On the rate of change of the sharp constant in the Sobolev–Poincaré inequality

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Abstract
The rate of change of the sharp constant in the Sobolev–Poincaré or Friedrichs inequality is estimated for a Euclidean domain that moves outward. The key ingredients are a Hadamard variation formula and an inequality that reverses the usual Hölder inequality.

KEYWORDS
Sobolev–Poincaré inequality, Hadamard variation, Reverse Hölder inequality

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1 | INTRODUCTION

The Sobolev inequality, in its many and varied forms, is a key functional geometric inequality by which integrability properties of a function are inferred from integrability properties of its derivative. The most basic form of the inequality states that in n-dimensional Euclidean space, \( n \geq 2 \), and for \( r \in [1, n) \), there is a finite constant \( S_r(\mathbb{R}^n) \) such that

\[
\|u\|_{L^r(\mathbb{R}^n)} \leq S_r(\mathbb{R}^n) \|\nabla u\|_{L^r(\mathbb{R}^n)},
\]

(1.1)

for any real-valued smooth function \( u \) of compact support in \( \mathbb{R}^n \). It is then of significance to determine, if possible, the sharp constants in such inequalities as well as the extremal functions. For example, the case \( r = 1 \) of (1.1) is equivalent to the isoperimetric inequality, and the sharp constant in (1.1) when \( r = 1 \) is the isoperimetric constant – this fact is due independently to Federer and Fleming and to Maz’ya, as described by Chavel [5].

In the setting of any open region \( \Omega \) of finite volume in \( \mathbb{R}^n \), it is a consequence of the basic Sobolev inequality (1.1) that, for \( r \in [1, n) \) and \( p \in [1, r^*] \), there is a finite constant \( S_{p,r}(\Omega) \) such that

\[
\|u\|_{L^p(\Omega)} \leq S_{p,r}(\Omega) \|\nabla u\|_{L^r(\Omega)},
\]

(1.2)

for any function \( u \) in the Sobolev space \( W^{1,r}_0(\Omega) \). We remark that, by scale invariance, \( S_{p,r}(\Omega) = S_r(\mathbb{R}^n) \) for any open set in \( \mathbb{R}^n \). The inclusion of the Sobolev space \( W^{1,r}(\mathbb{R}^n) \subseteq L^{r^*}(\mathbb{R}^n) \) in (1.1) is not a compact embedding whereas the embedding in (1.2) is compact (Rellich compactness) if \( p < r^* \). The sharp constant in the Sobolev–Poincaré inequality (1.2), now in the context of the region \( \Omega \), is in essence the number

\[
C_{p,r}(\Omega) = \inf \left\{ \frac{\int_\Omega |\nabla u|^r \, d\mu}{\left( \int_\Omega |u|^p \, d\mu \right)^{r/p}} : u \in C_0^\infty(\Omega), u \neq 0 \right\},
\]

(1.3)

where \( d\mu \) stands for Lebesgue measure in \( \mathbb{R}^n \). The reason for writing the sharp constant in the Sobolev inequality in this form is historical: in two dimensions, \( C_{2,2}(\Omega) \) is the classical Rayleigh quotient for the principal frequency or bass note of the planar
region $\Omega$ while $4/C_{1.2}$ corresponds to the torsional rigidity of the region, both important physical concepts in the context of solid mechanics. Pólya and Szegő’s monograph [18] is a standard reference from this viewpoint. The relationship between the sharp constant $S_{p,r}(\Omega)$ in the Sobolev inequality (1.2) and the eigenvalue $C_{p,r}(\Omega)$ given by (1.3) is then

$$S_{p,r}(\Omega) = C_{p,r}(\Omega)^{-1/r}.$$ 

The Sobolev inequality implies that $C_{p,r}(\Omega)$ is positive.

For $p \in [1, r]$, there is a unique positive minimizer $\phi$ normalized by

$$\int_{\Omega} \phi^p \, d\mu = 1, \quad (1.4)$$

(see Franzina and Lamberti [8, Theorem 3.1] or the work of Bonheure et al. [1, Examples 4.4 and 4.6] for the case $p \in (1, r]$ while uniqueness follows from the maximum principle in the case $p = 1$). We therefore restrict ourselves to $p$ in the range $1 \leq p \leq r$. Uniqueness can fail when $p > r$ as shown, for example, by Nazarov [15] when $\Omega$ is an annulus. The normalized minimizer when $p \in [1, r)$ depends on the particular region $\Omega$ in $\mathbb{R}^n$ and on the exponents $r$ and $p$.

The minimizer $\phi$ satisfies an Euler–Lagrange partial differential equation with zero boundary values, namely

$$0 = \Delta_r \phi + C_{p,r}(\Omega) \phi^{p-1} = \text{div} \left( |\nabla \phi|^{r-2} \nabla \phi \right) + C_{p,r}(\Omega) \phi^{p-1}, \quad \phi|_{\partial \Omega} = 0. \quad (1.5)$$

It will also be useful to record the scaling law

$$C_{p,r}(R \Omega) = R^{n-r-1} C_{p,r}(\Omega), \quad (1.6)$$

which is straightforward using the change of variables $y = x/R$.

It is clear from its definition (1.3) that if $\Omega \subseteq \tilde{\Omega}$ then $C_{p,r}(\tilde{\Omega}) \geq C_{p,r}(\Omega)$, so that bigger regions have smaller eigenvalues just as bigger drums have lower bass notes. Our intention herein is to quantify the rate of decrease of the eigenvalue $C_{p,r}$ as the region $\Omega$ expands. Assuming that $\Omega$ has $C^1$ boundary that moves with velocity $e^w \eta$, where $\eta = \eta(\zeta)$ is the unit outward normal to $\Omega$ and $w = w(\zeta)$ is a bounded, continuous function on the boundary of $\Omega$, we denote the resulting region at time $t$ by $\Omega_t$. Thus, $C_{p,r}(\Omega_t)$ is a non-increasing function of $t$, which begs the following question: can one bound $C_{p,r} = \frac{d}{dt} C_{p,r}(\Omega_t)|_{t=0}$? Uniqueness of a positive normalized minimizer in the case $1 \leq p \leq r$ leads to the differentiability of the Sobolev–Poincaré constant $C_{p,r}(\Omega)$ as the domain $\Omega$ varies smoothly.

