Redundancies in Dependently Typed Lambda Calculi and Their Relevance to Proof Search

Zachary Snow, David Baelde and Gopalan Nadathur

Department of Computer Science and Engineering
University of Minnesota
4-192 EE/CS Building, 200 Union Street SE, Minneapolis, MN 55455
snow@cs.umn.edu, david.baelde@gmail.com, gopalan@cs.umn.edu

Abstract. Dependently typed λ-calculi such as the Logical Framework (LF) are capable of representing relationships between terms through types. By exploiting the “formulas-as-types” notion, such calculi can also encode the correspondence between formulas and their proofs in typing judgments. As such, these calculi provide a natural yet powerful means for specifying varied formal systems. Such specifications can be transformed into a more direct form that uses predicate formulas over simply typed λ-terms and that thereby provides the basis for their animation using conventional logic programming techniques. However, a naive use of this idea is fraught with inefficiencies arising from the fact that dependently typed expressions typically contain much redundant typing information. We investigate syntactic criteria for recognizing and, hence, eliminating such redundancies. In particular, we identify a property of bound variables in LF types called rigidity and formally show that checking that instantiations of such variables adhere to typing restrictions is unnecessary for the purpose of ensuring that the overall expression is well-formed. We show how to exploit this property in a translation based approach to executing specifications in the Twelf language. Recognizing redundancy is also relevant to devising compact representations of dependently typed expressions. We highlight this aspect of our work and discuss its connection with other approaches proposed in this context.

1 Introduction

There is a significant, and growing interest in mechanisms for specifying, prototyping and reasoning about formal systems that are described by syntax-directed rules. Dependently typed λ-calculi such as the Logical Framework (LF) \cite{HH93} provide many conveniences from a specification perspective in this context: such calculi facilitate the use of a higher-order approach to describing the syntax of formal objects, they allow relationships between terms to be captured in an elegant way through type dependencies, and they allow proof-checking to be realized through type-checking. Such calculi can also be given a logic programming interpretation by exploiting the isomorphism between formulas and types \cite{How80}. The Twelf system \cite{PS99}, based on this idea, has been used successfully in specifying, prototyping, and reasoning about varied formal systems.
While a system like LF provides interesting and useful ways to factor typing properties of terms and relationships between terms, this ability is not essential to its specification applications. In particular, it is possible also to use predicate based descriptions over simply typed $\lambda$-terms to similar effect. In fact, it is possible to formally present a translation of dependently typed specifications into a predicate logic form that uses the properties of being a type and of being a term of a certain type \cite{Fel89,FM90}. Such a translation lends itself to the possibility of using an implementation of a conventional logic programming language like $\lambda$Prolog \cite{NM88,GHN+08} to animate specifications written in LF \cite{SBN10}. Moreover, if the translation preserves the structure of the original specification, it would be possible to view the dependently typed descriptions as meta-programs and to reason about them using techniques for reasoning about the generated predicate based specifications.

Unfortunately, the reality with respect to a straightforward translation does not quite fit this promise. The essential source of the problem is the fact that expressions in a dependently typed language typically contain much redundant type information. When such information is preserved in a translation, it leads to a predicate logic program that is not efficient to execute. The resulting extraneous typing constraints destroy also the transparency of the correspondence, and thereby interfering with the meta-program view and the reasoning possibilities.

These observations provide the motivation for the work we describe here: identifying redundancies in LF expressions. More specifically, we want to elucidate syntactic criteria for determining such unnecessary information that can, for instance, be exploited in a translation based approach to implementing LF specifications. We describe a property of bound variables in LF types called rigidity and show in a formal sense that knowledge of the specific instantiations of such variables is unnecessary from the perspective of checking if the expression is well-formed. While our observations are driven by a particular application, they also have a relevance in other contexts such as that of devising compact representation of proofs \cite{Ree08}. We discuss these connections in the paper.

In the next section we describe the dependently typed $\lambda$-calculus and LF. We then introduce a translation from LF to a predicate logic that preserves provability, and describe how redundancies in LF derivations can impact the performance of the generated logic program. In Section 4 we develop a technique for identifying and eliminating such redundancies. Then in Section 5 we show how it can be applied to improve the translation, and identify some important practical extensions to the translation. We conclude the paper with a discussion of possible future directions for this work.

