A Gauge Invariant Way to Evaluate Quasiparticle Effective Mass in Fermionic Systems with Gauge Interactions

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Abstract

In this paper, we propose a gauge-invariant way to define and calculate the effective mass for quasiparticles in systems with gauge interactions, and apply it to a model closely related to the half-filled Landau level problem. Our model is equivalent to the Halperin-Lee-Read $\nu = 1/2$ Hamiltonian with an ultraviolet cutoff $\Lambda$ for the gauge fields, and we expand our answer in powers of $\Lambda/k_F$, assuming it is small. In this definition the effective mass depends only on the gauge-invariant density and current response functions of the system. Within RPA, this definition yields a finite result for the effective mass in our model. We also comment on corrections to this effective mass formula when processes beyond RPA are included. Finally, we comment briefly on the observation that organizing the perturbation expansion in powers of $\Lambda/k_F$ is a way to systematically study the physics beyond RPA.
I. INTRODUCTION

Recently there has been considerable interest in understanding the novel physics of a two-dimensional (2D) electron gas in a high magnetic field, with Landau level filling factor \( \nu \) at or close to one-half. It was argued \cite{1,2} that a useful way to look at the problem is to view electrons as composite fermions (CF) \cite{3} carrying two flux quanta, such that the flux carried by the CFs cancel that of the external magnetic field in average. Therefore at mean field level the CFs see zero net magnetic flux, and are expected to form a Fermi-liquid like state, with a Fermi surface satisfying Luttinger’s theorem. Such a simple picture of weakly interacting CFs seems to agree very well with experiments \cite{4-7}, at least at a qualitative level, suggesting that CFs are stable quasiparticles in the system. Convincing evidence of the existence of a Fermi surface has also been found in numerical studies \cite{9}. However a more careful analysis \cite{1} (mostly perturbative) of quasiparticle properties found that quasiparticles near the Fermi surface have a divergent effective mass and very short life time (except when they interact with super-long range interactions), implying that the quasiparticles in this theory are not stable. Previous studies of quasiparticle properties have been based on the single particle Green function or self-energy, which is a \emph{gauge dependent} quantity. It has been shown \cite{8} that singularities encountered in self-energy calculations disappear in calculations of gauge-invariant response functions. Another difficulty of the standard perturbative approach is that the coupling constant for the gauge interaction is not small in this problem; therefore it is unclear whether or not the perturbation series in powers of the coupling constant is convergent.

In this paper, we propose a physically motivated definition for the effective mass of quasiparticles that is fully gauge-invariant and expressed in terms of the (gauge-invariant) response functions of the system. This definition is very much in the spirit of Landau’s original picture of quasiparticles in the Fermi liquid which arise from adiabatically dressing the particles (or holes) in the free Fermi gas with interactions. This new definition may be applied to models with gauge interactions in general. In this work we apply it to a
model that is closely related to the $\nu = 1/2$ problem; the only difference is that there is a finite momentum cutoff $\Lambda$ in both the gauge and scalar interactions of the CFs. In real space this corresponds to a flux that is distributed near the particle within a region of order $1/\Lambda$ (or a “fat” flux tube), instead of the original point-like distribution of flux. The original problem is recovered in the limit $\Lambda \to \infty$, while in the limit $\Lambda \to 0$ the mean field approximation becomes exact. We calculate the leading contribution to the effective mass from interactions, in the limit of small $\Lambda/k_F$, which is equivalent to calculating response functions in the Random Phase Approximation (RPA). We do not find any singularity at this level. We will also discuss how to go to higher orders in this approach. We also suggest that expanding in powers of $\Lambda/k_F$ may be a useful alternative way to organize the perturbation series.

We start by introducing the model we use, and then present the definition of the quasi-particle effective mass, and evaluate it at the lowest order in $\Lambda/k_F$. We conclude with a summary and discussion.

