(σ, τ)-AMENABILITY OF C∗-ALGEBRAS

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Abstract. Suppose that A is an algebra, σ, τ : A → A are two linear mappings such that both σ(A) and τ(A) are subalgebras of A and X is a (τ(A), σ(A))-bimodule. A linear mapping D : A → X is called a (σ, τ)-derivation if D(ab) = D(a)·σ(b) + τ(a)·D(b) (a, b ∈ A). A (σ, τ)-derivation D is called a (σ, τ)-inner derivation if there exists an x ∈ X such that D is of the form either D−x(a) = x·σ(a) − τ(a)·x (a ∈ A) or D+x(a) = x·σ(a) + τ(a)·x (a ∈ A). A Banach algebra A is called (σ, τ)-amenable if every (σ, τ)-derivation from A into a dual Banach (τ(A), σ(A))-bimodule is (σ, τ)-inner.

Studying some general algebraic aspects of (σ, τ)-derivations, we investigate the relation between amenability and (σ, τ)-amenability of Banach algebras in the case when σ, τ are homomorphisms. We prove that if A is a C∗-algebra and σ, τ are ∗-homomorphisms with ker(σ) = ker(τ), then A is (σ, τ)-amenable if and only if σ(A) is amenable.

1. Introduction and preliminaries

The notion of an amenable Banach algebra was introduced by B.E. Johnson in his definitive monograph [7]. This class of Banach algebras arises naturally out of the cohomology theory for Banach algebras, the algebraic version of which was developed by G. Hochschild [6]. For a comprehensive account on amenability the reader is referred to books [4, 13].

Suppose that A is an algebra, σ, τ : A → A are two linear mappings such that both σ(A) and τ(A) are subalgebras of A and X is a (τ(A), σ(A))-bimodule. A linear mapping D : A → X is called a (σ, τ)-derivation if

D(ab) = D(a)·σ(b) + τ(a)·D(b) (a, b ∈ A).

We say that a (σ, τ)-derivation D is a (σ, τ)-inner derivation if it is of the form either

D−x(a) = x·σ(a) − τ(a)·x (a ∈ A) or D+x(a) = x·σ(a) + τ(a)·x (a ∈ A) for some x ∈ X.

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For other approaches to generalized derivations and their applications see [1, 2, 3, 11] and references therein. In particular, the automatic continuity problem for \((\sigma, \tau)\)-derivations is considered in [8, 9, 10] and an achievement of continuity of \((\sigma, \tau)\)-derivations without linearity is given in [5].

A wide range of examples are as follows (see [9]):

(i) Every ordinary derivation is an \(id_A\)-derivation, where \(id_A\) is the identity map on the algebra \(A\).

(ii) Every endomorphism \(\alpha\) on \(A\) is an \(\alpha^2\)-derivation on \(A\).

(iii) A \(\theta\)-derivation is nothing than a point derivation \(d : A \to \mathbb{C}\) at the character \(\theta\).

A Banach algebra \(A\) is said to be \((\sigma, \tau)\)-amenable (resp. \((\sigma, \tau)\)-contractible) if every continuous \((\sigma, \tau)\)-derivation from \(A\) into a dual Banach \((\tau(A), \sigma(A))\)-bimodule \(X^*\) (a Banach \((\tau(A), \sigma(A))\)-bimodule \(X\), resp.) is \((\sigma, \tau)\)-inner. Recall that the bimodule structure of the dual space \(Y^*\) of a normed bimodule \(Y\) over a Banach algebra \(B\) is defined via

\[(b \cdot f)(y) = f(yb), \quad (f \cdot b)(y) = f(by) \quad (b \in B, y \in Y, f \in Y^*)\]

If \(\sigma = \tau\) we simply use the terminologies \(\sigma\)-derivation, \(\sigma\)-amenability, etc.

We establish some general algebraic properties of \((\sigma, \tau)\)-derivations and investigate the relation between amenability and \((\sigma, \tau)\)-amenability of Banach algebras. In particular, we prove that if \(\mathfrak{A}\) is a \(C^*\)-algebra and \(\sigma : \mathfrak{A} \to \mathfrak{A}\) is a \(\ast\)-homomorphism, then \(\mathfrak{A}\) is \(\sigma\)-amenable if and only if \(\sigma(\mathfrak{A})\) is amenable.

