BLOCKS AND SUPPORT VARIETIES

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1. Introduction

Block decompositions of module categories are well-known and much studied, especially in modular representation theory. When considering cohomological questions, it is often more convenient to work in the stable module category, but this makes little difference to the block theory: one simply loses the simple blocks for which all modules are projective. The theory of varieties for modules for finite groups gives a rich supply of interesting thick subcategories of the stable module category. There are block decompositions of these arising from the usual blocks of the group algebra, but it turns out that in general the blocks break up even further. In this paper we study this phenomenon, particularly in the case of the thick subcategory determined by a single line in the maximal ideal spectrum $V_G(k)$ of the cohomology algebra. We give simple examples where calculations can easily be done, and use a theorem of Benson to reduce the general case to examples of this kind.

Finally, we describe how the theory of block varieties developed by Linckelmann seems to shed more light on these phenomena.

2. Block decompositions of categories

If $C$ is an additive category, then by an additive subcategory of $C$ we mean a full subcategory $C'$ such that if $X$ is a finite coproduct, in $C$, of objects of $C'$, then $X$ is in $C'$. In particular, $C'$ is also an additive category, and contains all objects of $C$ isomorphic to objects of $C'$.

Let $C$ be an additive category. We say that $C$ is the direct sum of a family $\{C_i : i \in I\}$ of additive subcategories of $C$, and write

$$C = \bigoplus_{i \in I} C_i$$

if every object $X$ of $C$ can be expressed as a coproduct

$$X = \bigoplus_{i \in I} X_i$$

with $X_i$ an object of $C_i$, and

$$\text{Hom}(X, Y) = 0 = \text{Hom}(Y, X)$$

whenever $X$ is an object of $C_i$ and $Y$ is a coproduct of objects of $\{C_j : j \neq i\}$.

It follows that if $X = \bigoplus X_i$ and $Y = \bigoplus Y_i$, with $X_i, Y_i \in C_i$, then

$$\text{Hom}(X, Y) = \prod_{i \in I} \text{Hom}(X_i, Y_i).$$
It is easy to see that the projection $X \mapsto X_j$ from $\mathcal{C}$ to $\mathcal{C}_j$ is functorial for each $i \in I$, and is both left and right adjoint to the inclusion functor $\mathcal{C}_j \to \mathcal{C}$. It follows that the direct sum decomposition of $X$ is unique up to natural isomorphism, and that the subcategories $\mathcal{C}_i$ are closed under all limits and colimits in $\mathcal{C}$. In particular, they are closed under arbitrary products and coproducts and under taking direct summands.

If $\mathcal{C}$ is an abelian category then $\mathcal{C}_i$ is abelian, closed under taking extensions, subobjects and quotients, and the inclusion and projection functors are exact.

If $\mathcal{C}$ is a triangulated category, then if $\mathcal{C}_i$ is closed under the shift functor, it is triangulated, with the inclusion and projection functors exact. In this paper we shall mostly be considering the case where $\mathcal{C}$ is a triangulated subcategory of the stable module category $\text{stmod}(kG)$ of a finite group algebra, and in this case it follows automatically that each $\mathcal{C}_i$ is closed under the shift functor, by Tate duality.

For example, if $kG$ is a finite group algebra then it is easy to check that the module category $\text{mod}(kG)$ has a block decomposition into the module categories of the blocks, in the usual sense, of the group algebra. Also the stable module category $\text{stmod}(kG)$ has a block decomposition into the stable module categories of the nonsimple blocks of $kG$. We give the easy proof of this later in this section.

We shall now describe the block decomposition of an arbitrary thick subcategory of a stable module category.

**Definition 2.1.** Let $\mathcal{C}$ be a thick subcategory of the stable module category $\text{stmod}(kG)$ of a finite group, and let $\mathcal{I}$ be the class of indecomposable objects of $\mathcal{C}$. Define $\sim$ to be the smallest equivalence relation on $\mathcal{I}$ such that $M \sim N$ whenever $\text{Hom}(M, N) \neq 0$.

Thus, for $M, N \in \mathcal{I}$, $M \sim N$ if and only if there exist objects $M = L_0, \ldots, L_n = N$ of $\mathcal{I}$ such that for every $i = 1, \ldots, n$, either $\text{Hom}(L_{i-1}, L_i) \neq 0$ or $\text{Hom}(L_i, L_{i-1}) \neq 0$. By the remark on Tate duality above, it follows that $M \sim \Omega(M)$ for every $M \in \mathcal{I}$, and hence $M \sim N$ if

\[ 0 \neq \text{Ext}^i_{kG}(M, N) \cong \text{Hom}(\Omega^i(M), N) \]
for any $i \in \mathbb{Z}$.

If $I$ is the set of equivalence classes, and we define $C_i$ to be the full subcategory of $C$ consisting of direct sums of objects of $i \in I$, then it is easy to see that $C = \bigoplus_{i \in I} C_i$ is a block decomposition of $C$. We shall call the blocks $C_i$ ext-blocks of $C$ to distinguish them from the blocks of the group algebra. In fact, the main aim of this paper is to study the relationship between the two notions of block.

