Central Limit Theorems for Non-Symmetric Random Walks on Nilpotent Covering Graphs: Part II

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Abstract
In the present paper, as a continuation of our preceding paper (Ishiwata et al. 2018), we study another kind of central limit theorems (CLTs) for non-symmetric random walks on nilpotent covering graphs from a view point of discrete geometric analysis developed by Kotani and Sunada. We introduce a one-parameter family of random walks which interpolates between the original non-symmetric random walk and the symmetrized one. We first prove a semigroup CLT for the family of random walks by realizing the nilpotent covering graph into a nilpotent Lie group via discrete harmonic maps. The limiting diffusion semigroup is generated by the homogenized sub-Laplacian with a constant drift of the asymptotic direction on the nilpotent Lie group, which is equipped with the Albanese metric associated with the symmetrized random walk. We next prove a functional CLT (i.e., Donsker-type invariance principle) in a Hölder space over the nilpotent Lie group by combining the semigroup CLT, standard martingale techniques, and a novel pathwise argument inspired by rough path theory. Applying the corrector method, we finally extend these CLTs to the case where the realizations are not necessarily harmonic.

Keywords Central limit theorem · Non-symmetric random walk · Nilpotent covering graph · Discrete geometric analysis · Rough path theory

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1 Introduction

The long time behavior of random walks on graphs is one of principal themes among probability theory, harmonic analysis, geometry, graph theory, group theory and so on. In particular, the central limit theorem (CLT), a generalization of the Laplace–de Moivre theorem, has been studied intensively and extensively in various settings. Our main concern is such a long time behavior of random walks with periodic structures by using ideas from homogenization theory. Generally speaking, homogenization theory is a method which relates a periodic system to the corresponding homogenized system through a scaling relation between the time and the underlying space (cf. Bensoussan–Lions–Papanicolaou [4]). However, since the notion of the scale change on graphs is not defined, it is not possible to apply this method directly to the case where the underlying space is a graph. Therefore, it is necessary to find a realization of the graph, preserving the periodicity, in a space on which a scaling is defined.

In a series of papers [12–17], Kotani and Sunada studied long time asymptotics of symmetric random walks on a crystal lattice $X$, a covering graph of a finite graph $X_0$ whose covering transformation group $\Gamma$ is abelian, by placing a special emphasis on the geometric feature. We remark that the crystal lattice has inhomogeneous local structures though it has a periodic global structure. Especially, in [15], they introduced the notion of standard realization, which is a discrete harmonic map $\Phi_0$ from a crystal lattice $X$ into the Euclidean space $\Gamma \otimes \mathbb{R}$ equipped with the Albanese metric, to characterize an equilibrium configuration of crystals. In [14], they proved the CLT by applying the homogenization method mentioned above through the standard realization $\Phi_0$. As the scaling limit, they captured a homogenized Laplacian on $\Gamma \otimes \mathbb{R}$. In terms of probability theory, it means that, for fixed $0 \leq t \leq 1$, a sequence of $\Gamma \otimes \mathbb{R}$-valued random variables $\{n^{-1/2}\Phi_0(w_{[nt]})\}_{n=1}^{\infty}$ converges to $B_t$ as $n \to \infty$ in law. Here $\{w_n\}_{n=0}^{\infty}$ is the given random walk on $X$ and $\{(B_t)_{0 \leq t \leq 1}\}$ is a standard Brownian motion on $\Gamma \otimes \mathbb{R}$ equipped with the Albanese metric. In their proof, both the symmetry of the given random walk $\Phi_0$ and the harmonicity of the realization $\Phi_0$ play an important role to show the convergence of the sequence of infinitesimal generators associated with $\{n^{-1/2}\Phi_0(w_{[nt]})\};\ 0 \leq t \leq 1\}_{n=1}^{\infty}$. Indeed, these properties are effectively used to delete a diverging drift term as $n \to \infty$ from the homogenized Laplacian. See also Kotani [12]. Ishiwata [8] discussed a similar problem to [12, 14] for symmetric random walks on a $\Gamma$-nilpotent covering graph $X$, a covering graph of a finite graph $X_0$ whose covering transformation group $\Gamma$ is a finitely generated nilpotent group of step $r$. See [10, Section 6] for several examples of $\Gamma$-nilpotent covering graphs in the case where $\Gamma$ is the 3-dimensional discrete Heisenberg group $\mathbb{H}^3(\mathbb{Z})$. Needless to say, $X$ is a crystal lattice in the case $r = 1$. In the nilpotent case, $X$ is properly realized into a nilpotent Lie group $G$ such that $\Gamma$ is isomorphic to a cocompact lattice of $G$. Thanks to a basic property of the canonical dilation on $G$, the diverging drift term appears only in $\mathfrak{g}^{(1)}$-direction, where $\mathfrak{g}^{(1)}$ is the generating part of the Lie algebra $\mathfrak{g}$ of $G$. Hence, it is sufficient to introduce the notion of harmonicity of the realization $\Phi_0 : X \to G$ only on $\mathfrak{g}^{(1)}$ for proving the CLT in the nilpotent case.

If we consider a non-symmetric case, the above method does not work well because the diverging drift term from the non-symmetry of the given random walk does not vanish. To overcome this difficulty, in the case of crystal lattices, Ishiwata, Kawabi and Kotani [9] introduced two kinds of schemes. One is to replace the usual transition operator by the transition-shift operator, which “deletes” the diverging drift term. Combining this scheme with a modification of the harmonicity of the realization $\Phi_0$, they proved that a sequence $\{n^{-1/2}(\Phi_0(w_{[nt]}) - [nt]\rho_\mathfrak{g}(\gamma_p));\ 0 \leq t \leq 1\}_{n=1}^{\infty}$ converges in law to a $\Gamma \otimes \mathbb{R}$-valued...
standard Brownian motion \( (B_t)_{t \geq 0} \) as \( n \to \infty \). Here \( \rho_\mathcal{R}(\gamma_p) \in \Gamma \otimes \mathbb{R} \) is the so-called asymptotic direction which appears in the law of large numbers for the random walk \( \{\Phi_0(w_n)\}_{n=0}^{\infty} \) on \( \Gamma \otimes \mathbb{R} \). See [9, 16] for details. The other is to introduce a one-parameter family of \( \Gamma \otimes \mathbb{R} \)-valued random walks \( \{\xi^{(t)}\}_{0 \leq t \leq 1} \) which “weakens” the diverging drift term, where this family interpolates the original non-symmetric random walk \( \xi^{(1)}_n := \Phi_0(w_n) \) \( (n = 0, 1, 2, \ldots) \) and the symmetrized one \( \xi^{(0)}_n \). Putting \( \epsilon = n^{-1/2} \) and letting \( n \to \infty \), we capture a drifted Brownian motion \( \{B_t + \rho_\mathcal{R}(\gamma_p)t\}_{0 \leq t \leq 1} \) as the limit of a sequence \( \{n^{-1/2}\xi_{[n]}^{(t)}; 0 \leq t \leq 1\}_{n=1}^{\infty} \). See Trotter [35] for related early works. We note that this scheme is well-known in the study of the hydrodynamic limit of weakly asymmetric exclusion processes. See e.g., Kipnis–Landim [11], Tanaka [34] and references therein.

In our preceding paper [10], we proved a CLT for a non-symmetric random walk \( \{w_n\}_{n=0}^{\infty} \) on the \( \Gamma \)-nilpotent covering graph \( X \) by applying the former transition-shift scheme with a notion of modified harmonic realization \( \Phi_0 : X \to G \), which is a generalization of [8, 16]. More precisely, as the CLT-scaling limit mentioned above, we captured a diffusion process on \( G \) generated by a homogenized sub-Laplacian with a non-trivial \( g^{(2)} \)-valued drift \( \beta(\Phi_0) \) arising from the non-symmetry of the given random walk, where \( g^{(2)} := [g^{(1)}, g^{(1)}] \).

The main purpose of the present paper is to prove another kind of CLTs for the non-symmetric random walk \( \{w_n\}_{n=0}^{\infty} \) on the \( \Gamma \)-nilpotent covering graph \( X \) by applying the latter scheme to the nilpotent setting. We first introduce a one-parameter family of transition probabilities \( (p_\epsilon)_{0 \leq \epsilon \leq 1} \) on \( X \) as the linear interpolation between the given transition probability \( p_1 := p \) and the symmetrized one \( p_0 \), that is, \( p_\epsilon := p_0 + \epsilon(p - p_0) \) \( (0 \leq \epsilon \leq 1) \). For each \( \epsilon \), we next take a \( \Gamma \)-periodic realization \( \Phi^{(\epsilon)} : X \to G \) associated with the transition probability \( p_\epsilon \) and define a \( G \)-valued random walk \( \{\xi^{(\epsilon)}_n\}_{n=0}^{\infty} \) by \( \xi^{(\epsilon)}_n := \Phi^{(\epsilon)}(w^{(\epsilon)}_n) \) \( (n = 0, 1, 2, \ldots) \), where \( \{w^{(\epsilon)}_n\}_{n=0}^{\infty} \) is the random walk on \( X \) associated with the transition probability \( p_\epsilon \). Note that \( \Phi^{(\epsilon)} \) is not necessarily modified harmonic as above. Then by putting \( \epsilon = n^{-1/2} \) and letting \( n \to \infty \), we obtain a CLT for the family of \( G \)-valued random walks \( \{\xi^{(n^{-1/2})}_n\}_{n=1}^{\infty} \). More precisely, under suitable assumptions \( (A1) \) and \( (A2) \) on the family of \( \Gamma \)-periodic realizations \( \{\Phi^{(\epsilon)}\}_{0 \leq \epsilon \leq 1} \), we prove that a sequence \( \{\tau_{n^{-1/2}}(\xi^{(n^{-1/2})}_{[n]}); 0 \leq t \leq 1\}_{n=1}^{\infty} \) converges in law to a \( G \)-valued diffusion process \( \{Y_t\}_{0 \leq t \leq 1} \) as \( n \to \infty \), where \( \tau_{n^{-1/2}} : G \to G \) is the canonical dilation whose scale is \( n^{-1/2} \), and the diffusion process \( \{Y_t\}_{0 \leq t \leq 1} \) is generated by the homogenized sub-Laplacian with the \( g^{(1)} \)-valued drift \( \rho_\mathcal{R}(\gamma_p) \) defined on \( G \) equipped with the Albanese metric \( g^{(0)} \) associated with the symmetrized transition probability \( p_0 \).

Here we would like to emphasize that, to our best knowledge, the present paper provides the first result on CLTs in the nilpotent setting in which a \( g^{(1)} \)-valued drift appears in the infinitesimal generator of the limiting diffusion. On the other hand, as mentioned in [10], there are many papers on CLTs in which \( g^{(2)} \)-valued drift like \( \beta(\Phi_0) \) appears in the infinitesimal generator of the limiting diffusion. In view of these circumstances, the study of the long time asymptotics of random walks on more general graphs by applying our latter scheme would be an interesting problem.

The rest of the present paper is organized as follows: In Section 2, we introduce our framework and state the main results (Theorems 2.1 and 2.2). In Section 3, for readers’ convenience, we give a brief idea of Theorems 2.1 and 2.2 in the case where \( \Gamma \) is the 3-dimensional discrete Heisenberg group \( \mathbb{Z}^3 \), one of the most simplest discrete nilpotent groups. We make a preparation from nilpotent Lie groups and discrete geometric analysis in Section 4. We devote ourselves to prove the main results in Section 5. Since the realization map \( \Phi^{(\epsilon)} \) is not necessarily modified harmonic, several technical difficulties arise in the
proof. To overcome them, we take a modified harmonic realization \( \Phi_0^{(r)} : X \to G \) and show that the \((g^{(1)}-)\)-corrector, the difference between \( \Phi^{(r)} \) and \( \Phi_0^{(r)} \) in the \( g^{(1)} \)-direction, is not so big. This approach is the so-called corrector method in the context of stochastic homogenization theory, and it is effectively used in the study of random walks in random environments (see e.g., Biskup [5], Papanicolaou–Varadhan [26], Kozlov [18] and Kumagai [19]). In Section 5.1, we present a key property (Proposition 5.1) of the family of modified harmonic realizations \( (\Phi_0^{(r)})_{0 \leq r \leq 1} \). Combining this property with Trotter’s approximation theorem, we prove a semigroup CLT (Theorem 2.1) under only \((A1)\) in Section 5.2. In Section 5.3, we prove a functional CLT (Theorem 5.4) for a random walk \( \{\xi_n^{(n-1/2)} := \Phi_0^{(n-1/2)}(w_n^{(n-1/2)})\}_{n=1}^{\infty} \) under not only \((A1)\) but \((A2)\). To prove such kind of CLT which is much stronger than the semigroup CLT, we show in Lemma 5.6 tightness of the family of probability measures induced by the \( G \)-valued stochastic processes given by the geodesic interpolation of \( \{\tau_{n-1/2}(\xi_n^{(n-1/2)}) ; 0 \leq t \leq 1\}_{n=1}^{\infty} \). In the case \( r = 2 \), we can easily prove it by combining the modified harmonicity of each \( \Phi_0^{(r)} \) with standard martingale techniques. On the other hand, the same argument as in the case \( r = 2 \) is insufficient in the case \( r \geq 3 \). To handle the higher-step terms, we employ a pathwise argument similar to [10], which is inspired by the Lyons extension theorem (cf. Lyons [21], Lyons–Qian [22], Lyons–Caruana–Lévy [20] and Friz–Victoir [6]) in rough path theory. Since rough path theory is built on free nilpotent Lie groups and our nilpotent Lie group \( G \) is not necessarily free, we need a careful examination of the proof of the Lyons extension theorem. Combining Theorem 5.4 with several nice properties of the \((g^{(1)}-)\)-corrector, we then prove that a functional CLT (Theorem 2.2) also holds for the family of random walks \( \{\xi_n^{(n-1/2)}\}_{n=1}^{\infty} \). Finally, in Section 6, we give an example of non-symmetric random walks on nilpotent covering graphs with explicit computations.

Throughout the present paper, \( C \) denotes a positive constant that may change from line to line and \( O(\cdot) \) stands for the Landau symbol. If the dependence of \( C \) and \( O(\cdot) \) are significant, we denote them like \( C(N) \) and \( O_N(\cdot) \), respectively.

2 Framework and Results

Let \( \Gamma \) be a torsion free, finitely generated nilpotent group of step \( r \) \((r \in \mathbb{N}) \) and \( X = (V, E) \) a \( \Gamma \)-nilpotent covering graph, where \( V \) is the set of its vertices and \( E \) is the set of all oriented edges. For \( e \in E \), we denote by \( o(e) \), \( t(e) \) and \( \overline{e} \) the origin, the terminus and the inverse edge of \( e \), respectively. We set \( E_x = \{ e \in E \mid o(e) = x \} \) for \( x \in V \). Let \( \Omega_{x,n}(X) \) be the set of all paths \( c = (e_1, e_2, \ldots, e_n) \) of length \( n \in \mathbb{N} \cup \{\infty\} \) starting from \( x \in V \). For simplicity, we write \( \Omega_x(X) := \Omega_{x,\infty}(X) \). Denote by \( X_0 = (V_0, E_0) := \Gamma \backslash X \) the quotient graph of \( X \), which is finite by definition.

Let \( p : E \to [0, 1] \) be a \( \Gamma \)-invariant transition probability satisfying

\[
\sum_{e \in E_x} p(e) = 1 \quad (x \in V) \quad \text{and} \quad p(e) + p(\overline{e}) > 0 \quad (e \in E).
\]

The transition probability \( p \) induces, in a natural manner, the probability measure \( \mathbb{P}_x \) on the set \( \Omega_x(X) \) \((x \in V) \). The random walk associated with \( p \) is the time-homogeneous Markov chain \((\Omega_x(X), \mathbb{P}_x, \{w_n\}_{n=0}^{\infty})\) with values in \( X \) defined by \( w_n(c) := o(e_{n+1}) \) for
We define the transition operator $L$ associated with the random walk by

$$L f(x) := \sum_{e \in E_x} p(e) f(t(e)) \quad (x \in V, \ f : V \to \mathbb{R})$$

and the the $n$-step transition probability by $p(n, x, y) := L^n \delta_y(x)$ for $n \in \mathbb{N}$ and $x, y \in V$, where $\delta_y$ stands for the Dirac delta at $y$. Since $p$ is $\Gamma$-invariant, the random walk on $X$ associated with $p$ induces a random walk on the quotient graph $X_0$ through the covering map $\pi : X \to X_0$, and vice versa. By abuse of the notation, we denote the transition probability of the random walk on $X_0$ by $p : E_0 \to [0, 1]$. We define the $n$-step transition probability $p(n, x, y)$ for $n \in \mathbb{N}$, $x, y \in V_0$ as above.

Throughout the present paper, we assume that the random walk on $X_0$ is irreducible, that is, for every $x, y \in V_0$, there exists some $n = n(x, y) \in \mathbb{N}$ such that $p(n, x, y) > 0$. Note that the irreducibility of the random walk on $X$ implies this condition. Conversely, this condition does not imply the irreducibility on $X$ in general. Under the irreducibility condition on $X_0$, by applying the Perron-Frobenius theorem, we find a unique positive function $m : V_0 \to (0, 1]$, called the invariant probability measure on $X_0$, satisfying

$$\sum_{x \in V_0} m(x) = 1 \quad \text{and} \quad m(x) = \sum_{e \in (E_0)_x} p(\bar{e}) m(t(e)) \quad (x \in V_0).$$

Set $\tilde{m}(e) := p(e)m(\rho(e))$ for $e \in E_0$. The random walk on $X_0$ is called $(m)$-symmetric if $\tilde{m}(e) = \tilde{m}(\bar{e})$ holds for every $e \in E_0$. Otherwise, it is called $(m)$-non-symmetric. We also write $m : V \to (0, 1]$ for the $\Gamma$-invariant lift of $m : V_0 \to (0, 1]$. Let $H_1(X_0, \mathbb{R})$ be the first homology group of $X_0$. We define the homological direction of the random walk on $X_0$ by

$$\gamma_p := \sum_{e \in E_0} \tilde{m}(e) e \in H_1(X_0, \mathbb{R}).$$

We easily see that the random walk on $X_0$ is $(m)$-symmetric if and only if $\gamma_p = 0$. For more details, see Section 4.3.

We introduce a continuous state space in which the $\Gamma$-nilpotent covering graph $X$ is properly embedded. We find a connected and simply connected nilpotent Lie group $G_T = (G, \cdot)$ such that $\Gamma$ is isomorphic to a cocompact lattice in $G$ due to Mal’cèv’s theorem (cf. Mal’cèv [23]). We call a piecewise smooth $\Gamma$-equivariant map $\Phi : X \to G$ a periodic realization of $X$. We write $(g, [\cdot, \cdot])$ for the Lie algebra of $G$.

We construct a new product $* \cdot$ on $G$ in the following way. Set $n_1 := g$ and $n_{k+1} := [g, n_k]$ for $k \in \mathbb{N}$. Then we have $g = n_1 \supset \cdots \supset n_r \supseteq n_{r+1} = \{0 \}$ by definition. Define the subspace $g^{(k)}$ of $g$ by $n_k = g^{(k)} \oplus n_{k+1}$ for $k = 1, 2, \ldots, r$. Then the Lie algebra $g$ is decomposed as $g = g^{(1)} \oplus g^{(2)} \oplus \cdots \oplus g^{(r)}$ and each $Z \in g$ is uniquely written as $Z = Z^{(1)} + Z^{(2)} + \cdots + Z^{(r)}$, where $Z^{(k)} \in g^{(k)}$ for $k = 1, 2, \ldots, r$. Then we define a map $\tau_{\varepsilon}(g) : g \to g$ by $\tau_{\varepsilon}(g) : g \to g$ by $\varepsilon Z^{(1)} + \varepsilon^2 Z^{(2)} + \cdots + \varepsilon^r Z^{(r)}$ for $\varepsilon \geq 0$ and also define a Lie bracket product $[\cdot, \cdot]$ on $g$ by

$$[Z_1, Z_2] := \lim_{\varepsilon \searrow 0} \frac{\varepsilon}{\varepsilon} \left[ \tau_{\varepsilon}^{(0)}(Z_1), \tau_{\varepsilon}^{(0)}(Z_2) \right] \quad (Z_1, Z_2 \in g).$$

We introduce a map $\tau_{\varepsilon} : G \to G$, called the dilation operator on $G$, by

$$\tau_{\varepsilon}(g) := \exp \left( \frac{\varepsilon}{\varepsilon} \log (g) \right) \quad (\varepsilon \geq 0, \ g \in G), \quad \text{(2.1)}$$

which is the scalar multiplication on $G$. Then a Lie group product $* \cdot$ on $G$ is defined by

$$g * h := \lim_{\varepsilon \searrow 0} \frac{1}{\varepsilon} \left( \tau_{1/\varepsilon}(g) \cdot \tau_{1/\varepsilon}(h) \right) \quad (g, h \in G).$$
The Lie group $G_\infty = (G, \ast)$ is called the limit group of $(G, \cdot)$. We note that the Lie algebra of $(G, \ast)$ coincides with $(g, [\cdot, \cdot])$.

We equip $(G, \cdot)$ with the so-called Carnot–Carathéodory metric $d_{CC}$, which is an intrinsic metric on $G$ defined by

$$d_{CC}(g, h) := \inf \left\{ \int_0^1 \| \dot{w}_t \|_{g_0^{(0)}} dt \mid \begin{array}{c} w \in \text{Lip}([0, 1]; G), \ w_0 = g, \ w_1 = h, \\
\text{w is tangent to } g^{(1)} \end{array} \right\}, \quad (2.2)$$

for $g, h \in G$, where $\text{Lip}([0, 1]; G)$ denotes the set of all Lipschitz continuous paths and $\| \cdot \|_{g_0^{(0)}}$ the Albanese norm associated with $p_0$ on $g^{(1)}$ defined below. See [10, Section 3.2] for more details.

