Vorticity generation at second order in cosmological perturbation theory

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(Dated: June 26, 2009)

We show that at second order in cosmological perturbation theory vorticity generation is sourced by entropy gradients. This is an extension of Crocco’s theorem to a cosmological setting.

PACS numbers: 98.80.Cq, 98.80.Jk, 47.10.-g

Phys. Rev. D 79, 123523 (2009), arXiv:0904.0940v3

I. INTRODUCTION

It is well known in fluid dynamics that vorticity generation is sourced by entropy gradients. This was first pointed out by Crocco in 1937 [1]. Despite vorticity being ubiquitous in nature, studies of vorticity in the early universe and in cosmology have so far been rare.

At linear order in perturbation theory scalar and vector perturbations, classified according to their transformation behaviour on spatial 3-hypersurfaces, decouple from each other. Scalar perturbations are much easier to treat mathematically and play the dominant role in structure formation on super-horizon scales. At linear order vorticity, intrinsically of vectorial nature (in fluid dynamics simply the curl of the fluid velocity) cannot be constructed from scalar quantities, and the vorticity tensor constructed from vector perturbations is, in general, sourced only by anisotropic stress and decays in its absence [2, 3, 4, 5, 6].

Things are different though at second order in the perturbations. Only recently, with second order cosmological perturbation theory becoming mature (see e.g. Refs. [8, 9, 10, 11, 12, 13, 14]), has the issue of vorticity generation beyond the standard linear order begun to be addressed [15, 16, 17]. However, these previous studies have been restricted to barotropic fluids allowing for a non-adiabatic pressure perturbation gives qualitatively different results.

We focus here on the case of vorticity generation at second order in the perturbations, sourced only by scalar and vector perturbations. We consider a perfect fluid, namely a fluid whose energy-momentum tensor is diagonal (the inclusion of anisotropic stress will source vorticity already at first order in the perturbations). Such a fluid has an equation of state as \( \rho = \frac{\rho_0}{1 + c_s^2 \delta P / \rho_0} \), where \( c_s \) is the adiabatic sound speed, defined as \( c_s^2 = P'/\rho' \), and we have defined the non-adiabatic pressure perturbation \( \delta P_{\text{nad}} \equiv \frac{\partial P}{\partial S} \delta S \), in addition to the pressure and energy density perturbations. For a detailed discussion of the non-adiabatic pressure perturbation, see Ref. [18].

Our main result, derived in detail below and given in Eq. (3.4), can be written concisely as

\[
\omega_{2i} = \frac{c_s^2}{H^2} \frac{\partial P_{\text{nad}}}{\rho} \delta P_{\text{nad1},i},
\]

that is, the gradients in the non-adiabatic pressure perturbation coupled to gradients in the density act as a source for vorticity at second order.

We use cosmological perturbation theory throughout, which will make the numerical implementation of the results straightforward and, compared to other approaches, has the added benefit of actually being physically transparent. The Paper is organised as follows. In the next section we define and introduce the key variables. In Section III we define the vorticity tensor and give its evolution. We discuss our results and conclude in Section IV. All the governing equations necessary to derive the results in Section III are given in the appendix.

In this Paper, we use conformal time, \( \eta \), throughout, denoting derivatives with respect to conformal time with a prime. The scale factor is \( a \), and the Hubble parameter is \( \mathcal{H} \), where \( \mathcal{H} = a'/a \). Greek indices, \( \mu, \nu, \lambda \) run from 0, ..., 3, while lower case Latin indices, \( i, j, k \), take the value 1, 2, or 3. Covariant derivatives are denoted by a semi-colon, partial derivatives by a comma. The order of the perturbations is denoted with a subscript immediately after a perturbed quantity. We work in the uniform curvature gauge throughout.

