THE FUNCTIONAL DETERMINANT AND THE PARTITION FUNCTION
IN GEOMETRIC FLOWS

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Abstract

We propose the use of the functional determinant of geometric operators in constructing an entropy functional associated to geometric flows. Our approach is based on the direct computation of the partition function, with a well-defined set of microstates and macrostates in the canonical ensemble. The approach is motivated by a fundamental enigma in Perelman’s derivation of his famous $W$-entropy. The defining feature of our entropy is that the energy of each microstate in the partition function is invariant along the associated geometric flow - a clue that could be inferred from Perelman’s work. Moreover, the monotonicity of our entropy along the associated geometric flow is then a natural result in the statistical mechanics framework. While we will not argue in a completely rigorous manner, we will use the formalism to derive an explicit formula for an entropy associated to conformal flows on a closed surface based on the Polyakov formula for the determinant of the Laplacian. We also discuss possible extensions of our results to more general operators and manifolds.

1 Introduction

Functionals exhibiting monotonicity along solutions of partial differential equations have proven to be extremely useful. In a landmark paper \cite{11} that eventually contributed to the solution of the Poincaré conjecture, G. Perelman defined a functional he called the $W$-entropy, which is given by

$$W(g, f, \tau) = \int_M \left( \tau (R + |\nabla f|^2) + f - n \right) (4\pi \tau)^{-n/2} e^{-f} \, dv$$

(1)

on a closed manifold $M$ with any Riemannian metric $g$, any $f \in C^\infty(M)$, and any $\tau > 0$. Perelman showed that along any solution to the system of differential equations

$$\begin{align*}
\frac{\partial g}{\partial t} &= -2\mathrm{Rc} \\
\frac{\partial f}{\partial t} &= -\Delta f + |\nabla f|^2 - R + \frac{n}{2\tau} \\
\frac{d\tau}{dt} &= -1,
\end{align*}$$

(2)

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which includes the Ricci flow $\frac{\partial g}{\partial t} = -2Rc$, the $\mathcal{W}$-entropy is non-decreasing. More specifically, he showed

$$\frac{d\mathcal{W}}{dt} = 2\tau \int_M \left| Rc + \nabla^2 f - \frac{1}{2\tau} g \right|^2 (4\pi\tau)^{-n/2} e^{-f} dv$$

(3)

along (2).

The Lyapunov-type property of the $\mathcal{W}$-entropy above has far-reaching consequences in geometric analysis and in completing Hamilton’s program of resolving the Poincaré conjecture. However, a less talked-about, but nevertheless tantalizing fact is that Perelman derived his $\mathcal{W}$-entropy from the formulas for the entropy in statistical mechanics. To describe his derivation, let us first recall some statistical physics below.

Recall that in statistical mechanics, the partition function for a thermodynamic system $\Gamma$ in a canonical ensemble is defined as

$$Z = \sum_i e^{-\beta E_i},$$

(4)

where the summation is over all available “microstates” $\varphi_i$ of a system with $E_i$ the energy associated to $\varphi_i$, and $\beta = 1/\tau$ with $\tau$ as the temperature. The partition function is the normalization factor in the *Gibbs probability/measure*

$$p_i = \frac{e^{-\beta E_i}}{Z},$$

(5)

which is the probability that the system occupies the microstate $\varphi_i$ at thermal equilibrium (at temperature $\tau$). In general the microstates may not be discrete, in which case we have the general formula

$$Z = \int_{\Omega} e^{-\beta E(\omega)} d\omega,$$

(6)

where $\Omega$ is the space of microstates with $E: \Omega \rightarrow \mathbb{R}$ the associated energy function, and $d\omega$ is a density of states measure on $\Omega$. Both the energy values of microstates and the density of states are independent of temperature. The (equilibrium) entropy of $\Gamma$ at temperature $\tau$ is given by

$$S = \log Z - \beta \frac{\partial}{\partial \beta} \log Z.$$  

(7)

In addition to microstates, there are also “macrostates”. As a thermodynamic system, $\Gamma$ has a full set of macrostates describing its large-scale properties. The temperature is a macrostate itself. A certain subset of macrostates are constraints that determine the energy value of each microstate, physical examples of macrostates of this type include the volume and the external magnetic field. From (7) one readily computes that

$$\frac{\partial S}{\partial \tau} = \frac{\sigma(\tau)^2}{\tau^3},$$

(8)

where $\sigma(\tau) = \sqrt{\langle (E - \langle E \rangle)^2 \rangle}$ is the standard deviation of the energy with respect to the Gibbs measure (5). Thus the entropy $S$ is non-decreasing when *only* the temperature is increased.
In [11], Perelman treated $\tau$ as the temperature and it seems that he also conceived the Riemannian metrics $g$, smooth functions $f$ on an $n$-dimensional closed manifold as the energy-determining macrostates of some underlying thermodynamic system. Then he declared

$$\log Z = \int_M (-f + \frac{N}{2}) (4\pi \tau)^{-n/2} e^{-f} \, dv$$

without specifying what the underlying microstates and their energy values are (and hence also the partition function). He proceeded to substitute (9) directly into (7), and differentiated along (2) to arrive at the $W$-entropy [11]. Therefore Perelman did not propose the partition function $Z$, but instead its natural log. This problem does not seem to generate much interest in the differential geometry circle, as the author only knows of Xiang-Dong Li (see [10]) who has addressed this issue directly. One can of course choose to overlook it as pure coincidence, but the power that the $W$-entropy has demonstrated in geometric analysis begs for a sound examination of its original motivation from statistical mechanics.

Motivated by the problem discussed above, we propose here a general scheme of defining partition functions associated to relevant geometric flows - based on the functional determinant of differential operators. This scheme has well-defined microstates as members of a functions space, with the energy of each microstate being the “Dirichlet energy” associated to the underlying operator. The data that determines the operator (such as the Riemannian metric, additional smooth functions, etc.) then become the energy-determining macrostates. Therefore, this is a scheme that follows the statistical mechanics prescription thoroughly.

In view of formula (6), a good reason for considering the functional determinant is the well-known formula

$$\int_V e^{-\langle \varphi, A\varphi \rangle} \mathcal{D}\varphi = (\det A)^{-1/2}$$

for a nonnegative-definite, self-adjoint operator $A$ on a finite $n$-dimensional inner product space $(V, \langle \cdot, \cdot \rangle)$. The measure $\mathcal{D}\varphi$ is the Euclidean measure $\pi^{-n/2} dx_1 \cdots dx_n$, where $\langle \varphi, A\varphi \rangle = \sum_i \lambda_i x_i^2$ via the eigenfunction expansion $\varphi = \sum_i x_i \varphi_i$, $A\varphi_i = \lambda_i \varphi_i$. To make mathematically rigorous sense of (10) for an operator on an infinite dimensional function space, one needs regularization techniques such as the zeta-function regularization. Physicists have long been using (10) as a formal definition of the determinant of differential operators that they needed to compute in quantum field theory. They have also explored its connection to partition functions via the zeta-function regularization (see the works [4] and [5], for example). Therefore, the idea of using the functional determinant to define partition functions is not at all new. However, its use in defining entropy functionals in geometric evolution equations does not seem to have been explored before.

