\textbf{F(T) Models within Bianchi Type I Universe}

M. Sharif * and Shamaila Rani †
Department of Mathematics, University of the Punjab,
Quaid-e-Azam Campus, Lahore-54590, Pakistan.

Abstract

In this paper, we consider spatially homogenous and anisotropic Bianchi type I universe in the context of \( F(T) \) gravity. We construct some corresponding models using conservation equation and equation of state parameter representing different phases of the universe. In particular, we take matter dominated era, radiation dominated era, present dark energy phase and their combinations. It is found that one of the models has a constant solution which may correspond to the cosmological constant. We also derive equation of state parameter by using two well-known \( F(T) \) models and discuss cosmic acceleration.

**Keywords:** \( F(T) \) gravity; Bianchi type I universe; Torsion.

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1 Introduction

Modified theories of gravity have recently gained a lot of interest due to its possible explanation about dark energy (DE). Modern cosmology is in a state of crises in a sense that it started with dark matter and erupted with an indication that most of the universe is made up of DE. The discovery of the accelerated expansion of the universe \cite{1,2,3} indicates that the universe is nearly

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* msharif@math.pu.edu.pk
† shamailatoor.math@yahoo.com
spatially flat and consists of about 74% DE causing cosmic acceleration. This unknown energy (having negative pressure) is physically equivalent to vacuum energy and is almost equally distributed in the universe. It has been used as an essential factor in a recent attempt to formulate a cyclic model of the universe. The universe (in its developing process) passes transiently through the stiff fluid era ($\omega = 1$), the radiation dominated era ($\omega = 1/3$), matter dominated era ($\omega = 0$), transition era ($\omega = -1/3$) and then tends to the DE dominated era ($\omega = -1$). In General Relativity (GR), the simplest and most appealing candidate of DE is the cosmological constant. However, it suffers from two serious theoretical problems, the cosmological constant problem and coincidence problem [4].

In alternative theories, $f(R)$ theory has many applications in cosmology and gravity. This theory can directly be achieved by replacing the Ricci scalar $R$ by $f(R)$ in the Einstein-Hilbert action in GR. The study of the physics of $f(R)$ models is, however, hampered by the complexity of the fourth order field equations in the framework of metric formalism [5]-[14]. Following the same scheme of modification in action, $F(T)$ gravity can be obtained by replacing torsion scalar $T$ by its general function $F(T)$ in the Lagrangian of teleparallel gravity [15]-[21]. It helps to explain the accelerated universe without introducing any DE component [22]-[25].

The $F(T)$ gravity models use the Weitzenbök connection which has no curvature but only torsion. Here torsion is responsible for the accelerated expansion of universe and is formed using four parallel vector fields, called vierbiens [26], which are linearly independent. In this framework, the torsion tensor is formed from the products of the first derivatives of tetrad. An important advantage of this theory is that its field equations are of second order and hence easy to tackle as compared to $f(R)$ theory.

Myrzakulov [27] discussed different $F(T)$ models including scalar fields and gave analytical solutions for the scale factors and scalar fields. The same author studied the relationship between $F(T)$ gravity and $k$-essence [28] and also presented some new models of purely kinetic $k$-essence. Karami and Abdolmaleki [29] obtained equation of state (EoS) parameter of polytropic, standard, generalized and modified Chaplygin gas in this modified gravity scenario. Wu and Yu [30] discussed the two new $F(T)$ models and showed how the crossing of phantom divide line takes place. They also discussed the observational constraints corresponding to these models. Dent et al. [31] discussed $F(T)$ cosmology both at background and perturbed level. They derived expressions for growth factor, stability of this theory and vector-tensor
perturbations. Li et al. [32] explored this modified gravity and local Lorentz invariance and remarked that this theory is not local Lorentz invariant. Yang [33] introduced some new $F(T)$ models and described their physical implications and cosmological behavior. Bengochea [34] investigated the consequences of data sets in this modified gravity. All the above mentioned work have been carried out for the FRW metric.

In this paper, we would reconstruct the $F(T)$ gravity models using Bianchi type $I$ spacetime which is the generalization of FRW metric [35]. This theory becomes equivalent to GR if $F(T)$ is replaced by a constant [31]-[32]. The paper is organized as follows: In next section, we present some basics of teleparallel gravity and the corresponding field equations for Bianchi $I$ are given in section 3. A detailed construction of $F(T)$ gravity models is given using two approaches in section 4. Section 5 is devoted to study the EoS parameter for two particular models and also a discussion on cosmic acceleration is provided. In the last section, we summarize and conclude the results.

