Correlation of Dirac potentials and atomic inversion in cavity quantum electrodynamics

Agung Trisetyarso
School of Fundamental Science and Technology, Keio University
3-14-1 Hiyoshi, Kohoku-ku, Yokohama-shi, Kanagawa-ken 223-8522, Japan.
(Dated: 12 May 2010)

Controlling the time evolution of the population of two states in cavity quantum electrodynamics (QED) is necessary by tuning the modified Rabi frequency in which the extra classical effect of electromagnetic field is taken into account. The theoretical explanation underlying the perturbation of potential on spatial regime of bloch sphere is by the use of (Bagrov, Baldiotti, Gitman, and Shamsutdinova) BBGS-Darboux transformations on the electromagnetic field potential in one-dimensional stationary Dirac model in which the Pauli matrices are the central parameters for controlling the collapse and revival of the Rabi oscillations. It is shown that by choosing $\sigma_1$ in the transformation generates the parabolic potential causing the total collapse of oscillations; while $\{\sigma_2, \sigma_3\}$ yield the harmonic oscillator potentials ensuring the coherence of qubits.

PACS numbers: 42.50.Pq, 02.30.Uu
Keywords: Dirac potentials, atomic inversion, Darboux transformations

I. INTRODUCTION

Recent research on quantum theory scrutinizes the reconciliation between quantum and classical physics, and it attempts to take the advantages from the classical for constructing the new theories. For instance, the quantum information between sender and receiver can be teleported by involving classical information transmission generated by the joint measurement of EPR pair. In the modern research of experimental quantum information, the notions of Coulomb and photon blockades represent a recent resurgence of interest in the influence of classical in quantum physics. Here, we show that concurrent condition of the electromagnetic field behaving classically and quantumly in cavity QED provides the conditions for steering the dynamics of collapses and revival sequences of Rabi oscillations.

Rabi oscillations are the important features in the field of experimental quantum information, since the oscillations represent the coherence and decoherence of the qubits in various type of physical schemes of quantum computer. The famous Rabi oscillations are derived from the quantity associated as the atomic inversion, $W(t)$, defined as the variation in the excited- and ground-state populations

$$W(t) = P_e(t) - P_g(t).$$

When the electromagnetic field is classical, the oscillations in atomic inversion have constant amplitude. If the field is quantum, the oscillations consist of a sequence of collapses and revivals. The object of the control of the time evolution of the population of two states in cavity is to formulate the Hamiltonian $H$, which realizes the population quantity of excited- and ground-state.

In the recent years, there have been several efforts to control the collapses and revivals, in order to damp out or to ensure the stability of the oscillations by considering the trade-off of physical constraints in the related quantum computer scheme. Nevertheless, the general theory of how to control the atomic inversion is poorly comprehended.

On the theoretical side, Darboux transformations contributions into the wide range of physics fields have been widely known over centuries. Different with the other transforms such as the Fourier and Laplace transforms, the action of Darboux transform does not change the domain of the potential and state of the system, but it creates the new potential and state of the system. The output of Darboux transform is a set of new potential and state of the physical system. For instance, in nonlinear Schrödinger equation, Darboux transform generates the new solutions of the equation.

The Darboux transform is very useful to describe the evolution of a physical system due to the change of its Hamiltonian potential as found in wide range of nonlinear physics. However, its contribution to the field of quantum computation is still poorly understood.

Recently, there are several efforts in order to open the mystery of Darboux transformations into quantum computation. Samsonov et al., followed by Hussin et al., propose a novel technique to apply Darboux transformations into Jaynes-Cumming Hamiltonian. Bagrov et al. coined a Darboux transformation which does not violate the two-level system equation structure. These processes can be done by using the method of intertwining on the exactly solvable Dirac potentials which are equivalent to the Darboux transformations between Schrödinger potentials.

The application of Darboux transformations to one-dimensional Dirac equation is performed by intertwining...
the Hamiltonian in Dirac equation

\[ \hat{L}h_0 = h_1 \hat{L}, \]

where \( \hat{L} \) is intertwining operator and \( h_0 \) \((h_1)\) is the old (new) Dirac Hamiltonian. In the sense of perturbation theory, \( h_0 \) \((h_1)\) is unperturbed (perturbed) Dirac Hamiltonian, where \( h_1 = h_0 + V(\alpha, \beta) \), in which \( \{\alpha, \beta\} \) contribute into the perturbation terms. Recently, it has been shown that numerous problems in quantum mechanics can be solved under this technique.

To date, the closest theoretical result to the experimental was only successfully performed by Samsonov et al. by showing the contribution of pseudosymmetry potential in controlling the qubits dynamics of superconducting circuits based on Josephson tunnel junctions. In this paper, we would like to show that it is possible to control Rabi oscillations in cavity QED by considering the presence of extra classical electromagnetic field. It emerges the perturbation effect on the system and can be represented by one fold Darboux transformations on one-dimensional Dirac equation. The results suggest that there is a correlation of Dirac potentials and the atomic inversion.

The paper is organized as follows: in Subsection A, we introduce the method proposed by Bagrov et al., then we generalized the methods: Pauli matrix in operator \( B \) is not constrained in \( \sigma_3 \), but the use of \( \{\sigma_1, \sigma_2\} \) is also considered. Moreover, the potential is not only restricted in (pseudo)-scalar form, but also we allow the potential in the form of vector where the Pauli matrices act as the potential basis and the function as the potential coefficient. Subsection B contains the introduction of the mathematical model of atom-field interaction in cavity QED. Here, the Rabi frequency and oscillations of fully-quantum mechanical model are modified due to the presence of the extra classical effect of electromagnetic field. Section C provides the elucidation of the implementation of BBGS-Darboux transformations to semiclassical Rabi model. The outcome is to obtain the parameters in BBGS-Darboux transformations: \( \alpha(t), \beta(t) \), and \( |V(t)| \). The section covers two types of procedures: first, reformulation of the equation from Schrödinger equation into Dirac equation. Here, the representation of data is changed, but we do not control anything. Second, the action of Darboux transformation on the Dirac equation. In this step, we perturb the system to control the potential and state of the system. In Section D, it is shown that the choice of Pauli matrices in BBGS-Darboux transformations relates to the type of one-dimensional stationary Dirac potential of electromagnetic field: \( \{\sigma_1\} \) generates the parabolic potential (so-called Yukawa-Rabi oscillations) and \( \{\sigma_2, \sigma_3\} \) yields the harmonic oscillator potential.

