YUKAWA MATRICES FROM A SPONTANEOUSLY BROKEN ABELIAN SYMMETRY

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ABSTRACT

We classify all the phenomenologically viable fermion mass matrices coming from a spontaneously broken abelian symmetry $U(1)_X$, with one and two additional chiral fields of opposite charges $X = \pm 1$. We find that the non-trivial Kähler metric can fill zeroes of the fermion mass matrices up to phenomenologically interesting values. A general anomaly analysis shows that for one additional chiral field the only way to achieve anomaly cancellation is by use of the Green-Schwarz mechanism. For two additional fields with $X = \pm 1$ and negative charge differences in the lepton sector the anomalies can however be directly put to zero. This case gives a unique prediction for the ratio of the two Higgs scalars of MSSM, $\tan \beta \sim \frac{m_t}{m_b}(\sin \theta_c)^2$, where $\theta_c$ is the Cabibbo angle.

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1. The Froggat-Nielsen [1] idea of understanding fermion mass textures in terms of additional (to the Standard Model) $U(1)$ symmetries (global or gauged) has recently been reviewed in the context of supersymmetric models [2], [3], [4], [6]. The recent investigations go into two directions. First, several concrete models for the mass textures have been proposed [5]. Second, a potential connection has been pointed out between the mechanism for the fermion mass generation and the Green-Schwarz mechanism for anomaly cancellation, in case of gauged $U(1)$ symmetries [7]. This connection has been explored in some detail in the framework of models with “stringy” $U(1)$ symmetries spontaneously broken slightly below the string scale [8],[9] (another way of obtaining Froggatt-Nielsen structures in effective superstrings is described in [18]).

In this paper we generalize and systematize these results in several aspects. We begin with a general search for phenomenologically acceptable fermion mass textures in a model with one additional $U(1)$ symmetry and one or two additional chiral fields of opposite charge (singlets with respect to the SM gauge group but charged under the new $U(1)$). To make our search effective we derive general formulae for the CKM matrix elements in terms of the entries in the mass matrices (the formulae are given in the Appendix) which provide a useful supplement to the already existing expressions in the literature for the mass eigenvalues [4]. Equipped with both, we are able to find in our model all phenomenologically acceptable mass textures. It is interesting to observe that the number of acceptable textures is strongly limited: to four if we insist on getting all the physical observables with precisely the right order of magnitude and to few more if we allow for order $\lambda$ (Cabbibo angle) deviations in some of them. One can argue that this additional freedom is acceptable, given the fact that the model can predict the mass matrix entries only up to coefficients of order $O(1)$. The textures with some of the entries being zero are possible only if we accept to work with $O(\lambda)$ precision for some of the observables. It is interesting to notice that in such cases, in general, the zeroes disappear after renormalization which brings the Kähler potential to the canonical form. Thus, the final results for the mass matrices depend on those renormalization effects and do not have any zeroes.

The second part of the paper is devoted to a discussion of the potential anomalies introduced by the additional $U(1)$ symmetry. Starting with the observation that the quark mass matrices are invariant under shifts in $U(1)$ charges, it is legitimate to ask if this freedom can be used to cancel the anomalies. We derive a general formula for the anomalies in terms of the quark and lepton masses and prove that anomaly cancellation is impossible for physically acceptable textures with one singlet, or with two singlets if all the charge differences which enter the mass textures are positive. However, the Green-Schwarz mechanism of anomaly cancellation can always be realized (by the mentioned above redefinition of the $U(1)$ fermion charges) and, in fact, this requirement gives no constraint on the textures. Finally, we show that, with two singlets and negative charge differences, one finds solutions with anomaly cancellation achieved in the standard way, without asking for the Green-Schwarz mechanism. In this case, the prediction of the model is $\tan \beta \sim \frac{m_u}{m_b} \lambda^2 \sim 2$, independent of the possible $U(1)_X$ charge assignment for leptons.
2. The physical quantities, the fermion masses and the Kobayashi-Maskawa matrix are obtained by diagonalization of the mass matrices

\[
U_L m^U U_R^T = \text{diag} \left( m_u, m_c, m_t \right) \\
D_L m^D D_R^T = \text{diag} \left( m_d, m_s, m_b \right) \\
L_L m^L L_R^T = \text{diag} \left( m_e, m_\mu, m_\tau \right),
\]

(1)

the CKM matrix being given by \( V_{\text{CKM}} = U_L D_L^T \). There are 9 masses, 3 mixing angles and 1 CP violating phase to be compared with the experimental data. The known experimental data has a strongly hierarchical structure in the three families. Using the Cabibbo angle (\( \lambda = \sin \theta_c \sim 0.22 \)) as a small expansion parameter, the order of magnitude values of the mass ratios can be written in the form

\[
\frac{m_u}{m_t} \sim \lambda^8, \quad \frac{m_c}{m_t} \sim \lambda^4, \quad \frac{m_d}{m_b} \sim \lambda^4, \quad \frac{m_s}{m_b} \sim \lambda^2, \quad \frac{m_e}{m_\tau} \sim \lambda^4, \quad \frac{m_\mu}{m_\tau} \sim \lambda^2
\]

(2)
at a high energy scale \( M_{\text{GUT}} \sim 10^{16} \text{GeV} \). The hierarchical structure is also transparent in the CKM matrix. In the Wolfenstein parametrization of the known experimental data, \( V_{\text{CKM}} \) writes as

\[
V_{\text{CKM}} = \begin{pmatrix}
1 - \frac{\lambda^2}{2} & \lambda & \lambda^3 A \chi (\rho + i\eta) \\
-\lambda & 1 - \frac{\lambda^2}{2} & \lambda^2 A \chi \\
\lambda^3 A \chi (1 - \rho + i\eta) & -\lambda^2 A \chi & 1
\end{pmatrix},
\]

(3)

where \( A \sim 1, \lambda < \rho, \eta < 1 \) and the factor \( \chi \simeq 0.7 \) describes the running from \( M_Z \) to \( M_{\text{GUT}} \) [10]. The CP phase is neglected in this paper, so for our purposes \( V_{\text{CKM}} \) is just an orthogonal matrix.

