MORI DREAM K3 SURFACES OF PICARD NUMBER FOUR: PROJECTIVE MODELS AND COX RINGS

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Abstract. In this paper we study the geometry of the 14 families of K3 surfaces of Picard number four with finite automorphism group, whose Néron-Severi lattices have been classified by È. B. Vinberg. We provide projective models, we identify the degrees of a generating set of the Cox ring and in some cases we prove the unirationality of the associated moduli space.

Contents

Introduction 2
1. Preliminaries 3
1.1. Linear systems of K3 surfaces 3
1.2. Vinberg’s classification 4
1.3. Effective and nef cones 4
1.4. Cox rings 5
2. K3 surfaces of Picard number four 7
   The family $F_1$ 8
   The family $F_2$ 9
   The family $F_3$ 10
   The family $F_4$ 11
   The family $F_5$ 12
   The family $F_6$ 14
   The family $F_7$ 16
   The family $F_8$ 17
   The family $F_9$ 19
   The family $F_{10}$ 20
   The family $F_{11}$ 21
   The family $F_{12}$ 23
   The family $F_{13}$ 24
   The family $F_{14}$ 25
3. Magma code 26
4. Tables 26
References 40

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Introduction

It follows from the global Torelli theorem that the automorphism group of a complex K3 surface $X$ is finite if and only if the quotient group $O(S)/W^{(2)}(S)$ is finite, where $S$ is the Néron-Severi lattice of $X$, $O(S)$ is its isometry group and $W^{(2)}(S)$ is the subgroup generated by reflections relative to classes with self-intersection $-2$. The hyperbolic lattices with the latter property are called 2-reflective and there are a finite number of them for each rank $\geq 3$. Their classification is due to V.V. Nikulin [Nik79, Nik84, Nik00] for rank 3 and $\geq 5$, and to É. B. Vinberg for rank 4 [Vin07]. Complex K3 surfaces with finite automorphism group and Picard number at least three can be also characterized as those K3 surfaces having finitely many $(-2)$-curves, and in such case the classes of such curves generate the effective cone of the surface.

In this paper we study the geometry of K3 surfaces with finite automorphism group and having Picard number 4: we compute the effective cone and the nef cone (Theorem 2.1), we find projective models and we study their Cox ring. In particular, we prove the following result, for more details see Propositions 2.2, 2.4, 2.5, 2.8, 2.11, 2.13, 2.16, 2.18, 2.20, 2.22, 2.25, 2.27, 2.29, 2.31.

In what follows we will denote by $F_i$, $i = 1, \ldots, 14$, the family of Mori dream K3 surfaces whose Néron-Severi lattice is isometric to the lattice $V_i$ defined in Theorem 1.5 and by $M_i$ the associated moduli space.

Theorem 0.1. Let $X$ be a complex K3 surface with Picard number four and with finite automorphism group. Then up to isomorphism equivalence $X$ belongs to one of the following families:

$F_1$: complete intersections of three quadrics in $\mathbb{P}^5$ having three nodes;
$F_2$: minimal resolutions of quartic surfaces in $\mathbb{P}^3$ with a node and two hyperplane sections through the node which are union of two conics;
$F_3$: minimal resolutions of double covers of $\mathbb{P}^2$ branched along a plane sextic with a double point with two bitangent lines passing through the point;
$F_4$: double covers of the Hirzebruch surface $\mathbb{F}_4$ blown-up at two general points;
$F_5$: minimal resolutions of double covers of $\mathbb{P}^2$ branched along a plane sextic with three nodes;
$F_6$: double covers of $\mathbb{P}^2$ branched along a smooth plane sextic with two 3-tangent lines and two 6-tangent conics;
$F_7$: smooth quartic surfaces in $\mathbb{P}^3$ having four hyperplane sections which are the union of two conics;
$F_8$: minimal resolutions of double covers of $\mathbb{F}_3$ branched along the union of the smooth rational curve $E$ with $E^2 = -3$ and a reduced curve $B$ intersecting $E$ at one point $p$ and the fiber of $\mathbb{F}_3 \to \mathbb{P}^1$ through $p$ at one point with multiplicity two;
$F_9$: minimal resolutions of double covers of $\mathbb{P}^2$ branched along a plane sextic having two nodes and such that the line through the nodes is tangent to the sextic curve at one more point;
$F_{10}$: smooth quartic surfaces in $\mathbb{P}^3$ having one hyperplane section which is the union of four lines;
$F_{11}$: smooth quartic surfaces in $\mathbb{P}^3$ having three reducible hyperplane sections which are the union of two conics;
$F_{12}$: double covers of $\mathbb{P}^2$ branched along a smooth plane sextic with three 3-tangent lines such that the cover is trivial over the union of the three lines;
$F_{13}$: double covers of $\mathbb{P}^2$ branched along a smooth plane sextic with three 3-tangent lines such that the cover is not trivial over the union of the three lines;
$F_{14}$: double covers of $\mathbb{P}^2$ branched along a smooth plane sextic with one 3-tangent line and three 6-tangent conics.
Corollary 1.2. Let $D$ be an effective non-zero nef divisor of $X$. Assume that $X$ is a Mori dream K3 surface of Picard number four such that $\text{NS}(X)$ is not isometric to $V_{14}$. The degrees of a set of generators of the Cox ring $R(X)$ are given in Table 6 and all degrees in the Table are necessary to generate $R(X)$, except eventually for those marked with a star. In case $\text{NS}(X) \cong V_{14}$ the degrees of a set of generators of $R(X)$ are either classes of $(-2)$-curves or sums of at most three elements in the Hilbert basis of the nef cone. Moreover, the degrees given in Table 7 are necessary to generate $R(X)$.

Moreover, for a very general K3 surface in the families $\mathcal{F}_i$ with $i \in \{4, 5, 6, 7, 10, 12\}$ we provide a presentation for the Cox ring.

As a consequence of the first Theorem we also obtain the following result, see Corollary 2.3, 2.6, 2.10, 2.12, 2.14, 2.17, 2.19, 2.21, 2.23, 2.26 and 2.28.

Theorem 0.3. The moduli spaces $\mathcal{M}_i$ with $i \in \{1, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$ are unirational. Moreover, $\mathcal{M}_3$ is rational.

1. Preliminaries

We will work over the field $\mathbb{C}$ of complex numbers.

1.1. Linear systems of K3 surfaces. To be self-contained, let us recall the following results of Saint-Donat [SD74]. For a divisor $D$ such that the linear system $|D|$ is base point free, we denote by $\varphi_D$ the morphism associated to $|D|$.

Theorem 1.1. Let $D$ be a divisor on a K3 surface $X$.

a) if $D$ is effective, non-zero, and $D^2 = 0$, then $D \sim aF$ where $|F|$ is a free elliptic pencil and $a \in \mathbb{N}$.

b) If $D$ is big and nef, then $h^0(D) = 2 + \frac{1}{2}D^2$, and either $|D|$ has no fixed part, or $D \sim aF + E$ where $F$ is a smooth elliptic curve and $E$ is a ($-2$)-curve such that $F \cdot E = 1$.

c) If $D$ is big and nef and $|D|$ has no fixed part, then $|D|$ is base point free and $\varphi_D$ is either 2 to 1 onto its image (hyperelliptic case) or it maps $X$ birationally onto its image, contracting the ($-2$)-curves $\Gamma$ such that $D \cdot \Gamma = 0$ to singularities of type $ADE$.

Corollary 1.2. Let $X$ be a K3 surface such that none of its elliptic fibrations has sections and $D$ be an effective non-zero nef divisor of $X$, then $|D|$ is base point free.

Proof. Assume that $D$ is a nef divisor with non-empty base locus. By Theorem 1.1 $D$ is linearly equivalent to a sum involving two smooth curves $F, E$ such that $F$ is a smooth elliptic curve and $E$ a ($-2$)-curve with $F \cdot E = 1$. The morphism associated to $|F|$ defines an elliptic fibration $\varphi_F : X \to \mathbb{P}^1$ (see [SD74]) and $E$ is a section of it.

Theorem 1.3. (See [SD74, Proposition 5.7] and its proof). Let $|D|$ be a complete linear system without fixed components on a K3 surface $X$. Then $|D|$ is hyperelliptic if and only if one of the following cases occurs:

a) $D^2 \geq 4$ and there is a smooth elliptic curve $F$ such that $F \cdot D = 2$;

b) $D^2 \geq 4$ and there is an irreducible curve $D'$ with $D'^2 = 2$ and $D \sim 2D'$ (thus in that case
Remark 1.4. For brevity, we will say that a divisor $D$ is base point free or is hyperelliptic if the linear system $|D|$ is.

1.2. Vinberg’s classification. We now state the classification of K3 surfaces of Picard number four with finite automorphism group due to E. B. Vinberg [Vin07, Theorem 1]. As usual we denote by $A_m, D_n, E_6, E_7, E_8$ the negative definite lattices that correspond to the root systems of the same types ($m \geq 1, n \geq 4$), and by $U$ the lattice with Gram matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Moreover, $L_1 \oplus L_2$ denotes the direct sum of the lattices $L_1$ and $L_2$, $nL$ ($n \in \mathbb{N}$) denotes the direct sum of $n$ copies of the lattice $L$ and $L(k)$ ($k \in \mathbb{Z}$) is the lattice with the same underlying group as $L$, but with Gram matrix multiplied by $k$.

Theorem 1.5. Let $X$ be an algebraic complex K3 surface with Picard number $\rho(X) = 4$ and finite automorphism group. Then $\NS(X)$ is isometric to one of the following 14 lattices:

1. $V_1 = (8) \oplus 3A_1$, $V_2 = (-4) \oplus (4) \oplus A_2$, $V_3 = (4) \oplus A_3$, $V_{3+k} = U(k) \oplus 2A_1$, $k \in \{1, 2, 3, 4\}$,
2. $V_{7+k} = U(k) \oplus A_2$, $k \in \{1, 2, 3\}$, $V_{11} = U(6) \oplus A_2$, $V_{12} = \begin{pmatrix} 0 & -3 \\ -3 & 2 \end{pmatrix} \oplus A_2$,
3. $V_{13} = \begin{pmatrix} 2 & -1 & -1 & -1 \\ -1 & -2 & 0 & 0 \\ -1 & 0 & -2 & 0 \\ -1 & 0 & 0 & -2 \end{pmatrix}$,
4. $V_{14} = \begin{pmatrix} 12 & -2 & 0 & 0 \\ -2 & -2 & 1 & 0 \\ 0 & -1 & -2 & -1 \\ 0 & 0 & -1 & -2 \end{pmatrix}$.

1.3. Effective and nef cones. Let $X$ be a K3 surface and $\Eff(X) \subseteq \Cl(X) \otimes \mathbb{Q} \cong \NS(X) \otimes \mathbb{Q}$ be its effective cone. If $\rho(X) \geq 3$ and $\Eff(X)$ is polyhedral, then its extremal rays are generated by classes of $(-2)$-curves (see [AHL10, Proposition 2.13]). In order to identify the classes of all $(-2)$-curves we use the following algorithm, known as “Vinberg’s algorithm” [Vin75]:

- Fix a class $\alpha \in \Cl(X)$ with $\alpha^2 > 0$.
- Find all classes $w \in \Cl(X) \cap \alpha^+ \supseteq \Dim$ with self-intersection $-2$ (there are finitely many of them since the restriction of $Q$ to $\alpha^+$ is negative definite) and let $L$ be a list of such classes, which is a root system.
- If $L$ is not empty, fix a set $L^+ \subseteq L$ of positive roots as follows: choose randomly an integral combination $H$ of the vectors in $L$ having non zero intersection with all of them and let $L^+$ be the set of vectors in $L$ having positive intersection with $H$.
- Construct the list $R_0$ of simple roots in $L^+$ inductively as follows: let $L^+_0 \subseteq L^+$ be the set of vectors having minimal intersection $m_0$ with $H$ and, once $L^+_i$ is given for some $i \geq 0$, define $L^+_{i+1} \subseteq L^+$ as the set of vectors having intersection $m_0 + i + 1$ with $H$ and non-negative intersection with all vectors in $L^+_k$ for $0 \leq k \leq i$. The process stops when $m_0 + n = \max \{v \cdot H: v \in L^+\}$ and the set of simple roots in $L^+$ is $R_0 := \cup_{0 \leq i \leq n} L^+_i$.
- Construct a set of fundamental roots for $\Cl(X)$ inductively as follows: let $R_0$ be as in the previous item ($= \emptyset$ if $L$ is empty) and define $R_{i+1}$ as the union of $R_i$ with the set of classes $w \in \Cl(X)$ such that $w^2 = -2$, $w \cdot \alpha = i + 1$ and having non-negative intersection with all the elements of $R_i$. 

$D^2 = 8$).
c) $D^2 = 2$, and in this case $\varphi_D$ is a double cover of $\mathbb{P}^2$. 


The set $R_n$ is a set of fundamental roots of $\text{Cl}(X)$ if the following property holds for the convex polyhedral cone $C$ generated by the vectors in $R_n$: the intersection matrix of the vectors generating any facet of $C$ is negative semidefinite.

We recall that the nef cone $\text{Nef}(X) \subseteq \text{Cl}(X)_\mathbb{Q}$ of a smooth surface is the dual of its effective cone with respect to the intersection form. In what follows we will denote by $\text{BEff}(X)$ and $\text{BNef}(X)$ the Hilbert bases of $\text{Eff}(X)$ and $\text{Nef}(X)$, i.e. the unique minimal generating sets of $\text{Eff}(X) \cap \text{Cl}(X)$ and $\text{Nef}(X) \cap \text{Cl}(X)$ (see for example [MS00, Definition 7.17]).

The previous algorithm has been implemented in a Magma [BCP97] program which is described in [ACDL21] and included as ancillary file in the arXiv version of the same paper: https://arxiv.org/abs/1909.01267.

1.4. Cox rings. We now recall some preliminary facts on Cox rings of surfaces, based on [ADHL15]. The Cox ring of a smooth projective complex variety $X$ with finitely generated and free divisor class group $\text{Cl}(X)$ is defined as

$$R(X) := \bigoplus_{D \in K} \Gamma(X, \mathcal{O}_X(D)),$$

where $K \subseteq \text{WDiv}(X)$ is a subgroup of the group of Weil divisors such that $K \rightarrow \text{Cl}(X), \ D \mapsto [D]$ is an isomorphism. Observe that $R(X)$ is a $K$-graded algebra over $\mathbb{C}$. The variety $X$ is called Mori dream space if $R(X)$ is a finitely generated complex algebra.

To simplify notation, given $w \in \text{Cl}(X)$, we will denote by $R(X)_w$ the vector space $\Gamma(X, \mathcal{O}_X(D))$, where $D \in K$ has $[D] = w$. Moreover, in this case we will say that $f \in \Gamma(X, \mathcal{O}_X(D))$ is homogeneous of degree $w$. Given a set of homogeneous generators $\{f_i : i \in I\}$ of $R(X)$, it can be easily proved that their degrees $\{e_i : i \in I\} \subset K$ generate the effective cone of $X$ (see [AHL10, Proposition 2.1]). In particular, the effective cone of a Mori dream space is polyhedral. Moreover, if $D$ is an effective divisor whose class belongs to the Hilbert basis of the effective cone, then any generating set of $R(X)$ contains an element of degree $[D]$.

The following theorem [ADHL15, Theorem 5.1.5.1], together with the classification of $K3$ surfaces with finite automorphism group (see, [Nik79,Nik84,Nik00,PSŠ71,Vin07]) provides a complete classification of $K3$ surfaces with finitely generated Cox ring.

**Theorem 1.6.** Let $X$ be an algebraic $K3$ surface. Then the following statements are equivalent.

(i) $X$ is a Mori dream surface.

(ii) The effective cone $\text{Eff}(X) \subseteq \text{Cl}_\mathbb{Q}(X)$ is polyhedral.

(iii) The automorphism group of $X$ is finite.

In [ACDL21] we proved the following result on the degrees of a generating set of the Cox ring of a $K3$ surface. We say that $R(X)$ has a generator in degree $w$ when each minimal set of homogeneous generators of $R(X)$ contains an element of degree $w$.

**Theorem 1.7.** ([ACDL21, Theorem 2.7]) Let $X$ be a smooth projective $K3$ surface over $\mathbb{C}$. Then the degrees of a minimal set of generators of its Cox ring $R(X)$ are either

(i) classes of $(-2)$-curves,

(ii) sums of at most three elements of the Hilbert basis of the nef cone (allowing repetitions),

(iii) classes of divisors of the form $2(F + F')$, where $F,F'$ are smooth elliptic curves with $F \cdot F' = 2$. 

Moreover, we developed different techniques to show that $R(X)$ has no generators in a certain degree (see [ACDL21, Corollary 2.2, Corollary 2.4]). Given $f \in \mathbb{C}(X)^*$, we denote by $\text{div}_E(f)$ the divisor $\text{div}(f) + E$.

**Theorem 1.8.** Let $X$ be a smooth projective variety over $\mathbb{C}$, $E_1, E_2, E_3$ be effective divisors of $X$, $f_i \in H^0(X, E_i)$, $i = 1, 2, 3$ and $D \in \text{WDiv}(X)$.

