were not influenced by a compression. Figure 1 depicts the above example of one pair compression.

The example gives rise to a high-level method for the original problem with a dataset of size $n$ and memory capacity of $k$ items. Namely 1) keep adding items to the data structure until it is full; 2) choose any pair of items with the same weight and compress them. Notice that if we choose those pairs without care, in the worst case, we might end up representing the full dataset by its top $k$ elements, introducing an error of almost $n$ which is much larger than $n$. Intuitively, pairs being compacted (compressed) should have their ranks as close as possible, thereby affecting as few queries as possible.

![Figure 1: One pair compression for (a,b) introduces ±1 rank error to inner queries and no error to outer queries.](image)

This intuition is implemented via a compactor. First introduced by Manku et al. in [17], it defines an array of $k$ items with weight $w$ each, and a compaction procedure which compress all $k$ items into $k/2$ items with weight $2w$. A compaction procedure first sorts all items, then deletes either even or odd positions and doubles the weight of the rest. Figure 2 depicts the error introduced for different rank queries $q$, by a compaction procedure applied to an example array of items $\{1, 3, 5, 8\}$. Notice that the compactor utilizes the same idea as the one pair compression, but on the pairs of neighbors in the sorted array; thus by performing $k/2$ non-intersecting compressions it introduces an overall error of $w$ as opposed to $kw/2$.

The algorithm introduced in [17], defines a stack of $H = O(\log \frac{n}{w})$ compactors, each of size $k$. Each compactor obtains as an input a stream and outputs a stream with half the size by performing a compact operation each time its buffer is full. The output of the final compactor is a stream of length $k$ that can simply be stored in memory. The bottom compactor that observes items has a weight of 1; the next one observes items of weight 2 and the top one $2H-1$. The output of a compactor on $h$-th level is an input of the compactor on $(h+1)$-th level. Note that the error introduced on $h$-th level is equal to the number of compactions $m_h = \frac{n}{kw}$, times the error introduced by one compaction $w_h$. The total error can be computed as: $\text{Err} = \sum_{h=1}^{H} m_h w_h = H \frac{n}{k} = O(\frac{n}{\epsilon} \log \frac{1}{\delta})$.

Setting $k = O(\frac{1}{\epsilon} \log \epsilon n)$ will lead to an approximation error of $\epsilon n$. The space used by $H$ compactors of size $k$ each is $O(\frac{1}{\epsilon} \log^2 \epsilon n)$. Note that the algorithm is deterministic. Later, Agarwal et al. [3] suggested the compactor to choose the odd or even positions randomly and equiprobably, pushing the introduced error to zero in expectation. Additionally, the authors suggested a new way of feeding a subsampled streams into the data structure, recalling that $O(\frac{1}{\epsilon^2} \log \frac{1}{\delta})$ samples preserve quantiles with $\pm \epsilon n$ approximation error. The proposed algorithm requires $O(\frac{1}{\epsilon^2} \log^{3/2} \frac{1}{\delta})$ space and succeeds with high constant probability.

To prove the result the authors introduced a random variable $X_{i,h}$ denoting the error introduced on the $i$-th compaction at $h$-th level. Then the overall error is $\text{Err} = \sum_{i=1}^{H} \sum_{h=1}^{m_h} w_h X_{i,h}$, where $w_h X_{i,h}$ is bounded, has mean zero and is independent of the other variables. Thus, due to the Hoeffding’s inequality:

$$P(\text{Err} > \epsilon n) \leq 2 \exp \left( -\frac{\epsilon^2 n^2}{\sum_{h=1}^{H} \sum_{i=1}^{m_h} w_h^2} \right).$$

Setting $w_h = 2^{h-1}$ and $k = O\left( \frac{1}{\epsilon^2} \frac{1}{\sqrt{(1/\delta)}} \right)$ will keep the error probability bounded by $\delta$ for $O\left( \frac{1}{\epsilon^2} \right)$ quantiles.

The following improvements were made by Karnin et al. [13].
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(1) Use exponentially decreasing size of the compactor. Higher weighted items receive higher capacity compactors.
(2) Replace compactors of capacity 2 with a sampler. This retains only the top $O(\log \frac{1}{\delta})$ top compactors.
(3) Keep the size of the top $O(\log \log 1/\delta)$ compactors fixed.
(4) Replace the top $O(\log \log 1/\delta)$ compactors with a GK sketch [9].

(1) and (2) reduced the space complexity to $O(\frac{1}{\epsilon} \sqrt{\log 1/\epsilon})$, (3) pushed it further to $O(\frac{1}{\epsilon} \log^2 \frac{1}{\delta})$, and (4) led to an optimal $O(\frac{1}{\epsilon} \log \log \frac{1}{\delta})$. The authors also provided a matching lower bound. Note, the last solution is not mergeable due to the use of GK [9] as a subroutine.

While (3) and (4) lead to the asymptotically better algorithm, its implementation is complicated for application purposes and mostly are of a theoretical interest. In this paper we build upon the KLL algorithm of [13] using only (1) and (2).

