Robust Bayesian variable selection with sub-harmonic priors

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Abstract: This paper studies Bayesian variable selection in linear models with general spherically symmetric error distributions. We propose sub-harmonic priors which arise as a class of mixtures of Zellner's g-priors for which the Bayes factors are independent of the underlying error distribution, as long as it is spherically symmetric. Because of this invariance to spherically symmetric error distribution, we refer to our method as a robust Bayesian variable selection method. We demonstrate that our Bayes factors have model selection consistency and are coherent. We also develop Laplace approximations to Bayes factors for a number of recently studied mixtures of g-priors that have appeared in the literature (including our own) for Gaussian errors. These approximations, in each case, are given by the Gaussian Bayes factor based on BIC times a simple rational function of the prior's hyper-parameters and the $R^2$'s for the respective models. We also extend model selection consistency for several g-prior based Bayes factor methods for Gaussian errors to the entire class of spherically symmetric error distributions. A simulation study and an analysis of two real data sets indicates good performance of our robust Bayes factors relative to BIC and to other mixture of g-prior based methods.

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1. Introduction

Suppose the linear regression model is used to relate $Y$ to the $p$ potential predictors $x_1, \ldots, x_p$,

$$y = \alpha 1_n + X_F \beta_F + \sigma_F \epsilon_F,$$

where the subscript $F$ refers to the full model $M_F$. In the model (1.1), $\alpha$ is an unknown intercept parameter, $1_n$ is an $n \times 1$ vector of ones, $X_F = (x_1, \ldots, x_p)$ is an $n \times p$ design matrix, and $\beta_F$ is a $p \times 1$ vector of unknown regression coefficients. In the error term of (1.1), $\sigma_F$ is an unknown scalar and $\epsilon_F$ has a spherically symmetric (SS) distribution with the density $f(\|\epsilon_F\|^2)$, $E[\epsilon_F] = 0_n$ and $\text{Var}[\epsilon_F] = I_n$. We assume that the columns of $X_F$ have been standardized so that for $1 \leq i \leq p$, $x_i'1_n = 0$ and $x_i'x_i/n = 1$. "This work was partially supported by KAKENHI #21740065 & #23740067.

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We shall be particularly interested in the variable selection problem where we would like to select an unknown subset of the effective predictors. It will be convenient throughout to index each of these $2^p$ possible subset choices by the vector
\[ \gamma = (\gamma_1, \ldots, \gamma_p)' \]
where $\gamma_i = 0$ or 1. We use $q_\gamma = \gamma' 1_p$ to denote the size of the $\gamma$th subset. The problem then becomes that of selecting a submodel of (1.1)
\[ y = \alpha_1 n + X\gamma \beta_\gamma + \sigma_\gamma \epsilon_\gamma. \tag{1.2} \]
In (1.2), $X_\gamma$ is the $n \times q_\gamma$ matrix whose columns correspond to the $\gamma$th subset of $x_1, \ldots, x_p$, $\beta_\gamma$ is a $q_\gamma \times 1$ vector of unknown regression coefficients. Let $\mathcal{M}_\gamma$ denote the submodel given by (1.2). We assume the error term $\epsilon_\gamma$ has the SS density
\[ \epsilon_\gamma \sim f(\|\epsilon_\gamma\|^2) \tag{1.3} \]
for all $\gamma$, with $E[\epsilon_\gamma] = 0_n$ and $\text{Var}[\epsilon_\gamma] = I_n$. Further $\sigma_\gamma$ is an unknown scalar in the error term. We note that, in most earlier studies, the error terms in linear models have been assumed to have a Gaussian distribution, e.g. as in George and Foster (2000) and Liang et al. (2008).

In this paper, we assume that $n > p + 1$ (the so called classical setup) and $\{x_1, \ldots, x_p\}$ are linearly independent, which implies that
\[ \text{rank } X_F = p, \quad \text{rank } X_\gamma = q_\gamma. \tag{1.4} \]
We also assume in much of the paper that the null model $\mathcal{M}_N$ ($q_\gamma = 0$ or $\gamma = (0, \ldots, 0)'$) is not a possible model, that is, the number of possible models is $2^p - 1$, rather than, $2^p$, although in Section 3, we indicate an alternative development which allows all $2^p$ possible models.

A Bayesian approach to this problem entails the specification of prior distributions on the models $\pi_\gamma = \Pr(\mathcal{M}_\gamma)$, and on the parameters $\alpha, \beta_\gamma, \sigma_\gamma$ of each model. For each such specification, of key interest is the posterior probability of $\mathcal{M}_\gamma$ given $y$,
\[ \Pr(\mathcal{M}_\gamma | y) = \frac{\pi_\gamma m_\gamma(y)}{\sum_\gamma \pi_\gamma m_\gamma(y)} = \frac{\pi_\gamma BF_{\gamma:F}}{\sum_\gamma \pi_\gamma BF_{\gamma:F}}, \tag{1.5} \]
where $\pi_N = \Pr(\mathcal{M}_N) = 0$ is assumed as mentioned in the above (although see Remark 3.3). In (1.5), $m_\gamma(y)$ is the marginal density under $\mathcal{M}_\gamma$ and $BF_{\gamma:F}$ is the Bayes factor for comparing each of $\mathcal{M}_\gamma$ to the full model $\mathcal{M}_F$ which is defined as
\[ BF_{\gamma:F} = \frac{m_\gamma(y)}{m_F(y)}, \]
where $m_F(y)$ is the marginal density under the full model. In Bayesian model selection,
\[ \arg\max_\gamma \Pr(\mathcal{M}_\gamma | y) = \arg\max_\gamma \pi_\gamma m_\gamma(y) = \arg\max_\gamma \pi_\gamma BF_{\gamma:F}, \tag{1.6} \]
is typically selected as the best model.

In this paper, the main focus is on $BF_{\gamma,F}$, not $\pi_\gamma$. Hence our main aim is to propose and study specifications for the prior distribution of the parameters for each submodel $M_\gamma$. In particular, the joint density we consider for $M_\gamma$ has the form

$$p_\gamma(\alpha_\gamma, \theta_\gamma, \sigma_\gamma | \nu) \propto \sigma_\gamma^{-\nu - 1} \| \theta_\gamma \|^{-q_\gamma + \nu}$$

for $0 < \nu < q_\gamma$. The term including $\theta_\gamma$ in the prior above, $\| \theta_\gamma \|^{-q_\gamma + \nu}$ for $0 < \nu < \min(2, q_\gamma)$, is known as a sub-harmonic function, that is,

$$\sum_{i=1}^{q_\gamma} \left\{ \frac{\partial^2 / \partial \theta_i^2}{\| \theta_\gamma \|^{-q_\gamma + \nu}} \right\} > 0,$$

we call the prior given by (1.7) a sub-harmonic prior. We will show that such priors lead to robust Bayes factors, in the sense that each Bayes factor does not depend on the form of the underlying error distribution. We also show that the resulting procedure has model selection consistency and is also coherent.

The organization of this paper is as follows. In Section 2, we give details of the prior distribution. In Section 3, we show that the Bayes factor with respect to the above prior is given by

$$BF_{\gamma,F}(\nu) = BF_{\gamma,F}^G(\nu)$$

where

$$BF_{\gamma,F}^G(\nu) = \frac{\int_0^\infty g^{\frac{\nu}{2} - 1} (1 + g)^{\frac{n - q_\gamma - 1}{2}} \{ g(1 - R^2_\gamma) + 1 \}^{-\frac{n - 1}{2}) dg}{\int_0^\infty g^{\frac{\nu}{2} - 1} (1 + g)^{\frac{n - q_\gamma - 1}{2}} \{ g(1 - R^2_F) + 1 \}^{-\frac{n - 1}{2}) dg},$$

for $0 < \nu < \min,q_\gamma = 1$. In (1.9), $BF_{\gamma,F}(\nu)$ is the Bayes factor for standard Gaussian errors and $R^2_\gamma$ and $R^2_F$ are the coefficient of determination under the submodel $M_\gamma$ and the full model $M_F$, respectively. From (1.8), the Bayes factor does not depend on the same SS sampling density. Hence, even when there is no specific information about the form of the error distribution of each model (other than spherical symmetry), it is not necessary to specify the exact form of the sampling density. It suffices to assume it is Gaussian. As far as we know, in the area of Bayesian variable selection with shrinkage priors or Zellner’s $g$-priors, the sampling density has been assumed to be Gaussian and this kind of robustness result has not yet been studied. Originally analogous robustness results were derived by Maruyama (2003), Maruyama and Strawderman (2005) and others in the problem of estimating regression coefficients with the Stein effect and it was this type of result that lead us to search for a similar phenomenon in the context of model selection. Note that we use the term “robustness” in this sense of distributional robustness over the class of SS error distributions. We specifically are not using the term to indicate a high breakdown point. The use of the term “robustness” in our sense is however common (if somewhat misleading) in the context of insensitivity to the error distribution in the context of shrinkage
literature. In Section 4, by use of the Laplace approximation, we approximate the Bayes factor given by (1.8) as \( BF_{\gamma;F}^G(\nu) \approx BF_{\gamma;F}^G[\text{BIC}] \)

\[
BF_{\gamma;F}^G(\nu) = \frac{\varphi(q_{\gamma} - \nu, 1 - R_{\gamma}^2)}{\varphi(p - \nu, 1 - R_{F}^2)} BF_{\gamma;F}^G[\text{BIC}]
\]

as \( n \to \infty \) where

\[ \varphi(s, r) = rs^{s-1} \{ (1/r - 1)e \}^{s} \]

and \( BF_{\gamma;F}^G[\text{BIC}] \) is the BIC based alternative for standard Gaussian errors

\[
BF_{\gamma;F}^G[\text{BIC}] = \left\{ \frac{(1 - R_{\gamma}^2)^{-n_{\eta}}}{(1 - R_{F}^2)^{-n_{\eta} - p}} \right\}^{1/2}
\]

(See, e.g. Hastie, Tibshirani and Friedman (2009), Chapter 7.) Since \( \varphi(s, r) \) does not depend on \( n \), (1.10) is asymptotically equivalent to BIC with a simple \( O(1) \) rational correction function depending upon \( \nu \) as well as the \( R \)-squares and the numbers of predictors. Actually this is a special case of Theorem 4.1 in which several Bayes factors under Gaussian errors which have been proposed in earlier studies, are shown to have similar asymptotic approximations. While the main theme in this paper is to develop the relationship (1.8) under sub-harmonic priors, we believe that this asymptotic equivalence is another noteworthy contribution, in particular, from the computational point of view. In Section 5, we show that our Bayes factor has model selection consistency uniformly over the class of SS error distributions, as \( n \to \infty \) and \( p \) is fixed. It also follows from these results that several model selection methods recently studied in the literature for Gaussian errors have model selection consistency for the entire class of SS error distributions. We provide illustrations of the method and comparisons with other methods using both simulated and real data in Section 6. We give concluding remarks in Section 7. The Appendix presents some of the more technical proofs.

