ON THE COMBINATORICS OF THE GRAY CYLINDER

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Abstract. This paper develops some combinatorics of the lax Gray cylinder on the cells of \( \Theta \) understood as a full subcategory of the category of strict \( \omega \)-categories. More, we construct a span relating the Cartesian cylinder, the Gray cylinder, and the shift functor.

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OVERVIEW

Berger’s categorical wreath product and the cell categories \( \Theta_n \) and \( \Theta \). As a preliminary we recall how the wreath product of [Berger2] provides an elegant and explicit description of the categories \( \Theta_n \) and \( \Theta \) in terms of the simplex category. We also recall Berger’s description of
strict $n$-categories (resp. strict $\omega$-categories) as an orthogonal subcategory of the category of $n$-cellular sets $\hat{\Theta}$ (resp. cellular sets $\Theta$).

**Remark 0.1.** Throughout this work we will be concerned only with *strict* versions of higher categories. As such, by $n$-category we will mean the strict notion.

**The Gray cylinder.** The Gray tensor product for 2-categories, first defined in [Gray], is the monoidal structure $\otimes$ for the bi-closed structure

$$(\text{oplax }(-, -), \otimes, \text{lax }(-, -))$$

where, for two 2-categories $X$ and $Y$, $\text{oplax } (X, Y)$ is the 2-category:

- whose objects are functors;
- whose 1-morphisms are oplax-natural transformations; and
- whose 2-morphisms are modifications

and similarly $\text{lax } (X, Y)$ is essentially the same but with lax-natural transformations instead of oplax ones. While the Gray tensor product is not well known outside of the category theory community, within it, it is a cornerstone of formal category theory. The Gray (a.k.a. generalized Gray, Crans-Gray, etc.) tensor product for $\omega$-categories fills the analogous role for those entities. In this work we construct a pair of explicit combinatorial descriptions of the lax Gray cylinder over cells $T$ of $\Theta$, that is to say the tensor products $[1] \otimes T$.

**The lax shuffle decomposition.** As first observed in [Berger1], the cartesian product of cells $S$ and $T$ of $\Theta$, taken either in $\hat{\Theta}$ or $\omega\text{-Cat}$, is covered by all of their “shuffled bouquets”, which restrict on the subcategory $\Delta$ in $\Theta$ to the usual shuffles of simplices. For the case of the cartesian cylinder $[1] \times T$ this yields a description of that object as a wide pushout. Indeed, given a cell

$$T = [n]; (T_1, \ldots, T_n)$$

of $\Theta$, the cellular set $[1] \times T$ enjoys the universal property of the colimit below left. The lax shuffle decomposition makes the gluing along copies of $[n]; (T_1, \ldots, T_n) \text{ lax}$ by fattening the faces $[n]; (T_i)$ along which we glue, replacing them with spans

$$[n]; (T_i) \to [n]; (T_{<k}, [1] \otimes T_k, T_{>k}) \leftarrow [n]; (T_i)$$

as in the colimit below right.
Remark 0.2. Importantly the colimit computing the Gray cylinder needs to be performed in \( \omega\text{-Cat} \) and not on \( \hat{\Theta} \) for reasons we discuss in the text.

Steiner’s \( \omega\text{-categories} \) and the correctness of the formula. While the lax shuffle decomposition is a well formed combinatorial construction it remains to shown that the construction describes the Gray cylinder \([1] \otimes (\_\_)\). To do so we show that the lax shuffle decomposition preserve globular sums - certain wide pushouts of globes in \( \omega\text{-Cat} \) - whence to prove that formula correct for the Gray cylinder it suffices to prove that it correctly computes the Gray cylinder over a single globe. To prove this in turn we appeal to Steiner’s theory.

Steiner, in [Steiner1], develops a treatment of strict treatment of \( \omega\text{-categories} \) as directed augmented complexes. A directed augmented complex is a chain complex of abelian groups \( A_{\bullet} \), in the homological (positive degree) convention, together with further data. These further data rigidify the intuitive translation of chain complex \( A_{\bullet} \) into a \( \omega\text{-graph} \) into providing an \( \omega\text{-category} \). Specifically the extra data are used to encode the orientation of cells as “positivity” - this is the directed in directed augmented complexes - and to pick out which elements of

\[1\text{Although this author learned the material from [AraMaltsiniotis]}\]
the 0th group are "real" objects as opposed to merely formal linear combinations - this is the datum of the augmentation. The strength of Steiner’s theory on which we lean is that the lax Gray tensor product of Steiner’s ω-categories is easily written in terms of the tensor product of the underlying chain complexes.

**Shifted product rule.** We also develop a formula for $[1] \otimes T$ as an ω-category enriched category. We define, for any $n \in \mathbb{N}$ and objects $S_1, \ldots, S_n \in \text{Ob}(\Theta)$, the ω-category $P.R. \ (S_1, \ldots, S_n)$ as the colimit

$$
\lim \left\{ \begin{array}{c}
([1] \otimes S_1) \times S_2 \times \cdots \times S_n \\
\{1\} \otimes S_1 \times \text{id} \times \cdots \times \text{id} \\
S_1 \times S_2 \times \cdots \times S_n \\
\vdots \\
S_1 \times \cdots \times S_{n-1} \times S_n \\
\text{id} \times \cdots \times \text{id} \times \{0\} \otimes S_n \\
S_1 \times \cdots \times S_{n-1} \times ([1] \otimes S_n)
\end{array} \right\}
$$

taken in ω-categories. We observe that

$$
P.R. \left( (S_i)_{i \in \langle n \rangle} \right) = \left( ([1] \otimes S_1) \times \cdots \times S_n \right) \bigoplus_{\prod S_i} \cdots \bigoplus_{\prod S_i} \left( S_1 \times \cdots \times ([1] \otimes S_n) \right)
$$

justifying the name of product rule - and use these formula to describe the Hom-categories of $[1] \otimes [n]; (S_1, S_2, \ldots, S_n)$ - hence the (dimensionally) shifted part of the name.

**The Cartesian←Gray→Shift span.** Lastly we describe how the Gray cylinder serves as the apex of a span

$$
[1] \times (\_ \_) \leftrightarrow [1] \otimes (\_ \_) \Rightarrow [1] ; (\_ \_)
$$

We construct this span explicitly, by way of the shuffle decomposition and shifted product rule. We illustrate this span at the 2-simplex [2] in Figure 2.
Remark. While a pleasing construction in its own right, the Cartesian-Gray-shift span, in the forthcoming $\mathbf{Z}$-cats II this span is seen to be and or induce a span of natural weak equivalences for various model structures.
1. Berger’s Categorical Wreath Product and the Cell Categories Θ and Θn

What follows was developed first in [Berger2].⁴

1.1. Segal’s category Γ. Segal’s category Γ is a skeleton, in the sense of a small category of chosen representatives for each isomorphism class, for the opposite category of the category of finite pointed sets.

Definition 1.1. Let $\Gamma$, Segal’s gamma category, be the category specified thus: let

$$\text{Ob}(\Gamma) = \{\langle k \rangle = \{1, \ldots, k\} \mid k \geq 1\} \cup \{\langle 0 \rangle = \emptyset\},$$

and let $\Gamma(\langle n \rangle, \langle m \rangle)$ be defined by the expression

$$\Gamma(\langle n \rangle, \langle m \rangle) = \{\varphi : \langle n \rangle \rightarrow \text{Sub}_{\langle m \rangle} \mid \forall i \neq j \in \langle m \rangle, \varphi(i) \cap \varphi(j) = \emptyset\}$$

where, for any category $A$ and object $a$ thereof, $\text{Sub}_A(a)$ is the category of subobjects of $a$. Define the composition of morphisms in $\Gamma$ by setting

$$\langle \ell \rangle \xrightarrow{\varphi} \langle m \rangle \xrightarrow{\sigma} \langle n \rangle$$

to be the map

$$\sigma \circ \varphi : i \mapsto \bigcup_{j \in \varphi(i)} \sigma(j).$$

Remark 1.2. The equivalence of categories between $\Gamma$ and $\text{FinSet}^{\text{op}}$ is a particularly truncated analogue of the Grothendieck construction - a map of finite pointed sets is replaced with the data of the fibres it parameterizes (see Figure 3).

⁴However our presentation borrows more from [CisinskiMaltsiniotis].
1.2. Berger’s categorical wreath product.

**Definition 1.3.** Let \( A \) and \( B \) be small categories. Given a functor \( G : B \to \Gamma \), we define 
\[ B \int_G A = B \int A \] (with the second notation suppressing the functor \( G \) when the meaning is clear) be the category the objects of which are indexed tuples 
\[ b; (a_1, \ldots, a_m) \]
where:
- \( b \) is an object of \( B \), \( G(b) = \langle m \rangle \); and
- \((a_1, \ldots, a_m)\) describes a function \( G(b) \to \text{Ob}(A) \).

The morphisms of the wreath product \( B \int A \), denoted 
\[ g; f : (a_i)_{i \in G(b)} \to (c_i)_{i \in G(d)} \]
are comprised of a morphism 
\[ g : b \to d \]
of \( B \) and a morphism of \( \hat{A} \), 
\[ f = \left( (f_{ji} : a_i \to c_j)_{j \in G(g(i))} \right)_{i \in G(b)} : \prod_{i \in G(b)} A[a_i] \to \prod_{j \in G(g(i))} A[c_j]. \]

The composition 
\[ b; (a_i)_{i \in G(b)} \xrightarrow{g; f} d; (c_i)_{i \in G(d)} \xrightarrow{r; q} \ell; (k_i)_{i \in G(\ell)} \]
is denoted \( r \circ g; q \circ f \) where the meaning of \( r \circ g \) is clear and 
\[ q \circ f = \left( (q_{jk} \circ f_{ki})_{j \in G(r(g(i)))} \right)_{i \in G(b)} \]
with the values for \( k \in G(d) \) being those unique \( k \) in \( G(g(i)) \) such that \( j \in G(r(k)) \).

**Remark 1.4.** We’ll also use the more explicit notation \( g; (f_1, f_2, \ldots, f_n) \) where doing so simplifies the exposition.

**Example 1.5.** Define the functor \( F : \Delta \to \Gamma \) by setting 
\[ F([n]) = \langle n \rangle \]
and setting for each \( \varphi : [m] \to [n] \), 
\[ F(\varphi) : \langle m \rangle \to \langle n \rangle \]
to be the function 
\[ F(\varphi) : \langle m \rangle \to \text{Sub}_{\text{Set}}(\langle n \rangle) \]
given thus: 
\[ F(\varphi)(i) = \{ j \mid \varphi(i - 1) < j \leq \varphi(i) \}. \]
Remark 1.6. It is not hard to see that the wreath product defines a functor
\[ \int (\_ \downarrow \Gamma) \times \text{Cat} \rightarrow \text{Cat}. \]

1.3. The Categories $\Theta$ and $\Theta_n$.

Definition 1.7. Let
\[ \gamma : \Delta \rightarrow \Delta \int \Delta \]
be the obvious functor extending the assignment $\gamma([n]) = [n];([0], \ldots, [0])$. We define the categories $\Theta_n$ to be the $n^{th}$ wreath product of $\Delta$ with itself, i.e. we set
\[ \Theta_n = \Delta \int \left( \cdots \int \Delta \right). \]

We set $\Theta$ to be the conical colimit\(^3\)
\[ \lim \rightarrow \left\{ \Delta \xrightarrow{\gamma} \Delta \int \Delta \int \cdots \right\}. \]

Remark 1.8. It should be noted that
\[ \Theta \sim \Delta \int \Theta \sim \Delta \int \Delta \int \Theta \sim \cdots \]
so we may denote cells - where cells are the objects of $\Theta$ - in many compatible ways. For example for any $T$ a cell of $\Theta$ we may also write $T = [n];(T_1, \ldots, T_n)$ for some unique $n \in \mathbb{N}$ and unique $T_1, \ldots, T_n$ cells of $\Theta$.

