Conformal anomalies on Einstein spaces with Boundary

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Abstract

The anomalous rescaling for antisymmetric tensor fields, including gauge bosons, and Dirac fermions on Einstein spaces with boundary has been prone to errors and these are corrected here. The explicit calculations lead to some interesting identities that indicate a deeper underlying structure.

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I. INTRODUCTION

In this letter we would like to correct some results that we gave previously for the conformal anomaly of quantum fields on spaces with boundaries \[1\]. These results caused some consternation because there seemed to be a disagreement between general results obtained from heat kernel asymptotics \[1–3\] and direct calculations of the anomaly \[4–13\]. We can confirm now that the situation has been resolved with the discovery by Vassilevich \[14\] of corrections to the heat kernel asymptotics \[15–19\].

The physical motivation for these calculations is connected with quantum cosmology and the quantum state of the universe, where boundary effects play an important role. In the path integral approach boundary terms are important in one–loop corrections.

Besides the practical applications of these results there are also a number of mathematical coincidences that seem to 1) originate in an interesting identity. Before discussing these results we will use the calculation of the conformal anomaly of a Dirac fermion as a consistency check.

II. ONE LOOP AMPLITUDES

It is convenient to define quantum amplitudes using a path integral over field configurations on a Riemannian manifold \(\mathcal{M}\) with boundary \(\Sigma\),

\[
e^{-\Gamma} = \int d\mu[\phi] e^{-S[\phi]},
\]

where \(S[\phi]\) is the Riemannian action and \(\hbar = 1\).

In the semi–classical approximation the path integral is dominated by saddle point configurations \(\phi_s\) which satisfy the classical equations and agree with given boundary data on \(\Sigma\). The fields can be expanded about these saddle points to obtain a perturbative expansion of the quantum amplitude,

\[
\Gamma = \Gamma^{(0)} + \Gamma^{(1)} + \ldots
\]

where the leading term \(\Gamma^{(0)} = S[\phi_s]\). The one–loop term involves a linear operator \(\Delta\) depending on the background \(\phi_s\),

\[
\Gamma^{(1)} = \frac{1}{2} \log \det \Delta.
\]

Gauge fields can only be correctly accounted for if we introduce gauge–fixing terms \(\mathcal{F}[\phi]\) and corresponding ghost fields. Then,

\[
\Gamma^{(1)} = \frac{1}{2} \sum_j (-1)^{f_j} \log \det \Delta_j,
\]

where \(j\) labels the different types of field and \(f_j = 1\) for fermions and ghosts.

The determinant is infinite and has to be regulated. We shall use \(\zeta\)–function regularisation with,

\[
\zeta_j(s) = \sum_n \lambda_n^{-s},
\]
where $\lambda_n$ are the eigenvalues of the operator $\Delta_j$. This allows us to define

\[
\log \det \Delta_j = -\zeta'_j(0) - \zeta_j(0) \log \mu^2. \tag{6}
\]

where $\mu$ is the renormalisation scale.

Results on the heat kernel expansion of operators can be used to find the scale dependent term on an arbitrary bounded manifold. (The mathematical background can be found in [20]). If the heat kernel expansion coefficients of the operator $\Delta$ on $\mathcal{M}$ are denoted by $B_m(\Delta, \mathcal{M})$, then

\[
\zeta_j(0) = B_2(\Delta_j, \mathcal{M}). \tag{7}
\]

The renormalisation scale dependence of the path integral is therefore,

\[
\mu \frac{d\Gamma}{d\mu} = -\sum_j (-1)^f_j B_2(\Delta_j, \mathcal{M}). \tag{8}
\]

We will denote the sum by $B(\phi, \mathcal{M})$. In a theory that is otherwise independent of scale this term is the conformal anomaly.

