OPTIMAL ADAPTIVE MULTIDIMENSIONAL-TIME SIGNAL ENERGY
ESTIMATION ON THE BACKGROUND NOISE.

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Abstract.
We construct an adaptive asymptotically optimal in the classical norm of the space
$L^2(Ω)$ of square integrable random variables the Energy estimation of a signal (function)
observed in some points (plan of experiment) on the background noise.

Key words and phrases: Energy functional, adaptive estimations, loss function, Fourier
series, gradient, variation.

1. Statement of problem. Notations.

Let $V(n), n = 16, 17, \ldots$ be a sequence of a vector - valued (in general case) sets (plans of experiences) in the cube $[-π, π]^d, d = 1, 2, 3, \ldots$:

$V(n) = \{x = \vec{x}_i = \vec{x}_i(n),\}, \vec{x}_i \in [-π, π]^d.$

At the points $\vec{x}_i$ we observe the unknown signal (process, field) $f = f(x), x \in [-π, π]^d$ on the background noise:

$y(i) = f(\vec{x}_i) + σ ξ_i, \quad (1)$

where the sequence noise $\{ξ_i\}$, is the sequence of errors of measurements, is the sequence of independent (or weakly dependent) centered: $E ξ_i = 0$ normalized: $Var(ξ_i) = 1$ random variables, $σ = const > 0$ is a standard deviation of errors.

Let us denote by the $E_m$ the so - called Energy operator, i.e. such that the Energy of the signal $f$ may be defined as follows: (at last in the case $m = 0, 1, 2$)

$W_m(f) \overset{def}{=} ||E_m f||^2 L_2(−π, π)^d, \quad (2)$

where we define for an arbitrary $M = 1, 2, \ldots -$ dimensional vector $z = (z_1, z_2, \ldots, z_M) |z| = \sqrt{\sum_{l=1}^{M} z_l^2}$ and for the vector function $g = g(x) = (g_1, g_2, \ldots, g_M)$

$||g||^2 L_2[−π, π]^d = ||g||^2 \overset{def}{=} \int_{(−π, π)^d} |g(x)|^2 dx, \; dx = dx_1 dx_2 \ldots dx_d. \quad (3)$

Note that the functional
\[ \|f\|S(m) = \left( \sum_{l=0}^{m} W_l(f) \right)^{1/2} \]

is the classical Hilbert's-Sobolev's norm.

We assume that the operator \( E_m = E_m(f) \) is the spherical invariant homogeneous differential operator of order \( m, m = 0, 1, 2, \ldots \). For example:

\[
E_0(f) = f; \quad E_1(f) = \text{grad}(f), \quad E_2(f) = \Delta(f) = \sum_{l=1}^{M} \frac{\partial^2 f}{\partial x_i^2}. \quad (4)
\]

Note that if \( m = 2r \) is an even number, then \( E_m = \Delta^r \).

Note that our assumption (4) is true when the environment is homogeneous and isotropic.

Our goal is to offer and investigate an adaptive asymptotically as \( n \to \infty \) optimal in the probabilistic \( L_2(\Omega) \) sense estimation (measurement) \( W_m(f, n) = W_m(n) \) of an energy functional \( W_m(f) = W_m \), based on the observations \( \{y(i), i = 1, 2, \ldots, n\} \).

The probabilistic \( L_2(\Omega) \) sense denotes that we consider the following loss function:

\[
Z(W_m(f), W_m(f, n)) = \mathbb{E} [W_m(f, n) - W_m(f)]^2. \quad (5)
\]

Here \( W_m(f, n) = W_m(n) \) denotes the estimation of the functional \( W_m = W_m(f) \), adaptive or not.

The considered problem appears in the financial mathematic [4], technical diagnosis and geophysics [3], image processing [5] etc.

The one-dimensional case is consider in [1] - [3] etc. We notice that there are some essential differences between the one-dimensional and multidimensional cases; we will show, for example, that in the multidimensional case we need to use only the optimal experience design.

On the other words, this problem is called "filtration of a signal on the background phone", "adaptive noise canceler" or "regression problem".

In the one-dimensional case \( d = 1 \) this problem was considered in many publications ([1] - [5] etc). The case \( d = 2 \) is known as "picture processing" or equally "image processing".