1.4. Globular Sums. The decompositions of the preceding remark terminate in the observation that every cell $T$ of $\Theta$ admits a unique description as a wide pushout of globes. Indeed, for every $T$ of $\Theta$, there exist unique integers $n_0 \geq m_1 \leq n_1 \geq \cdots \leq n_{\ell-1} \geq m_\ell-1 \leq n_\ell$ for which we have a canonical comparison isomorphism
\[ \lim \rightarrow \left\{ \begin{array}{ccc}
\overline{n}_0 & \overline{n}_1 & \overline{n}_\ell \\
\overline{t}^{m_0-m_1} & \overline{s}^{n_1-m_1} & \overline{t}^{m_\ell-1-m_\ell-1} \\
\overline{m}_1 & \overline{s}^{n_1-m_1} & \overline{m}_\ell-1 \\
\end{array} \right\} \sim T. \]

\(^3\)An elementary argument about the projective-canonical and Reedy-canonical model structures on $\text{CAT} (\mathbb{N}, \text{Cat})$ provides that this conical colimit enjoys the universal property of the pseudo-colimit.
Codifying this relation between $T$ and integers $n_0 \geq m_1 \leq n_1 \geq \cdots \leq n_{\ell-1} \geq m_{\ell-1} \leq n_{\ell}$ we write $T = \overline{n_0} \oplus \overline{n_1} \oplus \cdots \oplus \overline{n_{\ell-1}} \oplus \overline{n_{\ell}}$. We will refer to the right-hand expression as a globular sum and the equality as the globular-sum decomposition.
2. The Gray Cylinder $[1] \otimes (\_): \hat{\Theta} \rightarrow \hat{\Theta}$

The lax Gray tensor product $\otimes$ for $\omega$-categories is defined by its enjoyment of the following universal property.\(^4\)

$$\omega\text{-Cat} (B, oplax (A, C)) \sim \rightarrow \omega\text{-Cat} (A \otimes B, C) \sim \rightarrow \omega\text{-Cat} (A, \text{Lax} (B, C))$$

Since for any object $X \in \hat{\Theta}$, the functor

$$X \otimes (\_): \Theta \rightarrow \omega\text{-Cat}$$

is the restriction of a left adjoint endofunctor on $\omega\text{-Cat}$ to the subcategory $\Theta$, and cells admit a canonical globular decomposition, it follows that for any functor $F$, to prove that $F \sim \rightarrow X \otimes (\_)$ it suffices to prove that:

- $F (\overline{n}) \sim \rightarrow X \otimes \overline{n}$ for all $n \in \mathbb{N}$; and
- that $F$ preserves globular sums.

We'll use this observation, with $X = [1]$, as follows:

- in Section 2.1 we provide a recursive formula - a lax version of the shuffle decomposition - assigning to every cell $T$ of $\Theta$, a wide push-out - which we'll denote (in a subtle and local (in scope) abuse of notation) by $[1] \otimes_{LG} T$;
- in Section 2.2 we prove that our formula preserves globular sums;
- in Section 2.3 we prove - by way of comparison to Steiner's treatment of $\omega$-categories - that $[1] \otimes_{LG} \overline{n}$ enjoys the universal property of the lax Gray cylinder on $\overline{n}$ for all $n \in \mathbb{N}$.

From these results it then follows that the lax shuffle decomposition presents the lax Gray cylinder.

2.1. The Lax shuffle decomposition. Recall that the cartesian cylinder admits the so-called shuffle decomposition. The lax Gray tensor product enjoys a similar expression, but the gluing along copies of $[n]; (T_i)$ is made lax.

Remark 2.1. As explained above in what follows we mean, by $[1] \otimes_{LG} T$, a particular colimit and only later will we prove that the colimit $[1] \otimes_{LG} T$ presents the Gray cylinder.

Definition 2.2. We define the function

$$[1] \otimes_{LG} (\_): \text{Ob} (\Theta) \rightarrow \text{Ob} (\omega\text{-Cat})$$

and a family of maps:

$$\{0\} \otimes_{LG} T : T \rightarrow [1] \otimes_{LG} T$$

and

$$\{1\} \otimes_{LG} T : T \rightarrow [1] \otimes_{LG} T$$

recursively as follows.

Set

$$[1] \otimes_{LG} [0] = [1]$$

\(^4\)For specific treatments in other models see [Verity],[CampionKapulkinMaehara], or [AraMaltsiniotis]
and set
\[ \{0\} \otimes [0] = \{0\} : [0] \to [1] = [1] \otimes [0] \]
and
\[ \{1\} \otimes [0] = \{0\} : [0] \to [1] = [1] \otimes [0] . \]

We then define \([1] \otimes [\ell] ; (A_i)\) to be the colimit of the diagram below left, taken in \(\text{Str-}\omega\text{-Cat}\), with the morphisms of that diagram being given in the diagram below right.

\[ \ldots \]

We then set
\[ [\ell] ; (A_i) \xrightarrow{\{0\} \otimes [\ell] ; (A_i)} [1] \otimes [\ell] ; (A_i) \]

to be the inclusion
\[ [\ell] ; (A_i) \xrightarrow{d^\ell ; \text{id} \ldots \text{id}} [\ell + 1] ; ([0] ; A_1, \ldots, A_n) \to [1] \otimes [\ell] ; (A_i) \]

and similarly set
\[ [\ell] ; (A_i) \xrightarrow{\{1\} \otimes [\ell] ; (A_i)} [1] \otimes [\ell] ; (A_i) \]

to be the inclusion
\[ [\ell] ; (A_i) \xrightarrow{d^\ell + 1 ; \text{id} \ldots \text{id}} [\ell + 1] ; (A_1, \ldots, A_n, [0]) \to [1] \otimes [\ell] ; (A_i) . \]

\textbf{Remark 2.3.} An example of the maps \(\{0\} \otimes (\_\_\_)\) and \(\{1\} \otimes (\_\_\_)\) is illustrated in Figure 5. The connection between these formulae and the pasting diagrams for \([1] \times [n] ; (T_1, T_2, \ldots, T_n)\) and \([1] \otimes [n] ; (T_1, T_2, \ldots, T_n)\) can be seen in Figure 6.
Figure 5. Illustration of the endpoint inclusions \( \{0\} \otimes [2] \) and \( \{1\} \otimes [2] \)

Figure 6. \( \Theta \)-graph pasting diagram for \( [1] \times [n] ; (T_1, T_2, \ldots, T_n) \)

Figure 7. \( \Theta \)-2-graph pasting diagram for \( [1] \otimes [n] ; (T_1, T_2, \ldots, T_n) \)
2.2. The Lax shuffle decomposition preserves globular sums.

**Theorem 2.4.** The lax shuffle decomposition, the assignment on objects
\[ [1] \otimes \_ : \text{Ob}(\Theta) \to \text{Ob}(\text{Str-}\omega\text{-Cat}) \]

preserves globular sums in the sense that for any \( \ell, n_0, \ldots, n_\ell, m_1, \ldots, m_\ell \) with \( n_{i-1} \geq m_i \leq n_i \) for all \( 1 \leq i \leq \ell \), the canonical comparison map between the \( \omega \)-categories
\[ [1] \otimes mLG [n_0] \bigoplus [1] \otimes mLG [n_1] \bigoplus \cdots \bigoplus [1] \otimes mLG [n_{\ell-1}] \bigoplus [1] \otimes mLG [n_\ell] \]

and
\[ [1] \otimes mLG \left( \frac{n_0}{m_1} \bigoplus \cdots \bigoplus \frac{n_{\ell-1}}{m_\ell} \right) \]

is an isomorphism.

The proof proceeds by induction. The base case of \( n = 0 \) is treated as Lemma 2.5, whereas the induction is treated as Lemma 2.7. The proofs themselves are long because of numerous large diagrams, though they are not difficult. Indeed the picture which underlies the formalism is rather intuitive. For example the 0-globular case follows from seeing the consideration of two possible decompositions of the pasting diagram for \([1] \otimes [2] ; (A, B)\) (See Figure 8).

We now attend to the promised Lemmata and their proofs.

**Lemma 2.5.** The functor \([1] \otimes \_ : \text{Ob}(\Theta) \to \text{Ob}(\text{Str-}\omega\text{-Cat})\)

preserves 0-globular sums in the following sense: for
\[ [n] ; (A_i) = [1] ; A_1 \oplus \cdots \oplus [1] ; A_n \]

we have that the canonical comparison map
\[ \left( [1] \otimes [1] ; (A_1) \right) \oplus \cdots \oplus \left( [1] \otimes [1] ; (A_n) \right) \rightarrow [1] \otimes [n] ; (A_i) \]
is an isomorphism as indicated.

Proof. We’ll explicitly show that we’ve a canonical isomorphism

$$[1] \otimes [2] \; (A, B) \sim [1] \otimes [1] \; (A) \bigoplus_1 [1] \otimes [1] \; (B)$$

A nearly identical argument provides that canonical map between

$$[1] \otimes [n] \; (A_1, \ldots, A_n) \bigoplus_1 [1] \otimes [m] \; (B_1, \ldots, B_m)$$

and

$$[1] \otimes [n + m] \; (A_1, \ldots, A_n, B_1, \ldots, B_m)$$

is an isomorphism, and the general case \(k\)-ary, instead of binary case, follows therefrom.

We begin with \([1] \otimes [1] \; (A) \bigoplus_1 [1] \otimes [1] \; (A)\). This \(\omega\)-category is but the colimit of the diagram below left. We observe that the colimit of the central span in the diagram is but \([3] \; (A, 0, B)\) so the diagram above has the very-same colimit as the diagram below right.

But this diagram may be partially completed (dotted arrows) into the commutative diagram below left which, by the universal property of the push-out can be further completed (dashed arrows). But we then find that the diagram below left shares it’s colimit with the diagram (solid arrows) below right.
But that above right diagram can be similarly completed (dotted arrows), and partially computed (dashed arrows), thereby sharing a colimit with the diagram

which is but \([1] \otimes [2] ; (A, B)\).

Prior to proving that \([1] \otimes \_\) preserves 1, 2, \ldots-globular sum, we introduce some simplifying notation.
Definition 2.6. Given a cell $T$ of $\Theta$ and some $n \in \mathbb{N}$, let $\pi; T$ denote the cell
\[
[1]; [1]; \cdots; [1]; T
\]

Lemma 2.7. The assignment
\[
[1] \otimes (\_): \text{Ob}(\Theta) \to \text{Ob}(\text{Str-}\omega\text{-Cat})
\]
preserves $n$-globular sums for all $n \geq 1$ in the sense that for any such $n$ and any cells $X, Y, \ldots, Z$ of $\Theta$, the canonical comparison map between
\[
[1] \otimes (\pi; X) \bigoplus [1] \otimes (\pi; Y) \bigoplus \cdots \bigoplus [1] \otimes (\pi; Z)
\]
and
\[
[1] \otimes \left(\pi; X \bigoplus \pi; Y \bigoplus \cdots \bigoplus \pi; Z\right)
\]
is an isomorphism.

Remark 2.8. The sophisticated reader may wonder why there is much left to prove here. Indeed as $\pi; (\_)$ takes 0-globular sums to $n$-globular ones and preserves colimits and $[1] \otimes (\_)$ is defined by way of colimits. We cannot however naively commute one past the other as the choice of diagram over which $[1] \otimes (\_)$ is a colimit is not functorial over all of $\Theta$, instead it is only functorial for the non-full subcategory $[1]; \Theta \hookrightarrow \Theta$. It should be possible for one to continue down this path and arrive at another proof of the lemma, but it is likely more trouble than it is worth.

Proof. As with the previous lemma, we’ll prove the claim for binary sums. The general case of $k$-ary sums merely requires larger diagrams, whence we leave it to the reader.

Let $n \geq 1$ be given and assume that for any cells $X$ and $Y$ of $\Theta$ and any $m \leq n$ that we have canonical isomorphisms
\[
[1] \otimes \left(\overline{m}; X \bigoplus \overline{m}; Y\right) \xleftarrow{\sim} [1] \otimes (\overline{m}; X) \bigoplus [1] \otimes (\overline{m}; Y)
\]
we will prove that, for any cells $Z$ and $W$ of $\Theta$, there is a canonical isomorphism
\[
[1] \otimes \left(\overline{n+1}; Z \bigoplus \overline{n+1}; W\right) \xleftarrow{\sim} [1] \otimes (\overline{n+1}; Z) \bigoplus [1] \otimes (\overline{n+1}; W)
\]
To that end see that the colimit of the diagram below left is $[1] \otimes \left(\overline{n+1}; Z \bigoplus \overline{n+1}; W\right)$, since $\overline{n+1}; Z \bigoplus \overline{n+1}; W = [1]; (\overline{\pi}; (Z \oplus W))$, and the colimit of the diagram below right
is $[1] \otimes (\overline{n+1}; Z) \bigoplus_{[1] \otimes \overline{n+1}} [1] \otimes (\overline{n} + 1; W)$.

The induction step is then completed by observing that the colimits of the rows of the diagram on the right are the entries in the diagram on the left. This is clear for the first, second, fourth, and fifth rows so it suffice to prove it for the third row. Since $[1]; (_{-})$ preserves connected colimits, such as spans, the colimit of the span

$$[1]; \left( \left[ \left[ \left[ 1 \otimes (\overline{n}; Z) \right] \otimes (\overline{n}); \overline{n} \right] \otimes (\overline{n}); W \right] \right)$$

is but $[1]; (_{-})$ of the colimit of the span

$$[1] \otimes (\overline{n}; Z) \leftarrow [1] \otimes (\overline{n}; W)$$

which is to say it is $[1] \otimes (\overline{n}; Z) \oplus [1] \otimes (\overline{n}; W)$ which is but $[1] \otimes (\overline{n}; Z \oplus W)$ by hypothesis. \hfill \Box

2.3. Steiner’s theory. We now develop enough of Steiner’s theory to use its elegant description of the Gray tensor product to prove the lax Gray shuffle decomposition correct.

2.3.1. Steiner Complexes and their relation to $\omega$-categories. In [Steiner1] (and nicely recovered in [AraMaltsiniotis] whose exposition we largely follow here) we find developed a treatment of $\omega$-categories as chain complexes of abelian groups, in the homological (positive degree) convention, together with further data required to encode the orientation of cells as “positivity”. Of particular utility to us here is that the lax gray tensor product of $\omega$-categories in this treatment is easily written in terms of the tensor product of the underlying chain complexes.

