CRITICAL EXPONENTS OF INDUCED DIRICHLET FORMS ON SELF-SIMILAR SETS

Shi-Lei Kong and Ka-Sing Lau

Abstract

In [26], we studied certain random walks on the hyperbolic graphs $X$ associated with the self-similar sets $K$, and showed that the discrete energy $\mathcal{E}_X$ on $X$ has an induced energy form $\mathcal{E}_K$ on $K$ that is a Gagliardo-type integral. The domain of $\mathcal{E}_K$ is a Besov space $\Lambda^{\alpha,\beta/2}$ where $\alpha$ is the Hausdorff dimension of $K$ and $\beta$ is a parameter determined by the “return ratio” of the random walk. In this paper, we study the functional relationship of $\mathcal{E}_X$ and $\mathcal{E}_K$ as well as the associated Besov spaces. In particular, we investigate the critical exponents of the $\beta$ in $\Lambda^{\alpha,\beta/2}$ in order for $\mathcal{E}_K$ to be a regular Dirichlet form. We provide some criteria to determine the critical exponents through the effective resistance of the random walk on $X$, and make use of certain electrical network techniques to calculate the exponents for some concrete examples.

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1 Introduction

Let \((K, \rho, \nu)\) be a metric measure space in which \((K, \rho)\) is a locally compact separable metric space and \(\nu\) is an \(\alpha\)-Ahlfors measure, i.e., \(\nu(B(x, r)) \propto r^\alpha\) for any ball \(B(x, r)\) with center at \(x \in K\) and radius \(r \in (0, 1)\) (by \(f \propto g\), we mean \(f\) and \(g\) are positive functions, and \(C^{-1} g \leq f \leq C g\) for some \(C > 0\)). We call such \(K\) an \(\alpha\)-set in the case that \(K\) is a compact subset in \(\mathbb{R}^d\) with the Euclidean metric [22].

The Besov space \(\Lambda^{\alpha, \beta/2}_{2, 2}\), \(\beta > 0\) is the Banach space contained in \(L^2(K, \nu)\) defined via the norm

\[
\|u\|_{\Lambda^{\alpha, \beta/2}_{2, 2}} = \|u\|_{L^2} + \left( \iint_{K \times K} \frac{|u(\xi) - u(\eta)|^2}{|\xi - \eta|^\alpha + \beta} d\nu(\xi) d\nu(\eta) \right)^{1/2},
\]

(note that \(\nu \times \nu\) vanishes on the diagonal) where the integral term is called the Gagliardo integral and denoted by \(E^{(\beta)}[u]\). Similarly we define another Besov space \(\Lambda^{\alpha, \beta/2}_{2, \infty}\) via

\[
\|u\|_{\Lambda^{\alpha, \beta/2}_{2, \infty}} = \|u\|_{L^2} + \left( \sup_{0 < r \leq 1} r^{-(\alpha + \beta)} \iint_{K} \int_{B(\eta, r)} |u(\xi) - u(\eta)|^2 d\nu(\xi) d\nu(\eta) \right)^{1/2}.
\]

On a classical domain (with \(\alpha = d\) in \(\mathbb{R}^d\), it is well-known that \(\Lambda^{\alpha, 1}_{2, \infty}\) equals the Sobolev space \(W^{1, 2}\), and for \(0 < \beta < 2\), \(\Lambda^{\alpha, \beta/2}_{2, 2}\) equals the fractional Sobolev space \(W^{s, 2}\) with \(s = \beta/2\) [1]. It is easy to see that \(\Lambda^{\alpha, \beta/2}_{2, 2} \subset \Lambda^{\alpha, \beta/2}_{2, \infty} \subset \Lambda^{\alpha, \beta/2}_{2, 2}\) for \(\beta < \beta^*\); \(\Lambda^{\alpha, \beta/2}_{2, 2}\) can be trivial for sufficiently large \(\beta\). We define a critical exponent \(\beta^*\) of \(K\) by

\[
\beta^* = \sup\{\beta > 0 : \Lambda^{\alpha, \beta/2}_{2, 2} \text{ contains nonconstant functions}\}.
\]

The Besov spaces \(\Lambda^{\alpha, \beta/2}_{2, \infty}\), \(\Lambda^{\alpha, \beta/2}_{2, 2}\) and the critical exponents play an important role in the study of the Dirichlet forms. For a classical domain \(\Omega\), the standard Dirichlet form \(\mathcal{E}(u, v) = \int_{\Omega} \nabla u(x) \nabla v(x) dx\) is defined on the domain \(\mathcal{D} = \Lambda^{\alpha, 1}_{2, \infty}(= \mathcal{W}^{1, 2})\), and \(\beta^* = 2\). The theory of Dirichlet forms on a metric measure space was originated in the seminal work of Beurling and Deny [6,12], in which a local regular Dirichlet forms \((\mathcal{E}, \mathcal{D})\) (if exists) gives a generalization of Laplacian. In [21], Jonsson showed that the domain \(\mathcal{D}\) of the local regular Dirichlet form on the Sierpinski gasket is the Besov space \(\Lambda^{\alpha, \beta^*/2}_{2, \infty}\), where \(\alpha = \log 3/\log 2\) and \(\beta^* = \log 5/\log 2(\approx 2.322)\). This consideration was extended by Pietruska-Paluba to nested fractals and \(\alpha\)-sets [31,32]. From the probabilistic point of view, the \(\beta^*\) is referred to as the “walk dimension”, which is the scaling exponent in the space-time relation of the diffusion process (Brownian motion) \(\{X_t\}\) on the underlying set \(K\): \(\mathbb{E}_x(|X_t - x|^2) \approx t^{2/\beta^*}\).

Typically, \(\{X_t\}\) has heat kernels that obey the sub-Gaussian estimate:

\[
p(t, \xi, \eta) \propto \frac{1}{t^{\alpha/\beta^*}} \exp \left( -c \left( \frac{|\xi - \eta|}{t^{1/\beta^*}} \right)^{\beta^*/(\beta^* - 1)} \right),
\]

(1.2)
(here the value of $c > 0$ varies in the upper and lower bounds). In particular, Barlow and Bass in [3, 5] proved the above heat kernel estimate of the Brownian motion on the Sierpinski carpet, and a numerical approximation $\beta^* \approx 0.97$ is highlighted in [4]. The estimates (1.2) on nested fractals were also obtained by Kumagai [27], in which he evaluated $\beta^*$ for some specific cases. Local regular Dirichlet forms have also been studied in the general setting of metric measure spaces together with the heat kernel estimates (e.g., [13, 15, 17, 18]). In particular, Grigor’yan, Hu and Lau [15] proved that $2 \leq \beta^* \leq \alpha + 1$ under the assumption that a sub-Gaussian heat kernel exists together with a chain condition (see also Stós [35] for the same inequality on the $\alpha$-sets). However, despite the various developments, there is no effective algorithm to determine the critical exponent $\beta^*$, and it is still an open question whether a Laplacian will exist on some more general fractal sets.

On a classical domain in $\mathbb{R}^d$, the Gagliardo integral $\mathcal{E}^{(\beta)}$ in (1.1) with $0 < \beta < 2$ defines a non-local regular Dirichlet form that corresponds to a fractional Laplacian and a symmetric $\beta$-stable process. In [35], assuming a Brownian motion exists on an $\alpha$-set $K$, Stós investigated the same type of non-local Dirichlet forms $\mathcal{E}^{(\beta)}$, $\beta < \beta^*$ from the associated stable-like processes that is subordinate to the Brownian motion, and he showed that the Besov spaces $\Lambda^{\alpha, \beta/2}_{2, 2}$ are the domains of $\mathcal{E}^{(\beta)}$. For such processes, the heat kernels were studied in detail by Chen and Kumagai [7] on an $\alpha$-set with $0 < \beta < 2$. Recently, there is a considerable interest devoted to the regular non-local Dirichlet forms and the jump processes on metric measure spaces (e.g., [8, 14, 16, 20]).

In [26], we studied the non-local Dirichlet forms with another approach. For a self-similar set $K$ in $\mathbb{R}^d$ with the open set condition (OSC), it is known that there is a hyperbolic graph $(X, \mathcal{E})$ (augmented tree) on the symbolic space $X$ of $K$, and the hyperbolic boundary and $K$ are Hölder equivalent [23, 28, 29]. On $(X, \mathcal{E})$, we introduced a class of transient reversible random walks with return ratio $\lambda \in (0, 1)$ (the conductance $c(x, y)$ depends on $\lambda$), and called them $\lambda$-natural random walks ($\lambda$-NRW) (see Section 2). The random walk was shown to satisfy the conditions of Ancona’s theorem in [2] so that the Martin boundary and the hyperbolic boundary (and hence $K$) are homeomorphic. Moreover, the hitting distribution $\nu$ is the normalized $\alpha$-Hausdorff measure where $\alpha$ is the Hausdorff dimension of $K$. By using a boundary theory of Silverstein on Markov chains [33], we proved that the graph energy

$$\mathcal{E}^{(\lambda)}_X[f] = \frac{1}{2} \sum_{x, y \in X} c(x, y) |f(x) - f(y)|^2$$

defined by the $\lambda$-NRW induces a non-negative definite bilinear form on $K$:

$$\mathcal{E}^{(\beta)}_K(u, v) \sim \int_{K \times K} \frac{(u(\xi) - u(\eta))(v(\xi) - v(\eta))}{|\xi - \eta|^\alpha + \beta} d\nu(\xi) d\nu(\eta)$$
with $\beta = \log \lambda / \log r$, where $r$ equals the minimal contraction ratio among the maps in the IFS that generates $K$. Clearly the domain $\mathcal{D}_K^{(\beta)} = \{ u \in L^2(K, \mu) : \mathcal{E}_K^{(\beta)}[u] < \infty \}$ is the Besov space $\Lambda_{2,2}^{\alpha,\beta/2}$.

As we see from the above, unlike the classical case, the Dirichlet forms in (1.4) can be obtained more directly on general self-similar sets without recourse to the local regular Dirichlet form (i.e., the Laplacian). In this paper we continue the investigation of the induced bilinear functional $\mathcal{E}_K^{(\beta)}$. We are aiming for a deeper understanding of the boundary theory of this $\lambda$-NRW, in particular on the critical exponents, so as to shed some light on the problem of the existence of Laplacians on the more general fractal sets. We will focus on two issues, namely, to establish the functional relationship of the discrete energy $\mathcal{E}_X^{(\lambda)}$ and the induced $\mathcal{E}_K^{(\beta)}$ where $\beta = \log \lambda / \log r$, then use it to study the critical exponents of $\{ \Lambda_{2,2}^{\alpha,\beta/2} \}_{\beta > 0}$, domains of $\mathcal{E}_K^{(\beta)}$.

Let $\mathcal{D}_X^{(\lambda)}$ be the domain of $\mathcal{E}_X^{(\lambda)}$, and let $\mathcal{H}\mathcal{D}_X^{(\lambda)}$ be the class of harmonic functions in $\mathcal{D}_X^{(\lambda)}$. For $u \in \mathcal{D}_K^{(\beta)}$, we use $Hu$ to denote the Poisson integral of $u$ on $X$, and for $f \in \mathcal{D}_X^{(\lambda)}$, we let $\text{Tr}f(\xi) = \lim_{x_n \to \xi} f(x_n)$. By imposing a norm on $\mathcal{D}_X^{(\lambda)}$, we prove a theorem analogous to the classical trace theorem (see Theorem 3.5, Corollaries 3.6 and 3.7).

**Theorem 1.1.** Suppose $K$ is a self-similar set and assume that the OSC holds. Then for a $\lambda$-NRW with $\lambda \in (0, r^\alpha)$, $\text{Tr}(\mathcal{D}_X^{(\lambda)}) = \mathcal{D}_K^{(\beta)}$. Moreover, $\text{Tr} : \mathcal{H}\mathcal{D}_X^{(\lambda)} \to \mathcal{D}_K^{(\beta)}$ is a Banach space isomorphism, and $\text{Tr}^{-1} = H$ on $\mathcal{D}_K^{(\beta)}$. (Here $\beta = \log \lambda / \log r$.)

The condition $\lambda \in (0, r^\alpha)$ in Theorem 1.1 will be used throughout the paper. It implies that $\beta > \alpha$, and functions in $\mathcal{D}_K^{(\beta)}$ are Hölder continuous (Proposition 2.5); moreover, the convergence rate $(\lambda/r^\alpha)^n$ is essential when we consider functions in $\mathcal{D}_X^{(\lambda)}$ that tend to the boundary $K$.

To consider the critical exponent of $\mathcal{D}_K^{(\beta)}$, we introduce some finer classification of the domains. We let

$$\beta_1^* := \sup \{ \beta > 0 : \mathcal{D}_K^{(\beta)} \cap C(K) \text{ is dense in } C(K) \},$$
$$\beta_2^* := \sup \{ \beta > 0 : \text{dim } \mathcal{D}_K^{(\beta)} = \infty \},$$
$$\beta_3^* := \sup \{ \beta > 0 : \mathcal{D}_K^{(\beta)} \text{ contains nonconstant functions} \},$$

Clearly we have $2 \leq \beta_1^* \leq \beta_2^* \leq \beta_3^* \leq \infty$, and $\beta_3^* = \beta^*$ for the $\beta^*$ defined previously. In the standard cases, these three exponents are equal, but there are also examples that they are different [19]. We will discuss these exponents and to provide some criteria to determine them. Our approach relies on the effective resistance. We use
$R^{(\lambda)}(\xi, \eta)$ to denote the limiting resistance for $\xi, \eta \in K$ (see Section 4), and note that the infinite word $i^\infty$ of $\{S_i\}_{i=1}^N$ will represent an element in $K$.

**Theorem 1.2.** With the assumptions as in Theorem 1.1, the domain $D^{(\beta)}_K$ consists of only constant functions if and only if $R^{(\lambda)}(i^\infty, j^\infty) = 0$ for all $i, j = 1, \cdots, N$.

Consequently, (i) if we let $\lambda^*_3 = \sup\{\lambda > 0 : R^{(\lambda)}(i^\infty, j^\infty) = 0, 1 \leq i, j \leq N\}$, then $\beta^*_3 = \log \lambda^*_3 / \log r$; (ii) if $\beta^*_3 > \alpha$ and $K$ is connected, then $\beta^*_2 = \beta^*_3$.

The theorem is proved in Theorems 5.4 and 5.6. The main idea is that the condition $R^{(\lambda)}(\cdot, \cdot) = 0$ on the finite set $\{i^\infty\}$ implies that it equals zero on a dense subset in $K$, and this leads to an infinite effective conductance on the dense subset. Then the continuity of $u \in D^{(\beta)}_K$ implies that $u$ can only be a constant function.

For $\beta^*_1$, we have a result on the post critically-finite (p.c.f.) sets [24]. We let $V_0$ denote the “boundary” of $K$.

**Theorem 1.3.** If in addition, $K$ is a p.c.f. set and satisfies another mild geometric condition (see Theorem 5.9). Then if

$$R^{(\lambda-\epsilon)}(\xi, \eta) > 0, \quad \forall \xi \neq \eta \in V_0,$$

for some $0 < \epsilon < \lambda$, then $D^{(\beta)}_K$ is dense in $C(K)$ with supremum norm.

Consequently, if $\lambda^*_1 := \inf\{\lambda > 0 : R^{(\lambda)}(\xi, \eta) > 0, \forall \xi \neq \eta \in V_0\} \in (0, r^\alpha)$, then $\beta^*_1 = \log \lambda^*_1 / \log r$.

A challenging task is to determine the limiting resistance $R^{(\lambda)}(i^\infty, j^\infty)$ (or $R^{(\lambda-\epsilon)}(\xi, \eta)$ for $\xi, \eta \in V_0$) to be $= 0$ or $> 0$ in the above theorems. For this we make use of the basic tools in the electrical network theory (series and parallel laws, $\Delta$-$Y$ transform, as well as cutting and shorting) for such estimation. We provide some special cases as examples.

