Impact on the power spectrum of Screening in Modified Gravity Scenarios

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(Dated: May 27, 2013)

We study the effects of screened modified gravity of the $f(R)$, dilaton and symmetron types on structure formation, from the quasi-linear to the non-linear regime, using semi-analytical methods. For such models, where the range of the new scalar field is typically within the Mpc range and below in the cosmological context, non-linear techniques are required to understand the deviations of the power spectrum of the matter density contrast compared to the Λ-CDM template. This is nowadays commonly tackled using extensive N-body simulations. Here we present new results combining exact perturbation theory at the one loop level (and a partial resummation of the perturbative series) with a halo model. The former allows one to extend the linear perturbative analysis up to $k \lesssim 0.15 h^{-1}$Mpc at the perturbative level while the latter leads to a reasonable, up to a few percent, agreement with numerical simulations for $k \lesssim 3h^{-1}$Mpc for large curvature $f(R)$ models, and $k \lesssim 1 h^{-1}$Mpc for dilatons and symmetrons, at $z = 0$. We also discuss how the behaviors of the perturbative expansions and of the spherical collapse differ for $f(R)$, dilaton, and symmetron models.

PACS numbers: 98.80.-k

I. INTRODUCTION

The acceleration of the expansion of the Universe\cite{1,2} can be accommodated with General Relativity (GR) by introducing a finely tuned cosmological constant. It could also be that the laws of gravity on sufficiently large scales are not well understood and need to be reexamined (see for instance \cite{3}). Scalar field models with a coupling to matter density lead to the existence of a fifth force depending on the gradient of the scalar field profile in the vicinity of dense bodies. Of course, the existence of long range scalar forces in the solar system is tightly constrained by the Cassini probe (fifth force test)\cite{4} and the Lunar Ranging experiment (test of the strong equivalence principle)\cite{5}. All in all, such a scalar force in the solar system must be highly suppressed, implying that screening mechanisms must be introduced to guarantee the compatibility of the solar system tests with long range scalar forces on cosmological scales. Three types of screening have been unraveled: the chameleon mechanism\cite{6-8} where the scalar field mass grows with the matter density and Yukawa suppresses the fifth force in dense environments, the Damour-Polyakov property\cite{9} where the coupling to matter itself is attracted towards zero in dense matter, and finally the Vainshtein mechanism\cite{10} whereby the normalised scalar fluctuations have a reduced coupling in dense environments. The first two are describable as scalar tensor theories with a non-linear potential $V(\phi)$ and a coupling function $A(\phi)$. In the presence of matter, the effective potential has a minimum where the mass and/or the coupling of the scalar field are large and/or small enough. The Vainshtein mechanism is characteristic of models with non-linear kinetic terms. In the rest of this paper, we will solely focus on scalar-tensor models with the chameleon or the Damour-Polyakov property, the latter for the dilatons\cite{11} and symmetrons\cite{12-14}. Interestingly, a seemingly unrelated model, the $f(R)$ theories\cite{15}, are viable when possessing the chameleon property\cite{16}. We will describe the behaviour of the large curvature $f(R)$ models too.

Models with chameleon or Damour-Polyakov signatures can all be described by a tomographic method\cite{17} whereby the potential and the coupling function of $\phi$ can be reconstructed from the time behaviour of the mass function $m(a)$ and the coupling $\beta(a)$ at the minimum of the effective potential for $\phi$, for the matter density $\rho(a)$, as a function of the cosmological scale factor $a$. This is a very physical way of defining models as it determines properties affecting the growth of structures. Indeed, the scalar force in the cosmological background modifies the geodesics of the Cold Dark Matter (CDM) particles implying a change of the growth rate of linear perturbations\cite{18}. This modification depends only on the range of the fifth force cosmologically, $\lambda(a) = m^{-1}(a)$, and on the coupling of matter particles to the scalar field, $\beta(a)$. The tomographic mapping allows one to relate these features of linear perturbation theory to the full non-linear Lagrangian description of the model, which is determined by the shape of $V(\phi)$ and $\beta(\phi)$. Explicit examples are known for the dilatons, symmetrons and large curvature $f(R)$ models\cite{19}. Here we shall consider $f(R)$ models in their Jordan frame, although an Einstein frame treatment would be just as adequate. For all these models, we perform a cosmological perturbative analysis, using the $m(a) - \beta(a)$ parameterisation, which can be pursued to all order and that we stop at the third order to take into account effects up to one loop in the power spectrum. We also tackle non-perturbative properties of the screened models with the same parameterisation in a spherical collapse approach. This $m(a) - \beta(a)$ way of defining screened models goes beyond the effective descriptions of linear perturbations which has been recently
developed by several groups\textsuperscript{20, 23} in as much as higher orders in perturbation theory and non-perturbative features are readily available and calculable as exemplified in this paper.

The analysis of these models in the non-linear regime of structure formation has been recently tackled using the tomographic mapping and large N-body simulations with the ECOSMOG code for dilatons and symmetrons\textsuperscript{24}. Earlier numerical simulations of \( f(R) \) models are also available\textsuperscript{25}. It turns out that the typical scale for the scalar range cosmologically must be lower than 1 Mpc\textsuperscript{17, 26}, which is already in the mildly non-linear regime, implying that such simulations are indeed necessary. They all reveal features on scales up to a few Mpc's which can be qualitatively understood as follows. On large and linear scales, hardly any modification of gravity occurs while on very small scales, where modified gravity is screened, the usual GR behaviour is recovered. In between these two limits, the power spectrum of the matter density contrast is largely model dependent. For large curvature \( f(R) \) models, the deviation from the Λ-CDM result has a “bump” at a scale corresponding to the range of the scalar force. For dilatons and symmetrons, there is a flattening of the discrepancy over the same range of non-linear scales. In general, the screening of \( f(R) \) models is less strong that the ones of the dilatons and symmetrons.

In this paper, we combine perturbation theory and a partial resummation of the perturbative series (more precisely, a regularization of the perturbative expansion) with a halo model in order to probe the same scales as the N-body simulations. At the perturbative level, we expand the scalar contribution to the effective Newtonian potential up to third order in the non-linear density contrast. Whereas the first order enhances the gravitational clustering (for the models we consider here) and is similar to a scale-dependent Newton constant, the second and third orders are the first signs of the chameleon or screening mechanisms that make these theories consistent with Solar-System constraints. This allows us to calculate exactly the one-loop power spectrum in the presence of a scalar modification of gravity. This provides a reasonable agreement with the simulations up to quasi-linear scales around 0.15hMpc\textsuperscript{-1} at \( z = 0 \). We discuss how the behavior of this perturbative expansion differs for the \( f(R) \), dilaton, and symmetron models. We also give the recipe to extend this treatment up to an arbitrary number of loops, but this approach becomes computationally increasingly complex.

Next, we go beyond the one-loop approximation by using an improved halo model\textsuperscript{27}. This includes both a Lagrangian-space regularization of the one-loop expansion, which automatically generates a partial account of higher orders (so that the probability distribution of relative particle displacements is well behaved, i.e., positive and well normalized, which is not the case in the sharp truncation associated with the standard one-loop result), and an account of non-perturbative terms, associated with pancake formation (large-scale structures on mildly non-linear scales) and halo formation (which governs the high-\( k \) behavior).

This necessitates to describe the spherical collapse in \( f(R) \) models and scalar-tensor modifications of gravity\textsuperscript{16, 28, 29}. This is done by assuming a profile ansatz (identical to the typical profile in the linear regime for Gaussian fields) for the calculation of the scalar force on a spherical shell and using the \( m(a) − β(a) \) parameterisation of dilatons and symmetrons. We find that the \( f(R) \) models differ from the dilatons and symmetrons in as much as the linear density threshold, which becomes mass dependent here due to the scale dependence of the scalar force, converges to the linear weak field limit for small masses, whereas it converges to the Λ-CDM value for dilatons and symmetrons.

Using this combined approach for the matter density power spectrum, we find a good agreement with numerical simulations up to 3hMpc\textsuperscript{-1} for \( f(R) \) models, and 1hMpc\textsuperscript{-1} for dilatons and symmetrons. For some cases, the difference with N-body simulations is up to a factor of 2: this occurs for certain symmetron models where the perturbative expansion does not converge rapidly enough (more precisely, the screening mechanism has not converged on weakly non-linear scales at third order in the scalar and density fields) or the scalar field quickly relaxes to a singular value in spherically symmetric large overdensities. In the other cases, where the third order terms in the perturbative expansion are smaller than the second order ones and the scalar field remains in a regular (analytic) domain, we find a good agreement.

In summary, our semi-analytical treatment of the power spectrum of screened models provides a much faster description of the non-linear power spectrum than N-body simulations. As such it can be used as a sieve to distinguish interesting models where deviations from Λ-CDM could be large enough to be within reach of future large galaxy surveys from less constrained ones which are out of reach. Indeed, we have found that the semi-analytical results always reproduce the correct order of magnitude given by N-body simulations and even correspond to the simulated results up to a few percent on large scales, when the perturbative series of the screened models converges fast enough. Increasing the accuracy of our method would certainly require to understand better the shape of the halos in modified gravity and the halo mass function. This is left for future work.

In section \textsuperscript{11}, we recall the main features of modified gravity models that are relevant for large scale structures and briefly discuss the quasi-static approximation. In section \textsuperscript{11}, we analyse the perturbative series and the power spectrum in the single stream approximation at the one loop order. In section \textsuperscript{14}, we consider the spherical collapse of \( f(R) \) and scalar-tensor models. In section \textsuperscript{15}, we define the halo model and give our results on the power spectrum of \( f(R) \), dilaton and symmetron models. Finally we conclude before an appendix where the construction of the halo model and its combination with
perturbation theory is recalled.

II. MODIFIED GRAVITY MODELS

We describe here the two classes of models that we consider in this article: i) \( f(R) \) theories, where the Einstein-Hilbert action is complemented by a term which depends on the Ricci curvature, and ii) scalar-tensor theories, where an additional scalar field \( \phi \) is conformally coupled to the ordinary matter.

A. \( f(R) \) models

The first class of models that we consider in this paper corresponds to \( f(R) \) theories\(^{,}\), where the Einstein-Hilbert action is supplemented by a term that depends on the Ricci scalar, which we choose of the form \(^{15, 16, 30}\)

\[
f(R) = -2\Lambda - \frac{f_{R0} c^2}{n} \frac{R_0^{n+1}}{R^n}.
\]

(1)

This involves two parameters, the normalization \( f_{R0} \) and the exponent \( n > 0 \). This is also the large-curvature regime of the model proposed in \(^{16}\), which is consistent with Solar-System and Milky-Way constraints thanks to the chameleon mechanism, for \( |f_{R0}| \lesssim 10^{-5} \). This modification of the Einstein-Hilbert action leads to a modified Poisson equation, which reads as \(^{25}\)

\[
\nabla^2 \Psi = \frac{16\pi G}{3} a^2 \delta \rho - \frac{a^2}{6} \delta R, \tag{2}
\]

where \( \Psi \) is the modified Newtonian potential, whose gradient governs the motion of particles, and \( \delta \rho \) is the matter density fluctuation. Here and in the following, we use comoving coordinates, \( \mathbf{x} = \mathbf{r}/a \) (and \( \nabla = \nabla_{\mathbf{x}} \)), where \( a \) is the cosmological scale factor, while we use the physical matter density \( \rho \) (its mean \( \bar{\rho} \) decreases with time as \( a^{-3} \)).

The fluctuation of the Ricci scalar, \( \delta R = R - \overline{R} \), is determined by the constraint \(^{25}\)

\[
\nabla^2 \delta f_R = \frac{a^2}{3} \left[ \delta R - 8\pi G \delta \rho \right] , \tag{3}
\]

where we have introduced:

\[
\delta f_R = f_R(R) - f_R(\overline{R}), \tag{4}
\]

and

\[
f_R(R) = \frac{df}{dR} = f_{R0} c^2 \frac{R_0^{n+1}}{R^{n+1}}. \tag{5}
\]

Here we have used the quasi-static approximation and have discarded a negligible \( f_{RR} \) term. Then, Eqs. (2) and (3) govern the dynamics of the system.

Even though Eq. (3) is nonlinear in \( \delta R \), we can check that it is self-averaging, in the sense that on large scales we recover \( \delta R \to 0 \). This is due to the fact that nonlinearities enter through the Laplacian \( \nabla^2 \) in the left hand side of Eq. (3). Integrating over a large volume \( V \) of boundary \( \mathcal{S} \) and using Ostrogradsky’s theorem gives

\[
\int_V \frac{dV}{V} \delta R = \frac{3}{a^2} \int_S \frac{dS}{V} (\mathbf{n} \cdot \nabla \delta f_R) + 8\pi G \frac{\delta M}{V}, \tag{6}
\]

where \( \mathbf{n} \) is the normal unit vector to \( \mathcal{S} \). Thanks to the conservation of matter and the finite amplitude of particle motions with respect to the Hubble flow, the last term \( \delta M/V \) goes to zero for \( V \to \infty \), while the surface term also goes to zero (typically faster than the inverse of the radius of the volume \( V \)). Therefore, the average over a large volume of \( \delta R \) goes to zero, which means that there is no cumulative contribution to the potential \(^{2}\) on large scales due to small-scale nonlinearities, and we recover the background Hubble flow on large scales \(^{31}\).

In a perturbative approach to the formation of large-scale structures, we expand in the fluctuations \( \delta \rho \) and \( \delta \rho \) with respect to the background, which gives rise to successive derivatives of \( f_R(R) \). Thus, we define the quantities

\[
n \geq 1: \quad \kappa_n(a) = H^{2n-2} \left( \frac{d^n f_R}{dR^n} \right)(\overline{R}), \tag{7}
\]

(7)

where \( \kappa_n \) have the dimension of a length squared, while the background Ricci scalar is given by

\[
\overline{R}(a) = 3H_0^2 \left[ \Omega_{m0} a^{-3} + 4 \Omega_{b0} \right]. \tag{8}
\]

(8)

To be consistent with previous works which focused on linear theory, we also define

\[
m(a) = \frac{1}{\sqrt{3\kappa_1}}. \tag{9}
\]

In this paper, we perform numerical computations for the three \( f(R) \) theories of the power-law form \(^{1}\) with \( n = 1 \) and \( f_{R0} = -10^{-4}, -10^{-5}, \) and \(-10^{-6}\) to compare our results with numerical simulations from \(^{24}\).

B. Scalar field models

1. Klein-Gordon and modified Poisson equations

We now turn to scalar-tensor theories, where the action defining the system in the Einstein frame has the general form \(^{31}\)

\[
S = \int d^4x \sqrt{-g} \left[ \frac{M_0^2}{2} R - \frac{1}{2} (\nabla \phi)^2 - V(\phi) \right] + \int d^4x \sqrt{-\tilde{g}} \mathcal{L}_m(\psi^{(i)}, \tilde{g}_{\mu\nu}) , \tag{10}
\]

where \( g \) is the determinant of the metric tensor \( g_{\mu\nu} \) and \( \psi^{(i)} \) are various matter fields. The additional scalar field
$\varphi$ is explicitly coupled to matter through the Jordan-frame metric $\tilde{g}_{\mu\nu}$, which is given by the conformal rescaling

$$\tilde{g}_{\mu\nu} = A^2(\varphi) g_{\mu\nu},$$

(11)

and $\tilde{g}$ is its determinant.

This coupling implies that matter particles of mass $m$ are sensitive to a “fifth force”, $F = -m c^2 \nabla \ln A$. This can be written as an additional contribution $\Psi_A$ to the Newtonian term $\Psi_N$ in the total gravitational potential,

$$\Psi = \Psi_N + \Psi_A,$$

(12)

with

$$\frac{1}{\alpha^2} \nabla^2 \Psi_N = 4\pi G \rho,$$

$$\Psi_A = c^2 (A - \bar{A}),$$

(13)

were we assumed $A(\varphi) \approx 1$, as required by experimental constraints on the variation of fermion masses.

On the other hand, the coupling also means that the equation of motion of the scalar field explicitly depends on the matter environment (which also enables screening mechanisms to appear), and in the quasi-static limit the Klein-Gordon equation reads as

$$\frac{c^2}{\alpha^2} \nabla^2 \varphi = \frac{dV}{d\varphi} + \rho \frac{dA}{d\varphi}.$$

(14)

Thus, Eqs. (12) and (14) play the same role as Eqs. (2) and (3) encountered in $f(R)$ theories and fully determine the dynamics of the system and the formation of large-scale structures. From the point of view of the matter dynamics (e.g., after integration over the field $\varphi$), this is indeed a “modified gravity” theory because the contribution $\Psi_A$ appears as a modification to the Poisson equation. Finally, each scalar field model will be specified by the choice of the scalar field potential $V(\varphi)$ and of the coupling function $A(\varphi)$.

Again, we can check that small-scale nonlinearities are self-averaging. Indeed, assuming for instance periodic boundary conditions for the matter density $\rho$, we can look for periodic solutions to the nonlinear Klein-Gordon equation (14). Then, the potential $\Psi_A$ is periodic and does not show a cumulative growth with $|\nabla|$, so that we recover the background Hubble flow on large scales.

