IS QUANTUM SPACETIME INFINITE DIMENSIONAL?

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Abstract

The Stringy Uncertainty relations, and corrections thereof, were explicitly derived recently from the New Relativity Principle that treats all dimensions and signatures on the same footing and which is based on the postulate that the Planck scale is the minimal length in Nature in the same vein that the speed of light was taken as the maximum velocity in Einstein’s theory of Special Relativity. A simple numerical argument is presented which suggests that Quantum Spacetime may very well be infinite dimensional. A discussion of the repercussions of this new paradigm in Physics is given. A truly remarkably simple and plausible solution of the cosmological constant problem results from the New Relativity Principle: The cosmological constant is not constant, in the same vein that Energy in Einstein’s Special Relativity is observer dependent. Finally, following El Naschie, we argue why the observed $D = 4$ world might just be an average dimension over the infinite possible values of the Quantum Spacetime and why the compactification mechanisms from higher to four dimensions in String theory may not be actually the right way to look at the world at Planck scales.

1. Preface

Before starting, we wish to say that readers already familiar with [1] may skip the second section entirely. We deem it absolutely necessary to repeat the calculations that led us to the String Uncertainty Relations, corrections thereof, and the direct link between the Regge behaviour of string theory with the area quantization [1]. In order to understand the main results of this work in section 3 one must follow closely section 2. We apologize for having repeated the results of [1].

The New Relativity Principle that encompasses the ideas of Noncommutative C-spaces, [1], Polydimensional Covariance [2] and Scale Relativity [3] offers, in addition to the straightforward derivation of the String Uncertainty Relations, corrections thereof, and the direct link between the Regge behaviour of string theory with the area quantization [1]. In order to understand the main results of this work in section 3 one must follow closely section 2. We apologize for having repeated the results of [1].

The New Relativity Principle that encompasses the ideas of Noncommutative C-spaces, [1], Polydimensional Covariance [2] and Scale Relativity [3] offers, in addition to the straightforward derivation of the String Uncertainty Relations, a truly remarkable and simple solution to the cosmological constant problem: the so called cosmological constant is not a constant. This is shown following the same arguments that Einstein gave when he showed that the energy was observer dependent: it is in the eye of the beholder. Finally, following El Naschie, we argue why the observed $D = 4$ world might just be an average dimension over the infinite possible values of the Quantum Spacetime and why the compactification mechanisms from higher to four dimensions in String theory may not be actually the right way to look at the world at Planck scales.

2. Introduction: String Uncertainty Relations from the New Relativity Principle

Recently we have proposed that a New Relativity principle may be operating in Nature which could reveal important clues to find the origins of $M$ theory [1]. We were forced to introduce this new Relativity principle, where all dimensions and signatures of spacetime are on the same footing, to find a fully covariant formulation of the $p$-brane Quantum Mechanical Loop Wave equations. This New Relativity Principle, or the principle of Polydimensional Covariance as has been called by Pezzaglia, has also been crucial in the derivation of Papapetrou’s equations of motion of a spinning particle in curved spaces that was a long standing problem which lasted almost 50 years [2]. A Clifford calculus was used where all the equations were written in terms of Clifford-valued multivector quantities; i.e. one had to abandon the use of vectors and tensors and replace them by Clifford-algebra valued quantities, matrices, for example.

In this section we will explicitly derive the String Uncertainty Relations, and corrections thereof, directly from the Quantum Mechanical Wave equations on Noncommutative Clifford manifolds or C-spaces [1]. There was a one-to-one correspondence between the nested hierarchy of point, loop, 2-loop, 3-loop,........p-loop histories encoded in terms of hypermatrices and wave equations written in terms of Clifford-algebra valued multivector quantities. This permits us to recast the QM wave equations associated with the hierarchy of
nested $p$-loop histories, embedded in a target spacetime of $D$ dimensions, where the values of $p$ range from $p = 0, 1, 2, 3, ..., D - 1$, as a single QM line functional wave equation whose lines live in a Noncommutative Clifford manifold of $2^D$ dimensions. $p = D - 1$ is the maximum value of $p$ that saturates the embedding spacetime dimension.