### A. BBGS-Darboux transformations

In this Subsection, we introduce (Bagrov, Baldiotti, Gitman, and Shamshutdinova) BBGS-Darboux transformations which has the notation \( \mathcal{D}(\sigma_i) \). The symbol \( \sigma_i \) is the Pauli matrix in element of intertwining operator \( \hat{L} = A \frac{\partial}{\partial t} + B \), \( A = \sigma_0 = \{0, 1\} \), and \( B = \alpha_i(t) + (f(t) - \beta_i(t))\sigma_i \).

The action of \( \mathcal{D}(\sigma_i) \) on a set of potential and state of a Hamiltonian is defined by \( \mathcal{D}(\sigma_i)[N] \{V, \Psi\} = \{V[N], \Psi[N]\} \) transforming the old Hamiltonian \( \hat{h}(V)[\Psi] = \varepsilon_0 \Psi \) transforming \( \hat{h}(V[1])\Psi[1] = \varepsilon_1 \Psi[1] \) to \( \varepsilon_2 \Psi[2] \). The potential is a vector in three dimensional Euclidean Bloch sphere using Pauli matrices as the basis and it can be extended into \( n \)-dimensional. The transformations affect the potential from \( V(t) = \sum_j \sigma(j)^f_j(t) \rightarrow V[1](t) = \sum_j \sigma(j)^f_j(t) + \Delta f_j(t) \) and \( \sigma(j) \in \{\sigma_1, \sigma_2, \sigma_3\} \).

Based on this formulation, the magnitude of potential is transformed \( |V(t)| = \sum_j |f_j(t)| \rightarrow |V[N][t]| = \sum_j |f_j(t)| \). The effect of the transformations to the state is \( \Psi \rightarrow \hat{L}\Psi = \Psi[N] \). In order to obtain the coefficients \( \alpha(t) \) and \( \beta(t) \), intertwining operation, \( \hat{L}_{\text{int}} = \hat{h}_{\text{new}} \hat{L} \), between the old and the new Hamiltonian is needed. For \( N \)-fold BBGS-Darboux transformations, \( \hat{L}_0 = \hat{h}_1 \hat{L} \rightarrow \hat{L}_1 = \hat{h}_2 \hat{L} \rightarrow ... \rightarrow \hat{L}_{N-1} = \hat{h}_N \hat{L} \).

After the above definitions, we can now express the following theorem.

**Theorem 1.** Let \( \{V, \Psi\} \) represents a physical system and \( \mathcal{D}(\sigma_i)[N] \) is a BBGS-Darboux transformation operator. The open-loop control mechanism can be constructed, where the BBGS-Darboux transformation operator can be assumed as a controller, \( \{V, \Psi\} \) as the initial system, \( \{V[N], \Psi[N]\} \) as the final system, and the eigenvalues \( \varepsilon_N(\sigma_i) \) are the output variables.

**Proof.** Let \( \{V, \Psi\} \) is the initial condition of a physical system, in which the initial eigenvalues \( \varepsilon_0 \) are belong to the system. The action of BBGS-Darboux transformation on this system is \( \mathcal{D}(\sigma_i)[N] \{V, \Psi\} = \{V[N], \Psi[N]\} \), in which the eigenvalues \( \varepsilon_N(\sigma_i) \) are belong to the new eigenstates \( \Psi[N] \).

The complete proofs of the above theorem are given in Section III and Section III. In this paper, we only consider \( N = 1 \) or one fold Darboux transformations.

### B. Cavity QED model

Following the Ref. [21], the system of a two level atom and photons in a single cavity mode, if the electromagnetic field is classical, is represented by \( H = h\omega_0 \frac{\partial}{\partial t} + h\Omega[\sigma^+ b(t) + \sigma^- b(t)^\dagger] \) where \( b(t) = b_0 e^{-i\omega t} \). If the electromagnetic field is quantum, the Hamiltonian is \( H = h\omega_0 \frac{\partial}{\partial t} + h\Omega[\sigma^+ B + \sigma^- B^\dagger] \), where \( B \) \((B^\dagger)\) is annihilation (creation) operator and \( |\gamma|^2 \) is the average photon number.
It is shown that the atomic inversion of Jaynes-Cummings Hamiltonian is represented by
\[
W(t) = \frac{\langle \gamma, \uparrow | \sigma_z(t) | \gamma, \uparrow \rangle}{\langle \gamma, \uparrow | \sigma_z(0) | \gamma, \uparrow \rangle} = e^{-i\gamma^2} \sum_{n=0}^{\infty} \frac{|\gamma|^{2n}}{n!} (1 - 2\Omega^2(n + 1)^{\sin^2(\Omega t)}/(\Omega^2 n^2)) (3)
\]

The superscript ‘m’ denotes the modification. The equation (3) represents the condition that the electromagnetic field concurrently behaves classically and quantumly: the term \(b^*b\) corresponds to classical field and \(\sqrt{n + 1}\) is the consequence of quantum effect of the field. This term can be easily derived from a Hamiltonian driven by a classical light source as mentioned in Ref.\(^{22}\). Another way, it also might be obtained by transforming Jaynes-Cummings Hamiltonian as explained in Ref.\(^{22}\). Also, the modified atomic inversion is
\[
\Omega_n^m = \sqrt{\Omega(\kappa^2 + b^*b + n + 1)}.
\]

Darboux transformation contributes to the \(b^*b\) term. For \(N\)-fold Darboux transformations, it changes \(b^*b \rightarrow b_i[1]b_i[1] \rightarrow ... \rightarrow b_i[N]b_i[N]\), where subscript \(i\) corresponds to the choice of Pauli matrix in the intertwining operator.

