Another estimating the absolute value of Mertens function

Rong Qiang Wei∗

Abstract

Through an inversion approach, we suggest a possible estimation for the absolute value of Mertens function $|M(x)|$ that $|M(x)| \sim \left[ \frac{1}{\pi \sqrt{x + \varepsilon}} \right] \sqrt{x}$ (where $x$ is an appropriately large real number, and $\varepsilon$ ($0 < \varepsilon < 1$) is a small real number which makes $2x + \varepsilon$ to be an integer). For any large $x$, we can always find an $\varepsilon$, so that $|M(x)| < \left[ \frac{1}{\pi \sqrt{x + \varepsilon}} \right] \sqrt{x}$.

keywords

Mertens Function, absolute value, Möbius transform, Hilbert transform

1 Introduction

The Mertens function $M(n)$ is defined as the cumulative sum of the Möbius function $\mu(k)$ for all positive integers $n$,

$$M(n) = \sum_{k=1}^{n} \mu(k)$$

Sometimes the above definition can be extended to real numbers as follows:

$$M(x) = \sum_{1 \leq k < x} \mu(k)$$

In many applications of $M(x)$, the estimation of its absolute value is very important and of interest. Mertens (1897) concluded that $|M(x)| < x^{1/2}$, which is known as Mertens conjecture. However, this conjecture was proved false through extensive computations (e.g., Odlyzko and te Riele, 1985; Pintz, 1987; Saouter and te Riele, 2014). These computations above are based or partially based on sieving, and $|M(x)|$ has been computed for all $x \leq 10^{22}$ (Kuznetsov, 2011). The isolated values of $M(2^n)$ has been computed for all positive integers $n \leq 73$ ($|M(2^{73})| = 6524408924$; Hurst, 2016). Moreover, Wei (2016) suggested a disproof for Mertens conjecture from the viewpoint of probability.

∗College of Earth and Planet Sciences, University of Chinese Academy of Sciences, Beijing, PRC, 100049. e-mail: wrq1973@ucas.edu.cn
There are many other explicit upper bounds for $|M(x)|$. For example, Walfisz (1963) showed that,

$$|M(x)| \leq x \exp \left[-c \ln^{3/5} x (\ln \ln x)^{-1/5} \right]$$

(3)

where $c$ is a constant.

MacLeod (1967; 1969) showed that,

$$|M(x)| \leq \frac{x + 1}{80} + \frac{11}{2} \quad (x \geq 1)$$

(4)

El Marraki (1995) proved that,

$$|M(x)| \leq \frac{0.002969}{(\log x)^{1/2}} x \quad (x \geq 142194)$$

(5)

and

$$|M(x)| \leq \frac{0.6437752}{\log x} x \quad (x > 1)$$

(6)

Ramaré (2013) showed that,

$$|M(x)| \leq \frac{0.0146 \log x - 0.1098}{(\log x)^2} x \quad (x \geq 464402)$$

(7)

Other explicit bound for $|M(x)|$ are:

$$|M(x)| < \frac{x}{4345} \quad (x > 2160535)$$

(8)

$$|M(x)| < \frac{0.58782x}{\ln^{11/9}(x)} \quad (x > 685)$$

(9)

Wei (2016) has discussed $|M(x)|$ based on the assumption that $\mu(n)$ is an independent random sequence. This assumption is inferred from the numerical consistency between empirical statistical quantities for $2 \times 10^7 \mu(n)$s and those from number theory. The following inequality (10) for $|M(x)|$ holds with a probability of $1 - \alpha$,

$$|M(x)| \leq \sqrt{\frac{6/\pi^2}{\sqrt{\alpha}}} \sqrt{x}$$

(10)

However, inequality (10) is argued for the reason that the prime factorization of integers is not random (the primes being intricately interdependent) and so the $\mu(n)$ is not random. Wei (2018) estimated the $|M(x)|$ by an approach of statistical mechanics, in which the $\mu(n)$ is taken as a particular state of a modified one-dimensional Ising model without the exchange interaction between the spins. If $M(x)$ is a quantity that can be measured, $|M(x)| \leq \sqrt{\frac{C}{\alpha} x}$ (where

\footnote{From https://en.wikipedia.org/wiki/Mertens_function.}
\( C \) is constant) with a probability \( 1 - \alpha \) (\( 0 < \alpha < 1 \)) from the viewpoint of the energy fluctuations in the canonical ensemble. Without the assumption above, Wei (2017) further explained that \( |M(x)| \leq Cx^{1/2} \) with a very large \( C \) (possible \( +\infty \)) based on three facts. Recently, Czopik (2019) studied on the estimation of the \( M(x) \).