2. Derived functions and tomography

In a perturbative approach, we now expand in powers of the fluctuations $\delta \varphi$ and $\delta \rho$ of the scalar field and of the matter density field, with respect to the uniform background $(\bar{\varphi}, \bar{\rho})$. This means that we expand the potential $V(\varphi)$ and the coupling function $A(\varphi)$, and we are led to define the successive derivatives

$$n \geq 2: \quad \beta_n(\varphi) = M_{Pl}^{-n} \frac{d^n A}{d\varphi^n}(\varphi),$$

(15)

$$n \geq 2: \quad \kappa_n(\varphi, \bar{\varphi}) = \frac{M_{Pl}^{-n+1}}{c^2} \frac{\partial^n V_{\text{eff}}}{\partial \varphi^n}$$

$$= \frac{M_{Pl}^{-n+1}}{c^2} \left[ \frac{d^n V}{d\varphi^n} + \bar{\rho} \frac{d^n A}{d\varphi^n} \right],$$

(16)

where $V_{\text{eff}} = V + \bar{\rho}(A - 1)$ is the effective potential which enters the Klein-Gordon equation (14). Thus, the coefficients $\beta_n$ are dimensionless while the coefficients $\kappa_n$ have the dimension of a wavenumber squared. To use consistent notations with previous works which focused on linear theory, we also define

$$\beta = \beta_1 \quad \text{and} \quad m^2 = \kappa_2.$$

(17)

Following [32, 33], it is convenient to write these functions in terms of the scale factor $a(t)$, by defining $\beta_n(a) \equiv \beta_n(\sqrt[3]{a})$ and $\kappa_n(a) \equiv \kappa_n(\sqrt[3]{a}, \bar{\rho}(a))$ (we use the same notations, to avoid introducing too many functions). Through the two functions $\beta(a)$ and $m(a)$ it is possible (at least in some regular domain) to reconstruct the two functions $V(\varphi)$ and $A(\varphi)$, so that one can also define each scalar field model through the former functions $\beta(a)$ and $m(a)$. This allows one to build for instance models which satisfy a specific pattern for the growth of large-scale structures at linear order. In any case, as in [17], we note that Eq. (14) reads at zeroth order (i.e., for the uniform background) as

$$\frac{dV}{d\varphi} + \bar{\rho} \frac{dA}{d\varphi} = 0,$$

(18)

and taking the derivative with respect to the scale factor $a$ yields

$$\left( \frac{d^2 V}{d\varphi^2} + \bar{\rho} \frac{d^2 A}{d\varphi^2} \right) \frac{d\varphi}{da} + \frac{d\bar{\rho}}{da} \frac{dA}{d\varphi} = 0.$$

(19)

Using $\bar{\rho} \propto a^{-3}$ and Eqs. (15)–(17) we obtain

$$\frac{d\varphi}{da} = \frac{3\beta \bar{\rho}}{c^2 M_{Pl} m^2 a}.$$

(20)

Then, we easily obtain high-order derivatives $\beta_n$ and $\kappa_n$ by recursion, as

$$\beta_{n+1}(a) = M_{Pl}^{-1} \frac{d\beta_n}{d\varphi} = \frac{c^2 M_{Pl}^2 m^2 a}{3\beta} \frac{d\beta_n}{da},$$

(21)

and

$$\kappa_{n+1}(a) = \frac{c^2 M_{Pl}^2 m^2 a}{3\beta} \frac{d\kappa_n}{da} + \frac{m^2 \beta_n}{\beta}.$$

(22)

a. Generalized dilaton models  The original dilaton model corresponds to the coupling function [2]

$$A(\varphi) = 1 + \frac{1}{2} \frac{A_2}{M_{Pl}^4} (\varphi - \varphi_s)^2$$

(23)

and the potential $V(\varphi) = V_0 \exp(-\varphi/M_{Pl})$. This can be generalized [24] by keeping the coupling function as in
but specifying the mass \(m(a)\) instead of the potential \(V(\varphi)\). Thus, the model is determined by the parameters \(\{m_0, r, A_2, \beta_0\}\), with
\[
m(a) = m_0 a^{-r},
\]
and \(\beta_0\) is the value of \(\beta(a)\) today \((a_0 = 1)\). Then, using Eqs. (18) and (20) we obtain
\[
\beta(a) = \beta_0 e^{-s(a^{2r-3}-1)/(3-2r)}, \quad \text{with} \quad s = \frac{9A_2\Omega_m H_0^2}{c^2 m_0^2}.
\]
Using Eqs. (21)-(22), the derivatives needed for second-order computations read as
\[
\beta_2 = A_2, \quad \kappa_3 = \frac{m^2 A_2}{\beta} \left( 1 - \frac{2r}{s} a^{3-2r} \right),
\]
and at third order,
\[
\beta_3 = 0, \quad \kappa_4 = -\frac{m^2 A_2^2}{\beta^2} \left( 1 + \frac{2r}{s} (3-4r) a^{6-4r} \right).
\]
To compare our results with the numerical simulations from [34], we consider the same set of parameters, which we recall in Table II.

TABLE I: List of dilaton models considered in this paper. They are the same as in [34].

| model name | \(m_0\)[h/Mpc] | \(r\) | \(\beta_0\) | \(s\) |
|------------|----------------|------|-----------|------|
| A1         | 0.334          | 1    | 0.5       | 0.6  |
| A2         | 0.334          | 1    | 0.5       | 0.24 |
| A3         | 0.334          | 1    | 0.5       | 0.12 |
| B1         | 0.334          | 1    | 0.25      | 0.24 |
| B2         | 0.334          | 1    | 0.75      | 0.24 |
| B4         | 0.334          | 1    | 1         | 0.24 |
| C1         | 0.334          | 1.33 | 0.5       | 0.24 |
| C2         | 0.334          | 0.67 | 0.5       | 0.24 |
| C3         | 0.334          | 0.4  | 0.6       | 0.24 |
| D1         | 0.667          | 1    | 0.5       | 0.06 |
| D2         | 0.167          | 1    | 0.5       | 0.96 |
| D4         | 0.111          | 1    | 0.5       | 2.16 |

\(b.\) Generalized symmetron models The symmetron model [12-14] corresponds to a phase transition from a single-well to a double-well effective potential, so that the modifications to gravity only appear after a finite time (with respect to the background), at a scale factor \(a_s\), where the field \(\varphi\) moves from the initial single minimum \(\varphi = 0\), in high-density environments with \(\rho > \bar{\rho}(a_s)\), to one of the two new minima \(\pm \varphi_c(\rho)\) which appear in low-density environments with \(\rho < \bar{\rho}(a_s)\). Following [34], we consider a generalization [24] defined by the functions
\[
\beta(a) = \beta_0 \left[ 1 - \left( \frac{a_s}{a} \right)^3 \right]^n, \quad m(a) = m_0 \left[ 1 - \left( \frac{a_s}{a} \right)^3 \right]^{\bar{n}},
\]
for \(a > a_s\), and \(\beta = 0\) and \(m = 0\) for \(a \leq a_s\). Thus, the model is now defined by the parameters \(\{\beta_0, n, m_0, \bar{n}\}\), with \(\bar{n} > 0\), \(\bar{n} > 0\), and \(\bar{n} - 2\bar{n} + 1 > 0\) (which arises from the requirement that \(\overline{\varphi}(a_s)\) be finite). Using again Eqs. (21)-(22), we obtain
\[
\beta_2 = \frac{\bar{n} c^2 a^3 m^2}{3\Omega_m H_0^2} \left[ 1 - \left( \frac{a_s}{a} \right)^3 \right]^{-1}, \quad \kappa_3 = \frac{n + 2\bar{n}}{\bar{n}} m^2 \frac{\beta_2}{\beta}, \quad \kappa_4 = \frac{8\bar{n}^2 - \bar{n}(2 + \bar{n}) + \bar{n} (4\bar{n} - 2) m^2 \beta_2^2}{\bar{n}^2}.
\]

To compare our results with the numerical simulations from [34], we consider the same set of parameters, which we recall in Table II.

TABLE II: List of symmetron models considered in this paper. They are the same as in [34].

| model name | \(a_s\) | \(m_0\)[h/Mpc] | \(\bar{n}\) | \(\beta_0\) | \(\bar{n}\) |
|------------|--------|----------------|-----------|-----------|--------|
| A1         | 0.5    | 0.033          | 0.5       | 1         | 0.5    |
| A2         | 0.5    | 0.033          | 0.5       | 1         | 0.25   |
| A3         | 0.5    | 0.017          | 0.5       | 1         | 0.25   |
| A4         | 0.5    | 0.017          | 1         | 1         | 1.5    |
| B1         | 0.33   | 0.033          | 0.5       | 1         | 0.5    |
| B2         | 0.33   | 0.033          | 0.5       | 1         | 0.25   |
| B3         | 0.33   | 0.017          | 0.5       | 1         | 1.5    |
| B4         | 0.33   | 0.017          | 1         | 1         | 1.5    |

3. Quasi-static approximation

As in most published works, throughout this article we use the quasi-static approximations [3] and [14]. However, for the symmetron models described above, the singularity of the functions \(\beta(a)\) and \(m(a)\) of Eqs. (21-22) at \(a_s\) could be expected to give rise to significant transients. We investigate here the magnitude of this effect, at the linear level over the fluctuation \(\delta \varphi\) due to the singularity of \(\beta(a)\). Without the quasi-static approximation, the Klein-Gordon equation (14) becomes
\[
\ddot{\varphi} + 3H \dot{\varphi} - \frac{c^2}{a^2} \nabla^2 \varphi = -\frac{dV}{d\varphi} - \rho \frac{dA}{d\varphi}.
\]
Then, the equation of motion for the field fluctuation, \(\delta \varphi = \varphi - \overline{\varphi}\), reads at linear order in \(\delta \varphi\) and \(\delta \rho\) as
\[
\delta \ddot{\varphi} + 3H \delta \dot{\varphi} - \frac{c^2}{a^2} \nabla^2 \delta \varphi = -c^2 m^2 \delta \varphi - \frac{\beta}{M_{Pl}} \delta \rho,
\]
where \( \delta \rho = \rho - \bar{\rho} \). Here we have absorbed possible transients of the background \( \bar{\rho} \), with respect to its quasi-static approximation, into a redefinition of the derivatives \( m^2 \) and \( \beta \). Introducing the rescaled field \( v = a \delta \varphi \) and the conformal time \( \tau = \int dt/a \), the Klein-Gordon equation \( (35) \) becomes

\[
v'' - \frac{a''}{a} v - c^2 \nabla^2 v = -c^2 m^2 a^2 v - \frac{\beta a^3 \delta \rho}{M_{Pl}},
\]

where primes denote derivatives with respect to \( \tau \). This reads in Fourier space as

\[
\tilde{v}'' + \omega^2(\tau) \tilde{v} = \tilde{S}(\tau),
\]

with

\[
\omega^2 = k^2 c^2 + c^2 a^2 m^2 - \frac{a''}{a}, \quad \tilde{S} = -\frac{\beta a^3}{M_{Pl}} \delta \rho.
\]

The quasi-static approximation is recovered by neglecting the time derivatives, which yields \( \tilde{v} = \tilde{S}/\omega^2 \) (at this linear order in \( \delta \varphi \)). Here we only investigate the impact of sudden changes or singularities of the coupling function \( \beta(a) \), whence of the source \( \tilde{S} \). For this purpose, we can go beyond the quasi-static approximation by keeping the term \( \tilde{v}'' \) in the linearized Klein-Gordon equation \( (37) \) but neglecting the time dependence of \( \omega^2 \) (this applies to cases where \( \beta(\tau) \) and \( \tilde{v}(\tau) \) vary on a shorter time-scale than the scale factor \( a(\tau) \) and the mass \( m^2(a) \)). This yields the approximation

\[
\tilde{v}(\tau) \simeq \int_0^\tau d\tau' \tilde{S}(\tau') \frac{\sin[\omega(\tau - \tau')] \sin[\omega(\tau + \tau')]}{\omega},
\]

and an integration by parts gives

\[
\tilde{v}(\tau) \simeq \frac{\tilde{S}(\tau)}{\omega^2} - \int_0^\tau d\tau' \tilde{S}'(\tau') \frac{\cos[\omega(\tau - \tau')] \cos[\omega(\tau + \tau')]}{\omega^2},
\]

where we assumed that the source decays for \( \tau \to 0 \). The first term in Eq. \( (40) \) is the quasi-static approximation, and further integrations by parts yield terms of increasing order in \( 1/\omega \). However, for singular coupling functions \( \beta(\tau) \) this stops at the order where the integral over the \( a \)-derivative \( \tilde{S}^{(m)} \) becomes divergent. In particular, for singular coupling functions of the form \( (28) \) we must stop at Eq. \( (10) \) if \( \hat{n} < 1 \). Let us consider the case

\[
\tau > \tau_s : \quad \tilde{S}(\tau) = \tilde{S}_s(\tau - \tau_s)^{\hat{n}} \quad \text{with} \quad \hat{n} < 1, \quad (41)
\]

and \( \tilde{S} = 0 \) for \( \tau < \tau_s \). Then, Eq. \( (10) \) gives at late times

\[
\tilde{v}(\tau) \simeq \frac{\tilde{S}(\tau)}{\omega^2} - \frac{\tilde{S}_s \Gamma(\hat{n} + 1)}{\omega^2 + \hat{n}} \cos[\tilde{n} \frac{\pi}{2} + \omega(\tau_s - \tau)], \quad (42)
\]

and the quasi-static approximation is valid if \( (\omega \tau_s)^{-\hat{n}} \gg 1 \). The modified gravity effects that we consider in this paper appear on scales where \( k \lesssim a \omega \), whence \( \omega \sim k c \), and a ten percent accuracy on \( \delta \varphi \) requires

\[
k > \frac{10^{1/\hat{n}}}{c \tau} \sim 3 \times 10^{-4+1/\hat{n}} \text{hMpc}^{-1}. \quad (43)
\]

Thus, we obtain \( k > 0.03h\text{Mpc}^{-1} \) for \( \hat{n} = 0.5 \) and \( k > 3h\text{Mpc}^{-1} \) for \( \hat{n} = 0.25 \). Therefore, on the scales of interest for modified gravity probes, \( k \gtrsim 0.1h\text{Mpc}^{-1} \), the quasi-static approximation is only valid up to a slightly lower accuracy than ten percent if \( \hat{n} = 0.25 \), and to better accuracy for higher values of \( \hat{n} \). In particular, for \( \hat{n} > 1 \) or for regular coupling functions as in dilaton models, or the generic case which includes the \( f(R) \) theories, the correction to the quasi-static approximation is suppressed by a factor \( 1/(\omega \tau) \) and a ten percent accuracy (at least) is reached as soon as \( k > 3 \times 10^{-3}h\text{Mpc}^{-1} \) (and better at higher \( k \)). For small \( k \) or the homogeneous background, the accuracy of the quasi-static approximation is set by \( 1/(\omega \tau) \approx 1/(c \tau) \ll 1 \). Thus, the quasi-static approximation is sufficient for our purposes throughout this paper.

A different issue is the fact that the symmetron models arise from a double well potential and that different domains may fall within different minima \( \beta(\tau) \). As seen above, soon after \( a \), the quasi-static approximation should become valid within each domain. However, at the boundaries between different regions, new phenomena associated with these domain walls take place and are not described in this paper. They would require specific methods suited to such topological defects.

### III. Perturbative Approach

The equations of motion are nonlinear and there are no explicit solutions in the general case. As in the usual \( \Lambda \)-CDM cosmology we can look for perturbative solutions, where we expand in fluctuations with respect to the uniform expanding background. We describe in this section this perturbative approach to the equations of motion, up to any order in all field fluctuations. We give explicit expressions up to third order. For scalar-tensor models, this is carried out in full generality using the \( m(a) - \beta(a) \) parameterisation.

#### A. Expansion of the modified "gravitational potential"

To compute the dynamics of the matter particles we need the modified gravitational potential \( \Psi \) given by either Eq. \( (2) \) or Eq. \( (12) \). In the quasi-static approximation, \( \Psi \) is a mere functional of the matter density fluctuations \( \delta \rho \) (i.e., it does not depend on the past evolution) and it is convenient to first solve for \( \Psi(\delta \rho) \). Next, this expression can be used in the equation of motion of the matter particles (the Euler equation in the single-stream approximation), which can be solved as in the standard \( \Lambda \)-CDM case by a perturbative expansion of the density and velocity fields in powers of the linear growing mode. A similar approach was already used in \( (37) \) for DGP \( (38) \) and \( f(R) \) models, up to one-loop order, and in \( (39) \) (where only the linear order was kept in the modified
gravitational potential). Here we describe how this perturbative approach, which relies on two successive expansions, applies in a similar fashion to \( f(R) \) theories and scalar-tensor models, with an explicit coupling to the matter density in the Klein-Gordon equations that governs this additional scalar field. We also show how the tomographic approach determines the higher-order terms from derivatives of the two coupling functions that appear at linear order.

