Research Article

On a Kind of Dirichlet Character Sums

Rong Ma,1 Yulong Zhang,2 and Guohe Zhang2

1 School of Science, Northwestern Polytechnical University, Xi’an, Shaanxi 710072, China
2 The School of Electronic and Information Engineering, Xi’an Jiaotong University, Xi’an, Shaanxi 710049, China

Correspondence should be addressed to Rong Ma; marong0109@163.com

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Let \( p \geq 3 \) be a prime and let \( \chi \) denote the Dirichlet character modulo \( p \). For any prime \( q \) with \( q < p \), define the set \( E(q, p) = \{ a | 1 \leq a, a \leq p, a \equiv 1 (\text{mod } p) \text{ and } a \equiv a (\text{mod } q) \} \). In this paper, we study a kind of mean value of Dirichlet character sums \( \sum_{a \leq p, a \in E(q, p)} \chi(a) \), and use the properties of the Dirichlet \( L \)-functions and generalized Kloosterman sums to obtain an interesting estimate.

1. Introduction

Let \( k \geq 3 \) be an integer and let \( \chi \) denote the Dirichlet character modulo \( k \), for any real number \( x \geq 1 \), many scholars have studied the following sums:

\[
\sum_{n \leq x} \chi(n),
\]

where \( n \) are positive integers.

Perhaps one of the most famous results is Pólya’s inequality [1]. That is, when \( \chi \) is the primitive character modulo \( k \), we have

\[
\sum_{n \leq x} \chi(n) < k^{1/2} \log k.
\]

In fact, the result can be extended to the nonprincipal character \( \chi \) modulo \( k \) [2]. Further details about the estimates of character sums can be found in the literature, for example, [3, 4].

For any fixed integer \( H > 0 \) and any positive integer \( k \geq 3 \), define the following set:

\[
L(H, k) = \{ a | 1 \leq a, a \leq k - 1, (a, k) = 1, a \equiv 1 (\text{mod } k), |a - \bar{a}| \leq H \},
\]

where \( \bar{a} \) denotes the inverse of \( a \) modulo \( k \).

Let \( r_2(k) \) be the number of solutions of the congruence equation \( ab \equiv 1 (\text{mod } k) \) for \( 1 \leq a, b \leq k \) in which \( a \) and \( b \) are of opposite parity, this can be expressed as follows:

\[
r_2(k) = \sum_{\substack{a \in \mathbb{Z}^* \\ gcd(a, k) = 1 \\ a \equiv b (\text{mod } k)}} 1.
\]

Richard [7] asks us to find \( r_2(k) \) or at least to say something nontrivial about it. About this problem, a lot of scholars have studied it [8–12]. Now we let \( m \) be another integer with \( m < k \) and let \( r_m(k) \) denote the number of all pairs of integers \( a, b \) satisfying \( ab \equiv 1 (\text{mod } k) \), \( 1 \leq a, b \leq k \),

Xi and Yi [5] studied the problem for \( \chi \) the nonprincipal Dirichlet character modulo \( k \), and got

\[
\sum_{n \in L(H, k)} \chi(n) \ll k^{1/2} d(k) \log H,
\]

where \( 0 < H \leq q \) was a constant and \( d(k) \) was the divisor function. Before this, Wenpeng [6] got an asymptotic formula for the case that \( \chi \) was the principal Dirichlet character modulo \( k \).

On the other hand, for each integer \( a \) with \( 1 \leq a \leq k \) and \( (a, k) = 1 \), we know that there exists one and only one \( b \) with \( 1 \leq b \leq k \) such that \( ab \equiv 1 (\text{mod } k) \). Let \( r_m(k) \) be the number of solutions of the congruence equation \( ab \equiv 1 (\text{mod } k) \) for \( 1 \leq a, b \leq k \) in which \( a \) and \( b \) are of opposite parity, this can be expressed as follows:

\[
r_m(k) = \sum_{\substack{a \in \mathbb{Z}^* \\ gcd(a, k) = 1 \\ a \equiv b (\text{mod } k)}} 1.
\]
and \( m \parallel (a + b) \). Lu and Yi [13] have obtained the asymptotic formula of generalized D. H. Lehmer problem as follows:

\[
r_m(k) = \sum_{a=1}^{k} 1 = \left(1 - \frac{1}{m}\right) \phi(k) + O\left(k^{1/2} \log^2 k\right),
\]

where the \( O \) constant only depends on \( m \).