A simple way to understand these structures is to postulate a family (horizontal) gauge symmetry spontaneously broken by the vacuum expectation values (vev’s) of some scalar fields \( \phi \) which are singlets under the Standard Model gauge group. The hierarchy of fermion masses and mixing angles is then explained by assignement of charges of the horizontal group such that invariant terms in the lagrangian (or superpotential in the supersymmetric case) have the form \( \left\langle \frac{\phi}{M} \right\rangle^{n_{ij}} \bar{\psi}_i \psi_j H \) (after decoupling of the heavy fields), where \( \psi_i \) are the SM fermions, \( H \) is a Higgs field and \( M \) is a large scale. Postulating \( \varepsilon \equiv \frac{\langle \phi \rangle}{M} \sim \lambda \) (the Cabibbo angle) one can easily explain hierarchies in the effective Yukawa couplings, even in the simplest case of abelian \( U(1)_X \) symmetry, with all the coefficients of the higher dimension operators of the order \( O(1) \).

3. In the following we investigate systematically all phenomenologically acceptable Yukawa matrices which can be obtained in a model with one horizontal \( U(1)_X \) symmetry and

a) one additional SM gauge singlet \( \phi \) of charge \( X = -1 \)

b) two additional SM singlets \( \phi \) and \( \bar{\phi} \) of opposite charges \( X = -1 \) and \( \bar{X} = +1 \) respectively.

To simplify the task it is very useful to have explicit formulae for the mass eigenvalues and the CKM mixing angles (i.e. for the rotation matrices \( U_L \) and \( D_L \)) directly in terms of the original Yukawa matrix entries. The former exist in the literature whereas the later are given in [11].
under the assumption of hierarchy between all rows and all columns, as in the original proposal of Froggatt-Nielsen [1]. In this case $U_L$ and $D_L$ are almost diagonal and the small mixing angles can be analytically computed. However, this assumption is too restrictive for a systematic study of all phenomenologically acceptable mass matrices in the model considered. Therefore, we present in the Appendix the formulae for the CKM mixing angles derived under a weaker assumption, namely that $m_{33} \geq m_{ij}$, $(i,j) \neq (3,3)$ for all the fermion matrices and that the rotation matrices $U_L$ and $D_L$ consist of small rotations, at most of order $O(\lambda)$. The latter assumption follows from the naturalness argument: the smallness of the CKM rotations should not be due to a relative fine-tuning of the $U_L$ and $D_L$.

The derivation is based on the formalism developed in [11]. The matrices $m$ can be diagonalized by three successive rotations in the $(2,3)$, $(1,3)$ and $(1,2)$ sectors (with the angles $S_{23}, S_{13}$ and $S_{12}$),

$$Q_L = \begin{pmatrix} 1 & -S_{12}^Q & 0 \\ S_{12}^Q & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -S_{13}^Q \\ 0 & 1 & 0 \\ S_{13}^Q & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -S_{23}^Q \\ 0 & S_{23}^Q & 1 \end{pmatrix}, \quad (4)$$

where $(Q_L = U_L, D_L)$. For all the textures to be discussed in the following, the matrices $U_R$ and $D_R$ (with the angles $S_{23}^{UQ}, S_{13}^{UQ}$ and $S_{12}^{UQ}$) are almost diagonal in the $(2,3)$ and $(1,3)$ sectors. In the $(1,2)$ sector they can be either almost diagonal or almost antidiagonal, the physical CKM matrix being the same in the two cases. Without loss of generality, we can consider the almost diagonal case; the antidiagonal one differs by just a permutation of it and need not a particular treatment.

We consider now case (a), with one SM gauge singlet. Due to our ignorance of the exact coefficients multiplying the powers of $\varepsilon$ in the Yukawa couplings and of the present experimental uncertainties of (2) and (3) we allow for mass matrices resulting in predictions compatible with these data up to few deviations $O(\lambda)$ in some observables. Our goal is to classify the best textures according to the number of these deviations. Supersymmetry is assumed, so that the Yukawa couplings are encoded in the superpotential. In this case, if the sum of the $U(1)_X$ charges of the fields corresponding to a specific Yukawa interaction is negative, the holomorphicity of the superpotential forbids this particular Yukawa coupling and the corresponding element is zero. Zeroes in the mass matrices are generically indicating the appearance of additional symmetries beyond the Standard Model and were largely investigated in the literature Refs.[12], [13], [14].

The part of the $U(1)_X$ invariant superpotential of the theory responsible for the quark and lepton masses is

$$W = \sum_{ij} \left[ Y_{ij}^{U} \theta (q_i + u_j + h_2) \left( \frac{\phi}{M} \right)^{q_i+u_j+h_2} Q^i U^j H_2 ight] + Y_{ij}^{D} \theta (q_i + d_j + h_1) \left( \frac{\phi}{M} \right)^{q_i+d_j+h_1} Q^i D^j H_1 + Y_{ij}^{E} \theta (l_i + e_j + h_1) \left( \frac{\phi}{M} \right)^{l_i+e_j+h_1} L^i E^j H_1 \right], \quad (5)$$
where $Y^{U,D,E}_{ij}$ are numbers of $O(1)$. We denote the fields and their $U(1)_X$ charges by the same capital and small letters, respectively.

