ON HIGGS BUNDLES ON ELLIPTIC SURFACES

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Abstract. The aim of this paper is to establish an equivalence of certain categories of Higgs bundles on a non-isotrivial elliptic surface \( \pi : X \to C \) with \( \chi(X) > 0 \) and certain categories of Parabolic Higgs bundles on \( C \).

1. Introduction

For an elliptic fibration \( \pi : X \to C \) (see Subsection 2.1), considering \( C \) as an orbifold with the orbifold structure above the points where \( \pi \) has multiple fibers, we have a natural isomorphism (induced by \( \pi \)) of groups

\[
\pi_1(X, x) \xrightarrow{\cong} \pi_1^{orb}(C, c)
\]

where \( x \in X \) and \( c = \pi(x) \in C \), when \( \chi(X) > 0 \) (8, 6, 11) The space of Jordan equivalence classes of \( GL(n, \mathbb{C}) \) representations of \( \pi_1(X, x) \) can be described as the moduli space of rank \( n \) Higgs bundles on \( X \) with vanishing rational Chern classes from the work of Simpson [13, 14]. In a similar fashion, the Jordan equivalence classes of \( GL(n, \mathbb{C}) \) representations of \( \pi_1^{orb}(C, c) \) has a description as the moduli space of rank \( n \) parabolic higgs bundles on \( C \) with vanishing parabolic degree. The isomorphism in 1.0.0.1 suggests a natural correspondence between these moduli spaces. If we consider only representations into the maximal compact subgroup \( U(n) \subset GL(n, \mathbb{C}) \), then the corresponding moduli spaces are that of moduli space of semi-stable vector bundles on \( X \) with vanishing rational Chern classes and parabolic semi-stable bundles on \( C \) of parabolic degree 0 (10) and an algebraic geometric correspondence between these moduli spaces was exhibited by Stefan Bauer in [2] (see also [4, 3], [7] for related questions). Motivated by the approach pursued in [2] (in particular the notion of vertical bundles), we exhibit in this article a correspondence between (categories of) semistable Higgs bundles on \( X \) and parabolic Higgs bundles on \( C \) with certain Chern class conditions (as in [2]) and assuming \( \pi \) to be a non-isotrivial fibration. The case of Higgs bundles has been studied in [12] as well, but their study is restricted to certain strata in the moduli space. Recall that in the case of elliptic surfaces with multiple fibers, after a base change (which can be assumed to be Galois), we get another elliptic surface which is an etale Galois cover of the original surface and further the new elliptic surface has no multiple fibers. Using this fact (to be made precise in Section 3), we reduce the problem to the situation where \( \pi \) has no multiple fibers and hence the
isomorphism in 1.0.0.1 is actually an isomorphism of the usual fundamental groups $\pi_1(X, x) \cong \pi_1(C, c)$. We have a natural inclusion map of sheaves

$$0 \to \pi^*(K_C) \to \Omega^1_X.$$  

(1.0.0.2)

Our problem then is to show that any Higgs bundle $(V, \theta)$ of the appropriate topological type and semistable, actually is of the form $V \cong \pi^*(W)$ and the Higgs field is the pull back of a Higgs field on $W$ (we get a Higgs field on $V$ using 1.0.0.2 and the isomorphism $V \cong \pi^*(W)$). It is not clear if every Higgs field on a bundle on $X$ of the form $\pi^*(W)$ where $W$ is a vector bundle on $C$ is actually pull-back of a Higgs field on $W$. The assumption of non-isotriviality comes of use here, since we have in this case an isomorphism (see Lemma 2.1.2)

$$K_C \cong \pi_*(\Omega^1_X)$$

As a consequence of this isomorphism and projection formula, we see that our task is now reduced to showing that the underlying vector bundle $V$ is a pull-back of a bundle from $C$, which we establish in Section 3. Before we state our main result, we recall some notation introduced in Section 3. We denote by $C^\text{Higgs}_X$ the category of Higgs bundles on $X$ semistable with respect to a fixed polarization together with $c_2 = 0$ and determinant a vertical divisor. Similarly $C^\text{Higgs}_C$ denote the category of semistable Higgs bundles on $C$. The main result of this article is the following:

**Theorem.** Every object $(V, \theta) \in \text{Obj}(C^\text{Higgs}_X)$ is isomorphic as a Higgs bundle to a Higgs bundle of the form $(\pi^*(W), d\pi(\phi))$, where $(W, \phi) \in \text{Obj}(C^\text{Higgs}_C)$. Further, the functor $\pi^*$ is essentially surjective, full, faithful and consequently $\pi^*$ is an equivalence of categories between $C^\text{Higgs}_X$ and $C^\text{Higgs}_C$.

The organization of this article is as follows: Section 2 is divided into two subsections, the first one contains basic facts on elliptic surfaces and intersection theory on elliptic surfaces. The second subsection deals with the notion of vertical bundles. In Section 3, we state and prove the main results of this article.

2. Preliminaries

We denote by $k$ an algebraically closed field of characteristic 0.

2.1. **Elliptic surfaces.** An elliptic surface over $k$ is a fibered surface $\pi : X \to C$, where the general fibers are genus 1 curves and $X, C$ are a smooth projective surface and a smooth projective curve over $k$ respectively. We call an elliptic surface as above relatively minimal if there are no exceptional curves (equivalently no self intersection $-1$ curves) on the fibers.

Just to be consistent with the definition of vertical bundles (see Definition 2.2.1) defined in the next subsection, we call a divisor $D$ vertical, if $D = \sum r_i F_i$ where $r_i \in \mathbb{Q}$ and $F_i$ for every $i$ is a divisor corresponding to a fiber at some point of $C$. We call a divisor $D$ vertically supported if $\text{Supp}(D)$ maps to a proper closed subset of $C$ under $\pi$. A vertically supported divisor
$D$ always satisfies $D^2 \leq 0$ and is *vertical* precisely when $D^2 = 0$. For two divisors $D_1$ and $D_2$, we write $D_1 \equiv D_2$ if $D_1$ is numerically equivalent to $D_2$. The vertical divisors corresponding to various fibers are all numerically equivalent. Hence as far as intersection theory is concerned, we may work with a fixed fiber, which we denote by $F$. The sheaf $R^1\pi_*(\mathcal{O}_X)$ is a line bundle on $C$ and in the case of relatively minimal elliptic fibrations, we have

