**Abstract**

Three-manifolds can be obtained through surgery of framed links in $S^3$. We study the meaning of surgery procedures in the context of topological strings. We obtain $U(N)$ three-manifold invariants from $U(N)$ framed link invariants in Chern-Simons theory on $S^3$. These three-manifold invariants are proportional to the Chern-Simons partition function on the respective three-manifolds. Using the topological string duality conjecture, we show that the large $N$ expansion of $U(N)$ Chern-Simons free energies on three-manifolds, obtained from some class of framed links, have a closed string expansion. These expansions resemble the closed string A-model partition functions on Calabi-Yau manifolds with one Kahler parameter. We also determine Gopakumar-Vafa integer coefficients and Gromov-Witten rational coefficients corresponding to Chern-Simons free energies on some three-manifolds.
# 1 Introduction

After the second superstring revolution, several useful relations have been discovered unifying various ideas of physics and mathematics. One such surprising discovery in the recent past has been the new connections between Chern Simons gauge theory and the physics of closed topological string theory in certain backgrounds.

The initial steps in this direction was taken by Gopakumar and Vafa in [1], [2], [3]. The conjecture put forward by these authors relate large $N$ Chern-Simons gauge theory on $S^3$, which is equivalent to A-twisted open topological string theory on $T^*S^3$ [4], to the A-type closed topological string theory on the resolved conifold. This conjecture was then tested at the level of the observables of the Chern Simons theory, namely the knot invariants. In [5], Ooguri and Vafa formulated the conjecture in terms of invariants for the unknot (a circle in $S^3$), and further checks were carried out for more nontrivial knots in [6], [7], [8], [9].

The evaluation of knot invariants in [5] actually led to very strong integrality predictions for the (instanton generated) A-model disc amplitudes, which were then verified from the more tractable mirror B-model side by several authors [10], [11], [12], [13].

Purely from gauge theory considerations, following the idea of ‘t Hooft [14], it looks to be a challenging problems to prove that the Feynmann perturbative expansion of any $U(N)$ gauge theory in the large $N$ limit is equivalent to a closed string theory. It is believed that the Gopakumar-Vafa duality conjecture can provide insight in determining the ‘t Hooft expansion of $U(N)$ Chern-Simons free energy on any three-manifold $M$.

As we have already mentioned, the Gopakumar-Vafa duality conjecture states that $U(N)$ Chern-Simons theory on $S^3$, which describes the topological $A$-model of $N$ D-branes on $X = T^*S^3$, is dual to topological closed string theory on $X^t = \mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbb{P}^1$. Having verified the conjecture at the level of Chern-Simons partition function on $S^3$ and Wilson loop observables (the knot invariants), we need to understand the meaning of surgery of framed links in $S^3$ within the context of topological strings, and this is one of the issues we set out to address in this paper.

From the fundamental theorem of Lickorish and Wallace [15], it is well known that any three-manifold $M$ can be obtained by surgery on a framed link in $S^3$. Further, two framed links related by a set of moves called Kirby moves determine the same manifold. In Chern-Simons theory, an algebraic expression has been derived [16], in terms of framed link invariants, which are unchanged under Kirby moves of
the framed links. Hence the algebraic expression represents three-manifold invariants which are proportional to the Chern-Simons partition function \((Z[M])\) on the three-manifold \(M\). Incorporating the results of the topological string duality conjecture, we determine large \(N\) expansion for \(\ln Z[M]\) for many manifolds. Surprisingly, the expansion looks like an \(A\)-model closed string partition function on a Calabi-Yau space with one Kahler parameter.

We point out an important subtlety here. As is well known, the classical solutions of the Chern-Simons action on a general three-manifold \(M\) are the flat connections on \(M\). In the weak coupling limit, the Chern-Simons partition function gets contributions from a perturbative expansion around all such stationary points. Noting that the space of flat connections may be either a collection of a set of stationary points or a set of connected pieces, this partition function can be appropriately written as a sum or an integral over the space of flat connections. The large \(N\) expansion of 't Hooft is expected to relate the \(1/N\) expansion of the Chern-Simons theory around a given flat connection to an \(A\)-type closed topological string theory. In \cite{17}, this has been shown from a matrix model approach, for the Lens space \(\mathcal{L}(p, 1)\). In this paper, however, we show that the full Chern-Simons partition function \((\ln Z[M])\) has a closed string interpretation for a class of three manifolds.

Indeed, proposing new duality conjectures between Chern-Simons theory on general three-manifolds \(M\) and the corresponding dual closed string theories will involve the extraction of the partition function around individual flat connections, in lines with \cite{17}. This will extend the original conjecture by Gopakumar and Vafa, for general manifolds \(M\). We believe that our results on the invariants, involving the full partition function, would be useful in proposing and understanding fully the nature of such dualities. We will elaborate on this point further in the concluding section.

The organisation of the paper is as follows. In section 2, we briefly recapitulate the framed link invariants in \(U(N)\) Chern-Simons theory, and present the \(U(N)\) three-manifold invariants obtained from framed link invariants in \(S^3\). In section 3, we show the relation between the three-manifold invariants and the observables in topological string theory. Further, we obtain closed string invariants for the Chern-Simons free energies. Section 4 contains some explicit results on the Gopakumar Vafa coefficients corresponding to the large \(N\) expansion of the Chern-Simons free energy on some manifolds. Section 5 ends with some discussions and scope for future research. In an appendix, we present some results on the integer invariants for the unknot with arbitrary framing, which are useful for the computation of the
Gopakumar Vafa coefficients.

2 \textbf{U(N) Chern-Simons Gauge theory}

Chern-Simons gauge theory on a three-manifold \(M\) based on the gauge group \(U(N)\) is a factored Chern-Simons theory of two gauge groups, \(SU(N)\) and \(U(1)\). That is, the action is simply a sum of two Chern-Simons actions, one for gauge group \(SU(N)\) and the other for \(U(1)\), each with an independent coupling constant \((k, k_1)\)

\[
S = \frac{k}{4\pi} \int_M Tr \left( A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right) + \frac{k_1}{4\pi} \int_M Tr \left( B \wedge dB \right)
\]  

(2.1)

where \(A\) is a gauge connection for gauge group \(SU(N)\) and \(B\) is the connection for \(U(1)\). Clearly, the \(U(N)\) partition function \(Z_{\{U(N)\}}[M] \equiv Z[M]\) is just the product of two partition functions \([Z_{\{SU(N)\}}[M], Z_{\{U(1)\}}[M]]\)

\[
Z[M] = \int[D\!A][D\!B] e^{iS}
\]  

(2.2)

We shall now briefly present the Wilson loop observables in the theory. The \(U(N)\) Wilson loop operators for a \(r\)-component link \(L\) made up of component knots \(K_i\)'s are simply factored Wilson operators of the \(U(1)\) and \(SU(N)\) theories

\[
W_{\{\{R_i,n_i\}\}}[L] = \prod_{i=1}^{r} Tr_{R_i} U^{(A)}[K_i] Tr_{n_i} U^{(B)}[K_i],
\]  

(2.3)

where \(U^{(A)}[K_i] = P \left[ \exp \oint_{K_i} A \right]\) denotes the holonomy of the \(SU(N)\) gauge field \(A\) around the component knot \(K_i\) of a link \(L\) carrying representation \(R_i\) and \(U^{(B)}[K_i] = P \left[ \exp \oint_{K_i} B \right]\) denotes the holonomy of the \(U(1)\) gauge field \(B\) around the component knot \(K_i\) carrying \(U(1)\) charge \(n_i\). The expectation value of these Wilson loop operators are the \(U(N)\) link invariants which are products of \(SU(N)\) and \(U(1)\) invariants

\[
V^{\{U(N)\}}_{\{R_1,n_1\}, \ldots, \{R_r,n_r\}}[L,M] = \langle W_{\{\{R_i,n_i\}\}}[L] \rangle = \frac{\int[D\!A][D\!B] e^{iS} W_{\{\{R_i,n_i\}\}}[L]}{\int[D\!A][D\!B] e^{iS}} = V^{\{SU(N)\}}_{R_1, \ldots, R_r}[L,M] V^{\{U(1)\}}_{n_1, \ldots, n_r}[L,M]
\]  

(2.4)

2.1 \textbf{U(N) Framed Link Invariants in S}^3

The observables in \(U(1)\) Chern-Simons theory on a three-sphere \(S^3\) capture only self-linking numbers (also called framing numbers) and the linking numbers between the
component knots of any link. Hence, the \( U(1) \) link invariant will be

\[
V_{n_1, n_2, \ldots, n_r}^{(U(1))} [L, S^3] = \exp \left( \frac{i \pi}{k_1} \sum_{i=1}^{r} n_i^2 p_i \right) \exp \left( \frac{i \pi}{k_1} \sum_{i \neq j} n_i n_j \ell_{k_{ij}} \right) .
\]

where \( p_i \)'s are the framing numbers of the component knots \( K_i \)'s and \( \ell_{k_{ij}} \) are the linking numbers between the component knots \( K_i, K_j \). From eqns. (2.4), (2.5), it is clear that the \( U(N) \) link invariants coincides with \( SU(N) \) invariants if and only if \( p_i \)'s and \( \ell_{k_{ij}} \) are zero.

The evaluation of \( SU(N) \) framed link invariants from Chern-Simons theory on \( S^3 \) makes use of the two ingredients: (i) the connection between Chern-Simons field theory and the corresponding Wess-Zumino conformal field theory (ii) the fact that knots and links can be obtained by closure or plaiting of braids. We refer the reader to [18] [16] for detailed description of obtaining \( SU(2) \) framed invariants and framed invariants from Chern-Simons theory on \( S^3 \) based on any arbitrary semi-simple group.

