THE HARMONICITY OF THE REEB VECTOR FIELD ON PARACONTACT METRIC 3-MANIFOLDS

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Abstract. In this paper, we characterize three dimensional paracontact metric manifolds whose Reeb vector field $\xi$ is harmonic. The paper is also a complete study of 3-dimensional paracontact metric $(\tilde{\kappa}, \tilde{\mu}, \tilde{\nu})$-manifolds. We investigated the properties of such manifolds according to the cases $\tilde{\kappa} > -1$, $\tilde{\kappa} = -1$, $\tilde{\kappa} < -1$. Finally examples about paracontact metric $(\kappa, \mu)$-manifolds in 3-dimension are presented.

1. Introduction

Let $(M, g)$ be smooth, oriented, connected pseudo-Riemannian manifold and $(TM, g^S)$ its tangent bundle endowed with the Sasaki metric (also referred to as Kaluza-Klein metric in Mathematical Physics) $g^S$. The energy of the smooth map $V : (M, g) \rightarrow (TM, g^s)$, that is as the energy vector field, is given by

$$E(V) = \frac{1}{2} \int_M (tr_g V^* g^s) dv = \frac{n}{2} vol(M, g) + \frac{1}{2} \int_M \|\nabla V\|^2 dv$$

(assuming $M$ compact, in non-compact case, one works over relatively compact domains). It can be shown that $V : (M, g) \rightarrow (TM, g^s)$ is harmonic map if and only if

$$tr [R(\nabla V, V)] = 0, \quad \nabla^* \nabla V = 0.$$  \hspace{1cm} (1.1)

where

$$\nabla^* \nabla V = \sum_i \varepsilon_i (\nabla e_i \nabla e_i V - \nabla \nabla e_i e_i V)$$

is the rough Laplacian with respect to a pseudo-orthonormal local frame $\{e_1, ..., e_n\}$ on $(M, g)$ with $g(e_i, e_i) = \varepsilon_i = \pm 1$ for all indices $i = 1, ..., n$.

If $(M, g)$ is a compact Riemannian manifold, only parallel vector fields define harmonic maps.

Next, for any real constant $\rho \neq 0$, let $\chi^\rho(M) = \{W \in \chi(M) : \|W\|^2 = \rho\}$. We consider vector fields $V \in \chi^\rho(M)$ which are critical points for the energy functional $E|_{\chi^\rho(M)}$, restricted to vector fields of the same length. The Euler-Lagrange equations of this variational condition yield that $V$ is a harmonic vector field if and only if

$$\nabla^* \nabla V \text{ is collinear to } V.$$  \hspace{1cm} (1.3)

This characterization is well known in the Riemannian case (3, 37, 39). Using same arguments in pseudo-Riemannian case, G.Calvaruso [8] proved that same result is still valid for vector fields of constant length, if it is not lightlike. G.Calvaruso [8] also investigated harmonicity properties for left-invariant vector fields on three-dimensional Lorentzian Lie groups, obtaining several classification results and new

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examples of critical points of energy functionals. G. Calvaruso \cite{9} studied harmonicity properties of vector fields on four-dimensional pseudo-Riemannian symmetric spaces.

A contact metric \((\kappa, \mu)\)-manifold is a contact Riemannian manifold \((M, \varphi, \xi, \eta, g)\) such that the Reeb vector field \(\xi\) belongs to the so-called \((\kappa, \mu)\)-nullity distribution, i.e. the curvature tensor field satisfies the condition

\begin{equation}
R(X, Y)\xi = \kappa (\eta(Y) X - \eta(X) Y) + \mu (\eta(Y) X \eta(Y) - \eta(Y) X hY) + \nu (\eta(Y) \phi hX - \eta(Y) \phi hY),
\end{equation}

for some real numbers \(\kappa\) and \(\mu\), where \(2h\) denotes the Lie derivative of \(\varphi\) in the direction of \(\xi\). This new class of Riemannian manifolds was introduced in \cite{6} as a natural generalization both of the Sasakian condition \(R(X, Y)\xi = \eta(Y) X - \eta(X) Y\) and of those contact metric manifolds satisfying \(R(X, Y)\xi = 0\) which were studied by D. E. Blair in \cite{4}. In \cite{24}, Koufogiorgos and Tsichlias considered contact metric manifolds satisfying \(1.4\) where \(\kappa\) and \(\mu\) are non constant smooth functions. These contact metric manifolds are called generalized \((\kappa, \mu)\)-contact metric manifolds. They showed that in dim \(M > 3\), \(\kappa\) and \(\mu\) must be constant and in dim \(M = 3\) gave an example for which \(\kappa\) and \(\mu\) are not constants; this case is studied further in \cite{25}. In \cite{27}, Koufogiorgos et al. proved the existence of a new class of contact metric manifolds: the so called \((\kappa, \mu, \nu)\)-contact metric manifolds. Such a manifold \(M\) is defined through the condition definition

\begin{equation}
R(X, Y)\xi = \kappa (\eta(Y) X - \eta(X) Y) + \mu (\eta(Y) X \eta(Y) - \eta(Y) X hY) + \nu (\eta(Y) \phi hX - \eta(Y) \phi hY),
\end{equation}

where \(\kappa, \mu\) and \(\nu\) are non constant smooth functions on \(M\). Furthermore, it is shown in \cite{27} that these manifolds exist only in the dimension 3, whereas such a manifold in dimension greater than 3 is a \((\kappa, \mu)\)-contact metric manifold.

A \((2n + 1)\)-dimensional contact metric manifold \((M, \varphi, \xi, \eta, g)\) whose characteristic vector field \(\xi\) is a harmonic vector field is called an \(H\)-contact metric manifold. D. Perrone \cite{33} proved that \((M, \varphi, \xi, \eta, g)\) is \(H\)-contact metric manifold if and only if \(\xi\) is an eigenvector of the Ricci operator, generalizing the same result of J. C. Gonzalez- Davila and L. Vanhecke \cite{19} for \(n = 1\). For example, \(\eta\)-Einstein contact metric manifolds, \(K\)-contact manifolds, \((\kappa, \mu)\)-contact metric manifolds, \((\kappa, \mu)\)-generalized contact metric manifolds and strongly locally \(\varphi\)-symmetric spaces are \(H\)-contact metric manifolds. Koufogiorgos et al. \cite{27} also showed that if \(M\) is a \((\kappa, \mu, \nu)\)-contact metric manifold, then \(M\) is an \(H\)-contact metric manifold. Conversely, if \(M\) is a 3-dimensional \(H\)-contact metric manifold, then \(M\) is a \((\kappa, \mu, \nu)\)-contact metric manifold on an everywhere open and dense subset of \(M\).

\(H\)-(almost) contact metric manifolds have been intensively studied for 2003 years and their geometric properties are well understood (see more details \cite{15}, \cite{27}, \cite{28}, \cite{31}, \cite{34} and \cite{36}).

Recently, in \cite{13}, an unexpected relationship between contact \((\kappa, \mu)\)-spaces and paracontact geometry was found. It was proved (cf. Theorem \cite{24}) below that any (non-Sasakian) contact \((\kappa, \mu)\)-space carries a canonical paracontact metric structure \((\tilde{\varphi}, \xi, \tilde{\eta}, \tilde{g})\) whose Levi-Civita connection satisfies a condition formally similar to \(1.4\)

\begin{equation}
\tilde{R}(X, Y)\xi = \tilde{\kappa} (\eta(Y) X - \eta(X) Y) + \tilde{\mu} (\eta(Y) X \eta(Y) - \eta(Y) X hY)
\end{equation}

where \(2\tilde{h} := \mathcal{L}_\xi \tilde{\varphi}\) and, in this case, \(\tilde{\kappa} = (1 - \mu/2)^2 + \kappa - 2\), \(\tilde{\mu} = 2\).

We recall that paracontact manifolds are smooth manifolds of dimension \(2n + 1\) endowed with a 1-form \(\eta\), a vector field \(\xi\) and a field of endomorphisms of tangent spaces \(\tilde{\varphi}\) such that \(\eta(\xi) = 1\), \(\tilde{\varphi}^2 = I - \eta \otimes \xi\) and \(\tilde{\varphi}\) induces an almost paracomplex structure on the codimension 1 distribution defined by the kernel of \(\eta\) (see §2 for more details). If, in addition, the manifold is endowed with a pseudo-Riemannian metric \(\tilde{g}\) of signature \((n, n + 1)\) satisfying

\[\tilde{g}(\tilde{\varphi}X, \tilde{\varphi}Y) = -\tilde{g}(X, Y) + \eta(X)\eta(Y),\quad d\eta(X, Y) = \tilde{g}(X, \tilde{\varphi}Y)\]
(\(M, \eta\)) becomes a contact manifold and \((\tilde{\varphi}, \xi, \eta, \tilde{g})\) is said to be a paracontact metric structure on \(M\). The study of paracontact geometry was initiated by Kaneyuki and Williams in [21] and then it was continued by many other authors. Very recently a systematic study of paracontact metric manifolds, and some remarkable subclasses like para-Sasakian manifolds, was carried out by Zamkovoy [40]. The importance of paracontact geometry, and in particular of para-Sasakian geometry, has been pointed out especially in the last years by several papers highlighting the interplays with the theory of para-Kähler manifolds and its role in pseudo-Riemannian geometry and mathematical physics (cf. e.g. [1], [2], [10], [17], [18]).

If a \((2n+1)\)-dimensional paracontact metric manifold \(M(\tilde{\varphi}, \xi, \eta, \tilde{g})\) whose curvature tensor satisfies \([1,0]\), it is called \textit{paracontact} \((\tilde{k}, \tilde{\mu})\)-manifold. The class of paracontact \((\tilde{k}, \tilde{\mu})\)-manifolds is very large. It contains para-Sasakian manifolds, as well as those paracontact metric manifolds satisfying \(\tilde{R}(X,Y)\xi = 0\) for all \(X, Y \in \Gamma(TM)\) (recently studied in [31]). But, unlike in the contact Riemannian case, a paracontact \((\tilde{k}, \tilde{\mu})\)-manifold such that \(\tilde{\kappa} = -1\) in general is not para-Sasakian. There are in fact paracontact \((\tilde{k}, \tilde{\mu})\)-manifolds such that \(\tilde{h}^2 = 0\) (which is equivalent to take \(\tilde{\kappa} = -1\)) but with \(\tilde{h} \neq 0\). Another important difference with the contact Riemannian case, due to the non-positive definiteness of the metric, is that while for contact metric \((\kappa, \mu)\)-spaces the constant \(\kappa\) can not be greater than 1, here we have no restrictions for the constants \(\tilde{\kappa}\) and \(\tilde{\mu}\).

Cappelletti Montano et al. [14] showed in fact that there is a kind of duality between those manifolds and contact metric \((\kappa, \mu)\)-spaces. In particular, they proved that, under some natural assumption, any such paracontact metric manifold admits a compatible contact metric \((\kappa, \mu)\)-structure (eventually Sasakian). Moreover, they proved that the nullity condition is invariant under \(D\)-homothetic deformations and determines the whole curvature tensor field completely.

Recently, in [11], Calvaruso and Perrone characterized \(H\)-contact pseudo-Riemannian manifolds. Although contact pseudo-Riemannian manifolds and paracontact metric manifolds have same metric, they have different structures (see [10] and [40] for more details).

As we pointed out there are natural relations beetween contact metric, paracontact and contact Lorentzian sense [10] with 3-dimensional manifolds and \(H\)-(para and Lorentzian) contact manifolds.

These considerations motivate us to study \(H\)-paracontact metric manifolds with dimension 3. §2 is devoted to preliminaries. In §3 we study the common properties of paracontact metric \((\tilde{k}, \tilde{\mu}, \tilde{\nu})\)-manifold (see §2 for definition) for the cases \(\tilde{k} < -1\), \(\tilde{k} = -1\), \(\tilde{k} > -1\). We prove for instance that while the values of \(\tilde{k}\), \(\tilde{\mu}\) and \(\tilde{\nu}\) change, the paracontact metric \((\tilde{k}, \tilde{\mu}, \tilde{\nu})\)-manifold remains unchanged under \(D\)-homothetic deformations. Moreover we prove that in \(\dim M > 3\), paracontact metric \((\tilde{k}, \tilde{\mu}, \tilde{\nu})\)-manifold must be paracontact \((\tilde{k}, \tilde{\mu})\)-manifold. In §4 we prove that a paracontact metric 3-manifold \(M\) is an \(H\)-contact metric manifold if and only if it is a paracontact metric \((\tilde{k}, \tilde{\mu}, \tilde{\nu})\)-manifold on an everywhere open and dense subset of \(M\). We also show that a paracontact metric \((\tilde{k}, \tilde{\mu}, \tilde{\nu})\)-manifold with \(\tilde{k} = -1\) is not necessary para-Sasakian (see the tensor \(\tilde{h}\) has the canonical form (II)). As stated above, this case is important difference with the contact Riemannian case. So we can construct non-trival examples of non-para Sasakian manifold with \(\tilde{k} = -1\). In §5 we give a relation between \((\kappa, \mu, \nu = \text{const.})\)-contact metric manifold with the Boeckx invariant \(I_M = \frac{1-\kappa}{\sqrt{1+\kappa}}\) is constant along the integral curves of \(\xi\) i.e. \(\xi(I_M) = 0\) and paracontact metric \((\tilde{k}, \tilde{\mu}, \tilde{\nu})\)-manifold. In §6 we construct some examples about paracontact metric \((\tilde{k}, \tilde{\mu}, \tilde{\nu})\)-manifolds according to the cases \(\kappa > -1\), \(\kappa = -1\), \(\kappa < -1\).

2. Preliminaries

A differentiable manifold \(M\) of dimension \(2n+1\) is said to be a \textit{contact manifold} if it carries a global 1-form \(\eta\) such that \(\eta \wedge (d\eta)^n \neq 0\). It is well known that then there exists a unique vector field \(\xi\) (called the Reeb vector field) such that \(i_\xi \eta = 1\) and \(i_\xi d\eta = 0\). The \(2n\)-dimensional distribution transversal to the Reeb vector field defined by \(D := \ker(\eta)\) is called the contact distribution. Any contact manifold \((M, \eta)\)
admits a Riemannian metric $g$ and a $(1,1)$-tensor field $\varphi$ such that
\begin{align}
\varphi^2 &= -I + \eta \otimes \xi, \quad \varphi \xi = 0, \quad \eta(X) = g(X, \xi) \\
(\varphi X, \varphi Y) &= g(X, Y) - \eta(X)\eta(Y), \quad g(X, \varphi Y) = d\eta(X, Y),
\end{align}
for any vector field $X$ and $Y$ on $M$. The contact manifold $(M, \eta)$ together with the geometric structure $(\varphi, \xi, \eta, g)$ is then called contact metric manifold (or contact Riemannian manifold). Let $h$ be the operator defined by $h = \frac{1}{2}\mathcal{L} \varphi$, where $\mathcal{L}$ denotes Lie differentiation. The tensor field $h$ vanishes identically if and only if the vector field $\varphi$ is Killing and in this case the contact metric manifold is said to be $K$-contact.

It is well known that $h$ and $\varphi h$ are symmetric operators, and
\begin{align}
\varphi h + h\varphi &= 0, \quad h\xi = 0, \quad \eta \circ h = 0, \quad \text{tr} h = \text{tr} \varphi h = 0,
\end{align}
where $\text{tr} h$ denotes the trace of $h$. Since $h$ anti-commutes with $\varphi$, if $X$ is an eigenvector of $h$ corresponding to the eigenvalue $\lambda$ then $\varphi X$ is also an eigenvector of $h$ corresponding to the eigenvalue $-\lambda$. Moreover, for any contact metric manifold $M$, the following relation holds
\begin{align}
\nabla_X \xi &= -\varphi X - \varphi h X
\end{align}
where $\nabla$ is the Levi-Civita connection of $(M, g)$. If a contact metric manifold $M$ is normal, in the sense that the tensor field $N_\varphi := [\varphi, \varphi] + 2d\eta \otimes \xi$ vanishes identically, then $M$ is called a Sasakian manifold. Equivalently, a contact metric manifold is Sasakian if and only if $R_{XY} \xi = \eta(Y)X - \eta(X)Y$. Any Sasakian manifold is $K$-contact and in dimension 3 the converse also holds (cf. [2]).

As a natural generalization of the above Sasakian condition one can consider contact metric manifolds satisfying
\begin{align}
R(X, Y)\xi &= \kappa(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)hX - \eta(X)hY)
\end{align}
for some real numbers $\kappa$ and $\mu$. \[\text{(2.5)}\] is called $(\kappa, \mu)$-nullity condition. This type of Riemannian manifolds was introduced and deeply studied by Blair, Koufogiorgos and Papantoniou in [6] and a few years earlier by Koufogiorgos for the case $\mu = 0$ [23]. Among other things, they proved the following result.

**Theorem 2.1** ([6]). Let $(M, \varphi, \xi, \eta, g)$ be a contact metric $(\kappa, \mu)$-manifold. Then necessarily $\kappa \leq 1$ and $\kappa = 1$ if and only if $M$ is Sasakian. Moreover, if $\kappa < 1$, the contact metric manifold $M$ admits three mutually orthogonal and integrable distributions $D_h(0)$, $D_h(\lambda)$ and $D_h(-\lambda)$ defined by the eigenspaces of $h$, where $\lambda = \sqrt{1 - \kappa}$.