II. RABI MODEL UNDER DARBOUX TRANSFORMATIONS

In this Section, we present the theoretical explanation underlying the perturbation effects of atom-field in cavity due to the presence of the classical fields. The perturbation causes the emergence of new potential in which it is perturbed on \(y\)-axis of Bloch sphere. To do so, it is accomplished through two step of procedures: first, the representation of the system is changed from Schrödinger equation into one-dimensional Dirac equation. The motivation of this change of representation is because one-dimensional Dirac equation is considered more promising rather than Schrödinger equation, since one-dimensional Dirac equation admitting vectors in their potential, instead of Schrödinger equation which its potential is in the scalar form only. Second, the action of Darboux transform is applied on the Dirac equation, physically meaning that the physical system is perturbed so that the potential and the state of the system can be controllable.

We rigorously evaluate BBGS-Darboux transformations in Rabi model, the results substitute into the modified Rabi frequency to obtain the dynamics of Rabi oscillation. The evaluation of BBGS-Darboux transformations in Rabi model involve three Pauli matrices: \(\{\sigma_1, \sigma_2, \sigma_3\}\). The outcomes of this evaluation are: first, the perturbation terms, i.e., the coefficients \(\alpha_1(t)\) and \(\beta_1(t)\) which are required to obtain the transformations operator, \(B\); second, the new Dirac potential of field, \(|V_1[1](t)|\). Moreover, we also obtain the computational result following the theory; to accomplish it, several methods are needed to simplify the differential equations form and also to obtain the exact solutions of the equations. We find that Homotopy Perturbation Method (HPM)\(^{32}\) is necessary when solving the coupled-nonlinear Ricatti differential equations problem for \(D(\sigma_1)[N = 1]\), while for \(D(\{\sigma_2, \sigma_3\})[N = 1]\), the choices produce ordinary differential equations only; however, one needs the Theorem in Ref.\(^{27}\) to simplify the equations in complex number form by finding their square roots, in case they can be easily simulated.

A. Rabi model in one-dimensional Dirac equation

The first procedure, i.e., the change of representation from Schrödinger equation into Dirac equation, is given in this Subsection. Rewrite Schrödinger equation into one-dimensional stationary Dirac equation of Rabi model
\[
\hat{h}\Psi = \varepsilon_0\Psi.
\]

\(\hat{h}\) is attained from \(V\) from \(\frac{d^2\Psi}{d^2x} + V\), \(V = -\hbar\Omega(\sigma^+b(t) - \sigma^-b^\dagger(t))\) and \(\varepsilon_0\) is \(\frac{\hbar\omega_0}{\Delta}\). We also use the following assumption \(b(t) = b_0e^{-i\omega t}\), and
\[
b_i[1](t) = 2\beta_i(t) - b(t).
\]

Because \(b_0\) is a constant and real, \(b_i[1] = b_0\), then.

The Dirac potential in the equation (5) can be easily changed into \(V(t) = \varepsilon_0\Omega(b_0\sigma_x\sin(\omega t) - \sigma_y\cos(\omega t))\) meaning that the potential is in vector form using Pauli matrices as orthogonal basis.

Although the idea of representing the physical system in one-dimensional Dirac equation may be better than Schrödinger equation, since the potential of Dirac Hamiltonian is in vector form, however it has the drawback: the atomic excitation term in the Hamiltonian is vanished in the new representation. One of justification for this drawback is that the new representation may be valuable to explain the phenomenon in which strong coupling is much greater than atomic excitation as observed in Ref.\(^{22}\).

B. The action of BBGS-Darboux transformation on Rabi model in one-dimensional Dirac equation

Following the procedure of representation is the action of Darboux transformation on the new representation. Below, the procedure is elucidated deeply.
The one fold BBGS-Darboux transformations change

\[ V(t) = \sum_{j=1}^{2} \sigma(j)^{\dagger} f_j = \sigma(1)^{\dagger} f_1(t) + \sigma(2)^{\dagger} f_2(t) \]

\[ = -\hbar \Omega (\sigma^+ b(t) - \sigma^- b^\dagger(t)) \]

\[ = i\hbar \Omega b_0 (\sigma_y \sin(\omega t) - \sigma_y \cos(\omega t)) \]

\[ \rightarrow V_1[t] = \sum_{j=1}^{2} \sigma(j)^{\dagger} f'_j = \sigma(1)^{\dagger} f'_1(t) + \sigma(2)^{\dagger} f'_2(t) \]

\[ = -\hbar \Omega (\sigma^+ b_0[1](t) - \sigma^- b_0^\dagger[1](t)) \]

\[ = -i\hbar \Omega (\sigma_y b_0(2\beta_1(t) - b_0 \cos(\omega t)) + \sigma_x b_0 \sin(\omega t))) \]

It is clear that the transformations change the initial electromagnetic field over the y-direction on the Bloch sphere in the magnitude of 2\beta_1(t). Due to the transformations, the potential magnitude changes

\[ |V(t)| = (\hbar \Omega b_0)^2 \rightarrow |V_1[1](t)| = 4\beta_1^2(t) - 4\beta_1(t)b_0 \cos(\omega t) + b_0^2. \tag{7} \]

(One can find by the similar manner that for Jaynes-Cummings model \( |V(t)| = (\hbar \Omega)^2(n + \frac{1}{2})^2 \).

From intertwining operation \( \hat{L} \hat{h} = \hat{h}_1 \hat{L} \), one can obtain the following equation:

\[ i\sigma_z \hat{B} + \hbar \Omega (\sigma^+ b_0[1](t) + \sigma^- b_0^\dagger[1](t)) \hat{B} - \hbar \Omega (B \sigma^+ b(t) + B \sigma^- b^\dagger(t)) - \hbar \Omega (\sigma^+ \hat{b}(t) + \sigma^- \hat{b}^\dagger(t))) = 0. \tag{8} \]

This is the master equation for the following Subsections.

