A NEW MODEL FOR MIXING BY DOUBLE-DIFFUSIVE CONVECTION (SEMI-CONVECTION),
III. THERMAL AND COMPOSITIONAL TRANSPORT THROUGH NON-LAYERED ODDC

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ABSTRACT

Oscillatory double-diffusive convection (ODDC; also known as semi-convection) refers to a type of double-diffusive instability that occurs in regions of planetary and stellar interiors that have a destabilizing thermal stratification and a stabilizing mean molecular weight stratification. In this series of papers, we use an extensive suite of three-dimensional (3D) numerical simulations to quantify the transport of heat and chemical species by ODDC. Rosenblum et al. first showed that ODDC can either spontaneously form layers that significantly enhance the transport of heat and chemical species compared to microscopic transport or remain in a state dominated by large-scale gravity waves, in which there is a more modest enhancement of the turbulent transport rates. Subsequent studies in this series focused on identifying under what conditions layers form and quantifying transport through layered systems. Here we proceed to characterize transport through systems that are unstable to ODDC, but do not undergo spontaneous layer formation. We measure the thermal and compositional fluxes in non-layered ODDC from both two-dimensional (2D) and 3D numerical simulations, and show that 3D simulations are well approximated by similar simulations in a 2D domain. We find that the turbulent mixing rate in this regime is weak and can, to a first-level approximation, be neglected. We conclude by summarizing the findings of papers I through III into a single prescription for transport systems unstable to ODDC.

Key words: convection – hydrodynamics – planets and satellites: general – stars: interiors

1. INTRODUCTION

1.1. Background

Stratified fluids that have a stabilizing compositional gradient (Ledoux stable) and a destabilizing thermal gradient (Schwarzschild unstable) are common in a variety of astrophysical objects and were first discussed by Schwarzschild & Härm (1958) in the context of the growing convection zones of massive stars (>10M⊙). Walin (1964) and Kato (1966) realized that fluid instabilities may develop in such an environment, driving the turbulent transport of heat and chemical species. They are now commonly referred to as oscillatory double-diffusive instabilities, and saturate into a nonlinear weakly turbulent regime called Oscillatory Double Diffusive Convection (ODDC) (Spiegel 1969). This nomenclature derives from the oscillatory nature of the basic unstable fluid motions which bear resemblance to over-stable internal gravity waves (Kato 1966). The efficiency of mixing (of heat and chemical species) in ODDC is potentially significant to evolution models for stars (Langer et al. 1985; Merryfield 1995) and giant planets (Stevenson 1982; Leconte & Chabrier 2012; Nettelmann et al. 2015).

The conditions for ODDC to occur are defined by three non-dimensional parameters (Baines & Gill 1969). The Prandtl number, Pr, and the diffusivity ratio, τ, are defined as

\[ Pr = \frac{\nu}{\kappa_T}, \quad \tau = \frac{\kappa_p}{\kappa_T}, \]

where \( \nu \) is the viscosity, and \( \kappa_T \) and \( \kappa_p \) are the thermal and compositional diffusivities, respectively. The third parameter, the inverse density ratio, is defined as

\[ R_0^{-1} = \frac{\partial}{\partial t} \nabla^2 \mu = \frac{\tau}{\Pr} \left( \frac{d \ln \mu}{d \ln p} \right), \]

where \( T(r), p(r), \) and \( \mu(r) \) are the background profiles of temperature, pressure, and mean molecular weight as a function of radius \( r \) in the star or planet being considered, and where \( \delta = \frac{\partial \ln \rho}{\partial \ln T} \mid_{\mu, \rho} \) and \( \phi = \frac{\partial \ln \rho}{\partial \ln T} \mid_{\mu, \rho} \) are thermodynamic derivatives of the equation of state. The subscript “ad” is used to denote an adiabatic gradient (a derivative at constant entropy). ODDC occurs when \( R_0^{-1} \) is within the following range (Walin 1964; Kato 1966).

\[ 1 < R_0^{-1} < R_e^{-1} \equiv \frac{Pr + 1}{Pr + \tau}. \]

When \( R_0^{-1} \) is less than one, the system is unstable to overturning convection, and when \( R_0^{-1} \) is greater than \( R_e^{-1} \), the system is stable to infinitesimal perturbations (but may still be unstable to finite amplitude ones; see Section 6.4). A useful proxy for the inverse density ratio is the so-called “reduced inverse density ratio” parameter (Mirouh et al. 2012), defined as

\[ r = \frac{R_0^{-1} - 1}{R_e^{-1} - 1}. \]

This parameter maps the natural ODDC range to the interval [0, 1] where 0 marks the onset of overturning convection and 1 marks the boundary between the ODDC parameter space and that of linear stability.
1.2. Recent Studies and Current Work

In previous papers of this series, we have set out to systematically characterize the different types of behavior demonstrated in ODDC. By analyzing three-dimensional (3D) direct numerical simulations, Rosenblum et al. (2011) first identified two general classes of behavior: one in which thermocompositional layers spontaneously emerge through a mechanism known as the γ-instability (Radko 2003); which occurs after the growth and saturation of the primary instability of ODDC and causes a significant increase in transport), and one in which layers do not form and where the dynamics are dominated by what they called “homogeneous turbulence.” Mirouh et al. (2012) built on the work of Rosenblum et al. (2011) by developing a semi-analytical rule for identifying the regions of parameter space where layers naturally form and where they do not. They found that layers can only spontaneously form if $1 < R_0^{-1} < R_L^{-1} < R_c^{-1}$, where $R_L^{-1}$ is typically of order $Pr^{-\frac{1}{2}}$ (see Section 6). They also determined that non-layered ODDC for $R_L^{-1} < R_0^{-1} < R_c^{-1}$ is ultimately dominated by large-scale internal gravity waves which (surprisingly) also augment thermal and compositional transport, though not as much as in the layered case. Wood et al. (2013) then studied the dynamics and transport properties of layered ODDC ($1 < R_0^{-1} < R_L^{-1}$). In this work we now focus on characterizing the behavior of non-layered ODDC when $R_L^{-1} < R_0^{-1} < R_c^{-1}$ and quantifying the associated thermocompositional mixing it induces.

In Section 2 we present our mathematical model for the dynamics of ODDC. We also briefly describe how we study it using DNS and discuss the metrics by which we measure thermal and compositional fluxes. In Section 3 we describe the evolution of a typical non-layered simulation and discuss the difficulties of running numerical simulations in extreme parameter regimes. In Section 4 we discuss the effectiveness of two-dimensional (2D) simulations for modeling ODDC compared with the full 3D DNS. We present the results of our numerical experiments in 2D and 3D, focusing in particular on the measurements of thermal and compositional fluxes in Section 5. Finally, in Section 6 we summarize the findings of papers I, II, and III of this series and discuss them in the context of previous work and applications to astrophysics.