2 The Edinburgh Logical Framework

The Edinburgh Logical Framework \cite{HHP93} (LF) is a dependently typed $\lambda$-calculus with three categories of expressions: kinds, types or type families that are classified by kinds and objects or terms that are classified by types. We assume two denumerable sets of variables, one for objects and the other for types. We
use $x$ and $y$ to denote object variables, $u$ and $v$ to denote type variables and $w$ to denote either. Letting $K$ range over kinds, $A$ and $B$ over types, and $M$ and $N$ over object terms, the syntax of LF expressions is given by the following rules:

$$
K := \text{Type} \mid \Pi x : A.K \\
A := u \mid \Pi x : A.B \mid \lambda x : A.M \mid A M \\
M := x \mid \lambda x : A.M \mid M N
$$

Expressions of any of these kinds will be denoted by $P$ and $Q$. Here, $\Pi$ and $\lambda$ are operators that associate a type with a variable and bind its free occurrences over the expression after the period. We write $P[N_1/x_1, \ldots, N_n/x_n]$ to denote a simultaneous substitution with renaming to avoid variable capture. We write $A \rightarrow P$ for $\Pi x : A.P$ when $x$ does not appear free in $P$, and abbreviate $\Pi x_1 : A_1. \ldots \Pi x_n : A_n. P$ by $\Pi x : A.P$.

\begin{align*}
\Gamma \vdash \cdot \quad &\text{null-ctx} \\
\Gamma \vdash K \text{ kind} &\quad \Gamma \vdash A : \text{Type} \quad &\text{kind-ctx} \quad u \notin \text{dom}(\Gamma) \\
\Gamma \vdash A : \text{Type} &\quad \Gamma \vdash x : A \text{ ctx} \quad &\text{type-ctx} \quad x \notin \text{dom}(\Gamma) \\
\Gamma \vdash \Gamma \text{ ctx} &\quad \Gamma \vdash \Pi x : A.K \text{ kind} \quad \Gamma \vdash A : \text{Type} \quad \Gamma \vdash x : A \text{ kind} \\
\Gamma \vdash \Gamma \text{ ctx} &\quad \Gamma \vdash u : K \text{ ctx} \quad \Gamma \vdash x : A ^\beta \quad \Gamma \vdash x : A \text{ var-fam} \quad \Gamma \vdash x : A \text{ var-obj} \\
\Gamma \vdash A : \text{Type} &\quad \Gamma \vdash (\Pi x : A.B) : \text{Type} \quad \Gamma \vdash (\Pi x : A.B) : \text{Type} \\
\Gamma \vdash A : \text{Type} &\quad \Gamma \vdash (\lambda x : A.M) : (\Pi x : A^\beta.B) \quad \Gamma \vdash \Gamma \text{ abs-fam} \\
\Gamma \vdash \Gamma \text{ abs-fam} &\quad \Gamma \vdash \Gamma \text{ abs-obj} \\
\Gamma \vdash A : \text{Type} &\quad \Gamma \vdash M : A \text{ abs-obj} \\
\Gamma \vdash A : \text{Type} &\quad \Gamma \vdash M : \text{Type} \quad \Gamma \vdash A : \text{Type} \quad \Gamma \vdash \Gamma \text{ app-fam} \\
\Gamma \vdash \Gamma \text{ app-fam} &\quad \Gamma \vdash \Gamma \text{ app-obj}
\end{align*}

\textbf{Fig. 1.} Rules for Inferring LF Assertions

The type correctness of LF expressions is assessed relative to contexts that are finite collections of assignments of types and kinds to variables (we use $\cdot$ to denote the empty context). LF deals with assertions of the following four forms:

\begin{align*}
\vdash \Gamma \text{ ctx} &\quad \Gamma \vdash K \text{ kind} \quad \Gamma \vdash A : K \quad \Gamma \vdash M : A
\end{align*}

The first assertion signifies that $\Gamma$ is a well-formed context. The remaining assertions mean respectively that, relative to a (well-formed) context $\Gamma$, $K$ is a well-formed kind, $A$ is a well-formed type of kind $K$ and $M$ is a well-formed object of type $A$. Figure 1 presents the rules for deriving such assertions. The
inference rules allow for the derivation of an assertion of the form $\Gamma \vdash M : A$ only when $A$ is in normal form. To verify such an assertion when $A$ is not in normal form, we first derive $\Gamma \vdash A : Type$ and then verify $\Gamma \vdash M : A^{\beta}$. A similar observation applies to $\Gamma \vdash A : K$.