2. Prior distributions

In this section, for each submodel, we give a prior joint density of a form

\[ p(\alpha, \beta, \eta_{\gamma}) = p(\alpha)p(\eta_{\gamma})p(\beta|\eta_{\gamma}), \]

where \( \eta_{\gamma} = 1/\sigma_{\gamma}^2 \). The natural choice of priors for location \((\alpha)\) and scale \((\eta_{\gamma})\) are

\[
p_{\alpha}^{I}(\alpha) = I_{(-\infty, \infty)}(\alpha), \tag{2.1}
\]

and

\[
p_{\eta}^{I}(\eta_{\gamma}) = \eta_{\gamma}^{-1}I_{(0, \infty)}(\eta_{\gamma}). \tag{2.2}
\]

In (2.1) and (2.2), the superscript \(^I\) means that the prior density is improper. Since (2.1) and (2.2) have invariance to location and scale transformation, respectively, they are considered by many as non-informative objective priors.
The problem is that they are improper and hence determined only up to an arbitrary multiplicative constant. In this paper, the use of improper priors is justified through sequences of proper priors approaching the target improper priors (2.1) and (2.2):

\[ p_\alpha(\alpha; h_\alpha) = \frac{1}{2h_\alpha} I_{(-h_\alpha, h_\alpha)}(\alpha) \]  

where \( h_\alpha \to \infty \) and

\[ p_\eta(\eta_\gamma; h_\eta) = \frac{1}{h_\eta} \frac{1}{\Gamma(q_\gamma/2)} \int_{-h_\eta}^{h_\eta} e^{\nu/2} I_{(1/h_\eta, h_\eta)}(\eta_\gamma) \eta_\gamma^{-1} I_{(1/h_\eta, h_\eta)}(\eta_\gamma) = \frac{\eta_\gamma^{-1} I_{(1/h_\eta, h_\eta)}(\eta_\gamma)}{2 \log h_\eta} \]  

where \( h_\eta \to \infty \). See the beginning of Section 3 for details of the justification.

Next we give a sequence of proper conditional priors on \( \beta_\gamma \) given \( \eta_\gamma \), which approach an improper conditional prior on \( \beta_\gamma \) given \( \eta_\gamma \):

\[ p_{\beta|\eta}(\beta_\gamma|\{\eta_\gamma, \nu\}; h_g) = \int_0^\infty p_g(g|\nu; h_g) \phi_q(\beta_\gamma|0, g\eta_\gamma^{-1}(X_\gamma^\prime X_\gamma)^{-1}) dg \]  

where \( \nu \) and \( h_g \) are non-random positive parameters. Further

\[ p_g(g|\nu; h_g) = \{\nu/2\}^{h_\nu/2} g^{h_\nu/2 - 1} I_{[0, h_g]}(g) \]  

and \( \phi_q(\cdot|\mu, \Sigma) \) denotes the \( q \)-variate Gaussian density with mean vector \( \mu \) and covariance matrix \( \Sigma \). The prior (2.5) clearly has a hierarchical structure and it can be interpreted as a scale mixture of Zellner’s \( g \)-priors. Similar priors have been considered by Liang et al. (2008) and Maruyama and George (2010) under the Gaussian linear regression setup. See Sub-Section 2.1 below for a review of priors on \( g \). For any fixed \( h_g > 0 \) and \( \nu > 0 \), the prior \( p_{\beta|\eta}(\beta_\gamma|\{\eta_\gamma, \nu\}; h_g) \) is clearly a proper probability density, that is,

\[ \int_{\mathbb{R}^{q_\gamma}} p_{\beta|\eta}(\beta_\gamma|\{\eta_\gamma, \nu\}; h_g) d\beta_\gamma = 1. \]

When \( h_g \to \infty \) and \( 0 < \nu < q_\gamma \), the limit of a variant of \( p_{\beta|\eta}(\beta_\gamma|\{\eta_\gamma, \nu\}; h_g) \) is given by

\[ p_{\beta|\eta}(\beta_\gamma|\{\eta_\gamma, \nu\}) = \lim_{h_g \to \infty} \left\{ h_g^{\nu/2} \right\} p_{\beta|\eta}(\beta_\gamma|\{\eta_\gamma, \nu\}; h_g) \]

\[ = \frac{\Gamma\left(\frac{q_\gamma - \nu}{2}\right) 2^{\nu/2} \pi^{q_\gamma/2}}{\nu \pi^{(q_\gamma-\nu)/2}} \Gamma\left(\frac{2}{q_\gamma-\nu}\right) \frac{\eta_\gamma^{-1}}{\nu^{q_\gamma/2}} \frac{1}{\nu} \exp \left( -\frac{\eta_\gamma^{-1} (\beta_\gamma^\prime X_\gamma^\prime X_\gamma \beta_\gamma) }{2\nu} \right) \]  

\[ \int_0^\infty g^{\nu/2 - 1} |X_\gamma^\prime X_\gamma|^{1/2} \eta_\gamma^{q_\gamma/2} \exp \left( -\frac{\eta_\gamma^{-1} (\beta_\gamma^\prime X_\gamma^\prime X_\gamma \beta_\gamma) }{2\nu} \right) dg \]

\[ = \frac{1}{\nu^{q_\gamma/2}} \frac{\eta_\gamma^{-1}}{\nu^{q_\gamma/2}} \frac{1}{\nu} \exp \left( -\frac{\eta_\gamma^{-1} (\beta_\gamma^\prime X_\gamma^\prime X_\gamma \beta_\gamma) }{2\nu} \right). \]
In summary, the proper prior joint density under $M$, which we will use in this paper, is given by

$$p_{\gamma}(\alpha, \beta, \eta; \nu; h) = p_{\alpha}(\alpha; h_\alpha)p_{\eta}(\eta; h_\eta)p_{\beta|\gamma}(\beta|\eta, \nu; h_\eta) \tag{2.8}$$

with $h = (h_\alpha, h_\eta, h_\beta)'$, which clearly satisfies

$$\int_{-\infty}^{\infty} \int_{\mathbb{R}^+} \int_{0}^{\infty} p_{\gamma}(\alpha, \beta, \eta; \nu; h) \, d\alpha \, d\beta \, d\eta = 1,$$

for any fixed $h$ and any $\nu > 0$. In Section 3, we will use the improper joint density of $\alpha$, $\beta$, and $\eta$, given by

$$p'_{\gamma}(\alpha, \beta, \eta; \nu) = \lim_{h \to \infty} V(h, \nu)p_{\gamma}(\alpha, \beta, \eta; \nu; h) \tag{2.9}$$

$$= \frac{\Gamma((q_\gamma - \nu)/2)}{2^{\nu/2} \pi^{
u/2}} |\mathbf{X}'_1 \mathbf{X}_1|^{1/2}(\beta_\gamma^2 \mathbf{X}'_1 \mathbf{X}_1 \gamma \beta_\gamma)^{-(q_\gamma - \nu)/2} \eta_{\gamma}^{\nu/2 - 1},$$

where $0 < \nu < q_\gamma$ and

$$V(h, \nu) = \{2h_\alpha\} \{(2 \log h_\eta) \{2 \nu/h_\gamma^{\nu/2}\} = (8/\nu)h_\alpha \{\log h_\eta\} h_\gamma^{\nu/2}. \tag{2.10}$$

Hence, in a certain sense, the larger $\nu$ is (as long as $q_\gamma - \nu > 0$), the more objective the joint prior $p'_{\gamma}(\alpha, \beta, \eta; \nu)$ is.

In this presentation of the improper joint density $p'_{\gamma}(\alpha, \beta, \eta; \nu)$, two facts, K1. $(\alpha, \beta)$ and $\eta$ are separable,

K2. the part depending on $\eta$, is given by the power function $\eta_{\gamma}^{\nu/2 - 1}$,

will be the key for calculating the marginal density in the next section. See also Remark 3.1.

If, in the above joint prior on $(\alpha, \beta, \eta)$, we make the change of variables, $\theta_\gamma = (\mathbf{X}'_1 \mathbf{X}_1)^{1/2} \beta_\gamma$, the joint prior of $(\alpha, \theta_\gamma, \eta)$ becomes

$$p'_{\gamma}(\alpha, \theta_\gamma, \eta; \nu) = \frac{\Gamma((q_\gamma - \nu)/2)}{2^{\nu/2} \pi^{
u/2}} ||\theta_\gamma||^{-(q_\gamma - \nu)/2} \eta_{\gamma}^{\nu/2 - 1}. \tag{2.11}$$

As noted in Section 1, the part depending on $\theta_\gamma$, $||\theta_\gamma||^{-(q_\gamma - \nu)}$ for $0 < \nu < \min(2, q_\gamma)$, is known as a sub-harmonic function, that is,

$$\sum_{i=1}^{q_\gamma} \frac{\partial^2}{\partial \theta_i^2} ||\theta_\gamma||^{-(q_\gamma - \nu)} = (q_\gamma - \nu)(2 - \nu) ||\theta_\gamma||^{-(q_\gamma - \nu)^2 - 2} > 0.$$

### 2.1. Review of priors on $g$

As noted above, the prior given by (2.5) is a scale mixture of Zellner’s $g$-priors. Actually the original Zellner’s $g$-priors were used for the Gaussian linear regression setup and historically the hyperparameter $g$ has been a priori fixed or
somewhat estimated. The first paper to effectively use a prior on $g$ was Zellner and Siow (1980); they stated things in terms of multivariate Cauchy densities, which can always be expressed as a mixture of $g$-priors. Here we review the prior on $g$, the second stage of $g$-priors for Gaussian linear regression. We hope that it makes the positioning of our prior on $g$, (2.6), which is for SS errors, clearer.

As a generalization of $p_g(g|\nu; h_g)$ given by (2.6), consider the prior on $g$,

$$p_g(g|\nu, k; h_g) = \frac{g^{\nu/2-1}(1 + g^{-1})^{-k/2}I_{[0, h_g]}(g)}{V_g(\nu, k; h_g)}, \quad (2.12)$$

(so $p_g(g|\nu, 0; h_g) = p_g(g|\nu; h_g)$). In (2.12), $k \geq 0$, $\nu > -k$, $h_g > 0$ and

$$V_g(\nu, k; h_g) = \int_0^{h_g} g^{\nu/2-1}(1 + g^{-1})^{-k/2} dg.$$

Note that it is improper at $g = 0$ when $\nu \leq -k$. As we will see in Section 3 and 4, Bayes factors are not well-defined when the prior on $g$ is improper at $g = 0$ and that is why $\nu > -k$ is assumed.

When $k > 0$ and $-k < \nu < 0$, without truncating by $h_g$ the prior of $g$ is proper under the support $g \in (0, \infty)$. In this case, the normalizing constant becomes

$$V_g(\nu, k; \infty) = B(-\nu/2, \nu + k/2).$$

Further even when $0 \leq \nu < q_\gamma$ (non-negative $\nu$ implies impropriety at $g = \infty$), the Bayes factor under $M_\gamma$ is well-defined as shown in Section 3.

As in (2.5) and (2.7), we are considering the (improper) case where $0 < \nu < q_\gamma$ and $k = 0$. Liang et al. (2008) considered the (proper) case where $-2 < \nu < 0$ and $k = 2 - \nu$, Guo and Speckman (2009) and Celeux et al. (2011) considered the (improper) case where $\nu = 0$ and $k = 2$.

