The matter Lagrangian of an ideal fluid.

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We show that the matter Lagrangian of an ideal fluid equals (up to a sign –depending on its definition and on the chosen signature of the metric) the total energy density of the fluid, i.e. rest energy density plus internal energy density.

I. INTRODUCTION

Since the pioneering work by Taub [1] and by Fock & Kenmer in their 1955 monograph about gravitation (with English translation versions in Fock and Kenmer [2] and Fock [3]), it has become important to calculate the value of the matter Lagrangian used in relativistic theories of gravity, particularly when dealing with extended theories of gravity with curvature-matter couplings as first explored by Goenner [4], later by Allemandi et al. [5] and Bertolami et al. [6] and the constructions of Harko and Lobo [7], Harko and Lobo [8] deriving in in couplings of the trace of the energy momentum tensor with the curvature of space-time [9, 10].

The value of the matter Lagrangian has been known to have values (up to a sign depending on the sign definition of the matter action and thus the energy-momentum tensor and the chosen signature of the metric) of [e.g 3, 11–14] e or [e e 15–17] p, where e is the total energy matter density of a barotropic fluid, i.e. mass energy density plus internal energy density and p the pressure of the fluid. For the case of the Hilbert action, both values for the matter Lagrangian yield the same Einstein field equations, but as explained by Harko & Lobo [18] the choice p = 0 yields a null matter Lagrangian and this appears to be an inconsistency for extended theories of gravity where non-null matter Lagrangians are to be selected.

Using well known techniques of variational calculus on the definition of the matter Lagrangian, we show in the present work that for the case of an ideal fluid the matter Lagrangian equals (up to a sign) the total energy density of the fluid which consists of its rest mass energy density plus its internal energy density. In Section II we define the energy-momentum tensor as a function of the variations of the matter Lagrangian. In Section III we derive in in couplings of the trace of the energy momentum tensor with the curvature of space-time and with this, equation (2) can be written as:

\[ T_{\alpha\beta} = \pm \frac{2}{\sqrt{-g}} \frac{\delta \left( \sqrt{-g} L_{\text{matt}} \right)}{\delta g^{\alpha\beta}}. \]  

Using the fact that [19]:

\[ \delta g = g g^{\alpha\beta} \delta g_{\alpha\beta} = -g g_{\alpha\beta} \delta g^{\alpha\beta}, \]  

it follows that equation (2) can be written as:

\[ T_{\alpha\beta} = \pm 2 \frac{\delta L_{\text{matt}}}{\delta g^{\alpha\beta}} \mp g_{\alpha\beta} L_{\text{matt}}. \]  

Note that since \( \delta (g^{\alpha\beta} g_{\alpha\beta}) = 0 \) then

\[ \delta g_{\alpha\beta} = -g_{\alpha\mu} g_{\beta\nu} \delta g^{\mu\nu}, \]

and with this, equation (2) can be written as:

\[ T_{\alpha\beta} = \pm \frac{2}{\sqrt{-g}} \frac{\delta \left( \sqrt{-g} L_{\text{matt}} \right)}{\delta g_{\alpha\beta}}. \]
and so:

\[ T^{\alpha\beta} = \pm 2 \frac{\delta \mathcal{L}_{\text{matt}}}{\delta \sigma_{\alpha\beta}} + g^{\alpha\beta} \mathcal{L}_{\text{matt}}. \]  

\[ \text{(7)} \]

**III. IDEAL FLUIDS**

The appropriate form of the first law of thermodynamics for relativistic fluids can be written as [see e.g. 20]:

\[ d \left( \frac{e}{\rho} \right) = T d \left( \frac{\sigma}{\rho} \right) - pd \left( \frac{1}{\rho} \right), \]

\[ \text{(8)} \]

where the total energy density \( e = \rho c^2 + \xi \) contains the rest energy density \( \rho c^2 \) and a pure internal energy density term \( \xi \). The pressure is represented by \( p \) and the fluid or gas (baryonic) mass density is \( \rho \). The entropy density of the fluid in the previous relation is written as \( \sigma \).

