Dissipative Magnetic Soliton in a Spinor Polariton Bose–Einstein Condensate

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Magnetic soliton is an intriguing nonlinear topological excitation that carries magnetic charges while featuring a constant total density. So far, it has only been studied in the ultracold atomic gases with the framework of the equilibrium physics, where its stable existence crucially relies on a nearly spin-isotropic, antiferromagnetic, interaction. Here, we demonstrate that magnetic soliton can appear as the exact solutions of dissipative Gross–Pitaevskii equations in a linearly polarized spinor polariton condensate with the framework of the non-equilibrium physics, even though polariton interactions are strongly spin anisotropic. This is possibly due to a dissipation-enabled mechanism, where spin excitation decouples from other excitation channels as a result of gain-and-loss balance. Such unconventional magnetic soliton transcends constraints of equilibrium counterpart and provides a novel kind of spin-polarized polariton soliton for potential application in opto-spintronics.

Keywords: exciton–polariton Bose-Einstein condensate, soliton, excitation, Bogoliubov-de Gennes equation, spinor

INTRODUCTION

Spinor polariton condensate in semiconductor microcavities [1–4] provides a unique out-of-equilibrium platform for exploring exotic nonlinear excitations with spin textures, which may even transcend usual restrictions of equilibrium systems. Formed from strong couplings between excitons and photons, polaritons possess peculiar spin properties: the $J_z = \pm 1$ (spin-up or spin-down) spin projections of the total angular momentum of excitons along the growth axis of the structure directly correspond to the right- and left-circularly polarized photons absorbed or emitted by the cavity, respectively [1]. Therefore, the properties of a spinor polariton fluid (e.g., density and phase distributions) can be probed from the properties of the emitted light [5]. In addition, the polariton–polariton interaction features a strong spin anisotropy [6–8], with a repulsive interaction between same spins ($g > 0$) and a weaker, attractive, interaction between opposite spins ($g_{12} < 0$). Furthermore, a polariton condensate is intrinsically open dissipative, distinguishing it fundamentally from its atomic counterpart [9]. Recently, half-soliton [10, 11] and half-vortices [12, 13] behaving like magnetic monopoles have been experimentally observed in spinor polariton condensates under coherent pumping. There, the key prerequisite for such excitation is the spin-anisotropic antiferromagnetic interaction, while dissipation only occurs as a perturbation. Instead, below, we present a new kind of polariton soliton that carries magnetic charges—dissipative...
magnetic soliton (DMS). In particular, whereas magnetic soliton cannot occur in equilibrium condensates with strongly spin-anisotropic (antiferromagnetic) interactions, it can nevertheless appear in non-equilibrium spinor polariton condensates harnessing dissipation as essential resources.

Magnetic soliton [14–16] is a localized nonlinear topological excitation, which exhibits a density dip in one component and a hump in the other, but featuring a constant total density. It is a fundamentally important entity in the nonlinear context, as it provides an exceptional example of exact vector soliton solution that can exist outside the paradigmatic Manakov limit ($g = g_{12}$); within this limit, a multicomponent nonlinear system is integrable [17–19]. It also attracts considerable interests in the condensed matter, offering interesting perspectives as regards many-body phenomenon of solitonic matter [20]. So far, magnetic soliton has only been realized in a spinor Bose–Einstein condensate (BEC) with nearly-isotropic spin interactions of antiferromagnetic type [14, 21, 22], $0 < g - g_{12} \ll g$. This requirement is essential because it makes the density depletion—invariably induced alongside spin excitation—strongly suppressed by a high energy cost, thus ensuring the characteristic constant density background of magnetic soliton. Beyond this regime, a stable magnetic soliton cannot occur in an atomic superfluid.

In this work, we theoretically show that a stable magnetic soliton can be formed in a linearly polarized polariton condensate under non-resonant excitations with a spatially homogeneous pump, even though $g - g_{12} > g$. It is an exact soliton solution to the multicomponent driven-dissipative Gross–Pitaevskii (GP) equation, preserving its energy over infinitely long times—so coined as DMS. It stems from a dissipation-enabled mechanism rather than an energetic mechanism (cf. Figure 1): the spin-polarization excitation, originally coupled to other dissipative excitations in a multicomponent quantum fluid, becomes decoupled conditionally on the local balance of gain and loss, thus allowing non-decaying localized spin texture far from the spin-isotropic Manakov limit. We remark that DMS exists for a time-independent and spatially uniform pump, which affords an appealing advantage in view of potential application [23, 24]: while polariton soliton has been well known to promise applications in opto-spintronics, present schemes for the generation and stabilization of solitons usually rely on complex engineering of the space–time profile of the pump [25–30], which requires optical isolation that has hitherto been challenging to integrate at acceptable performance levels and introduce redundant and power-hungry electronic components.

The structure of the paper is as follows. In Section II, we present our theoretical model of dissipative Gross–Pitaevskii equations, based on which we solve for the novel magnetic solitons that carry magnetic charges while featuring a constant total density. In Section IV, we present a comprehensive study of the physical mechanism of the magnetic solitons with the help of the dynamic structure factors. Finally, we conclude with a summary in Section V, and all the detailed mathematical derivations are outlined in Section A.

II DISSIPATIVE GROSS–PITAEVSKII EQUATIONS AT QUASI-1D

Motivated by Ref. [31], we consider a spinor polariton BEC formed under a homogeneous incoherent pumping in a wire-shaped microcavity that bounds the polaritons to a quasi-1D channel in the following geometry: In the $x$-direction, the polariton BEC is homogeneous; in the $y$-direction, the wire size $d$ is sufficiently small compared to the wire length, thus providing a small lateral quantum confinement. Moreover, the incoherent pump is also restricted to a small transverse size so that $h^2/(md^2) \gg gn_0$, where $m$ is the effective mass of polaritons and $n_0$ is the 1D polariton density, the polariton motion in the $y$ direction can be seen as frozen. In this case, the order parameter for the polariton BEC at quasi-1D can be effectively described by a complex vector $\psi(x,t) = [\psi_1(x,t), \psi_2(x,t)]^T$, in the circular basis. Here, $\psi_1$ and $\psi_2$ are the spin-up and spin-down wavefunctions, and we denote the density in each component by $n_1$ and $n_2$, respectively. The system dynamics is governed by driven-dissipative GP equations coupled to a rate equation for the reservoir density $n_R$ [37–41]:

$$i\hbar \frac{\partial \psi_1}{\partial t} = \left[ \frac{-\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + g|\psi_1|^2 + g_{12}|\psi_2|^2 \right] \psi_1 + g_{21}n_2 \psi_1 + D_i \psi_1,$$

$$i\hbar \frac{\partial \psi_2}{\partial t} = \left[ \frac{-\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + g|\psi_2|^2 + g_{12}|\psi_1|^2 \right] \psi_2 + g_{21}n_1 \psi_2 + D_i \psi_2,$$

$$\frac{\partial n_R}{\partial t} = P - [ \gamma_R + R(|\psi_1|^2 + |\psi_2|^2)] n_R. \tag{3}$$
Here, interactions between polaritons are typically $g_{12} < 0, g > 0$, and $|g_{12}| < g$. The interaction between the condensate and reservoir is modeled by constant $g_R$. Condensed polaritons decay at a rate $\gamma_c$, but are replenished from the reservoir at a rate $R$. This process is captured by $D_\nu = i\hbar (R\eta - \gamma_c)/2$. Reservoir polariton decays at a rate $\gamma_2$ and is driven by an off-resonant continuous-wave pump, which is spatially homogeneous. Note that, here, we have assumed that the reservoir lacks spin selectivity due to infinite fast spin relaxation [37].