The other key ingredient in our proposed scheme is the invariance of the Dirichlet energy $\langle \varphi, A\varphi \rangle$ of the differential operator $A$ along the associated geometric flow for each microstate $\varphi$. This is the absolute essential property that we require between the operator $A$ and the associated geometric flow, and it is also the driving mathematical motivation behind our approach. In general, we are seeking operators whose Dirichlet energy is invariant under a specified geometric flow, or vice-versa. This key ingredient stems from the observation that, in deriving the expression (11) for the $W$-entropy, Perelman took the total derivative of (9) with respect to $t$ (which

1his $W$-entropy is negative $S$.

2the author is not sure about the situation in the theoretical physics community

3For a mathematical background on the zeta-function regularization, see the book [3] or the paper [13].
is essentially $-\tau$) along the flow (2). Since only the partial derivative is taken in (7), we can infer that if there really were a partition function (with well-defined microstates and energies) underlying the $W$-entropy, then the system (2) should be a flow along which the energy of each microstate $\varphi$ remains unchanged. Moreover, this would mean that the monotonicity given by (3) is simply the natural result (8).

Despite the resemblance of (10) to (6), we cannot simply take (10) as the definition of our partition function. The reason is because the measure $d\omega$ in (6) must be independent of the macrostates, but (at least in the finite dimensional case) the measure $D\varphi$ in (10) does depend on the operator $A$ and hence also on the associated energy-determining macrostates. There is of course also the issue with the measure $D\varphi$ in (10) not even being mathematically well-defined, but we will see how by forgoing this issue and pretend that the measure is there, we can still derive sensible formulas. Our approach is to somehow modify the measure that appears in (10) into one that is macrostate-independent. We will do this by introducing a multiplicative factor to (10) whose exact form is motivated by the invariance of the Dirichlet energy mentioned above. The entropy is then calculated according to formula (7) by using the new partition function.

At the present, we cannot verify that the $W$-entropy indeed arises from the functional determinant approach described above. In fact, in a later section we will explain how the $W$-entropy cannot arise in exactly the way our scheme is carried out. However, we will apply our scheme to some special cases. First, we will work out the entropy formula for the Laplacian on closed surfaces, with the associated geometric flow being any conformal variation of the metric. This will not only yield a mathematically rigorous (at least within a conformal class) formula for the entropy, but it will also be explicitly computable. The highlight of the end result is that in this case the entropy will be monotonic along any conformal variation of the metric in the temperature variable, hence corroborating (8) and is consistent with the idea of the invariance of Dirichlet energy along the flow.

Secondly, we will apply the same formalism and come up with the entropy formula for the so-called “drifted Laplacian”, associated to any flow that contains a conformal deformation of the metric. The additional data here is a smooth function, which determines the Dirichlet energy of the drifted Laplacian together with the metric. The entropy formula in this case is still more-or-less rigorous, but more work is needed to make it explicitly computable - at least on a closed surface. We think that this entropy has potentially some connection to the $W$-entropy since there both a metric and a function appears as determining data.

The organization of this note is as follows. Section 2 contains work on the entropy for the Laplacian on closed surfaces - this section illustrates the methodology involved in the scheme described above, and should be considered to be the central focus of this note. In Section 3, we attempt to generalize the results in Section 2 to the drifted Laplacian operator by reiterating the techniques illustrated in Section 2. The discussions in Section 3 will actually be quite general, the drifted Laplacian is just one of the operators that we think is easier to tackle. A brief, but noteworthy mention of conformal geometry is also at the end of Section 3. In section 4 we return to our original motivation and discuss why our scheme needs to be modified in order that it even has a chance of giving rise to the $W$-entropy. Finally in section 5, we deem it necessary to give a physical interpretation of the objects appearing in our scheme. The author is not a physicist, so the reader is advised to be strongly sceptical of the content in Section 5. An appendix containing

\[4\text{besides just being independent of the temperature } \tau\]
some calculations for the determinant in finite dimensions is attached at the end, with the hope of supplying credibility to some of the more sketchy arguments used in Sections 2 and 3.

2 The case of the Laplacian on a Closed Surface

In this section, we will actually derive expressions for two related partition functions on a closed Riemann surface - one for metrics in the same conformal class, and one for all metrics on the surface. We will then compute entropy functions in the sense of (7) corresponding to each partition function. Up to a geometric constant, the partition function for all metrics generalizes that of metrics in the same conformal class. The route we take here may seem redundant, but our goal is to demonstrate that different formulas turn out to be all consistent with each other. This is particularly important due to the fact that we will be working with a measure on infinite dimensional function spaces that is not rigorously defined.

Incredibly, there are explicit mathematical results on the determinant of the Laplacian on a closed surface $M$. For the remainder of this section, $A_g$ will denote the Laplacian $-\Delta$ acting on functions. Note that $\Delta$ is defined as $\text{div} \nabla$ and thus nonpositive-definite. The following results appeared in various forms in the research literature, the versions below are extracted from the book [3].

**Theorem 1.** Suppose $g(t)$ is a smooth family of conformal metrics on a closed surface $M$, i.e. \( \frac{\partial g}{\partial t} = \psi g \) for some function $\psi(x,t)$. Then

\[
\frac{d}{dt} \log \det A_g(t) = - \int_M \psi \left( \frac{R_g(t)}{24\pi} - \frac{1}{\text{Area}(g(t))} \right) dA_g(t),
\]

where $R_g$ is the scalar (twice the Gauss) curvature of $g$.

**Corollary 1.** Suppose two metrics $g$ and $h$ on a closed surface $M$ are related by $g = e^{\psi} h$. Then

\[
\log \det A_g - \log \det A_h = - \frac{1}{48\pi} \int_M \left( |\nabla \psi|^2_h + 2\psi R_h \right) dA_h + \log \left( \frac{\text{Area}(g)}{\text{Area}(h)} \right).
\]

The formula in Corollary 1 was first derived by the Physicist A. Polyakov in [12], and such formulas have been called “Polyakov formulas”. In light of (10), we define the quantity

\[
Z_g(\beta) = \int_{\mathcal{H}} e^{-\beta E_g(\varphi)} \mathcal{D}\varphi_g = (\det \beta A_g)^{-1/2},
\]

where $\mathcal{H}$ is the domain of the Laplacian, $\beta = 1/\tau$, and $E_g(\varphi) = -\int_M \varphi \Delta \varphi \, dv = \int_M |\nabla \phi|^2 \, dv$ is the Dirichlet energy with all metric quantities dependent on the metric $g$. The notation $\beta A_g$ is for the operator consisting of applying $A_g$ first and then multiplying by $\beta$, although the order clearly does not matter. Since the space $C^\infty(M)$ of all smooth functions is dense in the domain of the Laplacian corresponding to any metric on $M$, we can assume $\mathcal{H} = C^\infty(M)$ for any metric $g$. 