$$\chi(X) = 12\deg(L)$$

where $L = R^1\pi_*(\mathcal{O}_X)^*$. For a relatively minimal elliptic surface, we have the following canonical bundle formula due to Kodaira

$$K_X \cong \pi^*(K_C \otimes L) \otimes \mathcal{O}_X(\Sigma_i (m_i - 1)F_i)$$

where $F_i$ are effective divisors on $X$ with g.c.d of the coefficients of the components being 1 and the multiple fibers of $\pi$ are precisely of the form $m_iF_i$. In particular note that the canonical divisor class on a relatively minimal elliptic surface is represented by vertical divisor. If $Y \to C$ is an elliptic surface which is not relatively minimal, then assume after blowing down the exceptional curves $\{E_1, \ldots, E_r\}$, we get a relatively minimal model, which we denote by $X$. We then have

$$K_Y \cong K_X \otimes \mathcal{O}_Y(E_1 + \ldots + E_r).$$

Hence $K_Y$ is represented by a vertically supported divisor. In particular we have for any elliptic surface $X \to C$,

$$K_X.F = 0.$$

We will need the following characterization of vertical divisors in the subsequent sections

**Lemma 2.1.1.** Assume $\chi(X) > 0$. Then for a divisor $D$,

$$D \text{ vertical } \iff D.F = 0 \text{ and } D^2 = 0$$

**Proof.** $D$ vertical clearly implies

$$D.F = 0 = D^2.$$

Conversely, assume $D.F = 0$ and $D^2 = 0$. Let $H$ be an ample divisor on $X$. Choose $m, n \in \mathbb{Z}$, such that $(mD + nF).H = 0$. Since now $(mD + nF)^2 = 0$, we get from Hodge index theorem on surfaces that $mD + nF \equiv 0$ and $D \equiv rF$ where $r \in \mathbb{Q}$. If $D$ is vertically supported then $D = aF$, with $a \in \mathbb{Q}$. This is the case if $D$ is effective. Hence it is enough to show $D_l = D + lF$ is effective where $l \in \mathbb{N}$ as $D_l$ satisfies the hypothesis of the lemma and the preceding discussion applied to $D_l$ says $D_l$ is vertical and hence so do $D = D_l - lF$. To see this choose $l >> 0$, so that $(D_l).H > (K_X).H$. Then

$$H^2(X, \mathcal{O}_X(D_l)) = H^0(X, \text{Hom}(D_l, K_X))^* = 0.$$

Applying Riemann-Roch theorem, we see that

$$H^0(X, \mathcal{O}_X(D_l)) = H^1(X, \mathcal{O}_X(D_l)) + \chi(\mathcal{O}_X) > 0.$$
and hence we have $D_i$ is effective. □

The assumption $\chi(X) > 0$ in the above lemma cannot be relaxed as the lemma clearly fails for an elliptic surface of the form $X \cong C \times E \to C$ where $E$ is a fixed elliptic curve.

The following lemma is necessary for our study

**Lemma 2.1.2.** Let $X$ be a non-isotrivial elliptic surface with no multiple fibers. Then the natural map

$$K_C \to \pi_* \Omega^1_X$$

is an isomorphism.

**Proof.** Consider the s.e.s

$$0 \to \pi^* (K_C) \to \Omega^1_X \to \Omega^1_{X/C} \to 0.$$  

Applying $\pi_*$, we get the following long exact sequence

$$0 \to K_C \to \pi_* (\Omega^1_X) \to \pi_* (\Omega^1_{X/C}) \to K_C \otimes R^1 \pi_* (\mathcal{O}_X) \to 0.$$  

The sheaf $\pi_* (\Omega^1_{X/C})$ is a rank 1 sheaf on $C$, while $K_C \otimes R^1 \pi_* (\mathcal{O}_X)$ is a rank 1 locally free sheaf. The map $\sigma$ restricted to the generic fiber is the Kodaira-Spencer map which is non-zero if $X$ is assumed to be non-isotrivial. Hence the kernel of $\sigma$ is precisely $\pi_* (\Omega^1_{X/C})_{tor} = \pi_* (\Omega^1_{X/C})_{tor}$. So we have

$$0 \to K_C \to \pi_* (\Omega^1_X) \to \pi_* (\Omega^1_{X/C})_{tor} \to 0.$$  

Now from [9, Proposition 1, Page 3] we get $H^0(C, \pi_* (\Omega^1_{X/C})_{tor}) = 0$. But as $\pi_* (\Omega^1_{X/C})_{tor}$ is a torsion sheaf on $C$, it has to be the 0-sheaf since a non-zero torsion sheaf on a curve always has sections. Hence we have

$$K_C \overset{\cong}{\to} \pi_* (\Omega^1_X).$$  

□

### 2.2. Vertical Bundles.

We keep the assumption that $\pi : X \to C$ is a non-isotrivial elliptic fibration. Denote by $K$ the function field $k(C)$ of $C$. For a vector bundle $V$ on $X$, we denote by $V_K$ the bundle on $X_K := X \times_C \text{spec}(K)$ given by pull back of $V$ to $X_K$ through the natural map $X_K \to X$. Similarly for an extension $L/K$ of fields we denote by $X_L := X \times_C \text{spec}(L)$ and $V_L$ the bundle on $X_L$ given by the pull back of $V$ to $X_L$ through the morphism $X_L \to X$.

Let us recall the definition of vertical bundles as defined in [2, Page 512, Definition 1.3]

**Definition 2.2.1.** A rank $n$ vector bundle $V$ on $X$ is called vertical if $V$ has a filtration

$$(0) = V_0 \subset V_1 \subset \ldots \subset V_n = V.$$  

by sub-bundles $V_i$, with $V_i/V_{i-1} \cong \mathcal{O}_X(D_i)$, where $D_i$ are vertical divisors.
The main result of this subsection is the following proposition which relates vertical bundles $V$ on $X$ and $V_K$.

**Proposition 2.2.1.** Let $V$ be a vector bundle with $c_2(V) = 0$ and $\det(V) \cong \mathcal{O}_X(D)$ with $D$ a vertical divisor. Then $V$ is vertical if and only if $V_K$ is semistable.

