Does Smoothing Matter?

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Abstract

We study how inhomogeneities modify large scale parameters in General Relativity. For a particular model, we obtain exact results: we compare an infinite string of extremal black holes to a corresponding smooth line with the same mass and charge in five dimensions. We find that the effective energy density does not differ significantly.
1 Introduction

Less than 5% of all matter and energy in the universe is visible—the origin of the rest is a mystery. Dark Energy (DE) constitutes perhaps 70% of this missing mass (see, e.g., [1]); this is the energy needed to accelerate the whole universe outwards at its measured rate. The remaining 25 ∼ 30% is Dark Matter (DM) (see, e.g., [2]), and is needed to account for the proper motions of celestial bodies.

This paper investigates how averaging out inhomogeneities in matter affects the measurement of mass and energy parameters. When we calculate interactions between celestial bodies, we generally use a smooth mass density profile in galaxies or clusters; that is, we average mass distributions to simplify our calculations. However, Einstein’s theory is nonlinear, and this averaging could potentially alter what energies we expect to see. The averaged stress tensor of the real metric need not be the same as the real stress tensor of the averaged metric; this discrepancy could be pertinent to the problem of DM.

Effects of inhomogeneities and DM seem to have some characteristics in common. Smoothing of mass fluctuations should become less important at greater distances from the fluctuations; thus the effects on energy density would be greater closer to the masses, which is where halos are identified. It also makes sense that the existence of halos is contingent upon some intrinsic properties of mass distributions in general, because the halos are found on such different scales (galaxies, galaxy clusters, even superclusters). This marks a difference from DE, which is seen as a constant energy density throughout the whole universe, a “cosmological constant” as Einstein originally called it.

This work ties into the “fitting problem” first described by George Ellis in 1983 [3]. Ellis discussed the problem of how to average out inhomogeneities with respect to cosmological models of the universe, and thus, from a modern perspective, his work has more relevance to DE than DM. The effect of inhomogeneities for cosmology has also been called “back-reaction.” There has been significant discussion of back-reaction and DE: see, e.g., [4]. Some of these works claim that back-reaction could account for a cosmological constant and obviate the need for DE; others refute this claim.

In the context of DM, F. Cooperstock and S. Tieu claimed to construct a model in which a galaxy’s need for a halo was canceled by including some of its possible rotational energy [5]; M. Korzynski and others have argued convincingly that this claim is erroneous [6].

All these calculations, however, are perturbative–based upon approximations; until now, there has not been an exact calculation of the impact of smoothing on energy [6]. When perturbative calculations do not lead to large effects, one can take them seriously; however, computations that claim large effects in perturbation theory are by their nature uncontrolled approximations and cannot be trusted without an exact treatment. This paper determines the exact energy for two scenarios, one of which is a ‘smoothed’ version of the other. Both scenarios use BPS black hole solutions because it is easy to find a solution for any sum of these special black holes, as the electromagnetic repulsions and gravitational attractions between them cancel [7]. Specifically, we compare an infinite

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1We thank S. Rasanen for making us aware of exact cosmological solutions in a specific model with inhomogeneous dust distributions considered by Kozaki and Nakao in [4].
string of discrete charged black holes to a smooth line of charge, with both distributions normalized to have the same average charge per unit length. We work in five dimensions because such distributions are divergent in four dimensions. We find that in our specific calculation, inhomogeneities cannot give rise to apparent DM.

2 The Metrics

The solution of the coupled Einstein-Maxwell field equations which describes a set of \( n = 1 \ldots N \) extremal black holes with charges \( q_n \) in five dimensions\(^2\) is \(^8\):

\[
d s^2 = -\Omega^{-2} dt^2 + \Omega d\vec{x}^2, \quad A_0 = \sqrt{3}\Omega^{-1}, \quad \Omega = 1 + \sum_{n=1}^{N} \frac{q_n}{|x_n - \vec{x}|^2}; \tag{2.1}
\]

as we are in five dimensions, \( d\vec{x}^2 = dx^2 + dy^2 + dz^2 + dv^2 \). We note that \( \Omega \) is time-independent and harmonic (it satisfies \( \nabla^2 \Omega = 0 \) outside the source), and that, in these coordinates, the horizon of each black hole is shrunk down to a point.

These are BPS solutions, with the property that gravitational attraction and electromagnetic repulsion between any set of them cancel \(^7\). This makes them time-independent, and allows for the simple solution in (2.1).