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5In $\mathbb{R}^n$ with the flat metric, $\Delta$ is thus the sum of second partial derivatives in each coordinate. Note that $\nabla$ is the gradient with respect to the metric.
This last assumption means that our microstates are fixed with respect to changes in the energy-determining macrostates - a theme that will be repeated throughout.

Note that the constant scaling of the metric by $\beta$ implies $\beta A_g = A_{\beta^{-1}g}$. The first line in (11) is a heuristic definition since we do not know how to construct the measure $\mathcal{D}_g$ rigorously, but the second line is well-defined mathematically by the zeta-function regularization. We use the subscript $g$ in $Z_g(\beta)$ to denote that the measure $\mathcal{D}_g$ depends on the metric $g$, and as explained in the introduction it means that (11) should not be considered as a true partition function. The rationale for the dependence of $\mathcal{D}_g$ on the metric is that this is true for the finite dimensional case (see Appendix). We will modify the measure $\mathcal{D}_g$ so that it eventually takes a form that is metric-invariant.

We want to point out a confusion that may arise as a result of the notations used here, and it reinforces the importance of the dependence of the measure $\mathcal{D}_g$ on $g$. As operators, $\beta A_g$ and $A_{\beta^{-1}g}$ are equal, and we are computing the Dirichlet energy of $A_{\beta^{-1}g}$ with respect to the volume form $dv_g$ in (11). We are not computing the Dirichlet energy of $A_{\beta^{-1}g}$ in (11) with respect to the volume form $dv_{\beta^{-1}g}$, in which case it would be equal to the Dirichlet energy of $A_g$ with respect to the volume form $dv_g$. To put things in perspective, from definition (11) we have

$$
(\det \beta A_g)^{-1/2} = \int_{\mathcal{H}} e^{-\beta \langle \varphi, A_g \varphi \rangle_g} \mathcal{D}_g
= \int_{\mathcal{H}} e^{-\langle \varphi, A_{\beta^{-1}g} \varphi \rangle_g} \mathcal{D}_g
$$

On the other hand, we must also have

$$
(\det \beta A_g)^{-1/2} = (\det A_{\beta^{-1}g})^{-1/2}
= \int_{\mathcal{H}} e^{-\langle \varphi, A_{\beta^{-1}g} \varphi \rangle_{\beta^{-1}g}} \mathcal{D}_{\beta^{-1}g}
= \int_{\mathcal{H}} e^{-\langle \varphi, A_g \varphi \rangle_g} \mathcal{D}_{\beta^{-1}g}.
$$

For consistency’s sake, we must then have

$$
\int_{\mathcal{H}} e^{-\langle \varphi, A_{\beta^{-1}g} \varphi \rangle_g} \mathcal{D}_g = \int_{\mathcal{H}} e^{-\langle \varphi, A_g \varphi \rangle_g} \mathcal{D}_{\beta^{-1}g},
$$

which can be easily verified in the finite-dimensional case (see Appendix).

First we consider metrics in the same conformal class on a closed surface. Suppose $g(t)$ is a smooth family of metrics in the same conformal class on a closed surface $M$. The vital observation here is that the Dirichlet energy is invariant under conformal deformations. To see this, suppose $\frac{\partial g}{\partial t} = \psi g$ is a conformal deformation on $M$. For a fixed function $\varphi$, we have the elementary formula

$$
\frac{\partial}{\partial t} |\nabla \varphi|_g^2 = -\frac{\partial g}{\partial t} (\nabla \varphi, \nabla \varphi)
$$

This is a special case of the conformal invariance of the Dirichlet energy of the Laplacian on surfaces, as we will see below.

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whose proof we will skip. Using (13) and another well-known formula $\frac{\partial}{\partial t}dv = \frac{1}{2}\text{tr}_g(\frac{\partial g}{\partial t})$, we easily compute that

$$\frac{d}{dt} \int_M |\nabla \varphi|^2 dv = -\int_M \psi |\nabla \varphi|^2 dv + \int_M |\nabla \varphi|^2 \psi dv = 0.$$ 

We will take advantage of this fact in the following argument. Since $E_g$ is invariant under conformal deformations of $g$ and therefore the only object dependent on $g$ in the left-hand side of (11) is the measure $D\varphi_g$, we must have

$$\frac{\partial Z_{g(t)}(\beta)}{\partial t} = \int_{\mathcal{H}} e^{-\beta E_g(\varphi)} \frac{\partial}{\partial t} D\varphi_g. \tag{14}$$

Then by the identification (11) and using Theorem 1, we compute that

$$\frac{\partial}{\partial t} \log Z_{g(t)}(\beta) = -\frac{1}{2} \frac{\partial}{\partial t} \log \det(\beta A_{g(t)})$$

$$= -\frac{1}{2} \left( -\int_M \psi \left( \frac{R_{\tilde{g}(t)}}{24\pi} - \frac{1}{\text{Area}(\tilde{g}(t))} \right) dv_{\tilde{g}(t)} \right)$$

$$= \frac{1}{2} \int_M \psi \left( \frac{R_{\tilde{g}(t)}}{24\pi} - \frac{1}{\text{Area}(\tilde{g}(t))} \right) dv_{\tilde{g}(t)}$$

$$= -\frac{1}{2} \frac{d}{dt} \log \det A_{g(t)}$$

$$= \frac{d}{dt} \log Z_{g(t)}(1), \tag{15}$$

where we defined $\tilde{g}(t) = \beta^{-1} g(t)$ which still satisfies $\frac{\partial \tilde{g}}{\partial t} = \psi \tilde{g}$, and note that the constant factor $\beta^{-1}$ scales out of the curvature and volume so as to cancel out with the scaling of the volume form.