In [13], the authors suggest the size of the compactor to decrease as $k_h = e^{cH-h}k$, for $c \in (0.5, 1)$, then $\sum_{h=1}^{H} \sum_{j=1}^{m_h} n_j^2 \leq n^2 / (k^2 C)$ and $P(|\text{Err}| > \epsilon n) \leq 2 \exp\left(-C \epsilon^2 k^2\right) \leq \delta$, where $C = 2e^c(2c - 1)$. Setting $k = O\left(\frac{1}{\sqrt{\epsilon}} \log 1/\epsilon\right)$ leads to the desired approximation guarantee for all $O(1/\epsilon)$ quantiles with constant probability. Note that the smallest meaningful compactor has size 2; thus the algorithm will require $k(1 + c + \ldots + e^{\log_{10}(h)k}) + O(\log n) = \frac{k}{\frac{c \ell}{2}} + O(\log n)$ compactors, where the last term is due to the stack of compactors of size 2. The authors suggested replacing that stack with a basic sampler, which picks one item out of every $2^{\log_{10}(h)k}$. This unused fraction accounts for $O(1/\epsilon)$ space. The resulting space complexity is $O(\frac{1}{\epsilon} \sqrt{\log 1/\epsilon})$. We provide the pseudocode for the core routine in Algorithm 1.

Algorithm 1 Core routines for KLL algorithm [13]

1. function KLL.update(item)
2. if Sampler(item) then KLL[0].append(item)
3. for $h = 1 \ldots H$ do
4. if len(KLL[h]) $\geq k_h$ then KLL.compact(h)
5. function KLL.compact(h)
6. KLL[h].sort(); rb = RANDOM([0,1]);
7. KLL[h + 1].extend(KLL[h][rb : : 2])
8. KLL[h] = []

3 OUR CONTRIBUTION

Although the asymptotic optimum is already achieved for the quantile problem, there remains room for improvement from a practical perspective. In what follows we provide novel modifications to the existing algorithms that improve both their memory consumption and run-time. In addition to the performance, we ensure the algorithm is easy to use by (1) having the algorithm require only a memory limit, as opposed to versions that must know the values of $\epsilon, \delta$ in advance, and (2) by extending the functionality of the sketching algorithm to handle weighted examples. We demonstrate the value of our algorithm in Section 5 with empirical experiments.

Figure 3: Portion of unsaturated memory when compacting the top layer.

3.1 Lazy compactions

Consider a simplified model, when the length of the stream $n$ is known in advance. One can easily identify the weight on the top layer of KLL data structure, as well as the sampling rate and the size of each compactor. Additionally, these parameters do not change while processing the stream. Then note that while we are processing the first half of the stream, the top layer of KLL will be at most half full, i.e. half of the top compactor memory will not be in use during processing first $\frac{n}{2}$ items. Let $s$ be the total amount of allocated memory and $c$ be the compactor size decrease rate. The top layer is of size $s(1-c)$, meaning that a fraction of $(1-c)/2$ is not used throughout that time period. The suggested value for $c$ is $1/\sqrt{2}$ which means that this quantity is 15%. This is of course a lower estimate as the other layers in the algorithm are not utilized in various stages of the processing. A similar problem arise when we do not know the final $n$ and keep updating it online. When the top layer is full the algorithm compacts it into a new layer; at this moment the algorithm basically doubles its guess of the final $n$. Although after this compaction $k/2$ items immediately appear on the top layer, we still have $1/4$ of the top layer not in use until the next update of $n$. This unused fraction accounts for 7% of the overall allocated memory.

We suggest all the compactors share the pool of allocated memory and perform a compaction only when the pool is fully saturated. This way each compaction is applied to a potentially larger set of items compared to the fixed budget setting, leading to less compactions. Each compaction introduces a fixed amount of error thus the total error introduced is lower. Algorithm 2 gives the formal lazy-compacting algorithm, and Figure 4 visualizes its advantage: in vanilla KLL all compactors have less items than their individual capacities, in lazy KLL this is not enforced due to sharing the pool of memory. In Figure 3 you can see that the memory is indeed unsaturated even when we compact the top level.

Figure 4: Compactor saturation: vanilla KLL vs. lazy KLL

*In fact [13] has a fixable mistake in their derivation. For the sake of completeness in Appendix A we clarify that the original results holds although with a slightly different constant terms.*
We represent the four possibilities of Algorithm 2 Update procedure for lazy KLL

Additionally, we conclude that instead of suffering an error of up to \(\frac{n}{k}\) in expectation, the error is cut in half compared to its worst-case analysis without the error-spreading improvement. We thus still have an unbiased estimator for the query’s rank but with probability at least \(\frac{1}{2}\) it is an inner query with error \(-w\); and with probability at most \(\frac{1}{4}\) it is an inner query with error \(+w\).

3.2 Reduced randomness via Anti-Correlations

Consider the process involving a single compactor layer. A convenient way of analyzing its performance is viewing it as a stream processing unit. It receives a stream of size \(n\) and outputs a stream of size \(n/2\). When collecting \(k\) items it sorts them, and outputs (to the output stream) either those with even or odd locations. A deterministic compactor may admit an error of up to \(n/k\). A compactor that decides whether to output the even or odds uniformly at random (to the output stream) either those with even or odd locations. A deterministic compactor may admit an error of up to \(n/k\). A compactor that decides whether to output the even or odds uniformly at random (to the output stream) either those with even or odd locations. A deterministic compactor may admit an error of up to \(n/k\). A compactor that decides whether to output the even or odds uniformly at random (to the output stream) either those with even or odd locations. A deterministic compactor may admit an error of up to \(n/k\). A compactor that decides whether to output the even or odds uniformly at random (to the output stream) either those with even or odd locations. A deterministic compactor may admit an error of up to \(n/k\). A compactor that decides whether to output the even or odds uniformly at random (to the output stream) either those with even or odd locations. A deterministic compactor may admit an error of up to \(n/k\).

