A FUNCTORIAL APPROACH TO THE STABILITY OF VECTOR BUNDLES

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Abstract. On a normal projective variety the locus of $\mu$-stable bundles that remain $\mu$-stable on all Galois covers prime to the characteristic is open in the moduli space of Gieseker semistable sheaves. On a smooth projective curve of genus at least 2 this locus is big in the moduli space of stable bundles. As an application we obtain a very different behaviour of the étale fundamental group in positive vs. characteristic 0.

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1. Introduction

Consider the stack of vector bundles on a smooth projective curve $C$ over an algebraically closed field $k$ of characteristic $p \geq 0$. Semistability is a property of vector bundles which is tailored to obtain a moduli space. Via the Harder-Narasimhan-filtration (HN-filtration for short) it also reveals additional structure of the category of vector bundles and immediately implies that semistability is functorial under pullback by finite separable morphisms. Even more structure is revealed via the Jordan-Hölder-filtration (JH-filtration for short). However, in contrast to the HN-filtration the JH-filtration is not unique and thus functoriality fails for stability.

Recently, those morphisms that preserve the stability of vector bundles under pullback have been identified: for curves these are exactly the genuinely ramified morphisms, see [3, Theorem 5.3]. In higher dimension, genuinely ramified morphisms also preserve stability, see [2, Theorem 1.2].

The main goal of this paper is to address a way to measure the failure of stability to be functorial under all finite separable pullbacks. As an application we obtain a very different behaviour of the étale fundamental groups in positive versus characteristic 0.

Representations of $\pi_{\text{ét}}(C)$ correspond to vector bundles of degree 0 which are trivialized on some étale cover over of $C$, see [10, 1.2 Proposition]. In positive characteristic these étale trivializable bundles are dense in the moduli space $M_{C}^{ss,r,0}$ of semistable bundles of rank $r$ and degree 0, see [7, Corollary 5.1]. This no longer holds in characteristic 0 as we show that the general bundle remains stable on all...
étale covers (avoiding the characteristic). Put another way, the étale fundamental group has enough information to recover the moduli space in positive characteristic but not in characteristic 0.

To make our results precise we need a definition. Call a vector bundle on $C$ prime to $p$ stable if it remains stable after pullback by all finite Galois covers $D \to C$ which have degree prime to $p$, see also Definition 2.7. The locus of prime to $p$ stable bundles is open - a direct consequence of the following theorem.

**Theorem 1** (Theorem 3.11 for curves). Let $r \geq 2$ and $C$ be a smooth projective curve over an algebraically closed field of characteristic $p \geq 0$. Then there exists a connected étale prime to $p$ Galois cover $\pi : C_{r, \text{good}} \to C$ such that a vector bundle $V$ of rank $r$ is prime to $p$ stable iff $\pi^* V$ is stable.

An analogous statement holds for $\mu$-stable bundles on a normal projective variety, see Theorem 3.11. Having identified this locus as open one should also address non-emptiness. Recall that an open subset $U$ of a variety $X$ is called big if $X \setminus U$ has codimension at least 2 in $X$.

**Theorem 2** (Theorem 4.9). Let $r \geq 2$. If $C$ has genus $g_C \geq 2$, then the prime to $p$ stable locus $M^p_{s, r, d} C$ is big in the moduli space of stable bundles $M_{s, r, d}^C$. More precisely, we have
\[
\dim(M^s_{r, d} \setminus M^{p^r-s, r, d}_{C}) \leq rr_0 (g_C - 1) + 1,
\]
where $r_0$ denotes the largest proper divisor of $r$. If $p$ is not the smallest proper divisor of $r$, then equality holds.

By considering $d = 0$ in Theorem 2, we obtain the different behaviour of the étale fundamental group, i.e., the non-density of the étale trivializable bundles in characteristic 0 versus their density in positive characteristic.

**Corollary 3.** Let $C$ be a smooth projective curve of genus $g_C \geq 2$. Let $r \geq 2$. Then the stable bundles of rank $r$ that are trivialized on a prime to $p$ étale cover are not dense in $M^s_{r, r, 0}$. In rank 2 and characteristic 0 such a non-density result has been independently obtained by Ghiasabadi and Reppen, see [10, Corollary 4.16].

We also note that the density of the étale trivializable bundles in positive characteristic means that we can not extend Theorem 1 nor Theorem 2 to include all covers; allowing only for covers of degree prime to the characteristic is crucial.

The key observation in proving Theorem 1 is that while stability is in general not preserved under pullback by a Galois cover $D \to C$ polystability is. In fact, we can say more: A stable vector bundle $V$ on $C$ decomposes on $D$ into a direct sum $\bigoplus_{e=1}^n W_{r, e}^{\text{stable}}$ of pairwise non-isomorphic stable bundles $W_i$ all appearing with the same multiplicity $e$. Furthermore, the Galois group of $D/C$ acts transitively on the isomorphism classes of the $W_i$, see Lemma 3.2.

The construction of the cover $C_{r, \text{good}}$ checking for prime to $p$ stability is then split into two parts: A cover $C_{r, \text{large}}$ checking for the decomposition behaviour if $n \geq 2$ and a cover $C_{r, \text{good}}$ including $n = 1$.

The cover $C_{r, \text{large}}$ is easily constructed using the transitive action of the Galois group. To include the case $n = 1$ the difficulty arises that while all the conjugates of $W = W_1$ by the Galois group are isomorphic these isomorphisms might not be compatible. We provide a workaround for descending simple invariant bundles.
Pretending that \( W \) descends for now allows for a comparison of the linearizations of \( V \) on \( D \) and \( W^{\oplus e} \). This gives rise to a \( \text{GL}_e \) representation of the Galois group. Finite subgroups prime to the characteristic of \( \text{GL}_e \) are well-understood. By Jordan’s theorem - which in positive characteristic is due Brauer and Feit - they are close to being abelian. This allows us to find a cover which also checks for this decomposition behaviour.

The same type of cover works in higher dimensions. However, the workaround for descend only works for curves. To obtain Theorem 1 in higher dimensions we carefully set up the requirements for the workaround of descend and then use a restriction theorem for stability to reduce to dimension 1.

Theorem 2 is obtained by a dimension estimate on the strata defined by the decomposition behaviour of a stable bundle.

The paper is structured as follows. In §2 we define functorial notions of stability and study them for genus \( g \leq 1 \). We also collect some preliminary properties of (semi)stable bundles under pullback as well as a descend lemma for (not necessarily étale) flat Galois covers of normal varieties.

In §3 we prove the key lemma. Then we construct the prime to \( p \) cover \( C_{r\text{−}good} \) that checks whether a vector bundle is prime to \( p \) stable.

In §4 we investigate certain strata which arise as the complement of the prime to \( p \) stable locus and estimate their dimension. We work with arbitrary étale Galois covers and obtain Theorem 2 by considering the cover constructed in Theorem 1. We also provide a descend lemma for étale cyclic covers which may be of independent interest, see Lemma 4.6.

Notation. We work over an algebraically closed field \( k \) of characteristic \( p \geq 0 \). A variety is a separated integral scheme of finite type over \( k \). A curve is a variety of dimension 1. The function field of a variety \( X \) is denoted by \( \kappa(X) \).

If \( X \) is a projective variety, then we implicitly choose an ample bundle \( \mathcal{O}_X(1) \) on \( X \). If we consider a finite morphism \( \pi : Y \to X \) we set \( \mathcal{O}_Y(1) = \pi^*\mathcal{O}_X(1) \). By (semi)stability we mean \( \mu \)- (semi)stability of reflexive sheaves with respect to \( \mathcal{O}_X(1) \).

We denote the moduli space of (semi)stable vector bundles of rank \( r \) and degree \( d \) on a smooth projective curve \( C \) by \( M^{\text{ss},r,d}_C \) (resp. \( M^{ss, r, d}_C \)).

Given a morphism \( \pi : Y \to X \) of varieties and a sheaf \( F \) on \( X \) we denote the pullback \( \pi^*F \) also by \( F|_Y \).

By a cover \( Y \to X \) of varieties we mean a finite separable morphism of varieties, i.e., a finite dominant morphism such that the extension of function fields \( \kappa(Y)/\kappa(X) \) is separable. A cover is called Galois if the extension of function fields \( \kappa(Y)/\kappa(X) \) is Galois. An étale (Galois) cover is a (Galois) cover \( Y \to X \) which is étale.

2. First Observations

We start by collecting some elementary results on pullback and semistability as well as descent theory for flat Galois covers, which is slightly trickier than for étale Galois covers. Then we introduce the functorial notions of stability and give a complete analysis for smooth projective curves of genus \( \leq 1 \).

2.1. Preliminaries on Stability and Pullback. In this subsection we recall several notions of stability as well as the basic properties of \( \mu \)-(semi)stable vector
bundles under pullback. We also include a descent lemma along (possibly non-étale) flat Galois covers in terms of linearizations.

We begin by recalling semistability, the reader is referred to [13, Chapter 1, 4] for a detailed account. On a smooth projective curve $C$ we have two numerical invariants attached to a vector bundle $V$: the rank $\text{rk}(V)$ and the degree $\text{deg}(V)$. This allows us to define the slope $\mu(V) := \text{deg}(V)/\text{rk}(V)$ which in turn is used to define (semi)stability. The vector bundle $V$ is called semistable if for all subbundles $0 \neq W \subsetneq V$ we have $\mu(W) \leq \mu(V)$. It is called stable if the inequality is strict for all subbundles $0 \neq W \subsetneq V$.

These notions are tailored to obtain a moduli space of semistable vector bundles of rank $r$ and degree $d$ which we denote by $M_{G}^{\text{ss},r,d}$. The closed points of $M_{G}^{\text{ss},r,d}$ correspond to the polystable vector bundles of rank $r$ and degree $d$, i.e., vector bundles which are a direct sum of stable bundles of the same slope $d/r$. The moduli space of stable bundles $M_{G}^{\text{ss},r,d}$ is an open subset of $M_{G}^{\text{ss},r,d}$.

On a normal projective variety $X$ of dimension $\geq 2$ there are several analogues to (semi)stability on a curve. On the one hand, we have more numerical invariants attached to a coherent sheaf $F$: (the coefficients of) the Hilbert polynomial $P(F)(n) = \dim(X) \sum_{i=0}^{\dim(X)} \frac{\alpha_i(F)}{i!} n^i$.

On the other hand, the Hilbert polynomial depends on the choice of a polarization $\mathcal{O}_X(1)$ of $X$. We implicitly fix the polarization - also see the notations.