The line functional wave equation in the Clifford manifold, C-space is:

$$\int d\Sigma \left( \frac{\delta^2}{\delta X(\Sigma)\delta X(\Sigma)} + \mathcal{E}^2 \right) \Psi[X(\Sigma)] = 0. \quad (1)$$

where $\Sigma$ is an invariant evolution parameter of $1^D$ dimensions generalizing the notion of the invariant proper time in Special Relativity linked to a massive point particle line (path) history:

$$(d\Sigma)^2 = (d\Omega_{p+1})^2 + \Lambda^2 p (dx^\mu dx_\mu) + \Lambda^{2(p-1)} (d\sigma^{\mu\nu} d\sigma_{\mu\nu}) + \Lambda^{2(p-2)} (d\sigma^{\mu\nu\rho} d\sigma_{\mu\nu\rho}) + \ldots \quad (2)$$

$\Lambda$ is the Planck scale in $D$ dimensions. $X(\Sigma)$ is a Clifford-algebra valued "line" living in the Clifford manifold (C-space):

$$X = \Omega_{p+1} + \Lambda^p x_\mu \gamma^\mu + \Lambda^{p-1} \sigma_{\mu\nu} \gamma^\mu \gamma^\nu + \Lambda^{p-2} \sigma_{\mu\nu\rho} \gamma^\mu \gamma^\nu \gamma^\rho + \ldots \quad (3a)$$

The multivector $X$ encodes in one single stroke the point history represented by the ordinary $x_\mu$ coordinates and the holographic projections of the nested family of 1-loop, 2-loop, 3-loop...$p$-loop histories onto the embedding coordinate spacetime planes given respectively by:

$$\sigma_{\mu\nu}, \sigma_{\mu\nu\rho}, \ldots, \sigma_{\mu_1 \mu_2 \ldots \mu_{p+1}} \quad (3b)$$

The scalar $\Omega_{p+1}$ is the invariant proper $p + 1 = D$-volume associated with the motion of the (maximal dimension) $p$-loop across the $D = p + 1$-dim target spacetime. There was a coincidence condition [1] that required to equate the values of the center of mass coordinates $x_\mu$, for all the $p$-loops, with the values of the $x^\mu$ coordinates of the point particle path history. This was due to the fact that upon setting $\Lambda = 0$ all the $p$-loop histories collapse to a point history. The latter history is the baseline where one constructs the whole hierarchy. This also required a proportionality relationship:

$$\tau \sim A \sim \frac{V}{\Lambda^2} \sim \ldots \sim \frac{\Omega_{p+1}}{\Lambda^p} \quad (4)$$

$\tau, A, V, \ldots, \Omega_{p+1}$ represent the invariant proper time, proper area, proper volume, proper $p + 1$-dim volume swept by the point, loop, 2-loop, 3-loop, ..., $p$-loop histories across their motion through the embedding spacetime, respectively. $\mathcal{E} = T$ is a quantity of dimension $(mass)^{p+1}$, the maximal $p$-brane tension ($p = D - 1$).

The wave functional $\Psi$ is in general a Clifford-valued, hypercomplex number. In particular it could be a complex, quaternionic or octonionic valued quantity. At the moment we shall not dwell on the very subtle complications and battles associated with the quaternionic/octonion extensions of Quantum Mechanics [14] based on Division algebras and simply take the wave function to be a complex number. The line functional wave equation for lines living in the Clifford manifold (C-spaces) are difficult to solve in general. To obtain the String Uncertainty Relations, and corrections thereof, one needs to simplify them. The most simple expression is to write the simplified wave equation in units $\hbar = c = 1$:

$$[-(\frac{\partial^2}{\partial x^\mu \partial x_\mu} + \frac{\Lambda^2}{2} \frac{\partial^2}{\partial \sigma^{\mu\nu} \partial \sigma_{\mu\nu}} + \frac{\Lambda^4}{3!} \frac{\partial^2}{\partial \sigma^{\mu\nu\rho} \partial \sigma_{\mu\nu\rho}} + \ldots) - \Lambda^{2p} \mathcal{E}^2] \Psi[x^\mu, \sigma^{\mu\nu}, \sigma^{\mu\nu\rho}, \ldots] = 0 \quad (5)$$

where we have dropped the first component of the Clifford multivector dependence, $\Omega^{p+1}$, of the wave functional $\Psi$ and we have replaced functional differential equations for ordinary differential equations. Had one kept the first component dependence $\Omega^{p+1}$ on $\Psi$ one would have had a cosmological constant contribution to the $\mathcal{E}$ term as we will see below. Similar types of equations in a different context with only the first two terms of eq-(5), have also been written in [2].

