LOWER DEFECT GROUPS AND VERTICES OF SIMPLE MODULES

AKIHIKO HIDA AND MASAO KIYOTA

ABSTRACT. We compare lower defect groups associated with \( p \)-regular classes and vertices of simple modules for a block of a finite group algebra. We show that lower defect groups are contained in vertices of simple modules after suitable reordering. Moreover, for a block of principal type, we show that a \( p \)-regular lower defect group which contains a vertex of simple module is a defect group of the block.

1. Introduction

Let \( G \) be a finite group. Let \( \mathcal{O} \) be a complete discrete valuation ring of characteristic 0 such that the residue field \( k = \mathcal{O}/J(\mathcal{O}) \) is an algebraically closed field of characteristic \( p > 0 \). Let \( Z_kG \) be the center of the group algebra \( kG \). We call a primitive idempotent of \( Z_kG \) a block of \( kG \). In this paper, we consider the relation between two important families of \( p \)-subgroups associated with a block \( b \) of \( kG \), lower defect groups of \( b \) and vertices of simple \( kGb \)-modules.

A conjugacy class of \( G \) is called \( p \)-regular if the order of its element is not divisible by \( p \). The \( p \)-regular conjugacy classes of \( G \) are distributed to blocks of \( kG \). Let \( \text{Cl}(G) \) be the set of all \( p \)-regular classes of \( G \) and \( \text{Cl}(G) = \bigcup_b \text{Cl}_b \) a decomposition of \( \text{Cl}(G) \) into blocks of \( kG \). Let \( \text{Cl}_b \) be a defect group of \( C_i \), namely, \( Q_i \) is a Sylow \( p \)-subgroup of \( C_G(x_i) \) where \( x_i \in C_i \). Then \( Q_i \) \((1 \leq i \leq l(b))\) are called lower defect groups of \( b \) associated with \( p \)-regular classes.

Let \( \{ S_i \}_{1 \leq i \leq l(b)} \) be a representatives of complete set of isomorphism classes of simple \( kGb \)-modules. It is known that this set and \( \text{Cl}_b \) have the same cardinality. We want to compare \( Q_i \) and a vertex \( \text{vx}(S_j) \) of \( S_j \). These are both subgroups of some defect group of \( b \). For example, if a defect group \( P \) of \( b \) is abelian, then \( \text{vx}(S_i) = P \) for all \( i \) by the theorem of Knörr [4]. Hence in this case, \( Q_i \leq \text{vx}(S_j) \) holds in general for all \( i \) if we renumber the indices suitably.

**Proposition 1.1.** Let \( b \) be a block of \( kG \). Let \( \{ S_i \}_{1 \leq i \leq l(b)} \) be a set of representatives of isomorphism classes of simple \( kGb \)-modules. Let \( \{ Q_i \}_{1 \leq i \leq l(b)} \) be the lower defect groups of \( b \) associated with \( p \)-regular classes. Then there exists a permutation \( \sigma \) of \( \{ 1, 2, \ldots, l(b) \} \) such that

\[ Q_i \leq \text{vx}(S_{\sigma(i)}) \]
for all $i$. In particular,

$$\prod_{i=1}^{l(b)} |Q_i| \leq \prod_{i=1}^{l(b)} |\text{vx}(S_i)|.$$  

Next we consider when the equality holds in Proposition 1.1. We show that, under some assumption, if the equality holds then $Q_i = \text{vx}(S_{\sigma(i)})$ is a defect group of $b$. A block $b$ is of principal type if the image of Brauer homomorphism with respect $Q$ is a block of $kC_G(Q)$ for every $p$-subgroup $Q$ contained in a defect group of $b$. For a $p$-subgroup $Q$ of $G$, we denote by $m_1^b(Q)$ the multiplicity of $Q$ as a lower defect group of $b$ associated with $p$-regular classes, that is, $m_1^b(Q)$ is a number of $i$ ($1 \leq i \leq l(b)$) such that $Q_i =_G Q$. The following theorem is our main result.

**Theorem 1.2.** Let $b$ be a block of principal type of $kG$ and $P$ a defect group of $b$. Let $S$ be a simple $kGb$-module and $R$ a vertex of $S$. If $R \leq Q < P$, then $m_1^b(Q) = 0$.