In this paper, let \( p \) be an odd prime and let \( q \) be a fixed prime with \( q < p \), define the set \( E(q, p) \) for \( a(1 \leq a \leq p) \) such that \( a \equiv 1 \pmod{p} \) and \( a \equiv \bar{a} \pmod{q} \), that is,

\[
E(q, p) = \{a \mid 1 \leq a \leq p-1, a \equiv 1 \pmod{p}, a \equiv \bar{a} \pmod{q}\}.
\]

As another case of (7), we will consider the mean value of Dirichlet character sums as follows:

\[
\sum_{a \in E(q, p)} \chi(a),
\]

and get an interesting estimate. That is, we will prove the following theorem.

**Theorem 1.** Let \( p \) be an odd prime and let \( q \) be a fixed prime with \( q < p \), and let \( \chi \) denote the Dirichlet character modulo \( p \). Let \( E(q, p) \) denote the following set:

\[
E(q, p) = \{a \mid 1 \leq a \leq p-1, a \equiv 1 \pmod{p}, a \equiv \bar{a} \pmod{q}\}.
\]

then, for any nonprincipal Dirichlet character \( \chi \) mod \( p \), we have the following estimate:

\[
\sum_{a \in E(q, p)} \chi(a) = O\left(p^{1/2 + \epsilon}\right),
\]

where the \( O \) constant only depends on \( q \).

From this Theorem we can get

\[
\sum_{a=1}^{q-1} \chi(a) = \sum_{a=1}^{p-1} \chi(a) - \sum_{a \equiv \bar{a} \pmod{q}} \chi(a) = O\left(p^{1/2 + \epsilon}\right).
\]

For any integer \( k \) and fixed integer \( m \) such that \( (m, k) = 1 \), whether or not there exists an estimate for

\[
\sum_{a \equiv \bar{a} \pmod{m}} \chi(a)
\]

is still an open problem.

2. Some Lemmas

In this section, we will give several lemmas which are necessary in the proof of the theorem.

**Lemma 2.** Let \( Q \) be an integer, and let \( \chi \) be a primitive character modulo \( Q \). Then, for any real number \( u \) and \( v \) with \( u < v \), we have

\[
\sum_{0 < n \leq Q} \chi(n) = \tau(\chi) \sum_{0 < |h| \leq H} \chi(h) \frac{e(-hu) - e(-hv)}{2\pi ih} + O\left(1 + \frac{Q \log Q}{H}\right),
\]

where \( e(x) = e^{2\pi i x} \) and \( \tau(\chi) = \sum_{a=1}^{Q} \chi(a) e(a/Q) \) are Gauss sums.

Especially, let \( u = 0 \), we have a slight modification

\[
\sum_{0 < n \leq Q} \chi(n) = \left\{ \begin{array}{ll} \tau(\chi) \sum_{n=1}^{\infty} \chi(n) \sin(2\pi n V) & \text{if } \chi(-1) = 1, \\ \tau(\chi) \sum_{n=1}^{\infty} \chi(n)(1 - \cos(2\pi n V)) & \text{if } \chi(-1) = -1. \end{array} \right.
\]

Proof. (See [1]).

**Lemma 3.** Let \( q \) be a prime, let \( Q \) be an integer with \( Q > q \), and let \( \chi \) be a primitive character modulo \( Q \), then, we have

\[
\sum_{n \leq Q/q} \chi(n) = \left\{ \begin{array}{ll} \tau(\chi) \sum_{n=1}^{\infty} \chi(n) \sin(2\pi n V) & + O(1), \\ \tau(\chi) \sum_{n=1}^{\infty} \chi(n)(1 - \cos(2\pi n V)) & + O(1), \end{array} \right.
\]

where \( \chi(\gamma) = e^{2\pi i x} \) and \( \tau(\chi) = \sum_{a=1}^{Q} \chi(a) e(a/Q) \) are Gauss sums.

Especially, let \( u = 0 \), we have a slight modification

\[
\sum_{0 < n \leq Q} \chi(n) = \left\{ \begin{array}{ll} \tau(\chi) \sum_{n=1}^{\infty} \chi(n) \sin(2\pi n V) & + O(1), \\ \tau(\chi) \sum_{n=1}^{\infty} \chi(n)(1 - \cos(2\pi n V)) & + O(1), \end{array} \right.
\]

where \( \chi(-1) = 1 \) or \( \chi(-1) = -1 \).