2. MATHEMATICAL MODEL AND NUMERICAL SIMULATIONS

In what follows and as in Papers I and II, we consider a region that is significantly smaller than the density scale height of a typical star or planet, and where flow speeds are much smaller than the sound speed of the medium. This allows us to use the Boussinesq approximation (Spiegel & Veronis 1960) and to ignore the spherical geometry of an actual stellar or planetary interior. We consider a Cartesian domain where $z$ is oriented in the radial direction and we assume locally constant background gradients of temperature, $T_0$, and mean molecular weight, $\mu_0$, that are related to $\nabla$ and $\nabla_\mu$ as follows:

$$T_0 = \frac{\partial T}{\partial r} = \frac{T}{p} \frac{\partial p}{\partial r} \nabla,$$

$$\mu_0 = \frac{\partial \mu}{\partial r} = \frac{\mu}{p} \frac{\partial p}{\partial r} \nabla_\mu,$$  

(5)

where $\nabla$ and $\nabla_\mu$ were defined in Equation (2). We use the following linearized equation of state,

$$\frac{\tilde{\rho}}{\rho_0} = -\alpha \tilde{T} + \beta \tilde{\rho},$$

(6)

where $\tilde{\rho}$, $\tilde{T}$ and $\tilde{\mu}$ are dimensional perturbations to their respective background profiles and where $\rho_0$ is the mean density of the region. The parameters $\alpha$ and $\beta$ are the coefficient of thermal expansion and coefficient of compositional contraction, respectively, and are related to $\delta$ and $\phi$ as

$$\alpha = \frac{\delta}{\tilde{T}} = -\frac{1}{\rho_0} \frac{\partial \rho}{\partial T} |_{\rho_0},$$

$$\beta = \frac{\phi}{\mu} = \frac{1}{\rho_0} \frac{\partial \mu}{\partial \mu} |_{\rho_0, \tilde{T}}.$$  

(7)

Using these assumptions, the standard non-dimensional governing equations for ODDC (Rosenblum et al. 2011; Mirouh et al. 2012) are

$$\nabla \cdot \tilde{u} = 0,$$

$$\frac{1}{Pr} \left( \frac{\partial \tilde{u}}{\partial t} + \tilde{u} \cdot \nabla \tilde{u} \right) = -\nabla \tilde{\rho} + (\tilde{T} - \tilde{\mu}) e_c + \nabla^2 \tilde{u},$$

$$\frac{\partial \tilde{T}}{\partial t} + \tilde{u} \cdot \nabla \tilde{T} - \tilde{\omega} = \nabla^2 \tilde{T},$$

$$\frac{\partial \tilde{\rho}}{\partial t} + \tilde{u} \cdot \nabla \tilde{\rho} - R_0^{-1} \tilde{\omega} = \tau \nabla^2 \tilde{\mu},$$

(8)

where $\tilde{u}$ is the velocity field, and all values with tildes are now non-dimensional. To arrive at these equations we have defined the unit length in terms of (dimensional) constants,

$$[l] = d = \left( \frac{\kappa_T \nu}{\alpha g |T_0 - T_{0c}|} \right)^{\frac{1}{2}} = \left( \frac{\kappa_T \nu}{\alpha g \frac{T}{p} \frac{\partial T}{\partial r} |\nabla - \nabla_{ad}|} \right)^{\frac{1}{2}}.$$  

(9)

where $g$ is the gravitational acceleration and where the parameter $T_{0c}$ is the background adiabatic temperature gradient which is defined as

$$T_{0c} = \frac{T}{p} \frac{\partial p}{\partial r} \nabla_{ad}.$$  

(10)

Using these constants, the units of time, $[t]$, temperature, $[\tilde{T}]$, and mean molecular weight, $[\tilde{\mu}]$, naturally emerge as

$$[t] = \frac{d^2}{\kappa_T},$$

$$[\tilde{T}] = d |T_0 - T_{0c}|,$$

$$[\tilde{\mu}] = \frac{\alpha}{\beta} d |T_0 - T_{0c}|.$$  

(11)

Furthermore, we can also rewrite the inverse density ratio $R_0^{-1}$ in terms of the parameters introduced above as

$$R_0^{-1} = \frac{\beta |\tilde{\mu}|}{\alpha |T_0 - T_{0c}|}.$$  

(12)

As the physical characteristics of stellar and planetary interiors cannot be easily reproduced in laboratory experiments,
we must rely on numerical simulations to gain insight about the nonlinear behavior of ODDC in these objects. As in Papers I and II, we solve the set of equations in (8) using a pseudospectral code (Traxler et al. 2011) where all perturbations are triply periodic in the domain. Here we focus on the region of parameter space where $R_0^{-1} > R_c^{-1}$. Some of the experimental data presented in subsequent sections was generated by Rosenblum et al. (2011) and Mirouh et al. (2012) in simulations run using this code. We have generated new data specifically for this work as well, including a series of 2D simulations which we will compare with the 3D ones.

The main quantities of interest we extract from these numerical simulations are the turbulent vertical fluxes of temperature and chemical species, $\langle \overline{\nu T} \rangle$ and $\langle \overline{\nu \mu} \rangle$, respectively, where the angle brackets represent a spatial integral over the entire computational domain. It is often useful to express these fluxes in terms of thermal and compositional Nusselt numbers ($\mathrm{Nu}_T$ and $\mathrm{Nu}_\mu$), which are the ratios of the total fluxes to the diffusive fluxes (of temperature and composition). The turbulent fluxes are defined as follows in terms of $\mathrm{Nu}_T$ and $\mathrm{Nu}_\mu$,

$$F_T^{\text{turb}} = \langle \overline{\nu T} \rangle = \mathrm{Nu}_T - 1,$$

$$F_\mu^{\text{turb}} = \langle \overline{\nu \mu} \rangle = \tau R_0^{-1}(\mathrm{Nu}_\mu - 1). \quad (13)$$

