How gravitational fluctuations degrade the high-dimensional spatial entanglement

Haorong Wu,1 Xilong Fan,2,* and Lixiang Chen1,†

1Department of Physics and Collaborative Innovation Center for Optoelectronic Semiconductors and Efficient Devices, Xiamen University, Xiamen 361005, China
2School of Physics and Technology, Wuhan University, Wuhan 430072, China

Twisted photons carrying orbital angular momentum (OAM) are competent candidates for future interstellar communications. However, the gravitational fluctuations are ubiquitous in spacetime. Thus a fundamental question arises naturally as to how the gravitational fluctuations affect the coherence and the degree of high-dimensional OAM entanglement when twisted photons travel across the textures of curved spacetime. Here, we consider the covariant scalar Helmholtz equations and the Minkowski metric with fluctuations of Gaussian distribution and formulate analytically the equations describing the motion for twisted light in the Laguerre-Gaussian mode space. It is seen that the OAM cannot remain conserved in the presence of gravitational fluctuations. Furthermore, two-photon density matrices are derived for interstellar OAM quantum entanglement distribution, and the degree of entanglement degradation is characterized by purity and negativity. It is revealed that the higher-dimensional OAM entanglement is more susceptible to spacetime fluctuations. We believe that our findings will be of fundamental importance for the future interstellar quantum communications with twisted photons.

I. INTRODUCTION

Multiple degrees of freedom, such as wavelength, polarization, and time-bins, have been utilized to convey information in optical communications [1, 2]. Additionally, entanglements of these degrees of freedom are the primary sources of quantum communications. However, since these entanglements may be altered during propagation under the impact of media, their transmission must be investigated to compensate for these modifications. Unlike polarization, which is specified in a two-dimensional space, orbital angular momentum (OAM) of light may assume well-defined values of $\ell h$ where $\ell = 0, \pm 1, \pm 2, \ldots$, that span an infinite-dimensional Hilbert space [3–14]. Due to the fact that information may be encoded in a high-dimensional space, light with OAM is a potential source for future quantum communications [15]. However, OAM entanglements are suffered from the adverse effects from the environment. For instance, some investigations have shown the degradation of OAM entanglements as a result of atmospheric turbulence [16–20]. As a result, the OAM light transmission distance remains inside the confines of a city [21, 22]. Fortunately, in outer space, OAM entanglement will be free from these harmful atmospheric effects.

In recent years, there were several attempts to connect the photon OAM detection with astronomical observations. Harwit briefly discussed the possible astronomical apparatus for detecting photon OAM as well as their limitations [23]. It was revealed by Tamburini et al. that it is possible to measure the twisted light for a direct observational demonstration of the existence of rotating black holes, arising from the relativistic effect of their surrounding space and time [24]. By extracting the OAM spectra from the radio intensity data collected by the Event Horizon Telescope, the first observational evidence of twisted light from a rotating black hole M87* was presented by Tamburini and coworkers [25]. Also, the effects of reduced spatial coherence of astronomical light sources and that of line-of-sight misalignment on the OAM detection were considered carefully by Hetharia et al [26].

Photons move not just through a medium in interstellar communications, but also across the textures of spacetime, requiring consideration of general relativity [27–29]. Recent years have also witnessed a growing research interest in how coherence and quantum entanglement was affected in curved spacetime [30–43]. To name a few, the event formalism was proposed to solve the closed timelike curves (CTC) problems, although a recent experiment with the quantum satellite $\text{Micius}$ did not support its prediction that time-energy entangled photons would decorrelate after passing through different regions of gravitational potential [30–33]; a master equation for photons was developed by modeling decoherence arising from a bath of stochastic gravitational disturbances [34, 35]; the quantum field theory of light in generally curved spacetime geometry was studied by Exirifard et al. and the leading corrections by the Riemann curvature to quantum optics are calculated [36]. Hitherto, to the best of our knowledge, the study for degradation of high-dimensional OAM entanglement due to gravitational fluctuations has not yet been conducted, which, however, is of fundamental importance for future interstellar communications with twisted photons.

Gravitational fluctuations may come from a variety of sources. Classically, they can be a remnant of primordial gravitational waves generated during the early stage of the universe, a remnant of the big bang as the cosmic microwave background, or the sum of the gravitational waves randomly produced by a vast variety of sources in the universe [42, 44]. Also, gravitational fluctuations can arise due to the quantum gravity, which is still in

* xilong.fan@whu.edu.cn
† chenlx@xmu.edu.cn
debate, in the form of quantum noise [45, 46]. Due to
the fact that the gravitational fluctuations are ubiqui-
tous in spacetime, it is of fundamental importance to
investigate the effect of gravitational fluctuations on the
high-dimensional OAM entanglement for future interstel-
lar communications. Here, we formulate analytically the
equations describing the motion for twisted light in the
Laguerre-Gaussian mode space based on the Minkowski
metric and fluctuations. By taking ensemble averages
over fluctuations, the evolution of two-photon density
matrices for high-dimensional OAM states is derived and
the degree of entanglement degradation is measured by
calculating its negativity. We reveal that the higher-
dimensional OAM entanglement is more susceptible to
spacetime fluctuations.