We consider two configurations: a discrete and a smooth solution. The discrete solution is for a set of black holes positioned along a line at equal distances \( L \) from one another. The smooth solution is for a smooth charged string along an axis. The continuous set can be viewed as a limiting case of the discrete set, namely the limit as the number of black holes per unit length tends to infinity and as the charge of each black hole tends towards zero, in such a way that the total charge per unit length remains the same.

2.1 The Discrete Solution

For the discrete case, we choose black holes with the same charge \( q_D \); thus \( \Omega_D \) is given by:

\[
\Omega_D = 1 + \sum_{n=-\infty}^{\infty} \frac{q_D}{r^2 + (v-n)^2}, \tag{2.2}
\]

where black holes are one unit length apart on the \( v \)-axis and \( r \) is the distance to the \( v \)-axis. We can restore the separation \( L \) by dimensional analysis (keeping the total charge per unit length constant):

\[
\Omega_D^L = 1 + \sum_{n=-\infty}^{\infty} \frac{Lq_D}{r^2 + (v-nL)^2} = 1 + \sum_{n=-\infty}^{\infty} \frac{q_D/L}{(r/L)^2 + (n-[v/L])^2}. \tag{2.3}
\]

Thus we can start with \( L = 1 \) and find the dependence on \( L \) simply by making the substitution \((r,v,q_D) \rightarrow (r/L,v/L,q_D/L)\).

\(^2\)In four dimensions, the solution \(^7\)\(^8\) has the form \( ds^2 = -\Omega^{-2} dt^2 + \Omega^2 d\vec{x}^2, A_0 = \Omega^{-1} \).
To evaluate the sum in (2.2), we use contour integration. We find
\[
\sum_{n=-\infty}^{\infty} \frac{1}{r^2 + (v-n)^2} = \frac{i\pi \cot(\pi(v+ir))}{2r} - \frac{i\pi \cot(\pi(v-ir))}{2r},
\] (2.4)
which simplifies to
\[
\sum_{n=-\infty}^{\infty} \frac{1}{r^2 + (v-n)^2} = \frac{\pi}{r} \frac{\sinh(2\pi r)}{\cosh(2\pi r) - \cos(2\pi v)}. \] (2.5)
Thus, using (2.2) and (2.5), the metric for the discrete solution is given by:
\[
\Omega_D = 1 + \frac{\pi q_D}{r} \frac{\sinh(2\pi r)}{\cosh(2\pi r) - \cos(2\pi v)}. \] (2.6)

2.2 The Smooth Solution

The metric for the smooth configuration is the same as the limit of the metric of the discrete configuration as the masses and the distances between them approach zero. From (2.3), we find:
\[
\Omega_S = 1 + \lim_{L \to 0} \sum_{n=-\infty}^{\infty} \frac{Lq_D}{r^2 + (v-nL)^2} = 1 + \int_{-\infty}^{\infty} dx \frac{q_D}{r^2 + (v-x)^2} = 1 + \frac{\pi q_D}{r}. \] (2.7)
As expected, the discrete harmonic function \(\Omega_D\) (2.6) approaches the smooth function \(\Omega_S\) (2.7) when the distance \(r\) to the \(v\)-axis is large.

3 Energy

Since we have normalized the two solutions so that they have the same average charge density along the \(v\) axis, and since the solutions are BPS, they must have the same total mass density as well; this does not immediately rule out a possible effect, as we discuss below.

We check this using Weinberg’s definition of total energy [9]: we rewrite the metric as a flat metric plus a correction term \(h\): \(g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}\). We split the Einstein tensor into terms linear and non-linear in \(h\): \(G^{(L)}_{\mu\nu}(h) + G^{(NL)}_{\mu\nu}(h) = T_{\mu\nu}\). Then the definition of the total (matter plus gravitational) energy density of a system is given by [9]:
\[
\text{Total Energy Density} = G^{(L)}_{00}(h) = T_{00} - G^{(NL)}_{00}(h). \] (3.1)
We integrate \(G^{(L)}_{00}(h)\) over all three directions \((x,y,z)\) orthogonal to the line of black holes. Integrating along the entire \(v\)-axis would make the integral divergent—this is not a problem: since the smooth configuration is independent of \(v\), and the discrete configuration is periodically dependent upon it, we need only integrate over one period of the discrete configuration (i.e., from 0 to 1 for \(L = 1\)). To perform the integral, we transform to 4-d cylindrical coordinates \((r, \theta, \varphi, v)\), and integrate over \(v, \theta, \varphi\), factoring in the Jacobian \(r^2 \sin(\varphi)\) for the coordinate change. Then the
total energy per unit length along the $v$-axis (describing an infinite slice of space orthogonal to the $v$-axis, with a width of one unit length) can be written as the following surface integral:

$$E = \int_{0}^{1} dv \int d\theta d\phi \, r^2 \sin(\phi) n^i (\partial_i h_{jj} - \partial_j h_{ij}) ,$$