By Proposition 1.1 and Theorem 1.2, we have the following Corollary.

**Corollary 1.3.** Let $\{x_i\}_{1 \leq i \leq l}$ be a set of representatives of $p$-regular conjugacy classes of $G$. Let $Q_i$ be a Sylow $p$-subgroup of $C_G(x_i)$. Let $\{S_i\}_{1 \leq i \leq l}$ be a set of representatives of isomorphism classes of simple $kG$-modules. Then

$$\prod_{i=1}^{l} |Q_i| \leq \prod_{i=1}^{l} |\text{vx}(S_i)|$$

and the equality holds if and only if $G$ is $p$-nilpotent.

We prove these results in section 3. In section 4 we consider blocks of $p$-solvable groups. We show a result which is slightly weaker than Theorem 1.2 without the assumption that $b$ is of principal type. Finally, in section 5 we add some results on the complexity of modules.

In this paper, all $kG$-modules are finite generated right modules. For an indecomposable $kG$-module $M$, we denote a vertex of $M$ by $\text{vx}(M)$. We refer to [6], [10], [11] and [12] for modular representations of finite groups.

2. **Lower defect groups of a block**

Lower defect groups of a block are defined by Brauer [3] and some related results can be found in [5], [8], [13] or [6, V, Section 10], [12, Chapter 5, Section 11]. In this section we quote some known results used in this paper. We consider lower defect groups associated with $p$-regular classes only.

Let $G_{p'}$ be the set of all $p$-regular elements of $G$. We denote by $kG_{p'}$ the $k$ subspace of $kG$ spanned by $G_{p'}$. If $C$ is a conjugacy class of $G$, we set $\hat{C} = \sum_{x \in C} x \in kG$. Then $\{\hat{C} \mid C \in \text{Cl}(G)\}$ is a basis of the center $ZkG$ of $kG$. Let $\text{Cl}(G_{p'})$ be the set of all $p$-regular conjugacy classes of $G$. Then $\{\hat{C} \mid C \in \text{Cl}(G_{p'})\}$ is a basis of $ZkG_{p'} = ZkG \cap kG_{p'}$. On the other hand,

$$ZkG_{p'} = \bigoplus_b ZkG_{p'} \cap ZkGb$$

where $b$ ranges over all blocks of $kG$ and there exists a disjoint decomposition

$$\text{Cl}(G_{p'}) = \bigcup_b \text{Cl}_{p'}(b)$$
such that \( \{ \hat{C}b \mid C \in Cl_{p'}(b) \} \) is a basis of \( ZkG_{p'}b \) for any block \( b \).

Let \( Cl_{p'}(b) = \{ C_i \}_{1 \leq i \leq l(b)} \) and \( Q_i \), a defect group of \( C_i \), that is, \( Q_i \) is a Sylow \( p \)-subgroup of \( C_G(x_i) \) where \( x_i \in C_i \). We denote by \( m_i^1(Q) \) the multiplicity of \( Q \) as a lower defect group of \( b \) associated with \( p \)-regular classes, that is, \( m_i^1(Q) \) is a number of \( i \ (1 \leq i \leq l(b)) \) such that \( Q = G Q_i \). Here, for subgroups \( H, K \) of \( G \), we write \( H = G K \) if \( H \) and \( K \) are \( G \)-conjugate.

Let \( M \) be a \( kG \)-module and \( H \subseteq K \) be subgroups of \( G \). Let \( M^H \) be the set of fixed points of \( H \) in \( M \). We denote by \( Tr^K_H \) the trace map from \( M^H \) to \( M^K \) defined by

\[
Tr^K_H(m) = \sum_{g \in H \backslash K} mg.
\]

We set \( M^K = Tr^K_H(M^H) \). The group \( G \) acts on \( kG \) and \( kG_{p'} \) by conjugation and we can define the trace maps

\[
Tr^K_H : (kG)^H \longrightarrow (kG)^K
\]

and

\[
Tr^K_H : (kG_{p'})^H \longrightarrow (kG_{p'})^K.
\]

The proof of the following basic properties of lower defect groups are found in [6, V, Section 10] and [12, Chapter 5, Section 11].

