UNBOUNDED WEIGHTED CONDITIONAL TYPE OPERATORS
ON $L^p(\Sigma)$

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Abstract. In this paper we consider unbounded weighted conditional type operators on the space $L^p(\Sigma)$, we give some conditions under which they are densely defined and we obtain a dense subset of the domain. Also, we get that a WCT operator is continuous if and only if it is every where defined. A description of polar decomposition, spectrum and spectral radius in this context are provided. Finally, we investigate hyperexpansive WCT operators on the Hilbert space $L^2(\Sigma)$. As a consequence hyperexpansive multiplication operators are investigated.

1. Introduction

In this paper we consider a class of unbounded linear operators on $L^p$-spaces having the form $M_wEM_u$, where $E$ is a conditional expectation operator and $M_u$ and $M_w$ are multiplication operators. What follows is a brief review of the operators $E$ and multiplication operators, along with the notational conventions we will be using.

Let $(\Omega, \Sigma, \mu)$ be a $\sigma$-finite measure space and let $A$ be a $\sigma$-subalgebra of $\Sigma$ such that $(\Omega, A, \mu)$ is also $\sigma$-finite. We denote the collection of (equivalence classes modulo sets of zero measure) $\Sigma$-measurable complex-valued functions on $\Omega$ by $L^0(\Sigma)$ and the support of a function $f \in L^0(\Sigma)$ is defined as $S(f) = \{t \in \Omega; f(t) \neq 0\}$. Moreover, we set $L^p(\Sigma) = L^p(\Omega, \Sigma, \mu)$. We also adopt the convention that all comparisons between two functions or two sets are to be interpreted as holding up to a $\mu$-null set. For each $\sigma$-finite subalgebra $A$ of $\Sigma$, the conditional expectation, $E_A(f)$, of $f$ with respect to $A$ is defined whenever $f \geq 0$ or $f \in L^p(\Sigma)$. In any case, $E_A(f)$ is the unique $A$-measurable function for which

$$\int_A f d\mu = \int_A E_A f d\mu, \quad \forall A \in A.$$

As an operator on $L^p(\Sigma)$, $E^A$ is an idempotent and $E^A(L^p(\Sigma)) = L^p(A)$. If there is no possibility of confusion we write $E(f)$ in place of $E^A(f)$ [10] [12]. This operator will play a major role in our work and we list here some of its useful properties:

- If $g$ is $A$-measurable, then $E(fg) = E(f)g$.
- $|E(f)|^p \leq E(|f|^p)$.
- If $f \geq 0$, then $E(f) \geq 0$; if $f > 0$, then $E(f) > 0$.
- $|E(fg)| \leq E(|f|^p)^{\frac{1}{p}}E(|g|^q)^{\frac{1}{q}}$, (Hölder inequality) for all $f \in L^p(\Sigma)$ and $g \in L^q(\Sigma)$, in which $\frac{1}{p} + \frac{1}{q} = 1$.

2010 Mathematics Subject Classification. 47B25, 47B38.

Key words and phrases. Conditional expectation, unbounded operators, hyperexpansive operators.
For each \( f \geq 0 \), \( S(f) \subseteq S(E(f)) \).

Let \( u \in L^0(\Sigma) \). The corresponding multiplication operator \( M_u \) on \( L^p(\Sigma) \) is defined by \( f \rightarrow uf \). Our interest in operators of the form \( M_uEM_u \) stems from the fact that such products tend to appear often in the study of those operators related to conditional expectation. This observation was made in [1, 2, 5, 8, 9]. In this paper, first we investigate some properties of unbounded weighted conditional type operators on \( L^p(\Sigma) \) and then we study hyperexpansive ones.

2. Unbounded weighted conditional type operators

Let \( X \) stand for a Banach space and \( B(X) \) for the Banach algebra of all linear operators on \( X \). By an operator in \( X \) we understand a linear mapping \( T : D(T) \subseteq X \rightarrow X \) defined on a linear subspace \( D(T) \) of \( X \) which is called the domain of \( T \). The linear map \( T \) is called densely defined if \( D(T) \) is dense in \( X \) and it is called closed if its graph \( (G(T)) \) is closed in \( X \times X \), where \( G(T) = \{(f, Tf) : f \in D(T)\} \). We studied bounded weighted conditional type operators on \( L^p \)-spaces in [4]. Also we investigated unbounded weighted conditional type operators of the form \( EM_u \) on the Hilbert space \( L^2(\Sigma) \) in [3]. Here we investigate unbounded weighted conditional type operators of the form \( M_uEM_u \) on \( L^p \)-spaces. Let \( f \) be a positive \( \Sigma \)-measurable function on \( \Omega \). Define the measure \( \mu_f : \Sigma \rightarrow [0, \infty) \) by

\[
\mu_f(E) = \int_E f d\mu, \quad E \in \Sigma.
\]

It is clear that the measure \( \mu_f \) is also \( \sigma \)-finite, since \( \mu \) is \( \sigma \)-finite. From now on we assume that \( u \) and \( w \) are conditionable (i.e., \( E(u) \) and \( E(w) \) are defined). Operators of the form \( M_uEM_u(f) = wE(u.f) \) acting in \( L^p(\mu) \) with \( D(M_uEM_u) = \{ f \in L^p(\mu) : wE(u.f) \in L^p(\mu) \} \) are called weighted conditional type operators (or briefly WCT operators). In the first proposition we give a condition under which the WCT operator \( M_uEM_u \) is densely defined.

**Theorem 2.1.** For \( 1 \leq p, q < \infty \) such that \( \frac{1}{p} + \frac{1}{q} = 1 \) and \( E(|u|^p)^\frac{1}{p}E(|u|^q)^\frac{1}{q} < \infty \) a.e., the linear transformation \( M_uEM_u \) is densely defined.

**Proof.** For each \( n \in \mathbb{N} \), define

\[
A_n = \{ t \in \Omega : n - 1 \leq E(|w|^p)(t)E(|u|^q)^\frac{1}{q}(t) < n \}.
\]

It is clear that each \( A_n \) is \( \mathcal{A} \)-measurable and \( \Omega \) is expressible as the disjoint union of sets in the sequence \( \{A_n\}_{n=1}^{\infty} \), \( \Omega = \cup_{n=1}^{\infty} A_n \).

