THE EXTENDING STRUCTURES PROBLEM FOR ALGEBRAS

A. L. AGORE AND G. MILITARU

Dedicated to Professor Constantin Năstăsescu on the occasion of his 70th birthday

Abstract. The paper is devoted to the dual of the Hochschild extension problem for associative algebras. Let $A$ be an algebra, $E$ a vector space containing $A$ as a subspace and $V$ a complement of $A$ in $E$. All algebra structures on $E$ containing $A$ as a subalgebra are described and classified by two non-abelian cohomological type objects which are explicitly constructed: $\mathcal{AH}_2^A(V,A)$ will classify all such algebras up to an isomorphism that stabilizes $A$ and $\mathcal{AH}_2^A(V,A)$ provides the classification up to an isomorphism of algebras that stabilizes $A$ and $V$. A new product, called the unified product, is introduced as a tool of our approach. Different types of split extensions of algebras are fully described in terms of special cases of unified products: in particular, the classical crossed product and its generalizations are special cases of the unified product. Examples and classification results are worked out in details in the case of flag extending structures and flag algebras: the latter being algebras $E$ that have a finite chain of subalgebras $E_0 := k \subset E_1 \subset \cdots \subset E_m := E$, such that each $E_i$ has codimension 1 in $E_{i+1}$. The results are obtained over an arbitrary base field, including those of characteristic two.

Introduction

The classification of associative algebras of a given dimension is an old problem initiated by Peirce [29] and independently by Study [34] who classified all algebras of dimension 2 over the field of complex numbers $\mathbb{C}$ and also gave a partial answer for dimension 3: the complete classification of all 22 types of 3-dimensional algebras over $\mathbb{C}$ can be found in [10, Section 5.1]. The classification of all associative algebras over an algebraically closed field of characteristic $\neq 2$ is known up to dimension 6. The 4-dimensional algebras were classified by Scorza [33] and reconfirmed by Gabriel [13]. In dimension 5 the complete list of all 59 types of algebras was given by Mazzola [24]. The 6-dimensional rigid algebras over a field of characteristic zero were classified by Goze and Makhlouf [16]. The classification of algebras over a field of characteristic 2, even in small dimension, remains a mystery as there are only a few results known in the literature. All two-dimensional non-associative algebras over a finite field were classified by Petersson and Scherer [30]. For the classification of nilpotent algebras of dimension 3 and 4, including the case of characteristic 2, we refer to [8] and [20, Chapter VI]. As explained in [10],

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nowadays there are two ways of approaching the problem: the algebraic classification which requires the explicit description of all isomorphism classes of algebras of a given dimension and the geometric classification, formulated in terms of the moduli space problem, where a description of these equivalence classes in a geometrical sense is to be given. Two other famous problems, intimately related to the classification problem, are the extension problem initiated by Hochschild [17], [18] and the deformation theory introduced by Gerstenhaber in a series of papers starting with [14], [15]. The deformation theory view point was adopted recently by Fialowski and Penkava [9], [10], [11], [12] for the classification of algebras of small dimension, including the construction of the moduli space. The extension problem for algebras in its full generality, can be restated in an elementary manner as follows:

The extension problem. Let \( A \) be a \( k \)-algebra, \( E \) a vector space and \( \pi : E \to A \) a \( k \)-linear epimorphism of vector spaces. Describe and classify all algebra structures \( \cdot \) that can be defined on the vector space \( E \) such that \( \pi : E \to A \) is a morphism of algebras.

As usual, by classification of two algebra structures \( \cdot \) and \( \cdot' \) on \( E \) we mean the classification up to an isomorphism of algebras \( (E, \cdot) \cong (E, \cdot') \) that stabilizes \( V := \text{Ker}(\pi) \) and co-stabilizes \( A \). The partial answer to the extension problem was given in [18, Theorem 6.2]: all algebra structures \( \cdot \) on \( E \) such that \( V \) is a two-sided ideal of null square (that is \( x \cdot y = 0 \), for all \( x, y \in V \)) are classified by the second cohomological group \( H^2(A, V) \). As a key tool of his approach, Hochschild introduced a new product that is a deformation of the trivial extension of an algebra by an \( A \)-bimodule \( V \) using a cocycle \( \vartheta : A \times A \to V \). In general, the extension problem is still open: in Section 1 we shortly review the problem as it is important for our goal. A general version of the product used in [18, Theorem 6.2], which we call the Hochschild product, is introduced as a way to approach the extension problem in its full generality. The elementary way in which we reformulated the extension problem allows us to consider the categorical dual of it: namely, we start with a \( k \)-linear monomorphism \( i : A \to E \) between an algebra \( A \) and a vector space \( E \) and we ask for the classification of all algebra structures on \( E \) such that \( i \) becomes an algebra map. Thus, we arrive at the following question:

Extending structures problem. Let \( A \) be an algebra and \( E \) a vector space containing \( A \) as a subspace. Describe and classify the set of all algebra structures \( \cdot \) that can be defined on \( E \) such that \( A \) becomes a subalgebra of \( (E, \cdot) \).

The extending structures (ES) problem was introduced and studied at the level of groups in [1] and for Lie (resp. Leibinz) algebras in [4] (resp. [5]). For these categories the ES-problem generalizes and unifies two famous problems: the extension problem of Hölder [19] and the factorization problem of Ore [26]. It was formulated for arbitrary categories in [3], where a partial answer in the context of quantum groups was obtained. The ES-problem is a very difficult one: if \( A = k \) then the ES-problem asks in fact for the classification of all algebra structures on a given vector space \( E \). For this reason, from now on we will assume that \( A \neq k \). Even though the ES-problem is a difficult one, we can still provide detailed answers to it in certain special cases which depend on the choice of the algebra \( A \) and mainly on the codimension of \( A \) in \( E \). Moreover, a new class of algebras, which we call flag algebras, appear on the route: these are finite dimensional
algebras $E$ for which there exists a finite chain of subalgebras of $E$

$$E_0 := k \subset E_1 \subset \cdots \subset E_m := E$$

such that $E_i$ has codimension 1 in $E_{i+1}$, for all $i = 0, \cdots, m-1$. The class of flag algebras of a given dimension can be classified using the method introduced in this paper by a recursive type algorithm in which the key step is the one for codimension 1: the crucial role will be played by the characters of the algebra $A$ and by some twisted derivations of $A$ satisfying certain compatibility conditions.

The paper is organized as follows: in Section 1 we recall the extension problem for algebras and the corresponding Hochschild product, presented in the most general setting. Considering that we are not able to indicate a reference where this problem was treated in its full generality we will mention here a few results that lead to the construction of the classifying object $\mathcal{H}^2(A, V)$ which generalizes the second Hochschild cohomology group $H^2(A, V)$. Our motivation is two fold: firstly, it turns out that the product responsible for the ES-problem, which will be called the unified product, is the categorical dual of the Hochschild product. Secondly, one of the classifying objects for the ES-problem, namely $\mathcal{AH}^2(V, A)$, is a sort of a dual of $\mathcal{H}^2(A, V)$.

In Section 2 we will perform the abstract construction of the unified product $A \ltimes V$: it is associated to an algebra $\Omega$, a vector space $V$ and a system of data $\Omega(A, V) = (\prec, \succ, \leftarrow, \rightarrow, f, \cdot)$ called an extending datum of $A$ through $V$. Theorem 2.2 establishes the set of axioms that need to be satisfied by $\Omega(A, V)$ such that $A \ltimes V$ with a certain multiplication becomes an associative unitary algebra, i.e. a unified product. In this case, $\Omega(A, V) = (\prec, \succ, \leftarrow, \rightarrow, f, \cdot)$ will be called an algebra extending structure of $A$ through $V$. We highlight here an important difference between the unified product for algebras and the corresponding unified products for groups [1] or Lie algebras [4]. The construction of the unified product $A \ltimes V$ for algebras requires two more actions $\leftarrow : A \times V \rightarrow A$ and $\rightarrow : A \times V \rightarrow V$ which connects $A$ and $V$. The two actions are missing in the case of groups because any inclusion of groups $H \leq G$ has a retraction map which is left $H$-linear, while in the case of Lie algebras the absence of the two maps is a consequence of the anti-symmetry on the bracket. Now let $A$ be an algebra, $E$ a vector space containing $A$ as a subspace and $V$ a given complement of $A$ in $E$. Theorem 2.4 provides the answer to the description part of the ES-problem: there exists an algebra structure $\cdot$ on $E$ such that $A$ is a subalgebra of $(E, \cdot)$ if and only if there exists an isomorphism of algebras $(E, \cdot) \cong A \ltimes V$, for some algebra extending structure $\Omega(g, V) = (\prec, \succ, \leftarrow, \rightarrow, f, \cdot)$ of $A$ through $V$. Moreover, the algebra isomorphism $(E, \cdot) \cong A \ltimes V$ can be chosen in such a way that it stabilizes $A$ and co-stabilizes $V$. Based on this result we are able to give the theoretical answer to the ES-problem in Theorem 2.8: all algebra structures on $E$ containing $A$ as a subalgebra are classified by two non-abelian cohomological type objects which are explicitly constructed. The first one is denoted by $\mathcal{AH}^2_A(V, A)$ and will classify all such structures up to an isomorphism that stabilizes $A$. We also indicate the bijection between the elements of $\mathcal{AH}^2_A(V, A)$ and the isomorphism classes of all extending structures of $A$ to $E$. Having in mind that we want to extend the algebra structure on $A$ to a bigger vector space $E$ this is in fact the object responsible for the classification of the ES-problem. If $V = k^n$ the significance of
the object $\mathcal{AH}_A^2 (k^n, A)$ is the following: it classifies up to an isomorphism all algebras which contain and stabilize $A$ as a subalgebra of codimension $n$. Hence, by computing $\mathcal{AH}_A^2 (k^n, A)$, for a given algebra $A$, we obtain important information regarding the classification of finite dimensional algebras. On the other hand, in order to comply with the traditional way of approaching the extension problem, we also construct a second classifying object, denoted by $\mathcal{AH}^2 (V, A)$, which provides the classification of the ES-problem up to an isomorphism of algebras that stabilizes $A$ and co-stabilize $V$. Thus, the object $\mathcal{AH}^2 (V, A)$, whose construction is simpler than the one of $\mathcal{AH}_A^2 (V, A)$, appears as a sort of dual of the classical Hochschild cohomological group. There exists a canonical projection $\mathcal{AH}^2 (V, A) \to \mathcal{AH}_A^2 (V, A)$ between these two classifying objects.

Section 3 deals with several special cases of the unified product and we emphasize the problem for which each of these products is responsible. Let $i : A \hookrightarrow E$ be an inclusion of algebras. Corollary 3.1, Corollary 3.3 and Corollary 3.5 give necessary and sufficient condition for $i$ to have a retraction which is a left/right $A$-linear map, an $A$-bimodule map and respectively an algebra map. In the latter case, the associated unified product has a simple form which we call the semidirect product by analogy with the groups and Lie algebra case since it describes the split monomorphisms in the category of algebras. We also show that the classical crossed products [28], Brzezinski’s product [7] or the Ore extensions are all special cases of the unified product. Definition 3.6 introduces a new concept, namely the matched pair of two algebras $A$ and $V$. As a special case of the unified product, we define the bicrossed product $A \bowtie V$ associated to a matched pair of algebras. Even if the definition differs from the one given in the case of groups or Lie algebras, Corollary 3.7 shows that it plays the same role, namely it provides the tool to answer the factorization problem for algebras. The end of the section deals with the commutative case: in this case the unified product and its axioms simplify considerably.

Computing the classifying objects $\mathcal{AH}_A^2 (V, A)$ and $\mathcal{AH}^2 (V, A)$ is a highly nontrivial problem. In Section 4 we shall identify a way of computing both objects for what we have called flag extending structures of $A$ to $E$ as defined in Definition 4.1. All flag extending structures of $A$ to $E$ can be completely described by a recursive reasoning where the key step is the case when $A$ has codimension 1 as a subspace of $E$. This case is solved in Theorem 4.4, where $\mathcal{AH}_A^2 (k, A)$ and $\mathcal{AH}^2 (k, A)$ are described. The key players in this context are the so-called flag datums of an algebra $A$ as defined in Definition 4.2: in the definition of a flag datum two characters of the algebra $A$ are involved as well as two twisted derivations satisfying certain compatibility conditions. Theorem 4.4 proves itself to be efficient for the classification of all flag algebras. Corollary 4.5 classifies and counts the number of types of isomorphisms of algebras of dimension 2 over an arbitrary field $k$. By iterating the algorithm, we can increase the dimension by 1 at each step, obtaining in this way the classification of all flag algebras in dimension 3, 4, etc. (see Theorem 4.10 for dimension 3). We mention that all classification results presented in this paper are over an arbitrary field $k$, including the case of characteristic 2 whose difficulty is illustrated.
1. Preliminaries

All vector spaces, linear or bilinear maps are over an arbitrary field \( k \). A map \( f : V \to W \) between two vector spaces is called the trivial map if \( f(v) = 0 \), for all \( v \in V \). By an algebra \( A \) we mean an associative and unitary algebra over \( k \). However, when the algebras are not unitary it will be explicitly mentioned. All algebra maps preserve units and any left/right \( A \)-module is unitary. \( \text{Alg} (A,k) \) denotes the space of all algebra maps \( A \to k \). For an algebra \( A \) we shall denote by \( _A \mathcal{M}_A \) the category of all \( A \)-bimodules, i.e. triples \((V, \to, \triangleleft)\) consisting of a vector space \( V \) and two bilinear maps \( \to: A \times V \to V \), \( \triangleleft: V \times A \to V \) such that \((V, \to)\) is a left \( A \)-module, \((V, \triangleleft)\) is a right \( A \)-module and \( a \mapsto (a \triangleleft b) = (a \to x) \triangleleft b \), for all \( a, b \in A \) and \( x \in V \). If \((V, \to, \triangleleft) \in _A \mathcal{M}_A \), then the trivial extension of \( A \) by \( V \) is the algebra \( A \times V \), with the multiplication given for any \( a, b \in A, x, y \in V \) by:

\[
(a, x) \cdot (b, y) := (ab, a \to y + x \triangleleft b)
\]

Let \( A \) be an algebra, \( E \) a vector space such that \( A \) is a subspace of \( E \) and \( V \) a complement of \( A \) in \( E \), i.e. \( V \) is a subspace of \( E \) such that \( E = A + V \) and \( A \cap V = \{0\} \). For a linear map \( \varphi : E \to E \) we consider the diagram:

\[
\begin{array}{c}
A \xymatrix{ \ar[r]^i & E \ar[r]^{\pi} & V } \ar@{=}[d] \\
A \ar[r]^{\varphi} & E \ar[r]^{\pi} & V \ar[d]_{\text{Id}} \\
A \ar[r]^{\text{Id}} & E \ar[r]^{\pi} & V \ar@{=}[u] \\
\end{array}
\]

where \( \pi : E \to V \) is the canonical projection of \( E = A + V \) on \( V \) and \( i : A \to E \) is the inclusion map. We say that \( \varphi : E \to E \) stabilizes \( A \) (resp. co-stabilizes \( V \)) if the left square (resp. the right square) of diagram (1) is commutative.

Two algebra structures \( \cdot \) and \( \cdot' \) on \( E \) containing \( A \) as subalgebra are called equivalent and we denote this by \((E, \cdot) \equiv (E, \cdot')\), if there exists an algebra isomorphism \( \varphi : (E, \cdot) \to (E, \cdot') \) which stabilizes \( A \). The algebra structures \( \cdot \) and \( \cdot' \) on \( E \) are called cohomologous and we denote this by \((E, \cdot) \approx (E, \cdot')\), if there exists an algebra isomorphism \( \varphi : (E, \cdot) \to (E, \cdot') \) which stabilizes \( A \) and co-stabilizes \( V \), i.e. diagram (1) is commutative.

\( \equiv \) and \( \approx \) are both equivalence relations on the set of all algebra structures on \( E \) containing \( A \) as subalgebra and we denote by \( \text{Ext}d(E, A) \) (resp. \( \text{Ext}d'(E, A) \)) the set of all equivalence classes via \( \equiv \) (resp. \( \approx \)). \( \text{Ext}d(E, A) \) is the classifying object for the ES problem: by explicitly computing \( \text{Ext}d(E, A) \) we obtain a parametrization of the set of all isomorphism classes of algebra structures on \( E \) which contain and stabilize \( A \) as a subalgebra. \( \text{Ext}d'(E, A) \) gives a more restrictive classification of the ES problem, similar to the approach used in the case of the extension problem. Any two cohomologous algebra structures on \( E \) are of course equivalent, hence there exists a canonical projection \( \text{Ext}d'(E, A) \to \text{Ext}d(E, A) \).

For two sets \( X \) and \( Y \) we shall denote by \( X \sqcup Y \) the coproduct in the category of sets of \( X \) and \( Y \), i.e. \( X \sqcup Y \) is the disjoint union of \( X \) and \( Y \).

