Knot cobordism and Lee’s Perturbation of Khovanov homology

Zipei Zhuang

January 7, 2022

Abstract

For a connected cobordism $S$ between two knots $K_1, K_2$ in $S^3$, we establish an inequality involving the number of local maxima, the genus of $S$, and the torsion orders of $Kh_t(K_1), Kh_t(K_2)$, where $Kh_t$ denotes Lee’s perturbation of Khovanov homology. This shows that the torsion order gives a lower bound for the band-unlinking number.

If $K_0, K_1$ are knots in $S^3$, a cobordism from $K_0$ to $K_1$ is a smooth compact oriented surface $F$ embedded in $S^3 \times [0, 1]$ with boundary $K_0 \times \{0\} \cup K_1 \times \{1\}$. If $F$ is an annulus, it is called a concordance. Any cobordism $F \subset S^3 \times [0, 1]$ is called ribbon if the projection from $S^3 \times [0, 1]$ to $[0, 1]$ restricts to a Morse function on $F$ with only index 0 and 1 critical points. A knot $K_0$ is ribbon concordant to $K_1$ if there exists a ribbon concordance from $K_0$ to $K_1$.

In the past few years there has been several works on applications of knot homology theories to knot cobordisms. For Khovanov homology $Kh$, it is shown [9] that a ribbon concordance between two links $(F, K_0, K_1)$ induces an injective map $\phi_F : Kh(K_0) \rightarrow Kh(K_1)$. Furthermore, Sarkar [11] showed that if $F$ has $k$ saddles, then $(2x)^d Kh_t(K_0) \cong (2x)^d Kh_t(K_1)$, where $Kh_t$ is Lee’s perturbation of Khovanov homology. Similar results are obtained in the context of knot Floer homology (see [12] [5]).

The main result of this paper is the following:

**Theorem 1.** Suppose that $(F, K_0, K_1)$ is a connected knot cobordism with $m$ births, $b$ saddles and $M$ deaths. Denote by $\bar{F}$ the cobordism from $K_1$ to $K_0$ obtained by horizontally mirroring $F$. Then up to a sign

$$(2x)^M \cdot \phi_{\bar{F}} \circ \phi_F = (2x)^{b-m} \cdot id_{Kh_t(K_0)}$$

This is analogous to the main result of [5], which is for knot Floer homology. In particular, if $F$ is a ribbon concordance, then $M = b - m = 0$, so $\phi_{\bar{F}} \circ \phi_F = id_{Kh_t(K_0)}$, $\phi_F$ is injective.

Let $T(K)$ be the torsion part of $Kh_t(K)$, and $xo(K)$ be the extortion order of $K$, i.e. the minimal $n$ such that $x^n T(K) = 0$. The same argument as in [5] gives the following corollary:
Corollary 1.

\[ xo(K_0) \leq \max\{M, xo(K_1)\} + 2g(F) \]  

(2)

where \( g(F) \) is the genus of \( F \).

The band-unlinking number \( ul_b(K) \) of a knot \( K \) is the minimum number of oriented band moves necessary to reduce \( K \) to an unlink. Corollary 1 furthermore gives

Corollary 2.

\[ xo(K) \leq ul_b(K) \]  

(3)

1 Lee’s perturbation of Khovanov homology

We will work over a field \( F \) in which 2 is invertible. Let \( A \) be the 2-dimensional Frobenius algebra over \( F[t] \) with basis \( \{1, x\} \), with the multiplication map \( m \), comultiplication map \( \triangle \), and counit map \( \epsilon \) defined as follows:

\[
\begin{align*}
1 \otimes 1 & \xrightarrow{m} 1 \\
x \otimes 1 & \xrightarrow{m} x \\
x \otimes x & \xrightarrow{m} t \\
1 & \xrightarrow{\triangle} 1 \otimes x + x \otimes 1 \\
x & \xrightarrow{\triangle} x \otimes x + t \cdot 1 \otimes 1 \\
1 & \xrightarrow{\epsilon} 0 \\
x & \xrightarrow{\epsilon} 1
\end{align*}
\]  

(4)

Applying this Frobenius algebra to the Kauffman cube of resolutions of a knot diagram, we get a chain complex over \( F[t] \), whose homology is called Lee’s perturbation of Khovanov homology [3]. In particular, setting \( t = 0 \), we get Khovanov’s original knot homology; taking \( t = 1 \), it becomes Lee homology [8].

