Moments of zeta and correlations of divisor-sums: Stratification and Vandermonde integrals

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Abstract
We refine a recent heuristic developed by Keating and the second author. Our improvement leads to a new integral expression for the conjectured asymptotic formula for shifted moments of the Riemann zeta-function. This expression is analogous to a formula, recently discovered by Brad Rodgers and Kannan Soundararajan, for moments of characteristic polynomials of random matrices from the unitary group.

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1 | INTRODUCTION

One of the most important problems in analytic number theory is to find an asymptotic formula for the $2k$th moment

$$M_k(T) := \int_0^T |\zeta(1/2 + it)|^{2k} \, dt$$

of the Riemann zeta-function $\zeta(s)$, where $k$ is an arbitrary positive integer. A folklore conjecture suggests that, for some unspecified constant $c_k$, we have $M_k(T) \sim c_k T (\log T)^{k^2}$ as $T \to \infty$. To date, this is known only for $k = 1, 2$, with $c_1 = 1$ and $c_2 = 1/(2\pi^2)$ [27]. The problem is so intractable that, up until recently, there had been no viable guess for the exact value of $c_k$. The approach of using correlations of divisor sums leads to conjectures for $c_3$ and $c_4$ [8, 9], and the process has
recently been examined in close detail and made more precise by Ng [24], Hamieh and Ng [20], and Ng, Shen, and Wong [25]. This approach seems to fail, however, to give a reasonable guess for \( c_k \) when \( k \geq 5 \) [9].

A breakthrough was made when Keating and Snaith [22] used random matrix theory to predict the exact value of \( c_k \) for all complex \( k \) with \( \Re(k) \geq -1/2 \). Remarkably, their predicted values of \( c_3 \) and \( c_4 \) agree with the conjectures in [8] and [9]. The conjectures for \( c_k \) for positive integers \( k \) were then refined by Farmer, Keating, Rubinstein, Snaith, and the second author [6] through a heuristic method called the recipe, which also applies to a general family of \( L \)-functions. At about the same time, Diaconu, Goldfeld, and Hoffstein [16] predicted the same values of \( c_k \) via a different approach using multiple Dirichlet series. Despite the differences between these approaches, all the conjectures agree.

The recipe arrives at the conjecture by using the approximate functional equation and then predicting that certain “off-diagonal” terms cancel in the evaluation of the moment. However, it does not indicate how these terms cancel or combine. In a recent series of papers [10–14], Keating and the second author address this problem by revisiting the now conventional approach of using Dirichlet polynomial approximations to \( \zeta^k(s) \) and examining correlations of divisor sums. This approach, which relies on the delta method of Duke, Friedlander, and Iwaniec [17], was previously employed by Gonek and the second author [9] to conjecture the values of \( c_3 \) and \( c_4 \) from a number theoretic perspective. The principal result of [14] is a new heuristic method that indicates how divisor sums may be combined to recover the prediction of the recipe. This new heuristic is inspired by ideas of Bogomolny and Keating [3, 4], and is reminiscent of the Hardy–Littlewood circle method.

In this paper, we refine the approach of [14] by using an integral form of the asymptotic formula for correlations of divisor sums that is predicted by the delta method. Through this and a generalization of the local calculations in [14], we predict that combining the divisor sums in the same way as in [14] leads to a certain “Vandermonde integral” expression (Conjecture 1.4). Evaluating this Vandermonde integral then immediately gives the sum of the \( \ell \)-swap terms from the recipe prediction (Conjecture 1.2), and thus removes the need to examine all possible decompositions of \( A \) and \( B \) as in [14, section 12].

We are interested in the shifted moments

\[
\frac{1}{T} \int_T^{2T} \prod_{\alpha \in A} \zeta \left( \frac{1}{2} + \alpha + it \right) \prod_{\beta \in B} \zeta \left( \frac{1}{2} + \beta - it \right) dt,
\]

where \( T \) is a parameter tending to \( \infty \) and \( A \) and \( B \) are finite multisets of complex numbers that have small moduli (say \( \ll 1/\log T \)). We study these moments by examining their Dirichlet polynomial approximations

\[
\mathcal{M}_{A,B}(T,X) := \frac{1}{T} \int_0^\infty \psi \left( \frac{t}{T} \right) \sum_{m=1}^\infty \sum_{n=1}^\infty \frac{\tau_A(m)}{m^z} Y \left( \frac{m}{X} \right) \frac{\tau_B(n)}{n^z} Y \left( \frac{n}{X} \right) dt,
\]

where \( X \) is another parameter tending to \( \infty \), \( \psi \) is a smooth, nonnegative function that is supported on \([1,2]\), say, and \( Y \) is a smooth, nonnegative function that is supported on \([0,1]\), say, and satisfies \( Y(0) > 0 \). Here, the coefficients \( \tau_A \) are defined for finite multisets \( A \) by

\[
\sum_{m=1}^\infty \frac{\tau_A(m)}{m^z} = \prod_{\alpha \in A} \zeta(s + \alpha)
\]
for all $s$ such that the left-hand side converges, where the product on the right-hand side is over all $\alpha \in A$, counted with multiplicity. Bringing the integration over $t$ inside the summation over $m, n$ in (1.1) gives

$$
\mathcal{M}_{A,B}(T,X) = \sum_{m=1}^{\infty} \frac{\tau_A(m)}{\sqrt{m}} Y\left(\frac{m}{X}\right) \sum_{n=1}^{\infty} \frac{\tau_B(n)}{\sqrt{n}} Y\left(\frac{n}{X}\right) \hat{\psi}\left(\frac{T}{2\pi \log \frac{m}{n}}\right),
$$

(1.3)

where

$$
\hat{\psi}(x) := \int_{-\infty}^{\infty} \psi(t) e^{-2\pi i t x} dt
$$

is the Fourier transform of $\psi$. The evaluation of (1.1) may therefore be viewed as a weighted counting of points $(m, n)$ on some algebraic variety. As suggested by Trevor Wooley, we may thus view our heuristic in terms of counting points on a variety via stratification and counting points on its subvarieties. We will discuss this further below Conjecture 1.4.

The recipe of [6] leads to the following prediction (see Section 2).

**Conjecture 1.1** (Conrey, Farmer, Keating, Rubinstein, and Snaith, 2005). If $\alpha, \beta \ll 1/ \log T$ for each $\alpha \in A$ and $\beta \in B$, then as $T \to \infty$ we have

$$
\mathcal{M}_{A,B}(T,X) \sim \sum_{U \subseteq A, V \subseteq B \atop |U| = |V|} \frac{1}{(2\pi i)^2} \int_{(\varepsilon)} \int_{(\varepsilon)} \tilde{Y}(\xi) \tilde{Y}(\eta) X^{\xi + \eta} \frac{1}{T} \int_{0}^{\infty} \psi\left(\frac{t}{T}\right) dt 
\times \prod_{\alpha \in U} \chi\left(\frac{1}{2} + \xi + \alpha + it\right) \prod_{\beta \in V} \chi\left(\frac{1}{2} + \eta + \beta - it\right) 
\times \sum_{n=1}^{\infty} \frac{\tau_A(\{x_a \cup (\eta)\}^{-} \tau_B(\{k \cup (\xi)\}^{-}))}{n} \right) \right) \right) \right) dt \ d\xi \ d\eta,
$$

(1.4)

where $\varepsilon > 0$ is an arbitrarily small constant, $\tilde{Y}$ denotes the Mellin transform of $Y$, $\chi$ is the factor from the functional equation $\zeta(s) = \chi(s)\zeta(1 - s)$, and the $n$-sum should be interpreted as its analytic continuation.
at 0 that is significant in size. On the other hand, if $X \ll T^\epsilon$, then we expect the $\ell'$-swap terms in (1.4) to be negligible. This is because in this case, moving the $\xi$- and $\eta$-lines to the right of 0 would make the value of the integral negligible without passing any poles.

Thus, for example, if $X < T^{1-\epsilon}$, then we expect only the 0-swap term to contribute to the main term. It is straightforward to prove this claim and evaluate (1.1) in this case because the rapid decay of the Fourier transform of $\psi$ implies that the terms with $m \neq n$ in (1.3) have negligible size when $X < T^{1-\epsilon}$. If $T^{1+\epsilon} < X < T^{2-\epsilon}$, then the situation is more complicated. In this case, the general results of Goldston and Gonek [19] indicate how we might be able to estimate (1.1) using correlations $\sum_{n \leq x} \tau_A(n)\tau_B(n+h)$ of divisor-sums. These sums can, in turn, be analyzed via the delta method of Duke, Friedlander, and Iwaniec [17]. This approach has been used by Gonek and the second author [9] to recover the random matrix theory prediction for the sixth and the eighth moments of zeta, and has been further developed by Keating and the second author [12] to heuristically evaluate (1.1) when $T^{1+\epsilon} < X < T^{2-\epsilon}$. Their ideas were made precise by Hamieh and Ng [20], who assumed an asymptotic formula for correlations of divisor sums that has a specific error term. Under this assumption, Hamieh and Ng proved that the 0- and 1-swap terms predicted by the recipe are indeed the main terms in the asymptotic formula for (1.1). Analogues of this result have also been proved for different families of L-functions [2, 7, 15]. These analogues hold assuming that the Generalized Lindelöf Hypothesis is true, and do not require any unproved hypothesis about correlations of divisor sums.

Estimating (1.1) for $X < T^{1-\epsilon}$ is straightforward, and the above mentioned works indicate that understanding correlations of divisor sums via the delta method of [17] may be the key to evaluating (1.1) when $T^{1+\epsilon} < X < T^{2-\epsilon}$. If $X > T^{2+\epsilon}$, then it seems that the delta method by itself is no longer enough. Indeed, using the Dirichlet polynomial approach, as in [9], to predict an asymptotic formula for the 10th moment of $\zeta(s)$ requires a Dirichlet polynomial approximation of length $X \approx T^{5/2}$ by the approximate functional equation, and the approach fails in this case because it predicts a negative leading term, as mentioned in [10, section 1]. Thus, we need new ideas to attack the problem of evaluating (1.1). Inspired by the ideas of Bogomolny and Keating [3, 4], Keating and the second author devised a new approach to the problem. The ideas behind their approach are based on the observation that the summand in (1.3) is negligible unless essentially

$$T \log \frac{m}{n} \ll 1,$$

because $\hat{\psi}(x)$ decays rapidly as $|x| \to \infty$. We thus expect the summand in (1.3) to contribute to the main term of the asymptotic formula if $m$ is close to $n$. More precisely, as

$$T \log \frac{m}{n} \approx T \left( \frac{m-n}{n} \right),$$

we see, at least intuitively, that the summand in (1.3) is negligible unless

$$m-n \ll \frac{X}{T}.$$

Hence, if $X < T^{1-\epsilon}$, then we expect only the terms with $m = n$ to contribute to the main term, and this can be proved in a straightforward way. If $T \ll X \ll T^2$, then $X/T \ll T$ and we expect the terms with $m-n \ll T$ to contribute to the main term. These terms may be analyzed using the delta method as discussed above. If $T^2 \ll X \ll T^3$, then we need to also examine the $m, n$ with
\( T \ll m - n \ll T^2 \). These seem to fall beyond the range of the delta method, but we may factor \( m = m_1 m_2 \) and \( n = n_1 n_2 \) and write

\[
T \log \frac{m}{n} = T \log \frac{m_1}{n_1} + T \log \frac{m_2}{n_2}.
\]

We then expect the terms with \( m_1/n_1 \) close to \( n_2/m_2 \) to contribute to the main term. A key realization in [11] and [13] is that it is useful to examine the \( m_1, n_1 \) such that \( m_1/n_1 \approx M/N \), where \( M/N \) is a fraction in lowest terms that has a small denominator. When \( m_1/n_1 \) is close to \( n_2/m_2 \), if \( m_1/n_1 \approx M/N \) then we must have \( m_2/n_2 \approx N/M \). This idea leads to summing over solutions of the systems

\[
M n_1 - N m_1 = h_1,
\]
\[
M m_2 - N n_2 = h_2,
\]

with \( M, N, h_1, h_2 \) ranging over all integers such that \( M, N > 0 \) and \( (M, N) = 1 \). We may view these solutions as points that give rise to new terms that were not accounted for in the previous heuristic in [9]. As pointed out by Wooley (see the discussion below Conjecture 1.4), this is analogous to counting points \((m_1, m_2, n_1, n_2)\) in the varieties

\[
M_1 m_1 - N_1 n_1 = h_1
\]
\[
M_2 m_2 - N_2 n_2 = h_2
\]

with \( |h| \leq H \), say, by examining its subvarieties. Keating and the second author [14] generalized the ideas in [11] and [13] to the situation with \( T^\ell \ll X \ll T^{\ell+1} \) for a general integer \( \ell \) by collecting the \( m_j, n_j \) with \( m_j/n_j \) close to the “rational direction” \( M_j/N_j \) and considering all possible directions subject to the conditions \((M_j, N_j) = 1 \) and \( M_1 \cdots M_\ell = N_1 \cdots N_\ell \). This leads naturally to a conjectural splitting of (1.1), as in the following discussion leading up to Conjecture 1.3.