**Proof.** If $V$ is vertical, then clearly $V_K$ is semistable. Now for the converse, let $\bar{K}/K$ be an algebraic closure of $K$. Consider the elliptic curve $X_{\bar{K}}$ and vector bundle $V_{\bar{K}}$ on $X_{\bar{K}}$. By assumption we have $V_{\bar{K}}$ is semistable with trivial determinant. From Atiyah’s classification results on vector bundles on elliptic curves $[\text{[1]}]$, we have

\[ V_{\bar{K}} \cong \bigoplus_i L_i \oplus I_m \]

where $L_i$ are degree 0 line bundles on $X_{\bar{K}}$ and $I_m$ denotes the unique indecomposable bundle on $X_{\bar{K}}$ of rank $m$ and trivial determinant. Let $L/K$ denote a finite Galois extension so that $L_i \in \text{Pic}^0(X_L), \ \forall i$. We then have a decomposition of $V_L$ as $\bigoplus_i L_i I_m$. Let $f : \bar{C} \to C$ be the finite galois cover of $C$ corresponding to $L/K$. Choose a minimal resolution $\bar{X}$ of $X \times_C \bar{C}$. Since $X$ was non-isotrivial, the same holds true for $\bar{X}$ and hence $\chi(\bar{X}) > 0$. Denote by $\bar{V}$ the pull back of $V$ to $\bar{X}$. The bundle $\bar{V}$ also satisfies $c_2(\bar{V}) = 0$ and $\bar{D} = \det(\bar{V})$ is a vertical divisor with $\bar{D}^2 = 0$. We have $\bar{V}_L = V_L$ and hence has a filtration by the line bundles $L_i$. We can extend this filtration on $\bar{V}_L$ to a filtration by torsion free subsheaves on $\bar{V}$,

\[ (0) = \bar{V}_0 \subset \ldots \bar{V}_{n-1} \subset \bar{V}_n. \]

such that $\bar{V}_i/\bar{V}_{i-1} \cong \mathcal{O}_X(D_i) \otimes I_{Z_i}$. Using the additivity of chern classes, we get

\[ \Sigma_i D_i = \bar{D}, \]

\[ \Sigma_{i<j} D_i D_j + \text{lt}(Z_i) = 0. \]  

Squaring $2.2.0.3$ and using the fact that $\bar{D}^2 = 0$, we get

\[ \Sigma_{i<j} D_i D_j = -\frac{1}{2} \Sigma_i D_i^2. \]  

Substituting $2.2.0.5$ in $2.2.0.4$ we get

\[ \Sigma_i \text{lt}(Z_i) = \frac{1}{2} \Sigma_i D_i^2. \]  

Since by assumption $D_i F = 0$, we have

\[ D_i^2 \leq 0, \forall i. \]

On the otherhand

\[ \text{lt}(Z_i) \geq 0, \forall i. \]

Hence from $2.2.0.6$ we get the only possibility is $\text{lt}(Z_i) = 0$ and $D_i^2 = 0$. Now from Lemma $[2.1.1]$ we can conclude $D_i$ are vertical divisors for all $i$. In particular

\[ L_i \cong \mathcal{O}_X, \forall i. \]
Now consider the s.e.s
\[ 0 \to L_0 \cong \mathcal{O}_X \to V_L \to V_L/L_0 \to 0. \]

Let \( G := Gal(L/K) \) be the Galois group. We have \( G \) acts on \( V_L \) and hence on the sections \( H^0(X, V_L) \). Now replace \( t_0 \) by \( Tr(t_0) = \Sigma_{g \in G} g(t_0) \) and we can assume \( t_0 \) is \( G \)-invariant and hence \( L_0 \) is a \( G \)-invariant trivial sub-bundle of \( V_L \). Hence by Galois descent we have a section \( s_0 : \mathcal{O}_X \to V_K \) with \( t_0 \) being the induced section of \( V_L \) via base change and \( V_L/L_0 \cong (V_K/s_0(\mathcal{O}_X))_L \). Replacing \( V_L \) by \( V_L/L_0 \) and \( t_0 \) by \( t_1 : L_1 \cong \mathcal{O}_X \to V_L/L_0 \) and repeating the argument we see that \( V_K \) has a filtration by sub-bundles with subquotients that are all trivial line bundles. Extend this filtration to a filtration of \( V \) and as in the case of \( \tilde{V} \), we see that \( V \) is vertical. \( \square \)

Let us recall now the definition of Higgs bundles on a projective variety \( Y \). A Higgs bundle on \( Y \) is a pair \((V, \theta)\) where \( V \) is a vector bundle on \( Y \) and \( \theta : V \to V \otimes \Omega^1_Y \) is a homomorphism with
\[ \theta \wedge \theta = 0. \]

We fix a polarization \( H \) on \( Y \) and let \( r \) be \( \dim(Y) \). We say a Higgs bundle \((V, \theta)\) on \( Y \) is semistable if for every subsheaf \( W \subset V \) preserved by \( \theta \) (i.e. \( \theta(W) \subset W \otimes \Omega^1_Y \)), we have
\[ c_1(W).H^{r-1}/\text{rank}(W) \leq c_1(V).H^{r-1}/\text{rank}(V). \]

Now consider the case when \( Y \) is a surface. For a vector bundle \( V \) on \( Y \), Denote by \( \Delta(V) \) the number \( c_1(V)^2 - 4c_2(V) \) which is called the Bogomolov Number of \( V \). If a vector bundle \( V \) admits a Higgs field \( \phi \) so that \((V, \phi)\) is a semistable Higgs bundle on \( Y \) (w.r.t \( H \)), then we have the Bogomolov inequality
\[ \Delta(V) \leq 0. \]

Further if \( \Delta(V) = 0 \), then the pair \((V, \phi)\) is semi-stable with respect to any other polarisation on \( Y \) [5 Theorem 1.3]. Hence now as a corollary of Proposition [2.2.1] we have the following generalization of [2] Lemma 1.4, page 512].

**Corollary 2.2.1.** If \((V, \theta)\) is a semistable Higgs bundle with \( c_2(V) = 0 \) and \( \det(V) \cong \mathcal{O}_X(D) \), where \( D \) is a vertical divisor, then \( V \) is a vertical bundle.