Remark 2.1. The noteworthy difference between $k > 0$ and $k = 0$ is that the (proper or improper) prior on $\beta_\gamma$ given $\eta_\gamma$,

$$p_{\beta_\gamma|\eta}(\beta_\gamma|\eta_\gamma, \{\nu, k\})
\overset{\text{def}}{=} \lim_{h_g \to \infty} V_g(\nu, k; h_g) \int_0^{h_g} p_g(g|\nu, k; h_g) \phi_{q_\gamma}(\beta_\gamma|0, g^{-1} \eta_\gamma^{-1} (X'_\gamma X_\gamma)^{-1}) dg
= \int_0^\infty \frac{g^{\nu/2-1} \eta_\gamma^{q_{\gamma}/2} |X'_\gamma X_\gamma|^{1/2}}{(1 + g^{-1})^{k/2} (2\pi g)^{q_{\gamma}/2}} \exp \left( -\frac{\eta_\gamma}{2g} \beta_\gamma' X_\gamma^{-1} X'_\gamma \beta_\gamma \right) dg
\quad (2.13)$$

is not separable for any $k > 0$. As will be seen in Section 3, the separability of the prior is the key for calculating the marginal density for SS error models other than Gaussian.

Remark 2.2. Here we discuss objectivity (or at least non-subjectivity) of the prior in terms of hyper-parameters of the prior on $g$. Consider the (proper or improper) prior on $\beta_\gamma$ given $\eta_\gamma$, which is given by (2.13). In order to obtain the asymptotic behavior of the density as $\eta_\gamma^{1/2} \|\beta_\gamma\| \to \infty$, we appeal to the
Tauberian theorem for the Laplace transform (see Geluk and de Haan (1987)). Since \((1 + g^{-1})^{k/2} \rightarrow 1\) for any \(k\) as \(g \rightarrow \infty\),

\[
\lim_{\nu_{1/2} \| \beta_2 \| \rightarrow \infty} \frac{p_{\beta|\eta}(\beta_2|\eta_1, \nu, k)}{\eta_{1/2}^{1}B_{1/2}(\gamma, \nu, \nu, \nu, k)} = \frac{\Gamma(\{q \gamma - \nu\}/2)}{2^{\nu/2}} |X_0^T X_0^{\gamma}|^{1/2} \tag{2.14}
\]

when \(\nu < q \gamma\). Hence the asymptotic order of (2.14) is the same as (2.7). The larger \(\nu(< q \gamma)\) is, the more objective the prior \(p_{\beta|\eta}(\beta_2|\eta_1, \nu, k)\) is.

3. Marginal density and Bayes factor under sub-harmonic priors

In this section we derive the marginal density under each submodel and the Bayes factor for comparing each \(M_\gamma\) to the full model \(M_F\). The marginal density of \(y\) under \(M_\gamma\), is given by

\[
m_\gamma(y|\nu; h) = \int_{-\infty}^{\infty} \int_{\nu^2}^{\infty} \int_{0}^{\infty} \eta_{\gamma}^{n/2} f(\eta_\gamma) \eta_{\gamma}^{1/2} \|y - \alpha 1_n - X_0 \beta_2 \|^2 \times p_\gamma(\alpha, \beta_2, \eta_\gamma|\nu; h) \, d\alpha \, d\beta_2 \, d\eta_\gamma,
\]

(3.1)

where the proper joint prior \(p_\gamma(\alpha, \beta_2, \eta_\gamma|\nu; h)\) is given by (2.8). In (3.1), \(h\) does not depend on the submodel, but is the same in all models. In the following, instead of \(m_\gamma(y|\nu; h)\) directly, we consider the limit of the product of \(m_\gamma(y|\nu; h)\) and \(V(h, \nu)\),

\[
M_\gamma(y|\nu) = \lim_{h \rightarrow \infty} V(h, \nu) m_\gamma(y|\nu; h)
\]

(3.2)

\[
= \int_{-\infty}^{\infty} \int_{\nu^2}^{\infty} \int_{0}^{\infty} \eta_{\gamma}^{n/2} f(\eta_\gamma) \eta_{\gamma}^{1/2} \|y - \alpha 1_n - X_0 \beta_2 \|^2 \times p_\gamma^m(\alpha, \beta_2, \eta_\gamma|\nu) \, d\alpha \, d\beta_2 \, d\eta_\gamma,
\]

which is the marginal density of \(y\) with respect to the improper joint prior \(p_\gamma^m(\alpha, \beta_2, \eta_\gamma|\nu)\) given by (2.9):

\[
p_\gamma^m(\alpha, \beta_2, \eta_\gamma|\nu) = \frac{\Gamma(\{q \gamma - \nu\}/2)}{2^{\nu/2}} |X_0^T X_0^{\gamma}|^{1/2} \eta_{\gamma}^{n/2} \frac{\eta_{\gamma}^{1/2} |X_0^T X_0^{\gamma}|^{1/2}}{\eta_{\gamma}^{1/2} |X_0^T X_0^{\gamma}|^{1/2}} \tag{3.3}
\]

The second equality in (3.2) follows from the monotone convergence theorem. Since \(V(h, \nu)\) does not depend on \(M_\gamma\) if we choose the same \(\nu\) in all submodels, the Bayes factor \(m_\gamma(y|\nu; h)/m_F(y|\nu; h)\) approaches \(M_\gamma(y|\nu)/M_F(y|\nu)\) as \(h \rightarrow \infty\). Hence the use of the improper joint prior is justified as long as \(M_\gamma(y|\nu)/M_F(y|\nu)\) is well-defined. As remarked in Section 1, the null-model is not a possible model. Since there is no \(\beta\) and hence no \(h_\gamma\), \(M_N(y|\nu)/M_F(y|\nu)\) is not well-defined. And thus \(\pi_N = \text{Pr}(M_N) = 0\) is assumed. However, see Remark 3.3 for an alternative development which allows \(M_N\) as a possible model.

Let \(M_\gamma^G(y|\nu)\) be the marginal density under \(M_\gamma\) with standard Gaussian errors \(e_G\). Before proceeding with the entire calculation of the marginal density, \(M_\gamma(y|\nu)\), when \(e_\gamma\) has a general SS distribution, we will provide a relationship between \(M_\gamma(y|\nu)\) and \(M_\gamma^G(y|\nu)\) as follows.
Lemma 3.1. Let $\nu$ be between 0 and $q_{\gamma}$. Assume the existence of $E[\|\epsilon_\gamma\|^\nu]$. Then

$$M_\gamma(y|\nu) = \frac{E[\|\epsilon_\gamma\|^\nu]}{E[\|\epsilon_G\|^\nu]} M^G_\gamma(y|\nu).$$

(3.4)

\textbf{Proof.} See Appendix.

Hence $M_\gamma(y|\nu)$ depends on the error distribution $\epsilon_\gamma$ only through the $\nu$-th moment of $\epsilon_\gamma$, $E[\|\epsilon_\gamma\|^\nu]$.

Remark 3.1. This relationship of Lemma 3.1 remains true under more general separable priors $\eta_{\nu/2-1}^{\nu/2-1} \pi(\alpha, \beta_\gamma)$, which satisfies K1 and K2. Unfortunately we could not find any other priors, which lead to analytically tractable Bayes factors under Gaussian errors, except for our sub-harmonic priors $\eta_\nu^{\nu}(\alpha, \beta_\gamma, \eta_\nu|\nu)$ given by (2.9). It follows that for any such separable prior, the distributional robustness of the Bayesian variable selection procedure we develop for our sub-harmonic prior will carry over. That is, whether MCMC or another computational method is used, it suffices to assume that the error distribution is Gaussian (with the same error variance) for all submodels, but the procedure will be simultaneously valid for all SS error distributions. Our prior has the advantage of analytic tractability, and as we show below, good performance.

We will make use of the following result which may be founded in Liang et al. (2008).

Lemma 3.2. Let $0 < \nu < q_{\gamma}$. Then

$$M^G_\gamma(y|\nu) = \frac{n^{1/2} \Gamma((n-1)/2)}{\|y - \bar{y}1_n\|^{n-1} \pi(n-1)/2} \int_0^\infty \frac{g^{\nu/2-1}(1+g)^{(n-q_{\gamma}-1)/2}}{(g(1-R^2_F) + 1)^{(n-1)/2}} dg,$$

(3.5)

where $R^2_F$ is the coefficient of determination under the submodel $M_\gamma$.

See equation (5) of Liang et al. (2008). Combining Lemmas 3.1 and 3.2, we have the main result of this paper.

Theorem 3.1. Assume the full model $M_F$ and the submodel $M_\gamma$ are given by (1.1) and (1.2), respectively. Also assume their error terms, $\epsilon_F$ and $\epsilon_\gamma$, have the same SS distribution (1.3) with mean zero and the identity covariance matrix. Let $0 < \nu < q_{\gamma}$. Assume that the proper joint prior densities of $(\alpha, \beta_\gamma, \eta_\gamma)$ and $(\alpha, \beta_\gamma, \eta_\mu)$ are given by (2.8) and assume also $E[\|\epsilon_\gamma\|^\nu] < \infty$. Then, for $M_\gamma \neq M_N$, the limit of the Bayes factor for comparing each of $M_\gamma$ to the full model $M_F$ is given by

$$BF_{\gamma:F}(\nu) = \lim_{h \to \infty} \frac{m_\gamma(y|\nu; h)}{m_F(y|\nu; h)} = BF^G_{\gamma:F}(\nu)$$

(3.6)

where

$$BF^G_{\gamma:F}(\nu) = \frac{\int_0^\infty g^{\nu/2-1}(1+g)^{(n-q_{\gamma}-1)/2}}{\int_0^\infty g^{\nu/2-1}(1+g)^{(n-q_{\gamma}-1)/2}} dg.$$
**Remark 3.2.** Maruyama and George (2010) considered Bayesian variable selection under Gaussian errors. They proposed Bayes factors with a simple analytic form under generalized ridge-type priors. The results heavily depend on special features of Gaussian distributions and hence the extension or generalization of Maruyama and George (2010) to the general SS case, may not be possible, or may not lead to analytically tractable procedures which are distributionally robust to SS error distributions.

**Remark 3.3.** Expression (3.7) again shows why the null model is not allowed as a possibility. For the null model $\mathcal{M}_N$, $\hat{R}_N^2 = 0$ so the numerator of (3.7) is infinite, and hence so would be $BF^G_{\gamma,F}(\nu)$. This situation may be avoided at a slight cost in complexity and in interpretability of the expressions. The required alteration in the prior distributions (proper and improper) is to treat the intercept parameter $\alpha$ as another $\beta$, (and not give it a “uniform” prior).

This results in replacing the improper prior in (2.9) by

$$p_1^G(\beta, \eta_\gamma | \nu) = \frac{\Gamma\left(\frac{n+1}{2} - \nu/2\right)}{\pi^{n/2} \Gamma(n/2)} |\mathbf{X}_\gamma^\prime \mathbf{X}_\gamma|^{1/2} (\beta^\prime \mathbf{X}_\gamma^\prime \mathbf{X}_\gamma \beta)_{\gamma}^{-\frac{n+\nu}{2}} \eta_\gamma^{-1},$$

where $\hat{\beta} = (\alpha, \beta^\prime, \eta_\gamma)$ and $\mathbf{X}_\gamma = (1_n | \mathbf{X}_\gamma)$. Similarly the marginal distribution in (3.5) and the Bayes factor given by (3.7) are replaced by

$$M_{\gamma}^G(y|\nu) = \frac{\Gamma(n/2)}{\|y\|^n \pi^{n/2}} \int_0^\infty g^{n}\left(1 + g\right)^{n-q_\gamma - 1/2} \left(1 - \hat{R}_\gamma^2\right)^{n/2} - dg,$$

and

$$BF_{\gamma,F}^G(\nu) = \frac{\int_0^\infty g^{n} \left(1 + g\right)^{n-q_\gamma - 1/2} \left(1 - \hat{R}_\gamma^2\right)^{n/2} - dg}{\int_0^\infty g^{n} \left(1 + g\right)^{n-q_\gamma - 1/2} \left(1 - \hat{R}_F^2\right)^{n/2} - dg},$$

(3.8)

where $\hat{R}_\gamma^2 = 1 - \text{RSS}_\gamma / \|y\|^2$, (the “coefficient of determination” of the model $\mathcal{M}_\gamma$, relative to the 0-intercept model). Hence with the substitution $R_\gamma^2 \rightarrow \hat{R}_\gamma^2$, $n - 1 \rightarrow n$, $q_\gamma \rightarrow q_\gamma + 1$, $y \rightarrow \hat{y} 1_n \rightarrow y$, all expressions and results in the paper remains valid and the result (Corollary 5.1) on model selection consistency in Section 5 holds also for the null model. Clearly $\hat{R}_\gamma^2$ is somewhat unusual, but if model selection consistency under the null-model is desirable, we can use $BF_{\gamma,F}^G(\nu)$. Actually, under the Gaussian regression setup, Guo and Speckman (2009) and Celeux et al. (2011) recommend use of the Bayes factor as a function of $\hat{R}_\gamma^2$, which just substitutes $g^{n}/2 - 1$ with $1/(1 + g)$ in $BF_{\gamma,F}^G(\nu)$ given by (3.8).

