Abstract. We compute explicit solutions $\Lambda^\pm_m$ of the Painlevé VI (PVI) differential equation from equivariant instanton bundles $E_m$ corresponding to Yang-Mills instantons with “quadrupole symmetry.” This is based on a generalization of Hitchin’s logarithmic connection to vector bundles with an $SL_2(\mathbb{C})$ action. We then identify explicit Okamoto transformation which play the role of “creation operators” for construction $\Lambda^\pm_m$ from the “ground state” $\Lambda^\pm_0$, suggesting that the equivariant instanton bundles $E_m$ might similarly be related to the trivial “ground state” $E_0$.

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1. Introduction

1.1. The Hierarchy of Equivariant Instanton Bundles. The work of Atiyah-Drinfeld-Hitchin-Manin (ADHM) on instantons \[4, 5\] is an important milestone for the highly productive interplay between mathematics and physics, and continues to inspire interesting work to this day. ADHM studied the correspondence between the Yang-Mills instantons of theoretical physics and instanton bundles. Yang-Mills instantons are self-dual connection on a vector bundle \( F \) over the four-sphere \( S^4 \). The Penrose twistor space of \( S^4 \) is the complex projective space \( \mathbb{C}P^3 \). Instanton bundles are a class of holomorphic vector bundles \( E \) over \( P^3 \) constructed from the linear algebra data of a self-dual linear monad, as discussed in Section 3.1 and Appendix 3.2.

The starting point for the present paper is a result of Gil Bor and the author \[11\] on equivariant Yang-Mills instantons. Specifically, we studied self-dual connections with a certain quadrupole symmetry. This work resolved open questions from \[58, 9\], where quadrupole symmetry played a key role in establishing the existence of non-minimal (neither self-dual nor anti-self-dual) Yang-Mills connections with nonzero instanton number \( c_2(E) \neq 0 \). The existence of non-minimal Yang-Mills connections had been established for the \( c_2(E) = 0 \) case in the seminal work \[66\]. Quadrupole symmetry of instantons on \( S^4 \) corresponds to the action of \( SU(2) \) on the complex twistor space \( \mathbb{C}P^3 \) via the unique irreducible representation on \( \mathbb{C}^4 \). The Fano three-fold \( P^3 \) is then almost-homogeneous under the complexification of the \( SU(2) \) action to an \( SL_2(\mathbb{C}) \) symmetry, meaning that the \( SL_2(\mathbb{C}) \) action has an open dense orbit. The key existence \[11\] and uniqueness \[63\] results from Section 3.1 are summarized in:

**Theorem 1.1.** For each nonnegative integer \( m \) there exists an \( SL_2(\mathbb{C}) \) equivariant instanton bundle \( E_m \) with rank \( \text{rk} E_m = 2 \) and instanton number \( c_2(E_m) = \frac{1}{2} m(m + 1) \). Every \( SL_2(\mathbb{C}) \) equivariant instanton bundle \( E \) of rank \( \text{rk} E = 2 \) is isomorphic to one of the \( E_m \).

Allowing for a modicum of poetic license, the hierarchical structure of Theorem 1.1 is reminiscent of the familiar quantum harmonic oscillator \[56\], where the energy eigenstates \( \psi_m \) are similarly indexed by a nonnegative integer \( m \). Furthermore, any eigenstate \( \psi_m \) is constructed from the ground state \( \psi_0 \) by successive applications of Dirac’s famous creation operator \( a^\dagger \). A motivation for this paper was to explore the possibility of an analogous “creation operator” \( S \) which by successive applications would construct the equivariant instanton bundle \( E_m \) from the trivial “ground state” equivariant instanton bundle \( E_0 \). This putative “creation operator” \( S \) is indicated by dashed arrows in the schematic:

\[
E_0 \xrightarrow{S} E_1 \xrightarrow{S} E_2 \xrightarrow{S} E_3 \xrightarrow{S} E_4. \tag{1.1}
\]

Alas we did not manage to construct such a “creation operator” \( S \) for the hierarchy \( E_m \) of equivariant instanton bundles in Eqn. 1.1.
1.2. The Shadow Hierarchy of PVI Solutions. We do however encounter some interesting “Platonic shadows” \[55\] of the putative “creation operator” \( S \), summarized in the following schematic subsuming eqn. 1.1:

\[
\begin{align*}
\Lambda^+_0 & \to Q \Lambda^+_1 & \Lambda^+_2 & \to Q \Lambda^+_3 & \Lambda^+_4 \\
E_0 & \to S E_1 & \to S E_2 & \to S E_3 & \to S E_4 \\
\Lambda^-_0 & \to Q^{-1} \Lambda^-_1 & \to Q^{-1} \Lambda^-_2 & \to Q^{-1} \Lambda^-_3 & \to Q^{-1} \Lambda^-_4
\end{align*}
\]

As further discussed in Sections 5 and 6, each of the equivariant instanton bundles \( E_m \) of Theorem 1.1 yields a pair of PVI solutions: \( \Lambda^+_m \leftarrow E_m \to \Lambda^-_m \). We have explicitly computed \( \Lambda^+_m \) for \( 0 \leq m \leq 4 \) in Section 4, following \[61\] and \[10\], see also \[45\]. Other geometric constructions of algebraic Painlevé VI solutions, and associated Frobenius manifolds, include \[27, 19, 20, 28, 29, 30, 62\]. This computation generalizes the work Hitchin in the influential paper \[27\], which gives the \( m = 0 \) “ground state” case \( \Lambda^+_0 \leftarrow E_0 \to \Lambda^-_0 \). Transformations between solutions of Painlevé VI have been known for many years, but deeper insights into their structure continue to emerge \[18, 8, 6, 22\]. In honor of Okamoto’s decisive contribution to this area \[51\], we will use the term Okamoto transformation for all transformations between Painlevé VI solutions. Section 4.2 contains additional discussion and references. As further discussed in Section 4.2, each solid horizontal arrow \( Q : \Lambda^+_m \to \Lambda^+_m+1 \) and \( Q^{-1} : \Lambda^-_m \to \Lambda^-_{m+1} \) indicates an explicit Okamoto transformation.

Then the operators \( Q \) and \( Q^{-1} \) each act as a “creation operator” on its respective hierarchy of PVI solutions, conjecturally for all nonnegative integers \( m \) although our case-by-case proof only covers a finite subset:

**Theorem 1.2.** For each nonnegative integer \( m \leq 4 \),

\[
\Lambda^+_m = Q^m \Lambda^+_0, \quad \Lambda^-_m = Q^{-m} \Lambda^-_0.
\]

We interpret the “creation operators” \( Q \) and \( Q^{-1} \) as “shadows” of a putative creation operator \( S \) for equivariant instanton bundles. Reconstructing the equivariant instanton \( E_m \) from the corresponding pair of Painlevé solutions \( \Lambda^+_m \) and \( \Lambda^-_m \) is one possible approach to constructing \( S \). It is theoretically possible to reconstruct an instanton bundle on \( \mathbb{P}^3 \) from its divisor of jumping lines and associated data \[52, 33, 25\]. In our equivariant setting, jumping lines are manifested as poles of the PVI solutions. However, the PVI solutions \( \Lambda^+_m \) have additional poles not corresponding to jumping lines, which have so far thwarted attempts to reconstruct the divisor of jumping lines.

1.3. Notes. One of the motivations for the present paper is a body of interesting work on the aetiology of the Okamoto transformations from the viewpoint of isomonodromic deformations focusing on Katz’s middle convolution and related ideas \[18, 6, 8, 22, 64\], complementing results on the classical Schlesinger transformations.
The hope is that a variant of middle convolution might produce the putative “creation operator” $S : E_m \to E_{m+1}$ on the hierarchy of equivariant instanton bundles.

Another motivation for the present paper is a wave of recent interest in instanton bundles and monads on Fano threefolds other than the classic case of $P^3$, see e.g. \cite{17, 39, 21, 59, 60, 14, 42, 2, 15, 57}. Like $P^3$, several of these Fano threefolds are almost-homogeneous for a natural $SL_2(\mathbb{C})$ action (see \cite{41, 49, 67, 13} for almost-homogenous threefolds). In one particularly promising instance, \cite{59} established existence and uniqueness of an equivariant instanton bundle with minimal nonzero instanton numbers. It is intriguing to speculate whether the equivariant instanton bundles might also constitute a hierarchy in this case, by analogy to Theorem 1.1 in the case of $P^3$. Fano threefolds have a plethora of rational curves since they are rationally connected \cite{16, 38, 26}. This could open the door for some of the ideas of Okamoto transformations, and possibly middle convolution, that play an important role for the $P^3$ results of the present paper.

2. Representation Theory Prerequisites

In this section we start with a unified review of the finite-dimensional representation theory of $SL_2(\mathbb{C})$, with emphasis is on the decomposition of tensor product representations into irreducibles. Although these results are rather standard, the formalism used in the present paper, and in \cite{11}, is rather nonstandard.

Let $\mathcal{V} := \mathbb{C}[x, y]$ be the infinite-dimensional vector space of polynomials with generated by the two indeterminates $x$ and $y$. $SL_2(\mathbb{C})$ acts on $\mathcal{V}$ in the usual way by linear substitutions in $x$ and $y$. The subspace $\mathcal{V}_d$ of degree-$d$ homogeneous polynomials is a vector space of dimension $d + 1$. It is well known that $SL_2(\mathbb{C})$ acts irreducibly on each $\mathcal{V}_d$, and that irreducible finite-dimensional representation of $SL_2(\mathbb{C})$ is isomorphic to some $\mathcal{V}_d$.

Tensor products of the irreducible representations $\mathcal{V}_d$ decompose according to the Clebsch-Gordan formula

$$\mathcal{V}_i \otimes \mathcal{V}_j = \mathcal{V}_{i+j} \oplus \mathcal{V}_{i+j-2} \oplus \cdots \oplus \mathcal{V}_{|i-j|}. \quad (2.1)$$

The equivariant linear projections implicit in the Clebsch-Gordan formula are simply described in terms of the “transvectants” (Überschiebungen) of classical invariant theory. For a non-negative integer $p$, the $p$-th transvectant is the $SL_2(\mathbb{C})$-equivariant bilinear map $\langle \cdot, \cdot \rangle_p : \mathcal{V} \times \mathcal{V} \to \mathcal{V}$ defined by

$$\langle u, v \rangle_p := \frac{1}{p!} \sum_{k=0}^p (-1)^k \binom{p}{k} \frac{\partial^p u}{\partial x^{p-k} \partial y^k} \frac{\partial^p v}{\partial x^k \partial y^{p-k}}. \quad (2.2)$$

Verifying the equivariance of the transvectants is not difficult, see e.g. \cite{11}. The 0-th transvectant is ordinary multiplication of polynomials, $\langle u, v \rangle_0 = u v$. To relate transvectants to the Clebsch-Gordan formula, one checks that the that a restriction of the $p$-th transvectant is a bilinear map $\mathcal{V}_i \times \mathcal{V}_j \to \mathcal{V}_{i+j-2p}$ which is nonzero if and only if $0 \leq p \leq \min(i, j)$.

From the obvious symmetry property

$$\langle u, v \rangle_p = (-1)^p \langle v, u \rangle_p, \quad (2.3)$$
we see that the nondegenerate bilinear form \( \langle \cdot, \cdot \rangle := \langle \cdot, \cdot \rangle_p : V_p \times V_p \to V_0 \) is symmetric if \( p \) is even, and antisymmetric (symplectic) if \( p \) is odd. We denote also by \( \langle \cdot, \cdot \rangle \) the bilinear extension to any direct sum of irreducible representation \( V_d \).

A basis for \( V_p \) may be constructed from a pair of linearly independent vectors \( a, b \in V_1 \). Linear independence of \( a \) and \( b \) is equivalent to \( \langle a, b \rangle \neq 0 \). One may check that \( \{ a^p, a^{p-1} b, \ldots, a b^{p-1}, b^p \} \) is a basis of \( V_p \), since

\[
\langle a^{p-j} b^j, a^i b^{p-j} \rangle = (-1)^j j! (p-j)! \langle a, b \rangle^p \neq 0, \quad (2.4)
\]

and \( \langle a^{p-j} b^j, a^i b^{p-k} \rangle = 0 \) if \( k \neq j \). This simultaneously proves the nondegeneracy of the bilinear form \( \langle \cdot, \cdot \rangle \) on \( V \).

