Character sheaves on unipotent groups in positive characteristic: foundations

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Dedicated to the memory of our friend Leonid Vaksman.

Published online: 11 July 2013
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Abstract In this article we formulate and prove the main theorems of the theory of character sheaves on unipotent groups over an algebraically closed field of characteristic $p > 0$. In particular, we show that every admissible pair for such a group $G$ gives rise to an $\mathbb{L}$-packet of character sheaves on $G$ and that conversely, every $\mathbb{L}$-packet of character sheaves on $G$ arises from a (nonunique) admissible pair. In the Appendices we discuss two abstract category theory patterns related to the study of character sheaves. The first Appendix sketches a theory of duality for monoidal categories, which generalizes the notion of a rigid monoidal category and is close in spirit to the Grothendieck–Verdier duality theory. In the second one we use a topological field theory approach to define the canonical braided monoidal structure and twist on the equivariant derived category of constructible sheaves on an algebraic group; moreover, we show that this category carries an action of the surface operad. The third Appendix proves that the “naive” definition of the equivariant $\ell$-adic derived category with respect to a unipotent algebraic group is equivalent to the “correct” one.

Keywords Unipotent group · Character sheaf · Equivariant derived category · Braided category · Fourier–Deligne transform · Little group method

Both authors were supported by the NSF grant DMS-0701106. M.B. was also supported by the NSF Postdoctoral Research Fellowship DMS-0703679 and by the NSF grant DMS-1001769. V.D. was also supported by the NSF grant DMS-1001660.

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Mathematics Subject Classification (2010) Primary 20G99; Secondary 20C15

1 Introduction

In a series of works beginning in the 1980s, George Lusztig developed a theory of character sheaves for reductive algebraic groups and explored its relation to the character theory of finite groups of Lie type. In 2003, he conjectured that there should also exist an interesting theory of character sheaves for unipotent groups in positive characteristic and calculated the first example of a nontrivial \( L \)-packet in this setting [42]. A general definition of an \( L \)-packet of character sheaves on a unipotent group over an algebraically closed field \( k \) of characteristic \( p > 0 \) was given in [14]. We also formulated there a list of conjectures related to this notion and discussed the orbit method for unipotent groups of nilpotence class \( < p \).

A brief introduction to the theory of character sheaves on unipotent groups was given in a joint talk by the authors, the slides for which are available online [15]. The present article contains the proofs of the results announced in Parts I and II of that talk (Part III was devoted to the relationship between character sheaves and characters of unipotent groups over finite fields, which is discussed in [13]).

Let us summarize the main features of the theory to which our work is devoted. Precise statements of the main results, as well as all the background definitions, are contained in Sect. 2, which can be viewed as an “extended introduction” to the paper.

Let \( G \) be a unipotent algebraic group over \( k \), and fix a prime \( \ell \neq p = \text{char } k \). Let \( \mathcal{D}_G(G) \) denote the \( G \)-equivariant derived category of constructible \( \mathbb{Q}_\ell \)-complexes on \( G \), where \( G \) acts on itself by conjugation. It is a braided monoidal category with respect to the functor of convolution with compact supports (Definition 2.7), which we denote by \( (M, N) \mapsto M \ast N \). Let us say that an object \( e \in \mathcal{D}_G(G) \) is a closed idempotent if there exists an arrow \( \mathbb{1} \to e \) that becomes an isomorphism after convolving with \( e \), where \( \mathbb{1} \) is the unit object in \( \mathcal{D}_G(G) \) (the delta-sheaf at the identity \( 1 \in G \)). The notion of a minimal closed idempotent is defined in the obvious way. If \( e \in \mathcal{D}_G(G) \) is a minimal closed idempotent, the \( L \)-packet of character sheaves corresponding to \( e \) is defined as the collection of objects \( M \in \mathcal{D}_G(G) \) such that \( e \ast M \simeq M \) and such that the underlying complex of \( M \) (obtained by discarding the \( G \)-equivariant structure) is an irreducible perverse sheaf on \( G \).

Some of the fundamental properties of character sheaves, proved in this article, are as follows. Every \( L \)-packet of character sheaves on \( G \) is finite. If \( M \) and \( N \) are two character sheaves on \( G \), then \( \text{Ext}^i_{\mathcal{D}_G(G)}(M, N) = 0 \) for all \( i > 0 \). Moreover, if \( M \) and \( N \) lie in the same \( L \)-packet defined by a minimal closed idempotent \( e \in \mathcal{D}_G(G) \), then \( M \ast N \) is perverse up to cohomological shift by an integer \( n_e \) determined only by \( e \) (if \( M \) and \( N \) lie in different \( L \)-packets, then \( M \ast N = 0 \)).

One of the ingredients in the proof is an explicit construction of minimal closed idempotents in \( \mathcal{D}_G(G) \), based on the notion of an admissible pair for a unipotent group (Sect. 2.10) and on the construction of the induction functor with compact supports \( \mathcal{D}_G(G') \to \mathcal{D}_G(G) \) for a closed subgroup \( G' \subset G \) (see Sect. 2.12).

These tools were used previously in [12] to develop a geometric approach to the study of characters of unipotent groups over finite fields. In particular, it was proved
there that every admissible pair for \( G \) gives rise to a minimal weak idempotent in \( D_G(G) \), where a weak idempotent is defined as an object \( e \in D_G(G) \) that satisfies \( e \ast e \cong e \). In the present article we complete the picture by showing that the classes of minimal closed idempotents and minimal weak idempotents in \( D_G(G) \) coincide (this result is parallel to the classical theorem that the coadjoint orbits of a unipotent group are closed). In addition, we prove that every minimal (weak or closed) idempotent in \( D_G(G) \) arises from an admissible pair for \( G \).

One of the motivations behind our work was to provide a foundation for the theory of character sheaves and characters on unipotent groups over finite fields, whose existence was conjectured by Lusztig. Suppose that \( k \) is an algebraic closure of a finite subfield \( \mathbb{F}_q \subset k \) and that \( G = G_0 \otimes_{\mathbb{F}_q} k \), where \( G_0 \) is a unipotent group over \( \mathbb{F}_q \). If \( M \) is a Frobenius-invariant character sheaf on \( G \), the corresponding trace function \( G_0(\mathbb{F}_q) \to \mathbb{Q}_\ell \) is invariant under conjugation. The relationship between these functions and the irreducible characters of the finite group \( G_0(\mathbb{F}_q) \) is studied in [13]. In the case when \( G \) is easy in the terminology of op. cit. (every point of \( G(k) \) is contained in the neutral connected component of its centralizer in \( G \)), the functions on \( G_0(\mathbb{F}_q) \) coming from Frobenius-invariant character sheaves on \( G \) coincide with the irreducible characters up to scaling. For the general case, we refer to Theorem 2.17 in op. cit. We remark that the present article relies on the results of [12] in a few places, while [13] depends significantly both on [12] and on the present article.

2 Main definitions and results

Most of this section is devoted to recalling several definitions and constructions that were discussed at length in [14] and/or [12]. In Sects. 2.1–2.3 we recall some facts about derived categories of constructible \( \ell \)-adic complexes, along with their equivariant versions. In Sects. 2.4–2.5 we introduce and discuss the monoidal categories \( D(G) \) and \( D_G(G) \) associated with any unipotent group \( G \). The definitions of character sheaves on unipotent groups in positive characteristic and their functional dimension are given in Sects. 2.6–2.7. They are followed by a digression in Sect. 2.8, where we recall several well-known results of character theory for finite groups that serve as a motivation behind our approach to the analysis of character sheaves. The main results of our work, along with various preliminaries, appear in Sects. 2.10–2.16. Finally, in Sect. 2.18 we explain the organization of the remaining sections of the article.

2.1 Basic definitions and notation

Throughout this article, we work over an algebraically closed field \( k \) of characteristic \( p > 0 \). We also fix, once and for all, a prime \( \ell \neq p \) and an algebraic closure \( \overline{\mathbb{Q}}_\ell \) of the field \( \mathbb{Q}_\ell \) of \( \ell \)-adic numbers.

By an algebraic group over \( k \), we will mean a smooth group scheme (equivalently, a reduced group scheme of finite type) over \( k \). A unipotent algebraic group (or “unipotent group,” for brevity) over \( k \) is an algebraic group over \( k \) that is isomorphic to a closed subgroup of the group \( U L_n(k) \) of unipotent upper-triangular matrices of size \( n \) over \( k \), for some \( n \in \mathbb{N} \).
Remark 2.1 Many of the definitions and results of our work can be formulated for unipotent groups over an arbitrary field $k$ of positive characteristic (sometimes it is necessary to assume that $k$ is perfect), and most of the auxiliary facts (see Sects. 4–5) remain valid in this more general setting. On the other hand, their proofs can be trivially reduced to the case where $k$ is algebraically closed. In addition, certain important properties of character sheaves require $k$ to be algebraically closed. For these reasons, we find it more convenient to assume that $k$ is algebraically closed from the very beginning.

If $X$ is an arbitrary scheme of finite type over $k$, one knows how to define the bounded derived category $D^b_c(X, \mathbb{Q}_\ell)$ of constructible complexes of $\mathbb{Q}_\ell$-sheaves on $X$ (see, e.g., [23, §§1.1.2–1.1.3]). We will denote it simply by $D(X)$, since $\ell$ is fixed. It is a triangulated $\mathbb{Q}_\ell$-linear category. Furthermore, it is equipped with a self-dual perverse $t$-structure $(\mathcal{D}^\leq(X), \mathcal{D}^\geq(X))$ [7], whose heart, $\text{Perv}(X) = \mathcal{D}^\leq(X) \cap \mathcal{D}^\geq(X)$, is called the category of perverse sheaves on $X$. It is an abelian category in which every object has finite length (op. cit., Thm. 4.3.1(i)).

2.2 Formalism of the six functors

In what follows, we will frequently employ Grothendieck’s “formalism of the six functors” for the category $\mathcal{D}(X)$ (as well as their equivariant versions, defined in Sect. 2.3 below). In particular, for any morphism $f : X \rightarrow Y$ of $k$-schemes of finite type, one has the pullback functor $f^* : \mathcal{D}(Y) \rightarrow \mathcal{D}(X)$, the pushforward functor $f_* : \mathcal{D}(X) \rightarrow \mathcal{D}(Y)$, the functor $f_! : \mathcal{D}(X) \rightarrow \mathcal{D}(Y)$ (pushforward with compact supports), and the functor $f^! : \mathcal{D}(Y) \rightarrow \mathcal{D}(X)$. We always omit the letters “$L$” and “$R$” from our notation for the six functors; thus, $f_!$ will stand for $Rf_!$ and $\otimes$ will stand for $\otimes^L_{\mathbb{Q}_\ell}$, etc.

Remark 2.2 In [24] the functor $f_!$ is defined only if $f$ is separated. This is enough for our purposes.

2.3 Equivariant derived categories

We remain in the setup of Sect. 2.1. Let $G$ be an algebraic group over $k$, let $X$ be a scheme of finite type over $k$, and suppose that we are given a left action of $G$ on $X$. In general, to get the correct definition of the “equivariant derived category” $\mathcal{D}_G(X)$, one must either adopt the approach of Bernstein and Lunts [11] (when $G$ is affine) or use the definition of $\ell$-adic derived categories for Artin stacks due to Laszlo and Olsson [38] and define $\mathcal{D}_G(X) = \mathcal{D}(G \backslash X)$, where $G \backslash X$ is the quotient stack of $X$ by $G$.

From now on, we assume that $G$ is unipotent. In this case, the naive definition of $\mathcal{D}_G(X)$ given below is equivalent to the correct one by Proposition 11.1.

\[1\] One does not have to assume that $f$ is separated (see, e.g., [38]).
Let us write $\alpha : G \times X \to X$ for the action morphism and $\pi : G \times X \to X$ for the projection. Let $\mu : G \times G \to G$ be the product in $G$. Let $\pi_{23} : G \times G \times X \to G \times X$ be the projection along the first factor $G$. The category $\mathcal{D}_G(X)$ is defined as follows.

**Definition 2.3** An object of the category $\mathcal{D}_G(X)$ is a pair $(M, \phi)$, where $M \in \mathcal{D}(X)$ and $\phi : \alpha^* M \cong \pi^* M$ is an isomorphism in $\mathcal{D}(G \times X)$ such that

$$\pi_{23}^*(\phi) \circ (\text{id}_G \times \alpha)^*(\phi) = (\mu \times \text{id}_X)^*(\phi),$$

i.e., the composition of the natural isomorphisms

$$(\text{id}_G \times \alpha)^* \alpha^* M \cong (\mu \times \text{id}_X)^* \alpha^* M \xrightarrow{(\mu \times \text{id}_X)^*(\phi)} (\mu \times \text{id}_X)^* \pi^* M \cong \pi_{23}^* \pi^* M$$
equals the composition

$$(\text{id}_G \times \alpha)^* \pi^* M \xrightarrow{(\text{id}_G \times \alpha)^*(\phi)} (\text{id}_G \times \alpha)^* \pi^* M \cong \pi_{23}^* \alpha^* M \xrightarrow{\pi_{23}^*(\phi)} \pi_{23}^* \pi^* M.$$

A morphism $(M, \phi) \to (N, \psi)$ in $\mathcal{D}_G(X)$ is a morphism $\nu : M \to N$ in $\mathcal{D}(X)$ satisfying $\phi \circ \alpha^*(\nu) = \pi^*(\nu) \circ \psi$. The composition of morphisms in $\mathcal{D}_G(X)$ is defined to be equal to their composition in $\mathcal{D}(X)$.

**Remark 2.4** With the aid of the proper base change theorem and the smooth base change theorem, one can easily “upgrade” the functors $f^*$, $f_*$, and $f!$ mentioned in Sect. 2.2 to functors between the equivariant derived categories $\mathcal{D}_G(X)$ and $\mathcal{D}_G(Y)$, in the case where a unipotent group $G$ acts on $X$ and $Y$, and the morphism $f$ commutes with the $G$-action (cf. [12, §4.4]).

**Remark 2.5** By Definition 2.3, we have a faithful forgetful functor $\mathcal{D}_G(X) \to \mathcal{D}(X)$. If $G$ is a connected unipotent group over $k$, one can show that the forgetful functor is fully faithful. In other words, being $G$-equivariant becomes a property of an $\ell$-adic complex on $X$ in this case. In particular, if $(M, \phi) \in \mathcal{D}_G(X)$, then $\phi$ is determined uniquely by $M$.

2.4 The monoidal categories $\mathcal{D}(G)$ and $\mathcal{D}_G(G)$

**Definition 2.6** If $G$ is a unipotent algebraic group over $k$, the equivariant derived category of $G$ is defined as the equivariant derived category $\mathcal{D}_G(G)$ with respect to the conjugation action of $G$ on itself.

**Definition 2.7** Let $G$ be a unipotent algebraic group over $k$. If $M$ and $N$ are objects of $\mathcal{D}(G)$ (respectively, $\mathcal{D}_G(G)$), the convolution with compact supports of $M$ and $N$ is the object of $\mathcal{D}(G)$ (respectively, $\mathcal{D}_G(G)$) defined by $M \ast N = \mu^!(p_1^* M \otimes p_2^* N)$, where $\mu : G \times G \to G$ is the multiplication morphism and $p_1$, $p_2 : G \times G \to G$ are the first and second projections.
Convention 2.8 From now on, the words “with compact supports” will be dropped, and the bifunctor $\ast$ will simply be referred to as the convolution of complexes on $G$. The other convolution bifunctor, obtained by replacing $\mu_!$ with $\mu_*$ in Definition 2.7, will not be used in the present work.²

It is easy to construct associativity constraints for the bifunctors $\ast$ on $\mathcal{D}(G)$ and on $\mathcal{D}_G(G)$, making each category monoidal, with unit object $\mathbb{1} = \mathbb{1}_G$ being the delta-sheaf at the identity element $1 \in G(k)$.

2.5 Properties and additional structures on $\mathcal{D}_G(G)$ and $\mathcal{D}(G)$

Here we give a brief outline (which is enough to read the most part of the article) and refer to Appendices 1 and 2 for the precise definitions and constructions.

2.5.1 A duality property weaker than rigidity

If $G$ is finite, the monoidal categories $\mathcal{D}_G(G)$ and $\mathcal{D}(G)$ are rigid. In general, they are not rigid but have a weaker property related to duality. Namely, $\mathcal{D}_G(G)$ and $\mathcal{D}(G)$ are $\mathcal{r}$-categories in the sense of Definition 9.6, see Example 9.9 and Lemma 9.10.

2.5.2 $\mathcal{D}_G(G)$ is a braided category

The braiding on $\mathcal{D}_G(G)$ is constructed in Definition 9.43. Although this braiding enters the formulation of some of our results (notably, Theorem 2.15(d)), in many situations it suffices to know its existence or even the following corollary of its existence.

Remark 2.9 The functors $(M, N) \mapsto M \ast N$ and $(M, N) \mapsto N \ast M$ on $\mathcal{D}_G(G)$ are isomorphic.

2.5.3 Pivotal structure on $\mathcal{D}(G)$ and $\mathcal{D}_G(G)$

As explained in Sect. 9.2.3, each of the monoidal categories $\mathcal{D}(G)$ and $\mathcal{D}_G(G)$ has a canonical pivotal structure in the sense of Definition 9.14.

2.5.4 Ribbon structure on $\mathcal{D}_G(G)$

According to Definition 9.39, a ribbon structure on a braided $\mathcal{r}$-category $\mathcal{M}$ is an automorphism $\theta$ of the functor $\text{Id}_\mathcal{M}$ satisfying certain conditions;⁴ in particular, $\theta$ has

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² In fact, convolution without compact supports, call it $\ast_*$ for now, can be expressed in terms of convolution with compact supports. Namely, if $M, N$ are objects of $\mathcal{D}(G)$ or $\mathcal{D}_G(G)$, one has a canonical isomorphism $M \ast_* N \cong D_G(D_G(N \ast D_G(M))$, where $D_G$ is the functor introduced in Definition 2.17. By Remark 2.18, the functor $D_G$ also has intrinsic meaning in terms of the monoidal structure given by convolution with compact supports.

³ As explained in Sect. 2.5.1, the usual framework of rigid braided categories is too restrictive for us.

⁴ One can also consider a ribbon structure as a pivotal structure satisfying a certain condition, see Corollary 9.42.
to be a twist in the sense of Definition 9.35. In Sect. 9.5 we define a canonical ribbon structure on $\mathcal{D}_G(G)$. For each $M \in \mathcal{D}_G(G)$ and $g \in G$, the action of the canonical twist $\theta_M : M \xrightarrow{\sim} M$ on the stalk $M_g$ equals the action of $g \in Z(g)$ on $M_g$. Here $Z(g) \subset G$ is the centralizer of $g$ and the action of $Z(g)$ on $M_g$ comes from the equivariant structure on $M$.

**Example 2.10** If $G$ is finite, then $\mathcal{D}_G(G)$ is the derived category of $A$, where $A$ is the category of $G$-equivariant constructible sheaves on $G$, also known as the category of modules over the quantum double of the group algebra of $G$. $A$ is a standard example of a ribbon category (and in fact, a modular one), see [9, §3.2].

### 2.5.5 Action of the surface operad

The category $\mathcal{D}_G(G)$ is equipped with a canonical action of the surface operad, see Sect. 10.2.3 of Appendix 2 and Remark 10.38(ii). The action of the genus 0 part of the surface operad amounts to the braided monoidal structure and twist mentioned in Sects. 2.4, 2.5.2, and 2.5.4.

### 2.6 Character sheaves and $L$-packets

The notion of a character sheaf on a unipotent algebraic group $G$ is defined in terms of certain “idempotents” in the category $\mathcal{D}_G(G)$. A more exhaustive study of idempotents in monoidal categories (which does not depend on any of the other results of the present article) appears in Sect. 3. Here we will briefly summarize some of the definitions given in that section, specialized to the monoidal category $(\mathcal{D}_G(G), \ast, 1)$.

**Definitions 2.11** (Weak and closed idempotents; minimal idempotents; Hecke subcategories) In the setup of Sect. 2.1, let $G$ be a unipotent algebraic group over $k$.

1. An object $e \in \mathcal{D}_G(G)$ is said to be a weak idempotent if $e \ast e \equiv e$. It is said to be a closed idempotent if there exists a morphism $\pi : 1 \longrightarrow e$ that becomes an isomorphism after convolving with $e$. Such a $\pi$ is called an idempotent arrow.
2. If $e \in \mathcal{D}_G(G)$ is a weak idempotent, the Hecke subcategory associated with $e$ is the full subcategory $e \mathcal{D}_G(G) \subset \mathcal{D}_G(G)$ consisting of objects $M \in \mathcal{D}_G(G)$ such that $e \ast M \cong M$ (equivalently, such that $M \cong e \ast N$ for some $N \in \mathcal{D}_G(G)$).
3. An object $e \in \mathcal{D}_G(G)$ is said to be a minimal weak idempotent (respectively, a minimal closed idempotent) if $e$ is a weak (respectively, closed) idempotent, $e \neq 0$, and for every weak (respectively, closed) idempotent $e'$ in $\mathcal{D}_G(G)$, we have either $e \ast e' = 0$ or $e \ast e' \equiv e$.

**Remarks 2.12** (i) If $e \in \mathcal{D}_G(G)$ is a closed idempotent, the Hecke subcategory $e \mathcal{D}_G(G)$ is a monoidal category with unit object $e$ (see Lemma 3.18).

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5 By Definition 2.3, for each $\gamma, g \in G$ one has $\phi_{\gamma, g} : M_{\gamma g^{-1}} \xrightarrow{\sim} M_g$. The (left) action of $Z(g)$ on $M_g$ is defined by $\gamma \mapsto \phi^{-1}_{\gamma, g} \cdot \gamma \in Z(g)$.

6 The origin of the adjective “closed” is explained in Sect. 3.4.
(ii) If $e \in \mathcal{D}_G(G)$ is a weak idempotent, the category $e\mathcal{D}_G(G)$ is not necessarily monoidal already for $G = G_a$ (see Remark 3.19, including its last sentence), although by Remark 2.9, the subcategory $e\mathcal{D}_G(G) \subset \mathcal{D}_G(G)$ is closed under convolution. There are also other reasons why the notion of weak idempotent is not really good. Nevertheless, the interplay between weak and closed idempotents, studied in Sect. 3, turns out to be useful for proving the main results of our work.

(iii) Let $e, e' \in \mathcal{D}_G(G)$ be minimal closed (respectively, minimal weak) idempotents. If $e \ast e' \neq 0$, then $e \cong e \ast e' \cong e' \ast e \cong e'$ (we have used Remark 2.9). So if $e \not\cong e'$, then $e \ast e' = 0$, and therefore, $e\mathcal{D}_G(G) \cap e'\mathcal{D}_G(G) = 0$.

Definitions 2.13 (Character sheaves and $\mathbb{L}$-packets) With the assumptions of Sect. 2.1, let $G$ be a unipotent algebraic group over $k$.

(1) Let $e \in \mathcal{D}_G(G)$ be a minimal closed idempotent. We write $\mathcal{M}_e^{\text{perv}}$ for the full subcategory of the Hecke subcategory $e\mathcal{D}_G(G)$ consisting of those objects for which the underlying $\ell$-adic complex is a perverse sheaf on $G$. It is clear that $\mathcal{M}_e^{\text{perv}}$ is an additive $\mathbb{Q}_\ell$-linear subcategory of $\mathcal{D}_G(G)$. The Lusztig packet of character sheaves on $G$ defined by $e$ is the set of (isomorphism classes of) indecomposable objects of the category $\mathcal{M}_e^{\text{perv}}$.

(2) An object of $\mathcal{D}_G(G)$ is a character sheaf if it lies in the Lusztig packet of character sheaves defined by some minimal closed idempotent in $\mathcal{D}_G(G)$.

From now on, for brevity, we write “$\mathbb{L}$-packet” in place of “Lusztig packet.”

Remark 2.14 If minimal closed idempotents $e, e' \in \mathcal{D}_G(G)$ are not isomorphic, then the corresponding $\mathbb{L}$-packets are disjoint by Remark 2.12 (iii).

The first main result of this article is the following:

Theorem 2.15 Let $G$ be a unipotent algebraic group over $k$, and let $e \in \mathcal{D}_G(G)$ be a minimal closed idempotent.

(a) Then $\mathcal{M}_e^{\text{perv}}$ is a semisimple abelian category with finitely many simple objects. In particular, $\mathbb{L}$-packets of character sheaves on $G$ are finite.

(b) There exists a (necessarily unique) integer $n_e$ such that $e[-n_e] \in \mathcal{M}_e^{\text{perv}}$. One has $0 \leq n_e \leq \dim G$. The subcategory $\mathcal{M}_e := \mathcal{M}_e^{\text{perv}}[n_e]$ of the monoidal category $e\mathcal{D}_G(G)$ is monoidal.

(c) The monoidal categories $e\mathcal{D}_G(G)$ and $\mathcal{M}_e$ are rigid; moreover, $\mathcal{M}_e$ is a fusion category\footnote{A fusion category over $\mathbb{Q}_\ell$ is a rigid $\mathbb{Q}_\ell$-linear monoidal category $\mathcal{C}$ such that the unit object of $\mathcal{C}$ is indecomposable, and as a $\mathbb{Q}_\ell$-linear category, $\mathcal{C}$ is equivalent to a direct sum of finitely many copies of the category of finite-dimensional vector spaces.} in the sense of [28].

(d) The restrictions to $\mathcal{M}_e$ of the braiding constructed in Definition 9.43 and of the twist constructed in Definition 9.45 define a ribbon structure on $\mathcal{M}_e$. This ribbon structure is modular.

(e) The perverse $t$-structure on $\mathcal{D}(G)$ induces a $t$-structure on $e\mathcal{D}_G(G)$, and the canonical functor $D^b(\mathcal{M}_e^{\text{perv}}) \rightarrow e\mathcal{D}_G(G)$ is an equivalence.
Remarks 2.16 (i) By Remark 2.12(i), the last part of (b) is equivalent to $\mathcal{M}_e$ being closed under convolution (the fact that $\mathcal{M}_e$ contains the unit object $e$ of the Hecke subcategory $e\mathcal{D}_G(G)$ is clear from the definition of $n_e$).

(ii) It is not hard to show that a minimal closed idempotent $e$ is indecomposable (cf. Corollary 3.49). So in the situation of Theorem 2.15(b), the object $e[-n_e]$ is a character sheaf. In particular, $\mathbb{L}$-packets are nonempty.

The proof of Theorem 2.15 is given in Sect. 8.11. It relies on a certain construction of all minimal closed idempotents in $e\mathcal{D}_G(G)$ (and the corresponding $\mathbb{L}$-packets), which is the keystone of our approach to the theory of character sheaves on unipotent groups. This construction, based on the notion of an admissible pair (Sect. 2.10) and on the induction functors for equivariant derived categories (Sect. 2.12), is given in Sect. 2.13.

The duality functor on the rigid monoidal category $e\mathcal{D}_G(G)$ can be calculated explicitly as follows.

Definition 2.17 For a unipotent algebraic group $G$ over $k$, we define the contravariant functor $\mathbb{D}_G^\sim : \mathcal{D}_G(G) \to \mathcal{D}_G(G)$ by $\mathbb{D}_G^\sim = \mathbb{D}_G \circ \iota^* = \iota^* \circ \mathbb{D}_G$, where $\iota : G \to G$ is given by $g \mapsto g^{-1}$ and $\mathbb{D}_G : \mathcal{D}_G(G) \to \mathcal{D}_G(G)$ is the Verdier duality functor. By abuse of notation, we also denote by $\mathbb{D}_G^\sim$ and $\mathbb{D}_G$ the corresponding functors $\mathcal{D}(G) \to \mathcal{D}(G)$ in the nonequivariant setting.

Remark 2.18 The functor $\mathbb{D}_G^\sim$ can be interpreted as inner Hom to the unit object in the monoidal category $\mathcal{D}_G(G)$ or $\mathcal{D}(G)$: see Lemma 9.10.

Proposition 2.19 In the situation of Theorem 2.15,

(a) There is a canonical isomorphism

$$\mathbb{D}_G^\sim e \cong e[-2n_e](-n_e) \otimes L_e,$$

where $L_e$ is a certain line\(^8\) over $\overline{\mathbb{Q}}_\ell$.

(b) Given an idempotent arrow $1 \to e$, the duality functor on $e\mathcal{D}_G(G)$ canonically identifies with $M \mapsto (\mathbb{D}_G^\sim M)[2n_e](n_e) \otimes L_e^{-1}$, where $L_e^{-1} = \text{Hom}_{\overline{\mathbb{Q}}_\ell}(L_e, \overline{\mathbb{Q}}_\ell)$.

The proposition is proved in Sect. 8.13.

Remarks 2.20 (i) By Theorem 2.15(a) and Remark 2.16(ii), the object $e \in \mathcal{M}_e$ is simple, so $\text{Hom}_{\mathcal{D}_G(G)}(e, e)$ is 1-dimensional. Thus, assertion (a) of Proposition 2.19 is equivalent to the statement that $\mathbb{D}_G^\sim e \cong e[-2n_e](-n_e)$.

(ii) We do not know whether the line $L_e$ can be canonically trivialized. In Proposition 2.46 and Sect. 8.12 below, we describe a trivialization of $L_e$ depending on an additional choice.

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\(^8\) That is, a 1-dimensional vector space.

\(^9\) It is not hard to show that $e\mathcal{D}_G(G)$ is stable under $\mathbb{D}_G^\sim$; see Lemma 9.49.
2.7 The notion of functional dimension

Theorem 2.15(b) allows us to introduce

**Definition 2.21** Let $G$ be a unipotent algebraic group over $k$, and let $e \in \mathcal{D}_G(G)$ be a minimal closed idempotent. The functional dimension of $e$ is the number $d_e = (\dim G - n_e)/2$. We also call $d_e$ the functional dimension of every character sheaf in the $\mathbb{L}$-packet defined by $e$.

By Theorem 2.15(b), $0 \leq d_e \leq (\dim G)/2$.

**Remark 2.22** This notion is analogous to the classical notion of functional dimension in the representation theory of real Lie groups. First, we show in Theorem 2.52(b) that functional dimension behaves as expected under induction functors. Second, if $G$ is connected and its nilpotence class is less than $p = \text{char } k$, then according to [14, Thm. 5.10], isomorphism classes of minimal closed idempotents $e \in \mathcal{D}_G(G)$ bijectively correspond to “coadjoint orbits” in the sense of [14, §4.1] and

$$d_e = \frac{1}{2} \cdot \dim \Omega_e, \quad n_e = \text{codim } \Omega_e,$$

where $\Omega_e$ is the orbit corresponding to $e$.

**Remark 2.23** The functional dimension may fail to be an integer in our setup. For example, let $G$ be a *fake Heisenberg group*, i.e., a noncommutative central extension of $G$ by $G$ (such extensions do exist in characteristic $p > 0$: cf. [14, §3.7]). Then there exist minimal closed idempotents $e \in \mathcal{D}_G(G)$ with $d_e = 1/2$ (loc. cit.).

The next theorem is proved in Sect. 8.11.

**Theorem 2.24** Let $G$ and $e$ be as in Theorem 2.15. The Gauss sum\(^{10}\) of the modular category $\mathcal{M}_e$ equals $\epsilon p^n$, where $n \geq 0$ is an integer, $\epsilon = 1$ when $d_e \in \mathbb{Z}$ and $\epsilon = -1$ when $d_e \notin \mathbb{Z}$. In particular, if the $\mathbb{L}$-packet of $e$ has only one element, then $d_e \in \mathbb{Z}$.

2.8 Elementary reminders

We now make a short digression to recall several constructions and results from character theory for finite groups that are, for the most part, very standard and well known. In the remainder of the article, we will see that all of them admit suitable geometric analogues in the world of equivariant $\ell$-adic complexes on unipotent groups, which are both interesting in their own right, and play an essential role in the proofs of our main results.

1. Let $\Gamma$ be a finite group, let $\text{Fun}(\Gamma)$ denote the algebra of functions $\Gamma \longrightarrow \mathbb{C}$ under pointwise addition and convolution, and let $\text{Fun}(\Gamma)^{\Gamma} \subset \text{Fun}(\Gamma)$ denote the subal-

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\(^{10}\) See [26, §6.2], formula (122), for the definition of the Gauss sum $\tau^+(\mathcal{M}_e)$ of $\mathcal{M}_e$.\]
gebra of conjugation-invariant functions.\(^{11}\) Then there is a bijection between the set of complex irreducible characters of \(\Gamma\) and the set of minimal (or “indecomposable”) idempotents in \(\text{Fun}(\Gamma)^{\Gamma}\), given by \(\chi \mapsto \frac{\chi^{(1)}}{\prod_{\gamma \in \Gamma} \chi} \cdot \chi\). This fact is one of the reasons why the definition of character sheaves involves minimal idempotents. Another motivation, coming from the orbit method for unipotent groups of “small” nilpotence class, is explained in detail in [14].

2. An important role in the character theory of finite groups is played by the operation of induction of class functions. If \(\Gamma' \subset \Gamma\) is a subgroup and \(f \in \text{Fun}(\Gamma')^{\Gamma'}\), the induced function \(\text{ind}_{\Gamma'}^{\Gamma} f \in \text{Fun}(\Gamma)^{\Gamma}\) can be obtained in two steps. First, we extend \(f\) by zero outside of \(\Gamma'\) to obtain a \(\Gamma'\)-invariant function \(\overline{f} : \Gamma \to \mathbb{C}\). Next, for every coset \(\gamma \Gamma' \subset \Gamma\), we form the corresponding conjugate, \(\overline{f}'\), of \(\overline{f}\) (it depends only on the coset \(\gamma \Gamma'\) and not on the particular element \(\gamma\)), and we define \(\text{ind}_{\Gamma'}^{\Gamma} f\) as the sum of all these conjugates, indexed by the elements of \(\Gamma / \Gamma'\).

3. If \(\chi\) is a complex irreducible character of \(\Gamma'\), there is a result, called Mackey’s irreducibility criterion, which gives necessary and sufficient conditions for the induced character \(\text{ind}_{\Gamma'}^{\Gamma} \chi\) to be irreducible as well. It is not hard to show that this result can be reformulated as follows: with the notation of #2 above, \(\text{ind}_{\Gamma'}^{\Gamma} \chi\) is irreducible if and only if \(\overline{\chi} \ast \delta \ast \overline{\chi} = 0\) for every \(x \in \Gamma \setminus \Gamma'\), where \(\delta\) denotes the delta-function at \(x\).

4. Some complex irreducible representations of a finite group \(\Gamma\) can be obtained by means of the following construction. Consider a pair \((H, \chi)\) consisting of a subgroup \(H \subset \Gamma\) and a homomorphism \(\chi : H \to \mathbb{C}^\times\). Let \(\Gamma'\) be the stabilizer of the pair \((H, \chi)\) with respect to the conjugation action of \(\Gamma\). We say that the pair \((H, \chi)\) is admissible if the following three conditions are satisfied:

1. \(\Gamma' / H\) is commutative;
2. The map \(B_\chi : (\Gamma' / H) \times (\Gamma' / H) \to \mathbb{C}^\times\) induced by \((\gamma_1, \gamma_2) \mapsto \chi(\gamma_1 \gamma_2 \gamma_1^{-1} \gamma_2^{-1})\) (which, in view of (1), is well defined and biadditive) is a perfect pairing, i.e., induces an isomorphism \(\Gamma' / H \to \text{Hom}(\Gamma' / H, \mathbb{C}^\times)\); and
3. For every \(g \in \Gamma\), \(g \notin \Gamma'\), we have \(\chi |_{H \cap H^g} \neq \chi^g |_{H \cap H^g}\), where \(H^g = g^{-1} H g\) and \(\chi^g : H^g \to \mathbb{C}^\times\) is obtained by transport of structure: \(\chi^g(h) = \chi(ghg^{-1})\).

Now, properties (1) and (2) above imply that the group \(\Gamma'\) has a unique complex irreducible representation \(\pi_{\chi}\) such that \(H\) acts by \(\chi\). Property (3) then implies that the induced representation \(\text{Ind}_{\Gamma'}^{\Gamma} \pi_{\chi}\) is irreducible, in view of Mackey’s irreducibility criterion (see #3).

5. If \(\Gamma\) is a finite nilpotent group, then every complex irreducible representation \(\rho\) of \(\Gamma\) arises from some admissible pair \((H, \chi)\) by means of the construction explained in #4; moreover, the “reduction process” described in one of the appendices to [14] allows to construct \((H, \chi)\) canonically (up to conjugation) for a given \(\rho\).

Note that the existence, for a given \(\rho\), of a noncanonical admissible pair \((H, \chi)\) (with \(\Gamma' = H\)) immediately follows from the classical theorem that every irreducible representation of a finite nilpotent group is induced from a 1-dimensional representation of a subgroup.

\(^{11}\) It coincides with the center of \(\text{Fun}(\Gamma)\). Its elements are often called “class functions” on \(\Gamma\).
One of the main technical tools used in the present article is the geometric notion of an admissible pair for a unipotent group over $k$, introduced in Sect. 2.10. The definition of this notion uses Serre duality for unipotent groups. However, as explained in [14], the Serre dual of a (connected) unipotent group over $k$ can only be defined canonically (i.e., by means of a universal property) as a group object in the category of perfect schemes over $k$. For this reason, it will be technically more convenient for us to place ourselves in the framework of perfect schemes and perfect group schemes from the very start.

Let us recall that a scheme $S$ in characteristic $p$ (i.e., such that $p$ annihilates the structure sheaf $\mathcal{O}_S$ of $S$) is said to be perfect if the morphism $\mathcal{O}_S \to \mathcal{O}_S$, given by $f \mapsto f^p$ on the local sections of $\mathcal{O}_S$, is an isomorphism of sheaves. In particular, a commutative ring $A$ of characteristic $p$ is perfect [30] if and only if $\text{Spec } A$ is a perfect scheme. We will denote by $\text{Sch}_k$ the category of all $k$-schemes, and by $\text{Perf}_k$ the full subcategory consisting of perfect schemes. The inclusion functor $\text{Perf}_k \hookrightarrow \text{Sch}_k$ admits a right adjoint, which we call the perfectization functor $X \mapsto X_{\text{perf}}$. (This follows from the results of Greenberg [30]. In our setup, $k$ is algebraically closed, but in fact, it suffices to assume that $k$ is perfect.)

It is not hard to see that a group object in the category $\text{Perf}_k$ is automatically a group scheme over $k$. Conversely, if $G$ is any group scheme over $k$, then $G_{\text{perf}}$ can be canonically equipped with a $k$-group scheme structure as well. For more details, we refer the reader to [12, §§A.3–A.4].

**Definitions 2.25** A perfect quasi-algebraic scheme over $k$ is an object of $\text{Perf}_k$ that is isomorphic to the perfectization of a scheme of finite type over $k$. A perfect quasi-algebraic group over $k$ is a group object of $\text{Perf}_k$ that is isomorphic to the perfectization of an algebraic group over $k$. For brevity, by a perfect unipotent group over $k$, we will mean a perfect quasi-algebraic group over $k$ that is isomorphic to the perfectization of a unipotent algebraic group over $k$.

**Remark 2.26** If $X$ is an arbitrary scheme over $k$, then, by adjunction, we obtain a canonical morphism $X_{\text{perf}} \to X$. It is known [30] to be a homeomorphism of the underlying topological spaces. Furthermore, if $f : U \to X$ is an étale morphism, the induced morphism $U_{\text{perf}} \to U \times_X X_{\text{perf}}$ is an isomorphism, and the functor $U \mapsto U_{\text{perf}}$ induces an equivalence between the étale topos of $X$ and that of $X_{\text{perf}}$.

It follows that every construction that can be formulated in terms of the étale topos of a scheme is “insensitive” to replacing a scheme over $k$ with its perfectization. In particular, if $X$ is a perfect quasi-algebraic scheme over $k$, the derived category $\mathcal{D}(X) = D^b_{\text{c}}(X, \mathcal{O}_X)$ of $X$ and the abelian category $\text{Perv}(X)$ of perverse sheaves on $X$ can be defined in the same way as for schemes of finite type over $k$. We also have the formalism of the six functors (cf. Sect. 2.2) for the derived categories of perfect quasi-algebraic schemes over $k$. Further, if $G$ is a perfect unipotent group acting on a quasi-algebraic scheme $X$ over $k$, the equivariant derived category $\mathcal{D}_G(X)$ can be defined as in Sect. 2.3, and character sheaves on $G$ can be defined as in Sect. 2.6. Moreover, every result about character sheaves on $G$ (or, more generally, about the
category $\mathcal{D}_G(G)$ that can be proved for perfect unipotent groups is also automatically valid for ordinary unipotent algebraic groups over $k$.

**Convention 2.27** From now on, unless explicitly stated otherwise, all schemes under consideration will be assumed to be perfect schemes over $k$. This convention will allow us to simplify the formulation of all of our results that depend on Serre duality.

### 2.10 Serre duality and admissible pairs

We remain in the setup of Sect. 2.1. From the viewpoint of the sheaves-to-functions correspondence, the next definition is a geometric analogue of the notion of a 1-dimensional representation of a finite group.

**Definition 2.28** A multiplicative local system on a perfect connected quasi-algebraic group $H$ over $k$ is a rank $\mathbb{Q}_\ell$-local system $L$ of rank 1 equipped with an isomorphism $\mu^*(L) \cong L \otimes L$, where $\mu : H \times H \rightarrow H$ denotes the multiplication morphism.

By abuse of notation, we usually denote a multiplicative local system by a single letter such as $L$. This is harmless in view of Remark 4.2(iii).

For our purposes, the **Serre dual** of $H$ should be thought of as the moduli space of multiplicative $\mathbb{Q}_\ell$-local systems on $H$ (a better approach is mentioned in Remark 4.6). The notion of Serre duality for unipotent groups (not to be confused with Serre duality in the cohomology theory for coherent sheaves) goes back to the article [47] and is discussed in more detail in Sect. 4 below. For the time being, it will suffice to know that the Serre dual, $H^*$, of an arbitrary perfect connected unipotent group $H$ exists as a (possibly disconnected) perfect commutative unipotent group over $k$ (see Proposition 4.3).

Furthermore, the construction of the biadditive pairing $B_\chi$ that enters condition (2) in the definition of an admissible pair for a finite group (see #4 in Sect. 2.8) admits a geometrization, which plays a role in the definition of an admissible pair for a perfect unipotent group. This geometric construction is explained in more detail in Sect. 4.3. For the time being, we will use this construction as a “black box.”

**Definition 2.29** Let $G$ be a perfect unipotent group over $k$, and let $(H, L)$ be a pair consisting of a connected subgroup $H \subset G$ and a multiplicative local system $L$ on $H$. The normalizer $N_G(H, L)$ of $(H, L)$ in $G$ is defined as the stabilizer of the isomorphism class $[L] \in H^*(k)$ in the normalizer $N_G(H) \subset G$ of $H$ in $G$.

**Definition 2.30** Let $G$ be a perfect unipotent group over $k$. An admissible pair for $G$ is a pair $(H, L)$ consisting of a connected subgroup $H \subset G$ and a multiplicative local system $L$ on $H$ such that the following three conditions are satisfied.

1. Let $G'$ be the normalizer of $(H, L)$ in $G$ (see Definition 2.29), and let $G'^\circ$ denote its neutral connected component. Then $G'^\circ/H$ is commutative.

12 Since $H^*$ is defined by a universal property, the conjugation action of $N_G(H)$ on $H$ induces an action of $N_G(H)$ on $H^*$. Note also that $[L]$ is a point of $H^*$ over $k$ by the definition of $H^*$. 

(2) The $k$-group morphism $\varphi_L : G^\circ/H \longrightarrow (G^\circ/H)^*$ defined by applying the construction of Sect. 4.3 below to $U = Z = G^\circ$, $N = H$, and $N^\circ = L$ is an isogeny.

(3) For every $g \in G(k)$ such that $g \notin G'(k)$, we have

$$L^\circ|_{(H \cap H g)^\circ} \not\cong L^g|_{(H \cap H g)^\circ},$$

where $H^g = g^{-1} H g$ and $L^g$ is the multiplicative local system on $H^g$ obtained from $L$ by transport of structure (via the map $h \mapsto g^{-1} h g$).

Remark 2.31 Since $\dim (G^\circ/H) = \dim (G^\circ/H)^*$ (see Sect. 4.2), condition (2) in the last definition is equivalent to finiteness of $\ker \varphi_L$.

2.11 Heisenberg minimal idempotents

The notion of an admissible pair allows us to construct a certain special class of minimal idempotents in equivariant derived categories of unipotent groups. Namely, let $G$ be a perfect unipotent group over $k$, let $(H, L)$ be an admissible pair for $G$, and let $G'$ be its normalizer in $G$, as defined above. Write $K_H$ for the dualizing complex of $H$, which in our setup is isomorphic to $\mathbb{Q}_\ell[2 \dim H]$, where $\mathbb{Q}_\ell$ is the constant $\ell$-adic local system of rank 1 on $H$ (because $k$ is algebraically closed). Finally, put $e_L = L \otimes K_H$, and let $e'_L$ denote the object of $\mathcal{D}_{G'}(G')$ obtained from $e_L$ by extension by zero. (Since $(H, L)$ is invariant under the conjugation action of $G'$, both $L$ and $K_H$ have canonical $G'$-equivariant structures.)

Lemma 2.32 The object $e'_L$ is a closed idempotent in $\mathcal{D}_{G'}(G')$, which, moreover, is minimal as a weak idempotent in $\mathcal{D}_{G'}(G')$.

The lemma results from [12, Prop. 8.1(a)–(b)]. In what follows, we refer to $e'_L$ as the Heisenberg minimal idempotent on $G'$ defined by the admissible pair $(H, L)$. As we will see shortly, one can obtain a minimal closed idempotent in $\mathcal{D}_G(G)$ from $e'_L$ via “induction with compact supports” (cf. Theorem 2.41).

The following result covers the base case in the inductive proof of Theorems 2.15 and 2.24 that will be given in this article.

Theorem 2.33 (S. Datta and T. Deshpande) If $G'$ is a perfect unipotent group over $k$ and $e'_L \in \mathcal{D}_{G'}(G')$ is a Heisenberg minimal idempotent, then Theorems 2.15 and 2.24 hold for $(G', e'_L)$ in place of $(G, e)$.

The proof given in Sect. 8.10 consists of references to the articles by Datta [19] and Deshpande [25].

2.12 Averaging functors and induction functors

We remain in the setup of Sect. 2.1. Let $G$ be a perfect unipotent group acting on a perfect quasi-algebraic scheme $X$ over $k$, and let $G' \subset G$ be a closed subgroup. There
is an obvious forgetful functor

\[ F : \mathcal{D}_G(X) \longrightarrow \mathcal{D}_{G'}(X). \]  

(2.3)

We will need its right and left adjoints. Let us recall their construction. It is based on the following remark.

**Remark 2.34** The functor (2.3) can be factored as

\[ \mathcal{D}_G(X) \xrightarrow{\text{pr}_2^*} \mathcal{D}_G((G/G') \times X) \xrightarrow{i^* \circ \Phi} \mathcal{D}_{G'}(X) \]  

(2.4)

Here \( \Phi : \mathcal{D}_G((G/G') \times X) \longrightarrow \mathcal{D}_{G'}((G/G') \times X) \) is the forgetful functor, \( G \) acts on both \( G/G' \) and \( X, \text{pr}_2 : (G/G') \times X \longrightarrow X \) is the projection, and \( i : X \hookrightarrow (G/G') \times X \) takes \( x \in X \) to \( (\bar{1}, x) \), where \( \bar{1} \in G/G' \) is the image of \( 1 \in G \).