II. DEFINITIONS

The Friedmann-Robertson-Walker (FRW) metric tensor, up to and including second order perturbations, has in the uniform curvature gauge the components (see,
e.g. Ref. [14])

\begin{align}
  g_{00} &= -a^2 (1 + 2\phi_1 + \phi_2) , \\
  g_{0i} &= a^2 \left( B_{1i} + \frac{1}{2} B_{2i} \right) , \\
  g_{ij} &= a^2 \delta_{ij} , \quad (2.1)
\end{align}

where we have assumed a flat \((K = 0)\) background without loss of generality. We neglect tensor perturbations, which will add another source term to the momentum conservation equation, but are beyond the scope of this work [19]. Here \(a = a(\eta)\) is the scale factor, \(\phi_1\) and \(\phi_2\) are the lapse functions at first and second order, respectively, and \(B_{1i}\) and \(B_{2i}\) represent the shear in this gauge. All perturbed quantities are function of \(x^\mu\). Note \(B_{1i}\) and \(B_{2i}\) can be further split into scalar and divergence-free vector parts [14], though this step is unnecessary here.

The fluid four velocity, \(u^\mu\), obeying the normalisation condition \(u^\mu u_\mu = -1\), has components

\begin{align}
  u_0 &= -a \left[ 1 + \phi_1 + \frac{1}{2} \phi_2 - \frac{1}{2} \phi_1^2 + \frac{1}{2} \phi_1 k_1^k \right] , \\
  u_i &= a \left[ V_{1i} + \frac{1}{2} V_{2i} - \phi_1 B_{1i} \right] , \quad (2.2)
\end{align}

up to second order, where \(v^i\) is the spatial three velocity, and \(V^i\) is the covariant spatial velocity, defined as \(V^i = v^i + B^i\).

### III. VORTICITY

The vorticity tensor is defined as the projected antisymmetrised covariant derivative of the fluid four velocity, that is [4]

\[ \omega_{\mu\nu} = \mathcal{P}_\mu \,^\alpha \mathcal{P}_\nu \,^\beta u_{[\alpha\beta]} , \quad (3.1) \]

where \(u_{[\alpha\beta]} \equiv \frac{1}{2} (u_{\alpha\beta} - u_{\beta\alpha})\), and the projection tensor \(\mathcal{P}_\mu\nu\) into the instantaneous fluid rest space is given by

\[ \mathcal{P}_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu . \quad (3.2) \]

The vorticity can then be decomposed, up to second order in perturbation theory, as \(\omega_{ij} \equiv \omega_{1ij} + \frac{1}{2} \omega_{2ij}\), where the first order part is simply

\[ \omega_{1ij} = a V_{1[i,j]} , \quad (3.3) \]

and the second order part is

\[ \omega_{2ij} = a V_{2[i,j]} + 2a \left[ V'_{[i,j]} + V_{[i,j]} (V_1 + B_1)_{[j]} - \phi_1 B_{1[i,j]} \right] . \quad (3.4) \]

Using the evolution equations given in Appendix [A] the first order part evolves as

\[ \omega'_{1ij} - 3H c_s^2 \omega_{1ij} = 0 , \quad (3.5) \]

which gives the well known result that at first order, in the absence of an anisotropic pressure source term, \(|\omega_{1ij} \omega_{1ij}^\dagger| \propto a^{-2}\) during radiation domination, where \(c_s^2 = 1/3\) [4]. Hence the vorticity remains zero in this case, if it is initially zero.

However, at second order we get a non-zero source term for the vorticity evolution equation, assuming zero anisotropic pressure.\(^2\) Even assuming zero first order vorticity, \(\omega_{1ij} = 0\), the second order vorticity evolves according to

\[ \omega_{2ij} - 3H c_s^2 \omega_{2ij} = \frac{2a}{\rho_0 + P_0} \left\{ 3H V_{1[i} \delta P_{\text{nad}1,j]} + \frac{\delta P_{1[i} \delta P_{\text{nad}1,j]}}{\rho_0 + P_0} \right\} . \quad (3.6) \]

Thus, we see that the non-adiabatic pressure perturbation gradients coupled to density perturbation gradients act as a source for second order vorticity. In the case of a vanishing entropy perturbation, as is the case for barotropic fluids, we recover the result of Ref. [17]. For completeness, we give the full second order vorticity evolution equation without assuming \(\omega_{1ij} = 0\) in the Appendix as Eq. [A].

