Entanglement entropy of random fractional quantum Hall systems

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Abstract. The entanglement entropy of the $\nu = 1/3$ and $\nu = 5/2$ quantum Hall states in the presence of short-range random disorder has been calculated using direct diagonalization. A microscopic model of electron–electron interaction is used and spin-polarized electrons are confined to a single Landau level and interact with long-range Coulomb interaction. In the case of very weak disorder, the values of topological entanglement entropy are roughly consistent with the expected theoretical results. By considering a broad range of disorder strengths, entanglement entropy was studied in an effort to detect quantum phase transitions. In particular, there is a signature of the transition as the function of the disorder strength for the $\nu = 5/2$ state. Prospects of using the density matrix renormalization group to compute the entanglement entropy for larger system sizes are discussed.
1. Introduction

This paper is a numerical study, using direct diagonalization, of the entanglement entropy of fractional quantum Hall systems in the presence of a delta correlated random potential. Entanglement entropy, quite different from thermodynamic entropy, is the Von Neumann entropy of the reduced density matrix of a subsystem and is a quantitative measure of the entanglement of the subsystem with the system. Our interest in this subject is twofold; firstly, it has been proposed that entanglement entropy can be used as a tool for characterizing fractional quantum Hall states. More precisely, Kitaev and Preskill [1] and Levin and Wen [2] have shown, for a topologically ordered state, that the entanglement entropy of a subsystem obeys an asymptotic relation

$$S \simeq a L - \gamma + O \left( \frac{1}{L} \right) + \cdots,$$

(1)

where $L$ is the linear size of the subsystem (the area law) and $\gamma$ is a universal quantity, the topological entanglement entropy, which is the natural logarithm of the quantum dimension. For this scaling law to apply, the system must be very large and the subsystem must be large (compared with a cutoff, but the subsystem must be small compared with the system). This is a rather formidable numerical requirement; however, there has been some success numerically [3–8] using (1) to extract the topological entanglement entropy of quantum Hall states. One may hope that by adding weak randomness, there will be less system size dependence and so it will be easier to obtain topological entanglement entropy. Of course, by adding randomness, momentum conservation is destroyed and one cannot treat as large a system by direct diagonalization. In any case, it is of interest to find out whether topological entanglement entropy can be calculated in the presence of weak disorder and whether the values obtained are consistent with previous numerical estimates.

The second motivation to undertake this study is to find out whether entanglement entropy can be used to detect transitions between the phases of quantum Hall systems. For example, experimentally, it is well known that fractional quantum Hall states are particularly sensitive to disorder. Can this sensitivity be detected in entanglement entropy? The two questions discussed above will be studied for two filling factors $\nu = 1/3$ in the lowest Landau level, representative of Laughlin states and the 5/2th state in the second Landau level. Currently, there is strong evidence, both experimentally and numerically [9], that the essential physics of the 5/2 state
is given by the Moore–Read wave function and thus the 5/2 state is representative of the more exotic states with non-abelian statistics.

The paper is organized as follows. In section 2, the model and the numerical method are briefly described and the results on topological entanglement entropy in the case of weak disorder are discussed. In section 3, entanglement entropy is calculated as a function of disorder strength for a wide range of disorders to determine whether transitions between the phases of Hall systems can be detected. In section 4, some preliminary results using the density matrix renormalization group (DMRG) to calculate entanglement entropy are described. Section 5 presents a summary and the conclusions. In the final section, a recent alternative method [23] for obtaining the topological entanglement entropy on the torus is discussed.