Let us define

$$Z(\beta) = Z_g(\beta)/Z_g(1) \tag{16}$$

and further define the new measure

$$D\varphi = \frac{D\varphi_g}{Z_g(1)} \tag{17},$$

so that we can formally write

$$Z = \int_{\mathcal{H}} e^{-\beta E_g(\varphi)} D\varphi. \tag{18}$$

Let us write $\Phi(t) = \frac{d}{dt} \log Z_{g(t)}(1)$, then it follows from (14) and (15) that

$$\int_{\mathcal{H}} e^{-\beta E_g(\varphi)} \frac{\partial}{\partial t} D\varphi_g = \Phi(t) Z_{g(t)}(\beta)$$

$$= \Phi(t) \int_{\mathcal{H}} e^{-\beta E_g(\varphi)} D\varphi_g. \tag{19}$$

\footnote{Note that this, and a more general variational formula to appear in the next section, is not computing the variation of eigenvalues of the operator. Relevant, and quite interesting work on the variation of eigenvalues along geometric flows can be found in [1], [7], and [8]. We think however, that this latter approach may figure in perhaps a different approach to establishing the $W$-entropy.}
We can go a step further and postulate from (19) that
\[ \frac{\partial}{\partial t} D\varphi_g = \Phi(t) D\varphi_g \] (20)
under any conformal deformation of \( g \). Then by differentiating (17) directly and using (20), we have
\[
\frac{\partial D\varphi}{\partial t} = \frac{\partial}{\partial t} \left( \frac{dZ_g(1)}{Z_g(1)^2} \frac{dD\varphi_g}{dt} + \frac{1}{Z_g(1)} \frac{\partial D\varphi_g}{\partial t} \right) = \frac{1}{Z_g(1)} \left( -\frac{d}{dt} \log Z_g(1) D\varphi_g + \Phi(t) D\varphi_g \right) = 0. \] (21)
Thus the measure (17) is invariant under conformal deformations of the metric as we had hoped to find, assuming that (20) holds. With this, \( Z \) as rigorously defined by (16) and formally given by (18) can now be considered as a true partition function.

Let us proceed to compute the entropy associated to the partition function defined by (16). By Corollary 1 and using the Gauss-Bonnet formula, we compute that
\[ \log Z = \left( \frac{1}{2} - \frac{\chi(M)}{12} \right) \log \beta. \] (22)
Then it follows readily that
\[ S = \left( \frac{1}{2} - \frac{\chi(M)}{12} \right) (\log \beta - 1). \] (23)
Therefore the partition function and its entropy only depend on the topology of the surface \( M \). This is a bit surprising in light of the fact that we only have the respective conformal invariance of the Dirichlet energy \( E_g \) and the measure \( D\varphi \). The correct interpretation here is that although the partition function \( Z \) (and hence the corresponding entropy \( S \)) defined above is a-priori dependent on the conformal class of the metric, as the metric moves across conformal classes the effect of the conformal variation in the Dirichlet energy \( E_g \) and that of the measure \( D\varphi \) cancel eachother.

From (23) we have
\[ \frac{\partial S}{\partial \tau} = \left( \frac{\chi(M)}{12} - \frac{1}{2} \right) \frac{1}{\tau}, \] (24)
which has a definite sign for all \( \tau > 0 \) except at the critical value of \( \chi(M) = 6 \) where it equals to zero. In particular, when \( M \) is in addition orientable we have
\[ \frac{\partial S}{\partial \tau} = -\left( \frac{2 + \text{genus}}{6} \right) \frac{1}{\tau} < 0. \] (25)
Thus for a closed Riemann surface, the entropy (23) is definitely monotonic in the temperature \( \tau = 1/\beta \). It is also important to point out that from our construction, (24) is also attained by

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8Note however, that by our identification (11) which resulted directly in (15), equation (19) is valid without assuming (20).

9Note that a measure like (17) can always be defined, but its conformal invariance is rooted in the result (18).
taking the total variation in $\tau$ along conformal variations of the metric, i.e. along a flow of the form
\[
\begin{align*}
\frac{\partial g}{\partial t} &= \psi g \\
\frac{d\tau}{dt} &= -1
\end{align*}
\] (26)
analogous to (2) for the $W$-entropy. We would like to point out that the Ricci flow is also a conformal flow in dimension 2, as it is given by $\frac{\partial g}{\partial t} = -R_g$, where $R$ is the scalar curvature of $g$.

The work above is just a warm-up to what we are really after, since the entropy (23) is for a thermodynamic system whose energy-determining macrostates (metrics in a fixed conformal class) all give the same microstate energy - quite a dull system indeed. We want to enlarge the space of energy-determining macrostates to all metrics on a closed surface. To this end, we look for a more general expression for the entropy in the form of (18) that contains a metric-independent (not just conformally-invariant) density measure $D\varphi$. The following argument generalizes the one that led to the previous entropy. Based on its validity in the finite dimensional case, we will assume that
\[
\frac{\partial}{\partial t} D\varphi_g = F D\varphi_g
\] (27)
under any variation of the metric, for a function $F$ independent of $\varphi \in \mathcal{H}$. Then we consider the general ansatz for the desired partition function as $Z = \eta(g) Z_g(\beta)$ for some function $\eta$ that depends only on the metric $g$, and where $Z_g(\beta)$ is as in (11). In terms of integration on $\mathcal{H}$, we can then write
\[
Z = \int_{\mathcal{H}} e^{-\beta E_g(\varphi)} \eta(g) D\varphi_g.
\]
Thus we have a new measure $\eta(g) D\varphi_g$, and the requirement that it be metric-independent means that
\[
\frac{d\eta}{dt} + F \eta = 0
\] (28)
for all variations $g(t)$ of the metric. To see what $F$ has to be, we compute
\[
\frac{\partial}{\partial t} \log Z_g(t)(\beta) = \left( -\beta \int_{\mathcal{H}} \frac{\partial E_g(\varphi)}{\partial t} e^{-\beta E_g(\varphi)} D\varphi_g + \int_{\mathcal{H}} e^{-\beta E_g(\varphi)} F D\varphi_g \right) / Z_g(t)(\beta)
= -\beta \left\langle \frac{\partial E_g(\varphi)}{\partial t} \right\rangle_\beta + F,
\]
where $\langle \ \rangle_\beta$ denotes the average (at temperature $\tau = \beta^{-1}$) with respect to the Gibbs measure, which can be used since the $\eta$ factors can be cancelled out. Then we see that
\[
F = \beta \left\langle \frac{\partial E_g(\varphi)}{\partial t} \right\rangle_\beta + \frac{\partial}{\partial t} \log Z_g(t)(\beta).
\] (29)
Since $F$ must also be independent of $\beta$, (29) must hold for all $\beta$. Setting $\beta = 1$ we then have
\[
F = \left\langle \frac{\partial E_g(\varphi)}{\partial t} \right\rangle_1 + \Phi(t),
\] (30)
where $\Phi(t)$ is as in (19). Note that when the variation in the metric is conformal, $\frac{\partial E_g(\varphi)}{\partial t} = 0$ and we immediately recover (20).
In order to incorporate (30) in solving for $\eta$ in (28), we need $\frac{\partial E_g(\varphi)}{\partial t}$ to be the variation of some function of metrics on the space of metrics\(^{10}\). However, it is difficult to show this in general. Instead, we seek to find another way to express $F$. To this end, we will now make an additional assumption that between any two metrics $g_0$ and $g$, we have

$$D\varphi_g = J(g_0, g)Dg_0$$

(31)

for some positive function $J$ dependent on $g_0$ and $g$ but independent of $\varphi \in \mathcal{H}$. The function $J$ is reminiscent of the Jacobian in calculus (see the Appendix for more on this). With a fixed metric $g_0$ chosen, $J(g_0, g)$ is a function of the metric $g$. Then by varying the metric, we compute that

$$\frac{\partial}{\partial t}D\varphi_{g(t)} = \frac{\partial J}{\partial t}D\varphi_{g_0} = \frac{\partial \log J}{\partial t}D\varphi_{g(t)}$$

in view of (31). Note that (31) also directly implies (27), out of which we can now conclude

$$F = \frac{\partial \log J}{\partial t}.$$  

(32)