2 Preliminaries

In this section, we introduce briefly the teleparallel theory of gravity and its generalization to $F(T)$ theory. The Lagrangian density for teleparallel and $F(T)$ gravity are, respectively, given as follows [32]

\[
L_T = \frac{e}{16\pi G} T, \tag{1}
\]
\[
L_{F(T)} = \frac{e}{16\pi G} F(T), \tag{2}
\]

where $T$ is the torsion scalar, $F(T)$ is a general differentiable function of torsion, $G$ is the gravitational constant and $e = \sqrt{-g}$. Mathematically, the torsion scalar is defined as

\[
T = S^{\mu\nu}_\rho T^\rho_{\mu\nu}, \tag{3}
\]

where $S^{\mu\nu}_\rho$ and torsion tensor $T^\rho_{\mu\nu}$ are given as follows

\[
S^{\mu\nu}_\rho = \frac{1}{2}(K^{\mu\nu}_\rho + \delta^\mu_\rho T^\theta_\theta - \delta^\nu_\rho T^\theta_\theta), \tag{4}
\]
\[
T^\lambda_{\mu\nu} = \Gamma^\lambda_{\nu\mu} - \Gamma^\lambda_{\mu\nu} = h^i_i (\partial_\mu h^i_\nu - \partial_\nu h^i_\mu). \tag{5}
\]
Here \( h^i_{\mu} \) are the components of the non-trivial tetrad field \( h^i \) in the coordinate basis. It is an arbitrary choice to choose the tetrad field related to the metric tensor \( g_{\mu\nu} \) by the following relation

\[
g_{\mu\nu} = \eta_{ij}h^i_{\mu}h^j_{\nu}, \tag{6}\]

where \( \eta_{ij} = diag(1, -1, -1, -1) \) is the Minkowski metric for the tangent space. For a given metric there exist infinite different tetrad fields \( h^i_{\mu} \) which satisfy the following properties:

\[
h^i_{\mu}h^j_{\nu} = \delta^i_j, \quad h^i_{\mu}h^j_{\nu} = \delta^\nu_{\mu}. \tag{7}\]

The procedure for evaluating the tetrad field has been given in many papers [13]-[21]. Notice that the Latin alphabets \((i, j, \ldots = 0, 1, 2, 3)\) will be used to denote the tangent space indices and the Greek alphabets \((\mu, \nu, \ldots = 0, 1, 2, 3)\) to denote the spacetime indices. The contorsion tensor \( K^{\mu\nu\rho} \) is defined as

\[
K^{\mu\nu\rho} = -\frac{1}{2}(T^{\mu\nu\rho} - T^{\nu\mu\rho} - T^{\rho\mu\nu}) \tag{8}\]

which is equal to the difference between Weitzenböck and Levi-Civita connections. The variation of Eq.(2) with respect to the vierbein field leads to the following field equations

\[
[e^{-1}\partial_{\mu}(eS^i_{\mu\nu}) - h^i_{\lambda}T^{\rho\lambda\nu}S_{\rho\mu}]F_T + S^i_{\mu\nu}\partial_{\mu}(T)F_{TT} + \frac{1}{4}h^r_{i}F = \frac{1}{2}\kappa^2 h^i_{\rho}T^\rho_{\nu}. \tag{9}\]

Here \( F_T = \frac{dF}{dT}, \quad F_{TT} = \frac{d^2F}{dT^2}, \quad \kappa^2 = 8\pi G, \quad S^i_{\mu\nu} = h^i_{\rho}S_{\rho\mu\nu} \) with antisymmetric property and \( T_{\mu\nu} \) is the energy-momentum tensor given as

\[
T^\nu_{\rho} = diag(\rho_m, -p_m, -p_m, -p_m), \tag{10}\]

where \( \rho_m \) is the density while \( p_m \) is the pressure of matter inside the universe.

### 3 The Field Equations

The line element for a flat, homogenous and anisotropic Bianchi type I universe is

\[
ds^2 = dt^2 - A^2(t)dx^2 - B^2(t)dy^2 - C^2(t)dz^2, \tag{11}\]
where the scale factors $A$, $B$ and $C$ are functions of cosmic time $t$ only. Using Eqs. (6) and (11), we obtain the tetrad components as follows \[36\]

$$
\begin{align*}
\dot{h}_i^\mu &= \text{diag}(1, A, B, C), \\
\dot{h}_i^\mu &= \text{diag}(1, A^{-1}, B^{-1}, C^{-1})
\end{align*}
$$

which obviously satisfies Eq. (7). Substituting Eqs. (4) and (5) in (3) and using (11), it follows after some manipulation

\[
T = -2 \left( \frac{\dot{A}B}{AB} + \frac{\dot{B}C}{BC} + \frac{\dot{C}A}{CA} \right).
\]

