ASYMPTOTIC BEHAVIOR OF ORTHOGONAL POLYNOMIALS.
SINGULAR CRITICAL CASE

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Abstract. Our goal is to find an asymptotic behavior as \( n \to \infty \) of the orthogonal polynomials \( P_n(z) \) defined by Jacobi recurrence coefficients \( a_n \) (off-diagonal terms) and \( b_n \) (diagonal terms). We consider the case \( a_n \to \infty, b_n \to \infty \) in such a way that \( \sum a_n^{-1} < \infty \) (that is, the Carleman condition is violated) and \( \gamma_n := 2^{-1} b_n (a_n a_{n-1})^{-1/2} \to \gamma \) as \( n \to \infty \). In the case \( |\gamma| \neq 1 \) asymptotic formulas for \( P_n(z) \) are known; they depend crucially on the sign of \( |\gamma| - 1 \). We study the critical case \( |\gamma| = 1 \). The formulas obtained are qualitatively different in the cases \( |\gamma_n| \to 1 - 0 \) and \( |\gamma_n| \to 1 + 0 \). Another goal of the paper is to advocate an approach to a study of asymptotic behavior of \( P_n(z) \) based on a close analogy of the Jacobi difference equations and differential equations of Schrödinger type.

1. Introduction

1.1. Orthogonal polynomials and Jacobi operators. Orthogonal polynomials \( P_n(z) \) can be defined by a recurrence relation

\[
a_{n-1} P_{n-1}(z) + b_n P_n(z) + a_n P_{n+1}(z) = z P_n(z), \quad n \in \mathbb{Z}_+, \quad z \in \mathbb{C},
\]

with the boundary conditions \( P_{-1}(z) = 0, P_0(z) = 1 \). We always suppose that \( a_n > 0, b_n = \bar{b}_n \). Determining \( P_n(z) \), \( n = 1, 2, \ldots \), successively from (1.1), we see that \( P_n(z) \) is a polynomial of degree \( n \): \( P_n(z) = p_n z^n + \cdots \) where \( p_n = (a_0 a_1 \cdots a_{n-1})^{-1} \).

For all \( z \) with \( \text{Im } z \neq 0 \), the equation (1.1) either has exactly one (up to a multiplicative constant) solution in \( \ell^2(\mathbb{Z}_+) \) or all its solutions are in \( \ell^2(\mathbb{Z}_+) \). The first instance is known as the limit point case and the second one – as the the limit circle case.

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It is natural (see, e.g., the book [1]) to associate with the coefficients \(a_n, b_n\) a three-diagonal matrix

\[
\mathcal{J} = \begin{pmatrix}
  b_0 & a_0 & 0 & 0 & 0 & \cdots \\
  a_0 & b_1 & a_1 & 0 & 0 & \cdots \\
  0 & a_1 & b_2 & a_2 & 0 & \cdots \\
  0 & 0 & a_2 & b_3 & a_3 & \cdots \\
  \vdots & \vdots & \vdots & \ddots & \ddots & \ddots
\end{pmatrix}
\]  

(1.2)

known as the Jacobi matrix. Then equation (1.1) with the boundary condition \(P_{-1}(z) = 0\) is equivalent to the equation \(\mathcal{J}P(z) = zP(z)\) for the vector \(P(z) = (P_0(z), P_1(z), \ldots)^T\). Thus \(P(z)\) is an “eigenvector” of the matrix \(\mathcal{J}\) corresponding to an “eigenvalue” \(z\).

Let us now consider Jacobi operators defined by matrix (1.2) in the canonical basis \(e_0, e_1, \ldots\) of the space \(\ell^2(\mathbb{Z}_+)\). The minimal Jacobi operator \(J_0\) is defined by the formula \(J_0 u = \mathcal{J}u\) on a set \(D \subset \ell^2(\mathbb{Z}_+)\) of vectors \(u\) with only a finite number of non-zero components. It is symmetric in the space \(\ell^2(\mathbb{Z}_+)\), and its adjoint operator \(J_0^*\) is given by the same formula \(J_0^* u = \mathcal{J}u\) on all vectors \(u \in \ell^2(\mathbb{Z}_+)\) such that \(\mathcal{J}u \in \ell^2(\mathbb{Z}_+)\). The operator \(J_0\) is essentially self-adjoint in the limit point case, and it has deficiency indices \((1, 1)\) in the limit circle case. In the limit point case, the closure \(\text{clos} J_0\) of \(J_0\) and its adjoint operator are defined on the same domain: \(\mathcal{D}(\text{clos} J_0) = \mathcal{D}(J_0^*)\).

The spectra of all self-adjoint extensions \(J\) of the minimal operator \(J_0\) are simple with \(e_0 = (1, 0, 0, \ldots)^T\) being a generating vector. Therefore it is natural to define the spectral measure of \(J\) by the relation \(d\rho_J(\lambda) = d(E_J(\lambda)e_0, e_0)\) where \(E_J(\lambda)\) is the spectral family of the operator \(J\). For all extensions \(J\) of the operator \(J_0\), the polynomials \(P_n(\lambda)\) are orthogonal and normalized in the spaces \(L^2(\mathbb{R}; d\rho_J)\):

\[
\int_{-\infty}^{\infty} P_n(\lambda)P_m(\lambda)d\rho_J(\lambda) = \delta_{n,m};
\]

as usual, \(\delta_{n,n} = 1\) and \(\delta_{n,m} = 0\) for \(n \neq m\).

The comprehensive presentation of the results described shortly above can be found in the books [1] [4] [17] and the surveys [11] [18] [21].

1.2. Asymptotic behavior of orthogonal polynomials. We are interested in the asymptotic behavior of the polynomials \(P_n(z)\) as \(n \to \infty\). It is of course to be expected that asymptotic formulas for \(P_n(z)\) depend crucially on the behavior of recurrence coefficients \(a_n\) and \(b_n\) for large \(n\). A study of this problem was initiated by P. Nevai in his book [14]. He (see also the papers [12] by A. Máté, P. Nevai, and V. Totik and [22] by W. Van Assche and J. S. Geronimo) investigated the case where \(a_n \to a_\infty > 0, b_n \to 0\) as \(n \to \infty\).

The case of the coefficients \(a_n \to \infty\) was later studied in [9] by J. Janas and S. Naboko and in [2] by A. I. Aptekarev and J. S. Geronimo. It was assumed in
these papers that the growth of $a_n$ is not too rapid. More precisely, the so called
Carleman condition
\begin{equation}
\sum_{n=0}^{\infty} a_n^{-1} = \infty
\end{equation}
(introduced by T. Carleman in his book [3]) was required. Under this assumption
for arbitrary $b_n$, the operators $J_0$ are essentially self-adjoint on $D$. With respect to
the coefficients $b_n$, it was assumed in [9] [2] that there exists a limit
\begin{equation}
\frac{b_n}{2\sqrt{a_{n-1}a_n}} =: \gamma_n \rightarrow \gamma, \quad n \rightarrow \infty;
\end{equation}
where $|\gamma| < 1$ so that $b_n$ are relatively small compared to $a_n$. A typical result of
these papers for a particular case $b_n = 0$ is stated below as Theorem 6.4 (it was
required in [2] that $b_n \rightarrow \infty$, but this is probably an oversight).

The case of rapidly increasing coefficients $a_n$ when the Carleman condition (1.3)
is violated, so that
\begin{equation}
\sum_{n=0}^{\infty} a_n^{-1} < \infty,
\end{equation}
was investigated in a recent paper [27]. It was assumed in [27] that $|\gamma| \neq 1$. The
asymptotic formulas for $P_n(z)$ turn out to be qualitatively different for $|\gamma| < 1$ and
$|\gamma| > 1$. Astonishingly, the asymptotics of the orthogonal polynomials in this a
priori highly singular case is particularly simple and general.

Let us briefly describe the results of [27]. In the case $|\gamma| < 1$ the orthogonal
polynomials are oscillating for large $n$:
\begin{equation}
P_n(z) = a_n^{-1/2} \left( \kappa_+(z)e^{-i\phi_n} + \kappa_-(z)e^{i\phi_n} + o(1) \right), \quad n \rightarrow \infty,
\end{equation}
where
\begin{equation}
\phi_n = \sum_{m=0}^{n-1} \arccos (-\gamma_m), \quad n \geq 1,
\end{equation}
and $\kappa_\pm(z) \in \mathbb{C}$ are some constants. The same (only the constants $\kappa_\pm(z)$ change)
formula is true for all solutions of equation (1.1). All of them belong to $\ell^2(\mathbb{Z}_+)$
because the factor $\{a_n^{-1/2}\} \in \ell^2(\mathbb{Z}_+)$ due to the condition (1.5). This implies that in
the case $|\gamma| < 1$, the Jacobi operators $J_0$ have deficiency indices $(1, 1)$.

In the case $|\gamma| > 1$, the operators $J_0$ are essentially self-adjoint (provided the
coefficients $a_n$ are polynomially bounded), and the orthogonal polynomials are ex-
ponentially growing:
\begin{equation}
P_n(z) = \kappa(z)a_n^{-1/2}(-\text{sgn } \gamma)^n e^{\phi_n}(1 + o(1)), \quad n \rightarrow \infty,
\end{equation}
(unless \( z \) is an eigenvalue of the self-adjoint operator \( J = \text{clos} \, J_0 \)) where

\[
\varphi_n = \sum_{m=0}^{n-1} \arccosh |\gamma_m|, \quad n \geq 1.
\]  

(1.9)

We emphasize that a finite number of terms in (1.7) and (1.9) are arbitrary; only the values of \( a_n, b_n \) for large \( n \) are essential.

According to (1.6) and (1.8) the asymptotic behavior of the polynomials \( P_n(z) \) is the same for all \( z \in \mathbb{C} \), both for real \( z \) and for \( z \) with \( \text{Im} \, z \neq 0 \). Only the coefficients \( \kappa_\pm(z) \) and \( \kappa(z) \) depend on \( z \).

We also note the paper [19] where the Carleman and non-Carleman cases were treated at an equal footing so that the difference in the corresponding asymptotic formulas for \( P_n(z) \) was not quite visible.

2. Main results

The paper has two goals. The first one is to present a general approach to a study of asymptotic behavior of orthogonal polynomials based on an analogy with the theory of differential equations of Schrödinger type. The second goal is to apply this scheme to a study of the critical case where \( |\gamma| = 1 \) in (1.4). We concentrate here on rapidly increasing coefficients when condition (1.5) is satisfied.

2.1. Critical case. A classical example where the critical case \( |\gamma| = 1 \) occurs is given by the Laguerre polynomials \( L_n^{(p)}(z) \); those are the orthogonal polynomials determined by the recurrence coefficients

\[
a_n = \sqrt{(n+1)(n+1+p)} \quad \text{and} \quad b_n = 2n+p+1, \quad p > -1.
\]

(2.1)

The corresponding Jacobi operators \( J = J^{(p)} \) have absolutely continuous spectra coinciding with \([0, \infty)\).

For sufficiently general coefficients \( a_n, b_n \), the critical case was studied in the papers [10, 13] (see also the references therein) where the Carleman condition (1.3) was required. Our goal is to study the critical case for rapidly growing coefficients \( a_n, b_n \) when the Carleman condition is not satisfied. The asymptotic formulas we obtain are quite different from those of the papers [10, 13].

To handle the critical case, we require more specific assumptions on the coefficients \( a_n \) and \( b_n \). To make our presentation as simple as possible, we assume that

\[
a_n = n^\sigma (1 + \alpha n^{-1} + O(n^{-2}))
\]

(2.2)

and

\[
b_n = 2\gamma n^\sigma (1 + \beta n^{-1} + O(n^{-2}))
\]

(2.3)

for some \( \alpha, \beta, \gamma \in \mathbb{R} \). The critical case is distinguished by the condition \( |\gamma| = 1 \). We set

\[
\nu = -\text{sgn} \, \gamma.
\]
As discussed in Sect. 1.2, in the non-critical case \(|\gamma| \neq 1\) asymptotic formulas are qualitatively different for \(\sigma < 1\) and for \(\sigma > 1\). In the critical case the borderline is \(\sigma = \frac{3}{2}\). Here we study the singular situation \(\sigma > \frac{3}{2}\).

Note that if \(P_n(z)\) are the orthogonal polynomials corresponding to coefficients \((a_n, b_n)\), then according to equation (1.1) the polynomials \((-1)^nP_n(-z)\) correspond to the coefficients \((a_n, -b_n)\). Therefore without loss of generality, we may suppose that \(\gamma = 1\) in (1.4) or (2.3).

The results below crucially depend on the value of \(\tau = 2\beta - 2\alpha + \sigma\). (2.4)

As an example, note that for the Jacobi coefficients (2.1), we have \(\sigma = 1, \alpha = 1 + p/2, \beta = (1 + p)/2\), so that \(\tau = 0\). In the main bulk of the paper we suppose that \(\tau \neq 0\).

The case \(\tau = 0\) (doubly critical) is discussed in the final Sect. 6.