Definition 2.9. A directed augmented complex $(K, K^*, e)$ is comprised of:

- a chain complex of abelian groups $K$,

$$\cdots \xrightarrow{d_{n+1}} K_n \xrightarrow{d_n} K_{n-1} \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_2} K_1 \xrightarrow{d_1} K_0$$
• a set of submonoids

\[ \{ K_n^* \subset K_n \}_{n \in \mathbb{N}} \]

(no compatibility between these submonoids and the differentials of \( K \) is assumed); and

• a morphism of groups \( e : K_0 \to \mathbb{Z} \) such that \( e \circ d_1 = 0 \).

Remark 2.10. The submonoids, which will define “positivity”, encode the direction of cells/group-elements, whereas the map \( e \), commonly known as an “augmentation”, identify the objects, as oppose to the formal sums of objects, as we will see.

A morphism of directed augmented complexes

\[ a : (K, K^*, e) \to (L, L^*, f) \]

is a morphism of augmented chain complexes \( a : (K, e) \to (L, f) \) which respects the positivity sub-monoids, i.e. \( a_n (K_n^*) \subset L_n^* \) for each \( n \in \mathbb{N} \). Let \( \mathcal{C}_{DA} \) denote the category of directed augmented complexes.

Steiner further defines functors

\[ \omega\text{-Cat} \to \mathcal{C}_{DA} \]

and

\[ \mathcal{C}_{DA} \to \omega\text{-Cat} \]

Definition 2.11. Let

\[ \lambda : \omega\text{-Cat} \to \mathcal{C}_{DA} \]

be the functor which sends a \( \omega \)-category \( X \), to the directed augmented chain complex

\[ (\lambda (X), \lambda^* (X^*), e_X) \]

where:

• the abelian groups \( \lambda (X)_n \) are:
  – generated by elements \([x]\) for each \( n \)-cells \( x : \overline{n} \to X \);
  – subject to the minimal relation such that, for any permissible composition of cells \( x, y : \overline{n} \to X \),

\[ x \bigoplus_m y : \overline{n} \bigoplus_m \overline{n} \to X \]

with composition cell

\[ x \star y : \overline{n} \to \overline{n} \bigoplus_m \overline{n} \to X \]

we have

\[ [x \star y] = [x] + [y] \]

• the differentials are generated by setting, for a generating element \([x]\),

\[ d ([x]) = [t (x)] - [s (x)] \]

• the positivity sub-monoids are the sub-monoids generated by those generating elements \([x]\); and

• the augmentation \( e_X \) is the unique map \( \lambda (X)_0 \to \mathbb{Z} \) which sends the generating elements \([x] \in \lambda (X)_0\), corresponding to objects of \( X \), to \( 1 \in \mathbb{Z} \).
Remark 2.12. Note that in our notation the composition cell \( x \star y \) is to be read left to right - not in the usual composition order - it is the composition

\[
s(x) \Rightarrow t(x) = s(y) \Rightarrow t(y)
\]

The action of \( \lambda \) on morphisms is precisely what one would expect:

- given a functor \( a : X \rightarrow Y \), an element \([x] \in \lambda(X)\) is sent by \( \lambda(a) \) to the element \([a(x)] \in \lambda(Y)\).

We define the functor

\[
\nu : \mathcal{C}_{DA} \rightarrow \omega\text{-Cat}
\]

as follows. Given a directed augmented complex \((K, K^*, e)\):

- the \( n \)-morphisms of \( \nu(K, K^*, e) \) are tables of graded group elements

\[
\begin{pmatrix}
x_0^0 & \cdots & x_{i-1}^0 & x_i^0 \\
x_0^1 & \cdots & x_{i-1}^1 & x_i^1
\end{pmatrix}
\]

where:

1. \( x_k^\varepsilon \in K_k \subset K \) for \( \varepsilon = 0, 1 \) and \( 0 \leq k \leq i \);
2. \( d(x_k^\varepsilon) = x_{k-1}^\varepsilon - x_{k-1}^0 \) for \( \varepsilon = 0, 1 \) and \( 0 \leq k \leq i \);
3. \( e(x_0^0) = 1 \) for \( \varepsilon = 0, 1 \); and
4. \( x_i^1 = x_i^0 \)

- the targets and sources of an \( i \)-morphism

\[
\begin{pmatrix}
x_0^0 & \cdots & x_{i-1}^0 & x_i^0 \\
x_0^1 & \cdots & x_{i-1}^1 & x_i^1
\end{pmatrix}
\]

are given:

\[
t \begin{pmatrix}
x_0^0 & \cdots & x_{i-1}^0 & x_i^0 \\
x_0^1 & \cdots & x_{i-1}^1 & x_i^1
\end{pmatrix} = \begin{pmatrix}
x_0^0 & \cdots & x_{i-2}^0 & x_{i-1}^1 \\
x_0^1 & \cdots & x_{i-2}^1 & x_{i-1}^1
\end{pmatrix}
\]

\[
s \begin{pmatrix}
x_0^0 & \cdots & x_{i-1}^0 & x_i^0 \\
x_0^1 & \cdots & x_{i-1}^1 & x_i^1
\end{pmatrix} = \begin{pmatrix}
x_0^0 & \cdots & x_{i-2}^0 & x_{i-1}^0 \\
x_0^1 & \cdots & x_{i-2}^1 & x_{i-1}^0
\end{pmatrix}
\]

- the identity \((i + 1)\)-cell for a table

\[
\begin{pmatrix}
x_0^0 & \cdots & x_{i-1}^0 & x_i^i \\
x_0^1 & \cdots & x_{i-1}^1 & x_i^i
\end{pmatrix}
\]

is the table

\[
\begin{pmatrix}
x_0^0 & \cdots & x_{i-1}^0 & x_i^0 & 0 \\
x_0^1 & \cdots & x_{i-1}^1 & x_i^1 & 0
\end{pmatrix}
\]

- the \( j \leq i \)-composition of \( i \)-cells

\[
\begin{pmatrix}
x_0^0 & \cdots & x_{i-1}^0 & x_i^0 \\
x_0^1 & \cdots & x_{i-1}^1 & x_i^1
\end{pmatrix} \text{ and } \begin{pmatrix}
y_0^0 & \cdots & y_{i-1}^0 & y_i^i \\
y_0^1 & \cdots & y_{i-1}^1 & y_i^i
\end{pmatrix}
\]

is the table

\[
\begin{pmatrix}
x_0^0 & \cdots & x_{j-1}^0 & x_j^0 & x_{j+1}^0 + y_{j+1}^0 & \cdots & x_i^0 + y_i^0 \\
y_0^1 & \cdots & y_j^1 & y_{j+1}^1 & x_{j+1}^1 + y_{j+1}^1 & \cdots & x_i^1 + y_i^1
\end{pmatrix}
\]
Theorem 2.13. (Theorem 2.11 of [Steiner1]) The functor
\[ \lambda : \omega\text{-Cat} \to C_{DA} \]
is left adjoint to the functor
\[ \omega\text{-Cat} \leftarrow C_{DA} : \nu \]

Example 2.14. Consider the \( n \)-globe \( \overline{\pi} \) as an \( \omega \)-category. The directed augmented complex
\( (\lambda(\pi), \lambda^*(\pi), e_{\pi}) \)
is comprised of:
- the chain complex
\[
\cdots \to 0 \to \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z} \to \cdots \to \mathbb{Z} \oplus \mathbb{Z}
\]
with the constituent morphisms
\[
\begin{bmatrix}
1 & -1
\end{bmatrix} : \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}
\]
and
\[
\begin{bmatrix}
1 & -1 \\
1 & -1
\end{bmatrix} : \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}
\]
- the obvious sub-monoids \( N \subset \mathbb{Z} \) and \( N \oplus N \subset \mathbb{Z} \oplus \mathbb{Z} \); and
- the augmentation
\[
\begin{bmatrix}
1 \\
0 & -1
\end{bmatrix} : \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z}
\]
A particularly nice fact about \( \overline{\pi} \) is that \( \overline{\pi} = \nu \circ \lambda(\overline{\pi}) \). Indeed, consider that a non-degenerate \( n \)-cell of \( \nu \circ \lambda(\overline{\pi}) \) is a table
\[
\begin{pmatrix}
(x_0^0, y_0^0) & \cdots & (x_{n-1}^0, y_{n-1}^0) & z_n \\
(x_1^0, y_1^0) & \cdots & (x_{n-1}^1, y_{n-1}^1) & z_n
\end{pmatrix}
\]
where (following Definition 2.11):
1. where:
   (a) \( z_n^0 = z_n^1 \geq 0 \) and
   (b) \( (x^0_0, y^0_0), (x^1_0, y^1_0), \cdots, (x^0_{n-1}, y^0_{n-1}), (x^1_{n-1}, y^1_{n-1}) \in \mathbb{N}^2 \)
2. \( d(z_n) = (x^1_{n-1}, y^1_{n-1}) - (x^0_{n-1}, y^0_{n-1}) \) and \( d(x^\varepsilon_k, y^\varepsilon_k) = x^\varepsilon_{k-1} - x^\varepsilon_k \) for \( \varepsilon = 0, 1 \) and \( 0 \leq k < n \);
3. \( e(x^\varepsilon_0, y^\varepsilon_0) = 1 \) for \( \varepsilon = 0, 1 \).

In light of (3) we find that \( (x^\varepsilon_0, y^\varepsilon_0) \) is either \( (1, 0) \) or \( (0, 1) \) for \( \varepsilon = 0, 1 \), and then in light of (2) and the hypothesis on non-degeneracy we find that \( (x^0_0, y^0_0) = (1, 0) \) and \( (x^1_0, y^1_0) = (0, 1) \). Likewise from (2), for \( 1 \leq k < n \) we may deduce that \( (x^\varepsilon_k, y^\varepsilon_k) \) is either \( (0, 1) \) or \( (1, 0) \), and if the table is to correspond to a non-degenerate cell, \( (x^1_k, y^1_k) = (0, 1) \) and \( (x^0_k, y^0_k) = (1, 0) \). Lastly, since \( d_n \) is the map
\[
\begin{bmatrix}
1 & -1
\end{bmatrix} : \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}
\]
it follows that \( z = 1 \). To wit, the unique non-degenerate \( n \)-cell of \( \nu \circ \lambda(\overline{\pi}) \) is the table
\[
\begin{pmatrix}
(1, 0) & \cdots & (1, 0) & 1 \\
(0, 1) & \cdots & (0, 1) & 1
\end{pmatrix}
\]
Similar arguments demonstrate that the only two non-degenerate $k$-cells, for $0 < k < n$ are
\[
\begin{pmatrix}
(1,0) & \cdots & (1,0) & (1,0) \\
(0,1) & \cdots & (0,1) & (1,0)
\end{pmatrix}
\]
and
\[
\begin{pmatrix}
(1,0) & \cdots & (1,0) & (0,1) \\
(0,1) & \cdots & (0,1) & (0,1)
\end{pmatrix}
\]
and that there are two 0-cells,
\[
\begin{pmatrix}
(1,0) \\
(1,0)
\end{pmatrix}
\]
and
\[
\begin{pmatrix}
(0,1) \\
(0,1)
\end{pmatrix}
\]

2.3.2. (Strong) Steiner complexes, (strong) Steiner $\omega$-categories, Berger’s Wreath product, and the tensor product. The co-unit $\eta : \lambda \circ \nu \Rightarrow \text{id}$ of the adjunction $\lambda \dashv \nu$ (Theorem 2.13) is not in general invertible, i.e. $\nu$ is not full-and-faithful. We can however identify a sub-category of $\text{StC}_{DA} \rightarrow C_{DA}$, the so-called Steiner complexes, on which $\nu$ restricts to a full-and-faithful functor.

**Theorem 2.15.** (Steiner$^5$) For all Steiner complexes $K$, the co-unit
\[
\eta_K : \lambda \circ \nu(K) \rightarrow K
\]
is an isomorphism. In particular, $\nu$ restricted to the full subcategory $\text{StC}_{DA}$ of $C_{DA}$, subtended by the Steiner complexes, is full-and-faithful.

More, on the subcategory of strong Steiner complexes, $\text{StrStC}_{DA}$, the adjunction $\lambda \dashv \nu$ restricts to an equivalence of categories
\[
\text{StrStC}_{DA} \overset{\sim}{\longrightarrow} \text{StrSt}_{\omega}\text{-Cat}
\]
between the subcategory $\text{StrStC}_{DA} \rightarrow \text{StC}_{DA} \rightarrow C_{DA}$ of strong Steiner complexes and the subcategory
\[
\text{StrSt}_{\omega}\text{-Cat} \rightarrow \text{St}_{\omega}\text{-Cat} \rightarrow \omega\text{-Cat}
\]

of strong Steiner $\omega$-categories.