For the organization of the paper, in Section 2 we summarize the needed results from [26]. In Section 3 we prove some basic results on the limits of functions in $D^{(\lambda)}_X$ as well as the extension of functions in $D^{(\beta)}_K$ via the Poisson integral, and prove Theorem 1.1. We define and justify the limiting resistance in Section 4 and prove Theorems 1.2 and 1.3 in Section 5. In Section 6 we make use of the electrical techniques to give some implementations of the theorems by some examples. Some remarks and open problems are provided in Section 7.
2 Preliminaries

We will give a brief summary of the background results in [26] for the convenience of the reader, and all the unexplained notations can be found there. Let \( \{S_i\}_{i=1}^N \), \( N \geq 2 \), be an iterated function system (IFS) of contractive similitudes on \( \mathbb{R}^d \) with contraction ratios \( \{r_i\}_{i=1}^N \), and let \( K \) be the self-similar set. Let \( \Sigma^* \) be the symbolic space of \( K \). Let \( r = \min\{r_i : i = 1, \cdots, N\} \). For \( n \geq 1 \), define

\[
\mathcal{J}_n = \{ x = i_1 \cdots i_k \in \Sigma^* : r_x \leq r^{n} < r_{i_1 \cdots i_{k-1}} \},
\]

and \( \mathcal{J}_0 = \{ \varnothing \} \) by convention. Consider the modified symbolic space \( X = \bigcup_{n=0}^{\infty} \mathcal{J}_n \), which has a tree structure with a set \( \mathcal{E}_v \) of vertical edges. The tree can be strengthened to a more structural hyperbolic graph by adding horizontal edges according to the neighboring cells on each level \( n \) [23,28,29]. According to [29], we define

\[
\mathcal{E}_h = \bigcup_{n=0}^{\infty} \{ (x,y) \in \mathcal{J}_n \times \mathcal{J}_n : x \neq y, \inf_{\xi,\eta \in K} |S_x(\xi) - S_y(\eta)| \leq \gamma \cdot r^n \},
\]

where \( \gamma > 0 \) is arbitrary but fixed. Let \( \mathcal{E} = \mathcal{E}_v \cup \mathcal{E}_h \), and call \( (X,\mathcal{E}) \) an augmented tree, coined by Kaimanovich in [23]. It was shown that \( (X,\mathcal{E}) \) is a hyperbolic graph in the sense of Gromov [37]. In this case, the lengths of horizontal geodesics are uniformly bounded, and for any \( x,y \in X \), the canonical geodesic \( [x,u,v,y] \) consists of three segments, where \([x,u],[v,y]\) are vertical paths in \( \mathcal{E}_v \), and \([u,v]\) is a horizontal geodesic in \( \mathcal{J}_r \) with the smallest \( \ell \). Using this geodesic, the Gromov product \( (x|y) \) has a simple and useful geometric interpretation:

\[
(x|y) = \ell - h/2,
\]

where \( h \) is the length of \([u,v]\) and \( h \) is uniformly bounded. For some \( a > 0 \), there is a Gromov metric \( \rho_a \) on \( X \) such that \( \rho_a(x,y) \asymp e^{-a(x|y)} \) for all \( x \neq y \). Let \( \hat{X}_H \) be the completion of \((X,\rho_a)\), and define the hyperbolic boundary \( \partial_H X = \hat{X}_H \setminus X \). Then \((\partial_H X,\rho_a)\) is a compact metric space.

A geodesic ray \((x_n)_{n=0}^{\infty}\) is a sequence of words with \( x_n = i_1i_2\cdots i_{k(n)} \in \mathcal{J}_n \). If \( \xi \in \partial_H X \) has a canonical representation \( i_1i_2\cdots \in \Sigma^\infty \), then \((x_n)_{n}\) converges to \( \xi \), and \( \xi \in S_{x_n}(K) \) for all \( n \). It follows that for any other geodesic ray \((y_n)_{n}\) converging to \( \xi \), we have \( x_n \sim_h y_n \). In the sequel, we will make use of the geodesic rays frequently to relate functions on \( X \) and \( K \). We call the sequence \( \{\kappa_n\}_{n=0}^{\infty} \) a \( \kappa \)-sequence if each \( \kappa_n \) is a selection map from \( K \) to \( \mathcal{J}_n \), such that for each \( \xi \in K \), \( (\kappa_n(\xi))_{n=0}^{\infty} \) is a geodesic ray converging to \( \xi \). It follows from the above that
Lemma 2.1. For any IFS \( \{S_i\}_{i=1}^N \), let \((X, \mathcal{E})\) be the hyperbolic graph as defined above. Let \( E \) be a closed subset \( K \). Then for any two \( \kappa \)-sequences \( \{\kappa_n\}_{n=0}^\infty \) and \( \{\kappa'_n\}_{n=0}^\infty \), we have

\[
\kappa'_n(E) \subset \{x \in \mathcal{J}_n : d(x, \kappa_n(E)) \leq 1\}
\]

for each \( n \), where \( d(\cdot, \cdot) \) is the graph metric on \((X, \mathcal{E})\).

Theorem 2.2. \cite{23,29} For any IFS \( \{S_i\}_{i=1}^N \), let \((X, \mathcal{E})\) be the hyperbolic graph as defined above. Then the hyperbolic boundary is Hölder equivalent to the self-similar set \( K \), i.e., for the canonical map \( \iota : \partial_H X \to K \),

\[
\rho_a(\xi, \eta)(\asymp e^{-a|x|}) \asymp |\iota(\xi) - \iota(\eta)|^{-a/\log r}.
\]

Throughout this paper, we will always assume that the IFS \( \{S_i\}_{i=1}^N \) satisfies the open set condition (OSC) \cite{11}. In this case, the self-similar set \( K \) has Hausdorff dimension \( \alpha \) which is uniquely determined by \( \sum_{i=1}^N r_i^\alpha = 1 \).

In \cite{26}, we introduced a class of reversible random walks on the augmented tree \((X, \mathcal{E})\): for \( \lambda \in (0, 1) \), we set the conductance \( c : \mathcal{E} \to (0, \infty) \) such that

\[
c(x, x^-) = r_x^\alpha \lambda^{-|x|}, \quad \text{and} \quad c(x, y) \asymp r_x^\alpha \lambda^{-|x|}, \quad x \sim_h y \in X \setminus \{\emptyset\}, \tag{2.2}
\]

where \( x^- \) is the parent of \( x \), \( r_x := r_{i_1} \cdots r_{i_m} \) for \( x = i_1 \cdots i_m \). (For example, for the Sierpinski gasket, \( r_x = 1/3 \), and \( c(x, x^-) = (3\lambda)^{-|x|} \).) We define the natural random walk with return ratio \( \lambda \in (0, 1) \) (\( \lambda \)-NRW) to be the Markov chain \( \{Z_n\}_{n=0}^\infty \) on \( X \) with transition probability \( P(x, y) = c(x, y)/m(x) \) if \( x \sim y \), and 0 otherwise, where \( m(x) = \sum_{y : x \sim y} c(x, y) \) is the total conductance at \( x \in X \). Note that the random walk has a return ratio \( \lambda \in (0, 1) \) with respect to the vertical direction; hence \( \{Z_n\}_{n=0}^\infty \) is transient. Let \( \mathcal{M} \) denote the Martin boundary, and let \( Z_\infty \) be the \( \mathcal{M} \)-valued random variable as the limit of \( \{Z_n\}_{n=0}^\infty \).

Theorem 2.3. \cite{26} Let \( \{S_i\}_{i=1}^N \) be an IFS satisfying the open set condition, and let \( \{Z_n\}_{n=0}^\infty \) be a \( \lambda \)-NRW. Then

(i) the Martin boundary \( \mathcal{M} \), the hyperbolic boundary \( \partial_H X \) and the self-similar set \( K \) are all homeomorphic;

(ii) the Martin kernel \( K(x, \xi) \asymp \lambda^{-|x|/(y|x|)} \); \( r^{-\alpha(y|x|)} \);

(iii) the distribution \( \nu \) of \( Z_\infty \) on \( \mathcal{M} \) equals the normalized \( \alpha \)-Hausdorff measure on \( K \) when \( Z_0 = \emptyset \).
We will fix \( \lambda \in (0,1) \), and when there is no confusion, we will omit the superscripts of \( \lambda \) and \( \beta := \log \lambda / \log r \) in the involved notations on \( X \) and \( K \). It follows from part (i) that we can carry Doob’s discrete potential theory onto the self-similar set \( K \). We denote the space of harmonic functions (w.r.t. \( P \)) on \( X \) by \( \mathcal{H}(X) = \{ f \in \ell(X) : Pf = f \} \), where \( \ell(X) \) is the collection of real functions on \( X \), and \( Pf(x) = \sum_{y \in X} P(x,y) f(y) \). The Poisson integral for \( u \in L^1(K,\nu) \) is

$$Hu(\cdot) = \int_K K(\cdot,\xi) u(\xi) d\nu(\xi) \in \mathcal{H}(X). \quad (2.3)$$

The graph energy of \( f \in \ell(X) \) is given by

$$\mathcal{E}_X[f] = \frac{1}{2} \sum_{x,y \in X : x \sim y} c(x,y) |f(x) - f(y)|^2, \quad (2.4)$$

and the domain of \( \mathcal{E}_X \) is \( \mathcal{D}_X = \{ f \in \ell(X) : \mathcal{E}_X[f] < \infty \} \). Using Theorem 2.3 together with Silverstein’s approach on the Naïm kernel \( \Theta(\xi,\eta) \) on \( K \), we obtain an induced quadratic form on \( K \) as follows.

**Theorem 2.4.** Under the assumptions in Theorem 2.3, the graph energy in (2.4) induces an energy form \( \mathcal{E}_K[u] := \mathcal{E}_X[Hu] \) given by

$$\mathcal{E}_K[u] = \frac{m(\partial)}{2} \int_{K \times K} |u(\xi) - u(\eta)|^2 \Theta(\xi,\eta) d\nu(\xi) d\nu(\eta), \quad u \in L^2(K,\nu), \quad (2.5)$$

where \( \Theta(\xi,\eta) \propto (\lambda r^\alpha)^{-\beta} |\xi - \eta|^{-(\alpha + \beta)} \) with \( \beta = \log \frac{\lambda}{\log r} \).

The domain of \( \mathcal{E}_K \) is \( \mathcal{D}_K = \{ u \in L^2(K,\nu) : Hu \in \mathcal{D}_X \} \). It follows from \( \mathcal{E}_K[u] := \mathcal{E}_X[Hu] \) that \( \mathcal{D}_K \) also equals \( \{ u \in L^2(K,\nu) : \mathcal{E}_K(u) < \infty \} \). Hence \( \mathcal{D}_K \) is the Besov space \( \Lambda_{2,2}^{\alpha,\beta/2} \). If we define \( \|u\|_{\mathcal{E}_K} = \mathcal{E}_K[u] + \|u\|_{L^2(K,\nu)}^2 \), then \( (\mathcal{D}_K, \| \cdot \|_{\mathcal{E}_K}) \) is a Banach space, and is equivalent to \( \Lambda_{2,2}^{\alpha,\beta/2} \). For \( \gamma > 0 \), we let

$$C^\gamma(K) = \{ u \in C(K) : \|u\|_{C^\gamma} := \|u\|_{C^\gamma} + \text{esssup}_{\xi,\eta \in K} \frac{|u(\xi) - u(\eta)|}{|\xi - \eta|^{\gamma}} < \infty \} \quad (2.6)$$

denote the Hölder space. We will use the following result frequently. It was proved in [15] (the assumption of heat kernel stated there is not needed in the proof) that

**Proposition 2.5.** If \( \beta > \alpha \), then for all \( u \in L^2(K,\nu) \),

$$\|u\|_{C^\gamma} \leq C \|u\|_{\Lambda_{2,2}^{\alpha,\beta/2}} \quad (2.7)$$

with \( \gamma = (\beta - \alpha)/2 \). Consequently, \( \Lambda_{2,2}^{\alpha,\beta/2} \hookrightarrow C^\gamma \) is an imbedding.

It follows that for \( \alpha < \beta < \beta_1^* \), \( \mathcal{D}_K \cap C(K) = \mathcal{D}_K \) is trivially dense in \( \mathcal{D}_K \) under the norm \( \| \cdot \|_{\mathcal{E}_K} \), and in \( C(K) \) under the supremum norm. This implies that \( (\mathcal{E}_K,\mathcal{D}_K) \) is a non-local regular Dirichlet form.
3 Harmonic functions and trace functions

In this section, we will set up a natural relation between the finite energy harmonic functions on $X$ and the finite induced energy functions on $K$ (Theorem 3.5). First we use Theorem 2.3(ii) to provide a “uniform tail estimate” of the Martin kernel. As in [26, Section 5], we introduce a projection $\iota: X \to K$ by selecting $\iota(x) \in S_X(O \cap K)$ arbitrarily, where $O$ is an open set in the OSC satisfying $O \cap K \neq \emptyset$.

**Proposition 3.1.** Let $\{S_i\}_{i=1}^N$ be an IFS satisfying the OSC, and let $\{Z_n\}_{n=0}^\infty$ be a $\lambda$-NRW on the augmented tree $(X, \mathcal{E})$. Then for any $\epsilon, \delta > 0$, there exists a positive integer $n_0$ such that for any $x \in X$ and $|x| \geq n_0$, $K(x, \xi) \leq \epsilon$ for any $\xi \in K \setminus B(\iota(x), \delta)$.

**Proof.** It follows from Theorem 2.3(ii) that

$$K(x, \xi) \leq C_1\lambda^{|x|}(\lambda r^\alpha)^{-|x|\xi}, \quad x \in X, \xi \in K.$$ 

Note that $(x|\xi) \leq (\iota(x)|\xi)$ by [26, Lemma 3.7(ii)]. Hence for $\xi \in K \setminus B(\iota(x), \delta)$,

$$r^{-|x|\xi} \leq r^{-|\iota(x)|\xi} \leq C_2|\iota(x) - \xi|^{-1} \leq C_2\delta^{-1}$$

(the second inequality follows from Theorem 2.2). Hence for $\epsilon > 0$, we can pick a large integer $n_0$ such that the last inequality in the following holds:

$$K(x, \xi) \leq C_1\lambda^{n_0}r^{-|\alpha + \log \lambda/\log r|\xi} \leq C_1\lambda^{n_0}(C_2\delta^{-1})^{\alpha + \log \lambda/\log r} \leq \epsilon.$$

\[ \square \]

Let $\nu_x$, $x \in X$, denote the hitting distribution of $Z_\infty$ on $K$, starting from $x$. As $K(x, \cdot) = d\nu_x/d\nu$, the above result shows that the mass of $\nu_x$ will concentrate around $\iota(x)$ (equivalently, $S_x(K)$) as $|x| \to \infty$. We have a Fatou-type theorem as a corollary.

**Corollary 3.2.** Suppose $\{S_i\}_{i=1}^N$ satisfies OSC, and let $\{Z_n\}_{n=0}^\infty$ be a $\lambda$-NRW on the augmented tree $(X, \mathcal{E})$. Then for $u \in C(K)$ and $\epsilon > 0$, there exists a positive integer $n_0$ such that

$$|Hu(x) - u(\xi)| \leq \epsilon, \quad \forall |x| \geq n_0, \xi \in S_x(K). \quad (3.1)$$

In particular, $\lim_{n \to \infty} Hu(x_n) = u(\xi)$ uniformly for $\xi \in K$, where $(x_n)_n$ is a geodesic ray converging to $\xi$. 