This way, each coin flip defines 2 consecutive compactions; with probability \(\frac{1}{2}\) it is even \(\rightarrow\) odd (\(e\rightarrow o\)), and with probability \(\frac{1}{2}\) it is odd \(\rightarrow\) even (\(o\rightarrow e\)).

Let’s analyze the error under this strategy. Recall from Section 2 that for a rank query \(q\) and a compaction operation, \(q\) is either an inner or outer query. If it is an outer query, it suffers no error. If it is an inner query, it suffers and error of \(+w\) if we output the odds and \(-w\) if we output evens. Consider the error associated with a single query after two consecutive and anti-correlated compactions. We represent the four possibilities of \(q\) as \(ii, io, oi, oo\). Clearly, in expectation every two compactions introduce 0 error. Additionally, we conclude that instead of suffering an error of up to \(\pm w\) for every single compaction operation, we suffer that error for every two compaction operations. It follows that the variance of the error is twice smaller, hence the mean error is cut by a factor of \(\sqrt{2}\).

3.3 Error spreading

Recall that in the analysis of all compactor based solutions [3, 13, 17, 25]. During a single compaction we can distinguish two types of rank queries: inner queries, for which some error is introduced, and outer queries, for which no error is introduced. Though the algorithms use this distinction in their analysis, they do not take an action to reduce the number of inner queries. It follows that for an arbitrary stream and an arbitrary query, the query may be an inner query the majority of the time, as it is treated in the analysis. In this section we provide a method that makes sure that a query has an equal chance of being inner or outer, thereby cutting in half the variance of the error associated with any query, for any stream. Consider a single compactor with a buffer of \(k\) slots, and suppose \(k\) is odd. On each compaction we flip a coin and then either compact the items with indices 1 to \(k - 1\) (prefix compaction) or 2 to \(k\) (suffix compaction) equiprobably. This way each query is either inner or outer equiprobably. Formally, for a fixed rank query \(q\) with probability at least \(\frac{1}{2}\) it is an outer query and then no error is introduced, with probability at most \(\frac{1}{4}\) it is an inner query with error \(-w\); and with probability at most \(\frac{1}{4}\) it is an inner with error \(+w\).

We still have an unbiased estimator for the query’s rank but the variance is cut in half. We note that the same analysis applies for two consecutive compactions using the reduced randomness improvement discussed in Section 2. The configuration \((ii, io, oi, oo)\) of a query in two consecutive compactions described in Table 1 will now happen with equal probability, hence we have the same distribution for the error: 0 with probability at least \(\frac{1}{2}\), \(+w\) and \(-w\) with probability at most \(\frac{1}{4}\) each, meaning that the variance is cut in half compared to its worse case analysis without the error-spreading improvement. Figure 5 visualizes the analysis of the error for a fixed query during a single compaction operation.

### Table 1: Error of a fixed rank query during two anti-correlated compactions

|         | even \(\rightarrow\) odd | odd \(\rightarrow\) even |
|---------|---------------------------|--------------------------|
| even    | 0                         | \(-w\)                    |
| \(-w\)  | 0                          | \(+w\)                    |

Figure 5: Error analysis for a single query during a compaction. There are now four possibilities: prefix/suffix compaction, keep even/odd positions
some data streams the disjoint batch size is strictly larger than \( k/2 \) resulting in a reduction in the overall error.

The modified compactor operates in phases we call sweeps. It maintains the same buffer as before and an additional threshold \( \theta \) initialized as special null value. The items in the buffer are stored in non-decreasing sorted order. When we reach capacity we compact a single pair. If \( \theta \) is null we set it to \(-\infty\) or to the value of the smallest item uniformly at random. This mimics the behavior of the prefix/suffix compressions of Section 3.3. The pair we compact is a pair of consecutive items where the smaller item is the smallest item in the buffer that is larger than \( \theta \). If no such pair exist due to \( \theta \) being too large, we start a new sweep, meaning we set \( \theta \) to null and act as detailed above. We note that a sweep is the equivalent to a compaction of a standard compactor. Due to this reason, we consistently keep either the smaller or larger item when compacting a single pair throughout a sweep. To keep true to the technique of Section 3.2 we have sweep number \( 2i + 1 \) draw a coin to determine if the small or large items are kept, and sweep number \( 2i + 2 \) does the opposite. The pseudo-code for the sweep-compactor is given in Algorithm 3 and Figure 6 visualizes the inner state of the sweep-compactor during a single sweep.

