The Energy Dependence of the Saturation Momentum from RG Improved BFKL Evolution

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Abstract

We study the energy dependence of the saturation momentum in the context of the collinearly improved Leading and Next to Leading BFKL evolution, and in the presence of saturation boundaries. We find that the logarithmic derivative of the saturation momentum is varying very slowly with Bjorken-\(x\), and its value is in agreement with the Golec-Biernat and Wüsthoff model in the relevant \(x\) region. The scaling form of the amplitude for dipole-dipole or dipole-hadron scattering in the perturbative side of the boundary is given.

1 Introduction

The idea of parton saturation was introduced twenty years ago from Gribov, Levin and Ryskin\(^1\) as a dual description of unitarity in high energy hard processes. Since then significant progress has been made in our understanding of saturation and unitarity\(^2\). Appropriate QCD evolution equations that describe high density partonic systems have been derived\(^3\),\(^4\),\(^5\),\(^6\), and saturation is gradually becoming well established theoretically. At the same time there has been considerable phenomenological success, as saturation based models are used to explain data at RHIC and HERA. The McLerran-Venugopalan model\(^7\),\(^8\),\(^9\) exhibits gluon saturation in a simple and intuitive way, and is used in order to describe the early stages of high energy heavy ion collisions. In deep inelastic scattering, the model of Golec-Biernat and Wüsthoff\(^10\) has been quite successful, as it gives a very good fit to the HERA data at small values of \(x\) and in a wide region of \(Q^2\) values.

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One of the important issues has been the energy (or rapidity) dependence of the saturation momentum $Q_s(Y)$. How can we determine $Q_s(Y)$ from the available QCD evolution equations? To be more specific let’s consider the scattering of a color dipole of size $1/Q$ on a dipole or even a hadron of size $1/\mu$ and with relative rapidity $Y$. Then, considering the forward scattering amplitude $T(Q, \mu, Y)$, the saturation momentum corresponds to a line in the $Q^2 - Y$ plane, along which $T$ becomes constant and of order $1/\mu^2$, that is unitarity effects have been reached.

One should expect BFKL evolution \cite{11, 12} (but not its double logarithmic limit) to be the relevant dynamics, since this is the one that leads to high density partonic systems. For fixed coupling and at leading level this was done in \cite{13}, and the leading exponential (in $Y$) behavior of $Q_s$ was found. In \cite{14} it was reconfirmed and extended to the running coupling case with the result $\log(Q_s^2/\Lambda^2) \propto \sqrt{Y}$. BFKL dynamics, being a linear evolution, suffers from the lack of unitarity. Therefore it would seem natural, that the use of a more general evolution equation would give the proper description of $Q_s$. This was done in \cite{15}, where a numerical study of the Balitsky-Kovchegov equation \cite{16, 17} was carried out.

It would be nice of course to have analytical results from such non-linear equations. However, the exact details of the non-linear evolution effects should not be important in determining the energy dependence of $Q_s$. As far as the system remains on the perturbative side of the saturation line, all the effects of the non-linear terms could be described by a simple absorptive boundary. Using leading BFKL with either fixed or running coupling, this procedure was followed in \cite{18}. In the more realistic case of running coupling, negative $Y^{1/6}$ corrections to the leading $\sqrt{Y}$ were found. Even though this correction is parametrically small when compared to the leading behavior, it becomes important for realistic values of $Y$, and it tames the growth of $Q_s$ which is overestimated in the absence of boundaries.

The goal in this paper is the extension of this procedure at the next to leading level. The calculation of the NL correction to the BFKL kernel was completed in \cite{19, 20}. However, this negative correction turned out to be larger in magnitude than the leading contribution for reasonable values of $\alpha_s = 0.2 \div 0.3$. Even worse, when $\bar{\alpha}_s \gtrsim 0.05$, the full kernel has two complex saddle points which lead to oscillatory cross sections \cite{21}. Thus there is no hope that one will get sensible and reliable results when using the standard NL BFKL dynamics. Perhaps the most appealing cure of these pathologies, has been given by Ciafaloni, Colferai and Salam \cite{22, 23, 24, 25}. It was recognized that the large NL corrections emerge from the collinearly enhanced physical contributions. A method, the $\omega$-expansion, was then developed to
resum collinear effects at all orders, and the resulting improved BFKL equation was consistent with renormalization group constraints and the DGLAP equations \[26, 27, 28\] by construction. The kernel is positive in a much larger region which includes the experimentally accessible one. One of the outcomes of this method was a very reasonable value for the hard Pomeron intercept \(\omega_P\). Therefore we expect the application of the \(\omega\)-expansion in the saturation problem, to give reliable results for the energy dependence of \(Q_s\) and for the form of the scattering amplitude in the perturbative side of the saturation line.

In Section 2 we give a brief review of the \(\omega\)-expansion approach, and the full form of the NL in \(\omega\) kernel that we use is given in the Appendix. In Section 3 we define a critical line \(Q_c(Y)\) (not the saturation line) which belongs in the perturbative region and which is a line of almost, but not exactly, constant amplitude; there is no exponential (in \(Y\)) dependence. In Section 4, by considering perturbations around this line, we approximately solve the RG improved BFKL equation under certain but valid assumptions. In Section 5 we introduce the boundary, which essentially cuts out all paths that enter the saturation region \(Q \lesssim Q_s\) during the evolution. We remove almost all the freedom left in the solution given in Section 4, by fixing the amplitude to be constant on the saturation line. The \(Y\)-dependence of \(Q_s\) is given and a multiplicative constant is free. However, the logarithmic derivative \(\lambda_s = d\log(Q_s^2/\Lambda^2)/dY\) is well defined. In Section 6 we give our results for \(\lambda_s\) and comment on the value of \(Q_s\). For dipole-hadron scattering and at NL level in \(\omega\) we find \(\lambda_s(Y = 5 \div 9) = 0.30 \div 0.29\) and that \(\lambda_s\) is decreasing very slowly with \(Y\). We also comment on the parametric form of the boundary correction to \(\lambda_s\), which is \(\propto [\bar{\alpha}_s(Q_s^2)]^{5/3}\) and well defined for any value of \(Y\), since \(Q_c^2\) is increasing during the evolution. For a suitable but reasonable choice of our free multiplicative constant in \(Q_s\), we can fix its value to be, for example, \(Q_s(x \simeq 3 \times 10^{-4}) \simeq 1\) GeV. In Section 7 we give the scaling form of the amplitude on the perturbative side of the saturation line in terms of \(Q^2/Q_s^2\). This is the same (but with a different \(Q_s\)) as the one obtained in leading BFKL with either fixed or running coupling \[18\], and therefore we discuss its generality. In Section 8 we check our approximations and estimate the errors which are small and under control, and finally in Section 9 we conclude.

2 Review of the RG improved BFKL kernel

In this section we describe as briefly as possible the collinear improvement of the BFKL equation. The full derivation and the details can be found in the papers of Ciafaloni, Colferai and Salam \[22, 23, 24, 25\]. The BFKL equation
may be written as
\[
\frac{\partial T}{\partial Y} = \bar{\alpha}_s K \otimes T, \tag{1}
\]
with \(\bar{\alpha}_s = \alpha_s N_c / \pi\), and where we can imagine, for example, that \(T\) corresponds to the forward scattering amplitude for a dipole of size \(1/Q\) on a dipole of size \(1/\mu\). The eigenfunctions of the leading kernel are \((Q^2/\mu^2)^{\gamma-1}\), with corresponding eigenvalues \([11, 12]\).

\[\chi_0(\gamma) = 2\psi(1) - \psi(\gamma) - \psi(1 - \gamma),\tag{2}\]
where \(\psi(\gamma) = d\log \Gamma(\gamma)/d\gamma\). The pole structure of \(\chi_0(\gamma)\) is
\[\chi_0(\gamma) = \frac{1}{\gamma} + \frac{1}{1 - \gamma} + \text{finite}.\tag{3}\]

The NL correction to the total characteristic function \(\chi(\gamma)\) is \(\bar{\alpha}_s \chi_1(\gamma)\), where \(\chi_1(\gamma)\) is a known function \([19, 20]\) and is analytic in the region \(0 < \gamma < 1\). For simplicity let us consider its pole structure when \(N_f = 0\) (we will keep \(N_f \neq 0\) later), which is
\[\chi_1(\gamma) = -\frac{1}{2\gamma^3} - \frac{1}{2(1 - \gamma)^3} + \frac{A_{gg}(0)}{\gamma^2} + \frac{A_{gg}(0) - b}{(1 - \gamma)^2} + \text{finite},\tag{4}\]
where
\[b = \frac{\pi}{N_c} \frac{11N_c - 2N_f}{12\pi},\tag{5}\]
and \(A_{gg}(\omega)\) is the Mellin transform of the non-pole part of the gluon-gluon splitting function (the pole \(1/z\) has been taken into account in the leading kernel), that is
\[A_{gg}(\omega) = \int_0^1 dz \ z^\omega P_{gg}(z) - \frac{1}{\omega}.\tag{6}\]
When \(N_f = 0\) one has \(A_{gg}(0) = -b = -11/12 < 0\), and one can see in \((4)\) the large negative corrections arising from both the double and triple poles.