The adjoint representation of \( SL_2(\mathbb{C}) \) on its Lie algebra \( \mathfrak{sl}_2(\mathbb{C}) \) is an irreducible representation of dimension three, so it is isomorphic to the representation \( V_2 \). The Lie bracket is an equivariant antisymmetric bilinear map \([\cdot, \cdot] : \mathfrak{sl}_2(\mathbb{C}) \times \mathfrak{sl}_2(\mathbb{C}) \to \mathfrak{sl}_2(\mathbb{C}) \). The first transvectant is an equivariant antisymmetric bilinear map

\[
\langle \cdot, \cdot \rangle_1 : V_2 \times V_2 \to V_2, \quad \text{modulo scalar multiplication}, \quad \text{is the only such map.}
\]

We can then fix an identification of the Lie algebras \( \mathfrak{sl}_2(\mathbb{C}) \) and \( V_2 \) by defining the Lie bracket on \( V_2 \) to be the first transvectant, i.e.

\[
[u, v] := \langle u, v \rangle_1, \quad u, v \in V_2.
\]

A convenient basis of the Lie algebra \( V_2 \) is

\[
g_0(a, b) := -\frac{a b}{\langle a, b \rangle}, \quad g_+(a, b) := \frac{a^2}{2 \langle a, b \rangle}, \quad g_-(a, b) := -\frac{b^2}{2 \langle a, b \rangle}, \quad (2.5)
\]

which will be abbreviated as \( \{ g_0, g_+, g_- \} \) when the choice of \( a \) and \( b \) is clear. The Lie brackets of these basis vectors are:

\[
[g_0, g_+] = 2g_+, \quad [g_0, g_-] = -2g_-, \quad [g_+, g_-] = g_0.
\]

More generally, the linearization of the \( SL_2(\mathbb{C}) \) action on \( V_p \) is an equivariant bilinear map \([\cdot, \cdot] : \mathfrak{sl}_2(\mathbb{C}) \times V_p \to V_p \). Identifying \( \mathfrak{sl}_2(\mathbb{C}) \) with \( V_2 \) as above, one checks that this map again coincides with the first transvectant,

\[
[u, v] : V_2 \times V_p \to V_p \quad (u, v) \mapsto \langle u, v \rangle_1.
\]

It is straightforward to compute the action of the Lie algebra on the monomials in \( V \),

\[
[g_0(a, b), a^i b^j] = (i-j)a^i b^j
\]

\[
[g_+(a, b), a^i b^j] = j a^{i+1} b^{j-1}
\]

\[
[g_-(a, b), a^i b^j] = i a^{i-1} b^{j+1}. \quad (2.6)
\]

3. The Equivariant Instanton Bundles \( E_m \)

3.1. Existence and Uniqueness. In this section we revisit the results of \[11\] on equivariant instanton bundles on the complex projective space \( \mathbb{P}^3 \). The standard
A monad is straightforward. The kernel of an outgoing map contains the image of the incoming map. This imposes conditions on the fundamental datum of the monad, which we will call an instanton bundle as the cohomology of the complex vector space endowed with a complex symplectic form \( \langle \cdot, \cdot \rangle \). Then a (self-dual linear) monad over \( P^3 \) is a complex of vector bundles

\[
0 \rightarrow W \otimes \mathcal{O}_{P^3}(-1) \xrightarrow{A} V \otimes \mathcal{O}_{P^3} \xrightarrow{A^\circ} W' \otimes \mathcal{O}_{P^3}(1) \rightarrow 0,
\]

where \( A \) and \( A^\circ \) are as follows. The fundamental datum for the monad is an injective linear map \( A : W \otimes Z \rightarrow V \). This defines the vector bundle map \( W \otimes \mathcal{O}_{P^3}(-1) \rightarrow V \otimes \mathcal{O}_{P^3} \). Composing the transpose \( A' : V' \rightarrow W' \otimes Z' \) with the canonical isomorphism \( s : V \rightarrow V' \), defined by \( s(v) = \langle v, \cdot \rangle \), yields the surjective linear map \( A^\circ := A' \circ s : V \rightarrow W' \otimes Z' \). This defines the vector bundle map \( V \otimes \mathcal{O}_{P^3} \rightarrow W' \otimes \mathcal{O}_{P^3}(1) \). These vector bundle maps are required to fit together as a complex, meaning that the kernel of an outgoing map contains the image of the incoming map. This imposes conditions on the fundamental datum \( A : W \otimes Z \rightarrow V \), which are discussed in Proposition 3.3.

Self-dual linear monads correspond to a class of holomorphic vector bundles over \( P^3 \), which we will call instanton bundles in this paper (the term mathematical instanton bundles is also used). The construction an instanton bundle as the cohomology of the monad is straightforward. The kernel \( U^o := \ker A^\circ \) is a subbundle of the trivial bundle \( V \otimes \mathcal{O}_{P^3} \), with \( \text{rk} \, U^o = \dim V - \dim W \). Similarly, the image \( U := \text{im} \, A \) is a subbundle of \( V \otimes \mathcal{O}_{P^3} \), with \( \text{rk} \, U = \dim W \). Furthermore since this is a complex, \( U \) is a subbundle of \( U^o \) and the cohomology of the complex is the instanton bundle \( E := U^o / U \) over \( P^3 \). The rank of the instanton bundle \( \text{rk} \, E = \dim V - 2 \dim W \) is even, since the symplectic vector space \( V \) has even dimension. Furthermore the holomorphic vector bundle \( E \) inherits a complex symplectic structure, so the line bundle \( \det E \) is trivial and \( c_1(E) = c_1(\det E) = 0 \). We are primarily interested in the instanton bundles of rank \( \text{rk} \, E = 2 \), where the instanton number (or charge) is \( 2 \) the integer \( c_2(E) = \dim W \geq 0 \).

The inverse correspondence, constructing a monad from an instanton bundle, is important for studying instanton bundles with a group action. Representation theory affords powerful tools for studying linear maps \( A : W \otimes Z \rightarrow V \) in the equivariant case. The starting point for the present paper is the following result of [11]. Here the group \( SL_2(\mathbb{C}) \) acts on \( P^3 = P(Z) \) by identifying \( Z := V_3 \) with the irreducible representation of on \( \mathbb{C}^4 \):

**Theorem 3.1.** ([11]) For each nonnegative integer \( m \) there exists an \( SL_2(\mathbb{C}) \) equivariant instanton bundle \( E_m \) with rank \( \text{rk} \, E_m = 2 \) and instanton number \( c_2(E_m) = \frac{1}{2} m(m + 1) \).

The explicit construction from [11] of the equivariant monads for these \( E_m \) is summarized in Appendix 3.2. The paper [11] focused on \( SU(2) \) equivariant “real” instanton bundles, where the monad is required to satisfy an additional “ADHM reality condition.” This is an equivariant case of the famous Atiyah-Drinfeld-Hitchin-Manin correspondence [5, 11] between “real” instanton bundles on \( P^3 \) and Yang-Mills instantons on \( S^4 \). The existence result Theorem 3.1 follows immediately from [11] by forgetting the real structures and complexifying the group action.
Uniqueness of equivariant instanton bundles depends on the notion of equivalence. In [11] it was shown that every $SU(2)$ equivariant real instanton bundle of rank 2 is equivalent to one of the bundles $E_m$ of Theorem 3.1. But this uniqueness result does not preclude the possibility that there could exist other $SL_2(\mathbb{C})$ equivariant rank 2 instanton bundles that do not admit a real structure. That possibility was eliminated in subsequent work:

**Theorem 3.2.** ([63]) Every $SL_2(\mathbb{C})$ equivariant instanton bundle $E$ of rank $\text{rk} E = 2$ is isomorphic to one of the instanton bundles $E_m$ of Theorem 3.1.

A corollary is that every $SL_2(\mathbb{C})$ equivariant instanton bundle $E$ of rank $\text{rk} E = 2$ admits a real structure. However that this is true only for $\text{rk} E = 2$, and fails for higher rank. The proof of Theorem 3.2 in [63] reworks the ideas of [11] within the context complex geometry of the Fano threefold $P^3$, bypassing any steps that require the reality condition or study of Yang-Mills instantons on $S^4$. The main ingredient is the Drinfeld-Manin identification [4, 5] of $A^0: V \to W' \otimes Z'$ with a natural map $H^1(E \otimes \Omega^1) \to H^1(E(-1)) \otimes H^1(\Omega(1))$ of sheaf cohomology groups, where $Z' = H^0(\mathcal{O}(1))$, $V = H^1(E \otimes \Omega^1)$, and $W' = H^1(E(-1))$ is the dual of $W = H^1(E \otimes \Omega^2(1))$. The analogous computation of [11] for the “real” case was based on the identification of $W$ and $V$ with the kernels of certain Dirac operators [31, 5] on $S^4$ coupled to the corresponding Yang-Mills instanton connection. In both cases the key step is using the Atiyah-Bott-Lefschetz fixed-point theorem to determine characters of the admissible group actions on $W$ and $V$.

3.2. Construction Overview. This section contains an overview of instanton bundles from [3, 4], and an overview of the equivariant instanton bundles $E_m$ from [11]. The following fundamental result of Atiyah-Drinfeld-Hitchin-Manin [1, 5] translates the self-dual monads of eqn. (3.1) to linear algebra data:

**Proposition 3.3.** Let $W$ be a complex vector space and let $V$ be a complex vector space with symplectic form $\langle \cdot , \cdot \rangle$. A linear map $A: W \otimes Z \to V$ defines a self-dual monad as in eqn. (3.1) iff the following conditions hold:

1. The “injectivity condition”: $A(w \otimes z) = 0$ only if $w \otimes z = 0$.
2. The “isotropy condition”: $\langle A(w \otimes z), A(w' \otimes z) \rangle = 0$ for all $z \in Z$ and all $w, w' \in W$.

**Proof.** The sequence is exact at $W \otimes \mathcal{O}_{P^3}(-1)$, meaning that the image of the incoming map is equal to the kernel of the outgoing map, iff the injectivity condition holds. By duality, the sequence is also exact at $W' \otimes \mathcal{O}_{P^3}(1)$ in this case. The sequence is a complex at $V \otimes \mathcal{O}_{P^3}$, meaning that the image of the incoming map is contained in the kernel of the outgoing map, iff the isotropy condition holds. We note that the cohomology of this complex is concentrated at $V \otimes \mathcal{O}_{P^3}$, the cohomology is zero at $W \otimes \mathcal{O}_{P^3}(-1)$ and at $W' \otimes \mathcal{O}_{P^3}(1)$. □

Yang-Mills instantons over $S^4$ correspond self-dual monads that are “real,” meaning that the self-dual monads satisfy an additional “reality condition” [3, 4, 11]. The paper [11] studied Yang-Mills instantons with a certain $SU(2)$ “quadrupole symmetry”, by constructing the corresponding $SU(2)$ equivariant real self-dual monads. Forgetting the real structure and complexifying the $SU(2)$ action on the holomorphic bundles yields the $SL_2(\mathbb{C})$ equivariant instanton bundles used for the construction of Painlevé VI solutions discussed in the present paper. As previously mentioned in Section 3.1 it is known a posteriori [63] that imposing and
subsequently forgetting the reality condition produces all \( SL_2(\mathbb{C}) \) equivariant instanton bundles for rank 2, although this is not true for higher rank. The rank of a self-dual monad is defined to be the rank of the corresponding instanton bundle, \( \text{rk} \ E = \dim V - 2 \dim W \).

