Miyamoto groups of code algebras

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Abstract

A code algebra \(A_C\) is a nonassociative commutative algebra defined via a binary linear code \(C\). In a previous paper, we classified when code algebras are \(\mathbb{Z}_2\)-graded axial algebras generated by small idempotents. In this paper, for each algebra in our classification, we obtain the Miyamoto group associated to the grading. We also show that the code algebra structure can be recovered from the axial algebra structure.

1 Introduction

Code algebras were introduced in \([2]\) as a new class of commutative nonassociative algebras defined from binary linear codes. They contain a family of pairwise orthogonal idempotents, called \textit{toral idempotents}, indexed by the length of the code, and have a nice Peirce decomposition relative to this family. Their definition was inspired by an axiomatic approach to code Vertex Operator Algebras \([3, 4, 5]\), and, in particular, their connections with axial algebras were explored.

Strengthening the relevance of this axiomatisation, code algebras were shown in \([3]\) to have a striking resemblance to the coordinate algebras associated with the optimal short \(SL_2^3\)-structure of the simple Lie algebras of types \(E_7, E_8,\) and \(F_4\).

In this paper, we continue to explore the links of code algebras with axial algebras \([6]\). These are a relatively new class of commutative nonassociative algebras generated by semisimple idempotents called \textit{axes}. We have partial control over the multiplication by requiring that the eigenvectors for the adjoint action of an axis multiply according to a so-called \textit{fusion law}. This gives us the key property that, when the fusion law is graded, we may

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naturally identify automorphisms, called Miyamoto automorphisms, to each axis. These in turn generate the Miyamoto group which is a subgroup of the automorphism group.

Let $C$ be a binary code of length $n$. A code algebra $A_C$ has a basis $\{t_i : i = 1, \ldots, n\} \cup \{e^\alpha : \alpha \in C^*\}$, where $C^* := C \setminus \{0, 1\}$, and multiplication which mimics the code structure (see Definition 2.6 for details). We call the $t_i$ toral idempotents and the $e^\alpha$ codeword elements.

In a code algebra $A_C$, the toral idempotents are not enough to generate the algebra. However, in [2], we give a construction, called the $s$-map, to obtain idempotents of $A_C$ from a subcode $D$ of $C$. For the smallest possible subcode, $D = \langle \alpha \rangle$ where $\alpha \in C$, the $s$-map idempotents are called small idempotents. They are of the form

$$e_{\alpha,\pm} := \lambda t_\alpha \pm \mu e^\alpha$$

where $t_\alpha = \sum_{i \in \text{supp}(\alpha)} t_i$ and $\lambda, \mu \in F$.

In general, it is difficult to analyse the eigenvalues and eigenvectors, let alone give the fusion law, for $s$-map idempotents. However, in [1], we do this for small idempotents. We show that if $C$ is a projective code and $S$ is a set of codewords which generate $C$, then the small idempotents generate the algebra $A_C$. We give the eigenvalues and eigenvectors for the small idempotents explicitly and describe the fusion law in general. Hence we show when $A_C$ is an axial algebra with respect to the small idempotents.

The main result in [1] classifies when the fusion law for the small idempotents is $\mathbb{Z}_2$-graded under the assumption of Hypothesis 1 (see page 8).

In the first part of this paper, we show that you can recover the code algebra structure from the axial algebra structure. We say a set $X$ of small idempotents is pair-closed if whenever $e_{\alpha,+} \in X$, then $e_{\alpha,-} \in X$ and vice versa.

**Theorem 1.** Suppose that $A$ is a code algebra which is also an axial algebra with respect to a pair-closed set $X$ of small idempotents. Then, we can recover the code $C$, up to permutation equivalence, and also the special basis $\{t_i : i = 1, \ldots, n\} \cup \{e^\alpha : \alpha \in C^*\}$.

In particular, since we know the special basis, we can also recover the parts of the fusion law, even if two different parts happen to have the same eigenvalue.

From the main classification result [1, Theorem 6.1], we may split the code algebras with a $\mathbb{Z}_2$-graded fusion law up into three cases (for more detail see Section 2.3):

1. $|\alpha| = 1, C = \mathbb{F}_2^n$, and
   - (a) $n = 2, a = -1$.
   - (b) $n = 3$.  

2.
2. \(|\alpha| = 2\) and \(C = \bigoplus_{i=1}^{r} C_i\) is the direct sum of even weight codes all of the same length \(m \geq 3\).

3. \(|\alpha| > 2\)

In the second part of the paper, we give an explicit description of the Miyamoto automorphism associated to a small idempotent \(e_{\alpha, \pm}\) and calculate the Miyamoto groups.

**Theorem 2.** Assume Hypothesis [1] and suppose that \(A\) is a code algebra which is an axial algebra with respect to a pair-closed set of small idempotents \(X\) indexed by a set \(S\) of codewords which has a \(\mathbb{Z}_2\)-graded fusion law. Then the Miyamoto group \(G\) of \(A\) is

1. (a) \(G = S_3 \times S_3\).
   (b) \(G = 2^2\).

2. \(G = \begin{cases} 
2^{r(m-1)} & \text{if } m \text{ is odd,} \\
2^{r(m-2)} & \text{if } m \text{ is even.}
\end{cases}\)

3. \(G\) is an elementary abelian 2-group of order at most \(2^{|S|}\).

From [1], case (2), where \(|\alpha| = 2\), the algebra is in fact \(\mathbb{Z}_2 \times \mathbb{Z}_2\)-graded when the structure parameters are chosen sufficiently ‘nice’ way (see Section 2.3 for details).

**Theorem 3.** Assume Hypothesis [1] and suppose that \(A\) is a code algebra which is an axial algebra with respect to a pair-closed set of small idempotents \(X\) indexed by a set \(S\) of codewords. Suppose further that \(|\alpha| = 2\) and the fusion law of the small idempotents is \(\mathbb{Z}_2 \times \mathbb{Z}_2\)-graded. Then \(C = \bigoplus_{i=1}^{r} C_i\) is the direct sum of even weight codes \(C_i\) each of length \(m\) and \(A\) has Miyamoto group

\[
G = \begin{cases} 
\prod_{i=1}^{r} 2^{m-1} : S_m & \text{if } m \text{ is odd} \\
\prod_{i=1}^{r} 2^{m-2} : S_m & \text{if } m \text{ is even.}
\end{cases}
\]

The structure of the paper is as follows. We give the definition of an axial algebra and some of their properties in Section 2. We also recall some properties of binary linear codes and recap the definition and main results on code algebras. In Section 3 the recovery of the code algebra structure from the axial algebra structure is proved. Finally, in Section 4 we explicitly describe the Miyamoto automorphisms and identify the Miyamoto groups in all cases.

## 2 Background

We begin by briefly describing axial algebras and how to obtain automorphisms from a grading. We then give the definition of a code algebra and recall some results.
2.1 Axial algebras

We review the basic definitions related to axial algebras. For further details, see [6, 7]. We will take a slightly more general approach to the definition, where we will allow a subdividing of eigenspaces; see for example [4].

**Definition 2.1.** A pair $\mathcal{F} := (\mathcal{F}, \star)$ is a fusion law if $\mathcal{F}$ is a non-empty set and $\star: \mathcal{F} \times \mathcal{F} \to 2^\mathcal{F}$ is a symmetric map, where $2^\mathcal{F}$ denotes the power set of $\mathcal{F}$.

We call a single instance $\lambda \star \mu$ a fusion rule. We will extend the operation $\star$ to arbitrary subsets $U, V \subseteq \mathcal{F}$ by $U \star V = \bigcup_{\lambda \in U, \mu \in V} \lambda \star \mu$.

Let $A$ be a commutative non-associative (i.e. not-necessarily-associative) algebra over $\mathcal{F}$. Let $X$ be a set of elements of $A$. We write $\langle X \rangle$ for the vector space spanned by the elements in $X$ and $\langle \langle X \rangle \rangle$ for the subalgebra generated by $X$. For an element $a \in A$, the adjoint endomorphism $\text{ad}_a$ is defined by $\text{ad}_a(v) := av, \forall v \in A$. Let $\text{Spec}(a)$ be the set of eigenvalues of $\text{ad}_a$, and for $\lambda \in \text{Spec}(a)$, let $A_\lambda(a)$ be the $\lambda$-eigenspace of $\text{ad}_a$. Where the context is clear, we will write $A_\lambda$ for $A_\lambda(a)$.

**Definition 2.2.** Let $(\mathcal{F}, \star)$ be a fusion law and $A$ a commutative non-associative algebra over $\mathcal{F}$. An element $a \in A$ is an $(\mathcal{F}, \lambda)$-axis if the following hold:

1. $a$ is an idempotent (i.e. $a^2 = a$),
2. There exists a decomposition of the algebra
   \[ A = \bigoplus_{f \in \mathcal{F}} A_f \]
   such that the algebra multiplication satisfies the fusion law. That is,
   \[ A_x A_y \subseteq A_{x \star y} := \bigoplus_{z \in x \star y} A_z \]
   for all $x, y \in \mathcal{F}$.
3. There exists a map $\lambda: \mathcal{F} \to \text{Spec}(a)$, called an evaluation map, such that
   \[ av = \lambda_x v \]
   for all $v \in A_x$.

If, in addition, $A_1 = \langle a \rangle$, then we say $a$ is primitive.

