TOPOLOGICAL DEGREE OF SHIFT SPACES ON MONOIDS

JUNG-CHAO BAN, CHIH-HUNG CHANG*, AND NAI-ZHU HUANG

Abstract. This paper considers the topological degree of $G$-shifts of finite type for the case where $G$ is a nonabelian monoid. Whenever the Cayley graph of $G$ has a finite representation and the relationships among the generators of $G$ are determined by a matrix $A$, the coefficients of the characteristic polynomial of $A$ are revealed as the number of children of the graph. After introducing an algorithm for the computation of the degree, the degree spectrum, which is finite, relates to a collection of matrices in which the sum of each row of every matrix is bounded by the number of children of the graph. Furthermore, the algorithm extends to $G$ of finite free-followers.

1. Introduction

Let $A$ be a finite alphabet. Given $d \in \mathbb{N}$, a configuration is a function from $\mathbb{Z}^d$ to $A$, and a pattern is a function from a finite subset of $\mathbb{Z}^d$ to $A$. A subset $X \subseteq A^{\mathbb{Z}^d}$ is called a shift space if $X$, denoted by $X = X_F$, consists of configurations which avoid patterns from some set $F$ of patterns. A shift space is called a shift of finite type (SFT) if $F$ is a finite set; $\mathbb{Z}^d$ acts on $X$ by translation of configurations making $X$ a symbolic dynamical system. One of the many motivations in studying symbolic dynamical systems is it helps for the investigation of hyperbolic topological dynamical systems. The interested reader can consult standard literature such as [10, 24].

While almost all properties of SFTs are decidable for $d = 1$ (cf. [19]), the investigation of SFTs is rife for $d \geq 2$ since many undecidability issues company with it. It is even undecidable if an SFT is nonempty [9]. Different kinds of mixing properties have been introduced for examining the existence and denseness of periodic configurations [11]. A straightforward generalization is considering SFTs on $G$ which is associated with some algebraic structure. Whenever $G$ is a free monoid, it has been demonstrated that many such issues do not arise. For instance, the conjugacy
between two irreducible $G$-SFTs is decidable \cite{1}; furthermore, the nonemptiness,
extensibility, and the existence of periodic configurations are decidable for $G$-SFTs
\cite{4, 6}. Aside from the qualitative behavior of $G$-SFTs, the phenomena from the
computational perspective are also fruitful \cite{3, 5, 22, 23}.

For the case where $G = \mathbb{Z}^1$, the topological entropy of an SFT relates to the
spectral radius of an integral matrix, and the entropy spectrum (i.e., the set of
entropies) of SFTs is the set of logarithms of Perron numbers \cite{18, 19}. When $G = \mathbb{Z}^d$
for $d \geq 2$, the entropy of an SFT is a right recursively enumerable number
which may not be algebraic and is not computable \cite{17, 20, 21}. However, the story
is quite different when $G$ is a free monoid.

Suppose that $G$ is a finitely generated free monoid. Let $\Sigma$ be a finite set which
generates $G$. An element $g \in G$ is called an \textit{i-word} provided the minimal expresion of
$g$ is $g = g_1g_2\cdots g_i$ for some $g_1, \ldots, g_i \in \Sigma$. For $n \in \mathbb{N}$, let $\Gamma_n$ denote the set of $n$-blocks in
a $G$-SFT; an $n$-block is a pattern whose support consists of all $i$-words in $G$ for $i \leq n$.
Suppose the cardinality of $\Gamma_n$ behaves approximately like $c(n)\lambda^n$ for some $\lambda, \kappa \in \mathbb{R}$
and $c(n) = o(\kappa^n)$. It follows that such a $G$-SFT carries the topological entropy
$\ln \lambda$ if $\kappa = d$ is the number of generators of $G$ and 0 otherwise, and the topological
degree (defined in \cite{2}) is $\ln \kappa$ \cite{7}. The degree spectrum (i.e., the set of degrees) of
$G$-SFTs relates to the set of Perron numbers \cite{2}, and Petersen and Salama reveal
an algorithm to estimate the entropy of a hom-shift \cite{22}. (A hom-shift, roughly
speaking, is a $G$-SFT which is isotropic and symmetric; alternatively, a hom-shift
is determined by one rule in each direction. For instance, a $d$-dimensional golden
mean shift is a hom-shift. The interested reader is referred to \cite{13}.) Furthermore,
there is an infinite series expression for the entropy provided $|A| = d = 2$ \cite{5}.

This paper considers the topological degree of $G$-SFTs for the case where $G$ is a
finitely generated nonabelian monoid. The topological degree of a $G$-shift reflects
the idea of entropy dimension. More specifically, the topological degree of a $G$-shift
having positive topological entropy is $\ln d$, where $d$ is the number of generators
of $G$ and $G$ is a free monoid (cf. \cite{2, 7, 6}). In other words, the investigation of
topological degree relates to discovering zero entropy systems. The importance of
zero entropy systems has been revealed recently; many $\mathbb{Z}^d$-actions with zero entropy
exhibit diverse complexities. See \cite{12, 13, 15, 16} and the references therein for more
details. This elucidation extends the computation of topological degree of $G$-SFT to the case where $G$ is a monoid with finite representation (Theorem 4.2) or $G$ has finite free-followers (defined in [4], see Section 6). Notably, a finitely generated free monoid has finite representation, and a monoid with finite representation has finite free-followers.

On the other hand, Ban et al. [8] reveal that, if $G$ is a free monoid with $d$ generators, the degree spectrum of $G$-SFTs is a finite subset of Perron numbers less than or equal to $d$. Theorems 5.1 and 5.3 elaborate that the topological degree of a $G$-SFT relates to the maximal spectral radius of a collection of integral matrices which are constrained by the structure of the Cayley graph of $G$. Meanwhile, the necessary and sufficient conditions for a $G$-SFT having full topological degree are also addressed, which provide a criterion for determining whether a $G$-SFT has zero entropy.

The introduction ends with a summary of the remainder of the paper. Whenever $G$ is a monoid such that a matrix $A$ determines the relationships among the generators of $G$ and $G$ has finite representation (see Section 2), the coefficients of the characteristic polynomial of $A$ relate to the number of children of the Cayley graph of $G$ (Theorem 3.1). After revealing an algorithm for the computation of the topological degree (Theorem 4.3), the degree spectrum (Theorems 5.1 and 5.3) extends the previous result under the hypothesis that $G$ is free (cf. [8]). Furthermore, Section 6 extends the algorithm to the case where $G$ has finite free-followers.

2. Definition and Notation

Although most results in this investigation extend to groups with finite representation, the present paper focuses on shift spaces on monoids for clarity. Let $d$ be a positive integer. A semigroup is a set $G = \langle \Sigma | R \rangle$ together with a binary operation which is closed and associative, where $\Sigma = \{s_1, \ldots, s_d\}$ is the set of generators and $R$ is a set of equivalences which describe the relationships among the generators. A monoid is a semigroup with an identity element $e$.