Let \( f \in L^p(\Sigma) \) and \( \epsilon > 0 \). Then, there exists \( N > 0 \) such that

\[
\int_{\cup_{n=N}^{\infty} A_n} |f|^p d\mu = \sum_{n=N}^{\infty} \int_{A_n} |f|^p d\mu < \epsilon.
\]

Define the sets

\[
B_N = \cup_{n=N}^{\infty} A_n, \quad C_N = \cup_{n=1}^{N-1} A_n.
\]

Then, \( \int_{B_N} |f|^p d\mu < \epsilon \) and \( C_N = \{ \{ t \in \Omega : E(|w|^p)(t)E(|u|^q)^\frac{1}{q}(t) < N - 1 \} \} \). Next, we define \( g = f \cdot \chi_{C_N} \). Clearly \( g \in L^p(\Sigma) \) and \( E(g) = E(f) \cdot \chi_{C_N} \). Now, we show
that $g \in \mathcal{D} = \mathcal{D}(M_wEM_n)$. Consider the following:

\[
\int_\Omega |wE(ug)|^p \, d\mu = \int_\Omega |wE(uf)\chi_{C_n}|^p \, d\mu \\
= \int_{C_n} E(|w|^p)|E(uf)|^p \, d\mu \\
\leq \int_{C_n} E(|w|^p)E(|u|^q)\frac{p}{q} |f|^p \, d\mu \\
\leq (N-1) \int_{C_n} |f|^p \, d\mu < \infty.
\]

Thus, $wE(uf) \in L^p(\Sigma)$. Now, we show that $\|g-f\|_p < \epsilon$:

\[
\|g-f\|_p^p = \int_{\mathcal{X}} |g-f|^p \, d\mu \\
= \int_{C_n} |g-f|^p \, d\mu < \epsilon.
\]

Thus, $\mathcal{D}$ is dense in $L^p(\Sigma)$. \hfill \Box

Here we obtain a dense subset of $L^p(\mu)$ that we need it to proof our next results.

**Lemma 2.2.** Let $1 < p, q < \infty$ such that $\frac{1}{p} + \frac{1}{q} = 1$, $J = 1 + E(|w|^p)E(|u|^q)\frac{1}{q}$, $E(|w|^p)^{\frac{1}{p}}E(|u|^q)^{\frac{1}{q}} < \infty \ a.e., \ \mu$, and $d\nu = J \, d\mu$. We get that $S(J) = \Omega$ and

(i) $L^p(\nu) \subseteq \mathcal{D}(M_wEM_n)$,

(ii) $\| \cdot \|_\mu = \mathcal{D}(M_wEM_n)^{\nu} = L^p(\mu)$.

**Proof.** Let $f \in L^p(\nu)$. Then

\[
\|f\|_p^p \, d\mu \leq \|f\|_p^p < \infty,
\]

so $f \in L^p(\mu)$. Also, by conditional-type Hölder-inequality we have

\[
\|M_wEM_n(f)\|_p^p \, d\mu \leq \int_{\Omega} E(|w|^p)E(|u|^q)^{\frac{1}{q}} E(|f|^p) \, d\mu \\
= \int_{\mathcal{X}} E(|w|^p)E(|u|^q)^{\frac{1}{q}} |f|^p \, d\mu \\
\leq \|f\|_p^p < \infty,
\]

this implies that $f \in \mathcal{D}(M_wEM_n)$. Now we prove that $L^p(\nu)$ is dense in $L^p(\mu)$. By Riesz representation theorem we have

\[
(L^p(\nu))^\perp = \{g \in L^p(\mu) : \int_{\Omega} f \, g \, d\mu = 0, \ \forall f \in L^p(\nu)\}.
\]

Suppose that $g \in (L^p(\nu))^\perp$. For $A \in \Sigma$ we set $A_n = \{t \in A : J(t) \leq n\}$. It is clear that $A_n \subseteq A_{n+1}$ and $\Omega = \cup_{n=1}^{\infty} A_n$. Also, $\Omega$ is $\sigma$-finite, hence $\Omega = \cup_{n=1}^{\infty} \Omega_n$ with $\mu(\Omega_n) < \infty$. If we set $B_n = A_n \cap \Omega_n$, then $B_n \not\subseteq A$ and so $g \chi_{B_n} \not\subseteq g \chi_A \ a.e. \ \mu$. Since $\nu(B_n) \leq (n+1)\mu(B_n) < \infty$, we have $\chi_{B_n} \in L^p(\nu)$ and by our assumption $\int_{B_n} f \, d\mu = 0$. Therefore by Fatou’s lemma we get that $\int_{A} g \, d\mu = 0$. Thus for all $A \in \Sigma$ we have $\int_{A} g \, d\mu = 0$. This means that $g = 0 \ a.e. \ \mu$ and so $L^p(\nu)$ is dense in $L^p(\mu)$. \hfill \Box
By the Lemma [2.2] we get that $L^p(\nu)$ is a core of $M_uEM_u$. Here we give a condition that we will use in the next theorem.

(\*) If $(\Omega, A, \mu)$ is a $\sigma$-finite measure space and $J - 1 = \{E(|u|^s)\}^{1/p} E(|u|^{q}) \leq \infty$ a.e. $\mu$, then there exists a sequence $\{A_n\}_{n=1}^{\infty} \subseteq A$ such that $\mu(A_n) < \infty$ and $J - 1 < n$ a.e. $\mu$ on $A_n$ for every $n \in \mathbb{N}$ and $A_n \not\subset \Omega$ as $n \to \infty$.

**Theorem 2.3.** If $u, w : \Omega \to \mathbb{C}$ are $\Sigma$-measurable and $1 < p, q < \infty$ such that $\frac{1}{p} + \frac{1}{q} = 1$, then the following conditions are equivalent:

(i) $M_uEM_u$ is densely defined on $L^p(\Sigma)$,

(ii) $J - 1 = E(|u|^p)(E(|u|^{q}))^{\frac{p}{q}} \leq \infty$ a.e., $\mu$.

