Entanglement Sudden Death in Band Gaps

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Using the pseudomode method, we evaluate exactly time-dependent entanglement for two independent qubits, each coupled to a non-Markovian structured environment. Our results suggest a possible way to control entanglement sudden death by modifying the qubit-pseudomode detuning and the spectrum of the reservoirs. Particularly, in environments structured by a model of a density-of-states gap which has two poles, entanglement trapping and prevention of entanglement sudden death occur in the weak-coupling regime.

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I. INTRODUCTION

Realistic quantum systems are affected by decoherence and entanglement losses because of the unavoidable interaction with their environments [1]. For example, in Markovian environments, in spite of an exponential decay of the single-qubit coherence, the entanglement between two qubits may completely disappear at a finite time [2, 3]. This phenomenon, known as entanglement sudden death and experimentally proven to occur [4, 5], limits the time when entanglement can be exploited for practical purposes. So the issue of how to avoid or control entanglement sudden death-type decorrelation in a realistic physical system is especially important. Up to now, several ways were proposed for keeping the atomic entanglement for a long time by suppressing spontaneous emission. One way widely applied is to place the qubits in a structured environment, say, microcavity [6, 7] or in the photonic band gap of photonic crystals [8]. Other methods considered use dynamic manipulation such as mode modulation [9] or the quantum Zeno effect [10], which can be regarded as extensions of the so-called bang-bang method [11]. In this paper we focus on the entanglement dynamics of two qubits, each coupled to a non-Markovian structured environment. We continue the investigation of physical effects that may control the occurrence time of entanglement sudden death, and find that the speed of disentanglement is closely related to qubit-pseudomode detuning and the spectrum of the reservoirs. In particular, we mainly consider two qubits respectively coupled to a bath of oscillators with a density of states described by a frequency dependent function $D(\omega)$. In some previous works [12, 13, 14], the spectrum of the structured reservoir is a Lorentzian, which only give a single pseudomode to replace the effect of the structured reservoir. Then the short-time behavior of the qubit system leads to exponential decay [15]. However, if there are two or more poles close to the real $\omega$ axis, it is not clear that a simple picture of exponential decay will apply because the poles can interfere with each other. So we focus on the structured reservoir model of a band gap in which both Lorentzians are centered at the same frequency, and the second is given a negative weighting in our paper, and make comparison with the model of the function $D(\omega)$ has one Lorentzian.

II. THEORETICAL MODEL AND EXACT DYNAMICS OF TWO QUBITS

Consider a model consisting of two qubits $A$ and $B$, each interacting with a common zero-temperature bosonic reservoir, denoted $a$ and $b$, respectively. We assume that each qubit-reservoir system is isolated and the reservoirs are initially in the vacuum state while two qubits are initially in an entangled state. Since each qubit evolves independently, we can learn how to characterize the evolution of the overall system from the qubit-reservoir dynamics. The interaction between a qubit and an $N$-mode reservoir is described through the Hamiltonian (under the rotating-wave approximation and setting $\hbar = 1$)

$$
\hat{H}_j = \omega_j \hat{\sigma}_z \hat{\sigma}_z + \sum_{k=1}^{N} \omega_k \hat{b}^\dagger_k \hat{b}_k + \sum_{k=1}^{N} g_k (\hat{\sigma}^z_k \hat{b}^\dagger_k + \hat{\sigma}^z_k \hat{b}_k),
$$

where $\hat{b}^\dagger_k$, $\hat{b}_k$ are the creation and annihilation operators of quanta of the reservoir $(a$ or $b)$, $\hat{\sigma}^z = |1\rangle \langle 0|$, $\hat{\sigma}^z = |0\rangle \langle 1|$, and $\omega_j$ are the frequency operators and transition frequency of the $j$-th qubit $(j=A, B$ and here $\omega_A = \omega_B = \omega_0$); $\omega_k$ and $g_k$ are the frequency of the mode $k$ of the reservoir and its coupling strength with the $j$-th qubit.