1. \( f(R) \) models

In the \( f(R) \) theories, the modified potential \( \Psi \) is given by equation \([2]\) (in the quasi-static approximation), which involves the fluctuation \( \delta R \) of the Ricci scalar. Therefore, we first need to solve the constraint equation \([9]\) to obtain the functional \( \delta R[\delta \rho] \). Expanding the function \( f_R \), with the help of the derivatives \( \kappa_n \) introduced in \([7]\), and moving linear terms in \( \delta R \) to the left-hand-side, Eq.\((3)\) becomes

\[
(1 - \frac{\nabla^2}{a^2 m^2}) \cdot \delta R = \frac{\delta \rho}{M_{Pl}^2} + \sum_{n=2}^{\infty} \frac{3H^2 - 2n \kappa_n}{a^2 n!} \nabla^2 (\delta R)^n. \tag{44}
\]

Then, we can solve Eq.\((44)\) for \( \delta R \) by looking for a perturbative expansion in powers of the nonlinear density fluctuation \( \delta \rho \). Going to Fourier space, with the normalization \( \delta R(\mathbf{x}) = \int d\mathbf{k} e^{i \mathbf{k} \cdot \mathbf{x}} \delta R(\mathbf{k}) \), we write this expansion as

\[
\delta R(\mathbf{k}) = \sum_{n=1}^{\infty} \int d\mathbf{k}_1..d\mathbf{k}_n \, \delta D(\mathbf{k}_1 + .. + \mathbf{k}_n - \mathbf{k})
\]
\[
\times h_n(\mathbf{k}_1,...,\mathbf{k}_n) \, \delta \rho(\mathbf{k}_1) \ldots \delta \rho(\mathbf{k}_n). \tag{45}
\]

As the linear operator in the left hand side in Eq.\((44)\) is diagonal in Fourier space, with the inverse \( a^2 m^2/(a^2 m^2 + k^2) \), we easily obtain the kernels \( h_n \) by recursion, after substituting the expansion \([15]\) into Eq.\((44)\). This yields for instance for the first two kernels

\[
h_1(\mathbf{k}) = \frac{a^2 m^2}{M_{Pl}^2 (a^2 m^2 + k^2)}, \tag{46}
\]

\[
h_2(\mathbf{k}_1, \mathbf{k}_2) = \frac{-3a^4 m^6 \kappa_2 k^2}{2H^2 M_{Pl}^4 (a^2 m^2 + k_1^2)(a^2 m^2 + k_2^2)(a^2 m^2 + k^2)}. \tag{47}
\]

The expansion \([46]\) in powers of the nonlinear density fluctuation \( \delta \rho \) should be distinguished from the expansion in powers of the linear density fluctuation \( \delta \rho_L \) (or \( \psi_L \)) that we introduce below to solve the Euler equation. In particular, the order and the range of validity of these two expansions are not necessarily identical. For instance, if the high-order derivatives \( \kappa_n \) are very small (or if \( f(R) \) is a polynomial) the expansion \([45]\) may be truncated at a low order, even on scales where the density field is highly nonlinear.

Next, substituting into Eq.\((2)\) we obtain the expansion in the nonlinear density fluctuation \( \delta \rho \) of the modified gravitational potential,

\[
\Psi(\mathbf{k}) = \sum_{n=1}^{\infty} \int d\mathbf{k}_1..d\mathbf{k}_n \, \delta D(\mathbf{k}_1 + .. + \mathbf{k}_n - \mathbf{k})
\]
\[
\times H_n(\mathbf{k}_1,...,\mathbf{k}_n) \, \delta \rho(\mathbf{k}_1) \ldots \delta \rho(\mathbf{k}_n). \tag{48}
\]

This gives

\[
H_1(\mathbf{k}) = -\frac{a^2 (3a^2 m^2 + 4k^2)}{6M_{Pl}^2 k^2 (a^2 m^2 + k^2)} \tag{49}
\]

and

\[
n \geq 2: \ H_n = \frac{a^2}{6k^2} h_n. \tag{50}
\]

2. Scalar field models

In scalar-tensor theories, the modified potential \( \Psi \) depends on the scalar field \( \varphi \), hence we first need to solve the Klein-Gordon equation \((14)\), to obtain the functional \( \delta \varphi[\delta \rho] \). Subtracting from Eq.\((14)\) the uniform background \((18)\) and expanding in \( \delta \varphi \), using the derivatives \([15]-[16]\), we obtain

\[
\left(\frac{\nabla^2}{a^2 - m^2}\right) \cdot \delta \varphi = \frac{\beta}{c^2 M_{Pl}^2} \delta \rho + \frac{\beta_2}{c^2 M_{Pl}^2} \delta \varphi
\]
\[
+ \sum_{n=2}^{\infty} \left( \frac{\kappa_{n+1}}{M_{Pl}^{n+1}} + \frac{\beta_{n+1}}{c^2 M_{Pl}^{n+1+1}} \right) \frac{(\delta \varphi)^n}{n!}. \tag{51}
\]

Again, we solve this Klein-Gordon equation as a perturbative expansion in the nonlinear matter density fluctuation \( \delta \rho \),

\[
\delta \varphi(\mathbf{k}) = \sum_{n=1}^{\infty} \int d\mathbf{k}_1..d\mathbf{k}_n \, \delta D(\mathbf{k}_1 + .. + \mathbf{k}_n - \mathbf{k})
\]
\[
\times h_n(\mathbf{k}_1,...,\mathbf{k}_n) \, \delta \rho(\mathbf{k}_1) \ldots \delta \rho(\mathbf{k}_n), \tag{52}
\]

using the fact that the linear operator in the left hand side in Eq.\((51)\) is diagonal in Fourier space and easily inverted as \(-a^2/(a^2 m^2 + k^2)\). This yields for instance for the first two kernels

\[
h_1(\mathbf{k}) = \frac{-a^2 \beta}{c^2 M_{Pl} (a^2 m^2 + k^2)}, \tag{53}
\]

\[
h_2(\mathbf{k}_1, \mathbf{k}_2) = \frac{a^4 \beta (a^2 m^2 + k_1^2) + 2 \beta_2 (a^2 m^2 + k_2^2)}{2c^2 M_{Pl}^4 (a^2 m^2 + k_1^2)(a^2 m^2 + k_2^2)(a^2 m^2 + k^2)}. \tag{54}
\]

Next, substituting into Eq.\((12)\) we obtain the expansion over \( \delta \rho \) of the modified gravitational potential, as in Eq.\((18)\), writing the fifth-force contribution as

\[
\Psi_A = \sum_{n=1}^{\infty} \frac{c^2 \beta_n}{M_{Pl}^2 n!} (\delta \varphi)^n. \tag{55}
\]
This yields for instance
\[ H_1(k) = -\frac{a^2(k^2m^2 + k^2(1 + 2\beta^2))}{2M^2_\text{Pl}k^2(a^2m^2 + k^2)}. \]

**B. Single-stream approximation for the matter fluid**

1. **Hydrodynamical equations of motion**

We have described in the previous sections how to compute the modified gravitational potential up to any order in \( \delta \rho \) for \( f(R) \) theories and scalar-tensor models. From such expansions [48] (which may also be associated with other models), we now derive the dynamics of large-scale structures in the perturbative regime. In the single-stream approximation which is valid on large scales, the dynamics of the matter fluid is given by the continuity and Euler equations,

\[
\frac{\partial \rho}{\partial \tau} + \nabla \cdot [(1 + \delta) \mathbf{v}] = 0, \quad (57)
\]

\[
\frac{\partial \mathbf{v}}{\partial \tau} + \mathcal{H} \mathbf{v} + (\mathbf{v} \cdot \nabla) \mathbf{v} = -\nabla \cdot \mathbf{F}, \quad (58)
\]

where \( \tau = \int dt/a \) is the conformal time, \( \mathcal{H} = aH = \dot{a} \) the conformal expansion rate, \( \delta = \delta \rho/\bar{\rho} \) the matter density contrast, and \( \mathbf{v} \) the peculiar velocity. Introducing the time variable \( \eta = \ln(a) \) and the two-component vector \( \psi \),

\[
\psi \equiv \left( \begin{array}{c} \psi_1 \\ \psi_2 \end{array} \right) \equiv \left( \begin{array}{c} \delta \\ -(\nabla \cdot \mathbf{v})/\dot{a} \end{array} \right), \quad (59)
\]

Eqs. (57)-(58) read in Fourier space as

\[
\frac{\partial \tilde{\psi}_1}{\partial \eta} - \tilde{\psi}_2 = \int d\mathbf{k}_1 d\mathbf{k}_2 \delta_D(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}) \tilde{\alpha}(\mathbf{k}_1, \mathbf{k}_2) \times \tilde{\psi}_2(\mathbf{k}_1) \tilde{\psi}_1(\mathbf{k}_2), \quad (60)
\]

\[
\frac{\partial \tilde{\psi}_2}{\partial \eta} + \frac{k^2}{a^2 H^2} \tilde{\psi}_2 = \int d\mathbf{k}_1 d\mathbf{k}_2 \delta_D(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}) \tilde{\beta}(\mathbf{k}_1, \mathbf{k}_2) \tilde{\psi}_2(\mathbf{k}_1) \tilde{\psi}_2(\mathbf{k}_2), \quad (61)
\]

with

\[
\tilde{\alpha}(\mathbf{k}_1, \mathbf{k}_2) = \frac{(\mathbf{k}_1 + \mathbf{k}_2) \cdot \mathbf{k}_1}{k_1^2}, \quad \tilde{\beta}(\mathbf{k}_1, \mathbf{k}_2) = \frac{|\mathbf{k}_1 + \mathbf{k}_2|^2(\mathbf{k}_1 \cdot \mathbf{k}_2)}{2k_1^2 k_2^2}. \quad (62)
\]

In the standard \( \Lambda \)-CDM cosmology, where the Newtonian gravitational potential is linear in the density field, the continuity and Euler equations [48]-[51] are quadratic. In modified gravity models, such as those studied in this paper, the potential \( \Psi \) is nonlinear and contains terms of all orders in \( \delta \rho \). Therefore, we must introduce vertices of all orders and we write Eqs. [60]-[61] under the more concise form

\[
O(x, x') \cdot \tilde{\psi}(x') = \sum_{n=2}^{\infty} K^n(x; x_1, ..., x_n) \cdot \tilde{\psi}(x_1) \ldots \tilde{\psi}(x_n), \quad (63)
\]

where we have introduced the coordinates \( x = (k, \eta, i) \), \( i = 1, 2 \) is the discrete index of the two-component vector \( \tilde{\psi} \), and repeated coordinates are integrated over. The matrix \( O \) reads as

\[
O(x, x') = \delta_D(\eta' - \eta) \delta_D(k' - k) \times \left( \begin{array}{cc} \frac{\partial}{\partial \eta} & -1 \\ -\frac{2}{3} \Omega_m(\eta) (1 + \epsilon(k, \eta)) \frac{\partial}{\partial \eta} + \frac{1-3w\Omega_m(\eta)}{2} & \end{array} \right), \quad (64)
\]

where \( \epsilon(k, \eta) \), which measures the deviation from the Newtonian gravitational potential at linear order, is given by

\[
1 + \epsilon(k, \eta) = -2M^2_\text{Pl}a^{-2}k^2 H_1(k, \eta). \quad (65)
\]

The vertices \( K^n \) are equal-time vertices of the form

\[
K^n(x; x_1, ..., x_n) = \delta_D(\eta_1 - \eta) \delta_D(\eta_n - \eta) \times \delta_D(k_1 + ... + k_n - k) \gamma^n_{1,1,...,n}(k_1, ..., k_n; \eta). \quad (66)
\]

The nonzero vertices are the usual \( \Lambda \)-CDM ones,

\[
\gamma^n_{1,1}(k_1, k_2) = \frac{\delta (k_1 - k_2)}{2}, \quad \gamma^n_{1,2}(k_1, k_2) = \frac{\delta (k_1 + k_2)}{2}, \quad \gamma^n_{2,2}(k_1, k_2) = \hat{\beta}(k_1, k_2), \quad (67)
\]

which are of order \( n = 2 \) and do not depend on time, and the new vertices associated with the modified gravitational potential [48],

\[
n \geq 2 : \quad \gamma^n_{2,1,...,1}(k_1, ..., k_n; \eta) = -\frac{k^2}{a^2 H^2} \left( 3 \Omega_m H^2 M^2_\text{Pl} \right)^n \times \frac{1}{n!} \sum_{\text{perm}} H_n(k_1, ..., k_n; \eta), \quad (68)
\]

where we sum over all permutations of \( \{k_1, ..., k_n\} \) to obtain symmetrized kernels \( \gamma^n \).

From the analysis in the previous sections, and in particular from Eqs. [44] and [51], we can check that at all orders the vertices decay as \( k^2 \) at low \( k \),

\[
n \geq 2, \quad k \to 0 : \quad \gamma^n_{2,1,...,1}(k_1, ..., k_n) \sim k^2, \quad (69)
\]

where the limit is taken by letting the sum \( k = k_1 + ... + k_n \) go to zero while the individual wavenumbers \( \{k_1, ..., k_n\} \) remain finite. This is related to the lack of backreaction on large scales from small scales nonlinearities, noticed in Sec. III. As for the usual Newtonian gravity, this means that if the initial conditions had very little power on large scales (i.e., the linear power spectrum \( P_L(k) \) would decay faster than \( k^4 \) at low \( k \)), nonlinearities would only generate a \( k^4 \) tail at low \( k \). For CDM initial conditions, where \( P_L(k) \sim k^{1.00} \) at low \( k \), this ensures that we recover linear theory on large scales.
a. f(R) theories: From Sec. III A 1 we obtain for the first three kernels the expressions
\[ \epsilon(k, \eta) = \frac{k^2}{3(a^2m^2 + k^2)}. \] (70)
and
\[ \gamma_{2,1,1}^s(k_1, k_2, \kappa_1) = \frac{9a^4\Omega_m^2m^6\kappa_1 k^2}{4(a^2m^2 + k^2)(a^2m^2 + k^1)(a^2m^2 + k^2)}. \] (71)
and
\[ \gamma_{2,1,1}^s(k_1, k_2, k_3) = \frac{9a^4H^2\Omega_m^2\beta_1k^2}{2(a^2m^2 + k^2)} \times \frac{a^2m^2 + (\kappa_3 - 9m^2\kappa_2^2)(k_3 + k_2^2)}{(a^2m^2 + k^2)(a^2m^2 + k_2^1)(a^2m^2 + k_2^2)}. \] (72)
where we give an expression for the nonsymmetrized kernel \( \gamma_{2,1,1}^s \) as it is more compact.

b. Scalar-tensor models: From Sec. III A 2 we obtain
\[ \epsilon(k, \eta) = \frac{2\beta k^2}{a^2m^2 + k^2}, \] (73)
and
\[ \gamma_{2,1,1}^s(k_1, k_2) = \frac{9a^4H^2\Omega_m^2\beta_1k^2}{2(a^2m^2 + k^2)} \times \frac{a^2m^2 + (\kappa_3 - 9m^2\kappa_2^2)(k_3 + k_2^2)}{(a^2m^2 + k^2)(a^2m^2 + k_2^1)(a^2m^2 + k_2^2)}. \] (74)
and
\[ \gamma_{2,1,1,1}(k_1, k_2, k_3) = 9a^4H^4\Omega_m^3\beta_2k^2 \times \left\{ 6\beta^2_2(a^2m^2 + k_2^2)(2a^2m^2 + k_1^2) + 3a^2\beta_2\kappa_2(4a^2m^2 + k_2^2 + 2k_2^1 + k^2) + \beta_2(3a^2\beta_2\kappa_2^2 - a^2\beta_2\kappa_4(a^2m^2 + |k_2 + k_3|^2) + \beta_2(3a^2m^2 + |k_2 + k_3|^2)(4a^2m^2 + 3k_2^1 + k_2^2) \right\} \times \left\{ 2c^2(a^2m^2 + k_1^2)(a^2m^2 + k_2^2)(a^2m^2 + k_2^2) \right\}^{-1}. \] (75)

2. One-loop matter power spectrum

From the equation of motion (63) we can now compute the matter density power spectrum up to the required order in perturbation theory. In this paper, we only go up to third order in the fields, which corresponds to one-loop diagrams.