1. \( \mathcal{D}(\sigma_1) \)

First, we consider \( \mathcal{D}(\sigma_1) \) to determine \( \alpha_1(t) \), \( \beta_1(t) \), and \( |V_1[1](t)| \). By substituting \( \hat{B} = \alpha_1(t) + i(b(t) - \beta_1(t)) \sigma_1 \) into the equation (8), it produces the coupled nonlinear Riccati differential equations as following

\[ \frac{\dot{\alpha}_1(t)}{2\hbar \Omega} - \beta_1^2(t) + \beta_1(t)(e^{i\omega t} + b_0 e^{-i\omega t}) + b_0 = 0, \tag{9a} \]

\[ -\dot{\beta}_1(t) + i\omega b_0 e^{-i\omega t} (\hbar \Omega - 1) + 2\alpha_1(t) \hbar \Omega \times (\beta_1(t) - \cos(\omega t)) = 0. \tag{9b} \]

which are subject to the following initial conditions

\( \beta_1(0) = (\beta_1)_0 = b_0 \) and \( \alpha_1(0) = (\alpha_1)_0 = \frac{ib_0(1-i\hbar \Omega)}{2\hbar \Omega (b_0 - 1)}. \)

In order to obtain the exact solution of these equations, we follow the suggestion for using the Homotopy Perturbation Method (HPM)\(^{39}\).

HPM is considered as a promising tool to solve exactly coupled nonlinear equations since the method successfully reconciles the homotopy theory with perturbation theory\(^{34,35}\). Prior to their amalgamation, it was generally difficult to obtain exact solutions of coupled nonlinear equations using separately homotopy theory or perturbation theory. Most perturbation methods face problems with parameters: the methods admit the existence of a small parameter, in contrary there is no small parameter in frequent nonlinear problems. On the other side, most homotopy methods fail in defining the deformation of most coupled nonlinear differential equations into simpler equations due to their complexities.

As pointed out by Sweilam et al., the modified HPM is an effort to cover some drawbacks in conventional HPM. In conventional HPM, the solution is found in a series in which often coincides with the Taylor series. The problems arise since the series have very slow convergent rate, therefore the solution under HPM may be inaccurate. For that reason, a new idea in this modified method is proposed by involving Padé approximant to enlarge the domain of convergence of the solution in which truncated in the Taylor series.

To do so, the truncated solution in Taylor series as found in conventional HPM is transformed by Laplace transform, followed by finding the Padé approximant, and finished by taking the inverse Laplace transform to obtain a more accurate solution of the problem. We try to apply these similar procedures in order to obtain the exact solutions of the equation (9). The complete algorithms are given in Appendix (A).

Starting from solving the system in the equation (9) by using conventional HPM, one can find that the linear equations in terms of \( p \), which is an embedding parameter, of these two equations are

\[ p^0 : \left( \frac{\dot{\alpha}_1}_0[t] \right) = 0, \]  \[ -\left( \frac{\dot{\beta}_1}_0[t] \right) = 0, \tag{10a} \]

\[ p^1 : \frac{\dot{\alpha}_1}_1[t]}{2\hbar \Omega} = b_0 + (\beta_1)_0[t] \]

\[ \times(e^{i\omega t} + e^{-i\omega t} b_0) + ((\beta_1)_0[t])^2; \tag{10b} \]

\[ (\beta_1)_1[t] = ie^{-i\omega t}(\hbar \Omega - 1) b_0 + 2(\alpha_1)_0[t] \hbar \Omega((\beta_1)_0[t] - \cos[\omega t]), \tag{11a} \]

\[ p^2 : \left( \frac{\dot{\alpha}_1}_2[t] \right) = (e^{i\omega t} + e^{-i\omega t} b_0)(\beta_1)_1[t] \]

\[ + 2(\beta_1)_1[t] (\beta_1)_1[t] \]

\[ + 2\hbar \Omega(\alpha_1)_0[t] (\beta_1)_1[t]. \tag{11b} \]

Furthermore, the series solution obtained by using the \( N \)-th order perturbation of HPM can be truncated as following
\[
\alpha_1(t) \approx \sum_{i=0}^{N} (\alpha_1)_i(t) = \frac{i \omega b_0 (1 - \hbar \Omega)}{2 \hbar \Omega (b_0 - 1)} - 2 \hbar \Omega (b_0^2 - b_0)(t + i \frac{1 - e^{-i \omega t}}{\omega}) + \mathcal{O}(t^2),
\]

(13a)

\[
\beta_1(t) \approx \sum_{i=0}^{N} (\beta_1)_i(t) = b_0 + (\hbar \Omega - 1)((1 - e^{-i \omega t}) + \frac{i \omega b_0}{(b_0 - 1)} (\frac{\sin(\omega t)}{\omega} - b_0)) + \mathcal{O}(t^2).
\]

(13b)

Then, the procedures involving Laplace transforms change \( \mathcal{L}[\alpha_1(t)] = \hat{\alpha}_1(s) \) and \( \mathcal{L}[\beta_1(t)] = \hat{\beta}_1(s) \) in which \( s \) is replaced by \( \frac{1}{t} \). The definition of Laplace transform \( \mathcal{L}[\alpha_1(t)] = \hat{\alpha}_1(s) \) can be found in the Appendix [A]. These procedures yield

\[
\alpha_1(t) \approx \sum_{i=0}^{N} (\alpha_1)_i(t) = \frac{i \omega b_0 (1 - \hbar \Omega)}{2 \hbar \Omega (b_0 - 1)} t - 2 \hbar \Omega (b_0^2 - b_0)t^2 (1 + i \frac{\omega}{1 + \omega^2 t^2} - b_0 t^2)) + \mathcal{O}(t^3),
\]

(14a)

\[
\beta_1(t) \approx \sum_{i=0}^{N} (\beta_1)_i(t) = b_0 t + (\hbar \Omega - 1)t^2 (\frac{\omega}{1 + \omega t} + \frac{i \omega b_0}{(b_0 - 1)} (\frac{1}{1 + \omega^2 t^2} - b_0 t^2)) + \mathcal{O}(t^3).
\]