II. MODEL WITH FINITE CUTOFF IN INTERACTIONS

Consider the following Hamiltonian for fermions at half-filling:

$$H = \sum_i \frac{1}{2m} (\vec{p}_i - e\vec{A}(\vec{r}_i) - \vec{a}(\vec{r}_i))^2 + \frac{1}{2} \sum_{i \neq j} V(\vec{r}_i - \vec{r}_j)$$

(1)

where the momenta and positions are for the composite fermions. $\vec{a}$ actually depends on the coordinates of the other particles and its form depends on the filling factor; it is much easier to deal with these quantities in momentum space, and we can in fact write (using the explicit expression for $\vec{a}$ in terms of the coordinates of the other particles)

$$\delta a^l(\vec{r}_i) \equiv eA^l(\vec{r}_i) + a^l(\vec{r}_i) = \sum_{j \neq i} \sum_{q \neq 0} i\Phi(q) \frac{e^{imq_m}}{q^2} \exp(i\vec{q} \cdot (\vec{r}_i - \vec{r}_j))$$

(2)

In the above we have explicitly used the Coulomb gauge for $\vec{a}$. This is where we introduce the ”fat” fluxes: in the exact model, we have $\Phi(q) = 4\pi$ for a system at half-filling, but in
our model we take $\Phi(q) = 4\pi$ for $q \leq \Lambda$ and equal to 0 for $q > \Lambda$. This corresponds to an ultraviolet cutoff on the gauge field momenta. The reason for doing this will become clear shortly, but for the moment let us remark that for small $\Lambda$ it becomes possible to treat $\Lambda/k_F$ as a small parameter in organizing the perturbation expansion of the theory.

We can use equations 1, 2 to rewrite the Hamiltonian in terms of the density and current operators as follows:

$$H = \sum_i \frac{1}{2m} \vec{p}_i^2 + \frac{1}{2} \sum_q (\rho(q) \rho(-q) - N)V(q) + \left( \frac{1}{2} \sum_{q \neq 0} i \frac{\Phi(q)}{q} \rho(q)(\vec{J}(q) \times \hat{q})^\dagger + c.c \right)$$

(3)

Figure 1 shows the Feynman rules for the theory derived from equation 3. For convenience of notation, the interaction lines can be encoded into a matrix $V$ s.t.

$$V = \begin{pmatrix} V(q) & -i\Phi(q)/q \\ i\Phi(q)/q & 0 \end{pmatrix}$$

(4)

In addition, we may also define the following 2-by-2 matrix to simplify the writing of the appropriate RPA expressions for the response functions:

$$K = \begin{pmatrix} K_{00} & K_{01} \\ K_{10} & K_{11} \end{pmatrix}$$

(5)

Here $K_{00}$ is the density-density response function, $K_{01}$ is the density-current response function, etc. They are physical response functions which are independent of the gauge choice of the Hamiltonian. As a reminder, only the correlation functions of the transverse current $\vec{J}(q) \times \hat{q}$ enter this matrix, since the longitudinal current is completely determined from the density (by current conservation) and therefore not independent. Note that in the above matrices, $K = K(q, \omega)$.

As usual, we may write the RPA response functions $K^{RPA}$ in terms of the free response functions $K^{(0)}$ as

$$K^{RPA} = K^{(0)}(1 - VK^{(0)})^{-1},$$

(6)

and the RPA propagator as

$$V^{RPA} = (V^{-1} - K^{(0)})^{-1}. $$

(7)
In general, the connected Green functions are obtained from a geometric sum over the one particle irreducible Green functions (the so-called 1PI Green functions), and the RPA response functions are obtained by replacing the general 1PI Green functions by the free 1PI contribution. In our model with "fat" flux quanta, we can show that the 1PI corrections to the free 1PI contribution to the response functions are suppressed by factors of $\Lambda/k_F$. This is because the internal interaction lines must have momentum $q \leq \Lambda$; this phase space restriction causes the appropriate 1PI gauge invariant corrections to the free response functions to be suppressed by $\Lambda/k_F$. In this sense, when $\Lambda/k_F \to 0$ the corrections to RPA vanish relative the the RPA contributions.

III. EFFECTIVE MASS

Usually, the effective mass of composite fermions is defined in terms of the self energy of the composite fermions, which depends on the gauge choice of the Hamiltonian. Rather than doing this, we would like to motivate a very different way to calculate the effective mass. This is based on an energy theorem discussed by Pines and Nozieres [10], and in the spirit of Landau’s original picture of quasiparticles in the Fermi liquid. If the system is a Fermi liquid, and its low energy excitations can be described as stable quasiparticle and quasihole excitations, we may label a low energy eigenstate of the system by an occupation configuration $n_\mathbf{k}$ (see figure 2); such a state is smoothly connected to the corresponding state of the free Fermi system, by adiabatically turning on the interactions. What we do is to evaluate the energy of such states, which by the Hellman-Feynman theorem may be expressed as coupling constant integrals of certain gauge invariant correlation functions. In particular, we calculate the difference between the ground state and the excited states with a single particle-hole pair excitations; by definition this energy difference is related to the quasiparticle effective mass.

