CHIRAL DIFFERENTIAL OPERATORS:
FORMAL LOOP GROUP ACTIONS & ASSOCIATED MODULES

POKMAN CHEUNG

Abstract

Chiral differential operators (CDOs) are closely related to string geometry and the quantum theory of 2-dimensional \(\sigma\)-models. This paper investigates two topics about CDOs on smooth manifolds.

In the first half, we study how a Lie group action on a smooth manifold can be lifted to a “formal loop group action” on an algebra of CDOs; this turns out to be a condition on the equivariant first Pontrjagin class. The case of a principal bundle receives particular attention and gives rise to a type of vertex algebras of great interest. In the second half, we introduce a construction of modules over CDOs using the said “formal loop group actions” and semi-infinite cohomology. Intuitively, these modules should have a geometric meaning in terms of “formal loop spaces”. The first example we study leads to a new conceptual construction of an arbitrary algebra of CDOs. The other example, called the spinor module, may be useful for a geometric theory of the Witten genus.

§1. Introduction

The algebra of chiral differential operators (CDOs) on an affine space \(\mathbb{A}^d\) is an elementary conformal vertex algebra, and it has a holomorphic and a smooth version as well. The global construction of CDOs, first given by Gorbounov, Malikov and Schechtman, is closely related to string geometry and the quantum theory of \(\sigma\)-models. In particular, the obstructions for a complex manifold \(M\) to admit a sheaf of holomorphic CDOs \(D^h_M\) are certain refinements of \(c_1(M)\) and \(c_2(M)\); and if \(M\) is compact, then the genus-1 partition function of the conformal vertex superalgebra \(H^*(M; D^h_M)\) is up to a factor the Witten genus of \(M\). [GMS00] In the algebraic and holomorphic settings, Kapranov and Vasserot have given a geometric interpretation of CDOs using the notion of formal loop spaces they introduced [KV04, KV06]; see also [BD04]. However, in the smooth setting, such an interpretation seems to be still missing. From the point of view of field theories, holomorphic CDOs are known to physicists as a description of the large-volume limit of the half-twisted \(\sigma\)-models [Kap06, Wit07, Tan06], and they are also closely related to Costello’s holomorphic Chern-Simons theory. [Cos10, Cos11]

This paper concerns CDOs on smooth manifolds. In the first half, we study how a Lie group action on a manifold can be lifted to a “formal loop group action” on an algebra of CDOs. It turns out that the existence of such actions is a condition on the equivariant first Pontrjagin class, and they are parametrised by certain Cartan cochains. The case of a principal bundle receives particular attention and gives rise to a type of vertex algebras of great interest. The study of “formal loop group actions” on CDOs is not entirely new. For example, it is well-known that the left and right multiplications of a simple Lie group on itself can be lifted to a commuting pair of affine Lie algebra actions on any algebra of CDOs. [GMS01] Also, the equivariant chiral de Rham complex of Lian and Linshaw may be viewed as a particular differential graded version of our construction. [LL07]

In the second half of the paper, we introduce a construction of modules over CDOs using the above “formal loop group actions” and semi-infinite cohomology. This construction is quite analogous to that of associated vector bundles with connection, and includes as a special case a construction of Wakimoto modules. [FF90, Vor99] The first example we study leads to a new conceptual description of an arbitrary algebra of CDOs. For some homogeneous spaces, this description of CDOs has been given before. [GMS01] Another example of our construction, called the spinor module, may be useful for understanding the geometric meaning of the Witten genus for smooth string manifolds.

The following is a more detailed overview of the paper.

§1.1. Overview of the paper. [2] recalls the definition of an algebra of CDOs on a smooth manifold (Definition 2.3), as well as its construction and classification (Theorems 2.5 & 2.6). The explicit generators-and-relations description should be compared to the more conceptual one in [6].
Let $G$ be a compact connected Lie group, $\mathfrak{g}$ its Lie algebra, and $\lambda$ an invariant symmetric bilinear form on $\mathfrak{g}$. Recall that $\lambda$ determines a centrally extended loop algebra $\hat{\mathfrak{g}}_{\lambda}$ (Example 4.2) and also a vertex algebra $V_{\lambda}(\mathfrak{g})$ (Example 4.10). If $\mathfrak{g}$ is simple and $\lambda$ is $k$ times the normalized Killing form, then we will write the vertex algebra more traditionally as $V_k(\mathfrak{g})$. Let us introduce some terminology: a \textit{formal loop group action} of level $\lambda$ on a vector space $W$ is a $\hat{\mathfrak{g}}_{\lambda}$-action on $W$ whose restriction to $\mathfrak{g} \subset \hat{\mathfrak{g}}_{\lambda}$ integrates into a $G$-action; and if $W$ is a vertex algebra then we also ask $\mathfrak{g}$ to act by inner derivations. (In the main text this is called an \textit{inner ($\hat{\mathfrak{g}}_{\lambda}, G$)-action}; see Definition 3.4) Notice that in the latter case the action is induced by a map of vertex algebras $V_{\lambda}(\mathfrak{g}) \to W$.

Suppose $P$ is a smooth manifold with a smooth $G$-action and $\mathcal{D}^{ch}(P)$ is an algebra of CDOs on $P$. Conjecturally, there should be an interpretation of $\mathcal{D}^{ch}(P)$ in terms of the “formal loop space of $P$”. This motivates the main result in §4.4.

**Theorem 3.11.** The $G$-action on $P$ lifts to a formal loop group action on $\mathcal{D}^{ch}(P)$ of level $\lambda$ if and only if

$$8\pi^2 p_1(P)_G = \lambda(P)$$

where $p_1(P)_G$ is the equivariant first Pontrjagin class and $\lambda(P)$ is the image of $\lambda$ under the characteristic map $H^2(BG) \to H^2_G(M)$. Moreover, $\mathcal{D}^{ch}(P)$ has a conformal vector whose induced Virasoro algebra action intertwines with any formal loop group action as described.

The key ideas behind this result are: the use of the Cartan model for equivariant de Rham cohomology (recalled in §3.2), and the observation that the $G$-action on $P$ and the vertex algebra structure of $\mathcal{D}^{ch}(P)$ together determine a Cartan cocycle for $p_1(P)_G$ (Lemma 3.10). In fact, there is a more refined statement describing a bijection between formal loop group actions and certain Cartan cochains.

In the example where $G$ is simple and $P = G$ with the $G \times G$-action by left and right multiplications, our construction recovers a well-known result. Namely, for any $k \in \mathbb{C}$, there is an algebra of CDOs $\mathcal{D}^{ch}_k(G)$ together with an embedding of vertex algebras $V_{-k-h^\vee}(\mathfrak{g}) \otimes V_{k-h^\vee}(\mathfrak{g}) \hookrightarrow \mathcal{D}^{ch}_k(G)$, where $h^\vee$ is the dual Coxeter number (Example 3.12 & Proposition 3.13). Moreove r, for $k \notin \mathbb{Q}$, we give a new proof of a “chiral Peter-Weyl Theorem” concerning the structure of $\mathcal{D}^{ch}_k(G)$ as a module over $V_{-k-h^\vee}(\mathfrak{g}) \otimes V_{k-h^\vee}(\mathfrak{g})$ (Proposition 3.15).

From now on, $P$ is the total space of a principal $G$-bundle $\pi : P \to M$ and $\mathcal{D}^{ch}(P)$ is an algebra of CDOs equipped with a formal loop group action $V_{\lambda}(\mathfrak{g}) \to \mathcal{D}^{ch}(P)$. §4 is a more detailed study of the structure of $\mathcal{D}^{ch}(P)$. First we consider the centralizer subalgebra

$$\mathcal{D}^{ch}(P)^{\hat{\mathfrak{g}}} = C(\mathcal{D}^{ch}(P), V_{\lambda}(\mathfrak{g}))$$

i.e. the subalgebra invariant under the formal loop group action. This is different from an algebra of CDOs on $P/G = M$. In particular, part of the structure of $\mathcal{D}^{ch}(P)^{\hat{\mathfrak{g}}}$ is the Lie algebroid $(C^\infty(M), T(M)^G)$ (4.4), while the corresponding part of the structure of an algebra of CDOs on $M$ is the Lie algebroid $(C^\infty(M), T(M))$. For this reason $\mathcal{D}^{ch}(P)^{\hat{\mathfrak{g}}}$ has more interesting modules (also see below).

In the case $\pi : P \to M$ is a principal frame bundle of $TM$, we find a better description of $\mathcal{D}^{ch}(P)$ such that (among other things) the formal loop group action becomes manifest. This occupies the second half of §4 and leads to the following result:

**Theorem 4.14.** Suppose we are given a principal $G$-bundle $\pi : P \to M$, a representation $\rho : G \to SO(\mathbb{R}^d)$ and an isomorphism $P \times_{\pi} \mathbb{R}^d \cong TM$; a connection 1-form $\Theta$ of $\pi$; and a basic 3-form $dH$ on $P$ satisfying $dH = (\lambda + \lambda_D + \lambda_H)(\Omega \wedge \Omega)$, where $\Omega = d\Theta + \frac{1}{2}([\Theta \wedge \Theta])$ is the curvature 2-form (see also §1.2 for notations). These data determine an algebra of CDOs $\mathcal{D}^{ch}_{\Theta, H}(P)$ with a formal loop group action of level $\lambda$. For more details see the main text.

Even though the vertex algebras defined as above are particular examples of algebras of CDOs, they can in fact be used to recover all algebras of CDOs as well as construct other interesting objects (see below). Also, they have an arguably more appealing generators-and-relations description than an arbitrary algebra of CDOs.
\[ \Gamma^{ch}(\pi, W) := H^\ast -\lambda_{ad}^{+} (\mathfrak{g} - \lambda_{ad}^{+}, D^{ch}(P) \otimes W) \]

(Definition 5.3 & Lemma 5.6) Here, \( H^\ast -\lambda_{ad}^{+} \) is the semi-infinite \((or BRST)\) cohomology of the loop algebra of \( \mathfrak{g} \), which is defined only at level \(-\lambda_{ad}^{+} \) (recalled in 5.2–5.3). This construction is analogous to that of associated vector bundles. In fact, the author believes there is an interpretation of (1.1) in terms of a vector bundle over the “formal loops of \( M \)” equipped with some “connection”. The example where \( \pi \) is the projection from a simple Lie group to its flag manifold has been studied before as a construction of Wakimoto modules. [FP90, Vor99] Notice that if \( W = \text{a vertex algebra} \), then so is (1.1).

For our first example of (1.1), \( \pi \) is any principal frame bundle of \( TM \) and \( W = \mathbb{C} \) (hence \( \lambda = -\lambda_{ad}^{+} \)). By Theorem 4.11, the construction of \( \Gamma^{ch}(\pi, W) \) requires the choice of some \( H \in \pi^{+} \Omega^{3}(M) \) that satisfies \( dH = \text{Tr} \rho(\Omega) \wedge \rho(\Omega) \). The main result in \( \S \) is a new description of CDOs as alluded to earlier:

**Theorem 6.11.** \( \Gamma^{ch}(\pi, \mathbb{C}) \) is an algebra of CDOs on \( M \). Moreover, up to isomorphism, every algebra of CDOs on \( M \) arises in this fashion.

The proof consists of two parts: first we identify the two lowest weights of \( \Gamma^{ch}(\pi, \mathbb{C}) \) (4.2 & 4.5) and work out their structure (Proposition 6.7); then we use a certain property of its conformal vector to deduce that \( \Gamma^{ch}(\pi, \mathbb{C}) \) is completely determined by its two lowest weights (Proposition 6.9 & Corollary 6.10). For certain homogeneous spaces, this description of the algebras of CDOs has been given before. [GMS01]

There is also an extension of the result to supermanifolds, including a special case that recovers the chiral de Rham complex (\( \S \)).

For our second example of (1.1), \( G = \text{Spin}_{2d} \), \( \pi \) is a spin frame bundle of \( M \) and \( W = S \) is the spinor representation of \( \mathfrak{so}_{2d} \). By Theorem 4.14, the construction of the spinor module \( \Gamma^{ch}(\pi, S) \) requires the choice of some \( H \in \pi^{+} \Omega^{3}(M) \) that satisfies \( dH = \frac{1}{2} \text{Tr} (\Omega \wedge \Omega) \) (\( \S \)). \( \S \) consists mainly of an analysis of the spinor module that parallels the one of \( \Gamma^{ch}(\pi, \mathbb{C}) \), resulting in a more explicit description in terms of generating data and relations (Theorem 7.14). In fact, we regard the work in \( \S \) largely as preparation for further study. It is hoped that the spinor module will serve as an ingredient in a geometric theory of the Witten genus.

The appendix reviews the notion of vertex algebroids and their relations with vertex algebras. Even though vertex algebroids have a rather complicated definition, they are useful for handling the vertex algebras in this paper.

**§ 1.2. Notations and conventions.** In this paper, every vertex algebra \( V \) is graded by nonnegative integers which we call \textit{weights}; its component of weight \( k \) is denoted by \( V_{k} \) and its weight operator by \( L_{0} \), i.e. \( L_{0}V_{k} = k \). For any \( u \in V \), we always write the Fourier modes of its vertex operator as \( u_{k}, k \in \mathbb{Z} \), such that \( u_{k} \) has weight \(-k \). For any conformal vector \( \nu \in V \) we consider, \( \nu_{0} = L_{0} \).

Given a smooth manifold \( M \), we write \( C^{\infty}(M), T(M), \Omega^{n}(M) \) for its spaces of smooth \( \mathbb{C} \)-valued functions, vector fields and \( n \)-forms. Also, all ordinary cohomology groups and their equivariant versions have complex coefficients. Given a Lie algebra \( \mathfrak{g} \) and a finite-dimensional representation \( \rho : \mathfrak{g} \rightarrow \mathfrak{gl}(V) \), we write \( \lambda_{\rho} \) for the invariant symmetric bilinear form on \( \mathfrak{g} \) given by \( \lambda_{\rho}(A, B) = \text{Tr} \rho(A)\rho(B) \). In particular, \( \lambda_{ad} \) is the Killing form. Square brackets \([ \ ]\) are used for supercommutators between operators of any parities, while curly brackets \( \{ \} \) are reserved for a different use (see \( \A.3 \)). Repeated indices are always implicitly summed over all possible values, unless a specific range is indicated.

**§ 1.3. Acknowledgements.** The author is currently supported by the ESPRC grant EP/H040692/1. He would like to thank Matthew Ando for a suggestion that initially motivated Theorem 5.11 and Dennis Gaitsgory for explaining some aspects of semi-infinite cohomology. He would also like to thank a reviewer for pointing out some related works in the literature.
§2. CDOs on Smooth Manifolds

A sheaf of CDOs is a sheaf of vertex algebras locally modelled on an elementary vertex algebra known as a $\beta\gamma$-system, and it provides an approximate mathematical formulation of the quantum theory of 2-dimensional $\sigma$-models. [GMS00, Kap06, Wit07] This section reviews the construction and classification of sheaves of CDOs on a smooth manifold. For the definition of a vertex algebra, see [Kac98, FB04].

2.1. The algebra of CDOs on $\mathbb{A}^d$. Let $d$ be a positive integer. Define a unital associative algebra $\mathcal{U}$ with the following generators and relations

\begin{equation}
\label{eq:2.2}
b^i_n, a_{i,n}, n \in \mathbb{Z}, i = 1, \ldots, d, \quad [a_{i,n}, b^j_m] = \delta^n_1 \delta_{n,-m}, \quad [b^i_n, b^j_m] = 0 = [a_{i,n}, a_{j,m}].
\end{equation}

The commutative subalgebra $\mathcal{U}_+$ generated by $\{b^i_n\}_{n \geq 0}$ and $\{a_{i,n}\}_{n \geq 0}$ has a trivial representation $\mathbb{C}$. The induced $\mathcal{U}$-module

\[ \mathcal{D}^{ch}(\mathbb{A}^d) := \mathcal{U} \otimes \mathcal{U}_+ \mathbb{C} \]

has the structure of a vertex algebra. The vacuum is $1 = 1 \otimes 1$. The infinitesimal translation operator $T$ and weight operator $L_0$ are determined by

\[ T1 = 0, \quad [T, b^i_n] = (1 - n)b^i_{n-1}, \quad [T, a_{i,n}] = -na_{i,n-1} \]
\[ L_01 = 0, \quad [L_0, b^i_n] = -nb^i_n, \quad [L_0, a_{i,n}] = -na_{i,n} \]

The fields (or vertex operators) of $b^i_0 1 \in \mathcal{D}^{ch}(\mathbb{A}^d)_0$ and $a_{i,-1} 1 \in \mathcal{D}^{ch}(\mathbb{A}^d)_1$ are respectively

\[ \sum_n b^i_n z^{-n}, \quad \sum_n a_{i,n} z^{-n-1} \]

which determine the fields of other elements by the Reconstruction Theorem [FB04]. This vertex algebra has a family of conformal vectors of central charge $2d$, namely

\[ a_{i,-1} b^i_{-1} 1 + T^2 f, \quad f \in \mathcal{D}^{ch}(\mathbb{A}^d)_0 = \mathbb{C}[b^1_0, \ldots, b^d_0] \cdot 1. \]

The vertex algebra $\mathcal{D}^{ch}(\mathbb{A}^d)$ is freely generated by its associated vertex algebroid (see §A.5 and §A.7). To describe the latter, consider the affine space $\mathbb{A}^d = \text{Spec} \mathbb{C}[b^1, \ldots, b^d]$ and identify the functions, 1-forms and vector fields on $\mathbb{A}^d$ with the following subquotients of $\mathcal{D}^{ch}(\mathbb{A}^d)$:

- $\mathcal{O}(\mathbb{A}^d) = \mathcal{D}^{ch}(\mathbb{A}^d)_0$ via $b^i = b^i_0 1$, $b^i b^j = b^j b^i 1$, etc.

- $\Omega^1(\mathbb{A}^d) \subset \mathcal{D}^{ch}(\mathbb{A}^d)_1$ via $db^i = b^i_{-1} 1$

- $\mathcal{T}(\mathbb{A}^d) = \mathcal{D}^{ch}(\mathbb{A}^d)_1/\Omega^1(\mathbb{A}^d)$ via $\partial_i = \partial/\partial b^i = \text{coset of } a_{i,-1} 1$

Under these identifications, the vertex algebroid associated to $\mathcal{D}^{ch}(\mathbb{A}^d)$ is of the form

\[ (\mathcal{O}(\mathbb{A}^d), \Omega^1(\mathbb{A}^d), \mathcal{T}(\mathbb{A}^d), \bullet, \{ \}, \{ \})_\Omega. \]

The extended Lie algebroid structure consists of the usual differential on functions, Lie bracket on vector fields, Lie derivations by vector fields on functions and 1-forms, and pairing between 1-forms and vector fields. Using the splitting

\begin{equation}
\label{eq:2.3}
s: \mathcal{T}(\mathbb{A}^d) \to \mathcal{D}^{ch}(\mathbb{A}^d)_1, \quad X = X^i \partial_i \mapsto a_{i,-1} X^i
\end{equation}

the rest of the vertex algebroid structure, according to [A2], is given by

\begin{equation}
\label{eq:2.4}
X \bullet f = (\partial_j X^i)(\partial_i f) db^j, \quad \{ X, Y \} = -(\partial_j X^i)(\partial_i Y^j), \quad \{ X, Y \}_\Omega = -(\partial_k \partial_j X^i)(\partial_i Y^j) db^k
\end{equation}

These expressions do not seem to have any obvious global meaning; however, see Theorem 2.6

\[ \text{[This vertex algebra also has other conformal vectors that define the weights differently., Kac98]}. \]
§2.2. The sheaf of CDOs on $\mathbb{R}^d$. Now regard $b^1, \ldots, b^d$ as the standard coordinates of $\mathbb{R}^d$. The smooth functions, 1-forms and vector fields on any open set $W \subset \mathbb{R}^d$ form an extended Lie algebroid just as in (2.7) and the expressions in (2.4) again define a vertex algebroid

$$(C^\infty(W), \Omega^1(W), \mathcal{T}(W), \bullet, \{ \}, \{ \}_\Omega).$$

The vertex algebra it freely generates (see (A.7)) will be denoted by $\mathcal{D}^{ch}(W)$. This vertex algebra also has a family of conformal vectors of central charge $2d$, namely

$$\partial_{i,-1}db^i + \frac{1}{2}T\omega, \quad \omega \in \Omega^1(W), \ d\omega = 0. \tag{2.5}$$

For any inclusion of open sets $W' \subset W$, there is an obvious restriction map $\mathcal{D}^{ch}(W) \to \mathcal{D}^{ch}(W')$. This defines a sheaf of conformal vertex algebras $\mathcal{D}^{ch}$ on $\mathbb{R}^d$.

**Definition 2.3.** A sheaf of CDOs on a smooth $d$-dimensional manifold $M$ is a sheaf of vertex algebras $\mathcal{V}$ with the following properties:

- its weight-zero component is $\mathcal{V}_0 = C^\infty_M$, and
- each point of $M$ has a neighborhood $U$ such that $(U, \mathcal{V}|_U)$ is isomorphic to $(W, \mathcal{D}^{ch}|_W)$ for some open set $W \subset \mathbb{R}^d$.

A conformal structure on $\mathcal{V}$ is an element $\nu \in \mathcal{V}(M)$ such that each of the isomorphisms postulated above takes $\nu|_U \in \mathcal{V}(U)$ to one of the conformal vectors (2.5) of $\mathcal{D}^{ch}(W)$.

In order to state the results on the construction and classification of sheaves of CDOs, let us introduce a notation that will also appear often in the sequel.

**Definition 2.4.** Let $M$ be a smooth manifold. Given a connection $\nabla$ on $TM$ and $X \in \mathcal{T}(M)$, define an operator $\nabla^i X \in \Gamma(\text{End} TM)$ by

$$(\nabla^i X)(Y) := \nabla_X Y - [X, Y], \quad Y \in \mathcal{T}(M).$$

Notice that if $\nabla$ is torsion-free, then $\nabla^i X = \nabla X$.

**Theorem 2.5.** (Che12) Let $M$ be a smooth $d$-dimensional manifold.

(a) Given a connection $\nabla$ on $TM$ with curvature $R$ and any $H \in \Omega^3(M)$ satisfying $dH = \text{Tr}(R \wedge R)$, we can define a sheaf of vertex algebroids $\left( C^\infty_M, \Omega^1_M, \mathcal{T}_M, \bullet, \{ \}, \{ \}_\Omega \right)$ on $M$ by

$$
\begin{align*}
X \bullet f &:= \langle \nabla X \rangle f \\
\{X, Y\} &:= -\text{Tr}(\nabla^i X \cdot \nabla^j Y) \\
\{X, Y\}_\Omega &:= \text{Tr} \left( -\nabla^i (\nabla^j X) \cdot \nabla^k Y + \nabla^i X \cdot \iota_Y R - \iota_X R \cdot \nabla^i Y \right) + \frac{1}{2}\iota_X \iota_Y H
\end{align*}
$$

The sheaf of vertex algebras it freely generates (see §A.7) is a sheaf of CDOs on $M$, which will be denoted by $\mathcal{D}^{\text{ch}}_{M, \nabla, H}$. Up to isomorphism, every sheaf of CDOs on $M$ is of this form.

(b) There is a one-to-one correspondence between $\omega \in \Omega^1(M)$ satisfying $d\omega = \text{Tr} R$ and the conformal structures on $\mathcal{D}^{\text{ch}}_{M, \nabla, H}$. For any such $\omega$, the corresponding conformal structure, which will be denoted by $\nu^\omega$, has the following local expression:

$$\nu^\omega|_U = \sigma_{i,-1} \sigma^i + \frac{1}{2} \text{Tr} \left( \Gamma^\alpha_{\beta\gamma} \Gamma^{\beta\gamma} - \Gamma^{\alpha\beta\gamma} 1 \right) + \sigma^i (\{\sigma_j, \sigma_k\}) \sigma_{\alpha i}^{k} (\Gamma^{\sigma})_j \sigma_j + \frac{1}{2} \omega_{-1} 1 \tag{2.7}$$

where $U \subset M$ is any sufficiently small open set; $\sigma = (\sigma_1, \ldots, \sigma_d)$ is any basis of $\mathcal{T}(U)$ over $\mathbb{C}^\infty(U)$; $(\sigma^1, \ldots, \sigma^d)$ is the dual basis of $\Omega^1(U)$; and $\Gamma^\sigma \in \Omega^1(U) \otimes \mathfrak{gl}_d$ is the connection 1-form of $\nabla$ with respect to $\sigma$, i.e. $\nabla \sigma_i = (\Gamma^\sigma)_i \otimes \sigma_j$. Also, $\nu^\omega$ has central charge $2d$ and the property that

$$\nu^\omega \alpha = 0 \text{ for } \alpha \in \Omega^1(M), \quad \nu^\omega X = \text{Tr} \nabla^i X - \omega(X) \text{ for } X \in \mathcal{T}(M). \tag{2.8}$$

\begin{flushright}$\square$\end{flushright}
Remarks. (i) By Theorem 2.5, a smooth manifold $M$ admits sheaves of CDOs if and only if $p_1(M)$ is trivial in de Rham cohomology, while conformal structures always exist. For example, if $\nabla$ is orthogonal with respect to a Riemannian metric, then $\text{Tr} \ R = 0$ and a conformal structure can be defined using, say, $\omega = 0$. However, this result generalizes to supermanifolds, in which case the obstruction to conformal structures may well be nontrivial. [Che12]

(ii) In [Che12], we only obtained a local expression of $\nu^\omega$ in terms of local coordinate vector fields, but that implies the more general expression in (2.7) by a straightforward calculation.

(iii) In the original work [GMS04] as well as in [Che12], sheaves of CDOs and conformal structures were constructed by gluing local data. For smooth manifolds, the result is the above description in terms of generators and relations (or generating fields and OPEs), but the expressions in (2.6) and (2.7) do not look very inspiring. In [Che12] we will obtain a more conceptual description of CDOs using semi-infinite cohomology.

Theorem 2.6. [Che12] Let $\mathcal{D}^\text{ch}_{\nabla,H}$ and $\mathcal{D}^\text{ch}_{\nabla,H'}$ be sheaves of CDOs on a smooth manifold $M$ constructed as in Theorem 2.5; denote by $\bullet$, $\{ \}$, $\{ \}$ $\Omega$ (resp. $\{ \}$, $\{ \}$ $\text{H}$) the structure maps determined by $\nabla$ and $H$ (resp. $H'$) as in (2.6).

(a) There is a one-to-one correspondence between $\beta \in \Omega^2(M)$ satisfying $d\beta = H' - H$ and isomorphisms $\mathcal{D}^\text{ch}_{\nabla,H} \sim \mathcal{D}^\text{ch}_{\nabla,H'}$ that restricts to the identity on $\mathcal{C}_M$. For any such $\beta$, the corresponding isomorphism is induced by an isomorphism of sheaves of vertex algebroids (see (A.8))

\[(id, \Delta) : (C^\infty_M, \Omega^1_M, \nabla, \{ \}, \{ \}) \to (C^\infty_M, \Omega^1_M, \nabla, \{ \}, \{ \})\]

where $\Delta : T_M \to \Omega^1_M$ is given by $\Delta(X) = \frac{1}{2} \iota_X \beta$.

(b) Every isomorphism described above respects the correspondence in Theorem 2.5.

