Existence and large-time asymptotics for solutions of semilinear parabolic systems with measure data

TOMASZ KLIMSIAK

Abstract. We study the Cauchy–Dirichlet problem for monotone semilinear uniformly elliptic second-order parabolic systems in divergence form with measure data. We show that under mild integrability conditions on the data, there exists a unique probabilistic solution of the system. We also show that if the operator and the data do not depend on time, then the solution of the parabolic system converges as $t \to \infty$ to the solution of the Dirichlet problem for an associated elliptic system. In fact, we prove some results on the rate of the convergence.

1. Introduction

Let $D \subset \mathbb{R}^d$, $d \geq 2$ be an open bounded domain. In the present paper, we study systems of the form

$$\begin{cases}
\frac{\partial u^k}{\partial t} - A_t u^k = f^k(t, x, u) + \mu^k & \text{in } D_T, \quad k = 1, \ldots, N, \\
u|_{\partial D} = 0, \quad t \in (0, T], \quad u(0, \cdot) = \varphi & \text{on } D.
\end{cases} \tag{1.1}$$

Here, $D_T \equiv [0, T] \times D$, $\mu^k$, $k = 1, \ldots, N$ are bounded soft measures on $\mathbb{R}^d$, i.e., bounded Borel measures absolutely continuous with respect to the parabolic capacity determined by the operator

$$A_t = \frac{1}{2} \sum_{i,j=1}^d \frac{\partial}{\partial x_j} \left( a_{ij}(t, x) \frac{\partial}{\partial x_i} \right). \tag{1.2}$$

In this paper, we assume that $f : D_T \times \mathbb{R}^N \to \mathbb{R}^N$ is a continuous monotone vector field, i.e., $f(t, x, \cdot)$ is continuous for a.e. $(t, x) \in D_T$ and for some $\alpha \in \mathbb{R}$,

$$(f(t, x, y) - f(t, x, y'), y - y') \leq \alpha |y - y'|^2 \tag{1.3}$$

for a.e. $(t, x) \in D_T$ and every $y, y' \in \mathbb{R}^N$. Concerning the growth of $f$, we merely require $f$ to satisfy the following condition

$$f(\cdot, \cdot, 0) \in L^1(D_T), \quad \forall r > 0, y \in \mathbb{R}^N \quad R^{0,T} \left( \sup_{|y| \leq r} |f(\cdot, \cdot, y)| \right) < \infty, \quad m_1\text{-a.s.}, \tag{1.4}$$

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where $R^{0,T}$ is the potential operator of $\frac{\partial}{\partial t} - A_t$ on $D_T$. As for $A_t$, we assume that its coefficient $a : D_T \to \mathbb{R}^d \otimes \mathbb{R}^d$ is a measurable symmetric matrix-valued function such that for some $\Lambda \geq 1$,

$$\Lambda^{-1}|\xi|^2 \leq \sum_{ij=1}^d a_{ij}(t,x)\xi_i\xi_j \leq \Lambda|\xi|^2, \quad \xi \in \mathbb{R}^d$$

(1.5)

for a.e. $(t,x) \in D_T$.

The aim of the present paper was twofold. In the first part of the paper, we focus on the problem of existence of solutions to (1.1). One of the possible approaches to the problem (1.1) is to use the framework of renormalized (or entropy) solutions. Such a framework was successfully applied to scalar equations of the type (1.1) with Leray-Lions-type operator $A_t$ (see, e.g., [2]). For a function $u$ on $D_T$, put

$$f_u(t,x) = f(t,x,u(t,x)), \quad (t,x) \in D_T.$$

We say that $u : D_T \to \mathbb{R}^k$ is a renormalized solution of (1.1) if $f_u \in L^1(D_T)$, $u$ belongs to the space $T_2^{0,1}$, i.e., for every $s \geq 0$, $T_s(u^k) = ((-s) \vee u^k) \wedge s \in W_0^{2,1}(D_T)$ for $k = 1, \ldots, N$, and if $u^k$, $k = 1, \ldots, N$ is a renormalized solution of the linear Cauchy–Dirichlet problem

$$\frac{\partial u^k}{\partial t} - A_t u^k = f_u^k(t,x) + \mu^k, \quad u^k(t,\cdot)|_{\partial D} = 0, \quad t \in (0,T], \quad u(0,\cdot) = \varphi.$$

Unfortunately, showing that $f_u \in L^1(D_T)$ under the growth condition (1.4) is in general a complicated, if ever possible, task. Consequently, we do not know whether $u \in T_2^{0,1}$, and the concept of renormalized solutions is not applicable. However, we are able to prove that under (1.4), the function $f_u$ is quasi-integrable, i.e., roughly speaking, integrable except possibly on sets of small capacity (see Sect. 5 for details). This in turn implies that $u$ belongs to the stochastic Sobolev space $W^{0,1}(X^{D_T})$ introduced in [14] to investigate the obstacle problem with possible nowhere Radon reflection measure. The space $W^{0,1}(X^{D_T})$ is wider than $T_2^{0,1}$ but their elements are regular enough to define correctly probabilistic solutions of (1.1) in $W^{0,1}(X^{D_T})$ and prove the uniqueness result.

For $u \in B(D_T)$, let $\bar{u}(t,x) = u(T-t,x)$, and for a bounded soft measure $\mu$ on $D_T$, let $\bar{\mu}$ be the bounded soft measure on $D_T$ such that $\int \bar{\eta} \, d\mu = \int \eta \, d\bar{\mu}$ for every $\eta \in B_b(D_T)$. Roughly speaking, $u \in W^{0,1}(X^{D_T})$ is a probabilistic solution of (1.1) if for quasi-every (q.e. for short) $(s,x) \in D_T$,

$$\bar{u}(t,X_t) = 1_{(\zeta^s > T)}\varphi(X_T) + \int_t^{T \wedge \zeta^s} \bar{f}_u(r,X_r) \, dr + \int_t^{T \wedge \zeta^s} \, dA_r \bar{\mu} - \int_t^{T \wedge \zeta^s} \tilde{\sigma} \nabla \bar{u}(r,X_r) \, dB_r, \quad t \in [0, T \wedge \zeta^s], \quad P_{s,x}-a.s.,$$

(1.6)
where $\sigma \cdot \sigma^T = a$, $\mathbb{X} = \{(X, P_{s,x}); (s, x) \in \mathbb{R}_+ \times D\}$ is a time-inhomogeneous Markov process associated with the operator $A_t$, $\zeta^s$ is the first exit time of $(X, P_{s,x})$ from $D$, i.e.,

$$\zeta^s = \inf\{t \geq s : X_t \notin D\},$$

and $A^\mu$ is the additive functional of $\mathbb{X}$ in the Revuz correspondence with $\mu$. In the last integral in (1.6), $B$ is a standard $d$-dimensional Brownian motion and $\nabla \bar{u}$ stands for the stochastic gradient of $\bar{u}$ (see [14] or Sect. 5). Formula (1.6) can be regarded as a nonlinear Feynman-Kac formula, because taking $t = s$ and integrating it with respect to the measure $P_{s,x}$, we get

$$\bar{u}(s, x) = E_{s,x} \left( 1_{\{\zeta^s > T\}} \varphi(X_T) + \int_s^T \int_{\mathbb{R}^d} \bar{u}(r, X_r) dr + \int_s^T A^\mu r dr \right)$$

(1.8)

whenever the above integrals exist. We would like to stress that our probabilistic solution $u$ to (1.1) may be considered as some generalization of the notion of renormalized (or entropy) solution, because if $f_u \in L^1(\mathbb{R}^d)$, then $u \in T^0_2$, and $u$ is a renormalized (and entropy as well) solution to (1.1) (see Remark 5.14). Perhaps, also the following comment is appropriate at this point, although the probabilistic solution of (1.1) is in general weak and at first glance its definition seems complicated, it is actually very convenient to deal with.