We shall be studying in detail the ext-blocks of subcategories of $\text{stmod}(kG)$ determined by varieties. If $V$ is a closed homogeneous subvariety of the maximal ideal spectrum $V_G(k)$ of $H^*(G,k)$, then we denote by $C_V$ the full subcategory of $\text{stmod}(kG)$ consisting of the modules $M$ whose variety $V_G(M)$ is contained in $V$. This is a thick subcategory of $\text{stmod}(kG)$. We denote by $\sim_V$ the equivalence relation described above on the class of indecomposable objects of $C_V$.

**Proposition 2.2.** Suppose that $V$ is a closed homogeneous subvariety of $V_G(k)$. Then we have the following.

1. The number of ext-blocks of $C_V$ is finite.
2. If $C_V = \bigoplus_{i \in I} C_i$ is a direct sum decomposition, then any ext-block of $C_V$ is contained in some $C_i$.
3. If $M$ and $N$ are nonprojective indecomposable modules in the same ext-block of $C_V$, then $M$ and $N$ are in the same ordinary block of $kG$.
4. If $M$ is a nonprojective module in $C_V$ and if $M$ lies in a block $B$ of $kG$ with defect group $D$, then $V_G(M) \cap \text{res}^*_{G,D}(V_D(k)) \neq \{0\}$.

**Proof.** Let $L$ be a finitely generated $kG$-module with the property that $V_G(L) = V$. The fact that such a module exists is a standard property of support varieties for finite groups as in [C2]. Suppose that $M$ is a nonprojective indecomposable module in $C_V$. Then there exists an irreducible $kG$-module $S$ such that $\text{Hom}(M \otimes L, S) \neq 0$. We know this from the tensor product theorem for support varieties which tells us that $M \otimes L$ is not projective since the varieties of $M$ and $L$ do not intersect trivially. Then we have that $\text{Hom}(M, L^* \otimes S) \neq 0$ and hence there is some indecomposable component $U$ of $L^* \otimes S$ such that $M \sim_V U$. We know that there are only a finite number of simple modules $S$ and only a finite number of components of $L^* \otimes S$ for any $S$. Consequently, there are only a finite number of equivalence classes for the relation $\sim_V$, and hence only a finite number of ext-blocks.

Statement (2) repeats a general property of blocks proved above, and (3) follows, since there is clearly a direct sum decomposition of $C_V$ according to the ordinary blocks of $kG$.

To prove statement (4), we just need to recall that every module $N$ in $B$ is a direct summand of a module that is induced from a $kD$-module. Therefore, $V_G(N)$ is contained in $\text{res}^*_{G,D}(V_D(k))$. \qed

**Remark 2.3.** We should point out that the statement (1) of the proposition is in contrast to the fact, shown in [BCR], that the category $C_V$ may have an infinite number of mutually orthogonal thick subcategories. By this we mean that there may be an infinite number of thick subcategories such that if $M$ is a module in one and $N$ is in another then $\text{Ext}^*_G(M,N) = 0$. However, $C_V$ is not the direct sum
of these subcategories, as they do not contain all indecomposable objects of $\mathcal{C}_V$. Indeed, in the examples considered in [BCR] it can be shown that $\mathcal{C}_V$ has only one ext-block.

Notice that Proposition 2.2(3) says that the ext-blocks are a refinement of the ordinary blocks of $kG$. This refinement can be seen another way.

Proposition 2.4. Suppose that $V$ and $V'$ are closed homogeneous subvarieties of $V_G(\kappa)$ with $V \subseteq V'$, and let $M$ and $N$ be indecomposable modules in $\mathcal{C}_V$. Then if $M \sim V N$, then also $M \sim V' N$. Moreover, if $V = V_G(\kappa)$, then the ext-blocks of $\mathcal{C}_V = st\text{mod}(kG)$ are precisely the blocks of $kG$ which have defect greater than zero.

Proof. The first statement is obvious from the definition. The second is a well known fact about blocks. That is, if $V = V_G(\kappa)$ and if $M$ and $N$ are nonprojective modules in the same block of $kG$, then clearly $M \sim V$ $S$ and $N \sim V S'$ for some nonprojective simple modules $S$ and $S'$ in the block. But then there are simple modules $S = S_0, \ldots, S_n = S'$ in the block with $\text{Ext}^1(S_i, S_j) \neq 0$, so $S \sim V S'$.

We end this section with some remarks on how things change if we consider the stable category $\text{StMod}(kG)$ of arbitrary (not necessarily finitely generated) modules. For many of the most familiar subcategories, it turns out that the ext-block structure is the same as in the finitely generated case.

To make this precise, for a thick subcategory $\mathcal{C}$ of $\text{st}\text{mod}(kG)$, let $\mathcal{C}^\oplus$ be the localizing subcategory of $\text{StMod}(kG)$ generated by $\mathcal{C}$; i.e., the smallest triangulated subcategory of $\text{StMod}(kG)$ that contains $\mathcal{C}$ and is closed under arbitrary coproducts.

Proposition 2.5. Suppose $\mathcal{C}$ is a thick subcategory of $\text{st}\text{mod}(kG)$ that has a block decomposition $\mathcal{C} = \bigoplus_{i \in I} \mathcal{C}_i$. Then $\mathcal{C}^\oplus = \bigoplus_{i \in I} \mathcal{C}_i^\oplus$ is a block decomposition. In particular, there is a natural bijection between the blocks of $\mathcal{C}$ and the blocks of $\mathcal{C}^\oplus$.