Proof. From Lemma 2, we take \( u = 0 \), and \( v = 1/q \) and get

\[
\sum_{n \leq Q/q} \chi(n) = \tau(\chi) \sum_{0 < h < H} \frac{1 - e(-h/q)}{2\pi ih} + O\left(1 + \frac{Q \log Q}{H}\right).
\]
Lemma 4. Let \( \chi \) denote the Dirichlet character modulo \( q \), the generalized Kloosterman sums are defined by
\[
S_\chi(m, n; q) = \sum_{a \mod q} \chi(a) e\left(\frac{m\bar{a} + na}{q}\right),
\]
where \( a\bar{a} \equiv 1 \mod q \) and \( e(y) = e^{2\pi iy} \).
Then, we have the following estimate:
\[
S_\chi(m, n; q) \ll q^{1/2+\varepsilon}(m, n, q)^{1/2},
\]
where \((m, n, q)\) denotes the gcd of \( m, n, \) and \( q \).

Proof. (See [15]).

Lemma 5. Let \( m, n \) be integers and let \( q \geq 3 \) be prime, let \( \chi \) denote the Dirichlet character modulo \( q \), the generalized Kloosterman sums are defined by
\[
S_\chi(m, n; q) = \sum_{a \mod q} \chi(a) e\left(\frac{m\bar{a} + na}{q}\right),
\]
where \( a\bar{a} \equiv 1 \mod q \) and \( e(y) = e^{2\pi iy} \).

Proof. (See [15]).

Lemma 6. Let \( p \geq 3 \) be an odd prime, let \( \chi, \chi_1 \) be a Dirichlet character modulo \( p \), \( \chi \chi_1 \). For any odd prime \( q \), let \( \chi_2, \chi_3, \chi_4 \) be any Dirichlet characters with \( \chi_2 \mod q \), \( \chi_3 \mod q \) and \( \chi_4 \mod q \), respectively, then, no matter \( \chi \) is odd character or even character modulo \( p \), we have
\[
\sum_{\chi \neq \chi_1, \chi_2, \chi_3} \chi(\chi_1) \chi(\chi_2) \chi(\chi_3) \chi(\chi_4) \ll p^{3/2+\varepsilon};
\]
\[
\sum_{\chi \neq \chi_1, \chi_2, \chi_3} \chi(\chi_1) \chi(\chi_2) \chi(\chi_3) \chi(\chi_4) \ll p^{3/2+\varepsilon};
\]
\[
\sum_{\chi \neq \chi_1, \chi_2, \chi_3} \chi(\chi_1) \chi(\chi_2) \chi(\chi_3) \chi(\chi_4) \ll p^{3/2+\varepsilon},
\]
where the \( \ll \) constant only depends on \( q \).

Proof. For any integer \( n \) with \( (n, k) = 1 \) (\( k \geq 3 \) is any positive integer), we have
\[
\sum_{\chi \mod k} \chi(n) = \begin{cases} 
\frac{1}{2} \phi(k), & n \equiv \pm 1 \pmod{k}, \\
0, & \text{otherwise}.
\end{cases}
\]
\[
\sum_{\chi \mod k} \chi(n) = \begin{cases} 
\frac{1}{2} \phi(k), & n \equiv 1 \pmod{k}, \\
-\frac{1}{2} \phi(k), & n \equiv -1 \pmod{k}, \\
0, & \text{otherwise}.
\end{cases}
\]
Now let \( y > p \) and let \( A(y, \chi) = \sum_{p \leq y} \chi(n) \). Then, from the Pólya-Vinogradov inequality, we obtain
\[
A(y, \chi) \ll \sqrt{p} \log p.
\] (30)

Hence, from Abel's identity, for any \( \text{Re}(s) \geq 1 \), we can easily get
\[
L(s, \chi) = \sum_{n \leq p} \frac{\chi(n)}{n^s} + \int_p^\infty \frac{A(y, \chi)}{y^s} \, dy
\]
\[
= \sum_{n \leq p} \frac{\chi(n)}{n^s} + O\left(\frac{\log p}{p^{s-1/2}}\right)
\] (31)
\[
= \sum_{n \leq p} \frac{\chi(n)}{n^s} + O\left(\frac{\log p}{p^{1/2}}\right).
\]

We will take (25), for example, to prove this lemma. For \( (q, p) = 1 \) and
\[
\sum_{\substack{a \leq (p-1)/q \\text{mod } p}} \chi(a) e\left(\frac{a+b}{p}\right)
\]
\[
\sum_{a \leq (p-1)/q \\text{mod } p} \chi(a) e\left(\frac{a+b}{p}\right)
\]

where \( abcq^{-kn} \equiv 1(\text{mod } p) \). From Lemma 5, we can easily obtain
\[
\sum_{\substack{x_1 \neq x_2 \\text{mod } p \\text{mod } \chi x_1(-1)\equiv 1}} xx_1(q) x_1(q) \tau(\chi x_1) \tau(x_1) \times L(1, \chi x_1 x_2 x_3) \ll p^{3/2 + \epsilon} \log p \ll p^{3/2 + \epsilon},
\] (33)

where the \( \ll \) constant only depends on \( q \). Therefore, this completes the proof of Lemma 6.