(14b)

Padé approximant, \( \left[ \frac{N}{M} \right]_{(\alpha_1, \beta_1)}(t) \) with \( \{M, N\} > 0 \) and \( M + N < 6 \), is applied on the equations (14) to enlarge the convergence of these solutions followed by replacing the \( \frac{1}{t} \) by \( s \). The last step is obtaining the inverse Laplace transforms, \( \mathcal{L}^{-1}[(\hat{\alpha}_1(s), \hat{\beta}_1(s))] = \{\alpha_1(t), \beta_1(t)\} \), to obtain the true solutions as following

\[
\alpha_1(t) \approx \frac{1}{12} (b_0 - b_0^2) \hbar \Omega (3 \omega (4 t \omega + t^2 \omega^2 - 2) (\hbar \Omega - 1) + 4 \hbar \Omega^2 (b_0 - 1)^2 t (9 i \omega t + t^2 \omega^2 - 12)) + \mathcal{O}(t^3),
\]

(15)

\[
\beta_1(t) \approx \frac{2 i (2 b_0 - 1) (\hbar \Omega - 1) - i (2 b_0 - 2) e^{-i \omega t} (\hbar \Omega - 1) - i b_0 e^{i \omega t} (\hbar \Omega - 1) - i b_0^2 t^2 \omega (\hbar \Omega - 1) + 2 (b_0 - 1) t (b_0 + \hbar \Omega - 1)}{2 (b_0 - 1)} + \mathcal{O}(t^3)
\]

(16)

The Dirac potential of this transformation can be obtained by substituting the equation (16) into the equation (11). The plots of Rabi oscillations and potential versus time of this transformation are shown in the figure [11].

For the following Subsections, we find that the differential equations are easier and similar.

2. \( D(\sigma_3) \)

Second, we evaluate \( D(\sigma_3) \) to obtain \( \alpha_3(t), \beta_3(t), \) and \( |V_3[1]|(t) \). By replacing \( B = \alpha_3(t) + i (b(t) - \beta_3(t)) \sigma_3 \) into the equation (8), it yields

\[
\dot{\beta}_3(t) + i \alpha_3(t) + i \omega b_0 e^{-i \omega t} = 0,
\]

(17a)

3. \( D(\sigma_2) \)

Third, we examine \( D(\sigma_2) \) determine \( \alpha_2(t), \beta_2(t), \) and \( |V_2[1]|(t) \). By changing \( B = \alpha_2(t) + i (b(t) - \beta_2(t)) \sigma_2 \) into the equation (8), it results
Correlation of Dirac potentials and atomic inversion in cavity quantum electrodynamics

The equations in (17) and (18) are ordinary differential equations and they are easy to be solved. However, one needs the theorem as given in Appendix (B) to simplify the solutions, in case they are easily modelled.

Thus, from the equations (17) and (18), it can be obtained that $\alpha(t)_{(2,3)}$ and $\beta(t)_{(2,3)}$ for $D(\sigma_1, \sigma_3)[N = 1]$. Since the expression for

\[
2\hbar\Omega((\beta_2(t))^2 - \beta_2(t)b_0e^{-i\omega t}) + i\dot{\alpha}_2(t) \\
- (i\omega b_0e^{-i\omega t} + \dot{\beta}_2(t)) = 0,
\]

\[
-\hbar b_0(i\omega)(\sin(\omega t) + \cos(\omega t)) \\
+\hbar\alpha_2(t)(2\beta_2(t) - b_0e^{i\omega t}) - \hbar\alpha_2(t)b_0e^{-i\omega t} = 0.
\]

The potentials can be obtained by substituting those parameters into the equation (7). The plots of the Rabi oscillations and the potentials versus time are given in the figures (2) and (3), respectively.

III. CORRELATIONS OF THE ONE-DIMENSIONAL STATIONARY DIRAC FIELD POTENTIAL AND THE RABI OSCILLATIONS

In this Section, we evaluate our results in the previous Section which are believed casting much new light on the underlying physical problem.

The calculations in previous Section result $\beta_i(t)$ which are useful to obtain the transformations from $b^\dagger b \rightarrow b_i^\dagger b_i$ as defined by the equation (19). Furthermore, these new classical fields, $\{b_i^\dagger b_i | i \}$, are substituted into the
A. The parabolic potential

The graphs in this Section. (C) the detailed of computational methods for producing be achieved by 

D\text{(1(b))} for the Jaynes-Cummings state can be accomplished due to the drive transition from the ground state to an excited Jaynes-Cummings state. The Rabi oscillations is collapsed and extremely damped out to monotonous line. This phenomenon is similar to the previous results in Ref. and Ref. due to the classical drive field on resonance, the population is shared coherently between the states \( |g, 0\rangle \) and \( |l, 1\rangle \). where \( |l, 1\rangle = (|e, 0\rangle + |g, 1\rangle)\).

The figure (1(b)) shows that the Rabi oscillations is \( \sim e^{-\alpha t} \), since the form remembers to Yukawa’s potential, thus we name it by Yukawa – Rabi oscillations. This oscillations can be generated if the Dirac potential is parabolic as shown in fig (1(b)).

As can be seen in figure [1(a)], the absence of Dirac potential causes the atom inclined to occupy the ground state: during the potential is turned off, i.e., for \( t < 10 \), the first oscillations form wells meaning that initially the atoms occupy ground state, \( |g|n\rangle \), since the wells occupy the negative regime.

Furthermore, the potential is turned on after \( t = 10 \). It causes that the atomic population in ground state is gradually vanished and followed by coherently sharing the atomic population between the excited- and ground-state due to the presence of the potential. Under the circumstances, the Rabi oscillations are collapsed and extremely damped out to monotonous line.

The results may be very important for the future of quantum information. The choice of \( \sigma \) causes the coherent share between the excited- and ground-state meaning that in the case that strong interaction of classical field is much greater than atomic excitation in cavity quantum electrodynamics, the parabolic Dirac potential can change the qubit basis from \( \sigma_z \)-eigenbasis into \( \sigma_x \)-eigenbasis.