In general, the renormalisation scale dependence can be expressed in terms of local invariants and the coefficients are one-loop beta functions. To be specific, consider the operator

\[-D^2 + X \tag{9}\]

where $D_a$ is a gauge derivative. The $B_2$ coefficients can be put into the form

\[
16\pi^2 B_2(\Delta, \mathcal{M}) = \int_{\mathcal{M}} b_2(\Delta) d\mu + \int_{\partial \mathcal{M}} c_2(\Delta) d\mu. \tag{10}
\]

Volume terms up to $b_3$ are available in the literature. The surface terms depend upon the choice of boundary conditions. We will use mixtures of Dirichlet and Neumann boundary conditions,

\[P_- \phi = 0, \quad (\psi + n \cdot \nabla) P_+ \phi = 0, \tag{11}\]

where $P_\pm$ are projection operators and $n$ is the normal vector.

The results can be expressed in terms of polynomials in the curvature tensor of the manifold $R_{abcd}$ and the extrinsic curvature of the boundary $k_{ab},$

\[q = \frac{8}{3} k^3 + \frac{16}{3} k^b_a k^c_b k^a_c \quad - 8k k_{ab} k^{ab} + 4kR - 8R_{ab}(kn^a n^b + k^{ab}) + 8R_{abcd}k^{ac} n^b n^d \tag{12}\]

and

\[g = k^b_a k^c_b k^a_c \quad - kk_{ab} k^{ab} + \frac{2}{9} k^3 \tag{13}\]

For Dirichlet boundary conditions,

\[c^D_2 = \text{tr} \left( -\frac{1}{360} q + \frac{2}{35} g \quad - \frac{1}{3}(X - \frac{1}{6} R)k \quad - \frac{1}{2} n. \nabla (X - \frac{1}{6} R) + \frac{1}{18} C_{abcd} k^{ac} n^b n^d \right), \tag{14}\]
 whilst for Robin boundary conditions,
\[ c^R_2 = \text{tr}(-\frac{1}{360}q + \frac{2}{15}g - \frac{1}{3}(X - \frac{1}{6}R)k + \frac{1}{2}n\nabla(X - \frac{1}{6}R) - \frac{4}{3}(\psi - \frac{1}{3}k)^3 + 2(X - \frac{1}{6}R)\psi - (\psi - \frac{1}{3}k)(\frac{2}{15}k^2 - \frac{2}{15}k_{ab}k^{ab}) + \frac{1}{15}C_{abcd}k^{ac}n^bn^d). \] (15)

For mixed boundary conditions,
\[ c^2 = \text{tr}(P + c^R_2 + P - c^D_2 - \frac{2}{15}P + |a|P + |a|k^ab + \frac{4}{3}P + |a|P + |n|F_{ab}) \]
\[ + \frac{4}{3}P + |a|P + |n|\psi - \frac{2}{3}P + |a|n^bF_{ab}) \] (17)

where \( P + |a| \) denotes the surface derivative of \( P + |a| \). Two of the terms include corrections by Vassilevich [14] of the results in Branson and Gilkey [19]. The final term also corrects a sign error in ref. [2].

Because one of the most important applications of this work is to quantum cosmology, we will take the spacetime curvature to satisfy vacuum Einstein equations with a cosmological constant, i.e. \( R_{ab} = \Lambda g_{ab} \). This condition removes some terms from the results but leaves a high degree of generality. The heat kernel coefficients will be of the form
\[ b_2(\Delta) = \alpha_0\Lambda^2 + \alpha_2R_{abcd}R^{abcd} \] (19)
and
\[ c_2(\Delta) = \beta_1\Lambda k + \beta_2k^3 + \beta_3kk_{ab}k^{ab} + \beta_4k^b_{a}k^c_{b}k^a_{c} + \beta_5C_{abcd}k^{ac}n^bn^d. \] (20)