Thus, as in the standard perturbation theory, we look for a solution of the nonlinear equation of motion (63) as a perturbative expansion in powers of the linear growing mode \( \psi_L \),
\[ \psi(x) = \sum_{n=1}^{\infty} \psi^{(n)}(x), \quad \text{with} \quad \psi^{(n)} \propto \psi_L^n. \] (76)
At linear order, the equation of motion (63) becomes \( \mathcal{O}\cdot \psi_L = 0 \) and we obtain two linear growing and decaying modes, \( D_\pm(k, \eta) \), which are solutions of
\[ \frac{\partial^2 D}{\partial \eta^2} + \frac{1 - 3\omega \Omega_{de}}{2} \frac{\partial D}{\partial \eta} - \frac{3}{2} \Omega_m(1 + \epsilon) D = 0. \] (77)
Because at early times we recover the Einstein-de Sitter universe (the dark energy component also becomes negligible) we have the usual behaviors:
\[ t \to 0 : \quad D_+ \to a = e^\eta, \quad D_- \propto a^{-3/2} = e^{-3\eta/2}. \] (78)
However, at finite redshift, because of the \( k \)-dependent factor \( \epsilon(k, \eta) \), the linear modes \( D_\pm(k, \eta) \) now depend on the wavenumbers \( k \). In any case, assuming as usual that the decaying mode has had time to become negligible we can write the first-order solution as
\[ \tilde{\psi}^{(1)} = \tilde{\psi}_L = \delta_{L0}(k) \left( \frac{D_+(k, \eta)}{\frac{\partial D_+}{\partial \eta}(k, \eta)} \right). \] (79)
Hence the initial conditions are fully defined by the linear density field \( \delta_{L0} \). We refer the reader to [39] for a detailed analysis of the linear growing and decaying modes.
Next, to compute the higher orders \( \psi^{(n)} \) by recursion from Eq. (63), we introduce the retarded Green function \( R_L \) of the linear operator \( \mathcal{O} \), also called the linear propagator or response function, which obeys:
\[ \mathcal{O}(x, x') \cdot R_L(x', x'') = \delta_D(x - x''), \] (80)
\[ \eta_1 < \eta_2 : \quad R_L(x_1, x_2) = 0, \] (81)
and reads as
\[ R_L(x_1, x_2) = \frac{\Theta(\eta_1 - \eta_2) \delta_D(k_1 - k_2)}{D_+D_-D_{-1}D_{+2}D_{+1} - D_+D_{-1}D_{+2}D_{-1}} \times \left( \frac{D_+D_{-2}D_{-1}D_{+3}D_{+1} - D_+D_{-1}D_{+2}D_{-1}}{D_+D_{-2}D_{-1}D_{+3}D_{+1} - D_+D_{-1}D_{+2}D_{-1}} \right). \] (82)
It involves both the linear growing and decaying modes, \( D_+ \) and \( D_- \), and \( \Theta(\eta_1 - \eta_2) \) is the Heaviside function, which ensures causality. Then, from Eq. (63) we obtain at second and third order
\[ \dot{\tilde{\psi}}^{(2)} = R_L \cdot K_2 \cdot \tilde{\psi}^{(1)} \tilde{\psi}^{(1)}, \] (83)
\[ \dot{\tilde{\psi}}^{(3)} = 2R_L \cdot K_2 \cdot \tilde{\psi}^{(2)} \tilde{\psi}^{(1)} + R_L \cdot K_3 \cdot \tilde{\psi}^{(1)} \tilde{\psi}^{(1)} \tilde{\psi}^{(1)}. \] (84)
We show the diagrams associated with Eqs. (79), (83), and (84) in Fig. 1. The last diagram, associated with the last term in Eq. (83), does not appear in the standard \( \Lambda \)-CDM case. It is due to the vertex \( \gamma_{2,1,1}^s(\tilde{\psi}^{(1)} \tilde{\psi}^{(1)} \tilde{\psi}^{(1)}) \) associated with the term of order \( (\delta \rho)^3 \) of the nonlinear modified gravitational potential \( \Psi \).
The diagrams $P_{22}$ (which arises from the average \( \langle \tilde{\psi}^{(2)} \tilde{\psi}^{(2)} \rangle \)) in Eq. (50) by gluing together two diagrams $\tilde{\psi}^{(2)}$ of Fig. 1 and $P_{31}$ (which arises from the average \( \langle \tilde{\psi}^{(3)} \tilde{\psi}^{(1)} \rangle \)) in Eq. (55), by gluing together the first diagram $\tilde{\psi}^{(3)}$ of Fig. 1 with the diagram $\psi^{(1)}$ already appear in the $\Lambda$-CDM cosmology (but with different linear propagators and vertices). The diagram $P_{33}^\Psi$ is a new term which arises from the second diagram $\tilde{\psi}^{(3)}$ in Fig. 1. It is again due to the new vertex $\gamma_{2;1,1,1}$ associated with the term of order $\langle \delta \rho \rangle^3$ of the nonlinear modified gravitational potential $\Psi$. To be more explicit, the contribution $P_{31}$ becomes

$$
P_{31}(k, \eta) = 8 \int d\kappa_1 d\kappa_2 \delta_D(k_1 + k_2 - k) \int_{-\infty}^{\eta_1} d\eta_1 \int_{-\infty}^{\eta_2} d\eta_2 \times \sum \sum R_{L,i_1i_2}(k; \eta, \eta_1) R_{L,j_1j_2}(k; \eta_1, \eta_2) \times C_{L;1,1}(k; \eta) \gamma_{i_2;1,1;j_2}(k, -k_2; \eta_2), \tag{89}
$$

while the new contribution $P_{33}^\Psi$ reads as

$$
P_{33}^\Psi(k, \eta) = 6 \int d\kappa_1 \int_{-\infty}^{\eta} d\eta_1 R_{L;1,2}(k; \eta, \eta_1) C_{L;1,1}(k; \eta, \eta_1) \gamma_{i_2;1,1,1}(k, -k_1, k; \eta_1) \times C_{L;1,1}(k_1; \eta_1, \eta_2) \gamma_{i_2;1,1,1}(k_1, -k_1, k; \eta_1), \tag{91}
$$

where we focus on the equal-time power spectrum, so that both ends of the diagrams in Fig. 2 are taken at the same time $\eta$. Here $R_{L;1,2}(k; \eta, \eta_1)$ is the linear propagator, given by Eq. (62), while $C_{L;1,1}(k; \eta, \eta_2)$ is the linear correlation, given by

$$
C_{L}(x_1, x_2) = \langle \tilde{\psi}_L(x_1) \tilde{\psi}_L(x_2) \rangle = \delta_D(k_1 + k_2) P_{L;0}(k_1) \left( D_{1+} D_{2+} D_{1+} D_{2+} \right), \tag{92}
$$

Thus, Eq. (59) provides the expression of the matter density power spectrum up to one-loop order (i.e., up to $P_L^2$), using Fig. 2. Apart from the new diagram $P_{31}^\Psi$, the difference from the $\Lambda$-CDM cosmology is that the linear propagator $R_L$ also depends on wavenumber while the vertices also depend on time, through functions which depend on the details of the modified gravity theory as described in the previous sections. This means that it is not possible to compute analytically the integrals over time and the summations over indices which appear in the diagrams. In particular, there is no factorization of the form $P_{1loop}(k, \eta) = D(\eta)^4 P_{1loop,0}(k)$.

Our perturbative approach, illustrated up to one-loop order in Fig. 2 differs from the usual computation of the standard perturbative expansion (see (40) for the $\Lambda$-CDM cosmology, (37) for the DGP model and (39, 41) for modified gravity in the weak field limit). In the usual presentation, the expansion (76) is written as $\tilde{\psi} = \sum_n F_n^s \tilde{\psi}_L \cdots \tilde{\psi}_L$, in a fashion similar to Eq. (45), and the kernels $F_n^s$ are explicitly computed by substituting this expansion into the
equation of motion \((63)\). Then, the power spectrum is obtained as in Eq.\((85)\). In our framework we do not explicitly compute these kernels \(F_n^s\) but \(F_0^s\) and \(F_L^s\) are implicitly determined by Eqs.\((83)\) and \((84)\) and we directly go from the diagrams of Fig.\(1\) for \(\psi\) to the diagrams of Fig.\(2\) for \(P(k)\) (the standard approach yields a different type of diagrams and there is not always a one-to-one correspondence between these two diagrammatic expansions, in particular for higher-order correlations). The practical advantage of our formulation is that the diagrams of Fig.\(2\) only involve two-point functions, \(C_L\) and \(R_L\), and the vertices \(\gamma^s\). This avoids the need to compute kernels \(F_n^s(k_1,..,k_n)\) with an increasing number of dependent wavenumbers \(k_i\) as we go to higher orders.

A difference with the “closure method” used in \([37]\) for \(f(R)\) models is that we obtain explicit expressions for the power spectrum, as in Eqs.\((80)\) and \((81)\), instead of differential equations over time. This is because the integration over time of the equation of motion \((63)\) has already been performed at the level of the expansion \((76)\), through the Green functions \(R_L\) in Eqs.\((83)\) and \((84)\). Another difference with the closely related “steepest-descent” expansion described in \([39]\) is that we do not compute “self-energy” diagrams in intermediate steps. (A description of different perturbative expansions and diagrams may be found in \([42]\) for the Λ-CDM cosmology, our current approach being equivalent to the one described in Sec.4 of that paper but with a different derivation.)

All these perturbative approaches coincide when they are eventually expanded up to a given order over \(P_L\). Our approach, associated with Figs.\(1\) and \(2\) is simple (no more complex than the standard perturbative expansion) and convenient for numerical computations (it only involves the linear two-point functions \(R_L\) and \(C_L\) and the bare vertices \(\gamma^s\)). In particular, it can be readily applied to any equation of motion of the form \((63)\), whatever the index-, scale-, and time-dependence and the order of the new vertices \(\gamma^s\). This would hold for any modified gravity model with the quasi-static approximation, where the new degree of freedom can be written in terms of the density and velocity fields.

If the quasi-static approximation is not valid, one can still use the perturbative approach described in this section. However, instead of first looking for an expansion in \(\delta \rho\) for the modified gravitational potential \(\Psi\), we extend the doublet \((59)\) to a triplet \((\delta \rho, -\nabla \cdot \mathbf{v}/\dot{a}, \delta \phi/M_P)\) and we treat on the same footing the density and velocity fields and the new scalar field \(\delta \phi\) (the modified gravitational potential \(\bar{\Psi}\) being written in terms of both \(\delta \rho\) and \(\delta \phi\)).

FIG. 3: Relative deviation from Λ-CDM of the power spectrum in \(f(R)\) theories, at redshift \(z = 0\), for \(n = 1\) and \(f_{R_0} = -10^{-4}, -10^{-5},\) and \(-10^{-6}\). In each case, the triangles and the squares are the results of the “no-chameleon” and “with-chameleon” simulations from \([25]\), respectively. We plot the relative deviation of the linear power (solid line), of the one-loop power without “chameleon” effect \(\gamma^s_{\bar{\Psi},1,1} = 0\) (dashed line), and with lowest-order “chameleon” effect \(\gamma^s_{\bar{\Psi},1,1} = 0\) (dotted line).

C. Numerical results

1. \(f(R)\) theories

We show in Fig.\(3\) the relative deviation from Λ-CDM of the matter power spectrum obtained in \(f(R)\) theories at \(z = 0\), using perturbation theory. The triangles correspond to the “no-chameleon” simulations of \([25]\), where the constraint equation \((3)\) is linearized in \(\delta \rho\). This corresponds to truncating the expansions \((15)\) and \((18)\) at the first order, \(n = 1\), and to discard the new vertices \(\gamma^s_{\bar{\Psi},1,1,1}\) in Eq.\((82)\), so that the only modification from Λ-CDM enters through the factor \(\epsilon\) in the matrix \(\mathcal{O}\) of Eq.\((51)\). The squares are the fully nonlinear simulations of \([25]\), where the constraint equation \((3)\) is exactly solved. Because of the “chameleon” effect, which only appears through the nonlinear terms of Eq.\((3)\), the squares lie somewhat below the triangles in Fig.\(3\) for the same value of \(f_{R_0}\). As is well known, this effect is somewhat larger for lower values of \(|f_{R_0}|\).

The solid lines are the relative difference of the linear power spectra, \((P_L - P_{L,\Lambda CDM})/P_{L,\Lambda CDM}\). By definition, this can only include the effect of the factor \(\epsilon\) in the matrix \(\mathcal{O}\) of Eq.\((52)\). We can check that this recovers the deviation of the full nonlinear power spectrum measured in the simulations on large scales, \(k \leq 0.1h\text{Mpc}^{-1}\).

The dashed lines are the relative difference of the one-loop power spectra, \((P_{\text{tree+1loop}} - P_{\text{tree+1loop,ΛCDM}})/P_{\text{tree+1loop,ΛCDM}}\), when we only
take into account the factor $\epsilon$ for the modification of gravity, as in the “no-chameleon” simulations. In terms of the relative deviation of the matter power spectrum, this does not significantly improve the range of validity of the predictions as compared to linear theory (and fares worse at $k > 0.2h\text{Mpc}^{-1}$).

The dotted lines are the relative difference of the one-loop power spectra when we also take into account the first nonlinear vertex $\gamma^{s}_{2;1,1}$ associated with modified gravity. This corresponds to truncating the modified gravitational potential (15) at second order $(\delta\rho)^2$ and neglecting the new contribution $P^{\text{c}}_{31}$ in Eq. (2). As expected, we can see the first clue of the chameleon effect and the one-loop power spectrum becomes closer to its $\Lambda$-CDM counterpart, in agreement with the trend shown by the simulations. This extends somewhat the range of validity of the predictions (in terms of the relative deviation for $P(k)$), up to $k \sim 0.2h\text{Mpc}^{-1}$. These results agree with (37).

We have also computed the results obtained at one-loop order when we go up to order $(\delta\rho)^3$ for the modified gravitational potential (15), that is, when we take into account the new diagram $P^{\text{c}}_{31}$ in Fig. 2. It happens that for these models the curves would not be distinguishable from the dotted lines in Fig. 3 (hence they are not plotted in the figure). Thus, for $f(R)$ theories the new contribution $P^{\text{c}}_{31}$ is negligible.

2. Scalar-tensor models

We show our results for dilaton models in Fig. 4 for the power spectrum up to one-loop order. Here the simulations include the “screening” effect to all orders, as they exactly solve the Klein-Gordon equation (14) (in contrast with the $f(R)$ theories, we do not have simulation results which follow the nonlinear evolution of the density field while keeping the Klein-Gordon equation at the linear order in $\delta\varphi$).

Let us describe our results for the case “A1”: four lower lines and symbols in the upper panel. The solid line is again the relative deviation from $\Lambda$-CDM for the linear power spectra and it only matches the simulations on very large linear scales, $k < 0.1h\text{Mpc}^{-1}$. The other three lines are the relative deviations of the one-loop power spectrum when we take into account the effect of modified gravity on the gravitational potential $\Psi$ up to first, second, and third order over $\delta\rho$.

The dashed line corresponds to truncation at first order (i.e., only the factor $\epsilon$ is taken into account in the equation of motion (15)), which implies that no screening occurs. The location with respect to the linear curve depends on the model, because the two curves correspond to different quantities (linear or one-loop power spectra).

The dotted line takes into account the term of order $(\delta\rho)^2$ in the modified potential, that is, the new vertex $\gamma^{s}_{2;1,1}$. This nonlinearity corresponds to the lowest order of the screening mechanism and as we can see in the figure it yields a power spectrum which becomes closer to the $\Lambda$-CDM one, as compared to the previous dashed line. In the case of model A1 this even leads to a power spectrum which is smaller than the $\Lambda$-CDM one for $k \sim 0.15h\text{Mpc}^{-1}$. In fact, a numerical computation of the spherical collapse using the equation of motion truncated at this order for the modified potential $\Psi$ shows that the collapse is slowed down and even stops before reaching very high densities. Indeed, whereas the term of order $\delta\rho$ associated to the modification of the potential $\Psi$ speeds up the collapse (like a scale-dependent amplifi-
where the resulting perturbation theory in powers of $\tilde{b}$ is valid. The standard 1-loop screening mechanism are included, over large perturbation scales (but as for the $\Lambda$-CDM case the standard perturbation theory in powers of $\tilde{b}$ is not expected to converge very well). Thus, in contrast with the $f(R)$ theories shown in Fig. 3, it appears that the new diagram $P_{\tilde{b}}^2$ cannot be neglected and significantly improves the results. This again extends the validity of the predictions up to $k \sim 0.2h$Mpc$^{-1}$ at $z = 0$. We do not plot the models C and D here because their deviation from $\Lambda$-CDM is very small on these scales and they show the same behaviors.

We show our perturbative results for the symmetron models in Fig. 5, using the same line styles as in Fig. 4. Again, at one loop order, including the quadratic term in $(\delta \rho)$ gives a first screening correction, with a decrease of the one-loop power spectrum with respect to the one obtained when we only take into account the linear factor $\epsilon$, whereas the next cubic term in $(\delta \rho)$ partly corrects this screening. This works best for the cases A1, A4, B1, B3, and B4, where these successive orders seem to converge, in the sense that the results obtained with the three new factors $\epsilon$, $\gamma_{2,1,1}^s$, and $\gamma_{2,1,1,1}^s$, lie in-between the curves obtained with only $\epsilon$ (no screening) and only $\epsilon$ and $\gamma_{2,1,1}^s$ (over-screening). There, although we tend to overestimate the deviation from $\Lambda$-CDM, there is a reasonable agreement with simulations on very large scales (but not as good as for the dilaton models). For the models A2, A3, and B2, we find on the contrary that the results obtained with the three new factors $\epsilon$, $\gamma_{2,1,1}^s$, and $\gamma_{2,1,1,1}^s$, lie above the curves obtained with only $\epsilon$ (no screening). This means that, at this order, the expansion in $\delta \rho$ of the screening mechanism has not yet started to converge. As could be expected, these cases are those with the lowest values for the exponents $\{\tilde{n}, \tilde{m}\}$, see Table 3, which are the most singular functions $\beta(a)$ and $m(a)$ from Eqs. (28)-(29). Then, their higher-order derivatives $\beta_n$ and $\kappa_n$ diverge faster for $a \rightarrow a_s$ and the perturbative expansion (31) of the Klein-Gordon equation shows a smaller range of validity. More generally, the symmetron scenario is associated with a phase transition, from a single-well potential $V_{\text{eff}} = V + \rho(A-1)$ for $a < a_s$ (i.e., for high densities), to a double-well potential for $a > a_s$ (i.e., for low densities). Then, it is clear that perturbative approaches, which are best suited for cases where the background is at the unique minimum of a deep and isolated potential well, cannot handle very well epochs close to the transition time.

Therefore, the validity of the perturbative approach depends on the modified gravity scenarios. Among the three models studied in this paper, the most favorable case is the $f(R)$ theory, where (at one-loop order) the
screening mechanism converges very fast and the modified potential \( \Psi \) can be truncated at quadratic order \((\delta \rho)^2\).