A lot of experiences show that it is difficult to estimate \( |M(x)| \) directly. However, if we can represent \( M(x) \) as other functions which can be easily studied, it is possible to estimate \( |M(x)| \).

2 An estimation for \( |M(x)| \)

**Theorem 1** Let \( f(t) \) be a real-valued and positive function which is monotonic decreasing and integrable on \([a, +\infty)\) (where \( a \) is an integer). Then there is a constant \( c \), so that for each real number \( A \geq c \), we have

\[
\left| \sum_{a \leq n \leq A} f(n) - \left[ c + \int_{a}^{A} f(t) dt \right] \right| \leq f(A) \tag{11}
\]

**Proof:** For the integer \( n \geq a \), let

\[
b_n = f(n) - \int_{n}^{n+1} f(t) dt \tag{12}
\]

Because \( f(t) \) is monotonic decreasing and positive, and there is a point \( \xi \in (n, n+1) \) at least at which \( \int_{n}^{n+1} f(t) dt = f(\xi) \), we have,

\[
f(n) \geq \int_{n}^{n+1} f(t) dt \geq f(n+1) \tag{13}
\]

Further, we have,

\[
f(n) - f(n+1) \geq b_n \geq 0 \tag{14}
\]

Let \( s_N = \sum_{n=a}^{N} b_n \) (where \( N = a, a+1, a+2, \ldots \)). Then,

\[
s_N \leq \sum_{n=a}^{N} [f(n) - f(n+1)] = f(a) - f(N+1) < f(a) \tag{15}
\]

(15) shows that \( s_N \) is an increasing sequence (\( s_N = f(a) - f(N+1) < s_{N+1} = f(a) - f(N+2) < f(a) \)) but it has an upper bound, so it has a limit,

\[
\lim_{N \to +\infty} s_N = c < f(a) \tag{16}
\]

\(^2\)This theorem is not proposed by us. We have forgotten its source and we prove it again with our understanding. Readers who are interested it please refer to Iwaniec and Kowalski (2004), in which there is a similar conclusion (p. 19-20).
Further, we have,

\[ 0 < (c - \sum_{a \leq n \leq A} b_n) = \lim_{N \to +\infty} \sum_{n = [A] + 1}^{N} b_n \leq f([A] + 1) \leq f(A) \]  

(17)

Then,

\[ c - f(A) \leq \sum_{a \leq n \leq A} b_n \leq c \]  

(18)

Notice that

\[ \sum_{a \leq n \leq A} b_n = \sum_{a \leq n \leq A} f(n) - \sum_{a \leq n \leq A} \int_{n}^{n+1} f(t) dt \]

\[ = \sum_{a \leq n \leq A} f(n) - \int_{a}^{[A] + 1} f(t) dt \]  

(19)

From (18) we have,

\[ c - f(A) \leq \sum_{a \leq n \leq A} f(n) - \int_{a}^{[A] + 1} f(t) dt \leq c \]  

(20)

That is,

\[ c - f(A) + \int_{a}^{[A] + 1} f(t) dt \leq \sum_{a \leq n \leq A} f(n) \leq c + \int_{a}^{[A] + 1} f(t) dt \]  

(21)

Notice in (21) that

\[ \int_{a}^{[A] + 1} f(t) dt = \int_{a}^{A} f(t) dt + \int_{A}^{[A] + 1} f(t) dt \]

\[ = \int_{a}^{A} f(t) dt + f(\xi)(A < \xi < [A] + 1) \]  

\[ \leq \int_{a}^{A} f(t) dt + f(A) \]  

(22)

Thus we know,

\[ \sum_{a \leq n \leq A} f(n) \leq c + \int_{a}^{A} f(t) dt + f(A) \]  

(23)

And finally,

\[ \left| \sum_{a \leq n \leq A} f(n) - \left[ c + \int_{a}^{A} f(t) dt \right] \right| \leq f(A) \]  

(24)

QED
Proposition 1

\[ \sum_{1 \leq n \leq A} \frac{1}{n - x} \sim \ln \frac{A}{x + \varepsilon} \quad (A \to +\infty) \tag{25} \]

where \( x \neq n \) is a real number, \( 0 < \varepsilon < 1 \), and \( 2x + \varepsilon \) is an integer.

