COHEN–MACAULAY HYBRID GRAPHS

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Abstract. We introduce a new family of graphs, namely, hybrid graphs. There are infinitely many hybrid graphs associated to a single graph. We show that every hybrid graph associated to a given graph is Cohen–Macaulay. Furthermore, we show that every Cohen–Macaulay chordal graph is a hybrid graph.

INTRODUCTION

Let $G$ be the simple graph on the vertex set $[n]$ and the edge set $E(G)$. We identify the vertex $i$ with the variable $x_i$ and consider the polynomial ring $S = K[x_1, \ldots, x_n]$ over an arbitrary field $K$. Let $I(G) = \langle x_ix_j : \{i, j\} \in E(G) \rangle \subset S$ denotes the edge ideal of $G$. We say that the graph $G$ is Cohen–Macaulay if $S/I(G)$ is Cohen–Macaulay. Using the Stanley–Reisner correspondence, one can associate to $G$ the simplicial complex $\Delta_G$, whose faces are the independent subsets of $[n]$ and whose Stanley–Reisner ideal coincides with the edge ideal of $G$, i.e. $I_{\Delta_G} = I(G)$.

According to [8], it is unlikely to have a general classification of Cohen–Macaulay graphs. Therefore, it is natural to study this question for some special classes of graphs. For instance, Villarreal in [11] gave classification of all Cohen–Macaulay trees. Herzog, Hibi and Zheng [8] classified all Cohen–Macaulay chordal graphs later Herzog and Hibi [6] did for all Cohen–Macaulay bipartite graphs.

The potential aim of this paper is to generate new graphs $G'$ from a given graph $G$ with the property that $G'$ is always Cohen–Macaulay. The classification and construction of Cohen–Macaulay graphs is one of the central problems and enjoys rich literature, for instance [2], [3], [4], [5], [10] and [13]. Here, we generalize the notion of "whiskering", introduced by Villareal [11]. By a whisker to a vertex $x$, we mean to add a new vertex $y \notin [n]$ and an edge $\{x, y\}$. Villareal proved that if we add whiskers to every vertex of any graph $G$, the resulting graph $G'$ is always Cohen–Macaulay. Cook and Nagel [2], extends this work and defined clique-whiskering. They proved that $G'$ obtained by clique-whiskering is always Cohen–Macaulay. Recently, J. Biermann and A. V. Tuyl [1] defined $s$–coloring $\chi$ on a simplicial complex $\Delta$ to construct a Cohen–Macaulay simplicial complex $\Delta_\chi$, generalizing the previous constructions for simplicial complexes. More recent, A. M. Liu and T. Wu [9] generalized this notion of clique-whiskering for obtaining families of sequentially Cohen–Macaulay graphs.

In this paper, we give a construction of hybrid graphs (see 2). The well-known constructions mentioned in [2] and [11] appear to be particular cases of our construction. Moreover, by using the construction discussed in [11], one may obtain one Cohen–Macaulay Graph corresponding to a given simple graph $G$. Similarly, by using the construction, corresponding to each clique partition of vertex set, one may obtain one Cohen–Macaulay

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graph. But in our case, corresponding to any simple graph $G$ and a given clique partition of the vertex set of $G$, one may obtain infinitely many Cohen–Macaulay graphs. We call these new graphs as the hybrid graphs associated to $G$, here is our main result.

**Theorem 0.1** (Theorem 2.5). Let $G$ be a graph on the vertex set $[n]$ and $A_1, \ldots, A_r$ be a partition of $[n]$ into disjoint subsets such that $A_i$ is a clique in $G$. Let $B_i = \{y_{i,1}, \ldots, y_{i,s_i}\}$ and $G' = G_{B_1,\ldots,B_r}$ be a hybrid graph of $G$ with respect to $B_1, \ldots, B_r$. Let $\Delta'$ be the independence complex of $G'$ and $S = \mathbb{K}[x_{i,i}: i = 1, \ldots, n] \cup \{y_{i,j}: i = 1, \ldots, r, j = 1, \ldots, s_i\}$ be the polynomial ring. Then,

(i) $\Delta'$ is pure shellable of dimension $r - 1$.