For $x \in \mathcal{F}$, we call $A_x$ the $x$-part of $A$ with respect to $a$. We also allow a part $A_x$ to be empty. Note that we use similar notation for the parts of an axial algebra as for the eigenspaces. Just as for the eigenspaces, if we
want to stress that the decomposition is with respect to an axis \( a \), we write \( A_x(a) \), but normally we will just write \( A_x \).

Note that this is a slightly more general definition of an axis to that which is usually given. In the usual definition, the parts of \( A \) are the eigenspaces and so the map \( \lambda : x \mapsto \lambda x \) is injective and the third part in the above definition is redundant. The above definition allows us to split eigenspaces into different parts and treat them separately. In [4] an even more general definition of decomposition algebras is given.

**Definition 2.3.** A pair \( A = (A, X) \) is an \((F, \lambda)\)-axial algebra if \( A \) is a commutative non-associative algebra and \( X \) is a set of \((F, \lambda)\)-axes which generate \( A \).

When the fusion law is clear from context we drop the \( F \) and \( \lambda \) and simply use the term *axial algebra*.

**Definition 2.4.** The fusion law \( F \) is \( T \)-graded, where \( T \) is a finite abelian group, if there exist a partition \( \{F_t : t \in T\} \) of \( F \) such that for all \( s, t \in T \),

\[
F_s \ast F_t \subseteq F_{st}
\]

We allow the possibility that some part \( F_t \) is the empty set. Let \( A \) be an algebra and \( a \in A \) an \((F, \lambda)\)-axis (we do not require that \( A \) is an axial algebra). If \( F \) is \( T \)-graded, then the axis \( a \) defines a \( T \)-grading on \( A \) where the \( t \)-graded subspace \( A_t \) of \( A \) is

\[
A_t = \bigoplus_{\lambda \in F_t} A_{\lambda}(a)
\]

When \( F \) is \( T \)-graded we may define some automorphisms of the algebra in the following way. Let \( T^* \) denote the set of linear characters of \( T \) (i.e., the set of all homomorphisms from \( T \) to \( F^* \)). For an axis \( a \) and \( \chi \in T^* \), consider the map \( \tau_a(\chi) : A \to A \) defined by the linear extension of

\[
u \mapsto \chi(t)u\quad\text{for } u \in A_t(a).
\]

Since \( A \) is \( T \)-graded, this map \( \tau_a(\chi) \) is an automorphism of \( A \), which we call a *Miyamoto automorphism*. Furthermore, the map sending \( \chi \) to \( \tau_a(\chi) \) is a homomorphism from \( T^* \) to \( \text{Aut}(A) \).

The subgroup \( T_a := \langle \tau_a(\chi) : \chi \in T^* \rangle \) is called the *axial subgroup* corresponding to \( a \).

**Definition 2.5.** Let \( X \) be a set of \((F, \lambda)\)-axes. The *Miyamoto group with respect to \( X \)* is

\[
G(X) := \langle T_a : a \in X \rangle \leq \text{Aut}(A)
\]
We are particularly interested in \( \mathbb{Z}_2 \)-graded fusion laws. In this case, we identify \( \mathbb{Z}_2 \) with the group \( \{+,-\} \) equipped with the usual multiplication of signs. When \( \text{char}(\mathbb{F}) \neq 2 \), the sign character \( \chi_{-1} \) is the only non-trivial character. We write \( \tau_a := \tau_a(\chi_{-1}) \) and call it the Miyamoto involution associated to \( a \).

A set of axes \( X \) is called closed if it is closed with respect to its Miyamoto group. That is, \( X = X^{G(X)} \). One can show that given any set of axes \( X \), there is a unique smallest set \( \bar{X} \supseteq X \), called the closure of \( X \), such that \( \bar{X} \) is closed. By [7, Lemma 3.5], \( \bar{X} = X^{G(X)} \) and \( G(\bar{X}) = G(X) \).

2.2 Codes

A binary linear code \( C \) of length \( n \) and rank \( k \) is a \( k \)-dimensional subspace of \( \mathbb{F}_2^n \) and we call elements of \( C \) codewords. We will write \( 0 \) for the all zeroes codeword \((0,\ldots,0)\) and \( 1 \) for the all ones codeword \((1,\ldots,1)\). Note that a code \( C \) always contains \( 0 \), but does not necessarily contain \( 1 \). If \( 1 \in C \), then each codeword \( \alpha \in C \) has a complement \( \alpha^c := \alpha + 1 \). Conversely, if \( \alpha \in C \) has a complement, that is there exists a vector \( \alpha^c \in C \) such that \( \alpha + \alpha^c = 1 \), then \( 1 \in C \) and all codewords have complements.

We denote the support of \( \alpha \in \mathbb{F}_2^n \) by \( \text{supp}(\alpha) := \{ i = 1, \ldots, n : \alpha_i = 1 \} \) and say it has Hamming weight \( |\alpha| := |\text{supp}(\alpha)| \). We denote by the set of weights in \( C \) by \( \text{wt}(C) := \{ |\alpha| : \alpha \in C \} \). For two codewords \( \alpha, \beta \in C \), we use the notation \( \alpha \cap \beta := \text{supp}(\alpha) \cap \text{supp}(\beta) \).

Two codes \( C \) and \( D \) of length \( n \) are permutation equivalent if there exists a permutation \( \sigma \in S_n \) such that \( D = C^{\sigma} \), where \( \sigma \) acts naturally on codewords by permuting their support.

The dual code \( C^\perp \) of a code \( C \) is the set of \( v \in \mathbb{F}_2^n \) such that \( (v, C) = 0 \) where \( (\cdot, \cdot) \) is the usual inner product. A binary linear code \( C \) is projective if the minimum weight of a codeword in its dual \( C^\perp \) is at least three.

For a subset \( S \) of \( \{1,\ldots,n\} \), we write \( \text{proj}_S : C \to \mathbb{F}_2^n \) for the usual projection map onto the subspace of \( \mathbb{F}_2^n \) spanned by the standard basis vectors \( e_i = (0,\ldots,0,1,0,\ldots,0) \) with \( i \in S \). Note that \( \text{proj}_S(C) \) is also a binary linear code. In more code theoretic language, it is the code punctured at \( [n] - S \). For \( \alpha \in C \), we will abuse our notation and write \( \text{proj}_\alpha(C) \) for \( \text{proj}_{\text{supp}(\alpha)}(C) \).

2.3 Code algebras

We write \( C^* := C \setminus \{0,1\} \).

**Notation.** Note that throughout this paper, we will often write conditions involving \( 1 \), or \( \alpha^c \). We do not assume that \( 1 \in C \), or complements exist, just that if they do, then the conditions must hold.
Definition 2.6. Let $C \subseteq \mathbb{F}_2^n$ be a binary linear code of length $n$, $\mathbb{F}$ a field and $\Lambda \subseteq \mathbb{F}$ be a collection of structure parameters

$$\Lambda := \{a_{i,\alpha}, b_{\alpha,\beta}, c_{i,\alpha} \in \mathbb{F} : i = 1, \ldots, n, \alpha, \beta \in C^*\}.$$  

The code algebra $A_C(\Lambda)$ is the commutative algebra over $\mathbb{F}$ with basis

$$\{t_i : i = 1, \ldots, n\} \cup \{e^\alpha : \alpha \in C^*\},$$

and multiplication given by

$$t_i \cdot t_j = \delta_{i,j} t_i$$

$$t_i \cdot e^\alpha = \begin{cases} a_{i,\alpha} e^\alpha & \text{if } \alpha_i = 1 \\ 0 & \text{if } \alpha_i = 0 \end{cases}$$

$$e^\alpha \cdot e^\beta = \begin{cases} b_{\alpha,\beta} e^{\alpha+\beta} & \text{if } \alpha \neq \beta, \beta^c \\ \sum_{i \in \text{supp}(\alpha)} c_{i,\alpha} t_i & \text{if } \alpha = \beta \\ 0 & \text{if } \alpha = \beta^c \end{cases}$$

We will call the basis elements $t_i$ toral elements and the $e^\alpha$ codewords elements. For $\alpha \in C^*$, we use the notation $t_\alpha$ for $\sum_{i \in \text{supp}(\alpha)} t_i$.

Given a subcode $D$ of $C$, the s-map construction can be used for defining idempotents. This was first given in [2] and subsequently revised in [1, Proposition 2.2]. In this paper, we will be interested in the so-called small idempotents coming from this s-map construction. Fix $\alpha \in C^*$ and assume that $a := a_{i,\alpha} = a_{j,\alpha}$ and $c_\alpha := c_{i,\alpha} = c_{j,\alpha}$, for all $i, j \in \text{supp}(\alpha)$. Then for the subcode $D := \langle \alpha \rangle$, the s-map construction gives two idempotents

$$e_{\alpha,\pm} := \lambda t_\alpha \pm \mu_\alpha e^\alpha$$

where $\lambda := \frac{1}{2a[\alpha]}$ and $\mu_\alpha^2 = \frac{\lambda - 2}{c_\alpha}$. In order for the small idempotents to exist, we assume that $\mathbb{F}$ contains the root $r$ of the quadratic equation. If this is not the case, we extend $\mathbb{F}$ to $\mathbb{F}(r)$.