**Proposition 3.4.** \([11, 63]\) If \( SL_2(\mathbb{C}) \) acts on a rank 2 self-dual monad with \( Z = \mathcal{V}_3 \), then \( W = W(m) \) and \( V = V(m) \) for some nonnegative integer \( m \), where:

\[
W(m) = \bigoplus_{0 \leq l \leq m-1, l \equiv m-1 \pmod{2}} \mathcal{V}_{2l}, \quad V(m) \oplus \mathcal{V}_{2m-1} = \bigoplus_{m'-1 \leq j \leq m} \mathcal{V}_{2j+1}.
\]

It is convenient to consider \( V(m) \) as a summand of \( \hat{V}(m) \) with:

\[
\hat{V}(m) := V(m) \oplus \mathcal{V}_{2m-1} = \mathcal{V}_1 \oplus \mathcal{V}_3 \oplus \cdots \oplus \mathcal{V}_{2m-1} \oplus \mathcal{V}_{2m+1}.
\]

For example, \( \hat{V}(3) = \mathcal{V}_1 \oplus \mathcal{V}_3 \oplus \mathcal{V}_5 \oplus \mathcal{V}_7 \), and cancellation of \( \mathcal{V}_{2m-1} = \mathcal{V}_5 \) yields \( V(3) = \mathcal{V}_1 \oplus \mathcal{V}_3 \oplus \mathcal{V}_7 \). Note that \( V(0) = V(0) = \mathcal{V}_1 \) because \( \mathcal{V}_{-1} := 0 \). Both \( \hat{V}(m) \) and \( V(m) \) have an \( SL_2(\mathbb{C}) \) invariant \( \mathbb{C} \)-bilinear symplectic form \( \langle \cdot, \cdot \rangle \) defined by eqn. \ref{eqn:2.3}. For \( W(m) \) it is convenient to separate the odd and even cases. For odd \( m = 2k+1 \geq 0 \),

\[
W(2k+1) = \mathcal{V}_0 \oplus \mathcal{V}_4 \oplus \cdots \oplus \mathcal{V}_{4k-4} \oplus \mathcal{V}_{4k}.
\]

For example, \( W(3) = \mathcal{V}_0 \oplus \mathcal{V}_4 \). For even \( m = 2k \geq 0 \),

\[
W(2k) = \mathcal{V}_2 \oplus \mathcal{V}_6 \oplus \cdots \oplus \mathcal{V}_{4k-4} \oplus \mathcal{V}_{4k}.
\]

Note that \( W(0) = 0 \) because the index set of the sum is empty.

It is similarly convenient to consider an equivariant instanton bundle \( \hat{E} \) of rank 2 as a summand of an equivariant instanton bundle \( \hat{E} \) corresponding to the equivariant self-dual monad

\[
0 \longrightarrow W(m) \otimes \mathcal{O}_{P^3}(-1) \xrightarrow{A} \hat{V}(m) \otimes \mathcal{O}_{P^3} \xrightarrow{\hat{A}^*} W'(m) \otimes \mathcal{O}_{P^3}(1) \longrightarrow 0.
\]

The rank is \( \text{rk} \ \hat{E} = \dim \hat{V}(m) - 2 \dim W(m) = 2 + \dim \mathcal{V}_{2m-1} \). Whenever the image of \( A : W(m) \otimes Z \to \hat{V}(m) \) is contained in \( V(m) \subseteq \hat{V}(m) \), this monad splits as the direct sum

\[
0 \longrightarrow W(m) \otimes \mathcal{O}_{P^3}(-1) \xrightarrow{A} V(m) \otimes \mathcal{O}_{P^3} \xrightarrow{\hat{A}^*} W'(m) \otimes \mathcal{O}_{P^3}(1) \longrightarrow 0
\]

\[
\oplus \oplus \oplus
\]

\[
0 \longrightarrow 0 \longrightarrow \mathcal{V}_{m-1} \otimes \mathcal{O}_{P^3} \longrightarrow 0 \longrightarrow 0.
\]

The equivariant instanton bundle \( \hat{E} \) then splits as the direct sum of the desired rank 2 equivariant instanton bundle \( E \) and the trivial equivariant bundle with fiber \( \mathcal{V}_{2m-1} \).

Following \([11]\), we describe equivariant self-dual monads in terms of a set of coefficients \( a_{t, p} \), which are complex numbers in the general case:
Proposition 3.5. (11) Let $W = W(m)$ and $\hat{V} = \hat{V}(m)$ and $Z = V_3$. Then an equivariant linear map $\hat{A} : W \otimes Z \rightarrow \hat{V}$ is the direct sum

$$\hat{A} = \bigoplus_{0 \leq l \leq m-1, \ell \equiv m-1 \pmod{2}} \hat{A}_l$$

of the component maps

$$\hat{A}_l : V_{2l} \otimes V_3 \rightarrow \hat{V}, \quad w \otimes z \mapsto \bigoplus_{0 \leq p \leq \min(2l,3)} a_{l,p} \langle w, z \rangle_p .$$

Proof. Using Clebsch-Gordan eqn. (11), we check that $V_{2j+1}$ is a summand of $\hat{V}(m)$ iff $V_{2j+1}$ is a summand of $W(m) \otimes V_3$. Noting that the multiplicity of each summand of $\hat{V}(m)$ is at most one, the assertion follows from Schur’s Lemma. \qed

Proposition 3.6 then translates to conditions on the coefficients $a_{l,p}$ of $\hat{A}$.

Theorem 3.6. (11) An equivariant linear $\hat{A} : W \otimes Z \rightarrow \hat{V}$ defines an equivariant self-dual linear monad iff its coefficients $a_{l,p}$ satisfy:

1. The “injectivity condition”: $a_{l,0} \neq 0$ for all $l$.
2. The “isotropy condition” consists of the two conditions,
   a. The “diagonal isotropy condition”:
      $$(2l - 1)^2 a_{l,2}^2 = (2l + 1) a_{l,0}^2 + 2l(2l - 3) a_{l,1}^2$$
      whenever $1 \leq l$,
      $$(2l - 1)^2 a_{l,3}^2 = (2l + 2)(2l + 5) a_{l,0}^2 - 9(2l + 1) a_{l,1}^2$$
      whenever $2 \leq l$.
   b. The “off-diagonal isotropy condition”:
      $$a_{l,0} a_{l+2,2} = a_{l,1} a_{l+2,3}$$
      whenever $1 \leq l$.

We did not include the reality condition, which translates to [11] the reality of the coefficients $a_{l,p}$.

The existence result Theorem 3.1 now follows from:

Proposition 3.7. (11) For any nonnegative integer $m$, the set of coefficients defined as follows satisfies the conditions of Theorem 3.6:

$$a_{l,0} = (2l - 1) \sqrt{9(2m + 1)^2 - (2l + 3)^2}, \quad a_{l,1} = (2l - 1) \sqrt{(2m + 1)^2 - (2l + 3)^2},$$
$$a_{l,2} = (2l + 3) \sqrt{(2m + 1)^2 - (2l - 1)^2}, \quad a_{l,3} = (2l + 3) \sqrt{9(2m + 1)^2 - (2l - 1)^2}.$$ 

Furthermore, $a_{m-1,1} = 0$.

For each $m$, this set of coefficients explicitly constructs an equivariant instanton bundle $\hat{E}_m$ of rank $\text{rk} \hat{E}_m = 2 + \dim V_{2m-1}$. Since $a_{m-1,1} = 0$, this splits as the direct sum of the desired equivariant instanton bundle $E_m$ of rank 2 and the trivial equivariant bundle of rank 2$m$. This explicit construction was used to compute the Painlevé solutions listed in Section 4.1.

The uniqueness result Theorem 3.2 asserts that the solutions of Theorem 3.7 are unique up to equivalence. For a fixed value of $m$, the coefficient set $\hat{a}_{l,p}$ is equivalent
to the coefficient set $a_{l,p}$ iff there exist constants $\gamma_l$ and $\kappa_{2l+1}$ such that

$$\tilde{a}_{l,p} = \gamma_l \ a_{l,p} \ k_{2l+3-2p},$$

where each $\gamma_l \neq 0$ and each $\kappa_{2l+1} = 1$. For the real case, in $\gamma_l$ are real because the coefficients $\tilde{a}_{l,p}$ and $a_{l,p}$ are real, and in this case the uniqueness up to equivalence was established in [11]. The analogous result also holds for the general case [63], where the coefficients and the $\gamma_l$ may be complex.

4. The PVI solutions $\Lambda^\pm_m$

The celebrated sixth equation of Painlevé, which, incidentally, was not found by Painlevé, is a nonlinear ordinary differential equation for a function $\lambda(t)$, and depending the complex parameter vector $\theta = (\theta_1, \theta_2, \theta_3, \theta_4) \in \mathbb{C}^4$:

$$\frac{d^2 \lambda}{dt^2} = \frac{1}{2} \left( \frac{1}{\lambda} + \frac{1}{\lambda - 1} + \frac{1}{\lambda - t} \right) \left( \frac{d\lambda}{dt} \right)^2 - \left( \frac{1}{t} + \frac{1}{t - 1} + \frac{1}{\lambda - t} \right) \frac{d\lambda}{dt} (4.1)$$

$$+ \frac{\lambda(\lambda - 1)(\lambda - t)}{2 t^2(t - 1)^2} \left( (\theta_4 - 1)^2 - \theta_1^2 \frac{t}{\lambda^2} + \theta_2^2 \frac{t - 1}{(\lambda - 1)^2} + (1 - \theta_3^2) \frac{t(t - 1)}{(\lambda - t)^2} \right).$$

This form of Painlevé VI will be denoted as $P_{VI}(\theta)$. We will say that the pair $\Lambda = [\lambda(t); \theta]$ solves $P_{VI}$ if the function $\lambda(t)$ is a solution of the differential equation $P_{VI}(\theta)$.

Much of the Painlevé literature, especially the older literature, parametrizes the four constants as:

$$C = (\alpha, \beta, \gamma, \delta) := \left( \frac{1}{2}(\theta_4 - 1)^2, -\frac{1}{2}\theta_1^2, \frac{1}{2}\theta_2^2, \frac{1}{2}(1 - \theta_3^2) \right) \in \mathbb{C}^4.$$

The resulting equivalent form of the differential equation will be called classic Painlevé VI and will be denoted as $P_{VI}(C)$:

$$\frac{d^2 \lambda}{dt^2} = \frac{1}{2} \left( \frac{1}{\lambda} + \frac{1}{\lambda - 1} + \frac{1}{\lambda - t} \right) \left( \frac{d\lambda}{dt} \right)^2 - \left( \frac{1}{t} + \frac{1}{t - 1} + \frac{1}{\lambda - t} \right) \frac{d\lambda}{dt} (4.2)$$

$$+ \frac{\lambda(\lambda - 1)(\lambda - t)}{2 t^2(t - 1)^2} \left( \alpha + \beta \frac{t}{\lambda^2} + \gamma \frac{t - 1}{(\lambda - 1)^2} + \delta \frac{t(t - 1)}{(\lambda - t)^2} \right).$$

It is often both necessary and frustrating to convert between the parameters $\theta \in \mathbb{C}^4$ and $C \in \mathbb{C}^4$ when comparing results from the Painlevé literature. A good general reference on Painlevé VI is [34].

For a discussion of algebraic solutions of $P_{VI}$, we refer to [10] [7], which includes an overview of important prior work such as [34] [17] [27] [28] [30] [19] [18]. One approach for constructing solutions is to exploit the relationship [24] [37] between $P_{VI}$ and “isomonodromic deformations” of linear ODE’s with singular points, or equivalently of meromorphic connections on vector bundles over the complex projective line $\mathbb{P}^1$. Hitchin [28] [29] developed the method of “$SL_2(\mathbb{C})$-equivariant compactifications” to construct a certain class of isomonodromic deformations, from which he was able to reconstruct some solutions of $P_{VI}$ explicitly.
As observed by G. Bor [10], isomonodromic deformations can also be constructed using nontrivial $SL_2(\mathbb{C})$-equivariant vector bundles over equivariant compactifications, thereby generalizing Hitchin’s method. Implementing this on the monads of Theorem 3.1 yields the following result, which had been reported in the unpublished manuscript [61]. Defining the parameter vector $\mu := (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}) \in \mathbb{C}^4$, we have:

**Theorem 4.1.** ([61]) For each nonnegative integer $m$, the equivariant instanton bundle $E_m$ yields a pair of explicitly computable algebraic Painlevé VI solutions $\Lambda^+_m = [\lambda^+_m(t); (2m + 1)\mu]$ and $\Lambda^-_m = [\lambda^-_m(t); -(2m + 1)\mu]$. Each of the algebraic functions $\lambda^\pm_m(t)$ is expressed implicitly in terms of the rational function

$$t(w) = \frac{(1 + w)(-3 + w)^3}{(-1 + w)(3 + w)^3}$$

and a rational function of the form

$$\lambda^\pm_m(w) = \left(\frac{(-3 + w)^2}{(-1 + w)(3 + w)}\right) \frac{(-1 + w^2)f^+_m(w) + 8g^+_m(w)}{(3 + w^2)f^-_m(w) - 24g^-_m(w)},$$

where $f^+_m$ and $g^+_m$ are even polynomials of degree at most $2m(m + 1)$.