2. Denotations. Assumptions. Construction of our estimations. Let \( \vec{z} = z = \{z_j\}, j = 1, 2, \ldots, d, z_j \in [-\pi, \pi] \) be a \( d \)-dimensional vector,

\[
F(\vec{z}) = (2\pi)^{-d} \prod_{j=1}^{d} (\pi + z_j), \quad \delta(n) = \delta(n, V(n)) = 
\]

\[
\sup_{z \in [-\pi, \pi^d]} |G_n(z) - F(z)|, \quad G_n(z) = n^{-1} \sum_{i=1}^{n} I(x_i < z),
\]

where

\[
I(\vec{x} < \vec{z}) = 1 \leftrightarrow \forall j = 1, 2, \ldots, d \Rightarrow x_j < z_j,
\]

and \( I(\vec{x} < \vec{z}) = 0 \) in other case.
The value, or more exactly, the function \( \delta = \delta(n) = \delta(n, V(n)) \) is called *discrepancy* of a sequence of the plans \( V(n) \).

We suppose that

\[
\delta(n) = \delta(n, V(n)) \leq C(1, d)[\log(n)]^d/n,
\]

Note that in the one-dimensional case the condition (6) is satisfied even without the member \( \log(n) \) if \( x_i = -\pi + 2\pi i/n \) (the uniform plan); but in general case \( d \geq 2 \) we need to use, e.g., the classical Van der Corput or the Niederreiters sequences (*experience design*) (see [8], p. 183 - 202), for which the condition (6) is satisfied.

Note in addition that the Niederreiters sequences allow us to elaborate the convenient for application *sequential* estimation of the energy of signal \( W_m(f) \).

Recall that for arbitrary sequences of plans \( \{V(n)\} \)

\[
\delta(n) = \delta(n, V(n)) \geq C(2, d)[\log(n)]^{d - 1}/n, n \geq 3.
\]

We define also \( \nu = 2^{1/d}, N_{d,m}(n) = 0.25[n^{2/(4m+d)}] \) and for \( N \in (1, N_{d,m}(n)) \) the "rectangles"

\[
R(N) = \{ \vec{k} : \max_j k_j \leq N \}, \quad R_2(N) = R([\nu N]) \setminus R(N),
\]

\[
R_3(N) = \{ \vec{k} : \min_j k_j \geq N + 1 \} = Z_+^d \setminus R(N).
\]

Here \([z]\) denotes the integer part of a (positive) variable \( z \).

Let us use the classical orthonormal trigonometric system of a functions defined on the circle \([-\pi, \pi]\):

\[
\psi_1(x) = (2\pi)^{-1/2}, \quad \psi_2(x) = \pi^{-1/2} \sin x, \quad \psi_3(x) = \pi^{-1/2} \cos x, \quad \psi_4(x) = \pi^{-1/2} \sin(2x),
\]

\[
\psi_5(x) = \pi^{-1/2} \cos(2x), \quad \psi_6(x) = \pi^{-1/2} \sin(3x), \quad \psi_7(x) = \cos(3x)
\]

etc. We define for the multivariate integer index \( k = \vec{k} = (k_1, k_2, \ldots, k_d) \), where \( \forall s = 1, 2, \ldots, k_s \geq 1, \)

\[
\phi(\vec{k}, \vec{x}) = \prod_{s=1}^{d} \psi_{k_s}(x_s).
\]

**Remark 1.** Note that the functions \( \{\phi_{\vec{k}}(\vec{x})\} \) are eigen functions for energy operator \( E_m \). This circumstance allow us a possibility to generalize our method.

Further, we introduce the sequence of real numbers \( \{\lambda(k)\} \) as a square norms of the functions \( \{\psi_k(\cdot)\} : \lambda(k) = ||d\psi_k(x)/dx||^2 \). It is easy to compute: \( \lambda(1) = 0; \)

\[
\lambda(2) = \lambda(3) = 1; \quad \lambda(4) = \lambda(5) = 4; \quad \lambda(6) = \lambda(7) = 9; \quad \lambda(8) = \lambda(9) = 16, \ldots.
\]

Let us define

\[
\Lambda_d(\vec{k}) = \Lambda_d(\vec{k}) \overset{df}{=} \left[ \sum_{j=1}^{d} \lambda^2(k_j) \right]^{1/2}.
\]
Note that

$$\Lambda_d(k) \asymp \left( \sum_{j=1}^{d} k_j^2 \right)^{1/2}.$$  

The estimating function $E_m f(\cdot)$ belongs to the space $L_2(-\pi, \pi)^d$ or equally $W_m(f) < \infty$ if and only if

$$W_m(f) = \sum_{\k} c^2(\k) \cdot [\Lambda_d(k)]^{2m} < \infty.$$  \hspace{1cm} (8)

**Remark 2.** The expression (8) has a sense still in the case when the number $m$ is non-integer. Further we will use only the expression (8) for the value $W_m(f)$ tacking into account all the values $m$, $m \in [0, \infty)$.