**Remark 3.4.** A collection of Bayes factors is called coherent if

$$BF_{\gamma_1:\gamma_2} = BF_{\gamma_1:0} BF_{\gamma_0:\gamma_2},$$

and $BF_{\gamma_1:\gamma_2} = 1/BF_{\gamma_2:\gamma_1}$, for all $\gamma_1$ and $\gamma_2$ (see, e.g. Robert (2007)). By (3.6), the Bayes factors corresponding to our sub-harmonic priors are coherent (with the exception of those involving the null model $\mathcal{M}_N$), which is why we require $\pi_N = 0$. Also with the adaptation of the alternative specification given in Remark 3.3, coherence holds for all Bayes factors including those involving the null model $\mathcal{M}_N$. 
As in Zellner and Siow (1980) and Liang et al. (2008), the posterior probability of any model is an expression of the form (1.5) given by

$$\Pr(M_\gamma|y) = \frac{\pi_\gamma BF_{\gamma|F}}{\sum_{\gamma} \pi_\gamma BF_{\gamma|F}}.$$  

In our development, we choose the full model $M_F$ as the base model rather than the null model $M_N$ employing the encompassing approach of Zellner and Siow (1980). Without employing the adaptation of our prior described in Remark 3.3, the choice of the full model $M_F$ as the base model, as opposed to the null model $M_N$, is forced on us since $\pi_N = 0$. Liang et al. (2008) argue that the null model is the superior choice as a base model in their setup (which also involves $g$-priors or mixture thereof) due to incoherence of the Bayes factors if $M_F$ is chosen as the base model (in their setup). This incoherence arises in the setup of Liang et al. (2008) because prior distribution on the full model $M_F$ depends on each nested alternative $M_\gamma$. This incoherence is not a problem in our setup since the choice of prior for each submodel depends only on the submodel, and we have taken care that all relevant (conditional) posteriors are well defined. As noted above, by (3.6) our Bayes factors are coherent. In fact our development (aside from eliminating the null model from the consideration) is very close in spirit to the null-based Bayes factors approach in Liang et al. (2008).

Remark 3.5. By Theorem 3.1, even when there is no specific information about the error distribution of each model (other than spherical symmetry), but we assume they are all the same, it is not necessary to specify the exact form of the sampling density. It suffices to assume they are all Gaussian. As far as we know, in the area of Bayesian variable selection with shrinkage priors, the sampling density has been assumed to be Gaussian and this kind of robustness result has not yet been studied. Analogous robustness results have been derived by Cellier, Fourdrinier and Robert (1989), Maruyama (2003), Maruyama and Strawderman (2005) and others in the problem of estimating regression coefficients with the Stein effect.

### 3.1. BIC under spherically symmetric error distributions

BIC (Schwarz (1978)) is a popular criterion for model selection. See e.g. Hastie, Tibshirani and Friedman (2009) Chapter 7. We will show in this subsection that BIC has a similar distributional robustness property to the above Bayes model selection procedure. In Section 4, we will develop Laplace approximations to our Bayes factors which relate them to BIC. In Section 5 we will show that both the BIC and our Bayes model selection procedures are consistent for the entire class of SS models.

BIC for the model $M_\gamma$ is defined as

$$[\text{BIC}]_\gamma = -2\ln \left\{ \max_{\alpha, \beta, \eta} \frac{\eta^{n/2} f(\eta \| y - \alpha 1_n - X_\gamma \beta_\gamma \| ^2) n^{-q/2}}{\pi_\gamma} \right\},$$  

(3.9)
and is derived by eliminating $O(1)$ terms from the approximate marginal densities. Here we denote

$$M_\gamma(y|\text{BIC}) = \exp(-[\text{BIC}]\gamma/2).$$

(3.10)

In general, the maximization with respect to unknown parameters in (3.9) is not always tractable. However when $\epsilon_\gamma$ has a unimodal SS distribution, the maximization is achieved by $\hat{\alpha} = \bar{y}$, $\hat{\beta}_\gamma = (X'_\gamma X_\gamma)^{-1} X'_\gamma y$, and

$$1/\hat{\eta}_\gamma = c\|y - \hat{\alpha}1_n - X_\gamma \hat{\beta}_\gamma\|^2 = c\|y - \bar{y}1_n\|^2(1 - R_\gamma^2)$$

(3.11)

where $c$ is the sole solution of

$$n/2 + cf'(c)/f(c) = 0.$$  

(3.12)

Hence $M_\gamma(y|\text{BIC})$ may be expressed as

$$M_\gamma(y|\text{BIC}) = \frac{c^{-n/2}f(c)}{c^{-n/2}f_G(c)} M_G^\gamma(y|\text{BIC})$$

(3.13)

where $M_G^\gamma(y|\text{BIC})$ is $M_\gamma(y|\text{BIC})$ with Gaussian errors, specifically

$$M_G^\gamma(y|\text{BIC}) = c_G^{-n/2}f_G(c_G)\{\|y - \bar{y}1_n\|^2(1 - R_\gamma^2)\}^{-n/2}n^{-q_\gamma/2}$$

$$= n^{-n/2}f_G(n)\{\|y - \bar{y}1_n\|^2(1 - R_\gamma^2)\}^{-n/2}n^{-q_\gamma/2}$$

(3.14)

(since $c_G$ is given by $n$). Clearly (3.13) and (3.14) correspond to (3.4) and (3.5), respectively. Hence we have the following result.

**Theorem 3.2.** Assume the full model $M_F$ and the submodel $M_\gamma$ are given by (1.1) and (1.2), respectively. Also assume their error terms, $\epsilon_F$ and $\epsilon_\gamma$ have a unimodal SS distribution (1.3) with the mean zero and the identity covariance matrix. Then the Bayes factor based on BIC for comparing each of $M_\gamma$ to the full model $M_F$ is given by

$$BF_{\gamma:F}[\text{BIC}] = \frac{M_\gamma(y|\text{BIC})}{M_F(y|\text{BIC})} = BF_G^\gamma_{\gamma:F}[\text{BIC}]$$

(3.15)

where $BF_G^\gamma_{\gamma:F}[\text{BIC}]$ is the BIC based Bayes factor under Gaussian errors,

$$BF_G^\gamma_{\gamma:F}[\text{BIC}] = \left\{\left(1 - R_\gamma^2\right)^{-n/2}n^{-q_\gamma/2}\right\}^{1/2}.$$  

(3.16)

Obviously (3.16) corresponds to (3.7). By Theorem 3.2, the Bayes factor based on BIC is also independent of the error distribution provided each distribution is unimodal and is the same for all models (c.f. Theorem 3.1). Note that $BF_G^\gamma_{\gamma:F}[\text{BIC}]$ is well defined if $M_\gamma = M_N$. 
Remark 3.6. In an earlier version of this paper we developed the results in the more general context wherein the SS distribution of $\epsilon_\gamma \sim f_\gamma(\|\epsilon_\gamma\|^2)$ could depend on $\gamma$, i.e. it could be different for each submodel $M_\gamma$. So $\epsilon_F \sim f_F(\|\epsilon_F\|^2)$ as well. All of the above results can be developed for the more general case. The only essential changes are that (3.6) in Theorem 3.1 becomes

$$BF_{\gamma:F}(\nu) = \frac{E[\|\epsilon_\gamma\|\|\nu\]}{E[\|\epsilon_F\|\|\nu\]} BF_{\gamma:F}^{\nu}(\nu)$$  \hspace{1cm} (3.17)$$

and that (3.15) in Theorem 3.2 becomes

$$BF_{\gamma:F}[\text{BIC}] = \frac{c_\gamma^{-n/2} f_\gamma(c_\gamma)}{c_F^{-n/2} f_F(c_F)} BF_{\gamma:F}^{\nu}[\text{BIC}]$$  \hspace{1cm} (3.18)$$

where $c_\gamma$ and $c_F$ are the corresponding solution of (3.12) respectively. We investigated the ranges of these “correction terms” in (3.17) and (3.18) respectively when $\epsilon_\gamma$ and $\epsilon_F$ have possibly different SS $t$-distributions with at least 3-degrees of freedom (so that the variances exist), both analytically and numerically. We found that $BF_{\gamma:F}(\nu)$ was independent of $n$ and reasonably stable for all $\nu$ in range $(0, 1)$ but that stability was greater for $\nu$ close to 0. This trade-off between stability (favoring $\nu \approx 0$) and objectivity (favoring larger $\nu$) led us initially to prefer the midpoint of the allowable values in $(0, 1)$, namely $\nu = 1/2$ as the default choice. However the examples presented in Section 6 indicate that the performance of the method seems insensitive to the choice of $\nu$ in the range of $(0, 1)$. The correction factor for $BF_{\gamma:F}[\text{BIC}]$ on the other hand depends on $n$ and is considerably less stable to changes in the distributions of $\epsilon_\gamma$ than $BF_{\gamma:F}(\nu)$.

It is interesting to note in connection with the above that the correction term $E[\|\epsilon_\gamma\|\|\nu\]/E[\|\epsilon_F\|\|\nu\]]$ for $BF_{\gamma:F}(\nu)$ approaches 1 as $\nu \to 0$. Hence choices of $\nu$ close to 0 are essentially completely robust to choice of SS error distribution for the submodels.