1. If $\cap_{i=1}^2 \text{div}_{E_i}(f_i) = \emptyset$ and $h^1(X, D - E_1 - E_2) = 0$, then the following morphism is surjective

$$H^0(X, D - E_1) \oplus H^0(X, D - E_2) \to H^0(X, D), \quad (g_1, g_2) \mapsto f_1 g_1 + f_2 g_2.$$

2. If $\cap_{i=1}^3 \text{div}_{E_i}(f_i) = \emptyset$, $h^1(X, D - E_i - E_j) = 0$ for all distinct $i, j \in \{1, 2, 3\}$ and $h^2(X, D - E_1 - E_2 - E_3) = 0$, then the following morphism is surjective

$$\bigoplus_{i=1}^3 H^0(X, D - E_i) \to H^0(X, D), \quad (g_1, g_2, g_3) \mapsto f_1 g_1 + f_2 g_2 + f_3 g_3.$$

The following result shows that $R(X)$ is not generated in degrees which are sums of a very ample class and the class of an elliptic fibration under certain conditions.

**Proposition 1.9.** ([ACDL21, Lemma 2.8]) Let $X$ be a K3 surface and $D = F + D'$ a nef divisor, where $F$ is nef with $F^2 = 0$ and $D'$ is very ample. Assume that $F \sim E_1 + E_2$, where $E_1, E_2$ are $(-2)$-curves and that the image of the natural map

$$\phi : H^0(D - E_1) \oplus H^0(D - E_2) \to H^0(D)$$

has codimension two. Then $R(X)$ has no generator in degree $[D]$.

The following is a result by Ottem [Ott13, Proposition 2.2].

**Proposition 1.10.** Let $X$ be a smooth projective K3 surface. Let $A$ and $B$ be nef divisors on $X$ such that $[B]$ is base point free. Then the multiplication map

$$H^0(X, A) \otimes H^0(X, B) \to H^0(X, A + B)$$

is surjective if $h^1(X, A - B) = h^1(X, A) = 0$ and $h^2(X, A - 2B) = 0$.

On the other hand, the following results show that under suitable conditions $R(X)$ must have a generator in a certain degree.

**Lemma 1.11.** Let $X$ be a K3 surface and let $D = E_1 + E_2 + E_3$ be a base point free divisor, where $E_1, E_2, E_3$ are $(-2)$-curves such that $h^1(E_i + E_j) = 0$ for all distinct $i, j$. Then the natural map

$$\psi : \bigoplus_{i=1,2,3} H^0(D - E_i) \to H^0(D)$$

is not surjective. Moreover, if $E_1 \cap E_2 \cap E_3 = \emptyset$, then the image of $\psi$ has codimension one.

**Proposition 1.12.** Let $G = \{w_0, \ldots, w_r\} \subset \text{Cl}(X)$ containing the degrees of a homogeneous generating set of $R(X)$ and suppose that the linear system associated to $w_0 \in G$ is base point free. Then $R(X)$ has a generator in degree $w_0 \in G$ if one of the following holds:

1. any linear combination $w_0 = \sum_{i=1}^r a_i w_i$ with $a_i \in \mathbb{Z}, a_i \geq 0$ contains the class of a $(-2)$-curve in its support;
2. any linear combination $w_0 = \sum_{i=1}^r a_i w_i$ with $a_i \in \mathbb{Z}, a_i \geq 0$ contains in its support one of the classes of two $(-2)$-curves $E_1, E_2$ with $E_1 \cdot E_2 > 0$.
iii. \( w_0 = w_1 + w_2 + w_3 \), where \( w_i \) are the classes of three \((-2)\)-curves with \( h^1(w_i + w_j) = 0 \) for all distinct \( i, j \), and any linear combination \( w_0 = \sum_{i=1}^r a_i w_i \) with \( a_i \in \mathbb{Z}, a_i \geq 0 \) contains one among \( w_1, w_2, w_3 \) in its support.

The previous results have been implemented in several Magma \cite{BCP97} programs included as ancillary files in the arXiv version of the paper https://arxiv.org/abs/1909.01267.

2. K3 surfaces of Picard number four

In this section we study the geometry of each family of Mori dream K3 surfaces of Picard number four. As a first step, we compute their effective cone and their nef cone.

**Theorem 2.1.** For each of the 14 families of Mori dream K3 surfaces \( X \) of Picard number four, Tables 1, 2 and 3 describe

(i) the set \( E(X) \) of extremal rays of the effective cone and its Hilbert basis \( BEff(X) \),

(ii) the set of extremal rays \( N(X) \) of the nef cone and its Hilbert basis \( BNef(X) \).

Moreover, Table 4 gives the intersection matrix of the \((-2)\)-curves for each family.

**Proof.** A set of fundamental roots for the Picard lattice is obtained by means of the algorithm described in section 1.3, implemented in the Magma library Find-2 in \cite{ACDL21}. The nef cone is thus obtained as the dual of the effective cone with respect to the intersection form of \( \text{Cl}(X) \). □

We now prove the result about the Cox ring of such surfaces stated in the Introduction.

**Proof of Theorem 0.2.** By the remarks in section 1.4 the Cox ring has a generator in each degree \( w \in BEff(X) \). Observe that if \( D \) is not nef, then there exists a \((-2)\)-curve \( C \) such that \( D \cdot C < 0 \), so that \( C \) is contained in the base locus of \( |D| \) and the multiplication map \( H^0(X, D - C) \to H^0(X, D) \) by a non-zero element of \( H^0(X, C) \) is surjective. Thus we can assume \( [D] \) to be nef.

By Theorem 1.7 it is enough to consider those nef degrees which are sums of at most three elements of the Hilbert basis of the nef cone or of the form \( [2(F + F')] \), where \( F, F' \) are smooth elliptic curves with \( F \cdot F' = 2 \). The last type of degrees only exist for the families of K3 surfaces with Néron-Severi lattice isometric to either \( V_5 = U(2) \oplus 2A_1 \) or \( V_9 = U(2) \oplus A_2 \). In case \( \text{NS}(X) \cong V_5 \) the Cox ring has been computed in \cite[Proposition 6.7, i)]{AHL10}.

This result allows to form a finite list of possible nef degrees, which is then analyzed using the techniques in section 1.4, which have been implemented in several Magma programs \cite[§3.3]{ACDL21}. In particular Theorem 1.8, Proposition 1.9 and Proposition 1.10 allow to discard certain degrees, while Lemma 1.11 and Proposition 1.12 are used to prove that certain degrees are necessary.

We recall that, by Theorem 1.1, the linear system of any effective nef divisor is base point free unless there exists a smooth elliptic curve \( F \) and a \((-2)\)-curve \( E \) such that \( E \cdot F = 1 \). This happens only for the families of K3 surfaces with Néron-Severi lattice isometric to either \( V_4 = U \oplus 2A_1 \) or \( V_8 = U \oplus A_2 \). In case \( \text{NS}(X) \cong V_4 \) the Cox ring has been computed in \cite[Proposition 6.7, ii)]{AHL10}. In case \( \text{NS}(X) \cong V_8 \) we apply the same techniques, but we exclude from the computations the only element in the Hilbert basis of the nef cone which is not base point free, which is the class \( BNef[4] \).

In case \( \text{NS}(X) \cong V_{14} \) we are unable to compute a minimal generating set of \( R(X) \) for computational reasons (the Hilbert basis of the nef cone contains 111 elements). To obtain the set of degrees in Table 6 we only apply item i) of Theorem 1.8 and Proposition 1.12. □
We now analyze each family of Mori dream K3 surfaces of Picard number four using the results in Theorem 2.1 and Theorem 0.2. We will denote by $F_i$ the family of K3 surfaces whose Néron-Severi lattice is isometric to $V_i$ for $i = 1, \ldots, 14$.

The family $F_1$. Let $X$ be a K3 surface with $\text{NS}(X) \cong V_1 = (8) \oplus 3A_1$. By Theorem 2.1 $X$ contains 12 $(-2)$-curves whose intersection matrix is given in Table 4. Moreover, the Hilbert basis of the nef cone of $X$ contains 51 classes, and exactly six of them are classes of elliptic fibrations: $\text{BNef}[1]$, $\text{BNef}[6]$, $\text{BNef}[11]$, $\text{BNef}[27]$, $\text{BNef}[29]$ and $\text{BNef}[35]$. Each elliptic fibration has two fibers of type $\tilde{A}_1$ and has no sections.

**Proposition 2.2.** Let $X$ be a K3 surface with $\text{NS}(X) \cong V_1 = (8) \oplus 3A_1$. Then

(i) $X$ is birational to a complete intersection $\overline{X}$ of three quadrics in $\mathbb{P}^5$ of the form

$$Q_1(x_1, x_2, x_3, x_4, x_5) = Q_2(x_0, x_2, x_3, x_4, x_5) = Q_3(x_0, x_1, x_3, x_4, x_5) = 0,$$

where $Q_1$, $Q_2$, $Q_3$ are homogeneous of degree two and do not contain the monomials $x_0^2, x_1^2, x_3^2$;

(ii) the projection of $\overline{X}$ from the plane $x_0 = x_1 = x_2 = 0$ induces a double cover $\pi : X \to \mathbb{P}^2$ branched along a smooth plane sextic with six 6-tangent conics $C_1, \ldots, C_6$;

(iii) the surface $X$ has twelve $(-2)$-curves: the curves $R_{ij}$, $i = 1, \ldots, 6$, $j = 1, 2$, such that $\pi(R_{i1}) = \pi(R_{i2}) = C_i$;

(iv) a minimal generating set of the Cox ring of $X$ is $s_1, \ldots, s_{12}$, where $s_1, \ldots, s_{12}$ are defining sections of the $(-2)$-curves and $s_{13}, s_{14}, s_{15}$ is a basis of $H^0(\pi^*O_{\mathbb{P}^2}(1))$.

**Proof.** The class $h = \text{BNef}[8]$ has square 8, it is base point free by Corollary 1.2 and is non-hyperelliptic (the intersection numbers with the fibers of the 6 elliptic fibrations are $\geq 8$). Thus it defines a morphism $\varphi_h : X \to \mathbb{P}^5$ whose image is a degree eight surface $\overline{X}$ in $\mathbb{P}^5$ with three nodes $p_1, p_2, p_3$ at the images of the three disjoint $(-2)$-curves of classes $f_3, f_5, f_{12}$. By [SD74, Theorem 7.2] the surface $\overline{X}$ is a complete intersection of three quadrics, since the intersection numbers of $h$ with the fibers of the elliptic fibrations of $X$ are $> 3$.

If the points $p_1, p_2, p_3$ were not in general position, then $\overline{X}$ would contain the line $L$ through them. The strict transform of $L$ and the exceptional divisors over the nodes would thus generate a rank four lattice isometric to $D_4$. However, looking at the intersection matrix of the $(-2)$-curves of $X$ (see Table 4) one can see that there are no four of them with intersection matrix isometric to $D_4$, giving a contradiction. We can therefore suppose that the points $p_1, p_2, p_3$ are in general position, and up to projectivities we can assume that:

$$p_1 = (1, 0, 0, 0, 0, 0), \ p_2 = (0, 0, 1, 0, 0, 0), \ p_3 = (0, 0, 1, 0, 0, 0).$$

Let $\mathcal{N}$ be the net of quadrics containing $\overline{X}$. It can be proved that there is a unique quadric $Q_i$ in $\mathcal{N}$ having a node at $p_i$ for $i = 1, 2, 3$. Thus that there are three possible cases, up to renumbering the three quadrics:

a) $Q_1, Q_2, Q_3$ are three distinct quadrics which generate $\mathcal{N}$;

b) $Q_1 = Q_2$, $Q_1$ is singular along the line $L$ between $p_1, p_2$ and $Q_3$ is singular along $p_3$ (the line $L$ is not contained in $\overline{X}$ since otherwise $\overline{X}$ would be singular along $L$);

c) $Q_1 = Q_2 = Q_3$ and $Q_1$ is singular along the plane containing the points $p_1, p_2, p_3$.

Case c) is not possible, since $Q_1$ would be singular along the plane $\Pi$ containing $p_1, p_2, p_3$ and $\Pi \cap \overline{X}$ would consist of $p_1, p_2, p_3$ and a fourth singular point. Moreover, case b) can be proved to
be a degeneration of case a) (in fact, complete intersections of three quadrics in $\mathbb{P}^5$ of type b) can be proved to have 14 moduli.

Thus we can assume to be in case a) and the defining equations of the quadrics $Q_1, Q_2, Q_3$ (denoted in the same way) are as in item i) of the statement.

Consider the class $\text{BNef}[4] = h - f_3 - f_5 - f_{12}$. Then $h^2 = 2$ and $h \cdot f_i = 2$ for all $i$, thus $h$ is ample and the associated linear system is base point free by Corollary 1.2. Thus $h$ defines a morphism $\pi : X \to \mathbb{P}^2$ branched along a smooth plane sextic $B$. Since $2h = f_7 + f_{10} = f_4 + f_{12} = f_8 + f_9 = f_3 + f_6 = f_5 + f_{11} = f_1 + f_2$, the image by $\pi$ of the twelve $(-2)$-curves of $X$ are six smooth conics $C_1, \ldots, C_6 \subseteq \mathbb{P}^2$ such that $\pi^{-1}(C_i)$ is the union of two smooth rational curves $R_{ij}$ with $j = 1, 2$, for any $i = 1, \ldots, 6$. This implies that the conics are tangent to $B$ at 6 points.

By Theorem 0.2 the Cox ring $R(X)$ is generated in the following degrees:

$$f_1, \ldots, f_{12}, \text{BNef}[4].$$

Observe that $\text{BNef}[4]$ is also an element of the Hilbert basis of the effective cone of $X$ (see Table 1). Clearly any minimal generating set of $R(X)$ must contain the sections $s_{11}, \ldots, s_{12}$ defining the $(-2)$-curves of $X$ and a basis $s_{13}, s_{14}, s_{15}$ of $H^0(\text{BNef}[4])$, thus $\{s_1, \ldots, s_{15}\}$ is a minimal generating set of $R(X)$.

**Corollary 2.3.** The moduli space of K3 surfaces with $\text{NS}(X) \simeq (8) \oplus 3A_1$ is unirational.

**Proof.** Let $Q$ be the projective linear system of quadrics through the points $p_1, p_2, p_3$ (of dimension 17) and let $Gr$ be the Grassmannian of nets in $Q$. Let $A \subset Gr$ be the space of nets of quadrics of type a). Observe that, by the proof of Proposition 2.2, giving a net in $A$ is equivalent to give three quadrics $Q_1, Q_2, Q_3 \subset Q$, such that $Q_i$ has a node at $p_i$ for $i = 1, 2, 3$. Observe that having a node at $p_i$ imposes 5 conditions on the coefficients of a quadric in $Q$. Thus $A$ is a rational variety of projective dimension $3(\dim(Q) - 5) = 36$. By item i) of Proposition 2.2 the moduli space of K3 surfaces with $\text{NS}(X) \simeq (8) \oplus 3A_1$ is birational to the quotient of $A$ by the subgroup $G$ of $\text{PGL}_6(\mathbb{C})$ fixing $p_1, p_2, p_3$, whose elements are matrices of the form

$$\begin{pmatrix}
* & 0 & 0 & * & * & *
0 & * & 0 & \vdots & \vdots & \\
0 & 0 & * & \vdots & \vdots & \\
0 & 0 & 0 & \vdots & \vdots & \\
0 & 0 & 0 & \vdots & \vdots & \\
0 & 0 & 0 & * & * & *
\end{pmatrix}$$

and thus has dimension 20.

**The family $F_2$.** Let $X$ be a K3 surface with $\text{NS}(X) \cong V_2 = (4) \oplus (-4) \oplus A_2$. By Theorem 2.1 $X$ contains six $(-2)$-curves whose intersection matrix is given in Table 4. The Hilbert basis of the nef cone contains 35 classes, with four classes of elliptic fibrations: $\text{BNef}[5], \text{BNef}[11], \text{BNef}[23]$ and $\text{BNef}[29]$. Each elliptic fibration has one singular fiber of type $A_2$ and has no sections.

**Proposition 2.4.** Let $X$ be a K3 surface with $\text{NS}(X) \cong V_2 = (4) \oplus (-4) \oplus A_2$. Then

(i) $X$ is birational to a quartic surface $\overline{X} \subset \mathbb{P}^3$ with a node and with two hyperplane sections passing through the node which are the union of two conics;
(ii) the projection of $\mathbf{X}$ from the node induces a double cover $\pi : X \to \mathbb{P}^2$ branched along a smooth plane sextic with two 3-tangent lines $L_1, L_2$ and one 6-tangent conic $C$;

(iii) the surface has six $(-2)$-curves: the curves $R_{ij}$, $i, j = 1, 2$ such that $\pi(R_{11}) = \pi(R_{12}) = L_i$ and the curves $S_1, S_2$ such that $\pi(S_1) = \pi(S_2) = C$;

(iv) the Cox ring of $X$ is generated in the degrees given in Table 6 and has at least 23 generators in such degrees: $s_1, \ldots, s_6$ defining the $(-2)$-curves, $s_7, \ldots, s_{10}$ defining each a smooth fiber of the four elliptic fibrations, $s_{11} \in H^0(h - f_3)$ independent from the elements defining $\pi^{-1}(L_1), \pi^{-1}(L_2), s_{12}, \ldots, s_{23}$ whose degrees are elements of the Hilbert basis of the nef cone with self-intersections 4 (for $i = 12, \ldots, 15$), 10 (for $i = 16, \ldots, 19$) and 12 (for $i = 20, \ldots, 23$).

