Abstract

A partially embedded graph (or Peg) is a triple \((G, H, \mathcal{H})\), where \(G\) is a graph, \(H\) is a subgraph of \(G\), and \(\mathcal{H}\) is a planar embedding of \(H\). We say that a Peg \((G, H, \mathcal{H})\) is planar if the graph \(G\) has a planar embedding that extends the embedding \(\mathcal{H}\).

We introduce a containment relation of Pegs analogous to graph minor containment, and characterize the minimal non-planar Pegs with respect to this relation. We show that all the minimal non-planar Pegs except for finitely many belong to a single easily recognizable and explicitly described infinite family. We also describe a more complicated containment relation which only has a finite number of minimal non-planar Pegs.

Furthermore, by extending an existing planarity test for Pegs, we obtain a polynomial-time algorithm which, for a given Peg, either produces a planar embedding or identifies an obstruction.

Keywords: Planar Graphs, Partially Embedded Graphs, Kuratowski Theorem

1 Introduction

A partially embedded graph (Peg) is a triple \((G, H, \mathcal{H})\), where \(G\) is a graph, \(H\) is a subgraph of \(G\), and \(\mathcal{H}\) is a planar embedding of \(H\). The problem PartiallyEmbeddedPlanarity(Pep) asks whether a Peg \((G, H, \mathcal{H})\) admits a planar (non-crossing) embedding of \(G\) whose restriction to \(H\) is \(\mathcal{H}\). In this case we say that the Peg \((G, H, \mathcal{H})\) is planar. Despite of this being a very natural generalization of planarity, this approach has been considered only recently \[1\]. It should be mentioned that all previous planarity testing algorithms have been of little use for Pep, as they all allow flipping of already drawn parts of the graph, and thus are not suitable for preserving an embedding of a given subgraph.

It is shown in \[1\] that planarity of Pegs can be tested in linear time. In this paper we complement the algorithm in \[1\] by a study of the combinatorial aspects of this question. In particular, we provide a complete characterization of planar Pegs via a small set of forbidden substructures, similarly to the celebrated Kuratowski theorem \[11\], which characterizes planarity via the forbidden subdivisions of \(K_5\) and \(K_{3,3}\), and the closely related theorem of Wagner \[13\], which characterizes planarity via forbidden \(K_5\) and \(K_{3,3}\) minors. Our characterization can then be used to modify the existing planarity test for partially embedded graphs into a certifying algorithm that either finds

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a solution or finds a certificate, i.e., a forbidden substructure, that shows that the instance is not planar.

Understanding the forbidden substructures may be particularly beneficial in studying the problem *simultaneous embedding with fixed edges*, or SEFE for short, which asks whether two graphs $G_1$ and $G_2$ on the same vertex set $V$ admit two drawings $\Gamma_1$ and $\Gamma_2$ of $G_1$ and $G_2$, respectively, such that (i) all vertices are mapped to the same point in $\Gamma_1$ and $\Gamma_2$, (ii) each drawing $\Gamma_i$ is a planar drawing of $G_i$ for $i = 1, 2$, and (iii) edges common to $G_1$ and $G_2$ are represented by the same Jordan curve in $\Gamma_1$ and $\Gamma_2$. J"unger and Schulz [7] show that two graphs admit a SEFE if and only if they admit planar embeddings that coincide on the intersection graph. In this sense, our obstructions give an understanding of which configurations should be avoided when looking for an embedding of the intersection graph.

For the purposes of our characterization, we introduce a set of operations, called PEG-minor operations, that preserve the planarity of PEGs. Note that it is not possible to use the usual minor operations, as sometimes, when contracting an edge of $G$ not belonging to $H$, it is not clear how to modify the embedding of $H$. Our minor-like operations are defined in Section 2.

Our goal is to identify all minimal non-planar PEGs in the minor-like order determined by our operations; such PEGs are referred to as obstructions. Our main theorem says that all obstructions are depicted in Fig. 1 or belong to a well described infinite class of so called alternating chains (the somewhat technical definition is postponed to Section 2). It can be verified that each of them is indeed an obstruction, i.e., it is not planar, but applying any of the PEG-minor operations results in a planar PEG.

We say that a PEG avoids a PEG $X$ if it does not contain $X$ as a PEG-minor. Furthermore, we say that a PEG is obstruction-free if it avoids all PEGs of Fig. 1 and all alternating chains of lengths $k \geq 4$. Then our main theorem can be expressed as follows.

**Theorem 1.** A PEG is planar if and only if it is obstruction-free.

Since our PEG-minor operations preserve planarity, and since all the listed obstructions are non-planar, any planar PEG is obstruction-free. The main task is to prove that an obstruction-free PEG is planar.

Having identified the obstructions, a natural question is if the PEG-planarity testing algorithm of [1] can be extended so that it provides an obstruction if the input is non-planar. It is indeed so.

**Theorem 2.** There is a polynomial-time algorithm that for an input PEG $(G, H, \mathcal{H})$ either constructs a planar embedding of $G$ extending $\mathcal{H}$, or provides a certificate of non-planarity, i.e., identifies an obstruction present in $(G, H, \mathcal{H})$ as a PEG-minor.

The paper is organized as follows. In Section 2 we first recall some basic definitions and results on PEGs and their planarity, and then define the PEG-minor order and the alternating chain obstructions. In Section 3 we show that the main theorem holds for instances where $G$ is biconnected. We extend the main theorem to general (not necessarily biconnected) PEGs in Section 4. In Section 5 we present possible strengthening of our PEG-minor relations, and show that when more complicated reduction rules are allowed, the modified PEG-minor order has only finitely many non-planar PEGs. In Section 6 we briefly provide an argument for Theorem 2 and then conclude with some open problems.

2 Preliminaries and Notation

**Embeddings** A drawing of a graph is a mapping of each vertex to a distinct point in the plane and of each edge to a simple Jordan curve that connects its endpoints. A drawing is planar if the
|   |   |   |
|---|---|---|
| 1 | 2 | 3 |
| 4 | 5 | 6 |
| 7 | 8 | 9 |
| 10 | 11 | 12 |
| 13 | 14 | 15 |
| 16 | 17 | 18 |
| 19 | 20 | 21 |
| 22 |   |   |

Figure 1: The obstructions not equal to the $k$-fold alternating chains for $k \geq 4$. The black solid edges belong to $H$, the light dashed edges to $G$, but not to $H$. All the vertices belong to both $G$ and $H$, except for $K_5$ and $K_{3,3}$, where $H$ is empty.
curves representing the edges intersect only in common endpoints. A graph is planar if it admits a planar drawing. Such a planar drawing determines a subdivision of the plane into connected regions, called faces, and a circular ordering of the edges incident to each vertex, called rotation scheme. Traversing the border of a face \( F \) in such a way that the face is to the left yields a set of circular lists of vertices, the boundary of \( F \). Note that the boundary of a face is not necessarily connected if the graph is not connected and that vertices can be visited several times if the graph is not biconnected. The boundary of a face \( F \) can uniquely be decomposed into a set of simple edge-disjoint cycles, bridges (i.e., edges that are not part of a cycle) and isolated vertices. We orient these cycles so that \( F \) is to their left to obtain the facial cycles of \( F \).

Two drawings are topologically equivalent if they have the same rotation scheme and, for each facial cycle, the vertices to its left are the same in both drawings. A planar embedding is an equivalence class of planar drawings. Let \( \mathcal{G} \) be a planar embedding of \( G \) and let \( H \) be a subgraph of \( G \). The restriction of \( \mathcal{G} \) to \( H \) is the embedding of \( H \) that is obtained from \( \mathcal{G} \) by considering only the vertices and the edges of \( H \). We say that \( \mathcal{G} \) is an extension of a planar embedding \( \mathcal{H} \) of \( H \) if the restriction of \( \mathcal{G} \) to \( H \) is \( \mathcal{H} \).

**Connectivity and SPQR-trees** A graph is connected if any pair of its vertices is connected by a path. A maximal connected subgraph of a graph \( G \) is a connected component of \( G \). A cut-vertex is a vertex \( x \in V(G) \) such that \( G - x \) has more components than \( G \). A connected graph with at least three vertices is 2-connected (or biconnected) if it has no cut-vertex. In a biconnected graph \( G \), a separating pair is a pair of vertices \( \{x, y\} \) such that \( G - x - y \) has more components than \( G \).

A biconnected graph with at least four vertices is 3-connected if it has no separating pair. We say that a PEG \((G, H, \mathcal{H})\) is connected, biconnected and 3-connected if \( G \) is connected, biconnected and 3-connected, respectively. An edge of a graph \( G \) is sometimes referred to as a \( G \)-edge, and a path in \( G \) is a \( G \)-path.

A connected graph can be decomposed into its maximal biconnected subgraphs, called blocks. Each edge of a graph belongs to exactly one block, only cut-vertices are shared between different blocks. This gives rise to the block-cutvertex tree of a connected graph \( G \), whose nodes are the blocks and cut-vertices of \( G \), and whose edges connect cut-vertices to blocks they belong to.

The planar embeddings of a 2-connected graph can be represented by the SPQR-tree, which is a data structure introduced by Di Battista and Tamassia [3, 4]. A more detailed description of the SPQR-tree can be found in the literature [3 4 5 12]. Here we just give a sketch and some notation.

The SPQR-tree \( \mathcal{T} \) of a 2-connected graph \( G \) is an unrooted tree that has four different types of nodes, namely S-, P-, Q- and R-nodes. The Q-nodes are the leaves of \( \mathcal{T} \), and they correspond to edges of \( G \). Each internal node \( \mu \) of \( \mathcal{T} \) has an associated biconnected multigraph \( \mathcal{S} \), its skeleton, which can be seen as a simplified version of the graph \( G \). The subtrees of \( \mu \) in \( \mathcal{T} \) when \( \mathcal{T} \) is rooted at \( \mu \) determine a decomposition of \( G \) into edge-disjoint subgraphs \( G_1, \ldots, G_k \), each of which is connected and shares exactly two vertices \( u_i, v_i \) with the rest of the graph \( G \). Each \( G_i \) is represented in the skeleton of \( \mu \) by an edge \( e_i \) connecting \( u_i \) and \( v_i \). We say that \( G_i \) is the pertinent graph of the edge \( e_i \). We also say that the skeleton edge \( e_i \) contains a vertex \( v \) or an edge \( e \) of \( G_i \), or that \( v \) and \( e \) project into \( e_i \), if \( v \) or \( e \) belong to the pertinent graph \( G_i \) of \( e_i \). For a subgraph \( G' \) of \( G \), we say that \( G' \) intersects a skeleton edge \( e_i \), if at least one edge of \( G' \) belongs to \( G_i \).

The skeleton of an S-node is a cycle of length \( k \geq 3 \), the skeleton of a P-node consists of \( k \geq 3 \) parallel edges, and the skeleton of an R-node is a 3-connected planar graph. The SPQR-tree of a planar 2-connected graph \( G \) represents all planar embeddings of \( G \) in the sense that choosing planar embeddings for all skeletons of \( \mathcal{T} \) corresponds to choosing a planar embedding of \( G \) and vice
Figure 2: An example of a planar Peg (left) in which a contraction of a $G$-edge may result in two distinct Pegs, one of which is non-planar.

Suppose that $e = uv$ is an edge of the skeleton of a node $\mu$ of an SPQR-tree of a biconnected graph $G$, and let $G_e$ be the pertinent graph of $e$. The graph $G_e$ satisfies some additional restrictions depending on the type of $\mu$: if $\mu$ is an S-node, then $G_e$ is biconnected, and if $\mu$ is a P-node, then either $G_e$ is a single edge $uv$ or $G_e - \{u, v\}$ is connected. Regardless of the type of $\mu$, every cut-vertex in $G_e$ separates $u$ from $v$, otherwise $G$ would not be biconnected.

**Peg-minor operations** We first introduce a set of operations that preserve planarity when applied to a Peg $I = (G, H, \mathcal{H})$. The set of operations is chosen so that the resulting instance $I' = (G', H', \mathcal{H}')$ is again a Peg (in particular, $H'$ is a subgraph of $G'$ and $\mathcal{H}'$ is a planar embedding of $H'$). It is not possible to use the usual minor operations, as sometimes, when contracting an edge of $G - H$, the embedding of the modified graph $H$ is not unique and some of the possible embeddings lead to planar Pegs, while some do not. This happens, e.g., when a contraction of a $G$-edge creates a new cycle of $H$-edges, in which case it is not clear on which side of this cycle the remaining components of $H$ should be embedded (see Fig. 2).

We will consider seven minor-like operations, of which the first five are straightforward.

1. Vertex removal: Remove from $G$ and $H$ a vertex $v \in V(G)$ with all its incident edges.
2. Edge removal: Remove from $G$ and $H$ an edge $e \in E(G)$.
3. Vertex relaxation: For a vertex $v \in H$ remove $v$ and all its incident edges from $H$, but keep them in $G$. In other words, vertex $v$ no longer has a prescribed embedding.
4. Edge relaxation: Remove an edge $e \in E(H)$ from $H$, but keep it in $G$.
5. $H$-edge contraction: Contract an edge $e \in E(H)$ in both $G$ and $H$, update $\mathcal{H}$ accordingly.

The contraction of $G$-edges is tricky, as we have to care about two things. First, we have to take care that the modified subgraph $H'$ remains planar and second, even if it remains planar, we do not want to create a new cycle $C$ in $H$ as in this case the relative positions of the connected components of $H$ with respect to this cycle may not be uniquely determined. We therefore have special requirements for the $G$-edges that may be contracted and we distinguish two types, one of which trivially ensures the above two conditions and one that explicitly ensures them.

6. Simple $G$-edge contraction: Assume that $e = uv$ is an edge of $G$, such that at least one of the two vertices $u$ and $v$ does not belong to $H$. Contract $e$ in $G$, and leave $H$ and $\mathcal{H}$ unchanged.
7. Complicated $G$-edge contraction: Assume that $e = uv$ is an edge of $G$, such that $u$ and $v$ belong to distinct components of $H$, but share a common face of $\mathcal{H}$. Assume further that both $u$ and $v$ have degree at most 1 in $H$. This implies that we may uniquely extend $\mathcal{H}$ to an embedding $\mathcal{H}^+$ of the graph $H^+$ that is obtained from $H$ by adding the edge $uv$. Afterwards we perform an $H$-edge contraction of the edge $uv$ to obtain the new Peg.
Figure 3: Two non-isomorphic 5-fold alternating chains.

If a contraction produces multiple edges, we only preserve a single edge from each such set of multiple edges, so that $G$ and $H$ remain simple. Note that the resulting embedding $\mathcal{H}$ may depend on which edge we decide to preserve.

Let $(G, H, \mathcal{H})$ be a PEG and let $(G', H', \mathcal{H}')$ be the result of one of the above operations on $(G, H, \mathcal{H})$. The extra conditions on $G$-edge contractions ensure that the embedding $\mathcal{H}'$ is uniquely determined from the embedding of $H$ after such contraction. The conditions on vertex degrees in $H$ ensure that the rotation scheme of the $H'$-edges around the vertex created by the contraction is unique. In the complicated $G$-edge contraction, the requirement that the endpoints need to lie in distinct connected components of $H$ that share a face ensures that the contraction does not create a new cycle in $H'$ and that $H'$ has a unique planar embedding induced by $\mathcal{H}$.

It is not hard to see that an embedding $\mathcal{G}$ of $G$ that extends $\mathcal{H}$ can be transformed into an embedding $\mathcal{G}'$ of $G'$ that extends $\mathcal{H}'$. Therefore, all the above operations preserve planarity of PEGs. If a PEG $A$ can be obtained from a PEG $B$ by applying a sequence of the above operations, we say that $A$ is a PEG-minor of $B$ or that $B$ contains $A$ as a PEG-minor.

Alternating chains Apart from the obstructions in Fig. 1, there is an infinite family of obstructions, which we call the alternating chains. To describe them, we need some terminology. Let $C$ be a cycle of length at least four, and let $u$, $v$, $x$ and $y$ be four distinct vertices of $C$. We say that the pair of vertices $\{u, v\}$ alternates with the pair $\{x, y\}$ on $C$, if $u$ and $v$ belong to distinct components of $C - x - y$.

Intuitively, an alternating chain consists of a cycle $C$ of $H$ and a sequence of internally disjoint paths $P_1, \ldots, P_k$ of which only the endpoints belong to $C$, such that for each $i = 1, \ldots, k - 1$, the endpoints of $P_i$ alternate with the endpoints of $P_{i+1}$ on $C$, and no other pair of paths has alternating endpoints. Now assume that $P_1$ contains a vertex that is prescribed inside $C$. Due to the fact that the endpoints of consecutive paths alternate this implies that all $P_i$ with $i$ odd must be embedded inside $C$, while all $P_i$ with $i$ even must be embedded outside. A $k$-fold alternating chain is such that the last path $P_k$ is prescribed in a way that contradicts this, i.e., it is prescribed inside $C$ if $k$ is even and outside, if $k$ is odd. Generally it is sufficient to have paths of length 1 for $P_2, \ldots, P_{k-1}$ and to have a single vertex (for the prescription) in each of $P_1$ and $P_k$. We now give a precise definition.

Let $k \geq 3$ be an integer. A $k$-fold alternating chain is a PEG $(G, H, \mathcal{H})$ of the following form:

- The graph $H$ consists of a cycle $C$ of length $k + 1$ and two isolated vertices $u$ and $v$. If $k$ is odd, then $u$ and $v$ are embedded on opposite sides of $C$ in $\mathcal{H}$, otherwise they are embedded on the same side.
- The graph $G$ has the same vertex set as $H$, and the edges of $G$ that do not belong to $H$ form $k$ edge-disjoint paths $P_1, \ldots, P_k$, whose endpoints belong to $C$. The path $P_1$ has two edges and contains $u$ as its middle vertex, the path $P_k$ has two edges and contains $v$ as its middle vertex, and all the other paths have only one edge.
• The endpoints of the path $P_i$ alternate with the endpoints of the path $P_j$ on $C$ if and only if $j = i + 1$ or $i = j + 1$.

• All the vertices of $C$ have degree 4 in $G$ (i.e., each of them is a common endpoint of two of the paths $P_i$), with the exception of two vertices of $C$ that have degree three. One of these two vertices is an endpoint of $P_2$, and the other is an endpoint of $P_{k-1}$.

Let $Ach_k$ denote the set of $k$-fold alternating chains. It can be checked that for each $k \geq 4$, the elements of $Ach_k$ are obstructions; see Lemma 21. Obstruction 4 from Fig. 1 is actually the unique member of $Ach_3$, and is an obstruction as well. However, we prefer to present it separately as an ‘exceptional’ obstruction, because we often need to refer to it explicitly. Note that for $k \geq 5$ we may have more than one non-isomorphic $k$-fold chain; see Fig. 3.

3 Biconnected Pegs

In this section we prove Theorem 1 for biconnected Pegs. We first recall a characterization of biconnected planar Pegs via SPQR-trees.

Definition 3. Let $(G, H, \mathcal{H})$ be a biconnected Peg.

A planar embedding of the skeleton of a node of the SPQR-tree of $G$ is edge-compatible with $\mathcal{H}$ if, for every vertex $x$ of the skeleton and for every three edges of $H$ incident to $x$ that project to different edges of the skeleton, their order determined by the embedding of the skeleton is the same as their order around $x$ in $\mathcal{H}$.

A planar embedding of the skeleton $\mathcal{S}$ of a node $\mu$ of the SPQR-tree of $G$ is cycle-compatible with $\mathcal{H}$ if, for every facial cycle $\bar{C}$ of $\mathcal{H}$ whose edges project to a simple cycle $\bar{C}'$ in $\mathcal{S}$, all the vertices of $\mathcal{S}$ that lie to the left of $\bar{C}$ and all the skeleton edges not belonging to $\bar{C}$ that contain vertices that lie to the left of $\bar{C}$ in $\mathcal{H}$ are embedded to the left of $\bar{C}'$; and analogously for the vertices to the right of $\bar{C}$.

A planar embedding of a skeleton of a node of the SPQR-tree of $G$ is compatible if it is both edge- and cycle-compatible.

Angelini et al. showed that a biconnected Peg is planar if and only if the skeleton of each node admits a compatible embedding [1, Theorem 3.1]. We use this characterization and show that any skeleton of a biconnected Peg that avoids all obstructions admits a compatible embedding. Since skeletons of S-nodes have only one embedding, and their embedding is always compatible, we consider P- and R-nodes only. The two types of nodes are handled separately in Subsections 3.1 and 3.2, respectively.

The following lemma will be useful in several parts of the proof.

Lemma 4. Let $(G, H, \mathcal{H})$ be a Peg, let $u$ be a vertex of a skeleton $\mathcal{S}$ of a node $\mu$ of the SPQR-tree of $G$, and let $e$ be an edge of $\mathcal{S}$ with endpoints $u$ and $v$. Let $F \subseteq E(H)$ be the set of edges of $H$ that are incident to $u$ and project into $e$. If the edges of $F$ do not form an interval in the rotation scheme of $u$ in $\mathcal{H}$ then $(G, H, \mathcal{H})$ contains obstruction 2.

Proof. If $F$ is not an interval in the rotation scheme, then there exist edges $f, f' \in F$ and $g, g' \in E(H) \setminus F$, all incident to $u$, and appearing in the cyclic order $f, g, f', g'$ around $u$ in $\mathcal{H}$. Let $x$ and $x'$ be the endpoints of $f$ and $f'$ different from $u$ and let $y$ and $y'$ be the endpoints of $g$ and $g'$ different from $u$. For any skeleton edge $f$, we denote with $G_f$ the pertinent graph of $f$.

If $\mu$ is an S-node, then $g$ and $g'$ project to the same skeleton edge $uw$ with $v \neq w$. Note that $G_{uw}$ and $G_{uw}$ share only the vertex $u$ and moreover, they are both connected even after removing
Lemma 5. Let \( P \in G_{uw} \) and \( Q \in G_{uw} \), connecting \( x \) to \( x' \) and \( y \) to \( y' \), respectively. We may relax all internal vertices and all edges of \( P \) and \( Q \), and then perform simple edge contractions to replace each of the two paths with a single edge. This yields obstruction 2.

If \( \mu \) is an R-node, then \( G_{uw} - u \) is connected, and hence it contains a path \( P \) from \( x \) to \( x' \). Moreover, since \( G - G_{uw} \) is connected, it has a path \( Q \) from \( y \) to \( y' \). As in the previous case, contraction of \( P \) and \( Q \) yields obstruction 2.

