Remarks on infinitely many solutions for a class of Schrödinger equations with sign-changing potential

Rong Cheng and Yijia Wu

Abstract
In this paper, we study the existence of infinitely many nontrivial solutions for the following semilinear Schrödinger equation:

\[
\begin{aligned}
-\Delta u + V(x)u &= f(x, u), \quad x \in \mathbb{R}^N, \\
u &= u \in H^1(\mathbb{R}^N),
\end{aligned}
\]

where the potential \( V \) is continuous and is allowed to be sign-changing. By using a variant fountain theorem, we obtain the existence of infinitely many high energy solutions under the condition that the nonlinearity \( f(x, u) \) is of super-linear growth at infinity. The super-quadratic growth condition imposed on \( F(x, u) = \int_0^u f(x, t) \, dt \) is weaker than the Ambrosetti–Rabinowitz type condition and the similar conditions employed in the references.

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Keywords: Variational method; Schrödinger equation; Multiple solutions; Fountain theorem

1 Introduction and main result
In the present paper, we are concerned with the existence of infinitely many nontrivial solutions for the following semilinear Schrödinger equation:

\[
\begin{aligned}
-\Delta u + V(x)u &= f(x, u), \quad x \in \mathbb{R}^N, \\
u &= u \in H^1(\mathbb{R}^N).
\end{aligned}
\]  

(1.1)

Due to the strong background of (1.1) in many fields, such as physics, optics, etc., it has attracted a great deal of interest in the past several decades. Here we point out that a variety of types of existent results on solutions for (1.1) have been considered by different assumptions on potential \( V(x) \) and nonlinearity \( f(x, u) \) by applying variational methods. For the case of \( V(x) \) and \( f(x) \) being nonperiodic, many authors have studied the existence of nontrivial solutions for problem (1.1) with \( f(x) \) being superlinear in [1–3], sublinear
in [4], and asymptotically linear in [5]. Some special types of solution to (1.1) have also been considered in this direction. For instance, the existence of bound state solutions was considered in [6]. Bump type nodal solutions and sign-changing solutions were studied in [7–10]. For the case of $V(x)$ and $f(x,u)$ being periodic, the authors in [11, 12] obtained the existence of nontrivial solutions for (1.1).

In this paper, we focus on the existence of infinitely many nontrivial solutions of (1.1) by applying variational methods. In variational theory, lots of tools can be used to obtain multiple solutions for equations which have variational structures, such as D.C. Clark’s theory for functionals bounded below [13, 14] and mountain pass theorem for even functionals [15]. Among these tools, fountain theorem is an important one. Fountain theorem and its dual form were established by Bartsch in [16] and by Bartsch and Willem in [17] (see also [18]) respectively. They are powerful tools to find multiple critical points for variational functionals. Zou in [19] (see also in [20]) established variant fountain theorems without (P.S.) condition, which is a crucial condition for fountain theorem and its dual form. By using the variant fountain theorems given in [19, 20], Zhang and Xu obtained the existence of infinitely many nontrivial solutions of (1.1) in [21], where they did not suppose the common used condition that $f(x,u) → 0$ as $u → 0$, which was used in [19].

Then in [22], Tang weakened some conditions in [21] and gave some more general super-quadratic conditions near infinity for the primitive of $f(x,u)$, and finally showed that (1.1) possesses infinitely many nontrivial solutions by applying a result in [15]. Motivated by the work in [21, 22], we try to establish the existence of infinitely many high energy solutions of (1.1) under a more general super-quadratic condition than those in [21, 22]. So-called infinitely high energy solutions for (1.1) are solution sequences $\{u_n\}$ of (1.1) such that the corresponding energy denoted by the energy functional of (1.1) goes to infinity as $n → ∞$.

To state our result of the present paper, we make the following assumptions on the potential $V$ and nonlinearity $f$:

$$(V_1) \quad V(x) ∈ C(\mathbb{R}^N, \mathbb{R}) \text{ is bounded below.}$$

$$(V_2) \quad \text{For every } M > 0,$$

$$(S_1) \quad f(x,u) ∈ C(\mathbb{R}^N × \mathbb{R}) \text{ and } f(x,−u) = −f(x,u), \forall (x,u) ∈ \mathbb{R}^N × \mathbb{R}.$$ 

$$(S_2) \quad \text{There exist constants } c_1, c_2 > 0, p, q ∈ (1, 2) \text{ and } 2 < p < 2^∗ \text{ such that }$$