Proof. Let $X$ be an object of $\mathcal{C}_i^\oplus$ and let $Y = \bigoplus_{j \neq i} Y_j$ be a coproduct of objects of $\{\mathcal{C}_j : j \neq i\}$. For any object $X'$ of $\mathcal{C}_i$ and any object $Y'_j$ of $\mathcal{C}_j$, where $i \neq j$, $\text{Hom}(X', Y'_j) = 0$. Since $X'$ is a compact object of $\text{StMod}(kG)$, the functor $\text{Hom}(X',-) \text{ preserves arbitrary coproducts, and so } \text{Hom}(X', Y) = 0$. Since the class of objects with no nonzero maps to $Y$ is a localizing subcategory of $\text{StMod}(kG)$ that contains $\mathcal{C}_i$, it contains $\mathcal{C}_i^\oplus$, and hence $\text{Hom}(X, Y) = 0$. A similar proof shows that $\text{Hom}(Y, X) = 0$. Since $\bigoplus_{i \in I} \mathcal{C}_i^\oplus$ is a localizing subcategory of $\mathcal{C}^\oplus$ that contains $\mathcal{C}$, it must be the whole of $\mathcal{C}^\oplus$. So $\mathcal{C}^\oplus = \bigoplus_{i \in I} \mathcal{C}_i^\oplus$ is a direct sum decomposition.

It remains to show that $\mathcal{C}_i^\oplus$ has no nontrivial direct sum decomposition. Suppose that $\mathcal{C}_i^\oplus = \mathcal{D} \oplus \mathcal{D'}$. Then since $\mathcal{C}_i$ has no nontrivial direct sum decomposition, either $\mathcal{D}$ or $\mathcal{D'}$ must contain every object of $\mathcal{C}_i$. But every object of $\mathcal{C}_i^\oplus$ has a nonzero map from some object of $\mathcal{C}_i$, so either $\mathcal{D}$ or $\mathcal{D'}$ contains all objects of $\mathcal{C}_i^\oplus$. \hfill \Box

3. Fixed lines in the variety of a normal elementary abelian subgroup

In this section, we shall show that there is one important situation where the ext-blocks of the category $\mathcal{C}_V$ coincide with the ordinary blocks.
If $V$ is a closed subvariety of $V_G(k)$ then we say that $V$ is **minimally supported** on an elementary abelian subgroup $E$ of $G$ if $E$ is a minimal elementary abelian subgroup such that $V \subseteq \text{res}^*_G,E(V_E(k))$. We know that if $V$ is an irreducible subvariety, then $V$ is minimally supported on some $E$, which is unique up to conjugacy in $G$.

**Theorem 3.1.** Suppose that the finite group $G$ has a normal elementary abelian subgroup $E$. Let $V'$ be a line in $V_E(k)$; i.e., an irreducible linear subspace of $V_E(k)$ of dimension one. Let $V = \text{res}^*_G,E(V')$, and assume that $V$ and $V'$ have the properties that

1. $V$ is minimally supported on $E$, and
2. $V'$ is stable under the action of $G/C_G(E)$.

Then the ext-blocks of $C_V$ coincide with the ordinary blocks.

For the remainder of this section we shall assume the hypotheses and notation of the theorem. Let $E = \langle x_1, \ldots, x_n \rangle$ have rank $n$.

**Lemma 3.2.** $G/C_G(E)$ is a cyclic group of order prime to $p$, and acts on $V'$ by a linear character $\chi$.

*Proof.* Suppose that $E$ has rank $n$ and $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a nonzero point of the rank variety of $E$ corresponding to the line $V'$. Then the element

$$u_\alpha = 1 + \sum_{i=1}^n \alpha_i(x_i - 1) \in kE$$

has the property that for any $kE$-module $M$, $V' \subset V_E(M)$ if and only if the restriction $M_{\langle u_\alpha \rangle}$ is not free as a $k\langle u_\alpha \rangle$-module. By condition (1), the elements $\alpha_1, \ldots, \alpha_n$ must be linearly independent over the prime subfield $\mathbb{F}_p \subseteq k$, since, if there existed an $\mathbb{F}_p$-dependence relation involving the elements $\alpha_1, \ldots, \alpha_n$, then we could find some proper linear subspace of $k$, defined over $\mathbb{F}_p$, that contained $\alpha$. But then this linear subspace would be $\text{res}^*_{E,F}(V_F(k))$ for some proper subgroup $F \subseteq E$, contradicting condition (1). It follows that, since elements of $G/C_G(E)$ act on $V_E(k)$ by an $\mathbb{F}_p$-linear transformation, no nontrivial element of $G/C_G(E)$ can fix the line $V'$ pointwise. Consequently, the action on $V'$ gives us a faithful representation $\chi: G/C_G(E) \longrightarrow \text{GL}(1,k)$ and so $G/C_G(E)$ must be cyclic. \hfill \square

The primary tool that we need is the following.

**Proposition 3.3.** There exists an element $u \in \text{Rad}(kE)$, $u \notin \text{Rad}^2(kE)$ having the following properties, where we denote by $U = \langle u \rangle$ the subgroup of the group of units of $kE$ generated by the element $u = u + 1$, so $U$ is cyclic of order $p$.