We shall now present the polynomials for various framed knots and links. For the unknot \( U \) with an arbitrary framing \( p \), carrying a representation \( R \) of \( SU(N) \), the polynomial is

\[
V_{R}^{(SU(N))} [0^p], S^3] = q^{(pC_R)} V_{R} [U] = q^{-\frac{N^2}{N^2}} (q^{p \kappa_R dim \_q R} ),
\]

where \( q = \exp \left( \frac{2 \pi i}{k + 1} \right), \ell \) refers to the total number of boxes in the Young-Tableau of the representation \( R \) and

\[
\kappa_R = \frac{1}{2} \left( N \ell + \ell + \sum_i (l_i^2 - 2il_i) \right) ,
\]

with \( l_i \) being the number of boxes in the \( i \)-th row of the Young-Tableau of the representation \( R \). One can verify that both the quantum dimension of a representation, \( dim \_q R \) and \( \kappa_R \) are polynomials in variables \( \lambda^{\pm \frac{1}{2}} \) and \( q^{\pm \frac{1}{2}} \) where \( \lambda = q^N \). The frame dependent term involves a variable \( e^z = q^{\frac{1}{2N}} \). Hence the unknot invariants with framing \( p \neq 0 \) are no longer polynomials in variables \( q^{\pm \frac{1}{2}}, \lambda^{\pm \frac{1}{2}} \) but also involve one more variable \( e^z = q^{\frac{1}{2N}} \).

In order to make the polynomials independent of the variable \( e^z \), we can multiply by \( U(1) \) invariant (2.5) with a specific choice of \( U(1) \) charge \( n \) and coupling constant \( k_1 \)

\[
n = \frac{\ell}{\sqrt{N}} ; \quad k_1 = k + N .
\]
Therefore the $U(N)$ invariant for the $p$-framed unknot in $S^3$ with the above choice of $U(1)$ representation and coupling constant is
\[
V^{\{U(N)\}}_{(R, \ell, \sqrt{N})}[0^{(p)}, S^3] = q^{p\kappa_R} \dim_q R.
\]
(2.9)

Now, we can write the $U(N)$ framed knot invariants for torus knots of the type $(2, 2m+1)$ and other non-torus knots like $4_1, 6_1$ in $S^3$. For example, the $U(N)$ invariant for a framed torus knot of type $K \equiv (2, 2m+1)$ with framing $[p-(2m+1)]$ will be
\[
V^{\{U(N)\}}_{(R, \ell, \sqrt{N})}[K, S^3] = q^{p\kappa_R} \sum_{R_s \in R \otimes R} \dim_q R_s (-1)^{\epsilon_s} \left(q^{\kappa_R - \kappa_{R_s}}\right)^{2m+1},
\]
(2.10)

where $\epsilon_s = \pm 1$ depending upon whether the representation $R_s$ appears symmetrically or antisymmetrically with respect to the tensor product $R \otimes R$ in the $SU(N)_k$ Wess-Zumino Witten model. Similarly, $U(N)$ invariants for framed torus links of the type $(2, 2m)$ can also be written. For example, the $U(N)$ invariant for a Hopf link with linking number $-1$ and framing numbers $p_1$ and $p_2$ on the component knots carrying representations $R_1$ and $R_2$ will be
\[
V^{\{U(N)\}}_{(R_1, \ell_1, \sqrt{N}), (R_2, \ell_2, \sqrt{N})}[H^*(p_1, p_2), S^3] = q^{p_1\kappa_{R_1}} q^{p_2\kappa_{R_2}} \sum_{R_s \in R_1 \otimes R_2} \dim_q R_s q^{\kappa_{R_1} + \kappa_{R_2} - \kappa_{R_s}},
\]
(2.11)

where $\ell_1$ and $\ell_2$ refers to total number of boxes in the Young-Tableau of the representations $R_1$ and $R_2$ respectively. From now on, we shall denote
\[
V^{\{U(N)\}}_{(R_1, \ell_1, \sqrt{N}), (R_2, \ell_2, \sqrt{N}), \ldots, (R_r, \ell_r, \sqrt{N})}[L, S^3] = V^{\{U(N)\}}_{R_1, R_2, \ldots, R_r}[L, S^3]
\]
supressing the $U(1)$ charges as they are related to the total number of boxes in the Young-Tableau of the representations $R_i$'s (2.8). Using the framed $SU(N)$ invariants for arbitrary framed links in $S^3$ [16], it is straightforward to obtain the corresponding $U(N)$ link invariants. We will now see how these $U(N)$ framed invariants in $S^3$ with such special $U(1)$ representation reflects on three-manifold invariants.

### 2.2 $U(N)$ Three-Manifold Invariants

The Lickorish-Wallace theorem states that any three-manifold $M$ can be obtained by a surgery of framed knots and links in $S^3$. Two framed links related by Kirby moves will determine the same three-manifold. In other words, three-manifold invariants must be constructed from framed link invariants in such a way that they
are preserved under Kirby moves. The \( SU(N) \) three-manifold invariant \( F[M] \) for a
manifold \( M \) obtained by surgery of a framed link in \( S^3 \) will be
\[
F[M] = \alpha^{-\sigma(L)} \sum_{R_{i1},R_{i2}...R_{ir}} \mu_{R_{i1},R_{i2}...R_{ir}}(\{p_i\}, \{\ell k_{ij}\}) V_{R_{i1},R_{i2}...R_{ir}}^{SU(N)}[L, S^3],
\]  
(2.12)
where \( \{p_i\}, \{\ell k_{ij}\} \) are the framing and linking numbers and \( \sigma(L) \) is the signature of
the linking matrix of framed link \( L \). It has been proven \[16\] that \( F[M] \) is unchanged
under the operation of Kirby moves on framed links if we choose
\[
\alpha = \exp \left( \frac{i\pi c}{4} \right), \quad \mu_{R_{i1},R_{i2}...R_{ir}}(\{p_i\}, \{\ell k_{ij}\}) = S_{0R_{i1}}S_{0R_{i2}}...S_{0R_{ir}},
\]  
(2.13)
where \( c = \frac{k(N^2-1)}{k+N} \) and \( S_{0R_{i}} \)'s denotes the modular transformation matrix elements.
We see that \( \mu_{R_{i1},R_{i2}...R_{ir}} \) is independent of framing and linking numbers. We can now
construct \( U(N) \) three-manifold invariants from the \( U(N) \) framed link invariants in
\( S^3 \) as follows
\[
\tilde{F}[M] = \beta^{-\sigma(L)} \sum_{\{R_i\}} \tilde{\mu}_{R_{i1},R_{i2}...R_{ir}}(\{p_i\}, \{\ell k_{ij}\}) V_{R_{i1},R_{i2}...R_{ir}}^{U(N)}[L, S^3],
\]  
(2.14)
where \( \beta \) and \( \tilde{\mu} \) must be chosen such that \( \tilde{F}[M] \) is unchanged under Kirby moves on
framed links. Further, for obtaining three-manifolds from knots and disjoint links
with zero framing and linking numbers, we require
\[
\tilde{F}[M] = F[M], \quad \text{as } V_{R_{i1},...,R_{ir}}^{U(N)}[L, S^3] = V_{R_{i1},...,R_{ir}}^{SU(N)}[L, S^3].
\]  
(2.15)
Therefore, for \( \{p_i\} = 0, \{\ell k_{ij}\} = 0 \)
\[
\tilde{\mu}_{R_{i1},R_{i2}...R_{ir}}(\{p_i\} = 0, \{\ell k_{ij}\} = 0) = \mu_{R_{i1},R_{i2}...R_{ir}}. \quad (2.16)
\]
The special choice of the \( U(1) \) representation and the above limiting conditions
suggests that
\[
\beta = \alpha, \quad \tilde{\mu}_{R_{i1},R_{i2}...R_{ir}}(\{p_i\}, \{\ell k_{ij}\}) = \mu_{R_{i1},R_{i2}...R_{ir}} e^{-z \left( \sum_i \ell_{i}^2 p_i + \sum_{i,j} \ell_{ij} \ell_{k_{ij}} \right)},
\]  
(2.17)
for \( \tilde{F}[M] \) to be preserved under Kirby moves. Rewriting the \( U(N) \) link invariants in
terms of the \( SU(N) \) invariants, it is obvious that \( \tilde{F}[M] \) is the same as \( F[M] \). Hence
the three-manifold invariants do not distinguish between \( U(N) \) and \( SU(N) \) gauge
groups and they are proportional to the partition function \( Z[M] \) \[16\]:
\[
F[M] = \frac{Z[M]}{Z[S^3]}, \quad (2.18)
\]
The partition function on $S^3$ is equal to
\[ Z[S^3] = S_{00}. \] (2.19)

Let us introduce a slight modification in notation which will be useful when we relate these three-manifold invariants with expectation values of topological operators in topological string theory
\[ V_{R_{1},R_{2},...,R_{r}}^{U(N)}[L,S^3] = (-1)^{\sum_{i} \ell_{i} p_{i}} \sum_{2} \tilde{V}_{R_{1},R_{2},...,R_{r}}^{U(N)}[L,S^3](q,\lambda). \] (2.20)

The $U(N)$ (or equivalently $SU(N)$) three-manifold invariants can be rewritten as
\[ \frac{Z[M]}{S_{00}} = \frac{Z_{0}[M]}{S_{00}} + F[M, z, \{p_{i}\}, \{\ell_{k_{ij}}\}] \] (2.21)
where $Z_{0}[M]/S_{00}$ is independent of $z, \{p_{i}\}, \{\ell_{k_{ij}}\}$ and is given by
\[ \frac{Z_{0}[M]}{S_{00}} = \alpha^{-\sigma[L]} \sum_{R_{1},R_{2},...,R_{r}} S_{0R_{1}} \ldots S_{0R_{r}} (-1)^{\sum_{i} \ell_{i} p_{i}} \sum_{2} \tilde{V}_{R_{1},R_{2},...,R_{r}}^{U(N)}[L,S^3](q,\lambda), \] (2.22)
and $F[M, z, \{p_{i}\}, \{\ell_{k_{ij}}\}]$ contains the remaining $z, \{p_{i}\}, \{\ell_{k_{ij}}\}$ dependent terms. For knots and disjoint links with zero framing and linking numbers, it is not difficult to see that $F[M, z, \{p_{i}\}, \{\ell_{k_{ij}}\}] = 0$ resulting in $Z[M] = Z_{0}[M]$. In the next section, we will show the natural appearance of $Z_{0}[M]$ in the context of topological strings.