**Proposition 2.1.** Let \( b \) be a block of \( kG \) and \( P \) a defect group of \( b \). Then the following holds.

1. If \( Q \) is a \( p \)-subgroup of \( G \), then \( m_i^1(Q) = \dim(kG_{p'})^{G}_{Q}/\sum_{R<Q}(kG_{p'})^{G}_{R} \).
2. If \( m_i^1(Q) > 0 \) then \( Q \) is conjugate to a subgroup of \( P \) and \( m_i^1(P) = 1 \).
3. Let \( Cl_{p'}(b) = \{ C_i \}_{1 \leq i \leq l(b)} \) and \( Q_i \), a defect group of \( C_i \). Then \( l(b) = \#(Cl_{p'}(b)) \) is the number of isomorphism classes of simple \( kGb \)-modules and \( \{|Q_i| \mid 1 \leq i \leq l(b)\} \) is the set of elementary divisors of the Cartan matrix of \( kGb \).
4. Let \( m \geq 0 \). Then \( \sum_{Q} m_i^1(Q) \) where \( Q \) ranges over the set of representatives of conjugacy classes of subgroups of \( G \) of order \( p^m \) is the multiplicity of \( p^m \) as an elementary divisor of the Cartan matrix of \( kGb \).

We need the following result in Lemma 3.1.4 and Theorem 3.5.

**Proposition 2.2 ([3, Proposition (II) 1.3]).** If \( Q \) is a \( p \)-subgroup of \( G \) then

\[
m_i^1(Q) = \dim(kC_G(Q)_{p'})^{N_G(Q)}_{Q} Br_Q(b)
\]

where \( Br_Q : ZkG \longrightarrow ZkN_G(Q) \) is the Brauer homomorphism.

3. PROOF OF MAIN RESULTS

The following Proposition is proved in [2] Theorem 10, Corollary. It is obtained from [3] p.243, Corollaire] or [6, IV, Theorem 2.3] also.

**Proposition 3.1.** Let \( M \) be an indecomposable \( kG \)-module.

1. If \( \varphi \) be the Brauer character corresponding to \( M \). Let \( x \in G_{p'} \) and \( Q \in \text{Syl}_{p'}(C_G(x)) \). If

\[
\varphi(x) \notin p\mathbb{O}
\]

then \( \text{Res}_Q^G M \) has an indecomposable direct summand \( N \) such that \( \dim N \neq 0 \mod p \).

2. If \( Q \) is a \( p \)-subgroup of \( G \) and \( \text{Res}_Q^G M \) has an indecomposable direct summand \( N \) such that \( \dim N \neq 0 \mod p \), then \( Q \leq_C \text{vx}(M) \).
Proposition 1.1 is immediate from this Proposition.

(Proof of Proposition 1.1)
Let \( \varphi_i \) be the Brauer character of \( S_i \). Let \( \text{Cl}_{p'}(b) = \{ C_i \}_{1 \leq i \leq l(b)} \) be the \( p \)-regular conjugacy classes distributed into \( b \) in the block decomposition of \( \text{Cl}(G_{p'}) \). Fix \( x_i \in C_i \) for each \( i \). We may assume that \( Q_i \) is a Sylow \( p \)-subgroup of \( C_G(x_i) \) where \( x_i \in C_i \). Then the determinant of the matrix \( (\varphi_i(x_j))_{1 \leq i, j \leq l(b)} \) is not contained in \( J(O) \) by [12, Chapter 5, Theorem 11.6] and there exists a permutation \( \sigma \) such that

\[
\prod_i \varphi_{\sigma(i)}(x_i) \notin J(O).
\]

Since \( pO \subset J(O) \), the result follows from Proposition 3.4.

Next we prove Theorem 1.2. First we study the case that \( Q \) is a normal subgroup of \( G \).

Lemma 3.2. Let \( G \triangleright H \). Let \( b \) be a block of \( kH \) of defect 0. Let \( T = T(b) \) be the inertial group of \( b \) in \( G \). Assume that \( |T : H| \equiv 0 \mod p \). Then \( b(kH)^T = 0 \).

