OPTIMAL INVESTMENT AND REINSURANCE WITH PREMIUM CONTROL

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(Communicated by Hailiang Yang)

Abstract. This paper studies the optimal investment and reinsurance problem for a risk model with premium control. It is assumed that the insurance safety loading and the time-varying claim arrival rate are connected through a monotone decreasing function, and that the insurance and reinsurance safety loadings have a linear relationship. Applying stochastic control theory, we are able to derive the optimal strategy that maximizes the expected exponential utility of terminal wealth. We also provide a few numerical examples to illustrate the impact of the model parameters on the optimal strategy.

1. Introduction. As investment is an important factor in contemporary insurance studies, insurance risk models with investment return have drawn a great deal of attention in the actuarial literature. Besides investment, an insurance company can take advantage of buying reinsurance to alleviate its potential loss. In fact, optimality study in the presence of investment and/or reinsurance has become one of the key research topics in the past few decades.

With the single control of investment, many authors have derived optimal results for various insurance risk models. Among them, [3] derived the optimal investment strategy that maximizes the exponential utility of terminal wealth and minimizes the ruin probability for a diffusion risk model within the classical Black-Scholes financial market; [5] investigated the problem of optimal investment with the objective of minimizing the ruin probability for the classical risk model with a risky asset following a geometric Brownian motion; [21] considered the optimal investment problem for a jump-diffusion risk model under the assumption that an insurance firm can

2020 Mathematics Subject Classification. Primary: 97M30, 93E20; Secondary: 91G80.
Key words and phrases. Exponential utility, investment, optimal strategy, premium control, reinsurance, safety loading.

The research of Xin Jiang and Kam Chuen Yuen was supported by a grant from the Research Grants Council of the Hong Kong Special Administrative Region, China (Project No. HKU17320916). The research of Mi Chen was supported by National Natural Science Foundation of China (Nos. 11701087 and 11701088), Natural Science Foundation of Fujian Province (Nos. 2018J05003, 2019J01673 and JAT160130), Program for Innovative Research Team in Science and Technology in Fujian Province University, and the grant “Probability and Statistics: Theory and Application (No. IRTL1704)” from Fujian Normal University.

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invest its surplus in a risk-free asset or a risky asset; and [19] dealt with a similar problem of optimal investment but for a diffusion risk model with transaction costs.

In the context of dynamic proportional reinsurance, [7] derived the optimal reinsurance strategy that maximizes the expected discounted future surpluses; [16] derived the optimal reinsurance strategy which minimizes the ruin probability in the classical risk model and its diffusion approximation; [10] studied optimal reinsurance under the criterion of maximizing adjustment coefficient for a diffusion risk model as well as a jump-diffusion risk model; [22] considered the objective of maximizing the expected exponential utility and derived optimal reinsurance strategy for a risk model with common shock dependence under the expected value premium principle; [11] examined the same optimal problem under the variance premium principle; and [14] studied optimal dividend problems in a diffusion risk model under constant interest force.

In the presence of both investment and proportional reinsurance controls, [17] developed a numerical method to find the optimal solution for the compound Poisson risk model within the Black-Scholes financial market; [15] derived explicit expressions for both optimal controls and the resulting minimized ruin probability in the diffusion setting; [2] studied the optimal problem with multiple risky assets and no-shorting constraint, and obtained explicit expression for the optimal strategy in the diffusion framework; and [12] investigated the optimal problem by assuming that the risky asset price follows the constant elasticity of variance model and that the claim process is modeled by a jump-diffusion risk model.

In real situation, for various reasons including changes in economic and market conditions, insurers adjust their premium rates from time to time. As a result, the sizes of their portfolios and hence the claim arrival rates will change correspondingly. This fact suggests that there exists some relationship between premium rate and claim arrival rate. In view of this, either of these two rates can be served as a control variable in optimization problems involving insurance risk. Some research on optimal premium control can be found in [13] and [20]. In recent years, [6] derived the optimal premium and dividend barrier strategy maximizing the expected discounted dividends; [1] proposed an optimal premium problem with portfolio size being a function of premium, and derived the optimal result that minimizes the ruin probability; [18] scrutinized the problem of optimal premium expressed as a function of the deductible; and [23] studied the optimal investment and premium problem for a non-homogeneous compound Poisson process with time-varying intensity.

Motivated by the aforementioned papers, we consider the optimal investment, premium and reinsurance problem under the objective of maximizing the expected utility of terminal wealth in this paper. It is assumed that the surplus process follows a diffusion approximation to a non-homogeneous Poisson process with time-varying intensity, and that there exists a relation between the safety loading and the claim arrival rate. Compared with [23], the problem considered in this paper is more difficult to solve because of the emergence of reinsurance variable, especially in the proof of the existence and uniqueness of the solution. In Section 2, we present the model of study and formulate the optimization problem. In Section 3, we study the optimal investment, premium and reinsurance strategy that maximizes the expected utility of terminal wealth, and prove the existence and uniqueness of the solution. In Section 4, we examine the net profit condition in a special case. Finally, we present some numerical examples to illustrate the impact of different parameters on the optimal result in Section 5.
2. The model. Under the classical risk model, the surplus process \( \{Y_t, t \geq 0\} \) of an insurer is given by

\[
dY_t = dC_t - d \sum_{k=1}^{N_t} X_k,
\]

where \( C_t \) represents the aggregate insurance premium up to time \( t \), \( N_t \) is a homogeneous Poisson process with a constant intensity \( \lambda \), and \( X_k \) is a sequence of independent and identically distributed claim-size random variables. It is assumed that \( X_k \) is independent of \( N_t \). The first and second moment of \( X_k \) are denoted by \( \mu \) and \( \sigma^2 \), respectively.

In the classical setup, the arrival rate of the claim is a constant. But as was mentioned before, a time-varying arrival rate is practically more significant. Therefore, it is natural to model the claim arrival rate as a function of \( t \), say \( \lambda_t \). As a result, according to the expected value principle, the amount of premium paid up to time \( t \) can be expressed as

\[
C_t = \mu \int_0^t (1 + \theta_s) \lambda_s ds,
\]

where \( \theta_s \) is the safety loading rate of the insurance company at time \( s \). Intuitively, there is a natural connection between the safety loading rate and the claim arrival rate. When the safety loading goes up, less customers are willing to buy the insurance due to higher premium. Consequently, a decrease in the claim arrival rate is expected to occur. This fact suggests that \( \lambda_t \) can be expressed as a strictly decreasing function \( h \) of \( \theta_t \), i.e., \( h(\theta_t) = \lambda_t \). Such a relationship between \( \lambda_t \) and \( \theta_t \) was also considered in [6] and [23].