$$|f(x,u)| ≤ c_1 |u| + c_2 |u|^{p−1}, \quad \forall (x,u) ∈ \mathbb{R}^N × \mathbb{R}.$$ 

$$(S_3) \quad \lim_{|u| → ∞} \frac{f(x,u)}{|u|} = \infty.$$ 

$$(S_4) \quad \text{There exist a constant } μ > 2 \text{ and a function } v(x) > 0 \text{ with } |v|_{L^∞} < ∞ \text{ such that }$$

$$μF(x,u) ≤ uf(x,u) + v(x)u^2, \quad \forall (x,u) ∈ \mathbb{R}^N × \mathbb{R}.$$ 

Then the main result of this paper can be read as follows.

**Theorem 1.1** Assume that $V$ and $f$ satisfy $(V_1)$, $(V_2)$ and $(S_1)$–$(S_4)$. Then problem (1.1) possesses infinitely many high energy solutions.
Remark 1.2 Comparing with the multiple existent results on nontrivial solutions of (1.1) in [21, 22], Theorem 1.1 obtains infinitely many high energy solutions for (1.1) not only for infinitely many nontrivial solutions. Moreover, we do not make more assumptions on primitive $F(x, u)$ of $f(x, u)$. Specifically, Zhang and Xu in [21] assumed that $F(x, u) ≥ 0$ for $\forall (x, u) ∈ \mathbb{R}^N × \mathbb{R}$ and $\lim_{|u| → ∞} \frac{F(x, u)}{|u|^p} = \infty$. In [22], Tang improved the conditions in [21] and employed a condition that $F(x, u) ≥ 0$ for $|u| ≥ r_0 ≥ 0$ and $\lim_{|u| → ∞} \frac{F(x, u)}{|u|^p} = \infty$. However, we only assume the condition in this paper that

$$\lim_{|u| → ∞} \frac{f(x, u)}{u} = \infty. \quad (1.2)$$

It is worth mentioning that the condition $\lim_{|u| → ∞} \frac{F(x, u)}{|u|^p} = \infty$ was also used in [12]. It was also supposed there that $\frac{f(x, t)}{|t|^p}$ was strictly increasing and $f(x, t) = o(|t|)$ as $|t| → 0$ uniformly for $x ∈ \mathbb{R}^N$.

Remark 1.3 To obtain the boundedness of (PS) sequence, Zhang and Xu in [21] introduced that there exists a constant $\vartheta ≥ 1$ such that

$$\vartheta \tilde{F}(x, u) ≥ \tilde{F}(x, su), \quad \forall (x, u) ∈ \mathbb{R}^N × \mathbb{R}, s ∈ [0, 1], \quad (1.3)$$

where $\tilde{F}(x, u) = uf(x, u) − 2F(x, u)$. Then Tang in [22] improved the super-quadratic conditions in [21] and supposed that there exist $c_0 > 0$ and $κ > \max\{1, \frac{N}{2}\}$ such that

$$|F(x, u)|^κ ≤ c_0|u|^{2κ} \mathcal{F}(x, u), \quad \forall (x, u) ∈ \mathbb{R}^N × \mathbb{R}, |u| ≥ r_0, \quad (1.4)$$

where $\mathcal{F}(x, u) = \frac{1}{2}uf(x, u) − F(x, u) ≥ 0$.

In [22], Tang also imposed another super-quadratic condition that there exist two constants $\mu > 2$ and $ν > 0$ such that

$$\mu F(x, u) ≤ u^2 + ρu^2, \quad \forall (x, u) ∈ \mathbb{R}^N × \mathbb{R}. \quad (1.5)$$

Condition (1.5) can be weakened. Thus in this paper we assume condition (S_4) where $ν > 0$ is a function instead of $ρ > 0$ being a constant.

Remark 1.4 It is not difficult to verify that the functions

$$f(x, u) = a(x)u \ln(1 + |u|), \quad (1.6)$$

$$f(x, u) = a(x)(u^2 + u^3 \cos u) \quad (1.7)$$

satisfy (S_1)–(S_4), where $a(x)$ is a continuous bounded function with positive lower bound.