(iii) $\mu_{J-1} |A$ is $\sigma$-finite.

**Proof.** (i) $\to$ (ii) Set $E = \{E(|u|^p)(E(|u|^{q}))^{\frac{p}{q}} = \infty\}$. Clearly by Lemma (i), $f |_{E} = 0$ a.e., $\mu$ for every $f \in L^p(\nu)$. This implies that $fJ |_{E} = 0$ a.e., $\mu$ for every $f \in L^p(\mu)$. So we have $J \chi_{A \cap E} = 0$ a.e., $\mu$ for all $A \subseteq \Sigma$ with $\mu(A) < \infty$. By the $\sigma$-finiteness of $\mu$ we have $J \chi_{E} = 0$ a.e., $\mu$. Since $S(J) = \Omega$, we get that $\mu(E) = 0$.

(ii) $\to$ (i) Evident.

(ii) $\to$ (iii) Let $\{A_n\}_{n=1}^{\infty}$ be in (\*). We have

$$\mu_{J-1} |A (A_n) = \int_{A_n} E(|u|^p)(E(|u|^{q}))^{\frac{p}{q}} d\mu \leq n \mu(A_n) < \infty, \quad n \in \mathbb{N}.$$ 

This yields (iii).

(iii) $\to$ (i) Let $\{A_n\}_{n=1}^{\infty} \subseteq A$ be a sequence such that $A_n \not\subset \Omega$ as $n \to \infty$ and $\mu_{J-1} |A (A_n) < \infty$ for every $k \in \mathbb{N}$. It follows from the definition of $\mu_{J-1}$ that $J - 1 = E(|u|^p)(E(|u|^{q}))^{\frac{p}{q}} < \infty$ a.e., $\mu$ on $A$. Applying Theorem 2.1 we obtain (i).

Let $X, Y$ be Banach spaces and $T : X \to Y$ be a linear operator. If $T$ is densely defined, then there is a unique maximal operator $T^*$ from $D(T^*) \subset Y^*$ into $X^*$ such that

$$y^*(Tx) = \langle Tx, y^* \rangle = \langle x, T^*y^* \rangle = T^*y^*(x), \quad x \in D(T), \quad y^* \in D(T^*).$$

$T^*$ is called the adjoint of $T$.

By Riesz representation theorem for $L^p$-spaces we have $\langle f, F \rangle = F(f) = \int_{\Omega} f \hat{F} d\mu$, when $f \in L^p(\Sigma), \quad F \in L^q(\Sigma) = (L^p(\Sigma))^*$ and $\frac{1}{p} + \frac{1}{q} = 1$. By the Theorem 2.3 easily we get that: the operator $M_uEM_u$ is densely defined if and only if the operator $M_uEM_u$ is densely defined. In the next proposition we obtain the adjoint of the WCT operator $M_uEM_u$ on the Banach space $L^p(\Sigma)$.

**Proposition 2.4.** If the linear transformation $T = M_uEM_u$ is densely defined on $L^p(\Sigma)$, then $M_uEM_u$ is a densely defined operators on $L^q(\Sigma)$ and $T^* = M_uEM_u$, where $\frac{1}{p} + \frac{1}{q} = 1$. 


Theorem 2.6. If \( w \in L^2(\Omega) \), then it is continuous if and only if it is everywhere defined.

Proof. Let \( f \in \mathcal{D}(T) \) and \( g \in \mathcal{D}(T^*) \). So we have

\[
(Tf, g) = \int_{\Omega} wE(uf)\overline{g}d\mu = \int_{\Omega} fuE(wg)d\mu = \langle f, \overline{M_w}g \rangle.
\]

Hence \( T^* = \overline{M_w} \).

Now we prove that every densely defined WCT operator is closed.

Proposition 2.5. If \( \int E(|w|^q)\mu < \infty \) a.e., \( \mu \). Then the linear transformation \( M_w \mathcal{E}_w : \mathcal{D}(M_w \mathcal{E}_w) \to \mathcal{L}(\Sigma) \) is closed.

Proof. Assume that \( f_n \in \mathcal{D}(M_w \mathcal{E}_w) \), \( f_n \to f \), \( wE(u_{f_n}) \to g \), and let \( h \in \mathcal{D}(\overline{M_w} \mathcal{E}_w) \). Then

\[
\langle f, \overline{M_w} \mathcal{E}_w h \rangle = \lim_{n \to \infty} \langle f_n, \overline{M_w} \mathcal{E}_w h \rangle = \lim_{n \to \infty} (wE(u_{f_n}), h) = \langle g, h \rangle.
\]

This calculation (which uses the continuity of the inner product and the fact that \( f_n \in \mathcal{D}(M_w \mathcal{E}_w) \)) shows that \( f \in \mathcal{D}(M_w \mathcal{E}_w) \) and \( wE(u_f) = g \), as required.

In the next theorem we get that if WCT operator \( M_w \mathcal{E}_w \) is densely defined, then it is continuous if and only it is everywhere defined.

Theorem 2.6. If \( \int E(|w|^q)\mu < \infty \) a.e., \( \mu \). Then the WCT operator \( M_w \mathcal{E}_w : \mathcal{D}(M_w \mathcal{E}_w) \to \mathcal{L}(\Sigma) \) is continuous if and only if it is everywhere defined i.e., \( \mathcal{D}(M_w \mathcal{E}_w) = \mathcal{L}(\Sigma) \).

Proof. Let \( M_w \mathcal{E}_w \) be continuous. By Lemma 2.2, it is closed. Hence easily we get that \( \mathcal{D}(M_w \mathcal{E}_w) \) is closed and so \( \mathcal{D}(M_w \mathcal{E}_w) = \mathcal{L}(\Sigma) \). The converse is easy by closed graph theorem.

We denote the range of the operator \( T \) as \( \mathbb{R}(T) \) i.e., \( \mathbb{R}(T) = \{T(x) : x \in \mathcal{D}(T)\} \).