It was observed that the poles at \(\gamma = 0\) (\(\gamma = 1\)) arise when one considers the collinear (anti-collinear) limit \(Q^2 \gg \mu^2\) \((Q^2 \ll \mu^2)\).

Since a collinear small-\(x\) branching resulted in a pole \(1/\gamma\) in the leading kernel, a sequence of a small-\(x\) and a non-small-\(x\) branching, as can be seen
from (6) will result in a pole

\[ \frac{1}{\gamma} \omega A_{gg}(\omega) \rightarrow \frac{1}{\gamma} \bar{\alpha}_s \chi_0(\gamma) A_{gg}(0) \rightarrow \frac{\bar{\alpha}_s A_{gg}(0)}{\gamma^2}, \]  

(7)

where we have truncated the series in $\bar{\alpha}_s$ at NL level by using $\omega = \bar{\alpha}_s \chi(\gamma, \omega)$. This is the double pole at $\gamma = 0$ in Eq.(4). Similarly we obtain the double pole at $\gamma = 1$, where there is an extra contribution coming from the running of the coupling, if the extracted coupling $\bar{\alpha}_s$ in Eq.(1) is evaluated at $Q^2$.

However, if we simply do not truncate the result in (7) we should obtain the all orders collinear contribution to the BFKL kernel. Thus, including the leading kernel poles, the total eigenvalue will become

\[ \chi(\gamma, \omega) = \frac{1 + \omega A_{gg}(\omega)}{\gamma} + \frac{1 + \omega [A_{gg}(\omega) - b]}{1 - \gamma} + \text{finite}, \]  

(8)

which is $\omega$-dependent. This form will be further modified after the following discussion, since it remains to interpret the triple poles in Eq.(4).

Let us write the amplitude $T$ as

\[ T = \int \frac{d\omega}{2\pi i} \frac{d\gamma}{2\pi i} T_{\omega\gamma} \exp \left[ \omega Y - (1 - \gamma) \log \frac{Q^2}{\mu^2} \right], \]  

(9)

where $Y = \log(s/s_0)$ and with $s$ the total energy of the scattering. $s_0$ is an energy scale which depends on the scales $Q$ and $\mu$; $s_0 = s_0(Q, \mu)$. At leading level its choice is irrelevant. At NL level one chooses $s_0$ to be the natural scale of the problem under consideration. In the calculation of $\omega_P$, a suitable choice is $s_0 = Q\mu$. In the collinear (anti-collinear) limit the right choice is $s_0 = Q^2$ ($s_0 = \mu^2$). In this limit $Y = \log(1/x) = \log(s/Q^2)$, where $x$ is now the familiar Bjorken variable. As a consequence, the eigenvalue of the NL kernel depends on $s_0$, and the one given in Eq.(4) is for the symmetric choice $s_0 = Q\mu$. From (11) we can see that we can switch to $s_0 = Q^2$ ($s_0 = \mu^2$) by a simple shift $\gamma \rightarrow \gamma - \frac{\omega}{2}$ ($\gamma \rightarrow \gamma + \frac{\omega}{2}$), since

\[ \omega \log \frac{s}{Q\mu} - (1 - \gamma) \log \frac{Q^2}{\mu^2} \rightarrow \omega \log \frac{s}{Q^2} - (1 - \gamma) \log \frac{Q^2}{\mu^2}. \]  

(10)

The change in the eigenvalue can be obtained from the following iteration

\[ \omega = \bar{\alpha}_s \chi(\gamma - \frac{\omega}{2}) \rightarrow \bar{\alpha}_s \chi(\gamma) - \frac{\bar{\alpha}_s}{2} \chi'(\gamma) \rightarrow \bar{\alpha}_s \chi(\gamma) - \frac{1}{2} \bar{\alpha}_s^2 \chi_0(\gamma) \chi_0'(\gamma) \]  

\[ \Rightarrow \delta \chi(\gamma) = \frac{\bar{\alpha}_s}{2\gamma^3} - \frac{\bar{\alpha}_s}{2(1 - \gamma)^3} + \text{finite}, \]  

(11)
where the result is again truncated at NL level. When we add this correction to (4) the triple pole at $\gamma = 0$ cancels, as it should in the collinear limit, in order to have consistency with the RG equations. This is the origin of the triple poles in the NL BFKL kernel. Now one can see that Eq.(8) has to be substituted by

$$\chi(\gamma, \omega) = 1 + \omega A_{gg}(\omega) - \frac{\omega}{1 - \gamma + \frac{\omega}{2}} + \text{finite}, \quad (12)$$

when $s_0 = Q\mu$, so that

$$\chi(\gamma, \omega) = 1 + \omega A_{gg}(\omega) - \frac{\omega}{1 - \gamma + \omega} + \text{finite}, \quad (13)$$

when $s_0 = Q^2$.

It remains to combine the behavior (12) or (13) with the known L and NL BFKL kernels. As explained in the next section, the correct choice in the saturation problem is $s_0 = Q^2$. In the $\omega$-expansion approach, where one does not truncate at NL in $\bar{\alpha}_s$ as in Eqs.(7) and (11), and $\omega$ appears as the expansion variable, the leading kernel is

$$\chi_0(\gamma, \omega) = 2\psi(1) - \psi(\gamma) - \psi(1 - \gamma + \omega), \quad (14)$$

or

$$\chi_0(\gamma, \omega) = \chi_0(\gamma) - \frac{1}{1 - \gamma} + \frac{1}{1 - \gamma + \omega}, \quad (15)$$

where $\chi_0(\gamma)$ is the one given in Eq.(3). Both descriptions are equally good. Then we consider the NL in $\omega$ kernel. We need to add in $\chi_1(\gamma)$ a term $-\chi_0(\gamma)\chi_0'(\gamma)/2$ since $s_0 = Q^2$ (see Eq.(11)). After subtracting the NL BFKL corrections that are already included in (14) or (15) to avoid double counting, we replace all remaining poles at $\gamma = 1$ with poles at $\gamma = 1 + \omega$, making at the same time the substitution $A_{gg}(0) \rightarrow A_{gg}(\omega)$ and we obtain $\chi_1(\gamma, \omega)$. Then the NL in $\omega$ eigenvalue is

$$\chi(\gamma, \omega) = \chi_0(\gamma, \omega) + \omega \frac{\chi_1(\gamma, \omega)}{\chi_0(\gamma, \omega)}, \quad (16)$$

which has the proper limit given in Eq.(13) and includes the NL BFKL corrections. The details in constructing $\chi_1(\gamma, \omega)$ are given in the appendix.

Our freedom in using either (14) or (15) for $\chi_0(\gamma, \omega)$ (and similarly for $\chi_1(\gamma, \omega)$) emerges from our freedom in how to shift the poles; one can always add a finite part. However, it was shown that any resulting difference is of
order $O(\omega^2)$. This collinear resummation independence was checked in the value of $\omega_P$ where the results differ not more than $2 - 3\%$. We expect this to be true in the saturation problem too, since the two kernels are very close to each other for all allowed values of $\gamma$ and $\omega$. For practical reasons we shall use a simple pole shift as in (15), since in this case the $\omega$-dependence becomes as simple as possible. Notice that because of this $\omega$-dependence, the operator $K$ in Eq.(1) contains $\partial/\partial Y$ derivatives.

3 Evolution along the critical line

As mentioned earlier we consider the forward amplitude $T(Q, \mu, Y)$ for dipole-dipole or dipole-hadron scattering at NL level in $\omega$. Since the kernel eigenvalues are $\chi(\gamma, \omega)$, the evolution equation may be written in an operator form

$$\frac{\partial(T/\alpha_s)}{\partial Y} = \frac{1}{b\rho} \chi \left( 1 + \frac{\partial}{\partial \rho}, \frac{\partial}{\partial Y} \right) (T/\alpha_s).$$

(17)

Here we have defined the logarithmic variable $\rho = \log(Q^2/\Lambda^2)$, with $\Lambda$ the usual QCD parameter. The rapidity $Y$ is given by $Y = \log(1/x)$, with $x = Q^2/s$ and $s$ the total energy of the scattering. It is obvious that we have chosen an asymmetric energy scale $s_0 = Q^2$. The reason is that we are going to evolve the system, following a particular line in the $Q^2 - Y$ plane along which $Q^2$ is increasing (we should emphasize though that the system is not in the collinear regime). Furthermore, this is the standard expression of Bjorken-$x$ which is also used in the HERA data. The factor $1/b\rho$ is the running coupling $\bar{\alpha}_s$, with $b$ from (5).