The idempotents are particularly interesting when they lead to automorphisms of the algebra. As described in Section 2.1, if the idempotents are in fact axes for some graded fusion law then they are naturally associated to Miyamoto automorphisms. The situation where the code algebra is an axial algebra and the fusion law is $\mathbb{Z}_2$-graded was characterised in [1]. Hence, we shall assume the necessary conditions which guarantee the existence of such a $\mathbb{Z}_2$-grading. Before quoting the relevant results, we first recall some notation from [1].

Definition 2.7. Given $\beta \in C^*$, we define the weight partition to be the unordered pair

$$p(\beta) := (|\alpha \cap \beta|, |\alpha \cap (\alpha + \beta)|)$$
and let

$$C_\alpha(p) := \{ \beta \in C^* \setminus \{ \alpha, \alpha^c \} : p(\beta) = p \}$$

be the set of all \( \beta \) which give the weight partition \( p \). We define

$$P_\alpha := \{ p(\beta) : \beta \in C^* \setminus \{ \alpha, \alpha^c \} \}$$

to be the set of all weight partitions of \( \alpha \).

In order to define the eigenvalues and eigenvectors, we need to further define some scalars which will be their coefficients. For \( \beta \in C^* \setminus \{ \alpha, \alpha^c \} \), we define

$$\xi_\beta := \frac{\lambda a}{2\mu a_{\alpha,\beta}} (|\alpha| - 2|\alpha \cap \beta|) = \frac{1}{4\mu a_{\alpha,\beta}} \left( 1 - \frac{2|\alpha \cap \beta|}{|\alpha|} \right)$$

and let \( \theta_\beta^\pm \) be the two roots of

$$x^2 + 2\xi_\beta x - 1 = 0.$$

We may now state our assumptions for the rest of the paper.

**Hypothesis 1.**

1. Let \( C \) be a projective code and \( S = \{ \alpha_1, \ldots, \alpha_l \} \) be a set of conjugates of \( \alpha \in C^* \) which generate the code \( C \).

2. We assume that the structure parameters \( \Lambda \) are all non-zero,

\[
\begin{align*}
a &:= a_{i,\beta} \quad \text{for all } i \in \text{supp}(\beta), \beta \in C^* \\
b_{\alpha_j,\beta} &= b_{\alpha_k,\gamma} \quad \text{for all } \beta \in C_{\alpha_j}(p), \gamma \in C_{\alpha_k}(p), p \in P_\alpha \\
b_{\alpha^c_j,\beta} &= b_{\alpha^c_k,\gamma} \quad \text{for all } \beta \in C_{\alpha_j}(p), \gamma \in C_{\alpha_k}(p), p \in P_\alpha \\
c_\beta &:= c_{i,\beta} \quad \text{for all } i \in \text{supp}(\beta), \beta \in C^* \\
\end{align*}
\]

and \( a \neq \frac{1}{2|\alpha|}, \frac{1}{3|\alpha|} \).

3. Let \( \mathbb{F} \) be a field of characteristic \( p \neq 2 \) and not dividing \( |\alpha| \) (allowing \( p = 0 \)). Assume further that the following quadratics

\[
\begin{align*}
i. & \quad c_{\alpha_i} x^2 - \lambda + \lambda^2 = 0, \text{ and} \\
ii. & \quad x^2 + 2\xi_\beta x - 1 = 0, \text{ for all } \beta \in C^* \setminus \{ \alpha, \alpha^c \},
\end{align*}
\]

all split over \( \mathbb{F} \) and have distinct roots.

Note that 3(i) of the Hypothesis is precisely the condition required for the small idempotents to exist. In order to describe the eigenspaces of the small idempotents and for them to be axes, the roots of equations 3(ii) must be distinct. Note also that all the quadratics splitting over \( \mathbb{F} \) can be achieved by simply replacing the field of definition of \( A \) by the splitting field of the quadratics, which is a finite extension of \( \mathbb{F} \).
Defining

\[ \nu_p^\pm := \frac{1}{4} + \mu_a b_{\alpha, \beta}(\theta_{\pm}^\beta + \xi_\beta) \]

for \( p \in P_\alpha, \beta \in P_\alpha(p) \), we may now state the following.

**Theorem 2.8** ([1, Proposition 3.5, Theorem 4.1]). We assume Hypothesis 1. Then, \( e = e_{\alpha, +} \) is an axis with fusion law given in Table 2. It has eigenvalues and eigenvectors given in Table 1. Moreover, it is a primitive axis if \( \nu_p^\pm \neq 1 \) for all \( p \in P_\alpha \).

| Eigenvalue | Eigenvector |
|------------|-------------|
| 1          | \( e = \lambda t_\alpha + \mu_a e^\alpha \) |
| 0          | \( t_i \) for \( i \not\in \text{supp}(\alpha) \) |
| \( \lambda \) | \( t_j - t_k \) for \( k \in \text{supp}(\alpha), k \neq j \) |
| \( \lambda - \frac{1}{2} \) | \( 2\mu_a c_a t_\alpha - e^\alpha \) |
| \( \nu_p^\pm \) | \( w_\beta = \theta_{\pm}^\beta e^\beta + e^{\alpha+\beta} \) for \( \beta \in C_{\alpha}(p), p \in P_\alpha \) |

where \( j \in \text{supp}(\alpha) \) is fixed.

Table 1: Eigenspaces for small idempotents

Now that we know when a small idempotent \( e_{\alpha, \pm} \) is an axis, we can consider when a set of these generate the whole of the algebra.

Since we want the code algebra to be an axial algebra, in addition we need the fusion law of each of these axes to be the same. So we consider a set \( S = \{\alpha_1, \ldots, \alpha_l\} \) of conjugates of \( \alpha \) under the automorphism group which generate \( C \). This ensures that the weight partitions of each \( \alpha_i \) are the same. Hence, the fusion law \( F \) for each small idempotent is the same. However, the eigenvalues \( \varphi(\nu_p^\pm) \) may still depend on the value of \( c_a \). In [1, Theorem 5.1], we implicitly assumed that \( c_{\alpha_i} = c_{\alpha_j} \) for all \( \alpha_i, \alpha_j \in S \). In the next section, we show that in all but one small case this is a necessary assumption.

**Theorem 2.9** ([1 Theorem 5.1]). Assume Hypothesis 1. Suppose that for each small idempotent \( e \) the evaluation map \( \varphi_e : F \to \mathbb{F} \) for \( e \) equals \( \varphi : F \to \mathbb{F} \). Then, the code algebra \( A_{C}(\Lambda) \) is an \((F, \varphi)\)-axial algebra with respect to the small idempotents \( e_{\alpha_i, \pm}, i = 1, \ldots, l \), and has fusion law given in Table 2.

In this paper we are interested in calculating the associated Miyamoto groups. These are non-trivial if and only if the algebra is graded.
Table 2: Fusion law for small idempotents

\[
\begin{array}{cccccc}
1 & 0 & \lambda & \lambda - \frac{1}{2} & \nu_{p1}^+ & \ldots & \nu_{pk}^+ \\
0 & 0 & \lambda & \lambda - \frac{1}{2} & \nu_{p1}^+ & \ldots & \nu_{pk}^+ \\
\lambda & 1, \lambda, \lambda - \frac{1}{2} & \nu_{p1}^+, \nu_{p1}^- & \ldots & \nu_{pk}^+, \nu_{pk}^- \\
\lambda - \frac{1}{2} & \lambda - \frac{1}{2} & 1, \lambda - \frac{1}{2} & \nu_{p1}^+, \nu_{p1}^- & \ldots & \nu_{pk}^+, \nu_{pk}^- \\
& & & & & & \\
\nu_{p1}^+ & \nu_{p1}^+ & \nu_{p1}^+, \nu_{p1}^- & \nu_{p1}^+, \nu_{p1}^- & X_1 & N(p_1, p_k) \\
& & & & & & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
& & & & & & \\
\nu_{pk}^+ & \nu_{pk}^+ & \nu_{pk}^+, \nu_{pk}^- & \nu_{pk}^+, \nu_{pk}^- & N(p_k, p_1) & X_k \\
\end{array}
\]

where

\[ N(p, q) := \{ \nu_{p}^{(\beta+\gamma)}, \nu_{-}^{(\beta+\gamma)} : \beta \in C_{\alpha}(p), \gamma \in C_{\alpha}(q), \gamma \neq \beta, \alpha + \beta, \beta^c, \alpha + \beta^c \} \]

and \(X_i\) represents the table

| \(\nu_{p}^+\) | \(\nu_{p}^-\) |
|------------------|------------------|
| 1, 0, \lambda, \lambda - \frac{1}{2}, \nu_{p1}^+, N(p_1, p_i) | 1, 0, \lambda, \lambda - \frac{1}{2}, N(p_i, p_1) |
| 1, 0, \lambda, \lambda - \frac{1}{2}, \nu_{p1}^+, N(p_1, p_i) | 1, 0, \lambda, \lambda - \frac{1}{2}, N(p_i, p_1) |

\[ \text{Table 2: Fusion law for small idempotents} \]

**Theorem 2.10** ([1, Theorem 6.1]). Assume Hypothesis [1] and suppose \(A_C\) is a code algebra which is an axial algebra with respect to the small idempotents. Let \(\alpha \in C^*\), and define \(D := \text{proj}_\alpha(C)\). Then the fusion law of the small idempotents \(e_{\alpha, \pm}\) is \(\mathbb{Z}_2\)-graded if and only if

1. \(|\alpha| = 1, C = F_2^n, \) and
   
   \(\text{(a) } n = 2, \alpha = -1.\)
   
   \(\text{(b) } n = 3.\)

2. \(|\alpha| = 2 \) and \(C = \bigoplus C_i\) is the direct sum of even weight codes all of the same length \(m \geq 3.\)

3. \(|\alpha| > 2 \) where \(D\) is a projective code, \(1 \in D\) and there exists a codimension one linear subcode \(D_+\) of \(D\) such that \(1 \in D_+\) and \(D_+\) is the union of weight sets of \(D\).
In cases (2) and (3), we have

\[ A_+ = A_1 \oplus A_0 \oplus A_\lambda \oplus A_{\lambda-1} \oplus \bigoplus_{m \in \text{wt}(D_+)} A_{\nu^+(m,|\alpha|-m)} \]

\[ A_- = \bigoplus_{m \in \text{wt}(D)-\text{wt}(D_+)} A_{\nu^-(m,|\alpha|-m)} \]

where in case (2), \( D = \mathbb{F}_2^2 \) and \( D_+ = \{0,1\} \).

