Darboux-Halphen-Brioshi system with rank four

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Abstract

We will determine the condition that a homogeneous quadratic equation with rank four becomes the Darboux-Halphen-Brioshi system in terms of the associated non-associative algebra.

1 Introduction

We will study a homogeneous quadratic nonlinear differential equation

\[
\frac{dX_i}{dt} = \sum_{j,k=1}^{n} a^i_{jk} X_j X_k.
\]  

(1)

This type of equations contains many important examples such as Euler’s spinning top motion equation or the Lotka-Volterra equation (see Example 2.1 and 2.3).

In recent years, another class of quadratic equations called Halphen’s first equation \[\text{[H1]}\]

\[
\frac{dX_2}{dt} + \frac{dX_3}{dt} = 2X_2X_3,
\]

\[
\frac{dX_1}{dt} + \frac{dX_3}{dt} = 2X_1X_3,
\]

\[
\frac{dX_1}{dt} + \frac{dX_2}{dt} = 2X_1X_2.
\]  

(2)

is studied in many areas, such as the Atiyah-Hitchin metrics, modular forms, moonshine \[\text{[HM]}\], or the Painlevé analysis. Halphen’s first equation is satisfied by the null values of elliptic theta functions \[\text{[Q]}\]. Since the null values of
elliptic theta functions are related to the hypergeometric equations, Halphen also studied Halphen’s second equation in 1881 [H2]:

\[
\frac{dX}{d\tau} = X^2 + c(X - Y)^2 + b(Z - X)^2 + a(Y - Z)^2;
\]

\[
\frac{dY}{d\tau} = Y^2 + c(X - Y)^2 + b(Z - X)^2 + a(Y - Z)^2,
\]

\[
\frac{dZ}{d\tau} = Z^2 + c(X - Y)^2 + b(Z - X)^2 + a(Y - Z)^2,
\]

which is solved by hypergeometric functions. Halphen’s second equation is the simplest example of the Darboux-Halphen-Brioshi system [O1] (see the section 4).

In this paper we will study the condition when quadratic equations are reduced to the Darboux-Halphen-Brioshi system. We will write this condition in terms of non-associative algebras, which are found by L. Markus [M].

In the case of rank three, this condition is completely determined in [O2]. In this case the corresponding Darboux-Halphen-Brioshi system is related to hypergeometric equations or confluent hypergeometric equations.

The rank four case is important in the study of Painlevé equations, since the Painlevé sixth equations represent isomonodromic deformations of linear differential equations with four regular singularities. Moreover one of the simplest Darboux-Halphen-Brioshi system with rank four is related to modular forms of level three [O3]. This system is also related to the quantum cohomology of \(\mathbb{CP}^2\).

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2 Homogeneous Quadratic Differential Equations

We take a homogeneous quadratic nonlinear differential equations

\[
\frac{dX_i}{dt} = \sum_{j,k=1}^{n} a_{jk}^{i} X_j X_k,
\]

where \(a_{jk} = a_{kj}\). This type of equations contains many important examples.
Example 2.1 Euler’s spinning top motion equation:

\[
\begin{align*}
\frac{dX_1}{dt} &= 2X_2X_3, \\
\frac{dX_2}{dt} &= 2X_1X_3, \\
\frac{dX_3}{dt} &= 2X_1X_2.
\end{align*}
\]

Remark 2.2 Even if a quadratic equation is not homogeneous, we can interpret it into homogeneous equation with dummy variables.

Example 2.3 The Lotka-Volterra equation

\[
\begin{align*}
\frac{dN_1}{dt} &= aN_1 - bN_1N_2, \\
\frac{dN_2}{dt} &= -cN_1 - dN_1N_2,
\end{align*}
\]

is an inhomogeneous equation, but, with a dummy variable \(N_3\), it can be represented as a homogeneous equation:

\[
\begin{align*}
\frac{dN_1}{dt} &= aN_1N_3 - bN_1N_2, \\
\frac{dN_2}{dt} &= -cN_1N_3 - dN_1N_2, \\
\frac{dN_3}{dt} &= 0.
\end{align*}
\]

We define a non-associative algebra \(\mathcal{A}\)

\[
\mathcal{A} = \sum_{k=1}^{n} \mathbb{C} \ x_k, \quad x_j \cdot x_k = \sum_{i=1}^{n} a^j_{ik} \ x_i,
\]

which is associated with the homogeneous equation \(1\). Since \(a_{jk} = a_{kj}\), \(\mathcal{A}\) is commutative.

Remark 2.4 A relation between non-associative algebra and homogeneous equation is found by L. Markus in 1960 [M]. That is used in the study of classification of topological behavior of solution orbits near the origin, which is a critical point of \(1\).