Well-typed LF expressions admit a $\beta\eta$-long form. Types of $\beta\eta$-long form $(u M_1 \ldots M_n)$ are called base types. In the following, we shall only consider LF derivations whose end assertion only contains expressions in $\beta\eta$-long form. Notice that every expression in the entire derivation must then also be in $\beta$-normal form. This in turn means that in judgments of the forms $(\lambda x : A.B) : (\Pi x : A'.K)$ and $(\lambda x : A.M) : (\Pi x : A'.B)$ it must be the case that $A$ and $A'$ are identical, and that normalization need not be considered in the use of the var-fam and var-obj rules. Additionally, we shall modify inference rules so that all expressions in the entire derivation remain in $\beta\eta$-long form — the resulting system is referred to as canonical LF. For objects, app-obj and var-obj are replaced by the following big-step application rule (which we refer to as backchaining due to its logic programming interpretation):

\[
\frac{(y : \Pi x : B.A) \in \Gamma \quad \Gamma \vdash N_1 : B_1 \quad \ldots \quad \Gamma \vdash N_n : B_n [N_1/x_1 \ldots N_{n-1}/x_{n-1}]}{\Gamma \vdash M : A [N_1/x_1 \ldots N_n/x_n]}
\]

The rules for type families are changed in the same way.

The logic programming interpretation of LF is based on viewing types as formulas. More specifically, a specification or program in this setting is given by a context. This starting context, also called a signature, essentially describes the vocabulary for constructing types and asserts the existence of particular inhabitants for some of these types. Against this backdrop, questions can be asked about the existence of inhabitants for certain other types. Formally, this amounts to asking if an assertion of the form $\Gamma \vdash M : A$ has a derivation. However, the object $M$ is left unspecified — it is to be extracted from a successful derivation. Thus, the search for a derivation of the assertion is driven by the structure of $A$ and the types available from the context.

A concrete illustration of the paradigm is useful for later discussions. Consider a signature or context $\Gamma$ comprising the following assignments in sequence:

\[
\begin{align*}
nat & : Type, \quad z : nat, \quad s : nat \rightarrow nat, \\
list & : Type, \quad nil : list, \quad cons : nat \rightarrow list \rightarrow list, \\
append & : list \rightarrow list \rightarrow list \rightarrow Type, \\
appNil & : \Pi K : list. append nil K K, \\
appCons & : \Pi X : nat. \Pi L : list. \Pi K : list. \Pi M : list. \\
& \quad (append L K M) \rightarrow (append (cons X L) K (cons X M))
\end{align*}
\]

We can ask if there is some term $M$ such that the judgment

\[
\Gamma \vdash M : append (cons z nil) (cons (s z) nil) (cons z (cons (s z) nil))
\]
is derivable.

Furthermore, as Twelf allows for instantiatable meta-variables in the type \( A \), we can ask which list results from appending \((\text{cons } z \text{ nil})\) and \((\text{cons } (s \ z) \text{ nil})\); in the following, \( L \) is such a variable:

\[
\Gamma \vdash M : \text{append} (\text{cons } z \text{ nil}) (\text{cons } (s \ z) \text{ nil}) L.
\]

Here Twelf responds by instantiating \( L \) to \((\text{cons } z (\text{cons } (s \ z) \text{ nil}))\) and instantiating \( M \) with an LF object (proof term) of type \text{append} (\text{cons } z \text{ nil}) (\text{cons } (s \ z) \text{ nil}) (\text{cons } z (\text{cons } (s \ z) \text{ nil})). \) Sometimes the logic program \( \Gamma \) does not constrain a meta-variable, and so it is left uninstantiated in the proof term. Here, the interpretation is that the proof term is valid for any instantiation of the meta-variable by a term of the right type.

