IWP Solutions for Heterotic String in Five Dimensions

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Abstract

We obtain extremal stationary solutions that generalize the Israel–Wilson–Perjés class for the low-energy limit of heterotic string theory with \( n \geq 3 \) \( U(1) \) gauge fields toroidally compactified from five to three dimensions. A dyonic solution is obtained using the matrix Ernst potential (MEP) formulation and expressed in terms of a single real \( 3 \times 3 \)-matrix harmonic function. By studying the asymptotic behaviour of the field configurations we define the physical charges of the field system. The extremality condition makes the charges to saturate the Bogomol’nyi–Prasad–Sommerfield (BPS) bound.
1 Introduction

In effective low energy theories of gravity derived from superstring theory Einstein gravity is supplemented by additional fields such as the Kalb–Ramond, gauge fields, and the scalar dilaton which couples in a non–trivial way to other fields. These string gravity models preserve the long–distance behaviour of the mysterious quantum gravity and in special (BPS saturated) cases exactly reproduce it \[1\]. The bosonic sector of heterotic string theory compactified to three dimensions on a torus can be parametrized by the \((d+1) \times (d+1)\) and \((d+1) \times n\) Matrix Ernst Potentials \(X\) and \(A\) \[2\]–\[3\], where \(d+3\) is the original space–time dimension and \(n\) is the number of Abelian vector fields.

In this letter we consider the five–dimensional case and suppose a linear dependence between the MEP \(X\) and \(A\) following the procedure indicated in our previous work \[3\]. If \(n \geq 3\) this leads to non–trivial field configurations which generalize the the IWP class of solutions of the Einstein–Maxwell (EM) theory. Note that in our approach the number of gauge fields is bounded from below. A similar ansatz arose in the four dimensional case considered in \[3\] and \[4\] when the minimal number of gauge fields was equal to two (in the general case \(n \geq d+1\) \[3\]).

Furthermore, it is shown that the physical charges of the obtained solutions saturate the BPS bound as a consequence of the extremality condition. Among them we identify rotating black hole–type solutions with both electric and magnetic charges (dyonic solutions).

Some classes of five–dimensional BPS solutions with trivial and non–trivial values of electromagnetic charges were obtained in \[5\]–\[7\].

2 Matrix Ernst Potentials

We start from the effective field theory of heterotic string in five dimensions. The action of this theory reads

\[
S^{(5)} = \int d^{(5)}x \left| G^{(5)} \right|^{\frac{1}{2}} e^{-\phi^{(5)}} \left( R^{(5)} + \phi^{(5)} \partial^{(5)} \phi^{(5)} - \frac{1}{12} H^{(5)} H^{(5)MNP} - \frac{1}{4} F^{(5)I} F^{(5)IMNP} \right), \]

where

\[
F^{(5)I}_{MN} = \partial_M A^{(5)I}_N - \partial_N A^{(5)I}_M, \]

\[
H^{(5)}_{MNP} = \partial_M B^{(5)}_{NP} - \frac{1}{2} A^{(5)I}_M F^{(5)I}_{NP} + \text{cycl. perms. of } M,N,P. \]

Here \(G^{(5)}_{MN}\) is the 5-dimensional metric, \(B^{(5)}_{MN}\) is the anti–symmetric Kalb-Ramond field, \(\phi^{(5)}\) is the dilaton and \(A^{(5)I}_M\) denotes a set \((I = 1, 2, ..., n)\) of \(U(1)\) gauge fields.

After the Kaluza-Klein compactification on a two–torus, one obtains the following set of three–dimensional fields \[3\]-\[8\]:

\[\]
a) scalar fields

\[ G = (G_{pq} \equiv G^{(5)}_{p+3,q+3}), \quad B = (B_{pq} \equiv B^{(5)}_{p+3,q+3}), \]
\[ A = (A^{(5)}_p \equiv A^{(5)}_{p+3}), \quad \phi = \phi^{(5)} - \frac{1}{2} \ln |\det G|, \]

where \( p, q = 1, 2 \).