2. Extracting the topological entanglement entropy for weak disorder

The numerical method we have used is direct diagonalization applied to square (aspect ratio 1) clusters with periodic boundary conditions (the square torus geometry). The Landau gauge is used for the vector potential. Spin-polarized electrons are confined to a single Landau level and interact with a pure Coulomb potential. One can approach the limit of very large system sizes through clusters of a fixed aspect ratio and since we are concerned with quantum liquid states, an aspect ratio of one has been chosen. This numerical approach has previously been used to study the entanglement entropy without a disorder potential [5, 7]. The random potential [10] \( U(r) \) is taken to be delta correlated, i.e. \( \langle U(r)U(r') \rangle = U_0 \delta(r - r') \) and the disorder strength will be given in terms of a parameter \( U_R = \sqrt{3}U_0/2 \). Since momentum is not conserved, one is limited to smaller system sizes than for a disorder-free system. In particular, the largest system size treated for \( \nu = 1/3 \) is 10 electrons in 30 orbitals with a state space of approximately \( 30 \times 10^6 \) and 14 electrons in 28 orbitals for \( \nu = 5/2 \) with a state space of approximately \( 40 \times 10^6 \). (This is in contrast to the disorder-free case, \( \nu = 1/3 \), 13 electrons in 39 orbitals, and \( \nu = 5/2 \), 18 electrons in 36 orbitals, are relatively straightforward to treat.)

To calculate the entanglement entropy, we take a subsystem consisting of \( l \) adjacent orbitals (recall that in the Landau gauge, these orbitals consist of strips oriented along, say the y-axis, of width of the order of the magnetic length). The reduced density matrix is straightforward to compute from the ground state wave function. It is then diagonalized, giving the eigenvalues \( \lambda_j \) from which the orbital entanglement entropy \( S(l) \), \( S(l) = -\sum_j \lambda_j \ln \lambda_j \), is obtained. This procedure is performed for every realization of the random potential; the results are then averaged to give \( \langle S(l) \rangle \), where \( \langle \rangle \) denotes average over the random potential. The position of the subsystem has been fixed; that is, for say \( S(l = 3) \), the subsystem always consists of the first, second and third orbitals. For the smallest systems (six electrons in 18 orbitals), we have averaged over 1000 realizations of the random potential; for the largest systems, we have averaged over as few as ten realizations. This choice was dictated by the time-consuming nature of the larger calculations.

In figure 1, we have plotted the entanglement entropy versus square root of \( l \) for ten electrons in 30 orbitals. The green circles are for no disorder (an average is taken over the three ground states with \( k_y = 5, 15 \) and 25), whereas the blue and red circles are for disorder strength \( U_R = 0.05 \) averaged over 10 and 100 samples, respectively. The error bars are given by the root mean square (rms) values of \( S(l) \), i.e. \( \sigma = \sqrt{\langle S(l)^2 \rangle - \langle S(l) \rangle^2} / \sqrt{N_s - 1} \) with \( N_s \) being the number of samples. From relation (1), we expect linear behavior versus \( \sqrt{l} \) for subsystems small compared with the system size; this behavior is seen in figure 1. In particular, the linear
Figure 1. Entanglement entropy versus $\sqrt{l}$ for ten electrons in 30 orbitals. The green circles are for no disorder, whereas the blue and red circles are for $U_R = 0.05$ averaged over 10 and 100 samples, respectively. The error bars are given by the rms values of $S(l)$.

regime is larger for the disordered case, indicating a smaller finite size effect for a given system size.

This suggests a linear fit to the initial part of the $S(l)$ versus $\sqrt{l}$ curve in order to obtain the topological entanglement entropy as the negative of the $y$-intercept. The results of the fit are plotted in figure 2 for $U_R = 0.05$. (The number of fitted values of $l$ was chosen to give a local maximum in the value of $R^2$.) The topological entanglement entropy $\gamma$ is found to be $1.10 \pm 0.070$, whereas a similar fit for $U_R = 0.01$ gives $\gamma = 1.13 \pm 0.078$, both values being in excellent agreement with the value for the Laughlin 1/3 state of $2(\ln \sqrt{3}) \approx 1.10$ [26], the factor of 2 coming from the two boundaries of the subsystem. However, as will be discussed below, the excellent agreement may be fortuitous in that, for the small system sizes considered, $\gamma$ tends to be overestimated at this filling.