Furthermore the tensor product of chain complexes, together with the obvious choice of positivity sub-monoids, defines a bi-closed monoidal product on $\text{StrStC}_{DA}$ which, under $\nu$, passes to the lax Gray tensor product of $\omega\text{-Cat}$. Indeed the adjunction
\[
\lambda \dashv \nu : \text{StrStC}_{DA} \overset{\sim}{\longrightarrow} \text{StrSt}_{\omega}\text{-Cat}
\]
is a monoidal equivalence of monoidal categories.

**Definition 2.16.** Let $(K, K^*, e)$ and $(L, L^*, f)$ be directed augmented complexes. We recall that $K \otimes L$ is the chain complex with
\[
(K \otimes L)_n = \bigoplus_{i+j=n} K_i \otimes L_j
\]
and differentials
\[
d_n = \bigoplus_{i+j=n} \left( d^K_i \otimes \text{id}_L + (-1)^i \left( \text{id}_K \otimes d^L_j \right) \right)
\]

$^5$as cited in [AraMaltsiniotis], see Theorem 5.6 of [Steiner1] and Paragraph 2.15 of [AraMaltsiniotis]
We extend this to augmented chain complexes by picking the map \( e \otimes f : K_0 \otimes L_0 \to \mathbb{Z} \) as the augmentation.

The promised result regarding the lax Gray tensor product follows.

**Theorem 2.17.** (Steiner - See A.13 and A.14 of [AraMaltziniotis]) For strong Steiner complexes \( K \) and \( L \) and \( \omega \)-categories \( X \) we have isomorphisms

\[
\omega\text{-Cat}(\nu(K), \text{Oplax}^\omega(\nu(L), X)) \xleftrightarrow{\sim} \omega\text{-Cat}(\nu(K \otimes L), X)
\]

and

\[
\omega\text{-Cat}(\nu(K \otimes L), X) \xrightarrow{\sim} \omega\text{-Cat}(\nu(L), \text{Lax}^\omega(\nu(K), X))
\]

We leave the definitions of Steiner complexes and strong Steiner complexes to Appendix A as they are rather long and involved - a Steiner complex is a directed augmented complex which admits a unital loop-free basis, and a strong Steiner complex is a Steiner complex whose unital loop-free basis is strongly so. Fortunate for us however is the fact that we require the theory only to prove that our formula for \([1 \otimes \overline{n}]\) is correct and, as it happens, all the objects of \( \Theta \) are strong Steiner \( \omega \)-categories. Indeed, in [Steiner2] we find the following far stronger claim, which relates Berger’s wreath product to Steiner’s treatment of \( \omega \)-categories.

**Theorem 2.18.** (Theorem 5.6 of [Steiner2]) The functor

\[
\triangle \int C_{DA} \to C_{DA}
\]

restricts to a full-and-faithful functor

\[
\triangle \int \text{StrStC}_{DA} \to \text{StrStC}_{DA}
\]

Equivalently

\[
\triangle \int \text{StrSt}\omega\text{-Cat} \to \text{StrSt}\omega\text{-Cat}
\]

is full-and-faithful.

**Remark 2.19.** To clarify precisely how this result provides the strong Steiner-ness of all of the objects in \( \Theta \), consider that \( \overline{0} \) is strong Steiner, so \([n] : (\overline{0}, \overline{0}, \ldots, \overline{0})\) is strong Steiner by the theorem above for every \( n \in \mathbb{N} \), so all of \( \triangle \) is strong Steiner. Iterating this it follows that all of \( \Theta \) is strong Steiner.
2.4. The lax shuffle decomposition computes the Gray cylinder.

**Proposition 2.20.** For all $n \geq 0$, the diagram

\[
\begin{array}{ccc}
[2] ; (\bar{\pi}, \bar{\nu}) & \xrightarrow{\sim} & \nu(\lambda(\bar{T}) \otimes \lambda(\bar{n} + \bar{1})) \\
[1] ; (\bar{\pi}) & \xrightarrow{\sim} & [1] ; ([1] \otimes \bar{\pi}) \xrightarrow{\sim} \nu(\lambda(\bar{T}) \otimes \lambda(\bar{n} + \bar{1})) \\
[1] ; (\bar{\pi}) & \xrightarrow{\sim} & [2] ; (\bar{0}, \bar{\pi})
\end{array}
\]

is a gluing diagram, meaning:
- the solid morphisms into $\lambda(\bar{T}) \otimes \lambda(\bar{n} + \bar{1})$ are monomorphisms;
- the solid morphisms cover $\lambda(\bar{T}) \otimes \lambda(\bar{n} + \bar{1})$; and
- the two squares are pullbacks.

Moreover, as a consequence, the induced map

\[
\lim_{\rightarrow} \begin{cases}
[2] ; (\pi, 0) \\
[1] ; (\pi) \xrightarrow{\sim} [1] ; ([1] \otimes \pi) \xrightarrow{\sim} \nu(\lambda(\bar{T}) \otimes \lambda(\bar{n} + \bar{1})) \\
[1] ; (\pi) \xrightarrow{\sim} [2] ; (\bar{0}, \pi)
\end{cases}
\]

is an isomorphism (as indicated).

**Proof.** The case for $n \geq 3$ is fully general, the cases for $n = 0, 1, 2$ require minimal modifications. As such, we will show only that

\[
\begin{array}{ccc}
[2] ; (\bar{\pi}, \bar{\nu}) & \xrightarrow{\sim} & \nu(\lambda(\bar{T}) \otimes \lambda(\bar{n} + \bar{1})) \\
[1] ; (\bar{\pi}) & \xrightarrow{\sim} & [1] ; ([1] \otimes \bar{\pi}) \xrightarrow{\sim} \nu(\lambda(\bar{T}) \otimes \lambda(\bar{n} + \bar{1})) \\
[1] ; (\bar{\pi}) & \xrightarrow{\sim} & [2] ; (\bar{0}, \bar{\pi})
\end{array}
\]
is a gluing diagram for all $n \geq 3$.

Observe first that $[2]; (\bar{n}, 0)$, $[1]; ([1] \otimes \bar{n})$, and $[2]; (0, \bar{n})$ are all strong Steiner $\omega$-categories; the first and last as $\Theta$ is a full subcategory of the category of strong Steiner categories (see Remark 2.19) and the middle one since $[1]; (_\circ \circ)$ preserves the strong Steiner-ness of $\omega$-categories (as with Remark 2.19, this a consequence of Theorem 2.18). Thus to prove the diagram above to be a gluing diagram is to prove that

\[
\begin{align*}
\lambda([2]; (\bar{n}, 0)) & \to \lambda([1]; (\bar{n})) \\
\lambda([1]; (\bar{n})) & \to \lambda([1]; ([1] \otimes \bar{n})) \\
\lambda([1]; ([1] \otimes \bar{n})) & \to \lambda(\bar{n}; \bar{n}) \\
\lambda(\bar{n}; \bar{n}) & \to \lambda([2]; (\bar{n}, \bar{n}))
\end{align*}
\]

is a gluing diagram. What’s more, since the adjunction $\nu \dashv \lambda$ restricts to an adjoint equivalence of categories on the strong Steiner subcategories, the diagram above is a gluing diagram if and only if

\[
\begin{align*}
\lambda([2]; (\bar{n}, 0)) & \to \lambda([1]; (\bar{n})) \\
\lambda([1]; (\bar{n})) & \to \lambda([1]; ([1] \otimes \bar{n})) \\
\lambda([1]; ([1] \otimes \bar{n})) & \to \lambda(\bar{n}; \bar{n}) \\
\lambda(\bar{n}; \bar{n}) & \to \lambda([2]; (\bar{n}, \bar{n}))
\end{align*}
\]

is gluing diagram. It is this claim which we will now prove.

Let the copy of $\lambda(\bar{n})$ in $\lambda(\bar{n}) \otimes \lambda(\bar{n}; \bar{n})$ be the obvious directed augmented complex on the chain complex

\[
\begin{align*}
\langle h \rangle & \to \langle \ell \rangle \oplus \langle r \rangle \\
h & \mapsto -\ell + r
\end{align*}
\]
and let the copy of $\lambda (n+1) = \lambda ([1]; (\overline{n}))$ be the obvious directed augmented complex on the chain complex

$$
\begin{align*}
\langle v_{n+1} \rangle &\to \langle b_n \rangle \oplus \langle t_n \rangle \to \langle b_{n-1} \rangle \oplus \langle t_{n-1} \rangle \to \cdots \to \langle b_1 \rangle \oplus \langle t_1 \rangle \to \langle b_0 \rangle \oplus \langle t_0 \rangle \\
v_{n+1} &\to -b_n + t_n \\
b_n, t_n &\to -b_{n-1} + t_{n-1} \\
\cdots &
\end{align*}
$$

Then we find that $\lambda (\overline{1}) \otimes \lambda ([1]; \overline{n})$ is the obvious directed augmented complex on the chain complex

$$
\begin{align*}
\begin{array}{c}
\langle h \otimes v_{n+1} \rangle \\
\langle h \otimes t_n \rangle \\
\langle h \otimes b_n \rangle \\
\langle h \otimes t_{n-1} \rangle \oplus \langle h \otimes t_{n-2} \rangle \oplus \langle r \otimes t_{n-1} \rangle \\
\langle h \otimes b_{n-1} \rangle \oplus \langle h \otimes b_{n-2} \rangle \oplus \langle r \otimes b_{n-1} \rangle \\
\langle h \otimes t_1 \rangle \oplus \langle h \otimes t_0 \rangle \oplus \langle r \otimes t_1 \rangle \\
\langle h \otimes b_1 \rangle \oplus \langle h \otimes b_0 \rangle \oplus \langle r \otimes b_1 \rangle \\
\langle h \otimes t_0 \rangle \oplus \langle r \otimes t_0 \rangle \\
\langle h \otimes b_0 \rangle \oplus \langle r \otimes b_0 \rangle
\end{array}
\end{align*}
$$
Using this description we find that $\lambda([1] ; ([1] \otimes \pi))$ is the obvious directed augmented complex on

\[
\begin{array}{c}
\begin{array}{c}
\langle h \otimes v_m' \rangle \\
\langle h' \otimes t_{m-1} \rangle \\
\langle \ell' \otimes v_m' \rangle \oplus \langle r' \otimes v_m' \rangle \\
\langle h' \otimes b_{m-1}' \rangle \\
\langle \ell' \otimes t_{m-1}' \rangle \oplus \langle h' \otimes t_{m-2}' \rangle \oplus \langle r' \otimes t_{m-1}' \rangle \\
\langle \ell' \otimes b_{m-1}' \rangle \oplus \langle h' \otimes b_{m-2}' \rangle \oplus \langle r' \otimes b_{m-1}' \rangle \\
\langle \ell' \otimes t_{m-2}' \rangle \oplus \langle h' \otimes t_{m-3}' \rangle \oplus \langle r' \otimes t_{m-2}' \rangle \\
\langle \ell' \otimes b_{m-2}' \rangle \oplus \langle h' \otimes b_{m-3}' \rangle \oplus \langle r' \otimes b_{m-2}' \rangle \\
\langle e \rangle \\
\langle s \rangle
\end{array}
\end{array}
\]

and we identify the map $\lambda([1] ; ([1] \otimes \pi)) \longrightarrow \lambda(\overline{1}) \otimes \lambda([1] \otimes \pi)$ with the obvious directed augmented sub-complex on

\[
\begin{array}{c}
\begin{array}{c}
\langle h \otimes v_{n+1} \rangle \\
\langle h \otimes t_n \rangle \\
\langle \ell \otimes v_{n+1} \rangle \oplus \langle r \otimes v_{n+1} \rangle \\
\langle h \otimes b_n \rangle \\
\langle \ell \otimes t_n \rangle \oplus \langle h \otimes t_{n-1} \rangle \oplus \langle r \otimes t_n \rangle \\
\langle \ell \otimes b_n \rangle \oplus \langle h \otimes b_{n-1} \rangle \oplus \langle r \otimes b_n \rangle \\
\langle \ell \otimes t_{n-1} \rangle \oplus \langle h \otimes t_{n-2} \rangle \oplus \langle r \otimes t_{n-1} \rangle \\
\langle \ell \otimes b_{n-1} \rangle \oplus \langle h \otimes b_{n-2} \rangle \oplus \langle r \otimes b_{n-1} \rangle \\
\langle \ell \otimes t_2 \rangle \oplus \langle h \otimes t_1 \rangle \oplus \langle r \otimes t_2 \rangle \\
\langle \ell \otimes b_2 \rangle \oplus \langle h \otimes b_1 \rangle \oplus \langle r \otimes b_2 \rangle \\
\langle \ell \otimes t_1 + h \otimes t_0 \rangle \oplus \langle h \otimes b_0 + r \otimes t_1 \rangle \\
\langle \ell \otimes b_1 + h \otimes t_0 \rangle \oplus \langle h \otimes b_0 + r \otimes b_1 \rangle \\
\langle \ell \otimes b_0 \rangle \oplus \langle h \otimes b_0 \rangle \oplus \langle r \otimes t_0 \rangle
\end{array}
\end{array}
\]
We may similarly identify the maps $\lambda ([2]; (\pi, 0)) \to \lambda (\bar{T}) \otimes \lambda (n + 1)$ and $\lambda ([2]; (0, \pi)) \to \lambda (\bar{T}) \otimes \lambda (n + 1)$ with the obvious directed augmented sub-complexes