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Proof. Since \( u \) is continuous on the compact set \( K \), \( u \) is bounded and uniformly continuous. We let \( \sup_{\xi \in K} |u(\xi)| = M_0 < \infty \) and choose \( \delta > 0 \) such that \( |u(\xi) - u(\eta)| < \varepsilon/3 \) whenever \( |\xi - \eta| < \delta \) on \( K \). Furthermore, by Proposition 3.1, we choose \( n_0 \) such that both \( \text{diam}(S_x(K)) \leq \delta \) and \( K(x, \xi) \leq \frac{\delta}{\lambda_{n_0}} \) hold for any \( x \in X \) with \( |x| \geq n_0 \) and \( \xi \in K \setminus B(i(x), \delta) \). Then for \( |x| \geq n_0 \), by using the usual technique of splitting the following integral on \( K \) into \( K \cap B(i(x), \delta) \) and \( K \setminus B(i(x), \delta) \), we can show that

\[
|Hu(x) - u(i(x))| \leq \int_K |K(x, \eta)(u(\eta) - u(i(x)))|d\nu(\eta) \leq \varepsilon
\]

Hence for \( \xi \in S_x(K) \),

\[
|Hu(x) - u(\xi)| \leq |Hu(x) - u(i(x))| + |u(i(x)) - u(\xi)| \leq \frac{2\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon,
\]

and (3.1) holds. For the last statement, let \( (x_n)_n \) be a geodesic ray converging to \( \xi \), then \( x_n = i_1 \cdots i_n \), and this \( i_1 i_2 \cdots \in \Sigma^\infty \) is a representation of some \( \xi' \) with \( \xi' \in S_{x_n}(K) \), and \( \xi' = \xi \) in \( \partial_H X \). Hence by (3.1), we have \( \lim_{n \to \infty} Hu(x_n) = u(\xi') = u(\xi) \), and the convergence is uniform on \( \xi \). \( \square \)

In the rest of this section, we assume that the \( \lambda \)-NRW has a return ratio \( \lambda \in (0, r^\alpha) \). Then \( \beta = \log \lambda / \log r > \alpha \), and Proposition 2.5 applies.

**Lemma 3.3.** Suppose \( \{S_i\}_{i=1}^N \) satisfies OSC, and let \( \{Z_n\}_{n=0}^\infty \) be a \( \lambda \)-NRW on the augmented tree \( (X, \mathcal{E}) \) with \( \lambda \in (0, r^\alpha) \). Then for \( f \in \mathcal{D}_X \),

(i) there exists \( C > 0 \) (depend on \( f \)) such that for any geodesic ray \( (x_n)_n \),

\[
|f(x_{n+1}) - f(x_n)| \leq C(\lambda/r^\alpha)^{n/2},
\]

and hence \( \lim_{n \to \infty} f(x_n) \) exists;

(ii) for two equivalent geodesic rays \( (x_n)_n \) and \( (y_n)_n \), \( \lim_{n \to \infty} f(x_n) = \lim_{n \to \infty} f(y_n) \).

**Proof.** (i) Let \( \tau = \lambda/r^\alpha < 1 \). For a geodesic ray \( (x_n)_n \), since

\[
|f(x_{n+1}) - f(x_n)| \leq \sqrt{\frac{\mathcal{E}_X[f]}{c(x_{n+1}, x_n)}} \leq C(\lambda/r^\alpha)^{n/2} = C\tau^{n/2}, \tag{3.2}
\]

hence the sequence \( (f(x_n))_n \) converges in an exponential rate.

(ii) For two equivalent geodesic rays \( (x_n)_n \) and \( (y_n)_n \) that converge to the same \( \xi \), if they are distinct, then \( x_n \sim_h y_n \) for all \( n \) (or by Lemma 2.4). Then

\[
|f(x_n) - f(y_n)| \leq \sqrt{\frac{\mathcal{E}_X[f]}{c(x_n, y_n)}} \leq C'\tau^{n/2}, \tag{3.3}
\]

which tends to 0 as \( n \to \infty \). Hence the two limits are equal. \( \square \)
With the assumption as in Lemma 3.3 we can define a linear map \( \text{Tr} : D_X \to \ell(K) \) (called it a trace map) by

\[
(\text{Tr} f)(\xi) = \lim_{n \to \infty} f(x_n), \quad \xi \in K,
\]

where \((x_n)_n\) is a geodesic ray that converges to \(\xi\). We call \(f\) the trace function of \(f\). By Lemma 3.3(ii), the limit in (3.4) is “uniform” in the sense that for \(f \in D_X\) and \(\varepsilon > 0\), there exists a positive integer \(n_0\) such that

\[
|f(x) - \text{Tr} f(\xi)| \leq \varepsilon, \quad \forall |x| \geq n_0, \xi \in S_X(K).
\]

**Lemma 3.4.** Suppose \(\{S_i\}_{i=1}^N\) satisfies OSC, and let \(\{Z_n\}_{n=0}^\infty\) be a \(\lambda\)-NRW with ratio \(\lambda \in (0, r^\alpha)\) on the augmented tree \((X, \mathcal{E})\). Then \(\text{Tr} f\) is continuous on \(K\).

**Proof.** For \(\varepsilon > 0\), by (3.5), there exists \(n_0\) such that \(|f(x) - \text{Tr} f(\xi)| < \varepsilon/3\) for \(|x| \geq n_0\) and \(\xi \in S_X(K)\). Let \(M\) be the uniform bound of the horizontal geodesics in \((X, \mathcal{E})\) [28], and let \(C\) be a constant such that \(c(x, y) \geq C^{-1}(r^\alpha/\lambda|x|\) for all \(x \sim_h y\). By assumption \(\tau := \lambda/r^\alpha < 1\). We choose \(n_1 \geq n_0\) such that \(M \sqrt{C E_X[f]|\tau^{n_1}} < \varepsilon/3\).

As \(\xi - \eta \propto r(\xi, \eta)\) (Theorem 2.2), we can pick \(\delta > 0\) such that \((\xi|\eta) \geq n_1\) whenever \(|\xi - \eta| < \delta\). Now for \(\xi, \eta \in K\) with \(|\xi - \eta| < \delta\), consider a canonical geodesic \([\xi, u, v, \eta]\) with horizontal geodesic \((u = u_0, u_1, \ldots, u_k = v)\) (see Section 2). Then \(|u| \geq (\xi|\eta) \geq n_1\), and hence

\[
|\text{Tr} f(\xi) - \text{Tr} f(\eta)| \leq |\text{Tr} f(\xi) - f(u)| + |f(u) - f(v)| + |f(v) - \text{Tr} f(\eta)|
\]

\[
< \frac{\varepsilon}{3} + \sum_{i=0}^{k-1} |f(u_i) - f(u_{i+1})| + \frac{\varepsilon}{3}
\]

\[
< \frac{2\varepsilon}{3} + M \sqrt{C E_X[f]|\tau^{n_1}} < \varepsilon. \quad \text{(by (3.3))}
\]

This concludes that \(\text{Tr} f \in C(K)\). \qed

**Theorem 3.5.** Suppose \(\{S_i\}_{i=1}^N\) satisfies the OSC, and let \(\{Z_n\}_{n=0}^\infty\) be a \(\lambda\)-NRW with ratio \(\lambda \in (0, r^\alpha)\) on the augmented tree \((X, \mathcal{E})\). Then \(\text{Tr}(\mathcal{H}D_X) = D_K\) where \(\mathcal{H}D_X\) is the class of harmonic functions in \(D_X\). More precisely, \(\text{Tr} Hu = u\) for \(u \in D_K\), and \(H \text{Tr} f = f\) for \(f \in \mathcal{H}D_X\).

**Proof.** For \(u \in D_K\), by definition we have \(Hu \in \mathcal{H}D_X\). Note that \(D_K \cap C(K) = D_K\), as \(D_K = \Lambda_{2, 2}^{\alpha, \beta/2}\) can be imbedded into the H"older space \(O(\beta - \alpha/2)(K)\) if \(\beta > \alpha\) (Proposition 2.5). By Corollary 3.2 we have \(\text{Tr} Hu = u\).
For $f \in \mathcal{HD}_X$, let $u = Tr f$. Then $u \in C(K)$ (Lemma 3.3). For any $\varepsilon > 0$, by Corollary 3.2, there exists a positive integer $n_0$ such that for $|x| \geq n_0$ and $\xi \in S_X(K)$,

$$|f(x) - u(\xi)| < \frac{\varepsilon}{2} \quad \text{and} \quad |Hu(x) - u(\xi)| < \frac{\varepsilon}{2}.$$  

(3.6)

We show that $f = Hu$ on $X$. Suppose otherwise, we can assume without loss of generality that $f(x_0) > Hu(x_0)$ for some $x_0 \in J_n$. Let $a_n = \max_{x \in J_n} (f(x) - Hu(x))$, $n \geq 1$. Note that $f - Hu$ is harmonic. By the maximum principle of harmonic functions, we regard $J_{n+1}$ as the boundary of $X_{n+1} = \bigcup_{k=0}^{n+1} J_k$. Then $a_{n+1} \geq \max_{x \in J_n} (f(x) - Hu(x)) = a_n$, thus the sequence $\{a_n\}$ is non-decreasing. Hence $\inf_{n \geq m} a_n = a_m > 0$. This contradicts that $\lim_{n \to \infty} a_n = 0$ by (3.6). We conclude that $f = Hu = HTr f$. 

Let $\vartheta$ be the root of $(X, \mathcal{E})$, then $\mathcal{D}_X$ is a Hilbert space under the inner product $\langle f, g \rangle_\vartheta = f(\vartheta)g(\vartheta) + \mathcal{E}_X(f, g)$. Let $\| \cdot \|_\vartheta$ denote the norm, and let $\mathcal{D}_{X,0}$ be the $\| \cdot \|_\vartheta$-closure of functions on $X$ with finite supports. It is known that for $f \in \mathcal{D}_X$, it admits a decomposition $f = f_H + f_0$ where $f_H \in \mathcal{HD}_X$ and $f_0 \in \mathcal{D}_{X,0}$ [34, Theorem 3.69].

**Corollary 3.6.** With the same assumption as in Theorem 3.5, then for $f \in \mathcal{D}_X$, we have $Tr f = Tr f_H$, and hence $Tr f \in \mathcal{D}_K$.

**Proof.** It suffices to show that $Tr f \equiv 0$ for $f \in \mathcal{D}_{X,0}$, then the corollary follows from the above decomposition and Theorem 3.5 that $Tr(\mathcal{HD}_X) = \mathcal{D}_K$.

First we claim that if $\{g_\ell\}_\ell \subset \mathcal{D}_X$ satisfies $g_\ell \| \|_\vartheta \to 0$, then $\lim_{\ell \to \infty} g_\ell(x) = 0$ for all $x \in X$ uniformly. Indeed for $x \in X$, let $(\vartheta = x_0, x_1, \ldots, x_n = x)$ be the geodesic from $\vartheta$ to $x$, then it follows from the same argument as in (3.2) that

$$|g_\ell(x) - g_\ell(\vartheta)| \leq \sum_{k=0}^{n-1} |g_\ell(x_{k+1}) - g_\ell(x_k)| \leq \left( \sum_{k=0}^{n-1} C_1 \tau^{k/2} \right) \sqrt{\mathcal{E}_X(g_\ell)} = C_2 \sqrt{\mathcal{E}_X(g_\ell)}.$$ 

Also observe that $\lim_{\ell \to \infty} g_\ell(\vartheta) = 0$, and hence the claim follows.

Now for $f \in \mathcal{D}_{X,0}$, let $\{f_\ell\}_\ell \subset \mathcal{D}_X$ be such that each $f_\ell$ has finite support and $f_\ell \| \|_\vartheta \to f$. For $\varepsilon > 0$, by the claim, there exists $\ell_0$ such that $|f - f_{\ell_0}|(x) \leq \varepsilon$ for all $x \in X$. For $\xi \in K$, let $(x_n)_n$ be a geodesic ray that converges to $\xi$. Then

$$|f(x_n)| \leq |(f - f_{\ell_0})(x_n)| + |f_{\ell_0}(x_n)| \leq |f_{\ell_0}(x_n)| + \varepsilon, \quad \forall n.$$ 

This implies $Tr(f)(\xi) := \lim_{n \to \infty} f(x_n) = 0$, and completes the proof. 

In Theorem 3.5, we can actually give another norm on $\mathcal{D}_X$ so that $H : \mathcal{D}_K \to \mathcal{HD}_X$ is a Banach space isomorphism. Indeed, by Corollary 3.6 and the continuity
of functions in $D_K$, we know that functions in $D_X$ are bounded. Fix $w \in (0, r^\alpha)$. Let $\|f\|^2_{\ell^2(X,w)} = \sum_{x \in X} |f(x)|^2 w^{|x|}$, and define $\| \cdot \|_{\mathcal{E}_X}$ on $D_X$ by

$$\|f\|^2_{\mathcal{E}_X} = \mathcal{E}_X[f] + \|f\|^2_{\ell^2(X,w)}. \quad (3.7)$$

Then it is direct to check that $\|f\|^2_{\mathcal{E}_X}$ defines a complete norm on $D_X$.

**Corollary 3.7.** With the same assumption as in Theorem 3.5, let $w \in (0, r^\alpha)$. Then for all $u \in L^2(K, \nu)$,

$$\|Hu\|_{\ell^2(X,w)} \leq C \|u\|_{L^2(K,\nu)}. \quad (3.8)$$

Consequently, $H : (D_K, \| \cdot \|_{\mathcal{E}_K}) \rightarrow (\mathcal{HD}_X, \| \cdot \|_{\mathcal{E}_X})$ is an isomorphism.

**Proof.** Let $F(\theta, y)$ denote the probability that the random walk ever visits $y$ from $x$. For $n \geq 1$ and $|y| > n$, by [26, Theorem 4.6],

$$F(\theta, y) = \sum_{x \in J_n} F_n(\theta, x)F(x, y) = \sum_{x \in J_n} r_{x}^{\alpha} F(x, y) \geq r_{x}^{\alpha} \sum_{x \in J_n} F(x, y).$$

Hence $\sum_{x \in J_n} K(x, \xi) \leq r_{x}^{\alpha} \sum_{x \in J_n} F(x, y)$, it follows that for $u \in L^2(K, \nu)$,

$$\|Hu\|^2_{\ell^2(X,w)} = \sum_{x \in X} (\mathcal{E}_X(u(Z_\infty)))^2 w^{|x|} \leq \sum_{x \in X} (\mathcal{E}_X(u(Z_\infty)))^2 w^{|x|}$$

$$= \sum_{n=0}^{\infty} w^{|x|} \sum_{x \in J_n} K(x, \xi) u(\xi)^2 d\nu(\xi) \leq C \|u\|^2_{L^2(K,\nu)}.$$  

where $C = r^{-\alpha} \sum_{n=0}^{\infty} (w/r^\alpha)^n$. As $w/r^\alpha < 1$, this yields [3.5]. In view of Theorem 3.5 the norm isomorphism of the map $H : D_K \rightarrow \mathcal{HD}_X$ follows from this and $\mathcal{E}_K(u, v) = \mathcal{E}_X(Hu, Hv)$, and the open mapping theorem. \quad $\square$

### 4 Effective resistances of $\mathcal{E}_X$

In this section, we will set up the limiting resistance for the $\lambda$-NRW on the augmented tree $(X, \mathcal{E})$ in order to prepare for the investigation of the critical exponents of $D_K$ in the next section.

We will start with a general situation. Let $V$ be a finite graph with a reversible Markov chain with conductance $c(x, y)$, $x, y \in V$. Let $\ell(V)$ denote the class of real valued functions on $V$, and let $\mathcal{E}_V(f)$ be the graph energy of $f$. For any $V_1 \subset V$, it is
well-known that each \( f \in \ell(V_1) \) has a harmonic extension \( \tilde{f} \) on \( V \); \( \tilde{f} \) has the minimal energy among all \( g \in \ell(V) \) with \( g|_{V_1} = f \), and the harmonicity for \( x \in V \setminus V_1 \) implies

\[
\sum_{y \sim x} c(x, y)(\tilde{f}(x) - \tilde{f}(y)) = 0, \quad x \in V \setminus V_1.
\] (4.1)

In the following, we give an expression of the minimal energy in terms of the conductance \( c(x, y) \) of the chain on \( V \).

**Proposition 4.1.** Let \( V \) be a finite set, and \( V = V_1 \cup V_2 \) with \( \#V_1 \geq 2 \). Assume that there is a reversible Markov chain on \( V \) with conductance \( c(\cdot, \cdot) \). Then for \( f \in V_1 \),

\[
\min \left\{ \mathcal{E}_V[g] : g \in \ell(V), g|_{V_1} = f \right\} = \frac{1}{2} \sum_{x, y \in V_1, x \neq y} c_*(x, y)(f(x) - f(y))^2,
\] (4.2)

where \( c_*(x, y) = c(x, y) + \sum_{z \in V_2} c(x, z)G_{V_2}(z, w)P(w, y) \), \( x, y \in V_1, x \neq y \) (here \( G_{V_2}(\cdot, \cdot) \) is the Green function of the random walk restricted to \( V_2 \)), and it defines a conductance function on \( V_1 \).