The dependence of the topological entanglement entropy on system size for filling 1/3 is shown in figure 3 for $U_R = 0.01$. In this figure, $\gamma$ is plotted versus $1/N$ ($N$ is the number of orbitals). Clearly, it would be desirable, even with disorder, to be able to treat larger system sizes. Another approach for obtaining the topological entanglement entropy is, for a given $\langle S(l) \rangle$, to perform a linear extrapolation in $1/N$ yielding $\langle S^*(l) \rangle$. The $\langle S^*(l) \rangle$ is then plotted versus $\sqrt{l}$, a linear least-squares fit is performed and the $y$-intercept gives $-\gamma$. A plot of $\langle S^*(l) \rangle$ versus $\sqrt{l}$
**Figure 2.** Linear fit to the initial part of the $\langle S(l) \rangle$ versus $\sqrt{l}$ curve. $U_R = 0.05$ and the filling $\nu = 1/3$.

**Figure 3.** Dependence of the topological entanglement entropy $\gamma$ on the system size for filling $1/3$. $U_R = 0.01$ and $\gamma$ is plotted versus $1/N$. ($N$ is the number of orbitals.)
is shown in figure 4 for \( \nu = 1/3 \) using systems with 21–30 orbitals to get the extrapolations. The negative of the \( y \)-intercept is given by 1.30 with an error of 0.24. (The 0.24 is due to the deviation of the fit from a line. The error in the extrapolations to obtain \( \langle S^*(l) \rangle \) was not taken into account.) This method also gives a topological entanglement entropy consistent with the Laughlin 1/3 state.

Turning now to the \( \nu = 5/2 \) filling at \( U_R = 0.01 \), figure 5 shows \( \gamma \) calculated from the initial linear part of \( \langle S(l) \rangle \) (i.e. figure 2) versus \( 1/N \). For the largest system size, namely 14 electrons in 28 orbitals, \( \gamma \approx 1.5 \), considerably less than the \( \gamma \) expected from the Moore–Read state (\( \gamma_{\text{MR}} \approx 2.08 \) for two boundaries [26]). If the \( 1/N \) dependence is fitted by a line, one finds, at \( N = \infty \), a \( \gamma \) value of 2.34 with an error of 0.08. However, without knowing the answer, one does not know to extrapolate in figure 5 but not to extrapolate in figure 3, where the largest system sizes give acceptable answers without extrapolation. In figure 6 \( \langle S^*(l) \rangle \) versus \( \sqrt{\tilde{t}} \) is plotted for \( \nu = 5/2 \), where \( \langle S^*(l) \rangle \) was obtained through extrapolation of system sizes 24, 26 and 28. The \( y \)-intercept of the linear fit gives a \( \gamma \) value of 2.58. Of course, this is no great success; however, one obtains better agreement with the expected value if the ratio \( \frac{\gamma_{5/2}}{\gamma_{1/3}} \) is considered. Using the \( \langle S^*(l) \rangle \) method (\( \langle S^*(l) \rangle \) obtained from the three largest system sizes), \( \frac{\gamma_{5/2}}{\gamma_{1/3}} \approx 1.97 \) as compared with \( \frac{\gamma_{\text{MR}}}{\gamma_{\text{Laughlin}}} \approx 1.89 \).

In any case, it appears that the expected more generic behavior with weak randomness is unable to overcome the advantage of additional system sizes available for disorder-free calculations. That is, by using the \( S^*(l) \) method and two more system sizes (without disorder), the paper [5] was able to obtain agreement with the expected theoretical results within the error.
bars when $\nu = 1/3$ and $\nu = 5/2$. Without disorder, fitting a line to the initial part of the curve $S(l)$ versus $\sqrt{l}$ is problematic when $\nu = 1/3$, since, in addition to a linear increase, there is also a superimposed oscillation (see the finite size scaling (FSS) result in figure 1(a) of [5]). We suggest that a similar oscillation, although less pronounced in case disorder is present, causes difficulty in the extrapolation of the data presented in figure 3. To get the points in figure 3, in the case of the smallest system size, namely six electrons in 18 states, the first four $l$-values were used to obtain the best linear fit (i.e. minimize $R^2$), whereas in the case of ten electrons in 30 states, six initial $l$ values were used. From the extrapolated values in figure 4, oscillation causes an overestimate of $\gamma$ in both cases. On the other hand, for $\nu = 5/2$ there is less oscillation in $S^*(l)$; see figure 3(a) of [5] and figure 6 of this paper. To obtain the points in figure 5, for ten electrons, seven $l$-values were used, whereas for 14 electrons, eight $l$-values were used in the fit. Due to less oscillation and a greater number of $l$-values used, it is perhaps not surprising that extrapolation of $\gamma$, obtained from the initial part of $S(l)$, is more successful for $\nu = 5/2$ than an extrapolation at $\nu = 1/3$.