$$
\begin{array}{ccc}
n+2 & \langle h \otimes v_{n+1} \rangle & n+1 \langle h \otimes t_n \rangle & \langle r \otimes v_{n+1} \rangle \\
n+1 & \langle h \otimes b_{n} \rangle \oplus \langle r \otimes v_{n+1} \rangle & \langle h \otimes b_{n} \rangle & \langle r \otimes v_{n+1} \rangle \\
n & \langle h \otimes t_{n-1} \rangle \oplus \langle r \otimes t_{n-1} \rangle & \langle h \otimes b_{n-1} \rangle \oplus \langle r \otimes b_{n-1} \rangle & \langle h \otimes b_{n-1} \rangle \oplus \langle r \otimes b_{n-1} \rangle \\
n-1 & \langle h \otimes t_{n-2} \rangle \oplus \langle r \otimes t_{n-2} \rangle & \langle h \otimes b_{n-2} \rangle \oplus \langle r \otimes b_{n-2} \rangle & \langle h \otimes b_{n-2} \rangle \oplus \langle r \otimes b_{n-2} \rangle \\
2 & \langle h \otimes t_{1} \rangle \oplus \langle r \otimes t_{1} \rangle & \langle h \otimes b_{1} \rangle \oplus \langle r \otimes b_{1} \rangle & \langle h \otimes b_{1} \rangle \oplus \langle r \otimes b_{1} \rangle \\
1 & \langle h \otimes t_{0} \rangle \oplus \langle r \otimes t_{0} \rangle & \langle h \otimes b_{0} \rangle \oplus \langle r \otimes b_{0} \rangle & \langle h \otimes b_{0} \rangle \oplus \langle r \otimes b_{0} \rangle \\
0 & \langle h \otimes t_{0} \rangle \oplus \langle r \otimes t_{0} \rangle & \langle h \otimes b_{0} \rangle \oplus \langle r \otimes b_{0} \rangle & \langle h \otimes b_{0} \rangle \oplus \langle r \otimes b_{0} \rangle \\
\end{array}
$$

Now, we have belabored their description so as to make it evident that:

- the maps are monomorphisms;
- the maps cover the target - obvious in degrees strictly greater than 1, requiring a subtraction in degree 1, and obvious again in degree 0; and
- that the requisite squares are indeed pullbacks - this is not hard to see: in terms of generators we simply add $h \otimes t_0$ (resp. $h \otimes b_0$) to the other generators in degree 1 and remove $\langle h \otimes t_0 \rangle$ (resp. $\langle r \otimes b_0 \rangle$) in degree 0).

which concludes the proof.

Then, synthesizing the proposition above, and our work regarding the globular sum preservation, we prove the following.

**Theorem 2.21.** For all cells $T$ of $\Theta$, the canonical morphism $[1] \otimes T \xrightarrow{\sim} [1] \otimes T$ is an isomorphism, as indicated.

**Proof.** Since $[1] \otimes (\_)$ preserves globular sums (see Section 2.2), it suffices to show that $[1] \otimes \pi \xrightarrow{\sim} [1] \otimes \pi$ and this is implied by the proposition above.

2.5. **Shifted Product Rule and a $\omega$-Cat-enriched categorical treatment of the lax shuffle decomposition.** Its not hard to see that the objects of the $\omega$-category $[1] \otimes [n]; (S_i)$, the Gray cylinder on $[n]; (S_i)$, are in bijection with the objects of $[1] \times [n]; (S_i)$, the cartesian
cylinder on \([n] ; (S_i)\). Indeed both are in bijection with the set \(\{0, 1\} \times \{0, 1, \ldots, n\}\). This observation in turn gives us easy access to a description of the cellular sets \([1] \otimes [n] ; (S_i)\) as \(\omega\)-category enriched categories.

**Definition 2.22.** For any \(n \in \mathbb{N}\) and objects \(S_1, \ldots, S_n \in \text{Ob} (\Theta)\), define \(P.R. (S_1, \ldots, S_n)\) to be the colimit

\[
\lim \longrightarrow \begin{cases}
([1] \otimes S_1) \times S_2 \times \cdots \times S_n \\
\{1\} \otimes S_1 \times \text{id} \times \cdots \times \text{id} \\
S_1 \times S_2 \times \cdots \times S_n \\
\vdots \\
S_1 \times \cdots \times S_{n-1} \times S_n \\
\text{id} \times \cdots \times \text{id} \times \{0\} \otimes S_1 \\
S_1 \times \cdots \times S_{n-1} \times ([1] \otimes S_n)
\end{cases}
\]

taken in \(\omega\)-categories.

**Remark 2.23.** The reader should observe that we could well have written the following expression.

\[
P.R. \left( (S_i)_{i \in [n]} \right) = \left( ([1] \otimes S_1) \times \cdots \times S_n \right) \bigoplus_{\Pi S_i} \left( S_1 \times \cdots \times ([1] \otimes S_n) \right)
\]

**Lemma 2.24.** For any \(A_1, A_2, \ldots, A_n\) and \(B_1, B_2, \ldots, B_m\) of \(\Theta\) there is a canonical isomorphism from the colimit

\[
\lim \longrightarrow \begin{cases}
P.R. (A_1, A_2, \ldots, A_n) \times B_1 \times B_2 \times \cdots \times B_m \\
A_1 \times A_2 \times \cdots \times A_n \times B_1 \times B_2 \times \cdots \times B_m \\
A_1 \times A_2 \times \cdots \times A_n \times \text{P.R.} (B_1 \times B_2 \times \cdots \times B_m)
\end{cases}
\]
to the object \(\text{P.R.} (A_1, A_2, \ldots, A_n, B_1, B_2, \ldots, B_m)\).

**Lemma 2.25.** Let \(n \in \mathbb{N}\) and let

\[
\begin{align*}
S_1 &= [s_1] ; (R^1_{s_1}) \\
S_2 &= [s_2] ; (R^2_{s_2}) \\
\vdots
\end{align*}
\]

\[
\begin{align*}
S_{n-1} &= [s_{n-1}] ; (R^{n-1}_{s_{n-1}}) \\
S_n &= [s_n] ; (R^n_{s_n})
\end{align*}
\]

be cells of \(\Theta\). Then, for each \(1 \leq \ell \leq k \leq n\) the \(\omega\)-categories

\[
P.R. \left( (S_i)_{i \in [\ell, k]} \right) = P.R. \left( \left( [s_i] ; (R^i_{s_i}) \right)_{i \in [\ell, k]} \right)
\]
admit the following description as categories enriched in $\omega$-categories.

- on objects we find
  \[
  \text{Ob}
  \left(P.R.\left(\left(S_i\right)_{i \in (\ell,k)}\right)\right)
  = \text{Ob}
  \left(P.R.\left(\left([s_i]\right)_{i \in (\ell,k)}\right)\right)
  \sim \{\ell, \ell + 1, \ldots, k\} \times \prod_{i \in (\ell,k)} \{1, 2, \ldots, s_i\}
  \]

- and the $\text{Hom}_{\omega}$-categories
  \[
  \text{Hom}_{P.R.}\left(\left(S_i\right)_{i \in (\ell,k)}\right)
  \left(\left(x, (z_i)_{i \in (\ell,k)}\right), \left(y, (w_i)_{i \in (\ell,k)}\right)\right)
  \]
  are given by the following formula(e):
  - if $\ell \leq k$ then
    \[
    \text{Hom}_{P.R.}\left(\left(S_i\right)_{i \in (\ell,k)}\right)
    \left(\left(x, (z_i)_{i \in (\ell,k)}\right), \left(y, (w_i)_{i \in (\ell,k)}\right)\right)
    \]
    is the $\omega$-category
    \[
    \left(\prod_{a \in (\ell,x)} \prod_{q \in (z_a, w_a)} R^a_q\right) \times P.R.\left(\left(R^b_q\right)_{q \in (z_b, w_b)}\right) \times \prod_{q \in (y,k)} \prod_{u \in (z_u, w_u)} R^c_q
    \]
  - if $\ell \not\leq k$ then
    \[
    \text{Hom}_{P.R.}\left(\left(S_i\right)_{i \in (\ell,k)}\right)
    \left(\left(x, (z_i)_{i \in (\ell,k)}\right), \left(y, (w_i)_{i \in (\ell,k)}\right)\right)
    = \emptyset
    \]

**Remark 2.26.** See that in the case $n = 1$ this formula computes $[1] \otimes S_1$ as $[1] \otimes S = P.R.\left(S\right)$. We note too that, since we consider only cells of finite height, the recursion in the definition above terminates.

**Proof.** The proof of the first claim comes from consideration of the pasting diagram for $[1] \otimes ([n]; (T_1, T_2, \ldots, T_n))$ given in Figure 6. The second is more complicated, but is 'essentially' the same.

What’s more, this shifted product rule description allows for a completely explicit description of the $\omega$-category enriched functors $[1] \otimes f$ for each $f : A \rightarrow B$ of $\Theta$.

### 2.5.1. The $\omega$-functors $[1] \otimes f : [1] \otimes T \rightarrow [1] \otimes S$ as $\omega$-category enriched functors.

As we’ll see, to characterize the action of $[1] \otimes (\_)$ on the morphisms of $\Theta$, it will suffice to recurse to full subcategory $\Delta f \Delta \subset \Theta$.

Suppose
\[
f : g : [n]; (T_i)_{i \in (n)} \longrightarrow [m]; (S_j)_{j \in (m)}
\]
where
\[
\text{g} = \left(g_{i \rightarrow j}\right)_{j \in F(f)(i)}_{i \in (n)}
\]
with
\[
g_{i \rightarrow j} : T_i \rightarrow S_j,
\]
be a morphism of $\Theta \xrightarrow{\sim} \Delta f \Theta$. Since
\[
\text{Ob}\left([n]; (T_i)_{i \in (n)}\right) \sim \text{Ob}\left([1] \otimes [n]\right) \sim \text{Ob}\left([1] \times [n]\right) \sim \text{Ob}\left([1]\right) \times \text{Ob}\left([n]\right)
\]
and the two functors \([1] \otimes (_)\) and \([1] \times (_)\) coincide on objects, we characterize the \(\omega\)-category enriched functor
\[
[1] \otimes f; g : [1] \otimes [n] ; (T_i)_{i \in \langle n \rangle} \to [1] \otimes [m] ; (S_j)_{j \in \langle m \rangle}
\]
as follows.

We observe that \(\text{Ob} ([1] \otimes f; g)\) is given
\[
\begin{array}{ccc}
\text{Ob} ([1] \otimes [n] ; (T_i)) & \to & \text{Ob} ([1] \otimes [m] ; (S_j)) \\
[1] \times [n] & \overset{id \times f}{\longrightarrow} & [1] \times [m]
\end{array}
\]

The types of the constituent \(\omega\)-functors
\[
\text{Hom}_{[1] \otimes [n] ; (T_i)} ((y, x), (w, z)) \to \text{Hom}_{[1] \otimes [m] ; (S_j)} ((y, f(x)), (w, f(z)))
\]
are then given case-wise by Corollary 2.25, depending on \(\langle y, w \rangle\):

- \(1 \in \langle y, w \rangle\),
  \[
  \begin{array}{ccc}
P.R. \left( (T_i)_{i \in \langle x, z \rangle} \right) & \overset{P.R.(f; g, \langle x, z \rangle)}{\longrightarrow} & \prod_{i \in \langle x, z \rangle} P.R. \left( (S_j)_{j \in F(f)(i)} \right) \\
\end{array}
  \]

- \(1 \notin \langle y, w \rangle = \emptyset\)
  \[
  \prod_{i \in \langle x, z \rangle} (T_i) \overset{\prod_{i \in \langle x, z \rangle} (g_i \rightarrow j)_{j \in F(f)(i)}}{\longrightarrow} \prod_{i \in \langle x, z \rangle, j \in F(f)(i)} S_j
  \]

- \(y \nleq w\)
  \[
  \emptyset \overset{}{\longrightarrow} \emptyset
  \]

with the meaning of \(P.R. (f; g, \langle x, z \rangle)\) given by recursion to the following case.

