A collection of integrable systems
of the Toda type in continuous and discrete time,
with $2 \times 2$ Lax representations

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Abstract. A fairly complete list of Toda–like integrable lattice systems, both in the continuous and discrete time, is given. For each system the Newtonian, Lagrangian and Hamiltonian formulations are presented, as well as the $2 \times 2$ Lax representation and $r$–matrix structure. The material is given in the "no comment" style, in particular, all proofs are omitted.
0 Introduction

This paper contains almost exclusively a large number of formulas presented in the "no comment"-style. I hope that they will speak for themselves and tell once more the story on how much contain two simple systems: the Toda lattice and the relativistic Toda lattice. I have only tried to collect many closely related results in the form that makes the interrelations quite transparent. All proofs are omitted, because they consist of a direct verification.

I present a large number of integrable Newtonian systems, both in the continuous and discrete time, and give for all of them the equivalent Lagrangian and, for the continuous time systems, also Hamiltonian formulations (generalities are given in the Sect. 1), and also present $2 \times 2$ Lax representations. All these systems are reparametrizations of the Toda and the relativistic Toda lattices or their discrete time analogs (reminded in the Sect. 2, 3). For these systems the "big" Lax representations are well known (they include $N \times N$ matrices for the $N$–particle lattices). So, the corresponding reparametrizations immediately imply the "big" Lax representations for all our systems. In principle, the transition from the "big" Lax representation to the $2 \times 2$ one is well understood, at least at the formal level: they correspond to two different ways to represent a spectral problem connected with the second order linear difference operator. However, such transition is not uniquely defined, it may be performed in infinitely many different ways leading to different but gauge equivalent $2 \times 2$ Lax matrices. The problem is to find the gauge leading to the most nice Lax matrices. In particular, it is usually required that the matrix $L_k$ depends only on $x_k, p_k$, the coordinates and momenta of the $k$th particle. To find such gauge is sometimes a nontrivial task. I give here a collection of Lax representations having this property.

The last Section contains several historical remarks.

1 Newtonian, Lagrangian, and Hamiltonian systems in continuous and discrete time

In what follows we consider continuous time lattice systems in the Newtonian form:

$$\ddot{x}_k = F_k(x, \dot{x})$$

(1.1)

They all may be put into the Euler–Lagrange form

$$\frac{d}{dt} \frac{\partial \mathcal{L}(x, \dot{x})}{\partial \dot{x}_k} + \frac{\partial \mathcal{L}(x, \dot{x})}{\partial x_k} = 0$$

(1.2)
We will call the Lagrangian formulation of such equations the system
\[
\begin{align*}
p_k &= \frac{\partial \mathcal{L}(x, \dot{x})}{\partial \dot{x}_k} \\
\dot{p}_k &= -\frac{\partial \mathcal{L}(x, \dot{x})}{\partial x_k}
\end{align*}
\] (1.3)

The Hamiltonian formulation of these equations:
\[
\begin{align*}
\dot{x}_k &= \frac{\partial H(x, p)}{\partial p_k} \\
\dot{p}_k &= -\frac{\partial H(x, p)}{\partial x_k}
\end{align*}
\] (1.4)

The well–known Legendre transform gives a relation between the Lagrange and the Hamilton functions:
\[
H(x, p) = \sum_k \dot{x}_k p_k - \mathcal{L}(x, \dot{x})
\] (1.5)

A discrete time analog of the Euler–Lagrange equations (1.2):
\[
\frac{\partial}{\partial x_k} \left( \Lambda(\bar{x}, x) + \Lambda(x, \bar{x}) \right) = 0
\] (1.6)

We will call the Lagrangian formulation of such equations the system
\[
\begin{align*}
p_k &= -\frac{\partial \Lambda(\bar{x}, x)}{\partial x_k} \\
\bar{p}_k &= \frac{\partial \Lambda(\bar{x}, x)}{\partial \bar{x}_k}
\end{align*}
\] (1.7)

The Hamiltonian formulation in the discrete time case is not well–defined. However, for the systems related to the relativistic Toda hierarchy, we will find nice approximations of the Hamiltonian equations of motion (1.4).

2 Simplest flow of the Toda hierarchy and its bi–Hamiltonian structure

The simplest flow of the Toda hierarchy (hereafter denoted by TL) is:
\[
\begin{align*}
\dot{a}_k &= a_k (b_{k+1} - b_k), \\
\dot{b}_k &= a_k - a_{k-1}
\end{align*}
\] (2.1)

Its discretization (called hereafter dTL):
\[
\begin{align*}
\bar{a}_k &= a_k \frac{\beta_{k+1}}{\beta_k}, \\
\bar{b}_k &= b_k + h \left( \frac{a_k}{\beta_k} - \frac{a_{k-1}}{\beta_{k-1}} \right)
\end{align*}
\] (2.2)
where $\beta_k = \beta_k(a, b)$ are defined as a unique set of functions satisfying the relations

$$\beta_k = 1 + hb_k - \frac{h^2a_{k-1}}{\beta_{k-1}} = 1 + hb_k + O(h^2) \quad (2.3)$$

Both the TL and the dTL are bi–Hamiltonian. The first ("linear") invariant Poisson bracket:

$$\{a_k, b_k\}_1 = -\{a_k, b_{k+1}\}_1 = a_k \quad (2.4)$$

The second ("quadratic") one:

$$\{a_k, a_{k+1}\}_2 = -a_{k+1}a_k, \quad \{b_k, b_{k+1}\}_2 = -a_k \quad \{a_k, b_{k+1}\}_2 = -a_kb_{k+1} \quad (2.5)$$

The Hamilton functions generating the flow TL in these brackets:

$$H_1(a, b) = \frac{1}{2} \sum_k b_k^2 + \sum_k a_k, \quad H_2(a, b) = \sum_k b_k \quad (2.6)$$

3 Simplest flows of the relativistic Toda hierarchy and their bi–Hamiltonian structure

The first flow of the relativistic Toda hierarchy (denoted hereafter RTL+):

$$\dot{d}_k = d_k(c_k - c_{k-1}), \quad \dot{c}_k = c_k(d_{k+1} + c_{k+1} - d_k - c_{k-1}) \quad (3.1)$$

The second flow of the relativistic Toda hierarchy (denoted hereafter RTL−):

$$\dot{d}_k = d_k \left( \frac{c_k}{d_kd_{k+1}} - \frac{c_{k-1}}{d_{k-1}d_k} \right), \quad \dot{c}_k = c_k \left( \frac{1}{d_k} - \frac{1}{d_{k+1}} \right) \quad (3.2)$$

An integrable discretization of the flow RTL+ (denoted hereafter dRTL+):

$$\bar{d}_k = d_k \frac{a_{k+1} - hd_{k+1}}{a_k - hd_k}, \quad \bar{c}_k = c_k \frac{a_{k+1} + hc_{k+1}}{a_k + hc_k} \quad (3.3)$$

where $a_k = a_k(c, d)$ is defined as a unique set of functions satisfying the relations

$$a_k = 1 + h d_k + \frac{h c_{k-1}}{a_{k-1}} = 1 + h(d_k + c_{k-1}) + O(h^2) \quad (3.4)$$

An integrable discretization of the flow RTL− (denoted hereafter dRTL−):

$$\bar{d}_k = d_{k+1} \frac{d_k - hd_{k-1}}{d_{k+1} - hd_k}, \quad \bar{c}_k = c_{k+1} \frac{c_k + h d_k}{c_{k+1} + h d_{k+1}} \quad (3.5)$$
where \( \varphi_k = \varphi_k(c, d) \) is defined as a unique set of functions satisfying the relations

\[
\varphi_k = \frac{c_k}{d_k - h - h \varphi_{k-1}} = \frac{c_k}{d_k} + O(h) \tag{3.6}
\]

An integrable difference equation (hereafter called explicit dRTL):

\[
\tilde{d}_k = d_{k-1}\left(\frac{d_k + c_k}{d_{k-1} + c_{k-1}}\right), \quad \tilde{c}_k = c_k\left(\frac{d_{k+1} + c_{k+1}}{d_k + c_k}\right) \tag{3.7}
\]

This map is not close to the identity and therefore cannot serve as a discretization of some flow of the RTL hierarchy. However, it becomes related to the flow TL, if

\[
d_k \approx 1 + hb_k, \quad c_k \approx h^2 a_k \tag{3.8}
\]

(see Sect. 7 for details).