A torsion-free coherent sheaf $F$ on $X$ is called Gieseker-semistable if for all saturated subsheaves $0 \neq G \subseteq F$ we have $p(G) \leq p(F)$, where $p(F) := P(F)/\text{rk}(F)$ is the reduced Hilbert polynomial. The ordering is via the lexicographic ordering on the coefficients of the polynomials starting in the highest degree. The torsion-free coherent sheaf $F$ is called Gieseker-stable if the above inequality is strict. As in the curve case these notions lend themselves to a construction of a moduli space of Gieseker semistable torsion-free sheaves.

In this paper we are mostly concerned with the the notion of $\mu$-stability which we also abbreviate to stability: the slope of a coherent sheaf $F$ which is torsion-free on a big open subset is defined as $\mu(F) := \text{deg}(F)/\text{rk}(F)$, where the degree is defined as $\text{deg}(F) := \alpha_{\dim(X)-1}(F) - \text{rk}(F)\alpha_{\dim(X)-1}(\mathcal{O}_X)$.

We call a reflexive sheaf $F$ semistable if for all saturated subsheaves $0 \neq G \subsetneq F$ of smaller rank we have $\mu(G) \leq \mu(F)$. A reflexive sheaf $F$ is stable if the above inequality is strict. Further, $F$ is polystable if it is a direct sum of stable sheaves of the same slope $\mu(F)$.

We note that the degree of $F$ only depends on its isomorphism class on some big open subset of $X$. In particular, we have $\mu(F) = \mu(F^{\vee\vee})$, where $F^{\vee\vee}$ denotes the reflexive hull of $F$.

A cover $Y \rightarrow X$ of normal varieties is flat on a big open subset. As the slope only depends on the isomorphism class on a big open subset, this is the right setting to study pullback. The basic results are as follows:
Lemma 2.1. Let $\pi : Y \to X$ be a cover of normal projective varieties of degree $d$. Let $F$ be a reflexive sheaf on $X$ and $G$ be a torsion-free sheaf on $Y$. Then the following hold:

(i) $\mu(G) = d \mu(\pi_\ast G - \mu(\pi_\ast O_Y))$.
(ii) $\mu((F|_Y)^{\vee \vee}) = d \mu(F)$.
(iii) $F$ is semistable iff $(F|_Y)^{\vee \vee}$ is semistable.
(iv) If $F$ is polystable and $Y \to X$ is Galois, then $(F|_Y)^{\vee \vee}$ is polystable. If $\pi$ is prime to $p$, then $F$ is polystable iff $(F|_Y)^{\vee \vee}$ is polystable.
(v) If $(F|_Y)^{\vee \vee}$ is stable, then so is $F$.

Proof. (i) - (iv) are proven in [13, Lemma 3.2.1 - 3.2.3]. Note that the proofs are independent of the characteristic except for [13, Lemma 3.2.3]. Here the additional prime to $p$ assumption saves the splitting of the trace.

These results use descent for Galois covers which is a bit trickier than for étale ones. We spell this out in Lemma 2.4 for flat Galois covers. While a Galois cover may be non-flat in general the flat locus is a big open subset. The slope only depends on the isomorphism class on an big open subset and Lemma 2.4 can then be applied to the destabilizing subsheaf as well as the socle after restricting to the flat locus.

(v): A proper subsheaf of $F$ of slope $\geq \mu(F)$ pulls back to a proper subsheaf of $F|_Y$ on a big open subset of $Y$ of slope $\geq \mu(F|_Y)$ by (ii). The claim follows. \qed

We recall the notions of $G$-invariance and $G$-linearization and prove a descend lemma under flat Galois covers for the latter.

Definition 2.2. Let $Y \to X$ be a Galois cover of normal varieties with Galois group $G$. Thinking of $Y$ as the normal closure of $X$ in $\kappa(Y)$ we obtain an action of $G$ on $Y/X$.

A $G$-invariant torsion-free sheaf $V$ on $Y$ is a torsion-free sheaf $V$ together with isomorphisms $\psi_\sigma : V \xrightarrow{\sim} \sigma^* V$ for all $\sigma \in G$. By a slight abuse of notation we suppress the choice of the isomorphisms and call $V$ a $G$-invariant torsion-free sheaf.

A subsheaf $W \subseteq V$ of a $G$-invariant torsion-free sheaf $V$ is called $G$-invariant if the isomorphisms $\psi_\sigma : V \xrightarrow{\sim} \sigma^* V$ induce isomorphisms $W \xrightarrow{\sim} \sigma^* W$ of subsheaves.

A torsion-free sheaf $V$ on $Y$ is said to admit a $G$-linearization if for all $\sigma \in G$ there exists an isomorphism $\psi_\sigma : V \xrightarrow{\sim} \sigma^* V$ such that $\tau^* \psi_\sigma \circ \psi_\tau = \psi_{\sigma \tau}$ for all $\sigma, \tau \in G$.

Remark 2.3. By definition a $G$-invariant subsheaf $W \subseteq V$ of a torsion-free sheaf admitting a $G$-linearization admits a $G$-linearization as well.

For an étale Galois cover a linearization is the same as a descent-datum. This is in general not true for Galois covers or even flat Galois covers, see Example 2.5. There is however a version for an invariant saturated subsheaf of a torsion-free sheaves which descends:

Lemma 2.4. Let $Y \to X$ be a flat Galois cover of normal varieties with Galois group $G$. Let $V$ be a torsion-free sheaf on $X$. Then a $G$-invariant saturated subsheaf of $V|_Y$ descends to a saturated subsheaf of $V$.

Proof. Let $\eta_Y$ be the generic point of $Y$ and $\eta_X$ the generic point of $X$.

Consider a $G$-invariant saturated subsheaf $W \subseteq V|_Y$. Restricting the inclusion to $\eta_Y$ we obtain a $G$-invariant subvector space $W|_{\eta_Y} \subseteq (V|_{\eta_X})_{|_{\eta_Y}}$. 
The field extension \( \kappa(Y)/\kappa(X) \) is a \( G \)-torsor and we can apply descent theory. We obtain \( W'_{\eta_X} \subseteq V_{\eta_X} \) such that \( W'_{\eta_X} \otimes_{\kappa(X)} \kappa(Y) = W_{\eta_Y} \), as subspaces of \( (V_{\eta_X})_{|_{V_Y}} \).

By [18, Proposition 1], which also holds for varieties not just smooth projective varieties, there is a unique saturated subsheaf \( W' \subseteq V \) inducing the inclusion \( W'_{\eta_X} \subseteq V_{\eta_X} \). Pulling back along the flat morphism \( Y \to X \) we obtain a saturated subsheaf \( W'_{|_{V_Y}} \subseteq V_{|_{V_Y}} \) which agrees with the inclusion \( W_{\eta_Y} \subseteq (V_{\eta_X})_{|_{V_Y}} \) on the generic point. By another application of [18, Proposition 1] we conclude \( W'_{|_{V_Y}} = W \). \( \square \)

We provide examples which show that neither 'saturated' nor 'subsheaf of a sheaf which descends' can be removed in Lemma 2.3.

Example 2.5. Let \( E \) be an elliptic curve and \( \pi : E \to \mathbb{P}^1 \) be a 2 : 1 Galois cover ramified at 4 points. Denote the non-trivial element of the Galois group \( G = \mathbb{Z}/2 \) by \( \sigma \).

Consider a line bundle \( L \) of degree 1 on \( E \). Then \( L \oplus \sigma^* L \) admits a \( G \)-linearization, but does not descend to \( \mathbb{P}^1 \). Indeed, if there was a vector bundle \( V \) on \( \mathbb{P}^1 \) such that \( V|_E \cong L \oplus \sigma^* L \), then \( V \) is semistable of slope \( \frac{1}{2} \) by Lemma 2.1 Grothendieck’s classification of vector bundles on \( \mathbb{P}^1 \) does not allow for such a bundle, see e.g. [11].

Consider a point \( e \in E \) at which \( \pi \) is ramified. Let \( I \) be the effective Cartier divisor which cuts out \( e \in E \). Then \( I \) is a \( G \)-invariant subsheaf of \( \mathcal{O}_E \) but does not descend to a subsheaf \( I' \) of \( \mathcal{O}_C \). Indeed, by Lemma 2.1 such a subsheaf \( I' \) would be a line bundle of slope \( \frac{1}{2} \), which is impossible.

2.2. Functoriality and Small Genus.

Definition 2.6. A finite group \( G \) is called prime to \( p \) if \( p \nmid \#(G) \). A finite separable cover (resp. étale cover) \( \pi : Y \to X \) of varieties is prime to \( p \) if the Galois hull of \( \kappa(Y)/\kappa(X) \) (resp. of \( Y/X \)) has Galois group prime to \( p \).

Observe that prime to \( p \) morphisms are well-behaved under composition, i.e., the composition of two such morphisms is again prime to \( p \). We now introduce our functorial notions of stability.

Definition 2.7. Let \( X \) be a projective variety. A sheaf \( V \) on \( X \) is called separable-stable, (resp. étale-stable, resp. prime to \( p \) stable) if for every finite separable, (resp. finite étale, resp. finite étale prime to \( p \)) morphism \( \pi : Y \to X \) of varieties the pullback \( \pi^* V \) is stable with respect to \( \pi^* \mathcal{O}_X(1) \).

Example 2.8. Every line bundle is separable-stable. If \( p > 0 \), then a semistable vector bundle of rank \( r = p^n \), \( n \geq 1 \), and degree coprime to \( p \) is prime to \( p \) stable.

A finite separable morphism has two parts, namely an étale part and a genuinely ramified part. We recall the definition:

Definition 2.9. Let \( f : Y \to X \) be a cover of varieties. We say that \( f \) is genuinely ramified if every factorization \( Y \to Y' \to X \) of \( f \) such that \( Y' \to X \) is an étale cover satisfies that \( Y' \to X \) is an isomorphism.

Biswas, Das, and Parameswaran show in [2, Theorem 1.2] that genuinely ramified morphisms of normal projective varieties preserve stability under pullback. As a direct consequence we obtain:

Corollary 2.10. On a normal projective variety the notions of étale-stability and separable-stability agree for vector bundles.
Remark 2.11. Being able to go back and forth between covers and étale covers yields several advantages. On the one hand, it is easier to construct Galois covers than étale Galois covers. On the other hand, descent theory is simpler for étale Galois covers and there are - up to isomorphism - only finitely many étale covers of fixed degree. To be precise we have:

Lemma 2.12. Let $X$ be a normal projective variety. Then for fixed degree $d$ there are only finitely many étale covers $Y \to X$ of degree $d$ (up to isomorphism).