The general Kähler potential consistent with the $U(1)_X$ symmetry reads

$$K = \sum_{\Phi=Q^i,U^i,D^i,L^i,E^i,H_1,H_2} Z^\Phi_{ij} \left[ \theta (\varphi_i - \varphi_j) \left( \frac{\phi}{M} \right)^{\varphi_i - \varphi_j} + \theta (\varphi_j - \varphi_i) \left( \frac{\phi^+}{M} \right)^{\varphi_j - \varphi_i} \right] \Phi^i \Phi^+ j , \quad (6)$$

where $Z^\Phi_{ij}$ are numbers. The physical Yukawa couplings are obtained by the canonical normalization of the kinetic terms and are given by

$$Y^U_{ij} = \left( K^{-1/2} \right)^{i'j'}_i \frac{\partial^3 W}{\partial Q^{i'} \partial U^{j'} \partial H_2} \left( K^{-1/2} \right)^{j'j}_j , \quad (7)$$

and similar expressions for $Y^{D,E}$. The potential effect of the kinetic terms in (7) is to remove the eventual zeroes. Consequently, they can change the physical predictions of the texture. Some examples with zero filling of phenomenological interest will be given below.

Using the shorthand notation $\xi^q = \theta(q)\xi^q$ and defining the parameter $x$ by $h_1 = -q_3 - d_3 + x$ , we can write

$$Y^D = \xi^x \begin{pmatrix}
\xi_{q_13 + d_13} & \xi_{q_13 + d_23} \\
\xi_{q_23 + d_13} & \xi_{q_23 + d_23} \\
\xi_{d_13} & \xi_{d_23} \\
\xi_{d_13} & 1
\end{pmatrix} , \quad (8)$$

where $q_{ij} = q_i - q_j$, $d_{ij} = d_i - d_j$. Similar expressions hold for $Y^E$ (with the same $x$ since $Y^D_{33} \sim Y^E_{33}$, see ref. [15], implying $h_1 = -l_3 - e_3 + x$) and for $Y^U$ (with $x = 0$ in order to accomodate a heavy top mass). Only the charge differences appear in (8), so a specific texture will fix them. The residual freedom in the values of the charges will be further restricted by the anomaly cancellation conditions, to be discussed later on.

Let us now turn to the allowed textures, according to the announced $O(\lambda)$ deviation rule. An important point to emphasize is that a permutation of the first two columns independently for $Y_U$, $Y_D$ (first two lines simultaneously for $Y_U$, $Y_D$) of the mass matrices has as only effect changing the right-handed (left-handed) diagonalizing angles $S^{ij}_l$ ($S^{ij}_l$), the masses and $V_{CKM}$ being unchanged. So all proposed solutions have 7 other possibilities, related to each other by the permutations $u_1 \leftrightarrow u_2$, $d_1 \leftrightarrow d_2$, $q_1 \leftrightarrow q_2$.

To search for all phenomenologically acceptable mass matrices we scan over all possible charge assignments and use the formulae in the Appendix to check the resulting matrices.

The results are:

- Only one solution which gives the right order of magnitude in $\lambda$ for all masses and mixing angles and it is given by

$$q_{13} = 3, \quad q_{23} = 2, \quad u_{13} = 5, \quad u_{23} = 2, \quad d_{13} = 1, \quad d_{23} = 0$$

(9)
and corresponds to the original proposal of Ref.[1]. It is characterized by having no zeroes in the mass matrices. Four other solutions with one \(O(\lambda)\) deviation are obtained by the changes\[u_{i3} \rightarrow u_{i3} \pm 1, \quad d_{i3} \rightarrow d_{i3} \pm 1, \quad i = 1, 2,\]
and others with two deviations in combining them.

- One solution with two \(O(\lambda)\) deviations corresponds to one zero (filled as in the Eq.(7)) in \(Y^D\). The charge assignments for \(q_{ij}, u_{ij}\) and \(d_{ij}\) are

\[
q_{13} = 4, \quad q_{23} = 3, \quad u_{13} = 4, \quad u_{23} = 1, \quad d_{13} = 1, \quad d_{23} = -1 .
\]

The two deviations are \(\frac{m_{u_{i3}}}{m_{b}} \sim \lambda^5\) and \(V_{cb} \sim \lambda^3\).

- One solution with two \(O(\lambda)\) deviations, with two filled zeroes in \(Y^D\). The charge assignments are

\[
q_{13} = 4, \quad q_{23} = 3, \quad u_{13} = 4, \quad u_{23} = 1, \quad d_{13} = d_{23} = -1 .
\]

The deviations are \(\frac{m_{u_{i3}}}{m_{b}} \sim \lambda^3\) and \(V_{cb} \sim \lambda^3\).

In the last two solutions, the zero filling affects only the right angles \(S'_{13}\) and \(S'_{23}\), with a possible effect in the analysis of the neutral currents constraints [16], [17].

- One solution with two \(O(\lambda)\) deviations with two filled zeroes in \(Y^U\) and two filled zeroes in \(Y^D\). The charge assignment is

\[
q_{13} = -2, \quad q_{23} = -3, \quad u_{13} = 10, \quad u_{23} = 7, \quad d_{13} = 6, \quad d_{23} = 5 .
\]

The two deviations are in \(V_{ub}\) and \(V_{cb}\). Before the zero filling \(V_{ub} \sim \lambda^{12}\) and \(V_{cb} \sim \lambda^{11}\) so the example would not fulfill our criterion of \(O(\lambda)\) deviation, the predictions being completely wrong. After the zeroes filling, the predictions change \(V_{ub} \sim \lambda^2, V_{cb} \sim \lambda^3\) and are much closer to the correct results, Eq.(3). This example shown that the kinetic terms in Eq.(7) can play an important role and cannot generally be neglected.