**Proof.** From Proposition [2.2.1] it is enough to show \( V_K \) is semistable. If \( V_K \) is not semistable, then since \( X_K \) is a genus 1 curve, the H-N filtration of \( V_K \) induces a decomposition \( V_K = \bigoplus_{i=1}^j W_i \) where \( W_i \) is the destabilizing subsheaf of \( V_K/W_{i-1} \) if we set \( W_0 = (0) \). In particular each \( W_i \) is semistable and
\[ \deg(W_0) > \ldots > \deg(W_j). \]

Now \((\Omega^1_X)_K\) is a rank 2 vector bundle on \( X_K \) which is an extension of \( \mathcal{O}_{X_K} \) by itself. In particular \((\Omega^1_X)_K\) is semistable of degree 0 and so the
bundles $W_i \otimes (\Omega^1_X)_K$ are semistable with $\deg(W_i \otimes (\Omega^1_X)_K) = 2\deg(W_i)$ and $rk(W_i \otimes (\Omega^1_X)_K) = 2rk(W_i)$. We have then 

$$\mu(W_0) = \deg(W_0)/rk(W_0) > \mu(W_i \otimes (\Omega^1_X)_K) = \deg(W_i)/rk(W_i), \ i \geq 2.$$ 

So 

$$H^0(X_K, \text{Hom}(W_0, W_i \otimes (\Omega^1_X)_K)) = (0), \forall i \geq 2.$$ 

Hence we have $\theta_K(W_0) \subseteq W_0 \otimes (\Omega^1_X)_K$. Now extend $W_0$ to a torsion free subsheaf $W \subset V$ with $V/W$ torsionfree as well. Since $\theta_K$ preserves $W_0$, the Higgs field $\theta$ preserves the subsheaf $W$. On the other hand as we have 

$$c_1(W).F = \deg(W_0) > 0,$$

for a suitable $m >> 0$ and the polarisation $H + mF$, the slope $W$ exceeds that of $V$. But since $\Delta(V) = 0$, this will contradict the semistability of $(V, \theta)$ with respect to $H$. 

Before we end this section we would like to address two natural questions regarding vertical bundles. The first one is about when a subsheaf of a vertical bundle $V$ is itself vertical. Clearly such a subsheaf $N \subset V$ satisfies $c_1(N).F = 0$. We will see below [Lemma 2.2.1] that this condition is in fact sufficient. The other question is specific to the case when $\pi$ has no multiple fibers. In this situation there is a natural class of vertical bundles, which are the pull backs of bundles on $C$ to $X$. If $V$ is such a bundle then we have $V_K = \mathcal{O}^\oplus_{\pi}^{\oplus r}$ where $r = rk(V)$. Once again this condition turns out to be sufficient [Lemma 2.2.2].

**Lemma 2.2.1.** Let $V$ be a vertical bundle and $N \subset V$ a subsheaf with torsion free quotient $V/N$. Then $N$ is vertical precisely when $c_1(N).F = 0$.

**Proof.** Since $N$ and $V/N$ are torsion free, we have $N_K$ and $(V/N)_K$ are locally free on $X_K$. Further by assumption both are of degree 0. On the other hand as $V$ is vertical, $V_K \cong \oplus_i I_{k_i}$, where $I_{k_i}$ denote the unique indecomposable bundles of rank $k_i$ and trivial determinant. Hence both $N_K$ and $(V/N)_K$ also admit filtrations where the successive quotients are trivial line bundles. Any such filtration on $N_K$ and $(V/N)_K$ can be extended to a filtration on $N$ and $V/N$ with successive quotients all of rank 1 of the form $\mathcal{O}_X(D_i) \otimes I_{Z_i}$ where $D_i$ is a vertically supported divisor and $Z_i$ is a closed set of points on $X$. But this filtration is also a filtration on $V$. Now an argument involving Chern classes as in the proof of Proposition 2.2.1 gives us $Z_i = \emptyset, \forall i$ and $D_i$ are vertical divisors. Hence both $N$ and $V/N$ are vertical.

**Lemma 2.2.2.** Assume $\pi$ has no multiple fibers. Then a vertical bundle $V$ is isomorphic to $\pi^*(W)$ where $W$ is a bundle on $C$ if and only if $V_K = \mathcal{O}^\oplus_{\pi}^{\oplus r}$ where $r = rk(V)$.

**Proof.** Let $V$ be a vertical bundle with $V_K = \mathcal{O}^\oplus_{\pi}^{\oplus r}$. Since we have assumed $\pi$ has no multiple fibers, a line bundle corresponding to a vertical divisor restricts to the trivial line bundle on any fiber of $\pi$. Hence $V$ restricted to
any fiber is an iterated extension of trivial line bundles. In particular if for \( c \in C \), we denote by \( X_c \) by the fiber (scheme theoretic) of \( \pi \) above \( c \) and \( V_c \) the restriction \( V |_{X_c} \), then as \( h^0(X_c, O_{X_c}) = 1 \), we have

\[
h^0(X_c, V_c) \leq r.
\]

The equality occurs precisely at the points \( c \in C \) where \( V_c \) is the trivial rank \( r \) bundle on \( X_c \). Now from semi-continuity principle the set \( Z = \{ c \in C \mid h^0(X_c, V_c) = r \} \) if non-empty is a closed subset of \( C \). But on the other hand we have for \( \zeta \) the generic point of \( C \), \( h^0(X_\zeta, V_\zeta) = h^0(X_K, V_K) = r \). Hence \( \zeta \in Z \) and thus \( Z = C \). Thus \( V \) restricts to the trivial rank \( r \) bundle on every fiber and consequently \( V \cong \pi^*(\pi_*(V)) \).

\[\square\]

3. MAN THEOREM

3.1. Case of no multiple fibers. We keep the assumption of non-isotriviality and positive euler characteristic for \( \pi : X \rightarrow C \) throughout this subsection.

Further let us assume \( \pi \) has no multiple fibers. As in the previous sections we fix a polarisation \( H \) on \( X \). Denote by \( C_X^{Higgs} \) the category whose objects are semistable Higgs bundles \((V, \theta)\) on \( X \) with \( c_2(V) = 0 \) and \( det(V) \) a vertical divisor and morphisms are Higgs bundle morphisms. Similarly we denote by \( C_C^{Higgs} \) to be the category of semistable Higgs bundles on \( C \). We have a natural map

\[d\pi : \pi^*(K_C) \rightarrow \Omega_X^1\]

Now for a Higgs bundle \((W, \phi)\) on \( C \), let \( V = \pi^*(W) \). We denote by \( d\pi(\phi) \in Hom(V, V \otimes \Omega_X^1) \) the composition \((Id_V \otimes d\pi) \circ (\pi^*(\phi))\). Clearly \( d\pi(\phi) : V \rightarrow V \otimes \Omega_X^1 \) is a Higgs field on \( V \). We have the following lemma