Proposition 2.7. If \( E(|w|^2)E(|w|^2) < \infty \) a.e., \( \mu \) and \( M_w \mathcal{E}_w : \mathcal{D}(M_w \mathcal{E}_w) \subset L^2(\Sigma) \to L^2(\Sigma) \), then \( \mathbb{R}(M_w \mathcal{E}_w) \) is closed if and only if \( \mathbb{R}(M_w \mathcal{E}_w) = \mathbb{L}(\Sigma) \).

Proof. Let \( P : L^2(\Sigma) \times L^2(\Sigma) \to G(M_w \mathcal{E}_w) \) be a projection and \( Q : L^2(\Sigma) \times L^2(\Sigma) \to \{0\} \times L^2(\Sigma) \) be the canonical projection. It is clear that \( \mathbb{R}(M_w \mathcal{E}_w) = \mathbb{R}(QP) \). Also, \( \mathbb{R}(M_w \mathcal{E}_w) \cong \mathbb{R}((I - Q)(I - P)) \). Since \( P \) and \( Q \) are orthogonal projections, then \( \mathbb{R}(QP) \) is closed if and only if \( \mathbb{R}((I - Q)(I - P)) \). Thus we obtain the desired result.

It is well-known that for a densely defined closed operator \( T \) of \( \mathcal{H}_2 \) into \( \mathcal{H}_2 \), there exists a partial isometry \( U_T \) with initial space \( \mathcal{N}(T) = \overline{\mathbb{R}(T^*)} = \overline{\mathbb{R}((T))} \) and final space \( \mathcal{N}(T^*) = \overline{\mathbb{R}(T)} \) such that

\[ T = U_T |T|. \]
Theorem 2.8. Suppose that $\mathcal{D}(M_wEM_u)$ is dense in $L^2(\Sigma)$. Let $M_wEM_u = U|M_wEM_u|$ be the polar decomposition of $M_wEM_u$. Then

(i) $|M_wEM_u| = M_w E M_u$, where $u' = \left(\frac{E(|u|^2)}{E(|u|^2)}\right)^\frac{1}{2} \chi_{S\cap G}$ and $S = S(E(|u|^2))$,

(ii) $U = M_wEM_u$, where $w' : \Omega \to \mathbb{C}$ is an a.e. $\mu$ well-defined $\Sigma$-measurable function such that

$$w' = \frac{w}{(E(|w|^2)E(|u|^2))^\frac{1}{2}} \chi_{S\cap G},$$

in which $G = S(E(|w|^2))$.

Proof. (i). For every $f \in \mathcal{D}(M_wEM_u)$ we have

$$\|M_wEM_u(f)(f)\|^2 = \|M_wEM_u(f)\|^2.$$

Also, by Lemma 2.2 we conclude that $\mathcal{D}(M_wEM_u) = \mathcal{D}(|M_wEM_u|)$ and it is easily seen that $M_wEM_u$ is a positive operator. These observations imply that $|M_wEM_u| = M_wEM_u$.

(ii). For $f \in L^2(\Sigma)$ we have

$$\int_{\Omega} |w' E(u f)|^2 d\mu = \frac{\chi_{S\cap G}}{E(|w|^2)E(|u|^2)} \int_{\Omega} |w E(u f)|^2 d\mu,$$

which implies that the operator $M_wEM_u$ is well-defined and $\mathcal{N}(M_wEM_u) = \mathcal{N}(M_wEM_u)$. Also, for $f \in \mathcal{D}(M_wEM_u) \oplus \mathcal{N}(M_wEM_u)$ we have

$$U(|M_wEM_u|(f)) = wE(u f).\chi_{S\cap G} = wE(u f).$$

Thus $\|U(f)\| = \|f\|$ for all $f \in \mathcal{R}(|M_wEM_u|)$ and since $U$ is a contraction, then it holds for all $f \in \mathcal{N}(M_wEM_u)^\perp = \mathcal{R}(|M_wEM_u|)$. □

Here we remind that: if $T : \mathcal{D}(T) \subset X \to X$ is a closed linear operator on the Banach space $X$, then a complex number $\lambda$ belongs to the resolvent set $\rho(T)$ of $T$, if the operator $\lambda I - T$ has a bounded everywhere on $X$ defined inverse $(\lambda I - T)^{-1}$, called the resolvent of $T$ at $\lambda$ and denoted by $R_\lambda(T)$. The set $\sigma(T) := \mathbb{C} \setminus \rho(T)$ is called the spectrum of the operator $T$.

It is known that, if $a, b$ are elements of a unital algebra $A$, then $1 - ab$ is invertible if and only if $1 - ba$ is invertible. A consequence of this equivalence is that $\sigma(ab) \setminus \{0\} = \sigma(ba) \setminus \{0\}$. Now, in the next theorem we compute the spectrum of WCT operator $M_wEM_u$ as a densely defined operator on $L^2(\Sigma)$.

Proposition 2.9. Let $M_wEM_u$ be densely defined and $A \subseteq \Sigma$, then

(i) $\text{essrange}(E(uw)) \setminus \{0\} \subseteq \sigma(M_wEM_u)$,

(ii) If $L^2(A) \subseteq \mathcal{D}(EM_uw)$, then $\sigma(M_wEM_u) \setminus \{0\} \subseteq \text{essrange}(E(uw)) \setminus \{0\}$.

Proof. Since $\sigma(M_wEM_u) \setminus \{0\} = \sigma(EM_uw) \setminus \{0\}$, then by using theorem 2.8 of \cite{Y. ESTAREMI}, we get the proof. □

By a similar method that we used in the proof of theorem 2.8 of \cite{Y. ESTAREMI} we have the same assertion for the spectrum of the densely defined operator $EM_u$ on the space $L^p(\Sigma)$, i.e.,

(i) $\text{essrange}(E(u)) \cup \{0\} \subseteq \sigma(EM_u)$,
If \( L^p(A) \subseteq D(EM_u) \), then \( \sigma(EM_u) \subseteq \text{essrange}(E(u)) \cup \{0\} \).

By these observations we have the next remark.

**Remark 2.10.** Let \( M_w EM_u \) be densely defined operator on \( L^p(\Sigma) \) and \( A \subseteq \Sigma \), then

(i) \( \text{essrange}(E(uw)) \setminus \{0\} \subseteq \sigma(M_w EM_u) \),

(ii) If \( L^p(A) \subseteq D(EM_{uw}) \), then \( \sigma(M_w EM_u) \setminus \{0\} \subseteq \text{essrange}(E(uw)) \cup \{0\} \).