As shown by Wood et al. (2013), these fluxes often exhibit fast oscillations with large amplitudes due to the gravity waves, so for the purposes of analysis it can be more useful to consider related quantities known as the thermal and compositional dissipations, $\langle \overline{\nabla T^2} \rangle$ and $\langle \overline{\nabla \mu^2} \rangle$. Indeed, as shown by Malkus (1954), taking a spatial integral of the thermal and chemical evolution equations and then assuming that the system is in a statistically stationary state, we find that the dissipations are related to the turbulent fluxes and Nusselt numbers by

$$\overline{\mathrm{Nu}_T} - 1 = \langle \overline{\nabla T^2} \rangle,$$

$$\overline{\mathrm{Nu}_\mu} - 1 = \frac{\langle \overline{\nabla \mu^2} \rangle}{\tau R_0^{-1}} = \frac{\langle \overline{\nabla \mu^2} \rangle}{\langle \overline{\nabla T^2} \rangle}, \quad (14)$$

where the bars indicate temporal averages over the entire statistically stationary period. In practice, these are good approximations even when the temporal averaging is done over relatively short periods (see Wood et al. 2013 for instance), so in what follows we assume similar relations between the instantaneous Nusselt numbers and dissipations as well. This is advantageous because the dissipations are less sensitive to the oscillations of gravity waves than the fluxes and are therefore easier to analyze.

3. BEHAVIORS OF ODDC

In what follows, we show the results of a typical ODDC simulation, which has $Pr = \tau = 0.03$ and $R_0^{-1} = 7.87$, and was run at an effective resolution of 192$^3$ (the simulation domain has dimensions (1000$^3$)). Note that for these parameters, $R_0^{-1} \approx 2.5$ (Mirouh et al. 2012) and $R_c^{-1} \approx 17.2$, so this value of $R_0^{-1}$ is indeed in the range $R_0^{-1} < R_c^{-1} < R_c^{-1}$ which is unstable to ODDC, but where layers do not spontaneously form through the $\gamma$-instability. We first describe these results purely qualitatively, then move on to a more quantitative analysis.

3.1. Qualitative Study

Figure 1 shows the vertical component of the velocity field at early times and Figure 2 shows the $\gamma$-component of the velocity field later on in the simulation. At very early times, we first see the development of the fastest growing modes of the basic instability of ODDC, which resemble tubes of vertically oscillating fluid (shown in Figure 1(a)). This primary growth phase ends when the basic instability saturates due to nonlinear interactions inherent to the problem (see Figure 1(b)). As previously discussed by Mirouh et al. (2012), after the primary saturation the small-scale fastest growing modes of the primary instability stop growing but other larger-scale modes slowly continue to gain energy, amounting to a secondary phase of growth. From Figure 2(a) we see that the latter (which ultimately come to dominate the energetics of the system) generally have the largest possible scale in the horizontal direction. As the system evolves, energy is funneled into modes of progressively larger vertical scale until the secondary growth phase saturates and the system reaches a statistically stationary state. The dynamics of this state are characterized by gravity wave oscillations on fast timescales whose amplitudes are modulated chaotically and intermittently. This intermittency appears to be caused by nonlinear interactions between large-scale gravity modes and large-scale horizontal shearing modes. Indeed, we regularly observe the emergence of a strong horizontal shear layer that temporarily suppresses the wave field (Figure 2(b)). The shear then dissipates and the system proceeds as before. Figure 2 shows the distinct differences in the $\gamma$ velocity field between a gravity-wave-dominated phase and a shear-dominated phase.

3.2. Quantitative Study

To study this system in a more quantitative way, we now investigate the energy contained in individual modes. We shall refer to specific spatial modes by the number of wavelengths in the $x$, $y$, and $z$ directions. Mode $(l, m, k)$ would refer to a mode with horizontal wave numbers $l \frac{2\pi}{L_x}$ and $m \frac{2\pi}{L_y}$ (in the $x$ and $y$ directions), and vertical wavenumber $k \frac{2\pi}{L_z}$ (in the $z$ direction). We can quantify the transfer of energy to larger scales by considering the amount of energy in a mode “family.” A family of modes consists of all the modes with equivalent spatial structures given the symmetries between the $x$ and $y$ directions in the domain, and negative and positive wave numbers in each spatial direction. For example, the mode family 102 contains the modes $\{1, 0, 2\}$, $\{-1, 0, 2\}$, $\{0, 1, 2\}$, $\{0, -1, 2\}$, $\{1, 0, -2\}$, $\{-1, 0, -2\}$, $\{0, 1, -2\}$, and $\{0, -1, -2\}$ (Traxler et al. 2011).

Consistent with the qualitative results in Figure 2(a), Figure 3 shows that at early times the dynamics of ODDC are dominated by modes that have small horizontal scales ($\sqrt{l^2 + m^2} \approx 8$), with no structure in the vertical direction (with vertical wavenumber $k = 0$). Around $t = 2500$, the primary instability saturates. Mirouh et al. (2012) demonstrated that the level at which the primary instability saturates can be used to identify regions of parameter space where layer formation is expected to occur. However, the primary saturation level is of little use for characterizing the long-term transport properties of non-layered ODDC because a secondary growth phase occurs after primary saturation, further augmenting the thermal and compositional fluxes. From Figure 4 we see that while the fastest growing modes of the primary instability stop growing
at saturation, the mode family \((1, 0, 5)\) continues to grow and for a brief time becomes the most energetic mode family in the system. As time goes on, however, the mode family \((1, 0, 4)\) supplants \((1, 0, 5)\) as the most energetic, which is in turn overtaken by the mode family \((1, 0, 3)\). For this particular simulation, the handoff of energy to progressively larger scales stops with mode family \((1, 0, 3)\); mode family \((1, 0, 2)\) never becomes dominant. Crucially, Figures 3 and 4 also reveal the growth of the energy in large-scale shearing modes with purely horizontal fluid motions (mode families \((0, 0, 1)\) and \((0, 0, 2)\)). This is unexpected because these modes are not directly excited by the primary instability of ODDC. Instead, their growth must arise from nonlinear interactions between rapidly growing ODDC modes.

Around \(t = 5000\) after the mode family \((1, 0, 3)\) becomes dominant, the secondary growth phase appears to end, saturating into a statistically steady turbulent state. However, Figure 4 shows periodic bursts of growth in the shearing mode energies, suggesting intermittent (yet powerful) interactions between the shearing modes and the dominant gravity wave.
modes. In these interactions, illustrated in more detail in Figure 5, the growth of gravity waves excites the rapid growth of horizontal shearing modes, which in turn causes a decay in the amplitude of the waves. Without the wave field to amplify it, the shear then decays viscously. This finally allows the energy in the gravity waves to ramp back up again. While this interaction does not always manifest itself as such a distinct relaxation oscillation, it is still present in one form or another in all gravity-wave-dominated ODDC simulations. Figure 5 also shows that the interaction between the shear and gravity waves leads to intermittent modulation of the thermal and compositional fluxes, a result which may have some observational implications (see Section 5).