Let us state our main results. We first distinguish solutions of the Jacobi equation

\[
\begin{align*}
\lambda_{n+1} f_{n+1}(z) + \gamma_n f_n(z) + \lambda_{n-1} f_{n-1}(z) &= z f_n(z), \\
n &\in \mathbb{Z}_+, \quad z \in \mathbb{C},
\end{align*}
\]  

(2.5)

by their behavior for \(n \to \infty\).

**Theorem 2.1.** Let the assumptions (2.2), (2.3) with \(|\gamma| = 1\) and \(\sigma > \frac{3}{2}\) be satisfied. Set \(\epsilon = \min\{\sigma - \frac{3}{2}, \frac{1}{2}\}\). For all \(z \in \mathbb{C}\), the equation (2.5) has a solution \(\{f_n(z)\}\) with asymptotics

\[
f_n(z) = \nu^n n^{-\sigma/2 + 1/4} e^{-2i\sqrt{|\tau|n}} \left(1 + O(n^{-\epsilon})\right), \quad n \to \infty,
\]

(2.6)

if \(\tau < 0\), and with asymptotics

\[
f_n(z) = \nu^n n^{-\sigma/2 + 1/4} e^{-2i\sqrt{\tau n}} \left(1 + O(n^{-\epsilon})\right), \quad n \to \infty,
\]

(2.7)

if \(\tau > 0\). Asymptotic relations (2.6) and (2.7) are uniform in \(z\) from compact subsets of the complex plane \(\mathbb{C}\). For all \(n \in \mathbb{Z}_+\), the functions \(f_n(z)\) are entire functions of \(z \in \mathbb{C}\) of minimal exponential type.

By analogy with differential equations, it is natural to use the term “Jost solutions” for solutions \(\{f_n(z)\}\) constructed in Theorem 2.1. Note that according to (2.6) and (2.7) the leading terms of their asymptotics do not depend on \(z\). This is a unusual phenomenon specific for rapidly growing coefficients \(a_n\).

For orthogonal polynomials, we have the following result.

**Theorem 2.2.** Let the assumptions of Theorem 2.1 be satisfied. Then, for all \(z \in \mathbb{C}\), the sequence of the orthogonal polynomials \(P_n(z)\) has asymptotics

\[
P_n(z) = \nu^n n^{-\sigma/2 + 1/4} \left(\kappa_+(z)e^{-2i\sqrt{|\tau|n}} + \kappa_-(z)e^{2i\sqrt{|\tau|n}} + O(n^{-\epsilon})\right), \quad n \to \infty,
\]

(2.8)

if \(\tau < 0\), and

\[
P_n(z) = \kappa(z)\nu^n n^{-\sigma/2 + 1/4} e^{2\sqrt{\tau n}} \left(1 + O(n^{-\epsilon})\right), \quad n \to \infty,
\]

(2.9)
if $\tau > 0$. Here $\kappa_{\pm}(z)$ and $\kappa(z)$ are some complex constants. Asymptotic relations \((2.8)\) and \((2.9)\) are uniform in $z$ from compact subsets of the complex plane $\mathbb{C}$.

Note that the right-hand sides of \((2.8)\) and \((2.9)\) depend on $z$ only through asymptotic constants $\kappa_{\pm}(z)$ and $\kappa(z)$. These constants can be expressed via the Wronskians of the polynomial $\{P_n(z)\}_{n=-1}^{\infty}$ and the Jost $\{f_n(z)\}_{n=-1}^{\infty}$ solutions of the Jacobi equation \((2.5)\) (see Sect. 5.1). Obviously, formulas \((2.8)\) and \((2.9)\) play the role of formulas \((1.6)\) and \((1.8)\) for the non-critical case.

Spectral results are stated in the following assertion.

**Theorem 2.3.** Let the assumptions of Theorem \(2.1\) be satisfied.

1° If $\tau < 0$, then the minimal Jacobi operator $J_0$ has deficiency indices $(1,1)$ so that the spectra of all its self-adjoint extensions are discrete.

2° If $\tau > 0$, then the operator $J_0$ is essentially self-adjoint and the spectrum of its closure is semi-bounded from below and discrete.

We emphasize that although for all $\tau \neq 0$ the spectra of Jacobi operators are discrete, the reasons for this are different in the cases $\tau < 0$ and $\tau > 0$. In the first case, the discreteness of the spectra follows from general results of N. Nevanlinna \[15\] on Jacobi operators that are not essentially self-adjoint. In the second case, diagonal elements $b_n$ dominate in some sense off-diagonal elements $a_n$, but diagonal operators always have discrete spectra.

Note that the critical situation studied here is morally similar to a threshold behavior of orthogonal polynomials for the case $a_n \to a_\infty > 0$, $b_n \to 0$ as $n \to \infty$. For such coefficients, the role of \((1.4)\) is played (see \[14, 12, 26\]) by the relation

$$\lim_{n \to \infty} \frac{b_n - \lambda}{2a_n} = -\frac{\lambda}{2a_\infty}.$$ 

Since the essential spectrum of the operator $J$ is now $[-2a_\infty, 2a_\infty]$, the values $\lambda = \pm 2a_\infty$ are the threshold values of the spectral parameter $\lambda$. The parameter $-\lambda/(2a_\infty)$ plays the role of $\gamma$ so that the cases $|\gamma| < 1$ (resp., $|\gamma| > 1$) correspond to $\lambda$ lying inside the essential spectrum of $J$ (resp., outside of it).

2.2. **Classification.** Generically, leading terms of the Jost solutions asymptotics depend on the spectral parameter $z$. We call such situation “generic” or “regular”. However, it might happen that the dependence on $z$ disappears in asymptotic formulas for $f_n(z)$. We call this situation “exceptional” or “singular”. It occurs if the coefficients $a_n \to \infty$ sufficiently rapidly. However the conditions on the growth of $a_n$ are different in non-critical and critical cases.

Let us classify possible cases accepting assumptions \((2.2)\) and \((2.3)\). In the non-critical case when $|\gamma| \neq 1$ according to \((1.6)\) and \((1.8)\), the situation is singular if $\sigma > 1$. For the regular situation where $\sigma \leq 1$, we refer to the papers \[9, 2\].

In the critical case when $|\gamma| = 1$ and $\tau \neq 0$, we use Theorems \(2.1\) or \(2.2\). Now formulas \((2.8)\) and \((2.9)\) are true (and hence we are in the singular case) if $\sigma > 3/2$. The regular situation was studied in the papers \[10, 13\].
In Sect. 6, we also consider the doubly critical case when $|\gamma| = 1$ and $\tau = 0$. Here the regular (singular) situation occurs if $\sigma \leq 2$ (resp., $\sigma > 2$). The orthogonal polynomials with Jacobi coefficients (2.1) fall into the regular case.

The discussion above is summarized in Figure 1.

| Non-critical: $|\gamma| \neq 1$ | Regular | Singular |
|-------------------------------|---------|----------|
| $\sigma \leq 1$ | $\sigma > 1$ |

| Critical: $|\gamma| = 1, \tau \neq 0$ | Regular | Singular |
|-------------------------------------|---------|----------|
| $\sigma \leq 3/2$ | $\sigma > 3/2$ |

| Doubly critical: $|\gamma| = 1, \tau = 0$ | Regular | Singular |
|-------------------------------------------|---------|----------|
| $\sigma \leq 2$ | $\sigma > 2$ |

**Figure 1.** Regular and singular cases

### 2.3. Discrete versus continuous.

Let us compare difference (2.5) and differential

$$
-(a(x)f'(x, z))' + b(x)f(x, z) = zf(x, z), \quad x > 0, \quad a(x) > 0, \quad (2.10)
$$

equations. To a large extent, $x$, $a(x)$ and $b(x)$ play the roles of the parameters $n$, $a_n$ and $b_n$ in the Jacobi equation (2.5). The regular solution $\psi(x, z)$ of the differential equation (2.10) is distinguished by the conditions

$$
\psi(0, z) = 0, \quad \psi'(0, z) = 1.
$$

It plays the role of the polynomial solution $P_n(z)$ of equation (2.5) distinguished by the conditions

$$
P_{-1}(z) = 0, \quad P_0(z) = 1.
$$

A study of asymptotics of the regular solution $\psi(x, z)$ relies on a construction of special solutions of the differential equation (2.10) distinguished by their asymptotics as $x \to \infty$. For example, in the case $a(x) = 1$, $b(x) = \mathscr{H}(x)$ in (2.10) has a solution $f(x, z)$, known as the Jost solution, behaving like $e^{i\sqrt{z}x}$, $\text{Im} \sqrt{z} \geq 0$, as $x \to \infty$. Under fairly general assumptions equation (2.10) has a solution $f(x, z)$ (we also call it the Jost solution) whose asymptotics is given by the classical Liouville-Green formula (see Chapter 6 of the book [16])

$$
f(x, z) \sim \mathcal{G}(x, z)^{-1/2} \exp\left(-\int_{x_0}^{x} \mathcal{G}(y, z) dy\right) =: Q(x, z) \quad (2.11)
$$
as $x \to \infty$. Here $x_0$ is some fixed number and

$$
\mathcal{G}(x, z) = \sqrt{\frac{b(x) - z}{a(x)}}, \quad \text{Re} \mathcal{G}(x, z) \geq 0.
$$

Note that the function $Q(x, z)$ (the Ansatz for the Jost solution $f(x, z)$) satisfies equation (2.10) with a sufficiently good accuracy. Sometimes (if $a(x) \to a_\infty$, $b(x) \to 0$ slowly) it is convenient (see [25]) to omit the pre-exponential factor $\mathcal{G}(x, z)^{-1/2}$ in (2.11).
For \( z = \lambda \in \mathbb{R} \), the regular solution \( \psi(x, \lambda) \) is a linear combination of the Jost solutions \( f(x, \lambda) \) and \( \overline{f(x, \lambda)} \) which yields asymptotics of \( \psi(x, \lambda) \) as \( x \to \infty \). For example, in the case \( a(x) = 1, b \in L^1(\mathbb{R}_+) \) one has

\[
\psi(x, \lambda) \sim \kappa(\lambda) \sin(\sqrt{\lambda}x + \eta(\lambda)) \tag{2.12}
\]

where \( \kappa(\lambda) \) and \( \eta(\lambda) \) are known as the scattering (or limit) amplitude and phase, respectively. If \( \text{Im} \, z \neq 0 \), then one additionally constructs, by an explicit formula, a solution \( g(x, z) \) of (2.10) exponentially growing as \( x \to \infty \). This yields asymptotics of \( \psi(x, z) \) for \( \text{Im} \, z \neq 0 \). This scheme was realized in [25].

Let us compare asymptotic formulas (2.8) and (2.12). The crucial difference between them is that the phase \( \sqrt{\lambda}x \) in (2.12) depends on \( \lambda \) while \( \sqrt{|\tau|n} \) in (2.8) does not depend on \( z \).

2.4. Scheme of the approach. An analogy between the equations (2.5) and (2.10) is of course very well known. However it seems to be never consistently exploited before. In particular, the papers cited above rely on specific methods of difference equations. Some of these methods are quite ingenious, but, in the author’s opinion, the standard approach of differential equations works perfectly well and allows one to study asymptotic behavior of orthogonal polynomials in a very direct way. This approach was already used in the cases of coefficients satisfying \( a_n \to a_\infty > 0, b_n \to 0 \) in [24, 25] and of coefficients satisfying conditions (1.4) with \( |\gamma| \neq 1 \) and (1.5) in [27].

We are applying the same scheme in the critical singular case when conditions (2.2) and (2.3) are satisfied with \( |\gamma| = 1 \) and \( \sigma > 3/2 \). Let us briefly describe the main steps of our approach.

A. First, we forget about the orthogonal polynomials \( P_n(z) \) and distinguish solutions (the Jost solutions) \( f_n(z) \) of the difference equation (2.5) by their asymptotics as \( n \to \infty \). This requires a construction of an Ansatz \( Q_n \) for the Jost solutions.

B. Under assumption (1.5) this construction (see Sect. 4) is very explicit and, in particular, does not depend on \( z \in \mathbb{C} \). In the case \( \tau < 0 \), we set

\[
Q_n = \nu^n n^{-\sigma/2+1/4} e^{-2i\sqrt{|\tau|n}}. \tag{2.13}
\]

In the case \( \tau > 0 \), the Ansatz equals

\[
Q_n = \nu^n n^{-\sigma/2+1/4} e^{-2\sqrt{\tau}n}. \tag{2.14}
\]

In both cases the relative remainder

\[
r_n(z) := (\sqrt{a_{n-1}a_n}Q_n)^{-1}(a_{n-1}Q_{n-1} + (b_n - z)Q_n + a_nQ_{n+1}), \quad n \in \mathbb{Z}_+, \tag{2.15}
\]

belongs to \( \ell^1(\mathbb{Z}_+) \). At an intuitive level, the fact that the Ansätze (2.13) and (2.14) do not depend on \( z \in \mathbb{C} \) can be explained by the fast growth of the coefficients \( a_n \) which makes the spectral parameter \( z \) negligible in (2.15).

Actually, the Ansätze we use are only distantly similar to the Liouville-Green Ansatz (2.11). On the other hand, for integer \( \sigma \), (2.13) and (2.14) are close to
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formulas of the Birkhoff-Adams method significantly polished in [23] (see also Theorem 8.36 in the book [6]).