3 A translation to predicate logic

We now consider translating LF specifications into the logic of higher-order hereditary Harrop formulas, also known as \( \text{hohh} \) logic [MNPS91]. Intuitively, this logic is similar to Horn clause logic, except that it allows predicates to have simply typed \( \lambda \)-terms as arguments, it permits quantification over (non-predicate) function variables and it includes universal quantifiers and embedded implications in goals and the bodies of clauses. Although \( \text{hohh} \) does not permit dependent types, it has been shown that these dependencies can be systematically encoded by using predicates over the simpler form of \( \lambda \)-terms. This encoding is particularly interesting because it provides a way to utilize efficient implementations of \( \text{hohh} \) logic like the Teyjus system [GHN+08] in realizing an LF-based logic programming language.

The general idea of the encoding, first proposed by Felty [Fel89,FM90], is to first encode LF objects and types in a way that erases type dependencies, and to recover those relationships in the encoding of LF type judgments. One can then prove that the encoding is sound and complete by providing a mappings between LF derivation and \( \text{hohh} \) derivations of the encoded judgments. However, it is important to note that in the perspective of using the encoding for logic programming, a stronger correctness result is needed. Instead of considering only closed LF assertions, i.e. checking a given type judgment, we are interested in finding objects inhabiting a particular type, i.e. searching for a derivation of an LF assertion with a meta-variable for the object. Therefore, our correctness result should also state that any instantiation of that meta-variable is actually an LF encoding.

We shall only give an intuition and example of our translation, referring the reader to [Sno10] for details and proofs. Our translation proceeds in the same general fashion as Felty’s: LF objects and types are first encoded as \( \text{hohh} \) terms. Next the \text{hastype} predicate relates \( \text{hohh} \) terms representing LF objects with \( \text{hohh} \) terms representing the LF types of those objects. For instance, given an LF object \( z \) of type \( \text{nat} \), we relate \( \text{hohh} \) encodings \( z' \) and \( \text{nat}' \) thus: \text{hastype} \( z' \text{ nat}' \). As an example, the Twelf specification of \text{append} translates into
the clauses in Figure 2. From these clauses, we can, for example, derive the goal
```plaintext
hastype (cons (s z) nil) list
```
and we could search for terms X satisfying the following goal:
```plaintext
hastype X (append (cons z nil) (cons (s z) nil) (cons z (cons (s z) nil)))
```
Unfortunately, this program does not correspond exactly to the usual append logic program in hohh. Specifically, whenever a goal `hastype p (append l k m)` is proved, each list l, k, and m is “type-checked” by deriving a proof of, for example, the subgoal `hastype l list`. This involves a recursion over the entire structure of the list, and thereby introduces a quadratic complexity to the fundamentally linear operation of appending lists.

As we shall see, a meta-theoretical analysis of LF derivations can be used to justify the removal of some of those typing constraints. This study of derivations is best carried out directly in LF, leading to interesting results in their own right, some of which may be applicable beyond our translation problem.

### 4 Redundancy in LF derivations

The redundancy evoked above can be viewed from the LF standpoint alone. Consider a variable y of type `Πx:B.A` which might be used to derive some judgment `Γ ⊢ M : A[N_1/x_1...N_n/x_n]`:

\[
\frac{(y : Πx:B.A) \in Γ \quad Γ ⊢ N_1 : B_1 \quad ... \quad Γ ⊢ N_n : B_n [N_1/x_1...N_{n-1}/x_{n-1}]}{Γ ⊢ M : A[N_1/x_1...N_n/x_n]}
\]

It is reasonable to assume that when checking that an object has a particular type, or when searching for objects of a particular type, the type has been checked to be valid first, i.e. that `Γ ⊢ A : Type` has a derivation. It is often the case that some of the typing judgments `Γ ⊢ N_i : B_i` are superfluous in the sense that they can be found almost verbatim in the derivation that A is a type. Furthermore, it is possible to detect statically many of those cases, just by examining the occurrences of `x_i` in A. The idea is that if `x_i` occurs in A in such
a way that \( N_i \) will be found in \( A[N_1/x_1 \ldots N_n/x_n] \), whatever the other \( N_j \) are, then the premise \( x_i : N_i \) can be safely omitted.