Below we provide some answers, particularly in the case $p = r$ and in the case $n = r = 2$.

**Theorem 1.1.** There is a positive constant $K$ depending only on $n$ and $p$ such that, if $1 < p < n$,

$$- \frac{d}{dt} \left( C_{p,r}(\Omega_t) \right)^{\frac{n-p}{p}} \bigg|_{t=0} \geq \frac{(n-p)^{\frac{1}{p}}}{p} \frac{K^{\frac{1}{p-1}}}{\int_{\partial \Omega_t} e^{(1-p)w} \, d\sigma} \bigg|^{\frac{1}{p-1}} \quad (1.7)$$

and equality can only occur if $\Omega$ is a round ball and $w$ is constant. Also,

$$- \frac{d}{dt} \log C_{n,r}(\Omega_t) \bigg|_{t=0} \geq \frac{(n-1)K^{\frac{1}{n-1}}}{\int_{\partial \Omega_t} e^{(1-n)w} \, d\sigma} \bigg|^{\frac{1}{n-1}} \quad (1.8)$$

and (as before) equality can only occur if $\Omega$ is a round ball and $w$ is constant. The constant $K$ is given explicitly by (4.3).

**Theorem 1.2.** In dimension $n = 2$ and for $p \in [1, 2]$,

$$- \frac{d}{dt} \log C_{p,2}(\Omega_t) \bigg|_{t=0} \geq \frac{8\pi}{p} \frac{1}{\int_{\partial \Omega_t} e^{-w} \, d\sigma}. \quad (1.9)$$

Equality implies that $\Omega$ is a round disk and $w$ is constant.

Note than in the case $p = 1$, Theorem 1.2 is an estimate of the shape derivative of the (Saint–Venant) torsion rigidity.

Following the work of the first and last authors in [4], we also obtain an inequality comparing the eigenvalue $C_{p,p}$ before and after a conformal diffeomorphism. Here $\mathcal{B}$ stands for the unit ball in $\mathbb{R}^n$ and $\mathcal{B}_r$ for the ball of radius $r$. 

...
Theorem 1.3. Let \( n \geq 3 \) and let \( F : B \to \mathbb{R}^n \) be a conformal diffeomorphism. Then, for \( 0 < t < 1 \),

\[
\frac{d}{dt} \log \frac{C_{n,n}(F(B_t))}{C_{n,n}(B_t)} \leq 0. \tag{1.10}
\]

Suppose that \( 1 < p < n \) and that

\[
\int_{\partial B_t} |DF|^{n-p} \, d\sigma \leq |\partial B_t|. \tag{1.11}
\]

Then, for \( 0 < t < 1 \),

\[
\frac{d}{dt} \left[ C_{p,p}(F(B_t))^{\frac{n}{p(n-p)}} - C_{p,p}(B_t)^{\frac{n}{p(n-p)}} \right] \leq 0. \tag{1.12}
\]

Equality in either case can only occur if \( F(B_t) \) is a round ball.

These results, in the special case \( p = 2 \), were first obtained by two of the present authors, Carroll and Ratzkin [4, Theorem 11]. It is straightforward to adapt the proofs below from the Euclidean space setting to the setting of a general class of Riemannian manifolds in which an isoperimetric inequality holds. The discussion in [4] provides details of this particular extension of the results. Also, as described therein, Theorems 1.1 and 1.2 apply to a large collection of geometric flows, such as curvature-driven flows, under appropriate convexity hypotheses. Additionally, one can always apply both results to Hele–Shaw flow. As described in [10], Hele–Shaw flow models a viscous fluid injected into the space between two plates. The boundary \( \partial \Omega \) of the region \( \Omega \) bounding the fluid then moves with velocity \(-\nabla G\), where \( G \) is Green’s function for the Laplacian with a pole inside \( \Omega \subset \mathbb{R}^2 \) corresponding to the injection site. Thus (1.7) reads

\[
- \frac{d}{dt} (C_{p,p}(\Omega_t))^{\frac{2-p}{p(n-p-1)}} \bigg|_{t=0} \geq \frac{(2-p)}{p} \left( \frac{K}{1-p} \right)^{\frac{1}{p-1}}, \quad 1 < p < 2,
\]

and (1.9) reads

\[
- \frac{d}{dt} \log C_{p,2}(\Omega_t) \bigg|_{t=0} \geq \frac{8\pi}{p} \int_{\partial \Omega} \frac{1}{|\nabla G|^{-1}} \, d\sigma, \quad p \geq 1.
\]

Theorem 1.3 can be viewed as a variation on the classical Schwarz Lemma from complex analysis. One might also envisage versions of the Schwarz Lemma for \( n = r = 2 \), using Theorem 1.2, but this is already done in [2] by means of a different technique.

One may reasonably ask what the appropriate version of Theorem 1.1 might be when \( r \to 1^+ \). In the limit the infimum which defines the eigenvalue \( C_{p,r}(\Omega) \) by (1.3) is usually not attained in the Sobolev space \( W^{1,1}_0(\Omega) \), but rather in the space of functions with bounded variation, and so the Hadamard variation formula (2.1) that we use is not valid in the case \( r = 1 \). The article [12] details this phenomenon, and describes some interesting relations with the Cheeger constant.

Our proofs depend on two ingredients: a Hadamard variation formula (2.1), and an inequality (4.2) which reverses the usual Hölder inequality in the case of extremal Sobolev functions. We obtain a reverse-Hölder inequality in the case \( p = r \). One can find the requisite reverse-Hölder inequality for the case \( n = r = 2 \) in [2]. It now seems clear that a reverse-Hölder inequality for Sobolev eigenfunctions, in particular for the exponents \( p - 1 \) and \( p \), is a key step in our technique. We set out in Section 4 the current state of play for reverse-Hölder inequalities in this context. It is tempting to ask for similar results in the remaining cases, when \( p \neq r \), but we lack a reverse-Hölder inequality similar to (4.2).