If \( \mu \) is a P-node, then \( G_e - \{u, v\} \) is connected, and therefore there is a path \( P \) connecting \( x \) to \( x' \) in \( G_e - \{u, v\} \). Analogous to the previous cases, a path \( Q \) from \( y \) to \( y' \) exists that avoids \( u \) and \( P \). Again their contraction yields obstruction 2. \( \square \)

In the following, we assume that the \( H \)-edges around each vertex of a skeleton that project to the same skeleton edge form an interval in the rotation scheme of this vertex.

3.1 P-Nodes
Throughout this section, we assume that \((G, H, \mathcal{H})\) is a biconnected obstruction-free PEG. We fix a P-node \( \mu \) of the SPQR-tree of \( G \), and we let \( \mathcal{P} \) be its skeleton. Let \( u \) and \( v \) be the two vertices of \( \mathcal{P} \), and let \( e_1, \ldots, e_k \) be its edges. Let \( G_i \) be the pertinent graph of \( e_i \). Recall that \( G_i \) is either a single edge connecting \( u \) and \( v \), or it does not contain the edge \( uv \) and \( G_i - \{u, v\} \) is connected.

The goal of this section is to prove that \( \mathcal{P} \) admits a compatible embedding. We first deal with edge-compatibility.

Lemma 5. Let \((G, H, \mathcal{H})\) be a biconnected obstruction-free PEG. Then every \( P \)-skeleton \( \mathcal{P} \) has an edge-compatible embedding.

Proof. If \( \mathcal{P} \) has no edge-compatible embedding, then the rotation scheme around \( u \) conflicts with the rotation scheme around \( v \). This implies that there is a triplet of skeleton edges \( e_a, e_b, e_c \), for which the rotation scheme around \( u \) imposes a different cyclic order than the rotation scheme around \( v \). We distinguish two cases.

Case 1. The graph \( H \) has a cycle \( C \) whose edges intersect two of the three skeleton edges, say \( e_a \) and \( e_b \). Then the edge \( e_c \) must contain a vertex \( x \) whose prescribed embedding is to the left of \( C \), as well as a vertex \( y \) whose prescribed embedding is to the right of \( C \). Since \( x \) and \( y \) are connected by a path in \( G_e - \{u, v\} \), we obtain obstruction 1.

Case 2. The graph \( H \) has no cycle that intersects two of the three \( \mathcal{P} \)-edges \( e_a, e_b, e_c \). Each of the three \( \mathcal{P} \)-edges contains an edge of \( H \) adjacent to \( u \) as well as an edge of \( H \) adjacent to \( v \). Since \( G_i - \{u, v\} \) is connected for each \( i \), it follows that each of the three skeleton edges contains a path from \( u \) to \( v \), such that the first and the last edge of the path belong to \( H \). Fix such paths \( P_a \), \( P_b \) and \( P_c \), projecting into \( e_a, e_b \) and \( e_c \), respectively.

At least two of these paths (\( P_a \) and \( P_b \), say) also contain an edge not belonging to \( H \), otherwise they would form a cycle of \( H \) intersecting two skeleton edges. By relaxations and simple contractions, we may reduce \( P_a \) to a path of length three, whose first and last edge belong to \( E(H) \) and the middle edge belongs to \( E(G) \setminus E(H) \). The same reduction can be performed with \( P_b \). The path \( P_c \) can then be contracted to a single vertex, to obtain obstruction 2. \( \square \)

Next, we consider cycle-compatibility. Assume that \( H \) has at least one facial cycle whose edges intersect two distinct skeleton edges. It follows that \( u \) and \( v \) belong to the same connected component of \( H \); denote this component by \( H_{uw} \). We call a \( uv \)-cycle any facial cycle of \( H \) that contains both \( u \) and \( v \). Note that any \( uv \)-cycle is also a facial cycle of \( H_{uw} \), and a facial cycle of \( H_{uw} \) that contains both \( u \) and \( v \) is a \( uv \)-cycle. Following the conventions of [1], we assume that all
facial cycles are oriented in such a way that a face is to the left of its facial cycles. The next lemma shows that the vertices of $H_{uv}$ cannot violate any cycle-compatibility constraints without violating edge-compatibility as well.

**Lemma 6.** Assume that $C$ is a $uv$-cycle that intersects two distinct $\mathcal{P}$-edges $e_a$ and $e_b$, and that $x$ is a vertex of $H_{uv}$ not belonging to $C$. In any edge-compatible embedding of $\mathcal{P}$, the vertex $x$ does not violate cycle-compatibility with respect to $C$.

**Proof.** The vertex $x$ belongs to a skeleton edge $e_x$ different from $e_a$ and $e_b$, otherwise it cannot violate cycle-compatibility. Note that since $x$ is in $H_{uv}$, $e_x$ must contain a path $P$ of $H$ that connects $x$ to one of the poles $u$ and $v$. In the graph $H$, all the vertices of $P$ must be embedded on the same side of $C$ as the vertex $x$. The last edge of $P$ may not violate edge-compatibility, which forces the whole edge $e_x$, and thus $x$, to be embedded on the correct side of the projection of $C$, as claimed. \[\square\]

The next lemma shows that for an obstruction-free PEG, all vertices of $H$ projecting to the same $\mathcal{P}$-edge impose the same cycle-compatibility constraints for the placement of this edge.

**Lemma 7.** Let $x$ and $y$ be two vertices of $H$, both distinct from $u$ and $v$. Suppose that $x$ and $y$ project to the same $\mathcal{P}$-edge $e_a$. Let $C$ be a cycle of $H$ that is edge-disjoint from $G_a$. Then $x$ and $y$ are embedded on the same side of $C$ in $\mathcal{H}$.

**Proof.** Since $G_a - \{u, v\}$ is a connected subgraph of $G$, there is a path $P$ in $G$ that connects $x$ to $y$ and avoids $u$ and $v$. Since $C$ is edge-disjoint from $G_a$, the path $P$ avoids all the vertices of $C$. If $x$ and $y$ were not embedded on the same side of $C$ in $\mathcal{H}$, we would obtain obstruction 1 by contracting $C$ and $P$. \[\square\]

We now prove the main result of this subsection.

**Proposition 8.** Let $(G, H, \mathcal{H})$ be a biconnected obstruction-free PEG. Then every $P$-skeleton $\mathcal{P}$ of the SPQR-tree of $G$ admits a compatible embedding.

**Proof.** Fix an edge-compatible embedding that minimizes the number of violated cycle-compatibility constraints; more precisely, fix an embedding of $\mathcal{P}$ that minimizes the number of pairs $(C, x)$ where $C$ is a facial cycle of $\mathcal{H}$ projecting to a cycle $C'$ of $\mathcal{P}$, $x$ is a vertex of $H - \{u, v\}$ projecting into a skeleton edge $e_x$ not belonging to $C'$, and the relative position of $C'$ and $e_x$ in the embedding of $\mathcal{P}$ is different from the relative position of $C$ and $x$ in $\mathcal{H}$. We claim that the chosen embedding of $\mathcal{P}$ is compatible.

For contradiction, assume that there is at least one pair $(C, x)$ that violates cycle-compatibility in the sense described above. Let $e_x$ be the $\mathcal{P}$-edge containing $x$. Note that $e_x$ does not contain any edge of $H$ adjacent to $u$ or $v$. If it contained such an edge, it would contain a vertex $y$ from the component $H_{uv}$, and this would contradict Lemma 6 or Lemma 7. Thus, the edge $e_x$ does not participate in any edge-compatibility constraints.

It follows that $x$ does not belong to the component $H_{uv}$. That means that in $\mathcal{H}$, the vertex $x$ is embedded in the interior of a unique face $F$ of $H_{uv}$. We distinguish two cases, depending on whether the boundary of $F$ contains both poles $u$ and $v$ of $\mathcal{P}$ or not.

**Case 1.** The boundary of $F$ contains at most one of the two poles $u$ and $v$; see Fig. 4. Without loss of generality, the boundary of $F$ does not contain $u$. Thus, $F$ has a facial cycle $D$ that separates $u$ from $x$. The pertinent graph $G_x$ of $e_x$ contains a path $P$ from $x$ to $u$ that avoids $v$. The path $P$ does not contain any vertex of $H_{uv}$ except $u$, and in particular, it does not contain any vertex of $D$. Contracting $D$ to a triangle and $P$ to an edge yields obstruction 1, which is a contradiction.
Case 2. The boundary of \( F \) contains both poles \( u \) and \( v \) of the skeleton. In this case, since \( u \) and \( v \) belong to the same block of \( H \), the face \( F \) has a unique facial cycle \( D \) that contains both \( u \) and \( v \). The cycle \( D \) is the only \( uv \)-cycle that has \( x \) to its left (i.e., inside its corresponding face).

The cycle \( D \) may be expressed as a union of two paths \( P \) and \( Q \) connecting \( u \) and \( v \), where \( P \) is directed from \( u \) to \( v \) and \( Q \) is directed from \( v \) to \( u \). We distinguish two subcases, depending on whether the paths \( P \) and \( Q \) project to different \( \mathfrak{P} \)-edges.

Case 2.a Both \( P \) and \( Q \) project to the same skeleton edge \( e_D \). This case is depicted in Fig. 5. Each of the two paths \( P \) and \( Q \) has at least one internal vertex. Since all these internal vertices are inside a single skeleton edge, there must be a path \( R \) in \( G \) connecting an internal vertex of \( P \) to an internal vertex of \( Q \) and avoiding both \( u \) and \( v \). By choosing \( R \) as short as possible, we may assume that no internal vertex of \( R \) belongs to \( D \). Furthermore, since \( \mathfrak{P} \) by hypothesis has at least one violated cycle-compatibility constraint, it must contain at least two edges that contain an \( H \)-path from \( u \) to \( v \). In particular, there must exist a \( \mathfrak{P} \)-edge \( e_S \) different from \( e_D \) that contains an \( H \)-path \( S \) from \( u \) to \( v \).

Necessarily, the path \( S \) is embedded outside the face \( F \), i.e., the right of \( D \). And finally, the edge \( e_x \) must contain a \( G \)-path \( T \) from \( u \) to \( v \) that contains \( x \). Note that \( e_x \) is different from \( e_D \) and \( e_S \), because \( e_x \) has no \( H \)-edge incident to \( u \) or \( v \). Thus, the paths \( P, Q, S, T \) are all internally disjoint. The five paths \( P, Q, R, S, \) and \( T \) can then be contracted to form obstruction 3.

Case 2.b The two paths \( P \) and \( Q \) belong to distinct skeleton edges \( e_P \) and \( e_Q \). That means that the facial cycle \( D \) projects to a cycle \( D' \) of the skeleton, formed by the two edges. Modify the embedding of the skeleton by moving \( e_x \) so that it is to the left of \( D' \). This change does not violate edge-compatibility, because \( e_x \) has no \( H \)-edge adjacent to \( u \) or \( v \).

We claim that in the new skeleton embedding, \( x \) does not participate in any violated cycle-compatibility constraint. To see this, we need to check that \( x \) is embedded to the right of any facial cycle \( B \neq D \) of \( H_{uv} \) that projects to a cycle in the skeleton. Choose such a cycle \( B \) and let \( B' \) be its projection; see Fig. 6. Let \( e^+ \) or \( e^- \) denote the edges of \( D \) incident to \( u \) with \( e^+ \) being oriented towards \( u \) and \( e^- \) out of \( u \). Similarly, let \( f^+ \) and \( f^- \) be the incoming and outgoing edges of \( B \) adjacent to \( u \). In \( \mathcal{H} \), the four edges must visit \( u \) in the clockwise order \((e^+, e^-, f^+, f^-)\), with the possibility that \( e^- = f^+ \) and \( e^+ = f^- \).
Figure 6: Illustration of Case 2.b in the proof of Proposition 8. The left part represents the embedding of $\mathcal{P}$ after after $e_x$ has been moved to the left of $D'$. Edge-compatibility guarantees that $x$ is now on the correct side of every facial cycle in $\mathcal{P}$.

Since the embedding of the skeleton is edge-compatible, this means that any skeleton edge embedded to the left of $D'$ is also to the right of $B'$, as needed. We conclude that in the new embedding of $\mathcal{P}$, the vertex $x$ does not violate any cycle-compatibility constraint, and by Lemma 7, the same is true for all the other $H$-vertices in $e_x$. Moreover, the change of embedding of $e_x$ does not affect cycle-compatibility of vertices not belonging to $e_x$, so the new embedding violates fewer cycle-compatibility constraints than the old one, which is a contradiction. This proves that $\mathcal{P}$ has a compatible embedding.

Let us remark that there are only finitely many obstructions that may arise from a $P$-skeleton that lacks a compatible embedding. In fact, if $(G, H, \mathcal{H})$ is a non-planar Peg and if $G$ is a biconnected graph with no $K_4$-minor (implying that the SPQR-tree of $G$ has no $R$-nodes), then we may conclude that $(G, H, \mathcal{H})$ contains obstruction 1 or 2, since all the other obstructions contain $K_4$ as (ordinary) minor.

3.2 R-Nodes

Let us now turn to the analysis of R-nodes. As in the case of P-nodes, our goal is to show that if a skeleton $\mathcal{R}$ of an R-node in the SPQR-tree of $G$ has no compatible embedding, then the corresponding Peg $(G, H, \mathcal{H})$ contains an obstruction. The skeletons of R-nodes have more complicated structure than the skeletons of P-nodes, and accordingly, our analysis is more complicated as well. Similar to the case of P-nodes, we will first show that an R-node of an obstruction-free Peg must have an edge-compatible embedding, and as a second step show that in fact it must also have an edge-compatible embedding that also is cycle-compatible.

The skeleton of an R-node is a 3-connected graph. We therefore start with some preliminary observations about 3-connected graphs, which will be used throughout this section. Let $\mathcal{R}$ be a 3-connected graph with a planar embedding $\mathcal{R}^+$, let $x$ be a vertex of $\mathcal{R}$. A vertex $y$ of $\mathcal{R}$ is visible from $x$ if $x \neq y$ and there is a face of $\mathcal{R}^+$ containing $x$ and $y$ on its boundary. An edge $e$ is visible from $x$ if $e$ is not incident with $x$ and there is a face containing both $x$ and $e$ on its boundary. The vertices and edges visible from $x$ form a cycle in $\mathcal{R}$. To see this, note that these vertices and edges form a face boundary in $\mathcal{R}^+ - x$, and every face boundary in an embedding of a 2-connected graph is a cycle. We call this cycle the horizon of $x$.

In the following, we will consider a fixed skeleton $\mathcal{R}$ of an R-node. Since $\mathcal{R}$ is 3-connected, it has only two planar embeddings, denoted by $\mathcal{R}^+$ and $\mathcal{R}^-$. Suppose that neither of the two embeddings is compatible. The constraints on the embeddings either stem from a vertex whose incident $H$-edges project to distinct edges of $\mathcal{R}$ or from a cycle of $\mathcal{R}$ that is a projection of an $H$-cycle whose cycle-compatibility constraints demand exactly one of the two embeddings. Since neither $\mathcal{R}^+$ nor $\mathcal{R}^-$ are compatible, there must be at least two such structures, one requiring embedding $\mathcal{R}^+$, and the other one requiring $\mathcal{R}^-$. If these structures are far apart in $\mathcal{R}$, for example, if no vertex
of the first structure belongs to the horizon of a vertex of the second structure, it is usually not too difficult to find one of the obstructions. However, if they are close together, a lot of special cases can occur. A significant part of the proof therefore consists in controlling the distance of objects and showing that either an obstruction is present or close objects cannot require different embeddings.

As before, we distinguish two main cases: first, we deal with the situation in which both embeddings of \( R \) violate edge-compatibility. Next, we consider the situation in which \( R \) has at least one edge-compatible embedding, but no edge-compatible embedding is cycle-compatible.

### 3.2.1 \( R \) has no edge-compatible embedding

Let \( u \) be vertex of \( R \) that violates the edge-compatibility of \( R^+ \), and let \( v \) be a vertex violating edge-compatibility of \( R^- \). If \( u = v \), i.e., if a single vertex violates edge-compatibility in both embeddings, the following lemma shows that we can find an occurrence of obstruction 2 in \((G, H, \mathcal{H})\).

**Lemma 9.** Assume that an \( R \)-node skeleton \( R \) has a vertex \( u \) that violates edge-compatibility in both embeddings of \( R \). Then \((G, H, \mathcal{H})\) contains obstruction 2.

**Proof.** Let \( e'_1, \ldots, e'_m \) be the \( R \)-edges incident to \( u \) that contain at least one \( H \)-edge incident to \( u \). Assume that these edges are listed in their clockwise order around \( u \) in the embedding \( R^+ \). Let \( e_i \) be an \( H \)-edge incident to \( u \) projecting into \( e'_i \). By Lemma 4, if a triple of edges \( e'_i, e'_j, e'_k \) violates edge-compatibility in \( R^+ \), then this violation is demonstrated by the edges \( e_i, e_j, e_k \), i.e., the cyclic order of \( e_i, e_j \) and \( e_k \) in \( \mathcal{H} \) is different from the cyclic order of \( e'_i, e'_j \) and \( e'_k \) in \( R^+ \).

Choose a largest set \( I \subseteq \{1, \ldots, m\} \) such that the edges \( \{e_i, i \in I\} \) do not contain any violation of edge-compatibility when embedded according to \( R^+ \). Clearly, \( 3 \leq |I| < m \) because if each triple violated edge-compatibility in \( R^+ \), then \( R^- \) would be edge-compatible with \( u \). Also \( |I| < m \), otherwise \( R^+ \) would be edge-compatible with \( u \).

Choose an index \( i \in \{1, \ldots, m\} \) not belonging to \( I \). By maximality of \( I \), there are \( j, k, \ell \in I \) such that, without loss of generality, \((e_i, e_j, e_k, e_\ell)\) appear clockwise in \( R^+ \) and \((e_j, e_i, e_k, e_\ell)\) appear clockwise in \( \mathcal{H} \) (recall that \((e_j, e_k, e_\ell)\) have the same order in \( R^+ \) and \( \mathcal{H} \), by the definition of \( I \)).

For \( a \in \{1, \ldots, m\} \) let \( x_a \) be the endpoint of the skeleton edge \( e'_a \) different from \( u \). The horizon of \( u \) in \( R^+ \) contains two disjoint paths \( P \) and \( Q \) joining \( x_i \) with \( x_\ell \) and \( x_j \) with \( x_k \). By obvious contractions we obtain obstruction 2.

Let us concentrate on the more difficult case when \( u \) and \( v \) are distinct. To handle this case, we introduce the concept of ‘wrung Pegs’. A wrung Peg is a Peg \((G, H, \mathcal{H})\) with the following properties.

- \( G \) is a subdivision of a 3-connected planar graph, therefore it has two planar embeddings \( G^+, G^- \).
- \( H \) has two distinct vertices \( u \) and \( v \) of degree 3. Any other vertex of \( H \) is adjacent to \( u \) or \( v \), and any edge of \( H \) is incident to \( u \) or to \( v \). Hence, \( H \) has five or six edges, and at most eight vertices.
- \( H \) is not isomorphic to \( K_{2,3} \) or to \( K^-_4 \) (i.e., \( K_4 \) with an edge removed). Equivalently, \( H \) has at least one vertex of degree 1.
- The embedding \( \mathcal{H} \) of \( H \) is such that its rotation scheme around \( u \) is consistent with \( G^+ \) and its rotation scheme around \( v \) is consistent with \( G^- \). Note that such an embedding exists due to the previous condition.
Clearly, a wrung Peg is not planar, because neither $G^+$ nor $G^-$ is an extension of $H$. A minimal wrung Peg is a wrung Peg that does not contain a smaller wrung Peg as a Peg-minor. A minimal wrung Peg is not necessarily a planarity obstruction—it may contain a smaller non-planar Peg that is not wrung (see Fig. 7). However, it turns out that minimal wrung Pegs are close to being planarity obstructions. The key idea in using wrung Pegs is that they are characterized by being subdivisions of 3-connected graphs, a property that is much easier to control than non-embeddability of Pegs.

The following proposition summarizes the key property of wrung Pegs. In particular, it shows that there are only finitely many minimal wrung Pegs.

**Proposition 10.** If $(G, H, \mathcal{H})$ is a minimal wrung Peg, then every vertex of $G$ also belongs to $H$ and the graph $H$ is connected.

The proof of this proposition relies heavily on the notion of ‘contractible edge’, which is an edge in a 3-connected graph whose contraction leaves the graph 3-connected. This notion has been intensely studied [9, 10], and we are able to use powerful structural theorems that guarantee that any ‘sufficiently large’ wrung Pegs must contain an edge that can be contracted to yield a smaller wrung Peg.

**Proof.** Let $G^*$ be the 3-connected graph whose subdivision is $G$. A subdivision vertex is a vertex of $G^*$ of degree 2. A subdivided edge is a path in $G^*$ of length at least two whose every internal vertex is a subdividing vertex and whose endpoints are not subdividing vertices. Therefore, each edge of $G^*$ either represents an edge of $G$ or a subdivided edge of $G$.

The proof of the proposition is based on several claims.

**Claim 1.** Every subdividing vertex of $G$ is a vertex of $H$. Every subdivided edge of $G$ contains at most one vertex adjacent to $u$ and at most one vertex adjacent to $v$. If $H$ is disconnected then $G$ has at most one subdivided edge, which (if it exists) connects $u$ and $v$ and is subdivided by a single vertex.

If $G$ had a subdividing vertex $x$ not belonging to $H$, we could contract an edge of $G$ incident to $x$ to get a smaller Peg, which is still wrung.

To see the second part of the claim, note that two vertices adjacent to $u$ in the same subdivided edge would imply the existence of a loop or a multiple edge in $G^*$.

For the last part of the claim, note that if $H$ is disconnected, then every vertex of $H$ except for $u$ and $v$ has degree 1 in $H$. If a subdividing vertex adjacent to $u$ were also adjacent to an $H$-neighbor of $v$, then the edge between them could be contracted. This proves the claim.

A fundamental tool in the analysis of minimal wrung Pegs is the concept of contractible edges. An edge $e$ in a 3-connected graph $F$ is contractible if $F - e$ is also 3-connected, where $F - e$ is the graph obtained from $F$ by contracting $e$. Note that an edge $e = xy$ in a 3-connected graph $F$ is contractible if and only if $F - \{x, y\}$ is biconnected.

The next fact is a special case of a theorem by Kriesell [9], see also [10, Theorem 3].
Fact 1. If $F$ is a 3-connected graph and $w$ a vertex of $F$ that is not incident with any contractible edge and such that $F - w$ is not a cycle, then $w$ is adjacent to four vertices $x_1, x_2, y_1, y_2$, all having degree 3 in $F$, which induce two disjoint edges $x_1y_1$ and $x_2y_2$ of $F$, and both these edges are contractible.