Given a finite set of generators $\Sigma = \{s_1, s_2, \ldots, s_d\}$ and a $d \times d$ binary matrix $A$. A monoid $G$ of the form $G = \langle \Sigma | R_A \rangle$ means that

$$s_is_j = s_i \quad \text{if and only if} \quad A(i, j) = 0.$$
The Cayley graph of monoid $G$ in Example 2.1 has a finite representation. The generators $s_1, s_2, s_3$ satisfy the equivalences $s_1^2 = s_1$ and $s_2s_1 = s_2s_2 = s_2$. A pseudo-identity $e$ makes the Cayley graph of $G$ strongly connected.

Alternatively, $s_i$ is a right (resp. left) free generator if and only if $A(i, j) = 1$ ($A(j, j) = 1$) for $1 \leq j \leq d$. Let $\Sigma_R$ (resp. $\Sigma_L$) denote the set of right (resp. left) free generators of $G$. For each $g \in G$, the length $|g|$ indicates the number of generators used in its minimal presentation; that is,

$$|g| = \min\{j : g = g_1g_2\cdots g_j, g_i \in \Sigma \text{ for } 1 \leq i \leq j\}.$$

Suppose that $C = (V, E)$ is the Cayley graph of $G$. Define a subgraph $F = (V_F, E_F) \subseteq C$ as follows.

(i) $g = g_1\cdots g_n \in V_F$ if $g_n$ is the unique right free generator in $g$;
(ii) if $g_1\cdots g_n \in V_F$, then $g_1\cdots g_j \in V_F$ for each $j \leq n$;
(iii) $(g, g') \in E_F$ if and only if $(g, g') \in E$ and $g, g' \in V_F$.

A monoid $G$ has a finite representation if $F$ is a finite graph. For the rest of this paper, $G = \langle \Sigma | R_A \rangle$ denotes a monoid with a finite representation unless otherwise stated. See Example 2.1.
Example 2.1. Let $d = 3$ and let

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}. $$

The generators $s_1, s_2, s_3$ satisfy the equivalences determined by $A$; that is,

$$s_1^2 = s_1, s_2 s_1 = s_2, \text{ and } s_2^2 = s_2.$$ 

It follows that $G$ has a finite representation, see Figure 1 for the Cayley graph of $G$ together with a pseudo-identity and the graph of its finite representation.

Let $A = \{1, 2, \ldots, k\}$ be a finite alphabet, where $k$ is a natural number greater than one. A labeled tree is a function $t : G \rightarrow A$. For each $g \in G$, $t_g = t(g)$ denotes the label attached to the vertex $g$ of the Cayley graph of $G$. The full shift $A^G$ collects the labeled trees, and the shift map $\sigma : G \times A^G \rightarrow A^G$ is defined as $(\sigma_g t)_{g'} = t_{gg'}$ for $g, g' \in G$.

For each $n \geq 0$, let $\Delta_n = \{g \in G : |g| \leq n\}$ denote the initial $n$-subgraph of the Cayley graph. An $n$-block is a function $\tau : \Delta_n \rightarrow A$. A labeled tree $t$ accepts an $n$-block $\tau$ if there exists $g \in G$ such that $t_{gg'} = \tau_{g'}$ for all $g' \in \Delta_n$; otherwise, $\tau$ is a forbidden block of $t$ (or $t$ avoids $\tau$). A $G$-shift space is a set $X \subseteq A^G$ of all labeled trees which avoid all of a certain set of forbidden blocks.

3. Characterization of Finite Representation

For each $n \in \mathbb{N}$, let

$$\xi_n = \#\{g \in G : |g| = n, g_n \text{ is the only right free generator}\}$$

be the number of $n$-words which contain exactly one right free generator and end in it. Theorem 3.1 reveals $\{\xi_n\}$ plays an important role in the characteristic polynomial of $A$.

**Theorem 3.1.** Suppose $G = \langle \Sigma | R_A \rangle$ is a monoid determined by a binary matrix $A$. Then the characteristic polynomial of $A$ is

$$f(\lambda) = \lambda^d - \sum_{i=1}^{d} \xi_i \lambda^{d-i}. $$

Before proving Theorem 3.1 it is essential to characterize the structure of the Cayley graph of $G$. Let

$$P_n = \{g \in G : |g| = n + 1 \text{ and } g_1 = g_{n+1}\}$$
and
\[ \Xi_n = \{ g \in P_n : g_n \text{ is the only right free generator} \} \]
be the sets of periodic \((n + 1)\)-words and periodic \((n + 1)\)-words whose second last symbol is the one and only right free generator, respectively. It follows immediately that \(|P_n| = \text{tr}(A^n)\) and \(|\Xi_n| = \xi_n\).

**Lemma 3.2.** For each \((n + 1)\)-word \(g = g_1g_2\cdots g_ng_1 \in P_n\), there exists \(1 \leq i \leq n\) such that \(g_i\) is a right free generator.

**Proof.** Suppose not, it comes immediately that \((g_1\cdots g_n)^m \notin G\) is a vertex of \(F\) for all \(m \in \mathbb{N}\), which contradicts to \(|V_F| < \infty\). The proof is complete. \(\square\)

**Lemma 3.3.** For each positive integer \(n > d\), \(\xi_n = 0\). That is, every \((n + 1)\)-word contains at least one right free generator.

**Proof.** Suppose there exists \(n > d\) and \(g = g_1\cdots g_n \in G\) such that \(g_n\) is the only free generator. The pigeonhole principle asserts that \(g_i = g_j\) for some \(1 \leq i < j \leq d + 1\). Lemma 3.2 demonstrates that there exists \(i \leq \ell \leq j\) such that \(g_i\) is a right free generator, which is a contradiction. This derives the desired result. \(\square\)

Let
\[ T(\Xi_n) = \{ u_1\cdots u_nu_1\cdots u_i : i = 1, \ldots, n, u \in \Xi_n \} \]
collect the translation of all elements of \(\Xi_n\). Observe that \(|T(\Xi_n)| = n\xi_n\). Let
\[ L(P_m, \Xi_n) = \{ u_1\cdots u_nv_1\cdots v_nu_{n+1}\cdots u_{m+1} : s = \min\{ i : u_i \in S_R \}, u \in P_m, v \in \Xi_n \}, \]
where \(S_R\) is the set of right free generators. That is, \(L(P_m, \Xi_n)\) consists of words obtained by inserting the initial \(n\)-subword of every \(v = v_1\cdots v_{n+1} \in \Xi_n\) in a periodic \((m+1)\)-word \(u\) right after the first right free generator of \(u\). Obviously, \(L(P_m, \Xi_n) \subseteq P_{m+n}\).