Remark 1.5 By ($V_1$), $V(x)$ is bounded below. Thus there exists a positive constant $\tilde{V}$ such that $\tilde{V}(x) = V(x) + \tilde{V} ≥ V_0 > 0$. Let $\tilde{f}(x, u) = f(x, u) + \tilde{V}u$ for all $(x, u) ∈ \mathbb{R}^N × \mathbb{R}$. Consider the following system:

$$\begin{cases}
-Δ u + \tilde{V}(x)u = \tilde{f}(x, u), & x ∈ \mathbb{R}^N; \\
u ∈ H^1(\mathbb{R}^N).
\end{cases} \quad (1.8)$$
It is easy to check that (1.8) is equivalent to (1.1). Therefore, in what follows we always assume that \( V(x) \) has a positive lower bound.

2 Proof of Theorem 1.1

We work in the Hilbert space \( E = \{ u \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2) \, dx < +\infty \} \) equipped with inner product \( \langle u, v \rangle = \int_{\mathbb{R}^N} (\nabla u \nabla v + V(x)uv) \, dx \). The associated norm is \( \|u\| = (\int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2) \, dx)^{\frac{1}{2}} \), which is equivalent to the standard norm in \( H^1(\mathbb{R}^N) \) by (\( V_1 \)) and Remark (1.5). We will find solutions of (1.1) in \( E \). In what follows, we use \( \|\cdot\| \) and \( \|\cdot\|_p \) to denote the norms in \( E \) and \( L^p(\mathbb{R}^N) \), respectively.

For \( \lambda \in [1, 2] \), we define a functional on \( E \)

\[
I_\lambda(u) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2) \, dx - \lambda \int_{\mathbb{R}^N} F(x, u) \, dx
= \frac{1}{2} \|u\|^2 - \lambda \int_{\mathbb{R}^N} F(x, u) \, dx
= A(u) - \lambda B(u). \tag{2.1}
\]

Since \( 2 < p < 2^* \), \( E \) can be imbedded continuously into \( L^p(\mathbb{R}^N) \). By conditions (\( S_1 \)) and (\( S_2 \)), \( I_\lambda(u) \) is well defined. By [23], \( B(u) \in C^1(\mathbb{R}, \mathbb{R}) \) and \( B'(u) : E \to E^* \) is compact, where \( E^* \) is the dual space of \( E \). Thus one has \( I_\lambda(u) \in C^1(\mathbb{R}, \mathbb{R}) \), and for \( u, v \in E \),

\[
I_\lambda'(u)v = \langle u, v \rangle - \lambda B'(u)v = \langle u, v \rangle - \lambda \int_{\mathbb{R}^N} f(x, u)v \, dx. \tag{2.2}
\]

By (2.2), solutions of (1.1) correspond to critical points of \( I_\lambda \) with \( \lambda = 1 \).

Let \( \{ e_j : j \in \mathbb{N} \} \) be an orthogonal basis of \( E \). If \( j \in \mathbb{N} \), write \( X_j = \text{span}\{e_j\} \), then \( E = \overline{\bigoplus_{j=1}^{\infty} X_j} \).

Let \( Y_k = \bigoplus_{j=1}^{k} X_j \), \( Z_k = \overline{\bigoplus_{j=k}^{\infty} X_j} \). Notice that \( \dim Y_k < \infty \). Now we rewrite Theorem 2.1 of [19], since the existence of infinitely many nontrivial solutions of (1.1) is based on it.

**Theorem 2.1**

Suppose that the functional \( I_\lambda \) satisfies:

\( (F_1) \) \( \forall (\lambda, u) \in [1, 2] \times E, I_\lambda(-u) = I_\lambda(u), I_\lambda \) maps a bounded set to a bounded set.

\( (F_2) \) \( \forall u \in E, B(u) \geq 0, \) and \( \|u\| \to \infty, A(u) \to \infty \) or \( B(u) \to \infty \).

\( (F_3) \) There exist \( r_k > \rho_k > 0 \) such that, for all \( \lambda \in [1, 2] \),

\[
\alpha_k(\lambda) = \inf_{u \in Z_k : \|u\| = r_k} I_\lambda(u) > \beta_k(\lambda) = \max_{u \in Z_k : \|u\| = r_k} I_\lambda(u).
\]

Then \( \forall \lambda \in [1, 2], \) one has

\[
\alpha_k(\lambda) \leq \xi_k(\lambda) = \inf_{\gamma \in \Gamma_k} \max_{u \in B_k} I_\lambda(\gamma(u))
\]

where \( B_k = \{ u \in Y_k : \|u\| \leq r_k \} \), \( \Gamma_k = \{ \gamma \in C(B_k, E) : \gamma \text{ is odd}, \gamma|_{\partial B_k} = \text{id} \} \). For \( \lambda \in [1, 2], \) there exists a sequence \( \{ u_m^k(\lambda) \}_{m=1}^{\infty} \) such that

\[
\sup_m \|u_m^k(\lambda)\| < \infty,
\]

and as \( m \to \infty, \)

\[
I_\lambda'(u_m^k(\lambda)) \to 0, \quad I_\lambda(u_m^k(\lambda)) \to \xi_k(\lambda).
\]
Now we give several useful lemmas. The first lemma on compact imbedding is well known and its proof can be found in [15].