(3.2)

where $\vec{n}$ is a unit vector normal to the $v$-axis. Since at large values of $r$ the integrands are the same, this integral yields the same result for our two cases— in agreement with the requirements of the BPS condition (which ensures that interaction energies cancel) and the equality of charge per unit length (which ensures that the energies which are not due to interactions are also equal).

The equality of the total masses does not in itself preclude the possibility of an effect on DM: the energy which enters Einstein’s equations, and so is relevant to DM, is the covariant matter energy $T_{00}$. This quantity is not necessarily the same for our two different distributions of black holes; comparing them shows us how matter energy is affected by smoothing.

### 3.1 Stress Tensor and Integration

The matter energy of a system is given by the integral of $T_{00}$ over all space. In a general spacetime with a time-like Killing vector $\xi^v$ the following total derivative vanishes:

$$0 = \int d^5x \, \sqrt{-g} \, D_\rho (g^{\rho\mu} T_{\mu\nu} \xi^v) = \int d^5x \, \partial_\rho (\sqrt{-g} g^{\rho\mu} T_{\mu\nu} \xi^v) ;$$

(3.3)

consequently, the contribution on a spacelike slice defines a conserved energy. In our case, since both metrics are static and diagonal, the total matter energy reduces to:

$$E = - \int d^4x \, \sqrt{-g} \, g^{00} T_{00} .$$

(3.4)

The stress-tensor is usually defined with a factor $\frac{1}{4\pi}$, but as we only study ratios of energies, we are not interested in the overall normalization.

We calculate $T_{00}$ using

$$T_{\mu\nu} = F_{\mu\alpha} F_{\nu\beta} g^{\alpha\beta} - \frac{1}{4} g^{\mu\nu} (F_{\alpha\beta} F_{\gamma\delta} g^{\alpha\gamma} g^{\beta\delta}) , \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu ,$$

(3.5)

where $A$ is the gauge potential (2.1). Once again, because the metrics are static and diagonal, and furthermore, because the gauge potential $A$ is purely timelike and static, this simplifies, and we find

$$T_{00} = F_{0j} F_{0j} g^{jj} - \frac{1}{4} g^{00} (2 F_{0j} F_{0j} g^{00} g^{jj}) .$$

(3.6)

Recalling $A_0 = \sqrt{3} \Omega^{-1}$, $g_{00} = -\Omega^{-2}$, $g_{jj} = \Omega$, and the determinant $g = -\Omega^2$ from equation (2.1), we find:

$$E = - \int d^4x \, \sqrt{-g} \, g^{00} T_{00} = 6 \int d^4x \left( \frac{\partial_j \Omega}{\Omega} \right)^2 .$$

(3.7)

Here $(\partial_j \Omega)^2 = (\partial_x \Omega)^2 + (\partial_y \Omega)^2 + (\partial_z \Omega)^2 + (\partial_v \Omega)^2$. 

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3.2 Exact Energies

As discussed above equation (3.2), we integrate stress-energy over all three directions \((x, y, z)\) orthogonal to the line of black holes and integrate over one period of the discrete configuration over the \(v\)-axis. The smooth configuration is the simpler of the two; we calculate it first. As discussed in section 2.1, we use \(L = 1\), and reintroduce \(L\) later by rescaling. We ignore integration over \(\theta\) and \(\varphi\) in both configurations because it only produces equal overall constants; we also drop the \(\sin(\varphi)\) factor from the Jacobian. Defining \(k = \pi q\) and \(\Omega_S = 1 + \frac{k}{r}\), we find \((\partial_j \Omega_S)^2 = k^2/r^4\), and then (3.7) gives

\[
E_S = \int_0^\infty dr \frac{k^2}{(r+k)^2} = k. \tag{3.8}
\]