Note also that if we force $\pi_\gamma = 0$ for all submodels such that $q_\gamma \leq 2$, then the allowable range of $\nu$ is $(0, 3)$ and hence $\nu = 2$ becomes a possible choice. In this case, again, the correction term $E[\|\epsilon_\gamma\|\|\nu\]/E[\|\epsilon_F\|\|\nu\]] = 1$ regardless of the choice of error distributions, since we have assumed the variance of each component is 1. Hence, again, in this case, the Bayes factor is completely robust to choice of SS error distribution. Additionally, the case $\nu = 2$ corresponds to the harmonic prior

$$p_\nu^I(\alpha, \beta_\gamma, \eta_\gamma | \nu) \propto \|\theta_\gamma\|^{2-q_\gamma}$$

where $\theta_\gamma = (X_\gamma'X_\gamma)^{1/2} \beta_\gamma$ and $q_\gamma \geq 3$. It is well-known that the harmonic prior plays important roles in estimation problems with the Stein effect. See Maruyama (2003) for the detail. It is interesting to observe the additional advantage of the harmonic prior in the model choice problem. See Section 7 for some additional discussion of such priors.
4. The Laplace approximation of BF under Gaussian errors

In Section 3, we saw that the Bayes factor $BF_{\gamma:F}(\nu)$ under SS errors is equal to $BF_{\gamma:F}^{G}(\nu)$, which is the Bayes factor under Gaussian errors. In this section, we consider the so-called Laplace approximation of some Bayes factors under Gaussian errors. We will approximate not only the function $BF_{\gamma:F}^{G}(\nu)$ but also Bayes factors with respect to more general priors where the prior on $g$ is (2.12);

$$p_{g}(g|\nu,k;\infty) \propto g^{\nu/2-1}(1 + g^{-1})^{-k/2}.$$  (4.1)

When the same prior on $g$ is used for $M_{\gamma}$ and $M_{F}$, improper choices of $\nu$ $(0 \leq \nu < q_{\gamma})$ as well as proper choices of $\nu$ $(-k < \nu < 0)$ are valid for use. Under Gaussian errors, the Bayes factor for comparing each of $M_{\gamma}$ to $M_{F}$ is well-defined as

$$BF_{\gamma:F}^{G}[\nu,k] = \int_{0}^{\infty} g^{\nu/2-1}(1 + g^{-1})^{-k/2}(1 + g)^{-a/2} \{g(1 - R_{\gamma}^{2}) + 1\}^{-a-1} \frac{dg}{(1 + g)^{n-p-1}}.$$  (4.2)

where $k \geq 0$, $-k < \nu < q_{\gamma}$.

First we provide a summary of Laplace approximations to the integral based on Tierney and Kadane (1986). For integrals of the form

$$\int_{-\infty}^{\infty} \exp(h(\tau,n))d\tau,$$

we make the use of the fully exponential Laplace approximation, based on expanding a smooth unimodal function $h(\tau,n)$ in a Taylor series expansion about $\hat{\tau}$, the mode of $h(\tau,n)$. The Laplace approximation is given by

$$\lim_{n \to \infty} \int_{-\infty}^{\infty} \exp(h(\tau,n))d\tau \approx (2\pi)^{1/2} \hat{\sigma}_{h} \exp(h(\hat{\tau},n)),$$  (4.3)

where

$$\hat{\sigma}_{h} = \left\{ -\frac{\partial^{2} h(\tau,n)}{\partial \tau^{2}} \bigg|_{\tau = \hat{\tau}} \right\}^{-1/2}.$$  

In the following, we will use the symbol $f(n) \approx g(n)$ $(n \to \infty)$ if

$$\lim_{n \to \infty} \frac{f(n)}{g(n)} = 1.$$  (4.4)

Hence the approximation given by (4.3) is written as

$$\int_{-\infty}^{\infty} \exp(h(\tau,n))d\tau \approx (2\pi)^{1/2} \hat{\sigma}_{h} \exp(h(\hat{\tau},n)), \ (n \to \infty).$$  (4.5)

The next result gives approximations of the Bayes factor (3.7) in terms of the Bayes factor based on BIC given in (3.16).
Theorem 4.1. Let the prior be given by (4.1). Assume that \( \{\nu, k\} \) does not depend on \( n \).

1. Assume \(-k < \nu < q\), and \(0 < r < 1\). Then
\[
\int_0^\infty \frac{g^{\nu/2-1}}{(1 + g^{-1})^{k/2}} \frac{(1 + g)^{\nu-1}}{(1 + rg)^{\nu+1}} dg \approx \left\{ \frac{4\pi \varphi(q - \nu, r)}{n^{\nu-r} r^n} \right\}^{1/2},
\]
where \( \varphi(s, r) = rs^{-1}(1/r - 1)\{1/r - 1\}^{-s} \).

2. Assume that \(-k < \nu < q\), and also \(0 < R_2^2 \leq R_F^2 < 1\). Then
\[
BF_{\gamma,F}^G[\nu, k] \approx \tilde{BF}_{\gamma,F}^G[\nu] \approx BF_{\gamma,F}[\nu, 2 - \nu] \approx \tilde{BF}_{\gamma,F}[BIC]
\]
and \(BF_{\gamma,F}[BIC]\) is the BIC based alternative under Gaussian errors.

Proof. See Appendix. \(\square\)

Clearly the function \( \varphi \) does not depend on \( n \) and hence Theorem 4.1 shows that \( BF_{\gamma,F}[\nu, k] \) is asymptotically equivalent to BIC with a simple \( O(1) \) correction function depending \( \nu \) as well as \( \{p, q_\gamma\} \) and the \( R \)-squares. Although several fully Bayes factors for the variable selection problem have been proposed in the literature, the relationship between the approximate Bayes factors and naive BIC has not been shown to the authors’ knowledge. In this sense, while the main contributions in this paper are given in Section 3, Theorem 4.1 may be a practically useful contribution because of the simplicity of the approximate Bayes factor.

In this section, we have considered general Bayes factors under Gaussian errors. Remember that \( BF_{\gamma,F}(\nu) \) the Bayes factor w.r.t. sub-harmonic priors, under SS errors, is equal to \( BF_{\gamma,F}[\nu, 0] \) for \( 0 < \nu < 1 \). Under Gaussian errors, Liang et al. (2008) recommended the use of \( BF_{\gamma,F}[\nu, 2 - \nu] \) with \(-2 < \nu < 0\). Guo and Speckman (2009) and Celeux et al. (2011) recommended the use of \( BF_{\gamma,F}[0, 2] \). By Theorem 4.1, these Bayes factors may be approximated as follows.

Corollary 4.1.

\[
BF_{\gamma,F}(\nu) = BF_{\gamma,F}[\nu, 0] \approx \tilde{BF}_{\gamma,F}(\nu) \text{ for } 0 < \nu < 1. \quad \text{sub-harmonic prior}
\]

\[
BF_{\gamma,F}[0, 2] \approx \tilde{BF}_{\gamma,F}(0). \quad \text{a version of Guo and Speckman (2009)}
\]

\[
BF_{\gamma,F}[\nu, 2 - \nu] \approx \tilde{BF}_{\gamma,F}(\nu) \text{ for } -2 < \nu < 0. \quad \text{Liang et al. (2008)}
\]

In Section 6, we will see how approximate Bayes factors \( \tilde{BF}_{\gamma,F}(\nu) \) work numerically and how sensitive they are to the choice of \( \nu \).
5. Model selection consistency

In this section, we consider model selection consistency in the case where $p$ is fixed and as $n$ approaches infinity. Let $\mathcal{M}_T$ be the true model,

$$y = \alpha_T 1_n + X_T \beta_T + \sigma_T \epsilon.$$

Consistency for model choice is defined as

$$\text{plim}_{n \to \infty} \Pr(\mathcal{M}_T | y) = 1,$$

where plim denotes convergence in probability and the probability distribution is the sampling distribution under the true model $\mathcal{M}_T$. We will show that Bayes factors considered in the previous sections has a model selection consistency under generally SS errors. The consistency property is clearly equivalent to

$$\text{plim}_{n \to \infty} BF_{\gamma : T} = \text{plim}_{n \to \infty} BF_{\gamma : F} = 0 \quad \forall \gamma \neq T. \quad (5.1)$$

For model selection consistency, we make the following assumptions;

A1. $U_n = \|\epsilon\|^2/n$ is bounded in probability from below and from above, that is, for any $c > 0$ and any positive integer $n$, there exists an $M$ such that

$$\Pr\left( M^{-1} < U_n < M \right) > 1 - c.$$  

A2. The limit of the correlation matrix of $x_1, \ldots, x_p$, $\lim_{n \to \infty} X_F^T X_F/n$, exists and is positive definite.

A1 seems more general than necessary. It appears that, by the law of large numbers, $U_n$ ought to converge to 1 in probability, but this is not necessarily true if the error distribution is not Gaussian. In the case of a scale mixture of Gaussians, $U_n$ approaches, in law, a random variable $\xi$ which has the distribution of the mixing variable of the variance. Even when the error distribution is not a scale mixture of Gaussians, A1 appears to be a reasonable and minimal assumption. A2 is the standard assumption which also appears in Knight and Fu (2000) and Zou (2006). Under these mild assumptions, we have following preliminary results for proving the consistency.

Lemma 5.1. Assume A1 and A2.

1. Assume $\mathcal{M}_\gamma \neq \mathcal{M}_N$. For any $0 < k < 1$ and any positive integer $n$, there exists a $c_1(\gamma, k) > 2$ such that

$$\Pr\left( \frac{1}{c_1(\gamma, k)} < R_\gamma^2 < 1 - \frac{1}{c_1(\gamma, k)} \right) > 1 - k.$$  

2. Let $\gamma \supseteq T$. Then $(1 - R_T^2)/(1 - R_\gamma^2) \geq 1$. Further for any $0 < k < 1$ and any positive integer $n$, there exists a $c_2(\gamma, T, k) > 0$ such that

$$\Pr\left( 1 \leq \left( \frac{1 - R_T^2}{1 - R_\gamma^2} \right)^n < 1 + c_2(\gamma, T, k) \right) > 1 - k.$$  

(5.3)
3. Let $\gamma \not\supseteq T$. Then for any $0 < k < 1$ and any positive integer $n$, there exists a $c_3(\gamma, T, k) > 1$ such that

$$\Pr\left(\frac{1 - R_T^2}{1 - R_T^2} < 1 - \frac{1}{c_3(\gamma, T, k)}\right) > 1 - k.$$  \hspace{1cm} (5.4)

Proof. See Appendix.

First we give a consistency result on BIC.

**Theorem 5.1.** Assume A1 and A2. The Bayes factor based on BIC under Gaussian errors

$$BF_{\gamma}^{G} = \left\{ \frac{(1 - R_T^2)^{-n} n^{-q_T}}{(1 - R_\gamma^2)^{-n}n^{-p}} \right\}^{1/2}$$

is consistent for model selection under SS errors (including $M_{\gamma} = M_N$).

Proof. we show that

$$\lim_{n \to \infty} BF_{\gamma}^{G} = \lim_{n \to \infty} \left\{ n^{q_T - q_\gamma} \left( \frac{1 - R_T^2}{1 - R_\gamma^2} \right)^n \right\}^{1/2} = 0.$$  \hspace{1cm} (5.5)

Consider the following two situations:

1. $\gamma \supseteq T$: By part 2 of Lemma 5.1, \{(1 - R_T^2)/(1 - R_\gamma^2)\} is bounded in probability. Since $q_T > q_\gamma$, (5.5) is satisfied.

2. $\gamma \not\supseteq T$: By part 3 of Lemma 5.1, \{(1 - R_T^2)/(1 - R_\gamma^2)\} is strictly less than 1 in probability. Hence \{(1 - R_T^2)/(1 - R_\gamma^2)\} converges to zero in probability exponentially fast with respect to $n$. Therefore, no matter what value $q_T - q_\gamma$ takes, (5.5) is satisfied.

These complete the proof.

Note that in Theorem 5.1 we do not exclude the null model $M_N$ and hence BIC has model selection consistency even when the null model is true. When we consider consistency of the Bayes factors treated in the previous sections, $BF_{\gamma,F}(\nu)$, $BF_{\gamma,F}[\nu,k]$, $BF_{\gamma,F}^{-\gamma}(\nu)$, we have to exclude the null model $M_N$, but they all still have model selection consistency among non-null models.