Therefore, we have succeeded in identifying a function $J$ of the metric whose variation in the space of metrics gives us $F$. Then it is clear that $\eta(g) = J(g_0, g)^{-1}$ solves (28) for all variations $g(t)$ of the metric. Note that by (31),

$$J(g_0, g) = J(g_0, g_1)J(g_1, g)$$

(33)

for any three metrics $g_0, g_1$, and $g$. Although $J(g_0, g)$ depends on the base metric $g_0$ chosen, (33) implies that (32) is independent of the base metric - as it should be. Therefore, the more general entropy can be written as

$$Z = \frac{Z_g(\beta)}{J(g_0, g)}$$

with $g_0$ some fixed metric, which generalizes (16). In fact, by (31) we see that

$$Z_g(1) = \int_{\mathcal{H}} e^{-E_0(\varphi)} J(g_0, g)D\varphi_{g_0}$$

and it allows us to rewrite the general partition function as

$$Z = \frac{Z_g(\beta)}{Z_g(1)} \int_{\mathcal{H}} e^{-E_0(\varphi)} D\varphi_{g_0}.$$  (34)

The partition function (34) is our final desired product, which involves a measure

$$D\varphi = \int_{\mathcal{H}} e^{-E_0(\varphi)} D\varphi_{g_0}D\varphi_g$$

that is invariant under the choice of macrostate $g$. We speculate that a perturbative technique might be available to calculate this general partition function, since one clearly sees from (31) the need to expand $E_g(\varphi)$ about $g_0$.

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\(^{10}\)This can be seen as an integrability requirement on the space of metrics.
On a closed surface with metric $g$, by taking the partial derivative with respect to $\beta$ and using (23) the corresponding entropy to (34) is then

$$S = \left(\frac{1}{2} - \frac{\chi(M)}{12}\right)(\log \beta - 1) + \log \int_{\mathcal{H}} e^{-E_g(\phi)} \mathcal{D}_\mathcal{F} \phi_{g_0},$$

(35)

and we see the presence of the extra geometric term. Note that via the conformal invariance of the Dirichlet energy $E_g(\phi)$, the extra geometric term must be constant along conformal variations of the metric. Thus we also arrive at the same formula (35) by taking the total derivative with respect to $\beta$ along the flow (26).

Furthermore, by taking the partial derivative with respect to $\tau$ we still have (24) and (25), which also hold by taking the total derivative with respect to $\tau$ along the flow (26). In particular, for a closed Riemann surface the entropy (35) is still monotonic along (26). The entropy (35) again reflects the key property that certain geometric flows (any conformal flow in this case) preserve the energy of each microstate of the underlying system. We also want to point out that even if the entropy (35) cannot be made mathematically rigorous due to the presence of the extra geometric term, its variation in temperature (24) is completely well-defined.

A few words about the computability of (35). Note that at the same temperature, the entropy (35) is constant for metrics $g$ in the same conformal class, with any base metric $g_0$. Moreover, if $g$ is in the same conformal class as $g_0$, then the entropy becomes

$$S = \left(\frac{1}{2} - \frac{\chi(M)}{12}\right)(\log \beta - 1) - \frac{1}{2} \log \det A_{g_0},$$

(36)

by the invariance of the Dirichlet energy. Although we cannot compute (36) explicitly, if we fix another metric $g_1 = e^{\psi} g_0$ for some function $\psi$, then by Corollary 1 we see that

$$S_{g_1} - S_{g_0} = \frac{1}{96\pi} \int_M \left(\frac{1}{2} \left| \nabla\psi \right|^2 + 2\psi R_{g_0}\right) dA_{g_0} - \frac{1}{2} \log \frac{\text{Area}(g_1)}{\text{Area}(g_0)}.$$

One last remark is in order before we end this section. We want to stress again that the monotonicity of our entropy (35) exhibited by (24) should be seen as a testament to the validity of our scheme, in spite of the lack of mathematical rigor in our arguments. It is also worth pointing out that if we had not modified $Z_g(\beta)$ by a multiplicative factor and just took the functional determinant in (11) as the definition of our partition function, then the resulting entropy $S$ according to (7) will not be monotonic along the flow (26). An obvious issue here is that the signs do not seem to match up, as the sign in (8) is always nonnegative while typically the sign in (24) is nonpositive. This issue is fundamentally rooted in the fact that if we scale a fixed metric $g$ by $\tilde{g}(\tau) = \tau g$, then

$$\frac{d}{d\tau} \log \det A = -\frac{1}{\tau} \left(\frac{\chi(M)}{6} - 1\right),$$

where $A = -\Delta \tilde{g}(\tau)$ and again we see the critical value of $\chi(M) = 6$. In particular, $\frac{d}{d\tau} \log \det A = (g+2)/3\tau > 0$ when $M$ is a closed Riemann surface. This is counter-intuitive, as scaling a metric

\footnotesize
\begin{itemize}
  \item \textit{Note that this is a relative geometric term that depends on a fixed metric $g_0$, hence (34) should be seen as a relative entropy.}
  \item for example, closed Riemann surfaces
  \item It is a simple exercise to check this using Theorem 1.
\end{itemize}

larger will lower the eigenvalues of the Laplacian and hence should also decrease the value of the determinant, since it is the product of eigenvalues. Somehow the zeta-function regularization has destroyed this intuition, but nevertheless the monotonicity of the entropy is still preserved.

3 Other Operators

The more ambitious task is to follow the same scheme as in the last section for obtaining the expression for entropy on higher dimensional manifolds with possibly different operators. More precisely, similar to (18) we want a partition function in the form

$$Z = \int_{\mathcal{H}} e^{-\beta E(\varphi)} \mathcal{D}\varphi,$$

where $E(\varphi) = \langle \varphi, A\varphi \rangle$ is the Dirichlet energy and $\mathcal{D}\varphi$ is a fixed measure on the domain $\mathcal{H}$ of a more general operator $A$. A natural operator to consider would be the drifted Laplacian $\Delta_{f} = \Delta - \langle \nabla f, \nabla \rangle$ on a Riemannian manifold $M$ with metric $g = \langle \cdot, \cdot \rangle$ and “potential function” $f \in C^\infty(M)$.