The Hochschild product and the extension problem revisited. We present a short review of the Hochschild extension problem for algebras [18] for two reasons: firstly,
we will approach it in its full generality, leaving aside the special case considered in [18] where only those extensions whose kernel is a null square ideal are considered. Secondly, we wish to illustrate the crucial role played by the product, which we will call the Hochschild product. A special case of this product was introduced in the proof of [18, Theorem 6.2]. Our main motivation is that the unified products appear in a natural way as duals, in the categorical sense, of these Hochschild products. As a consequence, we highlight that the null square condition imposed in [18] as a counterpart of the abelian case of the extension problem is essential for that theory.

Let $A$ be a $k$-algebra, $E$ a vector space and $\pi : E \to A$ a $k$-linear epimorphism of vector spaces and $V := \text{Ker}(\pi)$. The extension problem, in its full generality, asks for the classification of all algebra structures · that can be defined on the vector space $E$ such that $\pi : E \to A$ is a morphism of algebras. By classification of two algebra structures · and ·' on $E$ we mean the classification up to an isomorphism of algebras $(E, \cdot) \cong (E, \cdot')$ that stabilizes $V$ and co-stabilizes $A$. Two such algebra structures · and ·' on $E$ are called cohomologous and we denote by $\text{EXT}(E, A)$ the isomorphism classes of all such algebra structures on $E$ via this relation. The theoretical answer to the extension problem is provided by computing this classification object. In what follows we will prove that $\text{EXT}(E, A)$ is parameterized by a cohomological type object denoted by $\mathcal{H}^2(A, V)$ which generalizes the second Hochschild cohomological group $H^2(A, V)$.

To start with, we introduce the following:

**Definition 1.1.** Let $A$ be an algebra and $V$ a vector space. A Hochschild data between $A$ and $V$ is a system $\Theta(A, V) = (\rightarrow, \triangleleft, \vartheta, \cdot)$ consisting of four bilinear maps

\[
\rightarrow : A \times V \to V, \quad \triangleleft : V \times A \to V, \quad \vartheta : A \times A \to V, \quad \cdot : V \times V \to V
\]

A Hochschild data $\Theta(A, V) = (\rightarrow, \triangleleft, \vartheta, \cdot)$ will be called normalized if for any $a \in A$, $x \in V$ we have:

\[
\vartheta(a, 1_A) = \vartheta(1_A, a) = 0, \quad x \triangleleft 1_A = x, \quad 1_A \rightarrow x = x
\]

For a Hochschild data $\Theta(A, V) = (\rightarrow, \triangleleft, \vartheta, \cdot)$ we denote by $A \ast V = A \ast_{(\rightarrow, \triangleleft, \vartheta, \cdot)} V$ the vector space $A \times V$ with the multiplication given by

\[
(a, x) \ast (b, y) := (ab, \vartheta(a, b) + a \rightarrow y + x \triangleleft b + x \cdot y)
\]  

(2)

for all $a, b \in A$, $x, y \in V$. $A \ast V$ is called the Hochschild product associated to $A$ and $V$ if it is an associative algebra with the multiplication given by (2) and the unit $(1_A, 0_V)$. In this case $\Theta(A, V) = (\rightarrow, \triangleleft, \vartheta, \cdot)$ is called a Hochschild system and we denote by $\mathcal{H}S(A, V)$ the set consisting of all Hochschild systems between $A$ and $V$. The trivial example of a Hochschild system is the quadruple $(\rightarrow, \triangleleft, \vartheta, \cdot)$, for which $\vartheta$ and $\cdot$ are both the trivial maps and $(V, \rightarrow, \triangleleft)$ is an $A$-bimodule: in this case the Hochschild product $A \ast V$ is precisely the trivial extension of $A$ through an $A$-bimodule $V$. The multiplication defined by (2) is more general than the one that appears in the proof of [18, Theorem 6.2] – the latter is a special case of $A \ast V$ in the case when $\cdot : V \times V \to V$ is the trivial map, that is $x \cdot y = 0$, for all $x, y \in V$. In general, the necessary and sufficient conditions for $A \ast V$ to be a Hochschild product are given by the following:
Proposition 1.2. Let $A$ be an algebra, $V$ a vector space and $\Theta(A, V) = (\to, \triangleleft, \vartheta, \cdot)$ a Hochschild data between $A$ and $V$. Then $A \ast V$ is a Hochschild product if and only if the Hochschild data $\Theta(A, V)$ is normalized, the bilinear map $\cdot : V \times V \to V$ is associative and the following compatibility conditions hold for any $a, b \in A, x, y \in V$:

\[
\begin{align*}
(H1) \quad & (x \cdot y) \triangleleft a = x \cdot (y \triangleleft a); \\
(H2) \quad & (x \triangleleft a) \cdot y = x \cdot (a \to y); \\
(H3) \quad & a \to (x \cdot y) = (a \to x) \cdot y; \\
(H4) \quad & (a \to x) \triangleleft b = a \to (x \triangleleft b); \\
(H5) \quad & \vartheta(a, b) \triangleleft c = \vartheta(a, bc) - \vartheta(ab, c) + a \to \vartheta(b, c); \\
(H6) \quad & (ab) \to x = a \to (b \to x) - \vartheta(a, b) \cdot x; \\
(H7) \quad & x \triangleleft (ab) = (x \triangleleft a) \triangleleft b - x \cdot \vartheta(a, b)
\end{align*}
\]

Proof. To start with, we note that $(1_A, 0_V)$ is the unit of $A \ast V$ if and only if the Hochschild data $(\to, \triangleleft, \vartheta, \cdot)$ is normalized. The rest of the proof relies on a detailed analysis of the associativity condition for the multiplication given by (2). Since in $A \ast V$ we have $(a, x) = (a, 0) + (0, x)$, it follows that the associativity condition holds if and only if it holds for all generators of $A \ast V$, i.e. for the set $\{(a, 0) \mid a \in A\} \cup \{(0, x) \mid x \in V\}$. We will illustrate only a few cases, the rest of the details being left to the reader. For instance, the associativity condition for the multiplication given by (2) holds in $\{(0, x), (0, y), (a, 0)\}$ if and only if (H1) holds. Similarly, the associativity condition holds in $\{(0, x), (a, 0), (0, y)\}$ if and only if (H2) holds while, the associativity condition holds in $\{(0, x), (0, y), (0, z)\}$ if and only if $\cdot : V \times V \to V$ is associative.

Remark 1.3. By applying Proposition 1.2 we obtain that a Hochschild data $(\to, \triangleleft, \vartheta, \cdot)$ for which $\cdot$ is the trivial map is a Hochschild system if and only if $(V, \to, \triangleleft)$ is an $A$-bimodule and $\vartheta : A \times A \to V$ is a normalized cocycle (i.e. the compatibility condition $(H5)$ holds). This special case was considered in [18].

The Hochschild product is the tool to answer the extension problem in its full generality. Indeed, first we observe that $A \ast V$ is an extension of the algebra $A$ by $V$ via

\[
0 \longrightarrow V \overset{i_V}{\longrightarrow} A \ast V \overset{\pi_A}{\longrightarrow} A \longrightarrow 0
\]

where $i_V(v) = (0, v)$ and $\pi_A(a, v) := a$. Conversely, we have:

Proposition 1.4. Let $A$ be an algebra, $E$ a vector space and $\pi : E \to A$ a linear epimorphism of vector spaces with $V = \text{Ker}(\pi)$. Then any algebra structure $\cdot$ which can be defined on the vector space $E$ such that $\pi : (E, \cdot) \to A$ is a morphism of algebras is isomorphic to a Hochschild product $A \ast V$ and moreover, the isomorphism of algebras $(E, \cdot) \cong A \ast V$ can be chosen such that it stabilizes $V$ and co-stabilizes $A$.

Thus, any algebra extension of $A$ by $V$ is cohomologous to a Hochschild extension (3).

Proof. Indeed, let $\cdot$ be an algebra structure of $E$ such that $\pi : (E, \cdot) \to A$ is an algebra map. Since $k$ is a field we can pick a $k$-linear section $s : A \to E$ of $\pi$, i.e. $\pi \circ s = \text{Id}_A$
and \( s(1_A) = 1_E \). Using this section \( s \) we define three bilinear maps as follows:

\[
\begin{align*}
\preceq &= \preceq_s : V \times A \rightarrow V, \quad x \preceq a := x \cdot s(a) \\
\rightarrow &= \rightarrow_s : A \times V \rightarrow V, \quad a \rightarrow x := s(a) \cdot x \\
\vartheta &= \vartheta_s : A \times A \rightarrow V, \quad \vartheta(a, b) := s(a) \cdot s(b) - s(ab)
\end{align*}
\]

for all \( a, b \in A \) and \( x \in V \). Then

\[
\varphi : A \times V \rightarrow E, \quad \varphi(a, x) := s(a) + x
\]

is an isomorphism of vector spaces with the inverse \( \varphi^{-1}(y) = (\pi(y), y - s(\pi(y))) \), for all \( y \in E \). The key step is the following: using the system \((\varphi, \rightarrow, \vartheta)\) connecting \( A \) and \( V \) we can prove that the unique algebra structure \( \star \) that can be defined on the direct product of vector spaces \( A \times V \) such that \( \varphi : A \times V \rightarrow (E, \cdot) \) is an isomorphism of algebras is given by

\[
(a, x) \star (b, y) := (ab, \vartheta(a, b) + a \rightarrow y + x \preceq b + x \cdot y)
\]

for all \( a, b \in A, x, y \in V \). Indeed, we have:

\[
(a, x) \star (b, y) = \varphi^{-1}\left( \varphi(a, x) \cdot \varphi(b, y) \right) = \varphi^{-1}\left( (s(a) + x) \cdot (s(b) + y) \right) = \varphi^{-1}(s(a) \cdot s(b) + s(a) \cdot y + x \cdot s(b) + x \cdot y) = (ab, s(a) \cdot s(b) - s(ab) + s(a) \cdot y + x \cdot s(b) + x \cdot y) = (ab, \vartheta(a, b) + a \rightarrow y + x \preceq b + x \cdot y)
\]

as needed. Thus, \( \varphi : A \star V \rightarrow (E, \cdot) \) is an isomorphism of algebras that stabilizes \( V \) and co-stabilizes \( A \).

\[\square\]

Proposition 1.4 shows that the classification part of the extension problem reduces to the classification of all Hochschild products associated to all Hochschild systems between \( A \) and \( V \). This is what we do next by explicitly constructing a classification object which we denote by \( \mathcal{H}^2(A, V) \). To be best of our knowledge, this general case seems to be uncovered in the literature. First we need the following:

**Lemma 1.5.** Let \( \Theta(A, V) = (\rightarrow, \vartheta, \cdot) \) and \( \Theta'(A, V) = (\rightarrow', \vartheta', \cdot') \) be two Hochschild systems and \( A \star V \), respectively \( A \star' V \), the corresponding Hochschild products. Then there exists a bijection between the set of all morphisms of algebras \( \psi : A \star V \rightarrow A \star' V \) which stabilize \( V \) and co-stabilize \( A \) and the set of all linear maps \( r : A \rightarrow V \) with \( r(1_A) = 0_V \) satisfying the following compatibilities for all \( a, b \in A, x, y \in V \):

\[
\begin{align*}
(\text{CH1}) \quad x \cdot y &= x \cdot y; \\
(\text{CH2}) \quad x \preceq a &= x \preceq a \preceq a + x \cdot r(a); \\
(\text{CH3}) \quad a \rightarrow x &= a \rightarrow x + r(a) \cdot x; \\
(\text{CH4}) \quad \vartheta(a, b) + r(ab) &= \vartheta'(a, b) + a \rightarrow r(b) + r(a) \preceq b + r(a) \cdot r(b)
\end{align*}
\]

Under the above bijection the morphism of algebras \( \psi = \psi_r : A \star V \rightarrow A \star' V \) corresponding to \( r : A \rightarrow V \) is given for any \( a \in A \) and \( x \in V \) by:

\[
\psi(a, x) = (a, r(a) + x)
\]

Moreover, \( \psi = \psi_r \) is an isomorphism with the inverse given by \( \psi_r^{-1} = \psi_{-r} \).
Proof. A linear map \( \psi : A \star V \to A \star' V \) stabilizes \( V \) and co-stabilizes \( A \) if and only if there exists a uniquely determined linear map \( r : A \to V \) such that \( \psi(a, x) = (a, r(a) + x) \), for all \( a \in A \), \( x \in V \). Let \( \psi = \psi_r \) be such a linear map. We will prove that \( \psi \) is an algebra map if and only if \( r(1_A) = 0_V \) and the compatibility conditions (CH1)-(CH4) hold. To start with it is straightforward to see that \( \psi \) is a unitary map if and only if \( r(1_A) = 0_V \). The proof will be finished if we prove that the following compatibility holds for all generators of \( A \times V \):

\[
\psi((a, x) \star (b, y)) = \psi((a, x)) \star' \psi((b, y))
\]  

(5)

By a straightforward computation it follows that (5) holds for the pair \((a, 0)\), \((b, 0)\) if and only if (CH4) is fulfilled while (5) holds for the pair \((1_A, x)\), \((a, 0)\) if and only if (CH2) is satisfied. Finally, (5) holds for the pair \((a, 0)\), \((1_A, x)\) and respectively \((1_A, x)\), \((1_A, y)\) if and only if (CH3) and (CH1) hold. \( \square \)

Definition 1.6. Let \( A \) be an algebra and \( V \) a vector space. Two Hochschild systems \( \Theta(A, V) = (\cdotp, \triangleleft, \triangleright, \cdot) \) and \( \Theta'(A, V) = (\cdotp', \triangleleft', \triangleright', \cdot') \) are called cohomologous, and we denote this by \( \Theta(A, V) \approx \Theta'(A, V) \), if and only if \( \cdot = \cdot' \) and there exists a linear map \( r : V \to A \) such that \( r(1_A) = 0_V \) and for any \( a, b \in A \), \( x, y \in V \) we have:

\[
\begin{align*}
x \triangleleft a &= x \triangleleft' a + x \cdot' r(a) \\
ar \triangleright x &= a \triangleright' x + r(a) \cdot' x \\
\triangleright(a, b) &= \triangleright'(a, b) - r(ab) + a \triangleright' b + r(a) \cdot' b + r(a) \cdot' r(b)
\end{align*}
\]

As a conclusion, we obtain the following generalization of [18, Theorem 6.2].

Theorem 1.7. Let \( A \) be an algebra, \( E \) a vector space and \( \pi : E \to A \) a linear epimorphism of vector spaces with \( V = \ker(\pi) \). Then \( \approx \) is an equivalence relation on the set \( \mathcal{HS}(A, V) \) of all Hochschild systems between \( A \) and \( V \). If we denote by \( \mathcal{H}^2(A, V) \) := \( \mathcal{HS}(A, V)/\approx \), then the map

\[
\mathcal{H}^2(A, V) \to \text{EXT}(E, A), \quad (\cdotp, \triangleleft, \triangleright, \cdot) \mapsto A \star_{(\cdotp, \triangleleft, \triangleright, \cdot)} V
\]

is a bijection between \( \mathcal{AH}^2(V, A) \) and \( \text{EXT}(E, A) \).

Proof. Follows from Proposition 1.4 and Lemma 1.5. \( \square \)

Now we shall give an example which proves that the cohomological object \( \mathcal{H}^2(A, k) \) defined in Theorem 1.7 is a coproduct in the category of sets of the classical second Hochschild cohomological group \( H^2(A, k) \) and a hidden part that is responsible for classifying algebra extensions whose kernel is not a null square ideal. First we need the following:

Proposition 1.8. Let \( A \) be an algebra and \( V \) a vector space of dimension 1. Then the set of all Hochschild systems \( \mathcal{HS}(A, V) \) is the coproduct of the following sets:

\[
\mathcal{HS}(A, V) \cong \mathcal{H}_1(A) \sqcup \mathcal{H}_2(A),
\]

where:

\( \mathcal{H}_1(A) \) is the set of all triples \((\lambda, \Lambda, \theta)\) consisting of two algebra maps \( \lambda, \Lambda : A \to k \) and a bilinear map \( \theta : A \times A \to k \) satisfying the following compatibilities for any \( a, b, c \in A \):

\[
\theta(a, 1_A) = \theta(1_A, a) = 0, \quad \theta(ab, c) - \theta(ab, c) = \theta(a, b) \Lambda(c) - \theta(b, c) \lambda(a) \tag{6}
\]
\( \mathcal{H}_2(A) \) is the set of all triples \((\lambda, \theta, u)\) consisting of a linear map \(\lambda : A \to k\), a bilinear map \(\theta : A \times A \to k\) and a scalar \(u \in k^*\) such that

\[
\begin{align*}
\theta(a, 1_A) &= \theta(1_A, a) = 0, & \lambda(1_A) &= 1, \\
\theta(a, bc) - \theta(ab, c) &= \theta(a, b)\lambda(c) - \theta(b, c)\lambda(a), \\
\lambda(ab) &= \lambda(a)\lambda(b) - u\theta(a, b)
\end{align*}
\]

for all \(a, b \in k\).