1. For $x \in G/C_G(E)$, $xux^{-1} = \chi(x)u$, for $\chi$ as in Lemma 3.2.
2. If $M$ is a $kE$-module, then $V \subset V_E(M)$ if and only if $M_U$ is not a free $kU$-module.
3. If $M$ is a $kG$-module, then $V' \subset V_E(M)$ if and only if $M_U$ is not a free $kU$-module, where $V = \text{res}^*_G,E(V')$.\hfill \square
Proof. Corresponding to the line $V$ in $V_E(k)$, there is a cyclic shifted subgroup $U = \langle u_\alpha \rangle$, such that $\alpha = (\alpha_1, \ldots, \alpha_n) \in k^n$ and

$$u_\alpha = 1 + \sum_{i=1}^n \alpha_i (x_i - 1)$$

as in the previous proof. For any $kG$-module $M$, $V' \subseteq V_E(M)$ if and only if the restriction of $M$ to $U$ is not a free $kU$-module. Moreover, if we let $u = u_\alpha + w$ where $w \in \text{Rad}^2(kG)$, then the subgroup generated by $u$ has the same property (see [CI]).

The fact that $V$ is invariant under the action of $G/C_G(E)$, implies that $u_\alpha - 1$ must be an eigenvector in the space $\text{Rad}(kE)/\text{Rad}^2(kE)$ for the action of $G/C_G(E) = \langle x \rangle$ with the eigenvalue $\chi(x)$ for the element $x$. Because $G/C_G(E)$ has order prime to $p$, there is an element $u$ in $\text{Rad}(kG)$ where $x$ acts with eigenvalue $\chi(x)$ and with the property that

$$u \equiv u_\alpha - 1 \mod \text{Rad}^2(kG).$$

Taking $u = 1 + u$ and $U = \langle u \rangle$ proves parts (1) and (2). Part (3) follows from the fact that a $kG$-module has the property that $V \subseteq V_G(M)$ if and only if the restriction of $M$ to $E$ has the property that $V' \subseteq V_E(M_E)$ [AE].

Let $X = kG/ukG$ be the quotient module. We claim that $X \cong (kE/ukE)^G$. This is because clearly

$$(kE/ukE)^G \cong kG/(ukE)^G,$$

and

$$(ukE)^G \cong kG \otimes_{kE} ukE \cong kGu \otimes_{kE} kE \cong kGu,$$

which is the same as $ukG$ by Proposition 3.3(i).

Also $V_G(X) = V$. This is true by a rank variety argument. That is, $kE/ukE$ is an indecomposable periodic module, and hence its rank variety is a single line, which must be the line through $\alpha$.

Now we prove the following.

**Lemma 3.4.** Suppose that $M$ is a nonprojective $kG$-module in $C_V$. Then $\text{Ext}^n_{kG}(M, X)$ is nonzero for all $n$.

**Proof.** This follows from the fact that $X$ is induced from a $p$-subgroup:

$$\text{Ext}^n_{kG}(M, X) \cong \text{Ext}^n_{kG}(M, (kE/ukE)^G) \cong \text{Ext}^n_{kE}(M, kE/ukE) \neq 0$$

since the varieties of $M$ and $kE/ukE$ both contain $V'$. \qed

For each indecomposable projective summand $P$ of $kG$, $X$ has a corresponding summand $P/uP$, which is indecomposable (since it has a simple top) and non-projective (since its restriction to $kE$ is a direct sum of copies of $kE/ukE$). At this point we fix a $p$-block $B$ of $G$, let $\{P_1, \ldots, P_t\}$ be a complete set of representatives of the isomorphism classes of the projective indecomposable $kG$-modules in $B$, and let $X_i = P_i/uP_i$ for $i \in \{1, \ldots, t\}$, so $\{X_1, \ldots, X_t\}$ is a complete set of representatives of the isomorphism classes of indecomposable summands of $X$ in the block $B$.

So $X_1, \ldots, X_t$ are in $C_V \cap B$ (which is therefore nonzero), and Lemma 3.4 implies that if $M$ is any other module in $C_V \cap B$, then there is some $i \in \{1, \ldots, t\}$ such that
$M \sim_V X_i$. Consequently, in order to show that the modules in $\mathcal{C}_V \cap B$ are all in the same ext-block, and to prove Theorem 3.1, it remains to show that $X_i \sim_V X_j$ for all $i$ and $j$. The first step in this direction is the following. Recall that $\chi$ is the character having $C_G(E)$ as its kernel such that $gug^{-1} = \chi(g)u$. Let $\mathcal{Y}_i$ be the $kG$-module of dimension one which affords the character $\chi^i$.

