LONG TERM DYNAMICS OF SECOND ORDER-IN-TIME
STOCHASTIC EVOLUTION EQUATIONS WITH
STATE-DEPENDENT DELAY

IGOR CHUESHOV
Department of Mechanics and Mathematics
Kharkov National University
61077, Kharkov, Ukraine

PETER E. KLOEDEN AND MEIHUA YANG
School of Mathematics & Statistics
Huazhong University of Science & Technology
Wuhan 430074, China

Dedicated to the memory of Igor Chueshov

Abstract. The well-posedness and asymptotic dynamics of second-order-in-
time stochastic evolution equations with state-dependent delay is investigated.
This class covers several important stochastic PDE models arising in the theory
of nonlinear plates with additive noise. We first prove well-posedness in a
certain space of functions which are $C^1$ in time. The solutions constructed
generate a random dynamical system in a $C^1$-type space over the delay time
interval. Our main result shows that this random dynamical system possesses
compact global and exponential attractors of finite fractal dimension. To obtain
this result we adapt the recently developed method of quasi-stability estimates
to the random setting.

1. Introduction. The well-posedness and asymptotic dynamics of deterministic
evolution equations with state-dependent delays have been investigated recently by
Igor Chueshov and his colleague Alexander Rezounenko for both first and second-
order-in-time equations, see [15, 16], and form a small but important part of Igor
Chueshov’s life long work on the dynamical behavior of evolution equations in me-
chanics. In particular, they used the method of quasi-stability expounded in Igor’s
book [8] to establish the existence of attractors and exponential attractors.

In this paper we study the well-posedness and asymptotic dynamics of second-
order-in-time stochastic evolution equations with state-dependent delays of the form

\[ \ddot{u}(t) + k\dot{u}(t) + Au(t) + M(u_t) = \dot{W}, \quad t > 0, \]

in some Hilbert space $H$. Here the dot over a term indicates the time derivative, $A$ is a linear operator, $M(u_t)$ represents a (nonlinear) delay effect in the dynamics.

2010 Mathematics Subject Classification. Primary: 60H15, 35K90.
Key words and phrases. State-dependent delay, stochastic wave equation, pullback random
attractor, exponential attractor.

Partially supported by the Chinese NSF grant no. 1157112 and NCET-12-0204, and the Spanish
Ministerio de Economía y Competitividad project project MTM2015-63723-P.

*Died 23 April 2016*
and $\dot{W}$ is a trace-class Gaussian white noise. (All these objects will be specified later). A motivating model is the nonlinear plate equation of the form

$$\partial_{tt}u(t, x) + k \partial_t u(t, x) + \Delta^2 u(t, x) + au(t - \tau(u(t)), x) = \dot{W}, \ x \in \Omega, \ t > 0,$$

(2)

in a smooth bounded domain $\Omega \subset \mathbb{R}^2$ with appropriate boundary conditions on $\partial \Omega$. The delay mapping $\tau$ here is defined on segments of solutions over some interval $[-h, 0]$, while $k$ and $a$ are constants. We assume that the plate is placed on some foundation. In particular, the term $M(u_t) := au(t - \tau(u(t)), x)$ models the effect of a Winkler type foundation (see [45, 50]) with delay response. Our abstract model covers also the wave equation with state-dependent delay. It differs from the corresponding deterministic model in [15] by the inclusion of the noise term and also by the absence of the non-delayed nonlinearity term $F$, which introduces considerable technical difficulties and is omitted here so attention can be focussed on the effects of the noise term.

We note that the methods used here do not apply to first-order-in time equations, such as parabolic equations, when noise is added, since these methods require the solutions to be at least Lipschitz continuous in time.

The paper is organised as follows. Background material on random dynamical systems and random attractors is given in the next section 2. The well-posedness and generation of a random dynamical system is established in section 3. This uses a transformation involving an Ornstein-Uhlenbeck process for wave equations to convert the stochastic partial differential equation (1) into a random partial differential equation which can be investigated pathwise using deterministic methods. In sections 4-5 the quasi-stability of the generated random dynamical system is established, i.e., its dissipativity and asymptotic compactness, and then used in section 6 to show the existence and finite dimensionality of random and exponential attractors. Finally, several examples to specific systems are given in section 7. Some of the proofs closely follow their deterministic counterparts in [15] and are included here for the readers’ convenience to show how the additional noise terms are handled.

2. Random dynamical systems. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. We consider a measurable mapping $\theta : (\mathbb{R} \times \Omega, \mathcal{B}(\mathbb{R}) \otimes \mathcal{F}) \to (\Omega, \mathcal{F})$ satisfying the flow property $\theta(0, \cdot) = \text{id}_\Omega$, $\theta(t, \cdot) \circ \theta(\tau, \cdot) = \theta(t + \tau, \cdot)$ and, for simplicity, write $\theta(t, \omega) = \theta_t \omega$. We assume that $\theta$ preserves the measure $\mathbb{P}$, i.e., $\theta_t \mathbb{P} = \mathbb{P}$ for $t \in \mathbb{R}$. Then $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$ is called a metric dynamical system.

Let $H$ be a separable Banach space. A measurable mapping $\phi : (\mathbb{R}^+ \times \Omega \times H, \mathcal{B}(\mathbb{R}^+) \otimes \mathcal{F} \otimes \mathcal{B}(H)) \to (H, \mathcal{B}(H))$ satisfying the cocycle property

$$\phi(t + \tau, \omega, u) = \phi(t, \theta_t \omega, \phi(\tau, \omega, u)) \quad \text{for } t, \tau \geq 0, \ u \in H, \ \omega \in \Omega,$$

$$\phi(0, \omega, \cdot) = \text{id}_H \quad \text{for } \omega \in \Omega,$$

is called a random dynamical system (RDS). This RDS is called continuous if the mapping $u \mapsto \phi(t, \omega, u)$ is continuous for every $t > 0$ and $\omega \in \Omega$.

A mapping $\Omega \ni \omega \mapsto C(\omega) \neq \emptyset$ with closed values is called a random set (in $H$) if the mapping $\omega \ni \tau \mapsto \inf_{x \in C(\omega)} \|x - y\|_H$ is a random variable for any $y \in H$, see Castaing & Valadier [3], Chapter III.

A random variable $X \geq 0$ is called tempered if

$$\lim_{t \to \pm \infty} \frac{\log^+ X(\theta_t \omega)}{|t|} = 0.$$
This condition can be written (see [1]) in the form
\[
\forall \beta > 0, \forall \omega \in \Omega : \sup_{t \in \mathbb{R}} \{ X(\theta_t \omega) e^{\beta |t|} \} < +\infty.
\]
A random set \( C \) is called tempered if the random variable \( \omega \mapsto \sup_{y \in C(\omega)} \| y \| < \infty \) is tempered. We denote the family of all these tempered sets by \( D \).

**Definition 2.1.** A random set \( B \in D \) is called pullback absorbing for the random dynamical system \( \phi \) with respect to \( D \) if for every \( D \in D \) and \( \omega \in \Omega \) there exists a \( T = T_D(\omega) \geq 0 \) such that
\[
\phi(s, \theta_{-s} \omega, D(\theta_{-s} \omega)) \subset B(\omega), \; s \geq T.
\]
A random set \( B \) is called positively invariant if
\[
\phi(t, \omega, B(\theta_{-t} \omega)) \subset B(\omega) \quad \text{for all} \; \omega \in \Omega, \; t \geq 0.
\]
A random set \( C \in D \) is called pullback attracting for the random dynamical system \( \phi \) with respect to \( D \) if
\[
\lim_{t \to \infty} \text{dist}_H(\phi(t, \theta_{-t} \omega, D(\theta_{-t} \omega)), C(\omega)) = 0 \quad \text{for all} \; D \in D,
\]
where \( \text{dist}_H(X, Y) = \sup_{x \in X} \inf_{y \in Y} \| y - x \| \).

An RDS \( \phi \) is said to be asymptotically compact in \( D \) if there exists a compact pullback attracting random set.

**Definition 2.2.** A random set \( \mathfrak{A} = \{ \mathfrak{A}(\omega) : \omega \in \Omega \} \in D \) is called a random attractor if \( \mathfrak{A}(\omega) \) is compact for each \( \omega \in \Omega \), pullback attracting in the sense of Definition 2.1 and satisfies the invariance property:
\[
\phi(t, \omega, \mathfrak{A}(\omega)) = \mathfrak{A}(\theta_t \omega) \; \text{for} \; t \geq 0 \; \text{and} \; \omega \in \Omega.
\]

The existence of a random attractor is ensured by the conditions of the next theorem, a proof of which can be found, for instance, in Theorem 1.8.1 in [7].

**Theorem 2.3.** Let \( \phi \) be a continuous asymptotically compact RDS with a positive invariant compact pullback attracting random set \( B \in D \). Then there exists a random attractor \( \mathfrak{A} = \{ \mathfrak{A}(\omega) : \omega \in \Omega \} \) of \( \phi \) which is contained in \( B \) with
\[
\mathfrak{A}(\omega) = \bigcap_{n \in \mathbb{N}} \phi(nT, \theta_{-nT} \omega, B(\theta_{-nT} \omega)), \; \forall T > 0.
\]

3. **Well-posedness and generation of a random dynamical system.** The aim of this section is to show that problem (1) generates a random dynamical system in an appropriate linear phase space of \( C^{1} \) functions.

