Observational constraints on modified Chaplygin gas in Horava–Lifshitz gravity with dark radiation

B C Paul, P Thakur and M M Verma

Abstract. Cosmological models with modified Chaplygin gas (MCG) in the framework of Horava–Lifshitz (HL) theory of gravity, both with and without detailed balance, are obtained. The equation of state (EOS) for a MCG contains three unknown parameters namely, $A$, $\alpha$, $B$. The allowed values of some of these parameters of the EOS are determined using the recent astrophysical and cosmological observational data. Using observational data from $H(z)$, baryon acoustic oscillation (BAO) peak parameter and cosmic microwave background (CMB) shift parameter we study cosmologies in detailed-balance and beyond detailed-balance scenario. In this paper we take up the beyond detailed-balance scenario in totality and contribution of dark radiation in detailed-balance scenario on the parameters of the EOS. We explore the effect of dark radiation on the whole range of the effective neutrino parameter ($\Delta N_\nu$) to constrain matter contributing parameter $B$ in both the detailed-balance and the beyond detailed-balance scenarios. It has been observed that greater the dark radiation less the matter contribution in the MCG in both the scenario considered here. In order to check the validity of beyond detailed-balance scenario we plot supernovae magnitudes ($\mu$) with red-shift of Union2 data and then the variation of state parameter with redshift is studied. It is noted that beyond detailed-balance scenario is suitable for cosmological model in HL gravity with MCG.

Keywords. Modified Chaplygin gas; Horava–Lifshitz gravity; dark energy.

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1. Introduction

The Big-Bang cosmology has become the standard model for cosmology which accommodates the beginning of the Universe at some finite past. The discovery of CMBR [1,2] supports such a model of the Universe. However, Big-Bang cosmology based on perfect fluid assumption fails to explain some of the observed facts both in the early and at late Universe. The standard Big-Bang Model is known to have several limitations; for instance, (i) Horizon problem, (ii) flatness problem and (iii) the singularity problem,
It is known that these problems may be resolved by invoking a phase of inflation at a very early epoch. Most of these problems have, however, been resolved by invoking inflation [3–6] which may be obtained in the semiclassical theory of gravity. On the other hand, recent observations predict that our Universe is passing through a phase of acceleration [7]. This phase of acceleration is believed to be a late time phase of the Universe and it comes out that such a phase cannot be accommodated in the general theory of relativity with normal matter fields available in the standard model of particle physics. Since a Universe with inflation should give satisfactory explanation of what happens close to the Planck era, it is also necessary to consider a satisfactory theory which is valid near that epoch. It may be pointed out here that a quantum gravity effect becomes important at the Planck time. But a consistent theory of quantum gravity is yet to emerge. In this direction superstring theory may be considered as one of the promising candidate of quantum theory of gravity. Cosmological models are also proposed in loop quantum gravity (LQG) [8] which avoids initial singularity. However, a proper description of time evolution of quantum space-time in the LQG is not satisfactory. Several attempts have been made in the recent past to achieve a complete quantum gravitational theory (UV complete theory). Among many such attempts, Horava–Lifshitz (henceforth, HL) theory of gravity appears to be interesting. The success of the Lifshitz theory in solid-state physics motivated Horava to propose a theory of gravity, often called Horava–Lifshitz (HL) gravity [9] which may be important for exploring a viable cosmological model. In the ultraviolet (UV) limit, HL gravity has a Lifshitz-like anisotropic scaling as $t \rightarrow t^z t$ and $x^i \rightarrow l x^i$, between space and time, characterized by the dynamical critical exponent $z = 3$ and thus breaks the Lorentz invariance; while in the infra-red (IR) limit, the scale reduces to $z = 1$. So, it is expected that it may be reduced to a classical general relativistic theory of gravity in the low energy limit. The Friedmann equation gets modified by a $1/a^4$ term [10–12], where $a$ is the scale factor in a non-flat Universe in the HL gravity.

In the original HL gravity, Horava assumed two conditions: detailed balance and projectibility. More recently, Sotiriou, Visser and Weifurtner (SVW) [13], proposed a general HL theory with projectibility but without detailed-balance conditions. For a spatially curved Friedmann–Robertson–Walker Universe, the SVW generalization yields an extra $1/a^6$ term that modifies the coefficient of the $1/a^4$ term in the Friedmann equation compared to the HL theory. Therefore, it is important to look for cosmological models in Horava gravity considering projectibility with and without detailed-balance.

In the HL gravity, the initial Big-Bang singularity may not arise due to the presence of higher order terms in the spatial curvatures $R_{ij}$. There are many such novel features of HL gravity for which it is worth exploring different aspects of the observed Universe. A volume of literature in the framework of the HL gravity appeared containing the study of gravitational wave production [14–16], perturbation spectrum [17–19], black-hole properties [20–22], dark energy phenomenology [23,24], the problems of determining observational constraints in the theory [25], astrophysical phenomenology [26–28], thermodynamical properties etc. [29,30]. In spite of foundational and conceptual issues of HL gravity, cosmological scenario has been examined with generalized Chaplygin gas (GCG) [31]. One of the characteristic features of GCG is that it behaves as a pressureless fluid at the early stage of the evolution of the Universe, which however at a later stage, behaves like a cosmological constant, accounting for the observed features of the Universe. It is interesting to note here that an alternative to quintessence has been first proposed by
Observational constraints on modified Chaplygin gas

Kamenshchik et al [32] where the Chaplygin gas and the generalized Chaplygin gas cosmological models are obtained for the first time. Recently, a modified form of Chaplygin gas was also considered extensively in cosmology [33,34]. The modified Chaplygin gas (MCG) is more general and contains three free parameters. The idea is to interpolate states of standard fluids at high pressures and at high energy densities to a constant negative pressure at low energy densities [35]. In the present work we explore cosmological models with MCG in the framework of HL gravity and estimate the constraints on the parameters of MCG from recent cosmological observations. We examine the effect of effective neutrino parameter on both detailed-balance and beyond detailed-balance scenarios. Here compatibility of beyond detailed-balance scenario is also explored in detail using recent observational data in HL gravity with MCG. The objective of the paper is to estimate the constraints of EOS parameters using the observational data for viable cosmologies.

The equation of state parameter of the total cosmic fluid defined by $w(z) = p_{\text{tot}}/\rho_{\text{tot}}$, will be evaluated at different redshifts. Comparing the supernovae magnitudes ($\mu$) vs. redshift parameter ($z$) with Union2 data we explore the viability of beyond detailed-balance scenario. The suitability of the model is also examined using $w(z)$ vs. $z$ plot.

The paper is organized as follows: In §2, the basic equations for Horava–Lifshitz cosmology are presented and the Friedmann equations are obtained for detailed balance and beyond detailed-balance conditions. In §3, the energy density and EOS for MCG is presented. In §4, the constraints on detailed-balance condition and beyond detailed-balance condition from the observations is presented. In §5, numerical analysis to determine constraints on EOS parameters are obtained for detailed balance. In §6, numerical analysis to determine constraints on EOS parameters are obtained for beyond detailed-balance scenario. In §7, the viability of MCG in HL gravity is discussed. Finally, in §8, the results are summarized.