b) tensor fields

\[ g_{\mu\nu} = e^{-2\phi} \left( G^{(5)}_{\mu\nu} - G^{(5)}_{p+3,\mu} G^{(5)}_{q+3,\nu} G^{pq} \right), \quad B_{\mu\nu} = B^{(5)}_{\mu\nu} - 4 B_{pq} A^p_{\mu} A^q_{\nu} - 2 \left( A^p_{\mu} A^{p+2}_{\nu} - A^p_{\nu} A^{p+2}_{\mu} \right), \]

(following A. Sen \[1\] we consider the ansatz when \( B_{\mu\nu} = 0 \)).

c) vector fields \( A^{(a)}_\mu = \left( (A_1)_\mu^{(a)}, (A_2)_\mu^{(p+2)}, (A_3)^{I+4}_\mu \right) \) \( (a = 1, 2, 3, 4, 4 + I) \)

\[ (A_1)_\mu^p = \frac{1}{2} G^{pq} G^{(5)}_{q+3,\mu} \quad (A_3)^{I+4}_\mu = -\frac{1}{2} A^{(5)}_\mu^I + A^I_{q} A^q_{\mu}, \]
\[ (A_2)^{p+2}_\mu = \frac{1}{2} B^{(5)}_{p+3,\mu} - B_{pq} A^q_{\mu} + \frac{1}{2} A^I_{p} A^{I+4}_{\mu}, \]

which can be dualized on-shell as follows

\[ \nabla \times A^1 = \frac{1}{2} e^{2\phi} G^{-1} \left( \nabla u + (B + \frac{1}{2} AA^T) \nabla v + A \nabla s \right), \]
\[ \nabla \times A^3 = \frac{1}{2} e^{2\phi} (\nabla s + A^T \nabla v) + A^T \nabla \times A^1, \]
\[ \nabla \times A^2 = \frac{1}{2} e^{2\phi} G \nabla v - (B + \frac{1}{2} AA^T) \nabla \times A^1 + A \nabla \times A^3. \]

(3)

Here \( u \) and \( v \) are columns of dimension 2 with components \( u_1, u_2 \) and \( v_1, v_2 \), respectively; and the dimension of the column \( s \) is \( n \). So, the final system is defined by the quantities \( G, B, A, \phi, u, v \) and \( s \). As it had been established in \[2,3\], it is possible to introduce the matrix Ernst potentials \[10\]

\[ \mathcal{X} = \begin{pmatrix} -e^{-2\phi} + u^T X v + v^T A s + \frac{1}{2} s^T s & v^T X - u^T \\ X v + u + A s & X \end{pmatrix}, \quad \mathcal{A} = \begin{pmatrix} s^T + v^T A \\ A \end{pmatrix}, \]

(4)

where the \( 2 \times 2 \) matrix potential \( X = G + B + \frac{1}{2} AA^T \). This pair of potentials allows us to express the 3-dimensional action (2) in a quasi–EM form \[11,12\] :

\[ S^{(3)} = \int d^3 x \ | g | \frac{1}{2} \left\{ -R + \text{Tr} \left[ \frac{1}{4} \left( \nabla \mathcal{X} - \nabla \mathcal{X}^T - \mathcal{A} \nabla \mathcal{A}^T \right) \right] \right. \]
\[ + \frac{1}{2} \nabla \mathcal{A}^T G^{-1} \nabla \mathcal{A} \}, \]

(5)

where \( G = \frac{1}{2} \left( \mathcal{X} + \mathcal{X}^T - \mathcal{A} \mathcal{A}^T \right) \). This action leads to the following equations of motion

\[ \nabla^2 \mathcal{X} - 2 (\nabla \mathcal{X} - \nabla \mathcal{A}^T) (\mathcal{X} + \mathcal{X}^T - \mathcal{A} \mathcal{A}^T)^{-1} \nabla \mathcal{X} = 0, \]
\[ \nabla^2 \mathcal{A} - 2 (\nabla \mathcal{X} - \nabla \mathcal{A}^T) (\mathcal{X} + \mathcal{X}^T - \mathcal{A} \mathcal{A}^T)^{-1} \nabla \mathcal{A} = 0. \]