Algorithm 3 Sweep compaction procedure

1. function KLLsweep.compact(h)
2.   KLL[h].sort()
3.   \( i^* = \text{argmin}_i \{ \text{KLL}[h][i] \geq \text{KLL}[h].\theta \} \)
4.   if \( i^* == \text{None} \) then \( i^* = 0 \);
5.   KLL[h].\theta = KLL[i^* + 1];
6.   KLL[h].pop(i^* + randBit());
7. return KLL[h].pop(i^*)

Notice that for an already sorted stream the modified compactor performs only a single sweep, hence in this scenario the resulting error would not be a sum of \( n/k \) i.i.d. error terms, each of magnitude \( \pm w \) but rather a single error term of magnitude \( \pm w \). Though this extreme situation may not happen very often, it is likely that the data admits some sorted subsequences and the average sweep would contain more than \( k/2 \) pairs. We demonstrate this empirically in our experiments.

\[ \text{If we wish to ignore prefix/suffix compactions \( \theta \) should always be initialized to \(-\infty\).} \]

\[ \text{We ignore the case of items with equal value. Note that if that happens, these two items should be compacted together as this is guaranteed not to incur a loss.} \]

\[ \text{Notice that \(-\infty \) is still defined in the comparison model.} \]

\[ \text{If \( \theta \) is not chosen uniformly at random, or when we suffer a distribution drift and wish to give preference to more recent items.} \]

\[ \text{One can approach the problem naively: break down each update \( (a_i, w_i) \) into \( w_i \) unitary updates and feed it into lazy sweeping KLL.} \]

\[ \text{In the worst case, the time to process one unitary update is } O(\log \frac{1}{\epsilon}) \text{, and the time to process one weighted update is } O(\max(w_i) \log \frac{1}{\epsilon}). \]

\[ \text{However, in a common scenario, weights } w_i \text{ do not increase exponentially with } i. \text{ In this case for long enough streams, vast majority of updates } (a_i, w_i) \text{ would satisfy } w_i \ll w(i), \text{ and in particular } w_i < 2^{H_i}, \text{ where } 2^{H_i} \text{ is the sampling rate of the KLL sampler object.} \]

\[ \text{Recall, that in KLL the sampler maintains a reservoir sample of a single item until it observes } 2^{H_i} \text{ items and outputs the sample to the stream observed by the first compactor. Then compactor processes its input stream and provides an output stream to the next compactor and so on.} \]

\[ \text{In [13], in order to obtain mergeable sketches the sampler object is in fact defined in a way that it can accept weighted inputs. It feeds the inputs into a weighted reservoir sample until that weight is larger than } 2^{H_i}. \text{ At that point, the sampler has the reservoir sample of weight } w_1 \text{ and a new item of weight } w_2. \text{ One of these items is being outputted into the bottom compactor input stream with a weight of exactly } 2^k, \text{ with probabilities that ensure an unbiased error.} \]

\[ \text{Weighted reservoir sampler can process updates with the weights less than the sampling rate in } O(1) \text{ time. Therefore, if } w_i \text{ does not grow exponentially, then in the worst-case the update time for the majority of updates becomes } O(\log \frac{1}{\epsilon}). \]

Further, we provide two approaches for handling the weighted input scenario in the general case, where we do not assume the slow growth of \( w_i \). The first is achieved via a near black-box approach, wrapping the KLL algorithm and manipulating the input data, which introduces extra } O(\frac{1}{\epsilon} \sqrt{\log \frac{1}{\epsilon}}) \text{ overhead to the worst-case update time and } O(\log \frac{1}{\epsilon}) \text{ overhead to the amortized update time. In the second algorithm, we modify the core component, the compactor. It obtains a compactor that can handle items of different weight and uses the KLL paradigm with these new compactors to handle weighted inputs. The second approach does not suffer from the overhead of manipulating the incoming data and offers the same asymptotic run-time as the unweighted version.}
Algorithm 4 Base2update update procedure

1: function UPDATE((a, w))
2: \[ H = \arg \min_h (k^2 h > W) \]
3: \[ H_{\text{new}} = \arg \min_h (k^2 h > W + w) \]
4: if \( H_{\text{new}} > H \) then
5: delete bottom \( H_{\text{new}} - H \) compactors
6: KLL.pushItems (all items from the deleted compactors)
7: add \( H_{\text{new}} - H \) empty compactors on the top; \( H = H_{\text{new}} \)
8: let \( 2^H \) be the current sample rate
9: decompose \( w = w' + \sum_{h \in H_0} a_h k^h \)
10: for \( h < H \) s.t. \( a_h \neq 0 \) do KLL.pushItems(a, \( 2^h \))
11: KLL.pushItems(a, \( w' \))
12: for \( h < H \) s.t. \( a_h \neq 0 \) do KLL.pushItems(a, \( 2^h \))
13: \( a_H \) times repeat: KLL.pushItems(a, \( 2^H \))

Theorem 4.1. Algorithm 4 processes a stream of weighted updates and outputs all \( \epsilon \)-approximate quantiles with high probability using memory \( O(\frac{1}{\epsilon} \log 1/\epsilon) \). In the worst-case scenario, a single update invokes \( O(\log 1/\epsilon) \) update calls to compactors of the KLL sketch, and

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Figure 8: Compressing pair in the weighted compactor

\( O(\frac{1}{\epsilon} \log 1/\epsilon) \) calls to the sampler, resulting in a \( O(\frac{1}{\epsilon} \log^{3/2} \frac{1}{\epsilon}) \) worst-case run-time. The amortized run-time is \( O(\log^{2} \frac{1}{\epsilon}) \).

