MONOMIAL GOTZMANN SETS IN A QUOTIENT BY A PURE POWER

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Abstract. A homogeneous set of monomials in a quotient of the polynomial ring $S := \mathbb{F}[x_1, \ldots, x_n]$ is called Gotzmann if the size of this set grows minimally when multiplied with the variables. We note that Gotzmann sets in the quotient $R := \mathbb{F}[x_1, \ldots, x_n]/(x_i^a)$ arise from certain Gotzmann sets in $S$.

Then we partition the monomials in a Gotzmann set in $S$ with respect to the multiplicity of $x_i$ and show that if the growth of the size of a component is larger than the size of a neighboring component, then this component is a multiple of a Gotzmann set in $\mathbb{F}[x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n]$. We also adopt some properties of the minimal growth of the Hilbert function in $S$ to $R$.

Introduction

Let $S = \mathbb{F}[x_1, \ldots, x_n]$ be a polynomial ring over a field $\mathbb{F}$ with $\deg(x_i) = 1$ for $1 \leq i \leq n$. We use the lexicographic order on $S$ with $x_1 > \cdots > x_n$. For a homogeneous ideal $I$ in $S$, the Hilbert function $H(I, -) : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$ of $I$ is the numerical function defined by $H(I, t) = \dim_{\mathbb{F}} I_t$, where $I_t$ is the homogeneous component of degree $t$ of $I$. A set $M$ of monomials in $S$ is called lexsegment if for monomials $m \in M$ and $v \in S$ we have: If $\deg m = \deg v$ and $v > m$, then $v \in M$. A monomial ideal $I$ is called lexsegment if the set of monomials in $I$ is lexsegment. For a set of monomials $M$ in the homogeneous component $S_t$ of degree $t$ in $S$, let $\text{lex}_S(M)$ denote the lexsegment set of $|M|$ monomials in $S_t$. Also for a set of monomials $M$, $S_1 \cdot M$ denotes the set of monomials of the form $um$, where $u$ is a variable and $m \in M$. By a classical theorem of Macaulay \cite{Macaulay} we have

$$|(S_1 \cdot \text{lex}_S(M))| \leq |(S_1 \cdot M)|.$$  \hfill (1)

Since the Hilbert function of a homogeneous ideal is the same as the Hilbert function of its lead term ideal this inequality implies that for each homogeneous ideal in $S$ there is a lexsegment ideal with the same Hilbert function. One course of research inspired by Macaulay’s theorem is the study of the homogeneous ideals $I$ such that every Hilbert function in $S/I$ is obtained by a lexsegment ideal in $S/I$. Such quotients are called Macaulay-Lex rings. A well-known example is $S/(x_1^{a_1}, \ldots, x_n^{a_n})$ with $a_1 \leq \cdots \leq a_n \leq \infty$ and $x_i^\infty = 0$ which is due to Clements and Lindström \cite{ClementsLindstrom}. These rings have important applications in combinatorics and algebraic geometry. For a good account of these matters and basic properties of Macaulay-Lex rings we direct the reader to Mermin and Peeva \cite{MerminPeeva}, \cite{MerminPeeva2}. Some recently discovered classes of Macaulay-Lex rings can be found in Mermin and Murai \cite{MerminMurai}. 

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Monomial sets in $S$ whose sizes grow minimally in the sense of Macaulay’s inequality have also attracted attention: A homogeneous set $M$ of monomials is called Gotzmann if $\vert (S_1 \cdot \text{lex}_S(M)) \vert = \vert (S_1 \cdot M) \vert$ and a monomial ideal $I$ is Gotzmann if the set of monomials in $I_t$ is a Gotzmann set for all $t$. In [13], Gotzmann ideals in $S$ that are generated by at most $n$ homogeneous polynomials are classified in terms of their Hilbert functions. In [11] Murai finds all integers $j$ such that every Gotzmann set of size $j$ in $S$ is lexsegment up to a permutation. He also classifies all Gotzmann sets for $n \leq 3$. Gotzmann persistence theorem states that if $M$ is a Gotzmann set in $S$, then $S_1 \cdot M$ is also a Gotzmann set, see [1]. In [12] Murai gives a combinatorial proof of this theorem using binomial representations. He derives some properties of these representations which provide information on the growth of the Hilbert functions. Among other related works, Aramova, Herzog and Hibi obtains Macaulay’s and Gotzmann’s theorems for exterior algebras, [1]. More recently, Hoefel shows that the only edge ideals that are Gotzmann are the ones that arise from star graphs, see [6]. Also some results on generation of lexsegment and Gotzmann ideals by invariant monomials can be found in [14].