We change variable from $\rho$ to $\eta$ according to

$$\eta = \rho - \rho_c(Y),$$

(18)

so that

$$\frac{\partial}{\partial Y} \rightarrow \frac{\partial}{\partial Y} - \rho_c \frac{\partial}{\partial \eta}, \quad \frac{\partial}{\partial \rho} \rightarrow \frac{\partial}{\partial \eta}. \quad (19)$$

The dot represents a derivative with respect to $Y$. The function $\rho_c(Y)$ will be defined soon to be a line of almost, but not quite, constant amplitude; there will be no exponential dependence in $Y$. $\eta$ corresponds to fluctuations around this line. If we also expand the running coupling as

$$\frac{1}{b\rho} = \frac{1}{b\rho_c} - \frac{\eta}{b\rho_c^2} + O\left(\frac{\eta^2}{\rho_c^2}\right),$$

(20)

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Eq. (17) becomes

\[
b \rho c \left( \frac{\partial}{\partial Y} - \dot{\rho}_c \frac{\partial}{\partial \eta} \right) \left( T/\alpha_s \right) = \left( 1 - \frac{\eta}{\rho_c} \right) \chi \left( 1 + \frac{\partial}{\partial \eta} \frac{\partial}{\partial Y} \left( \frac{\partial}{\partial \eta} \right)^{-1} \right) \left( T/\alpha_s \right). \tag{21} \]

It is convenient to view the eigenvalue \( \chi(\gamma, \omega) \) as a function of \( \gamma \) and \( \lambda \), where \( \lambda \) is given by

\[
\omega = \lambda (1 - \gamma). \tag{22} \]

Then, if

\[
\phi(\gamma, \lambda) = \chi(\gamma, \lambda(1 - \gamma)), \tag{23} \]

Eq. (21) may be written as

\[
b \rho c \left( \frac{\partial}{\partial Y} - \dot{\rho}_c \frac{\partial}{\partial \eta} \right) \left( T/\alpha_s \right) = \left( 1 - \frac{\eta}{\rho_c} \right) \phi \left( 1 + \frac{\partial}{\partial \eta}, \dot{\rho}_c - \frac{\partial}{\partial Y} \left( \frac{\partial}{\partial \eta} \right)^{-1} \right) \left( T/\alpha_s \right). \tag{24} \]

We drop the \( \partial/\partial Y \) in the second argument of the \( \phi \) function. This is a valid approximation that will be checked at the end. Thus the second argument will be \( \dot{\rho}_c \). Even though \( \dot{\rho}_c \) is expected to be small, we do not expand around 0, since such an expansion would lead us to the usual BFKL evolution, where the coefficient in the expansion would be large at NL level. We do expand only with respect to the first argument, in a “diffusion approximation” much like that introduced in [29], around the point \( \gamma_c(Y) \) to obtain

\[
b \rho c \left( \frac{\partial}{\partial Y} - \dot{\rho}_c \frac{\partial}{\partial \eta} \right) \left( T/\alpha_s \right) = \left( 1 - \frac{\eta}{\rho_c} \right) \left[ \phi(\gamma_c, \dot{\rho}_c) + \phi^{1.0}(\gamma_c, \dot{\rho}_c) \left( 1 + \frac{\partial}{\partial \eta} - \gamma_c \right) + \frac{1}{2} \phi^{2.0}(\gamma_c, \dot{\rho}_c) \left( 1 + \frac{\partial}{\partial \eta} - \gamma_c \right)^2 \right] \left( T/\alpha_s \right), \tag{25} \]

where \( \phi^{m,n} \) is the \((m, n)\) derivative of \( \phi(\gamma, \lambda) \).
At this stage we need to determine the $Y$-dependence of $\gamma_c$ and $\rho_c$. We choose them, so that $T$ is as constant as possible. This will happen if the coefficients of the constant and $\partial/\partial\eta$ terms in (23) vanish. Thus

$$\phi(\gamma_c, \dot{\rho}_c) + (1 - \gamma_c)\phi^{1,0}(\gamma_c, \dot{\rho}_c) - 2\beta b \dot{\rho}_c = 0, \quad (26)$$

and

$$b\rho_c \dot{\rho}_c + \phi^{1,0}(\gamma_c, \dot{\rho}_c) = 0. \quad (27)$$

Because of the last term in (26) the coefficient of the constant term in (25) does not vanish. Nevertheless, its appearance is required in order to cancel $Y$-prefactors that will appear in the amplitude, by a suitable choice of the pure number $\beta$. At the end we will find $\beta = 1$. Eqs. (26) and (27) cannot be solved in general, except in the simplified cases of collinear models [30] where one keeps only the pole parts of the characteristic function $\chi(\gamma, \omega)$. These models could be useful in studying the form of $Q_s(Y)$ (which is not so simple even in this case), but its value is not very close with the one obtained when we use the full kernel. However, one can expand around the point $\gamma_0$, which is the solution to

$$\chi_0(\gamma_0) + (1 - \gamma_0)\chi_0'(\gamma_0) = 0, \quad (28)$$

and with numerical value $\gamma_0 = 0.372...$. Notice that (28) is the asymptotic form of (24) when $Y$ is very large, since from (27) we expect the asymptotic behavior $\dot{\rho}_c \propto 1/\sqrt{Y}$. Therefore, a first order expansion around the asymptotic value $\gamma_0$, simplifies our equations to

$$\gamma_c = \gamma_0 + \frac{-\phi(\gamma_0, \dot{\rho}_c) - (1 - \gamma_0)\phi^{1,0}(\gamma_0, \dot{\rho}_c) + 2\beta b \dot{\rho}_c}{(1 - \gamma_0)\phi^{2,0}(\gamma_0, \dot{\rho}_c)}, \quad (29)$$

and

$$\rho_c \dot{\rho}_c = \frac{\dot{\phi}(\gamma_0, \dot{\rho}_c)}{b(1 - \gamma_0)} - \frac{2\beta \dot{\rho}_c}{1 - \gamma_0}. \quad (30)$$

The numerical value of $\phi^{2,0}(\gamma_0, \dot{\rho}_c)$ is not too far from $\chi_0''(\gamma_0) = 48.5...$. The appearance of such a big number in the denominator of (24) suggests that the expansion around $\gamma_0$ should be valid even in the intermediate region $Y = 6 \div 8$, which is experimentally accessible.

One can see that for very large $Y$, the scattering can be described just by leading BFKL evolution with running coupling, since $\dot{\rho}_c$ goes to 0. This is easily understood. During the evolution $\rho_c$ is increasing and therefore the
value of the running coupling is decreasing. But this approach to leading BFKL dynamics will not be realized practically, until $Y$ reaches a very large and inaccessible value.

For what follows in the next two sections, we will not need the explicit form of $\rho_c(Y)$ and $\gamma_c(Y)$. We return to Eqs.(29) and (30) in Section 6.

4 Solving the BFKL equation

In this section we give an approximate solution of the RG improved BFKL equation in the vicinity of the line $\rho_c(Y)$. Making use of (26) and (27), the amplitude equation (25) becomes

$$
\left[ b\rho_c \frac{\partial}{\partial Y} - 2\beta b \dot{\rho}_c \left( 1 - \frac{\eta}{\rho_c} \right) + \phi^{1,0}(\gamma_c, \dot{\rho}_c) \frac{\eta}{\rho_c} \frac{\partial}{\partial \eta} \right.
$$

$$
- \frac{1}{2} \phi^{2,0}(\gamma_c, \dot{\rho}_c) \left( 1 - \frac{\eta}{\rho_c} \right) \left( 1 + \frac{\partial}{\partial \eta} - \gamma_c \right)^2 \left( \frac{T}{\alpha_s} \right) = 0. \tag{31}
$$

When $\eta \ll \rho_c$ one can drop the corresponding term in the first parenthesis of the last term. Now let’s isolate the leading exponential, in the scaling variable $\eta$, behavior of the amplitude and write $T$ as

$$
T = \alpha_s \exp\left[ -(1 - \gamma_c) \eta + 2\beta \log \rho_c \right] \psi(\eta, Y). \tag{32}
$$

We have neglected any target dependent ($\mu$-dependent) prefactors in $T$, which are proportional to $1/\mu^2$. We easily find that $\psi$ satisfies

$$
\left\{ -\phi^{1,0}(\gamma_c, \dot{\rho}_c) \frac{\partial}{\partial \rho_c} + \phi^{1,0}(\gamma_c, \dot{\rho}_c) \frac{\eta}{\rho_c} \frac{\partial}{\partial \eta} \right.
$$

$$
- \frac{1}{2} \phi^{2,0}(\gamma_c, \dot{\rho}_c) \frac{\partial^2}{\partial \eta^2} + \left[ \phi(\gamma_c, \dot{\rho}_c) + b\rho_c^2 \dot{\gamma}_c \right] \frac{\eta}{\rho_c} \right\} \psi = 0, \tag{33}
$$

which becomes now our basic equation. Since there is no explicit $Y$ dependence in (33), we have decided to view the function $\psi$ as a function of $\rho_c$ instead of $Y$, $\psi = \psi(\eta, \rho_c)$, by means of the transformation

$$
\frac{\partial}{\partial Y} \rightarrow \dot{\rho}_c \frac{\partial}{\partial \rho_c}, \tag{34}
$$

and we have simplified as much as possible the coefficients by using Eqs.(26) and (27).
Formally these coefficients should be expressed in terms of $\rho_c$, which can be done, in principle, when one uses Eqs. (29) and (31). Practically it is easier to have them as functions of $\lambda_c \equiv \dot{\rho}_c$. In any case, even though these are complicated functions, we will need only their asymptotic values in the approximate solution of (33).