Lemma 3.1.1. If \((W, \phi)\) is a semistable Higgs bundle on \( C \), then for any chosen polarisation on \( X \), the Higgs bundle \((V, d\pi(\phi))\) is semistable on \( X \)

Proof. Since \( \Delta(V) = 0 \), it is enough to prove that there exists a polarization with respect to which \((V, d\pi(\phi))\) is semistable. Assume the contrary and let \( H \) be a polarization for which the pair \((V, d\pi(\phi))\) is unstable. Since the bundle \( V_K \) is trivial and hence semistable, for any sub-sheaf of \( N \subset V \), we have \( c_1(N).F \leq 0 \). Hence changing the polarization from \( H \) to \( H + mF \) for \( m \gg 0 \), turns \((V, d\phi(\phi))\) into a semistable Higgs bundle in which case we are done or else the maximal destabilizing sub-sheaf \( V_{max} \) satisfies \( c_1(V_{max}).F = 0 \). But as \( V_K \) is trivial and \((V_{max})_K \) is a degree 0 sub-bundle of \( V_K \), the only possibility is \((V_{max})_K \) is itself trivial. Hence \((V/V_{max})_K \) is trivial as well. Now from Lemma 2.2.1 and Lemma 2.2.2 we have \( V_{max} \cong \pi^*(\pi_*(V_{max})) \). Hence \( \pi_*(V_{max}) \) is a sub-bundle of \( W \) of rank same as that of \( V_{max} \). Further we have

\[
\mu(\pi_*(V_{max})) = \frac{c_1(V_{max}).H.E.H}{rk(V_{max})} > \mu(W) = \frac{c_1(W).H.E.H}{rk(W)}.
\]

and \( \pi_*(V_{max}) \) is invariant under \( \phi \), which contradicts semistability of \((W, \phi)\). Hence \((V, d\pi(\phi))\) is semistable for the polarisation \( H \). \[\square\]
Thus we have a well defined functor

$$\pi^* : \mathcal{C}_C^{Higgs} \to \mathcal{C}_X^{vHiggs}$$

given by

$$(W, \phi) \mapsto (\pi^*(W), d\pi(\phi)).$$

From Lemma 2.1.2 we have the natural map $K_C \to \pi^*(\Omega^1_X)$ is an isomorphism. If $V = \pi^*(W)$ for $W$ a bundle on $C$, then from projection formulae every Higgs field $\theta$ on $V$ is of the form $d\pi(\phi)$ for $\phi$ a Higgs field on $W$. Hence the functor $\pi^*$ is full and faithful. Our main theorem in this section says $\pi^*$ is essentially surjective as well and hence is an equivalence of categories.

**Theorem 3.1.1.** Every object $(V, \theta) \in \text{Obj}(\mathcal{C}_X^{vHiggs})$ is isomorphic as a Higgs bundle to a Higgs bundle of the form $(\pi^*(W), d\pi(\phi))$ where $(W, \phi) \in \text{Obj}(\mathcal{C}_C^{Higgs})$. Further the functor $\pi^*$ is essentially surjective, full, faithful and consequently $\pi^*$ is an equivalence of categories between $\mathcal{C}_X^{vHiggs}$ and $\mathcal{C}_C^{Higgs}$.

**Remark 3.1.1.** The statement for line bundles (even without the assumption of non-isotriviality) is a consequence of Hodge theory for complex surfaces. Recall we have under the assumption of $\chi(X) > 0$,

$$g(C) = h^{1,0} = \dim_C(H^1(X, \mathcal{O}_X)) = \dim_C(H^0(X, \Omega^1_X)) = h^{0,1}.$$

On the other hand the dimension of the subspace $H^0(X, \pi^*(K_C)) \subseteq H^0(X, \Omega^1_X)$ is $g(C)$ as well. Hence we have the equality

$$H^0(X, \pi^*(K_C)) = H^0(X, \Omega^1_X).$$

In particular every 1-form on $X$ is the pull back of a 1-form on $C$. Now a rank 1 Higgs bundle of the form in the Theorem above is a pair $(L, \theta)$ where $L$ is isomorphic to a line bundle of the form $\mathcal{O}_X(D)$ with $D$ vertical (hence in the case of no multiple fibers, $D$ is the pull back of a divisor on $C$) and $\theta$ is a 1-form. So the statement holds true for rank 1 Higgs bundles as in the theorem.

The proof of Theorem 3.1.1 proceeds by analysing the restriction $(V_K, \phi_K)$ to $X_K$. Now since we have assumed $X$ to be non-isotrivial, we have from 2.1.2 $\pi_*(\Omega^1_X)$ is the line bundle $K_C$ on $C$. Hence

$$\dim_K(H^0(X_K, (\Omega^1_X)_K)) = 1$$

by semicontinuity principle. Consider the restriction of the s.e.s

$$0 \to \pi^*(K_C) \to \Omega^1_X \to \Omega^1_{X/C} \to 0.$$ 

to $X_K$. Since $(\pi^*(K_C))_K \cong \mathcal{O}_{X_K} \cong (\Omega^1_{X/C})_K$, we see that $(\Omega^1_X)_K$ is an extension of $\mathcal{O}_{X_K}$ by $\mathcal{O}_{X_K}$. Up to isomorphism, there are only 2 such bundles on $X_K$, the one being the trivial rank 2 bundle and the other the indecomposable bundle $J_2$. Since we have seen already that $\dim_K(H^0(X_K, (\Omega_X)_K)) = 1$, the bundle $(\Omega^1_X)_K$ cannot be the trivial bundle and hence it is isomorphic
to $I_2$. So the pair $(V_K, \phi_K)$ is a $I_2$-valued Higgs pair on $X_K$. Such an $I_2$ valued Higgs pair is equivalent to a morphism

$$I_2^* \to \text{End}(V,V)$$

such that fiberwise the image lands inside a family of commuting matrices. The following Lemma about $I_2$-valued Higgs pairs is what we need for our purposes

**Lemma 3.1.2.** Let $E$ be an elliptic curve over a field $k$ and $\phi : V \to V \otimes I_2$ be an $I_2$-valued Higgs field. We then have for any section $\alpha \in H^0(E, \text{End}(I_2, \mathcal{O}_E))$, the induced element $\beta = \alpha \circ \phi \in H^0(E, \text{End}(V,V))$ is Nilpotent.