When the safety loading is set to 0, the number of customer should reach its maximum, and hence the claim arrival rate increases to its maximum given other factors being unchanged. Denote this maximum rate by \( \lambda_{max} \) such that \( h(0) = \lambda_{max} \). On the other hand, apart from being positive, \( \theta_t \) should be bounded above since no customer is willing to buy when the safety loading is too high. Define \( \theta_{max} \) such that \( h(\theta_{max}) = 0 \). It is thus reasonable to assume that the value of \( \theta_t \) falls inside the interval \([0, \theta_{max}]\). Moreover, it is reasonable to assume that \( \theta_{max} \) is bounded below.

When \( \theta_t \) starts to increase from 0, the insurance is still cheap and customers are not so sensitive to early changes in the premium rate. Thus, the decreasing rate of \( \lambda_t \) is comparatively slow as \( \theta_t \) starts to increase. When \( \theta_t \) keeps going up and the premium rate increases to a certain level, more customers may find the insurance expensive and hence become sensitive to the change of premium rate. When this happens, the decrease in \( \lambda_t \) moves downward at a faster rate. On the whole, as \( \theta_t \) goes up, the decrease in \( \lambda_t \) is slow at the beginning and then drops faster at a later stage. In view of this fact, it is reasonable to let \( h(\theta_t) \) be a concave function. Then it follows from the definition of \( h \) that \( h^{-1} \) is a decreasing function with \( h^{-1}(0) = \theta_{max} \) and \( h^{-1}(\lambda_{max}) = 0 \). These together with the fact that \( h^{-1} \) and \( h \) are symmetric about the line where \( y = x \) imply that \( h^{-1} \) is a concave function on \([0, \lambda_{max}]\).

Unlike [23], this paper assumes that the insurer can purchase proportional reinsurance to reduce its potential risk. The retention level at time \( t \) is denoted by \( 1 - p_t \). According to the expected value principle, the reinsurance premium rate \( \delta \) is given by

\[
d\delta(p_t) = (1 + \eta_t) p_t \lambda_t \mu dt,
\]
where $\eta_t$ is the reinsurance safety loading. The surplus process with reinsurance of the insurer is then given by

$$dY_t = \mu Y_t(1 + \theta_t)dt - (1 + \eta_t)p_t \lambda_t \mu dt - d \sum_{k=1}^{N_t} (1 - p_{T_k})X_k,$$  
(3)

where $T_k$ is the time of the $k$th jump.

Here we assume that the reinsurance is non-cheap, i.e., $\eta_t > \theta_t$, and that there exists a linear relationship between $\eta_t$ and $\theta_t$. Specifically, the linear function has the form $\eta_t = a \theta_t + \eta_{\min}$ for some positive constant $a$, where $\eta_{\min} > 0$ is the minimum value of the reinsurance safety loading. This ensures that when the insurance safety loading becomes 0, the reinsurance company will not take the risk without potential profit. Because of non-cheap reinsurance and $\theta_t \in [0, \theta_{\max}]$, we have

$$a \theta_{\max} + \eta_{\min} - \theta_{\max} > 0,$$

which in turn gives a lower bound for $\eta_{\min}$, i.e., $\eta_{\min} > (1 - a) \theta_{\max}$. On the other hand, the insurer is not willing to purchase reinsurance most of the time if the reinsurance premium is too expensive. This implies that $\eta_{\min}$ cannot be too large. Hence, we assume that $\eta_{\min} \leq \theta_{\max}/2$ which is needed in the proof of Lemma 3.3. Since $\theta_{\max}/2 \geq \eta_{\min} > (1 - a) \theta_{\max}$, we have $a > 1/2$. Also, to make sure $\eta_{\min} > 0$, we set $a \leq 1$ so that $\eta_{\min} > (1 - a) \theta_{\max} \geq 0$. As a result, $a \in (1/2, 1]$.

Apart from reinsurance, it is assumed that the company can invest in the classical Black-Scholes financial market consisting of one risk-free asset and one risky asset. The prices of the risk-free asset $\{R_t, t \geq 0\}$ and the risky asset $\{S_t, t \geq 0\}$ are respectively given by

$$dR_t = r R_t dt,$$
$$dS_t = \alpha S_t dt + \beta S_t dW_t^S,$$

where $r, \alpha, \beta > 0$ and $\{W_t^S, t \geq 0\}$ is a standard Brownian motion. To avoid any triviality, we assume that $\alpha \geq r$.

Let $\pi_t$ denote the amount of money invested in the risky asset and the remaining wealth is invested in the risk-free asset. Then the surplus process of the insurer can be written as

$$dY_t = \pi_t \frac{dS_t}{S_t} + (Y_t - \pi_t) \frac{dR_t}{R_t} + \mu Y_t(1 + \theta_t)dt - (1 + \eta_t)p_t \lambda_t \mu dt - d \sum_{k=1}^{N_t} (1 - p_{T_k})X_k.$$  
(4)

It follows from [23] and [8] that the non-homogeneous compound Poisson process with a time-varying intensity can be approximated by

$$\sum_{k=1}^{N_t} X_k - \mu \int_0^t \lambda_s ds \overset{D}{\approx} \sigma \int_0^t \sqrt{\lambda_s} dW_s^Y,$$  
(5)

where the notation $\overset{D}{\approx}$ means approximately equal in distribution and $\{W_t^Y, t \geq 0\}$ is a standard Brownian motion independent of $\{W_t^S, t \geq 0\}$.