Let us derive an expression for the energy of a state $\Psi_0$ now [10]. Let $\langle \rangle$ denote expectation values with respect to this state. To start with, let us rewrite equation 3 by explicitly
Introducing a parameter $\alpha$ which accompanies the coupling constant:

$$H = \sum_i \frac{1}{2m} \vec{p}_i^2 + \alpha \left( \frac{1}{2} \sum_q (\rho(q)\rho(-q) - N)V(q) + \left( \frac{1}{2} \sum_{q \neq 0} i \frac{\Phi(q)}{q} \rho(q)(\vec{J}(q) \times \hat{q})^\dagger + c.c. \right) \right)$$  \hspace{1cm} (8)

and therefore we have the energy (as a function of $\alpha$)

$$E_0(\alpha) = \langle \Psi_0(\alpha) | \sum_i \frac{1}{2m} \vec{p}_i^2 | \Psi_0(\alpha) \rangle + E_{\text{int}}(\alpha)$$  \hspace{1cm} (9)

Differentiating this with respect to $\alpha$ we get

$$\frac{\partial E_0}{\partial \alpha} = \langle \Psi_0 | \frac{\partial H}{\partial \alpha} | \Psi_0 \rangle + \langle \frac{\partial \Psi_0}{\partial \alpha} | H | \Psi_0 \rangle + \langle \Psi_0 | H | \frac{\partial \Psi_0}{\partial \alpha} \rangle$$

$$= \frac{E_{\text{int}}}{\alpha} + E_0 \frac{\partial}{\partial \alpha} \langle \Psi_0 | \Psi_0 \rangle$$  \hspace{1cm} (10)

Noting that the state is normalized, we get

$$\frac{\partial E_0}{\partial \alpha} = \frac{E_{\text{int}}}{\alpha}$$  \hspace{1cm} (11)

This is nothing but a special form of the Hellman-Feynman theorem. We may then integrate over $\alpha$ and obtain:

$$E_0(\alpha = 1) - E_0(\alpha = 0) = \int_0^1 d\alpha \frac{E_{\text{int}}}{\alpha}$$  \hspace{1cm} (12)

Plugging in equation 8

$$\frac{E_{\text{int}}}{\alpha} = \frac{1}{2} \sum_q V(q)(\langle \rho(q)\rho^\dagger(q) \rangle - N) + \frac{1}{2} \sum_{q \neq 0} \left( \frac{i\Phi(q)}{q} \langle \rho(q)(\vec{J}(q) \times \hat{q})^\dagger + c.c. \right)$$

$$= -\frac{1}{2} \sum_{q \neq 0} V(q) \int_0^\infty \frac{d\omega}{\pi} \text{Im} K_{00}(q, \omega) - \frac{1}{2} \sum_{q \neq 0} \int_0^\infty \frac{d\omega}{\pi} \left( \frac{i\Phi(q)}{q} \text{Im} K_{01}(q, \omega) + c.c. \right)$$  \hspace{1cm} (13)

Let us note that we may do the $\omega$ integrals via contours (see figure 3; as the figure shows, the contour encloses the real excitations of the system):

$$\int_0^\infty d\omega \text{Im} K_{0\alpha}(q, \omega) = \frac{1}{2i} \oint K_{0\alpha}(q, \omega) d\omega$$  \hspace{1cm} (14)

We then perform the coupling constant integration (equation 12) to obtain:
\[ E_0(\alpha = 1) = E_0(\alpha = 0) - \frac{1}{4\pi i} \oint_{\Gamma} d\omega \sum_q \int_0^1 d\alpha \text{Tr}(V(q)K(q, \omega)) \]  

(15)

where we have taken the matrix forms for the interaction and response functions from equations 4 and 5 for compactness of notation.

The response functions \( K \) depend implicitly on \( \alpha \) because of interactions. As we mentioned earlier, in the limit \( \Lambda/k_F \rightarrow 0 \) we can replace the full response functions with the corresponding RPA expressions. Plugging in from equation 6 the form for \( K^{RPA} \) we get:

\[ E_0(\alpha = 1) = E_0(\alpha = 0) - \frac{1}{4\pi i} \oint_{\Gamma} d\omega \sum_q \int_0^1 d\alpha \text{Tr} \left( \frac{V(q)K^{(0)}(q, \omega)}{1 - \alpha V(q)K^{(0)}(q, \omega)} \right) \]

\[ = E_0(\alpha = 0) + \frac{1}{4\pi i} \oint_{\Gamma} d\omega \text{Tr} \log(1 - VK^{(0)}) \]  

(16)

Of course, \( E_0(\alpha = 0) \) is nothing but the energy for free fermions (the system being described by the state \( \Psi_0 \), or alternatively the shape of the fermi surface).