Proof. By the Mackey decomposition, we have

\[
(kH)^T \subset \sum_{t \in G/T} ((kH)^t)^T = (kH)^T.
\]

Since \( b \) is \( T \)-invariant, we have

\[
b(kH)^T = b \text{Tr}_T(kH) = \text{Tr}_T(bkH) = \text{Tr}_H(\text{Tr}_T(bkH)).
\]

Then

\[
\text{Tr}_H(bkH) = Z(bkH) = kb
\]

since \( b \) is a block of defect 0, and

\[
\text{Tr}_H(\text{Tr}_T(bkH)) = \text{Tr}_H(kb) = 0
\]

since \( |T : H| \equiv 0 \mod p \).

Lemma 3.3. Let \( Q \) be a normal \( p \)-subgroup of \( G \) and \( \overline{G} = G/Q \). Let \( \mu : kG \rightarrow k\overline{G} \) be the surjective algebra homomorphism induced by the natural surjective homomorphism \( G \rightarrow \overline{G} \). Let \( b \) be a block of \( kG \) and \( \overline{b} = \mu(b) \). Then \( \mu \) induces an isomorphism

\[
b \cdot (kC_G(Q))_{p'}^G \cong \overline{b} \cdot (k(QC_G(Q))/Q)_{p'}^G
\]

Proof. The natural surjective homomorphism \( G \rightarrow \overline{G} \) induces a bijection

\[
C_G(Q)_{p'} \rightarrow (QC_G(Q))_{p'} = (QC_G(Q)/Q)_{p'}.
\]

Hence \( \mu \) induces an isomorphism

\[
\mu_0 : kC_G(Q)_{p'} \cong k(QC_G(Q))_{p'}.
\]

On the other hand, since

\[
\mu(b \cdot (kC_G(Q))_{p'}^G) = \overline{b} \cdot (k(QC_G(Q))/Q)_{p'}^G
\]

and

\[
b \cdot (kC_G(Q))_{p'}^G \subset kC_G(Q)_{p'}, \quad \overline{b} \cdot (k(QC_G(Q))/Q)_{p'}^G \subset k(QC_G(Q))_{p'}^G
\]

it follows that the restriction of \( \mu_0 \) induces the desired isomorphism.
Lemma 3.4. Let $Q$ be a normal $p$-subgroup of $G$ and $b$ a block of $kG$. Let $P$ be a defect group of $b$. Assume that $Q < P$ and $C_P(Q) = Z(Q)$. Then $b \cdot (kC_G(Q)_{P'})_Q^G = 0$, in particular, $m^i_P(Q) = 0$.

Proof. Let $H = QC_G(Q)$. The block $b$ is a central idempotent of $kH$. Let $b = \sum b_i$ be the block decomposition of $b$ in $kH$. Let $T_i = T(b_i)$ be the inertial group of $b_i$ in $G$. Then there exists a defect group $P_i$ of $b$ such that $P_i \leq T_i$ and $P_i \cap H$ is a defect group of $b_i$. Since $P_i$ is conjugate to $P$ in $G$ and $C_P(Q) = Z(Q)$,

$$P_i \cap H = P \cap H = Q$$

and it follows that $Q$ is a defect group of $b_i$. Let $\mu : kG \rightarrow k(G/Q)$ be the natural surjective algebra homomorphism. Then $\mu(b_i) = \overline{b_i}$ is a block of defect 0 for all $i$. Since $Q < P$, we have $H \neq HP_i$ and $|T_i : H| \equiv 0 \mod p$. The inertial group of $\overline{b_i}$ in $G$ is $T_i = T_i/Q$ and $|T_i : H| \equiv 0 \mod p$. Hence

$$\overline{b_i}(k\overline{H})_{T_i}^i = 0$$

by Lemma 3.2 and

$$b(k\overline{H})_{T_i}^i = (\sum_i \overline{b_i})(k\overline{H})_{T_i}^i = 0.$$  

Then, by Lemma 3.3,

$$b(kC_G(Q)_{P'})_{Q}^G \simeq \overline{b}(k\overline{H})_{T_i}^i \subset \overline{b}(k\overline{H})_{T_i}^i = 0.$$  

Moreover, since $Q$ is a normal subgroup of $G$, $Br_Q(b) = b$ and the result follows from Proposition 2.2. \hfill \Box

In the following, we consider a $p$-subgroup $Q$ such that $C_P(Q) = Z(Q)$ for any defect group $P$ of $b$ containing $Q$.