Proof. This is an immediate consequence of the étale fundamental group $\pi_{\text{ét}}(X)$ of $X$ being topologically finitely generated. Let $Y \to X$ be an étale cover of degree $d$. Up to isomorphism $S = \{1, \ldots, d\}$ and the action of $\pi_{\text{ét}}(X)$ on $S$ is given by a continuous morphism $\pi_{\text{ét}}(X) \to S_d$, where $S_d$ denotes the symmetric group of $\{1, \ldots, d\}$ equipped with the discrete topology. As the étale fundamental group of a normal projective variety is topologically finitely generated, see [23, Satz 13.1], there are only finitely many continuous morphisms to a fixed finite group with the discrete topology. □

The notion of étale-stability on a smooth projective curve $C$ is only interesting if $g_C \geq 2$.

Lemma 2.13. Let $C$ be a smooth projective curve of genus $g_C \leq 1$. Then the following hold:

(i) If $g_C = 0$, then the only stable bundles are line bundles.
(ii) If $g_C = 1$, then a stable vector bundle of rank $r$ and degree $d$ is prime to $p$ stable iff $(r, d) = (1)$ and $r$ is a power of $p$.
(iii) If $g_C = 1$ and $C$ is an ordinary elliptic curve, then the only étale stable bundles are line bundles.
(iv) If $g_C = 1$ and $C$ is supersingular, then the notions of prime to $p$ stable and étale stable agree.

Proof. If $g_C = 0$, then (i) follows from Grothendieck’s classification of vector bundles on $\mathbb{P}^1$, see e.g. [11].

In the following we use that semistability is preserved under pullback by a cover and the behaviour of the degree under pullback, see Lemma 2.1.

If $g_C = 1$, we use [1, Theorem 5 and Theorem 7], which are both valid in arbitrary characteristic. These theorems immediately imply that there are no stable bundles of rank $r > 1$ and integral slope over an elliptic curve. In fact more can be said: a semistable vector bundle of rank $r$ and degree $d$ is stable iff $(r, d) = (1)$, a direct consequence of [22, Corollary 2.5].

Consider a stable bundle $V$ of rank $r > 1$ and degree $d$ such that $(r, d) = (1)$. On an étale cover of degree non-coprime to $r$ the pullback of $V$ can not be stable by the previous discussion. This proves the claim (iii) for ordinary elliptic curves as they have étale covers of any square degree. Indeed, for $d$ not divisible by $p$ multiplication by $d$ is of degree $d^2$. For $d = p$ the dual of the Frobenius $F^\vee : E \to E^{(p)}$ is étale of degree $p$.

If $r$ is a power of $p$ and $(r, d) = (1)$, then on all prime to $p$ covers we still have coprime rank and degree. This proves (ii).

If $C$ is supersingular, then every étale cover is prime to $p$ and we obtain (iv). □
3. Proof of Theorem 1

The idea to prove Theorem 1 is simple: There are two types of failure for a stable bundle to remain stable after pullback. Both of these failures can be detected on single cover. We make this more precise on a smooth projective curve $C$.

The key observation is that a stable bundle $V$ of rank $r$ on $C$ decomposes on an étale Galois cover $D \rightarrow C$ as $V_D \cong \bigoplus_{i=1}^n W_i^{\oplus e_i}$ for some pairwise non-isomorphic stable bundles $W_i$ on $D$ such that the Galois group acts transitively on the isomorphism classes of the $W_i$, see Lemma 3.2. This is somewhat similar to the decomposition of a prime ideal in a Galois extension of number fields; in particular $e$ does not depend on the index $i$.

If $n \geq 2$, this decomposition behaviour can already be detected on an étale Galois cover $C_{r-\text{large}}$, a cover dominating all étale covers of degree dividing $\operatorname{rk}(V) = r$, see Lemma 3.3.

If $V$ remains stable on $C_{r-\text{large}}$, then for any étale Galois cover $D \rightarrow C$ the decomposition is $V_D \cong W^{\oplus e}$. Pretending that $W$ descends to a stable bundle $M$ on $C$ (this is not clear at all but we provide a technical workaround, see Lemma 3.5) we can compare the descent data associated to $M^{\oplus e}$ and $V$ to obtain a $GL_r$-representation $\rho$ of the Galois group $\text{Gal}(D/C) = G$. The descent data agree on the kernel of $\rho$ and we are reduced to $G$ being a finite subgroup of $GL_r$. If $G$ is prime to $p$, then Jordan’s theorem - which also has a positive characteristic version due Brauer and Feit - has a particularly nice form:

**Theorem 3.1 ([14] p.114 for characteristic 0, [4] for positive characteristic).** Let $r$ be a natural number, $r \geq 1$. There exists a constant $J(r)$ such that for every finite prime to $p$ subgroup $G \subset GL_r$ there exists a normal abelian subgroup $N \leq G$ of index $\leq J(r)$.

Thus, there exists a normal abelian subgroup $N \subseteq G$ of index $\leq J(e)$, where $J(e)$ denotes the constant from Jordan’s theorem. As a finite abelian subgroup is simultaneously triagonalizable the decomposition $V_D \cong W^{\oplus e}$ can already be detected on $D/N$. We obtain a prime to $p$ étale Galois cover $C_{r-\text{good}}$ which detects the stability of $V_D$ as a cover dominating all prime to $p$ covers of degree $\leq rJ(r)$.

We split the construction of $C_{r-\text{good}}$ into two parts. First we show the key lemma and construct $C_{r-\text{large}}$. This construction can also be carried out over any normal projective variety.

Then we continue with the workaround for descending $W$ and finally construct $C_{r-\text{good}}$. The same type of cover works over a normal projective variety $X$. However, the workaround for descent only works for curves. Thus, one has to complete the descent setup on the level of $X$ and then restrict the setup to a large curve.

3.1. A large cover. The key observation for the (non-)functoriality of stability is the following lemma. A stable bundle can only decompose in a very special way after a Galois pullback.

**Lemma 3.2** (Key observation). Let $\pi : Y \rightarrow X$ be a Galois cover of normal projective varieties with Galois group $G$. Let $V$ be a stable vector bundle on $X$ of rank $r$. Then $V_Y \cong \bigoplus_{i=1}^n W_i^{\oplus e_i}$ for some pairwise non-isomorphic stable vector bundles $W_i$ on $Y$ and $n, e \geq 1$. Furthermore, $G$ acts transitively on the set of isomorphism classes $\{W_i \mid i = 1, \ldots, n\}$.

In particular, all the $W_i$ have the same rank $\frac{r}{ne}$. 

Proof. By Lemma 2.1 the bundle $V_Y$ is polystable. As $V_Y$ is a vector bundle we find that $V_Y \cong \bigoplus_{i=1}^{n} W_i^{\sigma_i}$ for pairwise non-isomorphic stable vector bundles $W_i$ on $Y$. Let $\iota : W \rightarrow V_Y$ denote the inclusion of one of the $W_i$. The image of $\bigoplus_{\sigma \in G} \sigma^* W \bigoplus_{\sigma_i} V_Y$ is a $G$-invariant subbundle and descends to a subbundle $E$ of $V$ by Lemma 2.4. As $E$ has the same slope as $V$, the stability of $V$ implies $E \cong V$. We obtain that $\bigoplus_{\sigma \in G} \sigma^* W \rightarrow V_Y$ is surjective. Using the stability of the $W_i$ we find that the group $G$ acts transitively on the isomorphism classes of the $W_i$. Clearly, $\text{rk}(\sigma^* W) = \text{rk}(W)$ for all $\sigma \in G$.

Let $e = e_{i_0}$ be the smallest index among the $e_i$ and $W = W_i$. For each $W_i$ there is a $\sigma_i \in G$ such that $\sigma_i^* W \cong W_i$. The inclusion $W_i^{\sigma_i} \rightarrow V_Y$ induces an inclusion $W_i^{\sigma_i} \rightarrow V_Y$ after pullback by $\sigma_i^{-1}$. We obtain $e_i \leq e$. By definition of $e$ we have equality. The computation of the rank of $W_i$ is now immediate. \hfill \Box

There are two fundamentally different ways for a stable bundle to decompose on a Galois cover: $n = 1$ or $n \geq 2$ in Lemma 3.2. We first find a cover that checks for $n \geq 2$ using that this decomposition can already be seen on a cover of degree $n$.

Lemma 3.3. Let $\pi : Y \rightarrow X$ be a Galois cover of normal projective varieties with Galois group $G$. Further, let $V$ be a stable vector bundle of rank $r$ on $X$ such that the decomposition $V_Y \cong \bigoplus_{j=1}^{n} W_j^{\sigma_j}$ of Lemma 3.2 satisfies $n \geq 2$. Then there is a factorization of $Y \rightarrow X$ into $Y \rightarrow Y' \overset{\pi'}{\rightarrow} X$ such that $\deg(\pi') = n$ and $V|_{Y'}$ is not stable.

More precisely, $V|_{Y'} \cong V' \oplus W'$, where $W'$ is of rank $r/n$ and $V|_{Y'}$ is isomorphic to a direct sum of conjugates of $W|_{Y'}$ under $G$.

Proof. By assumption there are at least two different $W_i$. Consider the stabilizer $H$ of $W := W_i^{\sigma_i}$ for some $i$ and fix an inclusion $\iota : W \rightarrow V_Y$. The image $E$ of $\bigoplus_{\sigma \in H} \sigma^* W \bigoplus_{\sigma_i} V_Y$ is an $H$-invariant subsheaf. Using the stability of the $W_i$ we find that $E$ is isomorphic to $W$. Therefore, the direct summand $W$ of $V_Y$ descends to a direct summand $W'$ of $V|_{Y'}$, where $Y' = Y/H$ and $Y \rightarrow Y' \overset{\pi'}{\rightarrow} X$ are the induced morphisms. Note that $\pi'$ has degree $\#(G/H) = n$.

Let $V|_{Y'} \cong W' \oplus V'$. As $G$ acts transitively on the isomorphism classes of the $W_i$ we have that $V|_{Y'}$ is a direct sum of $\sigma^* W|_{Y'}$ for some $\sigma \in G$. \hfill \Box

As a direct consequence we obtain the large cover checking for decomposition of a stable bundle into at least two non-isomorphic stable bundles on some cover:

Lemma 3.4. Let $X$ be a normal projective variety and $r \geq 2$. Then we have the following:

(i) There exists an étale Galois cover $X_{r_{-\text{large}}} \rightarrow X$ satisfying the following:

If $V$ is a vector bundle of rank $r$ on $X$ such that $V|_{X_{r_{-\text{large}}}}$ is stable, then for all étale Galois covers $Y \rightarrow X$ we have $V|_Y \cong W^{\oplus e}$ for some stable vector bundle $W$ on $Y$ and $e \geq 1$.