**Definition 2.27.** Suppose
\[
f; g : [n] ; ([t_i])_{i \in \langle n \rangle} \to [m] ; ([s_j])_{j \in \langle m \rangle}
\]
to be a morphism of \(\triangle \int \triangle\). Then set
\[
P.R. \left( ([t_i])_{i \in \langle x, z \rangle} \right) \overset{P.R.(f; g, \langle x, z \rangle)}{\longrightarrow} \prod_{i \in \langle x, z \rangle} P.R. \left( ([s_j])_{j \in F(f)(i)} \right)
\]
to be the functor given as follows. We set
\[
\begin{array}{ccc}
\text{Ob} \left( P.R. \left( ([t_i])_{i \in \langle x, z \rangle} \right) \right) & \overset{\text{Ob}(P.R.(f; g, \langle x, z \rangle))}{\longrightarrow} & \prod_{i \in \langle x, z \rangle} \text{Ob} \left( P.R. \left( ([s_j])_{j \in F(f)(i)} \right) \right) \\
[x, z] \times \left( \prod_{i \in \langle x, z \rangle} [t_i] \right) & \longrightarrow & \prod_{i \in \langle x, z \rangle} [f(i - 1), f(i)] \times \prod_{j \in F(f)(i)} [s_j]
\end{array}
\]
to be the product of:
• the composite multimorphism
\[
[x, z] \xrightarrow{\prod_{i \in \{x, z\}} [i - 1, i]} \prod_{i \in \{x, z\}} [f(i - 1), f(i)]
\]
\[p < i \quad \xrightarrow{i - 1} \quad p \geq i \quad \xrightarrow{i}
\]
\[(w_i)_{i \in \{x, z\}} \xrightarrow{\prod_{i \in \{x, z\}} (f|_{[i-1,i]}(w_i))_{i \in \{x, z\}}}
\]
which locates a 0–globular sum inside a product; and
• the product of multimorphisms
\[
\prod_{i \in \{x, z\}} (g_{i \rightarrow j})_{j \in F(f)(i)} : \prod_{i \in \{x, z\}} [t_i] \xrightarrow{\prod_{i \in \{x, z\}} (\prod_{j \in F(f)(i)} [s_j])}
\]
and on morphisms we define the constituent functors
\[
\text{Hom} \left( \left([b, (a_i)_{i \in \{x, z\}}], (d, (c_i)_{i \in \{x, z\}})\right) \right)
\]
\[
\prod_{i \in \{x, z\}} \text{Hom} \left( (f|_{[i-1,i]}(b), (g_{i \rightarrow j}(a_i))), (f|_{[i-1,i]}(d), (g_{i \rightarrow j}(c_i))) \right)
\]
by way of the following 2-functor computes to the product
\[
\prod_{i \in \{x, z\}} \left\{ \begin{array}{ll}
\text{P.R.} \left( ([0])_{j \in \{a_i, c_i\}} \right) & i \in \{b, d\} \neq \emptyset \\
\prod_{i \in \{a_i, b_i\}} [0] & i \notin \{b, d\} \neq \emptyset \\
\emptyset & i \notin \{b, d\} = \emptyset
\end{array} \right.
\]
which reduces to
\[
\prod_{i \in \{x, z\}} \left\{ \begin{array}{ll}
[a_i, c_i] & i \in \{b, d\} \neq \emptyset \\
[0] & i \notin \{b, d\} \neq \emptyset \\
\emptyset & i \notin \{b, d\} = \emptyset
\end{array} \right.
\]
Similarly, the target computes to the product
\[
\prod_{i \in \{x, z\}} \prod_{j \in F(f)(i)} \left\{ \begin{array}{ll}
\text{P.R.} \left( ([0])_{k \in \{g_{i \rightarrow j}(a_i), g_{i \rightarrow j}(c_i)\}} \right) & j \in \{f|_{[i-1,i]}(b), f|_{[i-1,i]}(d)\} \neq \emptyset \\
\prod_{k \in \{g_{i \rightarrow j}(a_i), g_{i \rightarrow j}(c_i)\}} [0] & j \notin \{f|_{[i-1,i]}(b), f|_{[i-1,i]}(d)\} \neq \emptyset \\
\emptyset & j \notin \{f|_{[i-1,i]}(b), f|_{[i-1,i]}(d)\} = \emptyset
\end{array} \right.
\]
which reduces to
\[
\prod_{i \in \{x, z\}} \prod_{j \in F(f)(i)} \left\{ \begin{array}{ll}
g_{i \rightarrow j}(a_i), g_{i \rightarrow j}(c_i) & j \in \{f|_{[i-1,i]}(b), f|_{[i-1,i]}(d)\} \neq \emptyset \\
[0] & j \notin \{f|_{[i-1,i]}(b), f|_{[i-1,i]}(d)\} \neq \emptyset \\
\emptyset & j \notin \{f|_{[i-1,i]}(b), f|_{[i-1,i]}(d)\} = \emptyset
\end{array} \right.
\]
Thus, we define

\[ \prod_{i \in (x, z)} \text{Hom} \left( \left( b, (a_i)_{i \in (x, z)} \right), \left( d, (c_i)_{i \in (x, z)} \right) \right) \]

to be the obvious map which is constituted of, component-wise, restrictions of maps \( g_{i \rightarrow j} \) and canonical maps \([0] \to [0] \) or \( \varnothing \to \varnothing \).

**Example 2.28.** While the formulae above are straightforward, it is not necessarily easy to see exactly what they are doing. As such it may be hard to convince one’s self that these formulae are: (1) correct and/or (2) interesting. As an example then, which should both clarify and justify the formulae, we’ll consider the map

\[ d^1; (s^1, s^0) = \{0, 2\}; ([0, 1, 1], \{0, 0, 1\}) : [1]; ([2]) \to [2]; ([1], [1]) \]

and describe the map

\[ \text{P.R.} \left( d^1; (s^1, s^0), \langle 0, 1 \rangle \right) : \text{P.R.} ([2]) \to \text{P.R.} ([1], [1]). \]

In Figure 9 we sketch \( \text{P.R.} ([2]) \) and in Figure 10 we sketch a pasting diagram for \( \text{P.R.} ([1], [1]) \). Then, by our formula, the functor

\[ \text{P.R.} \left( d^1; (s^1, s^0), \langle 0, 1 \rangle \right) : \text{P.R.} ([2]) \to \text{P.R.} ([1], [1]) \]

can be given explicitly by

\[ \begin{array}{ccc}
\text{Ob} (\text{P.R.} ([2])) & \xrightarrow{d^1 \times (s^1, s^0)} & \text{Ob} (\text{P.R.} ([1], [1])) \\
[1] \times [2] & \xrightarrow{\text{d}^1 \times (s^1, s^0)} & [2] \times [1] \times [1]
\end{array} \]

We first compute the functors between \( \text{Hom} \)-categories of the first sort in Lemma 2.25. We compute

\[ \text{Hom}_{\text{P.R.}([2])} \left( (0, 0), (1, 1) \right) \to \text{Hom}_{\text{P.R.}([1], [2])} \left( (0, 0), (2, 1, 0) \right) \]

\[ \begin{array}{ccc}
\text{P.R.} ([0]_1) & \xrightarrow{\text{P.R.} \left( \begin{array}{c}
0 < 1 \\
p = 1
\end{array} \right)} & \text{P.R.} \left( \begin{array}{c}
0 \not< 0 \\
p = 2
\end{array} \right) \\
[0, 1] & \xrightarrow{\{0, 1\}, \{0, 0\}} & [0, 1] \times [0, 0]
\end{array} \]

\[ \text{We hope that the reader will forgive us for only providing a picture of the pasting diagram in this second case; including the copy of [1] \times [1] as the Hom-category between (0, 0, 0) and (1, 1, 2) is difficult.} \]
where we note that \( \{0, 1\} = \{0, 1, 1\} \upharpoonright_{\{0,1\}} \) and \( \{0, 0\} = \{0, 0, 1\} \upharpoonright_{\{0,1\}} \). We compute

\[
\begin{array}{c}
\text{Hom}((1, 0), (2, 1)) \xrightarrow{\text{P.R.}([0]_2)} \text{Hom}((0, 1), (2, 1, 1)) \\
\text{P.R.}( [1, 2]) \xrightarrow{\{1,1\}, \{0,1\}} [1, 1] \times [0, 1]
\end{array}
\]

where we note that \( \{1,1\} = \{0, 1, 1\} \upharpoonright_{\{1,2\}} \) and \( \{0, 1\} = \{0, 0, 1\} \upharpoonright_{\{1,2\}} \). Lastly, we compute the following.

\[
\begin{array}{c}
\text{Hom}((0, 0), (1, 2)) \xrightarrow{\text{P.R.}([0], [0]_2)} \text{Hom}((0, 0), (2, 1, 1)) \\
\text{P.R.}( [0]_1, [0]_2) \xrightarrow{(\{0,0,1\}, \{0,1,1\})} [0, 1] \times [0, 1]
\end{array}
\]

We next compute the functors between Hom-categories of the second sort from Lemma 2.25. We compute

\[
\begin{array}{c}
\text{Hom}_{\text{P.R.}([2])}((0, 0), (0, 1)) \xrightarrow{\text{P.R.}([0], [0]_2)} \text{Hom}_{\text{P.R.}([1], [0]_2)}((0, 0, 0), (0, 1, 0)) \\
[0]_1 \xrightarrow{\text{P.R.}(\flat_p)} [0]_1 \times [0]_2
\end{array}
\]

we compute

\[
\begin{array}{c}
\text{Hom}_{\text{P.R.}([2])}((0, 1), (0, 2)) \xrightarrow{\text{P.R.}([0], [0]_2)} \text{Hom}_{\text{P.R.}([1], [0]_2)}((0, 1, 0), (0, 1, 1)) \\
[0]_2 \xrightarrow{\text{P.R.}(\flat_p)} [0]_1 \times [0]_2
\end{array}
\]

and we compute

\[
\begin{array}{c}
\text{Hom}_{\text{P.R.}([2])}((0, 0), (0, 2)) \xrightarrow{\text{P.R.}([0], [0]_2)} \text{Hom}_{\text{P.R.}([1], [0]_2)}((0, 0, 0), (0, 1, 1)) \\
[0]_1 \times [0]_2 \xrightarrow{\text{P.R.}(\flat_p)} [0]_1 \times [0]_1
\end{array}
\]

Changing the second index repeats the pattern and in all other cases the source Hom-category is the initial object.

Remark 2.29. That these formulae correct express \([1] \otimes f \circ g\) is, while complicated, completely formal and thus left to the reader.
Figure 9. The $\omega$-category $[1] \otimes [2]$

Figure 10. A pasting diagram for the strict-$\omega$-category $[1] \otimes ([1] \times [1])_L$
2.6. **An even more explicit decomposition for the lax-Gray Cylinder.** Now, the lax-shuffle decomposition for \([1] \otimes (\_ )\) enjoys only some of the useful properties that the shuffle decomposition does for \([1] \times (\_ )\). The formula requires taking colimits in the category of \(\omega\)-categories and not in the larger category of cellular sets precisely because not every cell of \([1] \otimes (\_ )\) factors through one of objects in that diagram. In light of the lemma of the previous section however, we can cover \([1] \otimes T\) with cellular sets in the image of \(\triangle \int \omega\text{-Cat}\).

**Corollary 2.30.** For any \([n] ; (A_i)\) of \(\Theta\), \([1] \otimes [n] ; (A_i)\), as a cellular set, is the colimit of diagram in Figure 11 in cellular sets.

**Proof.** Follows from the prior Lemma characterizing the \(\text{Hom} \ \omega\text{-categories of } [1] \otimes [n] ; (A_i)\).

2.7. **The Gray cylinder of hyperfaces.** The hyperfaces of cells \(T\) of \(\Theta\) hold a particularly important place in the Cisinski model categories which \(\widehat{\Theta}\) admits. In this section we use the formulae we’ve developed to describe how the Gray cylinder \([1] \otimes (\_ )\) acts on hyperfaces. Unsurprisingly, this action can be defined recursively on the height of the hyperface’s target.

2.7.1. **Vertical hyperfaces.** The action of \([1] \otimes (\_ )\) on the vertical hyperfaces

\[
[n] ; (A_{<k}, A'_{k}, A_{>k}) \longrightarrow [n] ; (A_{<k}, A_{k}, A_{>k})
\]

is easily described in terms of the lax shuffle decomposition.

**Corollary 2.31.** Given a vertical hyperface

\[
id; (id_{<k}, \nu, id_{>k}) : [n] ; (A_{<k}, A'_{k}, A_{>k}) \longrightarrow [n] ; (A_{<k}, A_{k}, A_{>k})
\]

the map

\[
[1] \otimes [n] ; (A_{<k}, A'_{k}, A_{>k}) \longrightarrow [1] \otimes [n] ; (A_{<k}, A_{k}, A_{>k})
\]
Figure 11. Complete wide-pushout for the Gray cylinder $[1] \otimes [n] ; (A_1, A_2, \ldots, A_n)$
is induced by the diagram

\[
\begin{array}{c}
\vdots \\
[0 \ldots 1] ; (A_{<k}, 0, A'_{k}, A_{>k}) \xrightarrow{\text{id} \otimes (id_{<k}, id_{>k}, id_{>k})} [0 \ldots 1] ; (A_{<k}, 0, A_{k}, A_{>k}) \\
\downarrow d^k \otimes (id_{<k}, 0, id_{>k}) \\
[n] ; (A_{<k}, A'_{k}, A_{>k}) \xrightarrow{\text{id} \otimes (id_{<k}, id_{>k})} [n] ; (A_{<k}, A_{k}, A_{>k}) \\
\downarrow d^k \otimes (id_{<k}, id_{k}, id_{>k}) \\
[n] ; (A_{<k}, [1] \otimes A'_{k}, A_{>k}) \xrightarrow{\text{id} \otimes (id_{<k}, [1] \otimes id_{>k})} [n] ; (A_{<k}, [1] \otimes A_{k}, A_{>k}) \\
\downarrow d^k \otimes (id_{<k}, [1] \otimes id_{>k}) \\
[n] ; (A_{<k}, A'_{k}, A_{>k}) \xrightarrow{\text{id} \otimes (id_{<k}, id_{>k})} [n] ; (A_{<k}, A'_{k}, A_{>k}) \\
\downarrow d^k \otimes (id_{<k}, id_{k}, id_{>k}) \\
[n + 1] ; (A_{<k}, A'_{k}, 0, A_{>k}) \xrightarrow{\text{id} \otimes (id_{<k}, id_{>k}, id_{>k})} [n + 1] ; (A_{<k}, A_{k}, 0, A_{>k}) \\
\downarrow \\
\vdots
\end{array}
\]

2.7.2. **Horizontal hyperfaces.** Above we used the fact that inner and outer vertical hyperfaces admit the same description in terms of the lax shuffle decomposition. This however is not the case for inner and outer horizontal hyperfaces - the formulae are subtly different.