2 | HADAMARD VARIATION FORMULA

The work of García Melián and Sabina de Lis [9] yields a Hadamard variation formula in a slightly more general setting than we require here. Take \( X : (-\varepsilon, \varepsilon) \times \bar{\Omega} \to \mathbb{R}^n \) to be a time-dependent vector field on the closure of \( \Omega \) and let \( \xi : (-\varepsilon, \varepsilon) \times \bar{\Omega} \to \mathbb{R}^n \) be its flow, so that

\[
\xi(0, x) = x, \quad \frac{\partial \xi}{\partial t}(t, x) = X(t, x).
\]

Set \( \Omega_t = \xi(t, \Omega) \).
Lemma 2.1. For $1 \leq p \leq r$,

$$\dot{C}_{p,r} = \frac{d}{dt} C_{p,r}(\Omega) \Big|_{t=0} = (1 - r) \int_{\partial\Omega} |\nabla \psi|^r (X, \eta) \, d\sigma. \quad (2.1)$$

The proof of the differentiability of $C_{p,r}(\Omega)$ w.r.t. $t$ at $t = 0$ follows from the work of García Melián and Sabina de Lis. One applies their Theorem 2 together with a classical regularizing argument necessitated by the possible degeneracy of the $p$-Laplace operator as on [9, pp. 896–897]. We don’t repeat their argument here.

3 | REARRANGEMENTS

We derive some preliminary rearrangement inequalities needed in Section 4 to prove reverse-Hölder inequalities for the eigenfunctions $\psi$. We set $M = \sup_{x \in \Omega} \psi(x)$ and, for $t \in [0, M]$, set

$$\Omega_t = \{ x \in \Omega : \psi(x) > t \}, \quad \mu(t) = |\Omega_t|.$$

This distribution function $\mu$ is nonincreasing, so it has an inverse function

$$\phi^* : [0, |\Omega|] \rightarrow [0, M], \quad \phi^*(v) = \sup\{ t \in [0, M] : \mu(t) > v \}.$$

Observe that both $\mu$ and $\phi^*$ are differentiable almost everywhere and (when they are both defined) we have

$$\mu'(t) = \frac{1}{(\phi^*)'(\mu(t))} = -\int_{\{\phi=t\}} \frac{d\sigma}{|\nabla \phi|}.$$

In this and the next section we will compare $\psi$ and $\phi^*$ to the corresponding extremal functions $\psi^*$ and $\psi^*$ for $B^*$, the round ball with $C_{p,r}(\Omega) = C_{p,r}(B^*)$, so we take this opportunity to record the equations which $\psi$ and $\psi^*$ satisfy. The function $\psi$ is radial and decreasing, so (see the introduction of [11])

$$-C_{p,r}(\Omega)\psi^{p-1} = -C_{p,r}(B^*)\psi^{p-1} = \Delta \psi$$

$$= \left( -\frac{d\psi}{d\rho} \right)^{r-2} \left[ (r-1) \frac{d^2\psi}{d\rho^2} + \frac{n-1}{\rho} \frac{d\psi}{d\rho} \right]$$

$$= -\rho^{1-n} \frac{d}{d\rho} \left( \rho^{p-1} \left( -\frac{d\psi}{d\rho} \right)^{r-1} \right).$$

We change variables to $v = \omega_n \rho^n$, and define $\psi^*(v) = \psi \left( \left( \frac{v}{\omega_n} \right)^{1/n} \right)$. Then

$$C_{p,r}(\Omega)(\psi^*(v))^{p-1} = n^r \omega_n^{r/n} \frac{d}{dv} \left[ v^{\frac{r(n-1)}{n}} \left( -\frac{d\psi}{d\psi} \right)^{r-1} \right],$$

which we can integrate once and rearrange to read

$$\left( -\left( \psi^*(v) \right)' \right)^{r-1} = n^r \omega_n^{r/n} v^{\frac{r(n-1)}{n}} C_{p,r}(\Omega) \int_0^v \left( \psi^*(\tau) \right)^{p-1} \, d\tau. \quad (3.1)$$

The following is an adaptation of Talenti’s inequality (see (34) of [19]).

Lemma 3.1. We have

$$\left( -\left( \phi^*(v) \right)' \right)^{r-1} \leq n^r \omega_n^{r/n} v^{\frac{r(n-1)}{n}} C_{p,r}(\Omega) \int_0^v \left( \phi^*(\tau) \right)^{p-1} \, d\tau \quad (3.2)$$

for almost every $v$, where $\omega_n$ is the volume of a unit ball in $\mathbb{R}^n$. Moreover, equality can only occur if $\Omega$ is a round ball.
Proof. The fact that $\phi$ is an extremal function implies that

$$C_{p,r}(\Omega) \int_{\Omega} \phi^{r-1} \, d\mu = -\int_{\Omega} \Delta_r \phi \, d\mu = -\int_{\partial \Omega} |\nabla \phi|^{r-2} \frac{\partial \phi}{\partial n} \, d\sigma = \int_{\partial \Omega} |\nabla \phi|^{r-1} \, d\sigma.$$  

We combine this inequality with Hölder’s inequality to obtain

$$|\partial \Omega| = \int_{\partial \Omega} \, d\sigma = \int_{\partial \Omega} |\nabla \phi|^{\frac{r-1}{r}} |\nabla \phi|^{\frac{1}{r}} \, d\sigma$$

$$\leq \left( \int_{\partial \Omega} |\nabla \phi|^{r-1} \, d\sigma \right)^{\frac{1}{r}} \left( \int_{\partial \Omega} |\nabla \phi|^{-1} \, d\sigma \right)^{\frac{r-1}{r}}$$

$$= (-\mu'(t))^{\frac{r-1}{r}} \left( C_{p,r}(\Omega) \int_{\Omega} \phi^{p-1} \, d\mu \right)^{\frac{1}{r}},$$

which we can rearrange to read

$$C_{p,r}(\Omega)(-\mu'(t))^{r-1} \int_{\Omega} \phi^{p-1} \, d\mu \geq |\partial \Omega|^r \geq \left[ \text{const} \left( \frac{1}{\omega_n} \mu(t) \right) \right]^r.$$  

The inequality (3.2) now follows once we change variables to $v = \mu(t)$ and recall $\mu' = \frac{1}{(\phi^*)'}$. Moreover, equality in (3.2) forces equality in our use of the isoperimetric inequality, which forces $\Omega$ to be a round ball.

It is crucial that the right hand sides of (3.1) and (3.2) are essentially the same.