For definitions and elementary properties of Banach algebras we refer the reader to [4, 12].

2. Algebraic Aspects of \((\sigma, \tau)\)-derivations

Throughout this section let \(\mathcal{A}\) be a unital algebra with unit \(e\), let \(\sigma, \tau : \mathcal{A} \to \mathcal{A}\) be linear mappings such that both \(\sigma(\mathcal{A})\) and \(\tau(\mathcal{A})\) be subalgebras of \(\mathcal{A}\) and \(\mathcal{X}\) is a \((\tau(\mathcal{A}), \sigma(\mathcal{A}))\)-bimodule. In the case where \(e \in \sigma(\mathcal{A}) \cup \tau(\mathcal{A})\), we assume that \(\mathcal{X}\) is unit linked, i.e. \(ex = xe = e\) \((x \in \mathcal{X})\). In this section we first give a sufficient condition under which each \((\sigma, \tau)\)-derivation on \(\mathcal{A}\) into \(\mathcal{X}\) is of the form \(D_x^+\) for some \(x \in \mathcal{X}\). We second show how the assumption that \(\sigma\) and \(\tau\) are homomorphisms ensures \(D_x^+ = 0\) for each \(x \in \mathcal{X}\), and provide a suitable setting for study of \((\sigma, \tau)\)-derivations as well. Let us start our work with the following definition.
Definition 2.1. Let $x$ be a fixed element of $\mathcal{X}$. A linear mapping $\varphi : \mathcal{A} \to \mathcal{A}$ is called a right (left, resp.) $x$-homomorphism if $x \cdot (\varphi(ab) - \varphi(a)\varphi(b)) = 0$ ( $(\varphi(ab) - \varphi(a)\varphi(b)) \cdot x = 0$, resp.) for each $a, b \in \mathcal{A}$.

Theorem 2.2. Suppose $\sigma(e) = \lambda e$ and $\tau(e) = (1 - \lambda)e$ for some nonzero number $\lambda \neq 1$. Then for each $(\sigma, \tau)$-derivation $D : \mathcal{A} \to \mathcal{X}$ there is an $x \in \mathcal{X}$ such that

$$D(a) = x \cdot \frac{\sigma(a)}{\lambda} = \frac{\tau(a)}{1 - \lambda} \cdot x \quad (a \in \mathcal{A}).$$

In this case, $\frac{\sigma}{x}$ is a right $x$-homomorphism and $\frac{\tau}{1-x}$ is a left $x$-homomorphism.

Proof. It follows from

$$D(a) = D(a) \cdot \sigma(e) + \tau(a) \cdot D(e) = \lambda D(a) + \tau(a) \cdot D(e)$$

that

$$(1 - \lambda)D(a) = \tau(a) \cdot D(e).$$

Now if $D(e) = 0$ then $D(a) = 0$ for each $a \in \mathcal{A}$. Hence we may suppose that $x := D(e) \neq 0$. Thus $D(a) = \frac{\tau(a) \cdot x}{1 - \lambda}$. A similar argument shows that $D(a) = \frac{x \cdot \sigma(a)}{\lambda}$. Now we have

$$\frac{\tau(ab)}{1 - \lambda} \cdot x = D(ab)$$

$$= D(a) \cdot \sigma(b) + \tau(a) \cdot D(b)$$

$$= \frac{\tau(a) \cdot x}{1 - \lambda} \cdot \sigma(b) + \tau(a) \cdot \frac{\tau(b) \cdot x}{1 - \lambda}$$

$$= \frac{\lambda \tau(a) \cdot x \cdot \sigma(b)}{1 - \lambda} + \frac{\tau(a) \tau(b) \cdot x}{1 - \lambda}$$

$$= \frac{\lambda \tau(a)}{1 - \lambda} \cdot \frac{\tau(b) \cdot x}{1 - \lambda} + \frac{\tau(a) \tau(b) \cdot x}{1 - \lambda}$$