The last equation contains the seeds of the String Uncertainty Relations and corrections thereof. Plane wave type solutions to eq-(5) are:
\[ \Psi = e^{i(k_\mu x^\mu + k_{\mu\nu} x^\mu x^\nu + k_{\mu\nu\rho} x^\mu x^\nu x^\rho + \ldots).} \]

where \( k_{\mu\nu}, k_{\mu\nu\rho} \ldots \) are the area-momentum, volume-momentum, \( p + 1 \)-volume-momentum conjugate variables to the holographic \( \sigma_{\mu\nu}, \sigma_{\mu\nu\rho} \ldots \) coordinates respectively. These are the components of the Clifford-algebra valued multivector \( \mathbf{K} \) that admits an expansion into a family of antisymmetric tensors of arbitrary rank like the Clifford-algebra valued "line" \( \mathbf{X} \) did earlier in eq-(3a). The multivector \( \mathbf{K} \) is nothing but the conjugate polymomentum variable to \( \mathbf{X} \) in \( \mathbf{C} \)-space. Inserting the plane wave solution into the simplified wave equation yields the generalized dispersion relation, after reinserting the suitable powers of \( \hbar \):

\[ \hbar^2 (k^2 + \frac{1}{2} \Lambda^2 (k_{\mu\nu})(k^{\mu\nu}) + \frac{1}{3!} \Lambda^4 (k_{\mu\nu\rho})(k^{\mu\nu\rho}) + \ldots) - \frac{\Lambda^2 \mathcal{E}^2}{\hbar^2 p} = 0. \]

This is just the generalization of the ordinary wave/particle dispersion relationship

\[ p^2 = \hbar^2 k^2. \quad p^2 - m^2 = 0. \]

Had one included the \( \Omega^{p+1} \) dependence on \( \Psi \): i.e an extra piece \( \exp [i \Omega_{p+1} \lambda] \), where \( \lambda \) is the cosmological constant of dimensions \( (mass)^{(p+1)} \). The required \( -\Lambda^{2p} \hbar^2 \mathcal{E} / (\partial \Omega_{p+1})^2 \) term of the simplified wave equation (5) would have generated an extra term of the form \( \Lambda^{2p} \mathcal{E}^2 \). After reinserting the suitable powers of \( \hbar \), the cosmological constant term will precisely shift the value of the \(-\Lambda^{2p} \mathcal{E}^2 / h^{2p} \) piece of eq-(7) to the value:

\[ -\left(\frac{\Lambda}{\hbar}\right)^2 \mathcal{E}^2 - \lambda^2 \],

which precisely has an overall dimension of \( m^2 \) as expected.

Hence, this will be then the " vacuum " contribution to maximal p-brane tension \( (p = D - 1) : \mathcal{E} = T_p \) has overall units \( (mass)^{p+1} \); i.e energy per \( p \)-dimensional volume. On dimensional grounds and to the coincidence condition \( [1] \) referred above one has that:

\[ (k_{\mu\nu})(k^{\mu\nu}) = \beta_2 (k^2)^2 = \beta_2 k^4. \quad (k_{\mu\nu\rho})(k^{\mu\nu\rho}) = \beta_3 (k^3)^2 = \beta_3 k^6 \ldots \]

where the proportionality factors in eq-(9) are the rank and dimension-dependent constants, \( \beta_2(D, r = 2), \beta_3(D, r = 3) \ldots \) associated with the 2-vector, 3-vector, \ldots components of the polymomentum \( \mathbf{K} \), respectively. \( \beta = 1 \) for the first term in eq-(7), a rank one tensor : vector. The coincidence condition implies that upon setting \( \Lambda = 0 \) all the \( p \)-loop histories collapse to a point history. In that case the areas, volumes, \ldots hypervolumes collapse to zero and the wave equation (5) reduces to the ordinary Klein-Gordon equation for a spin zero massive particle.