**Lemma 3.4.** Suppose \(x = x_1\cdots x_{n+1} \in P_n\) contains at least two right free generators.

Let \(x_s\) and \(x_r\) be the first and second free generators, respectively. Then
\[ r - s = l \quad \text{if and only if} \quad x \in L(P_{n-l}, \Xi_l) \]

**Proof.** If \(x \in L(P_{n-l}, \Xi_l)\), then \(x = u_1\cdots u_sv_1\cdots v_lu_{s+1}\cdots u_{n-l+1}\) for some \(u \in P_{n-l}, v \in \Xi_l\), where \(s = \min\{ i : u_i \in S_R \}\). Since \(v \in \Xi_l, x_{s+l} = v_l\) is the second right free generator in \(x\). This concludes that \(r - s = (s + l) - s = l\).
For each $x = x_1 \cdots x_{n+1} \in P_n$ which contains at least two right free generators, let $u = x_1 \cdots x_s x_{s+1} \cdots x_{n+1}$ and let $v = x_{s+1} \cdots x_{s+l} x_{s+1}$. Since $u_s = x_s \in S_R$, $x_s x_{s+l-1}$ is a two-word. Furthermore, $x \in P_n$ and $x_1 = x_{n+1}$ indicates that $u_1 = u_{n-l+1}$. Alternatively, $u = u_1 \cdots u_{n-l+1} \in P_{n-l}$. Similarly, $v_l = x_{s+l} \in S_R$ shows that $x_{s+l} x_{s+1}$ is also a two-word. The fact of $x_r = x_{s+l}$ being the second free generator elaborates that $v_1, \ldots, v_{l+1} \notin S_R$, $v_l \in S_R$, and $v_{l+1} = v_1$. Hence, $v \in \Xi$. Conclusively, $x \in L(P_{n-l}, \Xi)$. This completes the proof. □

Lemma 3.4 illustrates $L(P_{n-l}, \Xi_l) \cap L(P_{n-m}, \Xi_m) = \emptyset$ if and only if $l \neq m$. Proposition 3.5 additionally, reveals that the translation and insertion of $\Xi_n$ form a partition of periodic words.

**Proposition 3.5.** For each $n \in \mathbb{N}$, $\{L(P_{n-i}, \Xi_i)\}_{i=1}^{n-1} \cup \{T(\Xi_n)\}$ forms a partition of $P_n$.

**Proof.** Obviously, $L(P_{n-i}, \Xi_i) \cap T(\Xi_n) = \emptyset$ for $1 \leq i \leq n-1$ since every element of $T(\Xi_n)$ accepts exactly one generator while $L(P_{n-i}, \Xi_i)$ consists of words which contain at least two free generators. The desired result comes immediately from

$$P_n = \bigcup_{i=1}^{n-1} L(P_{n-i}, \Xi_i) \cup T(\Xi_n).$$

Indeed, the definitions of $L(P_{n-i}, \Xi_i)$ and $T(\Xi_n)$ indicate that

$$\bigcup_{i=1}^{n-1} L(P_{n-i}, \Xi_i) \cup T(\Xi_n) \subseteq P_n.$$

For each $x \in P_n$, $x \in T(\Xi_n)$ if $x$ has exactly one free generator. Otherwise, $x$ has $x_s$ and $x_r$ as its first two free generators for some $s < r$. Let $l = r - s$. Lemma 3.4 shows that $x \in L(P_{n-l}, \Xi_l)$. The proof is complete. □

**Example 3.6.** Continue with Example 2.1 recall that

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

and $G$ has only one right free generator $s_3$. Then $\Xi_1 = \{s_3 s_3\} = T(\Xi_1) = P_1$. Since $\Xi_2$ consists of words of the form $u_1 s_3 u_1$,

$$\Xi_2 = \{s_1 s_3 s_1, s_2 s_3 s_2\} \quad \text{and} \quad T(\Xi_2) = \{s_1 s_3 s_1, s_3 s_1 s_3, s_2 s_3 s_2, s_3 s_2 s_3\}.$$  

As defined above, $L(P_1, \Xi_1) = \{s_3 s_3 s_3\}$ collects the words obtained by inserting the first word of $\Xi_1$ in each word of $P_1$ right after the first right free generator. It
follows that
\[ P_2 = \{ s_1 s_3 s_1, s_2 s_3 s_2, s_3 s_1 s_3, s_3 s_2 s_3, s_3 s_3 s_3 \} = T(\Xi_2) \cup L(P_1, \Xi_1). \]

Similarly, \( \Xi_3 = \{ s_1 s_2 s_3 s_1 \} \) and \( T(\Xi_3) = \{ s_1 s_2 s_3 s_1, s_2 s_3 s_1 s_2, s_3 s_1 s_2 s_3 \} \).
\[ L(P_2, \Xi_1) = \{ s_1 s_3 s_3 s_1, s_2 s_3 s_3 s_2, s_3 s_3 s_1 s_3, s_3 s_3 s_2 s_3, s_3 s_3 s_3 s_3 \}, \]
\[ L(P_1, \Xi_2) = \{ s_3 s_1 s_3 s_3, s_2 s_3 s_3 s_3 \}. \]

Then
\[ P_3 = \{ s_1 s_2 s_3 s_1, s_1 s_3 s_3 s_1, s_2 s_3 s_3 s_2, s_3 s_3 s_3 s_3, s_3 s_3 s_3 s_3, s_3 s_3 s_3 s_3 \}, \]
\[ = T(\Xi_3) \cup L(P_2, \Xi_1) \cup L(P_1, \Xi_2). \]

Furthermore, \( \Xi_n = \emptyset \) for \( n \geq 4 \).

For each real \( n \times n \) matrix \( A \), there is a recursive formula for the coefficients of the characteristic polynomial of \( A \); more explicitly, \( f(\lambda) = \det(A - \lambda I) = \sum_{i=0}^{n} b_i \lambda^{n-i} \), where
\[
\begin{align*}
 b_0 &= (-1)^n, & b_1 &= -(-1)^n A_1, & b_2 &= -\frac{1}{2} (b_1 A_1 + (-1)^n A_2), \\
b_3 &= -\frac{1}{3} (b_2 A_1 + b_1 A_2 + (-1)^n A_3), & \ldots \\
b_n &= -\frac{1}{n} (b_{n-1} A_1 + b_{n-2} A_2 + \cdots + b_1 A_{n-1} + (-1)^n A_n),
\end{align*}
\]
and \( A_i \) is the trace of \( A^i \) for \( 1 \leq i \leq n \) (cf. [25] p.303-305)).

**Proof of Theorem 3.1.** Proposition 3.5 shows that, for \( n \in \mathbb{N} \),
\[ |P_n| = |T(\Xi_n)| + \sum_{i=1}^{n-1} |L(P_i, \Xi_{n-i})|; \]
that is, \( A_1 = \xi_1 \) and \( A_n = n\xi_n + \sum_{i=1}^{n-1} A_i \xi_{n-i} \) for \( n \geq 2 \). Since \( \xi_n = 0 \) for \( n > d \), (1) follows from
\[ \xi_n = \frac{1}{n} (A_n - \sum_{i=1}^{n-1} A_i \xi_{n-i}), \quad 1 \leq n \leq d, \]
and the recursive formula of the coefficients of the characteristic polynomial of \( A \). \( \square \)
4. Topological Degree of Shift Spaces on Monoids

Suppose that $X$ is a $G$-shift space. Let $\Gamma_n^{[g]}(X)$ denote the set of $n$-blocks of $X$ rooted at $g$; that is, the support of each block of $\Gamma_n^{[g]}(X)$ is $g\Delta_n$. Let $\gamma_n^{[g]}$ denote the cardinality of $\Gamma_n^{[g]}(X)$. The topological degree of $X$ is defined as

$$\text{(2)} \quad \deg(X) = \limsup_{n \to \infty} \frac{\ln \gamma_n(X)}{n},$$

where $\gamma_n(X) = \gamma_n^{[e]}(X)$. The rest of this paper omits the notation $X$ when it causes no confusion.