**Lemma 2.35** The forgetful functor (2.3) has a right adjoint and a left adjoint. The right adjoint is the functor

\[ \text{Av}_{G/G'} : \mathcal{D}_{G'}(X) \longrightarrow \mathcal{D}_G(X); \quad \text{Av}_{G/G'} := (\text{pr}_2)_*(i^* \circ \Phi)^{-1} \]  

(2.5)

(we are using the notation of Remark 2.34). The left adjoint is the functor

\[ M \longmapsto \text{av}_{G/G'}(M)[2d](d), \]

where \( d := \text{dim}(G/G') \) and \( \text{av}_{G/G'} : \mathcal{D}_{G'}(X) \longrightarrow \mathcal{D}_G(X) \) is defined by

\[ \text{av}_{G/G'} := (\text{pr}_2)!(i^* \circ \Phi)^{-1}. \]  

(2.6)

**Proof** Use the factorization (2.4) and the isomorphism

\[ \text{pr}_2^!(M) \xrightarrow{\sim} \text{pr}_2^*(M)[2d](d), \]

where \( d := \text{dim}(G/G') \). \( \Box \)

**Definition 2.36** The functor (2.5) will be called the averaging functor (for the pair \( G' \subset G \)). The functor (2.6) will be called averaging with compact supports. If \( G' = \{1\} \), we will write \( \text{av}_G \) and \( \text{Av}_G \) in place of \( \text{av}_{G/G'} \) and \( \text{Av}_{G/G'} \).

Note that the canonical morphism \( (\text{pr}_2)_! \longrightarrow (\text{pr}_2)_* \) induces a canonical morphism

\[ \text{av}_{G/G'} \longrightarrow \text{Av}_{G/G'}. \]  

(2.7)

**Remark 2.37** Applying the forgetful functor \( F \) to (2.7) and using the adjunction of Lemma 2.35 yields the following two morphisms:

\[ F \circ \text{av}_{G/G'} \longrightarrow F \circ \text{Av}_{G/G'} \longrightarrow \text{Id}_{\mathcal{D}_{G'}(X)}. \]  

(2.8)

These morphisms will be described more explicitly in Lemma 7.14.
Now let $G$ be a perfect unipotent group over $k$, and let $G' \subset G$ be a closed subgroup. Let $G$ (respectively, $G'$) act on itself by conjugation. Further, let $\iota : G' \hookrightarrow G$ denote the inclusion morphism. Then $\iota$ induces the functor

$$t! = \iota_* : \mathcal{D}_{G'}(G') \longrightarrow \mathcal{D}_G(G).$$

**Definition 2.38** The functors of induction and induction with compact supports are defined by

$$\text{Ind}^G_{G'} = \text{Av}_{G/G'} \circ \iota_* : \mathcal{D}_{G'}(G') \longrightarrow \mathcal{D}_G(G)$$

and

$$\text{ind}^G_{G'} = \text{av}_{G/G'} \circ \iota_* : \mathcal{D}_{G'}(G') \longrightarrow \mathcal{D}_G(G),$$

respectively.

The morphism (2.7) induces a canonical morphism $\text{ind}^G_{G'} \longrightarrow \text{Ind}^G_{G'}$. Lemma 2.35 implies the following statement.

**Corollary 2.39** (i) Define the functor $\text{Res}^G_{G'} : \mathcal{D}_G(G) \longrightarrow \mathcal{D}_{G'}(G')$ to be the composition $\mathcal{D}_G(G) \longrightarrow \mathcal{D}_{G'}(G) \longrightarrow \mathcal{D}_{G'}(G')$, where the first functor is the forgetful one. Then $\text{Ind}^G_{G'}$ is right adjoint to $\text{Res}^G_{G'}$.

(ii) Define the functor $\text{res}^G_{G'} : \mathcal{D}_G(G) \longrightarrow \mathcal{D}_{G'}(G')$ to be the composition $\mathcal{D}_G(G) \longrightarrow \mathcal{D}_{G'}(G) \longrightarrow \mathcal{D}_{G'}(G')$. Then $\text{ind}^G_{G'}$ is left adjoint to the functor

$$M \longmapsto \text{res}^G_{G'} M[2d](d), \quad d := \dim(G/G').$$

The functor $\text{Res}^G_{G'}$ defined in the corollary will be called the restriction functor.

**Remark 2.40** The functors $\text{av}_{G/G'}$, $\text{Av}_{G/G'}$, $\text{ind}^G_{G'}$, and $\text{Ind}^G_{G'}$ are triangulated. The canonical morphisms $\text{av}_{G/G'} \longrightarrow \text{Av}_{G/G'}$ and $\text{ind}^G_{G'} \longrightarrow \text{Ind}^G_{G'}$ are morphisms of triangulated functors.

2.13 Construction of $\mathbb{L}$-packets

The next result provides an explicit construction of all $\mathbb{L}$-packets of character sheaves on a unipotent group over $k$.

**Theorem 2.41** Let $G$ be a perfect unipotent group over $k$.

(a) Let $(H, \mathcal{L})$ be an admissible pair for $G$ (Definition 2.30), let $G'$ be its normalizer in $G$ (Definition 2.29), and let $e'_\mathcal{L} \in \mathcal{D}_{G'}(G')$ be the corresponding Heisenberg minimal idempotent (Sect. 2.11). Then $f = \text{ind}^G_{G'} e'_\mathcal{L}$ is a minimal closed idempotent in $\mathcal{D}_G(G)$. 


(b) With the notation of Theorem 2.15, \( n_{e'} = \dim H \) and \( n_f = \dim H - \dim (G/G') \).

c) Every minimal closed idempotent \( f \) in \( \mathcal{D}_G(G) \) arises from some admissible pair \((H, L)\) by means of the construction described in part (a).

For the proof, see Sect. 8.9.

**Corollary 2.42** Let \((H, L)\) be an admissible pair for \(G\), and let \(G'\) be its normalizer in \(G\). Then \( \dim H \geq \dim (G/G') \).

**Proof** Combine Theorem 2.41(b) with the inequality \( n_f \geq 0 \), which is a part of Theorem 2.15(b). \(\Box\)

**Remark 2.43** In Theorem 2.41(c) the pair \((H, L)\) is not unique, in general. However, we do have a weaker uniqueness statement:

**Proposition 2.44** Let \(G, H,\) and \(L\) be as in Theorem 2.41(a). Then there exists a connected normal subgroup \(A \subset G\) such that

\(a\) \(A \subset H\),

\(b\) \(L\bigr|_A\) is \(G\)-invariant, and

\(c\) \(A\) contains all connected normal subgroups of \(G\) satisfying (a) and (b).

Moreover, the \(G\)-orbit of \((A, L\bigr|_A)\) depends only on the idempotent \(f \in \mathcal{D}_G(G)\) constructed in Theorem 2.41(a).

The proposition is proved in Sect. 8.4.

**Remark 2.45** If \(G\) is connected, then \((A, L\bigr|_A)\) is \(G\)-invariant by property (b), so in this case the last statement of the proposition amounts to the assertion that the pair \((A, L\bigr|_A)\) is uniquely determined by \(f\).

2.14 The Verdier dual of a minimal idempotent

The next result is a more precise version of Proposition 2.19. In order to state it, we temporarily return to the framework of ordinary (as opposed to perfect) unipotent groups.

**Proposition 2.46** Let \(G\) be a unipotent algebraic group over \(k\), let \((H, L)\) be an admissible pair for \(G\), write \(G' = N_G(H, L)\), and let \(f = \text{ind}_{G'}^G e_L' \in \mathcal{D}_G(G)\) be the corresponding minimal closed idempotent. There is a natural isomorphism

\[ \mathbb{D}^{-}_{G} f \xrightarrow{\sim} f[-n_f](-n_f). \]  

(2.9)

This result is proved in Sect. 8.12.

**Remark 2.47** If we work in the framework of perfect unipotent groups, the assertion above fails already when \(H = G\). The reason is that the dualizing complex of a smooth perfect quasi-algebraic variety of dimension \(d\) over \(k\) cannot be canonically identified with \(\mathcal{O}_\ell[2d](d)\).
Proposition 2.46 states that with the notation of Proposition 2.19, a choice of an admissible pair \((H, L)\) for \(G\) giving rise to the minimal closed idempotent \(e\) defines an isomorphism 
\[
\mathbb{Q}_\ell \xrightarrow{\sim} L_e.
\]

**Quest 2.48** Does this isomorphism depend on the choice of \((H, L)\)?

### 2.15 Existence of minimal idempotents

**Theorem 2.49** Let \(G\) be a perfect unipotent group over \(k\). Then,

(a) Every minimal closed idempotent in \(\mathcal{D}_G(G)\) is minimal as a weak idempotent.
(b) Every minimal weak idempotent in \(\mathcal{D}_G(G)\) is closed.
(c) For every nonzero object \(N \in \mathcal{D}(G)\), there exists a minimal closed idempotent \(f \in \mathcal{D}_G(G)\) with \(f \ast N \neq 0\).

For the proof, see Sect. 8.7.

**Remark 2.50** Parts (a) and (b) of the last theorem can be reformulated together as follows: The class of minimal weak idempotents in \(\mathcal{D}_G(G)\) coincides with the class of minimal closed idempotents in \(\mathcal{D}_G(G)\).

### 2.16 Some Mackey theory

We remain in the setup of Sect. 2.1. Let \(G\) be a perfect unipotent group over \(k\), and let \(G' \subset G\) be a closed subgroup. The following notion is an obvious geometrization of the Mackey irreducibility criterion for induced characters of finite groups, as stated in #3 of Sect. 2.8.

**Definition 2.51** A weak idempotent \(e \in \mathcal{D}_G'(G')\) is said to satisfy the geometric Mackey condition with respect to \(G\) if for every \(x \in G(k) \setminus G'(k)\), we have \(e \ast \delta_x \ast e = 0\), where \(\delta_x \in \mathcal{D}(G)\) denotes the delta-sheaf at the point \(x\) and \(e\) denotes the object of \(\mathcal{D}(G)\) obtained by extending \(e\) to \(G\) by zero outside of \(G'\).

The last main result of our work, proved in Sect. 8.8, is

**Theorem 2.52** Let \(G\) be a perfect unipotent group over \(k\), let \(G' \subset G\) be a closed subgroup, let \(e \in \mathcal{D}_G'(G')\) be a minimal closed idempotent satisfying the geometric Mackey condition with respect to \(G\), and let \(f = \text{ind}^G_G e\). Then,

(a) \(f\) is a minimal closed idempotent in \(\mathcal{D}_G(G)\);
(b) we have \(n_f = n_e - \dim(G/G')\), or, equivalently, \(d_f = d_e + \dim(G/G')\);
(c) the functor \(\text{ind}^G_{G'}\) restricts to a monoidal equivalence \(e \mathcal{D}_G'(G') \xrightarrow{\sim} f \mathcal{D}_G(G)\), which is compatible with the braidings constructed in Definition 9.43 and the twists constructed in Definition 9.45;
(d) for every \(M \in e \mathcal{D}_G'(G')\), the canonical arrow \(\text{can}_M : \text{ind}^G_{G'} M \longrightarrow \text{Ind}^G_{G'} M\) is an isomorphism;
(e) if \(M \in e \mathcal{D}_G'(G')\) is perverse, then so is \([\text{dim}(G/G') \times \text{ind}^G_{G'} M])\].
Remark 2.53 The functor $\text{ind}^G_{G'} : \mathcal{D}_{G'}(G') \longrightarrow \mathcal{D}_G(G)$ is usually not monoidal.\textsuperscript{13} Nevertheless, in Sect. 8.1 we construct a weak semigroupal structure on $\text{ind}^G_{G'}$ in the sense of Definition 3.4. The first part of assertion (c) of Theorem 2.52 means that $\text{ind}^G_{G'}$ restricts to an equivalence $F : e\mathcal{D}'_G(G') \rightarrow f\mathcal{D}_G(G)$ and that the induced weak semigroupal structure on $F$ is actually strong (see Definition 3.4 and Remark 3.5). The other two statements of assertion (c) follow from the more general Lemma 8.6.

Remark 2.54 In the situation of Theorem 2.52, there is a canonical bijection between the set of idempotent arrows $\mathbb{1} \longrightarrow e$ and the set of idempotent arrows $\mathbb{1} \longrightarrow f$ (see Remark 8.4 for more details).

2.17 The main facts used in the proofs

The proofs of the main results of this article rely on the theory of Fourier–Deligne transform, the equality

$$\text{Ext}^2(\mathbb{G}_a, \mathbb{Q}_p/\mathbb{Z}_p) = 0 \quad (2.10)$$

proved by Breen [17], and the fact that the orbits of a unipotent group acting on an affine variety are closed. The latter is used in the proof of Proposition 8.17, and the equality (2.10) was used in [12] to prove the key Proposition 5.6.

2.18 Organization of the text

The rest of the article is organized as follows. In Sect. 3 we define and study the notion of a closed idempotent in an abstract monoidal category. We establish several general results that are used in the proofs of the main theorems of the paper.

In Sect. 4 we recall the notion of Serre duality [6,47] for perfect connected unipotent groups in characteristic $p > 0$ and that of Fourier–Deligne transform. We have to work in a setting slightly more general than the standard one (namely, to consider multiplicative local systems on non-necessarily commutative connected unipotent groups). The main results on the Fourier–Deligne transform in this setting are Proposition 4.15 and Theorem 4.16.

In Sect. 5 we establish Proposition 5.4 and Theorem 5.5, which are used in Sect. 8 to prove the main theorems of the article by induction. The proofs of the results of Sect. 5 are somewhat similar to the proof of the classical theorem that any irreducible representation of a finite nilpotent group is induced from a 1-dimensional representation of a subgroup.

We devote Sect. 6 to a general study of triples $(G, G', e)$ satisfying the geometric Mackey condition from Definition 2.51, while Sect. 7 establishes several properties of the functors $\text{av}_{G/G'}$ and $\text{Av}_{G/G'}$.

The main results of the article, stated in Sects. 2.6–2.16, are proved in Sect. 8. They are deduced without difficulty from the crucial Proposition 8.14, proved in Sect. 8.15.

\textsuperscript{13} Similarly, if $\Gamma' \subset \Gamma$ are finite groups, the induction map from class functions on $\Gamma'$ to class functions on $\Gamma$ does not preserve convolution.
In Appendices 1 and 2, we discuss certain abstract categorical formalisms and apply them to the category $D_G(G)$ from Sect. 2.4.

Appendix 1 is devoted to a duality formalism in the spirit of Grothendieck and Verdier. We introduce the notions of a Grothendieck–Verdier category and of an r-category. The latter generalizes the notion of a rigid monoidal category. We define the notions of a pivotal structure and a ribbon structure on a Grothendieck–Verdier category. (A more detailed exposition of the theory of Grothendieck–Verdier categories appears in [16]). Assuming that $G^\circ$ is unipotent, we prove in Appendix 1 that the monoidal category $D_G(G)$ is an r-category and describe a natural ribbon structure on it (without the unipotence assumption, this is done in Appendix 2). We also prove that every Hecke subcategory of a pivotal (respectively, ribbon) Grothendieck–Verdier category is also a pivotal (respectively, ribbon) Grothendieck–Verdier category.

Appendix 2 is devoted to a formalism of “pre-topological field theories” (the notion of a pre-TFT is a “lax” version of the notion of a TFT). Slightly modifying [10, §6], we associate a 2-dimensional pre-TFT to any algebraic stack $X$ of finite type over a field $k$. In the case where $X$ is the classifying stack of an algebraic group $G$, this pre-TFT allows us to solve two problems. First, we define a canonical ribbon structure on the r-category $D_G(G)$, where $G$ is an arbitrary algebraic group over $k$ and $D_G(G)$ is understood as the bounded derived category of constructible $\ell$-adic complexes on the stack quotient of $G$ by the adjoint action of $G$ [38]. Second, we prove a key Lemma 8.6 (on the compatibility of the functor $\text{ind}_{G'}^G : D_G'(G') \to D_G(G)$ with braidings and twists).

In Appendix 3 we show that if $G$ is an algebraic group over $k$ acting on a scheme $X$ of finite type over $k$, then the category $D_G(X)$ constructed in Definition 2.3 is naturally equivalent to the category $D^b_c(G \setminus X, \mathbb{Q}_\ell)$, where $G \setminus X$ is the stack quotient of $X$ by $G$, provided that the neutral connected component $G^\circ$ of $G$ is unipotent.

3 Idempotents in monoidal categories

3.1 Monoidal categories

There are several equivalent approaches to defining the notion of a monoidal category. The following approach will be the most convenient for us. If $\mathcal{M}$ is a category, a semigroupal structure on $\mathcal{M}$ is a pair consisting of a functor $\otimes : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ and a functorial collection, $\alpha$, of isomorphisms

$$\alpha_{X,Y,Z} : (X \otimes Y) \otimes Z \cong X \otimes (Y \otimes Z)$$

for all $X, Y, Z \in \mathcal{M}$, satisfying the “pentagon axiom” [43, p. 162]. One calls $\alpha$ the associativity constraint for $\otimes$. A semigroupal category is a triple $(\mathcal{M}, \otimes, \alpha)$ consisting of a category $\mathcal{M}$ and a semigroupal structure $(\otimes, \alpha)$ on $\mathcal{M}$. Often, by abuse of notation, one denotes a semigroupal category by a single letter such as $\mathcal{M}$.

**Definitions 3.1** (1) An object $E$ of a semigroupal category $\mathcal{M}$ is said to be unital if the functors $X \mapsto E \otimes X$ and $X \mapsto X \otimes E$ are isomorphic to the identity functor on $\mathcal{M}$. 
A semigroupal category is said to be monoidal if it has a unital object. A unit object in a monoidal category \( M \) is a pair \((E, u)\) consisting of a unital object \( E \) and an isomorphism\(^{14}\) \( u : E \otimes E \xrightarrow{\simeq} E \).

**Lemma 3.2** ([35], Rem. 4.2.7) If \( M \) is a monoidal category and \((E, u)\) and \((E', u')\) are two unit objects of \( M \), there exists a unique isomorphism \( f : E \xrightarrow{\simeq} E' \) such that \( u' \circ (f \otimes f) = f \circ u \).

The lemma allows us to speak of ‘the’ unit object in a monoidal category (rather than ‘a’ unit object). The unit object of a monoidal category will usually be denoted by \( 1 \).

**Lemma 3.3** ([35], Lem. 4.2.6) If \( M \) is a monoidal category and \( 1 = (1, u) \) is a unit object in \( M \), there exist unique functorial collections of isomorphisms

\[
\lambda = \{\lambda_X : 1 \otimes X \xrightarrow{\simeq} X\}_{X \in M} \quad \text{and} \quad \rho = \{\rho_X : X \otimes 1 \xrightarrow{\simeq} X\}_{X \in M}
\]

such that \( \lambda_1 = u = \rho_1 \) and the triangle axiom [43, p. 163] holds.

This lemma implies that the definition of a monoidal category given above is equivalent to the one from [43].

### 3.2 Semigroupal and monoidal functors

**Definition 3.4** Let \( M_1 \) and \( M_2 \) be semigroupal categories.

(a) A weak semigroupal structure on a functor \( F : M_1 \to M_2 \) between the underlying categories is a functorial collection, \( \phi \), of morphisms

\[
\phi(X, Y) : F(X) \otimes F(Y) \longrightarrow F(X \otimes Y) \quad \forall X, Y \in M_1
\]

that are compatible with the associativity constraints on \( M_1 \) and \( M_2 \).

(b) If each \( \phi(X, Y) \) is an isomorphism, then \( \phi \) is called a semigroupal structure (or a strong semigroupal structure).

(c) A semigroupal (respectively, weakly semigroupal) functor \( M_1 \to M_2 \) is a functor \( M_1 \to M_2 \) equipped with a semigroupal (respectively, weakly semigroupal) structure.

(d) If \( M_1 \) and \( M_2 \) are monoidal categories, a semigroupal functor \( M_1 \to M_2 \) is said to be monoidal if it takes unital objects to unital objects.

**Remark 3.5** Suppose that \( M_1 \) and \( M_2 \) are monoidal categories and \( F : M_1 \to M_2 \) is a semigroupal functor that is an equivalence of the underlying categories. Then \( F \) automatically takes unital objects to unital objects. In other words, a semigroupal equivalence between monoidal categories is the same as a monoidal equivalence.

\(^{14}\) By definition, there exists such an isomorphism for every unital object \( E \).
Remark 3.6 The notion of a weakly semigroupal functor is closely related to the notion of pseudo-tensor functor between pseudo-tensor categories from [8] (see §1.1.1 and §1.1.5 of op. cit.). Namely, by [8, §1.1.3], a semigroupal category can be considered as a special kind of pseudo-tensor category, and by [8, §1.1.6(ii)], a pseudo-tensor functor between semigroupal categories is the same as a weakly semigroupal functor.

3.3 Algebras

If \((\mathcal{M}, \otimes, \alpha)\) is a semigroupal category and \(A \in \mathcal{M}\), one defines the notion of an associative morphism \(\mu : A \otimes A \to A\) in the usual way. A pair \((A, \mu)\) consisting of an object \(A \in \mathcal{M}\) and an associative morphism \(\mu : A \otimes A \to A\) is called an algebra in \(\mathcal{M}\). A weakly semigroupal functor takes algebras to algebras.

If \(\mathcal{M}\) is monoidal, one also has the notion of a unital algebra (called a “monoid” in [43] and a “ring” in [35]) in \(\mathcal{M}\). In this case \(I_\mathcal{M}\) has a canonical structure of unital algebra, and any object of \(\mathcal{M}\) has a canonical structure of \(I_\mathcal{M}\)-bimodule.

Remark 3.7 A weakly semigroupal functor \(F : \mathcal{M}_1 \to \mathcal{M}_2\) between monoidal categories is said to be weakly monoidal if the algebra \(A := F(I_{\mathcal{M}_1})\) is unital and for every \(X \in \mathcal{M}_1\) the \(A\)-bimodule \(F(X)\) is unital. It is easy to show that a (strongly) semigroupal functor \(F\) is monoidal if and only if it is weakly monoidal.

3.4 Main definitions

Definitions 3.8 Let \(\mathcal{M}\) be a monoidal category.

(a) An object \(e \in \mathcal{M}\) is said to be a weak idempotent if \(e \otimes e \cong e\).
(b) An idempotent algebra in \(\mathcal{M}\) is an algebra \((e, \mu)\) such that \(\mu : e \otimes e \to e\) is an isomorphism.
(c) A morphism \(\textbf{1} \xrightarrow{\pi} e\) in \(\mathcal{M}\) is said to be an idempotent arrow if both morphisms

\[
\begin{align*}
\textbf{1} \otimes e & \xrightarrow{\pi \otimes \text{id}_e} e \otimes e \\
e \otimes \textbf{1} & \xrightarrow{\text{id}_e \otimes \pi} e \otimes e
\end{align*}
\]

are isomorphisms.
(d) An object \(e \in \mathcal{M}\) is said to be a closed idempotent if there exists an idempotent arrow \(\textbf{1} \rightarrow e\).

Remarks 3.9 (i) In the situation of (b), (c), or (d), the object \(e \in \mathcal{M}\) is a weak idempotent.
(ii) Lemma 3.10 below asserts that if both arrows (3.1) are isomorphisms, then they are equal as isomorphisms \(e \xrightarrow{\sim} e \otimes e\). On the other hand, it is not enough to require one of them to be an isomorphism: see Remark 3.13.
(iii) The term “closed idempotent” may be new, but the notion itself is not. For instance, a special class of closed idempotents, known as idempotent monads, was known for several decades (see Sect. 3.6 for more details). Moreover, closed idempotents in the general sense appear in [35, Exercise 4.2].
(iv) The origin of the adjective “closed” is explained in Sect. 3.5.
(v) A unit of an associative algebra is unique if it exists. In particular, this remark applies to idempotent algebras.
(vi) If \((e, \mu)\) is an idempotent unital algebra in \(\mathcal{M}\), then the unit \(\pi : 1 \longrightarrow e\) is an idempotent arrow in \(\mathcal{M}\). Indeed, the compositions \(1 \otimes e \xrightarrow{\pi \otimes \text{id}_e} e \otimes e \xrightarrow{\mu} e\) and \(e \otimes 1 \xrightarrow{\text{id}_e \otimes \pi} e \otimes e \xrightarrow{\mu} e\) are equal to \(\lambda_e\) and \(\rho_e\), respectively, and hence are isomorphisms. But, \(\mu\) is an isomorphism, hence so are the arrows (3.1).
(vii) We will see later (Proposition 3.36) that the notion of idempotent arrow is in fact equivalent to the notion of an idempotent unital algebra.
(viii) The notion of a weak idempotent is not very well behaved (see, for instance, Remark 3.19). The notion of a closed idempotent is much better behaved. For example, idempotent arrows have no automorphisms, and isomorphism classes of idempotent arrows bijectively correspond to isomorphism classes of closed idempotents (see Corollary 3.40).

**Lemma 3.10** If \(\pi : 1 \longrightarrow e\) is an idempotent arrow in a monoidal category \(\mathcal{M}\), then the isomorphisms \(e \sim e \otimes e\) corresponding to isomorphisms (3.1) are equal to each other.

**Proof** By [35, Lem. 4.1.2], the lemma holds if \(\mathcal{M}\) is the category of endofunctors of some category. To prove the lemma for any \(\mathcal{M}\), note that the left action of \(\mathcal{M}\) on itself defines a monoidal functor \(F : \mathcal{M} \longrightarrow \text{End}(\mathcal{M})\), where \(\text{End}(\mathcal{M})\) is the monoidal category of functors \(\mathcal{M} \longrightarrow \mathcal{M}\). Clearly, \(F(\pi)\) is an idempotent arrow in \(\text{End}(\mathcal{M})\), so by loc. cit., the isomorphisms (3.1) have equal images under \(F\). Since \(F\) is faithful, we are done. \(\square\)

### 3.5 An example

In this subsection we give an example that explains the origin of the adjective “closed” in Definition 3.8(c). At the same time, we show that in general, not every weak idempotent in a monoidal category is closed. Another important class of examples appears in Sect. 3.6 below.

Let \(X\) be a scheme, and let \(i : Y \longrightarrow X\) be a morphism of finite type such that the diagonal morphism \(Y \longrightarrow Y \times_X Y\) is an isomorphism.\(^{15}\) If \(A\) is any unital commutative ring, then sheaves of \(A\)-modules on the étale site \(X_{\text{ét}}\) form a monoidal category \(\mathcal{M}\) with respect to the usual tensor product. Consider the sheaf \(i_*A_Y \in \mathcal{M}\), where \(A_Y\) is the constant sheaf on \(Y_{\text{ét}}\) with stalk \(A\). Each stalk of \(i_*A_Y\) equals \(A\) or 0, so it has a canonical ring structure. It is easy to show that these structures come from a unique structure of idempotent algebra on \(i_*A_Y\). In particular, \(i_*A_Y\) is a weak idempotent. It is easy to show that \(i_*A_Y\) is a closed idempotent in \(\mathcal{M}\) if and only if \(i : Y \longrightarrow X\) is a closed embedding. Furthermore, if \(A\) is a field, then every closed idempotent in \(\mathcal{M}\) is isomorphic to \(i_*A_Y\) for some closed \(Y \subset X\).

\(^{15}\) Such an \(i\) is not necessarily a locally closed embedding. For instance (if \(X\) is Noetherian), one can take \(Y = Z \coprod (X \setminus Z)\), where \(Z \subset X\) is any closed subscheme.
3.6 Closed idempotents via adjoint functors

If $C$ is a (small) category, the category $\text{End}(C)$ of functors $C \rightarrow C$ is a strictly associative and strictly unital monoidal category with respect to composition of functors. Closed idempotents in this category were studied, under the name of idempotent monads, in a number of earlier works, among which we mention [1, 20, 21]. In this subsection we summarize the basic facts relating idempotent monads to the notions of adjoint functors and reflective subcategories, mostly following [35, §4.1] (the term “projector” is used in loc. cit. in place of “idempotent monad”).

Let us first recall the following

Definition 3.11 A subcategory $D$ of a category $C$ is said to be reflective if the inclusion functor $D \hookrightarrow C$ admits a left adjoint.

Proposition 3.12 ([35], Prop. 4.1.3–4.1.4)

(a) Let $C$ be a category, let $D \subset C$ be a full reflective subcategory, and let $L : C \rightarrow D$ be a left adjoint of the inclusion functor $I : D \hookrightarrow C$. Further, let us denote the adjunction morphisms by $\eta : L \circ I \rightarrow \text{Id}_D$ and $\epsilon : \text{Id}_C \rightarrow I \circ L$. Then $\eta$ is an isomorphism, and $\epsilon$ is an idempotent arrow in the category of endofunctors of $C$.

(b) Conversely, let $P : C \rightarrow C$ be a functor, and let $\epsilon : \text{Id}_C \rightarrow P$ denote an idempotent arrow in the category of endofunctors of $C$. Given an object $X \in C$, the following three statements are equivalent:

(i) The arrow $\epsilon_X : X \rightarrow P(X)$ is an isomorphism;
(ii) For any $Y \in C$, the map $\text{Hom}(P(Y), X) \rightarrow \text{Hom}(Y, X)$ given by $f \mapsto f \circ \epsilon_Y$ is bijective;
(iii) The map in (ii) is surjective for $Y = X$.

If $D$ is the full subcategory of $C$ consisting of objects $X \in C$ satisfying the equivalent conditions (i)–(iii), then $P(X) \in D$ for any $X \in C$. Furthermore, $P$ induces a functor $C \rightarrow D$ which is left adjoint to the inclusion functor $D \hookrightarrow C$. A fortiori, $D$ is a reflective subcategory of $C$.

Remark 3.13 The following example, taken from [35, Exercise 4.1], shows that in Definition 3.8(c) it is not enough to require one of the two arrows in (3.1) to be an isomorphism.

Let $C$ be the category with one object, $X$, such that $\text{End}_C(X) = \{\text{id}_X, p\}$ and the arrow $p : X \rightarrow X$ satisfies $p^2 = p$. Further, let $P : C \rightarrow C$ be the functor defined by $P(\text{id}_X) = \text{id}_X = P(p)$, and let $\epsilon : \text{Id}_C \rightarrow P$ be the morphism of functors given by $\epsilon_X = p$. Then $P \circ \epsilon : P \rightarrow P^2$ is an isomorphism, but $\epsilon \circ P : P \rightarrow P^2$ is not.

3.7 Hecke subcategories

Let $M$ be a monoidal category and $e \in M$ a weak idempotent. The essential image of the functor $M \rightarrow M$ defined by $X \mapsto e \otimes X$ (respectively, $X \mapsto X \otimes e$, $X \mapsto e \otimes X \otimes e$) will be denoted by $eM$ (respectively, $Me$, $eMe$). The full subcategories $eM$, $Me$, and $eMe$ of the category $M$ can also be described as follows:
\[ eM = \{ X \in \mathcal{M} \mid e \otimes X \cong X \}, \quad \mathcal{M}e = \{ X \in \mathcal{M} \mid X \otimes e \cong X \}, \quad (3.2) \]
\[ eMe = \{ X \in \mathcal{M} \mid e \otimes X \otimes e \cong X \}. \quad (3.3) \]

Lemma 3.14 \( eMe = eM \cap \mathcal{M}e. \)

**Proof** By definition, \( eMe \subseteq eM \cap \mathcal{M}e. \) By (3.2)–(3.3), \( eMe \subseteq eM \cap \mathcal{M}e. \) \( \square \)

If the idempotent \( e \) is closed, then \( eM \) and \( \mathcal{M}e \) have the following description.

Lemma 3.15 Let \( \pi : 1 \longrightarrow e \) be an idempotent arrow. An object \( X \in \mathcal{M} \) belongs to \( eM \) (respectively, \( \mathcal{M}e \)) if and only if the morphism \( \pi \otimes \text{id}_X : 1 \otimes X \longrightarrow e \otimes X \) (respectively, \( \text{id}_X \otimes \pi : X \otimes 1 \longrightarrow X \otimes e \)) is an isomorphism.

Corollary 3.16 If \( e \in \mathcal{M} \) is a closed idempotent, then the subcategories \( eM, \mathcal{M}e, \) and \( eMe \) of \( \mathcal{M} \) are closed under retracts.

**Proof** It suffices to consider \( eM. \) If \( \pi : 1 \longrightarrow e \) is an idempotent arrow, \( X \in eM, \) and \( Y \) is a retract of \( X, \) then the morphism \( \pi \otimes \text{id}_Y : 1 \otimes Y \longrightarrow e \otimes Y \) is a retract of \( \pi \otimes \text{id}_X : 1 \otimes X \longrightarrow e \otimes X. \) But a retract of an isomorphism is an isomorphism. \( \square \)

Definition 3.17 We call \( eMe \subseteq \mathcal{M} \) the Hecke subcategory of \( \mathcal{M} \) defined by \( e. \)

The full subcategory \( eMe \subseteq \mathcal{M} \) is closed under \( \otimes. \) So, it is a semigroupal category.

Lemma 3.18 If \( e \in \mathcal{M} \) is a closed idempotent, then the semigroupal category \( eMe \) is monoidal and \( e \) is a unital object of \( eMe. \)

**Proof** Lemma 3.15 shows that the functor \( X \mapsto e \otimes X \) is isomorphic to the identity functor on the subcategory \( eM \subseteq \mathcal{M} \) and \( a \text{ fortiori}, \) on the subcategory \( eMe \subseteq \mathcal{M}. \) Similarly, the functor \( X \mapsto X \otimes e \) is isomorphic to the identity functor on the subcategory \( \mathcal{M}e \subseteq \mathcal{M}, \) and hence also on \( eMe. \) \( \square \)

Remark 3.19 If \( e \in \mathcal{M} \) is a weak idempotent, then the semigroupal category \( eMe \) may fail to be monoidal.\(^{16}\) For instance, this happens if \( \mathcal{M} \) is the category \( D^b_c(\mathbb{A}^1, \mathbb{Q}_\ell) \) equipped with the usual tensor product and \( e = j_i\mathcal{Q}_\ell \oplus i!\mathcal{Q}_\ell, \) where \( j : \mathbb{A}^1 \setminus \{0\} \hookrightarrow \mathbb{A}^1 \) is the open embedding and \( i : \{0\} \hookrightarrow \mathbb{A}^1 \) is the corresponding closed embedding.\(^{17}\) Note that if \( G = \mathbb{G}_a, \) then the monoidal category \((D(\mathbb{A}^1), \otimes)\) is equivalent to \((\mathcal{O}_G(G), \ast)\) via the Fourier–Deligne transform.

Lemma 3.20 Let \( \mathcal{O} \xrightarrow{\pi} e \) be an idempotent arrow in a monoidal category \( \mathcal{M}, \) and let \( e \xrightarrow{\sigma} f \) be an idempotent arrow in the Hecke subcategory \( eMe. \) Then \( \sigma \circ \pi \) is an idempotent arrow in \( \mathcal{M}. \)

\(^{16}\) If \( e \) is an idempotent algebra, there is a remedy, see Remark 3.34.

\(^{17}\) Let \( e_1 = j_i\mathcal{Q}_\ell \) and \( e_2 = j_i\mathcal{Q}_\ell. \) Then \( e_1 \otimes e_1 \cong e_1, \) \( e_2 \otimes e_2 \cong e_2 \) and \( e_1 \otimes e_2 = 0. \) Thus, \( e_1, e_2 \in eMe. \) For \( n = 1, 2 \) there exist nonzero morphisms \( e_1 \rightarrow e_2[n] \) in \( \mathcal{M}. \) They are annihilated by the functors \( X \mapsto e_1 \otimes X \) and \( X \mapsto e_2 \otimes X, \) and hence also by the functor \( X \mapsto e \otimes X. \) In particular, the latter functor is not an autoequivalence of \( eM. \)
Proof Since \( f \in eM \subset eM \) the arrow \( \pi \otimes \text{id}_f : 1 \otimes f \to e \otimes f \) is an isomorphism by Lemma 3.15. The arrow \( \varpi \otimes \text{id}_f : e \otimes f \to f \otimes f \) is an isomorphism by assumption, so \( (\varpi \circ \pi) \otimes \text{id}_f : 1 \otimes f \to f \otimes f \) is an isomorphism. Similarly, \( \text{id}_f \otimes (\varpi \circ \pi) : f \otimes 1 \to f \otimes f \) is an isomorphism. \( \square \)

3.8 Adjunction properties

The following result (Proposition 3.22) will play an important role in the proof of Proposition 6.9.

Definition 3.21 A semigroupal category \( \mathcal{M} \) is said to be \textit{weakly symmetric} if the functors \( \mathcal{M} \times \mathcal{M} \to \mathcal{M} \) given by \( (X, Y) \mapsto X \otimes Y \) and \( (X, Y) \mapsto Y \otimes X \) are isomorphic. (The name “weakly symmetric” may be nonstandard.)

For example, a braided monoidal category is weakly symmetric. In general, if \( \mathcal{M} \) is weakly symmetric and \( e \in \mathcal{M} \) is a weak idempotent, it is clear that all three subcategories \( eM \), \( Me \), and \( eMe \) of \( \mathcal{M} \) coincide. Normally, we will write \( eM \) for the Hecke subcategory of \( \mathcal{M} \) defined by \( e \) in this situation.

Proposition 3.22 (a) If \( \xymatrix{1 \ar[r]^\pi & e} \) is an idempotent arrow in a monoidal category \( \mathcal{M} \), then the functor \( L : \mathcal{M} \to eM \) given by \( X \mapsto X \otimes e \) is left adjoint to the inclusion functor \( I : eM \hookrightarrow \mathcal{M} \). More precisely, the family of morphisms

\[
\{ \pi \otimes \text{id}_X : X \to e \otimes X \}_{X \in \mathcal{M}} \tag{3.4}
\]

defines an adjunction morphism \( \text{Id}_\mathcal{M} \to I \circ L \).

(b) Here is a partial converse. Let \( \mathcal{M} \) be a weakly symmetric monoidal category, and let \( e \in \mathcal{M} \) be a weak idempotent. Suppose that the functor \( \mathcal{M} \to eM \) given by \( X \mapsto e \otimes X \) is left adjoint to the inclusion functor \( eM \hookrightarrow \mathcal{M} \). Then \( e \) is a closed idempotent in \( \mathcal{M} \).

Proof (a) Consider the functor

\[
P = I \circ L : \mathcal{M} \to \mathcal{M}, \quad P(X) = e \otimes X, \tag{3.5}
\]

and note that (3.4) defines an idempotent arrow \( \epsilon : \text{Id}_\mathcal{M} \to P \) in the category of endofunctors of \( \mathcal{M} \). It remains to apply Proposition 3.12(b) and Lemma 3.15.

(b) By Proposition 3.12(a), there exists an idempotent arrow \( \epsilon : \text{Id}_\mathcal{M} \to P \), where \( P \) is defined by (3.5). Define \( \pi : 1 \to e \) to be the composition of \( \epsilon_1 : 1 \to e \otimes 1 \) and the canonical isomorphism \( \rho_e : e \otimes 1 \cong e \). Since \( \epsilon \) is an idempotent arrow, the morphism \( \text{id}_e \otimes \epsilon_X : e \otimes X \to e \otimes (e \otimes X) \) is an isomorphism for every \( X \in \mathcal{M} \). Setting \( X = 1 \), we see that \( \text{id}_e \otimes \pi : e \otimes 1 \to e \otimes e \) is an isomorphism. Since \( \mathcal{M} \) is weakly symmetric, \( \pi \otimes \text{id}_e : 1 \otimes e \to e \otimes e \) is also an isomorphism. \( \square \)
3.9 Minimal idempotents and Jacobson monoidal categories

**Definition 3.23** Let $\mathcal{M}$ be a weakly symmetric monoidal category that has a zero object. A minimal closed (respectively, weak) idempotent in $\mathcal{M}$ is a closed (respectively, weak) idempotent $e \in \mathcal{M}$ such that $e \neq 0$, and such that for every closed (respectively, weak) idempotent $e' \in \mathcal{M}$, we have either $e \otimes e' = 0$ or $e \otimes e' \cong e$.

**Example 3.24** Let $X$, $A$, and $\mathcal{M}$ be as in Sect. 3.5. Suppose that $A$ is a field. Then for every closed point $x \in X$, the closed idempotent $i_{i}A_{\{x\}} \in \mathcal{M}$ is minimal among weak idempotents. Moreover, all minimal closed idempotents in $\mathcal{M}$ have this form.

**Remark 3.25** A minimal closed idempotent might fail to be minimal as a weak idempotent. On the other hand, if a minimal weak idempotent happens to be a closed idempotent, then it is clearly minimal as a closed idempotent as well.

**Remark 3.26** We will show in Sect. 3.12 that minimal closed idempotents can also be defined in terms of a certain partial order on the set of isomorphism classes of all nonzero closed idempotents in $\mathcal{M}$ (cf. Remark 3.45 (iii)). Moreover, the construction of this partial order does not require the assumption that $\mathcal{M}$ is weakly symmetric.

The following observation is occasionally useful.

**Lemma 3.27** Let $\mathcal{M}$ be a weakly symmetric monoidal category with a zero object, $0$. A closed idempotent $e \in \mathcal{M}$ is minimal among closed idempotents if and only if $e \neq 0$ and the Hecke subcategory $e\mathcal{M}$ has no closed idempotents (up to isomorphism) apart from $0$ and its unit object, $e$. Similarly, a weak idempotent $e \in \mathcal{M}$ is minimal among weak idempotents if and only if $e \neq 0$ and every nonzero weak idempotent in the Hecke subcategory $e\mathcal{M} \subset \mathcal{M}$ is isomorphic to $e$.

**Proof** The “only if” statements are clear. Let us prove the “if” statements. Let $\mathcal{M}$ be a weakly symmetric monoidal category. If $\pi : 1 \to e$ and $\pi' : 1 \to e'$ are idempotent arrows in $\mathcal{M}$, the fact that $\mathcal{M}$ is weakly symmetric implies that $id_e \otimes \pi : e \to e \otimes e'$ is an idempotent arrow in $e\mathcal{M}$. This proves the first assertion. For the second assertion, observe that if $e$ and $e'$ are weak idempotents in a weakly symmetric semigroupal category $\mathcal{M}$, then $e \otimes e'$ is also a weak idempotent. \qed

**Definition 3.28** A weakly symmetric monoidal category $\mathcal{M}$ with a zero object is Jacobson if for every nonzero $N \in \mathcal{M}$ there exists a closed idempotent $e \in \mathcal{M}$ such that $e \otimes N \neq 0$ and $e$ is minimal among weak idempotents in $\mathcal{M}$.

**Example 3.29** In the situation of Example 3.24, $\mathcal{M}$ is Jacobson if and only if $X$ is a Jacobson scheme in the sense of [31, §10].

**Proposition 3.30** Let $\mathcal{M}$ be a Jacobson monoidal category.

(a) Every minimal closed idempotent in $\mathcal{M}$ is minimal as a weak idempotent.

(b) Every minimal weak idempotent in $\mathcal{M}$ is closed.

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18 In all the applications we have in mind, $\mathcal{M}$ will in fact be additive.
Proof We first prove (a) and (b) simultaneously. Let \( f \in \mathcal{M} \) be either a minimal closed idempotent or a minimal weak idempotent, and let \( e \in \mathcal{M} \) be a closed idempotent that is minimal as a weak idempotent and satisfies \( e \otimes f \neq 0 \). In either case, using the minimality of \( e \) and \( f \), we find \( e \cong e \otimes f \cong f \).

Let us prove (c). For any nonzero object \( N \in e\mathcal{M} \), we can find a closed idempotent \( f \in \mathcal{M} \) that is minimal as a weak idempotent and satisfies \( f \otimes N \neq 0 \). Since \( N \in e\mathcal{M} \) and \( f \otimes N \neq 0 \), we have \( f \otimes e \neq 0 \), so the minimality of \( f \) implies that \( f \otimes e \cong f \), i.e., \( f \in e\mathcal{M} \). Clearly, \( f \) is a minimal weak idempotent in \( e\mathcal{M} \). Moreover, \( f \) is a closed idempotent in \( e\mathcal{M} \) (if \( \pi : 1 \to f \) is an idempotent arrow in \( \mathcal{M} \) and \( e\mathcal{M} \) is monoidal, then \( \text{id}_e \otimes \pi : e \to e \otimes f \) is an idempotent arrow in \( e\mathcal{M} \)).

Proposition 3.31 Let \( \mathcal{M} \) be a weakly symmetric monoidal category with a zero object. Suppose that for every nonzero \( N \in \mathcal{M} \) there exists a closed idempotent \( \tilde{e} \in \mathcal{M} \) such that \( \tilde{e} \otimes N \neq 0 \) and \( \tilde{e}\mathcal{M} \) is Jacobson. Then \( \mathcal{M} \) is Jacobson.

Proof Let \( N \in \mathcal{M} \) be nonzero, and choose a closed idempotent \( \tilde{e} \in \mathcal{M} \) such that \( \tilde{e} \otimes N \neq 0 \) and \( \tilde{e}\mathcal{M} \) is Jacobson. Then \( \mathcal{M} \) is Jacobson.

3.10 Modules over an idempotent algebra

Statement (ii) of the following lemma allows us to interpret the adjunction from Proposition 3.22 as a “free-forget” adjunction.

Lemma 3.32 Let \( (e, \mu_e) \) be a unital idempotent algebra in a monoidal category \( \mathcal{M} \).

(i) A left \( e \)-module (respectively, right \( e \)-module, \( e \)-bimodule) \( X \) is unital if and only if the action morphism \( e \otimes X \to X \) (respectively, \( X \otimes e \to X \), \( e \otimes X \otimes e \to X \)) is an isomorphism.

(ii) Let \( e - \text{mod}_\mathcal{M} \) denote the category of unital left \( e \)-modules. Then the forgetful functor \( F : e - \text{mod}_\mathcal{M} \to \mathcal{M} \) is fully faithful, and its essential image equals \( e\mathcal{M} \). Similarly, the category of unital right \( e \)-modules (respectively, \( e \)-bimodules) identifies with \( \mathcal{M}e \) (respectively, \( e\mathcal{M}e \)).

Proof We will consider only left modules.

(i) The action morphism \( \alpha : e \otimes X \to X \) is an \( e \)-module morphism, so if \( \alpha \) is an isomorphism, then the \( e \)-module \( X \) is isomorphic to the free \( e \)-module \( e \otimes X \), which is clearly unital. Now suppose that \( X \) is unital, i.e., the composition

\[
X \xrightarrow{\cong} 1 \otimes X \xrightarrow{\pi \otimes \text{id}_X} e \otimes X \xrightarrow{\alpha} X
\]
equals \( \text{id}_X \). Then \( X \) is a retract of \( e \otimes X \in e\mathcal{M} \), so \( X \in e\mathcal{M} \) by Corollary 3.16. Therefore, \( \pi \otimes \text{id}_X \) is an isomorphism by Lemma 3.15. But, the composition (3.6) equals \( \text{id}_X \), so \( \alpha \) is an isomorphism.

(ii) By \([43, \text{p. 174}]\), \( F \) has a left adjoint, namely the “free module” functor \( \Phi : \mathcal{M} \rightarrow e - \text{mod} \mathcal{M} \), \( \Phi(X) := e \otimes X \). By (i), the adjunction \( \Phi F \rightarrow \text{id}_\mathcal{M} \) is an isomorphism, so \( F \) is fully faithful. It remains to show that the adjunction \( X \rightarrow \Phi F(X) \), \( X \in \mathcal{M} \), is an isomorphism if and only if \( X \in e\mathcal{M} \). This is a reformulation of Lemma 3.15. \( \square \)

Lemma 3.32 (i) allows us to give the following definition.

**Definition 3.33** Let \( e \) be a not necessarily unital idempotent algebra. A left \( e \)-module (respectively, right \( e \)-module, \( e \)-bimodule) \( X \) is said to be **unital** if the action morphism \( e \otimes X \rightarrow X \) (respectively, \( X \otimes e \rightarrow X \), \( e \otimes X \otimes e \rightarrow X \)) is an isomorphism.

**Remark 3.34** If \( e \) is not unital, then the category of unital left \( e \)-modules (respectively, unital right \( e \)-modules, unital \( e \)-bimodules) seems to be more reasonable than the category \( e\mathcal{M} \) (respectively, \( \mathcal{M}e \) and \( e\mathcal{M}e \)). For instance, the category of unital \( e \)-bimodules is automatically monoidal while \( e\mathcal{M}e \) may fail to be monoidal (see Remark 3.19).