Proof. Let $f_i$ the classes of the $(-2)$-curves for $i = 1, \ldots, 6$, and let $h = \text{BNeF}[7]$. Then $h^2 = 4$, $h \cdot f_i = 2$ for $i = 2, 4, 5, 6$, $h \cdot f_1 = 8$ and $h \cdot f_3 = 0$. By Corollary 1.2 the linear system associated to $h$ is base point free. Moreover, the intersections of $h$ with all classes of elliptic fibrations are $\geq 4$. Thus $h$ defines a birational morphism $\varphi : X \to \mathbb{P}^3$ whose image is a quartic surface $\mathbf{X} \subset \mathbb{P}^3$ with a node. Since $f_2 + f_3 = h - f_3$, then the $(-2)$-curves of classes $f_2, f_3$ are the proper transforms of two conics lying in a hyperplane section of $\mathbf{X}$ passing through the node. The same holds for $f_4, f_6$.

Now consider the class $k = \text{BNeF}[15] = h - f_3$. Then $k^2 = 2$, $k \cdot f_1 = k \cdot f_3 = 2$ and $k \cdot f_i = 1$ for $i \neq 1, 3$. Thus $k$ is ample and the associated linear system $|k|$ is base point free by Corollary 1.2. Thus it defines a double cover $\pi : X \to \mathbb{P}^2$ branched along a smooth plane sextic $B$. Since $k = f_2 + f_5 = f_4 + f_6$ and $2k = f_1 + f_3$, the image by $\pi$ of the six $(-2)$-curves of $X$ are a smooth conic $C \subset \mathbb{P}^2$ and two smooth lines $L_1, L_2 \subset \mathbb{P}^2$. The last statement follows from Theorem 0.2. Observe that we clearly need one generator for each $(-2)$-class, one generator for each elliptic fibration (since one of its fibers is reducible) and one generator in degree $k$. In the remaining degrees, we know that there is at least one generator of $R(X)$ by Proposition 1.12. \hfill \square

The family $\mathcal{F}_3$. Let $X$ be a K3 surface with $\text{Cl}(X) \cong V_3 = (4) \oplus A_3$. By Theorem 2.1 $X$ contains five $(-2)$-curves whose intersection matrix is given in Table 4 and described in Figure 1.

![Figure 1](image)

**Figure 1.** Intersection graph of $(-2)$-curves for $\mathcal{F}_3$

The Hilbert basis of the nef cone of $X$ contains 10 classes. Exactly one of them, BNeF[3], is the class of an elliptic fibration with two reducible fibers of type $A_1$ and without sections.

**Proposition 2.5.** Let $X$ be a K3 surface with $\text{NS}(X) \cong V_3 = (4) \oplus A_3$. Then

(i) $X$ is birational to a double cover $\pi : \mathbf{X} \to \mathbb{P}^2$ branched along a plane sextic $B$ with a double point $p$ with two bitangent lines $L_1, L_2$ passing through $p$;

(ii) the surface $X$ has five $(-2)$-curves: the exceptional divisor $E$ over the singular point $\pi^{-1}(p)$ and four curves $R_{ij}$, with $i, j = 1, 2$, such that $\pi(R_{ij}) = L_i$;

(iii) the Cox ring of $X$ is generated in the degrees given in Table 6 and has at least ten generators in such degrees: $s_1, \ldots, s_5$ defining the $(-2)$-curves, $s_6$ defining the pull-back of a line in $\mathbb{P}^2$ not passing through $p$ and $s_7, \ldots, s_{10}$ whose degrees are four distinct classes in the Hilbert basis of the nef cone with self-intersection 4.
Proof. Let \( f_1, \ldots, f_5 \) be the classes of the \((-2)\)-curves and \( h = \text{BNef}[2] \). Then \( h^2 = 2, h \cdot f_i = 0 \) and \( h \cdot f_i = 1 \) for \( i \neq 5 \). Thus \( h \) is nef. By Corollary 1.12, the linear system associated to \( h \) is base point free and thus defines a morphism \( \psi : X \to \mathbb{P}^2 \) which contracts the \((-2)\)-curve with class \( f_5 \) to a point \( p \in \mathbb{P}^2 \) and is branched along a plane sextic \( B \) with one node at \( p \). Since \( h = f_1 + f_4 + f_5 = f_2 + f_3 + f_5 \) the image by \( \psi \) of the four \((-2)\)-curves of classes \( f_1, \ldots, f_4 \) are two lines \( L_1, L_2 \) passing through \( p \) and tangent to \( B \) in two more points.

By Theorem 0.2, a minimal set of generators of the Cox ring \( R(X) \) has the following degrees:

\[
\rho = \{ f_1, \ldots, f_5, h, h_1, h_2, h_3, h_4 \}
\]

where \( h_i = \text{BNef}[i] \) with \( i = 1, 4, 7, 9 \). Clearly any minimal generating set of \( R(X) \) must contain the sections \( s_1, \ldots, s_5 \) defining the \((-2)\)-curves of \( X \) and an element \( s_6 \) such that \( s_1 s_4 s_5, s_2 s_3 s_5, s_6 \) is a basis of \( H^0(h) \). In the remaining degrees \( h_1, \ldots, h_4 \) we know that there is at least one generator by Proposition 1.12.

Proposition 2.6. The moduli space of K3 surfaces with \( \text{NS}(X) \cong (4) \oplus A_3 \) is rational.

Proof. Let \( q \) be the nodal point of the branch locus \( B \) and let \( p_1, p_2 \) (respectively \( r_1, r_2 \)) the other intersection points of the tangent line \( L_1 \) (respectively \( L_2 \)) to the sextic. Using automorphisms of \( \mathbb{P}^2 \), we can impose that

\[
q = (0, 1, 0), \ p_1 = (1, 0, 0), \ p_2 = (-1, 1, 0), \ r_1 = (0, 0, 1), \ r_2 = (0, 1, -1),
\]

and there are no more symmetries. One computes that the projective linear system of smooth plane sextics with a node at \( q \) and tangent to the lines \( L_1, L_2 \) at \( p_1, p_2, r_1, r_2 \) is an open set in the affine space \( \mathbb{A}^{16} \), therefore the moduli space of K3 surfaces with \( \text{NS}(X) \cong (4) \oplus A_3 \) is rational.

Remark 2.7. The surface \( X \) admits four different morphisms \( X \to \mathbb{P}^3 \) mapping onto a quartic surface with a singular point of type \( A_3 \), defined by the classes \( \text{BNef}[i], i = 2, 4, 7, 9 \).

The family \( F_4 \). Let \( X \) be a K3 surface with \( \text{NS}(X) \cong V_4 = U \oplus 2A_1 \). By Theorem 2.1, \( X \) contains five \((-2)\)-curves whose intersection matrix is given Table 4 and described in Figure 2. The Hilbert basis of the nef cone of \( X \) contains five classes, one of them is the class \( \text{BNef}[5] \) of an elliptic fibration \( \text{BNef}[5] \) with a section and two reducible fibers of type \( \tilde{A}_1 \).

![Figure 2. Intersection graph of \((-2)\)-curves for \( F_4 \)](image)

In this case, the Cox ring has been computed in [AHL10, Proposition 6.7, ii)], where \( X \) is described as a double cover of the Hirzebruch surface \( \mathbb{F}_4 \) blown-up at two general points \( p, q \). We will use the following notation for curves in \( \mathbb{F}_4 \): \( F_1, F_2 \) are the two fibers of the projection \( \mathbb{F}_4 \to \mathbb{P}^1 \) passing through \( p \) and \( q \), \( S_1 \) is the curve in \( \mathbb{F}_4 \) with \( S_1^2 = -4 \) and \( S_2 \) is a curve passing through \( p \) and \( q \) with \( S_2^2 = 4 \) and \( S_1 \cdot S_2 = 0 \). We will call \( S_1 \) also its pull-back in the blow-up of \( \mathbb{F}_4 \) at \( p, q \).

We recall these properties in the following proposition:

Proposition 2.8. Let \( X \) be a K3 surface with \( \text{NS}(X) \cong V_4 = U \oplus 2A_1 \). Then

(i) there is a double cover \( \pi : X \to Y \), where \( Y \) is a blow-up of \( \mathbb{F}_4 \) at two general points \( p, q \), branched along \( S_1 \) and a smooth curve \( B \) of genus 8;
(ii) the surface has five \((-2)\)-curves, which are the preimages by \(\pi\) of: \(S_1\), the proper transforms of \(F_1, F_2\) and the two exceptional divisors over \(p\) and \(q\); 
(iii) a minimal generating set of the Cox ring of \(X\) is \(s_1, \ldots, s_7\), where \(s_1, \ldots, s_5\) define the \((-2)\)-curves, \(s_6\) defines the preimage of the proper transform of \(S_2\) and \(s_7\) defines \(\pi^{-1}(B)\); 
(iv) for a very general \(X\) as before we have an isomorphism 
\[ R(X) \to \mathbb{C}[T_1, \ldots, T_7]/I, \ s_i \mapsto T_i, \]
where the degrees of the generators \(T_i\) for \(i = 1, \ldots, 7\) are given by the columns of the following matrix 
\[
\begin{pmatrix}
0 & -1 & 1 & -1 & 0 & -2 & -3 \\
0 & 0 & -1 & 0 & 0 & -2 & -3 \\
0 & 1 & 0 & 0 & -1 & 1 & 1 \\
-1 & 0 & 0 & 1 & 0 & 1 & 1
\end{pmatrix}
\]
and the ideal \(I\) is generated by the following polynomial 
\[ T_2^2 - f(T_1, \ldots, T_6), \]
where \(f(T_1, \ldots, T_6) = f(T_2^2, T_2, \ldots, T_6) \) and \(f\) is the defining polynomial of \(B\) in the Cox ring of \(Y\).

Remark 2.9. The class \(\text{BNef}[3]\) has self-intersection 4, is nef and has zero intersection with \(f_2, f_3, f_4\). Moreover, it has intersection 2 with the class \(\text{BNef}[5]\) of the unique elliptic fibration of \(X\), thus it is hyperelliptic. It can be easily checked that \(\text{BNef}[3]\) is not equal to \(3 \text{BNef}[5] + f_3\), thus it is base point free by Theorem 1.1. Moreover, \(\text{BNef}[3] = 2 \text{BNef}[5] + 2f_3 + f_2 + f_4\), thus by [SD74, Proposition 5.7, iii)] the image of the associated degree two morphism \(\varphi : X \to \mathbb{P}^3\) is a quadric cone. The morphism \(\varphi\) factorizes through a double cover of the Hirzebruch surface \(\eta : X \to \mathbb{F}_2\), and is branched along a divisor of the form \(S + B\), where \(S\) is the unique curve in \(\mathbb{F}_2\) with \(S^2 = -2\) and \(B \in |3S + 8F|\) (here \(F\) is a fiber). We denote by \(p, q\) the two intersection points of \(B\) and \(S\). The curves of classes \(f_1, f_2\) are mapped by \(\eta\) to the two fibers of \(\mathbb{F}_2\) through \(p, q\), the curve of class \(f_3\) has image \(S\), and the curves of classes \(f_2, f_4\) are contracted to the points \(p, q\). Observe that this model is related to the one described in Proposition 2.8 as follows: the blow-up of \(\mathbb{F}_2\) at \(p, q\) is isomorphic to the blow-up of \(\mathbb{F}_4\) at two general points, obtained contracting the proper transforms of the two fibers of \(\mathbb{F}_2\) through \(p, q\).

Corollary 2.10. The moduli space of K3 surfaces \(X\) with \(\text{NS}(X) \simeq U + 2A_1\) is unirational.

Proof. By Remark 2.9 we know that \(X\) is the double cover of \(\mathbb{F}_2\) branched along a curve of type \(B + S\), where \(S^2 = -2\) and \(B \in |3S + 8F|\) is smooth. By [Har77, Chapter V, Corollary 2.18], the divisors \(B\) and \(B - K_{\mathbb{F}_2} = 5S + 12F\) are ample. Thus \(h^2(\mathbb{F}_2, B) = 0\) and by Serre duality and Kodaira vanishing Theorem \(h^1(\mathbb{F}_2, B) = h^1(\mathbb{F}_2, K_{\mathbb{F}_2} - B) = 0\). It follows by the Riemann-Roch Theorem that \(h^0(\mathbb{F}_2, B) = 24\). The dimension of the automorphism group of \(\mathbb{F}_2\) is 7, thus the quotient of \(|B| \simeq \mathbb{P}^2\) by that automorphism group is a 16 dimensional unirational variety which is birational to the moduli space of K3 surfaces \(X\) with \(\text{NS}(X) \simeq U + 2A_1\). □

The family \(\mathcal{F}_{2s}\). Let \(X\) be a K3 surface with \(\text{NS}(X) \cong V_2 = U(2) \oplus A_2\). By Theorem 2.1 \(X\) contains six classes of \((-2)\)-curves whose intersection matrix is given in Table 4 and described in Figure 3. The Hilbert basis of the nef cone of \(X\) contains five classes. The classes \(\text{BNef}[3]\), \(\text{BNef}[4]\) and \(\text{BNef}[5]\) are classes of elliptic fibrations without sections and each having two reducible fibers of type \(A_1\).
Proposition 2.11. Let $X$ be a K3 surface with $\text{NS}(X) \cong \mathbb{V}_5 = U(2) \oplus 2A_1$. Then

(i) there is a minimal resolution $\varphi : X \to \overline{X}$ of a double cover $\pi : \overline{X} \to \mathbb{P}^2$ branched along a plane sextic $B$ with three nodes $p_1, p_2, p_3$;

(ii) the surface $X$ has six $(-2)$-curves: the exceptional divisors $E_1, E_2, E_3$ over the three nodes $p_1, p_2, p_3$ and the curves $R_1, R_2, R_3$ such that $\pi \varphi (R_i)$ is the line through $p_j, p_k$ with $j, k \neq i$;

(iii) a minimal generating set of the Cox ring of $X$ is $s_1, \ldots, s_7$, where $s_1, \ldots, s_6$ define the $(-2)$-curves and $s_7$ defines the ramification curve of $\pi \varphi$;

(iv) for a very general $X$ as before we have an isomorphism

$$\mathbb{C}[T_1, \ldots, T_7]/I \to R(X), \ T_i \mapsto s_i,$$

where the degrees of the generators $T_i$ for $i = 1, \ldots, 7$ are given by the columns of the following matrix

$$
\begin{pmatrix}
0 & 0 & 0 & -1 & -1 & 0 & -2 \\
-1 & -1 & 0 & 0 & 0 & 0 & -2 \\
0 & 1 & 0 & 1 & 0 & -1 & 1 \\
1 & 0 & -1 & 0 & 1 & 0 & 1
\end{pmatrix}
$$

and the ideal $I$ is generated by the following polynomial:

$$T_7^2 - f(T_1T_3T_5, T_2T_5T_6, T_1T_4T_6),$$

where $f(x_0, x_1, x_2) \in \mathbb{C}[x_0, x_1, x_2]$ is a defining polynomial of the plane sextic $B$.

Proof. Let $f_1, \ldots, f_5$ be the classes of the $(-2)$-curves (see Table 1) and let $h = \text{BNef}[1]$. Then $h^2 = 2$, $h \cdot f_i = 0$ for $i = 1, 5, 6$ and $h \cdot f_2 = 2$ for $i = 2, 3, 4$. Thus $h$ is nef. By Corollary 1.2 the linear system associated to $h$ is base point free and thus defines a degree two morphism to $\mathbb{P}^2$ branched along a plane sextic $B$ which contracts the $(-2)$-curves of classes $f_1, f_5, f_6$. Since such classes have zero intersection, then the three $(-2)$-curves are disjoint, thus are mapped to three distinct points $p_1, p_2, p_3 \in \mathbb{P}^2$. The plane sextic $B$ has three nodes at $p_1, p_2, p_3$. Moreover, since $h = f_1 + f_5 + f_6 = f_2 + f_3 + f_4 = f_1 + f_4 + f_6$, the $(-2)$-curves with classes $f_2, f_3, f_4$ are mapped to the three lines passing through $p_2, p_3, p_1, p_2$ and $p_1, p_3$ respectively.

The Cox ring of $X$ has been computed in [AHL10, Proposition 6.7, i)], where $X$ is also described as the double cover of a smooth quadric surface $F_0$ blown-up at two general points. \square

Corollary 2.12. The moduli space of K3 surfaces with $\text{NS}(X) \simeq U(2) \oplus 2A_1$ is unirational.