While the intermittency in the fluxes caused by the shear is interesting in its own right, for the purpose of modeling transport by ODDC in planetary or stellar evolution calculations, we are more concerned with estimating the mean fluxes over significant periods of time. These mean fluxes at secondary saturation depend on the parameters of the system ($R_0^{-1}$, Pr, and $\tau$). The results shown in this section, which were obtained at moderate $R_0^{-1}$ and moderate Pr and $\tau$, suggest that the turbulent transport in non-layered ODDC is weak. Indeed, Figure 5 shows that Nu_T and Nu_p remain of order unity throughout the simulations. To determine whether this is a generic property of ODDC at $R_0^{-1} > R_L^{-1}$, we need to run numerical experiments at larger $R_0^{-1}$ and smaller Pr and $\tau$. Probing this region of parameter space is difficult, however, because 3D simulations at low Pr and $\tau$ can be computationally very expensive, particularly for values of $R_0^{-1}$ that are close to marginal stability ($R_0^{-1} \rightarrow \frac{Pr + 1}{Pr + \tau}$). Small values of Pr and $\tau$ lead to small-scale turbulent features with steep gradients of velocity, temperature, and composition, which necessitate high spatial resolution. Furthermore, a larger $R_0^{-1}$ implies a larger buoyancy frequency ($N = \sqrt{Pr(R_0^{-1} - 1)}$) and leads to higher-frequency oscillations of the basic ODDC modes, necessitating smaller time steps. Given these challenges, in the next section we discuss the possibility of using 2D ODDC simulations as a potential surrogate for full 3D simulations at these extreme regions of parameter space.

4. 2D SIMULATIONS

Simulations of 2D ODDC are computationally inexpensive and are also less intensive in terms of data storage than 3D simulations. For this reason, we ran a series of tests to compare both the qualitative behavior and the quantitative estimates of the fluxes (and other system diagnostics) in 2D and 3D. Simulations in 2D and 3D often lead to very different types of dynamics, especially at low Pr (Schmalzl et al. 2004; van der Poel et al. 2013; Garaud & Brummell 2015). Fortunately, however, as we see from Figure 6 the secondary saturation level in the 2D simulation at Pr = $\tau = 0.03$ and $R_0^{-1} = 7.87$ is very similar to that of the 3D simulation analyzed in the previous section. This is, in fact, generally the case for each parameter set (Pr, $\tau$, $R_0^{-1}$) at $R_0^{-1} > R_L^{-1}$ where we have both 2D and 3D simulations.

Measurements of mode family energies show that key physical processes that dictate the behavior of 3D ODDC simulations are present in the 2D simulations as well. Figure 7
explores the energetics of the gravity waves and shear, and shows that the fractions of energy in each type of mode are of the same order in both cases. This is important because together these two types of modes contain most of the energy in non-layered systems after secondary saturation.

The computational economy of 2D simulations makes other types of analysis easier as well, such as running simulations in larger domains. In the previous section we showed that after primary saturation the dominant gravity wave modes have horizontal wavelengths commensurate with the domain size. We also showed that energy is transferred to modes with progressively larger vertical wavelengths. This raises the question of whether this energy transfer would always terminate at a vertical wavelength dependent on the domain size. For example, will the dominant mode after secondary saturation in a $200d^3$ domain have a vertical wavelength that is twice that of the dominant mode in a $100d^3$ domain? More importantly, do the fluxes depend on the domain size?

Using 2D data we find in all but one case that doubling the domain size in each direction leaves the vertical wavelength of the dominant mode unchanged. By contrast, the horizontal wavelength of the dominant mode always grows to the largest possible scale allowed by the domain. As a consequence, the dominant modes in the larger boxes are inclined more toward the horizontal than in the smaller ones. Importantly though, Figures 6(a) and (b) show that despite some qualitative differences between simulations with domains of different dimensions, we find that the time-averaged fluxes of temperature and chemical composition do not depend strongly on box size (they are within ~10% of one another). In the next section we therefore rely heavily on 2D simulations to draw conclusions about turbulent fluxes through non-layered ODDC.

5. RESULTS AND DISCUSSION

We now analyze the results of all numerical experiments done in 2D and 3D computational domains. We evaluate thermal and compositional fluxes in terms of the Nusselt numbers, which we calculate from thermal and compositional

Figure 6. (a) and (b) show the thermal and compositional dissipations vs. time for the simulation with $\text{Pr} = \tau = 0.03$ and $R_s^{-1} = 7.87$. Included are data from a 3D simulation, and two 2D simulations with differing domain sizes. While the larger 2D simulation takes longer to achieve its statistically stationary state, the mean fluxes are ultimately very similar for all three runs.

Figure 7. (a) Energy in gravity wave families of the form $(1, 0, n)$ as a percentage of the total energy in the system. (b) Energy in shearing modes families of the form $(0, 0, m)$ as a percentage of the total energy. We estimate errors according to the method described in Section 5.
transport in non-layered ODDC, the enhanced fluxes are only slightly larger than the transport due to thermal and molecular diffusion alone. Furthermore, this enhancement rapidly decreases with increasing $R_0^{-1}$. For the simulations that we ran with the smallest values of $R_0^{-1}$ (those closest to the layering threshold), the turbulent compositional flux is at most twice that of the flux due to diffusion alone and the turbulent heat flux is at most $\sim$20% of the diffusive flux. However, at larger values of $R_0^{-1}$, closer to marginal stability, the turbulent fluxes drop down to $\sim$5%-10% of the diffusive fluxes. Critically, we also find that the turbulent fluxes decrease as Pr and $\tau$ decrease. The simulations run at the parameter regime most similar to actual giant planetary interiors ($Pr = \tau = 0.003$) suggest that the mixing induced by non-layered ODDC at $R_0^{-1} > R_L^{-1}$ is effectively negligible in this case.