**Proof.** Let \( F^{V_1}(x, y) = \mathbb{P}_x(Z_{t_{V_1}} = y) \) where \( t_{V_1} \) is the first hitting time of \( V_1 \). Then

\[
F^{V_1}(z, y) = \sum_{w \in V_2} G_{V_2}(z, w)P(w, y), \quad \forall z \in V_2, y \in V_1.
\]

We can check directly from the definition that \( c_*(x, y) = c_*(y, x), x, y \in V_1 \), using the reversibility of the chain (i.e., \( m(x)P(x, z) = m(z)P(z, x) \) and \( m(z)G_{V_2}(z, w) = m(w)G_{V_2}(w, z) \)). Hence \( c_*(x, y) \) defines a conductance on \( V_1 \).

To prove (4.2), we let \( h(\cdot) = \sum_{y \in V_1} F^{V_1}(\cdot, y)f(y) \in \ell(V) \). Then it is easy to check that \( h \) is the unique function such that \( Ph = h \) on \( V_2 \) and \( h = f \) on \( V_1 \). Hence \( \mathcal{E}_V[h] = \min \{ \mathcal{E}_V[g] : g \in \ell(V), g|_{V_1} = f \} \). Observe that

\[
\mathcal{E}_V[h] = \frac{1}{2} \sum_{x, y \in V} c(x, y)(h(x) - h(y))^2 = \sum_{x, y \in V} c(x, y)(h(x) - h(y))h(x)
\]

Hence

\[
\mathcal{E}_V[h] = \sum_{x \in V_1} h(x) \sum_{y \in V} c(x, y)(h(x) - h(y)) \quad \text{(by \( Ph = h \) on \( V_2 \))}
\]

\[
= \sum_{x \in V_1} f(x)\left( \sum_{y \in V_1} c(x, y)(f(x) - f(y)) + \sum_{y \in V_2} c(x, y) \sum_{z \in V_1} F^{V_1}(z, y)(f(x) - f(z)) \right)
\]

\[
= \sum_{x, y \in V_1} f(x)(f(x) - f(y)) \left( c(x, y) + \sum_{z \in V_2} c(x, z)F^{V_1}(z, y) \right) \quad \text{(switch \( y \) and \( z \))}
\]

\[
= \sum_{x, y \in V_1} c_*(x, y)f(x)(f(x) - f(y))
\]

\[
= \frac{1}{2} \sum_{x, y \in V_1} c_*(x, y)(f(x) - f(y))^2. \quad \text{(use \( c_*(x, y) = c_*(y, x) \)}
\]

This yields (4.2). \( \square \)
For a finite connected graph \((X, \mathcal{E})\) with conductances, the effective resistance between two disjoint nonempty subsets \(E, F \subset X\) is given by
\[
R_X(E, F) = \left(\min\{\mathcal{E}_X[f] : f \in \ell(X) \text{ with } f = 1 \text{ on } E, \text{ and } f = 0 \text{ on } F\}\right)^{-1}. \tag{4.3}
\]
Also we set \(R_X(E, F) = 0\) if \(E \cap F \neq \emptyset\) by convention. Clearly \(R_X(\cdot, \cdot)\) is symmetric; the energy minimizer in (4.3) is unique, bounded in between 0 and 1, and is harmonic on \(X \setminus (E \cup F)\).

For the \(\lambda\)-NRW on \((X, \mathcal{E})\), for convenience and the simplicity in the estimations, we will assume slightly more that the conductance on the horizontal edges satisfies
\[
c(x, y) = r^\alpha|x| \lambda^{-|x|} \quad \text{for } x \sim_h y \in X \setminus \{\emptyset\}. \tag{4.4}
\]
(we use \(\sim\) in [22]), and there is no change of the results. Let \(\{\kappa_n\}_{n=0}^\infty\) be a \(\kappa\)-sequence defined in Section 2 i.e., each \(\kappa_n\) is a selection map from \(K\) to \(J_n\) such that for each \(\xi \in K\), \(\{\kappa_n(\xi)\}_{n=0}^\infty\) is a geodesic ray that converges to \(\xi\). For any two closed subsets \(\Phi, \Psi \subset K\), we define the level-\(n\) resistance between them (depend on \(\kappa_n\)) by
\[
R_n^{(\lambda)}(\Phi, \Psi) := R_X(\kappa_n(\Phi), \kappa_n(\Psi)), \tag{4.5}
\]
where \(X_n := \bigcup_{k=0}^n J_k\) and has same conductance restricted from \(X\).

**Theorem 4.2.** Suppose \(\{S_i\}_{i=1}^N\) satisfies the OSC, and let \(\{Z_n\}_{n=0}^\infty\) be a \(\lambda\)-NRW on the augmented tree \((X, \mathcal{E})\) with \(\lambda \in (0, r^\alpha)\). Then for any two closed subsets \(\Phi, \Psi \subset K\), the limit \(\lim_{n \to \infty} R_n^{(\lambda)}(\Phi, \Psi)\) exists, and is independent of the choice of the \(\kappa\)-sequence.

We will prove a technical lemma first. For \(E, F \subset J_n\) such that in the graph distance, \(\text{dist}(E, F) > 2\), we define
\[
\partial E = \{x \in J_n : \text{dist}(x, E) = 1\}, \quad \partial F = \{x \in J_n : \text{dist}(x, F) = 1\}. \tag{4.6}
\]
Let \(\mathcal{E}_n := \mathcal{E}_n(E, F) = \min\{\mathcal{E}_X[f] : f \in \ell(X_n), f = 1 \text{ on } E, f = 0 \text{ on } F\}\), and let \(f_n\) be the corresponding energy minimizing function.

**Lemma 4.3.** Consider the \(\lambda\)-NRW on \((X, \mathcal{E})\) with \(\lambda \in (0, r^\alpha)\). Let \(\{E_n\}_{n \geq 1}, \{F_n\}_{n \geq 1}\) be two sequences such that \(E_n, F_n \subset J_n\), and \(\lim \inf_{n \to \infty} \text{dist}(E_n, F_n) > 2\). If \(\sup_{n \geq 1} \mathcal{E}_n(E_n, F_n) < \infty\), then for any \(\varepsilon > 0\), there exists \(n_0\) such that for \(n \geq n_0\),
\[
\sum_{x \in \partial E_n} \sum_{y \in X_n \setminus E_n} c(x, y)(1 - f_n(y))^2 < \varepsilon, \quad \sum_{x \in \partial F_n} \sum_{y \in X_n \setminus F_n} c(x, y)f_n(y)^2 < \varepsilon,
\]
where \(f_n\) is the energy minimizer of \(\mathcal{E}_n := \mathcal{E}_n(E_n, F_n)\).
Proof. We write \( f = f_n \) for simplicity. We observe that
\[
\mathcal{E}_n = \frac{1}{2} \sum_{x, y \in X_n} c(x, y)(f(x) - f(y))^2
\]
\[
= \sum_{x, y \in X_n} c(x, y)f(x)(f(x) - f(y))
\]
\[
= \sum_{x \in E_n \cup F_n} f(x) \sum_{y \in X_n} c(x, y)(f(x) - f(y)) \quad \text{(by (4.1))}
\]
\[
= \sum_{x \in E_n \cup F_n} \sum_{y \in X_n} c(x, y)(1 - f(y)) \geq (r^n/\lambda)^n \sum_{y \in \partial E_n} (1 - f(y)). \quad (4.7)
\]

Thus \( f(x) \geq 1 - (\lambda/r^n)^n \mathcal{E}_n \) for \( x \in \partial E_n \). Using a similar argument, and that \( f \) is harmonic on \( X_n \setminus (E_n \cup F_n \cup \partial E_n) \), for large \( n \), we have
\[
\mathcal{E}_n \geq \frac{1}{2} \sum_{x, y \in X_n} c(x, y)(f(x) - f(y))^2 - \sum_{x \in \partial E_n} \sum_{y \in E_n} c(x, y)(f(x) - f(y))^2
\]
\[
= \left( \sum_{x \in E_n \cup F_n \cup \partial E_n \cup E_n} \sum_{y \in X_n} \sum_{x \in E_n \cup F_n \cup \partial E_n \cup E_n} c(x, y)f(x)(f(x) - f(y)) \right)
\]
\[
= \sum_{x \in E_n \cup F_n \cup \partial E_n \cup E_n} f(x) \sum_{y \in X_n \setminus E_n} c(x, y)(f(x) - f(y)) + \sum_{y \in E_n} c(y, y^-)(1 - f(y^-))
\]
\[
\geq (1 - (\lambda/r^n)^n \mathcal{E}_n) \sum_{x \in \partial E_n \cup F_n} \sum_{y \in X_n \setminus E_n} c(x, y)(f(x) - f(y)). \quad (4.8)
\]
(The last inequality holds because for \( x \in \partial E_n \), \( \sum_{y \in X_n \setminus E_n} c(x, y)(f(x) - f(y)) \geq 0 \), as by harmonicity, \( \sum_{y \in X_n \setminus E_n} \cdots = -\sum_{y \in E_n} \cdots = -\sum_{y \in E_n} c(x, y)(f(x) - 1) \geq 0 \).)

Now we use (4.7) and (4.8) to make the final estimate:
\[
\sum_{x \in \partial E_n \cup F_n \cup \partial E_n \cup E_n} \sum_{y \in X_n \setminus E_n} c(x, y)(1 - f(y))^2 \leq \left( \sum_{x \in \partial E_n \cup F_n \cup \partial E_n \cup E_n} \sum_{y \in X_n \setminus E_n} c(x, y)^{1/2}(1 - f(y)) \right)^2
\]
\[
\leq \frac{\lambda^n}{r^{\alpha(n+1)}} \left( \sum_{x \in \partial E_n \cup F_n \cup \partial E_n \cup E_n} \sum_{y \in X_n \setminus E_n} c(x, y)(1 - f(y)) \right)^2
\]
\[
= \frac{\lambda^n}{r^{\alpha(n+1)}} \left( \sum_{x \in \partial E_n \cup F_n \cup \partial E_n \cup E_n} \sum_{y \in X_n \setminus E_n} c(x, y)((1 - f(x)) + (f(x) - f(y))) \right)^2
\]
\[
\leq \frac{\lambda^n}{r^{\alpha(n+1)}} \left( k\mathcal{E}_n + \frac{\mathcal{E}_n}{1 - (\lambda/r^n)^n \mathcal{E}_n} \right)^2 =: \varepsilon(n) \quad \text{(by (4.7), (4.8))} \quad (4.9)
\]
where \( k = \sup_{x \in X} \# \{y : x \sim_h y\} \) (as the graph \( (X, \mathcal{E}) \) has bounded degree, and \( c(x, y) > 0 \) only when \( x \sim_h y \) or \( y = x^- \)). Hence we can choose \( n_0 \) such that \( \varepsilon(n) < \varepsilon \) for \( n > n_0 \). Analogously, using \( 1 - f \) instead of \( f \), we obtain the estimate for \( F \) as well. \( \square \)
Proof of Theorem 4.2: We fix a $\lambda \in (0, r^\alpha)$ and omit the superscript $(\lambda)$ in this proof. First, we fix a $\kappa$-sequence $\{\kappa_n\}_{n=0}^\infty$ and prove that $\lim_{n\to\infty} R_n(\Phi, \Psi)$ exists. For brevity, we write $\Phi_n := \kappa_n(\Phi)$ and $\Psi_n := \kappa_n(\Psi)$. If $\Phi \cap \Psi \neq \emptyset$, then by the property of geodesic rays in $(X, \xi)$, for any $n$, either $\Phi_n \cap \Psi_n \neq \emptyset$ or $\min\{d(x, y) : x \in \Phi_n, y \in \Psi_n\} \geq 1$ (by Lemma 2.1). In both situations, we have $\lim_{n\to\infty} R_n(\Phi, \Psi) \neq 0$ (for the second case, by (4.3), $R_n(\Phi, \Psi) \leq (\kappa_n \lambda^{-n})^{-1} = (\lambda/r^\alpha)^n$).

Hence we assume that $\Phi \cap \Psi = \emptyset$. Then there exists $\ell > 0$ such that for $n \geq \ell$, $\text{dist}(\Phi_n, \Psi_n) > 3$. By (4.3) and (4.5), for $n \geq \ell$,

$$R_n(\Phi, \Psi) = (\min\{\mathcal{E}_n[f] : f = 1 \text{ on } \Phi_n, \text{ and } f = 0 \text{ on } \Psi_n\})^{-1}. \quad (4.10)$$

Let $\mathcal{E}_n$ denote the minimal energy, and let $f_n \in \ell(X_n)$ be the energy minimizer in (4.10). Let $\{n_k\}_{k \geq 1}$ with $n_k \geq \ell$ be the subsequence such that $\lim_{k \to \infty} R_{n_k}(\Phi, \Psi) = \limsup_{n \to \infty} R_n(\Phi, \Psi) > 0$. (otherwise $\lim_{n \to \infty} R_n(\Phi, \Psi) = 0$). Then $\sup_k \mathcal{E}_{n_k} < \infty$.

For $n < n_k$ and $\xi \in K$, by Lemma 3.3(i), we have

$$|f_{n_k}(\kappa_n(\xi)) - f_{n_k}(\kappa_n(\xi))| \leq \sum_{m=1}^{n_k-n} |f_{n_k}(\kappa_{n+m-1}(\xi)) - f_{n_k}(\kappa_{n+m}(\xi))|$$

$$\leq C\left(\frac{\lambda}{r^\alpha}\right)^{n/2} := \varepsilon(n). \quad (4.11)$$

As $\lambda \in (0, r^\alpha)$, $\lim_{n \to \infty} \varepsilon(n) = 0$. Let $V_1 = \Phi_n \cup \Psi_n$. Then for sufficiently large $n$ and $n_k > n$, we have $\varepsilon(n) < \frac{1}{2}$, and

$$\mathcal{E}_{n_k} \geq \mathcal{E}_n[f_{n_k}] \geq \min\{\mathcal{E}_n[f] : f \in \ell(X_n), f = f_{n_k} \text{ on } V_1\}$$

$$= \frac{1}{2} \sum_{x, y \in V_1} c_\pi(x, y)(f_{n_k}(x) - f_{n_k}(y))^2 \quad \text{(by Proposition 4.1)}$$

$$\geq \sum_{x \in \Phi_n} \sum_{y \in \Psi_n} c_\pi(x, y)(f_{n_k}(x) - f_{n_k}(y))^2$$

$$\geq \sum_{x \in \Phi_n} \sum_{y \in \Psi_n} c_\pi(x, y)((1 - 2\varepsilon(n))^2 \quad \text{(by (4.11))}$$

$$= \min\{\mathcal{E}_n[f] : f \in \ell(X_n), f|_{\Phi_n} = 1 - 2\varepsilon(n), f|_{\Psi_n} = 0\}$$

$$= (1 - 2\varepsilon(n))^2 \mathcal{E}_n. \quad \text{(by Proposition 4.1)}$$

Therefore, $R_n(\Phi, \Psi) \geq (1 - 2\varepsilon(n))^2 R_{n_k}(\Phi, \Psi)$ for any large $n$ and $n_k > n$. Taking limit, we have

$$\liminf_{n \to \infty} R_n(\Phi, \Psi) \geq \lim_{k \to \infty} R_{n_k}(\Phi, \Psi) = \limsup_{n \to \infty} R_n(\Phi, \Psi).$$

Hence $\lim_{n \to \infty} R_n(\Phi, \Psi)$ exists.
Next we show that the above limit is independent of the choice of the \( \kappa \)-sequence. For this, we define

\[
\partial \Phi_n = \{ x \in J_n : d(x, \Phi_n) = 1 \}, \quad \partial \Psi_n = \{ x \in J_n : d(x, \Psi_n) = 1 \}
\]
as in (4.6). For any other \( \kappa \)-sequences \( \{ \kappa'_n \} \), it follows from Lemma 2.1 that \( \kappa'_n(\Phi) \subset \Phi_n \cup \partial \Phi_n \) and \( \kappa'_n(\Psi) \subset \Psi_n \cup \partial \Phi_n \). Hence it suffices to show that

\[
\lim_{n \to \infty} R_{X_n}(\Phi_n \cup \partial \Phi_n, \Psi_n \cup \partial \Phi_n) = \lim_{n \to \infty} R_n(\Phi, \Psi).
\] (4.12)

Without loss of generality, we assume \( \lim_{n \to \infty} R_n(\Phi, \Psi) > 0 \). Then \( \sup_n \mathcal{E}_n < \infty \). Let \( h_n \in \ell(X_n) \) with \( h_n = 1 \) on \( \partial \Phi_n \), \( h_n = 0 \) on \( \partial \Psi_n \), and \( h_n = f_n \) on \( X_n \setminus (\partial \Phi_n \cup \partial \Psi_n) \).

\[
0 \leq R_{X_n}(\Phi_n \cup \partial \Phi_n, \Psi_n \cup \partial \Phi_n)^{-1} - R_n(\Phi, \Psi)^{-1} \leq \mathcal{E}_{X_n}[h_n] - \mathcal{E}_{X_n}[f_n].
\]

Then by Lemma 4.3, for given \( \varepsilon \), and for large \( n \), \( \mathcal{E}_{X_n}[h_n] - \mathcal{E}_{X_n}[f_n] \leq 2\varepsilon \). This implies (4.12) and proves the theorem.