In this paper we study the Gotzmann sets and the minimal growth of the Hilbert function in the Macaulay-Lex quotient $R := F[x_1, \ldots, x_n]/(x_1^a)$, where $a$ is a positive integer. A set $M$ of monomials in $R$ can also be considered as a set of monomials in $S$ and by $R_t \cdot M$ we mean the set of monomials in $S_1 \cdot M$ that are not zero in $R$. A set $M$ of monomials in $R_t$ is Gotzmann if $\vert (R_t \cdot \text{lex}_R(M)) \vert = \vert (R_t \cdot M) \vert$, where $R_t$ is homogeneous component of degree $t$ of $R$ and $\text{lex}_R(M)$ denotes the lexsegment set of monomials in $R_t$ that has the same size as $M$. We show that Gotzmann sets in $R$ arise from certain Gotzmann sets in $S$: When a Gotzmann set in $R_t$ with $t \geq a$ is added to the set of monomials in $S_t$ that are divisible by $x_1^a$, one gets a Gotzmann set in $S_t$. Then we partition the monomials in a Gotzmann set in $S$ with respect to the multiplicity of $x_i$ and show that if the growth of the size of a component is larger than the size of a neighboring component, then this component is a multiple of a Gotzmann set in $F[x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n]$. Otherwise we obtain lower bounds on the size of the component in terms of sizes of neighboring components. We also note own adoptions of some properties concerning the minimal growth of the Hilbert function in $S$ to $R$.

For a general reference for Hilbert functions and Gotzmann ideals we recommend [2] and [5].

Gotzmann sets in $F[x_1, \ldots, x_n]/(x_1^a)$

We continue with the notation and the convention of the previous section. For a homogeneous lexsegment set $L$ in $S$ with $\vert L \vert = d$, the size of $S_1 \cdot L$ was computed by Macaulay. This number is very closely related to the $n$-th binomial representation of $d$ and is denoted by $d^{<n-1>}$. We refer the reader to [2] §4 for more information on this number. In contrast to the situation in $S$, for the homogeneous lexsegment set $L \subseteq R_t$ of size $d$, the size of the set $R_1 \cdot L$ depends also on $t$. We let $d_{n,t}$ denote this size. In the sequel when we talk about $d_{n,t}$ we will always assume that $d$ is smaller than the number of monomials in $R_t$ because otherwise $d_{n,t}$ is not defined. Notice that we have $d_{n,t} = d^{<n-1>}$ for $t < a - 1$. For a non-negative integer $i$, let $S_i^t$ and $R_i^t$ denote the set of monomials in $S_t$ and $R_t$ respectively that are divisible by $x_1^i$ but not by $x_1^{i+1}$. For a set of monomials $M$ in $R_t$, let $M^t$ denote the set $R_i^t \cap M$. Similarly, if $M$ is in $S_t$, then $M^t$ denotes $S_i^t \cap M$. Also let $I(M)$ denote
the smallest integer such that $M^{I(M)} \neq \emptyset$. Set $S' = F[x_2, \ldots, x_n]$ and let $S'_i \cdot M$ denote the set of monomials of the form $x_im$, where $2 \leq i \leq n$ and $m \in M$. For a monomial $u \in R$ and a monomial set $M$ in $R$ we let $u \cdot M$ denote the set of monomials in $R$ that are of the form $um$ with $m \in M$. We also let $M'/x_1$ denote the set of monomials $m$ in $S'$ such that $mx_1' \in M'$.

**Lemma 1.** Let $L$ be the lexsegment set of size $d$ in $R_t$ with $t \geq a - 1$ and $j$ denote $I(L)$. Then

$$d_{n,t} = \sum_{j \leq i \leq a-1} |L^i|^{<n-2>}.$$  

In particular, we have $d_{n,t} \geq d^{<n-2>}$ for $0 \leq h < a - 1$.

