Differential operators and Higher Specht polynomials

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Abstract. In this note, we study the action of the rational quantum Calogero-Moser system on polynomials rings. This a continuation of our paper [Ibrahim Nonkan 2021 J. Phys.: Conf. Ser. 1730 012129] in which we deal with the polynomial representation of the ring of invariant differential operators. Using the higher Specht polynomials we give a detailed description of the actions of the Weyl algebra associated with the ring of the symmetric polynomial \( \mathbb{C}[x_1, \ldots, x_n]^{S_n} \) on the polynomial ring \( \mathbb{C}[x_1, \ldots, x_n] \). In fact, we show that its irreducible submodules over the ring of differential operators invariant under the symmetric group are its submodules generated by higher Specht polynomials over the ring of the symmetric polynomials. We end up studying the polynomial representation of the ring of differential operators invariant under the actions of products of symmetric groups by giving the generators of its simple components, thus we give a differential structure to the higher Specht polynomials.

1. Introduction

The Hamiltonian is a classical operator in quantum mechanics and it is the main objet in the Schrödinger equation. In fact the solutions of the stationary schrödinger equation are the eigenfunctions of the Hamiltonian. We think that the best of the of dealing with Schrödinger equation is by understanding the actions the Hamiltonian on functions. A starting point may be to figure how the Hamiltonian operates on polynomials. The rational quantum Calogero-Moser system of \( n \) particles on the line of unit mass is of one of simplest case of Hamiltonian, so by observing that it is a differential operator invariant under the group of permutation of \( n \) letters, we constructed an appropriate ring and a module over that ring for the explicit study of its actions of polynomials [8]. We then studied the polynomials representation of rings of invariant differential under the symmetric group, we got an decomposition theorem by elaborating its simple component. Our study in [8] was not satisfactory, because we got a decomposition of the polynomial in terms modules generated by the Weyl algebra, in fact the ring of differential operator appears in the decomposition of the ring polynomial. We would like to express the actions the rational quantum Calogero-Moser system on polynomials by using only polynomials. In this paper we remedy what was lacking in [8] by using the higher Specht polynomials and we get a nice expression of the polynomial ring as a module over the ring invariant differential over the symmetric group. Finally we generalize the obtained results to the polynomials representation of rings of invariant differential under direct products symmetric group.
2. Preliminaries

2.1. The rational quantum Calogero-Moser system

Consider the differential operator

\[ H = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} - c(c + 1) \sum_{i \neq j} \frac{1}{(x_i - x_j)^2}. \]  

(2.1)

This is the quantum Hamiltonian for a system of \( n \) particles on the line of unit mass and the interaction potential (between particle 1 and 2) is \( c(c + 1)/(x_1 - x_2)^2 \). This system is called the rational quantum Calogero-Moser system.

It turns out that the rational quantum Calogero-Moser system is completely integrable. Namely, we have the following theorem.

**Theorem 2.1.** [4] There exist differential operators \( L_j \) with rational coefficients of the form

\[ L_j = \sum_{i=1}^{n} \left( \frac{\partial}{\partial x_i} \right)^j + \text{lower order terms}, j = 1, \ldots, n, \]

which are invariant under the symmetric group \( S_n \), homogeneous of degree \(-j\), and such that \( L_2 = H \) and \( [L_j, L_k] = 0, \forall j, k = 1, \ldots, n \).

It is clear that the differential operators \( L_j \) belong a localization of the ring of invariant differential operators under the symmetric group, so one way of understanding the actions of the \( L_j \) on polynomials is to study the polynomials representation of the ring of differential operators that are invariant under the symmetric group localized at \( \prod_{i \neq j} (x_i - x_j)^2 \).

2.2. Specht polynomials, and Higher Specht polynomials

In this subsection we recall some general facts about the actions of symmetric group on polynomials ring and give some useful notations. The symmetric group \( S_n \) is the group of permutations of the set of variables \( \{x_1, \ldots, x_n\} \). Let \( g \in \mathbb{C}[x_1, \ldots, x_n] \) be a polynomial, and \( \sigma \in S_n \) we define

\[ (\sigma g)(x_1, \ldots, x_n) = g(\sigma x_1, \ldots, \sigma x_n). \]

2.2.1. Specht polynomials

By a partition of \( n \) we mean a sequence \( \lambda = (\lambda_1, \ldots, \lambda_r) \) such that \( \lambda_1 \geq \lambda_2 \geq \ldots \lambda_r > 0 \) and \( \lambda_1 + \lambda_2 + \cdots + \lambda_r = n \).

We write \( \lambda \vdash n \) to mean that \( \lambda \) is partition of \( n \). Let \( \lambda = (\lambda_1, \ldots, \lambda_r) \vdash n \); we arrange the variables \( x_1, \ldots, x_n \) in an array, with \( r \) rows and \( \lambda_1 \) columns, containing a variable in the first \( \lambda_i \) positions of the \( i \)th row; each variable occurs exactly once in the array. For example, one such array for the partition \((4, 2, 1)\) of 7 is

\[
\begin{array}{ccc}
x_1 & x_4 & x_6 \\
x_2 & x_3 \\
x_5 \\
\end{array}
\]

such an array is called a \( \lambda \)-tableau. There are \( n! \) \( \lambda \)-tableaux for each partition \( \lambda \) of \( n \). We shall denote such tableaux by \( T \). Suppose that the variable \( a_1, \ldots, a_l \) occur in \( j \)th column of \( \lambda \)-tableau \( T \), with \( a_i \) in the \( i \)th row. We form the difference product \( \Delta(a_1, \ldots, a_l) = \prod_{i<k} (a_i - a_k) \), if
2.2.2. Higher Specht polynomials

The set of tableaux (resp standard tableaux) of shape $\lambda$ is denoted by $\text{Tab}(\lambda)$ (resp $\text{STab}(\lambda)$). For a standard $\lambda$-tableau $S$, one associates an index tableau $i(S)$ of the same shape in the following way. Define the word $w(S)$ reading $S$ from bottom to top in consecutive columns, from the left to the right. The number 1 in the word $w(S)$ has index 0. If the number $k$ in the word has index $p$, then $k+1$ has index $p$ if it is to the right of $k$ and $p+1$ if it is to the left. For example, if

$$
S = \begin{array}{ccc}
1 & 2 & 4 \\
3 & 5 & 
\end{array}
$$

then $w(S) = 31_01_52_02_04_1$, where the indices are attached as the subscript. Changing a number by its index in the tableau $S$, one obtains the index tableau $i(S)$ of $S$. Let $T$ be a $\lambda$-tableau, and let $c(\alpha, \beta)$ be the number in the cell $(\alpha, \beta)$ of $T$. Denote by $i_k$ the index in the cell $(\alpha, \beta)$ of $i(S)$ with $k = c(\alpha, \beta)$ and define $x_T^{i(S)} = x_1^{i_1} \cdots x_n^{i_n}$. The charge $c(S)$ of $S$ is defined to be the sum of the indices. For example,

if $S = \begin{array}{ccc}
1 & 2 & 4 \\
3 & 5 & 
\end{array}$ and $T = \begin{array}{ccc}
1 & 3 & 5 \\
2 & 4 & 
\end{array}$ then $i(S) = \begin{array}{ccc}
0 & 0 & 1 \\
1 & 2 & 
\end{array}$ and $x_T^{i(S)} = x_1^0x_2^1x_3^0x_4^2x_5^1$.