### 3 Topological Strings

Gopakumar and Vafa have conjectured that closed topological string theory on a resolved conifold is dual to large $N$ Chern-Simons gauge theory on $S^3$. The conjecture has been verified by comparing the large $N$ expansion of the free-energy of the Chern-Simons theory on $S^3$ with the closed topological string amplitude near the resolved conifold. This duality relates the Chern-Simons field theory variables $q$ and $\lambda$ with the string theory parameters
\[ q = e^{g_{s}}, \quad \lambda = e^{t} = e^{Ng_{s}}, \] (3.1)
where $g_{s}$ is the string coupling constant and $t$ is the Kahler parameter of the resolved conifold. With the above identification between the variables $q, \lambda$ with $g_{s}, t$, the Chern-Simons variable $e^{z}$ is
\[ e^{z} = e^{g_{s}/2N}. \] (3.2)
The large $N$ expansion is performed by taking the limits
\[ g_s \to 0 \text{ and } N \to \infty \] (3.3)

In this limit, the variable $z = g_s / 2N$ can be set to zero. This suggests that the $z$ independent part of the three-manifold invariant, namely, $Z_0[M]/S_{00}$ can be compared with quantities on the topological string side.

Ooguri and Vafa found another piece of evidence for this duality conjecture by showing that the Wilson loop operators in Chern-Simons theory correspond to certain observables in the topological string theory. The operators in the open topological string theory which contains information about links is given by

\[ Z(\{U_\alpha\}, \{V_\alpha\}) = \exp \left[ \sum_{\alpha=1}^{r} \sum_{d=1}^{\infty} \frac{1}{d} \text{Tr} U_\alpha^d \text{Tr} V_\alpha^d \right] \] (3.4)

where $U_\alpha$ is the holonomy of the gauge connection $A$ around the component knot $K_\alpha$ carrying the fundamental representation in the $U(N)$ Chern-Simons theory on $S^3$, and $V_\alpha$ is the holonomy of a gauge field $\tilde{A}$ around the same component knot carrying the fundamental representation in the $U(M)$ Chern-Simons theory on a Lagrangian three-cycle which intersects $S^3$ along the curve $K_\alpha$.

We can use some group theoretic properties to show that the expectation value of the operator (3.4) exactly matches $Z_0[M]/S_{00}$ provided we choose the rank of the two Chern-Simons gauge groups to be same ($N = M$). If we expand the exponential in eqn. (3.4), we will get

\[ Z(\{U_\alpha\}, \{V_\alpha\}) = 1 + \sum_{\vec{k}(\alpha)} \prod_{\alpha=1}^{r} \frac{1}{z_{\vec{k}(\alpha)} \gamma_{\vec{k}(\alpha)}(U_\alpha) \gamma_{\vec{k}(\alpha)}(V_\alpha)} \] (3.5)

where

\[ z_{\vec{k}(\alpha)} = \prod_{j} \gamma_j^{(\alpha)} \delta_j \gamma_j^{(\alpha)}(U_\alpha) = \prod_{j=1}^{\infty} \left( \text{Tr} U_\alpha^j \right)^{\gamma_j^{(\alpha)}} \] (3.6)

Here $\vec{k}(\alpha) = (k_1^{(\alpha)}, k_2^{(\alpha)}, \ldots)$ with $|\vec{k}(\alpha)| = \sum_j k_j^{(\alpha)}$ and the sum is over all the vectors $\vec{k}(\alpha)$ such that $\sum_{\alpha=1}^{r} |\vec{k}(\alpha)| > 0$. Using the group theoretic properties

\[ \gamma_{k_1}(U_1) \ldots \gamma_{k_r}(U_r) = \sum_{R_1, \ldots, R_r} \prod_{a=1}^{r} \chi_{R_a} (C(\vec{k}(\alpha))) \text{Tr}_{R_1} (U_1) \ldots \text{Tr}_{R_r} (U_r) \] (3.7)

\[ \sum_{\vec{k}} \frac{1}{z_{\vec{k}}} \chi_{R_1} (C(\vec{k})) \chi_{R_2} (C(\vec{k})) = \delta_{R_1 R_2} \] (3.8)
where $\chi_{R\alpha}(C(\tilde{k}^{(\alpha)}))$’s are characters of the symmetry group $S_{\ell\alpha}$ with $\ell\alpha = \sum_j jk_j^{(\alpha)}$ and $C(\tilde{k}^{(\alpha)})$ are the conjugacy classes associated to $\tilde{k}^{(\alpha)}$’s (denoting $k_j^{(\alpha)}$ cycles of length $j$), one can show that the eqn. (3.5) becomes

$$Z(\{U\alpha\}, \{V\alpha\}) = \sum_{\{R\alpha\}} r \prod_{\alpha=1}^r \text{Tr}_{R\alpha}(U\alpha)\text{Tr}_{R\alpha}(V\alpha) . \quad (3.9)$$

Ooguri and Vafa have conjectured a specific form for the vacuum expectation value (vev) of the topological operators (3.4) for knots [5] invoking the large $N$ topological string duality. This result was further refined for links [8, 19] which is generalisable for framed links as follows

$$\langle Z(\{U\alpha\}, \{V\alpha\}) \rangle_A = \exp \left[ \sum_{d=1}^{\infty} \sum_{\{R\alpha\}_d} \frac{1}{d} f_{(R_1,\ldots,R_r)}(q^d, \lambda^d) \prod_{\alpha=1}^r \text{Tr}_{R\alpha}V_{\alpha}^d \right] , \quad (3.10)$$

$$f_{(R_1,\ldots,R_r)}(q, \lambda) = \lambda^{\frac{1}{2}} \sum_{r} \frac{1}{\ell_{\alpha_0}} \sum_{Q,s} \frac{1}{(q^{1/2} - q^{-1/2})^2} N_{(R_1,\ldots,R_r),Q,s} q^s \lambda^Q \quad (3.11)$$

where the suffix $A$ on the vev implies that the expectation value is obtained by integrating the $U(N)$ gauge fields $A$’s on $S^3$, and $\ell\alpha$ is the total number of boxes in the Young Tableau of the representation $R\alpha$. Further, for framed links $N_{(R_1,\ldots,R_r),Q,s}$ are integers only if expectation value of the $U(N)$ Wilson loop operators in $S^3$ are defined as [9]

$$\langle \prod_{\alpha=1}^r \text{Tr}_{R\alpha}(U\alpha) \rangle = (-1)^{\sum_{\alpha=1}^r \ell_{\alpha_0}} V_{R_1,\ldots,R_r}^{(U(N))}[L, S^3] \quad (3.12)$$

$$= \lambda^{\frac{1}{2}} \sum_{r} \frac{1}{\ell_{\alpha_0}} \sum_{Q,s} \frac{1}{(q^{1/2} - q^{-1/2})^2} N_{(R_1,\ldots,R_r),Q,s} q^s \lambda^Q \quad (3.13)$$

This is justified since the holonomy $V_{\alpha}$ on the Lagrangian three-cycle $C$, under change of framing becomes $\tilde{V}_{\alpha} = (-1)^{p_{\alpha}} V_{\alpha}$ [9], which is equivalent to

$$\text{Tr}_{R\alpha} \tilde{V}_{\alpha} = (-1)^{\ell_{\alpha_0}} \text{Tr}_{R\alpha} V_{\alpha} . \quad (3.14)$$

If we also integrate the $\tilde{A}$ fields in the Chern-Simons field theory on the Lagrangian three-cycle then vev of the topological operator (3.9) is

$$\langle Z(\{U\alpha\}, \{V\alpha\}) \rangle_{A,\tilde{A}} = \sum_{R_1,\ldots,R_r} \langle \prod_{\alpha=1}^r \text{Tr}_{R\alpha} V_{\alpha} \rangle \langle \prod_{\alpha=1}^r \text{Tr}_{R\alpha} U_{\alpha} \rangle . \quad (3.15)$$

where we need to determine the expectation value of the Wilson loop operators on the Lagrangian three-cycle $C$ with Betti number $b_1 = r$ which are non-compact. For these non-compact Lagrangian three-cycles, it appears to be possible to deform knots.
and links into unknot and disjoint collection of unknots respectively. Therefore, it is convincing to assume
\[
\langle \text{Tr}_{R_\alpha} \tilde{V}_\alpha \rangle = \dim_q R_\alpha \equiv \frac{S_{00} R_\alpha}{S_{00}}. \tag{3.16}
\]

Even though the assumption looks logical, finding a proof still remains a challenging question. Substituting eqns. (3.12), (3.16) in (3.15) and comparing with eqns. (2.22), (2.20), we get the relation
\[
\alpha \sigma [L] \frac{Z_0[M]}{(S_{00})^{r+1}} = \langle Z(\{U_\alpha\}, \{V_\alpha\}) \rangle_{A, \tilde{A}}. \tag{3.17}
\]

Also, from eqn. (3.10), we get
\[
\ln \left( \alpha^{\sigma[L]} \frac{Z_0[M]}{(S_{00})^{r+1}} \right) = \sum_{d=1}^{\infty} \frac{1}{d} f_{R_1,\ldots,R_r}(q^d, \lambda^d) \prod_{\alpha=1}^{r} \langle \text{Tr}_{R_\alpha} V^d_\alpha \rangle. \tag{3.18}
\]

In the limit when \( N \to \infty \) and \( g_s \to 0 \), the Chern-Simons partition function \( Z[M] \) can be approximated to \( Z_0[M] \) (2.22) and the above equation relates the Chern-Simons free energy to the expectation value of the topological link operator (3.4) in topological closed string theory. It is appropriate to mention that the information of other three-manifolds obtained from surgery on framed links in \( S^3 \) in Chern-Simons theory is captured by integrating both the \( A \) and \( \tilde{A} \) gauge fields in the topological string theory. It will be interesting to see whether the Chern-Simons free energy (\( \ln Z_0[M] \)) can be shown to be equal to the closed string partition function in these cases. We will see later that the Chern-Simons free energy in fact resembles the A-model partition function.

Before proceeding to the large \( N \) expansion of the free energy, we briefly recapitulate some salient features of the A-model topological string partition function on Calabi-Yau manifolds.