The field equations (9) for $i = 0 = \nu$ and $i = 1 = \nu$ turn out to be

\[
F - 4 \left( \frac{\dot{A}B}{AB} + \frac{\dot{B}C}{BC} + \frac{\dot{C}A}{CA} \right) F_T = 2\kappa^2 \rho_m,
\]

\[14\]

\[
2 \left( \frac{\dot{A}B}{AB} + 2 \frac{\dot{B}C}{BC} + \frac{\dot{C}A}{CA} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C} \right) F_T - 4 \left( \frac{\dot{B}}{B} + \frac{\dot{C}}{C} \right) \\
\times \left[ \left( \frac{\dot{A}}{A} - \frac{\dot{A}^2}{A^2} \right) \left( \frac{\dot{B}}{B} + \frac{\dot{C}}{C} \right) + \left( \frac{\dot{B}}{B} - \frac{\dot{B}^2}{B^2} \right) \left( \frac{\dot{C}}{C} + \frac{\dot{A}}{A} \right) \right. \\
+ \left. \left( \frac{\dot{C}}{C} - \frac{\dot{C}^2}{C^2} \right) \left( \frac{\dot{A}}{A} + \frac{\dot{B}}{B} \right) \right] F_{TT} - F = 2\kappa^2 \rho_m.
\]

The conservation equation takes the form

\[
\dot{\rho}_m + \left( \frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C} \right) (\rho_m + p_m) = 0.
\]

The average scale factor $R$, the mean Hubble parameter $H$ and the anisotropy parameter $\Delta$ of the expansion respectively become

\[
R = (ABC)^{1/3},
\]

\[17\]

\[
H = \frac{1}{3} \left( \frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C} \right) = \frac{\dot{R}}{R},
\]

\[18\]

\[
\Delta = \frac{1}{3} \sum_{i=1}^{3} \left( \frac{H_i}{H} - 1 \right)^2,
\]

\[19\]
where \( H_i \) are the directional parameters in the direction \( x, y \) and \( z \) respectively given as
\[
H_1 = \frac{\dot{A}}{A}, \quad H_2 = \frac{\dot{B}}{B}, \quad H_3 = \frac{\dot{C}}{C}.
\] (20)

It is mentioned here that the isotropic expansion of the universe is obtained for \( \Delta = 0 \) which further depends upon the values of unknown scale factors and parameters involved in the corresponding models [37]-[39]. Equation (13) can be written as
\[
T = -9H^2 + J, \quad J = \frac{\dot{A}^2}{A^2} + \frac{\dot{B}^2}{B^2} + \frac{\dot{C}^2}{C^2}.
\] (21)

implying that
\[
H = \frac{1}{3}\sqrt{J - T}.
\] (22)

If we take \( F(T) = T \), then Eqs. (14) and (15) will reduce to
\[
\rho_m + \rho_T = \frac{1}{2\kappa^2} \left[ -4 \left( \frac{\dot{A}\dot{B}}{AB} + \frac{\dot{B}\dot{C}}{BC} + \frac{\dot{C}\dot{A}}{CA} \right) + T \right],
\] (23)
\[
p_m + p_T = \frac{1}{2\kappa^2} \left[ 2 \left( \frac{\dot{A}\dot{B}}{AB} + 2\frac{\dot{B}\dot{C}}{BC} + \frac{\dot{C}\dot{A}}{CA} + \frac{\ddot{B}}{B} + \frac{\ddot{C}}{C} \right) - T \right],
\] (24)

where \( \rho_T, p_T \) are the torsion contributions given by
\[
\rho_T = \frac{1}{2\kappa^2} \left[ -4 \left( \frac{\dot{A}\dot{B}}{AB} + \frac{\dot{B}\dot{C}}{BC} + \frac{\dot{C}\dot{A}}{CA} \right) (1 - F_T) + T - F \right],
\] (25)
\[
p_T = \frac{1}{2\kappa^2} \left[ 2 \left( \frac{\dot{A}\dot{B}}{AB} + 2\frac{\dot{B}\dot{C}}{BC} + \frac{\dot{C}\dot{A}}{CA} + \frac{\ddot{B}}{B} + \frac{\ddot{C}}{C} \right) (1 - F_T) 
+ 4 \left( \frac{\ddot{B}}{B} + \frac{\dot{C}}{C} \right) \left\{ \left( \frac{\dot{A}}{A} - \frac{\dot{A}^2}{A^2} \right) \left( \frac{\dot{B}}{B} + \frac{\dot{C}}{C} \right) + \left( \frac{\ddot{B}}{B} - \frac{\ddot{B}^2}{B^2} \right) \right. 
\times \left( \frac{\dot{C}}{C} + \frac{\dot{A}}{A} \right) + \left( \frac{\ddot{C}}{C} - \frac{\ddot{C}^2}{C^2} \right) \left( \frac{\dot{A}}{A} + \frac{\dot{B}}{B} \right) \right\} F_{TT} - T + F \right].
\] (26)