Let $\pi_1(X_0)$ be the fundamental group of $X_0$. Then we take a canonical surjective homomorphism $\rho : \pi_1(X_0) \to \Gamma$. This map then gives rise to a surjective homomorphism $\rho : H_1(X_0, \mathbb{Z}) \to \Gamma/[\Gamma, \Gamma]$ so that we have the canonical surjective linear map $\rho_\mathbb{R}$ from $H_1(X_0, \mathbb{R})$ onto $g^{(1)}$. We call $\rho_\mathbb{R}(\gamma_p) \in g^{(1)}$ the asymptotic direction. Note that $\gamma_p = 0$ implies $\rho_\mathbb{R}(\gamma_p) = 0$. However, the converse does not always hold. We induce a flat metric $g_0^{(1)}$ on $g^{(1)}$, called the Albanese metric associated with $p$ by the discrete Hodge-Kodaira theorem (cf. [16, Lemma 5.2]).

For the given transition probability $p$, we introduce a family of $\Gamma$-invariant transition probabilities $(p_\varepsilon)_{0 \leq \varepsilon \leq 1}$ on $X$ by

$$p_\varepsilon(e) := p_0(e) + \varepsilon q(e) \quad (e \in E), \quad (2.3)$$

where

$$p_0(e) := \frac{1}{2} \left( p(e) + \frac{m(t(e))}{m(o(e))} p(e) \right), \quad q(e) := \frac{1}{2} \left( p(e) - \frac{m(t(e))}{m(o(e))} p(e) \right).$$

We note that the family $(p_\varepsilon)_{0 \leq \varepsilon \leq 1}$ is given by the linear interpolation between the transition probability $p = p_1$ and the $m$-symmetric probability $p_0$. Moreover, the homological direction $\gamma_p$ equals $\varepsilon \gamma_p$ and the invariant measure $m_\varepsilon$ associated with $p_\varepsilon$ coincides with $m$ for $0 \leq \varepsilon \leq 1$ (cf. [16, Proposition 2.3]).

Let $L_\varepsilon$ be the transition operator associated with $p_\varepsilon$ for $0 \leq \varepsilon \leq 1$. We also denote by $g_0^{(e)}$ the Albanese metric on $g^{(1)}$ associated with $p_\varepsilon$. We write $G_\varepsilon$ for the nilpotent Lie group of step $r$ whose Lie algebra is $g = (g^{(1)}, g_0^{(e)}) \oplus g^{(2)} \oplus \cdots \oplus g^{(r)}$.

We now take the family of periodic realizations $(\Phi^{(r)})_{0 \leq \varepsilon \leq 1}$ satisfying

(A1): For every $0 \leq \varepsilon \leq 1$,

$$\sum_{x \in F} m(x) \log \left( \Phi^{(r)}(x)^{-1} \cdot \Phi^{(0)}(x) \right)_{g^{(1)}} = 0, \quad (2.4)$$

where $F$ denotes a fundamental domain of $X$.

We note that it is always possible to take $(\Phi^{(r)})_{0 \leq \varepsilon \leq 1}$ satisfying (A1).

For a metric space $T$, we denote by $C_\infty(T)$ the space of continuous functions on $T$ vanishing at infinity with the uniform topology $\| \cdot \|_\infty^T$. We define an approximation operator $P_\varepsilon : C_\infty(G_0^{(0)}) \to C_\infty(X)$ by $P_\varepsilon f(x) := f(\tau_\varepsilon \Phi^{(e)}(x))$ for $0 \leq \varepsilon \leq 1$ and $x \in V$. We extend each element $Z \in g$ to a left invariant vector field $Z_\varepsilon$ on the stratified Lie group $(G, \ast)$ in the usual manner. We take an orthonormal basis $\{V_1, V_2, \ldots, V_{d_1}\}$ of $(g^{(1)}, g_0^{(0)})$. Then the first main result is stated as follows:
Theorem 2.1

For $0 \leq s \leq t$ and $f \in C_{0\infty}(G(0))$, we have

$$\lim_{n \to \infty} \left\| L_{(n^{-1/2})}^{[nt] - [ns]} P_{n^{-1/2}} f - P_{n^{-1/2}} e^{-\tau_{n^{-1/2}}(t-s)A} f \right\|_{\infty}^X = 0,$$

(2.5)

where $(e^{-tA})_{t \geq 0}$ is the $C_0$-semigroup whose infinitesimal generator $A$ is given by

$$A = -\frac{1}{2} \sum_{i=1}^{d_1} V_{i*}^2 - \rho \Re(y_p).$$

(2.6)

(2) Let $\mu$ be a Haar measure on $G(0)$. Then, for any $f \in C_{0\infty}(G(0))$ and for any sequence $\{x_n\}_{n=1}^{\infty} \subset V$ satisfying $\lim_{n \to \infty} \tau_{n^{-1/2}}(\Phi(\alpha^{-1/2})(x_n)) =: g \in G(0)$, we have

$$\lim_{n \to \infty} L_{(n^{-1/2})}^{[nt] - [ns]} P_{n^{-1/2}} f(x_n) = e^{-tA} f(g) = \int_{G(0)} \mathcal{H}_t(h^{-1} \ast g) f(h) \mu(dh) \quad (t \geq 0),$$

(2.7)

where $\mathcal{H}_t(g)$ is a fundamental solution to $\partial u/\partial t + Au = 0$.

We now fix a reference point $x_\ast \in V$ such that $\Phi(0)(x_\ast) = 1_G$ and put

$$\bar{x}_n^{(\varepsilon)}(c) := \Phi(\varepsilon)(w_n^{(\varepsilon)}(c)) \quad (0 \leq \varepsilon \leq 1, \ n = 0, 1, 2, \ldots, c \in \Omega_{x_\ast}(X)).$$

Note that (A1) does not imply that $\Phi^{(\varepsilon)}(x_\ast) = 1_G$ for $0 < \varepsilon \leq 1$ in general. We then obtain a $G(0)$-valued random walk $(\Omega_{x_\ast}(X), \mathcal{P}^{(\varepsilon)}, \{\bar{x}_n^{(\varepsilon)}\}_{n=0}^{\infty})$ associated with the transition probability $p_\varepsilon$. For $t \geq 0$, $n = 1, 2, \ldots$ and $0 \leq \varepsilon \leq 1$, let $\bar{X}_t^{(\varepsilon, n)}$ be a map from $\Omega_{x_\ast}(X)$ to $G(0)$ given by

$$\bar{X}_t^{(\varepsilon, n)}(c) := \tau_{n^{-1/2}}(\bar{x}_n^{(\varepsilon)}(c)) \quad (c \in \Omega_{x_\ast}(X)).$$

We write $D_n$ for the partition $\{t_k = k/n \mid k = 0, 1, 2, \ldots, n\}$ of the time interval $[0, 1]$ for $n \in \mathbb{N}$. We define

$$\bar{Y}_t^{(\varepsilon, n)}(c) := \tau_{n^{-1/2}}(\bar{x}_n^{(\varepsilon)}(c)) = \tau_{n^{-1/2}}(\Phi^{(\varepsilon)}(w_k^{(\varepsilon)}(c))) \quad (t_k \in D_n, c \in \Omega_{x_\ast}(X))$$

(2.8)

and also define the $G(0)$-valued continuous stochastic process $(\bar{Y}_t^{(\varepsilon, n)})_{0 \leq t \leq 1}$ given by the $dCC$-geodesic interpolation of $(\bar{Y}_t^{(\varepsilon, n)})_{0 \leq t \leq 1}$. We consider a stochastic differential equation

$$dY_t = \sum_{i=1}^{d_1} V_{i*}^2(Y_t) \circ dB_t^i + \rho \Re(y_p) \ast (Y_t) dt, \quad Y_0 = 1_G,$$

(2.9)

where $(B_t)_{0 \leq t \leq 1} = (B_1, B_2, \ldots, B_{d_1})_{0 \leq t \leq 1}$ is an $\mathbb{R}^{d_1}$-valued standard Brownian motion starting from the origin. It is known that the infinitesimal generator of (2.9) coincides with $-\mathcal{A}$ defined by (2.6) (see Ishiwata–Kawabi–Namba [10, Section 5]). Let $(Y_t)_{0 \leq t \leq 1}$ be the $G(0)$-valued diffusion process which is the solution to (2.9). For $\alpha < 1/2$, we define the $\alpha$-Hölder distance $\rho_\alpha$ on $C([0, 1]; G(0))$ by

$$\rho_\alpha(w^1, w^2) := \sup_{0 \leq s < t \leq 1} \left( d_{CC}(u_s, u_t) + d_{CC}(1_G, u_0) \right), \quad \alpha := (w_t^1)^{1-\alpha} \ast w_t^2 \quad (0 \leq t \leq 1).$$

Then we set $C^{0, \alpha}_{\text{Hölder}}([0, 1]; G(0)) := \text{Lip}([0, 1]; G(0))^{\rho_\alpha}$, which is a Polish space (cf. Friz–Victoir [6, Section 8]). Let $\bar{P}^{(\varepsilon, n)}$ be the probability measure on $C^{0, \alpha}_{\text{Hölder}}([0, 1]; G(0))$ induced by the stochastic process $\bar{Y}_t^{(\varepsilon, n)}$ for $0 \leq \varepsilon \leq 1$ and $n \in \mathbb{N}$.
To establish the functional CLT for the family of non-symmetric random walks \( \{Z_n^{(k)}\}_{n=0}^{\infty} \), we need to impose an additional assumption.

(A2): There exists a positive constant \( C \) such that, for \( k = 2, 3, \ldots, r \),

\[
\sup_{0 \leq t \leq 1} \max_{x \in \mathcal{F}} \| \log \left( \Phi^{(k)}(x)^{-1} \cdot \Phi^{(0)}(x) \right) \|_{g^{(k)}} \leq C,
\]

where \( \| \cdot \|_{g^{(k)}} \) denotes a Euclidean norm on \( g^{(k)} = \mathbb{R}^{dk} \) for \( k = 2, 3, \ldots, r \).

Intuitively speaking, the situations that the distance between \( \Phi^{(k)} \) and \( \Phi^{(0)} \) tends to be too big as \( \varepsilon \searrow 0 \) are removed under (A2). By setting

\[
\log \left( \Phi^{(k)}(x) \right) |_{g^{(k)}} = \log \left( \Phi^{(0)}(x) \right) |_{g^{(k)}} \quad (x \in \mathcal{F}, k = 2, 3, \ldots, r)
\]

for \( \Phi^{(k)} : X \to G \) with (2.4), the family \( \{\Phi^{(k)}\}_{0 \leq k \leq 1} \) satisfies (A2). This means that it is always possible to take a family \( \{\Phi^{(k)}\}_{0 \leq k \leq 1} \) satisfying (A2) as well as (A1).

Then our main theorem is stated as follows:

**Theorem 2.2** We assume (A1) and (A2). Then the sequence \( \{\sqrt{n(1-\varepsilon)}Y_n\}_{n=1}^{\infty} \) converges in law to the diffusion process \( Y_{1_{0 \leq l \leq 1}} \) in \( C^{0,\alpha-HT}([0,1];G_0) \) as \( n \to \infty \) for all \( \alpha < 1/2 \).

In our preceding paper [10], we captured another \( G \)-valued diffusion process \( \{\sqrt{n(1-\varepsilon)}Y_n\}_{n=1}^{\infty} \) by applying the transition-shift scheme mentioned in Section 1. More precisely, the infinitesimal generator of \( \{\sqrt{n(1-\varepsilon)}Y_n\}_{0 \leq l \leq 1} \) is the homogenized sub-Laplacian on \( G \) with a non-trivial drift \( \beta(\Phi_0) \in g^{(2)} \) arising from the non-symmetry of the given random walk, where the Lie group \( G \) is equipped with the Albanese metric \( g_0 = g_0^{(1)} \). In particular, even in the centered case \( \rho_{R}(Y) = 0_{g} \), the non-trivial drift \( \beta(\Phi_0) \) remains in general. On the other hand, in this case, the limiting diffusion \( (Y_l)_{0 \leq l \leq 1} \) is generated by the homogenized sub-Laplacian on \( G_{(0)} \) equipped with the Albanese metric \( g_0^{(0)} \). See [10, Remark 5.3] for explicit expressions of the limiting diffusions \( (Y_l)_{0 \leq l \leq 1} \) and \( (\sqrt{n(1-\varepsilon)}Y_n)_{0 \leq l \leq 1} \).

### 3 A Brief Outline of the Proof Through a Simple Example

Some readers who are not familiar with discrete geometric analysis may find the argument in the present paper too complicated. Hence, to help them get a bird’s eye view of the proof of the main results (Theorems 2.1 and 2.2), we give a brief outline of the proof in the case where \( \Gamma \) is the 3-dimensional discrete Heisenberg group \( \Gamma = \mathbb{H}^3(\mathbb{Z}) \). Here \( \mathbb{H}^3(\mathbb{Z}) \) is one of the most simplest nilpotent groups defined by

\[
\mathbb{H}^3(\mathbb{Z}) := \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \bigg| x, y, z \in \mathbb{Z} \right\}.
\]

It is known that \( \Gamma^1 = \mathbb{H}^3(\mathbb{Z}) \) is a cocompact lattice in the 3-dimensional Heisenberg group \( G = \mathbb{H}^3(\mathbb{R}) = (\mathbb{R}^3, \ast) \), where the product \( \ast \) on \( \mathbb{R}^3 \) is given by

\[
(x, y, z) \ast (x', y', z') = (x + x', y + y', z + z' + xy').
\]

The corresponding Lie algebra \( g \) is given by \( (\mathbb{R}^3, [, ,]) \) generated by \( X_1 = (1, 0, 0), X_2 = (0, 1, 0) \) and \( X_3 = (0, 0, 1) \) with \( [X_1, X_2] = X_3 \) and \( [X_1, X_3] = [X_2, X_3] = 0_{g} \) under the matrix bracket \( [X, Y] := XY - YX \) for \( X, Y \in g \). We then see that the Lie algebra \( g \) is
decomposed as \( g = g^{(1)} \oplus g^{(2)} \), where \( g^{(1)} := \text{span}_R \{X_1, X_2\} \) and \( g^{(2)} := \text{span}_R \{X_3\} \). We should note that the nilpotent Lie group \( G = \mathbb{H}^3(\mathbb{R}) \) is free of step 2, which implies that the limit group \( G_\infty \) coincides with \( G \) itself. In general, the difference between \( G \) and \( G_\infty \) may appear in more general step cases.

Let \( X = (V, E) \) be a Cayley graph of \( \Gamma \) with a generating set \( S = \{ (\pm 1, 0, 0), (0, \pm 1, 0), (0, 0, \pm 1) \} \).

Namely, \( X \) is an oriented graph with vertex set \( V = \Gamma \) and for \( \gamma, \eta \in \Gamma \), \( \gamma \) is adjacent to \( \eta \) if \( \gamma^{-1} \star \eta \in S \). The quotient graph \( X_0 = (V_0, E_0) = \Gamma \backslash X \) is a 3-bouquet graph with \( V_0 = \{ x_0 \} \) and \( E_0 \) consisting of three loops \( e_1, e_2, e_3 \) and their inverse loops (see Fig. 1).

Consider a non-symmetric random walk on \( X_0 \) defined by

\[
\begin{align*}
p(e_1) &= \alpha, & p(e_2) &= \beta, & p(e_3) &= \gamma, & p(\bar{e}_1) &= \alpha', & p(\bar{e}_2) &= \beta', & p(\bar{e}_3) &= \gamma',
\end{align*}
\]

where \( \alpha, \beta, \gamma, \alpha', \beta', \gamma' > 0 \) and \( \alpha + \beta + \gamma + \alpha' + \beta' + \gamma' = 1 \). Then the one-parameter family of non-symmetric transition probabilities \( (p_\varepsilon)_{0 \leq \varepsilon \leq 1} \) is given by

\[
\begin{align*}
p_\varepsilon(e_1) &= \frac{(\alpha + \alpha') + \varepsilon(\alpha - \alpha')}{2}, & p_\varepsilon(e_2) &= \frac{(\beta + \beta') + \varepsilon(\beta - \beta')}{2}, \\
p_\varepsilon(e_3) &= \frac{(\gamma + \gamma') + \varepsilon(\gamma - \gamma')}{2}, & p_\varepsilon(\bar{e}_1) &= \frac{(\alpha + \alpha') + \varepsilon(\alpha' - \alpha)}{2}, \\
p_\varepsilon(\bar{e}_2) &= \frac{(\beta + \beta') + \varepsilon(\beta' - \beta)}{2}, & p_\varepsilon(\bar{e}_3) &= \frac{(\gamma + \gamma') + \varepsilon(\gamma' - \gamma)}{2}
\end{align*}
\]

for every \( 0 \leq \varepsilon \leq 1 \). Let \( \Phi^{(e)} : X \to G \) \( 0 \leq \varepsilon \leq 1 \) be a family of \( \Gamma \)-equivariant realizations. We should note that, since \( X_0 \) has only one vertex, each realization \( \Phi^{(e)} \) enjoys the modified harmonicity in the sense of (5.1). Thanks to this fact, the random walk

\[
\left\{ \log \left( \Phi^{(e)}(w_n^{(e)}) \right) \right\}_{g^{(1)}} - n \varepsilon \rho \mathbb{R} (\gamma_p) \right\}_{n=0}^\infty
\]

is a \( g^{(1)} \)-valued martingale, which will play a key role in the proof. Here, \( \{ w_n^{(e)} \}_{n=0}^\infty \) is the random walk on \( X \) associated with the transition probability \( p_\varepsilon \). See e.g., Namba [24, Lemma 2.5.3] for details.

**Step 1** (To show Theorem 2.1): In order to show the first main theorem (Theorem 2.1), we need to prove the convergence of the infinitesimal generator under the CLT-scaling.

[Fig. 1 A part of the Cayley graph \( X \) of \( \Gamma = \mathbb{H}^3(\mathbb{Z}) \) and its quotient graph \( X_0 \).]
We apply the Taylor expansion formula to \((I - L^N_{(\varepsilon)})P_t f\) in \(\varepsilon\). Then, the first order terms give rise to the constant drift of \(\rho_{\mathbb{R}}(\gamma_p)\) due to the modified harmonicity of \(\Phi^{(\varepsilon)}\) so that we formally have, for \(x \in V\) and \(f \in C^\infty_0(G)\),

\[
\frac{1}{N\varepsilon^2}(I - L^N_{(\varepsilon)})P_t f(x) = P_t\left(\mathcal{L} - \rho_{\mathbb{R}}(\gamma_p) - \beta_{(\varepsilon)}(\Phi^{(\varepsilon)})\right)f(x) + O\left(\frac{1}{N}\right) + O(N^2 \varepsilon)
\]

as \(N \to \infty, \varepsilon \searrow 0\) with \(N^2\varepsilon \searrow 0\), where \(\mathcal{L}\) is a sub-elliptic operator on \(G\) and \(\beta_{(\varepsilon)}(\Phi^{(\varepsilon)}) \in \mathfrak{g}^{(2)}\) is a quantity given by (5.5). We verify that (A1) implies that \(\beta_{(\varepsilon)}(\Phi^{(\varepsilon)})\) converges to zero as \(\varepsilon \searrow 0\) (Proposition 5.1). This immediately leads to Lemma 5.1. Finally, we combine the Trotter approximation theorem (cf. [35]) with Lemma 5.3. Then we arrive at Theorem 2.1 by putting \(\varepsilon = n^{-1/2}\) and letting \(n \to \infty\). Moreover, if we endow \(g^{(1)}\) with the Albanese metric \(g_0^{(0)}\) associated with \(p_0\), the sub-elliptic operator \(\mathcal{L}\) coincides with \(-\frac{1}{2}(V_1^2 + V_2^2)\) for some orthonormal basis \(\{V_1, V_2\}\) of \((g^{(1)}, g_0^{(0)})\). 

Step 2 (To show the functional CLT, Section 5.3): We impose the additional assumption (A2) to \((\Phi^{(\varepsilon)})_{0 \leq \varepsilon \leq 1}\). Let \((\mathbb{Y}^{(n-1/2,n)})_{n=1}^\infty\) be a sequence of stochastic processes defined by (2.8). In order to show the functional CLT, it is sufficient to prove the following two items:

- the convergence of the finite-dimensional distribution of \((\mathbb{Y}^{(n-1/2,n)})_{n=1}^\infty\),
- the tightness of the image measures \(\{P^{(n-1/2,n)} = \mathbb{P}_{\mathbb{X}^n} \circ (\mathbb{Y}^{(n-1/2,n)})_{n=1}^\infty\}

Since the former item is obtained by a simple application of Theorem 2.1, we mention the proof of the latter item. To deduce the item, it suffices to show the existence of a positive constant \(C > 0\) independent of \(n \in \mathbb{N}\) such that

\[
\mathbb{E}^{\mathbb{P}^{(n-1/2)}}_\mathbb{X}^n\left[d_{CC}(\mathbb{Y}^{(n-1/2,n)}_s, \mathbb{Y}^{(n-1/2,n)}_t)^{4m}\right] \leq C(t - s)^{2m}
\]

for \(m \in \mathbb{N}\) and \(0 \leq s \leq t \leq 1\). We emphasize that several martingale inequalities such as the Birkholder–Davis–Gundy inequality for the martingale \(\{\log(\Phi^{(\varepsilon)}(w_n^{(\varepsilon)}))\}_{n \geq 0} - n \varepsilon \rho_{\mathbb{R}}(\gamma_p)\}_{n \geq 0}\) play important roles to obtain (3.2). Consequently, we obtain that \(\mathbb{Y}^{(n-1/2,n)}\) converges in law to a \(G\)-valued diffusion process \(\mathbb{Y}\) which is a solution to the SDE

\[
dY_t = V_1^2(Y_t) \circ dB_t^1 + V_2^2(Y_t) \circ dB_t^2 + \rho_{\mathbb{R}}(\gamma_p)(Y_t) dt, \quad Y_0 = 1_G
\]

in the Hölder space \(C^{0,\alpha-\text{Hol}}([0, 1]; G^{(0)})\) for \(\alpha < 1/2\), where \((B_t^1, B_t^2)_{0 \leq t \leq 1}\) is a 2-dimensional standard Brownian motion starting from the origin.