**Proof.** We have to compute the set of all bilinear maps \(\rightarrow : A \times V \to V\), \(\triangleleft : V \times A \to V\), \(\vartheta : V \times V \to V\) satisfying the compatibility conditions (H1)-(H7) of Proposition 1.2. Let \(\{x\}\) be a basis of \(V\). Since \(V\) has dimension 1 there exists a bijection between the set of all Hochschild datums \((\rightarrow, \triangleleft, \vartheta)\) between \(A\) and \(V\) and the set of all 4-tuples \((\Lambda, \lambda, \theta, u)\) consisting of two linear maps \(\Lambda, \lambda : A \to k\), a bilinear map \(\theta : A \times A \to k\) and a scalar \(u \in k\). The bijection is given such that the Hochschild datum \((\rightarrow, \triangleleft, \vartheta)\) corresponding to \((\Lambda, \lambda, \theta, u)\) is defined as follows:

\[
a \mapsto x := \lambda(a)x, \quad x \triangleleft a := \Lambda(a)x, \quad \vartheta(a, b) := \theta(a, b)x, \quad x \cdot x := ux
\]

for all \(a \in A\). Now, the normalizing condition of Proposition 1.2 holds if and only if (7) holds. On the other hand the axioms (H1), (H3), (H4) are trivially fulfilled, while the axiom (H2) is equivalent to

\[
u\Lambda(a) = u\lambda(a)
\]

for all \(a \in A\). Axiom (H5) is equivalent to the last part of (6), axiom (H6) is equivalent to (9) while axiom (H7) is equivalent to (9) with \(\Lambda\) instead of \(\lambda\). A discussion on \(u\) is imposed by the compatibility \(u\Lambda(a) = u\lambda(a)\) and the conclusion follows easily:

\[
\mathcal{H}_1(A) \text{ corresponds to the case when } u = 0 \text{ (and this will induce algebra structures on } A \star V \text{ for which } V \text{ is a null square ideal) while } \mathcal{H}_2(A) \text{ corresponds to the case when } u \neq 0 \text{ and this will induce algebra structures on } A \star V \text{ for which } x \cdot x = ux \neq 0.
\]

The Hochschild product \(A \star V\) associated to \((\lambda, \Lambda, \theta) \in \mathcal{H}_1(A)\) has the form

\[
(a, x) \star (b, x) = (ab, (\theta(a, b) + \lambda(a) + \Lambda(b))x)
\]

while the Hochschild product \(A \ast V\) associated to \((\lambda, \theta, u) \in \mathcal{H}_2(A)\) has the multiplication

\[
(a, x) \ast (b, x) = (ab, (\theta(a, b) + \lambda(a) + \Lambda(b) + u)x)
\]

The latter are algebras for which \(V\) is not a null square ideal since \((0, x) \ast (0, x) = (0, ux)\).

**Corollary 1.9.** Let \(A\) be an algebra and \(V\) a vector space of dimension 1. Then, \(\mathcal{H}_2(A, V)\) is the coproduct of the following sets:

\[
\mathcal{H}_2(A, V) \cong \left( \mathcal{H}_1(A) / \approx_1 \right) \sqcup \left( \mathcal{H}_2(A) / \approx_2 \right),
\]

where:

\(\approx_1\) is the following equivalence relation on \(\mathcal{H}_1(A)\): \((\lambda, \Lambda, \theta) \approx_1 (\lambda', \Lambda', \theta')\) if and only if \(\lambda = \lambda', \Lambda = \Lambda'\) and there exists a linear map \(t : A \to k\) such that \(t(1_A) = 0\) and for any \(a, b \in A:\)

\[
\theta(a, b) = \theta'(a, b) - t(ab) + \lambda'(a)t(b) + \Lambda'(b)t(a)
\]
while ≈_2 is the following equivalence relation on \( H_2(A) \): \( (\lambda, \theta, u) ≈_2 (\lambda', \theta', u') \) if and only if \( u = u' \) and there exists a linear map \( t: A \to k \) such that \( t(1_A) = 0 \) and for any \( a, b \in A \):

\[
\lambda(a) = \lambda'(a) + t(a)u', \quad \theta(a, b) = \theta'(a, b) - t(ab) + u't(a)t(b) + \lambda'(a)t(b) + \lambda'(b)t(a)
\]

\[\text{Proof.}\] Follows from Proposition 1.8: the equivalence relation \( ≈_i \) is just \( ≈ \) of Definition 2.6 on \( H_i(A) \), for \( i = 1, 2 \).

\[\text{Example 1.10.}\] Let \( A = k_{(0,0)} \) be the 2-dimensional algebra having 1 and \( x \) as a basis and \( x^2 = 0 \). Then \( k_{(0,0)} \) has only one character, namely the algebra map sending \( x \) to 0. Based on this, using Corollary 1.9, it is straightforward to see that \( H^2(k_{(0,0)}, k) \cong k \sqcup k^*. \)

\( k \) corresponds to the classical second Hochschild cohomological group \( H^2(k_{(0,0)}, k) \cong k \), while \( k^* \), parameterizes all algebra structures on \( k_{(0,0)} \times k \) such that the two-sided ideal \( k \cong \{0\} \times k \) is not a null square ideal.

### 2. Unified products for algebras

In this section we shall give the answer to the ES-problem by explicitly constructing two cohomological type objects which will parameterize \( \text{Extd}(E, A) \) and \( \text{Extd}'(E, A) \). First we introduce the following:

**Definition 2.1.** Let \( A \) be an algebra and \( V \) a vector space. An extending datum of \( A \) through \( V \) is a system \( \Omega(A, V) = (\langle, \triangleright, \leftarrow, \rightarrow, f, \cdot) \) consisting of six bilinear maps

\[
\langle : V \times A \to V, \quad \triangleright : V \times A \to A, \quad \leftarrow : A \times V \to A, \quad \rightarrow : A \times V \to V \\
\quad f : V \times V \to A, \quad \cdot : V \times V \to V
\]

The extension datum \( \Omega(A, V) \) is called normalized if for any \( x \in V \) we have:

\[x \triangleright 1_A = 0, \quad x \langle 1_A = x, \quad 1_A \leftarrow x = 0, \quad 1_A \rightarrow x = x\] (10)

Let \( \Omega(A, V) = (\langle, \triangleright, \leftarrow, \rightarrow, f, \cdot) \) be an extending datum. We denote by \( A \rtimes_{\Omega(A, V)} V = A \ltimes V \) the vector space \( A \times V \) together with the bilinear map \( \bullet \) defined by:

\[(a, x) \bullet (b, y) := (ab + a \leftarrow y + x \triangleright b + f(x, y), \quad a \rightarrow y + x \langle b + x \cdot y)\] (11)

for all \( a, b \in A \) and \( x, y \in V \). The object \( A \rtimes V \) is called the unified product or the dual Hochschild product of \( A \) and \( \Omega(A, V) \) if it is an associative algebra with the multiplication given by (11) and the unit \((1_A, 0_V)\). In this case the extending datum \( \Omega(A, V) = (\langle, \triangleright, \leftarrow, \rightarrow, f, \cdot) \) is called an algebra extending structure of \( A \) through \( V \).

The maps \( \langle, \triangleright, \leftarrow \) and \( \rightarrow \) are called the actions of \( \Omega(A, V) \) and \( f \) is called the cocycle of \( \Omega(A, V) \).
The multiplication given by (11) has a rather complicated formula; however, for some specific elements we obtain easier forms which will be useful for future computations:

\[
(a, 0) \bullet (b, y) = (ab + a \leftarrow y, a \rightarrow y) \quad (12)
\]

\[
(0, x) \bullet (b, y) = (x \triangleright b + f(x, y), x \triangleleft b + x \cdot y) \quad (13)
\]

\[
(a, x) \bullet (0, y) = (a \leftarrow y + f(x, y), a \rightarrow y + x \cdot y) \quad (14)
\]

\[
(a, x) \bullet (b, 0) = (ab + x \triangleright b, x \triangleleft b) \quad (15)
\]

for all \(a, b \in A\) and \(x, y \in V\). In particular, for any \(a, b \in A\) and \(x, y \in V\) we have:

\[
(a, 0) \bullet (b, 0) = (ab, 0), \quad (0, x) \bullet (0, y) = (f(x, y), x \cdot y) \quad (16)
\]

\[
(a, 0) \bullet (0, x) = (a \leftarrow x, a \rightarrow x), \quad (0, x) \bullet (b, 0) = (x \triangleright b, x \triangleleft b) \quad (17)
\]

The next theorem provides the necessary and sufficient conditions that need to be fulfilled by an extending datum \(\Omega(A, V)\) such that \(A \ltimes V\) is a unified product.

**Theorem 2.2.** Let \(A\) be an algebra, \(V\) a vector space and \(\Omega(A, V) = (\triangleleft, \triangleright, \leftarrow, \rightarrow, f, \cdot)\) an extending datum of \(A\) by \(V\). The following statements are equivalent:

1. \(A \ltimes V\) is a unified product;
2. The following compatibilities hold for any \(a, b \in A\), \(x, y, z \in V\):

   \begin{align*}
   (A1) \quad & \Omega(A, V) \text{ is a normalized extending datum and } (V, \rightarrow, \triangleleft) \in A\mathcal{M}_A \text{ is an } A\text{-bimodule;} \\
   (A2) \quad & x \cdot (y \cdot z) - (x \cdot y) \cdot z = f(x, y) \rightarrow z - x \triangleleft f(y, z); \\
   (A3) \quad & f(x, y \cdot z) - f(x \cdot y, z) = f(x, y) \leftarrow z - x \triangleright f(y, z); \\
   (A4) \quad & a \rightarrow (x \cdot y) = (a \rightarrow x) \cdot y + (a \leftarrow x) \rightarrow y; \\
   (A5) \quad & (a \leftarrow x) \leftarrow y = a \leftarrow (x \cdot y) + af(x, y) - f(a \rightarrow x, y); \\
   (A6) \quad & (ab) \leftarrow x = a(b \leftarrow x) + a \leftarrow (b \rightarrow x); \\
   (A7) \quad & x \triangleright (ab) = (x \triangleright a)b + (a \triangleleft a) \triangleright b; \\
   (A8) \quad & x \triangleright (y \triangleright a) = (x \cdot y) \triangleright a + f(x, y)a - f(x, y \triangleleft a); \\
   (A9) \quad & (x \cdot y) \triangleleft a = x \triangleleft (y \triangleright a) + x \cdot (y \triangleleft a); \\
   (A10) \quad & a(x \triangleright b) + a \leftarrow (x \triangleleft b) = (a \leftarrow x)b + (a \rightarrow x) \triangleright b; \\
   (A11) \quad & x \triangleright (a \leftarrow y) + f(x, a \rightarrow y) = (x \triangleright a) \leftarrow y + f(x \triangleleft a, y); \\
   (A12) \quad & x \triangleleft (a \leftarrow y) + x \cdot (a \rightarrow y) = (x \triangleright a) \rightarrow y + (x \triangleleft a) \cdot y;
   \end{align*}

Before going into the proof of the theorem, we have a few observations on the compatibilities in Theorem 2.2. Although they look rather complicated at first sight, they are in fact quite natural and can be interpreted as follows: (A2) measures how far \((V, \cdot)\) is from being an associative algebra and is called the *twisted associativity condition*. The compatibility condition (A3) is a 2-cocycle type condition. (A5) and (A8) are deformations of the usual module conditions and they can be called *twisted module conditions* for the actions \(\leftarrow\) and \(\triangleright\). (A4), (A6), (A7), (A9), (A10) and (A12) are compatibility conditions between the actions \((\triangleleft, \triangleright, \leftarrow, \rightarrow)\) of \(\Omega(A, V)\). They will be used in the next section as a definition for the notion of *matched pair* of algebras.

**Proof.** We can easily check that \((1_A, 0_V)\) is a unit for the multiplication given by (11) if and only if the extending datum \(\Omega(A, V)\) is normalized. The rest of the proof relies on
a detailed analysis of the associativity condition for the multiplication given by (11):
\[
[(a, x) \cdot (b, y)] \cdot (c, z) = (a, x) \cdot [(b, y) \cdot (c, z)]
\] (18)
where \(a, b, c \in A\) and \(x, y, z \in V\). Furthermore, since in \(A \ltimes V\) we have \((a, x) = (a, 0) + (0, x)\) it follows that (18) holds if and only if it holds for all generators of \(A \ltimes V\), i.e. for the set \(\{(a, 0) \mid a \in A\} \cup \{(0, x) \mid x \in V\}\). However, since the computations are rather long but straightforward we will only indicate the main steps of the proof.
We will start by proving that (A2) and (A3) hold if and only if (18) holds for the triple \((0, x), (0, y), (0, z)\) with \(x, y, z \in V\). Indeed, we have:
\[
[(0, x) \cdot (0, y)] \cdot (0, z) = (f(x, y) \leftarrow z + f(x \cdot y, z), f(x, y) \rightarrow z + (x \cdot y) \cdot z)
\]
\[
(0, x) \cdot [(0, y) \cdot (0, z)] = (x \triangleright f(y, z) + f(x, y \cdot z), x \triangleleft f(y, z) + x \cdot (y \cdot z))
\]
Therefore, (A2) and (A3) hold if and only if (18) holds for the triple \((0, x), (0, y), (0, z)\) with \(x, y, z \in V\). In the same manner we can prove the following: (A4) and (A5) hold if and only if (18) holds for the triple \((a, 0), (0, x), (0, y)\) with \(a \in A, x, y \in V\). Furthermore, (18) holds for the triple \((a, 0), (b, 0), (0, x)\) if and only if (16) holds and \(\leftarrow\) is a left \(A\)-module structure on \(V\). (A7) holds, together with the fact that \(\triangleleft\) is a right \(A\)-module structure on \(V\), if and only if (18) holds for the triple \((0, x), (a, 0), (b, 0)\). (A8) and (A9) hold if and only if (18) holds for the triple \((0, x), (0, y), (a, 0)\). (18) holds for the triple \((a, 0), (0, x), (b, 0)\) if and only if (10) holds as well as the compatibility condition which makes \(V\) an \(A\)-bimodule with respect to the actions \(\leftarrow\) and \(\triangleleft\). Finally, (A11) and (A12) hold if and only if (18) holds for the triple \((0, x), (a, 0), (0, y)\).

From now on, an algebra extending structure of \(A\) through a vector space \(V\) will be viewed as a system \(\Omega(A, V) = (\triangleleft, \triangleright, \leftarrow, \rightarrow, f, \cdot)\) satisfying the compatibility conditions (A1)-(A12). We denote by \(\mathcal{AE}(A, V)\) the set of all algebra extending structures of \(A\) through \(V\).

**Example 2.3.** We provide a first example of an algebra extending structure and the corresponding unified product. More examples will be given in Section 3 and Section 4.
Let \(\Omega(A, V) = (\triangleleft, \triangleright, \leftarrow, \rightarrow, f, \cdot)\) be an extending datum of \(A\) through \(V\) such that \(\triangleright, \leftarrow\) and \(\cdot\) are the trivial maps. Then, \(\Omega(A, V)\) is an algebra extending structure of \(A\) through \(V\) if and only if \((V, \rightarrow, \triangleleft) \in \mathcal{A}M_A\) and the following compatibilities are fulfilled:
\[
f(x, y) \rightarrow z = x \triangleleft f(y, z), \quad af(x, y) = f(a \rightarrow x, y)
\]
\[
f(x, y \triangleleft a) = f(x, y)a, \quad f(x, a \rightarrow y) = f(x \triangleleft a, y)
\]
for all \(a \in A, x, y\) and \(z \in V\). In this case, the associated unified product \(A \ltimes V\) has the multiplication defined for any \(a, b \in A\) and \(x, y \in V\) by:
\[
(a, x) \cdot (b, y) := (ab + f(x, y), a \rightarrow y + x \triangleleft b)
\]
that is, \(A \ltimes V\) is a cocycle deformation of the usual trivial extension of \(A\) by \(V\), dual to the one considered in [18].

Let \(\Omega(A, V) = (\triangleleft, \triangleright, \leftarrow, \rightarrow, f, \cdot) \in \mathcal{AE}(A, V)\) be an algebra extending structure and \(A \ltimes V\) the associated unified product. Then the canonical inclusion
\[
i_A : A \rightarrow A \ltimes V, \quad i_A(a) = (a, 0)
\]
is an injective algebra map. Therefore, we can see $A$ as a subalgebra of $A \times V$ through
the identification $A \cong i_A(A) = A \times \{0\}$. Conversely, we have the following result which
provides the answer to the description part of the ES-problem:

**Theorem 2.4.** Let $A$ be an algebra, $E$ a vector space containing $A$ as a subspace and
* an algebra structure on $E$ such that $A$ is a subalgebra in $(E, *)$. Then there exists an
algebra extending structure $\Omega(A, V) = (\lhd, \rhd, \rhd \rhd, f, \cdot)$ of $A$ through a subspace $V$ of
$E$ and an isomorphism of algebras $(E, \cdot) \cong A \times V$ that stabilizes $A$ and co-stabilizes $V$.