We assume that:

(A1) The operator \( A \) in (1) is a positive operator with a discrete spectrum in a separable Hilbert space \( H \) with domain \( D(A) \subset H \). In particular, there exists an orthonormal basis \( \{ e_k \} \) of \( H \) such that
\[
Ae_k = \mu_k e_k, \quad \text{with} \; 0 < \mu_1 \leq \mu_2 \leq \ldots, \lim_{k \to \infty} \mu_k = \infty.
\]
We define the spaces \( D(A^\alpha) \) for \( \alpha > 0 \) (see, e.g., [33]) and an operator \( \mathcal{A} \) on the space \( Y = D(A^{1/2}) \times H \), where \( V(t) = (v(t), \dot{v}(t)) \) by
\[
\mathcal{A}V = (-w, Av + kw), \quad \text{for} \; V = (v, w) \in D(A) \equiv D(A) \times D(A^{1/2})
\]
(3)
The operator \( \mathcal{A} \) generates an exponentially stable \( C_0 \)-semigroup \( e^{-At} \) in \( Y \), see, e.g., [6].
To describe the delay term $M$ we need the following standard notation from the theory of delay differential equations. In (1) and below, if $z$ is a continuous function from $\mathbb{R}$ into a metric space $Y$, then, as in [26, 53], $z(t) \equiv z(t+s)$, $s \in [-h, 0]$, denotes the element of $C([-h, 0]; Y)$. Here $h > 0$ presents the (maximal) delay time. In addition, $C_\alpha = C([-h, 0]; D(A^{\alpha}))$ denotes the Banach space with the norm:

$$
|v|_{C_\alpha} \equiv \sup\{\|A^\alpha v(s)\| : s \in [-h, 0]\}.
$$

Here and below, $\| \cdot \|$ is the norm of $H$, and $\langle \cdot, \cdot \rangle$ is the corresponding hermitian inner product. We also write $C = C_0$.

In our considerations an important role is played by the choice of a phase space. Following Remark 2.1 in Chueshov & Rezounenko [15], we will use

$$
\mathcal{W} = C([-h, 0]; D(A^{1/2})) \cap C^1([-h, 0]; H),
$$

endowed with the norm $|\varphi|_{\mathcal{W}} = |\varphi|_{C_{1/2}} + |\dot{\varphi}|_{C_0}$.

We make the following basic hypothesis on the delay term (see [15]):

**(M1)** *The nonlinear delay term $M : \mathcal{W} \mapsto H$ is locally Lipschitz in the sense that*

$$
\|M(\varphi^1) - M(\varphi^2)\| \leq C_0 \left[ |\varphi^1 - \varphi^2|_{C_{1/2}} + |\dot{\varphi}^1 - \dot{\varphi}^2|_{C_0} \right]
$$

*for every $\varphi^j \in \mathcal{W}$ with $|\varphi^j|_{\mathcal{W}} \leq \varrho$, $j = 1, 2$.*

3.1. **Ornstein-Uhlenbeck processes generated by wave equations.** Consider the linear SPDE

$$
\ddot{u}(t) + \nu Au(t) + k\dot{u}(t) = \tilde{W}(t),
$$

where $A$ is a positive linear operator on $H$ and $\tilde{W}$ is a trace class white noise on $H$ with covariance operator $K$ with trace $\text{tr}_H K < \infty$.

More precisely, let $W$ be a continuous two-sided trace-class Brownian motion on $H$ with covariance $K$ with respect to a metric dynamical system $(\Omega, F, \mathbb{P}, \theta)$. Due to the properties of $K$ there is a Hölder continuous version of this Brownian motion with time set $\mathbb{R}$, which we will also denote by $W$. Let $(\mathcal{F}_t)_{t \in \mathbb{R}}$ be the natural filtration of $W$ given by $\mathcal{F}_t = \sigma\{W(s) : s \leq t\}$.

The deterministic part in (5) generates the exponentially stable strongly continuous semigroup $e^{-At}$ in $Y = D(A^{1/2}) \times H$ given by $e^{-At}(u_0, u_1) = (u(t), \dot{u}(t))$, where $u(t)$ solves the Cauchy problem

$$
\ddot{u}(t) + \nu Au(t) + k\dot{u}(t) = 0, \quad u|_{t=0} = u_0, \quad \dot{u}|_{t=0} = u_1.
$$

We consider the Ornstein-Uhlenbeck process $t \mapsto Z(t, \omega)$ on $Y$ given by

$$
Z(t, \omega) = \begin{pmatrix} Z^1(t, \omega) \\ Z^2(t, \omega) \end{pmatrix} = \int_{-\infty}^{t} e^{-A(t-s)} d\left( \begin{pmatrix} 0 \\ dW(t, \omega) \end{pmatrix} \right)
$$

for all $t \in \mathbb{R}$. It follows that $Z^1(t+ \cdot) \in \mathcal{W} = C([-h, 0]; D(A^{1/2})) \cap C^1([-h, 0]; H)$ for all $t \in \mathbb{R}$. Moreover, $Z(t)$ is $(\mathcal{F}_t)_{t \in \mathbb{R}}$-adapted and, by Chueshov & Scheutzow [17],

**Lemma 3.1.** *There exists a random variable $Z : \Omega \mapsto Y$ such that $t \mapsto Z(t, \omega)$ is a Hölder–continuous version of (6). In particular, this mapping is continuous and $\|Z\|_Y$ is a tempered random variable.*
3.2. Random evolution equations. Introducing the new variable \( v(t) = u(t) - Z^2(t) \) we can transform (1) to the random evolution equation
\[
\dot{v}(t) + k\dot{v}(t) + Av(t) + M(v_t + Z^1_t) = 0
\]
with the initial data
\[
v(0) = v_0 \equiv u_0 - Z^1(0, \omega), \quad \dot{v}(0) = v_1 \equiv u_1 - Z^2(0, \omega).
\]
We can then rewrite equation (7) as the first order random differential equation
\[
\frac{d}{dt} V(t) + \mathcal{A}V(t) = \mathcal{N}(V_t + Z_t) \quad t > 0,
\]
in the space \( Y = D(A^{1/2}) \times H \), where \( V(t) = (v(t), \dot{v}(t)) \) and \( \mathcal{A} \) is the operator defined by (3), while the map \( \mathcal{N} \) is defined by
\[
\mathcal{N}(\Phi) = (-0, M(\varphi)) \quad \text{for} \quad \Phi = (\varphi, \dot{\varphi}), \varphi \in \mathcal{W}.
\]

As seen above, the operator \( \mathcal{A} \) generates exponentially stable \( C_0 \)-semigroup \( e^{-\mathcal{A}t} \) in \( Y \).

**Definition 3.2.** A mild solution of (1) and (8) on an interval \([0, T]\) is defined as a function
\[
v \in C([-h, T]; D(A^{1/2})) \cap C^1([-h, T]; H),
\]
such that \( v(s) = \varphi(s), s \in [-h, 0] \), and \( V(t) \equiv (v(t), \dot{v}(t)) \) satisfies
\[
V(t) = e^{-tA}V(0) + \int_0^t e^{-(t-s)A}\mathcal{N}(V_s + Z_s)ds, \quad t \in [0, T].
\]

**Proposition 3.3.** Let (A1) and (M1) hold. Then for any \( \varphi \in \mathcal{W} \) there exist \( T_\varphi > 0 \) and a unique mild solution \( V(t) \equiv (v(t), \dot{v}(t)) \) of (10) on the interval \([0, T_\varphi]\). The solution depends continuously on the initial function \( \varphi \in \mathcal{W} \).

**Proof.** The argument for the local existence, uniqueness and continuous dependence of a mild solution is standard (see, e.g., [49, Proposition 2.1 and Corollary 2.2], [24]) and uses the Banach fixed point theorem for a contraction mapping in the space \( C([-h, T]; D(A^{1/2})) \cap C^1([-h, T]; H) \) with appropriately small \( T \). Note that \( M \) is locally Lipschitz by assumption (M1).

To obtain a global well-posedness result we need additional hypotheses on \( M \) (see [15]).

**M2** The nonlinear delay term \( M : \mathcal{W} \to H \) satisfies the linear growth condition:
\[
\|M(\varphi)\| \leq M_0 + M_1 \left\{ \max_{s \in [-h, 0]} \|A^{1/2}\varphi(s)\| + \max_{s \in [-h, 0]} \|\dot{\varphi}(s)\| \right\}, \quad \forall \varphi \in \mathcal{W}, \quad (11)
\]
for some \( M_j \geq 0, j = 0, 1 \).

The main result of this section is the following assertion.

**Theorem 3.4** (Well-posedness). Let (A1), (M1), and (M2) be valid. Then for any \( \varphi \in \mathcal{W} \) there exists a unique global mild solution \( V(t) \equiv (v(t), \dot{v}(t)) \) of (10) on the interval \([0, +\infty)\). These mild solutions satisfy a pathwise energy equality of the form
\[
E_0(v(t), \dot{v}(t)) + k \int_0^t \|\dot{v}(s)\|^2 ds = E_0(v(0), \dot{v}(0)) - \int_0^t (M(v_s + Z^1_s), \dot{v}(s)) ds, \quad (12)
\]
where
\[
E_0(u, v) \equiv \frac{1}{2} \left( \|v\|^2 + \|A^{1/2}u\|^2 \right).
\]
Moreover, for any \( \varrho > 0 \) and \( T > 0 \) there exists \( C_{\varrho,T} \) such that

\[
\|A^{1/2}(v^1(t) - v^2(t))\| + \|\dot{v}^1(t) - \dot{v}^2(t)\| \leq C_{\varrho,T} |\varphi^1 - \varphi^2|_W, \quad t \in [0,T],
\]

for any mild solutions \( v^1(t) \) and \( v^2(t) \) with initial data \( \varphi^1 \) and \( \varphi^2 \) such that \( |\varphi^j|_W \leq \varrho \) for \( j = 1 \) and 2.

