ON LOCAL TAMENESS OF CERTAIN GRAPHS OF GROUPS

RITA GITIK

Abstract. Let $G$ be the fundamental group of a finite graph of groups with Noetherian edges and locally tame vertices. We prove that $G$ is locally tame. It follows that if a finitely presented group $H$ has a non-trivial JSJ-decomposition over the class of its $VPC(k)$ subgroups for $k = 1$ or $k = 2$, and all the vertex groups in the decomposition are flexible, then $H$ is locally tame.

Keywords: Noetherian group, locally tame group, graph product, JSJ-decomposition, covering space, fundamental group.

1. Introduction

Let $H$ be a subgroup of a group $G$ given by the presentation $G = \langle X | R \rangle$. Let $K$ be the standard presentation 2-complex of $G$, i.e. $K$ has one vertex, $K$ has an edge, which is a loop, for every generator $x \in X$, and $K$ has a 2-cell for every relator $r \in R$. The Cayley complex of $G$, denoted by $\text{Cayley}_2(G)$, is the universal cover of $K$. Denote by $\text{Cayley}_2(G, H)$ the cover of $K$ corresponding to a subgroup $H$ of $G$.

Definition 1. cf. [2] and [7].

A finitely generated subgroup $H$ of a finitely presented group $G$ is tame in $G$ if for any finite subcomplex $C$ of $\text{Cayley}_2(G, H)$ and for any component $C_0$ of $\text{Cayley}_2(G, H) - C$ the group $\pi_1(C_0)$ is finitely generated.

A manifold $M$ is called a missing boundary manifold if it can be embedded in a compact manifold $\overline{M}$ such that $\overline{M} - M$ is a closed subset of the boundary of $\overline{M}$. Simon conjectured in [11] that if $M_0$ is a compact orientable irreducible 3-manifold, and $M$ is the cover of $M_0$ corresponding to a finitely generated subgroup of $\pi_1(M_0)$, then $M$ is a missing boundary manifold. Perelman’s solution of Thurston’s Geometrization Conjecture in 2003 implies that Simon’s conjecture holds for all compact orientable irreducible 3-manifolds, cf. [1] and [5].

Tucker proved in [12] that a non-compact orientable irreducible 3-manifold $M$ is a missing boundary manifold if and only if the trivial subgroup is tame in the fundamental group of $M$.

It is not known if there exists a finitely generated subgroup $H$ of a finitely presented group $G$ such that $H$ is not tame in $G$.

Tameness of a subgroup is connected to other properties which have been studied for a long time.

It is shown in [8] that if the trivial subgroup is tame in $G$ then $\pi_1^\infty(G)$ (the fundamental group at infinity of $G$) is pro-finitely generated. It is shown in [7]...
that if a finitely generated subgroup $H$ is tame in $G$ then $\pi_1^\infty(G, H)$ is pro-finitely generated.

It is shown in [8] that if the trivial subgroup is tame in $G$ then $G$ is QSF (Quasi-Simply-Filtrated).

The following definition was given in [3].

**Definition 2.** A group $G$ is locally tame if all finitely generated subgroups of $G$ are tame in $G$.

Recall that a group is called Noetherian or slender if all its subgroups are finitely generated. A group is polycyclic if it is Noetherian and solvable. For $n \geq 0$ a group $G$ is VPC($n$), (virtually polycyclic of length $n$) if it has $n+1$ subgroups, $G_0, \cdots, G_n$ such that $G_{i+1}$ is a normal subgroup of $G_i$ for $0 \leq i \leq n-1$, the quotient groups $G_i/G_{i+1}$ are isomorphic to $\mathbb{Z}$ for $0 \leq i \leq n-1$, $G_n$ is the trivial subgroup, and $G_0$ has finite index in $G$.

Note that VPC(0) groups are finite, VPC(1) groups are finite extensions of $\mathbb{Z}$, and VPC(2) groups are finite extensions of an extension of $\mathbb{Z}$ by $\mathbb{Z}$. There are only two non-isomorphic extensions of $\mathbb{Z}$ by $\mathbb{Z}$, namely the fundamental group of a torus and the fundamental group of a Klein bottle.