For each $a \in A$, let $\Gamma_{a,n} \subseteq \Gamma_n^{[g]}$ consist of $n$-blocks rooted at $g$ and labeled $a$ at root. A symbol $a$ is essential if $\gamma_{a,n} = |\Gamma_{a,n}| \geq 2$ for some $n \in \mathbb{N}$; otherwise, $a$ is an inessential symbol. Proposition 4.1 indicates that the limit in (2) exists provided $X$ is a $G$-SFT, and only the essential symbols matter for calculating the topological degree.

**Proposition 4.1** (See [3]). Suppose that $X$ is a $G$-SFT. Then the limit (2) exists and

$$\text{(3)} \quad \deg(X) = \lim_{n \to \infty} \frac{\sum_{i=1}^{k} \ln \gamma_{i,n}}{n} = \lim_{n \to \infty} \frac{\sum_{i \in E} \ln \gamma_{i,n}}{n},$$

where $E \subseteq A$ denotes the set of essential symbols.

Theorem 4.2 reveals that, whenever every symbol is essential, the degree of $G$-shift of finite type ($G$-SFT) is the logarithm of the spectral radius of $A$ (recall that $G = \langle \Sigma | R_A \rangle$ is determined by a $d \times d$ matrix $A$, see Section 2).

**Theorem 4.2.** Suppose that $X$ is a $G$-SFT and every symbol is essential. Then $\deg(X) = \ln \rho_A$, where $\rho_A$ is the spectral radius of $A$.

Ban and Chang [3] reveal an algorithm for computing the degree of $G$-SFTs, where $G$ is a Fibonacci set ($G = \langle \Sigma | R_A \rangle$ with $\Sigma = \{s_1, s_2\}$ and $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$). The algorithm extends to general $G$ via analogous argument. For the compactness and self-containment of the present paper, this section rephrases main ideas and propositions of the algorithm without detailed proofs, and Example 4.4 shows how

---

1In one-dimensional symbolic dynamical systems, a graph presentation of an SFT is called essential if there is no stranded vertex [19]. In other words, every vertex has its contribution in the corresponding SFT. This paper extends the idea to the alphabet of $G$-SFTs.
the algorithm works. For more details about the algorithm, see [3] and the references therein.

Since $G$ has a finite representation, each $G$-SFT relates to a recurrence representation (or system of nonlinear recurrence equations, SNRE) of the form

$$\gamma_{i,n} = \sum c_j \gamma_{i,n}^{j_1,1} \gamma_{k,n-1}^{j_1,2} \gamma_{k,n-2}^{j_2,2} \gamma_{k,n-l}^{j_l,i}$$

for some $l \in \mathbb{N}$, where $c_j \in \mathbb{N}$, $j = (j_1,1,\ldots,j_k,1,\ldots,j_1,i,\ldots,j_k,l)$, and $1 \leq i \leq k$. A simple subsystem of $X$ is of the form

$$\gamma_{i,n} = \gamma_{i,n-1}^{j_1,1} \gamma_{k,n-1}^{j_1,2} \gamma_{k,n-2}^{j_2,2} \gamma_{k,n-l}^{j_l,i}$$

for some $j_1,1,\ldots,j_k,1,\ldots,j_1,i,\ldots,j_k,l$ and $1 \leq i \leq k$. Take logarithm on the above equation and let

$$\theta_n = (\ln \gamma_{1,n}, \ldots, \ln \gamma_{k,n}, \ln \gamma_{1,n-1}, \ldots, \ln \gamma_{k,n-1}, \ldots, \ln \gamma_{1,n-l+1}, \ldots, \ln \gamma_{k,n-l+1})'$$

where $v'$ refers to the transpose of $v$. Then there exists a $kl \times kl$ matrix $M$ called adjacency matrix (of the simple subsystem) such that $\theta_n = M \theta_{n-1}$ for $n \geq l + 1$. Theorem 4.3 reveals that the degree of $X$ relates to the maximal spectral radius among the adjacency matrices of simple subsystems of $X$.

**Theorem 4.3 (See [3]).** Suppose that $X$ is a $G$-SFT. Then

$$\deg(X) = \max \{ \ln \rho_M : M \text{ is the adjacency matrix of a simple subsystem of } X \}$$

**Example 4.4.** Suppose that $X$ is a hom-shift on $G$ determined by a $k \times k$ binary matrix $T$; that is, for each labeled tree $t \in X$ and $g \in G$, a pattern $(t_g, t_{g^n})$ is allowable if and only if $T(t_g, t_{g^n}) = 1$. For instance, consider the case where $k = 2$ and $T = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. A hom-shift defined by $T$ is a full $G$-shift and $\deg(X) = \ln \rho_A$. This example shows that the above algorithm derives the desired result.

It follows from $s_3$ being a free generator that, for $i = 1, 2$, $\gamma_{i,n}^{[g]} = \gamma_{i,n}$ if $g = g' s_3$ for some $g, g' \in G$. Hence, for $i = 1, 2$,

$$\gamma_{i,n} = (\gamma_{1,n-1}^{[s_1]} + \gamma_{2,n-1}^{[s_2]})(\gamma_{1,n}^{[s_1]} + \gamma_{2,n}^{[s_2]})(\gamma_{1,n-1}^{[s_1]} + \gamma_{2,n-1}^{[s_2]})$$

$$= (\gamma_{1,n-1}^{[s_1]} + \gamma_{2,n-1}^{[s_2]})(\gamma_{1,n}^{[s_1]} + \gamma_{2,n}^{[s_2]})(\gamma_{1,n-1} + \gamma_{2,n-1})$$

Combining

$$\gamma_{1,n-1}^{[s_1]} = (\gamma_{1,n-2}^{[s_1]} + \gamma_{2,n-2}^{[s_2]})(\gamma_{1,n-2}^{[s_1]} + \gamma_{2,n-2}^{[s_2]}) = (\gamma_{1,n-2}^{[s_1]} + \gamma_{2,n-2}^{[s_2]})(\gamma_{1,n-2} + \gamma_{2,n-2})$$

$$\gamma_{1,n-1}^{[s_2]} = (\gamma_{1,n-2}^{[s_2]} + \gamma_{2,n-2}^{[s_2]})(\gamma_{1,n-2}^{[s_2]} + \gamma_{2,n-2}^{[s_2]}) = (\gamma_{1,n-2}^{[s_2]} + \gamma_{2,n-2}^{[s_2]})(\gamma_{1,n-2} + \gamma_{2,n-2})$$
with
\[ \gamma_{i,n} = \gamma_{i,n-1}^{[s_1s_2]} + \gamma_{i,n-2}^{[s_1s_2]} + \gamma_{i,n-3}^{[s_1s_2]} + \gamma_{i,n-4}^{[s_1s_2]} + \gamma_{i,n-5}^{[s_1s_2]} + \gamma_{i,n-6}^{[s_1s_2]} \]

derives that
\[ \gamma_{i,n} = (4\gamma_{i,n-3} + 4\gamma_{i,n-2} + 4\gamma_{i,n-1} + 4\gamma_{i,n-3} + 4\gamma_{i,n-2} + 4\gamma_{i,n-1}) \]
\[ = (2\gamma_{i,n-2} + 2\gamma_{i,n-1} + \gamma_{i,n-1} + \gamma_{i,n-1}) \]