Formally, we use the notion of a \textit{rigid occurrence} that is expressed by the judgment \( \vec{\gamma}: x_i \sqcup_o B \) defined in Figure 3 to characterize some of these cases.

\[
\begin{array}{l}
y_i \text{ distinct elements of } \delta \\
\Gamma; \delta; x \sqcup_o x \overrightarrow{\gamma} & \text{INIT}_o \quad \text{y} \notin \Gamma \text{ and } \Gamma; \delta; x \sqcup_o M_i \text{ for some } i \quad \text{APP}_o \\
\Gamma; \delta; x \sqcup_o y \overrightarrow{M} & \quad \text{y} / \in \Gamma \text{ and } \Gamma; \delta; x \sqcup_o M \\
\Gamma; \delta; y; x \sqcup_o M & \quad \Gamma; \delta; x \sqcup_o \lambda y: A.M \\
\end{array}
\]

\textbf{Fig. 3. Rigidly occurring variables in LF objects}

\textbf{Theorem 1.} Let \( \vec{\gamma} \) be a vector of LF objects, \( \vec{x} \) a vector of variables, and \( \vec{B} \) of canonical LF types, all of same length. Let \( \Gamma \) and \( \Delta \) be LF contexts, \( \delta \) be \( \text{dom}(\Delta) \). Let \( \Gamma_0 = x_1 : B_1, \ldots, x_n : B_n \). Let \( \Pi x:\vec{B}.A \) be a canonical type, where \( A \) is a base type. Suppose that there are derivations of:

\begin{itemize}
  \item \( \vec{x}; \delta; x_i \sqcup_o M \)
  \item \( \Gamma, \Gamma_0, \Delta \vdash M : A \)
  \item \( \Gamma, \Delta[\vec{N}/x] \vdash M[\vec{N}/x] : A[\vec{N}/x] \)
\end{itemize}

Then there is a derivation of \( \Gamma \vdash N_i : B_i[N_1/x_1, \ldots, N_{i-1}/x_{i-1}] \).

This theorem establishes a sort of substitution inversion: having an abstract and an instantiated derivation, we show that one can recover the derivation that was substituted, that is \( \Gamma \vdash N_i : B_i[N_1/x_1, \ldots, N_{i-1}/x_{i-1}] \). Given the nature of that statement, it is not surprising that we find in \textit{INIT}_o a condition reminiscent of higher-order patterns, a fragment of higher-order unification where most general unifiers are guaranteed, thanks to the ability to invert substitutions.

\textit{Proof (Theorem 1).} We proceed by induction on the rigidity derivation. Walking simultaneously through the two LF derivations, following the path given by the rigidity derivation, we eventually reach a point where we have on the one hand a derivation of \( x_i \overrightarrow{\gamma} : T[ y_1/z_1 \ldots y_k/z_k ] \) with \( B_i = \Pi z:\vec{C}.T \), and on the other a derivation of \( N_i \overrightarrow{\gamma} : T[ y_1/z_1 \ldots y_k/z_k ][N_1/x_1 \ldots N_{i-1}/x_{i-1}] \). The bound variables \( \overrightarrow{\gamma} \) being distinct, the substitution \( [ y_1/z_1 \ldots y_k/z_k ] \) is simply a renaming and can be inverted. We obtain a derivation of \( N_i \overrightarrow{\gamma} : T[ N_1/x_1 \ldots N_{i-1}/x_{i-1}] \) and finally \( N_i : B_i[N_1/x_1 \ldots N_{i-1}/x_{i-1}] \).

\textbf{Remark 1.} Note that it would be unsound to allow in \textit{INIT}_o any application \( x \overrightarrow{\gamma} \) rather than \( x \overrightarrow{\gamma} \) for distinct bound variables \( \overrightarrow{\gamma} \). With such a rule the rigidity lemma the above theorem is no longer true. For example, in a signature with
num : nat → Type and num_n : Π n : nat. (num n), we obtain a counter-example with \( M = λx. x z \) and \( N = t \): we have \( \Gamma \vdash (t z) : (num z) \) and 
\[
\Gamma, x : (nat → num z) \vdash (x z) : (num z)
\]
but not \( \Gamma \vdash t : nat → num z \).