Another result of our analysis is that gravity wave-dominated ODDC is responsible for the generation of large-scale shear. In all the simulations we have run so far, the main effect of the shear has been to modulate the wave-induced transport through strong nonlinear interactions with the wave field. One might wonder, however, whether the shear could become strong enough in some parameter regimes to trigger a shear instability which would then dramatically augment turbulent transport. To evaluate the likelihood of this happening, we consider the Richardson number, $Ri$, which is the ratio of the potential energy needed to cause overturn to the available kinetic energy in the shear. In the units of this paper, we define the Richardson number as

$$Ri(z) = \frac{N^2}{\left(\frac{\partial \bar{u}}{\partial z}\right)^2} \simeq Pr \left(\frac{R_0^{-1} - 1}{\left(\frac{dv}{dz}\right)^2 + \left(\frac{d\bar{v}}{dz}\right)^2}\right),$$

(15)

where $N$ is the buoyancy frequency, defined dimensionally as

$$N^2 = -g \frac{d \ln \rho}{dz},$$

where $\rho$ is the background density profile. The terms $\bar{u}$ and $\bar{v}$ are the horizontally averaged $x$ and $y$ components of velocity, respectively (for 2D simulations $v = 0$ everywhere, for all time). To calculate the typical minimum Richardson number for a simulation, we find the minimum of $Ri(z)$ for an individual time step and then take a time average of $Ri_{\text{min}}(z)$ over the period after secondary saturation. A plot of the time-averaged minimum Richardson number of the available simulations (Figure 9) shows that $Ri_{\text{min}}$ increases as $r$ (or equivalently, $R_0^{-1}$) increases. This is because by definition systems with higher $R_0^{-1}$ have a stronger stabilizing compositional stratification compared to their unstable thermal stratification, making them less susceptible to overturning. Also, recall from Figure 7 that simulations with higher values of $R_0^{-1}$ have a lower percentage of their total kinetic energy in shearing modes. This Richardson number data therefore suggests that if any shear-induced instabilities were to present themselves, they would do so at values of $R_0^{-1}$ that are very close to $R_L^{-1}$, the threshold for the layering instability.

To summarize our results so far, we found that mixing induced by non-layered ODDC at $R_0^{-1} > R_L^{-1}$ is mostly negligible with turbulent fluxes at most of the order of diffusive fluxes (usually several orders of magnitude smaller). The propensity for gravity wave-dominated ODDC to drive shear could be observationally interesting, however.

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3 We define the secondary saturation time to be the point at which the total kinetic energy of the system reaches a statistically stationary level and we identified it by inspection.
6. CONCLUSION

6.1. Synthesis of Results from Papers I, II, and III

This study marks the conclusion of a series of papers aimed at describing the thermal and compositional flux properties of ODDC (semi-convection), throughout the entire linearly unstable range for fluid parameters appropriate to stellar and planetary interiors. Rosenblum et al. (2011) laid the groundwork for this series by conducting a preliminary survey of the ODDC parameter space and identified that ODDC either spontaneously forms layers, or remains in a non-layered, mildly turbulent state, which Mirouh et al. (2012) later showed to be dominated by large-scale gravity waves. They also showed that the critical parameter to making predictions about layer formation is the inverse total flux ratio $\gamma_{\text{tot}}^{-1}$ defined as

$$\gamma_{\text{tot}}^{-1} = \frac{R_0^{-1} \mathrm{Nu}_{\mu} \gamma_{\text{turb}}^{-1}}{\mathrm{Nu}_{T}}. \quad (16)$$

More precisely, Rosenblum et al. (2011) showed that through the $\gamma$-instability (which was initially developed to describe the conditions that lead to layer formation in convection; Radko 2003) layers only form when

$$\frac{d\gamma_{\text{tot}}^{-1}}{dR_0^{-1}} < 0. \quad (17)$$

Next, Mirouh et al. (2012) (Paper I) produced a semi-analytical model for $\gamma_{\text{tot}}^{-1}$ given by

$$\gamma_{\text{tot}}^{-1} = \frac{\tau R_0^{-1} + \gamma_{\text{turb}}^{-1} (\mathrm{Nu}_T - 1)}{1 + (\mathrm{Nu}_T - 1)}. \quad (18)$$

They used the following empirically motivated prescription for $\mathrm{Nu}_T - 1$:

$$\mathrm{Nu}_T - 1 = (0.75 \pm 0.05) \left( \frac{\frac{Pr}{\tau}}{\tau} \right)^{0.25 \pm 0.15} \frac{1 - \tau}{R_0^{-1} - 1} (1 - r),$$

where $r$ is the quantity defined in Equation (4). Meanwhile, $\gamma_{\text{turb}}^{-1}$ is the inverse turbulent flux ratio and is expressed only in terms of parameters that can be calculated through a linear analysis of the original governing equations in (8). It is defined as

$$\gamma_{\text{turb}}^{-1} = \frac{\langle \overline{w \dot{\mu}} \rangle}{\langle w \dot{T} \rangle} = R_0^{-1} (\lambda_R + l^2) + \lambda^2 (\lambda_R + \tau l^2), \quad (19)$$

where $\lambda_R$ and $\lambda_1$ are the real and imaginary parts of the growth rate of the fastest growing mode of the primary instability of ODDC for a given parameter set $(R_0^{-1}, \mathrm{Pr}, \tau)$ and $l$ is the corresponding horizontal wavenumber of the fastest growing mode (see their appendix for an explanation on how $l$, $\lambda_R$ and $\lambda_1$ are calculated, as well as analytical approximations in the limit of small $\mathrm{Pr}$ and $\tau$).

Using their model for $\gamma_{\text{tot}}^{-1}$ Mirouh et al. (2012) were able to predict the range of $R_0^{-1}$ where spontaneous layer formation is possible. The function $\gamma_{\text{tot}}^{-1}$ is concave-up with a single minimum, so by identifying the value of $R_0^{-1}$ at which $\gamma_{\text{tot}}^{-1}$ reaches its minimum (referred to as $R_{L}^{-1}$) it is possible to identify the region of parameter space where layers naturally form from infinitesimal perturbations ($R_0^{-1} < R_{L}^{-1}$) and where they do not ($R_0^{-1} > R_{L}^{-1}$). They found that, typically

$$R_{L}^{-1} = \frac{R_{L}^{-1} - 1}{R_{G}^{-1} - 1} \sim \mathrm{Pr}^{\frac{1}{2}} \text{ and } R_{L}^{-1} \sim \mathrm{Pr}^{-\frac{1}{2}}. \quad (20)$$

Wood et al. (2013) (Paper II) then presented prescriptions for quantifying the thermal and compositional fluxes through layered systems in cases where $R_0^{-1} < R_{L}^{-1}$. They found that the thermal and compositional fluxes are

$$F_T = - \rho c_p \kappa_T \left[ \frac{\mathrm{Nu}_T - 1}{(T_{0T} - T_{0C})} + T_{0C} \right] = - \frac{\rho c_p \kappa_T}{p} \frac{\partial p}{\partial r} \left[ \frac{(\mathrm{Nu}_T - 1)}{(\nabla - \nabla_d) + \nabla_d} \right]. \quad (21)$$