**Proof.** Since $d_{n,h} = d^{<n-1>}$ for $h < a - 1$, the final statement is precisely [12, 1.7]. Since $L$ is lexsegment, we have $L^i = R^i_t$ for $j < i < a - 1$ giving $x_1 \cdot L^j \subseteq S'_i \cdot L^{i+1}$ for $j < i < a - 1$. Moreover $x_1 \cdot L^{a-1}$ is empty and so we get

$$R_1 \cdot L = \bigcup_{j \leq i \leq a-1} S'_i \cdot L^i.$$  

Note that $|L^i/x_1| = |L^i|$ and that $L^i/x_1$ is a lexsegment set in $S'$. Therefore $|S'_i \cdot L^i| = |S'_i \cdot (L^i/x_1)|$ and $|S'_i \cdot (L^i/x_1)| = |L^i|^{<n-2>}$. It follows that $d_{n,t} = |R_1 \cdot L| = \sum_{j \leq i \leq a-1} |L^i|^{<n-2>}$, as desired. Finally, since $\sum_{j \leq i \leq a-1} |L^i| = d$ we also have $\sum_{j \leq i \leq a-1} |L^i|^{<n-2>} \geq d^{<n-2>}$ by [12, 1.5] with equality only if $j = a - 1$. \qed

**Lemma 2.** Let $M$ be a set of monomials in $R_t$ with $t \geq a$. Let $B$ denote the set of monomials in $S_t$ that are divisible by $x_1^a$. We have the disjoint union

$$S'_1 \cdot (B \cup M) = S'_1 \cdot B \sqcup R_1 \cdot M.$$  

Therefore $d_{n,t} = (d + |B|)^{<n-1>} - |B|^{<n-1>}$. In particular, $d_{n,t} < d^{<n-1>}$.\bigskip

**Proof.** Since $t \geq a$, $B$ is non-empty. Note also that $B$ is a lexsegment set in $S$ because $x_1$ is the highest ranked variable. Meanwhile no monomial in $M$ is divisible by $x_1^a$ and hence $M$ is not a disjoint set. Since $R_1 \cdot M$ is the set of monomials in $S'_1 \cdot M$ that are not divisible by $x_1^a$, we clearly have $S'_1 \cdot (B \cup M) \supseteq S'_1 \cdot B \sqcup R_1 \cdot M$. Conversely, let $m$ be a monomial in $S'_1 \cdot (B \cup M)$. We may take $m \in (S'_1 \cdot M) \setminus (R_1 \cdot M)$. Then $m$ is divisible by $x_1^a$ and since the degree of $m$ is at least $a + 1$, $m/x_1^a$ is divisible by one of the variables, say $x_i$. Then $m = x_i(mx_{i}/x_1) \in S'_1 \cdot B$.

If $L$ is the lexsegment set of size $d$ in $R_t$, then we just showed that $|S'_1 \cdot (B \cup L)| = |S'_1 \cdot B| + |R_1 \cdot L|$. Moreover, since the maximal monomial in $R_t$ that is not in $B$ is the maximal monomial in $L$, we get that $L \cup B$ is a lexsegment set in $S$. It follows that $d_{n,t} = |R_1 \cdot L| = (d + |B|)^{<n-1>} - |B|^{<n-1>}$, as desired. The last statement now follows from [12, 1.5]. \qed

We show that Gotzmann sets in $R_t$ for $t \geq a$ arise from Gotzmann sets in $S_t$ that contain $B$.

**Theorem 3.** Let $M$ be a set of monomials in $R_t$ for $t \geq a$. Then $M$ is Gotzmann in $R_t$ if and only if $B \cup M$ is Gotzmann in $S_t$.

**Proof.** Let $L$ denote the lexsegment set in $R_t$ of the same size as $M$. Then Lemma 2 implies that $|R_1 \cdot L| = |R_1 \cdot M|$ if and only if $|S'_1 \cdot (B \cup M)| = |S'_1 \cdot (B \cup L)|$. Hence the statement of the proposition follows because $B \cup L$ is lexsegment in $S_t$ as we saw in the proof of Lemma 2. \qed
Lemma 5. The sizes of neighboring components. We first prove the following.

We consider the Gotzmann set $A := \{x_1^1, x_2^1, x_1^2, x_2^2, x_1^3, x_2^3\}$ in $F[x_1, x_2, x_3]/(x_1^1, x_2^2)$. Then the set $A \cup \{x_1^3\}$ is not Gotzmann in $F[x_1, x_2, x_3]/(x_1^1)$. Furthermore $A \cup \{x_1^1, x_2^3\}$ is not Gotzmann in $F[x_1, x_2, x_3]$.