(ii) The ring $S/I(G')$ is Cohen–Macaulay of dimension $r$.

As a quick consequence of the above theorem, we obtain following well-known results.

**Corollary 0.2** (Villareal). Suppose $G$ is a graph and let $G'$ be the graph obtained by adding a whisker at every vertex $v \in G$. Then the ideal $I(G')$ is Cohen–Macaulay.

**Corollary 0.3** (Cook and Nagel). Let $\pi = \{W_1, \ldots, W_t\}$ be a clique vertex partition of $G$. Then $I(G')$ is Cohen–Macaulay.

Finally, we give a complete characterization all Cohen–Macaulay chordal graphs by adding one more equivalent condition to a well known characterization of Cohen–Macaulay chordal graphs due to Herzog, Hibi and Zehng [8],

**Corollary 0.4.** Let $K$ be a field and $G$ be a chordal graph on the vertex set $[n]$. Let $F_1, \ldots, F_m$ be the facets of $\Delta(G)$ which admit a free vertex. Then the following are equivalent:

1. $G$ is Cohen–Macaulay;
2. $G$ is Cohen–Macaulay over $K$;
3. $G$ is unmixed;
4. $[n]$ is the disjoint union of $F_1, \ldots, F_m$.
5. $G$ is a hybrid graph.

1. Preliminaries

In this section, we recall some necessary definitions and results.

1.1. Simplicial complexes. A simplicial complex $\Delta$ on the vertex set $V = \{v_1, \ldots, v_n\}$ is a collection of subsets of $V$ such that $\{v_i\} \in \Delta$ for all $i$ and, $F \in \Delta$ implies that all subsets of $F$ are also in $\Delta$. The elements of $\Delta$ are called faces and the maximal faces under inclusion are called facets of $\Delta$. We denote by $\mathcal{F}(\Delta)$ the set of facets of $\Delta$. We say that a simplicial complex is pure if all its facets have the same cardinality. The dimension of a face $F$ is $\dim F = |F| - 1$, where $|F|$ denotes the cardinality of $F$. A simplicial complex is called pure if all its facets have the same dimension. The dimension of $\Delta$, $\dim(\Delta)$, is defined as:

$$\dim(\Delta) = \max\{\dim F: F \in \Delta\}.$$ 

Given a simplicial complex $\Delta$ on the vertex set $\{v_1, \ldots, v_n\}$. For $F \subseteq \{v_1, \ldots, v_n\}$ let $x_F = \prod_{v_i \in F} x_i$, and let $x_{\varnothing} = 1$. The non-face ideal or the Stanley-Reisner ideal of $\Delta$, denoted by $I_\Delta$, is an ideal of $S$ generated by square-free monomials $x_F$, where $F \not\in \Delta$. 
Definition 1.1 (Shellable simplicial complexes). A simplicial complex $\Delta$ is called shellable if there is a total order of the facets of $\Delta$, say $F_1, \ldots, F_t$, such that $\langle F_1, \ldots, F_{i-1} \rangle \cap \langle F_i \rangle$ is generated by a non-empty set of maximal proper faces of $F_i$ for $2 \leq i \leq t$. Any such order is called a shelling order of $\Delta$.

To say $F_1, F_2, \ldots, F_t$ is a shelling order of $\Delta$; it is equivalent to saying that for all $F_i$ and all $F_j < F_i$, there exists $x \in F_i \setminus F_j$ and $F_k < F_i$ such that $F_i \setminus F_k = \{x\}$.

The class of shellable simplicial complexes is important due to the following result.

Theorem 1.2. [Theorem 8.2.6, [7]] A pure shellable simplicial complex is Cohen–Macaulay over any arbitrary field.