4 | REVERSE-HÖLDER INEQUALITIES

The usual Hölder inequality may be reversed for extremal functions $\phi$ in a number of cases. The first such inequality is that of Payne and Rayner [16] who showed that the first Dirichlet eigenfunction $\phi$ of the Laplacian satisfies

$$\left( \int_{\Omega} \phi \, d\mu \right)^2 \geq \frac{4\pi}{\lambda(\Omega)} \int_{\Omega} \phi^2 \, d\mu,$$

where $\lambda(\Omega)$ is the principal frequency. Various extensions have since been obtained, most notably by Payne and Rayner themselves [17], by Köhler–Jobin [13,14], and by Chiti [6,7]. The following theorem recapitulates the results of all authors and completes some outstanding cases.

**Theorem 4.1.** Let $\phi$ be an extremal function for $C_{p,r}(\Omega)$ and let $0 < q_1 < q_2$. A reverse-Hölder inequality of the form

$$\left( \int_{\Omega} \phi^{q_1} \, d\mu \right)^{\frac{1}{q_1}} \geq C \left( \int_{\Omega} \phi^{q_2} \, d\mu \right)^{\frac{1}{q_2}}$$

holds in the following cases:

- $p = r$,
- $n = 2$ and $r = 2$, $q_1 = p - 1$, and $q_2 = p$,
- $q_1 = p$.

In all cases the constant $C$ depends only on $n$, $p$, $r$, $q_1$, $q_2$. Moreover, the inequality is isoperimetric in that equality implies that $\Omega$ is a round ball.

The case $p = r$ is proved in Proposition 4.4, and the case $q_1 = p$ in Proposition 4.6. One may find a proof of the case $n = 2$, $r = 2$, $q_1 = p - 1$, $q_2 = p$ in [2].
It transpires that the proof of Proposition 4.4 is easier than the proof of Proposition 4.6, primarily because the pde (1.5) is homogeneous only in the case \( p = r \). The homogeneity allows us to multiply the eigenfunctions \( \psi \) and \( \phi \) of the ball and of \( \Omega \) respectively, by convenient constants so that we can choose a scale on which to work. We find it curious that homogeneity proves to be even more important than linearity in this particular context.

We will compare \( \Omega \) to \( B^* \), the round ball with \( C_{p,r}(B^*) = C_{p,r}(\Omega) \). An important tool we use is the Faber–Krahn inequality, which implies that \( |\Omega| \geq |B^*| \), with equality if and only if \( \Omega = B^* \). Let \( \psi \) be an extremal function for \( C_{p,r}(B^*) \). In most of our computations for this section we will temporarily drop the normalizations that \( \|\phi\|_{L^p(\Omega)} = \|\psi\|_{L^p(B^*)} = 1 \). The method of proof here is due to Chiti [6,7].

**Proposition 4.2.** Let \( \phi \) be an extremal function on \( \Omega \) and let \( \psi \) be the extremal function on the ball \( B^* \) for which \( C_{p,r}(B^*) = C_{p,r}(\Omega) \). Normalize both \( \phi \) and \( \psi \) so that \( \|\phi\|_{L^\infty(\Omega)} = \|\psi\|_{L^\infty(B^*)} \). Then,

\[
\phi^*(v) \geq \psi^*(v) \quad \text{for} \quad 0 < v < |B^*|.
\]

Moreover, equality can occur for some \( v > 0 \) only if \( \Omega = B^* \).

**Proof.** If \( |\Omega| = |B^*| \) then \( \Omega = B^* \) by the Faber–Krahn inequality, and there is nothing to prove. Accordingly, we assume that \( |\Omega| > |B^*| \). In this case

\[
\phi^*(0) = \psi^*(0) = \|\phi\|_{L^\infty}, \quad \psi^*(|B^*|) = 0 < \phi^*(|B^*|).
\]

There must, therefore, exist some \( k > 1 \) such that \( k\phi^*(v) \geq \psi^*(v) \) on the interval \([0, |B^*|]\). We set

\[
k_0 = \inf \left\{ k > 1 : k\phi^* \geq \psi^* \text{ on } [0, |B^*|] \right\}
\]

and complete the proof by showing that \( k_0 = 1 \).

If \( k_0 > 1 \) then there exists \( v_0 \in (0, |B^*|) \) such that \( \psi^*(v) = k_0\phi^*(v) \) and \( \psi^*(v) < k_0\phi^*(v) \) on \((0, v_0)\). This permits the construction of a function \( u^* \) via

\[
u^* : [0, |B^*|] \longrightarrow \mathbb{R}, \quad u^*(v) = \begin{cases} k_0\phi^*(v), & 0 \leq v < v_0, \\ \psi^*(v), & v_0 \leq v \leq |B^*|, \end{cases}
\]

and subsequently a function \( u \) on the ball \( B^* \) by \( u(x) = u^*(|x|^n) \). It then follows from (3.1) and (3.2) that, for \( 0 < v < |B^*| \),

\[
\left(-\left( u^*(v) \right) \right)^{p-1} \leq n^{-r} \omega_n^{-p/q} \int_{B^*} u^{(p/q)(1+1/n)} \int_{v}^{u^*(v)} \left( u^*(\tau) \right)^{p-1} d\tau \ dx
\]

Hence,

\[
\int_{B^*} |\nabla u|^p \, d\mu = \int_{B^*} \left( \frac{du}{d\rho} \right)^p \, d\mu = \int_{0}^{|B^*|} n^p \omega_n^{p/n} \int \left( (u^*(v))^p \right)^{p-1} \int_{0}^{u^*(v)} v^p \, dv \, d\mu
\]

\[
\leq C_{p,r}(\Omega) \int_{0}^{|B^*|} \left(-\left( u^*(v) \right) \right)^{p-1} \int_{0}^{u^*(v)} \left( u^*(\tau) \right)^{p-1} d\tau \ dx
\]

\[
= C_{p,r}(\Omega) \int_{0}^{|B^*|} \left( u^*(\tau) \right)^{p-1} \int_{0}^{u^*(\tau)} \left( u^*(v) \right) d\tau \ dx
\]

\[
= C_{p,r}(\Omega) \int_{B^*} u^p \, d\mu.
\]

However, \( C_{p,r}(B^*) = C_{p,r}(\Omega) \), so this is only possible if \( u = \psi \), which cannot occur because \( u^* > \psi^* \) on \((0, v_0)\).