Remarks. (i) By Theorems 2.5 and 2.6, if $M$ admits sheaves of CDOs, their isomorphism classes form an $H^3(M)$-torsor. (ii) Since every sheaf of CDOs $\mathcal{D}^\text{ch}_{\nabla,H}$ is fine, for most purposes it suffices to (and we will) work with the vertex algebra of global sections $\mathcal{D}^\text{ch}_{\nabla,H}(M)$, which will be called an algebra of CDOs on $M$. The reader should keep in mind that by construction

\[(2.9) \quad \mathcal{D}^\text{ch}_{\nabla,H}(M)_0 = C^\infty(M), \quad \mathcal{D}^\text{ch}_{\nabla,H}(M)_1 = \Omega^1(M) \oplus T(M).\]

For a description of the higher weights, see (A.11)

Lemma 2.7. Consider an algebra of CDOs $\mathcal{D}^\text{ch}_{\nabla,H}(M)$ on a smooth manifold $M$. For any $\alpha \in \Omega^1(M)$ and $X \in T(M)$ viewed as elements of $\mathcal{D}^\text{ch}_{\nabla,H}(M)$, we have $\alpha_0 X = -\iota_X d\alpha$. Moreover, $\alpha_0 \equiv 0$ on $\mathcal{D}^\text{ch}_{\nabla,H}(M)$ if and only if $d\alpha = 0$.

Proof. By definition of $\mathcal{D}^\text{ch}_{\nabla,H}(M)$ and (A.3), we have $\alpha_0 f = 0$ for $f \in C^\infty(M)$ and

$$\alpha_0 X = -[X_{-1}, \alpha_0] = -L_X \alpha + d\alpha = -\iota_X d\alpha.$$ 

Since $\alpha_0$ is a derivation, it is trivial on the entire vertex algebra $\mathcal{D}^\text{ch}_{\nabla,H}(M)$ if and only if it is trivial on the generating subspaces $C^\infty(M)$ and $T(M)$. \(\square\)
§3. Formal Loop Group Actions on CDOs

In this section, we study how a Lie group action on a manifold can be lifted to a “formal loop group action” on an algebra of CDOs. This turns out to be a condition on the equivariant first Pontrjagin class of the manifold.

§ 3.1. Setting: manifold with a Lie group action. Throughout this section, let $G$ be a compact connected Lie group, $\mathfrak{g}$ its Lie algebra, and $\lambda$ an invariant symmetric bilinear form on $\mathfrak{g}$. Recall the loop algebra $L\mathfrak{g} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$: writing $A \otimes t^n$ as $A_n$, the Lie bracket is given by $[A_n, B_m] = [A, B]_{n+m}$. Also recall that $\lambda$ determines a central extension $\hat{\mathfrak{g}} = L\mathfrak{g} \oplus \mathbb{C}$ with

$[A_n, B_m] = [A, B]_{n+m} + n\lambda(A, B)\delta_{n+m, 0}, \ A, B \in \mathfrak{g}, \ n, m \in \mathbb{Z}.$

Let $P$ be a smooth manifold with a smooth right $G$-action. Later we will specialize to the case of a principal bundle, but at the moment $P$ can be any right $G$-manifold. The left $G$-action on $C^\infty(P)$ is completely determined by the induced map of Lie algebras $\mathfrak{g} \to T(P)$. The vector field generated by $A \in \mathfrak{g}$ will be written as $A^P \in T(P)$.

§ 3.2. The equivariant de Rham complex. Recall that $H^*_G(P) = H^*(EG \times_G P)$ can be computed by the Cartan model $(\Omega^*_G(P), d_G)$. \cite{GS99} The graded algebra of Cartan cochains is given by

$\Omega^*_G(P) = \bigoplus_{k \geq 0} \Omega^k_G(P), \quad \Omega^*_G(P) = \bigoplus_{2i+j=k} (\text{Sym}^i \hat{\mathfrak{g}}^\vee \otimes \Omega^j(P))^G.$

Let us follow a usual convention: regard any $\xi \in \Omega^*_G(P)$ as a $G$-equivariant polynomial map $\xi : \mathfrak{g} \to \Omega^*(P)$ and write its value at $A \in \mathfrak{g}$ as $\xi_A$. The Cartan differential then reads

$$(d_G \xi)_A = d\xi_A - \iota_{A^P} \xi_A.$$

The characteristic map $H^*(BG) \to H^*_G(P)$ is represented by the inclusion $(\text{Sym}^* \mathfrak{g}^\vee)^G \hookrightarrow \Omega^*_G(P)$, and the image of any $\eta \in (\text{Sym}^* \mathfrak{g}^\vee)^G = H^*(BG)$ will be denoted by $\eta(P) \in H^*_G(P)$.

§ 3.3. CDOs with a Lie group action. Choose a $G$-invariant connection $\nabla$ on $TP$, with curvature tensor $R$. This means for $A \in \mathfrak{g}$ we have $L_{A^P} \nabla = 0$, or equivalently

$$(\nabla^i A^P) = -\iota_{A^P} R$$

(see Definition \ref{def:invariant_connection}). Assume that $p_1(P) = 0$ and choose some $H \in \Omega^3(P)^G$ such that $dH = \text{Tr} (R \wedge R)$. By Theorem \ref{thm:lie_group_action}, $\nabla$ and $H$ determine an algebra of CDOs $\mathcal{D}^\text{ch}_{\nabla, H}(P)$, which is freely generated by a vertex algebroid (see also \ref{vertex_algebroids})

$$(C^\infty(P), \Omega^1(P), T(P), \bullet, \{\ }, \{\ }_\Omega);$$

when there is no risk of confusion, we simply write $\mathcal{D}^\text{ch}(P)$. Clearly, the $G$-invariance of $\nabla$ and $H$ implies the $G$-equivariance of the structure maps $\bullet, \{\ }, \{\ }_\Omega$, so that the $G$-action on $C^\infty(P) = \mathcal{D}^\text{ch}(P)_0$ extends to a $G$-action on $\mathcal{D}^\text{ch}(P)$.

Without loss of generality, we may assume that $\text{Tr} R = 0$. (For example, this is true if $\nabla$ is orthogonal with respect to a Riemannian metric.) Choose some $\omega \in \Omega^1(P)^G$ such that $d\omega = 0$. By Theorem \ref{thm:lie_group_action}, $\omega$ determines a $G$-invariant conformal vector $\nu^\omega$ in $\mathcal{D}^\text{ch}(P)$. The Fourier modes of the associated Virasoro field will be denoted by $L_{n}^\omega$, $n \in \mathbb{Z}$. Soon we will make a more specific choice of $\omega$.

Remark. The $G$-invariance of $\nabla$ and $H$, as well as that of other geometric data to appear later, can always be achieved by averaging over $G$ with respect to the Haar measure.

\footnote{In this paper we adopt the following habit: the kernel of $\hat{\mathfrak{g}} \to L\mathfrak{g}$ is always identified with $\mathbb{C}$ and, whenever $\hat{\mathfrak{g}} \to LG$ acts on some vector space, $1 \in \mathbb{C}$ always acts as identity.}

\footnote{It follows from a simple calculation that for any $X \in T(P)$ we have $L_X \nabla = \nabla(\nabla^i X) + \iota_X R$.}
Definition 3.4. An inner \((\hat{g}_A, G)\)-action on a vertex algebra \(V\) is a map of vertex algebras from \(V_\lambda(\hat{g})\) such that the induced \(g\)-action on \(V\) integrates into a \(G\)-action. An inner \((\hat{g}_A, G)\)-action on a conformal vertex algebra \(V\) is primary if the image of any \(A \in V_\lambda(\hat{g})_1\) is primary, or equivalently, if the induced \(\hat{g}_A\)-action on \(V\) is intertwined by the Virasoro algebra action.

Remarks. (i) By definition of \(V_\lambda(\hat{g})\) (see Example \([A.10]\)), any map \(V_\lambda(\hat{g}) \to V\) is determined by its component of weight one, i.e. a linear map \(g = V_\lambda(\hat{g})_1 \to V_1\). Taking the zeroth modes (resp. Fourier modes) then yields a map of Lie algebras from \(g\) to the inner derivations of \(V\) (resp. from \(\hat{g}_A\) to the endomorphisms of \(V\)). These are the induced actions mentioned above. (ii) Now we can state the goal of this section more precisely: find the condition under which the given \(G\)-action on \(D^{ch}(P)_0 = C^\infty(P)\) extends to an inner \((\hat{g}_A, G)\)-action on \(D^{ch}(P)\).

§3.5. Cartan cochains associated to CDOs. The \(G\)-action on \(P\) and the vertex algebroid structure associated to \(D^{ch}(P)\) together determine two Cartan cochains of degree 4, namely

\[
\begin{align*}
\chi^{2,2} \in (g^\vee \otimes \Omega^2(P))^G, & \quad \chi^{2,2}_A := \{A^P, -\} : \Omega^1(P) \to \Omega^2(P) = H^4_G(V), \quad \chi^{2,2}_A := \{A^P, A^P\} = \frac{1}{2} \chi^{2,2} - \frac{1}{2} \chi^{4,0}, \\
\chi^{4,0} \in (\text{Sym}^2 g^\vee \otimes C^\infty(P))^G, & \quad \chi^{4,0}_A := \{A^P, A^P\} = -\chi^{2,2} - \chi^{4,0}.
\end{align*}
\]

Indeed, by \([2.0]\) and \([3.1]\), the operator \(\{A^P, -\} : \Omega^1(P) \to \Omega^2(P)\) may be viewed as the indicated 2-form, and the \(G\)-invariance of \(\nabla, H\) implies the \(G\)-equivariance of \(\chi^{2,2}, \chi^{4,0}\). The \(G\)-action on \(P\) and the conformal vector \(\nu^\omega\) of \(D^{ch}(P)\) together determine a Cartan cochain of degree 2, namely

\[
\chi^{2,0} \in (g^\vee \otimes C^\infty(P))^G, \quad \chi^{2,0}_A := L_l^\omega A^P = \nabla^t A^P - \omega(A^P).
\]

Indeed, the \(G\)-invariance of \(\nabla, \omega\) implies the \(G\)-equivariance of \(\chi^{2,0}\).

Lemma 3.6. (a) The Cartan cochain \(2\chi^{2,2} + \chi^{4,0}\) is closed and represents \(8\pi^2 p_1(P)_G \in H^4_G(V)\). (b) The Cartan cochain \(\chi^{2,0}\) is exact. In fact, it is trivial with a suitable choice of conformal structure \(\nu^\omega\).

Proof. (a) According to \([BGV92]\), the Cartan cochain \(A \mapsto \text{Tr} (R - \nabla A^P)^2\) is closed and represents the class \(-8\pi^2 p_1(P)_G\). Then the claim follows from the calculation

\[
\text{Tr} (R - \nabla A^P)^2 = dH - 2 \text{Tr} (\nabla A^P \cdot R) + \text{Tr} (\nabla A^P \cdot \nabla A^P) = (dG H)_A - 2\chi^{2,2}_A - \chi^{4,0}_A
\]

where we have used \(dH = \text{Tr} (R \wedge R)\) and \((3.2)\).

(b) According to \([BGV92]\) again, the Cartan cochain \(A \mapsto \text{Tr} (R - \nabla A^P) = -\text{Tr} \nabla A^P\) is exact. Also we have \((dG \omega)_A = d\omega - \omega(A^P) = -\omega(A^P)\). This proves the first part of the claim. On the other hand, there exists some \(\omega' \in \Omega^1(P)^G\) such that

\[
-\text{Tr} \nabla A^P = (dG \omega')_A = d\omega' - \omega'(A^P)
\]

which is equivalent to two equations: firstly \(d\omega' = 0\), so that \(\omega'\) also determines a conformal vector \(\nu^\omega'\); and secondly \(L_l^\omega A^P = 0\). This proves the rest of the claim.

Remarks. (i) The closedness of \(2\chi^{2,2} + \chi^{4,0}\) is equivalent to a pair of equations

\[
d\chi^{2,2}_A = 0, \quad d\chi^{4,0}_A = 2\chi^{2,2}_A - \chi^{4,0}_A, \quad \text{for } A \in g
\]

which can also be verified directly from our assumptions on \(\nabla\) and \(H\). (ii) From now on we assume that the conformal vector \(\nu^\omega\) has been chosen such that \(L_l^\omega A^P = 0\) for \(A \in g\), and write the associated Virasoro operators simply as \(L_n\) for \(n \in \mathbb{Z}\).

Preparation. The following sequence of results all concern lifting the map of Lie algebras \(g \to T(P)\) to a linear map of the form

\[
g \to D^{ch}(P)_1 = T(P) \oplus \Omega^1(P), \quad A \mapsto A^P + h^2 A
\]

where \(h^2 : g \to \Omega^1(P)\) is assumed to be \(G\)-equivariant. In other words, \(h^2\) is a Cartan cochain.
Proposition 3.7. The linear map \((\mathfrak{g},\mathfrak{h})\) gives rise to a map of Lie algebras
\[(3.6) \quad \mathfrak{g} \to \text{Der} \mathcal{D}^{\text{ch}}(P), \quad A \mapsto (A^P + h^2,1)_0 \]
if and only if the closed 2-form \(\chi^2,2_A - dh^2,1_A\) is \(G\)-invariant for any \(A \in \mathfrak{g}\).

Proof. For \(A, B \in \mathfrak{g}\), consider the calculation
\[
\left[(A^P + h^2,1)_0, (B^P + h^2,1)_0\right] = \left([A^P, B^P] + \{A^P, B^P\}_\Omega + L_{A^P} h^2,1_B - L_{B^P} h^2,1_A\right)_0
\]
where we have used \((3.4)\) and the \(G\)-equivariance of \(h^2,1\). By Lemma \((3.7)\) the second term is trivial if and only if
\[
0 = d(\iota_{B^P} \chi^2,2_A - L_{B^P} h^2,1_A) = L_{B^P} (\chi^2,2_A - dh^2,1_A)
\]
where we have used \((3.4)\). This proves our claim. \(\square\)

The map of Lie algebras \(\mathfrak{g} \to T(P)\) extends in an obvious way to a map of extended Lie algebroids
\[
i : (\mathbb{C}, 0, \mathfrak{g}, 0, \lambda, 0) \to (C^\infty(P), \Omega^1(P), T(P)),
\]
and hence a map of vertex algebras \(V_\lambda(\mathfrak{g}) \to \mathcal{D}^{\text{ch}}(P)\), if and only if for any \(A \in \mathfrak{g}\) the closed 2-form
\[
\chi^2,2_A - dh^2,1_A
\]
is \(G\)-horizontal and
\[(3.7) \quad \chi^4,0_A + 2h^2,1_A(A^P) = \lambda(A, A).
\]

Proof. Let \(A, B \in \mathfrak{g}\). According to Definition \((3.1)\) the linear map \(h^2,1 : \mathfrak{g} \to \Omega^1(P)\) defines a map between the vertex algebroids in question if and only if
\[
\{A^P, B^P\} = \lambda(A, B) - h^2,1_B(B^P) - h^2,1_A(A^P)
\]
and
\[
\{A^P, B^P\}_\Omega = -L_{A^P} h^2,1_B + L_{B^P} h^2,1_A - dh^2,1_B(A^P) + h^2,1_B[A, B]
\]
By \((3.2)\) the first equation is equivalent to \((3.7)\). By \((3.2)\) again and the \(G\)-equivariance of \(h^2,1\), the second equation can be rewritten as \(\iota_{B^P} (\chi^2,2_A - dh^2,1_A) = 0\). This proves our claim. In fact, the \(G\)-horizontality of \(\chi^2,2_A - dh^2,1_A\) always implies that the left side of \((3.7)\) is locally constant, since
\[
d(\chi^4,0_A + 2\iota_{A^P} h^2,1_A) = 2\iota_{A^P} \chi^2,2_A - 2\iota_{A^P} dh^2,1_A + 2L_{A^P} h^2,1_A = 2\iota_{A^P} (\chi^2,2_A - dh^2,1_A)
\]
by virtue of \((3.3)\) and the \(G\)-equivariance of \(h^2,1\). Therefore if \(P\) is connected, the horizontal condition guarantees that \((3.7)\) must hold for some \(\lambda \in (\text{Sym}^2 \mathfrak{g}^\vee)^G\). \(\square\)

Proposition 3.8. The linear map \((\mathfrak{g},\mathfrak{h})\) determines a map of vertex algebroids
\[(i, h^2,1) : (\mathbb{C}, 0, \mathfrak{g}, 0, \lambda, 0) \to (C^\infty(P), \Omega^1(P), T(P)), \bullet, \{ \}, \{ \}_\Omega,
\]
and hence a map of vertex algebras \(V_\lambda(\mathfrak{g}) \to \mathcal{D}^{\text{ch}}(P)\), if and only if for any \(A \in \mathfrak{g}\) the closed 2-form
\[
\chi^2,2_A - dh^2,1_A
\]
is \(G\)-horizontal and
\[
\chi^4,0_A + 2h^2,1_A(A^P) = \lambda(A, A).
\]

Proof. First we compute the one-parameter subgroup of automorphisms generated by the inner derivation \((A^P + h^2,1)_0\). Since \(\mathcal{D}^{\text{ch}}(P)\) is freely generated by a vertex algebroid, it suffices to compute the automorphisms at weights 0 and 1. For the following computations, keep \((A, 3)\) in mind. For \(f \in C^\infty(P)\) and \(\alpha \in \Omega^1(P)\) we simply have
\[
(A^P + h^2,1)_0^n f = (A^P)_n f \quad \Rightarrow \quad e^{\iota(A^P + h^2,1)_0} f = e^{\iota A} \cdot f
\]
(3.8)
\[
(A^P + h^2,1)_0^n \alpha = L^{A^P}_n \alpha \quad \Rightarrow \quad e^{\iota(A^P + h^2,1)_0} \alpha = e^{\iota A} \cdot \alpha
\]
(3.9)
where \( \cdot \) refers to the given \( G \)-actions. For \( X \in T(P) \) we first have

\[
(A^P + h_A^{2,1})_0 X = [A^P, X] + \{A^P, X\}_\Omega - \iota_X dh_A^{2,1} = [A^P, X] + \iota_X (\chi_A^{2,2} - dh_A^{2,1})
\]

by Lemma 2.7 and 3.2. Then it follows by induction and the \( G \)-equivariance of \( \chi^{2,2}, h^{2,1} \) that

\[
(A^P + h_A^{2,1})_n X = L^n_A, P, X + nL_{A^P}^{n-1}\iota_X (\chi_A^{2,2} - dh_A^{2,1}), \quad n \geq 1
\]

(3.10)

\[
\Rightarrow \quad e^{t(A^P + h_A^{2,1})} X = e^{tA^P} X + te^{tA^P} \cdot \iota_X (\chi_A^{2,2} - dh_A^{2,1})
\]

This finishes the computation of the automorphisms.

If \( \chi_A^{2,2} - dh_A^{2,1} \) vanishes for all \( A \in g \), it follows from 3.8 – 3.10 that 3.6 integrates into the \( G \)-action described in 3.3. Conversely, assume that 3.6 integrates into a \( G \)-action. By 3.10, \( \chi_A^{2,2} - dh_A^{2,1} \) must vanish whenever \( e^A = 1 \). Since every element of \( G \) lies in a torus, the subset \( \{ A \in g \mid e^A = 1 \} \) spans \( g \), so that \( \chi_A^{2,2} - dh_A^{2,1} \) must in fact vanish for all \( A \in g \).

**Remark.** It follows from the proof that whenever 3.6 is integrable, the resulting \( G \)-action on \( D^{\text{ch}}(P) \) must be the one described in 3.3.

### § 3.10. Comparing the three conditions

Recall the conditions encountered respectively in Propositions 3.7, 3.8 and 3.9

(i) \( \chi_A^{2,2} - dh_A^{2,1} \) is \( G \)-invariant for \( A \in g \)
(ii) \( \chi_A^{2,2} - dh_A^{2,1} \) is \( G \)-horizontal for \( A \in g \)
(iii) \( \chi_A^{2,2} - dh_A^{2,1} = 0 \) for \( A \in g \)

In general, (iii) \( \Rightarrow \) (ii) \( \Rightarrow \) (i); the second implication follows from 3.4. From another point of view, a map of vertex algebras as in Proposition 3.8 always induces a map of Lie algebras as in Proposition 3.7, but the other implication may seem somewhat surprising. In the case \( g \) is semisimple, so that \( [g, g] = g \), all three conditions are equivalent.

Here is the main result of this section.

**Theorem 3.11.** The \( G \)-action on \( P \) lifts to an inner \((\hat{g}_\lambda, G)\)-action on \( D^{\text{ch}}(P) \) if and only if

\[
8\pi^2 p_1(P)_G = \chi(P).
\]

Moreover, this action is primary with respect to the chosen conformal vector \( \nu^\omega \).

**Proof.** Recall Definition 3.4 and Example 3.10 Any inner \((\hat{g}_\lambda, G)\)-action as described is determined by a linear map \( g \to D^{\text{ch}}(P)_1 \) of the form \( A \mapsto A^P + h_A^{2,1} \) for some \( h^{2,1} \in (g^t \otimes \Omega^1(P))^G \). By Propositions 3.8 and 3.9 the precise conditions on \( h^{2,1} \) are

\[
\begin{cases}
\chi_A^{2,2} = dh_A^{2,1} \\
\chi_A^{4,0} = \lambda(A, A) - 2h_A^{2,1}(A^P)
\end{cases} \iff 2\chi^{2,2} + \chi^{4,0} = \lambda + 2d_G h^{2,1}.
\]

(3.11)

By Lemma 3.6, this proves the first claim. The second claim is simply a restatement of Lemma 3.6) and the subsequent remark.

**Remark.** In the rest of the paper, \( h^{2,1} \) will be referred to as the **associated Cartan cochain** of the inner \((\hat{g}_\lambda, G)\)-action.

**Example 3.12.** CDOs on a Lie group. Let \( G \) be a simple compact Lie group. In this discussion, \( A, B \) will always mean general elements of \( g = T_e G \) and we will use the following notations:

- \( A^t \) (resp. \( A^l \)) is the left-invariant (resp. right-invariant) vector field on \( G \) that extends \( A \);
- \( \theta^t \) (resp. \( \theta^l \)) is the left-invariant (resp. right-invariant) Maurer-Cartan form on \( G \), i.e. the \( g \)-valued 1-form with \( \theta^t(A^t) = A \) (resp. \( \theta^l(A^l) = A \)).
The structure of $\lambda$ equations in (3.11) are now satisfied by $\theta_t$ choose a basis $T \rightarrow T$

By Theorem 2.5b, the trivial 1-form determines a conformal vector and $\nu$ respect to $D$(3.15)

Theorems 2.5a and 2.6a, with associated Cartan cochain

The transgression isomorphism $H^4(BG) \cong H^3(G)$, where both spaces are one-dimensional, is represented by the correspondence

$$\lambda \in (\text{Sym}^2 \mathfrak{g}^*)^G \mapsto -\frac{1}{4} \lambda(\theta^\ell \wedge [\theta^\ell \wedge \theta^\ell]) = -\frac{1}{4} \lambda(\theta^r \wedge [\theta^r \wedge \theta^r]) \in \Omega^3(G).$$

In particular, the closed 3-form corresponding to $\lambda_0$ will be denoted by $H_0$. Let us construct an algebra of CDOs together with a conformal vector. Let $A, B \in \mathfrak{g}$, $\nabla$ and $k H_0$ for any $k \in \mathbb{C}$ together determine an algebra of CDOs $\mathcal{D}_k^c(G) = \mathcal{D}_{k H_0}^c(G)$, and every algebra of CDOs on $G$ is up to isomorphism of this form. In the vertex algebroid structure of $\mathcal{D}_k^c(G)$, we find that

$$\nabla(\mathfrak{g}^*) \rightarrow T \mathfrak{g} \quad \text{and} \quad \mathfrak{g} \otimes \mathfrak{g} \rightarrow T \mathfrak{g}.$$ (3.12)

Indeed, both define the unique torsion-free connection that is compatible with any bi-invariant symmetric bilinear form.

The expressions in (3.12)–(3.15) are obtained by some computations that we have omitted.

Consider the action of $G$ on itself by right multiplication. The induced map of Lie algebras $\mathfrak{g} \rightarrow T(G)$ takes $A$ to $A^\ell$. Both $\nabla$ and $H_0$ are invariant under this action. In view of (3.12), the equations in (3.11) are satisfied by

$$h^\ell \in (\mathfrak{g}^* \otimes \Omega^1(G))^G, \quad h^\ell_\lambda := -\frac{k}{4} + \frac{h^\lambda}{4} \lambda_0(A, \theta^\ell)$$

and $\lambda = (-k - h^\lambda) \lambda_0$. Therefore we have an inner $(\mathfrak{g}, G)$-action

$$V_{-k - h^\lambda} : \mathfrak{g} \rightarrow \mathcal{D}_k^c(G)$$ (3.17)

with associated Cartan cochain $h^\ell$; its image will be denoted by $V_{-k - h^\lambda}(\mathfrak{g})^\ell$. This action is primary with respect to $\nu$.

Consider also the action of $G$ on itself by inverse left multiplication. The induced map of Lie algebras $\mathfrak{g} \rightarrow T(G)$ takes $A$ to $-A^r$. Both $\nabla$ and $H_0$ are invariant under this action as well. In view of (3.13), the equations in (3.11) are now satisfied by

$$h^r \in (\mathfrak{g}^* \otimes \Omega^1(G))^G, \quad h^r_\lambda := (\frac{k}{2} + \frac{h^\lambda}{4}) \lambda_0(A, \theta^r)$$

and $\lambda = (k - h^\lambda) \lambda_0$. Therefore we have another inner $(\mathfrak{g}, G)$-action

$$V_{k - h^\lambda} : \mathfrak{g} \rightarrow \mathcal{D}_k^c(G)$$ (3.19)

with associated Cartan cochain $h^r$; its image will be denoted by $V_{k - h^\lambda}(\mathfrak{g})^r$. This action is also primary with respect to $\nu$.

The next few propositions provide more details on these two $(\mathfrak{g}, G)$-actions on $\mathcal{D}_k^c(G)$.  

\footnote{Indeed, both define the unique torsion-free connection that is compatible with any bi-invariant symmetric bilinear form.}

\footnote{The expressions in (3.12)–(3.15) are obtained by some computations that we have omitted.}
Proposition 3.13. The vertex subalgebras $V_{-k} \cdot (\mathfrak{g})^\ell$ and $V_{k} \cdot (\mathfrak{g})^r$ commute with each other.

Proof. This amounts to showing that $(A^\ell + h_A^\ell)(-B^r + h_B^r) = 0$ for $A, B \in \mathfrak{g}$ and $i = 0, 1$. For $i = 1$, here is the calculation

$$(A^\ell + h_A^\ell)(-B^r + h_B^r) = -\{A^\ell, B^r\} - h_A^\ell(B^r) + h_B^r(A^\ell)$$

$$= \frac{1}{2}h^\ell \lambda_0(A, \theta^\ell(B^r)) + \left(\frac{1}{2}k + \frac{1}{2}h^\ell\right)\lambda_0(A, \theta^\ell(B^r)) + \left(-\frac{1}{2}k + \frac{1}{2}h^\ell\right)\lambda_0(\theta^\ell(B^r), A) = 0$$

which follows from (A.3), (3.14), (3.16) and (3.18). For $i = 0$, we have a similar calculation

$$(A^\ell + h_A^\ell)(-B^r + h_B^r) = -\{A^\ell, B^r\} - \{A^\ell, B^r\}_\Omega + t_{B^r}db_A^\ell + L_A h_B^r$$

$$= 0 - \left(\frac{1}{2}k + \frac{1}{2}h^\ell\right)\lambda_0([A, \theta^\ell(B^r)], \theta^\ell) + \left(\frac{1}{2}k + \frac{1}{2}h^\ell\right)\lambda_0(A, [\theta^\ell(B^r), \theta^\ell]) + 0 = 0$$

which also utilizes Lemma 2.7 and the commutativity between left and right multiplications.

