Abstract: We briefly describe some of our recent results for the mass spectrum and matrix elements using \(O(a)\) improved fermions for quenched \(QCD\). Where possible a comparison is made between improved and Wilson fermions.

1 Introduction

Upon discretising the \(QCD\)-Lagrangian in the Wilson formulation we find that the gluon piece has \(O(a^2)\) errors, but for the fermion part, the additional Wilson term, necessary to avoid the fermion ‘doubling’ problem gives us \(O(a)\) errors. Thus we would expect that looking at any physical quantity, for example a mass ratio, we have

\[
\frac{m_H}{m_{H'}} = r_0 + a r_1 + a^2 r_2 + O(a^3).
\]

(1)

Symanzik developed a systematic perturbative programme to \(O(a^n)\) in which a basis of irrelevant operators is added to the Lagrangian to completely remove \(O(a^{n-1})\) effects. Restricting this to on-shell (or physical) quantities, \[\] enables the equations of motion to be used to reduce the required set of operators, in both the action and for improved operators in matrix elements. In this talk we briefly report our progress

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on computing the hadron spectrum and matrix elements for $O(a)$ improved fermions, where in this case we expect for a physical quantity such as a mass ratio

$$\frac{m_H}{m_{H'}} = r_0 + a^2 r_2' + O(a^3).$$

(2)

The ground-work has been laid by the Alpha collaboration who succeeded in non-perturbatively calculating several improvement coefficients for $\beta \equiv 6.0/g^2 \geq 6.0$. Reviews and further references may be found in [2]. We shall here neither describe their method nor go into details of how the mass spectrum or matrix elements are computed on the lattice. A description of our spectrum results is given in ref. [3] while for Wilson matrix elements see, for example, [4]. Up to now we have generated configurations on $(16^3, 24^3) \times 32$ lattices at $\beta = 6.0$ and on $24^3 \times 48$ lattices at $\beta = 6.2$. We have used the string tension, $\sqrt{\kappa}$, to determine the lattice spacing $a$ at these two $g^2$ values. A physical value of $\sqrt{\kappa} = 0.427$GeV is taken to set the scale.

2 The mass spectrum

We have computed the $\rho$ ($J^{PC} = 1^{--}$) and nucleon ($N$) masses together with the $a_0$ ($0^{++}$), $a_1$ ($1^{++}$) and $b_1$ ($1^{+-}$) masses. The latter three mesons being $p$-wave states are difficult to measure with our symmetric sources and our results should only be taken as indicative of possible trends. In Fig. 1 we plot this hadron spectrum

![Figure 1: The hadron spectrum for light hadrons at $\beta = 6.0$ (circles), 6.2 (squares) using Wilson fermions (left picture) and using $O(a)$ improved fermions (right picture) against the lattice spacing, $a$. Experimental numbers are shown as stars.](image)

both for Wilson and $O(a)$ improved fermions. Roughly the same amount of CPU time has gone into producing each of these pictures. From eqs. (1,2) we expect that the dominant discretisation terms are such that linear extrapolations in $a$ for Wilson fermions and $a^2$ for $O(a)$ improved fermions are sufficient. Clearly with
only two points we must limit ourselves here to qualitative observations. First we note that due to the absence of the $O(a)$ term for the improved fermions, the convergence to the continuum limit is faster. On the other hand it does appear that, especially for heavier particles, the signal fluctuates more. It also seems that at least for the $\rho$ and $N$ particles, $O(a)$ improved fermions results lie closer to the continuum result – the Wilson results have a noticeable gradient. An alternative way of looking at these results is given in Fig. 2. While it is not necessary that quenched $QCD$ should reproduce exactly the continuum spectrum, in many present-day applications a hadron mass (typically the $\rho$ or $N$) is used to set the scale. Potential discrepancies between these scales can be revealed by normalising them against the string tension (which has $O(a^2)$ errors). Again this confirms the previous impression: $O(a)$ improved fermions for the $\rho$ or $N$ perform better than Wilson fermions.

3 Matrix elements

Matrix elements, such as decay constants or moments of structure functions, are more complicated to calculate than masses as the operators must be appropriately renormalised, so that $O_R = Z_O (1 + b_O am_q) O$, with $O = O + \sum_I c_I a O_I$. For present-day $\beta$ values, non-perturbative (NP) determinations of $Z_O$, $b_O$ and $\{c_I\}$ are preferable. (Here we are mainly interested in matrix elements in the chiral limit, so we do not need $b_O$; the other irrelevant operators $O_I$ are only required in the improved case to ensure complete $O(a)$ cancellation.) In Fig. 3 we show $Z_V$ ($V_\mu = \overline{q} \gamma_\mu q$) and $Z_A$ ($A_\mu = \overline{q} \gamma_\mu \gamma_5 q$) for both Wilson and $O(a)$ improved fermions. We see that in all