As already mentioned in Section 1, several essential difficulties appear in the case of more general covering graphs. The one difficulty comes from the number of vertices of the quotient graph \(X_0\). If it is larger than one, then each \(\Gamma\)-equivariant realization \(\Phi^{(\varepsilon)}, 0 \leq \varepsilon \leq 1\) is not always modified harmonic. Since Step 2 heavily relies on the modified harmonicity, we need to insert an additional step at which we prove the functional CLT for modified harmonic realizations. More precisely, we first show the functional CLT for \((p_{\varepsilon})\)-modified harmonic realizations \((\Phi^{(\varepsilon)})_{0 \leq \varepsilon \leq 1}\) in the same way as Step 2. Next, let \((\Phi^{(\varepsilon)} : X \to G)_{0 \leq \varepsilon \leq 1}\) be a family of \(\Gamma\)-equivariant realizations satisfying (A1) and (A2). Let \((\mathbb{Y}^{(n-1/2,n)})_{n=1}^\infty\) be a sequence of stochastic processes defined by (2.8) with the replacement of \(\Phi^{(\varepsilon)}_0\) by \(\Phi^{(\varepsilon)}\). To measure the difference between \(\Phi^{(\varepsilon)}\) and \(\Phi^{(\varepsilon)}_0\), we
introduce the \((g^{(1)})\)-corrector \(\text{Cor}^{(\varepsilon)}\) defined by (5.2). Since the nice estimation of the \(g^{(1)}\)-corrector (see (5.3)), we know that the same moment estimate as (3.2) can be established for \(\{\overline{Y}^{(n-1/2,n)}\}_{n=1}^{\infty}\) (see Lemma 5.1). Finally, we obtain that the sequence \(\{\overline{Y}^{(n-1/2,n)}\}_{n=1}^{\infty}\) converges in law to the same \(G\)-valued diffusion process \((Y_t)_{0\leq t\leq 1}\) in \(C^{0,\alpha-Hol}([0, 1]; G_{(0)})\) for \(\alpha < 1/2\). This completes the proof of Theorem 2.2.

Another difficulty arises when the nilpotency of \(\Gamma\) is greater than 2. If the nilpotent Lie group \(G\) is of step \(r\geq 3\), it is known that the dilation operators does not work well. This circumstance makes the analysis on \(G\) much difficult. The main reason of introducing the notion of limit group \(G_{\infty}\) is to overcome the difficulty. Nevertheless, there is a further difficulty in the proof of the tightness of \(\{P^{(n-1/2,n)}\}_{n=1}^{\infty}\), since it is hard to establish the moment estimate like (3.2) in higher-step cases. Therefore, we extend a novel pathwise argument inspired by the proof of Lyons’ extension theorem to the cases of (not necessarily free) nilpotent Lie groups (Lemma 5.8). Thanks to this extension, we can prove the tightness of \(\{P^{(n-1/2,n)}\}_{n=1}^{\infty}\) as well as the step-2 cases (Lemma 5.6), which is one of important contributions of the present paper.

4 Preliminaries

4.1 Some Properties on Nilpotent Lie Groups

Let us review some properties of the limit group. For more details, see e.g., Alexopoulos [1], Ishiwata [8] and Ishiwata–Kawabi–Namba [10]. Recall that \((G, \ast)\) is the limit group of a connected and simply connected nilpotent Lie group \((G, \cdot)\) of step \(r\) (see (2.1)). We note that \((G, \ast)\) is stratified. Namely, the decomposition of the corresponding Lie algebra \((g = g^{(1)} \oplus g^{(2)} \oplus \cdots \oplus g^{(r)}, [\cdot, \cdot, \cdot])\) satisfies that \(\|g^{(i)}, g^{(j)}\| \subset g^{(i+j)}\) when \(i + j \leq r\) and \(g^{(1)}\) generates \(g\). We also note that, for \(\varepsilon \geq 0\), the dilation \(\tau_\varepsilon : G \to G\) is a group automorphism on \((G, \ast)\) (see [8, Lemma 2.1]).

We introduce several notations that will be used throughout the present paper.

1. **Global coordinates on \(G\)**: We set \(d_k = \dim \mathbb{R}g^{(k)}\) for \(k = 1, 2, \ldots, r\) and \(d = \dim G\). For \(k = 1, 2, \ldots, r\), we denote by \(\{X^{(k)}_1, X^{(k)}_2, \ldots, X^{(k)}_{d_k}\}\) a basis of the subspace \(g^{(k)}\). We introduce two kinds of global coordinate systems in \(G\) through \(\exp : g \to G\). We identify the nilpotent Lie group \(G\) with \(\mathbb{R}^d\) as a differentiable manifold by

\[
\mathbb{R}^d \ni (g^{(1)}, g^{(2)}, \ldots, g^{(r)})
\]

\[
\mapsto g = \exp \left( g^{(r)}_1 X^{(r)}_1 \right) \cdot \exp \left( g^{(r)}_{d_r-1} X^{(r)}_{d_r-1} \right) \cdots \cdot \exp \left( g^{(r)}_1 X^{(r)}_1 \right) \]

\[
\cdot \exp \left( g^{(r-1)}_1 X^{(r-1)}_1 \right) \cdot \exp \left( g^{(r-1)}_{d_r-1} X^{(r-1)}_{d_r-1} \right) \cdots \cdot \exp \left( g^{(r-1)}_1 X^{(r-1)}_1 \right) \]

\[
\cdots \cdots \cdot \exp \left( g^{(1)}_1 X^{(1)}_1 \right) \cdot \exp \left( g^{(1)}_{d_1-1} X^{(1)}_{d_1-1} \right) \cdots \cdot \exp \left( g^{(1)}_1 X^{(1)}_1 \right) \in G,
\]

which is called the canonical (\(\ast\))-coordinates of the second kind. We also define the canonical (\(\ast\))-coordinates of the second kind just by replacing \(\cdot\) by the deformed product \(\ast\) in the above correspondence.

2. **Campbell–Baker–Hausdorff formula**: The relations between the deformed product and the given product on \(G\) is described by the following formula:

\[
\log \left( \exp(Z_1) \cdot \exp(Z_2) \right) = Z_1 + Z_2 + \frac{1}{2} [Z_1, Z_2] + \cdots \quad (Z_1, Z_2 \in g)\quad (4.1)
\]
Several formulas: It follows from the definition of the deformed product on $G$ that
\[
\log (g * h)_{g^{(k)}} = \log (g \cdot h)_{g^{(k)}} \quad (k = 1, 2).
\] (4.2)

However, this relation does not hold in general if $k = 3, 4, \ldots, r$. The following identities give a comparison between $(-)$-coordinates and $(\cdot)$-coordinates. For $g \in G$, we have the following:
\[
s_{i}^{(k)} = g_{i}^{(k)} \quad (i = 1, 2, \ldots, d_k, \ k = 1, 2),
\] (4.3)
\[
s_{i}^{(k)} = g_{i}^{(k)} + \sum_{0 < |K| \leq k-1} C_K \mathcal{P}^K(g) \quad (i = 1, 2, \ldots, d_k, \ k = 3, 4, \ldots, r)
\] (4.4)

for some constant $C_K$, where $K$ denotes a multi-index $((i_1, k_1), (i_2, k_2), \ldots, (i_{\ell}, k_{\ell}))$ with length $|K| := k_1 + k_2 + \cdots + k_{\ell}$ and $\mathcal{P}^K(g) := g_{i_1}^{(k_1)} \cdot g_{i_2}^{(k_2)} \cdots g_{i_{\ell}}^{(k_{\ell})}$. For $g, h \in G$, we also have the following:
\[
(g * h)_{i}^{(k)} = (g \cdot h)_{i}^{(k)} \quad (i = 1, 2, \ldots, d_k, \ k = 1, 2),
\] (4.5)
\[
(g * h)_{i}^{(k)} = (g \cdot h)_{i}^{(k)} + \sum_{|K_1| + |K_2| \leq k-1} C_{K_1, K_2} \mathcal{P}^{K_1^*}(g) \mathcal{P}^{K_2^*}(g \cdot h)
\] (4.6)

for $k = 3, 4, \ldots, r$.

Homogeneous norms on $G$: We introduce a norm $\| \cdot \|_{g^{(k)}}$ on $g^{(k)}$ by the usual Euclidean norm. If $Z \in g$ is uniquely decomposed by $Z = Z^{(1)} + Z^{(2)} + \cdots + Z^{(r)}$ ($Z^{(k)} \in g^{(k)}$), we define a function $\| \cdot \|_{g} : g \to [0, \infty)$ by $\|Z\|_{g} := \sum_{k=1}^{r} \|Z^{(k)}\|_{g^{(k)}}^{1/k}$. The following are the typical examples of homogeneous norms on $G$:
\[
\|g\|_{cc} := d_{cc}(1_G, g), \quad \|g\|_{Hom} := \|\log (g)\|_g \quad (g \in G).
\]

Note that these homogeneous norms are equivalent in the sense that there exists a positive constant $C$ such that $C^{-1} \|g\|_{cc} \leq \|g\|_{Hom} \leq C \|g\|_{cc}$ for $g \in G$ (cf. [10, Proposition 3.1]).

4.2 A Quick Review on Discrete Geometric Analysis

This subsection is concerned with some basics of discrete geometric analysis on graphs. We refer to Kotani–Sunada [16], Sunada [31–33] and references therein for details.

Let $X_0 = (V_0, E_0)$ be a finite graph and consider an irreducible random walk on $X_0$ associated with a transition probability $p : E_0 \to [0, 1]$. Let $m : V_0 \to (0, 1]$ be the normalized invariant probability measure on $X_0$. Set $\bar{m}(e) := p(e)m(\nu(e))$ for $e \in E_0$.

We define the 0-chain group, 1-chain group, 0-cochain group and 1-cochain group by
\[
C_0(X_0, \mathbb{R}) := \left\{ \sum_{x \in V_0} a_x x \mid a_x \in \mathbb{R} \right\}, \quad C_1(X_0, \mathbb{R}) := \left\{ \sum_{e \in E_0} a_e e \mid a_e \in \mathbb{R}, \bar{e} = -e \right\},
\]
\[
C^0(X_0, \mathbb{R}) := \{ f : V_0 \to \mathbb{R} \}, \quad C^1(X_0, \mathbb{R}) := \{ \omega : E_0 \to \mathbb{R} \mid \omega(\bar{e}) = -\omega(e) \},
\]
respectively. An element of $C^1(X_0, \mathbb{R})$ is called a 1-form on $X_0$. The boundary operator $\partial : C_1(X_0, \mathbb{R}) \to C_0(X_0, \mathbb{R})$ and the difference operator $d : C^0(X_0, \mathbb{R}) \to C^1(X_0, \mathbb{R})$ are defined by $\partial(e) = t(e) - o(e)$ for $e \in E_0$ and $df(e) = f(t(e)) - f(o(e))$ for $e \in E_0$, respectively. Then, the first homology group $H_1(X_0, \mathbb{R})$ and the first cohomology group $H^1(X_0, \mathbb{R})$ are defined by $\ker(\partial) \subset C_1(X_0, \mathbb{R})$ and $C^1(X_0, \mathbb{R})/\text{Im}(d)$, respectively. We
write \( L : C^0(X_0, \mathbb{R}) \to C^0(X_0, \mathbb{R}) \) for the transition operator associated with \( p \). We define a special 1-chain by

\[
\gamma_p := \sum_{e \in E_0} \tilde{m}(e)e \in C_1(X_0, \mathbb{R}).
\]

We easily verify that \( \partial(\gamma_p) = 0 \), that is, \( \gamma_p \in H_1(X_0, \mathbb{R}) \). Furthermore, it is clear that the random walk on \( X_0 \) is \((m-)\)-symmetric if and only if \( \gamma_p = 0 \). The 1-cycle \( \gamma_p \) is called the homological direction of the given random walk on \( X_0 \). A 1-form \( \omega \in C^1(X_0, \mathbb{R}) \) is said to be modified harmonic if \( \sum_{e \in E_0} p(e)\omega(x) = \langle \gamma_p, \omega \rangle \) for \( x \in V_0 \). Denote by \( \mathcal{H}^1(X_0) \) the set of modified harmonic 1-forms with the inner product

\[
\langle \omega_1, \omega_2 \rangle_p := \sum_{e \in E_0} \tilde{m}(e)\omega_1(e)\omega_2(e) - \langle \gamma_p, \omega_1 \rangle \langle \gamma_p, \omega_2 \rangle \quad (\omega_1, \omega_2 \in \mathcal{H}^1(X_0))
\]

associated with the transition probability \( p \). We may identify \( H^1(X_0, \mathbb{R}) \) with \( \mathcal{H}^1(X_0) \) by the discrete Hodge-Kodaira theorem (cf. [16, Lemma 5.2]). Then the inner product \( \langle \cdot, \cdot \rangle_p \) is induced on \( H^1(X_0, \mathbb{R}) \) through this identification.

Let \( \Gamma \) be a torsion free, finitely generated nilpotent group of step \( r \). Then a \( \Gamma \)-nilpotent covering graph \( X = (V, E) \) is defined by the \( \Gamma \)-covering of \( X_0 \). Let \( p : E \to [0, 1] \) and \( m : V \to (0, 1] \) be the \( \Gamma \)-invariant lifts of \( p : E_0 \to [0, 1] \) and \( m : V_0 \to (0, 1] \), respectively. Denote by \( \tilde{\pi} : G \to G/[G, G] \) the canonical projection. Since \( \Gamma \) is a cocompact lattice in \( G \), the subset \( \tilde{\pi}(\Gamma) \subset G/[G, G] \) is also a lattice in \( G/[G, G] \cong g^{(1)} \) (cf. Malcev [23] and Raghunathan [27]). Let \( \rho_{\mathbb{R}} : H_1(X_0, \mathbb{R}) \to \tilde{\pi}(\Gamma) \otimes \mathbb{R} \cong g^{(1)} \) be the canonical surjective linear map induced by the surjective homomorphism \( \rho : \pi_1(X_0) \to \Gamma \). We restrict the inner product \( \langle \cdot, \cdot \rangle_p \) on \( H^1(X_0, \mathbb{R}) \) to the subspace \( \text{Hom}(\tilde{\pi}(\Gamma), \mathbb{R}) \), thereafter take it up the dual inner product \( \langle \cdot, \cdot \rangle_{\text{alb}} \) on \( \tilde{\pi}(\Gamma) \otimes \mathbb{R} \). Consequently, we induce a flat metric \( g_0 \) on \( g^{(1)} \) and call it the Albanese metric on \( g^{(1)} \). This procedure can be summarized as follows:

\[
\begin{align*}
(g^{(1)}, g_0) & \cong (\tilde{\pi}(\Gamma) \otimes \mathbb{R}) \xrightarrow{\rho_{\mathbb{R}}} H_1(X_0, \mathbb{R}) \\
\text{Hom}(g^{(1)}, \mathbb{R}) & \cong \text{Hom}(\tilde{\pi}(\Gamma), \mathbb{R}) \xrightarrow{\rho_{\mathbb{R}}} H^1(X_0, \mathbb{R}) \cong (\mathcal{H}^1(X_0), \langle \cdot, \cdot \rangle_p).
\end{align*}
\]

5 Proof of Main Results

5.1 A One-Parameter Family of Modified Harmonic Realizations \((\Phi^{(\varepsilon)}_0)_{0 \leq \varepsilon \leq 1}\)

Recall that \((p_\varepsilon)_{0 \leq \varepsilon \leq 1}\) is the family of transition probabilities defined by (2.3). We now introduce the family of modified harmonic realizations \((\Phi^{(\varepsilon)}_0)_{0 \leq \varepsilon \leq 1}\). Namely, each \(\Phi^{(\varepsilon)}_0\) is the \(\Gamma\)-equivariant realization of \(X\) satisfying

\[
\sum_{e \in E_\varepsilon} p_\varepsilon(e) \log \left( \Phi^{(\varepsilon)}_0(\sigma(e))^{-1} \cdot \Phi^{(\varepsilon)}_0(\tau(e)) \right) \bigg|_{\mathfrak{g}^{(1)}} = \varepsilon \rho_{\mathbb{R}}(\gamma_p) \quad (x \in V).
\]

Moreover, we may assume that \(\Phi^{(0)}_0(x_*) = 1_G\) and

\[
\log \left( \Phi^{(\varepsilon)}_0(x) \right) \bigg|_{\mathfrak{g}^{(k)}} = \log \left( \Phi^{(\varepsilon)}(x) \right) \bigg|_{\mathfrak{g}^{(k)}} \quad (0 \leq \varepsilon \leq 1, x \in \mathcal{F}, k = 2, 3, \ldots, r)
\]
without loss of generality. We define the \((g^{(1)})\)-corrector \(\text{Cor}^{(e)}_{g^{(1)}} : X \rightarrow g^{(1)}\) by
\[
\text{Cor}^{(e)}_{g^{(1)}}(x) := \log \left( \Phi^{(e)}(x) \right) |_{g^{(1)}} - \log \left( \Phi^{(e)}_{0}(x) \right) |_{g^{(1)}} \quad (x \in V, \ 0 \leq e \leq 1).
\] (5.2)
We note that, thanks to (A1), we have
\[
\sum_{x \in \mathcal{F}} m(x) \text{Cor}^{(e)}_{g^{(1)}}(x) = \sum_{x \in \mathcal{F}} m(x) \text{Cor}^{(0)}_{g^{(1)}}(x) \quad (0 \leq e \leq 1).
\] (5.3)
In particular, there exists a positive constant \(M > 0\) independent of \(e \in [0, 1]\) such that
\[
\max_{x \in \mathcal{F}} \| \text{Cor}^{(e)}_{g^{(1)}}(x) \|_{g^{(1)}} \leq M \quad 0 \leq e \leq 1.
\] We also emphasize that, if \((\Phi^{(e)}_{0})_{0 \leq e \leq 1}\) satisfies (A1) and (A2), then \((\Phi^{(e)}_{0})_{0 \leq e \leq 1}\) also satisfies (A1) and (A2), respectively. Indeed, by combining (5.3) and (A1), we see that
\[
\sum_{x \in \mathcal{F}} m(x) \log \left( \Phi^{(e)}_{0}(x)^{-1} \cdot \Phi^{(0)}_{0}(x) \right) |_{g^{(1)}} = 0
\] for \(0 \leq e \leq 1\), which means that the family \((\Phi^{(e)}_{0})_{0 \leq e \leq 1}\) enjoys the assumption (A1).
Furthermore, by using (A2) and \(\Phi^{(e)}_{0}(x)^{(i)} = \Phi^{(e)}(x)^{(i)}\) for \(x \in V, \ 0 \leq e \leq 1\) and \(i = 2, 3, \ldots, r\), we have
\[
\sup_{0 \leq e \leq 1} \max_{x \in \mathcal{F}} \| \log \left( \Phi^{(e)}_{0}(x)^{-1} \cdot \Phi^{(0)}_{0}(x) \right) |_{g^{(2)}} \|_{g^{(2)}} \leq C
\] (5.4)
for some \(C > 0\), which implies that the family \((\Phi^{(e)}_{0})_{0 \leq e \leq 1}\) satisfies the assumption (A2).

We put \(d \Phi^{(e)}_{0}(e) = \Phi^{(e)}_{0}(o(e))^{-1} \cdot \Phi^{(e)}_{0}(t(e))\) for \(0 \leq e \leq 1\) and \(e \in E\). The aim of this subsection is to study the quantity
\[
\beta_{(e)}(\Phi^{(e)}_{0}) := \sum_{\xi \in E_{0}} \tilde{m}_{e}(\xi) \log \left( d \Phi^{(e)}_{0}(\xi) \right) |_{g^{(2)}} \quad (0 \leq e \leq 1),
\] (5.5)
where we put \(\tilde{m}_{e}(\xi) = p_{e}(\xi) m(o(e))\) for \(e \in E_{0}\). Note that, if the transition probability \(p_{0}\) is \(m\)-symmetric, then \(\beta_{(0)}(\Phi^{(0)}_{0}) = 0_{g}\). Loosely speaking, this quantity will appear as a coefficient of the second order term of the Taylor expansion of \((I - L^{(1)}_{(e)}) P_{e} f\) in \(e\), which is dealt in the proof of Lemma 5.3. In particular, we are interested in the short time behavior of \(\beta_{(e)}(\Phi^{(e)}_{0})\) as \(e \searrow 0\) for later use. Intuitively there seems to be little hope of knowing such behavior, because \(\Phi^{(e)}_{0}\) has the ambiguity in its \(g^{(2)}\)-components for every \(0 \leq e \leq 1\). However, the following proposition asserts that \(\beta_{(e)}(\Phi^{(e)}_{0})\) in fact approaches \(\beta_{(0)}(\Phi^{(0)}_{0}) = 0_{g}\) as \(e \searrow 0\) by imposing only (A1).

**Proposition 5.1** Under (A1), we have
\[
\lim_{e \searrow 0} \beta_{(e)}(\Phi^{(e)}_{0}) = \beta_{(0)}(\Phi^{(0)}_{0}) = 0_{g}.
\]

Fix a fundamental domain \(F\) of \(X\). Set \(\Psi^{(e)}(x) = \Phi^{(e)}_{0}(x)^{-1} \cdot \Phi^{(0)}_{0}(x)\) for \(0 \leq e \leq 1\) and \(x \in V\). Note that the map \(\Psi^{(e)} : V \rightarrow G\) is \(\Gamma\)-invariant. The following lemma is essential to prove Proposition 5.1.

\[\text{Springer}\]
Lemma 5.2 Under (A1), we have

$$\lim_{\varepsilon \searrow 0} \| \log (\Psi^{(\varepsilon)}(x)) \|_{g^{(1)}} = 0 \quad (x \in \mathcal{F}).$$

In particular, there exists a constant $C$ such that

$$\| \log (\Psi^{(\varepsilon)}(x)) \|_{g^{(1)}} \leq C \quad (0 \leq \varepsilon \leq 1, \ x \in \mathcal{F}).$$

Proof We set $\ell^2(\mathcal{F}) := \{ f : \mathcal{F} \to \mathbb{C} \}$ and equip it with the inner product and the corresponding norm defined by

$$\langle f, g \rangle_{\ell^2(\mathcal{F})} := \sum_{x \in \mathcal{F}} f(x)\overline{g(x)}, \quad \| f \|_{\ell^2(\mathcal{F})} := \left( \sum_{x \in \mathcal{F}} |f(x)|^2 \right)^{1/2} \quad (f, g \in \ell^2(\mathcal{F})),
$$

respectively. Since the invariant measure $m|_{\mathcal{F}} : \mathcal{F} \to (0, 1]$ is positive on the finite set $\mathcal{F}$, there are positive constants $c$ and $C$ such that

$$c \left( \sum_{x \in \mathcal{F}} m(x) |f(x)|^2 \right)^{1/2} \leq \| f \|_{\ell^2(\mathcal{F})} \leq C \left( \sum_{x \in \mathcal{F}} m(x) |f(x)|^2 \right)^{1/2} \quad (f \in \ell^2(\mathcal{F})).$$

(5.6)

It follows from the Perron–Frobenius theorem that $\ell^2(\mathcal{F}) = \langle \phi_0 \rangle \oplus \ell^1(\mathcal{F})$, where $\phi_0 = |\mathcal{F}|^{-1/2}$ is the normalized right eigenfunction corresponding to the maximal eigenvalue $\lambda_0 = 1$ of $L$. We define $\ell^2(\mathcal{F}) := \{ f \in \ell^2(\mathcal{F}) : |\mathcal{F}|^{1/2} \langle f, m \rangle_{\ell^1(\mathcal{F})} = 0 \}$. Note that $\ell^2(\mathcal{F})$ and the transition operator $L(\varepsilon)$ maps $\ell^2(\mathcal{F})$ to itself for all $0 \leq \varepsilon \leq 1$. Moreover, the inverse operator of $(I - L(\varepsilon))|_{\ell^2(\mathcal{F})} : \ell^2(\mathcal{F}) \to \ell^1(\mathcal{F})$ does exists since $L(\varepsilon)$ has a simple eigenvalue $\lambda_0(\varepsilon) = 1$ for $0 \leq \varepsilon \leq 1$. We define $Q : \ell^2(\mathcal{F}) \to \ell^2(\mathcal{F})$ by

$$Qf(x) := \sum_{e \in E_x} q(e) f(t(e)) \quad (f \in \ell^2(\mathcal{F}), \ x \in \mathcal{F}).$$

Then we verify that the transition operator $L(\varepsilon)$ has the decomposition of the form $L(\varepsilon) = L(0) + \varepsilon Q$ for every $0 \leq \varepsilon \leq 1$.