For each integer \( \ell \) with \( 1 \leq \ell \leq \min\{|A|, |B|\} \), let \( A = A_1 \cup \cdots \cup A_\ell \) be an arbitrary partition of \( A \) into \( \ell \) nonempty sets, and similarly let \( B = B_1 \cup \cdots \cup B_\ell \) be an arbitrary partition of \( B \) into \( \ell \) nonempty sets. By (1.2), we may then express \( \tau_A \) and \( \tau_B \) as the Dirichlet convolutions \( \tau_A = \tau_{A_1} * \cdots * \tau_{A_\ell} \) and \( \tau_B = \tau_{B_1} * \cdots * \tau_{B_\ell} \), and deduce from (1.1) that

\[
\mathcal{M}_{A,B}(T,X) = \frac{1}{T} \int_0^\infty \psi \left( \frac{t}{T} \right) \left\{ \sum_{1 \leq m_1 \ldots m_\ell < \infty \atop 1 \leq n_1 \ldots n_\ell < \infty} \frac{\tau_{A_1}(m_1) \cdots \tau_{A_\ell}(m_\ell) \tau_{B_1}(n_1) \cdots \tau_{B_\ell}(n_\ell)}{(m_1 \cdots m_\ell)^{1/2 + it}(n_1 \cdots n_\ell)^{1/2 - it}} \right\} \psi \left( \frac{t}{T} \right) \frac{Y\left( \frac{m_1 \cdots m_\ell}{X} \right) Y\left( \frac{n_1 \cdots n_\ell}{X} \right)}{X^2} \, dt.
\] (1.5)

Following the ideas in [14], for each \( \ell \) with \( 1 \leq \ell \leq \min\{|A|, |B|\} \), we collect the terms in (1.5) that have each \( m_j/n_j \) “close to, but not exactly” \( M_j/N_j \), and sum these terms over all possible \( M_j/N_j \). More precisely, we collect the terms in (1.5) such that \( m_j N_j - n_j M_j = h_j \) for each \( j = 1, \ldots, \ell \), and then sum over all integers \( h_1, \ldots, h_\ell \) and all positive integers \( M_1, \ldots, M_\ell, N_1, \ldots, N_\ell \) such that \( h_1 \cdots h_\ell \neq 0, M_1 \cdots M_\ell = N_1 \cdots N_\ell, \) and \( (M_j, N_j) = 1 \) for each \( j \). With a correction factor of \((\ell!)^{-2}\) that will be explained later, the sum of these terms is given by the definition

\[
S_\ell : = \frac{1}{(\ell!)^2} \sum_{M_1 \cdots M_\ell = N_1 \cdots N_\ell} \sum_{h_1, \ldots, h_\ell \in \mathbb{Z}} \sum_{(M_j, N_j) = 1 \forall j, h_1 \cdots h_\ell \neq 0} \frac{1}{T} \int_0^\infty \psi \left( \frac{t}{T} \right) \, dt.
\]
\[
\times \sum_{1 \leq m_1, \ldots, m_\ell < \infty} \frac{\tau_{A_1}(m_1) \cdots \tau_{A_\ell}(m_\ell) \tau_{B_1}(n_1) \cdots \tau_{B_\ell}(n_\ell)}{(m_1 \cdots m_\ell)^{1/2+i\tau} (n_1 \cdots n_\ell)^{1/2-\tau}}
\]
\[
\times Y\left(\frac{m_1 \cdots m_\ell}{X}\right) Y\left(\frac{n_1 \cdots n_\ell}{X}\right) dt. \tag{1.6}
\]

For \(\ell = 0\), we also define
\[
S_0 := \frac{1}{(2\pi i)^2} \int_{(\varepsilon)} \int_{(\varepsilon)} \tilde{Y}(\xi) \tilde{Y}(\eta) X^{\xi+\eta} \frac{1}{T} \int_{0}^{\infty} \psi\left(\frac{t}{T}\right) \sum_{n=1}^{\infty} \frac{\tau_B(n)}{n^{1+\xi+\eta}} dt d\xi d\eta. \tag{1.7}
\]

Our improvement of the approach in [14] stems from using an integral form of the prediction of the delta method of [17] to evaluate each \(m_j, n_j\)-sum in (1.6). Combining the results via the assumption of an independence hypothesis (see Section 3), we are led to make the following prediction.

**Conjecture 1.2.** Let \(\ell\) be an integer with \(1 \leq \ell < \min\{|A|, |B|\}\). If \(\alpha, \beta \ll 1/\log T\) for each \(\alpha \in A\) and \(\beta \in B\), then as \(T \to \infty\) we have
\[
S_\ell \sim \sum_{U \subset A, V \subset B \atop |U| = |V| = \ell} \frac{1}{(2\pi i)^2} \int_{(\varepsilon)} \int_{(\varepsilon)} \tilde{Y}(\xi) \tilde{Y}(\eta) X^{\xi+\eta} \frac{1}{T} \int_{0}^{\infty} \psi\left(\frac{t}{T}\right) \sum_{n=1}^{\infty} \frac{\tau_B(n)}{n^{1+\xi+\eta}} dt d\xi d\eta.
\]
\[
\times \prod_{\alpha \in U} \chi\left(\frac{1}{2} + \xi + \alpha + i\tau\right) \prod_{\beta \in V} \chi\left(\frac{1}{2} + \eta + \beta - i\tau\right)
\]
\[
\times \sum_{n=1}^{\infty} \frac{\tau_{A \setminus U}(n) \tau_{B \setminus V}(n)}{n} \psi\left(\frac{t}{T}\right) \sum_{n=1}^{\infty} \frac{\tau_B(n)}{n^{1+\xi+\eta}} dt d\xi d\eta. \tag{1.8}
\]

Conjectures 1.1 and 1.2 indicate that the following conjecture might be true. This predicts a “stratification” of the terms in (1.1) that may be viewed in terms of counting points on a variety via its subvarieties (see the paragraph containing (1.10) below) and gives an approach toward the problem of evaluating (1.1).

**Conjecture 1.3.** If \(\alpha, \beta \ll 1/\log T\) for each \(\alpha \in A\) and \(\beta \in B\), then as \(T \to \infty\) we have
\[
\mathcal{M}_{A,B}(T, X) \sim \sum_{\ell=0}^{\min\{|A|, |B|\}} S_\ell.
\]

The difference between this conjecture and Conjecture 1.1 is that Conjecture 1.1 predicts what the final answer should be, while Conjecture 1.3 proposes a sort of splitting of \(\mathcal{M}_{A,B}(T, X)\) that (we hope) may lead to a proof of Conjecture 1.1.

A few notes about Conjectures 1.2 and 1.3 are now in order. Note that the right-hand side of (1.8) is independent of the partitions \(A = A_1 \cup \cdots \cup A_\ell\) and \(B = B_1 \cup \cdots \cup B_\ell\), while the definition (1.6) of \(S_\ell\) may depend on the partitions. There is a slight abuse of notation in Conjecture 1.3 and (1.6) because, for example, the subset \(A_3\) in the definition of \(S_5\) may be different from the subset \(A_3\) in the definition of \(S_5\). However, this will not cause confusion because the arguments in this paper
will focus on only one (arbitrary) $\ell$. As mentioned earlier, we expect a power savings error term in the predicted asymptotic formula (1.4), as in [6]. We also expect power savings error terms in (1.8) and Conjecture 1.3, and also in Conjecture 1.4. As for the correction factor $(\ell!)^{-2}$ in the definition (1.6) of $S_\ell$, one may speculate that it may be needed to account for possible permutations of the $A_j$ and the $B_j$. If we do not include this correction factor in the definition (1.6) of $S_\ell$, then our heuristic calculations lead to Conjecture 1.2, except with an extra factor $(\ell!)^2$ multiplied to the right-hand side of (1.8).

Conjecture 1.3 looks similar to [14, Conjecture 1], but they are different. The main difference between Conjecture 1.3 and [14, Conjecture 1] is that the proposed splitting in [14, Conjecture 1] includes an average over all the ways to partition $A$ and $B$. This averaging seems to be necessary in [14] because if we use only one partition for each of $A$ and $B$, then the procedure in [14] does not give all the terms predicted by Conjecture 1.1 (see [14, Theorem 1]). This averaging necessitated multiplying the weights $(\ell!)^2\ell^{2k-2}\ell$ in [14, Conjecture 1], and a proposed explanation of these weights involving the averaging over partitions is written in [14, section 12]. Our improvement of the heuristic of [14] eliminates the need to average over the ways to partition $A$ and $B$. Moreover, the result of our heuristic calculations do not depend on the partitions of $A$ and $B$. We make these refinements by using an integral form of the asymptotic formula for correlations of divisor sums that is predicted by the delta method (see (3.1)) and rigorous evaluations of the local factors of a Euler product (Theorem 4.4) to predict the following “Vandermonde integral” expression for $S_\ell$.

Recall that $S_\ell$ is defined by (1.6).

**Conjecture 1.4.** Let $\ell$ be an integer with $1 \leq \ell \leq \min\{|A|, |B|\}$. If $\alpha, \beta \ll 1/\log T$ for each $\alpha \in A$ and $\beta \in B$, then as $T \to \infty$ we have

$$S_\ell \sim \frac{1}{(\ell!)^2(2\pi i)^2} \int_{(2\varepsilon)} \int_{(2\varepsilon)} \tilde{Y}(\xi)\tilde{Y}(\eta) \frac{X_{\xi+\eta}}{T} \int_0^\infty \psi(t) \frac{1}{(2\pi i)^{2\ell}} \oint_{|z_1|=\varepsilon} \cdots \oint_{|z_\ell|=\varepsilon} \left\{ \chi\left(\frac{1}{2} + \xi - z_j + it\right)\chi\left(\frac{1}{2} + \eta - w_j - it\right) \right\}$$

$$\times \prod_{\alpha \in A, \beta \in B} \zeta(1 + \alpha + \beta + \xi + \eta) \prod_{1 \leq j \leq \ell} \zeta(1 + \alpha + z_j) \prod_{1 \leq j \leq \ell} \zeta(1 + \beta + w_j)$$

$$\times \prod_{1 \leq j \leq \ell} (1/\zeta)(1 + \alpha + \xi + \eta - w_j) \prod_{1 \leq j \leq \ell} (1/\zeta)(1 + \beta + \xi + \eta - z_j)$$

$$\times \prod_{1 \leq i, j \leq \ell, i \neq j} (1/\zeta)(1 - z_i + z_j) \prod_{1 \leq i, j \leq \ell, i \neq j} (1/\zeta)(1 - w_i + w_j)$$

$$\times \prod_{1 \leq i, j \leq \ell} \zeta(1 + z_i + w_j - \xi - \eta) \zeta(1 - z_i - w_j + \xi + \eta)$$

$$\times A(A, B, Z, W, \xi + \eta) d\omega_\ell \cdots d\omega_1 dz_\ell \cdots dz_1 d\xi d\eta,$$

(1.9)

where $Z := \{z_1, \ldots, z_\ell\}$, $W := \{w_1, \ldots, w_\ell\}$, and $A(A, B, Z, W, \xi + \eta)$ is a Euler product that converges absolutely whenever $\Re(\xi) = \Re(\eta) = 2\varepsilon$ and $|\Re(\gamma)| \leq \varepsilon$ for all $\gamma \in A \cup B \cup Z \cup W$. Explicitly, $A$ is defined by (5.3).
Conjecture 1.4, with $\mathcal{A}$ defined by (5.3), is in fact equivalent to Conjecture 1.2. Indeed, we prove (rigorously) in Section 6 that the right-hand side of (1.8) is equal to the right-hand side of (1.9). Thus, (1.7) plus the sum of the right-hand side of (1.9) over all integers $\ell$ with $1 \leq \ell \leq \min\{|A|, |B|\}$ is equal to the right-hand side of (1.4). This gives a new alternative form of Conjecture 1.1, a “Vandermonde integral expression,” that is equivalent to Conjecture 1.1.

After discovering this Vandermonde integral expression through a rough “back-of-the-envelope” calculation, the second author informed Brad Rodgers of it in the summer of 2019. Rodgers responded that he and Kannan Soundararajan had previously found an analogous expression for moments of characteristic polynomials of random matrices. They had proved that if $U(N)$ is the group of $N \times N$ unitary matrices, then integrating with respect to the Haar measure gives

$$\int_{U(N)} \prod_{\alpha \in \mathcal{A}} \det (1 - e^{-\alpha} g) \prod_{\beta \in \mathcal{B}} \det (1 - e^{-\beta} g^{-1}) \, dg = \sum_{\ell = 0}^{\min\{|A|, |B|\}} J^{A,B}_{\ell},$$

where, for positive integers $\ell$, $J^{A,B}_{\ell}$ is defined by

$$J^{A,B}_{\ell} := \frac{(-1)^{\ell}}{(\ell!)^2 (2\pi i)^{2\ell + 1}} \oint_{|\xi|=1} \oint_{|z_1|=\ldots=|z_\ell|=\varepsilon} \oint_{|w_1|=\ldots=|w_\ell|=\varepsilon} \frac{2 \pi i \sum_{i=1}^{\ell}(z_i + w_i)}{1 - e^{-\xi}} \times \frac{Z(A, B)Z(A, Z^-)Z(B, W^-)}{Z(A, W_z)Z(B, Z_z)} \tilde{\Delta}_\ell(Z)\tilde{\Delta}_\ell(W)Z(W, Z_\xi)^2 \times dw_\ell \ldots dw_1 \, dz_\ell \ldots dz_1 \, d\xi,$$

and $J^{A,B}_{0}$ is defined by $J^{A,B}_{0} := Z(A, B)$, with

$$Z(C, D) := \prod_{\gamma \in C \atop \delta \in D} \frac{1}{1 - e^{-\gamma - \delta}}$$

and

$$\tilde{\Delta}_\ell(C) := \prod_{\gamma, \delta \in C \atop \gamma \neq \delta} (1 - e^\ell)^{$$

for finite multisets $C, D$ of complex numbers. It is quite remarkable that these two analogous Vandermonde integral expressions were discovered independently at about the same time through different approaches.