$$= \frac{\tau(a) \tau(b)}{(1 - \lambda)^2} \cdot x$$

$$= \frac{\tau(a) \cdot \tau(b)}{1 - \lambda 1 - \lambda} \cdot x.$$
there are an \( x \in X \) and two linear mappings \( \Sigma, T : A \to A \) such that \( 2\Sigma \) is a right \( x \)-homomorphism, \( 2T \) is a left \( x \)-homomorphism, \( D \) is a \((\Sigma, T)\)-derivation and

\[
D(a) = 2x \cdot \Sigma(a) = 2T(a) \cdot x = x \cdot \Sigma(a) + T(a) \cdot x \quad (a \in A).
\]

**Proof.** Put \( \Sigma = \frac{\sigma}{2\lambda} \) and \( T = \frac{\tau}{2(1-\lambda)} \).

Thus we have

**Theorem 2.4.** Let \( A \) be a unital algebra with unit \( e \). Suppose that \( \sigma, \tau : A \to A \) are linear mappings such that both \( \sigma(A) \) and \( \tau(A) \) are subalgebras of \( A \) and that \( \sigma(e) = \lambda e \) and \( \tau(e) = (1 - \lambda)e \) for some nonzero number \( \lambda \neq 1 \). Then every \((\sigma, \tau)\)-derivation is \((\sigma, \tau)\)-inner.

**Proposition 2.5.** Let \( x \) be a fixed element of \( X \). Then \( D_x^-(a) = x \cdot \sigma(a) - \tau(a) \cdot x \) is a \((\sigma, \tau)\)-derivation if and only if

\[
x \cdot (\sigma(ab) - \sigma(a)\sigma(b)) = (\tau(ab) - \tau(a)\tau(b)) \cdot x
\]

for all \( a, b \in A \).

**Proof.** \( D_x^- \) is a \((\sigma, \tau)\)-derivation if and only if

\[
x\sigma(ab) - \tau(ab)x = D_x^-(ab) = D_x^-(a)\sigma(b) + \tau(a)D_x^-(b) = (x\sigma(a) - \tau(a)x)\sigma(b) + \tau(a)(x\sigma(b) - \tau(b)x),
\]

or equivalently

\[
x \cdot (\sigma(ab) - \sigma(a)\sigma(b)) = (\tau(ab) - \tau(a)\tau(b)) \cdot x
\]

\( \square \)

Let \( X\sigma(A) := \{ x\sigma(a) : x \in X, a \in A \} = \{ 0 \} \) (resp. \( \tau(A)X = \{ 0 \} \)) and \( aX = \{ 0 \} \) (resp. \( \mathcal{X}a = \{ 0 \} \)) implies that \( a = 0 \). If all mappings \( D_x^- \) are \((\sigma, \tau)\)-derivation, then Proposition 2.5 implies that \( \sigma \) (resp. \( \tau \)) is necessarily a homomorphism. This pathological observation and the following proposition let us reasonably assume that \( \sigma \) and \( \tau \) are unital homomorphisms when we deal with \((\sigma, \tau)\)-amenability (or \((\sigma, \tau)\)-contractibility).
Proposition 2.6. Let $\sigma, \tau : A \rightarrow A$ be two unital homomorphisms. Then $D^+_x : A \rightarrow \mathcal{X}$ defined by $D^+_x(a) = x \cdot \sigma(a) + \tau(a) \cdot x$ is a $(\sigma, \tau)$-derivation if and only if $D^+_x = 0$.

Proof. We have
\[
x \cdot \sigma(a)\sigma(b) + \tau(a)\tau(b) \cdot x = x \cdot \sigma(ab) + \tau(ab) \cdot x = D^+_x(ab) = D^+_x(a) \cdot \sigma(b) + \tau(a) \cdot D^+_x(b) = (x \cdot \sigma(a) + \tau(a) \cdot x) \cdot \sigma(b) + \tau(a) \cdot (x \cdot \sigma(b) + \tau(b) \cdot x) = x \cdot \sigma(a)\sigma(b) + \tau(a) \cdot x \cdot \sigma(b) + \tau(a) \cdot x \cdot \sigma(b) + \tau(a)\tau(b) \cdot x.
\]
Thus $\tau(a) \cdot x \cdot \sigma(b) = 0$ for all $a, b \in A$. Putting $a = e$ we have $x \cdot \sigma(b) = 0$ and putting $b = e$ we have $\tau(a) \cdot x = 0$. Thus $D^+_x(a) = x \cdot \sigma(a) + \tau(a) \cdot x = 0$. \hfill \Box