Graphs. Throughout in this paper, $G$ will denote a simple graph on $[n]$ vertices which means $G$ has no loops or multiple edges. Given a subset $W$ of $[n]$, we define the induced subgraph of $G$ on $W$ to be the subgraph $G_W$ on $W$ consisting of those edges $\{i, j\} \in E(G)$ with $\{i, j\} \subseteq W$. A walk of length $m$ in $G$ is a sequence of vertices $\{i_0, \ldots, i_m\}$ such that $\{i_{j-1}, i_j\}$ are edges in $G$. A cycle of length $m$ is a closed walk $\{i_0, \ldots, i_m\}$ in which $n \geq 3$ and the vertices $i_1, \ldots, i_m$ are distinct. A graph $G$ on $[n]$ is connected if, for any two vertices $i$ and $j$, there is a walk between $i$ and $j$. A connected graph without cycles is said to be a tree. The complete graph $K_m$ has every pair of its $m$ vertices adjacent. A chord of a cycle $C$ is an edge $\{i, j\}$ of $G$ such that $i$ and $j$ are vertices of $C$ with $\{i, j\} \notin E(C)$. A graph is said to be chordal graph if each of its cycles of length $> 3$ has a chord, obviously every tree is a chordal graph. A subset $C$ of $V(G)$ is called a clique of $G$ if induced subgraph $G_C$ is complete. The clique complex $\Delta(G)$ of a finite graph $G$ on $V(G)$ is a simplicial complex whose faces are the cliques of $G$. For a finite simple graph $G$ on $n$ vertices, one may associate a square-free monomial ideal $I(G)$ in $S = K[x_1, \ldots, x_n]$, namely, edge ideal of $G$ defined as,

$$I(G) = (x_i x_j : \{i, j\} \in E(G))$$

We say that $G$ is Cohen–Macaulay over the field $K$, if the associated quotient ring $S/I(G)$ is Cohen–Macaulay. A subset $C \subseteq V(G)$ is called a vertex cover of $G$ if $C \cap E \neq \emptyset$ for all $E \in E(G)$. A vertex cover $C$ is minimal if no proper subset of $C$ is a vertex cover of $G$ and if all minimal vertex covers of $G$ have same cardinality, then we say that $G$ is unmixed. All Cohen–Macaulay graphs are unmixed but not vice versa. The following result characterizes all Cohen–Macaulay chordal graphs.

Theorem 1.3. [8, Theorem 2.1] Let $K$ be a field and $G$ be a chordal graph on the vertex set $[n]$. Let $F_1, \ldots, F_m$ be the facets of $\Delta(G)$ which admit a free vertex. Then the following are equivalent:

1. $G$ is Cohen–Macaulay;
2. $G$ is Cohen–Macaulay over $K$;
3. $G$ is unmixed;
4. $[n]$ is the disjoint union of $F_1, \ldots, F_m$.

2. Main result

The following definition is essential to understand the underlined construction.

Definition 2.1. Let $G$ be a graph on the vertex set $[n]$ and $A_1, \ldots, A_r$ be a partition of $[n]$ into disjoint subsets such that $A_i$ is a clique in $G(A_i)$ can be empty set). For each
Example 2.4. Let $B_i = \{y_{i,1}, \ldots, y_{i,s_i}\}$ be a non-empty set. Define the graph $G_{A_1, \ldots, A_r}^{B_1, \ldots, B_r}$ as follows:

$$G_{A_1, \ldots, A_r}^{B_1, \ldots, B_r} = G \cup \left( \bigcup_{i=1}^r \{F \subseteq A_i \cup B_i : |F| = 2\} \right). \quad (1)$$

We call the graph $G_{A_1, \ldots, A_r}^{B_1, \ldots, B_r}$ the hybrid graph of $G$ with respect to $B_1, \ldots, B_r$.

**Remark 2.2.** Corresponding to each partition of the vertex set, we have infinitely many choices to choose $B_i$’s thus there are infinitely many hybrid graphs corresponding to any given graph.