The relationship between energy density \( \rho \) and pressure of matter \( p \) is described by EoS, \( p = \omega \rho \), where \( \omega \) is the EoS parameter. For normal,
relativistic and non-relativistic matters, EoS parameter has different corresponding values. Using Eqs.\((14)\) and \((15)\), the EoS parameter is obtained as follows
\[
\omega = -1 + \frac{(4Y - 2E)F_T - 4ZF_{TT}}{-4UF_T + F},
\]
where
\[
E = \frac{\dot{A}\dot{B}}{AB} + 2\frac{\dot{B}\dot{C}}{BC} + \frac{\dot{C}\dot{A}}{CA} + \frac{\ddot{B}}{B} + \frac{\ddot{C}}{C},
\]
\[
Y = \frac{\dot{B}\dot{C}}{BC} + \frac{\ddot{B}}{B} + \frac{\ddot{C}}{C},
\]
\[
Z = \left(\frac{\dot{B}}{B} + \frac{\ddot{C}}{C}\right)\left[\left(\frac{\ddot{A}}{A} - \frac{\dot{A}^2}{A^2}\right)\left(\frac{\dot{B}}{B} + \frac{\dot{C}}{C}\right) + \left(\frac{\ddot{B}}{B} - \frac{\dot{B}^2}{B^2}\right)\right]
\times \left(\frac{\ddot{C}}{C} + \frac{\dot{A}}{A}\right) + \left(\frac{\ddot{C}^2}{C^2}\right)\left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B}\right),
\]
\[
U = \frac{\dot{A}\dot{B}}{AB} + \frac{\dot{B}\dot{C}}{BC} + \frac{\dot{C}\dot{A}}{CA}.
\]
It is mentioned here that the homogenous part of Eq.\((14)\) yields the following solution
\[
F(T) = \frac{c_0}{\sqrt{T}},
\]
where \(c_0\) is an integration constant. Using this equation in Eq.\((15)\), we obtain
\[
p_m = \frac{1}{2\kappa^2} \left(\frac{3M\dot{T}}{2T^2} - \frac{3\dot{H} + J + L}{T} - \frac{1}{2}\right) \frac{c_0}{\sqrt{T}}.
\]
where \(L = \frac{\ddot{B}\dot{C}}{BC} - \frac{\ddot{A}}{A}\) and \(M = \frac{\ddot{B}}{B} + \frac{\ddot{C}}{C}\). We would like to mention here that \(p_m\) vanishes for the FRW spacetime \([22]\).

### 4 Construction of Some \(F(T)\) Models

Here we construct some \(F(T)\) models with different cases of perfect fluid by using two approaches. In the first approach, we use the continuity equation \((16)\) while in the second approach, EoS parameter \((27)\) will be used. As the constituents of the universe are non-relativistic matter, radiations and DE, we consider the corresponding values of \(\omega\) in the following subsections.
4.1 Using Continuity Equation

In this approach, we use the following relation \[40\] for the Bianchi type I universe

\[
\frac{1}{9} \left( \frac{\dot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{C}}{C} \right)^2 = H_0^2 + \frac{\kappa^2 \rho_0}{3ABC},
\]

where \( H_0 \) is the Hubble constant having primary implication in cosmology and \( \rho_0 \) is an integration constant. The value of \( H_0 \) corresponds to the rate at which the universe is expanding today. This equation implies that

\[
(ABC)^{-1} = \frac{3}{\kappa^2 \rho_0} (H^2 - H_0^2). \tag{35}
\]

Using EoS in Eq.(16), it follows that

\[
\frac{\dot{\rho}_m}{\rho_m} + 3H(1 + \omega) = 0. \tag{36}
\]

The components of the universe are described by the terms dark matter and DE. We consider different cases of fluids and their combination to construct corresponding \( F(T) \) models. For example, for relativistic matter, \( \omega = 1/3 \), for non-relativistic matter, it is zero and for DE era, it is equal to \(-1\). [41]