For $t = a - 1$, have $d_{n-a-1} = d^{<n-2>} - 1$. Hence if a set of $M$ of monomials does not contain $x_1^{a-1}$, then $R_1 \cdot M = S_1 \cdot M$ and so $|R_1 \cdot M| > d_{n-a-1}$. On the other hand if $x_1^{a-1} \in M$, then $|R_1 \cdot M| = |S_1 \cdot M| - 1$. It follows that $M$ is Gotzmann in $R_{a-1}$ if and only if $M$ is Gotzmann in $S_{a-1}$ and $x_1^{a-1} \in M$.

We prove a result on Gotzmann sets in $S$. Let $M$ be a Gotzmann set in $S_t$. We show that $M'$ is a product of $x_1^1$ with a Gotzmann set in $S'$ if $|M'|^{<n-2>}$ is larger than $|M|^{t-1}$. Otherwise we provide lower bounds on the size of $M'$ depending on the sizes of neighboring components. We first prove the following.

**Lemma 5.** Let $M$ be a Gotzmann set of monomials in $S_t$ with $t \geq 0$. For $0 \leq i \leq t$ set $d_i = |M|$. For $0 \leq i \leq t + 1$ we have

$$|(S_1 \cdot M)^i| = \max\{d_i^{<n-2>}, d_{i-1}\}.$$  

**Proof.** For a set of monomials $K$ in $S_t$ and a monomial $u \in S$, let $u \cdot K$ denote the set of monomials $uk$, where $k \in K$. Note that a monomial in $(S_1 \cdot K)^i$ is either product of a variable in $S'$ with a monomial in $K^i$ or a product of $x_1$ with a monomial in $K^{i-1}$. It follows that $(S_1 \cdot K)^i = S_1^i \cdot K^i \cup x_1 \cdot K^{i-1}$. We also have $S_1^i \cdot K^i = (S_1^i \cdot (K^i/x_1)) = x_1^i \cdot (S_1^i \cdot (K^i/x_1))$. Applying this to the set $M$ we get that the size of the set $S_1^i \cdot M^i$ is equal to the size of $S_1^i \cdot (M^i/x_1)$ which is at least $d_i^{<n-2>}$ by Macaulay’s theorem. Meanwhile the size of the set $x_1 \cdot M_{i-1}$ is $d_{i-1}$. It follows for all $0 \leq i \leq t + 1$ that

$$|(S_1 \cdot M)^i| \geq \max\{d_i^{<n-2>}, d_{i-1}\}.$$  

Define

$$T = \bigcup_{0 \leq i \leq t} x_1^i \cdot (\text{lex}_{S'}(M^i/x_1)).$$  

Notice that we have $|T^i| = d_i$ for $0 \leq i \leq t$. We compute $|(S_1 \cdot T)^i|$ for $0 \leq i \leq t + 1$ as follows. From the first paragraph of the proof we have $(S_1 \cdot T)^i = S_1^i \cdot T^i \cup x_1 \cdot T^{i-1}$ and that $S_1^i \cdot T^i = x_1^i \cdot (S_1^i \cdot (T^i/x_1))$. But $T^i/x_1$ is a homogeneous lexsegment set by construction and so $S_1^i \cdot (T^i/x_1)$ is also a lexsegment set in $S_{t-i+1}'$, see [2] 4.2.5]. Hence $|S_1^i \cdot T^i| = d_i^{<n-2>}$. On the other hand $|x_1 \cdot T^{i-1}| = d_{i-1}$. Moreover, since $T^{i-1}/x_1$ is a lexsegment set in $S_{t-i+1}'$, the identity $x_1 \cdot T^{i-1} = x_1^i \cdot (T^{i-1}/x_1)$ gives that $x_1 \cdot T^{i-1}$ is obtained by multiplying each element in a homogeneous lexsegment set in $S'$ with $x_1^i$. Since $S_1^i \cdot T^i$ is also obtained by multiplying the lexsegment set $S_1^i \cdot (T^i/x_1)$ with $x_1^i$ we have either $S_1^i \cdot T^i \subseteq x_1 \cdot T^{i-1}$ or $S_1^i \cdot T^i \supseteq x_1 \cdot T^{i-1}$. Hence $(S_1 \cdot T)^i = S_1^i \cdot T^i$ if $d_i^{<n-2>} \geq d_{i-1}$ and $(S_1 \cdot T)^i = x_1 \cdot T^{i-1}$ otherwise. Moreover, $|(S_1 \cdot T)^i| = \max\{d_i^{<n-2>}, d_{i-1}\}$. Since the size of $M$ has the minimal possible growth, from Inequality [2] we get $|(S_1 \cdot M)^i| = \max\{d_i^{<n-2>}, d_{i-1}\}$ as desired. □