Factoring out the \( k^2 \) factor in (7), using the analog of the dispersion relation (8) and taking the square root, after performing the binomial/Taylor expansion of the square root, subject to the condition \( \Lambda^2 k^2 << 1 \), one obtains an effective energy dependent Planck " constant " that takes into account the Noncommutative nature of the Clifford manifold (\( \mathbf{C} \)-space) at Planck scales:

\[ \hbar_{\text{eff}}(k^2) = \hbar(1 + \frac{1}{2!} \beta_2 \Lambda^2 k^2 + \frac{1}{3!} \beta_3 \Lambda^4 k^4 + \ldots \ldots \ldots). \]

where we have included explicitly the \( D \) and rank dependent coefficients \( \beta_1, \beta_2, \beta_3 \ldots \) that arise in (9) due to the coincidence condition and on dimensional analysis.

Arguments concerning an effective value of Planck’s " constant " related to higher derivative theories and the modified uncertainty relations have been given by [8]. The advantage of this derivation based on the New Relativity Principle is that one automatically avoids the problems involving the ad hoc introduction of higher derivatives in Physics ( ghosts, \ldots ) .

The uncertainty relations for the coordinates-momenta follow from the Heisenberg-Weyl algebraic relation familiar in QM:

\[ \Delta x \Delta p \geq \mid < [\hat{x}, \hat{p} ] \mid. \quad [\hat{x}, \hat{p} ] = i\hbar \]

Now we have that in \( \mathbf{C} \)-spaces, \( x, p \) must not, and should not, be interpreted as ordinary vectors of spacetime but as one of the many components of the Clifford-algebra valued multivectors that " coordinatize " the Noncommutative Clifford Manifold, \( \mathbf{C} \)-space. The Noncommutativity is encoded in the effective value of
the Planck’s “constant” which modifies the Heisenberg-Weyl $x, p$ algebraic commutation relations and, consequently, generates new uncertainty relations:

$$\Delta x \Delta p \geq |\langle [\hat{x}, \hat{p}] \rangle| = \hbar(1 + \frac{1}{2.2!}\beta_2 \Lambda^2 < k^2 > + \frac{1}{2.3!}\beta_3 \Lambda^4 < k^4 > + .......)$$

(12)

Using the relations:

$$\hbar k = p. \quad < p^2 > \geq (\Delta p)^2. \quad < p^4 > \geq (\Delta p)^4 .......$$

(13)

one arrives at:

$$\Delta x \Delta p \geq \hbar + \frac{\beta_2 \Lambda^2}{4\hbar} (\Delta p)^2 + \frac{\beta_3 \Lambda^4}{12\hbar^4} (\Delta p)^4 + .......$$

(14)

Finally, keeping the first two terms in the expansion in the r.h.s of eq- (14) one recovers the ordinary String Uncertainty Relation [5] directly from the New Relativity Principle as promised:

$$\Delta x \geq \frac{\hbar}{\Delta p} + \frac{\beta_2 \Lambda^2}{4\hbar} (\Delta p).$$

(15)

which is just a reflection of the minimum distance condition in Nature [3,4,5,6,7,10] and an inherent Non-commutative nature of the Clifford manifold (C-space). Eq-(15) yields a minimum value of $\Delta x$ of the order of the Planck length $\Lambda$ that can be verified explicitly simply by minimizing eq-(15).

3. A Simple Argument Why Quantum Spacetime could be Infinite Dimensional

A Plausible Resolution of the Cosmological Constant Problem

So far the derivation of the String uncertainty relations from the New Relativity Principle has been straightforward. However, we wish to be more radical in our approach. An immediate question comes to mind:

Why did we truncate the series eqs – (5, 7) to a finite value of the Quantum Spacetime dimension?

If the New Relativity principle is true then we must include all dimensions for the Quantum Spacetime. It is all or nothing! Taking this radical view will generate instead of the finite series of eq-(7) an infinite series of the form:

$$\hbar^2 k^2 \sum_{r=1}^{\infty} \frac{\beta_r(D, r)}{r!} (k\Lambda)^{2(r-1)} = \lim_{p \to \infty} \frac{\Lambda^{2p}(c^2 - \lambda^2)}{\hbar^{2p}}.$$  

(16)

where $r = 1, 2, 3, ....D$ denotes the rank of the vector, 2-vector, 3-vector,...associated with the Clifford-algebra valued polymomentum $K$ conjugate to the Clifford-valued ” line ” in C-space : $X(\Sigma)$.