Let $G$ be a graph and $G’ = G_{A_1, \ldots, A_r}^{B_1, \ldots, B_r}$ be the hybrid graph of $G$ with respect to $B_1, \ldots, B_r$. Let $\Delta$ and $\Delta’$ be the independence complexes of $G$ and $G’$ respectively. Let

$$S = K[\{x_i : i = 1, \ldots, n\} \cup \{y_{i,j} : i = 1, \ldots, r, j = 1, \ldots, s_i\}]$$

be the polynomial ring. Let us define the ordering on the variables:

$$x_1 > \cdots > x_n > y_{1,1} > \cdots > y_{1,s_1} > \cdots > y_{r,1} > \cdots > y_{r,s_r} \quad (2)$$

As the facets of $\Delta’$ are maximal independent sets in $G’$, it is easy to see that the induced subgraph of $G’$ on $A_i \cup B_i$ is a complete graph. Thus in an independent set, we can select at most one element from $A_i \cup B_i$ for all $i$. In other words, if $T$ be a facet of $\Delta’$, then $|T \cap (A_i \cup B_i)| = 1$ for all $i$. Let $F = T \cap [n]$, then $F$ will be an independent set in $G$ and hence a face of $\Delta$. Let us consider $B = \bigcup_{i=1}^r B_i$ and $F’ = T \cap B$, then $T = F \cup F’$ and $F \cap F’ = \emptyset$. It is easy to note that $F’ = \bigcup_{j:A_j \cap F=\emptyset} \{y_{j,k_j}\}$ for some $1 \leq k_j \leq s_j$. Let us record this simple observation in the following proposition.

**Proposition 2.3.** The independence complex $\Delta’$ of the hybrid graph $G’$ is pure and every facet of $\Delta’$ is of the form $F \cup F’$, where $F$ is a face of $\Delta$ and $F’ = \bigcup_{j:A_j \cap F=\emptyset} \{y_{j,k_j}\}$ for some $1 \leq k_j \leq s_j$.

Note that there are $\prod_{j:A_j \cap F=\emptyset} |s_j|$ choices for $F’$, thus corresponding to each face $F$ of $\Delta$, there is a block of facets of $\Delta’$. Here, we explain this fact through the following example.

**Example 2.4.** Consider the graph $G$ with vertex set $V(G) = \{1, 2, 3, 4\}$ and edge set $E(G) = \{\{1, 2\}, \{1, 3\}, \{2, 3\}, \{2, 4\}, \{3, 4\}\}$. Consider the vertex partition as $V(G) = \{1\} \cup \{2\} \cup \{3\} \cup \{4\}$ and take $B_1 = \{5, 6\}, B_2 = \{7\}, B_3 = \{8\}, B_4 = \{9, 10, 11\}$ then $G’ = G_{A_1, \ldots, A_4}^{B_1, \ldots, B_4}$ will be,

![Diagram](image)

The facets of $\Delta’$ are shown in Table A.

If we consider another partition of vertex set as $V(G) = \{1, 3\} \cup \{2\} \cup \{4\}$ and $B_1 = \{5\}, B_2 = \{6\}, B_3 = \{7, 8, 9\}$, then $G’ = G_{A_1, \ldots, A_4}^{B_1, \ldots, B_3}$ will be,
polynomial ring. Then, $G$ complex of $G$ into disjoint subsets such that $A_i$ is a clique in $G$. Let $B_i = \{y_{i,1}, \ldots, y_{i,s_i}\}$ and $G' = G_{A_1, \ldots, A_r}$ be a hybrid graph of $G$ with respect to $B_1, \ldots, B_r$. Let $\Delta'$ be the independence complex of $G'$ and $S = K[\{x_i: \ i = 1, \ldots, n\} \cup \{y_{i,j}: \ i = 1, \ldots, r, \ j = 1, \ldots, s_i\}]$ be the polynomial ring. Then,