### 3.11 Idempotent unital algebras and closed idempotents

**Lemma 3.35** Let \( E \) be a unital object in a semigroupal category \( \mathcal{M} \). Choose an isomorphism \( u : E \otimes E \rightarrow E \). Then,

(a) \( u \) defines a structure of associative algebra on \( E \);

(b) for every pair of morphisms \( f, g : E \rightarrow E \), we have \( u \circ (f \otimes g) = (f \circ g) \circ u \).

**Proof** The pair \((E, u)\) is a unit object in \( \mathcal{M} \), see Definition 3.1 (3). It remains to apply Lemma 3.3. \( \square \)

The next result is converse to Remark 3.9 (vi).

**Proposition 3.36** If \( \pi : 1 \rightarrow e \) is an idempotent arrow in a monoidal category \( \mathcal{M} \), there exists a unique associative morphism \( \mu : e \otimes e \rightarrow e \), making \( e \) an idempotent algebra with unit \( \pi \).

**Proof** The condition that \( \pi : 1 \rightarrow e \) is a left unit means that the composition \( 1 \otimes e \rightarrow e \otimes e \rightarrow e \) equals \( \lambda_e : 1 \otimes e \rightarrow e \). Since \( \pi \otimes \text{id}_e : 1 \otimes e \rightarrow e \) is an isomorphism, there is a unique \( \mu : e \otimes e \rightarrow e \) with this property, and this \( \mu \) is an isomorphism. By Lemma 3.18, \( e \) is a unital object of \( e\mathcal{M}e \), so by Lemma 3.35, \( \mu : e \otimes e \rightarrow e \) is automatically associative. By construction, \( \pi : 1 \rightarrow e \) is a left unit for \( \mu \). By Lemma 3.10, \( \pi \) is also a right unit for \( \mu \). \( \square \)

**Corollary 3.37** Let \( e \in \mathcal{M} \) be a closed idempotent. Then

(a) every isomorphism \( \mu : e \otimes e \rightarrow e \) makes \( e \) into an idempotent algebra with unit;
(b) associating with \( \mu \) the corresponding unit \( \pi : 1 \to e \), one gets a one-to-one correspondence between isomorphisms \( e \otimes e \cong e \) and idempotent arrows \( 1 \to e \).

**Proof** By Proposition 3.36, there exists an isomorphism \( \mu : e \otimes e \cong e \), making \( e \) into an idempotent algebra with unit. By Lemma 3.18, \( e \) is a unital object of \( e \mathcal{M} e \).

So, if \( \mu' : e \otimes e \cong e \) is any isomorphism, then by Lemma 3.2, \( (e, \mu') \cong (e, \mu) \) and therefore \( (e, \mu') \) is also an idempotent algebra with unit. Thus, we have proved (a).

Statement (b) follows from (a) by Remark 3.9(vi) and Proposition 3.36. \( \square \)

### 3.12 A partial order on closed idempotents

In Sect. 3.12.1 we define a partial order on the set of isomorphism classes of closed idempotents in any monoidal category \( \mathcal{M} \). In Sect. 3.12.2 we show that if \( \mathcal{M} \) is weakly symmetric, then any two elements of this partially ordered set have an infimum (which equals their product).

#### 3.12.1 General case

If \( \pi : 1 \to e \) and \( \pi' : 1 \to e' \) are arrows in a monoidal category \( \mathcal{M} \), we write \( \text{Hom}(\pi, \pi') \) for the set of morphisms \( f : e \to e' \) such that \( f \circ \pi = \pi' \).

**Lemma 3.38** Let \( \pi \) and \( \pi' \) be idempotent arrows in a monoidal category \( \mathcal{M} \).

(a) The set \( \text{Hom}(\pi, \pi') \) has at most one element.

(b) If both \( \text{Hom}(\pi, \pi') \) and \( \text{Hom}(\pi', \pi) \) are nonempty, then the unique \( f \in \text{Hom}(\pi, \pi') \) and \( g \in \text{Hom}(\pi', \pi) \) are isomorphisms, inverse to each other.

**Proof** We apply the method used to prove Lemma 3.10. The functor \( F : \mathcal{M} \to \text{End}(\mathcal{M}) \) defined by the left action of \( \mathcal{M} \) on itself enjoys the following properties:

(i) \( F \) takes idempotent arrows to idempotent arrows;

(ii) \( F \) is faithful;

(iii) \( F \) reflects isomorphisms: If \( f \) is a morphism in \( \mathcal{M} \) and \( F(f) \) is an isomorphism in \( \text{End}(\mathcal{M}) \), then \( f \) is also an isomorphism.

In view of these facts, it suffices to prove the lemma when \( \mathcal{M} = \text{End}(\mathcal{C}) \) for some category \( \mathcal{C} \). In this case the lemma follows from Proposition 3.12. \( \square \)

**Lemma 3.39** Let \( \pi : 1 \to e \) be an idempotent arrow in a monoidal category \( \mathcal{M} \). If \( X \in e \mathcal{M} \), then the map \( \text{Hom}(e, X) \to \text{Hom}(1, X) \) defined by \( f \mapsto f \circ \pi \) is a bijection.

**Proof** Use the implication (i)\( \Rightarrow \) (ii) from Proposition 3.12 (b) in the following situation: \( \mathcal{C} = \mathcal{M} \), \( P(Y) := e \otimes Y \), and \( \epsilon : \text{Id}_\mathcal{M} \to P \) is the idempotent arrow corresponding to \( \pi : 1 \to e \). \( \square \)

**Corollary 3.40** If \( \pi, \pi' : 1 \to e \) are two idempotent arrows with the same codomain in a monoidal category \( \mathcal{M} \), there exists a unique automorphism \( f : e \to e \) such that \( f \circ \pi = \pi' \).
Remark 3.41 The corollary implies that the map \((1 \xrightarrow{\pi} e) \mapsto e\) induces a bijection between the set of isomorphism classes of idempotent arrows in \(\mathcal{M}\) and the set of isomorphism classes of closed idempotents in \(\mathcal{M}\).

Definition 3.42 Given idempotent arrows \(\pi\) and \(\pi'\) in a monoidal category \(\mathcal{M}\), we write \(\pi \geq \pi'\) if \(\text{Hom}(\pi, \pi') \neq \emptyset\). By Lemma 3.38, this pre-order induces a partial order on the set of isomorphism classes of idempotent arrows in \(\mathcal{M}\), or, equivalently (by Remark 3.41), on the set of isomorphism classes of closed idempotents in \(\mathcal{M}\).

Here are some equivalent descriptions of the partial order in Definition 3.42.

Proposition 3.43 Let \(e\) and \(e'\) be closed idempotents in a monoidal category \(\mathcal{M}\). The following conditions are equivalent:

(i) \(e \geq e'\) in the sense of Definition 3.42;
(ii) \(e' \in e\mathcal{M}e\);
(iii) \(e' \in e\mathcal{M}\);
(iv) \(e' \in \mathcal{M}e\).

Proof By symmetry, it suffices to check that (i) \(\iff\) (iii). Lemma 3.39 shows that (iii) \(\Rightarrow\) (i). Let us prove that (i) \(\Rightarrow\) (iii). Choose idempotent arrows \(\pi : 1 \rightarrow e\) and \(\pi' : 1 \rightarrow e'\). Suppose there is a morphism \(f : e \rightarrow e'\) with \(f \circ \pi = \pi'\). Since \(\pi' \otimes \text{id}_{e'} : 1 \otimes e' \rightarrow e' \otimes e'\) is an isomorphism and \(\pi' = f \circ \pi\), we see that \(e'\) is a retract of \(e \otimes e'\). Since \(e \otimes e' \in e\mathcal{M}\), we get \(e' \in e\mathcal{M}\) by Corollary 3.16. \(\Box\)

3.12.2 Weakly symmetric case

Lemma 3.44 Let \(\mathcal{M}\) be a weakly symmetric monoidal category.

(i) If \(\pi : 1 \rightarrow e\) and \(\pi' : 1 \rightarrow e'\) are idempotent arrows in \(\mathcal{M}\), then so is \(\pi \otimes \pi' : 1 \rightarrow e \otimes e'\).

(ii) If \(e, e' \in \mathcal{M}\) are closed idempotents, so is \(e \otimes e'\).

Proof It suffices to check (i). Since \(\text{id}_e \otimes \pi : e \otimes 1 \rightarrow e \otimes e\) and \(\text{id}_e \otimes \pi' : e' \otimes 1 \rightarrow e' \otimes e'\) are isomorphisms, so is \(\text{id}_e \otimes \pi \otimes \text{id}_{e'} \otimes \pi' : e \otimes 1 \otimes e' \otimes 1 \rightarrow e \otimes e \otimes e' \otimes e'\). So by weak symmetry, \(\text{id}_e \otimes \pi \otimes \pi' : e \otimes e' \otimes 1 \otimes 1 \rightarrow e \otimes e' \otimes e \otimes e'\) is an isomorphism, i.e., \(\pi \otimes \pi'\) is an idempotent arrow. \(\Box\)

Remarks 3.45 (i) The set of idempotents in a commutative monoid \(A\) can be equipped with the following partial order:

\[ e_1 \leq e_2 \quad \text{if} \quad e_1e_2 = e_1.\]

Any two idempotents \(e_1, e_2 \in A\) have an infimum, namely \(\text{inf}(e_1, e_2) = e_1e_2\). The unit of \(A\) is the biggest idempotent.

(ii) Now let \(\mathcal{M}\) be a weakly symmetric monoidal category and \(W(\mathcal{M})\) (respectively, \(C(\mathcal{M})\)) the set of isomorphism classes of weak (respectively, closed) idempotents in \(\mathcal{M}\). Applying the previous remark to the monoid of isomorphism classes of objects of \(\mathcal{M}\), one gets a partial order on \(W(\mathcal{M})\). The induced partial order on \(C(\mathcal{M}) \subset W(\mathcal{M})\) equals that from Definition 3.42 (to see this, use the equivalence (i) \(\iff\) (iv) from Proposition 3.42).
(iii) If $\mathcal{M}$ has a zero object, then minimal weak (respectively, closed) idempotents in $\mathcal{M}$ in the sense of Definition 3.23 are the same as minimal elements in the set of (isomorphism classes of) nonzero weak (respectively, closed) idempotents in $\mathcal{M}$ (in the closed case this follows from Lemma 3.44).

**Corollary 3.46** Let $\mathcal{M}$ be a weakly symmetric monoidal category and $C(\mathcal{M})$ the set of isomorphism classes of closed idempotents in $\mathcal{M}$ equipped with the partial order from Definition 3.42. Then any two elements of $C(\mathcal{M})$ have an infimum (which equals their product).

*Proof* Use Remarks 3.45 (i–ii) and Lemma 3.44. \hfill \Box

3.13 Retracts and direct sums of closed idempotents

**Lemma 3.47** Let $e \xrightarrow{i} 1 \xrightarrow{p} e$ be morphisms in a monoidal category $\mathcal{M}$ such that $pi = id_e$. Then $p : 1 \rightarrow e$ is an idempotent arrow.

*Proof* We have to show that $p \otimes id_e : 1 \otimes e \rightarrow e \otimes e$ and $id_e \otimes p : e \otimes 1 \rightarrow e \otimes e$ are isomorphisms. Let us show this for $p \otimes id_e$. Since $pi = id_e$, it suffices to show that $ip \otimes id_e : 1 \otimes e \rightarrow 1 \otimes e$ equals $id_{1 \otimes e}$. Representing $1 \otimes e$ as a retract of $1 \otimes 1$, we see that this is equivalent to showing that $ip \otimes id_e : 1 \otimes 1 \rightarrow 1 \otimes 1$ equals $id_{1 \otimes 1} \otimes ip$. For any $f \in \operatorname{End}(1)$, one has $f \otimes id_1 = id_1 \otimes f$, so $ip \otimes ip = (ip \otimes id_1) \cdot (id_1 \otimes ip) = id_1 \otimes ip ip = id_1 \otimes ip$. \hfill \Box

**Corollary 3.48** If $e$ is a closed idempotent in a monoidal category $\mathcal{M}$ and $e' \in \mathcal{M}$ is a retract of $e$, then $e'$ is a closed idempotent in $\mathcal{M}$, and $e' \leq e$ with respect to the partial order introduced in Definition 3.42.

*Proof* By Corollary 3.16, $e' \in eMe$. By Lemma 3.18, $eMe$ is a monoidal category with unital object $e$. Applying Lemma 3.47 to $eMe$, we get an idempotent arrow $e \rightarrow e'$ in $eMe$. So by Lemma 3.20, $e'$ is a closed idempotent in $\mathcal{M}$. Clearly, $e' \leq e$. \hfill \Box

Corollary 3.48 immediately implies the following statement.

**Corollary 3.49** In an additive monoidal category, every minimal closed idempotent $e$ is indecomposable.

Here “minimal” means “minimal among all nonzero closed idempotents in $\mathcal{M}$ with respect to the partial order introduced in Definition 3.42.” We note that by Remark 3.45 (iii), if $\mathcal{M}$ is weakly symmetric, this is the same as minimality in the sense of Definition 3.23.

**Proposition 3.50** Let $\mathcal{M}$ be an additive monoidal category. Let $e_1, \ldots, e_n \in \mathcal{M}$ and $e = e_1 \oplus \cdots \oplus e_n$. Then the following conditions are equivalent:

(i) $e$ is a closed idempotent;
(ii) Each $e_i$ is a closed idempotent and $e_i \otimes e_j = 0$ for $i \neq j$. 
If these conditions hold, then

\[ eM = \bigoplus_i e_i M, \quad Me = \bigoplus_i Me_i, \quad eMe = \bigoplus_{i,j} e_i Me_j \quad (3.7) \]

**Proof** It is clear that \( (ii) \Rightarrow (i) \). Let us show that \( (i) \Rightarrow (ii) \). By Corollary 3.48, each \( e_i \) is a closed idempotent in \( M \). Moreover, the proof of Corollary 3.48 shows that the projection \( p_i : e \to e_i \) is an idempotent arrow in the monoidal category \( eMe \). So, the morphism \( \text{id}_{e_i} \otimes p_i : e_i \otimes e \to e_i \otimes e_i \) is an isomorphism, which implies that \( e_i \otimes e_j = 0 \) for \( i \neq j \).

Let us prove the first decomposition in (3.7) (the other two are proved similarly). It is clear that \( e_i M = e e_i M \subset eM \) and that each object of \( eM \) is a direct sum of objects of \( e_1 M, \ldots, e_n M \). It remains to show that if \( X \in e_i M, Y \in e_j M \), and \( i \neq j \) then \( \text{Hom}(Y, X) = 0 \). By Proposition 3.22(a), \( \text{Hom}(Y, X) = \text{Hom}(e_i \otimes Y, X) \) and \( e_i \otimes Y = 0 \) because \( e_i \otimes e_j = 0 \). \( \square \)

### 4 Serre duality and Fourier–Deligne transform

#### 4.1 Definition of the Serre dual

We keep the assumptions of Sect. 2.1 (although, in fact, it would suffice to require the field \( k \) to be perfect throughout this section). In particular, \( \text{char} \ k = p > 0 \), and \( \ell \) is a fixed prime different from \( p \).

The notion of a multiplicative \( \overline{\mathbb{Q}}_\ell \)-local system on a connected perfect quasi-algebraic group over \( k \) was introduced in Definition 2.28. In order to formulate the definition of the Serre dual of a connected perfect unipotent group, we need a relative version:

**Definition 4.1** Let \( H \) be a connected perfect quasi-algebraic group over \( k \), and let \( S \) be a perfect quasi-algebraic scheme over \( k \). A family of multiplicative \( \overline{\mathbb{Q}}_\ell \)-local systems on \( H \) parametrized by \( S \) (or an \( S \)-family of multiplicative local systems on \( H \)) is a \( \overline{\mathbb{Q}}_\ell \)-local system \( L \) on \( H \times S \) of rank 1 equipped with an isomorphism

\[ \varphi : \mu_{12}^*(L) \xrightarrow{\sim} p_{13}^*(L) \otimes p_{23}^*(L), \quad (4.1) \]

where

\[ \mu_{12} = \mu \times \text{id}_S : H \times H \times S \to H \times S \]

(with \( \mu \) being the multiplication morphism for \( H \)) and

\[ p_{13}, p_{23} : H \times H \times S \to H \times S \]

are the projections along the second and first factors, respectively. The isomorphism \( \varphi \) is called a multiplicative structure on the local system \( L \).
Remarks 4.2 (i) Since $H$ is connected, an $S$-family of multiplicative local systems on $H$ has no nontrivial automorphisms.

(ii) For $s \in S(k)$ and $h_1, h_2, h_3 \in H(k)$, one has the diagram

\[
\begin{array}{ccc}
\mathcal{L}_{h_1h_2h_3,s} & \xrightarrow{\simeq} & \mathcal{L}_{h_1h_2,s} \otimes \mathcal{L}_{h_3,s} \\
\simeq & & \simeq \\
\mathcal{L}_{h_1,s} \otimes \mathcal{L}_{h_2h_3,s} & \xrightarrow{\simeq} & \mathcal{L}_{h_1,s} \otimes \mathcal{L}_{h_2,s} \otimes \mathcal{L}_{h_3,s}
\end{array}
\]

in which all arrows come from (4.1). It clearly commutes if $h_1 = h_2 = h_3 = 1$. Since $H$ is connected, this implies that the diagram commutes for all $h_1, h_2, h_3 \in H(k)$.

(iii) Suppose that $\mathcal{L}$ is a rank 1 $\mathbb{Q}_\ell$-local system on $H \times S$ such that

\[
\mu_{12}^*(\mathcal{L}) \cong p_{13}^*(\mathcal{L}) \otimes p_{23}^*(\mathcal{L}).
\]

Then the pullback of $\mathcal{L}$ to $[1] \times S \subset H \times S$ is trivial. Moreover, since $H$ is connected, specifying an isomorphism (4.1) is equivalent to specifying a trivialization of this pullback. In particular, $\mathcal{L}$ has a multiplicative structure, and any two multiplicative structures on $\mathcal{L}$ are canonically isomorphic.

In view of the last remark, we will always denote a family of multiplicative local systems by a single letter such as $\mathcal{L}$ or $\mathcal{E}$.

Proposition 4.3 Let $H$ be a (possibly noncommutative) connected perfect unipotent group over $k$. There exist a (possibly disconnected) perfect commutative unipotent group $H^*$ over $k$ and an $H^*$-family $\mathcal{E}$ of multiplicative $\mathbb{Q}_\ell$-local systems on $H$ with the following universal property.

If $S$ is a perfect quasi-algebraic scheme over $k$, the map $f \mapsto (\text{id}_H \times f)^* \mathcal{E}$ is an isomorphism between the group of $k$-morphisms $f : S \rightarrow H^*$ and the group of isomorphism classes of $S$-families of multiplicative $\mathbb{Q}_\ell$-local systems on $H$.

In the case where $H$ is commutative, the idea of this construction goes back to Serre’s article [47], and the result itself is proved in [6]. For the proof in general, we refer the reader to §A.12 of the appendix in [12].

Remark 4.4 If $\mathcal{L}$ is a rank 1 $\mathbb{Q}_\ell$-local system on $H \times S$ such that $\mu_{12}^*(\mathcal{L}) \cong p_{13}^*(\mathcal{L}) \otimes p_{23}^*(\mathcal{L})$, then by Remark 4.2(iii), we obtain a uniquely defined morphism $S \rightarrow H^*$ (it is independent of the choice of a multiplicative structure on $\mathcal{L}$).

Definition 4.5 The pair $(H^*, \mathcal{E})$ satisfying the conclusion of Proposition 4.3 is called a Serre dual of $H$. Of course, it is determined uniquely up to unique isomorphism. As usual, by abuse of terminology, we will often refer to $H^*$ itself as the Serre dual of $H$, in which case $\mathcal{E}$ will be called “the universal local system on $H \times H^*$.”

Remark 4.6 Following Serre [47], the authors of [6,44] define $H^*$ to be the moduli space of central extensions of $H$ by $\mathbb{Q}_p/\mathbb{Z}_p$ rather than the moduli space of
multiplicative $\mathbb{Q}_\ell$-local systems on $H$. In fact, a choice of a group homomorphism $\psi : (\mathbb{Q}_p, +) \longrightarrow \mathbb{Q}_\ell$ with kernel $\mathbb{Z}_p$ allows to identify the two spaces. Namely, if $\tilde{H}$ is a central extension of the group scheme $H$ by $\mathbb{Q}_p/\mathbb{Z}_p$, then $\tilde{H}$ is a $\mathbb{Q}_p/\mathbb{Z}_p$-torsor over $H$; moreover, the $\mathbb{Q}_\ell$-local system $\tilde{H}_\psi$ on $H \times S$ corresponding to $\psi : \mathbb{Q}_p/\mathbb{Z}_p \longrightarrow \mathbb{Q}_\ell$ and this torsor is multiplicative. Thus, one gets a map between the two moduli spaces, and it is easy to see that this is an isomorphism.

4.2 Properties of Serre duality

Serre duality for perfect connected unipotent groups has several properties that are similar to the properties of the functor $\Gamma^1 \mapsto \Gamma^1$, where $\Gamma$ is an arbitrary finite group and $\Gamma^* = \text{Hom}(\Gamma, \mathbb{C}^\times)$. The first one and the third one follow easily from the definition of the Serre dual.

(1) Serre duality, $H \mapsto H^*$, is a contravariant functor from the category $\text{pu}^\circ_k$ of perfect connected unipotent groups over $k$ to the category $\text{cpu}_k$ of commutative perfect unipotent groups over $k$.

(2) If $H \in \text{pu}^\circ_k$, the restriction homomorphism $H^* \longrightarrow [H, H]^*$ has finite image. This follows from [34, Theorem 1.2].

(3) If $G \longrightarrow H \longrightarrow K \longrightarrow 1$ is an exact sequence of perfect connected unipotent groups over $k$, the induced sequence $0 \longrightarrow K^* \longrightarrow H^* \longrightarrow G^*$ of commutative perfect unipotent groups is also exact.

(4) If $G \overset{f}{\longrightarrow} H$ is an isogeny between perfect connected unipotent groups over $k$, the induced homomorphism $H^* \longrightarrow G^*$ has finite kernel. Indeed, isomorphism classes of multiplicative $\mathbb{Q}_\ell$-local systems $\mathcal{L}$ on $H$ such that $f^*\mathcal{L}$ is trivial are parametrized by the finite group $\text{Hom}((\text{Ker } f)(k), \mathbb{Q}_\ell)$.

(5) The restriction of the functor $H \mapsto H^*$ to the category $\text{cpu}_k^\circ = \text{pu}_k^\circ \cap \text{cpu}_k^\circ$ of perfect connected commutative unipotent groups over $k$ is an exact equivalence between $\text{cpu}_k^\circ$ and its opposite category. Furthermore, the square of this restriction is naturally isomorphic to the identity functor on $\text{cpu}_k^\circ$ (cf. [6,47]).

(6) If $H \in \text{pu}_k^\circ$ and $H^{ab} = H/[H, H]$ is the abelianization of $H$, then the induced morphism $(H^{ab})^* \longrightarrow H^*$ is a monomorphism and identifies $(H^{ab})^*$ with the neutral connected component of $H^*$. To see this, combine (2), (3), and (5).

(7) If $H \in \text{cpu}_k^\circ$, then $\dim H^* = \dim H$. Hence, in view of (6), we deduce that if $H \in \text{pu}_k^\circ$, then $\dim H^* = \dim H^{ab}$.

4.3 An auxiliary construction

In this subsection we recall a construction, which plays an important role in the definition of an admissible pair (Sect. 2.10) and in the proofs appearing in Sect. 5. On the other hand, it is not used in the rest of Sect. 4.

Let $U$ be a (possibly disconnected and/or noncommutative) perfect unipotent group, and let $N \subset U$ be a normal connected subgroup. Since the Serre dual, $N^*$, of $N$ is

\footnote{Note that $\text{cpu}_k$ is an abelian category and $\text{cpu}_k^\circ$ is an exact subcategory of $\text{cpu}_k$.}
defined by a universal property, it is clear that $U$ acts on $N^*$ by $k$-group scheme automorphisms. Now let $N$ be a multiplicative $\mathbb{Q}_\ell$-local system on $N$ such that the corresponding element of $N^*(k)$ is $U$-invariant. Further, let $Z \subset U$ be a connected subgroup such that $N \subset Z$ and $[U, Z] \subset N$. According to the next lemma, these data define a $k$-group scheme morphism

$$\varphi_N : U/N \longrightarrow (Z/N)^*.$$  

**Lemma 4.7** (i) Let $c : U \times Z \longrightarrow N$ denote the commutator morphism, $c(u, z) = uzu^{-1}z^{-1}$. Then $L := c^*N$ has a structure of a $U$-family of multiplicative $\mathbb{Q}_\ell$-local systems on $Z$.

(ii) The morphism of $k$-schemes $\tilde{\varphi}_N : U \longrightarrow Z^*$ corresponding to $L$ factors as

$$U \twoheadrightarrow U/N \xrightarrow{\varphi_N} (Z/N)^* \hookrightarrow Z^*.$$  

Moreover, $\tilde{\varphi}_N$ and $\varphi_N$ are homomorphisms of group schemes over $k$.

For the proof, see [12, §A.13].

### 4.4 The Fourier–Deligne transform

Let us now define a version of the Fourier–Deligne transform that will be used in our article. We consider a more general setting than [22, 36, 44] by working with a possibly noncommutative connected unipotent group. In the commutative case, our definition amounts to the inverse of the usual Fourier–Deligne transform (up to a cohomological shift).

Let $H$ be a perfect connected unipotent group over $k$, and let $S$ be an arbitrary perfect quasi-algebraic scheme over $k$. Let $(H^*, \mathcal{E})$ be the Serre dual of $H$, and let $\mathcal{E}_S$ denote the pullback of $\mathcal{E}$ to $H \times H^* \times S$. Further, let $\text{pr} : H \times H^* \longrightarrow H$ and $\text{pr}' : H \times H^* \longrightarrow H^*$ denote the first and second projection morphisms. As usual, we write $\mathbb{K}_H$ for the dualizing complex of $H$.

**Definition 4.8** Define triangulated functors $\mathcal{F}'_S, \mathcal{F}'_{S,*} : \mathcal{D}(H^* \times S) \longrightarrow \mathcal{D}(H \times S)$ by

$$\mathcal{F}'_S(M) = \mathbb{K}_H \otimes (\text{pr} \times \text{id}_S)_! (\mathcal{E}_S \otimes (\text{pr}' \times \text{id}_S)^* M),$$

$$\mathcal{F}'_{S,*}(M) := \mathbb{K}_H \otimes (\text{pr} \times \text{id}_S)_* (\mathcal{E}_S \otimes (\text{pr}' \times \text{id}_S)^* M).$$

We write $\mathcal{F}' = \mathcal{F}'_{\text{Spec } k}$ and $\mathcal{F}'_* = \mathcal{F}'_{\text{Spec } k,*}$, so we have

$$\mathcal{F}' : \mathcal{D}(H^*) \longrightarrow \mathcal{D}(H), \quad \mathcal{F}'(M) := \mathbb{K}_H \otimes \text{pr}_!(\mathcal{E} \otimes \text{pr}'^* M), \quad (4.2)$$

$$\mathcal{F}'_* : \mathcal{D}(H^*) \longrightarrow \mathcal{D}(H), \quad \mathcal{F}'_*(M) := \mathbb{K}_H \otimes \text{pr}_* (\mathcal{E} \otimes \text{pr}'^* M). \quad (4.3)$$

20 Without loss of generality, one can take $Z$ to be the pre-image in $U$ of the neutral connected component of the center of $U/N$.

21 The morphism is well defined by Remark 4.4.
One has a canonical morphism of functors $\mathcal{F}_S' \to \mathcal{F}_{S,\ast}'$ (in particular, $\mathcal{F}' \to \mathcal{F}_{\ast}'$).

**Remark 4.9** We consider $\mathcal{D}(H^* \times S)$ as a monoidal category with respect to tensor product, and $\mathcal{D}(H \times S)$ as a monoidal category with respect to convolution with compact supports (see (4.5)). It is well known [22,36,44] that if $H$ is commutative, then $\mathcal{F}_S'$ is a monoidal equivalence and $\mathcal{F}_S' \to \mathcal{F}_{S,\ast}'$ is an isomorphism.

**Remark 4.10** One can show that in general, $\mathcal{F}_S'$ can be decomposed as

$$\mathcal{D}(H^* \times S) \sim \to \mathcal{D}(H^{ab,\text{st}} \times S) \xrightarrow{\pi^!} \mathcal{D}(H \times S), \tag{4.4}$$

where $H^{ab,\text{st}}$ is the “stacky abelianization” of $H$ introduced in [34, §4.1] with $\pi : H \to H^{ab,\text{st}}$ being the canonical projection, the first arrow in (4.4) is a monoidal equivalence (Rem. 3.5), and $\pi^!$ is a fully faithful semigroupal functor (Definition 3.4(c)).

In the case $S = \text{Spec } k$, we will construct a version of the decomposition (4.4) that does not use $H^{ab,\text{st}}$ explicitly, and we will prove that the morphism $\mathcal{F}_S' \to \mathcal{F}_{S,\ast}'$ is an isomorphism without assuming $H$ to be commutative, see Proposition 4.15 and Theorem 4.16. The same arguments allow us to treat the case of any quasi-algebraic $k$-scheme $S$, but we will not need this degree of generality.

### 4.5 Semigroupal structure on $\mathcal{F}'$

In Sects. 4.5.1–4.5.3 we will construct a semigroupal structure (Definition 3.4(b)) on a quite general class of functors. In Sects. 4.5.4–4.5.5 we will show that the functor $\mathcal{F}'$ defined by (4.2) belongs to this class.

#### 4.5.1. Let $S$ be a perfect quasi-algebraic scheme over $k$, and let $H$ be any perfect quasi-algebraic group over $k$. We equip the category $\mathcal{D}(H \times S)$ with the monoidal structure given by convolution with compact support:

$$M \ast N = \mu_{12}(p_{13}^* M \otimes p_{23}^* N), \tag{4.5}$$

where $\mu_{12}, p_{13}, p_{23} : H \times H \times S \to H \times S$ are as in Definition 4.1.

#### 4.5.2. Every object $\mathcal{C} \in \mathcal{D}(H \times S)$ defines a functor

$$F_{\mathcal{C}} : \mathcal{D}(S) \to \mathcal{D}(H), \quad F_{\mathcal{C}}(M) := \text{pr}_H!(\mathcal{C} \otimes \text{pr}_S^* M), \tag{4.6}$$

where $\text{pr}_S : H \times S \to S$ and $\text{pr}_H : H \times S \to H$ are the two projections. Similarly, each $\mathcal{K} \in \mathcal{D}(H \times S \times H \times S) = \mathcal{D}(H \times H \times S \times S)$ defines a functor

$$F_{\mathcal{K}} : \mathcal{D}(S \times S) \to \mathcal{D}(H \times H).$$

---

22 Not necessarily connected or unipotent.
One has a canonical functorial morphism

$$F_{C_1}(M_1) * F_{C_2}(M_2) \longrightarrow F_{C_1 \otimes C_2}(M_1 \otimes M_2) \quad (4.7)$$

for all $M_t \in \mathscr{D}(S)$ and $C_t \in \mathscr{D}(H \times S)$ (where $t = 1, 2$), namely the composition

$$F_{C_1}(M_1) * F_{C_2}(M_2) \xrightarrow{\sim} \mu_1 F_{C_1 \boxtimes C_2}(M_1 \boxtimes M_2) \longrightarrow \mu_1 F_{\Delta_1 \Delta_2}(M_1 \boxtimes M_2) \xrightarrow{\sim} F_{C_1 \circ C_2}(M_1 \otimes M_2), \quad (4.8)$$

where $\Delta : H \times H \times S \longrightarrow H \times S \times H \times S$ is defined by $\Delta(h_1, h_2, s) = (h_1, s, h_2, s)$ and the second morphism in (4.8) comes from the adjunction

$$C_1 \boxtimes C_2 \longrightarrow \Delta_1^* \Delta_2^*(C_1 \boxtimes C_2) = \Delta_1^* \Delta_2^*(C_1 \boxtimes C_2).$$

4.5.3. Now suppose we have a pair $(C, m)$, where $C \in \mathscr{D}(H \times S)$ and $m : C \times C \longrightarrow C$ is a morphism. Then one has a canonical functorial morphism

$$F_C(M_1) * F_C(M_2) \longrightarrow F_{C \circ C}(M_1 \otimes M_2), \quad M_1, M_2 \in \mathscr{D}(S), \quad (4.9)$$

namely the composition

$$F_C(M_1) * F_C(M_2) \longrightarrow F_{C \circ C}(M_1 \otimes M_2) \longrightarrow F_C(M_1 \otimes M_2),$$

where the first arrow is the morphism (4.7) and the second one is induced by $m$.

**Lemma 4.11** Suppose that $(C, m)$ is an associative algebra in $\mathscr{D}(H \times S)$. Then (4.9) makes $F_C$ into a weakly semigroupal functor (Definition 3.4(c)). It is strongly semigroupal if and only if the algebra $(C, m)$ is idempotent and

$$C_s \circ C_{s'} = 0 \quad \text{for any } s, s' \in S(k) \text{ such that } s' \neq s, \quad (4.10)$$

where $C_s \in \mathscr{D}(H)$ is the restriction of $C$ to $H \times \{s\} \subset H \times S$.

**Remark 4.12** If $(C, m)$ is an associative algebra in $\mathscr{D}_H(H \times S)$ (where $H$ acts by conjugation on the first factor and trivially on the second factor), then $F_C$ is a weakly semigroupal functor $\mathscr{D}(S) \longrightarrow \mathscr{D}_H(H)$.

4.5.4. From now on, we assume that $H$ is connected and unipotent. Let $\mathcal{L}$ be an $S$-family of multiplicative local systems on $H$. Set

$$C := \mathcal{L} \otimes \text{pr}^*_H \mathbb{K}_H = \mathcal{L} \otimes \text{pr}^*_S(\overline{\mathcal{L}}), \quad (4.11)$$

where $\mathbb{K}_H$ is the dualizing complex of $H$. Let $1 : S \longrightarrow H \times S$ denote the unit section; then $1^! C = (\overline{\mathcal{L}})_S$, so we get a canonical morphism

$$1^!(\overline{\mathcal{L}})_S \longrightarrow C. \quad (4.12)$$
Note that \(1:(\overline{Q}_\ell)_S\) is the unit object of \(\mathcal{D}(H \times S)\).

**Lemma 4.13**  
(i) The morphism (4.12) is an idempotent arrow.  
(ii) Condition (4.10) holds if and only if the map \(S(k) \to H^*(k)\) corresponding to \(\mathcal{L}\) is injective.

**Proof** In the case \(S = \text{Spec } k\), statement (i) is proved in [12, Prop. 8.1(a)], and the general case is similar. The “only if” part of (ii) is clear, and the “if” part follows from [12, Lem. 9.4]. \(\square\)

**Corollary 4.14** \(C\) has a unique structure of an idempotent algebra for which (4.12) is a unit.

**Proof** Use Lemma 4.13(i) and Proposition 3.36. \(\square\)

#### 4.5.5. Finally, let \(S = H^*\), and let \(\mathcal{L} = \mathcal{E} \in \mathcal{D}(H \times H^*)\) be the universal family of multiplicative local systems on \(H\). Define \(C\) by (4.11); then \(F_C = F'\) (to see this, compare (4.6) with (4.2)). Combining Lemma 4.11, Corollary 4.14, and Lemma 4.13(ii), we get a semigroupal structure (Definition 3.4(b)) on \(F' = F_C\).

#### 4.6 Construction of closed idempotents in \(\mathcal{D}(H)\)

In this subsection we will use the functor \(F' : \mathcal{D}(H^*) \to \mathcal{D}(H)\) to construct certain closed idempotents in the monoidal category \(\mathcal{D}(H)\), where \(H\) is a perfect connected unipotent group over \(k\). We begin by stating

**Proposition 4.15** The canonical morphism \(F' : \mathcal{D}(H^*) \to \mathcal{D}(H)\) is an isomorphism.

This proposition is proved in Sect. 4.7 below.

Set \(e_0 := F'((\overline{Q}_\ell)_{H^*})\), where \((\overline{Q}_\ell)_{H^*}\) is the constant sheaf on \(H^*\) with stalk \(\overline{Q}_\ell\). Since \((\overline{Q}_\ell)_{H^*}\) is the unit object of \(\mathcal{D}(H^*)\) and \(F'\) is a semigroupal functor (see Sect. 4.5), it is clear that \(e_0\) is a weak idempotent in \(\mathcal{D}(H)\). The next theorem says that \(e_0\) is a closed idempotent.

To construct an idempotent arrow \(1 \to e_0\), rewrite formula (4.3) as \(F'_*(M) = \text{pr}_*(\mathcal{E} \otimes \text{pr}'! M)\). This shows that \(F'_*\) is right adjoint to the functor

\[
\mathcal{F} : \mathcal{D}(H) \to \mathcal{D}(H^*), \quad N \mapsto \text{pr}'(\mathcal{E}^{-1} \otimes \text{pr}* N). \quad (4.13)
\]

We have a canonical isomorphism \(\mathcal{F}(1) \xrightarrow{\sim} (\overline{Q}_\ell)_{H^*}\), which by adjunction yields a morphism \(\pi_0 : 1 \to F'_*((\overline{Q}_\ell)_{H^*}) \xrightarrow{\sim} e_0\) (see Proposition 4.15).
Theorem 4.16 (a) The morphism $\pi_0 : 1 \rightarrow e_0$ is an idempotent arrow in $\mathcal{D}(H)$.
(b) The functor $\mathcal{F}'$ induces a monoidal equivalence
\[ \mathcal{D}(H^*) \sim e_0 \mathcal{D}(H) \subset \mathcal{D}(H). \]
(c) Let $i : [H, H] \hookrightarrow H$ denote the inclusion, and let $K \subset [H, H]^*$ denote the image of the restriction morphism $i^* : H^* \rightarrow [H, H]^*$. Then
\[ e_0 \cong \bigoplus_{[N] \in K(k)} i!*e_N, \]
the sum being finite by property Sect. 4.2(2). Here $e_N$ denotes the closed idempotent in $\mathcal{D}([H, H])$ defined by the multiplicative local system $N$ (cf. Sect. 2.11).

The theorem is proved in Sect. 4.7 below.

Remark 4.17 By Remark 4.12, the image of the functor $\mathcal{F}'$ lies in the subcategory $\mathcal{D}_H(H) \subset \mathcal{D}(H)$ of complexes equivariant under conjugation. In particular, $e_0 \in \mathcal{D}_H(H)$, so $e_0 \mathcal{D}(H) = \mathcal{D}(H)e_0$ is the Hecke subcategory of $\mathcal{D}(H)$ defined by $e_0$.

Corollary 4.18 Let $i : F \hookrightarrow H^*$ be a closed immersion, and let $\pi_F : (\mathbb{Q}_\ell)^* H \rightarrow i!(\mathbb{Q}_\ell)_F$ be the corresponding idempotent arrow in $\mathcal{D}(H^*)$. Put $e_F = \mathcal{F}'(i!(\mathbb{Q}_\ell)_F)$. Then
\[ \mathcal{F}'(\pi_F) \circ \pi_0 : 1 \rightarrow e_F \]
is an idempotent arrow in $\mathcal{D}(H)$.

Proof Use Theorem 4.16(b) and Lemma 3.20. \qed

4.7 Proof of Proposition 4.15 and Theorem 4.16

Let us fix a multiplicative local system $N$ on $H$ and write $N' = i^*N \overset{\text{def}}{=} N|[H,H]$. Let $H^*_N$ denote the fiber of the restriction morphism $i^* : H^* \rightarrow [H, H]^*$ over $[N'] \in [H, H]^*(k)$. Properties (2), (5), and (6) in Sect. 4.2 imply that $H^*_N$ is a connected component of $H^*$.

Further, let us write $e'_N := i!*e_N$; it is a closed idempotent in $\mathcal{D}(H)$.

We will now state an auxiliary lemma and deduce Proposition 4.15 and Theorem 4.16 from it. The lemma will be proved at the end of this subsection.

Lemma 4.19 (a) The restriction of $\mathcal{F}' : \mathcal{D}(H^*) \rightarrow \mathcal{D}(H)$ to the full subcategory $\mathcal{D}(H^*_N)$ is an equivalence onto the full subcategory $e'_N\mathcal{D}(H) \subset \mathcal{D}(H)$.
(b) If $M \in \mathcal{D}(H^*_N)$, the canonical map $\mathcal{F}'(M) \rightarrow \mathcal{F}'_*(M)$ is an isomorphism.

For the time being, let us assume that this lemma holds. Choose multiplicative local systems $N_1', \ldots, N_n'$ on $H$ such that $[N_1'], \ldots, [N_n']$ is a list of all elements of
$K^*(k)$, without repetitions. Then the $H^*_{N_j}$ are all the connected components of $H^*$, whence

$$\mathcal{D}(H^*) = \bigoplus_j \mathcal{D}(H^*_{N_j}).$$ \hfill (4.14)

Thus, Lemma 4.19(b) implies Proposition 4.15.

One has $e'_{N_j} \ast e'_{N_r} = 0$ for $j \neq r$ because the $e_{N_j}$ are pairwise nonisomorphic minimal closed idempotents in $\mathcal{D}([H, H])$. So by Proposition 3.50, $e := \bigoplus_{j=1}^n e'_{N_j}$ is a closed idempotent in $\mathcal{D}(H)$ and

$$e\mathcal{D}(H) = \bigoplus_j e'_{N_j} \mathcal{D}(H).$$ \hfill (4.15)

In Sect. 4.5 we constructed a semigroupal structure on the functor $\mathcal{F}' : \mathcal{D}(H^*) \longrightarrow \mathcal{D}(H)$. So by (4.14), (4.15), and Lemma 4.19(a), $\mathcal{F}'$ induces a semigroupal equivalence $\mathcal{D}(H^*) \simeq e\mathcal{D}(H)$. By Remark 3.5, this is the same as monoidal equivalence. By Lemma 3.18, $e$ is a unital object of $e\mathcal{D}(H)$ and $e_0$ was defined to be the image of the unit object of $\mathcal{D}(H^*)$, so $e_0 \cong e$. This proves parts (c) and (b) of Theorem 4.16.

Finally, to prove Theorem 4.16(a), note that by Proposition 4.15, the functor (4.13) is left adjoint to $\mathcal{F}'$, and the morphism $\pi_0 : \mathbb{1} \longrightarrow e_0$ comes by adjunction from the natural isomorphism $\mathcal{F}(\mathbb{1}) \sim (\mathcal{O}_\mathbb{1})_{H^*}$. We already saw that $e_0$ is a closed idempotent in $\mathcal{D}(H)$, so it remains to apply Theorem 4.16(b) together with Proposition 3.22(a) and the uniqueness of adjunctions.

**Proof of Lemma 4.19** As a first step, we reduce the lemma to the case where $N$ is trivial. If $\mathcal{L}$ is any multiplicative local system on $H$, we have an automorphism $\lambda_\mathcal{L} : H^* \sim H^*$ given by $[N] \mapsto [\mathcal{L} \otimes N]$, as well as a monoidal autoequivalence $\sigma_\mathcal{L} : \mathcal{D}(H) \longrightarrow \mathcal{D}(H)$ given by $M \longmapsto \mathcal{L} \otimes M$. There are natural isomorphisms

$$\mathcal{F}' \sim \sigma_\mathcal{L} \circ \mathcal{F}' \circ \lambda_\mathcal{L}^* \quad \text{and} \quad \mathcal{F}'_* \sim \sigma_\mathcal{L} \circ \mathcal{F}'_* \circ \lambda_\mathcal{L}^*$$

that are compatible with the canonical morphism $\mathcal{F}' \longrightarrow \mathcal{F}'_*$. Furthermore, if $N$ is another multiplicative local system on $H$, then $\lambda_\mathcal{L}$ restricts to an isomorphism $H^*_{N_j} \sim H^*_{N \otimes N_j}$, while $\sigma_\mathcal{L}(e'_{N_j}) \cong e'_{N \otimes N_j}$. Hence, Lemma 4.19 holds for $N$ if and only if it holds for $\mathcal{L} \otimes N$. In particular, from now on we may and do assume that $N$ is trivial. To simplify notation, we will write $H_1^* = H^*_{N}$ and $e'_1 = e'_{N}$ in this case.

Let $\pi : H \longrightarrow H_{ab} \overset{\text{def}}{=} H/[H, H]$ denote the natural projection. Note that $H_1^* \overset{\text{def}}{=} (H^*)^\circ$, so by property Sect. 4.2(6), the homomorphism $\pi^* : (H_{ab})^* \longrightarrow H^*$ induces an isomorphism $H_{ab}^* \sim H_1^*$ and therefore an equivalence $\mathcal{D}((H_{ab})^*) \sim \mathcal{D}(H_1^*)$. By Remark 4.9 (applied to $H_{ab}$ in place of $H$), the inverse Fourier transform $\mathcal{F}' : \mathcal{D}((H_{ab})^*) \longrightarrow \mathcal{D}(H_{ab})$ is an equivalence and the canonical morphism $\mathcal{F}'_* : \mathcal{D}(H_{ab}) \longrightarrow \mathcal{D}(H_{ab})^*$ is an isomorphism. We have commutative diagrams
\[ D((H^\text{ab})^*) \xrightarrow{\sim} D(H^\text{ab}) \quad \text{and} \quad D((H^\text{ab})^*) \xrightarrow{\sim} D(H^\text{ab}) \]

(\text{the right one is commutative by smooth base change}). Moreover, the canonical morphism \( F' \rightarrow F'_s \) agrees via these diagrams with the canonical morphism \( \text{ab} F' \rightarrow \text{ab} F'_s \), which is known to be an isomorphism. Now statement (b) of the lemma is clear, and statement (a) follows from the fact that the functor \( \pi^* : D(H^\text{ab}) \rightarrow D(H) \) induces an equivalence \( D(H^\text{ab}) \xrightarrow{\sim} \mathcal{D}(H(1)) \subset D(H) \).

\[ \square \]

5 Reduction process for constructible complexes

As before, we use the notation of Sect. 2.1. In particular, \( k \) denotes an algebraically closed field of characteristic \( p > 0 \), and \( \ell \) is a prime with \( \ell \neq p \).

In this section we establish Proposition 5.4 and Theorem 5.5, which will allow us to prove the main results of this article by induction. In particular, we show that if \( G \) is a perfect unipotent group over \( k \) and \( M \in D(G) \) is nonzero, then there exists an admissible pair \( (H, \mathcal{L}) \) for \( G \) such that \( (i^H_{1} \subset G \mathcal{L}) \star M \neq 0 \), where \( i^{H \subset G} : H \hookrightarrow G \) stands for the inclusion morphism.

The arguments used in the proofs of Proposition 5.4 and Theorem 5.5 are analogous to the reduction process for representations of finite nilpotent groups described in one of the appendices to [14], as well as to the “geometric reduction process” for representations of finite groups of the form \( G(\mathbb{F}_q) \), where \( G \) is a unipotent group over \( \mathbb{F}_q \), introduced in [12, §7].

5.1 Formulation of the results

If \( G \) is a (quasi-)algebraic group over \( k \), we write \( \mathcal{P}(G) \) for the set of pairs \( (H, \mathcal{L}) \) where \( H \subset G \) is a connected subgroup and \( \mathcal{L} \) is a multiplicative \( \overline{\mathbb{Q}}_\ell \)-local system on \( H \). We let \( \mathcal{P}_\text{norm}(G) \subset \mathcal{P}(G) \) denote the subset of pairs \( (H, \mathcal{L}) \in \mathcal{P}(G) \) such that \( H \) is normal in \( G \).