For mathematical convenience, we write $u_t = \sqrt{\lambda_t}$ as the control variable in relation to the premium rate. Let $\pi = \{\pi_t, t \geq 0\}$, $p = \{p_t, t \geq 0\}$, and $u = \{u_t, t \geq 0\}$. It follows from (4) and (5) that the controlled surplus process of the insurer with policy $(\pi, p, u)$ satisfies

$$dY_t^{\pi,p,u} = [r Y_t^{\pi,p,u} + \mu G(u_t)(1 - ap_t) - \mu u_t^2 p_t \eta_{\min} + \pi_t (\alpha - r)] dt$$
$$+ \pi_t \beta dW_t^S + (1 - p_t)\sigma u_t dW_t^Y,$$  
(6)
with the initial surplus \( Y_0^{\pi,p,u} = x \), where \( G(u) = u^2 h^{-1}(u^2) \). Let \( \Pi \) be the admissible policy set. A control policy \((\pi, p, u)\) \( \in \Pi \) is said to be admissible if the following three conditions hold: (i) \((\pi, p, u)\) is predictable such that for all \( t < \infty \), \( 0 \leq u_t \leq u^0 \) where \( u^0 \) is defined in Lemma 3.2; (ii) \( E[\int_0^t \pi_t^2 ds] < \infty \); and (iii) \( E[\int_0^t p_t^2 ds] < \infty \).

The generator of the surplus process (6) for any function \( f(t, x) \in C^{1,2} \) is then given by

\[
A^{\pi,p,u} f(t, x) = f_t + \frac{1}{2} \left[ \sigma^2 \pi_t^2 + \alpha^2 u_t^2 (1 - p_t)^2 \right] f_{xx} + [r x + \mu G(u_t)(1 - ap_t) - u_t^2 \mu p_t \eta_{\text{min}} + \pi_t (\alpha - r)] f_x.
\]

**3. Optimal results.** Suppose the insurer wants to maximize the expected utility of terminal wealth. Let the utility function \( U(Y_T) \) be a concave function, i.e., \( b'(x) > 0 \) and \( b''(x) < 0 \). For each control policy \( (\pi, p, u) \), the performance function is given by

\[
V^{\pi,p,u}(t, x) = E[b(Y_T^{\pi,p,u}) | Y_t^{\pi,p,u} = x],
\]

and the value function has the form

\[
V(t, x) = \sup_{(\pi, p, u) \in \Pi} V^{\pi,p,u}(t, x).
\]

Applying the techniques of dynamic programming principle, one can show that \( V(t, x) \) satisfies the following Hamilton-Jacobi-Bellman (HJB) equation

\[
\sup_{(\pi, p, u)} A^{\pi,p,u}(t, x) = 0, \quad t < T,
\]

with the boundary condition

\[
V(T, x) = b(x).
\]

Using the method of [4], one can come up with the following verification theorem.

**Theorem 3.1.** Suppose that \( w(t, x) \) is a strictly increasing and concave function with respect to \( x \), and \( w(t, x) \in C^{1,2} \) is a classical solution to the HJB equation (8) with the boundary condition (9). Then the value function \( V(t, x) \) must coincide with \( w(t, x) \), i.e.,

\[
V(t, x) = w(t, x), \quad t \leq T.
\]

Furthermore, if \((\pi^*, p^*, u^*)\) satisfies \( A^{\pi^*,p^*,u^*} V(t, x) = 0 \) for all \((t, x) \in [0, T] \times R\), then the policy is optimal.

It follows from (7) and Theorem 3.1 that \( w(t, x) \) is a solution to

\[
w_t + \sup_{(\pi, p, u) \in \Pi} \left[ \frac{1}{2} \sigma^2 u_t^2 (1 - p_t)^2 + \beta^2 \pi_t^2 \right] w_{xx} + [r x + \mu G(u_t)(1 - ap_t) - u_t^2 \mu p_t \eta_{\text{min}} + \pi_t (\alpha - r)] w_x = 0,
\]

and simple calculus yields

\[
\beta^2 \pi_t w_{xx} + (\alpha - r)w_x = 0,
\]

\[
\sigma^2 u_t^2 (1 - p_t) w_{xx} + (ap_t G(u_t) + \mu u_t^2 \eta_{\text{min}}) w_x = 0,
\]

\[
\sigma^2 u_t(1 - p_t)^2 w_{xx} + \mu G((u_t)(1 - ap_t) - 2 \mu u_t p_t \eta_{\text{min}}) w_x = 0.
\]
Solving the above system of equations, we obtain the optimal policy \((\pi^*, p^*, u^*)\) where

\[
\pi^*(t, x) = -\frac{(\alpha - r)}{\beta^2} \frac{w_x}{w_{xx}}, \quad (11)
\]

\[
p^*(t, x) = 1 + \frac{\alpha u G(u^*(t, x)) + \mu u^2(t, x)\eta_{\min}}{\sigma^2 u^2(t, x)} \frac{w_x}{w_{xx}}, \quad (12)
\]

\[
u^*(t, x) = \frac{\nu}{\sigma^2} \left[ ah^{-1}(u^*(t, x)) + \eta_{\min} \right] \frac{w_x}{w_{xx}} - 2\eta_{\min}, \quad (13)
\]

Suppose that the insurer has an exponential utility function given by

\[
b(x) = -\frac{\gamma}{m} e^{-mx}, \quad \gamma > 0, \ m > 0, \quad (14)
\]

where \(m\) is the so-called constant absolute risk aversion coefficient. Under (14), we try the solution with the form

\[
w(t, x) = -\frac{\gamma}{m} \exp \left\{ -mxe^{r(T-t)} + g(T-t) \right\}, \quad (15)
\]

where \(g\) is a measurable function such that (15) is a solution to (8). From the boundary condition \(w(T, x) = b(x)\), we have \(g(0) = 0\). Let \(w_t, w_x, w_{xx}\) be the partial derivatives of \(w(t, x)\). Then we have

\[
w_t = w(t, x)(mrxe^{r(T-t)} - g'(T-t)), \quad (16)
\]

\[
w_x = w(t, x)\left( -me^{r(T-t)} \right), \quad (17)
\]

\[
w_{xx} = w(t, x)(m^2e^{2r(T-t)}). \quad (18)
\]

It is obvious that \(w(t, x)\) is a \(C^{1,2}\) function, and that \(w_x/w_{xx} < 0\). Therefore, the optimal value \(\pi^*_t > 0\) and \(p^*_t < 1\) are both guaranteed. Inserting (17) and (18) into (11)-(13) yields the optimal strategies

\[
\pi^*_t = \frac{\alpha - r}{m\beta^2} e^{-r(T-t)}, \quad (19)
\]

\[
p^*_t = 1 - \frac{\alpha u G(u^*_t) + \mu u^2_t \eta_{\min}}{\sigma^2 u^2_t m} e^{-r(T-t)}, \quad (20)
\]

\[
u^*_t = \frac{\nu}{\sigma^2} \left[ ah^{-1}(u^*_t) + \eta_{\min} \right] e^{-r(T-t)} \left[ ah^{-1}(u^*_t) - \eta_{\min} \right] + 2\eta_{\min}, \quad (21)
\]