The flows RTL\( \pm \) and the maps dRTL\( \pm \) and explicit dRTL are bi–Hamiltonian. The first (linear) invariant Poisson bracket:

\[
\{c_k, d_{k+1}\}_1 = -c_k, \quad \{c_k, d_k\}_1 = c_k, \quad \{d_k, d_{k+1}\}_1 = c_k \tag{3.9}
\]

(only the non–vanishing brackets are written down), and Hamilton functions generating the flows (4.1), (4.2) in this bracket:

\[
H_1^{(+)}(c, d) = \frac{1}{2} \sum_k (d_k + c_k)^2 + \sum_k (d_k + c_k) c_k \tag{3.10}
\]

\[
H_1^{(-)}(c, d) = -\sum_k \log(d_k) \tag{3.11}
\]

The second (quadratic) invariant Poisson bracket:

\[
\{c_k, c_{k+1}\}_2 = -c_k c_{k+1}, \quad \{c_k, d_{k+1}\}_2 = -c_k d_{k+1}, \quad \{c_k, d_k\}_2 = c_k d_k \tag{3.12}
\]

the corresponding Hamilton functions:

\[
H_2^{(+)}(c, d) = \sum_k (d_k + c_k) \tag{3.13}
\]

\[
H_2^{(-)}(c, d) = \sum_k \frac{d_k + c_k}{d_k d_{k+1}} \tag{3.14}
\]

4 Systems TL and dTL:

**Parametrization of the linear bracket**

Canonical parametrization of the bracket \( \{\cdot, \cdot\}_1 \):

\[
a_k = e^{x_{k+1} - x_k}, \quad b_k = p_k \tag{4.1}
\]

The resulting system is the conventional Toda lattice.
4.1 The flow TL

Hamilton function:

\[ H(x, p) = H_1(a, b) = \frac{1}{2} \sum_k p_k^2 + \sum_k e^{x_k-x_{k-1}} \]  

(4.2)

Hamiltonian equations of motion:

\[ \begin{cases} 
\dot{x}_k = p_k \\
\dot{p}_k = e^{x_{k+1}-x_k} - e^{x_k-x_{k-1}} 
\end{cases} \]  

(4.3)

Newtonian equations of motion:

\[ \ddot{x}_k = e^{x_{k+1}-x_k} - e^{x_k-x_{k-1}} \]  

(4.4)

4.2 The map dTL

Lagrangian form of equations of motion:

\[ \begin{cases} 
hp_k = e^{\tilde{x}_k-x_k} - 1 + h^2 e^{x_k-\tilde{x}_{k-1}} \\
h\tilde{p}_k = e^{\tilde{x}_k-x_k} - 1 + h^2 e^{x_{k+1}-\tilde{x}_k} 
\end{cases} \]  

(4.5)

Newtonian form of equations of motion:

\[ e^{\tilde{x}_k-x_k} - e^{x_k-x_{k-1}} = h^2 \left( e^{x_{k+1}-x_k} - e^{x_k-\tilde{x}_{k-1}} \right) \]  

(4.6)

4.3 Lax representations and \( r \)-matrix structure

The both systems above allow a Lax representation with the Lax matrix

\[ L_k = \begin{pmatrix} p_k + \lambda & e^{x_k} \\
-e^{x_k} & 0 \end{pmatrix} \]  

(4.7)

This matrix satisfies the \( r \)-matrix ansatz

\[ \{ L_k(\lambda) \otimes L_j(\mu) \} = \left[ \frac{P}{\lambda - \mu}, L_k(\lambda) \otimes L_k(\mu) \right] \delta_{kj} \]  

(4.8)
with

\[ P = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 \\
\end{pmatrix} \quad (4.9) \]

which assures the complete integrability.

The Lax representation for the flow TL:

\[ \dot{L}_k = M_{k+1}L_k - L_kM_k \quad (4.10) \]

with

\[ M_k = \begin{pmatrix}
-\lambda & -e^{x_k} \\
e^{-x_k} & 0 \\
\end{pmatrix} \quad (4.11) \]

The Lax representation for the map dTL:

\[ \bar{L}_kV_k = V_{k+1}L_k \quad (4.12) \]

with

\[ V_k = \begin{pmatrix}
1 - h\lambda - h^2e^{x_k-x_{k-1}} - he^{x_k} \\
h e^{-x_{k-1}} & 1 \\
\end{pmatrix} \quad (4.13) \]

5 Systems TL and dTL:

Parametrization of the quadratic bracket

Canonical parametrization of the bracket \( \{\cdot, \cdot\}_2 \):

\[ a_k = e^{x_{k+1} - x_k + p_k}, \quad b_k = e^{p_k} + e^{x_k - x_{k-1}}. \quad (5.1) \]

The resulting system is the modified Toda lattice.

5.1 The flow TL

Hamilton function:

\[ H(x, p) = H_2(a, b) = \sum_k e^{p_k} + \sum_k e^{x_k - x_{k-1}} \quad (5.2) \]
Hamiltonian equations of motion:

\[
\begin{align*}
\dot{x}_k &= e^p_k \\
\dot{p}_k &= e^{x_{k+1} - x_k} - e^{x_k - x_{k-1}}
\end{align*}
\] (5.3)

Newtonian equations of motion:

\[
\ddot{x}_k = \dot{x}_k \left( e^{x_{k+1} - x_k} - e^{x_k - x_{k-1}} \right)
\] (5.4)

5.2 The map dTL

Lagrangian form of equations of motion.

\[
\begin{align*}
he^p_k &= \left( e^{\tilde{x}_k - x_k} - 1 \right) \left( 1 + he^{x_k - \tilde{x}_k - 1} \right) \\
he^{\tilde{p}_k} &= \left( e^{\tilde{x}_k - x_k} - 1 \right) \left( 1 + he^{x_{k+1} - \tilde{x}_k} \right)
\end{align*}
\] (5.5)

Newtonian form of equations of motion:

\[
\frac{\left( e^{\tilde{x}_k - x_k} - 1 \right)}{\left( e^{x_k - \tilde{x}_k} - 1 \right)} = \frac{\left( 1 + he^{x_{k+1} - x_k} \right)}{\left( 1 + he^{x_k - \tilde{x}_k - 1} \right)}
\] (5.6)

5.3 Lax representations and \( r \)-matrix structure

Lax representations for the both systems above include the Lax matrix

\[
L_k = \begin{pmatrix}
\lambda e^p_k - \lambda^{-1} & e^{x_k} \\
-e^{-x_k} & \lambda
\end{pmatrix}
\] (5.7)

satisfying the \( r \)-matrix ansatz

\[
\{ L_k(\lambda) \otimes L_j(\mu) \} = \left[ r(\lambda, \mu), L_k(\lambda) \otimes L_k(\mu) \right] \delta_{kj}
\] (5.8)

with

\[
r(\lambda, \mu) = \begin{pmatrix}
\frac{1}{2} \frac{\lambda^2 + \mu^2}{\lambda^2 - \mu^2} & 0 & 0 & 0 \\
0 & -\frac{1}{2} & \frac{\lambda \mu}{\lambda^2 - \mu^2} & 0 \\
0 & \frac{\lambda \mu}{\lambda^2 - \mu^2} & \frac{1}{2} & 0 \\
0 & 0 & 0 & \frac{1}{2} \frac{\lambda^2 + \mu^2}{\lambda^2 - \mu^2}
\end{pmatrix}
\] (5.9)
The Lax representation for the flow TL: (4.10) with

\[ M_k = \begin{pmatrix} \lambda^{-2} + e^{x_k - x_{k-1}} & -\lambda^{-1} e^{x_k} \\ \lambda^{-1} e^{-x_{k-1}} & 0 \end{pmatrix} \] (5.10)

The Lax representation for the map dTL: (4.12) with

\[ V_k = \begin{pmatrix} 1 + h\lambda^{-2} + he^{x_k - \bar{x}_{k-1}} & -h\lambda^{-1} e^{x_k} \\ h\lambda^{-1} e^{-\bar{x}_{k-1}} & 1 \end{pmatrix} \] (5.11)

6 Systems TL and dTL: Parametrization of the mixed bracket

For an arbitrary real \( \epsilon \) the linear combination \( \{\cdot, \cdot\}_1 + \epsilon\{\cdot, \cdot\}_2 \) allows the following canonical parametrization (a 1-parameter deformation of (4.1)):

\[ a_k = e^{x_{k+1} - x_k} + \epsilon p_k, \quad b_k = \frac{e^{\epsilon p_k} - 1}{\epsilon} + \epsilon e^{x_k - x_{k-1}} \] (6.1)

6.1 The flow TL

Hamilton function:

\[ H(x, p) = \epsilon^{-1} H_2(a, b) - \epsilon^{-1} \sum_k p_k = \sum_k e^{\epsilon p_k} - \frac{1}{\epsilon^2} + \sum_k e^{x_k - x_{k-1}} \] (6.2)

Hamiltonian equations of motion:

\[ \begin{align*}
\dot{x}_k &= \left( e^{\epsilon p_k} - 1 \right)/\epsilon \\
\dot{p}_k &= e^{x_{k+1} - x_k} - e^{x_k - x_{k-1}}
\end{align*} \] (6.3)

Newtonian equations of motion:

\[ \ddot{x}_k = (1 + \epsilon \dot{x}_k) \left( e^{x_{k+1} - x_k} - e^{x_k - x_{k-1}} \right) \] (6.4)
6.2 The map dTL

Lagrangian form of equations of motion:

\[
\begin{align*}
he^p_k &= \left( \epsilon \left( e^{x_k-x_{-1}} + h \right) \left( 1 + hee^{x_k-x_{-1}} \right) 
\right) \left( 1 + hee^{x_{k+1}-x_k} \right) \\
he^p_{\tilde{k}} &= \left( \epsilon \left( e^{\tilde{x}_k-x_{k}} + h \right) \left( 1 + hee^{x_{k+1}-\tilde{x}_k} \right) 
\right) \left( 1 + hee^{x_{-1}-x_k} \right)
\end{align*}
\]  

(6.5)