(ii) There is an étale prime to $p$ Galois cover $X_{r_{-\text{large}}} \rightarrow X$ such that:

If $V$ is a vector bundle of rank $r$ on $X$ such that $V|_{X_{r_{-\text{large}}}}$ is stable, then for all étale prime to $p$ Galois covers $Y \rightarrow X$ we have $V|_Y \cong W^{\oplus e}$ for some stable vector bundle $W$ on $Y$ and $e \geq 1$. 

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Proof. (i): Decomposing into different stable vector bundles descends to some étale cover of degree $n$ such that $n \mid r$, see Lemma 3.3. There are only finitely many such étale covers up to isomorphism, see Lemma 2.12. In particular, there is an étale Galois cover $X_{r\text{-large}}$ dominating all étale covers of degree dividing $r$. This is the desired cover.

(ii): Define $X'_{r\text{-large}}$ as an étale prime to $p$ Galois cover dominating all étale prime to $p$ covers of degree dividing $r$. This is the desired cover. □

3.2. A good cover. To construct the cover $X_{r\text{-good}}$ detecting prime to $p$ stability it remains to deal with decomposition behaviour of the form $V'_Y = W' \otimes \mathfrak{c}$, where $Y \to X$ is a Galois cover of normal projective varieties and $V$ a stable bundle. We start with the workaround for descent of $G$-invariant stable bundles. This requires working on curves and the mild assumption that $\det(W)$ already descends. The determinant-descent can be set up on arbitrary varieties and we are then able to derive the main theorem by reducing to the case of curves via a restriction theorem for stability.

We start with the workaround for descent. If one is only interested in the case of curves, then there is an honest descent lemma one could use instead, see Lemma 4.3. The workaround roughly says that a $G$-linearization of the determinant of an $G$-invariant simple bundle can lifted to a linearization for a slightly bigger Galois cover.

Lemma 3.5 (Workaround for descent). Let $D \to C$ be a Galois cover of smooth projective curves with Galois group $G$. Let $V$ be a simple $G$-invariant vector bundle of rank $r$ on $D$. Further, assume that $\det(V)$ admits a $G$-linearization.

Then there exists a lift of the $G$-linearization of $\det(V)$ to a system of isomorphisms $\psi_\sigma : V \simto \sigma^* V$. Furthermore, there exists a cyclic Galois cover $\varphi : D' \to D$ such that

(i) $\varphi$ is prime to $p$ of degree $\deg(\varphi) \mid r$,
(ii) $D' \to D \to C$ is a Galois cover,
(iii) $\Gal(D'/D) \subseteq \Gal(D'/C)$ is central, and
(iv) there exists a 1-cocycle $\alpha : \Gal(D'/C) \to \mu_r$ such that

$$\varphi^*(\psi_\sigma) \cdot \alpha(\sigma')^{-1} : V_{|D'} \simto \sigma'^* V_{|D'}$$

defines a $\Gal(D'/C)$-linearization of $V_{|D'}$, where $\sigma$ denotes the image of $\sigma'$ under the natural morphism $\Gal(D'/C) \to G$.

Proof. For two simple isomorphic bundles $V$ and $W$ we have a surjective morphism $\Hom(V,W) \xrightarrow{\det} \Hom(\det(V),\det(W))$. Thus, the $G$-linearization of $\det(V)$ lifts to isomorphisms $\psi_\sigma : V \simto \sigma^* V$ such that $\psi_{\sigma r}^{-1} \circ \tau^* \psi_\sigma \circ \psi_\tau = \lambda_{\sigma,\tau} \in \mu_r$. Indeed, after identifying $\Hom(V,V)$ with $k$ the determinant corresponds to the $r$-th power map.

A computation [5, Proposition 2.8] shows that the family $\lambda_{\sigma,\tau}$ defines a 2-cocycle. Let $p^r = r$ with $r'$ coprime to $p$ and $\lambda'_{\sigma,\tau} = \lambda_{\sigma,\tau}^p$. The 2-cocycle condition for $\lambda_{\sigma,\tau}$ implies the 2-cocycle condition for $\lambda'_{\sigma,\tau}$. We obtain $\lambda' := (\lambda'_{\sigma,\tau}) \in H^2(G,\mu_r)$.

Let $\Gal$ be the absolute Galois group of $\kappa(C)$. As $C$ is a curve over an algebraically closed field, $\kappa(C)$ is a $C_1$ field by Tsen’s Theorem, see [21, Corollary 6.5.5]. In particular, $H^2(\Gal, (\kappa(C)^{sep})^*)$ vanishes, see [21, Proposition 6.5.8]. By Hilbert 90 we also have vanishing of $H^1(\Gal, (\kappa(C)^{sep})^*)$, see [21, Theorem 6.2.1]. Applying these two vanishing results to the long exact cohomology sequence of the short
exact sequence

$$0 \to \mu_{r'} \to (\kappa(C)^{sep})^* \xrightarrow{x \mapsto x^{r'}} (\kappa(C)^{sep})^* \to 0$$

we obtain $H^2(\text{Gal}, \mu_{r'}) = 0$.

By [21, Theorem 1.2.4] the element $\lambda' \in H^2(G, \mu_{r'})$ corresponds to an extension

$$0 \to \mu_{r'} \to G' \to G \to 0$$

inducing the action of $G$ on $\mu_{r'}$. As the action of $G$ on $\mu_{r'}$ is trivial, we find that $\mu_{r'}$ is central in $G'$. Write $G$ as a quotient of $\text{Gal}$. Since $H^2(\text{Gal}, \mu_{r'}) = 0$, we obtain that the central extension

$$0 \to \mu_{r'} \to \text{Gal} \times_G G' \to \text{Gal} \to 0$$

is a central split extension and thus trivial.

The kernel $K$ of $H \to G$ is a subgroup of $\mu_{r'}$. In particular, $K \subseteq H$ is central and cyclic. Denote by $\kappa(D')$ the field extension of $\kappa(C)$ corresponding to $\text{Gal} \to H$ and by $D'$ the associated curve. We obtain Galois covers $D' \xrightarrow{\sim} D \to C$ such that $\text{Gal}(D'/D) \subseteq \text{Gal}(D'/C)$ is central and cyclic. Furthermore, the obstruction $\lambda' \in H^2(G, \mu_{r'})$ vanishes in $H^2(H, \mu_{r'})$.

The triviality of the 2-cocycle $\phi^* \lambda' \in H^2(H, \mu_{r'})$ means that there is a 1-cocycle $\alpha' : H \to \mu_{r'}$ such that $\partial(\alpha')(\sigma, \tau) = \lambda'_{f(\sigma), f(\tau)}$, where $f : H \to G$ denotes the surjection constructed above.

Recall that in positive characteristic $p$-th roots are unique. Thus, there is a 1-cocycle $\alpha : H \to \mu_{r'}, \sigma \mapsto \alpha'(\sigma)^{1/p}$ such that $\partial(\alpha)(\sigma, \tau) = \lambda'_{f(\sigma), f(\tau)}$. By construction the isomorphisms $\phi^* \psi_{f(\sigma)} \cdot \phi(\sigma)^{-1}, \sigma \in H$, define a linearization. Indeed, we have

$$(\phi^* \psi_{f(\sigma)} \cdot \phi(\sigma)^{-1})^{-1} \circ \tau^* \phi^* \psi_{f(\sigma)} \cdot \phi(\sigma)^{-1} \circ \phi^* \psi_{f(\tau)} \cdot \phi(\tau)^{-1} = \lambda'_{f(\sigma), f(\tau)} \cdot (\partial(\alpha)(\sigma, \tau))^{-1} = 1.$$ 

Remark 3.6. A shorter (but less precise) argument is the following: Recall that $H^2(\text{Gal}, \mu_{r'}) = \text{colim} H^2(G', \mu_{r'})$, see [21, Proposition 1.2.5], where the colimit is taken over all finite Galois extensions of $\kappa(C)$ and $G'$ denotes the Galois group. We obtain $\text{Gal} \to G' \to G$ such that the obstruction $\lambda$ vanishes on the associated curve. However, this does not give us a way to control the kernel which is crucial.

Note that Lemma 3.5 only works for curves and requires the mild assumption that the determinant descends. Given that the determinant descends we can detect decomposition on a cover of degree bounded by the constant of Jordan’s theorem. We do this in the following lemma. This would already allow us to deduce Theorem 1 for curves but we only give the general proof later.

Lemma 3.7. Let $D \to C$ be an étale prime to $p$ Galois cover with Galois group $G$. Let $V$ be a vector bundle on $C$ such that $V|_D \cong W^{\otimes e}$ for some simple $G$-invariant
vector bundle $W$ satisfying that $\det(W)$ descends to $C$. Denote the constant from Jordan’s theorem, see Theorem 3.11, by $J(e)$.

Then there exists a normal subgroup $N \subseteq G$ of index $\leq J(e)$ and $W' \subseteq V_{G'}$ such that $W'_{|D'} \cong W$, where $D \to C' := D/N \to C$ are the natural morphisms.

**Proof.** Denote the rank of $W$ by $r$. Let $\psi^W : W \xrightarrow{\sim} \sigma^*W, \sigma \in G$, be a system of isomorphisms lifting the descent datum of $\det(W)$, see Lemma 3.5. By the same lemma there is a Galois cover $D' \xrightarrow{\sim} D$ with prime to $p$ cyclic Galois group $H$ such that $D' \to D \to C$ is a Galois cover with Galois group $G'$. Further, there exists a 1-cocycle $\alpha : G' \to \mu_r$ such that $\varphi^*(\psi^W_\sigma) \cdot \alpha(\sigma')^{-1}$ is a $G'$-linearization, where $\sigma$ denotes the image of $\sigma'$ in $G$. Furthermore, $H \subseteq G'$ is central.

Our goal is to find a normal subgroup $N' \subseteq G'$ of index $\leq J(e)$ containing $H$ and an $N'$-invariant subbundle $W|_{D'} \subseteq V_{D'}$. By Lemma 2.4 the inclusion $W|_{D'} \subseteq V_{D'}$ descends to $C'$, where $C'$ denotes the normal closure of $C$ in the fixed field $k(D')^{N'}$. Then the lemma follows as $C' = C/N$, where $N$ is the image of $N'$ in $G$.