The dilaton model remains within reach of this perturbative approach, as the expansion in \( \delta \rho \) of the screening converges (at this order) and we find a gradual improvement as we go from first to third order in \( \delta \rho \) for \( \Psi \), with a good match to simulations at this order over the scales described by one-loop standard perturbation theory.

The symmetron model is the most difficult case, because of the singularity of the potentials and the coupling functions near the transition \( \alpha_s \), which limits the validity of a perturbative approach. Then, depending on the value of the parameters of the model, the expansion may have started to converge or not at order \((\delta \rho)^3\).

In any case, these results show that it is important to take into account non-linear effects of the modified gravity model. This allows one to extend somewhat the linear regime, which is limited to very large scales where the deviations from \( \Lambda \)-CDM are very small, by going to one-loop (or higher) order and including the first effects of screening mechanisms. By comparing the results obtained at different orders, one may also estimate the range of validity of the perturbative expansion, although a more direct and reliable approach is to compare the perturbative and non-perturbative contributions within a halo-model framework as described in Sec. IV C below. One drawback is that in a fully general parameterization of modified gravity, where for instance one considers all possible operators or degrees of freedom at a given order \([20-22]\), the number of combinations increases at higher order and most works have focused on the linear regime. Therefore, it remains useful to consider specific but still rather broad classes of models, such as the \( f(R) \) and scalar-tensor models studied in this paper. Indeed, using for instance the tomographic approach described in Sec. III which also applies to the fully non-linear spherical collapse described in the next section, the model is fully defined at the non-linear level. This allows us to compute the power spectrum on a broad range of scales, as shown in Sec. IV C below, and to go beyond linear theory, which has a rather limited application.

IV. SPHERICAL COLLAPSE

To go beyond the large scales described by one-loop standard perturbation theory, we wish to combine the perturbative expansion described in the previous section with a halo model. This requires a description of the halo mass function and density profiles. Unfortunately, even for the \( \Lambda \)-CDM cosmology, there is no well-controlled modelization of the low-mass tail of the halo mass function and of the halo density profiles. Therefore, as in \([39]\) we only include the effects of modified gravity on the large-mass tail of the halo mass function, which must fall as \( e^{-\delta L^2/(2 \sigma^2)} \), where \( \sigma^2 \) is the linear density variance at mass \( M \) and \( \delta L \) is the linear density threshold required to reach a given nonlinear density contrast, which we take as 200 to define virialized halos. This property derives from the Gaussian initial conditions and this rare-event tail is governed by spherical density fluctuations (because we define halos by a spherical overdensity criterion). Therefore, to compute the linear threshold \( \delta L(M) \) we first study the spherical dynamics in this section.

A. Spherical dynamics

If the initial conditions are spherically symmetric, the equation of motion of the physical radius \( r(t) \) of a given particle reads as usual as

\[
\dot{r} = -\frac{\partial \Psi}{\partial r} = -\frac{\partial \Psi_N}{\partial r} - \frac{\partial \Psi_A}{\partial r},
\]

where as in Eq. (12) we split the modified gravitational potential as \( \Psi = \Psi_N + \Psi_A \) and \( \Psi_N \) is the Newtonian potential given by the first equation in (13) (for the \( f(R) \) theories we also define \( \Psi_A \equiv \Psi - \Psi_N \)). Introducing the comoving Lagrangian coordinate \( q \) of each shell, which would enclose the same initial mass \( M \) in a uniform universe, and its normalized radius \( y(t) \),

\[
y(t) = \frac{r(t)}{a(t)q} \quad \text{with} \quad q = \left( \frac{3M}{4\pi\rho_0} \right)^{1/3}, \quad y(t=0) = 1,
\]

we obtain the equation of motion

\[
\frac{\partial^2 y}{\partial \eta^2} + \frac{1-3\omega \Omega_{de}}{2} \frac{\partial y}{\partial \eta} + \frac{\Omega_m}{2} (y^{-3} - 1)y = \frac{-3\Omega_m y}{8\pi G r} \frac{\partial \Psi_A}{\partial r}.
\]

This equation gives the evolution with time of the field \( y(q, \eta) \), and because the fifth force \(-\partial \Psi_A/\partial r\) usually depends on the shape of the density profile we must simultaneously follow the dynamics of all shells, \( 0 < q < \infty \).

In Eq. (95), to write the contribution associated with the Newtonian potential as \( \Omega_m (y^{-3} - 1)y/2 \), we have assumed that the mass within the shell \( q \) is constant. In principle, it would be possible to write the spherical dynamics without using this assumption, by following the crossings of different shells. However, it would be very time-consuming to follow the fast oscillations of the inner shells and not sufficient to reach a high accuracy because in these collapsed regions a strong radial orbit instability develops and leads to virialization (the dynamics are singular and infinitesimal deviations from spherical symmetry are amplified up to the magnitude of the radial motions \([43]\)).

To bypass this problem and the need to compute the motion of all shells, we follow \([39]\) and we simplify the equation of motion (95) by focusing on the shell associated with the mass \( M \) of interest and using an ansatz for the shape of the density profile. In other words, for a given mass \( M \), we follow the dynamics of the radius \( r_M(t) \) which contains this mass \( M \), using Eq. (95) as the equation of motion for \( y_M(t) \). However, in contrast to the
In contrast with the Λ-CDM cosmology, this function reads from Eq. (99) as we use the density profile ansatz
\[
\delta(x) = \frac{\delta_M}{\sigma_{x_M}^2} \int_{V_M} \frac{dV'}{V_M} \xi_L(x, x')
\]
(96)
Here \(\xi_L\) is the linear density correlation function, \(\sigma_{x_M}^2\) the variance of the linear density contrast at the comoving radius \(x_M\), which defines the sphere of volume \(V_M\), \(\delta_M = y_M^{-3} - 1\) the nonlinear density contrast at radius \(x_M\), and \(\tilde{W}(kx_M) = 3(\sin(kx_M) - kx_M \cos(kx_M)) / (kx_M)^3\) the Fourier transform of the top hat of radius \(x_M\). By definition, this profile is normalized so that the density contrast within radius \(x_M\) is equal to \(\delta_M\). It is also the typical profile of rare events in the linear regime [13, 44], and governs the large-mass tail of the halo mass function [15] (when we neglect the nonlinear distortion of the profile). As recalled above, this procedure only applies until the nonlinear density contrast reaches about 200, because at higher densities shell crossings modify the Newtonian force itself.

This approximation transforms Eq. (96) into an ordinary differential equation for \(y_M(t)\), and this defines a function \(\delta_M = F_M[\delta_L]\) which maps the linear density contrast \(\delta_L\) (which defines the initial amplitude of the density fluctuation) to the nonlinear density contrast \(\delta_M\). In contrast with the Λ-CDM cosmology, this function \(F_M\) now depends on the mass \(M\) because of the scale dependence of the fifth force. Next, we can invert this function to obtain the linear density contrast, \(\delta_L = F_M^{-1}(\delta)\), associated with a given nonlinear threshold \(\delta\). In particular, defining as in [91, 15] virialized halos by a nonlinear density threshold of 200, we obtain the associated linear threshold \(\delta_L(M) = F_M^{-1}(200)\). This function describes how the formation of massive halos is made easier by the fifth force, as a smaller linear threshold \(\delta_L\) is required as compared to the Λ-CDM case. We describe below our results for this characteristic function for \(f(R)\) theories and scalar-field models.

### B. \(f(R)\) theories

In the case of \(f(R)\) theories, the fifth force potential reads from Eq. (2) as
\[
\nabla^2 \Psi_A = \frac{4\pi G}{3} \rho a^2 \delta - \frac{a^2}{6} \delta R.
\]
(98)
Introducing the normalized fluctuation \(\alpha(x)\) of the Ricci scalar,
\[
\delta R = 8\pi G \rho \alpha(x),
\]
(99)
we obtain in spherical symmetry
\[
\frac{\partial \Psi_A}{\partial x} = \frac{4\pi G \rho a^2}{3x^2} \int_0^x dx' \frac{x'^2}{x^2} (\delta - \alpha),
\]
(100)
where as usual \(x = r/a\) is the comoving coordinate. Then, Eq. (95) writes as
\[
\frac{d^2 \delta y_M}{d\eta^2} + \frac{1 - 3\Omega_m \delta y_M}{2} \frac{d\delta y_M}{d\eta} + \frac{\Omega_m}{2} (y_M^{-3} - 1) y_M = -\Omega_m y_M \int_0^{x_M} dx x^2 x_M^3 (\delta - \alpha),
\]
(101)
where we focus on the shell associated with a given mass \(M\). On the other hand, the field \(R(x)\) (whence \(\alpha(x)\)) is given by the constraint equation [3]. In spherical symmetry, for the power-law models [11], this yields,
\[
\frac{d^2 \alpha}{dx^2} + \frac{2}{x} \frac{d\alpha}{dx} - \frac{(n + 2)\Omega_m}{\Omega_m(1 + \alpha) + 4\Omega_L \alpha^3} \left(\frac{\Omega_m a^{-3} + (1 + \alpha) + 4\Omega_A}{\Omega_m + 4\Omega_A}\right)^{n+2} = 0 - \alpha - \delta,
\]
(102)
with
\[
m_0 = \frac{H_0}{c} \sqrt{\frac{\Omega_m + 4\Omega_A}{(n + 1)|f_{R0}|}}.
\]
(103)
Thus, to compute the spherical dynamics we numerically solve Eqs. (101) and (102). At each time step we solve the constraint equation (102), using a multigrid relaxation algorithm and the density profile (97), normalized by \(\delta_M\) at radius \(x_M\), and we advance over time with Eq. (101).

It is interesting to consider the “weak field” regime, which has been studied in many previous works [24, 39], where the constraint equations (43) or (102) are linearized in \(\delta R\) or \(\alpha\). This gives in Fourier space the weak field expressions
\[
\tilde{\alpha}_{w.f.} = \frac{a^2 m_0^2}{a^2 m_0^2 + k^2} \tilde{\delta}, \quad \tilde{\Psi}_{A,w.f.} = \epsilon(k) \tilde{\Psi}_N,
\]
(104)
where \(\epsilon(k)\) was given in Eq. (100).

Equation (102) is nonlinear and clearly shows the “chameleon” mechanism which ensures convergence to General Relativity in dense environments. Indeed, the term \((\alpha - \delta)\) tends to make \(\alpha\) converge to \(\delta\), so that the fifth force vanishes as seen in Eqs. (100) and (101). This happens on large scales, where the spatial derivatives in Eq. (102) can be neglected, which corresponds to \(k \to 0\) in the weak field expression (104), and in very dense regions, where both \(\alpha\) and \(\delta\) are large. This latter chameleon mechanism cannot be seen in the linearized solution (101) and is due to the nonlinear character of Eq. (102). For large \(\alpha\) and \(\delta\), the left hand side scales linearly with \(\alpha\) whereas the right hand side scales as \(\alpha^{n+3}\), so that in sufficiently dense environments we recover \(\alpha \approx \delta\), up to corrections of order \(\delta^{n-1}\).
We show in Fig. 6 the linear density contrast $\delta_L(M)$ that we obtain at $z = 0$, as a function of the halo mass. Because of the fifth force in the right hand side in Eq. (101), the collapse is accelerated as compared to the $\Lambda$-CDM case, and increasingly so for larger $|f_{R_0}|$ and lower masses (whereas on large scales, we recover General Relativity as $\epsilon(k) \to 0$ for $k \to 0$). This leads to a linear threshold $\delta_L$, at fixed nonlinear density contrast $\delta = 200$, which is lower than in the $\Lambda$-CDM case and decreases at low mass.

We can check in Fig. 6 that the “chameleon” effect, associated with the nonlinearity of the constraint (1) or (102), decreases the deviation from $\Lambda$-CDM, as compared to the result which would be obtained using the weak field approximation (104) ($\alpha_{w.f.}$, dotted lines), or the nonlinear solution of Eq. (102) ($\alpha_{n. l.}$, solid lines). We consider the halo masses $M = 10^{12} h^{-1} M_\odot$ (upper panel) and $M = 10^{11} h^{-1} M_\odot$ (lower panel), at $z = 0$.

In Eq. (102), we can see in the upper panel of Fig. 7 that for massive and large halos the nonlinear field $\alpha(x)$ can follow the rise of the density contrast up to the halo center, if $|f_{R_0}|$ is not too large. This reduces the fifth force as compared to the weak field approximation, in agreement with Fig. 6. However, for low mass halos and small size objects (at fixed density), the “cost” associated with spatial gradients again becomes too important and $\alpha(x)$ cannot follow the rise of the density field. This implies that within the halo $\delta - \alpha \simeq \delta$ and the fifth force accelerates the collapse. Moreover, in this regime the fifth force no longer depends on $\alpha$ and the collapse follows the weak field approximation, as seen in Fig. 6.

These behaviors are illustrated in Fig. 8 where we show the ratio $F_\Lambda/F_N$ of the fifth force to the Newtonian force at $z = 0$. Although our density profiles are different from the NFW profiles used in [46], we recover the same features. As explained above, for the low-mass halo we recover the weak-field limit with a fifth force.
we change variables from the field $\phi$ to $M$ as in Eqs. (15) and (17) we defined $\beta$ for the massive halo, the chameleon effect becomes very important for both $f_R = -10^{-5}$ and $-10^{-6}$. This suppresses the fifth force in the high-density core, while at large radii we recover General Relativity, and the gravitational force is only modified at about $1h^{-1}Mpc$ for $M = 10^{11}h^{-1}M_\odot$.

### C. Scalar field models

In the scalar-tensor theories that we study in this paper, the fifth force is given by Eq. (13),

$$\frac{\partial \Psi_A}{\partial x} = \frac{c^2}{M_{Pl}} \beta(\varphi) \frac{\partial \varphi}{\partial x},$$

(105)

where as in Eqs. (10) and (17) we defined $\beta = dA/d\varphi$, but the derivative is taken at the local value of $\varphi$ instead of the background $\varphi$. Nevertheless, to express the equations in terms of the function $\beta(a)$ introduced in Sec. 11B.2 we change variables from the field $\varphi(x)$ to the field $\alpha(x)$ defined by

$$\alpha = a(\varphi),$$

(106)

where $a(\varphi)$ is the inverse of the function $\varphi(a)$, that is, $\alpha(x)$ is the scale factor which was observed when the background value $\varphi$ was equal to the present local value $\varphi(x)$. In particular, from Eq. (20) we have

$$\frac{d\varphi}{d\alpha} = \frac{3\beta_0 f_\alpha}{c^2 M_{Pl} m_\alpha^2 \alpha},$$

(107)

where we note with a subscript $\alpha$ the values of functions taken at point $\alpha$, such as $\beta_\alpha = \beta(\alpha)$, to distinguish from the background values, such as $\beta = \beta(a)$. Then, Eq. (99) reads as

$$\frac{d^2 y_M}{d\eta^2} + \frac{1 - 3w_\Omega M_{de}}{2} \frac{d y_M}{d\eta} + \frac{\Omega_M}{2} (y_M - 1) y_M = -9 \Omega_m \alpha \beta_\alpha^2 \frac{\partial \alpha}{m_\alpha^2 \alpha^4 x M} \frac{\partial \alpha}{x}$$

(108)

where we again focus on the dynamics of the shell associated with a given mass $M$. On the other hand, the field $\varphi(x)$ (whence $a(x))$ is given by the quasi-static Klein-Gordon equation [14]. Using Eq. (107), this reads in spherical symmetry as

$$\frac{d^2 \alpha}{d x^2} + 2 \frac{d \alpha}{x d x} + \left[ \frac{d \ln \beta_\alpha}{d \alpha} - 2 \frac{d \ln m_\alpha}{d \alpha} - \frac{4}{\alpha} \left( \frac{d \alpha}{d x} \right)^2 \right] = \frac{m_\alpha^2 \alpha^4}{3 \alpha} \left[ 1 + \delta - a^2/\alpha^2 \right].$$

(109)

Then, to compute the spherical dynamics we numerically solve Eqs. (108) and (109), using the ansatz (97) for the shape of the density profile.

The “weak field” limit corresponds to linearizing the Klein-Gordon equations (13) or (109) in $\delta \varphi = \varphi - \varphi_b$ or $\delta \alpha = \alpha - \alpha_b$. This gives in Fourier space the weak field expressions

$$\delta \hat{\alpha}_{w.f.} = \frac{-a^3 m^2}{3 (a^2 m^2 + k^2)} \hat{\delta}, \quad \hat{\Psi}_A, w.f. = \epsilon(k) \hat{\Psi}_N,$$

(110)

where $\epsilon(k)$ was given in Eq. (73).

On large scales, where the fluctuations are small, we recover the weak field regime (110) and we converge to General Relativity in the limit $k \to 0$ (the spatial gradient and the factor $1/x$ in Eq. (108) give rise to a factor $k^2$ as compared to the Newtonian force, which is also seen in the factor $\epsilon(k)$ in Eq. (73)).

On small scales, a “screening” mechanism associated with the nonlinearity of Eq. (109) again ensures that we recover General Relativity in dense environments, where $\delta \to +\infty$. However, the details can depend on the scalar field model.

For dilaton models, where $m(a)$ grows at low $a$, the right hand side in Eq. (109) makes $\alpha$ converge to $a^2/\alpha$, that is, $\varphi \to \varphi(b \to \rho)$. Indeed, in this limit of large densities the right hand side scales as $m_\alpha^2 \alpha$ whereas the left hand side only scales linearly with $\alpha$. Then, the fifth force on the right hand side of Eq. (108) is suppressed as compared to Newtonian gravity by a factor $\beta_\alpha^2 m_\alpha^2$.