**Theorem 2.2** ([27]). Let $(M, \varphi, \xi, \eta, g)$ be a $2n + 1$-dimensional contact metric $(\kappa, \mu, \nu)$-manifold. Then the following relations hold, for any $X, Y \in \Gamma(TM)$,
\begin{align}
\nabla h^2 &= (\kappa - 1)\varphi^2 \quad \text{for } \kappa \leq 1, \\
(\nabla_X h)Y - (\nabla_Y h)X &= (1 - \kappa)(\eta(X)\varphi Y - \eta(Y)\varphi X) + 2g(X, \varphi Y)\xi \\
&\quad + (1 - \mu)(\eta(X)\varphi h Y - \eta(Y)\varphi h X) \\
&\quad + \nu(\eta(X)hY - \eta(Y)hX),
\end{align}
\begin{align}
\xi(\kappa) &= 2\nu(\kappa - 1), \quad \xi(\lambda) = \nu(\lambda),
\end{align}
\begin{align}
\nabla h &= \mu h + \nu h, \\
(\nabla_X \varphi h)Y - (\nabla_Y \varphi h)X &= (1 - \kappa)(\eta(Y)X - \eta(X)Y) + (1 - \mu)(\eta(Y)hX - \eta(X)hY) \\
&\quad + \nu(\eta(X)\varphi h Y - \eta(Y)\varphi h X).
\end{align}

The standard contact metric structure on the tangent sphere bundle $T_1 M$ satisfies the $(\kappa, \mu)$-nullity condition if and only if the base manifold $M$ has constant curvature $c$. In this case $\kappa = c(2 - c)$ and $\mu = -2c^2 c$. Other examples can be found in [7].

Before mentioning almost paracontact manifolds, we will give some properties of 3-dimensional contact metric manifolds.
Let $M(\varphi, \xi, \eta, g)$ be a contact metric 3-manifold. Let
\[ U = \{ p \in M \mid h(p) \neq 0 \} \subset M \]
\[ U_0 = \{ p \in M \mid h(p) = 0 \} \subset M \]

That $h$ is a smooth function on $M$ implies $U \cup U_0$ is an open and dense subset of $M$, so any property satisfied in $U_0 \cup U$ is also satisfied in $M$. For any point $p \in U \cup U_0$, there exists a local orthonormal basis $\{e, \varphi e, \xi\}$ of smooth eigenvectors of $h$ in a neighborhood of $p$ (this we call a $\varphi$-basis). On $U$, we put $he = \lambda e$, $h\varphi e = -\lambda\varphi e$, where $\lambda$ is a nonvanishing smooth function assumed to be positive.

**Lemma 2.3** [20]. (see also [12]) On the open set $U$ we have
\[ \nabla e = a\varphi e, \quad \nabla e = b\varphi e, \quad \nabla \varphi e = -c\varphi e + (\lambda - 1)\xi, \]
\[ \nabla \varphi e = -ae, \quad \nabla \varphi \varphi e = -be + (1 + \lambda)\xi, \quad \nabla \varphi \varphi \varphi e = ce, \]
\[ \nabla \xi = 0, \quad \nabla \xi = -(1 + \lambda)\varphi e, \quad \nabla \varphi \varphi \xi = (1 - \lambda)e, \]
\[ \nabla \xi h = -2ah\varphi + \xi(\lambda)s \]

where $a$ is a smooth function,
\[ b = \frac{1}{2\lambda}(\varphi e(\lambda) + A) \quad \text{with} \quad A = \eta(\varphi e) = S(\xi, e), \]
\[ c = \frac{1}{2\lambda}(e(\lambda) + B) \quad \text{with} \quad B = \eta(\varphi \varphi e) = S(\varphi e, \varphi e), \]

and $s$ is the type $(1, 1)$ tensor field defined by $s\xi = 0$, $se = e$ and $s\varphi e = -\varphi e$.

Now we recall the notion of almost paracontact manifold (cf. [21]). An $(2n + 1)$-dimensional smooth manifold $M$ is said to have an almost paracontact structure if it admits a $(1, 1)$-tensor field $\tilde{\varphi}$, a vector field $\xi$ and a 1-form $\eta$ satisfying the following conditions:

(i) $\eta(\xi) = 1$, $\varphi^2 = I - \eta \otimes \xi$,

(ii) the tensor field $\tilde{\varphi}$ induces an almost paracomplex structure on each fibre of $D = \ker(\eta)$, i.e. the $\pm 1$-eigendistributions, $D_{\tilde{\varphi}} := D_{\varphi}(\pm 1)$ of $\tilde{\varphi}$ have equal dimension $n$.

From the definition it follows that $\tilde{\varphi} \xi = 0$, $\eta \circ \tilde{\varphi} = 0$ and the endomorphism $\tilde{\varphi}$ has rank $2n$. When the tensor field $N_{\tilde{\varphi}} := [\tilde{\varphi}, \tilde{\varphi}] - 2d\eta \otimes \xi$ vanishes identically the almost paracontact manifold is said to be normal. If an almost paracontact manifold admits a pseudo-Riemannian metric $\tilde{g}$ such that
\[ \tilde{g}(\tilde{\varphi}X, \tilde{\varphi}Y) = -\tilde{g}(X, Y) + \eta(X)\eta(Y), \]
for all $X, Y \in \Gamma(TM)$, then we say that $(M, \tilde{\varphi}, \xi, \eta, \tilde{g})$ is an almost paracontact metric manifold. Notice that any such a pseudo-Riemannian metric is necessarily of signature $(n, n + 1)$. For an almost paracontact metric manifold, there always exists an orthogonal basis $\{X_1, \ldots, X_n, Y_1, \ldots, Y_n, \xi\}$ such that $\tilde{g}(X_i, X_j) = \delta_{ij}$, $\tilde{g}(Y_i, Y_j) = -\delta_{ij}$ and $Y_i = \tilde{\varphi}X_i$, for any $i, j \in \{1, \ldots, n\}$. Such basis is called a $\tilde{\varphi}$-basis.

If in addition $d\eta(X, Y) = \tilde{g}(X, \tilde{\varphi}Y)$ for all vector fields $X, Y$ on $M$, $(M, \tilde{\varphi}, \xi, \eta, \tilde{g})$ is said to be a paracontact metric manifold. In a paracontact metric manifold one defines a symmetric, trace-free operator $\tilde{h} := \frac{1}{2}L_{\tilde{\varphi}}\tilde{\varphi}$. It is known [40] that $\tilde{h}$ anti-commutes with $\tilde{\varphi}$ and satisfies $\tilde{h}\xi = 0$ and
\[ \tilde{\nabla}\xi = -\tilde{\varphi} + \tilde{\varphi}\tilde{h}, \]

where $\tilde{\nabla}$ is the Levi-Civita connection of the pseudo-Riemannian manifold $(M, \tilde{g})$. Moreover $\tilde{h} \equiv 0$ if and only if $\xi$ is a Killing vector field and in this case $(M, \tilde{\varphi}, \xi, \eta, \tilde{g})$ is said to be a $K$-paracontact manifold. A normal paracontact metric manifold is called a para-Sasakian manifold. Also in this context the para-Sasakian condition implies the $K$-paracontact condition and the converse holds only in dimension 3.
Moreover, in any para-Sasakian manifold

(2.19) \( \tilde{R}(X, Y)\xi = -(\eta(Y)X - \eta(X)Y) \),

holds, but unlike contact metric geometry the condition (2.19) not necessarily implies that the manifold is para-Sasakian. Differentiating \( \tilde{\nabla}_Y \xi = -\tilde{\varphi}Y + \tilde{\varphi}\tilde{h}Y \), we get

(2.20) \( \tilde{R}(X, Y)\xi = -(\tilde{\nabla}_X \tilde{\varphi})Y + (\tilde{\nabla}_Y \tilde{\varphi})X + (\tilde{\nabla}_X \tilde{\varphi})\tilde{h}Y - (\tilde{\nabla}_Y \tilde{\varphi})\tilde{h}X \).

Zamkovoy [40] proved that

(2.21) \( (\tilde{\nabla}_\xi \tilde{h})X = -\tilde{\varphi}X + \tilde{h}^2\tilde{\varphi}X + \tilde{\varphi}\tilde{R}(\xi, X)\xi \).

Moreover, Zamkovoy [40] showed that Ricci curvature \( \tilde{S} \) in the direction of \( \xi \) is given by

(2.22) \( \tilde{S}(\xi, \xi) = -2n + tr\tilde{h}^2 \).

An almost paracontact structure \( (\tilde{\varphi}, \xi, \eta) \) is said to be integrable if \( N_\tilde{\varphi}(X, Y) \in \Gamma(\mathbb{R}\xi) \) whenever \( X, Y \in \Gamma(D) \). For paracontact metric structures, the integrability condition is equivalent to \( \tilde{\nabla}^{pc} \tilde{\varphi} = 0 \) [40].

J. Welyyczko [38] proved that any 3-dimensional paracontact metric manifold is always integrable. So for 3-dimensional paracontact metric manifold, we have

(2.23) \( (\tilde{\nabla}_X \tilde{\varphi})Y = -\tilde{g}(X - \tilde{h}X, Y)\xi + \eta(Y)(X - \tilde{h}X) \)

To finish this section, we recall the following Theorem.

**Theorem 2.4** ([13]). Let \( (M, \varphi, \xi, \eta, g) \) be a non-Sasakian contact metric \((\kappa, \mu)\)-space. Then \( M \) admits a canonical paracontact metric structure \((\tilde{\varphi}, \xi, \eta, \tilde{g})\) given by

(2.24) \( \tilde{\varphi} := \frac{1}{\sqrt{1 - \kappa}}h, \quad \tilde{g} := \frac{1}{\sqrt{1 - \kappa}}d\eta(\cdot, \cdot) + \eta \otimes \eta. \)

### 3. Preliminary results on 2n+1-dimensional paracontact \((\tilde{\kappa}, \tilde{\mu}, \tilde{\nu})\)-manifolds

Theorem 2.4 motivates the following definition.

**Definition 3.1.** A 2n + 1-dimensional paracontact metric \((\tilde{\kappa}, \tilde{\mu}, \tilde{\nu})\)-manifold is a paracontact metric manifold for which the curvature tensor field satisfies

(3.1) \( \tilde{R}(X, Y)\xi = \tilde{\kappa}(\eta(Y)X - \eta(X)Y) + \tilde{\mu}(\eta(Y)\tilde{h}X - \eta(X)\tilde{h}Y) + \tilde{\nu}(\eta(Y)\tilde{\varphi}\tilde{h}X - \eta(X)\tilde{\varphi}\tilde{h}Y) \)

for all \( X, Y \in \Gamma(TM) \), where \( \tilde{\kappa}, \tilde{\mu}, \tilde{\nu} \) are smooth functions on \( M \).

In this section, we discuss some properties of paracontact metric manifolds satisfying the condition (3.1). We start with some preliminary properties.
Lemma 3.2. Let \((M, \varphi, \xi, \eta, \tilde{g})\) be a \(2n + 1\)-dimensional paracontact metric \((\kappa, \mu, \nu)\)-manifold. Then the following identities hold:

\[
\begin{align*}
\tilde{h}^2 &= (1 + \tilde{k})\varphi^2, \\
\tilde{Q}\xi &= 2n\tilde{k}\xi, \\
(\nabla_X \varphi)Y &= -\tilde{g}(X - \tilde{h}X, Y)\xi + \eta(Y)(X - \tilde{h}X), \text{ for } \tilde{k} \neq -1, \\
(\nabla_X \tilde{h})Y - (\nabla_Y \tilde{h})X &= -(1 + \tilde{k})(2\tilde{g}(X, \varphi Y)\xi + \eta(X)\varphi Y - \eta(Y)\varphi X) \\
&\quad + (1 - \tilde{\mu})(\eta(X)\varphi \tilde{h}Y - \eta(Y)\varphi \tilde{h}X) \\
&\quad - \nu(\eta(X)\varphi \tilde{h}Y - \eta(Y)\varphi \tilde{h}X), \\
(\nabla_X \tilde{\varphi})Y - (\nabla_Y \tilde{\varphi})\tilde{h}X &= -(1 + \tilde{k})(\eta(X)Y - \eta(Y)X) \\
&\quad + (1 - \tilde{\mu})(\eta(X)\tilde{h}Y - \eta(Y)\tilde{h}X) \\
&\quad - \nu(\eta(X)\varphi \tilde{h}Y - \eta(Y)\varphi \tilde{h}X), \\
\tilde{R}_\xi \tilde{h}Y &= \tilde{\kappa}(\tilde{g}(X, Y)\xi - \eta(Y)X) + \tilde{\mu}(\tilde{g}(Y, X)\xi - \eta(X)\tilde{h}X) \\
&\quad + \nu(\eta(\varphi \tilde{h}X, Y)\xi - \eta(Y)\varphi \tilde{h}X), \\
\tilde{\xi}(\tilde{k}) &= -2\tilde{\nu}(1 + \tilde{k}).
\end{align*}
\]

for any vector fields \(X, Y\) on \(M\), where \(\tilde{Q}\) denotes the Ricci operator of \((M, \tilde{g})\).

Proof. The proof of (3.2) - (3.6) are similar to that of \cite{14}, Lemma 3.2. The relation (3.8) is an immediate consequence of (3.4), (3.5) and (\nabla_X \varphi)Y - (\nabla_Y \varphi)\tilde{h}X = (\nabla_X \varphi)\tilde{h}Y - (\nabla_Y \varphi)\tilde{h}X + \tilde{\varphi}((\nabla_X \tilde{h})Y - (\nabla_Y \tilde{h})X).

Putting \(\xi\) instead of \(X\) in the relation (3.5), we have (3.7). Using (3.8), (3.9) and (3.7), we obtain

\[
\tilde{\nabla}_\xi \tilde{h}^2 = (\tilde{\nabla}_\xi \tilde{h}) + \tilde{h}(\tilde{\nabla}_\xi \tilde{h} = (\tilde{\mu}\tilde{h} - \tilde{\nu}\tilde{h}) + \tilde{h}(\tilde{\mu}\tilde{h}\varphi - \tilde{\nu}\tilde{h}) = -2\tilde{\nu}(1 + \tilde{k})*2.
\]

Alternately, differentiating (3.2) along \(\xi\) and using (2.29), we obtain

\[
\tilde{\nabla}_\xi \tilde{h}^2 = \xi(\tilde{k})\varphi^2.
\]

Combining (3.10) and (3.11), we complete the proof \(\square\).

Remarkable subclasses of paracontact \((\tilde{k}, \tilde{\mu}, \tilde{\nu})\)-manifolds are given, in view of (2.19), by para-Sasakian manifolds, and by those paracontact metric manifolds such that \(\tilde{R}(X, Y)\xi = 0\) for all vector fields \(X, Y\) on \(M\). Notice that, because of (3.2), a paracontact \((\tilde{k}, \tilde{\mu}, \tilde{\nu})\)-manifold such that \(\tilde{k} = -1\) satisfies \(\tilde{h}^2 = 0\). Unlike the contact metric case, since the metric \(\tilde{g}\) is pseudo-Riemannian we can not conclude that \(\tilde{h}\) vanishes and so the manifold is para-Sasakian.

Given a paracontact metric structure \((\varphi, \xi, \eta, \tilde{g})\) and \(\alpha > 0\), the change of structure tensors

\[
\begin{align*}
\tilde{\eta} &= \alpha\eta, \\
\tilde{\xi} &= \frac{1}{\alpha}\xi, \\
\tilde{\varphi} &= \varphi, \\
\tilde{\tilde{g}} &= \alpha\tilde{g} + \alpha(\alpha - 1)\eta \otimes \eta
\end{align*}
\]

is called a \(D_\alpha\)-homothetic deformation. One can easily check that the new structure \((\varphi, \xi, \tilde{\eta}, \tilde{g})\) is still a paracontact metric structure \(\cite{10}\).

Proposition 3.3. Let \((\varphi, \xi, \tilde{\eta}, \tilde{g})\) be a paracontact metric structure obtained from \((\varphi, \xi, \tilde{\eta}, \tilde{g})\) by a \(D_\alpha\)-homothetic deformation. Then we have the following relationship between the Levi-Civita connections \(\nabla\) and \(\tilde{\nabla}\) of \(\tilde{g}\) and \(\tilde{\tilde{g}}\), respectively,

\[
\tilde{\nabla}_X Y = \nabla_X Y + \frac{\alpha - 1}{\alpha}\tilde{g}(\varphi \tilde{h}X, Y)\xi - (\alpha - 1)(\eta(Y)\varphi X + \eta(X)\varphi Y).
\]
Furthermore,
\begin{equation}
\tilde{h} = \frac{1}{\alpha} \tilde{h}.
\end{equation}

After a long but straightforward calculation one can prove the following proposition.