We will operate in position space, i.e., expanding OAM
states into Laguerre-Gaussian (LG) modes, LG_{l,p}(x). Se-
veral integrals involving LG modes and other functions
will be produced. Due to the fact that LG modes include
associated Laguerre polynomials, solving the integrals
will be challenging. Fortunately, this issue may be over-
come by using the generating-function approach. Unless
otherwise specified, geometrized units with \( c = G = 1 \)
are used. The signature of metrics will be chosen as
\((- , +, +, +)\). The following conventions apply to indices:
all Greek indices run in \( \{0, 1, 2, 3\} \), whereas Latin indices
run in \( \{1, 2, 3\} \). The following assumptions and approx-
imations are made for simplicity. First, the beam is as-
sumed to be monochromatic and the paraxial approxi-
mation is applied; second, the polarization degrees of
freedom will be neglected, since light fields with different
polarizations will not couple to each other in the following
analysis and polarizations obey the superposition princi-
ple; third, the geodesic via which photons propagate will
be approximated by a straight line due to the weak magni-
itudes of fluctuations. Throughout this paper, \( x \) or \( x^\mu \)
stand for the four position vector \( (x^0, x^1, x^2, x^3) \)
for its spatial components \( (x^1, x^2, x^3) \) and \( x_\perp \) for the spatial
transverse components \( (x^1, x^2) \).

II. LIGHT WAVE IN METRIC FLUCTUATIONS

We start by considering two entangled light beams
traveling between satellites as in Fig. 1. Suppose the
metric experienced by them can be expressed as
\( g_{\mu\nu}(x) = \eta_{\mu\nu} + \epsilon h_{\mu\nu}(x) \), where \( \epsilon \) is a systematic perturbation pa-
rameter which will be set to one in the end, \( \eta_{\mu\nu} \) is the
Minkowski metric and \( h_{\mu\nu}(x) \) is the metric fluctuations.
As known, there are a variety of fluctuation sources, e.g.,
a remnant of primordial gravitational waves generated
during the early stage of the universe, gravitational waves
randomly produced by a vast variety of sources in the uni-
verse, and the quantum gravity effects. Hence, according
to the law of large numbers, we can assume the space-
time to be fluctuating around the Minkowski metric, i.e.,
\( \langle h_{\mu\nu}(x) \rangle = 0 \), and isotropic such that its autocorrelation
obeys a Gaussian distribution [47, 48] \( \langle h_{\mu\nu}(x)h_{\mu\nu}(x') \rangle =
A^2\delta_{\mu\nu} \exp \left(-\sum_{\mu=0}^{3} (x^\mu - x'^\mu)^2 / L^2 \right) \), where \( A \) is the
fluctuation strength, and \( L \) is the correlation length of
the fluctuation. Points with distance less than \( L \) behave
coherently. Later, when we take ensemble averages over
Eq. (2), only the spatial fluctuations make an impact on
the OAM light, while the temporal parts have no effects.
Therefore, the choice of temporal correlation functions
will not affect our results of spatial OAM mode distri-
bution. Without loss of generality, we have followed Ref.
[42] to treat the temporal and spatial components of fluc-
tuations equally, namely, the same correlation length \( L \) is
assumed. Also, as in [47], we assumed that \( h_{\mu\nu}(x) \ll 1 \).
As a result, only the first-order perturbation is required
in our calculation.

In the scalar theory of light propagation in free space,
scalar light field distributions satisfy the Helmholtz equa-
tion [49], \( g^{\mu\nu}\psi_{\mu\nu} = 0 \), where commas represent partial
derivatives. This equation is already Lorentz invariant
and \( \psi(x^\mu) \) are scalar fields, whose second-order covariant
derivatives are invariant when the order of derivatives
change, so according to the comma-goes-to-semicolon rule
[50, 51], its counterpart in curved spacetime can be
written as \( \Box \psi = g^{\mu\nu}\psi_{\mu;\nu} = g^{\mu\nu}(\psi_{\mu,\nu} - \Gamma^\lambda_{\mu\nu}\psi_{,\lambda} = 0 \),
where the semicolons mean covariant derivatives, \( \Gamma^\lambda_{\mu\nu} \)
are the Christoffel symbols and we have used the fact
that \( \psi(x) \) is a scalar field. This equation can also be de-
rivied from the action of a scalar field in curved spacetime,
\( S[\psi] = \int dx^4 \sqrt{-g}g^{\mu\nu}\psi_{,\mu}\psi_{,\nu}/2 \), by varying it with respect
to \( \psi \), where \( g \) is the determinant of the metric. We have
assumed the light beam propagates along the \( x^3 \) direc-
tion so \( \psi(x) = T(x)e^{ik(x^0-x^3)} \), where \( k \) is the frequency
and \( T(x) \) is the spatial structure mode. For a Gaussian
beam, \( T(x) \) satisfies the paraxial wave equation, whose
solutions may be represented as Hermite-Gaussian (HG)
modes, Laguerre-Gauss (LG) modes, Ince-Gaussian (IG)
modes [36], etc. Substituting the fluctuation metric into
this equation and keeping terms up to the first order of
\( \epsilon \), we obtain the variations of the transverse modes given by

\[
\partial_3 T(x) = \partial_3^{(M)} T(x) + \epsilon \partial_3^{(F)} T(x),
\]

where \( \partial_3^{(M)} T(x) = (2ik)^{-1} \nabla_2^2 T(x) \) is the variations in
the Minkowski spacetime, and \( \partial_3^{(F)} T(x) = (2i)^{-1} k(h_{00} +

FIG. 1. Scheme for interstellar OAM entanglement trans-
mission. The image of satellites and planets are designed by
macrovector/Freepik.
$h_{33} T(x) + (2i k)^{-1} ((h_{33} - h_{11}) \partial_x^2 T(x) + (h_{33} - h_{22}) \partial_y^2 T(x))$

is the first-order corrections due to the gravitational fluctuations. We will use the superscripts (M) and (F) to distinguish these light field variations. To obtain this result, we have used the paraxial approximation to drop $\partial_y^2 T(x)$, and terms containing $k^{-2}$ are omitted due to the large magnitude of the frequency $k$. Later, we will find that only terms proportional to $k^{-1}$ have effects.