Let us consider the case when the $j$-th qubit is initially in the excited state and its corresponding reservoir is in the vacuum state. Using the pseudomode method, we focus on an idealized model of a band gap (or photon density of states gap) in which both Lorentzians are centered at the same frequency, and the second is given a negative weighting, so that

$$
D(\omega) = \frac{W_1 \Gamma_1}{(\omega - \omega_c)^2 + (\frac{\Gamma_1}{2})^2} - \frac{W_2 \Gamma_2}{(\omega - \omega_c)^2 + (\frac{\Gamma_2}{2})^2},
$$

where now the weights of the two Lorentzians are such that $W_1 - W_2 = 1$ and $\Gamma_2 < \Gamma_1$ ensure positivity of $D$. $\omega_c$ is
the center of the spectrum, and $\Gamma_1, \Gamma_2$ are the full width at half maximum of two Lorentzians, respectively. The effect of the Lorentzian with negative weight is to introduce a dip into the density of states function $D(\omega)$ where the coupling of the two pseudomodes decaying with decay rates $\alpha$ and $\beta$, respectively. The two pseudomodes are coupled and reservoirs. The set of ordinary differential equations associated with the density of states gap in Eq. (3) is given by

$$\frac{d\rho}{dt} = -i[H_0, \rho] - \frac{\Gamma_1}{2}[a_1\dagger a_1\rho - 2a_1a_1\rho\dagger + \rho a_1\dagger a_1]$$

$$- \frac{\Gamma_2}{2}[a_2\dagger a_2\rho - 2a_2a_2\rho\dagger + \rho a_2\dagger a_2], \quad (3)$$

where $H_0 = \omega_0\sigma_0^+\sigma_0^- + \omega_1a_1^\dagger a_1 + \omega_2a_2^\dagger a_2 + \Omega_0[a_1^\dagger\sigma^+_1 + a_2\sigma^+_2] + V(a_1^\dagger a_2 + a_1a_2^\dagger)$, (4)

and $\rho$ is the density operator for the j-th qubit and the pseudomodes of its corresponding reservoir; $a_1$ and $a_2$ are the annihilation operators of the two pseudomodes decaying with decay rates $\Gamma_1 = W_1\Gamma_2 - W_2\Gamma_1$ and $\Gamma_2 = W_1\Gamma_2 - W_2\Gamma_2$ respectively. The two pseudomodes are coupled and $V = \sqrt{W_1W_2(\Gamma_1 - \Gamma_2)/2}$ is the strength of the coupling. The qubit interacts coherently with the second pseudomode (the strength of the coupling $\Gamma_0$), which is in turn coupled to the first one. Both pseudomodes are leaking into independent Markovian reservoirs. The set of ordinary differential equations associated to the master equation (3) is

$$i\frac{dc_1}{dt} = \omega_0c_1 + \Omega_0b_2,$$

$$i\frac{db_1}{dt} = z_1^*b_1 + Vb_2,$$

$$i\frac{db_2}{dt} = z_2^*b_2 + Vb_1 + \Omega_0c_1, \quad (5)$$

where $c_1$, $b_1$, and $b_2$ are the complex amplitudes for the states with one excitation in the qubit, one excitation in the first pseudomode, and one excitation in the second pseudomode, respectively. The positions of the true poles are $z_1^* = \omega_c - i\Gamma_1/2$ and $z_2^* = \omega_c - i\Gamma_2/2$. Due to the initial state of the j-th qubit and its corresponding reservoir is $|1\rangle_j \otimes |0\rangle$ (with the state $|0\rangle = \prod_{k=1}^W |0_k\rangle$), then $c_1(0) = 1$, and $b_1(0) = b_2(0) = 0$, so we can acquire the exact solutions ($c_1$, $b_1$ and $b_2$) of the differential equations (5) easily through a computer program.