Due to space restrictions we defer the proof to Appendix B.

4.2 Weight-aware Compactor

Here we suggest a solution which does not require any stream transformation. Instead, we modify the main building block, the compactor, to handle different weights for its inputs. We define a weight aware compactor as an object that receives a stream of items of weights in \([w, 2w]\) for some pre-defined scalar \( w \), and the compaction procedure is similar to the unweighted case:

1. sort the array using \( a_j \) as an index
2. break the array into pairs of neighbors \( (a_i, w_i), (a_{i+1}, w_{i+1}) \)
3. compress each pair, using procedure described above

The intuition behind the weighted pair compression is depicted in the Figure 8 and the rest of the process is given in Algorithm 5, that due to space restrictions is available in the Appendix.

Lemma 4.2. Given a stream of \( n \) items of weights in \([w, 2w]\), a weight-aware compactor outputs a stream of \( n/2 \) items of weight \([2w, 4w]\). If the memory budget of the weight-aware compactor is \( k \) we have that for any query \( q \), the error in its rank in the output stream compared to the input stream is equal to \( \sum_{i=1}^k X_i \). Here, the \( X_i \)'s are independent random variables. For every \( X_i \) we have \( E[X_i] = 0 \) and \( |X_i| < 2w \) w.p. 1.

Due to space restrictions we defer the proof to Appendix B.

The new algorithm will operate as the unweighted version of KLL did. It maintains a hierarchy of compactors, and a sampler at the bottom hierarchy. A compactor at level \( h \) accepts inputs of weights in \([2^h, 2^{h+1}]\), instead of exactly \( 2^h \) as in the unweighted case. As before, the sampler outputs items of weight \( 2^H \) and accepts items of weight in range from \( 1 \) to \( 2^H \).

Theorem 4.3. Algorithm 5 processes a stream of weighted updates and outputs all \( \epsilon \)-approximate quantiles with high probability using space \( O(\frac{1}{\epsilon} \log 1/\epsilon) \) and has both worst-case runtime and amortized runtime equal \( O(\log 1/\epsilon) \).
Due to space restrictions we defer the proof to Appendix B.

5 EXPERIMENTAL RESULTS

5.1 Data Sets
To study the algorithms properties we tested it on both synthetic and real datasets, with various sizes, underlying distributions and orders. Note that all the approximation guarantees of the investigated algorithms do not depend on the order in the data, however in practice the order might significantly influence the precision of the output within the theoretical guarantees. Surprisingly the worst-case is achieved when the dataset is randomly shuffled. Therefore, we will pay more attention to randomly ordered data sets in this section. We also experiment with the semi-random orders that resemble more to real life applications. Due to the space limitations we could not possibly present all the experiments in the paper and present here only the most interesting findings.

Our experiments were carried on following synthetic datasets. Sorted is a stream with all unique items in ascending order. Shuffled is a randomly shuffled stream with all unique items. Trending is a stream with all unique items in ascending order. Trending stream mimics a statistical drift over time (widely used in ML). Brownian simulates a Brownian motion or a random walk which generates time series data not unlike CPU usage, stock market, traffic congestion, etc. The length of the stream varies from $10^5$ to $10^9$ for all the datasets.

In addition to synthetic data we use two publicly available datasets. The first contains text information and the second contains IP addresses. Both objects types have a natural order and can be fed as input to a quantile sketching algorithm.

1) Anonymized Internet Traces 2015 (CAIDA) [1] The dataset contains anonymized passive traffic traces from the internet data collection monitor which belongs to CAIDA (Center for Applied Internet Data Analysis) and located at an Equinix data center in Chicago, IL. For simplicity we work with the stream of pairs (IPsource, IPdestination). The comparison model is lexicographic. We evaluate the performance on the prefixes of the dataset of different sizes: from $7 \times 10^7$ to $9 \times 10^9$. Note that evaluation of the CDF of the underlying distribution for traffic flows helps optimize packet managing. CAIDA’s datasets are used widely for verifying different sketching techniques to maintain different statistics over the flow, and finding quantiles and heavy hitters specifically.

2) Page view statistics for Wikimedia projects (Wiki) [2] The dataset contains counts for the number of requests for each page of the Wikipedia project during 8 months of 2016. The data is aggregated by day, i.e. within each day data is sorted and each item is assigned with a count of requests during that day. Every update in this dataset is the title of a Wikipedia page. We will experiment with both the original dataset and with its shuffled version. Similarly to CAIDA we will consider for the Wiki dataset prefixes of size from $10^7$ to $10^9$. In our experiments, each update is a string containing the name of the page in Wikipedia. The comparison model is lexicographic.

5.2 Implementation and Evaluation Details
All the algorithms and experimental settings are implemented in Python 3.6.3. The advantage of using a scripting language is fast prototyping and readable code for distribution inside the community. Time performance of the algorithm is not the subject of the research in the current paper, and we leave its investigation for future work. This in particular applies to the sweep compactor KLL and the algorithms for weighted quantiles, which theoretically improve the worst-case update time exponentially in $\frac{1}{\varepsilon}$. All the algorithms in the current comparison are randomized, thus for each experiment the results presented are averaged over 50 independent runs. KLL and all suggested modifications are compared with each other and LWYC (the algorithm Random from [12]). In [25] the authors carried on the experimental study of the algorithms from [3, 9, 17, 18] and concluded that their own algorithm (LWYC) is preferable to the others: better in accuracy than [9] and similar in accuracy compared with [18] while LWYC has a simpler logic and easier to implement.