2. Horava–Lifshitz cosmology

In Horava–Lifshitz gravity [9–11], it is convenient to use the four-dimensional space-time metric of the Arnowitt–Deser–Misner (ADM) decomposition form which is given by

$$ \text{d}s^2 = -N^2\text{d}t^2 + g_{ij}(\text{d}x^i + N_i\text{d}t)(\text{d}x^j + N_j\text{d}t), $$

where the basic variables are lapse function $N$, shift vector $N_i$ and the spatial metric $g_{ij}$. The scaling transformation of the coordinates reads: $t \to l^3t$ and $x^i \to lx^i$. The shift $N^i$ and the 3d spatial metric $g_{ij}$ depend both on the time coordinate $t$ and the spatial coordinate $x^i$, the lapse $N$ is assumed to depend on time only. This condition on the lapse is called the projectibility condition. The action of HL gravity consists of kinetic and potential terms as follows:

$$ S_g = S_K + S_V = \int \text{d}t \text{d}^3x \sqrt{g} N (L_K + L_V). $$

The kinetic term is given by

$$ S_K = \int \text{d}t \text{d}^3x \sqrt{g} N \left[ \frac{2(K_{ij}K^{ij} - \lambda K^2)}{\kappa^2} \right], $$

where $K_{ij} = (g_{ij} - \nabla_i N_j - \nabla_j N_i)/2N$ is the extrinsic curvature and dot represents derivative with respect to time ($\dot{t}$).
2.1 Detailed-balance condition and projectibility

The symmetry property of the Lagrangian $L_V$, employed in the gravitational action, drastically reduces the number of invariants which one should actually consider in the action to begin with [36]. The above symmetry is known as detailed balance which follows from condensed matter systems and requires that the Lagrangian $L_V$ should be derivable from a superpotential $W$ [11]. Under the detailed-balance condition, the total action of HL gravity is given by

$$S_g = \int dt d^3x \sqrt{g} N \left[ \kappa^2 C_{ij} C^{ij} + \frac{2}{2\omega^4} (K_{ij} K^{ij} - \kappa K^2) \right] + \frac{2\omega^4}{2\omega^4} \kappa^2 \mu \epsilon^{ijk} R_{ij} \nabla_k R^l_k + \frac{2\omega^4}{2\omega^4} \kappa^2 \mu^2 R_{ij} R^{ij}$$

$$- \frac{2\omega^4}{2\omega^4} \kappa^2 \mu^2 R_{ij} R^{ij} \right] \times \left[ - \frac{2\omega^4}{2\omega^4} \kappa^2 \mu^2 \left( \frac{1 - 4\lambda}{4} R^2 + \Lambda r - 3\Lambda^2 \right) \right] \right], \quad (4)$$

where

$$C^{ij} = \epsilon^{ijk} \nabla_k \left( R^j_i - \frac{R \delta^j_i}{4} \right) \quad (5)$$

is known as the Cotton tensor, and the covariant derivatives are determined with respect to the spatial metric ($g_{ij}$), $\epsilon^{ijk}$ is a totally antisymmetric unit tensor, $\lambda$ is a dimensionless constant and the variables $\kappa$, $\omega$ and $\mu$ are constants.

In the above gravitational action, to include matter components, one needs to add a cosmological stress–energy tensor to the gravitational field equations, that recovers the usual general relativity formulation in the low-energy limit [13,23,37]. The matter–tensor is a hydrodynamical approximation that leads to the existence of energy density ($\rho_m$) and pressure ($p_m$) in the Friedmann equation, where $\rho_m$ represents the total matter density, that accounts for both the baryonic $\rho_b$ as well as the dark matter $\rho_{dm}$, including the normal matter (where $p_m$ is the pressure).

Horava obtained the gravitational action by assuming that the lapse function is just a function of time, i.e., $N = N(t)$. Here we use FRW metric with $N = 1$, $g_{ij} = a^2(t) \gamma_{ij}$, $N^i = 0$ with

$$\gamma_{ij} dx^i dx^j = \frac{dr^2}{1 - Kr^2} + r^2 d\Omega^2,$$  

where $K = -1, 1, 0$, corresponds to open, close and flat Universe respectively. By varying $N$ and $g_{ij}$ in the gravitational action (4), one obtains the following field equations:

$$H^2 = \frac{\kappa^2}{6(3\lambda - 1)} (\rho_m + \rho_c) + \frac{\kappa^2}{6(3\lambda - 1)} \left[ \frac{3\kappa^2 \mu^2 K^2}{8(3\lambda - 1)a^4} + \frac{3\kappa^2 \mu^2 \Lambda^2}{8(3\lambda - 1)} \right] - \frac{\kappa^4 \mu^2 \Lambda K}{8(3\lambda - 1)^2 a^2},$$  

(7)
Observational constraints on modified Chaplygin gas

\[ \dot{H} + \frac{3H^2}{2} = -\frac{\kappa^2}{4(3\lambda - 1)}(\rho_m \omega_m + \rho_r \omega_r) \]

\[ -\frac{\kappa^2}{4(3\lambda - 1)} \left[ \frac{\kappa^2 \mu^2 K^2}{8(3\lambda - 1)a^4} - \frac{3\kappa^2 \mu^2 \Lambda^2}{8(3\lambda - 1)} \right] \]

\[ -\frac{\kappa^4 \mu^2 \Lambda K}{16(3\lambda - 1)^2 a^4} \]  

where \( H = \dot{a}/a \). In the above field equations the term proportional to \( a^{-4} \) may be considered as the usual ‘dark radiation term’, characteristic of the HL cosmology \([10,11]\) and the constant term is identified with the usual cosmological constant. The conservation equation for matter is:

\[ \dot{\rho}_m + 3H(\rho_m + p_m) = 0, \]  

and that of radiation is:

\[ \dot{\rho}_r + 3H(\rho_r + p_r) = 0, \]  

where we denote

\[ G_{\text{cosmo}} = \frac{\kappa^2}{16\pi(3\lambda - 1)}, \]  

\[ \frac{\kappa^4 \mu^2 \Lambda}{8(3\lambda - 1)^2} = 1, \]  

\[ G_{\text{grav}} = \frac{\kappa^2}{32\pi}. \]  

2.2 Beyond detailed-balance condition with projectibility

As it is not known with certainty whether the detailed-balance condition is enough for extracting whole information of HL gravity \([10,11]\) or it is necessary to do something with this balance condition, one can investigate cosmological scenario in the HL gravity relaxing the detailed-balance condition. In this subsection we discuss the cosmology of HL gravity in the presence of MCG, baryon, radiation and dark radiation without detailed balance. The aim of the paper is to explore the effects of dark radiation on the parameters of the MCG model by taking into account the observational data. The Friedmann equations in this case can be written as \([13,37–40]\)

\[ H^2 = \frac{2\sigma_0}{(3\lambda - 1)}(\rho_m + \rho_r) \]

\[ + \frac{2}{(3\lambda - 1)} \left[ \frac{\sigma_1}{6} + \frac{\sigma_3 K^2}{6a^4} + \frac{\sigma_4 K}{6a^6} \right] \]

\[ + \frac{\sigma_2 K}{3(3\lambda - 1)a^2}. \]  

\[ \text{Pramana – J. Phys., Vol. 81, No. 4, October 2013} \]  

695
\[ \dot{H} + \frac{3}{2} H^2 = \frac{3\sigma_0}{(3\lambda - 1)} (\rho_m \omega_m + \rho_r \omega_r) \]

\[- \frac{1}{(3\lambda - 1)} \left\{ \frac{\sigma_1}{6} + \frac{\sigma_3 K^2}{18a^4} + \frac{\sigma_4 K}{6a^6} \right\} \]

\[ + \frac{\sigma_2 K}{6(3\lambda - 1)a^2}, \]  

(15)

where \( \sigma_0 = \kappa^2 / 12 \). We define some useful dimensionless parameters, which are given below:

\[ G_{\text{cosmo}} = \frac{6\sigma_0}{8\pi (3\lambda - 1)}, \]  

(16)

\[ \sigma_2 = -3(3\lambda - 1), \]  

(17)

\[ G_{\text{grav}} = \frac{6\sigma_0}{16\pi}, \]  

(18)

where \( \sigma_2 < 0 \) and \( \sigma_4 > 0 \). In the case of the detailed balance case, in the IR limit \( (\lambda = 1) \), the two parameters \( G_{\text{cosmo}} \) and \( G_{\text{grav}} \) coincide.