**Corollary 4.3.** For any \( q \geq 0 \) we have the scale-invariant inequality

\[
\frac{\|\phi\|_{L^q}}{\|\phi\|_{L^\infty}} \geq \frac{\|\psi\|_{L^q}}{\|\psi\|_{L^\infty}} \quad \text{with equality if and only if } \Omega = B^*.
\]
Proof. This follows from integration of the inequality in Proposition 4.2.

Proposition 4.4. Let \( 0 < q_1 < q_2 < \infty \). There exists \( K \) depending only on \( n, p, q_1, \) and \( q_2 \) such that

\[
\left( \int_{\Omega} \phi^{q_1} \, d\mu \right)^{q_2} \geq K(C_{p,p}(\Omega))^{-\frac{q_2}{p(q_2-q_1)}} \left( \int_{\Omega} \phi^{q_2} \, d\mu \right)^{q_1},
\]

and equality implies that \( \Omega \) is a round ball.

Proof. If \(|\Omega| = |B^*|\) there is nothing to prove, so we assume \(|\Omega| > |B^*|\). This time we choose the normalization

\[
\int_{\Omega} \phi^{q_1} \, d\mu = \int_{B^*} \psi^{q_1} \, d\mu,
\]

so that (4.1) implies that

\[
\psi^*(0) = \|\psi\|_{L^\infty} > \|\phi\|_{L^\infty} = \phi^*(0).
\]

We also know that \( \phi^*(|B^*|) > 0 = \psi^*(|B^*|) \). Consequently, the graphs of the functions \( \phi^* \) and \( \psi^* \) must cross somewhere in the interval \((0, |B^*|)\). Let

\[
v_0 = \inf \{ v \in [0, |B^*|] : \phi^*(v) \leq \psi^*(v) \text{ for all } \bar{v} \in (0, v) \}
\]

be the first crossing when viewed from the left. Then

\[
0 < v_0 < |B^*|, \quad \psi^* \geq \phi^* \text{ in } [0, v_0], \quad \psi^*(v_0) = \phi^*(v_0),
\]

and there exists \( v \in (v_0, |B^*|) \) such that \( \phi^*(v) > \psi^*(v) \). In fact, by continuity the inequality \( \phi^* > \psi^* \) must hold in a nontrivial interval \( I \) surrounding \( v \).

We claim that \( \phi^* > \psi^* \) on the entire interval \((v_0, |B^*|)\). Supposing otherwise, there must exist \( v_1 \in (v_0, |B^*|) \) with \( \phi^*(v_1) = \psi^*(v_1) \). We can then define

\[
u^*(v) = \begin{cases} 
\psi^*(v), & 0 \leq v \leq v_0, \\
\max\{\psi^*(v), \phi^*(v)\}, & v_0 \leq v \leq v_1, \\
\phi^*(v), & v_1 \leq v \leq |B^*|,
\end{cases}
\]

and \( u(x) = \nu^*(\omega_n|x|^n) \). Again by (3.1) and (3.2) we have

\[
(-(u^*)'(v))^{p-1} \leq n^{-\frac{1}{p}}\omega_n^{-\frac{1}{pn}}v^{\frac{\mu_1-n}{n}}C_{p,p}(\Omega) \int_{0}^{v} (u^*(\tau))^{p-1} \, d\tau
\]

so, as in the proof of Proposition 4.2, we have

\[
\int_{B^*} |\nabla u|^p \, d\mu = \int_{B^*} \left( \frac{du}{d\rho} \right)^p \, d\mu = \int_0^{|B^*|} n^p\omega_n^{\frac{1}{pn}}v^{\frac{\mu_1-n}{n}}((u^*)'(v))^p \, dv
\]

\[
\leq C_{p,p}(\Omega) \int_0^{|B^*|} (-(u^*)'(v)) \int_0^{v} (u^*(\tau))^{p-1} \, d\tau \, dv
\]

\[
= C_{p,p}(\Omega) \int_0^{|B^*|} (u^*(\tau))^{p-1} \int_\tau^{|B^*|} (-(u^*)'(v)) \, dv \, d\tau
\]

\[
= C_{p,p}(\Omega) \int_{B^*} u^p \, d\mu.
\]

That \( C_{p,p}(\Omega) = C_{p,p}(B^*) \) now implies \( u \) is a multiple of \( \psi \), which is impossible.

We conclude that \( \psi^* \geq \phi^* \) on \([0, v_0]\) and \( \psi^* < \phi^* \) on \((v_0, |B^*|)\). Then the argument used to prove [4, Theorem 7] shows that

\[
\left( \int_{\Omega} \phi^{q_2} \, d\mu \right)^{1/q_2} \leq \left( \int_{B^*} \psi^{q_2} \, d\mu \right)^{1/q_2} \left( \int_{B^*} \psi^{q_2} \, d\mu \right)^{1/q_1} \left( \int_{\Omega} \phi^{q_1} \, d\mu \right)^{1/q_1}.
\]
which we can rewrite as

\[
\left( \int_{\Omega} \phi^{q_1} \, d\mu \right)^{q_2} \geq \tilde{C} \left( \int_{\Omega} \phi^{q_2} \, d\mu \right)^{q_1}, \quad \tilde{C} = \frac{\left( \int_{B^*} \psi^{q_1} \, d\mu \right)^{q_2}}{\left( \int_{B^*} \psi^{q_2} \, d\mu \right)^{q_1}}.
\]

All that remains is to unravel the constant \( \tilde{C} \). Let \( R \) be the radius of \( B^* \) and define the function

\[
\tilde{\psi} : B_1 \longrightarrow \mathbb{R}, \quad \tilde{\psi}(x) = \psi(Rx).
\]

Then \( \tilde{\psi} \) is an extremal function for \( C_{p,p}(B_1) \), because the PDE (1.5) is homogeneous in the case \( p = r \). By (1.6) we have

\[
C_{p,p}(B^*) = C_{p,p}(B_R) = R^{-p}C_{p,p}(B_1)
\]

which implies that

\[
R = \left( \frac{C_{p,p}(B^*)}{C_{p,p}(B_1)} \right)^{-1/p} = \left( \frac{C_{p,p}(\Omega)}{C_{p,p}(B_1)} \right)^{-1/p}.
\]

Thus

\[
\tilde{C} = R^{n(q_2 - q_1)} \frac{\left( \int_{B_1} \tilde{\psi}^{q_1} \, d\mu \right)^{q_2}}{\left( \int_{B_1} \tilde{\psi}^{q_2} \, d\mu \right)^{q_1}} = \frac{\left( \int_{B_1} \tilde{\psi}^{q_1} \, d\mu \right)^{q_2}}{\left( \int_{B_1} \tilde{\psi}^{q_2} \, d\mu \right)^{q_1}}.
\]

We will use the case of \( q_1 = p - 1 \) and \( q_2 = p \) in the next section.