The charges of the fields for the textures which results from the most successful solution (9) are collected in Table 2.

Case (b), with a vector-like pair of singlets, offers (in addition to (9) which is a solution in this case, too) additional textures which fit exactly the experimental data (2) and (3). They are characterized by a big, negative charge difference, all the others being positive. For the up-quarks, the assignement is

\[
u_{13} = -11, \quad u_{23} = 2, \quad q_{13} = 3, \quad q_{23} = 2 .
\]

For the down-quark masses it is

\[
d_{13} + 2x = -7, \quad d_{23} = 0, \quad q_{13} = 3, \quad q_{23} = 2 ,
\]

where \(\varepsilon^x = \frac{\lambda^b}{\lambda^t}\). This corresponds to the matrix

\[
Y_D = \varepsilon^x \begin{pmatrix}
\varepsilon^4 & \varepsilon^3 & \varepsilon^3 \\
\varepsilon^5 & \varepsilon^2 & \varepsilon^2 \\
\varepsilon^7 & 1 & 1
\end{pmatrix}.
\]
The two assignements for \( u_{13} \) and \( u_{23} \) in (9) and (13) can be combined with those for \( d_{13} \) and \( d_{23} \) in (9) and (14) to give four solutions. Remark that the textures with negative charge differences are characterized by an anti-hierarchy structure in the first column, the two other columns being the same as in Eq.(9). There are other solutions with some \( O(\lambda) \) deviations the case (b) which are not displayed here. They are less interesting than the corresponding one in case (a) in the sense that the Kähler potential plays no role in their structure. All of the above solutions are collected in Table 1.

A universal prediction of all these models is \( V_{us} \sim V_{ub}/V_{cb} \). On the other hand, the parameter \( x \), related to the \( \tan\beta \) parameter of the MSSM by \( \tan\beta = \frac{m_t}{m_b} x \) is arbitrary, being a priori unconstrained by the experimental data (2),(3). Finally, we use the assignements (9)-(14) in the analysis of the gauge anomaly cancellation conditions to be discussed in the next paragraph.

| Number singlets | \( Y_U \) | \( Y_D \) | Number deviations |
|-----------------|----------|----------|------------------|
| 1 \( \lambda^8 \lambda^5 \lambda^3 \) \( \lambda^7 \lambda^4 \lambda^2 \) \( \lambda^6 \lambda^2 1 \) 0 |
| 2 \( \lambda^8 \lambda^5 \lambda^3 \) \( \lambda^9 \lambda^4 \lambda^2 \) \( \lambda^{11} \lambda^2 1 \) 0 |
| 1 \( \lambda^8 \lambda^5 \lambda^4 \) \( \lambda^7 \lambda^4 \lambda^3 \) \( \lambda^6 \lambda^3 1 \) 2 |
| 1 \( \lambda^8 \lambda^5 \lambda^2 \lambda^3 \) \( \lambda^7 \lambda^4 \lambda^2 \lambda^3 \) \( \lambda^{10} \lambda^7 1 \) 2 |

**TABLE 1:** Phenomenologically interesting quark mass matrices from a horizontal \( U(1)_X \) symmetry. The underlined entries are filled zeroes as in Eq.(7). The solutions \( (Y_U, Y_D) \) not separated by a horizontal line can be freely combined. The first two columns (independently for \( Y_U, Y_D \)) or the first two lines (simultaneously for \( Y_U, Y_D \)) can be interchanged without changing the physical predictions. Solutions with two singlets and some \( O(\lambda) \) deviations are not displayed.
Let us now turn to the question of anomaly cancellation. The new abelian gauge group $U(1)_X$ is potentially anomalous. It generates, through triangle diagrams, mixed anomalies with the standard model gauge group. The anomaly conditions in connection with mass textures were considered in [3] and [4] in the particular case of $h_1 + h_2 = 0$. Moreover, ref.[3] concentrates on left-right symmetric mass matrices, while ref.[4] assumes $h_1 = h_2 = 0$, and positive charge differences $q_i, u_i, d_i, l_i, e_i$ for $i = 1, 2$.

We relax here these restrictions and analyse the general case. Indeed, as remarked here above there exist acceptable textures with negative $X$-charge differences with two singlets. Moreover, the value of $h_1 + h_2$ constrains the $\mu$-term of the MSSM and there is no compelling reason to assume it to vanish. The mixed anomalies to be considered are the following

$$[SU(3)]^2 U(1)_X : \quad A_3 = \sum_{i=1}^{3} (2q_i + u_i + d_i) ,$$

$$[SU(2)]^2 U(1)_X : \quad A_2 = \sum_{i=1}^{3} (3q_i + l_i) + h_1 + h_2 ,$$

$$[U(1)_Y]^2 U(1)_X : \quad A_1 = \sum_{i=1}^{3} \left( \frac{1}{3} q_i + \frac{8}{3} u_i + \frac{2}{3} d_i + l_i + 2e_i \right) + h_1 + h_2 ,$$

$$U(1)_Y [U(1)]^2 X : \quad A'_1 = \sum_{i=1}^{3} \left[ q_i^2 - 2u_i^2 + d_i^2 - l_i^2 + e_i^2 \right] - h_1^2 + h_2^2 . \quad (16)$$

The last anomaly to be considered in principle is $[U(1)_X]^3$. It must take a precise value depending on the $U(1)_X$ Kac-Moody level if the anomalies are cancelled by the Green-Schwarz mechanism and it should vanish if the the other anomalies vanish. Since the theory could have other Standard Model singlets charged under $U(1)_X$ with no vev’s, we don’t consider $A_X$ in this paper. For the same reason we do not consider mixed gravitational anomalies.