Our results on the existence and uniqueness of solutions of (1.1) generalize known results in the sense that we consider semilinear parabolic systems with measure data (semilinear elliptic systems with measure data are considered in [15,23]). We should also stress that our results are proved for systems with $f$ satisfying quite general condition (1.3) for which the usual monotonicity methods do not apply and we only require $f$ to satisfy mild integrability condition (1.4) analogous to the integrability condition considered for elliptic equations or systems in [2,15,23]. We also allow $f$ to depend on $x$.

In the second part of the paper, we investigate the asymptotic behavior as $t \to \infty$ of probabilistic solutions of (1.1) in the case where $A_t = A$, $f$ and $\mu = (\mu^1, \ldots, \mu^N)$ do not depend on time. Let $\tilde{\mu}^k(B) = \mu^k([0, 1] \times B)$, $k = 1, \ldots, N$, for any Borel subset $B$ of $D$. Our main result says that under the assumptions ensuring the existence and uniqueness of a solution $v$ to the elliptic system

$$\begin{aligned}
-A v^k &= f^k(x, v) + \tilde{\mu}^k \quad \text{in } D, \\
v|_{\partial D} &= 0
\end{aligned}
(1.9)$$

we have

$$u(t, x) \to v(x) \quad \text{as } t \to \infty$$

for q.e. $x \in D$. As a matter of fact, we prove that there is $c$ depending only on $d$, $\Lambda$ such that for every $t > 0$ and $q \in (0, 1)$,
\[ |u(t, x) - v(x)| \leq c(1 - q)^{-1/q} t^{-d/2} \left( (P_x(\zeta^0 > t))^{(1-q)/q} \| \varphi \|_{L^1} + \| Rf(\cdot, 0) \|_{L^1} + \| R\tilde{\mu} \|_{L^1} \right) \]

(1.10)

for \( q.e. \, x \in D \), where \( R \) is the potential operator of \(-A\) on \( D \). From this, it follows in particular that there exists \( c = c(\Lambda, d, |D|) \) such that for every \( t > 0 \),

\[ |u(t, x) - v(x)| \leq c(1 - q)^{-1/q} t^{-d/2} (e^{-bt(1-q)/q} \| \varphi \|_{L^1} + \| f(\cdot, 0) \|_{L^1} + \| \tilde{\mu} \|_{TV}), \]

(1.11)

because it is known that \( P_x(\zeta^0 > t) \leq ae^{-bt} \) for some \( a, b > 0 \) depending only on \( d, \Lambda \) and \( |D| \) (the Lebesgue measure of \( D \)) and \( \| R\tilde{\mu} \|_{L^1} \leq C(d, \lambda, |D|) \| \tilde{\mu} \|_{TV} \) (\( \tilde{\mu} \) stands for the total variation of \( \tilde{\mu} \)). For instance, if \( f(\cdot, 0) = 0 \) and \( \mu(dx) = 0 \), then the rate of convergence in (1.11) is the same as in the classical case of one linear equation (see [8]). We also show that in fact (1.10) holds for every \( x \) from the set

\[ F_0 = \{ x \in D; \forall r > 0 \, R|f(\cdot, 0)|(x) + (R\tilde{\mu})(x) + R(\sup_{|y| \leq r} |f(\cdot, y)|)(x) < \infty \}. \]

(1.12)

In case \( N = 1 \), the large-time asymptotic behavior of solutions to parabolic equations with measure or \( L^1 \) data was investigated in [21,27–29] (in [27], the case of general, possibly singular measures is considered). In all these papers in proofs, some comparison results are used. Therefore, the methods of [21,27–29] cannot be applied to systems considered in the present paper. Let us also point out that these methods do not provide estimates of the difference between solutions to parabolic equations and the corresponding stationary solutions.

Although the main results of the paper concern systems of PDEs and are analytic in nature, the methods of proofs are those of stochastic analysis, Markov processes, and especially the theory of backward stochastic differential equations. Therefore, in Sects. 2–4, we give relevant background material concerning these topics. Then, in Sect. 5, we prove our results on the existence and uniqueness of solutions of (1.1), and in Sect. 6 results on their large-time behavior. Our idea of using the methods of backward stochastic differential equations to the study of large-time behavior of semilinear parabolic equations is new. It seems likely that it can be applied to wider that (1.1) class of equations.

### 2. Preliminary results

Let us fix a probability space \((\Omega, \mathcal{F}, P)\) equipped with a filtration \(\{\mathcal{F}_t\}\) satisfying the usual conditions.

By \( B \), we denote a standard \( d \)-dimensional \(\{\mathcal{F}_t\}\)-Brownian motion. By \( \mathcal{A} \), we denote the set of all \(\{\mathcal{F}_t\}\) progressively measurable real-valued processes and by \( \mathcal{Y} \) (respectively, \( \mathcal{Y}_c \)) the subspace of \( \mathcal{A} \) consisting of all increasing càdlàg (respectively, continuous) processes \( Y \) such that \( Y_0 = 0 \). \( M \) is the space of all processes \( Z \in \mathcal{A} \) such that \( P(\int_0^T |Z_t|^2 \, dt < \infty) = 1 \) for every \( T > 0 \). \( M^p, \, p > 0 \), is the subspace of
Let \( X \) be a progressively measurable process such that for every \( p \geq 1 \),

\[
|X_t|^p = |X_0|^p + p \int_0^t |X_{s-}|^{p-1} \langle \dot{X}_{s-}, dK_s \rangle + p \int_0^t |X_s|^{p-1} \langle \dot{X}_s, H_s dB_s \rangle \nonumber
\]

\[
\quad + \frac{p}{2} \int_0^t |X_s|^{p-2} 1_{X_s \neq 0} (2 - p) (|H_s|^2 - \langle \dot{X}_s, H_s H_s^* \dot{X}_s \rangle) + (p - 1) |H_s|^2 \rangle ds \nonumber
\]

\[
\quad + 1_{[p=1]} L_t + \sum_{0 < s \leq t} (\Delta |X_s|^p - \langle |X_s|^{p-1} \dot{X}_s, \Delta X_s \rangle), \quad t \geq 0.
\]

**Proof.** The proof is a modification of the proof of [3, Lemma 2.2]. Obviously, it suffices to prove the formula for \( t \in [0, T] \). Set \( u_\varepsilon(x) = (|x|^2 + \varepsilon^2)^{1/2}, \ x \in \mathbb{R}^N, \ \varepsilon > 0 \). A straightforward computation shows that

\[
\nabla u_\varepsilon^p(x) = pu_\varepsilon^{p-2}(x)x, \ D^2 u_\varepsilon^p(x) = pu_\varepsilon^{p-2}(x)I + p(p - 2)u_\varepsilon^{p-4}(x)(x \otimes x),
\]

where \( I \) is the \( n \)-dimensional identity matrix. By the Itô–Meyer formula,
\[ u_\varepsilon^p(X_s) = u_\varepsilon^p(X_0) + \int_0^t \langle \nabla u_\varepsilon^p(X_{s-}), dK_s \rangle + \int_0^t \langle \nabla u_\varepsilon^p(X_{s-}), H_s dB_s \rangle \\
+ \frac{1}{2} \sum_{i,j} \int_0^t \frac{\partial^2 u_\varepsilon^p}{\partial x_i \partial x_j} (X_{s-}) d[|X^i, X^j|] + \sum_{0 \leq s \leq t} (\Delta u_\varepsilon^p(X_s) - \langle \nabla u_\varepsilon^p(X_{s-}), \Delta X_s \rangle) \\
= u_\varepsilon^p(X_0) + \sum_{i=1}^4 I_i^\varepsilon(t). \]