**Corollary 4.5.** There exists a constant \( K \) depending only on \( n \) and \( p \) such that, in the case of any extremal function \( \phi \) for \( C_{p,p}(\Omega) \),

\[
\left( \int_{\Omega} \phi^{p-1} \, d\mu \right)^{p} \geq K(C_{p,p}(\Omega))^{-n/p} \left( \int_{\Omega} \phi^p \, d\mu \right)^{p-1}.
\]

The constant \( K \) is given explicitly as

\[
K = \left( \frac{C_{p,p}(B_1)}{C_{p,p}(\Omega)} \right)^{\frac{n}{p}} \frac{\left( \int_{B_1} \tilde{\psi}^{p-1} \, d\mu \right)^{p}}{\left( \int_{B_1} \tilde{\psi}^p \, d\mu \right)^{p-1}}.
\]

where \( \tilde{\psi} \) is an extremal function for \( C_{p,p} \) in the case of the unit ball \( B_1 \). Equality can only occur if \( \Omega \) is a round ball.

We close this section with a result generalizing the main theorem of [3].

**Proposition 4.6.** Let \( 1 \leq r < n, 1 \leq p \leq r, \) and \( q > p \). There exists \( K > 0 \) depending only on \( n, r, p, q \) such that

\[
\left( \int_{\Omega} \phi^{q} \, d\mu \right)^{1/p} \geq K(C_{p,p}(\Omega))^{\frac{n(q-p)}{p(n-p)+q}} \left( \int_{\Omega} \phi^{q} \, d\mu \right)^{1/q},
\]

for all extremal functions \( \phi \) for \( C_{p,p}(\Omega) \).

\[ \text{Proof 12.} \] As before we let \( B^* \) be the ball with \( C_{p,p}(\Omega) = C_{p,p}(B^*) \), and let \( \psi \) be the corresponding extremal function on \( B^* \). By the Faber–Krahn inequality, we have \( |\Omega| \geq |B^*| \), with equality if and only if \( \Omega = B^* \). In the case \( |\Omega| = |B^*| \) we must also have equality in (4.4), which will more precisely read

\[
\left( \int_{\Omega} \phi^q \, d\mu \right)^{1/p} \geq \tilde{K} \left( \int_{\Omega} \phi^q \, d\mu \right)^{1/q}, \quad \tilde{K} = \frac{\left( \int_{B^*} \psi^p \, d\mu \right)^{1/p}}{\left( \int_{B^*} \psi^q \, d\mu \right)^{1/q}};
\]
we will see that, in fact, this constant $\tilde{K}$ is optimal in general. Furthermore, if we let $R$ be the radius of $B^*$ then (1.6) implies that

$$R = \left( \frac{C_{p,r}(B^*)}{C_{p,r}(B)} \right)^{\frac{p}{n(p-r)p}}.$$ 

so that

$$\tilde{K} = \left( \frac{\int_{B^*} |\psi|^p \, d\mu}{\int_{B^*} |\psi|^q \, d\mu} \right)^{1/p} = R \left( \frac{\int_{B^*} |\tilde{\psi}|^p \, d\mu}{\int_{B^*} |\tilde{\psi}|^q \, d\mu} \right)^{1/q} = \left( \frac{C_{p,r}(B^*)}{C_{p,r}(B)} \right)^{\frac{p(q-p)}{n(q-p-r)p}} \left( \frac{\int_{B^*} |\tilde{\psi}|^p \, d\mu}{\int_{B^*} |\tilde{\psi}|^q \, d\mu} \right)^{1/q} = K \left( C_{p,r}(\Omega) \right)^{\frac{p(q-p)}{n(q-p-r)p}},$$

where $\tilde{\psi} : B \to \mathbb{R}$, $\tilde{\psi}(x) = \psi(Rx)$, is the extremal function on the unit ball $B$.

Next we treat the case $|\Omega| > |B^*|$. Normalize both extremal functions $\phi$ and $\psi$ so that

$$\int_{\Omega} \phi^p \, d\mu = 1, \quad \int_{B^*} \psi^p \, d\mu = 1.$$ 

Combining these normalizations with $|\Omega| > |B^*|$ we see that

$$1 = \int_0^{\Omega} (\psi^*)^p \, dv = \int_0^{\Omega} (\phi^*)^p \, dv > \int_0^{\Omega} (\phi^*)^p \, dv,$$

which implies we cannot have $\psi^* \leq \phi^*$ on the whole of the interval $[0, |B^*|]$. On the other hand, we know that

$$\psi^*(|B^*|) = 0 < \phi^*(|B^*|),$$

so the graphs of these two functions must cross. Define

$$v_1 = \inf \{ v \in [0, |B^*|] : \psi^*(\tilde{v}) < \phi^*(\tilde{v}) \text{ for all } \tilde{v} \in (v, |B^*|) \};$$

this is the first crossing of the two graphs, when viewed from the right hand side. By continuity, $\psi^*(v_1) = \phi^*(v_1)$ and $\psi^* < \phi^*$ on the interval $(v_1, |B^*|)$. We also cannot have $v_1 = 0$, as this would contradict (4.5).