Note that in the second case, provided we make some further assumptions on the structure parameters, the fusion law is in fact \( \mathbb{Z}_2 \times \mathbb{Z}_2 \)-graded, which leads to further Miyamoto automorphisms. This situation will be described later where we calculate the associated Miyamoto group.

3 Structure parameters

Suppose that \( A \) is an axial algebra with a \( \mathbb{Z}_2 \)-graded fusion law which we know to also be a code algebra coming from a projective code \( C \) and the axes \( X \) are a set of small idempotents. Can we recover the code algebra structure? That is, we do not assume that the code, or the special basis of the toral and codeword elements are known.

We say that a set \( X \) of small idempotents is pair-closed if whenever \( \lambda t_\alpha + \mu e_\alpha \in X \), then \( \lambda t_\alpha - \mu e_\alpha \in X \) and vice versa.

**Theorem 3.1.** Suppose that \( A \) is a code algebra which is also an axial algebra with respect to a pair-closed set \( X \) of small idempotents. Then, we can recover the code \( C \), up to permutation equivalence, and also the special basis \( \{ t_i : i = 1, \ldots, n \} \cup \{ e_\alpha : \alpha \in C^* \} \).

Note that in an axial algebra we can easily calculate the eigenspaces given any axis. However, if the evaluation map \( \varphi \) is not injective, then we may not be able to calculate the parts. In particular, it may be the case that for distinct \( x, y \in \mathcal{F} \) \( \varphi(x) = \varphi(y) \) and \( x \star z = y \star z \) for all \( z \in \mathcal{F} \). Then, the \( x \)- and \( y \)-parts would be indistinguishable from the algebra structure.

**Corollary 3.2.** Suppose that \( A \) is a code algebra which is also an axial algebra with respect to a pair-closed set \( X \) of small idempotents. The parts \( x \in \mathcal{F} \) are distinguishable.

**Proof.** By Theorem 3.1, we know the special basis and hence the parts. ◇

We prove Theorem 3.1 via a series of lemmas.

First we reduce the problem to partitioning the set \( X \) into pairs of idempotents that come from the same codeword.

Suppose that we have such a partition; our pairs are \( \{ \lambda t_\alpha \pm \mu e_\alpha \} \) with \( \alpha \in S \) for some set \( S \). We stress that the \( \alpha \) in \( S \) are just formal labels as
the code is not yet known. We show that we can indeed recover the code $C$ from this.

**Proposition 3.3.** Suppose that we can partition the set $X$ of axes into pairs $\{\lambda t_\alpha \pm \mu_\alpha e^\alpha\}$ with $\alpha \in S$ for some set $S$. Then, we can recover the code $C$, up to permutation equivalence, and we know the the special basis $\{t_i : i = 1, \ldots, n\} \cup \{e^\alpha : \alpha \in C\}$.

**Proof.** Given each pair $e_{\alpha,\pm} = \lambda t_\alpha \pm \mu_\alpha e^\alpha$, we add and subtract $t_\alpha$ and $e^\alpha$. Since $X$ generates the algebra, $\{t_\alpha, e^\alpha : \alpha \in S\}$ also generates the algebra.

Let $T = \{t_i : i = 1, \ldots, n\}$ be the set of toral elements. Since by assumption $S$ generates the code and the code is projective, we may multiply the $t_\alpha$ to get any $t \in T$. Explicitly, we know that the code we are trying to reconstruct is projective. Hence, for any $i \in 1, \ldots, n$, there exist a subset of codewords $Y \subseteq S$ such that $\{i\} = \bigcap_{\beta \in Y} \text{supp}(\beta)$. Note that $t_\alpha t_\beta = t_{\alpha \cap \beta}$ and so multiplication of the $t_\alpha$ corresponds to intersection of the corresponding codewords. Hence $\langle \langle t_\alpha : \alpha \in S \rangle \rangle = \langle \langle T \rangle \rangle$. The unique set of pairwise annihilating idempotents in $\langle \langle T \rangle \rangle$ is $T$. Note however, that since $S$ is just a set of formal labels, we can only recover $T$ up to a permutation of $\{1, \ldots, n\}$.

Similarly, we can construct each $e_\beta$ by multiplying appropriate $e^\alpha$ together. Hence, we can recover the special basis. Therefore we know the code $C$, as its additive structure is the multiplicative structure of the set $\{e_\beta : \beta \in C\}$. Note however, as argued above with the $t_i$, we only know $C$ up to a permutation action on the columns. \qed

We now show that we can indeed find the pairs.

**Lemma 3.4.** Given $\lambda t_\alpha + \mu_\alpha e^\alpha \in X$, we can identify $\lambda t_\alpha - \mu_\alpha e^\alpha \in X$ provided we are not in the case where $1 \in C$, $|\alpha| = \frac{n}{2}$, $a = \frac{2}{n} = \frac{1}{|\alpha|}$, $\lambda = \frac{1}{2}$ and $\lambda t_\alpha - \mu_\alpha e^\alpha \in X$.

**Proof.** We consider subalgebras generated by pairs of axes and study properties of this subalgebra. First consider the case that our pair is $\{\lambda t_\alpha \pm \mu_\alpha e^\alpha\}$ as wanted. Note that

$$(\lambda t_\alpha + \mu_\alpha e^\alpha)(\lambda t_\alpha - \mu_\alpha e^\alpha) = (\lambda - \mu_\alpha c_\alpha) t_\alpha$$

$$= \lambda(2\lambda - 1) t_\alpha$$

which is non-zero when $\lambda \neq \frac{1}{2}$ or equivalently $a \neq \frac{1}{|\alpha|}$. So in all cases $\langle \langle \lambda t_\alpha \pm \mu_\alpha e^\alpha \rangle \rangle$ is 2-dimensional, but in particular the two axes are not mutually orthogonal unless $\lambda \neq \frac{1}{2}$.

Suppose that our possible pair is $\{\lambda t_\alpha + \mu_\alpha e^\alpha, \lambda t_\beta + \mu_\beta e^\beta\}$ for some $\beta \in C^* - \{\alpha, \alpha^c\}$. Then,

$$(\lambda t_\alpha + \mu_\alpha e^\alpha)(\lambda t_\beta + \mu_\beta e^\beta) = \lambda^2 t_{\alpha \cap \beta} + \lambda a|\alpha \cap \beta|(\mu_\alpha e^\alpha + \mu_\beta e^\beta) + \mu_\alpha \mu_\beta b_{\alpha,\beta} e^{\alpha^c + \beta}$$

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In particular, since \( \mu_\alpha \mu_\beta \beta_{\alpha,\beta} \neq 0 \) and \( \alpha + \beta \neq \alpha, \beta \), the above is not in the span of \( \{ \lambda \mu_\alpha + \mu_\alpha e^\alpha, \lambda \mu_\beta + \mu_\beta e^\beta \} \). Hence, the dimension of \( \langle \lambda \mu_\alpha + \mu_\alpha e^\alpha, \lambda \mu_\beta + \mu_\beta e^\beta \rangle \) is strictly bigger than two.

Finally, consider the case where the pair is \( \{ \lambda \mu_\alpha + \mu_\alpha e^\alpha, \lambda \mu_\beta + \mu_\beta e^\beta \} \). These are mutually orthogonal idempotents and hence generate a subalgebra of dimension two. Hence, unless \( 1 \in C \), \( |\alpha| = \frac{n}{2} \) and \( \lambda = \frac{1}{2} \), we can correctly pair partition \( X \) into pairs \( \{ \lambda \mu_+ \pm \mu_\alpha e^\alpha \} \).

So, we can reduce to the case where \( 1 \in C \), \( |\alpha| = \frac{n}{2} \), \( a = \frac{2}{n} = \frac{1}{|\alpha|} \) and \( \lambda = \frac{1}{2} \). In this case, \( \varphi(\lambda - \frac{1}{2}) = 0 \) and so both the 0- and \( (\lambda - \frac{1}{2}) \)-parts are in the 0-eigenspace. Note however, that there may be additional parts in it too.

Given \( e_+ := \lambda \mu_\alpha + \mu_\alpha e^\alpha \) it remains to distinguish between \( e_- := \lambda \mu_\alpha + \mu_\alpha e^\alpha \) and \( e_+^\pm := \lambda \mu_\alpha \pm \mu_\alpha e^\alpha \). To do this, we study the 0-eigenspace of each axis.