**Proof.** The bundle $I_2$ is an extension of $\mathcal{O}_E$ by $\mathcal{O}_E$ and hence we have a s.e.s

$$0 \to \mathcal{O}_E \to I_2 \to \mathcal{O}_E \to 0.$$

Further

$$H^0(E, I_2) = k\langle s \rangle, \quad H^0(E, \text{End}(I_2, \mathcal{O}_E)) = k\langle t \rangle.$$

In particular for $a \in H^0(E, I_2)$ and $b \in H^0(E, \text{End}(I_2, \mathcal{O}_E))$, we always have

$$ba = 0 \in H^0(E, \mathcal{O}_E).$$

We also have

$$I_2 \cong I_2^*$$

Fix an ismorphism as above and then we have

$$H^0(E, I_2^*) = k\langle t^* \rangle, \quad H^0(E, \text{End}(I_2^*, \mathcal{O}_E)) = k\langle s^* \rangle.$$

Consider the morphism (which we denote by $\theta$ as well) induced by the Higgs field $\theta : I_2^* \to \text{End}(V,V)$. We have a trace map $\text{Tr}_V : \text{End}(V,V) \to \mathcal{O}_X$ and $\text{Tr}_V \circ \theta \in H^0(E, \text{End}(I_2^*, \mathcal{O}_E))$. Let $\text{Tr}_V \circ \theta = \lambda s^*$ and $\alpha = \gamma t^*$. Then

$$\text{Tr}(\beta) = \text{Tr}_V \circ \theta \circ \alpha = \lambda \gamma s^* t^* = 0.$$

Let $L/k$ be a finite extension so that we have a decomposition of $V_L$ as direct sum of generalized eigenspaces of $\beta$,

$$V_L = \bigoplus_{\delta_j \in L} V_L^{\delta_j}.$$

Since $\theta$ pointwise lands in a family of commuting endomorphisms, we have $V_L^{\delta_j}$ are preserved by $\theta$. Hence we have induced maps

$$\theta^{\delta_j} : I_2^* \to \text{End}(V_L^{\delta_j}, V_L^{\delta_j}).$$

and $\beta^{\delta_j} = \theta^{\delta_j} \circ \alpha$. In particular

$$\beta = \bigoplus \beta^{\delta_j},$$
Now as in the case of $V$, we get

$$\text{rank}(V^\delta_j)L_j = \text{Tr}(\beta^\delta_j) = 0.$$ 

Hence either $\beta = 0$ or all the eigenvalues are 0 and hence $\beta$ is nilpotent. $\square$

As a consequence of the above Lemma we have the following

**Lemma 3.1.3.** Let $(V, \theta)$ be an $I_2$-valued Higgs field with $V$ a semistable rank $r$ degree 0-bundle on $E$. Then either,

(a) $V = L \otimes O^E$ with $\text{deg}(L) = 0$, and $\theta \circ (\text{id}_V \otimes t) = 0$, or

(b) $\exists W \subset E$ with $\text{deg}(W) = 0$ and $\theta(W) \subset W \otimes I_2$.

**Proof.** Consider the endomorphism

$$T = \theta \circ t : V \to V$$

We have from Lemma 3.1.2 that $T$ is a nilpotent endomorphism. Let $W := \text{Ker}(T)$. Now as $V$ is semistable of degree 0, we have $\text{deg}(W) \leq 0$. On the other hand by the same reasoning $\text{deg}(\text{Im}(T)) \leq 0$. Hence $\text{deg}(W)$ is forced to be 0. So if $\phi \neq 0$, then $W$ is a proper degree 0 sub-bundle invariant under $\theta$ and we are done. If $\theta = 0$, then $\theta$ factors through $s : O_E \to I_2$, i.e we have an endomorphism $\phi : V \to V$ such that

$$\theta = (\text{id}_V \otimes s) \circ (\phi).$$

Using Atiyah’s classification results on bundles on elliptic curves, it is easy to see that unless $V = L \otimes O^E$ where $\text{deg}(L) = 0$, $\phi$ always leaves invariant a proper degree 0 sub-bundle of $V$. $\square$

We have now all the ingredients to prove Theorem 3.1.1. We provide below 2 different arguments. The first one though works only in the case when the Higgs bundle has no sub-Higgs sheaves.

**3.1.1. Higgs bundles with no sub-Higgs sheaves.**

**Proof.** Consider the spectral cover $Y \subset T^*X$ associated to a Higgs bundle $(V, \theta)$. Let rank of $V$ be $r$. The fact that $(V, \theta)$ has no sub-Higgs sheaves is equivalent to $Y$ being irreducible and the natural map $q : Y \to X$ is a finite map, which restricted to the smooth locus $Y^{sm}$ of $Y$ is a ramified $r$-sheeted cover of $q(Y^{sm})$. Further we have $V = q_*(L)$ where $L$ is a rank 1 torsion free sheaf on $Y$. Now think of the Higgs field $\theta$ as a morphism

$$\theta : T_X \to \text{End}(V, V).$$

For $x \in X$, the image of the induced morphism of vector spaces

$$\theta(x) : T_x X \to \text{End}(V_x, V_x)$$

by integrality condition on $\theta$ lies inside a commuting family of endomorphism. Hence the matrices in the image of $\theta(x)$ can be simultaneously triangularized and the eigenvalues correspond to linear maps $T_x X \to \mathbb{C}$ or equivalently elements of $T_x^* X$ which is precisely the set $q^{-1}(x) \subset Y$. Though there might not exist global sections of $T^* X$ which restrict to the eigenvalues
pointwise, we can find sections of suitable symmetric powers of $T^*X$ which correspond to the co-efficients of the characteristic polynomials. The discriminants of the pointwise characteristic polynomials can also be extended to a section of a suitable symmetric power of $T^*X$. Let us call it $\Delta(\theta)$. Now as we have seen already $T^*X$ restricts to the unique indecomposable rank 2 bundle of trivial determinant when restricted to the smooth fibers. Further it has a unique section which if non-zero is nowhere vanishing. Assume now $x \in X$ with fiber of $\pi$ over $y = \pi(x)$ smooth and $\Delta(\theta)(x) = 0$. Then $\Delta(\theta)$ vanishes on the entire fiber $\pi^{-1}(y)$. In particular as the vanishing locus of $\Delta(\theta)$ is a closed set, it has to be nowhere vanishing on an open set $\pi^{-1}(U)$ where $U \subset C$ is open. In particular we see that $q$ is unramified on $q^{-1}(U)$ and the ramification locus is a vertically supported Divisor on $X$. Denote the scheme theoretic fiber of $f$ over $X_K$ by $Y_K$ which is a disjoint union of elliptic curves over $K$. On the other hand the torsion free sheaf $L$ restricts to a line bundle $L_K$ on $Y_K$ and $V_K = (q_K)_*(L_K)$. If we denote by $G$ the Galois group (note here we do not assume $Y_K$ to be connected, but the Galois group makes sense), then we have