We claim that $\psi^* \geq \phi^*$ on the interval $[0, v_1]$. Indeed, if this inequality did not hold, then we would have $\psi^*(v_2) < \phi^*(v_2)$ for some $v_2 \in [0, v_1)$, and by continuity this inequality would extend to an interval containing $v_2$. In that case, the function

$$w^* : [0, |B^*|] \to [0, \infty), \quad w^*(v) = \begin{cases} \max \{ \phi^*(v), \psi^*(v) \}, & 0 \leq v \leq v_1, \\ \psi^*(v), & v_1 \leq v \leq |B^*|. \end{cases}$$

satisfies

$$-(w^*(v))^{r-1} \leq n^{-r(n-1)} C_{p,r}(\Omega)^{\frac{r(n-1)}{n}} \int_0^v (w^*(\tau))^{p-1} \, d\tau$$

by (3.1) and (3.2). Now we can define a function

$$w : B^* \to \mathbb{R}, \quad w(x) = w^*(\omega_n |x|^n),$$

for which

$$\int_{B^*} |\nabla w|^r \, d\mu = \int_{B^*} \left( \frac{d}{d\rho} \right)^r w \, d\mu = \int_0^{B^*} n^{r(n-1)} \omega_n^{-r(n-1)} ((w^*(v))^{r-1})^{\frac{n-1}{n}} \, dv.$$
≤ C_{p,r}(\Omega) \int_{0}^{B^*} (-u^*(v)) \int_{0}^{v} (u^*(\tau))^{p-1} d\tau dv
= C_{p,r}(\Omega) \int_{0}^{B^*} (u^*(\tau))^{p-1} \int_{\tau}^{B^*} (-u^*(v)) dv d\tau
= C_{p,r}(\Omega) \int_{B^*} w^p d\mu.

Since C_{p,r}(\Omega) = C_{p,r}(B^*), the function w must be extremal on B^*, which implies w must be a scalar multiple of \psi. This would contradict the fact that \psi < \phi on an interval containing v_2.

We’ve concluded that the graphs of \phi^* and \psi^* cross exactly once on the interval [0,|B^*|]. The remainder of the argument is exactly the same as that in [3], and we refer the reader to this treatment.

5 | PROOF OF THEOREM 1.2

The proof of Theorem 1.2 is easiest, so we present it first.

We will need a reverse-Hölder inequality proved in [2], which reads

\[ \left( \int_{\Omega} \phi^{p-1} d\mu \right)^{\frac{2}{p}} \geq \frac{8\pi}{p C_{p,2}(\Omega)} \left( \int_{\Omega} \phi^p d\mu \right)^{\frac{2p-2}{p}}. \] (5.1)

In our setting \( X|_{\partial\Omega} = e^{w} \eta \), so

\[ -\hat{C}_{p,2} = \int_{\partial\Omega} e^{w} \left( \frac{\partial \phi}{\partial \eta} \right)^2 d\sigma \]
\[ \geq \frac{1}{\int_{\partial\Omega} e^{-w} d\sigma} \left( \int_{\partial\Omega} \frac{\partial \phi}{\partial \eta} d\sigma \right)^2 \]
\[ = \frac{1}{\int_{\partial\Omega} e^{-w} d\sigma} \left( \int_{\partial\Omega} \Delta \phi d\mu \right)^2 \]
\[ = \frac{(C_{p,2}(\Omega))^2}{\int_{\partial\Omega} e^{-w} d\sigma} \left( \int_{\partial\Omega} \phi^{p-1} d\mu \right)^2 \]
\[ \geq \frac{8\pi C_{p,2}(\Omega)}{p \int_{\partial\Omega} e^{-w} d\sigma} \left( \int_{\partial\Omega} \phi^p d\mu \right)^{\frac{2p-2}{p}} = \frac{8\pi C_{p,2}(\Omega)}{p \int_{\partial\Omega} e^{-w} d\sigma}, \]

which proves (1.9). Here we have first used (2.1), followed by the Cauchy–Schwarz inequality, the divergence theorem, (1.5), (5.1), and (1.4). Furthermore, equality in (1.9) forces equality in all the inequalities we have used. Equality in our use of (5.1) can only occur if \( \Omega \) is a round disk and equality in our use of the Cauchy–Schwarz inequality can only occur if \( w \) is constant. □

6 | PROOF OF THEOREM 1.1

We first observe that

\[ |\nabla \phi|^{p-1} = e^{\frac{w(p-1)}{p}} |\nabla \phi|^{p-1} e^{\frac{w(1-p)}{p}}, \]

so use of Hölder’s inequality with exponents \( \frac{p}{p-1} \) and \( p \) yields

\[ \int_{\partial\Omega} |\nabla \phi|^{p-1} d\sigma \leq \left( \int_{\partial\Omega} e^{\frac{w(p-1)}{p}} |\nabla \phi|^{p-1} e^{\frac{w(1-p)}{p}} d\sigma \right)^{\frac{p}{p-1}} \left( \int_{\partial\Omega} e^{(1-p)w} d\sigma \right)^{\frac{1}{p}}. \]
This we can rewrite as
\[ \int_{\partial \Omega} e^u |\nabla \phi|^p \, d\sigma \geq \left( \frac{\int_{\partial \Omega} |\nabla \phi|^{p-1} \, d\sigma}{\int_{\partial \Omega} e^{(1-p)w} \, d\sigma} \right)^{\frac{p}{p-1}}. \]

Thus (2.1), Hölder’s inequality, the divergence theorem, and (4.2) combine to give us

\[ -\dot{C}_{p,p} = (p-1) \int_{\partial \Omega} e^u |\nabla \phi|^p \, d\sigma \]
\[ \geq \frac{p-1}{(\int_{\partial \Omega} e^{(1-p)w} \, d\sigma)^{\frac{1}{p-1}}} \left( \int_{\partial \Omega} |\nabla \phi|^{p-1} \, d\sigma \right)^{\frac{p}{p-1}} \]
\[ = \frac{p-1}{(\int_{\partial \Omega} e^{(1-p)w} \, d\sigma)^{\frac{1}{p-1}}} \left( \int_{\Omega} -|\nabla \phi|^p \, d\mu \right)^{\frac{1}{p-1}} \]
\[ = \frac{p-1}{(\int_{\partial \Omega} e^{(1-p)w} \, d\sigma)^{\frac{1}{p-1}}} \left( \int_{\Omega} \Delta \phi \, d\mu \right)^{\frac{1}{p-1}} \]
\[ = \frac{(p-1)(C_{p,p}(\Omega))^{\frac{p}{p-1}}}{(\int_{\partial \Omega} e^{(1-p)w} \, d\sigma)^{\frac{1}{p-1}}} \left[ \left( \int_{\Omega} \phi \, d\mu \right)^{\frac{1}{p-1}} \right] \]
\[ \geq \frac{(p-1)(C_{p,p}(\Omega))^{\frac{p}{p-1}}}{(\int_{\partial \Omega} e^{(1-p)w} \, d\sigma)^{\frac{1}{p-1}}} \left[ K(C_{p,p}(\Omega))^{-n/p} \left( \int_{\Omega} \phi \, d\mu \right)^{p-1} \right]^{\frac{1}{p-1}} \]
\[ = \frac{(p-1)K^{\frac{1}{p-1}}}{(\int_{\partial \Omega} e^{(1-p)w} \, d\sigma)^{\frac{1}{p-1}}} \left( C_{p,p}(\Omega) \right)^{\frac{1}{p-1}} \left( \phi \phi \right)^{\frac{1}{p-1}}. \]

