THE CAUCHY PROBLEM FOR DISSIPATIVE BENJAMIN-ONO EQUATION IN WEIGHTED SOBOLEV SPACES

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ABSTRACT. We study the well-posedness in weighted Sobolev spaces, for the initial value problem (IVP) associated with the dissipative Benjamin-Ono (dBO) equation. We establish persistence properties of the solution flow in the weighted Sobolev spaces $Z_{s,r} = H^s(\mathbb{R}) \cap L^2(|x|^{2r}dx)$, $s \geq r > 0$. We also prove some unique continuation properties in these spaces. In particular, such results of unique continuation show that our results of well posedness are sharp.

1. Introduction

This work is concerned with the initial-value problem (IVP) associated with the dissipative-Benjamin-Ono (dBO) equation

$$\begin{cases}
u_t + \mathcal{H}\partial_x^2 u + D^\alpha u + uu_x = 0, \; x \in \mathbb{R}, \; t > 0, \\
u(x,0) = \phi(x),
\end{cases}$$

(1.1)

where $0 \leq \alpha \leq 2$, $D^s$ denotes the fractional derivative of order $s$ defined, via Fourier transform as

$$D^s f(x) = (|\xi|^s \hat{f})^\vee(x),$$

and $\mathcal{H}$ is the Hilbert transform defined by

$$\mathcal{H}f(x) = \text{p.v.} \frac{1}{\pi} \int_{\mathbb{R}} \frac{f(y)}{x-y} dy = \mathcal{F}^{-1}(-\text{sgn}(\xi)\hat{f}(\xi))(x).$$

(1.2)

We see that (1.1) represents, for $\alpha = 0$, the well-known Benjamin-Ono (BO) equation, that was deduced by Benjamin [2] and later by Ono [38] as a model for long internal gravity waves in deep stratified fluids. The BO equation has been extensively studied, in the last years, with respect to regularity in Sobolev spaces. In this sense, issues about locally or globally well-posedness (LWP and GWP, resp.) are addressed. In general, the main goal is to find the minimal regularity in Sobolev spaces, see [5], [35] and [11]. Moreover, unique continuation principles are often investigated. For more details see, [25], [28] and [23]. Several papers on the study of local-well posedness or global-well posedness in weighted Sobolev has been devoted to BO equation, see [27], [23], [22] and [14]. Solutions without infinity decay in the initial data can be found in [29] and [21]. Two-dimensional versions are also of great interest in the literature, see for example [37], [8], [17], [1], [9], [15] and [40]. Recently, a higher order versions of the BO equation were studied, see [14], [32], [33], and references therein. Kenig, Ponce and Vega [31] obtained results of uniqueness solutions for the BO equation, see also [22].

For $\alpha = 2$, the (dBO) becomes the Benjamin-Ono-Burgers (BOB) equation

$$\partial_t u + \mathcal{H}\partial_x^2 u - \partial_x^2 u + uu_x = 0.$$  

(1.3)

This equation was derived by Edwin and Roberts in [10]. In [39] Otani obtained global-well posedness in $H^s(\mathbb{R})$, where $s > -1/2$. After that, Vento [41] showed
Theorem 1.1. Let \( a \in (0, 1] \), then the following statements are true.

i) Let \( s \geq r > 0 \) and \( r < 3/2 + a \). Then, the IVP (1.1) is GWP in \( Z_{s,r} \).

ii) Let \( r \in [3/2 + a, 5/2 + a) \) and \( r \leq s \). Then, the IVP (1.1) is GWP in \( Z_{s,r} \).

Theorem 1.2. Let \( u \in C([0, T]; Z_{1,1}) \) be a solution of the IVP (1.1), with \( a \in (0, 1] \).
If there exist two different times \( t_1, t_2 \in [0, T] \) such that \( u(t_j) \in Z_{3/2+a,3/2+a} \), \( j = 1, 2 \), then

\[
\dot{u}(0, t) = 0, \quad \text{for all} \quad t \in [0, T].
\]

Theorem 1.3. Let \( u \in C([0, T]; Z_{2,2}) \) be a solution of the IVP (1.1), with \( a \in (0, 1] \).
If there exist three different times \( t_1, t_2, t_3 \in [0, T] \) such that \( u(t_j) \in Z_{5/2+a,5/2+a} \), \( j = 1, 2, 3 \), then there exists \( t_1 < t < t_2 \) such that

\[
u(x, t) = 0, \quad \text{for all} \quad x \in \mathbb{R}, \ t \geq \bar{t}.
\]

Now we present some ingredients for the proof of Theorems 1.1–1.3.

For the Theorems 1.2 and 1.3 we will adapt the techniques introduced by Fonseca, Felipe and Ponce [24] for the study of the Benjamin-ono equation with a generalized dispersion. It consists in the use a commutator estimate for the fractional derivatives (see [24], Proposition 2.2) and the Stein derivative for obtaining the unique continuation principles. Our proof is a little different from the one presented in [24]. In fact, we made use of an additional commutator estimate for the derivative \( D^r \) (see Proposition 2.10 below), see also [23] and [7].

The proof of Theorem 1.1 will be obtained by using the ideas of Fonseca, Pastrán, and Rodríguez-blancos [21]. In this work the authors studied a version of the BO equation with a dissipative effect. We note that the dissipative term in (dBO) equation, has the effect of pushing down the indices \( r \) and \( s \) of space \( Z_{s,r} \) (see the definition below), in relation those obtained in [23] and [24].

This index is critical in the sense that the flow map \( \phi \mapsto u \) is of class \( C^3 \) from \( H^s(\mathbb{R}) \) to \( H^s(\mathbb{R}) \), for \( s < -1/2 \). By and large, issues such as well-posedness and asymptotic behavior of solutions, for the (BO) equation, has been widely studied in the last years, see [3], [9], [12], [15], [16], and references therein.

The well-posedness for the dBO equation was first examined by Vento [41]. More precisely he obtained, in the case \( 1 < \alpha < 2 \), global well-posedness in \( H^s(\mathbb{R}) \), where \( s > -\alpha/4 \), and Ill-posedness (holds when \( \alpha = 1 \), for \( s < -\alpha/4 \), in the sense that the mapping data-solution is not \( C^3 \) in a neighborhood of the origin. If \( 0 \leq \alpha < 1 \), he also obtained the Ill-posedness in \( H^s(\mathbb{R}) \), for all \( s \in \mathbb{R} \), in the sense that the mapping data-solution is not \( C^2 \) at origin. On the other hand, about the long-time behavior of the solutions, we can also add that the following perturbation of the IVP (1.1)

\[
\begin{cases}
u_t + \mathcal{H}\partial_x^2 u + D^\alpha u + wu_x = f, \ x \in \mathbb{R}, \ t > 0, \\
u(x, 0) = \phi(x),
\end{cases}
\]

where \( f \in L^2((1 + x^2)^{1/2} \ dx), \) has a global attractor with finite dimension, in the sense of Hausdorff, see [1].

As the usual, we are considering the well-posedness in the Kato’s sense, that is, includes, existence, uniqueness, persistence property and smoothness of the map data-solution. In this paper, we are mainly interested in to study the global well-posedness of the IVP (1.1) in weighted Sobolev spaces.

Our main results are the following:

\[
\begin{align*}
\text{Theorem 1.1.} & \quad \text{Let } a \in (0, 1], \text{ then the following statements are true.} \\
& \quad \text{i) Let } s \geq r > 0 \text{ and } r < 3/2 + a. \text{ Then, the IVP (1.1) is GWP in } Z_{s,r}. \\
& \quad \text{ii) Let } r \in [3/2 + a, 5/2 + a) \text{ and } r \leq s. \text{ Then, the IVP (1.1) is GWP in } Z_{s,r}. \\
\end{align*}
\]

\[
\begin{align*}
\text{Theorem 1.2.} & \quad \text{Let } u \in C([0, T]; Z_{1,1}) \text{ be a solution of the IVP (1.1), with } a \in (0, 1]. \\
& \quad \text{If there exist two different times } t_1, t_2 \in [0, T] \text{ such that } u(t_j) \in Z_{3/2+a,3/2+a}, \\
& \quad \text{then } u(0, t) = 0, \text{ for all } t \in [0, T]. \\
\end{align*}
\]

\[
\begin{align*}
\text{Theorem 1.3.} & \quad \text{Let } u \in C([0, T]; Z_{2,2}) \text{ be a solution of the IVP (1.1), with } a \in (0, 1]. \\
& \quad \text{If there exist three different times } t_1, t_2, t_3 \in [0, T] \text{ such that } u(t_j) \in Z_{5/2+a,5/2+a}, \\
& \quad \text{then there exists } t_1 < t < t_2 \text{ such that } \\
& \quad u(x, t) = 0, \text{ for all } x \in \mathbb{R}, \ t \geq \bar{t}. \\
\end{align*}
\]
Theorem 1.2 shows that the Theorem 1.1 (part i)) is sharp in the sense that it’s not possible to find an index \( r \) more than \( 3/2 + a \), so that the part i) still hold valid. Theorem 1.3 also shows that the Theorem 1.1 (part ii)) is sharp, in the same sense previous.

Reciprocally our well-posedness results show that the Theorems 1.2 and 1.3 are also sharp, in the sense that the indexes \( 3/2 + a \) and \( 5/2 + a \), respectively, cannot be pushed down.

The rest of this paper is as follows. Section 2 contains some preliminary estimates that will be useful in the coming sections. In the section 3 we prove the well-posedness. Theorem 1.2 will be proved in section 4. To finish, the proof of Theorem 1.3 will be present in section 5.

2. Preliminaries

Next, we introduce some results which will be useful to demonstrate our main results.

**Proposition 2.1.** Let \( \delta, \nu > 0 \) such that \( J^\delta f \in L^2(\mathbb{R}) \) and \( \langle x \rangle^\nu f \in L^2(\mathbb{R}) \). Then for any \( \beta \in (0, 1) \)
\[
\| J^{\beta \delta}(\langle x \rangle^{(1-\beta)\nu} f) \| \leq c \| \langle x \rangle^{\nu} f \|^{1-\beta} \| J^\delta f \|^\beta.
\] (2.5)

**Proof.** See [23]. \( \square \)

**Remark 2.2.** Assuming that \( u \) is sufficiently regular we obtain, for every \( t \) in which the solution there exists
\[
\int u(x,t)dx = \int \phi(x)dx.
\] (2.6)
This implies that
\[
\hat{u}(0,t) = \hat{\phi}(0).
\] (2.7)
Moreover
\[
\frac{d}{dt}\|u(t)\|^2 < 0,
\] (2.8)
and
\[
\frac{d}{dt}\int xu(x,t)dx = \frac{1}{2}\|u(t)\|^2.
\] (2.9)

By defining \( L^p_s := (1 - \Delta)^{-s/2}L^p(\mathbb{R}^n) \), the following result characterizes these spaces.