**Proposition 3.4 (\cite{14}).** Under the same assumptions of Proposition 3.3, the curvature tensor fields \( \tilde{R} \) and \( \hat{R} \) are related by
\begin{equation}
\alpha \tilde{R}(X,Y)\xi = \hat{R}(X,Y)\xi - (\alpha - 1)((\bar{\nabla}_X \tilde{\phi})Y - (\bar{\nabla}_Y \tilde{\phi})X + \eta(Y)(X - \tilde{h}X) - \eta(X)(Y - \hat{h}Y))
\end{equation}
\begin{equation}
- (\alpha - 1)^2(\eta(Y)X - \eta(X)Y)
\end{equation}

Using Proposition 3.4 one can easily get following

**Proposition 3.5.** If \((M, \tilde{\phi}, \tilde{\xi}, \eta, \tilde{g})\) is a paracontact \((\tilde{k}, \tilde{\mu}, \tilde{\nu})\)-manifold, then \((\hat{\phi}, \hat{\xi}, \hat{\eta}, \hat{g})\) is a paracontact \((\bar{k}, \bar{\mu}, \bar{\nu})\)-structure, where
\begin{equation}
\tilde{k} = \frac{\bar{k} + 1 - \alpha^2}{\alpha^2}, \quad \tilde{\mu} = \frac{\bar{\mu} + 2\alpha - 2}{\alpha}, \quad \tilde{\nu} = \frac{\bar{\nu}}{\alpha}.
\end{equation}

Using the same proof of (\cite{14}, Proposition 3.9.) we have

**Corollary 3.6.** Let \((M, \tilde{\phi}, \tilde{\xi}, \eta, \tilde{g})\) be a paracontact metric \((\tilde{k}, \tilde{\mu}, \tilde{\nu})\)-manifold such that \(\tilde{k} \neq -1\). Then the operator \(h\) in the case \(\tilde{k} > -1\) and the operator \(\tilde{\phi}h\) in the case \(\tilde{k} < -1\) are diagonalizable and admit three eigenvalues: 0, associate with the eigenvector \(\xi\), \(\lambda\) and \(-\lambda\), of multiplicity \(n\), where \(\lambda := \sqrt{1 + \tilde{k}}\).

The corresponding eigendistributions \(D_{\tilde{h}}(0) = \mathbb{R}\xi, D_{\tilde{h}}(\lambda), D_{\tilde{h}}(-\lambda)\) and \(D_{\tilde{\phi}h}(0) = \mathbb{R}\xi, D_{\tilde{\phi}h}(\lambda), D_{\tilde{\phi}h}(-\lambda)\) are mutually orthogonal and one has \(\tilde{\phi}D_{\tilde{h}}(\lambda) = D_{\tilde{h}}(-\lambda), \tilde{\phi}D_{\tilde{h}}(-\lambda) = D_{\tilde{h}}(\lambda)\) and \(\tilde{\phi}D_{\tilde{\phi}h}(\lambda) = D_{\tilde{\phi}h}(-\lambda), \tilde{\phi}D_{\tilde{\phi}h}(-\lambda) = D_{\tilde{\phi}h}(\lambda)\). Furthermore,
\begin{equation}
D_{\tilde{h}}(\pm \lambda) = \left\{ X \pm \frac{1}{\sqrt{1 + \tilde{k}}} \tilde{h}X | X \in \Gamma(D^\pm) \right\}
\end{equation}
in the case \(\tilde{k} > -1\), and
\begin{equation}
D_{\tilde{\phi}h}(\pm \lambda) = \left\{ X \pm \frac{1}{\sqrt{1 + \tilde{k}}} \tilde{\phi}hX | X \in \Gamma(D^\pm) \right\}
\end{equation}
where \(D^+\) and \(D^-\) denote the eigendistributions of \(\tilde{\phi}\) corresponding to the eigenvalues corresponding to the eigenvalues 1 and \(-1\), respectively.

In the sequel, unless otherwise stated, we will always assume the index of \(D_{\tilde{h}}(\pm \lambda)\) (in the case \(\tilde{k} > -1\)) and of \(D_{\tilde{\phi}h}(\pm \lambda)\) (in the case \(\tilde{k} < -1\)) to be constant.

Being \(\hat{h}\) (in the case \(\tilde{k} > -1\)) or \(\tilde{\phi}h\) (in the case \(\tilde{k} < -1\)) diagonalizable, one can easily prove the following lemma. Following similar steps in proof of the theorem (\cite{14}, Lemma 3.11), we can give following Lemma.

**Lemma 3.7.** Let \((M, \tilde{\phi}, \tilde{\xi}, \eta, \tilde{g})\) be a paracontact metric \((\tilde{k}, \tilde{\mu}, \tilde{\nu})\)-manifold such that \(\tilde{k} \neq -1\). If \(\tilde{k} > -1\) (respectively, \(\tilde{k} < -1\)), then there exists a local orthogonal \(\tilde{\phi}\)-basis \(\{X_1, \ldots, X_n, Y_1, \ldots, Y_n, \xi\}\) of eigenvectors of \(h\) (respectively, \(\tilde{\phi}h\)) such that \(X_1, \ldots, X_n \in \Gamma(D_{\tilde{h}}(\lambda))\) (respectively, \(Y_1, \ldots, Y_n \in \Gamma(D_{\tilde{\phi}h}(\lambda))\)) and
\begin{equation}
\tilde{g}(X_i, X_j) = -\hat{g}(Y_i, Y_j) = \begin{cases} 
1, & \text{for } 1 \leq i \leq r \\
-1, & \text{for } r + 1 \leq i \leq r + s 
\end{cases}
\end{equation}
where \(r = \text{index}(D_{\tilde{h}}(\lambda))\) (respectively, \(r = \text{index}(D_{\tilde{\phi}h}(\lambda))\)) and \(s = n - r = \text{index}(D_{\tilde{h}}(\lambda))\) (respectively, \(s = \text{index}(D_{\tilde{\phi}h}(\lambda))\)).
Lemma 3.8. The following differential equation is satisfied on every \((2n + 1)\)-dimensional paracontact metric \((\tilde{k}, \tilde{\mu}, \tilde{\nu})\)-manifold \((M, \tilde{\varphi}, \xi, \eta, \tilde{g})\):

\[
(3.20) \quad 0 = \xi(\tilde{k})(\eta(Y)X - \eta(X)Y) + \xi(\tilde{\mu})(\eta(Y)\tilde{h}X - \eta(X)\tilde{h}Y) + \xi(\tilde{\nu})(\eta(Y)\tilde{\varphi}hX - \eta(X)\tilde{\varphi}hY) \\
+ X(\tilde{k})\tilde{\varphi}^2Y - Y(\tilde{k})\tilde{\varphi}^2X + X(\tilde{\mu})\tilde{h}Y - Y(\tilde{\mu})\tilde{h}X + X(\tilde{\nu})\tilde{\varphi}hY - Y(\tilde{\nu})\tilde{\varphi}hX.
\]

Proof. Differentiating (2.5) along an arbitrary vector field \(Z\) and using the relation \(\hat{\nabla}\xi = -\tilde{\varphi} + \tilde{\varphi}h\) we find

\[
\hat{\nabla}_Z \tilde{R}(X, Y)\xi = Z(\tilde{k})(\eta(Y)X - \eta(X)Y) + Z(\tilde{\mu})(\eta(Y)\tilde{h}X - \eta(X)\tilde{h}Y) + Z(\tilde{\nu})(\eta(Y)\tilde{\varphi}hX - \eta(X)\tilde{\varphi}hY) \\
+ \hat{\nabla}_Z \tilde{R}(X, Y)\xi - \tilde{R}(\hat{\nabla}_Z X, Y) - \tilde{R}(\hat{\nabla}_Z Y, X) = \tilde{R}(X, Y)\hat{\nabla}_Z \xi
\]

The use of the last relation, (2.5), and \(\hat{\nabla}\xi = -\tilde{\varphi} + \tilde{\varphi}h\), we deduce

\[
(\hat{\nabla}_Z \tilde{R})(X, Y, \xi) = \tilde{R}(\hat{\nabla}_Z X, Y) - \tilde{R}(\hat{\nabla}_Z Y, X) = \tilde{R}(X, Y)\hat{\nabla}_Z \xi
\]

Using the last relation in second Bianchi identity, we obtain

\[
0 = Z(\tilde{k})(\eta(Y)X - \eta(X)Y) + Z(\tilde{\mu})(\eta(Y)\tilde{h}X - \eta(X)\tilde{h}Y) + Z(\tilde{\nu})(\eta(Y)\tilde{\varphi}hX - \eta(X)\tilde{\varphi}hY) \\
+ X(\tilde{k})(\eta(Z)Y - \eta(Z)Y) + X(\tilde{\mu})(\eta(Z)\tilde{h}Y - \eta(Z)\tilde{h}Z) + X(\tilde{\nu})(\eta(Z)\tilde{\varphi}hY - \eta(Z)\tilde{\varphi}hZ) \\
+ 2\hat{\nabla}(\tilde{k})(\eta(Y)(\tilde{h}X, \tilde{h}Y) - (\tilde{h}Y, \tilde{h}X) + 2\hat{\nabla}(\tilde{\mu})(\eta(Z)(\tilde{h}Y, \tilde{h}Z) - (\tilde{h}Z, \tilde{h}Y) \\
+ 2\hat{\nabla}(\tilde{\nu})(\eta(Z)(\tilde{\varphi}hY, \tilde{h}Z) - (\tilde{h}Z, \tilde{\varphi}hY) + 2\hat{\nabla}(\tilde{\varphi}hY, \tilde{h}Z - (\tilde{h}Z, \tilde{\varphi}hY) \\
+ \tilde{R}(X, Y)\tilde{\varphi}hZ - \tilde{R}(X, Y)\tilde{\varphi}hX - \tilde{R}(Y, Z)\tilde{\varphi}X + \tilde{R}(Z, X)\tilde{\varphi}Y
\]
for all $X, Y, Z \in \Gamma(TM)$. Putting $\xi$ instead of $Z$ in the last relation, we obtain

$$0 = \xi(\check{\kappa})(\eta(Y)X - \eta(X)Y) + \xi(\check{\mu})(\eta(Y)\check{h}X - \eta(X)\check{h}Y) + \xi(\check{\nu})(\eta(Y)\check{\phi}X - \eta(X)\check{\phi}Y) + X(\check{\kappa})Y - \eta(Y)\xi + X(\check{\mu})\check{h}Y + X(\check{\nu})\check{\phi}Y + Y(\check{\kappa})(\eta(X)\xi - X(\check{\mu})\check{h}X - Y(\check{\nu})\check{\phi}X + 2\check{\kappa}\check{g}(\check{h}X, Y)\xi + \check{\mu}[(\check{\nabla}_X\check{h})Y - (\check{\nabla}_Y\check{h})X + \eta(X)((\check{\nabla}_Y\check{\phi})\xi - (\check{\nabla}_X\check{\phi})\xi) + \eta(Y)((\check{\nabla}_Y\check{\phi})\xi - (\check{\nabla}_X\check{\phi})\xi)] + \check{\nu}[(\check{\nabla}_X\check{\phi})Y - (\check{\nabla}_Y\check{\phi})X + \eta(X)((\check{\nabla}_Y\check{\phi})\xi - (\check{\nabla}_X\check{\phi})\xi)] + \eta(Y)((\check{\nabla}_Y\check{\phi})\check{h}X - (\check{\nabla}_X\check{\phi})\check{h}X) + \check{R}(Y, \xi)\check{\phi}X + \check{R}(X, \check{\phi})\check{h}X - \check{R}(\xi, X)\check{\phi}Y.$$

Substituting (3.5), (3.6) and (3.8) in the last relation we finally get (3.20) and it completes the proof. □

**Theorem 3.9.** Let $(M, \check{\phi}, \xi, \eta, \check{g})$ be a $2n + 1$-dimensional paracontact metric $(\check{\kappa}, \check{\mu}, \check{\nu})$-manifold with $\check{\kappa} \neq -1$ and $n > 1$. Then $M$ is a paracontact $(\check{\kappa}, \check{\mu})$-manifold, i.e. $\check{\kappa}, \check{\mu}$ are constants and $\check{\nu}$ is the zero function.

**Proof.** Firstly we assume that $\check{\kappa} < -1$. The Corollary 3.9 and Lemma 3.7 implies the existence of a local orthogonal $\check{\phi}$-basis $\{X_1, \ldots, X_n, \check{\phi}X_1, \ldots, \check{\phi}X_n, \xi\}$ such that $\check{\phi}hX_i = \lambda X_i$, $\check{\phi}\check{h}X_i = -\lambda\check{\phi}X_i$ and $\check{h}\xi = 0, i = 1, \ldots, n$. Substituting $X = X_i$ and $Y = \check{\phi}X_j$ in (3.20), we obtain

$$0 = (\check{\phi}X_i)(\check{\kappa}) + \check{\lambda}X_i(\check{\mu}) + \check{\lambda}(\check{\phi}X_i)(\check{\nu}) = 0,$$

and

$$X_i(\check{\kappa}) - \check{\lambda}(\check{\phi}X_i)(\check{\mu}) = 0.$$  

Finally, substituting $X = \check{\phi}X_i$ and $Y = \check{\phi}X_j$, $(i \neq j)$, in (3.20) we get

$$0 = (\check{\phi}X_i)(\check{\kappa}) - \check{\lambda}(\check{\phi}X_i)(\check{\mu}) = 0.$$  

By virtue of (3.21), (3.22), (3.23) and (3.24) we deduce that

$$X_i(\check{\kappa}) = (\check{\phi}X_i)(\check{\kappa}) = X_i(\check{\mu}) = (\check{\phi}X_i)(\check{\mu}) = X_i(\check{\nu}) = (\check{\phi}X_i)(\check{\nu}) = 0.$$  

Using (3.25) we get $d\check{\mu} = \xi(\check{\mu})\eta$, and so

$$0 = d(d\check{\mu}) = d\xi(\check{\mu}) \wedge \eta + \xi(\check{\mu})d\eta.$$  

Acting (3.26) on the pairs $(X_i, \xi)$ and $(\check{\phi}X_i, \xi)$, respectively, we obtain

$$d\xi(\check{\mu}) = \xi(\check{\mu})\eta.$$  

Substituting (3.27) into (3.26), we get $\xi(\check{\mu}) = 0$, that is, the function $\check{\mu}$ is constant. Arguing in a similar manner one finds that the function $\check{\kappa}$ is constant. Using (3.9) we conclude that $\check{\nu} = 0$.

Next let us suppose that $\check{\kappa} > -1$. By virtue of the Corollary 3.6 and Lemma 3.7 one can construct a local $\check{\phi}$-basis $\{X_1, \ldots, X_n, \check{\phi}X_1, \ldots, \check{\phi}X_n, \xi\}$ such that $\check{h}X_i = \check{\lambda}X_i$, $\check{h}\check{\phi}X_i = -\check{\lambda}\check{\phi}X_i$ and $\check{h}\xi = 0, i = 1, \ldots, n$. Following same procedure, it can be shown that the functions $\check{\kappa}, \check{\mu}$ are constants and $\check{\nu}$ is zero function. □
4. Classification of the 3-dimensional Paracontact metric \((\tilde{\kappa}, \tilde{\mu}, \tilde{\nu})\)-manifolds

A self-adjoint linear operator \(A\) of a Riemannian manifold is always diagonalizable, but this is not the case for a self-adjoint linear operator \(A\) of a Lorentzian manifold. It is known ([35], pp. 50-55) that self-adjoint linear operator of a vector space with a Lorentzian inner product can be put into four possible canonical forms. In particular, the matrix representation \(\tilde{g}\) of the induced metric on \(M^3_1\) is of Lorentz type, so the self-adjoint linear \(A\) of \(M^3_1\) can be put into one of the following four forms with respect to frames \(\{e_1, e_2, e_3\}\) at \(T_p M^3_1\) [29, 30].

(I) 
\[
A = \begin{pmatrix}
\lambda_1 & 0 & 0 \\
0 & \lambda_2 & 0 \\
0 & 0 & \lambda_3
\end{pmatrix}, \quad \tilde{g} = \begin{pmatrix}
-1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix},
\]

(II) 
\[
A = \begin{pmatrix}
\lambda & 0 & 0 \\
1 & \lambda & 0 \\
0 & 0 & \lambda_3
\end{pmatrix}, \quad \tilde{g} = \begin{pmatrix}
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 1
\end{pmatrix},
\]

(III) 
\[
A = \begin{pmatrix}
\lambda & 0 & 0 \\
0 & \lambda & 1 \\
1 & 0 & \lambda
\end{pmatrix}, \quad \tilde{g} = \begin{pmatrix}
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 1
\end{pmatrix},
\]

(IV) 
\[
A = \begin{pmatrix}
\gamma & -\lambda & 0 \\
\lambda & \gamma & 0 \\
0 & 0 & \lambda_3
\end{pmatrix}, \quad \tilde{g} = \begin{pmatrix}
-1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}, \lambda \neq 0.
\]

The matrices \(\tilde{g}\) for cases (I) and (IV) are with respect to an orthonormal basis of \(T_p M^3_1\), whereas for cases (II) and (III) are with respect to a pseudo-orthonormal basis. This is a basis \(\{e_1, e_2, e_3\}\) of \(T_p M^3_1\) satisfying \(\tilde{g}(e_1, e_1) = \tilde{g}(e_2, e_2) = \tilde{g}(e_3, e_3) = 0\) and \(\tilde{g}(e_1, e_2) = \tilde{g}(e_3, e_3) = 1\).