From now on, we use uppercase letters for independent variables of a homogeneous quadratic equation, and lowercase letters for basis of the corresponding commutative non-associative algebra. The independent variables are dual to bases of the corresponding algebra.
Lemma 2.5 The commutative non-associative algebra $A$ and the homogeneous quadratic nonlinear differential equation admits a one to one correspondence. Moreover, $A$ determines a structure of homogeneous quadratic equation. If and only if two homogeneous quadratic nonlinear differential equations

$$\frac{dX_i}{dt} = \sum_{j,k=1}^{n} a_{jk}^i X_j X_k, \quad \frac{dY_i}{dt} = \sum_{j,k=1}^{n} b_{jk}^i Y_j Y_k,$$

are interchanged by a linear transformation

$$Y_j = \sum_{k=1}^{n} c_{jk} X_k,$$

then the corresponding commutative non-associative algebras are equivalent:

$$x_k = \sum_{k=1}^{n} c_{jk} y_j.$$

The lemma above is easily verified.

3 Halphen’s Equation and Hypergeometric Function

The Halphen’s second equation (3) is solved by hypergeometric functions.

Remark 3.1 If $a = b = c = -1/8$, then the equation (3) turns into the Halphen’s first equation (2), and then it is solved by theta functions.

In [O2] we demonstrated the condition that a third order quadratic system is solved by hypergeometric functions.

Theorem 3.2 If and only if a three dimensional commutative non-associative algebra admits the unit, then the corresponding quadratic system is solved by (confluent) hypergeometric functions or elementary functions.

If the automorphism group of the non-associative algebra is a finite group, the quadratic equation is solved by (confluent) hypergeometric functions. If the automorphism group of the non-associative algebra is an infinite group, the quadratic equation is solved by elementary functions.
Example 3.3 (Matrix Riccati equation) Let $A, X$ be $2 \times 2$ symmetric matrices which satisfy
$$\frac{dX}{dt} = XAX.$$ 
If $A$ is a constant matrix, then the equation is a third order quadratic system, and then the associated commutative non-associative algebra is a Jordan algebra. Since the algebra admits the unit and the automorphism is $O(2)$, the equation is solved by elementary functions. This algebra is independent of $A$ up to an isomorphism, if $\det A \neq 0$.

Example 3.4 For the Halphen’s second equation \( \{3\} \), we take the basis of the associated algebra as
$$e_1 = -x + y + z, \quad e_2 = x - y + z, \quad e_3 = x + y - z.$$  
Then the multiplication is as follows.
$$e_1 \cdot e_1 = (1 + 4(b + c))e, \quad e_1 \cdot e_2 = -e_3 - 4ce, \quad e_1 \cdot e_3 = -e_2 - 4be,$$
$$e_2 \cdot e_2 = (1 + 4(a + c))e, \quad e_2 \cdot e_3 = -e_1 - 4ae, \quad e_2 \cdot e_3 = -e_2 - 4be,$$
$$e_3 \cdot e_3 = (1 + 4(a + b))e, \quad e_3 \cdot e_1 = -e_2 - 4be, \quad e_3 \cdot e_2 = -e_3 - 4ce.$$

where $e = e_1 + e_2 + e_3$. This algebra has the unit $e$. Then \( \{3\} \) is solved by the hypergeometric function $F(\alpha, \beta, \gamma, z)$, where
$$a = \frac{1}{4}(2\alpha\beta - \gamma - \alpha\gamma - \beta\gamma + \gamma^2),$$
$$b = \frac{1}{4}(\alpha^2 + \beta^2 + \gamma - \alpha\gamma - \beta\gamma - 1),$$
$$c = \frac{1}{4}(-2\alpha\beta - \gamma + \alpha\gamma + \beta\gamma).$$

Let $p, q, r$ be an exponents of hypergeometric equation at singular points
$$p = 1 - \gamma, \quad q = -\alpha - \beta + \gamma, \quad r = \alpha - \beta,$$
then
$$1 + 4(a + c) = p^2, \quad 1 + 4(a + b) = q^2, \quad 1 + 4(b + c) = r^2.$$

If
$$\alpha = \frac{k + 9}{12k}, \quad \beta = \frac{k - 9}{12k}, \quad \gamma = \frac{1}{2},$$

5
then by the transformation given by Chazy, Halphen’s second equation turns into

\[ y''' = 2y'' - 3(y')^2 + \frac{4}{36 - k^2}(6y' - y^2)^2. \]  

(5)

Moreover, if \( k = 0 \), then the equation (5) is equivalent with

\[
\begin{align*}
\frac{dX}{dt} &= X^2 + (V - X)(W - X), \\
\frac{dW}{dt} &= W^2 - (X - W)^2 + (V - X)(W - X), \\
\frac{dV}{dt} &= V^2 + (W - X)^2 - (X - V)^2 + (V - X)(W - X),
\end{align*}
\]

which is solved by Airy functions. This is also commented in [CO].