The sum of the infinite series depends on the infinite family of rank and dimension dependent coefficients $\beta_r(D, r)$ appearing in eq-(9,10,12). For simplicity purposes, and for the sake of the argument, we will take all of the coefficients to have the simplest value of them all : 1. The infinite series yields:

$$\sum_{r=1}^{\infty} \frac{(k\Lambda)^{2(r-1)}}{r!} = \sum_{r=1}^{\infty} \frac{z^{2(r-1)}}{r!} = \sum_{r'=0}^{\infty} \frac{z^{2r'}}{(r'+1)!} = \frac{e^z - 1}{z^2} , \text{ where } z \equiv k\Lambda.$$  

(17)

One recovers one of the confluent hypergeometric functions as the value of the sum. Therefore, after recasting the sum in terms of hyperbolic functions

$$\hbar^2_{eff} = \hbar^2 \frac{e^{z^2} - 1}{z^2} \Rightarrow \hbar^2_{eff} = \hbar^2 e^{z^2/2} \frac{sinh(z^2/2)}{(z^2/2)}.$$  

(18)
following the exact same steps as in the previous section one gets the full blown Uncertainty Relations for Quantum Spacetime due to the contributions of all extended objects : \( p = 0, 1, 2, \ldots, \infty \) :

\[
\Delta z \equiv \Lambda k \Rightarrow \Delta x \geq \sqrt{2} \Lambda \frac{e^{(\Delta z)^2/4}}{(\Delta z)^2} \sqrt{\sinh \left[ \frac{(\Delta z)^2}{2} \right]}.
\]  

(19)

One can verify that the function :

\[
x = x(z^2) \equiv \sqrt{2} \Lambda \frac{e^{z^2/4}}{z^2} \sqrt{\sinh \left( \frac{z^2}{2} \right)}.
\]  

(20)
after differentiating it and equating it to zero, has a minimum/maximum for those values of \( z_0 \) such that satisfy :

\[
tanh [z^2/2] = \frac{z^2}{4 - z^2}.
\]  

(21)

When \( z = 0, \infty \Rightarrow x = \infty \) as expected in eq-(20), when the momentum is \( k = 0, \infty \). The minimum value of \( x \) occurs when

\[
1.2621 < z_0 < 1.2626 \quad x_{\min} \sim 1.2426 \ \Lambda.
\]  

(22)

Therefore, for momentum values precisely of the order of the Planck’s momentum : \( k_0 \sim (z_0/\Lambda) \) that gives 1.262 \( k_p \), one reaches the minimum distance of 1.2426 \( \Lambda \) ! as it is required from the New Relativity Principle : Polydimensional Covariance and Scale Relativity [2,3].

Of course, one can always tune the infinite number of coefficients (16) in an infinite number of ways to reproduce the Planck scale as the minimum scale for arbitrary values of the momentum. A smaller subclass of infinite tuning possibilities appears when the Planck scale minimum occurs precisely at Planck values of the momentum. It is remarkable that the simplicity arguments of setting all the values of the coefficients to 1 yields the desired results of attaining a minimum Planck scale for Planck values of the momentum. We believe this is not a numerical coincidence.

An immediate question arises : if there are many ways of selecting and tuning the coefficients \( \beta_i \) in (16) to satisfy the requirements of attaining minimal Planck scale uncertainty at Planck scale momentum, i.e there is an infinite range of possible values for the \([x, p] = i\hbar_{eff} \) commutation relations, is there a physical criteria to select a unique value of \( \hbar \) ?

We believe that the answer to this question may lie in Hopf algebraic structures at Planck scales [12]. Recently there has been a lot of activity pertaining the Hopf algebraic structure underlying to perturbative QFT [9] and the numerical " miracles" of the Renormalization Group process. As Kreimer has pointed out, the iterated removal of nested divergences while maintaining locality has to fulfill combinatorial properties summarized by Zimmermann’s forest formula. There is an underlying mathematical structure that is in no way accidental. For relations to low dimensional topology, number theory,...we refer to Kreimer et al [9]. The authors [12] have suggested that there is a Planck scale Hopf algebra as a particular example of a Noncommutative differential geometry at Planck scales where the Planck scale acts as a natural ultraviolet regulator. In fact, Majid found a \([x, p] = i\hbar_{eff} \) commutation relation that bears a striking resemblance to ours. Thiemann has also argued that Planck scale should serve as natural regulator for matter QFT [10]. Nottale [3] has argued in numerous occasions that there is a deep link between the Renormalization Group process, that is based in scaling arguments, and Scale Relativity. In fact he has given a resolution dependent effective Planck’s constant that Granik recently has shown to agree with eq-(10) up to the first terms [17].