Next, we recall that the curvature tensor of a 3-dimensional pseudo-Riemannian manifold satisfies

\[
(4.1) \quad \tilde{R}(X,Y)Z = \tilde{g}(Y,Z)\tilde{Q}X - \tilde{g}(X,Z)\tilde{Q}Y + \tilde{g}(\tilde{Q}Y, Z)X - \tilde{g}(\tilde{Q}X, Z)Y - \frac{\tau}{2}(\tilde{g}(Y, Z)X - \tilde{g}(X, Z)Y).
\]

The tensor \(\tilde{h}\) has the canonical form (I). Let \((M, \tilde{\varphi}, \xi, \eta, \tilde{g})\) be a 3-dimensional paracontact metric manifold and \(p\) is a point of \(M\). Then there exists a local orthonormal \(\tilde{\varphi}\)-basis \(\{\tilde{e}, \tilde{\varphi}e, \xi\}\) in a neighborhood of \(p\) where \(-\tilde{g}(\tilde{e}, \tilde{e}) = \tilde{g}(\tilde{\varphi}e, \tilde{\varphi}e) = \tilde{g}(\xi, \xi) = 1\). Now, let \(U_1\) be the open subset of \(M\) where \(\tilde{h} \neq 0\) and let \(U_2\) be the open subset of points \(p \in M\) such that \(\tilde{h} = 0\) in a neighborhood of \(p\). \(U_1 \cup U_2\) is an open subset of \(M\). For every \(p \in U_1\) there exists an open neighborhood of \(p\) such that \(\tilde{h}e = \tilde{\lambda}e, \tilde{h}\tilde{\varphi}e = -\tilde{\lambda}\tilde{\varphi}e\) and \(\tilde{h}\xi = 0\) where \(\tilde{\lambda}\) is a non-vanishing smooth function. So the matrix form of \(\tilde{h}\) is given by

\[
\tilde{h} = \begin{pmatrix}
\tilde{\lambda} & 0 & 0 \\
0 & -\tilde{\lambda} & 0 \\
0 & 0 & 0
\end{pmatrix}
\]

with respect to local orthonormal basis \(\{\tilde{e}, \tilde{\varphi}e, \xi\}\). Using same method with [27] and [32], we have
Lemma 4.1. Let \((M, \tilde{\varphi}, \xi, \eta, \tilde{g})\) be a 3-dimensional paracontact metric manifold. Then for the covariant derivative on \(U_1\) the following equations are valid

\[
i) \tilde{\nabla}_{\tilde{e}} \tilde{e} = \frac{1}{2\lambda} \left( \tilde{\sigma}(\tilde{e}) - (\tilde{\varphi}\tilde{e})(\lambda) \right) \tilde{\varphi}\tilde{e}, \quad ii) \tilde{\nabla}_{\tilde{e}} \tilde{\varphi}\tilde{e} = \frac{1}{2\lambda} \left( \tilde{\sigma}(\tilde{e}) - (\tilde{\varphi}\tilde{e})(\lambda) \right) \tilde{e} + (1 - \lambda)\xi, \\
iii) \tilde{\nabla}_{\tilde{e}} \xi = (\lambda - 1)\tilde{\varphi}\tilde{e}, \\
iv) \tilde{\nabla}_{\tilde{\varphi}\tilde{e}} \tilde{e} = -\frac{1}{2\lambda} \left( \tilde{\sigma}(\tilde{\varphi}\tilde{e}) + \tilde{e}(\lambda) \right) \tilde{\varphi}\tilde{e} - (\lambda + 1)\xi, \quad v) \tilde{\nabla}_{\tilde{\varphi}\tilde{e}} \tilde{\varphi}\tilde{e} = \frac{1}{2\lambda} \left( \tilde{\sigma}(\tilde{\varphi}\tilde{e}) + \tilde{e}(\lambda) \right) \tilde{e}, \\
vii) \tilde{\nabla}_{\tilde{\varphi}\tilde{e}} \xi = -\lambda + 1)\tilde{\varphi}\tilde{e}, \\
vii) \tilde{\nabla}_{\tilde{\varphi}\tilde{e}} \tilde{\varphi}\tilde{e} = b\tilde{\varphi}\tilde{e}, \\
ix) [\tilde{e}, \xi] = (\lambda - 1 - b)\tilde{\varphi}\tilde{e}, \quad x) [\tilde{e}, \tilde{\varphi}\tilde{e}] = (\lambda - 1 - b)\tilde{\varphi}\tilde{e}, \\
xii) [\tilde{e}, \tilde{\varphi}\tilde{e}] = \frac{1}{2\lambda} \left( \tilde{\sigma}(\tilde{e}) - (\tilde{\varphi}\tilde{e})(\lambda) \right) \tilde{e} + \frac{1}{2\lambda} \left( \tilde{\sigma}(\tilde{\varphi}\tilde{e}) + \tilde{e}(\lambda) \right) \tilde{\varphi}\tilde{e} + 2\xi,
\]

where

\[
\tilde{\sigma} = \tilde{S}(\xi, \cdot)_{\ker \eta}.
\]

Proposition 4.2. Let \((M, \tilde{\varphi}, \xi, \eta, \tilde{g})\) be a 3-dimensional paracontact metric manifold. On \(U_1\) we have

\[
\tilde{\nabla}_{\tilde{\varphi}\tilde{e}} \tilde{\varphi}\tilde{e} = \frac{1}{2\lambda} \left( \tilde{\sigma}(\tilde{\varphi}\tilde{e}) + \tilde{e}(\lambda) \right) \tilde{\varphi}\tilde{e} + 2\xi,
\]

where \(s\) is the (1,1)-type tensor defined by \(s\xi = 0, \ s\tilde{e} = \tilde{e}, \ s\tilde{\varphi}\tilde{e} = -\tilde{\varphi}\tilde{e}\).

Proof. Using (4.2), we get

\[
(\tilde{\nabla}_{\tilde{\varphi}\tilde{e}} \tilde{\varphi}\tilde{e})\xi = 0 = (-2b\tilde{\varphi}\tilde{e} + \xi(\lambda)s)\xi, \\
(\tilde{\nabla}_{\tilde{\varphi}\tilde{e}} \tilde{\varphi}\tilde{e})\tilde{e} = 2\lambda b\tilde{\varphi}\tilde{e} + \xi(\lambda)\tilde{e} = (-2b\tilde{\varphi}\tilde{e} + \xi(\lambda)s)\tilde{e}, \\
(\tilde{\nabla}_{\tilde{\varphi}\tilde{e}} \tilde{\varphi}\tilde{e})\tilde{\varphi}\tilde{e} = -2\lambda b\tilde{\varphi}\tilde{e} + \xi(\lambda)\tilde{\varphi}\tilde{e} = (-2b\tilde{\varphi}\tilde{e} + \xi(\lambda)s)\tilde{\varphi}\tilde{e}.
\]

Proposition 4.3. Let \((M, \tilde{\varphi}, \xi, \eta, \tilde{g})\) be a 3-dimensional paracontact metric manifold. Then the following equation holds on \(M\).

\[
\tilde{h}^2 - \tilde{\varphi}^2 = \frac{\tilde{S}(\xi, \xi)}{2} \tilde{\varphi}^2
\]

Proof. Using (4.2), we have \(\tilde{S}(\xi, \xi) = 2(\lambda^2 - 1)\). After calculating the statement of \(\tilde{h}^2 - \tilde{\varphi}^2\) with respect to the basis components, we get

\[
\tilde{h}^2\xi - \tilde{\varphi}^2\xi = \frac{\tilde{S}(\xi, \xi)}{2} \tilde{\varphi}^2\xi = 0, \quad \tilde{h}^2\tilde{e} - \tilde{\varphi}^2\tilde{e} = \frac{\tilde{S}(\xi, \xi)}{2} \tilde{\varphi}^2\tilde{e}, \quad \tilde{h}^2\tilde{\varphi}\tilde{e} - \tilde{\varphi}^2\tilde{\varphi}\tilde{e} = \frac{\tilde{S}(\xi, \xi)}{2} \tilde{\varphi}^2\tilde{\varphi}\tilde{e}
\]

Thus the last equation completes the proof of (4.4).

Lemma 4.4. Let \((M, \tilde{\varphi}, \xi, \eta, \tilde{g})\) be a 3-dimensional paracontact metric manifold. Then the Ricci operator \(\tilde{Q}\) is given by

\[
\tilde{Q} = a_1I + b_1\eta \otimes \xi - \tilde{\varphi}(\tilde{\nabla}_{\tilde{\varphi}\tilde{e}} \tilde{\varphi}\tilde{e}) + \tilde{\sigma}(\tilde{\varphi}\tilde{e}) \otimes \xi - \tilde{\sigma}(\tilde{e})\eta \otimes \tilde{e} + \tilde{\sigma}(\tilde{\varphi}\tilde{e})\eta \otimes \tilde{\varphi}\tilde{e}
\]

where \(a_1\) and \(b_1\) are smooth functions defined by \(a_1 = 1 - \lambda^2 + \frac{\varphi^2}{2}\) and \(b_1 = 3(\lambda^2 - 1) - \frac{\varphi^2}{2}\), respectively. Moreover the components of the Ricci operator \(\tilde{Q}\) are given by
\[
\begin{align*}
    \tilde{Q}\xi &= (a_1 + b_1)\xi - \tilde{\sigma}(\tilde{e})\tilde{e} + \tilde{\sigma}(\tilde{\varphi}\tilde{e})\tilde{\varphi}\tilde{e} \\
    \tilde{Q}\tilde{e} &= \tilde{\sigma}(\tilde{e})\xi + (a_1 - 2b\lambda)\tilde{e} - \xi(\tilde{\lambda})\tilde{\varphi}\tilde{e} \\
    \tilde{Q}\tilde{\varphi}\tilde{e} &= \tilde{\sigma}(\tilde{\varphi}\tilde{e})\xi + \xi(\tilde{\lambda})\tilde{e} + (a_1 + 2b\lambda)\tilde{\varphi}\tilde{e}
\end{align*}
\] (4.7)

**Proof.** From (4.1), we have

\[
\tilde{i}X = \tilde{R}(X,\xi)\xi = \tilde{S}(\xi,\xi)X - \tilde{S}(X,\xi)\xi + \tilde{Q}X - \eta(X)\tilde{Q}\xi - \frac{r}{2}(X - \eta(X)\xi),
\]

for any vector field \(X\). Using (2.21), the last equation implies

\[
\tilde{Q}X = -\tilde{\varphi}^2X + \tilde{h}^2X - \tilde{\varphi}(\tilde{\nabla}_\xi\tilde{h})X - \tilde{S}(\xi,\xi)X + \tilde{S}(X,\xi)\xi + \eta(X)\tilde{Q}\xi + \frac{r}{2}(X - \eta(X)\xi).
\]

Since \(\tilde{S}(\xi,\xi) = \tilde{S}(\tilde{\varphi}^2X,\xi) + \eta(X)\tilde{S}(\xi,\xi)\), we have

\[
(4.8) \quad \tilde{Q}X = \frac{\tilde{S}(\xi,\xi)}{2} - \tilde{\varphi}^2X + \tilde{\varphi}(\tilde{\nabla}_\xi\tilde{h})X - \tilde{S}(\xi,\xi)X + \tilde{S}(\tilde{\varphi}^2X,\xi) + \eta(X)\tilde{S}(\xi,\xi) + \eta(X)\tilde{Q}\xi + \frac{r}{2}\tilde{\varphi}^2X.
\]

One can easily prove that

\[
(4.9) \quad \tilde{Q}\xi = -\tilde{\sigma}(\tilde{e})\tilde{e} + \tilde{\sigma}(\tilde{\varphi}\tilde{e})\tilde{\varphi}\tilde{e} + \tilde{S}(\xi,\xi)\xi.
\]

Using (4.9) in (4.8), we have

\[
(4.10) \quad \tilde{Q}X = \left(1 - \tilde{\lambda}^2 + \frac{r}{2}\right)X + \left(3(\tilde{\lambda}^2 - 1) - \frac{r}{2}\right)\eta(X)\xi - \tilde{\varphi}(\tilde{\nabla}_\xi\tilde{h})X + \tilde{\varphi}(\tilde{\varphi}^2X)\xi - \eta(X)\tilde{\sigma}(\tilde{e})\tilde{e} + \eta(X)\tilde{\sigma}(\tilde{\varphi}\tilde{e})\tilde{\varphi}\tilde{e},
\]

for arbitrary vector field \(X\). Hence, proof comes from (1.10). By (4.3) and (4.10) we get (4.7). \(\square\)

**Theorem 4.5.** A 3-dimensional paracontact metric manifold is an \(H\)-paracontact manifold if and only if the characteristic vector field \(\xi\) is an eigenvector of the Ricci operator.

**Proof.** From (1.2) we have

\[
\tilde{\nabla}\tilde{\nabla}\xi = -\tilde{\nabla}_\xi\tilde{\nabla}\xi + \tilde{\nabla}\tilde{\varphi}_\xi\epsilon\xi + \tilde{\nabla}\tilde{\varphi}_\xi\tilde{\varphi}\epsilon\xi - \tilde{\nabla}\tilde{\varphi}_\xi\tilde{\varphi}\epsilon\xi
\]

\[
= -\tilde{\sigma}(\tilde{e})\tilde{e} + \tilde{\sigma}(\tilde{\varphi}\tilde{e})\tilde{\varphi}\tilde{e} + 2(\tilde{\lambda}^2 + 1)\xi.
\]

By using (1.3) we obtain \(\tilde{\sigma}(\tilde{e}) = \tilde{\sigma}(\tilde{\varphi}\tilde{e}) = 0\). \(\square\)

**Theorem 4.6.** Let \((M, \tilde{\varphi}, \xi, \eta, \tilde{g})\) be a 3-dimensional \(H\)-paracontact metric manifold. Then the \((\tilde{\kappa}, \tilde{\mu}, \tilde{\nu})\)-manifold always exists on every open and dense subset of \(M\) and also \(\tilde{\kappa} > -1\).

**Proof.** Since \(M\) is a \(H\)-paracontact metric manifold, \(\xi\) is an eigenvector of \(\tilde{Q}\). Hence we obtain that \(\tilde{\sigma} = 0\). Putting \(s = \frac{1}{\tilde{h}}\tilde{h}\) in (4.10) we have

\[
(4.11) \quad \tilde{Q} = a_1I + b_2\eta \otimes \xi - 2b\tilde{h} - \frac{\xi(\tilde{\lambda})}{\lambda}\tilde{\varphi}\tilde{h}.
\]

Setting \(Z = \xi\) in (1.1) and using (1.11), we obtain

\[
\tilde{R}(X,Y)\xi = (\tilde{\lambda}^2 - 1)(\eta(Y)X - \eta(X)\tilde{h}X - \eta(X)\tilde{h}Y) - \frac{\xi(\tilde{\lambda})}{\lambda}(\eta(Y)\tilde{\varphi}\tilde{h}X - \eta(X)\tilde{\varphi}\tilde{h}Y),
\]

where the functions \(\tilde{\kappa}, \tilde{\mu}\) and \(\tilde{\nu}\) defined by \(\tilde{\kappa} = \frac{\tilde{S}(\xi,\xi)}{2}, \tilde{\mu} = -2b, \tilde{\nu} = -\frac{\xi(\tilde{\lambda})}{\lambda}\), respectively. So, it is obvious that for this type \(\tilde{\kappa} > -1\). This completes proof of the Theorem. \(\square\)
Moreover, using \([1.11]\), we have
\[
\hat{\mathcal{Q}} \varphi - \tilde{\varphi} \hat{\mathcal{Q}} = 2 \tilde{\mu} \hat{\varphi} - 2 \tilde{\nu} \hat{h}.
\]

**The tensor \( \hat{h} \) has the canonical form (II).** Let \((M, \hat{\varphi}, \xi, \eta, \tilde{\gamma})\) be a 3-dimensional paracontact metric manifold and \(p\) is a point of \(M\). Then there exists a local pseudo-orthonormal basis \(\{e_1, e_2, e_3\}\) in a neighborhood of \(p\) where \(\tilde{g}(e_1, e_1) = \hat{g}(e_2, e_2) = \tilde{g}(e_1, e_3) = \hat{g}(e_2, e_3) = 0\) and \(\hat{g}(e_1, e_2) = \tilde{g}(e_3, e_3) = 1\).

**Lemma 4.7.** Let \(U\) be the open subset of \(M\) where \(\hat{h} \neq 0\). For every \(p \in U\) there exists an open neighborhood of \(p\) such that \(h_1 = e_1, h_2 = 0, h_3 = 0\) and \(\varphi e_1 = \pm e_1, \varphi e_2 = \mp e_2, \varphi e_3 = 0\) and also \(\xi = e_3\).