We are now ready to show that every vertex of $G$ also belongs to $H$. Suppose for a contradiction that $G$ has a vertex $w$ not belonging to $H$. By Claim 1, $w$ is not a subdivision vertex, so $w$ is also a vertex of $G^*$. If $w$ were incident to a contractible edge of $G^*$, we could contract this edge to obtain a smaller wrung $\overset{\star}{\text{PEG}}$. Hence, $w$ is not incident to any contractible edge of $G^*$. Fix now the four vertices from Fact 1 and let $e_1 = x_1y_1$ and $e_2 = x_2y_2$ be the two contractible edges. Necessarily all the four endpoints of $e_1$ and $e_2$ belong to $H$, otherwise we could contract one of them to get a smaller wrung $\overset{\star}{\text{PEG}}$. Moreover, the edges $e_1$ and $e_2$ cannot contain $u$ or $v$, because their endpoints have degree three and are adjacent to the vertex $w$ not belonging to $H$. Therefore, each endpoint of $e_1$ and $e_2$ is adjacent to either $u$ or $v$ in $G^*$ (and also in $G$ and in $H$).

Assume without loss of generality that $x_1$ is adjacent to $u$. Then $y_1$ cannot be adjacent to $u$, because then $u$ and $w$ would form a separating pair in $G^*$, hence $y_1$ is adjacent to $v$. Analogously, we may assume that $x_2$ is adjacent to $u$ and $y_2$ is adjacent to $v$. The graph $H$ must be connected, otherwise we could contract $e_1$ or $e_2$. This means that $H$, together with $e_1$ and $e_2$ and the two edges $wx_1$ and $wx_2$ form a subdivision of $K_4$, and therefore they form a wrung $\overset{\star}{\text{PEG}}$ properly contained in $(G, H, H)$. Therefore any vertex of $G$ also belongs to $H$.

It remains to prove that $H$ is connected. For this we need another concept for dealing with subdivisions of 3-connected graphs. Let $F$ be a 3-connected graph and let $e = xy$ be an edge of $F$. The cancellation of $e$ in $F$ is the operation that proceeds in the three steps 1) Remove $e$ from $F$, to obtain $F - e$. 2) If the vertex $x$ has degree 2 in $F - e$, then replace the subdivided edge containing $x$ by a single edge. Do the same for $y$ as well. 3) Simplify the graph obtained from step 2 by removing multiple edges.

Let $F \oplus e$ denote the result of the cancellation of $e$ in $F$. Note that $F \oplus e$ may contain vertices of degree 2 if they arise in step 3 of the above construction. An edge $e$ is cancellable if $F \oplus e$ is 3-connected. It is called properly cancellable if it is cancellable, and moreover, the first two steps in the above definition produce a graph without multiple edges.

Claim 2. A cancellable edge $e$ in a 3-connected graph $F$ is either properly cancellable or contractible.

Suppose that $e = xy$ is cancellable, but not properly cancellable. We show that it is contractible. Since $e$ is not properly cancellable, one of its endpoints, say $x$, has degree 3 in $F$ and its two neighbors $x'$ and $x''$ besides $y$ are connected by an edge. We show that between any pair of vertices $a$ and $b$ of $F - \{x, y\}$ there are two vertex-disjoint paths. In $F$ there exist three vertex-disjoint $a - b$-paths $P_1, P_2$ and $P_3$. If two of them avoid $x$ and $y$ then they are also present in $F - \{x, y\}$. Therefore, we may assume that $P_1$ contains $x$ and $P_2$ contains $y$. Then $P_1$ contains the subpath $x'x''y$ which can be replaced by the single edge $x'x''$. Again at most one of the paths contains vertices of $\{x, y\}$ and therefore we again find two vertex-disjoint $a - b$-paths in $F - \{x, y\}$. This shows that $F - \{x, y\}$ is biconnected and therefore $e = xy$ is contractible. This concludes the proof of the claim.

Moreover, we need the following result by Holton et al. [6], which we present without proof.

Fact 2. If $F$ is a 3-connected graph with at least five vertices, then every triangle in $F$ has at least two cancellable edges.

We proceed with the proof of Proposition 10 and show that $H$ is connected. Suppose for a contradiction that $H$ is disconnected, and let $H_u$ and $H_v$ be its two components containing $u$ and
Let \( x_1, x_2 \) and \( x_3 \) be the three neighbors of \( u \) in \( H \), and \( y_1, y_2 \) and \( y_3 \) the three neighbors of \( v \). Recall from Claim \( 1 \) that \( G \) has at most one subdividing vertex, and that the possible subdivided edge connects \( u \) and \( v \).

Since \( G^* \) is 3-connected, it has three disjoint edges \( e_1, e_2 \) and \( e_3 \), each of them connecting a vertex of \( H_u \) to a vertex of \( H_v \). At least one of them avoids both \( u \) and \( v \). Assume without loss of generality that \( e_1 = x_1y_1 \) is such an edge. If \( e_1 \) were a contractible edge of \( G^* \), we would get a smaller wrung \( \text{PEG} \). Therefore the graph \( G^* - \{x_1, y_1\} \) has a cut-vertex \( w \). Note that \( w \) is either \( u \) or \( v \). Otherwise, \( H_u - \{x_1, y_1, w\} \) would be connected, and \( H_v - \{x_1, y_1, w\} \) would be connected as well. However, since (by disjointness) at least one of the edges \( e_2, e_3 \) avoids \( w \), this implies that \( G^* - \{x_1, y_1, w\} \) would be connected as well, contradicting the choice of \( w \).

So, without loss of generality, \( G^* \) has a separating triplet \( \{x_1, y_1, u\} \). Since at least one of the two edges \( e_2, e_3 \) avoids this triplet, we see that one of the components of \( G^* - \{x_1, y_1, u\} \) consists of a single vertex \( x' \in \{x_2, x_3\} \). Since each vertex in a minimal separator must be adjacent to each of the components separated by the separator, \( G^* \) contains the two edges \( x'x_1 \) and \( x'y_1 \). Consequently, \( x', x_1 \) and \( y_1 \) induce a triangle in \( G^* \) (and in \( G \)), and by Fact \( 2 \) at least one of the two edges \( x_1y_1 \) and \( x'y_1 \) is cancellable, and by Claim \( 2 \) at least one of the two edges is contractible or properly cancellable, contradicting the minimality of \( (G, H, \mathcal{H}) \).

This completes the proof of Proposition \( 10 \). \( \square \)

Proposition \( 10 \) implies that a minimal wrung \( \text{PEG} \) has at most seven vertices. Therefore, to show that each wrung \( \text{PEG} \) contains one of the obstructions from Fig. \( 1 \) is a matter of a finite (even if a bit tedious) case analysis. We remark that a minimal wrung \( \text{PEG} \) does not contain any of the exceptional obstructions from Fig. \( 1 \) except obstructions 18–22, obstruction 3, \( K_5 \), and \( K_{3,3} \). A minimal wrung \( \text{PEG} \) does not contain any \( k \)-fold alternating chain for \( k \geq 4 \). As the analysis requires some more techniques, we defer the proof to Lemma \( 14 \).

Let us show how the concept of wrung \( \text{PEG}s \) can be applied in the analysis of \( R \)-skeletons.

Consider again the skeleton \( \mathcal{R} \), with two distinct vertices \( u \) and \( v \), each of them violating edge-compatibility of one of the two embeddings of \( \mathcal{R} \). This means that \( u \) is incident to three \( H \)-edges \( e_1, e_2, e_3 \) projecting into distinct \( \mathcal{R} \)-edges \( e'_1, e'_2, e'_3 \), such that the cyclic order of \( e_i \)'s in \( \mathcal{H} \) coincides with the cyclic order of \( e'_i \)'s in \( \mathcal{R}^- \), and similarly \( v \) is adjacent to \( H \)-edges \( f_1, f_2, f_3 \) projecting into \( \mathcal{R} \)-edges \( f'_1, f'_2, f'_3 \), whose order in \( \mathcal{R}^+ \) agrees with \( \mathcal{H} \). We have the following observation.

**Observation 1.** If all the \( e'_i \) and \( f'_i \) for \( i = 1, 2, 3 \) are distinct, then \( G \) contains a wrung \( \text{PEG} \).

If all the \( e'_i \) and \( f'_i \) for \( i = 1, 2, 3 \) are distinct, then it is fairly easy to see that \( G \) must contain a wrung \( \text{PEG} \), obtained simply by replacing each edge of \( \mathcal{R} \) with a path of \( G \), chosen in such a way that all the six edges \( e_i \) and \( f_i \) belong to these paths. Such a choice is always possible and yields a wrung \( \text{PEG} \). In particular, this is always the case if \( u \) and \( v \) are not adjacent in \( \mathcal{R} \). Thus the observation holds.

If, however, \( u \) and \( v \) are connected by an \( \mathcal{R} \)-edge \( g' \), and if, moreover, we have \( e'_i = g' = f'_j \) for some \( i \) and \( j \), the situation is more complicated, because there does not have to be a path in \( G \) that contains both edges \( e_i \) and \( f_j \) and projects into \( g' \). In such a situation, we do not necessarily obtain a wrung \( \text{PEG} \). This situation is handled separately in Lemma \( 13 \). Altogether, we prove the following proposition.

**Proposition 11.** Let \( (G, H, \mathcal{H}) \) be a biconnected obstruction-free \( \text{PEG} \), and let \( \mathcal{R} \) be the skeleton of an \( R \)-node of the SPQR-tree of \( G \). Then \( \mathcal{R} \) has an edge-compatible embedding.
In this case \((G, H, \mathcal{H})\) either contains a wrung \(\text{PEG}\), or it does not, and \(u\) and \(v\) are connected by an edge.

In the remainder of this subsection, we prove that in either case \((G, H, \mathcal{H})\) contains one of the obstructions from Fig. 1. We first show that if \(\mathcal{R}\) does not contain a wrung \(\text{PEG}\), then it contains one of the obstructions 4, 5 and 6; see Lemma 13. Finally, we also present a detailed analysis showing that any minimal wrung \(\text{PEG}\) (of which there are only finitely many) contains one of the obstructions from Fig. 1; see Lemma 14.

The following technical lemma is a useful tool, which we employ in both proofs. To state the lemma, we use the following notation: let \(x_1, x_2, \ldots, x_k\) be (not necessarily distinct) vertices of a graph \(F\). We say that a path \(P\) in \(F\) is a path of the form \(x_1 \rightarrow x_2 \rightarrow \cdots \rightarrow x_k\), if \(P\) is a simple path that is obtained by concatenating a sequence of paths \(P_1, P_2, \ldots, P_{k-1}\), where \(P_i\) is a path connecting \(x_i\) to \(x_{i+1}\) (note that if \(x_i = x_{i+1}\), then \(P_i\) consists of a single vertex).

**Lemma 12.** Let \(\mathcal{R}\) be a 3-connected graph with a fixed planar embedding \(\mathcal{R}^+\). Let \(u\) and \(v\) be two vertices of \(\mathcal{R}\) connected by an edge \(B\). Let \(u_1\) and \(u_2\) be two distinct neighbors of \(u\), both different from \(v\), such that \((v, u_1, u_2)\) appears counter-clockwise in the rotation scheme of \(u\). Similarly, let \(v_1\) and \(v_2\) be two neighbors of \(v\) such that \((u, v_1, v_2)\) appears clockwise around \(v\). (Note that we allow some of the \(u_i\) to coincide with some \(v_j\).) Then at least one of the following possibilities holds:

1. The graph \(\mathcal{R}\) contains a path of the form \(v \rightarrow u_1 \rightarrow u_2 \rightarrow v_1 \rightarrow v_2 \rightarrow u\).
2. The graph \(\mathcal{R}\) contains a path of the form \(u \rightarrow v_1 \rightarrow v_2 \rightarrow u_1 \rightarrow u_2 \rightarrow v\). (This is symmetric to the previous case.)
3. The graph \(\mathcal{R}\) has a vertex \(w\) different from \(u_2\) and three paths of the forms \(w \rightarrow u_1\) and \(w \rightarrow v_1\), respectively. These paths only intersect in \(w\), and none of them contains \(u\) or \(v\).
4. The graph \(\mathcal{R}\) has a vertex \(w\) different from \(v_2\) and three paths of the forms \(w \rightarrow v_2 \rightarrow u_2\), \(w \rightarrow v_1\) and \(w \rightarrow u_1\), respectively. These paths only intersect in \(w\), and none of them contains \(u\) or \(v\). (This is again symmetric to the previous case.)

**Proof.** Let \(C_u\) be the horizon of \(u\) and \(C_v\) the horizon of \(v\). Orient \(C_u\) counterclockwise and split it into three internally disjoint oriented paths \(v \rightarrow u_1\), \(u_1 \rightarrow u_2\) and \(u_2 \rightarrow v\), denoted by \(C_u^1\), \(C_u^2\), and \(C_u^3\) respectively. Similarly, orient \(C_v\) clockwise, and split it into \(C_v^1 = u \rightarrow v_1\), \(C_v^2 = v_1 \rightarrow v_2\), and \(C_v^3 = v_2 \rightarrow u\).

Let \(F_1\) and \(F_2\) be the two faces of \(\mathcal{R}\) incident with the edge \(uv\), with \(F_1\) to the left of the directed edge \(\vec{uv}\). Note that each vertex on the boundary of \(F_1\) except \(u\) and \(v\) appears both in \(C_u^1\) and in \(C_v^1\). Similarly, the vertices of \(F_2\) (other than \(u\) and \(v\)) appear in \(C_u^3\) and \(C_v^3\). There may be other vertices shared between \(C_u\) and \(C_v\) and we have no control about their position. However, at least their relative order must be consistent, as shown by the following claim.

**Claim 3.** Suppose that \(x\) and \(y\) are two vertices from \(C_u \cap C_v\), at most one of them incident with \(F_1\) and at most one of them incident with \(F_2\). Then \((v, x, y)\) are counter-clockwise on \(C_u\) if and only if \((u, x, y)\) are clockwise on \(C_v\); see Fig. 8.

To prove the claim, draw two curves \(\gamma_x\) and \(\gamma_y\) connecting \(u\) to \(x\) and to \(y\), respectively. Draw similarly two curves \(\delta_x\) and \(\delta_y\) from \(v\) to \(x\) and to \(y\). The endpoints of each of the curves appear in a common face of \(\mathcal{R}^+\), so each curve can be drawn without intersecting any edge of \(\mathcal{R}\). Also, the assumptions of the claim guarantee that at most two of these curves can be in a common face of \(\mathcal{R}^+\), and this happens only if they share an endpoint, so the curves can be drawn internally disjoint. Consider the closed curve formed by \(\gamma_x, \delta_x,\) and the edge \(uv\), oriented in the direction
Figure 8: The two directed horizontal lines represent $C_u$ and $C_v$. A vertex $x$ appearing on both $C_u$ and $C_v$ is represented by a dotted line connecting its position on $C_u$ with its position on $C_v$. Here is an example of a situation forbidden by the Claim, where two shared vertices $x$ and $y$ appear in different order on the directed cycles $C_u$ and $C_v$.

Figure 9: Case A of the proof of Lemma 12. The thick line represents the constructed walk.

$u \to x \to v \to u$. Suppose, without loss of generality, that $y$ is to the left of this closed curve. Then $\gamma_y$ is also to the left of it, and $(uv, \gamma_x, \gamma_y)$ appear in counter-clockwise order around $u$, so $(v, x, y)$ are counter-clockwise on $C_u$. By analogous reasoning, $(u, x, y)$ are clockwise around $v$. The claim is proved.

We now consider several cases depending on whether various parts of $C_u$ share vertices with parts of $C_v$.

**Case A.** $C^3_u$ shares a vertex $x$ with $C^1_v$. Consider a walk starting in $v$, following $C_u$ counter-clockwise through $u_1$ and $u_2$ until $x$, then following $C_v$ clockwise from $x$ through $v_1$ and $v_2$ till $u$; see Fig. 9. The above Claim 3 guarantees that this walk is actually a path (note that $x$ cannot belong to either $F_1$ or $F_2$). This path corresponds to the first case in the statement of the lemma. Similarly, if $C^1_u \cap C^3_v$ is nonempty, a symmetric argument yields the second case of the lemma.

Assume for the rest of the proof that $C^3_u \cap C^1_v = \emptyset = C^1_u \cap C^3_v$.

**Case B.** No internal vertex of $C^2_u$ belongs to $C^1_u \cup C^3_v$. Define a walk in $\mathcal{R}$ by starting in $v_1$, following $C_v$ counter-clockwise until we reach the first vertex (call it $x$) that belongs to $C_u$, then following $C_u$ counter-clockwise through $u_1$ and $u_2$, until we reach the first vertex $y$ from $C^3_v \cap C_u$, then following $C_v$ from $y$ towards $v_2$ while avoiding $v_1$ and $u$. Note that the vertices $x$ and $y$ must exist, because $F_1$ and $F_2$ each have at least one vertex from $C_u \cap C_v$. Note also that $x \in C^1_u$ and $y \notin C^1_v$, otherwise we are in Case A.

The walk defined above is again a path, it avoids $u$ and $v$, and by putting $w := u_1$, we are in the situation of the third case of the lemma. Symmetrically, if no internal vertex of $C^2_v$ belongs to $C^1_u \cup C^3_v$, we obtain the fourth case of the lemma.

Suppose that none of the previous cases (and their symmetric variants) occurs. What is left is the following situation.

**Case C.** $C^2_u$ has an internal vertex $x$ belonging to $C^1_u \cup C^3_v$, and $C^2_v$ has an internal vertex $y$ belonging to $C^1_v \cup C^3_u$. We cannot simultaneously have $x \in C^1_v$ and $y \in C^1_u$ as that would contradict Claim 3. So assume that $x \in C^1_v$ and $y \in C^3_u$, the other case being symmetric. Consider a walk $W_1$ from $u_1$ along $C_u$ counter-clockwise through $u_2$, and let $z$ be the first vertex on $C_u$ after $u_2$ that belongs to $C_v$. We must have $z \in C^2_v$, otherwise $z$ and $y$ violate Claim 3. Continue $W_1$ from $z$ until
Lemma 13. Let \((G, H, \mathcal{H})\) be a PEG and let \(\mathcal{R}\) be the skeleton of an \(R\)-node of \(G\) such that \(\mathcal{R}\) has two distinct vertices \(u\) and \(v\), each violating edge-compatibility of one of the embeddings of \(\mathcal{R}\). If \((G, H, \mathcal{H})\) does not contain a wrung PEG, then \((G, H, \mathcal{H})\) contains obstruction 4, 5, or 6.

Proof. As we have seen, \(u\) must be incident to three \(H\)-edges \(e_1 = ux_1, e_2 = ux_2, e_3 = ux_3\) projecting to distinct \(\mathcal{R}\)-edges \(e'_1 = uu_1, e'_2 = uu_2, e'_3 = uu_3\), such that the cyclic order of the \(e'_i\)'s in \(\mathcal{H}\) coincides with the cyclic order of \(e'_1, e'_2, e'_3\) in \(\mathcal{R}^+\). We may assume that \(e'_1, e'_2, e'_3\) appear counter-clockwise around \(u\) in \(\mathcal{H}\) and \((e'_1, e'_2, e'_3)\) appear counter-clockwise around \(u\) in \(\mathcal{R}^+\).

Similarly, \(v\) is incident to three \(H\)-edges \(f_1 = vy_1, f_2 = vy_2, f_3 = vy_3\) projecting to distinct \(\mathcal{R}\)-edges \(f'_1 = vv_1, f'_2 = vv_2, f'_3 = vv_3\) whose order in \(\mathcal{R}^-\) agrees with \(\mathcal{H}\). Assume that \((f'_1, f'_2, f'_3)\) appear counter-clockwise around \(v\) in \(\mathcal{H}\) and \((f'_1, f'_2, f'_3)\) appear counter-clockwise around \(v\) in \(\mathcal{R}^-\), and therefore clockwise around \(v\) in \(\mathcal{R}^+\).

If all the edges \(e'_1, e'_2, e'_3\) and \(f'_1, f'_2, f'_3\) are distinct, then \((G, H, \mathcal{H})\) contains a wrung PEG by Observation 1. Hence, one edge \(e'_i\) must coincide with an edge \(f'_j\). After possibly renaming the edges, we can assume \(e'_3 = f'_3\), and hence \(u_3 = v\) and \(v_3 = u\). Moreover, we may assume that the pertinent graph of \(e'_3\) does not contain a path from \(u\) to \(v\) that would contain both \(e_3\) and \(f_3\), otherwise we again obtain a wrung PEG. Since any vertex of the pertinent graph of \(e'_3\) lies on a \(G\)-path from \(u\) to \(v\) that projects into \(e'_3\), we conclude that the pertinent graph of \(e'_3\) has a cycle that contains the two edges \(e_3\) and \(f_3\).

After performing relaxations in \((G, H, \mathcal{H})\) if necessary, we may assume that the graph \(H\) only contains the edges \(e_i\) and \(f_i\) for \(i = 1, 2, 3\), and the vertices incident to these edges. We may also assume, after performing deletions and contractions if necessary, that the pertinent graph of \(e'_3\) is the four-cycle formed by the edges \(ux_3, xv, vy_3\) and \(yu\). Furthermore, we may assume that the pertinent graph of \(e'_1\) is a path of length at most two containing the edge \(e_1 = ux_1\), and similarly for \(e'_2, f'_1\) and \(f'_2\). In fact, if the four edges \(e'_1, e'_2, f'_1\) and \(f'_2\) do not form a four-cycle in \(\mathcal{R}\), we may assume that the pertinent graph of each of them is a single edge \(e_i\) or \(f_i\); however, if the four edges

\[ v \quad C^1_u \quad u_1 \quad C^2_u \quad u_2 \quad C^3_u \quad v \]

\[ v \quad C^1_v \quad v_1 \quad C^2_v \quad v_2 \quad C^3_v \quad v \]

Figure 10: Case B of the proof of Lemma 12. Note that the vertex \(y\) may also belong to \(C^3_v\).

\[ v \quad C^1_u \quad u_1 \quad C^2_u \quad u_2 \quad C^3_u \quad v \]

\[ v \quad C^1_v \quad v_1 \quad C^2_v \quad v_2 \quad C^3_v \quad v \]

Figure 11: Case C of the proof of Lemma 12. The thick lines represent \(W_1\) and \(W_2\).
form a four-cycle in $\mathcal{R}$, we may not contract their pertinent graphs to a single edge, as that would form a new cycle in $H$, which our contraction rules do not allow. Lastly, the pertinent graph of any edge of $\mathcal{R}$ different from $e'_i$ and $f'_i$ for $i = 1, 2, 3$ may be contracted to a single edge. This means, in particular, that any path in $\mathcal{R}$ that does not contain the edge $e'_3$ is the projection of a unique path of $G$.