Special cases include the four–sphere \( S \), disc \( D \) and spherical cap \( C \). The disc is a region of flat space bounded by a three–sphere and the cap a region of the four–sphere with maximum colatitude \( \theta \). The heat kernel coefficients in these cases simplify to,
\[ B_2(\Delta, S) = \frac{3}{2}\alpha_0 + 4\alpha_1, \quad B_2(\Delta, D) = \frac{77}{8}\beta_2 + \frac{9}{8}\beta_3 + \frac{3}{8}\beta_4 \] (21)
and
\[ B_2(\Delta, C) = B_2(\Delta, S)(\frac{1}{2} - \frac{2}{3}\cos\theta + \frac{1}{4}\cos^3\theta) + B_2(\Delta, D)\cos^3\theta + \frac{9}{8}\beta_1\cos\theta\sin^2\theta. \] (22)

The disc and the cap are related in the limit that \( \theta = 0 \), then \( B_2(\Delta, C) = B_2(\Delta, D) \).

### III. DIRAC FERMIONS

The conformal anomaly has been evaluated explicitly for Dirac fermions on a disc [4,5,10–12]. This gives us an opportunity to check the general results.

We will use boundary conditions that can be implemented locally by means of projection operators. There is only one possible choice [21], \( \psi = k/2 \) and
\[ P_+ = \frac{1}{2}(1 - \gamma_5\gamma \cdot n). \] (23)

(The gamma matrix conventions are such that \( \{\gamma_a, \gamma_b\} = -2g_{ab} \) and for the commutator of derivatives \( [D_a, D_b] = -\frac{1}{2}\{\gamma_a, \gamma_b\}R^{cd}_{ab} \).

The fermion operator is given by
\[ \Delta = (\gamma \cdot D)^2 = -D^2 + \frac{1}{4} R \] (24)

Substitution into the general formulae gives the values shown in the table. In the special cases of the disc and spherical cap,

\[ B_2(\Delta, C) = B_2(\Delta, D) = \frac{1}{2} B_2(\Delta, S) = \frac{11}{180} \] (25)

and all of the \( \theta \) dependence disappears.

**IV. ANTISYMMETRIC TENSORS**

In four dimensions antisymmetric tensor fields \( A_p \) can have rank \( p \), the number of indices, from zero to four. Fields with rank zero and one represent scalar and vector boson fields whilst a rank two field is locally equivalent to a scalar and ranks three and four have no physical degrees of freedom. The classical theory of the rank one field is conformally invariant in four dimensions but not the others.

The Riemannian action for a p–form will be taken to be

\[ S_p = \frac{1}{4} \int F_{a_1...a_{p+1}} F^{a_1...a_{p+1}} d\mu \] (26)

where \( F \) is the curvature \( dA_p \). The integral is taken over the volume of the manifold.

The action has a gauge invariance \( \delta_g A_p = dA_{p-1} \), where \( A_{p-1} \) is an arbitrary antisymmetric tensor of rank \( p-1 \). This means that we have to fix the gauge and introduce ghost fields. Following the literature [22–24], we use a gauge fixing condition \( \delta A_p = 0 \), then there are two anticommuting ghosts of rank \( p-1 \), three commuting ghosts of rank \( p-2 \) and so on.

The anomalous rescaling of the action becomes

\[ \mu \frac{d\Gamma_p}{d\mu} = -\sum_{q=0}^{p} (-1)^{p-q} (p-q+1) \log \det(\Delta_q) \] (27)

where \( \Delta_q \) is the Hodge–de Rahm operator \( d\delta + \delta d \), which results from gauge fixing.