It is useful to use the variable $x$

$$x = h_1 + q_3 + d_3 = h_1 + l_3 + e_3 , \quad h_2 + q_3 + u_3 = 0 \quad (17)$$

to write the following linear combinations of the anomalies

$$A_1 + A_2 - 2A_3 = 2x + \sum_{a=1}^{2} \left[ \frac{2}{3} (q_{a3} + u_{a3}) - \frac{4}{3} (q_{a3} + d_{a3}) + 2 (l_{a3} + e_{a3}) \right] ,$$

$$A_3 + 3(h_1 + h_2) = \sum_{a} [(q_{a3} + u_{a3}) + (q_{a3} + d_{a3})] + 3x . \quad (18)$$

For a given mass texture, the charge differences $(q_{a3} + u_{a3})$, $(q_{a3} + d_{a3})$, $(l_{a3} + e_{a3})$ are fixed for $a = 1, 2$, and so is the r.h.s. of eq.(18). The remaining freedom in the charge assignement can then be utilized to cancel the anomalies. It is appropriate to approach this issue in terms of the familiar invariances of the SM fermion mass matrices: $SU(3) \times SU(2) \times U(1)_Y$, the baryon number
B, the lepton number L and the Peccei-Quinn symmetry P, with a charge 1 for all the matter fields and −2 for the two Higgs doublets. Since the charge X of the U(1)_X has to commute with the Standard Model gauge group the allowed shifts in X are given by a linear combination of 4 abelian charges:

\[ \hat{X} = X + \beta Y + \gamma (B - L) + \delta (3B + L) + \zeta P \] (19)

without changing the mass textures. This redefinition can be used to achieve an anomaly free theory by implementing either the Green-Schwarz condition (with \( \sin^2 \theta_W = \frac{2}{3} \)) through the relations, \( \frac{4}{3}A_1 = A_2 = A_3 \), or through the usual one, \( A_1 = A_2 = A_3 = 0 \), provided, of course, the eqs.(18) are satisfied. In both cases one must also impose \( A'_1 = 0 \). Now, since \( (B - L) \) and, obviously, \( Y \) have no SM anomalies, one can only take advantage of the shifts \( \delta \) and \( \zeta \) to change \( A_1, A_2, A_3 \). Then, one can use a combination of \( (B - L) \) and \( Y \) to obtain \( A'_1 = 0 \). However \( A'_1 \) and, a fortiori, \( A_1, A_2, A_3 \), are all invariant under the following shift

\[ \hat{X} = X + \eta [A_1Y - (A_1 + A_2 + \frac{4}{3}A_3 - 2h_1 - 2h_2)(B - L)] . \] (20)

This residual freedom is tacitly assumed in our results from now on. It could eventually be fixed by studying other physical consequences of the U(1)_X gauge symmetry than the mass matrices themselves.

Therefore the redefinition (19) of the charge can be used to adjust the anomalies in (16) without affecting the combinations (18). As we shall see in the next paragraph, those combinations have in the model considered a direct physical meaning: they can be expressed in terms of the determinants of the mass matrices. This is certainly an interesting feature of the proposed solution for the Yukawa hierarchies. It is worth remarking that the equation \( A'_1 = 0 \) is linear in the shift parameters. This property follows from \( \text{Tr } Y B^2 = \text{Tr } Y L^2 = 0 \).

5. We now turn to the implementation of the anomaly cancellation in the specific models. Let us first assume that the charge differences are all positive, but without restricting the values of \( h_1 \) and \( h_2 \). Then we can readily write the relations between the Yukawa determinants and anomaly combinations as follows

\[ \det (Y_U Y_D^{-2} Y_L^2) = \varepsilon^{3/2(A_2 + A_1 - 2A_3)} , \]
\[ \det (Y_U Y_D) = \varepsilon^{A_3 + 3(h_1 + h_2)} , \] (21)

which can also be combined into the relation

\[ \det (Y_D^{-2} Y_L^2) = \varepsilon^{A_2 + A_1 - \frac{5}{3} A_3 - 2(h_1 + h_2)} . \] (22)

Clearly, the anomaly combinations appearing in these mass determinants are precisely those that are invariant under the symmetries of the mass matrices given by (19). The first of Eqs.(21) is
independent of the value of $h_1 + h_2$. It is the best-suited relation to decide if the anomalies can be put to zero or not, using the known fermion masses.

For positive charge differences, any mass matrices that approximately satisfy the phenomenological mass ratios (2) entail the following results:

\begin{align*}
A_1 + A_2 - 2A_3 & \simeq 12 + 2x , \\
A_1 + A_2 - \frac{8}{3}A_3 & \simeq 2 (h_1 + h_2) .
\end{align*}

The vanishing of the anomalies would require the physically excluded value $x = -6$. However, as noticed in [3] and generalized in [4], it is possible to implement the Green-Schwarz mechanism with $\sin^2 \theta_W = \frac{3}{8}$ at the unification scale [7] if $A_2 = A_3 = \frac{3}{5}A_1$ and $h_1 + h_2 = 0$. But, through the redefinition (19) of $X$, the parameters $\delta$ and $\zeta$ can always be chosen so that, $A_2 = \frac{3}{5}A_1 = 3(6 + X)$ and $h_1 + h_2 = 0$, as required. In this case, the Green-Schwarz mechanism will cancel the anomaly and $\sin^2 \theta_W = \frac{3}{8}$.