**Corollary 5.1.** Assume A1 and A2. \{\nu,k\} is assumed independent of $n$ and $\mathcal{M}_{\gamma}$. Assume also $M_N$ is excluded from possible models. Then

1. $BF_{\gamma,F}^{-\gamma}(\nu)$ for $\nu < 1$ is consistent for model selection under SS errors.

2. $BF_{\gamma,F}[\nu,k]$ for $-k < \nu < 1$ is consistent for model selection under SS errors.

3. $BF_{\gamma,F}(\nu)$ for $0 < \nu < 1$ is consistent for model selection under SS errors.
Proof. By part 1 of Lemma 5.1, when $M \gamma \neq M N$, both $R_{\gamma}^2$ and $R_{F}^2$ are positive and strictly less than 1 with probability 1. Hence both $\varphi(q_{\gamma} - \nu, 1 - R_{\gamma}^2)$ and $\varphi(p - \nu, 1 - R_{F}^2)$ where

$$\varphi(s, r) = rs^{s-1}(1/r - 1)e^{-s}$$

are positive and bounded from above with probability 1 provided $\nu < 1$ and $\nu$ is independent of $n$ and $M_{\gamma}$. (On the other hand, since $R_{N}^2 \equiv 0$, $\varphi(q_{\gamma} - \nu, 1 - R_{N}^2)$ is not defined.) As in Theorem 4.1,

$$\tilde{BF}^G_{\gamma:F}(\nu) = \left\{ \frac{\varphi(q_{\gamma} - \nu, 1 - R_{\gamma}^2)}{\varphi(p - \nu, 1 - R_{F}^2)} \right\}^{1/2} BF^G_{\gamma:F}[BIC].$$

Hence consistency of $\tilde{BF}^G_{\gamma:F}(\nu)$ follows from consistency of BIC.

Further as $n \to \infty$, we have

$$BF^G_{\gamma:F}[\nu, k] \approx \tilde{BF}^G_{\gamma:F}(\nu)$$

by Theorem 4.1 provided $-k < \nu < 1$ and $\{\nu, k\}$ are independent of $n$ and $M_{\gamma}$. Hence consistency of $BF^G_{\gamma:F}[\nu, k]$ follows from consistency of $\tilde{BF}^G_{\gamma:F}(\nu)$.

Remember that $BF^G_{\gamma:F}(\nu)$, the Bayes factor w.r.t. sub-harmonic priors, under SS errors, is equal to $BF^G_{\gamma:F}[\nu, 0]$ for $0 < \nu < 1$. Hence consistency of $BF^G_{\gamma:F}(\nu)$ follows from consistency of $BF^G_{\gamma:F}[\nu, k]$.

Remark 5.1. Liang et al. (2008) established model selection consistency for $\nu < 0$ and $k = 2 - \nu$ for Gaussian errors. Corollary 5.1 in conjunction with Theorem 4.1 extends their result to the entire class of SS distributions for a broader class of $\nu$ and $k$. Also as noted in Remark 3.3, $BF^G_{\gamma:F}(\nu)$ results in model selection consistency for all models including the null model for all SS distributions. Additionally a development along the lines of Theorem 4.1 allows an analogous BIC based approximation to $\tilde{BF}^G_{\gamma:F}(\nu)$.

This extension also allows an extension of model selection consistency to all SS error distributions for the method of Guo and Speckman (2009); Celeux et al. (2011) based on $\tilde{R}_{\gamma}^2$. Further as noted above, an alternative robust sub-harmonic prior based method, the model selection consistency for these $\tilde{R}_{\gamma}^2$ based methods also apply for the null model.

It should be emphasized in each of the above cases that is is the Bayes factor method developed for the Gaussian case that is shown to have model selection consistency for the entire class of SS error distributions. These Gaussian based Bayes factors, however, are not Bayes factors for error distributions which are not Gaussian, the sole exception being our robust Bayes factors which are based on separable priors in the sense described earlier, and which are simultaneously Bayes factors relative to the same prior.

Remark 5.2. The issue of model selection consistency in our setup, is somewhat complicated by the wide choice of possible error distributions. If all errors are
normally distributed, then under our assumptions A2 on the design matrix \( X_F \), imply that each \( R^2_\gamma \) approaches a constant, and that \( \|\epsilon\|^2/n \to 1 \). If on the other hand, all models are variance mixtures of Gaussians with mixture variance distributed as a positive random variable \( \xi \), then \( \|\epsilon\|^2/n \to \xi \) a random variable, and \( R^2_\gamma \) also approaches a random variable which is bounded above and below in probability provided that \( \xi \) is similarly bounded.

In general philosophical terms, it might be better to assume that the sequence of error terms \( \epsilon = (\epsilon_1, \ldots, \epsilon_n)' \) are exchangeable for all \( n \). By De Finetti’s Theorem, this would imply that the error terms all have a variance mixture of normal distributions. We have chosen a slightly weaker requirement on the sequence of error distributions, namely, that \( \|\epsilon\|^2/n \) remains bounded above and below in probability, which extracts the necessary limiting behavior of the error terms to ensure consistency of model selection. Interestingly, although we attain model selection consistency with these assumptions, it is not necessarily true that \( 1 = \text{var} \epsilon_i = \text{var} \xi \) is consistently estimated by \( \|\epsilon\|^2/n \).

6. Examples

In this section, we provide illustrations of the method using both simulated and real data. In each example, we compare several different versions of the Laplace approximated Bayes factors \( \tilde{BF}_G^\gamma \) and \( BF_G^\gamma \) \([\text{BIC}]\). The values of \( \nu \) are \(-2, -1, 0, 0.5, 0.95\). These choices correspond to our default choice, \( \nu = 1/2 \) and \( \nu = 0.95 \) which also satisfies our robustness condition \( 0 < \nu < 1 \). The choice \( \nu = 0 \) approximates \( BF_G^\gamma [0, 2] \) of Guo and Speckman (2009) and \( \nu = -2 \) and \( \nu = -1 \) approximates two choices of Liang et al. (2008) as presented in Corollary 4.1.

6.1. Simulation Studies

We compare numerical performance of our with BIC in a small simulation study. We generated 16 possible correlated predictors \((p = 16)\) as follows:

\[
\begin{align*}
\text{cor}=0.5 & \quad x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10} \sim \mathcal{N}(0, 1) \\
\text{cor}=0.3 & \quad x_{11}, x_{12}, x_{13}, x_{14}, x_{15}, x_{16} \sim \mathcal{N}(0, 1) \\
\text{cor}=0.1 & \quad \text{cor}=-0.4 \\
\text{cor}=-0.2 & \quad \text{cor}=-0.4
\end{align*}
\]

Here “cor” denotes the correlation of two Gaussian random variables. Also \((x_1, x_2), (x_3, x_4), (x_5, x_6), (x_7, x_8), (x_9, x_{10}), x_{11}, x_{12}, x_{13}, x_{14}, x_{15}, x_{16}\) are assumed to be independent. After generating pseudo random \(x_1, \ldots, x_{16}\), we centered and scaled them as noted in Section 1. We set \( n = 30 \) and consider 4 cases where the true predictors are

\[
\begin{align*}
q_T = 16 & \quad x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}, x_{11}, x_{12}, x_{13}, x_{14}, x_{15}, x_{16} \\
q_T = 12 & \quad x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}, x_{11}, x_{12}
\end{align*}
\]
\( q_T = 8 \) \( x_1, x_2, x_5, x_6, x_9, x_{10}, x_{11}, x_{12} \)
\( q_T = 4 \) \( x_1, x_2, x_5, x_6 \)
(where \( q_T \) denotes the number of true predictors) and the true model is given by

\[
y = 1_{30} + 2 \sum_{i \in \{\text{true}\}} x_i + \sigma \times \begin{cases} 
N_{30}(0, I_{30}), \\
\text{Multi-t}(0, I_{30}; 3, 30),
\end{cases}
\]  

with \( \sigma = 0.5, 1, 2 \). Tables 1 and 2 show how often the true model ranks first and how often it is in the top 3 among \( 2^{16} - 1 \) candidates when the number of replicates is \( N = 200 \). The error distributions are Gaussian (Table 1) and multivariate-t with 3 degrees of freedom (Table 2). For the case of normally distributed errors (Table 1), the Bayes factor methods performed well and stably for \( \sigma = 0.5 \) and \( \sigma = 1 \) and did reasonably well for \( \sigma = 2 \) for the smaller true models (\( q_T = 4, 8 \)). BIC seemed, generally, to have a preference for larger models, and performed much less well than the Bayes factor method for \( \sigma = 0.5 \) and \( \sigma = 1 \) for models of smaller size (\( q_T = 4, 8, 12 \)). For \( \sigma = 2 \), BIC did substantially better than BF for the largest model (\( q_T = 16 \)) and somewhat better for \( q_T = 12 \).

Performance of \( \tilde{\text{BF}}_{G; F}(\nu) \) seemed relatively insensitive to the choice of \( \nu \). When \( q_T \neq 16 \), the choice of \( \nu \) makes little difference. But when \( q_T = 16 \), positive \( \nu = (0.5, 0.95) \) seems to perform better especially for larger \( \sigma \).

Interestingly, for the case of a multivariate-t error distribution with 3 degrees of freedom (the minimum so that a variance exists), the numerical results were quite similar to those in the normal case for both \( \tilde{\text{BF}}_{G; F}(\nu) \) and BIC, both quantitatively and qualitatively. One possible aspect of the relative insensitivity of the results to choice of \( \nu \) in heavy tailed case is the extension of model selection consistency for the entire class of SS errors to a broad class mixture of \( g \)-prior based methods given by Corollary 5.1.

### 6.2. Analysis of real data

In this section, we apply our methods (approximate Bayes factor and BIC) to Hald data set presented and analyzed in Casella and Moreno (2006) and to the US Crime data set in Raftery, Madigan and Hoeting (1997). See those papers for detailed descriptions of the data sets. Table 3 and 4 present posterior probabilities based on \( \tilde{\text{BF}}_{G; F}(\nu) \) of the top three selected models (assuming equal prior probabilities on all models) for several different choices of \( \nu \) \( (0.95, 0.5, 0, -1, -2) \). BIC was also included in the study. In each case, the first, second and third ranked choices based on \( \tilde{\text{BF}}_{G; F}(\nu) \) were identical regardless of the choices of \( \nu \). Also in each case the top ranked submodel based on \( \tilde{\text{BF}}_{G; F}(\nu) \) was regarded as reasonable in the earlier papers. As in the simulation study, and as noted in several previous studies, BIC seems to choose bigger models. In particular for the Hald data, the top choice \( \{x_1, x_2\} \) agrees with that of Casella and Moreno (2006) and also of Berger and Pericchi (1996) and Draper and Smith (1998).
Table 1

Frequency of the true model (Gaussian error)

| $q_T$ | 16 | 12 | 8 | 4 |
|-------|----|----|---|---|
| rank | 1-3 | 1-3 | 1-3 | 1-3 |

$\sigma = 0.5$

| $\tilde{BF}_G(0.95)$ | 1.00 | 1.00 | 0.94 | 1.00 | 0.94 | 0.99 | 0.89 | 0.99 |
|-----------------------|------|------|------|------|------|------|------|------|
| $\tilde{BF}_G(0.5)$  | 1.00 | 1.00 | 0.95 | 1.00 | 0.94 | 0.99 | 0.88 | 0.99 |
| $\tilde{BF}_G(0)$    | 1.00 | 1.00 | 0.95 | 1.00 | 0.94 | 0.99 | 0.88 | 0.99 |
| $\tilde{BF}_G(-1)$   | 1.00 | 1.00 | 0.96 | 1.00 | 0.94 | 1.00 | 0.89 | 0.99 |
| $\tilde{BF}_G(-2)$   | 1.00 | 1.00 | 0.96 | 1.00 | 0.94 | 1.00 | 0.89 | 0.99 |
| BIC                  | 1.00 | 1.00 | 0.44 | 0.62 | 0.28 | 0.46 | 0.20 | 0.35 |

