Attribute-Driven Community Search

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ABSTRACT

Recently, community search over graphs has attracted significant attention and many algorithms have been developed for finding dense subgraphs from large graphs that contain given query nodes. In applications such as analysis of protein-protein interaction (PPI) networks, citation graphs, and collaboration networks, nodes tend to have attributes. Unfortunately, most previously developed community search algorithms ignore these attributes and result in communities with poor cohesion w.r.t. their node attributes. In this paper, we study the problem of attribute-driven community search, that is, given an undirected graph $G$ where nodes are associated with attributes, and an input query $Q$ consisting of nodes $V_q$ and attributes $W_q$, find the communities containing $V_q$, in which most community members are densely inter-connected and have similar attributes.

We formulate our problem of finding attributed truss communities (ATC), as finding all connected and close $k$-truss subgraphs containing $V_q$, that are locally maximal and have the largest attribute relevance score among such subgraphs. We design a novel attribute relevance score function and establish its desirable properties. The problem is shown to be NP-hard. However, we develop an efficient greedy algorithmic framework, which finds a maximal $k$-truss containing $V_q$, and then iteratively removes the nodes with the least popular attributes and shrinks the graph so as to satisfy community constraints. We also build an elegant index to maintain the known $k$-truss structure and attribute information, and propose efficient query processing algorithms. Extensive experiments on large real-world networks with ground-truth communities shows the efficiency and effectiveness of our proposed methods.

1. INTRODUCTION

Graphs have emerged as a powerful model for representing different types of data. For instance, unstructured data (e.g., text documents), semi-structured data (e.g., XML databases) and structured data (e.g., relational databases) can all be modeled as graphs, where the vertices(nodes) are respectively documents, elements, and tuples, and the edges can respectively be hyperlinks, parent-child relationships, and primary-foreign-key relationships [28]. In these graphs, communities naturally exist as groups of nodes that are densely interconnected. Finding communities in large networks has found extensive applications in protein-protein interaction networks, sensor/communication networks, and collaboration networks. Consequently, community detection, i.e., finding all communities in a given network, serves as a global network-wide analysis tool, and has been extensively studied in the literature. Specifically, various definitions of communities based on different notions of dense subgraphs have been proposed and studied: quasi-clique [10], densest subgraph [37], $k$-core [35, 29, 11, 3], and $k$-truss [20, 22]. More recently, a related but different problem called community search has generated considerable interest. It is motivated by the need to make answers more meaningful and personalized to the user [30, 20]. For a given set of query nodes, community search seeks to find the communities containing the query nodes.

In the aforementioned applications, the entities modeled by the network nodes often have properties which are important for making sense of communities. E.g., authors in collaboration networks have areas of expertise; proteins have molecular functions, biological processes, and cellular components as properties. Such networks can be modeled using attributed graphs [39] where attributes associated with nodes capture their properties. E.g., Figure 1 shows an example of a collaboration network. The nodes $q_1, q_2, \ldots$ represent authors. Node attributes (e.g., DB, ML) represent authors’ topics of expertise. In finding communities (with or without query nodes) over attributed graphs, we might want to ensure that the nodes in the discovered communities have homogeneous attributes. For instance, it has been found that communities with homogeneous attributes among nodes more accurately predict protein complexes [19]. Furthermore, we might wish to query, not just using query nodes, but also using query attributes. To illustrate, consider searching for communities containing the nodes $\{q_1, q_2\}$. Based on structure alone, the subgraph $H$ shown in Figure 1 is a good candidate answer for this search, as it is densely connected. However, attributes of the authors in this community are not homogeneous: the community is a mix of authors working in different topics – DB, DM, IR, and ML. Previous community search methods include those based on $k$-core [35, 29, 11], $k$-truss [22], and 1.0-quasi-$k$-clique-$\ell$-adjacent community [10]. A $k$-core [29] is
attribute IR, is a more homogeneous community than subgraph will report previous community search methods can be hard to interpret owing to H. A recent work [14] proposed an attribute community model. And one node without ML it is more densely connected than the chain collaboration structure among the authors in the community. Thus, \{1\}

graphs, which has been extensively studied \[1, 18, 17, 5, 23, 12\]. A

some sparser ones and a careful balance has to be struck between subgraphs may be less homogeneous in their node attributes than could model authors’ attributes also as nodes and directly connect a subgraph in which each vertex has at least k neighbors within the subgraph. A k-truss \[22\] is a subgraph in which each edge is contained in at least (k - 2) triangles within the subgraph. The 1.0-quasi-k-clique-\ell-adjacent community model \[10\] allows two k-cliques overlapping in \ell vertices to be merged into one community. In Figure 1 for \(k = 4\) and \(\ell = 3\), all these community models will report H as the top answer and are thus unsatisfactory. The subgraph \(H_2\) obtained from \(H\) by removing node \(v_7\) with unique attribute IR, is a more homogeneous community than \(H\) and is just as densely connected (see Figure 2(b)). Intuitively, it is a better answer than \(H\). Thus, in general, communities found by most previous community search methods can be hard to interpret owing to the heterogeneity of node attributes. Furthermore, the communities reported could contain smaller dense subgraphs with more homogeneity in attributes, which are missed by most previous methods. A recent work [14] proposed an attribute community model. A detailed comparison of [14] with our model can be found in Section 3. Consider now querying the graph of Figure 1 with query nodes \(\{q_1, q_2\}\) and attributes (i.e., keywords) \{DB, DM\}. We would expect this search to return subgraph \(H_2\) (Figure 2(b)). On the other hand, for the same query nodes, if we search with attribute \{DB\} (resp., \{DM\}), we expect the subgraph \(H_1\) (resp., \(H_3\)) to be returned as the answer (Figure 2(a) & (c)). Both \(H_1\) and \(H_3\) are dense subgraphs where all authors share a common topic (DB or DM).

Given a query consisting of nodes and attributes (keywords), one may wonder whether we can filter out nodes not having those attributes and then run a conventional community search method on the filtered graph. To see how well this may work, consider querying the graph in Figure 1 with query node \(q_1\) and query attribute ML. Filtering out nodes without attribute ML and applying community search yields the chain consisting of \(v_{10}, q_1, v_8\), which is not densely connected. On the other hand, the subgraph induced by \(\{q_1, v_8, v_9, v_{10}\}\) is a 3-truss in Figure 2(d). Even though it includes one node without ML it is more densely connected than the chain above and is a better answer than the chain as it brings out denser collaboration structure among the authors in the community. Thus, a simple filtering based approach will not work. As some denser subgraphs may be less homogeneous in their node attributes than some sparser ones and a careful balance has to be struck between density and attribute homogeneity.

Another topic related to our problem is keyword search over graphs, which has been extensively studied [1][15][17][5][23][12]. A natural question is whether we can model the information suitably and leverage keyword search to find the right communities. We could model authors’ attributes also as nodes and directly connect them to the author nodes and query the resulting graph with the union of the author id’s and the keywords. Figure 3 illustrates this for a small subgraph of Figure 1 and a query. Keyword search finds answers corresponding to trees or subgraphs with minimum communication cost that connect the input keywords/nodes, where the communication cost is based on diameter, query distance, weight of spanning tree or steiner tree. On this graph, if we search for the query node \(q_1\) and attribute DB, we will get the single edge connecting \(q_1\) and DB as the answer as this is the subgraph with minimum communication cost connecting these two nodes. Clearly, this is unsatisfactory as a community.

In sum, attributed graphs present novel opportunities for community search by combining dense structure of subgraphs with the level of homogeneity of node attributes in the subgraph. Most previous work in community search fails to produce satisfactory answers over attributed graphs, while keyword search based techniques do not find dense subgraphs. The main problem we study in this paper is finding top-r communities from attributed graphs, given a community search query consisting of query nodes and query attributes. This raises the following major challenges. Firstly, how should we combine dense connectedness with the distribution of attributes over the community nodes? We need a community definition that promotes dense structure as well as attribute homogeneity. However, there can be tension between these goals: as illustrated in the example above, some denser subgraphs may be less homogeneous in their node attributes than some sparser ones. Secondly, the definition should capture the intuition that the more input attributes that are covered by a community, the better the community. Finally, we need to find the answer communities from large input graphs in an efficient manner.

To tackle these challenges, we propose an attributed truss community (ATC) model. Given a query \(Q = (V_Q, W_Q)\) consisting of a set of query nodes \(V_Q\) and a set of query attributes \(W_Q\), a good community \(H\) must be a dense subgraph which contains all query nodes and attributes \(W_Q\) must be contained in numerous nodes of the community. The more nodes with attribute \(w \in W_Q\), the more importance to \(w\) commonly accorded by the community members. Additionally, the nodes must share as many attributes as possible. Notice that these two conditions are not necessarily equivalent. Capturing these intuitions, we define an attribute score function that strikes a balance between attribute homogeneity and coverage. Moreover, as a qualifying cohesive and tight structure, we define a novel concept of \((k, d)\)-truss for modeling a densely connected community. A \((k, d)\)-truss is a connected \(k\)-truss containing all query nodes, where each node has a distance no more than \(d\) from every query node. This inherits many nice structural properties, such as bounded diameter, \(k\)-edge connectivity, and hierarchical structure. Thus, based on attribute score function and \((k, d)\)-truss, we propose a novel community model as attributed truss community (ATC), which is a \((k, d)\)-truss with the maximum attribute score. In this paper, we make the following contributions.

- We motivate the problem of attributed community search, and identify the desiderata of a good attributed community (Section 2).
We propose a novel dense and tight subgraph, \((k,d)\)-truss, and design an attribute score function satisfying the desiderata set out above. Based on this, we propose a community model called attributed truss community (ATC), and formulate the problem of attributed community search as finding ATC (Section 4).

We analyze the structural properties of ATC and show that it is non-monotone, non-submodular and non-supermodular, which signal huge computational challenges. We also formally prove that the problem is NP-hard (Section 5).

We develop a greedy algorithmic framework to find an ATC containing given query nodes w.r.t. given query attributes. It first finds a maximal \((k,d)\)-truss, and then iteratively removes nodes with smallest attribute score contribution. For improving the efficiency and quality, we design a revised attribute marginal gain function and a bulk removal strategy for cutting down the number of iterations (Section 6).

For further improving efficiency, we explore the local neighborhood of query nodes to search an ATC. This algorithm first generates a Steiner tree connecting all query nodes, and then expands the tree to a dense subgraph with the insertion of carefully selected nodes, that have highly correlated attributes and densely connected structure (Section 7).

We conduct extensive experiments on 7 real datasets, and show that our attribute community model can efficiently and effectively find ground-truth communities and social circles over real-world networks, significantly outperforming previous work (Section 8).

We discuss related work in Section 3 and conclude the paper with a summary in Section 9.

2. PRELIMINARIES AND DESIDERATA

2.1 Preliminaries

We consider an undirected, unweighted simple graph \(G = (V,E)\) with \(n = |V(G)|\) vertices and \(m = |E(G)|\) edges. We denote the set of neighbors of a vertex \(v\) by \(N(v)\), and the degree of \(v\) by \(d(v) = |N(v)|\). We let \(d_{\max} = \max_{v \in V} d(v)\) denote the maximum vertex degree in \(G\). W.l.o.g. we assume that the graphs we consider are connected. Note that this implies that \(m \geq n-1\). We consider attributed graphs and denote the set of all attributes in a graph by \(A\). Each node \(v \in V\) contains a set of zero or more attributes, denoted by \(\text{attr}(v) \subseteq A\). The multiset union of attributes of all nodes in \(G\) is denoted \(\text{attr}(V)\). Note that \(\text{attr}(V) = \sum_{v \in V} |\text{attr}(v)|\). We use \(V_w \subseteq V\) to denote the set of nodes having attribute \(w\), i.e., \(V_w = \{v \in V \mid w \in \text{attr}(v)\}\).

2.2 Desiderata of a good community

Given a query \(Q = (V_q,W_q)\) with a set of query nodes \(V_q \subseteq V\) and a set of query attributes \(W_q\), the attributed community search (ACS) problem is to find a subgraph \(H \subseteq G\) containing all query nodes \(V_q\), where the vertices are densely inter-connected, cover as many query attributes \(W_q\) as possible and share numerous attributes. In addition, the communication cost of \(H\) should be low. We call the query \(Q = (V_q,W_q)\) an ACS query. Before formalizing the problem, we first identify the commonly accepted desiderata of a good attributed community.