**Definition 5.1** We define a partial order on \( \mathcal{P}(G) \) by \( (H_1, \mathcal{L}_1) \leq (H_2, \mathcal{L}_2) \) if \( H_1 \subset H_2 \) and \( \mathcal{L}_2|_{H_1} \cong \mathcal{L}_1 \).

**Definition 5.2** A pair \( (H, \mathcal{L}) \in \mathcal{P}(G) \) is said to be compatible with a given object \( M \in D(G) \) if \( (i^H_{1} \subset G \mathcal{L}) \star M \neq 0 \). Here \( i^{H \subset G} : H \hookrightarrow G \) is the inclusion morphism.

**Remark 5.3** If \( M \neq 0 \), then there exists a pair \( (H, \mathcal{L}) \in \mathcal{P}_\text{norm}(G) \) compatible with \( M \) (for instance, one can take \( H = \{1\} \)). Clearly, among all such pairs \( (H, \mathcal{L}) \), there is a maximal one.

**Proposition 5.4** Let \( G \) be a perfect unipotent group over \( k \) and \( M \in D(G), M \neq 0 \). Suppose that \( (H, \mathcal{L}) \in \mathcal{P}_\text{norm}(G) \) is maximal among all pairs in \( \mathcal{P}_\text{norm}(G) \) that are...
compatible with $M$. If $L$ is invariant under the conjugation action of $G$, then the pair $(H, L)$ is admissible for $G$.

The proposition is proved in Sect. 5.3.

**Theorem 5.5** Let $G$ be a perfect unipotent group over $k$, let $M \in \mathcal{D}(G)$, and let $(A, N) \in \mathcal{P}_{\text{norm}}(G)$ be compatible with $M$. Then there exists an admissible pair $(H, L) \in \mathcal{P}(G)$ that is compatible with $M$ and satisfies $(A, N) \leq (H, L)$ (Definition 5.1).

The proof of Theorem 5.5 is contained in Sect. 5.4 below.

5.2 An auxiliary extension result

Our next goal is to prove Corollary 5.9, which will be used in the proofs of Proposition 5.4 and Theorem 5.5. First, we recall the following key result from [12].

**Proposition 5.6** Let $G$ be a connected perfect unipotent group over $k$, let $A \subset G_{\text{c}}$ be a connected subgroup such that $[G, G] \subset A$, and let $N$ be a multiplicative local system on $A$. Then the following properties of $N$ are equivalent:

(i) $N$ can be extended to a multiplicative local system on $G$;

(ii) The pullback of $N$ by the commutator morphism $G \times G \rightarrow A$ is trivial.

This is [12, Prop. 7.7]. Its proof given in [12] is based on the equality $\text{Ext}^2(G_{\alpha}, \mathbb{Q}_p/\mathbb{Z}_p) = 0$, where $\text{Ext}^2$ is computed in the category of fppf sheaves on the category of $k$-schemes.

**Remark 5.7** According to [34, Example 3.2], condition (ii) from Proposition 5.6 is weaker than the triviality of $N|_{[G, G]}$; in fact, (ii) is equivalent to the triviality of the pullback of $N$ to the “true commutator” of $G$ defined in [34, §4.1].

**Proposition 5.8** Let $G, A,$ and $N$ be as in Proposition 5.6, and let $M \in \mathcal{D}(G)$. Suppose that $(A, N)$ is compatible with $M$ and that $N$ has the equivalent properties (i)–(ii) of Proposition 5.6. Then there exists a multiplicative local system $L$ on $G$ such that $L|_A \cong N$ and $L * M \neq 0$.

**Proof** We begin with the following observation. Applying the projection formula to the multiplication morphism $G \times G \rightarrow G$, we see that if $E$ is any multiplicative local system on $G$, then for all $M, N \in \mathcal{D}(G)$, we have

$$\left( E \otimes M \right) * \left( E \otimes N \right) \cong E \otimes (M * N). \quad (5.1)$$

Our $N$ extends to a multiplicative local system $\tilde{N}$ on $G$. Formula (5.1) allows us to replace $M$ with $M \otimes \tilde{N}^{-1}$ and to assume that $\tilde{N}$ is trivial. Then compatibility of $M$ with $(A, \tilde{N})$ means that $\pi_1M \neq 0$, where $\pi : G \rightarrow G/A$ is the projection. But, $G/A$ is a connected commutative perfect unipotent group over $k$, so the functor $\mathcal{F} : \mathcal{D}(G/A) \rightarrow \mathcal{D}(G/A)$ is an equivalence (see Sect. 4.4 and Rem. 4.9). Thus, $\mathcal{F}^{-1}(\pi_1M) \neq 0$. This means that $L' * \pi_1M \neq 0$ for some multiplicative local system $L'$ on $G/A$. Setting $L = \pi^*L'$, we obtain $L * M = \pi^*(L' * \pi_1M) \neq 0$. \qed
Corollary 5.9 Let $G$ be a perfect unipotent group over $k$, let $M \in \mathcal{D}(G)$, and let $(A, N) \in \mathcal{P}(G)$ be compatible with $M$. Let $A_1$ be a connected subgroup of $G$ such that $[A_1, A_1] \subset A \subset A_1$. If the pullback of $N$ by the commutator morphism $A_1 \times A_1 \longrightarrow A$ is trivial, then there exists a multiplicative local system $N_1$ on $A_1$ such that $N_1|_{A_1} \cong N$ and $(A_1, N_1)$ is compatible with $M$.

Proof Set $\overline{M} := (i_{\mathbb{A}^G \cap G}^*N) \ast M$. By definition, compatibility of $(A, N)$ with $M$ means that $\overline{M} \neq 0$, so there exists a coset $A_1g$, $g \in G(k)$, such that $\overline{M}|_{A_1g} \neq 0$. Without loss of generality, we may suppose that $g = 1$ (otherwise, replace $M$ with $M \ast \delta_{g^{-1}}$, where $\delta_{g^{-1}}$ is the delta-sheaf at $g^{-1}$). Then we can replace $G$ with $A_1$ and $M$ with $M|_{A_1}$, which reduces us to Proposition 5.8. \hfill \Box

5.3 Proof of Proposition 5.4

To prove that $(H, \mathcal{L})$ is an admissible pair, we have to check three conditions (see Definition 2.30). Condition (3) is vacuous because the normalizer of $(H, \mathcal{L})$ in $G$ equals $G$. To check conditions (1) and (2), we use

Lemma 5.10 Let $Z \subset G^\circ$ be the pre-image of the neutral connected component of the center of $G^\circ/H$. Let $\varphi_\mathcal{L} : G^\circ/H \longrightarrow (Z/H)^*$ be the group morphism defined in Sect. 4.3. Then the morphism $Z/H \longrightarrow (Z/H)^*$ induced by $\varphi_\mathcal{L}$ is an isogeny.

Proof Let $K \subset G^\circ$ be the pre-image of the neutral connected component of the kernel of $\varphi_\mathcal{L}|_{(Z/H)^*}$. We have to show that $K = H$.

Clearly, $K$ is connected and $[K, K] \subset H \subset K$. Since $\mathcal{L}$ is $G$-invariant, $K$ is normal in $G$. So by Corollary 5.9, there exists a multiplicative local system $\mathcal{L}'$ on $K$ such that $\mathcal{L}'|_H \cong \mathcal{L}$ and $(K, \mathcal{L}')$ is compatible with $M$. Therefore, the maximality assumption on $(H, \mathcal{L})$ implies that $K = H$. \hfill \Box

Set $\tilde{G} := G^\circ/H$. Let $\tilde{Z}$ be the neutral connected component of the center of $\tilde{G}$. Lemma 5.10 says that the homomorphism $\tilde{Z} \longrightarrow \tilde{Z}^*$ induced by $\varphi_\mathcal{L} : \tilde{G} \longrightarrow \tilde{Z}^*$ is an isogeny. On the other hand, conditions (1)–(2) of Definition 2.30 amount to the requirement that $\tilde{G}$ is commutative and $\varphi_\mathcal{L}$ is an isogeny. So to check these conditions, it remains to prove the following lemma.

Lemma 5.11 Let $f : \tilde{G} \longrightarrow \mathfrak{A}$ be a homomorphism of connected algebraic groups with $\tilde{G}$ nilpotent and $\mathfrak{A}$ commutative. Let $\tilde{Z}$ be the neutral connected component of the center of $\tilde{G}$. Suppose that $f|_{\tilde{Z}} : \tilde{Z} \longrightarrow \mathfrak{A}$ is an isogeny. Then $\tilde{G}$ is commutative.

Proof Assume the contrary. Then $\dim \tilde{G} > \dim \tilde{Z} = \dim \mathfrak{A}$, so $\dim \ker f > 0$. Let $K$ be the neutral connected component of $\ker f$. The last nontrivial term of the sequence $K, [\tilde{G}, K], [\tilde{G}, [\tilde{G}, K]], \ldots$ is a nontrivial connected subgroup of $\tilde{Z} \cap K$. So $\dim (\tilde{Z} \cap K) > 0$, contrary to the assumption that $f|_{\tilde{Z}} : \tilde{Z} \longrightarrow \mathfrak{A}$ is an isogeny. \hfill \Box

5.4 Proof of Theorem 5.5

Without loss of generality, we may assume that $(A, N)$ is maximal among all pairs in $\mathcal{P}_\text{norm}(G)$ that are compatible with $M$. If $N$ is $G$-invariant, then by Proposition
5.4, we can take \((H, \mathcal{L}) = (A, \mathcal{N})\). So let \(\mathcal{N}\) be not \(G\)-invariant, and define \(G_1\) as the normalizer of \(\mathcal{N}\) in \(G\). Using the functor \(X \mapsto X \ast \delta_{\mathcal{N}}^{-1}\) for a suitable \(g \in G(k)\), as in the proof of Corollary 5.9, we may assume that \((A, \mathcal{N})\) is compatible with \(M\) \(\mid_{G_1}\). Since \(G_1 \subseteq G\), we may assume by induction that there exists an admissible pair \((H, \mathcal{L})\) for \(G_1\) satisfying the conclusion of Theorem 5.5 for the quadruple \((G_1, M\mid_{G_1}, A, \mathcal{N})\) in place of \((G, M, A, \mathcal{N})\).

We claim that \((H, \mathcal{L})\) also satisfies the conclusion of Theorem 5.5 for the quadruple \((G, M, A, \mathcal{N})\). We have \((A, \mathcal{N}) \leq (H, \mathcal{L})\) by construction. Since \((H, \mathcal{L})\) is compatible with \(M\) \(\mid_{G_1}\), it is also compatible with \(M\). It remains to prove the following lemma.

**Lemma 5.12** \((H, \mathcal{L})\) is admissible for \(G\) and \(N_\mathcal{G}(H, \mathcal{L}) = N_{G_1}(H, \mathcal{L})\).

**Proof** Let \(G'\) be the normalizer of \((H, \mathcal{L})\) in \(G\). Since \((A, \mathcal{N}) \leq (H, \mathcal{L})\), we have \(G' \subseteq G_1\), proving the second statement. To show that \((H, \mathcal{L})\) is admissible for \(G\), we have to check conditions (1)–(3) from Definition 2.30. Since \(G' \subseteq G_1\), condition (3) holds automatically for every \(g \in G(k)\) such that \(g \notin G_1(k)\). Conditions (1)–(2) and condition (3) for \(g \in G_1(k)\) hold because \((H, \mathcal{L})\) is admissible for \(G_1\).

6 Idempotents satisfying the Mackey condition

Throughout this section, \(k\) denotes an algebraically closed field of characteristic \(p > 0\), and \(\ell\) denotes a prime different from \(p\). We also fix a perfect unipotent group \(G\) over \(k\) and a closed subgroup \(G' \subset G\). Our goal is to prove two auxiliary results (Propositions 6.7 and 6.9) on weak and closed idempotents in \(\mathcal{D}_G(G')\) satisfying the geometric Mackey condition with respect to \(G\) (Definition 2.51). They will be used in the proofs of the main results of the article, given in Sect. 8. We also review some results on induction functors that were obtained in [12] (see Sect. 6.3).

6.1 Setup

Let \(e \in \mathcal{D}_{G'}(G')\) be a weak idempotent satisfying the geometric Mackey condition with respect to \(G\) (see Definition 2.51). As usual, we will write \(\overline{e}\) for the object of \(\mathcal{D}(G)\) obtained by extending \(e\) to all of \(G\) by zero outside of \(G'\). It follows from Lemma 6.6 below that for every \(N \in \mathcal{D}_G(G)\), the convolution \(\overline{e} \ast N\) is supported on \(G'\). Thus, \(\overline{e} \ast N\) can be viewed as the extension of \(e \ast (N\mid_{G'})\) by zero. In view of this fact, and in order to save space, we introduce the notation \(e \ast N\) for \(e \ast (N\mid_{G'})\), and we view \(N \mapsto e \ast N\) as a functor \(\mathcal{D}_G(G) \longrightarrow e\mathcal{D}_{G'}(G')\).

6.2 Induction functors

Let \(G' \subset G\) be as above.

The induction functors

\[
\text{ind}_{G'}^{G} : \mathcal{D}_G(G') \longrightarrow \mathcal{D}_G(G) \quad \text{and} \quad \text{Ind}_{G'}^{G} : \mathcal{D}_G(G') \longrightarrow \mathcal{D}_G(G)
\]
were defined in Sect. 2.12. The functor $\text{Ind}_{G}^{G'}$ is right adjoint to the restriction functor $D_G(G) \to D_{G'}(G')$, and for every $M \in D_{G'}(G')$, we let

$$\eta_M : \left(\text{Ind}_{G}^{G'} M \right)|_{G'} \to M$$

denote the adjunction morphism. Furthermore, by construction (see Sect. 2.12), for every $M \in D_{G'}(G')$, we have a canonical arrow

$$\text{can}_M : \text{ind}_{G}^{G'} M \to \text{Ind}_{G}^{G'} M.$$

### 6.3 Summary of some results of [12]

Let us now summarize some of the results that were obtained in [12, §5] in the setup of Sect. 6.1.

**Lemma 6.1** The object $f = \text{ind}_{G}^{G'} e$ is a weak idempotent in $D_{G}(G)$. If $e$ is a minimal weak idempotent, then so is $f$.

This follows from [12, Prop. 5.11 and Cor. 5.13].

**Lemma 6.2** (a) We have $\text{ind}_{G}^{G'} M \in fD_{G}(G)$ for every $M \in eD_{G'}(G')$.

(b) The functor

$$\text{ind}_{G}^{G'}|_{eD_{G'}(G')} : eD_{G'}(G') \to fD_{G}(G)$$

(6.1)

is a bijection at the level of isomorphism classes of objects.

(c) If $e$ is a closed idempotent in $D_{G'}(G')$, then the functor (6.1) is faithful.

(d) If $e$ and $f$ are closed idempotents (in $D_{G'}(G')$ and $D_{G}(G)$, respectively), then the functor (6.1) is an equivalence, whose quasi-inverse is isomorphic to the functor $N \mapsto e \ast N$.

This follows from [12, Thm. 5.12]. In Remark 6.5 below, we will see that in the situation of part (d) of the lemma, a choice of an idempotent arrow $1_{G'} \to e$ provides a canonical isomorphism between the quasi-inverse of (6.1) and the functor $N \mapsto e \ast N$.

**Lemma 6.3** (See Prop. 5.14 in [12]) For every $N \in D_{G}(G)$, there exists an isomorphism $f \ast N \simeq \text{ind}_{G}^{G'}(e \ast N)$, functorial with respect to $N$.

**Lemma 6.4** For every $M \in eD_{G'}(G')$, the composition

$$e \ast \text{ind}_{G}^{G'} M \overset{\text{id}_e \ast \text{can}_M}{\longrightarrow} e \ast \text{Ind}_{G}^{G'} M \overset{\text{id}_e \ast \eta_M}{\longrightarrow} e \ast M$$

(6.2)

is an isomorphism in $D_{G'}(G')$. In particular,

$$e \ast f \cong e.$$

(6.3)
This result, which strengthens [12, Prop. 5.15], is proved in Sect. 7.5.

Remark 6.5 Suppose that the idempotent $e$ is closed. Choose an idempotent arrow $\pi : 1_{G'} \longrightarrow e$, then for every $M \in e\mathcal{D}_{G'}(G')$, the map $\pi \star \text{id}_M : M \longrightarrow e \star M$ is an isomorphism. So by Lemma 6.4, the composition of (6.2) and the isomorphism $(\pi \star \text{id}_M)^{-1} : e \star M \cong M$ gives a functorial isomorphism

$$e \star \text{Res}^G_{G'} \text{ind}^G_{G'} M \cong M, \quad M \in e\mathcal{D}_{G'}(G').$$

Lemma 6.6 (See Lemma 5.16 in [12]) Let $\overline{e} \in \mathcal{D}(G)$ denote the extension of $e$ by zero to $G$. If $N \in \mathcal{D}_G(G)$, then $(\overline{e} \star N)|_{G \setminus G'} = 0$.

6.4 Main results

The next result will be used in the proof of Proposition 8.14.

Proposition 6.7 Let $G, G', e, f$ be as in Sects. 6.1–6.3. Let $(A, N) \in \mathcal{P}_{\text{norm}}(G)$ (see Sect. 5.1) be such that $N$ is $G$-invariant, and let $e_1 := N \otimes K_A \in \mathcal{D}_G(A) \subset \mathcal{D}_G(G)$ be the corresponding idempotent (cf. Sect. 2.11). If $f \star e_1 \cong f$ and $f \neq 0$, then $G' \supset A$ and $e \star e_1 \cong e$.

Proof. By Lemma 6.6, $e \star e_1$ is supported on $G'$, and by Lemma 6.3, $\text{ind}^G_{G'}(e \star e_1) \cong f$. Now by Lemma 6.2(b), $e \star e_1 \cong e$. This clearly implies that $G' \supset A$ unless $e = 0$. But $f \neq 0$, so $e \neq 0$.

Remark 6.8 If $e$ is a minimal weak idempotent in $\mathcal{D}_{G'}(G')$, then $f$ is also minimal by Lemma 6.1, so in this case, it suffices to assume that $f \star e_1 \neq 0$.

The rest of the section is devoted to a proof of

Proposition 6.9 Let $G$ be a perfect unipotent group over $k$, let $G' \subset G$ be a closed subgroup, and let $e \in \mathcal{D}_G(G')$ be a closed idempotent satisfying the geometric Mackey condition (Definition 2.51). Then the following statements are equivalent.

(i) For every $M \in e\mathcal{D}_G(G')$, the canonical arrow $\text{ind}^G_{G'} M \longrightarrow \text{Ind}^G_{G'} M$ is an isomorphism.

(i') The canonical arrow $\varphi : \text{ind}^G_{G'} e \longrightarrow \text{Ind}^G_{G'} e$ is an isomorphism.

(ii) The object $f = \text{ind}^G_{G'} e$ is a closed idempotent.

Clearly, (i)⇒(i'). The implications (i')⇒(ii)⇒(i) will be proved in Sects. 6.5–6.6 using the adjointness between $\text{Ind}^G_{G'} : \mathcal{D}_G(G') \longrightarrow \mathcal{D}_G(G)$ and $\text{Res}^G_{G'} : \mathcal{D}_G(G) \longrightarrow \mathcal{D}_G(G')$, see Corollary 2.39.

Remark 6.10 Only the implication (ii)⇒(i) will be used in the proofs in Sect. 8.
6.5 Proof of the implication $(i') \Rightarrow (ii)$

Let $\mathbb{1}_G \in \mathcal{D}_G(G)$, $\mathbb{1}_{G'} \in \mathcal{D}_{G'}(G')$ be the unit objects. Fix an idempotent arrow $\text{Res}^{G'}_{G'}(\mathbb{1}_G) = \mathbb{1}_{G'} \rightarrow e$. By adjunction (see Corollary 2.39), we get a morphism $\theta : \mathbb{1}_G \rightarrow \text{Ind}^G_{G'} e$. Let us prove that $\varphi^{-1}\theta : \mathbb{1} \rightarrow f$ is an idempotent arrow. It suffices to construct a morphism

$$\mu : f \ast \text{Ind}^G_{G'} e \rightarrow f$$

such that the compositions

$$f \ast \mathbb{1}_G \xrightarrow{\text{id}_f \ast \theta} f \ast \text{Ind}^G_{G'} e \xrightarrow{\mu} f$$

and

$$f \ast f \xrightarrow{\text{id}_f \ast \varphi} f \ast \text{Ind}^G_{G'} e \xrightarrow{\mu} f$$

are isomorphisms (indeed, looking at (6.6), we see that $\mu$ is an isomorphism, and then looking at (6.5), we see that $\text{id}_f \ast \theta : f \ast \mathbb{1}_G \rightarrow f \ast \text{Ind}^G_{G'} e$ is an isomorphism).

We will construct (6.4) and prove the invertibility of (6.5) and (6.6) without assuming that $\varphi$ is an isomorphism. By Lemma 6.3, there is a functorial isomorphism

$$f \ast N \overset{\sim}{\rightarrow} \text{ind}^G_{G'}(e \ast \text{Res}^G_{G'} N), \quad N \in \mathcal{D}_G(G).$$

So to construct (6.4), it suffices to define a morphism $e \ast \text{Res}^G_{G'} \text{Ind}^G_{G'} e \rightarrow e$. We define it to be the composition

$$e \ast \text{Res}^G_{G'} \text{Ind}^G_{G'} e \rightarrow e \ast e \overset{\sim}{\rightarrow} e,$$

where the first morphism comes from the adjunction $\text{Res}^G_{G'} \text{Ind}^G_{G'} \rightarrow \text{id}$ and the second one from the idempotent arrow $\mathbb{1}_{G'} \rightarrow e$. Using (6.7), it is easy to see that (6.5) is an isomorphism. To show that (6.6) is an isomorphism, use (6.7) and apply Lemma 6.4 to $M = e$.

6.6 Proof of the implication $(ii) \Rightarrow (i)$

We keep the notation of Proposition 6.9 and of Sect. 6.1. We now assume that both $e$ and $f$ are closed idempotents. We must prove property (i) in Proposition 6.9.

**Lemma 6.11** If $M \in e\mathcal{D}_{G'}(G')$, then $\text{Ind}^G_{G'} M \in f\mathcal{D}_G(G)$. 

**Proof** It suffices to show that for every $N \in \mathcal{D}_G(G)$ the map

$$\text{Hom}(f \ast N, \text{Ind}^G_{G'} M) \rightarrow \text{Hom}(N, \text{Ind}^G_{G'} M)$$

(6.8)
induced by an idempotent arrow \( 1_G \to f \) is bijective. Then the lemma will follow by applying the implication (ii) \( \Rightarrow \) (i) of Proposition 3.12(b).

By Corollary 2.39(i),
\[
\text{Hom}(N, \text{Ind}_G^G M) = \text{Hom}(\text{Res}_G^G N, M).
\]

Since \( M \in e\mathcal{D}_G(G') \), an idempotent arrow \( 1_G \to e \) induces an isomorphism
\[
\text{Hom}(e \ast \text{Res}_G^G, N, M) \sim \text{Hom}(\text{Res}_G^G, N, M)
\]
by Proposition 3.22(a). Thus, we get a functorial isomorphism
\[
\text{Hom}(N, \text{Ind}_G^G M) \sim \text{Hom}(e \ast \text{Res}_G^G, N, M), \quad M \in e\mathcal{D}_G(G'), N \in \mathcal{D}_G(G).
\]

(6.9)

So to prove the bijectivity of (6.8), it suffices to show that the morphism
\[
e \ast \text{Res}_G^G, N \to e \ast \text{Res}_G^G(f \ast N)
\]
induced by an idempotent arrow \( 1_G \to f \) is an isomorphism. Let \( \bar{e} \in \mathcal{D}_G(G') \) be the extension of \( e \in \mathcal{D}_G(G') \) by zero, then it suffices to show that the morphism \( \bar{e} \ast N \to \bar{e} \ast f \ast N \) is an isomorphism. This is clear because by (6.3), \( \bar{e} \in f\mathcal{D}_G(G') = \mathcal{D}_G(G)f \) and therefore, the morphism \( \bar{e} \to \bar{e} \ast f \) is an isomorphism. \( \square \)

Now let us prove that for any \( M \in e\mathcal{D}_G(G') \), the arrow
\[
\text{can}_M : \text{ind}_G^G M \to \text{Ind}_G^G M
\]
is an isomorphism.\(^{23}\) By Lemmas 6.11 and 6.2(a), both \( \text{Ind}_G^G \) and \( \text{ind}_G^G \) can be considered as functors from \( e\mathcal{D}_G(G') \) to \( f\mathcal{D}_G(G) \). So, it suffices to show that for every \( N \in f\mathcal{D}_G(G) \) the map
\[
\text{Hom}(N, \text{ind}_G^G M) \to \text{Hom}(N, \text{Ind}_G^G M)
\]
is bijective.

Fix an idempotent arrow \( \pi : \mathbb{1}_{G'} \to e \). By Lemma 6.2(d) and Remark 6.5, \( \text{ind}_G^G : e\mathcal{D}_G(G') \to f\mathcal{D}_G(G) \) is an equivalence whose quasi-inverse is canonically isomorphic to the functor \( N \mapsto e \ast \text{Res}_G^G N \). Thus, we get a canonical isomorphism

\(^{23}\) The idea of the argument we present below was borrowed from a proof of the result that Fourier–Deligne transform commutes with Verdier duality, which was explained to us by Dennis Gaitsgory and is reproduced in the appendix on the Fourier–Deligne transform in [14].
\[
\Hom(N, \ind_{G'}^G M) \xrightarrow{\sim} \Hom(e \ast \Res_{G'}^G N, M). \tag{6.11}
\]

Let \( h : \Hom(N, \ind_{G'}^G M) \longrightarrow \Hom(e \ast \Res_{G'}^G N, M) \) be the composition of (6.10) with (6.9). To prove that the map (6.10) is bijective, it suffices to show that \( h \) is bijective. But \( h \) equals (6.11): Both maps take \( \beta \in \Hom(N, \ind_{G'}^G M) \) to the composition

\[
e \ast \Res_{G'}^G N \xrightarrow{\text{id}_e \ast \Res_{G'}^G (\beta)} e \ast \Res_{G'}^G \ind_{G'}^G M \xrightarrow{\text{id}_e \ast \Res_{G'}^G (\text{can}_M)} e \ast \Res_{G'}^G \Ind_{G'}^G M \xrightarrow{\text{id}_e \ast \eta_M} e \ast M \xrightarrow{(\pi \ast \text{id}_M)^{-1}} M.
\]

### 7 Properties of averaging functors

Throughout this section, \( k \) denotes an algebraically closed field of characteristic \( p > 0 \), and \( \ell \) denotes a prime different from \( p \). We also fix a perfect unipotent group \( G \) over \( k \) and a closed subgroup \( G' \subset G \), with the exception of Sect. 7.4, where \( G \) and \( G' \) are assumed to be ordinary unipotent algebraic groups over \( k \).

Our goal is to establish certain properties of the functors \( \operatorname{av}_{G/G'} \) and \( \ind_{G}^{G} \), that were introduced in Sect. 2.12 (they will be used in the proofs of the main theorems of our work, given in Sect. 8) and to prove Lemma 6.4.

#### 7.1 Compatibility of \( \mathcal{F}' \) with \( \operatorname{av}_{G/G'} \)

Let us fix a perfect connected unipotent group \( H \) over \( k \) equipped with a \( G \)-action by group automorphisms. The construction of the functor \( \mathcal{F}' \) given in Sect. 4.4 easily generalizes to the equivariant setting (note that the universal local system \( \mathcal{E} \) on \( H \times H^* \) has a canonical \( G \)-equivariant structure), which yields functors

\[
\mathcal{F}'^G : \mathcal{D}_{G'}(H^*) \longrightarrow \mathcal{D}_{G}(H) \quad \text{and} \quad \mathcal{F}^G : \mathcal{D}_{G}(H^*) \longrightarrow \mathcal{D}_{G}(H)
\]

Our goal is to prove that these functors are compatible with the averaging functors \( \operatorname{av}_{G/G'} : \mathcal{D}_{G'}(H) \rightarrow \mathcal{D}_{G}(H) \) and \( \operatorname{av}_{G/G'} : \mathcal{D}_{G'}(H^*) \rightarrow \mathcal{D}_{G}(H^*) \) (see Proposition 7.4).

**Definition 7.1** Let \( X \) and \( Y \) be perfect quasi-algebraic schemes over \( k \) equipped with a \( G \)-action, and let \( f : X \longrightarrow Y \) be a \( G \)-morphism. We define an isomorphism of functors\(^{24}\)

\[f_! : \mathcal{D}_{G'}(X) \rightarrow \mathcal{D}_{G}(Y) \quad \text{and} \quad f^* : \mathcal{D}_{G}(Y) \rightarrow \mathcal{D}_{G}(X)
\]

---

\(^{24}\) We are using a slight abuse of notation. On the left-hand side of (7.1), \( f_! \) is viewed as a functor from \( \mathcal{D}_{G'}(X) \) to \( \mathcal{D}_{G}(Y) \), and \( \operatorname{av}_{G/G'} \) is computed on \( X \). On the right-hand side, \( f^* \) is viewed as a functor from \( \mathcal{D}_{G}(Y) \) to \( \mathcal{D}_{G}(Y) \), and \( \operatorname{av}_{G/G'} \) is computed on \( Y \).
as follows. The Cartesian square

\[
\begin{array}{ccc}
X & \xrightarrow{i_X} & (G/G') \times X \\
\downarrow{f} & & \downarrow{id \times f} \\
Y & \xrightarrow{i_Y} & (G/G') \times Y
\end{array}
\]

where \(i_X(x) = (\bar{1}, x)\) and \(i_Y(y) = (\bar{1}, y)\) determines an isomorphism of functors\(^{25}\)

\[i_Y^* \circ \Phi_Y \circ (id \times f)! \sim f! \circ i_X^* \circ \Phi_X : \mathcal{D}_G((G/G') \times X) \to \mathcal{D}_{G'}(Y),\]

which induces an isomorphism of functors

\[(id \times f)! \circ (i_X^* \circ \Phi_X)^{-1} \sim (i_Y^* \circ \Phi_Y)^{-1} \circ f! : \mathcal{D}_G(X) \to \mathcal{D}_G((G/G') \times Y).\]

Composing the latter isomorphism with \(pr_Y^2\), where \(pr_Y^2 : (G/G') \times Y \to Y\) is the second projection, defines (7.1).

**Definition 7.2** Let \(X\) and \(Y\) be perfect quasi-algebraic schemes over \(k\) equipped with a \(G\)-action, and let \(f : X \to Y\) be a \(G\)-morphism. We define an isomorphism of functors\(^{26}\)

\[f^* \circ \text{av}_{G/G'} \sim \text{av}_{G/G'} \circ f^* : \mathcal{D}_{G'}(Y) \to \mathcal{D}_G(X)\] (7.2)

as follows. With the notation of Definition 7.1, we have an isomorphism

\[i_X^* \circ \Phi_X \circ (id \times f)^* \sim f^* \circ i_Y^* \circ \Phi_Y : \mathcal{D}_G((G/G') \times Y) \to \mathcal{D}_{G'}(X),\]

which induces an isomorphism

\[(id \times f)^* \circ (i_Y^* \circ \Phi_Y)^{-1} \sim (i_X^* \circ \Phi_X)^{-1} \circ f^* : \mathcal{D}_{G'}(Y) \to \mathcal{D}_G((G/G') \times X).\]

---

\(^{25}\) We are using the definition of \(\text{av}_{G/G'}\) given in Sect. 2.12. In particular, \(G\) acts on \((G/G') \times X\) and \((G/G') \times Y\) diagonally, via the translation action on \(G/G'\) and the given action on \(X\) and \(Y\). The functors \(\Phi_X : \mathcal{D}_G((G/G') \times X) \to \mathcal{D}_{G'}((G/G') \times X)\) and \(\Phi_Y : \mathcal{D}_G((G/G') \times Y) \to \mathcal{D}_{G'}((G/G') \times Y)\) are the forgetful ones.

\(^{26}\) We use an abuse of notation similar to that employed in Definition 7.1.
Composing the latter isomorphism with \( \text{pr}_2^X \), where \( \text{pr}_2^X : (G/G') \times X \rightarrow X \) is the second projection, and applying the proper base change theorem to the Cartesian square

\[
\begin{array}{ccc}
(G/G') \times X & \xrightarrow{\text{pr}_2^X} & X \\
\downarrow \text{id} \times f & & \downarrow f \\
(G/G') \times Y & \xrightarrow{\text{pr}_2^Y} & Y
\end{array}
\]

defines \((7.2)\).

**Definition 7.3** Let \( X \) be a perfect quasi-algebraic scheme equipped with a \( G \)-action, and let \( L \in \mathcal{D}_G(X) \). We define functorial isomorphisms

\[
L \otimes \text{av}_{G/G'}(M) \xrightarrow{\sim} \text{av}_{G/G'}(F(L) \otimes M), \quad M \in \mathcal{D}_{G'}(X),
\]

where \( F : \mathcal{D}_G(X) \rightarrow \mathcal{D}_{G'}(X) \) is the forgetful functor, as follows.

With the notation of Remark 2.34, we have \( \text{pr}_2 \circ i = \text{id}_X \), which yields functorial isomorphisms

\[
(i^* \circ \Phi)((\text{pr}_2^X L) \otimes N) \xrightarrow{\sim} F(L) \otimes ((i^* \circ \Phi)(N))
\]

for all \( N \in \mathcal{D}_G((G/G') \times X) \). Thus, we obtain functorial isomorphisms

\[
(\text{pr}_2^X L) \otimes ((i^* \circ \Phi)^{-1}(M)) \xrightarrow{\sim} (i^* \circ \Phi)^{-1}(F(L) \otimes M).
\]

Applying \( \text{pr}_{2!} \) to both sides and using the projection formula yields \((7.3)\).

**Proposition 7.4** Let \( H \) be as above. There is a functorial family of isomorphisms

\[
\mathcal{F}^{G'}(\text{av}_{G/G'}(M)) \xrightarrow{\sim} \text{av}_{G/G'}(\mathcal{F}^{G'}(M))
\]

in \( \mathcal{D}_G(H) \) for all \( M \in \mathcal{D}_{G'}(H^*) \).

**Proof** Use Definitions 7.1, 7.2, and 7.3.

\[ \square \]

7.2 Weak semigroupal structure on \( \text{av}_{G/G'} \)

In this subsection we let \( H \) be any perfect quasi-algebraic group over \( k \) equipped with an action of \( G \) by group automorphisms. Note that both \( \mathcal{D}_{G'}(H) \) and \( \mathcal{D}_G(H) \) are monoidal categories with respect to the functor of convolution with compact supports. The goal of this subsection is to construct a weak semigroupal structure (Definition 3.4(a)) on the functor \( \text{av}_{G/G'} : \mathcal{D}_{G'}(H) \rightarrow \mathcal{D}_G(H) \).
One ingredient in the construction is Definition 7.1. Another ingredient is

**Definition 7.5** Let \( X \) and \( Y \) be perfect quasi-algebraic schemes over \( k \) equipped with a \( G \)-action, and let \( G \) act on \( X \times Y \) diagonally. We construct a functorial collection of morphisms\(^{27}\) (not necessarily isomorphisms)

\[
(\text{av}_{G/G'} M) \otimes (\text{av}_{G/G'} N) \longrightarrow \text{av}_{G/G'}(M \boxtimes N) \tag{7.4}
\]

for all \( M \in \mathcal{D}_{G'}(X) \) and \( N \in \mathcal{D}_{G'}(Y) \) as follows.

1) Let \( i_X, i_Y \) be as in Definition 7.1, and let \( i_{X \times Y} : X \times Y \longrightarrow (G/G') \times X \times Y \) be given by \( (x, y) \mapsto (T, x, y) \). As before, \( G \) acts diagonally on all product \( k \)-schemes appearing in the definition.

2) Define \( \Delta : (G/G') \times X \times Y \longrightarrow (G/G') \times X \times (G/G') \times Y \) by \( (g, x, y) \mapsto (g, x, g, y) \).

3) Observe that we have a natural isomorphism\(^{28}\)

\[
\Delta^*( (i_X^* \circ \Phi_X)^{-1}(M) \otimes (i_Y^* \circ \Phi_Y)^{-1}(N) ) \cong (i_{X \times Y}^* \circ \Phi_{X \times Y})^{-1}(M \boxtimes N),
\]

which gives rise to a morphism

\[
( (i_X^* \circ \Phi_X)^{-1}(M) \otimes (i_Y^* \circ \Phi_Y)^{-1}(N) ) \longrightarrow \Delta_*( (i_{X \times Y}^* \circ \Phi_{X \times Y})^{-1}(M \boxtimes N) )
\]

\[
= \Delta_! ( (i_{X \times Y}^* \circ \Phi_{X \times Y})^{-1}(M \boxtimes N) ).
\]

4) Write \( \text{pr} : (G/G') \times X \times (G/G') \times Y \longrightarrow X \times Y \) for the projection. Applying \( \text{pr}_! \) to the last morphism defines (7.4).

Now we can give the main definition of this subsection:

**Definition 7.6** (Weak semigroupal structure on \( \text{av}_{G/G'} \)) Given \( M, N \in \mathcal{D}_{G'}(H) \), we define a morphism

\[
\phi(M, N) : (\text{av}_{G/G'} M) * (\text{av}_{G/G'} N) \longrightarrow \text{av}_{G/G'}(M \ast N)
\]

as follows. Let \( \mu : H \times H \longrightarrow H \) be the multiplication morphism. Applying \( \mu_! \) to the morphism constructed in Definition 7.5, we obtain a morphism

\[
(\text{av}_{G/G'} M) * (\text{av}_{G/G'} N) \longrightarrow \mu_!( \text{av}_{G/G'}(M \boxtimes N) ) .
\]

We define \( \phi(M, N) \) as the composition of the latter morphism and the isomorphism

\[
\mu_!( \text{av}_{G/G'}(M \boxtimes N) ) \cong \text{av}_{G/G'}(\mu_!(M \boxtimes N)) \overset{\text{def}}{=} \text{av}_{G/G'}(M \ast N)
\]

constructed in Definition 7.1. It is straightforward to check that \( \phi \) is a weak semigroupal structure on the functor \( \text{av}_{G/G'} : \mathcal{D}_{G'}(H) \longrightarrow \mathcal{D}_G(H) \).

---

\(^{27}\) Here, \( \boxtimes \) denotes the external tensor product, viewed either as a functor from \( \mathcal{D}_{G'}(X) \times \mathcal{D}_{G'}(Y) \) to \( \mathcal{D}_{G'}(X \times Y) \) or as a functor from \( \mathcal{D}_{G}(X) \times \mathcal{D}_{G}(Y) \) to \( \mathcal{D}_{G}(X \times Y) \).

\(^{28}\) For any \( G \)-scheme \( Z \), we have the forgetful functor \( \Phi_Z : \mathcal{D}_G((G/G') \times Z) \longrightarrow \mathcal{D}_{G'}((G/G') \times Z) \).
7.3 The functor \( \text{av}_{G/G'} \) as a bimodule functor

We remain in the setup of Sect. 7.2.

**Proposition 7.7** Let \( X \) and \( Y \) be perfect quasi-algebraic schemes over \( k \) equipped with a \( G \)-action, and let \( G \) act on \( X \times Y \) diagonally. There exist isomorphisms

\[
(\text{av}_{G/G'} M) \boxtimes N \xrightarrow{\sim} \text{av}_{G/G'}(M \boxtimes N),
\]

functorial with respect to \( M \in \mathcal{D}_{G'}(X) \) and \( N \in \mathcal{D}_G(Y) \).

**Proof** Let us use the notation of Definition 7.5 and observe that

\[
i^*_{X \times Y} = i_X \times \text{id}_Y : X \times Y \hookrightarrow (G/G') \times X \times Y.
\]

This implies that for \( M \in \mathcal{D}_{G'}(X) \), \( N \in \mathcal{D}_G(Y) \), there are functorial isomorphisms

\[
((i^*_X \circ \Phi_X)^{-1}(M)) \boxtimes N \xrightarrow{\sim} (i^*_X \times Y \circ \Phi_{X \times Y})^{-1}(M \boxtimes N).
\]

Similarly, if \( \text{pr}^X_2 : (G/G') \times X \longrightarrow X \) and \( \text{pr}^{X \times Y}_2 : (G/G') \times X \times Y \longrightarrow X \times Y \) are the natural projections, we have \( \text{pr}^{X \times Y}_2 = \text{pr}^X_2 \times \text{id}_Y \). This defines (7.5). \( \square \)

**Corollary 7.8** Let \( H \) be a perfect quasi-algebraic group over \( k \) equipped with an action of \( G \) by group automorphisms. There exist isomorphisms

\[
(\text{av}_{G/G'} M) * N \xrightarrow{\sim} \text{av}_{G/G'}(M * N),
\]

\[
N * (\text{av}_{G/G'} M) \xrightarrow{\sim} \text{av}_{G/G'}(N * M),
\]

functorial with respect to \( M \in \mathcal{D}_{G'}(H) \) and \( N \in \mathcal{D}_G(H) \).

**Proof** As in Definition 7.6, let \( \mu : H \times H \longrightarrow H \) be the multiplication morphism and apply \( \mu^* \) to the isomorphism 7.5. Composing the result with the isomorphism (7.1) constructed for \( f = \mu \) defines (7.6). The construction of (7.7) is similar. \( \square \)

One can check that isomorphisms (7.6)–(7.7) define the structure of a \( \mathcal{D}_G(H) \)-bimodule functor on \( \text{av}_{G/G'} : \mathcal{D}_{G'}(H) \to \mathcal{D}_G(H) \).

7.4 Averaging functors and duality

In this subsection we fix an ordinary (as opposed to perfect) unipotent group \( G \) over \( k \) and a closed subgroup \( G' \subset G \). Our goal is to establish Corollary 7.10, which will be used in Sect. 8.12 below.

Let \( X \) be a scheme of finite type over \( k \) equipped with a \( G \)-action, and let \( \mathbb{D}_X : \mathcal{D}_G(X) \longrightarrow \mathcal{D}_G(X) \) denote the Verdier duality functor. When viewed as a functor \( \mathcal{D}_{G'}(X) \longrightarrow \mathcal{D}_{G'}(X) \), the Verdier duality functor will be denoted by \( \mathbb{D}'_X \).
Lemma 7.9 There is a natural isomorphism of functors from $\mathcal{D}_G(X)$ to $\mathcal{D}_G(X)$,
\begin{equation}
\text{av}_{G/G'}[2d](d) \xrightarrow{\sim} \mathbb{D}_X \circ \text{Av}_{G/G'} \circ \mathbb{D}'_X,
\end{equation}
where $d = \dim(G/G')$.

Proof Recall that $\text{Av}_{G/G'}$ is right adjoint to the forgetful functor $F : \mathcal{D}_G(X) \rightarrow \mathcal{D}_G(X)$ (see Lemma 2.35). Since $\mathbb{D}'_X \circ F \xrightarrow{\sim} F \circ \mathbb{D}_X$, the functor $\mathbb{D}_X \circ \text{Av}_{G/G'} \circ \mathbb{D}'_X$ is left adjoint to $F$. But $\text{av}_{G/G'}$ is also left adjoint to $F$ (see Lemma 2.35). \qed

In Definition 2.17, we constructed a contravariant functor $\mathbb{D}^-_G : \mathcal{D}_G(G) \rightarrow \mathcal{D}_G(G)$.

Corollary 7.10 There is a natural isomorphism of functors $\mathcal{D}_{G'}(G') \rightarrow \mathcal{D}_G(G)$,
\begin{equation}
\text{ind}^G_{G'}[2d](d) \xrightarrow{\sim} \mathbb{D}^-_G \circ \text{Ind}^G_{G'} \circ \mathbb{D}^-_{G'},
\end{equation}
where $d = \dim(G/G')$.

Proof Use Definition 2.38 and Lemma 7.9. \qed

7.5 Proof of Lemma 6.4

The last assertion of the lemma follows from the first one by taking $M = e$. In turn, the first assertion results from the more general

Proposition 7.11 Let $G$ be a perfect unipotent group over $k$, let $G' \subset G$ be a closed subgroup, and let $N \in \mathcal{D}_{G'}(G)$ and $L \in \mathcal{D}(G)$ be such that
\begin{equation}
L \star \delta_g \star N = 0 \quad \text{for all } g \in G(k), g \notin G'(k).
\end{equation}

Then the composition
\begin{equation}
F(\text{av}_{G/G'} N) \rightarrow F(\text{Av}_{G/G'} N) \rightarrow N
\end{equation}
becomes an isomorphism after convolution with $L$ on the left; here
\begin{equation}
F : \mathcal{D}_G(G) \rightarrow \mathcal{D}_G(G)
\end{equation}
is the forgetful functor, the first morphism in (7.11) is induced by the canonical morphism $\text{av}_{G/G'} \rightarrow \text{Av}_{G/G'}$, and the second one is the adjunction morphism.

Indeed, if $e \in \mathcal{D}_{G'}(G')$ is a weak idempotent that satisfies the geometric Mackey condition with respect to $G$, then for every $M \in e\mathcal{D}_{G'}(G')$ and every $g \in G(k)$ such that $g \notin G'(k)$ we have $\bar{e} \star \delta_g \star \bar{M} \cong \bar{e} \star \delta_g \star \bar{e} \star \bar{M} = 0$, so the proposition can be applied to $L = \bar{e}$ and $N = \bar{M}$, which yields the first assertion of Lemma 6.4.

\hspace{1cm}29 Here convolution is interpreted as a functor $\mathcal{D}(G) \times \mathcal{D}_G(G) \rightarrow \mathcal{D}(G)$. 

The proposition is proved in Sect. 7.5.1 below. As a first step, we will find a more explicit description of the morphisms (7.11) in the slightly more general setting of Sect. 7.5.1.

7.5.1. We fix a perfect quasi-algebraic scheme $X$ over $k$ and an action of $G$ on $X$. Then we have the functors $\text{av}_{G/G'} : \mathcal{D}_{G'}(X) \to \mathcal{D}_G(X)$ defined by (2.5)–(2.6) and a canonical morphism $\text{av}_{G/G'} \to \text{Av}_{G/G'}$. By Lemma 2.35, $\text{Av}_{G/G'}$ is right adjoint to the forgetful functor $F : \mathcal{D}_G(X) \to \mathcal{D}_{G'}(X)$, so we get morphisms

$$F \circ \text{av}_{G/G'} \to F \circ \text{Av}_{G/G'} \to \text{Id}_{\mathcal{D}_{G'}(X)}.$$  (7.12)

They will be described explicitly in Lemmas 7.12 and 7.14 below.

7.5.2. First, let us explicitly describe the functors $F \circ \text{av}_{G/G'}$, $F \circ \text{Av}_{G/G'}$ and the morphism $F \circ \text{av}_{G/G'} \to F \circ \text{Av}_{G/G'}$ in terms of the $G'$-equivariant embedding $i : X \to (G/G') \times X$, $i(x) = (\bar{1}, x)$ and the projection $\text{pr}_2 : (G/G') \times X \to X$.

We have commutative diagrams

$$\mathcal{D}_G((G/G') \times X) \xrightarrow{i^* \circ \Phi} \mathcal{D}_{G'}(X) \xrightarrow{\text{pr}_{2*}} \mathcal{D}_G(X)$$  (7.13)

and

$$\mathcal{D}_{G'}(X) \xleftarrow{i^*} \mathcal{D}_G((G/G') \times X) \xrightarrow{\Phi} \mathcal{D}_{G'}(X)$$

and

$$\mathcal{D}_G((G/G') \times X) \xrightarrow{\text{pr}_{2*}} \mathcal{D}_{G'}(X)$$  (7.14)

and

$$\mathcal{D}_{G'}(X) \xleftarrow{\text{pr}_{2*}} \mathcal{D}_G((G/G') \times X) \xrightarrow{\Phi} \mathcal{D}_{G'}(X)$$

in which $F$ and $\Phi$ are the forgetful functors and $i^* \circ \Phi : \mathcal{D}_G((G/G') \times X) \to \mathcal{D}_{G'}(X)$ is an equivalence.