**Proof.** The local existence and uniqueness of mild solutions are given by Proposition 3.3. Let \( V = (v, \dot{v}) \) be a mild solution of (1) and (10) on the (maximal) semi-interval \([-h,T_\varphi]\) and

\[
f^v(t) \equiv M(v^t + Z^t) \in C([0,T_\varphi]; H).
\]

It is clear that we can consider \((v(t), \dot{v}(t))\) as a mild solution of the linear non-delayed problem

\[
\ddot{v}(t) + A v(t) + k \dot{v}(t) + f^v(t) = 0, \quad t \in [0,T_\varphi), \quad (v(0), \dot{v}(0)) = (\varphi(0), \dot{\varphi}(0)) \in Y, \quad (15)
\]

where \( Y = D(A^{1/2}) \times H \). Therefore (see, e.g., [6]), \( v(t) \) satisfies an energy relation of the form

\[
E_0(v(t), \dot{v}(t)) + k \int_0^t \| \dot{v}(s) \|^2 ds = E_0(v(0), \dot{v}(0)) - \int_0^t (f^v(s), \dot{v}(s)) ds, \quad t < T_\varphi. \quad (16)
\]

Using the structure of \( f^v \), after some calculations (firstly performed on smooth functions), we can show that

\[
\int_0^t (f^v(s), \dot{v}(s)) ds = \int_0^t (M(v_s + Z^s), \dot{v}(s)) ds.
\]

Therefore (16) yields (12) for every \( t < T_\varphi \).

Using (12) and (11) we thus obtain

\[
E_0(v(t), \dot{v}(t)) + k \int_0^t \| \dot{v}(s) \|^2 ds \leq E_0(v(0), \dot{v}(0)) + c_2 \int_0^t \left[ \max_{\tau \in [-h,0]} \| A^{1/2} v(s + \tau) \|^2 + \max_{\tau \in [-h,0]} \| \dot{v}(s + \tau) \|^2 \right] ds.
\]

\[
+ c_3 \int_0^t \left[ \max_{\tau \in [-h,0]} \| A^{1/2} Z^1(s + \tau) \|^2 + \max_{\tau \in [-h,0]} \| Z^2(s + \tau) \|^2 \right] ds.
\]

Now

\[
\max_{\tau \in [-h,0]} \| A^{1/2} v(s + \tau) \|^2 + \max_{\tau \in [-h,0]} \| \dot{v}(s + \tau) \|^2 \leq |\varphi|_W^2 + 2 \max_{\sigma \in [0,s]} E_0(v(\sigma), \dot{v}(\sigma))
\]

for every \( s \in [0,T_\varphi) \), so

\[
\max_{\sigma \in [0,t]} E_0(v(\sigma), \dot{v}(\sigma)) \leq c \left[ 1 + t + E_0(v(0), \dot{v}(0)) + t \cdot |\varphi|_W^2 + \int_0^t \hat{3}(s) ds \right]
\]

\[
+ c \int_0^t \max_{\sigma \in [0,s]} E_0(v(\sigma), \dot{v}(\sigma)) ds,
\]

where

\[
\hat{3}(t) = \max_{\tau \in [-h,0]} \| A^{1/2} Z^1(t + \tau) \|^2 + \max_{\tau \in [-h,0]} \| Z^2(t + \tau) \|^2.
\]

Note that \( \hat{3}(t) \) is a real valued tempered random variable for each \( t \in \mathbb{R} \), since it inherits its measurability and temperedness from that of the Ornstein Uhlenbeck process \( Z(t) \) in Lemma 3.1.
An application of Gronwall’s lemma to the function \( \max_{\sigma \in [0,t]} E_0(v(\sigma), \dot{v}(\sigma)) \) yields the (a priori) estimate
\[
\max_{\sigma \in [0,t]} E_0(v(\sigma), \dot{v}(\sigma)) \leq C \left( 1 + E_0(v(0), \dot{v}(0)) + |\phi_1|^2_{\mathcal{W}} + \int_0^{T} \frac{3}{3}(s) \, ds \right) \cdot e^{ct},
\]
\[0 \leq t < T_{\bar{\varphi}}.
\]
This allows us to extend the solution on the semi-axis \( \mathbb{R}^+ \) in the standard way.

To prove (14) we use the fact that the difference \( \delta(t) = v^1(t) - v^2(t) \) solves the problem in (15) with
\[
f^\delta(t) = M(v^1_t + Z^1_t) - M(v^2_t + Z^1_t),
\]
The Lipschitz property (M1) and standard estimates [49, Corollary 2.2] allow us to complete the proof of (14) and hence of Theorem 3.4.

\[\square\]

**Remark 3.5.** Note that the solutions above are stochastic processes, but for typographical simplicity the dependence on the sample path, i.e., \( \omega \), has been suppressed. Moreover, the solution operator \( S_t V(0) = V(t) \), where \( V(t) \) is the solution to problem (8), generates a random dynamical system in the space \( \bar{\mathcal{W}} := \{ \Phi = (\varphi, \dot{\varphi}) \mid \varphi \in \mathcal{W} \} \subset C([-h,0]; D(A^2) \times H) \). Its first component \( S_t v_0 := v(t) \) generates a random dynamical system on in the space \( \mathcal{W} \), which will be denoted by \( (S_t, \mathcal{W}) \) for brevity. In particular, the solution operator \( S_t \) is a cocycle on \( \mathcal{W} \) with respect the noise driving system \( \theta \).

We conclude this section with a discussion of the existence of smooth solutions to problems (7) and (10). In the following assertion we show that mild solutions become strong solutions under additional hypotheses.

**Corollary 3.6 (Smoothness).** Let the hypotheses of Theorem 3.4 hold with assumption (M1) in the following (stronger) form:
\[
\| M(\varphi^1) - M(\varphi^2) \| \leq C_{\rho} |\varphi^1 - \varphi^2|_{C_0}
\]
for every \( \varphi^j \in \mathcal{W} \) with \( |\varphi^j|_{\mathcal{W}} \leq \rho \) for \( j = 1, 2 \). If the initial function \( \varphi \) possesses the property
\[
\varphi(0) \in D(A), \ \dot{\varphi}(0) \in D(A^{1/2}),
\]
then for a.e. \( \omega \in \Omega \), the solution \( v(t) \) satisfies
\[
v(t) \in L_\infty(0,T; D(A)), \ \dot{v}(t) \in L_\infty(0,T; D(A^{1/2})), \ \ddot{v}(t) \in L_\infty(0,T; H)
\]
for every \( T > 0 \). Moreover,
\[
v(t) \in C(\mathbb{R}_+; D(A)), \ \dot{v}(t) \in C(\mathbb{R}_+; D(A^{1/2})), \ \ddot{v}(t) \in C(\mathbb{R}_+; H).
\]

**Proof.** Let \( v(t) \) be a solution. By Theorem 3.4 we have that
\[
\max_{t \in [-h,T]} \left( \| A^{1/2} v(t) \|^2 + \| \dot{v}(t) \|^2 \right) \leq R_T
\]
for some \( R_T \), which can be chosen large enough so that also
\[
\max_{t \in [-h,T]} \| Z^2(t) \|^2 \leq R_T
\]
Note that under condition (19) the function \( t \mapsto f(t) := M(v_t + Z^1_t) \) is Lipschitz on any interval \([0,T]\) with values in \( H \). Indeed, by (19) and the above bounds we have
\[
\| M(v_{t_1} + Z^1_{t_1}) - M(v_{t_2} + Z^1_{t_2}) \| \leq C_{2R_T} |v_{t_1} - v_{t_2}|_{C_0}.
\]
The nonlinear delay term \( \dot{H} \) provided discrete state-dependent delay of the mild solutions of problem (10), i.e., the solutions of problem (7). We focus on the long-time dynamics of the system (S4).

Asymptotic properties: Dissipativity. We now commence our investigation of the long-time dynamics of the system (S4). Asymptotic properties: Dissipativity. Hence, the conclusion in (21).

Property (22) follows from [11, Proposition 2.4.37].