3. Entanglement entropy as a function of disorder strength

In this section, the entanglement entropy is studied for a wide range of disorder strengths. Entanglement entropy has previously been used to investigate the phase diagram of quantum
Hall systems as a function of interaction potential [8], one-dimensional (1D) quantum spin systems with disorder [11] and one-particle entanglement entropy for Anderson transitions [12] and to study the phase diagram of the 1D extended Hubbard model [13]. In figures 7(a) and (b), \( l \)-entanglement entropy \( \langle S(l) \rangle \) versus \( U_R \) is plotted for filling \( 1/3 \). In figure 7(a), \( \langle S(2) \rangle \) is shown, this figure being representative of small subsystems \( l \). In figure 7(b), \( \langle S(12) \rangle \) is graphed, this figure being characteristic of larger subsystems. Both graphs show that the entropy strongly decreases with disorder, with \( S(12) \) exhibiting a slightly sharper decrease. In particular in the \( S(12) \) graph, the entropy appears to decrease and then level out at a disorder strength of approximately \( U_R = 0.25 \). Although it is hard to draw a definite conclusion, this is at least consistent with Wan et al [22] who found that the mobility gap vanishes for \( U_R > 0.25 \).

Let us now turn to filling \( 5/2 \). The same sequence of graphs is presented in figures 8(a) and (b). Here there appears to be, especially for the large \( l \) graph (figures 8(b)), a transition at a disorder strength \( U_R \approx 0.04 \). A natural interpretation of these graphs is a quantum phase transition from the Moore–Read state for disorder strength \( U_R \approx 0.04 \). Previous numerical studies [5, 14, 27, 28] show that the ground state for pure Coulomb potential (no disorder) is topologically equivalent to the Moore–Read state. We therefore suggest that the sharp drop-off in figure 8(a) and particularly in figure 8(b), in contrast with the smoother curves in figures 7(a) and (b), is a transition due to destruction of the Moore–Read state by disorder. A possible picture of this transition is the destruction of p-wave superconductivity of composite fermions [30] by disorder. That such a transition should happen at rather weak disorder is physically appealing [31].
Figure 7. $\langle S(l) \rangle$ versus $U_R$ for filling 1/3. In (a), $\langle S(2) \rangle$ is plotted, whereas (b) is a plot of $\langle S(12) \rangle$.

In an effort to characterize possible phase transitions with disorder, we have calculated the variance $\langle S(l)^2 \rangle - \langle S(l) \rangle^2$. In figures 9 and 10, the variance for $l = 12$ is plotted in the case of $\nu = 5/2$ and $\nu = 1/3$, respectively. For $\nu = 5/2$ (figure 9), the variance is nominal through the transition region (other than for the anomalous behavior of ten electrons in 20 orbitals). In contrast, for $\nu = 1/3$ (figure 10), there is a general increase of the variance starting at $U_R = 0.05$, reaching a plateau at $U_R \approx 0.2-0.25$, which may indicate a transition in this range, consistent with figure 7 and consistent with [22].