Criteria of a good attributed community: Given a graph \(G(V,E)\) and an ACS query \(Q = (V_q,W_q)\), an attributed community is a connected subgraph \(H = (V(H),E(H)) \subseteq G\) that satisfies:

1. (Participation) \(H\) contains all query nodes as \(V_q \subseteq V(H)\);
2. (Cohesiveness) A cohesiveness function \(\text{coh}(H)\) that measures the cohesive structure of \(H\) is high.
3. (Attribute Coverage and Correlation) An attribute score function \(f(H,W_q)\) that measures the coverage and correlation of query attributes in vertices of \(H\) is high.
4. (Communication Cost) A communication cost function \(\text{com}(H)\) that measures the distance of vertices in \(H\) is low.

The participation condition is straightforward. The cohesiveness condition is also straightforward since communities are supposed to be densely connected subgraphs. One can use any notion of dense subgraph previously studied, such as \(k\)-core, \(k\)-truss, etc. The third condition captures the intuition that more query attributes covered by \(H\), the higher \(f(H,W_q)\); also more attributes shared by vertices of \(H\), the higher \(f(H,W_q)\). This motivates designing functions \(f(\ldots)\) with this property. Finally, keeping the communication cost low helps avoid irrelevant vertices in a community. This is related to the so-called free rider effect, studied in [23, 37]. Intuitively, the closer the community nodes to query nodes, subject to all other conditions, the more relevant they are likely to be to the query. Notice that sometimes a node that does not contain query attributes may still act as a “bridge” between other nodes and help improve the density. A general remark is that other than the first condition, for conditions 2–4, we may either optimize a suitable metric or constrain that the metric be above a threshold (below a threshold for Condition 4). We formalize this intuition in Section 4 and give a precise definition of an attributed community and formally state the main problem studied in the paper.

3. RELATED WORK

Work related to this paper can be classified into community search, keyword search, team formation, and community detection in attributed graphs. Table 1 shows a detailed comparison of representative works on these topics.

| Method | Topic | Participation | Attribute Function | Cohesiveness | Communication Cost |
|--------|-------|---------------|-------------------|--------------|-------------------|
| KS     | X     | √             | X                 | √            | √                 |
| TF     | X     | √             | X                 | √            | √                 |
| CS     | √     | √             | √                 | √            | √                 |
| ACS    | X     | √             | X                 | √            | X                 |

Table 1: A comparison of representative works on keyword search (KS), team formation (TF), community search (CS) and attributed community search (ACS).

1. (Participation) A (Participation) H contains all query nodes as \(V_q \subseteq V(H)\);
2. (Cohesiveness) A cohesiveness function \(\text{coh}(H)\) that measures the cohesive structure of \(H\) is high.
3. (Attribute Coverage and Correlation) An attribute score function \(f(H,W_q)\) that measures the coverage and correlation of query attributes in vertices of \(H\) is high.
4. (Communication Cost) A communication cost function \(\text{com}(H)\) that measures the distance of vertices in \(H\) is low.
In this section, we develop a notion of attributed community by formalizing the desiderata discussed in Section 2. We focus our discussion on conditions 2–4.

4.1 \((k, d)\)-truss

In the following, we introduce a novel definition of dense and tight substructure called \((k, d)\)-truss by paying attention to cohesiveness and communication cost.

**Cohesiveness.** While a number of definitions for dense subgraphs have been proposed over the years, we adopt the \(k\)-truss model, proposed by Cohen [9], which has gained popularity and has been found to satisfy nice properties.

A subgraph \(H \subseteq G\) is a \(k\)-core, if every vertex in \(H\) has degree at least \(k\). A triangle in \(G\) is a cycle of length 3. We denote a triangle involving vertices \(u, v, w \in V\) as \(\triangle_{uvw}\). The support of an edge \(e(u, v) \in E\) in \(G\), denoted \(sup_G(e)\), is the number of triangles containing \(e\), i.e., \(sup_G(e) = |\{\triangle_{uvw} : w \in V\}|\). When the context is obvious, we drop the subscript and denote the support as \(sup(e)\). Since the definition of \(k\)-truss [9][50] allows a \(k\)-truss to be disconnected, we define a connected \(k\)-truss below.

**Definition 1** \((\text{Connected K-Truss})\). Given a graph \(G\) and an integer \(k\), a connected \(k\)-truss is a connected subgraph \(H \subseteq G\), such that \(\forall e \in E(H), sup_H(e) \geq (k - 2)\).

Intuitively, a connected \(k\)-truss is a connected subgraph in which each connection (edge) \((u, v)\) is "endorsed" by \(k - 2\) common neighbors of \(u\) and \(v\) [9]. A connected \(k\)-truss with a large value of \(k\) signifies strong inner-connections between members of the subgraph. In a \(k\)-truss, each node has degree at least \(k - 1\), i.e., it is a \((k - 1)\)-core, and a connected \(k\)-truss is also a \((k - 1)\)-edge-connected, i.e., it remains connected if fewer than \((k - 1)\) edges are removed [4].

**Example 1**. Consider the graph \(G\) (Figure 1). The edge \(e(v_1, v_2)\) is contained in three triangles \(\triangle_{v_1v_2v_3}, \triangle_{v_2v_3v_4}\), and \(\triangle_{v_3v_4v_1}\) in the graph, thus its support is \(sup_G(e) = 3\). Consider the subgraph \(H_3\) of \(G\) (Figure 2(c)). Every edge of \(H_3\) has support \(\geq 2\), thus \(H_3\) is a 4-truss. Note that even though the edge \(e(v_1, v_2)\) has support 3, there exists no 3-truss in the graph \(G\) in Figure 1.

**Communication Cost.** For two nodes \(u, v \in G\), let \(dist_G(u, v)\) denote the length of the shortest path between \(u\) and \(v\) in \(G\), where \(dist_G(u, v) = +\infty\) if \(u\) and \(v\) are not connected. The diameter of a graph \(G\) is the maximum length of a shortest path in \(G\), i.e., \(\text{diam}(G) = \max_{u, v \in V(G)} \{dist_G(u, v)\}\). We make use of the notion of graph query distance in the following.

**Definition 2** \((\text{Query Distance})\). Given a graph \(G\) and query nodes \(V_q \subseteq V\), the vertex query distance of vertex \(v \in V\) is the maximum length of a shortest path from \(v\) to a query node \(q \in V_q\) in \(G\), i.e., \(dist_G(v, V_q) = \max_{q \in V_q} dist_G(v, q)\). Given a subgraph \(H \subseteq G\) and \(V_q \subseteq V(H)\), the query graph distance of \(H\) is defined as \(dist_H(H, V_q) = \max_{u \in H} \{dist_H(u, V_q)\}\).

Given a subgraph \(H \subseteq G\) and \(V_q \subseteq V(H)\), the query distance \(dist_H(H, V_q)\) measures the communication cost between the members of \(H\) and the query nodes. A good community should have a low communication cost with small \(dist_H(H, V_q)\).

For the graph \(G\) in Figure 1 and query nodes \(V_q = \{q_1, q_2\}\), the vertex query distance of \(v_2\) is \(dist_G(v_2, V_q) = \max_{q \in V_q} \{dist_G(v_2, q)\} = 2\). Consider the subgraph \(H_1\) in Figure 2(a). Then query graph distance of \(H_1\) is \(dist_{H_1}(H_1, V_q) = dist_{H_1}(q_1, q_2) = 2\). The diameter of \(H_1\) is \(\text{diam}(H_1) = 2\).
((k, d)-truss. We adapt the notions of k-truss and query distance, and propose a new notion of (k, d)-truss capturing dense cohesiveness and low communication cost.

**Definition 3** ((k, d)-truss). Given a graph \( H \), query nodes \( V_q \), and numbers \( k \) and \( d \), we say that \( H \) is a (k, d)-truss if \( H \) is a connected k-truss containing \( V_q \) and \( \text{dist}_H(H, V_q) \leq d \).

By definition, the cohesiveness of a (k, d)-truss increases with \( k \) and its proximity to query nodes increases with decreasing \( d \). For instance, the community \( H_1 \) in Figure 2(a) for \( V_q = \{q_1, q_2\} \) is a (k, d)-truss with \( k = 4 \) and \( d = 2 \).

### 4.2 Attribute Score Function

We first identify key properties that should be obeyed by a good attribute score function for a community. Let \( f(H, W_q) \) denote the attribute score of community \( H \) w.r.t. query attributes \( W_q \). We say that a node \( v \) of \( H \) covers an attribute \( w \in W_q \), if \( v \in \text{attr}(v) \). We say that a node of \( H \) is irrelevant to the query if it does not cover any of the query attributes.

**Principle 1:** The more query attributes that are covered by some node(s) of \( H \), the higher should be the score \( f(H, W_q) \). The rationale is obvious.

**Principle 2:** The more nodes contain an attribute \( w \in W_q \), the higher the contribution of \( w \) should be toward the overall score \( f(H, W_q) \). The intuition is that attributes that are covered by more nodes of \( H \) signify homogeneity within the community w.r.t. shared query attributes.

**Principle 3:** The more nodes of \( H \) that are irrelevant to the query, the lower the score \( f(H, W_q) \).

We next discuss a few choices for defining \( f(H, W_q) \) and analyze their pros and cons, before presenting an example function that satisfies all three principles. Note that the scores \( f(H, W_q) \) are always compared between subgraphs \( H \) that meet the same structural constraint of \((k, d)\)-truss. An obvious choice is to define \( f(H, W_q) := \sum_{w \in W_q} \text{score}(H, w) \), where \( \text{score}(H, w) \), the contribution of attribute \( w \) to the overall score, can be viewed as the relevance of \( H \) w.r.t. \( w \). This embodies Principle 1 above. Inspired by Principle 2, we could define \( \text{score}(H, w) := \left| \mathcal{V}(H) \cap v_w \right| \), i.e., the number of nodes of \( H \) that cover \( w \). Unfortunately, this choice suffers from some limitations by virtue of treating all query attributes alike. Some attributes may not be shared by many community nodes while others are and this distinction is ignored by the above definition of \( f(H, W_q) \). To illustrate, consider the community \( H_1 \) in Figure 2(a) and the query \( Q = \{\{q_1\}, \{DB\}\} \); \( H_1 \) has 5 vertices associated with the attribute \( DB \) and achieves a score of 5. The subgraph \( H \) of the graph \( G \) shown in Figure 1 also has the same score of 5. However, while the community in Figure 2(a) is clearly a good community, as all nodes carry attribute \( DB \), the subgraph \( H \) in Figure 1 includes several irrelevant nodes without attribute \( DB \). Notice that both \( H_1 \) and \( H \) are 4-trusses so we have no way of discriminating between them, which is undesirable.

An alternative is to define \( \text{score}(H, w) = \frac{|V_{\mathcal{V}(H)} \cap \mathcal{V}(H)|}{|\mathcal{V}(H)|} \) as this captures the popularity of attribute \( w \). Unfortunately, this fails to reward larger communities. For instance, consider the query \( Q = \{\{q_1, v_4\}, \{DB\}\} \) over the graph \( G \) in Figure 1. The subgraph \( H_1 \) in Figure 2(a) as well as its subgraph obtained by removing \( q_2 \) is a 4-truss and both will be assigned a score of 1.

In view of these considerations, we define \( f(H, W_q) \) as a weighted sum of the score contribution of each query attribute, where the weight reflects the popularity of the attribute.

**Definition 4** (Attribute Score). Given a subgraph \( H \subseteq G \) and an attribute \( w \), the weight of an attribute \( w \) is \( \theta(H, w) = \frac{|V_{\mathcal{V}(H)} \cap \mathcal{V}(H)|}{|\mathcal{V}(H)|} \) i.e., the fraction of nodes of \( H \) covering \( w \). For a query \( Q = \{V_q, W_q\} \) and a community \( H \), the attribute score of \( H \) is defined as \( f(H, W_q) = \sum_{w \in W_q} \theta(H, w) \times \text{score}(H, w) \), where \( \text{score}(H, w) = \left| V_w \cap \mathcal{V}(H) \right| \) is the number of nodes covering \( w \).