III. EQUATIONS OF MOTION FOR OAM STATES

We will consider light beams whose transverse components have LG-mode profiles, so $T(x) = LG_{l,p}(x, x^3)$, where $l$ is called the azimuthal index (or topological charge), and $p$ is the radial index. Since we are mostly interested in the entanglement between different azimuthal numbers, we will set the radial number $p$ to zero, implicitly. Hence, $|l, x^3\rangle$ stands for an OAM state with azimuthal number $l$ and $x^3$ indicates that the mode state will be decomposed in the position space at point $x^3$ as $|l, x^3\rangle = \int d^2x_{\perp} LG_l(x_{\perp}, x^3) |x_{\perp}\rangle$.

In the Minkowski spacetime, since the LG modes are the solutions of the paraxial wave equation, the two OAM indices $l$ and $p$ will be conserved over the propagation. However, in the presence of gravitational fluctuations, the OAM state will evolve according to $\partial_t |l, x^3\rangle = \int d^2x_{\perp} \left[ \partial_x^{(M)} LG_m(x_{\perp}, x^3) + \partial_x^{(F)} LG_m(x_{\perp}, x^3) \right] |x_{\perp}\rangle$,

where the perturbation parameter is set to 1. Obviously an OAM state may spread to an OAM spectrum, i.e., $|l, x^3\rangle \rightarrow |\tilde{l}, x^3 + \Delta x^3\rangle = \sum_m C_m |m, x^3 + \Delta x^3\rangle$,

where $\{C_m\}$ is a set of constants. A tilde symbol is used to indicate that this is a nonstandard OAM state. Especially, $\{|\tilde{l}, x^3 + \Delta x^3\rangle\}$ are not generally orthonormal. In other words, the OAM for single-photon states after experiencing the fluctuations cannot remain conserved well. This is because in the Minkowski metric without fluctuations, the spacetime is isotropic and homogeneous. However, the presence of fluctuations will break the spacetime symmetry, thus leading to the non-conservation and even the decoherence of single OAM states.

Suppose, in the future, we would establish a quantum communication channel by distributing two entangled OAM beams as in Fig. 1. The density operator for them is given by $\rho(x^3) = \sum_{l_1, l_2, j_1, j_2} |l_1, x^3\rangle \langle l_1, x^3| \rho_{l_1, l_2, j_1, j_2}(x^3) |j_1, x^3\rangle \langle j_2, x^3|$, where $l_1, l_2, j_1, j_2$ are OAM indices. In a propagation over a small distance $\Delta x^3$, we let the OAM basis evolve and then project them back to the standard OAM basis. In supplemental material, it will be shown that the equations of motion (EOM) for density operator are given by $\partial_t \rho_{l_1, l_2, j_1, j_2} = L^{l_1, m_1}_{l_1, l_2} \rho_{m_1, l_2, j_1, j_2} + L^{l_2, m_2}_{l_2, l_1} \rho_{m_2, l_1, j_1, j_2} + L^{l_1, m_1}_{j_1, l_2} \rho_{m_1, l_2, j_1, j_2} + L^{l_2, m_2}_{j_2, l_1} \rho_{m_2, l_1, j_1, j_2}$, where repeated indices imply Einstein summation convention and indices $m_1, m_2, n_1, n_2$ run in all azimuthal numbers. The $L$ symbols are given by $L_{n,s} = \int d^2x_{\perp} LG_n(x_{\perp}, x^3) LG^*_s(x_{\perp}, x^3)$. Notice that in the absence of any gravitational fluctuations, or in Minkowski spacetime, $\partial_x^{(F)} LG_l(x_{\perp}, x^3)$ identically vanish and all $L$ symbols are zero. This leaves the EOM to be $\partial_t \rho_{l_1, l_2, j_1, j_2} = 0$, i.e., the density matrix will remain unchanged if no gravitational fluctuations affect it. Furthermore, due to the fact that the fluctuation strength is very weak, in the evolution of some mode $|l\rangle$, the crosstalk between $|l\rangle$ and other modes are much smaller than $|l\rangle$ itself. Besides, from the definition of $L_{n,s}$, and in the situation where the correlation length is larger than the beam radius, we can approximately calculate $L_{n,s} \sim \int d^2x_{\perp} LG_n(x_{\perp}, x^3) LG^*_s(x_{\perp}, x^3)$, which is nonzero only when $n = s$. Therefore, we will neglect those crosstalk terms. The EOM will reduce to $\partial_t \rho_{l_1, l_2, j_1, j_2} = \left( L^{l_1, l_1}_{l_1, l_2} + L^{l_1, j_1}_{l_1, j_1} + L^{j_2, j_2}_{l_2, j_2} \right) |l_1, l_2, j_1, j_2\rangle$, where the Einstein summation convention is suppressed. By substituting the expressions of $L$ symbols, one may verify that terms proportional to $k$ cancel each other.