4. Preliminary density matrix renormalization group (DMRG) studies of entanglement entropy

A common theme of the previous sections was the benefits of finding a method of accessing larger system sizes. A possible method for quantum Hall systems is the use of DMRG [15]. One expects that the number of states kept in DMRG blocks needs to scale as the exponential of the entanglement entropy of the block for an accurate calculation. Since by the area law, entropy scales as $\sqrt{s}$, where $s$ is the number of sites in the block, the number of states kept needs to scale as $e^{\sqrt{s}}$. The bad news is that this depends on the exponential of $\sqrt{s}$; however, the good news is that it does not depend on the exponential of $s$ as in direct diagonalization. Hence, at least in principle, one should (if one can avoid being stuck in local minimum) be able to treat larger system sizes for quantum Hall systems by DMRG [16–19]. In particular, the authors of [18]
were able to accurately calculate ground state energies at $\nu = 1/3$ for up to 20 electrons and at $\nu = 5/2$ for up to 26 electrons in spherical geometry. In spherical geometry, 14 electrons at $\nu = 1/3$ and 20 electrons at $\nu = 5/2$ are accessible to direct diagonalization. However, the excitation gap, a more difficult numerical quantity at $\nu = 5/2$, was only accurately calculable by DMRG for up to 22 electrons, 1 ‘non-aliased’ system size larger than that accessible to direct diagonalization. In this section, DMRG will be used to calculate the entanglement entropy for quantum Hall systems without disorder. We will be content, in this preliminary study, to use DMRG to study a large system size still accessible to direct diagonalization, that is, 12 electrons in 36 orbitals in the $n = 0$ and $n = 1$ Landau levels.

In table 1, we display the ground state energy versus $m$, with $m$ denoting the number of states kept in the block; the first column is for the lowest Landau level, and the second for the second Landau level (the Madelung energy, which can be calculated exactly, is not included).

One sees that for the lowest Landau level, a fairly accurate result can be obtained even without extrapolation; for the $n = 1$ Landau level, extrapolation is more important (we extrapolate in $1/m$). Let us now consider the calculation of $S(l)$ (recall that $l$ is the number of sites in the subsystem used when calculating the entanglement entropy). All $S(l)$s are computed at the end of the calculation when the left and right blocks have an equal number of sites (in addition, there are two sites in the middle [17]). This makes the calculations more complicated (i.e. clearly it is easier to get $S(l)$ when there are $l$ sites in the block) but it is necessary to get reliable results. Figure 11 is a plot of $S(l)$ versus $\sqrt{l}$ up to $l = 8$ for different numbers of states in

**Figure 8.** $\langle S(l) \rangle$ versus $U_r$ for filling $5/2$. In (a), $\langle S(2) \rangle$ is plotted, whereas (b) is a plot of $\langle S(12) \rangle$. 
the blocks for 1/3 filling. One notes that even for the smallest block sizes, DMRG does a good job in computing $S(l)$. This is consistent with DMRG calculations of entanglement entropy performed by Shibata [20]. Turning now to 1/3 filling in the second Landau level ($\nu = 7/3$), figure 12 is a plot of $S(l)$ versus $\sqrt{l}$ for this filling. One again sees that different from the first Landau level, extrapolation is very important for obtaining an accurate result. The larger $l$ values are underestimated (i.e. entanglement is underestimated), particularly for calculations with a smaller number of states in the blocks. Of course, the energy is also less accurately calculated in the second Landau level by DMRG. This is not the whole story, since the 800-state calculation in the second Landau level does better for the energy (on a relative basis) than the 200-state calculation in the first Landau level. However, the 200-state calculation still does better in calculating $S(l)$.

Even though it seems possible to use more states in the blocks (the paper [18] uses up to 5000), it appears difficult to go beyond direct diagonalization in calculating the entanglement entropy in the second Landau level. A simple estimate shows, based on the above calculations, why this is the case. The computation for $\nu = 7/3$ indicates that at least 1000 block states (and this may be an underestimate) are necessary to obtain a fairly accurate result. In going from 36 to 48 sites (12–16 electrons), the ‘worst’ block goes from 18 to 24 sites ($1/2$ the system size, since the entanglement entropy of the system and the environment is equal). Assuming that the number of states kept needs to scale as $e^c\sqrt{s}$, the number of states needed for 48 sites is at least $1000\sqrt{24/18} \approx 3000$ states.

**Figure 9.** The variance of $S(12)$, $\langle S(12)^2 \rangle - \langle S(12) \rangle^2$ versus $U_R$ for filling 5/2.
The variance of $S(12)$, $\langle S(12)^2 \rangle - \langle S(12) \rangle^2$ versus $U_R$ for filling $1/3$.