As in the last section, we wish to first identify invariance properties of the “Dirichlet energy” for such operators. The following generalizes the conformal invariance of the Dirichlet energy of the Laplacian on a closed surface. Let $M$ be a closed manifold equipped with a Riemannian metric $g$, and two functions $f, V \in C^\infty(M)$. We consider in general the drifted Schrödinger operator $A = -\Delta + \langle \nabla f, \nabla \rangle + V$. Suppose we have a system of evolution equations

$$\begin{align*}
\frac{\partial g}{\partial t} &= h(g, f, V) \\
\frac{\partial f}{\partial t} &= F_1(g, f, V) \\
\frac{\partial V}{\partial t} &= F_2(g, f, V)
\end{align*}$$

(37)

Then we compute

$$\frac{d}{dt} \int_M \phi A\phi \, du = \frac{d}{dt} \int_M (|\nabla \phi|^2 + V\phi^2) \, du$$

$$= \int_M (-h(\nabla \phi, \nabla \phi) + F_2 \phi^2) \, du + \int_M (|\nabla \phi|^2 + V\phi^2) \left( \frac{1}{2} \text{tr}_g(h) - F_1 \right) \, du. \quad (38)$$

Consider a conformal deformation of the metric $g$, i.e. $h = \psi g$ for some function $\psi(x, t)$. Then $\text{tr}_g(h) = n\psi$, where $n$ is the dimension of $M$. In view of (38), we see that by letting $F_1 = \left( \frac{n}{2} - 1 \right) \psi$ and $F_2 = -\psi V$ we can make the derivative of the Dirichlet energy vanish. Thus we have the following result.

**Proposition 1.** On a compact manifold $M$ of dimension $n$, if $(g, f, V)$ satisfies a system of evolution equations of the form

$$\begin{align*}
\frac{\partial g}{\partial t} &= \psi g \\
\frac{\partial f}{\partial t} &= \left( \frac{n}{2} - 1 \right) \psi \\
\frac{\partial V}{\partial t} &= -\psi V
\end{align*}$$

(39)

for some function $\psi(x, t)$, then the Dirichlet energy of the associated drifted Schrödinger operator $A$ is invariant under the system.
We would like to point out that when \( n \geq 3 \), \( \frac{\partial g}{\partial t} = -Rg \) is the Yamabe flow.\[^{14}\]

For simplicity we will just investigate the case of the drifted Laplacian operator \( A = -\Delta_f \). Let us denote the data \((g, f)\) by \( \theta \), so that we can more precisely denote \( A = A_\theta \). Moreover, we will only discuss the analogous formula to \((33)\), since what we really want is an invariant measure \( D\varphi \) within some class of \( \theta \), with \( E_\theta(\varphi) \) nonconstant within the class.

Again we assume that \( D\varphi_\theta = J(\theta_0, \theta) D\varphi_{\theta_0} \) for some function \( J(\theta_0, \theta) \) independent of \( \varphi \), for any \( \theta = (g, f) \) and a fixed \( \theta_0 = (g_0, f_0) \). Then by the same argument as in the last section, we readily arrive at the general partition function

\[
Z = \frac{Z_\theta(\beta)}{Z_\theta(1)} \int_{\mathcal{H}} e^{-E_\theta(\varphi)} D\varphi_{\theta_0} \tag{40}
\]

with a fixed macrostate \( \theta_0 \), and where \( Z_\theta(\beta) \) is defined by the functional determinant of \( \beta A_\theta \) just like in \((11)\).\[^{15}\] The partition function \((40)\) holds for \( \theta = (g, f) \) with any metric \( g \) and any smooth function \( f \) on \( M \). Therefore just as in the last section, the underlying (though not rigorous) measure in the canonical ensemble is

\[
D\varphi = \frac{\int_{\mathcal{H}} e^{-E_\theta(\varphi)} D\varphi_{\theta_0}}{Z_\theta(1)} D\varphi, \tag{41}
\]

which is independent of \( \theta = (g, f) \) with any metric \( g \) and any smooth function \( f \) on \( M \).

By Proposition \[^{11}\] we see that any variation in \( \theta \) consisting of the first two equations of system \((39)\) leaves the energy \( E_\theta(\varphi) \) of each microstate \( \varphi \) unchanged. Therefore we can consider the flow

\[
\begin{align*}
\frac{\partial g}{\partial t} &= \psi g \\
\frac{\partial f}{\partial t} &= (\frac{n}{2} - 1) \psi \\
\frac{dx}{dt} &= -1,
\end{align*} \tag{41}
\]

which is analogous to \((2)\). Then taking the total variation of \((40)\) in the variable \( \beta \) along \((41)\), the entropy according to \((7)\) becomes

\[
S = -\frac{1}{2} \left( \log \frac{\det \beta A_\theta}{\det A_\theta} - \beta \frac{d}{d\beta} \left( \log \frac{\det \beta A_\theta}{\det A_\theta} \right) \right) + \log \int_{\mathcal{H}} e^{-E_\theta(\varphi)} D\varphi_{\theta_0}. \tag{42}
\]

The entropy formula \((42)\) holds for the drift Laplacian \( A_\theta \) on any closed \( n \)-dimensional manifold. At the present we cannot compute \((42)\) further, because to our knowledge the Polyakov formula \( \log \frac{\det \beta A_\theta}{\det A_\theta} \) for the drifted Laplacian is not known - even on closed surfaces. However, we believe that such a Polyakov formula must exist for closed surfaces just as in the case of the Laplacian. If this Polyakov formula is computed, then we can also use it to prove that the entropy \((42)\) is monotonic along the flow \((41)\).\[^{16}\]

\[^{14}\] see \[^{3}\] for a background

\[^{15}\] The zeta-function regularized determinant actually works for any self-adjoint elliptic operator on a closed manifold of any dimension. The drifted Laplacian is such an operator.

\[^{16}\] Note that by the invariance of the energy \( E_\theta(\varphi) \), the last term in \((42)\) also vanishes along \((41)\). Therefore, the variation of the entropy again depends only on the Polyakov formula of the operator in question, just as in the previous case of the Laplacian.
Let us elaborate on the conjecture mentioned above for the drifted Laplacian on closed surfaces. It is elementary to check that

$$\beta A_\theta = -\beta \Delta g + \beta \langle \nabla g f, \nabla \rangle_g = -\Delta \beta^{-1} g + \langle \nabla \beta^{-1} g f, \nabla \beta^{-1} g \rangle_{\beta^{-1} g} = A(\beta^{-1} g, f)$$

(43)

since the gradient scales as $\nabla c g = c^{-1} \nabla g$ for any $c > 0$. Then taking the partial derivative(s) with respect to $\beta$ of $\log \det \beta A_\theta$ amounts to varying the metric $\beta^{-1} g$ conformally in $\beta$ with $g$ fixed (and $f$ fixed as well) just as in the case of the ordinary Laplacian. Thus we expect to get formulas similar to those in Theorem 1 and Corollary 1, and since the fixed potential function $f$ is outstanding throughout, we expect $f$ to also appear explicitly in the final formulas for $S$ and $\frac{\partial S}{\partial \tau}$. This suggests that if explicitly worked out, $\log Z_\theta = (-1/2) \log \det A_\theta$ could resemble Perelman’s formula.