**Corollary 2.32.** For outer hyperfaces

\[
d^0 \otimes (\text{id}_1) \otimes \text{id}_{>1} : [n - 1] ; (A_{0 < i \leq n - 1}) \longrightarrow [n] ; (0, A_{0 < i \leq n - 1})
\]

and

\[
d^n \otimes (\text{id}_{n - 1}, \text{id}_{n - 1}) : [n - 1] ; (A_{0 < i \leq n - 1}) \longrightarrow [n] ; (A_{0 < i \leq n - 1}, 0)
\]

the maps

\[
[1] \otimes d^0 \otimes (\text{id}_1) \otimes \text{id}_{>1} : [n - 1] ; (A_{0 < i \leq n - 1}) \longrightarrow [n] ; (0, A_{0 < i \leq n - 1})
\]

and

\[
[1] \otimes d^n \otimes (\text{id}_{n - 1}, \text{id}_{n - 1}) : [n - 1] ; (A_{0 < i \leq n - 1}) \longrightarrow [n] ; (A_{0 < i \leq n - 1}, 0)
\]

are induced by the diagrams below, with \([1] \otimes d^0 \otimes (\text{id}_1) \otimes \text{id}_{>1}\) induced by the diagram on the left in Figure 12 and \([1] \otimes d^n \otimes (\text{id}_{n - 1}, \text{id}_{n - 1})\) induced by the diagram on the right in Figure 12.

Lastly we may attend to inner horizontal hyperfaces.
Corollary 2.33. Let 
\[ d^k; (i_{d_0 < i < k}, (id, !), id_{k < i < n}) : [n - 1]; (A_{0 < i < n}) \to [n]; (A_{0 < i < k}, A_k, [0], A_{k < i < n}) \]
be an inner hyperface. Then the map 
\[ [1] \otimes d^k; (i_{d_0 < i < k}, (id, !), id_{k < i < n}) : [1] \otimes [n] ; (A_{0 < i < n}) \to [1] \otimes [n]; (A_{0 < i < k}, A_k, [0], A_{k < i < n}) \]
is induced by the diagram in Figure 13 (in which we leave the denotation of all but the simplicial aspects of maps implicit).

Remark 2.34. The case where we replace \((id, !) : (A_k) \to (A_k, [0])\) with \((!, id) : (A_k) \to ([0], A_k)\) is nearly the same; we leave it to the reader to make the change themselves where required.
Figure 13. Diagram inducing the Gray cylinder of an inner hyperface
3. The Cartesian-Gray-Shift span and the Gray Cylinder

We can now define an important span

\[
\begin{array}{c}
 [1] \otimes (_) \\
\end{array}
\begin{array}{c c}
\kappa & \sigma \\
[1] \times (_) & [1] ; (_) \\
\end{array}
\]

of functors \( \Theta \longrightarrow \hat{\Theta} \), again by recursion on height.

**Definition 3.1.** Let

\[
\kappa_{[0]} : \begin{array}{c} [1] \otimes [0] \\
\end{array} \longrightarrow \begin{array}{c} [1] \times [0] \\
\end{array}
\]

be the identity and let

\[
\sigma_{[0]} : \begin{array}{c} [1] \otimes [0] \\
\end{array} \longrightarrow \begin{array}{c} [1] ; [0] \\
\end{array}
\]

be the identity as well. Then, by recursion on the height of cells, we define the maps \( \kappa_T \) and \( \sigma_T \) as follows.

Let \( n \in \mathbb{N} \) and \( j \in [n] \) and define the map \( \text{split}^j_{[n]} \) by the following expression.

\[
\begin{array}{c}
\text{split}^j_{[n]} : [n] \longrightarrow [1] \\
i < j \longrightarrow 0 \\
i \geq j \longrightarrow 1
\end{array}
\]
For all \([n]; (A_1, \ldots, A_n)\) of \(\Theta\), the following diagram commutes (proven below as Proposition 3.3).

(1) \[
\begin{array}{ccc}
[n+1];(0,A_1,A_2,\ldots,A_n) & \xleftarrow{id} & [n+1];(0,A_1,A_2,\ldots,A_n) \\
\downarrow & & \downarrow \\
[n];(A_1,A_2,\ldots,A_n) & \xleftarrow{id} & [n+1];(0,A_1,A_2,\ldots,A_n) \\
\downarrow & & \downarrow \\
[n];(A_1,A_2,\ldots,A_n) & \xleftarrow{id} & [n+1];(0,A_1,A_2,\ldots,A_n) \\
\downarrow & & \downarrow \\
[n+1];(A_1,0,A_2,\ldots,A_n) & \xleftarrow{id} & [n+1];(A_1,0,A_2,\ldots,A_n) \\
\downarrow & & \downarrow \\
\vdots & & \vdots \\
\downarrow & & \downarrow \\
[n+1];(A_1,\ldots,A_{n-1},0,A_n) & \xleftarrow{id} & [n+1];(A_1,\ldots,A_{n-1},0,A_n) \\
\downarrow & & \downarrow \\
[n+1];(A_1,\ldots,A_{n-1},0,A_n) & \xleftarrow{id} & [n+1];(A_1,\ldots,A_{n-1},0,A_n) \\
\downarrow & & \downarrow \\
[n+1];(A_1,\ldots,A_{n-1},0,A_n) & \xleftarrow{id} & [n+1];(A_1,\ldots,A_{n-1},0,A_n) \\
\downarrow & & \downarrow \\
[n+1];(A_1,\ldots,A_{n-1},0,A_n) & \xleftarrow{id} & [n+1];(A_1,\ldots,A_{n-1},0,A_n) \\
\end{array}
\]

Moreover, since:

- the colimit of the first column is \([1] \times [n]; (A_1, A_2, \ldots, A_n)\);
- the colimit of the second column is \([1] \otimes [n]; (A_1, A_2, \ldots, A_n)\); and
- the colimit on the right is the globular sum decomposition for \([1]; [n]; (A_1, A_2, \ldots, A_n)\);
this diagram induces maps $\kappa_{[n];(A_i)}$ and $\sigma_{[n];(A_i)}$ (the solid arrows) for which the diagram (solid and dashed arrows) commutes as follows.

(2)
Lemma 3.2. For all $n \in \mathbb{N}$, the diagram

commutes. Moreover, since:

- the colimit of the first column is $[1] \times [n]$;
- the colimit of the second column is $[1] \otimes [n]$; and
- the colimit on the right is the globular sum decomposition for $[1]; [n]$;
this diagram induces maps (the solid arrows) for which the diagram (solid and dashed arrows) commutes as follows.

\[ (4) \]

\[
\begin{array}{c}
\{1\} \times [n] \\
\{1\} \otimes [n] \\
\{0\} \times [n] \\
\{0\} \otimes [n] \\
{[n]} & {[n]} & {[n]} & {[n]} \\
\end{array}
\]

Proof. To show that Diagram 3 commutes it suffices to check the commutativity of the squares (for \(1 \leq k \leq n\)) of four sorts; the two sorts

\[ \begin{array}{c}
[n+1] \\
\sigma^n \leftarrow \{0\} \times [n] \\
\{0\} \otimes [n] \\
\{0\} \times [n] \\
\{0\} \otimes [n] \\
\sigma^n \leftarrow {[n]} \\
\end{array} \]
and the two sorts

\[
\begin{array}{c}
\text{id} ; (\{1\}) \downarrow \\
\text{id} ; (\{0\}) \downarrow \\
\end{array}
\]

commute.

For sort (I) it suffices to observe that

\[
\text{id} \circ d^k = d^k \circ \text{id} \circ \text{id}
\]

and

\[
(!, !) \circ (\{1\} \otimes 0) = (\!\! \times !) \circ (!, !)
\]

and for sort (II) see that

\[
\text{id} \circ d^k = d^k \circ \text{id} \circ \text{id}
\]

and

\[
(!, !) \circ (\{0\} \otimes 0) = (\!\! \times !) \circ (!, !)
\]

For sort (III) it suffices to observe that

\[
\text{split}^k_{[n+1]} = \text{split}^k_{[n+2]} \circ d^k
\]

and

\[
\sigma_0 \circ \{1\} \otimes 0 = \{1\}
\]

since \(\sigma_0\) is defined to be \(\text{id}_{[1]}\). Likewise, for sort (IV) it suffices to observe that

\[
\text{split}^k_{[n+1]} = \text{split}^{k+1}_{[n+2]} \circ d^k
\]

and

\[
\sigma_0 \circ \{0\} \otimes 0 = \{0\}
\]

again since \(\sigma_0 = \text{id}_{[1]}\).

Lastly, to check that Diagram 4 commutes, it suffices to observe that the diagrams
and

\[
\begin{array}{c}
[n+1] \xleftarrow{\text{id}} [n+1] \xrightarrow{\text{split}^1_{[n+1];[!][!][!]}} [1] \xrightarrow{\text{id};\{0\}} [1];[1] \\
\downarrow^{d^{n+1}} \quad \downarrow^{d^{n+1}} \\
[n] \xrightarrow{\{0\}} \\
\end{array}
\]

Proposition 3.3. For all \([n]; (A_1, \ldots, A_n)\) of \(\Theta\), the following diagram commutes.
Moreover, since:

- the colimit of the first column is \( [1] \times [n] \); \((A_1, A_2, \ldots, A_n)\);
- the colimit of the second column is \( [1] \otimes [n] \); \((A_1, A_2, \ldots, A_n)\); and
- the colimit on the right is the globular sum decomposition for \( [1]; [n]; (A_1, A_2, \ldots, A_n)\);
this diagram induces maps (the solid arrows) for which the diagram (solid and dashed arrows) commutes as follows.

$$\begin{array}{cccc}
[1] × [n] : (A_1, A_2, ..., A_n) & \rightarrow & [1] ⊗ [n] : (A_1, A_2, ..., A_n) & \rightarrow [1] × [n] : (A_1, A_2, ..., A_n) \\
\{1\} \times [n] : (A_1, A_2, ..., A_n) & \rightarrow & \{1\} ⊗ [n] : (A_1, A_2, ..., A_n) & \rightarrow \{1\} \times [n] : (A_1, A_2, ..., A_n) \\
\{0\} \times [n] : (A_1, A_2, ..., A_n) & \rightarrow & \{0\} ⊗ [n] : (A_1, A_2, ..., A_n) & \rightarrow \{0\} \times [n] : (A_1, A_2, ..., A_n) \\
[n] : (A_1, A_2, ..., A_n) & \rightarrow & [n] : (A_1, A_2, ..., A_n) & \rightarrow [n] : (A_1, A_2, ..., A_n)
\end{array}$$