For symmetron models with $\hat{m} > 1/2$, in dense regions we have $\alpha \to a_s$ and more precisely $\alpha \sim \delta^{-1/2\hat{m} - 1}$. Then, the fifth force on the right hand side of Eq. (108) is suppressed as compared to Newtonian gravity by a factor $\delta^{-2\hat{m}/(2\hat{m} - 1)}$. If $\hat{m} < 1/2$ we exactly have $\alpha = a_s$ in very dense regions (with a singular growth at the boundary of the constant-$\alpha$ region of the form
We illustrate our results for some dilaton models in Fig. 9. We can check that the fifth force accelerates the collapse and leads to a smaller linear density threshold \( \delta_\Lambda(M) \), as compared to the \( \Lambda \)-CDM case. Again, the nonlinearities decrease the departure from the \( \Lambda \)-CDM case, as compared to the weak field approximation (110).

In contrast with the results for \( f(R) \) theories shown in Fig. 8 at very low mass we do not converge to the weak field result but to the \( \Lambda \)-CDM threshold. This is due to the fact that the fifth force depends on the fields \( \alpha \) in a very different fashion in Eqs. (101) and (108). In the \( f(R) \) case, a small value of \( \alpha \) implies a fifth force which is proportional to the Newtonian force and no longer depends on the precise value of \( \alpha \), whereas in the scalar field case the fifth force does not relate to the Newtonian force and remains sensitive to the local value and slope of \( \alpha(x) \).

Thus, as seen in the upper panel of Fig. 10 the weak field approximation yields larger deviations from unity for the ratio \( \alpha/a \), which can become negative on small scales, as compared to the fully nonlinear solution, which is restricted to \( 0 < \alpha < a \) (for overdense regions). These constraints imply a smaller range for the nonlinear value and slope of \( \alpha \), which leads to a smaller fifth force. On very small scales, this ensures a convergence back to General Relativity, which is thus recovered over a broader regime than in \( f(R) \) theories. The lower panel of Fig. 10 shows how the fifth force decreases, with respect to the Newtonian force, for larger objects in the range \( M > 10^{10} h^{-1} M_\odot \). At large radii it quickly decays as \( 1/x^2 \) as we recover General Relativity [the factors \( 1/x_M \delta/\delta x \) in Eq. (108) or \( k^2 \) in Eq. (73)]. Thus, as compared to Fig. 8 the modification of gravity is more localized than in the \( f(R) \) models for low-mass halos. This is because it depends on the local value and slope of the new field \( \varphi(x) \), or \( \alpha(x) \), which makes a fast convergence to General Relativity possible, following the relaxation of \( \varphi \) towards \( \overline{\varphi} \). In contrast, in the \( f(R) \) model, if there is a significant modification of gravity in inner regions, because of a non-zero value of \( (\delta - \alpha) \) in Eq. (101) in the core, its effect at large radii decays in the same manner as the Newtonian contribution itself (but we still recover the Hubble flow because this Newtonian force, associated with the overdensity with respect to the mean, also decays at large distance).

We show our results for some symmetron models in Fig. 11. The general behavior is similar to the one found for dilaton models in Fig. 9 with a linear density thresh-
old $\delta_L(M)$ which is smaller than the $\Lambda$-CDM one, towards which it converges at large mass. Again, the result obtained with the exact nonlinear solution of Eq. (109) is closer to the $\Lambda$-CDM one, as compared to the weak field approximation, and converges back to the $\Lambda$-CDM threshold at very low masses (this can only be seen for the case A3 in the figure but we checked that at smaller mass the curves A1 and A2 show the same upturn). However, as compared to the dilaton models of Fig. 9, the difference between the weak field approximation and the nonlinear result is much greater. In particular, at high mass the nonlinear result quickly becomes very close to the $\Lambda$-CDM threshold.

These features are due to the behavior of the field $\alpha(x)$, illustrated in the upper panel of Fig. 12. As noticed above, the screening mechanism is very efficient because of the lower limit $\alpha \geq a_s$. For the case $\bar{m} = 1/2$ shown in the figure, which is at the boundary between the regimes $\bar{m} < 1/2$ and $\bar{m} > 1/2$, the field $\alpha(x)$ in high density regions is neither equal to $a_s$ or above $a_s$ by a factor of order $\delta^{-1/(2m-1)}$, but becomes exponentially close as $(\alpha - a_s) \sim e^{-\sqrt{\delta}}O(L-x)$, where $L$ is the radius of the high-density region. This yields a fifth force which also decays with $\delta$ as $e^{-\sqrt{\delta}}O(L-x)$. This behavior is reached for massive halos, where spatial gradients are small and $\alpha(x)$ can follow the rise of the density contrast until it comes very close to $a_s$. For low mass halos, at fixed density, spatial gradients come into play and stop $\alpha(x)$ before it gets very close to $a_s$. In both cases, this greatly decreases the fifth force as compared to the weak field approximation.

This is also illustrated by the ratio $F_A/F_N$ shown in the lower panel of Fig. 12. The profile of the ratio $F_A/F_N$ has already been studied in [17] but in a very different regime as they consider a symmetry breaking scale factor $a_s = 1$. In such a case, at $z > 0$ there is no deviation from $\Lambda$-CDM at all orders of perturbation theory (because $\varphi = 0$ is the single minimum of the effective potential over a finite range of densities around the background $\bar{\rho}$) nor for the spherical collapse of the overdensity (97), which is typically overdense at all radii. In [17] they still find a nonzero fifth-force because they consider isolated NFW density profiles, with the boundary condition $\rho \rightarrow 0$ at $x \rightarrow \infty$, whereas our profile (97) satisfies $\rho \rightarrow \bar{\rho}$ at large distance, which is more realistic in the early stages of the collapse. Nevertheless, these remarks again show that symmetron models with $a_s \sim 1$ are difficult to describe by analytical means, because they in-

FIG. 11: Linear density threshold $\delta_L(M)$, associated with a nonlinear density contrast $\delta = 200$, for some symmetron models at $z = 0$ (cases A1, A2, and A3 from top to bottom). The dotted lines (w.f.) are the weak-field limit (110) and the solid lines (n.l.) the solution to the fully nonlinear constraint (109).

FIG. 12: Upper panel: Radial profile of the nonlinear density contrast $\delta(x)$ (black dashed lines) and of the field $\alpha(x)$, using the weak field approximation (110) ($\alpha_{w.f.}$, dotted lines), or the nonlinear solution of Eq. (109) ($\alpha_{n.l.}$, solid lines). We consider the halo masses $M = 10^{13}$ and $10^{14} h^{-1} M_{\odot}$ (where the blue curves for $\alpha/a$ show a smaller deviation from unity, which also appears at a smaller scale), for the symmetron model A3 at $z = 0$. Lower panel: Ratio $F_A/F_N$ of the fifth force to the Newtonian force, for the symmetron model A3 at $z = 0$ (with screening effect). We show our results for the halo masses $M = 10^{10}, 10^{11}, 10^{13}$, and $10^{14} h^{-1} M_{\odot}$, from top to bottom.
volve two different phases. An accurate treatment would require a specific method which explicitly takes into account these two phases but we do not consider it in this paper as we wish to investigate the general method which applies to generic modified gravity models.

In our case, where $a_s < 1$, for low-mass halos we recover a behavior which is similar to the one obtained for dilaton models in Fig. 10 because the field $\varphi(x)$, or $\alpha(x)$, only probes its regular domain. For high-mass halos, there is enough room (spatial gradients are less constraining) for the field $\varphi(x)$ to depart from the background value $\overline{\varphi}$ and to come to the singular limit $\overline{\varphi}(a_s)$ (i.e., $\alpha = a_s$). This leads to an almost constant field $\alpha(x) \simeq a_s$ in the core and a vanishing fifth force, as seen by the sharp decay at small radii in the two cases $M = 10^{13}$ and $10^{14} h^{-1} M_\odot$. In the latter case, this gives rise to a localized fifth force at the boundary of the constant-$\alpha$ region, whereas we always recover as for the dilaton models the $1/x^2$ decay at large radii. In this case, the symmetron shows features similar to the original chameleon model where a "thin shell" entirely responsible for modified gravity develops close to the surface of the body. It is likely that this sharp feature is unstable with respect to deviations from spherical symmetry or gives rise to small-scale perturbations and shell crossings at this radius. This suggests that in such singular models the collapse may be significantly modified in localized regions and that the spherical dynamics may not be as efficient as in the $\Lambda$-CDM cosmology to understand the formation of massive halos.

Thus, we obtain for the spherical collapse of overdensities up to $\delta = 200$ the same trends as those found in Sec. III C in the perturbative regime. The effects of non-linearities (associated with the chameleon mechanism) are moderate for the $f(R)$ theories, somewhat greater for the dilaton models, and very large for the symmetron models. Then, deviations from the $\Lambda$-CDM dynamics increase at a qualitative level as we go from $f(R)$ theories to dilaton models, and next to symmetron models.

V. MATTER POWER SPECTRUM

We have seen in Sec. III that standard one-loop perturbation theory does not allow us to go far in the non-linear regime, where most of the departure from General Relativity occurs for the models that we consider in this paper. Therefore, we need a model which applies to a broader range of scales. In this paper, we use the model developed in [27], which combines perturbation theory with halo models to provide the matter power spectrum from large linear scales down to small highly nonlinear scales (see the appendix for details). As in usual halo models, it splits the matter power spectrum as

$$P(k) = P_{1H}(k) + P_{2H}(k),$$

where $P_{1H}$ is the contribution associated with pairs of particles which belong to the same halo, whereas $P_{2H}$ is the contribution associated with pairs of particles which belong to two different halos.

Then, the first contribution reads as

$$P_{1H}(k) = \int_0^{\infty} \frac{d\nu}{\nu} f(\nu) \frac{M}{F(2\pi)^3} \left(\tilde{u}_M(k) - \tilde{W}(kq_M)\right)^2,$$

(112)

where $\tilde{u}_M(k)$ is the normalized Fourier transform of the halo radial profile, $\tilde{W}(kq_M)$ is the normalized Fourier transform of the top hat of radius $q_M$, and $f(\nu)$ is the normalized halo mass function, defined as

$$n(M) \frac{dM}{M} = \frac{\overline{\rho}}{\nu} f(\nu) \frac{d\nu}{\nu}, \text{ with } \nu = \frac{\delta_L(M)}{\sigma(M)}.$$

(113)

Here $\sigma(M)$ is the root mean square of the linear density contrast at scale $M$ and $\delta_L = F^{-1}(200)$ is the linear density contrast associated with the nonlinear density threshold which defines collapsed halos, which we choose equal to 200. As described in Sec. IV, $\delta_L(M)$ depends on the mass because of the scale-dependence introduced by the modifications to gravity, and it is lower than the linear density threshold obtained in the $\Lambda$-CDM case. This helps the formation of massive halos and increases the one-halo contribution (112). In numerical computations, we use for $f(\nu)$ the fit from [45], which has been shown to match numerical simulations while obeying the asymptotic large-mass tail $f(\nu) \sim e^{-\nu^2/2}$ [45]. For the halo profiles, we choose the usual NFW profile [48] and the mass-concentration relation from [27]. This means that we neglect the impact of modified gravity on the halo profiles and we only take into account its effect on the density threshold $\delta_L(M)$.

Next, the two-halo contribution becomes

$$P_{2H}(k) = \int \frac{d\Delta q}{(2\pi)^3} F_{2H}(\Delta q) \langle e^{i(k - \Delta x)_{\text{vir}}/\Delta q} \frac{1}{1 + A_1} \times e^{-\frac{1}{2}k^2(1-\mu^2)\sigma_\perp^2} \left\{ e^{-\varphi_1(-ik\mu_s\sigma_\parallel^2/\mu_s^2)} + A_1 \right. \right.$$  

$$+ \int_{0^+}^{i\infty} \frac{dy}{2\pi i} e^{-\varphi_1(y)/\sigma^2_s} \left( \frac{1}{y} - \frac{1}{y + i k \sigma_\Delta q \sigma_\parallel^2} \right) \right\}. \right)$$

(114)

Let us briefly explain the derivation of Eq. (114) (see the appendix too). It is based on the exact expression [49, 50],

$$P(k) = \int \frac{d\Delta q}{(2\pi)^3} \langle e^{i(k - \Delta x)} \rangle, \right)$$

(115)

which relates the matter power spectrum to the statistics of the Eulerian separation, $\Delta x = x_2 - x_1$, of pairs of particles with initial Lagrangian separation $\Delta q = q_2 - q_1$. Then, the factor $F_{2H}$ in Eq. (114) is the probability that a pair of separation $\Delta q$ belongs to two different halos, the factor $\langle e^{i(k - \Delta x)_{\text{vir}}/\Delta q} \rangle$ is the contribution to $e^{i(k - \Delta x)}$ due to internal motions within each halo, the factor $e^{-\frac{1}{2}k^2(1-\mu^2)\sigma_\perp^2}$...
is the contribution associated with large-scale motions transverse to the initial separation $\Delta q$ (which are taken from Lagrangian linear theory, whence the Gaussian result), and the factor $e^{-\varphi(y)}(-ik\Delta q \sigma^2_{\perp})/\sigma^2_{\parallel}$ is the contribution associated with large scale longitudinal motions. Here $\sigma^2_{\perp}$ and $\sigma^2_{\parallel}$ are the variances of the transverse and longitudinal relative displacements as given by linear theory (up to normalization factors). The factor $A_1$ (which depends on $\varphi(y)$) and the complex integral in the last term arise from an adhesion-like regularization to mimic the formation of pancakes, see [27] for details.

For our purposes, the main point is that the expression (114) satisfies the following constraints:

(a) It has a perturbative expansion in integer powers of $P_L$, as in standard perturbation theory (but it also includes some nonperturbative contributions of the form $e^{-1/\sigma^2}$).

(b) It is consistent with linear theory.

(c) It is consistent with one-loop perturbation theory, when the skewness $S_3$ of the scale-dependent characteristic function $\varphi_{\parallel}$ is given by

$$S_3(\Delta q) = -\frac{24\pi}{\sigma^2_{\parallel}} \int_0^\infty dk \frac{P_{1\text{loop}}(k) - P_{1\text{loop}}^2(k)}{(\Delta q)^4 k^2} \left[ 2 + \cos(k\Delta q) - 3 \frac{\sin(k\Delta q)}{k\Delta q} \right], \quad (116)$$

where $P_{1\text{loop}}$ is the one-loop power spectrum associated with the Zel’’dovich dynamics [51] while $P_{1\text{loop}}^2$ is the true one-loop power spectrum (thus, this is also a measure of the deviation from the simple Zel’’dovich dynamics, which is recovered at all perturbative orders when $S_3 = 0$). In practice, as in [27], this is implemented by choosing for the characteristic function $\varphi_{\parallel}$ the ansatz

$$\varphi_{\parallel}(y) = \frac{1 - \frac{y}{\alpha}}{\alpha} \left( 1 + \frac{y}{1 - \alpha} \right) - \frac{1 - \frac{y}{\alpha}}{\alpha}, \quad (117)$$

where the scale-dependent parameter $\alpha(\Delta q)$ is given by

$$\alpha(\Delta q) = \frac{2 - S_3}{1 - S_3}. \quad (118)$$

(d) The underlying non-Gaussian probability distribution $P(\Delta x_{\parallel})$ is everywhere positive, normalized to unity, and satisfies the constraint $\langle \Delta x_{\parallel} \rangle = \Delta q$.

(e) It is well behaved at high $k$, where it remains positive while becoming subdominant with respect to the one-halo contribution.

Standard perturbation theory clearly satisfies points (a) to (c) but not points (d) and (e). In particular, it is well known that when truncated at a finite order it can lead to power spectra which become unphysically large or negative at high $k$. In contrast, Eq. (114) is built as a regularization of perturbation theory which always remains consistent with some physical constraints such as point (d), which ensures that this contribution to the matter power spectrum remains well behaved at high $k$ (positive with typically a $k^{-2}$ decay). Together with the one-halo contribution (112), this provides a realistic description of the matter power spectrum from large to small scales, which has been compared to numerical simulations for the Λ-CDM cosmology in [27].

This model (111) can be at once applied to the modified gravity scenarios that we consider in this paper. For the two-halo contribution (114), we need to provide the linear power spectrum [35] and the one-loop contribution [39], which determines the characteristic function $\varphi_{\parallel}$ through Eqs. (116)-(118). For the one-halo contribution (112), we need to provide the threshold $\delta_L(M)$, obtained in Sec. [15] As compared to the Λ-CDM case, the main new sources of inaccuracy are that we neglect the impact of modified gravity on the low-mass slope of the halo mass function and on the shapes of halo profiles (whereas in the Λ-CDM case these parameters have already been fitted to numerical simulations, for instance by choosing the NFW profile).

![FIG. 13: Relative deviation from Λ-CDM of the power spectrum in $f(R)$ theories, at redshift $z = 0$, for $n = 1$ and $f_R = -10^{-4}, -10^{-5}$, and $-10^{-6}$. In each case, the triangles and the squares are the results of the “no-chameleon” and “with-chameleon” simulations from [25], respectively. We plot the relative deviation of the nonlinear power power spectrum without chameleon effect (w.f., dotted lines) and with chameleon effect (n.l., solid lines).](image)

### A. $f(R)$ theories

We show our results for the deviation from Λ-CDM of the nonlinear matter density power spectrum in Fig. [13] for $f(R)$ theories at $z = 0$. For each $f(R)$ model, we plot both the “no-chameleon” and “with-chameleon” cases studied in [25] through numerical simulations.