$\sigma = 1$

| $\tilde{BF}_G(0.95)$ | 0.85 | 0.92 | 0.87 | 0.99 | 0.86 | 0.97 | 0.76 | 0.95 |
|-----------------------|------|------|------|------|------|------|------|------|
| $\tilde{BF}_G(0.5)$  | 0.83 | 0.90 | 0.88 | 0.99 | 0.87 | 0.97 | 0.76 | 0.94 |
| $\tilde{BF}_G(0)$    | 0.80 | 0.87 | 0.89 | 1.00 | 0.87 | 0.97 | 0.75 | 0.94 |
| $\tilde{BF}_G(-1)$   | 0.73 | 0.81 | 0.89 | 1.00 | 0.88 | 0.97 | 0.74 | 0.94 |
| $\tilde{BF}_G(-2)$   | 0.55 | 0.72 | 0.90 | 1.00 | 0.89 | 0.98 | 0.76 | 0.95 |
| BIC                  | 1.00 | 1.00 | 0.44 | 0.62 | 0.28 | 0.46 | 0.20 | 0.35 |

$\sigma = 2$

| $\tilde{BF}_G(0.95)$ | 0.06 | 0.11 | 0.26 | 0.42 | 0.50 | 0.74 | 0.51 | 0.73 |
|-----------------------|------|------|------|------|------|------|------|------|
| $\tilde{BF}_G(0.5)$  | 0.05 | 0.10 | 0.25 | 0.41 | 0.51 | 0.74 | 0.50 | 0.72 |
| $\tilde{BF}_G(0)$    | 0.05 | 0.10 | 0.24 | 0.41 | 0.52 | 0.73 | 0.49 | 0.72 |
| $\tilde{BF}_G(-1)$   | 0.04 | 0.06 | 0.22 | 0.39 | 0.43 | 0.74 | 0.48 | 0.72 |
| $\tilde{BF}_G(-2)$   | 0.02 | 0.03 | 0.17 | 0.32 | 0.47 | 0.72 | 0.43 | 0.71 |
| BIC                  | 0.62 | 0.77 | 0.31 | 0.48 | 0.24 | 0.40 | 0.19 | 0.34 |
### Table 2

Frequency of the true model (multi-t error)

| $q_T$ | 16 | 12 | 8  | 4  |
|-------|----|----|----|----|
| rank  | 1  | 1-3| 1  | 1-3|
|       |    |    |    |    |
|       |    |    |    |    |
| $\sigma = 0.5$ |  |  |  |  |
| $\tilde{BF}_G^{(0.95)}$ | 0.94 | 0.95 | 0.92 | 0.98 |
| $\tilde{BF}_G^{(0.5)}$ | 0.94 | 0.95 | 0.92 | 0.98 |
| $\tilde{BF}_G^{(0)}$ | 0.93 | 0.95 | 0.93 | 0.98 |
| $\tilde{BF}_G^{(-1)}$ | 0.93 | 0.94 | 0.93 | 0.98 |
| $\tilde{BF}_G^{(-2)}$ | 0.91 | 0.92 | 0.95 | 0.98 |
| BIC   | 0.98 | 0.99 | 0.47 | 0.63 |
|       |    |    |    |    |
| $\sigma = 1$ |  |  |  |  |
| $\tilde{BF}_G^{(0.95)}$ | 0.67 | 0.70 | 0.75 | 0.84 |
| $\tilde{BF}_G^{(0.5)}$ | 0.64 | 0.68 | 0.76 | 0.84 |
| $\tilde{BF}_G^{(0)}$ | 0.62 | 0.67 | 0.77 | 0.84 |
| $\tilde{BF}_G^{(-1)}$ | 0.58 | 0.63 | 0.76 | 0.83 |
| $\tilde{BF}_G^{(-2)}$ | 0.49 | 0.58 | 0.76 | 0.83 |
| BIC   | 0.89 | 0.93 | 0.45 | 0.60 |
|       |    |    |    |    |
| $\sigma = 2$ |  |  |  |  |
| $\tilde{BF}_G^{(0.95)}$ | 0.14 | 0.20 | 0.28 | 0.37 |
| $\tilde{BF}_G^{(0.5)}$ | 0.14 | 0.18 | 0.29 | 0.36 |
| $\tilde{BF}_G^{(0)}$ | 0.13 | 0.17 | 0.29 | 0.36 |
| $\tilde{BF}_G^{(-1)}$ | 0.09 | 0.13 | 0.29 | 0.33 |
| $\tilde{BF}_G^{(-2)}$ | 0.07 | 0.11 | 0.24 | 0.34 |
| BIC   | 0.47 | 0.59 | 0.28 | 0.39 |
Table 3
Hald data: posterior probabilities of top 3 selected models

| ν  | 0.95 | 0.5 | 0   | -1  | -2  |
|----|------|-----|-----|-----|-----|
| 1 {1, 2, } | 0.66 | 0.63 | 0.61 | 0.57 | 0.54 |
| 2 {1, 4} | 0.16 | 0.17 | 0.17 | 0.18 | 0.20 |
| 3 {1, 2, 4 } | 0.06 | 0.07 | 0.07 | 0.08 | 0.08 |

BIC
| 1 {1, 2, } | 0.25 |
| 2 {1, 2, 4} | 0.23 |
| 3 {1, 2, 3} | 0.23 |

Table 4
US crime data: posterior probabilities of top 3 selected models

| ν  | 0.95 | 0.5 | 0 | -1  | -2  |
|----|------|-----|---|-----|-----|
| 1 {1, 3, 4, 9, 11, 13, 14 } | 0.020 | 0.019 | 0.018 | 0.016 | 0.015 |
| 2 {1, 3, 4, 9, 11, 13, 14, 15} | 0.018 | 0.018 | 0.017 | 0.015 | 0.014 |
| 3 {1, 3, 5, 9, 11, 13, 14 } | 0.013 | 0.013 | 0.012 | 0.011 | 0.010 |

BIC
| 1 {1, 3, 4, 9, 11, 13, 14, 15} | 0.035 |
| 2 {1, 3, 4, 9, 11, 13, 14} | 0.026 |
| 3 {1, 3, 4, 9, 11, 12, 13, 14, 15} | 0.019 |

For the US Crime data, our top ranked model agrees with that of the Occam’s window posterior in Table 2 of Raftery, Madigan and Hoeting (1997). Interestingly our second ranked model includes $x_{15}$ which does not occur in any of Raftery, Madigan and Hoeting’s (1997) Occam’s window model choices, but which does occur in several models chosen by such classical methods as Mallow’s $C_p$, adjusted $R^2$, etc. in their Table 1.

7. Concluding remarks

Bayesian model selection for linear regression models with Gaussian errors has been popular area of study for some time. There is also a substantial literature devoted to studying the extension of Stein-type shrinkage estimators from models with Gaussian errors to those with general SS errors. In particular, it has long been observed that certain shrinkage estimators which improve over the least squares (LS) estimator for Gaussian models also improve over the LS estimator simultaneously for all SS error models (See for example, Cellier, Fourdrinier and Robert (1989)). Maruyama (2003) and Maruyama and Strawderman (2005) found, in addition, that certain separable priors (in the sense described in Section 2) leads to generalized Bayes shrinkage estimators that do not depend on the form of the underlying SS distribution and that also simultaneously improve on the LS estimator, sometimes dramatically so. The original aim of
this research was to see if similar separable priors could be found that have this distributional robustness property in the variable selection problem (and, of course, also to perform well).

The generalized Bayes priors developed in sections 2 and 3 turned out to satisfy our requirements and also to be closely related to other so called \(g\)-priors (or mixtures of \(g\)-priors) in the literature (See e.g. Liang et al. (2008); Guo and Speckman (2009)). Our original development required that the null model be excluded from the class of possible models, however we observed (see Remark 3.3) that a slightly altered version which treats the intercept term the same as all other regression coefficients (as opposed to giving it a uniform prior), allows the null model to be included as well. This alternative development gives Bayes factor which depend on the \(R^2\) relative to the null model where all coefficients including \(\alpha\) is 0. This dependence on \(R^2\) is related to the model selection procedures of Guo and Speckman (2009); Celeux et al. (2011).

Our prior distribution on the regression parameters is sub-harmonic for each (non-null) model, and as such, leads to admissible estimators under quadratic loss in each of the non-null models when the variance is known (see Maruyama and Takemura (2008)) regardless of the SS error distribution. Also if \(q \geq 3\) and \(\nu = 2\) the generalized Bayes estimator for each submodel is minimax and dominates the James-Stein estimator (See Maruyama (2003)). Hence although we require \(0 < \nu < 1\), we nevertheless expect good performance of the corresponding Bayes estimators.

The expression of our Bayes factors, e.g. (3.6), are relatively simple involving the ratio of two 1-dimensional integrals. To further simplify calculations we investigated Laplace approximations to our Bayes factors, and more generally, to a collection of Bayes factors arising from mixtures of \(g\)-priors that have recently appealed (See Liang et al. (2008); Guo and Speckman (2009)). We show in Section 4 that in each case the Bayes factor can be approximated as the Bayes factor for the Gaussian model based on BIC times a simple rational function depending \(q, \nu\) and the \(R^2\) of the models.

Using these Laplace approximations we are able to establish model selection consistency of our robust procedure for the entire class of SS distributions and to extend the model consistency results of several earlier papers for the Gaussian case to the entire class of SS distributions.