The steady-state solutions of Eqs. 1–3 are given by

$$\psi_{1,2}^0 = \frac{P[y_c - y_2/R]}{2},$$

where $\psi_{1,2}^0$ and $n_R^0$ denote the steady-state condensate wavefunction of each component and reservoir density, respectively. As shown, the steady-state polariton BEC has a uniform density determined by $n_0 = 1, y_c - y_2/R$ and is linearly polarized with a stochastic polarization direction in the absence of pinning [1]. Note that Eqs. 1–3 in the limit of fast reservoir [3, 42] are of immediate relevance in the context of the complex Ginzburg Landau equations [23, 24]. In the following, we choose system parameters where such steady state is within the modulation stable regime [42–45] (this can be further seen in Section IV).

### III DISSIPATIVE MAGNETIC SOLITON

On top of the steady state, two kinds of excitations can occur: density excitation and spin-polarization excitation. These excitations are, in general, correlated with each other and with the reservoir, so that fluctuations in one channel can induce that in another and are dissipative. As shown below, the central result of this work is that under the condition

$$D_\nu \psi_{1,2} = 0,$$  \hspace{1cm} (5)

the spin-polarization excitation decouples from other dissipative channels, such that it can support a new kind of nonlinear excitation against the steady-state background in situations not allowed in the equilibrium case.

We look for an analytical solution

$$\psi(x - ut) = [\psi_1(x - ut), \psi_2(x - ut)]^T$$

(in the circular basis) satisfying Eqs. 1–3, which describes a moving soliton with velocity $u$. For simplicity, hereafter, we will denote $\eta = x - ut$.

To describe populations in each component, we rewrite $n_1 = n_0 (1 + \delta n_1)/2$ and $n_2 = n_0 (1 - \delta n_2)/2$ in terms of the total density $n_0$ and the dimensionless variables $\delta n_{1,2}$. We, moreover, define a linear polarization angle $\varphi$, and global phase $\varphi_g$. The order parameter can then be generically written as

$$\left( \begin{array}{c} \psi_1 \\ \psi_2 \end{array} \right) = \frac{\mu_0}{2} \left( \begin{array}{c} 1 + \delta n_1 e^{i\varphi} \\ 1 - \delta n_2 e^{i\varphi} \end{array} \right) e^{i\varphi_g / 2} e^{-i\varphi / 2},$$

with $\mu_0 = g_R y_c / R$. We consider general boundary conditions:

$$\lim_{\eta \to \infty} \delta n_{1,2}(\eta) = 0$$  \hspace{1cm} and  \hspace{1cm} \lim_{\eta \to \infty} \varphi_{1,2}(\eta) = 0.$$  \hspace{1cm} (6)

Exact solutions for Eqs. 1–3 can be found under condition (5). The detailed calculations can be found in Appendix A. The results are:

$$\delta n_1 = \delta n_2 = \sqrt{1 - U^2} \sech \left( \frac{\eta}{\xi_s} \sqrt{1 - U^2} \right),$$

$$\varphi_{n} = \arctan \left( \frac{\sinh \left( \frac{\eta}{\xi_s} \sqrt{1 - U^2} \right)}{U} \right) + \frac{\pi}{2},$$

$$\varphi_g = -\arctan \left( \frac{\sqrt{1 - U^2} \tanh \left( \frac{\eta}{\xi_s} \sqrt{1 - U^2} \right)}{U} \right) - \arctan \left( \frac{\sqrt{1 - U^2}}{U} \right).$$

Here, $U = \sqrt{\langle n_0 (g - g_{12}) / 2m \rangle}$ is a dimensionless velocity and $\xi_s = \hbar / \sqrt{2m n_0 (g - g_{12})}$ denotes the spin healing length.

A typical space–time profile of the above soliton solution is illustrated in Figure 2A for $g_{12} = 0.1g$. The density distribution $n_{1,2}(\eta)$ in each component and $\varphi$ and $\varphi_g$ at a chosen time are shown in Figure 2B. We see that, unlike half-solitons, the vector soliton here is characterized by a density notch in one component and a hump in the other, whereas $n_1 + n_2 \equiv n_0$ is constant, i.e., it is magnetic soliton (see Figure 2A and top panel of Figure 2B). The linear polarization angle $\varphi$, and the global phase $\varphi_g$ vary simultaneously in space (see bottom panel of Figure 2B): $\varphi$, always jumps by $\pi$ across the soliton, $\lim \varphi_r - \varphi_l = \pi$, regardless of soliton velocity. In contrast, the phase jump of $\varphi_g$ is velocity dependent, with the maximum shift $-\pi$ only for stationary case.

To verify the above analytical solution, we have numerically solved Eqs. 1–3 starting from an initial order parameter given by Eqs. 6–9 for $t = 0$ along with $n_R(0) = y_c / R$. Comparisons of numerical and analytical solutions show perfect agreement; see Figure 2B for $t/\tau = 15$. We have numerically verified the stability of our solution by time evolving an initial order parameter where $n_1(0) = n_2(0)$ is perturbed from Eq. 7 while keeping $n_R(0) = n_1(0) + n_2(0)$ fixed.

The polarization texture of polariton magnetic soliton in Eqs. 6–9 can be characterized by standard Stokes parameters [33, 46, 47], $S(\eta) = (S_x, S_y, S_z)$, with $S_x(\eta) = 2\Re(\psi_1^* \psi_2)/n_0$, $S_y(\eta) = 2\Im(\psi_1^* \psi_2)/n_0$, and $S_z(\eta) = (|\psi_1|^2 - |\psi_2|^2)/n_0$. Here, $\Re$ and $\Im$ denote the real and imaginary part, respectively. For the stationary case $U = 0$, $S(\eta)$ is entirely in the $(S_x, S_y)$ plane and presents an ingoing divergent spin texture whose direction defines the magnetic charge (see top panel of Figure 2C): The degree of circularization $S_z$ reaches unity at the center, while the linear polarization $S_y$ flips its direction crossing the soliton core due to the jump in $\varphi$. In comparison, a moving soliton (see bottom panel of Figure 2C) has a broadened localization width, $l_w = \xi_s / \sqrt{1 - U^2}$, and its polarization becomes strongly elliptical in the $(S_x, S_y)$ plane near the core, with a decreased circular polarization (i.e., magnetization) given by $S_z(\eta) = \sqrt{1 - U^2}$. However, the linear polarization still flips across the soliton, independent of $U$. 

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To see whether the polariton magnetic soliton in an open-dissipative spinor condensate decays with time, we calculated its energy \( E \) as \[41, 48, 49\].

\[
E = \int dx \psi^\dagger \left( \frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \right) \psi + \frac{g - g_{12}}{4} \int dx (n_1 - n_3)^2 + \frac{g + g_{12}}{4} \int dx (n_1 + n_2 - n_0)^2.
\]

Here, the second term corresponds to the spin–spin interaction associated with \( S_S \), and the third term is the energy associated with the density depletion. Once the gain balances loss, as formulated by Eq. 5, we derive straightforwardly (see Eq. B2 in Appendix)

\[
\frac{dE}{dt} = -2R \left[ D \left( \frac{\partial \psi^\dagger}{\partial t} + \frac{\partial \psi}{\partial t} \right) \right] dx = 0.
\]
Such non-decaying polariton magnetic soliton, therefore, belongs to dissipative solitons [50–52].