Remark 4.1. The term \( M(u_t) \) satisfying (M3) can be written in the form

\[
M(u_t) = a \left( u(t) - \int_{t-\tau(u_t)}^t \dot{u}(s) \, ds \right) \equiv au(t) + M_*(u_t)
\]

for \( u_t \in W \). Obviously,

\[
\| M_*(u_t) \| \leq a \int_{t-h}^t \| \dot{u}(s) \| \, ds,
\]

so (M3) implies (M2). To guarantee (M1) we need to assume \( \tau \) is locally Lipschitz on \( W \):

\[
|\tau(\varphi^1) - \tau(\varphi^2)| \leq C_\varphi \left[ |\varphi^1 - \varphi^2|c_{1/2} + |\varphi^1 - \varphi^2|c_0 \right]
\]

for every \( \varphi^j \in W \) with \( |\varphi^j|_W \leq \varrho \) for \( j = 1, 2 \). Indeed, from (23) we have

\[
\| M(u^1_t) - M(u^2_t) \| \\
\leq a \| u^1(s - \tau(u^1_t)) - u^1(s - \tau(u^2_t)) \| + \| u^1(s - \tau(u^2_t)) - u^2(s - \tau(u^2_t)) \| \\
\leq a \eta |\tau(u^1_s) - \tau(u^2_s)| + a \max_{\theta \in [-h,0]} |u^1(s + \theta) - u^2(s + \theta)| \\
\leq a(h + \varrho c_\varphi) |u^1_s - u^2_s|_W
\]

for all \( u^j_t \in W \) with \( |u^j_t|_W \leq \varrho \) for \( j = 1, 2 \).

In the transformed random evolution equation (10) we have the term \( M(v_t + Z^1_t) \). The above derivations hold with \( u(t) = v(t) + Z^1(t) \). In particular, by the local Lipschitz property of the delay term, we have

\[
|\tau(\varphi^1 + Z^1_t) - \tau(\varphi^2 + Z^1_t)| \leq C_\varphi \left[ |\varphi^1 - \varphi^2|c_{1/2} + |\varphi^1 - \varphi^2|c_0 \right]
\]

provided \( \varphi^j \in W \) with \( |\varphi^j|_W \leq \varrho \) for \( j = 1, 2 \) and \( |Z^1_t|_W \leq \varrho \). Thus

\[
\| M(v^1_t + Z^1_t) - M(v^2_t + Z^1_t) \| \\\leq a \left( h + \varrho + |Z^2_t(\omega)|c_\varphi \right) C_\varphi |u^1_s - u^2_s|_W.
\]
Proposition 4.2. Let assumptions (A1), (M1) and (M3) be valid. Then for any \( k_0 \) there exists \( h_0 = h(k_0) > 0 \) such that for every \((k, h) \in [k_0, +\infty) \times (0, h_0)\) the system is pullback dissipative.

Proof. We use the Lyapunov method to obtain the result. The presence of the delay term \( M \) requires some modifications of the standard functional \( V \) that is usually used for second order systems (see, e.g., the proof of Theorem 3.10 [10, p.43-46]). In particular, we will use the functional

\[
\tilde{V}(t) \equiv E_0(v(t), \dot{v}(t)) + \frac{a}{2} \|v(t)\|^2 + \gamma(v(t), \dot{v}(t)) + \frac{\mu}{h} \int_0^h \left\{ \int_{t-s}^t \|\dot{v}(\xi)\|^2 d\xi \right\} ds,
\]

where \( a > 0 \) is the constant from (M3), \( E_0 \) is defined in (13) and the positive parameters \( \gamma \) and \( \mu \) will be chosen later.

The main idea behind the inclusion of an additional delay term in \( \tilde{V} \) is to find a compensator for \( M(v_t + Z^1_t) \). The compensator is determined by the structure of the mapping \( M \). This idea was already applied in [11, p.480] and [13] in the study of a flow-plate interaction model.

One can see that there is \( 0 < \gamma_0 < 1 \) such that

\[
\frac{1}{2} E_0(v(t), \dot{v}(t)) \leq \tilde{V}(t) \leq 2 E_0(v(t), \dot{v}(t)) + \frac{a}{2} \|v(t)\|^2 + \mu \int_0^h \|\dot{v}(t - \xi)\|^2 d\xi \tag{25}
\]

for every \( 0 < \gamma \leq \gamma_0 \).

Let us consider the time derivative of \( \tilde{V} \) along a solution. One can easily check that

\[
\frac{d}{dt}(v(t), \dot{v}(t)) = \|\dot{v}(t)\|^2 - k(v(t), \dot{v}(t)) - \|A^2 v(t)\|^2 - a\|v(t)\|^2 
- a(v(t), Z^1(t)) - (v(t), M_*(v_t + Z^1_t)). \tag{26}
\]

Combining (26) with the energy relation in (12) and using the estimate

\[
k(v, \dot{v}) \leq \frac{k^2}{a} \|\dot{v}\|^2 + \frac{a}{4} \|v\|^2,
\]

we obtain

\[
\frac{d}{dt} \tilde{V}(t) \leq - \left( \frac{k}{2} - \gamma \left( 1 + \frac{k^2}{a} \right) - \mu \right) \|\dot{v}(t)\|^2 + \frac{1}{2k} \|Z^1(t)\|^2 - (M_*(v_t + Z^1_t), \dot{v}(t)) 
- \gamma \left( a \|v(t)\|^2 + \|A^2 v(t)\|^2 + (v(t), M_*(v_t + Z^1_t)) \right) + \gamma a \|Z^1\|^2 
- \frac{\mu}{h} \int_0^h \|\dot{v}(t - \xi)\|^2 d\xi.
\]

We also have

\[
\|(M_*(v_t + Z^1_t))\| \leq a \int_{t-h}^t \|\dot{v}(s)\| ds + ha \|Z^1_t\|_C,
\]

so

\[
|(M_*(v_t + Z^1_t), \dot{v}(t))| \leq \frac{1}{8} k \|\dot{v}(t)\|^2 + \frac{2}{k} \|M_*(v_t + Z^1_t)\|^2 
\leq \frac{1}{8} k \|\dot{v}(t)\|^2 + \frac{4ha^2}{k} \int_{t-h}^t \|\dot{v}(s)\|^2 ds + \frac{4h^2a^2}{k} \|Z^1\|^2_2,
\]

since \( Z^2(t) = \dot{Z}^1(t) \).
In a similar way we also have
\[
\|(M^*(v_t + Z_t^1), v(t))\| \leq \frac{a}{2} \|v(t)\|^2 + \frac{1}{2a} \|M^*(v_t + Z_t^1)\|^2
\leq \frac{a}{2} \|v(t)\|^2 + ha \int_{t-h}^t \|\dot{v}(s)\|^2 \, ds + h^2 a \|Z_t^2\|^2.
\]

We obtain
\[
\frac{d}{dt} \tilde{V}(t) \leq -\left( \frac{3k}{8} - \gamma \left( 1 + \frac{k^2}{a} \right) - \mu \right) \|\dot{v}(t)\|^2
- \gamma \|A^\frac{1}{2} v(t)\|^2 + \left[ -\frac{\mu}{h} + \frac{4ha^2}{k} + \gamma ha \right] \int_0^h \|\dot{v}(t - \xi)\|^2 \, d\xi + 3(t),
\]
where
\[
3(t) := \frac{1}{2k} (1 + k\gamma a) \|Z^1(t)\|^2 + h^2 a \left( 1 + \frac{4a}{k} \right) \|Z_t^2\|^2.
\]

Thus, using (25), we arrive at the relation
\[
\frac{d}{dt} \tilde{V}(t) + \gamma a_0 \tilde{V}(t) \leq -\left( \frac{3k}{8} - \gamma \left( 2 + \frac{k^2}{a} \right) - \mu \right) \|\dot{v}(t)\|^2
+ \left[ -\frac{\mu}{h} + \mu \gamma a_0 + \frac{4ha^2}{k} + \gamma ha \right] \int_0^h \|\dot{v}(t - \xi)\|^2 \, d\xi + 3(t).
\]

Take \( \mu = \frac{\kappa}{4} \) and \( \gamma = \frac{\sigma k}{432\kappa} \), where \( 0 < \sigma < 1 \) is chosen such that \( \gamma \leq \gamma_0 \) for all \( k > 0 \) (the bound \( \gamma_0 \) arises in (25)). Assume also that \( h \) is such that
\[
-\frac{k}{4h} + \frac{\gamma k}{4} a_0 + \frac{4ha^2}{k} + \gamma ha \leq 0.
\]

Then (27) implies that
\[
\frac{d}{dt} \tilde{V}(t) + \gamma a_0 \tilde{V}(t) \leq 3(t).
\]

One can see there is a \( \sigma_0 = \sigma_0(k_0) \) such that \( \sigma_0 \leq \gamma k \leq \sigma/2 \) for all \( k \geq k_0 \). Therefore from (29) we obtain
\[
\tilde{V}(t) \leq \tilde{V}(0) e^{-\gamma a_0 t} + e^{-\gamma a_0 t} \int_0^t e^{\gamma a_0 s} 3(s) \, ds
\]
provided
\[
-\frac{k_0}{4h} + \frac{1}{8} a_0 + ah \left( \frac{4a}{k_0} + \frac{1}{2} \right) \leq 0.
\]

Here we have used (28) and the properties \( \gamma k < \frac{1}{2}, \gamma < \frac{1}{2} \), which follow from the choice of \( \gamma \). One can see that there exists \( \beta > 0 \) such that (31) holds when \( h \leq \beta k_0 \). Note that, from Lemma 3.1, we see that \( \|Z\|_{D(A^{1/2})} \) is a tempered random variable, hence, \( e^{-\gamma a_0 t} \int_0^t e^{\gamma a_0 s} 3(s) \, ds \) is bounded as \( t \to \infty \). Therefore, under this condition relation (30) implies the desired dissipativity property and completes the proof of Proposition 4.2.
5. **Asymptotic properties: Quasi-stability.** In this section we show that the system \((S_t, W)\) generated by the delay equation in (1) possesses some asymptotic compactness property which is called "quasi-stability" (see, e.g., [11] and [12]) and means that any two trajectories of the system are convergent modulo a compact term. As was already seen at the level of non-delayed systems (see, e.g., [10, 11, 12] and the references therein) and in the models here without delays in [15] this property usually leads to several important conclusions concerning long-time dynamics of the system.