**Remark.** This recovers the well-known fact that every algebra of CDOs on $G$ provides a realization of a commuting pair of affine Lie algebras of dual levels. [GMS01]

Proposition 3.14. For any $k \neq 0$, the sum of the Sugawara vectors of $V_{-k} \cdot (\mathfrak{g})^\ell$ and $V_{k} \cdot (\mathfrak{g})^r$ equals the conformal vector (3.15) of $\mathcal{D}_k^\chi(G)$.

Proof. Let us outline the calculation without going through all the details. The assumption $k \neq 0$ is precisely the condition for either affine vertex algebra to admit a Sugawara vector. [Kac98] [FB04] In addition to the notations introduced for the expression in (3.15), also let $t^\ell_1, t^\ell_2, \ldots$ denote the dual basis of $\mathfrak{g}$ with respect to $\lambda_0$ and $\theta^\ell_1, \theta^ll_2, \ldots$ (resp. $\theta^r_1, \theta^r_2, \ldots$) the corresponding components of $\theta^\ell$ (resp. $\theta^r$). By definition, the Sugawara vectors of $V_{-k} \cdot (\mathfrak{g})^\ell$ and $V_{k} \cdot (\mathfrak{g})^r$ are respectively

$$\nu^\ell = -\frac{1}{2k} \left( t_1^\ell + h_a^\ell \right)_{-1} (t_{a, \ell} + h_a^\ell) = -\frac{1}{2k} t_{a, \ell}^\ell - t_{a, \ell}^\ell + \frac{2k + h^\ell}{4k} t_{a, -1}^\ell \theta_{a, -1}^{\alpha, \ell} - \frac{(2k + h^\ell)^2}{32k} \theta_{a, -1}^{\alpha, \ell}$$

$$\nu^r = \frac{1}{2k} \left( - t_{a, r}^r + h_a^r \right)_{-1} (- t_{a, r}^r + h_a^r) = \frac{1}{2k} t_{a, -1}^r \theta_{a, -1}^{\alpha, r} + \frac{2k + h^r}{4k} t_{a, -1}^r \theta_{a, -1}^{\alpha, r} + \frac{(2k + h^r)^2}{32k} \theta_{a, -1}^{\alpha, r}$$

In order to express the quantities with superscript $r$ in terms of those with superscript $\ell$, let $\rho_b^b = \theta^b(b t^\ell_b)$. By (A.3) and the definition of $\nabla$,

$$t_a^\ell = \rho_a^b t^\ell_b = t_{b, -1} \rho_a^b - t^\ell_b \bullet \rho_a^b = t_{b, -1} \rho_a^b - (\nabla t^\ell_b) \rho_a^b = t_{b, -1} \rho_a^b - h^\ell \theta_{a}^{\ell, r}$$

Then we have the following calculations:

$$\theta_{a, -1}^{\alpha, \ell} = (\rho_a^b \theta_b^b)_{-1} (\rho_{-1}^a \theta_c^c)_{-1} = \theta_c^{\ell, \ell}$$

$$t_{a, -1}^{\alpha, \ell} = (t_{b, -1} \rho_a^b - h^\ell \theta_{a}^{\ell, r})_{-1} = (t_{b, -1} \rho_a^b - h^\ell \theta_{a}^{\ell, r})_{-1}$$

$$t_{a, -1}^{\alpha, r} = (t_{b, -1} \rho_a^b - h^\ell \theta_{a}^{\ell, r})_{-1} = (t_{b, -1} \rho_a^b - h^\ell \theta_{a}^{\ell, r})_{-1}$$

$$t_{a, -1}^{\ell, r} = (t_{b, -1} \rho_a^b - h^\ell \theta_{a}^{\ell, r})_{-1} t_{a, -1}^{\alpha, r}$$

$$= t_{b, -1} \rho_a^b - h^\ell \theta_{a}^{\ell, r} = t_{b, -1} \rho_a^b - h^\ell \theta_{a}^{\ell, r}$$

This is now easy to see that $\nu^\ell + \nu^r$ coincides with either expression in (3.15).
Preparation. (i) For $k' \in \mathbb{C}$, we will write $\mathfrak{g}_{k'}$ for $\mathfrak{g}_{k'\lambda_0}$ (see (3.11) and $U(\mathfrak{g})_{k'}$ for its universal enveloping algebra. Let $\mathfrak{g}_+ = \mathfrak{g}[t]$ and $\mathfrak{g}_{++} = t \cdot \mathfrak{g}[t]$. (ii) Let $\mathcal{P}$ denote the set of dominant weights of $G$ (with respect to a choice of maximal torus and Weyl chamber). For $\mu \in \mathcal{P}$ and $k' \in \mathbb{C}$, let us write $\mu^*$ for $-w_{\text{max}}(\mu)$, where $w_{\text{max}}$ is the longest element of the Weyl group; $M_{\mu}$ for the irreducible $G$-module of highest weight $\mu$; and $M_{k',\mu}$ for the irreducible positive-energy $(\mathfrak{g}_{k'}, G)$-module of highest weight $\mu$. (iii) To avoid confusion, eigenvalues of $L_0$ will be referred to as conformal weights here.

**Proposition 3.15.** For any $k \notin \mathbb{Q}$, there is a $(\mathfrak{g}_{-k-h^\vee} \oplus \hat{\mathfrak{g}}_{-k-h^\vee}, G \times G)$-equivariant embedding

$$\bigoplus_{\mu \in \mathcal{P}} M_{-k-h^\vee,\mu} \otimes M_{k-h^\vee,\mu^*} \hookrightarrow D^\text{ch}_k(G)$$

and its image is “dense” in the sense described below. Notice that the summand with $\mu = 0$ is the tensor product of (3.17) and (3.19).

**Proof.** By Example 3.12 and Proposition 3.13, the $G \times G$-action on $C^\infty(G)$ coming from left and right multiplications extends to a $(\mathfrak{g}_{-k-h^\vee} \oplus \hat{\mathfrak{g}}_{-k-h^\vee}, G \times G)$-action on $D^\text{ch}_k(G)$. According to the Peter-Weyl theorem, there is a canonical $G \times G$-equivariant embedding

$$\bigoplus_{\mu \in \mathcal{P}} M_{\mu} \otimes M_{\mu^*} \cong \bigoplus_{\mu \in \mathcal{P}} M_{\mu} \otimes M_{\mu}^\vee \hookrightarrow C^\infty(G)$$

whose image is dense in the $L^\infty$-topology. This induces a $(\mathfrak{g}_{-k-h^\vee} \oplus \hat{\mathfrak{g}}_{-k-h^\vee}, G \times G)$-equivariant map

$$\left( U(\hat{\mathfrak{g}})_{-k-h^\vee} \otimes U(\hat{\mathfrak{g}})_{k-h^\vee} \right) \otimes_{U(\hat{\mathfrak{g}}_+) \otimes U(\hat{\mathfrak{g}}_+)} \left( \bigoplus_{\mu \in \mathcal{P}} M_{\mu} \otimes M_{\mu^*} \right) \rightarrow D^\text{ch}_k(G).$$

If $k' \notin \mathbb{Q}$, then it follows from a consideration of the Casimir operator that any $\mathfrak{g}_{k'}$-module of the form $U(\hat{\mathfrak{g}})_{k'} \otimes U(\hat{\mathfrak{g}}_+)$ contains no proper submodule. Therefore the above induced map is equivalent to a map of the form (3.20).

For $D^\text{ch}_k(G)$, the filtration described in (A.11) splits $G \times G$-equivariantly with the use of (say) left-invariant vector fields and 1-forms. This results in a $G \times G$-equivariant isomorphism

$$\bigoplus_{\nu \geq 0} q^\nu D^\text{ch}_k(G)_\nu \cong C^\infty(G) \otimes \left( \bigotimes_{i \geq 1} \text{Sym}_{q^i}(\mathfrak{g} \oplus \mathfrak{g}^\vee) \right)$$

where the first $G$ acts on the symmetric powers in the obvious way and the second $G$ acts trivially there. Using this isomorphism, the $L^\infty$-topology on $C^\infty(G)$ induces a topology on $D^\text{ch}_k(G)$ that is compatible with the vertex algebra structure. If the image of (3.20) is not dense, then we can find some nontrivial $G \times G$-submodule $\mathcal{M} \subset D^\text{ch}_k(G)$ with no intersection with the said image. Also, we may assume that $\mathcal{M}$ is irreducible and, among all such submodules, has the lowest conformal weight, which is necessarily positive. Notice that $\mathfrak{g}_{++}$ acts trivially on $\mathcal{M}$. By Proposition 3.14

$$L_0 = \nu_0 + \nu_0^* = \frac{1}{2\pi}(-\Omega_0' + \Omega_0'') \quad \text{on} \quad \mathcal{M}$$

where $\Omega_0'$, $\Omega_0''$ are the Casimir numbers of the two $\mathfrak{g}$-actions. Since $\Omega_0', \Omega_0'' \in \mathbb{Q}$ but $k \notin \mathbb{Q}$, the conformal weight of $\mathcal{M}$ must be zero, giving a contradiction. This proves that (3.20) has a dense image.

**Remark.** This is a reproduction of one of the “chiral Peter-Weyl theorems” in [FS06]. It will be very interesting to understand the representation-theoretic meaning of the vertex algebra structure on (3.20), as well as to find an appropriate extension of the result for $k \in \mathbb{Q}$; see op. cit. for some conjectures.
§ 4. CDOs on Principal Bundles

In this section, our object of study is an algebra of CDOs $\mathcal{D}^{ch}(P)$ on the total space of a principal bundle $P \to M$ when it is equipped with a “formal loop group action” in the sense of [3]. The first goal is to understand the subalgebra of $\mathcal{D}^{ch}(P)$ invariant under the action. In subsequent sections we will construct and study modules over this invariant subalgebra. The second (and more technical) goal is to give, when $P \to M$ is a principal frame bundle, an alternative description of $\mathcal{D}^{ch}(P)$ that makes the “formal loop group action” manifest. This type of vertex algebras will play a central role in the rest of the paper.

§ 4.1. Setting: principal bundle. Let $G$ be again a compact connected Lie group and $\pi : P \to M$ a smooth principal $G$-bundle. Identify $H^*_G(P)$ with $H^*(M)$ via $\pi^*$. Choose a connection $\Theta$ on $\pi$, i.e. some $\Theta \in (\Omega^1(P) \otimes \mathfrak{g})^G$ such that $\Theta(A^P) = A$ for $A \in \mathfrak{g}$; its curvature is $\Omega = d\Theta + \frac{1}{2}[\Theta \wedge \Theta]$. Keep in mind that $\Theta$ is equivalent to a $G$-equivariant vector bundle decomposition $TP = T_hP \oplus T_vP$, where $T_hP = \ker \Theta$ and $T_vP = \ker \pi_*$; and $\Omega$ measures the non-integrability of the subbundle $T_hP$. Given $X \in T(M)$, denote its horizontal lift by $\tilde{X} \in T_h(P)^G$. Let us extend the notation $(-)^P$ (see § 3.1) $C^\infty(P)$-linearly to denote the isomorphism $C^\infty(P) \otimes \mathfrak{g} \cong T_v(P)$. For $X, Y \in T(M)$ notice that

\[ [\tilde{X}, \tilde{Y}] - [\tilde{X}, Y] = -\Omega(\tilde{X}, \tilde{Y})^P. \]

Also recall the notations introduced in § 3.2.

Consider an algebra of CDOs $\mathcal{D}^{ch}(P) = \mathcal{D}^{ch}_{\nabla, H}(P)$ defined as in § 3.3. Let $\lambda \in (\text{Sym}^2 \mathfrak{g}^\vee)^G$. Recall from Definition 3.4 and the proof of Theorem 3.11 the meaning of an inner $(\hat{\mathfrak{g}}, G)$-action on $\mathcal{D}^{ch}(P)$ and its associated Cartan cochain. It will be understood without further comment that any inner $(\hat{\mathfrak{g}}, G)$-action on $\mathcal{D}^{ch}(P)$ considered below extends the given $G$-action on $\mathcal{D}^{ch}(P)_0 = C^\infty(P)$.

Lemma 4.2. $8\pi^2 p_1(P)_G = 8\pi^2 p_1(M) - \lambda_{ad}(P)$.

Proof. Recall the G-equivariant decomposition $TP = T_hP \oplus T_vP$. Since $T_hP \cong \pi^*TM$ and $\pi^*$ identifies $H^*_G(P)$ with $H^*(M)$, we have $p_1(T_hP)_G = p_1(TM)$. Since $T_vP \cong P \times \mathfrak{g}$ with $G$ acting on $\mathfrak{g}$ in the adjoint representation, we also have $-8\pi^2 p_1(T_vP)_G = \lambda_{ad}(P)$ (see § 4.3.2). This proves the lemma. □

By this lemma, Theorem 3.11 specializes to our current setting as follows.

Corollary 4.3. The $G$-action on $P$ lifts to an inner $(\hat{\mathfrak{g}}_\Lambda, G)$-action on $\mathcal{D}^{ch}(P)$ if and only if

\[ 8\pi^2 p_1(M) = (\lambda + \lambda_{ad})(P). \]

Remark. For convenience, when $\mathcal{D}^{ch}(P)$ is equipped with an inner $(\hat{\mathfrak{g}}_\Lambda, G)$-action, we will refer to it as a principal $(\hat{\mathfrak{g}}_\Lambda, G)$-algebra.

§ 4.4. The invariant subalgebra. Suppose $\mathcal{D}^{ch}(P)$ is a principal $(\hat{\mathfrak{g}}_\Lambda, G)$-algebra, i.e. it is given an inner $(\hat{\mathfrak{g}}_\Lambda, G)$-action $V_\Lambda(\mathfrak{g}) \hookrightarrow \mathcal{D}^{ch}(P)$, defined by some associated Cartan cochain $h \in (\mathfrak{g}^\vee \otimes \Omega^1(P))^G$. Consider the centralizer subalgebra

\[ \mathcal{D}^{ch}(P)_h := \text{C}(\mathcal{D}^{ch}(P), V_\Lambda(\mathfrak{g})) \subset \mathcal{D}^{ch}(P). \]

[Kac98] [FB04] This is the subalgebra whose fields are $\hat{\mathfrak{g}}_\Lambda$-invariant. The weight-zero component consists of $f \in C^\infty(P)$ satisfying $(A^P + h_\Lambda) f = A^P f = 0$ for $A \in \mathfrak{g}$, i.e.

\[ \mathcal{D}^{ch}(P)^0_\Lambda = C^\infty(P)^G = \pi^*C^\infty(M). \]

The weight-one component consists of $\alpha + X \in \Omega^1(P) \oplus T(P)$ satisfying

\[ \begin{cases} (A^P + h_\Lambda)_0(\alpha + X) = L_{A^P} \alpha + [A^P, X] = 0 \\ (A^P + h_\Lambda)_1(\alpha + X) = \alpha(A^P) + \{A^P, X\} + h_\Lambda(X) = 0 \end{cases} \]

for $A \in \mathfrak{g}$.
By Proposition 3.13, they contain the following subalgebras over the Atiyah algebroid (\(C\) will be used. Consider the centralizers of the inner (\(\hat{\mathfrak{g}}\)) Example 4.5. CDOs on a Lie group. Other elements or not. (However, see Example 4.5.) In subsequent sections we will construct and study such modules over \(P\), i.e. \(\pi^*\Omega^1(M)\). This implies the short exact sequence

\[
\begin{array}{c}
0 \longrightarrow \pi^*\Omega^1(M) \longrightarrow \mathcal{D}^{\text{ch}}(P)^{\hat{\mathfrak{g}}} \longrightarrow \mathcal{T}(P)^G \longrightarrow 0
\end{array}
\]

In view of (4.3) and (4.4), the vertex algebroid associated to \(\mathcal{D}^{\text{ch}}(P)^{\hat{\mathfrak{g}}}\) (see (4.5) is of the form

\[
(C^{\infty}(M), \Omega^1(M), \mathcal{T}(P)^G, \cdots).\]

Since \(\mathcal{D}^{\text{ch}}(P)\) is freely generated by its associated vertex algebroid (see (A.7), \(\mathcal{D}^{\text{ch}}(P)^{\hat{\mathfrak{g}}}\) must contain at least a subalgebra that is freely generated by (4.3). In general, it is not clear whether \(\mathcal{D}^{\text{ch}}(P)^{\hat{\mathfrak{g}}}\) may contain other elements or not. (However, see Example 4.5)

Remarks. (i) By the above discussion, any module over \(\mathcal{D}^{\text{ch}}(P)^{\hat{\mathfrak{g}}}\) contains in its lowest weight a module over the Atiyah algebroid \((C^{\infty}(M), \mathcal{T}(P)^G)\), such as the space of sections of an associated vector bundle. In subsequent sections we will construct and study such modules over \(\mathcal{D}^{\text{ch}}(P)^{\hat{\mathfrak{g}}}\). (ii) It is noteworthy that \(\mathcal{D}^{\text{ch}}(P)^{\hat{\mathfrak{g}}}\) is different from an algebra of CDOs on the base manifold \(M\) — compare (4.4) and (2.9) — and admits more interesting modules.

Example 4.5. CDOs on a Lie group. This is a continuation of Example 3.12 and the same notations will be used. Consider the centralizers of the inner \((\mathfrak{g}, G)\)-actions (3.17) and (3.19):

\[
\mathcal{D}^{\text{ch}}_k(G)^{\hat{\mathfrak{g}}, \ell} = C(\mathcal{D}^{\text{ch}}_k(G), V_{-k-h^\vee}(\mathfrak{g})^{\ell}), \quad \mathcal{D}^{\text{ch}}_k(G)^{\hat{\mathfrak{g}}, r} = C(\mathcal{D}^{\text{ch}}_k(G), V_{k-h^\vee}(\mathfrak{g})^r).
\]

By Proposition 3.13 they contain the following subalgebras

\[
V_{-k-h^\vee}(\mathfrak{g})^r \subset \mathcal{D}^{\text{ch}}_k(G)^{\hat{\mathfrak{g}}, \ell}, \quad V_{k-h^\vee}(\mathfrak{g})^{\ell} \subset \mathcal{D}^{\text{ch}}_k(G)^{\hat{\mathfrak{g}}, r}.
\]

For \(k \neq 0\), both affine vertex algebras admit Sugawara vectors. By the coset construction and Proposition 3.14, the Sugawara vectors are also conformal for the respective centralizers. \footnote{The author is grateful to a reviewer for pointing this out to him.} In fact, for \(k \notin \mathbb{Q}\), it follows from Proposition 3.15 that

\[
V_{-k-h^\vee}(\mathfrak{g})^r = \mathcal{D}^{\text{ch}}_k(G)^{\hat{\mathfrak{g}}, \ell}, \quad V_{k-h^\vee}(\mathfrak{g})^{\ell} = \mathcal{D}^{\text{ch}}_k(G)^{\hat{\mathfrak{g}}, r}.
\]

6 These may be viewed as examples of a version of the Borel-Weil construction.

§ 4.6. Setting refined: principal frame bundle. In addition to the data described in (4.1) suppose we also have a representation \(\rho : G \to SO(\mathbb{R}^d)\) together with an isomorphism \(P \times_\rho \mathbb{R}^d \cong TM\). This induces a Riemannian metric on \(M\) and an orthogonal connection on \(TM\), which (for simplicity) is assumed to be torsion-free. Let \([p,v] \in TM\) denote the coset of \((p,v) \in P \times \mathbb{R}^d\). For \(i = 1, \ldots, d\), let \(e_i \in \mathbb{R}^d\) be the standard basis vectors and \(\tau_i \in T_0(P)\) the tautological vector fields defined by

\[
\tau_i|_p := \text{horizontal lift of } [p, e_i],
\]
which constitute a framing of $T_h P$. For $A \in \mathfrak{g}$ and $i, j = 1, \ldots, d$, we have the Lie brackets

$$[A^\gamma, \tau_i] = \rho(A)_{ji} \tau_j, \quad [\tau_i, \tau_j] = -\Omega(\tau_i, \tau_j)^\gamma$$

where we use $\rho$ to also denote the induced map of Lie algebras $\mathfrak{g} \to \mathfrak{so}_d$.

Let us define an algebra of CDOs on $P$ as in [13] using some more specific data (and slightly different notations). Let $\nabla'$ be the $G$-invariant flat connection on $TP$ with respect to which all $A^\gamma$ and $\tau_i$ are parallel, and $H'$ be a $G$-invariant closed 3-form on $P$. By Theorem 2.5, $\nabla'$ and $H'$ determine an algebra of CDOs $\mathcal{D}^{ch}(P)' = \mathcal{D}^{ch}_{\nabla', H'}(P)$, which is freely generated by a vertex algebroid

$$(C^\infty(P), \Omega^1(P), T(P), \cdot', \{ \}$, $\}$)$$

(see [A.7]). The structure of this vertex algebroid is, in view of the axioms (see Definition A.3), entirely determined by the portion displayed in the ensuing lemma. By Theorem 2.6, the trivial 1-form determines a conformal vector in $\mathcal{D}^{ch}(P)'$, namely

$$\nu' = t_{a,-1}^P \Theta_a + \tau_i,_{-1} t^i$$

where $t_1, t_2, \ldots$ are a basis of $\mathfrak{g}$; $\Theta^1, \Theta^2, \ldots$ the corresponding components of $\Theta$, so that $\Theta = \Theta^a \otimes t_a$; $\tau_1, \ldots, \tau_d$ the horizontal vector fields defined above; and $\tau_1, \ldots, \tau_d$ the dual horizontal 1-forms. In the rest of this section, we will provide a more detailed description of this conformal vertex algebra in the presence of an inner $(\hat{\mathfrak{g}}\lambda, G)$-action.

**Lemma 4.7.** For $f \in C^\infty(P)$, $A, B \in \mathfrak{g}$ and $i, j = 1, \ldots, d$, we have

- $A^\gamma \cdot f = \tau_i \cdot f = 0$
- $\{A^\gamma, B^\gamma\}' = -\lambda_{ad}^\gamma(A, B)$; $\{A^\gamma, \tau_i\}' = 0$; $\{\tau_i, \tau_j\}' = 2\text{Ric}_{ij}$
- $\{A^\gamma, B^\gamma\}_\Omega = \frac{1}{2}t_{A^\gamma, B^\gamma}H'$; $\{A^\gamma, \tau_i\}_\Omega = \frac{1}{2}t_{A^\gamma, \tau_i}H'$; $\{\tau_i, \tau_j\}_\Omega = d\text{Ric}_{ij} + \frac{1}{2}t_{\tau_i, \tau_j}H'$

where $\text{Ric}_{ij} = \rho(\Omega(\tau_i, \tau_j))_{ik}$. Moreover, we also have $\nu_1^\gamma A^\gamma = \nu_1^\gamma \tau_i = 0$.

**Proof.** The structure maps are defined by (2.6) and (2.8). Then the calculations follow easily from our definition of $\nabla'$, the Lie brackets (4.7) and the symmetries of the Riemannian curvature tensor $\rho(\Omega)$. Theorem 3.11 specializes to our current setting as follows.

**Corollary 4.8.** The $G$-action on $P$ lifts to an inner $(\hat{\mathfrak{g}}\lambda, G)$-action on $\mathcal{D}^{ch}(P)'$ if and only if

$$(\lambda + \lambda_{ad} + \lambda_\rho)(P) = 0.$$  

Moreover, this action is primary with respect to the conformal vector $\nu'$.

**Proof.** By assumption, $\pi : P \to M$ is the lifting of the special orthogonal frame bundle $F_{SO}(TM) \to M$ along $\rho : G \to SO_d$. This gives us a commutative diagram

$$\begin{align*}
H^4(BSO_d) & \xrightarrow{\rho^*} H^4(BG) \\
H^4(M) & \xrightarrow{=} H^4_{SO_d}(F_{SO}(TM)) \xrightarrow{=} H^4_G(P)
\end{align*}$$

By definition, $\lambda_\rho \in H^4(BG)$ is the image of $-8\pi^2 p_1 \in H^4(BSO_d)$ under $\rho^*$ (see [12]), which implies that $-8\pi^2 p_1(M) = \lambda_\rho(P)$. Now the first claim becomes a special case of Corollary 4.3. The other claim is true by Lemma 1.7.

---

7 The absence of a horizontal component in $[\tau_i, \tau_j]$ is equivalent to the torsion-free assumption. Moreover, the Jacobi identity for any $\tau_i, \tau_j, \tau_k$ is equivalent to the two Bianchi identities.
Remark. In the sequel we will write $\lambda^* = \lambda + \lambda_{ad} + \lambda_{\rho}$.

§ 4.9. Formal loop group action. Suppose $D^{ch}(P)'$ is a principal $(\hat{g}_\lambda, G)$-algebra, i.e. it is given an inner $(\hat{g}_\lambda, G)$-action $V_\lambda(\mathfrak{g}) \rightarrow D^{ch}(P)'$, defined by some associated Cartan cochain $h \in (\mathfrak{g}^\vee \otimes \Omega^1(P))^G$. Notice that the Cartan cochains in (3.2) may now be written as $\chi^{2,2} = \frac{1}{2\iota}d_GH'$ and $\chi^{4,0} = -\lambda_{ad} - \lambda_{\rho}$ by Lemma 4.7. Hence the condition (4.11) on $h$ is now equivalent to the equation

\begin{equation}
(4.10) \quad d_G(H' - 2h) = \lambda^*.
\end{equation}

This of course agrees with Corollary 4.18. The purpose of the next two lemmas is to modify the description of the vertex algebra $D^{ch}(P)'$ in such a way that the inner $(\hat{g}_\lambda, G)$-action becomes manifest and the data $(H', h)$ involved are replaced by a basic 3-form that trivializes $\lambda^*(\Omega \wedge \Omega)$.

Lemma 4.10. There is a natural bijection between the following two sets of data:

(i) $(H', h) \in \Omega^3(P)^G \oplus (\mathfrak{g}^\vee \otimes \Omega^1(P))^G$ such that $d_G(H' - 2h) = \lambda^*$, and

(ii) $(H, \beta) \in \pi^*\Omega^3(M) \oplus \Omega^2(P)^G$ such that $dH = \lambda^*(\Omega \wedge \Omega)$ and $\beta|\tau_{\iota} P = 0$.