In an AIM Workshop in 2016, Trevor Wooley suggested that the heuristic developed by Keating and the second author [11, 13, 14] has an interpretation in terms of the counting of rational points in algebraic varieties that is the subject of Manin’s arithmetic stratification conjectures [5, 18, 23]. Thus, we suspect that the sums (1.6) and Conjecture 1.3 present a stratification of $\mathcal{M}_{A,B}(T, X)$ that has the same interpretation. We may think of the problem of evaluating (1.5) as involving counting
solutions (weighted by divisor functions) in the variety
\[ m_1 \cdots m_\ell - n_1 \cdots n_\ell = h, \quad |h| \leq H \]
for some parameter \( H \). In making the definition (1.6), we are essentially stratifying this variety into the subvarieties
\[ m_1 N_1 - n_1 M_1 = h_1 \]
\[ m_2 N_2 - n_2 M_2 = h_2 \]
\[ \vdots \]
\[ m_\ell N_\ell - n_\ell M_\ell = h_\ell \]
(1.10)
with \( |h_1 h_2 \cdots h_\ell| \leq H \). Thus, as suggested by Wooley, our approach is analogous to counting rational points on high-dimensional varieties by stratification and counting points on subvarieties. Note that this stratification introduces some overcounting of solutions. However, we believe that the factor \( 1/\ell!2 \) in the definition (1.6) accounts for this overcounting. This factor may be explained intuitively by an argument similar to the one in [14, section 12.2].

As mentioned earlier, we improve the method in [14] by using the integral form (3.1) of the prediction of the delta method. A key observation in our refinement is that the functions \( G_E(s, q) \), defined by (3.2), that appear in the prediction of the delta method are closely related to the coefficients \( I_{C,D}(m) \) in the Dirichlet series expansion
\[
\prod_{\gamma \in C} \zeta(s + \gamma) \prod_{\delta \in D} \zeta(s + \delta) = \sum_{m=1}^{\infty} \frac{I_{C,D}(m)}{m^s} \quad (\text{Re}(s) \to \infty)
\]
(1.11)
(see Lemma 4.3). Thus, we are able to adapt the local calculations in [14] without much difficulty, as the coefficients \( I_{C,D}(m) \) exhibit properties similar to those of the coefficients \( \tau_E \) defined by (1.2). Another way we refine the approach of [14] is in making some of their technical arguments more precise. This includes expressing Fourier transforms of test functions in terms of the gamma function (Lemma 3.1), as has been done in [20, 21], and [24].

In future work, we aim to adapt the method to other families of \( L \)-functions and make parts of it more rigorous.

2 DESCENDING THROUGH THE RECIPE

We first review the heuristic of [6] and apply it to predict Conjecture 1.1. For the reader’s convenience, in this section we outline the steps of the recipe of [6]. The full details of the recipe may be found in [6].

We apply Mellin inversion to deduce from the definition (1.1) that
\[
\mathcal{M}_{A,B}(T, X) = \frac{1}{(2\pi i)^2} \int_{(2)} \int_{(2)} \tilde{Y}(\xi) \tilde{Y}(\eta) X^{\xi+\eta} \frac{1}{T} \int_0^\infty \psi\left(\frac{t}{T}\right) dt
dequence
where $\tilde{Y}$ is the Mellin transform of $Y$, which is defined by

$$\tilde{Y}(s) := \int_0^\infty Y(x)x^{s-1} \, dx.$$  

We move the lines of integration to Re$(\xi) = \varepsilon$ and Re$(\eta) = \varepsilon$. There are residues at the poles at $\xi = \frac{1}{2} - \alpha - it$, $\alpha \in A$ and $\eta = \frac{1}{2} - \beta + it$, $\beta \in B$. These residues are negligible due to the rapid decay of the Mellin transform $\tilde{Y}$ and the fact that $t \asymp T$ because $\psi$ is supported on $[1,2]$. We thus arrive at

$$\mathcal{M}_{A,B}(T,X) = \frac{1}{(2\pi i)^2} \int_{(\varepsilon)} \int_{(\varepsilon)} \tilde{Y}(\xi)\tilde{Y}(\eta)X^{\xi+\eta} \frac{1}{T} \int_0^\infty \psi\left(\frac{t}{T}\right) \times \prod_{\alpha \in A} \zeta\left(\frac{1}{2} + it + \xi + \alpha\right) \prod_{\beta \in B} \zeta\left(\frac{1}{2} - it + \eta + \beta\right) \, dt \, d\eta \, d\xi + O_C(T^{-C}), \quad (2.1)$$

where $C > 0$ is arbitrarily large. Now the approximate functional equation implies that

$$\zeta(s) \approx \sum_n \frac{1}{n^s} + \chi(s) \sum_n \frac{1}{n^{1-s}},$$

where we recall that $\chi(s)$ is the factor from the functional equation $\zeta(s) = \chi(s)\zeta(1-s)$. We replace each zeta factor in $(2.1)$ with the right-hand side of its approximate functional equation. We multiply out the resulting product and keep only the terms that have the same number of factors of the form $\chi\left(\frac{1}{2} + it + \xi + \alpha\right), \alpha \in A$, as factors of the form $\chi\left(\frac{1}{2} - it + \eta + \beta\right), \beta \in B$. We then extend the sums from the approximate functional equations to infinity. We use the approximation

$$\int_0^\infty \psi\left(\frac{t}{T}\right) \prod_{\alpha \in U} \chi\left(\frac{1}{2} + it + \xi + \alpha\right) \prod_{\beta \in V} \chi\left(\frac{1}{2} - it + \eta + \beta\right) \left(\frac{m}{n}\right)^it \, dt \approx 0$$

for all pairs $m, n$ of positive integers with $m \neq n$ and all pairs $U, V$ of subsets such that $U \subseteq A$, $V \subseteq B$, and $|U| = |V|$. Using the definition (1.2), we see that this procedure leads us to conjecture Conjecture 1.1.

### 3 ASCENDING THROUGH CONVOLUTION SUMS: APPLYING THE DELTA METHOD

Our two key assumptions for this section are the delta method prediction and an “independence hypothesis,” which we state in more detail as follows. The delta method in [17] predicts for a suitable function $f$ that

$$\sum_{1 \leq m,n < \infty} \tau_A(m)\tau_B(n) f(mN,nM)$$
\begin{equation}
\sim \frac{1}{(2\pi i)^2} \oint_{|z| = \varepsilon} \oint_{|w| = \varepsilon} \frac{1}{N^{1+z}M^{1+w}} \prod_{\alpha \in A} \xi(1 + z + \alpha) \prod_{\beta \in B} \xi(1 + w + \beta) 
\times \sum_{q=1}^{\infty} \frac{c_q(h)(q, N)^{1+z}(q, M)^{1+w}}{q^{2+z+w}} G_A \left(1 + z, \frac{q}{(q, N)} \right) G_B \left(1 + w, \frac{q}{(q, M)} \right) 
\times \int_{-\infty}^{\infty} x^z(x - h)^w f(x, x - h) \, dx \, dw \, dz,
\end{equation}

where
\[ c_q(h) : = \sum_{a \text{ mod } q \atop (a,q)=1} e^{2\pi i an/q} \]
is the Ramanujan sum and \( G_E \) is defined for finite multisets \( E \) of complex numbers by
\begin{equation}
G_E(s, q) = \sum_{d \mid \phi(q)} \frac{\mu(d)d^s}{\phi(d)} \sum_{e \mid d} \frac{\mu(e)e^s}{e^s} g_E\left(s, \frac{eq}{d}\right),
\end{equation}
with \( g_E(s, n) \) defined by
\begin{equation}
g_E(s, n) = \prod_{p \mid n} \left\{ \prod_{\gamma \in E} (1 - p^{-s-\gamma}) \right\} \sum_{m=0}^{\infty} \frac{\tau_E(p^{m+\text{ord}_p(n)})}{p^{ms}}.
\end{equation}

In [17], the weight function \( f \) in (3.1) is required to be smooth, and its partial derivatives are required to satisfy certain bounds. However, in our heuristic, we simply assume that the delta method prediction applies to our situation. For details on how to derive the prediction (3.1), see [9, section 1]. Forms of this prediction are stated in [12, section 3], [13, eq. (4)], and [14, section 6]. We may put these forms into (3.1) by interpreting them as a sum of residues and writing the sum of residues as a contour integral. The left-hand side of (3.1) may be called a correlation of divisor-sums. (In the case \( M = N = 1 \), some authors call it a “shifted convolution sum.”) The independence hypothesis that we assume in this paper is the assumption that any error terms implied in the conjecture (3.1) do not contribute to the main term in the resulting expression for \( S_\varepsilon \). As mentioned below Conjecture 1.3, we expect power savings error terms in our predicted asymptotic formulae.

We now begin our heuristic evaluation of \( S_\varepsilon \) using the above assumptions. We apply Mellin inversion and interchange the order of summation to deduce from the definition (1.6) of \( S_\varepsilon \) that
where \( \hat{\psi} \) is the Fourier transform defined by

\[
\hat{\psi}(x) = \int_{-\infty}^{\infty} \psi(t)e^{-2\pi i xt} \, dt.
\]  
(3.4)

As \( M_1 \cdots M_\ell = N_1 \cdots N_\ell \), it follows that

\[
S_\ell = \frac{1}{(\ell)!^2} \sum_{M_1 \cdots M_\ell = N_1 \cdots N_\ell} \sum_{h_1, \ldots, h_\ell \in \mathbb{Z}} \frac{1}{(2\pi i)^2} \int_{(2)} \int_{(2)} \bar{\Upsilon}(\xi)\bar{\Upsilon}(\eta)X^{\frac{1}{2}+\xi}(N_1 \cdots N_\ell)^{\frac{1}{2}+\eta} \\
\times \int \int_{\max\{0,h_1\}}^{\infty} \int \int_{\max\{0,h_\ell\}}^{\infty} \hat{\psi}\left(\frac{T}{2\pi} \log \frac{m_1 N_1 \cdots m_\ell N_\ell}{n_1 M_1 \cdots n_\ell M_\ell}\right) \, dx \, dy.
\]  
(3.5)

We apply the prediction (3.1) to each \( m_j, n_j \)-sum in (3.5) by taking in (3.1) \( A = A_j, B = B_j \), \( m = m_j, n = n_j, M = M_j, N = N_j \), and

\[
f(mN, nM) = f(m_1 N_1, n_1 M_1; m_2 N_2, n_2 M_2; \ldots; m_\ell N_\ell, n_\ell M_\ell) \\
= (m_1 N_1 \cdots m_\ell N_\ell)^{-\frac{1}{2}-\xi}(n_1 M_1 \cdots n_\ell M_\ell)^{-\frac{1}{2}-\eta} \hat{\psi}\left(\frac{T}{2\pi} \log \frac{m_1 N_1 \cdots m_\ell N_\ell}{n_1 M_1 \cdots n_\ell M_\ell}\right).
\]

By the independence hypothesis mentioned in the above paragraph, we may ignore any error terms arising from the use of (3.1). This leads us to predict that

\[
S_\ell \sim \frac{1}{(\ell)!^2} \sum_{M_1 \cdots M_\ell = N_1 \cdots N_\ell} \sum_{h_1, \ldots, h_\ell \in \mathbb{Z}} \frac{1}{(2\pi i)^2} \int_{(2)} \int_{(2)} \bar{\Upsilon}(\xi)\bar{\Upsilon}(\eta)X^{\frac{1}{2}+\xi}(N_1 \cdots N_\ell)^{\frac{1}{2}+\eta} \\
\times (M_1 \cdots M_\ell)^{\frac{1}{2}+\eta} \int_{\max\{0,h_1\}}^{\infty} \cdots \int_{\max\{0,h_\ell\}}^{\infty} \hat{\psi}\left(\frac{T}{2\pi} \log \frac{x_1 \cdots x_\ell}{(x_1 - h_1) \cdots (x_\ell - h_\ell)}\right) \\
\times \prod_{j=1}^{\ell} \left\{ \frac{1}{(2\pi i)^2} \oint_{|z|=\varepsilon} \oint_{|w|=\varepsilon} x_j^{\frac{1}{2}-\xi+z}(x_j - h_j)^{-\frac{1}{2}-\eta+w} N_j^{1+z} M_j^{1+w} \prod_{\alpha \in A_j} \frac{\zeta(1 + z + \alpha)}{\zeta(1 + z)} \right\} \\
\times \prod_{\beta \in B_j} \zeta(1 + w + \beta) \sum_{q=1}^{\infty} c_q(h_j)(q,N_j)^{1+z}(q,M_j)^{1+w} G_{A_j} \left(1 + z, \frac{q}{(q,N_j)}\right) \\
\times G_{B_j} \left(1 + w, \frac{q}{(q,M_j)}\right) \, dw \, dz \right\} \, dx_1 \cdots dx_\ell \, d\xi \, d\eta.
\]  
(3.6)
We let \( \varepsilon_j = \text{sgn}(h_j) \) and relabel \( h_j \) as \( \varepsilon_j h_j \), where now \( h_j \) is positive. Then, we make the change of variables \( x_j \mapsto h_j y_j \) for each \( j \) to see that (3.6) implies

\[
S_{\varepsilon} \sim \frac{1}{(\ell!)^2} \sum_{M_1 \cdots M_{\ell} = N_1 \cdots N_{\ell} \varepsilon_1, \ldots, \varepsilon_{\ell} \in \{1,-1\}} \sum_{1 \leq h_1, \ldots, h_{\ell} < \infty} \frac{1}{(2\pi i)^2} \int_{(2)} \int_{(2)} \hat{\Upsilon}(\xi)\hat{\Upsilon}(\eta)X^{\frac{\xi}{2} + \eta}(N_1 \cdots N_{\ell})^{\frac{1}{2} + \xi} \\
\times (M_1 \cdots M_{\ell})^{\frac{1}{2} + \eta} \int_{\max(0,\varepsilon_1)}^{\infty} \cdots \int_{\max(0,\varepsilon_{\ell})}^{\infty} \phi \left( \frac{T}{2\pi} \log \frac{y_1 \cdots y_{\ell}}{(y_1 - \varepsilon_1) \cdots (y_{\ell} - \varepsilon_{\ell})} \right) \\
\times \prod_{j=1}^{\ell} \left\{ \frac{1}{(2\pi i)^2} \oint_{|z| = \varepsilon} \oint_{|w| = \varepsilon} \frac{h_j^{-\xi + z - \eta + w} - \frac{1}{2} - \xi + z}{y_j^{-\frac{1}{2} - \xi + z}} \frac{(y_j - \varepsilon_j)^{-\frac{1}{2} - \eta + w}}{N_j^{1 + z} M_j^{1 + w}} \right\} \prod_{\alpha \in A_j} \xi(1 + z + \alpha) \\
\times \prod_{\beta \in B_j} \xi(1 + w + \beta) \sum_{q=1}^{\infty} c_q(h_j)(q, N_j)^{1 + z}(q, M_j)^{1 + w} G_{A_j} \left( 1 + z, \frac{q}{(q, N_j)} \right) \\
\times G_{B_j} \left( 1 + w, \frac{q}{(q, M_j)} \right) \frac{1}{d^{1 + \xi + \eta}} \sum_{d=1}^{\infty} \mu(q)(qd, N_j)^{1 + z}(qd, M_j)^{1 + w} G_{A_j} \left( 1 + z, \frac{q}{(qd, N_j)} \right) \\
\times G_{B_j} \left( 1 + w, \frac{q}{(qd, M_j)} \right) \right\} dy_1 \cdots dy_{\ell} d\xi d\eta.
\]