Such dissipative magnetic soliton (DMS) is quite unconventional, as its creation cannot be understood along the line of the well-known example in the equilibrium context. In Bose condensed atomic gas, the key prerequisite for creating magnetic soliton is an antiferromagnetic interaction satisfying $g - g_{12} \ll g$, i.e., close to the spin-isotropic Manakov limit ($g = g_{12}$). This condition, as can be seen from Eq. 10, creates a large energy separation between the density and spin-polarization excitations: The density depletion from $n_0$ near the soliton core requires much more energy than that associated with $S_y$, making the former energetically suppressed and thus ensuring a constant total density that characterizes magnetic soliton. However, such scenario fails here because polaritons feature $g - g_{12} > g$.

IV DISSIPATION-ENABLED FORMATION MECHANISM

To understand this unconventional phenomenon, the fact that Eqs. 6–9 are exact solutions offer a “sweet point.” We see that the balance of gain and loss [Eq. 5] is the key for fixing the background density at $n_0 = P_{Y_C} - γ_{Y_R}/R$. Simultaneously, this gives rise to a closed real equation for the magnetization $\{dS_y(y)/dy\}^2 + \{S_y^2(y) - (1 - U^2) S_y^2(y)\} = 0$ with $y = \eta \xi$ that is, the spin polarization excitation is decoupled from other excitation channels. We emphasize that such conditionally coherent dynamics has a fundamentally different origin from that in a purely conservative system such as atomic BEC. As such, magnetic soliton in the former case can occur far from the Manakov limit, in contrast to the latter where it is only possible when the deviation from $g = g_{12}$ is small breaking slightly system integrability.

The above dissipation-enabled decoupling of excitations is at the heart of DMS formation, which also manifests itself in the linear excitation regimes, e.g., in the excitation spectrum and linear response function. Briefly, to describe a spinor polariton BEC linearly perturbed from the steady state, we substitute Eq. 6 into Eqs. 1–3 and follow the standard Bogoliubov–de Gennes (BdG) approach (see details in Appendix C). The eigen-energy $\hbar \omega_q$ of excitations solves the equation $[\hbar \omega_q^2 - (\hbar \omega_S^2) \times (\hbar \omega_q^2 + i (\hbar n_0 + \hbar \gamma_R) (\hbar \omega_q^2 - [\hbar n_0 Y_C + (\hbar \omega_B^2)]\hbar \omega_q + ic(q))] = 0$, where $\hbar \omega_q = \sqrt{\epsilon_0^2 [\epsilon_0^2 + (g + g_{12}) n_0]}$, $\hbar \omega_B = \sqrt{\epsilon_0^2 [\epsilon_0^2 + (g + g_{12}) n_0]}$, and $c(q) = - (\hbar n_0 + \hbar \gamma_R) (\hbar \omega_B^2) + 2 h n_0 Y_C \epsilon_0^2$, with $\epsilon_0^2 = h^2 q^2/(2m)$ being the free-particle energy. Two decoupled equations follow: the quadratic equation immediately yields $\hbar \omega_q = \pm \hbar \omega_S$ for the energy of the spin-polarization excitation, whereas the cubic equation reflects the coupled linear excitations in the reservoir and density channel of polariton BEC. Importantly, we see $\omega_q$ is purely real (Figure 3A), regardless of whether the reservoir is fast or slow compared to the polariton BEC. This feature of the linear spin polarization excitation contrasts to the linear density excitation that generically exhibits a complex energy $\hbar \omega_D$ and eventually damps out. The latter is most transparent in the fast reservoir limit $\hbar y_0/\hbar Y_C \gg 1$. There, an adiabatic elimination of the reservoir gives $\hbar \omega_D = - i \Gamma/2 \pm \hbar \omega_0$, with $\hbar \omega_0 = \sqrt{\epsilon_0^2 [\epsilon_0^2 + (g + g_{12}) n_0] - 2 g \hbar \Gamma R} - \Gamma^2/4$ and $\Gamma = n_0 n_0^R R^2 \hbar/(\gamma_R + n_0 R)$. Note that $\omega_0$ is purely imaginary for $|\omega| \leq q_c$ due to polariton losses, with $q_c = m(\sqrt{\alpha^2 + \Gamma^2}/4 - \alpha^2)^2$. Here, we introduced parameter $\alpha = P/P_{th} - 1$ and the threshold value $P_{th} = \gamma_R Y_C/R$. In Figures 3A,B, we show the complex spectrum of linear density excitation for $\gamma_R/\gamma_C \gg 1$ where the results agree with numerical solutions of BdG equations. Note that the damping spectrum in Figure 3B shows that the considered steady state is indeed modulationally stable.

To further visualize the decoupling of excitations as a result of the balance between gain and loss, we analyze the linear response of the system. Considering an external density perturbation described by $\lambda \hbar \xi e^{i(q x - \omega t)} + H. c$ with $\lambda \ll 1$ is acted on the exciton–polariton BEC, we calculate the density static structure factor $S_D(q)$ and the spin-density static structure factor $S_S(q)$ [53]. For simplicity of analytical derivation, we assume fast reservoir limit and obtain (see the details in Appendix C)

$$S_D(q) = \begin{cases} \frac{\epsilon_0^2}{\hbar \omega_0} & \text{if } q < q_c, \\ \frac{4 \epsilon_0^2}{\pi \hbar^2} & \text{if } q_c < q < q_g, \\ \frac{1}{2} + \frac{1}{\pi \tan^{-1} \left( \frac{4 \alpha^2 - \Gamma^2}{4 \alpha^2 \hbar^2} \right)} \frac{\epsilon_0^2}{\hbar \omega_0} & \text{if } q > q_g, \end{cases}$$

and we also find $S_S(q) = 0$. Figure 3C shows lim $S_D(q)$ → 1, meaning the response of a polariton BEC to a density perturbation is exhausted by the density excitation, without collateral generations of excitations in other excitation sectors. If the system is instead subjected to a spin-dependent perturbation $\lambda \hbar \xi e^{i(q x - \omega t)} + H. c$, we find $S_D(q) = \epsilon_0^2/2(2m \omega_D)$, which approaches unity for $q \rightarrow \infty$ and $S_D(q) = 0$ (see Figure 3D). This further verifies that a perturbation in the spin polarization sector only induces spin excitations.

V CONCLUDING DISCUSSIONS

Summarizing, we theoretically show that a new kind of soliton DMS can be created in a spinor polariton condensate. The value and significance of our work are twofold. First, DMS has no atomic counterpart and relies crucially on the open-dissipative property of the system, in contrast to solitons discussed in Refs. [25–30, 37, 54, 55] and half-solitons in Refs. [10, 11]. Second, DMS provides a rare example of exact solutions to the dissipative GP equations at quasi-1D. In the future, it is interesting to explore concrete proposals for the experimental observation of the predicted phenomenon within feasible facilities and to study the unique quantum many-body physics associated with a collection of DMSs with same (opposite) magnetic charges. Furthermore, in our present
theoretical illustration, the condition of Eq. 5 reduces to $D_s = 0$, but the concept of dissipation-enabled decoupled excitations applies for generic cases where $D_s \neq 0$ rather than $D_s = 0$ holds. Thus, it is also interesting to explore in a broader context other new kinds of dissipative solitons that can arise from excitation decoupling.