We are now in a position to give an expression for the effective mass of composite fermions. As we remarked earlier, the energy is a functional of the shape of the fermi surface. Explicitly, we have \( K = K[n_k] \) where \( n_k \) equals zero for \( k \) outside the fermi surface and equals one inside. Varying the shape of the fermi surface will vary \( K \) and thereby vary the energy as per equation 16.

Let us take the relaxed configuration to be the circular fermi surface, with \( n_k^{(0)} = \theta(k_F - |k|) \) and the distorted fermi surface be described by \( n_k = n_k^{(0)} + \delta n_k \) where \( \delta n_k = \delta n_{k,F+l} - \delta n_{k,F-l} \) (i.e, we have created a particle-hole pair with particle momentum \( k_F + l \) and hole momentum \( k_F - l \) : see figure 2). For free fermions (\( \alpha = 0 \)) the difference between the energies of the two states is simply \( \delta E_0(\alpha = 0) = 2k_F \cdot l/m \). It is natural to define the effective mass for interacting fermions in the same way, i.e, we have

\[ \delta E_0(\alpha = 1) = \frac{2k_F \cdot l}{m^*} \]  

(17)

We like this definition of the effective mass because it is defined in terms of gauge-invariant quantities. Let us also point out that in the sums over \( q \) in the above expressions there is a cutoff \( \Lambda \) in our model, because \( \Phi(q) = 0 \) for \( q > \Lambda \); in addition, we require \( V(q) = 0 \) for \( q > \Lambda \) as well. Then we get
\[
\frac{2k_F \cdot l}{m^*} = \frac{2k_F \cdot l}{m} - \frac{1}{4\pi} \sum_{q \leq \Lambda} \oint_{\Gamma} dq \omega \frac{V(q)\delta K_{00} + \frac{q^2}{4\pi}(K_{00}\delta K_{11} + \delta K_{00}K_{11})}{1 - V(q)K_{00} - \frac{q^2}{4\pi}K_{00}K_{11}}
\]

where \(K_{\alpha\beta}[n_k] = K_{\alpha\beta}[n_k](q, i\omega)\) and \(\delta K_{\alpha\beta} = K_{\alpha\beta}[n_k] - K_{\alpha\beta}[n_k(0)]\) expanded to linear order.

We can then evaluate the above expressions in the small \(\Lambda\) limit. As is clear, while doing the \(\omega\) integral, we will pick up (in general) contributions from all the poles in the integrand; however, in the small \(q\) limit, the cyclotron mode saturates the F-sum rule, and therefore this is the only pole we need to take into account in the limit \(\Lambda \to 0\). First, we evaluate the effective mass as a function of \(\Lambda\) when \(V = 0\). Then we write down the correction to the effective mass treating \(V\) as a small perturbation. The calculations are straightforward, and we get for \(V = 0\):

\[
\frac{1}{m^*} = \frac{1}{m} \left(1 + \frac{\Lambda^4}{2k_F^4}\right)
\]

For small \(V\) we will get

\[
\frac{1}{m^*} = \frac{1}{m} \left(1 + \frac{\Lambda^4}{2k_F^4} - \frac{3}{8\pi} \int_0^\Lambda mV(q)q^5 dq\right)
\]

We should emphasize that there is no divergence in the effective mass at this level. Also, we should clarify that the \(i\omega \sim q^3\) pole in the RPA interaction (found for \(q\) small and \(\omega \ll qv_F\)) does not contribute to our effective mass formula. This is easily seen because the energy depends only on physical excitations of the system, given by poles of the response functions that lie on the real axis, and the contour \(\Gamma\) is chosen to only include such physical excitations. Corrections to RPA could certainly, in principle, suffer a divergence due to this pole (when we have to do additional frequency and momentum integrals for internal interaction lines). However, we found no divergences in the physical response functions when we included the 2-loop self-consistent corrections to RPA for the response functions. We remark once again, that in our approach, corrections to the effective mass formula from such processes will be higher order in our expansion in \(\Lambda/k_F\).

