Rainbow matchings of size $m$ in graphs with total color degree at least $2mn$

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Abstract
The existence of a rainbow matching given a minimum color degree, proper coloring, or triangle-free host graph has been studied extensively. This paper generalizes these problems to edge colored graphs with given total color degree. In particular, we find that if a graph $G$ has total color degree $2mn$ and satisfies some other properties, then $G$ contains a matching of size $m$. These other properties include $G$ being triangle-free, $C_4$-free, properly colored, or large enough.

Mathematics Subject Classifications: 05C15, 05C70

1 Introduction

Given a graph $G$, let $V(G)$ denote the vertex set of $G$ and $E(G)$ denote the edge set of $G$. If $S \subseteq V$, then $G[S]$ denotes the subgraph induced by the vertices in $S$. A graph $G$ is an $m$-matching if $G$ contains exactly $m$ edges, $2m$ vertices, and $e \cap e' = \emptyset$ for all edges $e \neq e'$ in $E(G)$. An edge coloring $c : E(G) \to [r] = \{1, \ldots, r\}$ is an assignment of colors to edges. A proper edge coloring of a graph is an edge coloring such that $c(e) \neq c(e')$ whenever $e \cap e' \neq \emptyset$ and $e \neq e'$. The colors used on a graph will be denoted $c(G)$, and $R$ will denote a generic color class. If $X, Y \subseteq V(G)$, then $c(X, Y)$ will denote the set of colors used on edges of the form $xy$, where $x \in X$, $y \in Y$. A graph $G$ is rainbow under $c$ if $c$ is injective on $E(G)$. In particular, a rainbow matching is a matching where each edge receives a unique color within the matching. The color degree of a vertex $v$ is denoted $\hat{d}_G(v)$, which is the number of colors $c$ assigns to edges incident upon $v$ in $G$; when it is clear from the context what $G$ is, we will drop the subscript. Let $\hat{d}^R(v)$ denote the

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number of $R$ colored edges incident upon $v$. The total color degree of $G$ with respect to $c$ is the sum of all the color degrees in the graph and denoted

$$\hat{d}(G) = \sum_{v \in V(G)} \hat{d}(v).$$

The average color degree of a graph $G$ is obtained by dividing the total color degree by $|V(G)|$, and is an equivalent notion. The minimum color degree of $G$ is denoted $\hat{\delta}(G)$. Finally, let $G - v$ denote the graph $G$ with the vertex $v$ deleted, and $G - R$ denote the graph $G$ with the edges in color class $R$ deleted. When convenient, we will let $c(e)$ denote a color class so that $G - c(e)$ denotes the graph $G$ without the edges in color class containing the edge $e$.

Rainbow matchings in graphs were originally studied in connection to transversals of Latin squares [9, 10]. However, the existence of rainbow matchings has also been studied in its own right. In [6], Li and Wang conjectured that any graph with $\hat{\delta}(G) \geq m \geq 4$ contains a rainbow matching of size $\lceil \frac{m}{2} \rceil$. This conjecture was partially confirmed in [5], and fully confirmed in [4].

Wang asked for a function $f$ such that any properly edge colored graph $G$ with $|V(G)| \geq f(\hat{\delta}(G))$ contains a rainbow matching of size $\hat{\delta}(G)$ [11]. Diemunsch et al. determined that $|V(G)| \geq \frac{98}{23}\hat{\delta}(G)$ is sufficient [1]. This problem was generalized to find a function $f$ such that any edge colored graph $G$ with $|V(G)| \geq f(\hat{\delta}(G))$ contains a rainbow matching of size $\hat{\delta}(G)$. The authors of [3] found that $|V(G)| \geq \frac{17}{4}\hat{\delta}(G)^2$ sufficed. This was improved to $4\hat{\delta}(G) - 4$ for $\hat{\delta}(G) \geq 4$ in [2] and [8] independently.