We next insert the identity

\[
c_q(h_j) = \sum_{d|q} d \mu \left( \frac{q}{d} \right)
\]

for the Ramanujan sum, and then relabel \( h_j \) as \( h_j d \) and \( q \) as \( qd \) to find that (3.7) implies

\[
S_{\varepsilon} \sim \frac{1}{(\ell!)^2} \sum_{M_1 \cdots M_{\ell} = N_1 \cdots N_{\ell} \varepsilon_1, \ldots, \varepsilon_{\ell} \in \{1,-1\}} \sum_{1 \leq h_1, \ldots, h_{\ell} < \infty} \frac{1}{(2\pi i)^2} \int_{(2)} \int_{(2)} \hat{\Upsilon}(\xi)\hat{\Upsilon}(\eta)X^{\frac{\xi}{2} + \eta}(N_1 \cdots N_{\ell})^{\frac{1}{2} + \xi} \\
\times (M_1 \cdots M_{\ell})^{\frac{1}{2} + \eta} \int_{\max(0,\varepsilon_1)}^{\infty} \cdots \int_{\max(0,\varepsilon_{\ell})}^{\infty} \phi \left( \frac{T}{2\pi} \log \frac{y_1 \cdots y_{\ell}}{(y_1 - \varepsilon_1) \cdots (y_{\ell} - \varepsilon_{\ell})} \right) \\
\times \prod_{j=1}^{\ell} \left\{ \frac{1}{(2\pi i)^2} \oint_{|z| = \varepsilon} \oint_{|w| = \varepsilon} \frac{y_j^{-\frac{1}{2} - \xi + z}}{N_j^{1 + z} M_j^{1 + w}} \right\} \prod_{\alpha \in A_j} \xi(1 + z + \alpha) \\
\times \prod_{\beta \in B_j} \xi(1 + w + \beta) \sum_{q=1}^{\infty} \frac{1}{d^{1 + \xi + \eta}} \sum_{d=1}^{\infty} \mu(q)(qd, N_j)^{1 + z}(qd, M_j)^{1 + w} G_{A_j} \left( 1 + z, \frac{q}{(qd, N_j)} \right) \\
\times G_{B_j} \left( 1 + w, \frac{q}{(qd, M_j)} \right) \right\} dy_1 \cdots dy_{\ell} d\xi d\eta.
\]
because
\[ \sum_{1 \leq h_j < \infty} h_j^{-\xi + z - \eta + w} = \zeta(\xi + \eta - z - w) \]
for each \( j \).

We next move the \( \xi \)- and \( \eta \)-lines to \( \text{Re}(\xi) = 2\varepsilon \) and \( \text{Re}(\eta) = 2\varepsilon \). Doing so traverses the poles of the factors \( \zeta(\xi + \eta - z - w) \). However, we expect the residues of the integrand at these poles to be negligible because of the presence of the factor \( \chi(1 + z + w - \xi - \eta) \) in the consequence (3.9) of Lemma 3.1. This factor is zero at the pole of \( \zeta(\xi + \eta - z - w) \). We then insert the definition (3.4) of \( \hat{\psi} \) and interchange the order of summation to arrive at the prediction

\[
S_\varepsilon \sim \frac{1}{(\varepsilon!)^2} \prod_{j=1}^{\ell} \int_{[w_1]=\varepsilon}^{[w_{\ell}]=\varepsilon} \int_{[z_1]=\varepsilon}^{[z_{\ell}]=\varepsilon} \sum_{M_1 \cdots M_{\ell} = N_1 \cdots N_{\ell}} \prod_{j=1}^{\ell} \left( \zeta(\xi + \eta - z_j - w_j)N_j^{-\frac{1}{2} + \xi - z_j}M_j^{-\frac{1}{2} + \eta - w_j} \right) 
\times \sum_{\varepsilon_j = \pm 1} \int_{\max\{0, \varepsilon_j\}}^{\infty} \left( \frac{y_j - \varepsilon_j}{y_j} \right)^{\frac{1}{2} - \xi - z_j} \left( \frac{y_j - \varepsilon_j}{y_j} \right)^{\frac{1}{2} - \eta - w_j} \prod_{\alpha \in A_j} \zeta(1 + z_j + \alpha) 
\times \prod_{\beta \in B_j} \zeta(1 + w_j + \beta) \sum_{d=1}^{\infty} \frac{1}{d^{1 + \xi + \eta}} \sum_{q=1}^{\infty} \frac{\mu(q)(qd, N_j)^{1 + z_j}(qd, M_j)^{1 + w_j}}{q^{2 + z_j + w_j}} 
\times G_{A_j} \left( 1 + z_j, \frac{qd}{(qd, N_j)} \right) G_{B_j} \left( 1 + w_j, \frac{qd}{(qd, M_j)} \right) \int dy_j \, dw_j \, dz_j \right) \, dt \, d\xi \, d\eta. \tag{3.8} \]

We may now evaluate each \( y_j \)-integral through the following lemma.

**Lemma 3.1.** If \( a, b \) are complex numbers with \( 0 < \text{Re}(a), \text{Re}(b) < \frac{1}{2} \), then

\[
\sum_{\varepsilon = \pm 1} \int_{\max\{0, \varepsilon\}}^{\infty} y^{a-1}(y - \varepsilon)^{b-1} \, dy = \chi(a + b)\chi(1 - a)\chi(1 - b) \left( \frac{1 + \tan(\frac{\pi}{2}a)\tan(\frac{\pi}{2}b)}{2} \right),
\]

where \( \chi(s) \) is the factor in the functional equation \( \zeta(s) = \chi(s)\zeta(1 - s) \).

**Proof.** For brevity, define \( I(a, b) \) by

\[
I(a, b) := \sum_{\varepsilon = \pm 1} \int_{\max\{0, \varepsilon\}}^{\infty} y^{a-1}(y - \varepsilon)^{b-1} \, dy.
\]

We have

\[
I(a, b) = B(a, 1 - a - b) + B(b, 1 - a - b),
\]
where \( B(\cdot, \cdot) \) is the beta function. Writing the beta function in terms of the gamma function, we deduce that

\[
I(a, b) = \left( \frac{\Gamma(a)}{\Gamma(1 - b)} + \frac{\Gamma(b)}{\Gamma(1 - a)} \right) \Gamma(1 - a - b).
\]

As \( \Gamma(s)\Gamma(1 - s) = \pi/ \sin(\pi s) \), it follows that

\[
I(a, b) = \left( \frac{1}{\sin \pi a} + \frac{1}{\sin \pi b} \right) \frac{\pi \Gamma(1 - a - b)}{\Gamma(1 - a)\Gamma(1 - b)}.
\]

Now the identity \( \sin A + \sin B = 2 \sin\left(\frac{A}{2} + \frac{B}{2}\right) \cos\left(\frac{A}{2} - \frac{B}{2}\right) \) implies

\[
\frac{1}{\sin \pi a} + \frac{1}{\sin \pi b} = \frac{2 \sin(\frac{\pi}{2}(a + b)) \cos(\frac{\pi}{2}(a - b))}{\sin(\pi a) \sin(\pi b)}.
\]

Thus,

\[
I(a, b) = \frac{\chi(a + b) \cos(\frac{\pi}{2}(a - b))}{2^{a+b-1} \pi^{a+b-2} \Gamma(1 - a)\Gamma(1 - b) \sin(\pi a) \sin(\pi b)},
\]

where we have also used the definition \( \chi(s) = 2^{s} \pi^{s-1} \sin(\frac{\pi}{2} s) \Gamma(1 - s) \). The double-angle formulae then imply that

\[
I(a, b) = \frac{\chi(a + b) \cos(\frac{\pi}{2}(a - b))}{2\chi(a)\chi(b) \cos(\frac{\pi}{2}a) \cos(\frac{\pi}{2}b)}.
\]

The lemma now follows from this, the identity \( \chi(s)\chi(1 - s) = 1 \), and the fact that

\[
\frac{\cos(\frac{\pi}{2}(a - b))}{\cos(\frac{\pi}{2}a) \cos(\frac{\pi}{2}b)} = 1 + \tan(\frac{\pi}{2}a) \tan(\frac{\pi}{2}b).
\]

\[\square\]

If \( \text{Re}(\xi) = \text{Re}(\eta) = 2\varepsilon, |z_j|, |w_j| = \varepsilon, \) and \( t \) is real, then it follows from Lemma 3.1 with \( a = \frac{1}{2} - \xi + z_j - it \) and \( b = \frac{1}{2} - \eta + w_j + it \) that

\[
\sum_{\varepsilon_j = \pm 1} \int_{\max\{0,\varepsilon_j\}}^\infty \left( \frac{y_j - \varepsilon_j}{y_j} \right)^{it} y_j^{-\frac{1}{2} - \xi + z_j} (y_j - \varepsilon_j)^{-\frac{1}{2} - \eta + w_j} dy_j
\]

\[
= \chi(1 + z_j + w_j - \xi - \eta) \chi(\frac{1}{2} + \xi - z_j + it) \chi(\frac{1}{2} + \eta - w_j - it)
\]

\[
\times \left( 1 + \tan(\frac{\pi}{2}(\frac{1}{2} - \xi + z_j - it)) \tan(\frac{\pi}{2}(\frac{1}{2} - \eta + w_j + it)) \right).
\]

(3.9)

If it also holds that \( T \leq t \leq 2T \) and \( |\text{Im}(\xi)|, |\text{Im}(\eta)| \leq T/2, \) say, then

\[
\tan(\frac{\pi}{2}(\frac{1}{2} - \xi + z_j - it)) \tan(\frac{\pi}{2}(\frac{1}{2} - \eta + w_j + it)) = 1 + O(e^{-T/4}).
\]
From this, the functional equation

$$\zeta(\xi + \eta - z_j - w_j)\chi(1 + z_j + w_j - \xi - \eta) = \zeta(1 + z_j + w_j - \xi - \eta),$$

and (3.8), we arrive at the prediction

$$S_\epsilon \sim \frac{1}{(\epsilon!)^2(2\pi i)^2} \int_{(2\epsilon)} \int_{(2\epsilon)} \tilde{Y}(\xi)\tilde{Y}(\eta) \frac{X^{\xi + \eta} T}{T} \int_0^\infty \psi\left(\frac{t}{T}\right) \frac{1}{(2\pi i)^2} \oint_{|z_1| = \epsilon} \cdots \oint_{|z_\epsilon| = \epsilon} \left\{ \begin{array}{l}
\zeta(1 + z_j + w_j - \xi - \eta)\chi\left(\frac{1}{2} + \xi - z_j + it\right) \\
\chi\left(\frac{1}{2} + \eta - w_j - it\right) \prod_{\alpha \in A_j} \zeta(1 + z_j + \alpha) \prod_{\beta \in B_j} \zeta(1 + w_j + \beta) \\
\times N_j^{-\frac{1}{2} + \xi - z_j} M_j^{-\frac{1}{2} + \eta - w_j} \sum_{d=1}^\infty \frac{1}{d^{1+\xi + \eta}} \sum_{q=1}^\infty \frac{\mu(q)\chi(q)(q^d, N_j)^{1+z_j}(q^d, M_j)^{1+w_j}}{q^{2+z_j+w_j}} \\
\times G_{A_j}\left(1 + z_j, \frac{qd}{(qd, N_j)}\right) G_{B_j}\left(1 + w_j, \frac{qd}{(qd, M_j)}\right) dt \, d\xi \, d\eta. \end{array} \right.$$

(3.10)

We next use multiplicativity to write this prediction in terms of a Euler product. We do this by formally applying the conclusion of the following lemma, which is well-known and implicitly used in numerous places throughout the literature. For a proof of this lemma, see [1], for example.