To conclude this section, let us very briefly point out the relation between our definition of the effective mass versus the standard definition coming from the self energy of the composite
fermion. The exact theory, \textit{a la} Halperin, Lee and Read \cite{Halperin1986} has interacting fermions and
gauge fields. There are two ways to evaluate the free energy of the system. One can first
integrate out the gauge fields and obtain an effective action for the \textit{fermions}, from which
one then evaluates the free energy for system in the saddle point approximation. This
process basically involves calculating the self energy of the composite fermion, and seeing
how this self-energy changes the free energy of the system of composite fermions. This
gives one definition of the effective mass, which is the one we are used to. Alternately,
we may integrate out the fermions first, and obtain an effective action for the \textit{gauge fields},
and again one can find the free energy of this system in the saddle point approximation.
This would give us a formula for the free energy that is identical to the expression given in
equation \ref{eq:16}. If one integrated out the fermi and gauge fields exactly, one would get the same
answer. However, because we are making the saddle point approximation in one case with
the effective fermion action and in the other case with the effective gauge field action, there
is no guarantee that we will get the same result. The fact that in the latter approach (the
one we use) we only make reference to gauge invariant finite quantities makes it somewhat
more desirable and possibly more physically meaningful.

IV. DISCUSSION

In the above, we have described a different way to define the effective mass of quasi-
particles in the system, first described by Pines and Nozieres in their classic book \cite{Pines1973}, in
the context of degenerate electron gas. This method of calculating the effective mass is
especially suitable for systems with gauge degrees of freedom, because things are defined
purely in terms of gauge invariant observables.

We should comment that our expression \ref{eq:20} seems to indicate that for increasing $\Lambda$ (and
$V = 0$) the effective mass decreases monotonically. Of course, this is only for small $\Lambda$. In
general, we have fermions in a strong magnetic field (and more specifically, fermions in the
lowest Landau level) and as Shankar and Murthy \cite{Shankar1994} suggest, it should be possible to see
signs of this even if the Chern-Simons formulation reduces the theory to quasiparticles in weak magnetic field. For fermions in the lowest Landau level, however, the kinetic energy is quenched; initially, we were hoping that our calculation would show signatures of this quenching by a monotonic enhancement of the effective mass as a function of increasing $\Lambda$ (in the lowest Landau level, $m^* = \infty$ in the absence of interactions), but we find exactly the opposite to lowest order. Still, for larger $\Lambda \sim \mathcal{O}(k_F)$ we would have to include contributions to the effective mass beyond RPA, and it is possible that the effective mass could show a monotonic enhancement of the desired kind.

Lastly, let us remark that $\nu = 1/2$ model with "fat" flux quanta has the merit that we can treat perturbation theory in a systematic fashion despite the fact that the coupling constant in the theory is not small. In a loose sense, adding interaction lines to 1PI diagrams results in a phase space suppression $\Lambda/k_F$ for each internal interaction line (by interaction, we mean both $\rho\rho$ as well as $\rho J$). This is because the momentum of each such internal line is restricted to be smaller than $\Lambda$. We have explicitly checked this result for all 2-loop diagrams (using the RPA interaction instead of the bare interaction, in order to treat the problem self-consistently). Whether or not the results obtained using our small $\Lambda$ approximation are meaningful for large $\Lambda$ (the exact problem has $\Lambda = \infty$) is an issue we cannot address within the present framework. However, the method of defining the effective mass in systems with gauge interactions in terms of gauge-invariant response functions as above is certainly well-defined and completely general.

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FIGUREs

FIG. 1. Feynman rules for theory.

FIG. 2. On the left, the ground state of the system is shown, for which we have a circular fermi surface. On the right, we depict a quasiparticle-quasihole excitation of this state. The text describes how we calculate the energy difference between these states in our model, and the effective mass is defined in terms of this energy difference.

FIG. 3. The contour includes all the physical modes of the system. As explained in the text, poles in the response function may appear off the real axis, but these will not contribute to the integral.
\begin{align*}
0 \quad q(q) \quad 1 &= i\Phi(q)/q \\
1 \quad q(q) \quad 0 &= -i\Phi(q)/q \\
0 \quad q(q) \quad 0 &= V(q) \\
1 \quad q(q) \quad 1 &= 0
\end{align*}

\[q(q) \qquad \text{Figure 1: Chari et. al. Figure 1.}\]
Figure 1: Chari et al. Figure 2.
Figure 1: Chari et. al. Figure 3.