We remark that the statement of the following theorem stays true if we permute the variables and write the assertion with respect to another variable. It is also instructive to compare this with [11] 2.1.

**Theorem 6.** Assume the notation of the previous lemma. If $d_i^{<n-2>} \geq d_{i-1}$, then $M'/x_1$ is Gotzmann in $S'$. Moreover, if $d_i^{<n-2>} < d_{i-1}$, then we have either $(d_i + 1)^{<n-2>} > d_{i-1} - 1$ or $d_i + 1 > d_{i+1}^{<n-2>}$.  


Proof. Assume that $d_i^{<n-2>} \geq d_{i-1}$ for some $0 \leq i < t + 1$. Then from the first statement we have $|\{S_i \cdot M\}| = d_i^{<n-2>}$. But $S_i \cdot M^i$ is a subset of $(S_i \cdot M)^i$ and $|S_i' \cdot M^i| = |S_i' \cdot (S_i' \cdot (M^i/x_1))| = |S_i' \cdot (M^i/x_1)| \geq d_i^{<n-2>}$. It follows that $|S_i' \cdot (M^i/x_1)| = d_i^{<n-2>}$ and so $M^i/x_1$ is Gotzmann.

We now prove the second assertion of the theorem. Assume that there exists an integer $1 \leq q \leq t$ such that $d_q^{<n-2>} < d_{q-1}$. By way of contradiction assume further that $(d_q + 1)^{<n-2>} \leq d_{q-1} - 1$ and $d_q + 1 \leq d_{q+1}^{<n-2>}$. We obtain a contradiction by constructing a set $W$ in $S_t$ whose size grows strictly less than the size of $M$. Let $w_{q-1}$ be the minimal monomial in $T^{q-1}$. Notice also that $d_q^{<n-2>} < d_{q-1}$ implies that $S_q \setminus T^q \neq \emptyset$ and let $w_q$ be the monomial that is maximal among the monomials in $S_q \setminus T^q$. Define

$$W = \bigcup_{0 \leq i \leq t, \ i \neq q-1,q} T^i \cup (T^{q-1} \setminus \{w_{q-1}\}) \cup \bigcup_{t \neq q-1,q} \{T^q \cup \{w_q\}\}.$$ 

Notice that by construction $W^i/x_1$ is a lexsegment set in $S'$ for all $0 \leq i \leq t$. Therefore, just as we saw for $T$, we have $|(S_1 \cdot W)^i| = \max\{|W^i|^{<n-2>}, |W^{i-1}|\}$. We also have $|W^i| = d_i$ for $i \neq q-1, q$, and $|W^{q-1}| = d_{q-1} - 1$ and $|W^q| = d_q + 1$. It follows that $|(S_1 \cdot T)^i| = |(S_1 \cdot W)^i|$ for all $i \neq q-1, q, q+1$. We finish the proof by showing that

$$\sum_{q-1 \leq i \leq q+1} |(S_1 \cdot W)^i| < \sum_{q-1 \leq i \leq q+1} |(S_1 \cdot T)^i|.$$ 

We have $|(S_1 \cdot W)^{q-1}| = \max\{(d_{q-1} - 1)^{<n-2>}, d_{q-2}\} = \max\{(d_{q-1} - 1)^{<n-2>}, d_{q-2}\} = |(S_1 \cdot T)^{q-1}|$. Notice also that $|(S_1 \cdot W)^q| = \max\{d_{q+1}^{<n-2>}, d_{q-1} - 1\} = d_{q-1} - 1 < d_{q-1} = \max\{d_{q-2}^{<n-2>}, d_{q-1}\} = |(S_1 \cdot T)^q|$. Finally, $|(S_1 \cdot W)^{q+1}| = \max\{d_{q+2}^{<n-2>}, d_q + 1\} = d_q^{<n-2>} = |(S_1 \cdot T)^{q+1}|$. $\square$

We generalize some properties of the minimal growth of the Hilbert function in $S$ to $R$. Firstly, we show that $d_{n,t}$ is increasing in the first parameter and decreasing in the second parameter.