C. Then we make a multiplicative change of variables

\[ f_n(z) = Q_n u_n(z) \]  (2.16)

which permits us to reduce the Jacobi equation (2.5) for \( f_n(z) \) to a Volterra “integral” equation for the sequence \( u_n(z) \). This equation depends of course on the parameters \( a_n, b_n \). In particular, it is different in the cases \( \tau < 0 \) and \( \tau > 0 \). However in both cases the Volterra equation for \( u_n(z) \) is standardly solved by iterations in Sect. 3 which allows us to prove that it has a solution such that \( u_n(z) \to 1 \) as \( n \to \infty \). Then the Jost solutions \( f_n(z) \) are defined by formula (2.16).

D. The sequence

\[ \tilde{f}_n(z) = \overline{f_n(z)} \]  (2.17)

also satisfies the equation (2.5). In the case \( \tau < 0 \), the solutions \( f_n(z) \) and \( \tilde{f}_n(z) \) are linearly independent. Therefore it follows from (2.13) that all solutions of the Jacobi equation (2.5) have asymptotic behavior (2.8) with some constants \( \kappa_\pm \).

In the case \( \tau > 0 \), a solution \( g_n(z) \) of (2.5) linearly independent with \( f_n(z) \) can be constructed by an explicit formula

\[ g_n(z) = f_n(z) \sum_{m=n_0}^{n} (a_{m-1} f_{m-1}(z) f_m(z))^{-1}, \quad n \geq n_0, \]  (2.18)

where \( n_0 = n_0(z) \) is a sufficiently large number. This solution grows faster than any power of \( n \) as \( n \to \infty \),

\[ g_n(z) = \frac{\nu^n}{2\sqrt{\tau}} n^{-\sigma/2+1/4} e^{2\sqrt{\tau n}} (1 + o(1)). \]

Since \( g_n(z) \) is linearly independent with \( f_n(z) \), the polynomials \( P_n(z) \) are linear combinations of \( f_n(z) \) and \( g_n(z) \) which leads to the formula (1.8).

Our plan is the following. A Volterra integral equation for \( u_n(z) \) is introduced and investigated in Sect. 3; the Jost solutions \( f_n(z) \) are defined Sect. 4; asymptotics of the orthogonal polynomials \( P_n(z) \) is studied in Sect. 5. The doubly critical case is discussed in Sect. 6.

3. Difference and Volterra equations

Here we reduce a construction of the Jost solutions \( f_n(z) \) of the Jacobi equation (2.5) to a Volterra “integral” equation which is then solved by iterations. This construction works under very general assumptions with respect to the coefficients \( a_n \) and \( b_n \). In particular, conditions (2.2) and (2.3) are not required here.
3.1. Preliminaries. Let us consider equation (2.5). Note that the values of $f_{m-1}$ and $f_m$ for some $m \in \mathbb{Z}_+$ determine the whole sequence $f_n$ satisfying the difference equation (2.5).

Let $f = \{f_n\}_{n=-1}^{\infty}$ and $g = \{g_n\}_{n=-1}^{\infty}$ be two solutions of equation (2.5). A direct calculation shows that their Wronskian

$$W[f,g] := a_n(f_ng_{n+1} - f_{n+1}g_n)$$

does not depend on $n = -1, 0, 1, \ldots$. In particular, for $n = -1$ and $n = 0$, we have

$$W[f,g] = 2^{-1}(f_{-1}g_0 - f_0g_{-1}) \quad \text{and} \quad W[f,g] = a_0(f_0g_1 - f_1g_0)$$

(for definiteness, we put $a_{-1} = 1/2$). Clearly, the Wronskian $W[f,g] = 0$ if and only if the solutions $f$ and $g$ are proportional.

It is convenient to introduce a notation

$$x'_n = x_{n+1} - x_n$$

for the “derivative” of a sequence $x_n$. Then we have

$$(x_{n-1})' = -x_{n-1}x_{n+1}x_n'$$

and

$$(e^{x_n})' = (e^{x'_n} - 1)e^{x_n}.$$ (3.2)

Note also the Abel summation formula (“integration by parts”):

$$\sum_{n=N}^{M} x_n y'_n = x_M y_{M+1} - x_{N-1} y_N - \sum_{n=N}^{M} x_{n-1}' y_n;$$ (3.3)

here $M \geq N \geq 0$ are arbitrary, but we have to set $x_{-1} = 0$ so that $x'_{-1} = x_0$.

To emphasize the analogy between differential and difference equations, we often use “continuous” terminology (Volterra integral equations, integration by parts, etc.) for sequences labelled by the discrete variable $n$. Below $C$, sometimes with indices, and $c$ are different positive constants whose precise values are of no importance.

In constructions below, it suffices to consider the Jacobi equation (2.5) for large $n$ only.

3.2. Multiplicative change of variables. For construction of $f_n(z)$, we will reformulate the problem introducing a sequence

$$u_n(z) = Q_n^{-1} f_n(z), \quad n \in \mathbb{Z}_+.$$ (3.4)

In our construction, the Ansatz $Q_n$ does not depend on $z$. In this section, we do not make any specific assumptions about the recurrence coefficients $a_n$, $b_n$ and the Ansatz $Q_n$ except of course that $Q_n \neq 0$; for definiteness, we set $Q_{-1} = 1$. In proofs, we usually omit the dependence on $z$ in notation; for example, we write $f_n, u_n, r_n$. First, we derive a difference equation for $u_n(z)$.
Lemma 3.1. Let the remainder $r_n(z)$ be defined by formula (2.15). Set

$$\Lambda_n = \frac{a_n}{a_{n-1}} \frac{Q_{n+1}}{Q_n}$$

(3.5)

and

$$R_n(z) = -\sqrt{\frac{a_n}{a_{n-1}} \frac{Q_n}{Q_{n-1}}} r_n(z).$$

(3.6)

Then equation (2.5) for a sequence $f_n(z)$ is equivalent to the equation

$$\Lambda_n(u_{n+1}(z) - u_n(z)) - (u_n(z) - u_{n-1}(z)) = R_n(z) u_n(z), \quad n \in \mathbb{Z}_+,$$

(3.7)

for sequence (3.4).

Proof. Substituting expression $f_n = Q_n u_n$ into (2.5) and using the definition (2.15), we see that

\[
\left(\sqrt{\frac{a_n}{a_{n-1}} a_n Q_n}\right)^{-1}\left(a_{n-1} f_{n-1} + (b_n - z) f_n + a_n f_{n+1}\right)
= \sqrt{\frac{a_n}{a_{n-1}} \frac{Q_{n-1}}{Q_n}} u_{n-1} + \frac{b_n - z}{\sqrt{a_{n-1} a_n}} u_n + \sqrt{\frac{a_n}{a_{n-1}} Q_n Q_{n+1}} u_{n+1}
= \sqrt{\frac{a_n}{a_{n-1}} \frac{Q_{n-1}}{Q_n}} (u_{n-1} - u_n) + \sqrt{\frac{a_n}{a_{n-1}} Q_n} (u_{n+1} - u_n) + R_n u_n
= \sqrt{\frac{a_n}{a_{n-1}} \frac{Q_{n-1}}{Q_n}} \left((u_{n-1} - u_n) + \Lambda_n (u_{n+1} - u_n) - R_n u_n\right)
\]

where the coefficients $\Lambda_n$ and $R_n$ are defined by equalities (3.5) and (3.6), respectively. Therefore the equations (2.5) and (3.7) are equivalent. □

Our goal here to construct solutions of the equation (3.7) such that

$$\lim_{n \to \infty} u_n(z) = 1.$$  
(3.8)

Let us set

$$X_n = \Lambda_1 \cdots \Lambda_n$$

(3.9)

and

$$G_{n,m} = X_{m-1} \sum_{p=n}^{m-1} X_p^{-1}, \quad m \geq n + 1.$$  
(3.10)

The following result will be proven in the next subsection. Note that we make assumptions only on products $G_{n,m} R_m$ but not on factors $G_{n,m}$ and $R_m$ separately.

Theorem 3.2. Set

$$h_n(z) = \sup_{n \leq m-1} |G_{n,m} R_m(z)|$$

(3.11)

and suppose that

$$\{h_m(z)\} \in \ell^1(\mathbb{Z}_+).$$

(3.12)
Then equation (3.7) has a solution \( u_n(z) \) satisfying an estimate
\[
|u_n(z) - 1| \leq e^{H_n(z)} - 1, \quad n \geq 0,
\]
where
\[
H_n(z) = \sum_{p=n+1}^{\infty} h_p(z).
\]
In particular, condition (3.8) holds.

3.3. Volterra equation. A sequence \( u_n \) will be constructed as a solution of the Volterra “integral” equation
\[
u_n(z) = 1 + \sum_{m=n+1}^{\infty} G_{n,m} R_m(z) u_m(z), \quad (3.14)
\]
This equation can be solved by successive approximations.

**Lemma 3.3.** Let the assumptions of Theorem 3.2 be satisfied. Set \( u_n^{(0)} = 1 \) and
\[
u_n^{(k+1)}(z) = \sum_{m=n+1}^{\infty} G_{n,m} R_m(z) u_m^{(k)}(z), \quad k \geq 0,
\]
for all \( n \in \mathbb{Z}_+ \). Then estimates
\[
|u_n^{(k)}(z)| \leq \frac{H_n(z)^k}{k!}, \quad \forall k \in \mathbb{Z}_+,
\]
are true.

**Proof.** Suppose that (3.16) is satisfied for some \( k \in \mathbb{Z}_+ \). We have to check the same estimate (with \( k \) replaced by \( k + 1 \) in the right-hand side) for \( u_n^{(k+1)} \). According to definitions (3.11) and (3.15), it follows from estimate (3.16) that
\[
|u_n^{(k+1)}| \leq \frac{1}{k!} \sum_{m=n+1}^{\infty} h_m H_m^k.
\]
Observe that
\[
H_m^{k+1} + (k + 1) h_m H_m^k \leq H_{m-1}^{k+1},
\]
and hence, for all \( N \in \mathbb{Z}_+ \),
\[
(k + 1) \sum_{m=n+1}^{N} h_m H_m^k \leq \sum_{m=n+1}^{N} (H_{m-1}^{k+1} - H_m^{k+1}) = H_n^{k+1} - H_{N+1}^{k+1} \leq H_N^{k+1}.
\]
Substituting this bound into (3.17), we obtain estimate (3.16) for \( u_n^{(k+1)} \). \( \square \)

Now we are in a position to solve equation (3.14) by iterations.

**Theorem 3.4.** Under the assumptions of Theorem 3.2 the equation (3.14) has a bounded solution \( u_n(z) \). This solution satisfies an estimate (3.13).
Proof. Set

\[ u_n = \sum_{k=0}^{\infty} u_n^{(k)} \]  

(3.18)

where \( u_n^{(k)} \) are defined by recurrence relations (3.15). Estimate (3.16) shows that this series is absolutely convergent. Using the Fubini theorem to interchange the order of summations in \( m \) and \( k \), we see that

\[ \sum_{m=n+1}^{\infty} G_{n,m} R_m u_m = \sum_{k=0}^{\infty} \sum_{m=n+1}^{\infty} G_{n,m} R_m u_n^{(k)} = \sum_{k=0}^{\infty} u_n^{(k+1)} = -1 + \sum_{k=0}^{\infty} u_n^{(k)} = -1 + u_n. \]

This is equation (3.14) for sequence (3.18). Estimate (3.13) also follows from (3.16), (3.18). □

Remark 3.5. A bounded solution \( u_n(z) \) of (3.14) is of course unique. Indeed, suppose that \( \{v_n\} \in \ell^\infty(\mathbb{Z}_+) \) satisfies the homogeneous equation (3.14), that is,

\[ v_n = \sum_{m=n+1}^{\infty} G_{n,m} R_m v_m \]

whence

\[ |v_n| \leq \sum_{m=n+1}^{\infty} h_m |v_m|. \]

This estimate implies that

\[ |v_n| \leq \frac{1}{k!} \left( \sum_{m=n+1}^{\infty} h_m \right)^k \max_n \{|v_n|\}, \quad \forall k \in \mathbb{Z}_+. \]

It follows that \( v_n = 0 \). Note however that we do not use the unicity in our construction.

It turns out that the construction above yields a solution of the difference equation (3.7).

Lemma 3.6. Let \( G_{n,m} \) be given by formulas (3.9) and (3.10). Then a solution \( u_n(z) \) of the integral equation (3.14) satisfies also the difference equation (3.7).