**Lemma 2.2** By condition \((V)\) and \(2 \leq p < 2^*\), the imbedding \(E \hookrightarrow L^p(\mathbb{R}^N)\) is compact.

**Lemma 2.3** Assume that \((V_1), (V_2), \) and \((S_1)\) hold, then there exist a number \(k_1 \in \mathbb{Z}^+\) and a sequence \(\{\rho_k\}\) such that, as \(\rho_k \to \infty \) \((k \to \infty)\), it has

\[
\alpha_k(\lambda) = \inf_{u \in Z_k, ||u|| = \rho_k} I_k(u) > 0, \quad \forall k \geq k_1, \tag{2.3}
\]

where \(\alpha_k(\lambda) \geq \frac{\rho_k^2}{4}, Z_k = \bigoplus_{j-k} X_j = \text{span}\{e_1, \ldots\}\).

**Proof** Observe that by conditions \((V_1), (V_2)\) and Lemma 3.8 of [18], it holds

\[
l_k(q) = \sup_{u \in Z_k, ||u||=1} ||u||_q \to 0, \quad k \to \infty. \tag{2.4}
\]

By condition \((S_2)\), there exists a constant \(c_3, c_4 > 0\) such that, for \(\forall (x, u) \in \mathbb{R}^N\),

\[
|F(x, u)| \leq c_3(|u|^2 + |u|^p). \tag{2.5}
\]

(2.5) and (2.1) imply that, for \(\forall (\lambda, u) \in [1, 2] \times \mathbb{R}\),

\[
I_k(u) = \frac{1}{2} ||u||^2 - \lambda \int_{\mathbb{R}^N} F(x, u) \, dx
\]

\[
\geq \frac{1}{2} ||u||^2 - \lambda \int_{\mathbb{R}^N} c_3|u|^2 + c_4|u|^p \, dx
\]

\[
\geq \frac{1}{2} ||u||^2 - 2(c_3 ||u||_2^2 + c_4 ||u||_p^p)
\]

\[
= \frac{1}{2} ||u||^2 - 2(c_3 l_2^2(k) ||u||^2 + c_4 l_p^p(k) ||u||^p). \tag{2.6}
\]

By (2.4), there exists \(k_1\) such that, for any \(k > k_1\), one has \(l_2(k) < \frac{1}{16c_3}\). Now we take \(\rho_k = \frac{1}{2} \left(\frac{1}{16c_3 l_2^2(k)}\right)^{\frac{1}{p-2}}\). Then a direct computation implies \(I_k(u) > \frac{\rho_k^2}{4}\) for \(u \in Z_k\) with \(||u|| = \rho_k\). Moreover, by \(l_p^p(k) \to 0\), it holds \(\rho_k \to \infty\) as \(k \to \infty\). \(\Box\)

Next we prove the following lemma.

**Lemma 2.4** Suppose that \((V_1), (V_2), (S_1)-(S_3)\) hold. Then, for \(k_1 \in \mathbb{Z}^+\) and the sequence \(\{\rho_k\}\) in Lemma 2.3, there exists \(r_k > 0\) such that, for \(r_k > \rho_k\),

\[
\beta_k(\lambda) = \max_{u \in Y_k, ||u|| = r_k} I_k(u) < 0, \quad \forall k \geq k_1, \tag{2.7}
\]

where \(k \in \mathbb{N}, Y_k = \bigoplus_{j=1}^k X_j = \text{span}\{e_1, \ldots, e_k\}\).
Proof By the proof of Lemma 2.6 in [21], for any finite dimensional subspace \( \tilde{E} \subset E \), there exists a constant \( \epsilon > 0 \) such that

\[
\text{meas}\{ x \in \mathbb{R}^N : |u(x)| \geq \epsilon \|u\| \} \geq \epsilon, \quad \forall u \in \tilde{E}, u \neq 0.
\]  

(2.8)