B. The harmonic oscillator potentials

The figure [2(a)] shows that \( D(\sigma_3) \) generates the sequences of peaks in negative regime which means the population is driven into the ground state. The sequences of peaks in Rabi oscillations can be created by adjusting fast oscillations of the Dirac potential in the positive regime as shown in the figure [2(b)]. Under this scheme, the sharpness of the peak in atomic inversion is directly proportional to the oscillations speed of Dirac potential: for instance, when \( t \sim 2000 \), the extreme sharp peak is produced if the fast oscillations of Dirac potential occur. Contrarily, when \( t \sim 6000 \), the unsharp peak is caused because the oscillations of Dirac potential are smooth and slow.

The emergence of these peaks in Rabi oscillations and the correlations with the Dirac potentials, may be relevant in the case of circuit quantum electrodynamics consisting of two superconducting qubits coupled to an on-chip coplanar waveguide (CPW) and explored by the power dependence of the heterodyne transmission as shown in Ref. Bishop et al. show that the emergences of peaks identified with a multiphoton-transmon qubit transition from the ground state to an excited Jaynes-Cummings state can be accomplished due to the drive power of heterodyne transmission. In contrast with our
results, the transition begins to saturate as the drive increases.

Furthermore, the choice of \( (\sigma_2) \) produces Rabi oscillations which are proportional to the oscillations of Dirac potential.

It is shown in the figure (3(a)) that initially the atom is in the excited state. The atomic population is gradually driven into the ground state along with the oscillations of potential gradually change from the positive- into the negative-regime.

This choice demonstrates that it is possible to ensure the coherence of the qubits without the collapse of Rabi oscillations by tuning the Dirac potential in the oscillations form across the positive-negative regimes as shown in figure (3(b)). This type of Rabi oscillations may be similar to the case of an atom which is initially in the excited state and field initially in a thermal state\(^{21}\).

IV. PHYSICAL IMPLEMENTATION

The proposed method suggests that the scheme is working under the apparatus in which the electromagnetic fields concurrently behave quantumly and classically. The one of possibilities to perform this is by involving instrument performing external forces behaving as artificial classical field.

In the recent research of photon in cavity, the atom-cavity system can be excited by photon transmission followed by the photon blockade of an optical cavity enclosing one trapped ion in the regime of strong atom-cavity coupling\(^{11}\). A single atom path is also controllable by the feedback of photon-by-photon\(^{12}\). However, it is not clear how photon, the quantum of the electromagnetic field, contributes to control the excitation and decay of the atomic population.

The next constraint is the classical field contributes into the modified Rabi oscillations in which the magnitude is \( \sim \sqrt{n/b} \). The possible experiment of classical field driving the atomic population in cavity may relate to the experimental research realizing the Keldysh picture\(^{20}\). Recent explorations demonstrate that controlling of electronic motion is enabled in ultrafast laser sources in the mid-infrared region\(^{14}\).

V. CONCLUSION

In this work, we show that the perturbation of atom-field in cavity due to the extra classical electromagnetic field can be used to control the atomic population in cavity. The perturbation theory on the system is described by the application of one fold Darboux transformations for the potential transformations of Rabi model. For simplicity, we use the result of this transformation to obtain the Rabi oscillations transformation by involving classical effect of electromagnetic in the equation. This method shows that it is possible to control the collapse and resurgence of Rabi oscillations in cavity QED under Darboux transformations.

The Pauli matrices are the parameters in the BBGS-Darboux transformations to determine the one-dimensional stationary Dirac potential of electromagnetic field which are responsible for controlling the Rabi oscillations. The appropriate choice of the parameters may be necessary for elucidating the surprising responses of the oscillations due to the external perturbations as found in Ref.\(^{14}\).

Based on the results, it is possible to propose an open-loop control mechanism for Rabi oscillations under Darboux transformations: the operator \( D(\sigma_i) \) is assumed as the controller and the \( \sigma_i \) is the system input. The initial system is \( \{V, \Psi\} \) and the final system is \( \{V[N], \Psi[N]\} \). The output variables which are read by the sensor are the new eigenvalues of the new Hamiltonian, \( \varepsilon_N(\sigma_i) \). The next challenge is defining an appropriate Darboux transformations so that the theoretical explanation underlying complete closed-loop control mechanism, as proposed in Ref.\(^{24}\), can be realized.

For further studies, it may be interesting to investigate the influence of \( N \)-fold Darboux transformations to the multi-qubits system for various types of quantum system in addition to Bose-Einstein condensation of exciton polaritons\(^{19}\). Anyons in a weakly interacting system\(^{45}\), etc. Furthermore, it is also very interesting if the Dirac potential is extended into \( n \)-dimensions and related to the various quantum systems, since the extension of the potential dimensions can be exploited to describe quantum transistor\(^{42}\).

We expect that this paper open the possibility to involve Darboux transformations into the extensive research of quantum information.

Acknowledgments - This work was supported in part by Grant-in-Aid for Scientific Research by MEXT, specially Promoted Research No. 1801002 and in part by Special Coordination Funds for Promoting Science and Technology. We also would like to thank Prof. Kohhi M. Itoh and Rodney Van Meter Ph.D, Toyofumi Ishikawa, Akhtar Waseem, Luis Jon Garcia, and Pierre-Andre Mortemousque for fruitful discussion.