Alternately, the transport of chemical species can be expressed as an effective diffusivity defined as

$$D_{\text{eff}, \mu} = \mathrm{Nu}_\mu \kappa_\mu. \quad (22)$$

In the above equations, the Nusselt numbers $\mathrm{Nu}_T$ and $\mathrm{Nu}_\mu$ can be modeled as

$$\mathrm{Nu}_T = 1 - A_T \tau^{-\frac{2}{3}} \mathrm{Pr}^b,$$

$$\mathrm{Nu}_\mu = 1 - A_\mu \tau^{-\frac{1}{3}} \mathrm{Pr}^d,$$

where $A_T \approx 0.1$ and $A_\mu \approx 0.03$, and where $a = 0.34 \pm 0.01, b = 0.34 \pm 0.03, c = 0.37 \pm 0.01$, and $d = 0.27 \pm 0.04$. The parameter $R_\tau$ is the thermal Rayleigh number for layered convection and is defined as a function of the layer height, $H$,

$$R_\tau (H) = \left( \frac{H^4}{d} \right) = \frac{\alpha g}{\kappa_T \tau^\nu} \left| T_{0C} - T_{0C} \right| H^4 \tau^\nu = \frac{\alpha g H^4}{\kappa_T \tau^\nu} \left| \nabla - \nabla_d \right| \tau^\nu. \quad (24)$$

As discussed by Wood et al. (2013), it is not clear a priori what the value of $H$ should be because in simulations of naturally layered systems the layers always gradually merge until only a single interface remains. This suggests that some other physical mechanism outside the scope of our model of ODDC determines layer height or that layers may always merge.
indeinitely, leaving the fluid fully mixed. For now it is left as a free parameter, much like the mixing length in mixing-length theory (see also Moore & Garaud 2016).

Finally, from the work done in this paper (Paper III), we have found that the turbulent transport of heat and composition can more or less be neglected in the non-layered ODDC parameter space (i.e., $R_{\infty}^{-1} < R_{0}^{-1} < R_{l}^{-1}$). We therefore model the total heat and compositional fluxes as

$$F_T = -\rho c_p \kappa_T \frac{dT}{dr} = -\rho c_p \kappa_T \frac{dT}{dr} = -\rho c_p \kappa_T \frac{dT}{dr} = -\rho c_p \kappa_T \frac{dT}{dr},$$

$$F_\mu = -\mu \frac{d\mu}{dr} = -\mu \frac{d\mu}{dr} = -\mu \frac{d\mu}{dr},$$

and we can write the corresponding effective compositional diffusivity simply as

$$D_{\text{eff}, \mu} = \kappa_\mu.$$  \hspace{1cm} (26)

Note that as discussed by Moore & Garaud (2016), $R_{0}^{-1}$ is typically much smaller than $R_{l}^{-1}$ in stellar models where the semiconvective region is adjacent to a convection zone. Our conclusions for non-layered ODDC would therefore only apply to stellar models with a semi-convection zone that is detached from the convection zone, or possibly to planetary models.

6.2. Possible Caveats to Transport Prescriptions

The model presented in Section 6.1 derives from a combination of “first-principle” theory and numerical experiments and has been demonstrated to provide a good fit to the macroscopic transport properties of ODDC in all available simulations. However, there are still potential sources of uncertainty in each of the three components of the model. Here we discuss uncertainty that may arise from the prescription for the layering cut-off, $R_{\infty}^{-1}$, described by Mirosh et al. (2012), the prescription for mixing in the layered case ($R_{0}^{-1} < R_{l}^{-1}$) proposed by Wood et al. (2013), and the prescription for mixing in the non-layered case ($R_{l}^{-1} < R_{0}^{-1} < R_{\infty}^{-1}$) described in this work. We also mention several specific circumstances under which we do not expect our model to remain valid.

As discussed by Mirosh et al. (2012), their proposed formula for $R_{\infty}^{-1}$ is likely to overestimate the true $R_{\infty}^{-1}$ by 20\%-40\% (see their study for an in depth explanation). This is not likely to affect stellar model predictions because in the most commonly occurring stellar semi-convection zones $R_{0}^{-1}$ is typically much smaller than $R_{\infty}^{-1}$ (Moore & Garaud 2016), meaning that their interiors are unambiguously in the regime where ODDC leads to spontaneous layer formation. In giant planets the effect of this uncertainty is less clear because no reliable estimates of $R_{0}^{-1}$ exist.

Next are caveats on the Wood et al. (2013) prescription for layered ODDC. There are four sources of uncertainty in their scalings: that in the dependence on Pr and $\tau$, on $R_{\infty}$, and on $R_{0}^{-1}$. Due to computational constraints, a limited range of values of Pr and $\tau$ was used to construct and test the model (between $\text{Pr} = 0.01$ and $\text{Pr} = 0.3$). While these values may be appropriate for the interiors of ice giant planets such as Uranus and Neptune (where Pr and $\tau$ are of order unity), and approximately valid in degenerate regions of stars and giant planets, such as Jupiter and Saturn (where Pr and $\tau$ are of order $10^{-3} - 10^{-4}$), they are very far from non-degenerate stellar values, where $\text{Pr} \sim \tau \sim 10^{-6}$ or less. So using these diffusivities in a stellar parameter regime requires some degree of trust in the extrapolation from experimental values of Pr and $\tau$ down to stellar values. They should be appropriate, however, for models of giant planets.

The scaling of the Nusselt numbers with $R_{\infty}$ (which suggests that the fluxes scale with the temperature difference and the compositional difference across the layers to the power of $\tau$) is the most robust because it is derived from arguments based purely on dimensional analysis (Turner 1968; Radko 2003). Note, however, that some layered convection simulations with tall thin aspect ratios do deviate from this $\tau$ rule. This is seen in particular in the early work of Rosenblum et al. (2011), where the use of a tall, thin domain artificially caused $N_{\infty}$ to scale with a weaker power of $H$ than $\tau$. The wider simulations of Wood et al. (2013) do not suffer from this problem at low $R_{\infty}$. However, even in these simulations there inevitably comes a point where the horizontal dynamics of the layers become increasingly constrained by their aspect ratios because the layers themselves become taller and thinner as the mergers take place. This is an artificial effect created by the necessary constraint of the chosen simulation domain, and should not be expected to occur in an actual star or planet. In other words, $N_{\infty} \propto (R_{\infty} \text{Pr})^{\tau}$ and $N_{\infty} \propto \tau$ are expected to hold at any stellar value.