Table 1. Comparison of DMRG and direct diagonalization energies.

| $m$  | $N = 0$     | $N = 1$     |
|------|-------------|-------------|
| 200  | $-3.3675$   | $-2.4109$   |
| 300  | $-3.3691$   | $-2.4178$   |
| 400  | $-3.3699$   | $-2.4203$   |
| 600  | $-3.3717$   | $-2.4233$   |
| 700  | $-3.3723$   |             |
| 800  |             | $-2.4219$   |
| Extrapolation | $-3.3739 \pm 0.005$ | $-2.4252 \pm 0.003$ |
| Exact     | $-3.3734$   | $-2.4254$   |

5. Conclusion

The entanglement entropy of the $\nu = 1/3$ and $\nu = 5/2$ quantum Hall states in the presence of short-range disorder has been calculated by direct diagonalization. For very weak disorder, the value of topological entanglement entropy (a universal quantity) is roughly consistent with the expected theoretical results and disorder-free calculations. However, (in particular for $\nu = 5/2$) the advantages of having less system size dependence with weak disorder are outweighed by
the disadvantage of inaccessibility of larger system sizes. To investigate the possibility of using entanglement entropy to detect quantum phase transitions, the entanglement entropy has been calculated for a broad range of disorder strengths. For $\nu = 1/3$, the $l$-orbital entanglement entropy (figures 7(a) and (b)) shows a strong decrease, and the variance (figure 10) shows a strong increase through the range $U_R \approx 0.1–0.25$. For the range of disorders considered and the amount of averaging done, we suggest that this is a possible signature of a phase transition similar to that observed for the mobility gap in [22] at $U_R \approx 0.25$. For $\nu = 5/2$, we see a much sharper transition feature in the $l$-orbital entanglement entropy (figures 8(a) and (b)) and at a much smaller value of the disorder strength, $U_R \approx 0.04$. Despite the sharper transition, there is no corresponding feature in the variance (figure 9), as there is in the $\nu = 1/3$ case. The sensitivity of the $5/2$ state to disorder is well known from experimental studies where samples must have a high (zero field) mobility to see an incompressible state. Thus, there is qualitative agreement with experiment, taken with due caution in that a quantitative comparison likely requires considering longer-range disorder. In the present study, entanglement entropy has been used to characterize the reduced density matrix. There is possibly additional information on the full spectrum of the reduced density matrix [14], which has been shown to be related to the

**Figure 11.** $S(l)$ versus $\sqrt{l}$ for $\nu = 1/3$, calculated by DMRG. The different symbols correspond to different numbers of states in the DMRG blocks. The green squares represent the exact results.
conformal field theory describing the 1D edge state of the quantum Hall state [8, 14, 24]. It would definitely be of interest [11] to study the entanglement spectrum in the present system. Even though the topological entanglement entropy (derived from the entanglement entropy) is a complete invariant [25], numerically it may be much easier to see transitions using the entire spectrum [8, 14]. Finally, we have displayed some preliminary results obtained using DMRG for computing entanglement entropy. These results indicate that DMRG holds some promise in calculating the entanglement entropy in the lowest Landau level; it appears very difficult to perform calculations in the second Landau level and to go beyond systems that one can treat by direct diagonalization. This may indicate that more powerful numerical methods, for example tensor network states [21] or the methods of [29], will prove useful.

Although the present study concerns computational signatures of topological states using entanglement entropy, one may speculate on experimental measures of entropy that might reveal topological states. Work in this direction on the relationship between noise measurements and entropy [32] is potentially most realistic, although how topological entanglement entropy might be extracted from such measurements is an open question.

Figure 12. $S(l)$ versus $\sqrt{l}$ for $\nu = 7/3$, calculated by DMRG. The different symbols correspond to different numbers of states in the DMRG blocks. The green squares represent the exact results, whereas the green crosses are the values extrapolated from DMRG.
Figure 13. \( S(17) \) versus \( \sqrt{\frac{N}{\alpha}} \) for \( \nu = 1/3 \), 11 electrons in 33 orbitals. The line is a fit to \( S(17) \) for \( \sqrt{\frac{N}{\alpha}} > \sqrt{2\pi} \).