Proof. By Proposition 2.11 $X$ is the minimal resolution of a double cover of $\mathbb{P}^2$ branched along a plane sextic $B$ with three nodes, thus the moduli space of such K3 surfaces is birational to the moduli space of such plane sextics. Up to a projectivity we can assume that the three nodes of
The family $\mathcal{F}_6$. Let $X$ be a K3 surface with NS($X$) $\cong V_6 = U(3) \oplus 2A_1$. By Theorem 2.1, $X$ contains eight $(-2)$-curves whose intersection matrix is given in Table 4. The Hilbert basis of the nef cone of $X$ contains 19 classes, four of them are classes of elliptic fibrations: BNef[5], BNef[8], BNef[18] and BNef[19]. Each elliptic fibration has two fibers of type $\tilde{\text{nef}}$ cone of dimension 18. The quotient of such space by the action of the projectivities leaving invariant the set of fundamental points gives a 16-dimensional unirational moduli space.

\begin{proof}
(iii) a minimal generating set of the Cox ring of $X$ is $s_1, \ldots, s_9$, where $s_1, \ldots, s_8$ define the $(-2)$-curves and $s_9 \in H^0(\pi^*O_{\mathbb{P}^2}(1))$;

(v) for a very general $X$ as before we have an isomorphism \[ \mathbb{C}[T_1, \ldots, T_9]/I \to R(X), \quad T_i \mapsto s_i, \]

where the degrees of the generators $T_i$ for $i = 1, \ldots, 9$ are given by the columns of the following matrix

\[
\begin{pmatrix}
0 & -2 & 0 & 0 & -1 & -2 & 0 & -1 & -1 \\
0 & -2 & -1 & 0 & 0 & -2 & -1 & 0 & -1 \\
0 & -3 & 0 & 1 & 0 & -2 & -1 & -1 & -1 \\
1 & -2 & -1 & 0 & -1 & -3 & 0 & 0 & -1
\end{pmatrix}
\]

and the ideal $I$ is generated by the following polynomials:

\[ T_1T_6 - \tilde{G}_1(T_3T_8, T_5T_7, T_9), \]
\[ T_2T_4 - \tilde{G}_2(T_3T_8, T_5T_7, T_9), \]
\[ T_1T_2T_3T_5 + T_4T_6T_7T_8 - \tilde{F}(T_3T_8, T_5T_7, T_9), \]

where $\tilde{G}_i = G \circ \varphi^{-1}$, $i = 1, 2$, and $\tilde{F} = F \circ \varphi^{-1}$ are obtained from $G_1, G_2$ and $F$ in item ii) respectively composing with the coordinate change in $\mathbb{P}^2$ given by \[ \varphi(x_0, x_1, x_2) = (F_1, F_2, s_9). \]

\begin{proof}
Let $f_1, \ldots, f_8$ be the classes of the $(-2)$-curves (see Table 1) and $h = \text{BNef}[15]$. Then $h^2 = 2$, $h \cdot f_i = 2$ for $i = 1, 2, 4, 6$ and $h \cdot f_i = 1$ for $i = 3, 5, 7, 8$. Thus $h$ is ample. By Corollary 1.2 the associated linear system is base point free and thus defines a double cover $\pi : X \to \mathbb{P}^2$ branched along a smooth plane sextic $B$. Since $h = f_3 + f_8 = f_5 + f_7$ and $2h = f_1 + f_6 = f_2 + f_4$ the image by $\pi$ of the eight $(-2)$-curves of $X$ are two 3-tangent lines $L_1, L_2$ and two 6-tangent conics $C_1, C_2 \subseteq \mathbb{P}^2$. This proves items i) and iii).

To prove ii), observe that, looking at the intersection matrix of the $(-2)$-curves of $X$ one can see that the restriction of the double cover $\pi$ over $D = L_1 \cup L_2 \cup C_1 \cup C_2$ is trivial. This implies
that there exists a plane cubic \( C \) such that \( B \cdot D = 2C \cdot D \) (see for example [Ver83, Proposition 1.7, Ch.3]). Let \( F_1, F_2, G_1, G_2, F \) be defining polynomials for \( L_1, L_2, G_1, G_2 \) and \( F \) respectively. Consider the pencil of plane sextics generated by \( F_1F_2G_1G_2 \) and \( F^2 \), which contains \( B \cap D \) in its base locus. Let \( V \) be an element of the pencil intersecting \( B \) in one more point. By Bezout’s Theorem, since \( V \) intersects \( B \) in at least 37 points counting multiplicities and \( B \) is irreducible, we have that \( V = B \).

By Theorem 0.2 the Cox ring \( R(X) \) is generated in the following degrees:

\[
f_1, \ldots, f_s, h.
\]

Clearly any minimal generating set of \( R(X) \) must contain the sections \( s_1, \ldots, s_8 \) defining the \((-2)\)-curves of \( X \) and a section \( s_9 \) such that \( s_3s_8, s_5s_7, s_9 \) is a basis of \( H^0(h) \). This proves item iv).

The first two relations in item v) are obvious, due to the fact that \( s_1s_6 \) and \( s_2s_4 \) define the preimages of the conics \( C_1 = \{G_1 = 0\} \) and \( C_2 = \{G_2 = 0\} \). The last relation, follows from the fact that

\[
(x_3 + F)(x_3 - F) = F_1F_2G_1G_2,
\]

thus up to renumbering we can assume \( x_3 + F = s_1s_2s_3s_5, x_3 - F = s_4s_6s_7s_8 \), so that \( 2F = s_1s_2s_3s_5 - s_4s_6s_7s_8 \). Up to rescaling the generators \( s_i \) we obtain the last relation. It can be proved with the same type of argument used in the proof of [ACDL21, Theorem 3.5] that the ideal \( I \) is prime for general \( F_1, F_2, G_1, G_2, F \). Since the ring \( \mathbb{C}[T_1, \ldots, T_9]/I \) is an integral domain, it has Krull dimension 6 and it surjects onto \( R(X) \), then \( \mathbb{C}[T_1, \ldots, T_9]/I \cong R(X) \). \( \square \)

**Corollary 2.14.** The moduli space of K3 surfaces with \( \text{NS}(X) \simeq U(3) \oplus 2A_1 \) is unirational.

**Proof.** It follows from item ii) in Proposition 2.13 that \( X \) is a double cover of \( \mathbb{P}^2 \) branched along a plane sextic which can be defined by an equation of the form \( F_1F_2G_1G_2 + F^2 = 0 \), where \( F_1, F_2 \in \mathbb{C}[x_0, \ldots, x_2] \) are homogeneous polynomials of degree one, \( G_1, G_2 \in \mathbb{C}[x_0, \ldots, x_2] \) of degree two and \( F \in \mathbb{C}[x_0, \ldots, x_2] \) of degree three.

Conversely, a double cover \( Y \) of \( \mathbb{P}^2 \) branched along a smooth plane sextic \( B \) with an equation of that form has the property that the curves defined by \( F_i = 0 \) for \( i = 1, 2 \) or \( G_j = 0 \) for \( j = 1, 2 \) are everywhere tangent to \( B \). Moreover, the double cover is trivial over their union. This implies that the pull-back of each such curve is the union of two smooth rational curves and it can be easily proved that the classes of such curves generate a sublattice of \( \text{NS}(Y) \) isometric to \( V_6 \). Thus the moduli space of such K3 surfaces \( Y \) has dimension at most \( 20 - 4 = 16 \).

This shows that an open subset of the moduli space of K3 surfaces with \( \text{NS}(X) \simeq U(3) \oplus 2A_1 \) is isomorphic to the quotient by \( \text{PGL}(3, \mathbb{C}) \) of an open subset of the image of the map:

\[
\Phi : H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1))^\oplus 2 \oplus H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))^\oplus 2 \oplus H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(3)) \to H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(6))
\]

defined by

\[
(F_1, F_2, G_1, G_2, F) \mapsto F_1F_2G_1G_2 + F^2.
\]

In particular, it is unirational. \( \square \)

**Example 2.15.** Let us take

\[
F_1 = x_1 + x_2, \quad F_2 = 2(x_0 + x_1 + x_2)
\]

\[
G_1 = x_0^2 + x_0x_2 + x_1x_2 + x_2^2, \quad G_2 = x_0^2 + x_0x_1 + 2x_1^2 + x_2^2
\]

\[
F = 2x_0^2x_1 + 2x_0x_2^2 + 2x_1^3 + 2x_0x_1x_2 + 2x_1^2x_2 + x_2^3.
\]
Let Proposition 2.16. contains eight (elliptic fibration is without sections and has two fibers of type $\widetilde{\text{six}}$ classes of elliptic fibrations $\text{BNef}[4], \text{BNef}[7], \text{BNef}[9], \text{BNef}[13], \text{BNef}[14]$ and $\text{BNef}[15]$. Each $F$ such that $Q_2$ has no roots $\alpha$ such that $\frac{2}{3}$ is a root of unity. Therefore the geometric Picard number of $Y$ is 4 and $Y/C$ is a K3 surface with $\text{NS}(Y/C) \cong U(3) \oplus 2A_1$.

The family $F_7$. Let $X$ be a K3 surface with $\text{NS}(X) \cong V_7 = U(4) \oplus 2A_1$. By Theorem 2.1 $X$ contains eight $(-2)$-curves and the Hilbert basis of the nef cone of $X$ contains 15 classes, with six classes of elliptic fibrations $\text{BNef}[4], \text{BNef}[7], \text{BNef}[9], \text{BNef}[13], \text{BNef}[14]$ and $\text{BNef}[15]$. Each elliptic fibration is without sections and has two fibers of type $A_1$.

Proposition 2.16. Let $X$ be a K3 surface with $\text{NS}(X) \cong V_7 = U(4) \oplus 2A_1$. Then

(i) $X$ is isomorphic to a smooth quartic surface in $\mathbb{P}^3$ having four hyperplane sections which are the union of two conics;

(ii) $X$ can be defined by an equation of the form

$$G_1(x_0, \ldots, x_3)G_2(x_0, \ldots, x_3) + F_1(x_0, \ldots, x_3)F_2(x_0, \ldots, x_3)F_3(x_0, \ldots, x_3)F_4(x_0, \ldots, x_3) = 0,$$

where $G_1, G_2 \in \mathbb{C}[x_0, \ldots, x_3]$ are homogeneous of degree two and $F_i \in \mathbb{C}[x_0, \ldots, x_3]$ are homogeneous of degree one for $i = 1, 2, 3, 4$;

(iii) the surface has eight $(-2)$-curves: the eight conics;

(iv) a minimal generating set of the Cox ring of $X$ is $s_1, \ldots, s_8$, where $s_1, \ldots, s_8$ define the $(-2)$-curves (we assume that $s_6s_7, s_3s_5, s_1s_8$ and $s_2s_4$ define the four reducible hyperplane sections);

(v) for a very general $X$ as before we have an isomorphism

$$\mathbb{C}[T_1, \ldots, T_8]/I \to R(X), \quad T_i \mapsto s_i,$$

where the degrees of the generators $T_i$ for $i = 1, \ldots, 8$ are given by the columns of the following matrix

$$
\begin{pmatrix}
0 & 0 & -1 & -1 & 0 & -1 & 0 & -1 \\
-1 & 0 & -1 & -1 & 0 & 0 & -1 & 0 \\
0 & 0 & -2 & -1 & 1 & 0 & -1 & -1 \\
1 & -1 & 1 & 2 & 0 & 1 & 0 & 0
\end{pmatrix}
$$

and the ideal $I$ is generated by the following polynomials:

$$T_1T_2T_3T_6 - G_1(T_0T_7, T_3T_5, T_1T_8, T_2T_4),$$

$$T_4T_5T_7T_8 - G_2(T_0T_7, T_3T_5, T_1T_8, T_2T_4),$$

where $G_i = G \circ \varphi^{-1}$, $i = 1, 2$ are obtained from $G_1, G_2$ in item ii) respectively composing with the coordinate change in $\mathbb{P}^3$ given by

$$\varphi(x_0, x_1, x_2, x_3) = (F_1, F_2, F_3, F_4).$$

Proof. Let $f_1, \ldots, f_8$ be the classes of the $(-2)$-curves (see Table 1) and let $h = \text{BNef}[11]$. Then $h^2 = 4$ and $h \cdot f_i = 2$ for all $i$. Thus $h$ is ample. Moreover, $h$ is not hyperelliptic by Proposition 1.3. By Corollary 1.2 the associated linear system is base point free, thus it defines an embedding of $X$ in $\mathbb{P}^3$ as a smooth quartic surface. Since $h = f_1 + f_8 = f_2 + f_4 = f_3 + f_5 + f_6 + f_7$ and $h \cdot f_i = 2$ for all $i$, then $X$ has four hyperplane sections which decompose into the union of two conics. We will
denote by \( C_i \) the conic whose class is \( f_i \). Observe that \( f_1 + f_2 + f_3 + f_6 = f_4 + f_5 + f_7 + f_8 = 2h \). This means that the four conics \( C_1, C_2, C_3, C_6 \) are contained in a quadric \( Q_1 \). For the same reason the conics \( C_4, C_5, C_7, C_8 \) are contained in a quadric \( Q_2 \).

Let \( F_1, \ldots, F_4 \) be defining polynomials for the four planes containing the conics \( C_1, \ldots, C_8 \), and \( G_1, G_2 \) be defining polynomials for the quadrics \( Q_1, Q_2 \) respectively. Consider the pencil of quartic surfaces in \( \mathbb{P}^3 \) generated by \( G_1 G_2 \) and \( F_1 F_2 F_3 F_4 \). Any member of the pencil intersects \( X \) along the degree 16 curve \( C_1 \cup \cdots \cup C_8 \). Since \( X \) is irreducible, it follows from Bezout’s theorem that \( X \) belongs to the pencil.

By Theorem 0.2 the Cox ring \( R(X) \) is generated in the following degrees:

\[
f_1, \ldots, f_8, h.
\]

Clearly any minimal generating set of \( R(X) \) must contain the sections \( s_1, \ldots, s_8 \) defining the \((-2)\)-curves of \( X \). Since generically the four polynomials \( F_1, \ldots, F_4 \) can be taken to be independent, then they generate \( H^0(O_{\mathbb{P}^3}(1)) \). This implies that \( s_1 s_8, s_2 s_4, s_3 s_5, s_6 s_7 \) generate \( H^0(h) \), so that a generator in degree \( H^0(h) \) is not necessary. This proves item iv). The two relations in item v) are due to the fact that \( s_1 s_2 s_5 s_8 \) and \( s_4 s_5 s_7 s_8 \) define the intersections \( X \cap Q_1 \) for \( i = 1, 2 \) respectively.

It can be proved with the same type of argument used in the proof of [ACDL21, Theorem 3.5] that the ideal \( I \) is prime for general \( G_1, G_2 \). Since \( \mathbb{C}[T_1, \ldots, T_8]/I \) is an integral domain of dimension \( \dim R(X) = \dim(X) + \text{rank Cl}(X) = 6 \) and surjects onto \( R(X) \), then it is isomorphic to it.

**Corollary 2.17.** The moduli space of K3 surfaces \( X \) with \( \text{NS}(X) \cong U(4) \oplus 2A_1 \) is unirational.

**Proof.** It follows from item ii) in Proposition 2.16 that \( X \) can be defined by an equation of the form \( G_1 G_2 + F_1 F_2 F_3 F_4 = 0 \), where \( G_1, G_2 \in \mathbb{C}[x_0, \ldots, x_3] \) are homogeneous polynomials of degree two and \( F_i \in \mathbb{C}[x_0, \ldots, x_3] \) are homogeneous polynomials of degree one for \( i = 1, 2, 3, 4 \).

Conversely, a smooth quartic surface \( Y \) with an equation of that form has the property that for any \( i = 1, 2, 3, 4 \) its intersection with the plane \( F_i = 0 \) is the union of two conics \( C_{i1}, C_{i2} \) defined by \( G_1 = 0 \) and \( G_2 = 0 \). In particular, for a general choice of the polynomials \( G_1, G_2, F_1, \ldots, F_4 \) we have \( C_1 \cdots C_4 = 8 \), \( C_{ij} \cdot C_{i'j'} = 2 \) for \( i \neq i' \) and \( j = 1, 2 \) and \( C_{ij} \cdot C_{i'j'} = 0 \) for \( i \neq i' \) and \( j \neq j' \). An easy computation shows that the classes of \( C_{11} + C_{12} + C_{21} + C_{22} \) generate a sublattice of the Picard lattice of \( Y \) isometric to \( V_7 \). This implies that the moduli space of such quartic surfaces has dimension at most \( 20 - 4 = 16 \).

This shows that an open subset of the moduli space of K3 surfaces with \( \text{NS}(X) \cong U(4) \oplus 2A_1 \) can be obtained taking the quotient by \( \text{PGL}(4, \mathbb{C}) \) of an open subset of the image of the map:

\[
\Phi : H^0(\mathbb{P}^3, O_{\mathbb{P}^3}(1))^\oplus 4 \oplus H^0(\mathbb{P}^3, O_{\mathbb{P}^3}(2))^\oplus 2 \to H^0(\mathbb{P}^3, O_{\mathbb{P}^3}(4))
\]

defined by

\[
(F_1, F_2, F_3, F_4, G_1, G_2) \to F_1 F_2 F_3 F_4 + G_1 G_2.
\]

In particular, it is unirational.

**The family \( \mathcal{F}_8 \).** Let \( X \) be a K3 surface with \( \text{NS}(X) \cong V_8 = U \oplus A_2 \). By Theorem 2.1 \( X \) contains four \((-2)\)-curves whose intersection matrix is given in Table 4 and whose intersection graph is described in Figure 4. The Hilbert basis of the nef cone of \( X \) contains five classes. The class \( \text{BNe}_5 \) defines an elliptic fibration having a section and one fiber of type \( A_2 \).
**Proposition 2.18.** Let $X$ be a K3 surface with $\NS(X) \cong \mathbb{Z}^8 = U \oplus A_2$. Then

(i) there is a degree two morphism $\varphi : X \to \mathbb{P}^4$, whose image is a cone over a rational normal cubic in $\mathbb{P}^3$, which factors through a degree two morphism $\mu : X \to \mathbb{F}_3$ branched along the union of the smooth rational curve $E$ with $E^2 = -3$ and a reduced curve $B$ intersecting $E$ at one point $p$ and the fiber of $\mathbb{F}_3 \to \mathbb{P}^1$ through $p$ at one point with multiplicity two;

(ii) the surface has four $(-2)$-curves: $\mu^{-1}(E)$, $\mu^{-1}(p)$ and two smooth rational curves mapping to the fiber of $\mathbb{F}_3$ through $p$;

(iii) the Cox ring $R(X)$ is generated in the degrees given in Table 6, in particular it has at least 8 generators whose degrees are either classes of $(-2)$-curves or the classes $\BNef[i]$, for $i = 1, 2, 3, 5$, in the Hilbert basis of the nef cone.