On the other hand, instead of considering all metrics on a closed surface, it also makes sense to consider the set of energy-determining macrostates as $[g] \times C^\infty(M)$, with $[g]$ a fixed conformal class. This means that the Dirichlet energy $E_\theta(\varphi)$ will only vary nontrivially with the potential function $f$. In fact, by deforming the metric inside $[g]$ and simultaneously deforming the potential function $f$ arbitrarily, (38) implies

$$\frac{\partial E_\theta(t)}{\partial t}(\varphi) = -\int_M \frac{\partial f}{\partial t}|\nabla \varphi|^2 du.$$  

(44)

Therefore we think that this is the setting in which the entropy (12) is most amenable to computation, since (44) is a manageable first-order approximation to the Dirichlet energy. In any case, the precise form of formula (12) will hinge on the Polyakov formulas for the drifted Laplacian.

The success we have had with conformal deformations also points to another general direction. There is a Polyakov formula for any second-order, conformally covariant operator in dimension 4. The formula was discovered by Branson and Ørsted, and like the Polyakov formula in Corollary 1 it is an integral expression containing, now, new curvature terms. We refer to the lecture notes of A. Chang for an extensive coverage on this subject. It will be a good idea to apply our scheme to such conformally covariant operators and derive the corresponding entropy functionals associated to conformal deformations of the metric. This potentially fruitful direction may shed some new light on the field of conformal geometry by linking it to statistical mechanics.

4 Further Speculations on the $\mathcal{W}$-entropy

The $\mathcal{W}$-entropy has a salient feature when (3) vanishes, which occurs along a smooth one-parameter family of $(g, f, \tau)$ satisfying the equation

$$\text{Rc} + \nabla^2 f - \frac{1}{2\tau} g = 0.$$  

(45)

17but not equal to. See next section for more details.

18The short, but very informative review [9] by R. Mazzeo is a helpful start.
A triple \((g, f, \tau)\) satisfying (45) is called a shrinking Ricci soliton, the associated family \(g(t)\) of metrics along (2) is actually a self-similar solution\(^\text{19}\) of the Ricci flow.

We would now like to argue that the \(\mathcal{W}\)-entropy cannot arise from the scheme described in this note - at least not by following the scheme exactly. According to (8), the vanishing of \(\frac{dS}{d\tau}\) happens if and only if all microstates share the same energy value. On the other hand, if the Dirichlet energy \(\langle \varphi, A_\theta \varphi \rangle\) is constant for all \(\varphi\), then the constant must be zero. For a self-adjoint operator \(A_\theta\) this is impossible, since we immediately run into trouble for an eigenvector \(\varphi\) of \(A_\theta\). Therefore, not only is it impossible to produce the \(\mathcal{W}\)-entropy using the functional determinant of self-adjoint operators as we have done, any entropy produced from our scheme can also never contain "soliton-like" objects, i.e. solutions \(\theta(\tau)\) to the associated flow involved such that \(\frac{dS}{d\tau} = 0\) along \(\theta(\tau)\). The formula (24) corroborates this claim except for \(\chi(M) = 6\), although we do not know how one can construct a closed, connected surface with Euler Characteritic 6.

In our opinion, the scheme described in this paper seem to be the most direct approach to constructing a partition function (and the ensuing entropy) associated to a geometric flow. Therefore it is desirable that some modification of our scheme still works in constructing the partition function for the \(\mathcal{W}\)-entropy. One possible modification is to consider functional determinants of operators that are not self-adjoint, but we do not know of any regularization schemes for such operators\(^\text{20}\). Even if such an operator \(A\) exists, there is still a problem of ensuring that it be invariant under the flow (2). In view of the cases involving conformal deformations that we have dealt with, it is also unclear to us whether or not to consider (2) as part of a more general family of flows\(^\text{21}\). Fitting the \(\mathcal{W}\)-entropy into our scheme seems to be a difficult problem.

One last remark before we leave this section. In statistical mechanics, there is also the concept of the free energy, defined by

\[
F = -\tau \log Z
\]

with \(\tau\) the temperature and \(Z\) the partition function as before. Clearly, the entropy (17) can be expressed using the free energy as

\[
S = -\frac{\partial F}{\partial \tau}.
\]

Therefore, Perelman’s declaration of (19) can be seen as a declaration of the free energy (up to a factor of \(-\tau\)). A straight-forward exercise also confirms that the \(\mathcal{W}\)-entropy also follows from (46) by differentiating along the flow (2)\(^\text{22}\).

### 5 Physical Interpretation

So far we have not explicitly discussed the thermodynamic system underlying our scheme. Since the formulas we have conjured up are quite consistent with the statistical mechanics (in

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\(^{19}\)that is, a family of metrics moving by diffeomorphisms

\(^{20}\)Note that a finite-dimensional example would be rotation by 90 degrees about a fixed axis, whose Dirichlet energy is always zero since the rotated vector is orthogonal to the original vector. Thus macrostates that determine unitary operators could be "solitons" associated to some geometric flow, but due to their lack of real eigenvalues something like the zeta-function regularization seems elusive.

\(^{21}\)From all the work in the literature it seems that (2), at least for a general manifold dimension \(n\), is quite special in its own right.

\(^{22}\)Some mathematicians have already observed this mathematical fact, see for example [6].
particular, the fact that (24) does not change sign), a more detailed physical interpretation should be in order. Recall the general formula

\[ Z = \int_{\mathcal{H}} e^{-\beta E(\varphi)} \mathcal{D}\varphi \]

of the partition function which we have employed throughout. In the most general sense, the microstates \( \varphi \) should be interpreted as scalar fields and the Dirichlet energy \( E(\varphi) \) can then be interpreted as the corresponding field energy. \( \mathcal{H} \) is then just the field configuration space. This is the most abstract interpretation, as it covers all potential examples of operators that one can apply our scheme to. On the other hand, the energy-determining macrostates are external (or background) constraints. These energy-determining macrostates should not be thought of as being analogous to the volume of a system of gas particles, as there the volume affects both the nature and the number of microstates (in the quantum and classical description, respectively). Since operators in our scheme for different energy-determining macrostates share a dense intersection of their domains, the microstates can be fixed as this dense intersection. Therefore, the best analogy for the energy-determining macrostates in our scheme would be to the external magnetic field applied to a system of a fixed number of particles with spin. The spin states of the particles do not change with the strength of the magnetic field applied, but the ensuing energy does.