The greatest source of uncertainty in the Wood et al. (2013) scalings is the Nusselt numbers’ dependence on $R_{0}^{-1}$. For the layered simulations studied in that work, $R_{0}^{-1}$ was typically $\sim$1–3 (due to computational constraints). However, at stellar values of Pr, layers can occur at inverse density ratios much larger than 10. In fact, for non-degenerate regions of a typical stellar interior $R_{0}^{-1} \sim 1000$. This naturally leads to the question of how $N_{\infty}$ and $N_{\mu}$ scale with the inverse density ratio at large $R_{0}^{-1}$. While this line of inquiry is interesting in its own right from a hydrodynamic perspective, it is probably not relevant in semiconvective zones adjacent to convection zones in actual stars, where typical values of $R_{0}^{-1}$ are more likely to be much closer to one (Moore & Garaud 2016). For parameters relevant to giant planetary interiors, the layering threshold is $R_{0}^{-1} \sim 1$–10, so the simulations studied by Wood et al. (2013) spanned a substantial portion of the range of $R_{0}^{-1}$ where layered convection occurs. Therefore, the Nusselt numbers’ dependence on $R_{0}^{-1}$ is not a significant source of uncertainty in the context of giant planets.

Finally, there are also uncertainties in our flux model for ODDC in the regime $R_{l}^{-1} < R_{0}^{-1} < R_{\infty}^{-1}$. First, in Equations (25) and (26) we simply chose to ignore the turbulent contribution to the fluxes due to non-layered ODDC on the grounds that it is usually very small, although the turbulent fluxes can admittedly be nearly as large as the diffusive fluxes when $R_{0}^{-1}$ is close to the layering threshold $R_{l}^{-1}$ (see Figure 8). We did this for simplicity, but the reader who is interested in improving the estimates for the total fluxes in that limit should feel free to add the turbulent fluxes as needed. Second, there is a more important model uncertainty which comes from the possibility that ODDC might yet take the form of layered convection, even when $R_{0}^{-1} > R_{l}^{-1}$. Indeed, there is a well-known subcritical branch of solutions (Huppert & Moore 1976; Proctor 1981), where layered convection can be triggered by very carefully selected finite amplitude perturbations, even when the system is stable to infinitesimal ones. Preliminary
simulations show that if a numerical experiment is initialized in an already-layered state it will remain layered indefinitely. This is true even if the system is fully stable \((R_0^{-1} > R_L^{-1})\) or simply stable to the layering \(\gamma\)-instability \((R_0^{-1} < R_L^{-1} < R_R^{-1})\). Whether Equations (21)–(26) are still valid for layered convection at \(R_L^{-1} > R_R^{-1}\) remains to be determined. However, as we will discuss below, no compelling mechanism so far has been proposed to explain how these very specific kinds of layered initial conditions may naturally occur in stars of planets. Until such a mechanism is found, we argue that non-layered ODDC is likely to be far more prevalent at values of \(R_0^{-1}\) greater than the layering threshold \(R_L^{-1}\).

There are several other conditions under which our model (Equations (21)–(26)) cannot be expected to hold. For example, in the presence of strong rotation, the above prescriptions will probably not apply (Blies et al. 2014). Rotation is known to inhibit regular overturning convection, and preliminary results indicate that strong rotation suppresses transport of both temperature and composition in ODDC as well. A more in depth discussion of the effects of rotation will be the subject of a future paper. Magnetic fields could clearly also affect our results.

6.3. Implications for Astrophysical Modeling

The implication of our findings from this paper for planetary modeling is that non-layered ODDC leads to fluxes that are not significantly larger than thermal conduction or molecular diffusion. Consequently, it is not sufficient simply to know if regions in the interior of a giant planet are unstable to ODDC. The fact that layered and non-layered ODDC lead to different transport characteristics means that special attention must be paid to calculating the threshold \(R_L^{-1}\), to determine which type of behavior will manifest.

The dynamics of non-layered ODDC is not expected to be pertinent to intermediate mass main sequence stars where most semiconvective regions likely have values of \(R_0^{-1}\) that are in the layered regime (Moore & Garaud 2016). However, it may be important in more massive stars that have standalone semiconvective zones, i.e., regions unstable to ODDC that are well-separated from convective zones, and in giant planets, where higher values of Pr (compared to stars) indicate a smaller range of \(R_0^{-1}\) that is unstable to ODDC, and a lower layering threshold, \(R_L^{-1}\).

In particular, the near-zero luminosity of Uranus suggests that thermal transport through the planet’s interior is inefficient (Hubbard et al. 1995). Advances in equation of state research (Redmer et al. 2011) lend credence to the idea that convection is being inhibited by steep compositional gradients. If this is the case, the inefficient gravity wave-dominated ODDC discussed in this paper may potentially play a role in Uranus’s thermal evolution.

Also, though it is not yet known whether this phenomenon could produce observable signatures, the intermittent growth of shear layers discussed in this work is a potentially significant feature of non-layered ODDC in the deep interiors of giant planets. Indeed, in contrast to our simulations where there is a symmetry between the x and y directions, global rotation may provide a preferred direction for shearing motions, which could induce large-scale azimuthal flows.

6.4. Discussion of Prior Studies

In this series of papers we have proposed a new prescription for transport due to ODDC in both its layered and non-layered forms based on an analysis of a comprehensive suite of numerical experiments. We now compare our complete model summarized in Section 6.1 with existing work in the astrophysical literature. Typically, prior studies on this topic have only addressed either the layered or non-layered form of double-diffusive convection, being unaware, perhaps, that both regimes may in fact occur. Langer et al. (1983), for example, derived an expression for the effective diffusivity of composition through stellar semiconvective regions without invoking thermocompositional layers. They proposed that

\[
D_{\mu} = \frac{\alpha \mu \tau \nu}{6} \frac{\nabla - \nabla_d}{\nabla_d + \frac{\beta}{4\mu} \nabla_{\mu} - \nabla},
\]

where in this case, \(\alpha\) is an efficiency factor. By contrast, the majority of other studies have assumed that double-diffusive regions are always layered. Stevenson (1982), who was the first to import this notion to the astrophysical context, proposed a relationship between thermal and compositional transport in double-diffusive regions in Jupiter and Saturn,

\[
F_{\mu} \sim \tau^{-1/2} F_T.
\]