$$q_K(V_K) = \oplus_{\sigma \in G} \sigma(L_K)$$

Hence $\#(G)(deg(L_K)) = \#(G)deg(V_K) = 0$. On the other hand $H^0(Y_K, L_K) = H^0(X_K, V_K) \neq 0$ and hence the only possibility is $L_K \cong \mathcal{O}_{Y_K}$. But then since $q_K$ is unramified $(q_K)_*(\mathcal{O}_{Y_K}) = \oplus_{i=1}^m (\oplus_{j=1}^{n_i} K_i^j)$ where $K_i$ are torsion line bundles on $X_K$ defining a connected subcover $q_K^i : Y_K^i \subset Y_K \to X_K$. But as $V_K$ is already an extension by trivial line bundles, the only possibility is $K_i = \mathcal{O}_{Y_K}$ for every $i$ and hence $Y_K$ is a disjoint union of copies of $X_K$ and $V_K = \oplus_{i=1}^r \mathcal{O}_{X_K}$.

\[\square\]

3.1.2. The general case.

Proof. Consider the Higgs pair $(V_K, \theta_K)$ on $X_K$. We have $V_K$ is an iterated extension of trivial line bundles. Now from Lemma 3.1.3 we have either $V_K$ is trivial or has degree 0 (hence semistable) sub-bundle $W_K \subset V_K$ preserved by $\theta_K$. Clearly $W_K$ is also an iterated extension by trivial line bundles as $V_K$ is so. We can extend $W_K$ to a subsheaf $W$ of $V$ with torsion free quotient $V/W$ and $\theta(W) \subseteq W \otimes \Omega^1_{X_K}$. Further $det(W).F = 0$. Every subsheaf $Q \subset V$ preserved by $\theta$ satisfies $det(Q).F \leq 0$ as $V_K$ is semistable of degree 0. Now changing polarization from $H$ to $H + mF$ for a suitable $m \in \mathbb{N}$, we can assume the subsheaf $V_{max} \subset V$, which has maximum slope among the subsheaves preserved by $\theta$ satisfies $det(V_{max}).F = 0$. In particular $(V_{max})_K$ is also an iterated extension by trivial line bundles and so do the quotient $(V/V_{max})_K$. Chose a filtration by trivial line bundles on $(V_{max})_K$ and $(V/V_{max})_K$ and extend them to $X$ as filtration on $V_{max}$ and $(V/V_{max})$ where the subquotients are rank 1 torsion free sheaves of type $\mathcal{O}_X(D_i) \otimes I_{Z_i}$ where $D_i$ are vertically supported divisors. Now observe this filtration gives a filtration on $V$ and as in the proof of Proposition 2.2.1 we can see that
in fact $Z_i = \emptyset$ and $D_i$ are vertical divisors. Hence both $V_{\max}$ and $V/V_{\max}$ are vertical bundles. Denote the induced Higgs fields on $V_{\max}$ and $V/V_{\max}$ by $\theta_0$ and $\theta_1$ respectively. From the assumption both of them are semistable Higgs bundles on $X$ as well of rank smaller than that of $V$. Hence by induction we have semistable Higgs bundles $(W_0, \phi_0)$ and $(W_1, \phi_1)$ on $C$ such that

$$(V_{\max}, \theta_0) \cong (\pi^*(W_0), \pi^*(\phi_0)), \quad (V/V_{\max}, \theta_1) \cong (\pi^*(W_1), \pi^*(\phi_1)).$$

Note that we have

$$deg(W_0) = det(V_{\max}).H/F.H \leq deg(W_1) = det(V/V_{\max}).H/F.H.$$  

Now consider the s.e.s (in fact a s.e.s of Higgs bundles on $X$)

$$0 \to V_{\max} \to V \to V/V_{\max} \to 0. \quad (3.1.2.1)$$

Applying $\pi_*$ to $3.1.2$ we get a long exact sequence

$$0 \to W_0 \to \pi_*(V) \to W_1 \to \pi_*(V) \otimes L^{-1}$$

where $L = R^1\pi_*(\mathcal{O}_X)^{-1}$. Now recall since $X$ is relatively minimal and $\chi(X) > 0$, we have $deg(L) > 0$. The map $\eta$ is compatible with the Higgs fields $\phi_0$ and $\phi_1$ on $W_0$ and $W_1$ respectively. But

$$deg(W_0) > deg(W_1 \otimes L^{-1})$$

and hence as $(W_0, \phi_0)$ and $(W_1, \phi_1)$ are semistable as Higgs bundles on $C$, the morphism $\eta = 0$. Hence

$$rk(\pi_*(V)) = rk(V) \implies V_K \cong \mathcal{O}_{X_K}^{\oplus rk(V)}.$$

### 3.2. Case of multiple fibers.

Let $\pi : X \to C$ denote a non-isotrivial elliptic fibration with multiple fibers and $\chi(X) > 0$. As in the previous subsection we denote by $\mathcal{C}_{\text{Higgs}}^X$ the category whose objects are semistable Higgs bundles $(V, \theta)$ on $X$ with $c_2(V) = 0$ and $det(V)$ a vertical divisor and morphisms being Higgs bundle morphisms. Let $c := \{c_1, \ldots, c_l\}$ be the points on $C$ where the fibers of $\pi$ are multiple. Let the multiplicities of these fibers be $m := \{m_1, \ldots, m_l\}$ respectively. Recall the notion of a parabolic vector bundle on $C$. A parabolic Vector bundle on $C$ with a parabolic structure at a point $c \in C$, consists of a vector bundle $V$ together with a flag

$$F^\bullet(V_c) := (0) \subset F^1(V_c) \subset F^2(V_c) \subset \ldots \subset F^r(V_c) = V_c.$$

and weights $\alpha_i \in \mathbb{R}$ assigned to each subspace $F^i(V_c)$ such that

$$0 < \alpha_1 < \ldots < \alpha_r \leq 1.$$

To such a parabolic vector bundle $(V, F^\bullet(V_c), \{\alpha_i\})$ we can associate a real number called the parabolic degree given by

$$\text{Pardeg}(V) := deg(V) + \sum_i \alpha_i \dim(F^i(V_c)/F^{i-1}(V_c)).$$

In general if there are parabolic structures on more than one point, then the definition of parabolic degree has to be appropriately modified. There is a
natural induced parabolic structure on every sub-bundle of \( V \) and we have an obvious notion of semistability (stability) using the parabolic degree instead of the usual degree. We also can define a parabolic Higgs bundle. Since in literature there are two different notions of a Higgs field, we want to specify what we mean by a parabolic Higgs field. For a parabolic vector bundle \((V, F^*(V_c), \{\alpha_i\})\) as defined above, a parabolic Higgs field is a morphism

\[
\phi : V \to V \otimes K_C(c)
\]

such that we have

\[
\phi(F^i(V_c)) \subseteq F^{i-1}(V_c) \otimes K_C(c).
\]