Quasi-stability requires additional hypotheses concerning the system (see [15]). We assume that

\[ \textbf{(M4)} \text{ There exists } \delta > 0 \text{ such that the delay term } M \text{ satisfies a subcritical local Lipschitz property: for any } \varrho > 0 \text{ there exists } L(\varrho) > 0 \text{ such that} \]

\[ \|M(\varphi^1) - M(\varphi^2)\| \leq L(\varrho) \max_{\theta \in [-h,0]} \|A^{1/2-\delta}(\varphi^1(\theta) - \varphi^2(\theta))\| \]  

for any \( \varphi^j \) with \( \|\varphi^j\|_W \leq \varrho \) for \( j = 1 \) and \( 2 \).

As in Remark 4.1 one can see that (32) holds for \( M \) given by (23) if we assume that

\[ |\tau(\varphi^1) - \tau(\varphi^2)| \leq L_\varrho(\varrho) \max_{\theta \in [-h,0]} \|A^{1/2-\delta}(\varphi^1(\theta) - \varphi^2(\theta))\|. \]  

**Theorem 5.1** (Quasi-stability). Let assumptions \((A1), (M1), (M2)\) and \((M4)\) hold. Then there exists positive constants \( C_1(R), \bar{\lambda} \) and \( C_2(R) \) such that for any two solutions \( \dot{u}^j(t) \) with initial data \( \varphi^j \) and possessing the properties

\[ \|\dot{u}^j(t)\|^2 + \|A^2 \dot{u}^j(t)\|^2 \leq R^2 \quad \text{for all } t \geq -h, \quad j = 1, 2, \]

the following quasi-stability estimate

\[ \|\dot{u}^1(t) - \dot{u}^2(t)\|^2 + \|A^2 (u^1(t) - u^2(t))\|^2 \leq C_1(R) e^{-\bar{\lambda}t} \max_{\theta \in [0,t]} \|A^{1/2-\delta}(\varphi^1(\xi) - \varphi^2(\xi))\|^2 \]

holds with some \( \delta > 0 \).

**Proof.** Consider two solutions \( U^j = (u^j, \dot{u}^j) \), \( j = 1, 2 \), of (1). Then set \( V^j = (v^j, \dot{v}^j) \), where \( v^j(t) = u^j(t) - Z^1(t) \) are the solutions of equation (7) with the initial data

\[ v^j(0) = v^j_0 \equiv \varphi^j - Z^1(0, \omega), \quad \dot{v}^j(0) = v^j_1 \equiv \dot{\varphi}^j - Z^2(0, \omega), \quad j = 1, 2. \]

Using (10) and exponential stability of the semigroup \( e^{-At} \) in the space \( Y = D(A^{1/2}) \times H \) we have that

\[ \|U^1(t) - U^2(t)\|_Y \]

\[ = \|V^1(t) - V^2(t)\|_Y \]

\[ \leq e^{-\bar{\lambda}t} \|V^1(0) - V^2(0)\|_Y + \int_0^t e^{-\bar{\lambda}(t-s)} \|\mathcal{N}(V^1_s + Z^1_s) - \mathcal{N}(V^2_s + Z^1_s)\|_Y ds, \]

\[ = e^{-\bar{\lambda}t} \|U^1(0) - U^2(0)\|_Y + \int_0^t e^{-\bar{\lambda}(t-s)} \|\mathcal{N}(U^1_s) - \mathcal{N}(U^2_s)\|_Y ds, \quad t > 0, \]

with \( \bar{\lambda} > 0 \), where \( \mathcal{N} \) is given by (9). Since

\[ \|\mathcal{N}(U^1_s) - \mathcal{N}(U^2_s)\|_Y \leq \|M(u^1_s) - M(u^2_s)\|, \]
using properties (32) we obtain
\[ \| \mathcal{N}(U_1^t) - \mathcal{N}(U_2^t) \|_Y \leq C(R) \max_{\theta \in [-h,0]} \| A^{1/2 - \delta}(v^1(s + \theta) - v^2(s + \theta)) \| \]
for some \( \delta > 0 \). Thus (36) yields
\[ \| U^1(t) - U^2(t) \|_Y \leq e^{-\lambda t} \| U^1(0) - U^2(0) \|_Y + C(R)I(t, v^1 - v^2), \quad t > 0, \]
where
\[ I(t, z) = \int_0^t e^{-\tilde{\lambda}(t-s)} \max_{t \in [-h,0]} \| A^{1/2 - \delta} z(s + t) \| \, ds \] with \( z(s) = v^1(s) - v^2(s) \).

Now we split \( I(t, z) \) as \( I(t, z) = I^1(t, z) + I^2(t, z) \), where
\[ I^1(t, z) = \int_0^h e^{-\tilde{\lambda}(t-s)} \max_{t \in [-h,0]} \| A^{1/2 - \delta} z(s + t) \| \, ds \leq C_R, h |z(0)|_W \int_0^h e^{-\lambda(t-s)} \, ds \]
\[ = C_R, h |z(0)|_W \cdot e^{-\lambda t} (e^{\lambda h} - 1) \tilde{\lambda}^{-1} \]
and
\[ I^2(t, z) = \int_0^t e^{-\tilde{\lambda}(t-s)} \max_{t \in [-h,0]} \| A^{1/2 - \delta} z(s + t) \| \, ds \leq \int_0^t e^{-\lambda(t-s)} \max_{s \in [0,t]} \| A^{1/2 - \delta} z(\xi) \| \, ds = (1 - e^{-\lambda t}) \tilde{\lambda}^{-1} \cdot \max_{\xi \in [0,t]} \| A^{1/2 - \delta} z(\xi) \|. \]

Thus (37) yields the desired estimate in (35).

\[ \square \]

**Corollary 5.2.** Let conditions (A1) and (M3) with (33) hold and let \( B_0 = \{ B_0(\omega) : \omega \in \Omega \} \) be a forward invariant pullback absorbing random set for \( (S_t, W) \) such that \( B_0(\omega) \subset \{ \varphi \in W : |\varphi|_W \leq R(\omega) \} \). Then there exist \( C_1(R, \omega) > 0 \), \( C_2(R, \omega) > 0 \) and \( \lambda(\omega) > 0 \) such that (35) holds for any pair of solutions \( u^1(t) \) and \( u^2(t) \) starting in \( B_0 \).

Taking in (35) the maximum over the interval \([t - h, t]\) yields
\[ |S_t \varphi^1 - S_t \varphi^2|_W \leq C_1(R) h e^{\lambda h} e^{-\lambda t} |\varphi^1 - \varphi^2|_W + C_2(R) h \max_{s \in [0,t]} \mu_W(u^1_s - u^2_s), \quad t \geq h, \]
(38)

where \( \mu_W(\varphi) := \max_{\theta \in [-h,0]} \| A^{1/2 - \delta} \varphi(\theta) \| \) is a compact semi-norm on \( W \). (We recall that a semi-norm \( \tilde{n}(x) \) on a Banach space \( X \) is said to be compact iff for any bounded set \( B \subset X \) there exists a sequence \( \{ x^n \} \subset B \) such that \( \tilde{n}(x^n - x^k) \to 0 \) as \( m, k \to \infty \). The quasi-stability property in (38) has the structure which is different from the standard form (see, e.g., [10, 11, 12]) of quasi-stability inequalities for (non-delayed) second order in time equations. However, as in [15], the consequences in our case are the same as in the case of standard quasi-stable systems.

### 6. Pullback random and exponential attractors

In this section we use Proposition 4.1 and Theorem 5.1 to establish the existence of a pullback random attractor and study its properties. The proofs follow those in [15] for the corresponding deterministic problems, so only the essential features will be given here.

We recall that a **pullback random attractor** of the random dynamical system \((S_t, W)\) is defined as a bounded tempered random closed set \( \mathfrak{A} \subset W \) which is
invariant (i.e., $S_t(\omega)A(\omega) = A(\theta t \omega)$ for all $t > 0$) and uniformly attracts all other bounded tempered random sets $B$ in $\mathcal{W}$:

$$\lim_{t \to \infty} \sup_{y \in B(\theta t \omega)} \text{dist}_\mathcal{W}(S_t(\theta t \omega) y, A(\omega)) = 0$$

The main consequence of dissipativity and quasi-stability given by Proposition 4.2 and Theorem 5.1 is the following theorem.

**Theorem 6.1 (Pullback Random Attractor).** Let assumption (A1) hold and assume that the term $M(v_t)$ has form (23) with $\tau : \mathcal{W} \mapsto [0, h]$ possessing property (33). Then the random dynamical system $(S_t, \mathcal{W})$ generated by (7) has a pullback random attractor $\mathcal{A} = \{A(\omega) : \omega \in \Omega\}$. Moreover,

$$\check{v} \in L_\infty(\mathbb{R}, H), \ \check{v} \in L_\infty(\mathbb{R}, D(A^{1/2})), \ v \in L_\infty(\mathbb{R}, D(A))$$

and

$$\|\check{v}(t)\| + \|A^{1/2}\check{v}(t)\| + \|Av(t)\| \leq R_*(\omega), \ \forall t \in \mathbb{R},$$

for any entire trajectory $\{v(t) : t \in \mathbb{R}\}$ with $v_t(\omega) \in A(\omega)$ for all $t \in \mathbb{R}$ and $\omega \in \Omega$.