To conclude the claim, it suffices to show

$$\lim_{\varepsilon \searrow 0} \| \log (\Psi^{(\varepsilon)}(x)) \|_{\ell^2(\mathcal{F})} = 0 \quad (i = 1, 2, \ldots, d_1)$$

by noting (5.6). We remark that $\log (\Psi^{(\varepsilon)}(x))|_{\ell^2(\mathcal{F})}$ for $i = 1, 2, \ldots, d_1$ thanks to (2.4). In the following, we fix $i = 1, 2, \ldots, d_1$. The modified harmonicity of $\Phi^{(\varepsilon)}_0$ gives

$$(I - L(\varepsilon))\left( \log (\Psi^{(\varepsilon)}(x)) \right|_{\ell^2(\mathcal{F})}^1 \right) = \varepsilon \left[ Q \left( \log \Phi^{(0)}_0(x) \right|_{\ell^1(\mathcal{F})} - \rho R(\gamma p) \right|_{\ell^1(\mathcal{F})}^1 \right]
$$

for $0 \leq \varepsilon \leq 1$ and $x \in \mathcal{F}$. This identity implies

$$\begin{align*}
\| \log (\Psi^{(\varepsilon)}(\cdot)) \|_{\ell^2(\mathcal{F})}^1 & \leq \varepsilon \| (I - L(\varepsilon))\left|_{\ell^1(\mathcal{F})} \right|_{\ell^2(\mathcal{F})} + \left\{ \| \log \Phi^{(0)}_0(\cdot) \|_{\ell^2(\mathcal{F})}^1 + \rho R(\gamma p) \|_{\ell^1(\mathcal{F})} \right\},
\end{align*}$$

(5.8)

where we used $\| Q \| \leq 1$ for the final line. By combining (5.8) with the identity

$$(I - L(\varepsilon))\left|_{\ell^2(\mathcal{F})} \right|_{\ell^1(\mathcal{F})}^1 = (I - L(0))\left|_{\ell^2(\mathcal{F})} \right|_{\ell^1(\mathcal{F})}^1 \left[ I - \varepsilon Q \left( I - L(0) \right) \left|_{\ell^1(\mathcal{F})} \right|_{\ell^1(\mathcal{F})}^1 \right],$$
we obtain
\[
\left\| \log \left( \Psi^{(e)}(\cdot) \right) \right\|_{\bar{c}^{2}(\mathcal{F})} \leq \varepsilon \left\| (I - L(0))^{-1}_{\bar{c}^{2}(\mathcal{F})} \cdot \left( 1 - \varepsilon \left\| Q_{\bar{c}^{2}(\mathcal{F})} (I - L(0))^{-1}_{\bar{c}^{2}(\mathcal{F})} \right\| \right)^{-1} \times \left\{ \left\| \log \Phi_{0}^{(e)}(\cdot) \right\|_{\bar{c}^{2}(\mathcal{F})} + \left\| \rho_{\mathcal{R}}(\gamma_{p}) \right\|_{g^{(2)}} \right\}. 
\]

Here we can choose a sufficiently small constant \( \varepsilon_{0} > 0 \) such that
\[
\sup_{0 < \varepsilon \leq \varepsilon_{0}} \left( 1 - \varepsilon \left\| Q_{\bar{c}^{2}(\mathcal{F})} (I - L(0))^{-1}_{\bar{c}^{2}(\mathcal{F})} \right\| \right)^{-1} \leq 2.
\]
Then we have
\[
\left\| \log \left( \Psi^{(e)}(\cdot) \right) \right\|_{\bar{c}^{2}(\mathcal{F})} \leq 2\varepsilon \left\| (I - L(0))^{-1}_{\bar{c}^{2}(\mathcal{F})} \right\| \left\{ \left\| \log \Phi_{0}^{(e)}(\cdot) \right\|_{\bar{c}^{2}(\mathcal{F})} + \left\| \rho_{\mathcal{R}}(\gamma_{p}) \right\|_{g^{(2)}} \right\}
\]
for sufficiently small \( \varepsilon > 0 \) and this implies (5.7).

**Proof of Proposition 5.1** By recalling (2.3) and that \( p_{0} \) is \( m \)-symmetric, we have
\[
\beta_{(e)}(\Phi^{(e)}_{0}) = \sum_{e \in E_{0}} \left\{ \frac{1}{2} (\tilde{m}_{0}(e) - \tilde{m}_{0}(\tilde{e})) \log \left( d\Phi^{(e)}_{0}(\tilde{e}) \right) \right\}_{g^{(2)}} + \varepsilon m(o(e))q(e) \log \left( d\Phi^{(e)}_{0}(\tilde{e}) \right)_{g^{(2)}} \
\]
and (4.1) yield
\[
\beta_{(e)}(\Phi^{(e)}_{0}) = \varepsilon \sum_{e \in E_{0}} m(o(e))q(e) \left\{ \log \left( \Psi^{(e)}(o(\tilde{e})) \right) \right\}_{\bar{c}^{(e)}} - \log \left( \Psi^{(e)}(t(\tilde{e})) \right) \right\}_{g^{(2)}} \
\]
\[
+ \varepsilon \sum_{e \in E_{0}} m(o(e))q(e) \log \left( d\Phi^{(e)}_{0}(\tilde{e}) \right)_{\bar{c}^{(e)}} - \frac{\varepsilon}{2} \sum_{e \in E_{0}} m(o(e))q(e) \left\{ \mathcal{T}^{(e)}(\tilde{e}) + \mathcal{T}^{(e)}(\tilde{e}) + \mathcal{T}^{(e)}(\tilde{e}) \right\},
\]
where
\[
\mathcal{T}^{(e)}(\tilde{e}) = \mathcal{T}^{(e,\lambda,v)}_{1}(\tilde{e}) = \left[ \log \left( \Psi^{(e)}(o(\tilde{e})) \right) \right]_{g^{(1)}} - \log \left( d\Phi^{(e)}_{0}(\tilde{e}) \right)_{g^{(1)}}. 
\]

Let \( \{ X^{(2)}_{1}, X^{(2)}_{2}, \ldots, X^{(2)}_{d_{2}} \} \) be a basis of \( g^{(2)} \). For \( i = 1, 2, \ldots, d_{2} \), we define a function \( F^{(e)}_{i} : V \to \mathbb{R} \) by \( F^{(e)}_{i}(x) := \log \left( \Psi^{(e)}(x) \right) \right\}_{X^{(2)}_{i}} \) for \( 0 \leq \varepsilon \leq 1 \) and \( x \in V \). Then we see
that the function $F^\varepsilon_i(x) = \Gamma$-invariant. Hence, there exists a function $\tilde{F}^\varepsilon_i(x) = V_0 \to \mathbb{R}$ such that

$$F^\varepsilon_i(x) = \Gamma \circ \tilde{F}^\varepsilon_i(x)$$

for $0 \leq \varepsilon \leq 1$ and $x \in V$. Then, by noting $\delta(y_{p,e}) = 0$, we have

$$\varepsilon \sum_{e \in E_0} m(o(e))q(e) \left\{ \log \left( \Psi^\varepsilon_i(o(e)) \right) - \log \left( \Psi^\varepsilon_i(t(e)) \right) \right\}_{g^{(2)}}$$

$$= \sum_{e \in E_0} (\tilde{m}_e(e) - \tilde{m}_e(0)) \left\{ \log \left( \Psi^\varepsilon_i(o(e)) \right) - \log \left( \Psi^\varepsilon_i(t(e)) \right) \right\}_{g^{(2)}}$$

$$= -c_i(\text{X}, \mathbb{R}) \langle y_{p,e}, \tilde{F}^\varepsilon_i \rangle_{C^1(\text{X}, \mathbb{R})} + \frac{1}{2} \sum_{e \in E_0} (\tilde{m}_e(e) - \tilde{m}_e(0))d\tilde{F}^\varepsilon_i(e)$$

$$= -c_0(\text{X}, \mathbb{R}) \langle \delta(y_{p,e}), \tilde{F}^\varepsilon_i \rangle_{C^0(\text{X}, \mathbb{R})} = 0.$$
for $x \in V$ and some $\theta \in G_{(0)}$ satisfying
\[
|\theta_{i*}^{(k)}| \leq \left( \Phi_0^{(r)}(x)^{-1} \ast \Phi_0^{(r)}(t(c)) \right)_{i*}^{(k)} \quad (i = 1, 2, \ldots, d_k, k = 1, 2, \ldots, r),
\]
where the summation $\sum_{(i_1, k_1) \geq (i_2, k_2)}$ runs over all $(i_1, k_1)$ and $(i_2, k_2)$ with $k_1 > k_2$ or $k_1 = k_2$ and $i_1 \geq i_2$. We denote by $\text{Ord}_\varepsilon(k)$ the terms of the right-hand side of (5.11) whose order of $\varepsilon$ equals just $k$. Then, (5.11) is rewritten as
\[
\frac{1}{N\varepsilon^2} (I - L_{(\varepsilon)}^N) P_{e}^{H} f(x) = \text{Ord}_\varepsilon(-1) + \text{Ord}_\varepsilon(0) + \sum_{k \geq 1} \text{Ord}_\varepsilon(k) \quad (x \in V),
\]
where
\[
\text{Ord}_\varepsilon(-1) = -\frac{1}{N\varepsilon} \sum_{i=1}^{d_1} X_{i*}^{(1)} f(\tau_\varepsilon(\Phi_0^{(r)}(x))) \sum_{c \in \Omega_{\varepsilon,N}(X)} p_\varepsilon(c) \left( \Phi_0^{(r)}(x)^{-1} \ast \Phi_0^{(r)}(t(c)) \right)_{i*}^{(1)}
\]
and
\[
\text{Ord}_\varepsilon(0) = -\frac{1}{N} \sum_{i=1}^{d_2} X_{i*}^{(2)} f(\tau_\varepsilon(\Phi_0^{(r)}(x))) \sum_{c \in \Omega_{\varepsilon,N}(X)} p_\varepsilon(c) \left( \Phi_0^{(r)}(x)^{-1} \ast \Phi_0^{(r)}(t(c)) \right)_{i*}^{(2)}
\]
\[-\frac{1}{2} \sum_{1 \leq \lambda, \nu \leq d_1} \|X_{1*}^{(1)} \cdot X_{\nu*}^{(1)} \|_{X_{1*}^{(2)}}
\times \left( \Phi_0^{(r)}(x)^{-1} \ast \Phi_0^{(r)}(t(c)) \right)_{\lambda*}^{(1)} \left( \Phi_0^{(r)}(x)^{-1} \ast \Phi_0^{(r)}(t(c)) \right)_{\nu*}^{(1)}
\]
\[-\frac{1}{2N} \sum_{1 \leq i, j \leq d_1} \frac{1}{X_{i*}^{(1)} X_{j*}^{(1)} f(\tau_\varepsilon(\Phi_0^{(r)}(x)))}
\times \sum_{c \in \Omega_{\varepsilon,N}(X)} p_\varepsilon(c) \left( \Phi_0^{(r)}(x)^{-1} \ast \Phi_0^{(r)}(t(c)) \right)_{i*}^{(1)} \left( \Phi_0^{(r)}(x)^{-1} \ast \Phi_0^{(r)}(t(c)) \right)_{j*}^{(1)}
\]
\[=: T_1(\varepsilon, N) + T_2(\varepsilon, N).
\]

**Step 1.** We first estimate $\text{Ord}_\varepsilon(-1)$. By recalling (4.2) and (5.1), we have inductively
\[
\sum_{c \in \Omega_{\varepsilon,N}(X)} p_\varepsilon(c) \left( \Phi_0^{(r)}(x)^{-1} \ast \Phi_0^{(r)}(t(c)) \right)_{i*}^{(1)}
\]
\[= \sum_{c' \in \Omega_{\varepsilon,N-1}(X)} p_\varepsilon(c') \sum_{c \in E_{c'(c')}} p_\varepsilon(c) \left( \Phi_0^{(r)}(x)^{-1} \cdot \Phi_0^{(r)}(t(c')) \cdot \Phi_0^{(r)}(t(c'))^{-1} \cdot \Phi_0^{(r)}(t(c)) \right)_{i}^{(1)}
\]
\[= \sum_{c' \in \Omega_{\varepsilon,N-1}(X)} p_\varepsilon(c') \log \left( \Phi_0^{(r)}(x)^{-1} \cdot \Phi_0^{(r)}(t(c')) \right)_{X_1^{(1)}}^{(1)} + \varepsilon \rho_\varepsilon(y_p) \left| X_1^{(i)} \right|
\]
\[= N\varepsilon \rho_\varepsilon(y_p) \left| X_1^{(i)} \right| \quad (x \in V, i = 1, 2, \ldots, d_1). \quad (5.12)
\]
Step 2. Next we estimate $\text{Ord}_\varepsilon(0)$. Let us consider the coefficient of $X^{(2)}_{t^*} f\left(\tau_\varepsilon (\Phi_0^{(e)}(x))\right)$. It follows from (5.1) and (4.2) that

$$\frac{1}{N} \sum_{c \in \Omega_{x,N}(X)} p_\varepsilon(c) \left\{ \left( \Phi_0^{(e)}(x)^{-1} \ast \Phi_0^{(e)}(t(c)) \right)^{(2)} \right\}_{i*}$$

$$= \frac{1}{2} \sum_{1 \leq \lambda < \nu \leq d_1} \left( \Phi_0^{(e)}(x)^{-1} \ast \Phi_0^{(e)}(t(c)) \right)^{(1)} \left( \Phi_0^{(e)}(x)^{-1} \ast \Phi_0^{(e)}(t(c)) \right)^{(1)} \left[ X^{(1)}_{\lambda}, X^{(1)}_{\nu} \right]_{X^{(2)}_{i*}}$$

$$= \frac{1}{N} \sum_{c \in \Omega_{x,N}(X)} p_\varepsilon(c) \log \left( \Phi_0^{(e)}(x)^{-1} \ast \Phi_0^{(e)}(t(c)) \right) \bigg|_{X^{(2)}_{i*}}$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} \sum_{c' \in \Omega_{x,k}(X)} p_\varepsilon(c') \sum_{e \in E_t(c')} p_\varepsilon(e) \log \left( d \Phi_0^{(e)}(e) \right) \bigg|_{X^{(2)}_{i*}} (x \in V). \quad (5.13)$$

Since the function

$$M_i^{(e)}(x) := \sum_{e \in E_x} p_\varepsilon(e) \log \left( d \Phi_0^{(e)}(e) \right) \bigg|_{X^{(2)}_{i*}} \quad (0 \leq \varepsilon \leq 1, i = 1, 2, \ldots, d_2, x \in V)$$

satisfies $M_i^{(e)}(\gamma x) = M_i^{(e)}(x)$ for $\gamma \in \Gamma$ and $x \in V$ due to the $\Gamma$-invariance of $p$ and the $\Gamma$-equivariance of $\Phi_0$, there exists a function $\mathcal{M}_i^{(e)} : V_0 \rightarrow \mathbb{R}$ such that

$$\mathcal{M}_i^{(e)}(\pi(x)) = M_i^{(e)}(x) \quad (0 \leq \varepsilon \leq 1, i = 1, 2, \ldots, d_2, x \in V).$$

Moreover, we have

$$L^k_{(\varepsilon)} \mathcal{M}_i^{(e)}(\pi(x)) = L^k_{(\varepsilon)} M_i(x) \quad (k \in \mathbb{N}, 0 \leq \varepsilon \leq 1, i = 1, 2, \ldots, d_2, x \in V)$$

due to the $\Gamma$-invariance of $p$. Then, by applying the ergodic theorem (cf. [9, Theorem 3.4]) for the transition operator $L_{(\varepsilon)}$, we can choose a sufficiently small $\varepsilon_0 > 0$ such that

$$\frac{1}{N} \sum_{k=0}^{N-1} L^k_{(\varepsilon)} \mathcal{M}_i^{(e)}(\pi(x)) = \sum_{x \in V_0} m(x) \mathcal{M}_i^{(e)}(x) + O_{\varepsilon_0} \left( \frac{1}{N} \right). \quad (5.14)$$

By (5.14), we obtain that the left-hand side of (5.13) equals

$$\beta_\varepsilon \left( \Phi_0^{(e)} \right) \bigg|_{X^{(2)}_{i*}} + O_{\varepsilon_0} \left( \frac{1}{N} \right) \quad (0 \leq \varepsilon \leq \varepsilon_0, i = 1, 2, \ldots, d_2).$$

Then Proposition 5.1 implies that $\mathcal{I}_{(\varepsilon, N)} \to 0$ as $N \to \infty$ and $\varepsilon \searrow 0$ with $N^2 \varepsilon \searrow 0$.  

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We also consider the coefficient of $X_{j*}^{(1)} X_{j*}^{(1)} f(\tau_{\varepsilon}(\Phi_0^{(e)}(x)))$. We have

\[
\frac{1}{2N} \sum_{c \in \Omega, N} p_e(c) \left( \Phi_0^{(e)}(x)^{-1} \Phi_0^{(e)}(\tau(c)) \right)_{j*}^{(1)} \left( \Phi_0^{(e)}(x)^{-1} \Phi_0^{(e)}(\tau(c)) \right)_{j*}^{(1)}
\]

\[
= \frac{1}{2N} \left\{ \sum_{c' \in \Omega, N, N-1} p_e(c') \left( \Phi_0^{(e)}(x)^{-1} \Phi_0^{(e)}(\tau(c')) \right)_{j*}^{(1)} \left( \Phi_0^{(e)}(x)^{-1} \Phi_0^{(e)}(\tau(c')) \right)_{j*}^{(1)} \right\}
\]

\[
+ \sum_{c \in E(c')} p_e(e) \log \left| \left| X_i^{(1)} \log \left| \left| d\Phi_0^{(e)}(e) \right| \right| \right| X_j^{(1)} \right|
\]

\[
+ 2(N - 1) \rho_{\mathbb{R}}(\gamma_{p*}) \left| X_i^{(1)} \rho_{\mathbb{R}}(\gamma_{p*}) \left| X_j^{(1)} \right| \right|
\]

\[
= \frac{1}{2N} \sum_{k=0}^{N-1} \sum_{c' \in \Omega, k} p_e(c') \sum_{c \in E(c')} p_e(e) \log \left| \left| X_i^{(1)} \log \left| \left| d\Phi_0^{(e)}(e) \right| \right| \right| X_j^{(1)} \right|
\]

\[
+ \frac{1}{2} (N - 1) \varepsilon^2 \rho_{\mathbb{R}}(\gamma_p) \left| X_i^{(1)} \rho_{\mathbb{R}}(\gamma_p) \right| X_j^{(1)}
\]

\[
= \frac{1}{2N} \sum_{k=0}^{N-1} L_{(e)}^k N_{ij}^{(e)}(x) + \frac{1}{2} (N - 1) \varepsilon^2 \rho_{\mathbb{R}}(\gamma_p) \left| X_i^{(1)} \rho_{\mathbb{R}}(\gamma_p) \right| X_j^{(1)} \quad (x \in V) \tag{5.15}
\]

by using (5.1) and (4.3), where the function $N_{ij}^{(e)} : V \to \mathbb{R}$ is defined by

\[
N_{ij}^{(e)}(x) := \sum_{c \in E(x)} p_e(e) \log \left| \left| X_i^{(1)} \log \left| \left| d\Phi_0^{(e)}(e) \right| \right| \right| X_j^{(1)} \right|.
\]

for $0 \leq \varepsilon \leq 1$, $i, j = 1, 2, \ldots, d_1$ and $x \in V$. In the same argument as above, $N_{ij}^{(e)}$ is $\Gamma$-invariant and there exists a function $N_{ij}^{(e)} : V_0 \to \mathbb{R}$ such that $N_{ij}^{(e)}(\pi(x)) = N_{ij}^{(e)}(x)$ for $x \in V$. We also have

\[
L_{(e)}^k N_{ij}^{(e)}(\pi(x)) = L_{(e)}^k N_{ij}^{(e)}(x) \quad (k \in \mathbb{N}, 0 \leq \varepsilon \leq 1, i, j = 1, 2, \ldots, d_2, x \in V)
\]

by the $\Gamma$-invariance of $p$. Thus, we choose a sufficiently small $\varepsilon_0 > 0$ such that

\[
\frac{1}{2N} \sum_{k=0}^{N-1} L_{(e)}^k N_{ij}^{(e)}(x) = \frac{1}{2N} \sum_{k=0}^{N-1} L_{(e)}^k N_{ij}^{(e)}(\pi(x))
\]

\[
= \frac{1}{2} \sum_{x \in V_0} m(x) (N(\Phi_0^{(e)}))_{ij}(x) + O_{\varepsilon_0} \left( \frac{1}{N} \right)
\]

\[
= \frac{1}{2} \sum_{e \in E_0} \tilde{m}_e(e) \log \left| \left| d\Phi_0^{(e)}(e) \right| \right| \log \left| \left| d\Phi_0^{(e)}(e) \right| \right| \left| X_j^{(1)} \right|
\]

\[
+ O_{\varepsilon_0} \left( \frac{1}{N} \right) \quad (0 \leq \varepsilon \leq \varepsilon_0, i, j = 1, 2, \ldots, d_1) \tag{5.16}
\]
by the ergodic theorem. Recall that \( \{V_1, V_2, \ldots, V_{d_1}\} \) denotes the orthonormal basis in \( (g^{(1)}, g_0^{(0)}) \). In particular, put \( X_i^{(1)} = V_i \) for \( i = 1, 2, \ldots, d_1 \) and let \( \{\omega_1, \omega_2, \ldots, \omega_{d_1}\} \) be the dual basis of \( \{V_1, V_2, \ldots, V_{d_1}\} \). Then we have

\[
\frac{1}{2} \sum_{e \in E_0} \tilde{m}_e(e) \log (d \Phi_0^{(e)}(\tilde{e}))|_{V_i} \log (d \Phi_0^{(e)}(\tilde{e}))|_{V_j}
\]

\[
= \frac{1}{2} \left( \sum_{e \in E_0} \tilde{m}_e(e) \omega_i^{(e)}(e) \omega_j^{(e)}(e) - \langle y_{p_1}, \omega_i \rangle \langle y_{p_1}, \omega_j \rangle \right) + \frac{1}{2} \varepsilon^2 \langle y_p, \omega_i \rangle \langle y_p, \omega_j \rangle
\]

\[
= \frac{1}{2} \langle \langle \omega_i^{(e)}, \omega_j^{(e)} \rangle \rangle (\varepsilon) + \frac{1}{2} \varepsilon^2 \rho_\mathcal{R}(y_p) |_{V_i} \rho_\mathcal{R}(y_p) |_{V_j} \quad (i, j = 1, 2, \ldots, d_1). \tag{5.17}
\]