Now we can define semistability for a parabolic Higgs bundle \((V, F^*(V_c), \phi)\) as in the usual case by restricting the slope condition to sub-bundles preserved by the Higgs field \(\phi\). The definition of parabolic Higgs bundles in the case of parabolic structures at more than one point is the same as above except we have to replace \(K_C(c)\) by \(K_{C_{m_j}}\) where \(m_j\) are the parabolic points. Assume from now on that \(g(C) \geq 2\). Further assume the weights associated with the filtration \(F^*(V_{c_j})\) at \(c_j\) all are rational and lie in \(\frac{1}{m_j} \mathbb{Z} \cap [0, 1]\). Let \(\mathcal{C}^{ParHiggs}_{(C,c,m)}\) denote the category of parabolic semistable Higgs bundles with weights as described above. Since we have assumed \(g(C) \geq 2\), there exists a Galois cover \(p : \tilde{C} \to C\) with Galois group denoted by \(G\) and the local ramification groups above \(c_j\) being the cyclic group \(\frac{\mathbb{Z}}{m_j \mathbb{Z}}\) for every \(j\). Let \(\mathcal{C}^{G-Higgs}\) denote the category of \(G\)-equivariant Higgs bundles on \(\tilde{C}\). We then have a natural equivalent of categories

\[
p^*_G : \mathcal{C}^{G-Higgs}_{\tilde{C}} \rightarrow \mathcal{C}^{ParHiggs}_{(C,c,m)}
\]

Now consider the elliptic surface \(\pi : X \to C\) as defined at the beginning of this subsection. We then have a commutative diagram

\[
\begin{array}{ccc}
\tilde{X} & \xrightarrow{q} & X \\
\downarrow{\tilde{\pi}} & & \downarrow{\pi} \\
\tilde{C} & \xrightarrow{p} & C
\end{array}
\]

Where \(p : \tilde{C} \to C\) is a Galois cover (with Galois group denoted by \(G\)) ramified exactly above the points \(\{c_1, \ldots, c_l\}\) with ramification groups at these points being \(\frac{\mathbb{Z}}{m_j \mathbb{Z}}\). Further \(\tilde{\pi} : \tilde{X} \to \tilde{C}\) is a relatively minimal non-isotrivial elliptic surface with no multiple fibers. Lastly the map \(q\) is an etale Galois cover with Galois group \(G\). Denote by \(\mathcal{C}^{G-vHiggs}_{\tilde{X}}\) the category of \(G\)-semistable \(G\)-equivariant Higgs bundles on \(\tilde{X}\) with \(c_2 = 0\) and \(c_1\) vertical. From Galois descent we have an equivalence of categories

\[
q^* : \mathcal{C}^{vHiggs}_{\tilde{X}} \rightarrow \mathcal{C}^{G-vHiggs}_{\tilde{X}}.
\]

We have seen in the previous subsection (Theorem 3.1.1) that the category \(\mathcal{C}^{vHiggs}_{\tilde{X}}\) is equivalent to the category \(\mathcal{C}^{\tilde{C}}\). Since the map \(\tilde{\pi}\) is equivariant
for the $G$-action, the functor $\pi^*$ defined in the previous subsection induces a functor also denoted by $\pi^*$

$$\pi^* : \mathcal{C}_G^{vHiggs} \rightarrow \mathcal{C}_{\tilde{X}}^{vHiggs}$$

Now we claim the following

**Lemma 3.2.1.** The functor $\pi^*$ is an equivalence of categories.

**Proof.** The proof is an application of Theorem 3.1.1. Recall every $G$-semistable $G$-Higgs bundle on $\tilde{X}$ is also semistable in the usual sense. Hence from Theorem 3.1.1 we have any object $(V, \theta) \in \text{Obj}(\mathcal{C}_{\tilde{X}}^{vHiggs})$ is isomorphic to a Higgs bundle of the form $(\pi^*W, d\pi(\phi))$. The only thing to verify is whether this isomorphism can be obtained in the category of $G$-equivariant Higgs bundles on $\tilde{X}$. To see this for $g \in G$, denote by $\tau^g_X$ and $\tau^g_C$ the corresponding automorphisms of $X$ and $C$ respectively. Recall a $G$ equivariant vector bundle on $X$ is a vector bundle $V$ on $X$, together with isomorphisms

$$\alpha_g : V \xrightarrow{\simeq} (\tau^X_g)^*(V)$$

satisfying

$$(\tau^X_h)^*(\alpha_g) \circ \alpha_h = \alpha_{gh}.$$ 

Since we have

$$\pi \circ \tau^X_g = \tau^C_g \circ \pi$$

We have

$$\alpha_g \in Isom(\pi^*W, (\tau^X_g)^*(\pi^*W)) = Isom(\pi^*W, \pi^*(\tau^C_g)^*W) = Isom(W, (\tau^C_g)^*W)$$

Hence the isomorphisms $\alpha_g$ descend to isomorphisms $\beta_g \in Isom(W, (\tau^C_g)^*W)$ which satisfy

$$(\tau^C_h)^*(\beta_g) \circ \beta_h = \beta_{gh}.$$ 

and hence $W$ gets a $G$-structure such that the $G$ structure on $V$ agrees with the induced $G$ structure on $\pi^*(W)$. By an analogous argument one can see that the Higgs field $\phi$ on $W$ is $G$ equivariant.

The discussion so far can be formalized as follows

**Theorem 3.2.1.** There is a natural equivalence of categories $\mathcal{C}_X^{vHiggs}$ and $\mathcal{C}_{(C, c, m)}^{ParHiggs}$.

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