Local Anti-Ramsey theory asks Anti-Ramsey type questions with assumptions about the local structure of the host graph. In particular, Local Anti-Ramsey theory is about the minimum $k$ such that any coloring of $K_n$ with $\hat{\delta}(G) \geq k$ contains a rainbow copy of $H$. In this vein, Wang’s question can be posed as follows: given $k$, what is the smallest $N$ such that any properly edge colored graph $G$ with $|V(G)| \geq N$ and $\hat{\delta}(G) \geq k$ contains a rainbow matching of size $k$? Furthermore, proper edge-coloring and triangle-free properties play similar roles in restricting the structure of a host graph.

The local assumptions in Anti-Ramsey theory are interesting in so far as they highlight the relationship between a local parameter and the target graph. In much of the rainbow matching literature, there are confounding local assumptions. For example, [1], [7], and [11] all consider host graphs that have a prescribed minimum color degree and are properly edge colored. In this case, an intuitive interpretation is that the minimum color degree and proper edge-coloring properties spread the colors apart in the host graph. As one would expect, this makes it easier to find a large rainbow matching. However, it is unclear whether both the minimum color degree and proper edge coloring property are necessary to find a large matching.

The goal of this paper is to shed light on the relationship between local assumptions and rainbow matchings. Rather than considering host graphs with a prescribed minimum color degree, we will consider host graphs with a prescribed average color degree. This is motivated in part by a question posed during the Rocky Mountain and Great Plains Graduate Research Workshop in Combinatorics in 2017.
**Question 1.** If $G$ is an edge colored graph on $n$ vertices with $\hat{d}(G) \geq 2mn$, does $G$ contain a rainbow matching of size $m$?

Section 2 considers this question for triangle-free and $C_4$-free host graphs. In the case of triangle-free graphs, we will prove the slightly stronger statement that if $G$ is a graph with $\hat{d}(G) > 2mn$, then there exists a rainbow matching of size $m + 1$. Section 3 pertains to properly edge colored host graphs. Finally, Section 4 considers edge colored graphs with total color degree $2mn$, but with no further assumptions.

## 2 Triangle-free and $C_4$-free Graphs

In this section, we consider triangle-free and $C_4$-free graphs.

**Theorem 2.** Let $G$ be a triangle-free graph on $n$ vertices. Let $c$ be an edge coloring of $G$ with $\hat{d}(G) > 2mn$. Then $c$ admits a rainbow matching of size $m + 1$.

**Proof.** For the sake of contradiction, let $M$ be a maximum rainbow matching of size $k \leq m$ with edges $u_iv_i$ for $1 \leq i \leq k$, such that the number of colors appearing on $G[V(G) \setminus V(M)] = H$ is maximized. Without loss of generality, suppose that $c(u_iv_i) = i$. Since $G$ is triangle-free, $\hat{d}(u_i) + \hat{d}(v_i) \leq n$ for all $u_iv_i \in E(M)$. If $H$ has an edge $e$, then $c(e) \in [k]$. Without loss of generality, suppose that $c(H) = [j]$ for some $0 \leq j \leq k$. Then for all $v \in V(H)$, we have $\hat{d}(v) \leq k + j$. Notice that if there exists an edge $e \in H$ with $c(e) = i$, then we can swap $e$ and $u_iv_i$ to conclude that $\hat{d}(u_i) + \hat{d}(v_i) \leq 2(j + k)$.

Now consider

$$2mn < \sum_{i=1}^{k} \hat{d}(u_i) + \hat{d}(v_i) + \sum_{v \in H} \hat{d}_G(v)$$

$$\leq \sum_{i=1}^{j} \hat{d}(u_i) + \hat{d}(v_i) + \sum_{i=j+1}^{k} \hat{d}(u_i) + \hat{d}(v_i) + \sum_{v \in H} \left( \hat{d}_H(v) + k \right)$$

$$\leq 2j(k + j) + (k - j)n + (n - 2k)(j + k)$$

$$= 2j^2 + 2j^2 + 2nk - 2jk - 2k^2$$

$$\leq 2j^2 - 2k^2 + 2nk$$

$$\leq 2nm.$$  

This is a contradiction; therefore, $k \geq m + 1$.  

A key element to the proof of Theorem 2 is the bound $\hat{d}(v) + \hat{d}(u) \leq n$ where $uv$ is an edge in a maximal matching. We can obtain a similar bound in $C_4$-free graphs in order to prove the next theorem.