**Theorem 2.3.** Let \( b \in (0, 1) \) and \( 2n/(n+2b) < p < \infty \). Then \( f \in L^p_b(\mathbb{R}^n) \) if and only if
\begin{enumerate}
  \item \( f \in L^p(\mathbb{R}^n) \),
  \item \( D^b f(x) = \left( \int_{\mathbb{R}^n} \frac{|f(x) - f(y)|^2}{|x-y|^{n+2b}}dy \right)^{1/2} \in L^p(\mathbb{R}^n) \), with,
\end{enumerate}
\[
\|f\|_{b,p} \equiv \|(1 - \Delta)^{b/2}f\|_p = \|J^bf\|_p \simeq \|f\|_p + \|D^bf\|_p \simeq \|f\|_p + \|D^bf\|_p.
\] (2.10)

**Proof.** See [33]. \( \square \)
From the previous theorem, part b), with \( p = 2 \) and \( b \in (0, 1) \), we have
\[
\|D_h^b(fg)\|_2 \leq \|f\|D_h^b g\|_2 + \|g\|D_h^b f\|_2.
\] (2.11)

**Lemma 2.4.** Let \( b \in (0, 1) \) and \( h \) a measurable function on \( \mathbb{R} \) such that \( h, h' \in L^\infty(\mathbb{R}) \). Then, for all \( x \in \mathbb{R} \)
\[
D_h^b h(x) \lesssim \|h\|_{L^\infty} + \|h'\|_{L^\infty},
\] (2.12)
moreover
\[
\|D_h^b(hf)\| \leq \|D_h^b h\|_{L^\infty} \|f\| + \|f\|_{L^\infty} \|D_h^b f\|.
\] (2.13)

**Proof.** See [20]. \( \square \)

Now, we turn our attention to the (dBO) equation. The integral equation associated to the IVP \((1.1)\) is given by
\[
u(t) = U(t)\phi + \int_0^t U(t-\tau)z(\tau)d\tau,
\] (2.14)
where \( z = \frac{1}{2}\partial_x u^2 \), and the semigroup \( U(t) \) is defined by
\[
(U(t)\phi)^\wedge(\xi) = e^{it|\xi|\alpha}\hat{\phi}, \quad t \in [0, \infty).
\] (2.15)
Putting \( \psi(\xi, t) = e^{-it|\xi|\alpha}|\xi|^{1+a} \), where \( a \in (0, 1] \), the following computations will be useful in our proofs.
\[
\partial_\xi(\psi(\xi, t)\hat{\phi}) = \left[ \left( (1 + a)|\xi|^a \text{sgn}(\xi) + 2it|\xi| \right)\hat{\phi} + \partial_\xi\hat{\phi} \right] \psi(\xi, t),
\] (2.16)
\[
\partial_x^2(\psi(\xi, t)\hat{\phi}) = \left[ \left( t^2(1 + a)^2|\xi|^{2a} - t(1 + a)a|\xi|^{a-1} + 2it^2(1 + a)|\xi|^{a+1}\text{sgn}(\xi) \
- 4t^2\xi^2 - 2its\text{sgn}(\xi) \right)\hat{\phi} - \left( 2t(1 + a)|\xi|^a\text{sgn}(\xi) - 4it|\xi| \right)\partial_\xi\hat{\phi} \right.
+ \partial_x^2\hat{\phi} \] \( \psi(\xi, t), \)
(2.17)
and
\[
\partial_x^2(\psi(\xi, t)\hat{\phi}) = \left[ \left( t^2a(1 + a)^2|\xi|^{2a-1}\text{sgn}(\xi) - t(a^2 - 1)a|\xi|^{a-2}\text{sgn}(\xi) - 8t^2\xi - 2it\delta_\xi \right.ight.
\left. + (1 + a)^3t^3|\xi|^{3a}\text{sgn}(\xi) + 4it(1 + a)^2t^2|\xi|^{2a+1} - 
8(1 + a)t^3|\xi|^{a+2}\text{sgn}(\xi) - 8it^3|\xi|^{3} + 4t^2|\xi|\text{sgn}(\xi) \right)\hat{\phi} - \right.
\left. \left( t^2(1 + a)^2t^2|\xi|^{2a} + 6i(1 + a)t^2|\xi|^{a+1}\text{sgn}(\xi) + 3a(1 + a)t|\xi|^{a-1} - 
4t^2\xi^2 + 6its\text{sgn}(\xi) \right)\partial_\xi\hat{\phi} + \left. \left( (1 + a)t|\xi|^a\text{sgn}(\xi) + 2it|\xi| \right)\partial_x^2\hat{\phi} + \right.
+ \partial_x^2\hat{\phi} \] \( \psi(\xi, t), \)
(2.18)
where \( \delta_\xi \) stands the Dirac delta function with respect to \( \xi \).

**Lemma 2.5.** Let \( \lambda > 0 \), then
\[
\|\xi^{2\lambda}\psi(\xi, t)\|_{L^\infty} \leq c(a, \lambda)t^{-\frac{a}{2\lambda}},
\] (2.19)
where
\[ c(a, \lambda) = \left(\frac{2\lambda e}{a + 1}\right)^{-\frac{a}{2a+1}}. \]
Moreover
\[ \||\xi|^\sigma e^{-t|\xi|^{1+a}}\|_{L^2_x}^2 = c_{\sigma, a} t^{-\frac{2\sigma + a}{2(1+a)}}. \tag{2.20} \]

Proof. Since \( |\xi|^{2\lambda} \psi(\xi, t) \leq \xi^{2\lambda} e^{-t|\xi|^{1+a}} := \varphi(\xi, t) \), a simple computation give us
\[ \partial_\xi \varphi(\xi, t) = \xi^{2\lambda} e^{-t|\xi|^{1+a}} \left(\frac{2\lambda}{\xi} - (1+a)t|\xi|^a\right). \]

Then \( \partial_\xi \varphi(\xi, t) = 0 \) if, and only if, \( |\xi| = |\xi_0| = \left(\frac{2\lambda}{1+a}\right)^{\frac{1}{1+a}} t^{-\frac{1}{1+a}} \). In view of \( \varphi(0, t) = 0 \) and \( \varphi(\xi, t) \to 0 \) with \( \xi \to \infty \), we conclude that
\[ |\xi^{2\lambda} \psi(\xi, t)| \leq \varphi(\xi_0, t) = c(a, \lambda) t^{-\frac{2\lambda}{1+a}}. \]

About identity (2.20), by using the change of variables \( \xi = t^{-1/1+a} \)
\[ \||\xi|^\sigma e^{-t|\xi|^{1+a}}\|_{L^2_x}^2 = t^{-\frac{2\sigma - a}{2(1+a)}} \int w^{2\sigma} e^{-2|w|^{1+a}} dw = c_{\sigma, a} t^{-\frac{2\sigma - a}{2(1+a)}}. \]

Lemma 2.6. Let \( a \in (0, 1) \) and \( \lambda \in \mathbb{Z}^+ \) then
\[ \|D_\xi^k(\psi(\xi, t)|\xi|^{\lambda} \hat{f})\| \leq c_{a, \lambda}(t^{-\frac{1}{1+a}} + t^{\frac{1}{1+a}} + t^{\frac{2\lambda}{1+a}})\|f\| + t^{-\frac{1}{1+a}} \|x|^b f\|, \tag{2.21} \]
moreover (2.21) still hold if \( |\xi|^\lambda \) is substituted by \( |\xi|^\lambda \xi^{\lambda_2}, \lambda = \lambda_1 + \lambda_2, \lambda_1, \lambda_2 \in \mathbb{Z}^+ \).

Proof. First, we see that
\[ \partial_\xi \psi = -\left((1+a)|\xi|^{\lambda} \text{sgn}(\xi) + 2it|\xi|\right) \psi. \tag{2.22} \]

Therefore using (2.11), (2.19) and (2.12)
\[ \|D_\xi^k(\psi|\xi|^{\lambda} \hat{f})\| \lesssim \|D_\xi^k(\psi|\xi|^{\lambda} \hat{f})\| + \|\psi|\xi|^{\lambda} D_\xi^k \hat{f}\| \lesssim \|D_\xi^k(\psi|\xi|^{\lambda})\| \|f\| + \|\psi|\xi|^{\lambda}\| \|D_\xi^k \hat{f}\| \lesssim_b \left(\|\xi|^{\lambda}|\psi\| + \|D_\xi(\xi^{\lambda}|\psi|)\|\|f\| + \|\xi|^{\lambda}|\psi| \|D_\xi^k \hat{f}\| \right) \tag{2.23} \]
This finish the proof.

In our estimates we need of the following result about the Hilbert transform in weighted spaces.

Lemma 2.7. Let \(-1/2 < \nu < 1/2\), then the Hilbert transform \( \mathcal{H} \) is a bounded operator in \( L^2(|x|^{\nu} dx) \), i.e.
\[ \|\mathcal{H} f|x|^{\nu}\| \leq \|f|x|^{\nu}\|. \tag{2.24} \]

Proof. See [25], in which the more general version can be found.
Proposition 2.8. For any \( \theta \in (0, 1) \) and \( \gamma > 0 \),
\[
D^\theta(|\xi|^{-\theta} \chi(\xi))(\eta) \sim \begin{cases} 
\frac{c|\eta|^{-\theta} + c_1}{|\eta|^{1+2\gamma}}, & \gamma \neq \theta, |\eta| \ll 1, \\
\frac{c(-\ln |\eta|)^{1/2}}{|\eta|^{1+2\gamma}}, & \gamma = \theta, |\eta| \ll 1, \\
& |\eta| \gg 1,
\end{cases}
\]
with \( D^\theta(|\xi|^{-\theta} \chi(\xi))(\cdot) \) continuous in \( \eta \in \mathbb{R} - \{0\} \). In particular, one has that
\[
D^\theta(|\xi|^{-\theta} \chi(\xi))(\cdot) \in L^2(\mathbb{R}) \text{ if and only if } \theta < \gamma + 1/2.
\]
In a similar fashion
\[
D^\theta(|\xi|^{-\theta} \chi(\xi))(\cdot) \in L^2(\mathbb{R}) \text{ if and only if } \theta > \gamma + 1/2.
\]
Proof. See Proposition 2.9 in [24]. \( \square \)