Proof. It follows from (3.14) that

\[ u_{n+1} - u_n = \sum_{m=n+2}^{\infty} (G_{n+1,m} - G_{n,m}) R_m u_m - G_{n,n+1} R_{n+1} u_{n+1}. \]  

(3.19)

Since according to (3.10)

\[ G_{n+1,m} - G_{n,m} = -X_n^{-1} X_{m-1} \quad \text{and} \quad G_{n,n+1} = 1, \]

and (3.14), (3.18), (3.19) hold.
equality (3.19) can be rewritten as

$$u_{n+1} - u_n = -X_n^{-1} \sum_{m=n+1}^{\infty} X_{m-1} R_m u_m.$$  \hspace{1cm} (3.20)

Putting together this equality with the same equality for $n+1$ replaced by $n$, we see that

$$\Lambda_n(u_{n+1} - u_n) - (u_n - u_{n-1}) = -\Lambda_n X_n^{-1} \sum_{m=n+1}^{\infty} X_{m-1} R_m u_m + X_n^{-1} \sum_{m=n}^{\infty} X_{m-1} R_m u_m.$$  

Since $X_n = \Lambda_n X_{n-1}$ by (3.9), the right-hand side here equals $R_n u_n$, and hence the equation obtained coincides with (3.7). \hfill \square

Thus putting together Theorem 3.4 and Lemma 3.6, we conclude the proof of Theorem 3.2.

Remark 3.7. It follows from equality (3.20) that under the assumptions of Theorem 3.2 we have an estimate

$$|u'_n| \leq \max_{n \in \mathbb{Z}^+} \{|u_n|\} |X_n|^{-1} \sum_{m=n}^{\infty} |X_m R_{m+1}|.$$  \hspace{1cm} (3.21)

3.4. Dependence on the spectral parameter. The results above can be supplemented by the following assertion.

Lemma 3.8. Let, for some open set $\Omega \subset \mathbb{C}$, the coefficients $R_n(z)$ be analytic functions of $z \in \Omega$. Suppose that the assumptions of Theorem 3.2 are satisfied uniformly in $z$ on compact subsets of $z \in \Omega$. Then all functions $u_n(z)$ are also analytic in $z \in \Omega$. Moreover, if $R_n(z)$ are continuous up to the boundary of $\Omega$ and the assumptions of Theorem 3.2 are satisfied uniformly on $\Omega$, then the same is true for the functions $u_n(z)$.

Proof. Observe that if the functions $u_m^{(k)}(z)$ in (3.15) depend analytically (continuously) on $z$, then the function $u_m^{(k+1)}(z)$ is also analytic (continuous). Since the series (3.18) converges uniformly, its sum is also an analytic (continuous) function. \hfill \square

According to (2.15) and (3.6) the remainder $R_n(z)$ depends linearly on $z$. In this case it is easy to obtain a bound on $u_n(z)$ for large $lz$.

Proposition 3.9. Suppose that sequence (3.11) satisfies a condition

$$h_n(z) \leq \mathcal{H}_n(1 + |z|) \quad \text{where} \quad \{\mathcal{H}_n\} \in l^1(\mathbb{Z}^+).$$  \hspace{1cm} (3.22)

Then, for an arbitrary $\varepsilon > 0$ and some constants $C_n(\varepsilon)$ (that do not depend on $z \in \mathbb{C}$), every function $u_n(z)$ satisfies an estimate

$$|u_n(z)| \leq C_n(\varepsilon)e^{\varepsilon|z|}, \quad z \in \mathbb{C}.$$  \hspace{1cm} (3.23)
**Proof.** According to inequality (3.13) and condition (3.22) we have an estimate

\[ |u_m(z)| \leq e^{H_m(z)} \leq e^{\varepsilon_m(1+|z|)} \quad \text{where} \quad \varepsilon_m = \sum_{p=m+1}^{\infty} \mathcal{H}_p. \quad (3.24) \]

According to (3.12) \( \varepsilon_m \to 0 \) as \( m \to \infty \). On the other hand, it follows from equation (3.7) that

\[ |u_{n-1}(z)| \leq (1 + |\Lambda_n| + |R_n(z)|)|u_n(z)| + |\Lambda_n||u_{n+1}(z)|. \]

Iterating this estimate, we find that

\[ |u_n(z)| \leq C_n(1 + |z|)(|u_{n+1}(z)| + |u_{n+2}(z)|) \leq \cdots \leq C_{n,k}(1 + |z|^k(|u_{n+k}(z)| + |u_{n+k+1}(z)|)) \quad (3.25) \]

for every \( k = 1, 2, \ldots \). For a given \( \varepsilon > 0 \), choose \( k \) such that \( 2\varepsilon_{n+k} \leq \varepsilon, 2\varepsilon_{n+k+1} \leq \varepsilon \).

Then putting estimates (3.24) and (3.25) together, we see that

\[ |u_n(z)| \leq 4C_{n,k}(1 + |z|^k)e^{\varepsilon|z|/2}. \]

Since \( (1 + |z|^k) \leq c_k(z)e^{\varepsilon|z|/2} \), this proves (3.23). \( \square \)

Functions \( u_n(z) \) satisfying estimates (3.23) for all \( \varepsilon > 0 \) are known as functions of minimal exponential type.

### 4. Jost solutions

In this section, we first calculate the remainder (2.15) for the Ansatz \( Q_n \) defined by formulas (2.13) or (2.14). Then we make substitution (2.16) and use Theorem 3.2 to construct an appropriate solution of the corresponding equation (3.7). This leads to Theorem 2.1.

#### 4.1. Ansatz

Let us apply the results of the previous section to recurrence coefficients \( a_n, b_n \) satisfying conditions (2.2), (2.3) where \( |\gamma| = 1 \). First, we exhibit an Ansatz \( Q_n \) such that the corresponding remainder (2.13) satisfies the condition

\[ r_n(z) = O(n^{-\delta}), \quad n \to \infty, \quad (4.1) \]

for some \( \delta > 3/2 \). We emphasize that this estimate with \( \delta > 1 \) used in the non-critical case in [27] is not sufficient now. Until Sect. 6, we always suppose that \( \tau \neq 0 \).

We treat the cases \( \tau > 0 \) and \( \tau < 0 \) parallelly setting \( \sqrt{\tau} > 0 \) if \( \tau > 0 \) and (for definiteness) \( \sqrt{\tau} = i\sqrt{|\tau|} \) if \( \tau < 0 \).

Let us seek \( Q_n \) in the form

\[ Q_n = \nu^n n^s e^{-\varphi_n}, \quad n \geq 1, \quad \nu = \text{sgn} \gamma, \quad (4.2) \]

\( Q_0 = 1 \). We have to calculate the remainder \( r_n(z) \) and find an exponent \( s \) and a sequence \( \varphi_n \) such that estimate (4.1) is satisfied with \( \delta > 3/2 \). Set \( \theta_n = \varphi_{n+1} - \varphi_n \)
and \( \varphi_1 = 0 \). Then

\[
\varphi_n = \sum_{m=1}^{n-1} \theta_m, \quad n \geq 2.
\] (4.3)

The phases \( \theta_n \) which we choose below (see (4.12)) obey the conditions \( \theta_n = O(n^{-1/2}) \) and \( \text{Re} \theta_n \geq 0 \).

Put

\[
\kappa_n = \sqrt{\frac{a_{n+1}}{a_n}}.
\]

It follows from condition (2.2) that

\[
\kappa_n = 1 + \frac{\sigma}{2n} + O(n^{-2}),
\] (4.4)

and

\[
(a_n a_{n-1})^{-1/2} = n^{-\sigma}(1 - (\alpha - \sigma/2)n^{-1} + O(n^{-2})).
\]

Using also (2.3), we see that sequence (1.4) satisfies a relation

\[
\gamma_n = \nu (1 + (\tau/2)n^{-1} + O(n^{-2}))
\] (4.5)

where \( \tau \) is defined by equality (2.4).

Let us now calculate the remainder (2.15). We write it as

\[
r_n(z) = \sqrt{\frac{a_{n-1}}{a_n} \frac{Q_{n-1}}{Q_n}} + \sqrt{\frac{a_n}{a_{n-1}} \frac{Q_{n+1}}{Q_n}} + \frac{b_n - z}{\sqrt{a_{n-1}a_n}}.
\]

Since according to (4.2) and (4.3)

\[
Q_{n+1}/Q_n = \nu \left( \frac{n+1}{n} \right)^s e^{-\theta_n},
\]

easy calculations yield the following assertion.

**Lemma 4.1.** The relative remainder (2.15) can be rewritten as

\[
r_n(z) = -\nu \kappa_{n-1}^{-1} \left( \frac{n-1}{n} \right)^s e^{\theta_{n-1}} - \nu \kappa_{n-1} \left( \frac{n+1}{n} \right)^s e^{-\theta_n} + 2\gamma_n - z(a_n a_{n-1})^{-1/2}.
\] (4.6)

**4.2. Estimate of the remainder.** Here we estimate expression (4.6). Using relations

\[
\left( \frac{n+1}{n} \right)^s = 1 + \frac{s}{n} + O(n^{-2}),
\]

(4.7) and setting \( k = s + \sigma/2 \), we see that

\[
r_n = -\nu (1 - kn^{-1})e^{\theta_{n-1}} - \nu (1 + kn^{-1})e^{-\theta_n} + 2\gamma_n - z(a_n a_{n-1})^{-1/2} + O(n^{-2})
\] (4.7)

where \( \gamma_n \) satisfies (4.5). Since

\[
e^{-\theta_n} = 1 - \theta_n + 2^{-1}\theta_n^2 - 6^{-1}\theta_n^3 + O(n^{-2}),
\]
expression \((4.7)\) can be written as
\[
\nu r_n = r_n^{(1)} + r_n^{(2)} + r_n^{(3)} + \tau n^{-1} - z\nu(a_n a_{n-1})^{-1/2} + O(n^{-2})
\]  
(4.8)
where
\[
r_n^{(1)} = 2kn^{-1}\theta_n + (1 - kn^{-1})(\theta_n - \theta_{n-1}),
\]  
(4.9)
\[
r_n^{(2)} = -\theta_n^2 + 2^{-1}(1 - kn^{-1})(\theta_n^2 - \theta_{n-1}^2)
\]  
(4.10)
and
\[
r_n^{(3)} = 6^{-1}kn^{-1}\theta_n^3 + 6^{-1}(1 - kn^{-1})(\theta_n^3 - \theta_{n-1}^3).
\]  
(4.11)
Our goal is to find the numbers \(k\) and \(\theta_n\) such that expression \((4.8)\) satisfies estimate \((4.1)\).

Let us first consider the quadratic term \((4.10)\). It should cancel with \(\tau n^{-1}\), up to a term \(O(n^{-2})\). It is convenient to set
\[
\theta_n = 2\sqrt{\tau(n + 1)^{1/2} - n^{1/2}}.
\]  
(4.12)
Then
\[
\theta_n = \sqrt{\tau n^{-1/2} - 4^{-1}\sqrt{\tau n^{-3/2}} + O(n^{-5/2})}
\]  
(4.13)
and
\[
r_n^{(2)} + \tau n^{-1} = O(n^{-2}).
\]  
According to \((4.12)\) for the linear term \((4.9)\), we have
\[
r_n^{(1)} = 2k\sqrt{\tau} n^{-3/2}
\]  
\[
+ (1 - kn^{-1})\sqrt{\tau(n^{-1/2} - 4^{-1}n^{-3/2} - (n + 1)^{-1/2} + 4^{-1}(n + 1)^{-3/2})}
\]  
\[
+ O(n^{-5/2}) = (2k - 1/2)\sqrt{\tau n^{-3/2}} + O(n^{-5/2}).
\]
The coefficient at \(n^{-3/2}\) is zero if \(k = 1/4\), that is,
\[
s = -\sigma/2 + 1/4.
\]  
(4.14)
Finally, for cubic terms \((4.11)\), we use that
\[
\theta_n^3 - \theta_{n-1}^3 = (\theta_n - \theta_{n-1})(\theta_n^2 + \theta_n\theta_{n-1} + \theta_{n-1}^2) = O(n^{-3/2})O(n^{-1}) = O(n^{-5/2}),
\]
whence \(r_n^{(3)} = O(n^{-5/2})\).

Let us summarize these calculations and observe that for our choice \((4.12)\) the phase \((4.3)\) equals
\[
\varphi_n = 2\sqrt{\tau} n^{1/2}.
\]  
(4.15)

**Lemma 4.2.** Let conditions \((2.2)\) and \((2.3)\) where \(|\gamma| = 1\) be satisfied. Define the numbers \(\tau\) and \(s\) by equalities \((2.4)\) and \((4.14)\) and suppose that \(\tau \neq 0\). Define the sequence \(Q_n\) by formula \((4.2)\) where \(\varphi_n\) is given by \((4.15)\). Then the corresponding remainder \((2.15)\) satisfies the estimate \((4.1)\) with \(\delta = \min\{\sigma, 2\}\).
According to (3.6) we have

\[ R_n(z) = -\nu \sqrt{\frac{a_n}{a_{n-1}}} \left( \frac{n}{n-1} \right)^{\nu} e^{-\theta_n} r_n(z), \]

whence by Lemma 4.2

\[ R_n(z) = O(n^{-\delta}) \quad \text{where} \quad \delta = \min\{\sigma, 2\}. \] (4.16)

4.3. Estimate of the “integral” kernel. Here we estimate matrix elements \( G_{n,m} \) defined by formulas (3.9) and (3.10) where \( \Lambda_n \) is given by (3.5). Now the product (3.9) equals

\[ X_n = a_1 a_2 \cdots a_n \frac{Q_2 Q_3 \cdots Q_n Q_{n+1}}{Q_0 Q_1 \cdots Q_{n-2} Q_{n-1}} = \frac{a_n Q_n Q_{n+1}}{a_0 Q_0 Q_1}. \]

It follows from definition (4.2) that

\[ X_n = a_0^{-1} x_n e^{-\varphi_n - \varphi_{n-1}} \] (4.17)

where

\[ x_n = (n + 1)^{\nu} n^{\nu} a_n = n^{1/2} (1 + cn^{-1} + O(n^{-2})) \] (4.18)

according to (2.2) and (4.14); the precise value of the constant \( c \) here is inessential.