If $T$ is a $\lambda$-tableau, $R(T)$ and $C(T)$ the row-stabilizer and column-stabilizer of $T$ respectively, consider the Young symmetrizer

$$
\varepsilon_T = \sum_{\sigma \in R(T)} \sum_{\tau \in C(T)} (\text{sgn } \tau) \tau \sigma
$$

in $\mathbb{C}[S_n]$. Terasoma et Yamada [11] have defined $F_T^S$ by

$$
F_T^S(x_1, \ldots, x_n) = \varepsilon_T(x_T^{i(S)}), \quad (2.2)
$$

for $S \in \text{STab}(\lambda)$ and $T \in \text{Tab}(\lambda)$.

The polynomials $F_T^S$ are called higher Specht polynomials. For a standard tableau $T \in \text{STab}(\lambda)$ the polynomial $F_T^S$ is called standard. One has then $n!$ standard higher Specht polynomials $F = \{ F_T^S; S, T \in \text{STab}(\lambda), \lambda \vdash n \}$. The following gives the fundamental property of the standard Specht polynomials.

**Theorem 2.4.** [11] The higher Specht polynomials in $F$ form a basis of the $\mathbb{C}[x_1, \ldots, x_n]^{S_n}$-module $\mathbb{C}[x_1, \ldots, x_n]$. 

---

$l > 1$, and if $l = 1$, $\Delta(a_l) = 1$. Multiplying these difference products for all the columns of $T$, we obtain a polynomial which we denote by $F_T$. For $\sigma \in S_n$, let $\sigma T$ be the tableau obtained from $T$ by replacing $x_i$ in $T$ by $\sigma x_i$. Then $\sigma F_T = F_{\sigma T}$. It follows that the set of all linear combinations of the $n!$ polynomials $F_T$, obtained from the $\lambda$-tableaux $T$, is a cyclic $\mathbb{C}[S_n]$-module generated by any $F_T$. We denote this module by $V(\lambda)$. A $\lambda$-tableau is said to be standard if the variables occur in increasing order ($x_i > x_j$ if $i > j$) along each row from left to right and down each column. M.H. Peel proved the following in [10].

**Theorem 2.2.** [10] $B^\lambda = \{ F_T : T$ is a standard $\lambda$-tableau $\}$ is a basis of $V(\lambda)$.

We call $F_T$ the Specht polynomial corresponding to the $\lambda$-tableau $T$, we call $V(\lambda)$ the Specht module corresponding to the partition $\lambda$, and $F_T$ a standard Specht polynomial if $T$ is a standard tableau.

**Theorem 2.3.** [10] The $V(\lambda)$ for $\lambda \vdash n$ form a complete list of irreducible $S_n$-module over the complex field.
For $\lambda \vdash n$ and $S, T \in \text{Tab}(\lambda)$, the polynomial $F_T^{S}$ is an homogeneous polynomials which degree equal to the charge $c(S)$ of $S$.

**Theorem 2.5.** [11] Fix $\lambda \vdash n$ and $S \in \text{Tab}(\lambda)$. Then the space $V^{S}(\lambda) = \sum_{T \in \text{Tab}(\lambda)} \mathbb{C}F_T^{S}$ is an irreducible $S_n$-module isomorphic to $V(\lambda)$ equipped with a basis $S^{\lambda}(\lambda) = \{F_T^{S}; T \in \text{STab}(\lambda)\}$.

For the standard tableau $S_0$ of form $\lambda$ in which the cells are numbered from left to right in successive rows, starting from the top, $F_T^{S_0}$ is proportional to the Specht polynomial $F_T$ associated to $T$ and $V(\lambda) = V^{S}(\lambda)$.

### 3. Decomposition Theorem of the polynomial ring with respect to the higher Spect polynomials

In this section we establish a decomposition theorem of the polynomial ring in $n$ indeterminates as a module over the ring of invariant differential operators by means of the the higher Specht polynomials.

Let $\mathcal{O}_{X} := \mathbb{C}[x_{1}, \ldots, x_{n}]$ be the polynomial ring and $\mathcal{O}_{Y} := \mathbb{C}[x_{1}, \ldots, x_{n}]^{S_{n}} = \mathbb{C}[y_{1}, \ldots, y_{n}]$ be the ring of invariant under the symmetric group $S_{n}$ where $y_{j} = \sum_{i=1}^{n} \frac{x_{i}^{j}}{j}$ for $j = 1, \ldots, n$,

We adopt the following notations $\Delta := \prod_{i \neq j}(x_{i} - x_{j}), \tilde{\mathcal{O}}_{X} := \mathbb{C}[x_{1}, \ldots, x_{n}, \Delta^{-1}], \tilde{\mathcal{O}}_{Y} := \mathbb{C}[y_{1}, \ldots, y_{n}, \Delta^{-2}]$, and $\tilde{\mathcal{D}}_{Y} := \mathbb{C}[y_{1}, \ldots, y_{n}, \frac{\partial}{\partial y_{1}}, \ldots, \frac{\partial}{\partial y_{n}}, \Delta^{-2}]$. By [8] we know that $\tilde{\mathcal{O}}_{X}$ is a $\tilde{\mathcal{D}}_{Y}$-module.

For a partition $\lambda \vdash n$ and a $\lambda$-tableau $T$, let $f_{\lambda}$ be the number of standard $\lambda$-tableaux which is also equal to $\dim_{\mathbb{C}} V(\lambda)$. We set

$$e_{T} = \frac{f_{\lambda}}{n!} \epsilon_{T}. \quad (3.1)$$

It is well-known that $V(\lambda) \cong \mathbb{C}[S_{n}]e_{T}$, and the set $\{e_{T}; \exists T \in \text{STab}(\lambda), \lambda \vdash n\}$ is the complete set of primitive idempotents of the group algebra $\mathbb{C}[S_{n}]$ (see [5, Theorem 3.1.10]).

**Lemma 3.1.** Let $V(\lambda)$ be the Specht module associated to the partition $\lambda \vdash n$ and $\epsilon_{T}$ the Young symmetrizer associated to a $\lambda$-standard tableau $T$. Then $\epsilon_{T}(V(\lambda)) = \{\epsilon_{T}(m)|m \in V(\lambda)\}$ is a one dimensional $\mathbb{C}$-vector space.

**Proof.** In fact we have

$$\epsilon_{T}V(\lambda) \cong \epsilon_{T}\mathbb{C}[S_{n}]e_{T} \cong e_{T}\mathbb{C}[S_{n}]e_{T} \cong \mathbb{C}e_{T} \text{ by } [2, \text{Theorem 3.9}]$$

$\square$

Recall that if $M$ is a semi-simple module over a ring $R$, and $N$ is simple $R$-module, then the isotopic component of $M$ associated to $N$ is the sum $\sum N' \subset M$ of all $N' \subset M$ such that $N' \cong N$.