Proposition 2.9. Let \( \gamma \in [0, 1/2) \), then
\[
D^\gamma(|\xi|^{\gamma-1/2} \chi(\xi)) \notin L^2(\mathbb{R}).
\]
Proof. Putting \( \gamma_1 = \gamma - 1/2 \), let \( \eta \in (0, 1) \). Then using a change of variables
\[
D^\gamma(|\xi|^{\gamma_1} \chi(\xi))(\eta)^2 = \int \frac{|y|^{\gamma_1} \chi(y) - |\eta|^{\gamma_1} \chi(\eta)^2}{|y - \eta|^{1+2\gamma}} dy
\]
\[
= \int \frac{|\xi + \eta|^{\gamma_1} \chi(\xi + \eta) - |\eta|^{\gamma_1} \chi(\eta)^2}{|\xi|^{1+2\gamma}} d\xi
\]
\[
= \int_{\xi \in (-\eta,1-\eta)} + \int_{\xi \notin (-\eta,1-\eta)}
\]
\[
:= A(\eta) + B(\eta).
\]
In the first integral above we have \( 0 < \xi + \eta < \eta < 1 \), then \( \chi(\xi + \eta) = \chi(\eta) = 1 \). By the Mean Value Theorem there exists \( z \in (\xi + \eta, \eta) \) such that
\[
(\xi + \eta)^{\gamma_1} - \eta^{\gamma_1} = \gamma_1 z^{\gamma_1-1} \eta.
\]
Therefore
\[
A(\eta) = \int_{-\eta}^{1-\eta} \frac{((\xi + \eta)^{\gamma_1} - \eta^{\gamma_1})^2}{|\xi|^{1+2\gamma}} d\xi
\]
\[
\geq \int_{-\eta}^{0} \frac{((\xi + \eta)^{\gamma_1} - \eta^{\gamma_1})^2}{|\xi|^{1+2\gamma}} d\xi
\]
\[
= \gamma_1^2 \int_{-\eta}^{0} \frac{\xi^{2(\gamma_1-1)} \xi^2}{|\xi|^{1+2\gamma}} d\xi
\]
\[
\geq \gamma_1^2 \int_{-\eta}^{0} \eta^{2(\gamma_1-1)} \xi^{1-2\gamma} d\xi
\]
\[
= \frac{\gamma_1^2}{2(\gamma_1 - 1)} \eta^{-1}
\]
\[
:= g(\eta).
\]
In view of \( D^\gamma(|\xi|^{\gamma_1} \chi(\xi))(\eta)^2 \geq g(\eta) \) for any \( 0 < \eta < 1 \) and \( g \notin L^1_{loc}(\mathbb{R}) \), we conclude the proof.
Proposition 2.10. If $f \in L^2(\mathbb{R})$ and $\phi \in H^1(\mathbb{R})$, then
\[
\| [D^\gamma; \phi] f \| \leq c \| \phi \|_{H^2} \| f \|,
\]  
where $\gamma \in (0, 1)$.

Proof. We observe that
\[
([D^\gamma; \phi] f)(\xi) = (D^\gamma (\phi f) - \phi D^\gamma f)(\xi) 
= \int (|\xi|^{\gamma} - |\eta|^{\gamma}) \hat{\phi}(\xi - \eta) \hat{f}(\eta) d\eta.
\]
It is easy to see that
\[
| |\xi|^{\gamma} - |\eta|^{\gamma} | \leq |\xi - \eta|^{\gamma},
\]
so
\[
|([D^\gamma; \phi] f)(\xi)| \leq \int |\xi - \eta|^{\gamma} |\hat{\phi}(\xi - \eta)| |\hat{f}(\eta)| d\eta = c(|\hat{D^\gamma \phi}| * |\hat{f}|)(\xi).
\]
Then, by the Young's inequality
\[
\| [D^\gamma; \phi] f \| \leq c \| \hat{D^\gamma \phi} \|_{L^1} \| f \| 
\leq c \| \hat{D^\gamma \phi} \|_{L^1} \| \hat{f} \| 
\leq c \| \phi \|_{H^2} \| f \|,
\]
where above we use that
\[
\| \hat{D^\gamma \phi} \|_{L^1} \leq \| D^\gamma \phi \|_{H^1} \leq \| \phi \|_{H^2}.
\]
This finish the proof.

2.1. Notation. In this paper, we use the following notation. We say $a \lesssim b$ if there exists a constant $c > 0$ such that $a \leq cb$. We also write $a \lesssim_l b$ when the constant depends on only parameter $l$. The Fourier transform of $f$, is defined by
\[
\hat{f}(\xi) = \int_{\mathbb{R}} e^{-i\xi x} f(x) dx.
\]
If $s \in \mathbb{R}$, $H^s := H^s(\mathbb{R})$ represents the nonhomogeneous Sobolev space defined as
\[
H^s(\mathbb{R}) = \{ f \in S'(\mathbb{R}) : \| f \|_{H^s} < \infty \},
\]
where
\[
\| f \|_{H^s} = \| \langle \xi \rangle^s \hat{f} \|_{L^2_x},
\]
and $\langle \xi \rangle = (1 + \xi^2)^{1/2}$. In addition, we define the Bessel potential $J^s$ by
\[
(J^s f)(\xi) = \langle \xi \rangle^s \hat{f}(\xi), \text{ for all } f \in S'(\mathbb{R}),
\]
hence $\| J^s f \|_{L^2_x} = \| f \|_{H^s}$. The weighted Sobolev space is defined by
\[
Z_{s,r} = H^s(\mathbb{R}) \cap L^2_{r},
\]
where $L^2_r = L^2(\langle x \rangle^{2r} dx)$. The norm in $Z_{s,r}$ is given by $\| \cdot \|_{Z_{s,r}}^2 = \| \cdot \|_{H^s}^2 + \| \cdot \|_{L^2_{r}}^2$. We also introduced the notation
\[
\dot{Z}_{s,r} = \{ f \in Z_{s,r} : \hat{f}(0) = 0 \}.
For help in our estimates, we define the function $\chi \in C_0^\infty(\mathbb{R})$, with $\text{supp}\chi \subset [-2, 2]$ and $\chi \equiv 1$ in $(-1, 1)$.

In the rest of the paper, we will denote the $L^2$-norm in the $x$ variable by $\| \cdot \|_{L^2_x} := \| \cdot \|$. 

3. Well-posedness

In the following we will obtain the global well-posedness. First we note that the case $a = 1$ can be approach by using the ideas in [20]. Thus, we lead only the case $a \in (0, 1)$.

Proof of Theorem 1.1. Case 1). $r = \theta, \theta \in (0, 1)$, $s \geq r$ and $s > 1/2$. Let $\phi \in H^s(\mathbb{R})$, then as we already observed, the solution $u(t)$ of (1.1) is unique and satisfies $u \in C([0, T]; H^s(\mathbb{R}))$, for all $T > 0$. Also we have the continuous dependence on the initial data in $H^s(\mathbb{R})$. Thus, in the following we will prove the persistence property in $L^2_x$. Putting $M := \sup_{[0, T]} \| u(t) \|_{H^s}$ and using the integral equation (2.14), for all $t \in [0, T]$

$$\| |x|^\theta u(t)\| \leq \| |x|^\theta U(t)\phi\| + \int_0^t \| |x|^\theta U(t - \tau)z(\tau)\| d\tau,$$

(3.30)

where $z = \frac{1}{2} D_x u^2$. Now, using Fourier transform, Stein’s derivative and (2.21), we obtain

$$\| |x|^\theta U(t)\phi\| = \| D_x^\theta(\psi(\xi, t)\hat{\phi})\| \lesssim \| \psi(\xi, t)\hat{\phi} \| + \| D_x^\theta(\psi(\xi, t)\hat{\phi})\| \lesssim_a (1 + t^{\frac{a}{1+a}} + t^{\frac{a}{1+a}}) \| \phi \| + \| |x|^\theta \phi\|. \tag{3.31}$$

We also can write for the integral term

$$\| |x|^\theta U(t - \tau)z(\tau)\| = \frac{1}{2} \| D_x^\theta(\psi(\xi, t - \tau)\xi \hat{u}^2)\| \lesssim \| \psi(\xi, t - \tau)\hat{u}^2 \| + \| D_x^\theta(\psi(\xi, t - \tau)\xi \hat{u}^2)\| \lesssim_a (1 + (t - \tau)^{-\frac{a}{1+a}} + (t - \tau)^{-\frac{a}{1+a}}) \| u^2(\tau)\| + (t - \tau)^{-\frac{a}{1+a}} \| |x|^\theta u^2(\tau)\| \lesssim_a (1 + (t - \tau)^{-\frac{a}{1+a}} + (t - \tau)^{-\frac{a}{1+a}}) M^2 + (t - \tau)^{-\frac{a}{1+a}} M \| |x|^\theta u(\tau)\|,$$

(3.32)

where above we used $\| u^2(\tau)\| \leq \| u(\tau)\|_H^2$, and $\| u(\tau)\|_\infty \leq \| u(\tau)\|_H^s$. By (3.30) - (3.31)

$$\| |x|^\theta u(t)\| \lesssim \| \phi \| + \| |x|^\theta \phi\| + (t^{\frac{a}{1+a}} + t^{\frac{a}{1+a}}) \| \phi \| + c_1 t^{2a/a+1} + c_2 t^{a/a+1}$$

$$+ M \int_0^t (t - \tau)^{-\frac{a}{1+a}} \| |x|^\theta u(\tau)\|, \tag{3.33}$$

(3.33)

where $c_1 = -\frac{a+1}{2a}$, $c_2 = -(a+1)/a$. In view of (2.8)

$$\| u(t)\| \leq \| \phi \|. \tag{3.34}$$

Then using (3.33), (3.34) and an inequality of Gronwall’s type, see [13] Lemma 7.1.1, we conclude

$$\| |x|^\theta u(t)\| \lesssim_a \| |x|^\theta \phi\| + g(t), \quad t \in [0, T], \tag{3.35}$$

(3.35)
The last term in (4.80) can be decomposed as
\[ L^2 \rightarrow L^p \] follows by using (3.35), see [7].

Using similar arguments to [23] and [7] we can show the continuous dependence on \( r = \theta \in (0, 1) \), \( s \geq r \), where \( 0 < s \leq 1/2 \) we can use the same ideas in [11] and [20].

Case 2). \( r = 1 + \theta, \theta \in (1/2, 1/2 + a) \), \( s \geq r \).