**Proof.** Let $f_1, \ldots, f_4$ be the classes of the $(-2)$-curves $E_1, \ldots, E_4$, where $f_3$ is the class of the section of the elliptic fibration defined by $\BNef[5]$, and $h = \BNef[3]$. Then $h^2 = 6$, $h \cdot f_i = 1$ for $i = 1, 4$ and $h \cdot f_i = 0$ for $i = 2, 3$. Since $h \cdot \BNef[5] = 2$ then the associated linear system is hyperelliptic by Proposition 1.3. Moreover, it is base point free by Theorem 1.1, since $h \not\equiv 4 \BNef[5] + f_3$. Let $\varphi$ be the associated morphism. Since $h = 3 \BNef[5] + 2f_3 + f_2$ where $\BNef[5] \cdot f_2 = 0$, $\BNef[5] \cdot f_3 = 1$ and $f_2 \cdot f_3 = 1$, then $h$ satisfies the hypothesis of [SD74, Proposition 5.7, ii)] thus $\varphi(X)$ is a cone over a rational normal twisted cubic in $\mathbb{P}^3$. Moreover, by [SD74, (5.9.2)] we have the description of $\varphi$ given in the statement. Observe that $E_3$ is mapped to the section $S$, $E_1$ and $E_4$ to the fiber of $\mathbb{F}_3 \to \mathbb{P}^1$ through $p$ and $E_2$ is contracted to the point $p$.

By Theorem 0.2 the Cox ring $R(X)$ is generated in the following degrees:

$$f_1, \ldots, f_4, h_1, h_2, h_3, h_5,$$

where $h_i := \BNef[i]$ for $i = 1, \ldots, 5$. Moreover, the set of such degrees is minimal, thus $R(X)$ has at least 8 generators.

**Corollary 2.19.** The moduli space of K3 surfaces $X$ with $\NS(X) \cong U \oplus A_2$ is unirational.

**Proof.** By item i) of Proposition 2.18 we know that $X$ is a double cover of $\mathbb{F}_3$ branched along a curve of type $S + B$, where $S^2 = -3$ and $B \in |3S + 10F|$ is smooth, where $F$ is a fiber of $\mathbb{F}_3 \to \mathbb{P}^1$. Moreover, $B$ is tangent at one point to the fiber $F_p$ of $\mathbb{F}_3 \to \mathbb{P}^1$ through $\{ p \} = B \cap S$. By the Riemann-Roch Theorem

$$\chi(B) = \frac{1}{2}B(B - K_{\mathbb{F}_3}) + 1 = 26.$$

By [Har77, Chapter V, Corollary 2.18] $B$ is very ample, thus $h^2(X, B) = 0$, and by Serre duality: $h^1(X, B) = h^1(X, K_{\mathbb{F}_3} - B)$. The divisor $B - K_{\mathbb{F}_3} \sim 5(S + 3F)$ is not ample by [Har77, Chapter V, Corollary 2.18], it contains an irreducible non-singular curve, in particular $B - K_{\mathbb{F}_3}$ is nef. Since $(B - K_{\mathbb{F}_3})^2 > 0$, by Mumford vanishing Theorem, we have $h^1(X, K_{\mathbb{F}_3} - B) = 0$. The linear system $|3S + 10F|$ is thus 25 dimensional. Since $B$ is very ample, imposing a tangency condition on $B$ at a point gives a codimension 1 space. The automorphism group of $\mathbb{F}_3$ is 8 dimensional, thus the quotient of the space of curves of type $S + B$ with $B \in |3S + 10F|$ which are tangent to the

![Intersection graph of (-2)-curves for $\mathcal{F}_8$](image-url)
fiber $F_p$ is a 16 dimensional unirational variety. Thus the moduli space of K3 surfaces $X$ with $\text{NS}(X) \simeq U \oplus A_2$ is unirational.

**The family $\mathcal{F}_9$.** Let $X$ be a K3 surface with $\text{NS}(X) \cong V_9 = U(2) \oplus A_2$. By Theorem 2.1 $X$ contains four classes of $(-2)$-curves, whose intersection matrix is given by Table 4 and whose intersection graph is given in Figure 5. The Hilbert basis of the nef cone contains seven classes, two of them defining elliptic fibrations $B\text{Nef}[6]$ and $B\text{Nef}[7]$ without sections and with one fiber of type $\tilde{A}_2$.

![Figure 5. Intersection graph of $(-2)$-curves of $\mathcal{F}_9$](image)

**Proposition 2.20.** Let $X$ be a K3 surface with $\text{NS}(X) \cong V_9 = U(2) \oplus A_2$. Then

(i) there is a minimal resolution $\varphi : X \to \overline{X}$ of a double cover $\pi : \overline{X} \to \mathbb{P}^2$ branched along a plane sextic $B$ having two nodes $p_1, p_2$ and such that the line $L$ through the nodes is tangent to $B$ at one point;

(ii) the surface has four $(-2)$-curves: two curves $R_1, R_2$ such that $\pi \varphi(R_i) = L$, $i = 1, 2$ and two curves $E_1, E_2$ with $\pi \varphi(E_i) = p_i$, $i = 1, 2$;

(iii) the Cox ring $R(X)$ is generated in the degrees given in Table 6, in particular has at least 10 generators whose degrees are either classes of $(-2)$-curves or the classes $B\text{Nef}[i]$, for $i = 1, 2, 3, 4, 6, 7$, in the Hilbert basis of the nef cone.

**Proof.** Let $f_1, \ldots, f_4$ be the classes of the $(-2)$-curves and $h = B\text{Nef}[5]$. We have that $h^2 = 2$, $h \cdot f_i = 1$ for $i = 1, 4$ and $h \cdot f_i = 0$ for $i = 2, 3$. By Corollary 1.2 the associated linear system is base point free and thus defines a degree two morphism $\pi : X \to \mathbb{P}^2$ which contracts the $(-2)$-curves of classes $f_2, f_3$. Since $f_2 \cdot f_3 = 0$, the branch locus of $\pi$ is a plane sextic $B$ with two nodes at $p, q \in \mathbb{P}^2$. Moreover $f_1 + f_2 + f_3 + f_4 = h$ and $f_1 \cdot f_4 = 1$. Thus the $(-2)$-curves of classes $f_1, f_4$ are mapped to a line $L$ passing through $p, q$ and tangent to $B$ at one more point. The last item follows from Theorem 0.2. \hfill $\square$

**Proposition 2.21.** The moduli space of K3 surfaces $X$ with $\text{NS}(X) \cong U(2) \oplus A_2$ is unirational.

**Proof.** By item i) of Proposition 2.20 $X$ is the minimal resolution of a double cover of $\mathbb{P}^2$ branched along a plane sextic $B$ with two nodes $p_1, p_2$ such that the line $L$ through $p_1, p_2$ is tangent to $B$ at one more point $q$. Up to projectivities we can assume that $p_1 = (1, 0, 0)$, $p_2 = (0, 1, 0)$ and $q = (-1, 1, 0)$. Using explicit computations, we get that the projective linear system of sextics with nodes at $p_1, p_2$ and tangent to the line $L$ at $q$ is 19-dimensional. Modding out by the automorphisms of $\mathbb{P}^2$ preserving $p_1, p_2, q$, which are of the form

$$
\begin{pmatrix}
1 & 0 & * \\
0 & 1 & * \\
0 & 0 & *
\end{pmatrix},
$$

we obtain an unirational space of dimension 16. Since the dimension of the moduli space of the K3 surfaces with $\text{NS}(X) \cong U(2) \oplus A_2$ is also 16, the claims follows. \hfill $\square$
The family $\mathcal{F}_{10}$. Let $X$ be a K3 surface with $\text{NS}(X) \cong V_{10} = U(3) \oplus A_2$. By Theorem 2.1 $X$ contains four $(-2)$-curves whose intersection matrix is given in Table 4 and whose intersection graph is described in Figure 5. The Hilbert basis of the nef cone of $X$ contains five classes. The classes $\text{BNef}[2], \text{BNef}[3], \text{BNef}[4]$ and $\text{BNef}[5]$ define elliptic fibrations without sections and with one fiber of type $\hat{A}_2$.

![Figure 6](image)

**Figure 6.** Intersection graph of $(-2)$-curves of $\mathcal{F}_{10}$

Proposition 2.22. Let $X$ be a K3 surface with $\text{NS}(X) \cong V_{10} = U(3) \oplus A_2$. Then

(i) $X$ is isomorphic to a smooth quartic surface in $\mathbb{P}^3$ having one hyperplane section which is the union of four lines;

(ii) a general $X$ can be defined by an equation of the form

$$F_0(x_0, \ldots, x_3)G(x_0, \ldots, x_3) + F_1(x_0, \ldots, x_3)F_2(x_0, \ldots, x_3)F_3(x_0, \ldots, x_3)F_4(x_0, \ldots, x_3) = 0,$$

where $F_i \in \mathbb{C}[x_0, \ldots, x_3]$ are homogeneous of degree one for $i = 0, \ldots, 4$ and $G \in \mathbb{C}[x_0, \ldots, x_3]$ is homogeneous of degree three;

(iii) the surface has four $(-2)$-curves: the four lines;

(iv) the Cox ring of $X$ is generated by $s_1, \ldots, s_8$, where $s_1, \ldots, s_4$ define the $(-2)$-curves and $s_5, \ldots, s_8$ define each a smooth fiber of one of the elliptic fibrations of $X$;

(v) for a very general $X$ as before we have an isomorphism

$$\mathbb{C}[T_1, \ldots, T_8]/I \rightarrow R(X), T_i \mapsto s_i,$$

where the degrees of the generators $T_i$ for $i = 1, \ldots, 8$ are given by the columns of the following matrix

$$
\begin{pmatrix}
0 & 0 & -1 & 0 & -1 & -1 & -1 & 0 \\
0 & -1 & 0 & 0 & -1 & -1 & 0 & -1 \\
0 & 1 & 1 & -1 & 1 & 2 & 0 & 0 \\
-1 & 1 & 1 & 0 & 2 & 1 & 0 & 0
\end{pmatrix}
$$

and the ideal $I$ is generated by the following polynomials:

$$T_1T_2T_3T_4 - F_0(T_1T_5, T_3T_6, T_2T_7, T_3T_8),$$

$$T_5T_6T_7T_8 - \tilde{G}(T_1T_5, T_4T_6, T_2T_7, T_3T_8),$$

where $\tilde{F}_0 = F \circ \varphi^{-1}$ and $\tilde{G} = G \circ \varphi^{-1}$ are obtained from $F_0, G$ in item ii) respectively composing with the coordinate change in $\mathbb{P}^3$ given by

$$\varphi(x_0, x_1, x_2, x_3) = (F_1, F_2, F_3, F_4).$$

Proof. Let $f_1, \ldots, f_4$ be the classes of the $(-2)$-curves and let $h = \text{BNef}[1]$. Then $h^2 = 4$ and $h \cdot f_i = 1$ for all $i$. Thus $h$ is ample and is non-hyperelliptic by Proposition 1.3. By Corollary 1.2 the linear system associated to $h$ is base point free, thus it defines an embedding of $X$ in $\mathbb{P}^3$ as a smooth quartic surface. Observe that $h = f_1 + f_2 + f_3 + f_4$ with $h \cdot f_i = 1$ for all $i$. This means that $X$ has one hyperplane section which is the union of four lines.
By Theorem 0.2 the Cox ring \( R(X) \) is generated in the following degrees:
\[
f_1, \ldots, f_4, \ h, \ e_1, \ e_2, \ e_3, \ e_4,
\]
where \( e_1, e_2, e_3, e_4 \) define the four elliptic fibrations of \( X \). Clearly any minimal generating set of \( R(X) \) must contain the sections \( s_1, \ldots, s_4 \) defining the \((-2)\)-curves and generators \( s_5, \ldots, s_8 \) defining smooth fibers of the elliptic fibrations. Moreover, observe that
\[
h = e_1 + f_1 = e_2 + f_4 = e_3 + f_2 = e_4 + f_3.
\]

An argument similar to the one in the proof of Proposition 2.16 shows that \( X \) can be defined by an equation of the form \( F_0G + F_1F_2F_3F_4 = 0 \) in \( \mathbb{P}^3 \), where \( F_i \) are homogeneous of degree one for \( i = 0, \ldots, 4 \) and \( G \) of degree 3. Since \( F_1, \ldots, F_4 \) can be chosen to be independent, then \( s_1s_5, s_4s_6, s_2s_7, s_3s_8 \) are a basis of \( H^0(h) \). Thus a generator in degree \( h \) is not necessary. The first relation is due to the fact that the hyperplane section \( F_0 = 0 \) is the union of the four lines whose defining sections are \( T_i \) for \( i = 1, 2, 3, 4 \). The second relation is due to the fact that the four elliptic curves defined by \( s_i, i = 5, \ldots, 8 \), are cut out by the cubic \( G = 0 \).

It can be proved with the same type of argument used in the proof of [ACDL21, Theorem 3.5] that the ideal \( I \) is prime for general \( G, F_0 \). Since \( \mathbb{C}[T_1, \ldots, T_8]/I \) is an integral domain of dimension \( \dim R(X) = \dim(X) + \text{rank } \text{Cl}(X) = 6 \) which surjects onto \( R(X) \), then it is isomorphic to \( R(X) \).

The proof of the following result is similar to that of Corollary 2.17.

**Corollary 2.23.** The moduli space of K3 surfaces \( X \) with \( \text{NS}(X) \simeq U(3) \oplus A_2 \) is unirational.

**Example 2.24.** Consider the following quartic in \( \mathbb{P}^3 \):
\[
X : x_0x_1x_2x_3 - (x_0 + x_1 + x_3)(x_0^3 + x_0^2x_1 + x_1^3 + x_1x_2^2 + x_2^3 + x_0x_1x_3 + x_2x_3^2) = 0.
\]
This is a smooth surface, with good reduction \( X_2 \) at prime 2. Using the Tate and Artin-Tate conjectures, one finds that \( X_2 \) has Picard number 4 and
\[
|\text{Br}(X_2)| \cdot |\text{disc}(\text{NS}(X_2))| = 3^3.
\]
Since no K3 surface \( Y \) with \( \text{NS}(Y) \simeq U \oplus A_2 \) can be embedded as a quartic in \( \mathbb{P}^3 \) (this can be checked directly looking at the self-intersections of the elements in Hilbert basis of the nef cone of the family \( F_8 \)), we have \( \text{NS}(X_2) \simeq U(2) \oplus A_2 \). Since \( X \) has Picard number at least 4, we conclude that \( \text{NS}(X) \simeq U(3) \oplus A_2 \).

**The family** \( F_{11} \). Let \( X \) be a K3 surface with \( \text{NS}(X) \cong V_{11} = U(6) \oplus A_2 \). By Theorem 2.1 \( X \) contains six \((-2)\)-curves whose intersection matrix is given in Table 4. The Hilbert basis of the nef cone of \( X \) contains 27 classes, eight of them defining elliptic fibrations: \( \text{BNe}(i) \) with \( i = 4, 7, 13, 16, 18, 21, 26, 27 \) without sections and with one fiber of type \( A_2 \).

**Proposition 2.25.** Let \( X \) be a K3 surface with \( \text{NS}(X) \cong V_{11} = U(6) \oplus A_2 \). Then
\begin{itemize}
  \item[(i)] \( X \) is isomorphic to a smooth quartic surface in \( \mathbb{P}^3 \) having three reducible hyperplane sections which are the union of two conics;
  \item[(ii)] \( X \) contains six \((-2)\)-curves: the six conics;
  \item[(iii)] the Cox ring of \( X \) is generated in the degrees given in Table 6, in particular it has at least 20 generators.
\end{itemize}

**Proof.** Let \( f_1, \ldots, f_6 \) be the classes of the \((-2)\)-curves and \( h = \text{BNe}[25] \). Then \( h^2 = 4, h \cdot f_i = 2 \) for all \( i \) and by Proposition 1.3 it is non-hyperelliptic. By Corollary 1.2 the associated linear system is base point free. thus it defines an embedding of \( X \) in \( \mathbb{P}^3 \) as a smooth quartic surface. Observe
that \( h = f_1 + f_5 = f_2 + f_4 = f_3 + f_6 \). This means that \( X \) has three reducible hyperplane sections which are the union of two conics.