**Proof.** Since $k$ is a field, there exists a linear map $p : E \to A$ such that $p(a) = a$, for all
$a \in A$. Then $V := \text{Ker}(p)$ is a complement of $A$ in $E$. We define the extending datum
$\Omega(A, V) = (\lhd = \lhd p, \rhd = \rhd p, \rhd \rhd = \rhd \rhd p, \rhd = \rhd p, f = f p, \cdot = \cdot p)$ of $A$ through $V$ by the
following formulas:

- $\rhd : V \times A \to A, \quad x \rhd a := p(x \ast a), \quad \lhd : V \times A \to V, \quad x \lhd a := x \ast a - p(x \ast a)$
- $\lhd : A \times V \to A, \quad a \lhd x := p(a \ast x), \quad \rhd : A \times V \to V, \quad a \rhd x := a \ast x - p(a \ast x)$
- $f : V \times V \to V, \quad f(x, y) := p(x \ast y), \quad \cdot : V \times V \to V, \quad x \cdot y := x \ast y - p(x \ast y)$

for any $a \in A$ and $x, y \in V$. We shall prove that $\Omega(A, V) = (\lhd p, \rhd p, \rhd \rhd p, \cdot p)$ is
an algebra extending structure of $A$ through $V$ and

$$\varphi : A \times V \to E, \quad \varphi(a, x) := a + x$$

is an isomorphism of algebras that stabilizes $A$ and co-stabilizes $V$. Instead of proving
the compatibility conditions (A1)-(A12), which require a very long and laborious computation, we use the following trick combined with Theorem 2.2: $\varphi : A \times V \to E,$
$\varphi(a, x) = a + x$ is a linear isomorphism between the algebra $(E, \cdot)$ and the direct
product of vector spaces $A \times V$ with the inverse given by $\varphi^{-1}(x) := (p(x), y - p(y))$, for all $y \in E$.
Thus, there exists a unique algebra structure on $A \times V$ such that $\varphi$
is an isomorphism of algebras and this unique multiplication $\odot$ on $A \times V$ is given by
$(a, x) \odot (b, y) := \varphi^{-1}(\varphi(a, x) \ast \varphi(b, y))$, for all $a, b \in A$ and $x, y \in V$. The proof is finished
if we prove that this multiplication is the one defined by (11) associated to the system
$(\lhd p, \rhd p, \lhd \rhd p, \rhd p, f p, \cdot p)$. Indeed, for any $a, b \in A$ and $x, y \in V$ we have:

$$(a, x) \odot (b, y) = \varphi^{-1}(\varphi(a, x) \ast \varphi(b, y)) = \varphi^{-1}((a + x) \ast (b + y))$$

$= \varphi^{-1}(ab + a \ast y + x \ast b + x \ast y)$

$= (ab + p(a \ast y) + p(x \ast b) + p(x \ast y), \quad a \ast y - p(a \ast y) + x \ast b - p(x \ast b) + x \ast y - p(x \ast y))$

$= (ab + a \lhd y + x \rhd b + f(x, y), \quad a \rhd y + x \lhd b + x \cdot y)$

$= (a, x) \bullet (b, y)$

as needed. Moreover, the following diagram is commutative

$$\begin{array}{ccc}
A & \xrightarrow{i} & A \times V \\
\downarrow{\text{Id}} & & \downarrow{\varphi} \\
A & \xrightarrow{i} & E & \xrightarrow{\pi} & V \\
\end{array}$$
where \( \pi : E \to V \) is the projection of \( E = A + V \) on the vector space \( V \) and \( q : A \times V \to V \), 
\( q(a, x) := x \) is the canonical projection. The proof is now finished.

Based on Theorem 2.4, the classification of all algebra structures on \( E \) that contain \( A \) as a subalgebra reduces to the classification of all unified products \( A \leq V \), associated to all algebra extending structures \( \Omega(A, V) = (\langle, \triangleright, \triangleleft, f, \cdot \rangle) \), for a fixed complement \( V \) of \( A \) in \( E \). Next we will construct explicitly the non-abelian cohomological type objects \( AH^2_A(V, A) \) and \( AH^2(V, A) \) which will parameterize the classifying sets \( \text{Extd}(E, A) \) and respectively \( \text{Extd}'(E, A) \). First we need the following:

**Lemma 2.5.** Let \( \Omega(A, V) = (\langle, \triangleright, \triangleleft, f, \cdot \rangle) \) and \( \Omega(A, V) = (\langle', \triangleright', \triangleleft', f', \cdot \rangle) \) be two algebra extending structures of \( A \) through \( V \) and \( A \leq V \), respectively \( A \leq V' \) the associated unified products. Then there exists a bijection between the set of all morphisms of algebras \( \psi : A \leq V \to A \leq V' \) which stabilize \( A \) and the set of pairs \( (r, v) \), where \( r : V \to A, v : V \to V \) are linear maps satisfying the following compatibility conditions for any \( a \in A, x, y \in V \):

\[
\begin{align*}
(M1) & \quad r(x \cdot y) = r(x) r(y) + f'(v(x), v(y)) - f(x, y) + r(x) \triangleleft' v(y) + v(x) \triangleright r(y); \\
(M2) & \quad v(x \cdot y) = r(x) \triangleleft' v(y) + v(x) \triangleright' r(y) + v(x) \triangleright a; \\
(M3) & \quad r(x \triangleleft a) = r(x) a - x \triangleright a + v(x) \triangleright a; \\
(M4) & \quad v(x \triangleleft a) = v(x) \triangleleft a; \\
(M5) & \quad r(a \triangleright x) = ar(x) - a \triangleleft x + a \triangleleft v(x); \\
(M6) & \quad v(a \triangleright x) = a \triangleleft v(x)
\end{align*}
\]

Under the above bijection the morphism of algebras \( \psi = \psi_{(r, v)} : A \leq V \to A \leq V' \) corresponding to \( (r, v) \) is given for any \( a \in A \) and \( x \in V \) by:

\[
\psi(a, x) = (a + r(x), v(x))
\]

Moreover, \( \psi = \psi_{(r, v)} \) is an isomorphism if and only if \( v : V \to V \) is an isomorphism and \( \psi = \psi_{(r, v)} \) co-stabilizes \( V \) if and only if \( v = \text{Id}_V \).

**Proof.** For a linear map \( \psi : A \leq V \to A \leq V' \) which stabilizes \( A \) we have \( \psi(a, 0) = (a, 0) \) for all \( a \in A \). Therefore, \( \psi \) is uniquely determined by two linear maps \( r : V \to A, v : V \to V \) such that \( \psi(0, x) = (r(x), v(x)) \) for all \( x \in V \). Then, for all \( a \in A \) and \( x \in V \) we have \( \psi(a, x) = (a + r(x), v(x)) \). Let \( \psi = \psi_{(r, v)} \) be such a linear map. We will prove that \( \psi \) is an algebra map if and only if the compatibility conditions (M1)-(M6) hold. It is enough to prove that the following compatibility holds for all generators of \( A \leq V \):

\[
\psi((d, w) \cdot (e, t)) = \psi((d, w)) \cdot \psi((e, t)) \quad (19)
\]

By a straightforward computation it follows that (19) holds for the pair \( (a, 0), (b, 0) \) if and only if (M1) and (M2) are fulfilled while (19) holds for the pair \( (0, x), (a, 0) \) if and only if (M3) and (M4) are satisfied. Finally, (19) holds for the pair \( (a, 0), (0, x) \) if and only if (M5) and (M6) hold.

Assume now that \( v \) is bijective. Then \( \psi_{(r, v)} \) is an isomorphism of algebras with the inverse given by \( \psi_{(r, v)}^{-1}(a, x) = (a - r(v^{-1}(x)), v^{-1}(x)) \) for all \( a \in A \) and \( x \in V \). Conversely, assume that \( \psi_{(r, v)} \) is bijective. Then \( v \) is obviously surjective. Consider now \( x \in V \) such that \( v(x) = 0 \). We obtain \( \psi_{(r, v)}(0, 0) = (0, 0) = (0, v(x)) = \psi_{(r, v)}(-r(x), x) \). As \( \psi_{(r, v)} \) is
In order to construct the object that parameterizes \( \text{Ext}_d(E, A) \) we need the following:

**Definition 2.6.** Let \( A \) be an algebra and \( V \) a vector space. Two algebra extending structures of \( A \) by \( V \), \( \Omega(A, V) = (\triangleleft, \triangleright, \prec, \succ, f, \cdot) \) and \( \Omega(A, V) = (\triangleleft', \triangleright', \prec', \succ', f', \cdot') \) are called **equivalent**, and we denote this by \( \Omega(A, V) \equiv \Omega'(A, V) \), if there exists a pair \((r, v)\) of linear maps, where \( r: V \to A \) and \( v \in \text{Aut}_k(V) \) such that \((\triangleleft', \triangleright', \prec', \succ', f', \cdot')\) is implemented from \((\triangleleft, \triangleright, \prec, \succ, f, \cdot)\) using \((r, v)\):

\[
\begin{align*}
    a \to' x &= v(a \to v^{-1}(x)) \\
    a \prec' x &= r(a \to v^{-1}(x)) - ar(v^{-1}(x)) + a \prec v^{-1}(x) \\
    x \prec' a &= v(v^{-1}(x) \prec a) \\
    x \triangleright' a &= r(v^{-1}(x) \prec a) - r(v^{-1}(x))a + v^{-1}(x) \triangleright a \\
    x \cdot' y &= v(v^{-1}(x) \cdot v^{-1}(y)) - v(r(v^{-1}(x)) \to v^{-1}(y)) - v(v^{-1}(x) \prec r(v^{-1}(y))) \\
    f'(x, y) &= r(v^{-1}(x) \cdot v^{-1}(y)) + f(v^{-1}(x), v^{-1}(y)) - r(r(v^{-1}(x)) \to v^{-1}(y)) - r(v^{-1}(x)) \prec r(v^{-1}(y))) + r(v^{-1}(x))r(v^{-1}(y)) - v^{-1}(x) \triangleright r(v^{-1}(y))
\end{align*}
\]

for all \( a \in A, x, y \in V \).

On the other hand, in order to parameterize \( \text{Ext}_d'(E, A) \) we need the following:

**Definition 2.7.** Let \( A \) be an algebra and \( V \) a vector space. Two algebra extending structures \( \Omega(A, V) = (\triangleleft, \triangleright, \prec, \succ, f, \cdot) \) and \( \Omega(A, V) = (\triangleleft', \triangleright', \prec', \succ', f', \cdot') \) are called **cohomologous**, and we denote this by \( \Omega(A, V) \approx \Omega'(A, V) \) if and only if \( \triangleleft' = \triangleleft, \triangleright' = \triangleright, f', \cdot' \) and there exists a linear map \( r: V \to A \) such that

\[
\begin{align*}
    a \prec' x &= r(a \to x) - ar(x) + a \prec x \\
    x \triangleright' a &= r(x \prec a) - r(x)a + x \triangleright a \\
    x \cdot' y &= x \cdot y - r(x) \to y - x \prec r(y) \\
    f'(x, y) &= r(x \cdot y) + f(x, y) - r(r(x) \to y) - r(x) \prec y - r(x \prec r(y)) + r(x)r(y) - x \triangleright r(y)
\end{align*}
\]

for all \( a \in A, x, y \in V \).

As a conclusion of this section, the answer to the ES-problem now follows:

**Theorem 2.8.** Let \( A \) be an algebra, \( E \) a vector space which contains \( A \) as a subspace and \( V \) a complement of \( A \) in \( E \). Then:
(1) \(\equiv\) is an equivalence relation on the set \(\mathcal{AE}(A,V)\) of all algebra extending structures of \(A\) through \(V\). If we denote by \(\mathcal{AH}^2(A,V) := \mathcal{AE}(A,V)/\equiv\), then the map
\[
\mathcal{AH}^2(A,V) \rightarrow \text{Ext}(E,A), \quad (\lhd, \rhd, \leftarrow, f, \cdot) \mapsto (A \ltimes V, \cdot)
\]
is a bijection between \(\mathcal{AH}^2(A,V)\) and the isomorphism classes of all algebra structures on \(E\) that contain and stabilize \(A\) as a subalgebra.

(2) \(\approx\) is an equivalence relation on the set \(\mathcal{AE}(A,V)\). If we denote by \(\mathcal{AH}^2(A,V) := \mathcal{AE}(A,V)/\approx\), then the map
\[
\mathcal{AH}^2(A,V) \rightarrow \text{Ext}'(E,A), \quad (\lhd, \rhd, \leftarrow, f, \cdot) \mapsto (A \ltimes V, \cdot)
\]
is a bijection between \(\mathcal{AH}^2(A,V)\) and the isomorphism classes of all algebra structures on \(E\) which stabilize \(A\) and co-stabilize \(V\).

Proof. The proof follows from Theorem 2.2, Theorem 2.4 and Lemma 2.5 once we observe that \(\Omega(g,V) \equiv \Omega'(g,V)\) in the sense of Definition 2.6 if and only if there exists an isomorphism of algebras \(\psi : A \ltimes V \rightarrow A \ltimes' V\) which stabilizes \(A\). Therefore, \(\equiv\) is an equivalence relation on the set \(\mathcal{AE}(A,V)\) and the first part follows. In the same way \(\Omega(g,V) \approx \Omega'(g,V)\) as defined in Definition 2.7 if and only if there exists an isomorphism of algebras \(\psi : A \ltimes V \rightarrow A \ltimes' V\) which stabilizes \(A\) and co-stabilizes \(V\) and this proves the second part of the theorem.

\[
\square
\]

3. Special cases of unified products

In this section we deal with several special cases of unified products. We emphasize the problem for which each of these products is responsible. We use the following convention: if one of the maps \(\rhd, \leftarrow, f\) or \(\cdot\) of an extending datum is trivial then we will omit it from the system \(\Omega(A,V) = (\lhd, \rhd, \leftarrow, f, \cdot)\).

Relative split extensions and cocycle semidirect products of algebras. We will prove that several special cases of the unified product are responsible for the description of algebra extensions \(A \subset E\) which split as morphisms of left/right \(A\)-modules, \(A\)-bimodule or as algebra maps. If \(A \subset E\) is an inclusion of algebras, then \(E\) will be viewed as a left/right \(A\)-module via the restriction of scalars: \(a \cdot x \cdot a' := axa'\), for all \(a, a' \in A\) and \(x \in E\). If \((V, \rightarrow, \lhd) \in \mathcal{AM}_A\) is an \(A\)-bimodule, then \(V \times V\) is viewed as an \(A\)-bimodule in the canonical way, i.e. the left (resp. right) action of \(A\) on \(V \times V\) is implemented by \(\rightarrow\) (resp. \(\lhd\)). A bilinear map \(f : V \times V \rightarrow A\) is called \(A\)-balanced if \(f(x, a \rightarrow y) = f(x \lhd a, y)\), for all \(a \in A\), \(x, y \in V\). Of course, \(A\)-bimodules and \(A\)-balanced maps \(f : V \times V \rightarrow A\) are in bijection to the set of all \(A\)-bimodule maps \(\tilde{f} : V \otimes_A V \rightarrow A\), where \(\otimes_A\) is the tensor product over \(A\).

First, we shall describe extensions of algebras \(A \subset E\) that split in \(\mathcal{AM}(A)\) (resp. \(\mathcal{AM}_A\)), i.e. there exists a left (resp. right) \(A\)-module map \(p : E \rightarrow A\) such that \(p(a) = a\), for all \(a \in A\).

\(1\) \((\lhd, \rhd, \leftarrow, f, \cdot)\) (resp. \((\lhd, \rhd, \leftarrow, f, \cdot)\)) denotes the equivalence class of \((\lhd, \rhd, \leftarrow, f, \cdot)\) via \(\equiv\) (resp. \(\approx\)).
Corollary 3.1. An extension of algebras $A \subset E$ has a retraction that is a left (resp. right) $A$-module map if and only if there exists an isomorphism of algebras $E \cong A \ltimes V$, where $A \ltimes V$ is the unified product associated to an algebra extending structure $\Omega(A, V) \in \mathcal{AE}(A, V)$ having $\leftarrow: A \times V \to A$ (resp. $\triangleright: V \times A \to A$) the trivial map.

Proof. Let $A \ltimes V$ be a unified product associated to $\Omega(A, V) \in \mathcal{AE}(A, V)$, for which $\leftarrow$ is the trivial map. Then the canonical projection $p_A : A \ltimes V \to A$ is a retraction of the inclusion $i_A : A \to A \ltimes V$ that is also left $A$-linear since, using (12), we have:

$$p_A(a \cdot (b, x)) = p_A((a, 0) \cdot (b, x)) = p_A(ab, a \mapsto x) = ab = ap_A((b, x))$$

for all $a, b \in A$, $x \in V$. Conversely, let $A \subset E$ be an inclusion of algebras which has a retraction $p : E \to A$ that is also a left $A$-module map. It follows from the proof of Theorem 2.4 that the action $\leftarrow = \leftarrow_p$ associated to the retraction $p$ is the trivial map since $a \leftarrow x = p(ax) = ap(x) = 0$, for all $a \in A$ and $x \in V = \text{Ker}(p)$. Thus, there exists an isomorphism of algebras $E \cong A \ltimes V$, where $A \ltimes V$ is the unified product associated to $\Omega(A, V) \in \mathcal{AE}(A, V)$, for which $\leftarrow$ is the trivial map.