3. EOS for modified Chaplygin gas

The equation of state for GCG [41,42] is given by

\[ p = -\frac{A}{\rho^\alpha} \]  

(19)

with \( 0 \leq \alpha \leq 1 \). In the above original Chaplygin gas corresponds to \( \alpha = 1 [43] \). It may be pointed out here that Chaplygin introduced the equation of state [43] to study the lifting force on a plane wing in aerodynamics. Chaplygin’s equation of state (19) has raised a renewed interest recently [44,45] because of its many remarkable and, in some sense, intriguingly unique features related to cosmological context. It is interesting to note that it has amazing connection with string theory: it can be obtained from the Nambu–Goto action for \( d \)-branes moving in a \( (d + 2) \)-dimensional space-time in the light-cone parametrization [46]. However, in cosmological context, the generalized form of Chaplygin gas may be useful for describing the observed Universe. GCG has two free parameters \( A \) (positive), \( \alpha \). Recently, a further modification of GCG has been proposed in the framework of cosmology [33]. The modified Chaplygin gas (MCG) is more general and it contains one more free parameter \( B \). The model is consistent with (i) gravitational lensing test [47,48] and (ii) gamma-ray bursts [49]. The equation of state for the MCG is given by

\[ p = B\rho - \frac{A}{\rho^\alpha}, \]  

(20)

where \( A, B, \alpha \) are arbitrary constants to be determined from observation for model building with \( 0 \leq \alpha \leq 1 \).
The energy conservation equation for the MCG is
\[ \dot{\rho}_c + 3H(\rho_c + p_c) = 0, \tag{21} \]
where \( \rho_c \) and \( p_c \) correspond to energy density and pressure of MCG respectively. Using eq. (20) in eq. (21) we obtain
\[ \rho_c = \left[ \frac{A}{1 + B} + \frac{C}{a^{3n}} \right]^{1/(1+\alpha)}, \tag{22} \]
where \( C \) is an arbitrary constant and we denote \( (1 + B)(1 + \alpha) = n \). Equation (22) can be rewritten as
\[ \rho_c = \rho_0 \left[ A_S + \frac{1 - A_S}{a^{3n}} \right]^{1/(1+\alpha)}, \tag{23} \]
where \( A_S = A/(1 + B) \times (1/\rho_0^{\alpha+1}) \) with \( (a/a_0) = 1/(1 + z) \), \( z \) is the red-shift parameter and we choose \( a_0 = 1 \) for convenience. MCG reduces to GCG model when we set \( B = 0 \) in the above equation.

4. Observational constraints on EOS parameters

In general theory of relativity, cosmological models with MCG has been studied and the constraints on EOS parameters for viable cosmologies are determined using observational data [34,50]. The EOS parameters of MCG will be explored here for viable cosmologies in the framework of HL gravity using the recent observational data. For this, we have taken up data from observed Hubble data (OHD), BAO peak parameter and CMB shift parameter.

4.1 Constraints obtained from detailed balance

In this case, using eqs (7) and (8), the Friedmann equations can be rewritten as
\[ H^2 = \frac{8\pi G}{3} (\rho_b + \rho_c + \rho_r) + \left( \frac{K^2}{2\Lambda a^4} + \frac{\Lambda}{2} \right) - \frac{K}{a^2}, \tag{24} \]
\[ \dot{H} + \frac{3}{2} H^2 = -4\pi G \left( \rho_c + \frac{1}{3} \rho_r \right) - \left( \frac{K^2}{4\Lambda a^4} - \frac{3\Lambda}{4} \right) - \frac{K}{2a^2}. \tag{25} \]

Let us define the following dimensionless density parameters:
(i) For matter component
\[ \Omega_i \equiv \frac{8\pi G}{3H^2} \rho_i. \tag{26} \]
(ii) For curvature
\[ \Omega_K \equiv -\frac{K}{H^2a^2}. \tag{27} \]
(iii) For cosmological constant

\[ \Omega_0 \equiv \frac{\Lambda}{2H_0^2}. \]  

(28)

We define another dimensionless parameter for expansion rate as

\[ E(z) \equiv \frac{H(z)}{H_0}. \]  

(29)

Using the above definition of parameters, the Friedmann equation now can be rewritten as:

\[
E^2(z) = \Omega_{b0}(1+z)^3 + \Omega_{c0}F(z) + \Omega_{r0}(1+z)^4 \\
+ \Omega_{K0}(1+z)^2 + \left( \Omega_0 + \frac{\Omega_{K0}^2(1+z)^4}{4\Omega_0} \right),
\]  

(30)

where

\[ F(z) = \left[ A_S + \frac{1 - A_S}{a^{3(1+B)(1+\alpha)}} \right]^{1/(1+\alpha)}. \]

(31)

Let us assume \( E(z = 0) = 1 \) at the present epoch, which leads to

\[
\Omega_{b0} + \Omega_{c0} + \Omega_{r0} + \Omega_{K0} + \Omega_0 + \frac{\Omega_{K0}^2}{4\Omega_0} = 1,
\]  

(32)

where \( \Omega_{b0}, \Omega_{c0}, \Omega_{r0}, \Omega_{K0} \) represent the present day baryon, MCG, radiation and curvature energy density parameters respectively. Here \( \Omega_0 \) is the energy density associated with the cosmological constant. The last term in eq. (32) corresponds to dark radiation, which is a characteristic feature of the HL theory of gravity. The dark radiation component may be important during nucleosynthesis. Thus, a suitable bound from Big-Bang Nucleosynthesis (henceforth, BBN) may be incorporated in the above EOS. Using the upper limit on the total amount of HL dark radiation that is permitted during BBN era is expressed by the parameter \( \Delta N_\nu \), which represents the effective neutrino species [51,52]. \( \Delta N_\nu \) denotes the variation from the usual neutrino species and contributes to dark radiation. We obtain the following constraint equation [25]:

\[
\frac{\Omega_{K0}^2}{4\Omega_0} = 0.135\Delta N_\nu \Omega_{r0}.
\]  

(33)

The BBN upper limit on \( \Delta N_\nu \), \(-1.7 \leq \Delta N_\nu \leq 2.0 \), is taken from refs [52,53]. A negative value of \( \Delta N_\nu \) is usually associated with models involving the decay of massive particles which we do not consider here. \( \Delta N_\nu = 0 \) corresponds to zero curvature scenario which will not be considered here because the HL cosmology with zero curvature is indistinguishable from \( \Lambda \)CDM model. The curvature in dynamical dark energy models is important, and neglecting the curvature term imposes a serious problem [54,55]. Therefore, we consider the limiting values for \( \Delta N_\nu \) which satisfy the bound \( 0 < \Delta N_\nu \leq 2.0 \).