\textbf{Lemma 7.} Let \( q \) be a fixed odd prime and let \( p \) be a prime with \( p > q \), let \( \chi \) denote the nonprincipal Dirichlet character modulo \( p \), and \( \chi_1 \) denote the Dirichlet character modulo \( p \), then, we have
\[
\sum_{\substack{x_1 \neq x_2 \\text{mod } p \\text{mod } \chi x_1(-1)\equiv 1}} xx_1(q) x_1(q) \tau(\chi x_1) \tau(x_1) \times L(1, \chi x_1 x_2 x_3) \ll p^{3/2 + \epsilon} \log p \ll p^{3/2 + \epsilon},
\] (34)

where the \( \ll \) constant only depends on \( q \).
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Proof. For primes $p$ and $q$, $\chi_1$ and $\chi \chi_1$ are nonprincipal and primitive characters modulo $\rho$, hence, from Lemmas 3 and 6, when $\chi(-1) = 1$, we can get

$$\sum_{\chi \mod \rho} \chi \chi_1 (q) \chi_1 (q) \sum_{a \leq (p-1)/q} \chi \chi_1 (a) \sum_{b \leq (p-1)/q} \chi_1 (b)$$

$$= \sum_{\chi \mod \rho} \chi \chi_1 (q) \chi_1 (q)$$

$$\times \left( \frac{\tau(\chi \chi_1)}{(q-1) \pi} \sum_{u=1}^{q-1} \sum_{\chi \mod q} \left( \chi_3 (u) \sin \frac{2\pi u}{q} \right) \right)$$

$$\times L (1, \chi \chi_1 \chi_3) + O (1) \right)$$

$$+ \sum_{\chi \mod \rho} \chi \chi_1 (q) \chi_1 (q)$$

$$\times \left( \frac{\tau(\chi_1)}{(q-1) \pi} \sum_{u=1}^{q-1} \sum_{\chi \mod q} \left( \chi_4 (u) \left( 1 - \cos \frac{2\pi u}{q} \right) \right) \right)$$

$$\times L (1, \chi \chi_1 \chi_3) + O (1) \right)$$

$$\times \left( \frac{\tau(\chi_1)}{(q-1) \pi} \sum_{u=1}^{q-1} \sum_{\chi \mod q} \left( \chi_4 (v) \left( 1 - \cos \frac{2\pi v}{q} \right) \right) \right)$$

$$\times L (1, \chi \chi_1 \chi_3) + O (1) \right)$$

$$= \frac{1}{(q-1)^2 \pi^2}$$

$$\times \sum_{a=1}^{q-1} \sum_{b=1}^{q-1} \sum_{\chi \mod q} \chi_3 (u) \chi_4 (v) \sin \frac{2\pi u}{q} \sin \frac{2\pi v}{q}$$

$$\times \sum_{\chi_1 \neq \chi_p} \chi \chi_1 (q) \chi_1 (m) \tau(\chi \chi_1) \tau(\chi_1)$$

$$\times L (1, \chi \chi_1 \chi_3) L (1, \chi \chi_1 \chi_3)$$

$$+ \frac{-1}{(q-1)^2 \pi^2}$$

$$\times \sum_{a=1}^{q-1} \sum_{b=1}^{q-1} \sum_{\chi \mod q} \chi_3 (u) \chi_4 (v) \sin \frac{2\pi u}{q} \sin \frac{2\pi v}{q}$$

$$\times \left( 1 - \cos \frac{2\pi u}{q} \right) \left( 1 - \cos \frac{2\pi v}{q} \right)$$

$$\times \left( \sum_{\chi \mod \rho} \chi \chi_1 (q) \chi_1 (q) \tau(\chi \chi_1) \tau(\chi_1) \right)$$