**Lemma 3.5.** For any $i$ and $j$, we have that $\mathcal{Y}_i \otimes X_j \sim_V X_j$.

**Proof.** First notice that if we fix a nonzero element $y \in \mathcal{Y}_i$, the map

$$
\mu : \mathcal{Y}_i \otimes P_j \longrightarrow P_j
$$

given by $\mu(y \otimes x) = ux$ is a $kG$-module homomorphism, since for $g \in G$,

$$
\mu(g(y \otimes x)) = \mu(\chi(g)y \otimes gx) = \chi(g)ugx = gug^{-1}gx = g\mu(y \otimes x).
$$

The cokernel of $\mu$ is $X_j$, and because $P_j$ is free as a $kU$-module, the kernel of $\mu$ is $u^{p-1}(\mathcal{Y}_i \otimes P_j)$. Because $\mathcal{Y}_i \otimes P_j$ is projective, we have that

$$
\Omega(X_j) \cong (\mathcal{Y}_i \otimes P_j)/u^{p-1}(\mathcal{Y}_i \otimes P_j).
$$

If $p = 2$, this proves that $\mathcal{Y}_i \otimes X_j \sim_V X_j$, and we can iterate the argument to get the conclusion of the lemma. That is, $\mathcal{Y}_i \otimes X_j \cong \Omega^t(X_j)$ for all $j$ and all $t$.

So we can assume that $p > 2$. In this situation we observe, by similar means, that

$$
\Omega^2(X_j) \cong u^{p-1}(\mathcal{Y}_i \otimes P_j) \cong \mathcal{Y}_{p-1} \otimes X_j.
$$

So in this case we have that

$$
\Omega^2(X_j) \cong \mathcal{Y}_i \otimes X_j.
$$

The final fact we need to complete the proof of Theorem 3.1 is the following.

**Lemma 3.6.** For some $1 \leq i, j \leq t$, suppose that $\text{Hom}_{kG}(P_i, P_j) \neq 0$. Then $\text{Hom}_{kG}(\mathcal{Y}_k \otimes X_i, \mathcal{Y}_\ell \otimes X_j) \neq 0$ for some $k$ and $\ell$.

**Proof.** Let $\varphi : P_i \longrightarrow P_j$ be a nonzero homomorphism. Let $m$ be the greatest integer such that $u^m \varphi(P_i) \neq 0$. Then fix a nonzero element $y \in \mathcal{Y}_m$ and define $\psi : \mathcal{Y}_m \otimes P_i \longrightarrow P_j$ by $\psi(y \otimes a) = u^m \varphi(a)$ for $y \in \mathcal{Y}_m$ and $a \in P_i$. Then $\psi$ is a nonzero $kG$-module homomorphism, the kernel of $\psi$ contains $\mathcal{Y}_m \otimes uP_i$, and the image of $\psi$ is contained in $u^{p-1}P_j \cong \mathcal{Y}_{p-1} \otimes X_j$. Therefore $\psi$ induces a nonzero map $\psi' : \mathcal{Y}_m \otimes X_i \longrightarrow \mathcal{Y}_{p-1} \otimes X_j$. Finally we need only observe that $\psi'$ cannot factor through a projective $kG$-module because its restriction to $kU$ does not factor through a projective $kU$-module.

**Proof of Theorem 3.1.** As noted, before the proof of Lemma 3.5, we need only show that $X_i \sim_V X_j$ for every $i$ and $j$. Because $P_1, \ldots, P_t$ are the projective modules in the block $B$, for any $i$ and $j$ there is a sequence $i = i_0, \ldots, i_r = j$ such that for every $k = 1, \ldots, r$, $\text{Hom}_{kG}(P_{i_{k-1}}, P_{i_k}) \neq 0$. So, by Lemma 3.5 there exist $k$ and $\ell$ such that $\mathcal{Y}_k \otimes X_{i_{k-1}} \sim_V \mathcal{Y}_\ell \otimes X_i$. The theorem now follows by Lemma 3.4.
4. Some examples.

In this section we show some examples in which the ext-blocks corresponding to a subvariety $V \subseteq V_G(k)$ do not coincide with ordinary blocks.

For the first example let $H$ be an abelian group of order 28 generated by elements $g, x$ and $y$ such that $g^7 = 1$ and $x^2 = 1 = y^2$, and let $G = H \times C_3$ be the semidirect product of $H$ by a cyclic group $C_3 = \langle z \rangle$ of order 3 acting on $H$ by

$$gz^{-1} = g^2 \quad zx^{-1} = y \quad yz^{-1} = xy.$$ 

Let $k$ be a field of characteristic 2 that contains a primitive $7^{th}$ root of unity, which we denote $\zeta$. Then $kH$ has seven simple modules $N_0, \ldots, N_6$, each one dimensional, where $g$ acts on $N_i$ by multiplication by $\zeta^i$. Each simple module is in a different block of $kH$, and we denote by $b_i$ the block containing $N_i$.

Then $z$ permutes the simple $kH$-modules, and hence the blocks of $kH$. That is, $z \otimes N_1 \cong N_4$, $z \otimes N_4 \cong N_2$, etc. Moreover, $kG$ has exactly three irreducible modules, $k$, $M_1$ and $M_2$ where $(M_1)_H \cong N_1 \oplus N_2 \oplus N_3$ and $(M_2)_H \cong N_3 \oplus N_5 \oplus N_6$, and each one is the unique simple module in a block of $kG$. Let $B_0, B_1$ and $B_2$ be the blocks of $kG$ containing $k$, $M_1$ and $M_2$ respectively.

Now suppose that $V'$ is a line in $V_H(k) \cong k^2$ such that $V'$ is not stable under the action of $G/H$, and that $V = \text{res}^G_{C,H}(V') \subseteq V_G(k)$.