Let $\psi^V : V|_{D'} \xrightarrow{\sim} \sigma^*V_{D'}$ be the descent datum associated to $V$. Choose an isomorphism $V|_{D} \xrightarrow{\sim} W^{|D}$ which exists by assumption. Define a map

$$\rho : G' \to \text{Gl}_r, \sigma' \mapsto \text{diag}(\alpha(\sigma'))((\psi^W_\sigma)^{-1})^{\otimes_e} \circ \sigma^*(\psi) \circ \psi^V_{\sigma} \circ \psi^{-1},$$

where $\sigma$ denotes the image of $\sigma'$ in $G$, i.e., $\rho$ measures the failure of the following diagram

$$\begin{array}{c}
\begin{array}{ccc}
W & \xrightarrow{\psi} & V|_{D} \\
((\psi^W_\sigma)^{-1})^{\otimes_e} & \xrightarrow{\sigma^*W^{\otimes_e}} & \sigma^*V|_{D}
\end{array}
\end{array}$$

\begin{array}{c}
\begin{array}{c}
\sigma^*(\psi)
\end{array}
\end{array}
$$

to commute twisted by $\text{diag}(\alpha(\sigma'))$. Another way to put this is that $\rho$ compares the $G'$-linearizations $(\varphi^*(\psi^W_\sigma)^{-1})^{\otimes_e}\text{diag}(\alpha(\sigma'))$ and $\varphi^*(\psi^V_\sigma)$ on $D'$.

We claim that $\rho$ defines a group morphism. Indeed, for $\sigma', \tau' \in G'$ mapping to $\sigma$ (resp. $\tau$) in $G$ we have

$$\rho(\tau')\rho(\sigma') =
\begin{align*}
\text{diag}(\alpha(\tau'))((\psi^W_\tau)^{-1})^{\otimes_e} &\tau^*(\psi)\psi^V_{\tau} \psi^{-1} \text{diag}(\alpha(\sigma'))((\psi^W_\sigma)^{-1})^{\otimes_e} \sigma^*(\psi)\psi^V_{\sigma} \psi^{-1} = \\
\text{diag}(\alpha(\tau')) &\alpha(\sigma'))((\psi^W_\tau)^{-1})^{\otimes_e} \tau^*(\psi)\psi^V_{\tau} \psi^{-1}((\psi^W_\sigma)^{-1})^{\otimes_e} \sigma^*(\psi)\psi^V_{\sigma} \psi^{-1} = \\
\text{diag}(\alpha(\tau') &\alpha(\sigma'))((\psi^W_\tau)^{-1})^{\otimes_e} \sigma^*(((\psi^W_\tau)^{-1})^{\otimes_e} \tau^*(\psi)\psi^V_{\tau} \psi^{-1})\sigma^*(\psi)\psi^V_{\sigma} \psi^{-1} = \\
\text{diag}(\alpha(\tau') &\alpha(\sigma'))(((\psi^W_\tau)^{-1})^{\otimes_e} \sigma^*(((\psi^W_\tau)^{-1})^{\otimes_e} \tau^*(\psi)\sigma^*(\psi)\psi^V_{\tau} \psi^{-1}) = \\
\text{diag}(\alpha(\tau') &\alpha(\sigma'))(((\psi^W_\tau)^{-1})^{\otimes_e} \sigma^*\tau^*(\psi)\sigma^*(\psi)\psi^V_{\tau} \psi^{-1} = \\
\rho(\tau') &\rho(\sigma'),
\end{align*}$$

where only the third and fifth equality require an explanation. To obtain the third equality we use that $((\psi^W_\sigma)^{-1})^{\otimes_e}$ commutes with matrices and that matrices with entries in $k$ do not change under pullback. To obtain the fifth equality we note that by construction of $\alpha$ the isomorphisms $\varphi^*(\psi^W_\sigma)\alpha(\sigma')^{-1}$ define a $G'$-linearization, see Lemma 3.5.

Replacing $D'$ by $D' / \ker(\rho)$ we can assume that $G'$ is a subgroup of $\text{Gl}_r$. By Jordan’s theorem, see Theorem 3.11, there is a normal abelian subgroup $N' \subseteq G'$
such that $G'/N'$ has cardinality at most $|J(e)|$. As $H$ is central in $G'$ the subgroup $N' + H$ is normal, abelian, and contains $H$. As a finite abelian subgroup of $G$ is simultaneously triagonalizable, we find the desired $(N' + H)$-invariant inclusion $W|_{D'} \subseteq V|_{D'}$. \hfill $\square$

To be able to apply the previous lemma we need to find a way to descend the determinant bundle. For such a construction we need to take roots of line bundles. If we avoid the characteristic, then this is always possible up to a cyclic cover.

**Lemma 3.8.** Let $X$ be a normal projective variety. Let $d$ be an integer prime to $p$. Further, let $L$ be a line bundle on $X$. Then there exists a cyclic Galois cover $\varphi : X' \to X$ such that $\deg(\varphi) \mid d$ and $L|_{X'}$ admits a $d$-th root on a big open subscheme.

**Proof.** Let $\mathcal{O}_X(1)$ be an ample line bundle. Clearly, it suffices to find a morphism $X' \to X$ as in the statement such that $L|_{X'} \otimes \mathcal{O}_X(1)^\otimes N$ has a $d$-th root for some $N$. Thus, we can assume that $L$ admits a non-zero global section, i.e., $L = \mathcal{O}_X(D)$ for some effective Cartier divisor $D$. Observe that it suffices to prove the Lemma for $\mathcal{O}_X(D)$ instead of $L$.

Choose an affine open $U$ containing the generic point of $D$ in $X$ such that $D|_U = V(f)$ for some non-zero divisor $f \in \mathcal{O}_U$. Consider the field extension $K/\kappa(X)$ generated by a $d$-th root of $f$. As $p \nmid d$ the extension $K/\kappa(X)$ is cyclic of order $d' \mid d$. Let $X'$ denote the normalization of $X$ in $K$. Note that there is a canonical finite morphism $\varphi : X' \to X$ of normal projective varieties. It is also separable by construction. As we only want to find an $d$-th root on a big open and $X' \to X$ is flat at all codimension 1 points, we can assume that $X' \to X$ is flat.

Consider $U' := \varphi^{-1}(U) \cup \varphi^{-1}(X \setminus D)$. By construction $U'$ is big. We show that $\mathcal{O}_X(D)|_{X'}$ admits a $d$-th root on $U'$. Let $t$ be a $d$-th root of $f$ on $\varphi^{-1}(U)$. Then $t$ defines an effective Cartier divisor $D'$ on $U'$. We have $\mathcal{O}_{U'}(-D')^\otimes d = \mathcal{O}_X(-D)|_{U'}$, as $t^d = f$ on $\varphi^{-1}(U)$ by construction and both are trivial on $\varphi^{-1}(X \setminus D)$. \hfill $\square$

**Definition 3.9.** A morphism $\pi : Y \to X$ of varieties is called quasi-étale if there is some big open subset $U \subseteq X$ such that $\pi^{-1}(U) \to U$ is étale. If $\pi^{-1}(U) \to U$ is an étale Galois cover with Galois group $G$, then we also say that $\pi : Y \to X$ is a quasi-étale Galois cover with Galois group $G$.

We can now set up the determinant descent needed to apply Lemma 3.7 on a normal projective variety.

**Lemma 3.10.** Let $X$ be a normal projective variety. Let $Y \to X$ be an étale prime to $p$ Galois cover with Galois group $G$. Further, let $V$ be a stable vector bundle of rank $r$ on $X$ such that $V$ is stable on $X_{r, \text{large}}$. Then there exists a commutative diagram of normal projective varieties

\[
\begin{array}{c}
Y' & \longrightarrow & X' \\
\downarrow & & \downarrow \\
Y & \longrightarrow & X
\end{array}
\]

such that

(i) we have $V_{Y'} \cong W'^{\otimes e'}$ such that $W'$ is stable and $\det(W')$ descends along $Y' \to X'$ on some big open subscheme of $Y'$,

(ii) $Y' \to X$ is a prime to $p$ Galois cover,
\((iii)\) \(X' \to X\) is cyclic of degree dividing \(r\), and 
\((iv)\) \(Y' \to X'\) is a quasi-étale Galois cover.

**Proof.** Consider the decomposition \(V_{|Y'} \cong W^{\otimes e}\) of Lemma 3.2. Clearly, \(\det(W)^{\otimes e}\) and \(\bigotimes_{g \in G} \sigma^* \det(W) \cong \det(W)^{\otimes \#(G)}\) descend to \(X\). Therefore, \(\det(W)^{\otimes d}\) descends to \(X\) as well, where \(d = \gcd(e, \#(G))\). Thus, there exists a line bundle \(L\) on \(X\) such that \(L_{|Y'} \cong \det(W)^{\otimes d}\). Note that \(p \nmid d\) since \(G\) is prime to \(p\).

We can apply Lemma 3.10 to find an étale prime to \(p\) on \(X\) good such that \(\prod_{\kappa' \mid \kappa}\left[\kappa' / \kappa\right] \cong \prod_{\kappa' \mid \kappa} \left[\kappa' / \kappa\right]\), with \(\kappa' \mid \kappa\) cyclic of degree dividing \(e'\). Then we have \(V_{|Y'} \cong W^{\otimes e'}\) for \(e' > e\) and \(W'\) stable. As the integer \(e'\) is at most \(r\), this process stops after finitely many iterations. \(\square\)

We can now prove the main theorem.

**Theorem 3.11.** Let \(X\) be a normal projective variety of dimension at least 1. Let \(r \geq 2\). Then there exists an étale prime to \(p\) Galois cover \(X_{r-\text{good}} \to X\) such that a vector bundle \(V\) of rank \(r\) on \(X\) is prime to \(p\) stable iff \(V_{|X_{r-\text{good}}}\) is stable.

In particular, prime to \(p\) stability is an open property in the moduli space of Gieseker semistable sheaves on \(X\).

**Proof.** Let \(X_{r-\text{good}}\) be an étale prime to \(p\) Galois cover dominating \(X_{r-\text{large}}\) from Lemma 3.4 and all prime to \(p\) covers of degree \(\leq J(r)\), where \(J(r)\) is the bound from Jordan's theorem, see Theorem 3.1.