The numerical analysis taken up here contains nine parameters, namely, \( \Omega_{b0}, \Omega_{c0}, \Omega_{r0}, \Omega_{K0}, \Omega_0, \Delta N_\nu, H_0, A_S, B, \alpha \). As the number of unknowns are more than the number
of equations, we fix some of the parameters using the best-fit values from the seven-year WMAP data [56]. The fixed parameters are \( \Omega_{m0} (= \Omega_{b0} + \Omega_{c0}) \), \( \Omega_{b0} \), \( H_0 \), \( \Omega_{r0} \) and the corresponding values of the parameters are chosen as follows: \( \Omega_{m0} = 0.27 \), \( \Omega_{b0} = 0.04 \), \( H_0 = 71.4 \text{ km/s/Mpc} \), \( \Omega_{r0} = 8.14 \times 10^{-5} \). Therefore, one has now only six free parameters to be determined which are \( \Omega_{K0}, \Omega_0, A_S, B, \alpha, \Delta N_v \). Substituting eq. (33) in eq. (32) one obtains

\[
\Omega_0(K, \Delta N_v, A_S, \alpha) = 1 - \Omega_{m0} - (1 - 0.135 \Delta N_v) \Omega_{r0} \\
- 0.73(K) \sqrt{\Delta N_v \Omega_{r0} - \Omega_{m0} \Omega_{r0} - \Omega_{r0}^2},
\]

(34)

\[
\Omega_{K0}(\Delta N_v, A_S, \alpha) = \sqrt{0.54 \Delta N_v \Omega_{r0} \Omega_0(K, \Delta N_v, A_S, \alpha)},
\]

(35)

which may be employed for a close or in an open Universe depending on the values of \( K \). Now, it reduces to four free parameters, namely, \( A_S, B, \alpha, \Delta N_v \). To determine the effect of dark radiation on the constraints of the parameters of the MCG in detailed-balance scenario, we took two extreme values of \( \alpha (\alpha = 0.999, 0.001) \) satisfying \( 0 \leq \alpha \leq 1 \) for two extreme values of \( \Delta N_v \) (0.01, 2.0) in both close and open Universe respectively. In this case, each of these values of \( \alpha \) and \( \Delta N_v \) determined the best-fit values of the other two parameters (i.e., \( A_S, B \)). Thereafter, at the extreme values of \( \Delta N_v \) for two extreme values of \( \alpha \) we plot contours for the parameters \( A_S, B \) at different confidence levels. From the contours for the pair \((A_S, B)\) drawn at different values of \( \alpha \) and \( \Delta N_v \), we determine the admissible range of values of the \( B \)-parameter for the MCG in HL gravity in the framework of open or close Universe.

4.2 Constraints obtained from beyond detailed-balance

In beyond detailed-balance scenario using eqs (14) and (15), the Friedmann’s equations can be rewritten as

\[
H^2 = \frac{8\pi G}{3} (\rho_b + \rho_c + \rho_r) + \left[ \frac{\sigma_1}{6} + \frac{\sigma_3 K^2}{6a^4} + \frac{\sigma_4 K}{6a^6} \right] - \frac{K}{a^2},
\]

(36)

\[
\dot{H} + \frac{3}{2} H^2 = -4\pi G \left( p_c + \frac{1}{3} \rho_r \right) - \frac{3}{2} \left[ -\frac{\sigma_1}{6} + \frac{\sigma_3 K^2}{18a^4} + \frac{\sigma_4 K}{6a^6} \right] - \frac{K}{2a^2}.
\]

(37)

Now, the above equation can be rewritten in terms of dimensionless Hubble parameter as follows:

\[
E^2(z) = \Omega_{b0}(1 + z)^3 + \Omega_{c0}F(z) + \Omega_{r0}(1 + z)^4 + \Omega_{K0}(1 + z)^2 \\
+ [\Omega_1 + \Omega_3(1 + z)^4 + \Omega_4(1 + z)^6],
\]

(38)

where

\[
F(z) = \left[ A_S + \frac{1 - A_S}{a^{3(1+B)(1+\alpha)}} \right]^{1/(1+\alpha)}.
\]

(39)
The dimensionless parameters, namely, $\Omega_1$, $\Omega_3$, $\Omega_4$ are related to the model parameters $\sigma_1$, $\sigma_3$, $\sigma_4$ as follows:

\[ \Omega_1 = \frac{\sigma_1}{6H_0^2}, \quad (40) \]
\[ \Omega_3 = \frac{\sigma_3 H_0^2 \Omega_{K0}^2}{6}, \quad (41) \]
\[ \Omega_4 = -\frac{\sigma_4 \Omega_{K0}}{6}. \quad (42) \]

At the present epoch $E(z = 0) = 1$, which leads to

\[ \Omega_{b0} + \Omega_{c0} + \Omega_{K0} + \Omega_1 + \Omega_3 + \Omega_4 = 1. \quad (43) \]

In the above equations $\Omega_4$ is required to be a positive quantity so that the Hubble parameter and the gravitational perturbations \cite{13,38} are positive definite at all values of redshifts. For the conveneince of our analysis, $\Omega_3$ is also assumed to be positive definite.

The above constraint equation is used to replace $\Omega_1$ in terms of other parameters in our analysis. Following the procedure adopted in ref. \cite{25} for $\Delta N$, we consider the upper limit of dark radiation beyond Standard Model at the BBN. Consequently, the following constraints at the time of BBN emerged ($z = z_{BBN}$) \cite{51–53,57}:

\[ \Omega_3 + 2z_{BBN}^2 = \Omega_{3\text{max}} = 0.135\Delta N_\nu \Omega_{r0}, \quad (44) \]

where $\Omega_3$ represents the usual dark radiation and $\Omega_4$ represents a kinetic-like component (a quintessence field dominated by kinetic energy) \cite{58,59}. The above equation will be used to replace $\Omega_4$ in terms of other parameters in the analysis. For simplicity we define

\[ \beta = \frac{\Omega_3}{\Omega_{3\text{max}}}, \quad (45) \]

where $\Omega_{3\text{max}}$ is the upper limit on $\Omega_3$. Consequently, $\Omega_3$ can be expressed in terms of the other parameters.

Following the detailed-balance scenario we consider $\Delta N_\nu$, so that it satisfies the bound $0 < \Delta N_\nu \leq 2.0$, following the importance of curvature in dark energy models and treating $\Omega_{K0}$ as a free parameter \cite{54,55}.

To sum up, in the numerical analysis taken up here, there are thirteen parameters, namely, $\Omega_{b0}$, $\Omega_{c0}$, $\Omega_{r0}$, $\Omega_{K0}$, $\Omega_1$, $\Omega_3$, $\Omega_4$, $\Delta N_\nu$, $H_0$, $A_S$, $B$, $\alpha$, $\beta$. We fix some of the parameters using the best-fit values from seven-year WMAP data \cite{56}. The fixed parameters are $\Omega_{m0} = \Omega_{b0} + \Omega_{c0}$, $\Omega_{b0}$, $H_0$, $\Omega_{r0}$ and the corresponding values of the parameters are chosen as follows: $\Omega_{m0} = 0.27$, $\Omega_{b0} = 0.04$, $H_0 = 71.4$ km/s/Mpc, $\Omega_{r0} = 8.14 \times 10^{-5}$. Using the constraint eqs (36)–(42) one can replace $\Omega_1$, $\Omega_3$, $\Omega_4$ in terms of the other six free parameters for the numerical analysis. We now have six free parameters, $\Omega_{K0}$, $A_S$, $B$, $\alpha$, $\beta$, $\Delta N_\nu$.