It is convenient to make a final change of variables from $(\eta, \rho_c)$ to $(\xi, t)$, where

$$\xi = \frac{\eta}{D \rho_c^{1/3}},$$

and

$$t = 3D_0(1 - \gamma_0)\rho_c^{1/3},$$

with

$$D = \left\{ \frac{\phi^{2,0}(\gamma_c, \dot{\rho}_c)}{2[\phi(\gamma_c, \dot{\rho}_c) + b\rho_c^2]} \right\}^{1/3},$$

and

$$D_0 = \left[ \frac{\chi_0^2(\gamma_0)}{2\chi_0(\gamma_0)} \right]^{1/3}.$$ 

Clearly $D_0 = 1.99...$ is the asymptotic value of $D$. After some straightforward algebra Eq. (33) becomes

$$\left[ h_1(t) \frac{\partial}{\partial t} - 4h_2(t) \frac{\xi}{t} \frac{\partial}{\partial \xi} - \frac{\partial^2}{\partial \xi^2} + \xi \right] \psi = 0.$$ 

While the coefficients of the last two terms have been totally simplified, the ones $h_1(t)$ and $h_2(t)$ of the first two terms are extremely complicated functions, given by

$$h_1(t) = -\frac{D_0(1 - \gamma_0)\phi^{1,0}(\gamma_c, \lambda_c)}{D \phi(\gamma_c, \lambda_c)},$$

and

$$h_2(t) = h_1(t) \left[ 1 + \frac{3D'(\lambda_c)\rho_c(\lambda_c)}{4D\rho_c'(\lambda_c)} \right].$$

As said before, $h_1$ and $h_2$ have been expressed in terms of $\lambda_c \equiv \dot{\rho}_c$. The prime stands for a derivative with respect to $\lambda_c$, and $\gamma_c$ and $\rho_c$ are given as functions.
of \( \lambda \) from Eqs.\((29)\) and \((30)\). \( \dot{\gamma}_c \) which is contained in the definition of \( D \), can also be easily expressed as a function of \( \lambda \) through a chain rule differentiation giving

\[
\dot{\gamma}_c = \gamma'_c(\lambda_c)\lambda_c/\rho'_c(\lambda_c).
\]

In the asymptotic region of large \( t \), or equivalently of small \( \lambda \), \( h_1 \) and \( h_2 \) approach 1,

\[
\lim_{t \to \infty} h_1(t) = \lim_{t \to \infty} h_2(t) = 1,
\]

with corrections of order \( \mathcal{O}(1/t^3) \), which turn out to be small not only parametrically, but even numerically for intermediate values of \( Y \).

It appears difficult to find an exact solution of Eq.\((39)\) even if we set \( h_1 = h_2 = 1 \). However, we can find an approximate solution of the form

\[
\psi(\xi, t) = \text{Ai}(\xi - \sigma) \exp \left[ -\frac{\xi^2}{t} h_2(t) - \sigma \int \frac{dt}{h_1(t)} - 2 \int \frac{dt h_2(t)}{t h_1(t)} \right],
\]

where \( \sigma \) is a constant to be determined and \( \text{Ai} \) is the Airy function. This expression satisfies Eq.\((39)\) up to a term \( R(\xi, t)\psi \), where

\[
R(\xi, t) = \frac{\xi^2}{t^2} \left[ 4h_2^2 + h_1 h_2 - th_1 \frac{dh_2}{dt} \right] = \frac{5\xi^2}{t^2} \left[ 1 + \mathcal{O} \left( \frac{1}{t^3} \right) \right].
\]

It is important that this remainder contains no terms of the order \( f(\xi)/t \), as those terms would be important when one integrates \((39)\) over \( t \). The approximate solution given, should be the one that matches the exact solution in the intermediate and asymptotic region. It contains the Airy function \([31]\) and the diffusion type factor \( \exp(-\xi^2/t) \), which appear in general when one considers BFKL evolution with running coupling. We also mention that at this point we lose track of the overall constant which appears in front of the amplitude, as a consequence of our inability to solve \((39)\) exactly.

5 The saturation boundary

Up to target dependent factors, the amplitude \( T \) is given by Eq.\((32)\) with \( \psi \) as found in Eq.\((43)\). It contains two parameters \( \beta \) and \( \sigma \), which are uniquely determined from the saturation boundary conditions in the asymptotic region. To that end we express \( T \) in terms of the variables \( (\xi, t) \) in the asymptotic region, that is

\[
T = \text{Ai}(\xi - \sigma) \exp \left[ -\frac{\xi t}{3} - \sigma t + (6\beta - 5) \log t \right].
\]

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When \( \xi \) becomes negative, the amplitude grows strongly, it reaches a maximum value and then it decreases rapidly to vanish at \( \xi = \xi_1 + \sigma \), where \( \xi_1 = -2.33... \) is the first zero of the Airy function. A decreasing amplitude is of course an unphysical situation, which simply means that we lose our ability to describe the evolution of the system by using linear equations. Thus we need to put a saturation boundary before \( T \) hits its maximum value, along which we fix \( T \) to be constant, in order to illustrate the effects of the non-linear terms in the simplest possible way. Now writing \( \xi \) as

\[
\xi = \xi_1 + \sigma + \delta \xi, \tag{46}
\]

we have

\[
\text{Ai}(\xi - \sigma) = \text{Ai}'(\xi_1) \left[ \delta \xi + \frac{\xi_1}{6} \delta \xi^3 + \frac{1}{12} \delta \xi^4 + \mathcal{O}(\delta \xi^5) \right], \tag{47}
\]

and we keep only the linear term as we will shortly see that \( \delta \xi \) is small in the region of interest. Then Eq.(45) becomes (dropping again unimportant for our purposes multiplicative constants)

\[
T = \exp \left[ -\frac{(\xi_1 + 4\sigma)t}{3} + (6\beta - 5) \log t - \frac{\delta \xi t}{3} + \log \delta \xi \right]. \tag{48}
\]

The maximum of this expression occurs when \( \delta \xi = 3/t \), thus we assume a saturation boundary at

\[
\delta \xi_s = \frac{3(1 + c)}{t}, \tag{49}
\]

where the amplitude is a finite fraction of its maximum value. \( c \) is a positive constant of order \( \mathcal{O}(1) \), that we are not able to determine. On the boundary the amplitude is

\[
T_s = \exp \left[ -\frac{(\xi_1 + 4\sigma)t}{3} + 6(\beta - 1) \log t \right]. \tag{50}
\]

Clearly, we need to choose our two parameters to be

\[
\beta = 1, \quad \sigma = -\frac{\xi_1}{4}, \tag{51}
\]

so that \( T \) is well defined and \( t \)-independent on the boundary.
Now we are finally in a position to give an expression for the saturation momentum. Using the results stated in Eqs. (49) and (51) and recalling the definitions (18), (35), (36) and (46) we obtain

$$\rho_s = \rho_c + \frac{3\xi_1}{4} D_0 \rho_c^{1/3} + \frac{1+c}{1-\gamma_0},$$  \hspace{1cm} (52)

which gives

$$Q_s = \Lambda \exp \left[ \frac{\rho_c}{2} + \frac{3\xi_1}{8} D_0 \rho_c^{1/3} + \frac{1+c}{2(1-\gamma_0)} \right].$$  \hspace{1cm} (53)

where we have replaced $D$ by its asymptotic value $D_0$. It is obvious that in our approach there is a free multiplicative constant. However, for the logarithmic derivative of the saturation momentum, $\lambda_s = \dot{\rho}_s$, there is no such ambiguity. In Section 8 we will discuss possible uncertainties in Eqs. (52) and (53), as far as the energy dependence is considered, which turn out to be parametrically and numerically small. In order to make use of (52) or (53) we have to determine $\rho_c(Y)$, a task that we now turn to.