Let
\[ \theta_n = (\ln \gamma_{i,n}, \ln \gamma_{i,n-1}, \ln \gamma_{i,n-2}, \ln \gamma_{i,n-3}, \ln \gamma_{i,n-4})' \]
For every simple subsystem of \( X \), the corresponding adjacency matrix is of the form
\[ M = \begin{pmatrix} B_1 & B_2 & B_3 \\ I & 0 & 0 \\ 0 & I & 0 \end{pmatrix}, \]
where \( B_l \) is a \( 2 \times 2 \) matrix satisfies \( \sum_{pq=1}^2 B_l(p,q) = \xi_l \) for all \( l = 1, 2, 3, p = 1, 2 \). That is, \( \theta_n = M\theta_n-1 \) for \( n \geq 3 \). Let
\[ v = (\rho_A^2, \rho_A, 1)' \]
Observe that \( Mv = \rho_A v \). Perron-Frobenius Theorem demonstrates that \( \rho_A \) is also the spectral radius of \( M \). In other words, \( \deg(X) = \ln \rho_A \).

**Proof of Theorem 4.3** The proof focuses on the case where \( X \) is a \( G \)-SFT determined by \( k \times k \) binary matrices \( A_1, \ldots, A_d \) for clarification, the demonstration of the general case is analogous. In this case, for each labeled tree \( t \in X \) and \( g \in G \), \( (t_g, t_{g^u}) \) is allowable if and only if \( A_l(t_g, t_{g^u}) = 1 \) for \( 1 \leq l \leq d \).

Write \( A_l = (a_{i_1i_2}) \) for \( 1 \leq l \leq d, 1 \leq i_1, i_2 \leq k \). Since \( \gamma_{j,n}^{[g_{s_1}]} = \gamma_{j,n} \) for all \( 1 \leq j \leq k, n \in \mathbb{N}, g \in G \) provided \( s_1 \) is a free generator, for \( 1 \leq i \leq k \),
\[ \gamma_{i,n} = \prod_{s_l \in T_R} \left( \sum_{j_1=1}^k a_{s_l|s_1j_1} \gamma_{j_1,n-1}^{[s_1]} \right) \]
\[ = \prod_{s_l \in T_R} \left( \sum_{j_1=1}^k a_{s_l|j_1} \gamma_{j_1,n-1}^{[n]} \right) \prod_{s_l \in T_R} \left( \sum_{j_1=1}^k a_{s_l|j_1} \gamma_{j_1,n-1}^{[s_1]} \right) \]
\[ = \prod_{s_l \in T_R} \left( \sum_{j_1=1}^k a_{s_l|j_1} \gamma_{j_1,n-1}^{[n]} \right) \prod_{s_l \in T_R} \left( \sum_{j_1=1}^k a_{s_l|j_1} \gamma_{j_1,n-1}^{[s_1]} \right) \]
Observe that \( f_1 = \prod_{s_l \in T_R} (\sum_{j_1=1}^k a_{s_l|j_1} \gamma_{j_1,n-1}) \) is a polynomial of degree \( \xi_1 \) over \( \gamma_{i,n-1}, \ldots, \gamma_{k,n-1} \).
Similarly, for each $s_l$ which is not a free generator,

$$\gamma_{j_1,n-1}^{[s_1]} = \prod_{s_l s_m \in G} (\sum_{j_2=1}^k a_{s_m;j_1,j_2} \gamma_{j_2,n-2}^{[s_l s_m]})$$

infers that

$$\gamma_{i,n} = f_1 \cdot \prod_{s_l s_m \in G, s_l \notin S_R} (\sum_{j_2=1}^k a_{s_l i,j_1} a_{s_m;j_1,j_2} \gamma_{j_2,n-2}^{[s_l s_m]}) .$$

Let

$$f_2 = \prod_{s_l s_m \in G, s_l \notin S_R, s_m \in S_R} (\sum_{j_2=1}^k a_{s_l i,j_1} a_{s_m;j_1,j_2} \gamma_{j_2,n-2}^{[s_l s_m]})$$

Then $f_2$ is a polynomial of degree $\xi_2$. Repeating the same process decompose

$$\gamma_{i,n} = f_1 f_2 \cdots f_\ell,$$

where $\ell = \max\{j : \xi_j \neq 0\} \leq d$, and $f_j$ is a polynomial of degree $\xi_j$ over $\gamma_{i,n-j}, \ldots, \gamma_{k,n-j}$ for $1 \leq j \leq \ell$.

Let

$$\theta_n = (\ln \gamma_{1,n}, \ldots, \ln \gamma_{k,n}, \ln \gamma_{1,n-1}, \ldots, \ln \gamma_{k,n-1}, \ldots, \ln \gamma_{1,n-d+1}, \ldots, \ln \gamma_{k,n-d+1})'.$$

For each simple subsystem of $X$, there exists

$$M = \begin{pmatrix} B_1 & B_2 & B_3 & \cdots & B_\ell \\ I & 0 & \cdots & \cdots & 0 \\ 0 & I & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & I & 0 \end{pmatrix},$$

where $B_j$ is a $k \times k$ nonnegative integral matrix satisfies $\sum_{q=1}^k B_j(p,q) = \xi_i$ for all $1 \leq j \leq \ell, 1 \leq p \leq k$, such that $M$ is the corresponding adjacency matrix (note that $\xi_j = 0$ for $j > \ell$). That is, $\theta_n = M \theta_{n-1}$ is the designated simple subsystem. Let $v = (\rho_A^{j-1} \cdots \rho_A 1)' \otimes 1_k$, where $\otimes$ is the Kronecker product and $1_k \in \mathbb{R}^k$ is the vector consisting of 1’s. It comes immediately that $Mv = \rho_A v$. Perron-Frobenius Theorem infers that $\rho_A$ is also the spectral radius of $M$. Hence, $\deg(X) = \ln \rho_A$.

This completes the proof. \qed

Remark 4.5. For the general cases, Proposition 4.1 demonstrates that Theorem 4.3 holds if the rows and columns of matrix $M$ indexed by inessential symbols are eliminated.
5. Degree Spectrum of $G$-SFTs

Theorem 4.2 reveals that the degree of $G$-SFTs is $\ln \rho_A$ whenever every symbol is essential. This section extends to the general case and gives the complete characterization of degree spectrum of $G$-SFTs.

Let $\mathbb{Z}_+^d$ be the set of nonnegative integers. For $m, n \in \mathbb{Z}_+^d$, define $m \preceq n$ if $m_i \leq n_i$ for $1 \leq i \leq d$, and $m < n$ if $m \preceq n$ and $m \neq n$. Theorem 5.1 characterizes the degree spectrum of $G$-SFTs for the case where $k = 2$.