**Proposition 7.** Let $d, n, t$ be positive integers. Then the following statements hold:

(1) $d_{n+1,t} > d_{n,t}$.

(2) $d_{n,t+1} \leq d_{n,t}$.

Moreover, for $n \geq 3$ we have $d_{n,t} = d^{<n-2>}$ for $t$ sufficiently large.

**Proof.** Since $d_{n,t} = d^{<n-1>}$ for $t < a - 1$, the first statement is precisely [12, 1.7] for $t < a - 1$. Let $L$ be the lexsegment set of size $d$ in $R_t$. For $t = a - 1$, we have

$$R \cdot L = (S_1 \cdot L) \setminus \{x_1^a\}.$$ 

So $d_{n,a-1} = d^{<n-1>} - 1$ and the first statement again follows from [12, 1.7]. For $t \geq a$ by Lemma 2, we have $d_{n,t} < d^{<n-1>}$; on the other hand, $d_{n,t+1} \geq d^{<n-1>}$ by Lemma 1. This establishes the first statement.

Since $d_{n,t} = d^{<n-1>}$ for $t \leq a - 2$, the second statement holds trivially for $t < a - 2$. Moreover, we have $d_{n,a-1} = d^{<n-1>} - 1$ from the previous paragraph and so $d_{n,a-1} < d_{n,a-2}$ as well. Also we eliminate the case $n = 2$ because $d_{2,t} = d$ for $t \geq a - 1$. So we assume that $t \geq a - 1$ and $n > 2$. Note that $n > 2$ implies that $|R_t^1| < |R_{t+1}^1|$ and $|R_t^i| < |R_{t+1}^i|$ for $t \geq a - 1$ and $i \leq a - 1$. Let $L_1$ and $L_2$ be two lexsegment sets of equal sizes in $R_t$ and $R_{t+1}$ respectively. The rest of the proof of the second statement is devoted to showing $|R_1 \cdot L_1| \geq |R_1 \cdot L_2|$.
Set \( j_1 = I(L_1) \) and \( j_2 = I(L_2) \). Since \( L_1 \) and \( L_2 \) are lexsegment sets, we have \( L_1^i = R_0^i \) for \( j_1 < i < a - 1 \) and \( L_2^i = R_{i+1}^i \) for \( j_2 < i < a - 1 \). We also have \( |L_1^i| \leq |R_0^i| \) and \( |L_2^i| \leq |R_{i+1}^i| \). But since \( |R_0^i| < |R_{i+1}^i| \), we have \( |L_1^i| < |L_2^i| \) for all \( i \) that is strictly bigger than both \( j_1 \) and \( j_2 \). Therefore \( j_1 \leq j_2 \) because the sizes of \( L_1 \) and \( L_2 \) are the same. Also note that \( |R_0^i| = |R_{i+1}^i| \) and so \( |L_1^i| = |L_2^{i+1}| \) for \( \max(j_1, j_2 - 1) < i < a - 1 \). We claim that \( j_1 + 1 \geq j_2 \). Otherwise we obtain a contradiction as follows. We have \( \max(j_1, j_2 - 1) = j_2 - 1 \neq j_1 \) and \( |L_1^{j_2-1}| = |L_2^{j_2-1}| = |R_{j_2+1}^i| \geq |L_2^j| \). Therefore
\[
|L_2| = \sum_{j_2 \leq i \leq a-1} |L_2^i| \leq \sum_{j_2-1 \leq i \leq a-2} |L_1^i| < \sum_{j_2-1 \leq i \leq a-1} |L_1^i| < \sum_{j_1 \leq i \leq a-1} |L_1^i| = |L_1|.
\]
Thus we have either \( j_1 = j_2 \) or \( j_1 + 1 = j_2 \). We handle these cases separately.