DATA AVAILABILITY STATEMENT

The raw data supporting the conclusions of this article will be made available by the authors, without undue reservation.

AUTHOR CONTRIBUTIONS

ZL and YH have developed and supervised the research projects with the help of W-ML. CJ and RW have done the detailed calculations. All the authors contribute to writing the manuscript.

REFERENCES

1. Shelykh IA, Kavokin AV, Rubo YG, Liew TCH, Malpuech G. Polariton Polarization-Sensitive Phenomena in Planar Semiconductor Microcavities. *Semicond Sci Technol.* (2010) 25:013001. doi:10.1088/0268-1242/25/1/013001
2. Deng H, Haug H, Yamamoto Y. Exciton-polariton Bose–Einstein Condensation. *Rev Mod Phys.* (2010) 82:1489–537. doi:10.1103/revmodphys.82.1489
3. Carusotto I, Ciuti C. Quantum Fluids of Light. *Rev Mod Phys.* (2013) 85:299–366. doi:10.1103/revmodphys.85.299
4. Byrnes T, Kim NY, Yamamoto Y. Exciton-polariton Condensates. *Nat Phys.* (2014) 10:803–13. doi:10.1038/nphys3143
5. Lagoudakis KG, Wouters M, Richard M, Baas A, Carusotto I, André R, et al. Quantized Vortices in an Exciton-Polariton Condensate. *Nat Phys.* (2008) 4:706–10. doi:10.1038/nphys1051
6. Renucci P, Amand T, Marie X, Senellart P, Bloch J, Sermage B, et al. Micropolarity Polariton Spin Quantum Beats without a Magnetic Field: A Manifestation of Coulomb Exchange in Dense and Polarized Polariton Systems. *Phys Rev B.* (2005) 72:075317. doi:10.1103/physrevb.72.075317
7. Takemura N, Anderson MD, Navadeh-Touphi M, Oberli DY, Portella-Oberli MT, Deveaud B. Spin Anisotropic Interactions of Lower Polaritons in the Vicinity of Polaritonic Feshbach Resonance. *Phys Rev B.* (2017) 95:205303. doi:10.1103/physrevb.95.205303
8. Navadeh-Touphi M, Takemura N, Anderson MD, Oberli DY, Portella-Oberli MT. Polaritonic Cross Feshbach Resonance. *Phys Rev Lett.* (2019) 122:047402. doi:10.1103/physrevlett.122.047402
9. Franco D, Giorgini S, Pitaevskii LP, Stringari S. Theory of Bose-Einstein Condensation in Trapped Gases. *Rev Mod Phys.* (1999) 71:463–512. doi:10.1103/physrevb.71.075317
10. Flayac H, Solyushkov DD, Malpuech G. Oblique Half-Solitons and Their Generation in Exciton-Polariton Condensates. *Phys Rev B.* (2011) 83:193305. doi:10.1103/physrevb.83.193305
11. Hivet R, Flayac H, Solyushkov DD, Tanase D, Boulier T, Andreoli D, et al. Half-solitons in a Polariton Quantum Fluid Behave like Magnetic Monopoles. *Nat Phys.* (2012) 8:724–8. doi:10.1038/nphys2406
12. Yuri GR. Half Vortices in Exciton-Polariton Condensates. *Phys Rev Lett.* (2007) 99:106401. doi:10.1103/physrevlett.99.106401
13. Lagoudakis KG, ostatnickiy T, Kavokin AV, Rubo YG, André R, Deveaud-Pledran B. Observation of Half-Quantum Vortices in an Exciton-Polariton Condensate. *Science.* (2009) 326:974–6. doi:10.1126/science.1177980
14. Qu C, Pitaevskii LP, Stringari S. Magnetic Solitons in a Binary Bose–Einstein Condensate. *Phys Rev Lett.* (2016) 116:160402. doi:10.1103/physrevlett.116.160402
15. Qu C, Tylutchki M, Stringari S, Pitaevskii LP. Magnetic Solitons in Rabi-Coupled Bose-Einstein Condensates. *Phys Rev A.* (2017) 95:033614. doi:10.1103/physreva.95.033614
16. Fujimoto K, Hamazaki K, Ueda M. Fractal Strings of Magnetic Solitons and a Nonthermal Fixed point in a One-Dimensional Antiferromagnetic Spin-1 Bose Gas. *Phys Rev Lett.* (2019) 122:173001. doi:10.1103/physrevlett.122.173001
17. Malomed BA, Mihalache D, Wise F, Torner L. Spatiotemporal Optical Solitons. *J Opt B: Quant Sem misc.* (2005) 7:R33–R72. doi:10.1088/1464-4266/7/5/r02
18. Kartashov YV, Malomed BA, Torner L. Solitons in Nonlinear Lattices. *Rev Mod Phys.* (2011) 83:247–305. doi:10.1103/revmodphys.83.247
19. Konotop VV, Yang J, Zezyulin DA. Nonlinear Waves in $\mathcal{PT}$-Symmetric Systems. *Rev Mod Phys.* (2016) 88:035002. doi:10.1103/revmodphys.88.035002
20. Lev P. Magnetic Solitons in Binary Mixtures of Bose–Einstein Condensates. *Rendiconti Lincei Scienze Fisiche e Naturali* (2019) 30:269–76. doi:10.1007/s12210-019-00797-6
21. Farolfi A, Trpygozgos D, Mordici C, Lamporesi G, Ferrari G. Observation of Magnetic Solitons in Two-Component Bose-Einstein Condensates. *Phys Rev Lett.* (2020) 125:030401. doi:10.1103/PhysRevLett.125.030401
22. Chai X, Lao D, Fujimoto K, Hamazaki K, Ueda M, Raman C. Magnetic Solitons in a Spin-1 Bose-Einstein Condensates. *Phys Rev Lett.* (2020) 125:030402. doi:10.1103/PhysRevLett.125.030402
23. Haas HA, Wong WS. Solitons in Optical Communications. *Rev Mod Phys.* (1999) 68:423–44. doi:10.1103/revmodphys.68.423
24. Marin-Palomo P, Karpov M, Kordts A, Pfeiffer MHP, et al. Microresonator-based Solitons for Massively Parallel Coherent Optical Communications. *Nature* (2017) 546:274–9. doi:10.1038/nature22387
25. Yulin AV, Egorov OA, Lederer F, Skryabin DV. Dark Polariton Solitons in Semiconductor Microcavities. *Phys Rev A.* (2008) 78:061801(R). doi:10.1103/physreva.78.061801
26. Ostrovskaya EA, Abdullaev J, Desyatnikov AS, Fraser MD, Kivshar YS. Dissipative Solitons and Vortices in Polariton Bose-Einstein Condensates. *Phys Rev A.* (2012) 86:013636. doi:10.1103/physreva.86.013636
27. Sich M, Krizhanovskii DN, Skolnick MS, Gorbach AV, Hartley R, Skryabin DV, et al. Observation of Bright Polariton Solitons in a Semiconductor Microcavity. *Nat Photon.* (2012) 6:50–5. doi:10.1038/nphoton.2011.267
28. Egorov OA, Gorbach AV, Lederer F, Skryabin DV. Two-dimensional Localization of Exciton Polaritons in Microcavities. *Phys Rev Lett.* (2010) 105:073903. doi:10.1103/PhysRevLett.105.073903

