Understanding Alternating Minimization for Matrix Completion

Moritz Hardt

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Abstract

Alternating minimization is a widely used and empirically successful heuristic for matrix completion and related low-rank optimization problems. Theoretical guarantees for alternating minimization have been hard to come by and are still poorly understood. This is in part because the heuristic is iterative and non-convex in nature. We give a new algorithm based on alternating minimization that provably recovers an unknown low-rank matrix from a random subsample of its entries under a standard incoherence assumption. Our results reduce the sample size requirements of the alternating minimization approach by at least a quartic factor in the rank and the condition number of the unknown matrix. These improvements apply even if the matrix is only close to low-rank in the Frobenius norm. Our algorithm runs in nearly linear time in the dimension of the matrix and, in a broad range of parameters, gives the strongest sample bounds among all subquadratic time algorithms that we are aware of.

Underlying our work is a new robust convergence analysis of the well-known Power Method for computing the dominant singular vectors of a matrix. This viewpoint leads to a conceptually simple understanding of alternating minimization. In addition, we contribute a new technique for controlling the coherence of intermediate solutions arising in iterative algorithms based on a smoothed analysis of the QR factorization. These techniques may be of interest beyond their application here.
1 Introduction

Alternating minimization is an empirically successful heuristic for the matrix completion problem in which the goal is to recover an unknown low-rank matrix from a subsample of its entries. Matrix completion has received a tremendous amount of attention over the past few years due to its fundamental role as an optimization problem and its applicability in number of areas including collaborative filtering and quantum tomography. Alternating minimization has been used early on in the context of matrix completion [BK, HH] and continues to play an important role in practical approaches to the problem. The approach also formed an important component in the winning submission for the Netflix Prize [KBV].

Given a subset $\Omega$ of entries drawn from an unknown matrix $A$, Alternating minimization starts from a poor approximation $X_0 Y_0^\top$ to the target matrix and gradually improves the approximation quality by fixing one of the factors and minimizing a certain objective over the other factor. Here, $X_0, Y_0$ each have $k$ columns where $k$ is the target rank of the factorization. The least squares objective is the typical choice. In this case, at step $\ell$ we solve the optimization problem

$$X_\ell = \arg\min_X \sum_{(i,j) \in \Omega} \left| A_{ij} - (XY_{\ell-1}^\top)_{ij} \right|^2.$$ 

This optimization step is then repeated with $X_\ell$ fixed in order to determine $Y_\ell$ as

$$Y_\ell = \arg\min_Y \sum_{(i,j) \in \Omega} \left| A_{ij} - (X_{\ell-1} Y^\top)_{ij} \right|^2.$$ 

Separating the factors $X_\ell$ and $Y_\ell$ is what makes the optimization step tractable. This basic update step is usually combined with an initialization procedure for finding $X_0, Y_0$, as well as methods for modifying intermediate solutions, e.g., truncating large entries. More than a specific algorithm we think of alternating minimization as a framework for solving a non-convex low-rank optimization problem.

A major advantage of alternating minimization over alternatives is that each update is computationally cheap and has a small memory footprint as we only need to keep track of $2k$ vectors. In contrast, the nuclear norm approach to matrix completion [CR, Rec, CT] requires solving a semidefinite program. The advantage of the nuclear norm approach is that it comes with strong theoretical guarantees under certain assumptions on the unknown matrix and the subsample of its entries. There are two (by now standard) assumptions which together imply that nuclear norm minimization succeeds. The first is that the subsample $\Omega$ includes each entry of $A$ uniformly at random with probability $p$. The second assumption is that the first $k$ singular vectors of $A$ span an incoherent subspace. Informally coherence measures the correlation of the subspace with any standard basis vector. More formally, the coherence of a $k$-dimensional subspace of $\mathbb{R}^n$ is at most $\mu$ if the projection of each standard basis vector has norm at most $\sqrt{\mu k/n}$. The space spanned by the top $k$ singular space of various random matrix models typically satisfies this property with small $\mu$. But also real-world matrices tend to exhibit incoherence when $k$ is reasonably small.

Theoretical results on matrix completion primarily apply to the nuclear norm semidefinite program which is prohibitive to execute on realistic instance sizes. There certainly has been progress on practical algorithms for solving related convex programs [JY, MHT, JS, AKKS, HO]. Unfortunately, these algorithms are not known to achieve the same type of recovery guarantees attained by exact nuclear norm minimization. This raises the important question if there are fast algorithms for matrix completion that come with guarantees on the required sample size comparable to those achieved by nuclear norm minimization. In this work we make progress on
We begin with our result on the exact we expect if the singular values decay rapidly. Given a sample of size \( N \). Here, \( e_i \) is the \( i \)-th standard basis vector so that \( \| e_i \| = 1 \). Our result generalizes straightforwardly to rectangular matrices as we will see.

Our algorithm will output a pair of matrices \((X, Y)\) where \( X \) is an orthonormal \( n \times k \) matrix that approximates \( U \) in the strong sense that \( \|(I - UU^\top)X\| \leq \varepsilon \). Moreover, the matrix \( XY^\top \) is close to \( M \) in Frobenius norm. To state the theorem we formally define the coherence of \( U \) as
\[
\mu(U) \overset{\text{def}}{=} \max_{i \in [n]} \| (n/k)^i \|_2^2
\]
where \( e_i \) is the \( i \)-th standard basis vector.

**Theorem 1.1.** Given a sample of size \( \tilde{O}(pn^2) \) drawn from an unknown \( n \times n \) matrix \( M = UAU^\top \) of rank \( k \) by including each entry with probability \( p \), our algorithm outputs with high probability a pair of matrices \((X, Y)\) such that \( \|(I - UU^\top)X\| \leq \varepsilon \) and \( \|M - XY^\top\|_F \leq \varepsilon \|M\|_F \). provided that
\[
pn \geq k(k + \log(n/\varepsilon))\mu(U)(\|M\|_F/\sigma_k)^2.
\]

Our result should be compared with two remarkable recent works by Jain, Netrapalli and Sanghavi [JNS] and Keshavan [Kes] who gave rigorous sample complexity bounds for alternating minimization. [JNS] obtained the bound \( pn \geq k^7(\sigma_1/\sigma_k)^6 \mu(U)^2 \) and Keshavan obtained the incomparable bound \( pn \geq k(\sigma_1/\sigma_k)^5 \mu(U) \) that is superior when the matrix has small condition number \( \sigma_1/\sigma_k \).

Since \( \|M\|_F \leq \sqrt{k}\sigma_1 \) our result improves upon [JNS] by at least a factor of \( k^4(\sigma_1/\sigma_k)^4 \mu(U) \) and improves on [Kes] as soon as \( \sigma_1/\sigma_k \gg k^{1/3} \). The improvement is larger when \( \|M\|_F = O(\sigma_1) \) which we expect if the singular values decay rapidly.

**Theorem 1.1** is a special case of **Theorem 6.1.** We remark that the number of least squares update steps is bounded by \( O(\log(n/\varepsilon) \log n) \). The cost of performing these update steps is up to a logarithmic factor what dominates the worst-case running time of our algorithm. It can be seen that the least squares problem can be solved in time \( O\left(nk^3 + |\Omega| \cdot k\right) \) which is is linear in \( n + |\Omega| \) and polynomial in \( k \). The number of update steps enters the sample complexity since we assume (as in previous work) that fresh samples are used in each step. However, the logarithmic dependence on \( 1/\varepsilon \) guarantees exponentially fast convergence and allows us to obtain any inverse polynomial error with only a constant factor overhead in sample complexity.

**Noisy matrix completion.** In noisy matrix completion the unknown matrix is only close to low-rank, typically in Frobenius norm. Our results apply to any matrix of the form \( A = M + N \), where \( M = UAU^\top \) is a matrix of rank \( k \) as before and \( N = (I - UU^\top)A \) is the part of \( A \) not captured by the dominant singular vectors. Here, \( N \) can be an arbitrary deterministic matrix that satisfies the following constraints:
\[
\max_{i \in [n]} \| e_i^\top N \|^2 \leq \frac{\mu_N}{n} \cdot \sigma_k^2 \quad \text{and} \quad \max_{ij \in [n]} |N_{ij}| \leq \frac{\mu_N}{n} \cdot \|A\|_F.
\]

Here, \( e_i \) denotes the \( i \)-th standard basis vector so that \( \| e_i^\top N \| \) is the Euclidean norm of the \( i \)-th row of \( N \). The conditions state no entry and no row of \( N \) should be too large compared to the Frobenius norm.
norm of $N$. We can think of the parameter $\mu_N$ as an analog to the coherence parameter $\mu(U)$ that we saw earlier. Since $N$ could be close to full rank, $\mu(V)$ is not a meaningful parameter in general. If the rank of $V$ is $k$, then our assumptions roughly reduce to what is implied by requiring $\mu(V) \leq \mu_N$.

From here on we let $\mu' = \max\{\mu(U), \mu_N, \log n\}$. We have the following theorem.

**Theorem 1.2.** Given a sample of size $\tilde{O}(pn^2)$ drawn from an unknown $n \times n$ matrix $A = M + N$ where $M = UAU^\top$ has rank $k$ and $N = (I - UU^\top)M$ satisfies (2), our algorithm outputs with high probability $(X, Y)$ such that $\| (I - UU^\top)X \| \leq \varepsilon$ and $\| M - XY^\top \|_F \leq \varepsilon \| A \|_F$ provided that

$$pn \geq k(k + \log(n/\varepsilon))\mu' \left( \frac{\| M \|_F + \| N \|_F / \varepsilon}{\sigma_k} \right)^5 \left(1 - \frac{\sigma_{k+1}}{\sigma_k}ight)^5.$$  

(3)

The theorem is a strict generalization of the noise-free case which we recover by setting $N = 0$ in which case the separation parameter $\gamma_k := 1 - \sigma_{k+1}/\sigma_k$ is equal to 1. The result follows from Theorem 6.1 that gives a somewhat stronger sample complexity bound. Compared to our noise-free bound, there are two new parameters that enter the sample complexity. The first one is the separation parameter $\gamma_k$. The second is the quantity $\| N \|_F / \varepsilon$. To interpret this quantity, suppose that $A$ has a good low-rank approximation in Frobenius norm, formally, $\| N \|_F \leq \varepsilon \| A \|_F$ for $\varepsilon \approx 1/2$, then it must also be the case that $\| N \|_F / \varepsilon \leq 2 \| M \|_F$. Our algorithm then finds a good rank $k$ approximation with at most $\tilde{O}(k^3(\sigma_1 / \sigma_k)^2 \mu' n)$ samples assuming $\gamma_k = \Omega(1)$. Hence, assuming that $A$ has a good rank $k$ approximation in Frobenius norm and that $\sigma_k$ and $\sigma_{k+1}$ are well-separated, our bound recovers the noise-free bound up to a constant factor.

Note that if we’re only interested in the second error bound $\| M - XY^\top \|_F \leq \varepsilon \| M \|_F + \| N \|_F$, we can eliminate the dependence on the condition number in the sample complexity entirely. The reason is that any singular value smaller than $\varepsilon \sigma_1 / k$ can be treated as part of the noise matrix. Assuming the condition number is at least $k$ to begin with we can always find two singular values that have separation at least $\Omega(k)$. This ensures that the sample requirement is polynomial in $k$ without any dependence on the condition number and gives us the following corollary.

**Corollary 1.3.** Under the assumptions of Theorem 1.2, if $\sigma_1 \geq k \sigma_k / \varepsilon$, then we can find $X, Y$ such that $\| M - XY^\top \|_F \leq \varepsilon \| A \|_F$ provided that $pn \geq \text{poly}(k)\mu'$.

The previous corollary is remarkable, because small error in Frobenius norm is the most common error measure in the literature on matrix completion. The result shows that in this error measure, there is no dependence on the condition number. The result is tight for $k = O(1)$ up to constant factors even information-theoretically as we will discuss below.