We denote in the considered case $W_m(f) < \infty$

$$\rho_m(N) = \rho_m(f, N) = \sum_{\k \in \mathbb{R}^3(N)} c^2(\k) \cdot [\Lambda_d(k)]^{2m}.$$  

It is evident that $\rho_m(N) \downarrow 0$ as $N \uparrow \infty$.

The values $\rho_m(N) = \rho_m(f, N)$ are known and well studied in the approximation theory, see, e.g., ([9]). Namely, the variable $\rho_0^{1/2}(N)$ is the error in the $L_2[-\pi, \pi]^d$ norm of the best approximation of a function $f$ by means of $d$-dimensional trigonometric polynomials with the maximal degree less or equal than $N$:

$$\rho_0(N) = \inf_{T(N)} ||f - T(N)||^2,$$

where $T(N)$ is the $d$-dimensional trigonometric polynomials with the maximal degree less or equal than $N$; and are closely connected with a so-called Zygmund's $L_2$ modules of continuity:

$$\omega_2(f, \delta) = \sup_{||\k|| \leq \delta} ||f(x + \k) - 2f(x) + f(x - \k)||, \delta \in [0, 1],$$

where the algebraic operations $\x \pm \h$ are understood coordinatewise modulo $2\pi$.

Indeed, the following implication holds:

$$\rho_0^{1/2}(f, N) \asymp N^{-l-\nu}, \ l = 0, 1, 2 \ldots, \nu \in [0, 1) \iff$$

$$\omega_2(f^{(l)}, \delta) \asymp \delta^{\nu}.$$  

We impose the following condition on the signal $f(\cdot)$: for some $\beta > 0$,

$$\rho_m(f, N) \leq C N^{-2\beta},$$  \hspace{1cm} (9)

or more general condition:

$$\lim_{N \to \infty} \frac{\rho_m(2N)}{\rho(N)} < 1.$$  

This conditions are satisfied in many practical cases.
Further, we will consider only the infinite-dimensional case, i.e. when \( \forall N > 0 \Rightarrow \rho_m(N) > 0 \). Moreover, we suppose

\[
\exists h \in (0, 1), \forall N \geq 3 \Rightarrow \rho_m(f, N) \geq C h^N. \tag{10}
\]

As usually, we will denote

\[
A(N) = B(N), N = 1, 2, \ldots \text{ iff } 0 < C_1 = \inf_{N \geq 1} A(N)/B(N) \leq \sup_{N \geq 1} A(N)/B(N) = C_2 < \infty.
\]

2. Construction of an estimations.

We estimate the unknown coefficients \( c(k) = c(\vec{k}) \) as follows:

\[
c(\vec{k}, n) = n^{-1} \sum_{i=1}^{n} y(i) \phi(\vec{k}, \vec{x}_i). \tag{11}
\]

The consistent estimation \( \sigma^2(n) \) of the variance \( \sigma^2 \) is offered in [1]:

\[
\sigma^2(n) = n^{-1} \sum_{i=1}^{n} \left( y(i) - f_n^{(0)}(x(i)) \right)^2, \tag{12}
\]

where \( f_n^{(0)}(x) \) is any preliminary consistent estimation of the function \( f(\cdot) \).

Let us offer a following family of a "projection" estimations \( W_m(f, n, N) \) of the values \( W_m(f) \) of a view:

\[
W_m(f, n, N) = \sum_{\vec{k} \in \mathbb{R}(N)} \left[ A_d(k) \right]^{2m} \cdot \left( c^2(n, \vec{k}) - n^{-1} \sigma^2(n) \right). \tag{13}
\]

We obtain after some calculations under conditions (9) and the following condition (14):

\[
||f||S(\max(d, 2m)) < \infty : \tag{14}
\]

\[
Z(W_m(f, n, N), W_m(f)) \asymp n^{-1} + \rho_m^2(N) + n^{-2}N^{4m+d}. \tag{15}
\]

3. Non-adaptive estimation.

We suppose that the conditions (9) and (15) are satisfied. Taking the value \( N_0 \) as follows:

\[
N_0 = \arg\min_N \left( N^{-2\beta} + n^{-1}N^{2m+d/2} \right),
\]

i.e.