**Lemma 7.12** The functors $F \circ \text{av}_{G/G'}$ and $F \circ \text{Av}_{G/G'}$ canonically identify with the compositions

$$\mathcal{D}_{G'}(X) \xrightarrow{\Phi(i^* \circ \Phi)^{-1}} \mathcal{D}_G((G/G') \times X) \xrightarrow{\text{pr}_{2*}} \mathcal{D}_{G'}(X)$$  (7.15)

and

$$\mathcal{D}_{G'}(X) \xrightarrow{\Phi(i^* \circ \Phi)^{-1}} \mathcal{D}_G((G/G') \times X) \xrightarrow{\text{pr}_{2*}} \mathcal{D}_{G'}(X).$$  (7.16)

The morphism $F \circ \text{av}_{G/G'} \to F \circ \text{Av}_{G/G'}$ corresponds under these identifications to the canonical morphism $\text{pr}_{2*} \to \text{pr}_{2*}$. 

Proof This immediately follows from the definitions (2.5)–(2.6) together with the commutativity of the diagrams (7.13)–(7.14).

7.5.3. The diagram of $G'$-varieties

$$X \hookrightarrow (G/G') \times X \xrightarrow{pr_2} X, \quad pr_2 \circ i = \text{id}_X,$$  \hfill (7.17)

defines canonical morphisms

$$pr_2! \longrightarrow pr_{2*} \longrightarrow i^*, \hfill (7.18)$$

where $pr_2!$ and $pr_{2*}$ are viewed as functors $\mathcal{D}_{G'}((G/G') \times X) \longrightarrow \mathcal{D}_{G'}(X)$. Namely, the morphism $pr_{2*} \longrightarrow i^*$ in (7.18) is the composition

$$pr_{2*} \longrightarrow pr_{2*} \circ i_* \circ i^* = (pr_2 \circ i)_* \circ i^* = i^* \hfill (7.19)$$

and also the composition

$$pr_{2*} = (pr_2 \circ i)^* \circ pr_{2*} = i^* \circ pr_2^* \circ pr_{2*} \longrightarrow i^*. \hfill (7.20)$$

Remark 7.13 Formula (7.19) shows that the composition $pr_2! \longrightarrow pr_{2*} \longrightarrow i^*$ equals the morphism $pr_2! \longrightarrow pr_{2*} \circ i_* \circ i^* = pr_2! \circ i_! \circ i^* = i^*$.

Lemma 7.14 Identify $F \circ \text{Av}_{G/G'}$ and $F \circ \text{Av}_{G/G'}$ with the compositions (7.15) and (7.16). Then the diagram (7.12) comes from the morphisms (7.18) and the equality $i^*\Phi(i^*\Phi)^{-1} = \text{Id}_{\mathcal{D}_{G'}(X)}$.

Proof It suffices to show that the adjunction $F \circ \text{Av}_{G/G'} \longrightarrow \text{Id}_{\mathcal{D}_{G'}(X)}$ comes from the morphism $pr_{2*} \longrightarrow i^*$ defined by (7.20). By definition,

$$F \circ \text{Av}_{G/G'} = i^*\Phi \circ pr_2^* \circ pr_{2*} \circ (i^*\Phi)^{-1}, \hfill (7.21)$$

and the adjunction $F \circ \text{Av}_{G/G'} \longrightarrow \text{Id}_{\mathcal{D}_{G'}(X)}$ comes from the adjunction $pr_2^*pr_{2*} \longrightarrow \text{Id}$. It remains to consider the commutative diagram

$$
\begin{array}{c}
\mathcal{D}_G((G/G') \times X) \xrightarrow{pr_{2*}} \mathcal{D}_G(X) \xrightarrow{pr_2^*} \mathcal{D}_G((G/G') \times X) \\
\Phi \downarrow \quad \downarrow F \quad \downarrow \Phi \\
\mathcal{D}_{G'}(X) \xleftarrow{i^* \circ \Phi} \mathcal{D}_{G'}((G/G') \times X) \xrightarrow{pr_{2*}} \mathcal{D}_{G'}(X) \xrightarrow{pr_2^*} \mathcal{D}_{G'}((G/G') \times X) \\
\text{Id} \downarrow \quad \downarrow \text{Id} \\
\mathcal{D}_{G'}(X) \xrightarrow{i^*} \mathcal{D}_{G'}((G/G') \times X) \xrightarrow{i^*} \mathcal{D}_{G'}(X)
\end{array}
$$

\Box
7.5.1 Proof of Proposition 7.11

In the proof we specialize the earlier discussion to the case where \( X = G \) and \( G \) acts on itself by conjugation. In particular, we will use the diagram (7.17) in this setting.

Define \( \tilde{N} \in \mathcal{D}_{G'}((G/G') \times X) \) by

\[
\tilde{N} := \Phi((i^* \circ \Phi)^{-1}(N)). \tag{7.21}
\]

In view of Lemma 7.14 and Remark 7.13, we can reformulate Proposition 7.11 as follows:

**Claim** Under the assumptions of Proposition 7.11, the natural morphism

\[
\text{pr}_2^! \tilde{N} \longrightarrow \text{pr}_2^! i_*^* \tilde{N}
\]

becomes an isomorphism after convolution with \( L \) on the left.

To prove the claim, let \( U \) be the complement of \( \{1\} \) in \( G/G' \), and let \( j : U \times G \hookrightarrow (G/G') \times G \) denote the inclusion map. The exact triangle

\[
j : j^* \tilde{N} = j : j^! \tilde{N} \longrightarrow \tilde{N} \longrightarrow i_* i^* \tilde{N} \longrightarrow j : j^* \tilde{N}[1]
\]

implies that it is enough to check that \( L \ast \text{pr}_2^! j : j^* \tilde{N} = 0 \), i.e.,

\[
L \ast \pi_1 N' = 0, \tag{7.22}
\]

where \( \pi := \text{pr}_2 \circ j : U \times G \longrightarrow G \) is the second projection and \( N' = j^* \tilde{N} \).

**Lemma 7.15** (Projection formula) We have

\[
L \ast \pi_1 N' \cong \pi_1(\pi^* L \ast N'), \tag{7.23}
\]

where on the right-hand side we are using the convolution with compact support\(^{30}\)

\[
\mathcal{D}(U \times G) \times \mathcal{D}(U \times G) \longrightarrow \mathcal{D}(U \times G).
\]

**Proof** Let \( \mu : G \times U \longrightarrow G \) denote the multiplication morphism. Consider the commutative diagram

\[
\begin{array}{ccc}
G \times U \times G & \xrightarrow{\pi'} & G \times G \\
\downarrow \mu' & & \downarrow \mu \\
U \times G & \xrightarrow{\pi} & G
\end{array}
\]

\(^{30}\) It is defined by a formula essentially identical to (4.5), except that now \( G \) plays the role of \( H \) and \( U \) plays the role of \( S \) (and the order of the factors must be reversed) in (4.5).
where $\pi'(g_1, u, g_2) = (g_1, g_2)$ and $\mu'(g_1, u, g_2) = (u, g_1g_2)$. By the K"unneth formula,

$$L * \pi_1 N' \overset{\text{def}}{=} \mu_1(L \boxtimes \pi_1 N') \cong \mu_1(\pi'_1(L \boxtimes N'))$$

$$\cong \pi_1 \mu'_1(L \boxtimes N') \overset{\text{def}}{=} \pi_1(\pi^* L * N').$$

By Lemma 7.15, to prove (7.22), it suffices to check that $\pi^* L * N' = 0$. Equivalently, we must show that for every $u \in U(k)$, the restriction of $\pi^* L * N'$ to $(u) \times G \subset U \times G$ is equal to 0. This follows from (7.10) together with

**Lemma 7.16** Let $i_u : G \hookrightarrow U \times G$ be given by $x \mapsto (u, x)$, and let $g \in G(k)$ be any representative of $u$. Then

$$i_u^*(\pi^* L * N') \cong L * \delta_g * N * \delta_{g^{-1}}.$$ 

**Proof** By the proper base change theorem,

$$i_u^*(\pi^* L * N') \overset{\text{def}}{=} i_u^* \mu'_1(L \boxtimes N') \cong \mu_1(L \boxtimes i_u^* \widetilde{N}) \cong L * (i_u^* \widetilde{N}).$$

Define $\lambda_g : G/G' \longrightarrow G/G'$ and $c_g : G \longrightarrow G$ by $\lambda_g(x) := gx, c_g(y) := gyg^{-1}$. Then $i_u = (\lambda_g \times \text{id}_G) \circ i$, whence

$$i_u^* \widetilde{N} \cong i^*(\lambda_g \times \text{id}_G)^* \widetilde{N}$$

$$\cong i^*(\lambda_g \times \text{id}_G)^*(\lambda_{g^{-1}} \times c_{g^{-1}})^* \widetilde{N}$$

$$\cong i^*(\text{id} \times c_{g^{-1}})^* \widetilde{N} \cong c_{g^{-1}}^* i^* \widetilde{N} \cong c_{g^{-1}}^* N \overset{\text{def}}{=} \delta_g * N * \delta_{g^{-1}}$$

(the second isomorphism uses the fact that $\widetilde{N} \in \mathcal{D}_{G'}((G/G') \times X)$ comes from a $G$-equivariant complex $(i^* \circ \Phi)^{-1}(N)$, see (7.21)). The lemma follows. \hfill \Box

### 8 Proofs of the main results

Throughout this section, $k$ denotes an algebraically closed field of characteristic $p > 0$ and $\ell$ denotes a prime different from $p$. Our goal is to prove the five main theorems (2.15, 2.24, 2.41, 2.49, and 2.52) and the propositions stated in Sect. 2.

The section is organized as follows. In Sects. 8.1 and 8.3, we recall some results from [12]. In Sect. 8.2 we formulate a key compatibility lemma, which will be proved in Appendix 2. In Sect. 8.4 we prove Proposition 2.44. In Sect. 8.5 we review the exactness properties of pushforward and pullback functors $f_*, f^*$, and induction functors $\text{ind}_{G'}^G$, $\text{Ind}_{G'}^G$ with respect to perverse $t$-structures. In Sect. 8.6 we formulate a key result from which the five main theorems are deduced in Sects. 8.7–8.11 without difficulty. Propositions 2.46 and 2.19 are proved in Sects. 8.12 and 8.13, respectively (the proof of Proposition 2.19 uses Theorem 2.41). Finally, the aforementioned key
result is proved in Sect. 8.15, using an auxiliary proposition from Sect. 8.14 that relies on Proposition 7.4 and an equivariant version of Corollary 4.18.

We remark that the order in which the results of Sect. 2 were formulated is different from the order in which they are proved here. On the other hand, the structure of the present section is linear: Each argument we give relies only on the earlier proofs and/or the results of the preceding sections of the article.

8.1 Weak semigroupal structure on the functor \( \text{ind}_{G'}^G \)

**Lemma 8.1** Let \( G \) be a perfect unipotent group over \( k \), and let \( G' \subset G \) be a closed subgroup.

(a) The functor of induction with compact supports \( \text{ind}_{G'}^G : D_{G'}(G') \longrightarrow D_G(G) \) has a natural weak semigroupal structure

\[
(\text{ind}_{G'}^G M) \ast (\text{ind}_{G'}^G N) \longrightarrow \text{ind}_{G'}^G (M \ast N) \tag{8.1}
\]

(the morphism is defined for all \( M, N \in D_{G'}(G') \)).

(b) If \( M, N \in D_{G'}(G') \) satisfy \( \overline{M} \ast \delta_x \ast \overline{N} = 0 \) for all \( x \in G(k) \setminus G'(k) \), then (8.1) is an isomorphism (where \( \overline{M} \in D(G) \) is the extension of \( M \) by zero to \( G \), and \( \delta_x \) denotes the delta-sheaf at \( x \)).

(c) Suppose that \( e \in D_{G'}(G') \) is a weak idempotent satisfying the geometric Mackey condition with respect to \( G \). If \( M, N \in eD_{G'}(G') \), then \( \overline{M} \ast \delta_x \ast \overline{N} = 0 \) for all \( x \in G(k) \setminus G'(k) \).

**Proof** (a) If \( i : G' \hookrightarrow G \) is the inclusion morphism, then by definition (cf. Sect. 2.12), we have \( \text{ind}_{G'}^G = \text{av}_{G/G'} \circ i_* \), where \( G \) acts on itself by conjugation. The functor \( i_* : D_{G'}(G') \longrightarrow D_G(G) \) has an obvious strong semigroupal structure. Combining it with the weak semigroupal structure on \( \text{av}_{G/G'} \) constructed in Definition 7.6, we obtain a weak semigroupal structure on \( \text{ind}_{G'}^G \).

(b) It is not hard to check that the weak semigroupal structure on \( \text{ind}_{G'}^G \) constructed in part (a) coincides with that defined in [12, §5.5.2]. Hence, the desired assertion follows from Proposition 5.11 of op. cit.

(c) This follows immediately from the observation that \( \overline{M} \cong \overline{M} \ast \overline{e} \) and \( \overline{N} \cong \overline{e} \ast \overline{N} \) whenever \( M, N \in eD_{G'}(G') \), together with Definition 2.51. \( \square \)

**Remark 8.2** In the situation of Lemma 8.1(c), the object \( \text{ind}_{G'}^G e \in D_G(G) \) is a weak idempotent by Lemma 6.2(a). Moreover, Lemma 3.18, Lemma 6.2(d), and Lemma 8.1 imply

**Corollary 8.3** Suppose that in the situation of Lemma 8.1(c), the idempotents \( e \) and \( f = \text{ind}_{G'}^G e \) are closed. Then the semigroupal categories \( eD_{G'}(G') \) and \( fD_G(G) \) are monoidal, and the weakly semigroupal functor \( \text{ind}_{G'}^G \) restricts to a monoidal equivalence

\[
eD_{G'}(G') \simto fD_G(G). \tag{8.2}
\]
Remark 8.4 In the situation of Corollary 8.3, there is a canonical bijection $\varphi : A_r e \cong A_r f$, where $A_r e$ is the set of idempotent arrows $1 \rightarrow e$. Namely, $\varphi : A_r e \cong A_r f$ is the composition

$$A_r e \cong \text{Isom}(e * e, e) \cong \text{Isom}(f * f, f) \cong A_r f,$$

where $\text{Isom}(e * e, e)$ stands for the set of isomorphisms $e * e \cong e$, the middle bijection in (8.3) comes from the monoidal equivalence (8.2), and the other two bijections in (8.3) are provided by Corollary 3.37.

Remark 8.5 Unless $G / G'$ is finite, the semigroupal functor $\text{ind}_{G'}^{G}$ from Lemma 8.1(a) is not weakly monoidal in the sense of Remark 3.7 because the algebra $\text{ind}_{G'}^{G}(1_{G'})$ is not unital.

8.2 Compatibility of $\text{ind}_{G'}^{G}$ with braidings and twists

Lemma 8.6 Let $G$ be a perfect unipotent group over $k$, and let $G' \subset G$ be a closed subgroup.

(a) For all $M, N \in \mathcal{D}_{G'}(G')$, the diagram

$$\begin{array}{ccc}
\text{ind}_{G'}^{G}(M) \ast \text{ind}_{G'}^{G}(N) & \rightarrow & \text{ind}_{G'}^{G}(M \ast N) \\
\beta_{\text{ind}_{G'}^{G}(M), \text{ind}_{G'}^{G}(N)} \downarrow & & \downarrow \beta_{\text{ind}_{G'}^{G}(M), \text{ind}_{G'}^{G}(N)} \\
\text{ind}_{G'}^{G}(N) \ast \text{ind}_{G'}^{G}(M) & \rightarrow & \text{ind}_{G'}^{G}(N \ast M)
\end{array}$$

commutes, where $\beta$ (respectively, $\beta'$) is the braiding on $\mathcal{D}_G(G)$ (respectively, $\mathcal{D}_{G'}(G')$) constructed in Definition 9.43, and the horizontal arrows come from the weak semigroupal structure on $\text{ind}_{G'}^{G}$ constructed in Lemma 8.1(a).

(b) For all $M \in \mathcal{D}_{G'}(G')$, we have $\theta_{\text{ind}_{G'}^{G}(M)} = \text{ind}_{G'}^{G}(\theta'_M)$, where $\theta$ (respectively, $\theta'$) is the twist on $\mathcal{D}_G(G)$ (respectively, on $\mathcal{D}_{G'}(G')$) constructed in Definition 9.45.

The lemma will be proved in Appendix 2 (see Corollary 10.47 and Sect. 10.5.4).

8.3 Heisenberg idempotents and the geometric Mackey condition

Lemma 8.7 Let $G$ be a perfect unipotent group over $k$, let $(H, L)$ be an admissible pair for $G$, and let $G'$ denote its normalizer in $G$. Let $e'_{L}$ denote the Heisenberg minimal idempotent in $\mathcal{D}_{G'}(G')$ constructed in Sect. 2.11 (cf. Lemma 2.32). Then $e'_{L}$ satisfies the geometric Mackey condition with respect to $G$.

Proof This is shown in [12, §9.5].
8.4 Proof of Proposition 2.14

To prove the existence of $A$ satisfying properties (a)–(c) of the proposition, it suffices to show that if normal subgroups $A_1, A_2 \subset G$ satisfy (a) and (b), then so does $A_1 A_2$. To this end, it is enough to check that the homomorphism

\[ (A_1 A_2)^* \rightarrow A_1^* \times A_2^* \]  

has finite kernel. But

\[ \text{Ker}\left((A_1 A_2)^* \rightarrow A_1^*\right) = (A_1 A_2/A_1)^* = (A_2/(A_1 \cap A_2))^* \]

by property Sect. 4.2(3), the homomorphism

\[ (A_2/(A_1 \cap A_2))^* \rightarrow (A_2/(A_1 \cap A_2)^0)^* \]

has finite kernel by Sect. 4.2(4), and the homomorphism $(A_2/(A_1 \cap A_2)^0)^* \rightarrow A_2^*$ is injective by Sect. 4.2(3), so (8.4) has finite kernel.

To finish the argument, we will prove the following.

**Lemma 8.8** Let $A \subset G$ be a connected normal subgroup and $N$ a $G^\circ$-invariant multiplicative local system on $A$. Put $G_1 = NG(N)$, so that $G^\circ \subset G_1 \subset G$, and let $e_1 = \text{av}_{G/G_1}(N \otimes \mathbb{K}_A) \in \mathcal{D}(G)(A) \subset \mathcal{D}(G)$. The following properties are equivalent:

(i) $A \subset H$ and $\mathcal{L}|_A$ is $G$-conjugate to $N$;

(ii) $f^* e_1 \cong e_1$, where $f$ is constructed in Theorem 2.41(a).

**Remark 8.9** The lemma implies that the $G$-orbit of $(A, \mathcal{L}|_A)$ depends only on $f$ and hence completes the proof of Proposition 2.44. Indeed, property (ii) of the lemma manifestly depends only on $f$. On the other hand, if $g_1, \ldots, g_n$ are representatives of the cosets of $G_1$ in $G$ and $N g_j$ are the corresponding conjugates of $N$, then $e_1 \cong \bigoplus_{j=1}^n (N g_j \otimes \mathbb{K}_A)$ in $\mathcal{D}(A)$, which shows that the datum of $e_1$ is equivalent to the datum of the $G$-orbit of $(A, \mathcal{L}|_A)$.

**Proof of Lemma 8.8** If (i) holds, then $e_1' \ast e_1 \cong e_1'$, as $\mathcal{L}|_A \cong N g_j$ for a unique $j$ with the notation of Remark 8.9. By definition, $f = \text{ind}_{G/G'} e_1' = \text{av}_{G/G'}(i_* e_1')$, where $i : G' \hookrightarrow G$ is the embedding. So (ii) follows from Corollary 7.8.

Now assume (ii). By Lemma 8.7, $e_1' \in \mathcal{D}(G')$ satisfies the geometric Mackey condition with respect to $G$. The argument that was used to prove Proposition 6.7 can be repeated verbatim in this case, and it shows that $A \subset G'$ and $e_1' \ast e_1 \cong e_1'$. Property (i) follows.

8.5 Exactness properties of $\text{ind}_{G/G'}^G$ and $\text{Ind}_{G/G'}^G$

Let us recall a result of M. Artin:
Theorem 8.10 ([7], Thm. 4.1.1) If \( f : X \rightarrow Y \) is an affine morphism of schemes of finite type over \( k \), the functor \( f_* : \mathcal{D}(X) \rightarrow \mathcal{D}(Y) \) is left exact with respect to the perverse \( t \)-structures, i.e., takes \( p \mathcal{D}^\leq(X) \) into \( p \mathcal{D}^\leq(Y) \).

Corollary 8.11 (op. cit., Cor. 4.1.2) Under the same assumptions, the functor \( f^! : \mathcal{D}(X) \rightarrow \mathcal{D}(Y) \) is right \( t \)-exact, i.e., takes \( p \mathcal{D}^\geq(X) \) into \( p \mathcal{D}^\geq(Y) \).

We also recall

Proposition 8.12 If \( f : X \rightarrow Y \) is a smooth morphism of \( k \)-schemes everywhere of relative dimension \( d \), then \( f^*[d] \) takes \( \text{Perv}(Y) \) into \( \text{Perv}(X) \).

Using the construction of induction functors presented in Sect. 2.12 together with Theorem 8.10, Corollary 8.11, and Proposition 8.12, one obtains

Lemma 8.13 Let \( G \) be a perfect unipotent group over \( k \), and let \( G' \subset G \) be a closed subgroup. Then

\[
\text{ind}^{G}_{G'}(p \mathcal{D}^\geq(G')) \subset p \mathcal{D}^\geq(G)[−\dim(G/G')]
\]

and

\[
\text{Ind}^{G}_{G'}(p \mathcal{D}^\leq(G')) \subset p \mathcal{D}^\leq(G)[−\dim(G/G')].
\]

8.6 The key proposition

The next proposition will be proved in Sect. 8.15.

Proposition 8.14 Let \( G \) be a perfect unipotent group over \( k \). For every nonzero \( N \in \mathcal{D}(G) \), there exists a closed idempotent \( f \in \mathcal{D}_G(G) \) such that \( f \ast N \neq 0 \) and \( f \equiv \text{ind}^{G}_{G'} e' \), where \( G' \subset G \) is the normalizer of some admissible pair \( (H, \mathcal{L}) \) and \( e' \) is the Heisenberg minimal idempotent \( e'_\mathcal{L} \in \mathcal{D}_{G'}(G') \) corresponding to \( (H, \mathcal{L}) \).

Remark 8.15 In the situation of Proposition 8.14, \( f \) is automatically minimal as a weak idempotent in \( \mathcal{D}_G(G) \). This follows from Lemmas 2.32, 8.7, and 6.1.

8.7 Proof of Theorem 2.49

Let \( N \in \mathcal{D}(G) \) be nonzero, and let \( f \in \mathcal{D}_G(G) \) satisfy the conclusion of Proposition 8.14. By Remark 8.15, \( f \) is minimal as a weak idempotent in \( \mathcal{D}_G(G) \). A fortiori, we see that:

- \( f \) is minimal as a closed idempotent, which yields Theorem 2.49(c) and
- \( \mathcal{D}_G(G) \) is a Jacobson monoidal category (Definition 3.28), so parts (a) and (b) of Theorem 2.49 follow from Proposition 3.30.
8.8 Proof of Theorem 2.52

(a) By assumption, $e$ is a minimal closed idempotent in $\mathcal{D}_{G'}(G')$. By Theorem 2.49(a), $e$ is a minimal weak idempotent in $\mathcal{D}_{G'}(G')$. By Theorem 2.49(a), $f = \text{ind}_{G}^{G'} e$ is a minimal weak idempotent in $\mathcal{D}_{G}(G)$. By Theorem 2.49(b), $f$ is a minimal closed idempotent in $\mathcal{D}_{G}(G)$.

(c) Since $f$ is a closed idempotent by part (a), the assertion follows from Corollary 8.3 and Lemma 8.6.

(d) Use the fact that $f$ is closed and the implication (ii)⇒(i) of Proposition 6.9.

(e) Combine Lemma 8.13 with assertion (d) of the theorem.

(b) This follows from part (e).

8.9 Proof of Theorem 2.41

By Lemma 2.32, $e'_{\mathcal{L}} \in \mathcal{D}_{G'}(G')$ is a minimal closed idempotent. By Lemma 8.7, it satisfies the geometric Mackey condition with respect to $G$. So Theorem 2.41(a) follows from Theorem 2.52(a).

The equality $n e'_{\mathcal{L}} = \dim H$ follows from the definition of $e'_{\mathcal{L}}$. Using this equality and Theorem 2.52(b), we get Theorem 2.41(b).

Let $N \in \mathcal{D}_{G}(G)$ be a minimal closed idempotent. Let $f$ be as in Proposition 8.14 (i.e., $f \ast N \neq 0$ and $f$ is obtained from an admissible pair by induction with compact supports). To prove Theorem 2.41(c) for $N$, it suffices to show that $N \cong f$. This is clear because $f \ast N \neq 0$, and both $N$ and $f$ are minimal closed idempotents in $\mathcal{D}_{G}(G)$ (for $f$ this follows from assertion (a), proved above).

8.10 Proof of Theorem 2.33

Theorem 2.15 for $(G', e'_{\mathcal{L}})$ in place of $(G, e)$ was proved by Deshpande, see [25, Theorems 1.1–1.5]. To prove Theorem 2.24 for $(G', e'_{\mathcal{L}})$, we need some notation.

Let $(H, \mathcal{L})$ be the admissible pair for $G'$ that gives rise to the minimal Heisenberg idempotent $e'_{\mathcal{L}}$. We can view $e'_{\mathcal{L}}$ as a closed idempotent either in $\mathcal{D}_{G'}(G')$ or in $\mathcal{D}_{G\circ}(G')$. Accordingly, we have modular categories

$$\mathcal{M} := \{ M \in e'_{\mathcal{L}} \mathcal{D}_{G'}(G') \mid M[-\dim H] \text{ is perverse} \},$$

$$\mathcal{M}_1 := \{ M \in e'_{\mathcal{L}} \mathcal{D}_{G\circ}(G') \mid M[-\dim H] \text{ is perverse} \}.$$

Let $\tau^+(\mathcal{M})$ and $\tau^+(\mathcal{M}_1)$ be their Gauss sums. Datta [19, §2.4] proved Theorem 2.24 for $(G\circ, e'_{\mathcal{L}})$ in place of $(G, e)$. So to prove Theorem 2.24 for $(G', e'_{\mathcal{L}})$, it suffices to show that $\tau^+(\mathcal{M})/\tau^+(\mathcal{M}_1)$ is a power of $p$. In fact, we will prove that

$$\tau^+(\mathcal{M}_1) = \tau^+(\mathcal{M})/|\Gamma|,$$

where $\Gamma := \pi_0(G') = G'/G'\circ$ is a $p$-group by unipotence of $G'$. 


To this end, consider the full subcategory $\mathcal{E} := \{M \in \mathcal{M} \mid \text{supp } M \subset H\}$. Note that $\mathcal{E}$ is braided equivalent to the category of finite-dimensional representations of $\Gamma$ and the twist on $\mathcal{M}$ induces the trivial twist on $\mathcal{E}$. In this situation we can apply [26, Thm. 6.16], which interprets the r.h.s. of (8.5) as the Gauss sum of a certain modular category. The latter identifies with $\mathcal{M}_1$ (to see this, combine [25, Lem. 1.4], [26, Thm. 4.44], and [26, Prop. 4.56(i)]).

8.11 Proof of Theorems 2.15 and 2.24

Let $e \in \mathcal{D}_G(G)$ be a minimal closed idempotent. By Theorem 2.41(c), we have $e \cong \text{ind}^{G'}_G e'_{L'}$ for some admissible pair $(H, \mathcal{L})$ for $G$, where $G'$ is the normalizer of $(H, \mathcal{L})$ in $G$ and $e'_{L'} \in \mathcal{D}_{G'}(G')$ is the Heisenberg minimal idempotent defined by $\mathcal{L}$ (see Sect. 2.11).

By Theorem 2.52(c), $\text{ind}^{G'}_G$ restricts to a monoidal triangulated equivalence

$$e'_{L'} \mathcal{D}_{G'}(G') \sim \rightarrow e \mathcal{D}_G(G), \tag{8.6}$$

and by Theorem 2.52(e), the equivalence (8.6) restricts to an equivalence between $\mathcal{M}_{e'_{L'}}^{\text{per}}$ and $\mathcal{M}_{e_{L}}^{\text{per}}[\text{dim}(G/G')]$. By Lemma 8.6, the monoidal equivalence (8.6) is compatible with the canonical braidings (Definition 9.43) and twists (Definition 9.45) on both categories. In addition, by Theorem 2.52(b), $n_e \leq n_{e'_{L'}}$ and the functional dimensions $d_{e'_{L'}}$ and $d_e$ differ by an integer. Theorem 2.33 shows that Theorems 2.15 and 2.24 hold for the idempotent $e'_{L'} \in \mathcal{D}_{G'}(G')$. Hence, Theorems 2.15 and 2.24 also hold for $e \in \mathcal{D}_G(G)$ with a possible exception of the inequality

$$n_e \geq 0, \tag{8.7}$$

which is a part of Theorem 2.15(b).

Now let us prove (8.7). Recall that $e[-n_e]$ is perverse by the definition of $n_e$. So

$$\text{Ext}^i(1, e[-n_e]) = 0 \quad \text{for } i < 0 \tag{8.8}$$

(this follows from the definition of a perverse sheaf and the fact that $1$ is a delta-sheaf). On the other hand, an idempotent arrow $1 \rightarrow e$ is a nonzero element of $\text{Hom}(1, e)$, so $\text{Ext}^{n_e}(1, e[-n_e]) = \text{Hom}(1, e) \neq 0$. Comparing this with (8.8), we get (8.7).

8.12 Proof of Proposition 2.46

We begin by proving the proposition in the case where $H = G$. In this case $n_f = n_{e'_{L'}} = \text{dim } H = \text{dim } G$ by Theorem 2.41(b). Moreover, $f = \mathcal{L} \otimes \mathbb{K}_G$, where $\mathcal{L}$ is a multiplicative local system on $G$ and $\mathbb{K}_G$ is the dualizing complex of $G$. Since $G$ is smooth of dimension $n_f$, there is a canonical identification $\mathbb{K}_G \sim \rightarrow \mathbb{Q}_\ell[2n_f](n_f)$ and therefore

$$f \sim \rightarrow \mathcal{L}[2n_f](n_f). \tag{8.9}$$
By definition, $\mathbb{D}_G^- = \mathbb{D}_G \circ \iota^*$, where $\iota : G \rightarrow G$ is given by $g \mapsto g^{-1}$. Since $\mathcal{L}$ is multiplicative, $\iota^*\mathcal{L} = \mathcal{L}^{-1}$, so $\mathbb{D}_G^- f = \mathbb{D}_G(\mathcal{L}^{-1} \otimes K_G) = \mathcal{L} \otimes \mathbb{D}_G(K_G) = \mathcal{L}$. Combining this with (8.9), one gets a canonical isomorphism $\mathbb{D}_G^- f \xrightarrow{\sim} f[-2n_f](-n_f)$, completing the proof of the proposition when $H = G$.

Next, we treat the general case. Writing $d = \dim(G/G')$, we recall that $n_f = n_{e'_{\mathcal{L}}} - d$ by Theorem 2.41(b). The first part of the proof yields a natural isomorphism

$$\mathbb{D}_G^- e'_{\mathcal{L}} \xrightarrow{\sim} e'_{\mathcal{L}}[-2n_{e'_{\mathcal{L}}}](-n_{e'_{\mathcal{L}}}).$$

Applying the functor $\mathbb{D}_G^- \circ \text{Ind}^G_{G'}$, we obtain a natural isomorphism

$$((\mathbb{D}_G^- \circ \text{Ind}^G_{G'})(e'_{\mathcal{L}}[-2n_{e'_{\mathcal{L}}}](-n_{e'_{\mathcal{L}}})) \xrightarrow{\sim} (\mathbb{D}_G^- \circ \text{Ind}^G_{G'} \circ \mathbb{D}_G^-)(e'_{\mathcal{L}}).$$

Composing the latter with the inverse of the isomorphism (7.9) provided by Corollary 7.10, we obtain a natural isomorphism

$$((\mathbb{D}_G^- \circ \text{Ind}^G_{G'})(e'_{\mathcal{L}}[-2n_{e'_{\mathcal{L}}}](-n_{e'_{\mathcal{L}}})) \xrightarrow{\sim} \text{ind}^G_{G'}(e'_{\mathcal{L}})[2d](d),$$

which is the same thing as an isomorphism

$$((\mathbb{D}_G^- \circ \text{Ind}^G_{G'})(e'_{\mathcal{L}}) \xrightarrow{\sim} \text{ind}^G_{G'}(e'_{\mathcal{L}})[-2n_f](-n_f).$$

Finally, the natural morphism $f = \text{ind}^G_{G'}(e'_{\mathcal{L}}) \rightarrow \text{Ind}^G_{G'}(e'_{\mathcal{L}})$ is an isomorphism by Theorem 2.52(d) and Lemma 8.7. This yields a natural isomorphism

$$\mathbb{D}_G^- f \xrightarrow{\sim} f[-2n_f](-n_f),$$

as desired.

8.13 Proof of Proposition 2.19

By Theorem 2.41(c), every minimal closed idempotent $e \in \mathcal{D}_G(G)$ arises from some admissible pair for $G$. In view of Remark 2.20(i), we see that Proposition 2.19(a) follows from Proposition 2.46. Now we will prove Proposition 2.19(b) using the language of Grothendieck–Verdier categories (see Definitions 9.2, 9.4, and 9.6 from Appendix 1).

By Example 9.9, $\mathcal{D}_G(G)$ is an r-category, where the duality functor is the functor $\mathbb{D}_G^-$ from Definition 2.17. In particular, $\mathcal{D}_G(G)$ is a Grothendieck–Verdier category.

By Lemma 9.50, $e\mathcal{D}_G(G)$ is a Grothendieck–Verdier category with dualizing object $\mathbb{D}_G^- e$, and the corresponding duality functor can be identified with $\mathbb{D}_G^-$. By part (a) of Proposition 2.19, we have $\mathbb{D}_G^- e \cong e[-2n_e](-n_e) \otimes L_e$ for a certain line $L_e$ over $\overline{\mathbb{Q}}_\ell$. In particular, $\mathbb{D}_G^- e$ is an invertible object of the monoidal category $e\mathcal{D}_G(G)$, with inverse $e[2n_e](n_e) \otimes L_e^{-1}$. Hence, $e$ is also a dualizing object of $e\mathcal{D}_G(G)$, and the duality
functor that \( e \) defines is given by \( M \mapsto (\mathbb{D}_G^{-1}M)[2n_e](n_e) \otimes L_{e^{-1}}. \) This implies Proposition 2.19(b) since we already saw in Theorem 2.15(c) that \( eD_G(G) \) is a rigid monoidal category with unit object \( e \).

The rest of this section is devoted to a proof of Proposition 8.14. It will be given in Sects. 8.15.2–8.15.3 after various preliminaries.

8.14 Closed idempotents via averaging

Let \( H \) be a connected perfect unipotent group over \( k \), and let \( G \) be a perfect unipotent group acting on \( H \) by group automorphisms. Fix a multiplicative local system \( \mathcal{L} \) on \( H \), and let \( G' \subset G \) be the stabilizer of the corresponding point of \( H^*(k) \). Equip the categories \( D_G(H) \) and \( D_G(H) \) with the monoidal structure given by convolution with compact supports. Since \( G' \) stabilizes the (isomorphism class of) \( \mathcal{L} \), both \( \mathcal{L} \) and \( e_\mathcal{L} = \mathcal{L} \otimes \mathbb{K}_H \) can be viewed as objects of \( D_{G'}(H) \).

**Lemma 8.16** \( e_\mathcal{L} \) is a closed idempotent in \( D_{G'}(H) \).

**Proof** Let \( s : H \rightarrow \text{Spec}(k) \) and \( 1 : \text{Spec}(k) \rightarrow H \) denote the structure morphism and the identity of \( H \), respectively. Then \( \mathbb{K}_H \cong s^!\mathbb{Q}_k \) and \( 1 \cong 1^!\mathbb{Q}_k \) (the unit object of \( D_{G'}(H) \)). The natural isomorphism \( \mathbb{Q}_k \cong 1^!\mathbb{K}_H \) induces by adjunction a morphism \( 1 \rightarrow \mathbb{K}_H \). On the other hand, since \( \mathcal{L}^\vee \) is a multiplicative local system, its fiber at the identity element of \( H(k) \) has a canonical trivialization, which yields an isomorphism \( 1 \otimes \mathcal{L}^\vee \cong 1 \). The composition \( 1 \otimes \mathcal{L}^\vee \rightarrow \mathbb{K}_H \) corresponds to a morphism \( 1 \rightarrow e_\mathcal{L} \).

By [12, §8.3], this morphism becomes an isomorphism in \( D(H) \), and hence also in \( D_{G'}(H) \), after convolving with \( e_\mathcal{L} \). \( \square \)

Recall that the averaging functor \( \text{av}_{G/G'} : D_{G'}(H) \rightarrow D_G(H) \) was defined in Sect. 2.12. The next result is used in the proof of Lemma 8.18.

**Proposition 8.17** The object \( \text{av}_{G/G'}(e_\mathcal{L}) \in D_G(H) \) is a closed idempotent.

**Proof** In the proof we will use the functors

\[ \mathcal{F}^G : D_G(H^*) \rightarrow D_G(H) \quad \text{and} \quad \mathcal{F}^{G'} : D_G(H^*) \rightarrow D_G(H) \]

(see Sect. 7.1) together with Proposition 7.4 and Corollary 4.18.

Let \( \delta_\mathcal{L} \in D_{G'}(H^*) \) denote the delta-sheaf at the point \([\mathcal{L}] \in H^*(k)\), let \( \mathcal{O}_\mathcal{L} \) be the \( G \)-orbit of \([\mathcal{L}] \), and let \( i : \mathcal{O}_\mathcal{L} \hookrightarrow H^* \) denote the inclusion morphism. Since \( G \) is unipotent and \( H^* \) is affine, \( i \) is closed. Moreover, \( \text{av}_{G/G'}(\delta_\mathcal{L}) \cong i_!(\mathbb{Q}_k)\mathcal{O}_\mathcal{L} \) and \( e_\mathcal{L} \cong \mathcal{F}^{G'}(\delta_\mathcal{L}) \), whence by Proposition 7.4, we have

\[ \text{av}_{G/G'}(e_\mathcal{L}) \cong \text{av}_{G/G'}(\mathcal{F}^G(\delta_\mathcal{L})) \cong \mathcal{F}^G(\text{av}_{G/G'}(\delta_\mathcal{L})) \cong \mathcal{F}^G(i_!(\mathbb{Q}_k)\mathcal{O}_\mathcal{L}). \]

Now Corollary 4.18 yields an idempotent arrow

\[ \mathcal{F}^G(\pi_{\mathcal{O}_\mathcal{L}}) \circ \pi_0 : 1 \rightarrow \mathcal{F}^G(i_!(\mathbb{Q}_k)\mathcal{O}_\mathcal{L}) \]
Let us recall from Sect. 5.1 that \(\mathcal{D}(H)\), and we only need to check that this arrow is a morphism in \(\mathcal{D}_G(H)\). But \(\pi_{\mathcal{O}_C} : (\overline{\mathbb{Q}}_l)_{H^*} \to i_!((\overline{\mathbb{Q}}_l)_{\mathcal{O}_C})\) is clearly \(G\)-equivariant, and \(\pi_0\) is \(G\)-equivariant because its construction is canonical. This completes the proof. \(\square\)

8.15 Proof of Proposition 8.14

The argument has three stages. First, we will prove an auxiliary Lemma 8.18 (which does not involve the object \(N\)). Next, we will reduce the proposition to the case where \(N\) is \(G\)-equivariant, using Corollary 7.8. Finally, we will prove the proposition in this special case with the aid of Lemma 8.18 and Proposition 6.7.

8.15.1 An auxiliary lemma

Let us recall from Sect. 5.1 that \(\mathcal{D}_{\text{norm}}(G)\) denotes the set of pairs \((A, N)\), where \(A\) is a normal connected subgroup of \(G\) and \(N\) is a multiplicative local system on \(A\).

**Lemma 8.18** Let \((A, N) \in \mathcal{D}_{\text{norm}}(G)\). Denote by \(G_1\) the normalizer of \(N\) in \(G\), consider the closed idempotent \(N \otimes \mathbb{K}_A \in \mathcal{D}_A(A)\) determined by \(N\) (cf. Sect. 2.11), and let \(e_1 \in \mathcal{D}_{G_1}(G_1)\) be its extension by zero \(^{31}\) to \(G_1\).

(a) \(e_1\) is a closed idempotent in \(\mathcal{D}_{G_1}(G_1)\).
(b) \(e_1\) satisfies the geometric Mackey condition with respect to \(G\).
(c) Let \(e_0 = \text{ind}_{G_1}^{G} e_1\). Then \(e_0\) is a closed idempotent in \(\mathcal{D}_{G}(G)\).
(d) The semigroupal categories \(e_1 \mathcal{D}_{G_1}(G_1)\) and \(e_0 \mathcal{D}_{G}(G)\) are monoidal.
(e) The functor \(\text{ind}_{G_1}^{G}\) restricts to a monoidal equivalence \(e_1 \mathcal{D}_{G_1}(G_1) \sim e_0 \mathcal{D}_{G}(G)\).
(f) \(e_1 \ast e_0 \cong e_1\).

**Proof** (a) We have a canonical idempotent arrow \(1 \to N \otimes \mathbb{K}_A\) in \(\mathcal{D}_A(A)\), and it is clear that it is in fact \(G_1\)-equivariant. Thus, \(N \otimes \mathbb{K}_A\) is also a closed idempotent in \(\mathcal{D}_{G_1}(A)\), which proves (a).

(b) We must prove that \(\overline{e_1} \ast \delta_x \ast \overline{e_1} = 0\) for all \(x \in G(k)\) such that \(x \notin G_1(k)\). The last identity is equivalent to \(\overline{e_1} \ast \delta_x \ast \overline{e_1} \ast \delta_{x^{-1}} = 0\). But \(\delta_x \ast \overline{e_1} \ast \delta_{x^{-1}}\) is the extension by zero of the object \(N_x \otimes \mathbb{K}_A \in \mathcal{D}_A(A)\), where \(N_x\) denotes the pullback of \(N\) by the automorphism \(a \mapsto x^{-1}ax\) of \(A\). Since \(x \notin G_1(k)\), we have \(N_x \not\cong N\), whence \((N \otimes \mathbb{K}_A) \ast (N_x \otimes \mathbb{K}_A) = 0\). This implies that \(\overline{e_1} \ast \delta_x \ast \overline{e_1} \ast \delta_{x^{-1}} = 0\).

(c) With the notation of Proposition 8.17, \(e_0\) is the extension by zero of the object \(\text{av}_{G/G_1}(e_N) \in \mathcal{D}_G(A)\). Applying Proposition 8.17 to the conjugation action of \(G\) on \(A\) yields (c).

Statements (d) and (e) follow from (a)–(c) and Corollary 8.3. Finally, statement (f) follows from the last assertion of Lemma 6.4. \(\square\)

8.15.2 Reduction of Proposition 8.14 to the equivariant case

Let us assume for the moment that the proposition holds for any \(0 \neq N \in \mathcal{D}_G(G)\).

We will explain how to prove it in full generality.

\(^{31}\) \(e_1\) has a canonical \(G_1\)-equivariant structure because \(N\) is \(G_1\)-invariant.
Let \( N \in \mathcal{D}(G) \) be nonzero. It suffices to prove Proposition 8.14 assuming that the stalk of \( N \) at 1 \( \in G \) is nonzero (otherwise, we can replace \( N \) with its right translation by an appropriate \( g \in G(k) \)). Define \( \widetilde{N} \in \mathcal{D}_G(G) \) by \( \widetilde{N} := \text{av}_G(N) \), where \( \text{av}_G : \mathcal{D}(G) \to \mathcal{D}_G(G) \) is the functor of averaging with compact support (see (2.6) and Definition 2.36). Then \( \widetilde{N} \neq 0 \) (because the stalk of \( \widetilde{N} \) at 1 \( \in G \) is nonzero). So by assumption, \( \widetilde{N} \ast f \neq 0 \) for some closed idempotent \( f \in \mathcal{D}_G(G) \) satisfying the last requirement of Proposition 8.14. Since \( \widetilde{N} \ast f \cong \text{av}_G(N \ast f) \) by Corollary 7.8, we see that \( N \ast f \neq 0 \).

### 8.15.3 Proof of Proposition 8.14 in the equivariant case

We now fix a nonzero \( N \in \mathcal{D}_G(G) \) and complete the proof of Proposition 8.14 in this special case.

Let \((A, \mathcal{N})\) be maximal among all pairs in \( \mathcal{D}_{\text{norm}}(G) \) that are compatible with \( N \) in the sense of Definition 5.2. Let \( G_1, e_1 \), and \( e_\mathcal{O} \) have the same meaning as in Lemma 8.18. Compatibility of \((A, \mathcal{N})\) with \( N \) means that

\[
e_1 \ast N \neq 0, \tag{8.10}\]

where \( e_1 \) is viewed as an object of \( \mathcal{D}_G(G) \).

We consider two cases. If \( G_1 = G \), then \((A, \mathcal{N})\) is an admissible pair for \( G \) by Proposition 5.4. Thus, the idempotent \( f := e_1 \) has the properties required in the formulation of Proposition 8.14.

Now assume that \( G_1 \subsetneq G \). By Lemma 8.18(e), \( e_\mathcal{O} \ast N = \text{ind}^{G_1}_{G} N_1 \) for some \( N_1 \in e_1 \mathcal{D}_{G_1}(G_1) \). Combining (8.10) with Lemma 8.18(f), we see that \( e_\mathcal{O} \ast N \neq 0 \) and therefore \( N_1 \neq 0 \). Since \( G_1 \neq G \), we may assume by induction that there exists a closed idempotent \( f_1 \in \mathcal{D}_{G_1}(G_1) \) that satisfies

\[
f_1 \ast N_1 \neq 0 \tag{8.11}\]

and has the form \( f_1 \cong \text{ind}^{G_1}_{G} e'_L \) for some admissible pair \((H, L)\) for \( G_1 \), where \( G' \) is the normalizer of \((H, L)\) in \( G_1 \) and \( e'_L \in \mathcal{D}_{G'}(G') \) is the Heisenberg minimal idempotent corresponding to \((H, L)\).

Since \( N_1 \in e_1 \mathcal{D}_{G_1}(G_1) \), it follows from (8.11) that \( f_1 \ast e_1 \neq 0 \). By Remark 8.15, \( f_1 \) is minimal as a weak idempotent in \( \mathcal{D}_{G_1}(G_1) \), so

\[
f_1 \ast e_1 \cong f_1, \tag{8.12}\]

i.e., \( f_1 \in e_1 \mathcal{D}_{G_1}(G_1) \).

Define \( f \in e_\mathcal{O} \mathcal{D}(G) \) by \( f := \text{ind}^{G}_{G_1} f_1 \). Let us show that \( f \) has the properties required in the formulation of Proposition 8.14. By Lemma 8.18(e), \( f \in e_\mathcal{O} \mathcal{D}(G) \) is a closed idempotent, so by Lemma 8.18(c), \( f \) is also a closed idempotent in \( \mathcal{D}_G(G) \). To show that \( f \ast N \neq 0 \), note that since \( f \in e_\mathcal{O} \mathcal{D}(G) \), one has

\[
f \ast N \cong f \ast e_\mathcal{O} \ast N \cong (\text{ind}^{G}_{G_1} f_1) \ast (\text{ind}^{G}_{G_1} N_1),
\]

and \((\text{ind}^{G}_{G_1} f_1) \ast (\text{ind}^{G}_{G_1} N_1) \neq 0 \) by (8.11) and Lemma 8.18(e).
It remains to check that \( f \) satisfies the last requirement of Proposition 8.14. By (8.12) and Lemma 8.7, we can apply Proposition 6.7, replacing \( G, f, e \) with \( G_1, f_1, e'_L \), respectively. We deduce that \( G' \supset A \) and \( e'_L \ast e_1 \cong e'_L \), which implies that \((A, N) \leq (H, L)\) with respect to the partial order introduced in Definition 5.1. Lemma 5.12 shows that \((H, L)\) is admissible for \( G \) and that \( G' \) is equal to the normalizer of \((H, L)\) in \( G \).