To determine the constraints on the parameters of the MCG in beyond detailed-balance scenario, we consider three values of $\alpha$ satisfying $0 \leq \alpha \leq 1$ ($\alpha = 0.999, 0.500, 0.001$) and determine the best-fit values for the other five parameters (i.e., $A_S$, $B$, $\beta$, $\Omega_{K0}$, $\Delta N_\nu$). Thereafter, at the best-fit value of $\Delta N_\nu$, $\beta$, $\Omega_{K0}$ for three values of $\alpha$ we plot 2d contours.
for the pair of parameters \((A_S, B)\) at different confidence levels. The contours of \(A_S, B\) are drawn at different values of \(\alpha\), which determines the permissible range of values of the \(B\)-parameter for the MCG in HL gravity in the framework of beyond detailed-balance scenario.

To examine the effect of dark radiation (i.e., effective neutrino parameter) on the constraints on the parameters of the MCG, we took two extreme values of \(\alpha (\alpha = 0.999, 0.001)\) satisfying \(0 \leq \alpha \leq 1\) for two extreme values of \(\Delta N_\nu (0.01, 2.0)\). In this case, each value of \(\alpha, \Delta N_\nu\) determines the best-fit values of the other four parameters (i.e., \(A_S, B, \beta, \Omega_{K0}\)). Thereafter, at the extreme values of \(\Delta N_\nu\) for two extreme values of \(\alpha\) we plot 2d contours for the parameters \(A_S, B\) for the best-fit values of \(\beta, \Omega_{K0}\) at different confidence levels. From the contours of \(A_S, B\) drawn at different values of \(\alpha\) and \(\Delta N_\nu\), we determine the permissible range of values of the \(B\)-parameter for the MCG in HL gravity in the framework of beyond detailed-balance scenario. We note that the range of values of \(B\) is narrower due to the effect of effective neutrino parameter on \(B\).

5. Numerical analysis to determine constraints on the EOS parameters in detailed-balance scenario

In this section we use three sets of different observational data to constrain the parameters of the MCG. Stern dataset for \(H(z)-z\) data has been used along with BAO peak parameter and CMB shift parameter. Chi-square minimization technique has been used here to determine the limiting values of the EOS parameters in the next subsections.

5.1 \(H(z)-z\) data as a tool for constraining

The best-fit parameters of the model considered here can be obtained by minimizing the entity chi-square, which is defined as

\[
\chi^2_{\text{OHD}}(H_0, A_S, B, \alpha, \Delta N_\nu, z) = \sum \frac{(H(H_0, A_S, B, \alpha, \Delta N_\nu, z) - H_{\text{obs}}(z))^2}{\sigma_z^2},
\]

(46)

where \(H_{\text{obs}}(z)\) is the observed Hubble parameter at redshift \(z\) and \(\sigma_z^2\) is the associated error with that particular observation. The Hubble parameter can be rewritten as

\[
H(z) = H_0 E(z),
\]

(47)

where

\[
E(z) = \left( \Omega_{b0}(1+z)^3 + \Omega_{c0} F(z) + \Omega_{r0}(1+z)^4 + \Omega_{K0}(1+z)^2 + \left( \Omega_0 + \frac{\Omega_{K0}^2(1+z)^4}{4\Omega_0} \right)^{1/2} \right)
\]

(48)

with

\[
F(z) = \left[ A_S + \frac{1-A_S}{a^{3(1+B)(1+\alpha)}} \right]^{1/(1+\alpha)}.
\]

(49)
We now consider the set of data for \( H(z) - z \) from that given by Stern [60]. There are 12 data points of \( H(z) \) at different redshifts \( (z) \) which will be used to constrain the MCG model.

5.2 BAO peak parameter as a tool for constraining

A model-independent baryon acoustic oscillation (BAO) peak parameter can be defined for low redshift \((z_1)\) measurements as

\[
A = \frac{\sqrt{\Omega_m}}{[E(z_1)]^{1/3}} \left[ \int_0^{z_1} \frac{dz}{E(z)} \right]^{2/3},
\]

where \( \Omega_m \) is the matter density parameter for the Universe. For a detailed description of the above defined parameter and related approximations, the reader is referred to [61].

The chi square function can be defined as:

\[
\chi^2_{BAO} = \frac{(A - 0.469)^2}{(0.017)^2},
\]

where we have used the measured value for \( A (0.469 \pm 0.017) \) as obtained by [61] from the Sloan digital sky survey (SDSS) data for luminous red galaxies (LRG) survey.

5.3 CMB shift parameter as a tool for constraining

Here the CMB shift parameter is defined as

\[
R = \sqrt{\Omega_m} \int_0^{z_{ls}} \frac{dz}{E(z)},
\]

where \( z_{ls} \) is the \( z \) at last scattering. The WMAP7 data give us \( R = 1.726 \pm 0.018 \) at \( z = 1091.3 \) [56]. Chi square in this case is defined as

\[
\chi^2_{CMB} = \frac{(R - 1.726)^2}{(0.018)^2}.
\]

5.4 Joint analysis with \( H(z) - z + BAO + CMB \)

Total chi-square function for the joint analysis is

\[
\chi^2_{tot} = \chi^2_{OHD} + \chi^2_{BAO} + \chi^2_{CMB}.
\]

The statistical analysis with \( \chi^2_{tot} \) gives the bounds on the model parameter specially on \( B \).

The contours between \( B \) and \( A_S \) for closed Universe for \( \alpha = 0.999 \) and 0.001 are shown in figures 1 and 2 respectively. Figures 1a and 2a are drawn for \( \Delta N_v = 0.01 \) and figures 1b and 2b are drawn for \( \Delta N_v = 2.0 \). From figure 1a, which is plotted for \( \alpha = 0.999 \) and \( \Delta N_v = 0.01 \), for closed Universe it appears that the value of \( B \) lies in the range \(-0.05138 < B < 0.03351\), \(-0.08234 < B < 0.08444\), \(-0.1073 < B < 0.1274\) at 68.3\%, 95.4\%, 99.73\% confidence levels respectively. From figure 1b which is plotted for \( \alpha = 0.999 \) and \( \Delta N_v = 2.0 \) for closed Universe, the value of \( B \) lies in the range
Observational constraints on modified Chaplygin gas

Figure 1. Constraints for closed Universe for $\alpha = 0.999$ from OHD+SDSS+CMB shift data at 68.3% (solid line), 95.4% (dashed line) and 99.73% (dotted line) contours.

From figure 2a which is plotted for $\alpha = 0.001$ and $\Delta N\nu = 0.01$ for closed Universe, the value of $B$ lies in the range $-0.05042 < B < 0.03751, -0.09878 < B < 0.09906, -0.1274 < B < 0.1562$ at 68.3%, 95.4%, 99.73% confidence levels respectively. From
Figure 2. Constraints for closed Universe for $\alpha = 0.001$ from OHD+SDSS+CMB shift data at 68.3% (solid line), 95.4% (dashed line) and 99.73% (dotted line) contours.

