DOMAIN-STRUCTURED CHAOS FOR DISCRETE RANDOM PROCESSES

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Abstract. We introduce the notion of domain-structured chaos and apply it to establish a connection between stochastic dynamics and deterministic chaos.

In this paper, a new efficient method is used to show that a discrete random process exhibits chaotic dynamics. To achieve this goal, the concepts of abstract similarity map and domain-structured chaos, previously considered in our papers [1][4][8], are further developed. It must be noted that the concept of domain-structured chaos is more abstract in the present paper than in [4], since it is free of topological assumptions.

1. Domain structured chaos

Let the metric space \((F, d)\) be given, with the distance \(d\). Assume that there exists the following presentation for the set \(F\),

\[
F = \{\mathcal{F}_{i_1 i_2 \ldots i_m \ldots} : i_k = 1, 2, \ldots, m, k = 1, 2, \ldots\},
\]

where \(m\) is a natural number. The presentation means that each element of the set \(F\) is labeled through at least by one member of the set \(\mathcal{F}\), and each element of the set \(\mathcal{F}\) presents a member from \(F\). We assume that the uniqueness is not necessarily required for this relation. That is, if \(\mathcal{F}_{i_1 i_2 \ldots i_n \ldots}\) and \(\mathcal{F}_{j_1 j_2 \ldots j_n \ldots}\) present the same element \(f \in F\), it is not necessary that \(i_n = j_n\) for all \(n = 1, 2, \ldots\). Moreover, we keep the distance \(d\) for the set \(\mathcal{F}\) considering \(d(\mathcal{F}_{i_1 i_2 \ldots i_n \ldots}, \mathcal{F}_{j_1 j_2 \ldots j_n \ldots}) = d(f_1, f_2)\), if \(\mathcal{F}_{i_1 i_2 \ldots i_n \ldots}\) and \(\mathcal{F}_{j_1 j_2 \ldots j_n \ldots}\) are presentations of elements \(f_1\) and \(f_2\) of the set \(F\) respectively, such that \(d(\mathcal{F}_{i_1 i_2 \ldots i_n \ldots}, \mathcal{F}_{j_1 j_2 \ldots j_n \ldots}) = 0\) for different presentations of the same point in \(F\). In what follows, we call the set \(\mathcal{F}\) a pre-chaotic structure for the set \(F\). It is clear that there is a naturally determined map \(\Phi : F \rightarrow F\), which does not satisfy the uniqueness condition, and values \(\Phi(f)\) for a fixed \(f\) constitute the set \(\{\varphi(\mathcal{F}_{i_1 i_2 \ldots i_n \ldots})\}\), with all labels of the point \(f\). In what follows, we shall describe chaotic properties of the map \(\Phi\) in terms of the dynamics of the map \(\varphi\).

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The following sets are needed,
\[ F_{i_1 i_2 \ldots i_n} = \bigcup_{j_k=1,2,\ldots,m} F_{i_1 i_2 \ldots i_n j_1 j_2 \ldots} \] (1.2)
where indices \( i_1, i_2, \ldots, i_n \), are fixed.

It is clear that
\[ F \supseteq F_{i_1} \supseteq F_{i_1 i_2} \supseteq \ldots \supseteq F_{i_1 i_2 \ldots i_n} \supseteq F_{i_1 i_2 \ldots i_n i_{n+1}} \ldots, \]
that is, the sets form a nested sequence. Let us introduce the map \( \varphi : F \to F \)
such that
\[ \varphi(F_{i_1 i_2 \ldots i_n \ldots}) = F_{i_2 i_3 \ldots i_n \ldots}. \] (1.3)

Considering iterations of the map, one can verify that
\[ \varphi^n(F_{i_1 i_2 \ldots i_n \ldots}) = F, \] (1.4)
for arbitrary natural number \( n \) and \( i_k = 1,2,\ldots,m, \ k = 1,2,\ldots \). The relations (1.3) and (1.4) give us a reason to call \( \varphi \) a similarity map and the number \( n \) the order of similarity. The similarity map \( \varphi \) is known as the Bernoulli shift [10] for the symbolic dynamics. We will say that for the sets \( F_{i_1 i_2 \ldots i_n} \) the diameter condition is valid, if
\[ \max_{i_k=1,2,\ldots,m} \text{diam}(F_{i_1 i_2 \ldots i_n}) \to 0 \ as \ n \to \infty, \] (1.5)
where \( \text{diam}(A) = \sup\{d(x,y) : x, y \in A\} \), for a set \( A \) in \( F \).

Denote by \( d(A,B) = \inf\{d(x,y) : x \in A, y \in B\} \) the function of two bounded sets \( A \) and \( B \) in \( F \). Set \( F \) satisfies the separation condition of degree \( n \) if there exist a positive number \( \varepsilon_0 \) and a natural number \( n \) such that for arbitrary indices \( i_1 i_2 \ldots i_n \) one can find indices \( j_1 j_2 \ldots j_n \) such that
\[ d(F_{i_1 i_2 \ldots i_n}, F_{j_1 j_2 \ldots j_n}) \geq \varepsilon_0. \] (1.6)

It is clear that the couple \((F,d)\) is not, in general, a metric space. Nevertheless, it is true that the map \( \varphi \) is continuous with respect to the distance \( d \), and all the attributes of definitions for Poincaré, Li-Yorke and Devaney chaos can be extended for the dynamics. The similarity map \( \varphi \) possesses the three ingredients of the Devaney chaos, namely density of periodic points, transitivity and sensitivity. A point \( F_{i_1 i_2 \ldots i_n} \in F \) is periodic with period \( n \) if its index consists of endless repetitions of a block of \( n \) terms.