![FIG. 2. Purities for different dimensional entanglements.](image)

Now let the beams propagate from distance $x^3$ to $x^3 + \Delta x^3$. Normally, the gravitational fluctuation is very weak, and the beam has to propagate a great distance before its density operator changes significantly. Therefore, we may choose $\Delta x^3$ to be much greater than $L$, the correlation length of the fluctuation. By invoking the Dyson series to the second-order, and taking ensemble average,
one may have the density operator at \((x^3 + \Delta x^3)\) as
\[
\rho_{i_1 i_2 j_1, j_2}(x^3 + \Delta x^3) = \rho_{i_1 i_2 j_1, j_2}(x^3) \left[ 1 + \int_{x^3}^{x^3+\Delta x^3} dx^3 \right]
\]
\[
\times \int_{x^3}^{x^3+\Delta x^3} dx^3 \left\{ \left( L^*_{i_1, i_1}(x^3) + L^*_{i_2, i_2}(x^3) + L_{j_1, j_1}(x^3) + L_{j_2, j_2}(x^3) \right) \left( L^*_{i_1, i_1}(x^3) + L^*_{i_2, i_2}(x^3) + L_{j_1, j_1}(x^3) + L_{j_2, j_2}(x^3) \right) \right\},
\]
where \(\langle \cdot \rangle\) means ensemble average. Note that if one takes ensemble averages over crosstalk terms \(L_{i,j}(x^3_i) L_{m,n}(x^3_n)\), the numerical calculations will show that \(\langle L_{i,j}(x^3_i) L_{m,n}(x^3_n) \rangle \approx \langle L_{i,i}(x^3_i) L_{m,m}(x^3_n) \rangle\) if \(l \neq j\) or \(m \neq n\), so the crosstalk terms can be safely neglected. After the ensemble averages are taken, we may utilize the generating-function method to find derivatives for each element. In astrophysics and cosmology, fluctuations occur over large distances \([47]\), so we consider the situation where the correlation length is larger than the beam radius, i.e., \(L \gg w(z)\). In this case, Equation (2) can be approximated by
\[
\partial_t \rho_{i_1 i_2 j_1, j_2}(x^3) = -C_{i_1 i_2 j_1, j_2} \rho_{i_1 i_2 j_1, j_2}(x^3)/\kappa,
\]
where \(\kappa = 2k^2 w_0^2/(3LA^2)\) is a characteristic length and \(C_{i_1 i_2 j_1, j_2} = (|l_1| - |j_1|)^2 + (|l_2| - |j_2|)^2\). Therefore, at any distance \(x^3\), we have
\[
\rho_{i_1 i_2 j_1, j_2}(x^3) = \rho_{i_1 i_2 j_1, j_2}(0) \exp(-C_{i_1 i_2 j_1, j_2} x^3/\kappa).
\]

Obviously, we can see a trivial case that \(C_{i_1 i_2 j_1, j_2} = 0\) if \(|l_1| = |j_1|\) and \(|l_2| = |j_2|\) and hence that the trace of density operator is not altered along the propagation. This property can also be seen from the definition of \(L_{m,n}\), from which we have \(L^*_{i,i} = -L_{i,i}\), and \(L_{i,i} = L_{i,i}\). Hence when \(|l_1| = |j_1|\) and \(|l_2| = |j_2|\), the fluctuation-induced correction \(L^*_{i,i}\) will be canceled by another correction \(L_{i,i}\); similarly, \(L^*_{i,i}\) will be canceled by \(L_{i,i}\). Therefore, these elements will be shelled from the fluctuations, and be protected from those adverse effects.

**IV. HIGH-DIMENSIONAL ENTANGLEMENT IN GRAVITATIONAL FLUCTUATION**

We will consider a high-dimensional entangled state \(|\psi_{\text{OAM}}(0)\rangle = \sum_{m=-M}^{M} |m\rangle |-m\rangle/\sqrt{D}\), where \(D = 2M + 1\) is the dimension of the system. We use the purity, defined by \(P = \text{tr}(\rho^2)\) \([52-54]\), to measure the coherence of the system. It can be shown to be \(P(x^3) = \sum_{l,j} \exp(-2C_{l,-l,-j} x^3/\kappa)/D^2\). We can see that the coherence will drop exponentially as \(x^3\) grows. Also, a higher-dimensional system will lose coherence more rapidly. For measuring entanglement, we use the negativity \(N\) \([55-59]\), defined by \(N(\rho) = -\sum_{\lambda_k < 0} \lambda_k\) where \(\lambda_k\) are eigenvalues of the partial transpose of \(\rho\), e.g., with respect to the second particle, \(\rho^{PT} = \sum_{i_1, i_2 j_1, j_2} |i_1\rangle |j_2\rangle \rho_{i_1 i_2 j_1, j_2} |j_1\rangle |i_2\rangle\).

For our system, the partial transpose density, with respect to the second particle, is \(\rho^{PT}_{\text{OAM}} = \sum_{l,j} \exp(-C_{l,-l,-j} x^3/\kappa) |l\rangle - |j\rangle \langle -l| /D\). We could numerically solve its eigenvalues to compute its negativity.