**Proposition 4.1.** Each of the subcategories $C_V \cap B_1$ and $C_V \cap B_2$ is a direct sum of three ext-blocks.

**Proof.** Suppose that $X$ is any module in $B_1$. Then $X_H \cong X_1 \oplus X_2 \oplus X_4$, where $X_i$ is a module in $b_i$. Indeed, since $|G : H|$ is not divisible by 2, $X$ is a direct summand of $X_H^{G}$, and it is easy to see that $X \cong X_i^{G}$. If, in addition $X$ is an indecomposable object of $C_V$, then $X_1$ is in exactly one of the subcategories $C_{V'}, C_{z(V')} \cup C_{z^2(V')}$.

So let $U_i$ be the subcategory of $B_1 \cap C_V$ consisting of all $X$ such that $X_i \in C_{V'}$ for $i = 1, 2 \text{ or } 4$.

Now suppose that $X$ and $Y$ are both in $U_i$ for $i = 1, 2 \text{ or } 4$. Then $X \cong X_i^{G}$ and $Y \cong Y_i^{G}$ for some $X_i$ and $Y_i$ in $b_i \cap C_{V'}$. Hence

$$\text{Hom}_{kG}(X, Y) \cong \text{Hom}_{kH}(X_i, Y_1 \oplus Y_2 \oplus Y_4) \neq 0,$$

since

$$\text{Hom}_{kH}(X_i, Y_i) \cong \text{Hom}_{kH}(k, X_i^* \otimes Y_i) \neq 0,$$

because $X_i^* \otimes Y_i$ is in the principal block $b_0$.

Suppose on the other hand that $X \in U_i$ and $Y \in U_j$ for $i \neq j$. Then, as before, $X \cong X_i^{G}$ and $Y \cong Y_j^{G}$. In this case

$$\text{Hom}_{kG}(X, Y) \cong \text{Hom}_{kH}(X_i, Y_1 \oplus Y_2 \oplus Y_4) = 0,$$

since

$$\text{Hom}_{kH}(X_i, Y_i) = 0$$

because the varieties of $X_i$ and $Y_i$ intersect trivially, while

$$\text{Hom}_{kH}(X_i, Y_i) = 0$$
for $\ell \neq i$ because $X_i$ and $Y_\ell$ are in different blocks of $kH$. Hence we have proved that the subcategories $U_1$, $U_2$ and $U_4$ are the ext-blocks of $C_V \cap B_1$. 

We should remark that many examples can be constructed along the lines we have just presented. For example, suppose that $p = 3$ and that $H = C_5 \times C_3^2$ and $G \cong H \rtimes C_2$, where the generator of order 2 acts on the $C_5$ and the first $C_3$ by inverting the elements but acts trivially on the second $C_3$. Then $H$ has five blocks, but $G$ has only three. If $V' \subseteq V_H(k)$ is a line that is not fixed (setwise) by the $C_2$ and if $V = \text{res}^*_{G,H}(V') \subseteq V_G(k)$, then $B \cap C_V$ has two ext-blocks for each nonprincipal block $B$ of $G$. A similar thing happens in characteristic five when $G = (C_3 \times C_5^2) \rtimes C_2$, or in characteristic seven when $G = (C_2^2 \times C_7^2) \rtimes C_3$. We shall show that these examples are typical of what happens in general.

The examples also show some unusual behavior of the idempotent modules. Specifically, we have the following. Assume the hypothesis and notation of the example at the beginning of the section.

**Corollary 4.2.** Let $G = (C_7 \times C_3^2) \rtimes C_3$ and $V$ be as in the example. Suppose that $e_V$ is the idempotent module corresponding to the subvariety $V$. Recall that $M_1$ is the unique simple module in the block $B_1$. Then $e_V \otimes M_1$ is a sum of three modules, one in each ext-block.

**Proof.** Suppose that $X$ is a nonprojective module in $C_V \cap B_1$. Then, because $M_1$ is the unique simple module in $B_1$, we must have that $\text{Hom}_{kG}(M_1, X) \neq 0$ and hence also that $\text{Hom}_{kG}(e_V \otimes M_1, X) \neq 0$. It follows that $e_V \otimes M_1$ must have a component in every ext-block of $C_V \cap B_1$. □

5. **Lines in general**

In this section, we reduce the study of ext-blocks in $C_V$ for an arbitrary line $V$ to the case studied in Section 3 using the following theorem of Benson [B2].

**Theorem 5.1.** [B2] Suppose that $V$ is a line in $V_G(k)$ which is minimally supported on an elementary abelian subgroup $E$. Suppose that $V' \subseteq V_E(k)$ is a line such that $\text{res}^*_{G,E}(V') = V$. Let $H$ be the set-wise stabilizer of $V'$ in $N_G(E)$, and let $\hat{V} = \text{res}^*_{H,E}(V')$. Then the categories $C_V$ and $C_{\hat{V}}$ are equivalent.

The equivalence of categories is easy to describe. The functor $C_{\hat{V}} \longrightarrow C_V$ is simply induction from $H$ to $G$, and the inverse functor $C_V \longrightarrow C_{\hat{V}}$ is restriction to $H$ followed by choosing the largest direct summand of the restriction that has variety $\hat{V}$. This last operation is equivalent to taking the tensor product with $e_{\hat{V}}$, the idempotent module corresponding to $\hat{V}$. The key point of the proof that these functors are equivalences of categories is that the conditions guarantee that in the Mackey decomposition

$$ (M'^G)_H \cong \sum_{H \trianglelefteq H} x \otimes (M_{{H \cap x\backslash H} x^{-1}})^H, $$

where $x$ runs through a complete set of representatives for the double cosets $H \trianglelefteq H$. This is a key result in the theory of Mackey decomposition for finite groups.