We now apply Lemma 12 to the embedded graph $\mathcal{R}^+$ ($u, v, u_1, u_2, v_1, v_2$ are named as in the lemma). Let us treat the four cases of the lemma separately. In the first case, $\mathcal{R}$ contains a path $P'$ of the form $v \rightarrow u_1 \rightarrow u_2 \rightarrow v_1 \rightarrow v_2 \rightarrow u$. The existence of such path implies that $u_1$ is not the same vertex as $v_1$ or $v_2$, and $v_2$ is not the same as $u_1$ or $u_2$. In particular, the four edges $e_1, e_2, f_1$ and $f_2$ do not form a cycle, and each of them has for pertinent graph a single edge of $H$. The path $P'$ is a projection of a $G$-path $P$. Performing contractions as needed, we may assume that $P$ does not contain any other vertices apart from $v, u_1 = x_1, u_2 = x_2, v_1 = y_1, v_2 = y_2$, and $u$. Moreover, if the vertices $x_2$ and $y_1$ are distinct, this implies that they belong to different components of $H$, and we may contract the edge of $P$ that connects them, to create a single vertex $w$. After these contractions are performed, we are left with a PEG $(G', H', \mathcal{H}')$, where $V(G') = V(H') = \{u, v, x_1, x_2, v_1, v_2, y_1, y_2\}$, $E(H') = \{ux_1, ux_2, uv, vx_2, vy_1, vy_2\}$, and $E(G') = E(H') \cup \{vx_1, x_1w, wy_1, y_2v, vy_3\}$. This PEG is the obstruction 6.

Since the case 2 of Lemma 12 is symmetric to case 1, let us proceed directly to case 3. In this case, $\mathcal{R}$ has a vertex $w$ different from $u_2$, and three paths $P_1: w \rightarrow u_2 \rightarrow v_2, P_2: w \rightarrow u_1$, and $P_3: w \rightarrow v_1$ which only share the vertex $w$. We may again assume that the paths do not contain any other vertex except those listed above. Let us distinguish two subcases, depending on whether the four edges $e'_1, e'_2, f'_1$ and $f'_2$ form a cycle in $\mathcal{R}$ or not.

Assume first that the edges form a cycle $C \subseteq \mathcal{R}$, that is, $u_1 = v_1$; see Fig. 12(a). Suppose that at least one of the four edges of $C$ contains more than one edge of $G$. After performing contractions in $G$, we may assume that there is only one edge of $C$ that contains two $G$-edges, and three edges of $C$ containing one $G$-edge each. This produces obstruction 5.

Finally, consider the case when the four edges $e'_1, e'_2, f'_1$ and $f'_2$ do not form a cycle. Then each of the four edges is a projection of a unique edge of $G$. If $u_2 \neq v_2$ (Fig. 12(b)), then we may contract $P_2 \cup P_3$ into a single vertex and assume that $u_1 = v_1$. On the other hand, if $u_2 = v_2$ (Fig. 12(c)), we may contract $P_2 \cup P_3$ into a single edge $u_1v_1$. Both situations correspond to obstruction 5. Note that this case is independent of the vertex $w$ and whether it coincides with other vertices. Since case 4 of Lemma 12 is symmetric to case 3, this covers all possibilities. 

The following lemma shows that any minimal wrung PEG contains one of the obstructions in Fig. 1. Although the analysis is straight-forward, there exist many different cases. For the sake of completeness, we provide a detailed proof.
Proof. Let \((G, H, \mathcal{H})\) be a minimal wrung Peg, and let \(G^+\) be the planar embedding of \(G\). By the definition of a wrung Peg, the graph \(H\) contains a vertex \(u\) with three adjacent vertices \(x_1, x_2, x_3\) occurring in this counterclockwise order around \(u\) both in \(G^+\) and in \(\mathcal{H}\). Similarly, \(H\) has a vertex \(v\) distinct from \(u\) that has three neighbors \(y_1, y_2, y_3\) occurring in clockwise order around \(v\) in \(G^+\) but in counter-clockwise order in \(\mathcal{H}\). Moreover, \(G\) is a subdivision of a 3-connected graph \(G^*\). By Proposition 10, the graph \(H\) is connected, and all vertices of \(G\) also belong to \(H\).

The proof of the lemma is split into two parts. In the first part, we assume that \(H\) contains the edge \(uv\), whereas in the second part we assume that this edge does not belong to \(H\).

To prove the first part, assume without loss of generality that \(v = x_3\) and \(u = y_3\). Let \(G^*\) be the embedding of \(G^+\) inherited from \(G^+\). Clearly, both \(u\) and \(v\) are vertices of \(G^*\), but some of the vertices \(x_i\) and \(y_i\) may be subdivision vertices in \(G\) and therefore do not appear in \(G^*\). Let \(x'_1\) be the vertex of \(G^*\) defined as follows: if \(x_1\) belongs to \(G^*\), then put \(x'_1 = x_1\), otherwise choose \(x'_1\) in such a way that \(x_1\) is subdividing the edge \(ux'_1\) of \(G^*\). The vertices \(x'_2, y'_1\) and \(y'_2\) are defined analogously. Note that \(x'_1\) can only be equal to \(x_1\), \(y_1\) or \(y_2\); if \(x'_1\) were equal to \(u\), \(v\) or \(x_2\), then \(G^*\) would have a multiple edge or a loop. Analogous restrictions hold for \(x'_2, y'_1\) and \(y'_2\) as well. Also, \(x'_1\) is not equal to \(x'_2\), and \(y'_1\) is not equal to \(y'_2\), otherwise we again get a multiple edge in \(G^*\).

By construction, \(v, x'_1,\) and \(y'_2\) are neighbors of \(u\) in \(G^*\) and their order around \(u\) in \(G^*\) is the same as the order of \(v, x_1,\) and \(x_2\) in \(G^+\), and similarly for the \(y_i\)'s. We apply Lemma 12 to the graph \(G^*\), with \(x'_2\) playing the role of \(u_1\) and \(y'_1\) playing the role of \(v_1\).

If the first case of Lemma 12 occurs, we find in \(G^*\) a path \(P\) of the form \(v \rightarrow x'_1 \rightarrow x'_2 \rightarrow y'_1 \rightarrow y'_2 = u\). Let us distinguish several possibilities, depending on whether the vertices \(x'_2\) and \(y'_1\) are distinct or not. If the two vertices are distinct, this implies that all four vertices \(x'_1, x'_2, y'_1\) and \(y'_2\) are distinct, and therefore \(G^* = G\). The path \(P\) then shows that \((G, H, \mathcal{H})\) contains obstruction 7.

Suppose now that \(x'_2 = y'_1\). Then at least one of the two vertices \(x_2\) and \(y_1\) must be equal to \(x'_2\). If both are equal to \(x'_2\), we get obstruction 17, while if exactly one of them is equal to \(x'_2\), we obtain obstruction 10.

The second case of Lemma 12 is symmetric to the first one. Let us deal with the third case. We then have a vertex \(w \in G^*\) and three paths \(P_1: w \rightarrow x'_2 \rightarrow y'_2, P_2: w \rightarrow x'_1\) and \(P_3: w \rightarrow y'_1\). The three paths only share the vertex \(w\), which is distinct from \(x'_2\), and therefore also of \(y'_2\). This means, in particular, that neither \(x'_1\) nor \(y'_1\) may coincide with any of \(x'_2\) or \(y'_2\). Note also that the vertex \(w\) is equal to \(x'_1\) or to \(y'_1\), because the three paths \(P_1, P_2,\) and \(P_3\) avoid \(u\) and \(v\) by Lemma 12.

If \(x'_1\) and \(y'_1\) are distinct, then \(x_1 = x'_1\) and \(y_1 = y'_1\), and moreover \(x_1\) and \(y_1\) are connected by a \(G\)-edge. If \(x'_1 = y'_1\), then either \(x_1 = y_1\) or \(x_1\) is connected by a \(G\)-edge to \(y_1\). In any case, \(x_1\) and \(y_1\) are either identical or adjacent in \(G\). By a similar reasoning, \(x_2\) and \(y_2\) are either identical or adjacent in \(G\).

If \(x_1\) and \(y_1\) are identical, then \(x_2\) and \(y_2\) are different, because in a wrung Peg \((G, H, \mathcal{H})\) the graph \(H\) must have at least one vertex of degree one. Therefore, if \(x_1 = y_1\) then \(x_1y_2 \in E(G)\) and since we also have \(x_1x_2 \in E(G)\) because of path \(P_1\), we get obstruction 1. The case when \(x_2 = y_2\) is analogous. If \(x_1 \neq y_1\) and \(x_2 \neq y_2\), then \(x_1y_1 \in E(G)\) and \(x_2y_2 \in E(G)\), and by contracting the edge \(uv\) we get obstruction 2.

Since case 4 of Lemma 12 is symmetric to case 3, this completes the analysis of wrung Pegs with the property \(uv \in E(H)\).

In the second part of the proof, we consider minimal wrung Pegs where \(u\) and \(v\) are not adjacent in \(H\). Since \(H\) is connected, one of the \(x_i\) must coincide with one of the \(y_j\) and after renumbering them, we may assume that \(x_3 = y_3\). To obtain a more symmetric notation where this vertex is not notationally biased towards \(u\) or \(v\), we denote it by \(w\). We now make a case distinction, based on
which of the vertices \(x_1, x_2, y_1\) and \(y_2\) are distinct and which ones are subdivision vertices. Note that \(w\) may not be a subdivision vertex, otherwise we could contract its incident edges to the edge \(uv\) to obtain a smaller wrung Peg. The overall case analysis works as follows.

I) Some of the vertices \(x_1, x_2, y_1, y_2\) coincide.

As before, by symmetry we can assume that \(x_1\) coincides with one of \(y_1\) or \(y_2\).

a) If \(x_1 = y_1\), then \((G, H, \mathcal{H})\) contains obstruction 1 or 4.

b) If \(x_1 = y_2\), then \((G, H, \mathcal{H})\) contains obstruction 16.

II) The vertices \(x_1, x_2, y_1, y_2\) are all distinct and the wrung Peg has a vertex of degree 2. By symmetry, we may assume that \(x_1\) is a subdivision vertex. We consider several subcases.

a) If the vertex \(y_1\) is also subdividing and \(G\) contains the edge \(x_1y_1\), then \((G, H, \mathcal{H})\) contains obstruction 2.

b) If the vertex \(y_2\) is also subdividing and \(G\) contains the edge \(x_1y_2\), then \((G, H, \mathcal{H})\) contains obstruction 12.

By symmetry these two cases cover all situations in which two subdivision vertices are connected by an edge.

c) The graph \(G\) contains the edge \(x_1v\).

Here we distinguish several subcases, depending on the position of \(y_1\) and \(y_2\) relative to the cycle \(C = ux_1vuw\) in \(G^+\).

1) If \(y_1\) and \(y_2\) are separated from \(x_2\) by \(C\), we obtain obstruction 14.

2) If \(y_1\) is separated from \(x_2\) but \(y_2\) is not, we obtain obstruction 5.

3) If both \(y_1\) and \(y_2\) are on the same side of \(C\) as \(x_2\) we use Lemma 12 to get obstruction 5, 9 or 13.

d) The vertex \(x_1\) is subdividing and adjacent to \(y_1\), but \(y_1\) is not subdividing. Again, we consider subcases, based on other subdividing vertices and their adjacencies.

1) If \(x_2\) is subdividing and adjacent to \(y_2\), we find obstruction 2.

2) If \(y_2\) is subdividing and adjacent to \(x_2\), we find obstruction 2.

Note that this covers all the cases where any other vertex is subdividing. If \(x_2\) was subdividing and adjacent to \(v\), we could exchange \(x_2\) with \(x_1\) to obtain an instance of case IIb. Analogously for \(y_2\) subdividing and adjacent to \(u\). Further, \(x_2\) cannot be subdividing and adjacent to \(y_1\) as this would create parallel subdivided edges \(ux_1y_1\) and \(ux_2y_1\). Therefore this covers all subcases where another vertex except \(x_1\) is a subdivision vertex.

3) If no vertex besides \(x_1\) is subdividing, we find obstruction 5 or 15.

e) If \(x_1\) is subdividing and adjacent to \(y_2\), but \(y_2\) is not subdividing, we find obstruction 9 or 13.

This does not need specific subcases, as no other vertex can be subdividing. If \(x_2\) was subdividing, it would be adjacent to \(v\) and by mirroring the embedding and exchanging \(x_1\) with \(x_2\) and \(y_1\) with \(y_2\), we would arrive in case IIc. Analogously for \(y_2\), which would have to be adjacent to \(u\). Hence we can assume that \(x_1\) is the only subdivision vertex.

III) The vertices \(x_1, x_2, y_1, y_2\) are all distinct and all vertices in the wrung Peg have degree at least 3.
a) \( G \) contains the edge \( uv \).

We distinguish cases, based on the embedding of the edge \( uv \), by considering the relative positions of the cycle \( C = uvwu \) and the vertices \( x_1, x_2, y_1 \) and \( y_2 \) in \( G^+ \).

1) If all these vertices are on the same side of \( C \), we find obstruction 2, 7, 8 or 15.
2) If one of these vertices is on one side and the others are on the other side, we obtain obstruction 5, 9 or 13. In this case, we may assume without loss of generality that \( C \) separates \( x_1 \) from the other vertices.
3) If the cycle \( C \) separates \( x_1 \) and \( x_2 \) from the other vertices, we find obstruction 11.
4) If the cycle \( C \) separates \( x_1 \) and \( y_1 \) from the other vertices, we find obstruction 2.

All other cases are symmetric to one of these.

b) \( G \) does not contain the edge \( uv \).

Here, we use the fact that \( G \) is 3-connected, and thus contains three vertex-disjoint paths \( p_1, p_2 \) and \( p_3 \) from \( \{x_1, u, x_2\} \) to \( \{y_1, v, y_2\} \). We distinguish cases, based on which vertex is connected to which.

1) The path \( p_1 \) connects \( x_1 \) to \( y_1 \), \( p_2 \) connects \( u \) to \( v \); we obtain obstruction 2.
2) The path \( p_1 \) connects \( x_1 \) to \( y_1 \), \( p_2 \) connects \( u \) to \( y_2 \); we obtain obstruction 2, 5 or 9.
3) The path \( p_1 \) connects \( x_1 \) to \( v \), \( p_2 \) connects \( x_2 \) to \( y_1 \); we obtain obstruction 2, 9, 11, 12 or 13.

This covers all cases where the paths connect \( x_1 \) to \( y_1 \) or to \( v \). However, in the case where \( x_1 \) is connected to \( y_2 \), it is necessary that \( u \) connects to \( y_2 \) and \( x_2 \) to \( v \), which after renaming the vertices is symmetric to the last case.

Now we treat the above cases individually.

**Case I:** Some of the vertices \( x_1, x_2, y_1 \) and \( y_2 \) coincide. Since \( x_1 \neq x_2 \) and \( y_1 \neq y_2 \) and because \( H \) has at least one vertex of degree 1, at most two of these vertices can coincide. By symmetry, we can assume that \( x_1 \) coincides with one of the vertices \( y_1 \) or \( y_2 \).

**Case Ia:** We have \( x_1 = y_1 \). The rotation schemes at \( u \) and \( v \) imply that \( x_2 \) and \( y_2 \) are embedded on the same side of the cycle \( C \) formed by \( ux_1vwu \), and therefore are embedded on different sides of \( C \) in \( H \). Hence, if \( G \) contains the edge \( x_2y_2 \), we obtain obstruction 1.

Now assume that \( x_2y_2 \) is not in \( G \). Note that \( x_2 \) and \( y_2 \) cannot both be subdivision vertices, as both would be part of a subdivision of the edge \( uv \). Without loss of generality, we assume that \( x_2 \) is not a subdivision vertex. Assume first that \( G \) has the edge \( y_2u \). Then \( x_2 \) must connect to two vertices in the set \( \{x_1, v, w\} \). The embedding of the edge \( y_2u \) implies that \( x_2 \) cannot be adjacent to both \( w \) and \( x_1 \), and hence we either have edges \( x_2x_1 \) and \( x_2v \) or \( x_2w \) and \( x_2v \). In both cases, the fact that \( \{u, v\} \) is not a separator implies that the edge \( wx_1 \) is in \( G \). The cycle \( C \) together with the edges \( uy_2, y_2v, ux_2, x_2v, \) and \( wx_1 \) then forms obstruction 4.

We can therefore assume that \( y_2u \) is not an edge of \( G \), and hence \( y_2 \) is not a subdivision vertex. It follows that \( y_2 \) is adjacent to \( w \) and \( x_1 \). Planarity then implies that \( x_2 \) cannot be adjacent to \( v \), so \( x_2 \) is adjacent to \( w \) and to \( x_1 \). To prevent \( \{x_1, w\} \) from being a separator, \( G \) must contain the edge \( uv \). The cycle \( C \) together with the edges \( x_1y_2, y_2w, x_1x_2, x_2w \) and \( uv \) again forms obstruction 4. This closes the case \( x_1 = y_1 \).

**Case Ib:** We have \( x_1 = y_2 \). This means that \( H \) contains a four-cycle \( C \) formed by the edges \( uw, uv, vx_1 \) and \( x_1u \). We will show that \( C \) has two diagonally opposite vertices that are adjacent to \( x_2 \), while the other two of its vertices are adjacent to \( y_1 \). This will imply that \( (G, H, H) \) has obstruction 16.

We first show that every vertex of \( C \) must be adjacent to at least one of \( x_2 \) and \( y_1 \). This is clear for the vertices \( u \) and \( v \). To see this for \( x_1 \) and \( w \), note that neither of them may have degree
two in \(G\), as it could then be contracted, contradicting the minimality of \((G, H, \mathcal{H})\). Suppose now that \(wx_1\) is an edge of \(G\), and suppose without loss of generality that it is embedded on the same side of \(C\) as \(y_1\) in \(\mathcal{G}^+\). Then \(y_1\) is embedded inside the triangle \(vwx_1\), and therefore it is adjacent to both \(w\) and \(x_1\). We conclude that each vertex of \(C\) is adjacent to at least one of \(x_1\) and \(w\).

Consequently, if e.g., \(x_2\) is a subdivision vertex, then it is adjacent to two diagonally opposite vertices of \(C\), and therefore \(y_1\) must be adjacent to the other two vertices of \(C\) forming obstruction 16.

Suppose that neither \(x_2\) nor \(y_1\) is subdividing. If one of these two vertices is adjacent to all the vertices of \(C\), we easily obtain obstruction 16. If both \(x_2\) and \(y_1\) are adjacent to three vertices of \(C\), and the vertex of \(C\) not adjacent to \(x_2\) is diagonally opposite to the vertex of \(C\) not adjacent to \(y_1\), we get a contradiction with 3-connectivity. The only remaining possibility is that the vertex of \(C\) not adjacent to \(x_2\) is connected to the vertex of \(C\) not adjacent to \(y_1\) by an edge of \(C\). This also yields obstruction 16.

This concludes the treatment of the cases where \(x_1, x_2, y_1\) and \(y_2\) are not distinct.

**Case II:** The vertices \(x_1, x_2, y_1\) and \(y_2\) are distinct and one of the vertices is subdividing. Without loss of generality we assume that \(x_1\) is subdividing, and we consider subcases based on the adjacencies of \(x_1\).

**Case IIa:** The vertex \(y_1\) is also subdividing, and \(G\) contains the edge \(x_1y_1\). If \(x_2\) was a subdivision vertex, it could not be adjacent to \(v\), as the corresponding edge would be parallel to the edge subdivided by \(x_1\). Therefore it would have to be adjacent to \(y_2\), which would give obstruction 2, by contracting the path \(uvw\) to a single vertex. Hence, we can assume that \(x_2\) is not subdividing, and by a symmetric argument also that \(y_2\) is not subdividing. Hence, each of them needs degree 3 and since they are embedded on the same side of the cycle \(ux_1y_1vu\), they must be adjacent, which again results in obstruction 2; see Figure 13(a).

**Case IIb:** The vertex \(y_2\) is also subdividing and \(G\) contains the edge \(x_1y_2\). As in the previous case it can be seen that neither of \(x_2\) and \(y_1\) may be subdividing. Since they must be embedded on different sides of the cycle \(ux_1y_2vw\) in \(\mathcal{G}^+\), the graph \(G\) must contain the edges \(x_2w, x_2v, y_1u\) and \(y_1w\), in order for them to have degree 3. Altogether, this yields obstruction 12.

**Case IIc:** The vertex \(x_1\) is subdividing and adjacent to \(v\). Therefore, \(ux_1vwu\) is a four-cycle in \(G\). Let us call it \(C\) and orient it in the direction \(u \to x_1 \to v \to w \to u\). Note that the vertex \(x_2\) is to the left of \(C\). We distinguish several cases based on the position of \(y_1\) and \(y_2\) relative to \(C\) in \(\mathcal{G}^+\). Note that if \(y_1\) is left of \(C\) then so is \(y_2\), because \(y_1, y_2\) and \(w\) must be clockwise around \(v\).

**Case IIc1:** Both \(y_1\) and \(y_2\) are to the right of \(C\). This means that \(x_2\) cannot be subdividing and it must be adjacent to \(w\) and \(v\). We also easily check that \(G\) must contain all the edges \(y_1y_2, wy_1\) and \(uy_2\), otherwise we get contradiction with 3-connectivity. This creates obstruction 14.

**Case IIc2:** The vertex \(y_1\) is to the right of \(C\) but \(y_2\) is left. Then \(y_1\) must be adjacent to both \(u\) and \(w\). We now show that \(G\) has the edge \(x_2y_2\). If \(x_2\) is a subdividing vertex, it cannot be adjacent to \(v\) or \(w\), as that would form a multiple edge, so it must be adjacent to \(y_2\). Similarly, if \(y_2\) is subdividing, it must be adjacent to \(x_2\). If neither \(x_2\) nor \(y_2\) is subdividing, they must also be adjacent, otherwise they would both have to be adjacent to \(u\), \(v\) and \(w\), forming a \(K_{3,3}\) subgraph in \(G\). Thus, \(x_2y_2 \in E(G)\). Moreover, at least one of \(x_2\) and \(y_2\) is adjacent to \(w\), otherwise \(u\) and \(v\) form a 2-cut. We then obtain obstruction 5.

**Case IIc3:** Consider again the 3-connected graph \(G^*\), and define \(x_1', y_1'\) and \(y_2'\) as in the first part of the proof of this lemma. Note that the vertex \(w\) cannot be subdividing, and therefore belongs to \(G^*\). Apply Lemma 12 to \(G^*\), taking \(u_1 = x_2', v_1 = y_1'\) and \(v_2 = y_2'\).