We vary \(D\) for different dimensions and plot them in Figs. 2 and 3. These figures show that the coherence and entanglement of a high-dimensional system will decline exponentially as their propagation distances increase. We define the decay distance as the length by which the negativity of the system drops to 1/e of the initial value, i.e., \(N(x^3)/N(0)\). Numerically, we find that the decay distances are 1.48\(\kappa\), 0.64\(\kappa\), 0.40\(\kappa\), 0.20\(\kappa\), 0.08\(\kappa\) for \(D = 3, 5, 7, 11, 19\), respectively, which shows that higher-dimensional system will suffer more severe degradation. Moreover, from the property of \(C_{i_1 i_2 j_1, j_2}\), some specific elements in density operators are free from fluctuations. Entanglements stored in these safety islands will be preserved as depicted in Fig. 4 for \(D = 7\). Note in this figure, only nonzero elements are shown. Therefore, the entanglements of those systems will never totally vanish. This fact could be an advantage of OAM entanglements. For higher-dimensional systems, even if there are more safety islands, considerably more elements will be affected by fluctuations. Higher-dimensional systems, in other words, have more complex structures that are more susceptible to fluctuations.

The degradation strength is characterized by the characteristic length \(\kappa = 2k^2 w_0^2/(3LA^2)\). Therefore, if we would like to increase the stability of entanglements to transfer information, we could increase the waist radius \(w_0\) or the frequency \(k\), or lower the dimension of the system. On the contrary, a stronger degradation strength can be achieved by doing the opposite, in order to experimentally quantify the parameters, \(A\) and \(L\), of fluctuations. Even so, this kind of experiments are out of current technologies.
V. CONCLUSION

We have investigated the covariant scalar wave equations in gravitational fluctuations and constructed the equations of motion for light with OAM. The evolution of OAM states in gravitational fluctuations is achieved by taking its ensemble average. The high-dimensional entanglements are explored and characterized by purities and negativities. Our estimations indicate that the entanglements are explored and characterized by purities and negativities. The parameters of gravitational fluctuations, \( A \) and \( L \), are still unknown to us. We could use OAM entanglements to measure them, or protect these entanglements from them by varying the parameters of the light beams. Recently, quantum entanglement distribution, with the polarization state of light, has been reported over a distance of 1200 kilometers based on a satellite, towards a global scale [28]. In the future, if the high-dimensional OAM entanglement is utilized to establish interstellar quantum communications, our result may serve as a model for how it will evolve as it travels across the spacetime. Especially, higher dimensional entanglements have more information capacities, while being more susceptible to fluctuations, and this balance should be taken into account.

ACKNOWLEDGMENTS

We would like to thank Filippus S. Roux for insightful discussions. This work is supported by the National Natural Science Foundation of China (12034016, 11922303), the Fundamental Research Funds for the Central Universities at Xiamen University (20720190057, 20720200074), the National Science Foundation of Fujian Province of China for Distinguished Young Scientists (2015J06002), and the program for New Century Excellent Talents in University of China (NCET-13-0495).

Appendix A  DERIVATION OF EOM FOR DENSITY OPERATOR

In a small distance, let the OAM basis change while \( \rho_{l_1l_2,j_1,j_2}(z) \) are unaltered, such that

\[
\rho(z + \Delta z) = \sum_{l_1,l_2,j_1,j_2} |\hat{l}_1, z + \Delta z\rangle |\hat{l}_2, z + \Delta z\rangle \rho_{l_1l_2,j_1,j_2}(z) \langle \hat{j}_1, z + \Delta z| \langle \hat{j}_2, z + \Delta z|
\]

\[
= \sum_{l_1,l_2,j_1,j_2} \int d^2x_{1\perp} d^2x_{2\perp} d^2x_{1\perp}' d^2x_{2\perp}' \hat{L}G_{l_1}(x_{1\perp}, z + \Delta z) \hat{L}G_{l_2}(x_{2\perp}, z + \Delta z) \hat{L}G_{j_1}^*(x_{1\perp}', z + \Delta z) \hat{L}G_{j_2}^*(x_{2\perp}', z + \Delta z) \rho_{l_1l_2,j_1,j_2}(z) \langle x_{1\perp}'| \langle x_{2\perp}'|
\]

\[
= \sum_{l_1,l_2,j_1,j_2} \int d^2x_{1\perp} d^2x_{2\perp} d^2x_{1\perp}' d^2x_{2\perp}' [L\hat{G}_{l_1}(x_{1\perp}, z) + \Delta z \partial_3 \hat{L}G_{l_1}(x_{1\perp}, z)] [\hat{L}G_{l_2}(x_{2\perp}, z) + \Delta z \partial_3 \hat{L}G_{l_2}(x_{2\perp}, z)] \langle x_{1\perp}'| \langle x_{2\perp}'| \langle x_{1\perp}| \langle x_{2\perp}| \rho_{l_1l_2,j_1,j_2}(z) \langle x_{1\perp}'| \langle x_{2\perp}'|,
\]

(A.1)

where \( \hat{L}G_{l}(x_{\perp}, z) \) is the position-space decomposition of the nonstandard OAM states \( \{|\hat{l}, z + \Delta z\rangle\} \), and since the beam has experienced a small propagation, it can be expanded in terms of \( \hat{L}G_{l}(x_{\perp}, z) \) as

\[
\hat{L}G_{l}(x_{\perp}, z + \Delta z) = \hat{L}G_{l}(x_{\perp}, z) + \Delta z \partial_3 \hat{L}G_{l}(x_{\perp}, z).
\]