3. $\sigma$-amenability of $C^*$-algebras

Throughout this section, let $\mathcal{A}$ be a Banach algebra, let $\sigma, \tau : \mathcal{A} \rightarrow \mathcal{A}$ be homomorphisms and let $\mathcal{X}$ be a Banach $(\tau(\mathcal{A}), \sigma(\mathcal{A}))$-bimodule. We also assume that all mappings under consideration are linear and continuous. In this section we study some interrelations between $(\sigma, \tau)$-amenability and ordinary amenability. Let us start our works with the following easy observation.

Proposition 3.1. If $\mathcal{A}$ is amenable (contractible), then $\mathcal{A}$ is $(\sigma, \tau)$-amenable ($(\sigma, \tau)$-contractible) for every two homomorphisms $\sigma$ and $\tau$.

Proof. We only prove the statement concerning amenability. The other case can be similarly proved.

Let $\mathcal{X}$ be a Banach $(\tau(\mathcal{A}), \sigma(\mathcal{A}))$-bimodule and $D : \mathcal{A} \rightarrow \mathcal{X}^*$ be a $(\sigma, \tau)$-derivation. One can regard $\mathcal{X}$ as a Banach $\mathcal{A}$-bimodule via
\[
a \cdot x = \tau(a)x, \quad x \cdot a = x\sigma(a) \quad (a \in \mathcal{A}, \ x \in \mathcal{X}).
\]
Then $D(ab) = D(a)\sigma(b) + \tau(a)D(b) = D(a) \cdot b + \tau(a) \cdot D(b)$. Therefore $D$ is a derivation, and so, by amenability of $\mathcal{A}$, there exists $f \in \mathcal{X}^*$ such that $D(a) = f \cdot a - a \cdot f = f\sigma(a) - \tau(a)f$. \hfill \Box
An interesting question is whether we can have a similar result but in contrary to Proposition 3.1.

**Example 3.2.** Let $\mathcal{A}$ be a non amenable Banach algebra, let $\tilde{\mathcal{A}} = \mathbb{C} \oplus \mathcal{A}$ be its unitization and $\sigma : \tilde{\mathcal{A}} \to \tilde{\mathcal{A}}$ be defined by $\sigma(z \oplus a) = z$ for all $z \in \mathbb{C}$ and $a \in \mathcal{A}$. Then $\tilde{\mathcal{A}}$ is not amenable (see [13]) but is $\sigma$-amenable. To prove the $\sigma$-amenability of $\tilde{\mathcal{A}}$ let $\tilde{\mathcal{X}}$ be a Banach space and $D : \tilde{\mathcal{A}} \to \tilde{\mathcal{X}}$ be a $\sigma$-derivation. Define $d : \mathbb{C} \to \tilde{\mathcal{X}}$ by $d(z) = D(z \oplus 0)$. Then $D$ is an ordinary derivation on $\mathbb{C}$, which is 0. Thus $D(z \oplus a) = D(z \oplus 0) + D(0 \oplus a) = d(z) + D(0 \oplus a) = D(0 \oplus a)$. Now we have

$$D(zw \oplus zb + wa + ab) = D((z \oplus a)(w \oplus b))$$
$$= D(z \oplus a)\sigma(w \oplus b) + \sigma(z \oplus a)D(w \oplus b)$$
$$= D(0 \oplus a)w + zD(0 \oplus b).$$

Putting $z = w = 0$ and $b = 1$ we get $D(0 \oplus a) = 0$. Thus $D(z \oplus a) = D(0 \oplus a) = 0$ for all $z \in \mathbb{C}$ and $a \in \mathcal{A}$.