| Faces of $\Delta$ | Corresponding facets of $\Delta'$ |
|-------------------|---------------------------------|
| $\emptyset$       | $\{5, 7, 8, 9\}, \{5, 7, 10\}, \{5, 7, 11\}, \{6, 7, 8, 9\}, \{6, 7, 8, 10\}, \{6, 7, 8, 11\}$ |
| $\{1\}$           | $\{1\} \cup \{7, 8, 9\}, \{1\} \cup \{7, 8, 10\}, \{1\} \cup \{7, 8, 11\}$ |
| $\{2\}$           | $\{2\} \cup \{5, 8, 9\}, \{2\} \cup \{5, 8, 10\}, \{2\} \cup \{5, 8, 11\}, \{2\} \cup \{6, 8, 9\}, \{2\} \cup \{6, 8, 10\}, \{2\} \cup \{6, 8, 11\}$ |
| $\{3\}$           | $\{3\} \cup \{5, 7, 9\}, \{3\} \cup \{5, 7, 10\}, \{3\} \cup \{5, 7, 11\}, \{3\} \cup \{6, 7, 9\}, \{3\} \cup \{6, 7, 10\}, \{3\} \cup \{6, 7, 11\}$ |
| $\{4\}$           | $\{4\} \cup \{5, 7, 8\}, \{4\} \cup \{6, 7, 8\}$ |
| $\{1, 4\}$        | $\{1, 2\} \cup \{7, 8\}$ |

In this case, Table B describes the facets of $\Delta'$.

| Faces of $\Delta$ | Corresponding facets of $\Delta'$ |
|-------------------|---------------------------------|
| $\emptyset$       | $\{5, 6, 7\}, \{5, 6, 8\}, \{5, 6, 9\}$ |
| $\{1\}$           | $\{1\} \cup \{6, 7\}, \{1\} \cup \{6, 8\}, \{1\} \cup \{6, 9\}$ |
| $\{2\}$           | $\{2\} \cup \{5, 7\}, \{2\} \cup \{5, 8\}, \{2\} \cup \{5, 9\}$ |
| $\{3\}$           | $\{3\} \cup \{6, 7\}, \{3\} \cup \{6, 8\}, \{3\} \cup \{6, 9\}$ |
| $\{4\}$           | $\{4\} \cup \{5, 6\}$ |
| $\{1, 4\}$        | $\{1, 4\} \cup \{6\}$ |

Lastly, assume the partition of vertex set as, $V(G) = \{1, 2, 3\} \cup \{4\}$ and $B_1 = \{5\}, B_2 = \{6, 7\}$, then $G' = G_{A_1, \ldots, A_2}$ will be.

See the table C for complete list of facets of $\Delta'$ in this case.

Now, we present the main result of this paper.

**Theorem 2.5.** Let $G$ be a graph on the vertex set $[n]$ and $A_1, \ldots, A_r$ be a partition of $[n]$ into disjoint subsets such that $A_i$ is a clique in $G$. Let $B_i = \{y_{i,1}, \ldots, y_{i,s_i}\}$ and $G' = G_{A_1, \ldots, A_r}$ be a hybrid graph of $G$ with respect to $B_1, \ldots, B_r$. Let $\Delta'$ be the independence complex of $G'$ and $S = K[\{x_i: \ i = 1, \ldots, n\} \cup \{y_{i,j}: \ i = 1, \ldots, r, \ j = 1, \ldots, s_i\}]$ be the polynomial ring. Then,
Table 3. C

| Faces of $\Delta$ | Corresponding facets of $\Delta'$ |
|-------------------|----------------------------------|
| $\emptyset$       | $\{5, 6\}, \{5, 7\}$           |
| $\{1\}$           | $\{1\} \cup \{6\}, \{1\} \cup \{7\}$ |
| $\{2\}$           | $\{2\} \cup \{6\}, \{2\} \cup \{7\}$ |
| $\{3\}$           | $\{3\} \cup \{6\}, \{3\} \cup \{7\}$ |
| $\{4\}$           | $\{4\} \cup \{5\}$             |
| $\{1, 4\}$        | $\{1, 4\} \cup \emptyset$      |

(i) $\Delta'$ is pure shellable of dimension $r - 1$.
(ii) The ring $S/I(G')$ is Cohen-Macaulay of dimension $r$.