Theorem 4.2 implies the following definition is well defined.

**Definition 4.4.** With the same assumption as in Theorem 4.2, we define the limiting resistance between two closed subsets \( \Phi \) and \( \Psi \) in \( K \) by

\[
R^{(\lambda)}(\Phi, \Psi) := \lim_{n \to \infty} R^{(\lambda)}_n(\Phi, \Psi).
\] (4.13)

(We omit the superscript \( \lambda \) if there is no confusion.)

5 The critical exponents of \( \mathcal{D}_X \)

We first establish a basic result on the existence of nonconstant functions in \( \mathcal{D}_K \).

**Theorem 5.1.** With the same assumption as in Theorem 4.2, suppose \( \Phi, \Psi \) are two closed subsets of \( K \) satisfying \( R(\Phi, \Psi) > 0 \). Then there exists \( u := u_{\Phi, \Psi} \in \mathcal{D}_K \) such that \( u = 1 \) on \( \Phi \), and \( u = 0 \) on \( \Psi \). Moreover, \( u_{\Phi, \Psi} \) is the unique energy minimizer in \( \mathcal{D}_K \) in the following sense

\[
R(\Phi, \Psi)^{-1} = \mathcal{E}_K[u_{\Phi, \Psi}] = \inf \{ \mathcal{E}_K[u'] : u' \in \mathcal{D}_K \text{ with } u' = 1 \text{ on } \Phi, u' = 0 \text{ on } \Psi \}.
\] (5.1)
Proof. First we show that the set on the right hand side in (5.1) is non-empty. Clearly \( \Phi \cap \Psi = \emptyset \) (otherwise \( R(\Phi, \Psi) = 0 \)). Fix a \( \kappa \)-sequence \( \{\kappa_n\}_n \). As in the proof of Theorem 4.2, there exists a positive integer \( \ell \) such that \( \kappa_n(\Phi) \cap \kappa_n(\Psi) = \emptyset \) for all \( n \geq \ell \), let \( f_n \in \mathcal{L}(X_n) \) be the energy minimizer for \( \kappa_n(\Phi) \) and \( \kappa_n(\Psi) \) as in (4.10). We extend \( f_n \) to \( X \) by setting \( f_n(x) = 0 \) for \( x \in X \setminus X_n \), then \( f_n \) is harmonic on \( X_{n-1} \). Note that \( 0 \leq f_n \leq 1 \) for all \( n \geq \ell \). Hence for each \( x \in X \), there exists a convergent subsequence of \( \{f_n(x)\}_{n \geq \ell} \). By the diagonal argument, we can find a subsequence \( \{f_{n_k}\}_{k \geq 1} \) with \( n_1 \geq \ell \) such that \( f_{n_k} \) converges to a function \( f := f_{\Phi, \Psi} \in \mathcal{L}(X) \) pointwise. We claim that

(a) \( f \in \mathcal{H}D_X \) and \( 0 \leq f \leq 1 \) on \( X \);

(b) For any \( \xi \in \Phi \), \( \lim_{n \to \infty} f(\kappa_n(\xi)) = 1 \);

(c) For any \( \eta \in \Psi \), \( \lim_{n \to \infty} f(\kappa_n(\eta)) = 0 \).

In fact, as \( f_{n_k} \) is harmonic on \( X_{n_k-1} \), the pointwise limit \( f \) is harmonic on \( X \). For \( k \geq 1 \), let \( g_k \) be the function on the edge set \( E \) defined by: for \( (x, y) \in E \),

\[
    g_k(x, y) = \begin{cases} 
        c(x, y)(f_{n_k}(x) - f_{n_k}(y))^2, & \text{if } x, y \in X_{n_k}, \\
        0, & \text{otherwise}.
    \end{cases}
\]

Then \( \mathcal{E}_{n_k} := \mathcal{E}_{X_{n_k}}[f_{n_k}] = \frac{1}{2} \sum_{(x, y) \in E} g_k(x, y) \), and \( \lim_{k \to \infty} g_k(x, y) = c(x, y)(f(x) - f(y))^2 \). By Fatou’s Lemma, we have

\[
    \mathcal{E}_X[f] = \frac{1}{2} \sum_{(x, y) \in E} c(x, y)(f(x) - f(y))^2 = \frac{1}{2} \sum_{(x, y) \in E} \left( \lim_{k \to \infty} g_k(x, y) \right) \\
    \leq \frac{1}{2} \liminf_{k \to \infty} \sum_{(x, y) \in E} g_k(x, y) = \lim_{k \to \infty} \mathcal{E}_{n_k} = R(\Phi, \Psi)^{-1} < \infty. \tag{5.2}
\]

Hence (a) follows.

To prove (b), observe that \( R(\Phi, \Psi) > 0 \) implies that \( \sup_{k \geq 1} \mathcal{E}_{n_k} < \infty \). Hence for any \( k \geq 1 \), \( n < n_k \) and \( \xi \in \Phi \), by Lemma\( \text{[33]}(\text{i}) \)

\[
    |f_{n_k}(\kappa_n(\xi)) - 1| \leq \sum_{m=n}^{n_k-1} |f_{n_k}(\kappa_m(\xi)) - f_{n_k}(\kappa_{m+1}(\xi))| \leq C_1(\lambda/r^\alpha)^{n/2}.
\]

Letting \( k \to \infty \), we have \( |f(\kappa_n(\xi)) - 1| \leq C_2(\lambda/r^\alpha)^{n/2} \), hence (b) follows by letting \( n \to \infty \). With a similar argument, we can also conclude (c).

By the claim and Theorem\( \text{[35]} \), let \( u = \text{Tr}f \in \mathcal{D}_K \). Then \( 0 \leq u \leq 1 \) on \( K \), \( u(\xi) = \lim_{n \to \infty} f(\kappa_n(\xi)) = 1 \) for all \( \xi \in \Phi \), and \( u(\eta) = \lim_{n \to \infty} f(\kappa_n(\eta)) = 0 \) for all \( \eta \in \Psi \).
Now we complete the proof of the theorem. By (5.2), $E_K[u, \varphi, \psi] = E_X[f, \varphi, \psi] \leq R(\varphi, \psi)^{-1}$. For the reverse inequality, it suffices to show that $R(\varphi, \psi)^{-1} \leq E_K[u]$ for all $u \in D_K$ with $u = 1$ on $\varphi$ and $u = 0$ on $\psi$. Fix a $\kappa$-sequence $\{\kappa_n\}_n$. For any $e \in (0, \frac{1}{2})$, by Proposition 3.2, there exists a positive integer $n_0 = n_0(e)$ such that $\|H u(\kappa_n(\xi)) - u(\xi)\| \leq e$ whenever $n \geq n_0$ and $\xi \in K$. Taking $V_1 = \kappa_n(\varphi) \cup \kappa_n(\psi)$ and $g = H u|_{V_1}$ as in Proposition 4.1, then we have, for $n \geq n_0$,

$$\min_{x,y \in V_1} c(x, y)(Hu(x) - Hu(y))^2.$$ 

Hence

$$E_{X_n}[Hu] \geq \min_{x,y \in V_1} c(x, y)(Hu(x) - Hu(y))^2 \geq (1 - 2e)^2 R_n(\varphi, \psi)^{-1}.$$

As $e$ can be arbitrarily small, we have $R(\varphi, \psi)^{-1} \leq \lim_{n \to \infty} E_{X_n}[Hu] = E_K[u]$. Hence (5.1) follows.

The uniqueness of $u_{\varphi, \psi}$ as an energy minimizer follows from the fact that $E_K$ is strictly convex in $D_K$.

The function $f_{\varphi, \psi} \in H D_X$ thus constructed is called a harmonic function induced by $\varphi$ and $\psi$. The function $u_{\varphi, \psi} = \text{Tr} f_{\varphi, \psi} \in D_K$ is referred as the energy minimizer of $\varphi$ and $\psi$.

Corollary 5.2. With the same assumption as in Theorem 4.2, the following conditions are equivalent: for two distinct points $\xi, \eta \in K$,

(i) there exists $u \in D_K$ with range $[0, 1]$ such that $u(\xi) = 1$ and $u(\eta) = 0$;

(ii) there exists $u \in D_K$ such that $u(\xi) \neq u(\eta)$;

(iii) $R(\xi, \eta) > 0$.

In this case, $R(\xi, \eta) = \sup \left\{ \frac{|u(\xi) - u(\eta)|^2}{E_K(u, u)} : u \in D_K, E_K(u, u) > 0 \right\}$.

Proof. Note that (i) $\Rightarrow$ (ii) is trivial, and (iii) $\Rightarrow$ (i) follows from Theorem 5.1. We need only prove (ii) $\Rightarrow$ (iii). We observe that the given $u \in D_K$ is continuous (Proposition 2.5). Fix any $\kappa$-sequence, by Corollary 3.2 there exists $n_0 > 0$ such
that for \( n \geq n_0 \), \(|Hu(\kappa_n(\xi)) - u(\xi)| \leq \frac{1}{2}|u(\xi) - u(\eta)|\), and the same for \( \eta \). Hence 
\[ |Hu(\kappa_n(\xi)) - Hu(\kappa_n(\eta))| \geq \frac{1}{3}|u(\xi) - u(\eta)|. \]
Then by (4.3), for \( n > n_0 \)
\[ \frac{|u(\xi) - u(\eta)|}{9R_n(\xi, \eta)} \leq \frac{|Hu(\kappa_n(\xi)) - Hu(\kappa_n(\eta))|^2}{R_n(\xi, \eta)} \]
\[ \leq c(\kappa_n(\xi), \kappa_n(\eta)) \frac{|Hu(\kappa_n(\xi)) - Hu(\kappa_n(\eta))|^2}{|\xi - \eta|^2} \leq \mathcal{E}_{\kappa_n}[Hu]. \]
Taking the limit on \( n \), we have 
\[ \frac{|u(\xi) - u(\eta)|}{9R(\xi, \eta)} \leq \mathcal{E}_K[u] < \infty. \]
Hence \( R(\xi, \eta) > 0 \). \( \square \)

**Corollary 5.3.** With the same assumption as in Theorem 4.2, if \( R^{(\lambda)}(\xi, \eta) = 0 \) for some \( \xi, \eta \in K \), then \( \beta_1^* \leq \log \lambda/\log r \) where \( \beta_1^* := \sup\{\beta > 0 : D_0^{(\beta)} \cap C(K) \) is dense in \( C(K)\} \).

**Proof.** If \( R^{(\lambda)}(\xi, \eta) = 0 \), then every \( u \in D_K \) must satisfy \( u(\xi) = u(\eta) \), so \( D_K \) is not dense in \( C(K) \), which implies \( \beta_1^* \leq \log \lambda/\log r \). \( \square \)

**Remark.** For the implication of (ii) \( \Rightarrow \) (iii) in Corollary 5.2, we can omit \( \lambda \in (0, r^\alpha) \) (i.e., \( \beta > \alpha \)), instead consider \( u \in D_K \cap C(K) \), and replace \( R(\xi, \eta) \) by \( \mathcal{R}(\xi, \eta) := \liminf_{n \to \infty} R_n(\xi, \eta) \), then the implication still holds. Consequently, Corollary 5.3 is still valid.

In the following, we will apply Corollary 5.2 to give some criteria to determine the critical exponents for \( \beta_2^* := \sup\{\beta > 0 : \dim D_0^{(\beta)} = \infty\} \) and \( \beta_3^* := \sup\{\beta > 0 : D_0^{(\beta)} \) contains nonconstant functions\}.

Let \( i_n = i_i \cdots i_i \in J_n \) denote the unique word in level \( n \) consisting of symbol \( i \in \Sigma \), and let \( i^\infty = i_i \cdots \in \Sigma^\infty \) (identified with the unique point in \( \bigcap_{n \geq 0} S_{i_n}(K) \)). Then for two distinct symbols \( i, j \in \Sigma \), we use \( R(i^\infty, j^\infty) \) to denote the limiting resistance for the corresponding two points in \( K \), and \( R(i^\infty, j^\infty) = \lim_{n \to \infty} R_n(i^\infty, j^\infty) \).

**Theorem 5.4.** With the same assumption as in Theorem 4.2, \( D_K \) consists of only constant functions if and only if
\[ R^{(\lambda)}(i^\infty, j^\infty) = 0, \quad \forall i, j \in \Sigma. \]  
(5.4)

Consequently, \( \beta_2^* = \log \lambda^* / \log r \) if
\[ \lambda^* := \sup\{\lambda > 0 : R^{(\lambda)}(i^\infty, j^\infty) = 0, \quad \forall i, j \in \Sigma\} \in (0, r^\alpha), \]  
(5.5)
and \( \beta_3^* = \infty \) if the above set of \( \lambda \) is empty.
Proof. If for some \( i, j \in \Sigma, R(i^\infty, j^\infty) > 0 \), then there exists \( u \in \mathcal{D}_K \) with \( u(i^\infty) \neq u(j^\infty) \) by Proposition 5.1 (or by Corollary 5.2 (iii) \( \Rightarrow \) (ii)). Thus it suffices to show that (5.4) implies \( \mathcal{D}_K = \{ \text{constant functions} \} \).

First we claim that for \( u \in C(K) \), if \( u(\xi^\infty) = u(\eta^\infty) \) for any \( \xi, \eta \in \Sigma^* \) and \( i, j \in \Sigma \), then \( u \) is a constant function. Indeed, let \( c = u(1^\infty) \), then for \( \xi = \eta, \) by assumption we have \( u(i^\infty) = c \) for any \( i \in \Sigma \). Next for \( \xi = i, \) by assumption again, we have \( u(i^\infty) = u(j^\infty) = c \) hence \( u(\xi^\infty) = c \) for any \( \xi \in \Sigma^1 \) and \( j \in \Sigma \). Following the same argument inductively, we have \( u(\eta^\infty) = c \) for any \( \eta \in \Sigma^* \) and \( j \in \Sigma \). By continuity, \( u \equiv c, \) a constant function.