It is clear that

\[ u_\varepsilon^p(X_t) \to |X_t|^p, \quad I_i^\varepsilon(t) \to p \int_0^t |X_{s-}|^{p-1} \langle \dot{X}_{s-}, dK_s \rangle, \quad \varepsilon \searrow 0 \]

a.s. for \( t \in [0, T] \). Using arguments from the proof of [3, Lemma 2.2], one can show that

\[ I_2^\varepsilon(t) \to p \int_0^t |X_s|^{p-1} \langle \dot{\hat{X}}_s, H_s dB_s \rangle, \quad \varepsilon \searrow 0 \]

uniformly on \([0, T]\) in probability, and that

\[ I_3^\varepsilon(t) = I_{3,1}^\varepsilon(t) + I_{3,2}^\varepsilon(t), \]

where

\[ I_{3,1}^\varepsilon(t) = \frac{p(p-2)}{2} \int_0^t (|X_s| u_\varepsilon^{-1}(X_s))^{4-p} |X_s|^{p-2} 1_{\{X_s \neq 0\}} (|H_s|^2 - \langle \dot{\hat{X}}_s, H_s H_s^* \dot{\hat{X}}_s \rangle) \, ds \\
+ p(p-1) (|X_s| u_\varepsilon^{-1}(X_s))^{4-p} |X_s|^{p-2} 1_{\{X_s \neq 0\}} |H_s|^2 \]

and

\[ I_{3,2}^\varepsilon(t) = \varepsilon^2 \frac{p}{2} \int_0^t |H_s|^2 u_\varepsilon^{p-4}(X_s) \, ds. \]

Moreover, as \( \varepsilon \searrow 0 \),

\[ I_{3,1}^\varepsilon(t) \to \int_0^t |X_s|^{p-1} 1_{\{X_s \neq 0\}} \left\{ (2 - p) \left( |H_s|^2 - \langle \dot{\hat{X}}_s, H_s H_s^* \dot{\hat{X}}_s \rangle \right) - (p - 1) |H_s|^2 \right\} \, ds \]

a.s. for every \( t \in [0, T] \). We now show the convergence of \( I_4^\varepsilon(t) \). It is clear that

\[ \Delta u_\varepsilon^p(X_s) \to |X_s|^p, \quad \langle \nabla u_\varepsilon^p(X_{s-}), \Delta X_s \rangle \to \langle |X_s|^{p-1}\dot{\hat{X}}_s, \Delta X_s \rangle, \quad s \in [0, T]. \]

Observe also that

\[ |\Delta u_\varepsilon^p(X_s)| \leq \sup_{\theta \in [0,1]} |\nabla u_\varepsilon^p(\theta \Delta X_s + X_{s-})| |\Delta X_s| \leq 3 \sup_{0 \leq t \leq T} (|X_t|^{p-1} + 1) |\Delta X_s| \]

and

\[ |\nabla u_\varepsilon^p(X_{s-})| \leq \sup_{0 \leq t \leq T} (|X_t|^{p-1} + 1). \]
Since $\sum_{0 < s \leq t} |\Delta X_s| \leq |K_T|$, applying the Lebesgue dominated convergence theorem shows that for every $t \in [0, T]$,

$$I_4^t(t) \to \sum_{0 < s \leq t} \left( \Delta |X_s|^p - \langle |X_s|^{p-1} \hat{X}_s, \Delta X_s \rangle \right), \quad e' \leq 0.$$

By what has already been proved, it follows that $I_{\varepsilon,2}^s(t)$ is convergent. Put $L_t(p) = \lim_{\varepsilon \to 0} I_{\varepsilon,2}^s(t)$. Then, $L$ is a càdlàg increasing process, and as in the proof [3, Lemma 2.2], one can show that if $p > 1$, then $L_t(p) = 0$ for $t \in [0, T]$, which completes the proof.

**REMARK 2.2.** It is well known that the function $\nu(x) = p|x|^{p-1} \hat{x}$ is the subgradient of the function $\nu(x) = |x|^p$. Therefore, $u(x) - u(y) \geq (x - y, \nu(x))$, $x, y \in \mathbb{R}^N$. This implies that the process

$$I_t = \sum_{0 < s \leq t} \left( \Delta |X_s|^p - \langle |X_s|^{p-1} \hat{X}_s, \Delta X_s \rangle \right), \quad t \geq 0$$

is increasing.

**COROLLARY 2.3.** Under the assumptions of Proposition 2.1, for every $0 \leq t \leq T$ and $p \geq 1$,

$$|X_t|^p + c(p) \int_t^T |X_s|^{p-2} 1_{\{X_s \neq 0\}} |H_s|^p \, ds \leq |X_T|^p - p \int_t^T |X_s|^{p-1} \langle \hat{X}_s, dK_s \rangle$$

$$- \int_t^T |X_s|^{p-1} \langle \hat{X}_s, H_s \, dB_s \rangle,$$

where $c(p) = p[(p - 1) \wedge 1]/2$.

### 3. Backward stochastic differential equations

Let $B$ denote a standard $d$-dimensional $\{\mathcal{F}_t\}$-Brownian motion. Let $\sigma$ be a bounded $\{\mathcal{F}_t\}$-stopping time, $\xi$ be an $\mathcal{F}_{\sigma}$-measurable random variable, $A \in \mathcal{V}$ and let $f : \Omega \times \mathbb{R}_+ \times \mathbb{R}^N \to \mathbb{R}^N$ be a measurable function such that $f(\cdot, y)$ is progressively measurable for every $y \in \mathbb{R}^N$.

Let us recall that a pair $(Y, Z)$ consisting of an $\mathbb{R}^N$-valued process $Y$ and an $\mathbb{R}^{N \times d}$-valued processes $Z$ is called a solution of BSDE$(\xi, \sigma, f + dA)$ if

1. $(Y, Z)$ are $\{\mathcal{F}_t\}$-progressively measurable, $Y$ is càdlàg, $P(\int_0^\sigma |Z_t|^2 \, dt < \infty) = 1$,
2. $t \mapsto f(t, Y_t) \in L^1(0, \sigma)$, $P$-a.s.,
3. $Y_t = \xi + \int_0^\sigma f(s, Y_s) \, ds + \int_0^\sigma dA_s - \int_0^\sigma Z_s \, dB_s$, $0 \leq t \leq \sigma$, $P$-a.s.

Let $(Y, Z)$ be a solution of BSDE$(\xi, \sigma, f + dA)$. By putting $Y_t = \xi$, $t \geq \sigma$, $Z_t = 0$, $t \geq \sigma$, we may and will assume in the sequel that the processes $Y, Z$ are defined for $t \geq 0$. We also adopt the convention that $\int_a^b = 0$ for $a \geq b$. Then, the stochastic equation in (c) is satisfied for every $t \geq 0$.

Let us consider the following assumptions.
(A1) \( E (|\xi|^p + (\int_0^\sigma |f(t,0)| \, dt)^p + |A|_\sigma^p) < +\infty \).

(A2) There is \( \mu \in \mathbb{R} \) such that \( (y - y', f(t,y) - f(t,y')) \leq \mu |y - y'|^2 \) for every \( t \geq 0, \, y, y' \in \mathbb{R}^N \).

(A3) For every \( t \geq 0, \) \( y \mapsto f(t,y) \) is continuous.

(A4) For every \( r > 0, \) \( E \int_0^\sigma \sup_{|y| \leq r} |f(t,y)| \, dt < \infty \).

In [3, Theorem 4.2], it is proved that under (A1)–(A4) with \( p > 1 \), there exists a unique solution \((Y, Z) \in S^p \otimes M^p \) of BSDE(\( \xi, \sigma, f \)). We will show how to modify the proof of [3, Theorem 4.2] to get the existence and uniqueness in the general case, i.e., for \( p \geq 1 \) and nonzero process \( A \).

The proof of [3, Theorem 4.2] is based on Lemma 3.1 and Proposition 3.2 in [3], [26, Theorem 2.2] and [4, Lemma 2.2]. To state the generalizations of Lemma 3.1 and Proposition 3.2 in [3] to equations with nonzero \( dA \), we need the following hypothesis.

(A) There is \( \mu \in \mathbb{R} \) and a nonnegative progressively measurable process \( \{f_t, \, t \geq 0\} \) such that \( (\hat{y}, f(t,y)) \leq f_t + \mu |y| \) for all \( (t,y) \in \mathbb{R}_+ \times \mathbb{R}^N \).

For a process \( A \in \mathcal{V} \), we denote by \( |A|_t \) its variation on the interval \([0, t]\).