Differentiation of these algebraic functions is straightforward by implicit differentiation. The first derivative of $\lambda^\pm_m := \lambda^\pm_m(w)$ with respect to $t := t(w)$ is:

$$\frac{d\lambda^\pm_m}{dt} = \frac{d}{dt}t(w) = \frac{(-1 + w)^2(3 + w)^4}{16w^2(-3 + w)^2} \frac{d\lambda^\pm_m}{dw},$$

The second derivative is expressed similarly.

### 4.1. The Explicit PVI Solutions

We exhibit here a pair of explicit PVI solutions $\Lambda^\pm_m = [\lambda^\pm_m; \pm(2m + 1)\mu]$ for each integer $0 \leq m \leq 4$. Each of these was computed via the generalization of Hitchin’s logarithmic connection of section 5.2 from the equivariant instanton bundles $E_m$ discussed in Sections 3.1 and 3.2. The detailed steps are given in Section 6.

- $m = 0$: The equivariant instanton bundle $E_0$ is the trivial rank 2 vector bundle over $\mathbb{P}^3$, and Hitchin’s original method of equivariant compactifications applies without need for generalization. The corresponding Painlevé pair is $\Lambda^+_0 = [\lambda^+_0(t); \mu]$ and $\Lambda^-_0 = [\lambda^-_0(t); -\mu]$ since $\pm(2m + 1) = \pm 1$.

$$f^+_0(w) = 1, \quad g^+_0(w) = 0;$$

$$\lambda^+_0(w) = \frac{(-3 + w)^2(-1 + w^2)}{(-1 + w)(3 + w)(3 + w^2)}.$$  \hspace{1cm} (4.6)

$$f^-_0(w) = 0, \quad g^-_0(w) = 1;$$
\[ \lambda_0^-(w) = -\frac{(-3 + w)^2}{3 (-1 + w) (3 + w)}. \quad (4.7) \]

\( \lambda_0^+(t) \) is equivalent by a change of coordinates to Hitchin’s Poncelet polygon solution \([29]\) of Painlevé VI for \( k = 3 \).

- **\( m = 1 \):** The generalization of Hitchin’s method to nontrivial bundles, as outlined in Section 5.2, is necessary for all \( m \geq 1 \). The equivariant instanton bundle \( E_1 \) is the “null correlation bundle” \([5]\) on \( P^3 \), with \( c_2 = 1 \). The corresponding Painlevé pair is \( \Lambda_1^+ = [\lambda_1^+(t); 3\mu] \) and \( \Lambda_1^- = [\lambda_1^-(t); -3\mu] \) since \( \pm (2m + 1) = \pm 3 \).

\[ f_1^+(w) = 1, \quad g_1^+(w) = 0; \]

\[ \lambda_1^+(w) = \frac{(-3 + w)^2 (-1 + w^2)}{(-1 + w) (3 + w) (3 + w^2)}. \quad (4.8) \]

\[ f_1^-(w) = 4, \quad g_1^-(w) = 3 + w^2; \]

\[ \lambda_1^-(w) = \frac{(-3 + w)^2 (5 + 3 w^2)}{5 (-1 + w) (3 + w) (3 + w^2)}. \quad (4.9) \]

Comparing eqns. 4.6 and 4.8 reveals the coincidence \( \lambda_1^+(t) = \lambda_0^-(t) \); the same functions solves \( P_{VI}(\theta) \) for both \( \theta = 3\mu \) and \( \theta = \mu \), in fact for a one-parameter family of \( \theta \). There are no additional coincidences among the solutions \( \lambda_m^\pm(t) \) computed in this section. For example, comparing eqns. 4.7 and 4.9 shows that \( \lambda_1^- (t) = \lambda_0^- (t) \), and in fact \( \lambda_1^- (t) \) is a solution of \( P_{VI}(\theta) \) only for the expected value \( \theta = -3\mu \).

- **\( m = 2 \):** The equivariant instanton bundle \( E_2 \) has second Chern class \( c_2(E_2) = 3 \). The corresponding Painlevé pair is \( \Lambda_2^+ = [\lambda_2^+(t); 5\mu] \) and \( \Lambda_2^- = [\lambda_2^-(t); -5\mu] \) since \( \pm (2m + 1) = \pm 5 \).

\[ f_2^+(w) = 12 (3 + w^2)^2, \quad g_2^+(w) = (-1 + w^2)^2; \]

\[ \lambda_2^+(w) = \frac{(-3 + w)^2 (-1 + w^2) (5 + w^2) (5 + 3 w^2)}{3 (-1 + w) (3 + w) (1 + w^2) (25 + 6 w^2 + w^4)}. \]

\[ f_2^- (w) = 16 (7 + w^2) (4 + 3 w^2 + w^4), \quad g_2^- (w) = (3 + w^2) (77 + 89 w^2 + 23 w^4 + 3 w^6); \]

\[ \lambda_2^- (w) = -\frac{(-3 + w)^2 (5 + w^2) (35 + 63 w^2 + 25 w^4 + 5 w^6)}{7 (-1 + w) (3 + w) (1 + w^2) (3 + w^2) (25 + 6 w^2 + w^4)}. \]
\begin{itemize}
\item $m = 3$: For the sake of brevity, we shall from now on leave to the reader the task of substituting the polynomials $f_m^\pm(w)$ and $g_m^\pm(w)$ into eqn. 4.4 The equivariant instanton bundle $E_3$ has second Chern class $c_2(E_3) = 6$. The corresponding Painlevé pair is $\Lambda_3^+ = \{\lambda_3^+(t); 7\mu\}$ and $\Lambda_3^- = \{\lambda_3^-(t); -7\mu\}$ since $\pm(2m + 1) = \pm 7$.

\[ f_3^+(w) = 8 \left( 3381 + 7536 w^2 + 6291 w^4 + 2576 w^6 + 611 w^8 + 80 w^{10} + 5 w^{12} \right), \]
\[ g_3^+(w) = (-1 + w^2)^2 \left( 3 + w^2 \right) \left( 147 + 111 w^2 + 57 w^4 + 5 w^6 \right); \]
\[ f_3^-(w) = 12 \left( 3 + w^2 \right)^2 \left( 3528 + 7272 w^2 + 6453 w^4 + 2460 w^6 + 678 w^8 + 84 w^{10} + 5 w^{12} \right), \]
\[ g_3^-(w) = 164052 + 590328 w^2 + 831465 w^4 + 631260 w^6 + 294435 w^8 + 88938 w^{10} + 18207 w^{12} + 2520 w^{14} + 225 w^{16} + 10 w^{18}. \]

\item $m = 4$: The equivariant instanton bundle $E_4$ has second Chern class $c_2(E_4) = 10$. The corresponding Painlevé pair is $\Lambda_4^+ = \{\lambda_4^+(t); 9\mu\}$ and $\Lambda_4^- = \{\lambda_4^-(t); -9\mu\}$ since $\pm(2m + 1) = \pm 9$.

\[ f_4^+(w) = 4 \left( 14619528 + 69918552 w^2 + 140631309 w^4 + 159541866 w^6 + 116463663 w^8 + 5838152 w^{10} + 20911122 w^{12} + 5489100 w^{14} + 1072278 w^{16} + 154176 w^{18} + 15729 w^{20} + 1050 w^{22} + 35 w^{24} \right), \]
\[ g_4^+(w) = 3 \left( -1 + w^2 \right)^2 \left( 3 + w^2 \right) \left( 141372 + 402732 w^2 + 558819 w^4 + 432297 w^6 + 209331 w^8 + 71361 w^{10} + 16497 w^{12} + 2403 w^{14} + 189 w^{16} + 7 w^{18} \right); \]
\[ f_4^-(w) = 8 \left( 326559519 + 1822652766 w^2 + 4648210677 w^4 + 6998194368 w^6 + 7025103459 w^8 + 5035679226 w^{10} + 2678780673 w^{12} + 1084740444 w^{14} + 341288829 w^{16} + 84427122 w^{18} + 16389951 w^{20} + 2449224 w^{22} + 272257 w^{24} + 21430 w^{26} + 1099 w^{28} + 28 w^{30} \right), \]
\[ g_4^-(w) = \left( 3 + w^2 \right)^2 \left( 334968777 + 2143174869 w^2 + 5776302213 w^4 + 8923510233 w^6 + 8999893881 w^8 + 6350646645 w^{10} + 3281293773 w^{12} + 1278719217 w^{14} + 383574771 w^{16} + 8968431 w^{18} + 16510551 w^{20} + 2388627 w^{22} + 267339 w^{24} + 22239 w^{26} + 1239 w^{28} + 35 w^{30} \right). \]
\end{itemize}

This completes the list of $\Lambda_m^\pm$ that were explicitly computed from the equivariant instanton bundles $E_m$ of Section 3.1 and Appendix 3.2. It is possible to compute $Q^{\pm m}\Lambda_0^\pm$ for any $m$, but the equality $\Lambda_m^\pm = Q^{\pm m}\Lambda_0^\pm$ of Theorem 1.2 has been established only for $m \leq 4$ and remains conjectural for $m \geq 5$. If this equality does hold for all $m$, then any $\Lambda_m^\pm$ could be recursively computed, and for example $\Lambda_3^+$ would be given by eqn. 4.4 with:
\[ f^+_5 = 6 \left( 3 + w^2 \right)^2 \left( 4921440381 + 37977143490w^2 + 127613420649w^4 + 250673770776w^6 \\
+ 327148723176w^8 + 304141893048w^{10} + 210622703024w^{12} + 112091223944w^{14} \\
+ 6489439598w^{16} + 15666181052w^{18} + 4231083002w^{20} + 931314344w^{22} \\
+ 167841056w^{24} + 24669272w^{26} + 2918360w^{28} + 271032w^{30} + 18849w^{32} + 882w^{34} \\
+ 21w^{36} \right), \]
\[ g^+_5 = (-1 + w^2)^2 \left( 6947915832 + 44000040942w^2 + 133368411033w^4 \\
+ 248155844508w^6 + 316015211160w^8 + 294283529028w^{10} + 208710837720w^{12} \\
+ 11564336732w^{14} + 50998415472w^{16} + 18167192624w^{18} + 5280068058w^{20} \\
+ 1254027252w^{22} + 241371320w^{24} + 36993180w^{26} + 4399008w^{28} + 391700w^{30} \\
+ 24840w^{32} + 1026w^{34} + 21w^{36} \right). \]

4.2. **The Explicit Okamoto Transformations.** The diagram of eqn. ?? summarizes the constructions of this section:

\[
\begin{array}{cccccc}
  & B & \leftrightarrow & \Lambda^+_0 & \rightarrow & \Lambda^+_1 \\
\Lambda^-_0 & \downarrow & R_5 & \downarrow & R_5 & B \\
  & \Lambda^-_1 & \rightarrow & \Lambda^+_2 & \rightarrow & \Lambda^+_3 \\
  & B & \rightarrow & R_5 & \rightarrow & B \\
  & \Lambda^-_2 & \rightarrow & \Lambda^+_4 & \rightarrow & \Lambda^+_3 \\
\Lambda^-_3 & \downarrow & R_5 & \downarrow & R_5 & \downarrow \\
  & \Lambda^-_4 & \rightarrow & \Lambda^+_4 & \rightarrow & \\
\end{array}
\]

(4.10)

Each double-headed arrow \( \leftrightarrow \) denotes an involutive Okamoto transformation between the Painlevé solutions \( \Lambda^\pm \), meaning that each of the self-compositions \( B^2 = B \) and \( R_5^2 = R_5 \) acts as the identity transformation on \( \Lambda_m^\pm \). The Okamoto transformation of Theorem 1.2 are then given by the compositions:

\[ Q := B R_5, \quad Q^{-1} := R_5 B. \] (4.11)

We start with a general discussion of Okamoto transformations, largely following Boalch [6, 8]. For the description of the parameters \( \theta \in \mathbb{C}^4 \), we had already defined the parameter vector \( \mu = (1, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}) \) and we further define the standard vectors:

\[ \varepsilon_1 = (1, 0, 0, 0), \quad \varepsilon_2 = (0, 1, 0, 0), \quad \varepsilon_3 = (0, 0, 1, 0), \quad \varepsilon_4 = (0, 0, 0, 1). \]

Note that \( \mu \cdot \mu = 1 \) where the ordinary dot product on \( \mathbb{R}^4 \) extends to a \( \mathbb{C} \)-bilinear form on \( \mathbb{C}^4 \), and that the complex reflection \( \theta \mapsto \theta - 2(\mu \cdot \theta) \mu \) appearing below is the complexification of an ordinary reflection through the hyperplane \( \mu \cdot \theta = 0 \) in \( \mathbb{R}^4 \). We can now introduce the fundamental Okamoto transformation \( R_5 \):

**Proposition 4.2.** ([51]) Suppose \( \Lambda = [\lambda(t); \theta] \) solves \( P_{V1} \). Then \( R_5 \Lambda \) as defined below also solves \( P_{V1} \):

\[
R_5 \Lambda := \left[ \frac{\varepsilon_3 \cdot \lambda(t) \varepsilon_3}{\bar{\lambda}(t)} + \frac{2 (\mu \cdot \theta)}{\lambda(t) (1-\varepsilon_3 \cdot \mu)} + \frac{1 \lambda(t) + \varepsilon_3 \cdot \mu}{\lambda(t) (1-\varepsilon_3 \cdot \mu)}; \theta - 2(\mu \cdot \theta) \mu \right].
\]
Here $\lambda'(t)$ denotes the derivative of $\lambda(t)$ with respect to $t$. It is an important, but not immediately obvious, fact that $R_5R_5\Lambda = \Lambda$ when the left side is well-defined. This includes all the $P_{VI}$ solutions $\Lambda = \Lambda^\pm_m$ of this paper (which are not Riccati solutions, e.g., [68]).