The approach of Jain et al. was adapted to the noisy setting by Gunasekar et al. [GAGG] showing roughly same sample complexity in the noisy setting under some assumptions on the noise matrix. We achieve the same improvements over [GAGG] as we did compared to [JNS] in the noise-free case. Moreover, our assumptions in (2) are substantially weaker than the assumption of [GAGG]. The latter work required the largest entry of $N$ in absolute value to be bounded by $O(\sigma_k / n \sqrt{k})$. This directly implies that each row of $N$ has norm at most $O(\sigma_k / \sqrt{k})$ and that $\| N \|_F \leq O(\sigma_k / \sqrt{k})$. Moreover under this assumption we would have $\gamma_k \geq 1 - o_k(1)$. Keshavan’s result [Kes] also applies to the noisy setting, but it requires $\| N \| \leq (\sigma_k / \sigma_1)^3$ and $\max_j \| e_j^\top N \| \leq \sqrt{\mu(U)k / n} \| N \|$. In particular this bound does not allow $\| N \|_F$ to grow with $\| M \|_F$. Since neither result allows arbitrarily small singular value separation, we cannot use these results to eliminate the dependence on the condition number as is possible using our technique.
Remark on required sample complexity and assumptions. It is known that information-theoretically $\Omega(k \mu(U)n)$ measurements are necessary to recover the unknown matrix [CT] and this bound is achieved (up to log-factors) by the nuclear norm semidefinite program. Compared with the information-theoretic optimum our bound suffers a factor $O(k\|M\|_F/\sigma_k^2)$ loss. While we do not know if this loss is necessary, there is a natural barrier. If we denote by $P_\Omega(A)$ the matrix in which all unobserved entries are 0 and the others are scaled by $1/p$, then $\Omega(k\mu(U)(\|M\|_F/\sigma_k)^2)n$ samples are necessary to ensure that $P_\Omega(A)$ preserves the $k$-th singular value to within constant relative error. Formally, $\|P_\Omega(A) - A\|_2 \leq 0.1\sigma_k$. While this is not a necessary requirement for alternating least squares, it represents the current bottleneck for finding a good initial matrix.

It is also known that without an incoherence assumption the matrix completion problem can be ill-posed and recovery becomes infeasible even information-theoretically [CT]. Moreover, even on incoherent matrices it was recently shown that already the exact matrix completion problem remains computationally hard to approximate in a strong sense [HMRW]. This shows that additional assumptions are needed beyond incoherence to make the problem tractable.

2 Proof overview and techniques

Robust convergence of subspace iteration. An important observation of [JNS] is that the update rule in alternating minimization can be analyzed as a noisy update step of the well known power method for computing eigenvectors, also called subspace iteration when applied to multiple vectors simultaneously. The noise term that arises depends on the sampling error induced by the subsample of the entries. We further develop this point of view by giving a new robust convergence analysis of the power method.

To illustrate the technique, consider a model of numerical linear algebra in which an input matrix $A$ can only be accessed through noisy matrix vector products of the form $Ax + g$, where $x$ is a chosen vector and $g$ is a possibly adversarial noise term. Our goal is to compute the dominant singular vectors $u_1, \ldots, u_k$ of the matrix $A$. Subspace iteration starts with an initial guess, an orthonormal matrix $X_0 \in \mathbb{R}^{n \times k}$ typically chosen at random. The algorithm then repeatedly computes $Y_\ell = AX_{\ell-1} + G_\ell$, followed by an orthonormalization step in order to obtain $X_\ell$ from $Y_\ell$. Here, $G_\ell$ is the noise variable added to the computation.

Theorem 3.8 characterizes the convergence behavior of this general algorithm. An important component of our analysis is the choice of a suitable potential function that decreases at each step. Here we make use of the tangent of the largest principal angle between the subspace $U$ spanned by the first $k$ singular vectors of the input matrix and the $k$-dimensional space spanned by the columns of the iterate $X_\ell$. Principal angles are a very useful tool in numerical analysis that we briefly recap in Section 3. Our analysis shows that the algorithm essentially converges at the rate of $(\sigma_{k+1} + \Delta)/(\sigma_k - \Delta)$ for some $\Delta \ll \sigma_k$ under suitable conditions on the noise matrix $G_\ell$.

Alternating least squares. We recall the well-known least squares update:

$$Y_\ell = \arg \min_Y \|P_\Omega(A - X_{\ell-1}Y^\top)\|_F^2.$$  

(4)

Since we can focus on symmetric matrices without loss of generality, there is no need for an alternating update in which the left and right factor are flipped. We therefore drop the term “alternating”. We can express the optimal $Y_\ell$ as $Y_\ell = AX_{\ell-1} + G_\ell$ using gradient information about the least squares objective. The error term $G_\ell$ has an intriguing property. Its norm $\|G_\ell\|$ depends on the quantity $\|V^\top X_{\ell-1}\|$ which coincides with the sine of the largest principal angle between $U$
and $X_{\ell-1}$. This property ensures that as the algorithm begins to converge the norm of the error term starts to diminish. Near exact recovery is now possible (assuming the matrix has rank at most $k$). A novelty in our approach is that we obtain strong bounds on $\|G_{\ell}\|$ by computing $O(\log n)$ independent copies of $Y_\ell$ (using fresh samples) and taking the componentwise median of the resulting matrices. The resulting procedure called MedianLS is analyzed in Section 4.

A difficulty with iterating the least squares update in general is that it is unclear how well it converges from a random initial matrix $X_0$. In our analysis we therefore use an initialization procedure that finds a matrix $X_0$ that satisfies $\|V^TX_0\| \leq 1/4$. Our initialization procedure is based on (approximately) computing the first $k$ singular vectors of $P_{O}(A)$. To rule out large entries in the vectors we truncate the resulting vectors. While this general approach is standard, our truncation procedure first applies a random rotation to the vectors that leads to a tighter analysis than the naive approach.

**Smooth orthonormalization.** A key novelty in our approach is the way we argue about the coherence of each iterate $X_\ell$. Ideally, we would like to argue that $\mu(X_\ell) = O(\mu^\ell)$. A direct approach would be to argue that $X_\ell$ was obtained from $Y_\ell$ using the QR-factorization and so $X_\ell = Y_\ell R^{-1}$ for some invertible $R$. This gives the bound $\|e_\ell^T X_\ell\| \leq \|e_\ell^T Y_\ell\| \cdot \|R^{-1}\|$ that unfortunately is quite lossy and leads to a dependence on the condition number.

We avoid this problem using an idea that’s closely related to the smoothed analysis of the QR-factorization. Sankar, Spielman and Teng [SST] showed that while the perturbation stability of QR can be quadratic, it is constant after adding a sufficiently large amount of Gaussian noise. In the context of smoothed analysis this is usually interpreted as saying that there are “few bad inputs” for the QR factorization. In our context, the matrix $Y_\ell$ is already the outcome of a noisy operation $Y_\ell = AX_{\ell-1} + G_\ell$ and so there is no harm in actually adding a Gaussian noise matrix $H_\ell$ to $Y_\ell$ provided that the norm of that matrix is no larger than that of $G_\ell$. Roughly speaking, this will allow us to argue that there is no dependence on the condition number when applying the QR-factorization to $Y_\ell$. There are some important complications. The magnitude of $Y_\ell$ may be too large to apply the smoothed analysis argument directly to $Y_\ell$. Instead we observe that the columns of $X_\ell$ are contained in the range $S$ of the matrix $[U | (NX_{\ell-1} + G_\ell + H_\ell)]$. Since $S$ has dimension at most $2k$ it suffices to argue that this space has small coherence. Moreover we can choose $H_\ell$ to be roughly on the same order as $NX_{\ell-1}$ and $G_\ell$ so that the smoothed analysis argument leads to an excellent bound bound on the smallest singular value of $NX_{\ell-1} + G_\ell + H_\ell$. To prove that the coherence is small we need to exhibit a basis for $S$. This requires us to argue about the related matrix $(I - U U^T)(NX_{\ell-1} + G + H_\ell)$ since we need to orthonormalize the last $k$ vectors against the first when constructing a basis. Another minor complication is that we don’t know the magnitude of $G_\ell$ so we need to find the right scaling of $H_\ell$ on the fly. We call the resulting procedure that SmoothQR and analyze its guarantees in Section 5.

**Putting things together.** The final algorithm that we analyze is quite simple to describe as shown in Figure 1. The algorithm makes use of an initialization procedure Initialize that we defer to Section 7. In Section 6 we prove our main theorem. The generalization of our result to rectangular matrices follows from a standard “dilation” argument that we describe in Section D.

The description of the algorithm also uses a helper function called Split that’s used to split the subsample into independent pieces of roughly equal size while preserving the distributional assumption that our theorems use. We discuss Split in Section C.
Input: Observed set of indices $\Omega \subseteq [n] \times [n]$ of an unknown symmetric matrix $A \in \mathbb{R}^{n \times n}$ with entries $P_\Omega(A)$, number of iterations $L \in \mathbb{N}$, error parameter $\varepsilon > 0$, target dimension $k$, coherence parameter $\mu$.

Algorithm $\text{SAltLS}(P_\Omega(A), \Omega, L, k, \varepsilon, \mu)$:

1. $(\Omega_0, \Omega') \leftarrow \text{Split}(\Omega, 2)$, $(\Omega_1, \ldots, \Omega_L) \leftarrow \text{Split}(\Omega', L)$
2. $X_0 \leftarrow \text{Initialize}(P_{\Omega_0}(A), \Omega_0, k, \mu)$
3. For $\ell = 1$ to $L$:
   (a) $Y_\ell \leftarrow \text{MedianLS}(P_{\Omega_\ell}(A), \Omega_\ell, X_{\ell-1}, L, k)$
   (b) $X_\ell \leftarrow \text{SmoothQR}(Y_\ell, \varepsilon, \mu)$

Output: Pair of matrices $(X_{L-1}, Y_L)$

Figure 1: Smoothed alternating least squares ($\text{SAltLS}$)

2.1 Further discussion of related work

There is a vast literature on the topic that we cannot completely survey here. Most closely related is the work of Jain et al. [JNS] that suggested the idea of thinking of alternating least squares as a noisy update step in the Power Method. Our approach takes inspiration from this work by analyzing least squares using the noisy power method. However, our analysis is substantially different in both how convergence and low coherence is argued. The approach of Keshavan [Kes] uses a rather different argument.

As an alternative to the nuclear norm approach, Keshavan, Montanari and Oh [KMO1, KMO2] present two approaches, a spectral approach and an algorithm called $\text{OptSpace}$. The spectral approach roughly corresponds to our initialization procedure and gives similar guarantees. $\text{OptSpace}$ requires a stronger incoherence assumption, has larger sample complexity in terms of the condition number, namely $(\sigma_1/\sigma_k)^6$, and requires optimizing over the Grassmanian manifold. However, the requirement on $N$ achieved by $\text{OptSpace}$ can be weaker than ours in the noisy setting. In the exact case, our algorithm has a much faster convergence rate (logarithmic dependence on $1/\varepsilon$ rather than polynomial).

There are a number of fast algorithms for matrix completion based on either (stochastic) gradient descent [RR] or (online) Frank-Wolfe [JS, HK]. These algorithms generally minimize squared loss on the observed entries subject to a nuclear norm constraint and in general do not produce a matrix that is close to the true unknown matrix on all entries. In contrast, our algorithm guarantees convergence in domain, that is, to the unknown matrix itself. Moreover, our dependence on the error is logarithmic whereas in these algorithms it is polynomial.

Privacy-preserving spectral analysis. Our work is also closely related to a line of work on differentially private singular vector computation [HR1, HR2, Har]. These papers each consider algorithms based on the power method where noise is injected to achieve the privacy guarantee known as Differential Privacy [DMNS]. Hardt and Roth [HR1, HR2, Har] observed that incoherence could be used to obtain improved guarantees. This requires controlling the coherence of the iterates produced by the noisy power method which leads to similar problems as the ones faced here. What’s simpler in the privacy setting is that the noise term is typically Gaussian leading to a cleaner analysis. Our work uses a similar convergence analysis for noisy subspace iteration that was used
2.2 Preliminaries and Notation

We denote by $A^\top$ the transpose of a matrix (or vector) $A$. We use the notation $x \geq y$ do denote that the relation $x \geq Cy$ holds for a sufficiently large absolute constant $C > 0$ independent of $x$ and $y$. We let $\mathcal{R}(A)$ denote the range of the matrix $A$. The coherence of a subspace plays an important role in our analysis.