(A.2)
Since $|\hat{I}, z + \Delta z\rangle$ is not a standard OAM state, the elements will be extracted by using $\langle l_1, z + \Delta z|\langle l_2, z + \Delta z|\rho(z + \Delta z)|j_1, z + \Delta z\rangle|j_2, z + \Delta z\rangle$. Therefore,

$$\rho(z + \Delta z)_{l_1, l_2, j_1, j_2} = \langle l_1, z + \Delta z|\langle l_2, z + \Delta z|\rho(z + \Delta z)|j_1, z + \Delta z\rangle|j_2, z + \Delta z\rangle$$

$$= \int d^2x_{1\perp}d^2x_{2\perp}d^2x'_{1\perp}d^2x'_{2\perp}LG_{l_1}^*(x_{1\perp}, z + \Delta z) LG_{l_2}^*(x'_{1\perp}, z + \Delta z) LG_{j_1}(x'_{2\perp}, z + \Delta z) LG_{j_2}(x'_{2\perp}, z + \Delta z)$$

$$\times \langle x_{1\perp}|\rho(z + \Delta z)|x'_{1\perp}\rangle|\langle x'_{2\perp}|\rangle.$$  

(A.3)

Then we expand those products and only keep terms up to first order of $\Delta z$ and use Eq. (??). Also, the orthogonality of LG functions states that

$$\int d^2x_{1\perp}LG_m(x_{1\perp}, z)LG_n^*(x_{1\perp}, z) = \delta_{mn},$$

(A.4)

and by applying the two-dimensional Gauss’s theorem,

$$\int d^2x_{1\perp}[\partial_3^{(M)}LG_m(x_{1\perp}, z)LG_n^*(x_{1\perp}, z) + LG_m(x_{1\perp}, z)\partial_3^{(M)}LG_n^*(x_{1\perp}, z)]$$

$$= \frac{1}{2ik} \int d^2x_{1\perp}\nabla_T \cdot [\nabla_T LG_m(x_{1\perp}, z)LG_n^*(x_{1\perp}, z) - LG_m(x_{1\perp}, z)\nabla_T LG_n^*(x_{1\perp}, z)]$$

$$= \frac{1}{2ik} \int_{S_\infty} [\nabla_T LG_m(x_{1\perp}, z)LG_n^*(x_{1\perp}, z) - LG_m(x_{1\perp}, z)\nabla_T LG_n^*(x_{1\perp}, z)] \hat{n} ds$$

$$= 0,$$  

(A.5)

where $S_\infty$ is the boundary surface in infinity. Now Equation (A.3) can be simplified into

$$\rho(z + \Delta z)_{l_1, l_2, j_1, j_2} = \rho(z)_{l_1, l_2, j_1, j_2} + \Delta z \left[ \sum_{m_1} \rho_{m_1 l_2, j_1 j_2} (z) \int d^2x_{1\perp}LG_{l_1}^*(x_{1\perp}, z)\partial_3^{(F)}LG_{m_1}(x_{1\perp}, z) + \sum_{m_2} \rho_{l_1 m_2 j_1 j_2} (z) \int d^2x_{1\perp}LG_{l_1}^*(x_{1\perp}, z)\partial_3^{(F)}LG_{m_2}(x_{1\perp}, z) \right]$$

$$+ \sum_{n_1} \rho_{l_1 l_2, n_1 j_2} (z) \int d^2x_{1\perp}LG_{l_1}(x_{1\perp}, z)\partial_3^{(F)}LG_{n_1}^*(x_{1\perp}, z) + \sum_{n_2} \rho_{l_1 l_2, n_2 j_2} (z) \int d^2x_{1\perp}LG_{l_2}(x_{1\perp}, z)\partial_3^{(F)}LG_{n_2}^*(x_{1\perp}, z) \right] \bigg].$$

(A.6)

Therefore, the equations of motion for density operator are given by

$$\partial_3 \rho_{l_1 l_2, j_1 j_2} = L_{l_1 m_1}^* \rho_{l_1 l_2, j_1 j_2} + L_{l_2 m_2}^* \rho_{l_1 m_2, j_1 j_2} + L_{j_1 n_1} \rho_{l_1 l_2, n_1 j_2} + L_{j_2 n_2} \rho_{l_1 l_2, j_1 n_2},$$

(A.7)

where Einstein summation convention is used and $m_1, m_2, n_1, n_2$ run in all azimuthal numbers. The $L$ symbols are given by

$$L_{n, s} = \int d^2x_{1\perp}LG_n(x_{1\perp}, z)\partial_3^{(F)}LG_s^*(x_{1\perp}, z).$$

(A.8)

**Appendix B  LG MODES AND THE GENERATING-FUNCTION METHOD**

The Laguerre-Gaussian (LG) modes which satisfy the paraxial wave equation,

$$-2ik \frac{\partial T}{\partial z} + \frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} + \frac{1}{r^2} \frac{\partial^2 T}{\partial \phi^2} = 0,$$