The first case of Lemma 12 yields a path \(v \to x_2' \to w \to y_1' \to y_2' \to u\). This implies that \(x_2' = x_2, y_1' = y_1\) and \(y_2' = y_2\). This case however cannot occur, because the edges \(vw\) and \(vy_1\) together with the path \(w \to y_1\) form a cycle, and \(y_2\) is on the other side of this cycle than \(u\), making
it impossible to embed the edge $y_2u$ in a planar way. Thus this would contradict the assumption that $G^+$ is a planar embedding of $G$.

In the second case of Lemma 12 we get a path $u \to y'_1 \to y'_2 \to x'_2 \to w \to v$. This implies $y'_1 = y_1$. We may have $x'_2 = y'_2$ or not, and if $x'_2 = y'_2$, then we may have either $x_2 = x'_2$ or $y_2 = x'_2$, but in any case, we know that $x_2$ and $y_2$ must be adjacent in $G$ and at least one of them must be adjacent to $w$. The edges $x_1v$, $y_1u$, $x_2y_2$ and one of $wx_2$ or $wy_2$ together form obstruction 5.

In the third case of the lemma, we have a vertex $w' \in \{x'_2, y'_1\}$ and three paths $w' \to w \to y'_2$, $w' \to x'_2$ and $w' \to y'_1$ sharing only the vertex $w'$. This implies that $y'_2 = y_2$. Note also that the paths $w' \to w$ and $w' \to y'_1$ together with the (possibly subdivided) edges $y'_1v$ and $vw$ form a cycle that separates $y_2$ from $u$, showing that $y_2$ and $u$ cannot be adjacent. Consequently, $y_2$ must be adjacent to at least one of $x_2$ and $y_1$. Moreover, $x_2$ and $y_1$ must be adjacent to each other, as shown by the existence of a path $x'_2 \to y'_1$. Together with the edge $wy_2$, this forms obstruction 9 or 13.

In the fourth case of the lemma, we have again $w' \in \{x'_2, y'_1\}$, this time with paths $w' \to y'_2 \to w$, $w' \to y'_1$ and $w' \to x'_2$. This again shows that $y'_2 = y_2$, and that $w$ is adjacent to $y_2$, $x_2$ is adjacent to $y_1$, and $y_2$ is adjacent to $x_2$ or $y_1$. This forms again obstruction 9 or 13.

**Case IIId:** The vertex $x_1$ is subdividing and adjacent to $y_1$, but $y_1$ is not subdividing. We
distinguish subcases based on whether other vertices are subdividing.

**Case IIId1:** The vertex $x_2$ is subdividing and adjacent to $y_2$, then by contracting the path $uvw$ to a single vertex, we obtain obstruction 2.

**Case IIId2:** The vertex $y_2$ is subdividing and adjacent to $x_2$. As in the previous case, we obtain obstruction 2. As argued in the description of the case analysis, this covers all instances, where a vertex besides $x_1$ is subdividing.

**Case IIId3:** The vertex $x_1$ is subdividing and adjacent to $y_1$, and no other vertex is subdividing. Clearly, if $x_2y_2 \in G$, we obtain obstruction 2. We can therefore assume that this is not the case. Hence, $x_2$ must be adjacent to at least two vertices in the set $\{w, v, y_1\}$.

First, assume that $x_2w$ and $x_2v$ are in $G$. The edge $x_2v$ admits two distinct embeddings, however in one of them the cycle $x_2wvx_2$ would enclose only the vertex $y_2$, which would imply the existence of the excluded edge $x_2y_2$. We can therefore assume that all vertices that do not belong to the cycle are on the same side of it. Since $y_2$ has degree 3 in $G$, this implies the existence of the edges $y_1y_2$ and $y_2u$. Since $\{u, v\}$ still would form a separating pair, the edge $wy_1$ must also be present. Omitting the edge $x_2w$ yields obstruction 5.

Next, assume that $x_2w$ and $x_2y_1$ are in $G$. Since $y_2$ has degree 3, and $y_1y_2$ is excluded, $G$ must contain the edges $y_1y_2$ and $y_2w$. Until now $\{w, y_1\}$ would still form a separating pair. Since $x_2y_2$ is excluded $G$ contains the edge $uv$. Altogether, this is obstruction 15.

Finally, assume that $x_2v$ and $x_2y_1$ are in $G$. Again, only one of the embeddings of $x_2v$ does not force the edge $x_2y_1$ to be present. Since $y_2$ has degree 3, it must be adjacent to $u$ and $w$. As $\{u, v\}$ would still be a separating pair, we also obtain the edge $wy_1$. Omitting the edges $y_2w$ and $x_2y_1$ yields obstruction 5. This finishes the case where $x_1$ is subdividing and adjacent to $y_1$.

**Case IIe:** The vertex $x_1$ is subdividing and adjacent to $y_2$, which is not subdividing. No other vertex may be subdividing, as this would be symmetric to case IIc. Clearly, $y_1$ must be adjacent to two vertices in the set $\{u, w, y_2\}$. We consider several cases.

Assume that $y_1$ is adjacent to $y_2$ and $w$. If $G$ contains $x_2v$, this forms obstruction 9. Therefore, assume that $x_2v$ is not in $G$. It follows that $x_2$ is adjacent to $w$ and $y_2$. Since $\{w, y_2\}$ still forms a separator, we also get edge $y_1w$. By omitting $y_1y_2$ and $y_1w$, we obtain obstruction 13.

Next, assume that $y_1$ is adjacent to $y_2$ and $u$ but not $w$. Then $x_2$ must be adjacent to $y_2$, otherwise $uv$ would form a separator or $x_2$ would have degree 2. Also, $w$ must be adjacent to $x_2$, otherwise $x_2$ or $v$ would have degree 2. This yields obstruction 13.

Finally, assume that $y_1$ is adjacent to $u$ and to $w$, and not to $y_2$ (otherwise one of the previous cases would apply). Clearly, $x_2$ must be adjacent to two of the three vertices $w, v$, and $y_2$. It is not possible that $x_2$ is only adjacent to $w$ and $v$, since $\{u, v\}$ would still form a separating pair. Hence $x_2y_2 \in E(G)$. Also $x_2$ must be adjacent to $w$, otherwise $uv$ would be a separator or $x_2$ would have degree 2. This gives obstruction 13.

This closes the case where $x_1$ is subdividing and adjacent to $y_2$, and thus all cases where $G$ has a subdividing vertex.

**Case III:** The graph $G$ does not have any subdividing vertices, and thus is 3-connected. We distinguish two subcases, based on whether $G$ contains the edge $uv$.

**Case IIIa:** The graph $G$ is 3-connected and contains the edge $uv$. The edge $uv$ can be embedded in a variety of different ways in $G^+$. We distinguish cases, based on this embedding, in particular, the relative position of the cycle $C = uvwvu$ and the vertices $x_1, x_2, y_1$ and $y_2$.

**Case IIIa1:** First, assume that all these vertices are embedded on the same side of $C$. We apply Lemma 12 on the vertices $u$ and $v$ with $u_i = x_i$ and $v_i = y_i$ for $i = 1, 2$. Suppose that case 1 of the lemma applies. Then we obtain a simple path $v \rightarrow x_1 \rightarrow x_2 \rightarrow y_1 \rightarrow y_2 \rightarrow u$. If $w$ is not contained in any of the subpaths, we can contract $vw$ and obtain obstruction 7. Further, by planarity, $w$ cannot subdivide any of the subpaths $x_1 \rightarrow x_2, y_1 \rightarrow y_2$ or $x_2 \rightarrow y_2$. Hence, it
must subdivide \( x_1 \to v \) or \( y_2 \to u \), which is completely symmetric, and we assume without loss of generality that \( w \) subvides the path \( x_1 \to v \) and all other subpaths consist of a single edge, each. Again, contracting \( wv \) yields obstruction 7.

Case 2 of the lemma is completely symmetric, we therefore continue with case 3. We obtain a vertex \( w' \neq x_2, y_2, u, v \) and paths \( w' \to x_2 \to y_2, w' \to x_1, w' \to y_1 \). Further, \( w \) may subdivide one of these paths. First of all, \( w' \) must coincide with either of \( x_1, y_1 \) or \( w \).

If \( w' = x_1 \), we obtain obstruction 2, unless \( w \) subvides the subpath \( x_1 \to y_1 \) (or symmetrically \( x_2 \to y_2 \)). We therefore assume that \( w \) subvides \( x_1y_1 \), and thus all other paths consist of single edges; see Fig. 13(b). Since \( y_1 \) must have degree at least 3, at least one of the three edges \( y_1y_2, y_1x_2 \) or \( y_1x_1 \) (note that \( y_1u \) is not possible in a planar way) must be present, resulting in obstructions 8, 15 and 2, respectively.

If \( w' = y_1 \), we again obtain obstruction 2, unless \( w \) subvides either \( x_1 \to y_1 \) or \( x_2 \to y_2 \). Again these cases are symmetric, and we assume \( w \) subvides \( x_1 \to y_1 \). Since \( x_1 \) has degree 3, it is either adjacent to \( y_1 \) or to \( x_2 \). This leads to obstructions 2 and 15, respectively.

If \( w' = w \), the situation is illustrated in Fig. 13(c) and \( x_1x_2 \) must be in \( G \) because of planarity and since \( x_1 \) has degree at least 3. Further, \( y_1 \) has degree at least 3 and therefore \( G \) either contains \( y_1y_2 \) or \( y_1x_2 \), which yields obstructions 8 and 15, respectively.

Since case 4 of the Lemma 12 is completely symmetric, this closes the case where the cycle \( C = uwvu \) bounds an empty triangle.

Case IIIa2: The graph \( G \) is 3-connected, contains the edge \( uv \) and the cycle \( C = uwvu \) is embedded so that it separates \( x_1 \) from the vertices \( x_2, y_1 \) and \( y_2 \). In this case, \( x_1 \) must be adjacent to \( v \) and \( w \) in \( G \). We apply Lemma 12 to vertices \( u \) and \( v \) with \( u_1 = x_2, u_2 = w, v_1 = y_1 \) and \( v_2 = y_2 \).

In case 1 of the lemma, we obtain a path \( u \to x_2 \to u \to y_1 \to y_2 \to u \); see Fig. 13(d) where the edge \( uv \) is drawn as a dotted curve. This path cannot be embedded in a planar way into \( G \) (without changing the embedding of \( uv \), which is assumed to be fixed), so this case does not occur.

In case 2 of the lemma, we obtain edges \( uy_1, y_1y_2, y_2x_2, x_2w \) and \( uv \). Together with the edge \( x_1v \), this forms an instance of obstruction 5.

In case 3 of the lemma, we obtain a vertex \( w' \neq w \) paths \( w' \to w \to y_2, w' \to x_2 \) and \( w' \to y_1 \). The vertex \( x_1 \) cannot subdivide any of these paths and also \( w' = x_1 \) is not possible by planarity.

Suppose that \( w' = x_2 \), we thus obtain edges \( x_2w, y_2w \) and \( x_2y_1 \). We already know that \( x_1v \) is an edge of \( G \). Figure 13(e) illustrates the situation, where \( x_1v \) is drawn dotted, the remaining edges are drawn as dashed curves. Further, \( y_2 \) must have another adjacency, which can by planarity either be \( y_1 \) or \( x_2 \). In the former case, we find obstruction 9, in the latter obstruction 13.

Suppose that \( w' = y_1 \), we then have edges \( x_2y_1, y_2w \) and \( y_1w \); see Fig. 13(f). Planarity and the degree of \( y_2 \) imply that \( y_1y_2 \) is in \( G \), and as above \( x_1v \) is in \( G \). Together, this yields obstruction 9.

In case 4 of the lemma, we obtain a vertex \( w' \neq y_2 \) and paths \( w' \to y_2 \to w, w' \to y_1 \) and \( w' \to x_2 \). Again, by planarity, \( x_1 \) may neither coincide with \( w' \) nor subdivide any of the paths.

Suppose that \( w' = x_2 \), and we thus have edges \( x_2y_2, y_2w \) and \( x_2y_1 \). As above we find the edge \( x_1v \). This gives obstruction 13.

Suppose that \( w' = y_1 \). This yields edges \( x_2y_1, y_2w \) and \( y_1y_2 \). As above we find edge \( x_1v \), and thus obstruction 9. This closes the case where exactly one of the vertices is enclosed by the cycle \( uwvu \).

Case IIIa3: The edge \( uv \) is embedded so that the cycle \( C = uwvu \) separates \( x_1 \) and \( x_2 \) from \( y_1 \) and \( y_2 \). We may assume that \( x_1 \) and \( x_2 \) are to the right of the directed cycle \( u \to v \to w \to u \), while \( y_1 \) and \( y_2 \) are to its left. Note that in this case, the vertices \( x_1 \) and \( x_2 \) must be adjacent, because otherwise both of them have to be adjacent to \( u, v \) and \( w \), contradicting planarity. Figure 13(g) illustrates the current situation. By 3-connectedness, both \( v \) and \( w \) must be adjacent to at least one of \( x_1 \) and \( x_2 \), and each \( x_i \) must be adjacent to at least one of \( v \) and \( w \). Planarity

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implies that $x_1w$ and $x_2v$ must both be edges of $G$. An analogous argument for $y_1$ and $y_2$ implies that $uy_1, y_1y_2$ and $y_2w$ are all edges of $G$. This forms obstruction 11.

**Case IIIa4:** The edge $uv$ is embedded so that the cycle $C = uvwu$ separates $x_1$ and $y_1$ from $x_2$ and $y_2$, see Fig. 13(h). Clearly, $x_1$ and $y_1$ both need degree 3. However, $C$ has only vertices $u, w$ and $v$, and $x_1$ and $y_1$ cannot both be connected to all vertices of $C$ in a planar way (otherwise we could find a planar drawing of $K_{3,3}$). Hence, $G$ must contain the edge $x_1y_1$, and by a symmetric argument also $x_2y_2$, which results in obstruction 2.

**Case IIIb:** The graph $G$ is 3-connected and does not contain the edge $uv$. Since $G$ is 3-connected it contains three vertex-disjoint paths $p_1, p_2$ and $p_3$ from $\{x_1, u, x_2\}$ to $\{y_1, v, y_2\}$. We distinguish cases, based on which endpoints are connected by the paths.

**Case IIIb1:** The path $p_1$ connects $x_1$ to $y_1$ and $p_2$ connects $u$ to $v$. Clearly, $p_3$ must then connect $x_2$ to $y_2$. Since $G$ does not contain the edge $uv$, $p_2$ must contain the vertex $w$, which implies that $p_1$ and $p_3$ consist of a single edge, each. This yields obstruction 2.

**Case IIIb2:** The path $p_1$ connects $x_1$ to $y_1$ and $p_2$ connects $u$ to $y_2$. Clearly, $p_3$ must then connect $x_2$ to $v$. We distinguish cases, based on whether $w$ is contained in one of these paths.

First, suppose that none of these paths contains $w$, that is, each of them consists of a single edge. The edges $x_2v$ and $y_2u$ admit two different embeddings that are completely symmetric. We therefore assume that $uvwuy_2$ bounds a face in the graph consisting of $H$ and the paths $p_i, i = 1, 2, 3$ with the embedding inherited from $G^+$. Then the cycle $x_2vuux_2$ separates $y_2$ from $x_1$ and $y_1$. Since $y_2$ needs degree at least 3, we either have $x_2y_2$ or $y_2w$ in $G$. The former would yield obstruction 2, thus we assume the latter. However, still $\{u, v\}$ would form a separating pair, thus implying that $w$ needs to be adjacent to either $x_1$ or to $y_1$. In both cases we obtain obstruction 5.

Hence, we can assume that $w$ is contained in one of the paths $p_1, p_2$ and $p_3$. First, assume that $w$ is contained in $p_1$. Again, the embedding choices offered by the edges $x_2v$ and $y_2u$ are completely symmetric, and as above we assume that the cycle $x_2vuux_2$ separates $y_2$ from $x_1$ and $y_1$. If $x_1y_1$ was in $G$, we could replace the path $p_1$ by this edge and the previous case would apply; we therefore assume that this is not the case. If $x_1v$ was in $G$, then the fact that $y_1$ needs another adjacency would again imply that $x_1y_1$ is in $G$. Since $x_1$ needs one more adjacency, the only option is the edge $x_1x_2$. Similarly, for $y_1$, the only option is the edge $y_1x_2$. The graph then contains obstruction 9.

Second, assume that $w$ is contained in $p_2$. We then have edges $x_1y_1, y_2w$ and $x_2v$, which have a unique embedding, in which the cycle $x_2vuux_2$ separates $y_2$ from $x_1$ and $y_1$. Since $y_2$ needs degree at least 3, it must be adjacent to either $x_2$ or to $u$. In the former case, we again get obstruction 2. In the latter case, we can replace the path $p_2$ with the edge $uy_2$ and we are in the case that $w$ is not contained in any of the three paths.

Finally, the case that $w$ is contained in $p_3$ is completely symmetric to the previous case. This closes the case where $p_1$ connects $x_1$ to $y_1$ and $p_2$ connects $u$ to $y_2$.

**Case IIIb3:** The path $p_1$ connects $x_1$ to $v$, $p_2$ connects $u$ to $y_2$, and thus $p_3$ connects $x_2$ to $y_1$. Again, we consider cases based on whether $w$ is contained in one of these paths. First, suppose that $w$ is not contained in any of these paths. Then $G$ contains the edges $x_2y_1, x_1v$ and $y_2u$, whose embedding in $G$ is uniquely determined. Since $w$ has degree at least 3 it must be connected to $x_1$ or $y_2$ in $G$. Both cases are completely symmetric, and we assume $wy_2$ is in $G$; see Fig. 13(i). So far, the set $\{u, v\}$ would still form a separating pair. Hence $G$ contains at least one of the edges $y_1y_2, y_2x_2$ and $wx_1$, resulting in obstruction 9, 13 and 12, respectively.

We can therefore assume that $w$ subdivides one of the paths. Suppose that $w$ subdivides $p_1$, that is, we have edges $x_1w, uy_2$ and $x_2y_1$; see Fig. 13(j). Clearly, $x_1$ must be adjacent to one of $x_2, v$ and $y_1$. However, $x_1x_2$ produces obstruction 9, $x_1y_1$ produces obstruction 13, and if $x_1v$ is in $G$, we replace the path $p_1$ by $x_1v$ and are in the case where $w$ is not contained in any of the three paths.

The case that $w$ is contained in $p_2$ is completely symmetric, we therefore assume it is contained
in \( p_3 \), and we thus have edges \( x_1v, y_2u, x_2w \) and \( y_1w \). There are two possible planar embeddings. One, in which \( x_1v \) comes after \( vu \) and before \( vy_2 \) in counter-clockwise direction around \( v \), and one in which \( x_1v \) comes after \( vy_2 \) and before \( vy_1 \) in counter-clockwise direction.

We start with the first case; see Fig. 13(k). Suppose that \( G \) contains the edge \( x_1y_2 \). The vertex \( x_2 \) needs another adjacency, which can either be \( x_1 \) or \( v \). Analogously, \( y_1 \) needs to be adjacent to either \( u \) or \( y_2 \). The combination \( x_1x_2 \) and \( y_1y_2 \) gives obstruction 11, any of the combination \( x_2v, y_1y_2 \) and \( x_1x_2, y_1u \) gives obstruction 9, and the combination \( x_2v \) and \( y_2u \) gives obstruction 12.

Now assume that \( G \) does not contain the edge \( x_1y_2 \). Then \( x_1 \) needs to be adjacent to \( x_2 \); its only options are \( x_2 \) and \( v \), but in the latter case \( x_2 \) still needs an edge and the only possibility is \( x_1 \). Analogously, \( y_1y_2 \) must be in \( G \). Together this forms obstruction 11.

Now suppose that the second embedding applies, and \( x_1v \) comes after \( vy_2 \) and before \( vy_1 \) in counter-clockwise direction around \( v \). If \( G \) contains \( x_1y_1 \), then it cannot contain \( x_2y_2 \) as well, otherwise we would get obstruction 2. Hence, \( x_2 \) must be adjacent to \( v \). Since \( y_2 \) is not adjacent to \( x_2 \), it must be adjacent to \( x_1 \), and we thus obtain obstruction 13. We can hence assume that \( G \) does not contain \( x_1y_1 \), and symmetrically also not \( x_2y_2 \). We hence get edges \( x_2v \) and \( y_1u \). But then still \( \{u, v\} \) forms a separator, which shows that either \( x_1y_1 \) or \( x_2y_2 \) is in \( G \); a contradiction. This finishes the last case, and thus concludes the proof.

3.2.2 \( \mathcal{R} \) has an edge-compatible embedding

Assume now that the embedding \( \mathcal{R}^+ \) of the skeleton \( \mathcal{R} \) is edge-compatible but not cycle-compatible. We first give a sketch of our general proof strategy. Our analysis of this situation strongly relies on the concept of \( C \)-bridge, which has been previously used by Juvan and Mohar in the study of embedding extensions on surfaces of higher genus \( [8] \), and which is also employed (under the name fragment) by Demoucron, Malgrange and Pertuiset in their planarity algorithm \( [2] \).

Let \( F \) be a graph and \( C \) a cycle of \( F \). A \( C \)-bridge is either a chord of \( C \), (i.e., an edge not belonging to \( C \) whose vertices are on \( C \)) or a connected component of \( F - C \), together with all vertices and edges that connect it to \( C \). A vertex of \( C \) that is incident to an edge of a \( C \)-bridge is called an attachment of \( C \). Let \( \text{att}(X) \) denote the set of attachments of \( X \). A bridge that consists of a single edge is trivial.

In our argument, we focus on cycles in \( \mathcal{R} \) that are projections of cycles in \( H \). Notice that in this case, any nontrivial bridge in \( \mathcal{R} \) has at least three attachments, because \( \mathcal{R} \) is 3-connected. If \( \mathcal{R}^+ \) violates cycle-compatibility, it means that \( H \) must contain a cycle \( C' \) that projects to a cycle \( C \) of \( \mathcal{R} \), and \( \mathcal{R}^+ \) has a \( C \)-bridge that is embedded on the ‘wrong’ side of \( C \). We concentrate on the substructures that enforce such ‘wrong’ position for a given \( C \)-bridge, and use them to locate planarity obstructions.