**Lemma 3.1.** Assume (A). Let \((Y, Z)\) be a solution of BSDE(\( \xi, \sigma, f + dA \)). If \( Y \in \mathcal{D}^p \) and \( E (\int_0^\sigma |f_t| \, dt)^p + E |A|_\sigma^p < \infty \) for some \( p > 0 \), then \( Z \in M^p \) and there exists \( C_p \) depending only on \( p \) such that for every \( a \geq \mu \),

\[
E \left( \int_0^\sigma e^{2at} |Z_t|^2 \, dt \right)^{p/2} \leq C_p E \left( \sup_{0 \leq t \leq \sigma} e^{apt} |Y_t|^p + \left( \int_0^\sigma e^{at} f_t \, dt \right)^p + \left( \int_0^\sigma e^{at} d|A|_t \right)^p \right).
\]

**Proof.** The proof goes through as for [3, Lemma 3.1], with obvious changes. \(\square\)

**Proposition 3.2.** Assume (A). Let \((Y, Z)\) be a solution of BSDE(\( \xi, \sigma, f + dA \)). If \( Y \in \mathcal{D}^p \) and \( E (\int_0^\sigma |f_t| \, dt)^p + E |A|_\sigma^p < \infty \) for some \( p > 1 \), then there exists \( C_p \) depending only on \( p \) such that for every \( a \geq \mu \),

\[
E \left( \sup_{0 \leq t \leq \sigma} e^{apt} |Y_t|^p + \left( \int_0^\sigma e^{2at} |Z_t|^2 \, dt \right)^{p/2} \right) \leq C_p E \left( e^{ap\sigma} |\xi|^p + \left( \int_0^\sigma e^{at} f_t \, dt \right)^p + \left( \int_0^\sigma e^{at} d|A|_t \right)^p \right).
\]

**Proof.** It suffices to repeat, with obvious changes, arguments from the proof of [3, Proposition 3.2]. The only difference is that we use our Corollary 2.3 instead of [3, Corollary 2.3]. \(\square\)

We now prove the analogues of [26, Theorem 2.2] and [4, Lemma 2.2] for equations with nonzero \( dA \).

**Lemma 3.3.** Assume that

\[
(f(t,y), y) \leq c |y|^2, \quad y \in \mathbb{R}^N, \quad t \geq 0
\]  

(3.1)
for some $c \geq 0$ and $\|\xi\|_\infty + c + \|A\|_\sigma \leq r < \infty$. If $(Y, Z)$ is a solution of BSDE($\xi, \sigma, f + dA$) such that $Y$ is of class (D), then $\|Y\|_\infty \leq r$.

**Proof.** By Corollary 2.3,

$$|Y_t| \leq |\xi| + \int_t^\sigma (f(s, Y_s), \dot{Y}_s) \, ds + \int_t^\sigma (\dot{Z}_s, dA_s) - \int_t^\sigma Z_s \, dB_s, \quad t \geq 0.$$ 

Since $Y$ is of class (D), it follows from (3.1) that $|Y_t| \leq E^{\mathcal{F}_t}(|\xi| + c + \|A\|_\sigma), \quad t \geq 0$, which implies the desired estimate. \hfill \Box

**LEMMA 3.4.** Assume (A1)–(A3) are satisfied with $p = 2$ and that $|f(t, y)| \leq c + |f(t, 0)|$ for every $t \geq 0, y \in \mathbb{R}^N$. Then, there exists a unique solution $(Y, Z)$ of BSDE($\xi, \sigma, f + dA$) such that $(Y, Z) \in \mathcal{D}^2 \otimes M^2$.

**Proof.** Using standard arguments, one can prove the existence of a unique solution $(\overline{Y}, \overline{Z}) \in \mathcal{D}^2 \otimes M^2$ of the BSDE

$$\overline{Y}_t = \int_t^\sigma dA_s - \int_t^\sigma Z_s \, dB_s.$$ 

Furthermore, from [26], it follows that under the assumptions of the lemma, there exists a unique solution $(Y', Z') \in \mathcal{D}^2 \otimes M^2$ of the BSDE

$$Y'_t = \xi + \int_t^\sigma \tilde{f}(s, Y'_s) \, ds - \int_t^\sigma Z'_s \, dB_s, \quad t \in [0, \sigma]$$

with $\tilde{f}(t, y) = f(t, y + \overline{Y}_t)$. Set $(Y, Z) = (Y' + \overline{Y}, Z' + \overline{Z})$. Then, $(Y, Z)$ is a solution of BSDE($\xi, \sigma, f + dA$). \hfill \Box

We are now ready to formulate and prove the existence and uniqueness result in case $p > 1$.

**THEOREM 3.5.** Let $p > 1$. If (A1)–(A4) are satisfied, then there exists a unique solution $(Y, Z) \in \mathcal{D}^p \times M^p$ of BSDE($\xi, \sigma, f + dA$).

**Proof.** Step 1. We first assume that $a \equiv \|f(\cdot, 0)\|_\infty + \|\xi\|_\infty + \|A\|_\sigma \|_\infty < \infty$. Let us define $h_n$ as in the first step of the proof of [3, Theorem 4.2] (with $r > a$). Then, in much, the same way as in the proof of that theorem, but using Lemma 3.1 and Proposition 3.2 instead of [3, Lemma 3.1] and [3, Proposition 3.2] and Lemmas 3.3 and 3.4 instead of [26, Theorem 2.2] and [4, Lemma 2.2], one can prove that there exists a solution $(Y, Z) \in \mathcal{D}^2 \otimes M^2$ of BSDE($\xi, \sigma, f + dA$).

Step 2. We define $\xi_n$, $f_n$ as in the second step of the proof of [3, Theorem 4.2] and set $A_n^p = \int_0^t 1_{\|A\|_2 \leq n} dA_s$. The proof of the existence of a solution $(Y, Z) \in \mathcal{D}^p \otimes M^p$ of BSDE($\xi, \sigma, f + dA$) goes as the proof of [3, Theorem 4.2], the only difference being in the use of Lemma 3.1 and Proposition 3.2 instead of [3, Lemma 3.2] and [3, Proposition 3.2]. \hfill \Box

We now turn to the case $p = 1$. We first prove the uniqueness result.
THEOREM 3.6. Let $p = 1$. If (A2) is satisfied, then there exists at most one solution $(Y, Z)$ of BSDE($\xi$, $\sigma$, $f + dA$) such that $Y$ is of class (D).

Proof. Without the loss of generality, we may assume that $\mu \leq 0$. Let $(Y, Z)$, $(Y', Z')$ be the solutions of BSDE($\xi$, $\sigma$, $f + dA$) such that $Y, Y'$ are of class (D). Then, $(\overline{Y}, \overline{Z}) = (Y - Y', Z - Z')$ is a solution of the BSDE

$$
\overline{Y}_t = \int_t^\sigma f(s, Y_s) - f(s, Y'_s) \, ds - \int_t^\sigma Z_s \, dB_s, \quad t \geq 0.
$$

Let $\tau_k = \inf\{t \geq 0; \int_0^t |Z_s|^2 \, ds \geq k\}$. By the Itô–Meyer formula and (A2),

$$
|\overline{Y}_t| \leq |\overline{Y}_{\tau_k \wedge \sigma}| + \int_t^{\tau_k \wedge \sigma} \langle f(s, Y_s) - f(s, Y'_s), \text{sgn}(Y_s) \rangle \, ds
$$

$$
- \int_t^{\tau_k \wedge \sigma} \langle \text{sgn}(Y_s), Z_s \, dB_s \rangle \leq - \int_t^{\tau_k \wedge \sigma} \langle \text{sgn}(Y_s), Z_s \, dB_s \rangle, \quad t \geq 0.
$$

Taking the conditional expectation with respect to $\mathcal{F}_t$ on both sides of the above inequality and then letting $k \to \infty$ and using the fact that $\overline{Y}$ is of class (D), we conclude that $|\overline{Y}_t| = 0$, $t \geq 0$. □

For $k > 0$, let us put

$$
T_k(y) = \frac{ky}{k \vee |y|}, \quad y \in \mathbb{R}^N.
$$

THEOREM 3.7. Let $p = 1$. If (A1)–(A4) are satisfied, then there exists a solution $(Y, Z)$ of BSDE($\xi$, $\sigma$, $f + dA$) such that $(Y, Z) \in \mathcal{D}^q \otimes M^q$ for $q \in (0, 1)$ and $Y$ is of class (D).