Suppose that the 0-eigenspace for \( e_+ \) is just the union of the 0-part and the \( (\lambda - \frac{1}{2}) \)-part. Since we assume that each axis has the same fusion law, this is also true for the other axes. The 0-eigenspace with respect to \( e_+ \) is

\[
\langle i, e^\alpha, 2\mu_\alpha c_\alpha t_\alpha - e^\alpha : i \notin \text{supp}(\alpha) \rangle
\]

which has dimension \( \frac{n}{2} + 2 \). Moreover, it has intersection with the 0-eigenspace of \( e_- \) of dimension \( \frac{n}{2} + 1 \) since the 0-part with respect to \( e_+ \) and \( e_- \) coincide. However, its intersection with the 0-eigenspace of \( e_+^\pm \) is the \( (\lambda - \frac{1}{2}) \)-part of \( e_+^\pm \) which has dimension 1. Hence, in this case, we can distinguish between \( e_- \) and \( e_+^\pm \).

So now we suppose that \( \varphi(\nu_\delta) = 0 \) for some \( p \in P_\alpha \) and \( \varepsilon = \pm 1 \). It is now much more difficult to study the intersection of the 0-eigenspaces. Indeed, the set of codewords \( C_\phi(\alpha) \) which intersect \( \alpha \) in the weight partition \( p \) may have many different weight partitions in their intersection with \( \alpha^\varepsilon \).

So, we now employ a different strategy.

Note that since \( e_- \) is in \( A_{1}(e_+) \oplus A_{\frac{1}{2}}(e_+) \), but \( e_+^\pm \) is not, we may reduce the problem to identifying the \( \lambda - \frac{1}{2} \)-part.

**Lemma 3.5.** If the \( \lambda \)-part of \( e_+ \) is known, then the \( \lambda - \frac{1}{2} \)-part is known.

**Proof.** Since \( \lambda * \lambda = \{ 1, \lambda, \lambda - \frac{1}{2} \} \) and the 1- and \( \lambda \)-parts are known, the \( \lambda - \frac{1}{2} \)-part is the intersection of the 0-eigenspace with \( A_{\lambda * \lambda} \).

Clearly, if the \( \frac{1}{2} \)-eigenspace is just the \( \lambda \)-part, then it is distinguished. The only option for another part \( x \) such that \( \varphi(x) = \frac{1}{2} \) is if \( \varphi(\nu_q^\ell) = \frac{1}{2} \) for some \( q \in P_\alpha \) and \( \ell = \pm 1 \). In order to identify the \( \lambda \)-part, we study pairs \( (0, \bar{\lambda}) \) where \( 0 \) and \( \bar{\lambda} \) are each unions of parts which are candidates for the 0-part and \( \lambda \)-part respectively.
**Lemma 3.6.** If the real 0-part is in 0, then the only candidate for \( \tilde{\lambda} \) is \( \lambda \) and hence the \( \lambda \)-part is distinguished.

**Proof.** We know from the fusion law that we must have 0 \( \star \tilde{\lambda} = \emptyset \). However, since 1 \( \in C \), an easy calculation ([I, Lemma 4.6]) shows that, \( \emptyset \neq \emptyset \star \nu^r = \nu^r \), for all \( r \in P_\alpha \). So \( \nu^0 \) cannot be in \( \tilde{\lambda} \). Hence \( \tilde{\lambda} = \lambda \) and so the \( \lambda \)-part is distinguished. \( \square \)

**Lemma 3.7.** The \( \lambda \)-part is distinguished.

**Proof.** Suppose that the \( \lambda \)-part is not distinguished. Then \( \nu^q \in \tilde{\lambda} \) for some \( q \in P_\alpha \) and \( \nu = \pm 1 \). By Lemma 3.6, the 0-part is not in 0. So we must have \( \nu^p \in 0 \) for some \( p \in P_\alpha \), \( \nu = \pm 1 \) and the 0-part in \( \nu^p \), our candidate for the \( \nu^p \)-part.

From the fusion law, a straightforward calculation ([I, Lemma 4.9]) shows that we require

\[
\tilde{\lambda} \star \nu^p = \begin{cases} 
\emptyset & \text{if } p = (0,|\alpha|) \\
\nu^p_{\nu} & \text{if } \xi_\beta = 0 \text{ where } p(\beta) = p \\
\nu^p_+ \pm \nu^p_- & \text{otherwise}
\end{cases}
\]

However, \( \nu^q \star 0 = \nu^q \). That is, \( \tilde{\lambda} \in \tilde{\lambda} \star \nu^p_\nu \), a contradiction. \( \square \)

Therefore, given any \( e_+ \) we can always pair it with \( e_- \). So, we have the following.

**Corollary 3.8.** We can partition \( X \) into pairs \( \{ \lambda t_\alpha \pm \mu_\alpha e^\alpha \} \).

This completes the proof of Theorem 3.1.

We now turn our attention to the \( c \) structure constants in the algebra.

**Theorem 3.9.** Suppose that \( AC \) is a \( \mathbb{Z}_2 \)-graded axial algebra with respect to the small idempotents corresponding to \( S \) and \( C \neq F^2 \). Then, \( c := c_\alpha \) for all \( \alpha \in S \).

**Proof.** Consider the graph on \( S \) with edges \( \alpha \sim \beta \) if \( \beta \neq \alpha^c \). Since \( C \) is projective and \( C \neq F^2 \), this graph is connected. Hence, it suffices to show that \( c_\alpha = c_\beta \) for \( \alpha, \beta \in S \) with \( \beta \neq \alpha^c \).

We fix notation. Let \( \nu^p(\alpha) \) denote the \( \nu^p \)-part with respect to the small idempotent associated to \( \alpha \). Similarly, for \( \xi_\beta(\alpha), \theta^0_\epsilon(\alpha) \) and \( \nu^p(\alpha) \).

By Theorem 3.1, the parts of \( \lambda t_\alpha + \mu_\alpha e^\alpha \) and \( \lambda t_\beta + \mu_\beta e^\beta \) are known. Let \( p = p(\beta) \) be the weight partition of \( \beta \) with respect to \( \alpha \). In other words, \( \beta \in C_p(\alpha) \). Then, \( \alpha \in C_p(\beta) \) for the same weight partition \( p \). So \( \nu^p(\alpha) \) and \( \nu^p(\beta) \) both have the same eigenvalue.

\[
0 = \varphi(\nu^p(\alpha)) - \varphi(\nu^p(\beta)) = b_{\alpha,\beta} \left( \mu_\alpha \theta^0_\epsilon(\alpha) - \mu_\beta \theta^0(\beta) + \mu_\alpha \xi_\beta(\alpha) - \mu_\beta \xi_\alpha(\beta) \right)
\]
Note that $b_{\alpha,\beta} \neq 0$. We split into two cases: either $|\alpha| - 2|\alpha \cap \beta| = 0$ in $F$, or not.

Suppose that $|\alpha| - 2|\alpha \cap \beta| = 0$. By [1, Lemma 3.3], this holds if and only if $\xi_\beta(\alpha) = 0$ which is equivalent to $\theta_+^\beta(\alpha) = \pm 1$. In particular, we may choose our labelling so that $\theta_+^\beta(\alpha) = 1$. Note that $|\beta| - 2|\alpha \cap \beta| = 0$ too and so we have the analogous result with respect to $\beta$. In this case, Equation 1 yields $\mu_\alpha = \mu_\beta$ and hence $c_\alpha = c_\beta$.

Now, suppose that $|\alpha| - 2|\alpha \cap \beta| = |\beta| - 2|\alpha \cap \beta| \neq 0$. By rearranging the equations for $\xi_\beta(\alpha)$ and $\xi_\alpha(\beta)$ and combining, we obtain

$$\xi_\beta(\alpha) = \frac{\mu_\beta}{\mu_\alpha} \xi_\alpha(\beta)$$

So Equation 1 yields $\theta_+^\beta(\alpha) = \frac{\mu_\beta}{\mu_\alpha} \theta_+^\alpha(\beta)$. We write $\omega := \frac{\mu_\beta}{\mu_\alpha}$. By definition, $\theta_\pm^\beta(\alpha)$ are the two solutions to $x^2 + 2\xi_\alpha(\beta)x - 1 = 0$

So, $\theta_\pm^\beta(\alpha) = \omega \theta_\pm^\alpha(\beta)$ are the two solutions to

$$x^2 + 2\omega\xi_\alpha(\beta)x - \omega^2 = x^2 + 2\xi_\beta(\alpha)x - \omega^2 = 0$$

However, $\theta_\pm^\beta(\alpha)$ are already the two solutions to $x^2 + 2\xi_\beta(\alpha)x - 1 = 0$ and hence $\omega^2 = 1$. That is, $\mu_\alpha = \mu_\beta$ (again using choices of sign for roots) and hence $c_\alpha = c_\beta$.

In particular, except for case (1a) where $C = F_2^2$ and $a = -1$, all the $\mathbb{Z}_2$-graded algebras in Theorem 2.10 have $c_\alpha := c_{\alpha_i} = c_{\alpha_j}$ for $\alpha_i, \alpha_j \in S$ and hence $\mu := \mu_{\alpha_i} = \mu_{\alpha_j}$ also.