Let us describe the argument in more detail. Suppose again that \( C' \) is a cycle of \( H \) that projects to a cycle \( C \) of \( \mathcal{R} \). Let \( x \) be a vertex of \( H \) that does not belong to any \( \mathcal{R} \)-edge belonging to \( C \). We say that \( x \) is happy with \( C' \), if its embedding in \( \mathcal{R}^+ \) does not violate cycle-compatibility with respect to the cycle \( C' \), i.e., \( x \) is to the left of \( C' \) in \( H \) if and only if \( x \) is to the left of \( C \) in \( \mathcal{R}^+ \). Otherwise we say that \( x \) is unhappy with \( C' \). We say that a \( C \)-bridge \( B \) of \( \mathcal{R} \) is happy with \( C' \) if there is a vertex \( x \) happy with \( C' \) that projects into \( B \), and similarly for unhappy bridges. A \( C \)-bridge that is neither happy nor unhappy is indifferent.

In our analysis of cycle-incompatible skeletons, we establish the following facts.

- With \( C \) and \( C' \) as above, if a single \( C \)-bridge is both happy and unhappy with \( C' \), then \( (G, H, \mathcal{H}) \) contains obstruction 1 or 4 (Lemma 22).
- Let us say that the cycle \( C' \) is happy if at least one \( C \)-bridge is happy with \( C' \), and it is unhappy if at least one \( C \)-bridge is unhappy with \( C' \). If \( C' \) is both happy and unhappy,
then \((G, H, \mathcal{H})\) contains obstruction 4, obstruction 16, or an alternating-chain obstruction (Lemma 23).

- Assume that the situation described above does not arise. Assume further that \(C'\) is an unhappy cycle of \(\mathcal{H}\). Then any edge of \(H\) incident to a vertex of \(C'\) must project into an \(R\)-edge belonging to \(C\), unless \((G, H, \mathcal{H})\) contains obstruction 3 or one of the obstructions from the previous item. Note that this implies, in particular, that the vertices of \(C\) impose no edge-compatibility constraints (Lemma 25).

- If \(C'_1\) and \(C'_2\) are two facial cycles of \(\mathcal{H}\) whose projection is the same cycle \(C\) of \(R\), then any \(C\)-bridge is happy with \(C'_1\) if and only if it is happy with \(C'_2\), unless the graph \(G\) is non-planar, or the PEG \((G, H, \mathcal{H})\) contains obstruction 1 (Lemma 26).

- If \(H\) contains a happy facial cycle as well as an unhappy one, we obtain obstruction 18 (Lemma 27).

- If \(H\) contains an unhappy facial cycle, and if at least one vertex of \(R\) imposes any non-trivial edge-compatibility constraints, then \((G, H, \mathcal{H})\) contains one of the obstructions 19–22 (Lemma 28).

Note that these facts guarantee that if \((G, H, \mathcal{H})\) is obstruction-free then \(R\) has a compatible embedding. To see this, assume that \(R^+\) is an edge-compatible but not cycle-compatible embedding of \(R\). This means that at least one facial cycle of \(\mathcal{H}\) is unhappy. This in turn implies that no cycle may be happy, and no vertex of \(R\) may impose any edge-compatibility restrictions. Consequently, the embedding \(R^−\) is compatible. In order to prove the above claims we need some technical machinery, in particular the concept of conflict graph of \(C\)-bridges and its properties.

**Conflict graph of a cycle and minimality of alternating chains** For a cycle \(C\) and two distinct vertices \(x\) and \(y\) of \(C\), an arc of \(C\) with endvertices \(x\) and \(y\) is a path in \(C\) connecting \(x\) to \(y\). Any two distinct vertices of a cycle determine two arcs. Let \(u, v, x, y\) be four distinct vertices of a cycle \(C\). We say that the pair \(\{x, y\}\) alternates with \(\{u, v\}\) if each arc determined by \(x\) and \(y\) contains exactly one of the two vertices \(\{u, v\}\). If \(U\) and \(X\) are sets of vertices of a cycle \(C\), we say that \(X\) alternates with \(U\) if there are two pairs of vertices \(\{u, v\}\) \(\subseteq U\) and \(\{x, y\}\) \(\subseteq X\) that alternate with each other.

Let now \(F\) be a graph containing a cycle \(C\). Intuitively, a bridge represents a subgraph, whose internal vertices and edges must all be embedded on the same side of \(C\) in any embedding of \(F\). Thus, a \(C\)-bridge may be embedded in two possible positions relative to \(C\). Moreover, if two bridges \(B_1\) and \(B_2\) have three common attachments, or if the attachments of \(B_1\) alternate with the attachments of \(B_2\), then in any planar embedding, \(B_1\) and \(B_2\) must appear on different sides of \(C\). This motivates the definition of two types of conflicts between bridges. We say that two \(C\)-bridges \(X\) and \(Y\) of \(F\) have a three-vertex conflict if they share at least three common attachments, and they have a four-vertex conflict if \(\text{att}(X)\) alternates with \(\text{att}(Y)\). Two \(C\)-bridges have a conflict if they have a three-vertex conflict or a four-vertex conflict. This gives rise to a conflict graph of \(F\) with respect to \(C\). For a cycle \(C\), define the conflict graph \(K_C\) to be the graph whose vertices are the \(C\)-bridges, and two vertices are connected by an edge of \(K_C\) if and only if the corresponding bridges conflict. Define the reduced conflict graph \(K_C^-\) to be the graph whose vertices are bridges of \(C\), and two bridges are connected by an edge if they have a four-vertex conflict.

As a preparation, we first derive some basic properties of conflict graphs.

**Lemma 15.** If \(F\) is a planar graph, then for any cycle \(C\) of \(F\) the conflict graph \(K_C\) is bipartite (and hence \(K_C^-\) is bipartite as well).
Proof. In any embedding of $F$, each $C$-bridge must be completely embedded on a single side of $C$. Two conflicting bridges cannot be embedded on the same side of $C$. \hfill \Box

Consider now the situation when $C$ is a cycle of length at least 4 in a 3-connected graph $F$. The goal is to show that in this case also the reduced conflict graph $K_C'$ is connected. To prove this we need some auxiliary lemmas. The first one states that if the attachments of a set of bridges alternate with two given vertices $x$ and $y$ of $C$, then the set must contain a $C$-bridge whose attachments alternate with $x$ and $y$, provided that the set of bridges is connected in the reduced conflict graph $K_C'$.

**Lemma 16.** Let $F$ be a graph and let $C$ be a cycle in $F$. Let $K$ be a connected subgraph of the reduced conflict graph $K_C'$ and let $\text{att}(K)$ be the set of all attachment vertices of the $C$-bridges in $K$, that is, $\text{att}(K) = \bigcup_{X \in K} \text{att}(X)$. If $\{x, y\}$ is a pair of vertices of $C$ that alternates with $\text{att}(K)$, then there is a bridge $X \in K$ such that the pair $\{x, y\}$ alternates with $\text{att}(X)$.

**Proof.** Let $\alpha$ and $\beta$ be the two arcs of $C$ with endvertices $x$ and $y$. Let $K_\alpha$ be the set of $C$-bridges from $K$ whose all attachments belong to $\alpha$, and let $K_\beta$ be the set of bridges from $K$ with all their attachments in $\beta$. Note that both $K_\alpha$ and $K_\beta$ are proper subsets of $K$, because $\{x, y\}$ alternates with $\text{att}(K)$.

Since no bridge in $K_\alpha$ conflicts with any bridge in $K_\beta$, and since $K$ is a connected subgraph in the reduced conflict graph, there must exist a bridge $X \in K$ that belongs to $K \setminus (K_\alpha \cup K_\beta)$. Clearly, $X$ has at least one attachment in the interior of $\alpha$ as well as at least one attachment in the interior of $\beta$. Thus, $\text{att}(X)$ alternates with $\{x, y\}$. \hfill \Box

Next, we show that in a 3-connected graph, unless $C$ is a triangle, its reduced conflict graph $K_C'$ is connected.

**Lemma 17.** Let $C$ be a cycle of length at least 4 in a 3-connected graph $F$. Then the reduced conflict graph $K_C'$ is connected (and hence $K_C$ is connected as well).

**Proof.** We first show that for a cycle $C$ of length at least 4 and a set of $C$-bridges $K$ that form a connected component in $K_C'$, every vertex of $C$ is an attachment of at least one bridge in $K$.

**Claim 4.** Let $C$ be a cycle of length at least four in a 3-connected graph $F$. Let $K$ be a connected component of the graph $K_C'$, and let $\text{att}(K)$ be the set $\bigcup_{X \in K} \text{att}(X)$. Then each vertex of $C$ belongs to $\text{att}(K)$.

Suppose that some vertices of $C$ do not belong to $\text{att}(K)$. Then there is an arc $\alpha$ of $C$ of length at least 2, whose endvertices belong to $\text{att}(K)$, but none of its internal vertices belongs to $\text{att}(K)$. Let $x$ and $y$ be the endvertices of $\alpha$. Let $\beta$ be the other arc determined by $x$ and $y$. Observe that, since $|\text{att}(K)| \geq 3$ in any 3-connected graph, $\beta$ also has length at least 2.

Since $F$ is 3-connected, $F - \{x, y\}$ is connected, and in particular, there is a $C$-bridge $Y$ that has at least one attachment $u$ in the interior of the arc $\alpha$ and at least one attachment $v$ in the interior of $\beta$. Clearly $Y \notin K$, since $Y$ has an attachment in the interior of $\alpha$.

Since the pair $\{u, v\}$ alternates with $\{x, y\} \subseteq \text{att}(K)$, Lemma 16 shows that there is a bridge $X \in K$ whose attachments alternate with $\{u, v\}$. Then $X$ and $Y$ have a four-vertex conflict, which is impossible because $K$ is a connected component of $K_C'$ not containing $Y$. This finishes the proof of the claim.

We are now ready to prove the lemma. Let $K$ and $K'$ be two distinct connected components of $K_C'$. Choose a bridge $X \in K$. Let $u$ and $v$ be any two attachments of $X$ that are not connected by an edge of $C$. By Claim 4 each vertex of $C$ is in $\text{att}(K')$, so $\text{att}(K')$ alternates with $\{u, v\}$, and
Lemma 18. Let $C$ be a cycle of length at least 4 in a graph $F$ and let $P$ be an induced path with $k \geq 2$ vertices in the graph $K^-_C$. Let $X_1, X_2, \ldots, X_k$ be the vertices of $P$, with $X_i$ adjacent to $X_{i+1}$ for each $i = 1, \ldots, k - 1$. Then for each $i \in \{1, \ldots, k\}$ we may choose a pair of vertices \( \{x_i, y_i\} \subseteq \text{att}(X_i) \), such that for each $i = 1, \ldots, k - 1$ the pair \( \{x_i, y_i\} \) alternates with the pair \( \{x_{i+1}, y_{i+1}\} \).

Proof. For each $j \leq k$, select a set $S_j \subseteq \text{att}(X_j)$ in such a way that for each $i < k$ the set $S_i$ alternates with $S_{i+1}$. Such a selection is possible, e.g., by taking $S_j = \text{att}(X_j)$. Assume now that we have selected \( \{S_j \mid j = 1, \ldots, k\} \) so that their total size $\sum_{j \leq k} |S_j|$ is as small as possible. We claim that each set $S_j$ consists of a pair of vertices \( \{x_j, y_j\} \).

Assume for contradiction that this is not the case. Since obviously each $S_j$ has at least two vertices, assume that for some $j$ we have $|S_j| \geq 3$. Clearly, this is only possible for $1 < j < k$. Select a pair of vertices \( \{x_{j-1}, y_{j-1}\} \subseteq S_{j-1} \) and a pair of vertices \( \{x_{j+1}, y_{j+1}\} \subseteq S_{j+1} \) such that both these pairs alternate with $S_j$. The sets $S_{j-1}$ and $S_{j+1}$ do not alternate because $P$ was an induced path. Therefore, there is an arc $\alpha$ of $C$ with endvertices \( \{x_{j-1}, y_{j-1}\} \) that has no vertex from $S_{j+1}$ in its interior, and similarly there is an arc $\beta$ with endvertices \( \{x_{j+1}, y_{j+1}\} \) and no vertex of $S_{j-1}$ in its interior.

Since both \( \{x_{j-1}, y_{j-1}\} \) and \( \{x_{j+1}, y_{j+1}\} \) alternate with $S_j$, there must be a vertex $x_j \in S_j$ that belongs to the interior of $\alpha$, and a vertex $y_j \in S_j$ belonging to the interior of $\beta$. The pair \( \{x_j, y_j\} \) alternates with both $S_{j-1}$ and $S_{j+1}$, contradicting the minimality of our choice of $S_j$.

Our next goal is to link the conflict graph with the elements of $\text{Ach}_k$. Recall that an element of $\text{Ach}_k$ consists of an $H$-cycle of length $k+1$ and $k$ edge-disjoint paths $P_1, \ldots, P_k$ such that consecutive pairs have alternating endpoints on $C$. Moreover, $P_2, \ldots, P_{k-1}$ are single edges, while $P_1$ and $P_k$ are subdivided by a single isolated $H$-vertex. Note that for all elements $(G_k, H_k, \mathcal{H}_k)$ of $\text{Ach}_k$, the conflict graph of the unique $H_k$-cycle forms a path of length $k$. To establish a link, we consider pairs of a graph and a cycle such that the conflict graph forms a path. Let $F$ be a graph, and let $C$ be a cycle in $F$. We say that the pair $(F, C)$ forms a conflict path, if each $C$-bridge of $F$ has exactly two attachments and the conflict graph $K^-_C$ is a path. (Note that if each $C$-bridge has two attachments, then the conflict graph is equal to the reduced conflict graph.)

Note that if $(G_k, H_k, \mathcal{H}_k)$ is an element of $\text{Ach}_k$ and $C$ the unique cycle of $H_k$, then $(G_k, C)$ forms a conflict path. However, not every conflict path arises this way. Suppose that $(F, C)$ forms a conflict path. Let $e = uv$ be an edge of $C$. The edge $e$ is called shrinkable if no $C$-bridge attached to $u$ conflicts with any $C$-bridge attached to $v$. Note that a shrinkable edge may be contracted without modifying the conflict graph.

Before we can show that the elements of $\text{Ach}_k$ are minimal non-planar pegs, we first need a more technical lemma about conflict paths.

Lemma 19. Assume that $(F, C)$ forms a conflict path. Then each vertex of $C$ is an attachment for at most two $C$-bridges.
Proof. Suppose that \((F,C)\) forms a conflict path, and a vertex \(v \in C\) is an attachment of three distinct bridges \(X, Y\) and \(Z\). These three bridges do not alternate, so there must be at least five bridges to form a path in \(K_C\). Let \(x, y\) and \(z\) be the attachments of \(X, Y\) and \(Z\) different from \(v\). The three vertices \(x, y\) and \(z\) must be all distinct, because a pair of bridges with the same attachments would share the same neighbors in the conflict graph, which is impossible if the conflict graph is a path with at least five vertices.

Choose an orientation of \(C\) and assume that the four attachments appear in the order \((v, x, y, z)\) with respect to this orientation. Let \(\alpha_{vx}, \alpha_{xy}, \alpha_{yz},\) and \(\alpha_{zx}\) be the four internally disjoint arcs of \(C\) determined by consecutive pairs of these four attachments.

For a subgraph \(P'\) of \(P\), let \(\text{att}(P')\) denote the set of all the attachments of the bridges that belong to \(P'\). Let \(P_{xz}\) be the subpath of \(K_C\) that connects \(X\) to \(Z\). At least one vertex of \(\text{att}(P_{xz})\) must belong to the interior of \(\alpha_{vx}\) and at least one vertex of \(\text{att}(P_{xz})\) must belong in the interior of \(\alpha_{zx}\). Hence the set \(\text{att}(Y)\) alternates with \(\text{att}(P_{xz})\) and by Lemma 16, at least one bridge in \(P_{xz}\) conflicts with \(Y\). This means that \(Y\) is an internal vertex of \(P_{xz}\).

Consider now the graph \(P_{xz} - Y\). It consists of two disjoint paths \(P_x\) and \(P_z\) containing \(X\) and \(Z\) respectively. We know that \(P_x\) has a vertex adjacent to \(X\) as well as a vertex adjacent to \(Y\), but no vertex adjacent to \(Z\). Consequently, \(\text{att}(P_x)\) contains at least one vertex from the interior of \(\alpha_{vx}\) as well as at least one vertex from the interior of \(\alpha_{yz}\). Similarly, \(\text{att}(P_z)\) has a vertex from the interior of \(\alpha_{xy}\) and from the interior of \(\alpha_{zy}\). Hence, the set \(\text{att}(P_x)\) alternates with \(\text{att}(P_z)\). Using Lemma 16, we easily deduce that at least one bridge of \(P_x\) must conflict with a bridge of \(P_z\), which is a contradiction.

Next, we show that the attachment vertices on the cycle \(C\) of a conflict path \((F,C)\) without shrinkable edges have a structure very similar to that of an alternating chain.

**Lemma 20.** Assume that \((F,C)\) forms a conflict path with \(k \geq 4\) \(C\)-bridges. Let \(X_1, \ldots, X_k\) be the \(C\)-bridges, listed in the order in which they appear on the path \(K_C\). Let \(\{x_i, y_i\}\) be the two attachments of \(X_i\). Assume that \(C\) has no shrinkable edge. Then

1. The two attachments \(\{x_1, y_1\}\) of \(X_1\) determine an arc \(\alpha_1\) of length 2, and the unique internal vertex \(z_1\) of this arc is an attachment of \(X_2\) and no other bridge.
2. The two attachments \(\{x_k, y_k\}\) of \(X_k\) determine an arc \(\alpha_k\) of length 2 different form \(\alpha_1\), and the unique internal vertex \(z_k\) of this arc is an attachment of \(X_{k-1}\) and no other bridge.
3. All the vertices of \(C\) other than \(z_1\) and \(z_k\) are attachments of exactly two bridges.

**Proof.** We know from Lemma 19 that no vertex of \(C\) is an attachment of more than two bridges.

Let \(\alpha\) and \(\beta\) be the two arcs of \(C\) determined by \(\{x_1, y_1\}\). The bridges \(X_3, \ldots, X_k\) do not alternate with \(X_1\), so all their attachments belong to one of the two arcs, say \(\beta\). The arc \(\alpha\) then has only one attachment \(z_1\) in its interior, and this attachment belongs to \(X_2\) and no other bridge. It follows that \(\alpha\) has only one internal vertex. This proves the first claim; the second claim follows analogously.

To prove the third claim, note first that any vertex of \(C\) must be an attachment of at least one bridge. Suppose that there is a vertex \(v\) that is an attachment of only one bridge \(X_j\). Let \(u\) and \(w\) be the neighbors of \(v\) on \(C\). By assumption, both \(u\) and \(w\) are attachments of at least one bridge that conflicts with \(X_j\).

Assume first, that a single bridge \(Y\) conflicting with \(X_j\) is attached to both \(u\) and \(w\). Since the arc determined by \(u\) and \(w\) and containing \(v\) does not have any other attachment in the interior, this means that \(Y\) conflicts only with the bridge \(X_j\). Then \(Y \in \{X_1, X_k\}\) and \(v \in \{z_1, z_k\}\). Next, assume that the bridge \(X_{j-1}\) is attached to \(u\) but not to \(w\), and the bridge \(X_{j+1}\) is attached to \(w\) but not \(u\). We then easily conclude that \(X_{j-1}\) conflicts with \(X_{j+1}\), which is a contradiction. \(\square\)
This directly implies that non-planar PEGs that form a conflict path and do not have shrinkable edges are $k$-fold alternating chains.

**Corollary 1.** Let $(G, H, \mathcal{H})$ be a non-planar PEG for which $H$ consists of a single cycle $C$ of length at least 4 and two additional vertices $u$ and $v$ that do not belong to $C$, such that $(G, C)$ forms a conflict path with bridges $X_1, \ldots, X_k$ along the path, each with attachments $\{x_i, y_i\}$. Let further $X_i$ consist of the single edge $x_iy_i$ for $i = 2, \ldots, k - 1$ and let $X_1$ consist of $x_1y_1$ and $X_k$ of $x_kv_k$. If $C$ does not contain shrinkable edges then $(G, H, \mathcal{H})$ is an element of $\text{Ach}_k$.

**Proof.** The non-planarity of $G$ implies that $u$ and $v$ must be embedded on different sides of $C$ if $k$ is even, and on the same side if $k$ is odd.

Clearly, the graphs $G$ and $H$ have the same vertex set. By assumption, each bridge $X_i$ forms a path $P_i$, which satisfy the properties for $k$-fold alternating chains; they have the right lengths and contain the right vertices. Further, since $(G, C)$ forms a conflict path their endpoints alternate in the required way.

Finally, as $C$ has no shrinkable edges, Lemma 20 implies that all vertices of $C$ have degree 4, with the exception of one of the attachments of $X_2$ and $X_{k-1}$, which have degree 3. This also implies that the length of the cycle is $k + 1$, and thus $(G, H, \mathcal{H})$ thus is an element of $\text{Ach}_k$. \hfill \square

We now employ the observations we made so far to show that every element of $\text{Ach}_k$ is indeed an obstruction.

**Lemma 21.** For each $k \geq 3$, every element of $\text{Ach}_k$ is an obstruction.

**Proof.** As observed before, $\text{Ach}_3$ contains a single element, which is the obstruction 4. Assume $k \geq 4$, and choose $(G', H', \mathcal{H}') \in \text{Ach}_k$. Let $C$ be the unique cycle of $H$, and let $u$ and $v$ be the two isolated vertices of $H$. Observing that $(G', H', \mathcal{H}')$ is not planar is quite straightforward: since no two conflicting bridges can be embedded into the same region of $C$, all the odd bridges $X_1, X_3, X_5, \ldots$ must be in one region while all the even bridges must be in the other region, and this guarantees that $u$ or $v$ will be on the wrong side of $C$.

Let us prove that $(G', H', \mathcal{H}')$ is minimal non-planar. The least obvious part is to show that contracting an edge of a cycle $C$ always gives a planar PEG. If the cycle $C$ contained a shrinkable edge $e = xy$, we might contract the edge into a single vertex $x_e$. After the contraction, the new graph still forms a conflict path, but the vertex $x_e$ is an attachment of at least three bridges, which contradicts Lemma 19. We conclude that $C$ has no shrinkable edge.

By contracting a non-shrinkable edge $C$, we obtain a new PEG $(G'', H'', \mathcal{H}'')$ where $H$ consists of a cycle $C'$ and two isolated vertices. The conflict graph of $C'$ in $G''$ is a proper subgraph of the conflict graph of $C$ in $G'$. In particular, the bridges containing $u$ and $v$ belong to different components of the conflict graph of $C'$. We may then assign each bridge to one of the two regions of the cycle $C'$, in such a way that the bridges containing $u$ and $v$ are assigned consistently with the embedding $\mathcal{H}''$, and the remaining bridges are assigned in such a way that no two bridges in the same region conflict.