Proof. Lemma 2.3 guarantees that the independence complex $\Delta'$ of $G'$ is pure and has
dimension $r - 1$, thus it is sufficient to show that $\Delta'$ is shellable. From above discussion,
we know that corresponding to every face of $\Delta$, there is a block of facets of $\Delta'$. Now we
define an order the facets of $\Delta'$ to show that $\Delta'$ is shellable.

We order the faces of $\Delta'$ in terms of increasing dimensions. If two faces
have same dimension, we order them by (2). Thus, associated to each face
$F$ of $\Delta'$, we consider the block associated to $F$, ordered as in (2).

Let us assume $S$ and $T$ be two distinct facets of $\Delta'$ with $S < T$, here arises two cases:

**Case: 1. When $S$ and $T$ belongs to different blocks** We can write $S = F \cup F'$
and $T = G \cup G'$ where $F, G$ are different facets of $\Delta$ and $F' = \bigcup_{j:A_j \cap F = \emptyset} \{y_{j,k_j}\}$ for some
$1 \leq k_j \leq s_j$, $G' = \bigcup_{j:A_j \cap G = \emptyset} \{y_{j,p_j}\}$ for some $1 \leq p_j \leq s_j$.

As $F \neq G$, we must have some $x_t \in G \setminus F$. Let $G_1 = G \setminus \{x_t\}$, then $G_1$ will also be a face
of $\Delta$. As $G_1 \subseteq G$ so

$$\{j : A_j \cap G = \emptyset\} \subseteq \{j : A_j \cap G_1 = \emptyset\}$$

thus $G' \subseteq G_1' := \bigcup_{j:A_j \cap G_1 = \emptyset} \{y_{j,p_j}\}$ for some $1 \leq p_j \leq s_j$. In fact, if $x_t \in A_p$ then
$G_1' = G' \cup \{y_{p,k_p}\}$ for some $1 \leq k_p \leq s_p$. Let us take $T_1 = G_1 \cup G_1'$, then $T_1 < T$ with
$T \setminus T_1 = \{x_t\}$.

**Case: 2. When $S$ and $T$ belongs to same block** In this case, $S = F \cup F'$ and
$T = F \cup F''$. As $S \neq T$, we have $F' \neq F''$ and $T \setminus S \neq \emptyset$. Let $l$ be the least number, such
that $y_{l,k_l} \in F'' \setminus F'$, thus $y_{l,k_l} \in F''$ and $y_{l,k_l} \notin F'$ which further implies the existence of a
$y_{l,k_l'} \in F'$ for some $1 \leq k_l' \neq k_l \leq s_l$ with $y_{l,k_l'} < y_{l,k_l}$ as $S < T$.

If $y_{l,k_l}$ is the only vertex in $F'' \setminus F'$, we are done, otherwise suppose $y_{m,k_m} \in F'' \setminus F'$.
We have ordered $F'$ and $F''$ as defined in (2) and as we have assumed $l$ to be least such
number, thus the first $l - 1$ components in $F'$ and $F''$ will be the same. Thus $F'$ and $F''$
will be of the form,

$$F' = \{\ldots, y_{l,k_l'}, \ldots, y_{m,k_m}, \ldots\}$$

$$F'' = \{\ldots, y_{l,k_l}, \ldots, y_{m,k_m}, \ldots\}$$

where $y_{m,k_m} \neq y_{m,k_m}$. Let us consider,

$$F'' = \{\ldots, y_{l,k_l'}, \ldots, y_{m,k_m}, \ldots\}$$

and suppose $T_1 = F \cup F''$, then $T_1$ will be a facet of $\Delta'$ by Lemma 2.3 with $T_1 < T$. Note
that $y_{m,k_m} \notin F'' \setminus F''$ and $y_{l,k_l} \in F'' \setminus F''$, thus $y_{m,k_m} \notin T \setminus T_1$ and $y_{l,k_l} \in T \setminus T_1$. If $y_{l,k_l}$
is the only element in $T \setminus T_1$, we are done, otherwise we shall repeat the same process.
As we have finite element, this process will terminate in finite steps and we shall have a $T_i < T$ such that $T \setminus T_i = \{y_{i,k_i}\}$, as required.