### 3.1 The A-model Topological String Partition Function

The A-model topological partition function on a Calabi-Yau manifold \( X \) with some number of Kahler parameters denoted by \( t_i \)'s is defined as
\[
\hat{F}(X) = \sum_g g_s^{2g-2} \hat{F}_g(\{t_i\}), \tag{3.19}
\]
\[
\hat{F}_g(\{t_i\}) = \sum_{\{\beta_i\} \in H_2(X,\mathbb{Z})} N_g^{\{\beta_i\}} e^{-\beta \cdot i}. \tag{3.20}
\]
where \( \hat{F}_g(\{t_i\}) \) are the A-model topological string amplitudes at genus \( g \) and \( N^g_{\{\beta_i\}} \) are the closed string Gromov-Witten invariants associated to genus \( g \) curves in the homology class \( \vec{\beta} \). In Ref. [3], a strong structure result has been derived from M-theory for the topological string partition function

\[
\hat{F}(X) = \sum_{\{m_i\}, r \geq 0, d > 0} \frac{1}{d} n_{r,\{m_i\}} (2 \sinh \frac{dg_s}{2})^{2r-2} \exp[-d(\sum m_i t_i)]
\]  

(3.21)

where \( n_{r,\{m_i\}} \) are integers usually referred to as Gopakumar-Vafa invariants. Clearly, the genus \( g \) Gromov-Witten invariants \( N^g_{\{\beta_i\}} \) will involve the set of Gopakumar-Vafa invariants \( n_{r \leq g, \{m_i\} \leq \{\beta_i\}} \).

We would like to derive a large \( N \) closed string expansion for the Chern-Simons free energy \( \ln Z_0[M] \) on any three manifold \( M \) (3.18) and show that it has the structure (3.21). It is not a priori clear whether the Chern-Simons free energy on any \( M \) will have a closed string interpretation. Interestingly, using the duality connection between Chern-Simons theory on \( S^3 \) and topological strings and some properties of group theory, we will show that the free energy (3.18) does have the form (3.21) for a subset of knots and links.

We can write the RHS of the eqn. (3.18) as follows

\[
\sum_{R_1, \ldots, R_r} f_{(R_1, \ldots, R_r)}(q^d, \lambda^d) \prod_{\alpha=1}^r \langle \operatorname{Tr}_{R_\alpha} V_\alpha^d \rangle = \sum_{\vec{k}(1), \ldots, \vec{k}(r)} f_{\vec{k}(1), \ldots, \vec{k}(r)}(q^d, \lambda^d) \prod_{\alpha=1}^r \frac{1}{z_{\vec{k}(\alpha)}} (\gamma_{\vec{k}(\alpha)}(V_\alpha^d)),
\]  

(3.22)

where \( f_{\vec{k}(1), \ldots, \vec{k}(r)} \) is the character transform of \( f_{R_1, \ldots, R_r} \) whose form has been derived in [8], namely,

\[
f_{\vec{k}(1), \ldots, \vec{k}(r)}(q, \lambda) = \left( \prod_j (q^\frac{1}{2} - q^{-\frac{1}{2}})^{\sum_{\alpha=1}^r k_j^{(\alpha)}} (q^\frac{1}{2} - q^{-\frac{1}{2}})^2 \right) \left( \lambda^\frac{1}{2} \left[ \sum_{\alpha} p_\alpha \left( \sum_j jk_j^{(\alpha)} \right) \right] \right) \times \frac{1}{z_{\vec{k}(\alpha)}} (\gamma_{\vec{k}(\alpha)}(V_\alpha^d)) \times \sum_Q \sum_{g \geq 0} n_{(\vec{k}(1), \ldots, \vec{k}(r)), g, Q} (q^\frac{1}{2} - q^{-\frac{1}{2}})^{2g} \lambda^Q ,
\]  

(3.23)

where

\[
n_{(\vec{k}(1), \ldots, \vec{k}(r)), g, Q} = \sum_{R_1, \ldots, R_r} \prod_{\alpha=1}^r \chi_{R_\alpha}(C(\vec{k}(\alpha))) \hat{N}_{(R_1, \ldots, R_r), g, Q} .
\]  

(3.24)

\( \hat{N}_{(R_1, \ldots, R_r), g, Q} \) are integers which compute the net number of BPS domain walls of charge \( Q \) and spin \( g \) transforming in the representation \( R_\alpha \) of \( U(M) \) in the topological string theory. As \( V_\alpha \)'s correspond to disjoint unknots with appropriate sign
corresponding to the framing numbers, we can write
\[ \langle \prod_{\alpha=1}^{r} \gamma_{\alpha}^{(\alpha)}(V_{\alpha}^{d}) \rangle = (-1)^{d} \left[ \sum_{p_{\alpha}} \left( \sum_{j} jk_{j}^{(\alpha)} \right) \right] \left( \prod_{j} \left( \lambda_{j}^{d_{j}} - \lambda_{j}^{d_{j} - d} \right) \right) \sum_{\alpha=1}^{r} k_{j}^{(\alpha)}. \] (3.25)

Incorporating the above results in eqn. (3.18), we get
\[ \ln \left( \frac{\alpha^{\sigma[L]} Z_{0}[M]}{(S_{00})^{r+1}} \right) = \sum_{d=1}^{\infty} \sum_{g,Q} \frac{1}{d} \left( 2 \sinh \frac{dgs}{2} \right)^{2g-2} \{ \lambda^{dQ} \times \right. \]
\[ \sum_{\ell_{1} \cdots \ell_{r}} n_{(\ell_{1}, \ldots, \ell_{r})},g,Q^{r} \prod_{\alpha=1}^{r} \left( \frac{1}{z_{\ell_{\alpha}}} \right) \left( -\lambda_{j}^{\frac{1}{2}} \right)^{d_{j}} \left( \sum_{p_{\alpha}} \left( \sum_{j} jk_{j}^{(\alpha)} \right) \right) \prod_{j} \left( \lambda_{j}^{\frac{d_{j}}{2}} - \lambda_{j}^{\frac{d_{j}}{2} - d} \right) \} \] (3.27)

The RHS of the above equation has the structure of the free energy for a closed string provided we can prove that the expression within parenthesis satisfies
\[ \{ \lambda^{dQ} \sum_{\ell_{1} \cdots \ell_{r}} n_{(\ell_{1}, \ldots, \ell_{r}),g,Q}^{r} \prod_{\alpha=1}^{r} \left( \frac{1}{z_{\ell_{\alpha}}} \right) \left( -\lambda_{j}^{\frac{1}{2}} \right)^{d_{j}} \left( \sum_{p_{\alpha}} \left( \sum_{j} jk_{j}^{(\alpha)} \right) \right) \prod_{j} \left( \lambda_{j}^{\frac{d_{j}}{2}} - \lambda_{j}^{\frac{d_{j}}{2} - d} \right) \} = \sum_{\{m_{i}\}} n_{g,\{m_{i}\}} e^{-d \sum_{i} m_{i}}. \]

Identifying, \( \lambda = \exp(t) \), we see that the above relation will be true for Calabi-Yau spaces with one Kahler parameter after performing appropriate analytic continuation of the variable \( \lambda \to \lambda^{-1} \). We should be able to extract the integer invariants (the Gopakumar-Vafa invariants) from the open string invariants \( n_{(\ell_{1}, \ldots, \ell_{r}),g,Q} \). Using the results obtained in Ref. [8], one can show that
\[ (-\lambda_{j}^{\frac{1}{2}})^{d_{j}} \left( \sum_{p_{\alpha}} \left( \sum_{j} jk_{j}^{(\alpha)} \right) \right) \prod_{j} \left( \lambda_{j}^{\frac{d_{j}}{2}} - \lambda_{j}^{\frac{d_{j}}{2} - d} \right) = (-\lambda_{j}^{\frac{1}{2}})^{d_{j}} \sum_{R_{\alpha}} (-\lambda_{j}^{\frac{1}{2}})^{\ell_{\alpha} d_{\alpha}} C_{\lambda}^{R_{\alpha}}(C^{(\alpha)}) S_{R_{\alpha}}(\lambda^{d}) \]
where \( S_{R_{\alpha}}(\lambda) \) is zero if \( R_{\alpha} \) is not a hook representation and if \( R_{\alpha} \) is a hook representation with \( \ell_{\alpha} \) boxes with \( \ell_{\alpha} - s_{\alpha} \) boxes in the first row (and we denote the representation as \( R_{\ell_{\alpha},s_{\alpha}} \)) then we have
\[ S_{R_{\alpha}}(\lambda^{d}) = (-1)^{s_{\alpha}} \lambda^{d(-\frac{\ell_{\alpha}}{2} + s_{\alpha})} \] (3.29)

Using the properties (3.24), (3.28) in eqn. (3.20) we get
\[ \ln \left( \frac{\alpha^{\sigma[L]} Z_{0}[M]}{(S_{00})^{r+1}} \right) = \sum_{d=1}^{\infty} \sum_{g,Q} \frac{1}{d} \left( 2 \sinh \frac{dgs}{2} \right)^{2g-2} \{ \sum_{\ell_{1} \cdots \ell_{r}} \sum_{\{s_{\alpha}\}} \tilde{N}_{(\ell_{1}, s_{1}, \ldots, \ell_{r}, s_{r}),g,Q} \times \]
\[ (-1)^{s_{\alpha}} (-1)^{d s_{\alpha}} \ell_{\alpha}^{d_{\alpha}} \sum_{p_{\alpha}} \left( \lambda^{d} \{ Q + \sum_{s_{\alpha}} (-\frac{d}{2} + s_{\alpha}) \} - \lambda^{d} \{ Q + 1 + \sum_{s_{\alpha}} (-\frac{d}{2} + s_{\alpha}) \} \right) \} \]
\[ = \sum_{g,d,m} \frac{1}{d} \left( 2 \sinh \frac{dgs}{2} \right)^{2g-2} n_{g,d,m} e^{-dmt} \] (3.30)
We can obtain the Gopakumar-Vafa invariants \( n_{g,m} \), and also the Gromov-Witten invariants, by evaluating the \( \hat{N}_{(R_1,\ldots,R_r),g,Q} \) for various framed knots and links.

Before proceeding to outline the evaluation of the \( \hat{N} \)'s in the next subsection, we remind the reader that in our computation of the Gopakumar-Vafa and the Gromov-Witten invariants, we use the full Chern-Simons partition function in (3.30), and show that this has a closed string interpretation. For reasons that have been outlined in the introduction, our results do not constitute new dualities between Chern-Simons theory on \( M \) and closed A-type topological string theories, although we believe that these results would be very important for gaining a full understanding of the same.

Also, note that even though one can check that \( \sum_\alpha (-\ell_\alpha/2 + Q) \) is always an integer, the term \( (-1)^{\lambda/2} \sum_\alpha \ell_\alpha p_\alpha \) within the parenthesis of (3.30) can be integral powers of \( \lambda \) (for arbitrary \( \ell_\alpha \)'s) if and only if \( p_\alpha \)'s on the components knots are even. This suggests that the closed string expansion (3.27) is possible only for framed knots and links with even numbers of framing numbers \( p_\alpha \)'s on all the component knots.