4 Automorphisms

In this section, we calculate the automorphism groups for each of the $\mathbb{Z}_2$-graded algebras in Theorem 2.10. Throughout this section, let $S$ be a set of codewords and $X$ be the corresponding pair-closed set of small-idempotents. First note that, in all but case (1a), the structure constant is the same for all $\alpha \in S$ and we write $c_\alpha$ for this. We also write $\mu := \mu_\alpha$ when $C \neq F_2^2$.

For cases (2) and (3) in Theorem 2.10, the code $D = \text{proj}_\alpha(C)$ has a codimension one subcode $D_-$ which is the sum of weight sets and $1 \in D_+$. Let $D_- := D \setminus D_+$. Then in both these cases, the negatively graded part is given by

$$A_- = \bigoplus_{m \in \text{wt}(D_-)} A_{\mu,|m| - m}$$

Since $m \in \text{wt}(D_\pm)$ if and only if $|\alpha| - m \in \text{wt}(D_\pm)$, we abuse notation and just write $p(\beta) \in \text{wt}(D_\pm)$ for $m \in \text{wt}(D_\pm)$, where $p(\beta) = (m, |\alpha| - m)$. We call this the standard case.
As noted, in case (2), provided we choose the structure constants in a ‘nice’ way, the $\mathbb{Z}_2$-grading extends to a $\mathbb{Z}_2 \times \mathbb{Z}_2$-grading. In this situation, we also describe the extra Miyamoto automorphisms obtained in addition to those in the standard case.

The graded parts in cases (1a) and (1b) in Theorem 2.10 are different from the other two cases. Here the negative part is not described by a codimension one code, so we must deal with these cases separately.

### 4.1 The standard case

In this case, $|\alpha| \geq 2$ and let $e = e_{\alpha, \pm}$.

**Proposition 4.1.** The action of $\tau_e$ on $A$ is given by

\[
t_i \mapsto t_i
\]

\[
e^\beta \mapsto \begin{cases} e^\beta & \text{if } p(\beta) \in \text{wt}(D_+) \\ -e^\beta & \text{if } p(\beta) \in \text{wt}(D_-) \end{cases}
\]

**Proof.** By definition 1.0, $\lambda - \frac{1}{2}$ and $\lambda$ (where it exists) are in the positive part of the grading. Since $t_i$, $e^\alpha$ and $e^{\alpha c}$ are all in the span of $A_+$, they are all fixed by $\tau_e$. Now consider $\beta \in C^* \setminus \{\alpha, \alpha c\}$. We may write

\[
(\theta^\beta_+ - \theta^\beta_-)e^\beta = (\theta^\beta_+ e^\beta + e^{\alpha + \beta}) - (\theta^\beta_- e^\beta + e^{\alpha + \beta}) = w_+^\beta - w_-^\beta
\]

Recall that we assume in Hypothesis 1 that $\theta^\beta_+ \neq \theta^\beta_-$. Since the grading of $\nu^\beta_\pm$ depends only on the partition $p$ and not on $\pm$ and $\tau_e$ negates the $w_\pm^\beta$ for $\beta$ such that $p(\beta)$ is graded negatively, the result follows.

**Corollary 4.2.** The automorphism $\tau_e$ does not depend on the value of the structure constants.

Recall that for a given $\alpha \in C$, we have two different idempotents $e_\pm := \lambda t_\alpha \pm \mu e^\alpha$.

**Corollary 4.3.** $\tau_{e_+} = \tau_{e_-}$.

**Proof.** By Proposition 4.1, we need only consider the $e^\beta$. In particular, whether or not $e^\beta$ is in the positive or negative part depends only on its intersection with $\alpha$, not the value of $\mu$.

In particular, $A$ is an axial algebra where the $\tau$-map, which is the map on the axes given by $a \mapsto \tau_a$, is not bijection. It is, in general, a 2-fold cover.

In light of the above result, we will write $\tau\alpha$ for $\tau_{e_\pm}$, where $e = \lambda t_\alpha \pm \mu e^\alpha$.

We may now consider the Miyamoto group in the case where the grading is a $\mathbb{Z}_2$-grading, but not a $\mathbb{Z}_2 \times \mathbb{Z}_2$-grading.
Corollary 4.4. In the standard case, the Miyamoto group is an elementary abelian 2-group of order at most $2^{|S|}$.

Proof. Consider the decompositions into positive and negative parts for each Miyamoto involution $\tau_e$, $e \in X$. By the above proposition, we see that the standard basis of $t_i$, $i = 1, \ldots, n$ and $e^\alpha$, $\alpha \in C^*$ is a refinement of the intersection of all the decompositions given by the gradings for each Miyamoto involution $\tau_e$. In particular, since each $\tau_e$ acts as $\pm 1$ on each basis element, it is clear that any two Miyamoto involutions commute and hence $G$ is abelian. Since all the $\tau_e$ have order two, $G$ is an elementary abelian 2-group. By Corollary 4.3, it is clear that there are at most $|S|$ generators, hence the order is at most $2^{|S|}$.

In the case where $|\alpha| = 2$, we can say more. In this case, $C = \bigoplus_{i=1}^r C_i$ is the direct sum of even weight codes $C_i$ all of length $m$.

Corollary 4.5. Let $|\alpha| = 2$ and so $C = \bigoplus_{i=1}^r C_i$, where $C_i$ has length $m$. Then the Miyamoto group is

$$\langle \tau_\alpha : \alpha \in S \rangle \cong \begin{cases} 2^r(m-1) & \text{if } m \text{ is odd} \\ 2^r(m-2) & \text{if } m \text{ is even} \end{cases}$$

Proof. First, by Corollary 4.4, $\tau_\alpha$ and $\tau_\beta$ commute, where $\alpha, \beta \in S$ and $N$ is an elementary 2-group of order at most $2^{|S|}$. We consider the case where $C$ is an irreducible code; the general case follows immediately from this.

Define a map $\varphi : C \to N$ by $\alpha \mapsto \tau_\alpha$ for $\alpha \in S$ and extend linearly.

We first show that $\varphi$ is a well-defined homomorphism. Let $\alpha \in C$ and decompose $\alpha$ in two different ways as $\alpha = \alpha_1 + \cdots + \alpha_k$ and $\alpha = \beta_1 + \cdots + \beta_l$, where $\alpha_i, \beta_j \in S$. We compare the actions of $\tau_{\alpha_1} \cdots \tau_{\alpha_k}$ and $\tau_{\beta_1} \cdots \tau_{\beta_l}$. Both fix all the $t_i$. The codeword element $e^\gamma$ is inverted by some $\tau_\delta$, $\delta \in S$, if and only if $|\gamma \cap \delta| = 1$. Now, if $|\gamma \cap \alpha|$ equals 0, respectively 1, then $\gamma$ intersects an even, respectively an odd, number of $\alpha_i$. However, the same is true for the $\beta_j$. Hence $\tau_{\alpha_1} \cdots \tau_{\alpha_k} = \tau_{\beta_1} \cdots \tau_{\beta_l}$ and $\varphi$ is a well-defined homomorphism.

Since $\varphi$ is a homomorphism, $N$ has order at most $2^{\dim(C)}$. Suppose that $\tau_\alpha = 1$ for some $\alpha \in C \setminus \{1\}$. Then every $\beta$ must intersect $\alpha$ in a set of size 0. Observe that the parity of the intersection of two codewords is equal to their inner product. It is clear that the only possible vector which has inner product 0 with every codeword in an irreducible even weight code is $1$. However, $1 \in C$ if and only if $m$ is even. So, when $m$ is even, the kernel of $\varphi$ is $\langle 1 \rangle$, otherwise it is trivial.

Remark 4.6. Note that the above proof does not hold when $|\alpha| \neq 2$. When $|\alpha| \neq 2$, the map $\varphi$ may not be well-defined.

Lemma 4.7. The orbits of $G(X)$ on $X$ are the pairs $\{e_{\alpha,+}, e_{\alpha,-}\}$ for $\alpha \in S$. In particular, the set of axes $X = \{e_{\alpha,\pm} : \alpha \in S\}$ is closed.
Proof. Let $\alpha, \beta \in S$. By Lemma 4.1, $\tau_\beta$ fixes $e_{\alpha,+}$ and $e_{\alpha,-}$ if $p(\alpha) \in D_+$ and swaps $e_{\alpha,+}$ and $e_{\alpha,-}$ if $p(\alpha) \in D_-$. \hfill \Box

4.2 $|\alpha| = 1$

We now deal with the case where $|\alpha| = 1$ as this behaves differently to the standard case. There are two subcases here where $A$ is $\mathbb{Z}_2$-graded. Either $n = 3$, or $n = 2$ and $a = -1$.

4.2.1 $n = 2$ and $a = -1$

Note that, as $C \cong \mathbb{F}_2^2$, $A$ decomposes as:

$$A = A_1 \oplus A_2 = \langle t_1, e^{\alpha_1} \rangle \oplus \langle t_2, e^{\alpha_2} \rangle$$

In particular, there are four small idempotents

$$e_{i,\pm} := \lambda t_i \pm \mu_i e^{\alpha_i} \in A_i$$

where $\lambda = \frac{1}{2|\alpha_i|} = -\frac{1}{2}$ and $\mu_i^2 = \frac{\lambda - \lambda^2}{c_{\alpha_i}} = -\frac{3}{4c_{\alpha_i}}$ for $i = 1, 2$. However, since this is the exception to Theorem 3.9, $c_{\alpha_1}$ and $c_{\alpha_2}$ may differ, and so $\mu_1$ and $\mu_2$ need not be the same either. In fact, this does produce a valid $\mathbb{Z}_2$-graded algebra with a non-trivial Miyamoto group.