As we know the spectral radius of a densely defined operator \( T \) is denoted by \( r(T) \) and is defined as: \( r(T) = \sup_{\lambda \in \sigma(T)} |\lambda| \). Hence we have the next corollary.

**Corollary 2.11.** If the WCT operator \( M_w EM_u \) is densely defined on \( L^p(\Sigma) \) and \( L^p(A) \subseteq D(EM_{uw}) \), then \( \sigma(M_w EM_u) \setminus \{0\} = \text{essrange}(E(uw)) \cup \{0\} \) and \( r(M_w EM_u) = ||E(uw)||_\infty \).

A densely defined operator \( T \) on the Hilbert space \( \mathcal{H} \) is said to be **hyponormal** if \( D(T) \subseteq D(T^*) \) and \( ||T^*(f)|| \leq ||T(f)|| \) for \( f \in D(T) \). A densely defined operator \( T \) on the Hilbert space \( \mathcal{H} \) is said to be **normal** if \( T^*T = TT^* \). For the WCT operator \( T = M_w EM_u \) on \( L^2(\Sigma) \) we have \( T^* = M_w EM_u \) and we recall that \( T \) is densely defined if and only if \( T^* \) is densely defined. If \( T \) is densely defined, then by the Lemma 2.2 we get that \( L^2(\nu) \subseteq D(T) \), \( L^2(\nu) \subseteq D(T^*) \) and

\[
\frac{L^2(\nu)}{\|\mu\|} = \frac{D(T)}{\|\mu\|} = \frac{D(T^*)}{\|\mu\|} = L^2(\mu),
\]

in which \( d\nu = Jd\mu \) and \( J = 1 + E(|w|^2)E(|u|^2) \). Also, we have \( T^*T = M_w EM_u \) and \( TT^* = M_w EM_u \). Similarly, we have \( L^2(\nu) \subseteq D(T^*T) \), \( L^2(\nu) \subseteq D(TT^*) \) and

\[
\frac{L^2(\nu)}{\|\mu\|} = \frac{D(T^*T)}{\|\mu\|} = \frac{D(TT^*)}{\|\mu\|} = L^2(\mu),
\]

in which \( d\nu' = J'd\mu \) and \( J' = 1 + (E(|w|^2))^2(E(|u|^2))^2 \). By these observations we have next assertions.

**Proposition 2.12.** Let WCT operator \( M_w EM_u \) be densely defined on \( L^2(\Sigma) \). Then we have the followings:

(i) If \( u(E(|w|^2))^\frac{1}{2} = \tilde{w}(E(|u|^2))^\frac{1}{2} \) with respect to the measure \( \mu \), then \( T = M_w EM_u \) is normal.

(ii) If \( T = M_w EM_u \) is normal, then \( E(|w|^2)E(u)^2 = E(|u|^2)E(w)^2 \) with respect to the measure \( \mu \).

**Proof.** (i) Direct computations shows that

\[
T^*T - TT^* = M_w EM_u - M_w EM_u,
\]
on \( L^2(\nu') \). Hence for every \( f \in L^2(\nu') \) we have

\[
\langle T^*T - TT^*(f), f \rangle = \int_X E(|w|^2)E(u)\bar{w}f - E(|u|^2)E(\tilde{w})\bar{f}w d\mu
\]

\[
= \int_X |E(u(E(|w|^2))^\frac{1}{2}f)|^2 - |E((E(|u|^2))^\frac{1}{2}\bar{w}f)|^2 d\mu.
\]

This implies that if

\[
(E(|w|^2))^\frac{1}{2} = u(E(|w|^2))^\frac{1}{2},
\]
then for all \( f \in L^2(\nu') \), \( (T^*T - TT^*(f), f) = 0 \). Thus \( T^*T = TT^* \).
(ii) Suppose that $T$ is normal. By (i), for all $f \in L^2(\nu')$ we have
\[
\int_X |E(u(E(|w|^2)))^{\frac{1}{2}}f)|^2 - |E((E(|w|^2)))^{\frac{1}{2}}\bar{w}f)|^2 d\mu = 0.
\]
Let $A \in \mathcal{A}$, with $0 < \nu'(A) < \infty$. By replacing $f$ to $\chi_A$, we have
\[
\int_A |E(u(E(|w|^2)))^{\frac{1}{2}}f)|^2 - |E((E(|w|^2)))^{\frac{1}{2}}\bar{w}f)|^2 d\mu = 0
\]
and so
\[
\int_A |E(u)|^2E(|w|^2) - |E(w)|^2E(|u|^2)d\mu = 0.
\]
Since $A \in \mathcal{A}$ is arbitrary and $\mu \ll \nu'$ (absolutely continuous), then $|E(u)|^2E(|w|^2) = |E(w)|^2E(|u|^2)$ with respect to $\mu$. □

**Proposition 2.13.** Let the WCT operator $M_wEM_u$ be densely defined on $L^2(\Sigma)$. Then we have the followings:

(i) If $u(E(|w|^2)))^{\frac{1}{2}} \geq \bar{w}(E(|u|^2)))^{\frac{1}{2}}$ with respect to $\mu$, then $T = M_wEM_u$ is hyponormal.

(ii) If $T = M_wEM_u$ is hyponormal, then $E(|w|^2)|E(u)|^2 \geq E(|u|^2)|E(w)|^2$ with respect to the measure $\mu$.