Analogously we can prove that the algebra extensions $A \subset E$ that split as right $A$-module maps are parameterized by the unified products $A \ltimes V$ associated to algebra extending structures $\Omega(A, V) \in \mathcal{AE}(A, V)$ for which the action $\triangleright : V \times A \to A$ is the trivial map. \qed

Examples 3.2. 1. The basic example of an algebra extension which splits as in Corollary 3.1 is a group algebras extension. Let $H \leq G$ be a subgroup of a group $G$. Then the group algebras extension $k[H] \subset k[G]$ has a retraction which is a left $k[H]$-module map [1, Theorem 2.1]. Hence, there exists an isomorphism of algebras $k[G] \cong k[H] \ltimes V$, for some algebra extending structure $\Omega(k[H], V) \in \mathcal{AE}(k[H], V)$ having $\leftarrow : k[H] \times V \to A$ the trivial map.

2. The second class of split extensions in the sense of Corollary 3.1 are the classical crossed products of algebras [28]. Let $A$ be an algebra, $G$ be a group and $\alpha : G \to \text{Aut}(A)$, $f : G \times G \to U(A)$ be two maps. We shall denote by $g \triangleright a := \alpha(g)(a)$, for all $g \in G$ and $a \in A$. Let $\overline{G}$ be a copy as a set of the group $G$ and $A_{\alpha}^f[G]$ be the free left $A$-module having $\overline{G}$ as an $A$-basis with the multiplication given by:

$$(a \overline{g})(b \overline{h}) := a(g \triangleright b)f(g, h)\overline{gh}$$

for all $a, b \in A$ and $g, h \in G$. $A_{\alpha}^f[G]$ is called the crossed product of $A$ and $G$ if it is an associative algebra with the unit $1_{A\overline{1}G}$. This is equivalent ([25], [28]) to the fact that $f(1_G, 1_G) = 1_A$ and the following compatibilities hold for any $g, h, l \in G$ and $a \in A$:

$$g \triangleright (h \triangleright a) = f(g, h)((gh) \triangleright a)f(g, h)^{-1}, \quad f(g, h)f(gh, l) = (g \triangleright f(h, l))f(g, hl);$$

Any crossed product $A_{\alpha}^f[G]$ is an extension of $A$ via the canonical map $i_A : A \to A_{\alpha}^f[G]$, $i_A(a) := a\overline{1}_G$. This extension splits in the category of left $A$-modules: the left $A$-linear map that splits $i_A$ being the augmentation map $\pi_A : A_{\alpha}^f[G] \to A$, $\pi_A(a\overline{g}) := a$, for all $a \in A$ and $g \in G$. Thus, using Corollary 3.1 we obtain that any crossed product $A_{\alpha}^f[G]$ is isomorphic to a unified product $A \ltimes V$ associated to an algebra extending structure $\Omega(A, V) \in \mathcal{AE}(A, V)$ for which the action $\leftarrow : A \times V \to A$ is the trivial map.
3. Let \( A \) be an algebra and \((W, 1_W)\) a pointed vector space. All algebra structures \( \cdot \) on the vector space \( A \otimes W \) such that \((a \otimes 1_W) \cdot (b \otimes w) = ab \otimes w \), for all \( a, b \in A \) and \( w \in W \) and having \( 1_A \otimes 1_W \) as a unit are fully described in [7, Proposition 2.1]: they are parameterized by the set of all pairs \((\sigma, R)\) consisting of two linear maps \( \sigma : W \otimes W \to A \otimes W \), \( R : W \otimes A \to A \otimes W \), satisfying a laborious set of axioms. Such an algebra structure, which is a very general construction, is denoted by \( A \otimes_R \sigma W \) and is called the Brzezinski’s product; they are classified, up to an isomorphism of algebras that stabilizes \( A \), in [27, Theorem 2.3]. Now, \( i_A : A \to A \otimes_R \sigma W \), \( i_A(a) = a \otimes 1_W \) is an injective algebra map which has a retraction that is a left \( A \)-module map. Indeed, let \( B = \{ e_i | i \in I \} \) be a basis in \( W \) such that \( 1_W \in B \) and \( \varepsilon : W \to k \), \( \varepsilon (e_i) = 1 \), for all \( i \in I \). Then \( \pi_A : A \otimes_R \sigma W \to A \), \( \pi_A := \text{Id}_A \otimes \varepsilon \) is a left \( A \)-linear map and a retraction of \( i_A \). Thus,

\[
A \otimes_R \sigma W \cong A \times V
\]

for an algebra extending structure \( \Omega(A, V) \in \mathcal{AE}(A, V) \) for which the action \( \leftarrow : A \times V \to A \) is the trivial map.

4. The Ore extensions are also a special case of the unified product; in particular any Weyl algebra is a unified product. Indeed, let \( \sigma : A \to A \) be an automorphism of the algebra \( A \), \( \delta : A \to A \) a \( \sigma \)-derivation and \( A[X, \sigma, \delta] \) the Ore extension associated to \((\sigma, \delta)\), that is \( A[X, \sigma, \delta] \) is the free left \( A \)-module having \( \{ X^n | n \geq 0 \} \) as a basis and the multiplication given by \( Xa = \sigma(a)X + \delta(a) \), for all \( a \in A \). Then, the canonical embedding \( i_A : A \to A[X, \sigma, \delta] \), \( i_A(a) = a \), has a retraction \( p_A : A[X, \sigma, \delta] \to A \) that is a left \( A \)-module map given by \( p_A(\sum_{i=0}^n a_i X^i) := a_0 \). Thus, there exists an isomorphism of algebras \( A[X, \sigma, \delta] \cong A \times V \), for an algebra extending structure \( \Omega(A, V) \in \mathcal{AE}(A, V) \) having \( \leftarrow : A \times V \to A \) the trivial map.

Using Corollary 3.1 we can describe the algebra extensions \( A \subset E \) that admit a retraction \( p : E \to A \) which is an \( A \)-bimodule map. In this case the axioms (A1)- (A12) which describe the corresponding unified products simplify considerably. Indeed, let \( \Omega(A, V) = (\triangleleft, \triangleright, \leftarrow, \rightarrow, f, \cdot) \) be an extending datum such that \( \leftarrow \) and \( \triangleright \) are both the trivial maps. Then \( \Omega(A, V) = (\triangleleft, \rightarrow, f, \cdot) \) is an algebra extending structure of \( A \) through \( V \) if and only if \((V, \rightarrow, \triangleleft) \in \mathcal{AM}_A \) is an \( A \)-bimodule, \( f : V \times V \to A \) is an \( A \)-balanced \( A \)-bimodule map and the following compatibilities hold for any \( x, y, z \in V \):

\[
\begin{align*}
& x \cdot (y \cdot z) - (x \cdot y) \cdot z = f(x, y) \cdot z - x \cdot f(y, z) \\
& a \rightarrow (x \cdot y) = (a \rightarrow x) \cdot y, \quad (x \cdot y) \triangleleft a = x \cdot (y \triangleleft a) \\
& x \cdot (a \rightarrow y) = (x \triangleleft a) \cdot y, \quad f(x, y \cdot z) = f(x \cdot y, z)
\end{align*}
\]

A system \( \Omega(A, V) = (\triangleleft, \rightarrow, f, \cdot) \) satisfying these compatibilities will be called a cocycle semidirect system of algebras. The unified product associated to a cocycle semidirect system \( \Omega(A, V) = (\triangleleft, \rightarrow, f, \cdot) \) will be denoted by \( A \#^\Delta V \) and will be called the cocycle semidirect product of algebras. Explicitly, \( A \#^\Delta V = A \times V \) (as vector spaces) with the multiplication given by:

\[
(a, x) \bullet (b, y) := (ab + f(x, y), a \rightarrow y + x \triangleleft b + x \cdot y)
\]

for all \( a, b \in A \) and \( x, y \in V \). Corollary 3.1 provides the following result:
Corollary 3.3. An extension of algebras \( A \subset E \) has a retraction that is an \( A \)-bimodule map if and only if there exists an isomorphism of algebras \( E \cong A \#^f V \), where \( A \#^f V \) is a cocycle semidirect product of algebras.

Example 3.4. Examples of algebras that split in the sense of Corollary 3.3 are the twisted products of algebras. A twisted product is a crossed product \( A_f \square G \) as defined in Example 3.2 for which the action \( \alpha \) is the trivial action, that is \( g \triangleright a = a \), for all \( g \in G \) and \( a \in A \). In this case the augmentation map \( \pi_A \) is also a right \( A \)-module map and thus any twisted product \( A_f \square G \) is isomorphic to a cocycle semidirect product of algebras.

A cocycle semidirect system of algebras \( \Omega(A, V) = (\triangleright, \leftarrow, f, \cdot) \) for which \( f \) is the trivial map is called a semidirect system of algebras. Explicitly, \( \Omega(A, V) = (\triangleright, \leftarrow, \cdot) \) is a semidirect system of algebras if and only if \( (V, \leftarrow, \triangleright) \in \mathcal{A} \mathcal{M}_A \) is an \( A \)-bimodule, \( (V, \cdot) \) is an associative (not-necessarily unitary) algebra and

\[
\begin{align*}
\alpha(x \cdot y) &= (\alpha \triangleright x) \cdot y, & (x \cdot y) \triangleleft a &= x \cdot (y \triangleleft a), & x \cdot (a \leftarrow y) &= (x \triangleleft a) \cdot y
\end{align*}
\]

for all \( a \in A, x, y \in V \). The cocycle semidirect product of algebras associated to a semidirect system \( \Omega(A, V) = (\triangleright, \leftarrow, \cdot) \) is called the semidirect product of algebras and will be denoted by \( A \# V \). This is a classical construction: it appears in an equivalent form in [31, Lemma a, pg. 212]; moreover, the semidirect product is simultaneously a special case of the unified product as well as of the Hochschild product. We call it the semidirect product by analogy with the group and Lie algebra case where the semidirect product describes the split extensions. More precisely, we have:

Corollary 3.5. An extension of algebras \( A \subset E \) has a retraction that is an algebra map if and only if there exists an isomorphism of algebras \( E \cong A \# V \), where \( A \# V \) is a semidirect product of algebras.

Proof. Indeed, the canonical projection \( p_A : A \# V \to A, p_A(a, x) = a \) is a retraction of the inclusion \( i_A : A \to A \# V \) and an algebra map. Conversely, from Theorem 2.4 it follows that if \( p : E \to A \) is an algebra map then \( \triangleright_p, \leftarrow_p \) and \( f_p \) constructed in the proof are all the trivial maps, i.e. the corresponding unified product \( A \times V \) is a semidirect product \( A \# V \). □

Matched pairs, bicrossed products and the factorization problem for algebras.

The concept of a matched pair of groups was introduced in [35] while the one for Lie algebras in [22, Theorem 4.1] and independently in [21, Theorem 3.9]. To any matched pair of groups (resp. Lie algebras) a new group (resp. Lie algebra) called the bicrossed product is associated and it is responsible for the so-called factorization problem. Since then, the corresponding concepts were introduced for several types of categories such as groupoids, Hopf algebras, local compact quantum groups, etc. - we refer to [2] for details and references. In what follows we will introduce the corresponding notion for associative algebras. First, we set the terminology. If \( (V, \cdot) \) is an associative (not-necessarily unitary) algebra, then the concept of left/right \( V \)-module or \( V \)-bimodule is defined as in the case of unitary algebras except of course for the unitary condition.
Definition 3.6. A matched pair of algebras is a system \((A, V, \triangleleft, \triangleright, \leftarrow, \rightarrow)\) consisting of an unitary algebra \(A\), a (not-necessarily unitary) associative algebra \(V = (V, \cdot)\) and four bilinear maps

\[
\triangleleft : V \times A \rightarrow V, \quad \triangleright : V \times A \rightarrow A, \quad \leftarrow : A \times V \rightarrow A, \quad \rightarrow : A \times V \rightarrow V
\]

such that \((V, \rightarrow, \triangleleft) \in A\mathcal{M}_A\) is an \(A\)-bimodule, \((A, \triangleright, \leftarrow) \in V\mathcal{M}_V\) is a \(V\)-bimodule and the following compatibilities hold for any \(a, b \in A, x, y \in V\):

(MP1) \(a \rightarrow (x \cdot y) = (a \rightarrow x) \cdot y + (a \leftarrow x) \rightarrow y\);
(MP2) \((ab) \leftarrow x = a(b \leftarrow x) + a \leftarrow (b \rightarrow x)\);
(MP3) \(x \triangleright (ab) = (x \triangleright a)b + (x \triangleleft a) \triangleright b\);
(MP4) \((x \cdot y) \triangleleft a = x \triangleleft (y \triangleright a) + x \cdot (y \triangleleft a)\);
(MP5) \(a(x \triangleright b) + a \leftarrow (x \triangleleft b) = (a \leftarrow x)b + (a \rightarrow x) \triangleright b\);
(MP6) \(x \triangleleft (a \leftarrow y) + x \cdot (a \rightarrow y) = (x \triangleright a) \rightarrow y + (x \triangleleft a) \cdot y\);

We make a few comments on these compatibilities. If we apply (MP2) and (MP3) for \(a = b = 1_A\), we obtain that \(1_A \leftarrow x = 0\) and \(x \triangleright 1_A = 0\), for all \(x \in V\). The two compatibility conditions together with the unitary condition derived from the fact that \((V, \rightarrow, \triangleleft)\) is an \(A\)-bimodule show that the system \((\triangleleft, \triangleright, \leftarrow, \rightarrow)\) is normalized in the sense of Definition 2.1. Similarly, let \(\Omega(A, V) = (\triangleleft, \triangleright, \leftarrow, \rightarrow, f, \cdot)\) be an extending datum such that \(f\) is the trivial map. Then \(\Omega(A, V) = (\triangleleft, \triangleright, \leftarrow, \rightarrow, \cdot)\) is an algebra extending structure of \(A\) through \(V\) if and only if \((A, (V, \cdot), \triangleleft, \triangleright, \leftarrow, \rightarrow)\) is a matched pair of algebras. In this case, the associated unified product \(A \times V\) will be denoted, as in the case of groups, Lie algebras, Hopf algebras, etc. by \(A \bowtie V\) and will be called the bicrossed product associated to the matched pair \((A, V, \triangleleft, \triangleright, \leftarrow, \rightarrow)\). Thus, \(A \bowtie V = A \times V\), as a vector space, with an unitary associative algebra structure given by

\[
(a, x) \bullet (b, y) := (ab + a \leftarrow y + x \triangleright b, a \rightarrow y + x \triangleleft b + x \cdot y)
\]

for all \(a, b \in A\) and \(x, y \in V\). The bicrossed product of two algebras is the construction responsible for the so-called factorization problem, which in the case of associative algebras comes down to:

Let \(A\) be a unitary algebra and \(V\) a (not-necessarily unitary) associative algebra. Describe and classify all unitary algebras \(E\) that factorize through \(A\) and \(V\), i.e. \(E\) contains \(A\) and \(V\) as subalgebras such that \(E = A + V\) and \(A \cap V = \{0\}\).

Indeed, as a special case of Theorem 2.4 we have:

Corollary 3.7. Let \(A\) be an unitary algebra and \(V\) a (not-necessarily unitary) associative algebra. Then, an algebra \(E\) factorizes through \(A\) and \(V\) if and only if there exists a matched pair of algebras \((A, V, \triangleleft, \triangleright, \leftarrow, \rightarrow)\) such that \(E \cong A \bowtie V\).

Proof. To start with, it is easy to see that any bicrossed product \(A \bowtie V\) factorizes through \(A \cong A \times \{0\}\) and \(V \cong \{0\} \times V\), which are subalgebras in \(A \bowtie V\). Conversely, assume that \(E\) factorizes through \(A\) and \(V\). Let \(p : E \rightarrow A\) be the \(k\)-linear projection of \(E\) on \(A\), i.e. \(p(a + x) := a\), for all \(a \in A\) and \(x \in V\). Now, we apply Theorem 2.4
for \( V = \text{Ker}(p) \). Since \( V \) is a subalgebra of \( E \), the map \( f = f_p \) constructed in the proof of Theorem 2.4 is the trivial map as \( x \cdot y \in V = \text{Ker}(p) \). Thus, the algebra extending structure \( \Omega(A, V) \) constructed in the proof of Theorem 2.4 is precisely a matched pair of algebras and the unified product \( A \ltimes V = A \bowtie V \) is the bicrossed product of the matched pair \( (A, V, \ltimes, \bowtie) \). \( \square \)

Corollary 3.7 shows that the factorization problem for algebras can be restated in a purely computational manner as follows: Let \( A \) and \( V \) be two given algebras. Describe the set of all matched pairs of algebras \( (A, V, \ltimes, \bowtie, \leftarrow, \rightarrow) \) and classify up to an isomorphism all bicrossed products \( A \bowtie V \). A detailed study of this question will be given elsewhere.