**Proof.** Since the tensor has canonical form (II) (with respect to a pseudo-orthonormal basis \(\{e_1, e_2, e_3\}\)) then \(h_1 = \lambda e_1 + e_2, h_2 = \lambda e_2, h_3 = \lambda e_3\). The characteristic vector field \(\xi\) with respect to the pseudo-orthonormal basis \(\{e_1, e_2, e_3\}\) can be written as \(\xi = a_{11} e_1 + a_{12} e_2 + a_{13} e_3 \neq 0\), for some smooth functions \(a_{11}, a_{12}, a_{13}\). Since \(h \xi = 0\), we have \(0 = a_{11} \lambda e_1 + (a_{11} + a_{12} \lambda) e_2 - 2 \lambda a_{13} e_3\). It is obvious that \(a_{11} \lambda = a_{11} + a_{12} \lambda = \lambda a_{13} = 0\). In this case, there are two possibilities. The first one is \(\lambda = 0\) and the second one is \(\lambda \neq 0\). In the second case, we find \(a_{11} = a_{12} = a_{13} = 0\) which is a contradiction of chosen \(\xi\). So the second one is impossible. Hence \(\lambda\) must be 0 and \(\xi = a_{12} e_2 + a_{13} e_3\). From \(\tilde{g}(\xi, \xi) = 1 = a_{13}^2\), it follows that \(a_{13} = \pm 1\), thus \(\xi = a_{12} e_2 \pm e_3\). On the other hand using the anti-symmetry tensor field \(\varphi\) of type \((1, 1)\), with respect to the pseudo-orthonormal basis \(\{e_1, e_2, e_3\}\), takes the form

\[
\begin{pmatrix}
\varphi_{11} & 0 & \varphi_{13} \\
0 & \varphi_{22} & \varphi_{23} \\
-\varphi_{23} & -\varphi_{13} & 0
\end{pmatrix}
\]

By using \(\varphi \xi = 0\) one can easily obtain that \(\varphi_{13} = 0\) and \(a_{12} \varphi_{22} \pm \varphi_{23} = 0\). Now we will prove that \(a_{12} = 0\). We suppose that \(a_{12} \neq 0\). In this case, using \((2.17)\) and \((4.12)\) we have

\[
0 = \tilde{g}(\varphi e_1, \varphi e_1) = -\tilde{g}(e_1, \eta e_1) \eta(e_1) = (\varphi_{23})^2.
\]

From last relation, we obtain \(a_{12} \varphi_{22} = 0\). Since \(a_{12} \neq 0\) we have \(\varphi_{22} = 0\) and therefore the tensor field \(\varphi\) with respect to the pseudo-orthonormal basis takes the form

\[
\begin{pmatrix}
\varphi_{11} & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
\]

Due to \((2.17)\) and \((4.13)\)

\[
0 = \tilde{g}(\varphi e_1, \varphi e_2) = -\tilde{g}(e_1, e_2) + \eta(e_1) \eta(e_1) = -1.
\]

This is a contradiction. The coefficient \(a_{12}\) should be zero. Hence we have

\[
\varphi e_1 = \varphi_{11} e_1, \varphi e_2 = -\varphi_{11} e_2 \text{ and } \xi = \pm e_3
\]

By \(\varphi^2 = I - \eta \otimes \xi\), one can easily obtain \(\varphi_{11} = \pm 1\). On the other hand, since \(0 = tr \hat{h} \) and \(h e_3 = 0\) an easy computation shows that \(\lambda = 0 = \lambda_3\). This completes the proof.

**Remark 4.8.** In this case, we will suppose that \(\varphi e_1 = e_1, \varphi e_2 = -e_2\). Moreover one can easily get \(\hat{h}^2 = 0\) but \(\hat{h} \neq 0\).
Lemma 4.9. Let \((M, \tilde{\varphi}, \xi, \eta, \tilde{g})\) be a 3-dimensional paracontact metric manifold. Then for the covariant derivative on \(U\) the following equations are valid

\[
\begin{align*}
    i) \quad \tilde{\nabla}_{e_1} e_1 &= -b_2 e_1 + \xi, \\
    ii) \quad \tilde{\nabla}_{e_1} e_2 &= b_2 e_2 + \xi, \\
    iii) \quad \tilde{\nabla}_{e_2} \xi &= -e_1 - e_2, \\
    iv) \quad \tilde{\nabla}_{e_2} e_1 &= -b_2 e_1 - \xi, \\
    v) \quad \tilde{\nabla}_{e_2} e_2 &= b_2 e_2, \\
    vi) \quad \tilde{\nabla}_{e_3} \xi &= e_2, \\
    vii) \quad \tilde{\nabla}_{\xi} e_1 &= a_2 e_1, \\
    viii) \quad \tilde{\nabla}_{\xi} e_2 &= -a_2 e_2, \\
    \text{ix}) \quad [e_1, \xi] &= -(1 + a_2) e_1 - e_2, \\
    \text{x}) \quad [e_2, \xi] &= (1 + a_2) e_2,
\end{align*}
\]

(4.14)

where \(\tilde{b}_2 = -\frac{1}{4} \tilde{\sigma}(e_1) = -\frac{1}{4} \tilde{S}(\xi, e_1)\).

Proof. Replacing \(X\) by \(e_1\) and \(Y\) by \(e_2\) in equation \(\tilde{\nabla}\xi = -\tilde{\varphi} + \tilde{\varphi}\tilde{h}\), we have iii), vii).

For the proof of viii) we have

\[
\tilde{\nabla}_{\xi} e_2 = \tilde{g}(\tilde{\nabla}_{\xi} e_2, e_2)e_1 + \tilde{g}(\tilde{\nabla}_{\xi} e_2, e_1)e_2 + \tilde{g}(\tilde{\nabla}_{\xi} e_2, \xi)\xi
\]

\[
= -\tilde{g}(e_2, \tilde{\nabla}_{\xi} e_1)e_1,
\]

If the function \(a_2\) is defined as \(\tilde{g}(\tilde{\nabla}_{e_2} e_1, e_2)\) then \(\tilde{\nabla}_{e_2} e_2 = -a_2 e_2\). The proofs of other covariant derivative equalities are similar to ii).

Putting \(X = e_1\), \(Y = e_2\) and \(Z = \xi\) in the equation (4.1), we have

\[
(4.15)
\]

\[\tilde{R}(e_1, e_2)\xi = -\tilde{\sigma}(e_1)e_2 + \tilde{\sigma}(e_2)e_1.\]

On the other hand, by using (2.20), we get

\[
\tilde{R}(e_1, e_2)\xi = (\tilde{\nabla}_{e_1} \tilde{\varphi}\tilde{h})e_2 - (\tilde{\nabla}_{e_2} \tilde{\varphi}\tilde{h})e_1,
\]

(4.16)

Comparing (4.16) with (4.15), we obtain

\[\tilde{\sigma}(e_1) = -2\tilde{b}_2, \quad \tilde{\sigma}(e_2) = 0 = \tilde{S}(\xi, e_2).\]

Hence, the function \(\tilde{b}_2\) is obtained from the last equation. \(\square\)

Next, we derive a useful formula for \(\tilde{\nabla}_{\xi}\tilde{h}\).

Proposition 4.10. Let \((M, \tilde{\varphi}, \xi, \eta, \tilde{g})\) be a 3-dimensional paracontact metric manifold. On \(U\) we have

\[
(4.17)
\]

\[\tilde{\nabla}_{\xi}\tilde{h} = 2a_2\tilde{\varphi}\tilde{h}\]

Proof. Using (4.14), we get

\[
\begin{align*}
    (\tilde{\nabla}_{\xi}\tilde{h})\xi &= 0 = (2a_2\tilde{\varphi}\tilde{h})\xi, \\
    (\tilde{\nabla}_{\xi}\tilde{h})e_1 &= -2a_2 e_2 = (2a_2\tilde{\varphi}\tilde{h})e_1, \\
    (\tilde{\nabla}_{\xi}\tilde{h})e_2 &= 0 = (2a_2\tilde{\varphi}\tilde{h})e_2.
\end{align*}
\]

(4.17)

Using (2.22), we have \(\tilde{S}(\xi, \xi) = -2\). After calculating the statement of \(\tilde{\varphi}^2 - \tilde{\varphi}^2\) with respect to the basis components, we get

\[
(4.18)
\]

\[\tilde{\varphi}^2 \xi - \tilde{\varphi}^2 = \frac{\tilde{S}(\xi, \xi)}{2}\tilde{\varphi}^2 e_0 = 0, \quad \tilde{\varphi}^2 e_1 - \tilde{\varphi}^2 e_1 = \frac{\tilde{S}(\xi, \xi)}{2}\tilde{\varphi}^2 e_1, \quad \tilde{\varphi}^2 e_2 - \tilde{\varphi}^2 e_2 = \frac{\tilde{S}(\xi, \xi)}{2}\tilde{\varphi}^2 e_2
\]

So we have following
Proposition 4.11. Let $(M, \tilde{\varphi}, \xi, \eta, \tilde{g})$ be a 3-dimensional paracontact metric manifold. Then the following equation holds on $M$.

\[ \tilde{h}^2 - \tilde{\varphi}^2 = \frac{\tilde{S}(\xi, \xi)}{2} \tilde{\varphi}^2 \]  

Lemma 4.12. Let $(M, \tilde{\varphi}, \xi, \eta, \tilde{g})$ be a 3-dimensional paracontact metric manifold. Then the Ricci operator $\tilde{Q}$ is given by

\[ \tilde{Q} = \tilde{a}I + \tilde{b}\eta \otimes \xi - \tilde{\varphi}(\tilde{\nabla}_L \tilde{h}) + \tilde{\sigma}(\tilde{\varphi}^2) \otimes \xi + \tilde{\sigma}(e_2)\eta \otimes e_1 + \tilde{\sigma}(e_1)\eta \otimes e_2 \]

where $\tilde{a}$ and $\tilde{b}$ are smooth functions defined by $\tilde{a} = 1 + \frac{\tilde{\varphi}^2}{2}$ and $\tilde{b} = -3 - \frac{\tilde{\varphi}^2}{2}$, respectively.

Proof. For 3-dimensional case, we have

\[ \tilde{h}X = \tilde{S}(\xi, \xi)X - \tilde{S}(X, \xi)\xi + \tilde{Q}X - \eta(X)\tilde{Q}X - r(X - \eta(X)\xi), \]

for any vector field $X$. Using (2.21), the last equation implies

\[ \tilde{Q}X = -\tilde{\varphi}^2X + \tilde{h}^2X - \tilde{\varphi}(\tilde{\nabla}_L \tilde{h})X - \tilde{S}(\xi, \xi)X + \tilde{S}(X, \xi)\xi + \eta(X)\tilde{Q}X + \frac{r}{2}(X - \eta(X)\xi). \]

Using $\tilde{\varphi}^2 = I - \eta \otimes \xi$, we get $\tilde{S}(X, \xi) = \tilde{S}(\tilde{\varphi}^2X, \xi) + \eta(X)\tilde{S}(\xi, \xi)$. So (4.21) becomes

\[ \tilde{Q}X = \frac{\tilde{S}(\xi, \xi)}{2}\tilde{\varphi}^2X - \tilde{\varphi}(\tilde{\nabla}_L \tilde{h})X - \tilde{S}(\xi, \xi)X + \tilde{S}(\tilde{\varphi}^2X, \xi)\xi + \eta(X)\tilde{S}(\xi, \xi)\xi + \eta(X)\tilde{Q}X + \frac{r}{2}\tilde{\varphi}^2X. \]

As the basis pseudo-orthonormal $\{e_1, e_2, \xi\}$, it follows that

\[ \tilde{Q}e_1 = \tilde{\sigma}(e_1)e_2 + \tilde{\sigma}(e_2)e_1 + \tilde{S}(\xi, \xi)\xi \]

Using (4.23) in (4.22), we have

\[ \tilde{Q}X = \left(1 + \frac{\tilde{\varphi}^2}{2}\right)X + \left(-3 - \frac{\tilde{\varphi}^2}{2}\right)\eta(X)\xi \]

\[ -\tilde{\varphi}(\tilde{\nabla}_L \tilde{h})X + \tilde{\sigma}(\tilde{\varphi}^2)X + \eta(X)\tilde{\sigma}(e_1)e_2 + \eta(X)\tilde{\sigma}(e_2)e_1, \]

for arbitrary vector field $X$. Hence, proof comes from (4.24).

A consequence of Lemma 4.12, we can give the components of the Ricci operator $\tilde{Q}$

\[ \tilde{Q} \xi = (\tilde{a} + \tilde{b})\xi + \tilde{\sigma}(e_2)e_1 + \tilde{\sigma}(e_1)e_2 \]

\[ \tilde{Q}e_1 = \tilde{\sigma}(e_1)\xi + \tilde{\varphi}e_1 - 2\tilde{\varphi}e_2 \]

\[ \tilde{Q}e_2 = \tilde{\sigma}(e_2)\xi + \tilde{\varphi}e_2 \]

Theorem 4.13. A 3-dimensional paracontact metric manifold is an $H$-paracontact manifold if and only if the characteristic vector field $\xi$ is an eigenvector of the Ricci operator.

Proof. We construct an orthonormal basis $\{\tilde{\varphi}e, \tilde{\varphi}e, \xi\}$ from the pseudo-orthonormal basis $\{e_1, e_2, \xi\}$ such that

\[ \tilde{\varphi}e = \frac{e_1 - e_2}{\sqrt{2}}, \quad \tilde{\varphi}e = \frac{e_1 + e_2}{\sqrt{2}}, \quad \tilde{g}(\tilde{\varphi}e, \tilde{\varphi}e) = -1 \text{ and } \tilde{g}(\tilde{\varphi}e, \tilde{\varphi}e) = 1. \]

Then $\tilde{h}$ with respect to this new basis takes the form,

\[ \tilde{h}e = \tilde{h}\tilde{\varphi}e = \frac{1}{2}(-\tilde{e} + \tilde{\varphi}e) \]
By Lemma 4.9 we have

\[ \nabla_{\tilde{e}}\tilde{e} = \frac{1}{\sqrt{2}}(-b_2 - \frac{1}{2}\tilde{\sigma}(e_1))\tilde{e} + \frac{1}{2}\xi, \quad \nabla_{\tilde{e}}\tilde{\varphi}\tilde{e} = -\frac{1}{\sqrt{2}}(b_2 + \frac{1}{2}\tilde{\sigma}(e_1))\tilde{e} + \frac{3}{2}\xi, \]

\[ \nabla_{\xi}\tilde{e} = \frac{\tilde{e} - 3\tilde{\varphi}\tilde{e}}{2}, \quad \nabla_{\tilde{\varphi}\tilde{e}}\tilde{e} = -\frac{1}{\sqrt{2}}(b_2 - \frac{1}{2}\tilde{\sigma}(e_1))\tilde{e} - \frac{1}{2}\xi, \]

\[ \nabla_{\tilde{\varphi}\tilde{e}}\tilde{\varphi}\tilde{e} = -\frac{1}{\sqrt{2}}(b_2 - \frac{1}{2}\tilde{\sigma}(e_1))\tilde{e} + \frac{1}{2}\xi, \quad \nabla_{\tilde{\varphi}\tilde{e}}\xi = -\tilde{\varphi}\tilde{e} \]

Using (4.27) we have

\[ \tilde{\nabla}^*\tilde{\nabla}\xi = -\tilde{\nabla}\tilde{\nabla}\xi + \tilde{\nabla}\tilde{\varphi}\tilde{e}\xi + \tilde{\nabla}\tilde{\varphi}\tilde{e}\xi - \tilde{\nabla}\tilde{\varphi}\tilde{\varphi}\tilde{e}\xi. \]

\[ = \sqrt{2}(\tilde{e} - \tilde{\varphi}\tilde{e}) + 2\xi \]

From (1.23) and (4.27), we obtain \( \tilde{\sigma}(e_1) = 0 \). This completes the proof of the theorem.

**Theorem 4.14.** Let \((M, \tilde{\varphi}, \xi, \eta, \tilde{\gamma})\) be a 3-dimensional paracontact metric manifold. If \((M, \tilde{\varphi}, \xi, \eta, \tilde{\gamma})\) is an \( H \)-paracontact manifold then the \((\tilde{\kappa}, \tilde{\mu})\)-structure always exists on every open and dense subset of \( M \) and also \( \tilde{\kappa} = -1 \).

**Proof.** Putting \( \tilde{\sigma} = 0 \) in (4.20) we get

\[ \tilde{Q} = \tilde{a}I + \tilde{b}\eta \otimes \xi - 2a_2\tilde{h}, \]

which yields

\[ \tilde{Q}\xi = \tilde{S}(\xi, \xi)\xi, \]

for any vector fields on \( M \). Putting \( \xi \) instead of \( Z \) in (4.1) we obtain

\[ \tilde{R}(X, Y)\xi = -\tilde{S}(X, \xi) + \tilde{S}(Y, \xi) - \eta(X)\tilde{Q}Y + \eta(Y)\tilde{Q}X + \frac{r}{2}(\eta(X)Y - \eta(Y)X). \]

for any vector field \( X \). Using (4.28) and (4.29) in (4.30), we obtain

\[ \tilde{R}(X, Y)\xi = -(\eta(Y)X - \eta(X)Y) - 2a_2(\eta(Y)\tilde{h}X - \eta(X)\tilde{h}Y), \]

where the functions \( \tilde{\kappa} \) and \( \tilde{\mu} \) defined by \( \tilde{\kappa} = \frac{\tilde{S}(\xi, \xi)}{2}, \tilde{\mu} = -2a_2 \), respectively. So, it is obvious that for this type \( \tilde{\kappa} = -1 \). This completes proof of the Theorem.