### IV. DISCUSSION AND CONCLUSIONS

In this work we have shown that at second order in the perturbations vorticity is generated from first order scalar and vector perturbations for a perfect fluid. This is an extension of Crocco’s Theorem to an expanding, dynamical background, namely a FRW universe.

Whereas in previous works barotropic flow was assumed, allowing for entropy gives a qualitatively novel result. This implies that the description of the cosmic fluid as a potential flow, which works exceptionally well at first order in the perturbations, will break down at second order for non-barotropic flows. Similarly in barotropic flow Kelvin’s theorem guarantees conservation of vorticity. This is no longer the case if the flow non-barotropic.

Entropy perturbations will arise in many settings, here we only considered the case of a single fluid with non-zero intrinsic entropy. Even in simple single field inflation models, after the end of the slow-roll regime, there is a non-zero entropy perturbation (or non-adiabatic pressure; see, e.g., Ref. [18]). Furthermore, in the case of multiple fluids we get in addition to the intrinsic entropies of the individual fluids also an entropy component stemming from the mixing of the fluids (the relative entropy perturbation in the terminology of Ref. [14]), even if there is no energy and momentum transfer between the fluids. This relative

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2 The calculation of the vorticity evolution equation uses first order evolution and field equations which we give in Appendix [A].
entropy perturbation can be another source of vorticity in the multi-fluid case. However in the cosmic plasma it is likely that the relative entropy perturbation between fluids is subdominant compared to those mechanisms discussed below.

Our result, that the vorticity is non-zero at second order in the presence of non-adiabatic or entropy perturbations, has immediate implications for the generation of magnetic fields in the early universe, since Biermann showed that the generation of magnetic fields is related to vorticity \([20]\) (see also Harrison, Ref. \([21]\)).

Previous works either used momentum exchange between multiple fluids to generate vorticity, as in Refs. \([13, 16, 22, 23, 24, 25, 26]\), or used intermediate steps to first generate vorticity for example by using shock fronts as in Ref. \([27]\). The vorticity generated in the latter case was then used to source turbulence. As we have shown above, we do not require such extra steps. We will study the evolution of magnetic fields in the case of non-zero entropy perturbations, and including tensor perturbations, numerically in a forthcoming paper \([13]\).

At first glance, one might expect that because the vorticity is generated as a second order effect by products of first order density and entropy perturbations, the vorticity will be of order \(10^{-10}\). However, the source term is in fact constructed from gradients of the density and entropy perturbations and, because of this, the effect can readily be larger than \(O(10^{-10})\). In Fourier space, this source term will be ‘boosted’ by the wavenumbers on small scales. Since our mechanism will manifest itself on sub-horizon scales and hence the wavenumbers are large, this has the potential to increase the effect. For example, the authors of Ref. \([28]\) recently showed, in the context of second order tensor perturbations, that on sub-horizon scales this effect can ‘boost’ second order quantities to be of comparable amplitude as first order quantities. In fact, the WMAP five year results results \([29]\) give only a non-zero upper bound on entropy perturbations on horizon scales. Since our effect is on sub-horizon scales, following the argument of WMAP5 and assuming a blue spectrum for the entropy perturbation, it is likely that this effect is non-negligible.

One possible observational signature of vorticity in the early universe is that of B-mode polarisation of the cosmic microwave background (CMB) radiation. At linear order in perturbation theory, only tensor perturbations (or gravitational waves) will produce B-mode polarisation since scalar perturbations only produce E-modes \([30]\), and vector perturbations will become subdominant during inflation, and will decay with the expansion of the universe. However, at second order the former is no longer true and, as we have shown above, density perturbations can generate vorticity. This could then produce B-mode polarisation large enough to be observable by future CMB experiments such as Planck \([31]\).

Acknowledgments

The authors are grateful to Kishore Ananda, Chris Clarkson and Roy Maartens for useful discussions and comments. AJC is supported by the Science and Technology Facilities Council (STFC). We used the computer algebra package CADABRA \([32]\) to obtain some of the conservation equations, and thank Kasper Peeters for useful discussions on using the package.

APPENDIX A

Here we give the evolution and constraint equations necessary to derive the results of Section III. For more details see Ref. \([14]\).