By using (2.21) and (2.22)
\[
\| x \|^{1 + \theta} U(t) \phi &= \| D_x^{1 + \theta} (\psi(t, \xi) \hat{\phi}) \|
\leq \| D_x^\theta (\psi \partial_t \hat{\phi}) \| + \| D_x^\theta (\partial_t \psi \hat{\phi}) \|
\leq (1 + t^{-\frac{s}{1-a}} + t^{-\frac{s}{2-a}}) \| \phi \| + \| D_x^\theta \hat{\phi} \|
\leq (1 + t^{-\frac{s}{1-a}} + t^{-\frac{s}{2-a}}) \| \phi \| + \| x \|^{\theta} \| \phi \|.
\]

We also have, from (2.21)
\[
A_2 \lesssim t \left( (1 + t^{-\frac{s}{1-a}} + t^{-\frac{s}{2-a}}) \| \phi \| + t^{-\frac{s}{1-a}} \| D_x^\theta \hat{\phi} \| \right)
= (t + t^{-\frac{s}{1-a}} + t^{-\frac{s}{2-a}}) \| \phi \| + t^{-\frac{s}{1-a}} \| x \|^{\theta} \| \phi \|.
\]

and
\[
A_1 \lesssim t \| D_x^\theta (|\xi|^a \text{sgn}(\xi) \psi \hat{\phi} \chi) \| + t \| D_x^\theta (|\xi|^a \text{sgn}(\xi) \psi (1 - \chi) \hat{\phi}) \|
= A_{1,1} + A_{1,2}.
\]

Then by (2.13)
\[
A_{1,2} \lesssim t \left( \| |\xi|^a \psi (1 - \chi) \|_\infty + \| \partial_\xi (|\xi|^a \text{sgn}(\xi) \psi (1 - \chi)) \|_\infty \| \phi \|
+ t \| |\xi|^a \psi (1 - \chi) \|_\infty \| D_x^\theta \hat{\phi} \|
\leq (1 + t^{-\frac{s}{1-a}} + t^{-\frac{s}{2-a}}) \| \phi \| + t^{-\frac{s}{1-a}} \| x \|^{\theta} \| \phi \|.
\]

We can write
\[
A_{1,1} \lesssim t \| D_x^\theta (|\xi|^a \text{sgn}(\xi) (\psi - 1) \hat{\phi} \chi) \| + t \| D_x^\theta (|\xi|^a \text{sgn}(\xi) \hat{\phi} \chi) \|
= A_{1,1} + A_{1,2}.
\]

and again by (2.13) and (2.22)
\[
A_{1,1} \lesssim t \left( \| |\xi|^a (\psi - 1) \chi \|_\infty + \| \partial_\xi (|\xi|^a \text{sgn}(\xi) (\psi - 1) \chi) \|_\infty \| \phi \|
+ t \| |\xi|^a (\psi - 1) \chi \|_\infty \| D_x^\theta \hat{\phi} \|
\leq A_1 (t \| \phi \| + \| x \|^{\theta} \| \phi \|).
\]

The last term in (4.81) can be decomposed as
\[
A_{1,2} \lesssim t \| D_x^\theta (|\xi|^a \text{sgn}(\xi) \hat{\phi} (\xi - \hat{\phi}(0)) \chi) \| + t \| D_x^\theta (|\xi|^a \text{sgn}(\xi) \hat{\phi}(0) \chi) \|
= \hat{A} + \hat{\bar{A}}.
\]
Then
\[ \|L\| \lesssim \|\xi|^a \hat{\phi}(\xi)\chi\| + \|\xi|^a \hat{\phi}(0)\chi\| \lesssim \|\xi|^a \|\hat{\phi}\|_\infty \lesssim_a \|J\hat{\phi}\| = \|\langle x\rangle \phi\|, \] (3.43)
and
\[ \|\partial_\xi L\| \lesssim \|\xi|^a \left(\frac{\hat{\phi}(\xi) - \hat{\phi}(0)}{\xi}\right)\chi\| + \|\xi|^a \partial_\xi \hat{\phi}(\xi)\chi\| + \|\xi|^a (\hat{\phi}(\xi) - \hat{\phi}(0))\partial_\xi \chi\| \]
\[ \lesssim \|\xi|^a \|\hat{\phi}\|_\infty + \|\xi|^a \|\partial_\xi \hat{\phi}\| + \|\xi|^a \|\hat{\phi}\|_\infty +
\]
\[ + \|\xi|^a \|\partial_\xi \chi\|\|\hat{\phi}(0)\| \]
\[ \lesssim_a \|\hat{\phi}\|_\infty + \|\hat{\phi}\|_\infty \|D^\theta_\xi (\|\xi|^a \sign(\xi)\chi)\| \lesssim_a \|\langle x\rangle \phi\|. \] (3.44)
where above we used \( \theta > 1/2 \). From (3.43) and (3.44) we obtain
\[ \hat{A} \lesssim \|L\|_{H^1_t} \lesssim \|\langle x\rangle^{1+\theta} \phi\|. \]
Since that \( \theta < 1/2 + a \), by (2.25) we obtain \( D^\theta_\xi (\|\xi|^a \sign(\xi)\chi) \in L^2 \), then
\[ \hat{A} \lesssim_a \|\hat{\phi}\|_\infty + \|\hat{\phi}\|_\infty \|D^\theta_\xi (\|\xi|^a \sign(\xi)\chi)\| \lesssim_a \|\langle x\rangle \phi\|. \] (3.45)
Now we will estimate the integral term
\[ \|\langle x\rangle^{1+\theta} U(t - \tau)\partial_x u^2\| = \frac{1}{2} \|D^{1+\theta}_\xi (\psi(t - \tau, \xi)\hat{u}^2)\| \]
\[ \leq \|D^\theta_\xi (\partial_\xi \psi \hat{u}^2)\| + \|D^\theta_\xi (\psi \partial_\xi (\xi \hat{u}^2))\| \]
\[ := B_{1,1} + B_{1,2}, \] (3.46)
where by (2.21)
\[ B_{1,2} \lesssim ((t - \tau) + (t - \tau)^{-\frac{1}{\alpha}} + (t - \tau)^{-\frac{1}{\alpha^2}})\|u^2\| + \|D^\theta_\xi \hat{u}^2\| +
\]
\[ + (1 + (t - \tau)^{-\frac{1}{\alpha}} + (t - \tau)^{-\frac{1}{\alpha^2}})\||\partial_\xi \hat{u}^2\| + (t - \tau)^{-\frac{1}{\alpha}} \|D^\theta_\xi \partial_\xi \hat{u}^2\|. \] (3.47)
Using Plancherel Identity and Sobolev embedding the terms with fractional derivative in (3.46) can be estimate as follows
\[ \|D^\theta_\xi \hat{u}^2\| = \|\langle x\rangle^\theta u^2\| \lesssim M\|\langle x\rangle^\theta u\|, \] (3.48)
and
\[ \|D^\theta_\xi (\partial_\xi \hat{u}^2)\| = \|\langle x\rangle^\theta u^2\| \lesssim \|\langle x\rangle^{1+\theta} u\|_\infty \lesssim M\|\langle x\rangle^{1+\theta} u\|. \] (3.49)
For the first term in (3.46) we have
\[ B_{1,1} \lesssim (t - \tau)(\|D^\theta_\xi (\|\xi|^a \sign(\xi)\psi \xi \hat{u}^2)\| + \|D^\theta_\xi (\psi \xi \hat{u}^2)\|) := B_{1,1}^1 + B_{1,1}^2, \] (3.50)
where by using (2.21) and Sobolev embedding we obtain
Thus by (2.21)

\[
B_{1,1}^1 = (t-\tau)\|D_\xi^6(\langle \xi \rangle^{a+1}\psi u^2)\| \\
\lesssim (t-\tau)\left[\left((t-\tau)^{-1} + (t-\tau)^{-\frac{1}{1+\nu}} + (t-\tau)^{-\frac{1}{1+\nu}}\right)\|u^2\| + \\
+ (t-\tau)^{-1}\|D_\xi^6u^2\|\right] \\
\lesssim \left(1 + (t-\tau)^{-\frac{1}{1+\nu}} + (t-\tau)^{-\frac{1}{1+\nu}}\right)\|\xi u^2\| + \\
\lesssim \left(1 + (t-\tau)^{-\frac{1}{1+\nu}} + (t-\tau)^{-\frac{1}{1+\nu}}\right)\|\xi u^2\| + (t-\tau)^{-\frac{1}{1+\nu}}\|\partial_x u^2\|. \tag{3.51}
\]

and

\[
B_{2,1}^2 \lesssim \left((t-\tau)^{-\frac{1}{1+\nu}} + (t-\tau)^{-\frac{1}{1+\nu}}\right)\|\xi u^2\| + \\
+ (t-\tau)^{-\frac{1}{1+\nu}}\|D_\xi^6(\xi u^2)\| \\
\lesssim \left((t-\tau)^{-\frac{1}{1+\nu}} + (t-\tau)^{-\frac{1}{1+\nu}}\right)\|\partial_x u^2\| + (t-\tau)^{-\frac{1}{1+\nu}}\|x^{\theta}\partial_x u^2\|. \tag{3.52}
\]

The terms on the right hand in (3.52) can be estimated as follows

\[
\|\partial_x u^2\| \lesssim \|u_x\|\|u\|_\infty \\
\lesssim M^2, \tag{3.53}
\]

and by inequality (2.25) with \(\nu = \delta = 1 + \theta\) and \(\beta = 1/1+\theta\)

\[
\|x^{\theta}\partial_x u^2\| \lesssim \|u\|_\infty\|x^{\theta}\partial_x u\| \\
\lesssim M\|\|\langle x \rangle^{1+\theta} u\| + \|J^{1+\theta} u\|\|. \tag{3.54}
\]

Now we can proceed as in the Case 1) to obtain the result.

For the case \(r = 1 + \theta, \theta \in (0,1/2)\), we can use nonlinear interpolation theory (see [K] and references therein).

Case 3). \(r = 2 + \theta \in (5/2, 5/2 + a), \theta \in (1/2, 1/2 + a)\). Let \(\phi \in \mathcal{Z}_{s,r}\), where \(s \geq r\), then by Plancherel identity

\[
\|x^{\theta+2}U(t)\phi\| = \|D_\xi^6(\partial_t^2(\psi(\xi,t)\hat{\phi}))\| \\
\lesssim \|D_\xi^6(\partial_t^4\psi(\xi,t)\hat{\phi})\| + \|D_\xi^6(\partial_t\psi\partial_\xi\hat{\phi})\| + \|D_\xi^6(\psi\partial_\xi^2\hat{\phi})\| \\
:= C + D + E, \tag{3.55}
\]

thus by (2.22)

\[
E \lesssim (1 + t^{\frac{1}{1+\alpha}} + t^{\frac{1}{1+\alpha}})\|\partial_\xi^2\hat{\phi}\| + \|D_\xi^6\partial_\xi^2\hat{\phi}\| \\
\lesssim (1 + t^{\frac{1}{1+\alpha}} + t^{\frac{1}{1+\alpha}})\|x^{2}\hat{\phi}\| + \|x^{\theta+2}\hat{\phi}\|. \tag{3.56}
\]