**Lemma 3.2.** Let $f(n_1, \ldots, n_\epsilon)$ be a complex-valued function such that

$$f(h_1 j_1, \ldots, h_\epsilon j_\epsilon) = f(h_1, \ldots, h_\epsilon) f(j_1, \ldots, j_\epsilon)$$

whenever $h_1 \cdots h_\epsilon$ is coprime to $j_1 \cdots j_\epsilon$. If

$$\sum_{1 \leq n_1, \ldots, n_\epsilon < \infty} |f(n_1, \ldots, n_\epsilon)| < \infty,$$

then

$$\sum_{1 \leq n_1, \ldots, n_\epsilon < \infty} f(n_1, \ldots, n_\epsilon) = \prod_p \left( \sum_{0 \leq m_1, \ldots, m_\epsilon < \infty} f(p^{m_1}, \ldots, p^{m_\epsilon}) \right).$$

We formally apply the conclusion of Lemma 3.2 to factor the sum on the right-hand side of (3.10) and write (the application is formal because we ignore any issues about convergence)
\[ \times \oint_{|w_1|=\varepsilon} \cdots \oint_{|w_{\ell}|=\varepsilon} \prod_{j=1}^{\ell} \left\{ \zeta(1+z_j+w_j-\xi-\eta)\chi(\frac{1}{2}+\xi-z_j+it) \right\} \times \chi(\frac{1}{2}+\eta-w_j-it) \prod_{\alpha \in A_j} \zeta(1+z_j+\alpha) \prod_{\beta \in B_j} \zeta(1+w_j+\beta) \right\} \times \prod_{p} \left( \sum \prod_{j=1}^{\ell} \left\{ p^{N_j(-\frac{1}{2}+\xi-z_j)+M_j(-\frac{1}{2}+\eta-w_j)} \right\} \times \sum_{d=0}^{\infty} \left( \sum_{q=0}^{1} (-1)^q p^{\min\{q+d,N_j\}(1+z_j)+\min\{q+d,M_j\}(1+w_j)-q(2+z_j+w_j)-d(1+\xi+\eta)} \times G_{A_j}(1+z_j,p^{q+d-\min\{q+d,N_j\}}) G_{B_j}(1+w_j,p^{q+d-\min\{q+d,M_j\}}) \right\} \right) \times d w_{\ell} \cdots d w_1 \, dz_{\ell} \cdots \, dz_1 \, dt \, d \xi \, d \eta. \] (3.11)

Note that the variables of summation \( M_1, \ldots, M_{\ell}, N_1, \ldots, N_{\ell} \) in (3.11) run through the nonnegative integers. In the next section, we will relate (3.11) to the functions \( I_{C,D}(m) \) defined by (1.11).

### 4 LOCAL CALCULATIONS

In this section, we closely examine the local factor of the Euler product in (3.11). We prove an expression for this local factor in terms of the coefficients \( I_{C,D}(m) \), which are defined by (1.11) for finite multisets \( C, D \) of complex numbers. Our main result for this section is Theorem 4.4. All the arguments in this section are rigorous.

We first prove some basic properties of the coefficients \( I_{C,D}(m) \).

**Lemma 4.1.** Let \( C \) and \( D \) be finite multisets of complex numbers, let \( s \) be a complex number, and let \( p \) be a prime.

(i) If \( m \) is a positive integer, then

\[ m^{-s}I_{C,D}(m) = I_{C_s,D_s}(m), \]

where, for every multiset \( E \), \( E_s \) denotes the multiset \( \{ \gamma + s : \gamma \in E \} \).

(ii) If \( r \) is a nonnegative integer, then

\[ I_{C \cup \{s\},D}(p^r) = I_{C,D}(p^r) + p^{-s}I_{C \cup \{s\},D}(p^{r-1}), \]

where the last term is to be interpreted as 0 if \( r = 0 \).
(iii) If $R$ and $M$ are nonnegative integers, then

$$
\sum_{r=0}^{R} I_{C,D}(p^{r+M}) = I_{C\cup\{0\},D}(p^{R+M}) - I_{C\cup\{0\},D}(p^{M-1}),
$$

where the last term is to be interpreted as 0 if $M = 0$.

(iv) We have

$$
\sum_{k=0}^{\infty} I_{C,\{-s\}}(p^{k})p^{-k(1+s)} = \left(1 - \frac{1}{p}\right) \prod_{\gamma \in C} \left(1 - \frac{1}{p^{1+s+\gamma}}\right)^{-1}
$$

whenever the left-hand side converges absolutely.

Proof. If $C = \{\gamma_1, \ldots, \gamma_h\}$ and $D = \{\delta_1, \ldots, \delta_\ell\}$, then the definition (1.11) implies

$$
I_{C,D}(m) = \sum_{m_1 \cdots m_h n_1 \cdots n_\ell = m} n_1^{-\delta_1} \cdots n_\ell^{-\delta_\ell} \mu(n_1) n_1^{-\gamma_1} \cdots \mu(n_\ell) n_\ell^{-\gamma_\ell}, \tag{4.1}
$$

and (i) immediately follows.

To prove (ii), use (4.1) to write

$$
I_{C\cup\{s\},D}(p^{r}) = \sum_{\nu+m_1+\cdots+m_h+n_1+\cdots+n_\ell = r} p^{-\nu s} p^{-m_1 \gamma_1} \cdots p^{-m_h \gamma_h} \mu(p^{n_1}) p^{-n_1 \delta_1} \cdots \mu(p^{n_\ell}) p^{-n_\ell \delta_\ell}, \tag{4.2}
$$

where $\nu$ and the $m_i$'s and $n_i$'s run through nonnegative integers. By (4.1), we see that $I_{C,D}(p^{r})$ equals the sum of the terms on the right-hand side of (4.2) that have $\nu = 0$, while the sum of the terms with $\nu \geq 1$ equals $p^{-s} I_{C\cup\{s\},D}(p^{r-1})$. This proves (ii).

Next, to show (iii), we take $s = 0$ in (ii) to deduce that

$$
I_{C,D}(p^{r+M}) = I_{C\cup\{0\},D}(p^{r+M}) - I_{C\cup\{0\},D}(p^{r+M-1}).
$$

Summing both sides from $r = 0$ to $r = R$ gives (iii).

Finally, (iv) follows immediately from the definition (1.11) and the Euler product expansion of zeta. \hfill \square

We next prove a generalization of [14, Lemma 2]. To state it, we define

$$
\Sigma(A, B, z, w, M, N; p) := \sum_{q=0}^{\infty} \sum_{d=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} (-1)^q I_{A,\{-z\}} \left(p^{j+q+d-min\{q+d,N\}}\right)
\times I_{B,\{-w\}} \left(p^{k+q+d-min\{q+d, M\}}\right)
\times p^{-d-j(1+z)-k(1+w)-q(2+z+w)+min\{q+d,N\}(1+z)+min\{q+d,M\}(1+w)}. \tag{4.3}
$$

Lemma 4.2. Let $\epsilon > 0$ be arbitrarily small, and let $p$ be a prime. Suppose that $A$ and $B$ are finite multisets of complex numbers and $z$ and $w$ are complex numbers such that $|\text{Re}(\gamma)| \leq \epsilon$ for all
\[ \gamma \in A \cup B \cup \{ z, w \}. \text{If } M \text{ and } N \text{ are nonnegative integers such that } \min\{M, N\} = 0, \text{ then} \]

\[ \Sigma(A, B, z, w, M, N; p) = p^{Mw + Nz} \left( 1 - \frac{1}{p^{1+w+z}} \right) \sum_{r=0}^{\infty} \frac{I_{A \cup \{w\}, [-z]}(p^{r+M})I_{B \cup \{z\}, [-w]}(p^{r+N})}{p^r}. \]

**Proof.** First, observe that (4.1) and the divisor bound imply that if the elements of \( C \) and \( D \) each have real part \( \geq -c \), then

\[ I_{C, D}(m) \ll \varepsilon m^{c+\varepsilon} \] (4.4)

for arbitrarily small \( \varepsilon > 0 \). From this and the assumption that \( |\text{Re}(\gamma)| \leq \varepsilon \) for all \( \gamma \in A \cup B \cup \{ z, w \} \), we deduce the absolute convergence of the right-hand side of (4.3). Thus, the sum \( \Sigma(A, B, z, w, M, N; p) \) is well-defined.

To prove Lemma 4.2, we may assume without loss of generality that \( N = 0 \), as the case with \( M = 0 \) will follow by symmetry. If \( N = 0 \), then the definition (4.3) specializes to

\[ \Sigma(A, B, z, w, M, 0; p) = \sum_{q=0}^{\infty} \sum_{d=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} (-1)^q I_{A, \{ -z \}}(p^{j+q+d})I_{B, \{ -w \}}(p^{k+q+d-min\{q+d,M\}}) \times p^{-d-j(1+z)-k(1+w)-q(2+z+w)+min\{q+d,M\}(1+w)}. \]

Split this into

\[ \Sigma(A, B, z, w, M, 0; p) = \Sigma^- + \Sigma^+, \] (4.5)

where \( \Sigma^- \) is the sum of the terms with \( d < M \), and \( \Sigma^+ \) is the sum of the terms with \( d \geq M \).

We first evaluate \( \Sigma^- \). If \( d < M \) and \( q \in \{0, 1\} \), then \( q + d - \min\{q + d, M\} = 0 \) and \( d + q(2 + z + w) - \min\{q + d, M\}(1 + w) = -dw + q(1 + z) \). We use these and then carry out the summation over \( q \) to write

\[ \Sigma^- = \sum_{d<M} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} I_{A, \{ -z \}}(p^{j+d})I_{B, \{ -w \}}(p^k)p^{-j(1+z)-k(1+w)+dw} \]

\[ - \sum_{d<M} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} I_{A, \{ -z \}}(p^{j+1+d})I_{B, \{ -w \}}(p^k)p^{-(j+1)(1+z)-k(1+w)+dw}. \]

Factor out the \( k \)-sum to deduce that

\[ \Sigma^- = \sum_{k=0}^{\infty} I_{B, \{ -w \}}(p^k)p^{-k(1+w)} \left\{ \sum_{d<M} \sum_{j=0}^{\infty} I_{A, \{ -z \}}(p^{j+d})p^{-j(1+z)+dw} \right\} \]

\[ - \sum_{d<M} \sum_{j=0}^{\infty} I_{A, \{ -z \}}(p^{j+1+d})p^{-(j+1)(1+z)+dw}. \]
The $j$-sums telescope, while we may evaluate the $k$-sum using Lemma 4.1(iv). Hence,

$$\Sigma^- = \left(1 - \frac{1}{p}\right) \prod_{\beta \in B} \left(1 - \frac{1}{p^{1+w+\beta}}\right)^{-1} \sum_{d < M} I_{A,[-z]}(p^d)p^d w.$$

From (i) and (iii) of Lemma 4.1, we see that

$$\sum_{d < M} I_{A,[-z]}(p^d)p^d w = \sum_{d < M} I_{A_{-w},[-z-w]}(p^d) = I_{A_{-w} \cup \{0\},[-z-w]}(p^{M-1}) = p^{(M-1)w} I_{A_{[w],[-z]}(p^{M-1})}.$$

Thus,

$$\Sigma^- = \left(1 - \frac{1}{p}\right)p^{(M-1)w} I_{A_{[w],[-z]}(p^{M-1})} \prod_{\beta \in B} \left(1 - \frac{1}{p^{1+w+\beta}}\right)^{-1}.$$

Having evaluated $\Sigma^-$, we next consider the sum $\Sigma^+$ defined in (4.5). If $d \geq M$ and $q \geq 0$, then

$$\min\{q + d, M\} = M.$$ We use this and then carry out the summation over $q$ to deduce that

$$\Sigma^+ = \sum_{d \geq M} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} I_{A,[-z]}(p^{j+d})I_{B,[w]}(p^{k+d-M})p^{-d-j(1+z)-(k+1)w+M(1+w)}$$

$$- \sum_{d \geq M} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} I_{A,[-z]}(p^{j+1+d})I_{B,[w]}(p^{k+1+d-M})p^{-d-(j+1)(1+z)-(k+1)(1+w)+M(1+w)}.$$

We re-label $d$ as $d + M$ to see that

$$\Sigma^+ = p^{Mw} \sum_{d=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} I_{A,[-z]}(p^{j+d+M})I_{B,[w]}(p^{k+d})p^{-d-j(1+z)-k(1+w)}$$

$$- p^{Mw} \sum_{d=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} I_{A,[-z]}(p^{j+1+d+M})I_{B,[w]}(p^{k+1+d})p^{-d-(j+1)(1+z)-(k+1)(1+w)}.$$

We add and subtract an extra sum and arrive at

$$\Sigma^+ = p^{Mw} \sum_{d=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} I_{A,[-z]}(p^{j+d+M})I_{B,[w]}(p^{k+d})p^{-d-j(1+z)-k(1+w)}$$

$$- p^{Mw} \sum_{d=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} I_{A,[-z]}(p^{j+1+d+M})I_{B,[w]}(p^{k+d})p^{-d-(j+1)(1+z)-k(1+w)}$$

$$+ p^{Mw} \sum_{d=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} I_{A,[-z]}(p^{j+1+d+M})I_{B,[w]}(p^{k+1+d})p^{-d-(j+1)(1+z)-(k+1)(1+w)}$$

$$- p^{Mw} \sum_{d=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} I_{A,[-z]}(p^{j+1+d+M})I_{B,[w]}(p^{k+1+d})p^{-d-(j+1)(1+z)-(k+1)(1+w)}.$$
MOMENTS OF ZETA AND CORRELATIONS OF DIVISOR-SUMS

We may combine the first two sums on the right-hand side and see that the $j$-sums telescope to leave only the $j = 0$ terms of the first sum. Similarly, we may combine the third and fourth sums and see that the $k$-sums telescope to leave only the $k = 0$ terms of the third sum. Therefore,

\[
\Sigma^+ = p^{Mw} \left( \sum_{d=0}^{\infty} \sum_{k=0}^{\infty} I_{A,\{-z\}}(p^{d+M})I_{B,\{-w\}}(p^{k+d})p^{-d-k(1+w)} \right) \\
+ p^{Mw} \sum_{d=0}^{\infty} \sum_{j=0}^{\infty} I_{A,\{-z\}}(p^{j+1+d+M})I_{B,\{-w\}}(p^d)p^{-d-(j+1)(1+z)}.
\]

In the last sum, relabel $j + 1$ as $j$ and then add and subtract the $j = 0$ terms to deduce that

\[
\Sigma^+ = p^{Mw} \left( \Sigma_1^+ + \Sigma_2^+ - \Sigma_3^+ \right),
\]

where $\Sigma_1^+$, $\Sigma_2^+$, and $\Sigma_3^+$ are defined by

\[
\Sigma_1^+ = \sum_{d=0}^{\infty} \sum_{k=0}^{\infty} I_{A,\{-z\}}(p^{d+M})I_{B,\{-w\}}(p^{k+d})p^{-d-k(1+w)},
\]

\[
\Sigma_2^+ = \sum_{d=0}^{\infty} \sum_{j=0}^{\infty} I_{A,\{-z\}}(p^{j+d+M})I_{B,\{-w\}}(p^d)p^{-d-j(1+z)},
\]

and

\[
\Sigma_3^+ = \sum_{d=0}^{\infty} I_{A,\{-z\}}(p^{d+M})I_{B,\{-w\}}(p^d)p^{-d},
\]

respectively.