According to the differential equations (5), for an initial state of the form

$$|\Psi(0)\rangle = (\alpha|00\rangle_{AB} + \beta|11\rangle_{AB}) \otimes |\bar{0}\rangle_a|\bar{0}\rangle_b, \quad (6)$$

where $\alpha, \beta$ are real, then the time evolution of the total system is (using the above pseudomode method)

$$|\Psi(t)\rangle = (\beta(c_1|1\rangle_A|01\rangle_a + b_1|0\rangle_A|10\rangle_a + b_2|0\rangle_A|01\rangle_a) \times (c_1|1\rangle_B|00\rangle_b + b_1|0\rangle_B|10\rangle_b + b_2|0\rangle_B|01\rangle_b)$$

$$+ \alpha|00\rangle_{AB}|01\rangle_a|00\rangle_b, \quad (7)$$

with $|01\rangle_a|00\rangle_b$ and $|00\rangle A|0\rangle_b |01\rangle_a |1\rangle_b$ mean the structured reservoir $a$ or $b$ states with no excitation in two pseudomodes, only one excitation in the first pseudomode, and one excitation in the second pseudomode, respectively. Then we can determine the two qubits dynamics. In particular, in the standard two qubits basis $C = \{|00\rangle_{AB}, |01\rangle_{AB}, |10\rangle_{AB}, |11\rangle_{AB}\}$, the reduced density matrix of the two qubits at time $t$ result as

$$\rho_{AB}(t) = (\alpha^2 + \beta^2|1 - |c_1|^2|) |00\rangle_{AB}\langle 00|$$

$$+ (\beta^2|c_1|^2(1 - |c_1|^2)) |01\rangle_{AB}\langle 01|$$

$$+ (\beta^2|c_1|^2(1 - |c_1|^2)) |10\rangle_{AB}\langle 10|$$

$$+ |\beta|^2|\bar{0}\rangle_{AB}\langle |11|_{AB}|\bar{0}\rangle_{AB}$$

$$+ \alpha\beta|c_1|0\rangle_{AB}\langle 11|$$

$$+ \alpha\beta|c_2|0\rangle_{AB}\langle 00|, \quad (8)$$

We use the concurrence $C$ [19], which attains its maximum value 1 for maximally entangled states and vanishes for separable states, to analyze the two-qubit entanglement dynamics. For $\rho_{AB}(t)$, its concurrence can be derived from [19], as

$$C(t) = 2\max\{0, \alpha\beta|c_1|^2 - \beta^2 |c_1|^2|1 - |c_1|^2|\}. \quad (9)$$

III. NUMERICAL RESULTS AND DISCUSSIONS

Similar to the result in Ref. [20], it is easy to find that the two-qubit entanglement can occur sudden death for $\alpha < \beta$. In this non-Markovian system-reservoir coupling model, the concurrence of the two qubits will vanish forever at a finite interval in weak-coupling regime, but for the strong-coupling regime, there will be entanglement revival after entanglement sudden death. In the following, we mainly investigate that the speed of the occurrence of entanglement sudden death is related to the spectrum of the reservoirs ($\Gamma_2$) and the qubit-pseudomode detuning ($\Delta = \omega_c - \omega_0$).

First we study the relation between the entanglement sudden death and the spectrum of the reservoirs. Fig.1 shows the entanglement dynamics of two qubits in the non-Markovian weak-coupling regime with $\Gamma_1/2 = 10\Omega_0$. We compare the entanglement dynamics of the two qubits for three different values of the width of the second Lorentzian spectral function, namely, $\Gamma_2/2 = \Omega_0, 2\Omega_0, 9\Omega_0$. As in Fig.1(a), the qubits are on resonance with the center of the spectrum, $\Delta = 0$. The concurrence decreases monotonically down to zero in a finite time. It is interesting to find that the speed of occurrence of entanglement sudden death can increase with $\Gamma_2$ increasing. Similar behavior happens when two qubits are near resonance with the center of the spectrum. However, When the qubits are far off-resonant with the center of the spectrum, $\Delta \gg \Omega_0$, as shown in Fig.1(b), where we choose $\Delta = 10\Omega_0$, the speed of occurrence of entanglement sudden death decreases as $\Gamma_2$ increases. We can give an intuitive explanation for these results. As we can see in Fig.2, the density of the spectrum $D(\omega)$ increases monotonically as $\Gamma_2$ increases for $\Delta = 0$, while the density of the spectrum $D(\omega)$ decreases monotonically as $\Gamma_2$ increases when $\Delta = 10\Omega_0$. It is proved that the entanglement sudden death is determined by the modes of the spectrum which are on resonance with the qubits: the speed
of the occurrence of entanglement sudden death decreases (increases) as the density of these modes decreases (increases).