Acknowledgments We are indebted to George Lusztig, who originally suggested in 2003 that there should exist a theory of character sheaves on unipotent groups in positive characteristic and computed the first interesting examples in this theory. We thank A. Beilinson, K. Costello, J. Lurie, and U. Tillmann for valuable advice. We also thank the referees for pointing out several misprints and omissions in an earlier version of our article.

9 Appendix 1: Grothendieck–Verdier categories and r-categories

Let \( G \) be an algebraic group. The monoidal categories \((\mathcal{D}(G), \ast)\) and \((\mathcal{D}_G(G), \ast)\) are usually not rigid, but they have a weaker type of duality, which goes back to Grothendieck and Verdier. In this appendix, we give an axiomatic treatment of the Grothendieck–Verdier formalism in monoidal categories. A more complete exposition of the subject can be found in [16].

Throughout this appendix, with the exception of Sect. 9.5, we interpret \( \mathcal{D}_G(G) \) as the bounded derived category \( D^b_c((\text{Ad} G) \backslash G, \mathcal{O}_G) \) of the stack quotient of \( G \) with respect to its conjugation action on itself (cf. [38]).

9.1 Grothendieck–Verdier categories and r-categories

9.1.1 Definitions and examples

**Definition 9.1** Let \( \mathcal{C} \) be a category and \( \Phi : \mathcal{C} \times \mathcal{C} \to \text{Sets} \) a functor, which is contravariant in both arguments. We say that \( \Phi \) is a dualizing functor if for every \( Y \in \mathcal{C} \) the functor \( X \mapsto \Phi(X, Y) \) is representable by some object \( DY \in \mathcal{C} \) and the contravariant functor \( D : \mathcal{C} \to \mathcal{C} \) is an antiequivalence. \( D \) is called the duality functor with respect to \( \Phi \).

**Definition 9.2** An object \( K \) in a monoidal category \( \mathcal{M} \) is said to be dualizing if the functor \( \Phi(X, Y) = \text{Hom}(X \otimes Y, K) \) is dualizing. The corresponding duality functor is called the duality functor with respect to \( K \).

**Remark 9.3** One can show that if a dualizing object exists, then it is unique up to tensoring by an invertible object, see [16, Proposition 1.3(i)] for more details.

**Definition 9.4** A Grothendieck–Verdier category is a pair \((\mathcal{M}, K)\), where \( \mathcal{M} \) is a monoidal category and \( K \in \mathcal{M} \) is a dualizing object.

By abuse of language, we will usually say “Grothendieck–Verdier category \( \mathcal{M} \)” instead of “Grothendieck–Verdier category \((\mathcal{M}, K)\).”
Below, we give some examples of Grothendieck–Verdier categories. More examples of such categories can be found in [16] and in the works by Barr, who studied them under the name of *-autonomous categories (e.g., see [2–5]).

**Example 9.5** Let \( M = (\mathcal{D}(X), \otimes) \), where \( X \) is a scheme of finite type over a field \( k \) and \( \mathcal{D}(X) \) is the bounded derived category of constructible \( \ell \)-adic sheaves on \( X \), \( \ell \neq \text{char } k \). Let \( K_X \in \mathcal{D}(X) \) be the dualizing complex. Then \( (M, K_X) \) is a Grothendieck–Verdier category. In this case \( D \) is the usual Verdier duality functor \( D_X \).

**Definition 9.6** A monoidal category \( M \) is said to be an \( r \)-category if the unit object \( 1 \in M \) is dualizing.

So any \( r \)-category can be considered as a Grothendieck–Verdier category with \( K = 1 \). The letter ‘r’ in the name “\( r \)-category” is related to the words “rigid” and “regular,” see Examples 9.7–9.8 below.

**Example 9.7** Any rigid monoidal category is an \( r \)-category. The next example shows that the converse is false.

**Example 9.8** Let \( X \) be a smooth \( k \)-scheme (or if you wish, a regular scheme of finite type over \( k \)). Suppose that \( X \) has pure dimension \( d \). Then the monoidal category \( (\mathcal{D}(X), \otimes) \) is an \( r \)-category, and \( D : \mathcal{D}(X) \to \mathcal{D}(X) \) is the functor \( N \mapsto (\mathbb{D}XN)[-2d](−d) \). If \( d > 0 \), then \( (\mathcal{D}(X), \otimes) \) is not rigid because \( D(M_1 \otimes M_2) \not\simeq D(M_2) \otimes D(M_1) \) for some \( M_1, M_2 \in \mathcal{D}(X) \). For example, take \( M_1 = M_2 = i_\ast \mathbb{Q}_\ell \), where \( i : \text{Spec } k \to X \) is a point; then \( D(M_1 \otimes M_2) = D(i_\ast \mathbb{Q}_\ell) = i_\ast \mathbb{Q}_\ell[-2d](−d) \) while \( D(M_2) \otimes D(M_1) = i_\ast \mathbb{Q}_\ell[-4d](−2d) \).

**Example 9.9** Let \( G \) be any algebraic group (not necessarily unipotent or even affine) over a field \( k \). By Lemma 9.10 below, the monoidal categories \( \mathcal{D}(G) \) and \( \mathcal{D}_G(G) \) equipped with the functor of convolution with compact support (see Definition 2.7) are \( r \)-categories with \( D \) being the functor \( \mathbb{D}_G \) from Definition 2.17. One can show that these \( r \)-categories are rigid if and only if \( G \) is proper, see [16, Corollary 3.8].

**Lemma 9.10** Let \( M \) denote either \( (\mathcal{D}(G), \ast) \) or \( (\mathcal{D}_G(G), \ast) \). There is a family of isomorphisms \( \text{Hom}(M \ast N, 1) \cong \text{Hom}(M, \mathbb{D}_G N), \) functorial in \( M, N \in M \).

**Proof** By Example 9.5, there are canonical isomorphisms

\[
\text{Hom}(M, \mathbb{D}_G N) \cong \text{Hom}(M, \mathbb{D}_G (t^* N)) \cong \text{Hom}(M \otimes t^* N, \mathbb{K}_G)
\]

for all \( M, N \in M \), where \( t : G \to G \) is given by \( g \mapsto g^{-1} \). Hence, we need to identify \( \text{Hom}(M \otimes t^* N, \mathbb{K}_G) \) with \( \text{Hom}(M \ast N, 1) \).

Let \( p : G \to \text{Spec } k \) denote the structure map, and let \( 1 : \text{Spec } k \to G \) denote the unit of \( G \). Adjunction yields functorial isomorphisms

\[
\text{Hom}(M \ast N, 1) \cong \text{Hom}(1^*(M \ast N), \mathbb{Q}_\ell)
\]

---

32 For a general field \( k \), the definition of \( \mathcal{D}(X) \) is given in [27,33]. If \( k \) is algebraically closed, it is equivalent to the definition from [23, §§1.1.2–1.1.3].
for all \(M, N \in \mathcal{M}\), and the proper base change theorem identifies \(1^*(M \ast N)\) with \(p_!(M \otimes t^*N)\). Using adjunction again, we get isomorphisms

\[
\text{Hom}(p_!(M \otimes t^*N), \mathbb{Q}_\ell) \cong \text{Hom}(M \otimes t^*N, p_!\mathbb{Q}_\ell) = \text{Hom}(M \otimes t^*N, \mathbb{K}_G)
\]

functorial in \(M, N \in \mathcal{M}\), completing the proof. \(\square\)

**Example 9.11** Here is a generalization of the previous example. Suppose we have a groupoid in the category of schemes of finite type over a field \(k\). Let \(X\) denote its “scheme of objects” and \(\Gamma\) its “scheme of morphisms.” Then \(\mathcal{D}(\Gamma)\) has a natural structure of Grothendieck–Verdier category, see [16, Example 2.2] for details. If \(X\) is a point, we get the Grothendieck–Verdier category \(\mathcal{D}(G)\) from Example 9.9. On the other hand, one can take any \(X\) and set \(\mathcal{D}(\Gamma) = X \times X\).

One can get more examples of Grothendieck–Verdier categories by using Lemma 9.50(b) below.

9.1.2 Some canonical isomorphisms

**Remarks 9.12** (i) By definition, in any Grothendieck–Verdier category \(\mathcal{M}\), one has an isomorphism

\[
\text{Hom}(X \otimes Y, K) \xrightarrow{\sim} \text{Hom}(X, DY)
\]

functorial in \(X, Y \in \mathcal{M}\). Since \(D\) is an antiequivalence, the right-hand side of (9.1) identifies with \(\text{Hom}(Y, D^{-1}X)\). So one also has an isomorphism

\[
\text{Hom}(X \otimes Y, K) \xrightarrow{\sim} \text{Hom}(Y, D^{-1}X)
\]

functorial in \(X, Y \in \mathcal{M}\). Thus, a Grothendieck–Verdier category equipped with the opposite tensor product is still a Grothendieck–Verdier category, but \(D\) gets replaced by \(D^{-1}\).

(ii) By (9.2), in any Grothendieck–Verdier category \(\mathcal{M}\), one has a functorial isomorphism \(\text{Hom}(D^2Y \otimes X, K) \xrightarrow{\sim} \text{Hom}(X, DY)\). Combining it with (9.1), one gets a functorial isomorphism

\[
g : \text{Hom}(X \otimes Y, K) \xrightarrow{\sim} \text{Hom}(D^2Y \otimes X, K), \quad X, Y \in \mathcal{M}.
\]

Equivalently, \(g\) is characterized by the commutativity of the diagram

\[
\begin{array}{ccc}
\text{Hom}(X \otimes Y, K) & \xrightarrow{\sim} & \text{Hom}(D^2Y \otimes X, K) \\
\text{Hom}(X, DY) & \xrightarrow{\sim} & \text{Hom}(D^2Y, DX)
\end{array}
\]

whose vertical arrows come from (9.1).
(iii) In any Grothendieck–Verdier category, there exist right and left internal Hom’s. More precisely, if one sets
\[
\text{Hom}(X, Z) = D^{-1}(DZ \otimes X), \quad (9.5)
\]
\[
\text{Hom}'(Y, Z) = D(Y \otimes D^{-1}Z) \quad (9.6)
\]
then (9.1) and (9.2) yield functorial isomorphisms
\[
\text{Hom}(X \otimes Y, Z ) \simarrow \text{Hom}(Y, \text{Hom}(X, Z)), \quad (9.7)
\]
\[
\text{Hom}(X \otimes Y, Z ) \simarrow \text{Hom}(X, \text{Hom}'(Y, Z)). \quad (9.8)
\]

(iv) From (9.1) and (9.2), one gets canonical isomorphisms
\[
D\mathbb{1} \simarrow K, \quad D^{-1}\mathbb{1} \simarrow K. \quad (9.9)
\]
and therefore canonical isomorphisms
\[
\mathbb{1} \simarrow D^2\mathbb{1}, \quad (9.10)
\]
\[
K \simarrow D^2K \quad (9.11)
\]
(the latter is the composition \(K \simarrow D\mathbb{1} \simarrow D^2D^{-1}\mathbb{1} \simarrow D^2K\)).

9.1.3 \(D^2\) as a monoidal equivalence

By (9.3), for each \(X, Y_1, Y_2 \in \mathcal{M}\), one has a canonical isomorphism
\[
\text{Hom}(X \otimes Y_1 \otimes Y_2, K) \simarrow \text{Hom}(D^2(Y_1 \otimes Y_2) \otimes X, K). \quad (9.12)
\]
On the other hand, writing \(X \otimes Y_1 \otimes Y_2\) as \((X \otimes Y_1) \otimes Y_2\) and applying (9.3) twice, one gets an isomorphism
\[
\text{Hom}(X \otimes Y_1 \otimes Y_2, K) \simarrow \text{Hom}(D^2Y_1 \otimes D^2Y_2 \otimes X, K). \quad (9.13)
\]
Combining (9.12) and (9.13), one gets a functorial isomorphism
\[
\text{Hom}(D^2(Y_1 \otimes Y_2) \otimes X, K) \simarrow \text{Hom}(D^2Y_1 \otimes D^2Y_2 \otimes X, K), \quad X, Y_1, Y_2 \in \mathcal{M}. \quad (9.14)
\]
Using Yoneda’s lemma and the isomorphism
\[
\text{Hom}(Z \otimes X, K) \simarrow \text{Hom}(Z, DX)
\]
we see that the isomorphism (9.14) comes from a unique functorial isomorphism
\[ D^2(Y_1 \otimes Y_2) \xrightarrow{\sim} D^2 Y_1 \otimes D^2 Y_2, \quad Y_1, Y_2 \in \mathcal{M}. \quad (9.15) \]

**Proposition 9.13** The isomorphism (9.15) defines a monoidal structure on the functor 
\[ D^2 : \mathcal{M} \xrightarrow{\sim} \mathcal{M}. \]

A proof is given in [16, §11.1]. We do not use Proposition 9.13 in the main body of this article.

9.2 Pivotal structures on Grothendieck–Verdier categories

9.2.1 The notion of a pivotal structure

**Definition 9.14** A pivotal structure on a Grothendieck–Verdier category \( \mathcal{M} \) is a functorial isomorphism
\[ \psi_{X,Y} : \text{Hom}(X \otimes Y, K) \xrightarrow{\sim} \text{Hom}(Y \otimes X, K), \quad X, Y \in \mathcal{M} \quad (9.16) \]

such that
\[ \psi_{X \otimes Y, Z} \circ \psi_{Y \otimes Z, X} \circ \psi_{Z \otimes X, Y} = \text{id}, \quad X, Y \in \mathcal{M}; \quad (9.17) \]
\[ \psi_{X,Y} \circ \psi_{Y,X} = \text{id}, \quad X, Y \in \mathcal{M}. \quad (9.18) \]

In particular, one has the notion of a pivotal structure on an r-category (which can be considered as a Grothendieck–Verdier category with \( K = \mathbb{1} \)).

**Definition 9.15** A pivotal Grothendieck–Verdier category is a Grothendieck–Verdier category with a pivotal structure. A pivotal r-category is an r-category with a pivotal structure.

The name “pivotal category” goes back to [29, Definition 1.3].

**Example 9.16** A symmetric Grothendieck–Verdier category has a canonical pivotal structure: The isomorphisms \( \text{Hom}(M \otimes N, K) \xrightarrow{\sim} \text{Hom}(N \otimes M, K) \) are induced by the symmetry isomorphisms \( M \otimes N \xrightarrow{\sim} N \otimes M \). In particular, one thus gets a canonical pivotal structure on the Grothendieck–Verdier category \( (\mathfrak{S}(X), K_X) \) from Example 9.5.

**Lemma 9.17** Let \( \mathcal{M} \) be a Grothendieck–Verdier category and \( \psi \) an isomorphism (9.16) satisfying (9.17). Then \( \psi \) satisfies (9.18) if and only if \( \psi_{K, \mathbb{1}} = \text{id} \).

**Proof** Setting \( Z = \mathbb{1} \) in (9.17), we see that (9.18) holds if and only if the isomorphism \( \psi_{X, \mathbb{1}} : \text{Hom}(X, K) \to \text{Hom}(X, K) \) equals the identity for all \( X \). By Yoneda’s lemma, this happens if and only if \( \psi_{K, \mathbb{1}} = \text{id} \). \( \Box \)

**Corollary 9.18** If \( \mathcal{M} \) is an r-category, then (9.17) implies (9.18).
Remark 9.19 By (9.17) and (9.18), a pivotal structure on a Grothendieck–Verdier category defines for any integers \( n \geq m \geq 1 \) a canonical isomorphism

\[
\Hom(X_1 \otimes \cdots \otimes X_n, K) \xrightarrow{\sim} \Hom(X_m \otimes \cdots \otimes X_n \otimes X_1 \otimes \cdots \otimes X_{m-1}, K)
\]

for all \( X_1, X_2, \ldots, X_n \in \mathcal{M} \).

9.2.2 Pivotal structures and isomorphisms \( \text{id} \xrightarrow{\sim} D^2 \)

Remark 9.20 By (9.1)–(9.2) and Yoneda’s lemma, a functorial isomorphism (9.16) is the same as an isomorphism \( D^{-1} \xrightarrow{\sim} D \) or equivalently an isomorphism \( \text{id} \xrightarrow{\sim} D^2 \).

Remark 9.20 yields an injective map from the set of pivotal structures on a Grothendieck–Verdier category \( \mathcal{M} \) to the set of isomorphisms \( f : \text{id} \xrightarrow{\sim} D^2 \).

Proposition 9.21 An isomorphism \( f : \text{id} \xrightarrow{\sim} D^2 \) belongs to the image of this map if and only if it satisfies the following conditions:

(i) \( f \) is monoidal;
(ii) \( f_K : K \xrightarrow{\sim} D^2 K \) equals the isomorphism (9.11).

A proof is given in [16, §13]. We do not use Proposition 9.21 in the main body of this article.

Remarks 9.22 (i) If \( \mathcal{M} \) is an r-category, then condition (ii) from Proposition 9.21 clearly follows from condition (i). For arbitrary Grothendieck–Verdier categories, this is false, see [16, Remark 5.8(i)].

(ii) By the previous remark, in the case of r-categories a pivotal structure can equivalently be defined to be a monoidal isomorphism \( f : \text{id} \xrightarrow{\sim} D^2 \). It is this definition that was used in works on rigid monoidal categories (e.g., see [28, Definition 2.7]).

(iii) Here is a way to combine the two conditions on \( f \) from Proposition 9.21 into one. Let \( \mathfrak{A} \) be the 2-groupoid of pairs consisting of a monoidal category and an object in it. A Grothendieck–Verdier category \( (\mathcal{M}, K) \) is an object in \( \mathfrak{A} \). The monoidal structure on \( D^2 \) and the isomorphism \( K \xrightarrow{\sim} D^2 K \) defined in Remark 9.12(iv) allow us to consider \( D^2 \) as a 1-automorphism of \( (\mathcal{M}, K) \in \mathfrak{A} \). The two conditions on \( f \) from Proposition 9.21 mean that \( f : \text{id} \xrightarrow{\sim} D^2 \) is a 2-isomorphism in \( \mathfrak{A} \).

9.2.3 The canonical pivotal structure on \( \mathcal{D}(G) \) and \( \mathcal{D}_G(G) \)

Example 9.23 We will write \( \mathcal{M} \) for one of the r-categories \( \mathcal{D}(G) \) and \( \mathcal{D}_G(G) \) (cf. Example 9.9). Let us give a description of \( \Hom(M_1 \ast \cdots \ast M_n, 1) \), \( M_i \in \mathcal{M} \), which makes the pivotal structure on \( \mathcal{M} \) obvious. First,

\[
\Hom(M_1 \ast \cdots \ast M_n, 1) = \Hom(1^* (M_1 \ast \cdots \ast M_n), \overline{G}_f), \quad (9.19)
\]

where \( 1 : \text{Spec} \ k \longrightarrow G \) is the unit of \( G \) (of course, in the case \( \mathcal{M} = \mathcal{D}_G(G) \) the right-hand side of (9.19) should be understood as \( \Hom \) in the category \( \mathcal{D}_G(\text{Spec} \ k) \)).
Now, define $Z_n \subset G^n$ by the equation $g_1 \ldots g_n = 1$, and let $\pi_1, \ldots, \pi_n : Z_n \to G$ be the projections. Then by proper base change, 

$$1^*(M_1 \ast \cdots \ast M_n) = p! (\pi_1^* M_1 \otimes \cdots \otimes \pi_n^* M_n), \quad p : Z_n \to \text{Spec } k. \quad (9.20)$$

Combining (9.19) with (9.20) and using the invariance of $Z_n \subset G^n$ with respect to cyclic permutations of the $n$ coordinates, we get a canonical isomorphism

$$\text{Hom}(M_1 \ast \cdots \ast M_n, 1) \xrightarrow{\sim} \text{Hom}(M_2 \ast \cdots \ast M_n \ast M_1, 1)$$

whose $n$-th power (in the obvious sense) equals the identity.

It is easy to see that the isomorphisms $\text{Hom}(M_1 \ast M_2, 1) \xrightarrow{\sim} \text{Hom}(M_2 \ast M_1, 1)$ that we obtain in the case $n = 2$ of this construction define a pivotal structure on the $r$-category $\mathcal{M}$ (which is either $\mathcal{D}(G)$ or $\mathcal{D}_G(G)$).

Remarks 9.24

(i) The Grothendieck–Verdier category from Example 9.11 has a canonical pivotal structure (in the spirit of Example 9.23).

(ii) By Remark 9.20, the pivotal structure on $\mathcal{D}(G)$ (respectively, $\mathcal{D}_G(G)$) from Example 9.23 yields an isomorphism $f : \text{Id} \xrightarrow{\sim} (\mathcal{D}_G)^2$. Let us compute it.

Lemma 9.25 The isomorphism $f : \text{Id} \xrightarrow{\sim} (\mathcal{D}_G)^2$ coming from the pivotal structure of Example 9.23 is equal to the composition

$$\text{Id} \xrightarrow{\sim} \mathbb{D}_G^2 \xrightarrow{\sim} (t^*)^2 \circ \mathbb{D}_G^2 \xrightarrow{\sim} t^* \circ \mathbb{D}_G \circ t^* \circ \mathbb{D}_G = (\mathbb{D}_G^*)^2,$$

where the first isomorphism is the standard one and the other two come from the natural identifications $(t^*)^2 \cong \text{Id}$ and $\mathbb{D}_G \circ t^* \cong t^* \circ \mathbb{D}_G$.

This lemma will be deduced in Sect. 9.2.5 from a more general Lemma 9.29.

9.2.4 Quasi-pivotal structures

Let $\mathcal{C}$ be a category and $\Phi : \mathcal{C} \times \mathcal{C} \to \mathcal{Sets}$ a dualizing functor (see Definition 9.1). Let $D$ be the duality functor with respect to $\Phi$, i.e., $\text{Hom}(X, DY) = \Phi(X, Y)$ for $X, Y \in \mathcal{C}$.

Definition 9.26 A quasi-pivotal structure on $\Phi$ is a functorial family of isomorphisms

$$\psi_{X,Y} : \Phi(X, Y) \xrightarrow{\sim} \Phi(Y, X), \quad X, Y \in \mathcal{C}.$$

Remark 9.27 If $\psi$ is a quasi-pivotal structure on $\Phi$, then for $X, Y \in \mathcal{C}$ we obtain a functorial isomorphism

$$\text{Hom}(X, Y) \xrightarrow{D} \text{Hom}(DY, DX) = \Phi(DY, X)$$

$$\xrightarrow{\psi_{DY,X}} \Phi(X, DY) = \text{Hom}(X, D^2 Y)$$
and hence a functorial isomorphism $Y \xrightarrow{\simeq} D^2 Y$. This defines a bijection between quasi-pivotal structures on $\Phi$ and isomorphisms of functors $\text{Id}_C \xrightarrow{\simeq} D^2$.

From now on, we assume that we are given a triple $(\mathcal{C}, \Phi, \psi)$, where $\mathcal{C}$ is a category, $\Phi : \mathcal{C} \times \mathcal{C} \to \text{Sets}$ is a dualizing functor, and $\psi$ is a quasi-pivotal structure on $\Phi$. Suppose moreover that we are given an action of $\mathbb{Z}/2\mathbb{Z}$ on $(\mathcal{C}, \Phi, \psi)$. We write $\tau : \mathcal{C} \xrightarrow{\sim} \mathcal{C}$ for the autoequivalence defined by $1 \in \mathbb{Z}/2\mathbb{Z}$.

Remarks 9.28

1. The functor $\Phi^\tau : (X, Y) \mapsto \Phi(\tau X, Y)$ is also dualizing, and the corresponding duality functor is $\tau^{-1} \circ D \cong \tau \circ D$, where $D$ is the duality functor with respect to $\Phi$.

2. The functor $D$ is $(\mathbb{Z}/2\mathbb{Z})$-equivariant, so there is a natural isomorphism $\tau \circ D \cong D \circ \tau$.

3. The composition

$$
\psi^\tau_{X,Y} : \Phi(\tau X, Y) \xrightarrow{\simeq} \Phi(X, \tau Y) \xrightarrow{\simeq} \Phi(\tau Y, X), \quad X, Y \in \mathcal{C},
$$

defines a quasi-pivotal structure on $\Phi^\tau$, where the first isomorphism comes from the $\mathbb{Z}/2\mathbb{Z}$-equivariant structure on $\Phi$ and the second one is $\psi^\tau_{X,Y}$.

Lemma 9.29

The isomorphism $\text{Id}_C \xrightarrow{\simeq} (\tau \circ D)^2$ coming from the quasi-pivotal structure described in Remark 9.28(3) via the construction of Remark 9.27 is equal to the composition

$$
\text{Id}_C \xrightarrow{\simeq} D^2 \xrightarrow{\simeq} \tau^2 \circ D^2 \xrightarrow{\simeq} (\tau \circ D)^2,
$$

where the first isomorphism corresponds to $\psi$ as in Remark 9.27, the second one comes from the natural identification $\text{Id}_C \cong \tau^2$, and the third one is induced by the isomorphism $\tau \circ D \xrightarrow{\simeq} D \circ \tau$ of Remark 9.28(2).

The proof is completely straightforward, so we omit it.

9.2.5 Proof of Lemma 9.25

Let us specialize Sect. 9.2.4 to the following setting. Take $\mathcal{C}$ to be either $\mathcal{D}(G)$ or $\mathcal{D}_G(G)$, define $\Phi(M_1, M_2) = \text{Hom}(M_1 \otimes M_2, K_G)$, where $K_G \in \mathcal{C}$ is the dualizing complex, and let $\psi$ be induced by the standard symmetry isomorphism $M_1 \otimes M_2 \xrightarrow{\simeq} M_2 \otimes M_1$. The action of $\mathbb{Z}/2\mathbb{Z}$ on the triple $(\mathcal{C}, \Phi, \psi)$ comes from $\tau := i^*$, where $i : G \to G$ is the inversion map.

We claim that the new duality functor $\Phi^\tau$ can be naturally identified with the functor $(M_1, M_2) \mapsto \text{Hom}(M_1 \ast M_2, 1)$, so that $\psi^\tau$ becomes identified with the pivotal structure defined in Example 9.23. Indeed, with the notation of Example 9.23, we can
identify $G$ with $Z_2 \subset G \times G$ via the map $g \mapsto (g^{-1}, g)$. Under this identification, $\pi_1$ becomes $\iota$ and $\pi_2$ becomes the identity map on $G$. Hence,

$$\text{Hom}(M_1 \ast M_2, \mathbb{1}) \xrightarrow{\sim} \text{Hom}(p_!(\iota^* M_1 \otimes M_2), \overline{\mathbb{Q}}_\ell) \xrightarrow{\sim} \text{Hom}(\iota^* M_1 \otimes M_2, K_G).$$

This implies that $\Phi^\tau(M_1, M_2) = \text{Hom}(M_1 \ast M_2, \mathbb{1})$, and the fact that $\psi^\tau$ coincides with the pivotal structure described in Example 9.23 follows from the construction. Applying Lemma 9.29 completes the proof.

9.3 Braided Grothendieck–Verdier categories

9.3.1 The functors $D^2$ and $D^4$

The next result is proved in [16, Lemma 6.8].

**Lemma 9.30** Let $(\mathcal{M}, K, \beta)$ be a braided Grothendieck–Verdier category, and let $\varphi^{\pm} : D^{-1} \xrightarrow{\sim} D$ be the isomorphisms induced by the compositions

$$\text{Hom}(Y, D^{-1} X) \xrightarrow{\sim} \text{Hom}(X \otimes Y, K) \xrightarrow{(\beta_{X,Y}^\pm)^*} \text{Hom}(Y \otimes X, K) \xrightarrow{\sim} \text{Hom}(Y, DX)$$

for all $X, Y \in \mathcal{M}$, where $\beta_{X,Y}^+ := \beta_{X,Y}$ and $\beta_{X,Y}^- := \beta_{Y,X}^{-1}$. Then

$$D(\varphi_X^\pm) = (\varphi_{DX}^\mp)^{-1} \text{ for all } X \in \mathcal{M}.$$

**Definition 9.31** If $(\mathcal{M}, K, \beta)$ is a braided Grothendieck–Verdier category, we define isomorphisms of functors $\vartheta^{\pm} : \text{Id}\mathcal{M} \xrightarrow{\sim} D^2$ as follows:

$$\vartheta_X^\pm = \varphi_{DX}^\mp = D(\varphi_X^\mp)^{-1} : X \xrightarrow{\sim} D^2 X \quad \forall X \in \mathcal{M},$$

where the second equality holds by the lemma above.

**Remark 9.32** In general, the isomorphisms $\vartheta^{\pm}$ are not monoidal.

**Lemma 9.33** Let $(\mathcal{M}, K, \beta)$ be a braided Grothendieck–Verdier category.

Then the monoidal functor $D^2 : \mathcal{M} \xrightarrow{\sim} \mathcal{M}$ is braided.

This result is [16, Proposition 6.1(i)]. The lemma implies that the functor $D^4 : \mathcal{M} \xrightarrow{\sim} \mathcal{M}$ is also braided. Note that we can consider $\vartheta^+ \vartheta^-$ as an isomorphism of functors between $\text{Id}\mathcal{M}$ and $D^4$.

**Lemma 9.34** In any braided Grothendieck–Verdier category $(\mathcal{M}, K, \beta)$, the isomorphism $\vartheta^+ \vartheta^- : \text{Id}\mathcal{M} \xrightarrow{\sim} D^4$ is monoidal. One has $\vartheta^- \vartheta^+ = \vartheta^+ \vartheta^-$. For the proof, see Definition 6.11 and Remark 6.12 in [16].
9.3.2 Pivotal structures and twists

Let us recall the following.

**Definition 9.35** If \((\mathcal{M}, \beta)\) is a braided monoidal category, a **twist** on \((\mathcal{M}, \beta)\) is an automorphism \(\theta : \text{Id}_\mathcal{M} \overset{\sim}{\longrightarrow} \text{Id}_\mathcal{M}\) of the identity functor satisfying

\[\theta_{X \otimes Y} = \beta_{Y,X} \circ \beta_{X,Y} \circ (\theta_X \otimes \theta_Y) \quad \forall X, Y \in \mathcal{M}.\]

Now suppose \((\mathcal{M}, \mathcal{K}, \beta)\) is a braided Grothendieck–Verdier category equipped with a pivotal structure \(\psi\). There exists a unique automorphism \(\theta\) of \(\text{Id}_\mathcal{M}\) such that for all \(X, Y \in \mathcal{M}\) the isomorphism

\[\text{Hom}(Y, D^{-1}X) \overset{\sim}{\longrightarrow} \text{Hom}(Y, D^{-1}X), \quad g \mapsto g \circ \theta_Y,\]

is equal to the composition

\[\text{Hom}(Y, D^{-1}X) \overset{\sim}{\longrightarrow} \text{Hom}(X \otimes Y, K) \overset{\psi_{X,Y}}{\longrightarrow} \text{Hom}(Y \otimes X, K) \overset{(\beta^{-1}_Y)^*}{\longrightarrow} \text{Hom}(X \otimes Y, K) \overset{\sim}{\longrightarrow} \text{Hom}(Y, D^{-1}X).\]

**Remark 9.36** It is clear that \(\psi\) can be expressed in terms of \(\theta\) as follows:

\[\psi_{X,Y} = \beta^{*}_{Y,X} \circ (\text{id}_X \otimes \theta_Y)^*\]

as isomorphisms

\[\text{Hom}(X \otimes Y, K) \overset{\sim}{\longrightarrow} \text{Hom}(X \otimes Y, K) \overset{\sim}{\longrightarrow} \text{Hom}(Y \otimes X, K)\]

for all \(X, Y \in \mathcal{M}\).

**Lemma 9.37** The map \(\psi \mapsto \theta\) constructed above is a bijection between the set of pivotal structures on \((\mathcal{M}, \mathcal{K})\) and the set of twists \(\theta\) on \((\mathcal{M}, \beta)\) satisfying \(\theta_K = \text{id}_K\).

For the proof, see Proposition 7.1 and Remark 7.2 in [16].

**Remarks 9.38**

1. In the situation of Lemma 9.37, let \(f : \text{Id}_\mathcal{M} \overset{\sim}{\longrightarrow} D^2\) be the monoidal isomorphism corresponding to \(\psi\) as in Remark 9.20 (see also Proposition 9.21). Unwinding the definitions, one sees that in terms of \(f\), the twist corresponding to \(\psi\) is given by \(\theta = (\theta^+)^{-1} \circ f\), where \(\theta^+ : \text{Id}_\mathcal{M} \overset{\sim}{\longrightarrow} D^2\) is the isomorphism constructed in Definition 9.31.

2. Let \((\mathcal{M}, \mathcal{K}, \beta)\) be a braided Grothendieck–Verdier category, and let \(\theta\) be a twist on \((\mathcal{M}, \beta)\) such that \(\theta_K = \text{id}_K\). If one defines \(\theta_X' = D^{-1}(\theta_{DX})\) for all \(X \in \mathcal{M}\), then \(\theta'\) is also a twist on \((\mathcal{M}, \beta)\) satisfying \(\theta'_K = \text{id}_K\) [16, Prop. 7.3(iii)–(iv)].

3. The map \(\theta \mapsto \theta'\) is an involution of the set of twists of \((\mathcal{M}, \beta)\) that act as the identity on \(K\) [16, Prop. 7.3(i)–(ii)].
(4) The involution $\theta \mapsto \theta'$ can be described in different terms [16, Prop. 7.3(v)]. Let $\psi, \psi'$ be the pivotal structures corresponding to $\theta$ and $\theta'$, and let $f, f' : \text{Id}_\mathcal{M} \xrightarrow{\sim} D^2$ be the monoidal isomorphisms corresponding to $\psi, \psi'$. Then $ff' : \text{Id}_\mathcal{M} \xrightarrow{\sim} D^4$ is equal to the monoidal isomorphism $\vartheta^+ \vartheta^- : \text{Id}_\mathcal{M} \xrightarrow{\sim} D^4$ of Lemma 9.34.

9.4 Ribbon Grothendieck–Verdier categories

**Definition 9.39** A ribbon structure on a braided Grothendieck–Verdier category $(\mathcal{M}, K, \beta)$ is a twist $\theta$ on $(\mathcal{M}, \beta)$ such that

$$\theta_X = D^{-1}(\theta_{DX}) \quad \text{for all } X \in \mathcal{M}. \quad (9.21)$$

A ribbon Grothendieck–Verdier category is a braided Grothendieck–Verdier category with a ribbon structure. A ribbon r-category is an r-category with a ribbon structure.

**Remark 9.40** The identity (9.21) holds if and only if for any $X, Y \in \mathcal{M}$ and $B : X \otimes Y \longrightarrow K$ one has

$$B \circ (\text{id}_X \otimes \theta_Y) = B \circ (\theta_X \otimes \text{id}_Y). \quad (9.22)$$

Note that unlike (9.21), formula (9.22) makes sense in any braided category with a fixed object $K$ ($K$ does not have to be dualizing and $\mathcal{M}$ does not have to be Grothendieck–Verdier). We do not know whether condition (9.22) is really interesting in this generality.

**Lemma 9.41** A twist $\theta$ satisfies (9.21) if and only if $\theta_K = \text{id}_K$ and $\theta' = \theta$, where $\theta'$ is defined in Remark 9.38(2).

**Proof** We only have to show that the equality $\theta_K = \text{id}_K$ follows from (9.21). This is clear because $K = D1$ and $\theta_1 = \text{id}_1$.

**Corollary 9.42** The correspondence between twists and pivotal structures (see Lemma 9.37) induces a bijection between ribbon structures on $(\mathcal{M}, K, \beta)$ and those pivotal structures on $(\mathcal{M}, K)$ for which the corresponding monoidal isomorphism $f : \text{Id}_\mathcal{M} \xrightarrow{\sim} D^2$ is invariant under the involution $f \mapsto f'$ from Remark 9.38(4).

**Proof** This follows from Lemma 9.41 and Remark 9.38(4).

9.5 The canonical ribbon structure on $\mathcal{D}_G(G)$

The r-category $(\mathcal{D}_G(G), \ast)$ has a natural ribbon structure. For an arbitrary algebraic group $G$, it is described in Appendix 10 below. In this subsection we define it for algebraic groups $G$ such that $G^0$ is unipotent. This assumption allows us to use the ad hoc construction of $\mathcal{D}_G(G)$ given in Definition 2.3.
Definition 9.43 (Braiding on $\mathcal{D}_G(G)$) Let $M, N \in \mathcal{D}_G(G)$, then the braiding $\beta_{M,N} : M \times N \longrightarrow N \times M$ is defined as follows. Consider the commutative diagram

$$
G \times G \xrightarrow{\xi} G \times G \xrightarrow{\mu} G,
$$

where $\tau(g, h) := (h, g)$ and $\xi(g, h) := (g, g^{-1}h)$. We have $M \times N = \mu_!(M \boxtimes N)$, and the above diagram shows that $N \times M = (\mu \tau)_!(M \boxtimes N) = \mu_!\xi!(M \boxtimes N)$. We define $\beta_{M,N} : \mu_!(M \boxtimes N) \longrightarrow \mu_!\xi!(M \boxtimes N)$ by $\beta_{M,N} := \mu!(f)$, where $f : M \boxtimes N \longrightarrow \xi!(M \boxtimes N)$ comes from the $G$-equivariant structure on $N$.

Remark 9.44 Checking the axioms of a braiding for $\beta$ is straightforward and is similar to the well-known case where $G$ is finite. In this case $\mathcal{D}_G(G)$ is the derived category of modules over the so-called quantum double of the group algebra of $G$ (see, e.g., §3.2 of [9]). These modules form a braided category which is not symmetric unless $|G| = 1$.

Definition 9.45 (Twist on $\mathcal{D}_G(G)$) Let $c : G \times G \rightarrow G$ be the conjugation action morphism $c(g, h) = ghg^{-1}$, let $p_2 : G \times G \rightarrow G$ denote the second projection, and let $\Delta : G \rightarrow G \times G$ denote the diagonal. Then $c \circ \Delta = \text{id}_G = p_2 \circ \Delta$. For each $M \in \mathcal{D}_G(G)$, the $G$-equivariant structure on $M$ yields an isomorphism $p_2^*M \longrightarrow \Delta^*c^*M$.

Pulling it back by $\Delta$, we get an isomorphism $\theta_M : M = \Delta^*p_2^*M \longrightarrow \Delta^*c^*M = M$.

By construction, $\theta$ is an automorphism of the identity functor on $\mathcal{D}_G(G)$, and one can check that it is related to the braiding $\beta$ of Definition 9.43 as follows:

$$
\theta_{M \times N} = \beta_{N,M} \circ \beta_{M,N} \circ (\theta_M \otimes \theta_N) \quad \forall M, N \in \mathcal{D}_G(G).
$$

In fact, this follows from part (a) of

Proposition 9.46 (a) If the $r$-category $\mathcal{D}_G(G)$ is equipped with the pivotal structure of Example 9.23 and the braiding of Definition 9.43, the corresponding twist (cf. Lemma 9.37) is the automorphism $\theta$ constructed in Definition 9.45.

(b) The twist $\theta$ is a ribbon structure on the braided $r$-category $(\mathcal{D}_G(G), \beta)$.

By Remark 9.36, the proposition follows at once from the next two lemmas.

Lemma 9.47 If $M, N \in \mathcal{D}_G(G)$, then the isomorphism

$$
\psi_{M,N} : \text{Hom}(M \times N, 1) \longrightarrow \text{Hom}(N \times M, 1)
$$

from Example 9.23 is equal to the pullback map via the composition

$$(\text{id}_M \otimes \theta_N) \circ \beta_{N,M} : N \times M \longrightarrow M \times N.$$
Lemma 9.48 For all $M \in \mathcal{D}_G(G)$, we have $\theta_{\mathcal{D}_G G}(M) = \mathcal{D}_G(G)(\theta_M)$.

For Lemma 9.48 we refer to [25, Proposition 7.2].

Proof of Lemma 9.47 We use the notation of Example 9.23. In particular,

$$Z_2 = \{ (g, h) \in G \times G \mid gh = 1 \},$$

$\pi_1, \pi_2 : Z_2 \to G$ are the two projections, and $p : Z_2 \to \text{Spec } k$ is the structure morphism. The composition

$$\text{Hom}(M \ast N, 1_{\otimes \ell}) \xrightarrow{\sim} \text{Hom}(1^*(M \ast N), \mathcal{Q}_\ell)$$

$$\xrightarrow{\sim} \text{Hom}(p_!(\pi_1^*(M) \otimes \pi_2^*(N)), \mathcal{Q}_\ell)$$

yields an identification\(^{33}\)

$$\text{Hom}(M \ast N, 1) \xrightarrow{\sim} \text{Hom}(p_!(\pi_1^*(M) \otimes \pi_2^*(N)), \mathcal{Q}_\ell). \quad (9.23)$$

Similarly, we have an identification

$$\text{Hom}(N \ast M, 1) \xrightarrow{\sim} \text{Hom}(p_!(\pi_1^*(N) \otimes \pi_2^*(M)), \mathcal{Q}_\ell). \quad (9.24)$$

Let $\tau : G \times G \xrightarrow{\sim} G \times G$ and $\xi : G \times G \xrightarrow{\sim} G \times G$ be as in Definition 9.43, that is, $\tau(g, h) = (h, g)$ and $\xi(g, h) = (g, g^{-1}hg)$. Note that both $\tau$ and $\xi$ preserve $Z_2 \subset G \times G$; moreover, $\xi|_{Z_2} = \text{id}_{Z_2}$. We also have $p \circ \tau = p$ and $\pi_1 \circ \tau = \pi_2$.

The natural isomorphism $p_! \xrightarrow{\sim} p_!\tau_!$ yields an isomorphism

$$p_!(\pi_1^*(N) \otimes \pi_2^*(M)) \xrightarrow{\sim} p_!\tau_!(\pi_1^*(N) \otimes \pi_2^*(M)) \cong p_!(\pi_1^*(M) \otimes \pi_2^*(N)). \quad (9.25)$$

The induced isomorphism

$$\text{Hom}(p_!(\pi_1^*(M) \otimes \pi_2^*(N)), \mathcal{Q}_\ell) \xrightarrow{\sim} \text{Hom}(p_!(\pi_1^*(N) \otimes \pi_2^*(M)), \mathcal{Q}_\ell)$$

coincides with $\psi_{M, N}$ modulo the identifications (9.23) and (9.24).

On the other hand, consider the composition

$$\begin{array}{ccc}
N \boxtimes M & \xrightarrow{f} & \xi_!(N \boxtimes M) \\
& \xrightarrow{\xi!(\text{id}_N \boxtimes \theta_M)} & \xi_!(N \boxtimes M),
\end{array} \quad (9.26)$$

where $f : N \boxtimes M \xrightarrow{\sim} \xi_!(N \boxtimes M)$ is the isomorphism coming from the $G$-equivariant structure on $M$, used in Definition 9.43. If we restrict (9.26) to $Z_2$, we obtain the

\(^{33}\) The Hom on the left-hand side is computed in $\mathcal{D}_G(G)$, while the Hom on the right-hand side is computed in $\mathcal{D}_G(\text{Spec } k)$. 

identity automorphism of $\pi_1^*(N) \otimes \pi_2^*(M)$ (here we used the definition of $\theta_M$; recall also that $\xi|_{Z_2} = \text{id}_{Z_2}$). This implies that the isomorphism $1^*(N \ast M) \overset{\sim}{\longrightarrow} 1^*(M \ast N)$ induced by $(\theta_M \ast \text{id}_N) \circ \beta_{N,M} : N \ast M \overset{\sim}{\longrightarrow} M \ast N$ is equal to the composition (9.25). Equivalently, $\psi_{M,N}$ is equal to the pullback map via the composition

$$
(\theta_M \ast \text{id}_N) \circ \beta_{N,M} : N \ast M \overset{\sim}{\longrightarrow} M \ast N.
$$

Finally, observe that since $\theta_1 = \text{id}_1$, Lemma 9.48 implies that

$$
(\theta_M \ast \text{id}_N)^* = (\text{id}_M \ast \theta_N)^* : \text{Hom}(M \ast N, 1) \longrightarrow \text{Hom}(M \ast N, 1),
$$

which completes the proof.

9.6 Hecke subcategories of Grothendieck–Verdier categories

The notion of a closed idempotent in a monoidal category was introduced in Definition 3.8(d).

**Lemma 9.49** Let $(\mathcal{M}, K)$ be a Grothendieck–Verdier category. Let $e \in \mathcal{M}$ be a closed idempotent. Then

$$
D(e\mathcal{M}) = \mathcal{M}e,
$$

(9.27)

$$
D(\mathcal{M}e) = D^2e \cdot \mathcal{M}.
$$

(9.28)

Note that $D^2e$ is a closed idempotent: This follows from (9.10) and (9.15). The notation $e\mathcal{M}$ and $\mathcal{M}e$ was introduced in Sect. 3.7.

**Proof** Let $\pi : 1 \longrightarrow e$ be an idempotent arrow. If $Y \in e\mathcal{M}$, the morphism $\pi \otimes \text{id}_Y : Y \longrightarrow e \otimes Y$ is an isomorphism. By (9.1), for every $X \in \mathcal{M}$, the morphism $\text{id}_X \otimes \pi : X \longrightarrow X \otimes e$ induces a bijection $\text{Hom}(X \otimes e, DY) \overset{\sim}{\longrightarrow} \text{Hom}(X, DY)$. Now Proposition 3.12(b) implies that $DY \in \mathcal{M}e$. This proves that

$$
D(e\mathcal{M}) \subset \mathcal{M}e.
$$

(9.29)

Now apply (9.29) to $\mathcal{M}$ equipped with the opposite tensor product. Then the dualization functor equals $D^{-1}$, so we get $D^{-1}(\mathcal{M}e) \subset e\mathcal{M}$, i.e., $\mathcal{M}e \subset D(e\mathcal{M})$. Combining this with (9.29), we get (9.27).

To prove (9.28), apply $D$ to (9.27) and note that $D^2(e\mathcal{M}) = D^2e \cdot \mathcal{M}$ by (9.15).

**Lemma 9.50** Let $(\mathcal{M}, K)$ be a Grothendieck–Verdier category, and let $e \in \mathcal{M}$ be a closed idempotent such that $D^2e \simeq e$.

(a) We have $D(e\mathcal{M}) = \mathcal{M}e$ and $D(\mathcal{M}e) = e\mathcal{M}$.

(b) $De \in e\mathcal{M}e$ is a dualizing object of the monoidal category $e\mathcal{M}e$, so $(e\mathcal{M}e, De)$ is a Grothendieck–Verdier category.
(c) The corresponding duality functor \( eMe \sim eMe \) can be identified with the restriction of \( D \) to \( eMe \). This identification is canonical as soon as one chooses an idempotent arrow \( \pi : \mathbb{1} \to e \).

**Proof** Statement (a) follows from Lemma 9.49. Let us prove (b) and (c). By Lemma 3.18, \( eMe \) is a monoidal category. By part (a), \( D(eMe) = eMe \); in particular, \( D(e) \in eMe \). Fix an idempotent arrow \( \pi : \mathbb{1} \to e \). Given \( X, Y \in eMe \), we have canonical isomorphisms

\[
\text{Hom}(X, DY) \cong \text{Hom}(X \otimes Y, K) \cong \text{Hom}(X \otimes Y \otimes e, K) \cong \text{Hom}(X \otimes Y, De),
\]

where the middle one comes from \( \pi : \mathbb{1} \to e \) and the other two come from (9.1). This implies both (b) and (c). \( \square \)

One can ask which Grothendieck–Verdier categories can be realized as \( eMe \), where \( M \) is an r-category and \( e \in M \) is a closed idempotent such that \( D^2 e \simeq e \). An answer to this question is given in [16, §9].