**Proof.** By a similar method of 2.12 we can get the proof. □

### 3. Hyperexpansive WCT operators

In this section we are going to present conditions under which WCT operator $M_wEM_u$ on $L^2(\Sigma)$ is $k$-isometry, $k$-expansive, $k$-hyperexpansive and completely hyperexpansive. For an operator $T$ on the Hilbert space $\mathcal{H}$ we set

\[
\Theta_{T,n}(f) = \sum_{0 \leq i \leq n} (-1)^i \binom{n}{i} \|T^i(f)\|^2, \quad f \in D(T^n), \quad n \geq 1.
\]

By means of this definition an operator $T$ on $\mathcal{H}$ is said to be:

(i) $k$-isometry ($k \geq 1$) if $\Theta_{T,k}(f) = 0$ for $f \in D(T^k)$,

(ii) $k$-expansive ($k \geq 1$) if $\Theta_{T,k}(f) \leq 0$ for $f \in D(T^k)$,

(iii) $k$-hyperexpansive ($k \geq 1$) if $\Theta_{T,n}(f) \leq 0$ for $f \in D(T^n)$ and $n = 1, 2, ..., k$.

(iv) completely hyperexpansive if $\Theta_{T,n}(f) \leq 0$ for $f \in D(T^n)$ and $n \geq 1$.

For more details one can see [9, 7, 11]. It is easily seen that for every $f \in L^2(\Sigma)$

\[
\|M_wEM_u(f)\|_2 = \|EM_v(f)\|_2,
\]

where $v = u(E(|w|^2)))^{\frac{1}{2}}$. Thus without loss of generality we can consider the operator $EM_v$ instead of $M_wEM_u$ in our discussion. First we recall some concepts that we need them in the sequel. Now we present our main results. The next lemma is a direct consequence of Theorem [2, 3].
Lemma 3.1. For every $n \in \mathbb{N}$ the operator $(EM_v)^n$ on $L^2(\Sigma)$ is densely-defined if and only if the operator $EM_v$ is densely defined on $L^2(\Sigma)$.

In the Theorem 3.2 we give some necessary and sufficient conditions for $k$-isometry and $k$-expansive WCT operators $EM_v$.

Theorem 3.2. If $\mathcal{D}(EM_v)$ is dense in $L^2(\mu)$, then:

(i) If the operator $EM_v$ is $k$-isometry ($k \geq 1$), then $A_k^0(|E(v)|^2) = 0$;

(ii) If $(1 + E(|v|^2)A_k^1(|E(v)|^2)) = 0$ and $|E(vf)|^2 = E(|v|^2)E(|f|^2)$ for all $f \in \mathcal{D}(EM_v)$, then the operator $EM_v$ is $k$-isometry;

(iii) If the operator $EM_v$ is $k$-expansive, then $A_k^0(|E(v)|^2) \leq 0$;

(iv) If $(1 + E(|v|^2)A_k^1(|E(v)|^2)) \leq 0$ and $|E(vf)|^2 = E(|v|^2)E(|f|^2)$ for all $f \in \mathcal{D}(EM_v)$, then the operator $EM_v$ is $k$-expansive, where

$$A_k^0(|E(v)|^2) = \sum_{0 \leq i \leq k} (-1)^i \binom{k}{i} |E(v)|^{2i}, \quad A_k^1(|E(v)|^2) = \sum_{1 \leq i \leq k} (-1)^i \binom{k}{i} |E(v)|^{2(i-1)}.$$

Proof. Suppose that the operator $EM_v$ is $k$-isometry. So for all $f \in \mathcal{D}((EM_v)^k)$ we have

$$0 = \Theta_{T,k}(f)$$

$$= \sum_{0 \leq i \leq k} (-1)^i \binom{n}{i} \| (EM_v)^i(f) \|^2$$

$$= \int_\Omega |f|^2 d\mu + \sum_{1 \leq i \leq k} (-1)^i \binom{n}{i} \int_\Omega |E(v)|^{2(i-1)}|E(vf)|^2 d\mu,$$

and so for all $\mathcal{A}$-measurable functions $f \in \mathcal{D}((EM_v)^k)$

$$0 = \int_\Omega |f|^2 d\mu + \sum_{1 \leq i \leq k} (-1)^i \binom{n}{i} \int_\Omega |E(v)|^{2(i-1)}|E(v)|^2 |f|^2 d\mu$$

$$= \int_\Omega \left( \sum_{0 \leq i \leq k} (-1)^i \binom{n}{i} |E(v)|^{2i} \right) |f|^2 d\mu.$$

Since $(EM_v)^k$ is densely defined, then we get that $A_k(|E(v)|^2) = 0$.

(ii) Let $1 + E(|v|^2)A_k^1(|E(v)|^2) = 0$ and $|E(vf)|^2 = E(|v|^2)E(|f|^2)$ for all $f \in$
\( \mathcal{D}((EM_v)^k) \). Then for all \( f \in \mathcal{D}((EM_v)^k) \) we have

\[
\Theta_{T,k}(f) = \sum_{0 \leq i \leq k} (-1)^i \binom{n}{i} \|(EM_v)^i(f)\|^2
= \int_{\Omega} |f|^2 d\mu + \sum_{1 \leq i \leq k} (-1)^i \binom{n}{i} \int_{\Omega} |E(v)|^2 E(\pi^1 |E(v)|^2) |f|^2 d\mu
= \int_{\Omega} |f|^2 d\mu + \int_{\Omega} \left( \sum_{1 \leq i \leq k} (-1)^i \binom{n}{i} (E(|v|^2))^{2(i-1)} \right) E(|v|^2) E(|f|^2) d\mu
= \int_{\Omega} (1 + E(|v|^2) A_k(|E(v)|^2)) |f|^2 d\mu
= 0.
\]

This implies that the operator \( EM_v \) is \( k \)-isometry.

(iii), (iv). By the same method that is used in (i) and (ii), easily we get (iii) and (iv).

\[ \square \]

Here we recall that if the linear transformation \( T = EM_v \) is densely defined on \( L^2(\Sigma) \), then \( T = EM_v \) is closed and \( T^* = M_\pi E \). Also, if \( \mathcal{D}(EM_v) \) is dense in \( L^2(\Sigma) \) and \( v \) is almost every where finite valued, then the operator \( EM_v \) is normal if and only if \( v \in L^0(A) \) \[3\]. Hence we have the Remark 3.3 for normal WCT operators.