**Case 1 (\( \omega = 0 \))**

This is the case of non-relativistic matter, like cold dark matter (CDM) and baryons. It is well approximated as pressureless dust and called the matter dominated era. Inserting \( \omega = 0 \) in Eq.(36) and using (35), we have

\[
\rho_m = \rho_c (ABC)^{-1} = \frac{3\rho_c}{\kappa^2 \rho_0} (H^2 - H_0^2), \tag{37}
\]

where \( \rho_c \) is an integration constant. In terms of torsion scalar, above equation becomes

\[
\rho_m = \frac{\rho_c}{3\kappa^2 \rho_0} (J - 9H_0^2 - T). \tag{38}
\]

Substitution of this value of \( \rho_m \) in (14) implies that

\[
2TF_T + F = \frac{2\rho_c}{3\rho_0} (J - 9H_0^2 - T) \tag{39}
\]
which has the solution
\[
F(T) = \frac{\rho_c}{3\rho_0} \left( \frac{1}{\sqrt{T}} \int \frac{J}{\sqrt{T}} dT - 18H_0^2 - \frac{2}{3}T \right). \tag{40}
\]

This will have a unique solution if the value of \(J\) is known which corresponds to the unknown scale factors. Thus for matter dominated era, we obtain a model in the form of torsion scalar and Hubble constant.

**Case 2 (\(\omega = 1/3\))**:

Here we consider the relativistic matter, like photons and massless neutrinos, with EoS parameter \(\omega = 1/3\). This case represents the radiation dominated era of the universe. Substituting \(\omega = 1/3\) in Eq.(36) and using (21) and (35), we obtain
\[
\rho_m = \frac{\rho_r}{81\kappa^8\rho_0^4} (J - 9H_0^2 - T)^4, \tag{41}
\]
where \(\rho_r\) is another integration constant. Inserting this \(\rho_m\) in (14), we get
\[
2TF_T + F = \frac{2\rho_r}{81\kappa^6\rho_0^6} (J - 9H_0^2 - T)^4 \tag{42}
\]
implying that
\[
F(T) = \frac{\rho_r}{81\kappa^6\rho_0^4 \sqrt{T}} \int \left( \frac{J}{T^{1/8}} - \frac{9H_0^2}{T^{1/8}} - T^{1/8} \right)^4 dT. \tag{43}
\]
This also depends upon the value of \(J\) as well as torsion scalar and Hubble constant.

**Case 3 (\(\omega = -1\))**:

This case represents the present DE constituting 74% of the universal density. Dark energy is assumed to have a large negative pressure in order to explain the observed acceleration of the universe. It is also termed as energy density of vacuum or cosmological constant \(\Lambda\). Replacing \(\omega = -1\) in Eq.(36), we have
\[
\rho_m = \rho_d, \tag{44}
\]
where \(\rho_d\) is an integration constant. Consequently, Eq.(14) takes the form
\[
2TF_T + F = 2\kappa^2\rho_d \tag{45}
\]
with

$$F(T) = 2\kappa^2 \rho_d.$$  \quad (46)

This turns out to be a constant model which is consistent with cosmological constant.

**Case 4 (Combination of \(\omega = 0\) and \(\omega = 1/3\)):**

Let us now consider the case when the energy density is a combination of two different fluids, the dust fluid and radiations. Adding Eqs. (38) and (41), after simplification, it follows that

$$\rho_m = \frac{1}{3\kappa^2 \rho_0} (J - 9H_0^2 - T)[\rho_c + \frac{\rho_r}{27\kappa^6 \rho_0^3}(J - 9H_0^2 - T)^3].$$  \quad (47)

Substituting this \(\rho_m\) in (14), we get

$$2TF_T + F = \frac{2}{3\rho_0} (J - 9H_0^2 - T)[\rho_c + \frac{\rho_r}{27\kappa^6 \rho_0^3}(J - 9H_0^2 - T)^3]$$  \quad (48)

and hence

$$F(T) = \frac{\rho_c}{3\rho_0} \left( \frac{1}{\sqrt{T}} \int \frac{J}{\sqrt{T}} dT - \frac{18H_0^2}{3} - \frac{2}{3} T \right)$$

$$+ \frac{\rho_r}{81\kappa^6 \rho_0^4 \sqrt{T}} \int \left( \frac{J}{T^{1/8}} - \frac{9H_0^2}{T^{1/8}} - T^{1/8} \right)^4 dT.$$  \quad (49)

**Case 5 (Combination of \(\omega = 0\) and \(\omega = -1\)):**

The combination of EoS parameters for matter dominated era and DE yields

$$\rho_m = \frac{\rho_c}{3\kappa^2 \rho_0} (J - 9H_0^2 - T) + \rho_d.$$  \quad (50)