First assume that \( j_1 = j_2 \). Set \( j = j_1 \). If \( j = a - 1 \), then by Lemma \( 1 \) we have \( |R_1 \cdot L_1| = |L_1| = |n-2| = |L_1| < |n-2| \) and similarly we get \( |R_1 \cdot L_2| = |L_1| < |n-2| \) giving the desired inequality. So assume that \( j < a - 1 \). Since \( |L_1| = |L_2| \) and \( |L_1| = |L_2^{j+1}| \) for \( j + 1 < i < a - 1 \), we have \( |L_1| = |L_2| = |L_2| + |L_2^{j+1}| \). Moreover, by Lemma \( 1 \) we have
\[
|R_1 \cdot L_1| - |R_1 \cdot L_2| = |L_1| - |L_2| < |n-2| - |L_2^{j+1}| < |n-2|.
\]
If \( |L_2^j| = |L_2^{j+1}| \), then \( |L_1| = |L_2| \) as well and the right hand side of the equation above is zero giving \( |R_1 \cdot L_1| = |R_1 \cdot L_2| \). If \( |L_2^j| \neq |L_2^{j+1}| \), then we necessarily have \( |L_1^j| < |L_2^{j+1}| \) because \( L_1^j \) is a subset of \( R_0^j \) whose size is equal to the size of \( R_{j+1}^j \). Moreover \( |L_1^j| = |R_0^j| < |R_{j+1}^j| \leq |R_{j+1}^{j+1}| \) because \( j + 1 \leq a - 1 \) and so \( |L_1^j| < |L_2^{j+1}| \) as well. Now we get that \( |L_2^j| < |L_2^{j+1}| \) because both \( |L_1^j| \) and \( |L_1^a| \) is strictly smaller than \( |L_2^{j+1}| \) and \( |L_1^{a-1}| + |L_1^j| = |L_2^j| + |L_2^{j+1}| \). But \( |L_2^{j+1}| \) is the number of all monomials of degree \( t - j \) in \( n - 1 \) variables. Hence [12, 1.6] applies and we get
\[
|L_1| - |L_2^j| < |n-2| + |L_2^j| < |n-2| - |L_2^{j+1}| < |n-2|.
\]
Equivalently, \( |R_1 \cdot L_1| \geq |R_1 \cdot L_2| \) as desired.

If \( j_1 + 1 = j_2 \), then from \( |L_1| = |L_2^{j+1}| \) for \( j_1 < i < a - 1 \) and Lemma 1 we have \( |L_1^i| + |L_1^j| = |L_2^{j+1}| \) and
\[
|R_1 \cdot L_1| - |R_1 \cdot L_2| = |L_1^i| < |n-2| + |L_1^j| < |n-2| - |L_2^{j+1}| < |n-2|.
\]
So we get that \( |R_1 \cdot L_1| - |R_1 \cdot L_2| > 0 \) from [12, 1.5].

Finally, fix an integer \( d \). Then for a sufficiently large integer \( t' \), the number of all monomials of degree \( t' - a - 1 \) in \( S' \) is bigger than \( d \) \((n > 2 \) is essential here). Now let \( L \) be the lexsegment set of size \( d \) in \( R_0 \). Then we have \( L^{n-1} = L \) and therefore Lemma 1 gives \( d_{n,t'} = d^{<n-2} \) as desired.

Let \( L_1 \) and \( L_2 \) be two lexsegment sets in \( R_0 \) with sizes \( b \) and \( c \) respectively with \( b \geq c \). Let \( j_1 \) and \( j_2 \) denote \( I(L_1) \) and \( I(L_2) \). Note that we have \( j_1 \leq j_2 \). Define
\[
t' = \begin{cases} 
t - j_2 \text{ if } j_1 = j_2 \text{ and } j_1 \neq a - 1 \text{ and } x_1^{j_1} x_n^{t - j_1} \notin (L_1 \cup L_2); \\
t + 1 - j_2 \text{ otherwise.}
\end{cases}
\]

We finish by noting down an adoption of [12, 1.5] for the ring \( R \).
Proposition 8. Assume the notation of the previous paragraph. For $n \geq 3$ we have
\\[ b_{n,t} + c_{n,t} > (b + c)_{n,t+t}. \]

Proof. Note that for $t < a - 1$, the statement of the lemma follows from [12, 1.5] because then $b_{n,t} = b^{<n-1}$, $c_{n,t} = c^{<n-1}$ and $(b+c)_{n,t+t} \leq (b+c)^{<n-1}$. So we assume $t \geq a - 1$.