The contribution of an attribute \( w \) to the overall score is \( \theta(H, w) \times \text{score}(H, w) = \frac{|V_w \cap \mathcal{V}(H)|^2}{|\mathcal{V}(H)|} \). This depends not only on the number of vertices covering \( w \) but also on \( w \)'s popularity in the community \( H \). This choice discourages vertices unrelated to the query attributes \( W_q \) which decrease the relevance score, without necessarily increasing the cohesion (e.g., trussness). At the same time, it permits the inclusion of essential nodes, which are added to a community to reduce the cost of connecting query nodes. They act as an important link between nodes that are related to the query, leading to a higher relevance score. We refer to such additional nodes as steiner nodes. E.g., consider the query \( Q = \{\{q_3\}, \{ML\}\} \) on the graph \( G \) in Figure 1. As discussed in Section 1 the community \( H_4 \) in Figure 2(d) is preferable to the chain of nodes \( v_5, q_3, v_{10} \). Notice that it includes \( v_9 \) with attribute \( DM \) (but not \( ML \); \( v_9 \) is thus a steiner node. It can be verified that \( f(H_4, W_q) = \frac{3}{2} \) which is smaller than the attribute score of the chain, which is 3. However, \( H_4 \) is a 3-truss whereas the chain is a 2-truss. It is easy to see that any supergraph of \( H_4 \) in Figure 1 is at most a 3-truss and has a strictly smaller attribute score.

The more query attributes a community has that are shared by more of its nodes, the higher its attribute score. For example, consider the query \( Q = \{\{q_1\}, \{DB, DM\}\} \) on our running example graph of Figure 1. The communities \( H_1, H_2, H_3 \) in Figure 2 are all potential answers for this query. We find that \( f(H_1, W_q) = 5 \cdot 1 + 2 \cdot \frac{3}{2} = 5.8 \) by symmetry, \( f(H_2, W_q) = 5.8 \); on the other hand, \( f(H_3, W_q) = 5 \cdot \frac{3}{2} + 5 \cdot \frac{3}{2} = 6.25 \). Intuitively, we can see that \( H_1 \) and \( H_2 \) are mainly focused in one area (DB or DM) whereas \( H_3 \) has 5 nodes covering both DB and DM each and also has the highest attribute score.

**Remark 1.** We stress that the main contribution of this subsection is the identification of key principles that an attribute score function must satisfy in order to be effective in measuring the goodness of an attributed community. Specifically, these principles capture the important properties of high attribute coverage and high attribute correlation within a community and minimal number of nodes irrelevant to given query. Any score function can be employed as long as it satisfies these principles. The algorithmic framework we propose in Section 2 is flexible enough to handle an ATC community model equipped with any such score function.

We note that a natural candidate for attribute scoring is the entropy-based score function, defined as \( e_{\text{entropy}}(H, W_q) = \sum_{w \in W_q} -\frac{|V_w \cap \mathcal{V}(H)|}{|\mathcal{V}(H)|} \log \frac{|V_w \cap \mathcal{V}(H)|}{|\mathcal{V}(H)|} \). It measures homogeneity of attribute values very well. However, it fails to reward larger communities, specifically violating Principle 1. E.g., consider the query \( Q = \{\{q_1, v_3\}, \{DB\}\} \) on the graph \( G \) in Figure 1. The subgraph \( H_1 \) in Figure 2(a) and its subgraph obtained by removing \( q_2 \) are both 4-trusses and both are assigned a score of 0. Clearly, \( H_1 \) has more nodes containing the query attribute DB.

### 4.3 Attributed Truss Community Model

Combining the structure constraint of (k, d)-truss and the attribute score function \( f(H, W_q) \), we define an attributed truss community (ATC) as follows.

**Definition 5.** [Attributed Truss Community] Given a graph \( G \) and a query \( Q = \{V_q, W_q\} \) and two numbers \( k \) and \( d \), \( H \) is an attributed truss community (ATC), if \( H \) satisfies the following conditions:

1. \( H \) is a (k, d)-truss.
2. \( f(H, W_q) \) is maximized over all (k, d)-trusses in \( G \).
3. The weighted attribute score of \( H \) is maximized over all (k, d)-trusses in \( G \).

The unique ATC optimal community \( \mathcal{H} \) is the largest (k, d)-truss satisfying these principles, and the algorithmic framework we propose in Section 2 is flexible enough to handle an ATC community model equipped with any such score function.
1. \( H \) is a \((k, d)\)-truss containing \( V_q \).

2. \( H \) has the maximum attribute score \( f(H, W_q) \) among sub-graphs satisfying condition (1).

In terms of structure and communication cost, condition (1) not only requires that the community containing the query nodes \( V_q \) be densely connected, but also that each node be close to the query nodes. In terms of query attribute coverage and correlation, condition (2) ensures that as many query attributes as possible are covered by as many nodes as possible.

Example 2. For the graph \( G \) in Figure 1 and query \( Q = \{(q_1, q_2), (DB, DM)\} \) with \( k = 4 \) and \( d = 2 \), \( H_2 \) in 2(b) is the corresponding ATC, since \( H_2 \) is a \((4, 2)\)-truss with the largest score \( f(H, W_q) = 6.25 \) as seen before.

The ATC-Problem studied in this paper can be formally formulated as follows.

**Problem Statement:** Given a graph \( G(V, E) \), query \( Q = \{V_q, W_q\} \) and two parameters \( k \) and \( d \), find an ATC \( H \), such that \( H \) is a \((k, d)\)-truss with the maximum attribute score \( f(H, W_q) \).

We remark that in place of the \((k, d)\)-truss with the highest attribute score, we could consider the problem of finding the \((k, d)\)-trusses with the highest attribute score. Our technical results and algorithms easily generalize to this extension.

## 5. PROBLEM ANALYSIS

In this section, we analyze the complexity of the problem and show that it is NP-hard. We then analyze the properties of the structure and attribute score function of our problem. Our algorithms for community search exploit these properties.

### 5.1 Hardness

Our main result in this section is that the ATC-Problem is NP-hard (Theorem 2). The crux of our proof idea comes from the hardness of finding the densest subgraph with \( \geq k \) vertices \( \text{[2]} \).

Unfortunately, that problem cannot be directly reduced to our ATC-Problem. To bridge this gap, we extend the notion of graph density to account for vertex weights and define a harder problem called WDaK-Problem — given a graph, find the subgraph with maximum "weighted density" with at least \( k \) vertices.

We now show that it is NP-hard and then reduce the WDaK-Problem to our problem.

#### Weighted Density

Let \( G = (V, E) \) be an undirected graph. Let \( w(v) \) be a non-negative weight associated with each vertex \( v \in V \). Given a subset \( S \subseteq V \), the subgraph of \( G \) induced by \( S \) is \( G_S = (S, E(S)) \), where \( E(S) = \{(u, v) \in E \mid u, v \in S\} \). For a vertex \( v \) in a subgraph \( H \subseteq G \), its degree is \( \deg_H(v) = |\{(u, v) \mid (u, v) \in E(H)\}| \). Next, we define:

**Definition 6 (Weighted Density).** Given a subset of vertices \( S \subseteq V \) of a weighted graph \( G \), the weighted density of subgraph \( G_S \) is defined as \( \chi(G_S) = \sum_{v \in S} \frac{\deg_G(v) + w(v)}{|S|} \).

Recall that traditional edge density of an induced subgraph \( G_S \) is \( \rho(G_S) = \frac{|E(S)|}{|S|^2} = \frac{\sum_{v \in S} \deg_G(v)}{|S|^2} \text{[2]}. \) That is, \( \rho(G_S) \) is twice the average degree of a vertex in \( G_S \). Notice that in Definition 6 if the weight of \( v \) is \( w(v) = 0 \), then the weighted density \( \chi(G_S) = 2\rho(G_S) \). It is well known that finding a subgraph with the maximum edge density can be solved optimally using parametric flow or linear programming relaxation \( [2] \). However, given a number \( k \), finding the maximum density of a subgraph \( G_S \) containing at least \( k \) vertices is NP-hard \( \text{[2]}. \)

Define the weight of a vertex \( v \) in a graph \( G \) as its degree in \( G \), i.e., \( w(v) = \deg_G(v) \). Then, \( \chi(G_S) = \sum_{v \in S} \frac{\deg_G(v) + \deg_G(v)}{|S|} = 2\rho(G_S) + \sum_{v \in S} \frac{\deg_G(v)}{|S|} \text{[2]}. \) We define a problem, the WDaK-Problem, as follows: given a graph \( G \) with weights as defined above, and a density threshold \( \alpha \), check whether \( G \) contains an induced subgraph \( H \) with at least \( k \) vertices such that \( \chi(H) \geq \alpha \).

We show it is NP-hard in Theorem 1. To establish this, we first show that the WDK-Problem, i.e., finding whether \( G \) has a subgraph \( H \) with exactly \( k \) vertices with weighted density at least \( \alpha \), i.e., \( \chi(H) \geq \alpha \) is NP-hard \( \text{[2]}. \) We then extend this result to the hardness of the WDaK-Problem.

**Lemma 1.** WDK-Problem is NP-hard.

**Proof.** We reduce the well-known NP-complete problem, CLIQUE, to WDK-Problem. Given a graph \( G = (V, E) \) with \( n \) vertices and a number \( k \), construct a graph \( G' = (V \cup V', E \cup E') \) as follows. For each vertex \( v \in V \), add \( n - \deg_G(v) \) new dummy vertices. \( G' \) contains an edge connecting each \( v \in V \) with each of its associated dummy vertices. Notice that the maximum degree of any node in \( G' \) is \( n \). In particular, every vertex in \( V \) has degree \( n \) in \( G' \) whereas every dummy vertex in \( V' \) has degree 1. So for any \( S \subseteq V \cup V', \chi(G_S') = 2\rho(G_S') + \sum_{v \in V} \frac{\deg_G(v)}{|S|} \leq 2\rho(G_S') + n \).

Set \( \alpha = n + k - 1 \). We claim that \( G' \) contains a clique of size \( k \) iff \( \chi(G') \geq \alpha \).

(\( \Rightarrow \)): Suppose \( H \subseteq G' \) is a \( k \)-clique. Since each vertex \( v \) of \( G \) has degree \( n \) in \( G' \) and \( 2\rho(H) \) is the average degree of a vertex in \( H \), we have \( \chi(H) = 2\rho(H) + n = k - 1 + n \).

(\( \Leftarrow \)): Suppose \( G' \) contains an induced subgraph \( G_S' \) with \( |S| = k \) and with \( \chi(G_S') \geq n + k - 1 \). It is clear that for any \( S \) with \( |S| = k \), \( \chi(G_S') \geq n + k - 1 \). The reason is that vertices in \( V' \) have degree 1 \( n \) in \( G' \). Thus, we must have \( S \cap V \subseteq V' \).

Now, for any \( S \subseteq V \cup V' \), \( \chi(G_S') \geq n + k - 1 \). Thus, \( \chi(G_S') = n + k - 1 \), and we can infer that \( 2\rho(G_S') = k - 1 \), implying \( G_S' \subseteq G_S \) is a \( k \)-clique.

**Theorem 1.** WDaK-Problem is NP-hard.

**Proof.** We reduce WDK-Problem to WDaK-Problem, using the ideas similar to those used in reducing the densest \( k \)-subgraph problem to the densest \( k \)-subgraph problem \( \text{[2]}. \)

**Theorem 2.** ATC-Problem is NP-hard.

**Proof.** We reduce the WDK-Problem to ATC-Problem. Given a graph \( G = (V, E) \) with \( |V| = n \) vertices, construct an instance \( G' \) as follows. \( G' \) is a complete graph over \( n \) vertices. For simplicity, we use \( V \) to refer to the vertex set of both \( G \) and \( G' \), without causing confusion. For each edge \( (u, v) \in E(G) \), create a distinct attribute \( w_{uv} \) for \( G' \). We assume \( w_{uv} \) and \( w_{uv} \) denote the same attribute. Then, with each vertex \( v \in V \) in \( G' \), associate a set of attributes: \( \text{attr}(v) = \{w_{uv} : (u, v) \in E(G)\} \). Notice that the cardinality of \( \text{attr}(v) \) is \( |\text{attr}(v)| = \deg_G(v) \). Also, attribute \( w_{uv} \) is present only in the attribute sets of \( u, v \), i.e., \( V_{w_{uv}} = \{u, v\} \).