6 Results for $\lambda_s$

Given a kernel, we need to solve Eq. (30) to find the $Y$-dependence of $\rho_c$. In general it appears easier to solve (when a solution is possible) for $\lambda_c = \dot{\rho}_c$. Considering $\rho_c$ as a function of $\lambda_c$ and differentiating with respect to $Y$ using the chain rule, we have

$$\lambda_c = \rho'_c(\lambda_c) \frac{d\lambda_c}{dY},$$  \hspace{1cm} (54)

and by separating variables we obtain

$$\int_{\lambda_c(0)}^{\lambda_c(Y)} d\lambda_c \rho'_c(\lambda_c) \frac{\lambda_c}{\lambda_c} = Y,$$  \hspace{1cm} (55)

where

$$\rho_c(\lambda_c) = \frac{\phi(\gamma_0, \lambda_c)}{b(1-\gamma_0)\lambda_c} - \frac{2}{1-\gamma_0},$$  \hspace{1cm} (56)

as determined by Eq. (30).

It is clear that one needs an initial condition specified in order to proceed. In leading BFKL with fixed coupling, the initial condition is totally irrelevant for $\lambda_c$ since the leading behavior of $\rho_c(Y)$ in that case is purely linear. In
the running coupling case, either in BFKL or in $\omega$-expansion, there is an integration constant $Y_0$ or $\lambda_c(0)$ as it appears in (55), which affects the value of $\lambda_c(Y)$. However, as we discuss shortly after and in Section 8, small changes in the initial conditions will not change our results significantly and furthermore this change is predictable. The most natural initial condition to adopt is

$$\rho_c(Y = 0) = \log \frac{\mu^2}{\Lambda^2},$$

with $\mu^{-1}$ of the order of the target size. Then it is straightforward to find $\lambda_c(0)$ from (56). Now after determining $\lambda_c$ and/or $\rho_c$, one can finally get the logarithmic derivative of the saturation momentum from the differentiation of (52), which gives

$$\lambda_s = \lambda_c + \frac{\xi_1 D_0}{4\rho_c^{2/3}} \lambda_c.$$  

This is our main result for $\lambda_s$. In the following we examine the cases of Leading BFKL, Leading in $\omega$ and NL in $\omega$ dynamics.

### 6.1 Leading BFKL - review

In this case it is straightforward to solve directly for $\rho_c(Y)$ using Eq.(30). We find

$$\rho_c(Y) = \sqrt{\frac{2\chi_0(\gamma_0)}{b(1 - \gamma_0)} Y + \left[ \rho_c(0) + \frac{2}{1 - \gamma_0} \right]^2 - \frac{2}{1 - \gamma_0}},$$

from which it is trivial to obtain $\lambda_c(Y)$. Then (58) provides us with an analytical expression for $\lambda_s(Y)$.

It is very useful to examine two particular limits of Eq.(59), in order to get a better understanding of the initial conditions.

(i) Let’s imagine first that the target is a very small dipole, such that the second term of the square root in (59) dominates, that is

$$\rho_c^2(0) \gg \frac{2\chi_0(\gamma_0)}{b(1 - \gamma_0)} Y.$$  

Then one can easily see that (59) reduces to the exact fixed coupling result

$$\rho_c(Y) = \rho_c(0) + \frac{\chi_0(\gamma_0)}{b(1 - \gamma_0)\rho_c(0)} Y,$$
with \( \rho_c(0) \) as given in (57) and the natural identification \( 1/b\rho_c(0) \to \bar{\alpha}_s \). This is expected since (60) simply states the fact that, the rapidity is not large enough (as \( 1/\alpha_s^2 \) parametrically) and therefore the NL and the running coupling corrections are unimportant.

(ii) Now let’s consider that \( Y \) is large enough so that the first term of the square root dominates, that is

\[
\rho_c^2(0) \ll \frac{2\chi_0(\gamma_0)}{b(1-\gamma_0)} Y. \tag{62}
\]

Then (59) reduces to

\[
\rho_c(Y) = \sqrt{\frac{2\chi_0(\gamma_0)}{b(1-\gamma_0)}} Y, \tag{63}
\]

In this case, where the running coupling dynamics is necessary, it is clear that the initial conditions are lost in the region that (62) is satisfied. Parametrically, the uncertainty in \( \lambda_c(Y) \), due to the initial conditions, vanishes as \( 1/Y^{3/2} \).

Thus it becomes crucial to specify the initial condition, e.g. whether the target is a small dipole or a hadron, and the region of \( Y \) values that one is interested in, before making any kind of expansion in (59). Even though we would like to imagine in general that \( Y \) is large, let’s consider a realistic case where the target is of size \( \mu \approx 1 \text{GeV} \) and the rapidity is in the region \( Y = 6 \div 8 \).

It is easy to check that the two terms in (53) are of comparable size. Therefore we use this equation without making any approximations. But there are two important points emerging from the above analysis, which will also be true in the case of the \( \omega \) expansion. The first point is that a running coupling description has to be considered in a correct (at least conceptually) treatment of the saturation problem. The second point is that the initial condition is not lost in our case, that is the result depends on \( \rho_c(0) \), but any small changes in the initial conditions are almost unobservable since they parametrically vanish as \( 1/\rho_c^{3/2} \) when one considers \( \lambda_c \).

6.2 Leading in \( \omega \)

We consider the kernel given in Eq.(13). The \( \phi \) function is

\[
\phi(\gamma, \lambda) = \chi_0(\gamma) - \frac{\lambda}{(1-\gamma)(1+\lambda)}. \tag{64}
\]
The $\rho_c$ equation, Eq.(30), becomes

$$\rho_c \dot{\rho}_c = \frac{\chi_0(\gamma_0)}{b(1-\gamma_0)} - \frac{\dot{\rho}_c}{b(1-\gamma_0)^2(1+\rho_c)} - \frac{2 \dot{\rho}_c}{1-\gamma_0}. \quad (65)$$

Even though $\dot{\rho}_c$ is expected to be small, we prefer not to do any kind of expansion in the second term of the right-hand side, in order to be consistent with the $\omega$-expansion approach. If we expand this term, we lose at the same time the correct collinear behavior as explained in Section 2. We also notice the negative sign of this term, which is expected, since the leading in $\omega$ kernel contains a part of the NL BFKL kernel. Now, following the general strategy presented in the beginning of this section, we can do the integral in (55) to obtain

$$\frac{\chi_0(\gamma_0)}{2b(1-\gamma_0)\lambda_c^2} - \frac{1}{b(1-\gamma_0)^2} \left( \log \frac{1+\lambda_c}{\lambda_c} - \frac{1}{1+\lambda_c} \right) - [\lambda_c \rightarrow \lambda_c(0)] = Y. \quad (66)$$

Even though we cannot analytically solve this equation for $\lambda_c$ (we would like to avoid an iterative procedure for reasons that we explained before), this is a transcendental equation that can be solved numerically for a given $Y$. Then we proceed to determine $\rho_c$ from (56) and $\lambda_s$ from (58).

We mention here that if we use the “equivalent” kernel given in (14), we will not be able to reach a result as simple as the one stated in (66), as it appears difficult to integrate Eq.(55). However the two kernels should give the same results.

### 6.3 Next to leading in $\omega$

The kernel in this case is the one in (10), where $\chi_1(\gamma, \omega)$ is a known function that is constructed in the Appendix following the procedure of Refs.[22, 23, 24, 25]. Here it is not possible to give a relatively simple expression as Eq.(66) in the previous subsection. The reason is that $\chi_1(\gamma, \omega)$ contains a piece $A_{gg}(\omega)$, the Mellin transform of the gluon splitting function. This has a term $\psi(2+\omega)$ and Eq.(55) cannot be integrated. In the region of interest the expected value of $\omega$ is $\omega = \lambda_c(1-\gamma_c)$, which can be found (a posteriori) to be around 0.25. Thus we can expand $A_{gg}(\omega)$, or just its $\psi(2+\omega)$ part, around $\omega = 0$. Then, depending on the order of this Taylor expansion, we find a more and more complicated transcendental equation (even at 0th order) containing fractions of polynomials and logarithms of $\lambda_c$. This equation is the analogous of (66), that we found in the leading in $\omega$ case. There is no reason to present such a complicated expression here, but it is again an algebraic equation which
can be solved numerically. We find, for a given $Y$, that $\lambda_c$ reaches rapidly a fixed value as we increase the order of the expansion mentioned above; even if we keep just the 0th order term, the result is very close to the exact value, something which is expected since we have approximated only a small piece of the NL in $\omega$ kernel. In our results for $\lambda_s$, that appear in the table of the next subsection, we have kept enough terms in this expansion so that the obtained value of $\lambda_c$ is practically the exact one. $\lambda_s$ is obtained in the way we have already explained in the previous subsections.

There is another, but clearly approximate, iterative procedure, that we can follow in order to obtain a simple equation, which nevertheless gives the same results for the NL in $\omega$ corrections. We write the full kernel as

$$\chi(\gamma, \omega) = \chi_0(\gamma, \omega) + \omega \delta,$$

and treat the term $\omega \delta$ as a perturbation in the following sense. For a given $Y$ we solve the leading in $\omega$ problem. Then for the resulting value of $\lambda_c$ we determine the number $\delta$ appearing in (67) from $\delta = \chi_1(\gamma_c, \omega_c)/\chi_0(\gamma_c, \omega_c)$, where $\omega_c = \lambda_c(1 - \gamma_c)$ as dictated from (22). Then we insert this value of $\delta$ in (67) and solve the new problem, which is of equal difficulty as the leading in $\omega$ problem. We obtain a new value of $\lambda_c$ and we can repeat the procedure. This converges rapidly and after a couple of iterations $\delta$ and $\lambda_c$ become constant.