(6)
3 BPS Saturated Dyon

In this Sec. we obtain a class of extremal solutions for the equations of motion (6) which generalize the IWP class of the EM theory following the procedure indicated in [3]. We consider a linear dependence between the potentials $A$ and $X$, and require the matrix Ernst potentials to satisfy the asymptotic flatness conditions $X_\infty \to \Sigma$ and $A_\infty \to 0$, where $\Sigma = \text{diag}(-1, -1, 1, 1, \ldots, 1)$. This leads to the following relation between the matrix Ernst potentials

$$A = (\Sigma - X)b,$$  

(7)

where $b$ is an arbitrary constant $3 \times n$–matrix. By substituting (7) into the action (5) and setting the Lagrangian of the system to zero (it implies that $R_{ij} = 0$), we get the following condition to be satisfied

$$bb^T = -\Sigma/2.$$  

(8)

Indeed, both equations of motion (6) reduce to the Laplace equation in Euclidean 3–space

$$\nabla^2[(\Sigma + X)^{-1}] = 0$$  

(9)

which can be directly solved by the harmonic function (see [3] for details)

$$\frac{2}{\Sigma + X} = \Sigma + \frac{M}{R}, \quad \text{where} \quad \frac{1}{R} = R e^{-\frac{1}{\sqrt{x^2 + y^2 + (z - i\alpha)^2}}},$$  

(10)

$M$ is a real 3–dimensional arbitrary constant matrix and $\alpha$ is a real constant. We choose $R$ in this way in order to deal with rotating black hole solutions [13]–[14] (in this case we have a ring singularity). In order to obtain a real value of the potential $A$ (see Eqs. (7) and (8)) we can proceed as follows: Let us require just the first two rows of $b$ to be real (leaving the remaining row imaginary), then we perform the matrix product (7) and set the factors that multiply the imaginary components of $b$ to zero. It turns out that this condition imposes the following restriction on the matrix $M$

$$M = \begin{pmatrix} m_{11} & m_{12} & 0 \\ m_{21} & m_{22} & 0 \\ m_{31} & m_{32} & 0 \end{pmatrix} = \begin{pmatrix} M_{11} & 0 \\ M_{21} & 0 \end{pmatrix}$$  

(11)

leading to real solutions for the potential $A$.

Now we begin to write down the explicit form of the single point–like solution in terms of the five dimensional variables. In order to so, we must calculate all vector three–fields using the dualization formulae (3). Thus, after some algebraic manipulations we obtain

$$\nabla \times \vec{A}^{(a)} = m^{(a)}\nabla \left( \frac{1}{R} \right),$$  

(12)
where \( m^{(1)} = -(m_{12} - m_{21})/2, \ m^{(2)} = m^{(4)} = m_{31}/2, \ m^{(3)} = -(m_{12} + m_{21})/2 \) and \( m^{(4+n)} = (m_{11}b_{n1} + m_{12}b_{n2}). \)

The relation between physical parameters and integration constants becomes evident when we switch from Cartesian to oblate spheroidal coordinates defined by

\[
x = \sqrt{\rho^2 + \alpha^2 \sin^2 \varphi}, \quad y = \sqrt{\rho^2 + \alpha^2 \sin \theta \sin \varphi}, \quad z = \rho \cos \theta,
\]

In terms of these coordinates the 3–interval reads

\[
ds^2_3 = (\rho^2 + \alpha^2 \cos^2 \theta)(\rho^2 + \alpha^2)^{-1} d\rho^2 + (\rho^2 + \alpha^2 \cos^2 \theta) d\theta^2 + (\rho^2 + \alpha^2) \sin^2 \theta d\varphi^2
\]

and only the \( A^{(a)}_\varphi \) does not vanish\footnote{In fact we have imposed the axial symmetry with respect to \( z \).}

\[
A^{(a)}_\varphi = m^{(a)} \cos \theta \frac{\rho^2 + \alpha^2}{\rho^2 + \alpha^2 \cos^2 \theta} = m^{(a)} \epsilon.
\]