For a vertex set \( S \subseteq V \), we will show that \( f(G_S', V_S) = \chi(G_S) \), where \( G_S = (S, E(S)) \) is the induced subgraph of \( G \) by \( S \), \( G_S = (S, E(S)) \) is the induced subgraph of \( G' \) by \( S \), and \( W_q = \{w_{uv} : \}

\text{Notice that the hardness of finding the maximum density subgraph with } \geq k \text{ vertices does not imply hardness of WDK-Problem for a specific weight function over the vertices and thus it needs to be proved.}
\((v, u) \in E(G)\). That is, the query attributes are the set of attributes associated with every edge of \(G\). We have
\[
f(G'_S, W_q) = \sum_{w_{vu} \in W_q} \frac{|(V_{wvu} \cap S)|}{|S|}^2 + \sum_{w_{vu} \in W_q} \frac{|V_{wvu} \cap S|}{|S|}
\]
(1)

For every attribute \(w_{vu} \in W_q\), exactly one of the following conditions holds:
- \(v, u \in S\): In this case \((v, u) \in E(S)\). Clearly, \(|V_{wvu} \cap S| = 2\), so \(|(V_{wvu} \cap S)|/(|V_{wvu} \cap S| - 1) = 2\).
- exactly one of \(u, v\) belongs to \(S\) and \((u, v) \in E \setminus E(S)\). In this case, \(|V_{wvu} \cap S| = 1\), so \(|(V_{wvu} \cap S)|/(|V_{wvu} \cap S| - 1) = 0\).
- \(u, v \notin S\). In this case, clearly \((v, u) \notin E(S)\) and \(|V_{wvu} \cap S| = 0\), so \(|(V_{wvu} \cap S)|/(|V_{wvu} \cap S| - 1) = 0\).

Therefore,
\[
\sum_{w_{vu} \in W_q} \frac{|(V_{wvu} \cap S)|}{|S|} = \sum_{v, u \in E(S)} \frac{2}{|S|} = 2 |E(S)|/|S| = 2 \rho(G_S).
\]
(2)

On the other hand, we have
\[
\sum_{w_{vu} \in W_q} \frac{|V_{wvu} \cap S|}{|S|} = \sum_{v, u \in E(S)} \frac{2}{|S|} + \sum_{v \in S, (v, u) \notin E(S)} \frac{1}{|S|} = \sum_{v \in S} \deg_{G}(v).
\]
(3)

Overall, \(f(G'_S, W_q) = 2 \rho(G_S) + \sum_{v \in S} \deg_{G}(v) = \chi(G_S)\).

We next show that an instance of WDalkP-Problem is a YES-instance iff for the corresponding instance of ATC-Problem, has a weighted density above a threshold, w.r.t. the query \(Q = (V_q, W_q)\) where \(V_q = \emptyset\) and \(W_q = \{w_{vu} : (v, u) \in E\}\) and the parameter \(d = 0\). The hardness follows from this.

\(\leq\) : Suppose \(G\) is a YES-instance of WDalkP-Problem, i.e., there exists a subset \(S^* \subseteq V\) such that for the induced subgraph \(G_{S^*}\) of \(G\), we have \(\chi(G_{S^*}) \geq \alpha\). Then, the subgraph \(G_{S^*} = (S^*, E(S^*))\) has \(f(G_{S^*}, W_q) = \chi(G_{S^*}) \geq \alpha\). In addition, since \(|S^*| \geq k\) and \(G^*\) is an \(n\)-clique, \(G_{S^*}\) is a k-clique, and hence a k-truss. For \(V_q = \emptyset\), trivially \(V_q \subseteq S^*\) and \(G_{S^*}\) satisfies the communication constraint on query distance. Thus, \(G_{S^*}\) is a \((k, d)\)-truss with \(f(G_{S^*}, W_q) \geq \alpha\), showing \(G\) is a YES-instance of ATC-Problem.

\(\geq\) : Suppose there exists a \((k, d)\)-truss \(G'_{S^*} = (S^*, E(S^*))\), a subgraph of \(G^*\) induced by \(S^* \subseteq V\), with \(f(G_{S^*}, W_q) \geq \alpha\). Then, we have \(G_{S^*} = (S^*, E(S^*))\) and \(\chi(G_{S^*}) = f(G_{S^*}, W_q) \geq \alpha\). Since \(G_{S^*}\) is a k-truss and \(|S^*| \geq k\), we have \(\chi(G_{S^*}) \geq \alpha\), showing \(G\) is a YES-instance of WDalkP-Problem.

\(\nabla\) : Since \(V_q = \emptyset\), we can set \(d\) to any value; we choose to set it to the tightest value.

5.2 Properties of \((k, d)\)-truss

Our attribute truss community model is based on the concept of k-truss, so the communities inherit good structural properties of k-trusses, such as k-edge-connected, bounded diameter and hierarchical structure. In addition, since the attribute truss community is required to have a bounded query distance, it will have a small diameter, as explained below.

A k-truss community is \((k - 1)\)-edge-connected, since it remains connected whenever fewer than \(k - 1\) edges are deleted from the community. Moreover, a k-truss based community has hierarchical structure that represents the hearts of the community at different levels of granularity\(^{24}\), i.e., a k-truss is always contained in some \((k - 1)\)-truss, for \(k \geq 3\). In addition, for a connected k-truss with \(n\) vertices, the diameter is at most \(|\frac{n-2}{\alpha^2}|\). Small diameter is considered an important property of a good community\(^{13}\).

Since the distance function satisfies the triangle inequality, i.e., for all nodes \(u, v, w\), \(\text{dist}_G(u, v) \leq \text{dist}_G(u, w) + \text{dist}_G(w, v)\), we can express the lower and upper bounds on the community diameter in terms of the query distance as follows.

**Observation 1**. For a \((k, d)\)-truss \(H\) and a set of nodes \(V_q \subseteq H\), we have \(d \leq \text{diam}(H) \leq \min\{\frac{4|V(H)|-2}{k}, 2d\}\).

**Remark**. Besides k-truss, there exist several other definitions of dense subgraphs including: \((r, s)\)-nucleus\(^{24}\), quasi-clique\(^{10}\), densest subgraph\(^{37}\), and k-core\(^{35}\). A \((k, r, s)\)-nucleus, for positive integers \(k\) and \(r < s\), is a maximal union of \(s\)-cliques in which every \(r\)-clique is present in at least \(k\) \(s\)-cliques, and every pair of \(r\)-cliques in that subgraph is connected via a sequence of \(s\)-cliques containing them. Thus, \((k, r, s)\)-nucleus is a generalized concept of k-truss, which can achieve very dense structure for large parameters \(k, r\), and \(s\). However, finding \((k, r, s)\)-nucleus incurs a cost of \(\Omega(m^3)\) time where \(m\) is the number of edges, which is more expensive than the \(O(m^{1.5})\) time taken for computing k-trusses, whenever \(s > 3\). A detailed comparison of k-truss and other dense subgraph models can be found in\(^{27}\). In summary, k-truss is a good dense subgraph model that strikes a balance between good structural properties and efficient computation.

5.3 Properties of attribute score function

We next investigate the properties of the attribute score function, in search of prospects for an approximation algorithm for finding ATC. From the definition of attribute score function \(f(H, W_q)\), we can infer the following useful properties.

**Positive influence of relevant attributes**. The more relevant attributes a community \(H\) has, the higher the score \(f(H, W_q)\). E.g., consider the community \(H_1\) and \(W_q = \{M_L\}\) in Figure\(^2\) (d). If the additional attribute “ML” is added to the vertex \(v_9\), then it can be verified that the score \(f(H_1, \{M_L\})\) will increase. We have:

**Observation 2**. Given a ATC \(H\) and a vertex \(v \in H\), let a new input attribute \(w \in W_q\ \text{attr}(v)\) be added to \(v\), and \(H'\) denote the resulting community. Then \(f(H', W_q) > f(H, W_q)\).

In addition, we have the following easily verified observation.

**Observation 3**. Given a ATC \(H\) and query attribute sets \(W_q \subseteq W_q\), we have \(f(H, W_q) \leq f(H, W_q')\).

**Negative influence of irrelevant vertices**. Adding irrelevant vertices with no query attributes to a ATC will decrease its attribute
score. For example, for \( W_q = \{DB\} \), if we insert the vertex \( v_7 \) with attribute \( IR \) into the community \( H_1 \) in Figure 2b, it decreases the score of the community \( w.r.t. \) the above query attribute \( W_q = \{DB\} \), i.e., \( f(H_1 \cup \{v_7\}, \{DB\}) < f(H_1, \{DB\}) \). The following observation formalizes this property.

**Observation 4.** Given two ATC’s \( H' \) and \( H'' \) where \( H \subseteq H'' \), suppose \( v_\in V(H') \setminus V(H) \) and \( v_\in \setminus W_q. \) Then \( f(H', W_q) < f(H, W_q) \).

**Non-monotone property and majority attributes.** The attribute score function is in general non-monotone w.r.t. the size of the community, even when vertices with query related attributes are added. For instance, for the community \( H \), even when vertices with query attributes are added, the attribute score function is in general non-monotone w.r.t. the size of the community.

The attribute \( V\) score function is in general non-monotone w.r.t. the size of the community. The attribute \( V\) score is non-monotone w.r.t. the size of the community.

The key difference between the two examples above is that DB is a “majority attribute” in \( H_1 \), a notion we formalize next. Formally, given a community \( H \) and query \( W_q \), we say that a set of attributes \( X \) includes majority attributes of \( H \), and \( \theta(H, W_q \cap X) = \sum_{w \in W_q \cap X} \theta(H, v) \geq \frac{f(H, W_q)}{2 |V(H)|}. \) Recall that \( \theta(H, w) \) is the fraction of vertices of \( H \) containing the attribute \( w \). We have:

**Lemma 2.** Let \( H \) be a ATC of a graph \( G \). Suppose there is a vertex \( v \in V(H) \) such that the set of attributes \( W_q \cap \set{v} \) includes the majority attributes of \( H \) and that adding \( v \) to \( H \) results in a ATC \( H' \) of \( G \). Then \( f(H', W_q) > f(H, W_q) \).

**Proof.** Suppose \( W_q = \{w_1, ..., w_l\} \) and w.l.o.g., let \( W_q \cap \set{v} = \{w_1, ..., w_l\} \), where \( 1 \leq r \leq l \). Let \( |V(H)| = b \), and for each attribute \( w_i \), let \( |V(H) \cap W_q| = b_i \). Since \( W_q \cap \set{v} \) includes the majority attributes of \( H \), \( \theta(H, W_q \cap \set{v}) = \frac{b_i - 1}{2b_i} \geq \frac{f(H, W_q)}{2 |V(H)|} \), so we have \( \sum_{i=1}^r b_i \geq 2b \).

We have \( f(H, W_q) = \sum_{i=1}^r \frac{|V(H) \cap W_q|}{|V(H)|} \frac{b_i}{b} = \sum_{i=1}^r \frac{b_i^2}{b} \), and \( f(H', W_q) = \sum_{i=1}^r \frac{(b_i + 1)^2}{(b_i + 1)^2} + \sum_{i=r+1}^l \frac{b_i^2}{b} \). As a result, \( f(H', W_q) - f(H, W_q) = \sum_{i=r+1}^l \left( \frac{b_i^2}{b} - \frac{b_i^2}{b} \right) = \frac{b_i^2}{b} (H, W_q) > 0 \).

This lemma will be helpful in designing bottom-up algorithms, by iteratively adding vertices with majority attributes to increase attribute score.