By Theorem 0.2 the Cox ring \( R(X) \) is generated in the following degrees:

\[
\begin{align*}
    f_1, \ldots, f_6, & \ h_1, \ldots, h_{14}, \ h^*, \ h^*_1, \ldots, h^*_{12}
\end{align*}
\]

where \( h_i \) and \( h_i^* \) are classes in the Hilbert basis of the nef cone such that

\[
\begin{align*}
    h_i \in \{ \text{BNef}[j] : j = 4, 7, 9, 11, 13, 15, 16, 18, 20, 21, 23, 24, 26, 27 \}
\end{align*}
\]

\[
\begin{align*}
    h^*_i \in \{ \text{BNef}[j] : j = 1 - 3, 5, 6, 8, 10, 12, 14, 17, 19, 22, 25 \}.
\end{align*}
\]

By Proposition 1.3, the classes \( h_i \) and \( h_i^* \) are non-hyperelliptic for all \( i \), and if \( v := \text{BNef}[i] \) we have

\[
\begin{align*}
    v^2 &= 0 \text{ for } i = 4, 7, 13, 16, 18, 21, 26, 27 \\
    v^2 &= 6 \text{ for } i = 9, 11, 15, 20, 23, 24 \\
    v^2 &= 10 \text{ for } i = 1, 2, 3, 5, 6, 8, 10, 12, 14, 17, 19, 22.
\end{align*}
\]

By the minimality test (Proposition 1.12) the degrees not marked with a star are necessary to generate \( R(X) \). Thus \( R(X) \) has at least 20 generators. □

Let \( X \) be a K3 surface which has 3 hyperplane sections \( P_k \), \( k \in \{1, 2, 3\} \), each of which being union of two conics \( C_{2k-1}, C_{2k} \). Let us denote by \( L_k \) the line which is intersection of the planes \( P_i, P_j \), for \( \{k, i, j\} = \{1, 2, 3\} \). From the intersection matrix, we see that on each line \( L_k \) there are 4 points \( p_{4k-3}, p_{4k-2}, p_{4k-1}, p_{4k} \) such that each point is the intersection of two conics. Moreover, one can label the conics and the 12 points, so that

\[
\begin{align*}
    \{p_1, p_2, p_{11}, p_{12}\} & \subset C_1, \quad \{p_2, p_3, p_7, p_8\} \subset C_3, \quad \{p_5, p_8, p_{10}, p_{12}\} \subset C_5 \\
    \{p_3, p_4, p_9, p_{10}\} & \subset C_2, \quad \{p_1, p_3, p_5, p_6\} \subset C_4, \quad \{p_5, p_7, p_9, p_{11}\} \subset C_6.
\end{align*}
\]

The situation is quite symmetric since for any choice \( C \in \{C_1, C_2\}, C' \in \{C_3, C_4\}, C'' \in \{C_5, C_6\}, \) there are exactly 9 points among the 12 points that are on the union of \( C, C', C'' \). Moreover for any such choice and any line \( L_j \) there are exactly 3 points among the 4 points \( p_k \) on \( L_j \) which are on the union of \( C, C', C'' \).

Conversely, in order to construct a K3 surface \( X \) with the same properties, let us consider three planes \( P_1, P_2, P_3 \) and the lines \( L_k = P_i \cap P_j \). Let us fix on line \( L_1 \) the points \( p_1, p_2, p_4 \), on \( L_2 \) the points \( p_6, p_7, p_8 \), on \( L_3 \) the points \( p_{10}, p_{11}, p_{12} \). For a set \( \{a, b, c, d\} \) among

\[
\{1, 2, 11, 12\}, \quad \{2, 4, 7, 8\}, \quad \{6, 8, 10, 12\},
\]

the (projective) linear system of quadrics passing through points \( p_a, p_b, p_c, p_d \) is 5 dimensional. Taking such a quadric \( Q_k \) and intersecting it with the hyperplane section \( P_k \) containing points \( p_a, p_b, p_c, p_d \) gives a conic \( C_{2k-1} \) containing the points \( p_a, p_b, p_c, p_d \). That conic is irreducible if we choose \( Q_k \) generic. The intersections of the conic \( C_{2k-1} \) with the lines \( L_i, L_j \) \( \{i, j, k\} = \{1, 2, 3\} \) gives the remaining points \( p_3, p_5, p_9 \). The projective linear system of quartics containing the three conics \( C_1, C_3, C_5 \) we constructed is 10 dimensional. By choosing a generic quartic, we obtain a smooth K3 surface \( X \) containing conics \( C_1, C_3, C_5 \). We denote by \( C_2, C_4, C_6 \) the conics contained in \( X \) which are residual to \( C_1, C_3, C_5 \). These conics are smooth since we supposed \( X \) is generic. The 6 conics \( C_k \) and the points \( p_1, \ldots, p_{12} \) have the incidence relation we described before, thus the intersection matrix of the curves \( C_1, \ldots, C_6 \) is the same as for the surfaces with Néron-Severi group isomorphic to \( U(6) \oplus A_2 \). From that construction, it is clear that:

**Corollary 2.26.** The moduli of K3 surfaces with \( \text{NS}(X) \simeq U(6) \oplus A_2 \) is unirational.
The family $\mathcal{F}_{12}$. Let $X$ be a K3 surface with

$$NS(X) \cong V_{12} = \begin{bmatrix} 0 & -3 \\ -3 & 2 \end{bmatrix} \oplus A_2.$$  

By Theorem 2.1 $X$ contains six classes of $(-2)$-curves whose intersection matrix is given in Table 4 and described in Figure 7. The Hilbert basis of the nef cone of $X$ contains 33 classes, two of them defining elliptic fibrations ($\text{BNef}[30]$ and $\text{BNef}[31]$) without sections and with one fiber of type $A_2$.

![Figure 7. Intersection graph of $(-2)$-curves of $\mathcal{F}_{12}$](image)

**Proposition 2.27.** Let $X$ be a K3 surface with $NS(X) \cong V_{12}$. Then

(i) there is a double cover $\pi : X \to \mathbb{P}^2$ branched along a smooth plane sextic with three 3-tangent lines $L_1, L_2, L_3$;

(ii) $X$ can be defined by an equation of the following form in $\mathbb{P}(1,1,1,3)$:

$$x_3^2 = F_1(x_0, x_1, x_2)F_2(x_0, x_1, x_2)F_3(x_0, x_1, x_2)G_1(x_0, x_1, x_2) + G_2(x_0, x_1, x_2)^2,$$

where $F_i \in \mathbb{C}[x_0, x_1, x_2]$ are homogeneous of degree one for $i = 1, 2, 3$ and $G_1, G_2 \in \mathbb{C}[x_0, x_1, x_2]$ are homogeneous of degree three;

(iii) the surface has six $(-2)$-curves: the curves $R_{ij}, i = 1, 2, 3, j = 1, 2$, such that $\pi(R_{i1}) = \pi(R_{i2}) = L_i$;

(iv) a minimal generating set of the Cox ring of $X$ is $s_1, \ldots, s_8$, where: $s_1, \ldots, s_6$ define the $(-2)$-curves and $s_7, s_8$ define smooth fibers of the two elliptic fibrations of $X$;

(v) for a very general $X$ as before we have an isomorphism

$$\mathbb{C}[T_1, \ldots, T_8]/I \to R(X), \ T_i \mapsto s_i,$$

where the degrees of the generators $T_i$ for $i = 1, \ldots, 8$ are given by the columns of the following matrix

$$\begin{pmatrix}
-1 & 0 & -1 & 0 & 0 & -1 & -2 & -1 \\
-2 & 0 & -3 & 1 & 0 & -2 & -3 & -3 \\
1 & 0 & 1 & 0 & -1 & 2 & 3 & 0 \\
2 & -1 & 1 & 0 & 0 & 1 & 3 & 0
\end{pmatrix}$$

and the ideal $I$ is generated by the following polynomials:

$$T_7T_8 - \tilde{G}_1(T_1T_2, T_3T_4, T_5T_6),$$

$$T_1T_3T_5T_7 + T_2T_4T_6T_8 - \tilde{G}_2(T_1T_2, T_3T_4, T_5T_6),$$

where $\tilde{G}_i = G \circ \varphi^{-1}, i = 1, 2$ are obtained from $G_1, G_2$ in item ii) respectively composing with the coordinate change in $\mathbb{P}^2$ given by

$$\varphi(x_0, x_1, x_2) = (F_1, F_2, F_3).$$
Proof. Let $f_1, \ldots, f_6$ be the classes of the $(-2)$-curves and $h = \text{BNeF}[33]$. Then $h^2 = 2$, $h \cdot f_i = 1$ for all $i$. Thus $h$ is ample. By Corollary 1.2 the associated linear system is base point free, thus it defines a double cover $\pi : X \to \mathbb{P}^2$ branched along a smooth plane sextic $B$. Since $h = f_1 + f_2 = f_3 + f_4 = f_5 + f_6$, the image by $\pi$ of the six $(-2)$-curves of $X$ are three lines $L_1, L_2, L_3 \subseteq \mathbb{P}^2$ such that $\pi^{-1}(L_i)$ is the union of two smooth rational curves for each $i = 1, \ldots, 3$. This implies that the lines $L_i$ are 3-tangent to $B$.

By Theorem 0.2 the Cox ring $R(X)$ is generated in the following degrees:

$$f_1, \ldots, f_6, \ h^*, \ e_1, e_2$$

where $e_1$ and $e_2$ are the two elliptic fibrations of $X$. Clearly any minimal generating set of $R(X)$ must contain the sections $s_1, \ldots, s_6$ defining the $(-2)$-curves of $X$ and one section defining a smooth fiber for each elliptic fibration.

It can be proved as in the proof of Proposition 2.13 that $X$ is a double cover of $\mathbb{P}^2$ branched along a smooth plane sextic $B$ defined by an equation of the form $F_1 F_2 F_3 G_1 + G_2^2 = 0$, where $F_1, F_2, F_3 \in \mathbb{C}[x_0, x_1, x_2]$ are homogeneous of degree one and $G_1, G_2 \in \mathbb{C}[x_0, x_1, x_2]$ of degree 3. Since $F_1, F_2, F_3$ can be chosen to be independent, then $s_1 s_2, s_3 s_4, s_5 s_6$ give a basis of $H^0(h)$. Thus a generator of $R(X)$ in degree $h$ is not necessary.

The first element in $I$ is due to the fact that the preimage by $\pi$ of the plane cubic $G_1 = 0$ is the union of two smooth elliptic curves which are fibers of the two distinct elliptic fibrations of $X$ (in fact $\text{BNeF}[30] + \text{BNeF}[31] = 3h$). The last relation follows from the fact that

$$(x_3 + G_2)(x_3 - G_2) = F_1 F_2 F_3 G_1,$$

thus up to renumbering we can assume $x_3 + G_2 = s_1 s_3 s_5 s_7$, $x_3 - G_2 = s_2 s_4 s_6 s_8$, so that $2G_2 = s_1 s_3 s_5 s_7 - s_2 s_4 s_6 s_8$. Up to rescaling the generators $s_i$ we obtain the last relation.

It can be proved with the same type of argument used in the proof of [ACDL21, Theorem 3.5] that the ideal $I$ is prime for general $G_1, G_2$. Thus $\mathbb{C}[T_1, \ldots, T_8]/I \cong R(X)$, since it is an integral domain of dimension $\dim R(X) = \dim(X) + \text{rank} \text{Cl}(X) = 6$. $\square$

The proof of the following result is similar to the one of Corollary 2.14.

**Corollary 2.28.** The moduli space of K3 surfaces with $\text{NS}(X) \cong V_{12}$ is unirational.

**The family $\mathcal{F}_{13}$.** Let $X$ be a K3 surface with

$$\text{NS}(X) \cong V_{13} = \begin{bmatrix}
2 & -1 & -1 & -1 \\
-1 & -2 & 0 & 0 \\
-1 & 0 & -2 & 0 \\
-1 & 0 & 0 & -2
\end{bmatrix}.$$

By Theorem 2.1 $X$ contains six $(-2)$-curves whose intersection matrix is given in Table 4 and described in Figure 8. The Hilbert basis of the nef cone of $X$ contains 39 classes of positive self-intersection. Thus $X$ has no elliptic fibrations.
Proposition 2.29. Let $X$ be a K3 surface with $\text{NS}(X) \cong V_{13}$. Then

(i) there is a double cover $\pi : X \to \mathbb{P}^2$ branched along a smooth plane sextic with three 3-tangent lines $L_1$, $L_2$ and $L_3$;

(ii) the surface has six $(-2)$-curves: the six curves $R_{ij}$, $i = 1, 2, 3$, $j = 1, 2$ such that $\pi(R_{11}) = \pi(R_{12}) = L_i$;

(iii) the Cox ring is generated in the degrees given in Table 6, in particular has at least 24 generators.

Proof. Let $f_1, \ldots, f_6$ be the classes of the $(-2)$-curves and $h = \text{BNef}[1]$. Then $h^2 = 2$ and $h \cdot f_i = 1$ for all $i$. Thus $h$ is ample and the associated linear system is base point free by Corollary 1.2. Thus it defines a double cover $\pi : X \to \mathbb{P}^2$ branched along a smooth plane sextic $B$. Since $h = f_3 + f_5 = f_1 + f_2 = f_4 + f_6$, the image by $\pi$ of the six $(-2)$-curves of $X$ are three lines $L_1, L_2, L_3 \subseteq \mathbb{P}^2$ such that $\pi^{-1}(L_i)$ is the union of two smooth rational curves for each $i = 1, 2, 3$. This implies that $L_1, L_2, L_3$ are 3-tangent to $B$.

By Theorem 0.2 the Cox ring $R(X)$ is generated in the following degrees:

$$f_1, \ldots, f_6, h^*, h_1, h_2, \ldots, h_{18},$$

where $h_1, h_2, \ldots, h_{18}$ are classes in the Hilbert basis of the nef cone, non-hyperelliptic, with self-intersection 4 (six of them), 26 (six of them) and 28 (six of them). By the minimality test (Proposition 1.12) $R(X)$ has a generator in all the above degrees, except possibly for $h$. $\square$

Remark 2.30. In [Rou20] the author proved that a K3 surface which is the double cover of the plane branched over a generic sextic which has 3 tritangent lines has Néron-Severi group as above, or isomorphic to the lattice $V_{12}$. So one can see how differently the configurations of the 3 tritangent lines lift to the K3 surface.

The family $F_{14}$. Let $X$ be a K3 surface with

$$\text{NS}(X) \cong V_{14} = \begin{bmatrix} 12 & -2 & 0 & 0 \\ -2 & -2 & -1 & 0 \\ 0 & -1 & -2 & -1 \\ 0 & 0 & -1 & -2 \end{bmatrix}.$$ By Theorem 2.1 $X$ contains eight classes of $(-2)$-curves whose intersection matrix is given in Table 4. The Hilbert basis of the nef cone of $X$ contains 111 classes, none of them defining elliptic fibrations.

Proposition 2.31. Let $X$ be a K3 surface with $\text{NS}(X) \cong V_{14}$. Then
(i) there is a double cover \( \pi : X \to \mathbb{P}^2 \) branched along a smooth plane sextic with one 3-tangent line \( L \) and three 6-tangent conics \( C_1, C_2, \) and \( C_3; \)
(ii) the surface has eight \((-2)\)-curves: the curves \( R_{ij}, i = 1, 2, 3 \) such that \( \pi(R_{ij}) = \pi(R_{i2}) = C_i \) and the curves \( S_1, S_2 \) such that \( \pi(S_1) = \pi(S_2) = L; \)
(iii) the Cox ring of \( X \) has at least 71 generators whose degrees are either classes of \((-2)\)-curves or elements of the Hilbert basis of the nef cone.

\textbf{Proof.} Let \( f_1, \ldots, f_8 \) be the classes of the \((-2)\)-curves and \( h = \text{BNeff}[8] \). Then \( h^2 = 2, h \cdot f_i = 2 \) for \( i = 1, \ldots, 5, 8 \) and \( h \cdot f_i = 1 \) for \( i = 6, 7 \). By Corollary 1.2 the associated linear system is base point free, thus defines a double cover of \( \mathbb{P}^2 \) branched along a smooth plane sextic \( B \). Since \( h = f_6 + f_7 \), and \( 2h = f_4 + f_5 = f_2 + f_3 = f_1 + f_3 \), the image by \( \pi \) of the eight \((-2)\)-curves of \( X \) are three smooth conics \( C_1, C_2, C_3 \subseteq \mathbb{P}^2 \) and one line \( L \subseteq \mathbb{P}^2 \), such that \( \pi^{-1}(C_i), i = 1, 2, 3 \) and \( \pi^{-1}(L) \) is the union of two smooth rational curves. The last item follows from Theorem 0.2. \hfill \Box

3. Magma code

We mention here the Magma [BCP97] libraries which are used in the proof of Theorem 0.2. Section 3.3 of [ACDL21] contains a detailed the description of the functions in each library.

- \texttt{LSK3Lib.m}: to deal with linear systems on K3 surfaces,
- \texttt{Find-2.m}: to compute the set of \((-2)\)-curves of a Mori dream K3 surface,
- \texttt{TestLib.m}: contains functions based on the results in section 1.4,
- \texttt{MinimalLib.m}: contains functions which check the minimality of a generating set of \( R(X) \).

Moreover, we will add to the arXiv version of the present paper the following two files:

- \texttt{K3Rank4}: text file containing the intersection matrix and the list of classes of \((-2)\)-curves for all Mori dream K3 surfaces of Picard number 3
- \texttt{Gen(K3Rank4)}: text file containing the list of classes which pass all tests in \texttt{TestLib.m} (thus contains the degrees of a generating set of \( R(X) \)), for all Mori dream K3 surfaces of Picard number 4).

4. Tables

This section contains the tables with the relevant information about Mori dream K3 surfaces of Picard number four: Néron-Severi lattice, effective cone and its Hilbert basis, nef cone and its Hilbert basis. Moreover, we provide a nef and big divisor in the Hilbert basis of the nef cone with minimum self-intersection in each family of K3 surfaces and its intersection properties with \((-2)\)-curves. Finally, we will give the degrees of a set of generators of the Cox ring \( R(X) \).

We recall that in the tables we will adopt this notation: \( \text{NS}(X) \) denotes the Néron-Severi lattice of the surface, \( \text{Eff}(X) \) is the effective cone and \( \text{BEff}(X) \) is its Hilbert basis, \( \text{E}(X) \) is the set of generators of the extremal rays of the effective cone (i.e. the set of classes of the \((-2)\)-curves), \( \text{Nef}(X) \) is the nef cone and \( \text{BNeff}(X) \) is its Hilbert basis, \( \text{N}(X) \) is the set of generators of the extremal rays of \( \text{Nef}(X) \).