Figure 1: The elementary cobordisms

A knot cobordism \((F, K_0, K_1)\) induces a map (see [6] [4] [3])

\[ \phi_F : Kh_t(K_0) \longrightarrow Kh_t(K_1) \]  

(5)
by decomposing $F$ into a sequence of elementary cobordisms, each represented by a Reidemeister move or a Morse move of the link diagrams. The Reidemeister moves induce isomorphisms on homology. The 0-Morse move (corresponding to a birth), 1-Morse move (saddle), 2-Morse move (death) induce maps on $Kh_t$ according to the unit map, the multiplication/comultiplication map, the counit map, respectively. The map induced by $F$ is the composition of the maps induced by these elementary cobordisms. Furthermore, one can consider dotted cobordisms, where a dot induces a multiplication by $x$ on homology. Such assignment is well-defined and functorial, up to a sign.

The cobordism maps satisfy the following local properties (see [11]):

1. Suppose $F$ has an 1-handle $h$. Let $F'$ be obtained from $F$ by surgery along $h$, and let $\hat{F}_1$ and $\hat{F}_2$ be obtained by adding a dot to $F'$ at either of the feet of $h$. Then (see Fig.2) up to a sign,

$$\phi_F = \phi_{\hat{F}_1} + \phi_{\hat{F}_2}$$

(6)

![Figure 2](image)

Figure 2:

2. If $F$ is a cobordism which can be decomposed into a sequence $F_1, \ldots, F_n$ in which $F_i, F_{i+1}$ are two adjacent saddle moves reverse to each other, let $S$ be the cobordism obtained from $F$ by deleting $F_i, F_{i+1}$, and let $\hat{S}_1, \hat{S}_2$ be the cobordism obtained by adding a dot on one of the two sides of $S$ (see Fig.3). Then up to a sign

$$\phi_F = \phi_{\hat{S}_1} + \phi_{\hat{S}_1}$$

(7)

2 The cobordism distance

Proof of Theorem [7]

Proof. Let $F$ be a connected oriented knot (dotted) cobordism from $K_0$ to $K_1$ in $S^3 \times [0,1]$ with $m$ births, $b$ saddles and $M$ deaths. We can rearrange the critical points of $F$ so that $F$ can be decomposed into a sequence of elementary cobordisms ordered as below (we omit the cobordisms of Reidemeister moves since they induce isomorphisms on homology):
The cobordism $S$ decomposes as $K_1 \xrightarrow{F_1} K_1 \sqcup U^m \xrightarrow{F_2} K' \xrightarrow{F_3} K_2 \sqcup U^M \xrightarrow{F_4} K_2$, where the piece $F_i$ comes from item (i) above, and $U^m$ denotes the $m$-component planar unlink. The cobordism $\bar{F}$ decomposes as $K_2 \xrightarrow{F_4} K_2 \sqcup U^M \xrightarrow{F_3} K' \xrightarrow{F_2} K_1 \sqcup U^m \xrightarrow{F_1} K_1$. Then $\phi_{\bar{F}} \circ \phi_F$ is the composition of 8 maps. The composition of the fourth and fifth step is to delete the $M$ unknots via $M$ deaths, then add them back with $M$ births. Let $S$ be the cobordism obtained by deleting the forth and fifth step. Then $\bar{F} \circ F$ is obtained from $S$ by sugerying along the $M$ tubes. By (8), we have

$$\phi_S = (2x)^M \phi_{\bar{F} \circ F}$$

since $\bar{F} \circ F$ is connected.

Let $T$ be the cobordism obtained from $S$ by deleting the third and sixth steps, i.e. $T$ is the composition $K_1 \xrightarrow{F_1} K_1 \sqcup U^m \xrightarrow{F_2} K' \xrightarrow{F_3} K_1 \sqcup U^m \xrightarrow{F_1} K_1$. By (7), we have

$$\phi_S = (2x)^{b-m} \cdot \phi_T$$

Note that $F_2 \circ F_1$ is a ribbon concordance, by [9] Theorem 1.

$$\phi_T = id_{Kh_t(K_0)}$$

Theorem 1 now follows from equation 8 9 10.