Studying the asymptotic behaviour of the 3–fields we see that the integration constants and the physical parameters of the theory are related by

\[
G \sim \left( -\frac{1 + 2m_{22}}{m_{31}} \right) \frac{m_{32}}{\rho} \left( -\frac{1 - 2m_{11}}{N_B} \right) = \left( -\frac{1 - 2m_{11}}{N_B} \rho \right), \quad B \sim \frac{m_{32}}{\rho} \sigma = \frac{N_B}{\rho} \sigma_2, \\
A = \left( \frac{A_1^{(n)}}{A_5^{(n)}} \right) \sim \left( \frac{2(m_{21}b_{1n} + m_{22}b_{2n})}{\rho} / \rho \right) = \left( \frac{Q_e^{(n)}}{Q_5^{(n)}} \rho \right), \quad \phi \sim -\frac{m_{11}}{\rho} = D \rho,
\]

\[
u_1 \sim \frac{m_{12} - m_{21}}{\rho} = \frac{N}{\rho}, \quad v_1 \sim \frac{m_{12} + m_{21}}{\rho} = \frac{Q_B}{\rho},
\]

\[
u_2 = v_2 \sim \frac{m_{31}}{\rho} = \frac{N_5}{\rho}, \quad s \sim 2 \frac{m_{11}b_{n1} + m_{12}b_{n2}}{\rho} = \frac{Q_m^{(n)}}{\rho},
\]

where \( m \) is the ADM mass, \( D, N, Q_B \) are the dilaton, NUT and axion charges, respectively; \( N_B \) and \( N_5 \) are 5–dimensional scalar charges, \( Q_e^{(n)} \) and \( Q_m^{(n)} \) are two sets of \( n \) electric and magnetic charges, and \( Q_5^{(n)} \) are \( n \) charges that come from the extra dimension of the electromagnetic sector. The extremality character of the found solutions makes these charges to saturate the BPS bound

\[
4(D^2 + m^2) + 2(Q_B^2 + N_B^2) + \sum_n (Q_e^{(n)})^2 = \sum_n (Q_e^{(n)})^2 + \sum_n (Q_m^{(n)})^2 + 4(N_5^2 + N_B^2), \quad (17)
\]
this means that the attractive forces are precisely balanced by the repulsive forces in the field configuration.

Let us count the number of independent parameters which parametrize the physical charges of the solution. One of them is the rotational parameter $\alpha$. The contribution of matrix $M$ is equal to 6. Matrix $b$ provides $2n - 3$ independent parameters since only its first two rows affect the solution and these rows are normalized and orthogonal each other in view of Eq. (8). Thus we have $2n + 4$ integration constants which define the charges of the fields (in the case of arbitrary $d$, the total number of independent parameters is $2(d + n)$).

The explicit form of the solution is given by the following relations

$$ds^2 = G_{MN}dx^Mdx^N = G_{pq} \left(dx^{p+3} + \omega^{(p)}d \varphi \right) \left(dx^{q+3} + \omega^{(q)}d \varphi \right) + e^{2\phi}g_{\mu\nu}dx^\mu dx^\nu,$$

where the symmetric matrix $G_{pq}$ has the components

$$(P^2 + Q^2)^2 G_{11} = (Q^2 - P^2) \left((\rho^2 + 2D \rho + \delta_0 - \alpha^2 \cos^2 \theta)(\rho^2 - \alpha^2 \cos^2 \theta) - 4(\rho + D) \rho \alpha^2 \cos^2 \theta \right) - 4PQ \left((2\rho^2 + 3D \rho + \delta_0 - 2\alpha^2 \cos^2 \theta)\rho - D \alpha^2 \cos^2 \theta \right) \alpha \cos \theta,$$

$$(P^2 + Q^2)^2 G_{12} = N_B \left((P^2 - Q^2)(\rho^2 - 3\alpha^2 \cos^2 \theta)\rho + 2PQ(3\rho^2 - \alpha^2 \cos^2 \theta) \alpha \cos \theta \right) + (Q_B N_5 + 2D N_B) \left((P^2 - Q^2)(\rho^2 - \alpha^2 \cos^2 \theta) + 4PQ \rho \alpha \cos \theta \right) + (\delta_1 N_B - \delta_4 N_5) \left((P^2 - Q^2)\rho + 2PQ \alpha \cos \theta \right),$$

$$(P^2 + Q^2)^2 G_{22} = (P^2 + Q^2)^2 - \delta_3 \left((P^2 - Q^2)(\rho^2 - \alpha^2 \cos^2 \theta) + 4PQ \rho \alpha \cos \theta \right) - 2(Q_B N_5 N_B + D N_B^2 + m N_5^2) \left((P^2 - Q^2)\rho + 2PQ \alpha \cos \theta \right) - (\delta_1 N_B^2 + \delta_2 N_5^2 - 2\delta_4 N_5 N_B)(P^2 - Q^2),$$

the conformal multiplier is

$$e^{2\phi} = 1 + \frac{2D \rho}{\rho^2 + \alpha^2 \cos^2 \theta} + \frac{\delta_0(\rho^2 - \alpha^2 \cos^2 \theta)}{(\rho^2 + \alpha^2 \cos^2 \theta)^2}$$