In Table 1 we give \( \text{E}(X), \text{BEff}(X), \text{N}(X) \) and \( \text{BNeff}(X) \), in Table 4 we give the intersection matrix of \((-2)\)-curves for each family of Mori dream K3 surfaces of Picard number four. In Tables 2 and 3 we give the Hilbert basis of the nef cone of \( X \) when the lattice \( \text{NS}(X) \) is isometric to \( V_1, V_2, V_{13} \) or \( V_{14} \).
For each family of Mori dream K3 surfaces of Picard number three, in Table 5 we give a nef and big class \( H \in \text{BNe}(X) \) of minimal self-intersection and its intersection properties with the \((−2)\)-curves.

In Table 6 we give the degrees of a set of generators of the Cox ring \( R(X) \) when \( \text{NS}(X) \) is not isometric to \( V_{14} \). All degrees in the Table 6 are necessary to generate \( R(X) \), except possibly for those marked with a star. In case \( \text{NS}(X) \cong V_{14} \) we give a subset of the degrees of a minimal generating set of \( R(X) \).
| $N^*$ | $\text{NS}(X)$ | $E(X)$ | $\text{BEff}(X)$ | $N(X)$ | $\text{BNe}(X)$ |
|---|---|---|---|---|---|
| 1 | $V_1$ | $(1, -1, 0, -2)$, $(1, -1, 2, 0)$, $(0, 0, -1, 0)$, $(2, -2, 2, -3)$, $(0, 1, 0, 0)$, $(2, -2, 3, -2)$, $(1, 0, 1, -2)$, $(1, -2, 0, -1)$, $(1, 0, 2, -1)$, $(1, -2, 1, 0)$, $(2, -3, 2, -2)$, $(0, 0, 0, 1)$ | $E(X)$ $\cup \{(1, -1, 1, -1)\}$ | $(1, -2, 0, 0)$, $(1, 0, 0, -2)$, $(1, 0, 0, 0)$, $(1, 0, 2, 0)$, $(3, -4, 0, -4)$, $(3, -4, 2, -4)$, $(3, -4, 4, -2)$, $(3, -4, 4, 0)$, $(3, -2, 4, -4)$, $(3, 0, 4, -4)$, $(5, -8, 4, -4)$, $(5, -4, 4, -8)$, $(5, -4, 8, -4)$, $(7, -8, 8, -8)$ | See Table 2 |
| 2 | $V_2$ | $(1, 1, 1, 0)$, $(0, 0, 0, 1)$, $(-1, 1, 1, 0)$, $(0, 0, -1, -1)$, $(0, 1, 1, -1)$, $(0, 1, 2, 1)$ | $E(X)$ | $(-3, 6, 4, -4)$, $(-3, 6, 8, 4)$, $(-1, 1, 0, 0)$, $(-1, 3, 4, 0)$, $(1, 1, 0, 0)$, $(1, 3, 4, 0)$, $(3, 6, 4, -4)$, $(3, 6, 8, 4)$ | See Table 2 |
| 3 | $V_3$ | $(1, 0, 2, 1)$, $(0, 0, 0, -1)$, $(1, 1, 2, 2)$, $(0, 1, 0, 0)$, $(0, -1, -1, 0)$ | $E(X)$ | $(1, 0, 0, 0)$, $(1, 1, 2, 1)$, $(2, -1, 2, 1)$, $(2, 1, 2, 3)$, $(3, 0, 4, 4)$ | $(1, 0, 0, 0), (1, 0, 1, 1), (1, 1, 2, 1), (2, -1, 2, 1), (2, 0, 2, 1), (2, 1, 2, 2), (2, 1, 2, 3), (3, 0, 4, 3), (3, 0, 4, 4), (3, 1, 4, 4)$ |
| 4 | $V_4$ | $(0, 0, 0, -1)$, $(-1, 0, 1, 0)$, $(1, -1, 0, 0)$, $(-1, 0, 0, 1)$, $(0, 0, -1, 0)$ | $E(X)$ | $(-2, -2, 0, 1)$, $(-2, -2, 2, 1)$, $(-2, -2, 1, 1)$, $(-1, -1, 0, 0)$, $(-1, 0, 0, 0)$ | $(-2, -2, 0, 1)$, $(-2, -2, 1, 0)$, $(-2, -2, 1, 1)$, $(-1, -1, 0, 0)$, $(-1, 0, 0, 0)$ |
| 5 | $V_5$ | $(0, -1, 0, 1)$, $(0, -1, 1, 0)$, $(0, 0, 0, -1)$, $(-1, 0, 1, 0)$, $(-1, 0, 0, 1)$, $(0, 0, -1, 0)$ | $E(X)$ | $(-1, -1, 0, 1)$, $(-1, -1, 1, 0)$, $(-1, -1, 1, 1)$, $(-1, 0, 0, 0)$, $(0, -1, 0)$ | $(-1, -1, 0, 1)$, $(-1, -1, 1, 0)$, $(-1, -1, 1, 1)$, $(-1, 0, 0, 0)$, $(0, -1, 0, 0)$ |
| $N$ | NS(X) | $E(X)$ | BEff(X) | $N(X)$ | BNEf(X) |
|-----|-------|-------|--------|--------|---------|
| 6   | $V_6$ | $(0,0,0,1), \quad (−2,−2,−3,−2), \quad (0,−1,0,−1), \quad (0,0,1,0), \quad (−1,0,0,−1), \quad (−2,−2,−2,−3), \quad (0,−1,−1,0), \quad (−1,0,−1,0)$ | $E(X)$ | $N(X)$ | BNEf(X) |
| 7   | $V_7$ | $(0,0,0,1), \quad (−1,−1,−1,−1), \quad (0,0,0,1), \quad (−1,0,0,1), \quad (0,−1,−1,0), \quad (−1,0,−1,0)$ | $E(X)$ | $N(X)$ | BNEf(X) |
| 8   | $V_8$ | $(0,0,0,1), \quad (−1,0,1,0), \quad (1,−1,0,0), \quad (0,0,−1,−1)$ | $E(X)$ | $N(X)$ | BNEf(X) |
| 9   | $V_9$ | $(0,0,0,1), \quad (0,−1,1,0), \quad (−1,0,1,0), \quad (0,0,−1,−1)$ | $E(X)$ | $N(X)$ | BNEf(X) |
| 10  | $V_{10}$ | $(0,0,0,−1), \quad (0,−1,1,1), \quad (−1,0,1,1), \quad (0,0,−1,0)$ | $E(X)$ | $N(X)$ | BNEf(X) |
| $N^o$ | NS($X$) | $E(X)$ | BEff($X$) | $N(X)$ | BNEf($X$) |
|-------|--------|--------|-----------|-------|-----------|
| 11    | $V_{11}$ | (0, 0, 0, -1), (0, -1, 1, 1), (-1, -1, 3, 2), (-1, 0, 1, 1), (-1, -1, 2, 3), (0, 0, -1, 0) | $E(X)$ | (-3, -2, 6, 6), (-2, -1, 2, 4), (-1, 0, 0, 0), (0, -1, 0, 0) | (-3, -3, 7, 7), (-3, -2, 5, 6), (-3, -2, 6, 5), (-3, -2, 6, 6), (-2, -3, 5, 6), (-2, -3, 6, 5), (-2, -2, 4, 5), (-2, -2, 5, 3), (-2, -2, 5, 4), (-2, -1, 2, 3), (-2, -1, 2, 4), (-2, -1, 3, 3), (-2, -1, 4, 2), (-1, -2, 2, 3), (-1, -2, 2, 4), (-1, -2, 3, 2), (-1, -2, 3, 3), (-1, -2, 4, 2), (-1, -1, 1, 1), (-1, -1, 1, 2), (-1, -1, 2, 1), (-1, -1, 2, 2), (-1, 0, 0, 0), (0, -1, 0, 0) |
| 12    | $V_{12}$ | (-1, -2, 1, 2), (0, 0, 0, -1), (-1, -3, 1, 1), (0, 1, 0, 0), (0, 0, -1, 0), (-1, -2, 2, 1) | $E(X)$ | (-7, -15, 9, 9), (-5, -12, 3, 6), (-5, -12, 6, 3), (-4, -7, 3, 6), (-4, -6, 3, 6), (-2, -3, 0, 0), (-1, -3, 0, 0) | (-7, -15, 9, 9), (-5, -12, 3, 6), (-5, -12, 6, 3), (-4, -7, 3, 6), (-4, -6, 3, 6), (-2, -3, 0, 0), (-1, -3, 0, 0) |
| No | NS(X)   | E(X)                                      | BEff(X)          | N(X)           | BNef(X)          |
|----|---------|-------------------------------------------|------------------|----------------|------------------|
| 13 | $V_{13}$| (0, 0, 0, −1), (1, 0, 0, 1),              | $E(X)$           | (2, −1, −1, −1),| See Table 2      |
|    |         | (0, 0, −1, 0), (1, 1, 0, 0),             |                  | (5, 1, 1, 1),  |                   |
|    |         | (1, 0, 1, 0), (0, −1, 0, 0)              |                  | (6, −3, −3, 4),|                   |
|    |         |                                           |                  | (6, −3, 4, −3),|                   |
|    |         |                                           |                  | (6, 4, −3, −3),|                   |
|    |         |                                           |                  | (8, −4, 3, 3),  |                   |
|    |         |                                           |                  | (8, 3, −4, 3),  |                   |
|    |         |                                           |                  | (8, 3, 3, −4)   |                   |
| 14 | $V_{14}$| (1, 0, −1, −2), (0, −1, 1, −1),          | $E(X)$           | (2, −3, 2, −1),| See Table 3      |
|    |         | (1, 2, −1, 0), (2, 2, −3, −2),           |                  | (7, −3, −8, −11),|                   |
|    |         | (0, 0, 1, 0), (0, 0, −1, 1),             |                  | (7, 12, −8, 4),|                   |
|    |         | (1, 1, 0, −2), (2, 3, −3, −1)            |                  | (8, 3, 8, −19),|                   |
|    |         |                                           |                  | (13, 3, −2, −29),|                   |
|    |         |                                           |                  | (13, 18, −2, −14),|                   |
|    |         |                                           |                  | (17, 12, −28, −16),|                  |
|    |         |                                           |                  | (17, 27, −28, −1),|                   |
|    |         |                                           |                  | (22, 27, −38, −11),|                  |
|    |         |                                           |                  | (23, 18, −22, −34),|                  |
|    |         |                                           |                  | (23, 33, −22, −19),|                  |
|    |         |                                           |                  | (28, 33, −32, −29)|                  |
Table 2. BNe(X) for NS(X) = V₁, V₂ and V₁₃.

| No | BNe(X) |
|----|--------|
| 1  | (1, −2, 0, 0), (1, −1, 0, −1), (1, −1, 0, 0), (1, −1, 1, −1), (1, −1, 1, 0), (1, 0, 0, −2), (1, 0, 0, −1), (1, 0, 0, 0), (1, 0, 1, −1), (1, 0, 1, 0), (1, 0, 2, 0), (2, −3, 0, −2), (2, −3, 1, −2), (2, −3, 2, −1), (2, −3, 2, 0), (2, −2, 0, −3), (2, −2, 1, −3), (2, −2, 3, −1), (2, −2, 3, 0), (2, −1, 2, −3), (2, −1, 3, −2), (2, 0, 2, −3), (2, 0, 3, −2), (3, −5, 2, −2), (3, −4, 0, −4), (3, −4, 1, −4), (3, −4, 2, −4), (3, −4, 3, −3), (3, −4, 4, −2), (3, −4, 4, −1), (3, −4, 4, 0), (3, −3, 3, −4), (3, −3, 4, −3), (3, −2, 2, −5), (3, −2, 4, −4), (3, −2, 5, −2), (3, −1, 4, −4), (3, 0, 4, −4), (4, −6, 3, −4), (4, −6, 4, −3), (4, −4, 3, −6), (4, −4, 6, −3), (4, −3, 4, −6), (4, −3, 6, −4), (5, −8, 4, −4), (5, −6, 5, −6), (5, −6, 6, −5), (5, −5, 6, −6), (5, −4, 4, −8), (5, −4, 8, −4), (7, −8, 8, −8) |
| 2  | (−3, 6, 4, −4), (−3, 6, 8, 4), (−2, 4, 3, −2), (−2, 4, 5, 2), (−1, 1, 0, 0), (−1, 2, 1, −1), (−1, 2, 2, 0), (−1, 2, 2, 1), (−1, 3, 2, −2), (−1, 3, 3, −1), (−1, 3, 4, 0), (−1, 3, 4, 1), (−1, 3, 4, 2), (0, 1, 0, 0), (0, 1, 1, 0), (0, 2, 1, −1), (0, 2, 2, 1), (0, 3, 2, −2), (0, 3, 3, −1), (0, 3, 4, 0), (0, 3, 4, 1), (0, 3, 4, 2), (1, 1, 0, 0), (1, 2, 1, −1), (1, 2, 2, 0), (1, 2, 2, 1), (1, 3, 2, −2), (1, 3, 3, −1), (1, 3, 4, 0), (1, 3, 4, 1), (1, 3, 4, 2), (2, 4, 3, −2), (2, 4, 5, 2), (3, 6, 4, −4), (3, 6, 8, 4) |
| 13 | (1, 0, 0, 0), (2, −1, −1, −1), (2, −1, −1, 0), (2, −1, 1, −1), (2, −1, 0, −1), (2, −1, 0, 0), (2, −1, 0, 1), (2, −1, 1, −1), (2, −1, 1, 0), (2, 0, −1, −1), (2, 0, −1, 0), (2, 0, −1, 1), (2, 0, 0, −1), (2, 0, 1, −1), (2, 1, −1, −1), (2, 1, −1, 0), (2, 1, 0, −1), (3, −1, 1, 1), (3, 0, 0, 1), (3, 0, 1, 0), (3, 1, −1, 1), (3, 1, 0, 0), (3, 1, 1, −1), (4, 0, 1, 1), (4, 1, 0, 1), (4, 1, 1, 0), (5, −2, −2, 3), (5, −2, 3, −2), (5, 1, 1, 1), (5, 3, −2, −2), (6, −3, −3, 4), (6, −3, 2, 2), (6, −3, 4, −3), (6, 2, −3, 2), (6, 2, −3, −3), (6, 4, −3, −3), (8, −4, 3, 3), (8, 3, −4, 3), (8, 3, 3, −4) |
Table 3. BNe(X) for NS(X) = V_{14}.

| NS  | BNe(X)                              |
|-----|-------------------------------------|
| 14  | (1, −1, 0, −1), (1, −1, 1, −1), (1, 0, −1, −1), (1, 0, 0, −2), (1, 0, 0, −1),
    | (1, 0, 0, 0), (1, 0, 1, −2), (1, 1, −1, −1), (1, 1, −1, 0), (1, 1, 0, −1),
    | (2, −3, 2, −1), (2, −2, 1, −1), (2, −1, −2, −3), (2, −1, −1, −3), (2, 0, −2, −3),
    | (2, 0, −1, −4), (2, 1, −3, −2), (2, 1, −2, −3), (2, 1, 0, −4), (2, 1, 1, −4),
    | (2, 2, −3, −1), (2, 2, −1, 0), (2, 2, 0, −3), (2, 3, −3, 0), (2, 3, −2, −1),
    | (2, 3, −2, 0), (2, 3, −2, 1), (2, 3, −1, −1), (3, −1, −3, −5), (3, 0, −4, −4),
    | (3, 1, −1, −6), (3, 1, 2, −7), (3, 1, 3, −7), (3, 2, −2, −5), (3, 2, 2, −6),
    | (3, 3, −5, −2), (3, 3, −4, −3), (3, 3, −2, −4), (3, 4, −5, −1), (3, 4, −4, −2),
    | (3, 4, −2, −3), (3, 4, −1, −3), (3, 5, −4, 1), (3, 5, −3, 1), (4, 1, 0, −9),
    | (4, 1, 1, −9), (4, 3, −6, −4), (4, 3, −5, −5), (4, 4, −4, −5), (4, 4, 1, −6),
    | (4, 5, −4, −4), (4, 5, 0, −5), (4, 6, −6, −1), (4, 6, −5, −2), (5, 1, −1, −11),
    | (5, 2, −7, −6), (5, 2, −2, −10), (5, 3, −8, −5), (5, 4, −5, −7), (5, 6, −8, −3),
    | (5, 6, −7, −4), (5, 7, −5, −4), (5, 7, −2, −5), (5, 7, −1, −5), (5, 8, −8, 0),
    | (5, 8, −7, 0), (6, −2, −7, −9), (6, 5, −10, −5), (6, 7, −7, −6), (6, 9, −10, −1),
    | (6, 10, −7, 3), (7, −3, −8, −11), (7, 3, 6, −16), (7, 4, −5, −12), (7, 5, −11, −7),
    | (7, 5, −6, −11), (7, 8, −12, −4), (7, 9, −12, −3), (7, 10, −6, −6), (7, 10, −5, −6),
    | (7, 11, −11, −1), (7, 12, −8, 4), (8, 3, 8, −19), (8, 7, −8, −11), (8, 8, −13, −6), (8, 11, −13, −3),
    | (8, 11, −8, −7), (9, 7, −9, −13), (9, 10, −10, −10), (9, 11, −15, −5), (9, 11, −10, −9),
    | (9, 13, −9, −7), (10, 10, −11, −12), (10, 13, −11, −9), (11, 3, −2, −24), (11, 13, −13, −11),
    | (11, 15, −2, −12), (13, 3, −2, −29), (13, 18, −2, −14), (14, 10, −23, −13), (14, 22, −23, −1),
    | (17, 12, −28, −16), (17, 27, −28, −1), (18, 22, −31, −9), (19, 15, −18, −28), (19, 27, −18, −16),
    | (22, 27, −38, −11), (23, 18, −22, −34), (23, 27, −26, −24), (23, 33, −22, −19), (28, 33, −32, −29)|
Table 4. Intersection matrix of \((-2)\)-curves.