Proof of Corollary 1
Proof. Let $F$ be an oriented knot cobordism from $K_0$ to $K_1$ with $m$ births, $b$ saddles, $M$ deaths. For $l \geq 0, \alpha \in Kh_t(K_0)$, by Theorem 1 we have

$$\phi_{F \circ \bar{F}}((2x)^{l+M} \cdot \alpha) = (2x)^M \phi_{\bar{F} \circ F}((2x)^l \cdot \alpha) = (2x)^{l+b-m} \cdot \alpha$$

(11)

This shows that if $l \geq \max\{0, xo(K_1) - M\}$, then

$$(2x)^{l+b-m} \cdot \alpha = \phi_{\bar{F} \circ F}((2x)^{l+M} \cdot \alpha) = 0$$

(12)

Therefore $xo(K_0) \leq b - m + \max\{0, xo(K_1) - M\} = \max\{M, xo(K_1)\} + 2g$ (recall that we work over a base field in which 2 is invertible) since $2g(F) = -\chi(F) = b - m - M$.

Proof of Corollary 2

Proof. By definition, after attaching $ul_b(K)$ oriented bands to $K$, we obtain an unlink $L$. Suppose $L$ has $M$ components. By performing $M-1$ deaths, we obtain a cobordism $S$ from $K$ to an unknot $U$ with 0 births, $ul_B(K)$ saddles and $M-1$ deaths. Then by Corollary 1 we have

$$xo(K) \leq \max\{M-1, 0\} + 2g(S) = M - 1 + ul_b(K) - M + 1 = ul_b(K).$$

(13)

The following corollary generalizes [11] Theorem 1.1:

Corollary 3. If $F$ is a knot concordance from $K_0$ to $K_1$ with $b$ saddles. Then

$$(2x)^b Kh_t(K_0) \cong (2x)^b Kh_t(K_1)$$

(14)

Proof. Suppose $F$ has $m$ births and $M$ deaths. Since $F$ is an annulus, $\chi(F) = m - b + M = 0, M = b - m$. By Theorem 1

$$(2x)^M \phi_{\bar{F} \circ F} = (2x)^{b-m} \cdot Id_{Kh_t(K_0)} = (2x)^M Id_{Kh_t(K_0)}$$

(15)

Note that $b \geq M$, for any $\alpha \in Kh_t(K_0)$,

$$\phi_{\bar{F} \circ F}((2x)^b \cdot \alpha) = (2x)^M \phi_{\bar{F} \circ F}((2x)^b-M \cdot \alpha) = (2x)^b \cdot \alpha$$

(16)

Therefore $\phi_{\bar{F} \circ F}$ is the identity map on $(2x)^b \cdot Kh_t(K_0)$. Reversing the role of $F$ and $\bar{F}$, we get $\phi_{\bar{F} \circ F}$ is the identity map on $(2x)^b \cdot Kh_t(K_1)$. So $\phi_{\bar{F}} : (2x)^b \cdot Kh_t(K_0) \rightarrow (2x)^b \cdot Kh_t(K_1)$ is an isomorphism.

□
References

[1] Akram Alishahi. Unknotting number and Khovanov homology. *Pacific J. Math.*, 301(1):15–29, 2019.

[2] Akram Alishahi and Eaman Eftekhary. Knot Floer homology and the unknotting number. *Geom. Topol.*, 24(5):2435–2469, 2020.

[3] Dror Bar-Natan. Khovanov’s homology for tangles and cobordisms. *Geom. Topol.*, 9:1443–1499, 2005.

[4] Magnus Jacobsson. An invariant of link cobordisms from Khovanov homology. *Algebr. Geom. Topol.*, 4:1211–1251, 2004.

[5] András Juhász, Maggie Miller, and Ian Zemke. Knot cobordisms, bridge index, and torsion in Floer homology. *J. Topol.*, 13(4):1701–1724, 2020.

[6] Mikhail Khovanov. A categorification of the Jones polynomial. *Duke Math. J.*, 101(3):359–426, 2000.

[7] Mikhail Khovanov. An invariant of tangle cobordisms. *Trans. Amer. Math. Soc.*, 358(1):315–327, 2006.

[8] Eun Soo Lee. An endomorphism of the Khovanov invariant. *Adv. Math.*, 197(2):554–586, 2005.

[9] Adam Simon Levine and Ian Zemke. Khovanov homology and ribbon concordances. *Bull. Lond. Math. Soc.*, 51(6):1099–1103, 2019.

[10] Jacob Rasmussen. Khovanov homology and the slice genus. *Invent. Math.*, 182(2):419–447, 2010.

[11] Sucharit Sarkar. Ribbon distance and Khovanov homology. *Algebr. Geom. Topol.*, 20(2):1041–1058, 2020.

[12] Ian Zemke. Knot Floer homology obstructs ribbon concordance. *Ann. of Math. (2)*, 190(3):931–947, 2019.