Inserting in Eq. (14), it follows that

$$2TF_T + F = \frac{2\rho_c}{3\rho_0} (J - 9H_0^2 - T) + 2\kappa^2 \rho_d$$  \quad (51)

yielding

$$F(T) = \frac{\rho_c}{3\rho_0} \left( \frac{1}{\sqrt{T}} \int \frac{J}{\sqrt{T}} dT - 18H_0^2 - \frac{2}{3} T \right) + 2\kappa^2 \rho_d.$$  \quad (52)
Case 6 (Combination of $\omega = -1$ and $\omega = 1/3$):

This case gives the following form of the energy density

$$\rho_m = \rho_d + \frac{\rho_r}{81\kappa^8\rho_0^4}(J - 9H_0^2 - T)^4. \quad (53)$$

Substituting this value in Eq.(14), we get

$$2TF_T + F = 2\kappa^2\rho_d + \frac{2\rho_r}{81\kappa^6\rho_0^4}(J - 9H_0^2 - T)^4 \quad (54)$$

which gives

$$F(T) = 2\kappa^2\rho_d + \frac{\rho_r}{81\kappa^6\rho_0^4\sqrt{T}}\int \left(\frac{J}{T^{1/8}} - \frac{9H_0^2}{T^{1/8}} - T^{1/8}\right)^4 dT. \quad (55)$$

It is mentioned here that the cases 4-6 provide $F(T)$ models for combination of different matters. Normally, the dark matter and DE developed independently. However, there are attempts [42] to include an interaction amongst them so that one can get some insights and see the combined effect of different fluids. Dark matter plays a central role in galaxy evolution and has measurable effects on the anisotropies observed in the cosmic microwave background. Although, matter made a larger fraction of total energy of the universe but its contribution would fall in the far future as DE becomes more dominant. It may provide an interaction between dark matter and DE and can drive transition from an early matter dominated era to a phase of accelerated expansion. Using the same phenomenon, DE and different forms of matter are discussed in the framework of $F(T)$ theory which may help to discuss accelerated expansion of the universe.

4.2 Using EoS Parameter

Here we formulate some $F(T)$ models in a slightly different way. We substitute different values of parameter $\omega$ in Eq.(27) and solve it accordingly. Equation (27) can be written as

$$4ZF_{TT} + [-4U(\omega + 1) + 2E - 4Y]F_T + (\omega + 1)F = 0. \quad (56)$$

Now we construct $F(T)$ models in the following cases.
Case 1:

Putting $\omega = \frac{1}{3}$ in Eq.(56), we have

$$2ZF_{TT} + (E - 2Y - \frac{8}{3}U)F_T + \frac{2}{3}F = 0. \quad (57)$$

This has the following general solution

$$F(T) = c_3 \exp T \left[ \frac{8U - 3E + 6Y + \sqrt{(3E - 8U - 6Y)^2 - 48Z}}{12Z} \right] + c_4 \exp T \left[ \frac{8U - 3E + 6Y - \sqrt{(3E - 8U - 6Y)^2 - 48Z}}{12Z} \right], \quad (58)$$

where $c_3$ and $c_4$ are constants.

Case 2:

Consider the case when pressure is zero, i.e., $\omega = 0$. In this case, Eq.(56) implies

$$4ZF_{TT} + [-4U + 2E - 4Y]F_T + F = 0. \quad (59)$$

It has the following general solution

$$F(T) = c_5 \exp T \left[ \frac{6U - 3E + 6Y + \sqrt{(-6U + 3E - 6Y)^2 - 24Z}}{12Z} \right] + c_6 \exp T \left[ \frac{6U - 3E + 6Y - \sqrt{(-6U + 3E - 6Y)^2 - 24Z}}{12Z} \right], \quad (60)$$

where $c_5$ and $c_6$ are arbitrary constants.

Case 3:

For $\omega = -1$, Eq.(56) becomes

$$4ZF_{TT} + (2E - 4Y)F_T = 0 \quad (61)$$

which has two solutions. The first solution yields $F(T) = c_7$ while the second solution is given by

$$F(T) = c_8 \exp \left[ \left( \frac{2Y - E}{2Z} \right) T \right], \quad (62)$$
where \(c_7\) and \(c_8\) are constants. Equations (68), (60) and (62) represent \(F(T)\) models corresponding to radiation, matter and DE phases respectively. The exponential form of \(F(T)\) models represents a universe which always lies in phantom or non-phantom phase depending on parameters of the models [43].