Let us state a necessary estimate on \( G_{n,m} \).

**Lemma 4.3.** Let \( X_n \) be given by formulas (4.17), (4.18). Then for all \( m > n \geq 0 \), matrix elements (3.10) satisfy an estimate

\[ |G_{n,m}| \leq Cm^{1/2} \] (4.19)

with a constant \( C \) that does not depend on \( n \) and \( m \).

**Proof.** By definition (4.17), we have

\[ a_0^{-1} \sum_{p=n}^{m-1} X_p^{-1} = \sum_{p=n}^{m-1} x_p^{-1} e^{\varphi_p + \varphi_{p-1}} \] (4.20)

where according to (3.2)

\[ e^{\varphi_p + \varphi_{p-1}} = (e^{\theta_p + \theta_{p-1}} - 1)^{-1} (e^{\varphi_p + \varphi_{p-1}})' \] (4.21)

Set

\[ y_n = x_n^{-1} (e^{\theta_n + \theta_{n-1}} - 1)^{-1}. \]

Using formula (3.3), we can integrate by parts in (4.20) which yields

\[ a_0^{-1} \sum_{p=n}^{m-1} X_p^{-1} = y_{m-1} e^{\varphi_m + \varphi_{m-1}} - y_{n-1} e^{\varphi_n + \varphi_{n-1}} - \sum_{p=n}^{m-1} y_p' e^{\varphi_p + \varphi_{p-1}} \] (4.22)

Let us estimate the right-hand side of (4.22). It follows from formula (4.13) that

\[ e^{\theta_n + \theta_{n-1}} - 1 = 2\sqrt{\tau} n^{-1/2} (1 + c_1 n^{-1/2} + c_2 n^{-1} + O(n^{-3/2})) \] (4.23)
where precise values of the constants \( c_1, c_2 \) are inessential. In particular, \((4.23)\) implies that

\[
|e^{\theta_n + \theta_{n-1}} - 1| \geq cn^{-1/2}, \quad c > 0.
\]

Putting together relations \((4.18)\) and \((4.23)\), we find that

\[
y_n = (2\sqrt{T})^{-1} \left( 1 + d_1n^{-1/2} + d_2n^{-1} + O(n^{-3/2}) \right)
\]

for some constants \( d_1 \) and \( d_2 \), whence

\[
|y_n| \leq C < \infty \quad \text{and} \quad y'_n = O(n^{-3/2}).
\] \((4.24)\)

Let us multiply equality \((4.22)\) by \( S_{m-1} \) and take into account that

\[
|e^{\varphi_p}| \leq |e^{\varphi_m}|, \quad p \leq m,
\] \((4.25)\)

because \( \Re \theta_n \geq 0 \). By definition \((3.10)\), we now have

\[
|G_{n,m}| \leq |x_{m-1}| \left( |y_{m-1}| + |y_{n-1}| + \sum_{p=n}^{m-1} |y'_{p-1}| \right).
\]

According to \((4.18)\) the first factor here is estimated by \( Cm^{1/2} \), and according to \((4.24)\) the second factor is uniformly bounded. This yields \((4.19)\). \(\square\)

4.4. Jost solutions. Let us come back to the equation \((3.7)\) with \( \Lambda_n \) and \( R_n(z) \) given by \((3.5)\) and \((3.6)\).

**Theorem 4.4.** Let the assumptions \((2.2)\), \((2.3)\) with \( \sigma > 3/2 \) and \( |\gamma| = 1 \) be satisfied. Set \( \varrho = \min\{\sigma - 3/2, 1/2\} \). Suppose that \( \tau \neq 0 \). For all \( z \in \mathbb{C} \), the equation \((3.7)\) has a solution \( u_n(z) \) with asymptotics

\[
u_n(z) = 1 + O(n^{-\varrho}), \quad n \to \infty.
\] \((4.26)\)

Moreover,

\[
u'_n(z) = O(n^{-1/2-\varrho}), \quad n \to \infty.
\] \((4.27)\)

For all \( n \in \mathbb{Z}_+ \), the functions \( u_n(z) \) are entire functions of \( z \in \mathbb{C} \) of minimal exponential type.

**Proof.** Let us proceed from Theorem 3.2. Estimates \((4.16)\) and \((4.19)\) show the sequence \((3.11)\) satisfies a bound \( h_m = O(n^{-\sigma+1/2}) \). It is in \( \ell^1(\mathbb{Z}_+) \) if \( \sigma > 3/2 \). Therefore Theorem 3.2 yields a solution \( u_n \) satisfying estimate \((3.13)\) which implies \((4.26)\).

For estimates of derivatives \( u'_n \), we use Remark 3.7. By virtue of \((4.17)\), \((4.18)\) and \((4.25)\) inequality \((3.21)\) implies that

\[
|u'_n| \leq Cn^{-1/2} \sum_{m=n}^{\infty} m^{1/2} |R_{m+1}|.
\]

Using also \((4.16)\), we obtain estimate \((4.27)\). \(\square\)
Now we set
\[ f_n(z) = Q_n u_n(z). \]
According to Lemma 3.1 this sequence satisfies the Jacobi equation (2.5) and according to definition (4.2) of \( Q_n \) it has asymptotics (2.6) or (2.7). Therefore Theorem 2.1 is a direct consequence of Theorem 4.4. The asymptotic relations (2.6) and (2.7) mean that the solution \( f_n(z) \) is oscillating for \( \tau < 0 \) and tends to zero faster than any power of \( n^{-1} \) for \( \tau > 0 \).

In the case \( \tau > 0 \), the condition \( f_n = Q_n(1 + o(1)) \) as \( n \to \infty \) determines a solution of the Jacobi equation (2.5) uniquely. Indeed, suppose that \( f_n = Q_n u_n \) and \( \tilde{f}_n = Q_n \tilde{u}_n \) (4.28)
where \( u_n = 1 + o(1) \) and \( \tilde{u}_n = 1 + o(1) \). It follows that
\[ W[f, \tilde{f}] = \lim_{n \to \infty} \left( a_n Q_n Q_{n+1}(u_n \tilde{u}_{n+1} - u_{n+1} \tilde{u}_n) \right) \] 
equals zero because according to (4.2) and (4.15) \( Q_n \to 0 \) faster than any power of \( n^{-1} \). Therefore \( \tilde{f}_n = c f_n \) where \( c = 1 \) by virtue of (4.28).

Essentially similar result is true in the case \( \tau < 0 \). Now we have to suppose that \( u_n \) and \( \tilde{u}_n \) in (4.28) satisfy conditions (4.26) and (4.27). Then
\[ u_n \tilde{u}_{n+1} - u_{n+1} \tilde{u}_n = u_n \tilde{u}_n' - u_n' \tilde{u}_n = O(n^{-1/2-\nu}). \] 
(4.30)
According to (4.2) and (4.14) we have
\[ a_n Q_n Q_{n+1} = O(a_n n^{2s}) = O(n^{1/2}). \] 
(4.31)
Combining the last two relations, we see that the Wronskian (4.29) equals zero whence again \( \tilde{f}_n = f_n \).

5. ORTHOGONAL POLYNOMIALS

As usual, we suppose that conditions (2.2) and (2.3) are satisfied with \( \sigma > 3/2 \) and \( |\gamma| = 1 \).

5.1. Subcritical case. We first consider the case \( \tau < 0 \). In addition to the Jost solution \( f(z) = \{f_n(z)\} \) constructed in Theorem 2.1 we can define the conjugate Jost solution \( \tilde{f}_n(z) \) by formula (2.17). It also satisfies equation (2.5) since the coefficients \( a_n \) and \( b_n \) are real, and it has the asymptotics
\[ \tilde{f}_n(z) = \nu^n n^{-\sigma/2+1/4} e^{2i\sqrt{|\tau|}n} \left( 1 + O(n^{-\nu}) \right), \quad n \to \infty, \]

Lemma 5.1. The Wronskian of the solutions \( f(z) \) and \( \tilde{f}_n(z) = f_n(\bar{z}) \) of the Jacobi equation (2.5) equals
\[ W[f(z), \tilde{f}(z)] = 2i\nu \sqrt{|\tau|} \neq 0 \] 
(5.1)
so that these solutions are linearly independent.
Using definitions (4.2) and (4.15), we see that
\[
\tau < 0 \quad \text{and let}
\]
\[
\text{the sequences}
\]

Thus we arrive at the following assertion.

Proof. Let us use notation (1.2). We now have \( f_n = Q_n u_n \) and \( \tilde{f}_n = \tilde{Q}_n \tilde{u}_n \) where the sequences \( u_n \) and \( \tilde{u}_n \) satisfy conditions (4.26) and (4.27) so that
\[
f_n \tilde{f}_{n+1} - f_{n+1} \tilde{f}_n = Q_n \tilde{Q}_{n+1} u_n \tilde{u}_{n+1} - Q_{n+1} \tilde{Q}_n u_{n+1} \tilde{u}_n
\]

Observe also that
\[
Q_{n+1} \tilde{Q}_n (u_n \tilde{u}_{n+1} - u_{n+1} \tilde{u}_n) = O(n^{2s-1/2}(1 + O(n^{-1})).
\]

Thus we arrive at the following assertion.

Theorem 5.2. Let the assumptions (2.2), (2.3) with \(|\gamma| = 1, \sigma > 3/2\) be satisfied, and let \( \tau < 0 \). Choose some \( z \in \mathbb{C} \) and put \( \sigma = \min\{\sigma - 3/2, 1/2\} \). Then an arbitrary solution \( F_n \) of the Jacobi equation (2.5) has asymptotics
\[
F_n(z) = \kappa_+(z)f_n(z) + \kappa_-(z)\tilde{f}_n(z),
\]
where the constants can be expressed via the Wronskians:
\[
\kappa_+(z) = \frac{\{F(z), f(z)\}}{2i\nu\sqrt{2\tau}}, \quad \kappa_-(z) = \frac{-\{F(z), f(z)\}}{2i\nu\sqrt{2\tau}}.
\]

Thus we arrive at the following assertion.

Theorem 5.2. Let the assumptions (2.2), (2.3) with \(|\gamma| = 1, \sigma > 3/2\) be satisfied, and let \( \tau < 0 \). Choose some \( z \in \mathbb{C} \) and put \( \sigma = \min\{\sigma - 3/2, 1/2\} \). Then an arbitrary solution \( F_n \) of the Jacobi equation (2.5) has asymptotics
\[
F_n = n^n n^{-\sigma/2+1/4} \left( \kappa_1 e^{-2i\sqrt{|\tau|n}} + \kappa_2 e^{2i\sqrt{|\tau|n}} \right) \left(1 + O(n^{-\sigma})\right), \quad n \to \infty,
\]
for constants \( \kappa_\pm \) defined by (5.6). Conversely, for arbitrary \( \kappa_\pm \in \mathbb{C} \), there exists a solution \( F_n \) of the equation (2.5) with asymptotics (5.7).

Recall that the polynomials \( P(z) = \{P_n(z)\} \) are the solutions of the Jacobi equation (2.5) satisfying the conditions \( P_{-1}(z) = 0, P_0(z) = 1 \). Therefore formula (2.8) of Theorem 2.2 is a particular case of formula (5.7) in Theorem 5.2. Moreover, we now have \( \kappa_-(\lambda) = \kappa_+(\lambda) \) because \( P_n(z) = P_n(-z) \). In particular, \( \kappa_-(\lambda) = \kappa_+(\lambda) \) if \( \lambda \in \mathbb{R} \).

Theorem 5.2 can be supplemented by the following assertion.
Proposition 5.3. Under the assumptions of Theorem 5.2 suppose that a solution $F_n$ of the equation (2.5) satisfies a bound

$$F_n = o(n^{-\sigma/2+1/4})$$

(5.8)

as $n \to \infty$. Then $F_n = 0$ for all $n \in \mathbb{Z}_+$.