**Non-submodularity and Non-supersubmodularity.** A set function \( g: 2^U \rightarrow \mathbb{R}^{\geq 0} \) is said to be submodular provided for all sets \( S \subseteq T \subseteq U \) and element \( x \in U \setminus T \), \( g(T \cup \{x\}) \leq g(S) + g(T) \) if \( g(S \cup \{x\}) \geq g(S) \). The function \( g(\cdot) \) is said to be supermodular if \( -g(\cdot) \) is submodular. Optimization problems over submodular functions lend themselves to efficient approximation. We thus study whether our attribute score function \( f(\cdot, \cdot) \) is submodular w.r.t. its first argument, viz., set of vertices.

Consider the graph \( G \) in Figure 1 and query \( W_q = \{DB, DM\} \) with \( k = 2 \). Let the induced subgraphs of \( G \) by the vertex sets \( S_1 = \{q_1, q_2\} \) and \( S_2 = \{q_1, q_3\} \) respectively be denoted \( G_{1, 2} \) and \( G_{1, 3} \), \( G_{1, 2} \subseteq G_{1, 3} \). Let \( v^* \) be a vertex not in \( G_{1, 2} \). Let us compare the marginal gains \( f(G_1 \cup \{v^*\}, W_q) - f(G_1, W_q) \) and \( f(G_2 \cup \{v^*\}, W_q) - f(G_2, W_q) \), from adding the new vertex \( v^* \) to \( G_{1, 2} \) and \( G_{1, 3} \). Suppose \( v^* = v_9 \) with attribute “DB”, then we have \( f(G_2 \cup \{v_9\}, W_q) - f(G_2, W_q) = (4 + 1/4) - (3 + 1/3) = 11/12 > f(G_1 \cup \{v_9\}, W_q) - f(G_1, W_q) = (3 + 1/3) - (2 + 1/2) = 5/6, \) violating submodularity of the attribute score function \( f(\cdot, \cdot) \). On the other hand, suppose \( v^* = q_2 \) with attributes “DB” and “DM”. Then we have \( f(G_2 \cup \{q_2\}, W_q) - f(G_2, W_q) = (4 + 1) - (3 + 1/3) = 5/3 < f(G_1 \cup \{q_2\}, W_q) - f(G_1, W_q) = (3 + 4/3) - (2 + 1/2) = 11/6, \) which violates supermodularity. We just proved:

**Lemma 3.** The attribute score function \( f(H, W_q) \) is neither submodular nor supermodular.

In view of this result, we infer that the prospects for an efficient approximation algorithm are not promising.

### 6. Top-down Greedy Algorithm

In this section, we develop a greedy algorithmic framework for finding an ATC. It leverages the notions of attribute score contribution and attribute marginal gain that we define. Our algorithm first finds a (k, d)-truss, and then iteratively removes vertices with smallest attribute score contribution. Then, we analyze the time and space complexity of our algorithm. We also propose a more efficient algorithm with better quality, based on attribute marginal gain and bulk deletion.

#### 6.1 Basic Algorithm

We begin with attribute score contribution. Given a subgraph \( H \subseteq G \), a vertex \( v \in V(H) \), and attribute query \( W_q \), let us examine the change to the score \( f(H, W_q) \) from dropping \( v \).

\[
f(H, W_q) = \sum_{w \in W_q} \left( \frac{|V_w \cap (V(H) \setminus \set{v})|^2}{|V(H)| - 1} \right) + \sum_{w \in W_q \cap \set{v}} \left( \frac{|V_w \cap (V(H) \setminus \set{v})|^2}{|V(H)| - 1} \right) - \sum_{w \in \set{v}} \left( \frac{|V_w \cap (V(H) \setminus \set{v})|^2}{|V(H)| - 1} \right)
\]

The second term represents the drop in the attribute score of \( H \) from removing \( v \). We would like to remove vertices with the least drop in score. This motivates the following.

**Definition 7 (Attribute Score Contribution).** Given a graph \( H \) and attribute query \( W_q \), the attribute score contribution of a vertex \( v \in V(H) \) is defined as \( f_H(v, W_q) = \sum_{w \in W_q \cap \set{v}} \left( \frac{|V_w \cap (V(H) \setminus \set{v})|^2}{|V(H)| - 1} \right) - 1 \).

The intuition behind dropping a vertex \( v \) from \( H \) is as follows. Since \( f(H, W_q) \) is non-monotone (Section 5.3), the updated score
Algorithm 1 Basic $(G, Q)$

Input: A graph $G = (V, E)$, a query $Q = (V_q, W_q)$, numbers $k$ and $d$.
Output: A $(k, d)$-truss $H$ with the maximum $f(H, W_q)$.

1: Find a set of vertices $S_0$ having the query distance $\leq d$, i.e., $S_0 = \{u : \text{dist}_G(u, Q) \leq d\}$. 
2: Let $G_0$ be the induced subgraph of $S$, i.e., $G_0 = (S_0, E(S_0))$, where $E(S_0) = \{(v, u) : v, u \in S_0, (v, u) \in E\}$. 
3: Maintain $G_0$ as a $(k, d)$-truss. 
4: Let $l = 0$; 
5: while $\text{connect}_{G_0}(Q) = \text{true}$ do 
6: Compute the attribute score of $f(G_l, W_q)$; 
7: Compute $f_G(u, W_q) = \sum_{v \in W_q} \text{attr}_{G_l}(u, v) |V(H) \cap V_u| - 1$, for all $u \in G_l$; 
8: $u^* \leftarrow \arg\min_u \in V(G_l) - V_q f_G(u, W_q)$; 
9: Delete $u^*$ and its incident edges from $G_l$; 
10: Maintain $G_l$ as a $(k, d)$-truss. 
11: $G_{l+1} \leftarrow G_l$, $l \leftarrow l + 1$; 
12: $H \leftarrow \arg\max_{G' \in \{G_0, G_1, \ldots, G_{l-1}\}} f(G', W_q)$; 

from dropping $v$ from $H$ may increase or decrease, so we check if $f(H - v, W_q) > f(H, W_q)$.

Algorithm overview. Our first greedy algorithm, called Basic, has three steps. First, it finds the maximal $(k, d)$-truss of $G$ as a candidate. Second, it iteratively removes vertices with smallest attribute score contribution from the candidate graph, and maintains the remaining graph as a $(k, d)$-truss, until no longer possible. Finally, it returns a $(k, d)$-truss with the maximum score among all generated candidate graphs as the answer.

The details of the algorithm follow. First, we find the maximal $(k, d)$-truss of $G$ as $G_0$. Based on the given $d$, we compute a set of vertices $S$ having query distance no greater than $d$, i.e., $S_0 = \{u : \text{dist}_G(u, Q) \leq d\}$. Let $G_0 \subset G$ be the subgraph of $G$ induced by $S_0$. Since $G_0$ may contain edges with support $< (k - 2)$, we invoke the following steps to prune $G_0$ into a $(k, d)$-truss.

$(k, d)$-truss maintenance: repeat until no longer possible:
(i) $k$-truss: remove edges contained in $(k - 2)$ triangles;
(ii) query distance: remove vertices with query distance $> d$, and their incident edges;
Notice that the two steps above can trigger each other: removing edges can increase query distance and removing vertices can reduce edge support. In the following, we start from the maximal $(k, d)$-truss $G_l$ where $l = 0$, and find a $(k, d)$-truss with large attribute score by deleting a vertex with the smallest attribute score contribution.

Finding a $(k, d)$-truss with large attribute score. $G_0$ is our first candidate answer. In general, given $G_l$, we find a vertex $v \in V(G_l) \setminus V_q$ with the smallest attribute score contribution and remove it from $G_l$. Notice that $v$ cannot be one of the query vertices. The removal may violate the $(k, d)$-truss constraint so we invoke the $(k, d)$-truss maintenance procedure above to find the next candidate answer. We repeat this procedure until $G_l$ is not a $(k, d)$-truss any more. Finally, the candidate answer with the maximum attribute score generated during this process is returned as the final answer, i.e., $\arg\max_{G' \in \{G_0, \ldots, G_{l-1}\}} f(G', W_q)$.

Example 3. We apply Algorithm 1 on the graph $G$ in Figure 1 with query $Q = (\{q_1\}, \{DB, DM\})$, for $k = 4$ and $d = 2$. First, the algorithm finds the $(k, d)$-truss $G_0$ as the subgraph $H$ shown in Figure 1. Next, we select vertex $v_7$ with the minimum attribute score contribution $f_{G_0}(v_7, W_q) = 0$, and remove it from $G_0$. Indeed it contains neither of the query attributes. Finally, the algorithm finds the ATC $H_2$ with the maximum attribute score in Figure 1(b), which, for this example, is the optimal solution.

6.2 Complexity Analysis

Let $n = |V(G)|$ and $m = |E(G)|$, and let $d_{\text{max}}$ be the maximum vertex degree in $G$. In each iteration $i$ of Algorithm 1, we delete at least one vertex and its incident edges from $G_i$. Clearly, the number of removed edges is no less than $k - 1$, and so the total number of iterations is $t \leq \min\{\frac{n - k}{m(k - 1)}\}$. We have the following result on the time and space complexity of Algorithm 1. We note that we do not need to keep all candidate ATCs in the implementation, but merely maintain a removal record of the vertices/edges in each iteration.

Theorem 3. Algorithm 1 takes $O(mp + t(|W_q|n + |V_q|m))$ time and $O(m + |\text{attr}(V)|)$ space, where $t \in O(\min(n, m/k))$, and $\rho$ is the arboricity of graph $G$ with $\rho \leq \min(d_{\text{max}}, \sqrt{m})$.

Proof Sketch: The time cost of Algorithm 1 mainly comes from three key parts: query distance computation, $k$-truss maintenance, and attribute score computation.

For query distance computation, finding the set of vertices $V$ with query distance $d$ from $V_q$ can be done by computing the shortest distances using a BFS traversal starting from each query node $q \in V_q$, which takes $O(|V_q|m)$ time. Since the algorithm runs in $t$ iterations, the total time cost of this step is $O(t|V_q|m)$.

Second, consider the cost of $k$-truss identification and maintenance. Finding and maintaining a series of $k$-truss graphs $(G_0, \ldots, G_{l-1})$ in each iteration takes $O(\rho m)$ time in all, where $\rho$ is the arboricity of graph $G_0$. It has been shown that $\rho \leq \min(d_{\text{max}}, \sqrt{m})$.

Third, consider the cost of computing attribute score contribution. In each iteration, the computation of attribute score contribution for every vertex takes time $O(\sum_{v \in V(G)} \min\{\text{attr}(v), |W_q|\}) = O(\min\{|\text{attr}(V)|, |W_q| \cdot n\}) \subseteq O(|W_q|n)$. Thus, the total cost of attribute score computation is $O(t|W_q|n)$.

Therefore, the overall time complexity of Algorithm 1 is $O(mp + t(|W_q|n + |V_q|m))$.

Next, we analyze the space complexity. For graphs $(G_0, \ldots, G_l)$, we record the sequence of removed edges from $G_0$: attaching a corresponding label to graph $G_i$ at each iteration $i$, takes $O(m)$ space in all. For each vertex $v \in G_i$, we keep $\text{dist}(v, Q)$, which takes $O(n)$ space. Hence, the space complexity is $O(m + n + |\text{attr}(V)|)$, which is $O(m + |\text{attr}(V)|)$, due to the assumption $n \leq m$.

6.3 An improved greedy algorithm

The greedy removal strategy of Basic is simple, but suffers from the following limitations on quality and efficiency. Firstly, the attribute score contribution myopically considers the removal vertex $v$ only, and ignores its impact on triggering removal of other vertices, due to violation of $k$-truss or distance constraints. If these vertices have many query attributes, it can severely limit the effectiveness of the algorithm. Thus, we need to look ahead the effect of each removal vertex, and then decide which ones are better to be deleted. Secondly, Basic removes only one vertex from the graph in each step, which leads to a large number of iterations, making the algorithm inefficient.

In this section, we propose an improved greedy algorithm called BULK, which is outlined in Algorithm 2. BULK uses the notion of attribute marginal gain and a bulk removal strategy.