**Definition 2.1** (Coherence). The $\mu$-coherence of a $k$-dimensional subspace $U$ of $\mathbb{R}^n$ is defined as

$$
\mu(U) \overset{\text{def}}{=} \max_{i \in [n]} \frac{1}{k} \|P_U e_i\|_2^2,
$$

where $e_i$ denotes the $i$-th standard basis vector.

3 Robust local convergence of subspace iteration

Figure 2 presents our basic template algorithm. The algorithm is identical to the standard subspace iteration algorithm except that in each iteration $\ell$, the computation is perturbed by a matrix $G_\ell$. The matrix $G_\ell$ can be adversarially and adaptively chosen in each round. We will analyze under which conditions on the perturbation we can expect the algorithm to converge rapidly.

**Input:** Matrix $A \in \mathbb{R}^{n \times n}$, number of iterations $L \in \mathbb{N}$, target dimension $k$

1. Let $X_0 \in \mathbb{R}^{n \times k}$ be an orthonormal matrix.
2. For $\ell = 1$ to $L$:
   (a) Let $G_\ell \in \mathbb{R}^{n \times k}$ be an arbitrary perturbation.
   (b) $Y_\ell \leftarrow AX_{\ell-1} + G_\ell$
   (c) $X_\ell \leftarrow GS(Y_\ell)$

**Output:** Matrix $X_L$ with $k$ orthonormal columns

![Figure 2: Noisy Subspace Iteration (NSI)](image)

Principal angles are a useful tool in analyzing the convergence behavior of numerical eigenvalue methods. We will use the largest principal angle between two subspaces as a potential function in our convergence analysis.

**Definition 3.1.** Let $X, Y \in \mathbb{R}^{n \times k}$ be orthonormal bases for subspaces $\mathcal{X}, \mathcal{Y}$, respectively. Then, the sine of the largest principal angle between $\mathcal{X}$ and $\mathcal{Y}$ is defined as

$$
\sin \theta(\mathcal{X}, \mathcal{Y}) \overset{\text{def}}{=} \| (I - XX^\top) Y \|.
$$

We use some standard properties of the largest principal angle.

**Proposition 3.2** ([ZK]). Let $\mathcal{X}, \mathcal{Y}, X, Y$ be as in **Definition 3.1** and let $X_\perp$ be an orthonormal basis for the orthogonal complement of $\mathcal{X}$. Then, we have $\cos \theta(\mathcal{X}, \mathcal{Y}) = \sigma_k(X^\top Y)$. and assuming $X^\top Y$ is invertible,

$$
\tan \theta(\mathcal{X}, \mathcal{Y}) = \|X_\perp Y (X^\top Y)^{-1}\|.
$$

From here on we will always assume that $A$ has the spectral decomposition

$$
A = U \Lambda_U U^\top + V \Lambda_V V^\top,
$$

(5)
where \( U \in \mathbb{R}^{n \times k} \), \( V \in \mathbb{R}^{n \times (n-k)} \) corresponding to the first \( k \) and last \( n-k \) eigenvectors respectively. We will let \( \sigma_1 \geq \ldots \geq \sigma_n \) denote the singular values of \( A \) which coincide with the absolute eigenvalues of \( A \) sorted in non-increasing order.

Our convergence analysis tracks the tangent of the largest principal angles between the subspaces \( \mathcal{R}(U) \) and \( \mathcal{R}(X_\ell) \). The next lemma shows a natural condition under which the potential decreases multiplicatively in step \( \ell \). We think of this lemma as a local convergence guarantee, since it assumes that the cosine of the largest principal angle between \( \mathcal{R}(U) \) and \( \mathcal{R}(X_{\ell-1}) \) is already lower bounded by a constant.

**Lemma 3.3** (One Step Local Convergence). Let \( \ell \in \{1, \ldots, L\} \). Assume that

\[
\cos \theta_k(U, X_{\ell-1}) \geq \frac{1}{2} > \frac{\|U^T G_\ell\|}{\sigma_k}.
\]

Then,

\[
\tan \theta(U, X_\ell) \leq \tan \theta(U, X_{\ell-1}) - \frac{\sigma_{k+1} + \frac{2\|V^T G_\ell\|}{\sigma_k}}{\sigma_k - \frac{2\|U^T G_\ell\|}{\sigma_k}}.
\]

**Proof.** We first need to verify that \( X_\ell \) has rank \( k \). This follows if we can show that \( \sigma_k(Y_\ell) > 0 \). Indeed, \( \sigma_k(Y_\ell) \geq \sigma_k(U^T Y_\ell) = \sigma_k(A_U U^T X_{\ell-1} + U^T G_\ell) \geq \sigma_k \cdot \sigma_k(U^T X_{\ell-1}) - \|U^T G_\ell\| \).

The right hand side is strictly greater than zero by our assumption, because \( \sigma_k(U^T X_{\ell-1}) = \cos \theta_k(U, X_{\ell-1}) \). Further, we have \( X_\ell = Y_\ell R \) for some invertible transformation \( R \). Therefore, \( U^T X_\ell \) is invertible and we can invoke Proposition 3.2 to express \( \tan \theta(U, X_\ell) \) as:

\[
\left\| V^T X_\ell (U^T X_\ell)^{-1} \right\| = \left\| V^T Y_\ell R R^{-1} (U^T Y_\ell)^{-1} \right\| = \left\| V^T Y_\ell (U^T Y_\ell)^{-1} \right\|.
\]

Using the fact that \( Y_\ell = A X_{\ell-1} + G_\ell \),

\[
\left\| V^T Y_\ell (U^T Y_\ell)^{-1} \right\| = \left\| V^T Y_\ell (A_U U^T X_{\ell-1} + U^T G_\ell)^{-1} \right\| = \left\| V^T Y_\ell ((A_U + U^T G_\ell(U^T X_{\ell-1})^{-1})U^T X_{\ell-1})^{-1} \right\| = \left\| V^T Y_\ell (U^T X_{\ell-1})^{-1} (A_U + U^T G_\ell(U^T X_{\ell-1})^{-1})^{-1} \right\|.
\]

Putting \( S = A_U + U^T G_\ell(U^T X_{\ell-1})^{-1} \), we therefore get

\[
\left\| V^T Y_\ell (U^T X_{\ell-1})^{-1} \right\| \leq \left\| V^T Y_\ell (U^T X_{\ell-1})^{-1} S^{-1} \right\| \leq \left\| V^T Y_\ell (U^T X_{\ell-1})^{-1} \right\| \cdot \|S^{-1}\| = \frac{\left\| V^T Y_\ell (U^T X_{\ell-1})^{-1} \right\|}{\sigma_k(S)}.
\]

In the second inequality we used the fact that for any two matrices \( P, Q \) we have \( \|PQ\| \leq \|P\| \cdot \|Q\| \).

Let us bound the numerator of the RHS as follows:

\[
\left\| V^T Y_\ell (U^T X_{\ell-1})^{-1} \right\| = \left\| A_V V^T X_{\ell-1}(U^T X_{\ell-1})^{-1} + V^T G_\ell(U^T X_{\ell-1})^{-1} \right\| = \left\| A_V V^T X_{\ell-1}(U^T X_{\ell-1})^{-1} \right\| + \left\| V^T G_\ell(U^T X_{\ell-1})^{-1} \right\| = \left\| A_V \right\| \cdot \left\| V^T X_{\ell-1}(U^T X_{\ell-1})^{-1} \right\| + \left\| V^T G_\ell(U^T X_{\ell-1})^{-1} \right\| = \sigma_{k+1} \cdot \tan \theta(U, X_{\ell-1}) + \left\| V^T G_\ell(U^T X_{\ell-1})^{-1} \right\| = \sigma_{k+1} \cdot \tan \theta(U, X_{\ell-1}) + \frac{\|V^T G_\ell\|}{\cos \theta_k(U, X_{\ell-1})} = \sigma_{k+1} \cdot \tan \theta(U, X_{\ell-1}) + 2\|V^T G_\ell\|.
\]
Here we used the fact that
\[
\| (U^T X_{\ell-1})^{-1} \| = \frac{1}{\sigma_k(U^T X_{\ell-1})} = \frac{1}{\cos \theta_k(U, X_{\ell-1})}.
\]
We also need a lower bound on \( \sigma_k(S) \). Indeed,
\[
\sigma_k(S) \geq \sigma_k(\Lambda_U) - \| U^T G_\ell (U^T X_{\ell-1})^{-1} \| \\
\geq \sigma_k - \| U^T G_\ell \| \cdot \| (U^T X_{\ell-1})^{-1} \| = \sigma_k - 2 \| U^T G_\ell \|.
\]
Note that the RHS is strictly positive due to the assumption of the lemma. Summarizing what we have,
\[
\tan \theta(U, X_\ell) \leq \frac{\sigma_{k+1} \cdot \tan \theta(U, X_{\ell-1}) + 2 \| V^T G_\ell \|}{\sigma_k - 2 \| U^T G_\ell \|}.
\]
This is equivalent to the statement of the lemma as we can see from a simple rearrangement.

The next lemma essentially follows by iterating the previous lemma.

**Lemma 3.4** (Local Convergence). Let \( 0 \leq \varepsilon \leq 1/4 \). Let \( \Delta = \max_{1 \leq \ell \leq L} \| G_\ell \| \) and \( \gamma_k = 1 - \sigma_{k+1}/\sigma_k \). Assume that \( \| V^T X_0 \| \leq 1/4 \) and \( \sigma_k \geq 8\Delta/\gamma_k \varepsilon \). Then,
\[
\| V^T X_\ell \| \leq \max \{ \varepsilon, 2 \cdot \| V^T X_0 \| \cdot \exp(-\gamma_k L/2) \}.
\]

**Proof.** Our first claim shows that once the potential function is below \( \varepsilon \) at step \( \ell - 1 \), it cannot increase beyond \( \varepsilon \).

**Claim 3.5.** Let \( \ell \geq 1 \). Suppose that \( \tan \theta(U, X_{\ell-1}) \leq \varepsilon \). Then, \( \tan \theta(U, X_\ell) \leq \varepsilon \).

**Proof.** By our assumption, \( \cos \theta_k(U, X_{\ell-1}) \geq \sqrt{1 - \varepsilon^2} \geq 15/16 \). Together with the lower bound on \( \sigma_k \), the assumptions for Lemma 3.3 are met. Hence, using our assumptions,
\[
\tan \theta(U, X_\ell) \leq \frac{(1 - \gamma_k)\sigma_k \varepsilon + 2\Delta}{\sigma_k - 2\Delta} \leq \varepsilon.
\]

Our second claim shows that if the potential is at least \( \varepsilon \) at step \( \ell - 1 \), it will decrease by a factor \( 1 - \gamma_k/2 \).

**Claim 3.6.** Let \( \ell \geq 1 \) Suppose that \( \tan \theta(U, X_{\ell-1}) \in [\varepsilon, 1/2] \). Then,
\[
\tan \theta(U, X_\ell) \leq (1 - \gamma_k/2) \tan \theta(U, X_{\ell-1}).
\]

**Proof.** Using the assumption of the claim we have \( \cos \theta(U, X_{\ell-1}) \geq \frac{1}{\tan \theta(U, X_{\ell-1})} \geq 1/2 > \Delta/\sigma_k \). We can therefore apply Lemma 3.3 to conclude
\[
\tan \theta(U, X_\ell) \leq \tan \theta(U, X_{\ell-1}) \cdot \frac{(1 - \gamma_k)\sigma_k + 2\Delta}{\sigma_k - 2\Delta} \leq \tan \theta(U, X_{\ell-1}) \cdot \frac{(1 - \gamma_k)(1 + \gamma_k/4)}{1 - \gamma_k/4} \leq \tan \theta(U, X_{\ell-1})(1 - \gamma_k/2)
\]

The two previous claims together imply that
\[
\tan \theta(U, X_\ell) \leq \max \{ \tan \theta(U, X_0)(1 - \gamma_k/2)^L, \varepsilon \},
\]
provided that \( \tan \theta(U, X_0) \leq 1/2 \). This is the case since we assumed that \( \sin \theta(U, X_0) \leq 1/4 \). Note that \((1 - \gamma_k/2)^L \leq \exp(-\gamma_k L/2)\). It remains to observe that \( \| V^T X_\ell \| \leq \tan \theta(U, X_\ell) \) and further \( \tan \theta(U, X_0) \leq 2 \| V^T X_0 \| \) by our assumption on \( X_0 \).
In our application later on the error terms $\|G_\ell\|$ decrease as $\ell$ increases and the algorithm starts to converge. We need a convergence bound for this type of shrinking error. The next definition expresses a condition on $G_\ell$ that allows for a useful convergence bound.