### 4.1 Application to proof search

There are several ways to exploit this property about LF derivations, and not just in the context of a translation, but in the more general setting of proof search. We come back to the problem of eliminating redundancies in the rule corresponding to backchaining on some element of the LF context:

\[
(y : Π x : B. A) ∈ \Gamma \quad Γ \vdash N_1 : B_1 \quad \ldots \quad Γ \vdash N_n : B_n[N_1/x_1 \ldots N_{n-1}/x_{n-1}]
\]

\[
\Gamma \vdash M : A[N_1/x_1 \ldots N_n/x_n]
\]

Eliminating redundancies here corresponds to limiting the number of redundant subderivations investigated during search.

We first consider recognizing rigid occurrences of some variables \( x_i \) in the target type \( A \). We formalize this as \( \bar{x}; x_i \sqsubseteq \bar{A} \), defined by the following rules:

\[
\frac{}{\Gamma ; : x \sqsubseteq o M_i \text{ for some } M_i} \quad \text{APP}_t \quad \frac{}{\Gamma ; y \sqsubseteq t B} \quad \text{PI}_t
\]

**Theorem 2.** Let \( \bar{N} \) be a vector of LF objects, \( \bar{x} \) a vector of variables, and \( \bar{B} \) of canonical LF types, all of same length. Let \( \Gamma \) and \( \Delta \) be LF contexts, \( δ \) be \( \text{dom}(\Delta) \). Let \( \Gamma_0 = x_1 : B_1, \ldots, x_n : B_n \). Let \( Π x : B. A \) be a canonical type, where \( A \) is a base type. Suppose that there are derivation of:

\[
- \bar{x}; x_i \sqsubseteq A
- \Gamma, \Gamma_0, \Delta \vdash A : \text{Type}
- \Gamma, \Delta[N/x] \vdash A[N/x] : \text{Type}
\]

Then there is a derivation of \( \Gamma \vdash N_i : B_i[N_1/x_1, \ldots, N_{i-1}/x_{i-1}] \).

**Proof.** Similarly to Theorem 1 we walk through the type structure, following the path given by rigidity. Eventually, we reach APP, and invoke directly the previous theorem.

From a practical viewpoint, this theorem allows us to statically analyze an LF specification (which constitutes the initial LF context) and discard some premises of the backchaining rules derived from that specification, without losing soundness. This is currently done in our translation.

There are yet more redundancies in this same style. We have used a rigid occurrence of some variable \( x_i \) in \( A \) to retrieve a typing derivation for \( N_i \) from the derivation that \( A[N_1/x_1, \ldots, N_{i-1}/x_{i-1}] \) is a valid type, but
we might also extend the application of rigidity to retrieve some information from the typing derivation for some $N_j$. Given that we already have a derivation of $\Gamma \vdash \Pi x_1: B_1, \ldots, x_n: B_n, A : Type$, we clearly have a derivation of $\Gamma, x_1 : B_1, \ldots, x_{j-1} : B_{j-1} \vdash B_j : Type$. We also have a derivation of $\Gamma \vdash N_j : B_j[N_1/x_1, \ldots, N_{j-1}/x_{j-1}]$, either directly as one of the premises when $x_j$ is not rigid in $A$ or through Theorem 2 when the corresponding premise has been elided. From this derivation we can also conclude that $\Gamma \vdash B_j[N_1/x_1, \ldots, N_{j-1}/x_{j-1}] : Type$ has a derivation. We can hence finally apply Theorem 2 to these derivations to conclude that we do indeed have a derivation of $\Gamma \vdash N_i : B_i[N_1/x_1, \ldots, N_{i-1}/x_{i-1}]$.

4.2 Related work

Reed [Ree08] approaches the problem of eliminating redundancies in LF from a different perspective, and with a different goal: that of reducing the size of proof-terms yielded during logic programming search, motivated by the fact that in some applications proof-terms must be transmitted or manipulated. He does so by developing a technique for identifying redundancies in terms, through a notion of strictness that is similar to rigidity, that he uses to identify sub-terms of LF objects that can be reconstructed, either from the types of nearby sub-terms, or from the type of an object itself. He describes two modes for omitting sub-terms, synthesis based omission and inheritance based omission, and uses strictness to determine which kind of omission, if any, is possible. In omission by inheritance, knowledge of a term’s type is used to elide (and later reconstruct) type derivations for sub-terms. For example, if $x + y$ is known to have type $nat$, then we automatically known that $x$ has type $nat$, given that $+$ has type $nat \to nat \to nat$. This is similar to what we described in Theorem 2. In omission by synthesis, the types of nearby sub-terms are used to elide and eventually reconstruct a given sub-term, when the sub-term being omitted appears (in a sufficient manner) in said type. For example, if $x = y$ is well-typed and $x$ has type $A$ we can deduce that $y$ has type $A$ as well. This is similar to the additional application of rigidity that we have described.