The coefficient of \( X_i^{(1)} X_j^{(1)} f(\tau_\varepsilon(\Phi_0^{(e)}(x))) \) equals

\[
- \frac{1}{2} \langle \langle \omega_i^{(e)}, \omega_j^{(e)} \rangle \rangle (\varepsilon) + \frac{1}{2} N \varepsilon^2 \rho_\mathcal{R}(y_p) |_{V_i} \rho_\mathcal{R}(y_p) |_{V_j} + O_{r_0} \left( \frac{1}{N} \right) \quad (i, j = 1, 2, \ldots, d_1) \tag{5.18}
\]

by combining (5.15) with (5.16) and (5.17). Therefore, (5.18) and the continuity of \( \langle \cdot, \cdot \rangle (\varepsilon) \) as \( \varepsilon \searrow 0 \) (cf. [9, Lemma 5.2]) imply

\[
\text{Ord}_\varepsilon(0) = \mathcal{I}_1(\varepsilon, N) + \mathcal{I}_2(\varepsilon, N) = - \frac{1}{2} \sum_{i=1}^{d_1} V_i^2 f(\tau_\varepsilon(\Phi_0^{(e)}(x))) + O(N^2 \varepsilon) + O_{r_0} \left( \frac{1}{N} \right) \tag{5.19}
\]

as \( N \to \infty \) and \( \varepsilon \searrow 0 \) with \( N^2 \varepsilon \searrow 0 \).

We finally discuss the estimate of \( \sum_{k \geq 1} \text{Ord}_\varepsilon(k) \). At the beginning, we show that the coefficient of \( X_i^{(k)} f(\tau_\varepsilon(\Phi_0^{(e)}(x))) \) vanishes as \( N \to \infty \) and \( \varepsilon \searrow 0 \) with \( N^2 \varepsilon \searrow 0 \). Thanks to

\[
\left| \left( \Phi_0^{(e)}(x) - 1 \cdot \Phi_0^{(e)}(t(c)) \right) (x) \right| \leq CN^k \quad (0 \leq \varepsilon \leq 1, x \in V),
\]

Equations 5.1 and (4.6), we have

\[
e^{k-2} \frac{1}{N} \sum_{x \in \Omega_{\varepsilon, N}(X)} p_\varepsilon(c) \left( \Phi_0^{(e)}(x)^{-1} * \Phi_0^{(e)}(t(c)) \right)_{t*}
\]

\[
e^{k-2} \frac{1}{N} \sum_{x \in \Omega_{\varepsilon, N}(X)} p_\varepsilon(c) \left( \left( \Phi_0^{(e)}(x)^{-1} \cdot \Phi_0^{(e)}(t(c)) \right) \right)_{t}
\]

\[
+ \sum_{|K_1| + |K_2| \leq k-1 \atop |K_1| \geq 0} C_{K_1, K_2} \mathcal{D}_{K_1} \left( \Phi_0^{(e)}(x)^{-1} \right) \mathcal{D}_{K_2} \left( \Phi_0^{(e)}(x)^{-1} \cdot \Phi_0^{(e)}(t(c)) \right)
\]

\[
\leq CM_i^{(k)} \left( \tau_\varepsilon(\Phi_0^{(e)}(x)) \right) (x) e^{k-2} N^{-k-1} + \sum_{|K_1| \leq k-2} e^{k-1-|K_1|} + \sum_{|K_1| + |K_2| \leq k-1 \atop |K_2| \geq 2} e^{k-2-|K_1|} N^{2|K_2| - 1}
\]

for \( i = 1, 2, \ldots, d_k \) and some continuous function \( M_i^{(k)} : G \to \mathbb{R} \). This converges to zero as \( N \to \infty \) and \( \varepsilon \searrow 0 \) with \( N^2 \varepsilon \searrow 0 \). We also observe that the coefficient of \( X_i^{(k)} X_j^{(k)} f(\tau_\varepsilon(\Phi_0^{(e)}(x))) \) converges to zero as \( N \to \infty \) and \( \varepsilon \searrow 0 \) with \( N^2 \varepsilon \searrow 0 \) by following the same argument as above.
We also consider the coefficient of \((a_3 f / \partial g_{i_1}^{(k_1)} \partial g_{i_2}^{(k_2)} \partial g_{i_3}^{(k_3)})(\theta)\). Since \(f\) is compactly supported, it is sufficient to show by induction on \(k = 1, 2, \ldots, r\) that, if \(\varepsilon N < 1\), then
\[
\varepsilon^k \left( \Phi_0^{(e)}(x) - 1 \ast \Phi_0^{(e)}(t(c)) \right)_{i_1}^{(k)} \leq M_i^{(k)} \left( \tau_e (\Phi_0^{(e)}(x) \ast \theta) \right) \times \varepsilon N
\]  
(5.20)
for \(i = 1, 2, \ldots, d_k\) and some continuous function \(M_i^{(k)} : G \to \mathbb{R}\). The cases \(k = 1\) and \(k = 2\) are clear. Suppose that (5.20) holds for less than \(k\). We have
\[
\varepsilon^k \left( \Phi_0^{(e)}(x) - 1 \ast \Phi_0^{(e)}(t(c)) \right)_{i_1}^{(k)} = \varepsilon^k \left\{ \left( \Phi_0^{(e)}(x) - 1 \ast \Phi_0^{(e)}(t(c)) \right)_{i_1}^{(k)} + \sum_{|K_1| + |K_2| \leq k - 1 \atop |K_2| > 0} C_{K_1, K_2} \right. 
\]
\[
\times \mathcal{P}_{K_1} \left( \Phi_0^{(e)}(x) - 1 \right) \mathcal{P}_{K_2} \left( \Phi_0^{(e)}(x) - 1 \ast \Phi_0^{(e)}(t(c)) \right) \}
\]
by using (4.1) and (4.6). Then we see that
\[
\left( \Phi_0^{(e)}(x) - 1 \right)_{i_1}^{(k_1)} = \left( \theta \ast \left( \Phi_0^{(e)}(x) \ast \theta \right)^{-1} \right)_{i_1}^{(k_1)}
\]
\[
= \theta_{i_1}^{(k_1)} + \left( \Phi_0^{(e)}(x) \ast \theta \right)^{-1} \left( \Phi_0^{(e)}(x) \ast \theta \right) \left( \Phi_0^{(e)}(x) \ast \theta \right)^{-1} \right)_{i_1}^{(k_1)}
\]
\[
+ \sum_{|L_1| + |L_2| = k_1 \atop |L_1|, |L_2| > 0} C_{L_1, L_2} \mathcal{P}_{L_1} \left( \Phi_0^{(e)}(x) \ast \theta \right) \mathcal{P}_{L_2} \left( \Phi_0^{(e)}(x) \ast \theta \right)^{-1} \right)
\]
Thus, we have inductively
\[
\left\| \left( \Phi_0^{(e)}(x) - 1 \right)_{i_1}^{(k_1)} \right\| \leq M \left( \Phi_0^{(e)}(x) \ast \theta \right)
\]
for a continuous function \(M : G \to \mathbb{R}\) and \(k_1 < k - 1\). We then conclude
\[
\varepsilon^k \left( \Phi_0^{(e)}(x) - 1 \ast \Phi_0^{(e)}(t(c)) \right)_{i_1}^{(k)} \leq C \left( \varepsilon^k N^k + \sum_{|K_1| + |K_2| \leq k - 1 \atop |K_2| > 0} M \left( \tau_e (\Phi_0^{(e)}(x) \ast \theta) \right) \varepsilon^{-|K_1|} N^{0} N^{0} \right)
\]
\[
\leq M_i^{(k)} \left( \tau_e (\Phi_0^{(e)}(x) \ast \theta) \right) \times \varepsilon N
\]
for some continuous function \(M_i^{(k)} : G \to \mathbb{R}\). These estimates implies that \(\sum_{k \geq 1} \text{Ord}_e(k)\) converges to zero as \(N \to \infty\) and \(\varepsilon \searrow 0\) with \(N^2 \varepsilon \searrow 0\).

Consequently, we obtain
\[
\left\| \frac{1}{N \varepsilon^2} \left( I - L_N^{(e)} \right) p^H f(x) - p^H A f(x) \right\| \to 0
\]
as \(N \to \infty\) and \(\varepsilon \searrow 0\) with \(N^2 \varepsilon \searrow 0\) by combining (5.11) with (5.12) and (5.19). This completes the proof.

*Proof of Theorem 2.1* We partially follow the argument by Kotani [12, Theorem 4]. Let \(N = N(n)\) be the integer satisfying \(n^{1/5} \leq N < n^{1/5} + 1\) and let \(k_N\) and \(r_N\) be the quotient and the remainder of \((|n| - |n|)/N(n)\), respectively. We put \(\varepsilon_N := n^{-1/2} + h_N := N \varepsilon_N^2\). Then we have \(k_N h_N = (|n| - |n| - r_N) \varepsilon_N^2 \to t - s \quad (n \to \infty)\).

Since \(C_0^\infty(G_{(0)}) \subset \text{Dom}(A) \subset C_0^\infty(G_{(0)})\) and \(C_0^\infty(G_{(0)})\) is dense in \(C_0^\infty(G_{(0)})\), the linear operator \(A\) is densely defined in \(C_0^\infty(G_{(0)})\). Furthermore, \((\lambda - A) (C_0^\infty(G_{(0)}))\) is dense.
in $C_{0}(G(0))$ for some $\lambda > 0$ (cf. Robinson [29, p.304]). Hence, by combining Lemma 5.3 and Trotter’s approximation theorem (cf. [35]), we obtain

$$\lim_{n \to \infty} \left\| L_{(n^{-1/2})}^{\frac{Nk}{n^{1/2}}} P_{n^{-1/2}}^{H} f - P_{n^{-1/2}}^{H} e^{-\langle t-s \rangle A} f \right\|_{X}^{\infty} = 0 \quad (f \in C_{0}^\infty(G(0))). \quad (5.21)$$

On the other hand, Lemma 5.3 implies

$$\lim_{n \to \infty} \left\| \frac{1}{r_{N}^{2} \varepsilon_{N}^{2}} (I - L_{(n^{-1/2})}^{N}) P_{n^{-1/2}}^{H} f - P_{n^{-1/2}}^{H} A f \right\|_{X}^{\infty} = 0 \quad (f \in C_{0}^\infty(G(0))). \quad (5.22)$$

Here we have

$$\left\| L_{(n^{-1/2})}^{\frac{n}{r} - \frac{[ns]}{r}} P_{n^{-1/2}}^{H} f - P_{n^{-1/2}}^{H} e^{-\langle t-s \rangle A} f \right\|_{X}^{\infty} \leq \left\| (I - L_{(n^{-1/2})}^{N}) P_{n^{-1/2}}^{H} f \right\|_{X}^{\infty} + \left\| L_{(n^{-1/2})}^{\frac{Nk}{n^{1/2}}} P_{n^{-1/2}}^{H} f - P_{n^{-1/2}}^{H} e^{-\langle t-s \rangle A} f \right\|_{X}^{\infty}. \quad (5.23)$$

It follows from $\| P_{n^{-1/2}}^{H} \| \leq 1$ that

$$\left\| (I - L_{(n^{-1/2})}^{N}) P_{n^{-1/2}}^{H} f \right\|_{X}^{\infty} \leq r_{N}^{2} \varepsilon_{N}^{2} \left\| (I - L_{(n^{-1/2})}^{N}) P_{n^{-1/2}}^{H} f - P_{n^{-1/2}}^{H} A f \right\|_{X}^{\infty} + r_{N}^{2} \varepsilon_{N}^{2} \left\| P_{n^{-1/2}}^{H} A f \right\|_{X}^{\infty} \leq r_{N}^{2} \varepsilon_{N}^{2} \left\| (I - L_{(n^{-1/2})}^{N}) P_{n^{-1/2}}^{H} f - P_{n^{-1/2}}^{H} A f \right\|_{X}^{\infty} + r_{N}^{2} \varepsilon_{N}^{2} \left\| A f \right\|_{X}^{\infty}. \quad (5.24)$$

Then, we obtain

$$\lim_{n \to \infty} \left\| L_{(n^{-1/2})}^{\frac{n}{r} - \frac{[ns]}{r}} P_{n^{-1/2}}^{H} f - P_{n^{-1/2}}^{H} e^{-\langle t-s \rangle A} f \right\|_{X}^{\infty} = 0 \quad (5.25)$$

for $f \in C_{0}^\infty(G(0))$ by combining (5.21), (5.22), (5.23) and (5.24) with $r_{N}^{2} \varepsilon_{N}^{2} \to 0$ as $n \to \infty$. For $f \in C_{0}(G(0))$, we also obtain (5.25) by following [9, Theorem 2.1].

We are now ready to show (2.5). By the triangular inequality and $\| L \| \leq 1$, we have

$$\left\| L_{(n^{-1/2})}^{\frac{n}{r} - \frac{[ns]}{r}} P_{n^{-1/2}}^{H} f - P_{n^{-1/2}}^{H} e^{-\langle t-s \rangle A} f \right\|_{X}^{\infty} \leq \left\| P_{n^{-1/2}}^{H} f - P_{n^{-1/2}}^{H} e^{-\langle t-s \rangle A} f \right\|_{X}^{\infty} + \left\| L_{(n^{-1/2})}^{\frac{n}{r} - \frac{[ns]}{r}} P_{n^{-1/2}}^{H} f - P_{n^{-1/2}}^{H} e^{-\langle t-s \rangle A} f \right\|_{X}^{\infty} + \left\| P_{n^{-1/2}}^{H} e^{-\langle t-s \rangle A} f - P_{n^{-1/2}}^{H} e^{-\langle t-s \rangle A} f \right\|_{X}^{\infty} \quad (f \in C_{0}(G(0))). \quad (5.26)$$

We note that the functions $f$ and $e^{-\langle t-s \rangle A} f$ is uniformly continuous and

$$d_{CC}(\tau_{n^{-1/2}} \Phi^{(n^{-1/2})}(x), \tau_{n^{-1/2}} \Phi_{0}^{(n^{-1/2})}(x)) = \frac{1}{\sqrt{n}} d_{CC}(\Phi^{(n^{-1/2})}(x), \Phi_{0}^{(n^{-1/2})}(x)) \leq \frac{M}{\sqrt{n}}$$

for some $M > 0$. Therefore, by using (5.25) and by letting $n \to \infty$ in (5.26), we obtain the desired convergence (2.5) for $f \in C_{0}(G(0))$. The latter part of Theorem 2.1 is obtained in the same way as [9, Theorem 2.1]. This completes the proof.
5.3 Proof of Functional CLT Under Modified Harmonicities

We set \( \xi_n^{(e)}(c) \) for \( 0 \leq e \leq 1, n = 0, 1, 2, \ldots \) and \( c \in \Omega_{x_e}(X) \). Then \( G_{(0)} \)-valued random walk \((\Omega_{x_e}(X), \mathbb{P}_{x_e}^{(e)}, \{\xi_n^{(e)}\}_{n=0}^{\infty})\) associated with the transition probability \( p_e \) is induced. For \( t \geq 0, n = 1, 2, \ldots \) and \( 0 \leq e \leq 1 \), we set

\[
X_t^{(e,n)}(c) := \tau_{n^{-1/2}}\left(\xi_n^{(e)}(c)\right) \quad (c \in \Omega_{x_e}(X)).
\]

and define

\[
Y_t^{(e,n)}(c) := \tau_{n^{-1/2}}\left(\xi_n^{(e)}(c)\right) = \tau_{n^{-1/2}}\left(\Phi_0^{(e)}(w_k^{(e)}(c))\right) \quad (t_k \in D_n, c \in \Omega_{x_e}(X)).
\]

We also define the \( G_{(0)} \)-valued continuous stochastic process \( \{Y_t^{(e,n)}\}_{n=0}^{\infty} \) given by the \( d_{CC}\)-geodesic interpolation of \( \{\xi_n^{(e)}\}_{n=0}^{\infty} \). We denote by \( P^{(e,n)} \) the probability measure on \( C^{0,\alpha-H\ddot{o}l}_{\alpha,*}([0,1]; G_{(0)}) \) induced by the stochastic process \( Y_t^{(e,n)} \) for \( 0 \leq e \leq 1 \) and \( n \in \mathbb{N} \).

The aim of this subsection is to prove the following.

**Theorem 5.4** We assume \((A1)\) and \((A2)\). Then the sequence \( \{Y_t^{(n^{-1/2},n)}\}_{n=1}^{\infty} \) converges in law to the diffusion process \( \{Y_t\}_{0 \leq t \leq 1} \) in \( C^{0,\alpha-H\ddot{o}l}_{\alpha,*}([0,1]; G_{(0)}) \) as \( n \to \infty \) for all \( \alpha < 1/2 \).

We put

\[
\|d\Phi_0^{(e)}\|_{\infty} = \max_{\varepsilon \in E_k} \max_{k=1,2,\ldots,r} \left\| \log \left(\Phi_0^{(e)}(\mathcal{C})\right)\right\|_{\mathfrak{g}(k)}^{1/k} \quad (0 \leq e \leq 1).
\]

We describe a relation between \( \|d\Phi_0^{(e)}\|_{\infty} \) and \( \|d\Phi_0^{(0)}\|_{\infty} \) for every \( 0 \leq e \leq 1 \). Thanks to [9, Lemma 5.3 (3)], \((A2)\) and \((5.9)\), we easily obtain the following:

**Lemma 5.5** Under \((A2)\), there exists a positive constant \( C \) such that

\[
\sup_{0 \leq e \leq 1} \|d\Phi_0^{(e)}\|_{\infty} \leq C \|d\Phi_0^{(0)}\|_{\infty}.
\]

Denote by \( G_{(0)}^{(k)} \) the nilpotent Lie group of step \( k \) whose Lie algebra coincides with \((\mathfrak{g}^{(1)}, \mathfrak{g}_0^{(0)}) \oplus \mathfrak{g}^{(2)} \oplus \cdots \oplus \mathfrak{g}^{(k)}\). For the piecewise smooth stochastic process \( \{Y_t^{(e,n)}\}_{0 \leq t \leq 1} \), we define its truncated process by

\[
\gamma_t^{(e,n; k)} = (Y_t^{(e,n; 1)}, Y_t^{(e,n; 2)}, \ldots, Y_t^{(e,n; k)}) \in G_{(0)}^{(k)} \quad (k = 1, 2, \ldots, r)
\]

in the \((*)\)-coordinate system. By Lemma 5.5, we put

\[
\sup_{0 \leq e \leq 1} \left\{ \|d\Phi_0^{(e)}\|_{\infty} + \|\rho_\mathcal{R}(Y_p)\|_{\mathfrak{g}(1)} \right\} \leq C \|d\Phi_0^{(0)}\|_{\infty} + \|\rho_\mathcal{R}(Y_p)\|_{\mathfrak{g}(1)} =: M.
\]

As is well-known in probability theory, it suffices to show the tightness of \( \{P^{(n^{-1/2},n)}\}_{n=1}^{\infty} \) and the convergence of the finite dimensional distribution of \( \{Y_t^{(n^{-1/2},n)}\}_{n=1}^{\infty} \) to obtain Theorem 5.4. In the former part of this section, we show the following.

**Lemma 5.6** \( \{P^{(n^{-1/2},n)}\}_{n=1}^{\infty} \) is tight in \( C^{0,\alpha-H\ddot{o}l}_{\alpha,*}([0,1]; G_{(0)}) \), where \( \alpha < 1/2 \).

As the first step of the proof of Lemma 5.6, we need to show the following lemma.
Lemma 5.7 Let $m, n$ be positive integers. Then there exists a constant $C > 0$ independent of $n$ (however, it may depend on $m$) such that

$$
E^{p(n^{-1/2})}_{\bar{s}_*} \left[ d_{CC}(Y_s^{(n^{-1/2}, n; 2)}, Y_t^{(n^{-1/2}, n; 2)})^{4m} \right] \leq C(t - s)^{2m} \quad (0 \leq s \leq t \leq 1).
$$

(5.27)

Proof We split the proof into several steps, based on the argument in [10].