It is easy to see that any collection of $C'$-bridges that does not have a conflict can be embedded inside a single region of $C'$ without crossing. Thus, $(G'', H'', \mathcal{H}'')$ is planar.

By analogous arguments, we see that removing or relaxing an edge or vertex of $H'$ yields a planar PEG. Contracting an edge incident to $u$ or $v$ yields an planar PEG as well. Thus, $(G', H', \mathcal{H}')$ is an obstruction. \hfill \square

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At least one of the embeddings is edge-compatible  Finally, we use all this preparation to analyze the skeletons of R-nodes. In all the following lemmas we suppose that \((G, H, \mathcal{H})\) is a 2-connected obstruction-free Peg, and that \(\mathcal{R}\) is an R-skeleton of \(G\) with at least one edge-compatible embedding \(\mathcal{R}^+\), which we assume to be fixed. We denote this hypothesis (HP1).

Let \(C\) be a cycle of \(\mathcal{R}\) that is a projection of a cycle \(C'\) of \(\mathcal{H}\). Recall that a vertex \(x\) of \(H\) that does not belong to an edge of \(C\) is happy with \(C'\) if it is embedded on the correct side of \(C\) in \(\mathcal{R}^+\), and that it is unhappy otherwise. Recall further that a \(C\)-bridge is happy with \(C'\), if it contains a happy vertex, and it is unhappy if it contains an unhappy vertex and that a bridge that is neither happy nor unhappy is indifferent. We first show that a \(C\)-bridge cannot be happy and unhappy at the same time.

**Lemma 22.** In the hypothesis (HP1), if \(C\) is a cycle of \(\mathcal{R}\) that is a projection of a cycle \(C'\) of \(\mathcal{H}\), then no \(C\)-bridge can be both happy and unhappy with \(C'\).

**Proof.** Assume a \(C\)-bridge \(X\) contains a happy vertex \(u\) and an unhappy vertex \(v\). If there exists a \(G\)-path from \(u\) to \(v\) that avoids all the vertices of \(C'\), then we obtain obstruction 1. Assume then that there is no such path. This easily implies that the bridge \(X\) is a single \(\mathcal{R}\)-edge \(B\) with two attachments \(x\) and \(y\). Figure 14 shows this situation and illustrates the following steps. Since both \(u\) and \(v\) are connected to \(x\) and to \(y\) by a \(G\)-path projecting into \(B\), there is a cycle \(D\) of \(G\) containing both \(u\) and \(v\), and which is contained in \(B\). Since every \(G\)-path from \(u\) to \(v\) inside \(B\) intersects \(x\) or \(y\), we conclude that \(D\) can be expressed as a union of two \(G\)-paths \(P\) and \(Q\) from \(x\) to \(y\), with \(u \in P\) and \(v \in Q\).

Similarly, the cycle \(C\) of \(\mathcal{R}\) can be expressed as a union of two \(\mathcal{R}\)-paths \(R\) and \(S\), each with at least one internal vertex. The paths \(R\) and \(S\) are projections of two \(H\)-paths \(R'\) and \(S'\). Since \(\mathcal{R}\) is 3-connected, it has a path \(T\) that connects an internal vertex of \(R\) to an internal vertex of \(S\), and whose internal vertices avoid \(C\). The path \(T\) is a projection of a \(G\)-path \(T'\). The paths \(P\), \(Q\), \(R'\), \(S'\) and \(T'\) can be contracted to form obstruction 4.

Recall that a cycle \(C'\) in \(H\) that projects to a cycle \(C\) in \(\mathcal{R}\) is happy, if there is at least one \(C\)-bridge that is happy with \(C'\) and it is unhappy, if at least one \(C\)-bridge is unhappy with \(C'\). Again, as the following lemma shows, cycles cannot be both happy and unhappy at the same time.

**Lemma 23.** In the hypothesis (HP1), if \(C'\) is a cycle of \(H\) whose projection is a cycle \(C\) of \(\mathcal{R}\), then \(C'\) cannot be both happy and unhappy.

**Proof.** Suppose \(C\) has a happy bridge \(X\) containing a happy vertex \(u\), and an unhappy bridge \(Y\) with an unhappy vertex \(v\).

If \(C\) is a triangle, then \(X\) and \(Y\) cannot be chords of \(C\) and therefore they have three attachments, each. This implies that they are embedded on different sides of the triangle and all vertices of the
triangle are attachments of both $X$ and $Y$. Since $Y$ is unhappy, it contains a vertex that is
prescribed on the same side of $C'$ as $X$. This yields obstruction 17. Otherwise, $C$ has length
at least 4, and we know that the reduced conflict graph $K_C$ is connected by Lemma 17. We find a
shortest path $X_1, \ldots, X_k$ in $K_C^{-}$ connecting $X = X_1$ to $Y = X_k$. If the path is a single edge, we
obtain obstruction 16. Otherwise we use Lemma 18 to choose for each $X_i$ a pair of attachments
$\{x_i, y_i\} \subseteq \text{att}(X_i)$, such that $\{x_i, y_i\}$ alternates with $\{x_{i+1}, y_{i+1}\}$.

Since each $C$-bridge of the skeleton represents a connected subgraph of $G$, we know that for
every $i = 2, \ldots, k - 1$ the graph $G$ has a path from $x_i$ to $y_i$ whose internal vertices avoid $C'$ and
which projects to the interior of $X_i$. We also know that there is a $G$-path $Q_1$ from $x_1$ to $u$, and a
$G$-path $R_1$ from $y_1$ to $u$ whose internal vertices avoid $C'$ and which project into $X_1$. Similarly, there
are $G$-paths $Q_k$ and $R_k$ from $x_k$ to $v$ and from $y_k$ to $v$, internally disjoint with $C'$ and projecting
into $X_k$. Performing contractions if necessary, we may assume that all these paths are in fact single
edges.

Consider the sub-P EG $(G', H', \mathcal{H}')$, where $H'$ consists of the cycle $C'$ and the two vertices $u$ and
$v$, and $G$ has in addition all the edges obtained by contracting the paths defined above. If $C'$ has
shrinkable edges, we may contract them, until no shrinkable edges are left. Then we either obtain
obstruction 4 (if $k = 3$), or Corollary 1 implies that we have obtained an occurrence of Ach$_k$ for
some $k \geq 4$.

Next, we show that it is not possible that one cycle is happy and another one is unhappy. However,
this is complicated if the cycles are too close in $\mathcal{R}$, in particular if they share vertices.
Therefore, we first show that an unhappy cycle $C'$ projecting to a cycle $C$ may not have an incident
$H$-edge that does not belong to $C$. Such an edge $e$, if it existed, would either be a chord of $C'$, or
it would be part of a bridge containing a vertex of $H$ (e.g., the endpoint of $e$ not belonging to $C'$).
The next two lemmas exclude these two cases separately.

In the former case, where $e$ is a chord of $C'$ that hence projects to a chord of $C$, we also call
$e$ a relevant chord. Note that if $B$ is an edge of $\mathcal{R}$ containing a relevant chord, then in an edge-
compatible embedding of $\mathcal{R}$, $B$ must always be embedded on the correct side of $C$. For practical
purposes, such an edge $B$ behaves as a happy bridge, as shown by the next lemma.

Lemma 24. In the hypothesis (HP1), let $C'$ be a cycle of $H$ that projects to a cycle $C$ of $\mathcal{R}$. Let $e$
be a relevant chord of $C'$ that projects into an $\mathcal{R}$-edge $B$. Then $C'$ cannot be unhappy.

Proof. Let $u$ and $v$ be the two vertices of $e$, which are also the two poles of $B$. Let $\alpha$ and $\beta$
be the two arcs of $C'$ determined by the two vertices $u$ and $v$, and let $\alpha$ and $\beta$ be the two arcs of $C$
that are projections of $\alpha'$ and $\beta'$, respectively. Note that each of the two arcs $\alpha$ and $\beta$ has at least
one internal vertex, otherwise $B$ would not be a chord.

Suppose for contradiction that $C$ has an unhappy bridge $X$ containing an unhappy vertex $x$.
We distinguish two cases, depending on whether $B$ is part of $X$ or not.

First, assume that the bridge $X$ contains the $\mathcal{R}$-edge $B$. Then $X$ is a trivial bridge whose only
dge is $B$; see Fig. 15(a). The edge $B$ then contains a $G$-path $P$ from $u$ to $v$ containing $x$. The
graph $G$ also has a path $Q$ connecting an internal vertex of $\alpha$ to an internal vertex of $\beta$ and avoiding
both $u$ and $v$. Together, the edge $e$, the paths $P$ and $Q$, and the arcs $\alpha$ and $\beta$ can be contracted
to form obstruction 3.

Assume now that the bridge $X$ does not contain $B$. Consider two $H$-cycles $C'_\alpha = \alpha' \cup e$ and
$C'_\beta = \beta' \cup e$, and their respective projections $C_\alpha = \alpha \cup B$ and $C_\beta = \beta \cup B$. It is not hard to see that
the vertex $x$ must be unhappy with at least one of the two cycles $C'_\alpha$ and $C'_\beta$. Let us say that $X$
is unhappy with $C'_\alpha$; see Fig. 15(b). Thus, $C_\alpha$ has at least one unhappy bridge. We claim that $C_\alpha$
also has a happy bridge. Indeed, let $Y$ be the bridge of $C_\alpha$ that contains $\beta$. Since $\beta$ has at least
Lemma 25. In the hypothesis (HP1), let \( C' \) be a cycle of \( H \) that projects to a cycle \( C \) of \( R \). If \( C' \) is unhappy, then every edge of \( H \) that is incident to a vertex of \( C' \) projects into an \( R \)-edge that belongs to \( C \).

Proof. For contradiction, assume that an edge \( e = uv \) of \( H \) is incident to a vertex \( u \in C' \), but projects into an \( R \)-edge \( B \notin C \). If \( v \) is also a vertex of \( C' \), then \( e \) is a relevant chord and \( C \) may not have any unhappy bridges by Lemma 24. If \( v \notin C' \), then \( v \) is an internal vertex of a \( C \)-bridge, and from edge-compatibility it follows that \( v \) is happy with \( C' \). Thus \( C \) has both happy and unhappy bridges, contradicting Lemma 23.

The previous two lemmas show that for an unhappy cycle \( C' \) of \( H \) projecting to a cycle \( C \) of \( R^+ \), no \( C \)-bridge contains an \( H \)-edge incident to a vertex of \( C \). In particular, the projection of any happy \( H \)-cycle is either disjoint from \( C \) (that is they are far apart) or it is identical to \( C \). We now exclude the latter case.

Lemma 26. In the hypothesis (HP1), let \( C'_1 \) and \( C'_2 \) be two distinct facial cycles of \( H \), which project to the same (undirected) cycle \( C \) of \( R \). Then any \( C \)-bridge that is happy with \( C'_2 \) is also happy with \( C'_1 \).

Proof. Let \( F_1 \) and \( F_2 \) be the faces of \( H \) corresponding to facial cycles \( C'_1 \) and \( C'_2 \), respectively.

Suppose for contradiction that at least one \( C \)-bridge \( X \) is unhappy with \( C'_1 \) and happy with \( C'_2 \). In view of Lemma 22 we may assume that \( X \) contains in its interior a vertex \( x \in H \), such that \( x \) is unhappy with \( C'_1 \) and happy with \( C'_2 \). Refer to Figure 16.

Suppose that the two facial cycles \( C'_1 \) and \( C'_2 \) are oriented in such a way that their corresponding faces are to the left of the cycles. Note that any vertex of \( C \) is a common vertex of \( C'_1 \) and \( C'_2 \). This shows that the two facial cycles have at least three common vertices, which implies that they correspond to different faces of \( H \).

Let \( a, b \) and \( c \) be any three distinct vertices of \( C \), and assume that these three vertices appear in the cyclic order \((a, b, c)\) when the cycle \( C'_1 \) is traversed according to its orientation. The interior
Proof. Suppose that \( C'_1 \) is unhappy and \( C'_2 \) is happy. By Lemma 25, this means that no \( C_1 \)-bridge may contain an edge of \( H \) incident to a vertex of \( C_1 \). Consequently, the two cycles \( C_1 \) and \( C_2 \) are vertex-disjoint. Since \( \mathfrak{R} \) is 3-connected, it contains three disjoint paths \( P_1 \), \( P_2 \) and \( P_3 \), each connecting a vertex of \( C_1 \) to a vertex of \( C_2 \). Each path \( P_i \) is a projection of a \( G \)-path \( P'_i \) connecting a vertex of \( C'_1 \) to a vertex of \( C'_2 \). Note that \( C_1 \) is inside a happy bridge of \( C_2 \), and \( C_2 \) is inside an unhappy bridge of \( C_1 \). Thus, contracting the cycles \( C'_1 \) and \( C'_2 \) to triangles and contracting the paths \( P'_i \) to edges, we obtain obstruction 18. \( \square \)

We are now ready to show that \( \mathfrak{R}^+ \) may not have a happy and an unhappy cycle.

**Lemma 27.** In the hypothesis (HP1), let \( C'_1 \) and \( C'_2 \) be two cycles of \( H \) that project to two distinct cycles \( C_1 \) and \( C_2 \) of \( \mathfrak{R} \). If \( C'_1 \) is unhappy, then \( C'_2 \) cannot be happy.

Proof. Suppose that \( C'_1 \) is unhappy and \( C'_2 \) is happy. By Lemma 25, this means that no \( C_1 \)-bridge may contain an edge of \( H \) incident to a vertex of \( C_1 \). Consequently, the two cycles \( C_1 \) and \( C_2 \) are vertex-disjoint. Since \( \mathfrak{R} \) is 3-connected, it contains three disjoint paths \( P_1 \), \( P_2 \) and \( P_3 \), each connecting a vertex of \( C_1 \) to a vertex of \( C_2 \). Each path \( P_i \) is a projection of a \( G \)-path \( P'_i \) connecting a vertex of \( C'_1 \) to a vertex of \( C'_2 \). Note that \( C_1 \) is inside a happy bridge of \( C_2 \), and \( C_2 \) is inside an unhappy bridge of \( C_1 \). Thus, contracting the cycles \( C'_1 \) and \( C'_2 \) to triangles and contracting the paths \( P'_i \) to edges, we obtain obstruction 18. \( \square \)
The next lemma shows that if any vertex $u$ of $\mathcal{R}$ that requires the embedding $\mathcal{R}^+$, then no cycle can be unhappy.

**Lemma 28.** In the hypothesis (HP1), assume that $H$ has three edges $e_1, e_2$ and $e_3$ that are incident to a common vertex $u$ and project into three distinct $\mathcal{R}$-edges $B_1, B_2$ and $B_3$ of $\mathcal{R}$. Then no cycle of $H$ that projects to a cycle of $\mathcal{R}$ can be unhappy.

**Proof.** Proceed by contradiction. Assume that there is an unhappy cycle $\mathcal{C}'$ of $\mathcal{H}$, which projects to a cycle $\mathcal{C}$ of $\mathcal{R}$. From Lemma 25 it then follows that $u$ does not belong to $\mathcal{C}$, and hence $u$ must belong to an unhappy $C$-bridge. From the same lemma we also conclude that the vertex $u$ and the three edges $e_i$ belong to a different component of $\mathcal{H}$ than the cycle $\mathcal{C}'$.

For $i \in \{1, 2, 3\}$, suppose that the $H$-edge $e_i$ connects vertex $u$ to a vertex $v_i$, and is contained in an $\mathcal{R}$-edge $B_i$ that connects vertex $u$ to a vertex $w_i$. These vertices, $H$-edges and $\mathcal{R}$-edges are distinct, except for the possibility that $v_i = w_i$.

Let $D$ be the horizon of $u$ in $\mathcal{R}^+$. The three vertices $w_1, w_2$ and $w_3$ split $D$ into three internally disjoint arcs $\alpha_{12}, \alpha_{13}$ and $\alpha_{23}$, where $\alpha_{ij}$ has endvertices $w_i$ and $w_j$.

As $\mathcal{R}$ is 3-connected, it contains three disjoint paths $P_1, P_2$ and $P_3$, where $P_i$ connects $w_i$ to a vertex of $\mathcal{C}$. We now distinguish two cases, depending on whether the paths $P_i$ can avoid $u$ or not.

First, assume that it is possible to choose the paths $P_i$ in such a way that all of them avoid the vertex $u$. We may then add $B_i$ to the path $P_i$ to obtain three paths from $u$ to $\mathcal{C}$, which only share the vertex $u$. It follows that the graph $G$ contains three paths $R_1^i, R_2^i$ and $R_3^i$ from $u$ to $\mathcal{C}'$ which are disjoint except for sharing the vertex $u$, and moreover, each $R_i^i$ contains the edge $e_i$. This yields obstruction 19.

Next, assume that it is not possible to choose $P_1, P_2$ and $P_3$ in such a way that all the three paths avoid $u$.

For $i \in \{1, 2, 3\}$, let $x_i$ be the last vertex of $P_i$ that belongs to $D$, assuming the path $P_i$ is traversed from $w_i$ towards $\mathcal{C}$. Let $Q_i$ be the subpath of $P_i$ starting in $x_i$ and ending in a vertex of $\mathcal{C}$ (so $Q_i$ is obtained from $P_i$ by removing vertices preceding $x_i$). Let $y_1, y_2$ and $y_3$ be the endvertices of $P_1, P_2$ and $P_3$ that belong to $\mathcal{C}$. We may assume that $y_i$ is the only vertex of $P_i$ belonging to $\mathcal{C}$, otherwise we could replace $P_i$ with its proper subpath.

We claim that one of the three arcs $\alpha_{12}, \alpha_{13},$ and $\alpha_{23}$ must contain all the three vertices $x_i$, possibly as endvertices. If the vertices $x_i$ did not belong to the same arc, we could connect each $x_i$ to a unique vertex $w_j$ by using the edges of $D$, and we would obtain three disjoint paths from $w_i$ to $\mathcal{C}$ that avoid $u$. Assume then, without loss of generality, that $\alpha_{12}$ contains all the three vertices $x_i$.

We may also see that if the cycles $\mathcal{C}$ and $D$ share a common vertex $y$, then $y$ belongs to $\alpha_{12}$. If not, we could connect $w_3$ to $y$ by an arc of $D$ that avoids $w_1$ and $w_2$, and we could connect $w_1$ and $w_2$ to two distinct vertices $x_i$ and $x_j$ by disjoint arcs of $D$, thus obtaining three disjoint paths from $w_i$ to $\mathcal{C}$ avoiding $u$.

Fix distinct indices $p, q, r \in \{1, 2, 3\}$ so that the three vertices $x_1, x_2$ and $x_3$ are encountered in the order $x_p, x_q, x_r$ when $\alpha_{12}$ is traversed in the direction from $w_1$ to $w_2$. Let $\beta$ be the arc of $D$ contained in $\alpha_{12}$ whose endpoints are $x_p$ and $x_r$. Clearly $x_q$ is an internal vertex of $\beta$.

We claim that at least one internal vertex of $\beta$ is connected to $u$ by an edge of $\mathcal{R}$. Assume that this is not the case. Then we may insert into the embedding $\mathcal{R}^+$ a new edge $f$ connecting $x_p$ and $x_r$ and embedded inside the face of $\mathcal{R}^+$ shared by $x_q$ and $u$. Let $\gamma$ be the arc of $C$ with endvertices $y_p$ and $y_r$ that does not contain $y_q$. The arc $\gamma$, the paths $Q_p$ and $Q_r$ and the edge $f$ together form a cycle in the (multi)graph $\mathcal{R} \cup \{f\}$. The vertex $x_q$ and the vertex $w_i$ are separated from each other by this cycle. Thus, the path $P_q$ must share at least one vertex with this cycle, but that is impossible, since $P_q$ is disjoint from $Q_p$, $Q_r$ and $\gamma$. We conclude that $\mathcal{R}$ has an $\mathcal{R}$-edge $B_4$ connecting $u$ to a vertex $x_3$ in the interior of $\beta$. 38
Figure 17: Illustration of Lemma\textsuperscript{28} the paths constructed in the proof\textsuperscript{[a]} and an intermediate step in obtaining one of the obstructions 20, 21 or 22\textsuperscript{(b)}.

We define three paths $R_1$, $R_2$ and $R_3$ of the graph $G$ as follows. The path $R_1$ starts in the vertex $u$, contains the edge $e_1 = uv_1$, proceeds from $v_1$ to $w_1$ inside $B_1$, then goes from $w_1$ to $x_p$ inside the arc $\alpha_{12}$, then follows $Q_p$ until it reaches the vertex $y_p$. Similarly, the path $R_2$ starts in $u$, contains the edge $e_2$, follows from $v_2$ to $w_2$ inside $B_2$, from $w_2$ to $x_r$ inside $\alpha_{12}$, and then along $Q_r$ to $y_r$. The path $R_3$ starts at the vertex $w_3$, proceeds towards $v_3$ inside $B_3$, then using the edge $e_3$ it reaches $u$, proceeds from $u$ to $x_4$ inside $B_4$, then from $x_4$ to $x_q$ inside $\beta$, then from $x_q$ towards $y_q$ along $Q_q$. If any of the three paths $R_i$ contains more than one vertex of $C'$, we truncate the path so that it stops when it reaches the first vertex of $C'$.

We also define two more paths $S_1$ and $S_2$ of $G$, where each $S_i$ connects the vertex $w_i$ to the vertex $w_3$ and projects into the arc $\alpha_{i3}$; see Fig.\textsuperscript{[a]} for an illustration of the constructed paths.

Note that the three paths $R_i$ only intersect at the vertex $u$, a path $S_i$ may only intersect $R_j$ at one of the vertices $w_1$, $w_2$ or $w_3$, and the cycle $C'$ may intersect $S_i$ only in the vertex $w_i$.

Consider the PEG $(G', H', \mathcal{H}')$ formed by the union of the cycle $C'$, the three paths $R_i$, and the two paths $S_j$, where only the cycle $C'$ and the three edges $e_1$, $e_2$ and $e_3$ with their vertices have prescribed embedding, and their embedding is inherited from $\mathcal{H}$.