The figure 2b which is plotted for $\alpha = 0.001$ and $\Delta N_\nu = 2.0$ for closed Universe, the value of $B$ lies in the range $-0.05702 < B < 0.0419$, $-0.09988 < B < 0.1002$, $-0.1296 < B < 0.1595$ at 68.3%, 95.4%, 99.73% confidence levels respectively. It is clear from the above analysis that the range of values of $B$ increases with an increase in the effective neutrino parameter $\Delta N_\nu$. Thus, for a given value of $\alpha$, the increase in neutrino parameter $\Delta N_\nu$ increases the range of $B$ parameter in the positive side.
Observational constraints on modified Chaplygin gas

The contours between $B$ and $A_S$ for open Universe for $\alpha = 0.999$ and $\alpha = 0.001$ are drawn in figures 3 and 4 respectively. Figures 3a and 4a are drawn for $\Delta N_v = 0.01$ and figures 3b and 4b are drawn for $\Delta N_v = 2.0$. It is evident from figure 3a which is plotted for $\alpha = 0.999$ and $\Delta N_v = 0.01$ for open Universe, that the parameter $B$ satisfies the following inequalities: $-0.05141 < B < 0.0178$, $-0.08437 < B < 0.07932$, $-0.1063 < B < 0.1266$ at 68.3%, 95.4%, 99.73% confidence level respectively. In figure 3b, the allowed range of values of parameter $B$ for $\alpha = 0.999$ and $\Delta N_v = 2.0$

![Figure 3](image-url)

**Figure 3.** Constraints for open Universe for $\alpha = 0.999$ from OHD+SDSS+CMB shift data at 68.3% (solid line), 95.4% (dashed line) and 99.73% (dotted line) contours.
for open Universe are obtained which are given by $-0.04921 < B < 0.02879$, $-0.08437 < B < 0.07713$, $-0.1074 < B < 0.120$ at 68.3%, 95.4%, 99.73% confidence levels respectively. It is evident that the domain of $B$ decreases with an increase in the effective neutrino parameter $\Delta N_\nu$.

Figure 4a is plotted for $\alpha = 0.001$ and $\Delta N_\nu = 0.01$ for open Universe. In this case, $B$ lies in the following ranges: $-0.05262 < B < 0.03971$, $-0.09768 < B < 0.09686$, $-0.1274 < B < 0.1496$ at 68.3%, 95.4%, 99.73% confidence levels respectively. From
Observational constraints on modified Chaplygin gas

figure 4b which is plotted for $\alpha = 0.001$ and $\Delta N_\nu = 2.0$ for open Universe, one can obtain viable cosmologies where $B$ satisfies the following inequalities: $-0.05812 < B < 0.03751$, $-0.09768 < B < 0.09576$, $-0.1274 < B < 0.1584$ at 68.3%, 95.4%, 99.73% confidence levels respectively. It is evident from the contours drawn in figure 4 that the range of $B$ increases with an increase in the effective neutrino parameter $\Delta N_\nu$. But the positive range of values of $B$ decreases.

6. Numerical analysis to determine constraints on the EOS parameters in beyond detailed-balance scenario

In this section, we determine constraints on the parameters of the MCG that were used in detailed-balance scenario of the previous section using the observed data. Stern dataset for $H(z)$-$z$ data, BAO peak parameter and CMB shift parameter are taken up here for the analysis. We use chi-square minimization technique to determine the limiting values for the EOS parameters in the next subsections.

6.1 $H(z)$-$z$ data as a tool for constraining

The best-fit parameters of the model considered here can be obtained by minimizing the entity chi-square which is defined as

$$\chi^2_{\text{OHD}}(H_0, \Omega_{K0}, A_S, B, \alpha, \beta, \Delta N_\nu, z) = \sum \frac{(H(H_0, \Omega_{K0}, A_S, B, \alpha, \beta, \Delta N_\nu, z) - H_{\text{obs}}(z))^2}{\sigma_z^2},$$

(55)

where $H_{\text{obs}}(z)$ is the observed Hubble parameter at redshift $z$ and $\sigma_z^2$ is the associated error with that particular observation. Hubble parameter is given by

$$H(z) = H_0 E(z),$$

(56)

where we denote

$$E^2(z) = \Omega_{b0}(1 + z)^3 + \Omega_{c0}F(z) + \Omega_{\zeta0}(1 + z)^4 + \Omega_{K0}(1 + z)^2 + [\Omega_1 + \Omega_2(1 + z)^4 + \Omega_4(1 + z)^6]$$

(57)

with

$$F(z) = \left[A_S + \frac{1 - A_S}{a^{3(1+B)(1+\alpha)}}\right]^{1/(1+\alpha)}.$$  

(58)

In this case $H(z)$-$z$ data is taken from Stern data analysis [60]. There are 12 data points of $H(z)$ at redshift $z$ which are used to constrain the MCG model.

6.2 BAO peak parameter as a tool for constraining

A model-independent baryon acoustic oscillation (BAO) peak parameter can be defined for low redshift $(z_1)$ measurements as:

$$A = \frac{\sqrt{\Omega_m}}{[E(z_1)]^{1/3}} \left[ \int_0^{z_1} \frac{dz(E(z))}{z_1} \right]^{2/3},$$

(59)

Pramana – J. Phys., Vol. 81, No. 4, October 2013
where $\Omega_m$ is the matter density parameter for the Universe. For a detailed description of the above defined parameter and related approximations, the reader is referred to [61]. The chi-square function can be defined as usual:

$$\chi^2_{\text{BAO}} = \frac{(A - 0.469)^2}{(0.017)^2},$$

(60)

where we have used the measured value for $A$ (0.469 ± 0.017) as obtained by [61] from the SDSS data for LRG survey.

6.3 CMB shift parameter as a tool for constraining

Here the CMB shift parameter is defined as

$$R = \sqrt{\Omega_m} \int_0^{z_{ls}} \frac{dz}{E(z)},$$

(61)

where $z_{ls}$ is the redshift at last scattering. The WMAP7 data give us $R = 1.726 ± 0.018$ at $z = 1091.3$ [56]. Chi square is defined as

$$\chi^2_{\text{CMB}} = \frac{(R - 1.726)^2}{(0.018)^2}.$$  

(62)

6.4 Joint analysis with $H(z)$+$BAO$+$CMB$

We define total chi-square function for our joint analysis as:

$$\chi^2_{\text{tot}} = \chi^2_{\text{OHD}} + \chi^2_{\text{BAO}} + \chi^2_{\text{CMB}}.$$  

(63)

The statistical analysis with $\chi^2_{\text{tot}}$ gives the bounds on the model parameter specially on $B$.