Step 1. First we show

$$
E^{p(n^{-1/2})}_{\bar{s}_*} \left[ d_{CC}(Y_{ik}^{(n^{-1/2}, n; 2)}, Y_{lt}^{(n^{-1/2}, n; 2)})^{4m} \right] \leq C\left(\frac{\ell - k}{n}\right)^{2m} \quad (n, m \in \mathbb{N}, \, i_k, \, t \in D_n (k \leq \ell))
$$

(5.28)

for some $C > 0$ which is independent of $n$ (but depending on $m$). By noting the equivalence mentioned in Section 4.1, (5.28) is equivalent to the existence of positive constants $C^{(1)}$ and $C^{(2)}$ independent of $n$ such that

$$
E^{p(n^{-1/2})}_{\bar{s}_*} \left[ \| \log (Y_{ik}^{(n^{-1/2}, n)})^{-1} \cdot Y_{lt}^{(n^{-1/2}, n)} \|_{q(i)}^{4m} \right] \leq C^{(1)} \left(\frac{\ell - k}{n}\right)^{2m},
$$

(5.29)

$$
E^{p(n^{-1/2})}_{\bar{s}_*} \left[ \| \log (Y_{ik}^{(n^{-1/2}, n)})^{-1} \cdot Y_{lt}^{(n^{-1/2}, n)} \|_{q(2)}^{2m} \right] \leq C^{(2)} \left(\frac{\ell - k}{n}\right)^{2m}.
$$

(5.30)

Step 2. We here prove (5.29). We have

$$
E^{p(n^{-1/2})}_{\bar{s}_*} \left[ \| \log (Y_{ik}^{(\varepsilon, n)})^{-1} \cdot Y_{lt}^{(\varepsilon, n)} \|_{q(i)}^{4m} \right] = \left(\frac{1}{\sqrt{n}}\right)^{4m} \sum_{i=1}^{d_1} \log \left(\frac{\xi_{ik}^{(\varepsilon)} \cdot \xi_{lt}^{(\varepsilon)}}{|X_i^{(\varepsilon)}|}\right)^{2m}
$$

$$
\leq \left(\frac{1}{\sqrt{n}}\right)^{4m} \cdot d_1^{2m} \cdot \max_{i=1,2,\ldots,d_1} \max_{x \in \mathcal{F}} \sum_{c \in \Omega_{\varepsilon, i, X}} p_x(c)
$$

$$
\times \log \left(\Phi_0^{(\varepsilon)}(x)^{-1} \cdot \Phi_0^{(\varepsilon)}(t(c))\right)^{4m} \quad (0 \leq \varepsilon \leq 1),
$$

(5.31)

where $\mathcal{F}$ stands for the fundamental domain in $X$ containing the reference point $x_* \in V$. For $i = 1, 2, \ldots, d_1$, $x \in \mathcal{F}$, $N \in \mathbb{N}, 0 \leq \varepsilon \leq 1$ and $c = (e_1, e_2, \ldots, e_N) \in \Omega_{\varepsilon, N}(X)$, put

$$
\mathcal{J}_i^{(\varepsilon)}(j) := \log \left(\frac{d\Phi_0^{(\varepsilon)}(e_j)}{|X_i^{(\varepsilon)}|} - \varepsilon \rho_R(y_p)\right)|_{X_i^{(\varepsilon)}},
$$

$$
\mathcal{N}_N^{(i, x)}(\Phi_0^{(\varepsilon)}; c) := \log \left(\Phi_0^{(\varepsilon)}(x)^{-1} \cdot \Phi_0^{(\varepsilon)}(t(c))\right)|_{X_i^{(\varepsilon)}} - N\rho_R(y_p)|_{X_i^{(\varepsilon)}} = \sum_{j=1}^{N} \mathcal{J}_i^{(\varepsilon)}(j).
$$
Note that $|\mathcal{J}_i(e)(j)| \leq M$ for $0 \leq \varepsilon \leq 1$, $i = 1, 2, \ldots, d_1$ and $j = 1, 2, \ldots, N$. Then we see that $\{\mathcal{N}_i^{(x)}\}_{N=1}^\infty$ is a martingale for $i = 1, 2, \ldots, d_1$ and $x \in \mathcal{F}$ (cf. [10, Lemma 3.3]). Hence, we use the Burkholder–Davis–Gundy inequality with the exponent $4m$ to obtain

$$
\sum_{c \in \Omega_{x,N}(X)} p_\varepsilon(c) \log \left\langle \Phi_0^{(e)}(x)^{-1} \cdot \Phi_0^{(e)}(t(c)) \right\rangle_{X_i^{(e)}}^{4m} 
\leq 2^{4m-1} \sum_{c \in \Omega_{x,N}(X)} p_\varepsilon(c) \left\{ \left(\mathcal{N}_i^{(x)}(c)\right)^{4m} + \left(N \varepsilon \rho(\gamma_p)\right)_{X_i^{(e)}}^{4m} \right\} 
\leq 2^{4m-1} C_{4m}^{4m} \sum_{c \in \Omega_{x,N}(X)} p_\varepsilon(c) \left\{ \left(\sum_{j=1}^{N} \mathcal{J}_i^{(e)}(j)^2\right)^{2m} + 2^{4m-1} \varepsilon^{4m} N^{4m} \|\rho(\gamma_p)\|_{g^{(i)}}^{4m} \right\} 
\leq 2^{4m} C_{4m}^{4m} M^{2m} N^{2m} + 2^{4m-1} M^{4m} \varepsilon^{4m} N^{4m} 
(x \in \mathcal{F}, i = 1, 2, \ldots, d_1, 0 \leq \varepsilon \leq 1, N \in \mathbb{N}). \tag{5.32}
$$

In particular, (5.32) implies

$$
\sum_{c \in \Omega_{x,\ell-k}(X)} p_{n-1/2}(c) \log \left\langle \Phi_0^{(n-1/2)}(x)^{-1} \cdot \Phi_0^{(n-1/2)}(t(c)) \right\rangle_{X_i^{(e)}}^{4m} 
\leq \left\{ 2^{4m} C_{4m}^{4m} M^{2m} + 2^{4m-1} M^{4m} \right\} (\ell - k)^{2m} \tag{5.33}
$$

by putting $\varepsilon = n^{-1/2}$ and $N = \ell - k$, where we note $(\ell - k)/n < 1$ since $1 \leq k \leq \ell \leq n$. We then obtain

$$
\mathbb{E}^{\mathbb{P}^{(n-1/2)}} \left[ \left\| \log \left( \mathcal{Y}_i^{(n-1/2),n} \right)^{-1} \cdot \mathcal{Y}_i^{(n-1/2),n} \right\|_{g^{(i)}}^{4m} \right] 
\leq d_1^{2m} \left\{ 2^{4m} C_{4m}^{4m} M^{2m} + 2^{4m-1} M^{4m} \right\} \left( \frac{\ell - k}{n} \right)^{2m} =: C^{(i)} \left( \frac{\ell - k}{n} \right)^{2m}
$$

by combining (5.31) with (5.33), which leads to (5.29).

**Step 3.** We show (5.30) at this step. We also see

$$
\mathbb{E}^{\mathbb{P}^{(n)}} \left[ \left\| \log \left( \mathcal{Y}_i^{(e,n)} \right)^{-1} \cdot \mathcal{Y}_i^{(e,n)} \right\|_{g^{(i)}}^{2m} \right] 
\leq \left( \frac{1}{n} \right)^{2m} \cdot d_2^{2m} \max_{i=1,2,\ldots,d_2} \max_{x \in \mathcal{F}} \left\{ \sum_{c \in \Omega_{x,\ell-k}(X)} p_\varepsilon(c) \right\} \times \log \left\langle \Phi_0^{(e)}(x)^{-1} \cdot \Phi_0^{(e)}(t(c)) \right\rangle_{X_i^{(2)}}^{2m} \tag{0 \leq \varepsilon \leq 1}. \tag{5.34}
$$
in the similar way to (5.31). Then we have

\[
\begin{align*}
\log \left( \Phi_0^{(e)}(x) \cdot \Phi_0^{(e)}(t(c)) \right)_{\lambda_i^{(2)}}^{2m} &= \left( \sum_{j=1}^{\ell-k} \log \left( d \Phi_0^{(e)}(e_j) \right)_{X_i^{(2)}} \right) - \frac{1}{2} \sum_{1 \leq j_1 < j_2 \leq \ell - k} \sum_{1 \leq k < v \leq d_i} \left\| X^{(1)}_{\lambda}, X^{(1)}_{v} \right\|_{X_i^{(2)}} \\
& \times \left\{ \log \left( d \Phi_0^{(e)}(e_{j_1}) \right)_{X^{(1)}_{\lambda}} \log \left( d \Phi_0^{(e)}(e_{j_2}) \right)_{X^{(1)}_{v}} \right\}^{2m} \\
& = 3^{2m-1} \left\{ \left( \sum_{j=1}^{\ell-k} \log \left( d \Phi_0^{(e)}(e_j) \right)_{X_i^{(2)}} \right) \right\}^{2m} \\
& + L \max_{1 \leq \lambda < v \leq d_i} \left( \sum_{1 \leq j_1 < j_2 \leq \ell - k} \log \left( d \Phi_0^{(e)}(e_{j_1}) \right)_{X^{(1)}_{\lambda}} \log \left( d \Phi_0^{(e)}(e_{j_2}) \right)_{X^{(1)}_{v}} \right)^{2m} \\
& + L \max_{1 \leq \lambda < v \leq d_i} \left( \sum_{1 \leq j_1 < j_2 \leq \ell - k} \log \left( d \Phi_0^{(e)}(e_{j_1}) \right)_{X^{(1)}_{\lambda}} \log \left( d \Phi_0^{(e)}(e_{j_2}) \right)_{X^{(1)}_{v}} \right)^{2m} \\
& \leq (\ell - k)^{2m} \left( \sum_{j=1}^{\ell-k} \frac{1}{\ell - k} \log \left( d \Phi_0^{(e)}(e_j) \right)_{X_i^{(2)}} \right)^{2m} \\
& \leq \| d \Phi_0^{(e)} \|^4 \| (\ell - k)^{2m} \leq M^{4m} (\ell - k)^{2m}. \\
(5.35)
\end{align*}
\]

where we put

\[
L := \frac{1}{2} \max_{i=1,2,\ldots,d_2} \max_{1 \leq \lambda < v \leq d_i} \left\| X^{(1)}_{\lambda}, X^{(1)}_{v} \right\|_{X_i^{(2)}}
\]

We fix \( i = 1,2,\ldots,d_2 \). Then we have

\[
\left( \sum_{j=1}^{\ell-k} \log \left( d \Phi_0^{(e)}(e_j) \right)_{X_i^{(2)}} \right)^{2m} = (\ell - k)^{2m} \left( \sum_{j=1}^{\ell-k} \frac{1}{\ell - k} \log \left( d \Phi_0^{(e)}(e_j) \right)_{X_i^{(2)}} \right)^{2m} \\
\leq \| d \Phi_0^{(e)} \|^4 \| (\ell - k)^{2m} \leq M^{4m} (\ell - k)^{2m}. \\
(5.36)
\]

by applying the Jensen inequality. For \( 1 \leq \lambda < v \leq d_i, x \in F, 0 \leq \varepsilon \leq 1, N \in \mathbb{N} \) and \( c = (e_1, e_2, \ldots, e_N) \in \Omega_{N,\varepsilon}^{(e)}(X) \), we set

\[
\widetilde{N}^{(\varepsilon,v,x)}_N(\Phi_0^{(e)} ; c) := \sum_{1 \leq j_1 < j_2 \leq N} \mathcal{J}_\varepsilon^{(e)}(j_1) \mathcal{J}_\varepsilon^{(e)}(j_2) = \sum_{j_2=2}^{N} \mathcal{J}_\varepsilon^{(e)}(j_2) \sum_{j_1=1}^{j_2-1} \mathcal{J}_\varepsilon^{(e)}(j_1). 
\]
Then we also see that \( \{ \tilde{N}_N^{(\lambda, v, x)} \}_{N=1}^{\infty} \) is an \( \mathbb{R} \)-valued martingale for every \( 1 \leq \lambda < v \leq d \) and \( x \in \mathcal{F} \). By applying the Burkholder–Davis–Gundy inequality with the exponent \( 2m \), we have

\[
\sum_{c \in \Omega_{x, N}(X)} p_e(c) \left( \tilde{N}_N^{(\lambda, v, x)}(c) \right)^{2m} \leq C_{(2m)}^{2m} \sum_{c \in \Omega_{x, N}(X)} p_e(c) \left\{ \sum_{j_2=2}^{N} J_v^{(\varepsilon)}(j_2)^2 \left( \sum_{j_1=1}^{j_2-1} J_\lambda^{(\varepsilon)}(j_1) \right)^2 \right\}^m
\]

\[
\leq C_{(2m)}^{2m} \sum_{c \in \Omega_{x, N}(X)} p_e(c) (N - 1)^m \sum_{j_2=2}^{N} \frac{1}{N - 1} J_v^{(\varepsilon)}(j_2)^{2m} \left( \sum_{j_1=1}^{j_2-1} J_\lambda^{(\varepsilon)}(j_1) \right)^{2m}
\]

\[
\leq C_{(2m)}^{2m} N^m \sum_{j_2=2}^{N} \frac{1}{N - 1} \left( \sum_{c \in \Omega_{x, N}(X)} p_e(c) J_v^{(\varepsilon)}(j_2)^{4m} \right)^{1/2}
\times \left\{ \sum_{c \in \Omega_{x, N}(X)} p_e(c) \left( \sum_{j_1=1}^{j_2-1} J_\lambda^{(\varepsilon)}(j_1) \right)^{4m} \right\}^{1/2}
\]

\[
\leq C_{(2m)}^{2m} M^{2m} N^m \sum_{j_2=2}^{N} \frac{1}{N - 1} \left\{ \sum_{c \in \Omega_{x, N}(X)} p_e(c) \left( \sum_{j_1=1}^{j_2-1} J_\lambda^{(\varepsilon)}(j_1) \right)^{4m} \right\}^{1/2}, \quad (5.37)
\]

where we used Jensen’s inequality for the third line and Schwartz’s inequality for the final line. Then the again use of the Burkholder–Davis–Gundy inequality with the exponent \( 4m \) gives

\[
\sum_{c \in \Omega_{x, N}(X)} p_e(c) \left( \sum_{j_1=1}^{j_2-1} J_\lambda^{(\varepsilon)}(j_1) \right)^{4m} \leq C_{(4m)}^{4m} \sum_{c \in \Omega_{x, N}(X)} p_e(c) \left( \sum_{j_1=1}^{j_2-1} J_\lambda^{(\varepsilon)}(j_1) \right)^{2m}
\]

\[
= C_{(4m)}^{4m} (j_2 - 1)^{2m} \sum_{c \in \Omega_{x, N}(X)} p_e(c) \left( \sum_{j_1=1}^{j_2-1} \frac{1}{j_2 - 1} J_\lambda^{(\varepsilon)}(j_1) \right)^{2m}
\]

\[
\leq C_{(4m)}^{4m} j_2^{2m} \sum_{c \in \Omega_{x, N}(X)} p_e(c) \sum_{j_1=1}^{j_2-1} \frac{1}{j_2 - 1} J_\lambda^{(\varepsilon)}(j_1)^{4m} \leq C_{(4m)}^{4m} M^{4m} j_2^{2m}. \quad (5.38)
\]

It follows from (5.37) and (5.38) that

\[
\sum_{c \in \Omega_{x, N}(X)} p_e(c) \left( \tilde{N}_N^{(\lambda, v, x)}(c) \right)^{2m} \leq C_{(2m)}^{2m} M^{2m} N^m \sum_{j_2=2}^{N} \frac{1}{N - 1} \left( C_{(4m)}^{4m} M^{4m} j_2^{2m} \right)^{1/2}
\]

\[
\leq C_{(2m)}^{2m} C_{(4m)}^{2m} M^{4m} N^{2m}. \quad (5.39)
\]
Hence, (5.39) implies

$$
\sum_{c \in \Omega_{s,N}(X)} p_{\mathbb{C}}(c) \left( \sum_{1 \leq j_1 < j_2 \leq N} \log \left( \Phi_{0}^{(e)}(e_{j_1}) \right) \left| X_{\lambda}^{(i)} \right| \log \left( \Phi_{0}^{(e)}(e_{j_2}) \right) \left| X_{\nu}^{(i)} \right| \right)^{2m} \\
\leq 4^{2m-1} \sum_{c \in \Omega_{s,N}(X)} p_{\mathbb{C}}(c) \left\{ \left( \sum_{1 \leq j_1 < j_2 \leq N} \left( \mathcal{N}_{N}^{(e)}(c) \right)^{2m} \right) + \left( \varepsilon \rho_{\mathbb{P}}(y_{p}) \left| X_{\lambda}^{(i)} \right| \rho_{\mathbb{P}}(y_{p}) \left| X_{\nu}^{(i)} \right| \frac{N(N-1)}{2} \right)^{2m} \right. \\
+ \left( \varepsilon \rho_{\mathbb{P}}(y_{p}) \left| X_{\lambda}^{(i)} \right| \sum_{1 \leq j_1 < j_2 \leq N} \mathcal{J}_{\lambda}^{(e)}(j_1) \right)^{2m} + \left( \varepsilon \rho_{\mathbb{P}}(y_{p}) \left| X_{\nu}^{(i)} \right| \sum_{1 \leq j_1 < j_2 \leq N} \mathcal{J}_{\nu}^{(e)}(j_2) \right)^{2m} \right\} \\
\leq 4^{2m-1} \left\{ C_{(2m)}^{2m} C_{(4m)}^{2m} M^{4m} N^{2m} + 2^{-2m} M^{4m} \varepsilon^{4m} N^{4m} \right. \\
+ \left. 2M^{2m} \varepsilon^{2m} N^{2m} \max_{1 \leq i \leq d_{1}} \sum_{c \in \Omega_{s,N}(X)} p_{\mathbb{C}}(c) \left( \sum_{j=1}^{N} \mathcal{J}_{i}^{(n)}(j) \right)^{2m} \right\} \\
\leq 4^{2m-1} \left\{ C_{(2m)}^{2m} C_{(4m)}^{2m} M^{4m} N^{2m} + 2^{-2m} M^{4m} \varepsilon^{4m} N^{4m} \right. \\
+ \left. 2M^{2m} \varepsilon^{2m} N^{2m} \left( 2^{2m} C_{(2m)}^{2m} M^{m} N^{m} + 2^{2m-1} 2^{m} M^{2m} \varepsilon^{2m} N^{2m} \right) \right\}, \tag{5.40}
$$

where we used (5.32) for the final line.

We now put $\varepsilon = n^{-1/2}$ and $N = \ell - k$. Then we have, for $1 \leq \lambda < \nu \leq d_{1},$

$$
\sum_{c \in \Omega_{s,N}(X)} p_{\mathbb{C}}(c) \left( \sum_{1 \leq j_1 < j_2 \leq \ell-k} \log \left( \Phi_{0}^{(e)}(e_{j_1}) \right) \left| X_{\lambda}^{(i)} \right| \log \left( \Phi_{0}^{(e)}(e_{j_2}) \right) \left| X_{\nu}^{(i)} \right| \right)^{2m} \\
\leq 4^{2m-1} M^{4m} \left( C_{(2m)}^{2m} C_{(4m)}^{4m} + 2^{-2m} + 2^{m+1} C_{(2m)}^{2m} M^{-m} + 2^{m} \right) (\ell-k)^{2m} \tag{5.41}
$$
due to (5.40) and $(\ell-k)/n < 1$. We obtain

$$
\mathbb{E}_{s}^{g^{2m}(n^{-1/2})} \left[ \left\| \log \left( \mathcal{Y}_{\ell_{k}}^{(n^{-1/2}, n^{-1/2})} \right) \mathcal{J}_{\ell_{k}}^{(n^{-1/2}, n^{-1/2})} \left| g^{(\ell_{k})} n^{-2m} \right| \right\|_{g^{(\ell_{k})}} \right] \leq C^{(2)} \left( \frac{\ell-k}{n} \right)^{2m}.
$$

by combining (5.34) with (5.35), (5.36) and (5.41), where

$$
C^{(2)} := d_{2}^{2m} 3^{2m-1} \left\{ M^{4m} + 2L \cdot 4^{2m-1} M^{4m} \left( C_{(2m)}^{2m} C_{(4m)}^{4m} + 2^{-2m} + 2^{m+1} C_{(2m)}^{2m} M^{-m} + 2^{m} \right) \right\}.
$$

This means (5.30) and we thus obtain (5.28).

**Step 4.** We show (5.27) at the last step. Suppose that $t_{k} \leq s \leq t_{k+1}$ and $t_{t} \leq t \leq t_{t+1}$ for some $1 \leq k \leq n \leq \ell$. Then we have

$$
d_{CC}(\mathcal{Y}^{(n^{-1/2}, n^{-1/2})}, \mathcal{Y}^{(n^{-1/2}, n^{-1/2})}) = (k-n) d_{CC}(\mathcal{Y}^{(n^{-1/2}, n^{-1/2})}, \mathcal{Y}^{(n^{-1/2}, n^{-1/2})}),$$

$$
d_{CC}(\mathcal{Y}^{(n^{-1/2}, n^{-1/2})}, \mathcal{Y}^{(n^{-1/2}, n^{-1/2})}) = (nt-\ell) d_{CC}(\mathcal{Y}^{(n^{-1/2}, n^{-1/2})}, \mathcal{Y}^{(n^{-1/2}, n^{-1/2})}).$$
by noting that the piecewise smooth stochastic process \( \mathcal{Y}^{(n^{-1/2},n)} \) is given by the \( \text{dCC} \)-geodesic interpolation. Hence, \((5.28)\) and the triangle inequality yield

\[
\mathbb{E}^{\alpha(n^{-1/2})}_{\mathcal{F}_t} \left[ \text{dCC}(\mathcal{Y}_s^{(n^{-1/2},n)}, \mathcal{Y}_t^{(n^{-1/2},n)})^{4m} \right] \\
\leq 3^{4m-1} \left\{ (k + 1 - ns)^{4m} \cdot C \left( \frac{1}{n} \right)^{2m} + C \left( \frac{\ell - k - 1}{n} \right)^{2m} \right. \\
+ \left( nt - \ell \right)^{4m} \cdot C \left( \frac{1}{n} \right)^{2m} \right\} \\
\leq C \left\{ (t_{k+1} - s)^{2m} + (t_{\ell} - t_{k+1})^{2m} + (t - t_{\ell})^{2m} \right\} \leq C(t - s)^{2m}.
\]

This completes the proof of Lemma 5.7. \( \square \)

In what follows, we write \( \mathcal{Y}_s^{(e,n)} \) for brevity. We now show the following by using Lemma 5.7.