It can be easily checked that the graph $G'$ is a subdivision of a 3-connected graph, so it has a unique edge-compatible embedding $\mathcal{G}'$. Consider the subgraph $\mathcal{R}'$ of $\mathcal{R}$ formed by all the vertices of $\mathcal{R}$ belonging to $G'$ and all the $\mathcal{R}$-edges that contain at least one edge of $G'$. The graph $G'$ is a subdivision of $\mathcal{R}'$. Thus, the embedding of $\mathcal{R}'$ inherited from $\mathcal{R}^+$ must have the same rotation schemes as the embedding $\mathcal{G}'$. Let $z_i$ be the endpoint of $R_i$ belonging to $C'$. Orient $C'$ so that $z_1, z_2, z_3$ appear in this cyclic order on $C'$. Suppose that $e_1, e_2$ and $e_3$ appear in this clockwise order in $\mathcal{H}$. Then the four vertices $u, v_1, v_2$ and $v_3$ are to the left of $C'$ in $\mathcal{G}'$, and hence also in $\mathcal{R}^+$. Since the four vertices are in an unhappy $C$-bridge of $\mathcal{R}$, they are to the right of $C'$ in $\mathcal{H}'$. This determines $(G', H', \mathcal{H}')$ uniquely.

We now show that $(G', H', \mathcal{H}')$ contains one of the obstructions 20, 21 or 22. First, we contract each of $S_1$ and $S_2$ to a single edge. We also contract the cycle $C'$ to a triangle with vertices $z_1, z_2$ and $z_3$. We contract the subpath of $R_3$ from $w_3$ to $v_3$ to a single vertex, and we contract the subpath of $R_3$ from $u$ to $z_3$ to a single edge. After reversing the order of the vertices on the cycle to make it happy, we essentially obtain the PEG shown in Fig.\textsuperscript{[b]} except that for $i = 1, 2$ it may be that $w_i = v_i$ or $w_i = z_i$, but not both since $v_i \neq z_i$. This is already very close to obstructions 20–22.
To contract $R_1$, we distinguish two cases. First, assume that $w_1$ belongs to $C'$. This means that $z_1 = w_1 \neq v_1$, because we know that $v_1$ is not in the same component of $H$ as $C'$. In this case, we contract the subpath of $R_1$ from $v_1$ to $w_1$ to a single edge. On the other hand, if $w_1$ does not belong to $C$, we contract the subpath of $R_1$ from $v_1$ to $w_1$ to a single vertex, and we contract the subpath from $w_1$ to $z_1$ to a single edge.

The contraction of $R_2$ is analogous to the contraction of $R_1$, and it again depends on whether $w_2$ belongs to $C$ or not. After these contractions are performed, we end up with one of the three obstructions 20, 21, or 22.

With the lemmas proven so far, we are ready to prove the following proposition.

**Proposition 29.** Let $(G, H, \mathcal{H})$ be an obstruction-free Peg, with $G$ biconnected. Let $\mathcal{R}$ be a skeleton of an $R$-node of the SPQR tree of $G$. If $\mathcal{R}$ has at least one edge-compatible embedding, then it has a compatible embedding.

**Proof.** Let $\mathcal{R}^+$ be an edge-compatible embedding. If this embedding is not cycle-compatible, then $H$ has an unhappy facial cycle $C'$ projecting to a cycle $C$ of $\mathcal{R}$. The previous lemmas then imply that every facial cycle of $H$ projecting to a cycle of $\mathcal{R}$ can only have unhappy or indifferent bridges. Besides, Lemma 28 implies that no vertex $u$ of $\mathcal{R}$ can be incident to three $R$-edges, each of them containing an edge of $H$ incident to $u$. Hence, the skeleton $\mathcal{R}$ has no edge-compatibility constraints. Consequently, we may flip the embedding $\mathcal{R}^+$ to obtain a new embedding that is compatible. □

This concludes our treatment of $R$-nodes and thus also the proof of the main theorem for biconnected Pegs. We now turn to 1-connected Pegs, i.e., Pegs that are connected but not necessarily biconnected and to disconnected Pegs.

### 4 Disconnected and 1-Connected Pegs

We have shown that a biconnected obstruction-free Peg is planar. We now extend this characterization to arbitrary Pegs. To do this, we will first show that an obstruction-free Peg $(G, H, \mathcal{H})$ is planar if and only if each connected component of $G$ induces a planar sub-Peg. Next, we provide a more technical argument showing that a connected obstruction-free Peg $(G, H, \mathcal{H})$ is planar, if and only if all the elements of a certain collection of 2-connected Peg-minors of $(G, H, \mathcal{H})$ are planar.

**Reduction to $G$ connected** Angelini et al. [1] proved the following lemma.

**Lemma 30** (cf. Lemma 3.4 in [1]). Let $(G, H, \mathcal{H})$ be a Peg. Let $G_1, \ldots, G_t$ be the connected components of $G$. Let $H_i$ be the subgraph of $H$ induced by the vertices of $G_i$, and let $\mathcal{H}_i$ be $\mathcal{H}$ restricted to $H_i$. Then $(G, H, \mathcal{H})$ is planar if and only if

1) each $(G_i, H_i, \mathcal{H}_i)$ is planar, and

2) for each $i$, for each facial cycle $\overline{C}$ of $H_i$ and for every $j \neq i$, no two vertices of $H_j$ are separated by $\overline{C}$, in other words, all the vertices of $H_j$ are embedded on the same side of $\overline{C}$.

A Peg that does not satisfy the second condition of the lemma must contain obstruction 1. Thus, if Theorem [1] holds for Pegs with $G$ connected, it holds for all Pegs.
**Reduction to G biconnected** Next, we consider connected Pegs, i.e., Pegs \((G, H, \mathcal{H})\) where \(G\) is connected. In contrast to planarity of ordinary graphs, it is not in general true that a Peg is planar if and only if each sub-Peg induced by a biconnected component of \(G\) is planar. However, for Pegs satisfying some additional assumptions, a similar characterization is possible.

Let \((G, H, \mathcal{H})\) be a connected Peg and let \(v\) be a cut-vertex of \(G\). We say that \(v\) is \(H\)-separating if at least two connected components of \(G - v\) contain vertices of \(H\).

Let \((G, H, \mathcal{H})\) be a connected Peg that avoids obstruction 1. Let \(v\) be an \(H\)-separating cut-vertex of \(G\) that does not belong to \(H\). Let \(x\) and \(y\) be two vertices of \(H\) that belong to different connected components of \(G - v\), chosen in such a way that there is a path in \(G\) connecting \(x\) to \(y\) whose internal vertices do not belong to \(H\). The existence of such a path implies that \(x\) and \(y\) share a face \(F\) of \(\mathcal{H}\), otherwise \(H\) would contain a cycle separating \(x\) from \(y\), creating obstruction 1. The face \(F\) is unique, because \(x\) and \(y\) belong to distinct components of \(H\). It follows that any planar embedding of \(G\) that extends \(\mathcal{H}\) must embed the vertex \(v\) in the interior of the face \(F\). We define \(H' = H \cup v\) and let \(\mathcal{H}'\) be the embedding of \(H'\) obtained from \(\mathcal{H}\) by inserting the isolated vertex \(v\) inside the interior of the face \(F\). As shown above, any planar embedding of \(G\) that extends \(\mathcal{H}\) also extends \(\mathcal{H}'\). We say that \((G, H', \mathcal{H}')\) is obtained from \((G, H, \mathcal{H})\) by fixing the cut-vertex \(v\).

Let \((G, H^+, \mathcal{H}^+)\) be a Peg that is obtained from \((G, H, \mathcal{H})\) by fixing all the \(H\)-separating cut-vertices of \(G\) not belonging to \(H\). Note that each \(H^+\)-separating cut-vertex is also \(H\)-separating, and vice versa. A planar embedding of \(G\) that extends \(\mathcal{H}\) also extends \(\mathcal{H}^+\) and in particular, \((G, H, \mathcal{H})\) is planar if and only if \((G, H^+, \mathcal{H}^+)\) is planar. We now show that this operation cannot create a new obstruction in \((G, H^+, \mathcal{H}^+)\).

**Lemma 31.** Let \((G, H, \mathcal{H})\) be a connected Peg that avoids obstruction 1, and let \((G, H^+, \mathcal{H}^+)\) be the Peg obtained by fixing all the \(H\)-separating cut-vertices of \(G\). Then \((G, H, \mathcal{H})\) contains a minimal obstruction \(X\) if and only if \((G, H^+, \mathcal{H}^+)\) contains \(X\).

**Proof.** Since \((G, H, \mathcal{H})\) is a Peg-minor of \((G, H^+, \mathcal{H}^+)\), it suffices to prove that if \((G, H^+, \mathcal{H}^+)\) contains an obstruction \(X = (G_X, H_X, \mathcal{H}_X)\) then we can efficiently find the same obstruction in \((G, H, \mathcal{H})\). This clearly holds in the case when \(H_X\) does not contain isolated vertices, because then any sequence of deletions, contractions and relaxations that produces \(X\) inside \((G, H^+, \mathcal{H}^+)\) will also produce \(X\) inside \((G, H, \mathcal{H})\).

Suppose now that \(H_X\) contains isolated vertices. Assume first that \(G_X\) is 2-connected. Let \(G_1, \ldots, G_t\) be the 2-connected blocks of \(G\), let \(H_i\) be the subgraph of \(H\) induced by the vertices of \(G_i\), let \(\mathcal{H}_i\) be the embedding of \(H_i\) inherited from \(\mathcal{H}\), and similarly for \(H_i^+\) and \(\mathcal{H}_i^+\). If \((G, H^+, \mathcal{H}^+)\) contains \(X\), then for some \(i\), \((G_i, H_i^+, \mathcal{H}_i^+)\) contains \(X\) as well (here we use the fact that each \(H^+\)-separating cut-vertex of \(G\) belongs to \(H^+\)). However, each \((G_i, H_i^+, \mathcal{H}_i^+)\) is a Peg-minor of \((G, H, \mathcal{H})\) — this is because any vertex \(v\) of \(H_i^+\) that is not a vertex of \(H_i\) is connected to a vertex of \(H\) by a path that internally avoids \(G_i\). By contracting all such paths, we obtain a copy of \((G_i, H_i^+, \mathcal{H}_i^+)\) inside \((G, H, \mathcal{H})\). Since \((G_i, H_i^+, \mathcal{H}_i^+)\) contains \(X\), so does \((G, H, \mathcal{H})\).

It remains to deal with the case when \(X\) is not 2-connected and \(H_X\) contains an isolated vertex. This means that \(X\) is obstruction 1. By assumption, \((G, H, \mathcal{H})\) does not contain obstruction 1. Suppose for contradiction that \((G, H^+, \mathcal{H}^+)\) contains obstruction 1. This means that \(H^+\) contains a cycle \(C\) and a pair of vertices \(v\) and \(w\) separated by this cycle, and that there exists a path \(P\) of \(G\) that connects \(v\) and \(w\) and has no vertex in common with \(C\).

If \(v\) is not a vertex of \(H\), then \(v\) is an \(H\)-separating cut-vertex. Therefore, there are two vertices \(x\) and \(y\) of \(H\) in distinct components of \(G - v\) that both share a face \(F\) with \(v\) and are connected to \(v\) by paths \(P_x\) and \(P_y\) of \(G\) which do not contain any other vertex of \(H\). Since \(x\) and \(y\) are in distinct components of \(H\), at least one of them, say \(x\), does not belong to the cycle \(C\). Since \(x\) shares a face with \(v\), it must be on the same side of \(C\) as \(v\). By the same reasoning, the vertex
w either belongs to \( H \) or there is a vertex \( z \in H \) that appears on the same side of \( C \) as \( w \) and is connected to \( w \) by a \( G \)-path \( P_w \), whose internal vertices do not belong to \( H \). In any case, we find a pair of vertices of \( H \) that are separated by \( C \) and are connected by a \( G \)-path that avoids \( C \). This shows that \((G, H, \mathcal{H})\) contains obstruction 1, which is a contradiction. \( \square \)

Lemma \ref{lem:minor_operations} shows that we can without loss of generality restrict ourselves to PEGs \((G, H, \mathcal{H})\) in which every \( H \)-separating cut-vertex belongs to \( H \). For PEGs having this property, we can show that planarity can be reduced to planarity of biconnected components.

First, we need a definition. Let \( H \) be a graph with planar embedding \( \mathcal{H} \), let \( v \) be a vertex of \( H \), and let \( H_1 \) and \( H_2 \) be two edge-disjoint subgraphs of \( H \). We say that \( H_1 \) and \( H_2 \) alternate around \( v \) in \( \mathcal{H} \), if there exist edges \( e, e' \in E(H_1) \) and \( f, f' \in E(H_2) \) which are all incident with \( v \) and appear in the cyclic order \((e, f, e', f')\) in the rotation scheme of \( v \) in the embedding \( \mathcal{H} \).

The following lemma is analogous to Lemma 3.3 of \cite{1}, except that the assumption “every non-trivial \( H \)-bridge is local” is replaced with the weaker condition “every \( H \)-separating cut-vertex of \( G \) is in \( H \)”. This new assumption is weaker, because a separating cut-vertex not belonging to \( H \) necessarily belongs to a non-local \( H \)-bridge. However, the proof in \cite{1} uses only this weaker assumption and therefore we have the following lemma.

**Lemma 32.** Let \((G, H, \mathcal{H})\) be a connected PEG with the property that every \( H \)-separating cut-vertex of \( G \) is in \( H \). Let \( G_1, \ldots, G_t \) be the blocks of \( G \), let \( H_i \) be the subgraph of \( H \) induced by the vertices of \( G_i \), and let \( \mathcal{H}_i \) be \( \mathcal{H} \) restricted to \( H_i \). Then, \((G, H, \mathcal{H})\) is planar if and only if

1) \((G_i, H_i, \mathcal{H}_i)\) is a planar PEG for each \( i \),

2) no two distinct graphs \( H_i \) and \( H_j \) alternate around any vertex of \( \mathcal{H} \), and

3) for every facial cycle \( \tilde{C} \) of \( \mathcal{H} \) and for any two vertices \( x \) and \( y \) of \( \mathcal{H} \) separated by \( \tilde{C} \), any path in \( G \) connecting \( x \) and \( y \) contains a vertex of \( \tilde{C} \).

Note that the last two conditions are always satisfied when \((G, H, \mathcal{H})\) avoids obstructions 1 and 2. We can also efficiently test whether the two conditions are satisfied and produce an occurrence of an obstruction when one of the conditions fails. This concludes the proof of Theorem \ref{thm:main_theorem}.

## 5 Other minor-like operations

Let us remark that our definition of PEG-minor operations is not the only one possible. In this paper, we preferred to work with a weaker notion of PEG-minors, since this makes the resulting characterization theorem stronger. However, in many circumstances, more general minor-like operations may be appropriate, providing a smaller set of obstructions.

For example, the \( G \)-edge contraction rules may be relaxed to allow contractions in more general situations. Here is an example of such a relaxed \( G \)-edge contraction rule: given a PEG \((G, H, \mathcal{H})\), assume \( e = uv \) is an edge of \( G \) but not of \( H \), assume that \( u \) and \( v \) have a unique common face \( F \) of \( \mathcal{H} \), and assume furthermore that each of the two vertices is visited only once by the corresponding facial walk of \( F \). If \( u \) and \( v \) are in distinct components of \( H \), or if the graph \( H \) is connected, we embed the edge \( uv \) into \( F \) and then contract it, resulting in a new PEG \((G', H', \mathcal{H}')\).

It is not hard to see that this relaxed contraction preserves the planarity of a PEG, and that \( \mathcal{H}' \) is uniquely determined. It also subsumes the ‘complicated \( G \)-edge contraction’ we introduced. With this stronger contraction rule, most of the exceptional obstructions can be further reduced, leaving only the obstructions 1, 2, 3, 4, 6, 11, 14, 16, and 17, as well as \( K_5 \) and \( K_{3,3} \). However, even this stronger contraction cannot reduce the obstructions from \( A_{ch_k} \).
To reduce the obstructions to a finite set, we need an operation that can be applied to an alternating chain. We now present an example of such an operation. See Fig. 18.

Suppose that \((G, H, \mathcal{H})\) is a Peg, let \(F\) be a face of \(\mathcal{H}\), let \(C\) be a facial cycle of \(F\) oriented in such a way that the interior of \(F\) is to the left of \(C\), let \(x\) and \(y\) be two vertices of \(C\) that are not connected by an edge of \(G\), and let \(z\) be a vertex of \(H\) not belonging to \(C\). Assume that the following conditions hold.

1. The vertex \(z\) is adjacent to \(x\) and \(y\) in \(G\).
2. The vertex \(z\) is embedded to the left of \(C\) in the embedding \(\mathcal{H}\), and is incident to the face \(F\).
3. Any connected component of \(H\) that is embedded to the left of \(C\) in \(\mathcal{H}\) is connected to a vertex of \(C \setminus \{x, y\}\) by an edge of \(G\).
4. Any edge of \(H\) that is incident to \(x\) or \(y\) and does not belong to \(C\) is embedded outside of \(F\) (i.e., to the right of \(C\)) in \(\mathcal{H}\).

We define a new Peg by the following steps.

- Remove vertex \(z\) and all its incident edges from \(G\) and \(H\).
- Add to \(G\), \(H\) and \(\mathcal{H}\) a new edge \(e = xy\). The edge \(e\) is embedded inside \(F\). (Note that the position of \(e\) in the rotation schemes of \(x\) and \(y\) is thus determined uniquely, because of condition 4 above.)
- The edge \(e\) splits the face \(F\) into two subfaces \(F_1\) and \(F_2\). Let \(C_1\) and \(C_2\) be the facial cycles of \(F_1\) and \(F_2\) such that \(C_1 \cup C_2 = C \cup \{e\}\). For any connected component \(B\) of \(H\) that is embedded to the left of \(C\) in \(\mathcal{H}\), let \(w\) be a vertex of \(C \setminus \{x, y\}\) adjacent to a vertex of \(B\). Such a vertex \(w\) exists by condition 3 above. If there are more such vertices, we choose one arbitrarily for each \(B\). If \(w\) belongs to \(C_1\), then \(B\) will be embedded inside \(F_1\), otherwise it will be embedded inside \(F_2\).

Let \((G', H', \mathcal{H}')\) be the resulting Peg. We easily see that if \((G, H, \mathcal{H})\) was planar, then \((G', H', \mathcal{H}')\) is planar as well. In fact, if the vertex \(z\) has degree 2 in \(G\), then we may even say that \((G, H, \mathcal{H})\) is planar if and only if \((G', H', \mathcal{H}')\) is planar.

The operation described above allows to reduce each \(k\)-fold alternating chain with \(k \geq 4\) to a smaller non-planar Peg which contains a \((k - 1)\)-fold alternating chain. It also reduces obstruction 4 to obstruction 3, and obstruction 16 to a Peg that contains obstruction 1. Therefore, when the above operation is added to the permissible minor operations, there will only be a finite number of minimal non-planar Pegs. More precisely, exactly nine minimal non-planar Pegs remain in this case.
Let us point out that the obstructions from the infinite family $\bigcup_{k \geq 4} \text{Ach}_k$ only play a role when cycle-compatibility is important. For certain types of PEGs, cycle-compatibility is not a concern. For instance, if the graph $H$ is connected, it can be shown that $(G, H, \mathcal{H})$ is planar if and only if all the skeletons of $G$ have edge-compatible embeddings, and therefore such a PEG is planar if and only if it avoids the finitely many exceptional obstructions.

6 Conclusion

Note that Theorem 1 together with the linear-time algorithm for testing planarity of a PEG immediately implies Theorem 2. In any non-planar instance $I = (G, H, \mathcal{H})$ only linearly many PEG-minor operations are possible. We test each one individually and use the linear-time testing algorithm to check whether the result is non-planar. In this way we either find a smaller non-planar PEG $I'$ resulting from $I$ by one of the operations, or we have found an obstruction, which by Theorem 1 is contained in our list. The running time of this algorithm is at most $O(n^3)$.

In fact, in many cases, as indicated in the paper, obstructions can be found much more efficiently, often in linear time. In particular, the linear-time testing algorithm gives an indication of which property of planar PEGs is violated for a given instance. Is it possible to find an obstruction in a non-planar PEG in linear time? In general, given a fixed PEG $(G, H, \mathcal{H})$, what is the complexity of determining whether a given PEG contains $(G, H, \mathcal{H})$ as PEG-minor? The answer here may depend on the PEG-minor operations we allow.

It is not known whether the results on planar PEGs can be generalized to graphs that have a partial embedding on a higher-genus surface. In fact, even the complexity of recognizing whether a graph partially embedded on a fixed higher-genus surface admits a crossing-free embedding extension is still an open problem.

References

[1] P. Angelini, G. Di Battista, F. Frati, V. Jelinek, J. Kratochvıl, M. Patrignani, and I. Rutter. Testing planarity of partially embedded graphs. In Proceedings 21st ACM-SIAM Symposium on Discrete Algorithms (SODA’10), pages 202–221. SIAM, 2010.

[2] G. Demoucron, Y. Malgrange, and R. Pertuiset. Reconnaissance et construction de représentations planaires topologiques. Rev. Franc. Rech. Oper., 8:33–34, 1964.

[3] G. Di Battista and R. Tamassia. On-line maintenance of triconnected components with SPQR-trees. Algorithmica, 15(4):302–318, 1996.

[4] G. Di Battista and R. Tamassia. On-line planarity testing. SIAM J. Comput., 25(5):956–997, 1996.

[5] C. Gutwenger and P. Mutzel. A linear time implementation of SPQR-trees. In Graph Drawing (GD ’00), volume 1984 of LNCS, pages 77–90, 2001.

[6] D. A. Holton, B. Jackson, A. Saito, and N. C. Wormald. Removable edges in 3-connected graphs. Journal of Graph Theory, 14(4):465–473, 1990.

[7] M. Jünger and M. Schulz. Intersection graphs in simultaneous embedding with fixed edges. J. Graph Algorithms Appl., 13(2):205–218, 2009.
[8] M. Juvan and B. Mohar. 2-restricted extensions of partial embeddings of graphs. *European J. Comb.*, 26(3–4):339–375, 2005.

[9] M. Kriesell. Contractible subgraphs in 3-connected graphs. *J. Comb. Theory Ser. B*, 80:32–48, 2000.

[10] M. Kriesell. A survey on contractible edges in graphs of a prescribed vertex connectivity. *Graphs and Combinatorics*, 18:1–30, 2002.

[11] K. Kuratowski. Sur le problème des courbes gauches en topologie. *Fund. Math.*, 15:217–283, 1930.

[12] P. Mutzel. The SPQR-tree data structure in graph drawing. In *Proceedings of the 30th International Conference on Automata, Languages and Programming (ICALP’03)*, pages 34–46, Berlin, Heidelberg, 2003. Springer-Verlag.

[13] K. Wagner. Über eine Eigenschaft der ebenen Komplexe. *Mathematische Annalen*, 114:570–590, 1937. 10.1007/BF01594196.

[14] H. Whitney. Congruent graphs and the connectivity of graphs. *Amer. J. Math.*, 43:150–168, 1932.