In the special case of the Laplacian on closed surfaces, we speculate that a more concrete physical interpretation may be available. In this case, the microstates \( \varphi \) can be thought of as the vibrational profile of a membrane on the surface. In particular, the value of \( |\varphi| \) at a point on the surface represents the transverse displacement of the membrane from the surface. The transverse displacement of the membrane should correspond to some kind of “stretching” of the membrane and the quantity \( |\nabla \varphi|^2 \) should be a measure of the magnitude of such a stretching - as seen by the Riemannian metric on the surface. Then the Dirichlet energy can be seen as the total amount of energy stored as a result of such a stretching. This however, only takes care of the potential energy. The correct physical interpretation should also include a kinetic part of the energy, which is not included in the Dirichlet energy.

6 Appendix

By the finite-dimensional version of (12) we mean pretending that the function space \( \mathcal{H} \) is just \( \mathbb{R}^n \) for some finite \( n \). Suppose \( A_g \psi = \lambda \psi \), then we must have \( A_{\beta^{-1}g} \psi = \beta \lambda \psi \). It follows immediately that a normalized eigenvector for the operator \( A_{\beta^{-1}g} \) in the metric \( \beta^{-1}g \) is \( \tilde{\psi}_i = \sqrt{\beta} \psi \), where \( \psi \) is a normalized eigenvector for \( A_g \) in the metric \( g \). Then for a given vector \( \varphi \), we have the expansion

\[ \varphi = \tilde{c}_i \tilde{\psi}_i = \tilde{c}_i \sqrt{\beta} \psi_i = c_i \psi_i, \]

which implies that \( \tilde{c}_i = \beta^{-1/2} c_i \).

Expanding \( \varphi \) in \( \psi_i \), we compute that \( \langle \varphi, A_{\beta^{-1}g} \varphi \rangle = \beta \lambda \tilde{c}_i^2 \) and the left-hand side of (12)

\[ \text{Note that this is different from the stretching that one considers for a general harmonic map, where the stretching of the membrane is along the intrinsic, tangent directions of the manifold.} \]
becomes
\[
\int_{\mathcal{H}} e^{-(\varphi, A_{\beta-1} \varphi)}_g \mathcal{D}\varphi_g = \int_{\mathbb{R}^n} e^{-\beta \lambda_i c_i^2} \pi^{-n/2} dc_1 \cdots dc_n \\
= \pi^{-n/2} \int_{\mathbb{R}} e^{-\beta \lambda_i c_i^2} dc_1 \cdots \int_{\mathbb{R}} e^{-\beta \lambda_n c_n^2} dc_n \\
= \pi^{-n/2} \left( \frac{\pi}{\beta \lambda_1} \cdots \frac{\pi}{\beta \lambda_n} \right) = \frac{\beta^{-n/2}}{\sqrt{\lambda_1 \cdots \lambda_n}}.
\]

If we expand \( \varphi \) in \( \tilde{\psi}_i \) and use the formula \( \mathcal{D}\varphi_{\beta-1} g = \pi^{-n/2} \tilde{d}_i \cdots \tilde{d}_n \) with \( \tilde{d}_i = \beta^{-1/2} d_i \), then the right-hand side of (12) becomes
\[
\int_{\mathcal{H}} e^{-(\varphi, A_{\varphi} \varphi)}_g \mathcal{D}\varphi_{\beta-1} g = \int_{\mathbb{R}^n} e^{-\lambda_i c_i^2} \pi^{-n/2} \beta^{-n/2} dc_1 \cdots dc_n \\
= \pi^{-n/2} \beta^{-n/2} \int_{\mathbb{R}} e^{-\lambda_i c_i^2} dc_1 \cdots \int_{\mathbb{R}} e^{-\lambda_n c_n^2} dc_n \\
= \pi^{-n/2} \beta^{-n/2} \left( \frac{\pi}{\lambda_1} \cdots \frac{\pi}{\lambda_n} \right) = \frac{\beta^{-n/2}}{\sqrt{\lambda_1 \cdots \lambda_n}}.
\]

Thus equation (12) holds, at least in finite dimensions. In the calculations above, we have used the well-known integral
\[
\int_{-\infty}^{\infty} e^{-ax^2} dx = \sqrt{\frac{\pi}{a}}.
\]

Note that the coordinates \( (c_1, \ldots, c_n) \) coming from eigenvectors of \( A_g \) are global on the Euclidean space \( \mathbb{R}^n \) (viewed as a smooth manifold), and therefore \( \pi^{-n/2} dc_1 \cdots dc_n \) are different measures on \( \mathbb{R}^n \) for different choices of \( g \). This explains why we should a-priori assume that the infinite-dimensional version of the measure is also dependent on the energy-determining macrostates.

Next we examine the finite-dimensional validity of \( J(g_0, g) \) appearing in formula (31). The argument below generalizes the one above for conformally scaling a metric by a constant. As we did above, we are taking \( \mathcal{H} = \mathbb{R}^n \). Let \( g \) and \( g_0 \) be two different metrics, and for a given \( \varphi \in \mathcal{H} \) we can expand it as \( \varphi = c_i \tilde{\psi}_i \) with \( \tilde{\psi} \) the normalized eigenvectors according to \( g_0 \), or expand it as \( \varphi = \tilde{c}_i \tilde{\psi}_i \) with \( \tilde{\psi} \) the normalized eigenvectors according to \( g \). Then we can write \( \tilde{\psi}_i = a_{ij} \psi_j \) with the matrix \( (a_{ij}) \) with \( (a^{ij}) \) its inverse, which implies \( \tilde{c}_i = a^{ij} c_j \) and hence \( \tilde{d}_i = a^{ij} d_j \).

From this we see that
\[
\mathcal{D}\varphi_g = \pi^{-n/2} \tilde{d}_i \cdots \tilde{d}_n \\
= \pi^{-n/2} \det (a^{ij}) dc_1 \cdots dc_n = \det (a^{ij}) \mathcal{D}\varphi_{g_0}.
\]

Therefore (31) is valid in the finite-dimensional case, with \( J(g_0, g) = \det (a^{ij}) \). Going from \( \{ \tilde{\psi}_i \} \) to \( \{ \psi_i \} \) is a linear coordinate-change on \( \mathcal{H} \), and thus \( \det (a^{ij}) \) can be interpreted as the Jacobian of the coordinate-change. In particular, \( \det (a^{ij}) \) is a function that depends only on \( g_0 \) and \( g \) (and not on \( \varphi \)).

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