Also, substituting (16)-(18) into (10), we obtain

\[
g'(T-t) - mrxe^{r(T-t)} = \frac{1}{2} \left[ \sigma^2 (u^*_t)^2 (1 - ap^*_t)^2 + \beta^2 (\pi^*_t)^2 \right] m^2 e^{2r(T-t)}
\]

\[
+ \left[ rx + \mu G(u^*_t) (1 - ap^*_t) \right] (1 - mrxe^{r(T-t)}) 
\]

\[
- \mu (u^*_t)^2 p^*_t \eta_{\min} + \pi^*_t (\alpha - r) \left( -me^{r(T-t)} \right).
\]

From (19) and (20), we see that \(\pi^*_t\) is easy to compute and that \(p^*_t\) is also computable if we know the value of \(u^*_t\). To calculate the value function, it remains to check the existence and uniqueness of \(u^*_t\) in (21). To do so, we need to examine some properties of \(G(x)\).
Lemma 3.2. There exists a point \( u^0 \in (0, \sqrt{\lambda_{\text{max}}}) \) such that \( G'(u^0) = 0 \), and that \( G(u) \) attains its maximum at \( u^0 \).

Proof. Since \( G(u) = u^2 h^{-1}(u^2) \), it is easy to check that
\[
G'(u) = 2uh^{-1}(u^2) + 2u^3h^{-1'}(u^2) = 2u(h^{-1}(u^2) + u^2h^{-1'}(u^2)).
\]
Set \( g(s) = h^{-1}(s) + sh^{-1'}(s) \) with \( s = u^2 \). By the properties of \( h^{-1} \), it is clear that
\[
g'(s) = 2h^{-1'}(s) + sh^{-1''}(s) < 0,
\]
which implies that \( g(s) \) is a decreasing function. Also, \( g(0) = \theta_{\text{max}} > 0 \) and \( g(\lambda_{\text{max}}) = \lambda_{\text{max}}h^{-1}(\lambda_{\text{max}}) < 0 \). One can conclude that there exists a point \( s^0 \in (0, \lambda_{\text{max}}) \) such that \( g(s^0) = 0 \). Put \( u_0 = \sqrt{s^0} \). Then we have \( g((u_0)^2) = 0 \). Hence, \( G'(u_0) = 2u_0g((u_0)^2) = 0 \).

Since \( G'(u) = 2ug(u^2) \) and \( g \) is a decreasing function with \( g((u_0)^2) = 0 \), it is easy to check that \( G'(u) > 0 \) for \( u < u_0 \), and that \( G'(u) < 0 \) for \( u > u_0 \). Therefore, \( G(u) \) attains its maximum at \( u_0 \); and it is increasing in \([0, u_0)\) and decreasing in \([u_0, \sqrt{\lambda_{\text{max}}}]) \.

We now give justification to narrow down the range for \( u^* \) by considering the risk model of study in the absence of reinsurance and investment. In this case, the risk process (6) becomes
\[
d Y_t = \mu G(u_t)dt + \sigma u_t dW_t^Y.
\]
It follows from the form of (22) and the properties of \( G \) in Lemma 3.2 that for \( u \geq u_0 \), the volatility coefficient increases but the drift coefficient decreases as \( u \) increases. This is unacceptable to a risk-adverse insurer who prefers large drift coefficient and small volatility coefficient. Thus, we restrict our search of the optimal value \( u^* \) within the interval \([0, u_0)\).

In order to check the existence of \( u^* \) in (21), we first need to study some properties of the right-hand side of (21) denoted by \( z(u) \), i.e.,
\[
z(u) = \frac{1 - a + \frac{a\mu}{m^2}(ah^{-1}(u^2) + \eta_{\text{min}})e^{-r(T-t)}}{\frac{\mu}{m^2}(ah^{-1}(u^2) + \eta_{\text{min}})(ah^{-1}(u^2) - \eta_{\text{min}})e^{-r(T-t)} + 2\eta_{\text{min}}}. \tag{23}
\]
Differentiating (23) with respect to \( u \), we obtain
\[
z'(u) = \frac{2ua^2Mh^{-1'}(u^2)}{[M(ah^{-1}(u^2) + \eta_{\text{min}})(ah^{-1}(u^2) - \eta_{\text{min}}) + 2\eta_{\text{min}}]^2} 
\times \{ -a^2M(h^{-1}(u^2))^2 - 2[1 - a + aM\eta_{\text{min}}]h^{-1}(u^2) - \eta_{\text{min}}^2 M + 2\eta_{\text{min}} \},
\]
where \( M = (ma^2)^{-1}mu^{-r(T-t)} > 0 \). Define \( \phi(y) \) as
\[
\phi(y) = -a^2My^2 - 2[1 - a + aM\eta_{\text{min}}]y - \eta_{\text{min}}^2 M + 2\eta_{\text{min}}. \tag{24}
\]
Since \( a^2M > 0 \), \( \phi \) is a concave quadratic function and
\[
z'(u) = \frac{2ua^2Mh^{-1'}(u^2)\phi(h^{-1}(u^2))}{[M(ah^{-1}(u^2) + \eta_{\text{min}})(ah^{-1}(u^2) - \eta_{\text{min}}) + 2\eta_{\text{min}}]^2}, \tag{25}
\]
with \( h^{-1'}(u^2) < 0 \). Thus, the sign of \( z'(u) \) depends very much on the sign of \( \phi(h^{-1}(u^2)) \). Since we only consider values of \( u \) belonging to \([0, u_0)\), \( \phi(h^{-1}(u^2)) \) lies in the interval \( I = (h^{-1}((u_0)^2), \theta_{\text{max}}) \). Hence, to study the shape of \( z(u) \) for \( u \in [0, u_0] \), we need to consider the following three cases:
(i) If \(\phi(y) > 0\) for \(y \in I\), then \(z'(u)\) in (25) is negative. This means that \(z\) is a decreasing function in \([0,u^0]\);
(ii) If \(\phi(y) < 0\) for \(y \in I\), then \(z'(u)\) in (25) is positive. In this case, \(z\) is an increasing function in \([0,u^0]\);
(iii) Consider the case that there exists some \(y \in I\) such that \(\phi(y) = 0\). Since \(\phi(y)\) of (24) is a quadratic concave function, it has two roots \(y_1, y_2\) (not necessarily unequal). When it has two roots, there is one positive root at most because \(y_1 + y_2 = -2(1 - a + aM\eta_{\min})/a^2M < 0\). Suppose that the positive root say \(y_1\) lies in the interval \(I\). As \(u\) increases from 0 to \(u^0\), \(y = h^{-1}(u^2)\) decreases from \(\theta_{\max}\) to \(h^{-1}((u^0)^2)\). Therefore, \(\phi(y)\) moves from negative to positive as \(u\) increases from 0 to \(u^0\). It follows from (25) and \(h^{-1}'(u^2) < 0\) that the sign of \(z'(u)\) changes from positive to negative as \(u\) increases from 0 to \(u^0\). As a result, one can conclude that \(z(u)\) is a concave function in \([0,u^0]\). Note that (i) and (ii) cover the case that the positive root \(y_1\) does not lie in \(I\), and that the case with no roots belongs to (ii).