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ZL and YH have developed and supervised the research projects with the help of W-ML. CJ and RW have done the detailed calculations. All the authors contribute to writing the manuscript.
29. Ma X, Egorov OA, Schumacher S. Creation and Manipulation of Stable Dark Solitons and Vortices in Microcavity Polariton Condensates. Phys Rev Lett (2015) 112:035311. doi:10.1103/physrevlett.112.035311

30. Ma X, Schumacher S. Vortex Multistability and Bessel Vortices in Polariton Condensates. Phys Rev Lett (2017) 118:157401. doi:10.1103/physrevlett.118.157401

31. Wertz E, Ferrier L, Solnyshkov DD, Johne R, Sanvitto D, Lemaître A, et al. Spontaneous Formation and Optical Manipulation of Extended Polariton Condensates. Nat Phys (2010) 6:860–4. doi:10.1038/nphys1750

32. Borgh MO, Keeling J, Berloff NG. Spatial Pattern Formation and Polarization Dynamics of a Nonequilibrium Spinor Polariton Condensate. Phys Rev B (2010) 81:233502. doi:10.1103/physrevb.81.233502

33. Ohadi H, Dreismann A, Rubo YG, Pinsker F, delValle InclanRedondo Y, Tzirtzos SI, et al. Spontaneous Spin Bifurcations and Ferromagnetic Phase Transitions in a Spinor Exciton-Polariton Condensate. Phys Rev X (2015) 5:031002. doi:10.1103/physrevx.5.031002

34. Liew TCH, Egorov OA, Matuszewski M, Kyrienko O, Ma X, Ostrovskaya EA. Instability-induced Formation and Nonequilibrium Dynamics of Phase Defects in Polariton Condensates. Phys Rev B (2015) 91:085413. doi:10.1103/physrevb.91.085413

35. Li G, Liew TCH, Egorov OA, Ostrovskaya EA. Incoherent Excitation and Switching of Spin States in Exciton-Polariton Condensates. Phys Rev B (2015) 92:064304. doi:10.1103/physrevb.92.064304

36. Askitopoulos A, Kallinik K, Liew TCH, Cilibrizi P, Hatzopoulos Z, Savvidis PG, et al. Nonresonant Optical Control of a Spinor Polariton Condensate. Phys Rev B (2016) 93:205307. doi:10.1103/physrevb.93.205307

37. Pinsker F, Flayac H. On-demand Dark Soliton Train Manipulation in a Spinor Polariton Condensate. Phys Rev Lett (2014) 112:140405. doi:10.1103/physrevlett.112.140405

38. Pinsker F. Approximate Solutions for Half-Dark Solitons in Spinor Non-equilibrium Polariton Condensates. Ann Phys (2015) 362:726–38. doi:10.1016/j.aop.2015.09.008

39. Pinsker F, Flayac H. Bright Solitons in Non-equilibrium Coherent Quantum Matter. Proc R Soc A (2016) 472:20150592. doi:10.1098/rspa.2015.0592

40. Xu X, Hu Y, Zhang Z, Liang Z. Spinor Polariton Condensates under Nonresonant Pumping: Steady States and Elementary Excitations. Phys Rev B (2017) 96:144511. doi:10.1103/physrevb.96.144511

41. Xu X, Chen L, Zhang Z, Liang Z. Dark-bright solitons in spinor polariton condensates under nonresonant pumping. J Phys B: Mol Opt Phys (2019) 52:025303. doi:10.1088/1361-6455/aaf4dd

42. Bobrovska N, Matuszewski M. Adiabatic Approximation and Fluctuations in Exciton-Polariton Condensates. Phys Rev B (2015) 92:035311. doi:10.1103/physrevb.92.035311

43. Bobrovska N, Ostrovskaya EA. Matuszewski M. Stability and Spatial Coherence of Nonresonantly Pumped Exciton-Polariton Condensates. Phys Rev B (2014) 90:205304. doi:10.1103/physrevb.90.205304

44. Bobrovska N, Matuszewski M, Daskalakis KS, Maier SA, Kéna-Cohen S. Dynamical Instability of a Nonequilibrium Exciton-Polariton Condensate. ACS Photon (2018) 5:111–8. doi:10.1021/acsphotonics.7b00283

45. Baboux F, De Bernardis D, Goblot V, Gladin VN, Gomez C, Galopin E, et al. Unstable and Stable Regimes of Polariton Condensation. Optica (2018) 5:1163–70. doi:10.1364/optica.5.001163

46. Shelykh IA, Rubo YG, Malpuech G, Solnyshkov DD, Kavokin A. Polarization and Propagation of Polariton Condensates. Phys Rev Lett (2006) 97:066402. doi:10.1103/PhysRevLett.97.066402

47. Sicl M, Tapia-Rodriguez LE, Sigurdsson H, Walker PM, Clarke E, Shelykh IA, et al. Spin Domains in One-Dimensional Conservative Polariton Solitons. ACS Photon (2018) 5:3095–102. doi:10.1021/acsphotonics.8b01410

48. Kivshar YS, Yang X. Perturbation-induced Dynamics of Dark Solitons. Phys Rev E Stat Phys Plasmas Fluids Relat Interdiscip Top (1994) 49:1657–70. doi:10.1103/physreve.49.1657

49. Smirnov LA, Smirnova DA, Ostrovskaya EA, Kivshar YS. Dynamics and Stability of Dark Solitons in Exciton-Polariton Condensates. Phys Rev B (2014) 89:235310. doi:10.1103/physrevb.89.235310

50. Kippenberg TJ, Gaeta AL, Lipson M, Gorodetsky ML. Dissipative Kerr Solitons in Optical Microresonators. Science (2018) 361:567. doi:10.1126/science.aan8083

51. Greul P, Akhmediev N. Dissipative Solitons for Mode-Locked Lasers. Nat Photon (2012) 6:84–92. doi:10.1038/nphoton.2011.345

52. Purwins H-G, Bödeker HU, Amirinashvili S. Dissipative Solitons. Adv Phys (2010) 59:485–701. doi:10.1080/00018732.2010.498228

53. Nozieres P, Pines D. The Theory of Quantum Liquids: Vol. II. New York: Addison-Wesley (1990).

54. Amo A, Pigeon S, Sanvitto D, Sala VG, Hivet R, Carusotto I, et al. Polariton Superfluids Reveal Quantum Hydrodynamic Dynamics. Science (2011) 332:1167–70. doi:10.1126/science.1202307

55. Xue Y, Matuszewski M. Creation and Abrupt Decay of a Quasistationary Dark Soliton in a Polariton Condensate. Phys Rev Lett (2014) 112:216401. doi:10.1103/physrevlett.112.216401

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APPENDIX A: DERIVATIONS OF EXACT SOLITON SOLUTION

Here, we present detailed derivation of the exact solutions in Eqs. 4–7 in the main text. We want to solve the effective 1D driven-dissipative GP equations for the order parameter \([\psi_1, \psi_2]^T\) of spinor polaron BEC, which are coupled to the rate equation of the density \(n_R\) of the polaron reservoir, i.e.,

\[
\begin{align*}
\frac{i\hbar}{\Delta t} \psi_1 &= \left\{ -\frac{h^2}{2m} \frac{\partial^2}{\partial z^2} + g|\psi_1|^2 + g_{12}|\psi_2|^2 + g_2 n_R + \frac{ih}{2} \left[R_{nR} - \gamma_C\right]\right\} \psi_1, \\
\frac{i\hbar}{\Delta t} \psi_2 &= \left\{ -\frac{h^2}{2m} \frac{\partial^2}{\partial z^2} + g|\psi_1|^2 + g_{12}|\psi_2|^2 + g_2 n_R + \frac{ih}{2} \left[R_{nR} - \gamma_C\right]\right\} \psi_2,
\end{align*}
\]

(A1)

Here, \(g_{12}\) denotes the interaction constant between the same (different) spin component, \(g_R\) is the interaction constant between condensed polaritons and reservoir polaritons whose density is \(n_R\). Condensed polaritons decay at rate \(\gamma_C\) and are replenished at rate \(R\) from reservoir. Reservoir polaritons decay at rate \(\gamma_R\) and \(P\) is the rate of an off-resonant cw pumping.