In the case of $C_7 \rtimes C_3^2$, the Mackey decomposition gives

$$ (M'^G)_H \cong \sum_{H \trianglelefteq H} x \otimes (M_{{H \cap x\backslash H} x^{-1}})^H, $$

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the terms \((M_{H \cap x H x^{-1}})^{TH}\) for \(x \notin H\) have varieties that intersect \(\hat{V}\) trivially, so that if \(M \in \mathcal{C}_V\) then the largest direct summand of \((M^G)_H\) with variety \(\hat{V}\) is stably isomorphic to \(M\). See [B2] for details.

Benson’s equivalence and Theorem 3.1 easily imply the following.

**Proposition 5.2.** Let \(V\) be a line in \(V_G(k)\) minimally supported on \(E\), and let \(H\) and \(\hat{V}\) be as above. Then the ext-blocks of \(\mathcal{C}_V\) are parametrized by the ordinary blocks of \(kH\).

**Proof.** By Benson’s theorem, there is a natural bijection between ext-blocks of \(\mathcal{C}_V\) and of \(\mathcal{C}_{\hat{V}}\). But Theorem 3.1 applies to \(\mathcal{C}_{\hat{V}}\), so the ext-blocks are given by the ordinary blocks of \(kH\). □

Of course, there is also a direct sum decomposition of \(\mathcal{C}_V\) given by the blocks of \(kG\), and so each ext-block is contained in an ordinary block. The way this happens is controlled by Brauer correspondence. Note that since

\[ C_G(E) = EC_G(E) \leq H \leq N_G(E), \]

for each block \(b\) of \(kH\) there is a unique block \(b^G\) of \(kG\), the Brauer correspondent of \(b\), and the blocks of \(kG\) that occur in this way are those with defect groups containing \(E\).

**Proposition 5.3.** In the situation of Proposition 5.2, every non-projective indecomposable module \(M\) in the ext-block of \(\mathcal{C}_V\) corresponding to a block \(b\) of \(kH\) is in the ordinary block \(b^G\) of \(kG\).

**Proof.** This follows from Nagao’s module-theoretic form of Brauer’s Second Main Theorem, which tells us that if \(M\) is in the block \(B\) of \(kG\), then \(M_H = M' \oplus M''\), where \(M'\) is a direct sum of modules in blocks of \(kH\) which have \(B\) as their Brauer correspondent and \(M''\) is a direct sum of modules projective relative to subgroups of \(H\) which do not contain \(E\). But then \(\hat{V}\) is not contained in the variety of \(M''\), so the image of \(M\) under Benson’s equivalence must be a summand of \(M'\). □

### 6. Linckelmann’s block varieties

In this section, we shall consider how the previous results are related to Linckelmann’s notion of block varieties [L1].

Let us briefly recall the definition. Let \(B\) be a block of \(kG\) with defect group \(D\). Choose a maximal \(B\)-Brauer pair \((D, e_D)\), and for each \(Q \leq D\) let \(e_Q\) be the unique block idempotent of \(kC_G(Q)\) such that \((Q, e_Q) \leq (D, e_D)\). So in particular \(e_{\{1\}}\) is the block idempotent corresponding to the block \(B\).

Let \(\mathcal{F}_{G,B}\) be the fusion system of the block: i.e., the category whose objects are subgroups of \(D\) and where a morphism from \(Q\) to \(R\) is a group homomorphism induced by conjugation by some \(x \in G\) such that \(x(Q, e_Q) \leq (R, e_R)\). Then Linckelmann defines the block cohomology \(H^*(G, B)\) to be \(\lim \mathcal{F}_{G,B}\), where the inverse limit is over the category \(\mathcal{F}_{G,B}\). More concretely, \(H^*(G, B)\) is the subring of \(H^*(D, k)\) consisting of elements that are stable in a suitable sense. Then the variety \(V_{G,B}\) is the maximal ideal spectrum of \(H^*(G, B)\).
The inclusion $H^*(G, B) \to H^*(D, k)$, composed with restriction $H^*(D, k) \to H^*(Q, k)$ induces a map of varieties
\[ r_Q^*: V_Q(k) \to V_{G, B} \]
for each subgroup $Q \leq D$, and in particular
\[ r_D^*: V_D(k) \to V_{G, B} \]
is a finite surjective map.

Also, the image of the restriction map $H^*(G, k) \to H^*(D, k)$ is contained in $H^*(G, B)$, so there is a natural map of varieties
\[ \rho_B: V_{G, B} \to V_G(k). \]

Linckelmann also defines a subvariety $V_{G, B}(M)$ of $V_{G, B}$, the block variety, for every finitely generated module $M$ in the block $B$ in such a way that $\rho_B$ induces a finite surjective map
\[ \rho_B: V_{G, B}(M) \to V_G(M). \]
So, as an invariant of the module $M$, $V_{G, B}(M)$ may be regarded as a refinement of $V_G(M)$.