\[
N_0 \asymp \left[ n^{1/(2\beta+2m+d/2)} \right] \overset{def}{=} N_0(\beta),
\]

we obtain:
\[ Z(W_m(f, n, N_0), W_m(f)) \leq K(\beta, m, d) \max \left[ n^{-1}, n^{-4\beta/(2\beta + 2m + d/2)} \right]. \]  

(16)

Note that in the case
\[ \beta \geq m + d/4 \]

(17)

\[ Z(W_m(f, n, N_0), W_m(f)) \leq K_1(\beta, m, d) n^{-1}. \]

(18)

The inequality (18) show us that the estimation \( W_m(f, n, N_0) \) is asymptotically optimal. But it is non-adaptive, since it dependent on the parameter \( \beta \), which is unknown as usually in many practical cases.

In the next section we offer the so-called adaptive estimation, i.e. which dependent only on the observations \( \{y(i)\} \) and has again the optimal speed of convergence, but under more strictly conditions.

4. Adaptive estimation.

Let us introduce the following important functional:
\[ \tau(N) = \tau(N, n) = \sum_{\vec{k} \in B_2(N)} c^2(\vec{k}, n) [\Lambda_d(\vec{k})]^{2m}, \]

(19)

which dependent only on the source data \( \{y(i)\} \), and let us choose the following (random) value of harmonics:
\[ \hat{N} = \arg\min_{N \leq N_{d,m}(n)} \tau(N, n). \]

(20)

It may be proved that under condition (9) as in articles [3], [4] in the sense of convergence with probability one and in the \( L_2(\Omega) \) norm
\[ \lim_{n \to \infty} \hat{N}/n^{1/(2\beta + 2m + d)} = K_3(\beta, m, d), \]

(21)

where \( K_3 = K_3(\beta, m, d) \) is some positive finite non-random constant.

Correspondingly in the choice of \( N_0 \), we can offer the following still adaptive estimation of \( W_m(f) \):
\[ \hat{W}_m(f, n) = \hat{W}_m(f, n, \hat{N}). \]

(22)

Substituting into the expression for the loss function \( Z(\cdot, \cdot) \) and using the equality (21), we conclude that under conditions (9), (14)
\[ Z(\hat{W}_m(f, n), W_m) \asymp \max \left( n^{-1}, n^{-4\beta/(2\beta + 2m + d)} \right). \]

(23)

Notice that in the case when
\[ \beta \geq m + d/2 \]

(24)

(c.f. with the condition (17)), the adaptive estimation (22) \( \hat{W}_m(f, n) \) is asymptotically optimal:
\[ Z(\hat{W}_m(f, n), W_m) \leq K_4(\beta, m, d) \, n^{-1}. \]  

(25)

5. Concluding remark.

We conclude as in the case density energy estimations \[11\], \[12\] still under condition

\[ W_{\text{max}(d, 2m)}(f) < \infty : \]

A. In the case when \( \beta < m + d/4 \) the optimal estimation of \( W_m(f) \) is impossible.

B. In the case

\[ m + d/4 \leq \beta < m + d/2 \]

there exists only the non-adaptive estimation with optimal rate of convergence \( n^{-1/2} \).

C. If ultimately

\[ \beta \geq m + d/2, \]

then there exists the optimal adaptive estimation for the energy value \( W_m \).

The optimality denotes, by our definition, that the speed of of convergence is (asymptotically) \( 1/\sqrt{n} \).

D. Note in addition that all the offered in this report estimations \( W_m(f, n, N) \) are asymptotically normal distributed with parameters correspondingly

\[ \text{E}W_m(f, n, N) - W_m \sim \rho_m(N), \]

\[ \text{Var}(W_m(f, n, N)) \sim n^{-1} \|f\|S(\max(d, 2m)) + \rho_m^2(N) + N^{4m+d}/n^2. \]

The last circumstance may be used by building of confidence interval for \( W_m(f) \) and for verifications of different statistical hypothesis.

F. Evidently, we can use by the computing of the offered here estimations the famous Fast Fourier Transform (FFT) with the computation of complexity \( \asymp C(d) \, n \, \log n \).

E. The accuracy proof of our assertions and building of the confidence region for \( W_m(f) \) is at the same as in the articles \[3\] - \[5\]. It used the approximation theory, theory of martingales, for instance, the exponential bounds in the Law of Iterated Logarithm (LIL), theory of Banach spaces of random variables etc.

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