The main difference with Reed’s work lies in the motivation. Reed’s work focuses on optimizing an LF object (that is, a proof term) for size by eliminating redundant parts of the object itself, and without particular concern for how such a term is discovered. We are concerned with optimizing search, and we use the redundancy analysis to avoid searching for parts of the typing derivation, but we still produce a complete LF proof term.

5 Optimizing the Twelf translation

We have presented a technique for identifying redundancies in LF derivations and identified a few ways to use it in the context of proof-search. Carrying these observations to the context of our translation to hohh—we are also concerned with ensuring that all hohh objects discovered as instantiations of
meta-variables actually correspond to encodings of LF objects—is not entirely trivial.

5.1 Meta-variables in objects

Building on Theorem 2, we have developed in [SBN10] an optimized translation from LF specifications to hohh logic. The part of this translation that removes redundant typing judgments is based on the mapping on types presented in Figure 4. The translation of context items of the form $x : A$ in an LF specification is given by $\langle A \rangle^+ x$, where $\langle \rangle$ denotes an empty sequence of variables; this operation is lifted to LF specifications by distribution to each item in the specification. The translation of a type $A$ for which an inhabitant $M$ is sought is correspondingly given by $\langle A \rangle^- (M)$. Notice that these translations are guided solely by the type $A$; this is highlighted by the fact that the translation actually returns a formula abstracted over the proof-term. This translation is illustrated by its application to the example Twelf specification considered in Section 2 that yields the clauses shown in Figure 5, which should be contrasted with the ones in Figure 2.

\[
\begin{align*}
\llbracket P x : A . B \rrbracket^+ & := \begin{cases} 
\lambda M. \forall x. \top \supset \llbracket B \rrbracket_\Gamma^+ (M x) & \text{if } \Gamma ; x \sqsubset; B \\
\lambda M. \forall x. \llbracket A \rrbracket^- (x) \supset \llbracket B \rrbracket_\Gamma^+ (M x) & \text{otherwise}
\end{cases} \\
\llbracket N \rrbracket^+ & := \lambda M. \text{hastype } M \langle N \rangle & \text{if } N \text{ is a base type}
\end{align*}
\]

\[
\begin{align*}
\llbracket P x : A . B \rrbracket^- & := \lambda M. \forall x. \llbracket A \rrbracket^+ (x) \supset \llbracket B \rrbracket^- (M x) \\
\llbracket N \rrbracket^- & := \lambda M. \text{hastype } M \langle N \rangle & \text{if } N \text{ is a base type}
\end{align*}
\]

**Fig. 4.** Optimized translation of LF specifications and judgments to hohh

\[
\begin{align*}
\text{hastype } z \text{ nat}, & \forall n. \text{ hastype } n \text{ nat }\supset \text{ hastype } (s \ n) \text{ nat}, \\
\text{hastype } \text{nul list}, & \forall n. \text{ hastype } n \text{ nat }\supset \forall l. \text{ hastype } l \text{ list }\supset \text{ hastype } (\text{cons } n \ l) \text{ list}, \\
\forall l. & \top \supset \text{ hastype } (\text{appendNil } l) (\text{append } \text{nul } l \ l), \\
\forall x. & \top \supset \forall l. \top \supset \forall k. \top \supset \forall m. \top \supset \forall a. \text{ hastype } a (\text{append } l \ k \ m) \supset \\
& \text{hastype } (\text{appCons } x \ l \ k \ m \ a) (\text{append } (\text{cons } x \ l) k \ (\text{cons } x \ m))
\end{align*}
\]

**Fig. 5.** Optimized translation of the LF specification for append

We have proved the optimized translation correct. The statement of its correctness is slightly complicated by the fact that it requires that all hohh terms correspond to LF expressions, so that we can use the translation to generate actual LF proof terms.
Theorem 3 (Optimized translation correctness). Let \( \Gamma \) be an LF specification such that \( \vdash \Gamma \) ctx has a derivation, \( A \) an LF type such that \( \Gamma \vdash A : Type \) has a derivation. Then, for any LF object \( M \) such that \( \Gamma \vdash M : A \) has a derivation, \( \llbracket \Gamma \rrbracket^+ \rightarrow \llbracket M : A \rrbracket^- \) is derivable. Moreover, if \( \llbracket \Gamma \rrbracket^+ \rightarrow \llbracket A \rrbracket^-(M) \) for an arbitrary hohh term \( M \), then it must be that \( M = \langle M' \rangle \) for some canonical LF object such that \( \Gamma \vdash M' : A \) has a derivation.