4. **Correspondence Between $$(\sigma, \tau)$$-Derivations and Derivations on $C^*$$-Algebras**

In this section we fix a $C^*$-algebra $\mathfrak{A}$ and a Banach $(\tau(\mathfrak{A}), \sigma(\mathfrak{A}))$-bimodule $\mathcal{X}$ and give a correspondence between $(\sigma, \tau)$-derivations on $\mathfrak{A}$ and ordinary derivations on a certain algebra related to $\mathfrak{A}$. To state the result, we give some terminology.

Let $\sigma, \tau : \mathfrak{A} \to \mathfrak{A}$ be two $*$-homomorphisms. The mapping $\psi : \mathfrak{A} \to \mathfrak{A} \oplus \mathfrak{A}$ defined by $\psi(a) = (\sigma(a), \tau(a))$ is a $*$-homomorphism, where $\oplus$ denotes the $C^*$-direct sum. Hence $\mathfrak{A}_{(\sigma, \tau)} := \psi(\mathfrak{A}) = \{(\sigma(a), \tau(a)) : a \in \mathfrak{A}\}$ is a $C^*$-algebra which is isometrically isomorphic to $\mathfrak{A}/\ker(\sigma) \cap \ker(\tau)$. If $\mathcal{X}$ is a Banach $(\tau(\mathfrak{A}), \sigma(\mathfrak{A}))$-bimodule, then $\mathcal{X}$ can be regarded as an $\mathfrak{A}_{(\sigma, \tau)}$-bimodule via the operations

$$x \cdot (\sigma(a), \tau(a)) = x \cdot \sigma(a), \quad (\sigma(a), \tau(a)) \cdot x = \tau(a) \cdot x \quad (a \in \mathfrak{A}, x \in \mathcal{X}).$$

Then to each linear mapping $D : \mathfrak{A} \to \mathcal{X}$ there corresponds a linear mapping $d : \mathfrak{A}_{(\sigma, \tau)} \to \mathcal{X}$ by $d(\sigma(a), \tau(a)) = D(a)$ in such a way that $D$ is a $(\sigma, \tau)$-derivation if and only if $d$ is an ordinary derivation. Now we are in the situation to present the results of this section.
Proposition 4.1. Each $(\sigma, \tau)$-derivation $D : \mathcal{A} \rightarrow \mathcal{X}$ is $(\sigma, \tau)$-inner if and only each derivation $d : \mathcal{A}_{(\sigma, \tau)} \rightarrow \mathcal{X}$ is inner.

Proof. Let each $(\sigma, \tau)$-derivation from $\mathcal{A}$ to $\mathcal{X}$ be $(\sigma, \tau)$-inner. Suppose that $d : \mathcal{A}_{(\sigma, \tau)} \rightarrow \mathcal{X}$ is a derivation. Then as mentioned above $D(a) = d(\sigma(a), \tau(a))$ is a derivation from $\mathcal{A}$ to $\mathcal{X}$. Hence there exists an $x \in \mathcal{X}$ such that $D(a) = x \cdot \sigma(a) - \tau(a) \cdot x$ for all $a \in \mathcal{A}$. This implies that

$$d(\sigma(a), \tau(a)) = D(a) = x \cdot (\sigma(a), \tau(a)) - (\sigma(a), \tau(a)) \cdot x$$

for all $(\sigma(a), \tau(a)) \in \mathcal{A}_{(\sigma, \tau)}$. Therefore $d$ is inner.

Conversely, let each derivation from $\mathcal{A}_{(\sigma, \tau)}$ to $\mathcal{X}$ be $(\sigma, \tau)$-inner. Suppose that $D : \mathcal{A} \rightarrow \mathcal{X}$ is a $(\sigma, \tau)$-derivation. Then again as above $d(\sigma(a), \tau(a)) = D(a)$ is a derivation from $\mathcal{A}_{(\sigma, \tau)}$ to $\mathcal{X}$. So there is an $x \in \mathcal{X}$ such that $d(u) = x \cdot u - u \cdot x$ for all $u \in \mathcal{A}_{(\sigma, \tau)}$, or equivalently $D(a) = d(\sigma(a), \tau(a)) = x \cdot \sigma(a) - \tau(a) \cdot x$. Thus $D$ is $(\sigma, \tau)$-inner.