We shall show that Linckelmann’s varieties give another way of constructing direct sum decompositions of the categories $C_V$. First, we need to generalize some familiar properties of varieties for modules to this setting.

We shall use the following useful theorem of Linckelmann [L3, Theorem 2.1].

**Theorem 6.1.** Let $B$ be a block of $G$ with defect group $D$, and let $i$ be a source idempotent of $B$. Then for any finitely generated module $M$ in the block $B$,
\[ V_{G, B}(M) = r_D^*(V_D(iM)), \]
where $iM$ is considered as a $kD$-module.

**Lemma 6.2.** Let
\[ 0 \to M_1 \to M_2 \to M_3 \to 0 \]
be a short exact sequence of modules in the block $B$. Then
\[ V_{G, B}(M_\alpha) \subseteq V_{G, B}(M_\beta) \cup V_{G, B}(M_\gamma) \]
for $\{\alpha, \beta, \gamma\} = \{1, 2, 3\}$.

**Proof.** Using Theorem 6.1 this follows easily from the well-known corresponding statement for cohomological varieties. Multiplying the short exact sequence by the source idempotent $i$, we get a short exact sequence
\[ 0 \to iM_1 \to iM_2 \to iM_3 \to 0, \]
and so
\[ V_{G, B}(M_\alpha) = r_D^*(V_D(iM_\alpha)) \subseteq r_D^*(V_D(iM_\beta) \cup V_D(iM_\gamma)) = r_D^*(V_D(iM_\beta)) \cup r_D^*(V_D(iM_\gamma)) = V_{G, B}(M_\beta) \cup V_{G, B}(M_\gamma). \]
\[ \square \]
The following follows immediately for the stable module category.

**Corollary 6.3.** Let $W$ be a closed homogeneous subvariety of $V_{G,B}$. Then the finitely-generated modules $M$ in the block $B$ for which $V_{G,B}(M) \subseteq W$ form a thick subcategory of $\text{stmod}(kG)$.

We shall denote this thick subcategory by $C_{W,B}$.

**Proposition 6.4.** Let $M$ and $N$ be finitely generated modules in a block $B$ of a finite group $G$. If $V_{G,B}(M) \cap V_{G,B}(N) = \{0\}$, then $\text{Hom}(M,N) = 0$.

**Proof.** Suppose $\phi : M \rightarrow N$ is a homomorphism between modules whose varieties intersect trivially. We can complete this map to a triangle

$$M \rightarrow N \rightarrow L \rightarrow \Omega(M)$$

in $\text{stmod}(kG)$. By Corollary 6.3,

$$V_{G,B}(L) = V_{G,B}(M) \cup V_{G,B}(N).$$

In [BL, Corollary 1.2], Benson and Linckelmann prove that the block variety of an indecomposable module is connected, and by [L3, Corollary 2.2], the block variety of a direct sum of two modules is the union of their individual block varieties. It follows that $L$ is the direct sum of two modules $L_M$ and $L_N$, with $V_{G,B}(L_M) = V_{G,B}(M)$ and $V_{G,B}(L_N) = V_{G,B}(N)$.

The octahedral axiom gives a commutative diagram

$$\begin{array}{ccc}
\Omega(L) & \rightarrow & M \\
\downarrow & & \downarrow \\
\Omega(L_N) & \rightarrow & X \\
\downarrow & & \downarrow \\
L_M & = & L_M
\end{array}$$

where, by Corollary 6.3,

$$V_{G,B}(X) \subseteq V_{G,B}(M) \cap V_{G,B}(N) = \{0\},$$

and so $X$ is projective. But the map $\phi$ factors through $X$. \hfill \Box

The next corollary follows immediately.

**Corollary 6.5.** If $W = \cup_{i \in I} W_i$ is the union of finitely many closed subvarieties $W_i$, where $W_i \cap W_j = \{0\}$ for $i \neq j$, then $C_{W,B}$ has a direct sum decomposition

$$C_{W,B} = \bigoplus_{i \in I} C_{W_i,B}.$$
Considering Linckelmann’s block varieties sheds new light on this. Let $B$ be either of the two blocks, and recall that there is a natural map of varieties $\rho_B : V_{G,B} \to V_G(k)$. In this example, one can calculate that $\rho_B^{-1}(V)$ is the union of three lines $W_1, W_2, W_3$ in $V_{G,B}$, and the intersection of $\mathcal{C}_V$ with the block $B$ is the direct sum $\mathcal{C}_{W_1,B} \oplus \mathcal{C}_{W_2,B} \oplus \mathcal{C}_{W_3,B}$ of thick subcategories determined by block varieties.

Similar observations apply in all other examples we have calculated, and it is natural to ask whether it is true in general, given a block $B$ of a finite group $G$ and a line $V$ in the image of $\rho_B$, that the ext-blocks of the intersection of $\mathcal{C}_V$ with the block $B$ are precisely the categories $\mathcal{C}_{W,B}$, for $W$ an irreducible component of $\rho_B^{-1}(V)$. If this were the case, then it would follow by a fairly straightforward argument that for any closed homogeneous subvariety $V$ of $V_G(k)$, the ext-blocks of the intersection of $\mathcal{C}_V$ with the block $B$ are just the categories $\mathcal{C}_{W,B}$ for $W$ a connected component of $\rho_B^{-1}(V)$.

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