We aim to find a particular type of traveling soliton solution \(\psi_{1,2}(x, t) = \psi_{1,2}(x - vt)\), with \(v\) the velocity of soliton, which is characterized by \(|\psi_1|^2 + |\psi_2|^2 = n_0\) with \(n_0\) a constant and satisfies the condition \([R_{nR} - \gamma_C]\psi_{1,2} = 0\) (i.e., \(D_{nR} \psi = 0\)). Therefore, we consider the following ansatz:

\[
\psi_1 = \psi_2 = \frac{\sqrt{n_0}}{2} \left( \frac{\sqrt{1 + \delta n e^{\Delta t}}}{\sqrt{1 - \delta n e^{-\Delta t}}} \right) e^{i\varphi_{x, t}} e^{i\varphi_{x, t}^R}.
\]

(A4)

Here, \(\varphi_{x, t}\) and \(\varphi_{x, t}^R\) are the global and relative phases of the spin-up and spin-down wavefunctions. Without loss of generality, we will assume the boundary conditions: \(\varphi_{x, R} = 0\) at \(\eta = -\infty\) and \(\delta n = 0\) at \(\eta = +\infty\).

In order to determine the constant \(n_0\), we substitute Eq. (A4) into Eq. (A3) and find \(n_0 = P/(\gamma_R + R_{nR})\). Thus, for \(P = (\gamma_R + R_{nR})\gamma_C/R\), and hence \(n_0 = P/\gamma_C - \gamma_R/R\), the condition \(D_{nR} \psi = 0\) is fulfilled. With these, and denoting \(\eta = x - vt\), we obtain from Eqs. (A1), (A2) that

\[
\begin{align*}
-ih\frac{\partial \varphi_1}{\partial \eta} &= \left\{ -\frac{h^2}{2m} \frac{\partial^2}{\partial z^2} + \left(g - g_{12}\right)|\psi_1|^2 + g_{12}n_0 + \frac{g_2 \gamma_R}{R}\right\} \psi_1(\eta), \\
-ih\frac{\partial \varphi_2}{\partial \eta} &= \left\{ -\frac{h^2}{2m} \frac{\partial^2}{\partial z^2} + \left(g - g_{12}\right)|\psi_2|^2 + g_{12}n_0 + \frac{g_2 \gamma_R}{R}\right\} \psi_2(\eta).
\end{align*}
\]

(A5)

Substituting Eqs. (A4) (with \(n_0 = P/\gamma_C - \gamma_R/R\)) into Eqs. (A5), (A6) yields following equations for \(\delta n(\eta)\), \(\varphi_{x, t}(\eta)\), and \(\varphi_{x, t}^R(\eta)\), respectively, i.e.,

\[
\frac{1 - \delta n^2}{\Delta t} \frac{\partial \varphi_{x, t}}{\partial (\eta/\xi)} + U\delta n^2 = 0, \quad (A8)
\]

\[
\frac{1 - \delta n^2}{\Delta t} \frac{\partial \varphi_{x, t}^R}{\partial (\eta/\xi)} - U\delta n = 0. \quad (A9)
\]

where \(\xi = \hbar/\sqrt{2m\gamma_R(g - g_{12})}\) and \(U = v/\xi\), with \(c_s = \sqrt{(g - g_{12})\gamma_R/2m}\).

Equation (A7) is a closed equation and can be readily solved. Using the boundary conditions \(\delta n = 0\) at \(\eta = \pm\infty\), we finally arrive at the soliton solutions in Eqs. 4–7 in the main text.

APPENDIX B: ENERGY OF THE SOLITON

Here, we calculate the change rate of the energy of above soliton. The energy functional of the soliton can be calculated according to [48, 49].

\[
\begin{align*}
E &= \int dx \left[ \psi^* \left( \frac{h^2}{2m} \frac{\partial^2}{\partial x^2} \right) \psi + \frac{1}{4} dx \{ (g + g_{12})(n(x, t) - n_0) \}^2 + (g - g_{12}) S_2^2(x, t) \right] \\
&= \int dx \left[ \psi^* \left( \frac{h^2}{2m} \frac{\partial^2}{\partial x^2} \right) \psi + \frac{1}{4} dx \{ (g + g_{12})(n(x, t) - n_0) \}^2 + (g - g_{12}) S_2^2(x, t) \right]. 
\end{align*}
\]

(B1)

where we have denoted \(n(x, t) = |\psi_1(x, t)|^2 + |\psi_2(x, t)|^2\) and \(S_2(x, t) = |\psi_1(x, t)|^2 - |\psi_2(x, t)|^2\). Changing the integration variable from \(x\) to \(\eta = x - vt\), and using GP Eqs. (A1)–(A3), we can directly calculate the total time derivative of \(E\) associated with our soliton solution as

\[
\frac{dE}{dt} = v \int d\eta \left[ \psi_1^* \left( \frac{g_2 n_R + ih}{2} (R_{nR} - \gamma_C) \right) \psi_1 \frac{d}{d\eta} \psi_1 + \left( g_2 n_R - \frac{ih}{2} \left(R_{nR} - \gamma_C\right) \right) \psi_1^* \frac{d}{d\eta} \psi_1 \right] + \int d\eta \left[ \psi_2^* \left( \frac{g_2 n_R + ih}{2} (R_{nR} - \gamma_C) \right) \psi_2 \frac{d}{d\eta} \psi_2 + \left( g_2 n_R - \frac{ih}{2} \left(R_{nR} - \gamma_C\right) \right) \psi_2^* \frac{d}{d\eta} \psi_2 \right] \\
= \int d\eta \left[ \frac{n}{2} (R_{nR} - \gamma_C) \left( \frac{d}{d\eta} \varphi_{x, t} + \Delta n \frac{d}{d\eta} \varphi_{x, t}^R \right) \right] = 0. \quad (B2)
\]

APPENDIX C: LINEAR COLLECTIVE EXCITATIONS

In this section, we present detailed derivations of the linear excitations of the considered system using Bogoliubov approach. As \(g > 0\) and \(g_{12} < 0\) with \(|g_{12}| < g\), for \(P \geq \gamma_R\gamma_C\), the steady state of the model system is a linearly polarized...
BEC with \( n^0_t = n^0_z = n_0/2 \), where \( n_0 = p/\gamma_C - \gamma_R/R \) and \( n^0_R = \gamma_C/R \). We further have \( \mu_t = \frac{1}{2} (g + g_{1z}) n_0 + g_R n^0_R \). For linear excitations, we follow the standard procedures of Bogoliubov decomposition and write \([40]\).