The proof of the following relies on Theorem 2 to recover typing judgments that have been optimized away. In addition, it shows that hohh terms must be well-formed LF objects. Note that this theorem implies that proof-search for encoded LF typing judgments will always fully instantiate the meta-variable corresponding to the object — otherwise, a dummy instantiation of that variable would still yield a valid derivation invalidating our theorem.

Unfortunately, we have not been able to exploit the extended redundancy analysis to further optimize our translation; it has proven difficult to ensure that hohh meta-variables are instantiated by LF encodings while still maintaining an efficient translation. This is due to the fact that, in eliminating redundancies in this fashion, we must eventually obtain a typing derivation in a setting without these optimizations, which could reduce or even destroy the effectiveness of such eliminations.

5.2 Meta-variables in types

Note that, while we have proved Theorem 3 for closed LF types, we have not yet considered the meaning of meta-variables in such types, what it means when a meta-variable is not bound during search, nor whether bindings for them are correct. Here there are two approaches.

Recall the interpretation of remaining meta-variables after proof-search, in both \( \lambda \)Prolog and Twelf: the goal actually holds for any term \( t \) of the right type. In particular, upon successful \( \lambda \)Prolog search for an encoded LF query, remaining meta-variables in the type can be instantiated by any encoding of an LF object. This can be done after the main proof search, by searching for an inhabitant of the corresponding type. We can then extend this treatment even to meta-variables that are bound during search, by simply checking after search succeeds that the meta-variables have been properly instantiated. Once we have checked that the initial type has been instantiated into a closed valid type in that way, Theorem 3 applies. In practice, this process is less intensive than proof search proper, and tends not to be overly expensive.

Going further, it should in fact be the case that, under our translations, no meta-variable could possibly be bound to the encoding of an LF term of incorrect type or to something that is not even an encoding. The intuition here is that the only time a meta-variable is bound in the logic programs generated by the translation is when it is matched with the head of a clause. Since the original specification is valid, any such matching clause should impose only the correct type on the meta-variable. However, the statement and proof of this theorem is not at all obvious, and is further stymied by the fact that it isn’t clear how exactly this extension to Twelf, which we are seeking to emulate, should behave.
6 Conclusion and Future Work

We have considered in this paper a translation from specifications in the dependently typed $\lambda$-calculus LF to a predicate logic over simply typed $\lambda$-terms. This translation is motivated by a desire to utilize implementations of proof search in the latter logic to realize LF-based proof search. A key task in making such a translation effective is that of identifying and, subsequently, eliminating redundancies in LF expressions and derivations. Specifically, we have described a property of bound variables in types that makes it unnecessary to type-check their instantiations in ensuring that expressions that use such types are well-formed. We note that our proof of such redundancy is based directly on the properties of LF expressions and derivations. Thus, our observation is of larger interest than just the translation task at hand.

The work described here can be extended in at least two ways. First, it should be possible to enhance our techniques for identifying redundancies. We have presented one such extension already through a more inclusive definition of the rigidity property. However LF derivations contain significant redundancies and we believe it is possible to carry out a richer analysis towards identifying these based on syntactic properties. Second, we can think of applying the specific techniques developed for detecting such redundancies to contexts different from translation. We have already discussed the relationship between our work and that of Reed. An understanding of the differences between our system and his could eventually lead to a better, and provably correct, ability to shorten LF proof terms that are needed in applications such as that of proof-carrying code [Nec97]. Moreover the usefulness of these ideas need not be limited to translation and compact representation of LF expressions: any application of LF that requires type-checking, such as automatic meta-theorem proving, could benefit from methods for discovering repetitive type information.

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