**Lemma 3.2.** Let $V(\lambda)$ be the Specht module associated to the partition $\lambda \vdash n$ and $M := \tilde{\mathcal{O}}_{X}$ and $M^{\lambda}$ the isotopic component of $M$ (as $\tilde{\mathcal{O}}_{Y}$-module) associated to $V(\lambda)$. Then $M^{\lambda}$ is $\tilde{\mathcal{D}}_{Y}$-module.
Proof. We only have to proof that $\tilde{D}_Y \cdot M^\lambda \subset M^\lambda$. Let $D \in \tilde{D}_Y$ and $N$ be a $S_n$-module isomorphic to $V(\lambda)$, since $D$ commute with the elements of the group algebra $\mathbb{C}[S_n]$. $D$ is an $S_n$-homomorphism from $N$ into $D(N)$. Then by virtue of the Schur lemma $D(N) = 0$ or $D(N) \cong N$ as a $S_n$-module, and $D(N) \subset M^\lambda$. Hence $\tilde{D}_Y \cdot M^\lambda \subset M^\lambda$.

Lemma 3.3. Let $T$ be a $\lambda$-standard tableau where $\lambda \vdash n$, then $\varepsilon_T(M^\lambda)$ is a $\tilde{D}_Y$-module.

Proof. Let $D \in \tilde{D}_Y$, we have $D(\varepsilon_T(M^\lambda)) = \varepsilon_T(D(M^\lambda)) \subset \varepsilon_T(M^\lambda)$, so $\varepsilon(M^\lambda)$ is a $\tilde{D}_Y$-module.

Before we state our main first result, let us recall the correspondence between group representations and modules over differential ring that will be useful in what follows[7, Paragraph 2.4].

The correspondence between $G$-representations and $D$-modules Let $L$ and $K$ be two fields, say that a $T_{K/k}$-module $M$ is $L$-trivial if $L \otimes_K M \cong L^n$ as $T_{L/k}$-modules. Denote by $\text{Mod}^L(T_{K/k})$ the full subcategory of finitely generated $T_{K/k}$-modules that are $L$-trivial. It is immediate that it is closed under taking submodules and quotient modules. Using the lifting $\phi : L$ may be thought of as a $T_{K/k}$-module. If $G$ is a finite group let $\text{Mod}(k[G])$ be the category of finite-dimensional representations of $k[G]$. Let now $k \rightarrow K \rightarrow L$ be a tower of fields such that $K = L^G$. Note that the action of $T_{K/k}$ commutes with the action of $G$. If $V$ is a $k[G]$-module, $L \otimes_k V$ is a $T_{K/k}$-module by $D(l \otimes v) = D(l) \otimes v$, $D \in T_{K/k}$, and $(L \otimes_k V)^G$ is a $T_{K/k}$-submodule.

Proposition 3.4. The functor

$$\nabla : \text{Mod}(k[G]) \rightarrow \text{Mod}(T_{K/k}), \quad V \mapsto (L \otimes_k V)^G$$

is fully faithful, and defines an equivalence of categories

$$\text{Mod}(k[G]) \rightarrow \text{Mod}^L(T_{K/k}).$$

The quasi-inverse of $\nabla$ is the functor

$$\text{Loc} : \text{Mod}^L(T_{K/k}) \rightarrow \text{Mod}(k[G]), \quad \text{Loc}(M) = (L \otimes_K M)^{\phi(T_{K/k})}.$$

Proof. see [7, Proposition 2.4]

Proposition 3.5. Let $T$ be a $\lambda$-standard tableau where $\lambda \vdash n$, set $M_T := \mathbf{e}_T \tilde{O}_X = \varepsilon_T \tilde{O}_X$ and $\text{STab}(n) = \bigcup_{\lambda \vdash n} \text{STab}(\lambda)$. Then we have:

(i) $M_T = \nabla(V(\lambda))$ where $\nabla$ is equivalence of categories defined in Proposition 3.4 for $G = S_n$;

(ii) $M_T = \varepsilon_T(M^\lambda)$ is simple $\tilde{D}_Y$-module;

(iii) $M^\lambda = \bigoplus_{T \in \text{STab}(n)} \mathbf{e}_T(M^\lambda)$.

Proof. (i) Let us consider the right $\mathbb{C}[S_n]$-module $V = \varepsilon_T \mathbb{C}[S_n]$ where $T$ is a $\lambda$-standard tableau. This is the image of $\mathbb{C}[S_n]$ by right multiplication map $\varepsilon_T : \mathbb{C}[S_n] \rightarrow \mathbb{C}[S_n]$. By [7, Example 2.5], we may turn this map to a left multiplication $\mathbb{C}[S_n]^r \rightarrow \mathbb{C}[S_n]^r$ and get an image which is isomorphic to $V(\lambda)$. we get an induced map

$$\nabla(\mathbb{C}[S_n]^r) \rightarrow \nabla(V(\lambda)) \subset \nabla(\mathbb{C}[S_n]^r),$$

which is a multiplication by $\varepsilon_T$ according to [7, Example 2.5]. Then $\nabla(V(\lambda))$ is egal to $\varepsilon_T \tilde{O}_X = M_T$. 

\[\text{Page 5}\]
(ii) Since \( V(\lambda) \) is a simple \( \mathbb{C}[S_n] \)-module, \( \nabla(V(\lambda)) \) is also a simple \( \tilde{D}_Y \)-module.

(iii) follows from the fact that \( 1 = \sum_{T \in \text{STab}(n)} e_T \) and \( e_T(M^\lambda) = 0 \) if \( T \) is not a \( \lambda \)-tableau.

\[ \square \]

**Theorem 3.6.** Let \( T \) be a \( \lambda \)-standard tableau where \( \lambda \vdash n \) and \( M_T \) be as in the above proposition. Then

\[ (i) \quad M_T = \bigoplus_{s \in \text{STab}(\lambda)} \tilde{O}_Y F^S_T \text{ as } \tilde{D}_Y \text{-module}, \]

\[ (ii) \quad \tilde{O}_X = \bigoplus_{\lambda \vdash n} \left( \bigoplus_{s, t \in \text{STab}(\lambda)} \tilde{O}_Y F^S_T \right) \text{ as a } \tilde{D}_Y \text{-module}. \]

**Proof.** (i) We know that the polynomial \( F^S_T \) generate a \( \mathbb{C}[S_n] \)-module inside \( \tilde{O}_X \) which is isomorphic to \( V(\lambda) \) [11]. Then \( F^S_T \in M^\lambda \) and \( M^\lambda = \bigoplus_{s, t \in \text{STab}(\lambda)} \mathbb{C}[S_n] F^S_T \tilde{O}_Y \) by [11]. Moreover \( \varepsilon_T(F^S_T) = cF^S_T, c \in \mathbb{C} \) and by Lemma 3.1 \( \varepsilon_T(\mathbb{C}[S_n] F^S_T) = \mathbb{C} F^S_T \). Hence \( M_T = \varepsilon_T(M^\lambda) = \bigoplus_{s \in \text{STab}(\lambda)} \tilde{O}_Y F^S_T \).

(ii) follows from Proposition 3.5 and [8, Theorem 3.6].

\[ \square \]

**Remark 3.7.** (i) Proposition 3.5 is a refined version of [8, Theorem 3.6]. We use the equivalence of categories \( \nabla \) to prove that \( M_T \) is a simple \( \tilde{D}_Y \)-module but in [8, Theorem 3.6] we use representation theory of group to prove that.