(B.1)
in cylindrical coordinates, are given by

$$LG_{l,p}(x, y, z) = \mathcal{N} \left( \frac{1}{w(z)} \left( \frac{2r}{w(z)} \right)^{|l|} L_p^{|l|} \left( \frac{2r^2}{w^2(z)} \right) \exp \left( -\frac{r^2}{w^2(z)} + il\phi + i\frac{kr^2}{2R(z)} + i\Phi(z) \right) \right),$$  \hspace{1cm} (B.2)

where $\mathcal{N} \equiv \sqrt{\frac{2^{l+p}}{\pi^{l+p}}} \left( \frac{1}{w(z)} \right)$ is the normalization constant, $r = \sqrt{x^2 + y^2}$, $\phi = \arctan(y/x)$, $w(z) = w_0 \sqrt{1 + (z/z_R)^2}$ with $w_0$ being the waist radius and $z_R = kw_0^2/2$ being the Rayleigh range, $R(z) = z[1 + (z/z_R)^2]$ is the curvature radius of the wavefronts, $\Phi(z) = -(|l| + 2p + 1)\arctan(z/z_R)$ is the Gouy phase, $k$ is the wave number, and $L_p^{|l|}(x)$ is the associated Laguerre polynomials. The radial index $p = 0, 1, 2, \ldots$ indicates the number of radial nodes of the mode, while the azimuthal index $l = 0, \pm1, \pm2, \ldots$ corresponds to the topological charge. It can be shown that the azimuthal index $l$ represents the amount of orbital angular momentum carried by every photon in that mode, so the LG modes are the decomposition of OAM basis in position space.

Since the LG modes contain some associated Laguerre polynomials, it would be difficult to integrate them with themselves or other functions. However, those integrals can be calculated with corresponding generating functions, and then generate the correct result. The generating functions for LG modes are given by

$$G_{LG(l,p)}(a, b, c) = \frac{1}{\Omega(z, c)} \exp \left[ \left( \frac{x + iy}{aw_0} + \frac{x - iy}{bw_0} - (1 + c)(x^2 + y^2) \right) \right],$$  \hspace{1cm} (B.3)

where

$$\Omega(z, c) = 1 - c + \frac{z}{z_R} + ic \frac{z}{z_R},$$  \hspace{1cm} (B.4)

and $a, b, c$ are new variables. The generating procedure is given by

$$LG_{l,p}(x, y, z) = \mathcal{N} \left( \frac{\sqrt{2}^{|l|}}{w_0 \cdot p!} \left( \frac{w_0 \cdot p!}{\sqrt{2}^{|l|}} \right) \left( \frac{x - iy}{aw_0} + \frac{y}{bw_0} + \frac{1 + c}{z_R} \right) \right),$$  \hspace{1cm} (B.5)

where the partial derivative with respect to $a$ if $l > 0$, with respect to $b$ if $l < 0$, and no partial derivative is taken if $l = 0$. Now that the associated Laguerre polynomials are replaced by a exponential function, the integral can be easily calculated. For example, for the integral Eq. (A.8), the generating functions for LG modes can be substituted in it yielding the generating functions for $L_{n,s}$,

$$G_{L(n,s)} = \int d^2x G_{LG(n)}(x, z) \partial_{\phi} G_{LG(l,p)}(a, b, c)|_{a, b, c = 0},$$  \hspace{1cm} (B.6)

Then using the generating procedure, we have

$$L_{n,s} = \mathcal{N}_a \left( \frac{\sqrt{2}^{|l|}}{w_0 \cdot p_n!} \right) \mathcal{N}_a \left( \frac{\sqrt{2}^{|l|}}{w_0 \cdot p_n!} \right) \left( \frac{x - iy}{aw_0} + \frac{y}{bw_0} + \frac{1 + c}{z_R} \right) G_{L(n,s)}(x, y, z)|_{a, b, c = 0}. \hspace{1cm} (B.7)$$

[1] M. Giustina, M. A. M. Versteegh, S. Wengerowsky, J. Handsteiner, A. Hochrainer, K. Phelan, F. Steinlechner, J. Kofler, J.-A. Larsson, C. Abellán, W. Amaya, V. Pruneri, M. W. Mitchell, J. Beyer, T. Gerrits, A. E. Lita, L. K. Shalm, S. W. Nam, T. Scheid, R. Ursin, B. Wittmann, and A. Zeilinger, Significant-loophole-free test of bell’s theorem with entangled photons, Physical review letters 115, 250401 (2015).

[2] I. Marcikic, H. De Riedmatten, W. Tittel, H. Zbinden, and N. Gisin, Long-distance teleportation of qubits at telecommunication wavelengths, Nature 421, 509 (2003).

[3] L. Allen, M. W. Beijersbergen, R. J. C. Spreeuw, and J. P. Woerdman, Orbital angular momentum of light and the transformation of laguerre-gaussian laser modes, Physical review A 45, 8185 (1992).

[4] G. F. Calvo, A. Picón, and E. Bagan, Quantum field theory of photons with orbital angular momentum, Physical Review A 73, 013805 (2006).

[5] A. Mair, A. Vaziri, G. Weihs, and A. Zeilinger, Entanglement of the orbital angular momentum states of photons, Nature 412, 313 (2001).

[6] G. Molina-Terriza, J. P. Torres, and L. Torner, Management of the angular momentum of light: preparation of photons in multidimensional vector states of angular momentum, Physical review letters 88, 013601 (2001).