The boundary conditions on the ghost fields can be found by an application of BRS symmetry. The BRS transformations are given by

\[ \delta_{BRS} A_p = dA_{p-1}. \] (28)

In consequence we require a set of boundary conditions which are preserved under exterior differentiation. A suitable set has been described by Gilkey [20]. Projection operators \( P_\pm \) are defined in such a way that \( P_+ \) has only tangential components on the boundary \( \Sigma \), and \( P_+ + P_- = 1 \). Then absolute boundary conditions are defined by equation (11), with

\[ \psi^{b_1...b_p}_{a_1...a_p} = k^{b_1}_{a_1} \delta^{b_2...b_p}_{a_2...a_p} \] (29)

Another set of boundary conditions, called relative, can be defined by dualising this set. Relative boundary conditions are BRS invariant when they apply to eigenstates of the Hodge–de Rahm operator. In our earlier work, we called the two sets of boundary
conditions electric and magnetic, depending on which fields were held fixed on the boundary. The surface of a conductor leads to relative boundary conditions on the electromagnetic potential.

The values of $\alpha$ and $\beta$ for $B_2(\Delta_p, \mathcal{M})$ obtained from the general formulae are shown in table $I$. These results are for absolute boundary conditions. The results for relative boundary conditions are given by the results for the dual form.

The values of $\alpha$ and $\beta$ can be combined to produce the conformal anomaly of the rank $p$ tensor including the ghost terms. Results are given in table $III$ for absolute and table $IV$ for relative boundary conditions.

V. DISCUSSION

As we mentioned in the introduction, calculations of the anomaly using the heat kernel coefficients $[1,2,8]$ have been at odds with direct calculations. Different covariant techniques can now be reconciled, particularly for the disc and the spherical cap. $B_2(\mathcal{D})$ has been calculated by a number of people using direct calculations $[4,5,11]$ for majorana fermions with the local boundary conditions described above. They find that $B_2(\mathcal{D}) = \frac{11}{180}$ which agrees with the result reported here. For fermions $B_2(\mathcal{C})$ has also been calculated directly by $[11,12]$ who find that $B_2(\mathcal{C}) = \frac{11}{180}$, which is again in agreement with our calculations. Finally, in a recent paper $[13]$ Esposito and Kamenshchik have calculated the anomaly on a disc for the one–form including the ghosts. They find the result $-\frac{31}{90}$ (as we do). This still disagrees with a calculation using only the physical degrees of freedom, but they argue that the difference arises from the difficulty in defining a canonical decomposition for the disc.

An examination of tables reveals some interesting coincidences. The boundary terms for ranks three and four combine with the volume terms by the Gauss–Bonnet theorem and form an integral expression for the Euler number $\chi$ on an arbitrary manifold. Let us denote the combined heat kernel coefficients by $B^a(A_p, \mathcal{M})$ and $B^r(A_p, \mathcal{M})$, for absolute and relative boundary conditions, then

\[
B^a(A_2, \mathcal{M}) = B^r(A_0, \mathcal{M}) + \chi, \quad B^r(A_2, \mathcal{M}) = B^a(A_0, \mathcal{M}) + \chi
\]  
\[B^a(A_3, \mathcal{M}) = B^r(A_3, \mathcal{M}) = -2\chi, \quad B^a(A_4, \mathcal{M}) = B^r(A_4, \mathcal{M}) = 3\chi.\]  
\[(30)\]

These relationships can be derived directly using standard cohomology theory and a new identity

\[
2B_2(\Delta_4, \mathcal{M}) - B_2(\Delta_3, \mathcal{M}) + B_2(\Delta_2, \mathcal{M}) - 2B_2(\Delta_0, \mathcal{M}) = 0,
\]  
\[(32)\]

for absolute and relative boundary conditions separately. This same identity is also responsible for the fact that the vector gauge field results are independent of the choice of boundary conditions. The last identity is unlikely to be coincidental and indicates a deeper underlying structure.
| rank | $\alpha_0$ | $\alpha_1$ | $\beta_1$ | $\beta_2$ | $\beta_3$ | $\beta_4$ | $\beta_5$ | $B_2(S)$ | $B_2(D)$ |
|------|------------|------------|----------|----------|----------|----------|----------|--------|---------|
| 0    | 1/5        | 1/180      | 1/135    | 1/135    | 1/135    | 1/135    | 1/135    | 1/135  | 1/135   |
| 1    | 1/5        | 1/180      | 1/135    | 1/135    | 1/135    | 1/135    | 1/135    | 1/135  | 1/135   |
| 2    | 1/5        | 1/180      | 1/135    | 1/135    | 1/135    | 1/135    | 1/135    | 1/135  | 1/135   |
| 3    | 1/5        | 1/180      | 1/135    | 1/135    | 1/135    | 1/135    | 1/135    | 1/135  | 1/135   |