Furthermore, this result immediately generalizes if there are negative charge differences in case (a) of the preceding paragraph, where one additional SM gauge singlet is assumed. Indeed, the zeroes in the mass matrices will be filled after the renormalization by the Kähler metric. However, the determinant of the Kähler metrics is $O(1)$ and the determinants of the mass matrices are not changed by this renormalization at the leading order in $\varepsilon$. The mass relations (21) and (22) are consequently satisfied leading to the same conclusion as for positive charge differences.

In the two singlet case, if there are negative charge differences, the relations (21) and (22) will change. It is easy to show that with gauge singlets with opposite charges, $X = +1$ and $X = -1$, all the anomalies can be put to zero. Let us consider a simple example which illustrates the general situation, with the following lepton charge assignment:

\begin{align*}
l_{13} = 2 , \quad l_{23} = 3 , \quad e_{13} = 2 , \quad e_{23} = -9 , \quad x = 2 ,
\end{align*}

corresponding to the lepton mass matrix

\begin{align*}
Y_L = \varepsilon^x \begin{pmatrix}
\varepsilon^4 & \varepsilon^3 & \varepsilon^2 \\
\varepsilon^5 & \varepsilon^2 & \varepsilon^3 \\
\varepsilon^2 & \varepsilon^5 & 1
\end{pmatrix}.
\end{align*}

The quark charge differences are taken as in (9). Thus the predictions of this model are in perfect agreement with experimental data, including the relation $\det Y_D \simeq \det Y_L$. However, the relations in (21) and (22) get modified and read now

\begin{align*}
\det (Y_U Y_D^{-2} Y_L^2) &= \varepsilon^{\frac{4}{3}(A_2 + A_1 - 2A_3) + 12 + 6x} , \\
\det (Y_D^{-2} Y_L^2) &= \varepsilon^{A_1 + A_2 - \frac{4}{3}A_3 - 2(h_1 + h_2) + 8 + 4x} .
\end{align*}
Using the values in (2) which precisely correspond to the mass matrices, we obtain

\[
A_2 + A_1 - 2A_3 \simeq 4 - 2x, \\
A_3 + 3(h_1 + h_2) \simeq 18 + 3x. \tag{27}
\]

The Green-Schwarz mechanism and \(\sin^2 \theta_W = \frac{3}{8}\) can be obtained with the help of the relations

\[
\frac{3}{5}A_1 = A_2 = A_3 = 6 - 3x, \\
h_1 + h_2 = 4 + 2x. \tag{28}
\]

However, it is now possible to have all anomalies \(A_i = 0\) \((i = 1, 2, 3)\) if \(x = 2\) and \(h_1 + h_2 = 8\). Interestingly enough, this value of \(x\) gives \(\lambda_b/\lambda_t \simeq \varepsilon^2\), requiring \(\tan \beta \sim 2\) to fit the experimental masses. As a matter of fact this is the unique prediction for the solutions with vanishing anomalies, using the most successful textures (9), (13), (14), and lepton mass matrices that fulfill (2). More precisely, there are more assignments of charge differences for leptons corresponding to zero anomalies, but for all of them \(x = 2\) and \(h_1 + h_2 = 8\). Indeed, generalizing the relations (21, 22) to account for possible negative charge differences and requiring \(A_i = 0\) we get the equation

\[
16n_U - 4(4 + x)n_D + 6(\mu + x)n_L = 18 + 3x \tag{29}
\]

where \(\mu = 2, 4\) correspond to \(\varepsilon^\mu \sim (m_\mu/m_\tau), (m_e/m_\tau)\), respectively. In (29) \(n_i = 0\) \((i = U, D, L)\) if all charge differences are positive in \(Y_i\) and \(n_i = 1\) if there is one column of negative charges in \(Y_i\), leading to a modification of (23). A further case to be analyzed correspond to \(n_L = 2\) and \(\mu = 3\), corresponding to a lepton mass matrix with two columns of negative charge differences. The only solution of (29) is \(n_U = n_D = 0, n_L = 1, \mu = 2\) and \(x = 2\), which is precisely the case given by (9) and (24) hereabove.

All the other structures with two singlets and negative charge differences must use the Green-Schwarz mechanism to cancel anomalies and have generally \(h_1 + h_2 \neq 0\). Therefore, if the hierarchy in the mass matrices is a consequence of an horizontal \(U(1)\) symmetry, the vanishing of the mixed anomalies requires \(\tan \beta \sim 2\). This is to be added as a possibility to the previously considered Green-Schwarz mechanism which predicts \(\sin^2 \theta_W = \frac{3}{8}\). In either case, the shifts in (19) are instrumental, as they are also to make the \(A'_1\) anomaly to vanish.

6. We now turn to specific models with definite integral charges for the various fields. Even if, in general, abelian charges can be rational numbers, the string origin of the present \(U(1)\) gauge group gives integral charges in explicit model constructions [9].

We rewrite the anomalies as functions of the charges \((q_3, u_3, d_3, l_3, e_3)\) and charge differences to be taken from the allowed structures listed above. More specifically, for the quarks we use, as before, the best textures (9), (13) and (14) and for the leptons all the assignments compatible with the known masses. We consider two cases:
- 1 singlet, $\frac{3}{5}A_1 = A_2 = A_3, A'_1 = 0$ (Green-Schwarz case).