Theorem 3.5. Let $b$ be a block of $kG$ and $Q$ a $p$-subgroup of $G$. Assume that $Q$ is a proper subgroup of a defect group of $b$. If $C_P(Q) = Z(Q)$ for any defect group $P$ of $b$ containing $Q$, then $m^i_P(Q) = 0$.

Proof. Let $N = N_G(Q)$. Let

$$Br_Q : ZkG \rightarrow ZkN$$

be the Brauer homomorphism and

$$Br_Q(b) = \sum_i b_i$$

a decomposition into block idempotents of $kN$. Then $b_i^Q = b$ and $Q < D_i$ by Brauer’s First Main Theorem where $D_i$ is a defect group of $b_i$. There exists a defect group $P_i$ of $b$ such that $D_i \leq P_i$. Then

$$C_{D_i}(Q) \leq C_{P_i}(Q) = Z(Q)$$

by the assumption. Hence

$$b_i \cdot (kC_G(Q)_{P'})_Q^N = 0$$

by Lemma 3.4 and

$$(kC_G(Q)_{P'})_Q^N Br_Q(b) = (kC_G(Q)_{P'})_Q^N (\sum_i b_i) = 0.$$  

Hence the result follows by Proposition 2.2. \hfill \Box
Lemma 3.6. Let \( b \) be a block of \( kG \) and \((P, e)\) a maximal \((G, b)\)-Brauer pair. Let \( \mathcal{F} = \mathcal{F}_{(P, e)}(G, b) \) be the fusion system of \( b \) with respect to \((P, e)\). Suppose that \( \mathcal{F} = \mathcal{F}_P(G) \). Let \( Q \) be a proper subgroup of \( P \). If \( Q \) is \( \mathcal{F} \)-centric, then \( m_k^1(Q) = 0 \).

Proof. Let \( P_1 \) be a defect group of \( b \) such that \( Q < P_1 \). Then there exists \( g \in G \) such that \( P^g = P_1 \). Let \( Q_1 = g Q \). Then

\[
\varphi : Q \longrightarrow Q_1(\leq P), \quad \varphi(u) = gu^{-1}
\]

is an isomorphism in \( \mathcal{F} = \mathcal{F}_P(G) \). Hence \( C_P(Q_1) = Z(Q_1) \) since \( Q \) is \( \mathcal{F} \)-centric and it follows that \( C_{P_1}(Q) = Z(Q) \). Hence \( b \) satisfies the assumption of Theorem 3.6.

\[\square\]

Now we prove Theorem 1.2 and Corollary 1.3. A block \( b \) of \( kG \) is of principal type if \( Br_m(b) \) is a block of \( kC_G(Q) \) for every \( p \)-subgroup \( Q \) contained in a defect group of \( b \) (\cite[Definition 6.3.13]{knorr}).

(Proof of Theorem 1.2)

There exists a \((G, b)\)-Brauer pair \((R, f)\) such that \( Z(R) \) is a defect group of the block \( f \) of \( kC_G(R) \) by the theorem of Knörr (\cite[3.6 Corollary]{knorr}, \cite[Corollary 10.3.2]{knorr}). Let \((P, e)\) be a maximal \((G, b)\)-Brauer pair such that \((R, f) \leq (P, e)\). Let \( \mathcal{F} = \mathcal{F}_{(P, e)}(G, b) \) be the fusion system of \( b \) with respect to \((P, e)\). Then \( R \) is an \( \mathcal{F} \)-centric subgroup of \( P \) by \cite[Proposition 8.5.3]{knorr}. Moreover, since \( b \) is a block of principal type, we have \( \mathcal{F} = \mathcal{F}_P(G) \) by \cite[Proposition 8.5.5]{knorr}. Since \( R \leq Q \leq P \), \( Q \) is an \( \mathcal{F} \)-centric subgroup by \cite[Proposition 8.2.4]{knorr}. Hence the results follows from Lemma 3.6.

Remark 3.7. In Theorem 1.2, \( S \) does not necessarily need to be simple. Let \( b \) be a block of \( OG \) with defect group \( P \) and \( S \) an indecomposable \( OGb \)-module. If \( \text{End}_{OGb}(S) \cong O/J(O)^m \) for some \( m > 0 \), then there is a \((G, b)\)-Brauer pair \((R, f)\) such that \( Z(R) \) is a defect group of the block \( f \) of \( kC_G(R) \) as in the proof of Theorem 1.2 by \cite[Corollary 10.3.2]{knorr}. It follows that if \( b \) is of principal type and \( R \leq Q \leq P \), then \( m_k^1(Q) = 0 \).