(6.1)

Here \( K \) is given by (4.3). In the case \( 1 < p < n \) we can rearrange (6.1) to read

\[ -(C_{p,p}(\Omega))^{\frac{1}{p-1}} \phi \phi \geq \frac{(p-1)K^{\frac{1}{p-1}}}{(\int_{\partial \Omega} e^{(1-p)w} \, d\sigma)^{\frac{1}{p-1}}}, \]

which implies (1.7). In the case \( p = n \) we rewrite (6.1) as

\[ \frac{-\dot{C}_{n,n}}{C_{n,n}(\Omega)} \geq \frac{(n-1)K^{\frac{1}{n-1}}}{(\int_{\partial \Omega} e^{(1-n)w} \, d\sigma)^{\frac{1}{n-1}}}, \]

which implies (1.8). Moreover, equality in (6.1) forces equality in (4.2), which in turn forces \( \Omega \) to be a round ball. Also, equality in our use of the Hölder inequality can only occur if \( e^{\frac{u(p-1)}{p}} \) is a multiple of \( e^{\frac{u(p-1)}{p}} \), which forces \( w \) to be constant. \( \square \)

7 | PROOF OF THEOREM 1.3

For a conformal diffeomorphism \( F : B \to \mathbb{R}^n \) we compare, for \( 0 < t < 1 \), the eigenvalue of the ball \( B_t \) to that of its conformal image \( \Omega_t = F(B_t) \). We set \( C_{p,p}(t) = C_{p,p}(B_t) \) and write \( \phi_t \) for the associated extremal function. We set \( \hat{C}_{p,p}(t) = C_{p,p}(\Omega_t) \), with its associated extremal function \( \hat{\phi}_t \). As usual, we choose the normalization

\[ \int_{B_t} \phi^p \, d\mu = 1 = \int_{\Omega_t} (\hat{\phi})^p \, d\tilde{\mu}. \]
We also set \( \psi = \tilde{\phi} \circ F \) and notice that, because \( F \) is conformal, \( |\nabla \psi| = |DF| |\nabla \tilde{\phi}| \). In this setting, the Hadamard variation formula (2.1) reads

\[
\frac{d}{dt} \tilde{C}_{p,p}(t) = (1 - p) \int_{\partial \Omega_t} |DF| |\nabla \tilde{\phi}|^p d\tilde{\sigma} = (1 - p) \int_{\partial B_t} |DF|^{n-p} |\nabla \psi|^p d\sigma. 
\]

We first combine the normalization \( \| \tilde{\phi} \|_{L^p(\Omega_t)} = 1 \) with the reverse Hölder inequality (4.2) to see that, with \( K \) given by (4.3),

\[
K \tilde{C}_{p,p}(t)^{-n/p} = K \tilde{C}_{p,p}(t)^{-n/p} \left( \int_{\Omega_t} (\tilde{\phi})^p d\tilde{\mu} \right)^{p-1} \leq \left( \int_{\Omega_t} |DF|^{n-p} |\nabla \psi|^{p-1} d\sigma \right)^p \leq \left( \int_{\partial B_t} |DF|^{n-p} |\nabla \psi|^{p-1} d\sigma \right)^p \leq \left( \int_{\partial B_t} |DF|^{n-p} d\sigma \right) \left( \int_{\partial B_t} |DF|^{n-p} |\nabla \psi|^p d\sigma \right)^{p-1} \leq \left( \frac{1}{1 - p} \frac{d}{dt} \tilde{C}_{p,p}(t) \right)^{p-1} \int_{\partial B_t} |DF|^{n-p} d\sigma.
\]

From the case of equality in (4.2) and in Hölder's inequality, equality holds in (7.2) if and only if \( F(\mathbf{B}_t) \) is a round ball and \( |DF| \) is constant.

In the case \( p = n \), the estimate (7.2) may be rewritten as

\[
- \frac{d}{dt} \log \tilde{C}_{n,n}(t) \geq \frac{(n - 1)K^{-1/n}}{|\partial B_t|^{-1/n}}.
\]

with equality holding if and only if \( F(\mathbf{B}_t) \) is a round ball and \( |DF| \) is constant. Thus the right hand side of (7.3) equates to

\[
- \frac{d}{dt} \log C_{n,n}(t) \text{ and (1.10) follows.}
\]

If \( 1 < p < n \), then (7.2) may be rewritten as

\[
- \frac{d}{dt} \left( \tilde{C}_{p,p}(t) \right)^{\frac{n-p}{p-1}} \geq \left( \frac{\left( \frac{n-p}{p} \right) K^{-1/n} \left( \int_{\partial B_t} |DF|^{n-p} d\sigma \right)^{-1/p} } { \left( \int_{\partial B_t} |DF|^{n-p} d\sigma \right)^{1/p} } \right).
\]
with equality holding if and only if $F(B_t)$ is a round ball and $|DF|$ is constant. In particular,

$$-\frac{d}{dt} \left( C_{p,p} (t) \right)^{\frac{n-p}{p(p-1)}} = \frac{\left( \frac{n-p}{p} \right) K^{\frac{1}{p-1}}}{|\partial B_1|^{\frac{1}{p-1}}}.$$

The estimate (1.12) now follows once

$$\int_{\partial B_t} |DF|^{n-p} \, d\sigma \leq |\partial B_t|.$$

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