(ii) It clear that \( M_T \) is also a \( \tilde{O}_Y \)-module in Theorem 3.6.(i) In fact \( \tilde{D}_Y \) is the weyl algebra associated to \( \tilde{O}_Y \) (see [6]) and we express the irreducible \( \tilde{D}_Y \)-module of \( \tilde{O}_X \) as module over \( \tilde{O}_Y \).

(iii) For \( S, T \in \text{STab}(\lambda) \) There is a kind of reversed similarity between the basis of \( V^S(\lambda) \) as vector space and the basis of \( M_T \) as a \( \tilde{D}_Y \)-module. The basis of \( V^S(\lambda) \) is given by the \( F^S_T \) with \( T \in \text{STab}(\lambda) \) and the one of \( M_T \) is given by the \( F^S_T \) with \( S \in \text{STab}(\lambda) \).

(iv) In Theorem 3.6.(ii), we get the decomposition of the polynomial ring (as a \( \tilde{D}_Y \)-module) into modules over the ring symmetric polynomials. For \( \varphi \in \tilde{O}_X \), there exists \( f^S_T(\varphi) \in \tilde{O}_Y, S, T \in \text{STab}(n) \) such that

\[ H \varphi = \sum_{\lambda \vdash n} \sum_{S, T \in \text{STab}(\lambda)} f^S_T(\varphi) F^S_T, \tag{3.2} \]

where \( H \) is the Hamiltonian in (2.1) and \( F^S_T \) is the higher Specht polynomial for \( S, T \in \text{STab}(\lambda) \). This shows how the Hamiltonian operates on polynomials. The set of standard higher Specht polynomials gives a basis over the ring of symmetric polynomials of the actions of the Hamiltonian \( H \) on polynomials. *Is it possible the extend equality (3.2) to power series or convergente power series?*

### 3.1. Invariant differential operators and higher Specht polynomials for the symmetric group

In this subsection we investigate the actions of invariant differential operators on higher Specht polynomials. Let \( \lambda \vdash n, T \) a \( \lambda \)-tableau, and \( C(T) \) be column stabilizer of \( T \), by [8, Lemma 4.4] we know that for every differential operator \( D \) in \( \tilde{D}_Y \) such that \( D(F_T) \neq 0 \) then there exists a polynomial \( G \) in \( \mathbb{C}[x_1, \ldots, x_n]^{C(T)} \) the polynomial ring invariant under the subgroup \( C(T) \) such
that $D(F_T) = F_TG$, we show that this is also true for the higher Specht polynomials.

Let $\lambda \vdash n$, $S \in STab(\lambda)$ and $T \in Tab(\lambda)$, we have that for all $\sigma \in C(T), \sigma(F_T) = \text{sgn}(\sigma)F_T$ and $\sigma(F^S_T) = \text{sgn}(\sigma)F^S_T$.

**Lemma 3.8.** Let $\lambda \vdash n$, $S \in STab(\lambda)$ and $T \in Tab(\lambda)$. Then there exists a polynomial $G \in \mathbb{C}[x_1, \ldots, x_n]^{C(T)}$ such that $F^S_T = F_TG$.

**Proof.** Let us consider the linear application $\varphi : V(\lambda) \rightarrow V^S(\lambda)$ defined by $\varphi(F_T) = F^S_T$. For every $\sigma \in \mathbb{C}[S_n]$, we have that $\varphi(\sigma F_T) = \sigma \varphi(F_T)$, so that $\varphi$ is a $\mathbb{C}[S_n]$-linear map and by the Schur $\varphi$ is a $S_n$-isomorphism. Suppose that $x_i$ and $x_j$ occur in the same column of $T$, and let $\pi = (x_i, x_j)$ the transposition of $x_i$ and $x_j$. Then

$$\pi F^S_T = \pi \varphi(F_T) = \varphi(\pi F_T) = \varphi(-F_T) = -F^S_T.$$ 

This implies that $(x_i - x_j)$ is a factor of $F^S_T$. This hold for each linear factor of $F_T$, so that $F_T$ divides $F^S_T$. Hence there exists a polynomial $G \in \mathbb{C}[x_1, \ldots, x_n]$ such that $F^S_T = F_TG$. Let now $\sigma \in C(T)$, we get

$$\sigma G = \sigma \left(\frac{F^S_T}{F_T}\right) = \frac{\sigma F^S_T}{\sigma F_T} = \frac{\text{sgn}(\sigma)F^S_T}{\text{sgn}(\sigma)F_T} = G.$$ 

Then $G \in \mathbb{C}[x_1, \ldots, x_n]^{C(T)}$. \hfill\qed

**Lemma 3.9.** Let $\lambda \vdash n, S, T \in Tab(\lambda)$ and $D$ a derivation in $\bar{D}_Y$ such that $D(F^S_T) \neq 0$. Then there exists a polynomial $G \in \mathbb{C}[x_1, \ldots, x_n]^{C(T)}$ such that $D(F^S_T) = F_TG$.

**Proof.** Let $D$ a derivation in $\bar{D}_Y$, we have that

$$D(F^S_T) = D(F_TG') \quad \text{where} \quad G' \in \mathbb{C}[x_1, \ldots, x_n]^{C(T)} \quad \text{by Lemma 3.8}$$

$$= (D(F_T)G' + F_TD(G'))$$

$$= F_TG''G' + F_TD(G') \quad \text{where} \quad G'' \in \mathbb{C}[x_1, \ldots, x_n]^{C(T)} \quad \text{by [8, Lemma 4.4]}}$$

$$= F_T(G''G' + D(G')) \quad \text{with} \quad G', G'' \in \mathbb{C}[x_1, \ldots, x_n]^{C(T)}.$$ 

Now let $\pi \in C(T)$ we have

$$\pi(G''G' + D(G')) = \pi(G'')\pi(G') + \pi D(G')$$

$$= G''G' + D(\pi G') \quad \text{since} \quad G', G'' \in \mathbb{C}[x_1, \ldots, x_n]^{C(T)}$$

$$= G''G' + D(G').$$ 

Then $G''G' + D(G') \in \mathbb{C}[x_1, \ldots, x_n]^{C(T)}$. We set $G = G''G' + D(G')$ and get $D(F^S_T) = F_TG$. \hfill\qed

**Theorem 3.10.** Let $\lambda \vdash n$ and $D \in \bar{D}_Y$ such that $D(F^S_T) \neq 0$ for $S, T \in STab(\lambda)$. Then the image of the $\mathbb{C}[S_n]$-module $V^S(\lambda)$ by $D$ is an $\mathbb{C}[S_n]$-module isomorphic to $V^S(\lambda)$. In others words, the actions of the differential operators of $\bar{D}_Y$ on the higher Specht polynomials generate isomorphic copies of the corresponding Specht module.