Proof. Once the maps in both directions are described below, it will be clear that they are inverse to each other. Let $CS_{\lambda^*}(\Theta) = \lambda^*(\Theta \wedge \Omega) - \frac{1}{2}\lambda(\Theta \wedge [\Theta \wedge \Theta])$. Notice that as Cartan cochains $CS_{\lambda^*}(\Theta) \in \Omega^3(P)^G$ and $\lambda^*(-, \Theta) \in (\mathfrak{g}^\vee \otimes \Omega^1(P))^G$ have the following differentials

\begin{equation}
(4.11) \quad d_G CS_{\lambda^*}(\Theta) = \lambda^*(\Omega \wedge \Omega) - \lambda^*(-, d\Theta), \quad d_G \lambda^*(-, \Theta) = \lambda^*(-, d\Theta) - \lambda^*.
\end{equation}

From (i) to (ii). Let $(H', h)$ be a pair as in (i). First define a $G$-invariant 2-form on $P$ by

\begin{equation}
(4.12) \quad \beta(A^P, B^P) = h_A(B^P) - h_B(A^P), \quad \beta(A^P, \tau_i) = 2h_A(\tau_i), \quad \beta(\tau_i, \tau_j) = 0
\end{equation}

for $A, B \in \mathfrak{g}$ and $i, j = 1, \ldots, d$. Indeed, the $G$-equivariance of $h$ and (4.11) imply the $G$-invariance of $\beta$. It follows from $d_G(H' - 2h) = \lambda^*$ that

\begin{equation}
(4.13) \quad \iota_{A^P} \beta = 2h_A - \lambda^*(A, \Theta) \quad \Rightarrow \quad d_G \beta = d\beta - 2h + \lambda^*(-, \Theta).
\end{equation}

Then define a $G$-invariant 3-form on $P$ by

\begin{equation}
(4.14) \quad H = H' - d\beta + CS_{\lambda^*}(\Theta).
\end{equation}

It follows from $d_G(H' - 2h) = \lambda^*$, (4.13) and (4.11) that $d_G H = \lambda^*(\Omega \wedge \Omega)$. Equivalently, $H$ is horizontal (hence basic) and satisfies $d\Theta = \lambda^*(\Omega \wedge \Omega)$.

From (ii) to (i). Given $(H, \beta)$ as in (ii), define $h$ by (4.13) and $H'$ by (4.14).

Preparation. (i) Let $\beta \in \Omega^2(P)^G$ and $H \in \pi^*\Omega^3(M)$ be given by applying the construction in Lemma 4.10 to the data $(H', h)$ described in (4.6) and (4.9). (ii) Define a $C^\infty(P)$-linear map $\Delta : T(P) \rightarrow \Omega^1(P)$ as follows: $\Delta(A^P) = -h_A$ for $A \in \mathfrak{g}$ and $\Delta(\tau_i) = -\frac{1}{2}\iota_{\tau_i} \beta$ for $i = 1, \ldots, d$. By Lemma 4.9 $\Delta$ determines an isomorphism $(\iota d, \Delta)$ from the vertex algebroid (4.8) to a new vertex algebroid

\begin{equation}
(4.15) \quad (C^\infty(P), \Omega^1(P), T(P), \bullet, \{ \}, \{ \})_\Omega.
\end{equation}

This in turn induces an isomorphism from $D^{ch}(P)'$ to the vertex algebra freely generated by (4.15), which will be (temporarily) denoted by $D^{ch}(P)$. By composition, it has an inner $(\hat{g}_\lambda, G)$-action $V_\lambda(\mathfrak{g}) \rightarrow D^{ch}(P)$ whose associated Cartan cochain is trivial, i.e. it simply takes any $A \in \mathfrak{g} = V_\lambda(\mathfrak{g})_1$ to $A^P$.

Lemma 4.11. For $f \in C^\infty(P)$, $A, B \in \mathfrak{g}$ and $i, j = 1, \ldots, d$, we have

\begin{itemize}
  \item $A^P \bullet f = \tau_i \bullet f = 0$
  \item $\{A^P, B^P\} = \lambda(A, B); \quad \{A^P, \tau_i\} = 0; \quad \{\tau_i, \tau_j\} = 2\text{Ric}_{ij}$
  \item $\{A^P, B^P\}_\Omega = \{A^P, \tau_i\}_\Omega = 0; \quad \{\tau_i, \tau_j\}_\Omega = d\text{Ric}_{ij} + \frac{1}{2}\iota_{\tau_i} \iota_{\tau_j} H + \lambda^*(\Omega(\tau_i, \tau_j), \Theta)$
\end{itemize}
Proof. The structure of \((4.15)\) is determined by the structure of \((4.8)\) (see Lemma 4.7) together with the isomorphism \((\text{id}, \Delta) : (4.8) \rightarrow (4.15)\) (see Definition 4.4 and the definition of \(\Delta\)). The claim for \(\bullet\) is easy. Recall the map \((i, h)\) in Proposition 4.8. Since \((\text{id}, \Delta) \circ (i, h) = (i, 0)\), we have \(\{A^p, B^p\} = \lambda(A, B)\) and \(\{A^p, B^p\}_\Omega = 0\). The other values of \(\{ \cdot \} \) are computed as follows

\[
\{A^p, \tau_i\} = \{A^p, \tau_i\}' + h_A(\tau_i) + \frac{1}{2} \beta(\tau_i, A^p) = 0
\]

\[
\{\tau_i, \tau_j\} = \{\tau_i, \tau_j\}' + \frac{1}{2} \beta(\tau_i, \tau_j) + \frac{1}{2} \beta(\tau_j, \tau_i) = 2\text{Ric}_{ij}
\]

using the definition of \(\beta (4.12)\). The other values of \(\{ \cdot \}\) are computed as follows

\[
\{A^p, \tau_i\}_\Omega = \{A^p, \tau_i\}'_\Omega + \frac{1}{2} L_{A^p} \tau_i \beta - L_\tau h_A + dh_A(\tau_i) - \frac{1}{2} h_{[A^p, \tau_i]} \beta
\]

\[
= \frac{1}{2} L_{A^p} \tau_i \beta - \frac{1}{2} h_{[A^p, \tau_i]} \beta + \frac{1}{2} dh_A(\tau_i)
\]

\[
\{\tau_i, \tau_j\}_\Omega = \{\tau_i, \tau_j\}'_\Omega + \frac{1}{2} L_{\tau_i} \tau_j \beta - \frac{1}{2} L_{\tau_j} \tau_i \beta + \frac{1}{2} d\beta(\tau_i, \tau_j) + h_U(\tau_i, \tau_j)
\]

\[
= \frac{1}{2} L_{\tau_i} \tau_j \beta + \frac{1}{2} L_{\tau_j} \tau_i \beta + \frac{1}{2} d\beta(\tau_i, \tau_j) + h_U(\tau_i, \tau_j)
\]

using, in order: the \(G\)-invariance of \(\beta\); the component of \((4.10)\) in \((g^\vee \otimes \Omega^2(P))^G\); the identity

\[
L_\tau \tau_i - L_\tau \tau_i = L_\tau \tau_i - \tau_i d_\tau = L_\tau \tau_i - \tau_i d_\tau = L_\tau \tau_i - \tau_i d_\tau = \tau_i d_\tau = \tau_i d_\tau = \tau_i d_\tau = \tau_i d_\tau = \tau_i d_\tau = \tau_i d_\tau = \tau_i d_\tau = \tau_i d_\tau = \tau_i d_\tau = \tau_i d_\tau
\]

the Lie brackets \((4.7); equation (4.13); and finally the definition of \(H (4.14)\).

Remark. The bijection in Lemma 4.10 also means that any \(H \in \pi^* \Omega^1(M)\) satisfying \(dH = \lambda^*(\Omega \wedge \Omega)\) determines a principal \((\hat{\lambda}^*, G)\)-algebra whose associated vertex algebroid structure is as in Lemma 4.11

Corollary 4.12. In the vertex algebra \(D^{\text{ch}}(P)\) we have the following normal-ordered expansions and commutation relations

\[
(f A^p)_n = \sum_{k \geq 0} f_{n-k} A^p_k + \sum_{k < 0} A^p_k f_{n-k}, \quad (f \tau_i)_n = \sum_{k \geq 0} f_{n-k} \tau_i k + \sum_{k < 0} \tau_i k f_{n-k}
\]

\[
[A^p, B^p_m] = [A, B]_{n+m} + n \lambda(A, B) \delta_{n+m, 0}, \quad [A^p, \tau_i m] = \rho(A)_{ji} \tau_j, n+m
\]

\[
[\tau_{i,n}, \tau_{j,m}] = -\Omega(\tau_i, \tau_j)_{n+m} + (n-m) \text{Ric}_{ij} n+m + \left(\frac{1}{2} L_{\tau_i} \tau_j \beta + \frac{1}{2} \beta(\tau_i, \tau_j) \right) n+m
\]

for \(f \in C^\infty(P); A, B \in g; i, j = 1, \ldots, d; \) and \(m, n \in \mathbb{Z}\).

Proof. This follows immediately from Lemma 4.11 and Definition 4.7.

Lemma 4.13. The vertex algebra \(D^{\text{ch}}(P)\) has a conformal vector of central charge \(2 \dim P\):

\[
\nu = t_{a,-1}^p \Theta^a + \tau_{i,-1} \tau^i - \frac{1}{2} \lambda^*(\Theta_{-1} \Theta).
\]

Moreover, we have \(\nu_1 A^p = \nu_1 \tau_i = 0\) for \(A \in g\) and \(i = 1, \ldots, d\).

Proof. By construction, the isomorphism from \(D^{\text{ch}}(P)\)’ to \(D^{\text{ch}}(P)\) (see the paragraph preceding Lemma 4.11) takes the conformal vector \(\nu^1\) in \((\mathfrak{g}^\vee \otimes C^\infty(P))^G\) to

\[
\nu = (t_{a,-1}^p - h_{\nu_1}) \Theta^a + (\tau_i - \frac{1}{2} L_{\tau_i} \beta) \tau^i
\]

\[
= t_{a,-1}^p \Theta^a + \tau_{i,-1} \tau^i - (h_{\nu_1} (t^b_a) \Theta^b + h_{\nu_1} (\tau_i) \tau^i) \tau_{-1} \tau^i - \frac{1}{2} \left(\beta(\tau_i, t^b_a) \Theta^a + \beta(\tau_i, \tau_{-1}) \tau^i \right) \tau_{-1} \tau^i
\]

\[
= t_{a,-1}^p \Theta^a + \tau_{i,-1} \tau^i - \frac{1}{2} \lambda^*(t_a, t_b) \Theta^b \tau_{-1} \tau^i - h_{\nu_1} (\tau_i) \tau^i \tau_{-1} \Theta^a + h_{\nu_1} \Theta^a \tau^i \tau_{-1} \Theta^a + 0
\]

\[
= t_{a,-1}^p \Theta^a + \tau_{i,-1} \tau^i - \frac{1}{2} \lambda^*(\Theta_{-1} \Theta)
\]

where we have used, in order: the component of \((4.10)\) in \((\text{Sym}^2 g^\vee \otimes C^\infty(P))^G\); the symmetry \(\alpha_{-1} \alpha = \alpha_{-1} \alpha\) for 1-forms \(\alpha, \alpha\); and the definition of \(\beta (4.12)\). The said isomorphism also takes \(\nu_1 (A^p + h_A)\) to \(\nu_1 A^p\) and \(\nu_1 (\tau_i + \tau_{i,-1})\) to \(\nu_1 \tau_i\). This proves our second claim in view of Lemma 4.7 and (2.8).
The following summarizes all the discussions since §4.6.

**Theorem 4.14.** Suppose we have: a principal G-bundle \( \pi : P \to M \), a representation \( \rho : G \to SO(\mathbb{R}^d) \) and an isomorphism \( P \times_\rho \mathbb{R}^d \cong TM \); a connection \( \Theta \) on \( \pi \) that (for simplicity) induces a torsion-free connection on \( TM \); an invariant symmetric bilinear form \( \lambda \) on \( \mathfrak{g} \) and a basic 3-form \( H \) on \( P \) satisfying \( dH = \lambda^*(\Omega \wedge \Omega) \), where \( \Omega \) is the curvature of \( \Theta \) and \( \lambda^* = \lambda + \lambda_{\text{ad}} + \lambda_\rho \) (see §4.2).

These data determine a vertex algebra \( \mathcal{D}_{\Theta,H}^\text{ch}(P) \) as follows. There are generating fields

\[
\Phi_f(z) \text{ for } f \in C^\infty(P), \quad \Phi_{A^f}(z) \text{ for } A \in \mathfrak{g}, \quad \Phi_{\tau_i}(z) \text{ for } i = 1, \ldots, d
\]

(see §4.7 and §4.6) whose weights are 0, 1 and 1 respectively. The vacuum is \( 1 \in C^\infty(P) \). The OPEs with \( \Phi_f(z) \) have the following leading terms

\[
\Phi_f(z)\Phi_g(w) = \Phi_{fg}(w) + \Phi_{f\partial g}(w)(z - w) + O((z - w)^2), \quad g \in C^\infty(P)
\]

\[
\Phi_{A^f}(z)\Phi_{A^g}(w) = \Phi_{A^{fg}}(w) + \Phi_{A^{\partial g}}(w) + O(z - w)
\]

\[
\Phi_{\tau_i}(z)\Phi_f(w) = \Phi_{\tau_i f}(w) + \Phi_{f\tau_i}(w) + O(z - w)
\]

where the three second terms (plus linearity) are meant to be the definition of fields associated to arbitrary 1-forms and vector fields. The OPEs of the other generating fields have the following singular parts

\[
\Phi_{A^{\partial f}}(z)\Phi_{B^g}(w) \sim \frac{\lambda(A, B)}{(z - w)^2} \Phi_{[A, B]^{\partial g}}(w)
\]

\[
\Phi_{\partial f}(z)\Phi_{\partial g}(w) \sim \frac{\rho(A)_{ji} \Phi_{\tau_i}(w)}{z - w}
\]

\[
\Phi_{\tau_i}(z)\Phi_{\tau_j}(w) \sim \frac{2\Phi_{\text{Ric}_{ij}}((z + w)/2)}{(z - w)^2} - \Phi_{\Omega(\tau_i, \tau_j)^{\partial g}}(w) + \frac{1}{2} \Phi_{\partial (\tau_i, \tau_j)} H(w) + \Phi_{\lambda^*(\Omega(\tau_i, \tau_j), \Theta)}(w)
\]

where \( \text{Ric}_{ij} = \rho(\Omega(\tau_i, \tau_j))_{jk} \). In particular, the fields \( \Phi_{A^f}(z) \) for \( A \in \mathfrak{g} \) represent an inner \((\mathfrak{g}_\lambda, G)\)-action (see Definition §4.7). Moreover, a Virasoro field of central charge \( 2 \dim P \) is given by

\[
:\Phi^{\text{vir}}_{\lambda^*}(z)\Phi_{\Theta}(z) + : \Phi_{\tau_i}(z)\Phi_{\tau_j}(z) : = -\frac{1}{2} \lambda^*(\Phi_{\Theta}(z), \Phi_{\Theta}(z))
\]

(see §4.7) with respect to which all the generating fields are primary. \( \square \)

**Remark.** For convenience (and lack of imagination), we will refer to \( \mathcal{D}_{\Theta,H}^\text{ch}(P) \) as a principal frame \((\mathfrak{g}_\lambda, G)\)-algebra. While this type of vertex algebras are particular examples of algebras of CDOs, in §6 we will see that all algebras of CDOs can be recovered in a natural way from principal frame \((\mathfrak{g}_\lambda, G)\)-algebras with \( \lambda = -\lambda_{\text{ad}} \). Moreover, that is only a special case of a construction associated to general principal \((\mathfrak{g}_\lambda, G)\)-algebras (to be introduced in §4.15).

**§4.15. The invariant subalgebra.** This discussion is a more detailed version of §4.4 specifically for a principal frame \((\mathfrak{g}_\lambda, G)\)-algebra \( \mathcal{D}_{\Theta,H}^\text{ch}(P) \). Consider the centralizer subalgebra

\[
\mathcal{D}_{\Theta,H}^\text{ch}(P)^G := C(\mathcal{D}_{\Theta,H}^\text{ch}(P), V_\lambda(\mathfrak{g})).
\]

The weight-zero component is again \( C^\infty(P)^G \). Let us examine the weight-one component below.

By definition, \( \mathcal{D}_{\Theta,H}^\text{ch}(P)^G \) consists of those elements \( \alpha + X + Y \in \Omega^1(P) \oplus T_\lambda(P) \oplus T_\Theta(P) \) that are annihilated by \( A^\alpha_P \) and \( A^X_P \) for \( A \in \mathfrak{g} \). Let \( t^1, t^2, \ldots \) be a basis of \( \mathfrak{g} \) and \( t^1, t^2, \ldots \) the dual basis of \( \mathfrak{g}^\vee \).
Let us write $X = X^i \tau_i$ and $Y = Y^a t_a^P$. By Lemma 4.11 and Definition A.3, both $\{ A^P, X^i \tau_i \}_\Omega$ and $\{ A^P, Y^a t_a^P \}_\Omega$ vanish. Then by A.3 the first condition on $\alpha + X + Y$ reads

$$A^P_0 (\alpha + X + Y) = L_{A^P} \alpha + [A^P, X] + [A^P, Y] = 0 \quad \text{for } A \in g$$

i.e. $\alpha, X, Y$ are all $G$-invariant. By Lemma 4.11 and Definition A.3 again, we find that

$$\{ A^P, X^i \tau_i \} = \rho(A) \tau_i \tau_j X^j$$
$$\{ A^P, Y^a t_a^P \} = \lambda(A, t_a) Y^a + [A, t_a]^P Y^a$$
$$= \lambda(A, t_a) Y^a - t^a ([A, t_a], t_b) Y^b \quad \because [A^P, Y] = 0$$
$$= (\lambda + \lambda_{ad})(A, Y)$$

Then by A.3 the second condition on $\alpha + X + Y$ reads

$$A^P_1 (\alpha + X + Y) = \alpha(A^P) + \rho(A) \tau_i \tau_j X^j + (\lambda + \lambda_{ad})(A, Y) = 0 \quad \text{for } A \in g.$$  

The two conditions together imply that $\alpha + (\tau_i X^j) \rho(\Theta)_{ij} + (\lambda + \lambda_{ad})(\Theta, Y)$ is $G$-invariant and horizontal (i.e. basic). Therefore $D_{\Theta, H}^1 (P)^\gamma$ is the direct sum of the following subspaces of $\Gamma^1 (P) \oplus T(P)$:

$$\pi^* \Omega^1 (M), \{ X - (\tau_i X^j) \rho(\Theta)_{ij} : X \in T_h(P)^G \}, \{ Y - (\lambda + \lambda_{ad})(\Theta, Y) : Y \in T_v(P)^G \}.$$  

This provides an explicit description of the short exact sequence (4.4) together with a splitting.
§5. Associated Modules and Algebras over CDOs

Given a principal algebra \(D^\text{ch}(P)\) in the sense of §4, this section introduces a construction of modules over the invariant subalgebra of \(D^\text{ch}(P)\) using semi-infinite cohomology. This construction is analogous to that of associated vector bundles. Examples will be studied in §6 and §7.

The semi-infinite cohomology of a centrally extended loop algebra can be defined using the Feigin complex, which is a vertex algebraic analogue of the Chevalley-Eilenberg complex, but with some important differences. For more about semi-infinite cohomology, see e.g. [Fei84, Vor93, BD04].

§5.1. The Chevalley-Eilenberg complex. Consider a finite-dimensional Lie algebra \(\mathfrak{g}\). Let \(t_1, t_2, \ldots\) be a basis of \(\mathfrak{g}^*\); \(t_1, t_2, \ldots\) the dual basis of \(\mathfrak{g}\); and \(\phi^1, \phi^2, \ldots\) the corresponding coordinates of the supermanifold \(\Pi \mathfrak{g}\). By definition \(\mathcal{O}(\Pi \mathfrak{g}) = \wedge^* \mathfrak{g}^*\) and \(\mathcal{T}(\Pi \mathfrak{g})\) consists of the derivations on \(\wedge^* \mathfrak{g}^*\). Suppose

\[ J, q, \theta \in \mathcal{T}(\Pi \mathfrak{g}), \quad \theta \in \mathcal{O}(\Pi \mathfrak{g}) \otimes \mathfrak{g} \]

are the elements corresponding respectively to the exterior degree on \(\wedge^* \mathfrak{g}^*\), the Chevalley-Eilenberg differential on \(\wedge^* \mathfrak{g}^*\), and the Maurer-Cartan form on \(\mathfrak{g}\); or in coordinates,

\[ (5.1) \quad J = \phi^a \frac{\partial}{\partial \phi^a}, \quad q = -\frac{1}{2} t^a ([t_b, t_c]) \phi^b \phi^c \frac{\partial}{\partial \phi^a}, \quad \theta = \phi^a \otimes t_a. \]

Notice that \(J = J \otimes 1\), \(q = q \otimes 1\) and \(\theta\) satisfy

\[ (5.2) \quad [J, q] = q, \quad J \theta = \theta, \quad [q, q] = 0, \quad q \theta + \frac{1}{2} [\theta, \theta] = 0. \]

Let \(W\) be a \(\mathfrak{g}\)-module. If \(J, q, \theta\) are regarded as operators on \(\mathcal{O}(\Pi \mathfrak{g}) \otimes W\) and \(Q = q + \theta\), then by (5.2) they satisfy \([J, Q] = Q\) and \([Q, Q] = 0\). The Chevalley-Eilenberg complex of \(\mathfrak{g}\) with coefficients in \(W\) can be written as

\[ (\mathcal{O}(\Pi \mathfrak{g}) \otimes W, J, Q) \]

where \(J\) is the grading operator and \(Q\) is the differential.

§5.2. The Feigin complex. Given any invariant symmetric bilinear form \(\lambda\) on \(\mathfrak{g}\), recall the centrally extended loop algebra \(\hat{\mathfrak{g}}\) (see §3.1 as well as the vertex algebra \(V_\lambda(\mathfrak{g})\) (see Example §A.10). Also consider the algebra of CDOs \(D^\text{ch}(\Pi \mathfrak{g})\), which is a fermionic version of §2.1 [Che12]. Now regard \(J, q, \theta\) as elements of \(D^\text{ch}(\Pi \mathfrak{g}) \otimes V_\lambda(\mathfrak{g})\) of weight 1.

Here are some computations with these elements:

\[
\begin{align*}
J_0 q &= [J, q] + [J, q]_\Omega = q + 0 = q \\
J_0 \theta &= (J_0 \otimes 1)(\phi^a \otimes t_a) = J \phi^a \otimes t_a = \phi^a \otimes t_a = \theta \\
q_0 q &= [q, q] + [q, q]_\Omega = 0 - t^a ([t_b, t_c]) t^b ([t_a, t_d]) \phi^d \phi^c = -\lambda_{ad}(t_c, t_d) \phi^d \phi^c \quad (\text{see } (1.2)) \\
q_0 \theta &= (q_0 \otimes 1)(\phi^a \otimes t_a) = q \phi^a \otimes t_a = -\frac{1}{2} \phi^b \phi^c \otimes [t_b, t_c] \\
\theta_0 q &= (\phi^a_0 \otimes t_a, 0) \theta_0 (q_0 \otimes 1) = [q_0, \theta_0] = \phi^b \phi^c \otimes [t_a, t_b] - \lambda(t_a, t_b) \phi^b \phi^a \\
\theta_0 \theta &= (\phi^a_0 \otimes t_a, 0) \theta_0 (\phi^a_0 \otimes t_a, 0) = \phi^b \phi^c \otimes [t_a, t_b] - \lambda(t_a, t_b) \phi^b \phi^a \\
\end{align*}
\]

using (A.3), (5.1) and the super version of (2.4). It follows that \(J\) and \(Q = q + \theta\) satisfy

\[
\begin{align*}
J_0 Q &= Q, & Q_0 Q &= -(\lambda_{ad} + \lambda)(t_a, t_b) \phi^a \phi^b \\
\Rightarrow [J_0, Q_0] &= Q_0, & [Q_0, Q_0] &= (\lambda_{ad} + \lambda)(t_a, t_b) \cdot \sum_{n \in \mathbb{Z}} n \phi_{-n} \phi^n
\end{align*}
\]
In particular, $Q_0^2 = 0$ if and only if $\lambda = -\lambda_{ad}$.

Let $W$ be a $\hat{\mathfrak{g}}-\lambda_{ad}$-module. Regard $J$ and $Q$ as elements of $\mathcal{D}^{ch}(\Pi g) \otimes V_{-\lambda_{ad}}(\mathfrak{g})$. The Feigin complex of $\hat{\mathfrak{g}}-\lambda_{ad}$ with coefficients in $W$ is

$$\left( \mathcal{D}^{ch}(\Pi g) \otimes W, J_0, Q_0 \right)$$

where $J_0$ is the grading operator and $Q_0$ is the differential.

§ 5.3. Semi-infinite cohomology. For any $\hat{\mathfrak{g}}-\lambda_{ad}$-module $W$ as above, let

$$(5.3) \quad H_{\mathfrak{g}}^{\infty+}(\hat{\mathfrak{g}}-\lambda_{ad}, W) := H^\ast \left( \mathcal{D}^{ch}(\Pi g) \otimes W, Q_0 \right).$$

Notice that if $W$ is a vertex algebra and the $\hat{\mathfrak{g}}-\lambda_{ad}$-action is inner, i.e. induced by a map of vertex algebras $V_{-\lambda_{ad}}(g) \to W$, then $(5.3)$ has the structure of a $\mathbb{Z}$-graded vertex algebra. (In this case, $J$ and $Q$ will also denote their images in $\mathcal{D}^{ch}(\Pi g) \otimes W$.)

Remarks. (i) Unlike Lie algebra cohomology, the grading on semi-infinite cohomology is neither bounded above nor below. More precisely, the restriction of $J_0$ to $\mathcal{D}^{ch}(\Pi g)_k \otimes W$ takes values between $-k$ and $\dim g + k$. (ii) This is only a special case of semi-infinite cohomology. For expositions in more general settings, see the references mentioned above.

Lemma 5.4. Let $W$ be a $\hat{\mathfrak{g}}-\lambda_{ad}$-module.

(a) The $\hat{\mathfrak{g}}-\lambda_{ad}$-invariant operators on $W$ induce grading-preserving operators on $(5.3)$. Moreover, if $W$ is a vertex algebra and its $\hat{\mathfrak{g}}-\lambda_{ad}$-action is inner, then there is a map of vertex algebras from the centralizer subalgebra $W^g = C(W, V_{-\lambda_{ad}}(g))$ to the zeroth gradation of $(5.3)$.

(b) If there is a Virasoro action on $W$ of central charge $c$ such that the $\hat{\mathfrak{g}}-\lambda_{ad}$-action is primary, then it induces a grading-preserving Virasoro action on $(5.3)$ of central charge $c - 2\dim g$. Moreover, if $W$ is a conformal vertex algebra and its $\hat{\mathfrak{g}}-\lambda_{ad}$-action is inner and primary, then $(5.3)$ is also a conformal vertex algebra with a conformal vector in the zeroth gradation.

Proof. (a) For the first claim, suppose $U \in \text{End} W$ is $\hat{\mathfrak{g}}-\lambda_{ad}$-invariant. Since on $\mathcal{D}^{ch}(\Pi g) \otimes W$ we have

$$[Q_0, 1 \otimes U] = [q_0 \otimes 1 + \phi^a_{n} \otimes t_{a,n}, 1 \otimes U] = \phi^a_{n} \otimes [t_{a,n}, U] = 0,$$

the operator $1 \otimes U$ is well-defined on $(5.3)$. Clearly it preserves the grading. For the second claim, let $u \in W^g$, i.e. $u \in W$ such that $A_n u = 0$ for $A \in \mathfrak{g}$ and $n \geq 0$. Since in $\mathcal{D}^{ch}(\Pi g) \otimes W$ we have

$$Q_0(1 \otimes u) = (q_0 \otimes 1 + \phi^a_{n} \otimes t_{a,n})(1 \otimes u) = \sum_{n \geq 0} \phi^a_{n} 1 \otimes t_{a,n} u = 0,$$

the element $1 \otimes u$ represents a class $[1 \otimes u]$ in $(5.3)$. Clearly $u \mapsto [1 \otimes u]$ is a map of vertex algebras and the image is contained in the zeroth gradation.