**Definition 3.7 (Admissible).** Let $\gamma_k = 1 - \sigma_{k+1}/\sigma_k$. We say that the pair of matrices $(X_{\ell-1}, G_\ell)$ is $\varepsilon$-admissible for NSI if

$$\|G_\ell\| \leq \frac{1}{32} \gamma_k \sigma_k \|V^TX_{\ell-1}\| + \frac{\varepsilon}{32} \gamma_k \sigma_k. \tag{7}$$

We say that a family of matrices $\{(X_{\ell-1}, G_\ell)\}_{\ell=1}^L$ is $\varepsilon$-admissible for NSI if each member of the set is $\varepsilon$-admissible. We will use the notation $\{G_\ell\}$ as a shorthand for $\{(X_{\ell-1}, G_\ell)\}_{\ell=1}^L$.

We have the following convergence guarantee for admissible noise matrices.

**Theorem 3.8.** Let $\gamma_k = 1 - \sigma_{k+1}/\sigma_k$. Let $\varepsilon \leq 1/2$. Assume that the family of noise matrices $\{G_\ell\}$ is $(\varepsilon/2)$-admissible for NSI and that $\|V^TX_0\| \leq 1/4$. Then, we have $\|V^TX_\ell\| \leq \varepsilon$ for any $\ell \geq 4\gamma_k^{-1} \log(1/\varepsilon)$.

**Proof.** We prove by induction that for every $t \geq 0$ after $L_t = 4t\gamma_k^{-1}$ steps, we have

$$\|V^TX_t\| \leq \max\left\{2^{-(t+1)}, \varepsilon\right\}.$$ 

The base case ($t = 0$) follows directly from the assumption that $\|V^TX_0\| \leq 1/4$. We turn to the inductive step. By induction hypothesis, we have $\|V^TX_t\| \leq \max\left\{2^{-(t+1)}, \varepsilon\right\}$. We apply Lemma 3.4 with “$X_0 = X_t$,” and error parameter $\max\left\{2^{-t}, \varepsilon\right\}$ and $L = L_{t+1} - L_t$. The conditions of the lemma are satisfied as can be easily checked using the assumption that $\{G_\ell\}$ is $\varepsilon/2$-admissible. Using the fact that $L_{t+1} - L_t = 4/\gamma_k$, the conclusion of the lemma gives

$$\|V^TX_{t+1}\| \leq \max\left\{\varepsilon, 2 \cdot \max\left\{\varepsilon, 2^{-(t+1)}\right\} \exp\left(-\frac{\gamma_k(L_{t+1} - L_t)}{2}\right)\right\} \leq \max\left\{\varepsilon, 2^{-(t+2)}\right\}. \tag*{\blacksquare}$$

## 4 Least squares update rule

**Input:** Target dimension $k$, observed set of indices $\Omega \subseteq [n] \times [n]$ of an unknown symmetric matrix $A \in \mathbb{R}^{n \times n}$ with entries $P_\Omega(A)$, orthonormal matrix $X \in \mathbb{R}^{n \times k}$.

**Algorithm LS($P_\Omega(A), \Omega, X, L, k$):

\[
Y \leftarrow \arg \min_{Y \in \mathbb{R}^{n \times k}} \|P_\Omega(A - XY^T)\|_F
\]

**Output:** Pair of matrices $(X, Y)$

Figure 3: Least squares update

Figure 4 describes the least squares update step specialized to the case of a symmetric matrix. Our goal is to express this update step as an update step of the form $Y = AX + G$ so that we may apply our analysis of noisy subspace iteration. This syntactic transformation is explained in Section 4.1 followed by a bound on the norm of the error term $G$ in Section 4.2.
4.1 From alternating least squares to noisy subspace iteration

The optimizer $Y$ satisfies a set of linear equations that we derive from the gradient of the objective function.

**Lemma 4.1** (Optimality Condition). Let $P_i: \mathbb{R}^n \rightarrow \mathbb{R}^n$ be the linear projection onto the coordinates in $\Omega_i = \{ j : (i, j) \in \Omega \}$ scaled by $p^{-1} = n/(|\Omega|)$, i.e., $P_i = p^{-1} \sum_{j \in \Omega} e_j e_j^\top$. Further, define the matrix $B_i \in \mathbb{R}^{k \times k}$ as $B_i = X^T P_i X$ and assume that $B_i$ is invertible. Then, for every $i \in [n]$, the $i$-th row of $Y$ satisfies $e_i^\top Y = e_i^\top A P_i X B_i^{-1}$.

**Proof.** Call the objective function $f(Y) = \| P_\Omega (A - X Y^\top) \|^2_F$. We note that for every $i \in [n], j \in [k]$, we have $\frac{\partial f}{\partial y_{ij}} = -2 \sum_{s \in \Omega} A_{is} X_{sj} + 2 \sum_{r=1}^k Y_{ir} \sum_{s \in \Omega} X_{sj} X_{sr}$. From this we conclude that the optimal $Y$ must satisfy $e_i^\top A P_i X = e_i^\top Y X^T P_i X = e_i^\top Y B_i$. Hence, $e_i^\top Y = e_i^\top A P_i X B_i^{-1}$. \[\blacksquare\]

The assumption that $B_i$ is invertible is essentially without loss of generality. Indeed, we will later see that $B_i$ is invertible (and in fact close to the identity matrix) with very high probability. We can now express the least squares update as $Y = AX + G$ where we derive some useful expression for $G$.

**Lemma 4.2.** Let $E = (I - XX^\top) U$. We have $Y = AX + G$ where $G = G^M + G^N$ and the matrices $G^M$ and $G^N$ are so each row $i \in [n]$ if $B_i$ is invertible we have the following expressions:

$$
\begin{align*}
    e_i^\top G^M &= e_i^\top U \Lambda U^\top P_i X B_i^{-1} \\
    e_i^\top G^N &= e_i^\top (N P_i X B_i^{-1} - N X).
\end{align*}
$$

**Proof.** By **Lemma 4.1**, $e_i^\top Y = e_i^\top A P_i X B_i^{-1} = e_i^\top Y = e_i^\top M P_i X B_i^{-1} + e_i^\top N P_i X B_i^{-1}$. Let $C_i = U^T P_i X$ and put $D = U^T X$. On the one hand,

$$
\begin{align*}
    e_i^\top M P_i X B_i^{-1} &= e_i^\top U \Lambda U C_i B_i^{-1} = e_i^\top (U \Lambda D - U \Lambda (D B_i - C_i) B_i^{-1}) \\
    &= e_i^\top M X - e_i^\top U \Lambda U (D B_i - C_i) B_i^{-1}
\end{align*}
$$

On the other hand,

$$
C_i = U^T P_i X = (XX^\top + E) P_i X = (U^T X) X^T P_i X + E^\top (P_i X) = DB_i + E^\top P_i X.
$$

Hence, as desired, $e_i^\top M P_i X B_i^{-1} = e_i^\top M X - e_i^\top U \Lambda U E^\top P_i X B_i^{-1}$. Finally, it follows directly by definition that $e_i^\top N P_i X B_i^{-1} = e_i^\top N X + e_i^\top G^N$. Putting the previous two equations together, we conclude that $Y = MX + G^M + NX + G^N = AX + G^M + G^N$. \[\blacksquare\]

4.2 Deviation bounds for the least squares update

In this section we analyze the norm of the error term $G$ from the previous section. More specifically, we prove a bound on the norm of each row of $G$. Our bound uses the fact that the matrix $E$ appearing in the expression for the error term satisfies $\|E\| = \|V^T X\|$. This gives us a bound in terms of the quantity $\|V^T X\|$.

**Lemma 4.3.** Let $\delta \in (0, 1)$. Assume that each entry is included in $\Omega$ independently with probability

$$
p \geq \frac{k \mu(X) \log n}{\delta^2 n}.
$$

Then, for every $i \in [n]$, $\mathbb{P}(\|e_i^\top G\| > \delta \cdot (\|e_i^\top M\| \cdot \|V^T X\| + \|e_i^\top N\|)) \leq \frac{1}{5}$. 

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Proof. Fix $i \in [n]$. Lemma A.5 shows that with probability 9/10 we have $\|e_i^T G^M\| \leq \delta \cdot \|e_i^T M\| \cdot \|V^T X\|$. Similarly, Lemma A.6 shows that with probability 9/10 we have $\|e_i^T G^N\| \leq \delta \cdot \|e_i^T N\|$. Both events occur with probability 4/5 and in this case we have

$$\|e_i^T G\| \leq \|e_i^T G^M\| + \|e_i^T G^N\| \leq \delta \cdot (\|e_i^T M\| \cdot \|V^T X\| + \|e_i^T N\|).$$

\[\blacksquare\]

### 4.3 Median least squares update

Given the previous error bound we can achieve a strong concentration bound by taking the component-wise median of multiple independent samples of the error term.

**Lemma 4.4.** Let $G_1, \ldots, G_t$ be i.i.d. copies of $G$. Let $\overline{G} = \text{median}(G_1, \ldots, G_t)$ be the component-wise median of $G_1, \ldots, G_t$ and assume $p$ satisfies (8). Then, for every $i \in [n]$,

$$\mathbb{P}\{\|e_i^T \overline{G}\| > \delta (\|e_i^T M\| \cdot \|V^T X\| + \|e_i^T N\|)\} \leq \exp(-\Omega(t)).$$

**Proof.** Fix $i \in [n]$ and let $g_1, \ldots, g_t \in \mathbb{R}^k$ denote the $i$-th rows of $G_1, \ldots, G_t$. Let $S = \{j \in [t]: \|g_j\| \leq B\}$ where $B = (\delta/4)\left(\|e_i^T M\| \cdot \|V^T X\| + \|e_i^T N\|\right)$. Applying Lemma 4.3 with error parameter $\delta/4$ it follows that $\mathbb{E}|S| \geq 4t/5$. Moreover, the draws of $g_j$ are independent. So we can apply a Chernoff bound to argue that $|S| > 2t/3$ with probability $1 - \exp(-\Omega(t))$. Assuming that this event occurs, we claim that $\overline{g} = \text{median}(g_1, \ldots, g_t)$ satisfies $\|\overline{g}\| \leq 4B$ and this claim establishes the lemma.

To prove this claim, fix any coordinate of $r \in [k]$. By the median property $\{[j: (g_j)_r^2 \geq \overline{g}_r^2]\} \geq t/2$. Since $|S| > 2t/3$ this means that at least $t/3$ vectors with $j \in S$ have $(g_j)_r^2 > \overline{g}_r^2$. In particular, the average value of $(g_j)_r^2$ over all $j \in S$ must be at least $t\overline{g}_r^2/3|S| \geq \overline{g}_r^2/3$. This shows that the average of $\|g_j\|^2$ over all $j \in S$ must be at least $\|\overline{g}\|^2/3$. On the other hand, we also know that the average squared norm in $S$ is at most $B^2$ by definition of the set $S$. It follows that $\|\overline{g}\|^2 \leq 3B^2$. This implies what we needed to show.

We can now conclude a strong concentration bound for the median of multiple independent solutions to the least squares minimization step. This way we can obtain the desired error bound for all rows simultaneously. This leads to the following extension of the least squares update rule.

**Input:** Target dimension $k$, observed set of indices $\Omega \subseteq [n] \times [n]$ of an unknown symmetric matrix $A \in \mathbb{R}^{nxn}$ with entries $P_{\Omega}(A)$, orthonormal matrix $X \in \mathbb{R}^{n \times k}$.