In Section 2, we mentioned that it is reasonable to have a lower bound for \(\theta_{\max}\). In order to prove the next lemma, we assume that

\[
\theta_{\max} \geq \frac{m\sigma^2}{\mu e^{-r(T-t)}a^2}(a + \frac{1}{2}) - \frac{\eta_{\min}}{a},
\]

or

\[
a\theta_{\max} + \eta_{\min} \geq \frac{m\sigma^2}{\mu e^{-r(T-t)}} (1 + \frac{1}{2a}).
\]

Under this assumption, the following lemma gives the uniqueness of \(u^*\) in (21), where \(u^*\) is the solution to the equation

\[
\frac{u}{G'(u)} = z(u),
\]

with \(z(u)\) given in (23).

**Lemma 3.3.** Assume that (26) holds. Equation (28) has a unique solution in \([0,u^0]\).

**Proof.** From Lemma 3.2, we have \(G'(u^0) = 0\). Hence, the left-hand side of (28) (i.e., \(u/G'(u)\)) goes to infinity as \(u\) tends to \(u^0\). Also, \(G(u) = u^2h^{-1}(u^2)\) by definition so that the left-hand side of (28) has the form

\[
\frac{u}{G'(u)} = \frac{1}{2h^{-1}(u^2) + 2a^2h^{-1'}(u^2)} \rightarrow \frac{1}{2h^{-1}(0) + 2 \times 0} = \frac{1}{2\theta_{\max}} \text{ as } u \rightarrow 0.
\]

By the properties of \(h^{-1}\), we have \(h^{-1'} < 0\) and \(h^{-1''} < 0\). Thus,

\[
\left(\frac{u}{G'(u)}\right)' = \frac{-8u^3h^{-1'}(u^2) - 4u^5h^{-1''}(u^2)}{(G'(u))^2} > 0.
\]

Therefore, \(u/G'(u)\) strictly increases from \(1/(2\theta_{\max})\) to \(\infty\) on the interval \([0,u^0]\).

Putting \(u = 0\) into (23), we have

\[
z(0) = \frac{1 - a + \frac{\mu}{m\sigma^2}(a\theta_{\max} + \eta_{\min})e^{-r(T-t)}}{\frac{\mu}{m\sigma^2}(a\theta_{\max} + \eta_{\min})e^{-r(T-t)} + 2\eta_{\min}} = 1 - a + \frac{\mu}{m\sigma^2}(a\theta_{\max} + \eta_{\min})e^{-r(T-t)}
\]

\[
= \frac{1 - a + \frac{\mu}{m\sigma^2}(a\theta_{\max} + \eta_{\min})e^{-r(T-t)}}{\frac{\mu}{m\sigma^2}(a^2\eta_{\max} + \eta_{\min})e^{-r(T-t)} + 2\eta_{\min}}.
\]

\[
= 1 - a + \frac{\mu}{m\sigma^2}(a\theta_{\max} + \eta_{\min})e^{-r(T-t)}
\]

\[
= \frac{1 - a + \frac{\mu}{m\sigma^2}(a\theta_{\max} + \eta_{\min})e^{-r(T-t)}}{\frac{\mu}{m\sigma^2}(a^2\eta_{\max} + \eta_{\min})e^{-r(T-t)} + 2\eta_{\min}}.
\]
We next compare the two sides of (28) at \( u = 0 \). Or equivalently, we compare \( 1/(2\theta_{\text{max}}) \) with \( z(0) \). Denote the numerator and denominator of \( z(0) \) by \( K_1 \) and \( K_2 \), respectively. Then (29) can be rewritten as \( z(0) = K_1/K_2 \). Consider

\[
2\theta_{\text{max}}K_1 - K_2 = 2\theta_{\text{max}}(1 - a) + \frac{2a\theta_{\text{max}} \mu}{m\sigma^2} (a\theta_{\text{max}} + \eta_{\text{min}}) e^{-r(T-t)}
\]

\[
- \frac{\mu e^{-r(T-t)}}{m\sigma^2} (a^2\theta_{\text{max}}^2 - \eta_{\text{min}}^2) - 2\eta_{\text{min}}
\]

\[
= (a\theta_{\text{max}} + \eta_{\text{min}}) \left[ \frac{\mu e^{-r(T-t)}}{m\sigma^2} (a\theta_{\text{max}} + \eta_{\text{min}}) - 2 \right] + 2\theta_{\text{max}}
\]

\[
\geq (a\theta_{\text{max}} + \eta_{\text{min}}) \left( \frac{1}{2a} - 1 \right) + 2\theta_{\text{max}}
\]

\[
\geq \frac{m\sigma^2}{\mu e^{-r(T-t)}} \times \frac{1 - 4a^2}{4a^2} + 2\theta_{\text{max}},
\]

where the last two inequalities are due to (27). In Section 2, it is assumed that \( \eta_{\text{min}} \leq \theta_{\text{max}}/2 < a\theta_{\text{max}} \). Under this assumption, it is easy to show that (26) can be rewritten as

\[
a\theta_{\text{max}} \geq \frac{m\sigma^2}{\mu e^{-r(T-t)}} \left( 1 + \frac{1}{2a} \right) - \eta_{\text{min}}
\]

\[
> \frac{m\sigma^2}{\mu e^{-r(T-t)}} \left( 1 + \frac{1}{2a} \right) - a\theta_{\text{max}}.
\]