To prove the lemma it is enough to show that there exist monomials $m_1$ and $m_2$ of degree $t'$ in $S'$ such that $m_1 \cdot L_1$ and $m_2 \cdot L_2$ are disjoint and that $R_1 \cdot (m_1 \cdot L_1)$ and $R_1 \cdot (m_2 \cdot L_2)$ have non-empty intersection. Because then $(b+c)_{n,t+t} \leq |R_1 \cdot (m_1 \cdot L_1)| + |R_1 \cdot (m_2 \cdot L_2)| = b_{n,t} + c_{n,t}$.

In the following we use the fact that if a minimal element in a set is of higher rank than a maximal element in another set then these two sets do not intersect.

We handle the case $j_1 = j_2 = a - 1$ separately and the proof for this case essentially carries over from [12, 1.5]. Let $x_1^{a-1} w$ be the minimal element in $L_1$, where $w$ is a monomial in $S'$. Consider $x_1^{j_1} \cdot L_1$ and $w x_n \cdot L_2$. The minimal element of $x_1^{j_1} \cdot L_1$ is $x_1^{a-1} x_2 w$ and the maximal element of $w x_n \cdot L_2$ is $x_1^{a-1} x_2^{a+1} w x_n$. Then $x_1^{a-1} x_2^{j_2} w > x_1^{a-1} x_2^{a+1} w x_n$ since $t' = t - a + 2$. On the other hand $x_1^{a-1} x_2^{j_2} w x_n \in (R_1 \cdot (x_1^{j_1} \cdot L_1)) \cap (R_1 \cdot (w x_n \cdot L_2))$. So we compute the maximal monomial in $x_1^{j_1} \cdot L_1$ and minimal monomial in $x_1^{j_2} \cdot L_2$, respectively. Note that $u = x_1^{j_1} x_2^{j_2} x_3^{j_3} x_4^{j_4}$. We consider different cases and show that $v > u$ in each case. First assume that $j_2 > j_1$. Then we have $L_1 = R_1^j$ and $t' = t + 1 - j_2$. It follows that $v = x_1^{j_1} x_2^{j_2} x_3^{j_3} x_4^{j_4} = x_1^{j_1} x_2^{j_2+1-j_2} x_3^{j_3} x_4^{j_4} > u$. Now assume that $j_1 = j_2$ and $x_1^{j_2} x_3^{j_2} \in (L_1 \cup L_2)$. Then $t' = t + 1 - j_2$. Also, since $|L_1| \geq |L_2|$ and $x_1^{j_2} x_3^{j_2} \in (L_1 \cup L_2)$, we have $x_1^{j_2} x_3^{j_2} \in L_1^j$ and so $v = x_1^{j_1} x_2^{j_2} x_3^{j_2} x_4^{j_4} = x_1^{j_1} x_2^{j_2+1-j_2} x_3^{j_2} x_4^{j_4} > u$. Finally, if $j_1 = j_2$ and $x_1^{j_2} x_3^{j_2} \notin (L_1 \cup L_2)$, then the minimal monomial in $L_1^j$ is bigger than $x_1^{j_2} x_3^{j_2} x_4^{j_4}$ and so $v = x_1^{j_1} x_2^{j_2} x_3^{j_2} x_4^{j_4} = x_1^{j_1} x_2^{j_2+1-j_2} x_3^{j_2} x_4^{j_4} > u$.

We now show that $R_1 \cdot (x_1^{j_1} \cdot L_1)$ and $R_1 \cdot (x_1^{j_2} \cdot L_2)$ have non-empty intersection. If $j_1 < j_2$, then as we saw above $L_1 = R_1^j$ and so $x_1^{j_2} x_3^{j_3} x_4^{j_4}$ is the minimal element in $L_1^j$ and $t'$ is the maximal element in $L_1^j$. But since $x_1^{j_2} x_3^{j_3} x_4^{j_4} \notin (L_1 \cup L_2)$, then the minimal monomial in $L_1^j$ is bigger than $x_1^{j_2} x_3^{j_3} x_4^{j_4}$ and so $v = x_1^{j_1} x_2^{j_2} x_3^{j_3} x_4^{j_4} = x_1^{j_1} x_2^{j_2+1-j_2} x_3^{j_3} x_4^{j_4} > u$.
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