4 Generalized Darboux-Halphen-Brioshi System

For a Fuchsian equation

\[
\frac{d^2y}{dz^2} + Q(z)y = 0,
\]

\[ Q(z) = \sum_{j=1}^{m} \frac{\alpha_j}{(z - a_j)^2} + \sum_{j=1}^{m-1} \frac{\beta_j}{(z - a_j)(z - a_{j+1})}, \]

if we set

\[ \tau = \frac{y_2}{y_1}, \quad X_0 = \frac{d}{d\tau} \log y_1, \quad X_j = \frac{d}{d\tau} \log \frac{y_1}{x - a_j}, \]

then

\[
\begin{align*}
\frac{dX_k}{d\tau} &= X_k^2 - \sum_{j=1}^{m} \alpha_j (X_j - X_0)^2 - \sum_{j=1}^{m-1} \beta_j (X_j - X_0)(X_{j+1} - X_0), \\
(X_j, X_k, X_l, X_n) &= (a_j, a_k, a_l, a_n),
\end{align*}
\]

(6)

where \((a, b, c, d)\) is an anharmonic ratio

\[ (a, b, c, d) = \frac{a - b}{c - d} \frac{c - b}{a - d}. \]

We call (6) the generalized Darboux-Halphen-Brioshi system (together with the algebraic relations) [O2]. The variable \( X_j \) is called the Brioshi variable.
Remark 4.1 If \( m = 2 \), there is no algebraic relation. In this case the generalized Darboux-Halphen-Brioshi system is the same as Halphen’s second equation.

The generalized Darboux-Halphen-Brioshi system was found in the study of differential relations of modular forms \([O3]\). It plays an important role in the study of moonshine \([HM]\).

Example 4.2 (The level-three Halphen equation) The equation

\[
\begin{align*}
W' + X' + Y' &= WX + XY + YW, \\
W' + Y' + Z' &= WY + YZ + ZW, \\
W' + X' + Z' &= WX + XZ + ZW, \\
X' + Y' + Z' &= XY + YZ + ZX, \\
e^{\frac{4}{3}\pi i}(XZ + YW) + e^{\frac{2}{3}\pi i}(XW + YZ) + (XY + ZW) &= 0
\end{align*}
\]

is called the level-three Halphen equation. This is a generalized Darboux-Halphen-Brioshi system given by the Picard-Fuchs equation

\[(1 - t^3)y'' - 3t^2y' - ty = 0 \tag{7}\]

of the \( \Gamma(3) \)-modular surface

\[x^3 + y^3 + z^3 - 3txyz = 0.\]

The equation (7) is the Laplace transform of a linear equation

\[
\left( z \frac{d}{dz} \right)^3 \phi = 27z^3 \phi,
\]

which appear in the Frobenius structure \([D]\) which turns up in the quantum cohomology of \( \mathbb{C}P^2 \).

Now, we will study a quadric homogeneous equation with rank four

\[
(D_3) \left\{ \begin{array}{l}
\frac{dX_i}{d\tau} = \sum_{j,k=0}^{3} a_{jk} X_j X_k, \\
Q(X_0, X_1, X_2, X_3) = \sum_{j,k=0}^{3} b_{jk} X_j X_k,
\end{array} \right.
\]

where \( a_{jk} = a_{kj}, b_{jk} = b_{kj} \). We assume a compatibility condition

\[
\frac{d}{d\tau} Q(X_0, X_1, X_2, X_3) = L(X_0, X_1, X_2, X_3) Q(X_0, X_1, X_2, X_3).
\]
This condition means that the hypersurface \( Q = 0 \) is invariant under the
vector field defined by \((\mathcal{D}_3)\). If \( L = 0 \) then \( Q \) is a first integral. From now
on, we assume \( L \neq 0 \). Moreover, we assume that \( Q \) is irreducible.

We define a commutative non-associative algebra \( \mathcal{A}_3(c) \) with a parameter
c = \((c_1, c_2, c_3, c_4)\), which is associated with \( \mathcal{D}_3 \), as

\[
\mathcal{A}_3(c) = \sum_{j=0}^{3} c_j x_j,
\]

\[
x_j \cdot x_k = \sum_{i=0}^{3} a_{jk}^i x_i + b_{jk} x, \quad x = \sum_{i=0}^{3} c_i x_i.
\]

We use this algebra \( \mathcal{A}_3(c) \) to study the condition that the system \( \mathcal{D}_3 \) is a
generalized Darboux-Halphen-Brioshi system.