The following Proposition is only stated and proved only for the finite set of cases $0 \leq m \leq 4$ because the proof is based on computations with the explicit solutions exhibited in section 4.1. However, it appears reasonable to conjecture that the Proposition holds more generally for each nonnegative integer $m$:

**Proposition 4.3.** For each nonnegative integer $m \leq 4$:

$$R_5\Lambda^\pm_m = \Lambda^\mp_m.$$

**Proof.** Referring to Theorem 4.1, $\Lambda^\pm_m = [\lambda^\pm_m, \pm (2m + 1)\mu]$ where $\lambda^\pm_m := \lambda^\pm_m(w)$ is a rational function of $w$. The derivative $(\lambda^\pm_m)^\prime$ with respect to $t$ is then computed implicitly as in eqn. 4.5 and is also expressed as a rational function of $w$. From Proposition 4.2 we have:

$$R_5\left[\lambda^+_m, 2m + 1\right] = \begin{bmatrix} \lambda^+_m + \frac{2(2m + 1)}{(t - 1)(\lambda^+_m)^\prime - \frac{1}{2}(2m + 1)} \frac{1}{(\lambda^+_m)^\prime - 1 - \frac{1}{2}(2m + 1)} - \frac{t(\lambda^+_m)^\prime + \frac{1}{2}(2m + 1)}{\lambda^+_m - 1}; - (2m + 1)\mu \end{bmatrix}.$$  

To prove $R_5\Lambda^+_m = \Lambda^-_m$ it remains to check that

$$\lambda^-_m = \lambda^+_m + \frac{2(2m + 1)}{(t - 1)(\lambda^+_m)^\prime - \frac{1}{2}(2m + 1)} \frac{1}{(\lambda^+_m)^\prime - 1 - \frac{1}{2}(2m + 1)} - \frac{t(\lambda^+_m)^\prime + \frac{1}{2}(2m + 1)}{\lambda^+_m - 1},$$

which was verified by explicit computation for each $0 \leq m \leq 4$ using the solutions listed in 4.1. Now applying $R_5$ to each side yields $R_5R_5\Lambda^+_m = R_5\Lambda^-_m$, which establishes $R_5\Lambda^-_m = \Lambda^+_m$ for each $0 \leq m \leq 4$ since $R_5$ is involutive. 

The group of Okamoto transformations is generated by $R_5$ together with some obvious symmetries of the Painlevé VI equation eqn.4.1. Since $\varepsilon_j \cdot \varepsilon_j = 1$ for each $1 \leq j \leq 4$, each of the complex reflections of $\mathbb{C}^4$ appearing below is the complexification of an ordinary reflection through a hyperplane in $\mathbb{R}^4$.

**Proposition 4.4.** Suppose $\Lambda = [\lambda(t); \theta]$ solves $P_{VI}$. Then each of $R_1\Lambda$, $R_2\Lambda$, $R_3\Lambda$, and $R_4\Lambda$ as defined below also solves $P_{VI}$:

1. $R_1\Lambda := [\lambda(t); \theta - 2(\varepsilon_1 \cdot \theta)\varepsilon_1]$.
2. $R_2\Lambda := [\lambda(t); \theta - 2(\varepsilon_2 \cdot \theta)\varepsilon_2]$.
3. $R_3\Lambda := [\lambda(t); \theta - 2(\varepsilon_3 \cdot \theta)\varepsilon_3]$.
4. $R_4\Lambda := [\lambda(t); \theta - 2(\varepsilon_4 \cdot \theta - 1)\varepsilon_4]$.

**Proof.** In each case the function $\lambda(t)$ is unchanged, while the parameter $\theta \in \mathbb{C}^4$ is changed. But in each case the change of parameter $\theta$ does not change the differential equation eqn.4.1

1. $P_{VI}(\theta - 2(\varepsilon_1 \cdot \theta)\varepsilon_1) = P_{VI}(\theta)$ because $\theta - 2(\varepsilon_1 \cdot \theta)\varepsilon_1 = (-\theta_1, \theta_2, \theta_3, \theta_4)$ and $(-\theta_1)^2 = \theta_1^2$. 

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Okamoto observed that the group of affine automorphisms of $\mathbb{C}^4$ generated by the complex reflections of $R_1, \ldots, R_5$ is the affine Weyl group of type $D_4$. Note the complex reflections $R_1, R_2$ and $R_3$ preserve the origin, but $R_4$ does not since it reflects the affine plane $\varepsilon_4 \cdot \theta - 1 = 0$ does not pass through the origin.

We can now specify the Okamoto transformation $B$ in the diagram eqn. (4.10)

$$B := (R_1 R_2 R_3 R_5) R_4 (R_5 R_3 R_2 R_1).$$

It is easily seen that $B$ is involutive, since each $R_j$ is involutive. The following Proposition is only stated and proved for the finite set of case $0 \leq m \leq 3$ because the proof is based on computations with the explicit solutions exhibited in section 4.1.

However, it is reasonable to conjecture that the Proposition holds more generally for each nonnegative integer $m$:

**Proposition 4.5.** For each nonnegative integer $m \leq 3$:

$$B\Lambda_m^- = \Lambda_{m+1}^+.$$  \hspace{1cm} (4.12)

**Proof.** Applying $(R_5 R_3 R_2 R_1)$ to both sides of eqn. (4.12) yields the equivalent

$$R_4(R_5 R_3 R_2 R_1)\Lambda_m^- = (R_5 R_3 R_2 R_1)\Lambda_{m+1}^+.$$  \hspace{1cm} (4.13)

The left side of eqn. (4.13) expands to

$$R_4(R_5 R_3 R_2 R_1)\Lambda_m^- = R_4(R_5 R_3 R_2 R_1) \left[ \lambda_m^-; -(2m + 1) \mu \right]$$

$$= \left[ \lambda_m^- + \frac{(2m + 1)}{(t-1)(\lambda_m^-)^{-1}+(2m+1)} + \frac{(\lambda_m^-)^{-1}-(2m+1)}{\lambda_m^- t} - \frac{t(\lambda_m^-)^{-1}+(2m+1)}{\lambda_m^-} \right] (2m + 3)\varepsilon_4$$

and the right side expands to

$$(R_5 R_3 R_2 R_1)\Lambda_{m+1}^+ = (R_5 R_3 R_2 R_1) \left[ \lambda_{m+1}^+; (2m + 3) \mu \right]$$

$$= \left[ \lambda_{m+1}^+ + \frac{-(2m + 3)}{(t-1)(\lambda_{m+1}^+)^{-1}+(2m+3)} + \frac{(\lambda_{m+1}^+)^{-1}+(2m+3)}{\lambda_{m+1}^+ t} - \frac{t(\lambda_{m+1}^+)^{-1}+(2m+3)}{\lambda_{m+1}^+} \right] (2m + 3)\varepsilon_4.$$
To prove $B\Lambda_m^- = \Lambda_{m+1}^+$ it remains to check that

$$
\lambda_m^+ + \frac{(2m + 1)}{(t-1)(\lambda_m^-)^{t-\frac{1}{2}}(2m+1) - \frac{t(t+1)(2m+1)}{\lambda_{m+1}^+}} - \frac{(2m + 3)}{(t-1)(\lambda_{m+1}^+)^{t-\frac{1}{2}}(2m+3) + \frac{t(t+1)(2m+1)}{\lambda_{m+1}^+}}
$$

which was verified by explicit computation for each $0 \leq m \leq 3$ using the solutions listed in Section 4.1.

The proof of Theorem 1.2 now follows from Propositions 4.3 and 4.5 applied to the Okamoto transformations $Q := BR_5$ and $Q^{-1} := R_5B$ introduced in eqn. 4.11.

Note that $QQ^{-1}\Lambda = \Lambda = Q^{-1}QA$ as suggested by the notation, since $R_5$ and $B$ are both involutive.

5. Isomonodromic Deformations via Hitchin’s Logarithmic Connection

5.1. On Trivial Equivariant Vector Bundles. In this section, we generalize Hitchin’s method of constructing isomonodromic deformations and associated solutions of $P_{\Gamma}$. Hitchin constructs a natural meromorphic connection on a trivial vector bundle over an “equivariant compactification” $Z$ of a quotient $SL_2(\mathbb{C})/\Gamma$ by a finite subgroup $\Gamma$. The monodromy of the pullback connection on a rational curve in $Z$ is then invariant under deformations of the rational curve [28, 27]. Hitchin applies a result of Jimbo-Miwa [37] to extract $P_{\Gamma}$ solutions corresponding to such isomonodromic deformation. Following an observation of Bor [10], we describe the generalization of Hitchin’s construction to nontrivial vector bundles over $Z$ which admit lifts of the $SL_2(\mathbb{C})$ action. We refer to Malgrange [43] for the general theory of isomonodromic deformations.

We first review Hitchin’s construction of a flat logarithmic, meaning meromorphic with logarithmic singularities, connection on an equivariant compactification. Let $Z$ be a compact three-dimensional complex manifold on which $SL_2(\mathbb{C})$ acts holomorphically, let $Y$ be a (possibly singular) $SL_2(\mathbb{C})$-invariant hypersurface in $Z$, and suppose that $Z - Y$ is an $SL_2(\mathbb{C})$ orbit with stabilizer conjugate to a finite subgroup $\Gamma \subset SL_2(\mathbb{C})$. Then $Z$ is called an equivariant compactification of $SL_2(\mathbb{C})/\Gamma$. The linearization of the $SL_2(\mathbb{C})$ action on $Z$ defines a holomorphic vector-bundle map

$$
\alpha : Z \times sl_2(\mathbb{C}) \rightarrow TZ,
$$

(5.1)
where $sl_2(\mathbb{C})$ is the Lie algebra of $SL_2(\mathbb{C})$, and $TZ$ is the tangent bundle of $Z$. For a point $q \in Z$, the restriction to fibers over $q$ is a linear map $\alpha_q : sl_2(\mathbb{C}) \to T_qZ$, which is invertible if and only if $q \in Z - Y$. It follows that inverse vector bundle map

$$\alpha^{-1} : TZ \to Z \times sl_2(\mathbb{C}),$$

cannot be holomorphic since it is singular on $Y$, in fact $\alpha^{-1}$ is logarithmic. Note that $\alpha^{-1}$ may be thought of as a meromorphic $sl_2(\mathbb{C})$-valued one-form on $Z$. Let $V$ be an $SL_2(\mathbb{C})$-representation. The product $SL_2(\mathbb{C})$ action on $E := Z \times V$ is a lift of the $SL_2(\mathbb{C})$ action on $Z$ to the trivial vector bundle $\pi : E \to Z$, meaning that $SL_2(\mathbb{C})$ acts on $E$ by vector-bundle morphisms such that the induced action on the base space $Z$ coincides with the original action on $Z$. Identifying a section $f$ of the trivial bundle $E = Z \times V$ with the corresponding map $Z \to V$, we can define Hitchin’s meromorphic connection by

$$\nabla f := df - [\alpha^{-1}, f], \quad (5.2)$$

where $[\cdot, \cdot] : sl_2(\mathbb{C}) \times V \to V$ denotes the linearization of the representation $SL_2(\mathbb{C}) \times V \to V$.