The "only if" part is trivial. For the "if" part let \(V\) be a vector bundle of rank \(r\) on \(X\) such that \(V_{|X_{r-\text{good}}}\) is stable. Consider an étale prime to \(p\) Galois cover \(Y \to X\) and let \(V_{|Y} \cong W^{\otimes e}\) be the decomposition of Lemma 3.2. Applying Lemma 3.10 we obtain a commutative diagram

\[
\begin{array}{ccc}
Y' & \longrightarrow & X' \\
\downarrow & & \downarrow \\
Y & \longrightarrow & X
\end{array}
\]
satisfying the properties (i) - (iv) of Lemma 3.10. In particular, we have an isomorphism \( V_{|Y'} \cong W'^{\oplus e'} \) for some stable bundle \( W' \) such that \( \det(W') \) descends on some big open along \( Y' \to X' \).

Observe that \( V' := V_{|X'} \) is stable as the degree of \( X' \to X \) is at most \( r \).

By Bertini's theorem the general complete intersection curve \( C' \) in \( X' \) is irreducible and irreducible after pullback to \( Y' \), see [17, Corollary 6.11 (3)]. Furthermore, the general such \( C' \) is also normal by [24, Theorem 7]. The general hyperplane section intersects the locus where \( Y' \to X' \) is not étale transversally. As \( Y' \to X' \) is quasi-étale we obtain that the pullback \( D' \) of the general such \( C' \) is an étale cover of \( C' \). We also note that \( D' \to C' \) is an étale Galois cover with \( Y' \) with the same Galois group as \( Y' \to X' \).

Observe that there are only finitely many intermediate quasi-étale Galois covers \( Y' \to Y'' \to X' \), where \( Y'' \) is a normal projective variety. On \( Y'' \) the bundle \( V' \) decomposes as \( V'_{|Y''} \cong W''^{\oplus e''} \) for some stable bundle \( W'' \) on \( Y'' \). Iterating the restriction theorem in arbitrary characteristic for normal projective varieties, see [17, Theorem 0.1] for positive characteristic and [26, Theorem 7.17] for arbitrary characteristic, we find that restricting \( W'' \to D'' := Y'' \times_{X'} C' \) is stable, where \( C' \) is a general complete intersection curve in \( c_1(O_X(-N_1)) \ldots c_1(O_X(-N_{n-1})) \) for \( N_i \gg 0 \).

Restricting the decomposition \( V'_{|Y'} \cong W'^{\oplus e'} \) of \( V' \) on \( Y' \) to such a \( D' := Y' \times_{X'} C' \)
we obtain an isomorphism \( (V'_{|C'})_{|D'} \cong (W'_{|D'})^{\oplus e'} \). Note that \( W'_{|D'} \) is stable and for general \( C' \) its determinant \( \det(W'_{|D'}) \) descends to \( C' \) by property (i) of Lemma 3.10.

Hence, we are in a position to apply Lemma 3.11. Thus, there is an intermediate cover \( D' \to D'' \to C' \) of degree \( \leq J(e') \) such that there is a stable subbundle \( M'' \subseteq V''_{|D''} \) pulling back to \( W'_{|D'} \) on \( D' \).

The intermediate cover \( D' \to D'' \to C' \) can be lifted to a quasi-étale factorization of \( Y' \to Y'' \to X' \). Indeed, let \( K \) be the kernel of the natural morphism \( \text{Gal}(D'/D'') \to \text{Gal}(D'/C') \). As \( \text{Gal}(Y'/X') = \text{Gal}(D'/C') \) we can define \( Y'' \) to be the normalization of \( X' \) in the field extension \( k(Y')^{K}/k(X') \).

Note that \( Y'' \to X \) is prime to \( p \) of degree at most \( rJ(e') \leq rJ(r) \). Consider the factorization \( Y'' \to Y''' \to X \) into its étale and genuinely ramified part. We find that \( V_{|Y''} \) is stable by assumption. By [6, Theorem 2.5] genuinely ramified covers preserve stability and the bundle \( V_{|Y'''} = V_{|Y''} \) is stable as well. Thus, \( V_{|Y''} \cong W'' \) and we obtain the stability of \( V'_{|D''} \). Therefore, \( V_{|D''} \cong M'' \) and pulling back to \( D' \) we find \( V_{|D'} \cong W'_{|D'}, \) i.e., \( e' = 1 \). Clearly, \( e \leq e' \) and we conclude that \( V_{|Y'} \) is stable.

**Remark 3.12.** We can interpret Theorem 3.11 in terms of prime to \( p \) étale trivializable bundles of rank \( r \geq 2 \), i.e., bundles that become trivial after pullback to some étale prime to \( p \) Galois cover. Such bundles correspond to a \( \text{GL}_r \)-representation of the prime to \( p \) completion \( \pi_{\text{ét}}(X) \). Moreover, stable prime to \( p \) étale trivializable bundles correspond to irreducible representations of \( \pi_{\text{ét}}(X) \). Then Theorem 3.11 says that such an irreducible representation of rank \( r \) becomes reducible after restricting along \( \pi_{\text{ét}}(X_{r,good}) \to \pi_{\text{ét}}(X) \).

For such a statement an étale prime to \( p \) Galois cover dominating all étale covers of degree bounded by the constant of Jordan’s theorem would suffice. Indeed, any representation of \( p : \pi_{\text{ét}}(X) \to \text{GL}_r \) factors via a finite prime to \( p \) subgroup \( G \) of \( \text{GL}_r \) such that \( G \) is a quotient of \( \pi_{\text{ét}}(X) \). By (the analogue of) Jordan’s theorem,
Theorem 3.1, there exists a finite abelian subgroup \( N \) of \( G \) of index at most \( J(r) \). Then \( G \) corresponds to an \( \text{étale} \) Galois cover \( Y \to X \) with Galois group \( G \) and the subgroup \( N \) corresponds to an intermediate \( \text{étale} \) Galois cover \( Y \to Y/N \to X \). By construction the restriction of the representation \( \rho \) to \( \pi'_\text{ét}(Y/N) \) becomes reducible as the abelian group \( N \) does not admit irreducible representation of degree \( r \geq 2 \).

We summarize the argument in a commutative diagram of \( \text{étale} \) Galois covers

\[
\begin{array}{ccc}
X_{r,\text{good}} & \longrightarrow & Y \\
\downarrow & & \downarrow \\
Y \to Y/N & \overset{\rho}{\to} & X,
\end{array}
\]

where we obtained the horizontal factorization via Jordan’s theorem and the dotted arrow using the finiteness of \( \text{étale} \) covers of bounded degree.

4. Proof of Theorem 2

Consider a smooth projective curve \( C \) of genus \( g_C \geq 2 \). To obtain the non-emptiness of the locus of prime to \( p \) stable bundles \( M^{s, r, d}_{C} \) we find estimates for the dimension of the complement

\[
Z := M^{s, r, d}_{C} \setminus M^{p^l-s, r, d}_{C}.
\]

This complement decomposes into two strata \( Z = Z_1 \sqcup Z_2 \), where

\[
Z_1 := \{ V \in M^{s, r, d}_{C} | V|_{C_{r-\text{good}}} \cong W^\oplus e, W \text{ stable on } C_{r-\text{good}}, e \geq 2 \}
\]

and

\[
Z_2 := \{ V \in M^{s, r, d}_{C} | V|_{C_{r-\text{good}}} \cong \bigoplus_{i=1}^{n} W_i^\oplus e, W_i \text{ stable on } C_{r-\text{good}}, n \geq 2 \}
\]

are obtained via applying Lemma 3.2 to \( C_{r-\text{good}} \to C \).

To this end we first reprove a theorem due to Faltings asserting that pullback by a cover induces a finite morphism on the level of moduli spaces of semistable vector bundles. This gives us the flexibility to compute the dimension after such a pullback.

Finding an estimate for \( \dim(Z_2) \) is fairly simple: the transitive action of the Galois group allows us to essentially recover the decomposition \( V|_{C_{r-\text{good}}} \cong \bigoplus_{i=1}^{n} W_i^\oplus e \) from a semistable vector bundle \( W' \) on an intermediate cover \( D' \to C \) of degree \( n \).

To find an estimate for \( \dim(Z_1) \) one has to compare the notions of \( G \)-linearization and \( G \)-invariance. While a \( G \)-invariant vector bundles might not descend, a simple \( G \)-invariant bundle does so up to twist by a line bundle.

4.1. Pullback is finite.

**Lemma 4.1.** Let \( \pi : D \to C \) be an \( \text{étale} \) cover of smooth projective curves. Then we have the following:

(i) The pushforward of a semistable bundle on \( D \) to \( C \) is semistable.
(ii) Let \( V \) be a semistable vector bundle on \( C \). Then \( \pi_* \mathcal{O}_D \otimes V \) is semistable of slope \( \mu(V) \).

**Proof.** (i) This short argument can already be found in the proof of [3, Proposition 5.1] for line bundles of degree 1.
Let $W$ be a semistable bundle of slope $\mu$ and rank $r$ on $D$. The pushforward $\pi_*W$ has slope $\mu/\deg(\pi)$.

If $\pi_*W$ was not semistable, consider the maximal destabilizing subbundle $V$ of $\pi_*W$. By adjunction $\pi^*V \to W$ is a non-zero morphism of semistable bundles. As

$$\mu(\pi^*V) = \deg(\pi)\mu(V) > \deg(\pi)\mu(\pi_*W) = \mu(W)$$

this is a contradiction.

(ii) Let $V$ be a semistable bundle on $C$. As $\pi$ is étale the bundle $\pi_*\mathcal{O}_D$ is of degree 0 by Riemann-Hurwitz. We obtain

$$\mu(V) = \mu(\pi_*(\mathcal{O}_D)) + \mu(V) = \mu(\pi_*(\mathcal{O}_D) \otimes V).$$

By the projection formula we have $\pi_*\mathcal{O}_D \otimes V \cong \pi_*\pi^*V$ which is semistable by (i) and Lemma 2.1 (iii).

As semistable bundles stay semistable under pullback by a cover $\pi : D \to C$, we obtain a morphism $\pi^* : M_{C}^{ss,r,d} \to M_{D}^{ss,r,\deg(\pi)d}$. The finiteness of $\pi^*$ can be proven using the degree of the theta divisor. This can be found in [12, Theorem 4.2] and goes back to [3, Theorem 1.4].

Here we give a shorter proof only using [3, Lemma 4.3] and basic properties of finite étale morphisms.

**Theorem 4.2.** Let $\pi : D \to C$ be a cover of smooth projective curves. Let $r \geq 1$ and $d \in \mathbb{Z}$. Then the induced morphism

$$\pi^* : M_{C}^{ss,r,d} \to M_{D}^{ss,r,\deg(\pi)d}$$

is finite. If $e$ denotes the degree of the étale part of $\pi$, then the fibre of $\pi^*$ at a stable bundle $W$ on $D$ has cardinality at most $e$.