VI. REFERENCES

1The remarkable feature under this representation is that the limit between quantum and classical is simply \( |b_0| = \sqrt{n + \frac{1}{2}} \).
2Alsing, P., Guo, D.-S., and Carmichael, H. J., “Dynamic stark effect for the jaynes-cummings system,” Phys. Rev. A\(^{45}\), 5135–5143 (1992).
3Anderson, Arlen, “Intertwining of exactly solvable dirac equations with one-dimensional potentials,” Phys. Rev. A\(^{43}\), 4602–4610 (1991).
Correlation of Dirac potentials and atomic inversion in cavity quantum electrodynamics

18

20

23

24

Birnbaum, K.M., Boca, A., Miller, R., Boozer, A.D., Northup, T.E., and Kimble, H.J., "Photon blockade in an optical cavity

Bennett, Charles H., Brassard, Gilles, Crépeau, Claude, Jozsa, Richard, Peres, Asher, and Wootters, William K., “Teleporting an unknown quantum state via dual classical and

Birnbaum, K.M., Boca, A., Miller, R., Boozer, A.D., Northup, T.E., and Kimble, H.J., “Photon blockade in an optical cavity

Birnbaum, K.M., Boca, A., Miller, R., Boozer, A.D., Northup, T.E., and Kimble, H.J., “Photon blockade in an optical cavity

Birnbaum, K.M., Boca, A., Miller, R., Boozer, A.D., Northup, T.E., and Kimble, H.J., “Photon blockade in an optical cavity

Birnbaum, K.M., Boca, A., Miller, R., Boozer, A.D., Northup, T.E., and Kimble, H.J., “Photon blockade in an optical cavity

Boehme, C. and Lips, K., “Electrical detection of spin coherence in silicon,” Physical review letters 91, 246603 (2003).

Brown, M., Schmidt-Kaler, F., Maali, A., Dreyer, J., Hagley, E., Raimond, J. M., and Haroche, S., “Quantum rabi oscillation: A direct test of field quantization in a cavity,” Phys. Rev. Lett. 76, 1800–1803 (1996).

Colosimo, P., Donny, G., Blaga, I. C., Wheeler, J., Hauri, C., Catoire, F., Tate, J., Chirla, R., March, A. M., Paulus, G. G., Muller, H. G., Agostini, P., and DiMauro, L. F., “Scaling foreground interactions towards the classical limit,” Nat Phys 4, 386–389 (2008).

Flagg, EB, Müller, A., Robertson, JW, Founta, S., Deppe, DG, Xiao, M., Ma, W., Salerno, G.J, and Shih, CK, “Resonantly driven coherent oscillations in a solid-state quantum emitter,” Nature Physics 5, 205–207 (2009).

Förstner, J., Weber, C., Danckwerts, J., and Knorr, A., “Phonon-assisted damping of rabi oscillations in semiconductor quantum dots,” Phys. Rev. Lett. 91, 124701 (2003).

Gell-Mann, Murray and Hartle, James B., “Classical equations for quantum systems,” Phys. Rev. D 47, 3345–3382 (1993).

Imamoglu, A., Schmidt, H., Woods, G., and Deutsch, M., “Strongly interacting photons in a nonlinear cavity,” Phys. Rev. Lett. 79, 1467–1470 (1997).

He, J.H.,”Homotopy perturbation method: a new nonlinear analytical technique,” Applied Mathematics and Computation 135, 73–79 (2003).

Hennessey, K., Badolato, A., Winger, M., Gerace, D., Atature, M., Golde, S., Falt, S., Hu, El, and Imamoglu, A., “Quantum nature of a strongly coupled single quantum dot–cavity system,” Nature 445, 896–899 (2007).

Hussin, V., Kuru, Ş., and Negro, J., “Generalized Jaynes–Cummings Hamiltonians,” Journal of Physics A: Mathematical and General 39, 11301–11311 (2006).

Kasprzak, J., Richard, M., Kundermann, S., Baas, A., Jeambrun, P., Keeling, MJM, Marchetti, FM, Szymańska, MH, et al., “Bose–Einstein condensation of exciton polaritons,” Nature 443, 409–414 (2006).

Keldysh, L V, “Ionization in the field of a strong electromagnetic wave(Multiphonon absorption processes and ionization probability for atoms and solids in strong electromagnetic field),” Soviet Physics-JETP 20, 1307–1314 (1965).

Knight, P. and Gerry, GC, Introductory Quantum Optics (Cambridge University Press, 2005).

Kosugi, Norihiro, Matsuo, Shigemasa, Konno, Kohkichi, and Hatakenaka, Noriyuki, “Theory of damped rabi oscillations,” Phys. Rev. B 72, 172509 (2005).

Kubanek, A., Koch, M., Sames, C., Ourjoumtsev, A., Pinkse, P. W. H., Murr, K., and Rempe, G., “Photon-by-photon feedback control of a single-atom trajectory,” Nature 462, 896–901 (2009).

Mabuchi, H. and Doherty, A. C., “Cavity Quantum Electrodynamics: Coherence in Context,” Science 298, 1372–1377 (2002).

Matveev, VB and Salle, MA, “Darboux transformations and solitons,” Springer Ser. Nonlinear Dynam., Berlin(1991).

Mooij, JE, Orlando, TP, Levitov, L., Tian, L., Van der Wal, C.H., and Lloyd, S., “Josephson persistent-current qubit,” Science 285, 1036 (1999).

Mottelson, B. R. and S. M., Introduction to higher algebra (Pergamon, 1964).

Nieto, L. M., Pecheritsin, A. A., and Samsonov, Boris F., “Interacting quantum field theory for the one-dimensional stationary dirac equation,” Annals of Physics 305, 151 – 189 (2003), ISSN 0003-4916, http://www.sciencedirect.com/science/article/B6WE1-48H33-2/2/1fff

Nomura, M., Kumagai, N., Iwamoto, S., Ota, Y., and Arakawa, Y., “Laser oscillation in a strongly coupled single-quantum-dot-nanocavity system,” Nat Phys 6, 279–283 (2010), http://dx.doi.org/10.1038/nphys1518

Puri, R. R. and Agarwal, G. S., “Collapse and revival phenomena in the jaynes-cummings model with cavity damping,” Phys. Rev. A 33, 3610–3613 (1986).

Rabinowitz, S., “How to Find the Square Root of a Complex Number,” Mathematics and Informatics Quarterly 3, 54–56 (1993).

Rudin, S. and Reinecke, TL, “Anharmonic oscillator model for driven and vacuum-field Rabi oscillations,” Physical Review B 63, 75308 (2001).