(Proof of Corollary 1.3)

The inequality holds by Proposition 1.1. If \( G \) is \( p \)-nilpotent, then \( kGb \) has a unique simple module (up to isomorphism) for every block \( b \) of \( kG \). The vertex of the simple \( kGb \)-module is a defect group of \( b \) and that is a \( p \)-regular lower defect group of \( b \). Hence the equality holds. On the other hand, assume that the equality holds. If \( P \) is a defect group of a block \( b \) of \( kG \), then \( m_k^1(P) = 1 \). Hence the principal block \( b_0(kG) \) of \( kG \) has a unique simple module (up to isomorphism) by Proposition 1.1 and Theorem 1.2 since \( b_0(kG) \) is a block of principal type by Brauer’s Third Main Theorem (\cite[Theorem 6.3.14]{knorr}). Hence \( G \) is \( p \)-nilpotent.

4. Blocks of \( p \)-solvable groups

Let \( N \) be a normal subgroup of \( G \). Let \( b \) be a block of \( kG \). Let \( c \) be the block of \( kN \) such that \( bc \neq 0 \) and \( H \) the inertial group of \( c \) in \( G \). Then there exists a block \( d \) of \( kH \) such that \( db = d \) and \( Tr^G_H(d) = b \). The \((kGb, kHd)\)-bimodule \( kkGd = kGd \).
Proof. Let $U$ be a $p$-subgroup of $H$. Let
\[ U = \{ W \leq H \mid W =_{G} U \} \]
and let $\{ U_{j} \}$ be a set of representatives of $H$-conjugacy classes of $U$. Then
\[ \sum_{j} m_{1}(U_{j}) = m_{1}(U). \]

Proof. Let $\{ R_{ij} \}_{1 \leq i \leq m, 1 \leq j \leq r(i)}$ be a set of representatives of $H$-conjugacy classes of subgroups of $H$ of order $|U|$ such that
\[ R_{ij} =_{G} R_{i^{'}, j^{'}} \iff i = i^{'} . \]
We may assume $U = R_{11}$ and $\{ U_{j} \} = \{ R_{1j} \}$. We set $R_{i} = R_{i1}$. For each $i$, $\Tr_{H}^{G}$ induces a $k$-linear map
\[ \Phi_{i} : \bigoplus_{1 \leq j \leq r(i)} \left( (kH_{p'})_{R_{ij}}^{H} d / \sum_{R < R_{ij}} (kH_{p'})_{R}^{H} d \right) \longrightarrow (kG_{p'})_{R_{i}}^{G} b / \sum_{R < R_{i}} (kG_{p'})_{R}^{G} b. \]
We claim that $\Phi_i$ is an isomorphism. For $t \in G$, if $R_i^t \leq H$ then $R_i^t = H$ $R_{ij}$ for some $j$. On the other hand, if $R_i^t \cap H < R_i^t$ then
\[
\text{Tr}^G_H((kH_{p'})_{R_i^t \cap H} d) \subset (kG_{p'})_{R_i^t} R_i^t H b = (kG_{p'})_{R_i^t} H b \subset \sum_{R \prec R_i} (kG_{p'})_{R_i} b.
\]
Hence we have
\[
(kG_{p'})_{R_i} b = \sum_{t \in G} \text{Tr}^G_H((kH_{p'})_{R_i^t \cap H} d) \subset \text{Tr}^G_H( \sum_j (kH_{p'})_{R_i^t} R_i^t j H d) + \sum_{R \prec R_i} (kG_{p'})_{R_i} b
\]
by Lemma [4.1] and $\Phi_i$ is surjective. Hence
\[
\sum_{1 \leq j \leq r(i)} m^1_d(R_{ij}) \geq m^1_b(R_i)
\]
for each $i$ and
\[
\sum_{i,j} m^1_d(R_{ij}) \geq \sum_i m^1_b(R_i).
\]
But $\sum_{i,j} m^1_d(R_{ij})$ is the multiplicity of $|U|$ as an elementary divisor of the Cartan matrix $C_d$ of $kHd$ and $\sum_i m^1_b(R_i)$ is that of the Cartan matrix $C_b$ of $kGb$. Since $kGb$ and $kHd$ are Morita equivalent, $C_d = C_b$ and it follows that
\[
\sum_{i,j} m^1_d(R_{ij}) = \sum_i m^1_b(R_i).
\]
Hence
\[
\sum_{1 \leq j \leq r(i)} m^1_d(R_{ij}) = m^1_b(R_i)
\]
and $\Phi_i$ is an isomorphism for each $i$. In particular, for $i = 1$, we have
\[
\sum_j m^1_d(U_j) = m^1_b(U).
\]\[\square\]