Proof. Without the loss of generality, we may assume that $\mu \leq 0$. Set

$$
\xi^n = T_n(\xi), \quad f_n(t, y) = f(t, y) - f(t, 0) + T_n(f(t, 0)), \quad A^n_t = \int_0^t 1_{\{|A_s| \leq n\}} \, dA_s.
$$

By Theorem 3.5, for every $n \in \mathbb{N}$, there exists a solution $(Y^n, Z^n) \in \mathcal{D}^2 \otimes M^2$ of BSDE($\xi^n$, $\sigma$, $f_n + dA^n$). Let $m \geq n$. Write $\delta Y = Y^m - Y^n$, $\delta Z = Z^m - Z^n$, $\delta \xi = \xi^m - \xi^n$ and

$$
\tau_k = \inf \left\{ t \geq 0; \int_0^t |\delta Z_s|^2 \, ds > k \right\}.
$$

By the Itô–Meyer formula, for $t \geq 0$, we have

$$
|\delta Y_{t \wedge \tau_k}| \leq |\delta Y_{\tau_k \wedge \sigma}| + \int_t^{\tau_k \wedge \sigma} \langle \text{sgn}(\delta Y_s), f_m(s, Y_s^m) - f_n(s, Y_s^n) \rangle \, ds
$$

$$
+ \int_t^{\tau_k \wedge \sigma} \langle \text{sgn}(\delta Y_s), d(A_s^m - A_s^n) \rangle + \int_t^{\tau_k \wedge \sigma} \langle \text{sgn}(\delta Y_s), \delta Z_s \, dB_s \rangle
$$

$$
\leq |\delta Y_{\tau_k \wedge \sigma}| + \int_t^{\tau_k \wedge \sigma} |f_m(s, Y_s^n) - f_n(s, Y_s^n)| \, ds
$$

$$
+ \int_t^{\tau_k \wedge \sigma} d|A^m - A^n|_s + \int_t^{\tau_k \wedge \sigma} \langle \text{sgn}(\delta Y_s), \delta Z_s \, dB_s \rangle.
$$
the last inequality being a consequence of monotonicity of \( f_n \) with respect to \( y \). Conditioning with respect to \( \mathcal{F}_t \) and using the fact that \( \delta Y \) is of class (D), we conclude from the above inequality that
\[
|\delta Y_t| \leq E^{\mathcal{F}_t} \left( |\xi| 1_{|\xi| > n} + \int_0^\sigma |f(s, 0)| 1_{|f(s, 0)| > n} \, ds + \int_0^\sigma 1_{|A_s| > n} \, d|A_s| \right)
\]
for \( t \geq 0 \). To complete the proof, it suffices now to repeat step by step the arguments following Eq. (12) in the proof of [3, Proposition 6.4]. □

4. Markov processes and potential theory

To make our exposition in the next sections self-contained, in this section, we recall some useful facts about diffusions associated with the operator \( A_t \) defined by (1.2), their additive functionals and the Revuz correspondence between these functionals and soft measures.

4.1. Time-inhomogeneous diffusions

Let \( \Omega = C(\mathbb{R}_+; \mathbb{R}^d) \) be the space of continuous \( \mathbb{R}^d \)-valued functions on \( \mathbb{R}_+ = [0, \infty) \), \( X \) be the canonical process on \( \Omega \), \( \mathcal{F}^0_{s,t} = \sigma(X_u, u \in [s, t]) \). We define \( \mathcal{F}_{s,\infty} \) as the completion of \( \mathcal{F}^0_{s,\infty} \) with respect to the family \( \mathcal{P} = \{ P_{s,\mu} : \mu \) is a probability measure on \( B(\mathbb{R}^d) \} \), where \( P_{s,\mu}(\cdot) = \int_{\mathbb{R}^d} P_{s,x}(\cdot) \mu(dx) \), and then we define \( \mathcal{F}_{s,t} \) as the completion of \( \mathcal{F}^0_{s,t} \) in \( \mathcal{F}_{s,\infty} \) with respect to \( \mathcal{P} \).

Let \( p \) denote the fundamental solution for the operator \( A_t \) defined by (1.2). It is known (see [31]) that there exists a unique time-inhomogeneous Markov process \( X = \{ (X_t, P_{s,x}) : (s, x) \in \mathbb{R}_+ \times \mathbb{R}^d \} \) associated with \( A_t \). Namely, \( X \) is a unique Markov process for which \( p \) is the transition density function, i.e.,
\[
P_{s,x}(X_t = x; 0 \leq t \leq s) = 1, \quad P_{s,x}(X_t \in \Gamma) = \int_{\Gamma} p(s, x, t, y) \, dy, \quad t > s
\]
for any \( \Gamma \in B(\mathbb{R}^d) \). It is known (see [32]) that \( X \) admits the so-called strict Fukushima decomposition, i.e., for every \( (s, x) \in \mathbb{R}_+ \times \mathbb{R}^d \),
\[
X_t - X_s = A_{s,t} + M_{s,t}, \quad s \leq t, \quad P_{s,x} \text{-a.s.,}
\]
where \( M \) is a two-parameter martingale additive functional (MAF) of \( X \) of finite energy and \( A \) is a two-parameter continuous additive functional (CAF) of \( X \) of zero energy. Moreover,
\[
\langle M_{s,t}^i, M_{s,t}^j \rangle_t = \int_s^t a_{ij}(r, X_r) \, dr, \quad s \leq t, \quad \text{(4.1)}
\]
which implies in particular that the process
\[
B_{s,t} = \int_s^t \sigma^{-1}(r, X_r) \, dM_{s,t}, \quad t \geq s, \quad \text{(4.2)}
\]
where $\sigma \cdot \sigma^T = a$ is a Brownian motion under $P_{s,x}$. It is also known (see [20]) that $B_s$ is an $\{F_{s,t}\}_{t \geq s}$-Brownian motion.

4.2. Time-homogeneous diffusions

In what follows we will also make substantial use of time-homogeneous Markov process $X'$ associated with the operator $\frac{\partial}{\partial t} + A_t$ (see [24]). A brief sketch of a useful construction of $X'$ on an extension of $\Omega$ is given below.

We set
\[ \Omega' = \mathbb{R}_+ \times \Omega, \quad P'_{s,x}(B) = P_{s,x}(\{\omega \in \Omega : (s, \omega) \in B\}) \] (4.3)
and consider the process $X$ on $\Omega'$ defined as
\[ X_t(s, \omega) = (s + t, X_{s+t}(\omega)), \quad t \geq 0. \] (4.4)

Let $\mathcal{F}'_t = \sigma(X_u, u \leq t)$, $\mathcal{F}'_\infty = \sigma(X_u, u < \infty)$ and let $\mathcal{F}'_\infty$ denote the completion of $\mathcal{F}'_\infty$ with respect to the family $\mathcal{P}' = \{P'_\mu : \mu \text{ is a probability measure on } \mathbb{R}_+ \times \mathbb{R}^d\}$ and $\mathcal{F}'_t$ denote the completion of $\mathcal{F}'_0$ in $\mathcal{F}'_\infty$ with respect to $\mathcal{P}'$. Then, $X' = \{(X_t, P'_{s,x}) : (s, x) \in \mathbb{R}_+ \times \mathbb{R}^d\}$ is a time-homogeneous Markov process with respect to the filtration $\{\mathcal{F}'_t\}$ with the transition density
\[ P'(t, (s, x), \Gamma) = P(s, x, s + t, \Gamma_{s+t}), \] (4.5)
where $\Gamma_{s+t} = \{x \in \mathbb{R}^d : (s + t, x) \in \Gamma\}$.