5 Construction of EoS Parameter and Cosmic Acceleration

In this section, we derive EoS parameter by using two different \(F(T)\) models and also investigate cosmic acceleration. For this purpose, we evaluate \(\rho_m\) and \(p_m\) using the field equations and then construct the corresponding EoS parameter.

5.1 The First Model

Consider the following \(F(T)\) model [22]

\[
F = \alpha T + \frac{\beta}{T},
\]

(63)

where \(\alpha\) and \(\beta\) are positive real constants. Replacing this value of \(F\) in Eqs. (14) and (15), it follows that

\[
2\kappa^2 \rho_m = (-4U + T)\alpha + \beta(1 + 4UT^{-1})T^{-1},
\]

(64)

\[
2\kappa^2 p_m = (2E - T)\alpha - \beta(1 + 2ET^{-1} + 8ZT^{-2})T^{-1}.
\]

(65)

Dividing Eq. (65) by (64), the EoS parameter is obtained as follows

\[
\omega = -1 + \frac{2(Y - U)(\alpha - \beta T^{-2}) - 8Z\beta T^{-3}}{(-4U + T)\alpha + \beta(1 + 4UT^{-1})T^{-1}}.
\]

(66)

Now we discuss this equation for particular values of \(\alpha\) and \(\beta\). For \(\alpha \neq 0, \beta = 0\), we obtain

\[
\omega = -1 + \frac{1}{3} \left(1 - \frac{Y}{U}\right).
\]

(67)

This leads to three different cases of \(\omega\) representing different phases of the evolution of the universe as follows:
• If \( \frac{Y}{U} > 1 \), then \( \omega < -1 \) which corresponds to the phantom accelerating universe.

• When \( \frac{Y}{U} < 1 \), the EoS parameter will be slightly greater than \(-1\) which means that the universe stays in the quintessence region.

• If \( \frac{Y}{U} = 1 \), we obtain a universe whose dynamics is dominated by cosmological constant with \( \omega = -1 \).

It is interesting to mention here that model (67) reduces to GR spatially flat Friedmann equation in the limiting case when anisotropy vanishes. The case \( \alpha = 0, \beta \neq 0 \) does not provide meaningful results.

### 5.2 The Second Model
Assume \( F(T) \) has the form \[ F = \alpha T + \beta T^n, \] (68)
where \( n \) is a positive real number. The corresponding field equations will become

\[
2\kappa^2 \rho_m = (-4U + T)\alpha + \beta(-4UnT^{-1} + 1)T^n, \quad (69)
\]

\[
2\kappa^2 p_m = (2E - T)\alpha + 2En\beta T^{n-1} - 4Z\beta n(n - 1)T^{n-2} - \beta T^n. \quad (70)
\]

Consequently, the EoS parameter takes the form

\[
\omega = -1 - \frac{2(U - Y)\alpha + 2n\beta(U - Y)T^{n-1} + 4Z\beta n(n - 1)T^{n-2}}{(-4U + T)\alpha + \beta(-4UnT^{-1} + 1)T^n}. \quad (71)
\]

The case \( \alpha \neq 0, \beta = 0 \) leads to the same discussion as in the first case. For \( \alpha = 0, \beta \neq 0 \), we have

\[
\omega = -1 + \frac{n}{2n + 1} \left[ 1 - \left\{ \frac{Y}{U} + \frac{Z(n - 1)}{U^2} \right\} \right]. \quad (72)
\]

For any positive real number \( n \), we can discuss as follows:

• When \( \frac{Y}{U} + \frac{Z(n-1)}{U^2} > 1 \), Eq.(72) gives \( \omega < -1 \) which represents the phantom accelerating universe.
• For $\frac{Y}{U} + \frac{Z(n-1)}{U^2} = 1$, we obtain $\omega = -1$ and hence the universe rests in DE era dominated by cosmological constant.

• The case $\frac{Y}{U} + \frac{Z(n-1)}{U^2} < -1$ corresponds to the quintessence era because $\omega > -1$.