1. Energy-momentum conservation

In the background the energy conservation equation is

\[
\rho' = -3H (\rho_0 + P_0) .
\]

At first order we have for the energy conservation equation

\[
\delta \rho'_{\parallel} + 3H (\delta \rho_{\parallel} + \delta P_{\parallel}) + (\rho_0 + P_0) v^k_{i,k} = 0 ,
\]

and for the momentum conservation equation

\[
V^i_{\parallel} + H (1 - 3s^2) V_{1i} + \left[ \frac{\delta P_1}{\rho_0 + P_0} + \phi_1 \right]_{,i} = 0 .
\]

At second order we only need the momentum conservation equation

\[
\left[ (\rho_0 + P_0) V_{2i} \right]' + 4H (\rho_0 + P_0) V_{2i} + \left[ \delta P_2 + (\rho_0 + P_0) \phi_2 \right]_{,i} + 2 (\delta \rho_1 + \delta P_1) \phi_{1,i} + 2 \left[ (\delta \rho_1 + \delta P_1) V_{3i} \right]'
\]

\[
+ 8H (\delta \rho_1 + \delta P_1) V_{1i} - 2 (\rho_0 + P_0) \left[ \phi_1 B_{1i} + (\phi_1^2)_{,i} \right] - 2 \phi_1 \left[ \rho_0 + P_0 \right] V^i_{1i} + \left( \rho_0' + P_0' + 4H (\rho_0 + P_0) \right) V_{1i}
\]

\[
+ 2 (\rho_0 + P_0) v^k_{i,k} V_{1i} + 2 (\rho_0 + P_0) v^k_{1,k} V_{1i} - 2 (\rho_0 + P_0) v^k_{1,k} B^k_{i,1} = 0 .
\]

2. Field equations

At zeroth order the evolution of the scale factor is governed by the Friedmann equation, given by

\[
H^2 = \frac{8\pi G}{3} a^2 \rho_0 .
\]
second order lapse function in Eq. (A4) above cancels when calculating the vorticity evolution. The Einstein constraint equations at first order are

\[ 2\mathcal{H}B_{1,k}^{i} + 6\mathcal{H}^{2}\phi_{1} = -8\pi G\alpha^{2}\delta\rho_{1}, \quad (A6) \]

and

\[ \nabla^{2}B_{1i} - B_{1,i}^{k} - 4\mathcal{H}\phi_{1} = 16\pi G\alpha^{2}(\rho_{0} + P_{0})V_{1i} \quad (A7) \]

Using the definition of the vorticity tensor, Eq. (3.4) above, and the equations given in the previous subsection, we get the evolution equation for the vorticity tensor at second order

\[
\begin{align*}
\omega'_{2ij} - 3\mathcal{H}_{s}^{2}\omega_{2ij} + 2 & \left( \frac{\delta P_{1} + \delta \rho_{1}}{\rho_{0} + P_{0}} \right)^{'} + V_{1j}^{k} - X_{1,k}^{j} \right] \omega_{1ij} \\
+ 2 & (V_{1}^{k} - X_{1}^{k}) \omega_{1ij,k} - 2 (X_{1,j}^{k} - V_{1,j}^{k}) \omega_{1j,k} + 2 (X_{1,i}^{k} - V_{1,i}^{k}) \omega_{1j,k} \\
= & \frac{a}{\rho_{0} + P_{0}} \left\{ 3\mathcal{H} \left( V_{1ij}\delta P_{nabd1,j} - V_{1j}\delta P_{nabd1,i} + \frac{1}{\rho_{0} + P_{0}} (\delta\rho_{1,j}\delta P_{nabd1,j} - \delta\rho_{1,i}\delta P_{nabd1,j}) \right) \right\}, \\
\end{align*}
\]

where \( X_{1i} \) is given entirely in terms of matter perturbations as

\[ X_{1i} = \nabla^{-2} \left[ \frac{4\pi G}{H^{2}} (\mathcal{H}(\rho_{0} + P_{0})V_{1i} - \delta\rho_{1,i}) \right]. \quad (A9) \]
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