5.2. **On General Equivariant Vector Bundles.** We now generalize the construction of $\nabla$ to lifts of an $SL_2(\mathbb{C})$ action on $Z$ to any vector bundle $\pi : E \to Z$, without assuming that $E$ is a trivial bundle $Z \times V$. In general, a lift of the $SL_2(\mathbb{C})$ action to $E$ is an $SL_2(\mathbb{C})$ action on $E$ by vector-bundle morphisms such that the induced action on the base space $Z$ coincides with the original action on $Z$. Let $X \in T_qZ$ be a tangent vector at $q \in Z - Y$, and let $f$ be a section of $E$. Define $\nabla$ on $E$ by:

$$(\nabla_X f)(q) := \lim_{t \to 0} \frac{f(\exp(t \alpha_q^{-1}(X)) \cdot q) - \exp(t \alpha_q^{-1}(X)) \cdot f(q)}{t}, \quad (5.3)$$

where the one-parameter subgroup $\exp(t \alpha_q^{-1}(X)) \subset SL_2(\mathbb{C})$ acts on $q \in Z$ in the first term, and on $f(q) \in E$ in the second term. Since $SL_2(\mathbb{C})$ acts holomorphically on $E$ and $Z$, and since $\alpha^{-1}$ is meromorphic on $Z$, $\nabla$ defines a meromorphic connection on $Z$. The restriction of $\nabla$ to $Z - Y = SL_2(\mathbb{C})/\Gamma$ is a flat holomorphic connection. To verify this, note that $\Gamma$ is finite and that the pullback of $\nabla$ by the quotient map the quotient map $SL_2(\mathbb{C}) \to SL_2(\mathbb{C})/\Gamma$ is a flat holomorphic connection on the trivial bundle $SL_2(\mathbb{C}) \times V$.

It is easy to verify that the connection eqn.$(5.3)$ coincides with Hitchin’s connection eqn.$(2.3)$ when $E$ is trivial. On $E = Z \times V$, identify as before the section $f$ of $E$ with the corresponding map $Z \to V$, and compute

$$(\nabla_X f)(q) = \lim_{t \to 0} \frac{f(\exp(t \alpha_q^{-1}(X)) \cdot q) - \exp(t \alpha_q^{-1}(X)) \cdot f(q)}{t}$$

$$= \lim_{t \to 0} \frac{f(\exp(t \alpha_q^{-1}(X)) \cdot q) - (f(q) + t [\alpha_q^{-1}(X), f(q)] + O(t^2))}{t}$$

$$= (L_X f)(q) - [\alpha_q^{-1}(X), f(q)].$$
If $V$ is a tangent vector field, $\nabla_V f := i_V(\nabla f)$, and applying the contraction $i_V$ to eqn. 5.2 gives
\[ \nabla_V f = L_V f - [\alpha^{-1}(V), f], \]
equating the equivalence of the two connections on a trivial bundle.

The restriction of $\nabla$ to $Z - Y$ is flat, so it defines a representation of the fundamental group of $Z - Y$. More precisely, fixing a basepoint $q \in Z - Y$, the holonomy (monodromy) along a path that starts and ends at $q$ depends only on the homotopy class of the path, defining the monodromy representation $h(q)$. The key observation, Hitchin’s construction of isomonodromic deformations from the connection $\nabla$ carries over essentially without change to nontrivial bundles $E \to Z$. The basic idea is to pull back the connection $\nabla$ to a rational curve in $Z$, and to consider continuous deformations of the rational curve. A holomorphic map $\kappa : \mathbb{C}P^1 \to Z$ will be called a parametrized rational curve, and the image $\kappa(\mathbb{C}P^1) \subset Z$ will be called the underlying (unparametrized) rational curve. A curve will be called transverse if it is nonsingular and it intersects the hypersurface $Y$ transversely. Choose a basepoint $p \in \mathbb{C}P^1$ such that $\kappa(p) \in Z - Y$. The preimage $\kappa^{-1}(Y) = \{a_1, a_2, \ldots, a_n\}$ is a finite subset of $\mathbb{C}P^1$ which does not contain $p$. The pullback $\kappa^* \nabla$ is a meromorphic connection on the bundle $\kappa^* E \to \mathbb{C}P^1$. The meromorphic connection is logarithmic if the pullback to any transverse curve has a simple pole, and we will see the simple poles explicitly in our computations. The restriction of $\kappa^* \nabla$ to $\mathbb{C}P^1 - \{a_1, \ldots, a_n\}$ is a flat holomorphic connection with monodromy representation
\[ h(\kappa^* \nabla, p) : \pi_1(\mathbb{C}P^1 - \{a_1, \ldots, a_n\}, p) \to GL((\kappa^* E)_p) := GL(E_{\kappa(p)}). \]

By functoriality, the monodromy representation of $\kappa^* \nabla$ factors through the monodromy representation of $\nabla$,
\[ h(\kappa^* \nabla, p) = h(\nabla, \kappa(p)) \circ \kappa_*, \]
where $\kappa_*$ is the homomorphism of fundamental groups induced by $\kappa$. A continuous transversality-preserving deformation of a transverse parametrized rational curve $\kappa$ results in the points $\{p, a_1, a_2, \ldots, a_n\}$ moving on $\mathbb{C}P^1$ while remaining distinct. Such a deformation of $\kappa$ induces a continuous deformation of $\kappa_*$, and of $h(\nabla, \kappa(p))$, and consequently of the monodromy representation $h(\kappa^* \nabla, p)$. The key observation, [27], Prop. 6, is that the deformation preserves the isomorphism classes of $\kappa_*$ and $h(\kappa(p))$, and therefore the isomorphism class of the monodromy representation is preserved under the deformation of the connection $\kappa^* \nabla$. This is in essence the defining property of an isomonodromic deformation of a meromorphic connection, see [43] details.

If the bundle $E \to Z$ is nontrivial, the isomorphism class of the pullback bundle $\kappa^* E \to \mathbb{C}P^1$ need not be preserved under deformation of the rational curve $\kappa$. The
change of isomorphism type of \( \kappa^*E \) under deformation of \( \kappa \) is associated with the term “jumping line” in the terminology of holomorphic vector bundles [5, 53], and with the terms “\( \tau \)-function” or “\( \tau \)-divisor” in the terminology of isomonodromic deformations [43, 57].

5.3. On the Equivariant Instanton Bundles \( E_m \). In this section, we give an explicit formula for the meromorphic one-form \( \alpha^{-1} \) constructed from the \( SL_2(\mathbb{C}) \) action on the three-dimensional complex projective space \( Z := P(\mathcal{V}_3) \), where \( \mathcal{V}_3 \) is the irreducible four-dimensional representation.

A degree-\( d \) homogeneous polynomial on \( \mathcal{V}_3 \) corresponds to a linear map from the symmetric product \( S^d(\mathcal{V}_3) \) to \( \mathbb{C} \), and an \( SL_2(\mathbb{C}) \)-invariant homogeneous polynomial corresponds to an equivariant map \( S^d(\mathcal{V}_3) \to \mathcal{V}_0 \). From Schur’s lemma and the following decompositions into irreducibles,

\[
S^1(\mathcal{V}_3) \simeq \mathcal{V}_3, \\
S^2(\mathcal{V}_3) \simeq \mathcal{V}_2 \oplus \mathcal{V}_0, \\
S^3(\mathcal{V}_3) \simeq \mathcal{V}_3 \oplus \mathcal{V}_3 \oplus \mathcal{V}_3, \\
S^4(\mathcal{V}_3) \simeq \mathcal{V}_6 \oplus \mathcal{V}_4 \oplus \mathcal{V}_6 \oplus \mathcal{V}_8,
\]

we conclude that the vector space of invariant degree-\( d \) polynomials has dimension 0 if \( 1 \leq d \leq 3 \), and dimension 1 if \( d = 4 \). The degree-4 invariant polynomial \( p \) defined by

\[
p(u) := \langle u^2, u^2 \rangle, \quad u \in \mathcal{V}_3,
\]

is nonzero, as one can check by computing \( p(\mathbb{X}_Y(x + y)) \neq 0 \). Any degree-4 invariant polynomial on \( \mathcal{V}_3 \) is then a scalar multiple of \( p \).

**Lemma 5.1.**

1. By the fundamental theorem of algebra, a degree-3 homogeneous polynomial \( u \in \mathcal{V}_3 \) can be factored as a product \( u = ab \) of degree-1 homogeneous polynomials \( a, b \in \mathcal{V}_1 \), and we define

\[
q(u) := \langle (a, b) \rangle \langle (b, c) \rangle \langle (c, a) \rangle^2.
\]

Then \( q(u) = K_q p(u) \) for some nonzero constant \( K_q \in \mathbb{C} \).

2. Choose a basis \( \{e_1, e_2, e_3, e_4\} \) of \( \mathcal{V}_3 \) and a basis \( \{g_1, g_2, g_3\} \) of \( \mathfrak{sl}_2(\mathbb{C}) \). The one-dimensional representation \( \Lambda^4(\mathcal{V}_3) \simeq \mathcal{V}_0 \) has basis \( e_1 \wedge e_2 \wedge e_3 \wedge e_4 \), and we define \( r(u) \) by means of the equation

\[
[g_1, u] \wedge [g_2, u] \wedge [g_3, u] \wedge u = r(u) e_1 \wedge e_2 \wedge e_3 \wedge e_4.
\]

Then \( r(u) = K_r p(u) \) for some nonzero constant \( K_r \in \mathbb{C} \) (depending on the basis choices).

**Proof.** Since \( q(u) \) and \( r(u) \) are both degree-4 invariant polynomials on \( \mathcal{V}_3 \), they are scalar multiples of \( p(u) \). Comparing \( q(x y (x + y)) \) and \( p(x y (x + y)) \) we conclude that \( K_q = -1/48 \neq 0 \). Choosing the basis \( \{e_1 = x^3, e_2 = x^2 y, e_3 = x y^2, e_4 = y^3\} \) of \( \mathcal{V}_3 \) and the basis \( \{g_1 = g_0(x, y), g_2 = g_+(x, y), g_3 = g_-(x, y)\} \) of \( \mathfrak{sl}_2(\mathbb{C}) \), we compute
Lemma 5.2. Suppose that there exists a unique vector $c \in a \in V$. Then $c$ is given by the formulae

$$c = a + \beta \in sl_2(\mathbb{C}) \oplus \mathbb{C} \text{ such that } [a, u] + \beta u = v.$$  \hspace{1cm} (5.4)

The vector $c = c(u, v) = a(u, v) + \beta(u, v)$ is given by the formulae

$$a(u, v) = \frac{\mathbf{h}(u, v)}{p(u)}, \quad \beta(u, v) = \frac{\sigma(u, v)}{p(u)},$$

where

$$\mathbf{h}(u, v) = \frac{1}{2} \langle \langle u^2, u \rangle_3, v \rangle_2 - \frac{1}{2} \langle \langle u^2, u \rangle_2, v \rangle_3 \in V_2 = sl_2(\mathbb{C}),$$

$$\sigma(u, v) = \langle \langle u^2, u \rangle_3, v \rangle \in V_0 = \mathbb{C}.$$  

Proof. Since $p(u) \neq 0$, Lemma 5.1 part 2 implies that $\{[g_1, u], [g_2, u], [g_3, u], u\}$ is a basis of $V_3$, so $u$ can be expressed uniquely as a linear combination

$$v = c_1 [g_1, u] + c_2 [g_2, u] + c_3 [g_3, u] + c_4 u,$$  \hspace{1cm} (5.5)
with complex coefficients \( c_i = c_i(u, v) \). Defining \( a := c_1 \mathbf{g}_1 + c_2 \mathbf{g}_2 + c_3 \mathbf{g}_3 \) and \( \beta := c_4 \) proves the first assertion.