The models are characterized by the condition $l_{13} = l_{23} = 3n, n \in \mathbb{Z}$ necessary in order to obtain integral charges as solutions of the anomaly conditions. We obtain three one-parameter solutions, displayed in the Table 2.

- 2 singlets, $A_1 = A_2 = A_3 = A'_1 = 0, h_1 + h_2 = 8, x = 2$.

This case, characterized by $l_{13} + l_{23} = 3n - 8, n \in \mathbb{Z}$ has only a one-parameter solution with integral charges, displayed in the last column of the Table 2.

| Number singlets | 1   | 1   | 1   | 2   |
|-----------------|-----|-----|-----|-----|
| $q_1$           | $3 + q$ | $3 + q$ | $3 + q$ | $3 + q$ |
| $q_2$           | $2 + q$ | $2 + q$ | $2 + q$ | $2 + q$ |
| $q_3$           | $q$ | $q$ | $q$ | $q$ |
| $u_1$           | $4 + q$ | $4 + q$ | $4 + q$ | $-2 - 4q$ |
| $u_2$           | $1 + q$ | $1 + q$ | $1 + q$ | $-5 - 4q$ |
| $u_3$           | $-1 + q$ | $-1 + q$ | $-1 + q$ | $-7 - 4q$ |
| $d_1$           | $2 - 3q$ | $4 - 3q$ | $4 - 3q$ | $2 + 2q$ |
| $d_2$           | $1 - 3q$ | $3 - 3q$ | $3 - 3q$ | $1 + 2q$ |
| $d_3$           | $1 - 3q$ | $3 - 3q$ | $3 - 3q$ | $1 + 2q$ |
| $l_1$           | $1 - 3q$ | $4 - 3q$ | $2 - 3q$ | $-8 - 3q$ |
| $l_2$           | $1 - 3q$ | $2 - 3q$ | $6 - 3q$ | $-6 - 3q$ |
| $l_3$           | $1 - 3q$ | $3 - 3q$ | $1 - 3q$ | $-9 - 3q$ |
| $e_1$           | $4 + q$ | $3 + q$ | $-1 + q$ | $13 + 6q$ |
| $e_2$           | $2 + q$ | $3 + q$ | $5 + q$ | $1 + 6q$ |
| $e_3$           | $q$ | $q$ | $2 + q$ | $10 + 6q$ |
| $x$             | 0   | 2   | 2   | 2   |
| Anomaly cancellation | GS | GS | GS | Yes |

**TABLE 2**: The models with integral charges corresponding to one singlet (Green-Schwarz) and two singlets (zero anomalies). The parameter $x$ is defined as $\tan \beta = \frac{m_t}{m_b} \lambda^x$ and $q$ is an integer. Only the solutions (9), (13) and (14) were used for the table.

7. To conclude, we analyzed the connection between the mass matrices and the anomaly cancellation conditions for a horizontal $U(1)_X$ gauge symmetry, spontaneously broken by one or two Standard Model gauge singlets of opposite $X$ charges. We first classified the acceptable mass
matrices according to the known data. We remarked that the renormalization coming from the Kähler potential is generally important if the mass matrices initially had some zeroes. The anomaly analysis reveals that with two singlets and negative charge differences for the leptons there is a unique solution with integral charges and zero anomalies which predicts $\tan \beta \sim \frac{m_t}{m_b} \lambda^2$. Restricting to the one singlet case, corresponding to the solution (9) there are three solutions with integral charges corresponding to the Green-Schwarz mechanism. In this case either $\tan \beta \sim \frac{m_t}{m_b}$ or $\tan \beta \sim \frac{m_t}{m_b} \lambda^2$.

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We display here the explicit expressions of the diagonalizing angles for the mass matrices defined in eqs. (1) and (4), necessary in order to compute \( V_{CKM} \). In the \( Y_{33} \geq Y_{ij} \) hypothesis we find the following results, using the parametrization (4):

\[
S'_{13} \simeq \frac{Y_{12}Y_{23} - Y_{13}Y_{22} + (S_{13}Y_{22} - S_{23}Y_{12})Y_{33}}{Y_{11}Y_{22} - Y_{12}Y_{21}}, \\
S'_{23} \simeq \frac{Y_{13}Y_{21} - Y_{11}Y_{23} - (S_{13}Y_{21} - S_{23}Y_{11})Y_{33}}{Y_{11}Y_{22} - Y_{12}Y_{21}}.
\]

(A1)

The general expressions for \( S_{12} \) and \( S'_{12} \) are more involved. However, if one of the following conditions holds:

\[
Y_{12}Y_{32} \leq Y_{23}Y_{33}, \quad Y_{12}Y_{32} \leq Y_{13}Y_{33}, \quad Y_{12} \leq Y_{22},
\]

and defining, following Hall-Rašin

\[
\tilde{Y}_{22} = Y_{22} - S'_{23}Y_{23} - S_{23}Y_{32} + S_{23}S'_{23}Y_{33} \simeq Y_{22} - Y_{23}Y_{32} \\
\tilde{Y}_{11} = Y_{11} - S'_{13}Y_{13} - S_{13}Y_{31} - S_{13}S_{23}Y_{21} + S_{13}S'_{13}Y_{33} \simeq Y_{11} - Y_{13}Y_{31} - Y_{13}Y_{23}Y_{21} \\
\tilde{Y}_{12} = Y_{12} - S'_{23}Y_{13} - S_{13}S_{23}Y_{22} - S_{13}Y_{32} + S_{13}S'_{23}Y_{33} \simeq Y_{12} - Y_{13}Y_{32} - Y_{13}Y_{23}Y_{22} \\
\tilde{Y}_{21} = Y_{21} - S'_{13}S_{23}Y_{22} - S_{13}Y_{23} - S_{23}S_{13}Y_{31} + S_{23}S'_{13}Y_{33} \simeq Y_{21} - Y_{23}Y_{31} - Y_{22}Y_{31}Y_{32},
\]