The commutative case. The case of commutative algebras needs to be treated distinctly. On the one hand, we obtain from (31) and (32) that a unified product \( A \ltimes V \) is a commutative algebra if and only if \( A \) is commutative, \( f : V \times V \to A, \cdot : V \times V \to V \) are symmetric bilinear maps, \( a \leftarrow x = x \triangleright a \) and \( a \rightarrow x = x \triangleleft a \), for all \( a \in A \) and \( x \in V \). On the other hand, if we look at the construction of the algebra extending structure from Theorem 2.4 in the case when \( A \subseteq E \) is an extension of commutative algebras, we also obtain that \( a \leftarrow_p x = x \triangleright_p a \) and \( a \rightarrow_p x = x \triangleleft_p a \) for all \( a \in A \) and \( x \in V \). Thus, in the commutative case Definition 2.1 takes the following form:

**Definition 3.8.** Let \( A \) be a commutative algebra and \( V \) a vector space. A **commutative extending datum of \( A \) through \( V \)** is a system \( \Omega(A, V) = (\triangleleft, \triangleright, f, \cdot) \) consisting of four bilinear maps

\[
\triangleleft : V \times A \to V, \quad \triangleright : V \times A \to A, \quad f : V \times V \to A, \quad \cdot : V \times V \to V
\]

such that \( f \) and \( \cdot \) are symmetric. Let \( \Omega(A, V) = (\triangleleft, \triangleright, f, \cdot) \) be a commutative extending datum. Then the multiplication on \( A \ltimes V = A \times V \) given by (11) takes the form:

\[
(a, x) \bullet (b, y) := (ab + y \triangleright a + x \triangleright b + f(x, y), \ y \triangleleft a + x \triangleleft b + x \cdot y)
\]  

(23)

for all \( a, b \in A \) and \( x, y \in V \). \( A \ltimes V \) is a commutative unified product if it is a commutative associative algebra with the multiplication given by (23) and the unit \( (1_A, 0_V) \). In this case the extending datum \( \Omega(A, V) = (\triangleleft, \triangleright, f, \cdot) \) is called a **commutative algebra extending structure of \( A \) through \( V \)**.

In other words, a commutative algebra extending structure of a commutative algebra \( A \) through a vector space \( V \) is a commutative extending datum \( \Omega(A, V) = (\triangleleft, \triangleright, f, \cdot) \) satisfying the axioms (A1)-(A12) of Theorem 2.2 in which we replace \( a \leftarrow x := x \triangleright a \) and \( a \rightarrow x := x \triangleleft a \) for all \( a \in A \), \( x \in V \). That is, the following compatibility conditions hold for any \( a, b \in A \), \( x, y, z \in V \):

1. **(CA1)** \((V, \triangleleft)\) is an \( A \)-module and \( x \triangleright 1_A = 0 \);
2. **(CA2)** \( x \cdot (y \cdot z) = (x \cdot y) \cdot z = z \triangleleft f(x, y) = x \triangleleft f(y, z) \);
3. **(CA3)** \( (x \cdot y) \triangleleft a = (x \triangleleft a) \cdot y + y \triangleleft (x \triangleright a) \);
4. **(CA4)** \( x \triangleright (ab) = a(x \triangleright b) + (x \triangleleft b) \triangleright a \);
5. **(CA5)** \( (x \cdot y) \triangleright a = x \triangleright (y \triangleright a) + f(x, y \triangleleft a) - f(x, y)a \);
6. **(CA6)** \( f(x, y \cdot z) = f(x, y \cdot z) = f(x, y) - x \triangleright f(y, z) \).
The above axioms are derived from those of Theorem 2.2 by taking into account that $A$ is commutative, $f$ and $\cdot$ are symmetric bilinear maps and $a \leftarrow x = x \triangleright a$ and $a \rightarrow x = x \triangleleft a$ for all $a \in A$ and $x \in V$. Indeed, as $a \leftarrow x = x \triangleright a$ and $a \rightarrow x = x \triangleleft a$, the normalizing conditions come down to $x \triangleleft 1_A = 0$ while $(A1)$ reduces to $(V, \triangleleft)$ being an $A$-module. Moreover, $(A4)$ follows from $(A7)$ by using $(x \cdot y) \triangleleft a = (y \cdot x) \triangleleft a$. In the same manner we can derive $(A8)$ out of $(A3)$ by having in mind that $(x \cdot y) \triangleright a = (y \cdot x) \triangleright a$. $(A6)$ and $(A9)$ can be derived from $(A5)$ by using the commutativity of $A$, more precisely $x \triangleleft ab = x \triangleleft ba$. Finally, $(A10)$ comes out of $(A4)$ by having in mind that $(x \cdot y) \triangleright a = (y \cdot x) \triangleright a$ while $(A11)$ follows from $(A3)$ by using $(x \cdot y) \triangleleft a = (y \cdot x) \triangleleft a$. Therefore, we are left with the independent set of 6 axioms listed above.

4. Examples. Classifying flag algebras

The challenge that remains after the theoretical answer of the ES-problem given in Theorem 2.8 is a purely computational one: for a given algebra $A$ that is a subspace in a vector space $E$ with a given complement $V$, we have to compute the classifying object $\mathcal{A}\mathcal{H}^2_A(V, A)$ and then to list all algebra structures on $E$ which extend the one of $A$. For the sake of completeness, we can also compute the space $\mathcal{A}\mathcal{H}^2(V, A)$. In what follows we provide a way of answering this problem for a large class of such structures.

**Definition 4.1.** Let $A$ be an algebra and $E$ a vector space containing $A$ as a subspace. An algebra structure on $E$ is called a flag extending structure of $A$ to $E$ if there exists a finite chain of subalgebras of $E$

$$E_0 := A \subset E_1 \subset \cdots \subset E_m := E$$

such that $E_i$ has codimension 1 in $E_{i+1}$, for all $i = 0, \cdots, m - 1$. An algebra $E$ that is a flag extending structure of $k$ will be called a flag algebra.

In the context of Definition 4.1 we have that $\dim_k(V) = m$, where $V$ is the complement of $A$ in $E$. All flag extending structures of $A$ to $E$ can be completely described by a recursive reasoning where the key step is $m = 1$. This step describes and classifies all unified products $A \ltimes V_1$, for a 1-dimensional vector space $V_1$. Then, by replacing the initial algebra $A$ with such a unified product $A \ltimes V_1$, which will be explicitly described in terms of $A$ only, we can iterate the process: after $m$ steps, we obtain the description of all flag extending structures of $A$ to $E$. A special case of interest for the classification of finite dimensional algebras is the case when $A = k$, i.e. to classify all $m$-dimensional flag algebras. First we need to introduce the following concept which plays the key role in the classification of flag extending structures:

**Definition 4.2.** Let $A$ be an algebra. A flag datum of $A$ is a 6-tuple $(\Lambda, \lambda, D, d, a_0, u)$, where $\Lambda, \lambda : A \rightarrow k$ are morphisms of algebras, $D, d : A \rightarrow A$ are linear maps, $a_0 \in A$, $u \in k$. 
u \in k$ satisfying the following compatibilities:

\begin{align}
\Lambda(a_0) &= \lambda(a_0), \quad D(a_0) = d(a_0), \quad \lambda \circ d = 0, \quad \Lambda \circ D = 0 \quad (25) \\
d(ab) &= a d(b) + d(a) \lambda(b), \quad D(ab) = \Lambda(a) D(b) + D(a) b \quad (26) \\
d^2(a) &= u d(a) + a a_0 - \lambda(a) a_0, \quad D^2(a) = u D(a) + a_0 a - \Lambda(a) a_0 \quad (27) \\
D(d(a)) - d(D(a)) &= (\Lambda(a) - \lambda(a)) a_0 \quad (28) \\
\Lambda(d(a)) - \lambda(D(a)) &= (\Lambda(a) - \lambda(a)) u \quad (29) \\
a D(b) + \Lambda(b) d(a) &= d(a) b + \lambda(a) D(b) \quad (30)
\end{align}

for all $a, b \in A$. We denote by $\mathcal{F}(A) \subseteq \text{Alg}(A,k)^2 \times \text{Hom}_k(A,A)^2 \times A \times k$ the set of all flag datums of $A$.

$\mathcal{F}(A)$ can be the empty set: for instance if the algebra $A$ has no characters, like in the case of the matrix algebra $A = M_n(k)$, for $n \geq 2$. The compatibilities (26) show that $d$ and $D$ are twisted derivations of the algebra $A$. Applying these compatibilities for $a = b = 1_A$ we obtain that $D(1_A) = d(1_A) = 0$, for any $(\Lambda, \lambda, D, d, a_0, u) \in \mathcal{F}(A)$.

**Proposition 4.3.** Let $A$ be an algebra and $V$ a vector space of dimension 1 with a basis $\{x\}$. Then there exists a bijection between the set $\mathcal{AE}(A,V)$ of all algebra extending structures of $A$ through $V$ and the set $\mathcal{F}(A)$ of all flag datums of $A$.

Through the above bijection, the unified product corresponding to $(\Lambda, \lambda, D, d, a_0, u) \in \mathcal{F}(A)$ will be denoted by $A \times (\Lambda, \lambda, D, d, a_0, u)$ and has the multiplication given for any $a, b \in A$ by:

\begin{align}
(a, 0) \cdot (b, 0) &= (ab, 0), \quad (0, x) \cdot (0, x) = (a_0, u x) \quad (31) \\
(a, 0) \cdot (0, x) &= (d(a), \lambda(a) x), \quad (0, x) \cdot (a, 0) = (D(a), \Lambda(a)x) \quad (32)
\end{align}

i.e., $A \times (\Lambda, \lambda, D, d, a_0, u)$ is the algebra generated by the algebra $A$ and $x$ subject to the relations:

\begin{align}
x^2 &= a_0 + u x, \quad ax = d(a) + \lambda(a) x, \quad xa = D(a) + \Lambda(a) x \quad (33)
\end{align}

for all $a \in A$.

**Proof.** We have to compute the set of all bilinear maps $\triangleleft : V \times A \to V, \triangleright : V \times A \to A, \leftarrow : A \times V \to A, \rightarrow : A \times V \to V, f : V \times V \to A$ and $\cdot : V \times V \to V$ satisfying the compatibility conditions (A1)-(A12) of Theorem 2.2. Since $V$ has dimension 1 there exists a bijection between the set of all extending datums of $A$ through $V$ and the set of all 6-tuples $(\Lambda, \lambda, D, d, a_0, u)$ consisting of four linear maps $\Lambda, \lambda : A \to k, D, d : A \to A$ and two elements $a_0 \in A$ and $u \in k$. The bijection is given such that the extending datum $\Omega(A,V) = (\triangleleft, \triangleright, \leftarrow, \rightarrow, f, \cdot)$ corresponding to $(\Lambda, \lambda, D, d, a_0, u)$ is given by:

\begin{align}
x \triangleleft a &= \Lambda(a)x, \quad x \triangleright a := D(a), \quad a \leftarrow := d(a), \quad a \rightarrow x := \lambda(a)x \\
f(x, x) &= a_0, \quad x \cdot x := u x
\end{align}

for all $a \in A$. Now, by a straightforward computation one can see that the axioms (A1)-(A12) of Theorem 2.2 are equivalent to the fact that $\Lambda, \lambda : A \to k$ are algebra maps and the compatibility conditions (25)-(30) hold. For instance, the fact that $(V, \rightarrow, \triangleleft)$ is an $A$-bimodule is equivalent to the fact that $\lambda$ and $\Lambda : A \to k$ are algebra maps. The
axiom (A2) holds if and only if $\Lambda(a_0) = \lambda(a_0)$, while the axiom (A4) is equivalent to $\lambda(d(a)) = 0$, for all $a \in A$. The remaining details are left to the reader. \qed

Proposition 4.3 provides an explicit description of all algebras which contain $A$ as a subalgebra of codimension 1: they are isomorphic to an algebra defined by (33), for some flag datum $(\Lambda, \lambda, D, d, a_0, u)$ of $A$. The existence of this type of algebras depends essentially on the algebra $A$. Next we will classify this type of algebras by providing the first explicit classification result of the ES-problem:

**Theorem 4.4.** Let $A$ be an algebra of codimension 1 in the vector space $E$. Then:

1. $\text{Ext}_{d}(E, A) \cong \mathcal{AH}_A^2(k, A) \cong \mathcal{F}(A)/\equiv$, where $\equiv$ is the equivalence relation on the set $\mathcal{F}(A)$ defined as follows: $(\Lambda, \lambda, D, d, a_0, u) \equiv (\Lambda', \lambda', D', d', a'_0, u')$ if and only if $\Lambda = \Lambda'$, $\lambda = \lambda'$ and there exists a pair $(q, \alpha) \in k^* \times A$ such that:

   \[
   D(a) = qD'(a) + \alpha a - \Lambda(a) \alpha \quad (34)
   \]

   \[
   d(a) = qd'(a) + a \alpha - \lambda(a) \alpha \quad (35)
   \]

   \[
   a_0 = q^2a'_0 + \alpha^2 - u \alpha + qd'(\alpha) + qD'(\alpha) \quad (36)
   \]

   \[
   u = qu' + \lambda'(\alpha) + \Lambda'(\alpha) \quad (37)
   \]

   for all $a \in A$. The bijection between $\mathcal{F}(A)/\equiv$ and $\text{Ext}_{d}(E, A)$ is given by:

   $$\frac{(\Lambda, \lambda, D, d, a_0, u)}{\equiv} \mapsto A_{\sim}(\Lambda, \lambda, D, d, a_0, u) x$$

   where $\frac{(\Lambda, \lambda, D, d, a_0, u)}{\equiv}$ is the equivalence class of $(\Lambda, \lambda, D, d, a_0, u)$ via the relation $\equiv$ and $A_{\sim}(\Lambda, \lambda, D, d, a_0, u) x$ is the algebra defined by (33).

2. $\text{Ext'}_{d}(E, A) \cong \mathcal{AH}_A^2(k, A) \cong \mathcal{F}(A)/\approx$, where $\approx$ is the following relation on the set $\mathcal{F}(A)$: $(\Lambda, \lambda, D, d, a_0, u) \approx (\Lambda', \lambda', D', d', a'_0, u')$ if and only if $\Lambda = \Lambda'$, $\lambda = \lambda'$ and there exists $\alpha \in A$ such that (34)-(37) are fulfilled for $q = 1$. The bijection between $\mathcal{F}(A)/\approx$ and $\text{Ext'}_{d}(E, A)$ is given by:

   $$\frac{(\Lambda, \lambda, D, d, a_0, u)}{\approx} \mapsto A_{\sim}(\Lambda, \lambda, D, d, a_0, u) x$$

   where $\frac{(\Lambda, \lambda, D, d, a_0, u)}{\approx}$ is the equivalence class of $(\Lambda, \lambda, D, d, a_0, u)$ via $\approx$.

**Proof.** Let $(\Lambda, \lambda, D, d, a_0, u)$, $(\Lambda', \lambda', D', d', a'_0, u') \in \mathcal{F}(A)$ and $\Omega(A, V)$, respectively $\Omega'(A, V)$ the corresponding algebra extending structures constructed in the proof of Proposition 4.3. The proof relies on Proposition 4.3 and Theorem 2.8. Since $\dim_k(V) = 1$, any linear map $r : V \to A$ is uniquely determined by an element $\alpha \in A$ such that $r(x) = \alpha$, where $\{x\}$ is a basis in $V$. On the other hand, any automorphism $v$ of $V$ is uniquely determined by a non-zero scalar $q \in k^*$ such such $v(x) = qx$. Based on these facts, a little computation shows that the compatibility conditions from Definition 2.6 and respectively Definition 2.7 take precisely the form given in (1) and (2) above and hence the proof is finished. \qed

Next we will highlight the efficiency of Theorem 4.4 in classifying flag algebras. We start with $A = k$: by computing $\mathcal{AH}_k^2(k, k)$ we will classify in fact all 2-dimensional algebras over an arbitrary field $k$ since any algebra map between two 2-dimensional algebras
automatically stabilizes $k$. Thus the next corollary originates in [29] and [34], where all 2-dimensional algebras over the field of complex numbers $\mathbb{C}$ were classified. By replacing $\mathbb{C}$ with an arbitrary field $k$ the situation changes: the number of isomorphism types of 2-dimensional algebras depends heavily on the characteristic of $k$ as well as on the set $k \setminus k^2$, where $k^2 = \{q^2 \mid q \in k\}$. First we set the notations which will play the key role in the classification of flag algebras:

If $k^2 \neq k$, we shall fix $S \subseteq k \setminus k^2$ a system of representatives for the following relation on $k \setminus k^2$: $d \equiv d'$ if and only if there exists $q \in k^*$ such that $d = q^2 d'$. Hence, $|S| = [k^* : (k^2)^*] - 1$, where $[k^* : (k^2)^*]$ is the index of $(k^2)^*$ in the multiplicative group $(k^*, \cdot)$.