The fusion law has three parts $1, 0$ and $\lambda - \frac{1}{2}$, where $\lambda - \frac{1}{2}$ is in the negatively graded part and $1$ and $0$ are in the positive part. Since $\lambda = \frac{1}{2}$, $\varphi(\lambda - \frac{1}{2}) = -1$ and all the parts are distinct. Note that the $(\lambda - \frac{1}{2})$-part is 1-dimensional and is spanned by $2\mu_i c_{\alpha_i} t_i - e^{\alpha_i}$. We write $\tau_{i,\pm}$ for $\tau_{e_{i,\pm}}$.

Lemma 4.8. Let $i, j = 1, 2$ with $i \neq j$. The automorphism $\tau_{i,\pm}$ acts trivially on $A_j$ and acts on $A_i = \langle t_i, e^{\alpha_i} \rangle$ as

$$\begin{pmatrix}
-\frac{1}{2} & \mp \mu_i \\
\mp \frac{3}{4\mu_i} & \frac{1}{2}
\end{pmatrix}$$

Proof. We have $\langle \lambda t_i + \mu_i e^{\alpha_i}, 2\mu_i c_{\alpha_i} t_i - e^{\alpha_i} \rangle = \langle t_i, e^{\alpha_i} \rangle = A_i$ and so $A_i$ is spanned by the $1$- and $-1$-eigenspaces. In particular, $\tau_{i,\pm}$ must act trivially on $A_j$. One can check that $\tau_{i,\pm}$ acts on $A_i$ as given by the matrix above by checking the action on the two vectors $\lambda t_i + \mu_i e^{\alpha_i}$ and $2\mu_i c_{\alpha_i} t_i - e^{\alpha_i}$. \hfill \Box

Note that in this case, unlike the Miyamoto involution in the standard case, $\tau_{i,\pm}$ does depend on $\mu_i$ and hence on the structure constant $c_{\alpha_i}$. Also, $\tau_{i,+} \neq \tau_{i,-}$.

Lemma 4.9. The subgroup $G_i := \langle \tau_{i,+}, \tau_{i,-} \rangle \cong S_3$. 

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Proof. The automorphisms $\tau_{i,+}$ and $\tau_{i,-}$ are involutions so they generate a dihedral group. Calculating the action of $\tau_{i,+}\tau_{i,-}$ on $A_i$, we have
\[
\begin{pmatrix}
-\frac{1}{2} & -\mu_i \\
\frac{3}{2\mu_i} & \frac{1}{2}
\end{pmatrix}
\begin{pmatrix}
-\frac{1}{2} & \mu_i \\
\frac{3}{2\mu_i} & \frac{1}{2}
\end{pmatrix}
= \begin{pmatrix}
-\frac{1}{2} & -\mu_i \\
\frac{3}{2\mu_i} & -\frac{1}{2}
\end{pmatrix}
\]
It is straightforward to check that this has order three. \hfill \square

Note that although the action of $G_i$ on the space $A_i$ does depend on $\mu_i$ and so on the structure constant $c_{\alpha,i}$, the isomorphism type of the group $G_i$ does not.

Corollary 4.10. The Miyamoto group is isomorphic to $S_3 \times S_3$.

Lemma 4.11. The orbit of $e_{i,\pm}$ under $G_i$ is \{ $t_i, e_{i,+}, e_{i,-}$ \}. Hence the closure of $\{ e_{i,\pm} : i = 1, 2 \}$ is $\{ t_1, t_2 \} \cup \{ e_{i,\pm} : i = 1, 2 \}$.

Proof. By Lemma 4.8, it is easy to see that the involution $\tau_{i,\pm}$ maps $t_i$ to $e_{i,\mp}$. Since by definition it fixes $e_{i,\pm}$ and $G_i \cong S_3$, the orbit of $e_{i,\pm}$ is \{ $t_i, e_{i,+}, e_{i,-}$ \}. \hfill \square

4.2.2 $n = 3$

In this case, $e$ has five parts, 1, 0, $\lambda - \frac{1}{2}$ and $\nu_+^0$, where 0 = (0, 1) is the only partition. Here $\nu_+^0$ are in the negative part and 1, 0, $\lambda - \frac{1}{2}$ are in the positive part.

Lemma 4.12. We have $\tau_{e,+} = \tau_{e,-}$ and the Miyamoto group is the Klein four group $2^2$.

Proof. In a similar way to Proposition 4.1, we see that $\tau_{\alpha,\pm}$ acts trivially on the $t_i$ and on $e^\alpha$ and $e^{\alpha^c}$ and inverts all the $e^\beta$ where $\beta \in C^* - \{ \alpha, \alpha^c \}$. Hence we have $\tau_{\alpha,+} = \tau_{\alpha,-}$.

Let $\alpha_i$ be the codeword with a 1 in the $i^{th}$ position and 0s elsewhere. So, $S = \{ \alpha_i : i = 1, 2, 3 \}$. Note that $C^* = S \cup S^c$, where $S^c = \{ \alpha^c : \alpha \in S \}$.

Let $\{ i, j, k \} = \{ 1, 2, 3 \}$. Now, $\tau_{\alpha_i}$ inverts all $e^\beta$ where $\beta \in C^* - \{ \alpha_i, \alpha_i^c \}$. In particular, nothing is fixed by both $\tau_{\alpha_i}$ and $\tau_{\alpha_j}$ and only $e^{\alpha_k}$ and $e^{\alpha_k^c}$ are inverted by both. In other words, $\tau_{\alpha_i}\tau_{\alpha_j}$ inverts all $e^\beta$ where $\beta \in C^* - \{ \alpha_k, \alpha_k^c \}$. Hence $\tau_{\alpha_i}\tau_{\alpha_j} = \tau_{\alpha_k}$ and the Miyamoto group is a Klein four group. \hfill \square

Lemma 4.13. The orbits of $G(X)$ on $X$ are the pairs $\{ e_{\alpha,+}, e_{\alpha,-} \}$ for $\alpha \in S$. In particular, the set of axes $X = \{ e_{\alpha,\pm} : \alpha \in S \}$ is closed.

Proof. The proof is analogous to the proof of Lemma 4.7. \hfill \square
4.3 The $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded case

In this case, $|\alpha| = 2$ and $C := \bigoplus_{i=1}^r C_i$, where the $C_i$ all have length $m$. From the previous paper we have:

**Proposition 4.14.** \cite[Proposition 5.10]{1} Let $C = \bigoplus_{i=1}^r C_i$ be the direct sum of even weight codes $C_i$ all of length $m$, $n = m^r$ such that $n \geq 5$ and $m \geq 3$. Assume Hypothesis \cite[I]{$|\alpha| = 2$} and further suppose that $b_{\beta,\gamma} = b_{\alpha_i+\beta,\gamma}$ and $c_\beta = c_{\alpha_i+\beta}$ for all $\beta \in C_\alpha(1)$, $\alpha_i \in S$ and $\gamma \in C^* \setminus \{\alpha, \alpha^c\}$. Then, the axial algebra $A_C$ has fusion law given by Table 3 and has a $\mathbb{Z}_2 \times \mathbb{Z}_2$-grading given by

\[
A_{0,0} = A_1 \oplus A_0 \oplus A_{\lambda - \frac{1}{2}} \oplus A_{\nu_0^\alpha} \oplus A_{\nu_0^\beta} \\
A_{1,0} = A_{\nu_0^\alpha} \\
A_{0,1} = A_{\nu_0^\beta} \\
A_{1,1} = A_\lambda
\]

|       | 1   | 0   | $\lambda$ | $\lambda - \frac{1}{2}$ | $\nu_+^1$ | $\nu_-^1$ | $\nu_+^0$ | $\nu_-^0$ |
|-------|-----|-----|-----------|-----------------|----------|----------|----------|----------|
| 1     | 1   | $\lambda$ | $\lambda - \frac{1}{2}$ | $\nu_+^1$ | $\nu_-^1$ | $\nu_+^0$ | $\nu_-^0$ |
| 0     | 0   | $\lambda$ | $1, \lambda - \frac{1}{2}$ | $\nu_+^1$ | $\nu_-^1$ | $\nu_+^0$ | $\nu_-^0$ |
| $\lambda$ | $\lambda$ | $1, \lambda - \frac{1}{2}$ | $\nu_+^1$ | $\nu_-^1$ | $\nu_+^0$ | $\nu_-^0$ |
| $\nu_+^1$ | $\nu_+^1$ | $\nu_-^1$ | $\nu_+^0$ | $\nu_-^0$ | $\nu_+^0$ | $\nu_-^0$ |
| $\nu_-^1$ | $\nu_+^1$ | $\nu_-^1$ | $\nu_+^0$ | $\nu_-^0$ | $\nu_+^0$ | $\nu_-^0$ |
| $\nu_+^0$ | $\nu_+^0$ | $\nu_-^0$ | $\nu_+^0$ | $\nu_-^0$ | $\nu_+^0$ | $\nu_-^0$ |
| $\nu_-^0$ | $\nu_+^0$ | $\nu_-^0$ | $\nu_+^0$ | $\nu_-^0$ | $\nu_+^0$ | $\nu_-^0$ |

where $X = 1, 0, \lambda - \frac{1}{2}, \nu_0^\alpha, \nu_0^\beta$

|       | $\lambda - \frac{1}{2}$ | $1, \lambda - \frac{1}{2}$ | $\nu_+^1$ | $\nu_-^1$ | $\nu_+^0$ | $\nu_-^0$ |
|-------|-----------------|-----------------|----------|----------|----------|----------|
| $\nu_+^1$ | $\nu_+^1$ | $\nu_-^1$ | $\nu_+^0$ | $\nu_-^0$ | $\nu_+^0$ | $\nu_-^0$ |
| $\nu_-^1$ | $\nu_+^1$ | $\nu_-^1$ | $\nu_+^0$ | $\nu_-^0$ | $\nu_+^0$ | $\nu_-^0$ |
| $\nu_+^0$ | $\nu_+^0$ | $\nu_-^0$ | $\nu_+^0$ | $\nu_-^0$ | $\nu_+^0$ | $\nu_-^0$ |
| $\nu_-^0$ | $\nu_+^0$ | $\nu_-^0$ | $\nu_+^0$ | $\nu_-^0$ | $\nu_+^0$ | $\nu_-^0$ |