(A3)

we obtain the approximate relations

\[
S_{12} \simeq \frac{\tilde{Y}_{12}\tilde{Y}_{22} + \tilde{Y}_{11}\tilde{Y}_{21}}{Y_{22}^2 - Y_{11}^2 + Y_{21}^2 + Y_{22}^2} \simeq \frac{\tilde{Y}_{12}}{Y_{22}} + \frac{\tilde{Y}_{11}\tilde{Y}_{21}}{Y_{22}^2}, \\
S'_{12} \simeq \frac{\tilde{Y}_{21} + \tilde{Y}_{11}S_{12}}{Y_{22} + Y_{12}S_{12}} \simeq \frac{\tilde{Y}_{21}}{Y_{22}} + \frac{\tilde{Y}_{11}\tilde{Y}_{12}}{Y_{22}^2}.
\]

(A4)
REFERENCES

[1] C.D. Froggatt and H.B. Nielsen, Nucl. Phys. B 147 (1979) 277;
S. Dimopoulos, Phys. Lett. B 129 (1983) 417;
J. Bagger, S. Dimopoulos, E. Masso and M. Reno, Nucl. Phys. B 258 (1985) 565;
Z.G. Berezhiani, Phys. Lett. B 129 (1983) 99; B 150 (1985) 177;
J. Bijnens and C. Wetterich, Nucl. Phys. B 283 (1987) 237.

[2] M. Leurer, Y. Nir and N. Seiberg, Nucl. Phys. B 398 (1993) 319, Nucl. Phys. B 420 (1994) 468;
Y. Nir and N. Seiberg, Phys. Lett. B 309 (1993) 337.

[3] L.E. Ibáñez and G.G. Ross, Phys. Lett. B 332 (1994) 100.

[4] P. Binétruy and P. Ramond, LPTHE 94/115, hep-ph/9412385.

[5] P. Ramond, R.G. Roberts and G.G. Ross, Nucl. Phys. B 406 (1993) 19.

[6] V. Jain and R. Shrock, ITP-SB-94-55.
E. Papageorgiu, LPTHE Orsay 40/94.

[7] L.E. Ibáñez, Phys. Lett. B 303 (1993) 55.

[8] M. Dine, N. Seiberg and E. Witten, Nucl. Phys. B 289 (1987) 585;
J. Atick, L. Dixon and A. Sen, Nucl. Phys. B 292 (1987) 109;
M. Dine, I. Ichénoise and N. Seiberg, Nucl. Phys. B 293 (1987) 253.

[9] A. Font, L.E. Ibáñez, H.P. Nilles and F. Quevedo, Nucl. Phys. B 307 (1988) 109; Phys. Lett. B 210 (1988) 101;
J.A. Casas, E.K. Katehou and C. Muñoz, Nucl. Phys. B 317 (1989) 171;
J.A. Casas and C. Muñoz, Phys. Lett. B 209 (1988) 214; Phys. Lett. B 214 (1988) 63;
A. Font, L.E. Ibáñez, F. Quevedo and A. Sierra, Nucl. Phys. B 331 (1990) 421.

[10] M. Olechowski and S. Pokorski, Phys. Lett. B 257 (1991) 388.

[11] L.J. Hall and A. Rašin, Phys. Lett. B 315 (1993) 164.

[12] S. Weinberg, in “A Festschrift for I.I. Rabi” (Trans. N.Y. Acad. Sci., Ser.II (1977), v.38), p.185;
F. Wilczek and A. Zee, Phys. Lett. B 70 (1977) 418;
T. Maehara and T. Yanagida, Prog. Theor. Phys. 60 (1978) 822;
F. Wilczek and A. Zee, Phys. Rev. Lett. 42 (1979) 421.

[13] H. Fritsch, Phys. Lett. B 70 (1977) 436; Phys. Lett. B 73 (1978) 317;
P.J. Gilman and Y. Nir, Ann. Rev. Nucl. Part. Sci. 40 (1990) 213;
P. Kaus and S. Meshkov, Mod. Phys. Lett. A 3 (1988) 1251.

[14] J. Harvey, P. Ramond and D. Reiss, Phys. Lett. B 92 (1980) 309;
S. Dimopoulos, L.J. Hall and S. Raby, Phys. Rev. Lett. 68 (1992) 1984; Phys. Rev. D 45 (1992) 4195;
H. Arason, D.J. Castaño, J. Ramond and E.J. Piard, Phys. Rev. D 47 (1993) 232;
G.F. Giudice, Mod. Phys. Lett. A 7 (1992) 2429.

[15] M.S. Chanowitz, J. Ellis and M.K. Gaillard, Nucl. Phys. B 128 (1977) 506;
A. Buras, J. Ellis, M.K. Gaillard and D.V. Nanopoulos, Nucl. Phys. B 135 (1978) 66;
H. Arason, D.J. Castaño, B. Keszthelyi, S. Mikaelian, E.J. Piard, P. Ramond and B.D. Wright,
Phys. Rev. Lett. 67 (1991) 2933.

[16] Y. Grossman and Y. Nir, hep-ph/9502418.

[17] E. Dudas, S. Pokorski and C.A. Savoy, work in progress.

[18] P. Binétruy and E. Dudas, LPTHE 95/18, SPht T95/042.