If $\text{char}(k) = 2$ and $k^2 \neq k$ we denote by $R \subseteq k \setminus k^2$ a system of representatives for the following new equivalence relation on $k \setminus k^2$: $\delta \equiv_1 \delta'$ if and only if there exists $q \in k^*$ such that $\delta - q^2 \delta' \in k^2$. Then, $|R| = [k^* : (k^2)^*] - 1$.

If $\text{char}(k) = 2$ we also consider the following equivalence relation on $k$: $c \equiv_2 c'$ if and only if there exists $\alpha \in k$ such that $c - c' = \alpha^2 - \alpha$. We shall fix $T \subseteq k$ a system of representatives for this relation such that $0 \in T$.

Based on Theorem 4.4 we can prove the following results that classify all 2-dimensional algebras over an arbitrary field. Of course, (2) and (3) below are well known results. For (4) and (5) we are not able to indicate a reference in the literature.

**Corollary 4.5.** Let $k$ be an arbitrary field. Then:

1. There exists a bijection $\mathcal{AH}_2^k(k, k) \cong k \times k/\equiv$, where $\equiv$ is the equivalence relation on $k \times k$ defined as follows: $(a, b) \equiv (a', b')$ if and only if there exists a pair $(q, \alpha) \in k^* \times k$ such that:

   $a = q^2 a' + \alpha^2 - b\alpha, \quad b = q^2 b' + 2\alpha$ \hspace{1cm} (38)

   The bijection between $k \times k/\equiv$ and the isomorphism classes of all 2-dimensional algebras is given by $(a, b) \mapsto k_{(a, b)}$, where $k_{(a, b)}$ is the algebra having $\{1, x\}$ as a basis and the multiplication given by $x^2 = a + bx$.

2. If $\text{char}(k) \neq 2$ and $k = k^2$, then the factor set $k \times k/\equiv$ is equal to $\{(0, 0), (0, 1)\}$. Thus, there exist only two types of 2-dimensional algebras, namely $k_{(0, 0)}$ and $k_{(0, 1)} \cong k \times k$.

3. If $\text{char}(k) \neq 2$ and $k \neq k^2$ then the factor set $k \times k/\equiv$ is equal to $\{(0, 0), (0, 1)\} \cup \{(d, 0) \mid d \in S\}$. Thus, in this case there exist $1 + [k^* : (k^2)^*]$ types of isomorphisms of 2-dimensional algebras namely $k_{(0, 0)}$, $k_{(0, 1)}$ and $k_{(d, 0)}$, for some $d \in S$.

4. If $\text{char}(k) = 2$ and $k = k^2$ then the factor set $k \times k/\equiv$ is equal to $\{(0, 0)\} \cup \{(c, 1) \mid c \in T\}$. Thus, in this case there exist $1 + |T|$ types of isomorphisms of 2-dimensional algebras namely $k_{(0, 0)}$ and $k_{(c, 1)}$, for some $c \in T$.

5. If $\text{char}(k) = 2$ and $k \neq k^2$ then the factor set $k \times k/\equiv$ is equal to $\{(0, 0)\} \cup \{(c, 1) \mid c \in T\} \cup \{(\delta, 0) \mid \delta \in R\}$. Thus, in this case there exist $1 + |T| + |R|$ types of isomorphisms of 2-dimensional algebras namely $k_{(0, 0)}$, $k_{(c, 1)}$, $k_{(\delta, 0)}$, for some $c \in T$ and $\delta \in R$.

The algebra $k_{(d, 0)}$, for some $d \in S$ is denoted by $k(\sqrt{d})$ and is a quadratic field extension of $k$. 
Proof. (1) Since there exists only one algebra map $k \to k$, namely the identity map, and any linear map $D : k \to k$ with $D(1) = 0$ is the trivial map it follows that $F(k) \cong k \times k$. Through this identification it is straightforward to see that the equivalence relation in Theorem 4.4 takes the form given by (38).

(2) The fact that $k \times k / \equiv$ has only two elements, namely $\{(0,0), (0,1)\}$ follows trivially. Let $(a, b) \in k \times k$. If $a + 4^{-1}b^2 = 0$, then we can we denote $b = 2\alpha$, with $\alpha \in k$. Then it follows that $(a, b) = (-\alpha^2, 2\alpha)$, for some $\alpha \in k$ and thus $(a, b) \equiv (0, 0)$. On the other hand, if $a + 4^{-1}b^2 \neq 0$, then we can pick $T \in k^*$ such that $a + 4^{-1}b^2 = T^2$, since $k = k^2$. Again, we denote $b = 2\alpha$, with $\alpha \in k$ and we obtain $(a, b) = (T^2 - \alpha^2, 2\alpha)$, and hence $(a, b) \equiv (0, 1)$.

(3) If $k \neq k^2$ we will prove that $k \times k / \equiv$ coincides with $\{(0,0), (0,1)\} \cup \{(d,0) \mid d \in S\}$. Indeed, consider $(a, b) \in k \times k$. Besides from the two possibilities already studied in (2) we can also have $a + 4^{-1}b^2 = d$, with $d \in k \setminus k^2$. As before, we denote $b = 2\alpha$, with $\alpha \in k$. It follows that $(a, b) = (d - \alpha^2, 2\alpha)$ and therefore $(a, b) \equiv (d, 0)$.

(4) and (5) The proof is based on the following observations. $(a, b) \equiv (0, 0)$ if and only if $b = 0$ and $a \in k^2$. Thus, if $k = k^2$, then we have that $(a, 0) \equiv (0, 0)$, for any $a \in k$. If $k \neq k^2$, then $(a, 0)$ is either equivalent to $(0, 0)$ if $a \in k^2$ or to $(\delta, 0)$, for some $\delta \in R$ in the case that $a \in k \setminus k^2$. We take into account that $(\delta, 0) \equiv (\delta', 0)$ if and only if $\delta \equiv_1 \delta'$. Let now $(a, b) \in k$, with $b \neq 0$. Then $(a, b) \equiv (ab^{-2}, 1)$. If $a = 0$ the latter is equivalent to $(0, 1)$ and, if $a \neq 0$, $(ab^{-2}, 1)$ is equivalent to $(c, 1)$, for some $c \in T$ since $(c, 1) \equiv (c', 1)$ if and only if $c \equiv_2 c'$. This finishes the proof. \[ \square \]

Theorem 4.4 provides the necessary tool for describing and classifying flag algebras in a purely computational and algorithmic way. Having described all 2-dimensional algebras over arbitrary fields in Corollary 4.5 we can now take a step further and describe all flag algebras of dimension 3, i.e. all algebras $E$ for which there exists a chain of subalgebras

\[ k = E_0 \subset E_1 \subset E_2 = E \]  

(39)

such that $\text{dim}(E_1) = 2$. The algebras described in this way will be classified up to an isomorphism that stabilizes $E_1$ by using Theorem 4.4. First we should notice that since $E_1$ has dimension 2, it should coincide with one the following algebras: $k_{(0,0)}$, $k_{(1,0)}$, $k_{(d,0)} = k(\sqrt{d})$, or $k_{(c,1)}$. Now, the algebra $k(\sqrt{d})$ has no characters (i.e. there exist no algebra maps $k(\sqrt{d}) \to k$). This means that $\mathcal{F}(k(\sqrt{d}))$ is empty, i.e. there exist no 3-dimensional algebras containing $k(\sqrt{d})$ as a subalgebra. Thus, any 3-dimensional flag algebra $E$ has the middle term $E_1$ isomorphic to $k_{(0,0)}$, $k_{(0,1)}$ if $\text{char}(k) \neq 2$ or to $k_{(0,0)}$, $k_{(c,1)}$, for some $c \in T$ in the case when $\text{char}(k) = 2$. In what follows we describe all 3-dimensional flag algebras $E$ that contain and stabilize $k_{(0,0)}$ as a subalgebra. First, we need to describe the set of all flag datums of the algebra $k_{(0,0)}$:

**Lemma 4.6.** Let $k$ be a field. Then $\mathcal{F}(k_{(0,0)})$ is the coproduct of the following sets:

\[ \mathcal{F}(k_{(0,0)}) := \mathcal{F}_1(k_{(0,0)}) \cup \mathcal{F}_2(k_{(0,0)}) \cup \mathcal{F}_3(k_{(0,0)}), \]  

where:
\( F_1 (k_{(0,0)}) \cong k \times k^* \times k \), with the bijection given such that the flag datum \((\Lambda, \lambda, D, d, a_0, u)\) corresponding to \((D_1, a_01, u) \in k \times k^* \times k\) is defined by:
\[
\lambda(x) = \Lambda(x) := 0, \quad D(x) = d(x) := D_1 x, \quad a_0 := D_1^2 - u D_1 + a_01 x, \quad u := u
\]
\( F_2 (k_{(0,0)}) \cong k^2 \), with the bijection given such that the flag datum \((\Lambda, \lambda, D, d, a_0, u)\) corresponding to \((D_1, u) \in k^2\) is defined by:
\[
\lambda(x) = \Lambda(x) := 0, \quad D(x) = d(x) := D_1 x, \quad a_0 := D_1^2 - u D_1, \quad u := u
\]
\( F_3 (k_{(0,0)}) \cong \{(D_1, d_1) \in k \times k \mid D_1 \neq d_1\}\) with the bijection given such that the flag datum \((\Lambda, \lambda, D, d, a_0, u)\) corresponding to \((D_1, d_1) \in k \times k\) (with \(D_1 \neq d_1\)) is defined by:
\[
\lambda(x) = \Lambda(x) := 0, \quad D(x) := D_1 x, \quad d(x) := d_1 x, \quad a_0 := -D_1 d_1, \quad u := D_1 + d_1
\]
Proof. The algebra \( k_{(0,0)} \) has only one character, namely the map \( \Lambda : k_{(0,0)} \to k \), defined by \( \Lambda(1) = 1 \) and \( \Lambda(x) = 0 \). A straightforward computation shows that the set \( F (k_{(0,0)}) \) of all flag datums of \( k_{(0,0)} \) identifies with the set of all quadruples \((D, d, a_0, u) \in k^4\) satisfying the following two compatibilities:
\[
a_0 D_1 = a_01 d_1, \quad D_1^2 - u D_1 = d_1^2 - u d_1
\]
Under this bijection the flag datum \((\Lambda, \lambda, D, d, a_0, u)\) associated to the quadruple \((D_1, d_1, a_01, u) \in k^4\) is given by:
\[
\lambda(x) = \Lambda(x) := 1, \quad D(x) := D_1 x, \quad d(x) := d_1 x, \quad a_0 := D_1^2 - u D_1 + a_01 x
\]
A detailed discussion on these coefficients (if \( a_01 = 0 \) or \( a_01 \neq 0 \)) allows us to write \( F (k_{(0,0)}) \) as the disjoint union of the sets mentioned in the statement. \( \square \)

The next result classifies all 3-dimensional flag algebras that contain and stabilize \( k_{(0,0)} \) as a subalgebra. The classification does not depend on the characteristic of the field \( k \).

**Corollary 4.7.** Let \( k \) be an arbitrary field.

1. If \( k = k^2 \), then there exist five isomorphism classes of 3-dimensional flag algebras that contain and stabilize \( k_{(0,0)} \) as a subalgebra, each of them having the \( k \)-basis \( \{1, x, y\} \) and relations given as follows:
   \[
   A_0^{:} : \quad x^2 = 0, \quad y^2 = y, \quad xy = x, \quad yx = 0;
   A_0^1 : \quad x^2 = 0, \quad y^2 = y, \quad xy = yx = 0;
   A_0^2 : \quad x^2 = 0, \quad y^2 = 0, \quad xy = yx = 0;
   A_0^3 : \quad x^2 = 0, \quad y^2 = x, \quad xy = yx = 0;
   A_0^4 : \quad x^2 = 0, \quad y^2 = x + y, \quad xy = yx = 0
   \]
2. If \( k \neq k^2 \), then there exist \( 4 + [k^* : (k^2)^*] \) isomorphism classes of 3-dimensional flag algebras that contain and stabilize \( k_{(0,0)} \) as a subalgebra, namely those from (1) and an additional family defined for any \( d \in S \) by the relations:\(^2\)
   \[
   A_0^0 (d) : \quad x^2 = 0, \quad y^2 = dx, \quad xy = yx = 0
   \]

\(^2\)We recall that \( |S| = [k^* : (k^2)^*] - 1 \)
Proof. The proof is similar to the one of Corollary 4.5. Indeed, in Lemma 4.6 we have described the set of all flag datums \( \mathcal{F}(k(0,0)) \). The equivalence relation from Theorem 4.4 takes the following form for each of the sets \( \mathcal{F}_i(k(0,0)) \), \( i = 1, 2, 3 \):

- For \( \mathcal{F}_3(k(0,0)) \): \( (D_1, d_1) \equiv (D'_1, d'_1) \) if and only if there exists \( (q, \alpha) \in k^* \times k \) such that
  \[
  D_1 = qD'_1 + \alpha, \quad d_1 = qd'_1 + \alpha
  \]
  In this case the factor set \( \mathcal{F}_3(k(0,0))/\equiv \) is a singleton having \((0,1)\) as the only element. The algebra associated to \((0,1)\), i.e. the unified product \( k(0,0) \bowtie (\Lambda, \lambda, D, d, a_0, u) \) from Theorem 4.4, is precisely the noncommutative algebra \( A_1^0 \).

- For \( \mathcal{F}_2(k(0,0)) \): \( (D_1, u) \equiv (D'_1, u') \) if and only if there exists \( (q, \alpha) \in k^* \times k \) such that
  \[
  D_1 = qD'_1 + \alpha, \quad u = qu' + 2\alpha
  \]
  We can easily show that, regardless of the characteristic of \( k \), the factor set \( \mathcal{F}_2(k(0,0))/\equiv \) has two elements namely \( \{(0,0), (0,1)\} \). The associated unified products \( k(0,0) \bowtie (\Lambda, \lambda, D, d, a_0, u) \) are the algebras \( A_2^0 \) and \( A_3^0 \).

- For \( \mathcal{F}_1(k(0,0)) \): \( (D_1, a_0, u) \equiv (D'_1, a'_{01}, u') \) in \( k^* \times k \times k \) if and only if there exists \( (q, \alpha_0, \alpha_1) \in k^* \times k \times k \) such that
  \[
  D_1 = qD'_1 + \alpha_0, \quad u = qu' + 2\alpha_0, \quad a_{01} = q^2 a'_{01} - q\alpha_1 u' + 2q\alpha_1 D'_1 \tag{40}
  \]
  Suppose first that \( \text{char}(k) \neq 2 \). In order to compute the factor set \( \mathcal{F}_1(k(0,0))/\equiv \) we distinguish two cases. First, if \( k = k^2 \), we can easily show that \( \mathcal{F}_1(k(0,0))/\equiv \) has two elements namely \( \{(0,1,0), (0,1,1)\} \); the unified products \( k(0,0) \bowtie (\Lambda, \lambda, D, d, a_0, u) \) associated to \((0,1,0)\) and \((0,1,1)\) are precisely the algebras \( A_4^0 \) and \( A_5^0 \).

  Secondly, if \( k \neq k^2 \), it is straightforward to see that \( \mathcal{F}_1(k(0,0))/\equiv \) is precisely the set \( \{(0,1,0), (0,1,1)\} \cup \{(0, d, 0) \mid d \in S\} \). Moreover, the unified product associated to \((0, d, 0)\), for some \( d \in S \), is the algebra \( A^0(d) \).

Assume now that \( \text{char}(k) = 2 \). Then the equivalence relation given by (40) takes the form:

\[
D_1 = qD'_1 + \alpha_0, \quad u = qu', \quad a_{01} = q^2 a'_{01} - q\alpha_1 u'
\]

The factor set \( \mathcal{F}_1(k(0,0))/\equiv \) is the same as in the case \( \text{char}(k) \neq 2 \). This can be easily seen from the following observations: for any \( u \neq 0 \), we have that \((D_1, a_{01}, u) \equiv (0,1,1)\) and \((D_1, a_{01}, 0) \equiv (0,1,0)\) if and only if \( a_{01} \in (k^2)^* \). Finally, for \( d \in k \setminus k^2 \) we have that \((D_1, d, 0) \equiv (0, d, 0)\) and the proof is finished since \((0, d, 0) \equiv (0, d', 0)\) if and only if there exists \( q \in k^* \) such that \( d = q^2 d' \). \( \square \)

Next we will classify all flag algebras of dimension 3 that contain and stabilize \( k(0,1) \) as a subalgebra: remark that even in characteristic 2 this algebra still exists as an intermediary algebra corresponding to \( k(c,1) \), for \( c := 0 \in T \), since we have chosen a system of representatives \( T \) which contains 0. First we need the following:

**Lemma 4.8.** Let \( k \) be a field. Then \( \mathcal{F}(k(0,1)) \) is the coproduct of the following sets:

\[
\mathcal{F}(k(0,1)) := \mathcal{F}_1(k(0,1)) \sqcup \mathcal{F}_2(k(0,1)) \sqcup \mathcal{F}_3(k(0,1)) \sqcup \mathcal{F}_4(k(0,1)),
\]

where:
We distinguish four cases depending on the characters ($\Lambda$, $\lambda$, $\Lambda$, $\lambda$) of the maps that send $x$ to $\lambda x$.