The confluent hypergeometric function that results after summing the infinite series (16) is no numerical accident. Gamma functions have long been known to be essential in the dimensional regularization procedures and in Veneziano’s formula that spawned String Theory. In addition, the crux of including all dimensions in our calculation for the \([x, p] \) commutation relations is that one does not have to truncate/amputate the \([x, p] \) commutators to a finite number of terms like it was done in [7]. We have given in (19) the full blown Quantum Spacetime Uncertainty relations that are more general than the usual String Uncertainty relations; We are including the effects of all extended objects!
The main lesson from this numerical exercise is that Quantum Spacetime could be **infinite** dimensional if we invoke the New Relativity principle to the fullest potential within the context of Noncommutative Clifford manifolds, C-spaces and Quantum Groups (Hopf algebras). This result that the Quantum Spacetime is infinite-dimensional has been advocated many times by [3,4] within the context of Fractals, Scale Relativity and a Cantorian-Fractal spacetime: a transfinite infinite nested hierarchy of fractal Cantorian sets of infinite dimensionality. Quantum sets have been proposed long ago by Finkelstein in the formulation of Quantum Relativity [16]. This straightforward numerical analysis is a strong indication that Quantum Spacetime could be infinite-dimensional and that it may indeed be fractal at its very core. Nature is Fractal. It is not a big surprise that Quantum Spacetime could be as well. Being fractal supports the view of Majid [12] that Quantum Geometry is a Braided Categorical one. Since a Fractal Quantum Spacetime has fractal dimensions, it follows naturally that it should allow for fractional spins, charges, statistics, i.e. The Quantum Geometric world has Braided Statistics.

Ordinary real numbers are no longer useful to describe the infinite dimensional Fractal Quantum Spacetime we are proposing. It is meaningless to assume that we can measure a real number to infinite nonperiodic decimal places. It has been speculated for quite some time that due to the minimal Planck length, the geometry at Planck scales is Non-Archimedean. Therefore **p**-adic numbers are the natural ones to use at this scale. For a review of the mathematical applications of **p**-adic numbers in Physics and Fractals see [13]. For the role of **p**-adics in the construction of TGD see [15].

If Fractal Quantum Spacetime is indeed **infinite** dimensional we would have to drastically modify our naive perceptions that spacetime has a **fixed** dimension [4] and reconsider the validity of the compactification arguments studied so far from higher to low dimensions. Dimensions are resolution dependent [3,4]. Instead we may be obliged to view \( D = 4 \) only as an overall average dimension in the same way that the speed of the molecules inside a box at fixed temperature is distributed over a wide range of velocities and has an average one related to the temperature. El Naschie [4] using a Gamma distribution for the ensemble of dimensions, fractal arguments and Astrophysical data results has obtained average dimensions close to \( D = 4 \). The same ideas apply to the observed spacetime signature and to the resolution of the cosmological constant problem. The New Relativity principle treats all dimensions and signatures on the same footing.

To finalize we show why the cosmological "constant" should not be treated as such: it is in the eye of the beholder. The key to a plausible and remarkably simple solution to the cosmological constant problem lies within eq-(16) which is nothing but an generalization of Einstein's relation: \( E^2 - p^2 = m^2 \).

One simply shifts the cosmological constant term of (16) to the left hand side. Upon shifting it to the "left" we have:

\[
\left( \frac{\Lambda}{\hbar} \right)^{2p} \lambda^2 + \hbar^2 k^2 \sum_{r=1}^{\infty} \frac{\beta_r(r, D)}{r!} (k\Lambda)^{2(r-1)} = \lim_{p \to \infty} \frac{\Lambda^{2p} \mathcal{E}^2}{\hbar^{2p}}.
\]  

The New Relativity principle which is based on the principles of Polydimensional Covariance [2] which reshuffle a string history for a 5-brane history; a 9-brane history for a 5-brane history and so forth i.e the New Relativity principle is nothing but taking Chew’s bootstrap idea to the heart: each \( p \)-brane is made of **all** the others! It is in this fashion why **all** dimensions (and signatures) must be treated on the same footing. As Nottale and El Naschie have argued, dimensions are not absolute concepts in Quantum Spacetime, they are resolution dependent.