**Algorithm** MEDIANLS($P_{\Omega}(A), \Omega, X, L, k$):

1. $(\Omega_1, \ldots, \Omega_t) \leftarrow \text{Split}(\Omega, t)$ for $t = O(\log n)$.
2. $Y_t \leftarrow \text{LS}(P_{\Omega_1}(A), \Omega, X, L, k)$

**Output:** Pair of matrices $(X, \text{median}(Y_1, \ldots, Y_t))$

**Figure 4:** Median least squares update

**Lemma 4.5.** Let $\Omega$ be a sample in which each entry is included independently with probability $p \geq \frac{k\mu(X)\log^2 n}{\delta n}$. Let $Y \leftarrow \text{MEDIANLS}(P_{\Omega}(A), \Omega, X, L, k)$. Then, we have with probability $1 - 1/n^3$ that $\overline{Y} = AX + G$ and $\overline{G}$ satisfies for every $i \in [n]$ the bound $\|e_i^\top \overline{G}\| \leq \delta \|e_i^\top M\| \cdot \|V^\top X\| + \delta \|e_i^\top N\|$. 

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Proof. By Lemma C.1, the samples $Ω_1, \ldots, Ω_t$ are independent and each set $Ω_j$ includes each entry with probability at least $p/t$. The output satisfies $Y = \text{median}(Y_1, \ldots, Y_t)$, where each $Y_j$ is of the form $Y_j = AX + G_j$. It follows that $\text{median}(Y_1, \ldots, Y_t) = AX + \bar{G}$ where $\bar{G} = \text{median}(G_1, \ldots, G_t)$. We can therefore apply Lemma 4.4 to conclude the lemma using the fact that $t = O(\log n)$ allows us to take a union bound over all $n$ rows. ■

5 Incoherence via smooth QR factorization

As part of our analysis of alternating minimization we need to show that the intermediate solutions $X_\ell$ have small coherence. For this purpose we propose an idea inspired by Smoothed Analysis of the QR factorization [SST]. The problem with applying the QR factorization directly to $Y_\ell$ is that $Y_\ell$ might be ill-conditioned. This can lead to a matrix $X_\ell$ (via QR-factorization) that has large coordinates and whose coherence is therefore no longer as small as we desire. A naive bound on the condition number of $Y_\ell$ would lead to a large loss in sample complexity. What we show instead is that a small Gaussian perturbation to $Y_\ell$ leads to a sufficiently well-conditioned matrix $\tilde{Y}_\ell = Y_\ell + H_\ell$. Orthonormalizing $\tilde{Y}_\ell$ now leads to a matrix of small coherence. Intuitively, since the computation of $Y_\ell$ is already noisy the additional noise term has little effect so long as its norm is bounded by that of $G_\ell$. Since we don’t know the norm of $G_\ell$, we have to search for the right noise parameter using a simple binary search. We call the resulting procedure SmoothQR and describe in in Figure 5.

| Input: Matrix $Y \in \mathbb{R}^{n \times k}$, parameters $\mu, \varepsilon > 0$. |
| Algorithm SmoothQR($Y, \varepsilon, \mu$) |
| 1. $X \leftarrow \text{QR}(Y), H \leftarrow 0, \sigma \leftarrow \varepsilon \|Y\|/n$. |
| 2. While $\mu(X) > \mu$ and $\sigma \leq \|Y\|$: |
| (a) $X \leftarrow \text{GS}(Y + H)$ where $H \sim N(0, \sigma^2/n)$ |
| (b) $\sigma \leftarrow 2\sigma$ |
| Output: Pair of matrices $(X,H)$ |

Figure 5: Smooth Orthonormalization (SmoothQR)

To analyze the algorithm we begin with a lemma that analyzes the smallest singular value under a Gaussian perturbation. What makes the analysis easier is the fact that the matrices we’re interested in are rectangular. The square case was considered in [SST] and requires more involved arguments.

Lemma 5.1. Let $G \in \mathbb{R}^{n \times k}$ be any matrix with $\|G\| \leq 1$ and let $V$ be a $n - k$ dimensional subspace with orthogonal projection $P_V$. Let $H \sim N(0, \tau^2/n)^{n \times k}$ be a random Gaussian matrix. Assume $k = o(n/\log n)$. Then, with probability $1 - \exp(-\Omega(n))$, we have $\sigma_k(P_V(G + H)) \geq \Omega(\tau)$.

The proof follows from standard concentration arguments and is contained in Section B. To use this lemma in our context we’ll introduce a variant of $\mu$-coherent that applies to matrices rather than subspaces.

Definition 5.2 ($\rho$-coherence). Given a matrix $G \in \mathbb{R}^{n \times k}$ we let $\rho(G) \overset{\text{def}}{=} \frac{n}{k} \max_{i \in [n]} \|e_i^T G\|^2$. 

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The next lemma is our main technical tool in this section. It shows that adding a Gaussian noise term leads to a bound on the coherence after applying the QR-factorization.

**Lemma 5.3.** Let \( k = o(n/\log n) \) and \( \tau \in (0, 1) \). Let \( U \in \mathbb{R}^{n \times k} \) be an orthonormal matrix. Let \( G \in \mathbb{R}^{n \times k} \) be a matrix such that \( \|G\| \leq 1 \). Let \( H \sim N(0, \tau^2/n) \) be a random Gaussian matrix. Then, with probability \( 1 - \exp(-\Omega(n)) - n^5 \), there is an orthonormal matrix \( Q \in \mathbb{R}^{n \times 2k} \) such that:

1. \( \mathcal{R}(Q) = \mathcal{R}([U \mid G + H]) \).
2. \( \mu(Q) \leq O\left( \frac{1}{\tau^2} \cdot (\rho(G) + \mu(U) + \log n) \right) \).

**Proof.** First note that \( \mathcal{R}([U \mid G + H]) = \mathcal{R}([U \mid (I - UU^\top)(G + H)]) \). Let \( B = (I - UU^\top)(G + H) \). Applying the QR-factorization to \( [U \mid B] \), we can find two orthonormal matrices \( Q_1, Q_2 \in \mathbb{R}^{n \times k} \) such that have that \( [Q_1 \mid Q_2] = [U \mid BR^{-1}] \) where \( R \in \mathbb{R}^{k \times k} \). That is \( Q_1 = U \) since \( U \) is already orthonormal. Moreover, the columns of \( B \) are orthogonal to \( U \) and therefore we can apply the QR-factorization to \( U \) and \( B \) independently. We can now apply Lemma 5.1 to the \( (n - k) \)-dimensional subspace \( U^\perp \) and the matrix \( G + H \). It follows that with probability \( 1 - \exp(-\Omega(n)) \), we have \( \sigma_k(B) \geq \Omega(\tau) \). Assume that this event occurs.

Also, observe that \( \sigma_k(B) = \sigma_k(R) \). The second condition is now easy to verify

\[
\frac{n}{k} \left\| e_i^\top Q \right\|^2 = \frac{n}{k} \left\| e_i^\top U \right\|^2 + \frac{n}{k} \left\| e_i^\top BR^{-1} \right\|^2 = \mu(U) + \frac{n}{k} \left\| e_i^\top BR^{-1} \right\|^2
\]

On the other hand,

\[
\frac{n}{k} \left\| e_i^\top BR^{-1} \right\|^2 \leq \frac{n}{k} \left\| e_i^\top B \right\|^2 \left\| R^{-1} \right\|^2 \leq O\left( \frac{n}{k\tau^2} \left\| e_i^\top B \right\|^2 \right)
\]

where we used the fact that \( \left\| R^{-1} \right\| = 1/\sigma_k(R) = O(1/\tau) \). Moreover,

\[
\frac{n}{k} \left\| e_i^\top B \right\|^2 \leq 2 \frac{n}{k} \left\| e_i^\top (I - UU^\top)G \right\|^2 + 2\rho((I - UU^\top)H) \leq 2\rho(G) + 2\rho(UU^\top G) + 2\rho((I - UU^\top)H).
\]

Finally, \( \rho(UU^\top G) \leq \mu(U) \left\| U^\top G \right\|^2 \leq \mu(U) \) and, by Lemma B.1, we have \( \rho((I - UU^\top)H) \leq O(\log n) \) with probability \( 1 - 1/n^3 \). The lemma follows with a union bound over the failure probabilities. \( \blacksquare \)

The next lemma states that when SmoothQR is invoked on an input of the form \( AX + G \) with suitable parameters, the algorithm outputs a matrix of the form \( X' = QR(AX + G + H) \) whose coherence is bounded in terms of \( \mu(U) \) and \( \rho(G) \) and moreover \( H \) satisfies a bound on its norm. The lemma also permits to trade-off the amount of additional noise introduced with the resulting coherence parameter.

**Lemma 5.4.** Let \( \tau > 0 \) and assume \( k = o(n/\log n) \) and \( \mu(U)k \leq n \). There is an absolute constant \( C_{3,4} > 0 \) such that the following claim holds. Let \( G \in \mathbb{R}^{n \times k} \). Let \( X \in \mathbb{R}^{n \times k} \) be an orthonormal matrix such that \( \nu \geq \max(\|G\|, \|NX\|) \). Assume that

\[
\mu \geq \frac{C_{5.4}}{\tau^2} \left( \mu(U) + \frac{\rho(G) + \rho(NX)}{\nu^2} + \log n \right)
\]

Then, for every \( \varepsilon \leq \tau \nu \) satisfying \( \log(n/\varepsilon) \leq n \) and every \( \mu \leq n \), we have with probability \( 1 - O(n^{-4}) \), the algorithm SmoothQR(AX + G, \varepsilon, \mu) terminates in \( O(\log(n/\varepsilon)) \) steps and outputs \((X',H)\) such that \( \mu(X') \leq \mu \) and where \( H \) satisfies \( \|H\| \leq \tau \nu \).
While the first term has a quadratic dependence on \( \sigma^2 \leq \tau^2 \nu^2 / 4 \). We claim that in this case with probability \( 1 - \exp(-\Omega(n)) \) we must have that \( \|H\| \leq \tau \nu \). Indeed, assuming the algorithm terminates when \( \sigma^2 \leq \tau^2 \nu^2 / k \), the algorithm took at most \( t = O(\log(n/\epsilon)) \leq O(n) \) steps. Let \( H_1, \ldots, H_t \) denote the random Gaussian matrices generated in each step. We claim that each of them satisfies \( \|H_t\| \leq \tau \nu \). Note that for all \( t \) we have \( \mathbb{E} \|H_t\|^2 \leq \tau^2 \nu^2 / 4 \). The claim therefore follows directly from tail bounds for the Frobenius norm of Gaussian random matrices and holds with probability \( 1 - \exp(-\Omega(n)) \). The next claim now finishes the proof.

**Claim 5.5.** With probability \( 1 - O(1/n^4) \), the algorithm terminates in an iteration where \( \sigma^2 \leq \tau^2 \nu^2 / 4 \).

To prove the claim, consider the first iteration in which \( \sigma^2 \geq \tau^2 \nu^2 / 8 \). Let us define \( G' = (NX + G)/\nu \). We can now apply Lemma 5.3 to the matrix \( G' \) which satisfies the assumption of the lemma that \( \|G'\| \leq 1 \). The lemma then entails that with the stated probability bound there is an orthonormal \( n \times 2k \) matrix \( Q \) such that

\[
\mathcal{R}(Q) = \mathcal{R}([U \mid G' + H]) = \mathcal{R}([U \mid G + NX + H]),
\]

and moreover \( \mu(Q) \leq O\left( \frac{1}{\tau^2} \cdot (\rho(G) + \mu(U) + \log n) \right) \). On the one hand,

\[
\mathcal{R}(X') = \mathcal{R}(AX + G + H) = \mathcal{R}(MX + NX + G + H) \subseteq \mathcal{R}([U \mid NX + G + H]) = \mathcal{R}(V).
\]

The inclusion follows from the fact that \( U \) is an orthonormal basis for the range of \( MX = U\Sigma_U U^T X \). On the other hand, \( \rho(G') = O(\rho(G/\nu) + \rho(NX/\nu')) \). Hence, by Lemma B.2 and the fact that \( \dim(Q) \leq 2 \dim(X') \), we have \( \mu(X') \leq 2 \mu(Q) \leq \mu \). This shows that the termination criterion of the algorithm is satisfied provided we pick \( C_{5,4} \) large enough.