Proof: If we let \( f(t) = \frac{1}{1-t} (t \neq x) \), and \( a = 2x + \varepsilon \) in theorem, we have,

\[
\left| \sum_{2x+\varepsilon \leq n \leq A} \frac{1}{n - x} - \left[ c + \int_{2x+\varepsilon}^{A} \frac{1}{t-x} \, dt \right] \right| \leq \frac{1}{A-x}
\]

i.e.

\[
\left| \sum_{2x+\varepsilon \leq n \leq A} \frac{1}{n - x} - \left[ c + \ln \frac{A-x}{x+\varepsilon} \right] \right| \leq \frac{1}{A-x}
\]

i.e.

\[
\left| \sum_{2x+\varepsilon \leq n \leq A} \frac{1}{n - x} - \left[ c + \ln \frac{A(1-x/A)}{x+\varepsilon} \right] \right| \leq \frac{1}{A(1-x/A)} \tag{26}
\]

The last step of (26) shows that \( \sum_{2x+\varepsilon \leq n \leq A} \frac{1}{n - x} \) converges to \( c + \ln \frac{A}{x+\varepsilon} \) (uniformly), i.e.,

\[
\sum_{2x+\varepsilon \leq n \leq A} \frac{1}{n - x} = c + \ln \frac{A}{x+\varepsilon} \tag{27}
\]

Adding \( d_n = \sum_{1 \leq n \leq (2x+\varepsilon)-1} \frac{1}{n-x} \) to both sides of the (27) and because \( d_n \) is finite even \( d_n \to 0 \) if an appropriate \( x \) is selected (see details in Remark 3), we can get,

\[
d_n + \sum_{2x+\varepsilon \leq n \leq A} \frac{1}{n - x} = \sum_{1 \leq n \leq A} \frac{1}{n - x}
\]

\[
= c + \ln \frac{A}{x+\varepsilon} + d_n
\]

\[
\sim \ln \frac{A}{x+\varepsilon} \quad (A \to +\infty) \tag{28}
\]

QED

Remark 1.
In proposition the reason for introduction of the parameter \( \varepsilon \) is that Theorem requires \( a \) to be an integer, but \( x \) being an integer will cause \( \frac{1}{n-x} \) to
be meaningless. Therefore, we add an \( \varepsilon \) to \( 2x \) to make \( 2x + \varepsilon \) (\( x \) now is a real number) an integer. Moreover, it can be seen that, like \( A \), \( \varepsilon \) is hidden on the left side of the proposition [1] so the left and right sides of proposition [1] are "balanced" (the same below).

**Remark 2.**

In (28) \( c \) is omitted because it is finite and (far) less than the logarithmic part. In fact, from (16) it can be seen that \( c < f(a) = \frac{1}{x} \). In order to convince oneself this is plausible, it is instructive to consider the Euler constant \( \gamma \),

\[
\gamma = \lim_{N \to +\infty} \left( \sum_{n=1}^{N} \frac{1}{n} - \ln N \right) = 0.577215... \tag{29}
\]

**Remark 3.**

In (28) \( d_n \to 0 \) when \( x \) is very large, \( x = n + \theta/2 \) (\( 0 < \theta < 1 \)), and \( \theta \to 1 \). This can be seen from the following,

It is well known that,

\[
\psi(x - N) = \psi(x) - \sum_{n=1}^{N} \frac{1}{x - n} = \psi(x) + \sum_{k=1}^{N} \frac{1}{n - x} \tag{30}
\]

where \( \psi(x) \) is Psi function.

Therefore,

\[
\sum_{n=1}^{2x+\varepsilon} \frac{1}{n - x} = \psi(-x - \varepsilon) - \psi(x) \approx \psi(-x - \varepsilon) - \psi(x + \varepsilon) = \pi \cot[\pi(x + \varepsilon)] + \frac{1}{x + \varepsilon} \sim \pi \cot[\pi(x + \varepsilon)] \tag{31}
\]

where \( \varepsilon \) is very small, \( x \) is very large, and the upper limit of the sum is \( 2x + \varepsilon \) rather than \( 2x + \varepsilon - 1 \) for convenience.