**Proof.** First observe that $\pi^*$ is a morphism of projective varieties. Thus, it suffices to show that it is quasi-finite. Furthermore, it suffices to prove the quasi-finiteness for Galois covers as every cover is dominated by a Galois cover.

As each cover factors as an étale cover and a genuinely ramified cover it suffices to show the theorem for these two types of morphisms separately.

The genuinely ramified case immediately follows from [3, Lemma 4.3]. In fact, the lemma tells us that $\pi^*$ is injective on points: If two polystable bundles on $C$ become isomorphic on $D$, then they are already isomorphic on $C$.

It remains to consider the case where $\pi$ is an étale Galois cover. Let $V$ be a polystable bundle on $C$. Consider the polystable bundle $\pi^*V \cong \bigoplus W_i$, where the $W_i$ are stable on $D$, see Lemma 2.1. By Lemma 4.1 all bundles $\pi_*W_i$ are semistable of slope $\mu(V)$. The projection formula implies that $\pi_*(\mathcal{O}_D) \otimes V \cong \pi_*\pi^*V$. Thus, $V \subseteq \pi_*\pi^*V$ appears in the JH-filtration of $\bigoplus \pi_*W_i$. As the graded object associated to the JH-filtration is unique, there are only finitely many choices for $V$ if we fix $\bigoplus W_i$.

If $V_D \cong W$ is stable on $D$, then comparing the ranks of $V$ and $\pi_*W$ we find that there can be at most $\deg(\pi)$ many different such $V$. \qed

4.2. Strata and dimension. In this subsection we complete the proof of Theorem 2 by a dimension estimate on the complement of the prime to $p$ stable locus. Consider an étale Galois cover $D \to C$ of smooth projective curves with Galois group $G$. There are two different cases depending on whether $n = 1$ or $n \geq 2$ in the decomposition $V_D \cong \bigoplus_{i=1}^n W_i^{\oplus e}$ of Lemma 3.2. If $n = 1$, then there is only
Consider the short exact sequence

\[ 0 \to \mathbb{G}_m \to \text{Gl}_r \to \text{PGL}_r \to 0 \]

of étale sheaves on \( C_{\text{ét}} \). Applying the functors \( \Gamma(D, -) \) and \( \Gamma(C, -) \) we obtain a commutative diagram of exact sequences of pointed sets

\[
\begin{align*}
\hat{H}^1_{\text{ét}}(D, \mathbb{G}_m) & \to \hat{H}^1_{\text{ét}}(D, \text{Gl}_r) \to \hat{H}^1_{\text{ét}}(D, \text{PGL}_r) \to \hat{H}^2_{\text{ét}}(D, \mathbb{G}_m) = 0 \\
\hat{H}^1_{\text{ét}}(C, \mathbb{G}_m) & \to \hat{H}^1_{\text{ét}}(C, \text{Gl}_r) \to \hat{H}^1_{\text{ét}}(C, \text{PGL}_r) \to \hat{H}^2_{\text{ét}}(C, \mathbb{G}_m) = 0.
\end{align*}
\]

As \( \mathbb{G}_m \) lies in the center of \( \text{Gl}_r \) this sequence extends to \( \hat{H}^2 \) and exactness at \( \hat{H}^1_{\text{ét}}(D, \text{Gl}_r) \) is stronger than usual: If two \( \text{Gl}_r \)-torsors map to the same \( \text{PGL}_r \)-torsor they differ by a twist of a line bundle. In particular, we obtain that a \( \text{PGL}_r \)-torsor can be lifted to a \( \text{Gl}_r \)-torsor, which also can be found in [6, Chapter III].

The bundle \( W \) is an element in \( \hat{H}^1_{\text{ét}}(D, \text{Gl}_r) \). By definition of \( G \)-invariance we have isomorphisms \( \psi_\sigma : W \to \sigma^* W \) for all \( \sigma \in G \). The obstruction for descent \( \chi_{\sigma, \tau} := \psi_{\tau}^{-1} \circ \sigma^* \psi_\sigma \circ \psi_\tau \) is an isomorphism of \( W \). By assumption \( W \) is simple and \( \chi_{\sigma, \tau} \) lies in \( k^* \), i.e., considered as a \( \text{PGL}_r \)-torsor \( W \) descends to \( C \), see [6, Theorem 1.4.46]. By the surjectivity \( \hat{H}^1_{\text{ét}}(C, \text{Gl}_r) \to \hat{H}^1_{\text{ét}}(C, \text{PGL}_r) \) we find a vector bundle \( N \) on \( C \) such that \( N_{|D} \cong W \) as \( \text{PGL}_r \)-torsors. Thus, the vector bundles \( N_{|D} \) and \( W \) agree up to tensoring with a line bundle \( L \) on \( D \).

We are now ready to estimate the dimension of the complement of the prime to \( p \) stable locus. We formulate this for arbitrary étale Galois covers. To obtain the desired estimate for the prime to \( p \) stable locus we apply this to the cover \( C_{r-good} \) obtained in Theorem 3.3.11.

**Lemma 4.4.** Let \( \pi : D \to C \) be an étale Galois cover of a smooth projective curve \( C \) of genus \( g_C \geq 2 \). Let \( r \geq 2 \) and \( d \in \mathbb{Z} \). Denote by \( Z \) the closed subset of \( M_{C}^{s, r, d} \) given by stable bundles that do not remain stable after pullback to \( D \). Then \( Z = Z_1 \cup Z_2 \), where

\[
Z_1 := \{ V \in M_{C}^{s, r, d} \mid V_{|D} \cong W^{\oplus e}, W \in M_{D}^{s, \frac{\deg W}{\deg e}}, e \geq 2 \} \text{ and }
\]

\[
Z_2 := \{ V \in M_{C}^{s, r, d} \mid V_{|D} \cong W^{\oplus e}, W \in M_{D}^{s, \frac{\deg W}{\deg e}}, e < 2 \} \text{ and }
\]
\[ Z_2 := \{ V \in M^{s,r,d}_C \mid V_f^D := \bigoplus_{i=1}^n W_i^{\oplus e}, W_i \in M^{s,r,d}_{D^n, \deg(N) \cdot n}, n \geq 2 \} \]

are the strata induced by Lemma 3.3. Furthermore,
\[ \dim(Z_1) \leq r_0^2(g_C - 1) + 1 \quad \text{and} \quad \dim(Z_2) \leq r_0 r(g_C - 1) + 1, \]

where \( r_0 \) is the largest proper divisor of \( r \), i.e., \( r_0 \mid r \) and \( r_0 \neq r \).

If \( \pi \) is a prime to \( p \) cover and \( r \) is a power of \( p \), then \( Z_2 \) is empty.

**Proof.** Clearly, \( Z = Z_1 \cup Z_2 \) by Lemma 3.2.

We begin with the estimate for \( Z_1 \). Consider \( V \in Z_1 \) and \( W \) stable on \( D \) such that \( V_f^D \cong W^{\oplus e} \) for some \( e \geq 2 \). As the Galois group acts trivially on the isomorphism class of \( W \) we can apply Lemma 3.3. Thus, there is a line bundle \( L \) on \( D \) such that \( W \otimes L \cong N_{|D}^{\oplus e} \) for some stable vector bundle \( N \) on \( C \). After twisting \( N \) by a line bundle on \( C \), we can assume that \( 0 \leq \deg N < r \). Note that \( W \otimes L \) is stable and so is \( N \) by Lemma 2.1. Fixing the degree of \( N \) fixes the degree of \( L \) as \( \deg W + \frac{r}{2} \deg L = \deg(\pi) \deg(N) \).

We have \( \det(W)^{\oplus e} \cong \det(V)^{\oplus |D|} \) which implies that \( L^{\otimes r} \) descends to \( C \). As multiplication by \( r \) on \( \text{Pic}_D \otimes_k \) is a finite morphism, we obtain that the dimension of all possible line bundles \( L \) (with fixed degree) is at most \( g_C \). Write \( P(f) \) for the moduli space of line bundles \( D \) of degree \( (\deg(\pi)f - \deg(W)) \cdot \frac{r}{2} \) such that their \( r \)-th power descends to \( C \), where \( f \) is an integer.

Let \( 0 \leq f < r \) and fix a line bundle \( L' \) of degree \( f \) on \( C \). Denote the moduli space of stable bundles of rank \( r/e \) and determinant \( L' \) by \( M^{s,r,e}_C \). Consider the morphism
\[ M^{s,r,e}_C \times_k P(f) \to M^{s,r,d}_{D^n, \deg(N) \cdot n}, (N, L) \mapsto N^{\oplus e} \otimes L^{-1} \]
and denote the image by \( Z_{f,e} \). Observe that \( Z_{f,e} \) is closed and so is the finite union \( Z' = \bigcup_{f=0}^{r_0 - 1} \bigcup_{|r,e| < 1} Z_{f,e} \).

The above discussion shows that \( \pi^*(Z_1) \subseteq Z' \). By Theorem 4.2, we have that \( \pi^* \) is a finite morphism and obtain \( \dim(Z_1) = \dim(\pi^*(Z_1)) \). Computing the dimension we find
\[ \dim(Z_1) \leq \max_{|r,e| < 1}(r/e^2 - 1)(g_C - 1) + g_C = r_0^2(g_C - 1) + 1, \]
where \( r_0 \) is the largest proper divisor of \( r \). This concludes the estimate of \( \dim(Z_1) \).

To obtain a bound for \( \dim(Z_2) \) consider \( V \in Z_2 \). By Lemma 3.3 there is an intermediate cover \( D \to D' \to C \) of degree \( n \) such that \( V_{|D'} \cong V' \oplus W' \), where \( V' \) is semistable of rank \( r/n \) and \( W'_{|D} \) is a direct sum of conjugates of \( V'_D \).