It is unknown whether all finitely presented Noetherian groups are virtually polycyclic (question 11.38 from the Kourovka Notebook [6]), however there exist finitely generated Noetherian groups that are not virtually polycyclic, for example the Tarski monster.

The main results of this paper is the following theorem.

**Theorem 1.** Let $G$ be a finitely presented group which is the fundamental group of a finite graph of groups with Noetherian edge groups. If all the vertex groups of $G$ are locally tame then $G$ is locally tame.

Recall that a subgroup $H$ is elliptic in a graph of groups $G$ if $H$ is contained in a conjugate of a vertex group. A vertex group $K$ of a JSJ-decomposition of $G$ which fails to be elliptic in some other JSJ-decomposition of $G$ is called flexible, cf. [4].

Theorem 1 implies the following interesting result.

**Lemma 1.** If a finitely presented group $G$ has a non-trivial JSJ-decomposition over the class of its VPC($k$) subgroups for $k = 1$ or $k = 2$, and all the vertex groups in the decomposition are flexible, then $G$ is locally tame.

**Corollary 1.** Let $G$ be the fundamental group of a finite graph of groups which has all the vertex groups homeomorphic to $\mathbb{Z}^n \times$ (surface group) and all the edge groups homeomorphic to $\mathbb{Z}^{n+1}$. Then $G$ is locally tame.

**Remark 1.** Let $G$ be a finitely presented group which has a JSJ-decomposition over the class of its VPC($n+1$) subgroups. Let $K$ be a flexible vertex group of this decomposition. Then $K$ is either VPC($n+1$) or $K$ has a finite index subgroup $L$ such that $L$ has a normal VPC($n$) subgroup $N$ with $L/N$ the fundamental group of a surface. Furthermore, if $L/N$ is the fundamental group of a closed surface, then $K = G$.

**Conjecture.** If a finitely presented group $G$ has a non-trivial JSJ-decomposition over the class of its VPC($n+1$) subgroups for $n \geq 0$, and all the vertex groups in the decomposition are flexible, then $G$ is locally tame.
2. Proof of Theorem 1

We need the following notation.
Let $X^* = \{x, x^{-1} | x \in X\}$. For $x \in X$ define $(x^{-1})^{-1} = x$.

Let $G$ be a group generated by a set $X$ and let $H$ be a subgroup of $G$. Let $\{H^g\}$ be the set of right cosets of $H$ in $G$.

The coset graph of $G$ with respect to $H$, denoted $Cayley(G, H)$, is the oriented graph whose vertices are the cosets $\{H^g\}$, the set of edges is $\{H^g\} \times X^*$, and an edge $(H^g, x)$ begins at the vertex $H^g$ and ends at the vertex $H^{gx}$. Denote the Cayley graph of $G$ by $Cayley(G)$. Note that $Cayley(G, H)$ is the quotient of $Cayley(G)$ by left multiplication by $H$. Also note that the 1-skeleton of $Cayley_2(G)$ is $Cayley(G)$, and the 1-skeleton of $Cayley_2(G, H)$ is $Cayley(G, H)$.

Let $G$ be generated by a disjoint union of sets $X_i, 1 \leq i \leq n$. We call a connected subcomplex $C$ of $Cayley(G, H)$ an $X_i$-component, if all edges of $C$ have the form $(H^g, x)$ with $x \in X_i^*$.

**Proof of Theorem 1.**

Let $G$ be a finite graph of groups with vertex groups $V_i, 1 \leq i \leq n$ and edge groups $E_j, 1 \leq j \leq m$. As $G$ is finitely presented and all the edge groups are Noetherian, hence finitely generated, it follows that all the vertex groups are finitely presented. Let the vertex group $V_i$ be generated by a finite set $X_i$ such that the sets $X_i$ and $X_k$ are disjoint for $i \neq k$.

Consider a finitely generated subgroup $H$ of $G$. Note that $H$ is the fundamental group of a (possibly infinite) graph of groups which has the vertex groups isomorphic to subgroups of conjugates of $V_i$ and the edge groups isomorphic to subgroups of conjugates of $E_j$. [10].