For the discrete configuration, the calculation is more complicated. We define \(s = \sinh(2\pi r)\), \(c = \cosh(2\pi r)\), \(t = \sin(2\pi v)\), and \(d = \cos(2\pi v)\) and compute:

\[
(\partial_j \Omega_D)^2 = \frac{k^2 \left[s(d-c) + 2\pi r(1-cd)\right]^2 + (2\pi rst)^2}{(c-d)^4}. \tag{3.9}
\]

Then, with \(\Omega_D = 1 + \frac{k}{r} s \frac{s}{c-d}\), and the measure factor \(r^2\), the energy for the discrete configuration is given by:

\[
E_D = \int_0^1 dv \int_0^\infty dr \frac{k^2 \left[s(d-c) + 2\pi r(1-cd)\right]^2 + (2\pi rst)^2}{(c-d)^2(r(c-d) + ks)^2}. \tag{3.10}
\]

This is not easily evaluated as an indefinite integral, but we can compare \(E_S\) and \(E_D\) numerically. To understand the results, let us clarify what we should test. For \(L = 1\) and various \(k\) (charge), we want to find the differences in energy between the two configurations. However, a smaller \(k\) will certainly yield a smaller energy and smaller energy discrepancy, so to compare various \(k\)'s we divide by \(E_S\) because \(k \propto E_S\).

Thus we study the ratio \(\left(\int_0^\infty \int_0^1 (E_D - E_S) dv dr\right) / \left(\int_0^\infty \int_0^1 E_S dv dr\right)\) as a function of \(k\). Different \(k\) represent different densities, as we have scaled the separation \(L\) to \(L = 1\). The results, computed with Maple, are consistent with zero; the actual numerical integrations are somewhat delicate because of divergences in the integrand (\(E_D\) diverges when \(r = 0\) and \(v = 0, 1\)).

4 Physical Application

Though we did not find an effect, we discuss how any effect that we might have found could have been interpreted physically. The first step would have been to choose a scale, because DM is found on several different levels: galaxies, galaxy clusters, superclusters. Then we would determine which objects are smoothed for the situation chosen. For example, to see how smoothing might create a halo around a galaxy, we would smooth the stars inside the galaxies, as is the common practice when calculating intergalactic motions.

Our numerical results were calculated for \(L = 1\). We can use these results, however, if we rescale the situation by changing the mass so that the density of the galaxy is correct. The value of
$k$ in equations (3.8) to (3.10) would then be modified to provide the correct mass matching up with the real-life situation. The $k$ we used was $\pi q$, and $q = Gm$ for extremal black holes. The ratio to demand for similarity is $k/1 = \pi Gm/L$.

So, we need the mass ($m$) and average distance between masses ($L$) as input, and then we can see what the energy difference is for $k = \pi Gm/L$. Let us apply these ideas to the situation of galaxy halos just discussed, testing how the smoothing of stars within the galaxies could affect the energy. We use planck-units, so that $G = 1$. The average distance between adjacent stars is about 10 light-years, $10^{52}$ Planck lengths. The average mass of a star is a bit less than $10^{39}$ Planck masses. Our $k$ is then around $10^{-13}$.

Next let us apply the smoothing to galaxies within galaxy clusters. The average galaxy mass is $10^{45}$ Planck-masses. The average separation between galaxies is $10^{7}$ light-years, $10^{58}$ Planck-lengths. So $k$ for this situation is again $10^{-13}$.

A third scale at which to look is that of superclusters. Superclusters consist of about a dozen clusters, each with a mass of about $10^{55}$ Planck-masses. Usually they are arranged in strings with lengths of 10-100 Mpc, $10^{58}$ or $10^{59}$ Planck-lengths, so the distance between each is about $10^{57}$ or $10^{58}$. Our $k$ is then $10^{-2}$ or $10^{-3}$.

5 Conclusion

The calculation performed in this paper is by no means flawless. Five-dimensional effects are expected to show similar trends as their four-dimensional counterparts, but when one is discussing the extent of an effect, 5-d is less applicable. Also, these calculations were done with BPS solutions. They are easy to work with but certainly do not represent standard matter. They are maximally charged and describe point-like masses, which is of course physically unrealistic.

These are the first exact calculations of the effects of inhomogeneities on the apparent mass of astrophysical objects; as such, they complement and confirm perturbative arguments. There is room for development–our calculation was based on a particular model, and calculations in other models could help elucidate the exact effects of smoothing.

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