Using (2.22)

\[
D \lesssim t\|D_\xi^6(\langle \xi \rangle^{a}\text{sgn}(\xi)\partial_\xi\hat{\phi})\| + t\|D_\xi^6(\langle \xi \rangle^{a}\psi\partial_\xi\hat{\phi})\| := D_1 + D_2, \tag{3.57}
\]

where

\[
D_1 \lesssim t\|D_\xi^6(\langle \xi \rangle^{a}\text{sgn}(\xi)\psi\partial_\xi\hat{\phi})\| + t\|D_\xi^6(\langle \xi \rangle^{a}\text{sgn}(\xi)\psi(1-\chi)\partial_\xi\hat{\phi})\| \\
:= D_{1,1} + D_{1,2}. \tag{3.58}
\]
We also can write
\[
D_{1,1} \lesssim t(\|\xi^n \psi \partial_t \hat{\phi} - \partial_x \hat{\phi}(0)\| + \|D^\theta_x(\xi^n \psi \partial_t \hat{\phi})\|)
\]
where by (2.13)
\[
D^1_{1,1} \lesssim t(\|\xi^n \psi \partial_t \hat{\phi} - \partial_x \hat{\phi}(0)\| + \|\xi^n \psi \partial_t \hat{\phi}\| + \|\xi^n \psi \partial^2 \hat{\phi}\|)
\]
\[
\lesssim t((t^{-\frac{n}{2}} + t^{-\frac{n}{4}} + t^{-\frac{n}{4}})\|\partial_t \hat{\phi}\| + t^{-\frac{n}{4}} \|\partial^2 \hat{\phi}\|)
\]
and
\[
D^2_{1,1} \lesssim t \left(\|\psi\| + \|\partial_t \hat{\phi}\| + \|\xi^n \partial_x \hat{\phi}\| \right)
\]
\[
\lesssim \|\partial_t \hat{\phi}\| \left( t + t^{-\frac{n}{2}} + t^{-\frac{n}{4}} \right) \|\xi^n \chi\| + \|D^\theta_x(\xi^n \psi \partial_t \hat{\phi})\|
\]
From (2.25) follows that $N \in L^2$. By (2.13) we also obtain
\[
D_{1,2} \lesssim t \left(\|\xi^n (1 - \chi) \psi\| + \|\partial_t (\xi^n (1 - \chi) \psi)\| \right)
\]
\[
\lesssim t \left( t^{-\frac{n}{2}} + t^{-\frac{n}{4}} + t^{-\frac{n}{4}} \|\partial_t \hat{\phi}\| + t^{-\frac{n}{4}} \|\partial^2 \hat{\phi}\| \right)
\]
and by (2.21)
\[
D_2 \lesssim t \left( t^{-\frac{n}{2}} + t^{-\frac{n}{4}} + 1 \right) \|\partial_t \hat{\phi}\| + t^{-\frac{n}{4}} \|\partial^2 \hat{\phi}\|
\]
Finally, using (2.17)
\[
C \lesssim t^2 \|D^\theta_x(\xi^{n+1} \psi \hat{\phi})\| + t \|D^\theta_x(\xi^{n+1} \psi \hat{\phi})\| + t^2 \|D^\theta_x(\xi^{n+1} \psi \hat{\phi})\|
\]
where by (2.21)
\[
C_1 \lesssim t^2 \left( t^{-\frac{n}{2}} + t^{-\frac{n}{4}} + t^{-\frac{n}{4}} \right) \|\hat{\phi}\| + t^{-\frac{n}{4}} \|\partial^2 \hat{\phi}\|
\]
Now, using (2.13) the term \( C_2^2 \in H^1_\xi \), in fact

\[
t |||\xi|||^{a-1}\psi(1 - \chi)\hat{\phi}||| \lesssim t |||\xi|||^{a}\psi\hat{\phi} \frac{1 - \chi}{\xi} \lesssim t \frac{1 - \chi}{\xi} \|\phi\| t^{-\frac{1}{16}} \lesssim t^{\frac{1}{16}} \|\phi\|, \tag{3.67}
\]

and

\[
t |||\xi|||^{a-1}\psi(1 - \chi)\hat{\phi}||| \lesssim t \left( |||\xi|||^{a+1}\psi\hat{\phi} \frac{1 - \chi}{\xi} + \|\|\xi|||^{a+2}\psi\hat{\phi} \frac{1 - \chi}{\xi} \right)
+ \|\|\xi|||^{a+1}\psi\hat{\phi}\partial_\xi \hat{\phi} \frac{1 - \chi}{\xi} \| + \|\|\xi|||^{a}\psi\partial_\xi \hat{\phi} \frac{1 - \chi}{\xi} \|
+ \|\|\xi|||^{a}\hat{\phi}\partial_\xi \left( \frac{1 - \chi}{\xi} \right) \|
\lesssim t \left( \left( t^{-\frac{1}{16}} + t^{-\frac{2a}{16}} + t^{-\frac{1}{16}} \right) \|\phi\| + t^{-\frac{1}{16}} \|x\phi\| \right)
\lesssim \left( t^{\frac{1}{16}} + t^{\frac{2a}{16}} + t^{\frac{1}{16}} \right) + t^{\frac{1}{16}} \|x\phi\|.
\tag{3.68}
\]

For estimate \( C_2^1 \) we use that \( \hat{\phi}(0) = 0 \). By the Taylor formula

\[
\hat{\phi}(\xi) = \xi \partial_\xi \hat{\phi}(0) + \int_0^1 (\xi - \tau) \partial_\xi^2 \hat{\phi}(\tau) d\tau,
\tag{3.69}
\]

follows that

\[
C_2^1 = t \mathcal{D}_\xi^0 (|||\xi|||^{a-1}\xi^2\chi \partial_\xi \hat{\phi}(0)) + t \mathcal{D}_\xi^0 \left( \int_0^\xi (\xi - \tau) |||\xi|||^{a-1}\xi^2\chi \partial_\xi^2 \hat{\phi}(\tau) d\tau \right),
\tag{3.70}
\]

where \( R \in H^1_\xi \). In fact, using Sobolev embedding

\[
|||R||| \lesssim |||\xi|||^{a-1}\xi^2\chi \partial_\xi^2 \hat{\phi}||| R \|
\lesssim \|\partial_\xi^2 \hat{\phi}\|_{R} |||\xi|||^{a+1}\chi \|
\lesssim_a \|x|||^{2 + \theta} \|, \tag{3.71}
\]

and

\[
|||\partial_\xi R||| \lesssim_a \|\|\xi|||^{a-2}\xi^2\chi \partial_\xi^2 \hat{\phi}||| R \| + t |||\|\xi|||^{2a-1} + |||\xi|||^{a}|||\partial_\xi^2 \hat{\phi}||| R \|
+ \|\|\xi|||^{a-1}\chi \xi^2 \partial_\xi^2 \hat{\phi}||| R \| + \|\|\xi|||^{a-1}\chi \int_0^\xi \partial_\xi^2 \hat{\phi}(\tau) d\tau \|
\lesssim_a \|x|||^{2 + \theta} \|. \tag{3.72}
\]

Now, using (2.11)

\[
C_3 \lesssim t^2 \left( \left( \|\|\xi|||^{a+1}||| \chi \| + \|\partial_\xi (|||\xi|||^{a+1} \chi) \| \right) \|\phi\| + \|\|\xi|||^{a+1} ||| \mathcal{D}_\xi^0 \hat{\phi}||| \right)
\lesssim t^2 \left( t^{-1} + t^{-\frac{1}{16}} + t^{-\frac{1}{16}} + t^{-\frac{1}{16}} \right) \|\phi\| + t^{-1} \|x|||^{\theta} \| \phi \|
\lesssim (t + t^{\frac{1}{16}} + t^{\frac{1}{16}} + t^{\frac{1}{16}}) \|\phi\| + t \|x|||^{\theta} \|. \tag{3.73}
\]
From (2.21) we obtain

\[ C_4 = t^2 \| D_\xi^\theta (\xi^2 \psi \hat{\phi}) \| \lesssim t^2 \left( (t^{-\frac{2}{1+a}} + t^{-\frac{2a+2}{1+a}} + t^{-\frac{2}{1+a}}) \| \phi \| + t^{-\frac{2}{1+a}} \| x^\theta \phi \| \right) \]

\[ \lesssim \left( t^{-\frac{2}{1+a}} + t^{-\frac{2a+1}{1+a}} + t^{-\frac{2}{1+a}} \right) \| \phi \| + t^{-\frac{2}{1+a}} \| x^\theta \phi \|, \]

and

\[ C_5 = t \| D_\xi^\theta (\text{sgn}(\xi) \psi \hat{\phi}) \| \lesssim t \left( (1 + t^{\frac{1}{1+a}} + t^{-1}) \| \phi \| + \| D_\xi^\theta (\text{sgn}(\xi) \phi) \| \right). \]

We can estimate the last term with Stein derivative as follows. Since that \(-\frac{1}{2} < \theta - 1 < \frac{1}{2}\), by Lemma 2.7 and (2.10)

\[ \| D_\xi^\theta (\text{sgn}(\xi) \psi \hat{\phi}) \| \approx \| |x|^{\theta-1} x \mathcal{H} \phi \| \]

\[ = \| |x|^{\theta-1} x \mathcal{H} \phi \| \]

\[ = \| |x|^{\theta-1} \mathcal{H} (x \phi) \| \]

\[ \lesssim \| |x|^{\theta-1} x \phi \| \]

\[ = \| |x|^{\theta} \phi \|. \]

The estimates for the integral term follows by way similar. From way similar to the mentioned in Case 2), the case \( r = 2 + \theta, \theta \in (0, 1/2) \), follows by the nonlinear interpolation theory, see [3]. This finish the proof of Theorem 1.1.

**Remark 3.1.** It is possible to give another proof of the well-posedness by using a commutator estimate for the fractional derivatives, see [24, Proposition 2.2]. See also [30], for the more general version.

4. **Proof of Theorem 1.2**

**Proof of Theorem 1.2.** Case \( a \in (0, 1/2) \).

The main idea of the proof is observe that the terms in \((2.16)\) has an appropriate decay when \(|\xi|\) goes to infinity. First we consider the integral equation associated with IVP \((1.1)\)

\[ u(t) = U(t) \phi - \int_0^t U(t - \tau) u(\tau) \partial_x u(\tau) d\tau, \]

where

\[ \hat{U}(t) \phi(\xi) = \psi(\xi, t) \hat{\phi}(\xi). \]

Let \( 3/2 + a = 1 + \gamma \), where \( \gamma \in (1/2 + a, 1) \), then multiplying \((4.77)\) by \( 1 + \gamma \) and taking the Fourier transform lead to

\[ D_\xi^\gamma \partial_t \hat{u}(\xi) = D_\xi^\gamma \partial_t (\psi(\xi, t) \hat{\phi}) - \int_0^t D_\xi^\gamma \partial_t (\psi(\xi, t - \tau) \hat{z}) d\tau, \]

where \( z = \frac{1}{2} \partial_x u^2 \).