As the edge groups of $G$ are Noetherian, the edge groups of $H$ are also Noetherian and the vertex groups of $H$ are finitely generated.

Note that all maximal $X_i$-components of $Cayley_2(G, H)$ have fundamental groups which are subgroups of conjugates of $V_i$, hence the maximal $X_i$-components of $Cayley_2(G, H)$ are homeomorphic to $Cayley_2(V_i, U_i)$, with $U_i$ a finitely generated subgroup of $V_i$.

As $H$ is finitely generated, there exists a finite connected subcomplex $(K, H \cdot 1)$ of $Cayley_2(G, H)$ such that the inclusion map of $(K, H \cdot 1)$ in $Cayley_2(G, H)$ induces an isomorphism of $\pi_1(K, H \cdot 1)$ with $\pi_1(Cayley_2(G, H), H \cdot 1) = H$.

Let $C$ be a compact subcomplex of $Cayley_2(G, H)$. Note that $C$ has non-empty intersection with only finitely many maximal $X_i$-components of $Cayley_2(G, H)$. The complex $K$ can be enlarged to contain $C$. It can be enlarged more, so it consists of finitely many maximal $X_i$-components of $Cayley_2(G, H)$ which have non-trivial intersection with $C$ and the 2-cells with boundaries in the union of those $X_i$-components. By construction, $K = C$ has a finite number of connected components.

As the vertex groups $V_i$ are locally tame, the fundamental group of each component of the complement of $C$ in any maximal $X_i$-component is finitely generated, hence the fundamental group of each component of $K - C$ is finitely generated.

Note that $(Cayley_2(G, H) - C) = (Cayley_2(G, H) - K) \cup (K - C)$. Let $W$ be a connected component of the closure of $Cayley_2(G, H) - K$. Then $W \cap K$ is connected and $\pi_1(W \cap K)$ is isomorphic to $\pi_1(W)$ because $K$ carries the fundamental group of $Cayley_2(G, H)$. So for each component $K_i$ of $K - C$ which intersects $W$ non-trivially, $\pi_1(K_i \cap W) = \pi_1(W)$. Let $W^i$ be the (possibly infinite) union
of all components of $\text{Cayley}_2(G, H) - K$ which have non-trivial intersection with $K_i$. Then $\pi_1(W^i \cup K_i) = \pi_1(K_i)$ which is finitely generated. Hence the fundamental group of each component of $\text{Cayley}_2(G, H) - C$ is finitely generated, proving Theorem 1.

3. Proof of Lemma 1

Remark 2. The following result was proved in [3]. Let $K_0$ be a finite index subgroup of a finitely presented group $K$. A finitely generated subgroup $H$ of $K$ is tame in $K$ if and only if $H \cap K_0$ is tame in $K_0$.

It follows that virtually locally tame groups are locally tame.

Remark 3. Note that the fundamental group of a surface is locally tame.

It is shown in [3] that finitely generated free groups are locally tame. Indeed, for any free group $F$ and its finitely generated subgroup $H$ the complex $\text{Cayley}_2(F, H)$ is one-dimensional. When $H$ is finitely generated, $\text{Cayley}_2(F, H)$ is homotopic to a wedge of finitely many circles. It follows that the fundamental group of a non-closed surface is tame.

It is shown in [3] that finitely generated abelian groups are locally tame, hence the fundamental group of a torus is locally tame.

Note that the fundamental group of a closed orientable surface of genus greater than one can be written as a double of a free group over a cyclic subgroup. Hence Theorem 1 implies that fundamental groups of closed orientable surfaces of genus greater than one are locally tame.

As closed orientable surfaces are double covers of non-orientable closed surfaces of the same genus, Remark 2 implies that the fundamental groups of non-orientable closed surfaces are locally tame.

Proof of Lemma 1.

Consider, first, the case when a finitely presented group $G$ has a non-trivial JSJ-decomposition over the class of its $\text{VPC}(1)$ subgroups and all the vertex groups in the decomposition are flexible. Note that $\text{VPC}(1)$ groups are Noetherian.

The flexible vertex groups in such JSJ-decomposition are either $\text{VPC}(1)$ or virtually($\text{fundamental group of surfaces}$), cf. [9] and [4]. Furthermore, if a vertex group $M$ in that decomposition is virtually($\text{the fundamental group of a closed surface}$), then $G = M$.