As mentioned earlier we compared our algorithms under a fixed space restrictions. In other words, in all experiments we fixed the space allocated to the sketch and evaluated the algorithm based on the best accuracy it can achieve under that space limit. We measured the accuracy as the maximum deviation among all quantile queries, otherwise known as the Kolmogorov-Smirnov divergence, widely used to measure the distance between CDFs of two distributions. Additionally, we measure the introduced variance caused separately by the compaction steps and sampling. Its value can help the user to evaluate the accuracy of the output. Note that for KLL this value depends on the size of the stream, and is independent of the arrival order of the items. In other words, the guarantees of KLL are the same for all types of streams, adversarial and structured. Some of our improvements change this property; recall that the sweep compactor KLL, when applied to sorted input, requires only a single sweep per layer. For this reason, in our experiments we found variance to be dependent not only on the internal randomness of the algorithm but also the arrival order of the stream items.

5.3 Results
Note that the majority modifications presented in the current paper can be combined for better performance, due to the space limitations we present only some of them. For the sake of simplicity we will fix the order of suggested modification: as lazy from Section 3.1, reduced randomness from Section 3.2, error spreading from Section 3.3 and sweeping from Section 3.4, and denote all possible combinations as four 0/1 digits, i.e. 0000 would imply the vanilla KLL without any modifications, while 0011 would imply that we use KLL with error spreading trick and sweeping.

In Figures 9b and 9a we compare the size/precision trade-off for LWYC, vanilla KLL, and KLL with modifications. First, we can see that all KLL-based algorithms provide the approximation ratio significantly better than LWYC as the space allocation is growing, which confirms theoretical guarantees. Second, from the experiments it becomes clear that all algorithms behave worse on the data without any order, i.e. shuffled stream. Although the laziness give the most significant push to the performance of the Vanilla KLL, all other modifications improve the precision even further if combined. One can easily see it in the table 9g for shuffled dataset and table 9h for the sorted stream. Same experiments were carried
Figure 9: Figures 9b, 9a, 9d, and 9e depict the trade-off between maximum error over all queried quantiles and the sketch size: Figures 9b and 9a test the performance of the algorithms on shuffled and sorted data streams; Figures 9d and 9e on CAIDA and Wikipedia datasets correspondingly. Tables 9g and 9h show the same trade-off, but make it possible to see the difference between different combos. Figure 9f demonstrates independence of the algorithms performance from stream length, dashed lines indicate the sketch size equal 256 and the solid lines correspond to the sketch of size 1024. Finally, Figure 9c mix the trending data with a different amounts of a random noise and demonstrates the influence of the stream order on the algorithm precision.

Although, theoretically none of the algorithms should depend on the length of the dataset, we verified this property in practice, the results can be seen on Figure 9f.

In Figure 9c we verified that although all the theoretical bounds hold, KLL and LWYC performance indeed depend on the amount of randomness in the stream, more randomness leads to less precision. Our experiment were held on the trending dataset, i.e. the stream containing two components: \(A\times(\text{mean-zero random variable})\) and \(B\times(\text{trend t/n})\). Figure 9c shows how precision drops as \(A/B\) start to grow (X-axis). Note that modified algorithm does not drop in precision as fast as vanilla KLL or LWYC.

### 6 HIGH-PERFORMANCE IMPLEMENTATION

For simplicity of analysis, experimentation, and exposition, the pseudocode so far assumes the use of list-based data structures. In reality those would include link fields that would double the space usage for data types whose physical size is similar to that of pointers. Moreover, they are not very efficient in terms of update operations. In practice, a factor of two in space and update time is very significant.
We verified experimentally that the KLL algorithm proposed by Karnin et al. [13] has predicted asymptotic improvement over LWYC [25]. We proposed four modifications to KLL with provably better constants in the approximation bounds. Experiments verified that the approximation is roughly twice as good in practice compared to KLL and more than four times better compared to LWYC (and growing with the space allocated to the sketch). Moreover, the worst-case update time for the presented sweep-compactor based KLL is \( O(\log 1/\epsilon) \) which improves over the rest of the compactor based algorithms. Two algorithms proposed for the weighted streams improve over the naive extension from \( O((\max w_i) \log 1/\epsilon) \) to \( O(\log 1/\epsilon) \) while maintaining the same space complexity. Finally, we provide an very efficient data structure for maintaining compactor based structures such as the algorithms above.

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\*The core code can be found at github.com/apache/incubator-datasketches-java
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A  FIXING THE ORIGINAL KLL PROOF

The original paper by Karnin et al. [13] contains a mistake regarding the number of compactions performed at a single level. Correcting the mistake is trivial and does not change the authors claim. Nevertheless, we provide a correction of their argument. The authors use compactors of exponentially decreasing size. Higher weight items receive higher capacity compactors. The error appeared in the last inequality of the bound on \( m_h \) — the number of compaction made at level \( h \) (page 6 in [13]):

\[
m_h \leq \frac{n}{k_h w_h} \leq \frac{2n}{k^2 H} (2/c)^{H-h} \leq (2/c)^{H-h-1},
\]

where \( H \) is the height of the top compactor, \( k_h = k c^{H-h} \) is the size of the compactor at height \( h \). Note, that the last inequality implies \( n \leq ck^2 2^{H-2} = k_{h-1} w_{h-1} \), while from the definition of \( H \) it follows that at least one compaction happened on level \( H - 1 \). Therefore \( n \geq k_{h-1} w_{h-1} \). Fixing this slightly constant in the final upper bound.