Without loss of generality, we assume that \( t_1 = 0 < t_2 \). Let \( \phi \in \mathcal{Z}_{3/2+a, 3/2+a} \), then by the Theorem 1.1(i)) follows that

\[ u \in C([0, T]; H^{3/2+a}(\mathbb{R}) \cap L_r^2), \text{ where } 0 < r < 3/2 + a. \]
With help of function $\chi$ we write the linear part of integral equation (4.78) as follows

$$\chi D_\xi^\gamma \partial_\xi (\psi(\xi, t) \hat{\phi}) = [\chi; D_\xi^\gamma] \partial_\xi (\psi(\xi, t) \hat{\phi}) + D_\xi^\gamma (\chi \partial_\xi (\psi(\xi, t) \hat{\phi})) = A + B,$$

where by (2.22)

$$A = [\chi; D_\xi^\gamma] \left( (1 + a)\psi(\xi, t) |\xi|^\alpha \text{sgn}(\xi) \hat{\phi} + 2it\xi \psi(\xi, t) \hat{\phi} + \psi(\xi, t) \partial_\xi \hat{\phi} \right)$$

$$:= A_1 + A_2 + A_3,$$ (4.80)

and

$$B = tD_\xi^\gamma \left( \chi \psi(\xi, t) |\xi|^\alpha \hat{\phi} + 2itD_\xi^\gamma (\chi \psi(\xi, t) |\xi|^\alpha \hat{\phi}) + D_\xi^\gamma (\chi \psi(\xi, t) \partial_\xi \hat{\phi}) \right)$$

$$:= B_1 + B_2 + B_3.$$ (4.81)

The next result will be useful in our estimates.

**Claim 4.1.** $A_i, B_j \in L^2$, for $i = 1, 2, 3$ and $j = 2, 3$.

**Proof.** By using Proposition 2.10, inequality (2.21) and Plancherel’s identity, we obtain

$$\|A_1\| \lesssim \|\chi\|_{H^2_\xi} \|t(1 + a)\psi(\xi, t) |\xi|^\alpha \text{sgn}(\xi) \hat{\phi}\|$$

$$\lesssim_{t,a} \|\phi\|,$$ (4.82)

$$\|A_2\| \lesssim \|\chi\|_{H^2_\xi} \|2it\xi \psi(\xi, t) \hat{\phi}\|$$

$$\lesssim_{t,a} \|\phi\|,$$ (4.83)

and

$$\|A_3\| \lesssim \|\chi\|_{H^2_\xi} \|\psi(\xi, t) \partial_\xi \hat{\phi}\|$$

$$\lesssim \|x\phi\|.$$ (4.84)

For the $B_j$ terms, using (2.11) and (2.13) we get

$$\|B_2\| \lesssim_{t} \|\chi \psi(\xi, t) |\xi|^\alpha \hat{\phi}\| + \|D_\xi^\gamma (\chi \psi(\xi, t) |\xi|^\alpha \hat{\phi})\|$$

$$\lesssim_{t,a} \|\phi\| + \|D_\xi^\gamma \hat{\phi}\|$$

$$\lesssim_{t,a} \|\phi\| + \|x^{1/2 + \alpha} \phi\|,$$ (4.85)

and

$$\|B_3\| \lesssim_{t} \|\chi \psi(\xi, t) \partial_\xi \hat{\phi}\| + \|D_\xi^\gamma (\chi \psi(\xi, t) \partial_\xi \hat{\phi})\|$$

$$\lesssim_{t,a} (\|\partial_\xi \hat{\phi}\| + \|D_\xi^\gamma \partial_\xi \hat{\phi}\|)$$

$$\lesssim_{t,a} \|x\phi\| + \|x^{3/2 + \alpha} \phi\|.$$ (4.86)
Claim 4.2. Let
\[
\lim_{\xi \to 0} \frac{\partial}{\partial \xi} \left( \chi(\psi, t - \tau) (t - \tau) \langle 1 + a \rangle |\xi|^{\alpha} \mathbb{1} \right) \right] d\tau
\]
where above, we used the property $\Delta$.
This finish the proof of Claim 4.2.

B

The integral part in (4.78) can be written as
\[
\int_0^t \left[ \chi; D_\xi^\gamma \right] \left( \psi(\xi, t - \tau) (t - \tau) (1 + a) |\xi|^{\alpha} \text{sgn}(\xi) \right) d\tau + D_\xi^\gamma \left( \chi \left( \psi(\xi, t - \tau) (t - \tau) (1 + a) |\xi|^{\alpha} \text{sgn}(\xi) \right) d\tau \right)
\]
where $t \in [0, T]$, then $A_i, B_i \in L^2$, for $i = 1, 2, 3$.

Proof. First of all, by an examining of proof of the Claim 4.1, we see that was only used $\phi \in L^2((x)^{3/2+a} dx)$. Thus, we need to establish that the function $z$ also belongs to this space for every $t \in (0, T]$. By this, we observe that
\[
\partial_t \left( (x)^{3/2+a} u_z^2 \right) = (3/2 + a) (x)^{-1/2+a} u_x^2 + (x)^{3/2+a} \partial_x u_z^2.
\]
Then it’s enough to estimate the first side in the last identity. Therefore
\[
\left\| J((x)^{3/2+a} u_z^2) \right\| = \left\| J((x)^{3/2+a} u_z^2) \right\|
\]
\[
\lesssim \left\| J((x)^{3/2+a} u_z^2) \right\|^2
\]
(4.89)
where above, we used the property $u \in C((0, T]; H^\infty)$ and the Proposition 2.1 with $\beta = 1/4 - a/2$, $\nu = 1$ and $\delta = 1/4$. For the $B_1$ term, by (2.13) and (2.10) we can write
\[
\| B_1 \| \lesssim_{t, a} \left\| D_\xi^\gamma \left( \chi \psi(\xi, t - \tau) \right) \langle \xi \rangle^{a+1} \right\|
\]
\[
\lesssim_{t, a} \left\| u_x^2 \right\| + \left\| D_\xi^\gamma \hat{u}^2 \right\|
\]
(4.90)
This finish the proof of Claim 4.2.

With respect to $B_1$ term, we write
\[
B_1 = t(1 + a) |\xi|^{\alpha} \text{sgn}(\xi) \hat{\psi}(\xi) \chi(\hat{\psi}(\xi) - \hat{\phi}(0)) + t(1 + a) |\xi|^{\alpha} \text{sgn}(\xi) \psi(\xi) \chi(\xi) \hat{\phi}(0)
\]
\[
:= B_{1,1} + B_{1,2}.
\]
(4.91)
Then $B_{1,1} \in H^1(\mathbb{R})$, in fact
\[
\left\| B_{1,1} \right\| \lesssim_{t, a} \left\| |\xi|^{\alpha} \chi(\xi) \hat{\phi}(\xi) \right\| + \left\| |\xi|^{\alpha} \chi(\xi) \hat{\phi}(0) \right\|
\]
\[
\lesssim_{t, a} \left\| |\xi|^{\alpha} \chi(\xi) \right\| \left\| \hat{\phi} \right\| + \left\| |\xi|^{\alpha} \chi(\xi) \right\| \left\| \hat{\phi}(0) \right\|
\]
(4.92)
and
\[
\partial_t B_{1,1} = t(1 + a) |\xi|^{\alpha} \text{sgn}(\xi) \psi \left( \hat{\phi}(\xi) - \hat{\phi}(0) \right) \left\| \xi \right\|
\]
\[
+ t(1 + a) |\xi|^{\alpha} \text{sgn}(\xi) \partial_x \chi(\hat{\phi}(\xi) - \hat{\phi}(0)) + t(1 + a) |\xi|^{\alpha} \text{sgn}(\xi) \partial_x \hat{\phi}(\xi).
\]
(4.93)
Then
\[ \| \partial_\xi B_{1,1} \|_{t,a} \lesssim \| \xi^a \chi \|_{t,a} \| \partial_\xi \hat{\phi} \|_\infty + \| \xi^a \chi \partial_\xi \psi(\xi, t) \|_{L^\infty_\xi} \| \hat{\phi} \| + \| \partial_\xi \psi(\xi, t) \|_{L^\infty_\xi} \| \xi^a \chi \|_{t,a} \| \hat{\phi}(0) \| \]
\[ + \| \xi^a \partial_\xi \hat{\phi} \| + \| \xi^a \partial_\xi \chi \|_{t,a} \| \hat{\phi}(0) \| + \| \xi^a \chi \|_{t,a} \| \partial_\xi \hat{\phi} \| \]
\[ \lesssim \| (x)^{3/2+a} \phi \| + \| x \phi \|, \]  
(4.94)

where above we used the Sobolev embedding
\[ \| \partial_\xi \hat{\phi} \|_\infty \lesssim \| J^{3/2+a}_\xi \hat{\phi} \| = \| (x)^{3/2+a} \phi \|. \]

Therefore by Claim \[\ref{claim1}\] Claim \[\ref{claim2}\] and \[\ref{claim1.94} - \ref{claim1.94.4}\], follows that
\[ t(1 + a)\| \xi^a \text{sgn}(\xi) \psi(\xi, t) \chi(\xi) \hat{\phi}(0) \| \in H^{7}(\mathbb{R}). \]

Writing
\[ t(1 + a)\| \xi^a \text{sgn}(\xi) \psi \hat{\phi}(0) \| = t(1 + a)\| \xi^a \text{sgn}(\xi) (\psi - 1) \chi \hat{\phi}(0) \| + t(1 + a)\| \xi^a \text{sgn}(\xi) \chi \hat{\phi}(0) \|
\]
\[ := C_1 + C_2, \]  
(4.95)

follows that \( C_1 \in H^1_{\xi}(\mathbb{R}). \) Thus \( C_2 \in H^7_{\xi}, \) therefore from \[\ref{5.10}\]
\[ t(1 + a)D^7_{\xi}(\| \xi^a \text{sgn}(\xi) \chi \hat{\phi}(0) \|) \in L^2(\mathbb{R}). \]

Since that \( \gamma = 1/2 + a, \) by \[\ref{2.25}\] follows that \( \hat{\phi}(0) = 0. \) Therefore by \[\ref{2.4}\], we obtain
\[ \hat{u}(0, t) = 0, \]
for every \( t \) in which the solution exists.