Theorem 5.1. Suppose that $k = 2$. Let $\xi = (\xi_1, \ldots, \xi_d)$. The degree spectrum of $G$-SFTs is

$$H = \{ \ln \lambda : \lambda = \max \{ x : x^d - \sum_{i=1}^{d} \alpha_i x^{d-i} = 0 \} \text{ for some } \alpha \in \mathbb{Z}_+^d, \alpha \preceq \xi \}. $$

Proof. Obviously, two inessential symbols infers that the degree is 0; Theorem 4.2 indicates the degree is $\ln \rho_A$ and $\rho_A = \max \{ x : x^d - \sum_{i=1}^{d} \beta_i x^{d-i} = 0 \}$ if every symbol is essential. It suffices to consider the case where $1 \in \mathcal{A}$ is essential and $2 \in \mathcal{A}$ is inessential.

Without loss of generality, assume that $\xi_i > 0$ for $1 \leq i \leq d$. Similar to the discussion in Example 4.4 write $\gamma_{1,n} = f_1 f_2 \cdots f_d$, where

$$f_1 = \prod_{u_1 \in S_R} \left( \sum_{j_1=1}^{2} a_{u_1;1,j_1} \gamma_{j_1,n-1} \right)$$

and

$$f_i = \prod_{u_1 \cdots u_{i-1} \in G, u_i \in S_R, u_1 \cdots u_i \in S_R} \left( \sum_{j_1=1}^{2} a_{u_1;1,j_1} a_{u_2;j_1,j_2} \cdots a_{u_i,j_{i-1},j_i} \gamma_{j_i,n-i} \right)$$

for $2 \leq i \leq d$, and $f_i$ is a polynomial of degree $\xi_i$. Hence, every simple subsystem of $X$ is of the form

$$\gamma_{1,n} = \gamma_{1,n-1}^{\eta_1} \gamma_{2,n-1}^{\tau_1} \gamma_{1,n-2}^{\eta_2} \gamma_{2,n-2}^{\tau_2} \cdots \gamma_{1,n-d}^{\eta_d} \gamma_{2,n-d}^{\tau_d},$$

$$\gamma_{2,n} = \gamma_{2,n-1}^{\xi_1} \gamma_{3,n-2}^{\xi_2} \cdots \gamma_{2,n-d}^{\xi_d},$$

where $\eta_i + \tau_i = \xi_i$ for $1 \leq i \leq d$. 

Let $\theta_n = (\ln \gamma_{1,n}, \ln \gamma_{2,n}, \ln \gamma_{1,n-1}, \ln \gamma_{2,n-1}, \ldots, \ln \gamma_{1,n-d+1}, \ln \gamma_{2,n-d+1})'$, and let

$$M = \begin{pmatrix} 0 & \xi_1 & 0 & \xi_2 & \cdots & 0 & \xi_d \\ 1 & 1 & \cdots & \cdots & \cdots & \cdots & \cdots \\ \vdots & \vdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 0 & 0 & \cdots & \cdots & \cdots & \cdots \\ \end{pmatrix}.$$ 

Then the simple subsystem is $\theta_n = M\theta_{n-1}$. Since 2 is inessential, the degree of such a simple subsystem is $\ln \lambda$, where $\lambda$ is the spectral radius of $M$. A straightforward examination elaborates that $\lambda = \max \{x : x^d - \sum_{i=1}^d \alpha_i x^{d-i} = 0\}$. This derives

$$H \subseteq \{\ln \lambda : \lambda = \max \{x : x^d - \sum_{i=1}^d \alpha_i x^{d-i} = 0\} \text{ for some } \alpha \in \mathbb{Z}_+^d, \alpha \preceq \xi\}.$$ 

To show that, for each $\alpha \in \mathbb{Z}_+^d$ satisfying $\alpha \preceq \xi$, there exists a $G$-SFT such that $\deg X = \ln \lambda$ with $\lambda = \max \{x : x^d - \sum_{i=1}^d \alpha_i x^{d-i} = 0\}$, construct a one-step $G$-SFT as follows. Without loss of generality, assume that $\xi_i > 0$ for $i \leq d$. The symbol 2 is inessential in the following construction, thus it suffices to mention where to label 1. 

For $n \in \mathbb{N}$, let $S_1 = \{e\}$ and, for $n \geq 2$, let

$$S_n = \{g = g_1 \ldots g_{n-1} : gs \in G \text{ for some } s \in S_R, g_i \in S_R \text{ for } 1 \leq i \leq n-1\}.$$ 

Observe that $S_n = \emptyset$ if and only if $n > d$ (under the assumption that $\xi_n = 0$ if and only if $n > d$). Let

$$\overline{S}_n = \{gs : g \in S_n, s \in \Sigma, gs \in G\}.$$ 

Then $\bigcup_{n=1}^d \overline{S}_n$ is the set of supports of two-blocks of $X$ up to shift. For $n = 1$, let $B_1 \subseteq \overline{S}_1$ consists of 1-blocks $\phi$ which satisfy $\phi_g = 1$ if and only if $g \in S_1 \cup \Sigma \setminus S_R$ and $|\{g \in S_R : \phi_g = 1\}| = \alpha_1$. 

In other words, each pattern of $B_1$ labels 1 at, except from the root and non-free generators, arbitrary $\alpha_1$ free generators. This makes $\max \{p : \gamma_{1,n-1}^p | \gamma_{1,n} = \alpha_1\}$. 

Analogously, let \( B_2 \subseteq S_2^A \) consists of 1-blocks \( \phi \) which satisfy \( \phi_g = 1 \) if and only if
\[
g \in S_2 \cup \{ g' \in S_2 : s \notin S_R \} \quad \text{and} \quad |\{ g' \in S_2 : s \in S_R, \phi_{g'} = 1 \}| = \alpha_2.
\]
Then \( \max \{ p : \gamma_{1,n-2}^p \gamma_{1,n} \} = \alpha_2 \). Repeating the same process to construct \( B_i \) for \( i \leq d \) makes
\[
\max \{ p : \gamma_{1,n-i}^p \gamma_{1,n} \} = \alpha_i \quad \text{for} \quad 1 \leq i \leq d.
\]
For each subset \( H \subseteq G \) such that \( H \) forms the support of a one-block, observe that there exists \( g \in G \) ended in free generator and \( 1 \leq i \leq d \) such that \( H = gH' \) for some \( H' \subseteq S_i \). Then each labeled pattern of support \( H \) follows the same rule as determined in \( S_i \). Notably, Such a pattern is still in \( B_i \).

Therefore, every simple subsystem of \( X \) generated by \( B = \bigcup_{i=1}^d B_i \) is of the form
\[
\gamma_{1,n} = c \cdot \gamma_{1,n-1}^{\alpha_1} \gamma_{2,n-1}^{\alpha_2} \cdots \gamma_{2,n-d}^{\alpha_d} \gamma_{2,n-d+1}^{\beta_1} \cdots \gamma_{2,n-d}^{\beta_d} \quad \gamma_{2,n} = \gamma_{2,n-1}^{\beta_1} \cdots \gamma_{2,n-d}^{\beta_d},
\]
where \( c \) is a constant, and \( \alpha_i + \beta_i = \xi_i \) for all \( i \). A straightforward examination indicates that \( \deg X = \ln \lambda \) with \( \lambda = \max \{ x : x^d - \sum_{i=1}^d \alpha_i x^{d-i} = 0 \} \).