$\mathcal{F}_1(k_{(0,1)}) \cong k^3$ with the bijection given such that the flag datum $(\Lambda, \lambda, D, d, a_0, u)$ corresponding to $(D_1, a_0, u) \in k^3$ is defined by

$$\Lambda(x) = \lambda(x) := 0, \quad D(x) = d(x) := D_1 x, \quad a_0 := D_1^2 - u D_1 - a_0 x, \quad u := u$$

$\mathcal{F}_2(k_{(0,1)}) \cong k^3$ with the bijection given such that the flag datum $(\Lambda, \lambda, D, d, a_0, u)$ corresponding to $(D_1, a_0, u) \in k^3$ is defined by

$$\Lambda(x) = \lambda(x) := 0, \quad D(x) = d(x) := D_1 x, \quad a_0 := D_1^2 - u D_1 + a_0 x, \quad u := u$$

$\mathcal{F}_3(k_{(0,1)}) \cong k^2$ with the bijection given such that the flag datum $(\Lambda, \lambda, D, d, a_0, u)$ corresponding to $(D_1, d_1, u) \in k^2$ is defined by

$$\Lambda(x) := 0, \quad \lambda(x) := 1, \quad D(x) := d_1 x, \quad d(x) := d_1 (1 - x), \quad a_0 := D_1 d_1, \quad u := D_1 - d_1$$

$\mathcal{F}_4(k_{(0,1)}) \cong k^2$ with the bijection given such that the flag datum $(\Lambda, \lambda, D, d, a_0, u)$ corresponding to $(D_1, d_1, u) \in k^2$ is defined by

$$\Lambda(x) := 1, \quad \lambda(x) := 0, \quad D(x) := D_1 (1 - x), \quad d(x) := d_1 x, \quad a_0 := D_1 d_1, \quad u := d_1 - D_1$$

Proof. Since $x^2 = x$ it follows that the algebra $k_{(0,1)}$ has only two characters, namely the maps that send $x \mapsto 0$ and respectively $x \mapsto 1$. Thus, in order to compute $\mathcal{F}(k_{(0,1)})$ we distinguish four cases depending on the characters $(\Lambda, \lambda)$. More precisely, $\mathcal{F}_1(k_{(0,1)})$ will parameterize all flag datums for which $\Lambda(x) = \lambda(x) = 0$ while $\mathcal{F}_2(k_{(0,1)})$ those for which $\Lambda(x) = \lambda(x) = 1$. We are left with two more cases: $\mathcal{F}_3(k_{(0,1)})$ (resp. $\mathcal{F}_4(k_{(0,1)})$) parameterizes all flag datums for which $\Lambda(x) := 0$ and $\lambda(x) = 1$ (resp. $\Lambda(x) = 1$ and $\lambda(x) = 0$). The conclusion follows in a straightforward manner by applying Definition 4.2. \hfill \square

Now we shall classify all 3-dimensional flag algebras that contain and stabilize $k_{(0,1)}$ as a subalgebra. This time, the classification depends heavily on the field $k$.

Corollary 4.9. The isomorphism classes of all 3-dimensional flag algebras that contain and stabilize $k_{(0,1)}$ as a subalgebra are given as follows:

1. If $\text{char}(k) \neq 2$ and $k = k^2$, then there are five such isomorphism classes, namely the algebras with $k$-basis $\{1, x, y\}$ and relations given as follows:

$A_1 : \quad x^2 = x, \quad y^2 = 0, \quad xy = yx = 0$

$A_2 : \quad x^2 = x, \quad y^2 = x - 1, \quad xy = yx = 0$

$A_3 : \quad x^2 = x, \quad y^2 = 0, \quad xy = yx = y$

$A_4 : \quad x^2 = x, \quad y^2 = x, \quad xy = yx = y$

$A_5 : \quad x^2 = x, \quad y^2 = 0, \quad xy = y, \quad yx = 0$

2. If $\text{char}(k) \neq 2$ and $k \neq k^2$, then there exist $3 + 2[k^* : (k^2)^*]$ such isomorphism classes. These are the five types from (1) and two additional families defined by the following relations for any $d \in S$:

$B_1(d) : \quad x^2 = x, \quad y^2 = d(x - 1), \quad xy = yx = 0$

$B_2(d) : \quad x^2 = x, \quad y^2 = d x, \quad xy = yx = y$
(3) If \( \text{char}(k) = 2 \) and \( k = k^2 \), then there exist \( 3 + 2 \mid T \mid \) such isomorphism classes, namely the algebras having \( \{1, x, y\} \) as a \( k \)-basis and subject to the following relations for any \( c \in T \):

\[
\begin{align*}
C_1^1 : & \quad x^2 = x, \quad y^2 = 0, \quad xy = yx = 0 \\
C_1^2(c) : & \quad x^2 = x, \quad y^2 = c + (c + 1)x, \quad xy = yx = 0 \\
C_3^1 : & \quad x^2 = x, \quad y^2 = 0, \quad xy = yx = y \\
C_4^1(c) : & \quad x^2 = x, \quad y^2 = y + cx, \quad xy = yx = y \\
C_5^1 : & \quad x^2 = x, \quad y^2 = 0, \quad xy = y, \quad yx = 0
\end{align*}
\]

(4) If \( \text{char}(k) = 2 \) and \( k \neq k^2 \), then there exist \( 3 + 2 \mid T \mid + 2 \mid R \) such isomorphism classes. These are the types from (3) and two additional families defined by the following relations for any \( \delta \in R \):

\[
\begin{align*}
D_1(\delta) : & \quad x^2 = x, \quad y^2 = \delta (x + 1), \quad xy = yx = 0 \\
D_2(\delta) : & \quad x^2 = x, \quad y^2 = \delta x, \quad xy = yx = y
\end{align*}
\]

Proof. We shall use the description of \( \mathcal{F}(k_{(0,1)}) \) given in Lemma 4.8. First of all we remark that, due to symmetry, the unified products associated to the flag datums of \( \mathcal{F}_i(k_{(0,1)}) \) are isomorphic to the ones associated to \( \mathcal{F}_3(k_{(0,1)}) \). Thus we only have to analyze the cases \( \mathcal{F}_i(k_{(0,1)}) \), with \( i = 1, 2, 3 \). The equivalence relation from Theorem 4.4, applied for each of the sets \( k^3 \) and \( k^2 \), takes the following form:

- For \( \mathcal{F}_1(k_{(0,1)}) \equiv k^3 \): \( (D_1, a_{01}, u) \equiv (D_1', a_{01}', u') \) if and only if there exists \( (q, \alpha_0, \alpha_1) \in k^* \times k \times k \) such that

\[
\begin{align*}
D_1 &= q D_1' + \alpha_0 + \alpha_1 \\
a_{01} &= q^2 a_{01}' + \alpha_1^2 - q u' \alpha_1 + 2q \alpha_1 D_1' \\
u &= qu + 2 \alpha_0
\end{align*}
\]

Suppose first that \( \text{char}(k) \neq 2 \). Then we have: \( (D_1, a_{01}, u) \equiv (0,0,0) \) if and only if \( a_{01} = (D_1 - 2^{-1}u) \) and \( (D_1, a_{01}, u) \equiv (0,1,0) \) if and only if \( a_{01} = (D_1 - 2^{-1}u) \) are in \( k^* \). These two observations show that if \( k = k^2 \) then the factor set \( k^3/ \equiv \) has two elements, namely \( \{(0,0,0), (0,1,0)\} \). The unified products \( k_{(0,1)} \times (\Lambda, \lambda, D, d, a_0, u) \) are associated to \( (0,0,0) \) and respectively \( (0,1,0) \) are the algebras \( A_1^1 \) and \( A_2^1 \). On the other hand, if \( k \neq k^2 \), then the factor set \( k^3/ \equiv \) is equal to \( \{(0,0,0); (0,1,0)\} \cup \{(0,d,0) \mid d \in S\} \). This can be easily seen from the following observations: \( (D_1, a_{01}, u) \equiv (0,d,0) \) if and only if \( a_{01} = (D_1 - 2^{-1}u) \) are for some \( q \in k^* \) while \( (0,d,0) \equiv (0,d',0) \) if and only if \( d = q^2 d' \), for some \( q \in k^* \). The unified product associated to \( (0,d,0) \) is the algebra \( B_1(d) \).

Assume now that \( \text{char}(k) = 2 \). Then the equivalence relation on \( k^3 \) given by (41)-(43) takes the form: \( (D_1, a_{01}, u) \equiv (D_1', a_{01}', u') \) if and only if there exists \( (q, \alpha_0, \alpha_1) \in k^* \times k \times k \) such that

\[
D_1 = q D_1' + \alpha_0 + \alpha_1, \quad a_{01} = q^2 a_{01}' + \alpha_1^2 - q u' \alpha_1, \quad u = qu'
\]
We will prove the following: if $k = k^2$, then the factor set $k^3/\equiv$ is equal to $\{(0,0,0), (0,c,1)\} \cup \{(0,\delta,0) | \delta \in R\}$. Indeed, the above equalities are consequences of the following observations: $(D_1, a_{01}, u) \equiv (0,0,0)$ if and only if $u = 0$ and $a_{01} \in k^2$; $(D_1, a_{01}, u) \equiv (0,\delta,0)$ if and only if $u = 0$ and $a_{01} \equiv_1 d$. Now, for any $a 
eq 0$ we have that $(D_1, a_{01}, u) \equiv (u^{-1}D_1, u^{-2}a_{01}, 1)$, and moreover $(D_1, a_{01}, 1) \equiv (0,c,1)$, for some $c \in T$. Finally, $(0,c,1) \equiv (0,c',1)$ if and only if $c \equiv_2 c'$. The algebras associated to $(0,0,0)$, $(0,c,1)$ and $(0,\delta,0)$ are $C^1_3(c)$ and respectively $D_1(\delta)$.

- For $F_2(k_{(0,1)}) \cong k^3$: $(D_1, a_{01}, u) \equiv (D'_1, a'_{01}, u')$ if and only if there exists $(q,\alpha_0,\alpha_1) \in k^4 \times k \times k$ such that

\[
\begin{align*}
D_1 &= qD'_1 - \alpha_0 \quad a_{01} = q^2a'_{01} - \alpha_1^2 - q\alpha_1u' - 2q\alpha_1D'_1 \\
u &= qu' + 2(\alpha_0 + \alpha_1)
\end{align*}
\]

Suppose first that $\text{char}(k) \neq 2$. Then we have: $(D_1, a_{01}, u) \equiv (0,0,0)$ if and only if $a_{01} = -(D_1+2^{-1}u)^2$ and $(D_1, a_{01}, u) \equiv (0,1,0)$ if and only if $a_{01} = -(D_1+2^{-1}u)^2 + q^2$, for some $q \in k^*$. These two observations show that if $k = k^2$ then the factor set $k \times k \times k/\equiv$ has two elements, namely $(0,0,0)$ and $(0,1,0)$. The unified products $k_{(0,1)} \times (\Lambda, \lambda, D, d, a_0, u)$ $y$ associated to $(0,0,0)$ and $(0,1,0)$ are the algebras $A^1_3$ and respectively $A^1_4$. On the other hand, if $k \neq k^2$ then the factor set $k \times k \times k/\equiv$ is equal to $\{(0,0,0), (0,1,0)\} \cup \{(0,d,0) | d \in S\}$. Indeed, let $(D_1, a_{01}, u) \in k^3$ such that $a_{01} = -(D_1+2^{-1}u)^2 + d$, for some $d \neq 0$. Then $(D_1, a_{01}, u) \equiv (0,d,0)$ and moreover $(0,d,0) \equiv (0,d',0)$ if and only if there exists $q \in k^*$ such that $d = q^2d'$. The unified product $k_{(0,1)} \times (\Lambda, \lambda, D, d, a_0, u)$ $y$ associated to $(0,d,0)$, for some $d \in S$, is the algebra $B_2(d)$.

Consider now the case $\text{char}(k) = 2$. We have: $(D_1, a_{01}, u) \equiv (0,0,0)$ if and only if $u = 0$ and $a_{01} \in k^2$; $(D_1, a_{01}, u) \equiv (0,\delta,0)$ if and only if $u = 0$ and $a_{01} \equiv_1 \delta$. Let $(D_1, a_{01}, u) \in k^3$ with $u \neq 0$, then $(D_1, a_{01}, u) \equiv (u^{-1}D_1, u^{-2}a_{01}, 1)$. Finally, $(D_1, a_{01}, 1) \equiv (D'_1, a'_{01}, 1)$ if and only if $(D_1, a_{01}, u) \equiv (0,0,0)$. These observations show that if $k = k^2$, then the factor set $k^3/\equiv$ is equal to $\{(0,0,0), (0,c,1) | c \in T\}$ while if $k \neq k^2$, the factor set $k^3/\equiv$ coincides with $\{(0,0,0), (0,c,1) \mid c \in T\} \cup \{(0,\delta,0) | \delta \in R\}$. The algebras corresponding to $(0,0,0)$, $(0,c,1)$ and $(0,\delta,0)$ are $C^1_3, C^1_4(c)$ and respectively $D_2(\delta)$.

- For $F_3(k_{(0,1)}) \cong k^2$: $(D_1, d_1) \equiv (D'_1, d'_1)$ if and only if there exists a triple $(q,\alpha_0,\alpha_1) \in k^4 \times k \times k$ such that

\[
D_1 = qD'_1 + \alpha_0 + \alpha_1, \quad d_1 = qd'_1 - \alpha_0
\]

Regardless of the characteristic of $k$, the factor set $k \times k/\equiv$ contains only one element, namely $(0,0)$. The algebra associated to it is $A^1_3$, if $\text{char}(k) \neq 2$ or $C^1_4$, if $\text{char}(k) = 2$.

The proof is now finished. \qed

In order to complete the description of all 3-dimensional flag algebras we are left to study one more case, namely the one when $\text{char}(k) = 2$ and the intermediary algebra is isomorphic to $k_{(c,1)}$, for some $c \in T$. The case $c = 0$ is treated in Corollary 4.9. Suppose
now that $c \neq 0$. We will see that the set $\mathcal{F}(k_{(c,1)})$ of all flag datums is empty (i.e. there exists no 3-dimensional algebras containing $k_{(c,1)}$ as a subalgebra) since the algebra $k_{(c,1)}$ has no characters. Indeed, the algebra $k_{(c,1)}$ is generated by an element $x$ such that $x^2 = c + x$. Therefore, the characters of this algebra are in bijection with the solutions in $k$ of the equation $y^2 + y + c = 0$. Now, since $c \in T$ and $c \neq 0$ this equation has no solutions in $k$: if $\alpha$ is such a solution then $c = \alpha + \alpha^2$, i.e. $c \equiv 2 \mod 0$ and we obtain $c = 0$ since $c \in T$ a system of representatives for $\equiv 2$ containing 0.

As a conclusion of this section we obtain the classification of all 3-dimensional flag algebras:

**Theorem 4.10.** Let $k$ be an arbitrary field. Then any 3-dimensional flag algebra is isomorphic to an algebra from the following list:

1. $A_i^0$ or $A_i^1$, for all $i = 1, \cdots, 5$, if $\text{char}(k) \neq 2$ and $k = k^2$.
2. $A_i^0$, $A_i^1$, $A_i^0(d)$, $B_i(d)$ or $B_2(d)$, for all $i = 1, \cdots, 5$ and $d \in S$, if $\text{char}(k) \neq 2$ and $k \neq k^2$.
3. $A_i^0$, $C_i^1(c)$, $C_i^1(c)$ or $C_i^1$, for all $i = 1, \cdots, 5$ and $c \in T$, if $\text{char}(k) = 2$ and $k = k^2$.
4. $A_i^0$, $A_i^0(d)$, $C_i^1(c)$, $C_i^1(c)$, $D_i(d)$ or $D_2(d)$, for all $i = 1, \cdots, 5$, $d \in S$, $c \in T$ and $\delta \in R$, if $\text{char}(k) = 2$ and $k \neq k^2$.

In particular, if $k := \mathbb{C}$ the field of complex numbers, Theorem 4.10 shows that only 10 out of the 22 types of algebras of dimension 3 [10, Section 5.1] are flag algebras. Theorem 4.10 highlights the efficiency of Theorem 4.4 in classifying flag algebras over arbitrary fields $k$ turning the problem into a purely computational one.

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Faculty of Engineering, Vrije Universiteit Brussel, Pleinlaan 2, B-1050 Brussels, Belgium

E-mail address: ana.agore@vub.ac.be and ana.agore@gmail.com

Faculty of Mathematics and Computer Science, University of Bucharest, Str. Academiei 14, RO-010014 Bucharest 1, Romania

E-mail address: gigel.militaru@fmi.unibuc.ro and gigel.militaru@gmail.com