For nonconstant \( u \in C(K) \), by the claim we can pick \( \xi \in \Sigma^* \) and \( i, j \in \Sigma \) such that \( u(\xi^\infty) \neq u(\eta^\infty) \). We telescope \( u \) on the cell \( S_\xi(K) \) to get \( \tilde{u} = u \circ S_\xi \). Then \( \tilde{u}(i^\infty) \neq \tilde{u}(j^\infty) \). By Proposition 5.2 (or by Corollary 5.2 (ii) \( \Rightarrow \) (iii)) and assumption (5.4), we must have \( \tilde{u} \notin \mathcal{D}_K \). Note that

\[
\mathcal{E}_K[u] = \frac{c_1}{\mathcal{S}_K(K)} \int \int_{S_\xi(K)} \frac{|u(\xi) - u(\eta)|^2}{|\xi - \eta|^{\alpha + \beta}} d\nu(\xi) d\nu(\eta) \\
\geq c_2 \int \int_K \frac{|\tilde{u}(\xi) - \tilde{u}(\eta)|^2}{|\xi - \eta|^{\alpha + \beta}} d\nu(\xi) d\nu(\eta) \geq c_3 \mathcal{E}_K[\tilde{u}],
\]

hence \( u \notin \mathcal{D}_K \). Finally as \( \mathcal{D}_K \cap C(K) = \mathcal{D}_K \) by Theorem 2.2, \( \mathcal{D}_K \) contains constant functions only. \( \Box \)

Next we will show that \( \beta_2^* = \beta_3^* \) under the connectedness of the self-similar set. The following lemma is a key step to include more non-trivial functions in \( \mathcal{D}_K \).

Lemma 5.5. With the same assumption as in Theorem 4.2, suppose \( \xi \in K \) and \( \Psi \) is a closed subset in \( K \) satisfying \( R(\xi, \Psi) > 0 \). Let \( u = u_{\xi, \Psi} \in \mathcal{D}_K \) be the limiting harmonic function. Then for \( \eta \in K \) such that \( 0 < u(\eta) < 1 \), we have \( R(\eta, \Psi) > 0 \) and \( R(\xi, \Psi \cup \{ \eta \}) > 0 \).

Proof. Let \( f = f_{\xi, \Psi} = Hu \) and \( \varepsilon = \min\{u(\eta), 1 - u(\eta)\} > 0 \). Fix a \( \kappa \)-sequence \( \{\kappa_n\}_n \). By Proposition 3.2, there exists a positive integer \( m_0 \) such that

\[
|f(\kappa_n(\eta)) - u(\eta)| < \varepsilon/4, \quad \forall n \geq m_0.
\]

Following the same argument as in the proof of Proposition 5.1 let \( f_n \in \ell(X_n) \) be the energy minimizer in (4.10) with \( \Phi = \{\xi\} \). By passing to subsequence, we assume, without loss of generality, that \( f_n \in \ell(X) \) converges to \( f \) pointwise.

Note that for \( n \geq 1 \) and \( k < n \), by Lemma 3.3 (i),

\[
|f_n(\kappa_k(\eta)) - f_n(\kappa_n(\eta))| \leq \sum_{m=k}^{n-1} |f_n(\kappa_m(\eta)) - f_n(\kappa_{m+1}(\eta))| \leq C_1(\lambda/r^\alpha)^k/2.
\]
Thus we can pick a positive integer $m_1 \geq m_0$ such that
\[ |f_n(\kappa_{m_1}(\eta)) - f_n(\kappa_n(\eta))| < \epsilon/4, \quad \forall \ n \geq m_1. \]  \tag{5.8}

Since $f_n(\kappa_{m_1}(\eta)) \to f(\kappa_{m_1}(\eta))$ as $n \to \infty$, there exists a positive integer $n_0$ such that $n_0 \geq m_1$ and
\[ |f_n(\kappa_{m_1}(\eta)) - f(\kappa_{m_1}(\eta))| < \epsilon/4, \quad \forall \ n \geq n_0. \]  \tag{5.9}

Combining (5.7)–(5.9), we have $f_n(\kappa_n(\eta)) \in (\epsilon/4, 1 - \epsilon/4)$ for all $n \geq n_0$. Using (4.3) and (4.5), for $n \geq n_0$, we have
\[ R_n(\eta, \Psi) \geq \frac{f_n(\kappa_n(\eta))^2}{E_{X_n}[f_n]} > \frac{\epsilon^2}{16} R_n(\xi, \Psi). \]  \tag{5.10}

Hence $R(\eta, \Psi) > 0$ by passing limit.

To prove $R(\xi, \Psi \cup \{\eta\}) > 0$, let $g_n \in \ell(X_n)$ be the energy minimizer in (4.10) with $\Phi = \{\eta\}$. By passing to subsequence if necessary, we let $\gamma_{1,n} = f_n(\kappa_n(\eta))$ and $\gamma_{2,n} = g_n(\kappa_n(\xi))$. Then $\gamma_{1,n}\gamma_{2,n} \in [0, 1 - \epsilon/4]$ as $\gamma_{1,n} \in (\epsilon/4, 1 - \epsilon/4)$ (by last part) and $\gamma_{2,n} \in [0, 1]$. For $n \geq 1$, we can check that the function
\[ h_n := \frac{1}{1 - \gamma_{1,n}\gamma_{2,n}} f_n - \frac{\gamma_{1,n}}{1 - \gamma_{1,n}\gamma_{2,n}} g_n \in \ell(X_n) \]
satisfies $h_n(\kappa_n(\xi)) = 1$, and $h_n = 0$ on $\kappa_n(\Psi \cup \{\eta\})$. Moreover, $h_n$ is harmonic on $X_n \setminus \kappa_n(\Psi \cup \{\xi, \eta\})$, thus $E_{X_n}[h_n] = (R_n(\xi, \Psi \cup \{\eta\}))^{-1}$ by (4.5). Hence
\[ R(\xi, \Psi \cup \{\eta\}) = \lim_{n \to \infty} (E_{X_n}[h_n])^{-1} \geq \lim_{n \to \infty} \left( \frac{2E_{X_n}[f_n]}{(1 - \gamma_{1,n}\gamma_{2,n})^2} + \frac{2\gamma_{1,n}^2 E_{X_n}[g_n]}{(1 - \gamma_{1,n}\gamma_{2,n})^2} \right)^{-1} \]
\[ \geq \lim_{n \to \infty} \left( \frac{2}{(\epsilon/4)^2 R_n(\xi, \Psi)} + \frac{2(1 - \epsilon/4)^2}{(\epsilon/4)^2 R_n(\eta, \Psi)} \right)^{-1} \]
\[ = \left( \frac{2}{(\epsilon/4)^2 R(\xi, \Psi)} + \frac{2(1 - \epsilon/4)^2}{(\epsilon/4)^2 R(\eta, \Psi)} \right)^{-1} > 0. \]

**Theorem 5.6.** With the assumptions in Theorem 4.2, assume further $K$ is connected, and there exists $\beta > \alpha$ such that $\mathcal{D}_K(= \mathcal{D}^{(\beta)}_K)$ is non-trivial. Then $\beta^*_2 = \beta^*_3$.

**Proof.** It suffices to verify that for $\lambda \in (0, r^\alpha)$, dim $\mathcal{D}_K > 1$ (i.e., $\mathcal{D}_K$ contains nonconstant functions) implies that dim $\mathcal{D}_K = \infty$. We have $R(\xi, \eta) > 0$ for some $\xi, \eta \in K$ by Corollary 5.2 (ii) $\Rightarrow$ (iii). The energy minimizer $u_1 = u_{\xi, \eta} \in \mathcal{D}_K$ is continuous with $u_1(\xi) = 1$ and $u_1(\eta) = 0$, hence there exists $\eta_1 \in K$ such that
Lemma 5.7. With the same assumption as in Theorem 4.2, for a finite set $E \subset K$ with $\#E \geq 2$, if $R(\xi, \eta) > 0$ for all distinct $\xi \neq \eta$ in $E$, then $R(\xi, E \setminus \{\xi\}) > 0$ for all $\xi \in E$.

**Proof.** We prove the lemma by induction on $\#E$. It is trivial if $\#E = 2$. Suppose the lemma holds for $\#E = m$ ($m \geq 2$). Now let $\#E = m + 1$. We choose arbitrarily three distinct points $\xi_1, \xi_2, \xi_3 \in E$. Then it suffices to show that $R(\xi_1, E \setminus \{\xi_1\}) > 0$.

By induction hypothesis, we have three positive limiting resistances $R_1 := R(\xi_1, E \setminus \{\xi_1, \xi_2\})$, $R_2 := R(\xi_2, E \setminus \{\xi_2, \xi_3\})$ and $R_3 := R(\xi_3, E \setminus \{\xi_3, \xi_1\})$.

For sufficiently large $n$, let $f_{1,n}$, $f_{2,n}$, $f_{3,n} \in \ell(X_n)$ be the energy minimizer in (4.10) with $(\Phi, \Psi) = (\{\xi_1\}, E \setminus \{\xi_1, \xi_2\})$, $(\{\xi_2\}, E \setminus \{\xi_2, \xi_3\})$, $(\{\xi_3\}, E \setminus \{\xi_3, \xi_1\})$ respectively. Fix a $\kappa$-sequence $\{\kappa_n\}$. Let $\gamma_{1,n} = f_{1,n}(\kappa_n(\xi_2))$, $\gamma_{2,n} = f_{2,n}(\kappa_n(\xi_3))$, and $\gamma_{3,n} = f_{3,n}(\kappa(\xi_1))$. Then $\gamma_{i,n} \in [0,1]$ for $i = 1,2,3$. For sufficiently large $n$, we can check that the function

$$h_n := \frac{1}{1 + \gamma_{1,n}\gamma_{2,n}\gamma_{3,n}}(f_{1,n} - (\gamma_{1,n}f_{2,n} + \gamma_{1,n}\gamma_{2,n}f_{3,n}))$$

(5.11)
satisfies \( h_n(\kappa_n(\xi_1)) = 1 \), and \( h_n = 0 \) on \( \kappa_n(E \setminus \xi_1) \). Moreover, \( h_n \) is harmonic on \( X_n \setminus \kappa_n(E) \), thus \( \mathcal{E}_{X_n}[h_n] = (R_n(\xi_1, E \setminus \{\xi_1\}))^{-1} \) by [16, 5]. Hence

\[
R(\xi_1, E \setminus \{\xi_1\}) = \lim_{n \to \infty} (\mathcal{E}_{X_n}[h_n])^{-1} \\
\geq \lim_{n \to \infty} \left( \frac{3(\mathcal{E}_{X_n}[f_{1,n}] + \gamma^2_{1,n} \mathcal{E}_{X_n}[f_{2,n}] + \gamma_{1,n}^2 \gamma_{2,n}^2 \mathcal{E}_{X_n}[f_{3,n}])}{(1 + \gamma_{1,n} \gamma_{2,n} \gamma_{3,n})^2} \right)^{-1} \\
= \left( \frac{3(R_{1}^{-1} + \gamma_{1,n}^2 R_{2}^{-1} + \gamma_{1,n}^2 \gamma_{2,n}^2 R_{3}^{-1})}{(1 + \gamma_{1,n} \gamma_{2,n} \gamma_{3,n})^2} \right)^{-1} > 0.
\]

This completes the proof of the induction. \( \square \)

Following Kigami [24], for an IFS \( \{S_j\}_{j=1}^N \) with a self-similar set \( K \), we let \( C_K = \bigcup_{i \in \Sigma_i \neq j}(S_i(K) \cap S_j(K)) \), and define a critical set by \( \mathcal{C} = \pi^{-1}(C_K) \), a post critical set by \( \mathcal{P} = \bigcup_{n \geq 1} \sigma^n(\mathcal{C}) \). We call \( K \) post critically finite (p.c.f.) if \( \mathcal{P} \) is a finite set.

It is known that for the similitudes \( S_j = r_j(R_j x + b_j) \), \( j = 1, \ldots, N \), if the \( \{R_j\}_{j=1}^N \) are commensurable, then the p.c.f. property implies the OSC [9], and the statement is not true without the commensurable assumption [39]. We introduce two geometric conditions on the p.c.f. sets:

(C) for any family of distinct subcells \( S_{i_1}(K), \ldots, S_{i_k}(K) \) that intersects at a point \( p \), there exists \( 0 < \delta < 1 \) and closed cones \( \mathcal{C}_j, 1 \leq j \leq k \) with vertex at \( p \) such that

\[
S_{i_j}(K) \cap B(p, \delta) \subset \mathcal{C}_j, \quad \text{and} \quad \mathcal{C}_j \cap \mathcal{C}_\ell = \{p\} \quad \forall \ 1 \leq j, \ \ell \leq k, \ j \neq \ell;
\]

(H) there exists constant \( \gamma > 0 \) such that for any \( x, y \in X \) with \( |x| = |y| \), if \( S_x(K) \cap S_y(K) = \emptyset \), then

\[
dist(S_x(K), S_y(K)) > \gamma \cdot r^{|x|}.
\]

Condition (C) says the intersecting cells are separated by closed cones (except at the vertices), and the geometric meaning is clear. Condition (H) says that if two cells are disjoint, then they are “strongly” separate; it has been used in [21], [28] and [19]. Note that the familiar self-similar sets satisfies this condition, and it is proved in [19] the if the IFS is of the form \( S_j(x) = r(x + b_j) \) and is p.c.f., then \( K \) satisfies condition (H).

**Lemma 5.8.** Let \( K \) be a p.c.f. self-similar set that satisfies either (C) or (H). Suppose for \( \alpha < \beta < \beta' \), \( u \) satisfies \( u \circ S_i \in \Lambda_{2,2}^{\alpha,\beta/2} \) for each \( i \in \Sigma \), then \( u \in \Lambda_{2,2}^{\alpha,\beta/2} \).
Proof. First suppose that $K$ satisfies (C). By the separation of the cones, and the cosine law of a triangle, we can show that there exists $c > 0$ such that if $S_i(K)$ intersects $S_j(K)$ at $p$, and for $\xi \in S_i(K) \cap B(p, \delta)$, $\eta \in S_j(K) \cap B(p, \delta)$,

$$|\xi - \eta| \geq c(|\xi - p| + |\eta - p|) \geq 2c|\xi - p|^{1/2} \cdot |\eta - p|^{1/2}. \quad (5.12)$$

Since $u \circ S_i \in \Lambda_{2,2}^{\alpha,\beta'/2}$, it follows from Theorem 2.2 that $u \circ S_i \in C^{(\beta' - \alpha)/2}(K)$. As $u(\xi) = \sum_{i=1}^{N} u(\xi) \chi_{S_i(K)}(\xi)$, we show that $u$ is also Hölder continuous of order $(\beta' - \alpha)/2$ at any $p \in S_i(K) \cap S_j(K)$. Indeed we observe that for $\xi \in S_i(K) \cap B(p, \delta), \eta \in S_j(K) \cap B(p, \delta)$,

$$|u(\xi) - u(\eta)| \leq |u(\xi) - u(p)| + |u(\eta) - u(p)| \leq C(|\xi - p|^{(\beta' - \alpha)/2} + |\eta - p|^{(\beta' - \alpha)/2}) \leq 2C(|\xi - p| + |\eta - p|)^{(\beta' - \alpha)/2} \leq C_1|\xi - \eta|^{(\beta' - \alpha)/2} \quad (by \ (5.12)).$$

This together with (5.12) imply

$$\int_{S_i(K) \cap B(p, \delta)} \int_{S_j(K) \cap B(p, \delta)} \frac{|u(\xi) - u(\eta)|^2}{|\xi - \eta|^{\alpha + \beta}} d\nu(\xi) d\nu(\eta) \leq C_2 \int_{S_i(K) \cap B(p, \delta)} \frac{d\nu(\xi)}{|\xi - p|^{\alpha + \beta - \beta'}} \int_{S_j(K) \cap B(p, \delta)} \frac{d\nu(\eta)}{|\eta - p|^{\alpha + \beta' - \beta}} < \infty. \quad (5.13)$$

Now as $u(\xi) = \sum_{i=1}^{N} u(\xi) \chi_{S_i(K)}(\xi)$, we have

$$\mathcal{E}_K[u] = \sum_{i,j=1}^{N} \int_{S_i(K)} \int_{S_j(K)} \frac{|u(\xi) - u(\eta)|^2}{|\xi - \eta|^{\alpha + \beta}} d\nu(\xi) d\nu(\eta) = (\sum_{i,j=1}^{N} + \sum_{i \neq j}) \int_{S_i(K)} \int_{S_j(K)} \frac{|u(\xi) - u(\eta)|^2}{|\xi - \eta|^{\alpha + \beta}} d\nu(\xi) d\nu(\eta) := S_I + S_{II}.$$

By a change of variable,

$$S_I = \sum_{i=1}^{n} \int_{K} \int_{K} \frac{|u \circ S_i(\xi) - u \circ S_i(\eta)|^2}{|\xi - \eta|^{\alpha + \beta}} d\nu(\xi) d\nu(\eta) < \infty.$$

By (5.13), it is easy to check that $S_{II} < \infty$. This shows that $\mathcal{E}_K[u] < \infty$, so that $u \in \mathcal{D}_K = \Lambda_{2,2}^{\alpha,\beta'/2}$.