Assuming $n = 1$ as a particular case in Eqs.(69) and (70), we have

$$\rho_m = \frac{1}{2\kappa^2}(\alpha + \beta)(-4U + T), \quad (73)$$
$$p_m = \frac{1}{2\kappa^2}(\alpha + \beta)(2E - T). \quad (74)$$

In the following, we discuss the evolution of the scale factor for Bianchi type I universe. For this purpose, we assume

$$p_m = \frac{A_{-1}(T)}{\rho_m} + A_0(T) + A_1(T)p_m \quad (75)$$

such that $A_{-1}$, $A_0$, $A_1$ are constants. Substituting Eqs.(73) and (74) in the above equation, it follows that

$$2E - T = \frac{a}{-4U + T} + b + c(-4U + T), \quad (76)$$

where

$$a = \frac{4\kappa^4A_{-1}}{(\alpha + \beta)^2}, \quad b = \frac{2\kappa^2A_0}{\alpha + \beta}, \quad c = A_1. \quad (77)$$

This equation leads to

$$T = \frac{4U + 2E - b + 8Uc}{2(1 + c)} \pm \frac{1}{2(1 + c)}$$
$$\times \sqrt{b^2 - 4a - 4ac - 4Eb + 4E^2 + 8Ub - 16EU + 16U^2}]^{1/2}. \quad (78)$$

Substituting this value of torsion in Eq.(22), we have

$$H = \frac{1}{3} \left[ J - \frac{4U + 2E - b + 8Uc}{2(1 + c)} \pm \frac{1}{2(1 + c)} \right.$$
$$\times \sqrt{b^2 - 4a - 4ac - 4Eb + 4E^2 + 8Ub - 16EU + 16U^2}]^{1/2}. \quad (79)$$
The correspondingly average scale factor becomes

\[ R = R_0 \exp \left\{ \frac{1}{3} \int \left[ J - \frac{4U + 2E - b + 8Uc}{2(1 + c)} \mp \frac{1}{2(1 + c)} \times \sqrt{b^2 - 4a - 4ac - 4Eb + 4E^2 + 8Ub - 16EU + 16U^2} \right]^{1/2} dt \right\} \]  

(80)

As a special case of model (75), if we take \( A_{-1} \) as a constant while \( A_0 = 0 = A_1 \), we obtain standard Chaplygin gas EoS \([44]\). In this respect, Eqs.(78) and (79) give the following results respectively

\[ T = (E + 2U) \pm \sqrt{(E - 2U)^2 - a} , \]  

(81)

\[ H = \frac{1}{3} \left[ J - (E + 2U) \mp \sqrt{(E - 2U)^2 - a} \right] . \]  

(82)

The average scale factor for chaplygin gas has the form

\[ R = R_0 \exp \left\{ \frac{1}{3} \int \left[ J - (E + 2U) \mp \sqrt{(E - 2U)^2 - a} \right] dt \right\} . \]  

(83)

This represents an exponential expansion which may result a rapid increment between the distance of two non-accelerating observers as compared to the speed of light. As a result, both observers are unable to contact each other. Thus if our universe is forthcoming to a de Sitter universe \([23]\), then we would not be able to observe any galaxy other than our own Milky way system.

6 Summary

In this paper, we have investigated the recently developed \( F(T) \) gravity, where \( T \) is responsible for the cosmic acceleration without DE component. For this purpose, we have taken Bianchi type I spacetime which is one of the simplest models describing anisotropic, spatially homogenous and flat universe. Some \( F(T) \) gravity models have been constructed by using two different approaches. In the first approach, we have used the continuity equation while in the second method, EoS parameter is used. These \( F(T) \) gravity models represent three different phases of the universe exhibiting different values of EoS parameter. The matter, radiation and DE eras respectively correspond to \( \omega = 0 \), \( \omega = 1/3 \) and \( \omega = -1 \).
Matter dominated era describes expansion of the universe filled with non-interacting dust particles while radiation dominated era represents early universe after the hot big bang. The DE case corresponds to the universe dominated by a strong negative pressure causing late-time acceleration. If we consider combination of radiation and matter, we may have more interesting results to study the developing universe. We have also constructed some models by using different combinations of EoS parameter. Also, we have obtained $F(T)$ models in exponential form for some particular values of EoS parameter.

Since the evolution of EoS parameter is one of the biggest efforts in observational cosmology today. We have considered two well-known $F(T)$ models and found the corresponding expression for $\omega$. These have been discussed for some particular values of parameters $\alpha$, $\beta$ which yield fruitful results corresponding to realistic situations. The cosmic acceleration has been discussed by using some $F(T)$ models. We conclude that our universe would approach to de Sitter universe in the infinite future. It is interesting to mention here that one of the $F(T)$ models (case 3) inherits a constant solution which may correspond to the cosmological constant. Notice that the isotropic expansion of the universe is obtained for $\Delta = 0$ which depends upon the values of unknown scale factors and parameters involved in the corresponding models $[37]^{-}[39]$. It is worthwhile to point out here that our results correspond to FRW universe for the special case, i.e., $A(t) = B(t) = C(t) = a(t)$.

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