Remark 3.3. Suppose that the operator \( EM_v \) is normal and \( \mathcal{D}(EM_v) \) is dense in \( L^2(\mu) \) for a fixed \( k \geq 1 \). If \( |E(f)|^2 = E(|f|^2) \) on \( S(v) \) for all \( f \in \mathcal{D}((EM_v)^k) \), then:

(i) The operator \( EM_v \) is \( k \)-isometry \((k \geq 1)\) if and only if \( A_k(|v|^2) = 0 \);

(ii) The operator \( EM_v \) is \( k \)-expansive if and only if \( A_k(|v|^2) \leq 0 \).

\[ \text{Proof.} \] Since \( EM_v \) is normal, then \( |E(v)|^2 = E(|v|^2) = |v|^2 \). Thus by Theorem 3.2 we have (i) and (ii).

Here we give some properties of \( 2 \)-expansive WCT operators and as a corollary for \( 2 \)-expansive multiplication operators.

Proposition 3.4. If \( \mathcal{D}(EM_v) \) is dense in \( L^2(\mu) \) and \( EM_v \) is \( 2 \)-expansive, then:

(i) \( EM_v \) leaves its domain invariant:

(ii) \( |E(v)|^{2k} \geq |E(v)|^{2(k-1)} \) a.e. \( \mu \) for all \( k \geq 1 \).
Remark 3.1 and Theorem 3.2. (iii) of [7] we get that

\[d_{EM}(f) = \frac{\sum_{n=0}^{\infty} \phi(n)}{n} \leq 1\]  

so \(M_v(f) \in D(EM_v)\).

(ii) Since \(EM_v\) leaves its domain invariant, then \(D(EM_v) \subseteq D^\infty(EM_v)\). So by lemma 3.2 (iii) of [7] we get that \(\|EM_v(f)\| \geq \|EM_v(k-1)(f)\|^2\) for all \(f \in D(EM_v)\) and \(k \geq 1\) we have

\[\int_{\Omega} |E(v)|^{2(k-1)}|E(vf)|^2d\mu \geq \int_{\Omega} |E(v)|^{2(k-2)}|E(vf)|^2d\mu,
\]

and so

\[\int_{\Omega} (|E(v)|^{2(k-1)} - |E(v)|^{2(k-2)} |E(vf)|^2d\mu \geq 0,
\]

for all \(f \in D(EM_v)\). This leads to \(|E(v)|^{2k} \geq |E(v)|^{2(k-1)}\) a.e., \(\mu\). \(\square\)

**Corollary 3.5.** If \(D(M_v)\) is dense in \(L^2(\mu)\) and \(M_v\) is \(2\)-expansive, then:

(i) \(M_v\) leaves its domain invariant:

(ii) \(v^{2k} \geq v^{2(k-1)}\) a.e. \(\mu\) for all \(k \geq 1\).

Recall that a real-valued map \(\phi\) on \(\mathbb{N}\) is said to be completely alternating if

\[\sum_{0 \leq i \leq n} (-1)^i \binom{n}{i} \phi(m+i) \leq 0 \text{ for all } m \geq 0 \text{ and } n \geq 1.\]

The next remark is a direct consequence of Lemma 3.1 and Theorem 3.2.

**Remark 3.6.** If \(D(EM_v)\) is dense in \(L^2(\mu)\) and \(k \geq 1\) is fixed, then:

(i) If the operator \(EM_v\) is \(k\)-hyperexpansive \((k \geq 1)\), then \(A_n^0(|E(v)|^2) \leq 0\) for \(n = 1, 2, \ldots, k\);

(ii) If \((1 + E(|v|^2)A_n^1(|E(v)|^2)) \leq 0\) and \(|E(vf)|^2 = E(|v|^2)E(|f|^2)\) for all \(f \in D(EM_v)^n\) and \(n = 1, 2, \ldots, k\), then the operator \(EM_v\) is \(k\)-hyperexpansive \((k \geq 1)\);

(iii) If the operator \(EM_v\) is completely hyperexpansive, then

(a) the sequence \(|E(v)(t)|^2|^2_{n=0}^\infty\) is a completely alternating sequence for almost every \(t \in \Omega\),
(b) \(A_n^0(|E(v)|^2) \leq 0\) for \(n \geq 1\).

(iv) If \((1 + E(|v|^2)A_n^1(|E(v)|^2)) \leq 0\) and \(|E(vf)|^2 = E(|v|^2)E(|f|^2)\) for all \(f \in D((EM_v)^n)\) and \(n \geq 1\), then the operator \(EM_v\) is completely hyperexpansive.
By Remark 3.6 and some properties of normal WCT operators we get the next remark for \( k \)-hyperexpansive and completely hyperexpansive normal WCT operators.

**Remark 3.7.** Let the operator \( EM_v \) be normal, \( D(EM_v) \) be dense in \( L^2(\mu) \) and \( k \geq 1 \) be fixed. If \( |E(f)|^2 = E(|f|^2) \) on \( S(\nu) \) for all \( f \in D((EM_v)^k) \), then

(i) \( EM_v \) is \( k \)-hyperexpansive (\( k \geq 1 \)) if and only if \( A_n(|v|^2) \leq 0 \) for \( f \in D(T^n) \) and \( n = 1, 2, ..., k \).

(ii) \( EM_v \) is completely hyperexpansive if and only if the sequence \( \{|u(t)|^2\}_{n=0}^\infty \) is a completely alternating sequence for almost every \( t \in \Omega \).

If all functions \( v^{2i} \) for \( i = 1, ..., n \) are finite valued, then we set

\[
\triangle_{v,n}(x) = \sum_{0 \leq i \leq n} (-1)^i \binom{n}{i} |v|^{2i}(t).
\]

Also, if \( \mathcal{A} = \Sigma \), then \( E = I \). So we have next two corollaries.

**Corollary 3.8.** If \( D(M_v) \) is dense in \( L^2(\mu) \) for a fixed \( n \geq 1 \), then:

(i) \( M_v \) is \( k \)-expansive if and only if \( \triangle_{v,n}(x) \leq 0 \) a.e. \( \mu \).

(ii) \( M_v \) is \( k \)-isometry if and only if \( \triangle_{v,n}(x) = 0 \) a.e. \( \mu \).