Newtonian form of equations of motion:

\[
\begin{align*}
\frac{\epsilon \left( e^{x_k-x_{-1}} + h \right) \left( 1 + hee^{x_{k+1}-x_k} \right)}{\epsilon \left( e^{x_{-1}-x_k} + h \right) \left( 1 + hee^{x_{k+1}-x_k} \right)} &= \left( 1 + hee^{x_k-x_{-1}} \right) \\
\left( 1 + hee^{x_{k+1}-x_k} \right) 
\end{align*}
\]  

(6.6)

A special case $\epsilon = h$. If the parameter $\epsilon$ coincides with the (small) stepsize $h$, then the previous discrete time system simplifies and serves as a discretization of the Toda lattice. The corresponding Lagrangian equations of motion:

\[
\begin{align*}
e^{hp_k} &= e^{x_k-x_{-1}} \left( 1 + h^2 ee^{x_{k+1}-x_k} \right) \\
e^{\tilde{p}_k} &= e^{x_k-x_{k}} \left( 1 + h^2 ee^{x_{k+1}-\tilde{x}_k} \right)
\end{align*}
\]  

(6.7)

Newtonian form of equations of motion:

\[
\begin{align*}
e^{x_k-x_{-1}} - 2x_k + x_k &= \left( 1 + h^2 ee^{x_{k+1}-x_k} \right) \\
\left( 1 + h^2 ee^{x_{k+1}-x_k} \right)
\end{align*}
\]  

(6.8)

6.3 Lax representations and $r$–matrix structure

The both systems above have Lax representations with the Lax matrix

\[
L_k = \begin{pmatrix}
\lambda e^p_k - \lambda^{-1} & e^{x_k} \\
-ee^{-x_k} & \lambda^2
\end{pmatrix}
\]  

(6.9)

which satisfies the following $r$–matrix ansatz:

\[
\{L_k(\lambda) \otimes L_j(\mu)\} = \epsilon \left( r(\lambda, \mu), L_k(\lambda) \otimes L_k(\mu) \right) \delta_{kj}
\]  

(6.10)
with the $r$–matrix (5.9).

The Lax representation for the flow TL: (4.10) with

$$M_k = \begin{pmatrix} \frac{\lambda^{-2} - 1}{\epsilon} + \epsilon e^{x_k} - x_{k-1} & -\lambda^{-1} e^{x_k} \\ \lambda^{-1} e^{-x_{k-1}} & 0 \end{pmatrix}$$

(6.11)

The Lax representation for the map dTL: (4.12) with

$$V_k = \begin{pmatrix} 1 - \frac{h}{\epsilon} + \frac{h}{\epsilon} \lambda^{-2} + h(\epsilon - h)e^{x_k} - \bar{x}_{k-1} & -h\lambda^{-1} e^{x_k} \\ h\lambda^{-1} e^{-\bar{x}_{k-1}} & 1 \end{pmatrix}$$

(6.12)

Special case $\epsilon = h$. In this case also the Lax representation of the map dTL simplifies significantly: it reads (4.12) with

$$L_k = \begin{pmatrix} \lambda e^{hp_k} - \lambda^{-1} & h e^{x_k} \\ -h e^{-x_k} & \lambda h^2 \end{pmatrix}$$

(6.13)

$$V_k = \begin{pmatrix} \lambda^{-2} & -h\lambda^{-1} e^{x_k} \\ h\lambda^{-1} e^{-\bar{x}_{k-1}} & 1 \end{pmatrix}$$

(6.14)

7 Explicit dRTL: different parametrizations

7.1 Hirota’s discretization of the Toda lattice

This system corresponds to the following parametrization of the $(c, d)$ variables of the relativistic Toda hierarchy:

$$c_k = h^2 e^{x_{k+1}} - x_k, \quad d_k = 1 + hp_k - h^2 e^{x_k} - x_{k-1}$$

(7.1)

This results in the Poisson bracket $h\{\cdot, \cdot\}_1$.

The equations of motion of the explicit dRTL map in this parametrization may be put in the following Lagrangian form:

$$\begin{cases} 
hp_k = e^{\bar{x}_k} - x_k - 1 - h^2 e^{x_{k+1}} - x_k + h^2 e^{x_k} - x_{k-1} \\
h\bar{p}_k = e^{\bar{x}_k} - x_k - 1
\end{cases}$$

(7.2)
It follows a nice "Hamiltonian" form of equations of motion:

\[
\begin{align*}
\tilde{e}x_k - x_k - 1 &= h\tilde{p}_k \\
\tilde{p}_k - p_k &= he^{x_{k+1} - x_k} - he^{x_k - x_{k-1}}
\end{align*}
\] (7.3)

The Newtonian form of equations of motion:

\[
\begin{align*}
\tilde{e}x_k - x_k - e x_k &= h^2 (e^{x_{k+1} - x_k} - e^{x_k - x_{k-1}})
\end{align*}
\] (7.4)

The Lax representation: (4.12) with

\[
L_k = \begin{pmatrix}
p_k + \lambda & e^{x_k} \\
-(1 + hp_k)e^{-x_k} & -h
\end{pmatrix}
\] (7.5)

\[
V_k = \begin{pmatrix}
1 - h\lambda - h^2 e^{x_k - x_{k-1}} & -he^{x_k} \\
he^{-x_{k-1}} & 1
\end{pmatrix}
\] (7.6)

Note that the matrix \(V_k\) depends only on the \(x_j\)'s, and not on their discrete time updates \(\tilde{x}_j\) or on the momenta \(p_j\). This will be the common feature of all the results in this Section.

The Lax matrix (7.5) is a one–parameter deformation of the standard Toda Lax matrix (4.7), but still satisfies the \(r\)-matrix ansatz (4.8).

### 7.2 Standard discretization of the Toda lattice

This system corresponds to the following parametrization of the variables \((c,d)\) of the relativistic Toda hierarchy:

\[
c_k = h^2 e^{x_{k+1} - x_k} + hp_k, \quad d_k = e^{hp_k}
\] (7.7)

This results in the Poisson bracket \(h\{\cdot, \cdot\}_2\). The equations of motion of the explicit dRTL map may be presented in this parametrization in the following Lagrange form:

\[
\begin{align*}
e^{hp_k} &= \tilde{e}x_k - x_k \left(\frac{1 + h^2 e^{x_k - x_{k-1}}}{1 + h^2 e^{x_{k+1} - x_k}}\right) \\
e^{h\tilde{p}_k} &= \tilde{e}x_k - x_k
\end{align*}
\] (7.8)
This may be presented in a nice "Hamiltonian" form:

\[
\begin{align*}
\tilde{x}_k - x_k &= \hbar \tilde{p}_k \\
\tilde{e}^{-\hbar \tilde{p}_k} - \hbar \tilde{p}_k &= \frac{1 + h^2 e^{x_{k+1} - x_k}}{1 + h^2 e^{x_k - x_{k-1}}} \tag{7.9}
\end{align*}
\]

The Newtonian form of equations of motion:

\[
\begin{align*}
\tilde{e}^{-2x_k + x_k} &= \frac{1 + h^2 e^{x_{k+1} - x_k}}{1 + h^2 e^{x_k - x_{k-1}}} \tag{7.10}
\end{align*}
\]

The Lax representation: \((4.12)\) with

\[
L_k = \begin{pmatrix}
\lambda e^{\hbar p_k} - \lambda^{-1} & \hbar e^x \\
-h e^{-x_k + \hbar p_k} & 0
\end{pmatrix} \tag{7.11}
\]

\[
V_k = \begin{pmatrix}
\lambda^{-2} & -h \lambda^{-1} e^x \\
h \lambda^{-1} e^{-x_{k-1}} & 1
\end{pmatrix} \tag{7.12}
\]

The Lax matrix \((7.11)\) is also a 1–parameter deformation of the standard Toda Lax matrix \((4.7)\), but satisfies a different \(r\)–matrix ansatz:

\[
\{L_k(\lambda) \otimes L_j(\mu)\} = \hbar \left[r(\lambda, \mu), L_k(\lambda) \otimes L_k(\mu)\right] \delta_{kj} \tag{7.13}
\]

with the \(r\)–matrix \((5.9)\).

### 7.3 Explicit discretization of the modified Toda lattice

This system corresponds to the following parametrization of the variables \((c, d)\) of the relativistic Toda hierarchy:

\[
c_k = h^2 e^{x_{k+1} - x_k + p_k}, \quad d_k = 1 + \hbar e^{p_k} + h e^{x_k - x_{k-1}} \tag{7.14}
\]

This results in the Poisson bracket \(\{\cdot, \cdot\}_2 - \{\cdot, \cdot\}_1\). The equations of motion of the explicit dRTL map may be presented in this parametrization in the following Lagrange form:

\[
\begin{align*}
\hbar e^{p_k} &= \left(e^{\tilde{x}_k - x_k} - 1\right) \frac{1 + h e^{x_k - x_{k-1}}}{1 + h e^{x_{k+1} - x_k}} \tag{7.15} \\
\hbar e^{\tilde{p}_k} &= \left(e^{\tilde{x}_k - x_k} - 1\right)
\end{align*}
\]
This may be presented in a nice "Hamiltonian" form:

\[
\begin{cases}
\tilde{e}^{\tilde{x}_k-x_k} - 1 = \hbar \tilde{p}_k \\
\tilde{e}^{\tilde{p}_k-p_k} = \frac{1 + \hbar e^{x_{k+1}-x_k}}{1 + \hbar e^{x_k-x_{k-1}}}
\end{cases}
\] (7.16)

The Newtonian form of equations of motion:

\[
\begin{pmatrix}
\tilde{e}^{\tilde{x}_k-x_k} - 1 \\
\tilde{e}^{x_k-x_{k-1}} - 1
\end{pmatrix} = \frac{1 + \hbar e^{x_{k+1}-x_k}}{1 + \hbar e^{x_k-x_{k-1}}} 
\] (7.17)

The Lax representation: (4.12) with

\[
L_k = \begin{pmatrix}
\lambda e^{p_k} - \lambda^{-1} & e^{x_k} \\
-(1 + \hbar e^{p_k})e^{-x_k} & \lambda
\end{pmatrix}
\] (7.18)

\[
V_k = \begin{pmatrix}
1 + \hbar \lambda^{-2} + \hbar e^{x_{k-1}} - h\lambda^{-1}e^{x_k} \\
\hbar \lambda^{-1}e^{-x_{k-1}} & 1
\end{pmatrix}
\] (7.19)

The Lax matrix (7.18) is a 1–parameter deformation of the Lax matrix (5.7), but satisfies the same $r$–matrix ansatz (5.8).