### 6 Convergence bounds for alternating minimization

The total sample complexity we achieve is the sum of two terms. The first one is used by the initialization step that we discuss in Section 7. The second term specifies the sample requirements for iterating the least squares algorithm. It therefore makes sense to define the following two quantities:

\[
p_{\text{init}} = \frac{k^2 \mu^* ||A||_2^2 \log n}{\gamma_k^2 \sigma_k^2 n} \quad \text{and} \quad p_{\text{LS}} = \frac{k \mu^* (||M||_F^2 + ||N||_F^2 / \epsilon^2) \log(n/\epsilon) \log^2 n}{\gamma_k^5 \sigma_k^2 n}
\]

While the first term has a quadratic dependence on \( k \) it does not depend on \( \epsilon \) at all and it has single logarithmic factor. The second term features a linear dependence on \( k \). Our main theorem shows that if the sampling probability is larger than the sum of these two terms, the algorithm converges rapidly to the true unknown matrix.

**Theorem 6.1** (Main). Let \( k, \epsilon > 0 \). Let \( A = M + N \) be a symmetric \( n \times n \) matrix where \( M \) is a matrix of rank \( k \) with the spectral decomposition \( M = U A_U U^T \) and \( N = (I - U U^T)A = V A_V V^T \) satisfies (2). Let \( \gamma_k = 1 - \sigma_{k+1} / \sigma_k \) where \( \sigma_k \) is the smallest singular value of \( M \) and \( \sigma_{k+1} \) is the largest singular value of \( N \).

Then, there are parameters \( \mu = \Theta(\gamma_k^{-2} k (\mu^* + \log n)) \) and \( L = \Theta(\gamma_k^{-1} \log(n/\epsilon)) \) such that the output \( (X, Y) \) of \( \text{SALT}LS(P_{\Omega}(A), \Omega, k, L, \epsilon, \mu) \) satisfies \( \| (I - U U^T) X_L \| \leq \epsilon \) with probability \( 9/10 \).

Before we prove the theorem in Section 6.1, we will state an immediate corollary that gives bounds on the reconstruction error in the Frobenius norm.
Corollary 6.2 (Reconstruction error). Under the assumptions of Theorem 6.1, we have that the output \((X, Y)\) of \textsc{SaltLS} satisfies \(\|M - XY^T\|_F \leq \varepsilon \|A\|_F\) with probability \(9/10\).

Proof. Let \((X, Y)\) be the matrices given by our algorithm when invoked with error parameter \(\varepsilon/2\). By Theorem 6.1 we have \(\|UU^T - XX^T\| = \|(I - UU^T)X\| \leq \frac{\varepsilon}{2}\). Using the proof of Theorem 6.1 we also know that \(Y = AX + G\) where \(G\) is \((\varepsilon/4)\)-admissible so that \(\|G\|_F \leq \varepsilon \sigma_k/2\). Consequently,

\[
\|M - XY^T\|_F = \|M - XX^T A + XG\|_F \leq \|UU^T A - XX^T A\|_F + \|XG\|_F \\
\leq \|UU^T - XX^T\| \|A\|_F + \|G\|_F \\
\leq (\varepsilon/2)\|A\|_F + (\varepsilon/2)\sigma_k \leq \varepsilon\|A\|_F.
\]

In the second inequality we used that for all matrices \(P, Q\) we have \(\|PQ\|_F \leq \|P\| \cdot \|Q\|_F\).

6.1 Proof of Theorem 6.1

Proof. We first apply Theorem 7.1 (shown below) to conclude that with probability \(19/20\), the initial matrix \(X_0\) satisfies \(\|V^T X_0\| \leq 1/4\) and \(\mu(X_0) \leq 32\mu(U) \log n\). Assume that this event occurs. Our goal is now to apply Theorem 3.8. Consider the sequence of matrices \(\{(X_{\ell-1}, \tilde{G}_\ell)\}_{\ell=1}^L\) obtained by the execution of \textsc{SaltLS} starting from \(X_0\) and letting \(\tilde{G}_\ell = G_\ell + H_\ell\) where \(G_\ell\) is the error term corresponding to the \(\ell\)-step of \textsc{MedianLS}, and \(H_\ell\) is the error term introduced by the application of \textsc{SmoothQR} at step \(\ell\). To apply Theorem 3.8, we need to show that this sequence of matrices is \((\varepsilon/2)\)-admissible for \textsc{NSI} with probability \(19/20\). The theorem then directly gives that \(\|V^T X_L\| \leq \varepsilon\) and this would conclude our proof by summing up the error probabilities.

Let

\[
\tau = \frac{\gamma_k}{128} \quad \text{and} \quad \tilde{\mu} = \frac{C_{5,4}}{\tau^2} (20\mu^* + \log n).
\]

Let \(\mu\) be any number satisfying \(\mu \geq \tilde{\mu}\). Since \(\tilde{\mu} = \Theta(\gamma_k^{-2} k(\mu^* + \log n))\), this satisfies the requirement in the theorem. We prove that with probability \(9/20\), the following three claims hold:

1. \(\{(X_{\ell-1}, G_\ell)\}_{\ell=1}^L\) is \((\varepsilon/4)\)-admissible,
2. \(\{(X_{\ell-1}, H_\ell)\}_{\ell=1}^L\) is \((\varepsilon/4)\)-admissible,
3. for all \(\ell \in \{0, \ldots, L - 1\}\), we have \(\mu(X_\ell) \leq \mu\).

This implies the claim that we want using a triangle inequality since \(\tilde{G}_\ell = G_\ell + H_\ell\).

The proof of these three claims is by mutual induction. For \(\ell = 0\), we only need to check the third claim which follows from the fact that \(X_0\) satisfies the coherence bound. Now assume that all three claims hold at step \(\ell - 1\), we will argue that the with probability \(1 - n/100\), all three claims hold at step \(\ell\). Since \(L \leq n\), this is sufficient.

The first claim follows from Lemma 4.4 using the induction hypothesis that \(\mu(X_{\ell-1}) \leq \tilde{\mu}\). Specifically, we apply the lemma with \(\delta = c \min\{\gamma_k \sigma_k \|M\|_F, \varepsilon \gamma_k \sigma_k / \|N\|_F\}\) for sufficiently small constant \(c > 0\). The lemma requires the lower bound \(p \geq \frac{k\mu^* \log n}{\delta n}\). We can easily verify that the right hand side is a factor \(L = \Theta(\gamma_k^{-1} \log(n/\varepsilon))\) smaller than what is provided by the assumption of the theorem. This is because new samples are used in each of the \(L\) steps so that we need to divide the given bound by \(L\). Lemma 4.4 now gives with probability \(1 - 1/n^3\) the upper bound

\[
\|G_\ell\|_F \leq \frac{1}{4} \left( \frac{1}{32} \gamma_k \sigma_k \left\|V^T X_{\ell-1}\right\| + \frac{\varepsilon}{32} \gamma_k \sigma_k \right).
\]
In particular, this satisfies the definition of $\varepsilon/4$-admissibility. We proceed assuming that this event occurs as the error probability is small enough to ignore.

The remaining two claims follow from Lemma 5.4. We will apply the lemma to $AX_\ell + G_\ell$ with $v = \sigma_k(\|V^\top X_{\ell-1}\| + \varepsilon)$ and $\tau$ as above. Note that

$$\|NX_{\ell-1}\| \leq \sigma_k \|V^\top X_{\ell-1}\|.$$  

Hence we have $\nu \geq \max\{|G_\ell|, |NX_{\ell-1}|\}$ as required by the lemma. The lemma also requires a lower bound $\mu$. To satisfy the lower bound we invoke Lemma B.3 showing that with probability $1 - 1/n^2$, we have

$$\frac{1}{\nu^2} (\rho(G) + \rho(NX)) \leq 10\mu^*.$$  

We remark that this is the lemma that uses the assumption on $N$ provided by (2). Again we assume this event occurs. In this case we have

$$\mu \geq \mu = \frac{C_5 \mu}{\tau^2} (20\mu^* + \log n)$$  

and so we see that $\mu$ satisfies the requirement of Lemma 5.4. It follows that SmoothQR produces with probability $1 - 1/n^4$ a matrix $H_\ell$ such that

$$\|H_\ell\| \leq \tau \nu \leq \frac{\gamma_k \nu}{128} \leq \frac{1}{4} \left( \frac{1}{32} \gamma_k \sigma_k \|V^\top X_{\ell-1}\|_F + \frac{\varepsilon}{32} \gamma_k \sigma_k \right).$$  

In particular, $H_\ell$ satisfies the requirement of $(\varepsilon/4)$-admissibility. Moreover, the lemma gives that $\mu(X_\ell) \leq \mu$. This shows that also the second and third claim of our inductive claim continue to hold. All error probabilities we incurred were $o(1/n)$ and we can sum up the error probabilities over all $L \leq n$ steps to concludes the proof of the theorem.

7 Finding a good starting point

Figure 6 describes an algorithm that computes the top $k$ singular vectors of $P_\Omega(A)$ and truncates them in order to ensure incoherence. The algorithm serves as a fast initialization procedure for our main algorithm. This general approach is relatively standard in the literature. However, our truncation argument differs from previous approaches. Specifically, we use a random orthonormal transformation to spread out the entries of the singular vectors before truncation. This leads to a tighter bound on the coherence.

**Theorem 7.1 (Initialization).** Let $A = M + N$ be a symmetric $n \times n$ matrix where $M$ is a matrix of rank $k$ with the spectral decomposition $M = U \Lambda U^\top$ and $N = (I - UU^\top)A$ satisfies (2). Assume that each entry is included in $\Omega$ independently probability

$$p \geq \frac{Ck(\mu(U) + \mu_N)(\|A\|_F/\gamma_k \sigma_k)^2 \log n}{n}$$  

for a sufficiently large constant $C > 0$. Then, the algorithm Initialize returns an orthonormal matrix $X \in \mathbb{R}^{n \times k}$ such that with probability $9/10$, $\|V^\top X\|_F \leq 1/4$ and $\mu(X) \leq 32\mu(U) \log n$.

**Proof.** The proof follows directly from Lemma 7.3 and Lemma 7.4 below.
**Input:** Target dimension $k$, observed set of indices $\Omega \subseteq [n] \times [n]$ of an unknown symmetric matrix $A \in \mathbb{R}^{n \times n}$ with entries $P_\Omega (A)$, coherence parameter $\mu \in \mathbb{R}$.

**Algorithm** `Initialize($P_\Omega (A), \Omega, k, \mu$):

1. Compute the first $k$ singular vectors $W \in \mathbb{R}^{n \times k}$ of $P_\Omega (A)$.
2. $\tilde{W} \leftarrow WO$ where $O \in \mathbb{R}^{k \times k}$ is a random orthonormal matrix.
3. $T \leftarrow T_\epsilon (\tilde{W})$ with $\mu' = \sqrt{8 \mu \log(n)/n}$ where $T_\epsilon$ replaces each entry of its input with the nearest number in the interval $[-c, c]$.
4. $X \leftarrow QR(T)$

**Output:** Orthonormal matrix $X \in \mathbb{R}^{n \times k}$.

---

**Remark 7.2.** To implement `Initialize` it is sufficient to compute an approximate singular value decomposition of $P_\Omega (A)$. From our analysis it is easy to see that it is sufficient to compute the $k$-th singular value to accuracy, say, $\gamma_k \sigma_k / 100k$. This can be done efficiently using, for example, the Power Method (Subspace Iteration) with $O(k \gamma^{-1}) \log n$ iterations. See [Hig, Ste] for details on the Power Method. In particular, the running time of this step is dominated by the running time of LS.

**Lemma 7.3.** Assume that $\Omega$ satisfies Equation 9. Then, $\mathbb{P} \left\{ \| V^T W \|_2 \leq 1/16 \sqrt{k} \right\} \geq 1 - 1/n^2$.