Then we can repeat for different $Y$. We have decided to extract explicitly a factor of $\omega$ in (67), instead of treating all the NL correction as a constant, since this factor characterizes the very leading behavior of the NL term.

### 6.4 Summary of results and discussion

We assume an initial condition $\rho_c(0) = 2 \log 5$, so that the target is of size $\mu^{-1} \approx 1 \text{GeV}^{-1}$, when $\Lambda \approx 200 \text{MeV}$. We set the number of active flavors to be $N_f = 3$. In the table we summarize the values of $\lambda_s$, obtained for the different kernels considered and for various values of $Y$. 

\[
\text{Values of } \lambda_s = d \log(Q_s^2/\Lambda^2)/dY
\]

| Y  | BFKL | L in \(\omega\) | NL in \(\omega\) | \(\delta \chi/\chi_0\) |
|-----|------|------------------|------------------|------------------|
| 5   | 0.433| 0.366            | 0.300            | -0.20            |
| 6   | 0.419| 0.358            | 0.297            | -0.20            |
| 7   | 0.406| 0.350            | 0.294            | -0.19            |
| 8   | 0.393| 0.343            | 0.291            | -0.19            |
| 9   | 0.382| 0.336            | 0.288            | -0.18            |
| ... | ...  | ...              | ...              | ...              |
| 25  | 0.278| 0.258            | 0.238            | -0.13            |

Some discussion and a couple of comments need to follow here.

(i) In all cases, and particularly in the \(\omega\) expansion approach, \(\lambda_s\) is decreasing very slowly. We have included the inaccessible value \(Y = 25\) in the table, just to illustrate this slow decrease. In the case of the NL in \(\omega\) kernel, its value is practically constant in a certain \(Y\) region. For example, in the region of phenomenological interest \(10^{-2} \gtrsim x \gtrsim 10^{-4}\), which corresponds to \(5 \lesssim Y \lesssim 9\), we effectively have \(\lambda_s = 0.30 \pm 0.29\) and therefore the energy dependence of the saturation momentum is well described by the simple exponential law \(Q_s^2(Y) \propto \Lambda^2 \exp(\lambda_s Y)\). This is in agreement with saturation based models [10] that have successfully interpreted much of the HERA data. In these models such an ansatz for \(Q_s\) was taken, with \(\lambda_s\) as a free parameter, and after the fitting to the data was performed, the resulting value of \(\lambda_s\) was very close to the one we quoted above.

However, we should emphasize again that a fixed coupling description (which gives a constant value for \(\lambda_s\), but much larger than the one found here) is not the right approach to the saturation problem, at least conceptually. The variation of \(\lambda_s\) is rather slow partly because of the particular interplay between the leading behavior and the saturation corrections, as one can check in Eq.(58), and partly because of NL corrections which slow down the evolution.

Now we come to the issue of the actual value of \(Q_s\). As already mentioned at the end of Section 5, we do have control of the energy dependence of \(Q_s\), but there is an ambiguity of a free multiplicative constant as can be seen in Eq.(53). Thus we cannot give any accurate prediction for the value of \(Q_s\). We just mention though, that for a “reasonable” choice of \(c\) in (53), we obtain a phenomenologically accepted result. For example, with \(c = 3/4\) (and \(\Lambda = 200\) MeV), we find \(Q_s(x \simeq 3 \times 10^{-4}) \simeq 1\) GeV.

In a very recent work [32] a fit to the HERA data has been performed. The model relies on the combination of the Balitsky-Kovchegov and the DGLAP
equations. In such a case NL effects are neglected. As a part of their results, a best fit value of \( \lambda_s = 0.36 \pm 0.04 \) is given, where the form \( Q_s^2 \propto \Lambda^2 \exp(\lambda_s Y) \) has been assumed. It is interesting that the central value of this result coincides with our Leading in \( \omega \) one.

(ii) It is clear that as \( Y \) increases the results of the \( \omega \) expansion converge to those of leading BFKL. This is expected as already mentioned at the end of Section 3. But this kind of convergence is slow, and it will not happen until \( Y \) reaches a very large value. Thus, in the region of phenomenological interest, the effects of the collinear improvement of the BFKL kernel are important. The leading BFKL prediction is reduced by an amount around 25 – 30% when NL in \( \omega \) corrections are included.

It is also instructive to compare the leading and the NL in \( \omega \) results. In the last column of the table we give the ratio \( \delta \chi/\chi_0 \), which is the NL correction to the kernel divided by the leading one and evaluated at the point \( (\gamma_c, \omega_c) \). We notice that the correction has a “natural size” of the order of \( O(\alpha_s) \), when compared to the leading contribution, where the running \( \alpha_s \) is evaluated at the relevant value of \( Q^2 \), say somewhere between \( Q_c^2(Y) \) and \( Q_s^2(Y) \). Essentially this is a key element of the RG improvement of the BFKL equation. NL corrections are naturally small. We mention here that this observation is also true in the problem of the calculation of the hard Pomeron intercept \( \omega_P \) [23, 24].

(iii) In all cases, the second term in (58) which represents the effects of the saturation boundary, is smaller in magnitude than the leading term, but its contribution is extremely important. Even though \( \rho_c \) is big, this term will not be negligible until \( Y \), and therefore \( \rho_c \), reaches a huge value.

It is quite interesting to investigate the parametric form of this saturation correction. In view of Eq. (22) we naturally define \( \omega_c = (1 - \gamma_0)\lambda_c \) and \( \omega_s = (1 - \gamma_0)\lambda_s \). Expressing \( \lambda_c \) in terms of \( \rho_c \) in the 2nd term of (58), by keeping only the very leading behavior as determined from the dominant term in (65), we obtain

\[
\omega_s = \omega_c + \frac{\xi_1 D_0 \chi_0(\gamma_0)}{4b} \frac{1}{\rho_c^{5/3}} \]

\[
= \omega_c + \frac{\xi_1}{4} \left[ \frac{b^2 \chi_0^2(\gamma_0) \chi_0''(\gamma_0)}{2} \right]^{1/3} \left[ \tilde{\alpha}_s(\rho_c) \right]^{5/3}.
\]

Contributions of the type \( \tilde{\alpha}_s^{5/3} \) due to the running of the coupling have been observed and studied in the calculation of the hard Pomeron intercept [33, 34, 35, 36, 24]. There is a striking similarity between the above equation and
Eq.(4.9) in [24]. In our case the $\chi$ function is evaluated at $\gamma_0$ and not at its minimum, and the additional $1/4$ factor is understandable through Eq.(51). We need to mention though, that in the saturation problem there is no limit in the applicability of (58) or (68). On the contrary, since the line of evolution is one of increasing $\rho_c$ and decreasing $\bar{\alpha}_s$, the second term is always smaller than the leading one.

7 The amplitude and geometrical scaling

In this section we return to discuss the form of the amplitude. Since this is constant along the saturation line, we naturally expect a scaling behavior in a certain region around this line. Of course we restrict ourselves to the perturbative side of the boundary, since we do not know how to handle the details of the non-linear effects. The scaling form of the amplitude should be more and more exact as $Y$ increases, but it is also a reliable expression even in the region of phenomenological interest.

Returning to Eq.(45), with $\beta$ and $\sigma$ as obtained in (51), and expressing the variables ($\xi, t$) in terms of the more physical variables ($Q, Q_s$), following the definitions we have given throughout this paper, it is straightforward to find

$$T \propto \left(\frac{Q^2}{Q_s^2}\right)^{1-\gamma_0} \left[ \log \frac{Q^2}{Q_s^2} + \frac{1 + c}{1 - \gamma_0} \right].$$

The target dependent factors that are not written in this expression are of the form $(\text{const.}/\mu^2)$, with $\mu$ characterizing as always the target size. We notice also that the form of the amplitude will not change if we change $Q_s$ by a multiplicative constant, since such an operation can be absorbed in a redefinition of the constant $c$. Eq.(69) is valid in the region

$$\log \frac{Q^2}{Q_s^2} \lesssim \sqrt{\frac{6}{|\xi_1|} D_0 \rho_c^{1/3} - \frac{1 + c}{1 - \gamma_0}},$$

a constraint arising from the expansion of the Airy function. Parametrically this constraint is satisfied since $\rho_c$ is growing with $Y$. In the region of interest, the number on the right hand side is not large enough to allow scaling in a wide region of $Q^2$ values, as it has been observed in the HERA data [37]. Even at the very small value $x \simeq 10^{-4}$, the inequality in (70) will totally break at $Q^2 \gtrsim 50 \text{ GeV}^2$ (independent of the unknown constant $c$). This “scaling” window depends relatively strongly on the initial condition. Furthermore, a more complete solution of (39) could also change the above estimate, but we do
not know how to control this issue in a more precise way. Other approximations
that we have made, for example \( \eta \ll \rho_c \), are also parametrically satisfied and
are much weaker numerically.