### 3.2 Determination of the \( \hat{N} \)'s from framed link invariants

The general formula for \( f \) (3.11) in terms of framed link invariants (2.20) can be written as [19]

\[
\hat{f}_{R_1,R_2,\ldots,R_r}(q, \lambda) = \lambda \sum_\alpha \ell_\alpha p_\alpha/2 \sum_{d,m=1}^{\infty} (-1)^{m-1} \frac{\mu(d)}{dm} \sum_{\{\vec{k}(\alpha j)\}, R_{\alpha j}} \times \\
\prod_{\alpha=1}^{r} \chi_{R_{\alpha j}} \left(C \left( \sum_{j=1}^{m} \frac{C'(\vec{k}(\alpha j))}{\ell_{\alpha j}} \right) \right) \prod_{j=1}^{m} \frac{|C'(\vec{k}(\alpha j))|}{\ell_{\alpha j}!} \times \\
\chi_{R_{\alpha j}} (C(\vec{k}(\alpha j))) \hat{V}_{R_{R_{1j},R_{2j},\ldots,R_{rj}}} [L, S^3] (q^d, \lambda^d)
\]

where \( \mu(d) \) is the Moebius function defined as follows: if \( d \) has a prime decomposition \( \{p_i\} \), \( d = \prod_{i=1}^{n} p_i^{m_i} \), then \( \mu(d) = 0 \) if any of the \( m_i \) is greater than one. If all \( m_i = 1 \), then \( \mu(d) = (-1)^a \). The second sum in the above equation runs over all vectors \( \vec{k}(\alpha j) \), with \( \alpha = 1, \ldots r \) and \( j = 1, \ldots m \), such that \( \sum_{\alpha=1}^{r} |\vec{k}(\alpha j)| > 0 \) for any \( j \) and over representations \( R_{\alpha j} \). Further \( \vec{k}_d \) is defined as follows: \( (\vec{k}_d)_{d_i} = k_i \) and has zero entries for the other components. Therefore, if \( \vec{k} = (k_1, k_2, \ldots) \), then

\[
\vec{k}_d = (0, \ldots, 0, k_1, 0, \ldots, 0, k_2, 0, \ldots),
\]
where $k_1$ is in the $d$-th entry, $k_2$ in the $2d$-th entry, and so on. Hence, one can directly evaluate $f$ from $U(N)$ framed link invariants (2.20) and verify the conjecture (3.11). Using the following equations

$$
M_{R_1, \ldots, R_r; R'_1, \ldots, R'_r} = \sum_{R''_1, \ldots, R''_r} \prod_{\alpha=1}^{r} C_{R_\alpha R'_\alpha R''_\alpha} S_{R''_\alpha}(q),
$$

(3.33)

$$
\hat{f}_{(R'_1, \ldots, R'_r)}(q, \lambda) = (q^{-1/2} - q^{1/2})^{r-2} \sum_{g \geq 0, Q} \hat{N}_{(R'_1, \ldots, R'_r), g, Q}(q^{-1/2} - q^{1/2})^{2g \lambda} Q,
$$

we can write eqn.(3.31) as

$$
f_{R_1, R_2, \ldots, R_r}(q, \lambda) = \sum_{R_1, \ldots, R_r} M_{R_1, \ldots, R_r; R'_1, \ldots, R'_r} \hat{f}_{(R'_1, \ldots, R'_r)}(q, \lambda).
$$

(3.34)

In eqn.(3.33), $R_\alpha, R'_\alpha, R''_\alpha$ are representations of the symmetric group $S_{\ell_\alpha}$ which can be labelled by a Young-Tableau with a total of $\ell_\alpha$ boxes and $C_{R R' R''}$ are the Clebsch-Gordan coefficients of the symmetric group. In the next section, we will evaluate $\hat{N}$ for few framed knots, links and present the results of our computation of the Gopakumar-Vafa invariants and Gromov-Witten invariants.

## 4 Examples and explicit Results

Our aim in this section is to compute the Gopakumar-Vafa invariants and Gromov-Witten invariants corresponding to the Chern-Simons free energy on three manifolds obtained from surgery of the respective framed links in $S^3$.

### 4.1 Knots in standard framing

In standard framing, there is no distinction between $U(N)$ and $SU(N)$ knot invariants. Therefore for this class of knots with zero framing number, the Chern-Simons partition function will be

$$
Z[M] = Z_0[M],
$$

(4.1)

and the signature of the linking matrix $\sigma[L] = 0$. Substituting in eqn.(3.30), we get

$$
\ln Z[M] - 2\ln Z[S^3] = \sum_{d=1}^{\infty} \sum_{g} \frac{1}{d} \left(2 \sinh \frac{dg_s}{2}\right)^{2g-2} \times
$$

\[
\{ \sum_{Q} \sum_{\ell, s} \hat{N}_{(R_{\ell, s}), g, Q} (-1)^s \left(\lambda^{d(Q-\frac{d}{4}+s)} - \lambda^{d(Q+1-\frac{d}{4}+s)}\right) \}
\]

$$
= \sum_{g, d, m} \frac{1}{d} \left(2 \sinh \frac{dg_s}{2}\right)^{2g-2} n_{g, d} e^{-dm t}
$$

We will now compute the integer coefficients $n_{g, d}$ for few examples.
4.1.1 Unknot

The surgery of this simplest knot gives manifold $S^2 \times S^1$ whose Chern-Simons partition function $Z[S^2 \times S^1] = 1$. The non-zero $\hat{N}$'s for the simplest unknot is

$$\hat{N}_{0, \pm \frac{1}{2}} = \mp 1. \quad (4.3)$$

Substituting this result in eqn.(4.2) and doing the appropriate analytic continuation $\lambda \to \lambda^{-1}$, the non-zero $n_{g,m}$ is

$$n_{0,1} = 2. \quad (4.4)$$

Hence eqn.(4.2) reduces to

$$-2 \ln Z[S^3] = \sum_d \frac{1}{d(2 \sinh \frac{dg}{2})^2} 2e^{-dt}. \quad (4.5)$$

Clearly, the large $N$ expansion of Chern-Simons free-energy on $S^3$ gives Gopakumar-Vafa invariant $(n_{g,m}[S^3])$

$$n_{0,1}[S^3] = -1. \quad (4.6)$$

Using the integer invariants, we can evaluate the Gromov-Witten invariants

$$N^0_{m>0}[S^3] = \frac{-1}{m^3}, N^1_{m>0}[S^3] = \frac{1}{12m}, N^2_{m>0}[S^3] = \frac{-m}{240}. \quad (4.7)$$

These invariants in the closed topological string partition function imply that the target Calabi-Yau space is a resolved conifold.

4.1.2 Torus knots of type $(2, 2m+1)$

The knots obtained as a closure of two-strand braid with $2m + 1$ crossings are the type $(2, 2m + 1)$ torus knots. The surgery of these torus knots in $S^3$ will give Seifert homology spheres $X(\frac{2}{-1}, \frac{2m+1}{m+1}, \frac{-2m+1}{1})$. It will be interesting to determine the Gopakumar-Vafa integer invariants and the closed Gromov-Witten invariants corresponding to large $N$ expansion of Chern-Simons free-energy on such Seifert manifolds.

(i) The $\hat{N}_{R,g,Q}$ corresponding to the torus knot $(2,5)$ for represenations upto three boxes are tabulated below

| $Q$    | $g=0$ | $g=1$ | $g=2$ |
|--------|-------|-------|-------|
| 3/2    | 3     | 4     | 1     |
| 5/2    | -5    | -5    | -1    |
| 7/2    | 2     | 1     | 0     |

$\hat{N}_{R,g,Q}$ for the torus knot $(2,5)$
| $Q$ | $g=0$ | $g=1$ | $g=2$ | $g=3$ | $g=4$ | $g=5$ | $g=6$ | $g=7$ |
|-----|-------|-------|-------|-------|-------|-------|-------|-------|
| 3   | 20    | 60    | 69    | 38    | 10    | 1     | 0     | 0     |
| 4   | -80   | -260  | -336  | -221  | -78   | -14   | -1    | 0     |
| 5   | 120   | 400   | 534   | 366   | 136   | 26    | 2     | 0     |
| 6   | -80   | -260  | -336  | -221  | -78   | -14   | -1    | 0     |
| 7   | 20    | 60    | 69    | 38    | 10    | 1     | 0     | 0     |

$\hat{N}^{\square}_{g,Q}$ for the torus knot $(2, 5)$

| $Q$ | $g=0$ | $g=1$ | $g=2$ | $g=3$ | $g=4$ | $g=5$ | $g=6$ | $g=7$ |
|-----|-------|-------|-------|-------|-------|-------|-------|-------|
| 3   | 30    | 115   | 176   | 137   | 57    | 12    | 1     | 0     |
| 4   | -120  | -490  | -819  | -724  | -365  | -105  | -16   | -1    |
| 5   | 180   | 750   | 1286  | 1174  | 616   | 186   | 30    | 2     |
| 6   | -120  | -490  | -819  | -724  | -365  | -105  | -16   | -1    |
| 7   | 30    | 115   | 176   | 137   | 57    | 12    | 1     | 0     |