Table 3: Fusion law for $|\alpha| = 2$

**Remark 4.15.** We note that although there are three parts which are non-trivially graded, one of them is distinguishable from the rest. Indeed, $A_\lambda$ is 1-dimensional, whereas both $A_{\nu_0^\alpha}$ and $A_{\nu_0^\beta}$ have dimension $\frac{|C|}{4} \geq 1$.

If $|\alpha \cap \beta| = 1$, then $\xi_\beta = 0$ and $\theta_\beta = \pm 1$. So, $A_{\nu_0^\alpha}$ is spanned by $w_+^\beta = e^\beta + e^{\alpha+\beta}$ and $A_{\nu_0^\beta}$ is spanned by $w_-^\beta = -e^\beta + e^{\alpha+\beta}$.

We write $T := \mathbb{Z}_2 \times \mathbb{Z}_2$. Recall that we have a map $\tau : X \times T^* \to \text{Aut}(A)$ where $\tau_\alpha(\chi)$ acts by

\[
v \mapsto \chi(t)v
\]
for \( v \in A_t \). The non-trivial characters can be labelled by non-trivial elements of \( T \) and they are given by
\[
\chi_t : s \mapsto \begin{cases} 1 & \text{if } s = 1_T, t \\ -1 & \text{otherwise} \end{cases}
\]

In light of the above Proposition 4.14, we abuse notation and label our non-trivial characters by \( \chi_+ \), \( \chi_- \) and \( \chi_\lambda \) corresponding to the grading of those parts.

**Lemma 4.16.** Let \( e_\pm := \lambda t_\alpha \pm \mu e^\alpha \). Then, for all \( \chi \in T^* \)
\[
\tau_{e_+}(\chi) = \tau_{e_-}(\chi)
\]

**Proof.** Note that \( e_{\alpha,+} \) and \( e_{\alpha,-} \) differ just by the parity of their coefficient \( \mu \) of \( e^\alpha \). Observe that \( A_\lambda = \left\langle t_i - t_j \right\rangle \), where \( \text{supp}(\alpha) = \{i,j\} \), does not depend on \( \mu \). Similarly, \( A_{\nu_1} := \left\langle \pm e^\beta + e^{\alpha+\beta} \right\rangle \) which do not depend on \( \mu \). \( \square \)

In particular, just as in the standard case, the \( \tau \)-map given by \( a \mapsto \tau_a(\chi) \) for any \( \chi \in T^* \) is not a bijection. Similarly to the standard case, we will write \( \tau_\alpha(\chi) \) for \( \tau_{e_\pm}(\chi) \), where \( e_\pm = \lambda t_\alpha \pm \mu e^\alpha \).

**Lemma 4.17.** Let \( \text{supp}(\alpha) = \{i,j\} \). The action of \( \tau_\alpha(\chi_\pm) \) is given by
\[
\tau_\alpha(\chi_\pm) : t_i \mapsto t_j \\
t_j \mapsto t_i \\
t_k \mapsto t_k \quad \text{for } k \notin \text{supp}(\alpha) \\
e^\beta \mapsto \begin{cases} e^\beta & \text{if } \beta \in C_\alpha(0) \\ \pm e^{\alpha+\beta} & \text{if } \beta \in C_\alpha(1) \end{cases}
\]

**Proof.** Note that \( \tau_\alpha(\chi_\pm) \) inverts the \( \lambda \)-part, which is spanned by \( t_i - t_j \), and the \( \nu_1^\pm \)-part and fixes all the other parts. For \( \beta \in C_\alpha(1) \), by writing \( e^\beta = \frac{1}{2} \left( (e^\beta + e^{\alpha+\beta}) - (e^-\beta + e^{\alpha+\beta}) \right) = \frac{1}{2}(w^\beta_+ - w^\beta_-) \) and applying \( \tau_\alpha(\chi_\pm) \), we see the result follows. \( \square \)

It is easy to see from Lemma 4.17 that \( \tau_\alpha(\chi_+) \) acts as the involution \( (i,j) \) in the natural action, where \( \text{supp}(\alpha) = \{i,j\} \). Hence \( \langle \tau_\alpha(\chi_+) : \alpha \in S \rangle \cong \text{Aut}(C) \). From [2, Lemma 3.6], \( \text{Aut}(C) \) acts on \( A \) in the natural action if and only the structure constants are regular. That is, \( a_{i,\alpha} = a_{g^i,\alpha g^i}, b_{\alpha,\beta} = b_{\alpha g^i,\beta g^i} \) and \( c_{i,\alpha} = c_{g^i,\alpha g^i} \) for all \( g \in \text{Aut}(C), i \in \text{supp}(\alpha), \alpha, \beta \in C^*, \beta \neq \alpha, \alpha^c \).

**Corollary 4.18.** The group \( \langle \tau_\alpha(\chi_+) : \alpha \in S \rangle = \text{Aut}(C) \) and the structure constants in \( A \) are in fact regular.
We begin by considering the case where $C$ is an irreducible code. So, $\text{Aut}(C) = \langle \tau_\alpha(\chi) : \alpha \in S \rangle \cong S_n$. Note that $\tau_\alpha(\chi)$ are just the Miyamoto automorphisms from the standard case. Hence, by Corollary 4.5,

$$N := \langle \tau_\alpha(\chi) : \alpha \in S \rangle \cong \begin{cases} 2^{n-1} & \text{if } n \text{ is odd} \\ 2^{n-2} & \text{if } n \text{ is even} \end{cases}$$

**Lemma 4.19.** For axes $a, b \in X$ and $t \in T$, we have

$$\tau_a(\chi) \tau_b(\chi) = \tau_{a+b}(\chi)$$

**Proof.** This is a straightforward calculation. \qed

**Corollary 4.20.** $N$ is a normal subgroup of the Miyamoto group $G(X)$.

**Theorem 4.21.** Let $C$ be an irreducible code, then $G(X) \cong 2^{n-1} : S_n$ if $n$ is odd and $G(X) \cong 2^{n-2} : S_n$ if $n$ is even.

**Proof.** From the definition of the $\tau$ map, $\tau_e(\chi) = \tau_\alpha(\chi) \tau_\beta(\chi)$ and so $G(X)$ is generated by just $\tau_\alpha(\chi)$ and $\tau_\beta(\chi)$, for $e \in X$. It is clear that $\langle \tau_e(\chi) : e \in X \rangle$ intersects $N$ trivially, hence the result follows. \qed

We may now consider the general case, where $C$ is not irreducible.

**Corollary 4.22.** Let $C = \bigoplus_{i=1}^r C_i$ be a sum of even weight codes $C_i$, all of length $m \geq 3$. Then,

$$G(X) = \begin{cases} \prod_{i=1}^r 2^{m-1} : S_m & \text{if } m \text{ is odd} \\ \prod_{i=1}^r 2^{m-2} : S_m & \text{if } m \text{ is even} \end{cases}$$

**Proof.** Since $C = \bigoplus_{i=1}^r C_i$, each $\alpha \in S$ is contained in some $C_i$. Define $X_i$ as the set of axes in $X$ which come from codewords in $C_i$. We claim that $G(X)$ is a central product of $G(X_i)$.

For $i \neq j$, let $\tau_i(\chi)$ and $\tau_j(\psi)$ be the Miyamoto automorphisms associated to an axis in $X_i$ and in $X_j$ coming from codewords $\alpha_i$ and $\alpha_j$ respectively. Since $i \neq j$, it is clear that their actions commute on the toral elements and codewords elements $e^\beta$ such that $\beta \in C_{\alpha_i}(0) \cap C_{\alpha_j}(0)$.

Since $\alpha_i$ and $\alpha_j$ are disjoint, $\beta \in C_{\alpha_i}(1)$ if and only if $\alpha_i + \beta \in C_{\alpha_j}(1)$. Since $e^\beta \mapsto -e^\beta$ commutes with $e^\beta \mapsto \pm e^{\alpha_i+\beta}$ and addition of codewords is commutative, we see that $\tau_i(\chi)$ and $\tau_j(\psi)$ commute and hence the claim follows.

Since each $C_i$ is irreducible, by Theorem 4.21, $G(X_i)$ is isomorphic to either $2^{m-1} : S_m$, or $2^{m-2} : S_m$. Since these are centre-free, the result follows. \qed
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