Eq. (69) is the same (but with a different \( Q_s \)) as the one derived in Ref. [18],
in a leading BFKL description of the saturation problem with either fixed or
running coupling. Therefore it is interesting to try and interpret this result.
The power behavior, with the anomalous dimension \( 1 - \gamma_0 \), is a consequence
of the BFKL dynamics, either the “standard” or the collinearly improved.
The logarithm in the square bracket is the effect of the absorptive boundary.
Such a boundary imposes the condition that somewhere beyond the saturation
line (where of course (69) is not valid any more since non-linear effects are
important) the amplitude vanish. Clearly a logarithm satisfies this condition.
We can give a more mathematical argument that (69) is the only scaling form
allowed by the dynamics we have considered.

To illustrate the issue in a simple way let us consider fixed coupling leading
BFKL dynamics. In this case Eq. (17) simplifies to

\[
\frac{\partial T}{\partial Y} = \bar{\alpha}_s \chi_0 \left( 1 + \frac{\partial}{\partial \rho} \right) T. \tag{71}
\]

Following a similar procedure as in Section 3, we define now \( \eta = \rho - \rho_s(Y) \)
and write the amplitude as \( T = \exp\left[ -(1 - \gamma_0) \eta \right] \psi(\eta) \) to obtain

\[
\left[ \bar{\alpha}_s \chi_0 \left( \gamma_0 + \frac{\partial}{\partial \eta} \right) - \dot{\rho}_s(1 - \gamma_0) + \dot{\rho}_s \frac{\partial}{\partial \eta} \right] \psi(\eta) = 0, \tag{72}
\]

with \( \gamma_0 \) a constant undetermined for the moment and where we dropped the
\( \partial/\partial Y \) term in order to look for an exact scaling solution. Clearly \( \dot{\rho}_s \) has to be
constant. Expanding the \( \chi_0 \) function around \( \gamma_0 \), (72) becomes

\[
\left\{ \bar{\alpha}_s \chi_0(\gamma_0) - (1 - \gamma_0) \dot{\rho}_s + [\bar{\alpha}_s \chi_0'(\gamma_0) + \dot{\rho}_s] \frac{\partial}{\partial \eta} \right. \\
\left. + \bar{\alpha}_s \sum_{n=2}^{\infty} \frac{\chi_0^{(n)}(\gamma_0)}{n!} \left( \frac{\partial}{\partial \eta} \right)^n \right\} \psi(\eta) = 0. \tag{73}
\]

Let us assume that that \( \psi(\eta) \) is a finite polynomial of \( \eta \) and of order \( k \), that is

\[
\psi(\eta) = a_0 + a_1 \eta + a_2 \eta^2 + \ldots + a_k \eta^k. \tag{74}
\]
It is easy to see that the coefficients of the $\eta^k$ and $\eta^{k-1}$ terms in (73) will vanish if and only if

$$\bar{\alpha}_s\chi_0(\gamma_0) - (1 - \gamma_0)\dot{\rho}_s = 0, \quad (75)$$

$$\bar{\alpha}_s\chi_0'(\gamma_0) + \dot{\rho}_s = 0, \quad (76)$$

respectively and $\gamma_0$ becomes our usual one. But if $k \neq 0, 1$ the coefficients of the remaining terms will not vanish and (73) is not satisfied. Thus only the case $k = 1$, which is what we have found in (69), is allowed. The case $k = 0$ is allowed too, but it is not consistent with our saturation boundary conditions. Notice that (75) and (76) are equivalent to the reduction of (26) and (27) in the fixed coupling BFKL case, with $\gamma_c \rightarrow \gamma_0$ and $\beta \rightarrow 0$.

The above can be generalized in the case of an RG improved kernel with running coupling in a straightforward way. Then the solution with $k = 1$ is an approximate one when $\eta \ll \rho_s$ and we are lead to Eqs. (26) and (27), again with $\gamma_c \rightarrow \gamma_0$ and $\beta \rightarrow 0$. The one is satisfied in the asymptotic region and the other determines the leading behavior of $\rho_s$. Notice that this procedure gives us a very fast way to obtain (69), but only the leading behavior of $Q_s$, which is $Q_c$, is recovered. That is, we do not find in $Q_s$ the corrections due to the boundary, which turned out to be important as we saw in the previous sections.

8 Estimation of errors

Here we check the approximations that we have made and which could be possible sources of errors. We find that all these errors are small and under control.

(i) There is an ambiguity arising from the initial condition. $\rho_c(0)$ should characterize the target size and is given by Eq. (57). If we consider the scattering on a proton we may choose $\mu$ to be around 1 GeV but we do not know the exact value. The same applies for $\Lambda$ which can be taken to be around 200 MeV. However, changing $\rho_c(0)$ by small amounts does not affect our results. We have already discussed that the corresponding corrections for $\lambda_s$ are parametrically very small. From another point of view we first notice that Eq. (57) has “translational” invariance in $Y$. Thus, changing $\rho_c(0)$, for example from $2 \log 5$ to $2 \log 3$ which describes equally good a hadronic target, corresponds to a shift of the rapidity by a unit or so. Recalling our results from Section 6, where we found that $\lambda_s$ is effectively constant in a wide regions of $Y$ values, it
is clear the error is small. On the contrary, the value of \( Q_s \) is sensitive to the initial conditions assumed.

**(ii)** In Section 3 we have expanded our basic \( \rho_c \) and \( \gamma_c \) equations, (26) and (27), to first order in \( \delta \gamma = \gamma_c - \gamma_0 \). For the Leading in \( \omega \) kernel, we easily find that

\[
\delta \gamma = \frac{2\lambda_c(1 - \gamma_0)[1 + b(1 - \gamma_0)(1 + \lambda_c)]}{(1 - \gamma_0)^3 \chi''_0(\gamma_0)(1 + \lambda_c) - 2\lambda_c} = \frac{\lambda_c(1.85 + 0.591\lambda_c)}{12 + 10\lambda_c}.
\]

(77)

If we expand to second order in \( \delta \gamma \) we have to add on the right-hand side of Eq.(30) the term

\[
-\frac{(\delta \gamma)^2 \phi^{2,0}(\gamma_0, \lambda_c)}{2b(1 - \gamma_0)}.
\]

(78)

This term gives a very small contribution in (30). Numerically it is \( 1 - 2\% \) of the leading term when we plug in the expected value of \( \lambda_c \), the reason being the large numerical value of \( \chi''_0(\gamma_0) \) which appears in the denominator in (77). What is more interesting, is that this kind of correction can be neglected by making use of a more rigorous argument. From the equations above we see that the contribution is of order \( \mathcal{O}(\lambda_c^2) \), which is of order \( \mathcal{O}(\omega^2) \) in view of (22). Clearly this overlaps with NNL in \( \omega \) corrections.

**(iii)** In Section 5 we fixed the amplitude to be constant on the saturation line. This will be exactly true in the asymptotic region. In the intermediate region factors of the form \( \exp[\mathcal{O}(1/t^2)] \equiv \exp[g(t)] \) remain, because the functions \( h_1 \) and \( h_2 \) are not equal to 1. These functions though, turn out to be close to 1, and therefore it is harmless to ignore these terms. Equivalently, we could multiply our solution (13) with \( \exp[-g(t)] \), which gives an equally good approximate solution since the remainder (44) changes just by a term \( h_1(t)dg(t)/dt = \mathcal{O}(1/t^3) \).

There are also certain assumptions and approximations that we made when deriving \( \delta \xi_s \) and \( \rho_s \) as given in (19) and (22) respectively. We kept only the linear term in \( \delta \xi \) in the expansion of the Airy function in (47), we replaced \( D \) by its asymptotic value \( D_0 \) and we neglected \( \xi^2/t \) and \( \xi/t^2 \) terms in the exponent of the amplitude. If we relax these assumptions, we find that we should add in (22) terms of the form \( \mathcal{O}(\rho_c^{-2/3}, \rho_c^{-1}, ...) \). Notice that there are no \( \mathcal{O}(\rho_c^{-1/3}) \) terms. All these corrections vanish when \( \rho_c \) becomes large. Thus in (22) all the terms that survive in the asymptotic region, including the constant one, have been kept. For completeness, we have checked that if we include all the available terms up to order \( \mathcal{O}(\rho_c^{-1}) \), then \( \lambda_s \) is decreasing by \( \simeq 0.01 \) in the
intermediate $Y$ region. Therefore, we wouldn’t like to sacrifice the relative simplicity of (52) and (58) for such an unimportant error.