Attribute Marginal Gain. We begin with a definition.
Definition 8 (Attribute Marginal Gain). Given a graph \( H \), attribute query \( Q \), and a vertex \( v \in V(H) \), the attribute marginal gain is defined as \( \text{gain}_G(v, W_q) = f(H, W_q) - f(H - S_H(v), W_q) \), where \( S_H(v) \subseteq V(H) \) is \( v \) together with the set of vertices that violate (k, d)-truss after the removal of \( v \) from \( H \).

Notice that by definition, \( v \in S_H(v) \). For example, consider the graph \( G \) in Figure 4 and the query \( Q = \{q_1\}, \{ML\} \), with \( k = 3 \) and \( d = 2 \). The vertex \( v_9 \) has no attribute “ML” and \( \text{gain}_G(v_9, W_q) = 0 \) by Definition 8, indicating no attribute score contribution by vertex \( v_9 \). However, the fact is that \( v_9 \) is an important bridge for connections among the vertices \( q_1, v_8 \), and \( v_10 \) with attribute “ML”. The deletion of \( v_9 \) will thus lead to the deletion of \( v_8 \) and \( v_10 \), due to the 3-truss constraint. Thus, \( S_G(v_9) = \{v_8, v_9, v_{10}\} \). The marginal gain of \( v_9 \) is \( \text{gain}_G(v_9, W_q) = f(G, W_q) - f(G - S_G(v_9), W_q) = \frac{3}{7} > 0 \). This shows that the deletion of \( v_9 \) from \( G \) decreases the attribute score. It illustrates that attribute marginal gain can more accurately estimate the effectiveness of vertex deletion than score attribute contribution, by naturally incorporating look-ahead.

One concern is that \( \text{gain}_H(v, W_q) \) needs the exact computation of \( S_H(v) \), which has to simulate the deletion of \( v \) from \( H \) by invoking (k, d)-truss maintenance, which is expensive. An important observation is that if vertex \( v \) is to be deleted, its neighbors \( u \in N(v) \) with degree \( k - 1 \) will also be deleted, to maintain k-truss. Let \( P_H(v) \) be the set of \( v \)'s 1-hop neighbors with degree \( k - 1 \) in \( H \), i.e., \( P_H(v) = \{u \in N(v) : \deg_H(u) = k - 1\} \).

We propose a local attribute marginal gain, viz., \( \text{gain}_H(v, W_q) = f(H, W_q) - f(H - \bar{P}_H(v), W_q) \), to approximate \( \text{gain}_H(v, W_q) \). Continuing with the above example, in graph \( G \), for deleting vertex \( v_9 \), note that \( \deg(v_9) = \deg(v_{10}) = 2 = k - 1 \), so we have \( P_H(v_9) = \{v_8, v_9, v_{10}\} \), which coincides with \( S_G(v_9) \). In general, \( \text{gain}_H(v, W_q) \) serves as a good approximation to \( \text{gain}_H(v, W_q) \) and can be computed more efficiently.

Bulk Deletion. The second idea incorporated in BULK is bulk deletion. The idea is that instead of removing one vertex with the smallest attribute marginal gain, we remove a small percentage of vertices from the current candidate graph that have the smallest attribute marginal gain. More precisely, let \( G \) be the current candidate graph and let \( \epsilon > 0 \). We identify the set of vertices \( S \) such that \( |S| = \frac{\epsilon}{\max \{1, \deg_H(G)\}} \) and the vertices in \( S \) have the smallest attribute marginal gain, and remove \( S \) from \( G \), instead of removing a vertex at a time. Notice that the resulting algorithm \( \text{ATC}_{[1]} \) has size \( |V(G_{[1]})| \leq \frac{\epsilon}{\max \{1, \deg_H(G)\}} |V(G)| \) after the deletion of \( S \). We can safely terminate the algorithm once the size of \( G \) drops below \( k \) vertices and return the best \( \text{ATC} \) obtained so far, due to the constraint of \( k \)-truss. Thus, it follows that the number of iterations \( t \) drops from \( O(\min \{n, m/k\}) \) to \( O(\log_{\frac{1}{\epsilon}} \frac{1}{\epsilon}) \).

7. INDEX-BASED SEARCH ALGORITHM

While the BULK algorithm based on the framework of Algorithm 1 has polynomial time complexity, when the graph \( G \) is large and the query \( Q \) has many attributes, finding \( ATC \)s entails several \( ATC \) queries, which can be expensive. To help efficient processing of \( ATC \) queries, we propose a novel index called attributed-truss index (ATIndex). It maintains known graph structure and attribute information.

7.1 Attributed Truss Index

The ATIndex consists of three components: structural trussness, attribute trussness, and inverted attribute index.

Algorithm 2 BULK (\( G, Q \))

Input: A graph \( G = (V, E) \), a query \( Q = (V_q, W_q) \), numbers \( k \) and \( d \), parameter \( \epsilon \).

Output: A (k, d)-truss \( H \) with the maximum \( f(H, W_q) \).

1: Find the maximal (k, d)-truss \( G_0 \).
2: Let \( l = 0 \);
3: while connect\( G_l(Q) \) is true do
4: Find a set of vertices \( S \) of the smallest \( \text{gain}_G(v, W_q) \) with the size of \( |S| = \epsilon \). \( |V(G)| \);
5: Delete \( \bar{S} \) and their incident edges from \( G_l \);
6: Maintain the (k, d)-truss of \( G_l \);
7: \( G_{l+1} \leftarrow G_l; l \leftarrow l + 1 \);
8: \( H \leftarrow \text{arg max}_{G \subseteq G_l} \{f(G, W_q)\} \).

Structural Trussness. Recall that trusses have a hierarchical structure, i.e., for \( k \geq 3 \), a k-truss is always contained in some \( (k - 1) \)-truss. For any vertex or any edge, there exists a k-truss with the largest k containing it. We define the trussness of a subgraph, an edge, and a vertex as follows.

Definition 9 (Trussness). Given a subgraph \( H \subseteq G \), the trussness of \( H \) is the minimum support of an edge in \( H \) plus 2, i.e., \( \tau(H) = 2 + \min_{e \in E(H)} \text{supp}_H(e) \). The trussness of an edge \( e \in E(G) \) is \( \tau_G(e) = \max_{H \subseteq G \cap \text{code}(H)} \{\tau(H)\} \). The trussness of a vertex \( v \in E(G) \) is \( \tau_G(v) = \max_{H \subseteq G \cap \text{code}(H)} \{\tau(H)\} \).

Consider the graph \( G \) in Figure 4 and let the subgraph \( H \) be the triangle \( \Delta_{q_1, v_1, v_2} \). Then the trussness of \( H \) is \( \tau(H) = 2 + \min_{e \in E(H)} \text{supp}_H(e) = 3 \), since each edge is contained in one triangle in \( H \). However, the trussness of the edge \( e(q_1, v_1) \) is \( 4 \), because there exists a 4-truss containing \( e(q_1, v_1) \) in Figure 3b, and any subgraph \( H \) containing \( e(q_1, v_1) \) has \( \tau(H) \leq 4 \), i.e., \( \tau_G(e(q_1, v_1)) = \max_{H \subseteq G \cap \text{code}(H)} \{\tau(H)\} = 4 \). In addition, the vertex trussness of \( q_1 \) is also \( 4 \), i.e., \( \tau_G(q_1) = 4 \).

Based on the trussness of a vertex (edge), we can infer in constant time whether there exists a k-truss containing it. We construct the structural trussness index as follows. For each vertex \( v \in V \), we keep the vertex trussness of \( v \), and also maintain the edge trussness of its incident edges in decreasing order of trussness. This supports efficient checking of whether vertex \( v \) or its incident edge is present in a \( k \)-truss, avoiding expensive \( k \)-truss search. Also, it can efficiently retrieve \( v \)'s incident edges with a given trussness value. In addition, we use a hashtable to maintain all the edges and their trussness. Notice that for a graph \( G \), \( \tau(\emptyset) \) denotes the maximum structural trussness of \( G \).

Attributed Trussness. Structural trussness index is not sufficient for \( ATC \) queries. Given a vertex \( v \) in \( G \) with structural trussness \( \tau_G(v) \geq k \), there is no guarantee that \( v \) will be present in a (k, d)-truss with large attribute score w.r.t. query attributes. E.g., consider the graph \( G \) and vertex \( v_3 \) with \( \tau_G(v_3) = 4 \) in Figure 4. Here, \( v_3 \) will not be present in an \( ATC \) for query attributes \( W_q = \{\text{“ML”}\} \) since it does not have attribute “ML”. On the contrary, \( v_1 \) is in an \( ATC \) w.r.t. \( W_q = \{\text{“DM”}\} \). By contrast, \( v_9 \) is not present in a 4-truss w.r.t. attribute “DM” even though it has that attribute. To make such searches efficient, for each attribute \( w \in A \), we consider an attribute projected graph, which only contains the vertices associated with attribute \( w \), formally defined below.

Definition 10. (Attributed Projected graph & Attributed Trussness). Given a graph \( G \) and an attribute \( w \in A(V) \), the projected graph of \( G \) on attribute \( w \) is the induced subgraph of \( G \) by \( V_w \), i.e., \( G_w = (V_w, E_{V_w}) \subseteq G \). Thus, for each vertex \( v \) and edge \( e \)
in $G_w$, the attributed trussness of $v$ and $w$ in $G_w$ are respectively defined as $\tau_G(v) = \max_{H \subseteq \text{ATC}(G) \land \exists \theta \in V(H)} \{\tau(H)\}$ and $\tau_G(w) = \max_{H \subseteq \text{ATC}(G) \land \exists \theta \in E(H)} \{\tau(H)\}$.

For example, for the graph in Figure 1, the projected graph $G_w$ of $G$ on $w = \text{"DB"}$ is the graph $H_1$ in Figure 2(a). For vertices $v_1$ and $v_4$, even though both have the same structural trussness $\tau_H(v_1) = \tau_H(v_4) = 4$, in graph $H_1$, vertex $v_4$ has attribute trussness $\tau_{H_1}(v_4) = 4$ w.r.t. $w = \text{"DB"}$, whereas vertex $v_1$ is not even present in $H_1$, indicating that vertex $v_4$ is more relevant with \text{"DB"} than $v_1$.

Inverted Attribute Index. We propose an inverted index for each attribute $w \in A$, denoted $\text{invA}_w$. It maintains an inverted list of the vertices in $V_{w \theta}$, i.e., the vertices containing attribute $w$, in decreasing order of the vertex structural trussness. Thus, invA_w is in the format $\{(v_1, \tau_c(v_1)), ..., (v_j, \tau_c(v_j))\}$. $\tau_c(v_j) \geq \tau_c(v_{j+1})$, $j \in [1-\ell]$. The inverted attribute index and structural trussness index can both be used to speed up Algorithms 1 and 2.

ATIndex Construction. Algorithm 3 outlines the procedure of ATIndex construction. It first constructs the index of structural trussness using the structural decomposition algorithm of \text{[36]}, then constructs the index of attribute trussness and finally the index of attribute trussness and the arboricity of $G$, and $\rho \leq \min\{d_{\text{max}} \times \sqrt{m}\}$. Then, for each keyword $w \in \mathcal{A}$, it invokes the truss decomposition algorithm on the projected graph $G_w \subseteq G$, which takes $O(|E(G_w)|\rho)$ time and $O(m)$ space. In implementation, we deal with each $G_w$ separately, and release its memory after the completion of truss decomposition and write attribute trussness index to disk. Overall, ATIndex construction takes $O((m + \sum_{w \in \mathcal{A}}|E(G_w)|))$ time and $O(m)$ space, and the index occupies $O(m + \sum_{w \in \mathcal{A}}|E(G_w)|)$ space on disk.

7.2 Index-based Query Processing

In this section, we propose an ATIndex-based query processing algorithm by means of local exploration, called LocATC.

Algorithm overview. Based on the ATIndex, the algorithm first efficiently detects a small neighborhood subgraph around query vertices, which tends to be densely and closely connected with the query attributes. Then, we apply Algorithm 2 to shrink the candidate graph into a $(k, d)$-truss with large attribute score. The outline of the algorithm LocATC is presented in Algorithm 4. Note that, when no input parameters $k$ and $d$ are given in LocATC, we design an auto-setting mechanism for parameters $k$ and $d$, which will be explained in Section 5.