It is known (see [25]) that $X'$ admits the strict Fukushima decomposition, i.e., for every $(s, x) \in \mathbb{R}_+ \times \mathbb{R}^d$,
\[ X_t = X_0 + A_t + M_t, \quad t \geq 0, \quad P'_{s,x}\text{-a.s.}, \]
where $A$ is a CAF of $X'$ of zero energy and $M$ is a MAF of $X'$ of finite energy. It is also known (see [19]) that
\[ p(A_t)(\omega') = A_{s,s+t}(\omega), \quad p(M_t)(\omega') = M_{s,s+t}(\omega), \quad t \geq 0 \] (4.6)
for $\omega' = (s, \omega)$, where $p : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$ is the orthogonal projection. Set
\[ A_t = p(A_t), \quad M_t = p(M_t), \quad t \geq 0 \]
and for $t \geq 0$ define $\tau(t) : \Omega' \to \mathbb{R}_+$ by putting
\[ \tau(t)(\omega') = s + t = \tau(0)(\omega) + t \]
for $\omega' = (s, \omega)$. From now on, we adopt the convention that if $\xi$ is a random variable on $\Omega$, then $\xi(\omega') = \xi(\omega)$ for $\omega' = (s, \omega) \in \Omega'$. With this convention
\[ X_t = (\tau(t), X_{\tau(t)}), \quad t \geq 0. \]
We put
\[ \zeta = \inf\{t \geq 0; X_t \notin \mathbb{R}_+ \times D\}, \quad \zeta_t = \zeta \wedge T_t, \quad T_t = T - \tau(0). \]

Set
\[ B_t = \int_0^t \sigma^{-1}(X_r) \, dM_r, \quad t \geq 0. \] (4.7)

By (4.1) and (4.6),
\[ \langle M^i, M^j \rangle_t = \int_0^t a_{ij}(X_r) \, dr, \quad t \geq 0, \quad \text{P}_{s,s}'-\text{a.s.} \]

for every \((s, x) \in \mathbb{R}_+ \times \mathbb{R}^d\). Therefore, \(B\) is a Brownian motion under \(P_{s,s}'\) for every \((s, x) \in \mathbb{R}_+ \times \mathbb{R}^d\). In fact, by [20], it is an \(\{\mathcal{F}'_t\}_{t \geq 0}\)-Brownian motion.

4.3. Capacity, soft measures, and quasi-continuity

Let \(\mathcal{W}\) be the space of all \(u \in L^2(\mathbb{R}_+; H^1_0(D))\) such that \(\frac{\partial u}{\partial t} \in L^2(\mathbb{R}_+; H^{-1}(D))\) endowed with the usual norm \(\|u\|_{\mathcal{W}} = \|u\|_{L^2(\mathbb{R}_+; H^1_0(D))} + \|\frac{\partial u}{\partial t}\|_{L^2(\mathbb{R}_+; H^{-1}(D))}\).

We define the parabolic capacity of an open set \(U \subset \mathbb{R}_+ \times D\) as
\[ \text{cap}(U) = \inf\{\|u\|_{\mathcal{W}} : u \in \mathcal{W}, u \geq 1_U \text{ a.e. in } \mathbb{R}_+ \times D\}. \]

The parabolic capacity of a Borel set \(B \subset \mathbb{R}_+ \times D\) is defined as
\[ \text{cap}(B) = \inf\{\text{cap}(U) : U \text{ is an open subset of } \mathbb{R}_+ \times D, B \subset U\}. \]

We also consider the Newtonian capacity on \(D\). For an open set \(U \subset D\), we put
\[ \text{cap}_N(U) = \inf\{\|u\|_{H^1_0(D)} : u \in H^1_0(D), u \geq 1_U \text{ a.e. in } D\}, \]
and for a Borel set \(B \subset D\), we put
\[ \text{cap}_N(B) = \inf\{\text{cap}_N(U) : U \text{ is an open subset of } D, B \subset U\}. \]

From now on, we say that some property is satisfied for quasi-every (q.e, for short) \(x \in D\) (respectively, \((s, x) \in \mathbb{R}_+ \times D\) if it is satisfied except for some Borel subset of \(D\) (respectively, \(\mathbb{R}_+ \times D\)) of \(\text{cap}_N\) (respectively, \(\text{cap}\) capacity zero.

Let \(\mu\) be a Radon measure on \(D\) (respectively, \(\mathbb{R}_+ \times D\)). Following [6], we say that \(\mu\) is soft if \(\mu\) charges no set of \(\text{cap}_N\) (respectively, \(\text{cap}\) capacity zero. In what follows by \(\mathcal{M}_0(D)\) (respectively, \(\mathcal{M}_0\)), we denote the set of all soft measures on \(D\) (respectively, \(\mathbb{R}_+ \times D\)) and by \(\mathcal{M}_{0,b}(D)\) (respectively, \(\mathcal{M}_{0,b}\)) the set of all bounded soft measures on \(D\) (respectively, \(\mathbb{R}_+ \times D\)).

Let \(B\) be a Borel subset of \(\mathbb{R}_+ \times D\) (respectively, \(D\)) and \(u : B \rightarrow \mathbb{R}\) be a Borel measurable function. We say that \(u\) is quasi-continuous if for every \(\varepsilon > 0\), there exists a closed set \(F_\varepsilon \subset B\) such that \(\text{cap}(B \setminus F_\varepsilon) < \varepsilon\) (respectively, \(\text{cap}_N(B \setminus F_\varepsilon) < \varepsilon\)) such that \(u|_{F_\varepsilon}\) is continuous. In the paper, we shall mostly work with functions on \(B = D\)
or $B = D_T$. We adopt the convention that for $u$ defined on $B = D$ (respectively, $B = D_T$) $u(x) = 0$ (respectively, $u(t, x) = 0$) for $x \in \mathbb{R}^d \setminus D$ (respectively, $(t, x) \in (\mathbb{R}_+ \times \mathbb{R}^d) \setminus D_T$). It is well known that $u$ on $D_T$ (respectively, $D$) is quasi-continuous iff the process $[0, \tau u \equiv t \to u(X_t)] (respectively, [0, \tau \equiv t \to u(X_t))$ is continuous under $P_{s, x}$ (respectively, $P_{0, x}$) for q.e. $(s, x) \in D_T$ (respectively, q.e. $x \in D$). We will also consider quasi-càdlàg functions on $D_T$, i.e., Borel functions $u$ on $D_T$, such that the process $[0, \tau \equiv t \to u(X_t)$ is càdlàg for q.e. $(s, x) \in D_T$.

4.4. Additive functionals and soft measures

Let $E_{s, x}$ (respectively, $E'_{s, x}$) denote the expectation with respect to $P_{s, x}$ (respectively, $P'_{s, x}$) and let $m_1$ denote the Lebesgue measure on $\mathbb{R}_+ \times \mathbb{R}^d$. Let $\zeta^0$ be defined by (1.7) with $s = 0$, i.e.,

$$\zeta^0 = \inf\{t \geq 0, X_t \notin D\}.$$  

Let us recall that a positive AF $A$ of $\mathbb{X}'$ and a positive soft measure $\mu$ on $\mathbb{R}_+ \times \mathbb{R}^d$ are in the Revuz correspondence if

$$\langle \mu, f \rangle \equiv \int_0^\infty \int_D f \, d\mu = \lim_{a \to \infty} \int_0^\infty \int_D \left( E'_{s, x} \int_0^{\zeta} e^{-a t} f(X_t) \, dA_t \right) \, dm_1(s, x)$$

for every $f \in B^+(\mathbb{R}_+ \times D)$. If $\langle \mu, 1 \rangle < \infty$, then $A$ is called integrable. It is known (see [22,30]) that under (4.8), the family of all integrable positive AFs of $\mathbb{X}'$ and the family of all bounded positive soft measures on $\mathbb{R}_+ \times D$ are in one-to-one correspondence. In what follows the additive functional corresponding to a positive bounded soft measure $\mu$ will be denoted by $A^\mu$.