A small simulation study and an analysis of the Hald data (See Casella and Moreno (2006)) and the US Crime data (See Raftery, Madigan and Hoeting (1997)) indicates that our method performs well. It gives results consistent with the results of the cited papers for the real data sets and performs comparably and sometimes better than several of the mixture of \(g\)-prior methods.
Appendix A: Proof of Lemma 3.1

Under the submodel $\mathcal{M}_\nu$, the conditional marginal density of $y$ with respect to improper prior $\eta_\nu^{\nu/2-1}$ given $\alpha$ and $\beta_\nu$ is

\[
\int_0^\infty \eta_\nu^{\nu/2} f(\eta_\nu \| y - \alpha 1_n - X_\nu \beta_\nu \|) \eta_\nu^{\nu/2-1} d\eta_\nu \\
= \| y - \alpha 1_n - X_\nu \beta_\nu \|^{-\nu} \int_0^\infty t^{(n+\nu)/2-1} f(t) dt \\
= \int_0^\infty \frac{t^{(n+\nu)/2-1} f(t) dt}{\int_0^\infty t^{(n+\nu)/2-1} f_G(t) dt} \int_0^\infty f_G(\eta \| y - \alpha 1_n - X_\nu \beta_\nu \|) \eta_\nu^{(n+\nu)/2-1} d\eta_\nu
\]  

(A.1)

where $f_G(t) = (2\pi)^{-n/2} \exp(-t/2)$, provided

\[
\int_0^\infty t^{(n+\nu)/2-1} f(t) dt < \infty \iff E[\| \epsilon_\nu \|^{\nu}] < \infty. \tag{A.2}
\]

Therefore, we have

\[
M_\nu(y \| \nu) = \frac{E[\| \epsilon_\nu \|^{\nu}]}{E[\| \epsilon_G \|^{\nu}]} M^G_\nu(y \| \nu). \tag{A.3}
\]

Appendix B: Proof of Theorem 4.1

Denote the left-hand side of (4.6) by $H(n)$. When approximating $H(n)$, make the change of variables $\tau = \log g$. See Liang et al. (2008) for details. With this transformation, the integral becomes

\[
H(n) = \int_{-\infty}^\infty \frac{e^{(\nu/2-1)\tau} (1 + e^\tau)^{(n-\nu-1)/2} e^{\nu+\tau} d\tau}{(1 + e^{-\tau})^{k/2} (1 + re^\tau)^{(n-\nu-1)/2}}, \tag{B.1}
\]

where the extra $e^\tau$ comes from the Jacobian of the transformation of variables. Denote the logarithm of the integrand function in (B.1) by $h(\tau, n)$. We have

\[
\frac{\partial}{\partial \tau} h(\tau, n) = \frac{z}{2} \left( \frac{1}{1 + z} \left\{ \frac{(n-1)(1-r)}{1+rz} + \frac{\nu+k}{z} \right\} + (q-\nu) \right), \tag{B.2}
\]

where $z = e^\tau$. Since $0 < r < 1$ and $\nu + k > 0$, the equation $\{\partial/\partial \tau\} h(\tau, n) = 0$ has the only one positive root $\hat{\tau} = e^\tau$. It clearly satisfies

\[
\lim_{n \to \infty} \frac{n}{\hat{\tau}} = \frac{1}{r-1} - \frac{1}{q-\nu}. \tag{B.3}
\]
Hence we have
\[
e^{h(\tau,n)} = \left\{ z^{\nu} (1 + z)^{n-q-1}(1 + z^{-1})^{-k} (1 + r \hat{z})^{-n+1} \right\}^{1/2}
= \left\{ \frac{z^{q+\nu}}{r^{n-1}} \left( 1 + \frac{n/z}{n} \right)^{n-q-1-k} \left( 1 + \frac{n/(r \hat{z})}{n} \right)^{-n+1} \right\}^{1/2}
\approx \left\{ \left( \frac{q - \nu}{n(1/r - 1)} \right)^{q-\nu} r^{-n+1} \exp\left\{ (1 - 1/r)n/z \right\} \right\}^{1/2}
= \left\{ \left( \frac{q - \nu}{n(1/r - 1)} \right)^{q-\nu} r^{-n+1} \right\}^{1/2}.
\]

Similarly, as in (B.2), we have
\[
\frac{\partial^2}{\partial \tau^2} h(\tau,n) = \frac{\partial (\hat{\theta} / \partial \tau) h(\tau,n)}{1 + z} - \frac{z^2}{2(1 + z)} \left\{ \frac{(n-1)(1-r)r \hat{z}}{(1+r \hat{z})^2} + \frac{\nu + k}{z^2} \right\}
\]
and
\[
\frac{\partial^2}{\partial \tau^2} h(\tau,n)|_{\tau = \hat{\tau}} \approx -\frac{z}{2(1 + z)} \frac{(n-1)(1-r)r \hat{z}}{(1+r \hat{z})^2} \approx -\frac{n-1-r}{2z} \approx -\frac{q - \nu}{2}. \quad (B.5)
\]

Therefore we have
\[
H(n) \approx (2\pi)^{1/2} e^{h(\hat{\tau},n)} \left\{ -\partial^2 / \partial \tau^2 \right\} h(\tau,n)|_{\tau = \hat{\tau}}^{-1/2}
\approx \left\{ \frac{4\pi}{q - \nu} \left( \frac{q - \nu}{n(1/r - 1)} \right)^{q-\nu} r^{-n+1} \right\}^{1/2}. \quad (B.6)
\]
as \( n \to \infty \). Hence the part 1 of the theorem follows.

Since \( BF_{G,F}^{\nu,k}[\nu, k] \) in (4.2) is given by the ratio of such integrals, part 2 of the theorem follows.

Appendix C: Proof of Lemma 5.1

Let \( \mathcal{M}_T \) be the true submodel \( \mathbf{y} = \alpha_T \mathbf{1}_n + \mathbf{X}_T \mathbf{\beta}_T + \sigma_T \mathbf{\epsilon} \) where \( \mathbf{X}_T \) is the \( n \times q_T \) true design matrix and \( \mathbf{\beta}_T \) is the true \( (q_T \times 1) \) coefficient vector.

For the submodel \( \mathcal{M}_n, \mathbf{1}_n - R^T_z \) is given by \( \| \mathbf{Q}_\gamma (\mathbf{y} - \tilde{y} \mathbf{1}_n) \|^2 / \| \mathbf{y} - \tilde{y} \mathbf{1}_n \|^2 \) with \( \mathbf{Q}_\gamma = \mathbf{I} - \mathbf{X}_\gamma (\mathbf{X}_\gamma')^{-1} \mathbf{X}_\gamma' \). The numerator and denominator are rewritten as
\[
\| \mathbf{Q}_\gamma (\mathbf{y} - \tilde{y} \mathbf{1}_n) \|^2 = \| \mathbf{Q}_\gamma \mathbf{X}_T \mathbf{\beta}_T + \sigma_T \mathbf{\epsilon} \|^2
= \beta_T \mathbf{X}_T' \mathbf{Q}_\gamma \mathbf{X}_T \mathbf{\beta}_T + 2 \sigma_T \beta_T \mathbf{X}_T' \mathbf{Q}_\gamma \mathbf{\epsilon} + \sigma_T^2 \mathbf{\epsilon}' \mathbf{Q}_\gamma \mathbf{\epsilon}
\]
where \( \mathbf{\epsilon} = \mathbf{\epsilon} - \bar{\mathbf{e}} \mathbf{1}_n \) and similarly
\[
\| \mathbf{y} - \tilde{y} \mathbf{1}_n \|^2 = \beta_T \mathbf{X}_T' \mathbf{X}_T \mathbf{\beta}_T + 2 \sigma_T \beta_T \mathbf{X}_T' \mathbf{\epsilon} + \sigma_T^2 \| \mathbf{\epsilon} \|^2.
\]
Since \( \hat{\epsilon}'Q\gamma\hat{\epsilon} \leq \|\hat{\epsilon}\|^2 \), 1 - \( R^2 \) is bounded as

\[
\frac{\beta_T'\{X_T'Q\gamma X_T/n\}\beta_T + 2\sigma_T\beta_T'\{X_T'Q\gamma\epsilon/n\} + \sigma^2_TW\gamma V_n}{\beta_T'\{X_T'X_T/n\}\beta_T + 2\sigma_T\beta_T'\{X_T'\epsilon/n\} + \sigma^2_TV_n} \leq 1 - R^2 \leq \frac{\beta_T'\{X_T'Q\gamma X_T/n\}\beta_T + 2\sigma_T\beta_T'\{X_T'Q\gamma\epsilon/n\} + \sigma^2_TV_n}{\beta_T'\{X_T'X_T/n\}\beta_T + 2\beta_T'\{X_T'\epsilon/n\} + \sigma^2_TV_n} \tag{C.2}
\]

where \( V_n = \hat{\epsilon}'\hat{\epsilon}/n \) and \( W\gamma = \hat{\epsilon}'Q\gamma \|\hat{\epsilon}\|^2 \sim Be(\{n - q\gamma - 1\}/2, q\gamma/2) \). In (C.2), we have the following.

- Since \( E[\epsilon] = 0 \) and \( \text{Var}[\epsilon] = I_n \), \( E[X_T'\epsilon/n] = 0 \) and
  \[
  \text{var} (X_T'\epsilon/n) = n^{-1}\{X_T'X_T/n\} \rightarrow 0. \tag{C.3}
  \]
  Therefore \( \beta_T'X_T'\epsilon/n \) approaches 0 in probability.

- When \( \gamma \geq T \), \( Q\gamma X_T \) is a zero matrix. When \( \gamma \not\geq T \), \( \beta_T'\{X_T'Q\gamma\epsilon/n\} \rightarrow 0 \) in probability can be proved as (C.3).

- By the assumption A2, \( X_T'X_T/n - X_T'Q\gamma X_T/n \) is positive-definite for any \( n \) and hence
  \[
  \beta_T'\{X_T'X_T/n\}\beta_T > \beta_T'\{X_T'Q\gamma X_T/n\}\beta_T, \quad \text{for } \beta_T \neq 0.
  \]

- \( W\gamma \) converges to 1 in probability.

- By the assumption A1 on \( \epsilon'\epsilon/n \), \( V_n \) is also bounded in probability from below and from above.

Combining these facts, we see \( 0 < R^2 < 1 \) with strict inequalities in probability.

Since \( Q\gamma X_T = 0 \) for \( \gamma \geq T \) and using (C.1), \( (1 - R^2)/(1 - R^2) \) is given by \( \|Q\gamma\|^2/\|Q\gamma\|^2 \). Further we easily have

\[
1 \leq 1 - R^2 = \frac{\|Q\gamma\|^2}{\|Q\gamma\|^2} \leq \|\hat{\epsilon}\|^2 \leq \frac{1}{W\gamma}.
\]

Note \( W\gamma \sim Be(\{n - q\gamma - 1\}/2, q\gamma/2) \) is distributed as \( (1 + \chi^2_{q\gamma}/\chi^2_{n-q\gamma-1})^{-1} \) where \( \chi^2_{n-q\gamma-1} \) and \( \chi^2_{q\gamma} \) are independent. Hence

\[
\left\{1 + \frac{\chi^2_{q\gamma}/\chi^2_{n-q\gamma-1}}{n}\right\}^{-n} = \left\{1 + \left\{n/\chi^2_{n-q\gamma-1}\right\}\left\{\lambda_{q\gamma}/n\right\}\right\}^{-n} \sim \exp(-\chi^2_{q\gamma}) \text{ as } n \rightarrow \infty
\]

since \( \chi^2_{n-q\gamma-1}/n \rightarrow 1 \) in probability. Therefore \( W\gamma^{-n} \) is bounded in probability from above and hence the theorem follows.
\[
\frac{1 - R^2_T}{1 - R^2_\gamma} \text{ is written as}
\]
\[
1 - R^2_T = \frac{\sigma_T^2 ||Q_T \hat{\epsilon}||^2}{\beta_T' X_T' Q, X_T \beta_T + 2 \sigma_T \beta_T' X_T' Q, \epsilon + \sigma_T^2 Q, \hat{\epsilon}}
\]
\[
\leq \frac{\beta_T' X_T' Q, X_T \beta_T + 2 \sigma_T \beta_T' X_T' Q, \epsilon + \sigma_T^2 Q, \hat{\epsilon}}{\sigma_T^2 V_n}
\]
\[
= \left( \beta_T' \{ X_T' Q, X_T/n \} \beta_T + 2 \sigma_T \beta_T' \{ X_T' Q, \epsilon/n \} \right) V_n^{-1}.
\]

(C.4)

Clearly \( W_\gamma \rightarrow 1 \) in probability. Also since \( \gamma \notin T \), \( \beta_T' \{ X_T' Q, X_T/n \} \beta_T > 0 \) for any \( n \). Further as \( \{ X_T' Q, \epsilon/n \} \rightarrow 0 \) in probability, \( (1 - R^2_T)/(1 - R^2_\gamma) \) is strictly smaller than 1 in probability.

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