**Corollary 3.9.** Let \( D(M_v) \) be dense in \( L^2(\mu) \) and \( k \geq 1 \) be fixed. Then

(i) \( M_v \) is \( k \)-hyperexpansive (\( k \geq 1 \)) if and only if \( \triangle_{v,n}(t) \leq 0 \) a.e., \( \mu \) for \( n = 1, 2, ..., k \).

(ii) \( M_v \) is completely hyperexpansive if and only if the sequence \( \{|u(t)|^2\}_{n=0}^\infty \) is a completely alternating sequence for almost every \( t \in \Omega \).

Finally we give some examples.

**Example 3.10.** Let \( \Omega = [-1, 1] \), \( d\mu = \frac{1}{2}dx \) and \( \mathcal{A} = \mathcal{A} = \{(-a, a) : 0 \leq a \leq 1\} \) (Sigma algebra generated by symmetric intervals). Then

\[
E^A(f)(t) = \frac{f(t) + f(-t)}{2}, \quad t \in \Omega,
\]

where \( E^A(f) \) is defined. If \( v(t) = e^t \), then \( E^A(v)(t) = \cosh(t) \) and we have the followings:

1) \( E^A M_v \) is densely defined and closed on \( L^p(\Omega) \).

2) \( \sigma(E^A M_v) = R(\cosh(t)) \).

3) \( E^A M_v \) is not 2-expansive, since

\[
1 - 2|E(v)|^2(t) + |E(v)|^4(t) = 1 - 2\cosh^2(t) + \cosh^4(t) = (\cosh^2(t) - 1)^2 \geq 0.
\]
Example 3.11. Let $\Omega = \mathbb{N}$, $\mathcal{G} = 2^\mathbb{N}$ and let $\mu(\{t\}) = pq^{t-1}$, for each $t \in \Omega$, $0 \leq p \leq 1$ and $q = 1 - p$. Elementary calculations show that $\mu$ is a probability measure on $\mathcal{G}$. Let $A$ be the $\sigma$-algebra generated by the partition $B = \{\Omega_1 = \{3n : n \geq 1\}, \Omega_1^c\}$ of $\Omega$. So, for every $f \in \mathcal{D}(E^A)$ we have

$$E(f) = \alpha_1 \chi_{\Omega_1} + \alpha_2 \chi_{\Omega_1^c}$$

and direct computations show that

$$\alpha_1(f) = \frac{\sum_{n \geq 1} f(3n) pq^{3n-1}}{\sum_{n \geq 1} pq^{3n-1}}$$

and

$$\alpha_2(f) = \frac{\sum_{n \geq 1} f(n) pq^{n-1} - \sum_{n \geq 1} f(3n) pq^{3n-1}}{\sum_{n \geq 1} pq^{n-1} - \sum_{n \geq 1} pq^{3n-1}}.$$  

So, if $u$ and $w$ are real functions on $\Omega$. Then we have the followings:

1) If $\alpha_1((|w|^q)^{\frac{1}{q}}) \alpha_1((|w|^p)^{\frac{1}{p}}) < \infty$ and $\alpha_2((|w|^q)^{\frac{1}{q}}) \alpha_2((|w|^p)^{\frac{1}{p}}) < \infty$, then the operator $M_w EM_u$ is a densely defined and closed operator on $L^p(\Omega)$.

2) $\sigma(M_w EM_u) = \{\alpha_1(E(uw)), \alpha_2(E(uw))\}$.

Example 3.12. Let $\Omega = [0,1] \times [0,1]$, $d\mu = ddtdt'$, $\Sigma$ the Lebesgue subsets of $\Omega$ and let $\mathcal{A} = \{A \times [0,1] : A$ is a Lebesgue set in $[0,1]\}$. Then, for each $f$ in $L^2(\Sigma)$, $(Ef)(t,t') = \int_0^1 f(t,s)ds$, which is independent of the second coordinate. Hence for $v(t,t') = t^m$ we get that $v$ is $\mathcal{A}$-measurable and $EM_v$ is $k$-expansive and $k$-isometry if

$$\sum_{0 \leq i \leq k} (-1)^i \binom{k}{i} x^{2mi} \leq 0, \quad \sum_{0 \leq i \leq k} (-1)^i \binom{k}{i} t^{2mi} = 0,$$

respectively. This example is valid in the general case as follows:

Let $(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$ be two $\sigma$-finite measure spaces and $\Omega = \Omega_1 \times \Omega_2$, $\Sigma = \Sigma_1 \times \Sigma_2$ and $\mu = \mu_1 \times \mu_2$. Put $A = \{A \times \Omega_2 : A \in \Sigma_1\}$. Then $A$ is a sub-$\sigma$-algebra of $\Sigma$. Then for all $f$ in domain $E^A$ we have

$$E^A(f)(t_1) = E^A(f)(t_1, t_2) = \int_{\Omega_2} f(t_1, s)d\mu_2(s) \quad \mu - a.e.$$  

on $\Omega$.

Also, if $(\Omega, \Sigma, \mu)$ is a finite measure space and $k : \Omega \times \Omega \to \mathbb{C}$ is a $\Sigma \otimes \Sigma$-measurable function such that

$$\int_{\Omega} |k(., s)f(s)|d\mu(s) \in L^2(\Sigma)$$

for all $f \in L^2(\Sigma)$. Then the operator $T : L^2(\Sigma) \to L^2(\Sigma)$ defined by

$$Tf(t) = \int_{\Omega} k(t, s)f(s)d\mu, \quad f \in L^2(\Sigma),$$

is called kernel operator on $L^2(\Sigma)$. We show that $T$ is a weighted conditional type operator. Since $L^2(\Sigma) \times \{1\} \cong L^2(\Sigma)$ and $vf$ is a $\Sigma \otimes \Sigma$-measurable function,
when \( f \in L^2(\Sigma) \). Then by taking \( v := k \) and \( f'(t, s) = f(s) \), we get that
\[
E^A(vf)(t) = E^A(vf')(t, s) = \int_{\Omega} v(t, t') f'(t', d\mu(t') = \int_{\Omega} v(t, t') f(t') d\mu(t') = Tf(t).
\]

Hence \( T = EM_v \), i.e, \( T \) is a weighted conditional type operator. This means all assertions of this paper are valid for a class of integral type operators.

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