Thus, we have

\[
2\theta_{\text{max}} > \frac{m\sigma^2}{\mu e^{-r(T-t)}} \left( \frac{1}{a} + \frac{1}{2a^2} \right).
\]

Plugging this into (30), we obtain

\[
2\theta_{\text{max}} K_1 - K_2 > \frac{m\sigma^2}{\mu e^{-r(T-t)}} \left( \frac{1}{4a^2} - 1 \right) + \frac{1}{a} + \frac{1}{2a^2}
\]

\[
= \frac{m\sigma^2}{\mu e^{-r(T-t)}} \left( \frac{3}{4a^2} + \frac{1}{a} - 1 \right) > 0,
\]

where the last inequality holds because \( a \in (1/2, 1) \). Moreover, by definition, \( K_1 > 0 \) and \( K_2 > 0 \), and hence (31) gives

\[
\frac{1}{2\theta_{\text{max}}} < \frac{K_1}{K_2} = \frac{1 - a + \frac{a\mu}{m\sigma^2} (a\theta_{\text{max}} + \eta_{\text{min}}) e^{-r(T-t)}}{\mu \sigma^2 (a^2\theta_{\text{max}}^2 - \eta_{\text{min}}^2) e^{-r(T-t)} + 2\eta_{\text{min}}} = z(0).
\]

Finally, it can be concluded that \( u/G'(u) < z(u) \) at \( u = 0 \). On the other hand, it is clear from (23) that \( z(u^0) \) is finite. Therefore, \( u/G'(u) > z(u) \) as \( u \to u_0 \).

For \( u \in [0, u_0) \), the shape of \( z(u) \) is summarized in Cases (i)-(iii). In each of these three cases, the above analysis of \( u/G'(u) \) and \( z(u) \) guarantees that \( z(u) \) intercepts with \( u/G'(u) \) once in \([0, u_0)\). Hence, we conclude that there exists one unique solution \( u^* \) to (28) in \([0, u^0)\).

Given the form of \( h \), one can use some mathematical software such as MATLAB and Mathematica to compute the value of \( u^* \).
4. **Net profit condition in a special case.** Motivated by [12], we consider the net profit condition under reinsurance given by

\[ C_t - \delta(p_t) - E \left[ \sum_{k=1}^{N_c} (1 - p_{T_k}) X_k \right] > 0. \tag{32} \]

Recall the form of \( C_t \) in (1) and that of \( \delta(p_t) \) in (2). Under the diffusion approximation (5), (32) becomes

\[
\mu \int_0^t (1 + \theta_s) \lambda_s ds - \mu \int_0^t (1 + \eta_s) p_s \lambda_s ds - \mu \int_0^t (1 - p_s) \lambda_s ds = \mu \int_0^t (\theta_s - \eta_s p_s) \lambda_s ds > 0. \tag{33}
\]

For notational convenience, we suppress the subscript corresponding to time in all the related symbols in the rest of this section. In order to check the net profit condition (33) under the optimal policy, we need to specify the form \( h \), and derive explicit expressions for \( p^\star \) and \( \lambda^\star = u^2 \) by using (20) and (21).

Due to the complexity of (21), it is difficult to solve for \( u^\star \) explicitly in general. Here we consider a special case in which \( h \) is linear. Since \( h(0) = \lambda_{max} \) and \( h(\theta_{max}) = 0 \), \( h(\theta) \) has the form

\[ \lambda = h(\theta) = -\frac{\lambda_{max}}{\theta_{max}} \theta + \lambda_{max}, \tag{34} \]

and the inverse of \( h(\theta) \) is given by

\[ \theta = h^{-1}(\lambda) = \theta_{max} - \frac{\theta_{max}}{\lambda_{max}} \lambda. \tag{35} \]

Under (34), we have

\[ G(u) = u^2 h^{-1}(u^2) = \theta_{max} u^2 - \frac{\theta_{max}}{\lambda_{max}} u^4. \]

Differentiating \( G \) with respect to \( u \) yields

\[ G'(u) = 2u \theta_{max} - \frac{4 \theta_{max}}{\lambda_{max}} u^3. \tag{36} \]

Inserting (36) and (35) into (28), we obtain the equation

\[
\left( \frac{2 \theta_{max}}{\lambda_{max}} - 4u \frac{\theta_{max}}{\lambda_{max}} \right) \left[ 1 - a + \frac{a \mu}{m \sigma^2} \left( a \theta_{max} - \frac{a \theta_{max}}{\lambda_{max}} u^2 + \eta_{min} \right) e^{-r(T-t)} \right] = -\frac{\mu}{m \sigma^2} \left[ a^2 \theta_{max}^2 \left( 1 - \frac{u^2}{\lambda_{max}} \right)^2 - \eta_{min}^2 \right] e^{-r(T-t)} + 2 \eta_{min}. \tag{37}
\]

By letting \( s = u^2 \), one can rewrite (37) as

\[ As^2 + Bs + C = 0, \tag{38} \]
where

\[
A = -3\mu^2 \theta_{\text{max}}^2 \sigma^2 e^{-r(T-t)},
\]

\[
B = 4\theta_{\text{max}} \left[ \frac{\mu \sigma^2 e^{-r(T-t)}}{2\sigma^2} (a\theta_{\text{max}} + \eta_{\text{min}}) + 1 - a \right],
\]

\[
C = -\frac{\mu}{2\sigma^2} e^{-r(T-t)} (a\theta_{\text{max}} + \eta_{\text{min}})^2 + 2\eta_{\text{min}} - 2\theta_{\text{max}}(1-a).
\]

Let

\[
\Delta = B^2 - 4AC
\]

\[
= 4\theta_{\text{max}}^2 \left[ \frac{\mu^2 \sigma^2 e^{-2r(T-t)}}{2\sigma^2} (a\theta_{\text{max}} + \eta_{\text{min}})^2 + 2\mu a \sigma^2 e^{-r(T-t)} (a\theta_{\text{max}} + 4\eta_{\text{min}} - a^2\theta_{\text{max}} - a\eta_{\text{min}}) + 4(1-a)^2 \right] > 0.
\]

Then the two roots of (38) are given by

\[
s_1 = \frac{-B - \sqrt{\Delta}}{2A} \quad \text{and} \quad s_2 = \frac{-B + \sqrt{\Delta}}{2A}.
\]

It is obvious that \(\sqrt{\Delta}\) is not analytically tractable.