**Proof.** Since the system $(S_t, \mathcal{W})$ is dissipative (see Proposition 4.2), for the existence of a pullback random attractor we need to prove that $(S_t, \mathcal{W})$ is pullback asymptotically smooth. For this we can use a random generalization of the Ceron-Lopes type criteria (see, e.g., [27] or [10]), which states (see [10, p.19, Corollary 2.7]) that the quasi-stability estimate in (38) implies that $(S_t, \mathcal{W})$ is an asymptotically smooth dynamical system. Thus the existence of a compact pullback random attractor is established.

To determine the finite dimensionality of the attractor we apply the same idea as in [10] and [11], which originated from the Málek–Nečas method of “short” trajectories (see [34] and also [35]), but as in [15] we use a different choice of the space of “short” trajectories, which is motivated by the delay structure of the model and the choice of the phase space.

As in [10, 11] we rely on the abstract result [10, Theorem 2.15, p.23] on finite dimensionality of bounded closed sets in a Banach space which are invariant with respect to a Lipschitz mapping possessing some squeezing property. We consider the auxiliary space

$$\mathcal{W}(-h, T) \equiv C([-h, T]; D(A^{1/2})) \cap C^1([-h, T]; H), \ T > 0,$$

due to with the norm

$$[\varphi]_{\mathcal{W}(-h, T)} = \max_{s \in [-h, T]} \|A^{1/2} \varphi(s)\| + \max_{s \in [-h, T]} \|\dot{\varphi}(s)\|.$$  

We note that in the case $T = 0$ we have $\mathcal{W}(-h, 0) = \mathcal{W}$. Thus $\mathcal{W}(-h, T)$ is the space of extensions with the same smoothness of functions from $\mathcal{W}$ on the interval $[-h, T]$.

Let $\mathcal{B}$ be a random set in the phase space $\mathcal{W}$. We denote by $\mathcal{B}_T$ the set of functions $u \in \mathcal{W}(-h, T)$ which solve (1) with initial data $u_{t \in [-h, 0]} = \check{v}$ with $\check{v}(\omega) \in \mathcal{B}(\omega)$. We interpret $\mathcal{B}_T(\omega)$ as a set of “pieces” of trajectories starting from $\mathcal{B}(\omega)$. We also define the shift (along solutions to (1)) operator $\mathcal{R}_T(\omega) : \mathcal{B}_T(\omega) \mapsto \mathcal{W}(-h, T)$ by the formula

$$(\mathcal{R}_T u)(t) = u(T + t), \ t \in [-h, T],$$

where $u$ is the solution to (1) with initial data from $\mathcal{B}(\omega)$.

The following lemma states that the mapping $\mathcal{R}_T$ satisfies some contractive property modulo compact terms.
Lemma 6.2. Let $\mathcal{B} = \{ \mathcal{B}(\omega) : \omega \in \Omega \}$ be a forward invariant random set for the random dynamical system $(S_t, \mathcal{W})$ such that $\mathcal{B}(\omega) \in \{ \phi : |\phi| \leq R \}$ for some $R(\omega)$. Let $T > h$. Then $\mathcal{B}_T$ is forward invariant with respect to the shift operator $\mathcal{R}_T$ and

$$[\mathcal{R}_T \varphi^1 - \mathcal{R}_T \varphi^2]_{\mathcal{W}(-h, T)} \leq c_1(R)e^{-\tilde{\lambda}(T-h)} \left| \varphi^1 - \varphi^2 \right|_{\mathcal{W}(-h, T)} + c_2(R) \left[ n(\varphi^1 - \varphi^2) + n(\mathcal{R}_T \varphi^1 - \mathcal{R}_T \varphi^2) \right]$$

for every $\varphi^1, \varphi^2 \in \mathcal{B}_T$, where $n(\varphi) = \sup_{s \in [0, T]} \| A^{1/2} \varphi \|$ is a compact semi-norm on the space $\mathcal{W}(-h, T)$.

Proof. The invariance of $\mathcal{B}_T(\omega)$ is obvious due to the construction. The relation in (42) follows from Theorem 5.1. The compactness of the semi-norm $n$ is implied by the infinite dimensional version of Arzelà-Ascoli theorem, see, e.g., the Appendix in [11].

We choose $T > h$ such that $\eta_T = c_1(R)e^{-\tilde{\lambda}(T-h)} < 1$ and take $\mathcal{A} = \mathcal{W}$, where $\mathcal{A}$ is the random attractor. It is clear that the random set $\mathcal{A}_T$ is strictly invariant. Therefore we can apply [10, Theorem 2.15, p.23] to establish the finite dimensionality of the component subsets of $\mathcal{A}_T$ in $\mathcal{W}(-h, T)$. The final step is to consider the restriction mapping

$$r_h : \{ v(t), t \in [-h, T] \} \mapsto \{ v(t), t \in [-h, 0] \},$$

which is obviously Lipschitz continuous from $\mathcal{W}(-h, T)$ into $\mathcal{W}$. Since $r_h \mathcal{A}_T = \mathcal{A}$ and Lipschitz mappings do not increase fractal dimension of a set, we conclude that

$$\dim_{\mathcal{F}}^\mathcal{W}(\mathcal{A}(\omega)) \leq \dim_{\mathcal{F}}^{\mathcal{W}(-h, T)} \mathcal{A}_T(\omega) < \infty.$$

To prove the regularity properties in (39) and (40) we can use Theorem 5.1 and the same idea as in [10, 11], see also [12]. Indeed, let $\gamma = \{ v(t) : t \in \mathbb{R} \}$ be a full trajectory of the system, i.e., $(S_t v_s)(\rho) = v(t + s + \rho)$ for $\rho \in [-h, 0]$. Assume that $v_s(\omega) \in \mathcal{A}(\theta_t(\omega))$ for all $t \in \mathbb{R}$. Consider the difference of this trajectory and its small shift $\gamma_\varepsilon = \{ v(t + \varepsilon) : t \in \mathbb{R} \}$ and apply the inequality in (35) with starting point at $s \in \mathbb{R}:

$$\| \dot{v}(t + \varepsilon) - \dot{v}(t) \|^2 + \| A^{1/2} (v(t + \varepsilon) - v(t)) \|^2 \leq C_1(R)e^{-\tilde{\lambda}(t-s)} |v_{s+\varepsilon} - v_s|_{\mathcal{W}}^2 + C_2(R) \max_{\xi \in [s, t]} \| A^{1/2-\delta} (v(\xi + \varepsilon) - v(\xi)) \|^2.$$

Since $v_s(\omega) \in \mathcal{A}(\theta_s(\omega))$ for all $s \in \mathbb{R}$, in the limit $s \to -\infty$ we obtain that

$$\| \dot{v}(t + \varepsilon) - \dot{v}(t) \|^2 + \| A^{1/2} (v(t + \varepsilon) - v(t)) \|^2 \leq C_2(R) \sup_{\xi \in [-\infty, t]} \| A^{1/2-\delta} (v(\xi + \varepsilon) - v(\xi)) \|^2.$$

Now in the same way as in [10, p. 102, 103] or in [11, p. 386, 387] we can conclude that

$$\frac{1}{\varepsilon^2} \left[ \| \dot{v}(t + \varepsilon) - \dot{v}(t) \|^2 + \| A^{1/2} (v(t + \varepsilon) - v(t)) \|^2 \right]$$

is uniformly bounded in $\varepsilon \in (0, 1)$. This implies (passing with the limit $\varepsilon \to 0$) that

$$\| \ddot{v}(t) \|^2 + \| A^{1/2} \dot{v}(t) \|^2 \leq C_R.$$

Now using equation (1) we conclude that $\| Av(t) \|^2 \leq C_R$. This gives (39) and (40). The final statement follows from Corollary 3.6. This completes the proof of Theorem 6.1. □
Now we present a result on the existence of exponential attractors. We recall the following definition.

**Definition 6.3** (cf. [22]). A compact set $\mathcal{A}_{\exp} \subset \mathcal{W}$ is said to be (generalized) pullback random exponential attractor for the dynamical system $(S_t, \mathcal{W})$ if $\mathcal{A}_{\exp}$ is a positively invariant random set whose fractal dimension is finite (in some extended space $\hat{\mathcal{W}} \supset \mathcal{W}$) and for every bounded set $D \subset \mathcal{W}$ there exist positive constants $C_D = C(|D|_{\hat{\mathcal{W}}})$ (possibly random) and $\gamma$ such that

$$\text{dist}_\mathcal{W} (S_t(\omega) D, \mathcal{A}_{\exp}(\omega)) \equiv \sup_{x \in D} \text{dist}_\mathcal{W} (S_t(\omega)x, \mathcal{A}_{\exp}(\omega)) \leq C_D \cdot e^{-\gamma t}, \quad t \geq 0.$$  

(43)

This concept has been introduced for deterministic systems in [22] in the case when $\hat{\mathcal{W}}$ and $\mathcal{W}$ are the same. See the recent survey article [37, 46].