If the structured reservoir only contains a Lorentzian ($\Gamma_2 = 0$), from the Ref.[12], we note that the speed of disentanglement decreases as the width of the Lorentzian spectral ($\Gamma_1$) is replaced by $\lambda$ in Ref.[12] increases on/near the resonant couplings, and the speed of disentanglement increases as $\lambda$ increases for large deduning coupling. However, in our paper the density of the spectrum $D(\omega)$ is composed by two Lorentzians, and the second is given a negative weighting. Then the counter results can be acquired when we consider the width of the second Lorentzian spectral $\Gamma_2$ influence the speed of the occurrence of entanglement sudden death. That is to say, $\Gamma_1$ and $\Gamma_2$ have the opposite effects on the speed of the occurrence of entanglement sudden death. So we can keep two-qubit entanglement for a long time trough choosing suitable spectrum of the reservoirs.

As we know, the effect of the Lorentzian with negative weight is to introduce a dip into the density of states function $D(\omega)$. For a perfect gap, where $D(\omega_c) = 0$, we would also have $W_1/\Gamma_1 = W_2/\Gamma_2$, then the single-qubit excited-state population trapping can occurs [15]. According to Ref.[21], we can find that there is a direct link between the time-dependent entanglement and single-qubit excited-state population for independent qubits, each coupled to a zero-temperature bosonic environment. So there also will appear two-qubit entanglement trapping if the qubits are resonant with the gap in our paper. But it is surprising to see that the entanglement trapping only can occur in the weak-coupling regime, and entanglement sudden death still appears quickly in the intermediate-coupling regime and strong-coupling regime. For the intermediate-coupling regime, the two-qubit entanglement can vanish forever. Due to the strong non-Markovian effects, the entanglement of the two qubits can arise the phenomenon of entanglement sudden death and revival in the strong-coupling regime (as shown in Fig.3).

Then we fix the spectrum of the reservoirs to study the relation between the entanglement sudden death and qubit-pseudomode detuning $\Delta$. In Fig.4 we show the entanglement dynamics for two qubits in a structured reservoir, which only has one Lorentzian spectral. Fig.5 shows the entanglement dynamics when the qubits interact with two independent reservoirs structured by a model of a density-of-states gap which has two Lorentzians. A comparison between Fig.4 and Fig.5 reveals that, both for the weak-coupling regime, the speed of the occurrence of entanglement sudden death only can decrease as $|\Delta|$ increases in the one Lorentzian model. However, there exist a critical value $|\Delta_e|$ (with $\Delta_e^2 = \frac{3}{4} \frac{\sqrt{21} - 1}{\sqrt{10} - 1} W_1 \Omega_0$) in the band gap model, then the speed of occurrence of entanglement sudden death will increase as $|\Delta|$ increases when $|\Delta| < |\Delta_e|$, and entanglement sudden death can occur more slowly with $|\Delta|$ increasing at $|\Delta| > |\Delta_e|$.