8 Systems RTL± and dRTL±:

Parametrization of the quadratic bracket

Canonical parametrization of the bracket $\{ \cdot, \cdot \}_2$:

\[
d_k = e^{p_k}, \quad c_k = g^2 e^{x_{k+1}-x_k} + p_k
\] (8.1)

The resulting system is the relativistic Toda lattice.

8.1 The flow RTL+

Hamilton function:

\[
H(x, p) = H_2^{(+)}(c, d) = \sum_k e^{p_k} \left( 1 + g^2 e^{x_{k+1}-x_k} \right)
\] (8.2)
Hamiltonian equations of motion:

\[
\begin{align*}
\dot{x}_k &= e^p_k \left( 1 + g^2 e^{x_{k+1}-x_k} \right) \\
\dot{p}_k &= g^2 e^{x_{k+1}-x_k} + p_k - g^2 e^{x_k-x_{k-1}} + p_{k-1}
\end{align*}
\]  

(8.3)

Lagrangian equations of motion:

\[
\begin{align*}
e^p_k &= \frac{\dot{x}_k}{1 + g^2 e^{x_{k+1}-x_k}} \\
\dot{p}_k &= \dot{x}_k - \frac{g^2 e^{x_{k+1}-x_k}}{1 + g^2 e^{x_{k+1}-x_k}} - \frac{g^2 e^{x_k-x_{k-1}}}{1 + g^2 e^{x_k-x_{k-1}}}
\end{align*}
\]  

(8.4)

Newtonian equations of motion:

\[
\ddot{x}_k = \dot{x}_{k+1} \dot{x}_k - \frac{g^2 e^{x_{k+1}-x_k}}{1 + g^2 e^{x_{k+1}-x_k}} - \frac{g^2 e^{x_k-x_{k-1}}}{1 + g^2 e^{x_k-x_{k-1}}}
\]  

(8.5)

8.2 The map dRTL+

"Hamiltonian" form of equations of motion:

\[
\begin{align*}
e^{\bar{x}_k-x_k} - 1 &= h e^{\bar{p}_k} \left( 1 + g^2 e^{x_{k+1}-x_k} \right) \\
\bar{e}^{\bar{p}_k} &- p_k = \frac{\left( 1 + h g^2 e^{x_{k+1}-x_k} + \bar{p}_k \right)}{\left( 1 + h g^2 e^{x_k-x_{k-1}} + \bar{p}_{k-1} \right)} \quad (8.6)
\end{align*}
\]

Lagrangian form of equations of motion:

\[
\begin{align*}
h e^{p_k} &= \frac{\left( e^{\bar{x}_k-x_k} - 1 \right)}{\left( 1 + g^2 e^{x_k-x_{k-1}} \right)} \quad \frac{\left( 1 + g^2 e^{x_k-x_{k-1}} \right)}{\left( 1 + g^2 e^{x_{k+1}-x_k} \right)} \\
\bar{h} e^{\bar{p}_k} &= \frac{\left( e^{\bar{x}_k-x_k} - 1 \right)}{\left( 1 + g^2 e^{x_k-x_{k-1}} \right)}
\end{align*}
\]  

(8.7)
Newtonian form of equations of motion:

\[
\frac{(e^x_k - x_k - 1)}{(e^{x_{k+1}} - x_k - 1)} = \frac{(1 + g^2 e^{x_{k+1} - x_k})}{(1 + g^2 e^{x_{k+1} - x_k - 1})} \frac{(1 + g^2 e^{x_k - x_{k-1}})}{(1 + g^2 e^{x_k - x_{k-1} - 1})} \tag{8.8}
\]

8.3 The flow RTL−

Hamilton function:

\[
H(x, p) = H^2_2 (c, d) = \sum_k e^{-p_k} \left(1 + g^2 e^{x_k - x_{k-1}}\right) \tag{8.9}
\]

Hamiltonian equations of motion:

\[
\begin{align*}
\dot{x}_k &= -e^{-p_k} \left(1 + g^2 e^{x_k - x_{k-1}}\right) \\
\dot{p}_k &= g^2 e^{x_{k+1} - x_k} - p_{k+1} - g^2 e^{x_k - x_{k-1} - 1} - p_k
\end{align*} \tag{8.10}
\]

Lagrangian equations of motion:

\[
\begin{align*}
e^p_k &= -\left(1 + g^2 e^{x_k - x_{k-1}}\right) \\
\dot{p}_k &= -\dot{x}_{k+1} \frac{g^2 e^{x_{k+1} - x_k}}{1 + g^2 e^{x_{k+1} - x_k}} + \dot{x}_k \frac{g^2 e^{x_k - x_{k-1}}}{1 + g^2 e^{x_k - x_{k-1} - 1}}
\end{align*} \tag{8.11}
\]

Newtonian equations of motion – the same \([8.3]\) as for the RTL+ (!).

8.4 The map dRTL−

"Hamiltonian" form of equations of motion:

\[
\begin{align*}
e^{\bar{x}_k - x_k} - 1 &= -he^{-p_k} \left(1 + g^2 e^{x_k - \bar{x}_{k-1}}\right) \\
\bar{e}p_k - p_k &= \frac{\left(1 - hg^2 e^{x_k - \bar{x}_{k-1} - p_k}\right)}{\left(1 - hg^2 e^{x_{k+1} - \bar{x}_k - p_{k+1}}\right)}
\end{align*} \tag{8.12}
\]
Lagrangian form of equations of motion:

\[
\begin{align*}
\dot{e}p_k &= -\frac{h(1 + g^2 e^{x_k} - \bar{x}_{k-1})}{(e^{\bar{x}_k} - x_k - 1)} \\
\dot{e}\bar{p}_k &= -\frac{h(1 + g^2 e^{x_{k+1} - \bar{x}_k}) (1 + g^2 e^{\bar{x}_k} - \bar{x}_{k-1})}{(e^{\bar{x}_k} - x_k - 1) (1 + g^2 e^{\bar{x}_{k+1}} - \bar{x}_k)} 
\end{align*}
\]  

(8.13)

Newtonian form of equations of motion – (8.8), the same as for the dRTL+ (!).

8.5 Lax representations and \( r \)-matrix structure

Lax representations for the four systems of this Section is given in terms one and the same Lax matrix:

\[
L_k = \begin{pmatrix}
\lambda e^{p_k} - \lambda^{-1} e^{x_k} \\
-g^2 e^{-x_k} + p_k & 0
\end{pmatrix}
\]  

(8.14)

It satisfies the \( r \)-matrix ansatz (5.8) with the \( r \)-matrix (5.9).

Lax representation for the flow RTL+: (4.10) with

\[
M_k = \begin{pmatrix}
\lambda^{-2} + g^2 e^{x_k - x_{k-1} + p_{k-1}} - \lambda^{-1} e^{x_k} \\
\lambda^{-1} g^2 e^{-x_{k-1} + p_{k-1}} & 0
\end{pmatrix}
\]  

(8.15)

Lax representation for the map dRTL+: (4.12) with

\[
V_k = \begin{pmatrix}
1 + h\lambda^{-2} + h g^2 e^{x_k - \bar{x}_{k-1} + \bar{p}_{k-1}} - h\lambda^{-1} e^{x_k} \\
h\lambda^{-1} g^2 e^{-\bar{x}_{k-1} + \bar{p}_{k-1}} & 1
\end{pmatrix}
\]  

(8.16)

Lax representation for the flow RTL−: (4.10) with

\[
M_k = \begin{pmatrix}
0 & -\lambda e^{x_k} - p_k \\
\lambda g^2 e^{-x_{k-1}} & \lambda^2 + g^2 e^{x_k - x_{k-1} - p_k}
\end{pmatrix}
\]  

(8.17)

Lax representation for the map dRTL−:

\[
W_{k+1} L_k = L_k W_k
\]  

(8.18)

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with the same Lax matrix (8.14) and

$$W_k = \begin{pmatrix}
1 & h\lambda e^{x_k} - p_k \\
-h\lambda g^2 e^{-\bar{x}_{k-1}} & 1 - h\lambda^2 - h g^2 e^{x_k - \bar{x}_{k-1} - p_k}
\end{pmatrix} \quad (8.19)$$