**Proof.** Let $\lambda \vdash n, D \in \bar{D}_Y$ such that $D(F^S_T) \neq 0$ for $S, T \in STab(\lambda)$ and set $W^S_D(\lambda) = D(V^S(\lambda))$ the image of the module $V^S(\lambda)$ under the map $D$. Since the $\mathbb{C}$-vector space $V^S(\lambda)$ is equipped with a basis $F^S(\lambda) = \{F^S_T; T \in STab(\lambda)\}$, $W^S_D(\lambda)$ is the vector space spanned by the set $\{D(F^S_T); T \in STab(\lambda)\}$. The elements of $\{D(F^S_T); T \in STab(\lambda)\}$ are linearly independent over $\mathbb{C}$. Since $D$ commute with elements of $\mathbb{C}[S_n]$, $W^S_D(\lambda)$ is an $\mathbb{C}[S_n]$-module isomorphic to $V^S(\lambda)$. \hfill\qed
4. Differential operators and higher Specht polynomials for direct product of 2 symmetric groups

Direct product representation is very helpful in dealing with the applications of group theory in quantum chemistry particularly in connection with selection rules in spectroscopy. The direct product can be taken between any numbers of irreducible representations whether or not these are associated with any linear vectors. In many cases it is even not necessary to know their bases of representations.

4.1. Motivations and preliminaries

If two atoms of with \( n_1, n_2 \) particles respectively, come together to form a molecule. We may to a first approximation neglect the interaction between the two atoms so long as the distance between them is relatively large. In this approximation the two kinds of particles are dynamically different, for the particles of each atom are influenced only by the nucleus and the remaining particles of the same atom. The symmetry is therefore described by the sub-group \( S_{n_1} \times S_{n_2} \) of the symmetric group \( S_n \) of \( n = n_1 + n_2 \) things in which the first \( n_1 \) things and the last \( n_2 \) things are permuted among themselves. A similar situation arises when three or more atoms come together to form a molecule. Mathematically these consideration suggest the following facts. The coordinates \( x_1, \ldots, x_n \) of the particles under consideration may be separated into \( x_1, \ldots, x_{n_1} \) and \( x_{n_1 + 1}, \ldots, x_n \) where \( n = n_1 + n_2 \) and that the Hamiltonian (energy operator) \( H \) consists of two parts:

\[
H = H_1 + H_2,
\]

where \( H_1 \) contains only functions of \( x_1, \ldots, x_{n_1} \) and differentiations with respect to these variables, whereas \( H_2 \) contains only functions of \( x_{n_1+1}, \ldots, x_n \) and differentiations with respect to \( x_{n_1+1}, \ldots, x_n \). In this case \( H_1 \) and \( H_2 \) are the energy operators of the two atoms whose interaction is neglected. Then the eigenfunctions of \( H \) may be obtained as products

\[
\varphi = \varphi_1(x_1, \ldots, x_{n_1})\varphi_2(x_{n_1+1}, \ldots, x_n),
\]

where \( \varphi_1 \) is an eigenfunction of \( H_1 \) and \( \varphi_1 \) is an eigenfunction on \( H_2 \). It is quite clear that these products are eigenfunctions of \( H \) for we have

\[
H(\varphi) = (H_1 + H_2)\varphi_1\varphi_2 = (H_1\varphi_1)\varphi_2 + \varphi_1(H_2\varphi_2) = (E_1 + E_2)\varphi_1\varphi_2 = (E_1 + E_2)\varphi.
\]

Hence, the eigenvalue corresponding to \( \varphi \) is \( E = E_1 + E_2 \). All the eigenvalues of the system can be obtained in this way.

For convenience we will adopt the following notation \( H = H_1 \otimes H_2 \), \( \varphi = \varphi_1 \otimes \varphi_2 \) and \( E = E_1 \otimes E_2 \) such that

\[
H\varphi = (H_1 \otimes H_2)(\varphi_1 \otimes \varphi) = H_1\varphi_1 \otimes H_2\varphi_2 = E_1\varphi_2 \otimes E_2\varphi_2 = (E_1 \otimes E_2)(\varphi_1 \otimes \varphi_2) = E\varphi.
\]

We may consider

\[
H_1 = \sum_{i=1}^{n_1} \frac{\partial^2}{\partial x_i^2} - \sum_{1 \leq i < j \leq n_1} \frac{c(c+1)}{(x_i - x_j)^2}, \quad \text{and} \quad H_2 = \sum_{i=n_1+1}^{n} \frac{\partial^2}{\partial x_i^2} - \sum_{n_1+1 \leq i < j \leq n} \frac{c(c+1)}{(x_i - x_j)^2}.
\]
Let $\Delta_1 = \prod_{1 \leq i < j \leq n_1} (x_i - x_j)$, $\Delta_2 = \prod_{n_1 + 1 \leq i < j \leq n} (x_i - x_j)$ and $\Delta = \Delta_1 \times \Delta_2$.

The best way of understanding how $H$ operates on polynomials is by studying the polynomials representations of the ring of differential operators invariant under the direct product symmetric groups $S_{n_1} \times S_{n_2}$. For this we will make us of the theory of representation theory of product of symmetric groups.

4.2. Representation theory of product of symmetric groups, Higher Specht polynomials

In this subsection we recall some notions about irreducibles of representations of product of symmetric groups (see [1] for more details).

Let $\Lambda = (\lambda^1, \lambda^2)$ a pair of partitions (Young diagrams), with $\lambda^1 \vdash n_1$, $\lambda^2 \vdash n_2$ such that $n_1 + n_2 = n$, $\Lambda$ is called a 2-diagram. The sequence $(n_1, n_2) = (|\lambda^1|, |\lambda^2|)$ is called the type $\Lambda$ and denoted by $\text{type}(\Lambda)$. The sum $n = n_1 + n_2$ is called the size of $\Lambda$. The irreducible representations of $S_{n_1} \times S_{n_2}$ are indexed by the set of 2-diagrams of type $(n_1, n_2)$. By filling each "box" with a non-negative integer, we obtain a 2-tableau from a 2-diagram. The original 2-diagram is called the shape of the 2-diagram. A 2-tableau $T = (T^1, T^2)$ is said to be standard if the written sequence on each column and row of $T^1$ and $T^2$ is strictly increasing, and each number from 1 to $n$ appears exactly once. The set of all standard 2-tableaux of shape $\Lambda$ is denoted by $ST(\Lambda)$.

A standard 2-tableau $T = (T^1, T^2)$ is said to be natural if and only if the set of numbers written in $T^1$ is $\{1, \ldots, n_1\}$ and the set of numbers written in $T^2$ is $\{n_1 + 1, \ldots, n\}$. The set of all standard 2-tableaux of shape $\Lambda$ is denoted by $NST(\Lambda)$.