Proof. Let us proceed from Theorem 5.2. Comparing relations (5.7) and (5.8) we see that

$$\kappa_+ e^{2i\sqrt{|\tau|n}} + \kappa_- e^{2i\sqrt{|\tau|n}} = o(1), \quad n \to \infty. \tag{5.9}$$

Let us show that this implies equalities $\kappa_+ = \kappa_- = 0$. The modulus of the left-hand side of (5.9) is minorated by $||\kappa_+| - |\kappa_-||$ whence $|\kappa_+| = |\kappa_-|$. Let $\kappa_+ = \kappa_- e^{i\theta}$ where $\theta \in [0, 2\pi)$ so that

$$|\kappa_+ e^{-2i\sqrt{|\tau|n}} + \kappa_- e^{2i\sqrt{|\tau|n}}| = |\kappa_+| |e^{i\sqrt{|\tau|n+i\theta} + 1}|. \tag{5.10}$$

Observe that for an arbitrary sequence $\psi_n$ such that $\psi_n \to \infty$ and $\psi'_n \to 0$ as $n \to \infty$, the set $\{e^{i\psi_n}\}$ is dense on the unit circle $\mathbb{T}$. Indeed, choose an arc $\Delta \subset \mathbb{T}$. The points $e^{i\psi_n}$ rotate around $\mathbb{T}$ and they cannot jump over $\Delta$ if $|\psi_n| < |\Delta|$. It follows that $e^{i\psi_n} \in \Delta$ for some sufficient large $n$ (actually, for an infinite number of $n$). In particular, the sequence $e^{i\psi_n}$ where $\psi_n = 4\sqrt{|\tau|n + \theta}$ cannot converge to $-1$. Now it follows from (5.9) and (5.10) that $\kappa_+ = \kappa_- = 0$ whence $F_n = 0$ according to (5.5). \qed

5.2. Supercritical case. Here we consider the case $\tau > 0$. Now the Jost solutions $\{f_n(z)\}$ and $\{f_n(z)\}$ of equation (2.5) have the same asymptotic behavior (2.7) as $n \to \infty$. Therefore their Wronskian equals zero, and hence they coincide. So, we have to find another solution $\{g_n(z)\}$ linearly independent with $\{f_n(z)\}$. Choose an arbitrary $z \in \mathbb{C}$. Asymptotics (2.7) implies that $f_n(z) \neq 0$ for sufficiently large $n$, say, $n \geq n_0 = n_0(z)$. Let us define $\{g_n(z)\}$ by the formulas

$$g_n(z) = f_n(z)G_n(z) \tag{5.11}$$

and

$$G_n(z) = \sum_{m=n_0}^{n} (a_{m-1} f_{m-1}(z) f_m(z))^{-1}, \quad n \geq n_0. \tag{5.12}$$

First, we recall an elementary assertion of a general nature.

Theorem 5.4 ([27], Theorem 4.8). Suppose that a sequence $f(z) = \{f_n(z)\}$ satisfies the Jacobi equation (2.5). Then the sequence $g(z) = \{g_n(z)\}$ defined by formulas (5.11) and (5.12) satisfies the same equation and the Wronskian $W[f(z), g(z)] = 1$. In particular, the solutions $f(z)$ and $g(z)$ are linearly independent.
It remains to find asymptotics of the sequence $g_n(z)$ as $n \to \infty$. To that end, we will integrate by parts in (5.12). It follows from relations (2.16), (4.2) and (4.21) that

$$(a_{n-1} f_{n-1} f_n)^{-1} = -(a_{n-1}(n-1) s^n u_{n-1} u_n)^{-1} e^{\varphi_{n-1} + \varphi_n} = -t_n (e^{\varphi_{n-1} + \varphi_n})'$$

where

$$t_n = (a_{n-1}(n-1) s^n)^{-1} (e^{\theta_{n-1} + \theta_n} - 1)^{-1} (u_{n-1} u_n)^{-1}. \tag{5.13}$$

**Lemma 5.5.** Sequence (5.13) satisfies relations

$$t_n = \frac{1}{2\sqrt{\tau}} + O(n^{-\varphi}), \quad t'_n = O(n^{-1/2-\varphi}).$$

**Proof.** It suffices to use formula (4.18) for the first factor in the right-hand side of (5.13), formula (4.23) – for the second factor and apply Theorem 4.4 to the third factor. \hfill \Box

Now formula (3.3) of integration by parts yields a representation for the sequence (5.12):

$$G_n = -\sum_{m=n_0}^{n} t_m (e^{\varphi_{m-1} + \varphi_m})' = -t_n e^{\varphi_n + \varphi_{n+1}} + t_{n_0-1} e^{\varphi_{n_0-1} + \varphi_{n_0}} + \tilde{G}_n \tag{5.14}$$

where

$$\tilde{G}_n = \sum_{m=n_0}^{n} t'_m e^{\varphi_{m-1} + \varphi_m}. \tag{5.15}$$

Let us consider the right-hand side of (5.14). It follows from from formula (4.18) and Lemma 5.5 that the first term has asymptotics

$$t_n e^{\varphi_n + \varphi_{n+1}} = \frac{1}{2\sqrt{\tau}} e^{4\sqrt{\tau}m} (1 + O(n^{-\varphi})), \quad n \to \infty. \tag{5.16}$$

The second term does not depend on $n$. Let us show that the remainder $\tilde{G}_n$ is also negligible.

**Lemma 5.6.** Let $\tilde{G}_n(z)$ be given by formula (5.15) where $t_n$ is defined in (5.13). Then

$$|\tilde{G}_n(z)| \leq C n^{-\varphi} e^{4\sqrt{\tau}m}. \tag{5.17}$$

**Proof.** It follows from Lemma 5.5 that

$$|\tilde{G}_n| \leq C \sum_{m=n_0}^{n} m^{-1/2-\varphi} e^{4\sqrt{\tau}m} = C \sum_{m=n_0}^{n} p_m (e^{4\sqrt{\tau}m})' \tag{5.18}$$

where

$$p_m = m^{-1/2-\varphi} (e^{4\sqrt{\tau}m} - 1)^{-1}. \tag{5.19}$$
Integrating by parts in the right-hand side of (5.18), we obtain, similarly to (5.14), that
\[ \sum_{m=n_0}^{n} m^{-1/2} e^{i \sqrt{\tau m}} = p_n e^{i \sqrt{\tau(n+1)}} - p_{n_0-1} e^{i \sqrt{\tau n_0}} - \sum_{m=n_0}^{n} p'_{m-1} e^{i \sqrt{\tau m}} \tag{5.20} \]
According to (5.19) we have
\[ p_m = O(m^{-\sigma}) \quad \text{and} \quad p'_m = O(m^{-1-\sigma}). \]
Therefore the first term in the right-hand side of (5.20) satisfies estimate (5.17) and the sum is bounded by
\[ e^{2 \sqrt{2 \tau n}} \sum_{n_0 \leq m < n/2} m^{-1-\sigma} + e^{4 \sqrt{\tau n}} \sum_{n/2 \leq m \leq n} m^{-1-\sigma}. \]
The first sum in the right-hand side is bounded and the second one is \( O(n^{-\sigma}) \).

Let us come back to the representations (5.11) and (5.14). Putting together relations (5.16) and (5.17) and using asymptotics (2.6) for \( f_n(z) \), we obtain the following result.

**Theorem 5.7.** Let the assumptions of Theorem 2.1 be satisfied, and let \( \tau > 0 \). Choose some \( z \in \mathbb{C} \). Then the solution (2.18) of the Jacobi equation (2.5) satisfies the asymptotic relation
\[ g_n(z) = -\nu^n 2^{-\sigma/2 + 1/4} e^{2 \sqrt{\tau n}} (1 + O(n^{-\sigma})), \quad n \to \infty. \]

Since \( \{g_n(z)\} \notin \ell^2(\mathbb{Z}_+) \), we can state

**Corollary 5.8.** If the assumptions of Theorem 2.1 are satisfied and \( \tau > 0 \), then the minimal Jacobi operator \( J_0 \) is essentially self-adjoint.

Set \( P(z) = \{P_n(z)\}_{n=-1}^{\infty} \), \( f(z) = \{f_n(z)\}_{n=-1}^{\infty} \) and
\[ \Omega(z) := W[P(z), f(z)] = -2^{-1} f_{-1}(z) \tag{5.21} \]
where the first formula (3.1) has been used. The Wronskian \( \Omega(z) \) is also known as the Jost function. Since \( f \in \ell^2(\mathbb{Z}_+) \), we see that \( \Omega(z) = 0 \) if and only if \( z \) is an eigenvalue of the Jacobi operator \( J = \text{clos} J_0 \). Zeros of \( \Omega(z) \) are real because \( J \) is self-adjoint. By Theorem 5.4 the Wronskian \( W[f(z), g(z)] = 1 \) so that
\[ P_n(z) = \omega(z) f_n(z) - \Omega(z) g_n(z) \tag{5.22} \]
with \( \omega(z) = W[P(z), g(z)] \). Note that \( \omega(z) \neq 0 \) if \( \Omega(z) = 0 \). Therefore Theorems 2.1 and 5.7 yield formula (2.9) where \( \kappa(z) = -\Omega(z) \). This concludes the proof of Theorem 2.2. Moreover, equality (5.22) allows us to supplement it by the following result.
Proposition 5.9. Let the assumptions of Theorem 2.1 be satisfied, and let \( \tau > 0 \). If \( \Omega(z) = 0 \), then
\[
P_n(z) = W[P(z), g(z)] \nu^n n^{-\sigma/2 + 1/4} e^{-2\sqrt{\tau n}} (1 + O(n^{-\sigma})).
\]

The resolvent of the self-adjoint operator \( J = \text{clos} J_0 \) can be constructed by the standard (cf. Lemma 2.6 in [24]) formulas. Recall that \( e_n, n \in \mathbb{Z}^+ \), is the canonical basis in the space \( \ell^2(\mathbb{Z}^+) \).

Proposition 5.10. Under the assumptions of Theorem 5.7, the resolvent \((J - z)^{-1}\) of the Jacobi operator \( J \) is given by the equalities
\[
((J - z)^{-1} e_n, e_m) = \Omega(z)^{-1} P_n(z) f_m(z), \quad \text{Im} z \neq 0,
\]
if \( n \leq m \) and \((J - z)^{-1} e_n, e_m) = ((J - z)^{-1} e_m, e_n)\). Here \( \Omega(z) \) is the Wronskian (5.21).

In view of Theorem 2.1, \( f_n(z) \) and, in particular, \( \Omega(z) \) are entire functions of \( z \in \mathbb{C} \). This allows us to state

Corollary 5.11. The spectrum of the operator \( J \) is discrete, and its eigenvalues \( \lambda_1, \cdots, \lambda_k, \ldots \) are given by the equation \( \Omega(\lambda_k) = 0 \). The resolvent \((J - z)^{-1}\) is an analytic function of \( z \in \mathbb{C} \) with poles in the points \( \lambda_1, \cdots, \lambda_k, \ldots \).

In view of formula (5.21) and equation (2.5) for \( n = 0 \), the equation for eigenvalues of \( J \) can be also written as
\[
(b_0 - \lambda_k) f_0(\lambda_k) + a_0 f_1(\lambda_k) = 0.
\]

It follows from representation (5.23) for \( n = m = 0 \) that the spectral measure of \( J \) is given by the standard formula
\[
\rho(\{\lambda_k\}) = 2 \frac{f_0(\lambda_k)}{f_{-1}(\lambda_k)}
\]
where \( f_{-1}(\lambda) \) is the derivative of \( f_{-1}(\lambda) \) in \( \lambda \). Alternatively, we have
\[
\rho(\{\lambda_k\}) = \left(\sum_{n=0}^{\infty} P_n(\lambda_k)^2\right)^{-1}.
\]

A proof of this relation can be found, for example, in [27], Sect. 4.4.

5.3. Operators with discrete spectrum. Although the discreteness of the spectrum of the Jacobi operator \( J \) was already verified in Corollary 5.11 by variational technique this result can be obtained under fairly more general assumptions. The operator \( J \) will now be defined via its quadratic form
\[
J[u, u] = (J_0 u, u) = \sum_{n=0}^{\infty} b_n |u_n|^2 + \sum_{n=0}^{\infty} (a_{n-1} u_{n-1} + a_n u_{n+1}) \bar{u}_n
\]
where $a_n$ and $b_n$ are real numbers and $J_0$ is given by matrix (1.2). The form $J[u,u]$ is defined on the set $D$, and it is real.

**Proposition 5.12.** Suppose that $b_n \rightarrow \infty$ (or $b_n \rightarrow -\infty$) and that

$$s_n := |b_n| - |a_{n-1}| - |a_n| \rightarrow \infty$$

as $n \rightarrow \infty$. Then the form (5.24) is bounded from below (from above) and closable. The spectrum of the operator $J$ corresponding to this form is discrete.

**Proof.** By the Schwarz inequality, we have

$$2\sum_{n=0}^{\infty} a_n u_{n+1} \bar{u}_n \leq \sum_{n=0}^{\infty} (|a_{n-1}| + |a_n|)|u_n|^2.$$  

(5.26)

The sum $\sum a_{n-1} u_{n-1} \bar{u}_n$ satisfies the same estimate. Suppose, for example, that $b_n \rightarrow +\infty$. Then (5.26) yields an estimate

$$J[u,u] \geq \sum_{n=0}^{\infty} s_n |u_n|^2 = (Su,u)$$  

(5.27)

where $(Sf)_n = s_n f_n$. It now follows from condition (5.25) that the form $J[u,u]$ is bounded from below. Since the operator $J_0$ is symmetric on $D$, the form $J[u,u]$ is closable and hence it gives rise to a self-adjoint operator $J$. The inequality (5.27) implies that its spectrum is discrete because the operator $S$ has discrete spectrum. \hfill \Box

Proposition 5.12 holds, in particular, for Jacobi operators when $a_n > 0$. It applies directly to the Friedrichs’ extension of the operator $J_0$, but its conclusion remains true for all extensions $J$ of $J_0$ because the deficiency indices of $J_0$ are finite. Note that condition (5.25) does not guarantee that the operator $J_0$ is essentially self-adjoint. Indeed, suppose, for simplicity, that $2b_n = \gamma \sqrt{a_{n-1}a_n}$ where $|\gamma| > 1$. It is shown in [27] (see Corollary 4.22) that the operator $J_0$ is essentially self-adjoint if and only if

$$\sum_{n=0}^{\infty} a_n^{-1} (|\gamma| + \sqrt{\gamma^2 - 1})^{2n} = \infty.$$  

However this series converges if $a_n \rightarrow \infty$ sufficiently rapidly.