Let $p'_D$ denote the transition density of the process $\mathbb{X}'$ killed on exiting $\mathbb{R}_+ \times D$. It is known that $A^\mu$ corresponds to $\mu$ iff for q.e. $(s, x) \in \mathbb{R}_+ \times D$,

$$E'_{s, x} \int_0^{\zeta} f(X_t) \, dA_t^\mu = \int_0^\infty \left( \int_0^\infty \int_D f(z) p'_D(t, (s, x), z) \, d\mu(z) \right) \, dt$$

for every $f \in B^+_0(\mathbb{R}_+ \times D)$ (see [22]).

Suppose now that the coefficients of the operator (1.2) do not depend on time, i.e., $a_{ij}(t, x) = a_{ij}(x)$ for $(t, x) \in D_T$. We then set

$$A = \sum_{i, j=1}^d \frac{\partial}{\partial x_j} \left( a_{ij} \frac{\partial}{\partial x_i} \right).$$

It is well known that the fundamental solution for $A$ has the property that $p(s, x, t, y) = p(t - s, x, y)$ for any $t > s, x \neq y$. Consequently, $\mathbb{X} = \{(X, P_x) : x \in \mathbb{R}^d\}$, where $P_x = P'_{0, x}$ is a time-homogeneous Markov process with the transition density $p(t, x, y) = p(0, x, t, y)$. It is also known (see, e.g., [9]) that in the time-homogeneous
case, a positive CAF $A$ of $X$ and a positive soft measure $\mu$ on $D$ are in the Revuz correspondence if

$$\langle \mu, f \rangle = \lim_{\alpha \to \infty} \alpha \int_D \left( E_x \int_0^{\xi^0} e^{-\alpha t} f(X_t) \, dA_t \right) \, dm(x)$$  \hspace{1cm} (4.11)$$

for every $f \in B^+(D)$, where $E_x$ denotes the expectation with respect to $P_x$. If $\langle \mu, 1 \rangle < \infty$, then $A$ is called integrable. In [9], it is proved that the family of all integrable positive CAFs of $X$ and the family of all bounded positive soft measures on $D$ are in one-to-one correspondence via formula (4.11).

Let $A_\mu$ denote the additive functional corresponding to a positive bounded soft measure $\mu$, $p_D$ denote the transition density of the process $X$ killed on exiting $D$, and let $G_D(x, y) = \int_0^\infty p_D(t, x, y) \, dt$ be the Green function for $D$. Then, for q.e. $x \in D$,

$$E_x \int_0^{\xi^0} f(X_t) \, dA_t = \int_0^\infty \int_D f(y) p_D(t, x, y) \, d\mu(y) \, dt = \int_D f(y) G_D(x, y) \, d\mu(y)$$  \hspace{1cm} (4.12)$$

for every $f \in B^+(D)$. In the whole paper for a fixed Borel positive measure $\mu$ on $D_T$ (respectively, $D$), we denote

$$R^{0,T}_\mu(s, x) = \int_0^\infty \left( \int_0^T \int_D p'_D(t, (s, x), z) \, d\mu(z) \right) \, dt, \quad R_\mu(x) = \int_D G_D(x, y) \, d\mu(y).$$

Finally, let us recall that there is $c$ depending only on $d$, $\Lambda$ such that $p(t, x, y) \leq Ct^{-d/2}$ for $t > 0$, $x, y \in \mathbb{R}^d$ (see, e.g., [1]). Therefore, by [5, Theorem 1.17], there is $c$ depending only on $d$, $\Lambda$ such that

$$\sup_{x \in \mathbb{R}^d} E_x \xi^0 \leq c |D|^{d/2},$$  \hspace{1cm} (4.13)$$

where $|D|$ denotes the Lebesgue measure of $D$, whereas from Corollary to Proposition 1.18 in [5], it follows that there exists constants $a > 0$, $b > 0$ depending only on $d$, $\Lambda$ and $|D|$ such that for every $t > 0$,

$$\sup_{x \in \mathbb{R}^d} P_x (\xi^0 > t) \leq ae^{-bt}.$$  \hspace{1cm} (4.14)$$

5. Markov-type BSDEs and PDEs

Let us fix $T > 0$ and set $D_T = [0, T] \times D$. In this section, we show existence and uniqueness results for systems of PDEs of the form

$$\begin{align*}
\frac{\partial u_k}{\partial t} + A_t u_k &= -f^k(t, x, u) - \mu^k, \quad k = 1, \ldots, N, \\
\left. u \right|_{\partial D}(t, \cdot) &= 0, \quad t \in [0, T), \quad u(T, \cdot) = \varphi \quad \text{on } D,
\end{align*}$$

where $A_t$ is given by (1.2).

Let $f : D_T \times \mathbb{R}^N \to \mathbb{R}^N$. We consider the following hypotheses.
(H1) $f(\cdot, \cdot, y)$ is measurable for every $y \in \mathbb{R}^N$ and $f(t, x, \cdot)$ is continuous for a.e. $(t, x) \in D_T$.

(H2) There is $\alpha \in \mathbb{R}$ such that $(f(t, x, y) - f(t, x, y'), y - y') \leq \alpha |y - y'|^2$ for every $y, y' \in \mathbb{R}^N$ and a.e. $(t, x) \in D_T$.

(H3) $f(\cdot, \cdot, 0) \in L^1(D_T)$, $\mu \in \mathcal{M}_{0, b}(D_T)$, $\varphi \in L^1(D)$.

(H4) $\forall r > 0, y \in \mathbb{R}^N$ $R^{0, T} (\sup_{|y| \leq r} |f(\cdot, \cdot, y)|) < \infty$, $m_1$-a.e.

REMARK 5.1. It is known (see [17, Proposition 3.6]) that if $f \in L^1(D_T)$, then $R^{0, T} |f| < \infty$, $m_1$-a.e. Therefore, (H4) is satisfied if $\sup_{|y| \leq r} |f(\cdot, \cdot, y)| \in L^1(D_T)$ for every $y \in \mathbb{R}^N$ and $r \geq 0$. However, the class of functions $f \in \mathcal{B}(D_T)$ such that $R^{0, T} |f| < \infty$, $m_1$-a.e. is wider than $L^1(D_T)$. It includes in particular the space $L^1(D_T; \delta \cdot m_1)$, where $\delta(x) = \text{dist}(x, \partial D)$ (see [17, Example 5.2]).

It is known that in case $N = 1$, one can find solutions of problems of the form (5.1) in the (nonlinear) space $\mathcal{T}^{0,1}_2$ of all Borel measurable functions $u$ on $D_T$ such that $T_k(u) \in L^2(0, T; H^1_0(D))$ for every $k \geq 1$ (see [2, 6]). In the case of systems, the problem is more difficult, because we do not know whether the solutions or its truncation have gradients in the usual sense (i.e., locally in some Sobolev space). This is related to the lack of integrability of $f_u$. The same problem appears in the case of elliptic systems. In the scalar case, it is known that a solution of (5.1) belongs to $W^{1,q}_0(D)$ for every $q \in [1, \frac{d}{d-1})$ (see [34]) but in case $N > 1$, the problem whether a solution belongs to $W^{1,q}_0(D)$ for some $q \geq 1$ is open. To overcome the difficulty in [15], the existence and uniqueness of elliptic systems with measure data similar to (5.1) is proved in some wider than $\mathcal{T}^{0,1}_2$ (see Corollary 5.6) linear space. The space introduced in [15] makes essential use of the Markov process $\mathbb{X}$ associated with the operator $A$ defined by (4.10) and therefore may be called a stochastic Sobolev space. In what follows we extend the ideas from [15] to parabolic systems. We begin with the definition of the stochastic Sobolev space of functions depending on time.