The functions defined by

\[
\mathbf{h}(u, v) := p(u) a(u, v) = p(u) (c_1(u, v) \mathbf{g}_1 + c_2(u, v) \mathbf{g}_2 + c_3(u, v) \mathbf{g}_3) \in \mathfrak{sl}_2(\mathbb{C}) = \mathcal{V}_2, \\
\sigma(u, v) := p(u) \beta(u, v) = p(u) c_4(u, v) \in \mathbb{C} = \mathcal{V}_0.
\]

are equivariant because eqn.\(^5\) is equivariant and \( p \) is invariant. We now determine the homogeneity properties of these equivariant functions. From eqn.\(^5.5\) and the antisymmetry of the wedge product we have

\[
v \wedge [\mathbf{g}_2, u] \wedge [\mathbf{g}_3, u] \wedge u = c_1(u, v) [\mathbf{g}_1, u] \wedge [\mathbf{g}_2, u] \wedge [\mathbf{g}_3, u] \wedge u, \\
[\mathbf{g}_1, u] \wedge v \wedge [\mathbf{g}_3, u] \wedge u = c_2(u, v) [\mathbf{g}_1, u] \wedge [\mathbf{g}_2, u] \wedge [\mathbf{g}_3, u] \wedge u
\]

and similar expressions involving \( c_3 \) and \( c_4 \) (this is just Cramer’s rule), and using Lemma \(^5.1\) part 2 we obtain

\[
v \wedge [\mathbf{g}_2, u] \wedge [\mathbf{g}_3, u] \wedge u = K_r p(u) c_1(u, v) e_1 \wedge e_2 \wedge e_3 \wedge e_4, \\
[\mathbf{g}_1, u] \wedge v \wedge [\mathbf{g}_3, u] \wedge u = K_r p(u) c_2(u, v) e_1 \wedge e_2 \wedge e_3 \wedge e_4 \quad (5.6)
\]

and similar expressions for \( c_3 \) and \( c_4 \). The left-hand side of each equation eqn.\(^5.6\) is clearly bihomogeneous of bidegree \((3, 1)\) in the variables \((u, v)\), so \( p(u) c_1(u, v) \) must be bihomogeneous of bidegree \((3, 1)\), as must the linear combinations \( \mathbf{h}(u, v) \) and \( \sigma(u, v) \).

To analyze equivariant bihomogeneous maps of bidegree \((3, 1)\) we use the decomposition into irreducibles

\[
S^3(\mathcal{V}_3) \otimes \mathcal{V}_3 \simeq \mathcal{V}_0 \oplus 2 \mathcal{V}_2 \oplus 2 \mathcal{V}_4 \oplus 3 \mathcal{V}_6 \oplus 2 \mathcal{V}_8 \oplus \mathcal{V}_{10} \oplus \mathcal{V}_{12},
\]

and Schur’s lemma. The vector space of equivariant bidegree \((3, 1)\) maps \( \mathcal{V}_3 \times \mathcal{V}_3 \rightarrow \mathcal{V}_0 \) then has dimension one, and one checks that \( \langle \langle u^2, v \rangle \rangle_3 \) is a basis, so \( \sigma(u, v) \) must be scalar multiple. Similarly, the vector space of equivariant bidegree \((3, 1)\) maps \( \mathcal{V}_3 \times \mathcal{V}_3 \rightarrow \mathcal{V}_2 \) has dimension two, and one checks that \( \{ \langle \langle u^2, v \rangle \rangle_3, v \} \) is a basis, so \( \mathbf{h}(u, v) \) must be a linear combination. We omit the calculation of the numerical coefficients. \( \square \)

We identify the tangent bundle \( TP(\mathcal{V}_3) \) with the quotient of \( \mathcal{V}_3^* \times \mathcal{V}_3 \) by the equivalence relation

\[
(u, v) \sim (\lambda u, \lambda v + a u), \quad \lambda \in \mathbb{C}^*, \quad a \in \mathbb{C}, \quad (5.7)
\]

and we denote the quotient map by \( \gamma : \mathcal{V}_3^* \times \mathcal{V}_3 \rightarrow TP(\mathcal{V}_3) \), and an equivalence class by \( \gamma(u, v) \in TP(\mathcal{V}_3) \). The main result of this section is:

**Proposition 5.3.**

\[
\alpha^{-1} : TP(\mathcal{V}_3) \rightarrow P(\mathcal{V}_3) \times \mathfrak{sl}_2(\mathbb{C}) \\
\gamma(u, v) \mapsto (\delta(u), a(u, v))
\]
Proof. The vector bundle morphism eqn.\[5.9\] defined by the linearization of the $SL_2(\mathbb{C})$ action on $Z = P(V_3)$ takes the form
\[
\alpha : P(V_3) \times \mathfrak{sl}_2(\mathbb{C}) \to TP(V_3)
\]
\[
(\delta (u), g) \mapsto \gamma (u, [g, u]).
\] (5.9)
We need to show that the composition of the maps eqn.\[5.9\] and eqn.\[5.8\] is the identity on each fiber over $Z - Y$, which is equivalent to the statement
\[
\alpha \circ \alpha^{-1} : \gamma (u, v) \mapsto \gamma (u, v)
\]
whenever $p(u) \neq 0$. Substituting from eqn.\[5.8\] and eqn.\[5.9\], we obtain the desired result:
\[
\gamma (u, [a(u, v), v]) = \gamma (u, v - \beta (u, v) u)
\]
\[
= \gamma (u, v). \quad \square
\]

6. PVI Solutions via Isomonodromic Deformations

6.1. Jimbo-Miwa. The relationship between $PV_1$ and a certain class of isomonodromic deformations has its origins in the work of R. Fuchs [24] in the early part of this century, we refer to [34] for a modern discussion. Following Hitchin [28, 27], we will use a formula of Jimbo and Miwa [37] to extract Painlevé solutions from the isomonodromic deformations discussed in the previous section.

We start with a lift $E \to Z$ of the $SL_2(\mathbb{C})$ action to a rank-two vector bundle. Consider a parametrized rational curve $\kappa : \mathbb{C}P^1 \to Z$ which satisfies:

1. The underlying rational curve $\kappa(\mathbb{C}P^1)$ intersects the hypersurface $Y$ transversely in $Z$.
2. The pullback bundle $\kappa^*E \to \mathbb{C}P^1$ admits a holomorphic trivialization $\kappa^*E \cong \mathbb{C}P^1 \times V$.
3. The preimage $\kappa^{-1}(Y) = \{0, 1, t, \infty\}$. (Note that $t$ is the cross-ratio of the four points.)

Then express the pullback connection as $\kappa^*\nabla = d + Az$, where $A$ is a meromorphic zero-form taking values in the vector space $End_0(V)$ of traceless endomorphisms of $V$. Now if $A_u \in End_0(V)$ denotes the residue of the meromorphic one-form $Az$ at $u \in \mathbb{C}P^1$, we have
\[
A(z) = \frac{A_0}{z} + \frac{A_1}{z-1} + \frac{A_t}{z-t}
\]
\[
= \left(\frac{A_0 + A_1 + A_t}{z^2} - \frac{((1 + t) A_0 + t A_1 + A_t)}{z(z-1)(z-t)}\right) z + t A_0.
\] (6.1)
and since the sum residues must vanish, the pole at \( z = \infty \) has residue

\[ A_\infty = -(A_0 + A_1 + A_t) \in \text{End}_0(V). \]

Now suppose that the eigenvectors of \( A_\infty \) are \( \{k, -k\} \) with \( k \) real and positive, and let \( \{r^+, r^-\} \) be nonzero eigenvectors \( A_\infty r^\pm = \pm kr^\pm \). It is evident from eqn. [6.2] that there is (generically) exactly one value of \( z \in \mathbb{C} \) such that \( r^\pm \) is an eigenvector of \( A(z) \), which will be denoted as \( \lambda^\pm := z \).

Now consider a (holomorphic) family of parametrized rational curves \( \kappa(w) \) indexed by \( w \) in an open subset of \( \mathbb{C} \), such that every curve in the family satisfies conditions i), ii), and iii) above, and such that the cross-ratio \( t(w) \) is nonconstant. Then expressing the function \( \lambda^\pm(w) \) implicitly as the (generally multi-valued) function \( \lambda^\pm(t) \), we have:

**Proposition 6.1.** (Jimbo and Miwa [37]) The function \( \lambda(t) = \lambda^\pm(t) \) is a solution of the classic Painlevé VI \( P_{VI}(C) \), eqn. [4.2] with parameters

\[ C = (\alpha, \beta, \gamma, \delta) = \left( \frac{1}{2} (\pm 2k - 1)^2, 2 \det A_0, -2 \det A_1, \frac{1}{2} (1 + 4 \det A_t) \right) \in \mathbb{C}^4. \]

We will only need the following special case, translated in terms of the parameters of \( P_{VI}(\theta) \), eqn. [4.1]

**Corollary 6.2.** If each of the four residues \( A_0, A_1, A_t \) and \( A_\infty \) has eigenvalues \( \left\{ \frac{4}{7}, -\frac{4}{7} \right\} \) with \( s \) real and positive, then:

1. \( \lambda^+(t) \) solves \( P_{VI}(\theta) \) with \( \theta = s \mu = \left( \frac{4}{7}, \frac{4}{7}, \frac{4}{7}, \frac{4}{7} \right) \in \mathbb{C}^4 \).
2. \( \lambda^-(t) \) solves \( P_{VI}(\theta) \) with \( \theta = -s \mu = \left( -\frac{4}{7}, -\frac{4}{7}, -\frac{4}{7}, -\frac{4}{7} \right) \in \mathbb{C}^4 \).

With our notation, the conclusion of the Corollary becomes: \( \Lambda^\pm := [\lambda^\pm(t); s \mu] \) solves \( P_{VI} \). The corollary applies to the isomonodromic deformation used to construct the Painlevé solutions \( \Lambda^\pm_m \) of Theorem [4.1]. Here each of the four residues \( A_0, A_1, A_t \) and \( A_\infty \) has eigenvalues \( \pm \frac{4}{7} = \pm \frac{1}{7}(2m + 1) \). We will see this from our computations below when \( m = 0 \), and refer to Manasliiski [45] for \( m > 0 \).

### 6.2. Deforming the Cross-Ratio

We now construct a family of parametrized rational curves \( \kappa(w) : \mathbb{C}P^1 \to P(\mathcal{V}_3) \) which satisfy the necessary conditions for the application of Proposition 6.1. For each \( w \), the preimage \( \kappa^{-1}(w) \) should consist of four points, and since the degree of the hypersurface \( Y \subset P(\mathcal{V}_3) \) is four, \( \kappa(w) \) should be a family of projective lines. The natural framework for studying families of projective lines with nonconstant cross-ratio \( t \) is the Geometric Invariant Theory [50] quotient of the Grassmannian, but for our purposes a more elementary approach suffices.

A pair of linearly independent vectors \( u \) and \( v \) in \( \mathcal{V}_3 \) defines the parametrized projective line

\[ \kappa_{u,v} : \mathbb{C}P^1 \to P(\mathcal{V}_3) \]

\[ z \mapsto \delta(u + zv), \quad (6.2) \]

where \( z \in \mathbb{C} \cup \{\infty\} = \mathbb{C}P^1 \), and \( \delta(u + \infty v) := \delta(v) \).
Lemma 6.3. For all but finitely many \( w \in \mathbb{C}P^1 \), the parametrized projective line
\[
\kappa_{(w)} := \kappa_{\tilde{u}(w), \tilde{v}(w)} : \mathbb{C}P^1 \to P(Y) \text{ defined by the pair of vectors }
\]
\[
\tilde{u}(w) := \frac{(w + 1)}{(w + 3)^3} (x + y)^2 (8x + (w^2 - 1)y), \quad \tilde{v}(w) := x^2 y
\]
satisfies the properties (1)-(3) of section 3, with \( t = t(w) \) given by eqn. 4.3.

Proof. The result follows immediately from the following identity:
\[
p \left( -(w + 3)^3 (\tilde{u}(w) + z \tilde{v}(w)) \right) \\
= -192 (w + 1)^2 (w + 3)^6 z (z - 1) \left( (w - 1)(w + 3)^3 z - (w + 1)(w - 3)^3 \right).
\]
The identity is verified by straightforward computation. \( \square \)

6.3. Residue Computations. In this section we use elementary complex analysis to study the meromorphic one-form \( \kappa_{u,v}^\ast \alpha^{-1} \) on \( \mathbb{C}P^1 \). This plays a central role in the construction of the meromorphic connection \( \kappa_{u,v}^\ast \nabla \) on \( \mathbb{C}P^1 \) which in turn is used to construct the data for Proposition 5.1.