9 Systems RTL± and dRTL±:

Parametrization of the linear bracket

Canonical parametrization of the bracket \{·, ·\}_1:

$$d_k = p_k - e^{x_k - x_{k-1}}, \quad c_k = e^{x_{k+1} - x_k} \quad (9.1)$$

9.1 The flow RTL+

Hamilton function:

$$H(x, p) = H_1^{(+)}(c, d) = \frac{1}{2} \sum_k p_k^2 + \sum_k p_k e^{x_{k+1} - x_k} \quad (9.2)$$

Hamiltonian equations of motion:

$$\begin{cases}
\dot{x}_k = p_k + e^{x_{k+1} - x_k} \\
\dot{p}_k = p_k e^{x_{k+1} - x_k} - p_{k-1} e^{x_k - x_{k-1}}
\end{cases} \quad (9.3)$$

Lagrangian equations of motion:

$$\begin{cases}
p_k = \dot{x}_k - e^{x_{k+1} - x_k} \\
\dot{p}_k = (\dot{x}_k - e^{x_{k+1} - x_k}) e^{x_{k+1} - x_k} - (\dot{x}_{k-1} - e^{x_k - x_{k-1}}) e^{x_k - x_{k-1}}
\end{cases} \quad (9.4)$$

Newtonian equations of motion:

$$\ddot{x}_k = (\dot{x}_{k+1} - e^{x_{k+1} - x_k}) e^{x_{k+1} - x_k} - (\dot{x}_{k-1} - e^{x_k - x_{k-1}}) e^{x_k - x_{k-1}} \quad (9.5)$$
9.2 The map dRTL+

"Hamiltonian" form of equations of motion:

\[
\begin{aligned}
& e^{\bar{x}_k - x_k} - 1 = \hbar \tilde{p}_k + \frac{he^{x_{k+1} - \bar{x}_k}}{1 - he^{x_{k+1} - \bar{x}_k}} \\
& \tilde{p}_k - p_k = h\tilde{p}_k e^{x_{k+1} - \bar{x}_k} - h\tilde{p}_{k-1} e^{x_k - \bar{x}_{k-1}}
\end{aligned}
\] (9.6)

Lagrangian form of equations of motion:

\[
\begin{aligned}
& hp_k = e^{\bar{x}_k - x_k} - \frac{1}{1 - he^{x_k - \bar{x}_{k-1}}} + he^{x_k - x_{k-1}} - he^{x_{k+1} - x_k} \\
& h\tilde{p}_k = e^{\bar{x}_k - x_k} - \frac{1}{1 - he^{x_{k+1} - \bar{x}_k}}
\end{aligned}
\] (9.7)

Newtonian form of equations of motion:

\[
\begin{aligned}
& e^{\bar{x}_k - x_k} - e^{x_k - x_k} = he^{x_{k+1} - x_k} - \frac{he^{x_{k+1} - x_k}}{1 - he^{x_{k+1} - x_k}} - he^{x_k - x_{k-1}} + \frac{he^{x_k - \bar{x}_{k-1}}}{1 - he^{x_k - \bar{x}_{k-1}}}
\end{aligned}
\] (9.8)

9.3 The flow RTL−

Hamilton function:

\[
H(x, p) = H_1^{(-)}(c, d) = - \sum_k \log \left( p_k - e^{x_k - x_{k-1}} \right)
\] (9.9)

Hamiltonian equations of motion:

\[
\begin{aligned}
& \dot{x}_k = - \frac{1}{p_k - e^{x_k - x_{k-1}}} \\
& \dot{p}_k = \frac{e^{x_{k+1} - x_k}}{p_k + e^{x_{k+1} - x_k}} - \frac{e^{x_k - x_{k-1}}}{p_k - e^{x_k - x_{k-1}}}
\end{aligned}
\] (9.10)
Lagrangian equations of motion:

\[
\begin{align*}
\dot{p}_k &= -\frac{1}{x_k} + e^{x_k-x_{k-1}} \\
\dot{p}_k &= -\dot{x}_{k+1}e^{x_{k+1}-x_k} + \dot{x}_ke^{x_k-x_{k-1}} \\
\end{align*}
\] (9.11)

Newtonian form of equations of motion:

\[
\ddot{x}_k = -\dot{x}_k^2(\dot{x}_{k+1}e^{x_{k+1}-x_k} - \dot{x}_{k-1}e^{x_k-x_{k-1}})
\] (9.12)

9.4 The map dRTL–

"Hamiltonian" form of equations of motion:

\[
\begin{align*}
\dot{p}_k - p_k &= h\frac{e^{x_k-x_{k-1}}}{p_k - e^{x_k-x_{k-1}}} \\
\end{align*}
\] (9.13)

Lagrangian form of equations of motion:

\[
\begin{align*}
p_k &= -\frac{h}{e^{x_k-x_{k-1}}} + e^{x_k-x_{k-1}} \\
\end{align*}
\] (9.14)

Lagrange function of equations of motion:

\[
\frac{h}{e^{x_k-x_{k-1}}} - \frac{h}{e^{x_k-x_{k-1}}} = e^{x_k-x_{k-1}} - e^{x_k-x_{k-1}} - e^{x_k-x_{k-1}} + e^{x_k-x_{k-1}}
\] (9.15)

9.5 Lax representations and \( r \)-matrix structure

All four systems considered in this Section allow Lax representations with the Lax matrix

\[
L_k = \begin{pmatrix}
    p_k + \lambda & e^{x_k} \\
    -p_k e^{-x_k} & -1
\end{pmatrix}
\] (9.16)
which satisfies the $r$–matrix ansatz (4.8).

Lax representation for the flow RTL+: (4.10) with

$$M_k = \begin{pmatrix} -\lambda & -e^{x_k} \\ p_{k-1}e^{-x_{k-1}} & 0 \end{pmatrix}$$  \hspace{1cm} (9.17)$$

Lax representation for the map dRTL+: (4.12) with

$$V_k = \begin{pmatrix} 1 - h\lambda - h^2\tilde{p}_{k-1}e^{x_k-\tilde{x}_{k-1}} & -he^{x_k} \\ \tilde{h}\tilde{p}_{k-1}e^{-\tilde{x}_{k-1}} & 1 \end{pmatrix}$$  \hspace{1cm} (9.18)$$

Lax representation for the flow RTL−: (4.11) with

$$M_k = \frac{\lambda^{-1}}{p_k - e^{x_k-\tilde{x}_{k-1}}} \begin{pmatrix} p_k & e^{x_k} \\ -p_ke^{-x_{k-1}} & -e^{x_k-x_{k-1}} \end{pmatrix}$$  \hspace{1cm} (9.19)$$

Lax representation the map dRTL−: (8.18) with

$$W_k = I - h\frac{1}{\lambda + h} \begin{pmatrix} p_k & e^{x_k} \\ -p_ke^{-x_{k-1}} & -e^{x_k-\tilde{x}_{k-1}} \end{pmatrix}$$  \hspace{1cm} (9.20)$$

This last Lax representation may be also presented in the form (4.12) with matrix $V_k$ being more nice than $W_k$ (which is non-typical for the map dRTL−):

$$V_k = I + \frac{h\lambda^{-1}}{p_k - e^{x_k-\tilde{x}_{k-1}}} \begin{pmatrix} p_k & e^{x_k} \\ -p_ke^{-x_{k-1}} & -e^{x_k-\tilde{x}_{k-1}} \end{pmatrix}$$  \hspace{1cm} (9.21)$$

10 Systems RTL± and dRTL±:

The first mixed parametrization

We now consider the following canonical parametrization of the bracket $\{\cdot, \cdot\}_2 - \delta\{\cdot, \cdot\}_1$:

$$d_k = e^{p_k} + \delta\left(1 + g^2e^{x_k-x_{k-1}}\right) \quad c_k = g^2e^{x_{k+1}-x_k} + p_k$$  \hspace{1cm} (10.1)$$

This is, obviously, a 1–parameter deformation of (8.1). The results of this section in the limit $\delta \rightarrow 0$ imply the results of the Section 8.
10.1 The flow RTL+

Hamilton function:

\[ H(x, p) = H_2^{(+)}(c, d) = \sum_k e^{p_k}(1 + g^2 e^{x_{k+1} - x_k}) + \delta g^2 \sum_k e^{x_k - x_{k-1}} \]  

(10.2)

Hamiltonian equations of motion:

\[ \begin{cases} 
\dot{x}_k = e^{p_k}(1 + g^2 e^{x_{k+1} - x_k}) \\
\dot{p}_k = g^2(e^{p_k} + \delta) e^{x_{k+1} - x_k} - g^2(e^{p_{k-1}} + \delta) e^{x_k - x_{k-1}} 
\end{cases} \]  

(10.3)

Lagrangian equations of motion:

\[ \begin{cases} 
e^{p_k} = \frac{\dot{x}_k}{1 + g^2 e^{x_{k+1} - x_k}} \\
\dot{p}_k = e^{p_k} \frac{g^2 e^{x_{k+1} - x_k}}{1 + g^2 e^{x_{k+1} - x_k}} - \dot{x}_{k-1} \frac{g^2 e^{x_k - x_{k-1}}}{1 + g^2 e^{x_k - x_{k-1}}} + \delta g^2(e^{x_{k+1} - x_k} - e^{x_k - x_{k-1}}) 
\end{cases} \]  

(10.4)

Newtonian equations of motion:

\[ \ddot{x}_k = \dot{x}_{k+1} \frac{g^2 e^{x_{k+1} - x_k}}{1 + g^2 e^{x_{k+1} - x_k}} - \dot{x}_k \frac{g^2 e^{x_k - x_{k-1}}}{1 + g^2 e^{x_k - x_{k-1}}} + \delta g^2 \dot{x}_k(e^{x_{k+1} - x_k} - e^{x_k - x_{k-1}}) \]  

(10.5)

It is interesting to remark that the right-hand side of this equation coincides with the right-hand side of (8.5) plus an additive perturbation which is exactly the right-hand side of the modified Toda lattice (5.4).