**TABLE I.** Coefficients for the conformal anomaly of a Dirac fermion

| rank | $\alpha_0$ | $\alpha_1$ | $\beta_1$ | $\beta_2$ | $\beta_3$ | $\beta_4$ | $\beta_5$ | $B_2(S)$ | $B_2(D)$ |
|------|------------|------------|----------|----------|----------|----------|----------|--------|---------|
| 0    | 1/5        | 1/180      | 1/135    | 1/135    | 1/135    | 1/135    | 1/135    | 1/135  | 1/135   |
| 1    | 1/5        | 1/180      | 1/135    | 1/135    | 1/135    | 1/135    | 1/135    | 1/135  | 1/135   |
| 2    | 1/5        | 1/180      | 1/135    | 1/135    | 1/135    | 1/135    | 1/135    | 1/135  | 1/135   |
| 3    | 1/5        | 1/180      | 1/135    | 1/135    | 1/135    | 1/135    | 1/135    | 1/135  | 1/135   |

**TABLE II.** Coefficients for terms in the $B_2$ coefficient for p-forms with absolute boundary conditions

| rank | $\alpha_0$ | $\alpha_1$ | $\beta_1$ | $\beta_2$ | $\beta_3$ | $\beta_4$ | $\beta_5$ | $B_2(S)$ | $B_2(D)$ |
|------|------------|------------|----------|----------|----------|----------|----------|--------|---------|
| 0    | 1/5        | 1/180      | 1/135    | 1/135    | 1/135    | 1/135    | 1/135    | 1/135  | 1/135   |
| 1    | 1/5        | 1/180      | 1/135    | 1/135    | 1/135    | 1/135    | 1/135    | 1/135  | 1/135   |
| 2    | 1/5        | 1/180      | 1/135    | 1/135    | 1/135    | 1/135    | 1/135    | 1/135  | 1/135   |
| 3    | 1/5        | 1/180      | 1/135    | 1/135    | 1/135    | 1/135    | 1/135    | 1/135  | 1/135   |

**TABLE III.** Coefficients for terms in the conformal anomaly or scaling terms for p-forms with absolute boundary conditions including ghosts. $B_2(S)$ refers to the combination of $B_2$ coefficients on a sphere and $B_2(D)$ on a disk.

| rank | $\alpha_0$ | $\alpha_1$ | $\beta_1$ | $\beta_2$ | $\beta_3$ | $\beta_4$ | $\beta_5$ | $B_2(S)$ | $B_2(D)$ |
|------|------------|------------|----------|----------|----------|----------|----------|--------|---------|
| 0    | 1/5        | 1/180      | 1/135    | 1/135    | 1/135    | 1/135    | 1/135    | 1/135  | 1/135   |
| 1    | 1/5        | 1/180      | 1/135    | 1/135    | 1/135    | 1/135    | 1/135    | 1/135  | 1/135   |
| 2    | 1/5        | 1/180      | 1/135    | 1/135    | 1/135    | 1/135    | 1/135    | 1/135  | 1/135   |
| 3    | 1/5        | 1/180      | 1/135    | 1/135    | 1/135    | 1/135    | 1/135    | 1/135  | 1/135   |

**TABLE IV.** Coefficients for terms in the conformal anomaly or scaling terms for p-forms with relative boundary conditions including ghosts. $B_2(S)$ refers to the combination of $B_2$ coefficients on a sphere and $B_2(D)$ on a disk.
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