Recall that \( k \geq 4 \) and \( c \in (0.5,1) \). We reuse the notation and refer to the height of the top compactor as \( H \). Additionally, we introduce \( H' \) which denotes the height of the top compactor of size 2. Due to the choice of \( k \) and \( c \) we can conclude that \( H' \leq H - 1 \).

Every compactor of size 2 contains at most one item, otherwise it would be compacted. Therefore, the bottom \( H' \) compactors have total weight \( \sum_{h=1}^{H'} H' w_h = \sum_{h=1}^{H'} 2^{H'-h} \leq 2^{H'} \). Similarly, every compactor of size \( k_h \) contains at most \( k_h - 1 = k c^{H-h} - 1 \leq (k - 1) c^{H-h} \) items. Then the total weight of compactors from level \( H' + 1 \) to \( H \) is:

\[
\sum_{h=H+1}^{H} (k_h - 1) w_h \leq \sum_{h=1}^{H} (k-1) c^{H-h} 2^{h-1} = (k-1) c^{H-1} \sum_{h=1}^{H} (2/c)^{h-1} = \frac{(k-1) c^{H-1} 2}{c-1} \leq (k-1) c^{H-1}.
\]

Putting together the total weight of the bottom \( H' \) and top \( H - H' \) compactors we get the upper bound on the number of items processed:

\[
n \leq (k-1)c^{H-1} + 2^{H'} \leq (k-1 + 1/2) 2^{H} \leq k 2^{H}.
\]

Plugging \( n \leq k 2^{H} \) into the last inequality of Equation 1 leads to \( m_h \leq 2(2/c)^{H-h} \) which is \( 4/c \) times worse than the initial derivation. Repeating the argument as in [13] and in the Section 2 of the current paper, we get \( \sum_{h=1}^{H} \sum_{i=1}^{m_h} w^2_h \leq \frac{2n^2 / k^2}{c(2c-1)} \). As in [13] applying Hoeffding’s inequality gives

\[
P(|\text{Err}| > \varepsilon n) \leq 2 \exp \left( -C c^3 k^2 \right) \leq \delta.
\]

However, the constant \( C \) has changed from \( 2c^3(2c-1) \) to \( C = \frac{1}{2} c^3(2c-1) \). Note that all asymptotic guarantees stay the same as in [13].

B  MISSING PROOFS

**Theorem 4.1** Algorithm 4 processes a stream of weighted updates. It outputs all \( \varepsilon \)-approximate quantiles with high probability using memory \( O(1/\varepsilon) \). In the worst-case scenario, a single update invokes \( O(\log 1/\varepsilon) \) update calls to compactors of the KLL sketch and \( O(1/\varepsilon \log 1/3) \) calls to the sampler. This results in a \( O(1/\varepsilon \log 1/3) \) worst-case update run-time. The amortized run-time is \( O(\log^2 1/\varepsilon) \).

**Proof.** The analysis of the error of this algorithm is straightforward, as an item of weight \( w \) is broken into several weights summing to \( w \). For the runtime analysis, we decompose it into three parts:

1. \( \text{increase of } H \) (lines 4-7 in Algorithm 4)
2. \( \text{push } w' \) to sampler and all \( a_k \) to levels \( h < H \) (lines 11,12)
3. \( \text{push } a_H \) to the top compactor (line 13)

For the first part, the worst-case happens when all compactors are full and all should be deleted due to increase of \( H \). Therefore, line 6 of Algorithm 4 will push \( O(\log 1/\varepsilon) \) items into the data structure at the total cost \( O(1/\varepsilon \log 1/3) \). Since each of these items must have been inserted earlier, the amortized runtime for the first part is \( O(\log^2 1/\varepsilon) \).

The second part is associated with the different components of \( w \) except the largest one, \( a_H \). In the worst-case, \( a_k = 1 \) for all \( h \in (H_k, H) \). Recall, that the worst-case runtime of lazy sweeping KLL is \( O(1/\varepsilon) \). Therefore the worst-case runtime is \( O(\log 1/\varepsilon) \). The amortized running time is \( O(\log^2 1/\varepsilon) \) as well.

Finally, the third part is taking into account adding the same element to the top layer \( a_H \) times. In the worst-case, in total \( O(k \log 1/\varepsilon) = O(1/\varepsilon \log 1/3) \) time. For the amortized case, although \( a_H \) could be equal to \( k - 1 \), we do not really need to add \( a_H \) copies of the item but rather remember the number of times the item is inserted. It follows that the amortized case is the same as that of inserting an item to a compactor which is \( O(\log(k)) = O(\log^2 1/\varepsilon) \).