**Proof.** By our assumption on $A$, we have $\max_{i\in[n]} \| e_i^T N \|_2^2 \leq (\mu N/n) \| A \|_F^2$ and $\max_{i,j\in[n]} | N_{ij} | \leq (\mu N/n) \| A \|_F$. Moreover, $\max_i \| e_i^T M \|_2^2 \leq (\mu (U) k/n) \| M \|_F^2$ and $\max_{i,j} | M_{ij} | \leq (\mu (U) k/n) \| M \|_F$. This shows that

$$\max_i \| e_i^T A \|_2^2 \leq \frac{\mu (U) k + \mu N}{n} \| A \|_F^2 \quad \text{and} \quad \max_{i,j} | A_{ij} | \leq \frac{\mu (U) k + \mu N}{n} \| A \|_F.$$ 

Plugging these upper bounds into Lemma A.3 together with our sample bound in Equation 9, we get that

$$\mathbb{P} \left\{ \| A - P_\Omega (A) \|_F > \frac{\gamma_k \sigma_k}{32 \sqrt{k}} \right\} \leq 1/n^2.$$ 

Put $\varepsilon = \gamma_k \sigma_k / 32 \sqrt{k}$. Now assume that $\| A - P_\Omega (A) \|_F \leq \varepsilon$ and let $W$ be the top $k$ singular vectors of $P_\Omega (A)$. On the one hand, $\sigma_k (P_\Omega (A)) \geq \sigma_k (A) - \varepsilon \geq \sigma_k - \gamma_k \sigma_k / 2$. One the other hand, by definition, $\sigma_{k+1} (A) = \sigma_k - \gamma_k \sigma_k$. Hence, by the Davis-Kahan sin $\theta$-theorem [DK, Ste] we have that

$$\| V^T W \| = \sin \theta_k (U, W) \leq \frac{\varepsilon}{\sigma_k (P_\Omega (A)) - \sigma_{k+1} (A)} \leq \frac{2 \varepsilon}{\gamma_k \sigma_k} = \frac{1}{16 \sqrt{k}}. \tag*{\blacksquare}$$

**Lemma 7.4.** Assume that $\| V^T W \|_2 \leq 1/16 \sqrt{k}$. Then, with probability $99/100$ we have $\| V^T X \|_F \leq 1/4$ and $\mu (X) \leq 32 \mu (U) \log n$.

**Proof.** By our assumption on $W$, there exists an orthonormal transformation $Q \in \mathbb{R}^{k \times k}$ such that $\| U Q - W \|_F \leq 1/16$. Moreover, $\mu (U Q) = \mu (U) \leq \mu$. In other words, $W$ is close in Frobenius norm to an orthonormal basis of small coherence. A priori it could be that some entries of $U Q$ are as large as $\sqrt{\mu k/n}$. However, after rotating $U Q$ be a random rotation, all entries will be as small as $\sqrt{\mu \log (n)/n}$. This is formalized in the next claim.
Claim 7.5. Let $Y \in \mathbb{R}^{n \times k}$ be any orthonormal basis with $\mu(Y) \leq \mu$. Then, for a random orthonormal matrix $O \in \mathbb{R}^{k \times k}$, we have $\mathbb{P}\left\{ \max_{i,j}|(YO)_{ij}| > \sqrt{8\mu \log(n)/n} \right\} \leq \frac{1}{n^2}$.

Proof. Consider a single entry $Z = (YO)_{ij}$. Observe that $Z$ is distributed like a coordinate of a random vector in $\mathbb{R}^k$ of norm at most $\sqrt{\mu k/n}$. By measure concentration, we have

$$\mathbb{P}\left\{ |Z| > \varepsilon \sqrt{\mu k/n} \right\} \leq 4\exp(-\varepsilon^2 k/2).$$

This follows from Levy’s Lemma (see [Mat]) using the fact that projection onto a single coordinate in $\mathbb{R}^k$ is a Lipschitz function on the $(k - 1)$-dimensional sphere. The median of this function is 0 due to spherical symmetry. Hence, the above bound follows. Putting $\varepsilon = \sqrt{8\log(n)/k}$, we have that

$$\mathbb{P}\left\{ |Z| > \sqrt{8\mu \log(n)/n} \right\} \leq 4\exp(3\log(n)) = 4n^{-4}.$$

Taking a union bound over all $kn \leq n^2/4$ entries, we have that with probability $1 - 1/n^2$,

$$\max_{i,j}|(YO)_{ij}| \leq \sqrt{8\mu \log(n)/n}.$$

Applying the previous claim to $UQ$, we have that with probability $1 - 1/n^2$, for all $i, j$, $(UQO)_{ij} \leq \mu'$. Furthermore, because a rotation does not increase Frobenius norm, we also have $\|UQO - WO\|_F \leq 1/16$. Truncating the entries of $WO$ to $\mu'$ can therefore only decrease the distance in Frobenius norm to $UQO$. Hence, $\|UQO - T\|_F \leq 1/16$. Also, since truncation is a projection onto the set $\{B: |B_{ij}| \leq \mu'\}$ with respect to Frobenius norm, we have

$$\|WO - T\|_F \leq \|UQO - T\|_F \leq \frac{1}{16}.$$

We can write $X = TR^{-1}$ where $R$ is an invertible linear transformation with the same singular values as $T$ and thus satisfies

$$\|R^{-1}\| = \frac{1}{\sigma_k(T)} \leq \frac{1}{\sigma_k(WO) - \sigma_1(WO - T)} \leq \frac{1}{1 - 1/16} \leq 2.$$

Therefore,

$$\|e_i^T X\| = \|e_i^T TR^{-1}\| \leq \|e_i^T T\| \|R^{-1}\| \leq 2\|e_i^T T\| \leq 2\sqrt{8\mu(U)\log(n)/n}.$$

Hence,

$$\mu(X) \leq \frac{n}{k} \cdot \frac{32k\mu(U)\log(n)}{n} \leq \frac{32\mu(U)\log(n)}{1}.$$

Finally,

$$\|V^T X\|_F = \|V^T TR^{-1}\|_F \leq \|V^T T\|_F \|R^{-1}\| \leq 2\|V^T T\|_F \leq 2\|V^T W\|_F + 2\|WO - T\|_F \leq 2\|V^T W\|_F + \frac{1}{8} \leq \frac{1}{4}.$$

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A Large deviation bounds

We need some matrix concentration inequalities. Turn to [Tro] for background.

**Theorem A.1** (Matrix Bernstein). Consider a finite sequence \( \{Z_k\} \) of independent random matrices with dimensions \( d_1 \times d_2 \). Assume that each random matrix satisfies \( \mathbb{E} Z_k = 0 \) and \( \|Z_k\| \leq R \) almost surely. Define \( \sigma^2 \overset{\text{def}}{=} \max \left\{ \| \sum_k \mathbb{E} Z_k Z_k^\top \|, \| \sum_k \mathbb{E} Z_k^\top Z_k \| \right\} \). Then, for all \( t \geq 0 \),

\[
P \left( \| \sum_k Z_k \| \geq t \right) \leq (d_1 + d_2) \cdot \exp \left( \frac{-t^2/2}{\sigma^2 + Rt/3} \right).
\]
Theorem A.2 (Matrix Chernoff). Consider a finite sequence \( \{X_k\} \) of independent self-adjoint matrices of dimension \( d \). Assume that each random matrix satisfies \( X_k \geq 0 \) and \( \lambda_{\text{max}}(X_k) \leq R \) almost surely. Define \( \mu_{\text{min}} \overset{\text{def}}{=} \lambda_{\text{min}}(\sum_k E X_k) \). Then,

\[
P \left( \lambda_{\text{min}}(\sum_k X_k) \leq (1 - \delta)\mu_{\text{min}} \right) \leq d \cdot \exp \left( \frac{-\delta^2 \mu_{\text{min}}}{2R} \right)
\]

A.1 Error bounds for initialization

Lemma A.3. Suppose that \( A \in \mathbb{R}^{m \times n} \) and let \( \Omega \subset [m] \times [n] \) be a random subset where each entry is included independently with probability \( p \). Then

\[
P(\|P_\Omega(A) - A\| > u) \leq n \exp \left( \frac{-u^2/2}{\sigma^2 + \frac{u}{3}(1/p - 1)\max_{ij}|A_{ij}|} \right)
\]

where \( \sigma^2 = (1/p - 1)\max \left( \max_i \|e_i^T A\|^2, \max_j \|A e_j\|^2 \right) \).

Proof. Let \( \xi_{ij} \) be independent Bernoulli-\( p \) random variables, which are 1 if \((i, j) \in \Omega \) and 0 otherwise. Consider the sum of independent random matrices \( P_\Omega(A) - A = \sum_{i,j} \left( \frac{\xi_{ij}}{p} - 1 \right) A_{ij} e_i e_j^T \). Applying Theorem A.1, we conclude that

\[
P(\|P_\Omega(A) - A\| > u) \leq n \exp \left( \frac{-u^2/2}{\sigma^2 + Ru/3} \right)
\]

where \( \sigma = (1/p - 1)\max \left( \max_i \|e_i^T A\|^2, \max_j \|A e_j\|^2 \right) \).

Here we use that

\[
\left\| \mathbb{E} \sum_{i,j} \left( \frac{\xi_{ij}}{p} - 1 \right)^2 A_{ij} e_i e_j^T e_j e_i^T \right\| = \left( \frac{1}{p} - 1 \right) \max_i \|e_i^T A\|^2
\]

and similarly

\[
\left\| \mathbb{E} \sum_{i,j} \left( \frac{\xi_{ij}}{p} - 1 \right)^2 A_{ij}^2 e_i e_j^T e_j e_i^T \right\| = \left( \frac{1}{p} - 1 \right) \max_i \|A e_j\|^2
\]

Further, \( \left\| \left( \frac{\xi_{ij}}{p} - 1 \right) A_{ij} e_i e_j^T \right\| \leq R = \left( \frac{1}{p} - 1 \right) \max_{ij} |A_{ij}|. \) This concludes the proof.

A.2 Error bounds for least squares

Lemma A.4. Let \( 0 < \delta < 1 \) and let \( i \in [n] \). Assume that \( p \geq \frac{k p(X) \log n}{\delta^2 n} \). Then, \( \mathbb{P} (\|B_i^{-1}\| > \frac{1}{1 - \delta}) \leq \frac{1}{n^\delta} \).

Proof. Let \( B = B_i \) and \( p = p_i \). Clearly, \( \|B^{-1}\| = 1/\lambda_{\text{min}}(B) \). We will use Theorem A.2 to lower bound the smallest eigenvalue of \( B \) by \( 1 - \delta \). Denoting the rows of \( X \) by \( x_1, \ldots, x_n \in \mathbb{R}^k \) we have \( B = \sum_{i=1}^n \frac{1}{p} Z_i x_i x_i^T \), where \( \{Z_i\} \) are independent Bernoulli(\( p \)) random variables. Moreover, \( \mathbb{E} B = X^T X = \mathbb{I}_{k \times k} \). Therefore, in the notation of Theorem A.2, this is \( \mu_{\text{min}}(B) = 1 \). Moreover, using our lower bound on \( p, \|\frac{1}{p} Z_i x_i x_i^T\| \leq \frac{1}{p} \|x_i\|^2 \leq \frac{\mu(X_{i, i})}{p n} \leq \frac{\delta^2}{20 \log n} \). Hence, by Theorem A.2, \( \mathbb{P} (\lambda_{\text{min}}(B) \leq 1 - \delta) \leq k \exp(-10 \log n) \). The claim follows.