\[
\begin{align*}
\psi_1(x,t) &= e^{-i\omega t}\sqrt{\frac{n_0}{2}} \left[ \frac{1}{1 + \sum_q \left( u_q \psi_q \epsilon^{(q_3-w_q)} + v_q \psi_q^{-1}(q_3-w_q) \right) } \right], \\
\psi_2(x,t) &= e^{-i\omega t}\sqrt{\frac{n_0}{2}} \left[ \frac{1}{1 + \sum_q \left( u_q \psi_q \epsilon^{(q_3-w_q)} + v_q \psi_q^{-1}(q_3-w_q) \right) } \right],
\end{align*}
\]

(C1)

and

\[
n_R(t) = n^0_R \left[ 1 + \sum_q \left( u_q \psi_q \epsilon^{(q_3-w_q)} + v_q \psi_q^{-1}(q_3-w_q) \right) \right].
\]

(C2)

It’s convenient to rewrite the excited components in Eq. (C1) in terms of \( u_q = u_{1q} + u_{2q} \) and \( v_q = v_{1q} + v_{2q} \) and \( u_q = u_{1q} - u_{2q} \) and \( v_q = v_{1q} - v_{2q} \), which are then subsequently substituted into Eqs. A1–A3. Retaining only the first-order terms of the fluctuations, we obtain the Bogoliubov-de Gennes (BdG) equation as

\[
\begin{pmatrix}
  u_{1q} + u_{2q} \\
  v_{1q} + v_{2q} \\
  u_{1q} - u_{2q} \\
  v_{1q} - v_{2q}
\end{pmatrix}
= \hbar \omega \begin{pmatrix}
  u_{1q} + u_{2q} \\
  v_{1q} + v_{2q} \\
  u_{1q} - u_{2q} \\
  v_{1q} - v_{2q}
\end{pmatrix},
\]

(C3)

with

\[
\begin{pmatrix}
  \epsilon^0_q + \frac{g + g_{1z}}{2} n_0 - \frac{g + g_{1z}}{2} n_0 \\
  0 \\
  -\frac{g + g_{1z}}{2} n_0 - \frac{g + g_{1z}}{2} n_0 \\
  0
\end{pmatrix}
\begin{pmatrix}
  0 \\
  0 \\
  0 \\
  0
\end{pmatrix}
\]

(C4)

Since the matrix \( \mathbf{G}_q \) is block diagonal, we obtain two decoupled BdG equations

\[
\begin{pmatrix}
\epsilon^0_q + \frac{g + g_{1z}}{2} n_0 & \frac{g + g_{1z}}{2} n_0 & \frac{g + g_{1z}}{2} n_0 \\
-\frac{g + g_{1z}}{2} n_0 & -\frac{g + g_{1z}}{2} n_0 & \frac{g + g_{1z}}{2} n_0 \\
-i R n_0 & i R n_0 & -i R n_0
\end{pmatrix}
\begin{pmatrix}
  u_{1q} + u_{2q} \\
  v_{1q} + v_{2q} \\
  u_{1q} - u_{2q} \\
  v_{1q} - v_{2q}
\end{pmatrix}
= \hbar \omega \begin{pmatrix}
  u_{1q} + u_{2q} \\
  v_{1q} + v_{2q} \\
  u_{1q} - u_{2q} \\
  v_{1q} - v_{2q}
\end{pmatrix},
\]

which describes coupled fluctuations in the density of condensed polaritons and reservoir, and

\[
\begin{pmatrix}
\epsilon^0_q + \frac{g + g_{1z}}{2} n_0 & \frac{g + g_{1z}}{2} n_0 & \frac{g + g_{1z}}{2} n_0 \\
-\frac{g + g_{1z}}{2} n_0 & -\frac{g + g_{1z}}{2} n_0 & \frac{g + g_{1z}}{2} n_0 \\
-i R n_0 & i R n_0 & -i R n_0
\end{pmatrix}
\begin{pmatrix}
  u_{1q} + u_{2q} \\
  v_{1q} + v_{2q} \\
  u_{1q} - u_{2q} \\
  v_{1q} - v_{2q}
\end{pmatrix}
= \hbar \omega \begin{pmatrix}
  u_{1q} + u_{2q} \\
  v_{1q} + v_{2q} \\
  u_{1q} - u_{2q} \\
  v_{1q} - v_{2q}
\end{pmatrix},
\]

which corresponds to linear excitation in spin polarization.

The eigen-energy can be directly calculated by solving BdG equations giving

\[
\left[ \left( \hbar \omega - \left( \hbar \omega \right)^2 \right) \right]
\times \left( \left( \hbar \omega \right)^3 + i \left( R n_0 + \gamma_R \right) \left( \hbar \omega \right)^2 - \left( R n_0 \gamma_C + \left( \hbar \omega \right)^2 \right) \right) = 0,
\]

with \( \hbar \omega_\delta = \sqrt{\left( \epsilon^0_q [g + g_{1z}] n_0 \right)^2 + \left( g + g_{1z} \right) n_0}, \) \( \hbar \omega_B = \sqrt{\left( \epsilon^0_q [g + g_{1z}] n_0 \right)^2}, \) and \( c(q) = -\left( R n_0 + \gamma_R \right) \left( \hbar \omega \right)^3 + 2 g_R n_0 \gamma_C^{-1} \).

**APPENDIX D: DENSITY AND SPINDENSITY RESPONSE FUNCTION**

Based on the knowledge of linear excitations in Section C, here, we derive the density and spin-density response functions of the considered system. We will present detailed calculations for the density response function. The spin-density function are derived in a similar fashion; we therefore only outline main steps.

1. **Dynamic Density Response Function**

Suppose the quasi-1D spinor polariton BEC is subjected to a time-dependent external perturbation in a form \( V_3 = -\lambda e^{i (q R t - \omega t)} e^{i \delta} + h.c \) with \( \lambda \ll 1 \) and \( \epsilon \ll 1 \), representing a density perturbation. In the presence of \( V_3 \), Eqs. A1–A3 are modified as

\[
\begin{align*}
\hbar \frac{\partial \psi_1}{\partial t} &= \left[ -\frac{\hbar^2}{2 m} \frac{\partial^2}{\partial x^2} + V_1 + \left( g |\psi_1|^2 + g_{1z} |\psi_2|^2 \right) \right] + g_R n_R, \\
\hbar \frac{\partial \psi_2}{\partial t} &= \left[ -\frac{\hbar^2}{2 m} \frac{\partial^2}{\partial x^2} + V_1 + \left( g |\psi_1|^2 + g_{1z} |\psi_2|^2 \right) \right] + g_R n_R, \\
\hbar \frac{\partial n_R}{\partial t} &= P - \left( \gamma_R + R \left( |\psi_1|^2 + |\psi_2|^2 \right) \right) n_R.
\end{align*}
\]