For a standard 2-tableau $T$, we associate a $w(T)$ in the following way which is similar to the one in subsection 2.3. First we read each column of the tableau $T^1$ from the bottom to the top starting from the left and we continue the procedure for $T^2$. Assigning the index $i(w)$ to the corresponding of $w(T)$ to the corresponding box, we get a new 2-tableau $i(T)$ which is called the index 2-table of $T$, respectively. For example $n = 8$, $n_1 = 5$, $n_2 = 3$, $\lambda^1 = (3, 2)$, $\lambda^2 = (2, 1)$

$$T = \begin{array}{ccc} 1 & 4 & 6 \\ 2 & 7 & 3 \\ 8 & & 5 \end{array}$$

and

$$i(T) = \begin{array}{ccc} 0 & 2 & 3 \\ 1 & 4 & 2 \end{array}$$

For words $u = (u_1, \ldots, u_n)$ and $v = (v_1, \ldots, v_n)$, we define $x_v^u = x_{v_1}^{u_1} \cdots x_{v_n}^{u_n}$. For standard 2-tableaux $S, T$, we define $x^{i(S)}_T = x^{i(w(S))}_w$. For a standard 2-tableau $T = (T^1, T^2)$ of shape $\Lambda$, $e_{T^i}$ is defined in the same way as (3.1). $e_{T^i}$ is an element in the group algebra in the permutation of the numbers in $T^i$. We set $e_T = e_{T^1} \cdot e_{T^2} = e_{T^1} \times e_{T^2}$. For $T, S \in ST(\Lambda)$, Ariki, Terasoma and Yamada have defined the higher Specht polynomial for $(T, S)$ in [1] by

$$F_T^S = F_T^S(x_1, \ldots, x_n) = e_T(x^{i(S)}_T)$$

In fact for $T = (T^1, T^2) \in ST(\Lambda)$ and $S = (S^1, S^2) \in ST(\Lambda)$ we have $F_T^S = F_{T^1}^{S^1} \cdot F_{T^2}^{S^2}$ where $F_{T^1}^{S^1}$ and $F_{T^2}^{S^2}$ are the higher Specht polynomials defined in (2.2)
Theorem 4.1. [1] For a 2-diagram $\Delta$ of type $(n_1,n_2)$ and $S \in ST(\Delta)$, the set $\{F_T^S | T \in \text{NST}(\Delta)\}$ forms a $C$-basis of a $S_{n_1} \times S_{n_2}$-submodule denoted by $V^S(\Delta)$, which affords to be the irreducible representation of $C[S_{n_1} \times S_{n_2}]$ corresponding to $\Delta$. All the other irreducible representations of $S_{n_1} \times S_{n_2}$ are obtained by same procedure.

4.3. Decomposition Theorem

In this subsection we establish a decomposition theorem of the polynomial ring in $n$ indeterminates.

4.3.1. Actions description

As we want to study the polynomials representation of the ring of invariant differential operators under the product of symmetric groups $S_{n_1} \times S_{n_2}$ localized at $\Delta$. It is convenient to precisely describe the action of that ring of invariant differential operators on the polynomials ring.

Let $V_i$ be a $C[S_{n_i}]$-module ($i = 1, 2$), we define the actions $C[S_{n_1} \times S_{n_2}]$ on $V_1 \otimes V_2$ by

$$(s_1 \times s_2)(v_1 \otimes v_2) = s_1 v_1 \otimes s_2 v_2,$$

for $s_i \in S_{n_i}$, $v_i \in V_i$, $i = 1, 2$. This makes $V_1 \otimes V_2$ into $C[S_{n_1} \times S_{n_2}]$. If $V_i$ is an irreducible $C[S_{n_i}]$-module ($i = 1, 2$), then $V_1 \otimes V_2$ is an irreducible $C[S_{n_1} \times S_{n_2}]$-module and all irreducible $C[S_{n_1} \times S_{n_2}]$-modules have this form [3, Chapter IV §27].

Let $D_X := C\langle x_1, \ldots, x_n, \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n} \rangle$ be the ring of differential operators associated with the polynomial ring $O_X := C[x_1, \ldots, x_n]$, and $O_Y := C[x_1, \ldots, x_n]^S_{n_1 \times S_{n_2}} = C[y_1, \ldots, y_n]$ be the ring of invariant under the symmetric group $S_{n_1} \times S_{n_2}$.

Let $O_{X_1} := C[x_1, \ldots, x_{n_1}], O_{X_2} := C[x_{n_1+1}, \ldots, x_n], O_{Y_1} := C[x_1, \ldots, x_{n_1}]^{S_{n_1}} = C[y_1, \ldots, y_{n_1}$ and $O_{Y_2} := C[x_{n_1+1}, \ldots, x_n]^{S_{n_2}} = C[y_{n_1+1}, \ldots, y_n]$ where

$$y_j = \sum_{i=1}^{n_1} \frac{x_i^j}{j} \text{ for } j = 1, \ldots, n_1, \text{ and } y_{n_1+j} = \sum_{i=1}^{n_2} \frac{x_{n_1+i}^j}{j} \text{ for } j = 1, \ldots, n_2$$

It is clear that $O_X = O_{X_1} \otimes O_{X_2}$ and $O_Y = O_{Y_1} \otimes O_{Y_2}$.

We denote by $D_Y = C\langle y_1, \ldots, y_n, \frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial y_n} \rangle$ the ring of differential operators associated with $O_Y = C[y_1, \ldots, y_n]$.

Let $D_{Y_1} = C\langle y_1, \ldots, y_{n_1}, \frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial y_{n_1}} \rangle$ and $D_{Y_2} = C\langle y_{n_1+1}, \ldots, y_n, \frac{\partial}{\partial y_{n_1+1}}, \ldots, \frac{\partial}{\partial y_n} \rangle$ so that $D_Y = D_{Y_1} \otimes D_{Y_2}$.

By localization, $O_X$ is turned into a $D_Y$-module, as the following lemma states.

Notations We adopt the following notations

$$\tilde{O}_X := C[x_1, \ldots, x_n, \Delta^{-1}], \tilde{O}_Y := C[y_1, \ldots, y_n, \Delta^{-2}], \tilde{D}_Y := C\langle y_1, \ldots, y_n, \frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial y_n}, \Delta^{-2} \rangle$$

$$\tilde{O}_{X_1} := C[x_1, \ldots, x_{n_1}, \Delta_{-1}], \tilde{O}_{Y_1} := C[y_1, \ldots, y_{n_1}, \Delta_{-1}^{-2}], \tilde{D}_{Y_1} := C\langle y_1, \ldots, y_{n_1}, \frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial y_{n_1}} \rangle$$

$$\tilde{O}_{X_2} := C[x_{n_1}, \ldots, x_n, \Delta_{-2}], \tilde{O}_{Y_2} := C[y_{n_1+1}, \ldots, y_n, \Delta_{-2}^{-2}], \tilde{D}_{Y_2} := C\langle y_{n_1+1}, \ldots, y_n, \frac{\partial}{\partial y_{n_1+1}}, \ldots, \frac{\partial}{\partial y_n} \rangle$$

$$\tilde{O}_{X_1} := C[x_1, \ldots, x_{n_1}, \Delta_{-1}], \tilde{O}_{Y_1} := C[y_1, \ldots, y_{n_1}, \Delta_{-1}^{-2}], \tilde{D}_{Y_1} := C\langle y_1, \ldots, y_{n_1}, \frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial y_{n_1}} \rangle$$

$$\tilde{O}_{X_2} := C[x_{n_1}, \ldots, x_n, \Delta_{-2}], \tilde{O}_{Y_2} := C[y_{n_1+1}, \ldots, y_n, \Delta_{-2}^{-2}], \tilde{D}_{Y_2} := C\langle y_{n_1+1}, \ldots, y_n, \frac{\partial}{\partial y_{n_1+1}}, \ldots, \frac{\partial}{\partial y_n} \rangle$$
For $M_1$ a $\tilde{D}_Y$-module, $M_2$ a $\tilde{D}_Y$-module, we make $M_1 \otimes M_2$ into a $\tilde{D}_Y \otimes \tilde{D}_Y$-module by setting

$$(D_1 \otimes D_2)(m_1 \otimes m_2) = D_1 m_1 \otimes D_2 m_2$$

for $D_1 \in \tilde{D}_Y$, $D_2 \in \tilde{D}_Y$, $m_1 \in \tilde{O}_X$, and $m_2 \in \tilde{O}_X$.