Using the quasi-stability estimate and ideas presented in [10, 11] we can construct exponential attractors for the system considered.

**Theorem 6.4.** Let the hypotheses of Theorem 6.1 hold. Then the dynamical system $(S_t, \mathcal{W})$ possesses a (generalized) exponential attractor whose dimension is finite in the space $\hat{\mathcal{W}} \equiv C([-h, 0]; \mathcal{W})(\hat{\mathcal{W}}_s)$ for each $s > 0$, where $H_{-s}, s > 0$, denotes the closure of $H$ with respect to the norm $\|A^{-s} \|$.

**Proof.** Let $\mathcal{B} = \{B(\omega) : \omega \in \Omega\}$ be a forward invariant bounded absorbing set for $(S_t, \mathcal{W})$. We apply Lemma 6.2 to obtain the (discrete) quasi-stability property for the shift mapping $\mathcal{R}_t$ defined in (41) on $\mathcal{B}_T$. We choose $T > h$ in (42) such that $\eta_T = c_1 \rho e^{-\lambda(t-h)} < 1$ and apply [10, Corollary 2.23] which gives us that the mapping $\mathcal{R}_T$, possesses a random exponential attractor with component sets $\mathcal{A}_T(\omega)$. Next, using (1) we can see that $\|\hat{v}(t)\|_{-2} < C_R$ for all $t \in \mathbb{R}^+$. This allows us to show that $S_t \varphi$ is a Hölder continuous in $t$ in the space $\hat{\mathcal{W}}$, i.e.,

$$|S_{t_1} \varphi - S_{t_2} \varphi|_{\hat{\mathcal{W}}} \leq C_{\mathcal{B}} |t_1 - t_2| \gamma, \quad t_1, t_2 \in \mathbb{R}^+, \quad y \in \mathcal{B},$$  

(44)

for some positive $\gamma > 0$. Now we consider the restriction map $r_h$ (see above) and the sets $r_h \mathcal{A}_T = \mathcal{A} \subset \mathcal{W}, \mathcal{A}_{\exp} \equiv \bigcup \{S_t \mathcal{A} : t \in [0, T]\} \subset \mathcal{W}$. It is clear that $\mathcal{A}_{\exp}$ is forward invariant. Since $r_h$ is Lipschitz from $\mathcal{W}(-h,T)$ into $\mathcal{W}$, $\mathcal{A}$ is finite dimensional. Therefore, the property in (44) implies that $\mathcal{A}_{\exp}$ has a finite fractal dimension in $\hat{\mathcal{W}}$. As in [10, p.123] we can see that $\mathcal{A}_{\exp}$ is an exponentially random attracting set for $(S_t, \mathcal{W})$. This completes the proof of Theorem 6.4. □

7. **Examples.** In this section we discuss several possible applications of the above results. Our main applications are related to nonlinear plate models, but also include wave equations and finite dimensional Itô stochastic differential equations.

7.1. **Plate models.** Let $\Omega \subset \mathbb{R}^2$ be a bounded smooth domain. In the space $H = L_2(\Omega)$ we consider the following problem

$$\partial_t u(t, x) + k \partial_t u(t, x) + \Delta^2 u(t, x) + a u(t - \tau[u_t], x) = \hat{W}, \quad x \in \Omega, \quad t > 0, \quad (45a)$$

$$u = \frac{\partial u}{\partial n} = 0 \text{ on } \partial \Omega, \quad u(\theta) = \varphi(\theta) \text{ for } \theta \in [-h, 0].$$  

(45b)

We assume that $\tau$ is a continuous mapping from $C([-h, 0]; H^2_0(\Omega)) \cap C^1(-h, 0; L_2(\Omega))$ into the interval $[0, h]$ and that $\hat{W}$ is a trace class white noise on $L_2(\Omega)$ with covariance operator $K$ with trace $\text{tr}_{L_2(\Omega)} K < \infty$. 

STOCHASTIC EVOLUTION EQUATIONS WITH STATE-DEPENDENT DELAY 1005
The model in (45) can be written in the abstract form (1) with $A = \Delta^2$ defined on the domain $D(A) = H^4 \cap H_0^2(\Omega)$. Here and below $H^s(\Omega)$ is the Sobolev space of the order $s$ and $H_0^s(\Omega)$ is the closure of $C_c^\infty(\Omega)$ in $H^s(\Omega)$. In this case we have $D(A^s) = H_0^s(\Omega)$ for $0 \leq s \leq 1/2$ with $s \neq 1/8, 3/8$.

As the simplest example of delay terms satisfying all hypotheses in (M1)–(M4) we can consider

$$\tau[u_t] = g(Q[u_t]),$$

(46)

where $g$ is a smooth mapping from $\mathbb{R}$ into $[0, h]$ and

$$Q[u_t] = \sum_{i=1}^N c_i u(t - \sigma_i, a_i).$$

Here $c_i \in \mathbb{R}$, $\sigma_i \in [0, h]$, $a_i \in \Omega$ are arbitrary constants. We could also consider the term $Q$ with the Stieltjes integral over delay interval $[-h, 0]$ instead of the sum. Another possibility is to consider combination of averages like

$$Q[u_t] = \sum_{i=1}^N \int_{\Omega} u(t - \sigma_i, x)\xi_i(x)dx,$$

(47)

where $\sigma_i \in [0, h]$ and $\{\xi_i\}$ are arbitrary functions from $L_2(\Omega)$.

We could also consider linear combinations of these $Q$’s and also their powers and products. The corresponding calculations are simple and related to the fact that for every $s > 1/4$ the space $D(A^s)$ is an algebra belonging to $C(\Omega)$.

### 7.2. Wave model.

Let $\Omega \subset \mathbb{R}^n$, $n = 2, 3$, be a bounded domain with a sufficiently smooth boundary $\Gamma$ and let $\nu$ denote the exterior normal on $\Gamma$. We consider the following wave equation

$$\partial_{tt} u - \Delta u + k\partial_t u + u(t - \tau[u_t]) = \dot{W} \quad \text{in} \quad Q = [0, \infty) \times \Omega$$

subject to boundary condition of either Dirichlet type

$$u = 0 \quad \text{on} \quad \Sigma \equiv [0, \infty) \times \Gamma,$$

(48)

or Robin type

$$\partial_\nu u + u = 0 \quad \text{on} \quad \Sigma.$$  

(49)

The initial conditions are given by $u(\theta) = \varphi(\theta)$, $\theta \in [-h, 0]$. In this case $H = L_2(\Omega)$ and $A$ is $-\Delta$ with either the Dirichlet (48) or Robin (49) boundary conditions, so $D(A^{1/2})$ is either $H_0^1(\Omega)$ or $H^1(\Omega)$ here.

We assume that $k$ is a positive parameter and for the delay term $u(t - \tau[u_t])$ we can assume that, as in the plate models above, $\tau[u_t]$ has the form (46) with $Q[u_t]$ given by (47). Moreover, instead of the averaging we can consider an arbitrary family of linear functionals on $H^{1-\delta}(\Omega)$ for some $\delta > 0$, i.e., we can take

$$Q[u_t] = \sum_{i=1}^N c_i l_i[u(t - \sigma_i)],$$

where $c_i \in \mathbb{R}$, $\sigma_i \in [0, h]$ and $l_i \in [H^{1-\delta}(\Omega)]'$ are arbitrary elements.
7.3. \textbf{Itō stochastic differential equations.} The results above can be also applied in the finite dimensional case when $H = \mathbb{R}^n$, $A$ is a symmetric $n \times n$ matrix and the nonlinear mapping $M : C([-h, 0]; \mathbb{R}^n) \to \mathbb{R}^n$ obeys appropriate requirements. In particular, the stochastic evolution equation (1) reduces to the noisy differential equation in $\mathbb{R}^n$,

$$\ddot{x}(t) + k\dot{x}(t) + Ax(t) + M(x_t) = \dot{W}, \quad t > 0,$$

which can be rewritten as a $2n$-dimensional Itō stochastic differential equation. The space of initial states becomes $W = C^1([-h, 0]; \mathbb{R}^n)$ (c.f. (4)) and hence possesses a linear structure.

In the deterministic ODE, i.e., with no noise term, the approach in this paper contrasts with the solution manifold method suggested in [51] (see also [28]). It does not assume any nonlinear compatibility conditions and provides us with a well-posedness result in a linear phase space. In addition, both approaches produce the same class of solutions after some time. See [15].