The map \( \Phi \) admits domain structured Devaney chaos on \( F \), if the map \( \varphi \) is Devaney chaotic on \( F \).

If the diameter and separation conditions are valid, then \( F \) is said to be a chaotic structure for \( F \).

The proof of the next theorem extends the technique for symbolic dynamics [10].
Theorem 1. If $F$ is a chaotic structure for $F$, then the dynamics of $\Phi$ admits domain structured chaos in the sense of Devaney.

Proof. Fix a member $F_{i_1i_2...i_n...}$ of $F$ and a positive number $\varepsilon$. Find a natural number $k$ such that $\text{diam}(F_{i_1i_2...i_k}) < \varepsilon$ and choose a $k$-periodic element $F_{i_1i_2...i_k}$ of $F_{i_1i_2...i_k}$. It is clear that the periodic point is an $\varepsilon$-approximation for the considered member. The density of periodic points is thus proved.

Next, utilizing the diameter condition, the transitivity will be proved if we show the existence of an element $F_{i_1i_2...i_n...}$ such that for any subset $F_{i_1i_2...i_k}$ there exists a sufficiently large integer $p$ so that $\varphi^p(F_{i_1i_2...i_n...}) \in F_{i_1i_2...i_k}$. This holds true since we can construct a sequence $i_1i_2...i_n...$ such that it contains all the sequences of the type $i_1i_2...i_k$ as blocks.

For sensitivity, fix a point $F_{i_1i_2...i_n...} \in F$ and an arbitrary positive number $\varepsilon$. Due to the diameter condition, there exist an integer $k$ and element $F_{i_1i_2...i_kj_{k+1}j_{k+2}...} \neq F_{i_1i_2...i_k}$ such that $d(F_{i_1i_2...i_k}, F_{i_1i_2...i_kj_{k+1}j_{k+2}...}) < \varepsilon$. We choose $j_{k+1}, j_{k+2}, ...$ such that $d(F_{i_1i_2...i_kj_{k+1}j_{k+2}...j_{k+n}}, F_{i_1i_2...i_kj_{k+1}j_{k+2}...j_{k+n}}) > \varepsilon_0$, by the separation condition. This proves the sensitivity. □

In [5, 6], Poisson stable motion is utilized to distinguish chaotic behavior from periodic motions in Devaney and Li-Yorke types. The dynamics is given the named Poincaré chaos.

The map $\Phi$ admits on $F$ domain structured Poincaré chaos, if the map $\varphi$ is Poincaré chaotic on $F$.

The next theorem shows that the Poincaré chaos is valid for the similarity dynamics.

Theorem 2. If $F$ is a chaotic structure for $F$, then the map $\Phi$ possesses domain structured Poincaré chaos.

The proof of the last theorem is based on the verification of Lemma 3.1 in [6], applied to the similarity map.

In addition to the Devaney and Poincaré chaos, it can be shown that the Li-Yorke chaos is also present in the dynamics of the map $\Phi$. The proof of the theorem is similar to that of Theorem 6.35 in [9] for the shift map defined in the space of symbolic sequences. In what follows we will say that the dynamics of the map $\Phi$ admits the domain structured chaos.

2. Chaotic random processes

Consider a discrete time random process discrete finite state space as a family of random variables $X(n), n = 1, 2, \ldots$. Denote by $S$ the state space of the process, and consider it with a distance $d$. We assume positive probability for all members of the state space, and for each experiment one of the members of the state space must necessarily happen.
Suppose that there exists a chaotic structure
\[ S = \{ S_{i_1 i_2 \ldots i_n} : i_k = 1, 2, \ldots, p, \ k = 1, 2, \ldots \}, \]  
for the state space. In the light of the last section’s results, this means that domain-structured chaos, and, consequently, Poincaré, Li-Yorke and Devaney chaos are present. Nevertheless, it does not mean that the random process is chaotic, since its dynamics may be richer than that of the map \( \phi \).

Next, we will consider the stochastic process, which is chaotic, owing to the coincidence of stochastic and deterministic dynamics.

We assume that the state space is finite and denote \( S = \{ s_1, s_1, \ldots, s_p \} \).

Assign for each member \( s_i, i = 1, \ldots, p, \) of the state space infinitely many presentation elements \( S_{i_1 i_2 \ldots i_k} \ldots \) where \( i_j = 1, \ldots, p, j = 2, \ldots \). Thus, one can consider the pre-chaotic structure, \( S = \{ S_{i_1 i_2 \ldots i_k} \ldots, i_j = 1, \ldots, p, j = 1, \ldots \} \).

Consider the distance \( d(S_{i_1 i_2 \ldots i_k} \ldots, S_{j_1 j_2 \ldots j_k} \ldots) = d(s_{i_1}, s_{j_1}) \) if \( i_1, j_1 = 1, \ldots, p \). It is easy to verify, that the diameter and separation conditions are valid for the set \( S \). Since the set of all realizations coincides with the set of all sequences on the finite set of numbers, and each experiment is equivalent to the iteration of the shift, the random dynamics admits the same chaotic properties as the trajectories of the similarity map dynamics. Therefore, the following theorem is valid.

**Theorem 3.** If the state space is finite, then the random process \( X(n) \) admits the domain-structured chaos.

That is, random processes such as coin tossing, dice rolling, traffic lights, tetrahedron dice rolling, and five city entrance, [7], are chaotic, since the collections of their realizations (sample sequences) are chaotic sets.

3. **Conclusion**

We have discussed the appearance of deterministic chaos in the discrete random process. We introduce an instrument that we call a domain structured chaos, which can have many more future applications. The approach can be used for non-stationary processes, continuous-time random processes, as well as processes with Markov chains. Our research suggests that the methods historically developed for chaotic dynamics, e.g. synchronization and control, can be extended to dynamics with probability.

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