**Lemma 4.2.** $\tilde{O}_X$ is a $\tilde{D}_Y$-module.

**Proof.** Let us make clear the action of $\tilde{D}_Y$ on $\tilde{O}_X$.

We have $y_j = \sum_{i=1}^{n_1} \frac{x^i_j}{j}$, $j = 1, \ldots, n_1$, hence $\frac{\partial}{\partial x_i} y_j = \sum_{j=1}^{n_1} x^j_i \frac{1}{j}$, $i = 1, \ldots, n_1$ and $y_{n_1+j} = \sum_{i=1}^{n_2} \frac{x^i_{n_1+i}}{j}$, $j = 1, \ldots, n_2$, hence $\frac{\partial}{\partial x_{n_1+i}} y_{n_1+j} = \sum_{j=1}^{n_2} x^j_{n_1+i} \frac{1}{j}$, $i = 1, \ldots, n_2$.

Let $A_1 = (x^i_j)_{1 \leq i, j \leq n_1}$, and $A_2 = (x^i_j)_{1 \leq i, j \leq n_2}$ and $A = \begin{pmatrix} A_1 & \vdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \vdots & A_2 \end{pmatrix}$ so that $\det(A_1) = \Delta_1$, $\det(A_2) = \Delta_2$ and $\det(A) = \Delta = \Delta_1 \times \Delta_2$ we get the following equation

$$
\begin{pmatrix}
\frac{\partial}{\partial x_1} \\
\vdots \\
\frac{\partial}{\partial x_n}
\end{pmatrix}
= A
\begin{pmatrix}
\frac{\partial}{\partial y_1} \\
\vdots \\
\frac{\partial}{\partial y_n}
\end{pmatrix}
$$

Since $\Delta \neq 0$, it follows that

$$
\begin{pmatrix}
\frac{\partial}{\partial y_1} \\
\vdots \\
\frac{\partial}{\partial y_n}
\end{pmatrix}
= A^{-1}
\begin{pmatrix}
\frac{\partial}{\partial x_1} \\
\vdots \\
\frac{\partial}{\partial x_n}
\end{pmatrix}
$$

and it clear that $\tilde{O}_X$ is a $\tilde{D}_Y$-module. \(\square\)

Let $\Lambda$ be a 2-diagram of size $n$ and $T \in NST(\Lambda)$, the element $e_T$ is a primitive idempotent for the group algebra $\mathbb{C}[S_{n_1} \times S_{n_2}]$ [12, chapter V, § 10], all the idempotent primitive idempotents of $\mathbb{C}[S_{n_1} \times S_{n_2}]$ have this form.

The following theorem is the analogue of [8, Theorem] for product of symmetric groups.

**Theorem 4.3.** Let $\Lambda$ be a 2-diagram of size $n$, $T \in NST(\Lambda)$, and $e_T$ is a primitive idempotent associated to $T$. Then we have:

(i) $e_T \tilde{O}_X$ is a nontrivial $\tilde{D}_Y$-submodule of $\tilde{O}_X$,

(ii) The $\tilde{D}_Y$-module $e_T \tilde{O}_X$ is simple,

(iii) There exist a $S \in ST(\Lambda)$ and a higher Specht polynomial $F_T^S$ for $(T, S)$ such that $e_T \tilde{O}_X = \tilde{D}_Y F_T^S$.

**Proof.** Let $\Lambda$ be a 2-diagram of size $n$ and $T \in NST(\Lambda)$, There exists $n_1, n_2 \in \mathbb{N}$, $\lambda^1 \vdash n_1$, $\lambda^2 \vdash n_2$, $T_1 \in STab(\lambda_1)$ and $T_2 \in STab(\lambda_2)$ such that

$$n_1 + n_2 = n, \lambda = (\lambda^1, \lambda^2) \text{ and } e_T = e_{T_1} \times e_{T_2}.$$
(i) we know that $e_T$ is an primitive idempotent for $\mathbb{C}[S_{n_i}], i = 1, 2$ and

$$e_T \tilde{O}_X = (e_T \times e_T) \tilde{O}_X = (e_T \times e_T)(\tilde{O}_{X_1} \otimes \tilde{O}_{X_2}) = e_T \tilde{O}_{X_1} \otimes e_T \tilde{O}_{X_2}.$$  

By [8, Theorem 3.6], $e_T \tilde{O}_{X_1}$, and $e_T \tilde{O}_{X_2}$ are nontrivial modules over $\tilde{D}_{Y_1}$ and $\tilde{D}_{Y_2}$ respectively. Hence $e_T \tilde{O}_X$ is a nontrivial module over $\tilde{D}_Y = \tilde{D}_{Y_1} \otimes \tilde{D}_{Y_2}$.

(ii) $e_T$ being primitive idempotent for $\mathbb{C}[S_{n_i}], i = 1, 2$, by [8, Theorem 3.6], we have that $e_T \tilde{O}_{X_1}$ and $e_T \tilde{O}_{X_2}$ are irreducible modules over $\tilde{D}_{Y_1}$ and $\tilde{D}_{Y_2}$ respectively. Then $e_T \tilde{O}_{X_1} \otimes e_T \tilde{O}_{X_2}$ is an irreducible $\tilde{D}_{Y_1} \otimes \tilde{D}_{Y_2}$-module. Hence $e_T \tilde{O}_X$ is a simple $\tilde{D}_Y$-module.

(iii) Let $S^i$ be the conical standard tableau of shape $\lambda_i, i = 1, 2$ and we know higher Specht polynomial $F_{S^i}^i$, is proportional to the Specht polynomial of $T_i, i = 1, 2$. Then By [8, Theorem 3.6] we that $e_T \tilde{O}_{X_1} = \tilde{D}_{Y_1} F_{S^1}^1$ and $\tilde{O}_{X_2} = \tilde{D}_{Y_2} F_{S^2}^2$ so that

$$e_T \tilde{O}_{X_1} = e_T \tilde{O}_{X_1} \otimes e_T \tilde{O}_{X_1}$$  

$$= \tilde{D}_{Y_1} F_{S^1}^1 \otimes \tilde{O}_{X_1} \tilde{D}_{Y_2} F_{S^2}^2$$  

$$= (\tilde{D}_{Y_1} \otimes \tilde{D}_{Y_2})(F_{S^1}^1 \otimes F_{S^2}^2)$$  

$$= \tilde{D}_Y F_{S}^T$$ with $T = (T^1, T^2)$ and $S = (S^1, S^2)$.

\[\Box\]

**Proposition 4.4.** Let $\Lambda$ be a 2-diagram of size $n$, $T \in NST(\Lambda)$ and $M^\Lambda$ be the isotopic component associated to $V(\Lambda)$ in $\tilde{O}_X$. Let us set $M_T := e_T \tilde{O}_X$. Then we have:

(i) $M_T = \bigtriangledown(V(\Lambda))$ where $\bigtriangledown$ the equivalence categories defined in Proposition 3.4 for $G = S_{n_1} \times S_{n_2}$

(ii) $M_T = e_T(M^\Lambda)$ is simple $\tilde{D}_Y$-module;

(iii) $M^\Lambda = \sum_{T \in NSTab(\Lambda)} e_T(M^\Lambda)$.