![FIG. 1: Time evolution of the concurrence of two qubits as a function of the dimensionless quantity $\Omega_0 t$ in non-Markovian weak-coupling regime, with (a) $\Delta = 0$ and (b) $\Delta = 10\Omega_0$. For the cases of (i) $\Gamma_2/2 = \Omega_0$ (solid dark curve), (ii) $\Gamma_2/2 = 2\Omega_0$ (dashed red curve), (iii) $\Gamma_2/2 = 9\Omega_0$ (dotted blue curve). The parameters used are: $\Gamma_1/2 = 10\Omega_0$, $W_1 = 1.1$, $W_2 = 0.1$ and $\alpha = \frac{1}{2}$, $\beta = \frac{\sqrt{2}}{2}$.](image1)

![FIG. 2: The density of the spectrum $D(\omega)$ as a function of the dimensionless quantity $\Gamma_2/2$ in non-Markovian weak-coupling regime, with (a) $\Delta = 0$ and (b) $\Delta = 10\Omega_0$. The parameters used are: $\Gamma_1/2 = 10\Omega_0$, $W_1 = 1.1$, $W_2 = 0.1$.](image2)
A physical interpretation of the result is that the density of the spectrum $D(\omega)$ decreases monotonically as $|\Delta|$ increases in the one Lorentzian model, while in the band gap model, $D(\omega)$ increases as $|\Delta|$ increases when $|\Delta| < |\Delta_c|$ and decreases as $|\Delta|$ increases at $|\Delta| > |\Delta_c|$. The relation between $D(\omega)$ and $\Delta$ is clearly shown in Fig. 6, for example, when $\Gamma_1/2 = 10\Omega_0, \Gamma_2/2 = \Omega_0$, we can calculate $\Delta_c = 3.53\Omega_0$.

**IV. CONCLUSION**

In summary, we have presented a non-Markovian model describing the exact entanglement dynamics of two qubits, each interacted with a structured reservoir. We have brought to light new relations between the speed of the occurrence of entanglement sudden death and the spectrum of the reservoirs/the qubit-pseudomode detuning for qubits prepared in entangled state. We firstly find that the speed of occurrence of entanglement sudden death is a increasing (decreasing) function of the width of the second Lorentzian spectral $\Gamma_2$ when the qubits are on/near resonance (large detuning) with the pseudomodes of the reservoirs. Due to the density of the spectrum $D(\omega)$ is composed by two Lorentzians in our paper, and the second is given a negative weighting, it is interesting to find that $\Gamma_1$ and $\Gamma_2$ have the opposite effects on the speed of the occurrence of entanglement sudden death. Then for a perfect gap, where $D(\omega_c) = 0$, we would also have $W_1/\Gamma_1 = W_2/\Gamma_2$, the qubits' entanglement trapping and prevention of entanglement sudden death can occur in the weak-coupling regime, but entanglement sudden death still appears quickly in the intermediate-coupling regime and strong-coupling regime. Next, a comparison between the one Lorentzian model and the band gap model, we find the speed of the occurrence of entanglement sudden death only can decrease as $|\Delta|$ increases in the one Lorentzian model and a critical value $|\Delta_c|$ exist in the band gap model. If $|\Delta| < |\Delta_c|$, the speed of occurrence
of entanglement sudden death will increase as $|\Delta|$ increases, and when $|\Delta| > |\Delta_c|$, entanglement sudden death can occur more slowly with $|\Delta|$ increasing. The results here obtained evidence the entanglement can be preserved or controlled by modifying the spectrum of the environment and highlight the potential of reservoir engineering for controlling and manipulating the dynamics of quantum systems.

Our results would apply to cavity QED experiments with trapping ions, and to circuit QED experiments. Entanglement between two remotely located trapped atomic ions has been recently demonstrated [22] and multiparticle-entangled states can be generated and fully characterized via state tomography [23]. Moreover, field coupling and coherent quantum state storage between two Josephson phase qubits has been achieved through a microwave cavity on chip [24]. Due to the possibilities for realizing strong coupling conditions between atoms and a high finesse cavity [25], a deep understanding of the non-Markovian dynamics is now indispensable.

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