Sachdev, Subir, “Atom in a damped cavity,” Phys. Rev. A 29, 2627–2633 (1984).

Samsonov, Boris F and Negro, Javier, “Darboux transformations of the jaynes-cummings hamiltonian,” Journal of Physics A: Mathematical and General 37, 10115–10127 (2004), http://stacks.iop.org/0305-4470/37/10115

Samsonov, Boris F and Shamshtudinova, V V, “Dynamical qubit controlling via pseudo-supersymmetry of two-level systems,” Journal of Physics A: Mathematical and Theoretical 41, 244023 (2009), http://stacks.iop.org/1751-8121/41/244023

Shao, J.L., Suqing, D., and Zhao, X.G., “Dynamics of coupled chains in an ac electric and a magnetic fields,” Physics Letters A 330, 267–273 (2004).

Steck, Daniel A., Jacobs, Kurt, Mabuchi, Hideo, Hideo, Bhattacharya, Tammy, and Habib, Salmon, “Quantum feedback control of atomic motion in an optical cavity,” Phys. Rev. Lett. 92, 223004 (2004).

Stievater, TH, Li, X., Steel, DG, Gainmon, D., Katzer, DS, Park, D., Piermarocchi, C., and Sham, LJ. “Rabi oscillations of excitons in single quantum dots,” Physical Review Letters 87, 133603 (2001).

Swelham, NH and Khader, MM, “Exact solutions of some coupled nonlinear partial differential equations using the homotopy perturbation method,” Computers and Mathematics with Applications (2009).

Tian, L and Carmichael, H. J., “Quantum trajectory simulations of two-state behavior in an optical cavity containing one atom,” Phys. Rev. A 46, R6801–R6804 (1992).

Trisetyarso, Äûng, “Dirac four-potential tunings-based quantum transistor,” (2010), arXiv:1003.4590

Walls, DF and Milburn, G.J., Quantum optics (Springer, 2006).

Weeks, C., Rosenberg, G., Seradjeh, B., and Franz, M., “Anyons in a weakly interacting system,” Nat Phys 3, 796–801 (2007), http://dx.doi.org/10.1038/nphys830

Yukawa, H., “On the interaction of elementary particles,” in Proc. Phys. Math. Soc. Jap. Vol. 17 (1935) p. 48.

Appendix A: Homotopy Perturbation Methods

In this appendix, we provide the algorithm for obtaining exact solution of coupled nonlinear equations.

1. Unravel the differential equations using conventional HPM. In this step, the widely known Tay-
lor series is needed to expand the solution of the nonlinear differential equations, \( f(x) \), at the point \( x = 0 \). The series reads:

\[
f(x) = \sum_{i=0}^{\infty} c_i x^i = \sum_{i=0}^{\infty} \frac{f^{(i)}(0)}{i!} x^i.
\]  

(A1)

2. Shortened the series solution by conventional HPM. This step is accomplished by obtaining the embedding parameter of the solutions.

3. Obtain the Laplace transform of the shortened series. The well-known Laplace transform of a function \( y(t) \) reads

\[
L[y(t)] = \hat{y}(s) = \int_{0}^{\infty} y(t) e^{-st} dt.
\]  

(A2)

4. Acquire the Padé approximant of the prior step. The Padé approximant is a rational function to approach the Taylor series expansion as best as possible. It gives

\[
\left[ \frac{M}{N} \right] \frac{f(x)}{1 + \sum_{i=0}^{N} b_i x^i}
\]  

(A3)

where \( M \) and \( N \) are given positive integers.

5. Obtain the inverse Laplace transform.

Appendix B: Explicit representation for square root of a complex number

Following the Ref.\(^{27,31}\), we provide the theorem to obtain the explicit representation for square root of a complex number.

**Theorem.** A complex number \( \sqrt{a + ib} \) can be simplified into

\[
p + iq
\]  

(B2)

by defining

\[
p = \frac{1}{\sqrt{2}} \sqrt{a^2 + b^2 + a}
\]  

(B3)

and

\[
q = \frac{\text{sgn}(b)}{\sqrt{2}} \sqrt{a^2 + b^2 - a}
\]  

(B4)

where \( \{a, b, p, q \in \mathbb{R} \text{ and } (b \neq 0)\} \) and \( \text{sgn}(b) \) is defined as the sign of \( b \) (to be +1 if \( b > 0 \) and -1 if \( b < 0 \)).

Appendix C: Computational simulation by Maple

This Appendix provides the computational methods to generate graphs in the Section (III). In this work, we use Maple. The following codes show the codes for modified atomic inversion simulation.

\[
\text{FIG. 4: Maple codes for the modified atomic inversion.}
\]

The meaning of the parameters in the figure (1) are \( \text{Rm} \) is the modified Rabi oscillations, \( \text{sim} \) is the modified atomic inversion, \( \text{adj} \) is the adjustment for atomic inversion in case it is needed, \( b[0] \) is the amplitude of classical electromagnetic field, \( h[\bar{b}] \) is \( h \), \( \kappaappa \) is \( \kappa \), \( \omegaomega \) is \( \omega \), \( \OmegaOmega \) is \( \Omega \), and \( \text{sol} \) is the code for setting the value of the constants.

Below, the codes of \( \{D(\sigma_1), D(\sigma_3), D(\sigma_2)\} \) solutions are shown, respectively.

\[
\text{FIG. 5: Maple codes for the solution of } D(\sigma_1).
\]

\[
\text{FIG. 6: Maple codes for the solution of } D(\sigma_3).
\]

Especially for \( D(\sigma_2) \), we find the oscillation amplitudes are extremely huge, therefore the amplitude is scaled down in case it is possible to be simulated.
FIG. 7: *Maple* codes for the modified atomic inversion in the case of $D(\sigma_2)$.

\[
\begin{align*}
&\text{FIG. 8: The scheme for obtaining the correlations between atomic inversion and Dirac potentials.}
\end{align*}
\]