Obviously \( \cot[\pi(x + \varepsilon)] = 0 \) when \( x + \varepsilon = n + 1/2 \). However, \( x \neq n \), and we can let \( x = n + \theta/2 \) (\( 0 < \theta < 1 \)). So that \( \cot[\pi(x + \varepsilon)] \to 0 \) when \( \theta \to 1 \). At this time, \( \varepsilon \to 0 \). Notice that \( 2x + \varepsilon = 2n + \theta + \varepsilon \) is an integer.

**Theorem 2** Let \( f(x) \) and \( g(x) \) be any functions. If

\[
g(x) = \sum_{k=1}^{\infty} f(k - x) \tag{32}
\]
then

\[ f(x) = \sum_{k=1}^{\infty} \mu(k)g(k-x) \]  

(33)

**Proof:** Please see Theorem 2 and the Propositions related in Chen (1997). \[ (32) \] and \[ (33) \] are special cases of this theorem which are named the parametric Möbius transform formulas by Chen (1997). Or please refer to Theorem 1 in Knockaert (1994). It should be pointed out that the "absolutely convergent" in Theorem 2 of Chen (1997) means that "every infinite series converges to an element which is independent of the combinations of the terms in the sum".

QED

**Theorem 3**

\[ M(x) \sim -\frac{\sqrt{x}}{\pi} \int_{0}^{\infty} \frac{dt}{(t+\varepsilon)^{\frac{3}{2}}(t-x)} \]  

(34)

**Proof:**

From \[ (28) \], we can see that \[ (35) \] is true, ie., \[ \sum_{1 \leq n \leq A} \frac{1}{n-x} \] converges to \[ \ln \frac{A}{x+\varepsilon} \] (uniformly) with the increasing \( A \) to \( +\infty \),

\[ \ln \frac{A}{x+\varepsilon} \sim \sum_{1 \leq n \leq A} \frac{1}{n-x} \quad (A \rightarrow +\infty) \]  

(35)

Thus from Theorem 2 we can write out \[ (36) \],

\[ \frac{1}{x} \sim \sum_{1 \leq n \leq A} \mu(n) \ln \frac{A}{n-(x-\varepsilon)} \]  

(36)

Let \( x' = x - \varepsilon \), and \( x \) denotes \( x' \) for simplicity. We have,

\[
\frac{1}{x+\varepsilon} \sim \sum_{1 \leq n \leq A} \mu(n) \ln \frac{A}{n-x}
\]

\[
= \sum_{1 \leq n \leq A} [M(n) - M(n-1)] \ln \frac{A}{n-x}
\]

\[
= \sum_{1 \leq n \leq A} \left[ M(n) \ln \frac{A}{n-x} - M(n-1) \ln \frac{A}{n-x} \right]
\]

\[
= \sum_{1 \leq n \leq A} \left[ M(n) \ln \frac{A}{n-x} - M(n) \ln \frac{A}{(n+1)-x} \right]
\]

\[
= \sum_{1 \leq n \leq A} M(n) \left[ \ln \frac{A}{n-x} - \ln \frac{A}{(n+1)-x} \right]
\]

\[
= \sum_{1 \leq n \leq A} M(n) [\ln((n+1)-x) - \ln(n-x)]
\]

\[
= \sum_{1 \leq n \leq A} M(n) \int_{n}^{n+1} \frac{dt}{t-x}
\]  

(37)
Noticing that $M(n)$ is constant in each $(n, n+1)$, and if we assume $M(0) = 0$, we can get,

\[
\frac{1}{x + \varepsilon} \sim \sum_{1 \leq n \leq A} M(n) \int_n^{n+1} \frac{dt}{t - x}
\]

\[
= \sum_{n=1}^{A} \int_n^{n+1} M(t) \frac{dt}{t - x}
\]

\[
= \int_1^{A} M(t) \frac{dt}{t - x}
\]

\[
= \int_0^{A} M(t) \frac{dt}{t - x} \quad (M(0) = 0)
\]

\[
= \int_0^{\infty} M(t) \frac{dt}{t - x} \quad (A \to +\infty)
\]

The last step of (38) is Hilbert transform of the $M(t)$ on the semiaxis and its inverse transform (e.g., Polyanin and Manzhirov, 2008; Antipov and Mkhitaryan, 2018. See Appendix in detail) is,

\[
M(x) \sim -\int_0^{\infty} \frac{dt}{(t + \varepsilon)(t - x)^{3/2}}
\]

(39)