Furthermore, by (4.28), we have

\[ \tilde{Q}\tilde{\varphi} - \tilde{\varphi}\tilde{Q} = 2\tilde{\mu}\tilde{h}\tilde{\varphi}. \]

**The tensor \( \tilde{h} \) has the canonical form (IV).** Let \((M, \tilde{\varphi}, \xi, \eta, \tilde{\gamma})\) be a 3-dimensional paracontact metric manifold and \( p \) is a point of \( M \). Then there exists a local orthonormal \( \tilde{\varphi} \)-basis \( \{\tilde{e}, \tilde{\varphi}\tilde{e}, \xi\} \) in a neighborhood \( p \) where \(-\tilde{g}(\tilde{e}, \tilde{e}) = \tilde{g}(\tilde{\varphi}\tilde{e}, \tilde{\varphi}\tilde{e}) = \tilde{g}(\xi, \xi) = 1 \). Now, let \( U_1 \) be the open subset of \( M \) where \( \tilde{h} \neq 0 \) and let \( U_2 \) be the open subset of points \( p \in M \) such that \( \tilde{h} = 0 \) in a neighborhood of \( p \). \( U_1 \cup U_2 \) is an open subset of \( M \). For every \( p \in U_1 \) there exists an open neighborhood of \( p \) such that \( \tilde{h}\tilde{e} = \lambda\tilde{\varphi}\tilde{e}, \tilde{h}\tilde{\varphi}\tilde{e} = -\tilde{\lambda}\tilde{e} \) and \( \tilde{h}\xi = 0 \) where \( \lambda \) is a non-vanishing smooth function. Since \( tr\tilde{h} = 0 \), the matrix form of \( \tilde{h} \) is given by

\[ \tilde{h} = \begin{pmatrix} 0 & -\tilde{\lambda} & 0 \\ \tilde{\lambda} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \]

with respect to local orthonormal basis \( \{\tilde{e}, \tilde{\varphi}\tilde{e}, \xi\} \).
Lemma 4.15. Let \((M, \tilde{\varphi}, \xi, \eta, \tilde{g})\) be a 3-dimensional paracontact metric manifold. Then for the covariant derivative on \(U_1\) the following equations are valid

\[
\begin{align*}
\text{i)} & \quad \tilde{\nabla}_e \tilde{e} = a_3 \tilde{\varphi} \tilde{e} + \lambda \xi, \\
\text{ii)} & \quad \tilde{\nabla}_e \tilde{\varphi} \tilde{e} = a_3 \tilde{\varphi} + \xi, \\
\text{iii)} & \quad \tilde{\nabla}_e \lambda = -\tilde{\varphi} \tilde{e} + \lambda \tilde{e}, \\
\text{iv)} & \quad \tilde{\nabla}_e \tilde{\varphi} \tilde{e} = b_3 \tilde{\varphi} - \xi, \\
\text{v)} & \quad \tilde{\nabla}_e \tilde{\varphi} \tilde{e} = b_3 \tilde{\varphi} + \lambda \xi, \\
\text{vi)} & \quad \tilde{\nabla}_e \xi = -\tilde{\varphi} \tilde{e} - \lambda \tilde{\varphi} \tilde{e}, \\
\text{vii)} & \quad \tilde{\nabla}_e \tilde{\varphi} \tilde{e} = b_3 \tilde{e}, \\
\text{viii)} & \quad \tilde{\nabla}_e \lambda = b_3 \tilde{\varphi}.
\end{align*}
\]

(4.31) \([\tilde{e}, \xi] = \tilde{\lambda} \tilde{e} - (1 + b_3) \tilde{\varphi} \tilde{e}, \quad x) \quad [\tilde{\varphi} \tilde{e}, \xi] = -(1 + b_3) \tilde{e} - \tilde{\lambda} \tilde{\varphi} \tilde{e}, \quad xi) \quad [\tilde{e}, \tilde{\varphi} \tilde{e}] = a_3 \tilde{e} - b_3 \tilde{\varphi} \tilde{e} + 2 \xi.

where \(a_3\) and \(b_3\) are defined by

\[
\begin{align*}
a_3 &= -\frac{1}{2\lambda} \left[ \sigma(\tilde{\varphi}) + (\tilde{\varphi}) \tilde{\lambda} \right], \\
b_3 &= \frac{1}{2\lambda} \left[ \sigma(\tilde{\varphi}) - \tilde{\varphi}(\tilde{\lambda}) \right],
\end{align*}
\]

respectively.

Proof. Replacing \(X\) by \(\tilde{e}\) and \(Y\) by \(\tilde{\varphi} \tilde{e}\) in equation \(\tilde{\nabla}_e \lambda = -\tilde{\varphi} + \tilde{\varphi} \tilde{h}\), we have \(\text{vii), vi)\).

For the proof of \(\text{viii)}\) we have

\[
\begin{align*}
\tilde{\nabla}_e \tilde{\varphi} \tilde{e} &= -\tilde{g}(\tilde{\nabla}_e \tilde{\varphi} \tilde{\varphi}, \tilde{e}) + \tilde{g}(\tilde{\nabla}_e \tilde{\varphi} \tilde{\varphi}, \tilde{\varphi} \tilde{e}) + \tilde{g}(\tilde{\nabla}_e \tilde{\varphi} \tilde{\varphi}, \xi) \\
&= \tilde{g}(\tilde{\varphi} \tilde{e}, \tilde{\nabla}_e \tilde{\varphi} \tilde{e}),
\end{align*}
\]

where \(b_3\) is defined by \(b_3 = \tilde{g}(\tilde{\nabla}_e \tilde{\varphi}, \tilde{\varphi} \tilde{e})\). So we obtain \(\tilde{\nabla}_e \tilde{\varphi} \tilde{e} = b_3 \tilde{e}\). The proofs of other covariant derivative equalities are similar to \(\text{vii)}\).

Putting \(X = \tilde{e}, Y = \tilde{\varphi} \tilde{e}, Z = \xi\) in the equation \(4.1\), we have

\[
\tilde{R}(\tilde{e}, \tilde{\varphi} \tilde{e}) \xi = -\tilde{g}(\tilde{Q} \xi, \tilde{\varphi} \tilde{e}) + \tilde{g}(\tilde{Q} \tilde{\varphi} \tilde{e}, \xi) \tilde{e}.
\]

Since \(\tilde{\varphi}(\tilde{X})\) is defined as \(\tilde{g}(\tilde{Q} \xi, X)\), we have

\[
\tilde{R}(\tilde{e}, \tilde{\varphi} \tilde{e}) \xi = -\tilde{g}(\tilde{e}) \tilde{\varphi} \tilde{e} + \tilde{g}(\tilde{\varphi} \tilde{e}) \tilde{e}.
\]

On the other hand, by using \(\text{2.20}\), we have

\[
\tilde{R}(\tilde{e}, \tilde{\varphi} \tilde{e}) \xi = (\tilde{\nabla}_e \tilde{\varphi} \tilde{h}) \tilde{\varphi} \tilde{e} - (\tilde{\nabla}_e \tilde{\varphi} \tilde{h}) \tilde{e},
\]

(4.33)

\[
= (-2a_3 \lambda - (\tilde{\varphi}) \tilde{\lambda}) \tilde{e} + (-2b_3 \lambda - \tilde{e} \tilde{\varphi}) \tilde{\varphi} \tilde{e}.
\]

Comparing \(\text{4.33}\) with \(\text{4.32}\), we get

\[
\tilde{\varphi}(\tilde{e}) = \tilde{e}(\tilde{\lambda}) - 2b_3 \lambda, \\
\tilde{\varphi}(\tilde{\varphi} \tilde{e}) = -\tilde{\varphi}(\tilde{\lambda}) - 2a_3 \lambda.
\]

Hence, the functions \(a_3\) and \(b_3\) are obtained from the last equation. \(\square\)

Next, we derive a useful formula for \(\tilde{\nabla}_e \tilde{h}\).

Proposition 4.16. Let \((M, \tilde{\varphi}, \xi, \eta, \tilde{g})\) be a 3-dimensional paracontact metric manifold. On \(U_1\) we have

\[
\tilde{\nabla}_e \tilde{h} = -2b_3 \tilde{h} \tilde{\varphi} + \tilde{\xi} \tilde{\lambda} s
\]

where \(s\) is the \((1,1)\)-type tensor defined by \(s \xi = 0, s \tilde{e} = \tilde{\varphi} \tilde{e}, s \tilde{\varphi} \tilde{e} = -\tilde{e}\).

Proof. Using \(\text{4.2}\), we get

\[
\begin{align*}
(\tilde{\nabla}_e \tilde{h}) \xi &= 0 = (-2b_3 \tilde{h} \tilde{\varphi} + \tilde{\xi} \tilde{\lambda} s) \xi, \\
(\tilde{\nabla}_e \tilde{h}) \tilde{e} &= 2b_3 \lambda \tilde{e} + \xi(\tilde{\lambda}) \tilde{\varphi} \tilde{e} = (-2b_3 \tilde{h} \tilde{\varphi} + \xi(\tilde{\lambda}) s) \tilde{e}, \\
(\tilde{\nabla}_e \tilde{h}) \tilde{\varphi} \tilde{e} &= -2b_3 \lambda \tilde{\varphi} \tilde{e} - \xi(\tilde{\lambda}) \tilde{e} = (-2b_3 \tilde{h} \tilde{\varphi} + \xi(\tilde{\lambda}) s) \tilde{\varphi} \tilde{e}.
\end{align*}
\]

\(\square\)
Proposition 4.17. Let \((M, \tilde{\varphi}, \xi, \eta, \tilde{g})\) be a 3-dimensional paracontact metric manifold. Then the following equation holds on \(M\).

\[
\tilde{h}^2 - \tilde{\varphi}^2 = \frac{\hat{S}(\xi, \xi)}{2}\tilde{\varphi}^2
\]  \(\text{(4.35)}\)

Proof. Using (2.22), we have \(\hat{S}(\xi, \xi) = 2(1 + \tilde{\lambda}^2)\). After calculating the statement of \(\tilde{h}^2 - \tilde{\varphi}^2\) with respect to the basis components, we get

\[
\tilde{h}^2\xi - \tilde{\varphi}^2\xi = \frac{\hat{S}(\xi, \xi)}{2}\tilde{\varphi}^2\xi = 0, \quad \tilde{h}^2\tilde{\varphi} - \tilde{\varphi}^3\tilde{\varphi} = \frac{\hat{S}(\xi, \xi)}{2}\tilde{\varphi}^2\tilde{\varphi}
\]  \(\text{(4.36)}\)

Completions the proof of (4.35). \(\square\)

Lemma 4.18. Let \((M, \tilde{\varphi}, \xi, \eta, \tilde{g})\) be a 3-dimensional paracontact metric manifold. Then the Ricci operator \(\tilde{Q}\) is given by

\[
\tilde{Q} = \tilde{a} I + \tilde{b}\eta \otimes \xi - \tilde{\varphi}(\nabla_\xi \tilde{h}) + \tilde{\sigma}(\tilde{\varphi}^2) \otimes \xi - \tilde{\sigma}(\tilde{\varphi})\eta \otimes \tilde{\varphi} + \tilde{\sigma}(\tilde{\varphi}\tilde{\varphi})\eta \otimes \tilde{\varphi}
\]  \(\text{(4.37)}\)

where \(\tilde{a}\) and \(\tilde{b}\) are smooth functions defined by \(\tilde{a} = 1 + \tilde{\lambda}^2 + \tilde{\varphi}\) and \(\tilde{b} = -3(\tilde{\lambda}^2 + 1) - \tilde{\varphi}\), respectively. Moreover the components of the Ricci operator \(\tilde{Q}\) are given by

\[
\tilde{Q}\xi = (\tilde{a} + \tilde{b})\xi - \tilde{\sigma}(\tilde{\varphi})\tilde{\varphi} + \tilde{\sigma}(\tilde{\varphi}\tilde{\varphi})\tilde{\varphi}
\]  \(\text{(4.38)}\)

Proof. By (4.1), we have

\[
\tilde{R}(X, \xi)\xi = \hat{S}(\xi, \xi)X - \hat{S}(X, \xi)\xi + \tilde{Q}X - \eta(X)\tilde{Q}\xi - \frac{r}{2}(X - \eta(X)\xi),
\]

for any vector field \(X\). Using (2.21), the last equation implies

\[
\tilde{Q}X = -\tilde{\varphi}^2X + \tilde{h}^2X - \tilde{\varphi}(\nabla_\xi \tilde{h})X - \tilde{S}(\xi, \xi)X + \tilde{S}(X, \xi)\xi + \eta(X)\tilde{Q}\xi + \frac{r}{2}(X - \eta(X)\xi).
\]  \(\text{(4.39)}\)

By writing \(\hat{S}(X, \xi) = \hat{S}(\tilde{\varphi}^2X, \xi) + \eta(X)\hat{S}(\xi, \xi)\) in (4.39), we obtain

\[
\tilde{Q}X = \frac{\hat{S}(\xi, \xi)}{2}\tilde{\varphi}^2X - \tilde{\varphi}(\nabla_\xi \tilde{h})X - \tilde{S}(\xi, \xi)X + \tilde{S}(\tilde{\varphi}^2X, \xi)\xi + \eta(X)\hat{S}(\xi, \xi)\xi + \eta(X)\tilde{Q}\xi + \frac{r}{2}\tilde{\varphi}^2X.
\]  \(\text{(4.40)}\)

We know that the Ricci tensor \(\hat{S}\) with respect to the orthonormal basis \(\{\tilde{\varphi}, \tilde{\varphi}\tilde{\varphi}, \xi\}\) is given by

\[
\tilde{Q}\xi = -\tilde{\sigma}(\tilde{\varphi})\tilde{\varphi} + \tilde{\sigma}(\tilde{\varphi}\tilde{\varphi})\tilde{\varphi} + \hat{S}(\xi, \xi)\xi
\]  \(\text{(4.41)}\)

Using (4.41) in (4.40), we have

\[
\tilde{Q}X = \left(1 + \tilde{\lambda}^2 + \frac{r}{2}\right)X + \left(-3(\tilde{\lambda}^2 + 1) - \frac{r}{2}\right)\eta(X)\xi
\]

\[
-\tilde{\varphi}(\nabla_\xi \tilde{h})X + \tilde{\sigma}(\tilde{\varphi}^2X)\xi - \eta(X)\tilde{\sigma}(\tilde{\varphi})\tilde{\varphi} + \eta(X)\tilde{\sigma}(\tilde{\varphi}\tilde{\varphi})\tilde{\varphi},
\]

for arbitrary vector field \(X\). This completes proof of the theorem. \(\square\)

Theorem 4.19. A 3-dimensional paracontact metric manifold is an \(H\)-paracontact manifold if and only if the characteristic vector field \(\xi\) is an eigenvector of the Ricci operator.
Proof. From (4.2) we have
\begin{align*}
\tilde{\nabla}^* \tilde{\nabla} \xi &= -\tilde{\nabla}_\xi \tilde{\nabla} \xi + \tilde{\nabla}_{\tilde{\nabla}_\xi} \xi + \tilde{\nabla}_{\tilde{\nabla} \xi} \xi - \tilde{\nabla}_{\tilde{\nabla}_\xi \xi} \xi.
\end{align*}
By using (4.3), we obtain \( \tilde{\sigma}(\tilde{\varepsilon}) = \tilde{\sigma}(\tilde{\varphi} \tilde{\varepsilon}) = 0 \).

**Theorem 4.20.** Let \((M, \varphi, \xi, \eta, \tilde{g})\) be a 3-dimensional \(H\)-paracontact metric manifold. Then the \((\tilde{\kappa}, \tilde{\mu}, \tilde{\nu})\)-manifold always exists on every open and dense subset of \(M\) and also \(\tilde{\kappa} < -1\).

Proof. Since \(M\) is a \(H\)-paracontact metric manifold we have \(\tilde{\sigma} = 0\). Putting \(s = \frac{1}{\lambda} \tilde{h}\) in (4.37) we get
\begin{align*}
(4.43) \quad \tilde{Q} = \tilde{a} I + \tilde{b} \eta \otimes \xi - 2 \tilde{b}_3 \tilde{h} - \left( \frac{\xi(\tilde{\lambda})}{\lambda} \right) \tilde{\varphi} \tilde{h},
\end{align*}
which yields
\begin{align*}
(4.44) \quad \tilde{Q} \xi = \tilde{S}(\xi, \xi),
\end{align*}
for any vector fields on \(M\). Setting \(\xi = Z\) in (4.1) we find
\begin{align*}
(4.45) \quad \tilde{R}(X, Y) \xi &= -\tilde{S}(X, \xi) + \tilde{S}(Y, \xi) - \eta(X) \tilde{Q} Y \\
&\quad + \eta(Y) \tilde{Q} X + \frac{\lambda}{2} (\eta(Y) \tilde{\varphi} \tilde{h} X - \eta(X) \tilde{\varphi} \tilde{h} Y),
\end{align*}
for any vector field \(X\). Using (4.43) and (4.44) in (4.14), we get
\begin{align*}
\tilde{R}(X, Y) \xi &= (-1 - \lambda^2)(\eta(Y)X - \eta(X)Y) - 2 \tilde{b}_3 (\eta(Y)X - \eta(X)Y) - \frac{\xi(\tilde{\lambda})}{\lambda} (\eta(Y) \tilde{\varphi} \tilde{h} X - \eta(X) \tilde{\varphi} \tilde{h} Y),
\end{align*}
where the functions \(\tilde{\kappa}, \tilde{\mu}\) and \(\tilde{\nu}\) defined by \(\tilde{\kappa} = \frac{\tilde{S}(\xi, \xi)}{2}, \tilde{\mu} = -2 \tilde{b}_3, \tilde{\nu} = \frac{\xi(\tilde{\lambda})}{\lambda}\), respectively. So, it is obvious that for this type \(\tilde{\kappa} < -1\). This completes proof of the Theorem.