**Theorem 3.** Let $G$ be a $C_4$-free graph on $n$ vertices. Let $c$ be an edge coloring of $G$ with $\hat{d}(G) \geq 2mn$. Then $c$ admits a rainbow matching of size $m$.  

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Proof. For the sake of contradiction, let $M$ be a maximum rainbow matching of size $k < m$ with edges $u_iv_i$ for $1 \leq i \leq k$, such that the number of colors appearing on $G[V(G) \setminus V(M)] = H$ is maximized. Without loss of generality, suppose that $c(u_iv_i) = i$. Since $G$ is $C_4$-free, $\hat{d}(u_i) + \hat{d}(v_i) \leq n + 1$ for all $u_iv_i \in E(M)$. If $H$ has an edge $e$, then $c(e) \in [k]$. Without loss of generality, suppose that $c(H) = [j]$ for $0 \leq j \leq k$.

Claim 4. If $xy \in E(H)$ with $c(xy) = i \leq j$, then $\hat{d}(u_i) + \hat{d}(v_i) \leq 2j + 2k$.

Notice that $x, y$ each see at most $j$ colors in $H$. Since $xy$ can share at most two edges with any edge in $M$ without creating a $C_4$ subgraph, we have $|c(\{u_i, v_i\}, xy)| \leq 2$ for every $1 \leq i \leq k$. Thus, $\hat{d}(x) + \hat{d}(y) \leq 2j + 2k$. By swapping $u_iv_i$ and $xy$, we obtain the desired bound on $\hat{d}(u_i) + \hat{d}(v_i)$.

Furthermore, $\sum_{v \in H} \hat{d}_G(v) \leq (n - 2k)(j + k) + k$. The $(n - 2k)j$ term comes from the fact that $H$ has $n - 2k$ vertices, each of which can see every color in $[j]$. We will show that there are at most $(n - 2k)k + k$ color degrees in $H$ that do not come from a color in $[j]$ by contradiction. Suppose that there are $(n - 2k)k + k + 1$ edges from $H$ to $M$. By the pigeon hole principle, there exists an edge $u_iv_i \in M$ that receives at least $n - 2k + 2$ edges from $H$. Notice that each vertex in $H$ can send at most two edges to $u_iv_i$. Therefore, there must exist two vertices in $H$ that each send two edges to $u_iv_i$, witnessing a $C_4$ subgraph; this is a contradiction.

Now consider

$$2mn \leq \sum_{i=1}^{k} \hat{d}(u_i) + \hat{d}(v_i) + \sum_{v \in H} \hat{d}_G(v)$$

$$\leq \sum_{i=1}^{j} \hat{d}(u_i) + \hat{d}(v_i) + \sum_{i=j+1}^{k} \hat{d}(u_i) + \hat{d}(v_i) + \sum_{v \in H} (\hat{d}_H(v) + k)$$

$$\leq j(2k + 2j) + (k - j)(n + 1) + (n - 2k)(j + k) + k$$

$$= 2kj + 2j^2 + nk + k - nj - j + nj + nk - 2kj - 2k^2 + k$$

$$\leq 2j^2 + 2nk - j + 2k - 2k^2$$

$$\leq 2j^2 - 2k^2 + 2k - j - 2n + 2mn$$

$$< 2mn.$$

This is a contradiction; therefore, $k \geq m$. 

3 Properly Edge Colored Graphs

In this section, we consider properly edge colored graphs. The idea to analyze a greedy algorithm that constructs a matching appears in [1] and [3]. The algorithm employed in this section is similar, with some adjustments to take into account the weaker degree assumption.

Theorem 5. Let $c$ be a proper edge coloring of $G$ with $n \geq 8m$ and $\hat{d}(G) \geq 2mn$. Then $c$ admits a rainbow matching of size $m$. 