$\hat{N}^{\square}_{g,Q}$ for the torus knot $(2, 5)$

| $g$ | $Q=9/2$ | $Q=11/2$ | $Q=13/2$ | $Q=15/2$ | $Q=17/2$ | $Q=19/2$ | $Q=21/2$ |
|-----|----------|----------|----------|----------|----------|----------|----------|
| 0   | 232      | -1652    | 4820     | -7400    | 6320     | -2852    | 532      |
| 1   | 1436     | -11626   | 37290    | -61400   | 55140    | -25726   | 4886     |
| 2   | 4046     | -38060   | 135824   | -241510  | 228824   | -110390  | 21266    |
| 3   | 6781     | -75590   | 303114   | -584700  | 584729   | -290550  | 56216    |
| 4   | 7384     | -100086  | 456013   | -958591  | 1011218  | -514589  | 98651    |
| 5   | 5384     | -92128   | 483836   | -1115009 | 1240265  | -642511  | 120163   |
| 6   | 2636     | -60064   | 370471   | -943863  | 1107524  | -580839  | 104135   |
| 7   | 851      | -27853   | 206727   | -589169  | 730275   | -385792  | 64961    |
| 8   | 173      | -9107    | 83995    | -272258  | 357280   | -189269  | 29186    |
| 9   | 20       | -2048    | 24548    | -92689   | 129164   | -68333   | 9338     |
| 10  | 1        | -301     | 5020     | -22898   | 34006    | -17899   | 2071     |
| 11  | 0        | -26      | 681      | -3984    | 6331     | -3304    | 302      |
| 12  | 0        | -1       | 55       | -462     | 789      | -407     | 26       |
| 13  | 0        | 0        | 2        | -32      | 59       | -30      | 1        |
| 14  | 0        | 0        | 0        | -1       | 2        | -1       | 0        |
| 15  | 0        | 0        | 0        | 0        | 0        | 0        | 0        |
| 16  | 0        | 0        | 0        | 0        | 0        | 0        | 0        |
| 17  | 0        | 0        | 0        | 0        | 0        | 0        | 0        |

$\hat{N}^{\square}_{g,Q}$ for the torus knot $(2, 5)$
\[
\mathcal{N}_{g,Q} \text{ for the torus knot (2, 5).}
\]

| \(g\) | \(Q=9/2\) | \(Q=11/2\) | \(Q=13/2\) | \(Q=15/2\) | \(Q=17/2\) | \(Q=19/2\) | \(Q=21/2\) |
|-----|------|------|------|------|------|------|------|
| 0   | 778  | -5483 | 15755 | -23750 | 19880 | -8783 | 1603  |
| 1   | 5929 | -46514| 145060| -232875| 204460| -93539| 17479 |
| 2   | 20986| -186222| 636631| -1095250| 1012006| -479592 | 91441 |
| 3   | 44960| -458386| 1732046| -3205225| 3116781| -1523901| 293725 |
| 4   | 64066| -764419| 3219215| -6427475| 6569210| -3295156| 634559 |
| 5   | 63300| -905137| 4290087| -9275345| 9951953| -5092334| 967476 |
| 6   | 44151| -780483| 4213699| -9914620| 11161326| -5796079| 1072006 |
| 7   | 21814| -496526| 3099128| -7990833| 9440203| -4952673| 878887 |
| 8   | 7564 | -233985| 1719912| -4905169| 6086843| -3213286| 538121 |
| 9   | 1795 | -81304 | 720501 | -2301847| 3004880| -1590473| 246448 |
| 10  | 277  | -20526 | 226187 | -823741 | 1135323 | -599603 | 83883 |
| 11  | 25   | -3656  | 52319  | -222684 | 3237849 | -170666 | 20876 |
| 12  | 1    | -435   | 8643   | -44635  | 68760   | -36018  | 3684  |
| 13  | 0    | -31    | 964    | -6421   | 10510   | -5458   | 436   |
| 14  | 0    | -1     | 65     | -626    | 1092    | -561    | 31    |
| 15  | 0    | 0      | 2      | -37     | 69      | -35     | 1     |
| 16  | 0    | 0      | 0      | -1      | 2       | -1      | 0     |
| 17  | 0    | 0      | 0      | 0       | 0       | 0       | 0     |

\[
\mathcal{N}_{5-g,Q} \text{ for the torus knot (2, 5)}
\]
The surgery of the torus knot $(2, 5)$ gives the Seifert manifold $M_1 \equiv X(\frac{2}{1}, \frac{5}{3}, -\frac{5}{1})$.

Comparing powers of $\lambda^{-1}$ (after analytic continuation) in eqn.(4.2) we can obtain Gopakumar-Vafa integer invariants $(n_{g,m}[M_1])$, corresponding to large $N$ expansion of the Chern-Simons free energy on $M_1$, in terms of $\hat{N}$’s as follows

\[
\begin{align*}
n_{g,1}[M_1] &= \hat{N}_{g,3/2} - 2\delta_{g,0}, \\
n_{g,2}[M_1] &= \hat{N}_{g,5/2} - \hat{N}_{g,3/2} + \hat{N}_{g,3}, \\
n_{g,3}[M_1] &= \hat{N}_{g,7/2} - \hat{N}_{g,5/2} - \hat{N}_{g,3} + \hat{N}_{g,4} - \hat{N}_{g,3} + \hat{N}_{g,9/2}.
\end{align*}
\]

From these invariants, we can extract Gromov-Witten invariants which are rational numbers. A few of them are given below

\[
\begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|}
\hline
m & g=0 & g=1 & g=2 & g=3 & g=4 & g=5 & g=6 & g=7 & g=8 & g=9 & g=10 & g \geq 11 \\
\hline
1 & 1 & 4 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\
2 & 12 & 51 & 67 & 38 & 10 & 1 & 0 & 0 & 0 & 0 & 0 & \\
3 & 109 & 1407 & 3466 & 6385 & 7239 & 5357 & 2634 & 851 & 173 & 20 & 1 & 0 \\
\hline
\end{array}
\]

(ii) The $\hat{N}_{R,g,Q}$ computation of the torus knot $(2, 7)$ will be

\[
\begin{array}{|c|c|c|c|c|}
\hline
Q & g=0 & g=1 & g=2 & g=3 \\
\hline
5/2 & -4 & -10 & -6 & -1 \\
7/2 & 7 & 14 & 7 & 1 \\
9/2 & -3 & -4 & -1 & 0 \\
\hline
\end{array}
\]

$\hat{N}_{g,Q}$ for the torus knot $(2, 7)$
From eqn. (4.2), we can obtain Gopakumar-Vafa integer invariants $n_{g,m} \big[ M_2 \big]$ corresponding to Chern-Simons theory on Seifert manifold $M_2 = X(\frac{2}{3}, \frac{7}{4}, \frac{2}{1})$. We present few of them in terms of $\hat{N}_{R,g,Q}$

$$
\begin{align*}
  n_{g,1}[M_2] &= -2\delta_{g,0} \\
  n_{g,2}[M_2] &= \hat{N}_{g,5/2} \\
  n_{g,3}[M_2] &= \hat{N}_{g,7/2} - \hat{N}_{g,5/2} \\
  n_{g,4}[M_2] &= \hat{N}_{g,9/2} - \hat{N}_{g,7/2} + \hat{N}_{g,5} \\
  n_{g,5}[M_2] &= -\hat{N}_{g,9/2} - \hat{N}_{g,5} + \hat{N}_{g,6} - \hat{N}_{g,5}
\end{align*}
$$

From these integer invariants, it is straightforward to obtain Gromov-Witten rational numbers.

It appears from the computation of $\hat{N}_{R,g,Q}$’s for the two torus knots that the range of $Q$ is $\ell(2m - 1)/2 \leq Q \leq \ell(2m + 3)/2$. This range of $Q$ allows finite number of $\hat{N}$’s to contribute to $n_{g,m}$, $N_{g,m}$ invariants. Thus we see that the large $N$ expansion of the Chern-Simons free energy on Seifert manifolds obtained from surgery of torus knots of type $(2, 2m + 1)$ can be given a closed string interpretation. The target Calabi-Yau space $Y$ corresponding to the closed topological string theory must have the Gopakumar-Vafa integer invariants and closed Gromov-Witten invariants determined from Chern-Simons theory. So far, we considered only knots in standard framing. In the next subsection we address three-manifolds obtained from framed knots in $S^3$. 

| $g$ | $Q=5$ | $Q=6$ | $Q=7$ | $Q=8$ | $Q=9$ |
|-----|-------|-------|-------|-------|-------|
| 0   | -84   | 336   | -504  | 336   | -84   |
| 1   | -574  | 2380  | -3612 | 2380  | -574  |
| 2   | -1652 | 7182  | -11060| 7182  | -1652 |
| 3   | -2623 | 12144 | -19042| 12144 | -2623 |
| 4   | -2529 | 12739 | -20420| 12739 | -2529 |
| 5   | -1536 | 8673  | -14274| 8673  | -1536 |
| 6   | -589  | 3892  | -6606 | 3892  | -589  |
| 7   | -138  | 1141  | -2006 | 1141  | -138  |
| 8   | -18   | 210   | -384  | 210   | -18   |
| 9   | -1    | 22    | -42   | 22    | -1    |
| 10  | 0     | 1     | -2    | 1     | 0     |
| 11  | 0     | 0     | 0     | 0     | 0     |
4.2 Framed Knots

We have seen that the Chern-Simons partition function $Z[M]$, corresponding to manifolds obtained from knots with non-zero framing number, can be approximated to $Z_0[M]$ in the limit $N \to \infty$, $g_s \to 0$. We shall now determine $\ln Z_0[M]$ for framed unknot with even framing number $2p$.

**Unknot with framing $p$:** The surgery of $p$-framed unknot results in Lens spaces $L(p,1)$. The $\hat{N}$’s for the unknot with framing $p = 4$ are

$$
\begin{array}{c|c}
Q & g=0 \\
-1/2 & 1 \\
1/2 & -1 \\
\end{array}
$$

$\hat{N}_{g,Q}$ for the unknot with framing $p = 4$

$$
\begin{array}{c|cccc}
Q & g=0 & g=1 & g=2 & g=3 \\
-1 & 2 & 1 & 0 & 0 \\
0 & -6 & -5 & -1 & 0 \\
1 & 4 & 4 & 1 & 0 \\
\end{array}
\begin{array}{c|cccc}
Q & g=0 & g=1 & g=2 & g=3 \\
-1 & 4 & 4 & 1 & 0 \\
0 & -10 & -15 & -7 & -1 \\
1 & 6 & 11 & 6 & 1 \\
\end{array}
$$