Therefore, by the fact that the dimension of \( Y_k \) is finite, there exists a constant \( \epsilon_k > 0 \) such that

\[
\text{meas}\{ \Omega_u^k \} \geq \epsilon_k, \quad \forall u \in Y_k, u \neq 0,
\]  

(2.9)

where \( \Omega_u^k = \{ x \in \mathbb{R}^N : |u(x)| \geq \epsilon_k \|u\| \} \) for all \( k \in \mathbb{N} \). By (S\_3), for each \( k \in \mathbb{N} \) and \( m_1 > \frac{1}{2} \), there exists a constant \( \zeta_k > 0 \) such that

\[
f(x, u) \geq \frac{2m_1 u}{\epsilon_k^3}, \quad \forall x \in \mathbb{R}^N, |u| \geq \zeta_k,
\]  

(2.10)

which yields that

\[
F(x, u) \geq \frac{m_1 u^2}{\epsilon_k^3}, \quad \forall x \in \mathbb{R}^N, |u| \geq \zeta_k.
\]  

(2.11)

Then, for any \( k \in \mathbb{N} \) and \( \lambda \in [1, 2] \), one has

\[
I_\lambda(u) = \frac{1}{2} \|u\|^2 - \lambda \int_{\mathbb{R}^N} F(x, u) \, dx \\
\leq \frac{1}{2} \|u\|^2 - \lambda \int_{\Omega_u^k} \frac{m_1 |u|^2}{\epsilon_k^3} \, dx \\
\leq \frac{1}{2} \|u\|^2 - \epsilon_k^2 \|u\|^2 m_1 \frac{\text{meas}(\Omega_u^k)}{\epsilon_k^3} \\
\leq \frac{1}{2} \|u\|^2 - m_1 \|u\|^2 = -\left( m_1 - \frac{1}{2} \right) \|u\|^2,
\]  

(2.12)

where \( u \in Y_k \) with \( \|u\| \geq \frac{\zeta_k}{ \epsilon_k^2} \). Choose \( r_k > \max\{ \xi_k, \frac{\zeta_k}{ \epsilon_k} \} \) for any \( k \geq k_1 \). Then, by \( m_1 > \frac{1}{2} \), it holds

\[
\beta_k(\lambda) = \max_{u \in Y_k, \|u\| = r_k} I_\lambda(u) \leq -\left( m_1 - \frac{1}{2} \right) r_k^2 < 0.
\]  

(2.13)

\[\square\]

Now we are ready to give the proof of Theorem 1.1.

Proof of Theorem 1.1 First of all, by (2.1), (2.2), and (2.5), \( I_k \) maps bounded sets to bounded sets uniformly for \( \lambda \in [1, 2] \). By (S\_1), we have \( I_k(-u) = I_k(u) \) for \( (\lambda, u) \in [1, 2] \times E \), which implies that condition (F\_1) of Theorem 2.1 holds. It is easy to see that (F\_2) holds by (S\_3) and (2.1). Condition (F\_3) is also true due to Lemma 2.3 and Lemma 2.4. Thus, by Theorem 2.1, there exists a sequence \( \{u_m^k(\lambda)\}_{m=1}^\infty \) for every \( k \geq k_1 \) and \( \lambda \in [1, 2] \) such that

\[
\sup_m \|u_m^k(\lambda)\| < \infty, \quad I_k(u_m^k(\lambda)) \to 0, \quad I_k(u_m^k(\lambda)) \to \xi_k(\lambda), \quad \text{a.e. on } \mathbb{R}^N
\]  

(2.14)
as \( m \to \infty \), where \( \xi_k(\lambda) = \inf_{\gamma \in \Gamma_k} \max_{u \in B_k} I_k(\gamma(u)) \), \( B_k = \{ u \in Y_k : \| u \| \leq r_k \} \), \( \Gamma_k = \{ \gamma \in C(B_k, E) : \gamma \text{ odd, } \gamma|_{\partial B_k} = \text{id} \} \). By (2.14), one can choose a sequence \( \{ \lambda_n \} \) satisfying \( \lambda_n \to 1 \) for each \( k \geq k_1 \). Moreover, we obtain a sequence \( \{ u_n^k(\lambda_n) \}_{m=1}^\infty \) which satisfies as \( m \to \infty \)

\[
\sup_m \| u_n^k(\lambda_n) \| < \infty, \quad I_{\lambda_n}'(u_n^k(\lambda_n)) \to 0. \tag{2.15}
\]