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Proof. Assume that $G$ is an edge minimal counter example to Theorem 5. Consider the following algorithm:

1. set $G_0 := G$
2. if there exists $v \in V(G_{i-1})$ with $\hat{d}(v) \geq 3(m-i)+1$, then $G_i = G_{i-1} - v$ and return to 2
3. else, if there exists color class $R$ with $|R| \geq 2(m-i)+1$ in $G_{i-1}$, then $G_i = G_{i-1} - R$ and return to 2
4. else, if there exists $uv \in E(G_{i-1})$, then $G_i = G_{i-1} - u - v - c(uv)$ and return to 2
5. return $i - 1$

Claim 6. Suppose the algorithm returns $k \leq m$. Then $G_i$ contains a matching of size $k - i$ for $0 \leq i \leq k$.

We will prove the claim by reverse induction on $i$. If $i = k$, then $G_i$ is empty, and the claim is true. Assume that the claim is true for $i$. We will prove the claim for $i - 1$. By the induction hypothesis, there exists a matching $M \subseteq G_i$ of size $k - i$. There are three cases:

Case 1: Assume $G_i = G_{i-1} - v$ where $\hat{d}(v) \geq 3(m-i)+1$. By construction, $v \notin V(M)$. Since $\hat{d}(v) \geq 3(m-i)+1$, there exists $u \in N(v)$, such that $u \notin V(M)$ and $c(uv) \notin c(M)$. Then $M' = M \cup \{uv\}$ is a rainbow matching of size $k - i + 1$.

Case 2: Assume $G_i = G_{i-1} - R$ for some color class $R$ with $|R| \geq 2(m-i)+1$. This implies that $c(e) \neq R$ for all $e \in E(M)$. Since $c$ is a proper coloring and $|R| \geq 2(m-i)+1$, there exist $e \in G_{i-1}$ such that $c(e) = R$ and $M' = M \cup \{e\}$ is a rainbow matching.

Case 3: Assume that $G_i = G_{i-1} - v - u - c(uv)$ for some $uv \in E(G_{i-1})$. By construction $N[u] \cup N[v]$ is disjoint from $V(M)$ and $c(e) \neq c(uv)$ for all $e \in M$. Therefore, $M' = M \cup \{uv\}$ is a rainbow matching.

This concludes the proof of the claim. Since $G$ is an edge minimal counter example, the algorithm applied to $G$ will return $k < m$. We will now derive a contradiction.

Let $W(G_i)$ denote the difference of total color degree between $G_i$ and $G_{i-1}$ under $c$.

Claim 7. For all $1 \leq i \leq k$, we have $W(G_i) \leq 2n$.

Case 1: Assume $G_i = G_{i-1} - v$ where $\hat{d}(v) \geq 3(m-i)+1$. Notice that $v$ is incident to at most $n - 1$ edges. Therefore, deleting $v$ will remove at most $2(n-1)$ color degrees.

Case 2: Assume $G_i = G_{i-1} - R$ for some color class $R$ with $|R| \geq 2(m-i)+1$. Because $c$ is proper, $|R| \leq \lfloor n/2 \rfloor$. Deleting all edges of color $R$ reduces the total color degree by at most $n$.

Case 3: Assume that $G_i = G_{i-1} - v - u - c(uv)$ for some $uv \in E(G_{i-1})$. Since $G_i$ is not constructed by step 2, we know that $\hat{d}(u), \hat{d}(v) \leq 3(m-i)$. Furthermore, since $G_i$ is
not constructed by step 3, we know that $|c(uv)| \leq 2(m - i)$. This implies that

$$
W(G_i) = 2(\hat{d}(v) + \hat{d}(u)) + 2|c(uv)| \\
\leq 16(m - i) \\
\leq 2n.
$$

This concludes the proof of the claim. Now we have

$$
2nm \leq \hat{d}(G) = \sum_{i=1}^{k} W(G_i) \leq 2nk,
$$

which is a contradiction since $k < m$. Therefore, the theorem is proven. 

\[\square\]

### 4 General Edge-Colored Graphs

Theorem 8 provides contrast for Theorems 2, 3, and 5. The proof of Theorem 8 is similar to the proof of Theorem 5. However, the greedy algorithm has been modified to accommodate graphs that are not properly colored.

**Theorem 8.** Let $c$ be an edge coloring of $G$ be a graph with $\hat{d}(G) \geq 2mn$ and $n \geq 12m^2 + 4m$. Then $c$ admits a rainbow matching of size $m$.