If assumptions (2.2), (2.3) are satisfied with $\gamma = 1$ and $\tau$ is defined by (2.4), then

$$s_n = \tau n^{\sigma-1} (1 + O(n^{-1})).$$

Therefore condition (5.25) holds true if $\tau > 0$ and $\sigma > 1$. This concludes the proof of Theorem 2.3. Moreover, we see that the spectra of all self-adjoint extensions $J$ of the minimal operator $J_0$ are discrete for all $\sigma > 1$.

Note that Proposition 5.12 looks similar to Theorem 8 in [8], where, by some reasons, it was assumed that $a_n n^{-2} \rightarrow \infty$ as $n \rightarrow \infty$. 


Example 5.13. Recall that the spectrum of the self-adjoint Jacobi operator $J$ with the coefficients $\{\alpha_n\}$ is absolutely continuous and coincides with $[0, \infty)$. Suppose now that $a_n$ are defined by (2.1) for some $p > -1$ but $b_n - 2n \to \infty$ as $n \to \infty$. According to Proposition 5.12 the corresponding Jacobi operator $J$ has discrete spectrum.

6. Doubly critical case

Here we consider the case where the number $\tau$ defined by (2.4) is zero. We do not treat this problem in its full generality. Instead, we exhibit two classes of Jacobi operators satisfying the condition $\tau = 0$ and admitting rather effective spectral analysis. Operators from these classes can be reduced to Jacobi operators with zero diagonal elements. Probably, the scheme used in the main part of the paper works also in the doubly critical case with the Ansatz defined by formulas similar to those of Theorem 8.36 (c) of the book [23] where $\sigma$ is integer.

6.1. Dediagonalization of Jacobi operators. Let us proceed from the well known construction (see, e.g., [5]) which puts into correspondence to an arbitrary self-adjoint Jacobi operator $J$ with zero diagonal elements $b_n$ a couple of Jacobi operators $J^{(\pm)}$. It turns out that under fairly general assumptions on off-diagonal elements $a_n$ of the operator $J$, the elements $a_n^{(\pm)}$ and $b_n^{(\pm)}$ of the operators $J^{(\pm)}$ satisfy the conditions (2.2) and (2.3) with $\gamma = 1$ and $\tau = 0$.

Let $e_n, n = 0, 1, \ldots$, be the canonical basis in the space $\ell^2(\mathbb{Z}_+)$ and let

$$Je_n = a_{n-1}e_{n-1} + a_ne_{n+1}$$

(6.1)

(as usual we put $e_{-1} = 0$). Set $Ue_n = (-1)^ne_n$. Then $JU = -JU$ so that the operators $J$ and $-J$ are unitarily equivalent. In the case $b_n = 0$, the orthogonal polynomials $P_n(z)$ satisfy the identity $P_n(-z) = (-1)^nP_n(z)$.

It follows from the definition (6.1) that the operator $J^2$ acts by the formula

$$J^2e_n = a_{n-2}a_{n-1}e_{n-2} + (a_{n-1}^2 + a_n^2)e_n + a_na_{n+1}e_{n+2}.$$  

(6.2)

Although $J^2$ is not a Jacobi operator, it can be reduced to Jacobi operators $J^{(\pm)}$ on the subspaces $H^{(\pm)}$ and $H^{(-)}$ of $\ell^2(\mathbb{Z}_+)$ spanned by the elements $e_n$ with even and odd $n$, respectively. The elements $e_n^{(\pm)} := e_{2n}$ and $e_n^{(-)} := e_{2n+1}$ where $n = 0, 1, \ldots$ are the bases in the spaces $H^{(\pm)}$ and $H^{(-)}$, so that each of the subspaces $H^{(\pm)}$ can be identified with the space $\ell^2(\mathbb{Z}_+)$. According to (6.2) $H^{(\pm)}$ are the invariant subspaces of the operator $J^2$ and

$$J^2e_n^{(\pm)} = a_{n-1}^{(\pm)}e_{n-1}^{(\pm)} + b_n^{(\pm)}e_n^{(\pm)} + a_n^{(\pm)}e_{n+1}^{(\pm)}$$

(6.3)

where

$$a_n^{(\pm)} = a_{2n}a_{2n+1} \quad \text{and} \quad b_n^{(\pm)} = a_{2n-1}^2 + a_{2n}^2$$

(6.4)

and

$$a_n^{(-)} = a_{2n+1}a_{2n+2} \quad \text{and} \quad b_n^{(-)} = a_{2n}^2 + a_{2n+1}^2.$$  

(6.5)
Note that, formally,
\[ a_n^{(-)} = a_{n+1/2}^{(+)}, \quad b_n^{(-)} = b_{n+1/2}^{(+)} \, . \]

According to (6.3) the restriction \( J^{(\pm)} = J^2 \big|_{\mathcal{H}^{(\pm)}} \) of the operator \( J^2 \) on the subspace \( \mathcal{H}^{(\pm)} \) is a Jacobi operator with matrix elements (6.4) or (6.5). In particular, we see that the spectral families of the operators \( J \) and \( J^{(\pm)} \) are linked by the formula
\[ E_J(\lambda, \mu) \big|_{\mathcal{H}^{(\pm)}} = E_{J^{(\pm)}}(\lambda^2, \mu^2) \]
where \((\lambda, \mu) \in \mathbb{R}_+\) is an arbitrary interval.

Thus we are led to the following assertion.

**Lemma 6.1.** Suppose that a self-adjoint Jacobi operator \( J \) has zero diagonal elements. Let \( a_n \) be its off-diagonal elements, and let the Jacobi operator \( J^{(\pm)} \) have matrix elements (6.4) or (6.5). Let \( P_n(z) \) and \( P_n^{(\pm)}(z) \) be the orthogonal polynomials corresponding to the matrix elements \( a_n, b_n = 0 \) and \( a_n^{(\pm)}, b_n^{(\pm)} \). Then
\[ P_n^{(+)}(z) = P_{2n}(\sqrt{z}) \quad \text{(6.6)} \]
and
\[ P_n^{(-)}(z) = (a_0 \sqrt{z})^{-1} P_{2n+1}(\sqrt{z}). \quad \text{(6.7)} \]

**Proof.** Let us proceed from the Jacobi equation
\[ a_{n-1} P_{n-1}(z) + a_n P_{n+1}(z) = z P_n(z) \]
for the polynomials \( P_n(z) \). Putting together this equation with the same equations where \( n \) is replaced either by \( n - 1 \) or \( n + 1 \), we find that
\[ a_{n-1} (a_{n-2} P_{n-2}(z) + a_{n-1} P_n(z)) + a_n (a_n P_n(z) + a_{n+1} P_{n+2}(z)) = z^2 P_n(z). \quad \text{(6.8)} \]
Setting here \( n = 2m \) and using notation (6.4), we see that
\[ a_m^{(+)} P_{2m-2}(z) + b_m^{(+)} P_{2m}(z) + a_m^{(+)} P_{2m+2}(z) = z^2 P_{2m}(z). \]
This is the equation for the orthogonal polynomials \( P_n^{(+)}(z^2) \) which proves (6.6).

Quite similarly, setting \( n = 2m + 1 \) in (6.8) and using notation (6.5), we obtain relation (6.7). \( \square \)

The following example is classical. Recall that the Laguerre polynomials \( L_n^{(p)}(z) \) where the parameter \( p > -1 \) are defined by the recurrence coefficients (2.1).

**Example 6.2.** Let \( a_n = \sqrt{(n + 1)/2}, b_n = 0 \). Then \( P_n(z) =: H_n(z) \) are the Hermite polynomials. The corresponding coefficients (6.4) and (6.5) are given by the formula (2.1) where \( p = -1/2 \) and \( p = 1/2 \), respectively. Lemma 6.1 implies that
\[ H_{2n}(z) = L_n^{(-1/2)}(z^2) \quad \text{and} \quad H_{2n+1}(z) = \frac{1}{\sqrt{2}} z L_n^{(1/2)}(z^2). \]
These relations are of course very well known (see, e.g., formulas (10.13.2) and (10.13.3) in the book [7]).
The following observation shows that under fairly general assumptions on $a_n$, the asymptotic behavior of the coefficients of the operators $J(\pm)$ is doubly degenerate.

**Lemma 6.3.** Suppose that

$$a_n = (n/2)^{\sigma/2}(1 + \alpha n^{-1} + O(n^{-2})) \quad (6.9)$$

for some $\sigma \geq 0$ and $\alpha \in \mathbb{R}$. Define the coefficients $a_n^{(\pm)}$ and $b_n^{(\pm)}$ by formulas (6.4) or (6.5). These coefficients satisfy conditions (2.2) and (2.3) with $\gamma^{(\pm)} = 1$, $\sigma^{(\pm)} = \sigma$, $\alpha^{(+)} = \alpha + \sigma/4$, $\beta^{(+)} = \alpha - \sigma/4$ and $\alpha^{(-)} = \alpha + \sigma/2$, $\beta^{(-)} = \alpha$. For both signs, we have $\tau^{(\pm)} = 0$.

**6.2. Jacobi operators with zero diagonal elements.** To use the results of the previous subsection, we need some information on orthogonal polynomials satisfying relation (1.1) where $b_n = 0$. In the Carleman case (1.3), this class of orthogonal polynomials was investigated in [9, 2]. It is convenient to state necessary results in the same form as in Sect. 5.2 of [27].

**Theorem 6.4.** Suppose that $b_n = 0$ for all $n \in \mathbb{Z}_+$. Let the Carleman condition (1.3) hold and

$$\frac{a_n}{\sqrt{a_{n-1}a_{n+1}}} - 1 \in \ell^1(\mathbb{Z}_+). \quad (6.10)$$

Set

$$\theta_n = 2^{-1}(a_n a_{n-1})^{-1/2}$$

and assume that

$$\theta'_n \in \ell^1(\mathbb{Z}_+) \quad (6.11)$$

and

$$\theta^3_n \in \ell^1(\mathbb{Z}_+) \quad (6.12)$$

Let

$$\varphi_n = \sum_{m=0}^{n-1} \theta_m. \quad (6.13)$$

Then:

1. The equation (2.5) where $\pm \text{Im} z \geq 0$ has a solution $\{f_n(z)\}$ with asymptotics

$$f_n(z) = (\mp i)^n a_n^{-1/2} e^{\pm iz \varphi_n} (1 + o(1)), \quad n \to \infty. \quad (6.14)$$

In particular, $\{f_n(z)\} \in \ell^2(\mathbb{Z}_+)$ for $\text{Im} z \neq 0$. Asymptotics (6.14) is uniform in $z$ from compact subsets of the half-planes $\pm \text{Im} z \geq 0$. For all $n \in \mathbb{Z}_+$, the functions $f_n(z)$ depend analytically on $z$ for $\pm \text{Im} z > 0$ and are continuous up to the real axis $\text{Im} z = 0$. 

Let the Wronskian \( \Omega(z) \) be defined by formula (5.21). Then \( \Omega(z) \neq 0 \) for all \( z \) with \( \pm \text{Im} \ z \geq 0 \), and the asymptotic behavior as \( n \to \infty \) of the orthogonal polynomials \( P_n(z) \) is given by the relations
\[
P_n(z) = \Omega(z) \frac{(\pm i)^{n+1}}{2\sqrt{a_n}} e^{\mp iz\varphi_n}(1 + o(1)), \quad \pm \text{Im} \ z > 0, \tag{6.15}
\]
and
\[
P_n(\lambda) = -a_n^{-1/2} \left( |\Omega(\lambda+i0)| \sin(\pi n/2 - \lambda \psi_n + \arg \Omega(\lambda+i0)) + o(1) \right), \quad \lambda \in \mathbb{R}. \tag{6.16}
\]

The operator \( J = \text{clos} J_0 \) is self-adjoint, and its resolvent \( (J-z)^{-1} \) is determined by the general formula (5.23). The spectrum of the operator \( J \) is absolutely continuous, coincides with the whole real axis and the spectral measure is given by the expression
\[
d\rho(\lambda) = \pi^{-1} |\Omega(\lambda+i0)|^{-2} d\lambda.
\]

Remark 6.5. Theorem 6.4 remains essentially true for slowly increasing coefficients \( a_n \) when condition (6.12) is violated. However, formulas (6.14), (6.15) and (6.16) become more complicated in this case.