(19)

and the components of the rotational vector are defined by $\omega^{(1)} = -N \epsilon$ and $\omega^{(2)} = N_5 \epsilon$. Here we have introduced the following quantities $P = \rho^2 + (m + D) \rho + \Delta_1 - \alpha^2 \cos^2 \theta,$
Q = (2ρ + (m + D))αcosθ, δ₀ = δ₁ - N₅², δ₁ = D² + \(\frac{1}{2}(Q_B - N)²\), δ₂ = m² + \(\frac{1}{4}(Q_B + N)²\), δ₃ = N₃² + N₀², δ₄ = \(\frac{1}{2}(m(N - Q_B) - D(N + Q_B))\) and \(\Delta₁ = mD + \frac{1}{4}(N² - Q_B²)\).

The only non–vanishing components of the five–dimensional matter fields are

\[ B = \frac{N_B(Pρ + Qαcosθ) - Δ₃P}{P² + Q²}, \]

\[ A_i^{(n)} = \frac{Q_e^{(n)}(Pρ + Qαcosθ) + (DQ_e^{(n)} + \frac{1}{2}(Q_B - N)Q_T^{(n)})P}{P² + Q²}, \]

\[ A_5^{(n)} = \frac{Q_e^{(n)}(Pρ + Qαcosθ) - 2(Δ₂b₁ₙ - Δ₃b₂ₙ)P}{P² + Q²}, \]

\[ ϕ^{(5)} = \ln \left( \frac{P(ρ² + 2Dρ + δ₀ - α²cos²θ) + 2Qαcosθ(ρ + D)}{P² + Q²} \right), \]

\[ B_{iφ}^{(5)} = -\frac{(N_BN₅ + \frac{1}{2}Q_e^{(n)}Q_m^{(n)})(Pρ + Qαcosθ) - (Δ₃N₅ + Δ₁b₂ₙQ_m^{(n)})P}{P² + Q²}ε - Q_Bε, \]

\[ B_{5φ}^{(5)} = \frac{(Δ₂b₁ₙQ_m^{(n)} + Δ₃(N - Q_m^{(n)}b₂ₙ))P - (\frac{1}{2}Q_e^{(n)}Q_l^{(n)} + N_BN)(Pρ + Qαcosθ)}{P² + Q²}ε + N₅ε \]

\[ A_φ^{(5)}l = \frac{(N₅Q_T^{(n)} - NQ_e^{(n)})(Pρ + Qαcosθ) - 2(N₅Δ₂b₁ₙ - (NΔ₁ + N₅Δ₃)b₂ₙ)P}{P² + Q²}ε - Q_m^{(n)}ε, \]

where \(Δ₂ = mN₅ + \frac{1}{2}N_B(Q_B - N), Δ₃ = -[DN_B + \frac{1}{2}N₅(Q_B + N)], b₁ₙ = \frac{1}{4Δ₁}[(Q_B + N)Q_e^{(n)} - 2mQ_T^{(n)}] \) and \(b₂ₙ = -\frac{1}{4Δ₁}[(Q_B - N)Q_T^{(n)} + 2DQ_e^{(n)}].\)

### 4 Conclusions

In this letter we have obtained a class of stationary extremal solutions that generalize the IWP class of EM theory for the five–dimensional heterotic string compactified to three dimensions on a two–torus. These solutions are expressed in terms of \(2n + 4\) \((n ≥ 3\) being the number of Abelian vector fields) real parameters uniquely related to the physical charges that saturate the BPS bound.

Among these solutions we identify rotating dyonic solutions with non–trivial value of NUT parameter. If one requires the asymptotic flatness condition to be satisfied in order to get a black hole configuration, one must set the NUT parameter to zero; however, in this case the found solutions become static.
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