$$\times L (1, \chi \chi_1 \chi_3) L (1, \chi \chi_1 \chi_3) + O (1) \right)$$

$$\times \left( \frac{\tau(\chi_1)}{(q-1) \pi} \sum_{u=1}^{q-1} \sum_{\chi \mod q} \left( \chi_4 (u) \left( 1 - \cos \frac{2\pi u}{q} \right) \right) \right)$$

$$\times L (1, \chi \chi_1 \chi_3) + O (1) \right)$$

$$\times \left( \frac{\tau(\chi_1)}{(q-1) \pi} \sum_{u=1}^{q-1} \sum_{\chi \mod q} \left( \chi_4 (v) \left( 1 - \cos \frac{2\pi v}{q} \right) \right) \right)$$

$$\times L (1, \chi \chi_1 \chi_3) + O (1) \right)$$

$$\times \left( \frac{\tau(\chi_1)}{(q-1) \pi} \sum_{u=1}^{q-1} \sum_{\chi \mod q} \left( \chi_4 (u) \left( 1 - \cos \frac{2\pi u}{q} \right) \right) \right)$$

$$\times L (1, \chi \chi_1 \chi_3) + O (1) \right)$$

where the $\ll$ constant is only concerned with $q$.

When $\chi(-1) = -1$, by the similar method, we can also obtain

$$\sum_{\chi \mod \rho} \chi \chi_1 (q) \chi_1 (q) \sum_{a=1}^{q-1} \chi \chi_1 (a) \sum_{b=1}^{q-1} \chi_1 (b)$$

$$\ll p^{3/2+\epsilon},$$

(35)

where the $\ll$ constant is only concerned with $q$. Combining (35) and (36), we can obtain Lemma 7. This completes the proof of Lemma 7. \( \square \)

Lemma 8. Let $q$ be a fixed odd prime and let $p$ be a prime with $p > q$, let $\chi$ denote the nonprincipal Dirichlet character modulo $\rho$, let $\chi_1, \chi_2$ denote the Dirichlet character modulo $p, q$ respectively, then, we have

$$\sum_{\chi \mod \rho} \sum_{\chi \mod q} \sum_{a=1}^{q-1} \sum_{b=1}^{q-1} \chi \chi_1 \chi_2 (a) \chi_1 \chi_2 (b) \ll p^{3/2+\epsilon},$$

(37)

where the $\ll$ constant is only concerned with $q$.

Proof. According to the properties of Dirichlet character, we can get

$$\sum_{\chi \mod \rho} \sum_{\chi \mod q} \sum_{a=1}^{q-1} \sum_{b=1}^{q-1} \chi \chi_1 \chi_2 (a) \chi_1 \chi_2 (b)$$

$$= \sum_{\chi \mod \rho} \sum_{\chi \mod q} \left( \sum_{a=1}^{q-1} \chi \chi_1 \chi_2 (a) - \sum_{a=1}^{q-1} \chi \chi_1 \chi_2 (a) \right)$$

$$\times \left( \sum_{b=1}^{q-1} \chi_1 \chi_2 (b) - \sum_{b=1}^{q-1} \chi_1 \chi_2 (b) \right)$$
\[
= \sum_{x_1 \mod p, x_2 \mod q} \left( \sum_{a \equiv p-1} \chi x_1 x_2 (a) \right) \times \sum_{a \equiv (p-1)/q} \chi x_1 x_2 (a)
\]

\[
\times \left( \sum_{b \equiv (p-1)/q} x_1 x_2 (b) - x_1 x_2 (q) \sum_{b \equiv (p-1)/q} x_1 x_2 (b) \right)
\]

\[
= \sum_{x_1 \mod p, x_2 \mod q} \sum_{x_1 \neq x_2, x_1 \neq x_2} \sum_{a \equiv p-1} \chi x_1 x_2 (a) \times x_1 x_2 (b).
\]

(38)

For primes \( p \) and \( q \), let \( \chi' = \chi x_1 \) be a nonprincipal and primitive character modulo \( p \), \( x_2 \) is also a primitive character modulo \( q \), so \( \chi x_1 x_2 \) is a primitive character modulo \( pq \); therefore, from Lemmas 3, 4, and 6 and from (38), it is clear that when \( \chi (-1) = 1 \), we can obtain

\[
\sum_{x_1 \mod p, x_2 \mod q} \sum_{x_1 \neq x_2, x_1 \neq x_2} \sum_{a \equiv p-1} \chi x_1 x_2 (a) \times x_1 x_2 (b)
\]