The “no-chameleon” case corresponds to the weak field approximation discussed in Secs. [13] and [14] the constraint equation [39] is linearized in the fluctuation $\delta R$.
of the Ricci scalar. This means that in the perturbative approach which provides the power spectrum \([87]\), up to one-loop order, we only include the factor \(\epsilon(k, \eta)\) which modifies the linear matrix \(\mathcal{O}\) in Eq.\((64)\) and we neglect the new quadratic and cubic vertices \(\gamma_{2;11}^2\) and \(\gamma_{2;11}^2\). Next, in the computation of the spherical collapse which provides the linear density threshold \(\delta_L(M)\), we use the same linearization in \(\delta R\), which corresponds to the weak field expression \([101]\) for the fifth force. In other words, the “no-chameleon” case corresponds to using the linear approximation in \(\delta \rho\) for the fifth force, i.e. truncating the expansion \((18)\) at \(n = 1\), [but \(\delta \rho\) itself is nonlinear, in the sense of the expansion \((70)\).]

The “with-chameleon” case corresponds to keeping the fully nonlinear constraint equation \((3)\). In the perturbative approach at one-loop order, this means that we include the new quadratic and cubic vertices \(\gamma_{2;11}^2\) and \(\gamma_{2;11}^2\), in addition to the linear kernel \(\epsilon\), in the equation of motion \((43)\). (As noticed in Sec.\(\text{III C 1}\) the cubic vertex \(\gamma_{2;11}^2\) can actually be neglected at this order, but not the quadratic vertex \(\gamma_{2;11}^2\).) In the spherical collapse dynamics we solve the exact nonlinear constraint equation \((102)\).

We can see in Fig.\(\text{13}\) that our approach is able to reproduce reasonably well the deviations from the \(\Lambda\)-CDM power spectrum up to \(k \sim 3h\text{Mpc}^{-1}\). In particular, it captures both the dependence on \(f_{R_0}\) and the impact of the chameleon mechanism. We do not have simulation results on smaller scale, to which we may compare our predictions, and the agreement may deteriorate at higher \(k\). Indeed, on small scales the power spectrum is sensitive to the shape of halo profiles and their mass-concentration relation, which are expected to be modified at some level as compared to \(\Lambda\)-CDM. Then, if these changes are large enough they cannot be neglected as in this paper, if one is interested in small scales. On the other hand, it may be possible to improve our modelization if one could build a reliable model to predict such modifications to halo profiles.

As compared with the PPF approximation introduced in \([52]\), which interpolates between the linear regime, where the modification of gravity is taken into account at the linear level without chameleon effect, and the nonlinear regime where one uses the \(\Lambda\)-CDM prediction, our framework does not introduce additional interpolation parameters. Moreover, the convergence to General Relativity on smaller scales is obtained by explicitly taking into account the chameleon mechanism (at one-loop order in the perturbative regime and exactly in the spherical dynamics used in the one-halo term). Therefore, the rate of convergence is truly governed by this non-linear effect, which depends on the modified gravity model, rather than by an independent parameterization which requires some tuning (on the coefficient \(c_{\text{nl}}\) or the function \(\Sigma^2(k)\) that enter the interpolation \([37, 52]\).

In any case, the comparison with Fig.\(\text{3}\) shows that our simple approach, which combines one-loop perturbation theory with the halo model, is already able to go significantly beyond the perturbative regime. Indeed, the range of the agreement with the simulations increases from \(k \sim 0.2\) to \(k \sim 3h\text{Mpc}^{-1}\) at least, as we go from Fig.\(\text{3}\) to Fig.\(\text{13}\). This is especially important as most of the signal occurs on the mildly nonlinear scales \(k \sim 1h\text{Mpc}^{-1}\). Moreover, smaller, higher nonlinear, scales suffer from other sources of uncertainties, which already appear in the \(\Lambda\)-CDM case, due to the inaccuracy of the halo profiles and concentrations, and to the impact of the baryon physics.

### B. Scalar-tensor models

We show our results for the deviation from \(\Lambda\)-CDM of the nonlinear power spectrum for dilaton models at \(z = 0\) in Fig.\(\text{14}\). Although we only have results from simulations which use the fully nonlinear Klein-Gordon equation \((14)\), as in Fig.\(\text{13}\) for the \(f(R)\) theories, we plot both our “no-screening” and “with-screening” predictions.

Again, the “no-screening” result corresponds to truncating the expansion \((18)\) at \(n = 1\), that is, using the linear approximation in \(\delta \rho\) of the fifth force or the linearized Klein-Gordon equation. This approximation is used for both the perturbative one-loop power spectrum and the spherical collapse threshold \(\delta_L(M)\).

The “with-screening” result solves the exact nonlinear Klein-Gordon equation \((109)\) in the spherical collapse. In the perturbative part, we consider the results obtained when we only include the new quadratic vertex \(\gamma_{2;11}^2\) (in addition to the linear factor \(\epsilon\), or both the quadratic and cubic vertices \(\gamma_{2;11}^2\) and \(\gamma_{2;11}^2\) (higher-order vertices do not contribute at one-loop order). Indeed, as seen in Sec.\(\text{III C 2}\) in contrast with the case of \(f(R)\) theories, the cubic vertex \(\gamma_{2;11}^2\) is not negligible on perturbative scales.

In agreement with the behaviors found in Sec.\(\text{III C}\) at the perturbative level and in Sec.\(\text{IV}\) for the spherical collapse, the comparison of Fig.\(\text{14}\) with Fig.\(\text{13}\) shows that the impact of the screening effect is greater for these dilaton models than for the \(f(R)\) theories. This greatly reduces the deviation of the power spectrum from the \(\Lambda\)-CDM case. We can check that our approach is able to recover this effect and to provide a reasonable match with the numerical simulations. At high \(k\) we tend to underestimate the deviation from \(\Lambda\)-CDM. This may be due to our neglect of modifications to the halo profiles. This discrepancy appears at a larger scale, \(k \sim 1h\text{Mpc}^{-1}\), for the models C4, D3, and D4, which are those where our model fares worse. However, they correspond to very small deviations from the \(\Lambda\)-CDM power spectrum, a few percents at \(k \sim 1h\text{Mpc}^{-1}\), which is at the limit of the accuracy of our modelization and amplifies the errors associated with our approximations (such as keeping NFW profiles). Nevertheless, even in these difficult cases we recover the order of magnitude of the deviation from \(\Lambda\)-CDM and of the screening effect. In particular, we again
significantly extend the range of validity of the analytical predictions, as compared to the one-loop perturbative results shown in Fig. 4, from $k \sim 0.2$ to $k \sim 1\ h\text{Mpc}^{-1}$ (the precise values depend somewhat on the dilaton model). Although we can distinguish the effect of the cubic vertex $\gamma_{2;111}$ on weakly nonlinear scales, its impact remains rather small and could be neglected in view of the overall accuracy of our modelization.

We show our results for symmetron models in Fig. 15 in the same fashion as in Fig. 14. As in Sec. III C 2 we can see on perturbative scales that the screening effect has not converged yet at one-loop order for the cases A2, A3, and to a small extent B2. Indeed, for these cases, on large scales, whereas including the first nonlinear (quadratic) vertex $\gamma_{2;11}$ decreases the deviation from $\Lambda$-CDM as compared to the “no-screening” prediction, including the next (cubic) vertex $\gamma_{2;111}$ over-corrects and yields a larger deviation than the “no-screening” prediction. This leads to an overestimation of the deviation from $\Lambda$-CDM on perturbative scales. To improve the modelization for these difficult cases, it may be necessary to go beyond one-loop order in the perturbative part, and more precisely up to the order where the screening effect is seen to converge. In practice, this requires heavier computations, especially since the time and space integrations do not factorize (in contrast with the $\Lambda$-CDM case where this is true up to a very good approximation). Moreover, the perturbative expansion of the screening effect may not converge very well (for instance, because of the singularity of the coupling functions $\beta_n(a)$ and $\kappa_n(a)$ at $a_s$).

Then, especially for the models A2 and A3 where these effects are the largest, our model gives a spurious oscillation for $\Delta P(k)/P(k)$ at $k \sim 0.4\ h\text{Mpc}^{-1}$ with a significant underestimation of the signal at $k > 1\ h\text{Mpc}^{-1}$. As for the other models, which are reasonably well reproduced by our approach, some of this discrepancy may be due to the changes of halo profiles. As we discuss in Sec. V C below, this underestimation at high $k$ for the $\Lambda$ models is related to the strong effect of the singular boundary $\mathcal{F}(a_s)$ on the behavior of the scalar field $\varphi(x)$ and of the fifth force $F_5$ noticed in Fig. 12. Then, numerical simula-
The contribution from the modification to the two-halo term peaks on weakly non-linear scales, \( k \sim 0.2 h \text{Mpc}^{-1} \), because on very large scales we recover General Relativity whereas on small scales the two-halo term gives a negligible contribution to the full power spectrum. For the same reason, the modification to the one-halo term only plays a role on non-linear scales, \( k \gtrsim 0.5 h \text{Mpc}^{-1} \), where the one-halo term becomes the dominant contribution to the power spectrum. Therefore, systematic perturbative expansions can only describe the modifications to the power spectrum below \( k \lesssim 0.5 h \text{Mpc}^{-1} \) (and actually, slightly below, because as explained in the appendix, our two-halo term already contains some small non-perturbative contributions associated with pancake formation). At higher \( k \), one must rely on more phenomenological approaches as we probe the inner shells of virialized halos. This also means that the theoretical accuracy of the full power spectrum is higher and better controlled for \( k \lesssim 0.5 h \text{Mpc}^{-1} \), down to a few percent \( 27 \), than for \( k \gtrsim 0.5 h \text{Mpc}^{-1} \), where it should be about 10%. However, as seen in Sec. IIIC2 and VB this accuracy, which holds for \( \Lambda \)-CDM-like cosmologies, is not reached in peculiar cases such as some symmetron models because of the screening mechanism. Indeed, we have seen that this involves an additional expansion scheme (as compared with \( \Lambda \)-CDM) which can converge more slowly than the usual expansion in the linear density and velocity fields. This can be the limiting factor of the perturbative approach but as shown in Sec. IIIC2 this can be detected from the comparison between different orders (i.e., as we include successive vertices \( \gamma_{2,11} \)).

The decomposition displayed in Fig. 10 shows that the behavior above \( k \gtrsim 0.5 h \text{Mpc}^{-1} \) is due to the one-halo term, hence to the spherical-collapse threshold \( \delta_L(M) \) studied in Sec. IV because this is the only effect that we include in this regime. At smaller scales, \( k \gtrsim 3 h \text{Mpc}^{-1} \), we can expect modifications to the halo profiles (e.g., to the mass-concentration relation) to come into play \( 27 \). Nevertheless, our results already explain the behaviors found in Figs. 13 and 14 where it is seen that in \( f(R) \) theories with \( |f_{R_0}| \gtrsim 10^{-5} \) the deviation from \( \Lambda \)-CDM of the power spectrum decreases with \( k \) in the range \( 1 < k < 5 h \text{Mpc}^{-1} \), whereas it is roughly constant for the dilaton models. Indeed, as noticed in Sec. IV from the comparison of Figs. 6 and 9 the dilaton screening mechanism is more efficient than the \( f(R) \) chameleon effect (in this regime), and the linear density threshold \( \delta_L(M) \) is significantly lower for these \( f(R) \) models than for these dilaton models, on mass scales \( 10^{14} < M < 10^{16} h^{-1}\text{M}_\odot \). In particular, \( \delta_L(M) \) converges more slowly to the \( \Lambda \)-CDM threshold at large mass for the \( f(R) \) models with \( |f_{R_0}| \gtrsim 10^{-5} \) than for these dilaton models. Then, because of the exponential factor \( e^{-\delta_L(M)^2/(2\sigma_d^2)} \) of the large-mass tail of the halo mass function, the deviation
For each model, we show the contribution from the modification to the two-halo term, \( \Delta \).

**FIG. 16:** Relative deviation from Λ-CDM of the power spectrum in \( k \) around \( k \sim 0.2 h \text{Mpc}^{-1} \), and the contribution from the modification to the one-halo term, \( \Delta P_{1H}/P \) (curves with a peak around \( k \sim 0.2 h \text{Mpc}^{-1} \)), and the contribution from the modification to the one-halo term, \( \Delta P_{2H}/P \) (curves with a peak around \( k \sim 2 h \text{Mpc}^{-1} \) or which keep growing at high \( k \)). We only consider the results with the full chameleon or screening effects.

from Λ-CDM of \( P_{1H}(k) \) grows faster at lower \( k \) (which corresponds to more massive and larger halos) in these \( f(R) \) models. This leads to the faster increase of the deviation from Λ-CDM of the full power spectrum at lower \( k \) in the range \( 1 < k < 5 h \text{Mpc}^{-1} \), until the one-halo contribution becomes subdominant at \( k \lesssim 0.5 h \text{Mpc}^{-1} \). In contrast, for the case \( f(R_0) = -10^{-6} \) the deviation due to the one-halo term keeps increasing with \( k \) in the range \( 1 < k < 5 h \text{Mpc}^{-1} \), because at large mass the linear threshold \( \delta_L(M) \) is very close to the Λ-CDM result, as seen in Fig. 6. Thus, the behavior of the deviation from Λ-CDM in this range of wavenumbers depends on the balance between the increased sensitivity at large mass of the exponential factor \( e^{-\delta_L(M)^2/(2\sigma^2)} \) and the convergence to General Relativity. This cannot be predicted a priori for a given class of models and we must evaluate this effect by explicit computations, as in Fig. 16.

The two lower panels of Fig. 16 also explain part of the discrepancy found for symmetron models in Fig. 15. Indeed, we can see that the one-halo contribution seems too small as compared to the two-halo contribution for the “symmetron-A” models, if we compare with the “symmetron-B” models and the \( f(R) \) and dilaton models. This is due to the very efficient screening effect noticed in Figs. 11 and 12, where we found that at large mass the linear threshold \( \delta_L(M) \) becomes very close to the Λ-CDM prediction. This was due to the lower bound \( \alpha_x \) (or \( \varphi(a_s) \)) which diminishes the range of values of the new degree of freedom \( \alpha(x) \) and greatly reduces the fifth force, as compared to the weak field limit where this constraint is discarded. This is not the case for the “symmetron-B” models (in this regime) because their phase transition takes place earlier, at \( a_s = 0.33 \) instead of 0.5. This gives more room for the new degree of freedom \( \alpha(x) \) within the regular domain \( a_s < \alpha \leq a \), and we checked that their linear threshold \( \delta_L(M) \) is halfway between the Λ-CDM and weak-field results. In other words, the scalar field \( \varphi \) is not so strongly pinned down to the singular value \( \varphi(a_s) \) and a significant fifth force can appear. This explains why our one-halo contribution

\( \Delta P_{1H}/P \) (curves with a peak around \( k \sim 0.2 h \text{Mpc}^{-1} \)), and the contribution from the modification to the one-halo term, \( \Delta P_{2H}/P \) (curves with a peak around \( k \sim 2 h \text{Mpc}^{-1} \) or which keep growing at high \( k \)). We only consider the results with the full chameleon or screening effects.

The two lower panels of Fig. 16 also explain part of the discrepancy found for symmetron models in Fig. 15. Indeed, we can see that the one-halo contribution seems too small as compared to the two-halo contribution for the “symmetron-A” models, if we compare with the “symmetron-B” models and the \( f(R) \) and dilaton models. This is due to the very efficient screening effect noticed in Figs. 11 and 12, where we found that at large mass the linear threshold \( \delta_L(M) \) becomes very close to the Λ-CDM prediction. This was due to the lower bound \( \alpha_x \) (or \( \varphi(a_s) \)) which diminishes the range of values of the new degree of freedom \( \alpha(x) \) and greatly reduces the fifth force, as compared to the weak field limit where this constraint is discarded. This is not the case for the “symmetron-B” models (in this regime) because their phase transition takes place earlier, at \( a_s = 0.33 \) instead of 0.5. This gives more room for the new degree of freedom \( \alpha(x) \) within the regular domain \( a_s < \alpha \leq a \), and we checked that their linear threshold \( \delta_L(M) \) is halfway between the Λ-CDM and weak-field results. In other words, the scalar field \( \varphi \) is not so strongly pinned down to the singular value \( \varphi(a_s) \) and a significant fifth force can appear. This explains why our one-halo contribution
is greater for the “B” models than for the “A” models in the lower panels of Fig. [10] This also explains why we obtained smooth curves in the lower panel of Fig. [10] with a reasonable agreement with the simulations, whereas we obtained a spurious oscillation in the upper panel with a significant underestimation of the power spectrum at high $k$. This discrepancy for the “A” models suggests that in these cases our approach of the spherical collapse dynamics is not sufficient to give an accurate account of the impact of this modified gravity on virialized objects. It is likely that deviations from spherical symmetry and perturbations to the radial density profile break down the fast relaxation towards $\overline{\gamma}(a_s)$ observed in our very symmetric case, because of spatial gradients. This would increase the fifth force and explain the rise with $k$ measured in the simulations for the deviation from $\Lambda$-CDM of the power spectrum. Fortunately, these difficult cases could be detected a priori from Figs. [11] and [12] by looking for the cases where such singular behaviors (sticking to a singular value) occur.