### 9.6.1 Hecke subcategories of pivotal Grothendieck–Verdier categories

**Lemma 9.51** Let \( (M, K) \) be a pivotal Grothendieck–Verdier category and \( e \in M \) a closed idempotent. Then \( (eMe, De) \) is a Grothendieck–Verdier category. Moreover, it has a unique pivotal structure \( \tilde{\psi} \) such that for all \( X, Y \in eMe \) and every idempotent arrow \( \pi : \mathbb{1} \to e \) the diagram

\[
\begin{array}{ccc}
\text{Hom}(X \otimes Y, De) & \xrightarrow{\psi_{X,Y}} & \text{Hom}(Y \otimes X, De) \\
\downarrow & & \downarrow \\
\text{Hom}(X \otimes Y, K) & \xrightarrow{\psi_{X,Y}} & \text{Hom}(Y \otimes X, K)
\end{array}
\]

in which the vertical arrows come from \( D\pi : De \to D\mathbb{1} = K \), commutes.

**Proof** The first statement follows from Lemma 9.50 because in a pivotal category \( D^2 \simeq \text{Id} \) (see Remark 9.20). Now fix an idempotent arrow \( \pi : \mathbb{1} \to e \). Then for every \( Z \in Me \) the map \( \text{Hom}(Z, De) \to \text{Hom}(Z, K) \) induced by \( D\pi : De \to D\mathbb{1} = K \) is bijective because it equals the composition \( \text{Hom}(Z, De) \xrightarrow{\sim} \text{Hom}(Z \otimes e, K) \xrightarrow{\sim} \text{Hom}(Z, K) \), where the first arrow comes from (9.1) and the second one from \( \text{id}_Z \otimes \pi : Z \to Z \otimes e \). Since the vertical arrows in (9.30) are bijections, there is a unique pivotal structure \( \tilde{\psi} \) on \( eMe \) such that the diagram (9.30) corresponding to our fixed idempotent arrow \( \pi : \mathbb{1} \to e \) commutes. We have to show that it commutes for any idempotent arrow \( \pi' : \mathbb{1} \to e \). By Corollary 3.40, \( \pi' = f \circ \pi \) for some \( f \in \text{Aut} e \), so it remains to show that the map

\[
\tilde{\psi}_{X,Y} : \text{Hom}(X \otimes Y, De) \to \text{Hom}(Y \otimes X, De), \quad X, Y \in eMe
\]
commutes with $\text{End} \, e$ acting on both sides of (9.31) via $D : \text{End} \, e \rightarrow \text{End}(De)$. It is easy to check that this action of $\text{End} \, e$ equals the one that comes from the map $\varphi : \text{End} \, e \rightarrow \text{End} \, X$ ($\varphi$ is defined because $e$ is a unit object of the monoidal category $eM$ and $X \in eM$). So the commutation of $\tilde{\psi}_{X,Y}$ with $\text{End} \, e$ follows from functoriality of $\tilde{\psi}_{X,Y}$ with respect to $X$. \qed

9.6.2 Hecke subcategories of braided Grothendieck–Verdier categories

**Lemma 9.52** Let $(\mathcal{M}, K, \beta)$ be a braided Grothendieck–Verdier category, let $D : \mathcal{M} \rightarrow \mathcal{M}$ be the corresponding duality functor, and let $e \in \mathcal{M}$ be a closed idempotent. The Hecke subcategory $eM = eM = eM \subset \mathcal{M}$ is stable under $D$ and is a braided Grothendieck–Verdier category with dualizing object $De$. The corresponding dualizing functor can be identified with the restriction of $D$ to $eM$.

**Proof** In Definition 9.31, we described an isomorphism of functors $\text{Id}_M \rightarrow D^2$. In particular, $D^2e \cong e$. All statements of the lemma now follow from Lemma 9.50. \qed

9.6.3 Hecke subcategories of ribbon Grothendieck–Verdier categories

**Lemma 9.53** Let $(\mathcal{M}, K, \beta)$ be a braided Grothendieck–Verdier category, fix a closed idempotent $e \in \mathcal{M}$, and let $\tilde{\mathcal{M}} = eM = eM = eM \subset \mathcal{M}$ be the Hecke subcategory defined by $e$.

(a) Suppose $\psi$ is a pivotal structure on $\mathcal{M}$ and $\tilde{\psi}$ is the induced pivotal structure on $\tilde{\mathcal{M}}$ (see Lemma 9.51). If $\theta$ and $\tilde{\theta}$ are the twists on $\mathcal{M}$ and $\tilde{\mathcal{M}}$ corresponding to $\psi$ and $\tilde{\psi}$ as in Lemma 9.37, then $\tilde{\theta} = \theta |_{\tilde{\mathcal{M}}}$.

(b) In the situation of (a), if $\theta$ is a ribbon structure on $\mathcal{M}$, then $\tilde{\theta}$ is a ribbon structure on $\tilde{\mathcal{M}}$.

**Proof** (a) Choose an idempotent arrow $\pi : 1 \rightarrow e$. The diagram

$$
\begin{array}{ccc}
\text{Hom}(X \otimes Y, De) & \xrightarrow{\tilde{\psi}_{X,Y}} & \text{Hom}(Y \otimes X, De) \\
\downarrow & & \downarrow \\
\text{Hom}(X \otimes Y, K) & \xrightarrow{\psi_{X,Y}} & \text{Hom}(Y \otimes X, K)
\end{array}
$$

in which the vertical arrows come from $D\pi : De \rightarrow D1 = K$, commutes for all $X, Y \in \tilde{\mathcal{M}}$. Now the claim follows from the definitions of $\theta$ and $\tilde{\theta}$ and the fact that $\pi$ identifies the duality functor for $(\tilde{\mathcal{M}}, De)$ with the restriction of $D$ to $\tilde{\mathcal{M}}$ (see Lemma 9.50(c)).

(b) This follows from Definition 9.39 and the fact that the duality functor for $(\tilde{\mathcal{M}}, De)$ can be identified with the restriction of $D$ to $\tilde{\mathcal{M}}$. \qed
10 Appendix 2: The structures on $\mathcal{D}_G(G)$ (a topological field theory approach)

Convention 10.1 Throughout this appendix, $\mathcal{D}_G(G)$ denotes the bounded derived category of constructible $\ell$-adic complexes [38] on the stack $\text{Ad}(G)\backslash G$ obtained by taking the quotient of $G$ by the conjugation action of $G$ on itself.

The convention above is necessary because we do not require $G$ to be unipotent. On the other hand, to be able to apply the results of this appendix to Lemma 8.6, we need to know that in the unipotent case the naive definition of $\mathcal{D}_G(G)$ is equivalent to the correct one. This is proved in Proposition 11.1 in Appendix 11.

10.1 Overview

To every algebraic stack $\mathcal{X}$ satisfying a certain “perfectness” condition, D. Ben-Zvi, J. Francis, and Nadler [10, §6] associate a 2-dimensional topological field theory (TFT), denoted by $Z_{\mathcal{X}}$. If $G$ is an algebraic group and $\mathcal{X}$ is its classifying stack $BG$, then $Z_{\mathcal{X}}(S^1)$ (i.e., the value of $Z_{\mathcal{X}}$ on the standard circle $S^1$) is the equivariant derived category of quasicoherent sheaves on $G$. This implies that the latter category is equipped with a braided structure and a twist. Note that using the language of 2-dimensional TFT to define a braided structure is natural because the braid groups are most naturally defined in terms of $\mathbb{R}^2$.

In this appendix we describe a similar construction for constructible sheaves instead of quasicoherent ones. In particular, for any algebraic group $G$ over any field, we define in Sect. 10.4 a canonical braided structure and a twist on $\mathcal{D}_G(G)$. Moreover, we define an action of the surface operad on $\mathcal{D}_G(G)$.

The main difference between the constructible case and the quasicoherent one is that the constructible derived category $\mathcal{D}(X_1 \times X_2)$ is usually not generated by objects of the form $M_1 \boxtimes M_2$, $M_i \in \mathcal{D}(X_i)$. Because of this, we get not a TFT but a pre-TFT in the sense of Sect. 10.2.2 (this is a “lax” version of the notion of TFT).

In Sect. 10.5 we study how the pre-TFT corresponding to an algebraic stack $\mathcal{X}$ depends on $\mathcal{X}$. This allows us to prove Lemma 8.6 (on the compatibility of the functor $\text{ind}_G^{G'} : \mathcal{D}_G(G') \to \mathcal{D}_G(G)$ with braidings and twists).

Section 10.6 is devoted to Grothendieck–Verdier duality in $\mathcal{D}_G(G)$ and more generally, in $Z_\mathcal{X}(S^1)$, where $\mathcal{X}$ is any algebraic stack of finite type over a field. We construct a dualizing object $K \in Z_\mathcal{X}(S^1)$ and show that the braiding and twist from Sect. 10.4 define on $(Z_\mathcal{X}(S^1), K)$ a structure of ribbon Grothendieck–Verdier category in the sense of Sect. 9.4.

Unlike [10, §6], we use $n$-categories only for $n \leq 2$. Some remarks on the $\infty$-categorical setting are given in Sect. 10.7.

Convention 10.2 The words “2-category” and “2-functor” are always understood in the “weak” sense (as opposed to the “strict” one).

34 There is also another difference, see Sect. 10.7.2.
10.2 The notion of pre-TFT

10.2.1 The 2-categories $\text{Cob}$, $\text{Cob}_\text{in}$, $\text{Cob}_\text{out}$

We follow [41, §1.1 and §1.4]. In this subsection “manifold” means “$C^\infty$-manifold.” If $M$ and $N$ are $(n - 1)$-dimensional closed oriented manifolds, then a bordism from $M$ to $N$ is an $n$-dimensional oriented manifold $B$ equipped with an oriented diffeomorphism $\alpha : \overline{M} \coprod N \to \partial B$ (here $\overline{M}$ is the manifold $M$ with the opposite orientation). If $(B', \alpha')$ is another bordism from $M$ to $N$, then by a diffeomorphism between $(B, \alpha)$ and $(B', \alpha')$, we mean an oriented diffeomorphism $f : B \to B'$ such that $f \circ \alpha = \alpha'$.

**Definition 10.3** A $(2,1)$-category is a 2-category whose 2-morphisms are invertible.

**Definition 10.4** $\text{Cob}$ is the following $(2,1)$-category:

- Its objects are closed, oriented 1-dimensional $C^\infty$-manifolds;
- For any $M, N \in \text{Cob}$, the category of 1-morphisms $\text{Mor}(M, N)$ is the groupoid whose objects are bordisms from $M$ to $N$ and whose isomorphisms are isotopy classes of diffeomorphisms between bordisms;
- 1-morphisms are composed by gluing bordisms.

**Remark 10.5** In [41] and [10] the above $(2,1)$-category is denoted by $\text{Cob}^{(2)}$ and $2\text{Cob}$, respectively (here “2” indicates the dimension of the bordisms).

**Definition 10.6** Let $\text{Cob}_\text{in}$ (respectively, $\text{Cob}_\text{out}$) denote the $(2,1)$-category that one gets from $\text{Cob}$ by considering only those bordisms $B$ from $M$ to $N$ for which the map $\pi_0(M) \to \pi_0(B)$ (respectively, $\pi_0(N) \to \pi_0(B)$) is surjective.

We have obvious 2-functors $\text{Cob}_\text{in} \to \text{Cob}$ and $\text{Cob}_\text{out} \to \text{Cob}$. The $(2, 1)$-categories $\text{Cob}$, $\text{Cob}_\text{in}$, and $\text{Cob}_\text{out}$ are symmetric monoidal with respect to disjoint union. (The precise meaning of this statement is explained in Remark 10.15 below.)

10.2.2 Pre-definition of a pre-TFT

Let $\text{Cat}$ denote the 2-category of categories.

**Pre-definition 10.7** A 2-dimensional pre-TFT (respectively, 2-dimensional incoming pre-TFT, 2-dimensional outgoing pre-TFT) with values in $\text{Cat}$ is the following collection of data:

(i) a 2-functor $Z : \text{Cob} \to \text{Cat}$ (respectively, $Z : \text{Cob}_\text{in} \to \text{Cat}$, $Z : \text{Cob}_\text{out} \to \text{Cat}$);

(ii) for every $n \geq 0$ and every closed oriented 1-manifolds $X_1, \ldots, X_n$, a functor

$$\prod_i Z(X_i) \to Z\left(\bigcup_i X_i\right);$$

(10.1)

(iii) certain compatibility data and conditions for the functors (10.1).
We skip the precise list of the compatibilities mentioned in (iii) (the reader can easily guess it). Instead, in Sect. 10.2.4 we give a definition of pre-TFT in the format of [40]; the idea is to combine data (i)–(iii) into a single 2-functor.

**Remark 10.8** In Pre-definition 10.7 “incoming” and “outgoing” are abbreviations for the names “positive incoming boundary” and “positive outgoing boundary,” which were suggested (in the case of TFTs) by Ralph Cohen and used by Chas and Sullivan in [18,48]. The synonym for “incoming” used by Lurie in Definition 4.2.10 and Theorem 4.2.11 from [41] is “noncompact.”

### 10.2.3 The structure on the category $Z(S^1)$

Let $Z$ be a pre-TFT. Then for any $X, X_1, \ldots X_n \in \text{Cob}_{\text{out}}$ and any 1-morphism $f : \coprod_i X_i \rightarrow X$ in $\text{Cob}_{\text{out}}$, one gets a canonical functor $\prod_i Z(X_i) \rightarrow Z(X)$ by composing the functor (10.1) with $Z(f)$. In particular, for every finite set $I$ any connected bordism from $S^1 \times I$ to $S^1$ defines a functor $Z(S^1)^I \rightarrow Z(S^1)$. It is clear how such functors are composed: They define an action of the *surface operad* on $Z(S^1)$ (this operad was introduced in [50]). As explained, e.g., in [49, §3.1], an action of the genus 0 part of the surface operad on a category $C$ defines a structure of braided monoidal category with a twist on $C$. In particular, if $Z$ is a pre-TFT, then the category $Z(S^1)$ is equipped with a canonical braided monoidal structure and twist. The same is true if $Z$ is an outgoing pre-TFT. If $Z$ is an incoming pre-TFT, then $Z(S^1)$ is a braided semigroupal category (see Sect. 3.1) equipped with a twist.

### 10.2.4 Precise definition of a pre-TFT

We recommend to skip this subsection. It is merely an exegesis of certain parts of Lurie’s article [40] (this article will be incorporated into his book “Higher algebra”). The idea is to combine data (i)–(iii) from Pre-definition 10.7 into a single 2-functor.

**Definition 10.9** Let $I, J$ be sets. A partially defined map $I \rightarrow J$ is a pair $(I_f,f)$, where $I_f \subset I$ and $f : I_f \rightarrow J$ is a usual map.

For partially defined maps, there is an obvious notion of composition.

**Definition 10.10** Segal’s category, denoted by $S$, is the category whose objects are finite sets and whose morphisms are partially defined maps.

**Remarks 10.11** (i) According to [40, Definition 1.1.7], Segal’s category (denoted by $\Gamma$) is the category of finite sets equipped with a based point. One has an equivalence $\Gamma \cong S$ (removing the base point).

(ii) The category introduced in Segal’s original work [46, Definition 1.1] is dual to $S$.

---

35 In fact, it is known that an action of the genus 0 surface operad on a category $C$ is the same as a structure of braided monoidal category with a twist on $C$. This follows from [45, Proposition 7.6] and the fact that the genus 0 surface operad is equivalent to the framed disk operad.
Now define a (2, 1)-category $\text{Cob}^\otimes$ as follows. Its objects are triples $(M, I, \pi)$, where $M \in \text{Cob}$, $I$ is a finite set, and $\pi : M \to I$ is a locally constant map. Given such a triple and an element $i \in I$, we set $M_i := \pi^{-1}(i)$. Define a 1-morphism $(M, I, \pi) \to (M', I', \pi')$ to be the following collection of data:

- a partially defined map $f : I \to I'$;
- for each $j \in I'$, a 1-morphism in $\text{Cob}$ from $\bigsqcup_{i \in f^{-1}(j)} M_i$ to $M_j$.

The 2-morphisms in $\text{Cob}^\otimes$ come from $\text{Cob}$. The composition of 1-morphisms and 2-morphisms in $\text{Cob}^\otimes$ is clear.

**Example 10.12** Let $(M, I, \pi) \in \text{Cob}^\otimes$. Then for each $i \in I$ one has in $\text{Cob}^\otimes$ a canonical 1-morphism

$$(M, I, \pi) \to (M_i, \{i\}, \pi_i), \quad (10.2)$$

where $\pi_i$ is the unique map $M_i \to \{i\}$. To define (10.2), use the identity 1-morphism $M_i \to M_i$ in $\text{Cob}$ and the partially defined map $f_i : I \to \{i\}$ such that $f(i) := i$ and if $i' \neq i$ then $f(i')$ is not defined.

**Definition 10.13** A 2-dimensional pre-TFT with values in $\text{Cat}$ is a 2-functor $Z : \text{Cob}^\otimes \to \text{Cat}$ with the following Segal property: For every $(M, I, \pi) \in \text{Cob}^\otimes$, the functor $Z(M, I, \pi) \to \prod_{i \in I} Z(M_i, \{i\}, \pi_i)$ induced by the 1-morphisms (10.2) is an equivalence.

Replacing in Definition 10.13 $\text{Cob}^\otimes$ by similar (2,1)-categories $\text{Cob}_\text{in}^\otimes$ and $\text{Cob}_\text{out}^\otimes$, one gets the precise notions of 2-dimensional incoming pre-TFT and outgoing pre-TFT.

Let us explain the relation between Definition 10.13 and the informal Definition 10.7. Considering in $\text{Cob}^\otimes$ only objects $(M, I, \pi)$ such that $I$ has a single element and only those 1-morphisms $(M, I, \pi) \to (M', I', \pi')$ for which the partially defined map $f : I \to I'$ is defined everywhere, we get a (2, 1)-category equivalent to $\text{Cob}$. If $Z : \text{Cob}^\otimes \to \text{Cat}$ is a 2-dimensional pre-TFT in the sense of Definition 10.13, then the restriction of $Z$ to $\text{Cob}$ is a 2-dimensional pre-TFT in the sense of Pre-definition 10.7. Conversely, if $Z : \text{Cob} \to \text{Cat}$ is a 2-dimensional pre-TFT in the sense of Definition 10.7, then one extends $Z$ to a 2-functor $\text{Cob}^\otimes \to \text{Cat}$ by setting $Z(M, I, \pi) := \prod_{i \in I} Z(M_i)$ and using (10.1) to define $Z$ on 1-morphisms.

**Remark 10.14** As explained by Grothendieck in exposé VI of SGA 1, given a 2-functor from a category $C$ to $\text{Cat}$, it is convenient to pass to the corresponding category cofibered over $C$. Similarly, in Definition 10.13 one could pass from the 2-functor $Z$ to the corresponding 2-category cofibered in categories over $\text{Cob}^\otimes$. This is what J. Lurie does systematically in [40].

**Remark 10.15** The pair consisting of the (2,1)-category $\text{Cob}^\otimes$ and the functor $\text{Cob}^\otimes \to S$ defined by $(M, I, \pi) \mapsto I$ is a symmetric monoidal (2, 1)-category in the sense of [40, Definition 1.2.11].
10.3 The notion of pre-sTFT

In [10, Definition 6.4] Ben-Zvi, Francis, and Nadler introduce a version of the (2, 1)-
category \( \text{Cob} \) in which manifolds are replaced by topological spaces satisfying a
finiteness condition. Similarly, we will consider a version of \( \text{Cob} \) in which mani-
folds are replaced by groupoids satisfying a finiteness condition. This will lead us to
the notion of pre-sTFT, where “s” stands for “strong” (and maybe for “stupid,” see
Remark 10.41 below).

10.3.1 The 2-categories \( \text{sCob} \), \( \text{sCob}_\text{in} \), \( \text{sCob}_\text{out} \)

**Definition 10.16** A groupoid \( \Gamma \) has *finite presentation* if it has finitely many isomor-
phism classes of objects, and the automorphism group of each object of \( \Gamma \) has finite
presentation. The (2,1)-category of groupoids of finite presentation is denoted by \( \mathcal{G} \).

**Definition 10.17** Let \( \Gamma_1, \Gamma_2 \in \mathcal{G} \). A *bordism* from \( \Gamma_1 \) to \( \Gamma_2 \) is a diagram

\[
\Gamma_1 \longrightarrow \Gamma \leftarrow \Gamma_2, \quad \Gamma \in \mathcal{G}.
\]

Bordisms from \( \Gamma_1 \) to \( \Gamma_2 \) form a 2-groupoid. Namely, a 1-morphism from a bordism
\( \Gamma_1 \xrightarrow{f_1} \Gamma \xleftarrow{f_2} \Gamma_2 \) to a bordism \( \Gamma_1 \xrightarrow{f'_1} \Gamma' \xleftarrow{f'_2} \Gamma_2 \) is defined to be a triple consisting
of an equivalence \( F : \Gamma \xrightarrow{\sim} \Gamma' \) and isomorphisms \( F \circ f_1 \xrightarrow{\sim} f'_1, F \circ f_2 \xrightarrow{\sim} f'_2 \);
such triples clearly form a groupoid.\(^{36}\) Now truncate the 2-groupoid of bordisms to a
1-groupoid.\(^{37}\)

**Definition 10.18** This groupoid is called the *groupoid of bordisms* from \( \Gamma_1 \) to \( \Gamma_2 \).

**Definition 10.19** \( \text{sCob} \) is the following (2,1)-category:

- Its objects are groupoids of finite presentation;
- For any \( \Gamma_1, \Gamma_2 \in \text{sCob} \), the category of 1-morphisms \( \text{Mor}(\Gamma_1, \Gamma_2) \) is the groupoid
  of bordisms \( \Gamma_1 \longrightarrow \Gamma \leftarrow \Gamma_2 \);
- The composition of bordisms \( \Gamma_1 \longrightarrow \Gamma_12 \leftarrow \Gamma_2 \) and \( \Gamma_2 \longrightarrow \Gamma_23 \leftarrow \Gamma_3 \) is the
  bordism \( \Gamma_1 \longrightarrow \Gamma_13 \leftarrow \Gamma_3 \), where \( \Gamma_13 \) is the categorical pushout \( \Gamma_12 \bigsqcup \Gamma_2 \Gamma_23 \).

**Definition 10.20** Let \( \text{sCob}_{\text{in}} \) (respectively, \( \text{sCob}_{\text{out}} \)) denote the (2,1)-category that
one gets from \( \text{Cob} \) by considering only those bordisms \( \Gamma_1 \longrightarrow \Gamma \leftarrow \Gamma_2 \) for which
the map \( \pi_0(\Gamma_1) \longrightarrow \pi_0(\Gamma) \) (respectively, \( \pi_0(\Gamma_2) \longrightarrow \pi_0(\Gamma) \)) is surjective.

10.3.2 Definition of a pre-sTFT

Let \( \text{Cat} \) denote the 2-category of categories.

\(^{36}\) This groupoid is often a set. This happens if and only if every object \( \gamma' \in \Gamma' \) such that \( \text{Aut} \gamma' \) has
nontrivial center belongs to the essential image of \( \Gamma_1 \bigsqcup \Gamma_2 \).

\(^{37}\) This truncation (which is not very barbarous by the previous footnote) allows us to avoid \( n \)-categories
for \( n > 2 \).
Pre-definition 10.21 A pre-sTFT (respectively, incoming pre-sTFT, outgoing pre-sTFT) with values in \( \text{Cat} \) is the following collection of data:

(i) a 2-functor \( Z : \text{sCob} \to \text{Cat} \) (respectively, \( Z : \text{sCob}_{\text{in}} \to \text{Cat} \), \( Z : \text{sCob}_{\text{out}} \to \text{Cat} \));

(ii) for every \( n \geq 0 \) and every groupoids of finite presentation \( \Gamma_1, \ldots, \Gamma_n \), a functor

\[
\prod_i Z(\Gamma_i) \to Z\left( \bigsqcup_i \Gamma_i \right);
\]

(iii) certain compatibility data and conditions for the functors (10.3) similar to those in Definition 10.7.

To formulate a complete definition of pre-sTFT, define a \((2,1)\)-category \( \text{sCob}^{\otimes} \) similarly to the \((2,1)\)-category \( \text{Cob}^{\otimes} \) from Sect. 10.2.4 and then, just as in Definition 10.13, define a pre-sTFT to be a 2-functor \( \text{sCob}^{\otimes} \to \text{Cat} \) having the Segal property.

10.3.3 From a pre-sTFT to a pre-TFT

Associating with a manifold its fundamental groupoid, one gets 2-functors

\[
\Pi : \text{Cob} \to \text{sCob}, \quad \Pi_{\text{in}} : \text{Cob}_{\text{in}} \to \text{sCob}_{\text{in}}, \quad \Pi_{\text{out}} : \text{Cob}_{\text{out}} \to \text{sCob}_{\text{out}}.
\]

If \( Z : \text{sCob} \to \text{Cat} \) is a pre-sTFT, then \( Z \circ \Pi : \text{Cob} \to \text{Cat} \) is a pre-TFT. Similarly, an incoming (respectively, outgoing) pre-sTFT defines an incoming (respectively, outgoing) boundary pre-TFT.

10.3.4 The canonical braiding and twist on \( Z(\mathbb{B} \mathbb{Z}) \)

For any group \( H \), let \( BH \) denote the corresponding groupoid (i.e., \( BH \) has one object with automorphism group \( H \)). The fundamental groupoid of the standard circle \( S^1 \) equals \( \mathbb{B} \mathbb{Z} \). Combining this with Sects. 10.3.3 and 10.2.3, we see that if \( Z \) is a pre-sTFT (or an outgoing pre-sTFT), then the category \( \mathcal{M} := Z(\mathbb{B} \mathbb{Z}) \) is equipped with a canonical braided monoidal structure and twist, and if \( Z \) is an incoming pre-sTFT, then \( \mathcal{M} \) is a braided semigroupal category equipped with a twist. In this subsection we will describe the same braided semigroupal structure and twist in concrete algebraic terms, without referring to Sect. 10.2.3. The reader may prefer to skip this description and go directly to Sect. 10.4.

Let \( F_n \) be the group freely generated by \( x_1, \ldots, x_n \). For each \( u \in F_n \), let \( \phi_u : \mathbb{B} \mathbb{Z} \to BF_n \) be the functor induced by the homomorphism \( \mathbb{Z} \to F_n \) that takes 1 to \( u \). For each \( n > 0 \), we have a bordism
\[ \Gamma_n \xrightarrow{f} BF_n \leftarrow g B\mathbb{Z} , \quad \Gamma_n := B\mathbb{Z} \underbrace{\cdots \underbrace{\cdots}_{n}}_{\cdots B\mathbb{Z}} \] (10.5)

where the restriction of \( f \) to the \( i \)-th copy of \( B\mathbb{Z} \) equals \( \phi_{x_i} \) and \( g := \phi_{x_1 \ldots x_n} \). Let \( \Phi_n : M^n \longrightarrow M \) be the composition

\[ M^n = Z(B\mathbb{Z})^n \longrightarrow Z(\Gamma_n) \longrightarrow Z(B\mathbb{Z}) = M, \] (10.6)

where the first arrow comes from the fact that \( Z \) is a pre-sTFT and the second one comes from the bordism (10.5). Define the tensor product on \( M \) to be \( \Phi_2 : M \times M \longrightarrow M \).

Our next goal is to define the braiding and the twist. We will use two obvious

Remarks 10.22 Let \( G \) be an arbitrary group.

(1) Giving a functor \( \Gamma_1 = B\mathbb{Z} \longrightarrow B\mathcal{G} \) is the same as giving an element \( g \in \mathcal{G} \) (namely, \( g \) is the image of \( 1 \in \mathbb{Z} \)). If functors \( \Psi, \Psi' : B\mathbb{Z} \longrightarrow B\mathcal{G} \) correspond to \( g, g' \in \mathcal{G} \), then an isomorphism \( \Psi \sim \Psi' \) is an element \( \gamma \in \mathcal{G} \) such that \( \gamma g \gamma^{-1} = g' \).

(2) Similarly, giving a functor \( \Gamma_2 \longrightarrow B\mathcal{G} \) is the same as giving a pair \( (g_1, g_2) \in \mathcal{G}^2 \). For two pairs \( (g_1, g_2), (g'_1, g'_2) \in \mathcal{G}^2 \), an isomorphism between the corresponding functors \( \Gamma_2 \longrightarrow B\mathcal{G} \) is the same thing as a pair \( (\gamma_1, \gamma_2) \in \mathcal{G}^2 \) such that \( \gamma_1 g_1 \gamma_1^{-1} = g'_1 \) and \( \gamma_2 g_2 \gamma_2^{-1} = g'_2 \).

Definition 10.23 Consider the following autoequivalence of the bordism (10.5) with \( n = 1 \):

\[
\begin{array}{ccc}
\Gamma_1 & \xrightarrow{f} & BF_1 & \leftarrow g & B\mathbb{Z} \\
\downarrow \text{Id} & & \downarrow \text{Id} & & \downarrow \text{Id} \\
\Gamma_1 & \xrightarrow{f} & BF_1 & \leftarrow g & B\mathbb{Z}
\end{array}
\]

where the isomorphism \( \text{Id} \circ f \sim f \circ \text{Id} \) is given by the element \( x_1 \in F_1 \) (cf. Remark 10.22(1)) and the isomorphism \( \text{Id} \circ g \sim g \circ \text{Id} \) is the identity map. This autoequivalence defines an automorphism \( \theta \) of the functor \( \Phi_1 = \text{Id}_M : M \longrightarrow M \), which is the twist on \( M \).

Definition 10.24 Consider the following autoequivalence of the diagram (10.5) with \( n = 2 \):
where \( \tau \) interchanges the two copies of \( B\mathbb{Z} \), \( \nu \) is induced by the homomorphism \( F_2 \rightarrow F_2 \) such that \( x_1 \mapsto x_2 \) and \( x_2 \mapsto x_2^{-1} x_1 x_2 \), the isomorphism \( \nu \circ f \rightleftarrows f \circ \tau \) is given by the pair \( (1, x_2) \in F_2^2 \) (cf. Remark 10.22(2)), and the isomorphism \( \nu \circ g \rightleftarrows g \circ \text{Id} \) is the identity. This autoequivalence defines a functorial isomorphism

\[
\beta_{M_1, M_2} : M_1 \otimes M_2 \rightleftarrows M_2 \otimes M_1, \quad M_1, M_2 \in \mathcal{M},
\]

which is the \textit{braiding} on \( \mathcal{M} \).

The fact that \((\mathcal{M}, \beta, \theta)\) is indeed a braided category with a twist follows from Sects. 10.2.3 and 10.3.3.

10.4 The pre-sTFT associated with an algebraic stack

We fix a field \( k \), and we will say “stack” or “scheme” instead of “stack over \( k \)” or “scheme over \( k \).”

To any algebraic stack \( \mathcal{X} \) of finite type, we will associate a pre-sTFT \( Z^{-}_{\mathcal{X}} \) and an outgoing\(^{38}\) pre-sTFT \( Z^{+}_{\mathcal{X}} \). By Sect. 10.3.4, each of the categories \( Z^{-}_{\mathcal{X}}(B\mathbb{Z}) \) and \( Z^{+}_{\mathcal{X}}(B\mathbb{Z}) \) is monoidal and equipped with a braiding and a twist. If \( \mathcal{X} \) is the classifying stack of an algebraic group \( G \), then \( Z^{-}_{\mathcal{X}}(B\mathbb{Z}) = \mathcal{D}^{-}_{G}(G) \) and \( Z^{+}_{\mathcal{X}}(B\mathbb{Z}) \) is the bounded above derived category \( \mathcal{D}^{-}_{G}(G) \). Moreover, the monoidal structure on \( Z^{-}_{\mathcal{X}}(B\mathbb{Z}) = \mathcal{D}^{-}_{G}(G) \) is defined by convolution with compact support (see Example 10.36 below). So, we get a braiding and a twist on the category \( \mathcal{D}^{-}_{G}(G) \) (or \( \mathcal{D}^{-}_{G}(G) \)) equipped with this monoidal structure. In the case where \( G^{\circ} \) is unipotent a braiding and a twist on \( \mathcal{D}^{-}_{G}(G) \) were already defined in Sect. 9.5; it is straightforward to check that the two braidings and twists are the same.

10.4.1 The stack \( \mathcal{X}^{-\Gamma} \)

Let \( \Gamma \) be a groupoid and \( \mathcal{X}^{-} \) a stack. Define the stack \( \mathcal{X}^{-\Gamma} \) as follows: For any scheme \( S \), an \( S \)-point of \( \mathcal{X}^{-\Gamma} \) is a functor \( \Gamma \rightarrow \mathcal{X}^{-}(S) \), where \( \mathcal{X}^{-}(S) \) is the groupoid of \( S \)-points of \( \mathcal{X}^{-} \).

Groupoids form a 2-category. So do stacks (see [39]). The 2-functor \((\mathcal{X}^{-}, \Gamma) \mapsto \mathcal{X}^{-\Gamma} \) is covariant in \( \mathcal{X}^{-} \) and contravariant in \( \Gamma \).

\(^{38}\) If \( \mathcal{X} \) satisfies a certain condition (which holds, e.g., for classifying stacks of unipotent groups), then the word “outgoing” is unnecessary here, see Corollary 10.35 and Definition 10.31.
Remark 10.25 Given a diagram of groupoids $\Gamma_1 \leftarrow \Gamma_2 \rightarrow \Gamma_3$, one can form the categorical pushout $\Gamma = \Gamma_1 \bigsqcup_{\Gamma_2} \Gamma_3$. The above definition of $\mathcal{X}^\Gamma$ immediately implies that $\mathcal{X}^\Gamma$ is the fiber product of $\mathcal{X}^\Gamma_1$ and $\mathcal{X}^\Gamma_3$ over $\mathcal{X}^\Gamma_2$.

Example 10.26 Let $G$ be an algebraic group and $\mathcal{X} = BG$. Let $A$ be an abstract group and $\Gamma = BA$. Then $\mathcal{X}^\Gamma$ is the quotient stack $\text{Hom}(A, G)/G$, where $G$ acts on the scheme $\text{Hom}(A, G)$ by conjugation. In particular, if $\Gamma = B\mathbb{Z}$, then $\mathcal{X}^\Gamma$ is the quotient stack of $G$ by the adjoint action of $G$.

Remark 10.27 If $\Gamma$ is the fundamental groupoid of a topological space $T$ and $\mathcal{X}$ is the classifying stack of an algebraic group $G$, then $\mathcal{X}^\Gamma$ is often called the stack of $G$-local systems on $T$.

Recall that according to [39], a morphism (i.e., a 1-morphism) of stacks $f : \mathcal{X} \rightarrow \mathcal{Y}$ is said to be representable if for any scheme $S$ equipped with a morphism to $\mathcal{Y}$, the stack $\mathcal{X} \times_{\mathcal{Y}} S$ is an algebraic space.

Lemma 10.28 Let $\mathcal{X}$ be an algebraic stack of finite type.

(i) If $\Gamma$ is a groupoid of finite presentation (see Definition 10.16), then $\mathcal{X}^\Gamma$ is an algebraic stack of finite type.

(ii) If a functor $\Gamma' \rightarrow \Gamma$ between groupoids of finite presentation is essentially surjective, then the corresponding morphism $\mathcal{X}^\Gamma \rightarrow \mathcal{X}^{\Gamma'}$ is representable.

Remark 10.29 The above lemma and Lemmas 10.34 and 10.44 below remain valid if the automorphism groups of the objects of $\Gamma$ are assumed to be of finite type but not necessarily of finite presentation. We do not need this fact.

Proof of Lemma 10.28 Statement (i) can be deduced from (ii) as follows. First choose an essentially surjective functor $I \rightarrow \Gamma$, where $I$ is a finite set (viewed as a discrete groupoid). Then use (ii) and the fact that $\mathcal{X}^I$ is an algebraic stack of finite type.

To prove (ii), it suffices to consider the following two cases.

(a) $\Gamma$ is obtained from $\Gamma'$ by freely adding an isomorphism $\gamma_1 \xrightarrow{\sim} \gamma_2$, $\gamma_1, \gamma_2 \in \Gamma'$. In other words, $\Gamma$ is the categorical pushout $\Gamma' \bigsqcup_{\{1, 2\}} \{1\}$, where the sets $\{1, 2\}$ and $\{1\}$ are considered as groupoids and the functor $\{1, 2\} \rightarrow \Gamma'$ takes $i$ to $\gamma_i$.

(b) $\Gamma$ is obtained from $\Gamma'$ by killing some $f \in \text{Aut}\gamma$, $\gamma \in \Gamma$. In other words, $\Gamma$ is the categorical pushout $\Gamma' \bigsqcup_{\beta\mathbb{Z}} \{1\}$, where the functor $B\mathbb{Z} \rightarrow \Gamma'$ takes the single object of $B\mathbb{Z}$ to $\gamma$ and the element $1 \in \mathbb{Z}$ to $f$. (As before, $\{1\}$ is the groupoid with one object and one morphism.)

In case (a), it suffices to use Remark 10.25, and the fact that the diagonal morphism $\mathcal{X} \rightarrow \mathcal{X} \times \mathcal{X}$ is representable. In case (b), Remark 10.25 shows that it suffices to prove the representability of the morphism $\alpha : \mathcal{X} \rightarrow \mathcal{X}^{B\mathbb{Z}}$ corresponding to the functor $B\mathbb{Z} \rightarrow \{1\}$. But, we have already considered case (a), so we know that the morphism $\beta : \mathcal{X}^{B\mathbb{Z}} \rightarrow \mathcal{X}$ corresponding to the functor $\{1\} \rightarrow B\mathbb{Z}$ is representable. Since $\beta\alpha = \text{id}_\mathcal{X}$, it follows that $\alpha$ is representable. \qed
10.4.2 The \(\ell\)-adic derived category of a stack

In this subsection we follow Laszlo and Olsson [38].

Convention 10.30 From now on, all algebraic stacks are assumed to be of finite type over \(k\). By a morphism of stacks we mean a 1-morphism.

For every algebraic stack \(\mathcal{X}\), Laszlo and Olsson [38] define the bounded derived category \(\mathcal{D}^b_c(\mathcal{X}, \overline{\mathbb{Q}}_\ell)\) and the unbounded derived categories \(\mathcal{D}^-_c(\mathcal{X}, \overline{\mathbb{Q}}_\ell)\), \(\mathcal{D}^+_c(\mathcal{X}, \overline{\mathbb{Q}}_\ell)\).

We will use the notation \(\mathcal{D}(\mathcal{X}) := \mathcal{D}^b_c(\mathcal{X}, \overline{\mathbb{Q}}_\ell)\), \(\mathcal{D}^-(\mathcal{X}) := \mathcal{D}^-_c(\mathcal{X}, \overline{\mathbb{Q}}_\ell)\), \(\mathcal{D}^+(\mathcal{X}) := \mathcal{D}^+_c(\mathcal{X}, \overline{\mathbb{Q}}_\ell)\).

Given a morphism \(f : \mathcal{X} \to \mathcal{Y}\), they define the functors \(f^* : \mathcal{D}(\mathcal{Y}) \to \mathcal{D}(\mathcal{X})\) and similar functors for \(\mathcal{D}^-\) and \(\mathcal{D}^+\). They also define \(f_* : \mathcal{D}^-(\mathcal{X}) \to \mathcal{D}^-(\mathcal{Y})\) and \(f_* : \mathcal{D}^+(\mathcal{X}) \to \mathcal{D}^+(\mathcal{Y})\).

The assignment \(\mathcal{X} \mapsto \mathcal{D}^-(\mathcal{X})\), \(f \mapsto f_*\) is a 2-functor from the 2-category of algebraic stacks to that of triangulated categories. With obvious changes, this is also true for \(f^*\), \(f^!\), and \(f_*\). One also has base change isomorphisms, just as for schemes.

In general, \(f_*\) and \(f_*\) do not map \(\mathcal{D}(\mathcal{X})\) to \(\mathcal{D}(\mathcal{Y})\) (e.g., take \(\mathcal{X}\) to be the classifying stack of \(\mathbb{G}_m\) and \(\mathcal{Y} = \text{Spec } k\)). However, this phenomenon does not occur for the following class of morphisms.

Definition 10.31 An algebraic stack \(\mathcal{X}\) is \textit{safe} if for every geometric point \(x\) of \(\mathcal{X}\), the algebraic group \((G_x)^{\circ}_{\text{red}}\) is unipotent (here \(G_x\) is the automorphism group of \(x\) and \((G_x)^{\circ}_{\text{red}}\) is the neutral component of the reduced scheme \((G_x)_{\text{red}}\)). A morphism of algebraic stacks is \textit{safe} if all its fibers are safe.

Remarks 10.32 (i) Representable morphisms are safe.
(ii) Morphisms from a safe stack to any algebraic stack are safe.

Lemma 10.33 If a morphism \(f : \mathcal{X} \to \mathcal{Y}\) of algebraic stacks is safe, then \(f_* : \mathcal{D}^-_c(\mathcal{X}, \overline{\mathbb{Q}}_\ell) \to \mathcal{D}^-_c(\mathcal{Y}, \overline{\mathbb{Q}}_\ell)\) and \(f_* : \mathcal{D}^+_c(\mathcal{X}, \overline{\mathbb{Q}}_\ell) \to \mathcal{D}^+_c(\mathcal{Y}, \overline{\mathbb{Q}}_\ell)\) map \(\mathcal{D}(\mathcal{X})\) to \(\mathcal{D}(\mathcal{Y})\).

Proof By base change and [39, Theorem 11.5], it suffices to consider the case where \(\mathcal{Y} = \text{Spec } k\) and \(\mathcal{X}\) is the classifying stack of a group scheme \(G\) such that \((G)^{\circ}_{\text{red}}\) is unipotent.

10.4.3 The theory \(Z^-_{\mathcal{X}}\)

For any algebraic stack \(\mathcal{X}\) of finite type, we will define a pre-sTFT \(Z^-_{\mathcal{X}} : \text{sCob} \longrightarrow \text{Cat}\).

By definition (see Sect. 10.3.1), an object of \(\text{sCob}\) is a groupoid \(\Gamma\) of finite presentation. Set \(Z^-_{\mathcal{X}}(\Gamma) := \mathcal{D}^-_c(\mathcal{X}^-\Gamma)\).

Now let us define \(Z^-_{\mathcal{X}}\) on 1-morphisms. Recall that a 1-morphism in \(\text{sCob}\) is a bordism of groupoids, i.e., a diagram of groupoids of finite presentation

\[
\Gamma_1 \longrightarrow \Gamma_{12} \leftarrow \Gamma_2.
\]
This diagram defines a correspondence

\[
\begin{array}{ccc}
\mathcal{D}^- \mathcal{X}^\Gamma_{12} & \xleftarrow{g} & \mathcal{D}^- \mathcal{X}^\Gamma_{2} \\
\mathcal{D}^- \mathcal{X}^\Gamma_{1} & \xrightarrow{f} & \mathcal{D}^- \mathcal{X}^\Gamma_{2} \\
\end{array}
\]

and therefore a functor

\[g! f^* : \mathcal{D}^- (\mathcal{X}^\Gamma_{1}) \longrightarrow \mathcal{D}^- (\mathcal{X}^\Gamma_{2}).\]  

(10.9)

Recall that the composition of bordisms \(\Gamma_1 \longrightarrow \Gamma_{12} \leftarrow \Gamma_2\) and \(\Gamma_2 \longrightarrow \Gamma_{23} \leftarrow \Gamma_3\) is the bordism \(\Gamma_1 \longrightarrow \Gamma_{13} \leftarrow \Gamma_3\), where \(\Gamma_{13}\) is the categorical pushout \(\Gamma_{12} \bigsqcup \Gamma_2 \Gamma_{23}\). Thus, we have a commutative diagram of stacks

\[
\begin{array}{ccc}
\mathcal{D}^- (\mathcal{X}^\Gamma_{1}) & \xleftarrow{\tilde{g}} & \mathcal{D}^- (\mathcal{X}^\Gamma_{13}) \\
\mathcal{D}^- (\mathcal{X}^\Gamma_{12}) & \xrightarrow{\tilde{f}} & \mathcal{D}^- (\mathcal{X}^\Gamma_{23}) \\
\mathcal{D}^- (\mathcal{X}^\Gamma_{1}) & \xrightarrow{f} & \mathcal{D}^- (\mathcal{X}^\Gamma_{2}) \\
\mathcal{D}^- (\mathcal{X}^\Gamma_{2}) & \xrightarrow{g} & \mathcal{D}^- (\mathcal{X}^\Gamma_{23}) \\
\end{array}
\]

in which the square is Cartesian by Remark 10.25. So the base change isomorphism \((f')^*g! \xrightarrow{\sim} \tilde{g}! \tilde{f}^*\) provides a canonical isomorphism between the composition

\[
\mathcal{D}^- (\mathcal{X}^\Gamma_{1}) \xrightarrow{g^! f^*} \mathcal{D}^- (\mathcal{X}^\Gamma_{2}) \xrightarrow{\tilde{g}'(f')^*} \mathcal{D}^- (\mathcal{X}^\Gamma_{3})
\]

and the functor \(g'! \tilde{g}^* \tilde{f}^* : \mathcal{D}^- (\mathcal{X}^\Gamma_{1}) \longrightarrow \mathcal{D}^- (\mathcal{X}^\Gamma_{3})\).

Finally, if \(\Gamma\) is a disjoint union of \(\Gamma_1, \ldots, \Gamma_n\), then \(\mathcal{D}^- \Gamma = \prod_i \mathcal{D}^- \mathcal{X}^\Gamma_i\), so we get a canonical functor

\[
\prod_i Z^{-} (\mathcal{X}^\Gamma_i) = \prod_i \mathcal{D}^- (\mathcal{X}^\Gamma_i) \xrightarrow{\otimes} \mathcal{D}^- \left( \prod_i \mathcal{X}^\Gamma_i \right) = \mathcal{D}^- (\mathcal{X}^\Gamma) = Z_{\mathcal{D}^-} \left( \bigsqcup_i \Gamma_i \right).
\]

10.4.4 The theory \(Z_{\mathcal{X}}\)

For \(\Gamma \in \textbf{sCob}\) set \(Z_{\mathcal{X}} (\Gamma) := \mathcal{D}^- (\mathcal{X}^\Gamma)\); this is a full subcategory of \(Z^- (\Gamma)\). To define \(Z_{\mathcal{X}}\) as a 2-functor, we have to ensure that the functor (10.9) preserves the class of bounded complexes. By Lemma 10.33, this is true if the morphism \(g\) in diagram (10.8) is safe in the sense of Definition 10.31.

**Lemma 10.34** Let \(\mathcal{X}\) be an algebraic stack of finite type and \(\alpha : \Gamma' \longrightarrow \Gamma\) a functor between groupoids of finite presentation. Suppose that either \(\mathcal{X}\) is safe or \(\alpha\) is essentially surjective. Then the morphism \(\mathcal{D}^- \Gamma \rightarrow \mathcal{D}^- \Gamma'\) induced by \(\alpha\) is safe.
Proof  Just as in the proof of Lemma 10.28(i), one shows that if $\mathcal{X}$ is safe, then so is $\mathcal{X}^T$. By Remark 10.32(ii), this implies that the morphism $\mathcal{X}^T \to \mathcal{X}^{T'}$ is safe. If $\alpha$ is essentially surjective, use Lemma 10.28(ii) and Remark 10.32(i). □

**Corollary 10.35** If $\mathcal{X}$ is safe, then $Z_{\mathcal{X}}$ is a well-defined pre-sTFT. If $\mathcal{X}$ is any algebraic stack of finite type, then $Z_{\mathcal{X}} : \mathbf{sCob}_{\text{out}} \to \mathbf{Cat}$ is an outgoing pre-sTFT.