$\hat{N}_{g,Q}$ for the unknot with $p = 4$

$$
\begin{array}{c|cccccccccccc}
Q & g=0 & g=1 & g=2 & g=3 & g=4 & g=5 & g=6 & g=7 & g=8 & g=9 & g=10 \\
-3/2 & 12 & 26 & 22 & 8 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1/2 & -58 & -181 & -246 & -175 & -67 & -13 & -1 & 0 & 0 & 0 & 0 \\
1/2 & 86 & 335 & 582 & 550 & 298 & 92 & 15 & 1 & 0 & 0 & 0 \\
3/2 & -40 & -180 & -358 & -383 & -232 & -79 & -14 & -1 & 0 & 0 & 0 \\
\end{array}
\begin{array}{c|cccccccccccc}
Q & g=0 & g=1 & g=2 & g=3 & g=4 & g=5 & g=6 & g=7 & g=8 & g=9 & g=10 \\
-3/2 & 46 & 155 & 224 & 167 & 66 & 13 & 1 & 0 & 0 & 0 & 0 \\
-1/2 & -206 & -915 & -1836 & -2057 & -1377 & -561 & -136 & -18 & -1 & 0 & 0 \\
1/2 & 290 & 1545 & 3768 & 5226 & 4446 & 2394 & 817 & 171 & 20 & 1 & 0 \\
3/2 & -130 & -785 & -2156 & -3336 & -3135 & -1846 & -682 & -153 & -19 & -1 & 0 \\
\end{array}
$$

$\hat{N}_{g,Q}$ for the unknot with framing $p = 4$
Using these \( \hat{N}_{R,g,Q} \) for the unknot with framing \( p = 4 \), we can determine Gopakumar-Vafa integer invariants \( n_{g,m}[\mathcal{L}(4, 1)] \) from the expansion of \( \ln Z_0[\mathcal{L}(p, 1)] \). We present some of the non-zero coefficients

\[
\begin{align*}
  n_{g,1}[\mathcal{L}(4, 1)] &= \hat{N}_{g,-1/2} - 2\delta_{g,0} , \\
  n_{g\leq2}[\mathcal{L}(4, 1)] &= \hat{N}_{g,1/2} - \hat{N}_{g,-1/2} + \hat{N}_{g,-1} , \\
  n_{g\leq3}[\mathcal{L}(4, 1)] &= -\hat{N}_{g,1/2} + \hat{N}_{g,1/2} - \hat{N}_{g,-1} - \hat{N}_{g,-1} + \hat{N}_{g,3/2} , \\
  n_{g,4}[\mathcal{L}(4, 1)] &= \hat{N}_{g,1} - \hat{N}_{g,0} - \hat{N}_{g,0} + \hat{N}_{g,-1} + \hat{N}_{g,-1/2} , \\
  &- \hat{N}_{g,3/2} - \hat{N}_{g,-3/2} + \hat{N}_{g,-3/2} ,
\end{align*}
\]

and the closed Gromov-Witten rational numbers can be deduced from the integer invariants. In the appendix, we have the \( \hat{N}_{R,g,Q} \) for representations up to two boxes for the unknot with arbitrary framing. They will be useful to determine \( n_{g,m} \) corresponding to Chern-Simons theory on Lens spaces \( \mathcal{L}(2p, 1) \). In the following subsection, we will consider framed links.

### 4.3 Framed Links

We take Hopf link \( H^*(p_1, p_2) \) with linking number \( \ell k = -1 \) and the framing on the two component knots as \( p_1 = p_2 = 4 \). Surgery of such a framed link in \( S^3 \) will give Lens space \( \mathcal{L}(15, 4) \). The \( \hat{N}_{(R_1,R_2),g,Q} = \hat{N}_{(R_2,R_1),g,Q} \) for this example is tabulated below:

| \( Q \) | \( g=0 \) | \( g=1 \) | \( g=2 \) | \( g=3 \) |
|---|---|---|---|---|
| \(-3/2\) | 3 | 1 | 0 | 0 |
| \(-1/2\) | -9 | -6 | -1 | 0 |
| \(1/2\) | 6 | 5 | 1 | 0 |

\( \hat{N}_{(\square,\Box),g,Q} \) for the framed Hopf link \( H^*(4, 4) \)

| \( Q \) | \( g=0 \) | \( g=1 \) | \( g=2 \) | \( g=3 \) |
|---|---|---|---|---|
| \(-3/2\) | 6 | 5 | 1 | 0 |
| \(-1/2\) | -16 | -20 | -8 | -1 |
| \(1/2\) | 10 | 15 | 7 | 1 |

\( \hat{N}_{(\square,\Box),g,Q} \) for \( H^*(4, 4) \)
| $Q$ | $g=0$ | $g=1$ | $g=2$ | $g=3$ | $g=4$ | $g=5$ | $g=6$ | $g=7$ | $g=8$ | $g=9$ | $g=10$ |
|-----|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| -2  | 23    | 41    | 29    | 9     | 1     | 0     | 0     | 0     | 0     | 0     | 0     |
| -1  | -117  | -312  | -367  | -230  | -79   | -14   | -1    | 0     | 0     | 0     | 0     |
| 0   | 180   | 606   | 920   | 771   | 376   | 106   | 16    | 1     | 0     | 0     | 0     |
| 1   | -86   | -335  | -582  | -550  | -298  | -92   | -15   | -1    | 0     | 0     | 0     |

$\hat{N}_{\square}, g, Q$ for the framed Hopf link $H^*(4, 4)$

| $Q$ | $g=0$ | $g=1$ | $g=2$ | $g=3$ | $g=4$ | $g=5$ | $g=6$ | $g=7$ | $g=8$ | $g=9$ | $g=10$ |
|-----|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| -2  | 94    | 271   | 338   | 221   | 78    | 14    | 1     | 0     | 0     | 0     | 0     |
| -1  | -438  | -1697 | -3001 | -3003 | -1820 | -680  | -153  | -19   | -1    | 0     | 0     |
| 0   | 634   | 2971  | 6431  | 8008  | 6188  | 3060  | 969   | 190   | 21    | 1     | 0     |
| 1   | -290  | -1545 | -3768 | -5226 | -4446 | -2394 | -817  | -171  | -20   | -1    | 0     |

$\hat{N}_{\square}, g, Q$ for the framed Hopf link $H^*(4, 4)$

| $Q$ | $g=0$ | $g=1$ | $g=2$ | $g=3$ | $g=4$ | $g=5$ | $g=6$ | $g=7$ | $g=8$ | $g=9$ | $g=10$ |
|-----|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| -2  | 86    | 335   | 582   | 550   | 298   | 92    | 15    | 1     | 0     | 0     | 0     |
| -1  | -376  | -1880 | -4350 | -5776 | -4744 | -2486 | -832  | -172  | -20   | -1    | 0     |
| 0   | 520   | 3080  | 8514  | 13672 | 13820 | 9142  | 4013  | 1158  | 211   | 22    | 1     |
| 1   | -230  | -1535 | -4746 | -8446 | -9374 | -6748 | -3196 | -987  | -191  | -21   | -1    |

$\hat{N}_{\square}, g, Q$ for the framed Hopf link $H^*(4, 4)$

| $Q$ | $g=0$ | $g=1$ | $g=2$ | $g=3$ | $g=4$ | $g=5$ | $g=6$ |
|-----|-------|-------|-------|-------|-------|-------|-------|
| -2  | 15    | 17    | 7     | 1     | 0     | 0     | 0     |
| -1  | -60   | -83   | -45   | -11   | -1    | 0     | 0     |
| 0   | 81    | 126   | 75    | 20    | 2     | 0     | 0     |
| 1   | -36   | -60   | -37   | -10   | -1    | 0     | 0     |

$\hat{N}_{\square}, g, Q$ for the framed Hopf link $H^*(4, 4)$

| $Q$ | $g=0$ | $g=1$ | $g=2$ | $g=3$ | $g=4$ | $g=5$ | $g=6$ |
|-----|-------|-------|-------|-------|-------|-------|-------|
| -2  | 27    | 45    | 30    | 9     | 1     | 0     | 0     |
| -1  | -105  | -206  | -165  | -66   | -13   | -1    | 0     |
| 0   | 138   | 301   | 262   | 113   | 24    | 2     | 0     |
| 1   | -60   | -140  | -127  | -56   | -12   | -1    | 0     |

$\hat{N}_{\square}, g, Q$ for the framed Hopf link $H^*(4, 4)$
| $g$  | $g=0$ | $g=1$ | $g=2$ | $g=3$ | $g=4$ | $g=5$ | $g=6$ |
|------|-------|-------|-------|-------|-------|-------|-------|
| -2   | 48    | 106   | 99    | 47    | 11    | 1     | 0     |
| -1   | -184  | -466  | -501  | -287  | -91   | -15   | -1    |
| 0    | 236   | 660   | 767   | 470   | 159   | 28    | 2     |
| 1    | -100  | -300  | -365  | -230  | -79   | -14   | -1    |

$\hat{N}_{(\cdot,\cdot,g,Q)}$ for the framed Hopf link $H^*(4,4)$

When one of the representations is trivial, the $\hat{N}_{(\cdot,\cdot,g,Q)}$’s will be equal to the invariants $\hat{N}_{R,g,Q}$’s computed for unknot with framing $p = 4$.

The Gopakumar-Vafa invariants from the expansion of $\ln Z_0[\mathcal{L}(15, 4)]$ are

\[
\begin{align*}
n_{g,1}[\mathcal{L}(15, 4)] & = 2n_{g,1}[\mathcal{L}(4, 1)] + \delta_{g,0}, \\
n_{g,2}[\mathcal{L}(15, 4)] & = \delta_{g,0} \hat{N}_{(\cdot,\cdot),g,-1} + 2n_{g,2}[\mathcal{L}(4, 1)], \\
n_{g,3,1}[\mathcal{L}(15, 4)] & = \hat{N}_{(\cdot,\cdot),g,0} - \hat{N}_{(\cdot,\cdot),g,-1} + 2\hat{N}_{(\cdot,\cdot),g,3/2} + 2n_{g,3}[\mathcal{L}(4, 1)], \\
n_{g,3,4}[\mathcal{L}(15, 4)] & = -2\hat{N}_{(\cdot,\cdot),g,0} - 2\hat{N}_{(\cdot,\cdot),g,3/2} + 2\hat{N}_{(\cdot,\cdot),g,-1/2} + 2\hat{N}_{(\cdot,\cdot),g,-3/2} + 2\hat{N}_{(\cdot,\cdot),g,-2} + 2n_{g,4}[\mathcal{L}(4, 1)].
\end{align*}
\]

These examples suggest that the Chern-Simons partition function on various manifolds can be given a A-model closed string theory interpretation. From the prediction of Gopakumar-Vafa integer invariants it should be possible to determine the nature of the Calabi-Yau background.