[7] J. Leach, M. J. Padgett, S. M. Barnett, S. Franke-Arnold, and J. Courtial, Measuring the orbital angular momentum of a single photon, Physical review letters 88, 257901.
General Relativ-
Physical Review D
Op-
Progress in optics
New
Communications
10660 (International Society
Advances in op-
10.1038/nature.2014.16328
The Euro-
view A
16
F. S. Roux, Infinitesimal-propagation equation for de-
21
M. Peplow, Twisted light sends mozart image over
23
M. Harwit, Photon orbital angular momentum in astro-
32
Q. Exirifard, E. Culf, and E. Karimi, Towards communi-
37
Q. Xu and M. Blencowe, Toy models for gravitational and
35
M. Lagouvardos and C. Anastopoulos, Gravitational de-
30
T. C. Ralph, G. J. Milburn, and T. Downes, Quantum
29
J. Yin, Y. Cao, H.-L. Li, S.-K. Liao, L. Zhang, J.-G. Ren,
28
J. Yin, Y. Cao, Y.-H. Li, J.-G. Ren, S.-K. Liao, L. Zhang,
27
D. Hetharia, M. P. van Exter, and W. Löffler, Spatial
26
D. Hetharia, M. P. van Exter, and W. Löffler, Spatial
25
A. Vaziri, B. Thidé, G. Molina-Terriza, and G. An-
24
F. Tamburini, B. Thidé, B. Thidé, G. Molina-Terriza, and G.
23
F. Tamburini, B. Thidé, and M. Della Valle, Measurement
22
M. Peplow, Twisted light sends mozart image over
21
M. Peplow, Twisted light sends mozart image over
20
F. S. Roux, T. Wellens, and V. N. Shatokhin, Entangle-
19
C. Paterson, Atmospheric turbulence and orbital angular
18
A. Vaziri, J.-W. Pan, T. Jennewein, G. Weihs, and A.
17
F. S. Roux, T. Wellens, and V. N. Shatokhin, Entangle-
16
Y. S. Kivshar and E. A. Ostrovskaya, Optical vortices
14
G. Molina-Terriza, E. M. Wright, and L. Torner, Propa-
13
A. M. Yao and M. J. Padgett, Orbital angular momentum:
12
G. Molina-Terriza, J. P. Torres, and L. Torner, Twisted
11
Y. S. Kivshar and E. A. Ostrovskaya, Optical vortices
10
L. Allen, M. Padgett, and M. Babiker, Iv the orbital
growth and twisting waves of light, Optics and photonics
9
A. Vaziri, J.-W. Pan, T. Jennewein, G. Weihs, and A.
8
A. Vaziri, G. Weihs, and A. Zeilinger, Experimental
7
A. Vaziri, G. Weihs, and A. Zeilinger, Experimental
6
A. Vaziri, G. Weihs, and A. Zeilinger, Experimental
5
A. Vaziri, G. Weihs, and A. Zeilinger, Experimental
4
A. Vaziri, G. Weihs, and A. Zeilinger, Experimental
3
A. Vaziri, G. Weihs, and A. Zeilinger, Experimental
2
A. Vaziri, G. Weihs, and A. Zeilinger, Experimental
1
A. Vaziri, G. Weihs, and A. Zeilinger, Experimental

[44] N. Christensen, Stochastic gravitational wave backgrounds, Reports on Progress in Physics 82, 016903 (2018).

[45] H. Zhang, X. Fan, and Y. Lai, Gravitons probing from stochastic gravitational waves background, arXiv preprint arXiv:2105.05083 10.48550/arXiv.2105.05083.

[46] M. Parikh, F. Wilczek, and G. Zahariade, The noise of gravitons, International Journal of Modern Physics D 29, 2042001 (2020).

[47] L. Asprea, A. Bassi, H. Ulbricht, and G. Gasbarri, Gravitational decoherence and the possibility of its interferometric detection, Physical Review Letters 126, 200403 (2021).

[48] L. Asprea, G. Gasbarri, and A. Bassi, Gravitational decoherence: A general nonrelativistic model, Physical Review D 103, 104041 (2021).

[49] Q. Cao and X. Deng, Power carried by scalar light beams, Optics communications 151, 212 (1998).

[50] S. M. Carroll, Spacetime and geometry (Cambridge University Press, 2019).

[51] C. W. Misner, K. S. Thorne, and J. A. Wheeler, Gravitation (Macmillan, 1973).

[52] S. Chatterjee and N. Makri, Density matrix and purity evolution in dissipative two-level systems: Ii. relaxation, Physical Chemistry Chemical Physics 23, 5125 (2021).

[53] W. H. Zurek, S. Habib, and J. P. Paz, Coherent states via decoherence, Physical Review Letters 70, 1187 (1993).

[54] A. Isar, A. Sandulescu, and W. Scheid, Purity and decoherence in the theory of a damped harmonic oscillator, Physical Review E 60, 6371 (1999).

[55] R. Horodecki, P. Horodecki, M. Horodecki, and K. Horodecki, Quantum entanglement, Reviews of modern physics 81, 865 (2009).

[56] P. Rungta, V. Bužek, C. M. Caves, M. Hillery, and G. J. Milburn, Universal state inversion and concurrence in arbitrary dimensions, Physical Review A 64, 042315 (2001).

[57] J. A. Anaya-Contreras, H. M. Moya-Cessa, and A. Zúñiga-Segundo, The von neumann entropy for mixed states, Entropy 21, 49 (2019).

[58] F. Mintert, A. R. Carvalho, M. Kuš, and A. Buchleitner, Measures and dynamics of entangled states, Physics Reports 415, 207 (2005).

[59] O. Gühne and G. Tóth, Entanglement detection, Physics Reports 474, 1 (2009).