10.2 The map dRTL+

"Hamiltonian" form of equations of motion:

\[ \begin{cases} 
e^{\bar{x}_k - x_k} - 1 = \frac{he^{\bar{p}_k}(1 + g^2 e^{x_{k+1} - \bar{x}_k})}{1 + h\delta g^2 e^{x_{k+1} - \bar{x}_k}} \\
e^{\bar{p}_k - p_k} = \frac{(1 + hg^2(e^{\bar{p}_k} + \delta) e^{x_{k+1} - \bar{x}_k})}{(1 + hg^2(e^{\bar{p}_{k-1}} + \delta) e^{x_k - \bar{x}_{k-1}}) \]  

(10.6)
Lagrangian form of equations of motion:

\[
\begin{align*}
heP_k &= \frac{(e^{\bar{x}_k - x_k} - 1)}{(1 + g^2 e^{x_{k+1} - \bar{x}_k})} \frac{(1 + g^2 e^{x_k - x_{k-1}})}{(1 + g^2 e^{x_{k+1} - x_k})} \left(1 + h\delta g^2 e^{x_k - \bar{x}_{k-1}}\right) \\
he\tilde{p}_k &= \frac{(e^{\bar{x}_k - x_k} - 1)}{(1 + g^2 e^{x_{k+1} - \bar{x}_k})} \left(1 + h\delta g^2 e^{x_{k+1} - \bar{x}_k}\right)
\end{align*}
\] (10.7)

Newtonian form of equations of motion:

\[
\begin{align*}
\frac{(e^{\bar{x}_k - x_k} - 1)}{(e^{x_k - x_{k-1}} - 1)} &= \frac{(1 + g^2 e^{x_{k+1} - x_k})}{(1 + g^2 e^{x_{k+1} - x_k})} \frac{(1 + g^2 e^{x_k - x_{k-1}})}{(1 + g^2 e^{x_{k+1} - x_k})} \left(1 + h\delta g^2 e^{x_k - \bar{x}_{k-1}}\right) (10.8)
\end{align*}
\]

Interesting enough, the right–hand side of the last equation is the product of the right–hand sides of (8.8) and (5.6).

10.3 The flow RTL–

Hamilton function:

\[
H(x, p) = -\delta^{-1} H_1^{(-)}(c, d) - \delta^{-1} \sum_k p_k = \delta^{-1} \sum_k \log(d_k) - \delta^{-1} \sum_k p_k
\] (10.9)

where, recall,

\[
d_k = e^{p_k} + \delta \left(1 + g^2 e^{x_k - x_{k-1}}\right) = e^{p_k} + O(\delta)
\] (10.10)

Hamiltonian equations of motion:

\[
\begin{align*}
\dot{x}_k &= -d_k^{-1} \left(1 + g^2 e^{x_k - x_{k-1}}\right) \\
\dot{p}_k &= d_k^{-1} g^2 e^{x_{k+1} - x_k} - d_k^{-1} g^2 e^{x_k - x_{k-1}}
\end{align*}
\] (10.11)

Lagrangian equations of motion:

\[
\begin{align*}
e^{p_k} &= -\frac{\left(1 + g^2 e^{x_k - x_{k-1}}\right)}{\dot{x}_k} \left(1 + \delta \dot{x}_k\right) \\
\dot{p}_k &= -\dot{x}_{k+1} \frac{g^2 e^{x_{k+1} - x_k}}{\left(1 + g^2 e^{x_{k+1} - x_k}\right)} + \dot{x}_k \frac{g^2 e^{x_k - x_{k-1}}}{\left(1 + g^2 e^{x_k - x_{k-1}}\right)}
\end{align*}
\] (10.12)
Newtonian equations of motion:

\[
\ddot{x}_k = (1 + \delta \dot{x}_k) \left( \ddot{x}_{k+1} \frac{g^2 x_{k+1} - x_k}{1 + g^2 e x_{k+1} - x_k} - \ddot{x}_{k-1} \frac{g^2 x_{k} - x_{k-1}}{1 + g^2 e x_{k} - x_{k-1}} \right)
\]

(10.13)

This is a multiplicative perturbation of (8.5).

10.4 The map dRTL—

"Hamiltonian" equations of motion:

\[
\begin{align*}
\dot{e} \ddot{x}_k - x_k - 1 &= -h D_k^{-1} (1 + g^2 e x_k - \ddot{x}_{k-1}) \\
\dot{e} \ddot{p}_k &= \left( 1 - h D_k^{-1} g^2 e x_k - \ddots_{k-1} \right) \\
&\quad \left( 1 - h D_{k+1}^{-1} g^2 e x_{k+1} - \ddots_k \right) \\
\end{align*}
\]

(10.14)

where

\[
D_k = e p_k + \delta \left( 1 + g^2 e x_k - \ddot{x}_{k-1} \right) = e p_k + O(\delta)
\]

(10.15)

Lagrangian form of equations of motion:

\[
\begin{align*}
\dot{e} \ddot{p}_k &= -\left( \delta (e \ddot{x}_k - x_k - 1) + h \right) \\
&\quad \left( e \ddot{x}_k - x_k - 1 \right) \\
\dot{e} \ddot{p}_k &= -\left( \delta (e \ddot{x}_k - x_k - 1) + h \right) \\
&\quad \left( e \ddot{x}_k - x_k - 1 \right) \\
\end{align*}
\]

(10.16)

Newtonian form of equations of motion:

\[
\begin{align*}
\frac{e \ddot{x}_k - x_k - 1}{e \ddot{x}_k - x_k - 1} \left( \delta (e \ddot{x}_k - x_k - 1) + h \right) &= \left( 1 + g^2 e x_{k+1} - x_k \right) \\
&\quad \left( 1 + g^2 e x_{k-1} - x_k \right)
\end{align*}
\]

(10.17)
Special case $\delta = h$. If the parameter $\delta$ is equal to the (small) stepsize $h$, the map dRTL simplifies, providing discretization of the relativistic Toda lattice \((8.8)\) different from \((8.8)\). In this case the ”Hamiltonian” equations of motion read:

\[
\begin{cases}
    e\bar{x}_k - x_k = \frac{1}{1 + he^{-p_k} (1 + g^2 e^{x_k} - \bar{x}_{k-1})} \\
    e\bar{p}_k - p_k = \frac{(1 + h g^2 (e^{p_{k+1}} + h)^{-1} e^{x_{k+1}} - \bar{x}_k)}{(1 + h g^2 (e^{p_k} + h)^{-1} e^{x_k} - \bar{x}_{k-1})}
\end{cases}
\]

(10.18)

The Lagrangian equations of motion read:

\[
\begin{cases}
    e^{p_k} = -h \frac{(1 + g^2 e^{x_k} - \bar{x}_{k-1})}{(1 - e^{-\bar{x}_k + x_k})} \\
    e^{\bar{p}_k} = -h \frac{(1 + g^2 e^{-\bar{x}_k} - \bar{x}_{k-1})}{(1 - e^{-\bar{x}_k + x_k})} \frac{(1 + g^2 e^{x_{k+1}} - \bar{x}_k)}{(1 + g^2 e^{x_k} - \bar{x}_{k-1})}
\end{cases}
\]

(10.19)

The Newtonian equations of motion:

\[
\frac{(1 - e^{-\bar{x}_k + x_k})}{(1 - e^{-\bar{x}_k + x_k})} = \frac{(1 + g^2 e^{x_{k+1}} - x_k)}{(1 + g^2 e^{x_{k+1}} - x_k)} \frac{(1 + g^2 e^{x_k} - \bar{x}_{k-1})}{(1 + g^2 e^{x_k} - \bar{x}_{k-1})}
\]

(10.20)

The system resembles very much the previous discretization of the relativistic Toda lattice \((8.8)\), however the relation between the two is far from trivial.

10.5 Lax representations and $r$–matrix structure

The Lax representations for the four systems considered in this Section may be given in terms of the following Lax matrix:

\[
L_k = \begin{pmatrix}
    \lambda e^{p_k} - \lambda^{-1} & e^{x_k} \\
    -g^2 (e^{p_k} + \delta) e^{-x_k} & \lambda g^2
\end{pmatrix}
\]

(10.21)
which satisfies the $r$–matriz ansatz \((5.8)\) with the $r$–matrix \((5.9)\).