We next evaluate each of $\Sigma_1^+$ and $\Sigma_2^+$. We first consider $\Sigma_1^+$. Group together terms that have the same $k + d$ to write

\[
\Sigma_1^+ = \sum_{r=0}^{\infty} I_{B,\{-w\}}(p^r)p^{-r} \sum_{k+d=r} I_{A,\{-z\}}(p^{d+M})p^{-kw}.
\]

In this expression, we have $kw = rw - dw = rw - (d + M)w + Mw$. Thus,

\[
\Sigma_1^+ = p^{-Mw} \sum_{r=0}^{\infty} I_{B,\{-w\}}(p^r)p^{-r(1+w)} \sum_{k+d=r} I_{A,\{-z\}}(p^{d+M})p^{(d+M)w}.
\]

By (i) and (iii) of Lemma 4.1, we have

\[
\sum_{k+d=r} I_{A,\{-z\}}(p^{d+M})p^{(d+M)w} = \sum_{k+d=r} I_{A_{-w\cup\{0\},\{-z-w\}}}(p^{d+M})
\]

\[
= I_{A_{-w\cup\{0\},\{-z-w\}}}(p^{r+M}) - I_{A_{-w\cup\{0\},\{-z-w\}}}(p^{M-1})
\]

\[
= p^{(r+M)w} I_{A\cup\{0\},\{-z\}}(p^{r+M}) - p^{(M-1)w} I_{A\cup\{0\},\{-z\}}(p^{M-1}).
\]
Hence,
\[
\Sigma_1^+ = \sum_{r=0}^{\infty} I_{B,[w]}(p^r) I_{A\cup[z],[-z]}(p^{r+M}) p^{-r} - p^{-w} I_{A\cup[w],[-z]}(p^{M-1}) \sum_{r=0}^{\infty} I_{B,[w]}(p^r) p^{-r(1+w)}.
\]

From this and Lemma 4.1(iv), it follows that
\[
\Sigma_1^+ = \sum_{r=0}^{\infty} I_{B,[w]}(p^r) I_{A\cup[z],[-z]}(p^{r+M}) p^{-r} - p^{-w} \left(1 - \frac{1}{p}\right) I_{A\cup[w],[-z]}(p^{M-1}) \prod_{\beta \in B} \left(1 - \frac{1}{p^{1+w+\beta}}\right)^{-1}.
\]

(4.10)

Having evaluated \(\Sigma_1^+\), we next turn our attention to the sum \(\Sigma_2^+\) defined by (4.8). Gather terms with the same \(j + d\) to write
\[
\Sigma_2^+ = \sum_{r=0}^{\infty} I_{A,[z]}(p^{r+M}) p^{-r} \sum_{j + d = r} I_{B,[w]}(p^d) p^{-jz}.
\]

In this sum, we have \(jz = rz - dz\). Thus,
\[
\Sigma_2^+ = \sum_{r=0}^{\infty} I_{A,[z]}(p^{r+M}) p^{-r(1+z)} \sum_{j + d = r} I_{B,[w]}(p^d) p^dz.
\]

From (i) and (iii) of Lemma 4.1, we see that
\[
\sum_{j + d = r} I_{B,[w]}(p^d) p^dz = \sum_{j + d = r} I_{B,[w],[-w-z]}(p^d) = I_{B,[w],[-w-z]}(p^r) = p^z I_{B\cup[z],[-w]}(p^r).
\]

Thus,
\[
\Sigma_2^+ = \sum_{r=0}^{\infty} I_{A,[z]}(p^{r+M}) I_{B\cup[z],[-w]}(p^r) p^{-r}.
\]

(4.11)

Combining now (4.7), (4.9), (4.10), and (4.11), we arrive at
\[
\Sigma^+ = -p^{(M-1)w} \left(1 - \frac{1}{p}\right) I_{A\cup[w],[-z]}(p^{M-1}) \prod_{\beta \in B} \left(1 - \frac{1}{p^{1+w+\beta}}\right)^{-1}
\]
\[
+ p^{Mw} \sum_{r=0}^{\infty} I_{A\cup[w],[-z]}(p^{r+M}) I_{B,[w]}(p^r) p^{-r} + p^{Mw} \sum_{r=0}^{\infty} I_{A,[z]}(p^{r+M}) I_{B\cup[z],[-w]}(p^r) p^{-r}
\]
\[
- p^{Mw} \sum_{r=0}^{\infty} I_{A,[z]}(p^{r+M}) I_{B,[w]}(p^r) p^{-r}.
\]

From this, (4.5), and (4.6), we arrive at
\[
\Sigma(A, B, z, w, M, 0; p) = p^{Mw} \sum_{r=0}^{\infty} I_{A\cup[w],[-z]}(p^{r+M}) I_{B,[w]}(p^r) p^{-r}
\]
\[
+ p^{Mw} \sum_{r=0}^{\infty} I_{A,[z]}(p^{r+M}) I_{B\cup[z],[-w]}(p^r) p^{-r} - p^{Mw} \sum_{r=0}^{\infty} I_{A,[z]}(p^{r+M}) I_{B,[w]}(p^r) p^{-r}.
\]

(4.12)
To write this in a more concise form, we apply the trick on [12, p. 746], as follows. By Lemma 4.1(ii), it holds that

\[
I_{A \cup \{w\}, [-z]}(p^{r+M}I_{B, [-w]}(p^r) + I_{A, [-z]}(p^{r+M})I_{B \cup \{z\}, [-w]}(p^r) - I_{A, [-z]}(p^{r+M})I_{B, [-w]}(p^r)
\]

\[
= (a_1 + a_2)b_1 + a_1(b_1 + b_2) - a_1b_1,
\]

where

\[
a_1 = I_{A, [-z]}(p^{r+M})
\]
\[
a_2 = p^{-w}I_{A \cup \{w\}, [-z]}(p^{r+M-1})
\]
\[
b_1 = I_{B, [-w]}(p^r)
\]
\[
b_2 = p^{-z}I_{B \cup \{z\}, [-w]}(p^{r-1}).
\]

Cancel the term \(-a_1b_1\), then add and subtract \(a_2b_2\) to deduce that

\[
I_{A \cup \{w\}, [-z]}(p^{r+M}I_{B, [-w]}(p^r) + I_{A, [-z]}(p^{r+M})I_{B \cup \{z\}, [-w]}(p^r) - I_{A, [-z]}(p^{r+M})I_{B, [-w]}(p^r)
\]

\[
= (a_1 + a_2)b_1 + a_1b_2 + a_2b_2 - a_2b_2
\]

\[
= (a_1 + a_2)(b_1 + b_2) - a_2b_2.
\]

From this, the definitions of \(a_1, a_2, b_1, b_2\), and Lemma 4.1(ii), we arrive at

\[
I_{A \cup \{w\}, [-z]}(p^{r+M}I_{B, [-w]}(p^r) + I_{A, [-z]}(p^{r+M})I_{B \cup \{z\}, [-w]}(p^r) - I_{A, [-z]}(p^{r+M})I_{B, [-w]}(p^r)
\]

\[
= I_{A \cup \{w\}, [-z]}(p^{r+M})I_{B \cup \{z\}, [-w]}(p^r) - p^{-w-z}I_{A \cup \{w\}, [-z]}(p^{r+M-1})I_{B \cup \{z\}, [-w]}(p^{r-1}).
\]

This and (4.12) imply

\[
\Sigma(A, B, z, w, M, 0; p) = p^{Mw} \sum_{r=0}^{\infty} I_{A \cup \{w\}, [-z]}(p^{r+M})I_{B \cup \{z\}, [-w]}(p^r) p^{-r}
\]

\[
- p^{Mw} p^{-w-z} \sum_{r=0}^{\infty} I_{A \cup \{w\}, [-z]}(p^{r+M-1})I_{B \cup \{z\}, [-w]}(p^{r-1}) p^{-r}.
\]

The \(r = 0\) term of the last sum is zero by the convention mentioned in Lemma 4.1(ii). Hence, we may relabel \(r - 1\) in this last sum as \(r\) to deduce that

\[
\Sigma(A, B, z, w, M, 0; p) = p^{Mw} \left(1 - \frac{1}{p^{1+w+z}}\right) \sum_{r=0}^{\infty} I_{A \cup \{w\}, [-z]}(p^{r+M})I_{B \cup \{z\}, [-w]}(p^r) p^{-r}. \quad (4.13)
\]

This proves Lemma 4.2 with the additional assumption that \(N = 0\). Now the definition (4.3) of \(\Sigma\) implies that

\[
\Sigma(A, B, z, w, M, N; p) = \Sigma(B, A, w, z, N, M; p).
\]
It follows from this and \((4.13)\) that
\[
\Sigma(A, B, z, w, 0, N; p) = \Sigma(B, A, w, z, N, 0; p) = p^N z \left(1 - \frac{1}{p^{1+w+z}}\right) \sum_{r=0}^{\infty} I_{A \cup \{w\}, \{z\}}(p^r) I_{B \cup \{z\}, \{w\}}(p^{r+N}) p^{-r}
\]
This proves Lemma 4.2 with the additional assumption that \(M = 0\). As \(\min\{M, N\} = 0\) means either \(M = 0\) or \(N = 0\), the proof of Lemma 4.2 is complete. \(\square\)

To use Lemma 4.2 to evaluate the sum in \((3.10)\), we need to relate the function \(G\), defined by \((3.2)\), with the function \(I\), which is defined by \((1.11)\).

**Lemma 4.3.** Let \(p\) be a prime, and let \(r\) be a nonnegative integer. Suppose that \(E\) is a finite multiset of complex numbers and \(s\) is a complex number such that \(\text{Re}(s + \gamma) > 0\) for every \(\gamma \in E\). Then
\[
G_E(s, p^r) = \frac{p}{p-1} \prod_{\gamma \in E} (1 - p^{-s-\gamma}) \sum_{j=0}^{\infty} \frac{I_{E,\{1-s\}}(p^{j+r})}{p^{js}}. \quad (4.14)
\]
Furthermore, we have
\[
G_E(s, p^r) \ll |E| \varepsilon p^{r(c+\varepsilon)} \quad (4.15)
\]
for arbitrarily small \(\varepsilon\), where \(c = -\min\{1 - \text{Re}(s)\} \cup \{\text{Re}(\gamma) : \gamma \in E\}\).

**Proof.** The assumption \(\text{Re}(s + \gamma) > 0\) and the bound \((4.4)\) imply the absolute convergence of both the right-hand side of \((4.14)\) and the definition \((3.3)\) of \(g_E(s, p^r)\) (note that the analogue of \((4.4)\) also holds for \(\tau_E\) because \(\tau_E = I_{E,\emptyset}\), where \(\emptyset\) is the empty set). If \(r = 0\), then the definition \((3.2)\) implies \(G_E(s, 1) = 1\), while the right-hand side of \((4.14)\) also equals 1 because
\[
\sum_{j=0}^{\infty} \frac{I_{E,\{1-s\}}(p^{j+r})}{p^{js}} = \left(1 - \frac{1}{p}\right) \prod_{\gamma \in E} \left(1 - \frac{1}{p^{s+\gamma}}\right)^{-1}
\]
by the definition \((1.11)\) and the Euler product expression for zeta. Thus, \((4.14)\) holds for \(r = 0\).

Now suppose that \(r \geq 1\). The definition \((3.2)\) of \(G_E(s, p^r)\) implies that
\[
G_E(s, p^r) = \left(\frac{p}{p-1}\right) g_E(s, p^r) - \left(\frac{p^s}{p-1}\right) g_E(s, p^{r-1}). \quad (4.16)
\]
The definition \((3.3)\) implies that
\[
g_E(s, p^j) = \left\{ \prod_{\gamma \in E} \left(1 - \frac{1}{p^{s+\gamma}}\right) \right\} \sum_{m=0}^{\infty} \frac{\tau_E(p^{m+j})}{p^{ms}} \quad (4.17)
\]
for \(j \geq 1\). Note that this also holds for \(j = 0\) because the definition \((1.2)\) and the Euler product expression for zeta imply that
\[
\sum_{m=0}^{\infty} \frac{\tau_E(p^m)}{p^{ms}} = \prod_{\gamma \in E} \left(1 - \frac{1}{p^{s+\gamma}}\right)^{-1}. \quad (1.12)
\]
It follows from these and (4.16) that

\[
G_E(s, p^r) = \left(\frac{p}{p-1}\right) \prod_{\gamma \in E} \left(1 - \frac{1}{p^{s+\gamma}}\right) \sum_{m=0}^{\infty} \frac{\tau_E(p^{m+r})}{p^{ms}} - \left(\frac{p^s}{p-1}\right) \prod_{\alpha \in A} \left(1 - \frac{1}{p^{s+\gamma}}\right) \sum_{m=0}^{\infty} \frac{\tau_E(p^{m+r-1})}{p^{ms}} \\
= \left(\frac{p}{p-1}\right) \prod_{\gamma \in E} \left(1 - \frac{1}{p^{s+\gamma}}\right) \sum_{m=0}^{\infty} \frac{\tau_E(p^{m+r}) - p^{s-1}\tau_E(p^{m+r-1})}{p^{ms}}. \tag{4.17}
\]

Now (4.1), which also holds for \( \tau_E \) because \( \tau_E = I_{E, \emptyset} \), implies

\[
\tau_E(p^{m+r}) - p^{s-1}\tau_E(p^{m+r-1}) = I_{E, \{1-s\}}(p^{m+r}).
\]

The identity (4.14) follows from this and (4.17).