Applying Proposition 6.3 to the parametrized rational curve defined in eqn. 6.2 immediately yields:

Corollary 6.4. The meromorphic \( \mathfrak{sl}_2(\mathbb{C}) \)-valued one-form \( \kappa_{u,v}^\ast \alpha^{-1} \) on \( \mathbb{C}P^1 \) is given by:
\[
\kappa_{u,v}^\ast \alpha^{-1} = a(u + zv, v) dz = \frac{h(u + zv, v)}{p(u + zv)} dz.
\]

Note that if \( p(v) = 0 \), then \( p(u + zv) \) is a degree-three polynomial in the variable \( z \), and compare with eqn. 6.1. To analyze the residues, consider instead the projective line \( \kappa_{u,v} \) with \( p(u) = 0 \), and expand in a Laurent series at \( z = 0 \):

Lemma 6.5. Suppose that \( u = a^2 b \) with \( (a, b) \neq 0 \). Then \( \kappa_{a^2 b, v}^\ast \alpha^{-1} \) has a simple pole at \( z = 0 \), and the residue can be read off from:
\[
a(a^2 b + zv, v) dz = \left( \frac{1}{4} g_0(a, b) z^{-1} + O(1) \right) dz.
\]

Proof. For brevity, we give the proof only for the “generic” case \( \langle v, a^3 \rangle \neq 0 \). Compute the leading term in the Taylor series of the numerator, and simplify:
\[
h(a^2 b + zv, v) = h(a^2 b, v) + O(z) \\
= -8 (a, b)^2 \langle v, a^3 \rangle a b + O(z) \\
= 8 (a, b)^3 \langle v, a^3 \rangle g_0(a, b) + O(z).
\]

The zeroth order Taylor coefficient of the denominator is \( p(a^2 b) \), which vanishes by Lemma 5.1 part 1. Computing the derivative
\[
\frac{d}{dz} p(a^2 b + zv) = \frac{d}{dz} \langle (a^2 b + zv)^2, (a^2 b + zv)^2 \rangle_6 \\
= \langle 2 (a^2 b + zv) v, (a^2 b + zv)^2 \rangle_6 + \langle (a^2 b + zv)^2, 2 (a^2 b + zv) v \rangle_6 \\
= \langle 2 (a^2 b + zv) v, (a^2 b + zv)^2 \rangle + (-1)^6 \langle 2 (a^2 b + zv) v, (a^2 b + zv)^2 \rangle
\]
and evaluating at $z = 0$ gives the first-order Taylor coefficient, so
\[
p(a^2 b + z v) = 4 \langle a^2 b, a^4 b^2 \rangle z + O(z^2) \\
= -32 \langle a, b \rangle^3 \langle v, a^3 \rangle z + O(z^2). \quad \Box
\]

We now analyze the numerator $h(u + z v, v)$ of the expression for $\kappa^*_{u, v} \alpha - 1$ given in Corollary 6.4, compare with the numerator of eqn. 6.1.

Lemma 6.6. For fixed $u$, $a$, and $b$, the expression
\[
\mathfrak{t}(z) := h(u + z a^2 b, a_2^2 b) + 8 \langle a, b \rangle^3 \langle u, a^3 \rangle g_0(a, b) z^2
\]
is a degree-one ($\mathfrak{sl}_2(\mathbb{C})$-valued) polynomial function of the variable $z$.

Proof. The identity $\kappa_{u, v}(z^{-1}) = \kappa_{v, u}(z)$ is immediate from eqn. 6.2, so Corollary 6.4 implies
\[
a(u + z^{-1} v, v) d z^{-1} = a(v + u, v) d z.
\]
(6.3)

Now substituting this into the tautology
\[
0 = \frac{h(u + z^{-1} a^2 b, a_2^2 b)}{p(u + z^{-1} a^2 b)} - a(u + z^{-1} a^2 b, a^2 b)
\]
\[
= z^2 \left( h(u + z^{-1} a^2 b, a_2^2 b) - p(u + z^{-1} a^2 b) a(u + z^{-1} a^2 b, a_2 b) \right) d z^{-1}
\]
and using the homogeneity of $p$,
\[
0 = z^2 h(u + z^{-1} a^2 b, a^2 b) d z^{-1} - z^{-2} p(a^2 b + z u) a(a^2 b + z u, u) d z,
\]
and Lemma 6.5 gives
\[
0 = z^2 h(u + z^{-1} a^2 b, a_2^2 b) d z^{-1}
\]
\[
- z^{-2} \left( -32 \langle a, b \rangle^3 \langle u, a^3 \rangle z + O(z^2) \right) \left( \frac{1}{4} g_0(a, b) z^{-1} + O(1) \right) \left( -z^2 d z^{-1} \right)
\]
\[
= \left( z^2 \mathfrak{t}(z^{-1}) + O(z) \right) d z^{-1}.
\]
Now $z^2 z^{-n} = O(z)$ only if $2 - n \geq 1$, so the polynomial $\mathfrak{t}(z)$ must have degree $n \leq 1$. \quad \Box

For fixed $u$, $a$, and $b$, we write the numerator in terms of the $\mathfrak{sl}_2(\mathbb{C})$ basis eqn. 2.5
\[
h(u + z a^2 b, a_2^2 b) = h_0(z) g_0(a, b) + h_+(z) g_+(a, b) + h_-(z) g_-(a, b),
\]
(6.4)
where the coefficient functions $h_0(z)$, $h_\pm(z)$ depend on the choice of $u$, $a$, and $b$. 
Proposition 6.7.
\[
\langle a, b \rangle^3 h_0(z) = 3 \langle u, a^2 b \rangle^2 \langle u, a b^2 \rangle - 2 \langle u, a^3 \rangle \langle u, a b^2 \rangle^2 - \langle u, a^3 \rangle \langle u, a^2 b \rangle \langle u, b^3 \rangle
\]
\[
+ 2 \langle a, b \rangle^3 (4 \langle u, a^3 \rangle \langle u, a b^2 \rangle - 3 \langle u, a^2 b \rangle^2) z
\]
\[
- 8 \langle a, b \rangle^6 \langle u, a^3 \rangle z^2,
\]
\[
\langle a, b \rangle^3 h_+(z) = 4 \langle u, b^3 \rangle \left( \langle u, a^2 b \rangle^2 - \langle u, a^3 \rangle \langle u, a b^2 \rangle + 2 \langle a, b \rangle^3 \langle u, a^3 \rangle z \right),
\]
\[
\langle a, b \rangle^3 h_-(z) = 2 \langle u, a^3 \rangle \left( \langle u, a^3 \rangle \langle u, b^3 \rangle - \langle u, a^2 b \rangle \langle u, a b^2 \rangle + 2 \langle a, b \rangle^3 \langle u, a^2 b \rangle z \right).
\]

Proof. Applying eqn. 2.4 to Lemma 6.6,
\[
h_0(z) + 8 \langle a, b \rangle^3 \langle u, a^3 \rangle z^2 = - \langle \mathfrak{f}(z), g_0 \rangle = O(z),
\]
\[
h_+(z) = -2 \langle \mathfrak{f}(z), g_- \rangle = O(z),
\]
\[
h_-(z) = -2 \langle \mathfrak{f}(z), g_+ \rangle = O(z).
\]

It remains compute and simplify the zeroth and first-order Taylor coefficients, as in the proof of Lemma 6.5. We omit the details for the sake of brevity. □

6.4. The Example of $\Lambda_0^\pm$ in detail. We now combine the results of previous sections to compute the two solutions $\Lambda_0^+ = [\lambda_0^+(t), \mu]$ and $\Lambda_0^- = [\lambda_0^-(t), -\mu]$ of $P_{\mathcal{V}I}$ arising from the the trivial bundle $E = P(V_3) \times V_1$ with the product $SL_2(\mathbb{C})$ action. The pullback bundle $\kappa_{u,v}^\ast E = \mathbb{CP}^1 \times V_1$ is trivial for any parametrized projective line $\kappa_{u,v} : \mathbb{CP}^1 \rightarrow P(V_3)$, and the pullback connection is described by:

Lemma 6.8.
\[
\kappa_{u,v}^\ast \nabla = d + A dz
\]
where the $End_0(V_1)$-valued zero-form $A$ is given by
\[
A(z) = - [a(u + zv, v), \cdot].
\]

Proof. The pullback of the connection $\nabla$ defined by eqn. 5.2 acting on a section $h$ of $P^1 \times V_1$, is
\[
(\kappa_{u,v}^\ast \nabla)h = dh - [\kappa_{u,v}^\ast a^{-1}, h],
\]
and by Corollary 6.4
\[
(\kappa_{u,v}^\ast \nabla)h = dh - [a(u + zv, v), h] dz. \quad \Box
\]

We now apply this result to the family of projective lines described in Lemma 6.3 and then use the results of Section 7 to calculate the data that enters into the Jimbo-Miwa formula Proposition 6.1 for $P_{\mathcal{V}I}$ solutions (although not strictly in this order).
We may then suppose that \( v = a^2 b \) with \( (a, b) \neq 0 \). From Lemma 6.5 (see also eqn. 6.3) we compute the residue of \( A dz \) at \( z = \infty \) to be the endomorphism
\[
A_\infty = \frac{1}{4} [g_0(a, b), \cdot] : V_1 \to V_1,
\]
and from eqn. 2.6 we see that \( A_\infty \) has eigenvalues \( \{ \frac{1}{4}, -\frac{1}{4} \} \) with corresponding eigenvectors \( \{ r^+ = a, r^- = b \} \). Next we need to solve for \( z = \lambda^\pm \in \mathbb{C} \) such that \( r^\pm \) is an eigenvector of \( A(\lambda^\pm) \).

**Lemma 6.9.** A vector \( r \in V_1 \) is an eigenvector of the linear map \( B : V_1 \to V_1 \) if and only if \( \langle r, Br \rangle = 0 \).

**Proof.** This holds for any two-dimensional vector space with a nondegenerate antisymmetric bilinear form. \( \square \)

So we need to solve for \( z \) the equation
\[
0 = \langle r^\pm, [A(z), r^\pm] \rangle,
\]
which by Lemma 6.8 and the definition of \( a \) (Lemma 5.2) is equivalent to
\[
0 = \langle r^\pm, [h^-((u + za^2 b, a^2 b), r^\pm)] \rangle.
\]
Using the basis expansion eqn. 6.4 and the fact that \( r^\pm \) is an eigenvector of \( [g_0(a, b), \cdot] \), this is equivalent to
\[
0 = h^+(z) f_0^\pm - h^-(z) g_0^\pm, \quad (6.5)
\]
where
\[
f_0^\pm := \langle r^\pm, [g^-((a, b), r^\pm)] \rangle, \\
g_0^\pm := -\langle r^\pm, [g^+(a, b), r^\pm] \rangle.
\]

Now specialize to the parameterized projective line \( \kappa(w) \) of Lemma 6.3. Substituting \( u = \tilde{u}(w) \), \( a = x \), \( b = y \) into the formulas for \( h^\pm(z) \) from Proposition 6.7 yields
\[
h^+(z) = -\frac{96 (w + 1)^3}{(w + 3)^7} 8 \left( (w - 3)^2 + 3 (w - 1) (w + 3) z \right) \\
h^-(z) = \frac{96 (w + 1)^3}{(w + 3)^7} (w - 1) \left( (w - 3)^2 (w + 1) - (w + 3) (3 + w^2) z \right).
\]
Then eqn. 6.5 is solved by \( z = \lambda_0^\pm(w) \), where (compare with eqn. 4.4)
\[
\lambda_0^\pm(w) := \left( \frac{(w - 3)^2}{(w - 1) (w + 3)} \right) \left( \frac{1}{3 + w^2} \right) \left( -1 + w^2 \right) f_0^\pm + 8 g_0^\pm \\
\left( \frac{1}{3 + w^2} \right) f_0^\pm - 24 g_0^\pm. \quad (6.6)
\]
Now $r^+ = a = x$ and $r^- = b = y$, so computing

$$f_0^+ = \langle x, [g_-(x, y), x] \rangle = \langle x, y \rangle = 1,$$

$$g_0^+ = -\langle x, [g_+(x, y), x] \rangle = 0,$$

and substituting into eqn. 6.6 we into we obtain eqn. 4.6 and similarly from

$$f_0^- = \langle y, [g_-(x, y), y] \rangle = 0,$$

$$g_0^- = -\langle y, [g_+(x, y), y] \rangle = -\langle y, x \rangle = 1.$$

we obtain eqn. 4.7. From Proposition 6.1 we conclude that $\Lambda_0^\pm = [\lambda_0^\pm(t); \mp \mu]$ solves $P_{VI}$. Here the parameter $\theta = \pm \mu \in \mathbb{C}^4$ is computed from the residues.

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