Figure 5a is plotted for $\alpha = 0.999$ for the best-fit values of $\beta$, $\Delta N_v$ and $\Omega_{K0}$. The parameter $B$ satisfies the following inequalities: $−0.05498 < B < 0.03808$, $−0.07989 < B < 0.07346$, $−0.1035 < B < 0.1128$ at 68.3%, 95.4%, 99.73% confidence levels respectively. Figure 5b is plotted for $\alpha = 0.500$ for the best-fit values of $\beta$, $\Delta N_v$ and $\Omega_{K0}$. The parameter $B$ in this case satisfies the following inequalities: $−0.05741 < B < 0.04421$, $−0.08349 < B < 0.08287$, $−0.1060 < B < 0.1305$ at 68.3%, 95.4%, 99.73% confidence levels respectively. Figure 5c is plotted for $\alpha = 0.001$ for best-fitted value of $\beta$, $\Delta N_v$ and $\Omega_{K0}$. We note that the parameter $B$ satisfies the following inequalities: $−0.09257 < B < 0.0707$, $−0.1326 < B < 0.1493$, $−0.1727 < B < 0.2247$ at 68.3%, 95.4%, 99.73% confidence levels respectively. It is evident that the allowed range of values of the parameter $B$ becomes larger compared to that of the detailed balance scenario [62].

Figure 6a is plotted for $\alpha = 0.999$ and $\Delta N_v = 0.01$ for the best-fit values of $\beta$ and $\Omega_{K0}$. It is evident that $B$ can take any value in the following ranges: $−0.05338 < B < 0.03351$, $−0.07935 < B < 0.07445$, $−0.1013 < B < 0.1224$ at 68.3%, 95.4%, 99.73% confidence levels respectively. Figure 6b is plotted for $\alpha = 0.999$ and $\Delta N_v = 2.0$, and it is evident that the value of $B$ lies in the range $−0.05462 < B < 0.03617$, $−0.07978 < B < 0.07662$, $−0.1032 < B < 0.1162$ at 68.3%, 95.4%, 99.73% confidence levels.
Observational constraints on modified Chaplygin gas

Figure 5. Constraints in beyond detailed-balance for $\alpha = 0.999, 0.500, 0.001$, from OHD+SDSS+CMB shift data at 68.3% (solid line), 95.4% (dashed line) and 99.73% (dotted line) contours.
B C Paul, P Thakur and M M Verma respectively. Figures 6a and 6b show that the range of permissible values of $B$ decreases with an increase in the effective neutrino parameter.

Figure 7a is plotted for $\alpha = 0.001$ and $\Delta N_\nu = 0.01$ for the best-fit values of $\beta$ and $\Omega_{K0}$. It is evident that the permissible values of $B$ now lie in the range $-0.06682 < B < 0.0407$, $-0.09434 < B < 0.08918$, $-0.1206 < B < 0.1416$ at 68.3%, 95.4%, 99.73% confidence levels respectively. Figure 7b is plotted for $\alpha = 0.001$ and $\Delta N_\nu = 2.0$, and it

![Figure 6](image_url)

**Figure 6.** Constraints in beyond detailed-balance for $\alpha = 0.999$ from OHD+SDSS+CMB shift data at 68.3% (solid line), 95.4% (dashed line) and 99.73% (dotted line) contours.
is evident that the values of $B$ lie in the range $-0.06347 < B < 0.04844$, $-0.09244 < B < 0.09241$, $-0.1194 < B < 0.1414$ at 68.3%, 95.4%, 99.73% confidence levels respectively. The contours drawn in figures 7a and 7b show that the range of permissible values of $B$ now decreases with an increase in the effective neutrino parameter. We note that the allowed range of values of the parameter $B$ decreased appreciably here compared to that obtained from figures 5a–5c. This signifies the fact that as the contribution of dark

![Figure 7](image)

**Figure 7.** Constraints in beyond detailed-balance for $\alpha = 0.001$ from OHD+ SDSS+CMB shift data at 68.3% (solid line), 95.4% (dashed line) and 99.73% (dotted line) contours.
radiation increases (through effective neutrino parameter) the range of admissible values of \( B \) decreases in beyond detailed-balance scenario which is the same as the one we obtain in detailed-balance scenario.

7. Viability of MCG in HL gravity

In this section we discuss some of the implications of the present scenario. Here we determine the evolution of the EOS parameter of the total cosmic fluid of the Universe which is defined as \( w(z) = p_{\text{tot}}/\rho_{\text{tot}} \), with the total pressure and energy density in detailed-balance scenario. The pressure and energy density are given by

\[
p_{\text{tot}} = p_c + \frac{1}{3} \rho_r + \frac{2}{k^2} \left[ \frac{K^2}{\Lambda a^4} - 3\Lambda \right],
\]

\[
\rho_{\text{tot}} = \rho_c + \rho_b + \rho_r + \frac{2}{k^2} \left[ \frac{3K^2}{\Lambda a^4} + 3\Lambda \right].
\]

In beyond detailed-balance scenario, the total pressure and the energy density are given respectively as

\[
p_{\text{tot}} = p_c + \frac{1}{3} \rho_r + \left[ -\frac{\sigma_1}{6\sigma_0} + \frac{\sigma_3K^2}{18\sigma_0a^4} + \frac{\sigma_4K}{6\sigma_0a^6} \right],
\]

\[
\rho_{\text{tot}} = \rho_c + \rho_b + \rho_r + \left[ \frac{\sigma_1}{6\sigma_0} + \frac{\sigma_3K^2}{6\sigma_0a^4} + \frac{\sigma_4K}{6\sigma_0a^6} \right].
\]

Here we replace the scale factor by redshift parameter, and the expressions for density parameter and the Hubble parameter are expressed in terms of the redshift parameter in the EOS. The EOS parameter can be expressed as a function of redshift parameter \( z \) which is given by

\[
w(z) = \frac{p_{\text{tot}}}{\rho_{\text{tot}}},
\]

From the plot of \( w(z) \) with \( z \), for beyond detailed-balance scenario we note that at high redshift (i.e., early times) it attains a fixed value \( \frac{1}{3} \) since radiation dominates in that epoch. In the intermediate redshift it behaves as dust for quite a long time. It is observed that the EOS parameter picks up negative values at small redshift, i.e., at very recent past. In the case of closed or open Universe the present value of the EOS parameter is found to be negative (\( -0.7 \)) which admits a late accelerating Universe.

In order to check the validity of the scenario we employed the best-fit values of the parameters of the MCG to find supernovae magnitudes \( \mu \) at different redshift \( z \) and plotted \( \mu \) vs. \( z \) curve. We compared these with original curves of Union2 data [63] and observed an excellent agreement.
8. Discussion

In this paper we present cosmologies with modified Chaplygin gas (MCG) in HL gravity scenario taking into account detailed-balance and beyond detailed-balance conditions both in the presence and absence of dark radiation. The equation of state of MCG has three unknown parameters. The permissible values of these parameters are explored from the observed data in the HL theory of gravity. Using data from different observations, namely, $H(z)-z$ (OHD), BAO peak and CMB shift parameter data, we determine the admissible values of the EOS parameters. For the MCG, the parameter $B$ represents the matter part. We analyse and determine the allowed range of values of $B$ for viable cosmologies. The analysis is carried out here both in open and closed Universe at 68.3%, 95.4%, 99.73% confidence levels.

In a close Universe we note that the permissible values of $B$ parameter lie in the range $-0.05702 < B < 0.0419, -0.09988 < B < 0.1002, -0.1296 < B < 0.1595$ at 68.3%, 95.4%, 99.73% confidence levels respectively for a maximum value of effective neutrino parameter. The range of $B$ obtained here is less than that obtained for best-fit value of effective neutrino parameter when dark radiation is not taken into account [62].