Let \( \Sigma \) be a subset of \( G \) of cardinality \( n \). Consider the morphism
\[ M^{s,r,d}_{D'} \to M^{s,r,d}_{D', \deg(\pi) \cdot d}, V' \mapsto \bigoplus_{\sigma \in \Sigma} \sigma^* V'_{|D} \]
and denote the image by \( Z_{D', \Sigma} \). Observe that \( Z_{D', \Sigma} \) is closed as the image of a finite morphism and by construction \( V_{|D'} \in Z_{D', \Sigma} \) for some \( \Sigma \) and \( D' \). Thus, \( \pi^* Z_2 \) contained in the union of all such \( Z_{D', \Sigma} \), where \( D \to D' \to C \) is an intermediate cover and \( \Sigma \) is a subset of \( G \) of cardinality \( n \). Up to isomorphism there are only finitely many intermediate covers \( D \to D' \to C \) and clearly there are only finitely many \( \Sigma \). Thus, we can estimate the dimension
\[ \dim(Z_2) = \dim(\pi^* Z_2) \leq \max_{D', \Sigma} \dim(Z_{D', \Sigma}), \]
where $D'$ and $\Sigma$ are as above. Applying Theorem 4.2 we have
\[ \dim(Z_{D',\Sigma}) \leq r^2_n(g_{D'} - 1) + 1. \]
By Riemann-Hurwitz we obtain
\[ \dim(Z_{D',\Sigma}) \leq n^2(r_n^2(g_C - 1) + 1 = r_n^0(g_C - 1) + 1, \]
and conclude
\[ \dim(Z_2) \leq \max_{n|r, n \neq 1} r_n^0(g_C - 1) + 1 = r_n0(g_C - 1) + 1. \]

If $G$ is prime to $p$ and $r$ is a power of $p$, then a decomposition of the form $V|D \sim = \bigoplus_{i=1}^n W_i \otimes e_i$, $n \geq 2$, can not happen. Indeed, $n \text{rk}(W_i)e = r$ and we find that $n$ is a power of $p$ as well. By Lemma 3.3 there is an intermediate cover of $D \to C$ of degree $n$. However, $G$ being prime to $p$ only allows for such an intermediate cover if $n = 1$. \hfill \Box

As a direct consequence of the dimension estimate we obtain the existence of stable bundles that remain stable on a fixed (étale) cover.

**Lemma 4.5.** Let $\pi : D \to C$ be an étale cover of a smooth projective curve $C$ of genus $g_C \geq 2$. Let $r \geq 2$ and $d$ be integers. Let $Z$ be the closed subset $Z$ of $M_{C}^{s,r,d}$ of stable bundles that are not stable after pullback to $D$. Then $\text{codim}(M_{C}^{s,r,d}(Z)) \geq 2$.

In particular, there are stable bundles of rank $r$ and degree $d$ on $C$ that remain stable after pull back to $D$.

**Proof.** Observe that we can replace $D \to C$ by its Galois closure. By Lemma 4.4 we have
\[ \dim(Z) \leq r_0^r(g_C - 1) + 1, \]
where $r_0$ is the largest proper divisor of $r$. As
\[ r^2(g_C - 1) + 1 = \dim(M_{C}^{s,r,d}), \]
gc $\geq 2$, and $r \geq 2$, we conclude
\[ \text{codim}(Z) \geq r(r - r_0)(g_C - 1) \geq 2. \]
\hfill \Box

For the cover $C_{r-\text{good}}$ the estimate obtained in Lemma 4.5 is sharp if the rank is prime to $p$. To show this we need a way to construct stable bundles with prescribed decomposition behaviour after pullback. This can be done for cyclic covers. We start with a descent lemma for such covers.

**Lemma 4.6.** Let $Y \to X$ be a cyclic étale cover of proper varieties with Galois group $G$. Let $V$ be a simple sheaf on $Y$. Then $V$ descends to $X$ iff $V$ is $G$-invariant.

**Proof.** The "only if" implication is trivial. For the "if" implication let $\sigma$ be a generator of $G$ of order $n$. Fix an isomorphism $\varphi_\sigma : V \xrightarrow{\sim} \sigma^*V$. For $2 \leq l \leq n$ define $\varphi_{\sigma^l} : V \xrightarrow{\sim} (\sigma^l)^*V$ inductively as the composition $\sigma^*\varphi_{\sigma^{l-1}} \circ \varphi_\sigma$. Further define $\varphi_e = \text{id}_V$, where $e$ denotes the identity of $G$.

Consider $\sigma^*\varphi_{\sigma^{n-1}} \circ \varphi_\sigma$. This is an automorphism of $V$. As $V$ is simple it corresponds to a scalar $\lambda \in k^*$. Since $k$ is algebraically closed we can find an $n$-th
root $\lambda^{1/n}$ of $\lambda$. The automorphisms $\psi_{\sigma i} := \lambda^{-1/n}\varphi_{\sigma i}$ define a $G$-linearization of $V$. Indeed, for $1 \leq l, l'$ such that $l' + l < n$ we have

$$(\sigma^l)^*\psi_{\sigma i} \circ \psi_{\sigma i} = \lambda^{(-1-l')/n} \cdot ((\sigma^l)^*)^\ast \varphi_{\sigma} \circ \cdots \circ \sigma^* \varphi_{\sigma} \circ \varphi_{\sigma} = \psi_{\sigma i+l'},$$

by definition. It remains to check this property for $l + l' = n$. We have

$$(\sigma^n)^*\psi_{\sigma i} \circ \psi_{\sigma i} = \lambda^{-1} \cdot (\sigma^{n-1})^* \varphi_{\sigma} \circ \cdots \circ \sigma^* \varphi_{\sigma} \circ \varphi_{\sigma} = \lambda^{-1} \lambda = 1$$

by definition of $\lambda$.

We are now able to show that the estimate in Lemma 4.4 is sharp for $C_{r-good}$ in most cases. It suffices to find a prime to $p$ cover where the decomposition locus has the right dimension as the decomposition locus for $C_{r-good}$ is the largest one.

**Lemma 4.7.** Let $C$ be a smooth projective curve of genus $g_C \geq 2$. Let $r \geq 2$ be such that $p$ is not the smallest proper divisor of $r$ if $\text{char}(k) = p > 0$. Then there is an étale prime to $p$ cyclic cover $D \rightarrow C$ such that $\dim(Z_2) = rr_0(g_C - 1) + 1$, where $r_0$ denotes the largest proper divisor of $r$ and $Z_2 \subset M_C^{s,r,d}$ is defined as in Lemma 4.4.

**Proof.** As $p$ is not the smallest divisor of $r$ we have that $r/r_0$ is prime to $p$. Let $\pi : D \rightarrow C$ be an étale cyclic cover of degree $r/r_0$, i.e., with Galois group $\mu_{r/r_0}$. Note that such covers correspond to torsion points of order $r/r_0$ in $\text{Pic}^0_{C/k}$ and always exist. Also note that $r/r_0$ is prime. Thus, there are no intermediate covers.

Consider

$$U := M_D^{s,\deg(\pi)d} \cap (M_C^{s,r_0,\deg(\pi)d} \setminus \pi^* M_C^{s,r_0,d}).$$

By Theorem 4.2 pullback along $\pi$ is a finite morphism and the set $U$ is open and non-empty. Thus, by Riemann-Hurwitz $U$ has dimension

$$\dim(U) = \frac{r^2}{r_0} \frac{1}{g_D} - 1 + 1 = rr_0(g_C - 1) + 1.$$

Consider a closed point $W' \in U$. Then the orbit $O$ of $W'$ under the action of $\mu_{r/r_0}$ is contained in $M_D^{s,\deg(\pi)d}$. By Lemma 4.5 the orbit $O$ has cardinality $r/r_0$ as otherwise $W'$ would descend to $C$. Clearly, no conjugate of $W'$ can descend to $C$ as well, i.e., $O \subset U$.

Consider the bundle $W := \bigoplus_{\sigma \in \mu_{r/r_0}} \sigma^* W'$. Then $W$ has rank $r$, admits a $\mu_{r/r_0}$-linearization, and no polystable summand of $W$ admits a $\mu_{r/r_0}$-linearization. Thus, there exists $V \in M_C^{s,r,d}$ such that $V|_D \cong W$. By construction $V$ lies in $Z_2$. In particular, $\pi^* Z_2$ contains the image of

$$U \rightarrow M_D^{s,\deg(\pi)d}, W' \mapsto \bigoplus_{\sigma \in \mu_{r/r_0}} \sigma^* W',$$

which has dimension $\dim(U)$. We obtain that

$$\dim(Z_2) \geq rr_0(g_C - 1) + 1.$$

As we already have the other inequality from Lemma 4.4 we conclude. □

**Remark 4.8.** Let $D \rightarrow C$ be an étale Galois cover. One can further decompose $Z_2$ into the strata $Z_2(n, e) := \{ V \in M_C^{s,r,d}| V|_D \cong \bigoplus_{i=1}^n W_i^{\text{triv}} \}$, where $V|_D \cong \bigoplus_{i=1}^n W_i^{\text{triv}}$ is the decomposition of Lemma 3.2. One can compute the dimension of $Z_2(n, 1)$ in an analogous manner if $D \rightarrow C$ is prime to $p$ and cyclic of degree $n$. The only change being that one has to remove all bundles of rank $r/n$ arising from an intermediate cover $D \rightarrow D' \rightarrow C$, $D' \neq D$. 

Applying the results of this subsection for arbitrary Galois covers to the cover $C_{r-good} \to C$, see Theorem 3.11, we obtain Theorem 2:

**Theorem 4.9.** Let $C$ be a smooth projective curve of genus $g_C \geq 2$. Let $r \geq 2$ and $d \in \mathbb{Z}$. Then the prime to $p$ stable bundles of rank $r$ and degree $d$ form a big open $M_{s,r,d}^{p,s.r.d}$ in $M_{s,r,d}^{s.r.d}$. More precisely, we have

$$\dim(M_{s,r,d}^{s,r.d} \setminus M_{s,r,d}^{p,s,r,d}) \leq r_0(g_C - 1) + 1,$$

where $r_0$ denotes the largest proper divisor of $r$. Moreover, if $p$ is not the smallest proper divisor of $r$, then equality holds.

Extending a prime to $p$ stable vector bundle from a large curve to a surrounding smooth projective variety using Mathur’s extension theorem, [19] Theorem 1, we obtain the existence of prime to $p$ stable vector bundles in higher dimensions. However, we can not control the numerical data, i.e., which components of the stack of bundles admit prime to $p$ stable bundles.

**Corollary 4.10.** Let $X$ be a smooth projective variety of dimension $\geq 2$. There are prime to $p$ stable vector bundles of rank $r \geq \dim(X)$ on $X$.

As the general bundle is prime to $p$ stable, we obtain:

**Corollary 4.11.** Let $C$ be a smooth projective curve of genus $g_C \geq 2$. Let $r \geq 2$. Then the stable bundles of rank $r$ that are trivialized on a prime to $p$ cover are not dense in $M_{s,r,0}^{s,r,0}$.

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