(b) The graded vertex algebra $\mathcal{D}^{ch}(\Pi g)$ has a conformal vector $\nu^{\Pi g}$ of central charge $-2\dim g$ (see § 6.1); denote its Virasoro operators by $L_n^{\Pi g}$, $n \in \mathbb{Z}$. Since $q$ and $\phi^a$ are primary, we have

$$[L_n^{\Pi g}, q_0] = 0, \quad [L_n^{\Pi g}, \phi^a_{m}] = -(n + m)\phi^a_{n+m}, \quad n, m \in \mathbb{Z}.$$

Since $\nu^{\Pi g}$ belongs to the zeroth gradation, every $L_n^{\Pi g}$ preserves the grading.

For the first claim, suppose $L_n^W \in \text{End} W$ for $n \in \mathbb{Z}$ define a Virasoro action of central charge $c$ and satisfy $[L_n^W, A_m] = -mA_{n+m}$ for $A \in \mathfrak{g}$ and $n, m \in \mathbb{Z}$. Then $L_n^{\Pi g} \otimes 1 + 1 \otimes L_n^W$ for $n \in \mathbb{Z}$ define a grading-preserving Virasoro action on $\mathcal{D}^{ch}(\Pi g) \otimes W$ of central charge $c - 2\dim g$. Also, since

$$[Q_0, L_n^{\Pi g} \otimes 1 + 1 \otimes L_n^W] = [q_0, L_n^{\Pi g}] \otimes 1 + [\phi^a_{n-m}, L_n^{\Pi g}] \otimes t_{a,m} + \phi^a_{n-m} \otimes [t_{a,m}, L_n^W] \quad = 0 + (n - m)\phi^a_{n-m} \otimes t_{a,m} + m\phi^a_{n-m} \otimes t_{a,n+m} \quad = 0$$

22
the said Virasoro action is well-defined on \( (\mathfrak{g}, \mathbb{C}) \) as well.

For the second claim, suppose \( \nu^W \in W \) is a conformal vector whose Virasoro operators \( L_n^W, n \in \mathbb{Z} \), satisfy \( [L_n^W, A_m] = -mA_{n+m} \) for \( A \in \mathfrak{g} \) and \( m, n \in \mathbb{Z} \). Then \( \nu^{\mathfrak{g}}_0 \otimes 1 \otimes \nu^W \) is a conformal vector of \( \mathcal{D}^{\text{ch}}(\mathfrak{g}) \otimes W \) belonging to the zeroth gradation. Also, since we have

\[
Q_0(\nu^{\mathfrak{g}}_0 \otimes 1 \otimes \nu^W) = Q_0(L_{-2}^{\mathfrak{g}} \otimes 1 + 1 \otimes L_{-2}^W)(1 \otimes 1) = 0
\]

by the previous computation, \( \nu^{\mathfrak{g}}_0 \otimes 1 + 1 \otimes \nu^W \) represents a conformal vector of \( (\mathfrak{g}, \mathbb{C}) \) as well. \( \square \)

The main purpose of this section is to introduce the following construction. As before, \( G \) is a compact connected Lie group and \( \mathfrak{g} \) its Lie algebra. Recall Definition 3.4 and Corollary 4.3.

**Definition 5.5.** Let \( \pi : P \to M \) be a smooth principal \( G \)-bundle and \( \lambda, \lambda' \) invariant symmetric bilinear forms on \( \mathfrak{g} \), such that \( \lambda + \lambda' = -\lambda_{\text{ad}} \) and \( \lambda'(\mathfrak{p}) = -8\pi^2 \alpha_1(M) \). For any principal \((\mathfrak{g}_\lambda, G)\)-algebra \( \mathcal{D}^{\text{ch}}(P) \) and positive-energy \((\mathfrak{g}_\lambda, G)\)-module \( W \), we define

\[
\Gamma^{\text{ch}}(\pi, W) := H^{\mathfrak{g}_\lambda + \alpha_1}(\pi, P \otimes \mathcal{D}^{\text{ch}}(P) \otimes W).
\]

Notice that if \( W \) is a vertex algebra and its \((\mathfrak{g}_\lambda, G)\)-action is inner, then \( \Gamma^{\text{ch}}(\pi, W) \) also has the structure of a vertex algebra (see 5.3).

**Remarks.** (i) The positive-energy condition means that there is a diagonalizable operator \( L_0^W \) on \( W \) such that \( [L_0^W, A_n] = -nA_n \) for \( A \in \mathfrak{g}, n \in \mathbb{Z} \), and the eigenvalues of \( L_0^W \) are bounded below. Whenever \( W \) admits a Virasoro action, it will be understood that \( L_0^W \) coincides with the zeroth Virasoro operator. For consistency, the eigenvalues of \( L_0^W \) will also be called weights. Let \( L_0^{\mathfrak{g}} \) (resp. \( L_0^P \)) be the weight operator on \( \mathcal{D}^{\text{ch}}(\mathfrak{g}) \) (resp. \( \mathcal{D}^{\text{ch}}(P) \)). Notice that \( L_0^{\mathfrak{g}} + L_0^P + L_0^W \) commutes with \( Q_0 \) (see the proof of Lemma 5.4), so that \( \Gamma^{\text{ch}}(\pi, W) \) inherits the notion of weights.

(ii) The above definition still makes sense without the integrability and/or the positive-energy conditions. On the other hand, under these two conditions, we can say that the lowest-weight component \( W_0 \) of \( W \) is a \( G \)-representation and then the lowest-weight component of \( \Gamma^{\text{ch}}(\pi, W) \) is \( (C^\infty(P) \otimes W_0)^G \), i.e. the space of sections of the associated vector bundle \( P \times_G W_0 \to M \).

**Lemma 5.6.** Consider again the data in Definition 5.5.

(a) For any \( W \) as described, \( \Gamma^{\text{ch}}(\pi, W) \) is a module over \( \mathcal{D}^{\text{ch}}(P) \) \( \mathfrak{g} \) (see 5.4). Moreover, if \( W \) is a vertex algebra and its \((\mathfrak{g}_\lambda, G)\)-action is inner, then there is a map of vertex algebras \( \mathcal{D}^{\text{ch}}(P) \) \( \mathfrak{g} \) \( \Gamma^{\text{ch}}(\pi, W) \).

(b) If there is a Virasoro action on \( W \) of central charge \( c \) such that the \((\mathfrak{g}_\lambda, G)\)-action is primary, then it induces a Virasoro action on \( \mathcal{D}^{\text{ch}}(\pi, W) \) of central charge \( c + 2 \dim M \). Moreover, if \( W \) is in fact a conformal vertex algebra such that its \((\mathfrak{g}_\lambda, G)\)-action is inner and primary, then \( \Gamma^{\text{ch}}(\pi, W) \) is also a conformal vertex algebra.

**Proof.** This follows from Lemma 5.4 together with the fact that \( \mathcal{D}^{\text{ch}}(P) \) is a conformal vertex algebra with central charge \( 2 \dim P \) and a primary inner \((\mathfrak{g}_\lambda, G)\)-action (see Theorem 3.11). \( \square \)
§6. Example: Recovering Algebras of CDOs

In this section, we analyze the zeroth semi-infinite cohomology of a particular type of principal algebra (in the sense of [4]) and as a result identify it with an algebra of CDOs. This provides a more conceptual description of a general algebra of CDOs.

§6.1. Goal: a new description of algebras of CDOs. Consider the special case of Definition 5.5 associated to a principal frame $(\hat{g} - \lambda_{ad}, G)$-algebra $D^c_{\Omega, H}(P)$ and the trivial $(\hat{g}_0, G)$-module $\mathbb{C}$:

$$\Gamma^c(\pi, \mathbb{C}) = H^{\hat{\pi}}_{+1}((\hat{g} - \lambda_{ad}, D^c_{\Theta, H}(P)) = H^0(D^c_{\hat{g}}(\Pi \hat{g}) \otimes D^c_{\Theta, H}(P), Q_0).$$

By Lemma 5.6, $\Gamma^c(\pi, \mathbb{C})$ is a conformal vertex algebra of central charge $2 \dim M$. For convenience, let us recall some of the data involved.

- Let $(t_1, t_2, \cdots)$ be a basis of $\hat{g}$; $(t^1, t^2, \cdots)$ the dual basis of $\hat{g}^\vee$; $(\phi^1, \phi^2, \cdots)$ the corresponding coordinates of the supermanifold $\Pi \hat{g}$; and $(\partial_1, \partial_2, \cdots)$ their coordinate vector fields.

- For the detailed definition of the vertex superalgebra $D^c_{\Pi \hat{g}}$, see e.g. [Che12]. Let us mention that, as a fermionic analogue of (2.1), it is generated by such elements as $\phi^1, \phi^2, \cdots$ and $\partial_1, \partial_2, \cdots$, and a conformal vector of central charge $-2 \dim \hat{g}$ is given by

$$\nu^\Pi_{\hat{g}} = -\partial_{a,-1}d\phi^a.$$

- For the detailed definition of the vertex algebra $D^c_{\Theta, H}(P)$, see Theorem 4.14 with $\lambda = -\lambda_{ad}$ in mind. Let us mention that it is defined using: a smooth principal $G$-bundle $\pi : P \to M$ with an isomorphism $P \times_{\hat{g}} \mathbb{R}^d \cong TM$ for some representation $\rho : G \to SO(\mathbb{R}^d)$; a connection $\Theta$ on $\pi$ that induces the Levi-Civita connection on $TM$; and a basic 3-form $H$ on $P$ that satisfies $dH = \lambda^i_\rho(\Omega \wedge \Omega)$, where $\Omega$ is the curvature of $\Theta$. Also, it is generated by such elements as $f \in C^\infty(P)$, $A^\rho \in T^\rho(P)$ for $A \in \hat{g}$ and $\tau_i \in T^\rho_i(P)$ for $i = 1, \cdots, d$, and a conformal vector of central charge $2 \dim P$ is given by

$$\nu^\rho = t^\rho_{-1}A^\rho + \tau_i_{-1} t^\rho_i - \frac{1}{2} \lambda^i_\rho(\Theta_{-1}\Theta).$$

For the meaning of various notations, see [1.2], [3.1] and [4.6].

- For details of the Feigin complex, see 5.2. Let us mention that the grading operator $J_0$ is determined by $J_0(\phi^a \otimes u) = \phi^a \otimes u$ and $J_0(\partial_a \otimes u) = -\partial_a \otimes u$ for any $u$; and the differential $Q_0 = (q + \theta)_{10}$ is induced by the elements

$$q = q \otimes 1 = -\frac{1}{2} t^a([t_b, t_c])\phi^b \phi^c \partial_a \otimes 1, \quad \theta = \phi^a \otimes t^\rho_a.$$

The conformal vector $\nu^\Pi_{\hat{g}} \otimes 1 + 1 \otimes \nu^\rho$ belongs to the zeroth gradation and is $Q_0$-closed. By the assumption on $\pi$, $M$ is an oriented Riemannian manifold. Let $\nabla$ be the Levi-Civita connection and $R$ the Riemannian curvature of $M$. The assumption on $H$ can then be written as $dH = Tr (R \wedge R)$. By Theorem 2.3, the data $(\nabla, H)$ determine an algebra of CDOs $D^c_{\Theta, H}(M)$ with a conformal vector $\nu = \nu^0$ of central charge $2d = 2 \dim M$. The goal of this section is to prove that

$$\Gamma^c(\pi, \mathbb{C}) \cong D^c_{\nabla, H}(M)$$

as conformal vertex algebras. In fact, we will analyze $\Gamma^c(\pi, \mathbb{C})$ without using any prior knowledge of $D^c_{\nabla, H}(M)$, and effectively rediscover the latter in the end.

Throughout this section we identify $\Omega^*(M)$ with the basic subspace of $\Omega^*(P)$.  

24
§ 6.2. The component of weight zero. Consider the Feigin complex that defines (6.1):

\[
D^{\text{ch}}(\Pi g) \otimes D_{\Theta, H}^{\text{ch}}(P), Q_0.
\]

Since its weight-zero component is simply the Chevalley-Eilenberg complex

\[
(\mathcal{O}(\Pi g) \otimes C^\infty(P), Q_0)
\]

(see §5.1), the weight-zero component of (6.1) is

\[
\Gamma^{\text{ch}}(\pi, \mathbb{C})_0 = H^0(\mathfrak{g}, C^\infty(P)) = C^\infty(P)^G = C^\infty(M).
\]

Understanding the rest of \(\Gamma^{\text{ch}}(\pi, \mathbb{C})\) requires more work.

**Lemma 6.3.** The element \(\gamma = \partial_a \otimes \Theta^a\) in \(D^{\text{ch}}(\Pi g)_1 \otimes D_{\Theta, H}^{\text{ch}}(P)_1\) satisfies

\[
Q_0 \gamma = \nu^1_{\Pi g} \otimes 1 + 1 \otimes t_{a, -1}^P \Theta^a, \quad \gamma_0 \gamma = 0
\]

\[
\Rightarrow [Q_0, \gamma_0] = L^1_{\Pi g} \otimes 1 + 1 \otimes (t_{a, -1}^P \Theta^a), \quad [\gamma_0, \gamma_0] = 0
\]

**Proof.** Recall the element \(Q = q + \theta\) from (6.3). Let us first write

\[
Q_0 \gamma = (q_0 \otimes 1 + \phi_a \otimes t_{a, 1}^P + \phi_a \otimes t_{a, 0}^P + \phi_a \otimes t_{a, -1}^P)(\partial_b \otimes \Theta^b).
\]

Then each of these terms is computed as follows:

\[
q_0 \partial_b \otimes \Theta^b = ([q, \partial_b] + \{q, \partial_b\}_\Theta) \otimes \Theta^b = -t^c([t_b, t_d]) \phi_c \partial_c \otimes \Theta^b
\]

\[
\phi_a \partial_b \otimes t_{a, 1}^P \Theta^b = -\partial_b \otimes t_{a, 1}^P \Theta^a
\]

\[
\phi_a \partial_b \otimes t_{a, 0}^P \Theta^b = \phi_a \partial_b \otimes [t_{a, 0}^P, \Theta^b] = -\phi_a \partial_b \otimes [t_{a, 0}^P, \Theta^c] \Theta^c
\]

\[
\phi_a \partial_b \otimes t_{a, -1}^P \Theta^b = 1 \otimes t_{a, -1}^P \Theta^a
\]

using (A.3), the super version of (2.2)–(2.4) and the \(G\)-invariance of \(\Theta = \Theta^a \otimes t_a\). In view of (6.2), this proves the claimed expression for \(Q_0 \gamma\). On the other hand, we have

\[
\gamma_0 \gamma = (\partial_{a, 1} \otimes \Theta^{-1}_a)(\partial_b \otimes \Theta^b) = \partial_{a, 1} \partial_b \otimes \Theta^{-1}_a \Theta^b = 0
\]

by the super version of (2.2). \(\square\)

**Lemma 6.4.** The weight-one component of the Feigin complex (6.4) is quasi-isomorphic to a subcomplex

\[
(\mathcal{O}(\Pi g) \otimes D^{\text{ch}}_h(P)_1, Q_0),
\]

where \(D^{\text{ch}}_h(P)_1\) is the direct sum of the following subspaces of \(\Omega^1(P) \oplus T(P)\):

\[
\Omega^1_h(P), \quad \{ \mathcal{X} - (\tau_i \mathcal{X}) \rho(\Theta)_{ij} : \mathcal{X} \in \Omega^1 \tau_i \in T_h(P) \}.
\]

**Proof.** First we compute the operator \(t_{a, -1}^P \Theta^a\) on \(D^{\text{ch}}_{\Theta, H}(P)_k, k = 0, 1\). Notice that (A.3)–(A.4) will be used repeatedly. For \(f \in C^\infty(P)\) and \(\alpha \in \Omega^1(P)\), we have

\[
(t_{a, -1}^P \Theta^a)_0 f = \Theta^a t_{a, 0}^P f = 0
\]

\[
(t_{a, -1}^P \Theta^a)_0 \alpha = (\Theta^a t_{a, 0}^P + \Theta^{-1}_a \alpha) = 0 + \alpha(t_{a, 1}^P \Theta^a = \alpha_v
\]
where $\alpha_v$ means the vertical part of $\alpha$. For $\mathcal{X} = \mathcal{X}^i \tau_i \in T_h(P)$, we have
\[
(t_{a,-1}^P \Theta^a)_0 \mathcal{X} = (t_{a,-1}^P \Theta^a + \Theta^e t_{a,0}^P + \Theta^a t_{a,1}^P) \mathcal{X} = 0 - t_{[\mathcal{E}^i, \mathcal{X}]} d\Theta^e - \rho(t_a)_{ij} (\tau_i \mathcal{X}^j) \Theta^a
\]
using Lemma 2.7 and the computation of $\{A^P, \mathcal{X}\}$ in 4.15. For $\mathcal{Y} = \mathcal{Y}^a t_a^P \in T_v(P)$, we have
\[
(t_{a,-1}^P \Theta^a)_0 \mathcal{Y} = \Theta^e (\mathcal{Y}^a t_a^P - t_{[\mathcal{E}^i, \mathcal{Y}]} d\Theta^e + (t_a^P \mathcal{Y}^b - \lambda_{ad}(t_a, \mathcal{Y})) \Theta^a
\]
\[
= \mathcal{Y} - \Theta^e ([t_a^P, \mathcal{Y}]) \Theta^a + \Theta^e ([t_a^P, \mathcal{Y}]) \Theta^a
\]
\[
= \mathcal{Y}
\]
using Lemma 4.11, Lemma 2.7, and the computation of $\{A^P, \mathcal{X}\}$ in 4.15 (with $\lambda = -\lambda_{ad}$ but without the assumption that $\mathcal{Y}$ is $G$-invariant).

Now consider the operator on the Feigin complex \ref{A} given by either of the following expressions, which are all equal by Lemma 6.3:
\[
[Q_0, \gamma_0 Q_0 \gamma_0] = [Q_0, \gamma_0] = \left( L_0^{\mathfrak{h}} \otimes 1 + 1 \otimes (t_{a,-1}^P \Theta^a)_0 \right)^2.
\]
Let $e$ denote its restriction to the weight-one component. By the calculations above, we have
\[
e(u_1 \otimes f) = u_1 \otimes f, \quad u_1 \in D^c(\Pi g), \quad f \in C^c(P)
\]
\[
e(u_0 \otimes \alpha) = u_0 \otimes \alpha_v, \quad u_0 \in \mathcal{O}(\Pi g), \quad \alpha \in \Omega^1(P)
\]
\[
e(u_0 \otimes \mathcal{X}) = u_0 \otimes (\tau_i \mathcal{X}^j) \rho(\Theta)_{ij}, \quad u_0 \in \mathcal{O}(\Pi g), \quad \mathcal{X} \in T_h(P)
\]
\[
e(u_0 \otimes \mathcal{Y}) = u_0 \otimes \mathcal{Y}, \quad u_0 \in \mathcal{O}(\Pi g), \quad \mathcal{Y} \in T_v(P)
\]
Notice that $e$ is a null-homotopic idempotent. Therefore the image of $1 - e_1$ is a quasi-isomorphic subcomplex. This is the subcomplex stated in the lemma.

\section{The component of weight one.} By Lemma 6.3, the weight-one component of (4.3) is
\[
\Gamma^c(\pi, \mathbb{C})_1 \cong H^0(\mathfrak{g}, D^c_h(P)_1) = D^c_h(P)_1^G
\]
\[
= \Omega^1(P)^G \oplus \{ \mathcal{X} - (\tau_i \mathcal{X}^j) \rho(\Theta)_{ij} : \mathcal{X} \in T_h(P)^G \}
\]
\[
= \Omega^1(M) \oplus \{ \tilde{X} - (\tau_i \tilde{X}^j) \rho(\Theta)_{ij} : X \in T(M) \}
\]
where $\tilde{X} \in T_h(P)^G$ denotes the horizontal lift of any $X \in T(M)$. Notice that by (4.7) the $G$-invariance of $\tilde{X} = \tilde{X}^i \tau_i$ means that
\[
A^P \tilde{X}^i = -\rho(A)_{ij} \tilde{X}^j \quad \text{for } A \in \mathfrak{g}.
\]
Also notice that the operator $\nabla X \in \Gamma(\text{End} \mathcal{M})$ lifts horizontally to
\[
\nabla \tilde{X} = (d \tilde{X}^j + \rho(\Theta)_{jk} \tilde{X}^k) \otimes \tau_j = (\tau_i \tilde{X}^j) \tau_i \otimes \tau_j
\]
where the first equality simply expresses the relation between $\nabla$ and $\Theta$, and the second equality follows from (6.7). The odd-looking term in (6.6) can be given a global expression using (6.8).

\footnote{By Lemma 5.6 there is a map of vertex algebras $D^c_{h, \mathfrak{g}}(P)^G \rightarrow \Gamma^c(\pi, \mathbb{C})$. Comparing (4.10) and (6.6), we see that the weight-one component of the said map is surjective and its kernel is $T_v(P)^G$ (since $\lambda = -\lambda_{ad}$).}
§ 6.6. The associated vertex algebroid. In view of (6.5) and (6.6), the extended Lie algebroid associated to \( \Gamma^{ch}(\pi, \mathbb{C}) \) is \( (C^\infty(M), \Omega^1(M), \mathcal{T}(M)) \) with the usual structure maps (see \( \text{A.5} \)). More precisely, we are identifying each \( X \in \mathcal{T}(M) \) with
\[
\ell X := \text{class of } \tilde{X} - (\tau_i \tilde{X}^j) \rho(\Theta)_{ij} \in \Gamma^{ch}(\pi, \mathbb{C})_1.
\]

Consider the vertex algebroid associated to \( \Gamma^{ch}(\pi, \mathbb{C}) \) (see \( \text{A.5} \) again):
\[
(C^\infty(M), \Omega^1(M), \mathcal{T}(M), \bullet, \{ \}, \{ \}_\Omega).
\]
Explicit expressions of \( \bullet, \{ \}, \{ \}_\Omega \) are given in the lemma below. Let \( V^f \) be the vertex algebra freely generated by this vertex algebroid, and
\[
\Psi : V^f \to \Gamma^{ch}(\pi, \mathbb{C})
\]
the resulting universal map (see \( \text{A.7} \)). By construction, \( \Psi \) is an isomorphism in the two lowest weights.

**Proposition 6.7.** For \( f \in C^\infty(M) \) and \( X, Y \in \mathcal{T}(M) \), we have
\[
\ell X \bullet f = (\nabla X) f \quad \{\ell X, \ell Y\} = -\text{Tr}(\nabla X \cdot \nabla Y)
\]
\[
\{\ell X, \ell Y\}_\Omega = \text{Tr}\left( -\nabla(\nabla X) \cdot \nabla Y + \nabla X \cdot \tau Y \cdot \nabla Y + \frac{1}{2} \ell X \cdot \ell Y \cdot H \right)
\]

**Proof.** For the calculations below, keep in mind \( \text{A.3} - \text{A.4} \) and Lemma \( 4.11 \). By definition \( \text{A.2} \), \( \ell X \bullet f \) is represented in \( \mathcal{D}^\infty_{ch, H}(P) \) by
\[
(\tilde{X} - \tau_i \tilde{X}^j \rho(\Theta)_{ij})_1 f = f(f \tilde{X} - \tau_i (f \tilde{X})^j \rho(\Theta)_{ij})
\]
\[
= (\tau_{i-1} \tilde{X}_0^i + \tilde{X}_0^i \tau_{i,0}) f - f \tilde{X} + (\tau_i (f \tilde{X})^j - f \tau_i \tilde{X}^j) \rho(\Theta)_{ij}
\]
\[
= f \tilde{X} + (\tau_i f) \tilde{X}^i - f \tilde{X} + (\tau_i f) \tilde{X}^j \rho(\Theta)_{ij}
\]
\[
= (\tau_i f) (\tilde{X}^i + \rho(\Theta)_{ij} \tilde{X}^j)
\]
This proves the first claim according to \( \text{6.8} \). By definition \( \text{A.2} \), \( \{\ell X, \ell Y\} \) is represented by
\[
(\tilde{X} - \tau_i \tilde{X}^j \rho(\Theta)_{ij})_1 (\tilde{Y} - \tau_k \tilde{Y}^\ell \rho(\Theta)_{kl})
\]
\[
= (\tilde{X}_0^i \tau_{i,0} + \tilde{X}_0^i \tau_{i,1}) \tilde{Y}^k
\]
\[
= -[\tau_{k-1}, \tilde{X}_0^i] \tilde{Y} - [[\tau_{k,0}, \tau_{k-1}], \tilde{X}_0^i] \tilde{Y}^k + \tilde{X}_0^i [\tau_{k,1}, \tau_{k-1}] \tilde{Y}^k
\]
\[
= (\tau_k (\tilde{X}^i) (\tilde{Y}^k)) + (\tau_k (\tilde{Y}^k) (\tilde{X}^i)) + \tilde{X}^i \Omega(\tau_{k,0}) \tilde{Y}^k + \tilde{X}^i \Omega(\tau_{k,1}) \tilde{Y}^k + 2 \tilde{R}c_{ik} \tilde{X}^i \tilde{Y}^k
\]
where we have used \( \text{6.7} \). This proves the second claim again thanks to \( \text{6.8} \). Let us only sketch the calculation of \( \{\ell X, \ell Y\}_\Omega \). First of all, by definition \( \text{A.2} \), it is represented by
\[
(\tilde{X} - \tau_i \tilde{X}^j \rho(\Theta)_{ij})_0 (\tilde{Y} - \tau_k \tilde{Y}^\ell \rho(\Theta)_{kl}) - (\tilde{X}, \tilde{Y}) - \tau_i (\tilde{X}, \tilde{Y})^j \rho(\Theta)_{ij}
\]
\[
= (\tau_{i-1} \tilde{X}_0^i + \tilde{X}_0^i \tau_{i,0} + \tilde{X}_0^i \tau_{i,1}) \tilde{Y}^k - (\tilde{X}, \tilde{Y})
\]
\[
+ \ell Y (\tilde{X}^i \rho(\Theta)_{ij}) - L_{\tilde{X}} (\tilde{Y}^j \rho(\Theta)_{ij}) + \tau_i (\tilde{X}, \tilde{Y})^j \rho(\Theta)_{ij}
\]

\[\text{10 For example, it follows from } \text{4.11} \text{ that } \ell X \cdot \ell Y = \ell [X, Y], \text{i.e. the Lie bracket in the said extended Lie algebroid indeed agrees with the usual Lie bracket on } \mathcal{T}(M).\]
where we have used Lemma 2.7. With some work, we can rewrite the first line as

\[-\Omega(\bar{X}, \bar{Y})^p - \tau_i \bar{Y}_k d(\tau_k \bar{X}^1) + \frac{1}{2} \xi \nu \epsilon \nu H + \lambda^*(\Omega(\bar{X}, \bar{Y}), \Theta)\]

and the second line, thanks to the first Bianchi identity, as

\[-(\tau_i \bar{X}^k)(\tau_k \bar{Y}^j) - (\tau_i \bar{Y}^k)(\tau_k \bar{X}^j)) \rho(\Theta)_{ij} + \tau_i \bar{X}^j \epsilon \nu \rho(\Omega)_{ij} - \tau_i \bar{Y}^j \epsilon \nu \rho(\Omega)_{ij} - \lambda_\nu(\Omega(\bar{X}, \bar{Y}), \Theta).\]

Notice that \(\lambda^* = \lambda_\nu\) in our current setting (see 6.1) and by the proof of Lemma 6.3 we may ignore the term \(-\Omega(\bar{X}, \bar{Y})^p\). Then it follows from (4.7) and (6.7) that \((\ell X, \ell Y)_\bar{\Omega}\) is also represented by

\[-(\tau_i \bar{X}^k)(\tau_k \bar{Y}^j)\sigma^\ell + \tau_i \bar{X}^j \epsilon \nu \rho(\Omega)_{ij} - \tau_i \bar{Y}^j \epsilon \nu \rho(\Omega)_{ij} + \frac{1}{2} \xi \nu H.\]

In view of (6.8), this proves the last claim.