The Lax representation for the flow $\text{RTL}^+$: \((4.10)\) with

$$M_k = \begin{pmatrix} \lambda^{-2} + g^2(e^{P_{k-1}} + \delta)e^{x_k-x_{k-1}} - \lambda^{-1}e^{x_k} \\ \lambda^{-1}g^2(e^{P_{k-1}} + \delta)e^{-x_{k-1}} - \lambda^{-1}e^{x_k} \\ 0 \end{pmatrix}$$

Lax representation for the map $\text{dRTL}^+$: \((4.12)\) with

$$V_k = \begin{pmatrix} 1 + h\lambda^{-2} + h g^2(e^{P_{k-1}} + \delta)e^{x_k-x_{k-1}} - h\lambda^{-1}e^{x_k} \\ \lambda^{-1}g^2(e^{P_{k-1}} + \delta)e^{-x_{k-1}} - \lambda^{-1}e^{x_k} \\ 1 \end{pmatrix}$$

Lax representation for the flow $\text{RTL}^-$: \((4.10)\) with

$$M_k = \frac{d_k^{-1}}{1 + \delta\lambda^2} \begin{pmatrix} -\lambda^2(e^{P_k} + \delta) - \lambda e^{x_k} \\ \lambda g^2(e^{P_k} + \delta)e^{-x_{k-1}} - \lambda g^2 e^{x_k-x_{k-1}} \end{pmatrix}$$

Lax representation for the map $\text{dRTL}^-$: \((8.18)\) with

$$W_k = I + \frac{h D_k^{-1}}{1 + (\delta - h)\lambda^2} \begin{pmatrix} \lambda^2(e^{P_k} + \delta) + \lambda e^{x_k} \\ -\lambda g^2(e^{P_k} + \delta)e^{-x_{k-1}} - \lambda g^2 e^{x_k-x_{k-1}} \end{pmatrix}$$

Special case $\delta = h$. In this case, for the map $\text{dRTL}^-$, the dependence of the matrix $W_k$ on $\lambda$ simplifies, because the denominator $1 + (\delta - h)\lambda^2$ becomes equal to $1$.

11 Systems $\text{RTL}^\pm$ and $\text{dRTL}^\pm$:

The second mixed parametrization

We consider the following canonical parametrization of the variables $(c, d)$:

$$d_k = \frac{e^{P_k} - 1}{\epsilon} - e^{x_k-x_{k-1}}, \quad c_k = e^{x_k+1-x_k+e^{P_k}}$$

By small $\epsilon$ it serves as a deformation of \((9.1)\). The corresponding Poisson bracket is $\{\cdot, \cdot\}_1 + \epsilon\{\cdot, \cdot\}_2$. The results of this Section allow to recover that of the section 9 in the limit $\epsilon \to 0$ (in order to reproduce the Lax representations of the Sect.9, the spectral parameter $\lambda$ of this Section has to be replaced by $1 + \epsilon\lambda/2 + O(\epsilon^2)$ before performing the limit $\epsilon \to 0$).
11.1 The flow RTL+

Hamilton function:
\[ H(x, p) = \epsilon^{-1} H_2^{(+)}(c, d) - \epsilon^{-1} \sum_k p_k = \sum_k \frac{e^\epsilon p_k - 1 - \epsilon p_k}{\epsilon^2} + \sum_k \frac{e^\epsilon p_k - 1}{\epsilon} e^x_{k+1} - x_k \] (11.2)

Hamiltonian equations of motion:
\[
\begin{align*}
\dot{x}_k &= \frac{(e^\epsilon p_k - 1)}{\epsilon} + e^x_{k+1} - x_k + \epsilon p_k \\
\dot{p}_k &= \frac{(e^\epsilon p_k - 1)}{\epsilon} e^x_{k+1} - x_k - \frac{(e^\epsilon p_k - 1)}{\epsilon} e^x_k - x_{k-1}
\end{align*}
\] (11.3)

Lagrangian equations of motion:
\[
\begin{align*}
e^\epsilon p_k &= \frac{(1 + \epsilon \dot{x}_k)}{(1 + \epsilon e^x_{k+1} - x_k)} \\
\dot{p}_k &= \frac{(\dot{x}_k - e^x_{k+1} - x_k)}{(1 + \epsilon e^x_{k+1} - x_k)} e^x_{k+1} - x_k - \frac{(\dot{x}_k - e^x_{k-1})}{(1 + \epsilon e^x_k - x_{k-1})} e^x_k - x_{k-1}
\end{align*}
\] (11.4)

The corresponding Newtonian equations of motion read:
\[
\ddot{x}_k = (1 + \epsilon \dot{x}_k) \left( \frac{\dot{x}_{k+1} - e^x_{k+1} - x_k}{(1 + \epsilon e^x_{k+1} - x_k)} e^x_{k+1} - x_k - \frac{\dot{x}_{k-1} - e^x_{k-1}}{(1 + \epsilon e^x_k - x_{k-1})} e^x_k - x_{k-1} \right)
\] (11.5)

11.2 The map dRTL+

"Hamiltonian" equations of motion:
\[
\begin{align*}
e^\epsilon \tilde{x}_k - x_k - 1 &= h \frac{(e^\epsilon \tilde{p}_k - 1)}{\epsilon} + h e^x_{k+1} - \tilde{x}_k + \epsilon \tilde{p}_k \\
\epsilon^\epsilon \tilde{p}_k - \epsilon p_k &= \frac{(1 + h(e^\epsilon \tilde{p}_k - 1)) e^x_{k+1} - \tilde{x}_k}{(1 + h(e^\epsilon \tilde{p}_k - 1)) e^x_k - \tilde{x}_{k-1}}
\end{align*}
\] (11.6)
Lagrangian equations of motion:

\[
\begin{align*}
\frac{h e^p k}{\epsilon (e^x - x - 1) + h} &= \frac{(1 - h e^{x_k - x_{k-1}})}{(1 + (\epsilon - h) e^{x_{k-1} - x_k})} \\
\frac{h e^p k}{\epsilon (e^x - x - 1) + h} &= \frac{(1 - h e^{x_k - x_{k-1}})}{(1 + (\epsilon - h) e^{x_{k-1} - x_k})} \\
\end{align*}
\]

Newtonian equations of motion:

\[
\frac{\epsilon (e^x - x - 1) + h}{\epsilon (e^x - x - 1) + h} = \frac{(1 + \epsilon e^{x_{k+1} - x_k}) (1 - h e^{x_{k+1} - x_k}) (1 + (\epsilon - h) e^{x_k - x_{k-1}})}{(1 + \epsilon e^{x_k - x_{k-1}}) (1 - h e^{x_k - x_{k-1}}) (1 + (\epsilon - h) e^{x_{k-1} - x_k})}
\]

**Special case** $\epsilon = h$. If the parameter $\epsilon$ is equal to the (small) stepsize $h$, then the equations above greatly simplify, delivering another discretization of the system (11.3). In this case we obtain the following ”Hamiltonian” equations of motion:

\[
\begin{align*}
e^{x_k - x_k} &= \frac{e^{hp_k}}{(1 - h e^{x_{k+1} - x_k})} \\
e^{hp_k - hp_k} &= \frac{(1 + h(e^{hp_k} - 1)e^{x_{k+1} - x_k})}{(1 + h(e^{hp_k} - 1)e^{x_k - x_{k-1}})}
\end{align*}
\]

Lagrangian equations of motion:

\[
\begin{align*}
e^{hp_k} &= e^{x_k - x_k} (1 - h e^{x_k - x_{k-1}}) (1 + h e^{x_{k+1} - x_{k-1}}) \\
e^{hp_k} &= e^{x_k - x_k} (1 + h e^{x_{k+1} - x_k})
\end{align*}
\]
Newtonian equations of motion:
\[
\tilde{x}_k - 2x_k + x_k = \frac{(1 + \epsilon e^{x_{k+1} - x_k})}{(1 + h e^{x_{k} - x_{k-1}})} \left(1 - h e^{x_{k+1} - x_k}\right)
\]

(11.11)

11.3 The flow RTL–

Hamilton function:
\[
H(x, p) = H_1(c, d) + \epsilon \sum_k p_k = -\sum_k \log(d_k) + \epsilon \sum_k p_k
\]

(11.12)

where, recall,
\[
d_k = \frac{e^{\epsilon p_k} - 1}{\epsilon} - e^{x_k - x_{k-1}} = p_k - e^{x_k - x_{k-1}} + O(\epsilon)
\]

(11.13)

Hamiltonian equations of motion:
\[
\begin{align*}
\dot{x}_k &= -d_k^{-1} \left(1 + \epsilon e^{x_k - x_{k-1}}\right) \\
\dot{p}_k &= d_{k+1}^{-1} e^{x_{k+1} - x_k} - d_k^{-1} e^{x_k - x_{k-1}}
\end{align*}
\]

(11.14)

Lagrangian equations of motion:
\[
\begin{align*}
ed \dot{p}_k &= \left(1 + \epsilon e^{x_k - x_{k-1}}\right) \frac{\dot{x}_k - \epsilon}{x_k} \\
\dot{p}_k &= -\dot{x}_{k+1} \frac{e^{x_{k+1} - x_k}}{1 + \epsilon e^{x_{k+1} - x_k}} + \dot{x}_k \frac{e^{x_k - x_{k-1}}}{1 + \epsilon e^{x_k - x_{k-1}}}
\end{align*}
\]

(11.15)

Newtonian equations of motion:
\[
\ddot{x}_k = -\dot{x}_k (\dot{x}_k - \epsilon) \left(\frac{x_{k+1} - x_k}{1 + \epsilon e^{x_{k+1} - x_k}}\right) - \dot{x}_{k-1} \left(\frac{x_k - x_{k-1}}{1 + \epsilon e^{x_k - x_{k-1}}}\right)
\]