**Theorem 4.3** The system \( \mathcal{D}_3 \) is a generalized Darboux-Halphen-Brioshi sys-
tem, if and only if there exist basis \( x_0, x_1, x_2, x_3 \) of \( \mathcal{A}_3(c) \) which satisfies the
following conditions:

1. \( e = x_0 + x_1 + x_2 + x_3 \) is the unit for any \( c \).
2. \( (\pm x_0 \pm x_1 \pm x_2 \pm x_3)^2 \) is proportional to the unit \( e \) for some \( c \).
3. \( (-x_0 + x_1 + x_2 + x_3)^2, (x_0 - x_1 + x_2 + x_3)^2, (x_0 + x_1 - x_2 + x_3)^2, \)
   \( (x_0 + x_1 + x_2 - x_3)^2 \) are proportional to the unit \( e \) for any \( c \).

In the light of the algebra associated with Halphen’s second equation, this theorem is an extension of theorem 3.2.

Proof.

We set a basis of the algebra \( \mathcal{A}_3(c) \) associated with the system \( \mathcal{D}_3 \):

\[
e_0 = x_0 + x_1 + x_2 + x_3,
\]

\[
e_1 = x_0 - x_1 + x_2 + x_3,
\]

\[
e_2 = x_0 + x_1 - x_2 + x_3,
\]

\[
e_3 = x_0 + x_1 + x_2 - x_3.
\]

Since \( e = e_0 \) is the unit for any \( c \), we obtain \( Q = Q(E_1, E_2, E_3) \). Moreover,
\( e_0^2 \) and \((x_0 - x_1 - x_2 - x_3)^2\) are proportional to the unit \( e \) for any \( c \), therefore
we obtain

\[
Q = \gamma_1 E_1 E_2 + \gamma_2 E_2 E_3 + \gamma_3 E_1 E_3,
\]
where \( \gamma_1 + \gamma_2 + \gamma_3 = 0 \). Since \( Q \) is irreducible, we obtain \( \gamma_j \neq 0 \) \((j = 1, 2, 3)\).

We set
\[
e_j^2 = \tilde{\alpha}_j e_0 \quad (j = 1, 2, 3).
\]

Then, since we have
\[
(x_0 + x_1 - x_2 - x_3)^2 = (e_2 + e_3 - e_0)^2 \equiv e_2 \cdot e_3 - (e_2 + e_3) \pmod{e_0},
\]
we obtain
\[
e_2 \cdot e_3 \equiv e_2 + e_3 \pmod{e_0}.
\]

In the same way, we obtain
\[
e_1 \cdot e_2 \equiv e_1 + e_2 \pmod{e_0},
\]
\[
e_1 \cdot e_3 \equiv e_1 + e_3 \pmod{e_0}.
\]

Therefore, we obtain
\[
e_1 \cdot e_2 = e_1 + e_2 + \tilde{\beta}_1 e_0, \quad (9)
\]
\[
e_2 \cdot e_3 = e_2 + e_3 + \tilde{\beta}_2 e_0, \quad (10)
\]
\[
e_1 \cdot e_3 = e_1 + e_3 + \tilde{\beta}_3 e_0, \quad (11)
\]

for some \( \tilde{\beta}_j \) \((j = 1, 2, 3)\). Let \( \mathcal{A} \) be a algebra determined by (8), (9), (10), (11). Here, we set
\[
\alpha_j = \frac{1}{4}(1 - \tilde{\alpha}_j) \quad (j = 1, 2, 3),
\]
\[
c_1 = c_2 = c_3 = c_4 = \frac{1}{4\gamma_3}(1 + \tilde{\beta}_3),
\]
\[
\beta_1 = -\frac{1}{2}(1 + \tilde{\beta}_1) - \frac{\gamma_1}{2\gamma_3}(1 + \tilde{\beta}_3),
\]
\[
\beta_2 = -\frac{1}{2}(1 + \tilde{\beta}_2) - \frac{\gamma_2}{2\gamma_3}(1 + \tilde{\beta}_3),
\]

and we determine a algebra \( \mathcal{A}_3(c) \) associated with generalized Darboux-Halphen-Brioshi system (11). Then \( \mathcal{A}_3(c) \) is equivalent with \( \mathcal{A} \). Therefore \( \mathcal{D}_3 \) is a generalized Darboux-Halphen-Brioshi system.

The converse is proved by a straightforward calculation.

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