**Proof.** To show that Diagram 5 commutes it suffices to check the commutativity of four sorts of squares, for $1 \leq k \leq n$: the two sorts

$$\begin{array}{cccc}
[n+1] : (A_1, A_{k-1}, A_k, A_{k+1}, ..., A_n) & \rightarrow & \text{id} & \rightarrow [n+1] : (A_1, A_{k-1}, A_k, A_{k+1}, ..., A_n) \\
d^k : (\text{id}, ..., \text{id}, (1, \text{id}), \text{id}, ..., \text{id}) & \rightarrow & [n] : (A_1, ..., A_{k-1}, A_k, A_{k+1}, ..., A_n) & \rightarrow \text{id} : (\text{id}, ..., \text{id}, (1, \text{id}), \text{id}, ..., \text{id}) \\
[n] : (A_1, ..., A_{k-1}, A_k, A_{k+1}, ..., A_n) & \rightarrow & \text{id} : (\text{id}, ..., \text{id}, K, \text{id}, ..., \text{id}) & \rightarrow [n] : (A_1, ..., A_{k-1}, [1] ⊗ A_k, A_{k+1}, ..., A_n) \\
d^1 : (\text{id}, ..., \text{id}, (1), \text{id}, ..., \text{id}) & \rightarrow & \text{id} : ([0] ⊗ A_1, ..., !) & \rightarrow [n] : (A_1, ..., A_n) \\
[n+1] : (A_1, ..., A_{k-1}, A_k, 0, A_{k+1}, ..., A_n) & \rightarrow & \text{id} & \rightarrow [n+1] : (A_1, ..., A_{k-1}, A_k, 0, A_{k+1}, ..., A_n) \\
d^1 : (\text{id}, ..., \text{id}, (1), \text{id}, ..., \text{id}) & \rightarrow & [n+1] : (A_1, ..., A_{k-1}, A_k, 0, A_{k+1}, ..., A_n) & \rightarrow \text{id} : (\text{id}, ..., \text{id}, (1), \text{id}, ..., \text{id})
\end{array}$$
and the two sorts

\[
\begin{align*}
&[n+1]: (A_1, \ldots, A_{k-1}, 0, A_k, A_{k+1}, \ldots, A_n) \\
&\xrightarrow{d^k: (\text{id}, \ldots, \text{id}, \text{id}, \text{id}, \ldots, \text{id})} [n]: (A_1, \ldots, A_{k-1}, A_k, A_{k+1}, \ldots, A_n) \\
&\xrightarrow{\text{id}: (\text{id}, \ldots, \text{id}, \{1\} \otimes A_k, \text{id}, \ldots, \text{id})} [n]: (A_1, \ldots, A_{k-1}, 1 \otimes A_k, A_{k+1}, \ldots, A_n) \\
&\xrightarrow{\text{id}: (\text{id}, \ldots, \text{id}, \{0\} \otimes A_1, \text{id}, \ldots, \text{id})} [n]: (A_1, \ldots, A_{k-1}, A_k, A_{k+1}, \ldots, A_n) \\
&\xrightarrow{d^k: (\text{id}, \ldots, \text{id}, (\text{id}, 1), \text{id}, \ldots, \text{id})} [n+1]: (A_1, \ldots, A_{k-1}, A_k, 0, A_{k+1}, \ldots, A_n) \\
&\xrightarrow{\text{id}: (\text{id}, \ldots, \text{id}, \{0\} \otimes A_k, A_{k+1}, \ldots, A_n) \\
&\xrightarrow{\text{id}: (\text{id}, \ldots, \text{id}, \{0\} \otimes A_1, \text{id}, \ldots, \text{id})} [n+1]: (A_1, \ldots, A_{k-1}, A_k, 0, A_{k+1}, \ldots, A_n) \\
&\xrightarrow{\text{id}: (\text{id}, \ldots, \text{id}, \{1\} \otimes A_k, \text{id}, \ldots, \text{id})} [1]: (0)
\end{align*}
\]

The commutation argument for sorts (I) and (II) are nearly identical. See that squares of sort (I) commute as (left-hand side - counter-clockwise, right-hand side - clockwise):

\[
\begin{align*}
\text{id}_{[n+1]} \circ d^k &= \text{id}_{[n+1]} \circ \text{id}_{[n+1]} \circ d^k \\
\text{id}_{A_1} \circ \text{id}_{A_1} &= \text{id}_{A_1} \circ \text{id}_{A_1} \circ \text{id}_{A_1} \\
\vdots \\
\text{id}_{A_{k-1}} \circ \text{id}_{A_{k-1}} &= \text{id}_{A_{k-1}} \circ \text{id}_{A_{k-1}} \circ \text{id}_{A_{k-1}} \\
(! \times \text{id}_{A_k}) \circ (!, \text{id}_{A_k}) &= (!, \text{id}) \circ (\kappa_{A_k}) \circ (\{1\} \otimes A_k) \\
\text{id}_{A_{k+1}} \circ \text{id}_{A_{k+1}} &= \text{id}_{A_{k+1}} \circ \text{id}_{A_{k+1}} \circ \text{id}_{A_{k+1}} \\
\vdots \\
\text{id}_{A_n} \circ \text{id}_{A_n} &= \text{id}_{A_n} \circ \text{id}_{A_n} \circ \text{id}_{A_n}
\end{align*}
\]

where the only non-obvious equality will follow by recursion to the case where \(A_k\) is of height 1 - Lemma 3.2. Nearly the same computation provides the commutativity of the squares of sort (II), with

\[
\begin{align*}
(! \times \text{id}_{A_k}) \circ (!, \text{id}_{A_k}) &= (!, \text{id}) \circ (\kappa_{A_k}) \circ (\{0\} \otimes A_k)
\end{align*}
\]

in place of

\[
\begin{align*}
(! \times \text{id}_{A_k}) \circ (!, \text{id}_{A_k}) &= (!, \text{id}) \circ (\kappa_{A_k}) \circ (\{1\} \otimes A_k)
\end{align*}
\]

is the computation. Again, this follows by recursion to the case where \(A_k\) is of height 1 - Lemma 3.2.
The commutation for sorts (III) and (IV) are likewise nearly identical. See that sort (III) commutes as

$$\text{id}_{[1]} \circ \text{split}^{k}_{[n+1]} \circ d^{k} = \text{split}^{k}_{[n+1]}$$

and

$$\{1\} \circ (!, \text{id}_{A_{k}}) = \sigma_{A_{k}} \circ \{1\} \otimes A_{k}$$

by recursion to the case where $A_{k}$ is of height 1 Lemma 3.2. The commutation for squares of sort (IV) is similar, as promised. Indeed squares of sort (IV) commute as

$$\text{id}_{[1]} \circ \text{split}^{k+1}_{[n+1]} \circ d^{k} = \text{split}^{k}_{[n+1]}$$

and

$$\{0\} \circ (!, \text{id}_{A_{k}}) = \sigma_{A_{k}} \circ \{0\} \otimes A_{k}$$

which again follows by recursion to $A_{k}$ of height 1 - Lemma 3.2.

Lastly, to check that Diagram 4 commutes, it suffices to observe that the diagrams commute.

\[\text{Diagram 4}\]

通贴

APPENDIX A. STEINER COMPLEXES AND STEINER \(\omega\)-CATEGORIES: BASES, LOOP-FREE BASES, ETC.

This appendix is a translation, done by Yuki Maehara and this author, from French to English, of [AraMaltsiniotis] Paragraphs 2.6 through 2.10.

2.6. A **basis** for an augmented directed complex $K$ is a graded set $B = (B_{i})_{i \geq 0}$ such that, for all $i \geq 0$,

1. $B_{i}$ is a basis for the $\mathbb{Z}$-module $K_{i}$;
2. $B_{i}$ generates the sub-monoid $K_{i}^{*}$ of $K_{i}$.

We will sometimes identify a basis $B = (B_{i})_{i \geq 0}$ with the set $\bigsqcup_{i \geq 0} B_{i}$.

Let $K$ be an augmented directed complex. We define a preorder $\leq$ on $K_{i}$ by setting

$$x \leq y \quad \iff \quad y - x \in K_{i}^{*}.$$

It follows immediately that if $K$ admits a basis, the preorder is in fact a partial order, and the elements of $B_{i}$ are the minimal elements of $(K_{i}^{*} \setminus \{0\}, \leq)$. Thus if $K$ admits a basis then that basis is unique.

We will say that an augmented directed complex $K$ is **based** if it admits a (necessarily unique) basis.
2.7. Fix an abelian group \( A \) freely generated by a basis \( B \). Let
\[
x = \sum_{b \in B} x_b b
\]
be an element in \( A \). We define the support of \( x \) to be the set
\[
\text{supp}(x) = \{ b \in B \mid x_b \neq 0 \}.
\]
Denote by \( A^* \) the sub-monoid of \( A \) generated by \( B \). We define two elements \( x_+ \) and \( x_- \) in \( A^* \) by
\[
x_+ = \sum_{x_b > 0} x_b b \quad \text{and} \quad x_- = -\sum_{x_b < 0} x_b b.
\]
We have \( x = x_+ - x_- \).

In particular, if \( K \) is an augmented directed complex admitting a basis \( B = (B_i)_{i \geq 0} \), for all \( x \in K_i \) with \( i \geq 0 \), we have, in applying the preceding paragraph to the abelian group \( K_i \) given by the basis \( B_i \), a notion of support of \( x \) and elements \( x_+ \) and \( x_- \) in \( K_i^* \).

2.8. Let \( K \) be an augmented directed complex with a basis \( B = (B_i)_{i \geq 0} \). For \( i \geq 0 \) and for \( x \in K_i \), we define the matrix
\[
\langle x \rangle = \begin{pmatrix}
\langle x \rangle_0^0 & \cdots & \langle x \rangle_0^i \\
\langle x \rangle_1^0 & \cdots & \langle x \rangle_1^i \\
\end{pmatrix},
\]
where \( \langle x \rangle_k^i \) are defined by recursion on \( k \) from \( i \) to 0:

- \( \langle x \rangle_k^i = x = \langle x \rangle_1^i; \)
- \( \langle x \rangle_k^0 = d(\langle x \rangle_k^0) = d(\langle x \rangle_k^1) = 0 < k \leq i. \)

It is easy to see that this matrix is an \( i \)-arrow in \( \nu(K) \) if and only if, \( x \) appears in \( K_i^* \), and we have \( e(\langle x \rangle_0^0) = 1 = e(\langle x \rangle_0^1) \). We will also set \( \langle x \rangle_i = x \) and \( \langle x \rangle_k^i = 0 \) for \( k > i \) and \( \epsilon = 0, 1 \).

In the case when \( \langle x \rangle \) is an \( i \)-arrows, this is compatible with the convention described in Paragraph 2.4 of [AraMaltziniotis].

We will say that a basis \( B \) for \( K \) is unital if, for all \( i \geq 0 \) and for all \( x \in B_i \), the matrix \( \langle x \rangle \) is an \( i \)-arrow in \( \nu(K) \), which amounts to \( e(\langle x \rangle_0^0) = 1 = e(\langle x \rangle_0^1) \).

We say that an augmented directed complex is unitally based if it is based and its unique basis is unital. If an augmented directed complex \( K \) admits a unital basis, then for all elements \( x \) in the basis for \( K \), we call the cell \( \langle x \rangle \) in \( \nu(K) \) the atom associated to \( x \).

2.9. Let \( K \) be an augmented directed complex with a basis \( B \). For \( i \geq 0 \), denote by \( \leq_i \) the smallest preorder on \( B(= \bigsqcup_j B_j) \) satisfying:
\[
x \leq_i y \quad \text{if} \quad |x| > i, |y| > i, \quad \text{and} \quad \text{supp}(\langle x \rangle_i) \cap \text{supp}(\langle y \rangle_i) \neq \emptyset.
\]

We will say that the basis \( B \) is loop-free if, for all \( i \geq 0 \), the preorder \( \leq_i \) is a partial order.

We say an augmented directed complex admits a loop-free basis if it admits a basis and its unique basis is loop-free.

2.10. We refer to an augmented directed complex with a unital, loop-free basis as a Steiner complex.

We call an \( \omega \)-category a Steiner \( \omega \)-category if it is in the essential image of the functor \( \nu : \text{CDA} \rightarrow \text{\omega-Cat} \) restricted to the category of Steiner complexes. The following theorem affirms that the functor \( \nu \) induces an equivalence between the categories of Steiner complexes and of Steiner \( \omega \)-categories.
Theorem A.1. (Steiner, 2.11 in [AraMaltsiniotis]) For all Steiner complexes $K$, the counit
\[ \lambda(\nu(K)) \to K \]
is an isomorphism. In particular, the restriction of the functor $\nu : CDA \to \omega\text{-}Cat$ to the full subcategory of Steiner complexes is full-and-faithful.

Proof. As cited in [AraMaltsiniotis], See Theorem 5.6 of [Steiner1]. □

Theorem A.2. ([Steiner], 2.12 in [AraMaltsiniotis]) Let $K$ be a Steiner complex. Then the $\infty$-category $\nu(K)$ is freely generated in the sense of polygraphs by the atoms $\langle x \rangle$ where $x$ varies over the basis for $K$.

Proof. See Theorem 6.1 of [Steiner1]. □

2.13. Let $K$ be an augmented directed complex admitting a basis $B$. Denote by $\leq_N$ the smallest preorder on $B$ satisfying
\[ x \leq_N y \quad \text{if} \quad x \in \text{supp}(d(y)_-) \text{ or } y \in \text{supp}(d(x)_+), \]
where, by convention, $d(b) = 0$ if $b \in B_0$. We will say that a basis $B$ is strongly loop-free if the preorder $\leq_N$ is a partial order.

Proposition A.3. ([Steiner], 2.14 in [AraMaltsiniotis]) Let $K$ be an augmented directed complex with a basis $B$. If $B$ is strongly loop-free, then it is also loop-free.

Proof. See Proposition 3.7 of [Steiner]. □

2.15. An augmented directed complex with a unital, strongly loop-free basis will be called a strong Steiner complex. By virtue of the preceding paragraph, a strong Steiner complex is a Steiner complex. We denote by $\text{StrStC}_{DA}$ the full subcategory of $C_{DA}$ spanned by the strong Steiner complexes.

By a strong Steiner $\omega$-category, we mean an $\omega$-category in the essential image of the functor $\nu : C_{DA} \to \omega\text{-}Cat$ restricted to the strong Steiner complexes. By virtue of Theorem 2.11, the functor $\nu$ induces an equivalence between the category of strong Steiner complexes and the category of strong Steiner $\omega$-categories.

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