Remark 6.6. It is easy to see that under condition (2.2) where \( \sigma \in (1/3, 1] \) all assumptions of Theorem 6.4 on \( a_n \) are satisfied. In this case, we have
\[
\varphi_n = \frac{n^{1-\sigma}}{2(1-\sigma)} + C_\sigma + o(1) \quad \text{if } \sigma < 1 \quad \text{and} \quad \varphi_n = 2^{-1} \log n + C_1 + o(1) \quad \text{if } \sigma = 1
\]
for some constants \( C_\sigma \). This allows us to simplify formulas (6.14), (6.15) and (6.16). For example, (6.14) where \( \pm \text{Im} \ z \geq 0 \) yields
\[
f_n(z) = (\mp i)^n n^{-\sigma/2} \exp \left( \pm i z \frac{n^{1-\sigma}}{2(1-\sigma)} \right) \left( 1 + o(1) \right) \quad \text{if } \sigma < 1,
\]
and
\[
f_n(z) = (\mp i)^n n^{(-1+\sigma)/2} \left( 1 + o(1) \right) \quad \text{if } \sigma = 1.
\]
Similar simplifications can be made in formulas (6.15) and (6.16) for the orthogonal polynomials.

Example 6.7 (Stieltjes-Carlitz polynomials). One of the families of Stieltjes-Carlitz polynomials is defined (see formula (9.3) and (9.4) in the book [4], Chapter VI, Sect. 9) by the recurrence coefficients \( b_n = 0 \) and
\[
a_n = k(n+1) \quad \text{for } n \text{ even} \quad \text{and} \quad a_n = n+1 \quad \text{for } n \text{ odd} \tag{6.17}
\]
where a parameter \( k > 0 \). Theorem 6.4 applies for \( k = 1 \); in this case the spectrum of the self-adjoint Jacobi operator \( J = \text{clos} J_0 \) is absolutely continuous and coincides with the whole real axis. On the contrary, if \( k \neq 1 \), then (see [4], Chapter VI, Sect. 9) the spectrum of the operator \( J \) is discrete and consists of eigenvalues \( c(k)(j+1/2) \) if \( k < 1 \) and eigenvalues \( c(k)j \) if \( k > 1 \); here \( j \in \mathbb{Z} \) and \( c(k) \) are some constants.
Note that for $k \neq 1$ conditions (6.10) and (6.11) are violated. Example (6.17) shows that these conditions cannot be omitted in Theorem 6.4.

This example exhibits a curious phenomenon: the absolutely continuous spectrum of $J$ for $k = 1$ is transformed into a purely discrete spectrum by an arbitrary small perturbation of $k$.

Formulas (6.15) and (6.16) are of course consistent with the classical asymptotic expressions for the Hermite polynomials when $a_n = \sqrt{(n+1)/2}$ and $b_n = 0$ (see, e.g., Theorems 8.22.6 and 8.22.7 in the G. Szegő’s book [20]).

The Carleman condition (1.3) is not very important here. Under assumption (1.5) the sequence (6.13) has a finite limit as $n \to \infty$, and hence it can be omitted in asymptotic formulas of Theorem 6.4. On the other hand, spectral results are drastically different in these cases. The following assertion is a particular case of Theorems 3.9 and 4.2 in [27].

**Theorem 6.8.** Let $b_n = 0$. Suppose that conditions (1.5) and (6.10) are satisfied. Then:

1° For all $z \in \mathbb{C}$, the equation (2.5) has a solutions $\{f_n(z)\}$ with asymptotics

$$f_n(z) = (-i)^n a_n^{-1/2} (1 + o(1)), \quad n \to \infty.$$  

The sequence $\tilde{f}_n(z) = \overline{f_n(\overline{z})}$ also satisfies (2.5). The functions $f_n(z)$ and $\tilde{f}_n(z)$ are analytic in the whole complex plane $\mathbb{C}$. Both solutions $\{f_n(z)\}, \{\tilde{f}_n(z)\}$ are in $\ell^2(\mathbb{Z}^+)$. 

2° The asymptotic behavior as $n \to \infty$ of the orthogonal polynomials $P_n(z)$ is given by the relation

$$P_n(z) = \frac{1}{\sqrt{a_n}} (\kappa_+(z)(-i)^n + \kappa_-(z)i^n) (1 + o(1))$$ 

for some constants $\kappa_+(z) \in \mathbb{C}$ and $\kappa_-(z) = \overline{\kappa_+(\overline{z})}$.

3° The symmetric operator $J_0$ has a one parameter family of self-adjoint extensions $J$. All operators $J$ have discrete spectra.

### 6.3. Regular and singular cases.

Here we combine the results of Lemma 6.1 and Theorems 6.4 or 6.8. We do not discuss the Jost solutions and state asymptotic results in terms of the orthogonal polynomials only. Let us distinguish regular and singular cases. The first result, for the regular case, follows from Theorem 6.4.

**Theorem 6.9.** Let condition (6.9) be satisfied with some $\sigma \in (2/3, 2]$. Define the coefficients $a_n^{(\pm)}$ and $b_n^{(\pm)}$ by one of the formulas (6.4) or (6.5). Let $J^{(\pm)}$ be the Jacobi operators with these coefficients, and let $P_n^{(\pm)}(z)$ be the corresponding orthogonal polynomials. Then:

1° If $z \in \mathbb{C} \setminus [0, \infty)$ and $\text{Im} \sqrt{z} > 0$, then

$$P_n^{(\pm)}(z) = \kappa^{(\pm)}(z)(-1)^n n^{-\sigma/4} \exp \left( -i \sqrt{z} \frac{n^{1-\sigma/2}}{2 - \sigma} \right) (1 + o(1)), \quad \sigma < 2,$$
and
\[ P_n^{(\pm)}(z) = \kappa^{(\pm)}(z)(-1)^nn^{-1/2-i\sqrt{z}}(1+o(1)) , \quad \sigma = 2 , \] (6.18)
as \( n \to \infty \) for some constants \( \kappa^{(\pm)}(z) \).

2. If \( \lambda \geq 0 \), then
\[ P_n^{(\pm)}(\lambda) = \kappa^{(\pm)}(\lambda)(-1)^nn^{-\sigma/4}\left(\sin\left(\sqrt{\lambda}n^{1-\sigma/2} + \eta^{(\pm)}(\lambda)\right) + o(1)\right), \quad \sigma < 2 , \] and
\[ P_n^{(\pm)}(\lambda) = \kappa^{(\pm)}(\lambda)(-1)^nn^{-\sigma/4}\left(\sin\left(\sqrt{\log n + \eta^{(\pm)}(\lambda)}\right) + o(1)\right), \quad \sigma = 2 , \] (6.19)
as \( n \to \infty \) for some constants \( \kappa^{(\pm)}(\lambda) \) and \( \eta^{(\pm)}(\lambda) \).

3. The operators \( J_0^{(\pm)} \) are essentially self-adjoint on the set \( D \). The spectra of their closures are absolutely continuous and coincide with \([0, \infty)\).

Remark 6.10. According to Lemma 6.3 condition (6.9) ensures that the coefficients \( a^{(\pm)}_n \) and \( b^{(\pm)}_n \) satisfy relations (2.2) and (2.3) with \( \gamma^{(\pm)} = 1 \), \( \sigma^{(\pm)} = \sigma \) and \( \tau^{(\pm)} = 0 \).

We emphasize that, in the doubly critical case, the role of the Carleman condition is played by the assumption \( \sigma \leq 2 \). Asymptotic phases in formulas of Theorem 6.9 depend on the spectral parameter. So, we are in the regular situation here.

Remark 6.11. We emphasize that Theorem 6.9 yields asymptotics as \( n \to \infty \) of the orthogonal polynomials \( P_n(z) \) uniformly in a neighborhood of the point \( z = 0 \). This is a very strong but specific result which cannot be generically true. For example, for the Laguerre polynomials \( L_n^{(p)}(z) \) it is true for \( p = -1/2 \) and \( p = 1/2 \) only. For other values of \( p \), the asymptotics of \( L_n^{(p)}(z) \) in a neighborhood of the point \( z = 0 \) is given by a more complicated Hilbs formula (see, e.g., formula (10.15.2) in the book [7]).

Example 6.12. Let
\[ a_{2n-1} = a_{2n} = \sqrt{n+1} \] (6.20)
for all \( n \in \mathbb{Z}_+ \). Then
\[ a^{(+)n} = \sqrt{(n+1)(n+2)} , \quad b^{(+)n} = 2n+2 \]
and the corresponding orthogonal polynomials are Laguerre polynomials \( L_n^{(1)}(z) \).

Therefore formulas (6.14), (6.15) and (6.16) are not true in this case for \( z = 0 \). This does not contradict Theorem 6.4 because condition (6.10) is violated for the coefficients (6.20).

Example 6.13. Let
\[ a_n = \sqrt{(n+1)(n+x)(n+y)(n+x+y)} \quad \text{and} \quad b_n = 2n^2 + (2x+2y-1)n + xy \] (6.21)
where parameters \( x, y \in \mathbb{R}_+ \). The corresponding orthogonal polynomials are known as continuous dual Hahn polynomials. Now assumptions (2.2) and (2.3) are satisfied...
with \( \gamma = 1, \sigma = 2, \alpha = x + y + 1/2 \) and \( \beta = x + y - 1/2 \) so that \( \tau = 0 \) and we are in the doubly critical case. Note that relations (6.4) hold true for the coefficients \( a_n^{(+)} = a_n \) and \( b_n^{(+)} = b_n \) defined by (6.21) if
\[
a_{2n} = \sqrt{(n + x)(n + y)} \quad \text{and} \quad a_{2n+1} = \sqrt{(n+1)(n+x+y)}.
\]
(6.22)
Since
\[
a_n = 2^{-1} n \left( 1 + (x + y)n^{-1} + O(n^{-2}) \right)
\]
both for even and odd \( n \in \mathbb{Z}_+ \), the conditions of Theorem 6.4 are satisfied for the coefficients (6.22). Therefore asymptotics of the continuous dual Hahn polynomials are given by formulas (6.18) and (6.19) and hence the regular case occurs. This is consistent with the condition \( \sigma = 2 \). The Jacobi operators \( J_0 \) with matrix elements (6.21) are essentially self-adjoint on the set \( \mathcal{D} \), and their closures have absolutely continuous spectra coinciding with \([0, \infty)\).

In the singular doubly critical case, we have the following result. It is a consequence of Theorem 6.8.

**Theorem 6.14.** Let condition (6.9) be satisfied with some \( \sigma > 2 \). Define the coefficients \( a_n^{(\pm)} \) and \( b_n^{(\pm)} \) by one of the formulas (6.4) or (6.5). Let \( J^{(\pm)} \) be the Jacobi operators with these coefficients, and let \( P_n^{(\pm)}(z) \) be the corresponding orthogonal polynomials. Then:

1. The asymptotic behavior as \( n \to \infty \) of the orthogonal polynomials \( P_n^{(\pm)}(z) \) is given by the relation
\[
P_n^{(\pm)}(z) = \kappa^{(\pm)}(z)(-1)^n n^{-\sigma/4} \left( 1 + o(1) \right)
\]
for some constants \( \kappa^{(\pm)}(z) \in \mathbb{C} \).

2. The operators \( J_0^{(\pm)} \) have deficiency indices \( (1, 1) \). All their self-adjoint extensions \( J^{(\pm)} \) have discrete spectra.

6.4. **Comments.** Let us come back to Figure 1 and summarize the results obtained. Suppose that assumptions (2.2), (2.3) are satisfied.

In the non-critical situation \( |\gamma| \neq 1 \), the singular case occurs if \( \sigma > 1 \). The corresponding Jacobi operators \( J_0 \) are essentially self-adjoint for \( |\gamma| > 1 \), and they have deficiency indices \( (1, 1) \) for \( |\gamma| < 1 \). In both cases the spectra of self-adjoint extensions of \( J_0 \) are discrete.

In the critical situation where \( |\gamma| = 1 \) but \( \tau \neq 0 \), the singular case occurs if \( \sigma > 3/2 \). The corresponding Jacobi operators \( J_0 \) are essentially self-adjoint for \( \tau > 0 \), and they have deficiency indices \( (1, 1) \) for \( \tau < 0 \). In both cases the spectra of self-adjoint extensions of \( J_0 \) are discrete. Surprisingly, this is, in general, no longer true in the intermediary case \( \tau = 0 \).

In the doubly critical situation where \( |\gamma| = 1 \) and \( \tau = 0 \), the singular case occurs if \( \sigma > 2 \). Under the assumptions of Theorem 6.14, the corresponding operators \( J_0 \) have deficiency indices \( (1, 1) \) and the spectra of their self-adjoint extensions are
discrete. According to Theorem 6.9 we are in the regular case if $\sigma \leq 2$. The corresponding operators $J_0$ are essentially self-adjoint and have absolutely continuous spectra coinciding with the half-axis $[0, \infty)$ if $\gamma = 1$ (or with the half-axis $(-\infty, 1]$ if $\gamma = -1$).

We emphasize that in the doubly critical case, the condition $\sigma > 2$ plays the role of $\sigma > 3/2$ in the (simply) critical case and of $\sigma > 1$ in the non-critical case.

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