**Proof.** Assume that $G$ is an edge minimal counter example to Theorem 8. Since $G$ is edge minimal, no color class can induce a $P_4$ (path on 4 vertices) or a triangle. This follows from the fact that if a color class $R$ induces a $P_4$ or triangle, then an edge can be deleted without reducing the total color degree of the graph. Therefore, each color class in $G$ induces a forest of stars. Let $s(R)$ denote the number of components induced by the color class $R$. Consider the following algorithm:

1. set $G_0 := G$
2. if there exists $v \in V(G_{i-1})$ with $\hat{d}(v) \geq 3(m - i) + 1$, then $G_i = G_{i-1} - v$ and return to 2
3. else, if there exists color $R$ with $s(R) \geq 2(m - i) + 1$ in $G_{i-1}$, then $G_i = G_{i-1} - R$ and return to 2
4. else, if there exists a vertex $v$ and a color $R$ such that $\hat{d}_R(v) \geq 3(m - i) + 1$ in $G_{i-1}$, then $G_i = G_{i-1} - v - R$ and return to 2
5. else, if there exists $uv \in E(G_{i-1})$, then $G_i = G_{i-1} - u - v - c(uv)$ and return to 2
6. return $i - 1
Since this algorithm is so similar to the algorithm featured in the proof of Theorem 5, the only things that remain to be checked are that step 4 lets us extend a matching, and that the bounds on steps 4 and 5 are still good.

Assume that \( G_i = G_{i-1} - v - R \) where \( \hat{d}^R(v) \geq 3(m - i) + 1 \). Let \( M \) be a rainbow matching of size \( k - i \), contained in \( G_i \). Since \( v \notin V(G_i) \), \( v \notin V(M) \). Furthermore, \( M \) does not contain an edge with color \( R \). Since \( \hat{d}^R(v) \geq 3(m - i) + 1 \), there exists an edge \( uv \) with \( c(uv) = R \) and \( u \notin M \). Then \( M \cup \{uv\} \) is a rainbow matching of size \( k - i + 1 \) contained in \( G_{i-1} \).

If \( G_i = G_{i-1} - v - R \) where \( \hat{d}^R(v) \geq 3(m - i) + 1 \), then 2 and 3 must have been rejected. The color \( R \) contributes at most \( n - 3(m - i) \) color using edges that are not incident upon \( v \). Since \( \hat{d}(v) \leq 3(m - i) \) and \( d(v) \leq n \), it follows that \( W(G_i) \leq n - 3(m - i) + \hat{d}(v) + d(v) \leq n - 3(m - i) + 3(m - i) + n = 2n \).

Suppose \( G_i = G_{i-1} - v - u - c(uv) \). Then steps 2, 3, and 4 must have been rejected. This implies that \( \hat{d}(v), \hat{d}(u) \leq 3(m - i) \). Furthermore, each color at \( v, u \) can be represented at most \( 3(m - i) \) times. Finally, the edges of color \( c(uv) \) can induce at most \( 2(m - i) \) stars with \( 3(m - i) \) edges each. Therefore, deleting all \( c(uv) \) colored edges reduces the color degree by at most \( 6m^2 + 2m \). Thus, \( W(G_i) \leq 24m^2 + 8m \leq 2n \).

Suppose that the algorithm terminates in \( k < m \) steps. Now we have

\[
2nm \leq \hat{d}(G) = \sum_{i=1}^{k} W(G_i) \leq 2nk,
\]

which is a contradiction since \( k < m \). Therefore, the theorem is proven.

\[\square\]

5 Future Work

Though we were not able to resolve Question 1 for all graphs, we believe the answer is affirmative:

**Conjecture 9.** All edge colored graphs \( G \) with \( \hat{d}(G) \geq 2mn \) contain a rainbow matching of size \( m \).

It would also be interesting to know under which conditions there exists a matching of size \( m + 1 \). It seems that a small improvement in the estimates in the proofs of Theorems 2 and 5 could yield this result for edge colored graphs \( G \) with \( \hat{d}(G) \geq 2mn \). In fact, it may be that the proper question to ask is whether any graph \( G \) with \( \hat{d}(G) \geq 2mn \) contains a rainbow matching of size \( m + 1 \).

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