**Lemma 5.8** For \( m, n \in \mathbb{N}, k = 1, 2, \ldots, r \) and \( \alpha < \frac{2m-1}{4m} \), there exist a measurable set \( \Omega^{(n)}_k \subset \Omega_{x_s}(X) \) and a non-negative random variable \( \mathcal{K}^{(n)}_k \in L^{4m}(\Omega_{x_s}(X) \to \mathbb{R}; \mathbb{P}^{(n^{-1/2})}_{x_s}) \) such that \( \mathbb{P}^{(n^{-1/2})}_{x_s}(\Omega^{(n)}_k) = 1 \) and

\[
\text{dCC}(\mathcal{Y}_s^{(n^{-1/2},n,k)}(c), \mathcal{Y}_t^{(n^{-1/2},n,k)}(c)) \leq \mathcal{K}^{(n)}_k(c)(t-s)^{\alpha} \quad (c \in \Omega^{(n)}_k, 0 \leq s \leq t \leq 1).
\]

*(5.42)*

**Proof** We partially follow Lyons’ original proof (cf. [21, Theorem 2.2.1]) of the extension theorem in rough path theory. We prove \((5.42)\) by induction on the step number \( k = 1, 2, \ldots, r \).

**Step 1.** In the cases \( k = 1, 2 \), we have already obtained \((5.42)\) in Lemma 5.7. In fact, \((5.42)\) for \( k = 1, 2 \) are obtained by a simple application of the Kolmogorov–Chentsov criterion with the bound

\[
\| \mathcal{K}^{(n)}_k \|_{L^{4m}(\mathbb{P}^{(n^{-1/2})}_{x_s})} \leq \frac{5C}{(1 - 2^{-\theta})(1 - 2\alpha - \theta)} \quad (n, m \in \mathbb{N}, k = 1, 2), \quad (5.43)
\]

where \( \theta = (2m - 1)/4m \) and \( C \) is a constant independent of \( n \) which appears on the right-hand side of \((5.27)\). See e.g., Stroock [30, Theorem 4.3.2] for details.

**Step 2.** We now fix \( n \in \mathbb{N} \). Assume that \((5.42)\) holds up to step \( k \). We note that this assumption is equivalent to the existence of measurable sets \( \hat{\Omega}^{(n)}_{j=1} \) and non-negative random variables \( \hat{\mathcal{K}}^{(n)}_{j=1} \) such that \( \mathbb{P}^{(n^{-1/2})}_{x_s}(\hat{\Omega}^{(n)}_{j=1}) = 1 \) and

\[
\| (d\mathcal{Y}^{(n^{-1/2},n)}_{s,t})(c) \|_{\mathbb{R}^{4m}} \leq \hat{\mathcal{K}}^{(n)}_{j=1}(c)(t-s)^{\alpha} \quad (c \in \hat{\Omega}^{(n)}_{j=1}, 0 \leq s \leq t \leq 1)
\]

*(5.44)*

with \( \hat{\mathcal{K}}^{(n)}_{j=1} \in L^{4m/j}(\Omega_{x_s}(X) \to \mathbb{R}; \mathbb{P}^{(n^{-1/2})}_{x_s}) \) for \( n, m \in \mathbb{N} \) and \( j = 1, 2, \ldots, k \).
We fix $0 \leq s \leq t \leq 1$, $n \in \mathbb{N}$ and write $\hat{\Omega}_{k+1}^{(n)} = \bigcap_{j=1}^{k} \hat{\Omega}_{j}^{(n)}$. We denote by $\Delta$ the partition $(s = t_{0} < t_{1} < \cdots < t_{N} = t)$ of the time interval $[s, t]$ independent of $n \in \mathbb{N}$. We now define two $G_{(0)}^{(k+1)}$-valued random variables $Z_{s,t}^{(n)}$ and $Z(\Delta)_{s,t}^{(n)}$ by

$$(Z_{s,t}^{(n)})^{(j)} := \begin{cases} (dY_{s,t}^{(n-1/2,i)})^{(j)} & (j = 1, 2, \ldots, k), \\ 0 & (j = k + 1), \end{cases}$$

$$Z(\Delta)_{s,t}^{(n)} := Z_{t_{0},t_{1}}^{(n)} \ast Z_{t_{1},t_{2}}^{(n)} \ast \cdots \ast Z_{t_{N-1},t_{N}}^{(n)};$$

respectively. For $i = 1, 2, \ldots, d_{k+1}$, (4.1) and (5.44) imply

$$\left| \left( Z(\Delta)_{s,t}^{(n)}(c) \right)_{t_{i}} \right|^{(k+1)} = \left( Z(\Delta \setminus \{t_{N-1}\})_{s,t}^{(n)}(c) \right)_{t_{i}}^{(k+1)}$$

$$= \left( Z_{t_{N-2},t_{N-1}}^{(n)}(c) \ast Z_{t_{N-1},t_{N}}^{(n)}(c) \right)_{t_{i}}^{(k+1)} - \left( Z_{t_{N-2},t_{N}}^{(n)}(c) \right)_{t_{i}}^{(k+1)}$$

$$\leq C \sum_{|K_{1}|+|K_{2}|=k+1} P_{t_{i}} \left( Z_{t_{N-2},t_{N-1}}^{(n)}(c) \ast Z_{t_{N-1},t_{N}}^{(n)}(c) \right)_{t_{i}}^{(k+1)}$$

$$\leq \hat{\kappa}_{k+1}^{(n)}(c)(t_{N} - t_{N-2})^{(k+1)} \leq \hat{\kappa}_{k+1}^{(n)}(c) \left( \frac{2}{N-1} (t-s) \right)^{(k+1)\alpha} (c \in \hat{\Omega}_{k+1}^{(n)}),$$

where the random variable $\hat{\kappa}_{k+1}^{(n)} : \Omega_{s,t}(X) \rightarrow \mathbb{R}$ is given by

$$\hat{\kappa}_{k+1}^{(n)}(c) := C \sum_{|K_{1}|+|K_{2}|=k+1} Q^{(n,K_{1})}(c) Q^{(n,K_{2})}(c),$$

$$Q^{(n,K)}(c) := \hat{\kappa}_{k_{1}}^{(n)}(c) \cdots \hat{\kappa}_{k_{\ell}}^{(n)}(c) \quad (K = (i_{1}, k_{1}), (i_{2}, k_{2}), \ldots, (i_{\ell}, k_{\ell})).$$

We emphasize that $\hat{\kappa}_{k+1}^{(n)}$ is non-negative and has the following integrability:

$$\mathbb{E}_{x_{*}}^{\|n/2\}} \left[ (\hat{\kappa}_{k+1}^{(n)})^{4m/(k+1)} \right] \leq C \sum_{k_{1},\ldots,k_{\ell} \geq 0} \mathbb{E}_{x_{*}}^{\|n/2\}} \left[ (\hat{\kappa}_{k_{1}}^{(n)} \cdots \hat{\kappa}_{k_{\ell}}^{(n)})^{4m/(k+1)} \right]$$

$$\leq C \sum_{k_{1},\ldots,k_{\ell} \geq 0} \prod_{k_{1}=1}^{\ell} \mathbb{E}_{x_{*}}^{\|n/2\}} \left[ (\hat{\kappa}_{k_{1}}^{(n)})^{4m/k_{1}} \right]_{k_{1}+1}^{(k+1)} < \infty,$$

where we used the generalized Hölder inequality for the second line. We then have

$$\left| \left( Z(\Delta)_{s,t}^{(n)}(c) \right)_{t_{i}} \right|^{(k+1)}$$

$$\leq \left| \left( Z(\Delta \setminus \{t_{N-1}\})_{s,t}^{(n)}(c) \right)_{t_{i}}^{(k+1)} \right| + \hat{\kappa}_{k+1}^{(n)}(c) \left( \frac{2}{N-1} (t-s) \right)^{(k+1)\alpha}$$

$$\leq \left( Z_{s,t}^{(n)}(c) \right)_{t_{i}}^{(k+1)} + \hat{\kappa}_{k+1}^{(n)}(c) \left( (k+1) \alpha \right)(t-s)^{(k+1)\alpha}$$

$$\leq \hat{\kappa}_{k+1}^{(n)}(c)(t-s)^{(k+1)\alpha} \quad (i = 1, 2, \ldots, d_{k+1}, c \in \hat{\Omega}_{k+1}^{(n)}).$$

(5.45)
by successively removing points until the partition $\Delta$ coincides with $\{s, t\}$, where $\xi(z)$ denotes the Riemann zeta function $\xi(z) := \sum_{n=1}^{\infty} (1/n^z)$ for $z \in \mathbb{R}$.

We now show that the family $\{Z(\Delta)_{s,t}^{(n)}\}$ is a Cauchy sequence. Let $\delta > 0$ and we take two partitions $\Delta = \{s = t_0 < t_1 < \cdots < t_N = t\}$ and $\Delta' = \{s, t\}$ independent of $n \in \mathbb{N}$ satisfying $|\Delta|, |\Delta'| < \delta$. We set $\hat{\Delta} := \Delta \cup \Delta'$ and write

$$\hat{\Delta}_\ell = \hat{\Delta} \cap [t_\ell, t_{\ell+1}) = \{t_\ell < s_{\ell 0} < s_{\ell 1} < \cdots < s_{\ell L_\ell} = t_{\ell+1}\} \quad (\ell = 0, 1, \ldots, N - 1).$$

Then (4.1) and (5.45) give

$$\left|\left(Z(\Delta)_{s,t}^{(n)}(c)\right)_{i_s}^{(k+1)} - \left(Z(\hat{\Delta})_{s,t}^{(n)}(c)\right)_{i_s}^{(k+1)}\right| = \left|\left(Z_{t_0, t_1}^{(n)}(c) \ast \cdots \ast Z_{t_{N-1}, t_N}^{(n)}(c)\right)_{i_s}^{(k+1)} - \left(Z(\Delta_0)_{t_0, t_1}^{(n)}(c) \ast \cdots \ast Z(\Delta_{N-1})_{t_{N-1}, t_N}^{(n)}(c)\right)_{i_s}^{(k+1)}\right|$$

$$= \left|\left(Z_{t_0, t_1}^{(n)}(c)\right)_{i_s}^{(k+1)} + \left(Z_{t_1, t_2}^{(n)}(c) \ast \cdots \ast Z_{t_{N-1}, t_N}^{(n)}(c)\right)_{i_s}^{(k+1)} - \left(Z(\Delta_0)_{t_0, t_1}^{(n)}(c)\right)_{i_s}^{(k+1)} - \left(Z(\Delta_1)_{t_1, t_2}^{(n)}(c) \ast \cdots \ast Z(\Delta_{N-1})_{t_{N-1}, t_N}^{(n)}(c)\right)_{i_s}^{(k+1)}\right|$$

$$\leq \hat{K}_{k+1}^{(n)}(c)(t_1 - t_0)^{(k+1)\alpha} + \left|\left(Z_{t_1, t_2}^{(n)}(c) \ast \cdots \ast Z_{t_{N-1}, t_N}^{(n)}(c)\right)_{i_s}^{(k+1)} - \left(Z(\Delta_0)_{t_1, t_2}^{(n)}(c) \ast \cdots \ast Z(\Delta_{N-1})_{t_{N-1}, t_N}^{(n)}(c)\right)_{i_s}^{(k+1)}\right| \quad (i = 1, 2, \ldots, d_{k+1}, c \in \hat{\Omega}_{k+1}^{(n)}).$$

By repeating this kind of estimate and noting $(k + 1)\alpha > 1$, we obtain

$$\left|\left(Z(\Delta)_{s,t}^{(n)}(c)\right)_{i_s}^{(k+1)} - \left(Z(\hat{\Delta})_{s,t}^{(n)}(c)\right)_{i_s}^{(k+1)}\right| \leq \sum_{\ell=0}^{N-1} \hat{K}_{k+1}^{(n)}(c)(t_\ell + 1 - t_\ell)^{(k+1)\alpha}$$

$$\leq \hat{K}_{k+1}^{(n)}(c)\left(\max_{\Delta}(t_\ell + 1 - t_\ell)^{(k+1)\alpha-1}\right) \sum_{\ell=0}^{N-1} (t_\ell + 1 - t_\ell)$$

$$\leq \hat{K}_{k+1}^{(n)}(c)(t - s) \cdot \delta^{(k+1)\alpha-1} \quad (i = 1, 2, \ldots, d_{k+1}, c \in \hat{\Omega}_{k+1}^{(n)}).$$

Thus, (5.46) leads to

$$\left|\left(Z(\Delta)_{s,t}^{(n)}(c)\right)_{i_s}^{(k+1)} - \left(Z(\hat{\Delta})_{s,t}^{(n)}(c)\right)_{i_s}^{(k+1)}\right| \leq 2\hat{K}_{k+1}^{(n)}(c)(t - s) \cdot \delta^{(k+1)\alpha-1} \rightarrow 0 \quad (i = 1, 2, \ldots, d_{k+1}, c \in \hat{\Omega}_{k+1}^{(n)})$$

as $\delta \searrow 0$ uniformly in $0 \leq s \leq t \leq 1$. Therefore, there exists, for $0 \leq s \leq t \leq 1$,

$$\bar{Z}_{s,t}^{(n)}(c) := \begin{cases} \lim_{|\Delta| \searrow 0} Z(\Delta)_{s,t}^{(n)}(c) \quad (c \in \hat{\Omega}_{k+1}^{(n)}), \\ 1_{G} \quad (c \in \Omega_{\pi}(X) \setminus \hat{\Omega}_{k+1}^{(n)}) \end{cases}$$

satisfying

$$\|Z_{s,t}^{(n)}(c)\|_{\mathbb{R}^{d_{k+1}}}^{(k+1)} \leq \bar{K}_{k+1}^{(n)}(c)(t - s)^{(k+1)\alpha} \quad (c \in \hat{\Omega}_{k+1}^{(n)}),$$

due to (5.45). We will show

$$\bar{Z}_{s,t}^{(n)}(c) = \gamma_{s}^{(n-1/2, n; k+1)}(c)^{-1} * \gamma_{t}^{(n-1/2, n; k+1)}(c) \quad (0 \leq s \leq t \leq 1, c \in \hat{\Omega}_{k+1}^{(n)}).$$
as the last step. For this, it is sufficient to check that
\[
\left( \overline{Z}_{s,t}^{(n)}(c) \right)^{(k+1)} = (dY_{s,t}^{(n^{-1/2},n)}(c))^{(k+1)} \quad (0 \leq s \leq t \leq 1, \, c \in \overline{\Omega}_{k+1}^{(n)}) \quad (5.47)
\]
by the definition of $\overline{Z}_{s,t}^{(n)}$. We fix $i = 1, 2, \ldots, d_{k+1}$ and $c \in \overline{\Omega}_{k+1}^{(n)}$. Put
\[
\Psi_{s,t}^{i}(c) := \left( dY_{s,t}^{(n^{-1/2},n)}(c) \right)^{(k+1)} - \left( \overline{Z}_{s,t}^{(n)}(c) \right)^{(k+1)} \quad (0 \leq s \leq t \leq 1).
\]
Then we easily see that $\Psi_{s,t}^{i}(c)$ is additive in the sense that
\[
\Psi_{s,t}^{i}(c) = \Psi_{s,u}^{i}(c) + \Psi_{u,t}^{i}(c) \quad (0 \leq s \leq u \leq t \leq 1). \quad (5.48)
\]
Since the piecewise smooth stochastic process $(Y_{t}^{(n^{-1/2},n)})_{0 \leq t \leq 1}$ is given by the $dCC$-geodesic interpolation of $\{X_{k}^{(n^{-1/2},n)}\}_{k=0}^{n}$, we have
\[
\left\| (dY_{s,t}^{(n^{-1/2},n)}(c))^{(k+1)} \right\|_{R_{k+1}} \leq \overline{K}_{k+1}^{(n)}(c)(t-s)^{(k+1)\alpha} \quad (c \in \overline{\Omega}_{k+1}^{(n)})
\]
for some set $\overline{\Omega}_{k+1}^{(n)}$ with $\mathbb{P}_{x_{*}}(\overline{\Omega}_{k+1}^{(n)}) = 1$ and random variable $\overline{K}_{k+1}^{(n)} : \Omega_{x_{*}}(X) \to \mathbb{R}$. Thus, we have
\[
\left| \Psi_{s,t}^{i}(c) \right| \leq \left( \overline{K}_{k+1}^{(n)}(c) + \overline{K}_{k+1}^{(n)}(c) \right)(t-s)^{(k+1)\alpha} \quad (0 \leq s \leq t \leq 1, \, c \in \overline{\Omega}_{k+1}^{(n)}). \quad (5.49)
\]
We may write $\overline{\Omega}_{k+1}^{(n)}$ instead of $\overline{\Omega}_{k+1}^{(n)} \cap \overline{\Omega}_{k+1}^{(n)}$ by abuse of notation. Because its probability equals one. For any small $\varepsilon > 0$, there is a sufficiently large $N \in \mathbb{N}$ such that $1/N < \varepsilon$. We then obtain as $\varepsilon \searrow 0$,
\[
\left| \Psi_{0,t}^{i}(c) \right| = \left| \Psi_{0,N}^{i}(c) + \Psi_{N/2,N}^{i}(c) + \ldots + \Psi_{[Nt]/N,N}^{i}(c) \right| \leq \left( \overline{K}_{k+1}^{(n)}(c) + \overline{K}_{k+1}^{(n)}(c) \right)(t-s)^{(k+1)\alpha-1} \left\{ \frac{1}{N} + \ldots + \frac{1}{N} + \left( t - \frac{[Nt]}{N} \right) \right\}^{(k+1)\alpha-1} \to 0 \quad (0 \leq t \leq 1, \, c \in \overline{\Omega}_{k+1}^{(n)})
\]
by (5.48) and $(k+1)\alpha - 1 > 0$. This implies that $\Psi_{0,t}^{i}(c) = 0$ for $0 \leq t \leq 1$ and $c \in \overline{\Omega}_{k+1}^{(n)}$. Hence, it follows from (5.48) that $\Psi_{s,t}^{i}(c) = \Psi_{0,t}^{i}(c) - \Psi_{0,s}^{i}(c) = 0$ for $0 \leq s \leq t \leq 1$ and $c \in \overline{\Omega}_{k+1}^{(n)}$, which leads to (5.47). Consequently, we know that there are a measurable set $G_{k+1}^{(n)} \subset \Omega_{x_{*}}(X)$ with probability one and a non-negative random variable $K_{k+1}^{(n)} \in L^{4m}(\Omega_{x_{*}}(X) \to \mathbb{R}; \mathbb{P}_{x_{*}}^{(n^{1/2})})$ satisfying
\[
dCC(Y_{s}^{(n^{-1/2},n); k+1}(c), Y_{t}^{(n^{-1/2},n); k+1}(c)) \leq K_{k+1}^{(n)}(c)(t-s)^{\alpha} \quad (c \in \Omega_{k+1}^{(n)}, \, 0 \leq s \leq t \leq 1).
\]
This completes the proof of Lemma 5.8.

\textbf{Proof of Lemma 5.6} For $m, n \in \mathbb{N}$ and $\varepsilon < \frac{2m-1}{4m}$, Lemma 5.8 implies
\[
\mathbb{E}_{x_{*}}^{(n^{1/2})} \left[ dCC(Y_{s}^{(n^{-1/2},n); r}, Y_{t}^{(n^{-1/2},n); r})^{4m} \right] \leq \mathbb{E}_{x_{*}}^{(n^{1/2})} \left[ (K_{r}^{(n)})^{4m} \right](t-s)^{4m\alpha}.
\]
for $0 \leq s \leq t \leq 1$. We thus have, by (5.43),
\[
\mathbb{E}_{x_{*}}^{(n^{1/2})} \left[ dCC(Y_{s}^{(n^{-1/2},n); r}, Y_{t}^{(n^{-1/2},n); r})^{4m} \right] \leq C(t-s)^{4m\alpha} \quad (0 \leq s \leq t \leq 1)
\]
for some constant $C > 0$ independent of $n \in \mathbb{N}$. Furthermore, thanks to (A2) and $\Phi_0^{(0)}(x_*) = 1_G$, there is a sufficiently large constant $C > 0$ such that

$$\sup_{n \in \mathbb{N}} \| \log (\Phi_0^{(n-1/2)}(x_*)) \|_{g^{(k)}} \leq C \quad (k = 1, 2, \ldots, r).$$

By applying the Kolmogorov tightness criterion, it follows that the family $\{P^{(n-1/2)}\}_{n=1}^\infty$ is tight in $C^{0, \alpha-Hölder}([0, 1]; G_{(0)})$ for $\alpha < \frac{4m^2 - 1}{4m} < \frac{1}{2} - \frac{1}{2m}$. By letting $m \to \infty$, we complete the proof.

By using Theorem 2.1, Lemma 5.8 and repeating the same argument as in [10, Lemma 4.8], we easily obtain the convergence of finite dimensional distribution of $(\mathcal{Y}^{(n-1/2)}_{s_1}, \mathcal{Y}^{(n-1/2)}_{s_2}, \ldots, \mathcal{Y}^{(n-1/2)}_{s_\ell}) \overset{(d)}{\to} (Y_{s_1}, Y_{s_2}, \ldots, Y_{s_\ell})$ as $n \to \infty$.

Finally, by combining this with Lemma 5.6, we complete the proof of Theorem 5.4.

### 5.4 Proof of Theorem 2.2

In this subsection, we give a proof of Theorem 2.2. We show that the same pathwise Hölder estimate as Lemma 5.8 also holds for the stochastic process $\{\mathcal{Y}^{(n-1/2)}\}_{n=1}^\infty$ by applying the corrector method.