\[
= \sum_{x_1 \mod p, x_2 \mod q} \sum_{x_1 \neq x_2, x_1 \neq x_2} \sum_{a \equiv p-1} \chi x_1 x_2 (a) \times x_1 x_2 (b)
\]

\[
\times \left( \frac{\tau (\chi x_1 x_2)}{q} \sum_{u=1, \chi u \mod q} \chi (u) \sin \frac{2\pi u}{q} \right)
\]

\[
\times L(1, \chi x_1 x_2 x_3) + O(1)
\]

\[
\times \left( \frac{\tau (\chi x_1 x_2)}{q} \sum_{v=1, \chi v \mod q} \chi (v) \sin \frac{2\pi v}{q} \right)
\]

\[
\times L(1, \chi x_1 x_2 x_4) + O(1)
\]

\[
\times \left( \frac{\tau (\chi x_1 x_2)}{q} \sum_{u=1, \chi u \mod q} \chi (u) \left( 1 - \cos \frac{2\pi u}{q} \right) \right)
\]

\[
\times L(1, \chi x_1 x_2 x_3) + O(1)
\]

(39)

where the \( \ll \) constant is only concerned with \( q \).

When \( \chi (-1) = -1 \), in the similar way, we can also obtain

\[
\sum_{x_1 \mod p, x_2 \mod q} \sum_{x_1 \neq x_2, x_1 \neq x_2} \sum_{a \equiv p-1} \chi x_1 x_2 (a) \times x_1 x_2 (b) \ll p^{3/2+\epsilon},
\]

(40)

where the \( \ll \) constant is only concerned with \( q \).

Therefore, from (39) and (40), we can easily get Lemma 8. This completes the proof of Lemma 8. \( \square \)

3. Proof of Theorem

In this section, we will complete the proof of the theorem. According to the orthogonality relation for character sums, we have

\[
\sum_{a \equiv a \mod p} \chi(a) = \sum_{b \equiv b \mod p} \chi(a)
\]

\[
= \frac{1}{p-1} \sum_{b \equiv b \mod p} \sum_{a \equiv a \mod p} \chi x_1 a \chi_1 (b)
\]

where
\[
\frac{1}{p-1} \sum_{\substack{a=1 \bmod q \atop \chi \neq \chi_p^a}} \sum_{b=1 \bmod q} \chi(a)\chi(b) + \frac{1}{p-1} \sum_{\chi \bmod p} \sum_{a=1 \bmod q \atop \chi \neq \chi_p^a} \sum_{b=1 \bmod q} \chi(a)\chi(b) = O(1) + \frac{1}{p-1} \sum_{a \leq (p-1)/q} \sum_{b \bmod (p-1)/q} \chi(a)\chi(b)
\]

Note that for any nonprincipal Dirichlet character \(\chi\) modulo \(k\) (\(k \geq 3\) is an positive integer), we have \(\sum_{n=1}^{k} \chi(n) = 0\), hence, we obtain

\[
\sum_{a \leq p-1} \sum_{b \leq p-1} \chi_1(a)\chi_1(b) \times \sum_{a \leq p-1} \sum_{b \leq p-1} \chi_1(a)\chi_1(b) = \left( \sum_{a \leq p-1} \chi_1(a) - \sum_{a \leq p-1} \chi_1(a) \right) \times \left( \sum_{b \leq p-1} \chi_1(b) - \sum_{b \leq p-1} \chi_1(b) \right)
\]

From (41) and (42) and Lemmas 7 and 8, we get

\[
\sum_{\chi \bmod p} \chi(a) = \frac{1}{p-1} \left( 1 + \frac{1}{\phi(q)} \right)
\]
\[ \times \sum_{\chi_1 \mod p \atop \chi_1 \neq \chi_0} \chi(q) \chi_1(q) \sum_{a \leq (p-1)/q} \chi_1(a) \sum_{b \leq (p-1)/q} \chi_1(b) \]
\[ + \frac{1}{(p-1)\phi(q)} \]
\[ \times \sum_{\chi_1 \mod p \atop \chi_1 \neq \chi_0} \sum_{\chi_2 \mod q \atop \chi_2 \neq \chi_0} \sum_{a \leq p-1 \atop q\mid a} \sum_{b \leq p-1 \atop q\mid b} \chi_1X_2(a) \chi_1X_2(b) + O(1) \]
\[ \ll p^{1/2+\varepsilon}, \]
(43)

where the \( \ll \) constant only depends on \( q \). This completes the proof of Theorem.

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