An ideal fluid is that for which no heat is exchanged between its components and so, the flow moves adiabatically, i.e. \( d (\sigma/\rho) = 0 \) and so, the first law of thermodynamics for an ideal fluid is given by:

\[ d \left( \frac{e}{\rho} \right) = -pd \left( \frac{1}{\rho} \right), \]

\[ \text{(9)} \]

or:

\[ \frac{de}{e + p} = \frac{d\rho}{\rho}. \]

\[ \text{(10)} \]

For the case of an ideal fluid, the energy-momentum tensor takes the form [see e.g. 19, 20]:

\[ T_{\alpha\beta} = (e + p) u_\alpha u_\beta - pg_{\alpha\beta}, \]

\[ \text{(11)} \]

where the four velocity \( u_\alpha \) satisfies the relation:

\[ u_\alpha u^\alpha = 1. \]

\[ \text{(12)} \]

**IV. MATTER LAGRANGIAN FOR AN IDEAL FLUID**

If there are no (baryonic) mass sources, the fluid satisfies a continuity equation given by [20]:

\[ \nabla_\alpha (\rho u^\alpha) = 0, \]

\[ \text{(13)} \]

where \( \nabla_\alpha \) is the covariant derivative. The previous equation can be written as:

\[ \frac{1}{\sqrt{-g}} \frac{\partial (\sqrt{-g} \rho u^\alpha)}{\partial x^\alpha} = 0, \]

\[ \text{(14)} \]

and so

\[ \delta \left( \sqrt{-g} \rho u^\alpha \right) = 0. \]

\[ \text{(15)} \]

In other words:

\[ u^\alpha \delta \rho = -\frac{\rho}{\sqrt{-g}} u^\alpha \delta \sqrt{-g} - \rho \delta u^\alpha. \]

\[ \text{(16)} \]

Using the fact that \( \delta (g_{\alpha\beta} u^\alpha u^\beta) = 0 \) and with the help of equation (5) it follows that:

\[ 2g_{\alpha\beta} \delta u^\beta = -u^\beta \delta g_{\alpha\beta} = u^\beta g_{\alpha\mu} g_{\beta\nu} \delta g^{\mu\nu} = u_\nu g_{\alpha\beta} \delta g^{\beta\nu}, \]

and so:

\[ 2\delta u^\alpha = u_\beta \delta g^{\alpha\beta}. \]

\[ \text{(17)} \]

Substitution of this last result in equation (10) and with the help of relation 3 yields:

\[ \delta \rho = \frac{1}{2} \rho (g_{\alpha\beta} - u_\alpha u_\beta) \delta g^{\alpha\beta}, \]

\[ \text{(18)} \]

which means that:

\[ \frac{\delta}{\delta g^{\alpha\beta}} \frac{\delta \rho}{\delta g^{\alpha\beta}} = \frac{\delta \rho}{\delta g^{\alpha\beta}} \frac{d}{d\rho} = \frac{1}{2} \rho (g_{\alpha\beta} - u_\alpha u_\beta) \frac{d}{d\rho}, \]

\[ \text{(19)} \]

for an adiabatic flow. Using this relation, equation (4)—or equivalently relation 7—can be written as:

\[ T_{\alpha\beta} = \left( \pm \rho \frac{d\mathcal{L}_{\text{matt}}}{d\rho} \mp \mathcal{L}_{\text{matt}} \right) g_{\alpha\beta} \mp \rho u_\alpha u_\beta \frac{d\mathcal{L}_{\text{matt}}}{d\rho}, \]

\[ \text{(20)} \]

for an ideal fluid.

From equations (11) and (20) it follows that:

\[ (e + p) u_\alpha u_\beta - pg_{\alpha\beta} = \left( \pm \rho \frac{d\mathcal{L}_{\text{matt}}}{d\rho} \mp \mathcal{L}_{\text{matt}} \right) g_{\alpha\beta} \mp \rho u_\alpha u_\beta \frac{d\mathcal{L}_{\text{matt}}}{d\rho}, \]

\[ \text{(21)} \]

In order to find a differential equation for the matter Lagrangian \( \mathcal{L}_{\text{matt}} \) we proceed in the following ways:

(i) Equate terms with \( u_\alpha u_\beta \) and the ones with \( g_{\alpha\beta} \) in equation (21), to obtain:

\[ e + p = \mp \rho \frac{d\mathcal{L}_{\text{matt}}}{d\rho}, \]

\[ \text{(22)} \]

\[ -p = \pm \rho \frac{d\mathcal{L}_{\text{matt}}}{d\rho} \mp \mathcal{L}_{\text{matt}}. \]

\[ \text{(23)} \]

(ii) Multiply equation (21) by the metric tensor \( g^{\alpha\beta} \) to obtain:

\[ e - 3p = \pm 3\rho \frac{d\mathcal{L}_{\text{matt}}}{d\rho} \mp 3\mathcal{L}_{\text{matt}}. \]

\[ \text{(24)} \]
(iii) Multiply equation (21) by the four velocity $u^\alpha$ to obtain:

$$L_{\text{matt}} = \mp e. \quad (25)$$

(iv) Multiply equation (21) by the projection tensor $P^\alpha{}_\lambda := u^\alpha u_\lambda - \delta^\alpha{}_\lambda$, which is orthogonal to the four velocity $u^\alpha$, to obtain exactly equation (23). With the value of $L_{\text{matt}} = \mp e$ equations (22)-(24) become:

$$\frac{dc}{d\rho} = e + p, \quad (26)$$

which is exactly the differential equation satisfied by the first law of thermodynamics for an ideal fluid as expressed in equation (10).

V. FINAL REMARKS

In this work we have shown that the value of the matter Lagrangian

$$L_{\text{matt}} = \mp e, \quad (27)$$

is valid for an ideal fluid with an energy-momentum tensor given by $T_{\alpha\beta} = (e + p) u_\alpha u_\beta - pg_{\alpha\beta}$.

For the case of dust, i.e. $p = 0$ and so $e = \rho c^2$, it follows that:

$$e = \rho c^2, \quad \text{and} \quad L_{\text{matt}} = \mp \rho c^2. \quad (28)$$

The total energy density of a barotropic fluid, one for which the pressure $p(\rho)$ is a function of the mass density $\rho$ only, is given by [see e.g. (21)]:

$$e = \rho c^2 + p \int p(\rho) d\rho / \rho^2, \quad (29)$$

and so:

$$L_{\text{matt}} = \mp \rho c^2 \mp p / \kappa - 1. \quad (30)$$

which coincides with the results obtained by Minazzoli and Harko [13]. For the case of a polytropic fluid for which

$$p \propto \rho^\kappa, \quad (31)$$

with constant polytropic index $\kappa$, it follows that [21]

$$e = \rho c^2 + \frac{p}{\kappa - 1}, \quad (32)$$

and so:

$$L_{\text{matt}} = \mp \rho c^2 \mp \frac{p}{\kappa - 1}. \quad (33)$$

The energy density equation (32) becomes the Bondi-Wheeler equation of state [21] for the case in which the pressure of the fluid is much greater than its rest mass density, i.e. $p \gg \rho c^2$ and so:

$$e = \frac{p}{(\kappa - 1)}. \quad (34)$$

For this Bondi-Wheeler case the matter Lagrangian is given by:

$$L_{\text{matt}} = \mp \frac{p}{\kappa - 1}. \quad (35)$$

In cosmological applications the previous two relations are valid for the cases of radiation with $\kappa = 4/3$, a monoatomic gas with $\kappa = 5/3$ and for cosmological vacuum with $\kappa = 0$. The case of cosmological dust is represented by a matter Lagrangian given by equation (28).

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\footnote{For the choice of a (−, +, +, +) signature, since the right-hand side of equation (12) equals −1, and the energy-momentum tensor in equation (11) is given by:

$$T_{\alpha\beta} = (e + p) u_\alpha u_\beta + pg_{\alpha\beta},$$

it follows that equation (18) turns into:

$$\delta \rho = \frac{1}{12} (g_{\alpha\beta} + u_\alpha u_\beta) \delta g^{\alpha\beta}.$$}

With all this, the ± and ± signs that appear in equations (22)-(24) are inverted so that:

$$L_{\text{matt}} = \pm e, \quad (36)$$
which leaves equation (26) unchanged. This means that for this metric signature, the ± and ∓ signs on equations (28)-(30) and (33)-(35) are also inverted.