Here, we give the descriptive definition of a hybrid graph.

**Definition 2.6.** A graph $G$ is said to be **hybrid** if there exists some graph $H$ such that $G$ is a hybrid graph associated to $H$.

The following result shows that the whiskering of a graph, given by Villarreal in [11] is a particular case of our construction.

**Corollary 2.7 (Villareal).** Suppose $G$ is a graph and let $G'$ be the graph obtained by adding a whisker at every vertex $v \in G$. Then the ideal $I(G')$ is Cohen–Macaulay.

**Proof.** Suppose $V(G) = \{v_1, \ldots, v_n\}$ and consider the trivial clique partition of $V(G)$ into singleton sets as, $V(G) = \{v_1\} \cup \ldots \cup \{v_n\}$. If we take $B_i = \{y_i\}$ for all $1 \leq i \leq n$, then $G' = G_{B_1, \ldots, B_n}$, thus $\Delta'$ is pure and $I(G')$ is Cohen–Macaulay by Theorem 2.5.

R. Woodroof [13] and D. Cook and U. Nagel [2] generalized the concept of whiskering, and defined the terms clique-whiskering and fully clique-whiskering respectively. Recall from [2] that a vertex clique-partition $\pi$ of a graph $G$ is a partition $\pi = \{W_1, \ldots, W_t\}$ of $V(G)$ such that each subgraph induced on $W_i$ is a nonempty clique, see [13]. The clique-whiskering is particular case of our construction.

**Corollary 2.8 (Theorem 3.3, [2]).** Let $\pi = \{W_1, \ldots, W_t\}$ be a vertex clique-partition of $G$. Then $I(G^\pi)$ is Cohen–Macaulay.

**Proof.** If $\pi = \{W_1, \ldots, W_t\}$ be a clique vertex partition of $G$. Let us take $B_i = \{y_i\}$, singleton sets for all $i$. Then $G^\pi = H_{W_1, \ldots, W_t}^{B_1, \ldots, B_t}$ is a particular hybrid graph associated to $G$ and hence Cohen–Macaulay by Theorem 2.5.

Herzog et al. [8] characterize Cohen–Macaulay chordal graphs. One can see easily that every Cohen–Macaulay chordal graph is in fact a hybrid graph associated to some graph. Thus, it characterizes all Cohen–Macaulay chordal graphs. By a free vertex in a simplicial complex, we mean a vertex which belongs to exactly on facet of the simplicial complex.

**Corollary 2.9.** Let $K$ be a field and $G$ be a chordal graph on the vertex set $[n]$. Let $F_1, \ldots, F_m$ be the facets of $\Delta(G)$ which admit a free vertex. Then the following are equivalent:

1. $G$ is Cohen–Macaulay;
2. $G$ is Cohen–Macaulay over $K$;
3. $G$ is unmixed;
4. $[n]$ is the disjoint union of $F_1, \ldots, F_m$.
5. $G$ is a hybrid graph.

**Proof.** (4) ⇒ (5) As, $[n] = F_1 \cup \ldots \cup F_m$ where $F_i$ are cliques of $\Delta(G)$ containing free vertices. Let $A_i$ and $B_i$ denote the non-free and free vertices of $F_i$ respectively. Let $A = \bigcup_{i=1}^{r} A_i$ and $H := G_A$ be the induced graph. Then $G = H_{A_1, \ldots, A_m}^{B_1, \ldots, B_t}$.

(5) ⇒ (1) Follows from Theorem 2.5.

In particular, one obtains
Corollary 2.10 (Corollary 6.3.5, [12]). If $G$ is a tree then the following are equivalent:

1. $G$ is Cohen–Macaulay.
2. $G$ is unmixed.
3. $G$ is a hybrid graph.

Remark 2.11. In small graphs, diagrammatically it is quite easy to check whether a given graph is hybrid or not. It is pertinent to mention that there exist graphs that are not hybrid but still Cohen–Macaulay. For example, 5–cycle is not hybrid but Cohen–Macaulay.

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