6. Final remarks

After an earlier version of this paper was posted at arXiv.org, we became aware of an interesting paper that calculates topological entanglement entropy using a different method in the flat torus geometry. (We thank Dr Haque for bringing this reference to our attention.) In essence the paper [23] calculates the entanglement entropy \( S(N/2) \) taking the subsystem to be half the system size. The scaling law \( S(N/2) \sim c_1\sqrt{N/\alpha} - 2\gamma \) is then used, where \( \alpha \) is the aspect ratio and \( N \) the number of orbitals in the system; this approach was also used by Shibata [20]. In the method described in section 2 (see also [5, 7]), the scaling law \( S(l) \sim c_2\sqrt{l} - 2\gamma \) is used, where \( l \) (the number of orbitals in the subsystem) is much smaller than \( N \). In this method, the subsystem for fixed \( l \) becomes increasingly thin since the number of states per unit length (the magnetic length) scales as \( \sqrt{N} \).

Let us examine this point [23] in greater detail. Suppose that there is a subsystem consisting of a fixed number of orbitals \( l \), and \( N \) becomes very large. Consider the square torus geometry, a ‘box’ of dimensions \( a \times a \); here \( a = \sqrt{2\pi N} \). The width of a box of \( l \) orbitals is \( \frac{l}{N} \sqrt{2\pi N} \), i.e. \( l/\sqrt{N} \), so the width goes to zero as \( \frac{1}{\sqrt{N}} \). However, as the width goes to zero, the length goes...
Figure 14. $S(17)$ versus $\sqrt{\frac{N}{\alpha}}$ for $\nu = 7/3$, 11 electrons in 33 orbitals.

as $\sqrt{2\pi N}$. Although the width and length are both ‘singular’ as $N$ goes to infinity, the area is perfectly well defined, $2\pi l$ (again in units of the magnetic length squared). Since the area law relates the entanglement entropy to a linear dimension of the subsystem, it is reasonable that $S(l)$ scales as the square root of the area, $S(l) \sim c \sqrt{l}$, and this is verified by explicit calculations.

It should be emphasized that neither approach is fully justified by the considerations in [1, 2]. At least for the current state of knowledge, the best justification for either method is that they give reasonable results where the physics is well understood, as Laughlin states. This is true for both techniques; hence, in principle, either technique can be used to calculate the topological entanglement entropy. That being said, since system sizes are limited, one technique may well be superior depending on the filling fraction in question. In particular, the method of [20, 23] allows one to extract $\gamma$ more accurately for $\nu = 1/3$ from finite size calculations. As an illustration of this method, in figure 13, $S(17)$ is plotted versus $\sqrt{\frac{N}{\alpha}}$ for 11 electrons in 33 states in the lowest Landau level (no disorder). For this system size the orbitals begin to strongly overlap at $\sqrt{\frac{N}{\alpha}} = \sqrt{2\pi} \approx 2.51$. (At this value of $\sqrt{\frac{N}{\alpha}}$ the orbitals are one magnetic length apart.) If one uses this value as a cutoff and fits the $S(17)$ curve to a line for $\sqrt{\frac{N}{\alpha}} > \sqrt{2\pi}$ a topological entanglement entropy $\gamma$ of $1.14 \pm 0.02$ is obtained. For a comparison, $S(17)$ versus $\sqrt{\frac{N}{\alpha}}$ is plotted in figure 14 for the same filling on the second Landau level. This plot is calculated from

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the lowest energy states in the momentum sectors $k_y = 11, 22$ and $33$, which are the ground states at the aspect ratio of one. There is evidence of several transitions as the thin torus is transformed into a square. It is interesting that the transitions in this figure as a function of $\sqrt{N/\alpha}$ bear some resemblance to the ‘plateau’ transition as a function of disorder strength as seen in figure 8.

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