The following theorem is the main result of this section. If the block $b$ is of principal type, then this is a consequence of Theorem [1,2].

**Theorem 4.3.** Let $G$ be a $p$-solvable group. Let $b$ be a block of $kG$. Let $Cl_{p'}(b) = \{C_i\}_{1 \leq \ell(b)}$ and $Q_i$ a defect group of $C_i$. Let $\{S_i\}_{1 \leq \ell(b)}$ be a set of representatives of isomorphism classes of simple $kGb$-modules. Then there exists a permutation $\sigma$ of $\{1, \ldots, \ell(b)\}$ such that
\[
Q_i \leq_G \text{vx}(S_{\sigma(i)})
\]
for all $i$ and
\[
Q_i <_G \text{vx}(S_{\sigma(i)})
\]
unless $Q_i$ is a defect group of $b$.

*Proof.* Let $c$ be a block of $kO_{p'}(G)$ such that $bc \neq 0$ and $H$ the inertial group of $c$ in $G$. Then there exists a block $d$ of $kH$ such that $db = d$ and $\text{Tr}^G_H(d) = b$. Moreover $(kGb, kHd)$-bimodule $bkGd = kGd$ induces a Morita equivalence between $kGb$ and $kHd$. In particular, $\ell(b) = \ell(d)$. Moreover if $P$ is a defect group of $d$ then $P$ is a defect group of $b$.

If $G = H$, then the result holds by Proposition [1] and Theorem [1,2] since $b = d$ is a block of principal type by [11, Lemma 10.6.5].
Suppose that $G > H$. Let $\text{Cl}_P(b) = \{C_i\}$ and $\text{Cl}_P(d) = \{\bar{C}_i\}$. Let $Q_i$ (resp. $\bar{Q}_i$) be a defect group of $C_i$ (resp. $\bar{C}_i$). If $Q$ is a $P$-subgroup of $P$,

$$|\{1 \leq j \leq l(d) \mid \bar{Q}_j \cong_G Q\}| = m^1_b(Q) = |\{1 \leq i \leq l(b) \mid Q_i \cong_G Q\}|$$

by Lemma 4.2. Hence we may assume $Q_i = \bar{Q}_i$ for every $1 \leq i \leq l(b)$. Let $\bar{S}_i = S_i \otimes_{kGb} kGd$ be the simple $kHd$-module corresponding to $S_i$. Then $\text{vx}(\bar{S}_i) = G \text{vx}(S_i)$. By induction there exists a permutation $\sigma$ of $\{1, \ldots, l(d)\}$ such that

$$Q_i \leq_H \text{vx}(\bar{S}_{\sigma(i)})$$

for all $i$ and

$$Q_i <_H \text{vx}(\bar{S}_{\sigma(i)})$$

if $Q_i <_H P$. Since $P$ is a defect group of $b$, it follows that

$$Q_i \leq_G \text{vx}(S_{\sigma(i)})$$

for all $i$ and

$$Q_i <_G \text{vx}(S_{\sigma(i)})$$

unless $Q_i$ is a defect group of $b$. \hfill \square

The following corollary is a block version of Corollary 1.3 for $p$-solvable groups.

**Corollary 4.4.** Let $G$ be a $p$-solvable group. Let $b$ be a block of $kG$. Let $\{S_i\}_{1 \leq i \leq l(b)}$ be a set of representatives of isomorphism classes of simple $kGb$-modules. Then

$$\det C_b \leq \prod_{i=1}^{l(b)} |\text{vx}(S_i)|$$

where $C_b$ is the Cartan matrix of $kGb$ and the equality holds if and only if $l(b) = 1$.