**Remark.** This result recovers the vertex algebroid described in Theorem 2.5. (as \(\nabla\) is now torsion-free). In other words, we have \(V^f = \mathcal{D}^{\mathbf{ch}, H}(M)\).

§ 6.8. The conformal vector. According to 6.1 the vertex algebra \(\mathcal{G}^{\mathbf{ch}}(\pi, \mathbb{C})\) has a conformal vector \(\nu^M\) of central charge \(2d = 2 \dim M\), represented by the element \(\nu^M \otimes 1 + 1 \otimes \nu^P\) of (6.4). By Lemma 6.3, \(\nu^M\) can also be represented by

\[(6.9) \quad \nu^M \otimes 1 + 1 \otimes \nu^P - Q_0 \gamma = 1 \otimes \left(\tau_{i, -1} \tau^i - \frac{1}{2} \lambda_\nu(\Theta)_{-1} \Theta\right).\]

The key to understanding the entire structure of \(\mathcal{G}^{\mathbf{ch}}(\pi, \mathbb{C})\) is the observation that \(\nu^M\) is generated by the associated vertex algebroid (in a certain way).

**Proposition 6.9.** The vertex algebra \(V^f\) has a conformal vector \(\nu^f\) with \(\Psi(\nu^f) = \nu^M\) (see (6.9)). Given an open subset \(U \subset M\) and a smooth section \(\sigma : U \rightarrow \pi^{-1}(U) \subset P\) of \(\pi\), there is a local expression

\[\nu^f|_U = (a_{\sigma})(-1) \sigma^i + \frac{1}{2} \epsilon \nu \left(\Gamma^\sigma_{-1} \Gamma^\sigma\right) + \sigma^i(\sigma^j, \sigma^k) \sigma^j_{k} \Gamma^\sigma_{ji}\]

where \((\sigma_1, \ldots, \sigma_d)\) is the \(C^\infty(U)\)-basis of \(T(U)\) induced by \(\sigma^1, \ldots, \sigma^d\) the dual basis of \(\Omega^1(U)\); and \(\Gamma^\sigma = \rho(\sigma^* \Theta)\). In particular, \(\nu^f\) belongs to \(\mathcal{F}_{\pi^{-1}(U)} V^f\) (see (4.11)).

**Proof.** By assumption, \(\pi : P \rightarrow M\) is a lifting of the usual frame bundle of \(TM\) (see 6.1), so that any local section of \(\pi\) indeed induces a local framing of \(TM\). Consider the smooth map \(g : \pi^{-1}(U) \rightarrow G\) defined by \(\sigma(\pi(p)) = p \cdot g(p)\) for \(p \in \pi^{-1}(U)\). To ease notations, let us also write \(\rho(g) : \pi^{-1}(U) \rightarrow SO(\mathbb{R}^d)\) simply as \(g\). Then we have

\[(6.10) \quad \sigma_i = g_{ri} \tau_r, \quad \sigma^i = g_{ri} \tau^r, \quad \tau_r = g_{ri} \sigma_i, \quad \tau^r = g_{ri} \sigma^i.\]

Since \(\nabla \sigma_i = \Gamma^\sigma_{ji} \odot \sigma_j\), it follows from (6.8) and (6.10) that

\[(6.11) \quad dg + \rho(\Theta) \cdot g = (\tau_{k} g) \tau^k = g \cdot \Gamma^\sigma\]

where \(\cdot\) denotes matrix multiplication.

Consider the representative of \(\nu^M\) in (6.9). Our main task is to express that element entirely in terms of \(\sigma\). First we can write

\[\tau_{r, -1} \tau^r = \tau_{r, -1} g_{ri} \sigma^i = (\tau_{r, -1} g_{ri}) \sigma^i - (g_{ri, -1} \tau_r, 0 + g_{ri, -2} \tau^r, 0) g_{si} \tau^s = \sigma_{r, -1} \sigma^i - \tau_{r, g_{si}} \tau^i \sigma^s - \epsilon \nu \left(\Gamma^\sigma_{-1} \Gamma^\sigma\right) + \frac{1}{2} \epsilon \nu \left(\Gamma^\sigma_{-1} \Gamma^\sigma\right)\]

\[\]
using (6.10), Corollary 4.12 and the Lie derivative \( L_\tau, \tau^* = \rho(\Theta)_{sr} \) implied by (4.17). Then we work on each term separately, with repeated use of (6.10) and (6.11):

\[
\text{2nd term} = (\tau_r g_{st}) \tau^*_{s1} (\rho(\Theta)_{r1} g_{tt} - g_{rr} \Gamma^r_{ji}) \\
= -g_{st}(\tau_r g_{su}) \tau^*_{s1} \rho(\Theta)_{rt} + g_{su} g_{rj} (\tau_r g_{sk}) \sigma^k_{j1} \Gamma^r_{ji} \\
= - (\tau_r \tilde{\sigma}^i_{j1} \rho(\Theta)_{rt} + g_{su} g_{rk} (\tau_r g_{sj}) \sigma^k_{j1} \Gamma^r_{ji} + \sigma^i (\tilde{\sigma}^j, \tilde{\sigma}^k)) \sigma^k_{j1} \Gamma^r_{ji} \\
= - (\tau_r \tilde{\sigma}^i_{j1} \rho(\Theta)_{rt} + \tilde{T} (\tilde{\Gamma}^r_{ji} \Gamma^r_{ji}) + \sigma^i (\tilde{\sigma}^j, \tilde{\sigma}^k)) \sigma^k_{j1} \Gamma^r_{ji} \\
\text{3rd term} = \lambda_\rho(\Theta_{-1} \Theta) - \tilde{T} (g^{-1} \cdot \rho(\Theta)_{-1} \cdot g \cdot \Gamma^r_{ji}) \\
\text{4th term} = -\frac{1}{2} \lambda_\rho(\Theta_{-1} \Theta) + \tilde{T} (g^{-1} \cdot \rho(\Theta)_{-1} \cdot g \cdot \Gamma^r_{ji}) - \frac{1}{2} \tilde{T} (\tilde{\Gamma}^r_{ji} \Gamma^r_{ji})
\]

These calculations together yield an identity in \( \mathcal{D}^\text{ch}_{\Theta, H}(\pi^{-1}(U)) \):

\[
\tau_{r-1} \tau^r - \frac{1}{2} \lambda_\rho(\Theta_{-1} \Theta) = (\tilde{\sigma}^i - \tau_r \tilde{\sigma}^i_{j1} \rho(\Theta)_{rt} - \frac{1}{2} \tilde{T} (\tilde{\Gamma}^r_{ji} \Gamma^r_{ji}) + \sigma^i (\tilde{\sigma}^j, \tilde{\sigma}^k)) \sigma^k_{j1} \Gamma^r_{ji}.
\]

The left hand side is defined globally on \( P \) and represents \( \nu^M \in \Gamma^\text{ch}(\pi, \mathbb{C}) \). Since the right hand side is manifestly generated by the subspace (6.9), it defines an element \( \nu^f \in V^f \) such that \( \Psi(\nu^f) = \nu^M \).

It remains to show that \( \nu^f \) is a conformal vector of \( V^f \). According to [Kac98, FBF94], this amounts to checking: (i) \( \nu^f = T \), (ii) \( \nu_0^f = L_0 \), and (iii) \( \nu^f \leq \nu^f \in \mathbb{C} \). For (i) and (ii) it suffices to check them on \( C^\infty(U) \cup \{ \ell \sigma_1, \ldots, \ell \sigma_d \} \) because \( V^f \) is locally generated by these elements. In fact, since \( \Psi(\nu^f) = \nu^M \) is known to be conformal and \( \Psi \) is an isomorphism in weights 0 and 1, everything we need to check automatically holds except for the equation \( \nu^f \leq \ell \sigma_i = T(\ell \sigma_i) \) in weight 2. This equation can be verified by a straightforward calculation, which we omit.

**Remark.** Since we have already observed that \( V^f = \mathcal{D}^\text{ch}_{\Theta, H}(\pi^{-1}(U)) \), this result recovers the conformal vector \( \nu^0 \) described in Theorem 2.5 (as \( \Gamma^r \) is now traceless).

**Corollary 6.10.** The map of vertex algebras \( \Psi : V^f \to \Gamma^\text{ch}(\pi, \mathbb{C}) \) (see (6.6)) is an isomorphism.

**Proof.** By Proposition 6.9, the conformal vector \( \nu^M \in \Gamma^\text{ch}(\pi, \mathbb{C}) \) belongs to \( \mathcal{F}_{\mathbb{Z}, (-1,-1)} \). Then Lemma A.12 applies so that \( \Psi \) is surjective. By construction, the ideal ker \( \Psi \subset V^f \) is trivial in weights 0 and 1. Then by Proposition 6.9 again and Lemma A.13, ker \( \Psi \) is in fact trivial in all weights.

Propositions 6.7 6.9 and Corollary 6.10 together show that \( \Gamma^\text{ch}(\pi, \mathbb{C}) \cong \mathcal{D}^\text{ch}_{\Theta, H}(\pi) \) as conformal vertex algebras, thus fulfilling the goal of this section. Let us summarize our work as follows.

**Theorem 6.11.** Suppose \( \pi : P \to M \) is a smooth principal \( G \)-bundle and \( \rho : G \to SO(\mathbb{R}^d) \) is a representation such that there is an isomorphism \( P \times_{\rho} \mathbb{R}^d \cong TM \). Given a principal frame \( (\tilde{\mathfrak{g}} - \lambda_{\tilde{\omega}}, G) \)-algebra \( \mathcal{D}^\text{ch}_{\Theta, H}(P) \) (see Theorem 4.14), the zeroth semi-infinite cohomology

\[
\Gamma^\text{ch}(\pi, \mathbb{C}) = H^{+0}(\tilde{\mathfrak{g}} - \lambda_{\tilde{\omega}}, \mathcal{D}^\text{ch}_{\Theta, H}(P))
\]

is an algebra of CDOs on \( M \). Up to isomorphism, every algebra of CDOs on \( M \) arises this way. This vertex algebra is freely generated by its weight-zero and weight-one components, which are represented bijectively by the following subspaces of \( \mathcal{D}^\text{ch}_{\Theta, H}(P) \):

\[
\pi^* C^\infty(M), \quad \pi^* \Omega^1(M) \oplus \{ \tilde{X} - (\tilde{\tau} - \tilde{X}^i \rho(\Theta)_{ij}) : X \in \mathcal{T}(M) \}.
\]

(For details on the weight-one component, see Lemma 6.4.) Moreover, \( \Gamma^\text{ch}(\pi, \mathbb{C}) \) has a conformal vector of central charge \( 2d = 2 \dim M \), represented in \( \mathcal{D}^\text{ch}_{\Theta, H}(P) \) by

\[
\tau_{i-1} \eta^i - \frac{1}{2} \lambda_\rho(\Theta_{-1} \Theta).
\]
To conclude this section, let us describe an extension of Theorem 6.1

§ 6.12. Generalization to supermanifolds. Suppose \( \pi : P \to M \) and \( \rho : G \to SO(\mathbb{R}^d) \) are the same as above; also let \( \rho' : G \to U(\mathbb{C}^r) \) be another representation, \( E = P \times_{\rho'} \mathbb{C}^r \) the associated vector bundle and \( \Pi E \) the corresponding cs-manifold. [DM99] The \( G \)-action on \( \mathcal{O}(\Pi \mathbb{R}^r) \otimes \mathbb{C} = \wedge^* (\mathbb{C}^r)^\vee \) induced by \( \rho' \) lifts to an inner \((\hat{\mathfrak{g}}_{\Lambda'}, G)\)-action on \( \mathcal{D}_{\text{ch}}(\Pi \mathbb{R}^r) \), a fermionic analogue of §2.1 (see Definition 3.4). By Theorem 4.14 there exists a principal frame \((\hat{\mathfrak{g}}_{\Lambda}, G)\)-algebra \( \mathcal{D}_{\Theta, H}(P) \) with \( \lambda + \lambda_{\rho'} = -\lambda_{\text{ad}} \) if and only if \[
\lambda^*(P) = (\lambda_{\rho} - \lambda_{\rho'})(P) = 0 \quad \iff \quad p_1(M) - \text{ch}_2(E) = 0.
\]

In this case, we can apply Definition 5.5 to construct a vertex superalgebra

\[
\Gamma_{\text{ch}}(\pi, \mathcal{D}_{\text{ch}}(\Pi \mathbb{R}^r)) = H^\mathbb{Z}_{\mathbb{Z}^+0} \left( \hat{\mathfrak{g}}_{-\lambda_{\text{ad}}}, \mathcal{D}_{\Theta, H}(P) \otimes \mathcal{D}_{\text{ch}}(\Pi \mathbb{R}^r) \right).
\]

Moreover, the \((\hat{\mathfrak{g}}_{\Lambda'}, G)\)-action on \( \mathcal{D}_{\text{ch}}(\Pi \mathbb{R}^r) \) is primary if and only if \[
\rho'(\mathfrak{g}) \subset \mathfrak{su}_r \quad \iff \quad c_1(E) = 0.
\]

In this case, \( \Gamma_{\text{ch}}(\pi, \mathcal{D}_{\text{ch}}(\Pi \mathbb{R}^r)) \) has a conformal vector of central charge \( 2(d - r) \) by Lemma 5.6b. It follows from a similar analysis that \( \Gamma_{\text{ch}}(\pi, \mathcal{D}_{\text{ch}}(\Pi \mathbb{R}^r)) \) is an algebra of CDOs on \( \Pi E \) in the sense of [Che12]. In particular, in the case \( \rho' = \rho_\mathbb{C} \) (i.e. \( E = TM_\mathbb{C} \)), both obstructions are trivial and \( \Gamma_{\text{ch}}(\pi, \mathcal{D}_{\text{ch}}(\Pi \mathbb{R}^d)) \) is the chiral de Rham algebra of \( M \).
§7. Example: Spinor Module over CDOs

In this section, our object of study is another example of the construction in §5 called the spinor module. Following a similar strategy as in §6 we analyze its structure in order to give an explicit description of generating data and relations. It is hoped that this spinor module has an interpretation in terms of a “spinor bundle with connection on the formal loops of a string manifold”, and a deeper understanding of it, including the identification of an appropriate Dirac operator, will lead to a useful geometric theory of the Witten genus. (This was in fact the original motivation of the paper.)

Recall that the normalized Killing form on so₄ is given by \( \lambda_0(A, B) = \frac{1}{2} \text{Tr} AB \). Notice that \( \lambda_p = 2\lambda_0 \) for the standard representation \( \rho \) and \( \lambda_{2d} = (2d - 4)\lambda_0 \) (see §12). For \( k \in \mathbb{C} \), we will write \( (\mathfrak{so}_d)_{k\lambda_0} \) more simply as \( (\mathfrak{so}_d)_{k} \) (see §3).

§7.1. The Ramond Clifford algebra. Let \( C\ell \) be the unital \( \mathbb{Z}/2\mathbb{Z} \)-graded associative \( \mathbb{C} \)-algebra with the following generators and relations

\[
\tag{7.1}
i_{i,n, \text{odd}}, \quad i = 1, \ldots, d, \quad n \in \mathbb{Z}, \quad [e_{i,n}, e_{j,m}] = e_{i,n}e_{j,m} + e_{j,m}e_{i,n} = -2\delta_{ij}\delta_{n+m,0}.
\]

\[12\) Suppose \( W \) is a \( C\ell \)-module with the property that any \( w \in W \) is annihilated by \( e_{i,n} \) for sufficiently large \( n \). The operators given by

\[
\tag{7.2}A^{C\ell}_n = \frac{1}{4} A_{ji} e_{i,n-r} e_{j,r} \quad \text{for } A \in \mathfrak{so}_d, \ n \in \mathbb{Z}
\]

define an \( (\mathfrak{so}_d)_1 \)-action on \( W \). On the other hand, the operators given by

\[
\tag{7.3}L^{C\ell}_n = -\frac{1}{8} \sum_{r \geq 0} (2r - n) e_{i,n-r} e_{i,r} + \frac{1}{8} \sum_{r < 0} (2r - n) e_{i,r} e_{i,n-r} + \frac{d}{16} \delta_{n,0} \quad \text{for } n \in \mathbb{Z}
\]

define a Virasoro action on \( W \) of central charge \( d/2 \). This is in fact the Sugawara construction associated to the above \( (\mathfrak{so}_d)_1 \)-action, and accordingly satisfies

\[
[L^{C\ell}_n, A^{C\ell}_m] = -mA^{C\ell}_{n+m} \quad \text{for } n, m \in \mathbb{Z}.
\]

The eigenvalues of \( L^{C\ell}_0 \) are called weights (as usual) and \( e_{i,n} \) changes weights by \(-n\). For more detailed explanations, see e.g. [Fuc95].

For the rest of the section, \( d = 2d' \) is even.

§7.2. The spinor representation of \( \mathfrak{so}_{2d'} \). Let \( C\ell_0 \) (resp. \( C\ell_+ \)) be the subalgebra of \( C\ell \) generated by those \( e_{i,n} \) with \( n = 0 \) (resp. \( n \geq 0 \)). The finite-dimensional Clifford algebra \( C\ell_0 \) has a unique irreducible \( \mathbb{Z}/2\mathbb{Z} \)-graded representation \( S_0 \). Regarding \( S_0 \) as a \( C\ell_+ \)-module on which \( \{e_{i,n}\}_{n>0} \) act trivially, we define a \( C\ell \)-module by

\[
S = C\ell \otimes C\ell_+ S_0.
\]

By (7.1), \( S \) as a vector space is spanned by elements of the form

\[
e_{i_p,n_p} \cdots e_{i_1,n_1} s, \quad n_1 < 0, \quad (n_p, i_p) < \cdots < (n_1, i_1), \quad s \in S_0
\]

where the indicated pairs are ordered lexicographically. According to (7.1) \( S \) admits a \( ((\mathfrak{so}_{2d'}), \text{Spin}_{2d'}) \)-action together with an intertwining Virasoro action of central charge \( d' \). Notice that the element displayed above has weight

\[
d'/8 + \lfloor n_1 \rfloor + \cdots + \lfloor n_p \rfloor.
\]

For convenience, we will write \( S_k \subset S \) for the component of weight \((d'/8) + k\).

\[12\) The notation and the normalization of the generators are chosen to resemble those in e.g. [LM89].
The following is our main object of study in this section.

§ 7.3. The spinor module over CDOs. Consider the special case of Definition 5.5 associated to a principal frame \(((\mathfrak{so}_{2d})_3, -4d'), \text{Spin}_{2d'}\text{-algebra} \mathcal{D}_{\mathcal{H}}^{\text{ch}}(P)\) and the \(((\mathfrak{so}_{2d})_1, \text{Spin}_{2d'}\text{-module} S:\)

\[
\Gamma^{\text{ch}}(\pi, S) = H^0\left( (\mathfrak{so}_{2d})_1, \mathcal{D}_{\mathcal{H}}^{\text{ch}}(P) \otimes S \right)
\]

(7.4)

By Lemma 5.6 and 7.2, \(\Gamma^{\text{ch}}(\pi, S)\) is a module over the vertex algebra \(\mathcal{D}_{\mathcal{H}}^{\text{ch}}(P)\text{-algebra}\) and admits a Virasoro action of central charge \(5d' = \frac{5}{2} \dim M\). For convenience, let us recall some of the data involved.

- Let \((t_1, t_2, \ldots)\) be a basis of \(\mathfrak{so}_{2d'}\); \((t^1, t^2, \ldots)\) the dual basis of \((\mathfrak{so}_{2d'})^\vee\); \((\phi^1, \phi^2, \ldots)\) the corresponding coordinates of the supermanifold \(\Pi\mathfrak{so}_{2d'}\); and \((\partial_1, \partial_2, \ldots)\) their coordinate vector fields.

- For comments on the conformal vertex superalgebra \(\mathcal{D}^{\text{ch}}(\Pi\mathfrak{so}_{2d'})\), see § 6.1.

- For the detailed definition of the vertex algebra \(\mathcal{D}_{\mathcal{H}}^{\text{ch}}(P)\), see Theorem 4.1 with \(\lambda = (3 - 4d')\lambda_0\) in mind. Let us mention that it is defined using: a principal Spin\(_{2d'}\)-frame bundle \(\pi : P \to M\); the Levi-Civita connection \(\Theta\) on \(\pi\); and a basic 3-form \(dH\) on \(P\) that satisfies \(dH = \lambda_0(\Omega \wedge \Omega)\), where \(\Omega\) is the curvature of \(\Theta\). Also, it is generated by such elements as \(f \in C^\infty(P)\), \(A^\kappa \in T_\kappa(P)\) for \(A \in \mathfrak{so}_{2d'}\) and \(\tau_i \in T_\kappa(P)\) for \(i = 1, \ldots, 2d'\), and a conformal vector of central charge \(2 \dim P\) is given by

\[
\nu^\kappa = t_{a_i} \Theta^a + \tau_i \tau^i - \frac{1}{2} \lambda_0(\Theta - \Theta).
\]

For the meaning of various notations, see § 3.1 and § 4.6.

§ 7.4. The component of the lowest weight. Consider the Feigin complex that defines (7.4):

\[
\left( \mathcal{D}^{\text{ch}}(\Pi\mathfrak{so}_{2d'}) \otimes \mathcal{D}_{\mathcal{H}}^{\text{ch}}(P) \otimes S, Q_0 \right).
\]

(7.7)

Since its component of weight \(d' / 8\) is simply the Chevalley-Eilenberg complex

\[
\left( \mathcal{O}(\Pi\mathfrak{so}_{2d'}) \otimes C^\infty(P) \otimes S_0, Q_0 \right)
\]

(see § 5.1), the component of (7.4) of weight \(d' / 8\) is

\[
\Gamma^{\text{ch}}(\pi, S)_0 = H^0(\mathfrak{so}_{2d'}, C^\infty(P) \otimes S_0) = (C^\infty(P) \otimes S_0)^{\mathfrak{so}_{2d'}} = S(M)
\]

(7.8)

i.e. the space of smooth sections of the spinor bundle on \(M\). Understanding the rest of \(\Gamma^{\text{ch}}(\pi, S)\) requires more work.

32
Lemma 7.5. Let \( \gamma \) be the same element as in Lemma 6.3 (with \( g = \mathfrak{so}_{2d'} \)) and \( \mathcal{V} = \mathcal{V}^a t_a^P \in \mathcal{T}_0(\mathcal{P})^{\mathfrak{so}_{2d'}} \). For \( n \in \mathbb{Z} \) we have the following equations of operators on the Feigin complex (7.4):

\[
\begin{align*}
[Q_0, \gamma_n \otimes 1] &= L_n^{\Pi a} \otimes 1 + 1 \otimes (t_a^P)_{n-1}^P \Theta_a^{\otimes} \otimes t_a^C \otimes t_a^C \otimes t_a^C \\
[Q_0, (\gamma_0 \otimes \mathcal{V})_n \otimes 1] &= 1 \otimes \mathcal{V}_a^{\otimes} \otimes 1 + 1 \otimes \mathcal{V}_{n-1}^{\otimes} \otimes t_a^C \\
\end{align*}
\]

Proof. Recall the differential \( Q_0 \) from (7.5). The first anticommutator is computed as follows:

\[
\begin{align*}
[Q_0, \gamma_n \otimes 1] &= [(q \otimes 1 + \phi^a \otimes t_a^P)_0, \gamma_n] \otimes 1 + [\phi^a_{r-1}, \partial_{b_n-r}] \otimes \Theta_s^{\otimes} \otimes t_a^C \\
&= L_n^{\Pi a} \otimes 1 + 1 \otimes (t_a^P)_{n-1}^P \Theta_a^{\otimes} \otimes t_a^C \\
\end{align*}
\]

using Lemma 6.3 and the fermionic analogue of (2.2). On the other hand, since \( \gamma_0 (1 \otimes \mathcal{V}) = \partial_b \otimes \mathcal{V}^b \), the second anticommutator can be written as

\[
\begin{align*}
[Q_0, (\partial_b \otimes \mathcal{V}^b)_n \otimes 1] &= [(q \otimes 1 + \phi^a \otimes t_a^P)_0, (\partial_b \otimes \mathcal{V}^b)_n] \otimes 1 + [\phi^a_{r-1}, \partial_{b_n-r}] \otimes \mathcal{V}^b \otimes t_a^C \\
&= ((q \otimes 1 + \phi^a \otimes t_a^P)_0)(\partial_b \otimes \mathcal{V}^b)_n \otimes 1 + \mathcal{V}^a_{n-1} \otimes t_a^C \\
\end{align*}
\]

using again the analogue of (2.2). It remains to compute the element \( (q \otimes 1 + \phi^a \otimes t_a^P)_0(\partial_b \otimes \mathcal{V}^b) \), which is the sum of the following:

\[
\begin{align*}
(q_0, \partial_b) \otimes \mathcal{V}^b &= [q, \partial_b] \otimes \mathcal{V}^b = -t^q([t_b, t_d]) \phi^a \partial_c \otimes \mathcal{V}^b \\
(\partial_b \otimes t_a^0, \mathcal{V}) &= \phi^a \partial_b \otimes t_a^P \mathcal{V}^b = -t^b([t_a, t_b]) \phi^a \partial_b \otimes \mathcal{V}^c \\
(\partial_b \otimes t_a^0, \mathcal{V}^b) &= \phi^a \otimes t_a^P \mathcal{V}^b = 1 \otimes \mathcal{V} \\
\end{align*}
\]

Indeed, the first line is similar to a computation in the proof of Lemma 6.3; the second follows from the \( \mathfrak{so}_{2d'} \)-invariance of \( \mathcal{V} \); and the last follows from the analogue of (2.2) and Corollary 4.12.

Preparation. Let \( \text{Cl} (M) = (\mathcal{C}^\infty (P) \otimes \text{Cl}(\mathfrak{q}))^{\mathfrak{so}_{2d'}} \), which can be viewed as the algebra of \( \mathcal{C}^\infty (M) \)-linear endomorphisms of \( S(M) \). For \( X \in \mathcal{T}(M) \), we denote its horizontal lift by \( \tilde{X} = \tilde{X}^i \tau_i \in \mathcal{T}_h(\mathcal{P})^{\mathfrak{so}_{2d'}} \) and its Clifford action by \( cX = \tilde{X}^i \otimes e_{i,0} \in \text{Cl}(M) \). Also, since each \( \mathcal{V} = \mathcal{V}^a t_a^P \in \mathcal{T}_0(\mathcal{P})^{\mathfrak{so}_{2d'}} \) satisfies

\[
\mathcal{V} \otimes 1 + \mathcal{V}^a \otimes t_a = 0 \quad \text{on} \quad (\mathcal{C}^\infty (P) \otimes \mathfrak{so}_{2d'})^{\mathfrak{so}_{2d'}} = S(M),
\]

it represents an endomorphism \( c\mathcal{V} = -\mathcal{V}^a \otimes t_a \in \text{Cl}(M) \).