QED

**Theorem 4**

\[
|M(x)| \sim \left[\frac{1}{\pi \sqrt{\varepsilon(x + \varepsilon)}}\right] \sqrt{x}
\]

(40)

**Proof:** Because

\[
\text{PV} \int_0^{\infty} \frac{t^{\mu-1}dt}{(t + \varepsilon)(t - x)} = \frac{\pi}{x + \varepsilon} \left[\frac{\varepsilon^{\mu-1}}{\sin(\mu \pi)} + x^{\mu-1} \cot(\mu \pi)\right]
\]

(41)

where $|\text{arg} \varepsilon| < \pi, x > 0, 0 < \text{Re} \mu < 2$ (Erd\'ely et al., 1954; Gradshteyn and Ryzhik, 2014).

Let $\mu = 1/2$ in (41). We have,

\[
- \int_0^{\infty} \frac{t^{-1/2}dt}{(t + \varepsilon)(t - x)} = -\frac{\pi}{\sqrt{\varepsilon(x + \varepsilon)}}
\]

(42)

From Theorem (3), we know,

\[
|M(x)| \sim \left[\frac{1}{\pi \sqrt{\varepsilon(x + \varepsilon)}}\right] \sqrt{x}
\]

(43)

QED

8
3 Discussions

3.1 Similarity of Theorem 4 to (44)

We can change the form of Theorem (4) to (44),

\[ |M(x)| \sim \left( \frac{1}{\pi \sqrt{1-\alpha(x+1-\alpha)}} \right) \sqrt{x} \quad (44) \]

Since \( \alpha (0 < \alpha < 1) \) is not fixed, for a reasonable \( x (= n + \theta / 2, \theta \to 1) \), an \( \alpha \) can always be found so that the following holds,

\[ |M(x)| < \left( \frac{1}{\pi \sqrt{1-\alpha(x+1-\alpha)}} \right) \sqrt{x} \quad (45) \]

(45) is similar to those like (10) in the introduction, in which \( |M(x)| \leq \sqrt{\frac{C}{\alpha}} x \) (where \( C \) is constant) holds with a probability \( 1 - \alpha \) (0 < \( \alpha < 1 \)). The later are estimated based on the assumption that \( \mu(n) \) is an independent random sequence, or from the view point of the energy fluctuations in the canonical ensemble of statistical mechanics. This similarity shows that \( \mu(n) \) could be taken as an independent random sequence. Although \( \mu(n) \) is determined according to that the prime factorization of integers is not random, but it can be produced by an independent and random function. This can be found from another definition of \( \mu(n) \),

\[ \mu(n) = \begin{cases} 0 & \text{if } n \text{ is non-squarefree} \\ (-1)^{\omega(n)} & \text{if } n \text{ is squarefree} \end{cases} \quad (46) \]

where \( \omega(n) \) is the number of distinct prime factors. According to Erdös-Kac Theorem, \( \omega(n) \) is independent and random when \( n \) is large.

On the other hand, the determined function could produce random distribution, for example, the logistic map (47) will generate random numbers (Trott, 2004),

\[ x_{n+1} = \lambda (1 - x_n) \quad (47) \]

where \( x_n \in [0, 1], n = 1, 2, 3, \ldots, 0 < \lambda \leq 4 \).

Therefore, the deterministic problems can be handled by the methods of probability to some extent, or vice versa. For example, some partial differential equations can be studied with the approaches of probability, or an deterministic definite integral can be calculated by the Monte Carlo method.

3.2 On the Theorem 4

It is an Fredholm integral equation of the first kind to obtain \( M(x) \) from \( \frac{1}{x+1} \) in Eq. (38). Such an integral equation is a typical ill-posed problem. Its solution is not unique. Therefore Theorem 4 is one of the possible solution. In fact, the introduction of free parameters \( \varepsilon \) can give a glimpse to this.
4 Conclusion

We suggest a possible estimation to $|M(x)|$ by the inverse Hilbert Transform on the real semiaxis of $\frac{1}{x^\varepsilon}$ ($0 < \varepsilon < 1$). For an appropriately $x (= n + \theta / 2, 0 < \theta < 1, \theta \to 1)$, we can always find a positive real number $\varepsilon$ which makes $2x + \varepsilon$ to be an integer, so that $|M(x)| \sim \frac{1}{\pi \sqrt{\varepsilon(x+\varepsilon)}} \sqrt{x}$. Further, we can select a reasonable $\varepsilon$, so that $|M(x)| < \frac{1}{\pi \sqrt{\varepsilon(x+\varepsilon)}} \sqrt{x}$.