| \(N^\circ\) | Lattice | intersection matrix of \((-2)\)-curves |
|---------|--------|----------------------------------|
| 1       | \(V_1\) | \[
\begin{bmatrix}
-2 & 6 & 0 & 0 & 2 & 4 & 0 & 0 & 4 & 4 & 2 & 4 \\
6 & -2 & 4 & 4 & 2 & 0 & 4 & 4 & 0 & 0 & 2 & 0 \\
0 & 4 & -2 & 4 & 0 & 6 & 2 & 0 & 4 & 2 & 4 & 0 \\
0 & 4 & 4 & -2 & 4 & 0 & 0 & 2 & 2 & 4 & 0 & 6 \\
2 & 2 & 0 & 4 & -2 & 4 & 0 & 4 & 0 & 4 & 6 & 0 \\
4 & 0 & 6 & 0 & 4 & -2 & 2 & 4 & 0 & 2 & 0 & 4 \\
0 & 4 & 2 & 0 & 0 & 2 & -2 & 4 & 0 & 6 & 4 & 4 \\
0 & 4 & 0 & 2 & 4 & 4 & 4 & -2 & 6 & 0 & 0 & 2 \\
4 & 0 & 4 & 2 & 0 & 0 & 0 & 6 & -2 & 4 & 2 & 4 \\
4 & 0 & 2 & 4 & 4 & 4 & 2 & 6 & 0 & 4 & -2 & 0 & 0 \\
2 & 2 & 4 & 0 & 6 & 0 & 4 & 4 & 0 & 4 & -2 & 4 \\
4 & 0 & 0 & 6 & 0 & 4 & 4 & 2 & 2 & 0 & 4 & -2
\end{bmatrix}
\] |
| 2       | \(V_2\) | \[
\begin{bmatrix}
-2 & 1 & 6 & 1 & 1 & 1 \\
1 & -2 & 1 & 1 & 3 & 0 \\
6 & 1 & -2 & 1 & 1 & 1 \\
1 & 1 & 1 & -2 & 0 & 3 \\
1 & 3 & 1 & 0 & -2 & 1 \\
1 & 0 & 1 & 3 & 1 & -2
\end{bmatrix}
\] |
| 3       | \(V_3\) | \[
\begin{bmatrix}
-2 & 0 & 0 & 2 & 1 \\
0 & -2 & 2 & 0 & 1 \\
0 & 2 & -2 & 0 & 1 \\
2 & 0 & 0 & -2 & 1 \\
1 & 1 & 1 & 1 & -2
\end{bmatrix}
\] |
| 4       | \(V_4\) | \[
\begin{bmatrix}
-2 & 0 & 0 & 2 & 0 \\
0 & -2 & 1 & 0 & 2 \\
0 & 1 & -2 & 1 & 0 \\
2 & 0 & 1 & -2 & 0 \\
0 & 2 & 0 & 0 & -2
\end{bmatrix}
\] |
| N° | Lattice | intersection matrix of (−2)-curves |
|----|---------|-----------------------------------|
| 5  | $V_5$   | $\begin{bmatrix} -2 & 0 & 2 & 2 & 0 & 0 \\ 0 & -2 & 0 & 0 & 2 & 2 \\ 2 & 0 & -2 & 0 & 2 & 0 \\ 2 & 0 & 0 & -2 & 0 & 2 \\ 0 & 2 & 2 & 0 & -2 & 0 \\ 0 & 2 & 0 & 2 & 0 & -2 \end{bmatrix}$ |
| 6  | $V_6$   | $\begin{bmatrix} -2 & 4 & 2 & 0 & 2 & 6 & 0 & 0 \\ 4 & -2 & 2 & 6 & 2 & 0 & 0 & 0 \\ 2 & 2 & -2 & 0 & 1 & 0 & 0 & 3 \\ 0 & 6 & 0 & -2 & 0 & 4 & 2 & 2 \\ 2 & 2 & 1 & 0 & -2 & 0 & 3 & 0 \\ 6 & 0 & 0 & 4 & 0 & -2 & 2 & 2 \\ 0 & 0 & 0 & 2 & 3 & 2 & -2 & 1 \\ 0 & 0 & 3 & 2 & 0 & 2 & 1 & -2 \end{bmatrix}$ |
| 7  | $V_7$   | $\begin{bmatrix} -2 & 2 & 2 & 0 & 2 & 0 & 2 & 0 \\ 2 & -2 & 2 & 4 & 0 & 2 & 0 & 0 \\ 2 & 2 & -2 & 0 & 4 & 2 & 0 & 0 \\ 0 & 4 & 0 & -2 & 2 & 0 & 2 & 2 \\ 0 & 0 & 4 & 2 & -2 & 0 & 2 & 2 \\ 2 & 2 & 2 & 0 & 0 & -2 & 4 & 0 \\ 0 & 0 & 0 & 2 & 2 & 4 & -2 & 2 \\ 4 & 0 & 0 & 2 & 2 & 0 & 0 & -2 \end{bmatrix}$ |
| 8  | $V_8$   | $\begin{bmatrix} -2 & 1 & 0 & 1 \\ 1 & -2 & 1 & 1 \\ 0 & 1 & -2 & 0 \\ 1 & 1 & 0 & -2 \end{bmatrix}$ |
| 9  | $V_9$   | $\begin{bmatrix} -2 & 1 & 1 & 1 \\ 1 & -2 & 0 & 1 \\ 1 & 0 & -2 & 1 \\ 1 & 1 & 1 & -2 \end{bmatrix}$ |
| $N^\circ$ | Lattice | Intersection matrix of $(-2)$-curves |
| --- | --- | --- |
| 10 | $V_{10}$ | $\begin{bmatrix} -2 & 1 & 1 & 1 \\ 1 & -2 & 1 & 1 \\ 1 & 1 & -2 & 1 \\ 1 & 1 & 1 & -2 \end{bmatrix}$ |
| 11 | $V_{11}$ | $\begin{bmatrix} -2 & 1 & 1 & 1 & 4 & 1 \\ 1 & -2 & 1 & 4 & 1 & 1 \\ 1 & 1 & -2 & 1 & 1 & 4 \\ 1 & 4 & 1 & -2 & 1 & 1 \\ 4 & 1 & 1 & 1 & -2 & 1 \\ 1 & 1 & 4 & 1 & 1 & -2 \end{bmatrix}$ |
| 12 | $V_{12}$ | $\begin{bmatrix} -2 & 3 & 0 & 1 & 0 & 1 \\ 3 & -2 & 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 3 & 1 & 0 \\ 1 & 0 & 3 & -2 & 0 & 1 \\ 0 & 1 & 1 & 0 & -2 & 3 \\ 1 & 0 & 0 & 1 & 3 & -2 \end{bmatrix}$ |
| 13 | $V_{13}$ | $\begin{bmatrix} -2 & 3 & 0 & 1 & 1 & 0 \\ 3 & -2 & 1 & 0 & 0 & 1 \\ 0 & 1 & -2 & 1 & 3 & 0 \\ 1 & 0 & 1 & -2 & 0 & 3 \\ 1 & 0 & 3 & 0 & -2 & 1 \\ 0 & 1 & 0 & 3 & 1 & -2 \end{bmatrix}$ |
| 14 | $V_{14}$ | $\begin{bmatrix} -2 & 0 & 6 & 0 & 4 & 1 & 1 & 4 \\ 0 & -2 & 4 & 4 & 0 & 1 & 1 & 6 \\ 6 & 4 & -2 & 4 & 0 & 1 & 1 & 0 \\ 0 & 4 & 4 & -2 & 6 & 1 & 1 & 0 \\ 4 & 0 & 0 & 6 & -2 & 1 & 1 & 4 \\ 1 & 1 & 1 & 1 & -2 & 3 & 1 \\ 1 & 1 & 1 & 1 & 3 & -2 & 1 \\ 4 & 6 & 0 & 0 & 4 & 1 & 1 & -2 \end{bmatrix}$ |
Table 5. Intersection of a nef and big divisor $H$ with $(-2)$-curves.

| $N^\circ$ | $\text{NS}(X)$ | $H$ | Intersection properties |
|-----------|-----------------|-----|-------------------------|
| 1         | $V_1$ $(1,1,1,-1)$ | $H^2 = 2$, $H \cdot E_i = 2$, $i = 1, \ldots, 12$ |
| 2         | $V_2$ $(0,1,1,0)$ | $H^2 = 2$, $H \cdot E_1 = H \cdot E_3 = 2$  $H \cdot E_i = 1$, $i = 2, 4, 5, 6$ |
| 3         | $V_3$ $(1,0,1,1)$ | $H^2 = 2$, $H \cdot E_5 = 0$  $H \cdot E_i = 1$, $i = 1, 2, 3, 4$ |
| 4         | $V_4$ $(-2,-2,1,1)$ | $H^2 = 4$, $H \cdot E_1 = H \cdot E_5 = 2$  $H \cdot E_i = 0$, $i = 2, 3, 4$ |
| 5         | $V_5$ $(-1,-1,0,1)$ | $H^2 = 2$, $H \cdot E_i = 0$, $i = 1, 5, 6$  $H \cdot E_i = 2$, $i = 2, 3, 4$ |
| 6         | $V_6$ $(-1,-1,-1,-1)$ | $H^2 = 2$, $H \cdot E_i = 2$, $i = 1, 2, 4, 6$  $H \cdot E_i = 1$, $i = 3, 5, 7, 8$ |
| 7         | $V_7$ $(-1,1,-1,1)$ | $H^2 = 4$, $H \cdot E_i = 2$, $i = 1, \ldots, 8$ |
| 8         | $V_8$ $(-2,-2,1,0)$ | $H^2 = 6$, $H \cdot E_1 = H \cdot E_4 = 1$  $H \cdot E_2 = H \cdot E_3 = 0$ |
| 9         | $V_9$ $(-1,1,1,0)$ | $H^2 = 2$, $H \cdot E_1 = H \cdot E_4 = 1$  $H \cdot E_2 = H \cdot E_3 = 0$ |
| 10        | $V_{10}$ $(-1,1,1,1)$ | $H^2 = 4$, $H \cdot E_i = 1$, $i = 1, 2, 3, 4$ |
| 11        | $V_{11}$ $(-1,1,2,2)$ | $H^2 = 4$, $H \cdot E_i = 2$, $i = 1, \ldots, 6$ |
| 12        | $V_{12}$ $(1,2,1,1)$ | $H^2 = 2$, $H \cdot E_i = 1$, $i = 1, \ldots, 6$ |
| 13        | $V_{13}$ $(1,0,0,0)$ | $H^2 = 2$, $H \cdot E_i = 1$, $i = 1, \ldots, 6$ |
| 14        | $V_{14}$ $(1,1,-1,-1)$ | $H^2 = 2$, $H \cdot E_i = 2$, $i = 1, 2, 3, 4, 5, 8$  $H \cdot E_6 = H \cdot E_7 = 1$ |
Table 6. Degrees of a set of generators of $R(X)$.

| No | NS(X) | Degrees of generators of $R(X)$ |
|----|-------|---------------------------------|
| 1  | $V_1$ | BEff                           |
| 2  | $V_2$ | $E, \text{BNef}[i], i = 1 - 5, 7, 11, 14, 15, 20, 23, 25, 29, 32 - 35$ |
| 3  | $V_3$ | $E, \text{BNef}[i], i = 1, 2, 4, 7, 9$ |
| 4  | $V_4$ | $E, \text{BNef}[3], \text{BNef}[3] + \text{BNef}[4]$ |
| 5  | $V_5$ | $E, \text{BNef}[1] + \text{BNef}[2]$ |
| 6  | $V_6$ | $E, \text{BNef}[15]$ |
| 7  | $V_7$ | $E$ |
| 8  | $V_8$ | $E, \text{BNef}[i], i = 1, 2, 3, 5$ |
| 9  | $V_9$ | $E, \text{BNef}[i], i = 1 - 4, 6, 7$ |
| 10 | $V_{10}$ | $E, \text{BNef}[i], i = 2 - 5$ |
| 11 | $V_{11}$ | $E, \text{BNef}[i], i = 4, 7, 9, 11, 13, 15, 16, 18, 20, 21, 23, 24, 26, 27$  
|    |       | $\text{BNef}[i^*], i = 1 - 3, 5, 6, 8, 10, 12, 14, 17, 19, 22, 25$ |
| 12 | $V_{12}$ | $E, \text{BNef}[30], \text{BNef}[31]$ |
| 13 | $V_{13}$ | $E, \text{BNef}[i], i = 1^*, 7, 9, 12, 14, 16, 17, 27, 28, 30 - 39$ |
| 14 | $V_{14}$ | Contains the degrees in Table 7 |
Table 7. Degrees of generators of the Cox ring of the family $\mathcal{F}_{14}$.

| $N^\circ$ | degrees of generators |
|-----------|-----------------------|
| 14        | (1, 0, −1, −2), (0, −1, 1, −1), (1, 2, −1, 0), (2, 2, −3, −2), (0, 0, 1, 0), (0, 0, −1, 1), (1, 1, 0, −2), (2, 3, −3, −1), (1, −1, 0, −1), (1, 0, −1, −1), (1, 0, 0, 0), (1, 1, −1, −1), (1, 1, −1, 0), (2, −3, 2, −1), (2, −2, 1, −1), (2, −1, −2, −3), (2, 1, −3, −2), (2, 1, 0, −4), (2, 2, −3, −1), (2, 2, 0, −3), (2, 3, −3, 0), (2, 3, −2, 1), (3, 0, −4, −4), (3, 1, 2, −7), (3, 2, −2, −5), (3, 2, 2, −6), (3, 3, −5, −2), (3, 3, −2, −4), (3, 4, −5, −1), (3, 4, −2, −3), (3, 5, −4, 1), (4, 1, 0, −9), (4, 4, −4, −5), (4, 5, −4, −4), (4, 5, 0, −5), (5, 2, −2, −10), (5, 3, −8, −5), (5, 7, −2, −5), (5, 8, −8, 0), (6, −2, −7, −9), (6, 5, −10, −5), (6, 9, −10, −1), (6, 10, −7, 3), (7, −3, −8, −11), (7, 3, 6, −16), (7, 5, −6, −11), (7, 8, −12, −4), (7, 9, −12, −3), (7, 10, −6, −6), (7, 12, −8, 4), (8, 3, 8, −19), (8, 7, −8, −11), (8, 11, −8, −7), (9, 10, −10, −10), (9, 11, −10, −9), (11, 3, −2, −24), (11, 15, −2, −12), (13, 3, −2, −29), (13, 18, −2, −14), (14, 10, −23, −13), (14, 22, −23, −1), (17, 12, −28, −16), (17, 27, −28, −1), (18, 22, −31, −9), (19, 15, −18, −28), (19, 27, −18, −16), (22, 27, −38, −11), (23, 18, −22, −34), (23, 27, −26, −24), (23, 33, −22, −19), (28, 33, −32, −29) }
References

[ACDL21] Michela Artebani, Claudia Correa Deisler, and Antonio Laface, Cox rings of K3 surfaces of Picard number three, J. Algebra 565 (2021), 598–626.

[ADHL15] Ivan Arzhantsev, Ulrich Derenthal, Jürgen Hausen, and Antonio Laface, Cox rings, Cambridge Studies in Advanced Mathematics, vol. 144, Cambridge University Press, Cambridge, 2015.

[AHL10] Michela Artebani, Jürgen Hausen, and Antonio Laface, On Cox rings of K3 surfaces, Compos. Math. 146 (2010), no. 4, 964–998.

[BCP97] Wieb Bosma, John Cannon, and Catherine Playoust, The Magma algebra system. I. The user language, J. Symbolic Comput. 24 (1997), no. 3-4, 235–265.

[Har77] R. Hartshorne, Algebraic geometry, Graduate Texts in Mathematics, vol. 52, Springer-Verlag, New York, 1977.

[MS00] Ezra Miller and Bernd Sturmfels, Combinatorial commutative algebra, Graduate Texts in Mathematics, vol. 227, Springer-Verlag, New York, 2000.

[Nik00] Viacheslav V. Nikulin, A remark on algebraic surfaces with polyhedral Mori cone, Nagoya Math. J. 157 (2000), 73–92.

[Nik79] Viacheslav V. Nikulin, Quotient-groups of groups of automorphisms of hyperbolic forms of subgroups generated by 2-reflections, Dokl. Akad. Nauk SSSR 248 (1979), no. 6, 1307–1309.

[Nik84] Viacheslav V. Nikulin, K3 surfaces with a finite group of automorphisms and a Picard group of rank three, Trudy Mat. Inst. Steklov. 165 (1984), 119–142.

[Ott13] John Christian Ottem, Cox rings of K3 surfaces with Picard number 2, J. Pure Appl. Algebra 217 (2013), no. 4, 709–715.

[PS71] I. I. Pjatecki˘ı-ˇSapiro and I. R. ˇSafarevi˘ı, Torelli’s theorem for algebraic surfaces of type K3, Izv. Akad. Nauk SSSR Ser. Mat. 35 (1971), 530–572.

[Vin75] È. B. Vinberg, Some arithmetical discrete groups in Lobačevski˘ı spaces, Discrete subgroups of Lie groups and applications to moduli (Internat. Colloq., Bombay, 1973), 1975, pp. 323–348. MR0422505

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