§ 7.6. Fields of low weights. Recall Lemma 5.4b for the special case (7.4): the \( \mathfrak{so}_{2d'} \)-invariant fields on \( D^{\mathfrak{ch}}_{\theta, H}(P) \otimes S \), including the vertex operators of \( D^{\mathfrak{ch}}_{\theta, H}(P)^{\mathfrak{so}_{2d'}} \), induce fields on \( \Gamma^{\mathfrak{ch}}(\pi, S) \). Let us now describe a collection of such fields (or rather their Fourier modes); we will later see that their actions on the subspace \( \Gamma^{\mathfrak{ch}}(\pi, S)_0 = S(M) \) generate the entire space \( \Gamma^{\mathfrak{ch}}(\pi, S) \).

For \( f \in \mathcal{C}^\infty (M), X \in \mathcal{T}(M) \) and \( n \in \mathbb{Z} \), define the following operators on \( \Gamma^{\mathfrak{ch}}(\pi, S) \):

\[
\begin{align*}
f_n := & \text{the operator induced by } f_n \otimes 1 \\
\ell X_n := & \text{the operator induced by } (\tilde{X} - \tau_i \tilde{X}^j \Phi_{ij})_n \otimes 1 \\
c X_n := & \text{the operator induced by } \tilde{X}^i \otimes e_{i,r} \\
\end{align*}
\]

Indeed, the first two are well-defined because both \( f \) and \( \tilde{X} - \tau_i \tilde{X}^j \Phi_{ij} \) belong to \( D^{\mathfrak{ch}}_{\theta, H}(P)^{\mathfrak{so}_{2d'}} \) according to (4.13) and so is the last one because for \( A \in \mathfrak{so}_{2d'} \) and \( m \in \mathbb{Z} \) we have

\[
\begin{align*}
[A_m^P \otimes 1 + 1 \otimes \mathcal{C}^\ell, \tilde{X}^i \otimes e_{i,r}] &= (A_m^P \tilde{X}^i)_{m+n-r} \otimes e_{i,r} + \tilde{X}^i \otimes (A_m \tilde{X})_{m+n-r} + (A_m \tilde{X})_{m+n-r} \otimes e_{i,r} + (A_m \tilde{X})_{m+n-r} \otimes e_{i,r} \otimes e_{j,m+r} \\
&= (A_{m+r} \tilde{X})_{m+n-r} \otimes e_{i,r} + A_{m+r} \tilde{X}^i \otimes e_{i,r} \otimes e_{j,m+r} = 0
\end{align*}
\]
thanks to (A.3), the $\mathfrak{so}_{2d}$-invariance of $\bar{X}$ and (A.1)–(A.2). These three sequences of operators should be regarded as the Fourier modes of certain fields of weights $0$, $1$ and $\frac{1}{2}$ respectively.

It will be convenient to also consider some other operators that are generated by those in (A.3). For $\alpha \in \Omega^1(M)$, $\mathcal{Y} \in T_v(P)^{\mathfrak{so}_{2d'}}$ and $n \in \mathbb{Z}$, define the following operators on $\Gamma^\ast(\pi, S)$:

$$
\alpha_n := \text{the operator induced by } \alpha_n \otimes 1
$$

$$
\mathcal{Y}_n := \text{the operator induced by } (\mathcal{Y} + \lambda_0(\Theta, \mathcal{Y}))_n \otimes 1
$$

These are well-defined because both $\alpha$ and $\mathcal{Y} + \lambda_0(\Theta, \mathcal{Y})$ belong to $D_{\Theta,H}^\text{ch}(P)^{\mathfrak{so}_{2d'}}$ according to (4.15) to see that $\alpha_n$ can be expressed in terms of (7.9), it suffices to assume $\alpha = fgd$ and notice that

$$
(fgd)_n = -\sum_{r \in \mathbb{Z}} rf_{n-r}g_r
$$

by (A.3)–(A.4). The next lemma describes how $c\mathcal{Y}_n$ can be expressed in terms of (7.9).

**Lemma 7.7.** The operator $c\mathcal{Y}_n$ defined above equals a sum of operators of the form

$$
\sum_{r \geq 0} cX_{n-r} cY_r - \sum_{r < 0} cY_r cX_{n-r} - 2\langle X, \nabla Y \rangle_n + \langle X, Y \rangle_n, \quad X, Y \in \mathcal{T}(M)
$$

where $\langle \cdot \rangle$ and $\nabla$ denote the Riemannian metric and Levi-Civita connection on $TM$.

**Proof.** For the calculations below, keep in mind (A.3)–(A.4). Let us write $\mathcal{Y} = \mathcal{Y}^a t^p_a$ and $\mathcal{Y}_{ij} = \mathcal{Y}^a (t^a)_{ij}$. By Lemma 7.5 and (7.2), we can also represent $c\mathcal{Y}_n$ on $D_{\Theta,H}^\text{ch}(P) \otimes S$ by

$$
-c\mathcal{Y}_n \otimes t^C_{\ell, r} + \lambda_0(\Theta, \mathcal{Y})_n \otimes 1 = \frac{1}{2} (\mathcal{Y}_{ij,n-r} \otimes 1) (1 \otimes e_{i,r-s}e_{j,s} - 2\Theta_{ij,r} \otimes 1).
$$

Since $c\mathcal{Y} = \frac{1}{2} \mathcal{Y}_{ij} \otimes e_{i,0} e_{j,0}$ corresponds to a 2-vector field under the isomorphism $C_{\ell}(M) \cong \Gamma(\wedge^2 TM)$, the above operator can be written as a sum of operators of the form

$$
\frac{1}{2} ((\bar{X}^i \bar{Y}^j - \bar{Y}^i \bar{X}^j)_{n-r} \otimes 1) (1 \otimes e_{i,r-s}e_{j,s} - 2\Theta_{ij,r} \otimes 1), \quad X, Y \in \mathcal{T}(M)
$$

where the factor of $\frac{1}{2}$ is included only for convenience. It remains to show that (7.13) represents (7.12).

Consider the first two terms in (7.12). By definition (7.9) they are represented by

$$
\sum_{r \geq 0} (\bar{X}^i_{n-r-s} \otimes e_{i,s}) (\bar{Y}^j_{r-t} \otimes e_{j,t}) - \sum_{r < 0} (\bar{Y}^j_{r-t} \otimes e_{j,t}) (\bar{X}^i_{n-r-s} \otimes e_{i,s}).
$$

In view of (7.11), let us split this sum into three parts. The first part consists of terms with $i \neq j$:

$$
\sum_{r \geq 0} \sum_{i \neq j} \sum_{s,t} \bar{X}^i_{n-r-s} \bar{Y}^j_{r-t} \otimes e_{i,s}e_{j,t} = \sum_{i \neq j} \sum_{s,t} (\bar{X}^i \bar{Y}^j)_{n-s-t} \otimes e_{i,s}e_{j,t}.
$$

The second part consists of terms with $i = j$ and $s + t \neq 0$, which vanishes by symmetry:

$$
\sum_{r \geq 0} \sum_{i} \sum_{s + t \neq 0} \bar{X}^i_{n-r-s} \bar{Y}^i_{r-t} \otimes e_{i,s}e_{i,t} = \sum_{i} \sum_{s + t \neq 0} (\bar{X}^i \bar{Y}^i)_{n-s-t} \otimes e_{i}s e_{i,t} = 0.
$$

The last part consists of terms with $i = j$ and $s + t = 0$, which is computed as follows:

$$
\sum_{r \geq 0} \sum_{i,s} \bar{X}^i_{n-r-s} \bar{Y}^i_{r+s} \otimes e_{i,s}e_{i,s} - \sum_{r < 0} \sum_{i,s} \bar{X}^i_{n-r-s} \bar{Y}^i_{r+s} \otimes e_{i,-s}e_{i,s}
$$

$$
= -\sum_{i,u} (2u + 1) \bar{X}^i_{n-u} \bar{Y}^i_u \otimes 1
$$

$$
= -(\bar{X} \bar{Y})_n \otimes 1 + 2 \langle X, \nabla Y \rangle_n \otimes 1
$$

$$
= -(X, Y)_n \otimes 1 + 2 \langle X, \nabla Y \rangle_n \otimes 1 - 2 (\bar{X} \bar{Y}_{i,j})_n \otimes 1
$$

where we have used (0.5) in the last equality. It follows from these calculations and a little reorganization that (7.12) is indeed represented by (7.13).
Next we record the relations between the lowest-weight component $\Gamma^{\text{ch}}(\pi, S)_0 = S(M)$ and the fields introduced in (7.4) as well as the relations between the fields themselves.

**Proposition 7.8.** For $f \in C^\infty(M), X \in T(M)$ and $s \in S(M)$, we have

$$f_0 s = f s, \quad cX_0 s = cX \cdot s, \quad \ell_0 s = \nabla_X s, \quad f_n s = cX_n s = \ell_X s = 0 \quad \text{for } n > 0$$

where $\cdot$ denotes Clifford multiplication and $\nabla$ the Levi-Civita connection.

**Proof.** The first three equations follow immediately from (7.8) and (7.9). The rest are true simply because $S(M) \subset \Gamma^{\text{ch}}(\pi, S)$ is the component of the lowest weight.

**Proposition 7.9.** Recall the maps $\bullet, \{ \}, \{ \}_\Omega$ in Proposition 6.7. For $f, g \in C^\infty(M), X, Y \in T(M)$ and $n, m \in \mathbb{Z}$, we have the normal-ordered expansions

$$(fg)_n = f_{n-r} g_r, \quad c(fX)_n = f_{n-r} cX_r, \quad \ell(fX)_n = \sum_{r \geq 0} f_n \ell_X + \sum_{r < 0} \ell_X f_n - (\ell_X \bullet f)_n$$

as well as the supercommutation relations

$$[f_n, g_m] = 0, \quad [cX_n, f_m] = 0, \quad [\ell_n, f_m] = (X f)_n + m$$

where $\langle \rangle$ and $\nabla$ denote the Riemannian metric and Levi-Civita connection on $TM$.

**Proof.** Let us verify the relations at the level of $\mathcal{D}^{\text{ch}}_{\partial, H}(P) \otimes S$. For the calculations below, keep in mind (7.9) and (A.3)–(A.4). The first two normal-ordered expansions are easy, e.g.

$$(\overline{fX})^i_{n-s} \otimes e_{i,s} = f_{n-s-t} \overline{X}^i_{t} \otimes e_{i,s} = (f_{n-r} \otimes 1)(\overline{X}^i_{s} \otimes e_{i,s}).$$

The remaining normal-ordered expansion follows immediately from the equation

$$\overline{fX} - \tau_i (\overline{fX})^i \Theta_{ij} = (\overline{X} - \tau_j \overline{X}^j \Theta_{ij}) - f - \ell X \bullet f$$

which is the result of the first calculation in the proof of Proposition 6.7.

For the supercommutators, the first three are easy. The next two are computed as follows, using (7.8) and (7.1) respectively:

$$[(\overline{X} - \tau_i \overline{X}^i \Theta_{ij}) \otimes 1, \overline{\nabla}^k_{m-r} \otimes e_{k,r}]$$

$$= [\overline{X}_n, \overline{\nabla}^k_{m-r}] \otimes e_{k,r} = (\overline{X}^i \tau_i \overline{\nabla}^k_{n+m-r} \otimes e_{k,r} = (\overline{\nabla}^k_{n+m-r} \otimes e_{k,r})$$

$$[\overline{X}^i_{n-r} \otimes e_{i,r}, \overline{\nabla}^j_{m-s} \otimes e_{j,s}]$$

$$= \overline{X}^i_{n-r} \overline{\nabla}^j_{m-s} \otimes [e_{i,r}, e_{j,s}] = -2 \overline{n} \overline{n}^i_{m-r} \nabla^j_{m} \otimes 1 = -2(X, Y)_{n+m} + 1$$

The last (super)commutator follows immediately from the equations

$$((\overline{X} - \tau_i \overline{X}^i \Theta_{ij})_1 (\overline{Y} - \tau_k \overline{\nabla}^k \Theta_{kt}) = \{\ell X, \ell Y\}$$

$$((\overline{X} - \tau_i \overline{X}^i \Theta_{ij})_0 (\overline{Y} - \tau_k \overline{\nabla}^k \Theta_{kt})$$

$$= ((\overline{X}, \overline{Y}) - \tau_i (\overline{X}, \overline{Y})^i \Theta_{ij}) - \Omega(\overline{X}, \overline{Y})^i - \lambda_0 (\overline{\Omega(\overline{X}, \overline{Y}), \Theta}) + \{\ell X, \ell Y\}$$

together with (7.10); these equations are the results of the second and third calculations in the proof of Proposition 6.7 but applied to our current setting where $\lambda^* = \lambda_0$ and $\lambda_0 = 2\lambda_0$ (see 7.3).
The following construction is manufactured using precisely the information about $\Gamma^{\text{ch}}(\pi, S)$ we have gathered so far: the subspace $S(M)$, the fields in §7.9 and their relations in Propositions 7.8 and 7.9.

§ 7.10. Comparing with a generators-and-relations construction. Let $U$ be a unital $\mathbb{Z}/2\mathbb{Z}$-graded associative algebra with generators

\begin{equation*}
\text{even: } f_n, \ell X_n, \alpha_n, c\mathcal{H}_n; \quad \text{odd: } cX_n
\end{equation*}

for $f \in C^\infty(M)$, $X \in T(M)$, $\alpha \in \Omega^1(M)$, $\mathcal{Y} \in T_v(P)^{\mathbb{Z}_2}$ and $n \in \mathbb{Z}$, such that (i) $1_n = \delta_{n,0}$, (ii) $f \mapsto f_n$, $X \mapsto \ell X_n$, \ldots are linear and (iii) they satisfy the supercommutation relations in Proposition 7.8. The subalgebra $U_+ \subset U$ generated by \{ $f_n, \ell X_n, cX_n$ \}_{n \geq 0} has an action on $S(M)$ as described in Proposition 7.8. Let $W^f = U \otimes U_+ S(M)$ and $W^f = W^f/\sim$ be the quotient obtained by imposing the normal-ordered expansions in (7.11), Lemma 7.7 and Proposition 7.9. Define an operator $L_0^f$ on $W^f$ by

\begin{equation}
L_0^f|_{S(M)} = \frac{d'}{8}, \quad [L_0^f, f_n] = -nf_n, \quad [L_0^f, \ell X_n] = -n\ell X_n, \quad [L_0^f, cX_n] = -ncX_n
\end{equation}

which are consistent with all the above relations; its eigenvalues are called weights.

By construction, there is a unique linear map

\begin{equation*}
\Psi : W^f \to \Gamma^{\text{ch}}(\pi, S)
\end{equation*}

that restricts to the identity on $S(M)$ and respects all the above operators. In fact, we will see that $\Psi$ is an isomorphism.

§ 7.11. The Virasoro action. According to 7.13, the operators on $\Gamma^{\text{ch}}(\pi, S)$ induced by 7.10 define a Virasoro action of central charge $5d'$; let us denote them by $L_n^M$ for $n \in \mathbb{Z}$. By Lemma 7.7 each $L_n^M$ can also be represented on $\mathcal{D}^\text{ch}_{\Theta,H}(P) \otimes S$ by

\begin{equation}
(\tau_{i,1-i}^\alpha)_n \otimes 1 + 1 \otimes L_n^C\ell - \Theta_{n-r}^a \otimes t_{a,r}^C\ell - \frac{d'}{2} \lambda_0(\Theta_{-1}\Theta)_n \otimes 1.
\end{equation}

The key to understanding the entire structure of $\Gamma^{\text{ch}}(\pi, S)$ is the observation that $L_0^M$ is generated by the fields described in 7.6 (in a certain way).

Proposition 7.12. Let $U \subset M$ be an open subset and $\sigma : U \to \pi^{-1}(U) \subset P$ a smooth section of $\pi$. Both of the weight operators $L_0^f$ on $W^f$ (see §7.10) and $L_0^M$ on $\Gamma^{\text{ch}}(\pi, S)$ have the local expression:

\begin{equation}
\sum_{r \geq 0} \sigma^+_{-r}(\ell \sigma_i)_r + \sum_{r < 0} (\ell \sigma_i)_r \sigma^-_{-r} + \frac{3}{4} \text{Tr} (\Gamma_{-r}^\sigma) + \sigma^+|_{-r, k} \sigma^-_{-s} (\Gamma_{ji})_{r+s}
\end{equation}

\begin{equation*}
- \frac{1}{2} \sum_{r \geq 0} r(\sigma^+_{-r})_r (\sigma^-_{-r})_r + \frac{1}{4} (\sigma^+_{-r})_r (\sigma^-_{-r})_r (\Gamma^\sigma_{0, r})_{r+s} + \frac{d'}{8}
\end{equation*}

where $(\sigma_1, \ldots, \sigma_2d)$ is the $C^\infty(U)$-basis of $T(U)$ induced by $\sigma$; $(\sigma^1, \ldots, \sigma^{2d})$ the dual basis of $\Omega^1(U)$; and $\Gamma^\sigma = \sigma^+ \Theta$.

Proof. By assumption, $\pi : P \to M$ is a lifting of the usual frame bundle of $TM$ (see 7.6), so that any local section of $\pi$ indeed induces a local framing of $TM$. Let us first study $L_0^M$. According to 7.11 $L_0^M$ is represented by the operator $\ell' + \ell''$ on $\mathcal{D}^\text{ch}_{\Theta,H}(P) \otimes S$, where

\begin{equation*}
\ell' = (\tau_{i,1-i}^\alpha - \frac{1}{2} \text{Tr}(\Theta_{-1}\Theta))_0 \otimes 1, \quad \ell'' = 1 \otimes L_0^C\ell - \Theta_{a,r}^a \otimes t_{a,r}^C\ell + \frac{1}{4} \text{Tr}(\Theta_{-1}\Theta)_0 \otimes 1.
\end{equation*}

\text{13} Since the composition $\widetilde{W}^f \to W^f \to \Gamma^{\text{ch}}(\pi, S)$ restricts to the identity on $1 \otimes S(M) \subset \widetilde{W}^f$, the quotient map must be injective there. Hence we may indeed identify $S(M)$ as a subspace of $W^f$.

\text{14} Recall from 2 that, like many other constructions in this paper, both $W^f$ and $\Gamma^{\text{ch}}(\pi, S)$ are the spaces of global sections of some underlying sheaves.
Since the calculation in the proof of Proposition 6.3 is valid for any principal frame algebra, \( \ell' \) can be expressed as the zeroth mode of (6.12). Our main task is to express \( \ell'' \) entirely in terms of \( \sigma \) as well.

For the calculations below, adopt again the notations in the proof of Proposition 6.3. Let \( \theta = g^{-1} dg \). Also let us introduce a (slight abuse of) notation:

\[
(\sigma_i)_r = \tilde{\sigma}_i,\quad i = 1, \ldots, 2d', \quad r \in \mathbb{Z}.
\]

By \((7.1)\) these operators satisfy the anticommutation relations

\[(7.17) \quad (\sigma_i)_r (\sigma_j)_r = -2\delta_{ij}\delta_{r+s,0}.\]

Now the first term of \( \ell'' \) can be written as follows

\[
1 \otimes L^\ell_0 = \frac{1}{4} \left( - \sum_{r \geq 0} r (\sigma_j)_s (\sigma_k)_t + \sum_{t < 0} r (\sigma_j)_t (\sigma_k)_r \right) (g_{ij,r-s} g_{ik,r-t} \otimes 1) + \frac{d'}{8}.
\]

By \((7.1)\) these operators satisfy the anticommutation relations

\[(7.18) \quad [(\sigma_i)_r (\sigma_j)_s] = -2\delta_{ij}\delta_{r+s,0}.\]

The first term follows from \((7.2)\) and \((7.14)\); the second step is a rearrangement that is valid because every sum in sight is finite when applied to arbitrary elements; and the last step is the result of some algebra and an application of \((7.18)\). The second term of \( \ell'' \) can be written as follows

\[
-\Theta_{-r} \otimes t^\ell_a = \frac{1}{4} \left( \sum_{s \geq 0} \Theta_{ij,-r} \otimes e_{i,r-s} e_{j,s} - \sum_{s < 0} \Theta_{ij,-r} \otimes e_{j,s} e_{i,r-s} \right) + \frac{1}{4} \sum_{s < 0} \Theta_{ij,-r} \otimes e_{j,s} e_{i,r-s} = \frac{1}{4} \left( \sum_{s \geq 0} (\sigma_k)_t (\sigma_i)_u - \sum_{s < 0} (\sigma_i)_t (\sigma_k)_u \right) (g^{-1} \Theta)_{kj,-s-t} g_{j,s-u} (\otimes 1)
\]

The first step uses \((7.2)\); the third step follows from \((7.10)\) and is valid thanks to the normal ordering of \( e_{i,r-s} e_{j,s} \); and the last step is largely similar to the above treatment of \( 1 \otimes L^\ell_0 \). These calculations, together with \((6.11)\), yield the following expression:

\[
\ell'' = -\frac{1}{2} \sum_{t > 0} t (\sigma_j)_t (\sigma_i)_t + \frac{1}{4} (\sigma_i)_t (\sigma_k)_t (\Gamma_{sj,k-s-t}^\sigma \otimes 1) + \frac{1}{4} \text{Tr} (\Gamma_{-1}^\sigma \Gamma^\sigma) \otimes 1 + \frac{d'}{8}.
\]

If we combine the earlier comment on \( \ell' \) with this expression of \( \ell'' \) and recall \((7.9)\)--\((7.10)\), then we find that \( \ell' + \ell'' \) represents the operator shown in \((7.10)\). This proves the claim for \( L^0_M \).

Since the expression in \((7.10)\) is generated by the operators in \((7.9)\)--\((7.10)\), it determines an operator \( L^0_0 \) on \( \mathcal{W}_f \). To show \( L^0_0 = L^0_f \), we need to verify that \( L^0_M \) satisfies \((7.14)\) using only the relations in \((7.11)\), Lemma \((7.7)\) and Propositions \((7.8)-(7.9)\). The calculations, which we omit, are straightforward.

**Corollary 7.13.** The linear map \( \Psi : \mathcal{W}_f \to \Gamma^{ch}(\pi, S) \) (see \((7.10)\)) is an isomorphism.

**Proof.** Let \( f \in C^\infty(M), \alpha \in \Omega^2(M) \) and \( X \in \mathcal{T}(M) \). By definition, \( \Psi \) is an isomorphism on the lowest weight \( d'/8 \). Assume that \( \Psi \) is an isomorphism on all weights up to \((d'/8) + k - 1 \) for some \( k > 0 \). Let \( u \in \Gamma^{ch}(\pi, S) \). For reason of weight as well as \((7.11)\), elements of the form

\[
(7.19) \quad f_n u, \ell X_n u, c X_n u \quad \text{for} \quad n > 0, \quad \alpha_n u \quad \text{for} \quad n \geq 0
\]
are in the image of \(\Psi\); then by Proposition 7.12, so is \(ku = (L_0^d - d'/8)u\). This proves the surjectivity of \(\Psi\) on weight \((d'/8) + k\). On the other hand, let \(u \in W_d^f \cap \ker \Psi\). Since the elements \((7.19)\) are also in the kernel of \(\Psi\), for reason of weight and \((7.11)\) again they must be trivial; then by Proposition 7.12, so is \(ku = (L_0^d - d'/8)u\). This proves the injectivity of \(\Psi\) on weight \((d'/8) + k\). By induction, \(\Psi\) is an isomorphism on all weights.

The following summarizes our analysis of \(\Gamma^{\text{ch}}(\pi, S)\).

**Theorem 7.14.** Suppose \(M^{2d'}\) is a Riemannian manifold with a spin structure \(\pi : P \to M\) and a 3-form \(H\) satisfying \(dH = \frac{1}{2} \text{Tr}(R \wedge R)\), where \(R\) is the Riemannian curvature. Let \(D^\text{ch}_{\Theta, H}(P)\) be the associated principal frame \((\mathfrak{so}_{2d'}, \text{Spin}_{2d'})\)-algebra (see Theorem 7.14), \(S\) the spinor representation of \(\mathfrak{so}_{2d'}\), and

\[
\Gamma^{\text{ch}}(\pi, S) = H \hat{\pi} + 0 \big(\mathfrak{so}_{2d'}, D^\text{ch}_{\Theta, H}(P) \otimes S\big).
\]

The lowest-weight component of \(\Gamma^{\text{ch}}(\pi, S)\) is \(S(M)\), the space of sections of the spinor bundle on \(M\). For \(f \in C^\infty(M)\) and \(X \in \mathcal{T}(M)\), there are associated fields \(\Phi_f(z)\), \(\Phi_{\ell X}(z)\) and \(\Phi_{\ell\chi}(z)\) on \(\Gamma^{\text{ch}}(\pi, S)\) of weights 0, 1 and \(\frac{1}{2}\) respectively; their Fourier modes are represented on \(D^\text{ch}_{\Theta, H}(P) \otimes S\) by

\[
f_n \otimes 1, \quad (\tilde{X} - \tau_i \tilde{X}^i \Theta_{ij})_n \otimes 1, \quad X^i_{n-r} \otimes e_{i,r}, \quad \text{for } n \in \mathbb{Z}.
\]

The actions of these fields on \(S(M)\) generate the entire space \(\Gamma^{\text{ch}}(\pi, S)\), subject only to the relations stated in Propositions 7.18 and 7.19. Moreover, there is a Virasoro field on \(\Gamma^{\text{ch}}(\pi, S)\) of central charge \(5d'\), whose Fourier modes are represented on \(D^\text{ch}_{\Theta, H}(P) \otimes S\) by

\[
(\tau_{i-\ell^2})_n \otimes 1 + 1 \otimes L^\ell_n \Theta^\alpha_{n-r} \otimes \ell^\ell_{n-r} - \frac{1}{2} \text{Tr}(\Theta_{-1} \Theta)_n \otimes 1, \quad \text{for } n \in \mathbb{Z}.
\]

This description of \(\Gamma^{\text{ch}}(\pi, S)\) allows us to define a natural filtration as follows.

### § 7.15. PBW filtration.

Given an increasing sequence of negative integers \(n = \{n_1 \leq \cdots \leq n_s < 0\}\), possibly empty, let us write

\[|n| = |n_1| + \cdots + |n_s| \quad (0 \text{ if } n = \{\}), \quad n(i) = \text{number of times } i \text{ appears in } n\]

and regard \(n\) as a partition of \(-|n|\). For any nonnegative integer \(w\), let \(\mathcal{A}_w\) be the set of triples \((n; m; p)\) of such sequences with \(|n| + |m| + |p| = w\) and \(m(i) \leq 2d'\) for all \(i < 0\). Define a partial ordering on \(\mathcal{A}_w\) by declaring that \((n; m; p) \prec (n'; m'; p')\) if one of the following is true:

- \(|n| < |n'|\), or \(|n| = |n'|\) and \(|m| < |m'|\)
- \(n'\) is a proper subpartition of \(n\), \(m = m'\) and \(p = p'\)
- \(n = n', m = m'\) and \(p\) is a proper subpartition of \(p'\)

For example in \(\mathcal{A}_3\), a particular ascending chain is

\[
(:, -2, -1) \prec (; -3) \prec (-1, -1, -1) \prec (-2; -1) \prec (-1, -1, -1)
\]

while \((; -1, -1, -1)\) and \((-1, -1, -1;)\) are the unique minimal and maximal elements.