Next we suppose that $K$ satisfies (H). Assume without loss of generality that $\text{diam}(K) = 1$. For $p \in S_i(K) \cap S_j(K), i \neq j \in \Sigma$, let $\delta = \frac{1}{2} \min\{|p - q| : q \in S_i(K) \cap
$S_j(K), q \neq p \}$. Following the same argument in the last paragraph, it suffices to show that (5.12) holds for $\xi \in S_j(K) \cap B(p, \delta)$ and $\eta \in S_j(K) \cap B(p, \delta)$. Indeed, suppose that $|\eta - p| \leq |\xi - p| \in (r^k, r^{k-1}]$ for some positive integer $k$. Let $x, y \in J_k$ with $\xi \in S_x(K) \subset S_i(K)$ and $\eta \in S_y(K) \subset S_j(K)$. As $\text{diam}(S_x(K)), \text{diam}(S_y(K)) \leq r^k$, $S_x(K) \cap S_y(K) = \emptyset$. Hence by condition (H),

$$|\xi - \eta| \geq \gamma \cdot r^k \geq \gamma r |\xi - p| \geq \frac{\gamma r}{2} (|\xi - p| + |\eta - p|).$$

This completes the proof. □

We let $V_0 = \pi(P)$ be the “boundary” of a p.c.f. set $K$, and let $V_n = \cup_{x \in \Sigma^n} S_x(V_0)$, $n \geq 1$.

**Theorem 5.9.** With the same assumption as in Theorem 4.2, assume further $K$ is a p.c.f. set with boundary $V_0$ and satisfies (C) or (H). Suppose

$$R^{(\lambda - \varepsilon)}(\xi, \eta) > 0, \quad \forall \xi \neq \eta \in V_0,$$

for some $\varepsilon \in (0, \lambda)$, then $D_K$ is dense in $C(K)$ (with the supremum norm).

Consequently, $\beta^*_1 = \log \lambda_1^* / \log r$ if

$$\lambda^*_1 := \inf \{ \lambda > 0 : R^{(\lambda)}(\xi, \eta) > 0, \forall \xi \neq \eta \in V_0 \} \in (0, r^\alpha),$$

and $\beta^*_1 \leq \alpha$ otherwise.

**Proof.** Let $V_0 = \{\xi_1, \xi_2, \ldots, \xi_m\}$. If $R(\xi_i, \xi_j) = 0$ for some $i \neq j$, then $D_K$ is not dense in $C(K)$ by Corollary 5.3. Now suppose that (5.14) holds and let $\beta_0 = \log(\lambda - \varepsilon) / \log r$. Then $R^{(\lambda - \varepsilon)}(\xi_i, V_0 \setminus \{\xi_i\}) > 0$ for all $i$ by Lemma 5.7. Thus we can obtain a “basis” of functions $\{u_i\}_{1 \leq i \leq m} \subset \Lambda^{\alpha, \beta_0/2}_1$ with $u_i(\xi_j) = \delta_{ij}$ following from Proposition 5.1. Using linear combinations, for any $v \in \ell(V_0)$, one can check that $u = \sum_{i=1}^m v(\xi_i)u_i \in \Lambda^{\alpha, \beta_0/2}_1 \subset D_K$ satisfies $u|_{V_0} = v$.

We use induction on $n$ to claim that for $\beta_n = \log(\lambda - \frac{\varepsilon}{r^n}) / \log r$ and for any $v \in \ell(V_n)$, there exists $u \in \Lambda^{\alpha, \beta_n/2}_1 \subset D_K$ such that $u|_{V_n} = v$. Indeed, the above verifies the case $n = 0$. Assume the statement holds for some $n$. Let $v \in \ell(V_{n+1})$. Note that $V_n = S_i^{-1}(V_{n+1} \cap S_i(K))$ for all $i \in \Sigma$. By induction hypothesis, for each $i$, there exists $w_i \in \Lambda^{\alpha, \beta_n/2}_1$ such that $w_i|_{V_n} = v|_{V_{n+1} \cap S_i(K) \circ S_i}$. Let $u(\xi) = \sum_{i=1}^N (w_i \circ S_i^{-1})(\xi) \chi(S_i(K))(\xi)$. Then $u|_{V_{n+1}} = v$ and $u \circ S_i = w_i \in \Lambda^{\alpha, \beta_n/2}_1$. By Lemma 5.8, $u \in \Lambda^{\alpha, \beta_n/2+1/2}_1 \subset D_K$. This completes the proof of induction.

As $n$ tends to infinity, $\beta_n$ decreases to $\beta = \log \lambda / \log r$, and $\bigcup_{n \geq 0} V_n$ is dense in $K$. Hence $D_K = \Lambda^{\alpha, \beta_2/2}_1$ is dense in $C(K)$.

□

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6 Network reduction and examples

In this section, we will provide a device to calculate the limiting resistances and the critical exponents of the Besov spaces on $K$. We first recall some formal notions and techniques on electric network theory [10,30].

Let $\mathcal{N} = (V, c)$ denote the (electric) network with vertex set $V$ (finite or countably infinite) and conductance $c : V \times V \to [0, \infty)$ ($c(x, y) = c(y, x)$ for all $x, y \in V$). The edge set $E = \{(x, y) \in (V \times V) \setminus \Delta : c(x, y) > 0\}$. An edge $(x, y) \in E$ is referred as a resistor (or conductor) with resistance $r_{xy} = r(x, y) = c(x, y)^{-1}$. The energy of $f \in \ell(V)$ on $\mathcal{N}$ is given by
\[
E_\mathcal{N}[f] = \frac{1}{2} \sum_{x, y \in V} c(x, y)(f(x) - f(y))^2
\]
as in (6.1). Also we can define the effective resistance $R_\mathcal{N}(A, B)$ between two nonempty subsets $A, B \subset V$ as in (4.3).

**Definition 6.1.** For two networks $\mathcal{N}_1 = (V_1, c_1)$ and $\mathcal{N}_2 = (V_2, c_2)$ with a set of common vertices $U \subset V_1 \cap V_2$, \#$U \geq 2$, we say that $\mathcal{N}_1$ and $\mathcal{N}_2$ are equivalent on $U$ if for any $f \in \ell(U)$,
\[
\inf\{E_{\mathcal{N}_1}[g_1] : g_1 \in \ell(V_1), g_1|_U = f\} = \inf\{E_{\mathcal{N}_2}[g_2] : g_2 \in \ell(V_2), g_2|_U = f\}. \tag{6.2}
\]

It is easy to show that if $\mathcal{N}_1$ and $\mathcal{N}_2$ are equivalent on $U$, then they are also equivalent on any $U' \subset U$. As a result, $R_{\mathcal{N}_1}(A, B) = R_{\mathcal{N}_2}(A, B)$ for any nonempty $A, B \subset U$.

The two most basic transformations to reduce networks to equivalent ones are the series law and the parallel law of resistance. The third one is the $\Delta$-Y transform (or star-triangle Law): let $\mathcal{N}_1$ be the triangle shaped network with $V_1 = \{x, y, z\}$ as on the left of Figure 1 and let $\mathcal{N}_2$ be the starlike network on the right with $V_2 = V_1 \cup \{p\}$; for the two network to be equivalent, the resistances are related by

\[
\begin{align*}
\Delta & \rightarrow Y \\
R_{xy} & = r_{xy} \\
R_{xz} & = r_{zx} \\
R_{yz} & = r_{yz}
\end{align*}
\]

![Figure 1: $\Delta$-Y transform](image-url)
\[ R_x = \frac{r_{xy}r_{zx}}{r_{xy} + r_{yz} + r_{zx}}, \quad R_y = \frac{r_{xy}r_{yz}}{r_{xy} + r_{yz} + r_{zx}}, \quad R_z = \frac{r_{zx}r_{yz}}{r_{xy} + r_{yz} + r_{zx}} \]

respectively. For some network \( \mathcal{N} = \{V, c\} \), \( \#V > 3 \) with proper symmetry, we can add one vertex and transform it to an equivalent starlike network (see the examples in the sequel and [25] for more details); we regard such transformation as a generalized \( \Delta-Y \) transform.

More generally, we have from Proposition 4.1, that if \( V = V^o \cup \partial V, \#\partial V \geq 2 \) then for \( f \in \ell(\partial V) \),

\[ \min \{ E_N[g] : g \in \ell(V), g|_{\partial V} = f \} = \frac{1}{2} \sum_{x,y \in \partial V, x \neq y} c^*(x, y)(f(x) - f(y))^2. \] (6.3)

Then the network \( \mathcal{N}_* = \{\partial V, c^*\} \) is equivalent to \( \mathcal{N} \) on \( \partial V \). For proper \( \partial V \), the graph of network \( \mathcal{N}_* \) may contain a complete subgraph \( K_n \). In this case, we say that the transform \( \mathcal{N} \to \mathcal{N}_* \) is a local completion. For example, as in Figure 2, let \( \partial V = \{x_1, x_2, \ldots, x_5\} \), then the graph of \( \mathcal{N}_* \) is a complete graph \( K_5 \).

![Figure 2: Local completion](image)

Besides the above mentioned transformations, there are other basic tools in network reduction we will use: cutting and shorting, and the Rayleigh’s monotonicity law, namely, if some resistances of resistors in a network are increased (decreased), then the effective resistance between any two points in the graph can only increase (decrease).

**Example 6.2. Cantor middle third set** Let \( S_1(\xi) = \frac{1}{3}\xi \) and \( S_2(\xi) = \frac{1}{3}(\xi + 2) \) on \( \mathbb{R} \). Then the self-similar set \( K \) is the Cantor middle-third set with ratio \( r = \frac{1}{3} \). It is totally disconnected and the Hausdorff dimension is \( \alpha = \frac{\log 2}{\log 3} \). The critical exponents \( \beta_1^* = \beta_2^* = \beta_3^* = \infty \).

Indeed, for \( \lambda \in (0, \frac{1}{2}) \) \( (r^\alpha = \frac{1}{2}) \), the limiting resistance between 0 and 1 (see Figure
The limiting resistance for Cantor set

\[ R^{(\lambda)}(0, 1) = R^{(\lambda)}(1^\infty, 2^\infty) = \lim_{n \to \infty} R^{(\lambda)}_n(1^n, 2^n) \]

\[
= \lim_{n \to \infty} \left( \sum_{k=1}^{n} c(1^k, 1^{k-1})^{-1} + \sum_{k=1}^{n} c(2^k, 2^{k-1})^{-1} \right) \\
= 2 \lim_{n \to \infty} \sum_{k=1}^{n} (2\lambda)^{-k} = \frac{4\lambda}{1 - 2\lambda},
\]

and Theorem 5.9 implies the result.

Example 6.3. Sierpiński gasket It is the self-similar set \( K \) generated by the maps \( S_i(\xi) = \frac{1}{2}(\xi - e_{i-1}) + e_i \) where \( e_0 = 0 \) and \( e_i, i = 1, \ldots, N-1 \) are the standard basis vectors in \( \mathbb{R}^{N-1} \). It is a p.c.f. set with \( \mathcal{P} = \{1^\infty, 2^\infty, \ldots, N^\infty\} \), and \( \alpha = \dim_H K = \frac{\log N}{\log 2} \). For the \( \lambda \)-NRW \( (r^\alpha = \frac{1}{N}) \), the conductance is \( c(x, x^-) = c(x, y) = (\lambda N)^{-|x|} \) where \( x \sim_h y \). The critical exponent of \( \Lambda^{\alpha, \beta/2}_{2, 2} \) is

\[ \beta_1^* = \beta_2^* = \beta_3^* = \frac{\log(N + 2)}{\log 2} \quad \text{at} \quad \lambda = \frac{1}{N + 2}. \]

(The critical exponent is known in [27].)
We only prove the case \( N = 3 \) (the other cases are quite similar; the reader is also advised to use \( N = 2 \) to get a clearer picture). By symmetry, it suffices to find the limiting resistance \( R^{(\lambda)}(1^\infty, 2^\infty) \). We denote \( R_n^{(\lambda)} = R_n^{(\lambda)}(1^n, 2^n) \) for short.

To estimate the upper bound, we delete the edges \((\vartheta, i), (ij^{k}, ji^{k})\), for \( i \neq j \in \Sigma\), \( k = 0, 1, \ldots, n-2\) in the subgraph of \( X_n \) (see Figure 4). Then we get a new subgraph consisting of 3 copies of \( X_{n-1} \) with 3 horizontal edges \((ij^{n-1}, ji^{n-1})\), \( i \neq j \in \Sigma\) at level \( n \) connecting them; we label these copies by 1, 2, 3 such that the copy \( i \) contains the vertex \( i^n \). Then apply the \( \Delta-Y \) transform to the three vertices in \( A_i := \{ij^{n-1} : j \in \Sigma\} \) at the \( n \)-th level of each copy to get a starlike tree with center \( i^n \), \( i \in \Sigma \) respectively. As the resistance between any pair of vertices in \( A_i \) equals \( 3\lambda R_{n-1} \), it follows that the resistance between \( i^n \) and a vertex in \( A_i \) in the corresponding starlike tree is \( \frac{3\lambda}{2} R_{n-1} \). Moreover, between any pair \( i^n, j^n \), \( i \neq j \), there is a 3-step path \([i^n, ij^{n-1}, ji^{n-1}, j^n]\). Replacing these paths with resistors, we get a triangle with vertices \( \{i^n \in \Sigma\} \) and each side has resistance \( 3\lambda R_{n-1} + (3\lambda)^n \).

By applying the monotonicity law and the series law,
\[
R_n^{(\lambda)} \leq R(1^n, 1^n) + R(1^n, 2^n) + R(2^n, 2^n) \\
= \frac{3\lambda}{2} R_{n-1} + \frac{2}{3} (3\lambda R_{n-1}^{(\lambda)} + (3\lambda)^n) + \frac{3\lambda}{2} R_{n-1}^{(\lambda)} \\
= 5\lambda R_{n-1}^{(\lambda)} + 2 \cdot 3^{n-1} \lambda^n.
\]
Hence \( R^{(\lambda)}(1^\infty, 2^\infty) = \lim_{n \to \infty} R_n^{(\lambda)} = 0 \) for \( \lambda \in (0, \frac{1}{3}) \). By Proposition 5.3 and Theorem 5.4, we have \( \beta_1^* \leq \beta_3^* \leq \frac{\log 5}{\log 2} \).

To obtain the lower bound of the critical exponent, we need another technique. We reassign the conductance on the \( n \)-th level of the subgraph \( X_n \) \((n \geq 1)\): for \( \mu > 0 \), let \( \overline{c}(x, x^-) = (3\lambda)^{-|x|} \) for \( x \in X_n \), and let
\[
\overline{c}(x, y) = \begin{cases} 
(3\lambda)^{-|x|}, & \text{if } |x| < n, \\
\mu^{-1}(3\lambda)^{-n}, & \text{if } |x| = n,
\end{cases}
\text{ for } x \sim_h y \in X_n.
\]
Denote the level-\( n \) resistance between \( 1^n \) and \( 2^n \) with respect to the above \( \overline{c} \) by \( R_n^{(\lambda, \mu)} \). Then apply the generalized \( \Delta-Y \) transforms to each triangle \((x, x_1, x_2, x_3)\) for \( x \in J_n-1 \), and then replace each pair \( \{x, x'\} \) by a single \( x \) (see Figure 5 for \( N = 2 \) for a clearer illustration; Figure 6 for \( N = 3 \) corresponds to the dotted box in Figure 5).