(iv) A crucial assumption that we made in order to simplify the BFKL equation, was to drop the $\partial/\partial Y$ term that appeared inside the argument of the BFKL kernel in (24) (of course we did that after switching to the scaling variable $\eta$; the argument cannot be applied directly in (17)). One can understand that this term has a small effect, since we are looking for a scaling solution. The dominant $\partial/\partial Y$ term which appears on the left-hand side of (24) is already one of the weak terms of our equation and therefore all other derivatives of this kind, which originate from NL corrections will be even weaker. It is interesting to understand this through a more rigorous argument. Let us consider the most important contribution of such a term, which is obtained by letting $\partial/\partial \eta \to -(1 - \gamma_0)$ and expanding to 1st order in the 2nd argument in (24). We get

$$- \left(1 - \frac{\eta}{\rho_c}\right) \phi^{0,1}(\gamma_0, \lambda_c) \frac{\partial}{\partial Y}(T/\alpha_s)$$

as a contribution to the left-hand side of our equation. For the case of the leading in $\omega$ kernel, $\phi^{0,1}$ is

$$\phi^{0,1}(\gamma_0, \lambda_c) = -\frac{1}{(1 - \gamma_0)(1 + \lambda_c)^2} < 0. \quad (80)$$

We already see from (79) that the correction is suppressed by $1/\rho_c$, when compared to the dominant $\partial/\partial Y$ term. Then we just follow the procedure of Sections 3 and 4 and we find that $\psi(\xi, t)$ is still given by (18), where now the functions $h_1(t)$ and $h_2(t)$ are more complicated but their asymptotic limit remains the same. At the same time the $\rho_c$ equation changes because of the $\eta \partial/\partial Y$ in (79). This will add on the right-hand side of Eq.(30) a term

$$\lambda_c^2 \frac{2\phi^{0,1}(\gamma_0, \lambda_c)}{\phi(\gamma_0, \lambda_c)} - 2b\lambda_c. \quad (81)$$

However, as in case (ii) this is of order $O(\omega^2)$ which overlaps with the NNL in $\omega$ corrections.

(v) We have considered three active flavors. A fourth quark, the charm, cannot be treated as massless as required by BFKL dynamics. Even if we set $N_f = 4$, then $\lambda_s$ will change by $\simeq 0.01$. 

25
9 Summary

We have studied the energy dependence of the saturation momentum, using the collinearly (RG) improved BFKL equation as our QCD dynamics. A saturation boundary of constant amplitude for dipole-hadron scattering has been used to describe the non-linear effects in a simple but effective way. Following the evolution of the system, we were able to determine this boundary as far as the energy (or rapidity $Y$) is considered. The saturation momentum $Q_s^2$ and its logarithmic derivative $\lambda_s$ depend on $Y$ in a complicated way, particularly at the Next to Leading Level. However, we have found that $\lambda_s$ is decreasing very slowly, so that it is practically constant for a wide region of $Y$ values and therefore $Q_s^2$ is effectively described by the simple exponential law $Q_s^2 \propto \Lambda^2 \exp(\lambda_s Y)$. In the region of current phenomenological interest we found at the NL level $\lambda_s = 0.30 \div 0.29$ with very reliable accuracy. The boundary (which imitates the non-linear effects), the RG improvement of the BFKL kernel at Leading and NL level, and the running of the coupling, were all important in obtaining $\lambda_s(Y)$. The exact value of $Q_s$ cannot be determined in our approach, because of a (not totally though) free multiplicative constant. The amplitude exhibits geometrical scaling in a certain region on the perturbative side of the boundary, and its form is independent of the particular BFKL dynamics considered.

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Appendix

Here we construct the NL in $\omega$ kernel with an energy scale choice $s_0 = Q^2$ following Refs.\[22, 23, 24, 25\]. We start with the NL BFKL kernel \[19, 20\],

\begin{equation}
\chi_1(\gamma) = -\frac{b}{2}[\chi_0^2(\gamma) + \chi_0'(\gamma)] - \frac{1}{4} \chi_0''(\gamma) \\
- \frac{\pi^2 \cos(\pi\gamma)}{4(1-2\gamma)\sin^2(\pi\gamma)} \left[ 3 + \left( 1 + \frac{N_f}{N_c^3} \right) \frac{2 + 3\gamma(1-\gamma)}{(3-2\gamma)(1+2\gamma)} \right] \\
+ \frac{3}{2} \zeta(3) + \left( \frac{67}{36} - \frac{\pi^2}{12} - \frac{5}{18} \frac{N_f}{N_c} \right) \chi_0(\gamma) + \frac{\pi^3}{4 \sin(\pi\gamma)} - \Phi(\gamma),
\end{equation}

(A.1)
where
\[
\Phi(\gamma) = \frac{\pi^3}{6 \sin(\pi \gamma)} - \int_0^1 dx \frac{x^{-\gamma} + x^{\gamma-1}}{1 + x} \text{Li}_2(x), \tag{A.2}
\]
with the leading BFKL kernel $\chi_0(\gamma)$ and $b$ as given in Eqs. (2) and (3) in the text. The pole structure is different that the one given in (4) because now we will include the $N_f$ dependence. We need first to give the splitting functions and their Mellin transforms, which are

\[
P_{gg}(z) = \frac{z}{(1+z)_+} + \frac{1-z}{z} + z(1-z) + b \delta(1-z), \tag{A.3}
\]

\[
P_{qg} = \frac{N_f}{2N_c} [z^2 + (1-z)^2], \tag{A.4}
\]

\[
A_{gg}(\omega) = \int_0^1 dz \, z^\omega P_{gg}(\omega) - \frac{1}{\omega} = b - \frac{1}{1+\omega} + \frac{1}{2+\omega} - \frac{1}{3+\omega} - [\psi(2+\omega) - \psi(1)], \tag{A.5}
\]

\[
A_{qg}(\omega) = \int_0^1 dz \, z^\omega P_{qg}(\omega) = \frac{N_f}{2N_c} \left( \frac{1}{1+\omega} - \frac{2}{2+\omega} + \frac{2}{3+\omega} \right). \tag{A.6}
\]

Define the function $A_T(\omega)$ by

\[
A_T(\omega) = A_{gg}(\omega) + \frac{C_F}{N_c} A_{qg}(\omega), \tag{A.7}
\]

with $C_F = (N_c^2 - 1)/2N_c$. We can write the pole structure of $\chi_1(\gamma)$ as

\[
\chi_1(\gamma) = -\frac{1}{2\gamma^3} - \frac{1}{2(1-\gamma)^3} + \frac{A_T(0)}{\gamma^2} + \frac{A_T(0) - b}{(1-\gamma)^2} - F \left( \frac{1}{\gamma} + \frac{1}{1-\gamma} \right) + \text{finite}, \tag{A.8}
\]

where $F$, which vanishes when $N_f = 0$ causing the simple poles to go away, is

\[
F = \frac{N_f}{6N_c} \left( \frac{5}{3} + \frac{13}{6N_c^2} \right). \tag{A.9}
\]
So far this refers to a symmetric energy scale $s_0 = Q\mu$ and in order to transform $\chi_1(\gamma)$ to the scale $s_0 = Q^2$ we need to add the piece $-\chi_0(\gamma)\chi'_0(\gamma)/2$ \[19, 20\]. This will remove the triple pole at $\gamma = 0$ and will double the one at $\gamma = 1$.

Now, our leading in $\omega$ kernel $\chi_0(\gamma, \omega)$ as given in \[13\] contains NL corrections which must be subtracted from our NL kernel to avoid double counting. Iterating $\omega = \bar{\alpha}_s\chi_0(\gamma, \omega)$ once, we easily find that we must add to the NL kernel the term $\chi_0(\gamma)/(1 - \gamma)^2$. This will cancel the remaining triple pole at $\gamma = 1$.

There are still double and single poles\[23\]. Putting all the previous pieces together and subtracting these remaining poles we find

\[
\tilde{\chi}_1(\gamma) = \chi_1(\gamma) - \frac{1}{2} \chi_0(\gamma)\chi'_0(\gamma) + \frac{\chi_0(\gamma)}{(1 - \gamma)^2}
\]

\[
- \frac{1}{\gamma} + F\left(\frac{1}{\gamma} + \frac{1}{1 - \gamma}\right) - \frac{A_T(0)}{\gamma^2} - \frac{A_T(0) - b}{(1 - \gamma)^2},
\]

(A.10)

a function with no poles at all at $\gamma = 0$ and $\gamma = 1$. Of course we need now to add the correct form of all these subtracted terms that appear in the second line of the previous equation. We obtain

\[
\chi_1(\gamma, \omega) = \tilde{\chi}_1(\gamma) + \frac{1}{\gamma} - F\left(\frac{1}{\gamma} + \frac{1}{1 - \gamma + \omega}\right)
\]

\[
+ \frac{A_T(\omega)}{\gamma^2} + \frac{A_T(\omega) - b}{(1 - \gamma + \omega)^2},
\]

(A.11)

Finally the full kernel is given as in Eq.(16).

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