**Lemma 5.9** For $m, n \in \mathbb{N}$ and $\alpha < \frac{2m-1}{4m}$, there exist an $\mathcal{F}_\infty$-measurable set $\Omega^{(n)}_r \subset \Omega(x_*)$ and a non-negative random variable $C^{(n)}_r \in L^{4m}(\Omega(x_*) \to \mathbb{R}; \mathbb{P}_x)$ such that

$$\mathbb{P}^{(n-1/2)}(\Omega^{(n)}_r) = 1 \quad \text{and} \quad d_{cc}(\mathcal{Y}^{(n-1/2)}_s(c), \mathcal{Y}^{(n-1/2)}_t(c)) \leq C^{(n)}_r(c)(t-s)^\alpha \quad (c \in \Omega^{(n)}_r, 0 \leq s < t \leq 1).$$

**Proof** Fix $n \in \mathbb{N}$ and $1 \leq k \leq \ell \leq n$. By triangular inequality, we have

$$d_{cc}(\mathcal{Y}^{(n-1/2)}_{t_k}, \mathcal{Y}^{(n-1/2)}_{t_\ell}) \leq d_{cc}(\mathcal{Y}^{(n-1/2)}_{t_k}, \mathcal{Y}^{(n-1/2)}_{t_\ell}) + d_{cc}(\mathcal{Y}^{(n-1/2)}_{t_k}, \mathcal{Y}^{(n-1/2)}_{t_\ell}) + d_{cc}(\mathcal{Y}^{(n-1/2)}_{t_\ell}, \mathcal{Y}^{(n-1/2)}_{t_\ell}).$$

We set $Z^{(n)}_t := (\mathcal{Y}^{(n-1/2)}_{t_k})^{-1} * \mathcal{Y}^{(n-1/2)}_t$ for $0 \leq t \leq 1$ and $n \in \mathbb{N}$. By definition, we have

$$\log \left( Z^{(n)}_t \right)_{g^{(i)}} = \frac{1}{\sqrt{n}} \text{Cor}^{(n)}_{g^{(i)}}(w_k) \quad (n \in \mathbb{N}, k = 0, 1, \ldots, n)$$

so that there is a constant $C > 0$ such that $\| \log \left( Z^{(n)}_t \right)_{g^{(i)}} \|_{g^{(i)}} \leq Cn^{-1/2}$ for $n \in \mathbb{N}$ and $k = 0, 1, 2, \ldots, n$. Moreover, it follows from the choice of the components of $\Phi_0^{(x)}(x)$ $(0 \leq \varepsilon \leq 1, x \in V)$ that $\| \log \left( Z^{(n)}_t \right)_{g^{(i)}} \|_{g^{(i)}} \leq Cn^{-1/2}$ for $n \in \mathbb{N}$ and $k = 0, 1, 2, \ldots, n$. By noting the equivalence of homogeneous norms, we have

$$d_{cc}(\mathcal{Y}^{(n-1/2)}_{t_k}, \mathcal{Y}^{(n-1/2)}_{t_\ell}) \leq C \| Z^{(n)}_t \|_{\text{Hom}} = C \sum_{i=1}^r \| \log \left( Z^{(n)}_t \right)_{g^{(i)}} \|_{g^{(i)}}^{1/i} \leq \frac{C}{\sqrt{n}}$$

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for \( n \in \mathbb{N} \) and \( k = 0, 1, 2, \ldots, n \). Then it follows from Lemma 5.8 and (5.50) that there exist an \( \mathcal{F}_\infty \)-measurable set \( \Omega_r^{(n)} \subset \Omega_{x_s}(X) \) and a non-negative random variable \( \bar{K}_r^{(n)} \in L^{4m}(\Omega_{x_s}(X) \rightarrow \mathbb{R}; \mathbb{P}_{x_s}) \) such that \( \mathbb{P}_{x_s}^{(n)}(\Omega_r^{(n)}) = 1 \) and

\[
\begin{align*}
d_{\text{CC}}(\bar{Y}_{t_k}^{(n-1)/2}(c), \bar{Y}_{t_\ell}^{(n-1)/2}(c)) &\leq C + K_r^{(n)}(c)\left(\frac{\ell - k}{n}\right)^\alpha + C \\
&\leq K_r^{(n)}(c)\left(\frac{\ell - k}{n}\right)^\alpha \quad (5.51)
\end{align*}
\]

for \( c \in \Omega_r^{(n)} \) and \( 0 \leq k \leq \ell \leq n \). For \( 0 \leq s < t \leq 1 \), take \( 0 \leq k \leq \ell \leq n \) such that \( k/n \leq s < (k+1)/n \) and \( \ell/n \leq t < (\ell+1)/n \). Since the stochastic process \( (\bar{Y}_t^{(n-1)/2}, 0 \leq t \leq 1) \) is also given by the \( CC \)-geodesic interpolation, we have

\[
\begin{align*}
d_{\text{CC}}(\bar{Y}_s^{(n-1)/2}, \bar{Y}_{t_{k+1}}^{(n-1)/2}) &= (k - ns)d_{\text{CC}}(\bar{Y}_{t_k}^{(n-1)/2}, \bar{Y}_{t_{k+1}}^{(n-1)/2}) \psi_G(\bar{Y}_s^{(n-1)/2}, \bar{Y}_{t_{k+1}}^{(n-1)/2}), \\
d_{\text{CC}}(\bar{Y}_t^{(n-1)/2}, \bar{Y}_{t_{k+1}}^{(n-1)/2}) &= (nt - \ell)d_{\text{CC}}(\bar{Y}_{t_\ell}^{(n-1)/2}, \bar{Y}_{t_{k+1}}^{(n-1)/2}).
\end{align*}
\]

We then use the triangular inequality and (5.51) to obtain

\[
\begin{align*}
d_{\text{CC}}(\bar{Y}_s^{(n-1)/2}, \bar{Y}_t^{(n-1)/2})(c) &\leq (k - ns)K_r^{(n)}(c)\left(\frac{1}{n}\right)^\alpha + K_r^{(n)}(c)\left(\frac{\ell - k - 1}{n}\right)^\alpha + (nt - \ell)K_r^{(n)}(c)\left(\frac{1}{n}\right)^\alpha \\
&\leq K_r^{(n)}(c)\left\{\left(\frac{k + 1}{n} - s\right)^\alpha + \left(\frac{\ell - k - 1}{n}\right)^\alpha + \left(t - \frac{\ell}{n}\right)^\alpha\right\} \leq K_r^{(n)}(c)(t - s)^\alpha \quad (c \in \Omega_r^{(n)}),
\end{align*}
\]

which completes the proof.

**Proof of Theorem 2.2** We split the proof into two steps.

**Step 1.** We show that the sequence \( (\bar{Y}_t^{(n-1)/2})_{n=1}^{\infty} \) converges in law to \( (Y_t)_{0 \leq t \leq 1} \) in \( C([0, 1]; G(0)) \) as \( n \rightarrow \infty \). For \( 0 \leq t \leq 1 \), take an integer \( 0 \leq k \leq n \) such that \( k/n < t < (k+1)/n \). Then, by the triangular inequality, (5.42), (5.49) and (5.50), we have, \( \mathbb{P}_{x_s}^{(n-1)/2} \)-almost surely,

\[
\begin{align*}
d_{\text{CC}}(\bar{Y}_t^{(n-1)/2}, \bar{Y}_t^{(n-1)/2})(c) &\leq d_{\text{CC}}(\bar{Y}_{t_k}^{(n-1)/2}, \bar{Y}_t^{(n-1)/2}) + d_{\text{CC}}(\bar{Y}_t^{(n-1)/2}, \bar{Y}_{t_{k+1}}^{(n-1)/2}) \\
&\leq K_r^{(n)}\left(t - \frac{k}{n}\right)^\alpha + C + \bar{K}_r^{(n)}\left(t - \frac{k}{n}\right)^\alpha \\
&\leq \left\{K_r^{(n)} + \bar{K}_r^{(n)}\right\}(t - s)^\alpha \quad (c \in \Omega_r^{(n)}), \quad m \in \mathbb{N}, \quad \alpha < \frac{2m - 1}{4m}.
\end{align*}
\]

Let \( \rho \) be a metric on \( C([0, 1]; G(0)) \) defined by

\[
\rho(w^{(1)}, w^{(2)}) := \max_{0 \leq t \leq 1} d_{\text{CC}}(w_t^{(1)}, w_t^{(2)}) + d_{\text{CC}}(1_G, (u_0^{(1)})^{-1} * w_0^{(2)}).
\]
By applying the Chebyshev inequality, (5.4) and (5.52), we have, for \( \varepsilon > 0 \) and \( m \in \mathbb{N} \),
\[
\begin{align*}
\mathbb{P}_{x,s}^{(n^{1/2})} \left( \rho \left( \mathcal{Y}_{f}^{(n^{1/2}), n} \right), \mathcal{Y}^{(n^{1/2}), n} \right) &> \varepsilon \\
&\leq \left( \frac{1}{\varepsilon} \right)^{4m} \mathbb{P}_{x,s}^{(n^{1/2})} \left[ \rho \left( \mathcal{Y}_{f}^{(n^{1/2}), n} \right), \mathcal{Y}^{(n^{1/2}), n} \right]^{4m} \\
&\leq \left( \frac{2}{\varepsilon} \right)^{4m} \left\{ \mathbb{P}_{x,s}^{(n^{1/2})} \left[ \max_{0 \leq t \leq 1} d_{CC} \left( \mathcal{Y}_{f}^{(n^{1/2}), n} \right), \mathcal{Y}^{(n^{1/2}), n} \right]^{4m} + C \cdot \left( \frac{1}{\sqrt{n}} \right)^{4m} \right\} \\
&\leq 3^{4m-1} \left( \frac{2}{\varepsilon} \right)^{4m} \left( \frac{1}{\sqrt{n}} \right)^{4m} \left\{ \mathbb{P}_{x,s}^{(n^{1/2})} \left[ (\mathcal{C}^{(n)})^{4m} \right] + \mathbb{P}_{x,s}^{(n^{1/2})} \left[ (\mathcal{K}_{f}^{(n)})^{4m} \right] \right\} \\
&\quad + C \cdot \left( \frac{2}{\varepsilon} \right)^{4m} \left( \frac{1}{\sqrt{n}} \right)^{4m} \rightarrow 0 \quad (n \rightarrow \infty).
\end{align*}
\]
Therefore, by Slutzky’s theorem, we obtain the convergence in law of \( \{\mathcal{Y}_{f}^{(n^{1/2}), n}\}_{n=1}^{\infty} \) to the diffusion process \( \{Y_{f}\}_{0 \leq t \leq 1} \) in \( C([0, 1]; G_{(0)}) \) as \( n \rightarrow \infty \).

**Step 2.** The previous step implies the convergence of the finite-dimensional distribution of \( \{\mathcal{Y}_{f}^{(n^{1/2}), n}\}_{n=1}^{\infty} \). On the other hand, we can show that the sequence of probability measures \( \{\mathbb{P}_{x,s}^{(n^{1/2})} \circ \mathcal{Y}^{(n^{1/2}), n}^{-1}\}_{n=1}^{\infty} \) is tight in \( \mathcal{C}^{0, \alpha-\text{Hol}} ([0, 1]; G_{(0)}) \), by noting Lemma 5.1 and by following the same argument as the proof of Lemma 5.6. Therefore, we complete the proof by combining these two.

\[ \square \]

### 6 Example

In this section, we discuss an example of \( \Gamma \)-nilpotent covering graph in the case where \( \Gamma \) is the 3-dimensional discrete Heisenberg group \( \mathbb{H}^{3}(\mathbb{Z}) \) defined by (3.1). Suppose that \( \Gamma = \mathbb{H}^{3}(\mathbb{Z}) \) is generated by two elements \( \gamma_{1} = (1, 0, 0) \) and \( \gamma_{2} = (0, 1, 0) \). We put \( g = (1/3, 1/3, 1/3) \in G = \mathbb{H}^{3}(\mathbb{R}) \) and define
\[
\begin{align*}
V_{1} &:= \{ g = \gamma_{i}^{e_{1}} \cdots \gamma_{i}^{e_{\ell}} \ast 1_{G} \mid i_{k} \in \{1, 2\}, e_{k} = \pm 1 (1 \leq k \leq \ell), \ell \in \mathbb{N} \cup \{0\}\}, \\
V_{2} &:= \{ g = \gamma_{i}^{e_{1}} \cdots \gamma_{i}^{e_{\ell}} \ast g \mid i_{k} \in \{1, 2\}, e_{k} = \pm 1 (1 \leq k \leq \ell), \ell \in \mathbb{N} \cup \{0\}\}.
\end{align*}
\]
Consider an \( \mathbb{H}^{3}(\mathbb{Z}) \)-nilpotent covering graph \( X = (V, E) \) defined by \( V = V_{1} \sqcup V_{2} \) and \( E := \{ (g, h) \in V_{1} \times V_{2} \mid g^{-1} \ast h = g, \gamma_{1}^{-1} \ast g, \gamma_{2}^{-1} \ast g \} \).

We note that \( X \) is invariant under the actions \( \gamma_{1} \) and \( \gamma_{2} \). We call the graph \( X \) a 3-dimensional Heisenberg hexagonal lattice, which is a generalization of the classical hexagonal lattice to the nilpotent setting (see Fig. 2). We have discussed two \( \mathbb{H}^{3}(\mathbb{Z}) \)-nilpotent covering graphs called 3-dimensional Heisenberg triangular lattice and 3-dimensional Heisenberg dice lattice, respectively. It should be noted that a 3-dimensional Heisenberg dice lattice is regarded as a hybrid of 3-dimensional Heisenberg triangular lattice and a 3-dimensional Heisenberg hexagonal lattice. See [10, Section 6] for details of them. The quotient graph \( X_{0} = (V_{0}, E_{0}) = \Gamma \backslash X \) is given by \( V_{0} = \{x, y\} \) and \( E_{0} = \{e_{i}, \tilde{e}_{i} \mid 1 \leq i \leq 3\} \) (see Fig. 3).
Fig. 2 A part of 3-dimensional Heisenberg hexagonal lattice and the projection of it on the $xy$-plane

We now discuss a non-symmetric random walk on $X$ with a transition probability $p : E \to (0, 1]$ given by

$$
p((g, g \star g)) = \alpha, \quad p((g, g \star \gamma_1^{-1} \star g)) = \beta, \quad p((g, g \star \gamma_2^{-1} \star g)) = \gamma.
$$

$$
p((g, g \star g)) = \alpha', \quad p((g, g \star \gamma_1^{-1} \star g)) = \beta', \quad p((g, g \star \gamma_2^{-1} \star g)) = \gamma',
$$

for every $g \in V_1$, where $\alpha, \beta, \gamma, \alpha', \beta', \gamma' > 0$, $\alpha + \beta + \gamma = 1$ and $\alpha' + \beta' + \gamma' = 1$. Then, the transition probability induces a non-symmetric random walk on $X_0$ with

$$
p(e_1) = \alpha, \quad p(e_2) = \beta, \quad p(e_3) = \gamma, \quad p(\vec{e}_1) = \alpha', \quad p(\vec{e}_2) = \beta', \quad p(\vec{e}_3) = \gamma'.
$$

The invariant measure $m : V_0 = \{x, y\} \to (0, 1]$ is given by $m(x) = 1/2$ and $m(y) = 1/2$. The first homology group $H_1(X_0, \mathbb{R})$ is spanned by two 1-cycles $[c_1] := [e_1 \star \vec{e}_2]$ and $[c_2] := [e_3 \star \vec{e}_2]$. Then the homological direction is calculated as $\gamma_p = (\hat{\alpha} / 2)[c_1] + (\hat{\gamma} / 2)[c_2]$. Here, we use the notations

$$
\hat{\alpha} = \alpha + \alpha', \quad \hat{\beta} = \beta + \beta', \quad \hat{\gamma} = \gamma + \gamma', \quad \hat{\alpha} = \alpha - \alpha', \quad \hat{\beta} = \beta - \beta', \quad \hat{\gamma} = \gamma - \gamma'
$$

for simplicity. The canonical surjective linear map $\rho_\mathbb{R} : H_1(X_0, \mathbb{R}) \to g^{(1)}$ is given by $\rho_\mathbb{R}([c_1]) = X_1$ and $\rho_\mathbb{R}([c_2]) = X_2$ so that we obtain $\rho_\mathbb{R}(\gamma_p) = (\hat{\alpha} / 2)X_1 + (\hat{\gamma} / 2)X_2$.

Let $\{\omega_1, \omega_2\} \subset \text{Hom}(g^{(1)}, \mathbb{R})$ be the dual basis of $\{X_1, X_2\}$. By identifying each $\omega_i$ with $\rho_\mathbb{R}(\omega_i) \in H^1(X_0, \mathbb{R})$ for $i = 1, 2$, we can see that $\{\omega_1, \omega_2\}$ is regarded as the dual basis of $\{[c_1], [c_2]\} \subset H_1(X_0, \mathbb{R})$.

We now discuss the family of transition probabilities $(p_\varepsilon)_{0 \leq \varepsilon \leq 1}$. By definition, we have

$$
p_\varepsilon(e_1) = \frac{1}{2} (\hat{\alpha} + \varepsilon \hat{\alpha}), \quad p_\varepsilon(e_2) = \frac{1}{2} (\hat{\beta} + \varepsilon \hat{\beta}), \quad p_\varepsilon(e_3) = \frac{1}{2} (\hat{\gamma} + \varepsilon \hat{\gamma}),
$$

$$
p_\varepsilon(\vec{e}_1) = \frac{1}{2} (\hat{\alpha} - \varepsilon \hat{\alpha}), \quad p_\varepsilon(\vec{e}_2) = \frac{1}{2} (\hat{\beta} - \varepsilon \hat{\beta}), \quad p_\varepsilon(\vec{e}_3) = \frac{1}{2} (\hat{\gamma} - \varepsilon \hat{\gamma})
for $0 \leq \varepsilon \leq 1$. Direct computations give us

\[
\langle \omega_1, \omega_1 \rangle_{p_\varepsilon} = \frac{\hat{\alpha}(\hat{\beta} + \hat{\gamma}) - \varepsilon^2 \hat{\alpha}^2}{4}, \quad \langle \omega_1, \omega_2 \rangle_{p_\varepsilon} = \frac{\hat{\alpha} \hat{\gamma} + \varepsilon^2 \hat{\alpha} \hat{\gamma}}{4}.
\]

We define the $\Gamma$-Albanese torus $\text{Alb}^\Gamma$ by $(g^{(1)}/\mathcal{L}, g_0)$, where $\mathcal{L} = \{aX_1 + bX_2 | a, b \in \mathbb{Z}\}$. Then we have

\[
v^{-1}_\varepsilon \equiv \text{vol}_{g_0} (\text{Alb}^\Gamma)^{-1} = \sqrt{\det (\langle \omega_i, \omega_j \rangle_{p_\varepsilon})_{i,j=1}^2} = \frac{1}{4} \sqrt{2 \hat{\alpha} \hat{\beta} \hat{\gamma} + \varepsilon^2 \hat{\alpha} \hat{\beta} (\hat{\beta} \hat{\gamma} + \hat{\beta} \hat{\gamma}) + \varepsilon^2 \hat{\beta} \hat{\beta} (\hat{\alpha} \hat{\gamma} + \hat{\alpha} \hat{\gamma}) + \varepsilon^2 \hat{\gamma} \hat{\gamma} (\hat{\alpha} \hat{\beta} + \hat{\alpha} \hat{\beta})}
\]

and

\[
\langle X_1, X_1 \rangle_{g_0}^{(\varepsilon)} = \frac{v^2}{4} \{ (\hat{\alpha} + \hat{\beta}) \hat{\gamma} - \varepsilon^2 \hat{\gamma} \hat{\gamma} \}, \quad \langle X_1, X_2 \rangle_{g_0}^{(\varepsilon)} = \frac{v^2}{4} (\hat{\alpha} \hat{\gamma} + \varepsilon^2 \hat{\alpha} \hat{\gamma}),
\]

\[
\langle X_2, X_2 \rangle_{g_0}^{(\varepsilon)} = \frac{v^2}{4} \{ (\hat{\alpha} + \hat{\gamma}) - \varepsilon^2 \hat{\alpha} \hat{\gamma} \}.
\]

We here give the family of modified harmonic realizations $(\Phi_0^{(\varepsilon)})_{0 \leq \varepsilon \leq 1}$. Introduce four arbitrary families $(x_\varepsilon)_{0 \leq \varepsilon \leq 1}$, $(y_\varepsilon)_{0 \leq \varepsilon \leq 1}$, $(z_\varepsilon)_{0 \leq \varepsilon \leq 1}$ and $(\kappa_\varepsilon)_{0 \leq \varepsilon \leq 1}$ in $\mathbb{R}$ such that $x_0 = y_0 = z_0 = 0$. Then it follows from (5.1) that the realization given by

\[
\Phi_0^{(\varepsilon)}(g) = \left( x_\varepsilon + \frac{\hat{\alpha}}{2}, y_\varepsilon + \frac{\hat{\gamma}}{2}, z_\varepsilon + \kappa_\varepsilon \right)
\]

is $(p_\varepsilon)$-modified harmonic for $0 \leq \varepsilon \leq 1$. We can see that the family $(\Phi_0^{(\varepsilon)})_{0 \leq \varepsilon \leq 1}$ satisfies (A1) if and only if $x_\varepsilon + y_\varepsilon = 0$ for $0 \leq \varepsilon \leq 1$. In addition, if $(z_\varepsilon)_{0 \leq \varepsilon \leq 1}$ and $(\kappa_\varepsilon)_{0 \leq \varepsilon \leq 1}$ satisfy $\sup_{0 \leq \varepsilon \leq 1} |z_\varepsilon + \kappa_\varepsilon| \leq C$ for a positive constant $C$, then the family $(\Phi_0^{(\varepsilon)})_{0 \leq \varepsilon \leq 1}$ satisfies (A2) as well as (A1).

Let $\{V_1, V_2\}$ be the Gram–Schmidt orthogonalization of $\{X_1, X_2\}$ with respect to $g_0^{(0)}$, that is,

\[
V_1 = \frac{\sqrt{\hat{\alpha} (\hat{\beta} + \hat{\gamma})}}{2} X_1 - \frac{\hat{\alpha} \hat{\beta}}{2 \sqrt{\hat{\alpha} (\hat{\beta} + \hat{\gamma})}} X_2, \quad V_2 = \frac{2v_0}{\sqrt{\hat{\alpha} (\hat{\beta} + \hat{\gamma})}} X_2.
\]
Then the infinitesimal generator \( \mathcal{A} \) defined by (2.6) is given by
\[
\mathcal{A} = -\frac{1}{2} (V_1^2 + V_2^2) - \left( \frac{\partial}{2} X_1 + \frac{\partial}{2} X_2 \right).
\]
\[
= -\frac{1}{2} (V_1^2 + V_2^2) - \frac{\partial}{\sqrt{\partial + \gamma}} V_1 - \frac{\partial \gamma}{4 \sqrt{\partial + \gamma}} V_2.
\]

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