Remark 4.21. The tensor \(\tilde{h}\) has the canonical form (III). Let \((M, \varphi, \xi, \eta, \tilde{g})\) be a 3-dimensional paracontact metric manifold and \(p\) is a point of \(M\). Then there exists a local pseudo-orthonormal basis \(\{e_1, e_2, e_3\}\) in a neighborhood of \(p\) where \(\tilde{g}(e_1, e_1) = \tilde{g}(e_2, e_2) = \tilde{g}(e_3, e_3) = 0\) and \(\tilde{g}(e_1, e_2) = \tilde{g}(e_3, e_3) = 1\). Since the tensor has canonical form (III) (with respect to a pseudo-orthonormal basis \(\{e_1, e_2, e_3\}\)) then \(\tilde{h}e_1 = \lambda e_1 + e_3, \tilde{h}e_2 = \lambda e_2 + \lambda e_3, \tilde{h}e_3 = e_2 + \lambda e_3\). Since \(0 = \tilde{tr}h = \tilde{g}(\tilde{h}e_1, e_2 + \tilde{g}(\tilde{h}e_2, e_1) + \tilde{g}(\tilde{h}e_3, e_3) = 3\lambda\), then \(\lambda = 0\). We write \(\xi = \tilde{g}(\xi, e_1)e_1 + \tilde{g}(\xi, e_2)e_2 + \tilde{g}(\xi, e_3)e_3\) respect to the pseudo-orthonormal basis \(\{e_1, e_2, e_3\}\). Since \(\tilde{h} \xi = 0\), we have \(0 = \tilde{g}(\xi, e_1)e_2 + \tilde{g}(\xi, e_2)e_1 + \tilde{g}(\xi, e_3)e_2\). Hence we get \(\xi = \tilde{g}(\xi, e_1)e_1 + \tilde{g}(\xi, e_2)e_2 + \tilde{g}(\xi, e_3)e_3\) which leads to a contradiction with \(\tilde{g}(\xi, \xi) = 1\). Thus we can not investigate this case.

5. An Application

In [7] Boeckx provided a local classification of non-Sasakian \((\kappa, \mu)\)-contact metric manifold respect to the number
\begin{align*}
(5.1) \quad I_M = \frac{1 - \mu}{\sqrt{1 - \kappa}},
\end{align*}
which is an invariant of a \((\kappa, \mu)\)-contact metric manifold up to \(D_\alpha\)-homothetic deformations.

Konfogjorgos and Tsichlias [20] gave a local classification of a non-Sasakian generalized \((\kappa, \mu)\)-contact metric manifold which satisfies the condition " the function \(\mu\) is constant along the integral curves of the characteristic vector field \(\xi\), i.e. \(\xi(\mu) = 0\)." One can easily prove that this condition is equivalent to
\(\xi(I_M) = 0\) for a non-Sasakian generalized \((\kappa, \mu)\)-contact metric manifold. It can be shown that \(\xi(I_M) = 0\) satisfies the condition \(\xi(\mu) = \nu(\mu - 2)\) for non-Sasakian \((\kappa, \mu, \nu)\)-contact metric manifolds. Moreover, the converse is also true. Recently, the authors gave the following local classification of a non-Sasakian \((\kappa, \mu, \nu)\)-contact metric manifolds with \(\xi(I_M) = 0\).\(^{22}\)

**Theorem 5.1** \(^{22}\). Let \((M, \varphi, \xi, \eta, g)\) be a non-Sasakian \((\kappa, \mu, \nu = \text{const.})\)-contact metric manifold and \(\xi(I_M) = 0\), where \(\nu = \text{const.} \neq 0\). Then

1) At any point of \(M\), precisely one of the following relations is valid: \(\mu = 2(1 + \sqrt{1 - \kappa})\), or \(\mu = 2(1 - \sqrt{1 - \kappa})\).

2) At any point \(P \in M\) there exists a chart \((U, (x, y, z))\) with \(P \in U \subseteq M\), such that

i) the functions \(\kappa, \mu\) depend only on the variables \(x, z\).

ii) if \(\mu = 2(1 + \sqrt{1 - \kappa})\), \((\text{resp. } \mu = 2(1 - \sqrt{1 - \kappa}))\), the tensor fields \(\eta, \varphi, g, h\) are given by the relations,

\[
\xi = \frac{\partial}{\partial x}, \quad \eta = dx - adz
\]

\[
g = \begin{pmatrix}
1 & 0 & -a \\
0 & 1 & -b \\
-a & -b & 1 + a^2 + b^2
\end{pmatrix}
\]

\[
(\text{resp. } g = \begin{pmatrix}
1 & 0 & -a \\
0 & 1 & -b \\
-a & -b & 1 + a^2 + b^2
\end{pmatrix}),
\]

\[
\varphi = \begin{pmatrix}
0 & a & -ab \\
0 & b & -1 - b^2 \\
0 & 1 & -b
\end{pmatrix}
\]

\[
(\text{resp. } \varphi = \begin{pmatrix}
0 & a & ab \\
0 & b & 1 + b^2 \\
0 & 1 & b
\end{pmatrix}),
\]

\[
h = \begin{pmatrix}
0 & 0 & -a\lambda \\
0 & \lambda & -2ab \\
0 & 0 & -\lambda
\end{pmatrix}
\]

\[
(\text{resp. } h = \begin{pmatrix}
0 & 0 & a\lambda \\
0 & -\lambda & 2ab \\
0 & 0 & \lambda
\end{pmatrix}),
\]

with respect to the basis \(\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\), where \(a = 2y + f(z)\) \((\text{resp. } a = -2y + f(z))\), \(b = \frac{\nu^2}{2} \nu - \frac{\nu f(z)}{2} \nu - \frac{\nu r'(z)}{2} \nu + \frac{\nu^2}{2} \nu + \frac{\nu r'(z)}{2} \nu + \frac{\nu}{2} r(z) e^{ux} + s(z)\) \((\text{resp. } b = \frac{\nu^2}{2} \nu - \frac{\nu f(z)}{2} \nu - \frac{\nu r'(z)}{2} \nu + \frac{\nu^2}{2} \nu + \frac{\nu r'(z)}{2} \nu + \frac{\nu}{2} r(z) e^{ux} + s(z))\) \(\lambda = \lambda(x, z) = r(z) e^{ux}\) and \(f(z), r(z), s(z)\) are arbitrary smooth functions of \(z\).

Now we are going to give a natural relation between a non-Sasakian \((\kappa, \mu, \nu)\)-contact metric manifolds with \(\xi(I_M) = 0\) and 3-dimensional paracontact metric manifolds.

If Theorem \(^{24}\) is adapted for the 3-dimensional non-Sasakain \((\kappa, \mu, \nu)\)-contact metric manifold and used same procedure for proof then we have same result. Hence we can give following Theorem.

**Theorem 5.2** \(^{13}\). Let \((M, \varphi, \xi, \eta, g)\) be a non-Sasakian \((\kappa, \mu, \nu)\)-contact metric manifold. Then \(M\) admits a canonical paracontact metric structure \((\tilde{\varphi}, \tilde{\xi}, \tilde{\eta})\) is given by

\[
(\ref{5.2}) \quad \tilde{\varphi} := \frac{1}{\sqrt{1 - \kappa}} h, \quad \tilde{\eta} := \frac{1}{\sqrt{1 - \kappa}} d\eta(\cdot, h) + \eta \otimes \eta.
\]

By \(^{13}\) we will prove following proposition.
Proposition 5.3. Let \((M, \varphi, \xi, \eta, g)\) be a non-Sasakian \((\kappa, \mu, \nu)\)-contact metric manifold. Then the Levi-Civita connections \(\nabla\) and \(\tilde{\nabla}\) of \(g\) and \(\tilde{g}\) are related as

\[
\tilde{\nabla}_XY = \nabla_XY + \frac{1}{2(1-\kappa)} \varphi h(\nabla_X \varphi h)Y - \frac{1}{\sqrt{1-\kappa}} \eta(Y)hX - \frac{1}{\sqrt{1-\kappa}} \eta(X)hY
\]

\[
- \frac{1}{2} \eta(Y)\varphi hX - \frac{(1-\mu)}{2} \eta(Y)\varphi X - \frac{\nu}{2} \eta(Y)\varphi^2 X
\]

\[
+ \frac{1}{2\sqrt{1-\kappa}} g(hX,Y) + \sqrt{1-\kappa} g(X,Y) - \sqrt{1-\kappa} \eta(X)\eta(Y)
\]

\[
+ \frac{(1-\mu)}{2\sqrt{1-\kappa}} g(hX,Y) - g(X, \varphi Y) + X(\eta(Y)) - \eta(\nabla_X Y)\xi
\]

\[
- \frac{1}{2} (1-\kappa)(X(\frac{1}{\sqrt{1-\kappa}})\varphi^2 Y + Y(\frac{1}{\sqrt{1-\kappa}})\varphi^2 X +
\]

\[
+ \frac{1}{(1-\kappa)} g(X, \varphi h Y)\varphi h \text{grad}(\frac{1}{\sqrt{1-\kappa}}),
\]

for any \(X, Y \in \Gamma(TM)\).

Proof. Using \((5.2)\) and applying the Koszul formulas for \(\tilde{g}\) respect \(\tilde{\nabla}\) to we obtain

\[
2\tilde{g}(\tilde{\nabla}_X Y, Z) = X \left( \frac{1}{\sqrt{1-\kappa}} g(Y, \varphi h Z) + \eta(Y)\eta(Z) \right) + Y \left( \frac{1}{\sqrt{1-\kappa}} g(X, \varphi h Z) + \eta(X)\eta(Z) \right)
\]

\[
- Z \left( \frac{1}{\sqrt{1-\kappa}} g(X, \varphi h Y) + \eta(X)\eta(Y) \right) + \left( \frac{1}{\sqrt{1-\kappa}} g([X, Y], \varphi h Z) + \eta([X, Y])\eta(Z) \right)
\]

\[
+ \left( \frac{1}{\sqrt{1-\kappa}} g([X, Z], \varphi h Y) + \eta([X, Z])\eta(Y) \right) - \left( \frac{1}{\sqrt{1-\kappa}} g([Y, Z], \varphi h X) + \eta([Y, Z])\eta(X) \right).
\]

Since \(\varphi \circ h\) is symmetric respect to \(g\) and \(\nabla g = 0\), we get

\[
2\tilde{g}(\tilde{\nabla}_X Y, Z) = \frac{1}{\sqrt{1-\kappa}} \left( 2g(\varphi h \nabla_X Y, Z) + g((\nabla_X \varphi h)Z, Y) + g((\nabla_Y \varphi h)Z, X) - g((\nabla_Z \varphi h)Y, X) \right)
\]

\[
+ 2(2\eta(X, Z)\eta(Y) + d\eta(X, Z)\eta(Y) - d\eta(X, Y)\eta(Z) + X(\eta(Y))\eta(Z))
\]

\[
+ X\left( \frac{1}{\sqrt{1-\kappa}} g(Y, \varphi h Z) + Y\left( \frac{1}{\sqrt{1-\kappa}} g(X, \varphi h Z) - Z\left( \frac{1}{\sqrt{1-\kappa}} g(X, \varphi h Y) \right) \right) \right)
\]

so that by using \((5.1)\), after a long but straightforward calculation

\[
\tilde{g}(\tilde{\nabla}_X Y, Z) = g \left( \frac{1}{\sqrt{1-\kappa}} (\varphi h \nabla_X Y + \frac{1}{2}(\nabla_X \varphi h)Y - \frac{(1-\kappa)}{2} \eta(Y)X - \frac{(1-\mu)}{2} \eta(Y)hX + \frac{\nu}{2} \eta(Y)\varphi h X)
\]

\[
- \eta(Y)\varphi X - \eta(X)\varphi Y + \frac{1}{2} X\left( \frac{1}{\sqrt{1-\kappa}} \varphi h Y \right)
\]

\[
+ Y\left( \frac{1}{\sqrt{1-\kappa}} \varphi h X - g(X, \varphi h Y) \text{grad}(\frac{1}{\sqrt{1-\kappa}}), Z \right)
\]

\[
(5.4) + \frac{1}{2\sqrt{1-\kappa}} g \left( \frac{1-\kappa}{2} g(X, Y)\xi + (1-\mu) g(hY, X) - \nu g(\varphi h Y, X)\xi
\]

\[
- 2\sqrt{1-\kappa} \eta g(X, \varphi h Y)\xi + 2\sqrt{1-\kappa} \eta (\eta(Y))\xi, Z \right).
\]
It is easy to see that \( \tilde{g}(\tilde{\nabla}_X Y, \xi) = \eta(\tilde{\nabla}_X Y) \) and then using (2.6) and (2.9) we have
\[
\varphi h \tilde{\nabla}_X Y = \varphi h \nabla_X Y + \frac{1}{2}(\nabla_X \varphi h) Y - \frac{(1 - \kappa)}{2}(\eta(Y) X - (1 - \mu)\eta(Y) h X + \nu \eta(Y) \varphi h X \\
- \sqrt{1 - \kappa}(\eta(Y) \varphi X + \eta(X) \varphi Y) + \sqrt{1 - \kappa}(X(\frac{1}{\sqrt{1 - \kappa}}) \varphi h Y + Y(\frac{1}{\sqrt{1 - \kappa}}) \varphi h X \\
- g(X, \varphi h Y) \text{grad}(\frac{1}{\sqrt{1 - \kappa}})) \\
+ \frac{(1 - \kappa)}{2}((-g(X, Y) + 2\eta(X) \eta(Y) - \nu g(\varphi h X, Y) - g(h X, Y)) \xi).
\]
By applying \( \varphi h \) to both sides of previous identity, using \( h \varphi = -\varphi h \) and (2.6), we obtain the assertion.

Now we will give a relation between \( \tilde{h} \) and \( h \) by following Lemma.

**Lemma 5.4.** Let \((M, \varphi, \xi, \eta, g)\) be a non-Sasakian \((\kappa, \mu, \nu)\)-contact metric manifold and let \((\tilde{\varphi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})\) be the canonical paracontact metric structure induced on \(M\), according to Theorem 5.2. Then we have
\[
(5.5) \quad \tilde{h} = \frac{1}{2\sqrt{1 - \kappa}}((2 - \mu)\varphi \circ h + 2(1 - \kappa)\varphi).
\]

**Proof.** By help the equation (2.8), (5.2) and the definitons of \( \tilde{h} \) and \( h \) we have
\[
2\tilde{h} = L_\xi \tilde{\varphi} = L_\xi \left( \frac{1}{\sqrt{1 - \kappa}} h \right)
\]
(5.6)
\[
= -\frac{\nu}{\sqrt{1 - \kappa}} h + \frac{1}{2\sqrt{1 - \kappa}} L_\xi (L_\xi \varphi).
\]
Using the identities \( \nabla \xi = -\varphi - \varphi h, \nabla \xi \varphi = 0 \) and \( \varphi^2 h = -h \), and following same procedure cf. [13] pag.270, we get
\[
2\tilde{h} = -\frac{\nu}{\sqrt{1 - \kappa}} h + \frac{1}{2\sqrt{1 - \kappa}}((2\nabla_\xi h + 4h^2 \varphi - 4h \varphi).
\]
By using (2.6) and (2.9), we obtain the claimed relation. \( \square \)

Let \((M, \varphi, \xi, \eta, g)\) be a non-Sasakian \((\kappa, \mu, \nu)\)-contact metric manifold. Choosing a local orthonormal \( \varphi \)-basis \( \{e, \varphi e, \xi\} \) on \((M, \varphi, \xi, \eta, g)\) and using Proposition 5.3, Lemma 5.4 and Lemma 2.3, one can give following Proposition.

**Proposition 5.5.** Let \((M, \varphi, \xi, \eta, g)\) be a non-Sasakian \((\kappa, \mu, \nu)\)-contact metric manifold and let \((\tilde{\varphi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})\) be the canonical paracontact metric structure induced on \(M\), according to Theorem 5.2. Then we have
\begin{align*}
&i) \quad \tilde{\nabla}_e e = -\frac{1}{2\lambda} e(\lambda) e + (\lambda + 1 - \frac{\mu}{2})\xi \\
&ii) \quad \tilde{\nabla}_e \varphi e = \frac{1}{2\lambda} e(\lambda) \varphi e + \xi \\
&iii) \quad \tilde{\nabla}_e \xi = -e + (\frac{\mu}{2} - 1 - \lambda)\varphi e, \\
&iv) \quad \tilde{\nabla}_{\varphi e} e = \frac{1}{2\lambda} (\varphi e(\lambda)e - \xi, \varphi e\varphi e = -\frac{1}{2\lambda} (\varphi e(\lambda)\varphi e + (\lambda + 1 - \frac{\mu}{2})\xi, \\
&v) \quad \tilde{\nabla}_{\varphi e} \xi = -e - (\lambda + 1 + \frac{\mu}{2})\varphi e, \\
&vi) \quad \tilde{\nabla}_e \xi = -e, \\
&vii) \quad \tilde{\nabla}_e \varphi e = \varphi e, \\
&ix) \quad [e, \xi] = (\frac{\mu}{2} - 1 - \lambda)\varphi e, [\varphi e, \xi] = -e - (\lambda + \frac{\mu}{2})\varphi e, [e, \varphi e] = -\frac{1}{2\lambda} (\varphi e(\lambda)e + \frac{\mu}{2} e(\lambda)\varphi e + 2\xi.
\end{align*}
Moreover, \( \tilde{g}(e, e) = \tilde{g}(\varphi e, \varphi e) = \tilde{g}(\varphi e, \xi) = 0 \) and \( \tilde{g}(e, \varphi e) = \tilde{g}(\xi, \xi) = 1 \)
We assume that \((M, \varphi, \xi, \eta, g)\) be a non-Sasakian \((\kappa, \mu, \nu)\)-contact metric manifold, \((\tilde{\varphi}, \xi, \eta, \tilde{g})\) be the canonical paracontact metric structure induced on \(M\), according to Theorem 5.2. Let \(\{e, \varphi e, \xi\}\) be an orthonormal \(\varphi\)-basis in neighborhood of \(p \in M\). Then one can always construct an orthonormal \(\tilde{\varphi}\)-basis \(\{\tilde{e}_1, \tilde{\varphi}e_1 = \tilde{e}_2, \xi\}\), for instance \(\tilde{e}_1 = (e - \varphi e)/\sqrt{2}, \tilde{\varphi}e_1 = (e + \varphi e)/\sqrt{2}\), such that \(\tilde{g}(\tilde{e}_1, \tilde{e}_1) = -1, \tilde{g}(\tilde{\varphi}e_1, \tilde{\varphi}e_1) = 1, \tilde{g}(\xi, \xi) = 1\). Moreover, from Lemma 5.4 the matrix form of \(h\) is given by

\[
(5.7) \quad \tilde{h} = \begin{pmatrix} -1 + \frac{\mu}{2} & -\lambda & 0 \\ \lambda & 1 - \frac{\mu}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix}
\]

with respect to local orthonormal basis \(\{\tilde{e}_1, \tilde{\varphi}e_1, \xi\}\). By using Proposition 5.3, we have following Proposition.