The case \( a \in [1/2, 1) \) follows by putting \( 3/2 + a = 2 + \gamma, \) where \( \gamma = a - 1/2 \) and using the derivative \( \partial^2_{\xi}(\psi(\xi, t)\hat{\phi}(\xi)). \) Additionally we observe that by \[\ref{2.4}\]
\[ D^5_{\xi}(\| \xi^{a-1} \chi(\xi) \|) \notin L^2(\mathbb{R}). \]

This finish the proof.

\[\square\]

5. PROOF OF THEOREM \[\ref{1.3}\]

The proof of the Theorems \[\ref{1.2}\] and \[\ref{1.3}\] in the case \( a = 1 \) can be obtained by the same approach of \[\ref{20.}\]. Thus, in the following we deal only with the case \( a \in (0, 1). \)

\[\text{Proof of Theorem}\ \[\ref{1.3}\] Case \( a \in (0, 1/2). \) First of all, we assume without loss of generality that \( 0 = t_1 < t_2 < t_3. \) Since \( \phi \in Z_{3/2+a,5/2+a}, \) using the Theorem \[\ref{1.1}\] (ii)), we see that
\[ u \in C([0, T]; H^{5/2+a}(\mathbb{R}) \cap L^2_{x}), \]  
(5.96)

where \( 0 < r < 5/2 + a. \)

Let \( 5/2 + a = 2 + \gamma, \) where \( \gamma \in (0, 1), \) then multiplying \[\ref{1.7.7}\] by \( |x|^{5/2+a} \) we obtain
\[ D^2_{\xi}(\partial^2_{\xi}(\widetilde{u^2(t)}) = D^2_{\xi}(\partial^2_{\xi}(\psi(\xi, t)\hat{\phi} - \frac{1}{2} \int_0^t (t - \tau)D^2_{\xi}(\partial^2_{\xi}(\psi(\xi, t - \tau)\hat{\phi})d\tau) \]
(5.97)

Then by help of the \( \chi \) function, we can write
\[ \chi D^2_{\xi}(\partial^2_{\xi}(\psi(\xi, t)\hat{\phi}) = \{ \chi; D^2_{\xi}(\partial^2_{\xi}(\psi(\xi, t)\hat{\phi}) + D^2_{\xi}(\chi \partial^2_{\xi}(\psi(\xi, t)\hat{\phi}))
\]
\[ := C + D, \]
where, using the second derivative in (2.17) we obtain
\[
C = [\chi; D^2_\xi]\left( (t^2(1+a)^2|\xi|^2a - t(1+a)a|\xi|^a - 2it^2(1+a)|\xi|^a \text{sgn}(\xi) \\
- 4t^2\xi^2 - 2its\text{sgn}(\xi)) \hat{\phi} - (2t(1+a)|\xi|^a \text{sgn}(\xi) - 4it|\xi|) \partial_\xi \hat{\phi} + \partial_\xi^2 \hat{\phi} \right) \psi(\xi, t) \\
:= C_1 + \ldots + C_8,
\]
(5.98)

and
\[
D = D^2_\xi\left( (t^2(1+a)^2|\xi|^2a - t(1+a)a|\xi|^a - 2it^2(1+a)|\xi|^a \text{sgn}(\xi) \\
- 4t^2\xi^2 - 2its\text{sgn}(\xi)) \hat{\phi} - (2t(1+a)|\xi|^a \text{sgn}(\xi) - 4it|\xi|) \partial_\xi \hat{\phi} + \partial_\xi^2 \hat{\phi} \right) \psi(\xi, t) \\
:= D_1 + \ldots + D_8.
\]
(5.99)

We need of the following result.

**Claim 5.1.** The above terms $C_j, D_j \in L^2(\mathbb{R})$, where $j = 1, \ldots, 8$. Except $D_2$ and $D_6$.

**Proof.** First of all, we deal with the $C_2$ term.

Since that $\hat{\phi}(0) = 0$, by the Taylor Formula
\[
\hat{\phi}(\xi) = \xi \partial_\xi \hat{\phi}(0) + \int_0^t (\xi - \tau) \partial_\xi^2 \hat{\phi}(\tau) d\tau,
\]
(5.100)

and Proposition 2.10 we obtain
\[
\|C_2\| \lesssim \|\chi\|_{H^1_\xi} \|t(1+a)a|\xi|^{a-1}\hat{\phi}\| \\
\lesssim_a t \left( \|\xi|^{a-1}\partial_\xi \hat{\phi}(0)\| + \|\xi|^{a-1}\hat{\phi}\| \int_0^t (\xi - \tau) \partial_\xi^2 \hat{\phi}(\tau) d\tau \right) \\
\lesssim_a t \left( \|\xi|^{a-1}\hat{\phi}(0)\| + \|\xi|^{a-1}\hat{\phi}\| \int_0^t (\xi - \tau) \partial_\xi^2 \hat{\phi}(\tau) d\tau \right) \\
\lesssim_a t \frac{1}{1+\frac{1}{\alpha}} \|\hat{\phi}\|_{H^{1/2+a}}.
\]
(5.101)

where above we used (2.21) and Sobolev embedding. To deal with the terms $C_i$, $i \neq 2$, we can proceed similarly to terms $A_i$, $i = 1, \ldots, 3$, in the Claim 4.1.

About the $D_j$ terms, using 2.10, 2.13 and 2.21
\[
\|D_1\| \lesssim \|t^2(1+a)^2|\xi|^2a \psi(\xi, t) \hat{\phi}\| + \|D^2_\xi(\chi t^2(1+a)^2|\xi|^2a \psi(\xi, t) \hat{\phi})\| \\
\lesssim_a t^2(\|\hat{\phi}\| + \|D^2_\xi(\chi t^2(1+a)^2|\xi|^2a \psi(\xi, t) \hat{\phi})\|) \\
\lesssim_a t^2 \left( \|\hat{\phi}\| + \|D^2_\xi(\chi t^2(1+a)^2|\xi|^2a \psi(\xi, t) \hat{\phi})\| \right) \\
\lesssim_a t^2 \left( \|\hat{\phi}\| + \left( t^{-\frac{1}{2+a}} + t^{-\frac{1}{2+a}} + t^{-\frac{2a}{1+2a}} \right) \|\hat{\phi}\| + t^{-\frac{2a}{1+2a}} \|\hat{\phi}\| \right) \\
\lesssim_{t,a} \|\hat{\phi}\| + \|\hat{\phi}\|^2 + \|\hat{\phi}\|^{1/2+a} \hat{\phi},
\]
(5.102)
The last term can be estimated by similar way to (3.75).

Claim 5.2. The terms $D_6$ finish the proof.

By way analogous to the linear part, we can write

$$
\|D_3\| \lesssim a t^2 \left( \|\xi|^{a+1} \psi(\xi, t) \hat{\phi}\| + \|D_\xi^2 (\xi|^{a+1} \psi(\xi, t) \hat{\phi})\| \right)
$$

$$
\lesssim a t^2 \left( (t^{-1} + t^{-1} + t^{-1}) \|\phi\| + t^{-1} \|\gamma \phi\| \right)
\lesssim t, a \|\phi\| + \|\gamma \phi\| \tag{5.103}
$$

$$
\|D_4\| \lesssim a t^2 \left( \|\xi^2 \psi(\xi, t) \hat{\phi}\| + \|D_\xi^2 (\xi^2 \psi(\xi, t) \hat{\phi})\| \right)
$$

$$
\lesssim a t^2 \left( (t^{-1} + t^{-1} + t^{-1}) \|\phi\| + t^{-1} \|\gamma \phi\| \right)
\lesssim t, a \|\phi\| + \|\gamma \phi\| \tag{5.104}
$$

$$
\|D_7\| \lesssim a t \left( \|\xi \psi(\xi, t) \partial_\xi \hat{\phi}\| + \|D_\xi^2 (\xi \psi(\xi, t) \partial_\xi \hat{\phi})\| \right)
$$

$$
\lesssim a \left( (t^{-1} + 1 + t^{-1}) \|\phi\| + t^{-1} \|\gamma \phi\| \right)
\lesssim t, a \|\phi\| + \|\gamma \phi\| \tag{5.105}
$$

$$
\|D_8\| \lesssim a \left( \|\psi(\xi, t) \partial_\xi^2 \hat{\phi}\| + \|D_\xi^2 (\psi(\xi, t) \partial_\xi^2 \hat{\phi})\| \right)
$$

$$
\lesssim t, a \|\phi\| + \|\gamma \phi\| \tag{5.106}
$$

and finally

$$
\|D_9\| \lesssim a \left( \|\text{sgn}(\xi) \psi(\xi, t) \hat{\phi}\| + \|D_\xi^2 (\text{sgn}(\xi) \psi(\xi, t) \hat{\phi})\| \right). \tag{5.107}
$$

The last term can be estimated by similar way to (3.75).

This finish the proof.

By way analogous to the linear part, we can write

$$
\int_0^t \left[ \chi; D_\xi^2 \right] \left( \psi(t - \tau, \xi) ((t - \tau)^2(1 + a)^2|\xi|^{2a} - (t - \tau)(1 + a)a|\xi|^{a-1} +
+ 2i(t - \tau)^2(1 + a)|\xi|^{a+1} \text{sgn}(\xi) - 4(t - \tau)^2|\xi|^2 - 2i(t - \tau) \text{sgn}(\xi) \right] \tilde{z}
- \left( 2(t - \tau)(1 + a)|\xi|^a \text{sgn}(\xi) - 4i(t - \tau)|\xi| \right) \partial_\xi \tilde{z} + \partial_\xi^2 \tilde{z}
+ D_\xi^2 \left( \chi \psi(t - \tau, \xi) ((t - \tau)^2(1 + a)^2|\xi|^{2a} - (t - \tau)(1 + a)a|\xi|^{a-1} +
+ 2i(t - \tau)^2(1 + a)|\xi|^{a+1} \text{sgn}(\xi) - 4(t - \tau)^2|\xi|^2 - 2i(t - \tau) \text{sgn}(\xi) \right) \tilde{z}
- \left( 2(t - \tau)(1 + a)|\xi|^a \text{sgn}(\xi) - 4i(t - \tau)|\xi| \right) \partial_\xi \tilde{z} + \partial_\xi^2 \tilde{z} \right) \tilde{d} \tau
:= C_1 + \ldots + C_8 + D_1 + \ldots + D_8. \tag{5.108}
$$

Claim 5.2. The terms $C_j, D_j \in \mathcal{L}^2$, for all $t \in [0, T]$, $j = 1, ..., 8$. Except $D_2$ and $D_6$. 
Proof. Similar to proof of Claim 4.2. In fact, looking at the proof of Claim 5.1 we see that we only used that \( \psi \in L^2((x)^{5/2+a}dx) \). Then its enough to show that

\[
z = uu_x \in L^2((x)^{5/2+a}dx), \text{ for all } t \in (0,T].
\]

We observe that

\[
\partial_t((x)^{5/2+a}u^2) = (5/2 + a)(x)^{1/2+a}u^2 + (x)^{5/2+a}\partial_x u^2.
\]  (5.109)

Therefore we will estimate the first side of the last inequality

\[
\|J((x)^{5/2+a}u^2)\| = \|J((x)^{5/4+a/2}u)^2\| \lesssim \|J((x)^{5/4+a/2}u)\|^2
\]  (5.110)

\[
\lesssim \|J^\beta u\|\|\langle x \rangle^{1/2}u\|^{1-\beta},
\]

where above we used \( u \in C((0,T];H^\infty) \) and Lemma 2.3 with \( \nu = 3/2, \delta = \frac{1}{\beta} \) and \( \beta = \frac{1-2a}{4} \).

This finish the proof.