Appendix

Hilbert transform pair on the semiaxis (eg., Polyanin and Manzhirov, 2008; Antipov and Mkhitaryan, 2018) are as follows,

\begin{align*}
    f(x) &= \int_0^\infty \frac{y(t)}{t-x} \, dt \quad (48) \\
    y(x) &= \frac{\sqrt{x}}{\pi^2} \int_0^\infty \frac{f(t)}{\sqrt{t}(t-x)} \, dt \quad (49)
\end{align*}

References

[1] Antipov, Y. A., Mkhitaryan, M. S., 2018. Integral relations associated with the semi-infinite Hilbert transform and applications to singular integral equations, Quarterly of Applied Mathematics, 76: 739-766.

[2] Chen, B. F., 1997. Parametric Möbius inversion formulas, Discrete Mathematics, 169: 211-215.

[3] Czopik, J., 2019. The Estimation of the Mertens Function. Advances in Pure Mathematics, 9, 415-420. https://doi.org/10.4236/apm.2019.94019.

[4] El Marraki, M., 1995. Fonction sommatoire de la fonction $\mu$ de Möbius, majorations effectives fortes, J. Théorie Nombres Bordeaux 7: 407–433.

[5] Erdèlyi, A., Magnus, W., Oberhettinger, F., Tricomi, F. G., Tables of Integral Transforms, vols. I and II. McGraw Hill, New York, 1954.

[6] Gradshteyn, I. S., Ryzhik, I. M., Table of Integrals, Series, and Products (8th Edition). Academic press, 2014. http://dx.doi.org/10.1016/B978-0-12-384933-5.00013-8
[7] Hurst, G., 2016. Computations of the Mertens Function and Improved Bounds on the Mertens Conjecture, https://arxiv.org/abs/1610.08551v2.

[8] Iwaniec, H., Kowalski, E., Analytic number theory, Colloquium publications (American Mathematical Society), 2004.

[9] Knackert, L., 1994. A Generalized Möbius Transform and Arithmetic Fourier Transforms, IEEE Transactions on signal processing, 42(11), 2967-2971.

[10] Kuznetsov, E., 2011. Computing the Mertens function on a GPU, arXiv:1108.0135.

[11] MacLeod, R. A., 1967. A new estimate for the sum $M(x) = \sum_{n \leq x} \mu(n)$, Acta Arith. 13: 49-59. Corrigendum: Acta Arith. 16 (1969): 99-100.

[12] Mertens, F., 1897. Über eine zahlentheoretische Funktion, Sitzungsber. Akad. Wiss. Wien., 106 (IIa): 761-830.

[13] Odlyzko, A. M. and te Riele, H., 1985. Disproof of the Mertens Conjecture, J. reine angew. Math., 357: 138-160.

[14] Pintz, J., 1987. An effective disproof of the Mertens conjecture. Astérisque, (147-148): 325–333.

[15] Polyanin, A. D., Manzhirov, A. V., Handbook of Integral Equations (2nd Edition), Chapman & Hall/CRC Taylor & Francis Group, 2008.

[16] Ramaré, O., 2013. From explicit estimates for primes to explicit estimates for the Möbius function, Acta Arithmetica, 157 (4): 365-379.

[17] Saouter, Y., te Riele, H., 2014. Improved results on the Mertens conjecture, Mathematics of computation, 83(285): 421–433.

[18] Trott, M., The Mathematica GuideBook for Programming. New York: Springer-Verlag, 2004.

[19] Walfisz, A., Weylsche Exponentialsummen in der neuer Zahlentheorie, Mathematische Forschungsberichte, XV, VEB Deutscher Verlag der Wissenschaften, Berlin, 1963. MR 36 #3737.

[20] Wei, R. Q., 2016. A recursive relation and some statistical properties for the Möbius function, International Journal of Mathematics and Computer Science, 11(2): 215-248.

[21] Wei, R. Q., 2017. Two elementary formulae and some complicated properties for Mertens function, Journal of Algebra, Number Theory: Advances and Applications, 18 (1-2): 15-33.

[22] Wei, R. Q., 2018. The upper bound of the Mertens function from the viewpoint of statistical mechanics, https://arxiv.org/abs/1807.09085v2.