**Proposition 5.6.** Let \((M, \varphi, \xi, \eta, g)\) be a non-Sasakian \((\kappa, \mu, \nu)\)-contact metric manifold and let \((\tilde{\varphi}, \xi, \eta, \tilde{g})\) be the canonical paracontact metric structure induced on \(M\), according to Theorem 5.2. Then we have

\[i) \quad \tilde{\nabla}_{\tilde{e}_1} \tilde{e}_1 = -\frac{1}{2\lambda}(\tilde{\varphi}\tilde{e}_1)(\lambda)\tilde{\varphi}\tilde{e}_1 + \lambda \xi, \quad \tilde{\nabla}_{\tilde{\varphi}e_1} \tilde{e}_1 = -\frac{1}{2\lambda}(\tilde{\varphi}\tilde{e}_1)(\lambda)\tilde{e}_1 + (2 - \frac{\mu}{2})\xi, \]

\[\tilde{\nabla}_{\tilde{\varphi}e_1} \xi = \lambda \tilde{e}_1 + (\frac{\mu}{2} - 2)\tilde{\varphi}\tilde{e}_1, \quad \tilde{\nabla}_{\xi} \tilde{e}_1 = -\frac{1}{2\lambda}\tilde{e}_1(\lambda)\tilde{\varphi}\tilde{e}_1 - \frac{\mu}{2}\xi, \]

\[v) \quad \tilde{\nabla}_{\tilde{\varphi}e_1} \tilde{\varphi}e_1 = -\frac{1}{2\lambda}(\tilde{\varphi}\tilde{e}_1)(\lambda)\tilde{\varphi}e_1 + \lambda \xi, \quad \tilde{\nabla}_{\xi} \tilde{\varphi}e_1 = -\tilde{\varphi}e_1, \quad \tilde{\nabla}_{\tilde{\varphi}e_1} \xi = -\tilde{\varphi}e_1, \quad \lambda \tilde{e}_1 + (\frac{\mu}{2} - 1)\tilde{\varphi}\tilde{e}_1\]

\[\tilde{\nabla}_{\xi} \tilde{\varphi}e_1 = -\tilde{e}_1, \quad \tilde{\nabla}_{\tilde{\varphi}e_1} \xi = (1 - \frac{\mu}{2})\tilde{e}_1 - \lambda \tilde{\varphi}\tilde{e}_1, \quad \tilde{\nabla}_{\tilde{e}_1} \tilde{\varphi}e_1 = -\frac{1}{2\lambda}(\tilde{\varphi}\tilde{e}_1)(\lambda)\tilde{e}_1 + \frac{1}{2\lambda}\tilde{e}_1(\lambda)\tilde{\varphi}\tilde{e}_1 + 2\xi.\]

**Proposition 5.7.** Let \((M, \varphi, \xi, \eta, g)\) be a non-Sasakian \((\kappa, \mu, \nu = \text{const.})\)-contact metric manifold with \(\xi(\mathcal{I}_M) = 0\) and suppose that \((\tilde{\varphi}, \xi, \eta, \tilde{g})\) be the canonical paracontact metric structure induced on \(M\). Then

\[
(5.9) \quad \tilde{\nabla}_{\xi} \tilde{h} = 2\tilde{h} \tilde{\varphi} + \nu \tilde{h}
\]

**Proof.** Taking \(\xi(\mu) = \nu(\mu - 2)\) and \(\xi(\lambda) = \nu \lambda\) into account and using the relations (5.7), (5.8) we get requested relation. \(\square\)

We suppose that \((M, \varphi, \xi, \eta, g)\) be a non-Sasakian \((\kappa, \mu, \nu)\)-contact metric manifold with \(\xi(\mathcal{I}_M) = 0\). Let \((\tilde{\varphi}, \xi, \eta, \tilde{g})\) be the canonical paracontact metric structure induced on \(M\), according to Theorem 5.2. By (2.20) and (5.8), and after very long computations we obtain that

\[
(5.10) \quad \tilde{R}(\tilde{e}_1, \tilde{\varphi}e_1)\xi = \left(\frac{1}{\lambda}(\frac{\mu}{2} - 1)\tilde{e}_1(\lambda) - \frac{1}{2}\tilde{e}_1(\mu)\right)\tilde{e}_1 + \left(\frac{1}{\lambda}(\frac{\mu}{2} - 1)(\tilde{\varphi}\tilde{e}_1)(\lambda) - \frac{1}{2}(\tilde{\varphi}\tilde{e}_1)(\mu)\right)\tilde{\varphi}\tilde{e}_1.
\]

By using Theorem 5.1 and \(\tilde{e}_1 = (e - \varphi e)/\sqrt{2}, \tilde{\varphi}e_1 = (e + \varphi e)/\sqrt{2}\) in (5.10) one can check that in fact \(\tilde{R}(\tilde{e}_1, \tilde{\varphi}e_1)\xi = 0\). Now if we use (5.11), we have

\[
(5.11) \quad \tilde{\sigma}(\tilde{\varphi}e_1)\xi = -\tilde{\sigma}(\tilde{e}_1)\tilde{\varphi}e_1 + \tilde{\sigma}(\tilde{\varphi}e_1)\tilde{e}_1.
\]

Comparing \(\tilde{R}(\tilde{e}_1, \tilde{\varphi}e_1)\xi = 0\) with (5.11), we get

\[
(5.12) \quad \tilde{\sigma}(\tilde{e}_1) = \tilde{\sigma}(\tilde{\varphi}e_1) = 0
\]

So \(\xi\) is an eigenvector of Ricci operator \(\tilde{Q}\).
Remark 5.8. From \((1.2)\) and \((5.8)\) we have
\[
\begin{align*}
\nabla^* \nabla \xi &= -\nabla_{\xi} \nabla_{\xi} \xi + \nabla_{\nabla_{\xi} \xi} \xi + \nabla_{\nabla_{\nabla_{\xi} \xi} \xi} \xi - \nabla_{\nabla_{\nabla_{\nabla_{\xi} \xi} \xi} \xi} \\
&= \left(\frac{1}{\lambda} \left(\frac{\mu}{2} - 1\right) \left(\varphi \xi_{1}\right)(\lambda) - \frac{1}{2} \left(\varphi \xi_{1}\right)(\mu)\right) \xi_{1} \\
&+ \left(\frac{1}{\lambda} \left(\frac{\mu}{2} - 1\right) \xi_{1}(\lambda) - \frac{1}{2} \xi_{1}(\mu)\right) \varphi \xi_{1} \\
&+ \left(2 - \frac{\mu}{2}\right)^2 + \mu^2 - 2\lambda^2) \xi
\end{align*}
\]
\((5.13)\)
Then by using again Theorem \((5.1)\) and \(\tilde{\xi}_{1} = (e - \varphi e)/\sqrt{2}, \varphi \xi_{1} = (e + \varphi e)/\sqrt{2}\) in \((5.13)\) and after long computations one can prove that the last relation reduces to
\[
\nabla^* \nabla \xi = 2 \xi
\]
\((5.14)\)
By virtue of \((5.14)\), we see immediately that \(\xi\) is harmonic vector field on \((M, \varphi, \xi, \eta, \tilde{g})\).

So we can give following Theorem:

**Theorem 5.9.** Let \((M, \varphi, \xi, \eta, \tilde{g})\) be a non-Sasakian \((\kappa, \mu, \nu = \text{const.})\)-contact metric manifold with \(\xi|_{I_{M}} = 0\) and let \((\tilde{\varphi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})\) be the canonical paracontact metric structure induced on \(M\), according to Theorem \((5.2)\). Then \((M, \tilde{\varphi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})\) is \(H\)-paracontact metric manifold.

By using previous Theorem and same procedure as in the Theorem \((1.20)\), we obtain following Theorem.

**Theorem 5.10.** Let \((M, \varphi, \xi, \eta, g)\) be a non-Sasakian \((\kappa, \mu, \nu = \text{const.})\)-contact metric manifold with \(\xi|_{I_{M}} = 0\) and let \((\tilde{\varphi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})\) be the canonical paracontact metric structure induced on \(M\), according to Theorem \((5.2)\). Then the curvature tensor field of the Levi Civita connection of \((M, \tilde{g})\) verifies the following relation
\[
\tilde{R}(X, Y)\xi = (\kappa - 2)(\eta(Y)X - \eta(X)Y) + 2(\eta(Y)\tilde{h}X - \eta(X)\tilde{h}Y) - \nu(\eta(Y)\tilde{\varphi}X - \eta(X)\tilde{\varphi}Y).
\]

6. **Examples**

Now, we are going to construct examples of 3-dimensional \((\tilde{\kappa}, \tilde{\mu}, \tilde{\nu})\)-paracontact metric manifolds.

**Example 6.1.** We consider the 3-dimensional manifold
\[
M = \{ (x, y, z) \in \mathbb{R}^3 \mid 2y + z \neq 0, \ z \neq 0 \}
\]
and the vector fields
\[
\tilde{e}_1 = \frac{\partial}{\partial x}, \quad \tilde{e}_2 = \frac{\partial}{\partial y}, \quad \tilde{e}_3 = (2y + z) \frac{\partial}{\partial x} - (2z - 2z^2) \frac{\partial}{\partial y} + \frac{\partial}{\partial z},
\]
where \(c\) is non zero real constant. The 1-form \(\eta = dx - (2y + z)dz\) defines a contact structure on \(M\) with characteristic vector field \(\xi = \frac{\partial}{\partial z}\). We define the structure tensor \(\varphi \tilde{e}_1 = 0, \varphi \tilde{e}_2 = \tilde{e}_3\) and \(\varphi \tilde{e}_3 = \tilde{e}_2\). Let \(\tilde{g}\) be Lorentzian metric defined by \(\tilde{g}(e_1, e_1) = -\tilde{g}(e_2, e_2) = \tilde{g}(e_3, e_3) = 1\) and \(\tilde{g}(e_1, e_2) = \tilde{g}(e_1, e_3) = \tilde{g}(e_2, e_3) = 0\). Then \((\varphi, \xi, \eta, \tilde{g})\) is a paracontact metric structure on \(M\). Using Lemma \((4.2)\) and Theorem \((4.20)\), we conclude that \(M\) is a generalized \((\tilde{\kappa}, \tilde{\mu})\) paracontact metric manifold with \(\tilde{\kappa} = -1 + z^2, \tilde{\mu} = 2(1 - z)\)

**Example 6.2.** Consider the 3-dimensional manifold
\[
M = \{ (x, y, z) \in \mathbb{R}^3 \mid 2y - z \neq 0, \},
\]
where \((x, y, z)\) are the cartesian coordinates in \(\mathbb{R}^3\). We define three vector fields on \(M\) as
\[
e_1 = (-2y + z) \frac{\partial}{\partial x} + (x - 2y - z) \frac{\partial}{\partial y} + \frac{\partial}{\partial z}, \quad e_2 = \frac{\partial}{\partial y}, \quad e_3 = \xi = \frac{\partial}{\partial x}
\]
Let $\tilde{g}$, $\tilde{\varphi}$ be the Riemannian metric and the $(1,1)$-tensor field given by

$$
\tilde{g} = \begin{pmatrix}
1 & 0 & (2y-z)/2 \\
0 & 0 & 1/2 \\
(2y-z)/2 & 1/2 & -(2y+z)^2 - (x-2y-z)
\end{pmatrix},
\tilde{\varphi} = \begin{pmatrix}
0 & 0 & -2y+z \\
0 & -1 & x-2y-z \\
0 & 0 & 1
\end{pmatrix}.
$$

So we easily obtain $\tilde{g}(e_1,e_1) = \tilde{g}(e_2,e_2) = \tilde{g}(e_1,e_3) = \tilde{g}(e_2,e_3) = 0$ and $\tilde{g}(e_1,e_2) = \tilde{g}(e_3,e_3) = 1$. Moreover we have $\eta = dx + (2y-z)dz$ and $\tilde{h} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ with respect to the basis $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$. By direct calculations we get

$$
\tilde{R}(X,Y)\xi = -\eta(Y)X - \eta(X)Y + 2\eta(Y)\tilde{h}X - \eta(X)\tilde{h}Y
$$

Finally we deduce that $M$ is a $(-1,2,0)$-paracontact metric manifold. Hence this example satisfies Theorem 4.14.

**Remark 6.3.** To our knowledge, the above example is the first numerical example satisfying $\tilde{\kappa} = -1$ and $\tilde{h} \neq 0$ in $\mathbb{R}^3$.

**Example 6.4.** In [27] Koufogiorgos et al. construct following example.

Consider 3-dimensional manifold

$$
M = \{ (x,y,z) \in \mathbb{R}^3 \mid 2x + e^{y+z} > 0, \ y \neq z \}
$$

and the vector fields

$$
e_1 = \frac{\partial}{\partial x},
$$

$$
e_2 = \left( -\frac{y^2 + z^2}{2} \right)^{\frac{1}{2}} \left( 2x + e^{y+z} \right)^{\frac{1}{2}} \frac{\partial}{\partial x} + \frac{z(2x + e^{y+z})^{\frac{1}{2}}}{y} + \frac{(2x + e^{y+z})^{-\frac{1}{2}}}{y} \frac{\partial}{\partial y} + \frac{y(2x + e^{y+z})^{\frac{1}{2}}}{z} + \frac{(2x + e^{y+z})^{-\frac{1}{2}}}{z} \frac{\partial}{\partial z},
$$

$$
e_3 = \left( \frac{y^2 + z^2}{2} \right)^{\frac{1}{2}} \left( 2x + e^{y+z} \right)^{\frac{1}{2}} \frac{\partial}{\partial x} + \frac{z(2x + e^{y+z})^{\frac{1}{2}}}{z} + \frac{(2x + e^{y+z})^{-\frac{1}{2}}}{z} \frac{\partial}{\partial y} + \frac{y(2x + e^{y+z})^{\frac{1}{2}}}{y} + \frac{(2x + e^{y+z})^{-\frac{1}{2}}}{y} \frac{\partial}{\partial z}.
$$

Let $\eta$ be the 1-form dual to $e_1$. The contact Riemannian structure is defined as follows

$$
\xi = e_1, \varphi e_1 = 0, \varphi e_2 = e_3 and \varphi e_3 = -e_2,
$$

$$
g(e_i, e_j) = \delta_{ij} \text{ for any } i, j \in \{1,2,3\}.
$$

Thus it can be deduced that $(M, \varphi, \xi, \eta, g)$ is a $(\kappa, \mu, \nu)$-contact metric manifold with $\kappa = 1 - \frac{1}{2x + e^{y+z}}$, $\mu = 2$ and $\nu = \frac{-2}{(2x + e^{y+z})}$.

Next, using (4.2) we can construct paracontact structure as follows

$$
\tilde{e}_1 = \xi, \tilde{e}_2 = \frac{1}{\sqrt{2}}(e_2 - e_3), \tilde{e}_3 = \frac{1}{\sqrt{2}}(e_2 + e_3) \text{ such that } \varphi \tilde{e}_1 = 0, \varphi \tilde{e}_2 = \tilde{e}_3 \text{ and } \varphi \tilde{e}_3 = \tilde{e}_2
$$

$$
\tilde{g}(\tilde{e}_1, \tilde{e}_1) = 1, \tilde{g}(\tilde{e}_2, \tilde{e}_2) = -1, \tilde{g}(\tilde{e}_3, \tilde{e}_3) = 1 \text{ and } \tilde{g}(\tilde{e}_1, \tilde{e}_2) = \tilde{g}(\tilde{e}_1, \tilde{e}_3) = \tilde{g}(\tilde{e}_2, \tilde{e}_3) = 0.
$$
Moreover, from Lemma 5.4 the matrix form of $\tilde{h}$ is given by

\begin{equation}
\tilde{h} = \begin{pmatrix}
0 & -\lambda & 0 \\
\lambda & 0 & 0 \\
0 & 0 & 0 \\
\end{pmatrix},
\end{equation}

where $\lambda = \sqrt{1 - \kappa}$. By Proposition 5.7 and Theorem 4.20 we finally deduce that $(M, \tilde{\varphi}, \xi, \eta, \tilde{\gamma})$ is a \((\tilde{\kappa}, \tilde{\mu}, \tilde{\nu})\)-paracontact metric manifold $\tilde{\kappa} = \kappa - 2$, $\tilde{\mu} = 2$ and $\tilde{\nu} = -\nu$.

**Remark 6.5.** In the last example $\nu$ is non-constant smooth function.

**Remark 6.6.** Choosing $\nu$ is constant function and using Theorem 5.10 we can construct a family of \((\tilde{\kappa} < -1, \tilde{\mu} = 2, \tilde{\nu} = -\nu)\)-paracontact metric manifolds.

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