The New Relativity Principle (Polydimensional Covariance [2]) states that the r.h.s (23) is truly an invariant (like the proper time or proper rest mass of a particle) while the terms on the l.h.s are just the analogs of the squared-norm of a four-vector \( E^2 - p^2 = m^2 \). Therefore, based on this simple analogy we propose that \( \lambda \) is **not** a constant but instead is just one of the many observer dependent components of the polymomentum multivector \( \mathbf{K} \) referred earlier in section 2. Hence we have that the combination:

\[
\left( \frac{\Lambda}{\hbar} \right)^{2p} \lambda^2 + \hbar^2 k^2 \sum_{r=1}^{\infty} \frac{\beta_r(r, D)}{r!} (k\Lambda)^{2(r-1)} = \left( \frac{\Lambda}{\hbar} \right)^{2p} \lambda^2 + \hbar^2 k^2 \sum_{r=1}^{\infty} \frac{\beta_r(r, D)}{r!} (k'\Lambda)^{2(r-1)} = \ldots = \frac{\Lambda^{2p} \mathcal{E}^2}{\hbar^{2p}}.
\]  

is an invariant of this New Relativity Theory like...
\[ E^2 - p^2 = E'^2 - p'^2 = E''^2 - p''^2 = \ldots = m^2. \] (25)

was in Special Relativity, where the squared of the maximal \( p \)-brane tension, associated with the spacetime filling \( p \)-brane, \( E^2 \) plays identical role to the one played by \( m^2 \) in Einstein’s Relativity. Eq-(24) is remarkably simple. It relates the microscopic world quantities: Planck scale, cosmological "constant" on the left, with the total Quantum Spacetime Energy per Unit \( p \)-volume (Elasticity of Spacetime quoting Zaharov) associated with the Quantum Spacetime-filling maximal \( p \)-brane \((p + 1 = D)\) on the right. The essential terms required to match the left with the right are precisely provided by the infinite number of modes associated with the point-history, loop-history, 2-loop history, 3-loop history, \ldots \( p \)-loop history excitations OF the Quantum Spacetime given precisely by

\[
\hbar^2 k^2 \sum_{r=1}^{\infty} \frac{\beta_r(r,D)}{r!} (k\Lambda)^{2(r-1)}.
\] (26)

that led to the full blown Quantum Spacetime Uncertainty Relation (19). For interesting work on the cosmological constant we refer to [19,20]. What remains is to find out what is the right Hopf Planck scale algebra that selects a unique value of the effective Planck "constant". Perhaps there are several?

Since in \( D = 4 \) the Planck length is given by:

\[
\Lambda = \sqrt{\frac{\hbar G}{c^3}}. \] (27)

the immediate question, similar to the one proposed long ago by Dirac, arises: isn’t it possible that \( \hbar, G, c \) are not "constants" in eq-(27) but they could vary in such a way as to leave the value of \( \Lambda \) invariant??? In the past years there has been a lot of research activity in the Astrophysics community pondering if the speed of light varies in Cosmology [18]. Nottale [3] has given another explanation for the resolution of the cosmological constant problem based on Scale Relativity [3]. His argument is essentially that it is meaningless to compare two values of energy densities at two different scales without including Scale-Relativistic effects. He explains why the \( 10^{50}, 10^{60} \) discrepancy is due to the Scale-Relativistic "Lorentz" dilation factors.

It seems that one may be forced to demolish the old established "idols" (using Finkelstein’ terminoloy) of spacetime, dimension, cosmological constant,\ldots in the same way that Relativity and Quantum Mechanics replaced the Cartesian-Newtonian paradigm. As we end the century, it is time perhaps to embrace a new paradigm in Physics that demolishes the concept of dimension as an idol. Number theory, Topology, Fractals, Cantor sets, \( p \)-adic analysis, QFT, Quantum Groups, Hopf algebras, Noncommutative Geometry\ldots seem all to be converging in disguised forms at the Planck scale upon looking at Quantum Spacetime with the magnifying glass of the New Relativity Theory based on Noncommutative C-spaces [1], the principle of Polydimensional Covariance [2] and Scale Relativity [3]: a magnifying glass lying deep inside the fuzzy crystal ball of imagination signaling what it may turn out to be a new paradigm in Physics.

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