10.4.5 Examples of functors (10.9)

**Example 10.36** Let $\mathcal{X} = BG$, where $G$ is an algebraic group. Then the functor (10.9) corresponding to the diagram (10.5) equals $\mu : D^{-}_{G^n(G^n)} \to D^{-}_G(G)$, where $\mu : G^n \to G$ is the map $(g_1, \ldots, g_n) \mapsto g_1 \cdots g_n$. So the composition

$$(D^{-}_G(G))^n \boxtimes D^{-}_{G^n(G^n)} \xrightarrow{10.9} D^{-}_{G^n(G^n)} \xrightarrow{\mu} D^{-}_G(G)$$

is the convolution with compact support $(M_1, \ldots, M_n) \mapsto M_1 \ast \cdots \ast M_n$.

The previous example was based on diagram (10.5). One can consider the diagram of fundamental groupoids corresponding to the bordism between $nS^1$ and $S^1$ given by the sphere with $n+1$ holes (here $nS^1$ stands for the disjoint union of $n$ copies of the standard circle $S^1$). In the next example, we consider a more general situation of a bordism between $mS^1$ and $nS^1$ given by a connected compact oriented surface of genus $g$ with $m+n$ holes.

**Example 10.37** As before, let $\mathcal{X} = BG$, where $G$ is an algebraic group. Let $\pi$ be the group with generators $A_1, B_1, \ldots, A_g, B_g, x_1, \ldots, x_m, y_1, \ldots, y_n$ and the defining relation

$$x_1 \cdots x_m A_1 B_1 A_1^{-1} B_1^{-1} \cdots A_g B_g A_g^{-1} B_g^{-1} = y_1 \cdots y_n.$$  (10.10)

For each $i \in \{1, \ldots, m\}$ consider the homomorphism $\mathbb{Z} \to \pi$ such that $1 \mapsto x_i$. For each $j \in \{1, \ldots, n\}$ consider the homomorphism $\mathbb{Z} \to \pi$ such that $1 \mapsto y_j$. These homomorphisms define a diagram of groupoids

$$B \mathbb{Z} \sqcup \cdots \sqcup B \mathbb{Z} \to B \pi \leftarrow B \mathbb{Z} \sqcup \cdots \sqcup B \mathbb{Z}.$$  

This is a bordism in $\mathbf{sCob}$, which clearly comes from a bordism in $\mathbf{Cob}$ (the definitions of $\mathbf{Cob}$ and $\mathbf{sCob}$ were given in Sects. 10.2.1 and 10.3.1). The functor (10.9) corresponding to this bordism is the composition

$$D^{-}_{G^n(G^n)} \xrightarrow{f^*} D^{-}_G(M) \xrightarrow{h_1} D^{-}_{G^n(G^n)} \xrightarrow{\text{av}} D^{-}_{G^n(G^n)}.  \quad (10.11)$$

Here $M$ is the variety of homomorphisms $\pi \to G$ (on which $G$ acts by conjugation), the map $f : M \to G^m$ (respectively, $h : M \to G^n$) corresponds to $x_1, \ldots, x_m \in \pi$ (respectively, to $y_1, \ldots, y_n \in \pi$), and $\text{av} = \text{av}_{G^n/G}$ is the functor of averaging with compact support (see Definition 2.36).
Remarks 10.38 (i) In the case $G = PGL(N), m = n = 0$ the composition (10.11) was studied in [32].

(ii) Composing $\Box : (D_G^-(G))^m \to D_G^-(G^m)$ with a functor of the form (10.11) for $n = 1$, one gets a functor $(D_G^-(G))^m \to D_G^-(G)$. All such functors form an action of the surface operad on $D_G^-(G)$. We already mentioned it in Sect. 10.2.3.

In the next examples we consider some 1-morphisms in $sCob$ that do not come from $Cob$.

Example 10.39 Consider the following diagram (10.7): $\Gamma_2 = \Gamma_1 = \Gamma_{12} = B\mathbb{Z}$, the functor $\Gamma_2 \to \Gamma_{12}$ is the identity, $\Gamma_1$ is the disjoint union of $n$ copies of $B\mathbb{Z}$, and the restriction of the functor $\Gamma_1 \to \Gamma_{12}$ to each copy of $B\mathbb{Z}$ is the identity. Let $\mathcal{X}$ be any algebraic stack of finite type over $k$. Set $\mathcal{Y} := \mathcal{X}^{B\mathbb{Z}}$. Then (10.9) is the functor $\Delta^* : D^-((\mathcal{Y})^n) \to D^-((\mathcal{Y}))$, where $\Delta : \mathcal{Y} \to \mathcal{Y}^n$ is the diagonal morphism. So the composition

$$D^-((\mathcal{Y})^n) \xrightarrow{\Box} D^-((\mathcal{Y})) \xrightarrow{10.9} D^-((\mathcal{Y}))$$

is the usual tensor product $(M_1, \ldots, M_n) \mapsto M_1 \otimes \cdots \otimes M_n$. Note that if $\mathcal{X}$ is the classifying stack of an algebraic group $G$, then $D^-(\mathcal{X}) = D_G^-(G), D^-((\mathcal{Y})^n) = D_G^-(G^n)$. $\mathcal{Y}$

Example 10.40 Consider the following diagram (10.7): $\Gamma_2 = \Gamma_1 = \Gamma_{12} = B\mathbb{Z}$, the functor $\Gamma_2 \to \Gamma_{12}$ is the identity, and the functor $\Gamma_1 \to \Gamma_{12}$ comes from the homomorphism $\mathbb{Z} \to \mathbb{Z}$ given by $n \mapsto mn$. Let $\mathcal{X} = BG$, where $G$ is an algebraic group. Then (10.9) is the functor $\psi_m^* : D_G(G) \to D_G(G)$, where $\psi_m : G \to G$ is the map $g \mapsto g^m$. Note that if $m = -1$ then $\psi_m^*$ comes from a nonoriented bordism between 1-manifolds.

Remark 10.41 Examples 10.36 and 10.39 show that the pre-sTFT $Z_{BG}$ encodes both the convolution on $Z_{BG}(B\mathbb{Z}) = D_G^-(G)$ and the usual tensor product. The pre-TFT corresponding to $Z_{BG}$ encodes the convolution but not the tensor product. Probably, this means that from the representation theorist’s point of view, the pre-TFT is more adequate than the pre-sTFT.

10.5 A “lax” functoriality of $Z_{\mathcal{X}}$ in $\mathcal{X}$

We will show that a separated morphism $f : \mathcal{X} \to \mathcal{Y}$ between algebraic stacks of finite type induces a “lax 1-morphism” $f_! : Z_{\mathcal{X},in} \to Z_{\mathcal{Y},in}$ in the sense of Sect. 10.5.1, where $Z_{\mathcal{X},in}$ is the incoming pre-sTFT that one gets by restricting $Z_{\mathcal{X}}$ to $Cob$\textsubscript{in}. This implies that $f_!^{B\mathbb{Z}} : D^-((\mathcal{X},B\mathbb{Z})) \to D^-((\mathcal{Y},B\mathbb{Z}))$ is a weakly semigroupal functor compatible with the braidings and twists. In particular, this holds for $\text{ind}^G_{G'} : D_G^-(G') \to D_G^-(G)$, where $G$ is an algebraic group and $G' \subset G$ is a closed subgroup.

39 “Weakly” is related to “lax,” and “semigroupal” (as opposed to “monoidal”) is related to “incoming.”
10.5.1 Lax 1-morphisms between pre-sTFT’s

**Pre-definition 10.42** Let \( Z, Z' : \sCob \to \Cat \) be pre-sTFTs. A **lax 1-morphism** \( F : Z \to Z' \) is the following collection of data:

(i) for each \( \Gamma \in \sCob \), a functor \( F^\Gamma : Z(\Gamma) \to Z'(\Gamma) \);

(ii) for each 1-morphism \( \alpha : \Gamma_1 \to \Gamma_2 \) in \( \sCob \), a morphism

\[
\xi_\alpha : Z'(\alpha) \circ F^{\Gamma_1} \to F^{\Gamma_2} \circ Z(\alpha) \quad (10.12)
\]

(note that both \( Z'(\alpha) \circ F^{\Gamma_1} \) and \( F^{\Gamma_2} \circ Z(\alpha) \) are functors \( Z(\Gamma_1) \to Z'(\Gamma_2) \));

(iii) for any \( n \geq 0 \) and any \( \Gamma_1, \ldots, \Gamma_n \in \sCob \), a morphism from the composition \( \prod_i Z(\Gamma_i) \to \prod_i Z'(\Gamma_i) \) to the composition \( \prod_i Z(\Gamma_i) \to Z(\bigsqcup_i \Gamma_i) \to Z'(\bigsqcup_i \Gamma_i) \).

These data should satisfy certain compatibility conditions. In particular, data (i)–(ii) should define a **lax natural transformation** \(^{40}\) between 2-functors \( Z \) and \( Z' \).

A complete definition of lax 1-morphism can be concisely formulated in terms of Sect. 10.2.4: Namely, a pre-sTFT is a 2-functor \( \sCob^{\otimes} \to \Cat \) with the Segal property, and a lax 1-morphism is a lax natural transformation between such functors.

Similarly, one defines the notion of lax 1-morphism between incoming pre-sTFTs (or, say, outgoing pre-TFTs).

**Remark 10.43** If \( F : Z \to Z' \) is a lax 1-morphism between incoming pre-sTFTs, then \( F^{B\Z} : Z(B\Z) \to Z'(B\Z) \) has a natural structure of weakly semigroupal functor in the sense of Definition 3.4 (“weakly” corresponds to “lax,” and “semigroupal” corresponds to “incoming”). This weakly semigroupal functor is compatible with the braidings and twists.

10.5.2 \( f_1 \) as a lax 1-morphism

Let \( f : \sX \to \sY \) be a separated morphism between algebraic stacks of finite type. Let \( \tilde{Z}_{\sX,\text{in}} \) denote the restriction of \( Z_{\sX} \) to \( \sCob_{\text{in}} \); this is an incoming pre-sTFT. We will define a lax 1-morphism \( f_1 : \tilde{Z}_{\sX,\text{in}} \to \tilde{Z}_{\sY,\text{in}} \). For any groupoid \( \Gamma \in \sCob_{\text{in}} \), one has a morphism \( f : \sX^\Gamma \to \sY^\Gamma \) and therefore a functor \( f^\Gamma_1 : Z_{\sX}^-(\Gamma) \to Z_{\sY}^-(\Gamma) \) (recall that \( Z_{\sX}^-(\Gamma) := \sP^-(\sX^\Gamma) \)). Thus, one has datum (i) from Pre-definition 10.42. Datum (iii) is the Küneth morphism

\[
f^\Gamma_1 M_1 \boxtimes \cdots \boxtimes f^\Gamma_1 M_n \overset{\sim}{\to} f^\Gamma_1 (M_1 \boxtimes \cdots \boxtimes M_n), \quad M_i \in \sP^-(\sX^\Gamma_i),
\]

where \( \Gamma := \bigsqcup_i \Gamma_i \) (and therefore \( f^\Gamma \) is a morphism \( \prod_i \sX^\Gamma_i \to \prod_i \sY^\Gamma_i \)).

---

\(^{40}\) According to [37, §2], this means the following. First, \( \xi_\alpha \) should be functorial in \( \alpha \) (this condition makes sense because all 1-morphisms \( \alpha : \Gamma_1 \to \Gamma_2 \) in \( \sCob_{\text{in}} \) form a category and \( Z(\alpha), Z'(\alpha) \) depend functorially on \( \alpha \)). Second, the assignment \( \alpha \mapsto \xi_\alpha \) should be compatible with the composition of \( \alpha \)'s, and if \( \Gamma_1 = \Gamma_2, \alpha = \Id \), then one should have \( \xi_\alpha = \Id \).
To define datum (ii), we will use that \( f : \mathcal{X} \to \mathcal{Y} \) is separated. By [39, Definition 7.6], this means that the diagonal morphism \( \mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X} \) is proper. The next lemma is proved just as Lemma 10.28(ii).

**Lemma 10.44** Let \( f : \mathcal{X} \to \mathcal{Y} \) be a morphism of algebraic stacks of finite type and \( \Phi : \Gamma' \to \Gamma \) a functor between groupoids of finite presentation. If \( f \) is separated and \( \Phi \) is essentially surjective, then the morphism \( \nu : \mathcal{X}_{\Gamma} \to \mathcal{X}_{\Gamma} \times_{\mathcal{Y}_{\Gamma}} \mathcal{Y}_{\Gamma} \) corresponding to \( f \) and \( \Phi \) is proper. If, in addition, \( f \) is representable, then \( \nu \) is a closed embedding.

Now let us construct the morphism (10.12) corresponding to a 1-morphism \( \alpha \) in \( \text{sCob}_{\text{in}} \). Such \( \alpha \) is, in fact, a diagram of groupoids of finite presentation \( \Gamma_1 \to \Gamma \leftarrow \Gamma_2 \).

We have to construct a canonical morphism

\[
\xi_{\alpha} : Z_{\mathcal{X}}(\alpha) \circ f_{1, \Gamma_1}^* \to f_{1, \Gamma_2}^* \circ Z_{\mathcal{Y}}(\alpha),
\]

i.e., a morphism from the composition

\[
\mathcal{D}^- (\mathcal{X}_{\Gamma_1}) \xrightarrow{f_{1, \Gamma_1}^*} \mathcal{D}^- (\mathcal{Y}_{\Gamma_1}) \xrightarrow{Z_{\mathcal{Y}}(\alpha)} \mathcal{D}^- (\mathcal{Y}_{\Gamma_2})
\]

(10.14) to the composition

\[
\mathcal{D}^- (\mathcal{X}_{\Gamma_1}) \xrightarrow{Z_{\mathcal{X}}(\alpha)} \mathcal{D}^- (\mathcal{X}_{\Gamma_2}) \xrightarrow{f_{1, \Gamma_2}^*} \mathcal{D}^- (\mathcal{Y}_{\Gamma_2}).
\]

(10.15)

Consider the diagram

\[
\begin{array}{ccc}
\mathcal{X}_{\Gamma} & \xrightarrow{\bar{v}} & \mathcal{X}_{\Gamma_2} \\
\downarrow \nu & & \downarrow \nu \\
\mathcal{Y}_{\Gamma_1} & \xrightarrow{\bar{u}} & \mathcal{Y}_{\Gamma_2} \\
\end{array}
\]

in which \( \nu \) is proper by Lemma 10.44. The compositions (10.14) and (10.15) equal, respectively, \( v_1 u^* f_{1, \Gamma_1}^* \) and \( f_{1, \Gamma_2}^* v_1 v^* \bar{u}^* \). The required morphism from \( v_1 u^* f_{1, \Gamma_1}^* = v_1 p_1 \bar{u}^* \) to \( f_{1, \Gamma_2}^* v_1 v^* \bar{u}^* = v_1 p_1 v_1 v^* \bar{u}^* \) comes from the adjunction \( \text{Id} \to \nu_1 v^* = \nu v^* \).

10.5.3 The functor \( \text{ind}_G^G \)

**Example 10.45** Let \( G \) be an algebraic group and \( G' \subset G \) a closed subgroup. Let \( \mathcal{X} := BG' \) and \( \mathcal{Y} := BG \) be the classifying stacks and \( f : \mathcal{X} \to \mathcal{Y} \) the natural morphism. Then \( Z_{\mathcal{X}}(BG) = \mathcal{D}^- (\mathcal{X} BZ) = \mathcal{D}^-_{G'}(G') \), so \( f_{1, BZ}^* \) is a functor \( \mathcal{D}^-_{G'}(G') \to \mathcal{D}^-_G(G) \).
Note that \( f \) is representable: Indeed, after base change \( \text{Spec} \, k \to BG \), it becomes the morphism \( G/G' \to \text{Spec} \, k \). So by Lemma 10.33, \( f^{BG} \) maps \( \mathcal{D}_{G'}(G') \) to \( \mathcal{D}_{G}(G) \).

**Definition 10.46** Each of the functors \( \mathcal{D}_{G'}(G') \to \mathcal{D}_{G}(G) \) and \( \mathcal{D}_{G'}(G') \to \mathcal{D}_{G}(G) \) from Example 10.45 is called induction with compact support and denoted by \( \text{ind}_{G'}^{G} \).

The morphism \( f : BG' \to BG \) from Example 10.45 is separated, so combining the construction from Sect. 10.5.2 with Remark 10.43, one gets the following.

**Corollary 10.47** Each of the functors

\[
\text{ind}_{G'}^{G} : \mathcal{D}_{G'}(G') \to \mathcal{D}_{G}(G), \quad \text{ind}_{G'}^{G} : \mathcal{D}_{G'}(G') \to \mathcal{D}_{G}(G)
\]

has a canonical structure of weakly semigroupal functor compatible with the braidings and twists.

**10.5.4 Conclusion**

If \( G \) is unipotent, a functor \( \text{ind}_{G'}^{G} : \mathcal{D}_{G'}(G') \to \mathcal{D}_{G}(G) \) and a weak semigroupal structure on it were defined already in Sects. 2.12 and 8.1. It is easy to see that this weakly semigroupal functor is equal to the functor \( \text{ind}_{G'}^{G} : \mathcal{D}_{G'}(G') \to \mathcal{D}_{G}(G) \) from Corollary 10.47. Moreover, the construction of \( \text{ind}_{G'}^{G} \) given in Sects. 10.5.2–10.5.3 is essentially identical to the one given in Sects. 2.12, 7.1–7.2, and 8.1, the only difference being the language used. A serious advantage of the language used in this section is that it makes the compatibility of \( \text{ind}_{G'}^{G} \) with the braidings and twists obvious: This compatibility immediately follows from the fact that the morphism (10.13) is functorial in \( \alpha \).

**10.6 Grothendieck–Verdier duality and ribbon structure on \( \mathcal{D}(\mathcal{X}^{BG}) \)**

Let \( \mathcal{X} \) be an algebraic stack of finite type over \( k \) and \( M := Z_{\mathcal{X}}(BG) = \mathcal{D}(\mathcal{X}^{BG}); \) e.g., if \( \mathcal{X} \) is the classifying stack of an algebraic group \( G \), then \( M = \mathcal{D}_{G}(G) \). In Sects. 10.3.4 and 10.4, we defined a canonical braided monoidal structure and twist on \( M \) using the structure of outgoing pre-TFT on \( Z_{\mathcal{X}} \). Our next goal is to define a canonical structure of ribbon Grothendieck–Verdier on \( M \). The construction given in Sects. 10.6.1–10.6.4 below uses \( Z_{\mathcal{X}} \) as well as \( Z_{\mathcal{X}} \subset Z_{\mathcal{X}}^{-} \). The advantage of \( Z_{\mathcal{X}}^{-} \) is that it is a “full” pre-TFT (not merely an outgoing one).

**Remark 10.48** If the stack \( \mathcal{X} \) is safe in the sense of Definition 10.31 (e.g., if \( \mathcal{X} = BG \), where \( G^{\circ} \) is unipotent), then considering \( Z_{\mathcal{X}}^{-} \) is not necessary because by Corollary 10.35, already \( Z_{\mathcal{X}} \) is a pre-TFT.

---

41 Note that the key construction of the morphism (7.4) is based on the equality \( \Delta_{\ast} = \Delta_{1} \), used in step 3 of the construction, i.e., on the separatedness of \( G/G' \) (which is equivalent to the separatedness of the morphism \( BG' \to BG \)).
10.6.1 The counit functor

Set $\mathcal{M}_0 := \mathbb{Z}_\mathcal{X}(\emptyset) = \mathcal{D}^-(\text{Spec } k)$. Let $\varepsilon : \mathcal{M} \to \mathcal{M}_0$ be the functor corresponding to the bordism\(^{42}\)

$$B\mathbb{Z} \to B(0) \leftarrow \emptyset$$

in $\text{sCob}$. Explicitly,

$$\varepsilon(M) = R\Gamma_c(\mathcal{X}, 1^* M), \quad M \in \mathcal{M} = \mathcal{D}(\mathcal{X}^{B\mathbb{Z}}),$$

where $1 : \mathcal{X} \to \mathcal{X}^{B\mathbb{Z}}$ comes from the homomorphism $\mathbb{Z} \to 0$. (Note that if $\mathcal{X} = BG$ then $1 : \mathcal{X} \to \mathcal{X}^{B\mathbb{Z}}$ is obtained from $1 : \text{Spec } k \to G$ by passing to the quotient with respect to the action of $G$ by conjugation.) Informally, we think of the functor $\varepsilon : \mathcal{M} \to \mathcal{M}_0$ as a “counit” or “augmentation.”

10.6.2 The dualizing object in $\mathcal{M}$

Set $K_{\mathcal{M}_0} := \overline{Q}_\ell \in \mathcal{M}_0$. There is a unique object $K_{\mathcal{M}} \in \mathcal{M}$ such that

$$\text{Hom}(M, K_{\mathcal{M}}) = \text{Hom}(\varepsilon(M), K_{\mathcal{M}_0}), \quad M \in \mathcal{M}.$$  

(10.19)

Explicitly,

$$K_{\mathcal{M}} = 1_* K_\mathcal{X},$$

(10.20)

where $K_{\mathcal{X}} \in \mathcal{D}(\mathcal{X})$ is the dualizing object.

**Example 10.49** If $\mathcal{X} = BG$, then $K_{\mathcal{M}} \simeq 1_{\mathcal{M}}[-2d]$, where $d := \text{dim } G = -\text{dim } BG$.

The next lemma is similar to Lemma 9.10.

**Lemma 10.50** $K_{\mathcal{M}}$ is a dualizing object in $\mathcal{M}$. The corresponding dualizing functor is $\mathcal{D}^- := \mathcal{D} \circ i^* = i^* \circ \mathcal{D}$, where $\mathcal{D} : \mathcal{D}(\mathcal{X}^{B\mathbb{Z}}) \xrightarrow{\sim} \mathcal{D}(\mathcal{X}^{B\mathbb{Z}})$ is the Verdier duality functor and $i \in \text{Aut}(\mathcal{X}^{B\mathbb{Z}})$ corresponds to $-1 \in \text{Aut}(\mathbb{Z})$.

**Proof** We have to construct a functorial isomorphism

$$\text{Hom}(M_1 \ast M_2, K_{\mathcal{M}}) \xrightarrow{\sim} \text{Hom}(M_1 \otimes i^* M_2, K_{\mathcal{X}^{B\mathbb{Z}}}), \quad M_i \in \mathcal{M} = \mathcal{D}(\mathcal{X}^{B\mathbb{Z}}),$$

(10.21)

\(^{42}\) This bordism corresponds to the bordism $S^1 \subset \{\text{disk}\} \supset \emptyset$ in $\text{Cob}$. 

\[
\]
where \( K_{\mathcal{X}^{\mathbb{Z}}_B} \in \mathcal{D}(\mathcal{X}^{\mathbb{Z}}_B) \) is the dualizing object. It follows from (10.19) that for \( M_1, M_2 \in \mathcal{M} = \mathcal{D}(\mathcal{X}^{\mathbb{Z}}_B) \) one has

\[
\operatorname{Hom}(M_1 \ast M_2, K_{\mathcal{M}}) = \operatorname{Hom}(N, K_{\mathcal{M}_0}), \quad N := R\Gamma_c(\mathcal{X}^{\mathbb{Z}}_B, M_1 \otimes t^* M_2).
\]

(10.22)

By usual Verdier duality, \( \operatorname{Hom}(N, K_{\mathcal{M}_0}) = \operatorname{Hom}(M_1 \otimes t^* M_2, K_{\mathcal{X}^{\mathbb{Z}}_B}) \).

Remark 10.51 If \( \mathcal{X} = BG \), then the functors \( \mathbb{D}^- \) and \( \mathbb{D} \) from Lemma 10.50 differ by a shift from the functors \( \mathbb{D}_G^- : \mathcal{D}_G(G) \xrightarrow{\sim} \mathcal{D}_G(G) \) and \( \mathbb{D}_G : \mathcal{D}_G(G) \xrightarrow{\sim} \mathcal{D}_G(G) \) used in the main part of the article. This difference is not essential for our purposes. In particular, Proposition 10.53 below implies that the braiding and twist from Sect. 10.3.4 make \( \mathcal{M} = \mathcal{D}_G(G) \) into a ribbon category even if one uses \( \mathbb{D}_G \) rather than \( K_{\mathcal{M}} \) as a dualizing object in \( \mathcal{M} \).

10.6.3 The pivotal structure on \( \mathcal{M} \)

As before, let \( t \in \operatorname{Aut}(\mathcal{X}^{\mathbb{Z}}_B) \) denote the automorphism corresponding to \(-1 \in \operatorname{Aut}(\mathbb{Z})\). For \( M_1, M_2 \in \mathcal{M} \) let

\[
\psi_{M_1, M_2} : \operatorname{Hom}(M_1 \ast M_2, K_{\mathcal{M}}) \xrightarrow{\sim} \operatorname{Hom}(M_2 \ast M_1, K_{\mathcal{M}})
\]

be the isomorphism corresponding via (10.21) to the isomorphism

\[
t^* : \operatorname{Hom}(M_1 \otimes t^* M_2, K_{\mathcal{X}^{\mathbb{Z}}_B}) \xrightarrow{\sim} \operatorname{Hom}(t^* M_1 \otimes M_2, K_{\mathcal{X}^{\mathbb{Z}}_B}).
\]

Proposition 10.52 (a) The isomorphism (10.23) is a pivotal structure on \( (\mathcal{M}, K_{\mathcal{M}}) \) (see Definition 9.14).

(b) The isomorphism \( (\mathbb{D}^-)^2 \xrightarrow{\sim} \operatorname{Id} \) corresponding to this pivotal structure by Remark 9.20 and Lemma 10.50 is equal to the obvious isomorphism

\[
(\mathbb{D}^-)^2 = (\mathbb{D} \circ t^*)^2 \xrightarrow{\sim} \mathbb{D}^2 \circ (t^*)^2 \xrightarrow{\sim} \operatorname{Id}.
\]

The proof will be given in Sects. 10.6.6–10.6.7. One can also deduce Proposition 10.52(a) from Proposition 10.53 and Lemma 9.37.

10.6.4 The ribbon structure on \( \mathcal{M} \)

By Sect. 10.3.4, \( \mathcal{M} \) is a braided category equipped with a twist \( \theta \). In Sect. 9.4 we defined the notion of ribbon structure.

Proposition 10.53 (a) \( \theta \) corresponds (in the sense of Lemma 9.37) to the pivotal structure from Sect. 10.6.3.

(b) \( \theta \) defines a ribbon structure on \( (\mathcal{M}, K_{\mathcal{M}}) \).

The proof will be given in Sects. 10.6.8–10.6.9.
10.6.5 A formula for $\text{Hom}(M_1 \cdots \cdot M_n, K_M)$, $M_i \in \mathcal{M}$

Consider the following bordism\(^{43}\) in $s\text{Cob}$:

$$
\Gamma_n \xrightarrow{f'_n} BF'_n \xleftarrow{\emptyset}, \quad \Gamma_n := 
\begin{array}{c}
B \mathbb{Z} \bigcup \cdots \bigcup B \mathbb{Z}, \\
\end{array}
\quad (10.24)
$$

where $F'_n$ is the group generated by $x_1, \ldots, x_n$ with the defining relation $x_1 \cdots x_n = 1$ and the restriction of $f'$ to the $i$-th copy of $B \mathbb{Z}$ takes $1 \in B \mathbb{Z}$ to $x_i$.

Since $Z^{-\varphi}$ is a pre-sTFT, the bordism (10.24) defines a functor

$$
\Phi'_n : \mathcal{M}^n = (Z^{-\varphi}(B \mathbb{Z}))^n \xleftarrow{\emptyset} (Z^{-\varphi}(B \mathbb{Z}))^n \xrightarrow{\emptyset} Z^{-\varphi}(\emptyset) = \mathcal{M}_0.
$$

**Lemma 10.54** One has a functorial isomorphism

$$
\text{Hom}(M_1 \cdots \cdot M_n, K_M) \xrightarrow{\sim} \text{Hom}(\Phi'_n(M_1, \ldots, M_n), K_{M_0}), \quad M_i \in \mathcal{M}.
$$

**Proof** By (10.19), it suffices to check that $\varepsilon(M_1 \cdots \cdot M_n) = \Phi'_n(M_1, \ldots, M_n)$. This is clear since composing the bordisms (10.5) and (10.17), one gets (10.24). \qed

10.6.6 Proof of Proposition 10.52(a)

**Remarks 10.55**

(i) The diagram (10.24) is acted on by the cyclic subgroup $C_n$ of the symmetric group $S_n$ generated by the cycle $(2, 3, \ldots, n, 1)$. Namely, $C_n$ acts on $\Gamma_n$ (respectively, on $F'_n$) by permuting the $n$ copies of $B \mathbb{Z}$ (respectively, the generators $x_1, \ldots, x_n$ of $F'_n$), and the functor $f' : \Gamma_n \longrightarrow BF'_n$ from (10.24) is $C_n$-equivariant (in the strict sense).

(ii) The previous remark yields a functorial isomorphism

$$
\Phi'_n(M_n, M_1, \ldots, M_{n-1}) \xrightarrow{\sim} \Phi'_n(M_1, M_2, \ldots, M_n), \quad M_i \in \mathcal{M} \quad (10.25)
$$

whose $n$-th power (in the obvious sense) equals the identity.

To prove Proposition 10.52(a), we have to show that the isomorphism $\psi$ defined in Sect. 10.6.3 has properties (9.17)–(9.18). Using Remark 10.55(ii) for $n = 2$, 3 and Lemma 10.54, we obtain functorial isomorphisms

$$
\text{Hom}(M_1 * M_2, K_M) \xrightarrow{\sim} \text{Hom}(M_2 * M_1, K_M), \quad M_1, M_2 \in \mathcal{M}, \quad (10.26)
$$

$$
\text{Hom}(M_1 * M_2 * M_3, K_M) \xrightarrow{\sim} \text{Hom}(M_3 * M_1 * M_2, K_M), \quad M_1, M_2, M_3 \in \mathcal{M} \quad (10.27)
$$

\(^{43}\) It corresponds to the following bordism in $\text{Cob}$: $S^1 \bigcup \cdots \bigcup S^1 \subset \{S^2 \text{ with } n \text{ holes}\} \supset \emptyset$. 
such that the square of (10.26) and the cube of (10.27) are equal to the identity. It is easy to see that (10.26) equals the isomorphism $\psi_{M_1, M_2}$ defined by (10.23) and (10.27) equals $\psi_{M_1 \ast M_2, M_3}$. Properties (9.17)–(9.18) follow.

10.6.7 Proof of Proposition 10.52(b)

One proves the assertion using Lemma 9.29 in exactly the same way as explained in Sect. 9.2.5. We skip the details.

10.6.8 Proof of Proposition 10.53(a)

We have to show that for each $M_1, M_2 \in \mathcal{M}$ the isomorphism

$$\psi_{M_1, M_2} : \text{Hom}(M_1 \ast M_2, K_\mathcal{M}) \xrightarrow{\sim} \text{Hom}(M_2 \ast M_1, K_\mathcal{M})$$

is equal to

$$\beta^*_{M_2, M_1} \circ (\text{id}_{M_1} \ast \theta_{M_2})^* : \text{Hom}(M_1 \ast M_2, K_\mathcal{M}) \xrightarrow{\sim} \text{Hom}(M_2 \ast M_1, K_\mathcal{M}).$$

To this end, we will describe the isomorphisms (10.28)-(10.29) in terms of $s\text{Cob}$.

By Lemma 10.54, we have

$$\text{Hom}(M_1 \ast M_2, K_\mathcal{M}) = \text{Hom}(\Phi'_2(M_1, M_2), K_\mathcal{M}_0),$$

where $\Phi'_2(M_1, M_2) = \varepsilon(M_1 \ast M_2)$ comes from the bordism

$$\Gamma_2 \xrightarrow{f'} BF_2' \leftarrow \emptyset,$$

which is a special case of (10.24). The isomorphism (10.28) comes from the auto-equivalence

$$\Gamma_2 \xrightarrow{f'} BF_2'$$

of diagram (10.30) described in Remark 10.55(i); namely, $\tau$ interchanges the two copies of $B\mathbb{Z}$ and $\xi'$ comes from the automorphism $F_2'$ interchanging the generators $x_1, x_2 \in F_2'$.

On the other hand, the isomorphism (10.29) comes from the composition

$$M_2 \ast M_1 \xrightarrow{\beta_{M_2, M_1}} M_1 \ast M_2 \xrightarrow{\text{id}_{M_1} \ast \theta_{M_2}} M_1 \ast M_2.$$

(10.32)
Recall that the functor \((M_1, M_2) \mapsto M_1 \ast M_2\) comes from the bordism

\[
\Gamma_2 \xrightarrow{f} BF_2 \xleftarrow{g} B\mathbb{Z},
\]

which is a special case of (10.5).

**Lemma 10.56** The composition (10.32) comes from the following autoequivalence

\[
\begin{array}{ccc}
\Gamma_2 & \xrightarrow{f} & BF_2 \xleftarrow{g} B\mathbb{Z} \\
\tau & & \xi \\
\Gamma_2 & \xrightarrow{f} & BF_2 \xleftarrow{g} B\mathbb{Z}
\end{array}
\]

of diagram (10.33): The left vertical arrow is the same as in (10.31), \(\xi\) comes from the automorphism of \(F_2\) interchanging the generators \(x_1, x_2 \in F_2\), the isomorphism \(\xi \circ f \simeq f \circ \tau\) equals the identity, and the isomorphism \(\xi \circ g \simeq g\) is given by the element \(x_2^{-1} \in F_2\) (see Remark 10.22(1)).

The desired equality between (10.28) and (10.29) follows from Lemma 10.56 because the autoequivalence of the diagram (10.30) induced by (10.34) equals (10.31). Thus, it remains to prove Lemma 10.56.

**Proof of Lemma 10.56** Combining Definitions 10.23 and 10.24, it is easy to check that the composition (10.32) comes from the autoequivalence

\[
\begin{array}{ccc}
\Gamma_2 & \xrightarrow{f} & BF_2 \xleftarrow{g} B\mathbb{Z} \\
\tau & & \xi \\
\Gamma_2 & \xrightarrow{f} & BF_2 \xleftarrow{g} B\mathbb{Z}
\end{array}
\]

of diagram (10.33) in which the notation is the same as in Definition 10.24, the isomorphism \(\nu \circ f \simeq f \circ \tau\) is given by the pair \((x_2, x_2) \in F_2^2\) (cf. Remark 10.22(2)), and the isomorphism \(\nu \circ g \simeq g \circ \text{Id}\) is the identity map. Both (10.34) and (10.35) define 1-isomorphisms between the bordisms \(\Gamma_2 \xrightarrow{f} BF_2 \xleftarrow{g} B\mathbb{Z}\) and \(\Gamma_2 \xrightarrow{f \circ \tau} BF_2 \xleftarrow{g} B\mathbb{Z}\). To prove the lemma, it suffices to show that these 1-isomorphisms are 2-isomorphic. This means constructing an isomorphism \(\nu \simeq \xi\) such that the corresponding isomorphisms \(\nu \circ f \simeq \xi \circ f\) and \(\nu \circ g \simeq \xi \circ g\) are equal to the compositions

\[
\nu \circ f \simeq f \circ \tau \xrightarrow{\xi} \xi \circ f, \quad \nu \circ g \simeq g \xrightarrow{\xi} \xi \circ g
\]
(in each of the compositions the first arrow comes from (10.34) and the second one from (10.35)). The isomorphism \( \nu \xrightarrow{\sim} \xi \) corresponding to the element \( x_2 \in F_2 \) has the desired properties. \( \square \)

10.6.9 Proof of Proposition 10.53(b)

By Remark 9.40, it suffices to check that for all \( M_1, M_2 \in \mathcal{M} \) the automorphism \( \theta_{M_1} \ast \theta_{M_2}^{-1} \in \text{Aut}(M_1 \ast M_2) \) induces the identity map from \( \text{Hom}(M_1 \ast M_2, K_M) \) to itself. By Lemma 10.54, it is enough to show that the automorphism

\[
\Phi_2(\theta_{M_1}, \theta_{M_2}^{-1}) \in \text{Aut} \Phi_2(M_1, M_2)
\]

(10.36)

is trivial. By the definition of \( \Phi_2 \) (see Sect. 10.6.5), the automorphism (10.36) comes from a certain automorphism of the bordism

\[
\Gamma_2 \xrightarrow{f'} BF_2' \leftarrow \emptyset,
\]

(10.37)

which is a special case of the bordism (10.24). A general automorphism of the bordism (10.37) is defined by a pair \((\alpha, a)\) consisting of an equivalence \( \alpha : BF_2' \xrightarrow{\sim} BF_2' \) and an isomorphism of functors \( a : \alpha \circ f' \xrightarrow{\sim} f' \); a pair \((\alpha, a)\) corresponds to the identity automorphism of (10.24) if \( a \) comes from an isomorphism \( \alpha \xrightarrow{\sim} \text{Id}_{BF_2'} \). In view of Definition 10.23, the automorphism in question corresponds to \( \alpha = \text{Id}_{BF_2'}, a_{\gamma_1} = x_1, a_{\gamma_2} = x_2^{-1}, \) where \( \gamma_1, \gamma_2 \) are the two objects of \( \Gamma_2 \) and \( x_1, x_2 \) are the generators of \( F_2' \). Since \( x_2^{-1} = x_1 \), this automorphism is trivial.

10.7 Some remarks on the \( \infty \)-categorical setting

10.7.1. It is becoming customary to define \( \mathcal{D}(\mathcal{V}) \) and \( \mathcal{D}^{-}(\mathcal{V}) \) as (stable) \( \infty \)-categories rather than merely as categories. In this setting the construction of the theories \( Z^{-}_{\mathcal{X}} \) and \( Z_{\mathcal{X}} \) given in Sects. 10.4.3–10.4.4 still goes through if the definition of pre-sTFT is modified accordingly. Namely, the 2-groupoid of bordisms defined in Sect. 10.3.1 should not be truncated to a 1-groupoid; then \( \textbf{sCob} \) becomes a \((3, 1)\)-category rather than a \((2, 1)\)-category.

10.7.2. In [10] Ben-Zvi, Francis, and Nadler consider the quasicoherent derived category of \( \mathcal{X}^Y \), where \( \mathcal{X} \) is a derived stack (rather than a “classical” one) and \( Y \) is any topological space (rather than a classifying space of a groupoid). This degree of generality would be useless to treat the categories (or \( \infty \)-categories) \( \mathcal{D}(\mathcal{X}^Y) \) and \( \mathcal{D}^{-}(\mathcal{X}^Y) \). Reason: unlike the quasicoherent case, for any derived stack \( \mathcal{X} \), one has \( \mathcal{D}(\mathcal{X}) = \mathcal{D}(\mathcal{X}^{cl}) \) and \( \mathcal{D}^{-}(\mathcal{X}) = \mathcal{D}^{-}(\mathcal{X}^{cl}) \), where \( \mathcal{X}^{cl} \) stands for the classical stack underlying \( \mathcal{X} \). On the other hand, if \( \mathcal{X} \) is a derived stack and \( Y \) is a topological space, then \( (\mathcal{X}^{Y})^{cl} \) depends only on \( \mathcal{X}^{cl} \) and the fundamental groupoid \( \Pi(Y) \). To see this, note that for any classical scheme \( S \) one has...
\[ \mathcal{D}^Y(S) := \text{Mor}(Y, \mathcal{D}(S)) = \text{Mor}(Y, \mathcal{D}^{cl}(S)) = \text{Mor}(\Pi(Y), \mathcal{D}^{cl}(S)); \]

the latter equality holds because \( \mathcal{D}^{cl}(S) \) is a usual groupoid rather than an \( \infty \)-groupoid.

11 Appendix 3: Equivalence of two definitions of \( \mathcal{D}_G(X) \)

In this appendix \( k \) denotes an algebraically closed field of arbitrary characteristic and \( G \) is an algebraic group over \( k \) acting on a scheme \( X \) of finite type over \( k \). We form the quotient stack \( \mathcal{Y} := G \setminus X \), write \( \mathcal{D}(\mathcal{Y}) = D^\beta_{et}(\mathcal{Y}, \overline{\mathbb{Q}}_\ell) \), and let \( \mathcal{D}^{\text{naive}}_G(X) \) denote the category constructed in Definition 2.3 (where it was denoted \( \mathcal{D}_G(X) \)). If \( q : X \to \mathcal{Y} \) is the quotient morphism, we obtain a Cartesian diagram

\[
\begin{array}{ccc}
G \times X & \to & X \\
\pi \downarrow & & \downarrow q \\
X & \to & \mathcal{Y}
\end{array}
\tag{11.1}
\]

where \( \alpha \) is the action map and \( \pi \) is the second projection. Hence, given \( N \in \mathcal{D}(\mathcal{Y}) \), the pullback \( q^*(N) \in \mathcal{D}(X) \) acquires an isomorphism \( \phi : \alpha^*q^*(N) \isom \pi^*q^*(N) \), which is easily seen to satisfy condition (2.1) of Definition 2.3. Therefore, \( q^* \) can be viewed as a functor \( \mathcal{D}(\mathcal{Y}) \to \mathcal{D}^{\text{naive}}_G(X) \).

**Proposition 11.1** If \( G^\circ \) is unipotent, the functor \( q^* : \mathcal{D}(\mathcal{Y}) \to \mathcal{D}^{\text{naive}}_G(X) \) is an equivalence.

To prove the proposition, we will construct a functor \( q^*_G : \mathcal{D}^{\text{naive}}_G(X) \to \mathcal{D}(\mathcal{Y}) \) and show that when \( G^\circ \) is unipotent, the functors \( q^*_G \) and \( q^* \) are quasi-inverse to each other. Given \( g \in G(k) \), by a slight abuse of notation, we will also denote by \( g : X \isom X \) the automorphism \( x \mapsto \alpha(g, x) \). We have \( q \circ g = q \). If \( (M, \phi) \in \mathcal{D}^{\text{naive}}_G(X) \), then for each \( g \in G(k) \), the isomorphism \( \phi \) induces an isomorphism \( M \isom g_\ast(M) \) and hence an automorphism of \( q_\ast(M) \). In this way we obtain an action of \( G(k) \), viewed as an abstract group, on the object \( q_\ast(M) \). Note that since \( q \) is representable, the functor \( q_\ast \) preserves boundedness, so \( q_\ast(M) \in \mathcal{D}(\mathcal{Y}) \).

**Lemma 11.2** If \( g \in G^\circ(k) \), then \( g \) acts trivially on \( q_\ast(M) \).

The lemma is proved by a standard continuity argument, which we include for completeness and the lack of a suitable reference.

**Proof** The statement becomes obvious if one rephrases the definition of the \( G(k) \)-action on \( q_\ast(M) \) as follows. Set \( N := q_\ast(M) \in \mathcal{D}(\mathcal{Y}) \). The \( G \)-equivariant structure on \( M \) induces a \( G \)-equivariant structure on \( N \) with respect to the trivial \( G \)-action on \( \mathcal{Y} \), that is, an isomorphism \( \phi_{\mathcal{Y}} : \pi^\ast_{\mathcal{Y}}(N) \isom \pi^*_{\mathcal{Y}}(N) \), where \( \pi_{\mathcal{Y}} : G \times \mathcal{Y} \to \mathcal{Y} \) is the projection. Rewrite \( \phi_{\mathcal{Y}} \) as a morphism

\[
N \to \pi_{\mathcal{Y}}^\ast \pi^\ast_{\mathcal{Y}}(N) = N \otimes R\Gamma(G, \overline{\mathbb{Q}}_\ell)
\tag{11.2}
\]
Then the automorphism of \( N \) corresponding to \( g \in G(k) \) is the composition of (11.2) with the morphism \( \text{id}_N \otimes \text{ev}_g : N \otimes R\Gamma(G, \mathbb{Q}_\ell) \to N \), where \( \text{ev}_g : R\Gamma(G, \mathbb{Q}_\ell) \to R\Gamma(\text{Spec } k, \mathbb{Q}_\ell) = \mathbb{Q}_\ell \) is induced by \( g : \text{Spec } k \to G \). Clearly, \( \text{ev}_g \) depends only on the image of \( g \) in \( G(k)/G^0(k) \).

By Lemma 11.2, for every \( M = (M, \phi) \in \mathcal{D}^\text{naive}_G(X) \), we obtain an action of the finite group \( \pi_0(G) \cong G(k)/G^0(k) \) on \( q_\ast(M) \). Since \( \mathcal{D}(\mathcal{Y}) \) is a \( \mathbb{Q}_\ell \)-linear Karoubi-complete category, the endomorphism \( P_M := \frac{1}{|\pi_0(G)|} \sum_{g \in \pi_0(G)} g \) of \( q_\ast(M) \) has an image, which is also the kernel of \( \text{id} - P_M \). We denote it by \( \text{q}_\ast^a(M) \), and it is clear that this construction defines a functor \( q_\ast^a : \mathcal{D}^\text{naive}_G(X) \to \mathcal{D}(\mathcal{Y}) \).

The definition of \( \text{q}_\ast^G \) shows that for each \( M \in \mathcal{D}^\text{naive}_G(X) \), we have natural morphisms \( \text{q}_\ast^G(M) \to \text{q}_\ast(M) \to \text{q}_\ast^G(M) \). In particular, if \( M \in \mathcal{D}^\text{naive}_G(X) \), then the adjunction morphism \( \text{q}_\ast \text{q}_\ast^a(N) \to M \) induces a morphism

\[ \text{q}_\ast \text{q}_\ast^a(M) \to M, \quad (11.3) \]

and if \( N \in \mathcal{D}(\mathcal{Y}) \), then the adjunction morphism \( N \to \text{q}_\ast^a \text{q}_\ast^a(N) \) induces a morphism

\[ N \to \text{q}_\ast^a \text{q}_\ast^a(N), \quad (11.4) \]

Proposition 11.1 follows from the next.

**Lemma 11.3** If \( G^0 \) is unipotent, then (11.3) and (11.4) are isomorphisms.

**Remark 11.4** It is not hard to show in general that \( \text{q}_\ast^G : \mathcal{D}^\text{naive}_G(X) \to \mathcal{D}(\mathcal{Y}) \) is right adjoint to \( \text{q}_\ast^a : \mathcal{D}(\mathcal{Y}) \to \mathcal{D}^\text{naive}_G(X) \), but we do not need this fact.

**Proof of Lemma 11.3** Smooth base change [38, §12] with respect to the morphism \( q : X \to \mathcal{Y} \) (cf. diagram (11.1)) reduces the proof of the lemma to the special case where \( X = G \times Y \) for some scheme \( Y \) of finite type over \( k \) and the \( G \)-action on \( X \) is given by the left multiplication action of \( G \) on itself. In this case the quotient morphism \( q : X \to \mathcal{Y} \) can be identified with the second projection \( \text{pr}_2 : G \times Y \to Y \). It is straightforward to check that the functor \( \text{pr}_2^* : \mathcal{D}(Y) \to \mathcal{D}^\text{naive}_G(G \times Y) \) is an equivalence, with quasi-inverse \( i^* : \mathcal{D}^\text{naive}_G(G \times Y) \to \mathcal{D}(Y) \), where \( i : Y \to G \times Y \) is given by \( y \mapsto (1, y) \). On the other hand, since \( G^0 \) is unipotent, it is isomorphic to an affine space as a variety over \( k \), so \( H^j(G, \mathbb{Q}_\ell) = 0 \) for \( j \geq 1 \) and \( H^0(G, \mathbb{Q}_\ell) \) can be identified with the space of functions \( \pi_0(G) \to \mathbb{Q}_\ell \) on which \( \pi_0(G) \) acts by translations. These observations and the Künneth formula imply that the maps (11.3) and (11.4) are isomorphisms. 

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