## 5 Summary and Conclusions

We have studied $U(N)$ Chern-Simons gauge theory and framed link invariants in $S^3$, for specific choice of $U(1)$ representations placed on the component knots of the link. From these $U(N)$ framed link invariants in $S^3$, we have constructed three-manifold invariants which are the same as $SU(N)$ three-manifold invariants. These invariants are proportional to the Chern-Simons partition function $Z[M]$ on the corresponding three-manifolds.

In this paper, we have used the results of the topological string duality conjecture relating Chern-Simons theory on $S^3$ to closed A model topological string theory on the resolved conifold, to obtain large $N$ expansions of the Chern-Simons free-energy ($\ln Z[M]$) on some non-trivial manifolds. The closed string theory expansion resembles the A-model topological string theory on a Calabi-Yau space with one
Kahler parameter. We have computed the Gopakumar-Vafa integer invariants and the Gromov Witten invariants associated with Chern-Simons partition function on three-manifolds like Seifert manifolds, Lens spaces etc.

We need to understand some subtle issues about the Chern-Simons partition function on any three-manifold \( M \). As we have pointed out in the introduction, the classical solutions of the Chern-Simons action are the flat connections on \( M \), and in the weak coupling (large \( k \)) limit, the partition function, which may be a sum or integral over the space of flat connections, can be written as

\[
Z[M] = \sum_c Z_c[M] \equiv \int_{\mu_c} Z_c[M],
\]

(5.1)

where \( Z_c[M] \) is obtained from perturbative expansion around a stationary point \( A = A_c \). The large \( N \) expansion proposed by ‘t Hooft requires \( \ln Z_c[M] \) to have a closed string interpretation whereas we have shown in this paper that \( \ln Z[M] \) has a closed string expansion for many manifolds.

Note that for the case of the three-sphere \( S^3 \), there is only one stationary point, which is the trivial connection. Therefore \( Z[S^3] \) is also equal to the perturbative expansion around the trivial connection and hence the closed string interpretation is expected from ‘t Hooft’s formulation.

In the context of Lens spaces \( \mathcal{L}(p, 1) \), the space of flat connections is a set of points. In ref. [17], \( Z[M] \) has been rewritten as a sum over all flat connections, enabling the extraction of the perturbative partition function around a non-trivial flat connection \( (Z_c[M]) \) for \( \mathcal{L}(p, 1) \). It has been shown from the matrix model approach that \( Z_c[M] \) can be given a closed string theoretic interpretation. The results establish the duality between Chern-Simons theory on Lens spaces \( \mathcal{L}(p, 1) \) and closed string theory on a \( A_{p-1} \) singularity fibred over \( P^1 \). The Gopakumar Vafa integer invariants that we have computed for Lens spaces \( \mathcal{L}(2p, 1) \) correspond to \( \ln (\sum_c Z_c[M]) \). There must be some relation between these integer invariants and the corresponding invariants on \( A_{2p-1} \) singularity fibred over \( P^1 \) at some special values of the Kahler parameters. We hope to decipher such interesting relations in future.

It appears to be a difficult task to rewrite \( Z[M] \) as a sum or integral over flat connections to determine \( Z_c[M] \) for other three-manifolds like Seifert-manifolds. The challenge lies in determining \( Z_c[M] \) and the closed string expansion to precisely state new duality conjectures between Chern-Simons theory on \( M \) and closed string theory. We leave the study of these aspects for a future publication.
Acknowledgements

PR would like to thank M. Marino and C. Vafa for discussions during the initial stages of the project, and is grateful to N. Habegger for comments. She would like to thank the Abdus Salam ICTP for providing local hospitality and an excellent academic atmosphere during her visit under the ICTP Junior Associateship scheme. We would like to thank S. Govindarajan, T. R Govindarajan, M. Marino, K. Ray and G. Thompson for discussions and clarifications. PB would like to thank CSIR for the grant.
Appendix

A $\hat{N}_{R,g,Q}$ for the unknot with arbitrary framing $p$

For an unknot with arbitrary framing $p$, the $\hat{N}_{R,g,Q}$ for fundamental representation is

$$\hat{N}_{R,0,Q=\pm1/2} = \mp(-1)^p, \hat{N}_{R,g\neq0,Q} = 0.$$  \hspace{1cm} (A.1)

For representations involving two boxes in the Young-Tableau, $\hat{N}$’s for arbitrary $g$ can be written as follows

$$\hat{N}_{\square,p-s,-1} = \left\{ \frac{\theta(s-3)}{(s-3)!} (2p-2s+2)(2p-2s+3)\ldots(2p-s-2) \right.$$  
$$\left. + \frac{\theta(s-5)}{(s-5)!} (2p-2s+2)(2p-2s+3)\ldots(2p-s-4) + \ldots \right\}$$  
$$+ x\theta(s-2), \hspace{1cm} (A.2)$$

$$\hat{N}_{\square,p-s,0} = -\delta_{s,2} - \frac{\theta(s-2)}{(s-2)!} (2p-2s+3)(2p-2s+4)\ldots(2p-s)$$  \hspace{1cm} (A.3)

$$\hat{N}_{\square,p-s,1} = \left\{ \frac{\theta(s-2)}{(s-2)!} (2p-2s+2)(2p-2s+3)\ldots(2p-s-1) \right.$$  
$$\left. + \frac{\theta(s-4)}{(s-4)!} (2p-2s+2)(2p-2s+3)\ldots(2p-s-3) + \ldots \right\}$$  
$$+ y\theta(s-1), \hspace{1cm} (A.4)$$

$$\hat{N}_{\square,p-s,-1} = \left\{ \frac{\theta(s-2)}{(s-2)!} (2p-2s+2)(2p-2s+3)\ldots(2p-s-1) \right.$$  
$$\left. + \frac{\theta(s-4)}{(s-4)!} (2p-2s+2)(2p-2s+3)\ldots(2p-s-3) + \ldots \right\}$$  
$$+ y\theta(s-1), \hspace{1cm} (A.5)$$

$$\hat{N}_{\square,p-s,0} = -\delta_{s,1} - \frac{\theta(s-1)}{(s-1)!} (2p-2s+3)(2p-2s+4)\ldots(2p-s+1)$$  \hspace{1cm} (A.6)

$$\hat{N}_{\square,p-s,1} = \left\{ \frac{\theta(s-1)}{(s-1)!} (2p-2s+2)(2p-2s+3)\ldots(2p-s) \right.$$  
$$\left. + \frac{\theta(s-3)}{(s-3)!} (2p-2s+2)(2p-2s+3)\ldots(2p-s-2) + \ldots \right\}$$  
$$+ x\theta(s), \hspace{1cm} (A.7)$$

with $x = 1 \& y = 0$ for odd $s$, $x = 0 \& y = 1$ for even $s$ and $\theta(u)$ is the usual theta function which is equal to one if $u > 0$ and zero if $u \leq 0$. The negative framing $(-p)$
BPS integers are related to the positive framing integers \( \hat{N} \)

\[
\hat{N}^{(p)}_{(R_1, \ldots, R_r), g, Q} = (-1)^{\alpha-1} \hat{N}^{(-p)}_{(R_t^1, \ldots, R_t^r), g, -Q},
\]

(A.8)

where \( R_t^i \)'s are obtained by transposing the rows and columns in the Young-Tableau representations. These integers are consistent with Marino-Vafa results for \( g = 0, 1, 2 \).

References

[1] R. Gopakumar, C. Vafa, “M-Theory and Topological Strings, I,” [hep-th/9809187]

[2] R. Gopakumar, C. Vafa, “On the Gauge Theory/ Geometry Correspondence,” [hep-th/9811131]

[3] R. Gopakumar, C. Vafa, “M-Theory and Topological Strings, II,” [hep-th/9812127]

[4] E. Witten, “Chern-Simons Gauge Theory as a String Theory,” [hep-th/9207094]

[5] H. Ooguri, C. Vafa, “Knot Invariants and Topological Strings,” Nucl. Phys. B577, 419, (2000), [hep-th/9912123]

[6] J. M. F. Labastida, M. Marino, “Polynomial Invariants for Torus Knots and Topological Strings,” [hep-th/0004196]

[7] P. Ramadevi, T. Sarkar, “On Link Invariants and Topological String Amplitudes,” Nucl. Phys. B 600 (2001) 487.

[8] J. M. F Labastida, M. Marino, C. Vafa, “Knots, Links and Branes at Large N,” JHEP11 (2000) 007, [hep-th/0010102]

[9] M. Marino, C. Vafa, “Framed Knots at Large N,” [hep-th/0108064]

[10] M. Aganagic, C. Vafa, “Mirror symmetry, D-branes and counting holomorphic discs,” [hep-th/0012041]

[11] M. Aganagic, A. Klemm, C. Vafa, “Disk instantons, mirror symmetry and the duality web,” Z. Naturforsch. A 57 (2002), [hep-th/0105045].
[12] S. Govindarajan, T. Jayaraman and T. Sarkar, “Disc instantons in linear sigma models,” Nucl. Phys. B646, (2002) 498, hep-th/0108234.

[13] W. Lerche, P. Mayr and N. Warner, ‘N = 1 special geometry, mixed Hodge variations and toric geometry,” hep-th/0208039.

[14] G. ’t Hooft, “A Planar Diagram Theory for Strong Interactions,” Nucl. Phys. B 72 (1974) 461.

[15] W. B. R Lickorish, “3-manifolds and the Temperley Lieb Algebra,” Math. Ann 290 (1991), 657; “Three-manifold invariants from combinatorics of Jones polynomial,” Pac. J. Math, 149 (1991) 337.

[16] R.K. Kaul, P. Ramadevi, “ Three-Manifold Invariants from Chern-Simons Field Theory with Arbitrary Semi-Simple Gauge Groups,” Commun.Math.Phys. 217 (2001) 295, hep-th/0005096.

[17] M. Aganagic, A. Klemm, M. Marino and C. Vafa, “Matrix model as a mirror of Chern-Simons theory,” hep-th/0211098.

[18] P.Ramadevi, S. Naik, “Computation of Lickorish’s Three Manifold Invariants using Chern-Simons Theory,” Commun. Math. Phys. 209 (2000) 29, hep-th/9901061.

[19] J.M.F. Labastida, M. Marino, “ A New Point of View in the Theory of Knot and Link Invariants,” math.QA/0104180. J. Knot Theory Ramifications 11 (2002) 173.