Note that \( \{ u_n^k(\lambda_n) \}_{m=1}^\infty \) is bounded for \( m \). So it follows from the compactness of \( B' \) that \( \{ u_n^k(\lambda_n) \}_{m=1}^\infty \) has a strongly convergent subsequence. We denote it still by \( \{ u_n^k(\lambda_n) \} \). We can suppose that

\[
\lim_{m \to \infty} u_n^k(\lambda_n) = u_n^k \in E, \quad \forall n \in \mathbb{N}, k \geq k_1. \tag{2.16}
\]

Then by (2.15) it holds

\[
I_{\lambda_n}'(v_n^k) = 0, \quad I_{\lambda_n}(u_n^k) \geq \bar{\rho}_k = \frac{\rho_k^2}{4}. \tag{2.17}
\]

Now we claim that \( \{ u_n^k \}_{n=1}^\infty \) is bounded. We prove the claim by contradiction. For convenience, we write \( u_n^k = u_n \) for \( n \in \mathbb{N} \). Suppose that \( \| u_n \| \to \infty \). Write \( v_n = \frac{u_n}{\| u_n \|} \). Then it is obvious that \( \| v_n \| = 1 \) and \( \| v_n \|_p \leq \tau \| v_n \| = \tau \) for \( 2 \leq p < 2^* \), where \( \tau \) is a positive constant by Lemma 2.2. Then by (2.1), (2.2), (2.14), (2.15), and assumption \((S_4)\), one has, for \( n \) being large,

\[
c + 1 \geq I_\lambda(u_n) - \frac{1}{\mu} I_\lambda'(u_n) u_n
\]

\[
= \frac{\mu - 2}{\mu} \| u_n \|^2 + \int_{\mathbb{R}^N} \left( \frac{1}{\mu} f(x, u_n) u_n - F(x, u_n) \right) dx
\]

\[
\geq \frac{\mu - 2}{\mu} \| u_n \|^2 - \frac{|v|_{L^\infty}}{\mu} \| u_n \|_2^2,
\]

which means

\[
1 \leq \frac{2|v|_{L^\infty}}{\mu - 2} \limsup_{n \to \infty} \| v_n \|_2^2. \tag{2.19}
\]

Passing to a subsequence, we can assume that

\[
v_n \to v_0 \quad \text{in } E;
\]

\[
v_n \to v_0 \quad \text{in } L^p(\mathbb{R}^N) \text{ for } 2 \leq p < 2^*;
\]

\[
v_n(x) \to v_0(x) \quad \text{a.e. on } \mathbb{R}^N.
\]

Therefore it follows from (2.19) and (2.20) that \( v_0 \neq 0 \), which implies that \( \text{meas}(\Sigma) > 0 \), where \( \Sigma = \{ x \in \mathbb{R}^N : v_0(x) \neq 0 \} \). Since \( \| u_n \| \to \infty \) as \( n \to \infty \), we get \( |u_n| \to \infty \) as \( n \to \infty \) on the set \( \Sigma \). Then, by \((S_4)\) and (2.17), one has

\[
a(1) = I_{\lambda_n}'(u_n) u_n = \| u_n \|^2 - \lambda_n \int_{\mathbb{R}^N} f(x, u_n) u_n dx,
\]

\[
(2.21)
\]
which implies that
\[
o(1) + 1 \geq \lambda_n \int_{\Sigma} \frac{f(x,u_n)u_n}{\|u_n\|^2} dx = \lambda_n \int_{\Sigma} \frac{f(x,u_n)u_n}{|u_n|^2} |v_n|^2 dx = \lambda_n \int_{\Sigma} \frac{f(x,u_n)}{u_n} |v_n|^2 dx \to \infty, \quad (2.22)
\]
as \( n \to \infty \). This contradiction concludes the boundedness of \( \{u_n\}_{n=1}^{\infty} \). Thus we get the claim. By the claim and (2.17), it can be shown by a standard way that there exists a strongly convergent subsequence of \( \{u^k_n\} \). Suppose that \( u^k_n \to u^k_0 \) as \( n \to \infty \). Then by (2.17) one has \( I_1(u^k_0) = 0 \), i.e., \( u^k_0 \) is a critical point of \( I_1(u) \). By (2.17) again, one has \( I_1(u^k_0) \geq \tilde{\rho}_k \to \infty \) as \( k \to \infty \). This fact means that equation (1.1) possesses infinitely many high energy solutions. The proof of Theorem 1.1 is complete. \( \square \)

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