The proof is complete. \( \square \)

**Remark 5.2.** Notably, \( \xi_n = 0 \) for \( n \geq 2 \) if and only if \( G \) is a free monoid. In this case, \( H = \{0, \ln 2, \ldots, \ln d\} \) is revealed in [8].

**Theorem 5.3.** Let \( \mathcal{M} \) be the set consisting of
\[
M = \begin{pmatrix}
C_1 & C_2 & C_3 & \cdots & C_d \\
I & 0 & \cdots & \cdots & 0 \\
0 & I & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \cdots & \vdots \\
0 & \cdots & 0 & I & 0
\end{pmatrix}
\]
for some \( l \times l \) matrices \( C_i \), \( l \leq k \), satisfying \( \sum_{q=1}^l C_i(p,q) \leq \xi_i \) for all \( 1 \leq i \leq d \), \( 1 \leq p \leq l \). The degree spectrum of \( G \)-SFTs is
\[
H = \{ \ln \lambda : \lambda \text{ is the spectral radius of } M \in \mathcal{M} \}.
\]

Corollary 5.4, follows from the proof of Theorem 5.1, elaborates a necessary and sufficient condition of a \( G \)-SFT achieved full degree.
Corollary 5.4. Suppose that $X$ is a $G$-SFT. Then $\deg(X) = \ln \rho_A$ if and only if the essential symbols form a subshift on right free generators; that is, for each $s \in S_R$ and $\phi$ is a one-block with support $\text{supp}(\phi) = s\Sigma$, $\phi_g$ is essential for $g \in \text{supp}(\phi)$.

Proof. It suffices to consider the case where $k = 2$ since the demonstration of the general case is analogous but more complicated. Recall that, in the proof of Theorem 5.1, every simple subsystem of $X$ is of the form

$$\gamma_{1,n} = \gamma_{1,n-1}^{\eta_1} \gamma_{2,n-1}^{\tau_1} \gamma_{1,n-2}^{\eta_2} \gamma_{2,n-2}^{\tau_2} \cdots \gamma_{1,n-d}^{\eta_d} \gamma_{2,n-d}^{\tau_d},$$

$$\gamma_{2,n} = \gamma_{1,n-1}^{\delta_1} \gamma_{2,n-1}^{\iota_1} \gamma_{1,n-2}^{\delta_2} \gamma_{2,n-2}^{\iota_2} \cdots \gamma_{1,n-d}^{\delta_d} \gamma_{2,n-d}^{\iota_d},$$

where $\eta_i + \tau_i = \delta_i + \iota_i = \xi_i$ for $1 \leq i \leq d$. In other words, $\deg(X) = \ln \lambda$, where $\lambda$ is the spectral radius of one of the following matrix, which depends on the essential of symbols.

$$M_1 = \begin{pmatrix}
\eta_1 & \tau_1 & \eta_2 & \tau_2 & \cdots & \eta_d & \tau_d \\
\delta_1 & \iota_1 & \delta_2 & \iota_2 & \cdots & \delta_d & \iota_d \\
1 & 0 & 0 & 0 & \cdots & 0 & 1 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \cdots & \ddots & \vdots \\
1 & 0 & 0 & 0 & \cdots & 0 & 1
\end{pmatrix},$$

$$M_2 = \begin{pmatrix}
\eta_1 & \eta_2 & \cdots & \eta_d \\
1 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
1 & 0 & \cdots & 1
\end{pmatrix},$$

$$M_3 = \begin{pmatrix}
\tau_1 & \tau_2 & \cdots & \tau_d \\
1 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
1 & 0 & \cdots & 1
\end{pmatrix}.$$
Set $F = \{F_g : g \in G\}$. Then $G$ has finite free-followers if $F$ is finite. It is easily seen that every finitely generated free monoid $G$ has finite -free-followers since $F_g = G$ for each $g \in G$. The investigation in Sections 4 and 5 extends to the case where $G$ has finite free-followers via analogous elaboration. This section, rather than rephrases every result in the previous two sections, presents an example to address how to compute the degree of a $G$-SFT ($G$ has finite free-followers herein) for the compactness of the paper.

Suppose that $d = k = 2$. In this case, $\Sigma = \{s_1, s_2\}$ and $A = \{1, 2\}$. Let $G = (\Sigma|R)$ be the monoid with $R = \{s_2s_1^{2i+1}s_2 = s_2\}_{i \geq 0}$. It follows that $G$ has finite free-followers. Indeed, let

$$F_{s_1} = \{s_1, s_2, s_1^2, s_1s_2, s_2s_1, s_2^2, \ldots\} = G,$$

$$F_{s_2} = \{s_1, s_2, s_1^2, s_1s_2, s_2s_1, s_1^3, s_1^2s_2, s_2s_1^2, s_1s_2^2, \ldots\} = \{s^n_1\}_{n \geq 1} \cup \{(s_1^{2i}s_2g : g = s^n_2, s_1^{2j}, i, j, n \geq 0)\},$$

$$F_{s_2s_1} = \{s_1, s_2, s_1^2, s_1s_2, s_1^3, s_1^2s_2, s_2s_1^2, s_1s_2^2, \ldots\} = \{s^n_1\}_{n \geq 1} \cup \{(s_1^{2i+1}s_2g : g = s^n_2, s_1^{2j}, i, j, n \geq 0)\}.$$ 

An examination indicates that, for each $g \in G$,

$$F_g = \begin{cases} 
F_{s_1}, & g = s^n_1; \\
F_{s_2}, & g \text{ ends in } s_2s_1^{2i}, i \geq 0; \\
F_{s_2s_1}, & g \text{ ends in } s_2s_1^{2i+1}, i \geq 0.
\end{cases}$$

A straightforward examination elaborates that there is a one-to-one correspondence between the monoid $G$ and the set of finite words of one-dimensional even-shift.

Let $X$ be a hom-shift on $G$ determined by $T = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. Alternatively, $X$ is a full $G$-shift; it follows immediately that $\deg(X) = \ln \lambda$, where $\lambda = \frac{1 + \sqrt{5}}{2}$ satisfies $\lambda^2 - \lambda - 1 = 0$. The following shows that the algorithm in Section 4 derives the same result.