Corollary 4.2. Let $\mathcal{A}$ be a $C^*$-algebra, $\sigma : \mathcal{A} \rightarrow \mathcal{A}$ be a $*$-homomorphism and $\mathcal{X}$ be a Banach $\sigma(\mathcal{A})$-bimodule. Then each $\sigma$-derivation $D : \mathcal{A} \rightarrow \mathcal{X}$ is $\sigma$-inner if and only each derivation $d : \sigma(\mathcal{A}) \rightarrow \mathcal{X}$ is inner.

Proof. Use the fact that $\mathcal{A}_{(\sigma, \sigma)} = \mathcal{A} / \ker(\sigma) = \sigma(\mathcal{A})$.

Under some restrictions on $\sigma$ and $\tau$ we mention our last result concerning $(\sigma, \tau)$-amenability of $\mathcal{A}$.

Theorem 4.3. Let $\mathcal{A}$ be a $C^*$-algebra and let $\sigma, \tau : \mathcal{A} \rightarrow \mathcal{A}$ be two $*$-homomorphisms with $\ker(\sigma) = \ker(\tau)$. Then $\mathcal{A}$ is $(\sigma, \tau)$-amenable ($(\sigma, \tau)$-contractible, resp.) if and only if $\sigma(\mathcal{A})$ is amenable (contractible, resp.).

Proof. Since $\ker(\sigma) = \ker(\tau)$ the relations

$$x \cdot \sigma(a) = x \cdot (\sigma(a), \tau(a)),$$

$$\tau(a) \cdot x = (\sigma(a), \tau(a)) \cdot x$$

ensure us to say that $\mathcal{X}$ is a Banach $(\tau(\mathcal{A}), \sigma(\mathcal{A}))$-bimodule if and only if it is a $\mathcal{A}_{(\sigma, \tau)}$-bimodule. Thus the correspondence between $(\sigma, \tau)$-derivations on $\mathcal{A}$ and derivations on $\mathcal{A}_{(\sigma, \tau)}$ helps us to complete the proof. Note that in this case $\mathcal{A}_{(\sigma, \tau)}$ is

$$\mathcal{A} / \ker(\sigma) \cap \ker(\tau) = \mathcal{A} / \ker(\sigma) = \sigma(\mathcal{A}).$$
The following corollary includes a converse to Proposition 3.1 in the framework of $C^*$-algebras. The general case where $\mathcal{A}$ is an arbitrary Banach algebra remains open.

**Corollary 4.4.** Let $\mathfrak{A}$ be a $C^*$-algebra and let $\sigma : \mathfrak{A} \to \mathfrak{A}$ be a $*$-homomorphism. Then $\mathfrak{A}$ is $\sigma$-amenable ($\sigma$-contractible, resp.) if and only if $\sigma(\mathfrak{A})$ is amenable (contractible, resp.).

**Remark 4.5.** If $\sigma$ is not homomorphism, then the above result is not true in general. To see this, let $\mathfrak{A}$ be an amenable $C^*$-algebra and $\sigma = \frac{1}{2}id_{\mathfrak{A}}$, where $id_{\mathfrak{A}}$ is the identity mapping on $\mathfrak{A}$. Note that $\sigma(\mathfrak{A}) = \mathfrak{A}$. Let $\mathcal{X}$ be a symmetric $\mathfrak{A}$-bimodule and define $D : \mathfrak{A} \to \mathcal{X}^*$ by $D(a) = f \cdot a$, where $f$ is a fixed element of $\mathcal{X}^*$. Then $D$ is a $\sigma$-derivation, since

$$D(ab) = f \cdot ab = (f \cdot a) \cdot \frac{b}{2} + a \cdot (f \cdot b) = D(a) \cdot \sigma(b) + \sigma(a) \cdot D(b).$$

But $D$ is not inner. In contrary, suppose that $D(a) = g \cdot \sigma(a) - \sigma(a) \cdot g$ for some fixed element $g$ of $\mathcal{X}^*$. Since $\mathcal{X}$ is symmetric, we have $D = 0$ which contradicts to the definition of $D$. Thus $\mathfrak{A}$ is not $\sigma$-amenable.

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