5. Complexity of modules

Let $Q$ be a $p$-subgroup of $G$ and $M$ an indecomposable $kG$-module. Suppose that $\text{Res}_Q^G M$ has a direct summand $N$ such that $\dim N \not\equiv 0 \mod p$. We set $|\text{vx}(M)| = p^{v(M)}$ and $|Q| = p^a$. Then by Proposition 5.1, $Q \leq_G \text{vx}(M)$ and in particular $a \leq v(M)$. If the dimension of a source of $M$ is divisible by $p$, then proper inequality $a < v(M)$ holds. We consider another information on this inequality related to the complexity of $M$. For the complexity of a module, we refer to [1] and [2, Section 5].

Let $c(M)$ be the complexity of $M$ and $r(M)$ the $p$-rank of $\text{vx}(M)$. Since $M$ is a direct summand of $\text{Ind}_{\text{vx}(M)}^G \text{Res}_{\text{vx}(M)}^G M$ and $c(\text{Res}_{\text{vx}(M)}^G M) \leq r(M)$, we have

$$c(M) \leq r(M) \leq v(M).$$

**Proposition 5.1.** Let $Q$ be a $p$-subgroup of $G$ and $M$ an indecomposable $kG$-module. Suppose that $\text{Res}_Q^G M$ has a direct summand $N$ such that $\dim N \not\equiv 0 \mod p$. If $|Q| = p^a$ then

$$a \leq v(M) + c(M) - r(M).$$

**Proof.** Since $N$ is a direct summand of $\text{Res}_Q^G M$ and $\dim N \not\equiv 0 \mod p$,

$$\text{rank}(Q) = c(N) \leq c(\text{res}_Q^G M) \leq c(M)$$

and

$$\text{rank}(Q) + v(M) - r(M) \leq v(M) + c(M) - r(M)$$
where $\text{rank}(Q)$ is the $p$-rank of $Q$. We may assume that $Q \leq \text{vx}(M)$ by Proposition 3.1(2). Let $E$ be an elementary abelian $p$-subgroup of $\text{vx}(M)$ of maximal rank and $F$ an elementary abelian $p$-subgroup of $Q$ of maximal rank. The class of elementary abelian $p$-groups satisfies the condition in Lemma 5.2 below. Hence $|Q|/|F| \leq |\text{vx}(M)|/|E|$ by Lemma 5.2 and we have

$$a \leq \text{rank}(Q) + v(M) - r(M).$$

Lemma 5.2. Let $X$ be a class of finite $p$-groups which satisfies the following property:

$$P \in X, \ P \triangleright R \Rightarrow R \in X.$$ 

Let $P$ be a $p$-group and $Q$ be a subgroup of $P$. Suppose $E \leq P$, $F \leq Q$ and $E, F \in X$. If $|L| \leq |F|$ for any subgroup $L \leq Q$ such that $L \in X$, then $|Q:F| \leq |P:E|$. 

Proof. We proceed by induction on $|P : Q|$. If $P = Q$ then $|E| \leq |F|$ by the assumption. Hence $|Q : F| \leq |P : E|$. Assume that $P > Q$ and let $R$ be a maximal subgroup of $P$ such that $Q \leq R < P$. Then $R \cap E \in X$ since $P \triangleright R$ and $E \triangleright R \cap E$. If $R \geq E$, then $P = RE$ and $P/R = RE/R \cong E/R \cap E$. It follows that $|P|/|E| = |R|/|R \cap E|$ and $|R|/|R \cap E| \geq |Q|/|F|$ by induction. If $R > E$, then $|P|/|E| > |R|/|E|$ and it follows that $|R|/|E| \geq |Q|/|F|$ by induction. 

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AKIHKO HIDA, FACULTY OF EDUCATION, SAITAMA UNIVERSITY, SHIMO-OOKUBO 255, SAKURA-KU, SAITAMA-CITY, SAITAMA, 338-8570, JAPAN
Email address: ahida@mail.saitama-u.ac.jp

MASAO KIYOTA, COLLEGE OF LIBERAL ARTS AND SCIENCES, TOKYO MEDICAL AND DENTAL UNIVERSITY, KONNODAI 2-8-30, ICHIKAWA, CHIBA, 272-0827, JAPAN
Email address: kiyota.las@tmd.ac.jp