Given a sequence \(n = \{n_1 \leq \cdots \leq n_s < 0\}\) as above and an \(s\)-tuple \(\alpha = (\alpha_1, \ldots, \alpha_s)\) in \(\Omega^1(M)\), or an \(s\)-tuple \(X = (X_1, \ldots, X_s)\) in \(\mathcal{T}(M)\), let us introduce the notations

\[
\alpha_n = \alpha_{n_1} \cdots \alpha_{n_s}, \quad cX_n = (cX_1)_n \cdots (cX_s)_n, \quad \ell X_n = (\ell X_1)_n \cdots (\ell X_s)_n \quad (1 \text{ if } n = \{\})
\]

which are operators on \(\Gamma^{\text{ch}}(\pi, S)\) (see 7.0). It follows from Theorem 7.14 that for \(w \geq 0\) we have

\[
\Gamma^{\text{ch}}(\pi, S)_w = \text{span}\left\{\ell X_n cY_m \alpha_p s : (n; m; p) \in \mathcal{A}_w; \text{ all suitable } \alpha, X, Y; s \in S(M)\right\}.
\]
Indeed, as $f_n = \frac{1}{n}(df)_n$ for $f \in C^\infty(M)$ and $n \neq 0$, Propositions 7.8 and 7.9 allow us to express every element of $\Gamma^{ch}(\pi, S)$ in the indicated form. For $(n; m; p) \in \mathcal{F}_w$ consider the subspaces

$$F_{\preceq}(n; m; p) = \text{span}\{\ell X_n \cdot c Y_m \cdot \alpha_{p}^s : (n'; m'; p') \preceq (n; m; p)\} \subset \Gamma^{ch}(\pi, S)_w$$

$$F_{\prec}(n; m; p) = \text{span}\{\ell X_n \cdot c Y_m \cdot \alpha_{p}^s : (n'; m'; p') \prec (n; m; p)\} \subset \Gamma^{ch}(\pi, S)_w$$

For the next statement, let $O$ (and $O'$) stand for one of the operators of the form $\alpha_n$, $c X_n$ or $\ell X_n$ with $n < 0$, and $fO$ the corresponding operator $(f\alpha)_n$, $(c fX)_n$ or $(\ell fX)_n$, where $f \in C^\infty(M)$. The subspaces $F_{\preceq}(n; m; p)$ and $F_{\prec}(n; m; p)$ have the following properties:

(i) $\cdots O O' \cdots \in F_{\preceq}(n; m; p) \Rightarrow \cdots [O, O'] \cdots \in F_{\prec}(n; m; p)$

(ii) $\cdots Os \in F_{\preceq}(n; m; p) \Rightarrow \cdots (fO)s - O(fs) \in F_{\prec}(n; m; p)$, $s \in S(M)$

Indeed (i) follows from the supercommutation relations in Proposition 7.9 and (ii) from the normal-ordered expansions there as well as Proposition 7.8. Consequently, there is a natural isomorphism

$$F_{\preceq}(n; m; p)/F_{\prec}(n; m; p) \cong \left(\bigotimes_{i < 0} \text{Sym}^{p(i)}_1(M)\right) \otimes \left(\bigotimes_{i < 0} \text{Sym}^{n(i)}_1 T(M)\right) \otimes \left(\bigotimes_{i < 0} \wedge^{m(i)}_1 T(M)\right) \otimes S(M)$$

where all tensor, symmetric and exterior products are over $C^\infty(M)$. Let $q$ be a formal variable. When all $(n; m; p) \in \mathcal{F}_w$ and all $w \geq 0$ are considered, we obtain

$$\bigoplus_{w \geq 0} q^w \left(\bigoplus_{(n; m; p) \in \mathcal{F}_w} F_{\preceq}(n; m; p)/F_{\prec}(n; m; p)\right) \cong \left(\bigotimes_{k \geq 1} \text{Sym}_{q^k} \Omega^1(M)\right) \otimes \left(\bigotimes_{k \geq 1} \text{Sym}_{q^k} T(M)\right) \otimes \left(\bigotimes_{k \geq 1} \wedge_{q^k} T(M)\right) \otimes S(M)$$

where $\text{Sym}_t = \sum_{n=0}^{\infty} t^n \text{Sym}^n$ and $\wedge_t = \sum_{n=0}^{2d'} t^n \wedge^n$ as usual.
Appendix §A. Vertex Algebroids

The notion of a vertex algebroid captures the part of structure of a vertex algebra that involves only its two lowest weights (see §1.2). Even though its definition is rather complicated, it serves as a convenient tool for handling the vertex algebras in this paper. This appendix reviews, mostly without proof, the category of vertex algebroids and an adjoint pair of functors between vertex algebras and vertex algebroids. For more details, the reader is referred to the original work [GMS04].

Definition A.1. An extended Lie algebroid \((A, \Omega, T)\) consists of:
- a commutative, associative \(\mathbb{C}\)-algebra with unit \((A, 1)\)
- an \(A\)-module \(\Omega\), together with an \(A\)-derivation \(d : A \to \Omega\) such that \(\Omega = A \cdot dA\)
- another \(A\)-module \(T\), equipped with a Lie bracket \([\ ]\)
- an \(A\)-linear map of Lie algebras \(T \to \text{End} \ A\), denoted by \(X \mapsto X\)
- a \(\mathbb{C}\)-linear map of Lie algebras \(T \to \text{End} \ A\), denoted by \(X \mapsto L_X\)
- an \(A\)-bilinear pairing \(\Omega \times T \to A\), denoted by \((\alpha, X) \mapsto \alpha(X)\)

Furthermore, it is required that:
- the \(T\)-actions on \(A\) and \(\Omega\) commute with \(d\)
- the \(T\)-actions on \(A, \Omega\) and \(T\) (via \([\ ]\)) satisfy the Leibniz rule with respect to \(A\)-multiplication
- \(df(X) = Xf\) for \(f \in A\) and \(X \in T\)

Definition A.2. A map of extended Lie algebroids \(\varphi : (A, \Omega, T) \to (A', \Omega', T')\) is simply a map of ordered triples that respects the extended Lie algebroid structures. Composition of maps is defined in the obvious way.

Definition A.3. A vertex algebroid \((A, \Omega, T, \bullet, \{\}, \{\})\) consists of an extended Lie algebroid \((A, \Omega, T)\) together with three \(\mathbb{C}\)-bilinear maps

\[
\bullet : T \times A \to \Omega, \quad \{\} : T \times T \to A, \quad \{\} : T \times T \to \Omega
\]

that satisfy the following identities:
- \(\{X, Y\} = \{Y, X\}\)
- \(d\{X, Y\} = \{X, Y\}_\Omega + \{Y, X\}_\Omega\)
- \(X \bullet (fg) - (gX) \bullet f - f(X \bullet g) = -(Xf)dg\)
- \(\{X, fY\} - f\{X, Y\} = -(Y \bullet f)(X) + [X, Y]f\)
- \(\{X, fY\}_\Omega - f\{X, Y\}_\Omega = -L_X(Y \bullet f) + [X, Y] \bullet f + Y \bullet (Xf)\)
- \(X\{Y, Z\}_\Omega - \{[X, Y], Z\} - \{Y, [X, Z]\} = \{X, Y\}_\Omega(Z) + \{X, Z\}_\Omega(Y)\)
- \(L_X\{Y, Z\}_\Omega - L_Y\{X, Z\}_\Omega + L_Z\{X, Y\}_\Omega + \{X, [Y, Z]\}_\Omega - \{Y, [X, Z]\}_\Omega - \{[X, Y], Z\}_\Omega = d\left(\{X, Y\}_\Omega(Z)\right)\)

for \(f, g \in A\) and \(X, Y, Z \in T\).

Remark. This definition is equivalent to, but slightly different from both the original one in [GMS04] with the notations \((\gamma, \{\}, c)\), and the one in [Che12] with the notations \((\ast, \{\}, \{\})\). The various notations are related as follows:

\[
X \bullet f = -\gamma(f, X) + dXf = f \ast X + dXf\]
\[
\{X, Y\} = \langle X, Y \rangle, \quad \{X, Y\}_\Omega = -c(X, Y) + \frac{1}{2}\langle X, Y \rangle.
\]

In this paper we adopt the above definition to simplify the description of certain vertex algebras.
Definition A.4. A map of vertex algebroids

\[(\varphi, \Delta) : (A, \Omega, T, \bullet, \{ \}, \{ \})_{\Omega} \to (A', \Omega', T', \bullet', \{ \}, \{ \})_{\Omega}'\]

consists of a map of extended Lie algebroids \(\varphi : (A, \Omega, T) \to (A', \Omega', T')\) together with a \(\mathbb{C}\)-linear map \(\Delta : T \to \Omega'\) such that

- \(\varphi X \bullet' \varphi f - \varphi(X \bullet f) = \Delta(f X) - (\varphi f)\Delta(X)\)
- \(\{\varphi X, \varphi Y\}' - \varphi\{X, Y\} = -\Delta(X)(\varphi Y) - \Delta(Y)(\varphi X)\)
- \(\{\varphi X, \varphi Y\}_{\Omega}' - \varphi\{X, Y\}_{\Omega} = -L_{\varphi X}\Delta(Y) + L_{\varphi Y}\Delta(X) - d(\Delta(X)(\varphi Y)) + \Delta([X, Y])\)

for \(f \in A\) and \(X, Y \in T\). Composition of maps is defined by

\[(\varphi', \Delta') \circ (\varphi, \Delta) = (\varphi' \circ \varphi, \varphi' \Delta + \Delta' \varphi|_{T}).\]

§ A.5. From vertex algebras to vertex algebroids: objects. Given a vertex algebra \((V, 1, T, Y)\), consider the following subquotient spaces

\[A := V_0, \quad \Omega := A_0(TA), \quad T := V_1/\Omega.\]

Choose a splitting \(s : T \to V_1\) of the quotient map to obtain an identification of vector spaces

\[(A.1) \quad \Omega \oplus T \cong V_1, \quad (\alpha, X) \mapsto \alpha + s(X).\]

The part of vertex algebra structure on \(V\) involving only the two lowest weights consists of an element \(1 \in V_0\), a linear map \(T : V_0 \to V_1\), and eight bilinear maps

\[V_i \times V_j \to V_k, \quad (u, v) \mapsto u_{j-k} v, \quad \text{for } i, j, k = 0, 1\]

satisfying a set of (Borcherds) identities. All these data, when rephrased in terms of the identification \((A.1)\), are equivalent to a vertex algebroid \((A, \Omega, T, \bullet, \{ \}, \{ \})\). The extended Lie algebroid \((A, \Omega, T)\) consists of precisely those data that are independent of the choice of \(s\), namely

\[
\begin{align*}
fg &:= f_0 g & f\alpha &:= f_0 \alpha & fX &:= f_0 s(X) \mod \Omega \\
Xf &:= s(X) f_0 & L_{X\alpha} &:= s(X)_{0\alpha} & [X, Y] &:= s(X)_{0} s(Y) \mod \Omega \\
\delta f &:= Tf & \alpha(X) &:= \alpha_1 s(X)
\end{align*}
\]

\[\text{[15] for } f, g \in A, \quad \alpha \in \Omega \quad \text{and} \quad X, Y \in T. \quad \text{The rest of the vertex algebroid structure is given by}
\]

\[(A.2) \quad X \bullet f := s(X)^{-1} f - s(fX)\]

\[\{X, Y\} := s(X) s(Y) - s([X, Y])\]

for \(f \in A\) and \(X, Y \in T\).

§ A.6. From vertex algebras to vertex algebroids: morphisms. Let \(\Phi : V \to V'\) be a map of vertex algebras. Then let \((A, \Omega, T, \bullet, \cdots)\) be the vertex algebroid associated to \(V\), and a splitting \(s : T \to V_1\); and similarly \((A', \Omega', T', \bullet, \cdots)\) associated to \(V'\) and \(s' : T' \to V'_1\). The part of data of \(\Phi\) involving only the two lowest weights, when rephrased in terms of identifications like \((A.1)\), are equivalent to a map of vertex algebroids \((\varphi, \Delta)\). It consists of the obvious map of ordered triples \(\varphi : (A, \Omega, T) \to (A', \Omega', T')\) induced by \(\Phi\), and a map \(\Delta : T \to \Omega'\) defined by

\[
\Delta(X) = \Phi s(X) - s'(\varphi X).
\]

\[\text{[15] For example, the definition of } Xf \text{ is indeed independent of } s \text{ because } \alpha_0 f = 0 \text{ for } f \in A \text{ and } \alpha \in \Omega.\]
§ A.7. From vertex algebroids to vertex algebras: objects. Let \((A, \Omega, \mathcal{T}, \bullet, \{ \}, \{ \})_\Omega\) be a vertex algebroid. In this discussion, \(f, g\) (resp. \(\alpha, \beta\)) (resp. \(X, Y\)) always denote general elements of \(A\) (resp. \(\Omega\)) (resp. \(\mathcal{T}\)). Let \(\mathcal{U}\) be a unital associative algebra with generators \(f_n, \alpha_n, X_n, n \in \mathbb{Z}\), and the relations

\[
\begin{align*}
1 & \mapsto f_n, \quad \alpha \mapsto \alpha_n, \quad X \mapsto X_n \quad \text{are linear} \\
\mathbf{1} & = \partial_n f_n, \quad (df)_n = -n f_n, \quad [f_n, g_m] = [f_n, \alpha_m] = [\alpha_n, \beta_m] = 0 \\
[X_n, f_m] & = (X f)_{n+m}, \quad [X_n, \alpha_m] = (L_X \alpha)_{n+m} + n \alpha(X)_{n+m} \\
[Y_n, f_m] & = [X, Y]_{n+m} + (\{X, Y\})_{n+m} + n \{X, Y\}_{n+m}
\end{align*}
\]

for \(n, m \in \mathbb{Z}\). The subalgebra \(\mathcal{U}_+ \subset \mathcal{U}\) generated by \(\{f_n\}_{n>0}\) and \(\{\alpha_n, X_n\}_{n \geq 0}\) has a trivial action on \(\mathbb{C}\). Let \(\tilde{V} := \mathcal{U} \otimes_{\mathcal{U}_+} \mathbb{C}\) be the induced \(\mathcal{U}\)-module and \(V := \tilde{V} / \sim\) the quotient module obtained by imposing the following relations for \(v \in \tilde{V}\):

\[
\begin{align*}
(f g)_n v & \sim \sum_{k \in \mathbb{Z}} f_{n-k} g_k v \\
(f \alpha)_n v & \sim \sum_{k \in \mathbb{Z}} f_{n-k} \alpha_k v \\
(f X)_n v & \sim \sum_{k \geq 0} f_{n-k} X_k v + \sum_{k < 0} X_k f_{n-k} v - (X \bullet f)_n v
\end{align*}
\]

Notice that the summations are always finite. It follows from the axioms of a vertex algebroid that (A.3)–(A.4) are consistent. \(^\square\) Define a vertex algebra structure on \(V\) as follows. The vacuum \(1 \in V\) is the coset of \(1 \otimes 1 \in \tilde{V}\). The infinitesimal translation \(T\) and weight operator \(L_0\) are determined by

\[
\begin{align*}
T 1 & = 0, \quad [T, f_n] = (1-n) f_{n-1}, \quad [T, \alpha_n] = -n \alpha_{n-1}, \quad [T, X_n] = -n X_{n-1} \\
L_0 1 & = 0, \quad [L_0, f_n] = -n f_n, \quad [L_0, \alpha_n] = -n \alpha_n, \quad [L_0, X_n] = -n X_n
\end{align*}
\]

which are consistent with (A.3)–(A.4); notice that actions of \(f_n, \alpha_n, X_n\) change weights by \(-n\). Identify \(A, \Omega, \mathcal{T}\) as subspaces of \(V\) via \(f = f_0 1, \alpha = \alpha_{-1} 1, X = X_{-1} 1\), and associate to them the fields

\[
\sum_n f_n z^{-n}, \quad \sum_n \alpha_n z^{-n-1}, \quad \sum_n X_n z^{-n-1}
\]

which are mutually local by (A.3). Now apply the Reconstruction Theorem \([PB04]\).

Remark. If \(V'\) is another vertex algebra whose associated vertex algebroid is \((A, \Omega, \mathcal{T}, \bullet, \{ \}, \{ \})_\Omega\), then by construction there is a canonical map of vertex algebras \(V \rightarrow V'\). If it is surjective (resp. bijective), then \(V'\) is said to be generated (resp. freely generated) by the vertex algebroid.

§ A.8. From vertex algebroids to vertex algebras: morphisms. A map of vertex algebroids

\[(\varphi, \Delta) : (A, \Omega, \mathcal{T}, \cdots) \rightarrow (A', \Omega', \mathcal{T}', \cdots)\]

induces a map \(\Phi : V \rightarrow V'\) between the freely generated vertex algebras by

\[
\begin{align*}
\Phi f & = \varphi f, \quad \Phi \alpha = \varphi \alpha, \quad \Phi X = \varphi X + \Delta(X) \\
\Phi f_n & = (\Phi f)_n \circ \Phi, \quad \Phi \alpha_n = (\Phi \alpha)_n \circ \Phi, \quad \Phi X_n = (\Phi X)_n \circ \Phi
\end{align*}
\]

for \(f \in A, \alpha \in \Omega, X \in \mathcal{T}, n \in \mathbb{Z}\). Indeed, these equations are consistent with (A.3)–(A.4).

Lemma A.9. Given a vertex algebroid \((A, \Omega, \mathcal{T}, \bullet, \{ \}, \{ \})_\Omega\), an isomorphism of extended Lie algebroids \(\varphi : (A, \Omega, \mathcal{T}) \rightarrow (A', \Omega', \mathcal{T}')\) and a \(\mathbb{C}\)-linear map \(\Delta : \mathcal{T} \rightarrow \Omega'\), if we define

\[
\bullet' : A' \times \mathcal{T}' \rightarrow \Omega', \quad \{ \}' : \mathcal{T}' \times \mathcal{T}' \rightarrow A', \quad \{ \}'_\Omega : \mathcal{T}' \times \mathcal{T}' \rightarrow \Omega'
\]

by the equations in Definition \([A,4]\), then \((A', \Omega', \mathcal{T}', \bullet', \{ \}', \{ \}'_\Omega)\) is a vertex algebroid and \((\varphi, \Delta)\) is by construction an isomorphism between the two vertex algebroids. \(\square\)

\(^{16}\) For example, \([X_n, (f Y)_m]\) can be evaluated by either taking the commutator first or expanding \((f Y)_m\) first. The resulting identity is already implied by the vertex algebroid axioms and does not lead to a new relation.
Example A.10. The vertex algebroids associated to a Lie algebra. Consider a Lie algebra \(\mathfrak{g}\) over \(\mathbb{C}\) and a vertex algebroid of the form \((\mathbb{C}, 0, \mathfrak{g}, 0, \lambda, 0)\) with \(\mathfrak{g}\) acting trivially on \(\mathbb{C}\). The second, fourth and last components are trivial by necessity. By Definition A.3 the conditions on \(\lambda : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}\) are

\[
\lambda(X, Y) = \lambda(Y, X), \quad \lambda([X, Y], Z) + \lambda(Y, [X, Z]) = 0
\]

i.e. it is an invariant symmetric bilinear form on \(\mathfrak{g}\). Let

\[
V_{\lambda}(\mathfrak{g}) = \text{vertex algebra freely generated by } (\mathbb{C}, 0, \mathfrak{g}, 0, \lambda, 0).
\]

In the case \(\mathfrak{g}\) is finite-dimensional, simple and \(\lambda\) equals \(k\) times the normalized Killing form, this is the same as the affine vertex algebra \(V_{\lambda}(\mathfrak{g})\). [Kac98, FB04]

§ A.11. PBW filtration of a freely generated vertex algebra. Given an increasing sequence of negative integers \(n = \{n_1 \leq \cdots \leq n_s < 0\}\), possibly empty, we write

\[
|n| = |n_1| + \cdots + |n_s| \quad (0 \text{ if } n = \{\}) \quad \text{number of times } i \text{ appears in } n
\]

and regard \(n\) as a partition of \(-|n|\). For \(w \geq 0\), let \(\mathcal{I}_w\) be the set of pairs \((n; m)\) of such sequences that satisfy \(|n| + |m| = w\). Define a partial ordering on \(\mathcal{I}_w\) by declaring that \((n; m) < (n'; m')\) if

- \(n < n'\), or
- \(n'\) is a proper subpartition of \(n\) and \(m = m'\), or
- \(n = n'\) and \(m\) is a proper subpartition of \(m'\)

For example, in \(\mathcal{I}_3\) a particular chain is given by \(\langle -2, -1 \rangle < \langle -3 \rangle < \langle -2, -1 \rangle < \langle -1, -1, -1 \rangle\), while \(\langle -1, -1, -1 \rangle\) and \(\langle -1, -1, -1 \rangle\) are the unique minimal and maximal elements.

Consider the vertex algebra \(V\) constructed in [A.7]. Given a sequence \(n = \{n_1 \leq \cdots \leq n_s < 0\}\) as above and an \(s\)-tuple \(\alpha = (\alpha_1, \ldots, \alpha_s)\) in \(\Omega\), or an \(s\)-tuple \(X = (X_1, \ldots, X_s)\) in \(T\), let

\[
\alpha_n = \alpha_{n_1} \cdots \alpha_{n_s}, \quad X_n = X_{1,n_1} \cdots X_{s,n_s} \quad (1 \text{ if } n = \{\})
\]

as operators on \(V\). It follows from the relations in [A.3] that

\[
V_w = \text{span} \left\{ X_n \alpha_m f : (n; m) \in \mathcal{I}_w; \text{ all suitable } \alpha, X; f \in A \right\}.
\]

For each \((n, m) \in \mathcal{I}_w\) define the subspaces

\[
\mathcal{F}_{\preceq(n; m)} = \text{span} \left\{ X_n \alpha_m f : (n'; m') \preceq (n; m) \right\} \subset V_w
\]

\[
\mathcal{F}_{\prec(n; m)} = \text{span} \left\{ X_n \alpha_m f : (n'; m') \prec (n; m) \right\} \subset V_w
\]

For the next statement, let \(O\) (and \(O'\)) stand for an operator of the form \(\alpha_n\) or \(X_n\) with \(n < 0\), and \(fO\) the corresponding operator \((f\alpha)_n\) or \((fX)_n\), where \(f \in A\). Observe that the subspaces just defined have the following properties:

(i) \(\cdots O'O \cdots \in \mathcal{F}_{\preceq(n; m)} \Rightarrow \cdots [O', O'] \cdots \in \mathcal{F}_{\prec(n; m)}\)

(ii) \(\cdots Og \in \mathcal{F}_{\preceq(n; m)} \Rightarrow \cdots ((fO)g - O(fg)) \in \mathcal{F}_{\prec(n; m)}, \quad f, g \in A\)

Indeed, (i) follows from (A.3) and (ii) from (A.4). These properties imply that

\[
\mathcal{F}_{\preceq(n; m)}/\mathcal{F}_{\prec(n; m)} \cong \left( \bigotimes_{i < 0} \text{Sym}^{n(i)} T \right) \otimes \left( \bigotimes_{i < 0} \text{Sym}^{m(i)} \Omega \right)
\]
where the tensor and symmetric products are over $A$. Let $q$ be a formal variable. When all $(n;m) \in \mathcal{I}_w$ and all $w \geq 0$ are considered, we obtain an isomorphism

$$
\bigoplus_{w \geq 0} \left( \bigoplus_{(n;m) \in \mathcal{I}_w} \mathcal{F}_{\leq (n;m)} / \mathcal{F}_{< (n;m)} \right) \cong \left( \bigotimes_{k \geq 1} \text{Sym}_q \mathcal{T} \right) \otimes \left( \bigotimes_{k \geq 1} \text{Sym}_q \Omega \right)
$$

where $\text{Sym}_q = \sum_{n=0}^{\infty} t^n \text{Sym}^n$ as usual. The subspaces $\mathcal{F}_{\leq (n;m)}$, $\mathcal{F}_{< (n;m)}$ and the above isomorphisms are natural, i.e. respected by maps described in Lemma A.12.

**Remark.** The definition of the subspaces $\mathcal{F}_{\leq (n;m)}$ and $\mathcal{F}_{< (n;m)}$ make sense for any vertex algebra.

**Lemma A.12.** If a vertex algebra $V$ has a conformal vector $\nu$ contained in $\mathcal{F}_{\leq (-1,-1)} \subset V_2$, then it is generated by its associated vertex algebroid.

**Proof.** Consider the vertex algebroid $(A, \Omega, T, \cdots)$ associated to $V$ and the vertex subalgebra $V' \subset V$ it generates. Let $f, g \in A$; $\alpha, \beta \in \Omega$; and $X \in T$. By definition, $V'_0 = V_0$ and $V'_1 = V_1$. Suppose $V'_i = V_i$ for $i \leq k - 1$, for some positive $k$, and let $u \in V_k$. It suffices to prove that $u \in V'$.

Clearly $\alpha_r u, X_r u \in V_{k-r} \subset V'$ for $r > 0$. Also, it follows from

$$
(fdg)_0 = \sum_{s \in \mathbb{Z}} f_{-s} (dg)_s = - \sum_{s > 0} s f_{-s} g_s + \sum_{s > 0} s g_{-s} f_s
$$

that we have $(fdg)_0 u \in V'$. Hence in fact

$$
\alpha_r u \in V' \text{ for } r \geq 0, \quad X_r u \in V' \text{ for } r > 0.
$$

This easily implies that $(\alpha_{-1} \beta)_0 u$, $(\alpha_{-2} 1)_0 u$ and $(X_{-1} \alpha)_0 u$ must all belong to $V'$. By assumption, $\nu$ is a sum of elements of the form $\alpha_{-1} \beta$, $\alpha_{-2} 1$ and $X_{-1} \alpha$, so that $ku = L_0 u = \nu_0 u \in V'$. Since $k > 0$, we have $u \in V'$ as desired.

**Lemma A.13.** If a vertex algebra $V$ has a conformal vector $\nu$ contained in $\mathcal{F}_{\leq (-1,-1)} \subset V_2$, then it has no nontrivial ideal consisting only of positive weights.

**Proof.** Use the same notations as in the proof of Lemma A.12. Let $I \subset V$ an ideal with $I_0 = 0$. Suppose $I_i = 0$ for $i \leq k - 1$, for some positive $k$, and let $u \in I_k$. It suffices to prove that $u = 0$.

Clearly $\alpha_r u, X_r u \in I_{k-r} = 0$ for $r > 0$. Also, it follows from (A.5) that $(fdg)_0 u = 0$. Hence in fact

$$
\alpha_r u = 0 \text{ for } r \geq 0, \quad X_r u = 0 \text{ for } r > 0.
$$

This easily implies that $(\alpha_{-1} \beta)_0 u = (\alpha_{-2} 1)_0 u = (X_{-1} \alpha)_0 u = 0$. By assumption, $\nu$ is a sum of elements of the form $\alpha_{-1} \beta$, $\alpha_{-2} 1$ and $X_{-1} \alpha$, so that $ku = L_0 u = \nu_0 u = 0$. Since $k > 0$, we have $u = 0$ as desired.

---

\[17\] If we extend the partial ordering on $\mathcal{I}_w$ to a total ordering, the latter will induce in an obvious way a filtration on $V_w$ whose associated graded space is the coefficient of $q^w$.
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School of Mathematics & Statistics, University of Sheffield, Hicks Building, Sheffield S3 7RH, U.K.
Email address: p.cheung@sheffield.ac.uk

45