Our goal is to calculate the density response function \([53]\) as defined by

\[
\chi(q, \omega) = \lim_{\Delta \to 0} \delta \rho_q / (\lambda e^{-i \omega \Delta}).
\]

where \( \delta \rho_q \) are the Fourier component of the density fluctuation induced by the external perturbation.
For $\lambda \to 0$, we follow standard procedures and look for solutions corresponding to small amplitude oscillations around the unperturbed steady-state polaron BEC and the reservoir, i.e., we write

$$\psi_{1k} = e^{-\omega_{1k}t/h} \left[ \frac{n_0}{2} + u_{1k} e^{i(qx-\omega_{1k}t)} + v_{1k} e^{-i(qx-\omega_{1k}t)} \right],$$

$$\psi_{2k} = e^{-\omega_{2k}t/h} \left[ \frac{n_0}{2} + u_{2k} e^{i(qx-\omega_{2k}t)} + v_{2k} e^{-i(qx-\omega_{2k}t)} \right],$$

where $u_{1k}$ and $v_{1k}$ ($i = 1, 2$) and $w_k$ are small coefficients due to the perturbation, and will be determined subsequently. Substituting Eq. D5 into Eq. D4 and retaining terms at the first order of $u_{1k}$ and $v_{1k}$, we obtain the linear density response function as

$$\chi(\omega, q) = \lambda^{-1} \sqrt{\frac{n_0}{V}} \int dx e^{-iqx} (u_{11} + v_{11} + u_{21} + v_{21}).$$

In order to determine $u_{1k}$ and $v_{1k}$, we insert Eq. D5 into Eqs. D1–D3, we obtain the density excitation satisfying following equations

$$\begin{pmatrix}
\frac{\partial}{\partial t} + \left( \frac{g + gi_1}{{\sigma}} \right)n_0 - \omega_{1k} & \frac{\partial}{\partial t} + \left( \frac{g + gi_1}{{\sigma}} \right)n_0 + \omega_{1k} \\
\frac{\partial}{\partial t} - \omega_{1k} & \frac{\partial}{\partial t} + \omega_{1k}
\end{pmatrix}
\begin{pmatrix}
u_{10} \frac{(2g_k + iR)}{2} \\
\frac{(2g_k - iR)}{2}
\end{pmatrix}
\begin{pmatrix}
u_{10} \frac{(2g_k + iR)}{2} \\
\frac{(2g_k - iR)}{2}
\end{pmatrix}
\begin{pmatrix}
u_{10} \frac{(2g_k + iR)}{2} \\
\frac{(2g_k - iR)}{2}
\end{pmatrix}
\begin{pmatrix}
u_{10} \frac{(2g_k + iR)}{2} \\
\frac{(2g_k - iR)}{2}
\end{pmatrix}
$$

(D7)

For analytical simplicity, we assume fast reservoir limit of $\gamma_k/\gamma_c \gg 1$. In this case, we find

$$w_i = -\frac{Rn_0}{2(Rn_0 + q_0)}(u_{1k} + v_{1k}) - \frac{Rn_0}{2(Rn_0 + q_0)}(u_{2k} + v_{2k})$$

(D8)

which is substituted back into the first two lines of Eq. D7 to yield.

$$u_{1i} = u_{2i} = -\frac{\nu_{10} \frac{(2g_k + iR)}{2}}{\left( \omega_{1k} + \omega_{0i} + \frac{iR}{2} \right) \left( \omega_{1k} + \omega_{0i} + \frac{iR}{2} \right)}$$

(D9)

$$v_{1i} = v_{2i} = \frac{\nu_{10} \frac{(2g_k + iR)}{2}}{\left( \omega_{1k} + \omega_{0i} + \frac{iR}{2} \right) \left( \omega_{1k} + \omega_{0i} + \frac{iR}{2} \right)}$$

(D10)

with $\omega_{0i} = \sqrt{\nu_{10} \left[ \nu_{10} \left( \frac{g + gi_1}{\sigma} \right)n_0 - 2g_k \hbar \Gamma R \right] - \Gamma^2/4}$ and $\Gamma = n_0 \nu_{10} R^2 \hbar (\gamma_k + \gamma_k R)$.

Using Eqs. D8–D10, the density response function in Eq. D6 is found as

$$\chi(\omega, q) = \lambda^{-1} \sqrt{\frac{n_0}{V}} \int dx e^{-iqx} (u_{11} + v_{11} + u_{21} + v_{21}).$$

(D11)

The dynamic structure factor is defined in terms of the imaginary part of the density response function, i.e., $S_D(q, \omega) = \frac{i}{\pi} \mathfrak{F}(\chi(q, \omega))$, where $\mathfrak{F}$ denotes the imaginary part. We have

$$S_D(q, \omega) = \lambda^{-1} \frac{\nu_{10} \frac{(2g_k + iR)}{2}}{\left( \omega_{1k} + \omega_{0i} + \frac{iR}{2} \right) \left( \omega_{1k} + \omega_{0i} + \frac{iR}{2} \right)}$$

(D12)

Finally, we calculate the static structure factor according to $S_D(\sigma) = \frac{1}{\pi} \int_{-\infty}^{\infty} S_D(q, \omega)d\omega$ and arrive at Eq. 11 in the main text. We note that in the limit of $\Gamma \to 0$, $q = 0$ and our result recovers the well-known result $S_D(q) = h\sigma q^2/2m \sqrt{\nu_{10} \left[ \nu_{10} \left( \frac{g + gi_1}{\sigma} \right)n_0 - 2g_k \hbar \Gamma R \right] - \Gamma^2/4}$ familiar from the atomic condensate.

### 2. Spin-Density Response Function

We now suppose that the model system is subjected to a time-dependent perturbation $\sigma_z V_i$ with $V_i$ defined in Section D1, where $\sigma_z$ is the $z$-component of the standard Pauli matrix. The modified dynamical equations in the presence of spin-dependent perturbation are given by

$$i\hbar \frac{\partial}{\partial t} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} -\hbar^2 / 2m \frac{\partial^2}{\partial x^2} + V_1 + g|\psi_1|^2 + g_2 |\psi_2|^2 + g_2 n_k \\ i\frac{Rn_k}{2} + g_2 n_k \end{pmatrix}$$

(D13)

$$i\hbar \frac{\partial}{\partial t} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} -\hbar^2 / 2m \frac{\partial^2}{\partial x^2} - V_1 + g|\psi_1|^2 + g_2 |\psi_2|^2 + g_2 n_k \\ i\frac{Rn_k}{2} - g_2 n_k \end{pmatrix}$$

(D14)

Following similar steps as before, we find that the spin-density response can be calculated as

$$\chi_s(q, \omega) = \frac{1}{\lambda} \sqrt{\frac{n_0}{V}} \int dx e^{-iqx} (u_{11} + \sqrt{\frac{n_0}{2V}} \nu_{11} - \sqrt{\frac{n_0}{2V}} \nu_{12})$$

(D16)

$$= \left[ \frac{1}{\hbar \omega_k + i\eta} - \frac{1}{\hbar \omega_k - i\eta} \right] e^{-i\omega_k \hbar \gamma N} 2m \omega_k$$

(D17)

with $\hbar \omega_k$ being the spectrum of spin-density as given previously. The spin-density static structure factor is found from $S_S(q) = \frac{1}{\pi} \int_{-\infty}^{\infty} S_S(q, \omega)d\omega$ as

$$S_S(q) = \frac{\hbar^2 q^2}{2m \sqrt{\nu_{10} \left[ \nu_{10} \left( \frac{g + gi_1}{\sigma} \right)n_0 - 2g_k \hbar \Gamma R \right] - \Gamma^2/4}}$$

(D18)

Obviously, $S_S(q) \to 1$ for $q \to \infty$. 

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