To prove the net profit condition (33), we make use of a smaller value \(\hat{\Delta}\) such that its square root is easy to get. Define

\[
\hat{\Delta} = \frac{4\theta_{\text{max}}^2}{\lambda_{\text{max}}^2} \left[ \frac{\mu \sigma^2 e^{-r(T-t)}}{2\sigma^2} (a\theta_{\text{max}} + \eta_{\text{min}}) + 1 - a \right]^2.
\]

It is easy to see that \(\hat{\Delta} < \Delta\). Let

\[
\hat{s}_1 = \frac{-B - \sqrt{\hat{\Delta}}}{2A} = \frac{a\mu e^{-r(T-t)} (a\theta_{\text{max}} + \eta_{\text{min}}) + (1-a)\sigma^2 \lambda_{\text{max}}}{\mu e^{-r(T-t)} a^2 \theta_{\text{max}}} \lambda_{\text{max}}, \quad (39)
\]

\[
\hat{s}_2 = \frac{-B + \sqrt{\hat{\Delta}}}{2A} = \frac{a\mu e^{-r(T-t)} (a\theta_{\text{max}} + \eta_{\text{min}}) + (1-a)\sigma^2 \lambda_{\text{max}}}{3\mu e^{-r(T-t)} a^2 \theta_{\text{max}}} \lambda_{\text{max}}. \quad (40)
\]

Since \(A\) is negative, (39) and (40) imply that \(\hat{s}_1 < s_1\) and \(\hat{s}_2 > s_2\). As \(s = a^2 = \lambda < \lambda_{\text{max}}\) by definition, we have \(\hat{s}_1 < s_1 < \lambda_{\text{max}}\). On the other hand, it follows from (39) that \(\hat{s}_1 > \lambda_{\text{max}}\) which leads to a contradiction. So, we reject \(s_1\), and take \(s_2\) as the solution to (38), i.e., \(s_2 = u^{*2} = \lambda^*\) is the optimal strategy associated with the premium rate. Note that the optimal \(\pi^*\) of (19) remains unchanged as it does not depend on \(u^{*2}\) or \(\lambda^*\), and that the resulting optimal reinsurance strategy in (20) has the form

\[
p^* = 1 - \frac{a\mu G(u^*) + \mu u^* \eta_{\text{min}}}{\sigma^2 u^2 m} e^{-r(T-t)}
\]

\[
= 1 - \frac{a\mu h^{-1}(\lambda^*) + \mu \eta_{\text{min}}}{\sigma^2 m} e^{-r(T-t)}, \quad (41)
\]

where \(G(u^*) = u^{*2} h^{-1}(u^{*2}), \ u^{*2} = \lambda^*,\) and \(h^{-1}\) is given in (35).

Put \(\tilde{s}_2 = \tilde{\lambda}\). Define \(\tilde{\theta} = h^{-1}(\lambda)\) where \(h^{-1}\) is given in (35). Since \(\tilde{s}_2 > s_2, \ \tilde{\lambda} > \lambda^*\) which in turn implies that \(\tilde{\theta} = h^{-1}(\lambda) < h^{-1}(\lambda^*) = \theta^*\) as \(h^{-1}\) is a decreasing
function. Similar to (41), we define

\[ \hat{p} = 1 - \frac{a \mu h^{-1}(\hat{\lambda}) + \mu \eta_{\min} e^{-r(T-t)}}{m \sigma^2}. \]

Using (35) and (40) with \( \tilde{\lambda} = \lambda \), we obtain

\[ \tilde{p} = -2a \mu e^{-r(T-t)}(a \theta_{\max} + \eta_{\min}) + (2a + 1)m \sigma^2. \]

(42)

It is clear that \( \tilde{p} \) is larger than \( p^* \) simply because \( \hat{\lambda} > \lambda^* \) and that \( h^{-1} \) is a decreasing function. With \( \tilde{p} > p^* \) and \( \hat{\lambda} < \lambda^* \), if we can prove that \( \hat{\lambda} - \tilde{\eta} \geq 0 \) where \( \tilde{\eta} = a \hat{\theta} + \eta_{\min} \), then \( \lambda^* - p^* \eta^* \geq 0 \) where \( \eta^* = a \theta^* + \eta_{\min} \). The main result in this section is summarized in the following theorem.

**Theorem 4.1.** If \( h(\theta) \) is a linear function with the form (34), then the net profit condition (33) holds.

**Proof.** To prove the theorem, we need to prove

\[ \hat{\lambda} - \tilde{\eta} = (1 - a \tilde{\eta}) \tilde{\lambda} - \tilde{\eta}_{\min} \geq 0. \]

(43)

It follows from the lower bound for \( \theta_{\max} \) in (27) that

\[ 2a \mu e^{-r(T-t)}(a \theta_{\max} + \eta_{\min}) - (1 + 2a)m \sigma^2 \geq 0, \]

which can be rewritten as

\[ [2a \mu e^{-r(T-t)}(a \theta_{\max} + \eta_{\min}) + 2(1 - a)m \sigma^2] \lambda_{\max} \geq 3a \mu e^{-r(T-t)}a \sigma^2 \lambda_{\max}, \]

(44)

by adding \( 3a \mu e^{-r(T-t)}a \sigma^2 \lambda_{\max} \) to each side of the inequality and then multiplying both sides by \( \lambda_{\max} \). Dividing both sides of (44) by \( 6a \mu e^{-r(T-t)}a \sigma^2 \lambda_{\max} \) yields

\[ \frac{2a \mu e^{-r(T-t)}(a \theta_{\max} + \eta_{\min}) + (1 - a)m \sigma^2}{3a \mu e^{-r(T-t)}a \sigma^2 \lambda_{\max}} \lambda_{\max} \geq \frac{m \sigma^2 \lambda_{\max}}{2a \mu e^{-r(T-t)}a \sigma^2 \lambda_{\max}}, \]

which can be rewritten as

\[ \hat{\lambda} \geq \frac{m \sigma^2 \lambda_{\max}}{2a \mu e^{-r(T-t)}a \theta_{\max}} \eta_{\min}, \]

due to (40) and \( \hat{\lambda} = \tilde{s}_2 \). Note that \( \tilde{\eta} \geq \eta_{\min} \), we have

\[ \hat{\lambda} \geq \frac{m \sigma^2 \lambda_{\max} \eta_{\min}}{2a \mu e^{-r(T-t)}a \theta_{\max} \tilde{\eta}}. \]