This kind of law naturally arises when assuming that the fluxes are limited to diffusive transport through thin, stably stratified interfaces (see Linden & Shirtcliffe 1978, and below). Later, Spruit (1992) also developed a parametrization for transport through layers separated by stable interfaces of this kind and proposed that the thermal Nusselt number and compositional diffusivity scale as

\[
N_{\mu} \sim \left(Pr Ra_T\right)^{1/2},
\]

\[
D_{\mu} \sim \left(\alpha \mu \tau \nu \right)^{1/2} \left(\frac{4}{\beta} - 3\right) \frac{\nabla - \nabla_d}{\nabla_{\mu}},
\]

where \(\nabla_{\nu}\) is the radiative gradient. More recently, works by Spruit (2013) and Zaussinger & Spruit (2013) have revised this model to account for the theoretically expected and experimentally observed, \(4/3\) flux law that Spruit’s original model did not satisfy. Also, recently Leconte & Chabrier (2012) have developed a formalism for determining the transport of heat and chemical composition in a giant planetary interior composed of convective thermocompositional layers. With their formalism, which is similar to that of mixing-length theory, they calculate the number of layers that compose a giant planetary interior, assuming as in previous studies that the interfaces between the layers are diffusive.

As we have shown in this paper, however, layers do not always necessarily form. The specious notion pervading recent astrophysical models that double-diffusive regions must always take the form of layers separated by thin, diffusive interfaces likely stems historically from earlier experimental work on double-diffusive convection in the geophysical context. In geophysical fluids where \(Pr \sim 1\), the range of \(R_0^{-1}\) that is unstable to infinitesimal perturbations (see Equation (3)) is very small \((R_0^{-1} = 1.14\) for \(Pr = 7\) and \(\tau = 0.01\), and the layering threshold, \(R_L^{-1}\), presumably lies between 1 and 1.14, if it even exists). Meanwhile, double-diffusive layering is observed fairly
frequently in volcanic lakes and in the arctic ocean, and the mean temperature and compositional gradients through these staircases have corresponding density ratios well in excess of $R^{-1}_L$ (i.e., the linearly stable regime). While the necessary finite amplitude mechanism by which these staircases form in this case remain to be determined even to date, such a mechanism must clearly exist in the geophysical case since the layers are observed. Laboratory experiments to study double-diffusive layers at high Pr therefore nearly always start with layers already present and merely focus on measuring the fluxes through the staircase.

Turner (1965) was the first to conduct laboratory experiments of thermod compositional double-diffusive convection in such a layered regime. In those experiments a layer of cold, low-salinity water was deposited carefully on top of a layer of high-salinity water that was then heated from below. The fluxes of temperature and salt across the sharp interface were then measured. In that study the following prescription was proposed for the fluxes across the interface

$$F_T \propto (\Delta T)^{3/2},$$
$$F_S \propto (\Delta S)^{3/2},$$

(30)

where $\Delta T$ and $\Delta S$ are the temperature and salinity differences between the two layers. As discussed earlier, these scaling laws can be derived from simple dimensional arguments (see Section 6.2). Later, Shirtcliffe (1973) conducted similar laboratory experiments of double-diffusive convection where the diffusive quantities were sugar and salt dissolved in water. Linden & Shirtcliffe (1978) then used Shirtcliffe’s results to develop the prescription for the relationship between the thermal and compositional fluxes that Stevenson (1982) later applied to the astrophysical case (see Equation (28)) and which is also at the heart of the models presented by Spruit (1992, 2013).

Given that geophysical and astrophysical double-diffusive convection are governed by the same basic equations, and without the help of numerical simulations at low Pr, it was natural for Stevenson (1982) and Spruit (1992) to extrapolate from the results of geophysical experiments to draw conclusions about astrophysical systems. This is, in fact, what led them to assume that astrophysical double-diffusive convection takes the form of stacked convective layers separated by thin, quiescent interfaces. However, with the help of modern numerical simulations and thanks to the work presented in this series of papers, we have now established that such an extrapolation results from geophysical double-diffusive convection to the astrophysical parameter regime is not only inadvisable, but in many cases incorrect.

It is crucial to remember that by contrast to the geophysical case, in giant planets $Pr \sim \tau \sim 10^{-2}-10^{-3}$, and $R^{-1}_L \approx 10–100$, meaning that there is a much wider range of $R_0^{-1}$ for which infinitesimal perturbations may trigger ODDC. In fact, $R^{-1}_L$ is so large that to achieve $R_0^{-1} > R^{-1}_L$ requires in many cases an unphysically large compositional gradient. In other words, unlike the Earth’s oceans and lakes, it is likely that the linear instability is a ubiquitous feature of double-diffusive regions in stars and planet. Far from being merely an intellectual distinction, the way in which the instability is excited has two important effects on the dynamics of double-diffusive systems in both the layered and non-layered parameter regimes.

First, in ODDC when $1 < R_0^{-1} < R^{-1}_L$, the layers that spontaneously form are fundamentally different from the layers observed in geophysics. In laboratory experiments done at geophysical parameters (where finite amplitude perturbations are necessary to initiate layered thermod compositional convection) (Noguchi & Niino 2010; Carpenter et al. 2012), layers appear to persist indefinitely and the layer interfaces are quiescent and diffusive. By contrast, in the layered simulations presented in this series of papers, the interfaces between layers are dynamic and turbulent and the layers tend to merge until a single one remains in the domain. More than being merely a qualitative difference, the different character of the layers in each context leads to significant quantitative differences in thermal and compositional transport. In particular, Wood et al. (2013) found that contrary to Equation (28) proposed by Stevenson (1982) and used by Spruit (1992, 2013), $\gamma^{-1}_{tot}$ is not proportional to $\tau^2$ in systems where layers form spontaneously.

Second, the non-layered ODDC regime is never observed in geophysical experiments, which is not surprising as (1) the range of parameters for which it could theoretically be observed is tiny, and (2) most experiments are initialized with layers in the first place. By contrast, non-layered ODDC is found so far to be ubiquitous in astrophysical simulations initialized with infinitesimal perturbations for $R_0^{-1} < R_0^{-1} < R^{-1}_L$. While it is possible that finite amplitude layering may naturally occur in astrophysics, no separate layering mechanism has yet been proposed to suggest that these finite amplitude instabilities would be as commonplace in astrophysics as the linear instability is likely to be. Until such a mechanism is found, it is therefore preferable to focus on the dynamics of ODDC that naturally develop from random infinitesimal perturbations rather than artificially imposed initial conditions consisting of layers of unspecified origin. Nevertheless, the behavior of high-density ratio layered ODDC at astrophysical parameters is an interesting problem that deserves further study and will therefore be the topic of a future paper.

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