We have
\[
R_n^{(\lambda, \mu)} \geq \frac{2\mu}{\mu + 3} (3\lambda)^n + R_n^{(\lambda, \phi(\mu))},
\tag{6.4}
\]
where \( \phi \) is given by the parallel resistance formula
\[
\phi(\mu)^{-1} = \left[ 3\lambda \left( \frac{2\mu}{\mu + 3} + \mu \right) \right]^{-1} + 1.
\tag{6.5}
\]
The equation \( \phi(\mu) = \mu \) has a solution \( \mu \in (0, 1) \) if and only if \( \lambda > \frac{1}{5} \). With such fixed point \( \mu \), by (6.4), we have \( R(\lambda, \mu) \geq \lim_{n \to \infty} R_{n}(\lambda, \mu) \geq R_{1}(\lambda, \mu) > 0 \). By Theorem 5.7, we have \( \log 5 \leq \beta_{1} \leq \beta_{2} \) and completes the proof. \( \Box \)

In the next example, we adjust the above method slightly for the new situation with two different limiting resistances of \((i^{\infty}, j^{\infty})\).

Example 6.4. Pentagasket The pentagasket is the attractor \( K \) of the five similarities \( S_{i}(\xi) = \frac{3 - \sqrt{5}}{2}(\xi - p_{i}) + p_{i} \), here we identify \( \mathbb{R}^{2} \) with \( \mathbb{C} \), and \( p_{i} = e^{2\pi i/5} \). It is a p.c.f. set with \( P = \{1^{\infty}, 2^{\infty}, \ldots, 5^{\infty}\} \), and \( \alpha := \dim_{H} K = -\frac{\log 5}{\log((3 - \sqrt{5})/2)} \). As \( r^{\alpha} = \frac{1}{5} \), the \( \lambda \)-NRW has conductance \((5\lambda)^{-n}\) on level \( n \). The critical exponent is

\[
\beta_{1}^{*} = \beta_{2}^{*} = \beta_{3}^{*} = \frac{\log((\sqrt{161} - 9)/40)}{\log((3 - \sqrt{5})/2)} \quad \text{at} \quad \lambda = \frac{\sqrt{161} - 9}{40}.
\]

(The critical exponent is known in \([27]\).)

To determine the critical exponent, we need to calculate the limiting resistances

\[
R(\lambda)(1^{\infty}, 2^{\infty}) \quad \text{and} \quad R(\lambda)(1^{\infty}, 3^{\infty}).
\]
We denote $A_n = R_n^{(λ)}(1^n, 3^n)$ and $B_n = R_n^{(λ)}(1^n, 2^n)$ for short. By referring to Figure 7 and using the same technique as before, we have

$$A_n \leq R(1^n, 1^n) + R(1^n, 3^n) + R(3^n, 1^n)$$

$$= 5λ(2A_{n-1} - B_{n-1}) + [(10λB_{n-1} + 2(5λ)^n)^{−1} + (15λB_{n-1} + 3(5λ)^n)^{−1}]^{−1}$$

$$= 10λA_{n-1} + λB_{n-1} + \frac{6}{5}(5λ)^n.$$ 

Analogously, we have $B_n \leq 10λA_{n-1} - λB_n + \frac{4}{5}(5λ)^n$. As the coefficient matrix

$$
\begin{pmatrix}
10λ & λ \\
10λ & −λ
\end{pmatrix}
$$

has eigenvalues $\frac{9±\sqrt{161}}{2}λ$, we have $\lim_{n→∞} A_n = \lim_{n→∞} B_n = 0$ if $λ < (\frac{9+\sqrt{161}}{2})^{-1} = \frac{\sqrt{161}−9}{40}$. Hence $R^{(λ)}(1^n, 2^n) = R^{(λ)}(1^n, 3^n) = 0$ for $λ \in (0, \frac{\sqrt{161}−9}{40})$.

By Proposition 5.3 and Theorem 5.4 we have $β_3^∗ \leq β_2^∗ \leq \frac{\log((\sqrt{161}−9)/40)}{\log((\sqrt{5}/2)^{−1})}$.

To obtain the lower bound of the critical exponents, we reassign the conductance on the bottom of the subgraph $X_n$ ($n \geq 1$) with two parameters $μ_1$ and $μ_2$: for $μ_1, μ_2 \in (0, 1)$, let $\tilde{c}(x, x^-) = (5λ)^{−|x|}$ for $x \in X_n$, and let

$$\tilde{c}(x, y) = \begin{cases} (5λ)^{−|x|}, & \text{if } |x| < n, \\ μ_1^{−1}(5λ)^{−n}, & \text{if } |x| = n \text{ and } x^- = y^-, \text{ for } x ∼_h y \in X_n, \\ μ_2^{−1}(5λ)^{−n}, & \text{if } |x| = n \text{ and } x^- \neq y^-, \end{cases}$$

Denote the level-n resistance between $1^n$ and $3^n$ (or $2^n$) with respect to above $\tilde{c}$ by $A_n^{(μ_1, μ_2)}$ (or $B_n^{(μ_1, μ_2)}$). We apply the local completion to each cone $(x_1, x_{11}, x_{13}, x_{14})$, $(x_2, x_{22}, x_{24}, x_{25})$, $(x_{33}, x_{35}, x_{31})$, $(x_4, x_{44}, x_{41}, x_{42})$, $(x_5, x_{55}, x_{52}, x_{53})$ for $x \in J_{n-2}$, and then replace each complete subgraph $K_4$ by a starlike network with greater energy (Figure 8). By a direct calculation, the conductance $c_4$ in $K_4$ is given by

$$c_4(x_1, x_{11}) = \frac{μ_1 + 4}{μ_1 + 2}, \quad c_4(x_1, x_{13}) = c_4(x_1, x_{14}) = \frac{μ_1 + 3}{μ_1 + 2},$$

Figure 7: Cutting in pentagasket
Example 6.5. Cantor set \times interval

Let $\Sigma = \{1, 2, 3, 4, 5, 6\}$ and let $p_1 = 0, p_2 = (0, \frac{1}{2}), p_3 = (0, \frac{2}{3}), p_4 = (\frac{2}{3}, 0), p_5 = (\frac{2}{3}, \frac{1}{2}), p_6 = (\frac{2}{3}, \frac{2}{3})$ in $\mathbb{R}^2$. For $i \in \Sigma$, let $S_i(\xi) = \frac{1}{3} \xi + p_i$ on $\mathbb{R}$. Then the self-similar set $K$ is the product of a Cantor middle-third set and a unit interval (see the associated augmented tree in Figure 7), and $\alpha = \dim_H K = \frac{\log 2}{\log 3} + 1 = \frac{\log 6}{\log 3}$. The $\lambda$-NRW has conductance $(6\lambda)^{-n}$ on the $n$-th level ($r^\alpha = \frac{1}{6}$). The critical exponents are

$$\beta_1^* = 2 \quad \text{at} \quad \lambda = \frac{1}{9}; \quad \beta_2^* = \beta_3^* = \infty$$

and the resistances in the star are given by

$$\rho_1 = \frac{\mu_1(\mu_1 + 2)}{\mu_1^2 + 5\mu_1 + 5}, \quad \rho_2 = \frac{\mu_1(\mu_1 + 2)^2}{(\mu_1 + 1)(\mu_1^2 + 5\mu_1 + 5)}.$$
First we show that $R^{(\lambda)}(1^\infty, 4^\infty) > 0$ for any $\lambda > 0$. For $n \geq 1$, consider a function $f_n$ on $X_n$ defined by

$$f_n(x) = \begin{cases} 
1/2, & \text{if } x = \emptyset, \\
1, & \text{if } i_1 = 1, 2, 3, \text{ for } x = i_1 i_2 \cdots i_k \in X_n, \\
0, & \text{if } i_1 = 4, 5, 6,
\end{cases}$$

Then by 4.3, $R_n^{(\lambda)}(1^n, 4^n) \geq (E_{X_n}[f_n])^{-1} = (6 \cdot (\frac{1}{2})^2 \cdot \frac{1}{\lambda})^{-1} = 4\lambda$. Thus for any $\lambda > 0$, $R^{(\lambda)}(1^\infty, 4^\infty) = \lim_{n \to \infty} R_n^{(\lambda)}(1^n, 4^n) \geq 4\lambda > 0$. By Theorem 5.4 we have $\beta^*_3 = \infty$. Also it is easy to see that $\beta^*_2 = \infty$.

Next we consider the limiting resistance $R^{(\lambda)}(1^\infty, 3^\infty)$ by using a similar shorting device as in previous examples. Denote $R_n^{(\lambda, \mu)} = R_n^{(\lambda)}(1^n, 3^n)$ for short. As in Example 6.4, we reassign the conductance on the bottom of the subgraph $X_n$ by an additional factor $\mu^{-1}$, and by the same method applied to triangles $(x, x_1, x_3)$ (also to $(x, x_4, x_6)$, see Figure 10), we have

$$R_n^{(\lambda, \mu)} \geq \frac{2\mu}{\mu + 1} (6\lambda)^n + R_n^{(\lambda, \phi(\mu))}, \quad (6.8)$$
where \( \phi \) is given by

\[
\phi(\mu)^{-1} = 2 \left[ 6\lambda \left( \frac{2\mu}{\mu + 1} + \mu \right) \right]^{-1} + 1. \tag{6.9}
\]

The equation \( \phi(\mu) = \mu \) has a solution \( \overline{\mu} \in (0,1) \) if and only if \( \lambda > \frac{1}{3} \).

With such fixed point \( \overline{\mu} \), by (6.8), we have \( R^{(\lambda)}(1^\infty,3^\infty) \geq \lim_{n \to \infty} R_n^{(\lambda,\overline{\mu})} \geq R_1^{(\lambda,\overline{\mu})} > 0 \). On the other hand, we show that if \( R^{(\lambda)}(1^\infty,3^\infty) > 0 \), then \( \lambda \geq \frac{1}{9} \). Without loss of generality, we assume that \( 0 < \lambda < 1/6 \). For \( n \geq 1 \), let \( f_n \) be the energy minimizer (harmonic function) on \( X_n \) with boundary conditions \( f_n(1^n) = 1 \) and \( f_n(3^n) = 0 \). Then \( R_n(1^n,3^n) = \mathcal{E}_{X_n}[f_n]^{-1} \). By Corollary 5.2 (iv) \( \Rightarrow \) (iii), let \( C_1 := \sup_{n \geq 1} \mathcal{E}_{X_n}[f_n] = \left( \inf_{n \geq 1} R_n(1^n,3^n) \right)^{-1} < \infty \). Pick a positive integer \( n_1 \) such that \( \sum_{n=n+1}^{n+1}(6\lambda)^n < \frac{1}{36C_1} \). Then for \( n \geq n_1 \),

\[
|f_n(1^n) - f_n(1^{n_1})|^2 \leq \mathcal{E}_{X_n}[f_n] R_{X_n}(1^n,1^{n_1}) \leq C_1 \sum_{k=n+1}^{n}(6\lambda)^k \leq \frac{1}{36}, \tag{6.10}
\]

which implies \( f_n(1^{n_1}) \geq \frac{5}{6} \). Analogously we have \( f_n(3^{n_1}) \leq \frac{1}{6} \). Let \( m = n - n_1 \). With a similar argument as in (6.10), for \( z \in \{1,4\}^m \),

\[
|f_n(1^{n_1}z) - f_n(1^{n_1})|^2 \leq \mathcal{E}_{X_n}[f_n] R_{X_n}(1^{n_1}z,1^{n_1}) \leq \frac{1}{36},
\]

which implies \( f_n(1^{n_1}z) \geq \frac{2}{3} \). Analogously we have \( f_n(3^{n_1}w) \leq \frac{1}{3} \) for all \( w \in \{3,6\}^m \). Now, for \( z = i_1 i_2 \cdots i_m \in \{1,4\}^m \), denote the word \( j_1 j_2 \cdots j_m \in \{3,6\}^m \) with \( j_k = i_k + 2 \) for all \( k \) by \( z' \). Note that for each \( z \in \{1,4\}^m \), there is a horizontal path with length \( 3^m - 1 \) from \( 1^{n_1}z \) to \( 3^{n_1}(z') \). The resistance on such path is given by \( R_{X_n}(1^{n_1}z,3^{n_1}(z')) = (3^m - 1)(6\lambda)^n \). Counting the energy on these \( 2^m \) disjoint horizontal paths, we get

\[
C_1 \geq \mathcal{E}_{X_n}[f_n] \geq \sum_{z \in \{1,4\}^m} \left[ f_n(1^{n_1}z) - f_n(3^{n_1}(z')) \right]^2 \geq \frac{2^{m-n_1}}{9(3^m - 1)(6\lambda)^n}
\]

for arbitrary \( n \geq n_1 \). Hence \( \lambda \geq \frac{1}{9} \) and the claim follows. By Proposition 5.3, we have \( \beta^*_1 = 2 \). \( \Box \)

**Remark.** To investigate the situation that \( \beta^* \neq \beta^*_3 \), it is natural to study the products of self-similar sets. But in general, if \( K_1 \) and \( K_2 \) are connected self-similar sets, then the critical exponent of the product \( K_1 \times K_2 \) satisfies

\[
\beta^*_1 \leq \max\{\dim_H K_1, \dim_H K_2\} + 1 \leq \dim_H K_1 + \dim_H K_2 = \alpha.
\]

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Although the criteria in the last section cannot be applied directly, it still has a similar link between the effective resistance of $E_X$ and the energy on the product (see [25] for more details). For example, in the product $[0,1] \times SG$, the limiting resistances $R^{(i)}(i^\infty,j^\infty)$ have two critical exponents $\lambda_1^* = \frac{1}{3}$ and $\lambda_3^* = \frac{1}{5}$ for various $i,j$, while $2 = \beta_1^* < \frac{\log 5}{\log 2} = \beta_3^* < \alpha = \frac{\log 6}{\log 2}$. With a similar technique as in Example 6.5 it follows that $\beta_1^* = 2$ if one of $K_i$ is a unit interval. To generalize the results above, we may leave a conjecture as
\[
\beta_1^*(K_1 \times K_2) = \min\{\beta_1^*(K_1), \beta_1^*(K_2)\}, \quad \text{and} \quad \beta_3^*(K_1 \times K_2) = \max\{\beta_3^*(K_1), \beta_3^*(K_2)\}.
\]

7 Remarks and open problems

The calculation of the critical exponents in Section 6 depends very much on the p.c.f. property. It is challenging to find an effective technique to estimate the non-p.c.f. sets like the Sierpiński carpet.

In our discussions, we assumed the return ratio $\lambda \in (0,r^\alpha)$ (hence $\alpha < \beta_1^*$) in order to guarantee functions in the domain of the induced bilinear form on $K$ are continuous (Proposition 2.5). While the condition is satisfied by the well-known fractals, it also excludes the situation that $\beta_1^* \leq \alpha$, which contains important examples (e.g., the classical domain, and product of fractals). We conjecture that the consideration in the paper is possible to adjust to this case. We also like to know if there is a nice sufficient condition for $\alpha < \beta_1^*$ based on the geometry of the self-similar sets.

We call a self-similar set $K$ mono-critical if it has a single critical exponent $\beta^* = \beta^*(K)$, i.e., $\beta^* = \beta_1^* = \beta_3^*$. It is known that all nested fractals, Cantor-type sets, and some non-p.c.f. sets including Sierpiński carpet are mono-critical (see [3–5]). For these sets, the critical exponent plays an important role. It is well-known that $\Lambda_{2, 2}^{\alpha, \beta^*/2}$ is trivial (see [21,31]) while $\Lambda_{2, \infty}^{\alpha, \beta^*/2}$ admits a local regular Dirichlet form on $L^2(K)$. On the other hand, it is constructed in [19] a modified Vicsek set that is mono-critical; on this set, $\Lambda_{2, \infty}^{\alpha, \beta^*/2}$ is dense in $L^2(K, \nu)$, but is not dense in $C(K)$, and there is a local regular Dirichlet form on $K$ which does not define on $\Lambda_{2, \infty}^{\alpha, \beta^*/2}$ or satisfies the energy self-similar identity in [24].

In conclusion, the question of constructing a local Dirichlet form on a self-similar set is still unsettled. It has much to do with the functional behavior of the Besov spaces at the critical exponents. Our study offers an alternative approach of using the return ratio $\lambda$ of the random walk and the induced Dirichlet form to study these critical cases. It will be interesting to carry out this initiation to a greater extension.
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