Replacing \( \tilde{\eta} = a \hat{\theta} + \eta_{\min} \) gives

\[ 2a \mu e^{-r(T-t)}a \sigma^2 \theta_{\max} \hat{\lambda}(a \hat{\theta} + \eta_{\min}) \geq m \sigma^2 \lambda_{\max} \eta_{\min}. \]

Dividing both sides of the last inequality by \( a \sigma^2 \lambda_{\max} \eta_{\min} \), one can show that

\[ \frac{2a \mu e^{-r(T-t)}a \theta_{\max} \hat{\lambda}}{m \sigma^2 \lambda_{\max} \eta_{\min}} \hat{\theta} \geq \frac{1}{a} - \frac{2a \mu e^{-r(T-t)}a \theta_{\max}}{m \sigma^2 \lambda_{\max}}. \]

Substituting \( \hat{\lambda} = \tilde{s}_2 \) on the right-hand side of the above inequality, where \( \tilde{s}_2 \) is defined in (40), we have

\[ \frac{2a \mu e^{-r(T-t)}a \theta_{\max} \hat{\lambda}}{m \sigma^2 \lambda_{\max} \eta_{\min}} \hat{\theta} \geq -2a \mu e^{-r(T-t)}(a \theta_{\max} + \eta_{\min}) + (2a + 1)m \sigma^2. \]

(45)

Note that the right-hand side of (45) equals \( \tilde{p} \) in (42), and that

\[ 1 - a \tilde{p} = \frac{2a \mu e^{-r(T-t)}a \theta_{\max}}{m \sigma^2 \lambda_{\max}} \hat{\lambda}. \]
due to (42) and (40) with \( \hat{s}_2 = \hat{\lambda} \), (45) essentially implies that
\[
(1 - a\hat{\theta})\hat{\theta} \geq \hat{\eta}_{min}.
\]
Therefore, the proof of (43) is complete.

Since \( \hat{p}_s > p^*_s \) and \( \hat{\theta}_s < \theta^*_s \) for all \( s \), one can conclude that
\[
\theta^*_s - \eta^*_s \theta^*_s > \hat{\theta}_s - \hat{\eta}_s \hat{p}_s \geq 0
\]
for all \( s \), which in turn implies that the net profit condition (33) under the optimal control policy holds.

5. Numerical examples. In this section, three numerical examples are given. Recall that the first and second moment of the claim-size distribution are \( \mu \) and \( \sigma^2 \), respectively. The linear function \( h \) has the form of (34) with \( \theta_{max} = 5 \) and \( \lambda_{max} = 10 \) so that \( h(\theta) = -2\theta + 10 \). The numerical results are obtained using MATLAB and the figures are plotted using 100 time points in \( [0, T = 10] \). Let \( \alpha = 0.3, r = 0.1, \beta = 0.4, \) and \( a = 0.8, \mu = m = \eta_{min} = 1. \)

**Example 5.1.** This example illustrates how the second moment of the claim size random variable \( \sigma^2 \) affects the optimal strategies \( p^*_s \) and \( u^*_s \). The results are presented in Figures 1 and 2. In each figure, \( \sigma^2 \) are set to be 0.5, 0.8, and 1.

![Figure 1. Effect of \( \sigma^2 \) on \( p^*_s \)](image)

In Figure 1, we see that the optimal value \( p^*_s \) decreases with time. This suggests that the reinsurance proportion decreases as the insurance risk becomes smaller. We also see that a bigger \( \sigma^2 \) yields a bigger \( p^*_s \). This result illustrates that when the claim gets riskier, the insurer would like to reinsure more so as to transfer a larger portion of the risk to the reinsurer.

Figure 2 examines the impact of \( \sigma^2 \) on the optimal value \( u^*_s \). Since \( u^2_s = \lambda^*_s = h(\theta^*_s) \), Figure 2 essentially illustrates the impact of \( \sigma^2 \) on the optimal safety loading. It is clear from the figure that the optimal value \( u^*_s \) increases with time. Thus, the optimal safety loading \( \theta^*_s \) decreases with time since \( h \) is a decreasing function. This suggests that the insurer prefers charging less to attract more customers when time moves closer to the maturity time \( T \). Furthermore, when \( \sigma^2 \) increases, the optimal value \( u^*_s \) decreases, which indicates an increasing optimal safety loading.
This reveals the fact that it is safer to charge more premium so as to compensate for a greater insurance risk.

**Example 5.2.** With the values of other variables being unchanged, we put $\sigma^2 = 1$ and study the impact of $\beta$ on the optimal value $\pi^*$. Figure 3 shows how the uncertainty of the risky asset $\beta$ affects the optimal investment strategy $\pi_t^*$. The values of $\beta$ are set to be 0.4, 0.6, and 0.8.

In Figure 3, we see that the optimal $\pi_t^*$ increases with time. In other words, a larger amount of risky asset is preferred when time approaches $T$. Also, we see that a bigger $\beta$ yields a smaller $\pi_t^*$. It means that a larger uncertainty in the risky asset leads to a smaller amount of money invested in the risky investment.
Example 5.3. Let $\beta = 0.4$, $\sigma^2 = 1$, and the values of other variables remain unchanged. In this example, we study how the reinsurance variables affect the optimal safety loading and the optimal reinsurance strategy. Since $\eta_t = a\theta_t + \eta_{\min}$, we investigate the impact of $a$ on $u^*_t$ in Figure 4 and illustrate the impact of $\eta_{\min}$ on $p^*_t$ in Figure 5.

![Figure 4. Effect of $a$ on $u^*_t$](image1)

Put $a = 0.6, 0.8, 1$. In Figure 4, it is clear that $u^*_t$ increases as $a$ decreases. This means that the optimal safety loading decreases as $a$ decreases. This result indicates that when reinsurance becomes expensive, insurer charges more by increasing the safety loading.
Let $a = 1$ and $\eta_{\text{min}} = 0.9, 1.1, 1.3$. In Figure 5, we see that the larger $\eta_{\text{min}}$, the smaller the optimal reinsurance strategy $p^*_t$. This fact suggests that as the reinsurer charges more, the insurer purchases less reinsurance to avoid paying a large amount of reinsurance premium.

Acknowledgments. The authors would like to thank the referees and the editor for carefully reading the article and valuable comments and suggestions.

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Received October 2018; revised March 2019.

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