Observe that $\gamma_{i,n}^{[s]} = \gamma_{i,n}$ for $i = 1, 2$ since $F_{s_1} = G$ for $j \in \mathbb{N}$. For $i = 1, 2$,

$$\gamma_{i,n} = (\gamma_{i,n}^{[s_1]} + \gamma_{i,n}^{[s_2]})(\gamma_{i,n-1}^{[s_1]} + \gamma_{i,n-1}^{[s_2]}),$$

$$= (\gamma_{i,n-1}^{[s_1]} + \gamma_{i,n-1}^{[s_2]})(\gamma_{i+1,n-1}^{[s_1]} + \gamma_{i+1,n-1}^{[s_2]}).$$

Also, $F_{s_2} = F_{s_2}$ and $F_{s_2s_1} = F_{s_2}$ infer that

$$\gamma_{i,n}^{[s_2]} = (\gamma_{i,n-1} + \gamma_{i,n-2}^{[s_2s_1]})(\gamma_{i,n-1} + \gamma_{i,n-2}^{[s_2s_1]}),$$

$$= (\gamma_{i,n-1}^{[s_2s_1]} + \gamma_{i,n-2}^{[s_2s_1]})(\gamma_{i+1,n-1} + \gamma_{i+1,n-1}^{[s_2s_1]}),$$

$$\gamma_{i,n}^{[s_2s_1]} = (\gamma_{i,n-1}^{[s_2s_1]} + \gamma_{i,n-2}^{[s_2s_1]} + \gamma_{i,n-1}^{[s_2s_1]} + \gamma_{i,n-2}^{[s_2s_1]}).$$
Hence, the SNRE of $X$ is
\[
\gamma_{i,n} = 2(\gamma_{1,n-1} + \gamma_{2,n-1})(\gamma_{1,n-2} + \gamma_{2,n-2})(\gamma_{1,n-3} + \gamma_{2,n-3})
\]
for $i = 1, 2$. Let $\theta_n = (\ln \gamma_{1,n}, \ln \gamma_{2,n}, \ln \gamma_{1,n-1}, \ln \gamma_{2,n-1})'$ and let
\[
M = \begin{pmatrix}
\eta_1 & \tau_1 & \eta_2 & \tau_2 \\
\delta_1 & \iota_1 & \delta_2 & \iota_2 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0
\end{pmatrix}.
\]

Then, every simple subsystem of the invariant system $\ln \gamma_{1,n}, \ln \gamma_{2,n}$ is of the form $\theta_n = M\theta_{n-1}$ with $\eta_j + \tau_j = \delta_j + \iota_j = 1$ for $1 \leq j \leq 2$. It follows that $\ln \gamma_{i,n} \approx e^{\lambda n}$ for $i = 1, 2$ and $n$ large enough.

Furthermore, every simple subsystem of $X$ is of the form
\[
\ln \gamma_{1,n} \approx \eta \ln \gamma_{1,n-1} + \tau \ln \gamma_{2,n-1} + e^{(n-2)\lambda} + e^{(n-3)\lambda}
\]
\[
\ln \gamma_{2,n} \approx \delta \ln \gamma_{1,n-1} + \iota \ln \gamma_{2,n-1} + e^{(n-2)\lambda} + e^{(n-3)\lambda}
\]
where $\eta + \tau = \delta + \iota = 1$. A straightforward examination shows that
\[
\deg(X) = \lim_{n \to \infty} \frac{\ln(\ln \gamma_{1,n} + \ln \gamma_{2,n})}{n} = \ln \lambda.
\]
This concludes the desired result.

References

1. N. Aubrun and M.-P. Béal, Tree-shifts of finite type, Theor. Comput. Sci. 459 (2012), 16–25.
2. J.-C. Ban and C.-H. Chang, Characterization for entropy of shifts of finite type on Cayley trees, 2017, arXiv:1705.03138.
3. , Coloring Fibonacci-Cayley tree: An application to neural networks, 2017, arXiv:1707.02227.
4. , Mixing properties of tree-shifts, J. Math. Phys. 58 (2017), 112702.
5. , On the topological entropy of subshifts of finite type on free semigroups, preprint, 2017.
6. , Tree-shifts: Irreducibility, mixing, and chaos of tree-shifts, Trans. Am. Math. Soc. 369 (2017), 8389–8407.
7. , Tree-shifts: The entropy of tree-shifts of finite type, Nonlinearity 30 (2017), 2785–2804.
8. J.-C. Ban, C.-H. Chang, and N.-Z. Huang, *Entropy bifurcation of neural networks on Cayley trees*, 2017, arXiv:1706.09283.

9. R. Berger, *The undecidability of the domino problem*, Mem. Amer. Math. Soc. 66 (1966).

10. R. Bowen, *Equilibrium states and the ergodic theory of Anosov diffeomorphisms*, Springer-Verlag, Berlin-New York, 1975.

11. M. Boyle, R. Pavlov, and M. Schraudner, *Multidimensional sofic shifts without separation and their factors*, Trans. Am. Math. Soc. 362 (2010), 4617–4653.

12. M. Carvalho, *Entropy dimension of dynamical systems*, Port. Math. 54 (1997), 19–40.

13. N. Chandgotia and B. Marcus, *Mixing properties for hom-shifts and the distance between walks on associated graphs*, Pacific J. Math. 294 (2018), 41–69.

14. W.-C. Cheng and B. Li, *Zero entropy systems*, J. Stat. Phys. 140 (2010), 1006–1021.

15. D. Dou, W. Huang, and K. K. Park, *Entropy dimension of topological dynamical systems*, Trans. Am. Math. Soc. 363 (2011), 659–680.

16. D. Dou, W. Huang, and K. K. Park, *Entropy dimension of measure preserving systems*, 2018.

17. M. Hochman and T. Meyerovitch, *A characterization of the entropies of multidimensional shifts of finite type*, Ann. of Math. 171 (2010), 2011–2038.

18. D. Lind, *The entropies of topological Markov shifts and a related class of algebraic integers*, Ergodic Theory Dynam. Systems 4 (1984), no. 02, 283–300.

19. D. Lind and B. Marcus, *An introduction to symbolic dynamics and coding*, Cambridge University Press, Cambridge, 1995.

20. B. Marcus and R. Pavlov, *Approximating entropy for a class of $\mathbb{Z}^2$ Markov random fields and pressure for a class of functions on $\mathbb{Z}^2$ shifts of finite type*, Ergodic Theory Dynam. Systems 33 (2013), 186–220.

21. R. Pavlov and M. Schraudner, *Classification of sofic projective subdynamics of multidimensional shifts of finite type*, Trans. Am. Math. Soc. 367 (2015), 3371–3421.

22. K. Petersen and I. Salama, *Tree shift complexity*, Theoret. Comput. Sci. (2018), https://doi.org/10.1016/j.tcs.2018.05.034.
23. S. T. Piantadosi, *Symbolic dynamics on free groups*, Discrete Contin. Dyn. Syst. **20** (2008), 725–738.

24. D. Ruelle, *Thermodynamic formalism: The mathematical structures of classical equilibrium statistical mechanics*, Addison-Wesley Publishing Co., Reading, Mass., 1978.

25. L. A. Zadeh and C. A. Desoer, *Linear system theory: The state approach*, McGraw-Hill, New York, 1963.

(Jung-Chao Ban) **DEPARTMENT OF APPLIED MATHEMATICS, NATIONAL DONG HWA UNIVERSITY, HUALIEN 97401, TAIWAN, ROC.**

E-mail address: jcban@gms.ndhu.edu.tw

(Chih-Hung Chang and Nai-Zhu Huang) **DEPARTMENT OF APPLIED MATHEMATICS, NATIONAL UNIVERSITY OF KAOSIUNG, KAOSIUNG 81148, TAIWAN, ROC.**

E-mail address: chchang@nuk.edu.tw; naizhu7@gmail.com