(11.16)

11.4 The map dRTL–

"Hamiltonian" equations of motion:
\[
\begin{align*}
e^{\tilde{x}_k - x_k} - 1 &= -h D_k^{-1} e^{x_k - \tilde{x}_{k-1}} \\
e^{\tilde{p}_k - \epsilon p_k} &= \frac{1 - h e D_k^{-1} e^{x_k - \tilde{x}_{k-1}}}{1 - h e D_{k+1}^{-1} e^{x_{k+1} - \tilde{x}_k}}
\end{align*}
\]

(11.17)
where

\[ D_k = \frac{e^{\varepsilon p_k} - e^{x_k - \tilde{x}_{k-1}}}{\varepsilon} = p_k - e^{x_k - \tilde{x}_{k-1}} + O(\varepsilon) \quad (11.18) \]

Lagrangian form of equations of motion:

\[
\begin{cases}
    e^{\varepsilon p_k} = \frac{(e^{\tilde{x}_k - x_k} - 1 - h\varepsilon)}{(e^{\tilde{x}_k - x_k} - 1)} \left( 1 + e^{x_k - \tilde{x}_{k-1}} \right) \\
    e^{\varepsilon \tilde{p}_k} = \frac{(e^{\tilde{x}_k - x_k} - 1 - h\varepsilon)}{(e^{x_k - \tilde{x}_{k-1}} - 1)} \left( 1 + e^{x_{k+1} - \tilde{x}_k} \right) \left( 1 + e^{x_k - \tilde{x}_{k-1}} \right) \left( 1 + e^{x_{k+1} - \tilde{x}_k} \right) \quad (11.19)
\end{cases}
\]

Newtonian form of equations of motion:

\[
\begin{align*}
    (e^{\tilde{x}_k - x_k} - 1)(e^{\tilde{x}_k - x_k} - 1 - h\varepsilon) &= \frac{(1 + e^{x_{k+1} - \tilde{x}_k})}{(1 + e^{x_k - \tilde{x}_{k-1}})} \left( 1 + e^{x_{k+1} - \tilde{x}_k} \right) \left( 1 + e^{x_k - \tilde{x}_{k-1}} \right) \quad (11.20)
\end{align*}
\]

11.5 Lax representations and \( r \)-matrix structure

The Lax representations for the four systems considered in this Section may be given in terms of the following Lax matrix:

\[
L_k = \begin{pmatrix}
    \lambda e^{\varepsilon p_k} - \lambda^{-1} & e^{x_k} \\
    -(e^{p_k} - 1)e^{-x_k} & -\lambda^{-1}
\end{pmatrix}
\quad (11.21)
\]

which satisfies the \( r \)-matrix ansatz (6.10) with the \( r \)-matrix (5.9).

The Lax representation for the flow RTL+:

\[
M_k = \begin{pmatrix}
    \lambda^{-2} - 1 + (e^{p_{k-1}} - 1)e^{x_k - x_{k-1}} & -\lambda^{-1}e^{x_k} \\
    \lambda^{-1}(e^{p_{k-1}} - 1)e^{-x_{k-1}} & \epsilon
\end{pmatrix}
\quad (11.22)
\]

Lax representation for the map dRTL+:

\[
V_k = \begin{pmatrix}
    1 - \frac{h}{\epsilon} + \frac{h}{\epsilon} \lambda^{-2} + \frac{h(\epsilon - h)}{\epsilon} (e^{p_{k-1}} - 1)e^{x_k - \tilde{x}_{k-1}} - h\lambda^{-1}e^{x_k} \\
    \frac{h}{\epsilon} \lambda^{-1}(e^{p_{k-1}} - 1)e^{-\tilde{x}_{k-1}} & 1
\end{pmatrix}
\quad (11.23)
\]
Lax representation for the flow RTL$-$: (4.10) with
\[
M_k = \frac{d_k^{-1}}{(\lambda^2 - 1)} \begin{pmatrix}
\lambda^2 (e^{\epsilon p} - 1) & \lambda \epsilon e^{x_k} \\
-\lambda (e^{\epsilon p} - 1)e^{-x_{k-1}} & -\epsilon e^{x_k - x_{k-1}}
\end{pmatrix}
\] (11.24)

Lax representation for the map dRTL$-$: (8.18) with
\[
W_k = I + \frac{hD_k^{-1}}{(1 - (1 + h\epsilon)\lambda^2)} \begin{pmatrix}
\lambda^2 (e^{\epsilon p} - 1) & \lambda \epsilon e^{x_k} \\
-\lambda (e^{\epsilon p} - 1)e^{-\bar{x}_{k-1}} & -\epsilon e^{x_k - \bar{x}_{k-1}}
\end{pmatrix}
\] (11.25)

**Special case** $\epsilon = h$. In this case the Lax representation for the map dRTL$+$ simplifies significantly: it reads (4.12) with
\[
L_k = \begin{pmatrix}
\lambda e^{hp_k} - \lambda^{-1} & he^{x_k} \\
-(e^{hp_k} - 1)e^{-x_k} & -h\lambda
\end{pmatrix}
\] (11.26)
\[
V_k = \begin{pmatrix}
\lambda^{-2} & -h\lambda^{-1}e^{x_k} \\
\lambda^{-1}(e^{hp_{k-1}} - 1)e^{-\bar{x}_{k-1}} & 1
\end{pmatrix}
\] (11.27)

12 Bibliographical remarks

The Toda lattice (4.4) was discovered by Toda, see:
- M.Toda. Theory of nonlinear lattices. Springer, 1981.

The relativistic Toda lattice (8.5) was invented by Ruijsenaars:
- S.N.M.Ruijsenaars. Relativistic Toda systems. *Commun. Math. Phys.*, 133 (1990) 217–247.

The modified Toda lattice (5.4) seems to appear for the first time in the work by Yamilov:
- R.I.Yamilov. Classification of Toda–type scalar lattices. In: *Nonlinear Evolution Equations and Dynamical Systems, Proc. of the 8th International Workshop*, World Scientific Publishing, 1993, pp. 423–431 (translation of the preprint of 1989 in Russian).
The Newtonian continuous time systems (9.5), (9.12) were introduced in:

- Yu.B.Suris. New integrable systems related to the relativistic Toda lattice. *J. Phys. A: Math. and Gen.* 30 (1997) 1745–1761.

and their generalizations (10.3), (10.13), (11.3), (11.16) seem to be introduced in the present paper for the first time.

The $2 \times 2$ Lax representation for the Toda lattice with the matrices (4.7), (4.11) belongs to Faddeev with co-authors, see:

- L.D.Faddeev, L.A.Takhtajan. Hamiltonian methods in the theory of solitons. Springer, 1987.

Analogous representations for the two flows of the relativistic Toda lattice with the matrices (8.14), (8.15), (8.17) appeared for the first time in:

- Yu.B.Suris. Discrete time generalized Toda lattices: complete integrability and relation with relativistic Toda lattices. *Phys. Lett. A* 145 (1990) 113–119.

For the modified Toda lattice (5.4) and for the flow (9.5) the $2 \times 2$ Lax representations were derived by Deconinck using an analog of the Walquist–Estabrook approach (they are equivalent to (5.7), (5.10) and to (9.16), (9.17), respectively):

- B.Deconinck. A constructive test for integrability of semi–discrete systems. *Phys. Lett. A* 223 (1996) 45–54.

For the flow (9.12), as well as for the flows from the Sect. 10, 11, such Lax representations are introduced here for the first time.

Turning to the discrete time systems. The Hirota’s discretization of the Toda lattice (Sect. 7.1) was found in:

- R.Hirota. Nonlinear partial difference equations. II. Discrete time Toda equation. *J. Phys. Soc. Japan* 43 (1977) 2074–2078; III Bäcklund transformations for the discrete time Toda equations. *J. Phys. Soc. Japan* 45 (1978) 321–332.

(However, the Lax representation given in the present paper seems to be new). The standard discretization of the Toda lattice (Sect. 7.2), together with the Lax representation, was found in the cited above paper of 1990 by the author, and even slightly earlier in the Russian version of another paper:

- Yu.B.Suris. Generalized toda chains in discrete time. *Leningrad Math. J.* 2 (1991) 339–352.

The discretizations from the Sect. 4 appeared in:
• Yu.B.Suris. Bi–Hamiltonian structure of the \textit{qd} algorithm and new discretizations of the Toda lattice. \textit{Physics Letters A} \textbf{206} (1995) 153–161.

the discretizations from the Sect. 5, 6, 7.3 – in:

• Yu.B.Suris. On some integrable systems related to the Toda lattice. \textit{J. Phys. A: Math. and Gen.} \textbf{30} (1997)

the discretizations from the Sect. 8 – in:

• Yu.B.Suris. A discrete–time relativistic Toda lattice. \textit{J. Phys. A: Math. and Gen.} \textbf{29} (1996) 451–465.

the discretizations from the Sect. 9 – in the already cited paper

• Yu.B.Suris. New integrable systems related to the relativistic Toda lattice. \textit{J. Phys. A: Math. and Gen.}, \textbf{30} (1997) 1745–1761.

The $2 \times 2$ Lax representations for all these discrete time systems seem to appear in the present paper for the first time, as well as the discrete time systems from the Sect. 10, 11.