For the terms in (5.99) we can write

\[
D_2 = c_a D_\xi^2(\chi t|\xi|^{1-a}\psi(\xi,t)\hat{\phi}(\xi)) := c_a D_\xi^2 \left( t|\xi|^{1-a}\psi(\xi,t)(\chi + 1 - \chi) \right)
\]  (5.111)

\[
:= D_{2,1} + D_{2,2},
\]

where \( c_a = -(1+a)\alpha \). Then \( D_{2,2} \in L^2_\xi \), in fact, in view of (2.13)

\[
\|D_{2,2}\| \lesssim a \left( 1 + \|t\| \langle x \rangle^{1-a} \psi(\xi,t)(1-\chi) \| \right) \|\phi\| +
\]

\[
+ t \|t\| \langle x \rangle^{1-a} \psi(\xi,t)(\chi - 1 - \chi) \| D_\xi^2 \hat{\phi} \|
\]

\[
\lesssim a \left( 1 + \|t\| \langle x \rangle^{1-a} \psi(\xi,t)(1-\chi) \| \right) \|\phi\| +
\]

\[
+ t \|t\| \langle x \rangle^{1-a} \psi(\xi,t)(\chi - 1 - \chi) \| D_\xi^2 \hat{\phi} \|
\]

\[
\lesssim a \left( 1 + \|t\| \langle x \rangle^{1-a} \psi(\xi,t)(1-\chi) \| \right) \|\phi\| +
\]

\[
t \|t\| \langle x \rangle^{1-a} \psi(\xi,t)(\chi - 1 - \chi) \| D_\xi^2 \hat{\phi} \|
\]

\[
\lesssim a \left( 1 + t \|t\| \langle x \rangle^{1-a} \psi(\xi,t)(1-\chi) \| \right) \|\phi\| +
\]

\[
t \|t\| \langle x \rangle^{1-a} \psi(\xi,t)(\chi - 1 - \chi) \| D_\xi^2 \hat{\phi} \|. \]  (5.112)

Since \( \hat{\phi}(0) = 0 \), using the Taylor’s formula

\[
\hat{\phi}(\xi) = \xi \partial_\xi \hat{\phi}(0) + \int_0^\xi (\xi - \sigma) \partial_\xi \hat{\phi}(\sigma) d\sigma
\]  (5.113)

in (5.111), we obtain

\[
D_{2,1} = t c_a D_\xi^2 \left( |\xi|^{a-1}\psi(\xi) \partial_\xi \hat{\phi}(0) \right) +
\]

\[
c_a D_\xi^2 \left( t|\xi|^{1-a}\psi(\xi) \int_0^\xi (\xi - \sigma) \partial_\xi \hat{\phi}(\sigma) d\sigma \right)
\]  (5.114)

\[
:= \tilde{D}_2 + R(\xi),
\]

where \( R \in H^1_\xi \), in fact
\[ \|F\| \lesssim_a t\|\xi|^a - 1 \chi \xi^2 \| \partial_\xi^2 \hat{\phi}\|_\infty \]
\[ \lesssim_{a,t} \|J_{\xi}^{1/2 + a} \partial_\xi^2 \hat{\phi}\| \]
\[ \lesssim_{a,t} \|(x)^{\delta/2 + a} \phi\|. \]

(5.115)

and

\[ \|\partial_\xi F\| \lesssim t \left(\|\xi|^{a - 2} \chi \xi^2 \| \partial_\xi^2 \hat{\phi}\|_\infty + \|\xi|^{a - 1} \partial_\xi \psi \xi^2 \chi \| \partial_\xi \hat{\phi}\|_\infty + \right. \]
\[ + \left. \|\xi|^{a - 1} \psi \xi^2 \| \partial_\xi^2 \hat{\phi}\|_\infty + \right. \]
\[ + \left. \|\xi|^{a - 1} \chi \int_0^\xi \partial_\xi^2 \hat{\phi}(\sigma) d\sigma\right) \]
\[ \lesssim_{a,t} \|\langle x\rangle^{5/2 + a} \phi\|. \]

(5.116)

We can also write

\[ \bar{D}_2 = t c_a D_\gamma (\|\xi|^{a} \text{sgn}(\xi) \chi \partial_\xi \hat{\phi}(0)(\psi - 1 + 1)) \]
\[ = \bar{D}_{2,1} + \bar{D}_2, \]

(5.117)

where \( \bar{D}_{2,1} \in L_2^\xi. \)

In fact, since

\[ \partial_\xi \psi(\xi,t) = - \left( t(1 + a)\|\xi|^{a} \text{sgn}(\xi) + 2t|\xi| \right) \psi, \]

we get

\[ |\psi(\xi,t) - 1| \lesssim (t^{1+a} + t^{1+a})|\xi|. \]

(5.118)

Then

\[ \|\xi|^{a} \text{sgn}(\xi) \chi \partial_\xi \hat{\phi}(0)(\psi - 1)\| \lesssim_{a,t} \|\partial_\xi \hat{\phi}(0)\| \|\xi|^{a + 1} \chi\|, \]

(5.119)

and

\[ \|\partial_\xi (\|\xi|^{a} \text{sgn}(\xi) \chi \partial_\xi \hat{\phi}(0)(\psi - 1))\| \lesssim_{a,t} t|\partial_\xi \hat{\phi}(0)| \left(\|\xi|^{a} \chi \psi - 1 \xi \right) \]
\[ + \|\xi|^{a + 1} \chi^\prime \| + \|\xi|^{a + 1} \chi^\prime \| \]
\[ \lesssim_{a,t} \|\partial_\xi \hat{\phi}(0)\| \left(\|\xi|^{2a} + |\xi|^{1+a} \chi\| + \|\xi|^{1+a} \chi^\prime \| \right). \]

(5.120)

For the \( D_6 \) term we estimate as follows

\[ -D_6^\gamma(2t(1 + a)|\xi|^{a} \psi \text{sgn}(\xi) \partial_\xi \hat{\phi}) = -D_6^\gamma(2t(1 + a)|\xi|^{a} \text{sgn}(\xi) \psi \partial_\xi \hat{\phi}(\chi + 1 - \chi)) \]
\[ := D_{6,1} + D_{6,2}, \]

(5.121)

where, by analogous to \((5.112), D_{6,2} \in L_2^\xi. \) We also write

\[ D_{6,1} = D_6^\gamma(2t(1 + a)|\xi|^{a} \text{sgn}(\xi) \chi \partial_\xi \hat{\phi}(1 + |\psi - 1|)) \]
\[ := D_{6,1}^1 + D_{6,2}^2, \]

(5.122)

where \( D_{6,2}^2 \in L_2^\xi, \) by the Lemma \( 2.4. \)
The $D^1_{6,1}$ term can be written as

$$D^1_{6,1} = D^7_\xi \left( 2t(1 + a)|\xi|^\alpha \text{sgn}(\xi) \chi \partial_\xi \hat{\phi}(\xi) \right) = 2t(1 + a)D^7_\xi \left( |\xi|^\alpha \text{sgn}(\xi) \chi (\partial_\xi \hat{\phi}(\xi) - \partial_\xi \hat{\phi}(0)) + |\xi|^\alpha \text{sgn}(\xi) \chi \partial_\xi \hat{\phi}(0) \right)$$

$$:= D_6 + D_2,$$

where $\tilde{D}_6 \in L^2_\xi$.

From the equalities above, we see that

$$D_2 + D_6 = \tilde{D}_{2,1} + R + D_{2,2} + \tilde{D}_6 + D_{6,2} +$$

$$+ 2t|\xi|^\alpha \text{sgn}(\xi) \chi \partial_\xi \hat{\phi}(0).$$

For the integral terms, by similar way we obtain

$$D_2 + D_6 = \tilde{D}_{2,1} + R + D_{2,2} + \tilde{D}_6 + D_{6,2} +$$

$$+ 2 \int_0^t (t - \tau)|\xi|^\alpha \text{sgn}(\xi) \chi \partial_\xi \hat{x}(0, \tau) d\tau.$$

From our hypotheses, Claim 5.1, Claim 5.2, 5.123 and 5.125 follows that

$$D^7_\xi \left( |\xi|^\alpha \text{sgn}(\xi) \chi \left( t \partial_\xi \hat{x}(0) - \int_0^t (t - \tau) \partial_\xi \hat{x}(0, \tau) d\tau \right) \right) \in L^2$$

if and only if

$$D^7_\xi \partial^2_\xi \hat{u}(\cdot, t) \in L^2(\mathbb{R}).$$

Also, from 2.9, we get

$$\partial_\xi \hat{x}(0, \tau) = -i \hat{x}(0, \tau)$$

$$= -\frac{i}{2} \int x \partial_x u^2(x, \tau) dx$$

$$= \frac{i}{2} \|u(\tau)\|^2$$

$$= i \frac{d}{d\tau} \int xu(x, \tau) dx.$$
Therefore, integrating by parts
\[
t\partial_t \hat{z}(0) - \int_0^t (t - \tau) \partial_t \hat{z}(0, \tau) d\tau = t\partial_t \hat{z}(0) - i \int_0^t (t - \tau) \frac{d}{d\tau} \int xu(x, \tau) dx d\tau \\
= -it \int x\phi(x) dx - it \int x\phi(x) dx \bigg|_{\tau=0}^{\tau=t} \\
- \int_0^t \int xu(x, \tau) dx d\tau \\
= -it \int x\phi(x) dx + it \int x\phi(x) dx - \\
- i \int_0^t \int xu(x, \tau) dx d\tau \\
= -i \int_0^t \int xu(x, \tau) dx d\tau. 
\]
(5.129)

Then, putting \( t = t_2 \), in (5.126) and (5.129)
\[
D^\gamma (|\xi|^a \text{sgn}(\xi) \chi) \int_0^{t_2} \int xu(x, \tau) dx d\tau \in L^2, 
\]
and by Stein derivative
\[
D^\gamma (|\xi|^a \text{sgn}(\xi) \chi) \int_0^{t_2} \int xu(x, \tau) dx d\tau \in L^2. 
\]
(5.131)

Then, in view of \( \gamma = 1/2 + a \), from (2.23) we obtain
\[
\int_0^{t_2} \int xu(x, \tau) dx d\tau = 0. 
\]
(5.132)

By Rolle’s lemma, there exists \( \tau_1 \in (0, t_2) \) such that
\[
\int xu(x, \tau_1) dx = 0. 
\]
(5.133)

Analogously, using that \( u(t_2), u(t_3) \in Z_{5/2+a, 5/2+a} \) we can show that exists \( \tau_2 \in (t_2, t_3) \) such that
\[
\int xu(x, \tau_2) dx = 0. 
\]
(5.134)

Finally from (5.133), (5.134) and identity (2.3) we obtain \( u(t) = 0 \), for all \( t \in [\tau_1, \tau_2] \).

In view of \( \|u(t)\| \) is decreasing in \( t \), we conclude that
\[
u(t) = 0, \forall t \geq \tau_1 = \bar{t}. 
\]

The case \( a \in [1/2, 1] \) can be deal by putting \( 5/2 + a = 3 + \gamma \), where \( \gamma = a - 1/2 \), using the derivative \( \partial^\gamma (\psi(\xi, t) \hat{\phi}(\xi)) \) and observing that by (2.20)
\[
D^\gamma (|\xi|^{a-1} \chi(\xi)) \notin L^2(\mathbb{R}).
\]
\( \square \)
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