The bound (4.15) follows from (4.14) and (4.4).

We now prove the main result of this section. Recall the notations \( E_s := \{\gamma + s : \gamma \in E\} \) and \( E^- := \{-\gamma : \gamma \in E\} \) that were stated below Conjecture 1.1.

**Theorem 4.4.** Suppose that \( \ell \) is a positive integer, \( \xi, \eta \) are complex numbers, and \( A, B, Z, W \) are finite multisets of complex numbers with \( Z = \{z_1, \ldots, z_\ell\} \) and \( W = \{w_1, \ldots, w_\ell\} \). Let \( A = A_1 \cup \cdots \cup A_\ell \) and \( B = B_1 \cup \cdots \cup B_\ell \) be partitions of \( A \) and \( B \), respectively. Let \( \epsilon > 0 \) be arbitrarily small, and suppose that \( \Re(\xi) = \Re(\eta) = 2\epsilon \) and \( |\Re(\gamma)| \leq \epsilon \) for all \( \gamma \in A \cup B \cup Z \cup W \). Then

\[
\sum_{M_1 + \cdots + M_\ell = N_1 + \cdots + N_\ell} \prod_{j=1}^{\ell} \left\{ p^{N_j(-\frac{1}{2} + \xi - z_j) + M_j(-\frac{1}{2} + \eta - w_j)} \right\} \\
\times \sum_{d=0}^{\infty} \sum_{q=0}^{d} (-1)^{q} p^{\min\{q + d, N_j\}(1 + z_j) + \min\{q + d, M_j\}(1 + w_j) - q(2 + z_j + w_j) - d(1 + \xi + \eta)}
\times G_{A_j}\left(1 + z_j, p^{q + d - \min\{q + d, N_j\}}\right) G_{B_j}\left(1 + w_j, p^{q + d - \min\{q + d, M_j\}}\right) \bigg) \\
= \left(1 - \frac{1}{p}\right)^{-2\ell} \prod_{j=1}^{\ell} \left(1 - \frac{1}{p^{1 + w_j + z_j - \xi - \eta}}\right) \prod_{\alpha \in A_j} \left(1 - \frac{1}{p^{1 + z_j + \alpha}}\right) \prod_{\beta \in B_j} \left(1 - \frac{1}{p^{1 + w_j + \beta}}\right) \\
\times \sum_{m=0}^{\infty} I_{A_{\xi+\eta} \cup W, (Z^-)_{\xi+\eta}}(p^m) I_{B \cup Z_{-\xi-\eta} W^-}(p^m) \frac{I_{B \cup Z_{-\xi-\eta} W^-}(p^m)}{p^m}.
\]
Proof. For brevity, let LHS denote the left-hand side of the conclusion of Theorem 4.4. We apply Lemma 4.3 and deduce that

\[
\text{LHS} = \sum_{M_1 + \cdots + M_\ell = N_1 + \cdots + N_\ell \atop \min(M_j, N_j) = 0 \forall j} \prod_{j=1}^\ell \left\{ p^{N_j(-\frac{1}{2} + \xi - z_j) + M_j(-\frac{1}{2} + \eta - w_j)} \right\} \times \left( 1 - \frac{1}{p} \right)^{-2} \prod_{\alpha \in A_j} \left( 1 - \frac{1}{p^{1 + z_j + \alpha}} \right) \prod_{\beta \in B_j} \left( 1 - \frac{1}{p^{1 + w_j + \beta}} \right) \times \sum_{d=0}^\infty \sum_{q=0}^\infty \sum_{i=0}^\infty \sum_{k=0}^\infty (1-\gamma_j) I(A_j, (-z_j) \{ p^{i+q+d-min(q+d,N_j)} \} I(B_j \{ -w_j \}) (p^{k+q+d-min(q+d,M_j)}) \times p^{-(1+z_j) -(1+w_j) + min(q+d,N_j)(1+z_j) + min(q+d,M_j)(1+w_j) - q(2+z_j+w_j) - d(1+\xi + \eta) } \}
\]

The d, q, i, k-sum equals \( \Sigma((A_j)_{\xi+\eta}, B_j, z_j - \xi - \eta, w_j, M_j, N_j; p) \) by (4.3) and Lemma 4.1(i). Thus, Lemma 4.2 gives

\[
\text{LHS} = \sum_{M_1 + \cdots + M_\ell = N_1 + \cdots + N_\ell \atop \min(M_j, N_j) = 0 \forall j} \prod_{j=1}^\ell \left\{ p^{N_j(-\frac{1}{2} - \eta) + M_j(-\frac{1}{2} + \eta)} \right\} \times \left( 1 - \frac{1}{p} \right)^{-2 \ell} \prod_{\alpha \in A_j} \left( 1 - \frac{1}{p^{1 + z_j + \alpha}} \right) \prod_{\beta \in B_j} \left( 1 - \frac{1}{p^{1 + w_j + \beta}} \right) \times \sum_{r=0}^\infty \frac{I(A_j)_{\xi+\eta \cup \{ w_j \} \cup \{ -z_j + \xi + \eta \} \{ p^r + M_j \} I(B_j \cup \{ z_j - \xi - \eta \} \{ -w_j \}) (p^{r+N_j}) \}}{p^r} \}
\]

We rearrange the factors and use the fact that \( M_1 + \cdots + M_\ell = N_1 + \cdots + N_\ell \) to arrive at

\[
\text{LHS} = \left( 1 - \frac{1}{p} \right)^{-2\ell} \prod_{j=1}^\ell \left\{ \left( 1 - \frac{1}{p^{1 + u_j + z_j - \xi - \eta}} \right) \prod_{\alpha \in A_j} \left( 1 - \frac{1}{p^{1 + z_j + \alpha}} \right) \prod_{\beta \in B_j} \left( 1 - \frac{1}{p^{1 + w_j + \beta}} \right) \right\} \times \sum_{M_1 + \cdots + M_\ell = N_1 + \cdots + N_\ell \atop \min(M_j, N_j) = 0 \forall j} \prod_{j=1}^\ell \left\{ \left( 1 - \frac{1}{p^{1 + z_j + \alpha}} \right) \prod_{\beta \in B_j} \left( 1 - \frac{1}{p^{1 + w_j + \beta}} \right) \right\} \times \sum_{r=0}^\infty \frac{I(A_j)_{\xi+\eta \cup \{ w_j \} \cup \{ -z_j + \xi + \eta \} \{ p^r+mj \} I(B_j \cup \{ z_j - \xi - \eta \} \{ -w_j \}) (p^{r+N_j}) \}}{p^r} \}
\]

To evaluate the sum over the \( M_j, N_j \), we interchange the order of summation to write it as

\[
\sum_{r_1=0}^\infty \cdots \sum_{r_\ell=0}^\infty \sum_{M_1 + \cdots + M_\ell = N_1 + \cdots + N_\ell \atop \min(M_j, N_j) = 0 \forall j} \prod_{j=1}^\ell \left\{ \frac{I(A_j)_{\xi+\eta \cup \{ w_j \} \cup \{ -z_j + \xi + \eta \} \{ p^r+mj \} I(B_j \cup \{ z_j - \xi - \eta \} \{ -w_j \}) (p^{r+N_j}) \}}{p^{r_1+\frac{1}{2}N_j+\frac{1}{2}M_j}} \right\}. \]
We make the change of variables $m_j = M_j + r_j$ and $n_j = N_j + r_j$, and then evaluate the $r_j$-sums to see that this sum equals

$$
\sum_{r_1=0}^{\infty} \cdots \sum_{r_{\ell}=0}^{\infty} \sum_{\min(m_j,n_j)=r_j \forall j} \prod_{j=1}^{\ell} \left\{ \frac{I(A_j)\zeta(\xi + \eta)\zeta(-z_j + \xi + \eta)}{p^{\frac{1}{2}n_j + \frac{1}{2}m_j}} \right\} \frac{I(B_j)\zeta(-\xi - \eta)\zeta(-w_j)}{p^{\frac{1}{2}n_j + \frac{1}{2}m_j}}
$$

$$= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \prod_{j=1}^{\ell} \left\{ \frac{I(A_j)\zeta(\xi + \eta)\zeta(-z_j + \xi + \eta)}{p^{\frac{1}{2}n_j + \frac{1}{2}m_j}} \right\} \frac{I(B_j)\zeta(-\xi - \eta)\zeta(-w_j)}{p^{\frac{1}{2}n_j + \frac{1}{2}m_j}}
$$

because, by (1.11), $I_{A_+ + (Z^-)\zeta + (\eta)}$ is the Dirichlet convolution

$$I_{A_+ + (Z^-)\zeta + (\eta)} = I_{(A_1)\zeta + (\eta)} * \cdots * I_{(A_\ell)\zeta + (\eta)}\zeta(-z_j + \xi + \eta),$$

and similarly for $I_{B_\cup Z^- \zeta - \eta W^-}$.

\[ \square \]

## 5 VANDERMONDE INTEGRAL EXPRESSIONS

We now return to the task of evaluating (3.11). We use Theorem 4.4 to evaluate each local factor on the right-hand side of (3.11) and predict that

$$S_\epsilon \sim \frac{1}{(\epsilon!)^2 (2\pi i)^2} \int_{(2\epsilon)} \int_{(2\epsilon)} \tilde{Y}(\xi)\tilde{Y}(\eta) \frac{X^{\xi + \eta}}{T} \int_0^{\infty} \psi \left( \frac{t}{T} \right) \frac{1}{(2\pi i)^2} \oint_{|z_1|=\epsilon} \cdots \oint_{|z_{\ell}|=\epsilon} \left\{ \chi \left( \frac{1}{2} + \xi - z_j + it \right) \chi \left( \frac{1}{2} + \eta - w_j - it \right) \right\}
$$

$$\times \chi \left( 1 + z_j + w_j - \xi - \eta \right) \prod_{\alpha \in A_j} \zeta \left( 1 + z_j + \alpha \right) \prod_{\beta \in B_j} \zeta \left( 1 + w_j + \beta \right)
$$

$$\times \prod_p \left\{ \prod_{j=1}^{\ell} \left\{ 1 - \frac{1}{p^{1 + z_j + \xi + \eta}} \right\} \prod_{\alpha \in A_j} \left( 1 - \frac{1}{p^{1 + z_j + \alpha}} \right) \prod_{\beta \in B_j} \left( 1 - \frac{1}{p^{1 + w_j + \beta}} \right) \right\}
$$

$$\times \left( 1 - \frac{1}{p} \right)^{-2\epsilon} \sum_{m=0}^{\infty} \frac{I_{A_+ + (Z^-)\zeta + (\eta)}\zeta(-z_j + \xi + \eta)}{p^{\frac{1}{2}n_j + \frac{1}{2}m_j}} \frac{I_{B_\cup Z^- \zeta - \eta W^-}}{p^{\frac{1}{2}n_j + \frac{1}{2}m_j}}
$$

$$\frac{d\omega_\epsilon \cdots d\omega_1 d\zeta_\epsilon \cdots d\zeta_1 dt d\xi d\eta.}
$$

By the Euler product expression for zeta, we may write our prediction more concisely as

$$S_\epsilon \sim \frac{1}{(\epsilon!)^2 (2\pi i)^2} \int_{(2\epsilon)} \int_{(2\epsilon)} \tilde{Y}(\xi)\tilde{Y}(\eta) \frac{X^{\xi + \eta}}{T} \int_0^{\infty} \psi \left( \frac{t}{T} \right) \frac{1}{(2\pi i)^2} \oint_{|z_1|=\epsilon} \cdots \oint_{|z_{\ell}|=\epsilon} \left\{ \chi \left( \frac{1}{2} + \xi - z_j + it \right) \chi \left( \frac{1}{2} + \eta - w_j - it \right) \right\}
$$

$$\times \chi \left( 1 + z_j + w_j - \xi - \eta \right) \prod_{\alpha \in A_j} \zeta \left( 1 + z_j + \alpha \right) \prod_{\beta \in B_j} \zeta \left( 1 + w_j + \beta \right)
$$

$$\times \prod_p \left\{ \prod_{j=1}^{\ell} \left\{ 1 - \frac{1}{p^{1 + z_j + \xi + \eta}} \right\} \prod_{\alpha \in A_j} \left( 1 - \frac{1}{p^{1 + z_j + \alpha}} \right) \prod_{\beta \in B_j} \left( 1 - \frac{1}{p^{1 + w_j + \beta}} \right) \right\}
$$

$$\times \left( 1 - \frac{1}{p} \right)^{-2\epsilon} \sum_{m=0}^{\infty} \frac{I_{A_+ + (Z^-)\zeta + (\eta)}\zeta(-z_j + \xi + \eta)}{p^{\frac{1}{2}n_j + \frac{1}{2}m_j}} \frac{I_{B_\cup Z^- \zeta - \eta W^-}}{p^{\frac{1}{2}n_j + \frac{1}{2}m_j}}
$$

$$\frac{d\omega_\epsilon \cdots d\omega_1 d\zeta_\epsilon \cdots d\zeta_1 dt d\xi d\eta.}
\[
\times \prod_{\ell} \left\{ \left(1 - \frac{1}{p}\right)^{-2\ell} \sum_{m=0}^{\infty} \frac{I_{A_{\xi+\eta}}(Z_{\ell})_{\xi+\eta}(p^m)}{I_{B_{\xi+\eta}}(Z_{\ell}-\eta)_{\xi+\eta}(p^m)} \right\} \\
\times d\omega_\ell \ldots d\omega_1 \, dz_\ell \ldots dz_1 \, dt \, d\xi \, d\eta,
\]

where the latter Euler product is to be interpreted as its analytic continuation.