To find a small neighborhood candidate subgraph, the algorithm starts from the query vertices $V_q$, and finds a Steiner tree connecting the query vertices. It then expands this tree by adding attribute-related vertices to the graph. Application of standard Steiner tree leads to poor quality, which we next explain and address.

Finding attributed Steiner tree $T$. As discussed above, a Steiner tree connecting query vertices is used as a seed for expanding into a $(k, d)$-truss. A naive method is to find a minimal weight Steiner tree to connect all query vertices, where the weight of a tree is the number of edges. Even though the vertices in such a Steiner tree achieve close distance to each other, using this tree seed may produce a result with a small trussness and low attribute score. For example, for the query $Q = \{(q_1, q_2), \{\text{"DB"}\}\}$ (see Figure 1), the tree $T_1 = \{(q_1, v_1), (v_1, q_2)\}$ achieves a weight of 2, which is optimal. However, the edges $(q_1, v_1)$ and $(v_1, q_2)$ of $T_1$ will not be present in any 2-truss with the homogeneous attribute of “DB”. Thus it suggests growing $T_1$ into a larger graph will yield a low attribute score for $W_q = \text{"DB"}$. On the contrary, the Steiner tree $T_2 = \{(q_1, v_1), (v_1, q_2)\}$ also has a total weight of 2, and both of its edges have the attribute trussness of 4 w.r.t. the attribute “DB”, indicating it could be expanded into a community with large attribute score. For discriminating between such Steiner trees, we propose a notion of attributed truss distance.

**Definition 11 (Attributed Truss Distance).** Given an edge $e = (u, v) \in G$ and query attributes $W_q$, let $G = \{G_w : w \in W_q\} \cup \{G\}$. Then the attributed truss distance of $e$ is defined as $\text{dist}_{ATC}(e) = 1 + \gamma(\sum_{w \in \mathcal{A}}(\tau(\emptyset) - \tau_w(e)))$, where $\tau(\emptyset)$ is the maximum structural trussness in graph $G$.

The set $G$ consists of $G$ together with all its attribute projected graphs $G_w$, for $w \in W_q$ and the difference $(\tau(\emptyset) - \tau_w(e))$ measures the shortfall in the attribute trussness of edge $e$ w.r.t. the maximum trussness in $G$. The sum $\sum_{w \in \mathcal{A}}(\tau(\emptyset) - \tau_w(e))$ indicates the overall shortfall of $\emptyset$ across $G$ as well as all its attribute projections. Smaller the shortfall of an edge, lower its distance. Finally, $\gamma$ controls the extent to which small value of structural and attribute trussness, i.e., a large shortfall, is penalized. Using ATIndex, for any edge $e$ and any attribute $w$, we can access the structural trussness $\tau_G(e)$ and attribute trussness $\tau_{G_w}(e)$ in $O(1)$ time. Since finding minimum weight Steiner tree is NP-hard, we apply the well-known algorithm of [25, 31] to obtain a 2-approximation, using attributed truss distance. The algorithm takes $O(m|W_q| + m + n \log n) \subseteq O(m(W_q) + n \log n)$ time, where $O(m(W_q))$ is the time taken to compute the attributed truss distance for $m$ edges.

Expand attributed Steiner tree $T$ to Graph $G_t$. Based on the attribute Steiner tree $T$ built above, we locally expand $T$ into a graph $G_t$ as a candidate $(k, d)$-truss with numerous query attributes. Lemma 2 gives a useful principle to expand the graph with insertion of a vertex at a time, while increasing the attribute score. Specifically,
if \( \theta(G_t, W_q \cap \text{attr}(v)) \geq \frac{\theta(G_t, W_q)}{\theta(G_t, \text{attr}(v))} \), then graph \( G_T \cup \{v\} \) has a larger attribute score than \( G_T \). We can identify such vertices whose attribute set includes majority attributes of the current candidate graph and add them to the current graph.

Now, we discuss the expansion process, conducted in a BFS manner. We start from vertices in \( T \), and iteratively insert adjacent vertices with the largest vertex attribute scores into \( G_t \) until the vertex size exceeds a threshold \( \eta \), i.e., \( |V(G_t)| \leq \eta \), where \( \eta \) is empirically tuned. After that, for each vertex \( v \in V(G_t) \), we add all its adjacent edges e into \( G_t \).

**Apply BULK on \( G_t \) with auto-setting parameters.** Based on the graph \( G_t \) constructed above, we apply Algorithm 3 with given parameters \( k \) and \( d \) on \( G_t \) to find an ATC. If input parameters \( k \) and \( d \) are not supplied, we can set them automatically as follows. We first compute a \( k \)-truss with the largest \( k \) connecting all query vertices. Let \( k_{\text{max}} \) denote the maximum trussness of the subgraph found. We set the parameter \( k \) to be \( k_{\text{max}} \). We also compute the query distance of \( G_t \), and assign it to \( d \), i.e., \( d := \text{dist}_{G_t}(G_t, V_q) \). We then invoke the BULK algorithm on \( G_t \) with parameters \( k, d \) to obtain a ATC with large trussness and high attribute cohesiveness.

**Friendly mechanism for query formulation.** Having to set values for many parameters for posing queries using LocATC can be daunting. To mitigate this, we make use of the auto-setting of parameters \( k \) and \( d \). Additionally, we also allow the user to omit the query attribute parameter \( W_q \) in a query \( Q(V_q, W_q) \) and write \( Q(V_q) \). Thus, only query nodes need to be specified. Our algorithm will automatically set \( W_q := \bigcup v \in V_q \text{attr}(v) \) by default. The rationale is that the algorithm will take the the whole space of all possible attributes as input, and leverage our community search algorithms to find communities with a proper subspace of attributes, while achieving high scores. For example, consider the query \( Q = \{(q_1, q_2), \ldots\} \) on graph \( G \) in Figure 1 (LocATC automatically sets \( W_q := \{DB, DM, ML\} \). The discovered community is shown in Figure 3(b), which illustrates the feasibility of this strategy. This auto-complete mechanism greatly facilitates query formulation.

This auto-complete query formulation is useful to identify relative attributes for discovered communities, which benefits users in a simple way.

**Handling bad queries.** In addition to auto-complete query formulation, we discuss how to handle bad queries issued by users. Bad queries contain query nodes and query attributes that do not constitute a community. Our solution is to detect outliers of bad queries and then suggest good candidate queries for users. The whole framework includes three steps. First, it identifies bad queries. Based on the structural constraint of \((k, d)\)-truss, if query nodes span a long distance and loosely connected in graphs, it tends to be a bad query. In addition, if none of query attributes present in the proximity of query nodes, it suggests to have no communities with homogeneous attributes, indicating bad queries as its. Instances of bad queries \( Q(V_q, W_q) \) have no \((k, d)\)-truss neither containing \( V_q \) nor achieving non-zero score for attributes \( W_q \). Second, it recommends candidates of good queries. Due to outliers existed in bad queries, we partition the given query into several small queries. Based on the distribution of graph distance, graph cohesiveness, and query attribute, we partition given query nodes into several disjoint good queries. Specifically, we start from one query node, and find the \((k, d)\)-truss community containing it. The query nodes and query attributes present in this community are formed as one new query. The process is repeated until all query nodes are presented in one community or no such one community containing it. Thus, we have several new queries that are good to find ATC. Third, our approach quickly terminates by returning no communities, due to the violation of \((k, d)\)-truss and irrelevant query attributes.

8. **EXPERIMENTS.** In this section, we evaluate the efficiency and effectiveness of our proposed ATC model and algorithms. All algorithms are implemented in C++, and the experiments are conducted on a Linux Server with Intel Xeon CUP X5570 (2.93 GHz) and 50GB main memory. In this section, we test all proposed algorithms on a Linux Server with Intel Xeon CUP X5570 (2.93 GHz) and 50GB main memory.

8.1 **Experimental Setup.**

**Datasets.** We conduct experimental studies using 7 real-world networks. The network statistics are reported in Table 4.

The first dataset is PPI network, Krogan 2006, from the BioGRID database, where the PPI data are related to the yeast Saccharomyces cerevisiae [19]. Each protein has three kinds of attributes: biological processes, molecular functions, and cellular components. There are 255 known protein complexes for Saccharomyces cerevisiae in the MIPS/CYGD [19], which we regard as ground-truth communities.

The second dataset is Facebook ego-networks. For a given user id \( X \) in Facebook network \( G \), the ego-network of \( X \), denoted ego-face-

works. The network statistics are reported in Table 2. A smaller attribute pool of attributes to it. Except Krogan, all other datasets are available from the Stanford Network Analysis Project [8].

**http://linqs.cs.umd.edu/projects/projects/lbc/ snap.stanford.edu**
Table 2: Network statistics ($K = 10^3$ and $M = 10^5$)

| Network   | $|V|/K$ | $|E|/K$ | $d_{max}$ | $|V(G)|$ | $|A|$ | $|atv(V)|$ |
|-----------|--------|--------|-----------|---------|------|----------|
| Krogan    | 2.8K   | 1.1K   | 140       | 1.9     | 352  | 10373866 |
| Facebook  | 1.9K   | 0.9K   | 416       | 29      | 228  | 28151    |
| Cornell   | 1.9K   | 0.8K   | 94        | 4       | 1588 | 18496    |
| Texas     | 1.97K  | 0.82K  | 104       | 9       | 1201 | 15437    |
| Amazon    | 3.35K  | 2.26K  | 549       | 7       | 1674 | 1803446  |
| DBLP      | 3.17K  | 1.1M   | 342       | 114     | 1584 | 15454590 |
| Youtube   | 1.1M   | 1.1M   | 28,754    | 19      | 5327 | 21632424 |
| LiveJournal | 4M   | 3.5M   | 14,815    | 352     | 11104| 12426352 |
| Orkut     | 3.1M   | 1.7M   | 33,313    | 76      | 9926 | 10373866 |

![Figure 4: Quality evaluation ($F_1$ score) on networks with real-world attributes and ground-truth communities](image)

8.2 Quality Evaluation

To evaluate the effectiveness of different community models, we compare LocATC with three state-of-the-art methods – ACC, MDC and LCTC on attributed networks with ground-truth communities.

Networks with real-world attributes. We experiment with the Krogan network and the 10 Facebook ego-networks, all having real-world attributes. For every ground-truth community, we randomly select a set of query nodes with size drawn uniformly at random from [1, 16]. We use 2 representative attributes from the community as query attributes. We choose attributes occurring most frequently in a given community and rarely occurring in other communities as representative attributes. We evaluate the accuracy of detected communities and report the averaged $F_1$-score over all queries on each network.

Figure 4 shows the $F_1$-score on Krogan, Cornell, Texas, and the 10 Facebook ego-networks. Our method (LocATC) achieves the highest $F_1$-score on most networks, except for facebook ego-networks f104 and f1684. The reason is that vertices of ground-truth communities in f104 and f1684 are strongly connected in structure, but are not very homogeneous on query attributes. LCTC has the second best performance, and outperforms MDC on all networks. We can see that MDC and LCTC do not perform as well as LocATC, because those community models only consider structure metrics, and ignore attribute features. Note that for each query with multiple query vertices, the attribute community search method ACC randomly takes one query vertex as input. We make this explicit and denote it as ACC-Q1 in Figure 4. For comparison, we apply the same query on our method LocATC, and denote it as LocATC-Q1. LocATC-Q1 clearly outperforms ACC-Q1 in terms of $F_1$-score, showing the superiority of our ATC model. In addition, LocATC achieves higher score than ACC-Q1, indicating our method can discover more accurate communities with more query vertices. Furthermore, we also compare the precision and recall of all methods on f414 network in Figure 5. MDC performs the worst on precision, since it considers no query attributes and includes many nodes that are not in ground-truth communities. ACC-Q1 is the winner on precision, which is explained by the strict attribute constraints in its definition. On the other hand, in terms of recall, ACC-Q1 is the worst method as it only identifies a small part of ground-truth communities. Overall, LocATC achieves a good balance between precision and recall. This is also reflected in LocATC achieving the best $F_1$-score on most datasets (Figure 4).