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig2.png}
\caption{$a_H^{-1}$ estimates using $H = \rho$ or $N$ as a scale normalised to the string tension for Wilson fermions (circles) compared to $O(a)$ improved masses (squares).}
\end{figure}
cases, while first order perturbation theory lies $\geq 10\%$ away from NP computations, tadpole improvement (TI) always gives results closer to the NP line. For the Wilson case we expect to find larger discrepancies (of $O(a)$) between the various NP estimates of $Z$ than for improved fermions (of $O(a^2)$). This seems most obvious for $Z_V$. We next note that both the Alpha, [10], and our non-perturbative determination of $Z_V$ for $O(a)$ improved fermions are in very good agreement with each other. From current conservation we expect $\langle N|V_{\mu}^R|N\rangle = p_\mu / E_N \chi_q$ with $\chi_u = 2$, $\chi_d = 1$. This at (eg) zero momentum allows a non-perturbative determination of $Z_V$ (and $b_V$), for both Wilson and $O(a)$ improved fermions. This result is exemplified in Fig. 4 where

Figure 3: Various determinations of $Z_V$ (upper row) and $Z_A$ (lower row) for Wilson (left column) and $O(a)$ improved fermions (right column). The tadpole improved (TI) results have been obtained from first order perturbation results, $Z_O(g^2) = 1 + c_O g^2 + O(g^4)$, [7], using the procedure given in [8]. Also plotted are results using the local vector current between nucleon, [6] (filled squares) and $\rho$ states, [8] (filled circles), and the local axial current between quark states, [12] (filled squares). Where appropriate, a simple Padé interpolation/extrapolation has been applied, $Z_O(g^2) = (1 + p_O g^2 + q_O g^4)/(1 + r_O g^2)$ with $p_O - r_O = c_O$. For the $O(a)$ improved fermions we also give the Alpha results (filled circles) and Padé extrapolation, [10] and from [11] (open circle).
Figure 4: $\langle N|V_R^4|N \rangle$, left picture, $\langle N|V_R^1|N \rangle \times E_N/p_1$, right picture, at $\vec{p} = (p_1, 0, 0)$ with $p_1$ being the minimum possible momenta, namely $p_1 = 2\pi/16a \approx 765\text{MeV}$ or $2\pi/24a \approx 694\text{MeV}$ at $\beta = 6.0, 6.2$ respectively. The expected values are $\chi_u = 2, \chi_d = 1$. Filled circles, squares denote $O(a)$ improved fermions at $\beta = 6.2, 6.0$ respectively, while open squares are for $\beta = 6.0$ Wilson fermions. To guide the eye, constant fits are made to the chiral limit for the right hand picture.

we perform a check of momentum effects and restoration of rotational invariance, by computing $\langle N|V_R^4|N \rangle$, $\langle N|V_R^1|N \rangle \times E_N/p_1$. While all results for $\langle N|V_R^4|N \rangle$ are consistent with each other, $\langle N|V_R^1|N \rangle \times E_N/p_1$ shows some spread; indeed at $\beta = 6.0$, Wilson fermions seem to perform better than the $O(a)$ improved fermions.

In the left picture of Fig. 5 we apply the Alpha result for $Z_A$ to a determination of $g_A$. There seems to be an approach to the experimental value of $g_A$ (but with rather large $O(a^2)$ corrections). For the moments of the quark distributions in the nucleon, we obtain the results shown in the right hand picture in Fig. 5. The lowest moment, $\langle x \rangle$, derived from the operator $\overline{q}\gamma_\mu D_\nu q$ is interesting, not only because it contains one derivative, but also because there are two distinct lattice representations involving diagonal or off-diagonal elements, the latter requiring for its evaluation a non-zero spatial momentum. (Again we always used the minimum possible.) This potentially allows a better study of the approach to the continuum limit. At present a non-perturbative renormalisation of $\langle x \rangle$ is not known; however no tadpole term contributes to first order perturbation theory, so that this automatically lies close to the $TI$ result, so we might hope that any error in the renormalisation constant is small. A further problem for the $O(a)$ improved operators is the addition of at least one irrelevant operator. However, as discussed in 8, we hope that its effect is also small. For $\langle x \rangle$, from Fig. 5, all results appear to be reasonably consistent with each other, and for $O(a)$ improved fermions there does not seem to be any large $O(a^2)$ effect. However, there seems to be a larger deterioration of the signal (in comparison to the Wilson case) when introducing a momentum into the operator.

In conclusion, it would seem that $O(a)$ Symanzik improvement does indeed bring us closer to the goal of calculating continuum masses and matrix elements in QCD. The numerical calculations were performed on the Quadrics facility at DESY-IfH.
Figure 5: $g_A = \Delta u - \Delta d$, left picture, and $\langle x^n \rangle^{(u)} - \langle x^n \rangle^{(d)}$ moments at a scale of $\mu \sim 1.95$ GeV, right picture. The phenomenological results are denoted with stars ($g_A = 1.26$ and $\langle x^n \rangle^{(u)} - \langle x^n \rangle^{(d)}$ by the MRS parametrisation). $O(a)$ improved fermion results are given by filled symbols. For $\langle x \rangle$, results using a zero-momentum diagonal matrix element are given by a square while the non-zero momentum off-diagonal results are denoted with a circle.

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