**Proof.** (i) Let consider the right $\mathbb{C}[S_{n_1} \times S_{n_2}]$-module $V = e_T \mathbb{C}[S_{n_1} \times S_{n_2}]$ where $T \in NST(\Lambda)$. This is the image of $\mathbb{C}[S_{n_1} \times S_{n_2}]$ by right multiplication map $e_T : \mathbb{C}[S_{n_1} \times S_{n_2}] \rightarrow \mathbb{C}[S_{n_1} \times S_{n_2}]$. By [7, Example 2.5], we may turn this map to a left multiplication $\mathbb{C}[S_{n_1} \times S_{n_2}]^r \rightarrow \mathbb{C}[S_{n_1} \times S_{n_2}]^r$ and get an image which is isomorphic to the irreducible $\mathbb{C}[S_{n_1} \times S_{n_2}]$-module $V^S(\Lambda)$ (for some $S \in ST(\Lambda)$) associated to $\Lambda$. We get an induced map

$$\bigtriangledown(\mathbb{C}[S_{n_1} \times S_{n_2}]^r) \rightarrow \bigtriangledown(V^S(\Lambda)) \subset \bigtriangledown(\mathbb{C}[S_{n_1} \times S_{n_2}]^r),$$

which is a multiplication by $e_T$ according to [7, Example 2.5]. Then $\bigtriangledown(V^S(\Lambda))$ is egal to $e_T \tilde{O}_X = M_T$.

(ii) Since $V^S(\Lambda)$ is a simple $\mathbb{C}[S_{n_1} \times S_{n_2}]$-module, $\bigtriangledown(V^S(\Lambda))$ is also a simple $\tilde{D}_Y$-module.

(iii) follows from the fact that $1 = \sum_{\Lambda} \sum_{T \in NSTab(\Lambda)} e_T$ where $\Lambda$ over all 2-diagram of size $n$ and $e_T(M^\Lambda) = 0$ if $T$ is not in $NST(\Lambda)$.

\[\Box\]

**Theorem 4.5.** Let $\Lambda$ be a 2-diagram of size $n$, $T \in NST(\Lambda)$ and $M_T = e_T \tilde{O}_X$. Then

(i) $M_T = \bigoplus_{S \in STab(\Lambda)} \tilde{D}_Y F_{S}^T$ as $\tilde{D}_Y$-module,
(ii) $\hat{O}_X = \bigoplus_{\Lambda} \bigoplus_{S \in STab(\Lambda)} \hat{O}_T F^S_T$ as a $\hat{D}_Y$-module, where sum runs over all the $2$-diagram $\Lambda$ of size $n$.

**Proof.** (i) Let $\Lambda$ be a $2$-diagram of size $n$, $T \in NST(\Lambda)$ and $M_T = e_T \hat{O}_X$. Suppose that $\Lambda = (\lambda^1, \lambda^2), T = (T^1, T^2)$, with $\lambda_i \vdash n_i, T^i \in STab(\lambda^i), i = 1, 2$ and $n = n_1 + n_2$. We have that

$$M_T = e_T \hat{O}_X = (e_{T^1} \times e_{T^2})(\hat{O}_{X_1} \otimes \hat{O}_{X_2}) = e_{T^1} \hat{O}_{X_1} \otimes e_{T^2} \hat{O}_{X_2} = M_{T^1} \otimes M_{T^2}$$

$$= \left( \bigoplus_{S^1 \in STab(\lambda^1)} \hat{O}_{T^1} F^{S^1}_{T^1} \right) \otimes \left( \bigoplus_{S^2 \in STab(\lambda^2)} \hat{O}_{T^2} F^{S^2}_{T^2} \right)$$

by Theorem 3.6

$$= \bigoplus_{S \in STab(\Lambda)} \hat{O}_T F^S_T$$

with $S = (S^1, S^2)$.

(ii) follows from the fact that $\hat{O}_X = \hat{O}_{X_1} \otimes \hat{O}_{X_2}$ and $\hat{O}_X = \bigoplus_{\lambda^1 \vdash n_1} \bigoplus_{S^1, T^1 \in STab(\lambda^1)} \hat{O}_{T^1} F^{S^1}_{T^1}$

and $\hat{O}_X = \bigoplus_{\lambda^2 \vdash n_2} \bigoplus_{S^2, T^2 \in STab(\lambda^2)} \hat{O}_{T^2} F^{S^2}_{T^2}$.

$\blacksquare$

**Theorem 4.6.** Let $\Lambda$ be a $2$-diagram of size $n$, $T \in NST(\Lambda)$ and $D \in \hat{D}_Y$ such that $D(F^S_T) \neq 0$ for $S \in STab(\Lambda)$. Then the image of the $\mathbb{C}[S_{n_1} \times S_{n_2}]$-module $V^S(\Lambda)$ by $D$ is an $\mathbb{C}[S_{n_1} \times S_{n_2}]$-module isomorphic to $V^S(\Lambda)$. In other words, the actions of the differential operators of $\hat{D}_Y$ on the higher Specht polynomials generate isomorphic copies of the corresponding module.

**Proof.** Let $\Lambda$ be a $2$-diagram of size $n$, $T \in NST(\Lambda)$ and $D \in \hat{D}_Y$ such that $D(F^S_T) \neq 0$ for $S \in STab(\Lambda)$. Then $D$ may be written as $D = D_1 \otimes D_2$ where $D_1 \in \hat{D}_{Y_1}$ and $D_2 \in \hat{D}_{Y_2}$. Suppose that $\Lambda = (\lambda^1, \lambda^2), T = (T^1, T^2), S = (S^1, S^2)$ with $\lambda_i \vdash n_i, T^i \in STab(\lambda^i), i = 1, 2$ and $n = n_1 + n_2$. We have that $F^S_T = F^{S^1}_{T^1} \otimes F^{S^2}_{T^2}$ and $D(F^S_T) = (D_1 \otimes D_2)(F^{S^1}_{T^1} \otimes F^{S^2}_{T^2}) = D_1 F^{S^1}_{T^1} \otimes D_2 F^{S^2}_{T^2} \neq 0$.

so that $D_1 F^{S^1}_{T^1} \neq 0$ and $D_2 F^{S^2}_{T^2} \neq 0$. Then by Theorem 3.10, $D_1 F^{S^1}_{T^1}$ generate a $\mathbb{C}[S_{n_1}]$-module isomorphic to $V^{S^1}(\lambda_i), i = 1, 2$. Hence $D_1 F^{S^1}_{T^1} \otimes D_2 F^{S^2}_{T^2}$ generate a $\mathbb{C}[S_{n_1} \times S_{n_2}]$-module isomorphic to $V^{S^1}(\lambda_1) \otimes V^{S^2}(\lambda_2) \cong V^{S}(\Lambda)$.

$\blacksquare$

All the results of this section can be generalized to products of finite number of symmetric groups.
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