To evaluate the \(z_j\) and \(w_j\)-integrals, we need to write out this analytic continuation and determine its poles and residues. The local factor

\[
\left(1 - \frac{1}{p}\right)^{-2\ell} \sum_{m=0}^{\infty} \frac{I_{A_{\xi+\eta}}(Z_{\ell})_{\xi+\eta}(p^m)}{I_{B_{\xi+\eta}}(Z_{\ell}-\eta)_{\xi+\eta}(p^m)}
\]

may be written as a power series in \(1/p\). The coefficient of \(1/p\) in this power series is

\[
2\ell + I_{A_{\xi+\eta}}(Z_{\ell})(p)I_{B_{\xi+\eta}}(Z_{\ell}-\eta),
\]

which, by (4.1), equals

\[
2\ell + \left( \sum_{\alpha \in A} p^{-\alpha - \xi - \eta} + \sum_{j=1}^{\ell} p^{-w_j} - \sum_{j=1}^{\ell} p^{-z_j - \xi - \eta} \right) \left( \sum_{\beta \in B} p^{-\beta} + \sum_{j=1}^{\ell} p^{-\xi + \eta} - \sum_{j=1}^{\ell} p^{-w_j} \right)
\]

\[
\sum_{\alpha \in A} \sum_{\beta \in B} p^{-\alpha - \beta - \xi - \eta} + \sum_{\alpha \in A} \sum_{j=1}^{\ell} p^{-\alpha - w_j} + \sum_{i=1}^{\ell} p^{-\xi + \eta} - \sum_{1 \leq i \neq j \leq \ell} p^{-w_i - w_j}
\]

\[
- \sum_{\alpha \in A} \sum_{\beta \in B} p^{-\alpha - \xi + \eta + w_j} + \sum_{1 \leq i \neq j \leq \ell} p^{-\xi - \eta} + \sum_{i=1}^{\ell} p^{-\xi + \eta}
\]

Therefore, the (analytic continuation of the) Euler product in (5.1) may be written as

\[
\prod_{\alpha \in A} \zeta(1 + \alpha + \beta + \xi + \eta) \prod_{\beta \in B} \zeta(1 + \alpha + z_j) \prod_{1 \leq i \leq \ell} (1/\zeta)(1 + \alpha + \xi + \eta - w_j) \prod_{\beta \in B} (1/\zeta)(1 + \alpha + \xi + \eta - z_j)
\]

\[
\times \prod_{1 \leq i, j \leq \ell} (1/\zeta)(1 - z_i + z_j) \prod_{\beta \in B} (1/\zeta)(1 - w_i + w_j)
\]

\[
\times \prod_{1 \leq i, j \leq \ell} (1/\zeta)(1 - z_i - w_j + \xi + \eta) \zeta(1 + z_i + w_j - \xi - \eta) \zeta(1 - z_i - w_j + \xi + \eta)
\]

\[
\times A(A, B, Z, W, \xi + \eta),
\]

(5.2)
where \( A(A, B, Z, W, \xi + \eta) \) is a Euler product that converges absolutely whenever \( \text{Re}(\xi) = \text{Re}(\eta) = 2 \varepsilon \) and \( |\gamma| \leq \varepsilon \) for all \( \gamma \in A \cup B \cup Z \cup W \). Explicitly, \( A \) is defined by

\[
A(A, B, Z, W, \xi + \eta) := \prod_p \left( 1 - \frac{1}{p} \right)^{-2\ell} \prod_{\alpha \in A, \beta \in B} \left( 1 - \frac{1}{p^{1+\alpha+\beta+\xi+\eta}} \right) \prod_{1 \leq j \leq \ell} \left( 1 - \frac{1}{p^{1+\alpha+z_j}} \right) 
\times \prod_{1 \leq j, k \leq \ell, \beta \in B} \left( 1 - \frac{1}{p^{1+\beta+\xi+\eta-z_j}} \right) \prod_{1 \leq l, k \leq \ell, i \neq j} \left( 1 - \frac{1}{p^{1-z_l+z_j}} \right) 
\times \prod_{1 \leq l, k \leq \ell, i \neq j} \left( 1 - \frac{1}{p^{1-w_l+w_j}} \right) \prod_{1 \leq l, k \leq \ell} \left( 1 - \frac{1}{p^{1+z_l+w_j-\xi-\eta}} \right) 
\times \prod_{1 \leq l, k \leq \ell} \left( 1 - \frac{1}{p^{1-z_l-w_j+\xi+\eta}} \right) \sum_{m=0}^{\infty} \frac{I_{A\xi+\eta\cup W}(Z_{\xi+\eta})(p^m)I_{B\cup Z_{-\xi-\eta}, W_{-}(p^m)}}{p^{m}} \right),
\]

(5.3)

where we recall that \( I_{C,D} \) is defined by (1.11). This leads us to conjecture Conjecture 1.4.

6 | EVALUATING THE VANDERMONDE INTEGRAL EXPRESSION

We now evaluate the \( z_j \)- and \( w_j \)-integrals in Conjecture 1.4 (or equivalently in (5.1)) and show that the right-hand side of (1.8) is equal to the right-hand side of (1.9). For convenience, we assume that the elements of \( A \) are distinct from each other, and the elements of \( B \) are distinct from each other. The general result will then follow from analytic continuation. If \( \text{Re}(\xi) = \text{Re}(\eta) = 2 \varepsilon \) and \( |\alpha|, |\beta| \leq \varepsilon/2 \) for all \( \alpha \in A \) and \( \beta \in B \), then the poles of the integrand in (1.9), viewed as a function of \( z_j \) (resp., \( w_j \)), that are enclosed by the circle \( |z_j| = \varepsilon \) (resp., \( |w_j| = \varepsilon \)) are at the points \( z_j = -\alpha \), where \( \alpha \in A \) (resp., \( w_j = -\beta \), where \( \beta \in B \)). Thus, the value of the \( z_1, \ldots, z_{\ell}, w_1, \ldots, w_{\ell} \)-integral in (1.9) equals the sum of the residues of the integrand at the points

\[
(z_1, \ldots, z_{\ell}, w_1, \ldots, w_{\ell}) = (-\alpha_1, \ldots, -\alpha_{\ell}, \ldots, -\beta_1, \ldots, -\beta_{\ell}),
\]

(6.1)

where \( \alpha_1, \ldots, \alpha_{\ell} \in A \) and \( \beta_1, \ldots, \beta_{\ell} \in B \). If \( \alpha_i = \alpha_j \) for some \( i \neq j \), then the residue is zero because of the presence of the factor \( 1/\zeta(1 - z_i + z_j) \) in (1.9). Hence, the residue is nonzero only at the points (6.1) such that \( \{\alpha_1, \ldots, \alpha_{\ell}\} \) is an \( \ell \)-element subset \( U \), say, of \( A \), and \( \{\beta_1, \ldots, \beta_{\ell}\} \) is an \( \ell \)-element subset \( V \), say, of \( B \). At such a point, the residue of (5.2) equals

\[
\prod_{\alpha \in A, \beta \in B} \zeta(1 + \alpha + \beta + \xi + \eta) \prod_{\alpha \in A, \beta \in U} \zeta(1 + \alpha - \beta) \prod_{\beta \in B, \beta \not\in V} \zeta(1 + \beta - \beta).
\]
\[ \times \prod_{\beta \in V} (1/\zeta)(1 + \alpha + \xi + \eta + \hat{\beta}) \prod_{\alpha \in A} (1/\zeta)(1 + \beta + \xi + \eta + \hat{\alpha}) \]
\[ \times \prod_{\alpha, \hat{\alpha} \in U \atop \alpha \neq \hat{\alpha}} (1 - 1/p^{1 + \alpha - \hat{\alpha}}) \prod_{\beta \in B \atop \beta \neq \hat{\beta}} (1 - 1/p^{1 + \beta - \hat{\beta}}) \]
\[ \times \prod_{\alpha \in A \atop \beta \in B} (1 - 1/p^{1 + \alpha + \xi + \eta + \hat{\beta}}) \prod_{\hat{\alpha} \in U \atop \hat{\beta} \in V} (1 - 1/p^{1 - \hat{\alpha} - \hat{\beta} - \xi - \eta}) \]
\[ \times \prod_{\alpha \in A \atop \beta \in B} (1 - 1/p^{1 + \alpha + \xi + \eta + \hat{\alpha}}) \prod_{\hat{\alpha} \in U \atop \hat{\beta} \in V} (1 - 1/p^{1 + \hat{\alpha} + \hat{\beta} + \xi + \eta}) \]
\[ \times A(A, B, U, V, \xi + \eta), \quad (6.2) \]

with

\[ A(A, B, U, V, \xi + \eta) \]
\[ = \prod_{p} \left\{ \prod_{\alpha \in A \atop \beta \in B} \left(1 - \frac{1}{p^{1 + \alpha + \beta + \xi + \eta}}\right) \prod_{\alpha \in A, \hat{\alpha} \in U \atop \alpha \neq \hat{\alpha}} \left(1 - \frac{1}{p^{1 + \alpha - \hat{\alpha}}}\right) \right. \]
\[ \times \prod_{\beta \in B, \hat{\beta} \in V \atop \beta \neq \hat{\beta}} \left(1 - \frac{1}{p^{1 + \beta - \hat{\beta}}}\right) \prod_{\alpha \in A \atop \beta \in B} \left(1 - \frac{1}{p^{1 + \beta + \xi + \eta + \alpha}}\right)^{-1} \]
\[ \times \prod_{\hat{\alpha} \in U \atop \beta \in V} \left(1 - \frac{1}{p^{1 + \hat{\alpha} + \hat{\beta} + \xi + \eta}}\right)^{-1} \prod_{\hat{\alpha} \in U \atop \hat{\beta} \in V} \left(1 - \frac{1}{p^{1 - \hat{\alpha} - \hat{\beta} - \xi - \eta}}\right)^{-1} \]
\[ \times \prod_{\hat{\alpha} \in U \atop \hat{\beta} \in V} \left(1 - \frac{1}{p^{1 + \hat{\alpha} + \hat{\beta} + \xi + \eta}}\right) \sum_{m=0}^{\infty} \frac{\tau(A \cup U)_{\xi + \eta} V - (p^m) \tau(B \cup (U_{\xi + \eta}) - (p^m))}{p^m} \right\} \quad (6.3) \]

because

\[ \left(1 - \frac{1}{p}\right)^{-2\ell} \prod_{\alpha \in A, \hat{\alpha} \in U \atop \alpha \neq \hat{\alpha}} \left(1 - \frac{1}{p^{1 + \alpha - \hat{\alpha}}}\right) \prod_{\beta \in B, \hat{\beta} \in V \atop \beta \neq \hat{\beta}} \left(1 - \frac{1}{p^{1 + \beta - \hat{\beta}}}\right) \]
\[ = \prod_{\alpha \in A, \hat{\alpha} \in U \atop \alpha \neq \hat{\alpha}} \left(1 - \frac{1}{p^{1 + \alpha - \hat{\alpha}}}\right) \prod_{\beta \in B, \hat{\beta} \in V \atop \beta \neq \hat{\beta}} \left(1 - \frac{1}{p^{1 + \beta - \hat{\beta}}}\right) \]

and

\[ I_{A(\xi + \eta) U \setminus U_{\xi + \eta}} (p^m) I_{B(U^-)_{\xi - \eta} V} (p^m) = \tau(A \cup U)_{\xi + \eta} V - (p^m) \tau(B \cup (U_{\xi + \eta}) - (p^m)) \]
by (1.11) and (1.2). As the reciprocals of the Euler product expressions for the zeta functions in (6.2) are precisely those found in (6.3), we may write (6.2) more concisely as the formal expression

$$
\prod_p \left\{ \sum_{m=0}^{\infty} \frac{\tau(A \setminus U)_{\xi+\eta \cup V}^{-}(p^m)\tau_B \setminus V \cup (U \_ \xi+\eta)^{-}(p^m)}{p^m} \right\},
$$

which we interpret as its analytic continuation. By multiplicativity (or a formal application of Lemma 3.2), we may write this as

$$
\sum_{n=1}^{\infty} \frac{\tau(A \setminus U)_{\xi+\eta \cup V}^{-}(n)\tau_B \setminus V \cup (U \_ \xi+\eta)^{-}(n)}{n},
$$

which we also interpret as its analytic continuation, given by (6.2). Now for each pair $U, V$ of sets such that $U \subseteq A$ and $V \subseteq B$ with $|U| = |V| = \ell'$, the number of points (6.1) with $U = \{\alpha_1, \ldots, \alpha_\ell\}$ and $V = \{\beta_1, \ldots, \beta_\ell\}$ is $(\ell')^2$. Thus, evaluating the $z_j$- and $w_j$-integrals in (1.9) leads to the prediction

$$
S_{\ell'} \sim \sum_{U \subseteq A, V \subseteq B \atop |U| = |V| = \ell'} \frac{1}{(2\pi i)^2} \int_{(2c)} \int_{(2c)} \tilde{\Upsilon}(\xi) \tilde{\Upsilon}(\eta) X_{\xi+\eta} \frac{1}{T} \int_{0}^{\infty} \psi \left( \frac{t}{T} \right) \int \prod_{\alpha \in U} \chi \left( \frac{1}{2} + \xi + \alpha + it \right) \prod_{\beta \in V} \chi \left( \frac{1}{2} + \eta + \beta - it \right) \int \sum_{n=1}^{\infty} \frac{\tau(A \setminus U)_{\xi+\eta \cup V}^{-}(n)\tau_B \setminus V \cup (U \_ \xi+\eta)^{-}(n)}{n} dt \ d\xi \ d\eta.
$$

This is the same as (1.8) because Lemma 4.1(i) implies

$$
\tau(A \setminus U)_{\xi+\eta \cup V}^{-}(n)\tau_B \setminus V \cup (U \_ \xi+\eta)^{-}(n) = \tau(A \setminus U)_{\xi \cup (V \_ \eta)}^{-}(n)\tau_B \setminus V \cup (U \_ \xi)^{-}(n).
$$

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