Hence Remark 2 and Remark 3 imply that the group $G$ satisfies the conditions of Theorem 1, therefore it is locally tame.

Next, consider the case when a finitely presented group $G$ has a non-trivial JSJ-decomposition over the class of its $\text{VPC}(2)$ subgroups and all the vertex groups in the decomposition are flexible. Note that $\text{VPC}(2)$ subgroups are Noetherian.

The flexible vertex groups in such JSJ-decomposition are either $\text{VPC}(2)$ or virtually-($\text{cyclic-by-a surface group}$), cf. [9] and [4]. Furthermore, if a flexible vertex group $K$ in that decomposition is virtually-($\text{cyclic-by-a closed surface group}$), then $G = K$.

If a group $L$ is ($\text{cyclic-by-a surface group}$) then there exists a surface $M$ and a normal cyclic subgroup $N$ of $L$ such that the following sequence is exact.

$$1 \to N \to L \to \pi_1(M) \to 1$$

and $L$ is the fundamental group of a bundle $X$ over $M$ with fiber $S^1$. 


If $H$ is a finitely generated subgroup of $L$ then either $H \cap N = \{1\}$ or $H \cap N$ is isomorphic to $\mathbb{Z}$. Let $K$ be the image of $H$ in $\pi_1(M)$. Note that $K$ is finitely generated. Let $M_K$ be the cover of $M$ with fundamental group $K$. Then $H$ is the fundamental group of a bundle $X_H$ over $M_K$ with fiber either $S^1$ if $H \cap N = \mathbb{Z}$ or fiber $\mathbb{R}$ if $H \cap N = \{1\}$. As $K$ is finitely generated, $M_K$ is a missing boundary surface. It follows that, in either case, $X_H$ is a missing boundary 3-manifold, so $L$ is locally tame.

If a group $L$ is VPC(2) then it is virtually either the fundamental group of a torus or the fundamental group of a Klein bottle, hence Remark 3 implies that $L$ is locally tame.

Therefore, Remark 2 implies that the group $G$ satisfies the conditions of Theorem 1, so it is locally tame.

4. Acknowledgement

The author would like to thank Mike Mihalik and Peter Scott for helpful discussions.

References

[1] L. Bessieres, G. Besson, M. Boileau, S. Maillot, and J. Porti, Geometrization of 3-manifolds, EMS Tracts in Mathematics, 13(2010), European Mathematical Society, Zurich.

[2] R. Gitik, Tameness and Geodesic Cores of Subgroups, J. Austral. Math. Soc (Series A), 69(2000), 153-161.

[3] R. Gitik, On Tame Subgroups of Finitely Presented Groups, to appear.

[4] V. Guirardel and G. Levitt, JSJ Decompositions of Groups, to appear in Astérisque.

[5] B. Kleiner and J. Lott, Notes on Perelman’s papers, Geometry and Topology, 12(2008), pp. 2587-2858.

[6] E. I. Khukhro and V. D. Mazurov, Unsolved Problems in Group Theory. The Kourovka Notebook, arXiv:1401.0300v13.

[7] M. Mihalik, Compactifying Coverings of 3-Manifolds, Comment. Math. Helv., 71(1996), 362-372.

[8] M. Mihalik and S. Tschantz, Tame Combinations of Groups, Trans. AMS, 349(1997), 4251-4264.

[9] P. Scott and G. A. Swarup, Regular Neighborhoods and Canonical Decompositions for Groups, Astérisque, 289(2003).

[10] G. P. Scott and C. T. C. Wall, Topological Methods in Group Theory, in Homological Group Theory, London Math. Soc. Lecture Notes Series, 36(1979), 137-214.

[11] J. Simon, Compactification of Covering Spaces of Compact 3-Manifolds, Mich. Math. J., 23(1976), 245-256.

[12] T. W. Tucker, Non-Compact 3-Manifolds and the Missing Boundary Problem, Topology, 13(1974), 267-273.

E-mail address: rita@gmail.com

Department of Mathematics, University of Michigan, Ann Arbor, MI, 48109