Summing the three components, we get a worst-case runtime of \( O(1/\varepsilon \log 1/3) \) and amortized time of \( O(1/\varepsilon \log^2 1/\varepsilon) \).

**Lemma 4.2** Given a stream of \( n \) items of weights in \([w, 2w]\), a weight-aware compactor outputs a stream of \( n/2 \) items of weight \([2w, 4w]\). For a size \( k \) weight-aware compactor the added error in rank between input and output stream (for any query \( a \)) is equal to \( \sum_{i=1}^{k} X_i \). Here, \( X_i \)'s are independent random variable such that \( E[X_i] = 0 \) and \( |X_i| < 2w \).

**Proof.** The claim regarding the stream length is trivial as every two items become a single item in the compact operation. Also, since the weight of an output is \( w_a + w_b \), with \( w_a, w_b \in [w, 2w] \) it follows that the output weights are in \([2w, 4w]\). For the error, consider an arbitrary query \( q \). In a single compact operation, \( q \) is an inner query if \( q < q_{a+1} \) for some even \( j \) and an outer query otherwise. If \( q \) is an outer query, the error associated to it is 0. Otherwise, the error is \( w_a \) with probability \( w_a/w \) and \( -w_b \) with probability \( w_b/w \). Denoting the error for \( q \) at compaction \( j \) as \( X_j \) we get that \( E[X_j] = 0 \) and \( |X_j| < 2w \) as claimed. Finally, since the size of the compactor is \( k \), a compact operation will occur for every \( k \) items and indeed the number of error variables \( X_i \) is \( n/k \).

**Theorem 4.3** Algorithm 5 processes a stream of weighted updates and outputs all \( \varepsilon \)-approximate quantiles with high probability using space \( O(1/\varepsilon \log 1/\varepsilon) \). Its worst-case update time is \( O(\log 1/\varepsilon) \).
We conclude that if we have a large number of compactor drops, the weighted case. Let \( X_{i,h} \) be a random variable which indicates the sign of the error introduced during the \( i \)-th compaction on the \( h \)-th level, and let it be equal to zero if no error is introduced. Note, that in Lemma 4.2 we showed that \( E(X_{i,h}) = 0 \) and \( X_{i,h} \leq 2w \), therefore the total error introduced is

\[
Err = \sum_{h=1}^{H} \sum_{i=1}^{m_h} 2\omega_h X_{i,h}.
\]

Repeating the argument as in Appendix A we conclude that:

\[
P(|\text{Err}| > \varepsilon W) \leq 2 \exp \left(-\frac{C}{4} \varepsilon^2 k^2 \right) < \delta.
\]

To reach the same approximation guarantees with the same probability of failure, one need to set \( k_{\text{new}} = 2k_{\text{old}} \), i.e. this algorithm will use twice as much space as the naive implementations. Additionally it stores a weight for each item explicitly which doubles the space complexity (this depends on the memory footprint of a stream item).

Note, that the error introduced in line 4 is not 0 in expectation and might accumulate over time. However, line 4 is only executed when an item of weight more than \( k2^H \) is processed. We can bound the overall weight of the items in the bottom compactors that were discarded as a small fraction of the overall number of items processed. The cumulative weight of items dropped will turn out to be a geometric sequence dominated by its last element, which in turn is a small fraction of the overall weight. In Appendix A we show that the total weight of items in compactor of height \( h \) is at most \( k2^h \). For weighted compactors it is \( k2^{h+1} \). The number of bottom compactors that are to be dropped is \( H_{\text{new}} - H - 1 \) and the level of the highest dropped compactor is \( H_s + H_{\text{new}} - H - 1 \). Therefore, the total weight dropped is less than \( k2^{H_s+H_{\text{new}}-H-1} \). Our goal is to bound the portion of total weight we dropped. Therefore, we will estimate the ratio of dropped weight to the weight of added items.

\[
\frac{k2^{H_s+H_{\text{new}}-H-1}}{k2^{H_{\text{new}}-1}} = 2^{H_s-H} - 2^{-\log_{2/c} k} = k^{-1/\log_2 1/c} \leq \varepsilon
\]

The last equation holds since \( c > 0.5 \) and \( \varepsilon = \Omega(1/k\sqrt{\log(k)}) \). It follows that with each compactor drop we discard at most an \( \varepsilon \) portion of the stream. At the same time before every such drop the total weight increase by at least 50%, added weight \( k2^{H_{\text{new}}-1} \geq 0.5 \). We conclude that if we have a large number of compactor drops, the final error introduced is \( \varepsilon(1+(2/3)+(2/3)^2+\ldots) \leq 3\varepsilon \). Adjusting the input memory allowance by a constant factor leads to the desired approximation.

To process any weighted update, Algorithm 5 applies lines 8, 10 and 12. If we use lazy compactions with sweeping, lines 8 and 10 in the worst case require \( O(\log \frac{1}{\varepsilon}) \) running time. As for line 12, we store a single item \( (a, 2^h) \) and its multiplicity \( w/2^h \leq k \), instead of pushing up to \( k \) items into the top compactor. Hence, in the worst case line 12 accounts for \( O(1) \) run-time. □
Figure 11: Average update time per item in nanoseconds for the lazy KLL algorithm described in Section 6. After the sketch has filled up, the algorithm spends roughly half its time sorting items in the level 0 buffer.