It is evident from figures 2a and 2b that the range of $B$ increases with an increase in the effective neutrino parameter $\Delta N_\nu$. In an open Universe, the parameter $B$ lies in the range $-0.05812 < B < 0.03751, -0.09768 < B < 0.09576, -0.1274 < B < 0.1584$ at 68.3%, 95.4%, 99.73% confidence levels respectively for maximum value of effective neutrino parameter. The domain of $B$ is found to be less than that obtained for best-fit value of effective neutrino parameter without dark radiation [62].

It appears from the analysis of a close and an open Universe that in a close Universe the domain of admissible values $B$ is comparatively narrower than that of an open Universe at 68.3%, 95.4%, 99.73% confidence levels respectively. We also note that $B$ may take negative values in this case. The negative value of $B$ implies the existence of exotic matter. In [64] the acceptable value of $B$ was predicted to be very small, which gets support from our analysis.

In the beyond detailed-balance scenario there are six free parameters, namely $\Omega_{K0}, A_s, B, \alpha, \beta, \Delta N_\nu$. It is found that the entire range of effective neutrino parameter is consistent with observations from our numerical analysis. The dependence of the extreme values of the neutrino parameter on other parameters are also shown in figures 6 and 7.

The contours drawn in figure 5 for beyond detailed-balance with different $\alpha$ for best-fit values of $\beta, \Delta N_\nu$ and $\Omega_{K0}$ project the admissible values of $B$ which lie in the range $-0.09257 < B < 0.0707, -0.1326 < B < 0.1493, -0.1727 < B < 0.2247$ at 68.3%, 95.4%, 99.73% confidence levels respectively. Thus, the range of $B$ in this case becomes larger than that of the detailed-balance scenario without dark radiation [62].

The contours drawn in figure 6a which are plotted for $\alpha = 0.999$ and $\Delta N_\nu = 0.01$ for best-fit values of $\beta$ and $\Omega_{K0}$ project the admissible values of $B$ which lie in the range $-0.05338 < B < 0.03351, -0.07935 < B < 0.07445, -0.1013 < B < 0.1224$ at 68.3%, 95.4%, 99.73% confidence levels respectively. The contours drawn in figure 6b which are plotted for $\alpha = 0.999$ and $\Delta N_\nu = 2.0$, permit the values of the parameter $B$ which lie in the range $-0.05462 < B < 0.03617, -0.07978 < B < 0.07662, -0.1032 < B < 0.1162$ at 68.3%, 95.4%, 99.73% confidence levels respectively. It is
clear that the range of values of $B$ decreases with an increase of the effective neutrino parameter.

The contours drawn in figure 7a which are plotted for $\alpha = 0.001$ and $\Delta N_\nu = 0.01$ for best-fit values of $\beta$ and $\Omega_{K0}$ give the allowed values of $B$ which lie in the range $-0.06682 < B < 0.0407$, $-0.09434 < B < 0.08918$, $-0.1206 < B < 0.1416$ at 68.3%, 95.4%, 99.73% confidence levels respectively. The contours drawn in figure 7b which are plotted for $\alpha = 0.001$ and $\Delta N_\nu = 2.0$ give the allowed values of $B$ which lie in the range $-0.06347 < B < 0.04844$, $-0.09244 < B < 0.09241$, $-0.1194 < B < 0.1414$ at 68.3%, 95.4%, 99.73% confidence levels respectively. It is clearly visible from the above analysis that the range of values of $B$ decreases with an increase in effective neutrino parameter.

We note that the range of values of $B$ decreases appreciably here compared to that obtained from figures 5a–5c. This signifies that as the contribution of dark radiation increases (through effective neutrino parameter), the contribution to the permissible range of values of $B$ decreases in beyond detailed-balance scenario like the one obtains in the case of detailed-balance scenario [62]. In figure 8 we plot the variation of the EOS parameter $w(z)$ with redshift parameter $z$ for beyond detailed-balance scenario. The curve shows the evolutionary phases of the Universe efficiently. It is evident that at high redshift (early times) the EOS parameter attains $1/3$, indicating radiation domination in that epoch. However, in the intermediate redshift we note that dust dominates through MCG for quite a long period of time. The best-fit values of $B$, $A_S$ and $\Delta N_\nu$ are shown in tables 1 and 2 for close Universe and in tables 3 and 4 for open Universe.

Using the best-fit values in beyond detailed-balance scenario $\mu$ vs. redshift curve is plotted in figure 9 and the figure is compared with union compilation data [63]. It is evident from the figure that cosmologies in HL gravity with MCG fits well with the experimental result.

![Figure 8](image_url)  
*Figure 8.* Equation of state parameter in beyond detailed-balance scenario.
Table 1. Best-fit values for MCG when $K = 1$.

| Model  | $B$   | $A_S$  | $\Delta N_v$ |
|--------|-------|--------|--------------|
| $\alpha = 0.999$ | 0.003745 | 0.062817 | 0.232994 |
| $\alpha = 0.500$  | 0.016592 | 0.110548 | 0.099996 |
| $\alpha = 0.001$  | 0.006192 | 0.052076 | 0.807051 |

Table 2. Best-fit values for MCG for extreme neutrino parameter when $K = 1$.

| Model  | $B$   | $A_S$  |
|--------|-------|--------|
| $\alpha = 0.999, \Delta N_v = 0.01$ | 0.00374357 | 0.0593243 |
| $\alpha = 0.999, \Delta N_v = 2.00$ | 0.00434955 | 0.0841938 |
| $\alpha = 0.001, \Delta N_v = 0.01$ | 0.00504535 | 0.0400150 |
| $\alpha = 0.001, \Delta N_v = 2.00$ | 0.00600186 | 0.0555941 |

Table 3. Best-fit values for MCG when $K = -1$.

| Model  | $B$   | $A_S$  | $\Delta N_v$ |
|--------|-------|--------|--------------|
| $\alpha = 0.999$ | 0.007498 | 0.107866 | 0.100055 |
| $\alpha = 0.500$  | 0.010499 | 0.110576 | 0.100002 |
| $\alpha = 0.001$  | 0.016478 | 0.114298 | 0.100005 |

Table 4. Best-fit values for MCG for extreme neutrino parameter when $K = -1$.

| Model  | $B$   | $A_S$  |
|--------|-------|--------|
| $\alpha = 0.999, \Delta N_v = 0.01$ | 0.00398596 | 0.0629336 |
| $\alpha = 0.999, \Delta N_v = 2.00$ | 0.00782675 | 0.133407 |
| $\alpha = 0.001, \Delta N_v = 0.01$ | 0.00528027 | 0.0418902 |
| $\alpha = 0.001, \Delta N_v = 2.00$ | 0.00937448 | 0.0817538 |

In this analysis we studied the dependance of extreme values of $\Delta N_v$ on other parameters in detailed-balance scenario with MCG. We also present here results obtained in the beyond detailed-balance scenario. In beyond detailed-balance scenario (BDB), there are two more free parameters and we found that the theory is rich and all the features of cosmologies can be accommodated with MCG. Earlier in the Einstein frame, MCG is employed to obtain viable cosmological models [34,50]. Here MCG is employed in the HL gravity and various physical parameters of the Universe are determined which get
support from observations. However, the present analysis does not enlighten the conceptual issues in HL gravity. It is important to look into details as to why the neutrino parameter becomes very small in HL gravity with MCG, which will be discussed elsewhere.

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Observational constraints on modified Chaplygin gas

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