In any case, Fig. [10] shows that it remains useful to consider modified gravity models that, even though not fully general, are well defined at the non-linear level while covering a broad range of models. Indeed, to obtain reliable estimates of the deviations from $\Lambda$-CDM and to assess the range of validity of these results, it is useful to go beyond linear theory and even beyond the perturbative regime. This allows us to take into account key screening mechanisms and to estimate the relative importance of different contributions, which show different degrees of accuracy (depending on whether they derive from systematic perturbative expansions or more phenomenological halo models).

VI. CONCLUSION

In this paper, we have studied semi-analytically the screening mechanisms which are necessary to make modified gravity models consistent with observations on small scales. As these mechanisms rely on the nonlinearity of the equations of motion, it is necessary to go beyond linear theory. We have presented a general approach, using perturbation theory (which applies to large scales) and the spherical collapse dynamics (which allows us to handle mildly nonlinear scales with the help of a halo model), to tackle these effects. Our approach applies to a large class of modified gravity scenarios, including $f(R)$ theories and scalar-tensor theories, such as dilaton and symmetron models.

The new degree of freedom, as compared to the $\Lambda$-CDM case, which may be associated with the fluctuations $\delta R$ of the Ricci scalar (in $f(R)$ theories) or of the new scalar field, $\delta \phi$, in scalar-tensor theories, gives rise to a fifth force which can be written as a new contribution $\Psi_A$ to the gravitational potential. This new field is nonlinearly coupled to the matter density, for instance through a Klein-Gordon equation with an effective potential which depends on $\rho$. Fortunately, in many cases this equation can be simplified by using the quasi-static approximation, so that this new degree of freedom is fully determined by the current density field (and one does not need to keep track of the past history of the field). We have checked that this approximation is valid for the cases that we consider in this paper. It would break down for very singular coupling functions [such as $\beta(a) \sim (a - a_s)^n$ with $n < 0.25$]. Then, the equation for the new degree of freedom takes the form of a constraint equation (i.e., without time derivatives) and implicitly determines the new field, whence the fifth force, as a nonlinear functional of the density field.

First, we have described how to compute the matter density power spectrum within a perturbative approach. One first solves the constraint equation for the new field as a perturbative expansion in the nonlinear density fluctuation $\delta \rho$. This also yields the fifth force potential $\Psi_A$ as a perturbative expansion in powers of $\delta \rho$. Then, using the usual single-stream approximation, which applies on large perturbative scales, one solves the new equations of motion for the density and velocity fields as a second perturbative expansion in powers of the linear density fluctuation $\delta L$. Because the fifth force is a nonlinear, non-polynomial functional of the density field, the Euler equation is no longer quadratic but contains vertices of all orders. In this paper, we have computed the matter density power spectrum up to one-loop order, which corresponds to third order in the fields. At this order, only three new vertices are relevant, $\epsilon$, $\gamma_{2,11}^s$, and $\gamma_{2,111}^s$, which arise from the linear, quadratic, and cubic terms in $\delta \rho$ of $\Psi_A$. The linear vertex $\epsilon$ corresponds to the weak field limit, where we linearize in the fluctuations of the new degree of freedom. The quadratic and cubic vertices $\gamma_{2,11}^s$ and $\gamma_{2,111}^s$ contain the first signs of the screening mechanism.

Thus, for the modified gravity models that we investigate here, the quadratic vertex $\gamma_{2,11}^s$ decreases the deviation from the $\Lambda$-CDM dynamics, as compared with the weak field approximation. In fact, if we use this truncated equation of motion up to high densities, for instance within a spherical collapse study, we find that this quadratic vertex even stops the collapse at finite density. The next nonlinear vertex, the cubic one $\gamma_{2,111}^s$, is a higher-order correction that somewhat diminishes the amplitude of this screening mechanism. We have found that for the $f(R)$ theories, the cubic vertex can actually be safely neglected on the perturbative scales described by one-loop order perturbation theory. For the dilaton models, the cubic vertex makes a small but noticeable correction. For the symmetron models, the situation is less favorable and in some cases, associated with the most singular coupling functions, the screening mechanism has not converged yet at this order, on these large scales. Indeed, it may happen that the cubic vertex $\gamma_{2,111}^s$ overcorrects the screening mechanism and yields a deviation from $\Lambda$-CDM which is larger than the one obtained at linear order. It is not obvious whether the expansion
would show a good convergence at higher orders. In all cases, the one-loop perturbative predictions only applies
to rather large scales, \( k \lesssim 0.15h\text{Mpc}^{-1} \) at \( z = 0 \), where

the signal is not very large.

Next, to go beyond the perturbative regime, we have studied the spherical collapse dynamics. More precisely, we have focused on the linear density threshold \( \delta_L(M) \) that is required to reach a fixed nonlinear density con-
trast of 200, which we use to define virialized halos. Be-
cause the fifth force accelerates the collapse (in the mod-
els studied here), this threshold is lower than the \( \Lambda \text{-CDM} \) prediction, especially for small or moderate masses. We
have also studied in details the behavior of the new degree freedom and the differences between the \( f(R) \), dilaton,
and symmetron models.

As in the perturbative regime, we find that the impact of the screening mechanism is smallest for the \( f(R) \) the-
ories. In particular, for small halos, which corresponds
to small scales at fixed density, the chameleon mecha-
nism is no longer relevant. This is because in this regime
the fifth force is not sensitive to the exact value of the
rescaled field \( \alpha \), which is much smaller than \( \delta \). More pre-
cisely, for small halos the density at the virial radius is
not large enough to overcome the effect of spatial gra-
dients. This prevents the chameleon mechanism to take
place (the field cannot follow the rise of the density field)
and the total gravitational force is equal to the Newto-
nian force, multiplied by a factor 4/3 as in the weak field
limit.

In the dilaton models, the screening mechanism is more
efficient and can become important again for very low mass halos. This is because the fifth force builds up in a
very different manner in scalar-tensor theories as com-
pared to \( f(R) \) models. It is no longer produced by the
integral over smaller radii of the difference between the
new field and the matter density contrast, but by the lo-
cal spatial derivative of the new field. Then, for small
objects at finite density, the new rescaled field \( \alpha \) again
becomes small and flat (because it cannot accommodate
too strong gradients), but instead of a large fifth force
this now yields a small fifth force. The situation is simi-
lar in symmetron models, with an even stronger screen-
ing mechanism because of the singularity of the coupling
functions, which pinpoints the new field \( \alpha \) (close) to the
singular value \( a_s \).

Finally, we have combined the perturbative expansion
with the spherical collapse dynamics to obtain a mod-
elization of the matter power spectrum from large linear
scales to mildly nonlinear scales, using a recently devel-
oped approach which uses the halo model. At this stage,
we cannot expect to describe very small, highly nonlinear
scales, because we neglect the impact of modified gravity
on halo shapes. We again find that our approach fares best for \( f(R) \) theories, where it reproduces both
“no-chameleon” and “with-chameleon” simulations. This
allows us to extend the validity of semi-analytical predic-
tions up to \( k \sim 3h\text{Mpc}^{-1} \) at least, at \( z = 0 \). This is a sig-
nificant improvement over the linear or one-loop pertur-
bative results, which are restricted to \( k \lesssim 0.15h\text{Mpc}^{-1} \).

For the dilaton models, the accuracy of our modelization
depends somewhat on the model, but we usually ob-
tain a reasonable agreement with simulations up to \( k \sim
1h\text{Mpc}^{-1} \). In particular, we capture the impact of the
screening mechanism. On small scales, \( k > 1h\text{Mpc}^{-1} \),
we tend to underestimate the power spectrum. This may
be due to our neglect of any change to the halo profiles.
The amplitude of this discrepancy worsens for models
where the deviations from \( \Lambda \text{-CDM} \) are small, which are
more sensitive to such approximations.

Again, the situation appears most difficult for the symmetron models, especially in those cases where the screening mechanism had not converged at one loop or the spherical collapse of massive halos is governed by the singular value \( \varphi(a_s) \) of the scalar field. There, our predic-
tions can differ from the simulations by a factor 2, in the
range \( k \leq 1h\text{Mpc}^{-1} \). Nevertheless, we still predict the
correct order of magnitude of the deviation of the power
spectrum from the \( \Lambda \text{-CDM} \) one. In more favorable cases,
we obtain a reasonable agreement. Fortunately, the diffi-
cult cases can be detected a priori from the bad behavior
of the perturbative expansion and the impact of the sin-
gular boundary \( \varphi(a_s) \), or from a spurious oscillation in
the prediction for the power spectrum (which is related
to these two problems).

Therefore, we have found that it is possible to build an
efficient semi-analytical modelization of the matter den-
sity power spectrum, which takes into account the non-
linear screening mechanism, for a large class of modified
gravity theories. This is most accurate for the \( f(R) \) the-
ories, where the chameleon effect is moderate (but this
still requires a fully nonlinear analysis). This is also due
to the fact that in these theories the gravitational poten-
tial remains of the same form as the Newtonian one in
the two asymptotic regimes, with a multiplicative fac-
tor of unity on large scales and of 4/3 on small scales.
This is not far from a moderately varying effective New-
ton’s constant, and we can expect the dynamics (e.g.,
halo shapes) to remain similar to the \( \Lambda \text{-CDM} \) ones. Our
model remains valid for scalar-tensor theories, where the
fifth force shows a very different behavior and the screen-
ing mechanism is stronger, except for some symmetron
models associated with singular coupling functions.

Such semi-analytic modelizations, which go beyond lin-
ear theory, are necessary because the linear regime is re-
stricted to very large scales where the deviations from \( \Lambda 
\text{-CDM} \) are small. They allow us to probe a broader range
of scales, up to the mildly non-linear regime where the de-
parture from \( \Lambda \text{-CDM} \) is largest (e.g., for some \( f(R) \) mod-
els) or more significant (especially as smaller scales are
less reliable because of the lower accuracy of theoretical
predictions, for instance because of the impact of baryon
physics). Moreover, we can compare the relative con-
tributions from perturbative and non-perturbative terms
and detect features associated with phase transitions as
in some symmetron models. This allows one to estimate
the validity of the predictions. This is an advantage of
modified gravity models that are fully defined at the non-linear level, while covering a broad range of models.

In order to improve our modelization, it may be useful to consider the impact of modified gravity on halo profiles. However, it is not obvious a priori how to devise a robust analytical approach. This would probably require detailed numerical simulations, to see for instance whether these effects may be described through a small set of parameters or to serve as a guideline for analytical modeling.

In order to handle the problematic cases of some symmetron models, where one-loop perturbation theory has not converged yet, one should devise more efficient methods to take into account the screening mechanism. The spherical collapse itself should also be improved. An accurate treatment accurately the screening mechanism. The spherical collapse itself should also be improved. An accurate treatment of such models, which involve two different phases around a critical density $\rho_s$, would probably require a specific method that explicitly takes into account these two phases.

Another topic would be the study of models where modified gravity models that are fully defined at the non-linear level, while covering a broad range of models.

In this appendix we provide some more details about the halo model and the calculation of the power spectrum in the Zel’dovich approximation by including non-linearities in the distribution of the parallel displacement field, while we keep linear theory for the transverse one. Thus, introducing the rescaled longitudinal displacement $\kappa_\parallel$ and its linear variance,

$$\kappa_\parallel = \frac{\Delta x_\parallel}{\Delta q}, \quad \sigma^2_\parallel = \frac{\sigma^2_\parallel}{(\Delta q)^2},$$

we define its cumulant generating function $\varphi_\parallel(y)$ by

$$\langle e^{-y\kappa_\parallel/\sigma^2_\parallel} \rangle = e^{-\varphi_\parallel(y)/\sigma^2_\parallel},$$

where the average is over the parallel displacements. The ansatz (117) used for $\varphi_\parallel$, which depends on the scale $\Delta q$, agrees with the expansion

$$\varphi_\parallel(y) = y - \frac{y^2}{2} + S_3 \frac{y^3}{6} + \ldots,$$

where $S_3(\Delta q)$ is constructed from

$$S_3(\Delta q) = -\frac{28\pi}{\sigma^2_\parallel} \int_0^\infty dk \frac{P_{1\text{loop}}(k) - P^Z_{1\text{loop}}(k)}{(\Delta q)^2 k^2}$$

$$\times \left[ 2 + \cos(k\Delta q) - 3 \frac{\sin(k\Delta q)}{k\Delta q} \right],$$

and $P_{1\text{loop}}(k)$ is the exact one-loop power spectrum constructed with perturbation theory. This ensures that the
associated power spectrum is exact up to one-loop order and it reads as

\[ P||_\parallel(k) = \int \frac{d\Delta q}{(2\pi)^{3/2}} e^{-\frac{1}{2}k^2(1-\mu^2)\sigma^2_{\parallel}}. \]

(127)

If we truncate \( \varphi || \) at quadratic order, \( \varphi || = y - y^2/2 \), we recover the Zel’dovich power spectrum \((121)\). Thus, the power spectrum \((127)\) is a generalization of the Zel’dovich power spectrum. It is consistent with the exact perturbative expansion up to one-loop order (i.e., \( P^2 \)), whereas the Zel’dovich power spectrum only agrees at linear order, and it also contains some perturbative terms at all higher orders in both Eulerian and Lagrangian spaces (generated through the non-polynomial function \( \varphi || \) and the exponential in Eq.\((127)\)).

The perturbative expressions \((127)\) and \((128)\) do not take into account non-perturbative phenomena such as shell crossings, which can be approximated using a simplified adhesion model whereby particles coalesce when \( \kappa_\parallel < 0 \). This is described by modifying the probability distribution function of \( \kappa_\parallel \),

\[ P_{\parallel}(\kappa_\parallel) = \int_{-\infty}^{\infty} \frac{d y}{2\pi i e^{\kappa_\parallel y}} e^{\frac{1}{2}y^2(1-\mu^2)\sigma^2_{\parallel}}. \]

(128)

To go to highly non-linear scales, we use the halo model and the power spectrum is split over one-halo and two-halo components as in Eq.\((111)\). Then, the probability that two particles belong to the same halo is \( P_{1H}(\Delta q) \)

\[ P_{1H}(\Delta q) = \frac{\int_{\nu_{\Delta q}/2}^{\infty} d\nu f(\nu) (2q_M - \Delta q)^2(4q_M + \Delta q)}{16q_M^3}, \]

(131)

where \( \nu = \delta_L(M)/\sigma_M \) as in Eq.\((113)\) and \( M = 4\pi\rho_M^3/3 \). The linear density contrast \( \delta_L(M) \), which is the one which leads to a halo of non-linear density contrast 200, depends on \( M \) as the modified gravity dynamics is scale dependent. The lower bound of the integral corresponds to the mass enclosed within a radius \( \Delta q/2 \). The probability of belonging to two halos is \( P_{2H} = 1 - P_{1H} \). Finally, the average of the component of the particle displacements which is associated with small-scale virialized motions within halos reads as

\[ \langle e^{i k \cdot \Delta x} \rangle_\Delta q = \left[ \frac{\int_{\nu_{\Delta q}/2}^{\infty} d\nu f(\nu) u_M(k)}{\int_{0}^{\nu_{\Delta q}/2} d\nu f(\nu)} \right]^2, \]

(132)

because we assume that virialized motions within two different halos are uncorrelated. We have defined the Fourier transform of the halo profile as

\[ u_M(k) = \frac{\int d\mathbf{x} e^{-ik\cdot\mathbf{x}} \rho_M(x)}{\int d\mathbf{x} \rho_M(x)}. \]

(133)

where \( M = \int d\mathbf{x} \rho_M(x) \).

Then, the two-halo part \( P_{2H}(k) \) of the power spectrum is given by Eq.\((114)\), where we recognize the “cosmic web” power spectrum \((130)\), to which we have added the factor \( P_{2H} \), to avoid double-counting with the one-halo term, and the small-scale motions factor \((132)\), to take into account the finite width of halos. The one-halo part \( P_{1H}(k) \) is given as usual by Eq.\((112)\), with the counter-term \( W^2 \) associated with mass and momentum conservation, which ensures that \( P_{1H}(k) \propto k^4 \) at low \( k \). Again, this gives a “halo-model” power spectrum \((111)\) which is identical to Eq.\((127)\) at all orders of perturbation theory. In particular, thanks to the choice \((120)\), it agrees with standard perturbation theory up to one-loop order (and contains partial terms at all higher orders, generated through the function \( \varphi ||(y) \), as well as non-perturbative terms of the form \( e^{-1/\sigma^2} \)). These are all the ingredients which are necessary to evaluate the power spectrum in our improved formulation of the halo model.

[1] S. Perlmutter et al. (Supernova Cosmology Project), Astrophys.J. 517, 565 (1999), astro-ph/9812133.

[2] A. G. Riess et al. (Supernova Search Team), Astron.J. 116, 1009 (1998), astro-ph/9805201.
In contrast, if nonlinearities such as $\delta R^2$, without a Laplacian prefactor, entered Eq. (3), this would yield contributions such as $\langle (\delta \rho)^2 \rangle$ for large volume averages which do not vanish, whence a component $\propto |\delta \rho|/\epsilon$, such as a modification of the Hubble expansion rate itself, that is, a backreaction on large scales from the initial (2) and a modification of the Hubble expansion rate.