Figure 6 shows the running time performance of all methods. In terms of supporting multiple query vertices, LocATC runs up to two orders of magnitude faster than MDC and LCTC on small ego-networks in Facebook, and LCTC is the winner on Cornell.
and Texas networks. For one query vertex, ACC-Q1 runs faster than LocATC-Q1, since k-cores can be computed quicker than k-trusses.

Networks with synthetic attributes. In this experiment, we test on 5 large networks – DBLP, Amazon, Youtube, LiveJournal, and Orkut, with ground-truth communities and synthetic attributes. We randomly select 1000 communities from 5000 top-quality ground-truth communities as answers. For each community, we generate a query \( Q = (V_q, W_q) \), where query vertices \( V_q \) are randomly selected from this community with a size randomly drawn from \([1, 16] \), and query attributes \( W_q \) are the 3 community attributes. Figure 7(a) shows the F1-score. Our method LocATC achieves the best F1-score among all compared methods on all networks, and MDC is the worst. The results clearly show the effectiveness and superiority of our ATC model for attributed community search. Moreover, LocATC-Q1 outperforms ACC-Q1 on most networks.

Figure 7(b) reports the running times of all methods on all networks. As we can see, LocATC runs much faster than MDC, and is close to LCTC. This indicates that LocATC can achieve high quality over large networks with an efficiency comparable to LCTC. Thus, compared to LCTC, the additional overhead of reasoning with attribute cohesiveness is small while the improvement in quality of communities discovered is significant. In addition, LocATC-Q1 runs much faster than LocATC, which shows the high efficiency of local exploration for one query vertex.

8.3 Efficiency Evaluation

We evaluate the various approaches using different queries on ego-facebook-414 (aka f414) and DBLP.

Varying query vertex size \( |V_q| \). We test 5 different values of \( |V_q| \), i.e., \([1, 2, 4, 8, 16] \) with the default query attribute size \( |W_q| = 2 \). For each value of \( |V_q| \), we randomly generate 100 sets of queries, and report the average running time in seconds. The results for f414 and DBLP are respectively shown in Figure 8(a) and (b). LocATC achieves the best performance, and increases smoothly with the increasing query vertex size. BULK is more effective than Basic, thanks to the bulk deletion strategy. Most of the cost of BULK and Basic comes from computing the maximal (k, d)-truss \( G_0 \). All methods takes less time on f414 than on DBLP network, due to the small graph size of f414.

Varying query attribute size \( |W_q| \). We test 5 different values of \( |W_q| \), from 1 to 5. For each value of \( |W_q| \), we randomly generate 100 sets of queries, and report the average running time. We show the result for f414 and DBLP respectively in Figure 9(a) and (b). Figure 9 shows all methods register only a modest increase in running time as \( |W_q| \) increases. Again, the local exploration method LocATC significantly outperforms other methods.

8.4 Index Construction

![Figure 7: Evaluation on networks with synthetic attributes and ground-truth communities](image)

![Figure 8: Varying query vertex size |V_q|: Query Time](image)

![Figure 9: Varying query attribute size |W_q|: Query Time](image)

Table 3: Comparison of index size (in Megabytes) and index construction time (wall-clock time in seconds)

| Network   | Graph Size | Index Size | Index Time |
|-----------|------------|------------|------------|
| Krogan    | 0.24       | 0.13       | 0.01       | 0.006      |
| Amazon    | 24         | 19         | 75         | 6.7        | 21.7       |
| DBLP      | 23         | 20         | 57         | 14.2       | 35.2       |
| Youtube   | 52         | 59         | 105        | 75.6       | 113.8      |
| LiveJournal | 808   | 806       | 1920      | 2192       | 3596      |
| Orkut     | 1710      | 2190      | 8451      | 231011     | 260454     |

Table 3 reports the size (MB) and construction time (seconds) of the structural k-truss index (K-Truss) and ATIndex, along with the size of each network. The 10 Facebook ego-networks have similar results, omitted from Table 3 for brevity. The size of ATIndex is comparable to the original graph size and structural k-truss index. It confirms that the ATIndex scheme has \( O(m + \sum_{w \in A} |E(G_w)|) \) space complexity. Given \(|A| > 1000\) on all these networks, it shows the projected attribute graphs are very sparse. The ATIndex construction time is comparable to k-truss index construction and is nearly as efficient. It can be seen that query processing efficiency is greatly aided by the ATIndex. For instance, consider the index construction and query processing times on DBLP network. In Figure 8(b) and Figure 9(b), the query time of BULK and Basic without ATIndex scheme take nearly 20 seconds, while the construction time of ATIndex is only 35.2 seconds (Table 3). That is, merely processing two queries more than pays off for the index construction effort.

8.5 Parameter Sensitivity Evaluation

In this experiment, we vary various parameters in used in the synthetic data generation, query generation, and in algorithm definitions, and evaluate the quality and efficiency performance of LocATC.

Varying homogeneous attributes in synthetic datasets. For each ground-truth community in Amazon, we randomly select 3 attributes, and assign each of these attributes to each of \( Z \) vertices in the community, where \( Z \) is a random number in \([50, Y]\). Note that different attributes may have different value of \( Y \). The parameter \( Y \) is varied from 60 to 90. As \( Y \) is increased, intuitively the level of homogeneity in the network and in its communities increases. The results of F1-score are shown in Figure 10. As homogeneous attributes in communities increase, MDC and LCTC maintain the same F1-score, while the F1-score of all methods of attributed community search – LocATC, LocATC-Q1, and ACC-Q1 – increases as homogeneity increases. Once again, LocATC is the best method even when the proportion of homogeneous attributes falls in [50,
LocATC-Q1 beats ACC-Q1 for all settings of homogeneity. Similar results can be also observed on other synthetic datasets.

Varying the average number of attribute/vertex \( \frac{|A|}{|V|} \) in synthetic datasets. In this experiment, we vary the average number of attribute/vertex \( \frac{|A|}{|V|} \) to generate different attribute sets in Amazon. The results are shown in Figure 11. With the increased \( \frac{|A|}{|V|} \), LocATC performs better. This is because the size of attribute set \( A \) becomes larger, which makes homogeneity of synthetic attributes in different communities more likely. Finally, it brings more challenges to detected accurate communities for a smaller \( \frac{|A|}{|V|} \). We can obtain similar results on other synthetic datasets.

Varying query vertex size \(|V_q|\) and query attribute size \(|W_q|\). We test the quality performance of LocATC using different queries by varying \(|V_q|\) and \(|W_q|\). The results are shown in Figure 11 (a) and (b). As we can see, given more information of query vertices and query attributes within communities, our algorithm accordingly performs better.

Varying parameters \( \epsilon \), \( \gamma \), and \( \eta \). We test the performance of LocATC by varying \( \epsilon \), \( \gamma \), and \( \eta \). We used the same query nodes that are selected in Sec. 8.2 on f414 network. Similar results can be also observed on other networks with real attributes. The results of F1-score and query time by varying \( \epsilon \) are respectively reported in Figure 12 (a) and (b). As we can see, LocATC removes a smaller portion of nodes, which achieves a higher F1-score using more query time. In addition, we test different values of \( \gamma \) and \( \eta \) and report the results in Figure 13 (a) and (b). The F1-score remains stable as \( \gamma \) increases from 0.1 to 0.5, and then decreases a little bit for a larger value of \( \gamma = 1.0 \). Thus, the default choice of \( \gamma = 0.2 \) is good at balancing the cohesive structure and homogenous attributes in an efficiency way. Furthermore, we also report the results by varying the parameter \( \eta \) in Figure 13 (a) and (b). As can be seen, the F1-score remains stable with increasing \( \eta \), while the running time increases a little with larger \( \eta \). The results show that the default setting \( \eta = 1000 \) is large enough for achieving a good balance of efficiency and quality.

8.6 Bad Query Evaluation

In this experiment, we use bad queries to test the performance of LocATC on f414 and DBLP networks. We generate bad queries by randomly choosing query nodes and query attributes from different ground truth communities. We test a set of 100 bad queries generated in this manner. We also test 100 queries that are selected in Sec. 8.2 as good queries. We intuitively expect bad queries to result in discovered communities with poor density, compared to good queries. We compare the quality of discovered communities \( C \) in terms of edge density, i.e., \( \frac{|E(C)|}{|V(C)|} \), averaged across the 100 queries. Figure 10(a) shows the results of edge density. Compared with bad queries, LocATC can find communities with larger densities for good queries. Figure 10(b) shows the average running times on good and bad queries. LocATC processes bad queries much faster than good queries, which achieves 6.8 times of efficiency improvement on DBLP network. Intuitively, LocATC can quickly return empty answers for bad queries, if the algorithm can determine the weak structure of query nodes and heterogeneous attributes of neighbors in the proximity of query nodes.

8.7 Case Study on PPI network

Besides the quality evaluation measured by F1-score, we also apply the LocATC algorithm on the protein-protein interaction (PPI) network Krogan. Recall that the performance of LocATC handling bad queries has been tested in Section 8.6, and we test good queries here. We examine the details of the discovered protein complexes to investigate biologically significant clues, which help us to better understand the protein complexes. Figure 17(a) shows one complex “transcription factor TFIIIC complex” in sccharomyces cerevisiae, which is identified by biologists previously. The graph contains 6 nodes and 12 edges, with density 0.8 and diameter 2. We adopt the following procedure for checking whether a protein is present in a complex. Taking gene id “854277” as an example, we can go to the NCBI input “854277” in the search box, and select the category of “Gene”, then we will obtain information related to this gene, from which we can check whether this gene is one of the proteins in the protein complex. Similar with the procedure of good query generation in Sec. 8.2, we randomly sample a query as \( Q = (V_q, W_q) \) where \( V_q = \{854277, 856100\} \) and \( W_q = \{"GO:0001099", "GO:0001041\} \), and set the parameters \( k = 3 \) and \( d = 3 \). To illustrate the importance of the consideration of protein attributes in detecting protein complexes, we simply use the structure and find the (3,3)-truss shown in Figure 17(b). This community contains 11 proteins including 6 proteins of the ground-truth complex of Figure 17(a). The other 5 proteins not present in the ground-truth complex are associated with no query attributes, but have other attributes \( w_3 \) and \( w_4 \), as shown in Figure 17(b). When we look up the database of Gene

https://www.ncbi.nlm.nih.gov/
http://wodaklab.org/cyc2008/resources/CYC2008_complex.tab
Ontolog[4], we know that the attributes of “biological processes” as “GO:0001009” and “GO:0001041” respectively represent “transcription from RNA polymerase III type 2 promoter” and “transcription from RNA polymerase III type 2 promoter”. Except query attributes, we omitted details of other attributes from Figure[17] for simplicity. LocATC is able to identify all proteins that preform the same biological process of transcription from RNA polymerase. Overall, LocATC successfully identifies all proteins that constitute the ground-truth complex in Figure[17](a). Other than these two homogeneous attributes, interestingly, we also discover another two attributes shared by all proteins in terms of “molecular functions”. Specifically, the attributes “GO:0001003” and “GO:0001005” respectively perform DNA binding activity as “RNA polymerase III type 2 promoter sequence-specific DNA binding” and “RNA polymerase III type 1 promoter sequence-specific DNA binding”. Overall, this complex exists in the cell nucleus, according to the same attribute “cellular components” of “GO:0005634” in all proteins.

9. CONCLUSION

In this work, we propose an attributed truss community (ATC) model that allows to find a community containing query nodes with cohesive and tight structure, also sharing homogeneous query attributes. The problem of finding an ATC is NP-hard. We also show that the attribute score function is not monotone, submodular, or supermodular, indicating approximation algorithms may not be easy to find. We propose several carefully designed strategies to quickly find high-quality communities. We design an elegant and compact index, ATIndex, and implement an efficient query processing algorithm, which exploits local exploration and bulk deletion. Extensive experiments reveal that ground-truth communities and social circles can be accurately found by our model and algorithms significantly outperform previous approaches. Several interesting questions remain. Some examples include attributed community search over heterogeneous graphs and edge-weighted graphs, and w.r.t. weighted query attributes.

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