REFERENCES

[1] L. Arnold, \emph{Random Dynamical Systems}, Springer-Verlag, Berlin, 1998.
[2] A. V. Babin and M. I. Vishik, \emph{Attractors of Evolutionary Equations}, Amsterdam, North-Holland, 1992.
[3] C. Castaing and M. Valadier, \emph{Convex Analysis and Measurable Multifunctions}, Lecture Notes in Mathematics, vol. 580, Springer-Verlag, Berlin, 1977.
[4] L. Boutet de Monvel, I. Chueshov and A. Rezounenko, Long-time behaviour of strong solutions of retarded nonlinear PDEs, \emph{Communications in Partial Differential Equations}, 22 (1997), 1453–1474.
[5] I. Chueshov, On a system of equations with delay that arises in aero-elasticity (Russian), \emph{Teor. Funktsii Funktsional. Anal. i Prilozhen.}, 54 (1990), 123–130; translation in \emph{J. Soviet Math.}, 58 (1992), 385–390.
[6] I. Chueshov, \emph{Introduction to the Theory of Infinite-Dimensional Dissipative Systems}, Acta, Kharkov, 1999, English translation, 2002; \url{http://www.emis.de/monographs/Chueshov/}
[7] I. Chueshov, \emph{Monotone Random Systems: Theory and Applications}, Lecture Notes Math. 1779, Springer, Berlin 2002.
[8] I. Chueshov, \emph{Dynamics of Quasi-Stable Dissipative Systems}, Springer-Verlag, Berlin 2015.
[9] I. Chueshov and I. Lasiecka, Attractors for second-order evolution equations with a nonlinear damping, \emph{J. Dyn. Diff. Eqns.}, 16 (2004), 469–512.
[10] I. Chueshov and I. Lasiecka, Long-time behavior of second order evolution equations with nonlinear damping, \emph{Memoirs Amer. Math. Soc.}, 195 (2008), viii+183 pp.
[11] I. Chueshov and I. Lasiecka, \emph{Von Karman Evolution Equations. Well-posedness and Long-time Dynamics}, Springer-Verlag, New York, 2010.
[12] I. Chueshov and I. Lasiecka, \emph{Well-posedness and long time behavior in nonlinear dissipative hyperbolic-like evolutions with critical exponents}, In: \emph{Nonlinear Hyperbolic PDEs, Dispersive and Transport Equations (HCDTE Lecture Notes, Part I)}, AIMS on Applied Mathematics, G. Alberti et al. (Eds.) AIMS, Springfield, 6 (2013), 1–96.
[13] I. Chueshov, I. Lasiecka and J. T. Webster, Attractors for delayed, non-rotational von Karman plates with applications to flow-structure interactions without any damping, \emph{Communications in Partial Differential Equations}, 39 (2014), 1965–1997.
[14] I. Chueshov and A. V. Rezounenko, Global attractors for a class of retarded quasilinear partial differential equations, \emph{C.R. Acad. Sci. Paris, Ser.I}, 321 (1995), 607-612; (detailed version: \emph{Math.Physics, Analysis, Geometry}, 2 (1995), 363–383).
[15] I. Chueshov and A. Rezounenko, Dynamics of second order in time evolution equations with state-dependent delay, \emph{Nonlinear Analysis TMA}, 123/124 (2015), 126–149.
[16] I. Chueshov and A. Rezounenko, Finite-dimensional global attractors for parabolic nonlinear equations with state-dependent delay, \emph{Communications on Pure and Applied Analysis}, 14 (2015), 1685–1704.
[17] I. Chueshov and M. Scheutzow, Inertial manifolds and forms for stochastically perturbed retarded semilinear parabolic equations, \emph{J. Dyn. Diff. Eqns.}, 13 (2001), 355–380.
[18] M. Conti, E. M. Marchini and V. Pata, Semilinear wave equations of viscoelasticity in the minimal state framework, *Discrete Contin. Dyn. Syst.*, **27** (2010), 1535–1552.

[19] K. L. Cooke and Z. Grossman, Discrete delay, distributed delay and stability switches, *J. Math. Anal. Appl.* **86** (1982), 592–627.

[20] V. Danese, P. G. Geredeli and V. Pata, Exponential attractors for abstract equations with memory and applications to viscoelasticity, *Discrete Contin. Dyn. Syst.*, **35** (2015), 2881–2904, arXiv:1410.5051.

[21] O. Diekmann, S. van Gils, S. Verduyn Lunel and H.-O. Walther, *Delay Equations: Functional, Complex, and Nonlinear Analysis*, Springer-Verlag, New York, 1995.

[22] A. Eden, C. Foias, B. Nicolaenko and R. Temam, *Exponential Attractors for Dissipative Evolution Equations*, Research in Appl. Math. 37, Masson, Paris, 1994.

[23] P. Fabrie, C. Galusinski, A. Miranville and S. Zelik, Uniform exponential attractors for a singularly perturbed damped wave equation, *Discrete Cont. Dyn. Systems*, **10** (2004), 211–238.

[24] W. E. Fitzgibbon, Semilinear functional differential equations in Banach space, *J. Differential Equations*, **29** (1978), 1–14.

[25] M. J. Garrido-Atienza and J. Real, Existence and uniqueness of solutions for delay evolution equations of second order in time, *J. Math. Anal. Appl.*, **283** (2003), 582–609.

[26] J. Hale, *Theory of Functional Differential Equations*, Springer-Verlag, Berlin, 1977.

[27] J. Hale, *Asymptotic Behavior of Dissipative Systems*, Amer. Math. Soc., Providence, RI, 1988.

[28] F. Hartung, T. Krisztnin, H.-O. Walther and J. Wu, Functional differential equations with state-dependent delays: Theory and applications. In: Canada, A., Drabek., P. and A. Fonda (Eds.) *Handbook of Differential Equations, Ordinary Differential Equations*, vol. 3, Elsevier Science B. V., North Holland, 2006, 435–545.

[29] A. G. Kartsatos and L. P. Markov, An $L^2$-approach to second-order nonlinear functional evolutions involving $m$-accretive operators in Banach spaces, *Differential Integral Equations*, **14** (2001), 833–866.

[30] T. Krisztin and O. Arino, The two-dimensional attractor of a differential equation with state-dependent delay, *J. Dynam. Diff. Eqns.*, **13** (2001), 453–522.

[31] K. Kunisch and W. Schappacher, Necessary conditions for partial differential equations with delay to generate $C_0$-semigroups, *J. Differential Equations*, **50** (1983), 49–79.

[32] J. L. Lions, *Quelques Méthodes de Résolution des Problèmes aux Limites Non Linéaires*, Dunod, Paris, 1969.

[33] J. L. Lions, E. Magenes, *Problèmes aux Limites Non Homogènes et Applications*, Dunon, Paris, 1968.

[34] J. Málek and J. Nečas, A finite dimensional attractor for three dimensional flow of incompressible fluids, *J. Differential Equations*, **127** (1996), 498–518.

[35] J. Málek and D. Pražák, Large time behavior via the method of $I$-trajectories, *J. Differential Equations*, **181** (2002), 243–279.

[36] J. Mallet-Paret, R. D. Nussbaum and P. Paraskevopoulos, Periodic solutions for functional-differential equations with multiple state-dependent time lags, *Topol. Methods Nonlinear Anal.*, **3** (1994), 101–162.

[37] V. Pata, Exponential stability in linear viscoelasticity with almost flat memory kernels, *Commun. Pure Appl. Anal.*, **9** (2010), 721–730.

[38] A. V. Rezounenko, Partial differential equations with discrete and distributed state-dependent delays, *J. Math. Anal. Appl.*, **326** (2007), 1031–1045.

[39] A. V. Rezounenko, Differential equations with discrete state-dependent delay: Uniqueness and well-posedness in the space of continuous functions, *Nonlinear Analysis: Theory, Methods and Applications*, **70** (2009), 3978–3986.

[40] A. V. Rezounenko, Non-linear partial differential equations with discrete state-dependent delays in a metric space, *Nonlinear Analysis: Theory, Methods and Applications*, **73** (2010), 1707–1714.

[41] A. V. Rezounenko, A condition on delay for differential equations with discrete state-dependent delay, *Journal of Mathematical Analysis and Applications*, **385** (2012), 506–516.
[43] A. V. Rezounenko and P. Zagalak, Non-local PDEs with discrete state-dependent delays: Well-posedness in a metric space, *Discrete Cont. Dyn. Syst.*, **33** (2013), 819–835.

[44] W. M. Ruess, Existence of solutions to partial differential equations with delay. In: *Theory and Applications of Nonlinear Operators of Accretive Monotone type*, Lecture Notes Pure Appl. Math., **178** (1996), 259–288.

[45] A. P. S. Selvadurai, *Elastic Analysis of Soil Foundation Interaction*, Elsevier, Amsterdam, 1979.

[46] A. Shirikyan and S. Zelik, Exponential attractors for random dynamical systems and applications, *Stoch PDE: Anal Comp*, **1** (2013), 241–281.

[47] R. E. Showalter, *Monotone Operators in Banach space and Nonlinear Partial Differential Equations*, AMS, Mathematical Surveys and Monographs, vol. 49, 1997.

[48] R. Temam, *Infinite Dimensional Dynamical Systems in Mechanics and Physics*, Springer-Verlag, Berlin, 1988.

[49] C. C. Travis and G. F. Webb, Existence and stability for partial functional differential equations, *Transactions of AMS*, **200** (1974), 395–418.

[50] V. Z. Vlasov and U. N. Leontiev, *Beams, Plates, and Shells on Elastic Foundation*, Israel Program for Scientific Translations, Jerusalem, 1966 (translated from Russian).

[51] H.-O. Walther, The solution manifold and $C^1$-smoothness for differential equations with state-dependent delay, *J. Differential Equations*, **195** (2003), 46–65.

[52] H.-O. Walther, On Poisson’s state-dependent delay, *Discrete Contin. Dyn. Syst.*, **33** (2013), 365–379.

[53] J. Wu, *Theory and Applications of Partial Functional Differential Equations*, Springer-Verlag, New York, 1996.

Received September 2016; revised January 2017.

E-mail address: kloeden@math.uni-frankfurt.de
E-mail address: yangmeih@hust.edu.cn