Lemma A.5. Let \( 0 < \delta < 1 \). Assume that \( p \geq \frac{k p(X) \log n}{\delta^2 n} \). Then, for every \( i \in [n] \), we have

\[
\mathbb{P} (\|e_i^T U A_{ij} E^T P_i X\| \geq \delta \|e_i^T M\| : \|V^T X\|) \leq \frac{1}{10}.
\]
Proof. Let $\Omega \in \mathbb{R}^{k \times k}$ be an orthonormal transformation such that all columns of $X' = X \Omega$ have $\ell_\infty$-norm at most $\sqrt{8\mu(X) \log(n)/n}$. Such a transformation exists as shown in Claim 7.5. Then,
\[
\mathbb{E}\left\| e_i^T U \Lambda U E^T P_i X \right\|^2 = \mathbb{E}\left\| e_i^T U \Lambda U E^T P_i x'_r \right\|^2 = \sum_{r=1}^{k} \mathbb{E}(e_i^T U \Lambda U E^T P_i x'_r)^2
\]
where $x'_r$ is the $r$-th column of $X'$. Let $w^T = e_i^T U \Lambda U E^T$. We have
\[
\|w\|^2 \leq \|e_i^T U \Lambda U\| \cdot \|E\| = \|e_i^T M\| \cdot \|V^T X\|.
\]
Finally,
\[
\mathbb{E}(w^T P_i x'_r)^2 = \sum_{j=1}^{n} \mathbb{E}(w_j(P_i)_{jj}(x'_r))^2 \leq \frac{1}{p} \cdot \|w\|_2^2 \cdot \|x'_r\|_\infty \leq \frac{8\mu(X) \log n}{pn}.
\]
Hence,
\[
\mathbb{E}\left\| e_i^T U \Lambda U E^T P_i X \right\|^2 \leq \frac{8kp(X) \log n}{pn} \left(\|e_i^T M\|^2 \cdot \|V^T X\|^2 \right) = \frac{\delta^2}{10} \|e_i^T M\|^2 \cdot \|V^T X\|^2.
\]
The claim now follows from Markov’s inequality. \hfill \blacksquare

**Lemma A.6.** Let $0 < \delta < 1$. Assume that $p \geq \frac{k \mu(X) \log n}{\delta^2 n}$. Then, for every $i \in [n],
\[
\mathbb{P}\left\{ \|e_i^T G^N\| > \delta \|e_i^T N\| \right\} \leq \frac{1}{10}.
\]
Proof. Recall, by Lemma 4.2, we have
\[
e_i^T G^N = e_i^T N P_i X B_i^{-1} - e_i^T N X = (e_i^T N P_i X - e_i^T N X B_i)B_i^{-1}
\]
Plugging in the lower bound on $p$ into Lemma A.4, we get that $\|B_i - I\| \leq \delta/4$ for all $i \in [n]$ with probability 19/20.

We will show that for every fixed $i \in [n]$, we have with probability 19/20 that
\[
\|e_i^T N P_i X - e_i^T N X\| \leq (\delta/4) \|e_i^T N\| \tag{10}
\]
Both events simultaneously occur with probability 9/10 and in this case we have:
\[
\|e_i^T G^N\| \leq \|e_i^T N P_i X - e_i^T N X\| \|B_i^{-1}\| + \|e_i^T N X\| \|B_i - I\| \|B_i^{-1}\| \leq (\delta/2) \|e_i^T N\| + (\delta/2) \|e_i^T N\| = \delta \|e_i^T N\|
\]
So, it remains to show Equation 10. Fix $i \in [n]$. Let $\Omega \in \mathbb{R}^{k \times k}$ be an orthonormal transformation such that all columns of $X' = X \Omega$ have $\ell_\infty$-norm at most $\sqrt{8\mu(X) \log(n)/n}$. Let $w^T = e_i^T N$ be the $i$-th row of $N$. Let us denote by $x'_1, \ldots, x'_k$ the $k$ columns of $X'$. We have that
\[
\mathbb{E}\left\| w^T P_i X - w^T X \right\|^2 = \mathbb{E}\left\| w^T (P_i - I) X \right\|^2 = \sum_{r=1}^{k} \mathbb{E}(w, (P_i - I)x'_r)^2 = \sum_{r=1}^{k} \sum_{j=1}^{n} \mathbb{E}((P_i)_{jj} - 1)^2 w_j^2(x'_r)^2 \leq \|w\|^2 \cdot \frac{8\mu(X) \log n}{pn} \|e_i^T N\|.
\]
The bound in Equation 10 now follows from Markov’s inequality. \hfill \blacksquare
B Additional lemmas and proofs for smooth QR factorization

Proof of Lemma 5.1. Fix a unit vector $x \in \mathbb{R}^k$. We have

$$\|P \nu (G + H)x\|^2 > \|P \nu Hx\|^2 - \|P \nu Gx, P \nu Hx\|$$

Note that $g = Hx$ is distributed like $N(0, \tau^2/n)$ and $y = P \nu Cx$ has norm at most 1. Due to the rotational invariance of the Gaussian measure, we may assume without loss of generality that $V$ is the subspace spanned by the first $n-k$ standard basis vectors in $\mathbb{R}^n$. Hence, denoting $h \sim N(0, \tau^2/n)^{n-k}$, our goal is to lower bound $\|h\|^2 - \langle y, h \rangle$. Note that $\mathbb{E}\|h\|^2 \geq \tau^2/2$ and by standard concentration bounds for the norm of each row of $H$ follows straightforwardly that

$$\mathbb{P}\{\|h\|^2 \leq \tau^2/4\} \leq \exp(-\Omega(n)).$$

On the other hand, $(y, h)$ is distributed like a one-dimensional Gaussian variable of variance at most $\tau^2/n$. Hence, by Gaussian tail bounds, $\mathbb{P}\{(y, h)^2 > \tau^2/8\} \leq \exp(-\Omega(n))$. Hence, with probability $1 - \exp(-\Omega(n))$, we have $\|P \nu (G + H)x\| \geq \Omega(\tau)$. We can now take a union bound over a net of the unit sphere in $\mathbb{R}^k$ of size $\exp(O(k \log k))$ to conclude that with probability $1 - \exp(O(k \log k))\exp(-\Omega(n))$, we have for all unit vectors $x \in \mathbb{R}^k$ that $\|P \nu (G + H)x\| \geq \Omega(\tau)$. Therefore $\sigma_k(P \nu (G + H)) \geq \Omega(\tau)$. By our assumption $\exp(O(k \log k)) = \exp(o(n))$ and hence this event occurs with probability $1 - \exp(-\Omega(n))$.

Lemma B.1. Let $P$ be the projection onto an $(n-k)$-dimensional subspace. Let $H \sim N(0, 1/n)^{n \times k}$. Then, $\rho(PH) \leq O(\log n)$ with probability $1 - 1/n^5$.

Proof. We have that $P = (I - UU^\top)$ for some $k$-dimensional basis $U$. Hence,

$$\rho(PU) \leq O(\rho(H)) + O(\rho(UU^\top H)).$$

Using concentration bounds for the norm of each row of $H$ and a union bound over all rows it follows straightforwardly that $\rho(H) \leq O(\log n)$ with probability $1 - 1/2n^5$. The second term satisfies

$$\rho(UU^\top) \leq O(1)\|U^\top H\|^2 = \mu(U)\|U^\top H\|^2.$$ But $U^\top H$ is a Gaussian matrix $N(0, 1/n)^{k \times k}$ and hence its largest singular value satisfies $\|U^\top H\|^2 \leq O(k \log(n)/n)$ with probability $1 - 1/2n^5$.

Lemma B.2. Let $X, Y$ be $k$ and $k'$ dimensional subspaces, respectively, such that $\mathcal{R}(X) \subseteq \mathcal{R}(Y)$. Then, $\mu(X) \leq \frac{k'}{k} \mu(Y)$.

Proof. We know that $\mu(Y)$ is rotationally invariant. Therefore, without loss of generality we may assume that $Y = [X \mid X']$ for some orthonormal matrix $X'$. Here, we identify $X$ and $Y$ with orthonormal bases. Hence,

$$\mu(X) = \frac{n}{k} \max_{i \in [n]} \|e_i^\top X\|^2 \leq \frac{n}{k} \max_{i \in [n]} (\|e_i^\top X\|^2 + \|e_i^\top X'\|^2) = \frac{n}{k} \max_{i \in [n]} \|e_i^\top Y\|^2 = \frac{k'}{k} \mu(Y).$$

The following technical lemma was needed in the proof of Theorem 6.1.

Lemma B.3. Under the assumptions of Theorem 6.1, we have for every $\ell \in [L]$ and $\nu = \frac{\alpha}{32} (\|V^\top X_{\ell-1}\| + \epsilon)$ with probability $1 - 1/n^2$,

$$\frac{1}{\nu^2} (\rho(G) + \rho(NX_{\ell-1})) \leq 3\mu^*.$$
Proof. Given the lower bound on $p$ in Theorem 6.1 we can apply Lemma 4.4 to conclude that $\|e_i^T G^M\| \leq \sqrt{k \mu(U)/n} \cdot v$ and $\|e_i^T G^N\| \leq \sqrt{\mu^2/n} \cdot v$. Hence, $\rho(G)\nu^2 \leq \mu^*$. Further, we claim that $\|e_i^T NX\|^2 \leq (\mu^*/n)\sigma_k \|V^TU\|$ for all $i \in [n]$, because

$$\|e_i^T NX\|^2 \leq \|e_i^T V\Sigma V^T\| \cdot \|V^TU\| = \|e_i^T N\| \cdot \|V^TX_{\ell-1}\|.$$  

Here we used the fact that

$$\|e_i^T N\|^2 = \|e_i^T NV\|^2 + \|e_i^T NU\|^2 = \|e_i^T NV\|^2 = \|e_i^T V\Sigma V^T\|^2.$$  

Using Equation 2, this shows that $\rho(NX_{\ell-1})\nu^2 \leq \mu^*$ and finishes the proof. \hfill \blacksquare

C Splitting up the subsample

We needed a procedure \textsc{Split}(Ω, t) that takes a sample Ω and splits it into $t$ independent samples that preserve the distributional assumption that we need. The next lemma is standard.

Lemma C.1. There is a procedure \textsc{Split}(Ω, t) such that if Ω is sampled by including each element independently with probability $p$, then \textsc{Split}(Ω, t) outputs independent random variables $\Omega_1, \ldots, \Omega_t$ such that each set $\Omega_i$ includes each element independently with probability $p_i \geq p/t$.

Proof sketch. Consider independent random samples $\Omega'_1, \ldots, \Omega'_t$ where each set contains every element independently with probability $p/2t$. Consider the multi-set $\Omega'$ obtained from taking the union of these sets (counting multiplicities). Each element occurs in $\Omega'$ at least once with probability $p' = 1 - (1 - p/t)^t \leq p$. The multiplicity is distributed according to a binomial random variable. Hence, we can simulate the distribution of $\Omega'$ given the random sample $\Omega$ by subsampling so that each entry is included with probability $p'$ and then introducing multiplicities randomly according to the correct Binomial distribution. On the other hand, given the random variable $\Omega'$ we can easily simulate $\Omega'_1, \ldots, \Omega'_t$ by assigning each element present in $\Omega'$ with multiplicity $k$ to a random subset of $k$ out $t$ sets. \hfill \blacksquare

D Generalization to rectangular matrices

For our purposes it will suffice to consider symmetric square matrices. This follows from a simple transformation that preserves the matrix coherence and singular vectors of the matrix. Indeed, given a matrix $B \in \mathbb{R}^{m \times n}$ and $m \leq n$ with singular value decomposition $B = \sum_{i=1}^t \sigma_i u_i v_i^\top$, we may consider the symmetric $(m + n) \times (m + n)$ matrix $A = \begin{bmatrix} 0 & B \\ B^\top & 0 \end{bmatrix}$. The matrix $A$ has the following properties: $A$ has a rank $2 \cdot \text{rank}(B)$ and singular values $\sigma_1(B), \ldots, \sigma_t(B)$ each occuring with multiplicity two. The singular vectors corresponding to the singular value $\sigma_i$ are spanned by the vectors $\{(u_i, 0), (0, v_i)\}$. In particular, an algorithm to find a rank $2k$ approximation to $A$ also finds a rank $2k$ approximation to $B$ up to the same error.

Moreover, let $\bar{U}$ denote the space spanned by the top $2k$ singular vectors of $A$, and let $U$, respectively $V$, denote the space spanned by the top $k$ left, respectively right, singular vectors of $B$. Then $\mu(\bar{U}) \leq \frac{n+m}{2k} \left( \frac{\mu(U)}{m} + \frac{\mu(V)}{n} \right) \leq \frac{n+m}{m} \max \{\mu(U), \mu(V)\}$. Note that we can assume that $(n + m)/m$ is constant by splitting $B$ into a sequence of $m \times O(m)$ matrices and recovering each matrix separately. It will also be important for us that we can turn a uniformly random subsample of $B$ into a uniformly random subsample of $A$. This is easily accomplished by splitting the sample into two equally sized halves, using one for $B$ and one for $B^\top$. The remaining quadrants of $A$ are 0 and can be subsampled trivially of any given density.