Let $W^{0,1}(\mathbb{X}^{D_T})$ denote the set of all $u \in FM$ (definition below) for which there exists a sequence $\{u_n\} \subset C^\infty_c(D_T)$ such that for q.e. $(s, x) \in D_T$,

$$
\int_0^{\xi_t} |(u_n - u)(X_t)|^2 \, dt \to 0 \quad \text{in probability } P_{s,x} \text{ as } n \to \infty
$$

and

$$
\int_0^{\xi_t} |\nabla(u_n - u_m)(X_t)|^2 \, dt \to 0 \quad \text{in probability } P_{s,x} \text{ as } n, m \to \infty.
$$

In [14], it is proved that for every $u \in W^{0,1}(\mathbb{X}^{D_T})$, there exists a unique (a.e.) function $v \in \mathcal{B}(D_T)$ such that for every $\{u_n\} \subset C^\infty_c(D_T)$ satisfying (5.2) and (5.3),

$$
\int_0^{\xi_t} |\nabla u_n(X_t) - v(X_t)|^2 \, dt \to 0 \quad \text{in probability } P_{s,x} \text{ as } n \to \infty
$$

for q.e. $(s, x) \in D_T$. 

Given \( u \in W^{0,1}(\mathbb{X}^{D_T}) \), we denote by \( \nabla \chi u \) the unique function \( v \) satisfying (5.4). Notice that directly from the construction of \( \nabla \chi u \), it follows that \( \nabla \chi u = \nabla u \) a.e. if \( u \in L^2(0, T; H_0^1(D)) \).

By \( FM \), we denote the space of Borel measurable functions \( u \) on \( DT \) such that for q.e. \((s, x) \in DT, P'_{s,x}(\int_0^{\xi} |u(X_r)|^2 \, dr < \infty) = 1 \). We say that \( u_n \to u \) in \( FM \) if (5.2) holds for q.e. \((s, x) \in DT \).

**DEFINITION.** Let \( u_n, u \in W^{0,1}(\mathbb{X}^{D_T}) \). We say that \( u_n \to u \) in \( W^{0,1}(\mathbb{X}^{D_T}) \) if

\[
\exists \text{ a subsequence } u_{nk_l} \to u \text{ in } FM \text{ and } \nabla \chi u_n \to \nabla \chi u \text{ in } FM.
\]

**REMARK 5.2.** (i) Let \( \mathcal{O}(\to) \) be the topology generated by the convergence in \( W^{0,1}(\mathbb{X}^{D_T}) \). This topology is metrizable by the F-norm

\[
|u|_1 = E_m\left[\left(\int_0^{\xi} u^2(X_r) \, dr\right)^{1/2} \right] + E_m\left[\left(\int_0^{\xi} |\nabla \chi u|^2(X_r) \, dr\right)^{1/2} \right].
\]

Indeed, by the Kantorovich–Kisyński theorem (see [11,12]),

\[
u_n \to u \text{ in } \mathcal{O}(\to) \text{ iff } \forall (n_k) \subset (n) \exists (n_{k_l}) \subset (n_k) \text{ } u_{nk_l} \to u \text{ in } W^{0,1}(\mathbb{X}^{D_T}). \tag{5.5}
\]

Assume that \( u_n \to u \) in \( \mathcal{O}(\to) \) and \( (u_n) \) does not converge to \( u \) in \( |\cdot|_1 \). Then, there exists \( \varepsilon > 0 \) and subsequence \( (n_k) \subset (n) \) such that

\[
|u - u_{nk}|_1 > \varepsilon, \quad k \geq 1. \tag{5.6}
\]

On the other hand, by (5.5), there exists a subsequence \( (n_{k_l}) \subset (n_k) \) such that \( u_{nk_l} \to u \) in \( W^{0,1}(\mathbb{X}^{D_T}) \). Hence, by the Lebesgue dominated convergence theorem, \( |u_{nk_l} - u|_1 \to 0 \) as \( l \to \infty \), which contradicts (5.6). Now, assume that

\[
|u_n - u|_1 \to 0 \text{ as } n \to \infty \text{ and let } (n_k) \subset (n). \text{ By [16, Proposition 3.3], there exists a subsequence } (n_{k_l}) \subset (n_k) \text{ such that } u_{nk_l} \to u \text{ in } W^{0,1}(\mathbb{X}^{D_T}). \text{ Since } (n_k) \subset (n) \text{ was arbitrary, (5.5) implies that } u_n \to u \text{ in } \mathcal{O}(\to).
\]

(ii) By [14, Proposition 4.6], the space \( (W^{0,1}(\mathbb{X}^{D_T}), \mathcal{O}(\to)) \) is complete.

**LEMMA 5.3.** If \( u \in W^{0,1}(\mathbb{X}^{D_T}), \theta \in C^1(\mathbb{R}) \text{ and there is } c > 0 \text{ such that } |\theta'(t)| \leq c \text{ for } t \in \mathbb{R}, \text{ then } \theta(u) \in W^{0,1}(\mathbb{X}^{D_T}) \text{ and } \nabla \chi \theta(u) = \theta'(u) \nabla \chi u.
\]

**Proof.** Since \( u \in W^{0,1}(\mathbb{X}^{D_T}), \text{ there exists a sequence } \{\eta_n\} \subset C_c(D_T) \text{ such that } \eta_n \to u \text{ and } \nabla \eta_n \to \nabla \chi u \text{ in } FM. \text{ Using the assumptions on } \theta, \text{ one can easily show that } \theta(\eta_n) \to \theta(u) \text{ and } \nabla \theta(\eta_n) = \theta'(\eta_n) \nabla \eta_n \to \theta'(u) \nabla \chi u \text{ in } FM, \text{ which proves the desired result.} \]

**LEMMA 5.4.** Let \( k \in \mathbb{R} \text{ and } u \in W^{0,1}(\mathbb{X}^{D_T}). \text{ Then, } u \land k, u \lor k \in W^{0,1}(\mathbb{X}^{D_T}) \text{ and}
\]

\[
\nabla \chi (u \land k) = 1_{(-\infty,k]}(u) \nabla \chi u = 1_{(-\infty,k]}(u) \nabla \chi u, \quad \text{a.e.,}
\]

\[
\nabla \chi (u \lor k) = 1_{(k,\infty)}(u) \nabla \chi u = 1_{[k,\infty)}(u) \nabla \chi u, \quad \text{a.e.}
\]


Proof. We will prove the lemma for \( u \wedge k \). The proof for \( u \vee k \) is analogous. Set

\[
\sigma_n(t) = \begin{cases} 
1, & \text{if } t \leq k, \\
n(k - t) + 1, & \text{if } t \in (k, k + \frac{1}{n}), \\
0, & \text{if } t \geq k + \frac{1}{n}.
\end{cases}
\]

and \( \theta_n(t) = \int_0^t \sigma_n(r) \, dr, \ t \in \mathbb{R} \). Standard arguments show that \( \theta_n(u) \to u \wedge k \) in \( FM \) and, by the use of Lemma 5.3, that \( \nabla_X \theta_n(u) = \sigma_n(u) \nabla_X u \to 1_{(-\infty, k)}(u) \nabla_X u \) in \( FM \). Hence, \( u \wedge k \in W^{0,1}(X^{DT}) \) and \( \nabla_X (u \wedge k) = 1_{(-\infty, k)}(u) \nabla_X u \). If we repeat the above arguments with \( \sigma_n \) replaced by \( \hat{\sigma}_n \) defined as

\[
\hat{\sigma}_n(t) = \begin{cases} 
1, & \text{if } t \leq k - \frac{1}{n}, \\
n(k - \frac{1}{n} - t) + 1, & \text{if } t \in (k - \frac{1}{n}, k), \\
0, & \text{if } t \geq k.
\end{cases}
\]

we will obtain \( \nabla_X (u \wedge k) = 1_{(-\infty, k)}(u) \nabla_X u \). \( \square \)

Let \( FB \) denote the space of Borel measurable functions on \( DT \) such that for q.e. \((s, x) \in DT \), \( P'_{s,x}(\text{ess sup}_{r \in [0, \xi_t]} |u(X_r)| < \infty) = 1 \). Observe that every quasi-càdlàg function belongs to \( FB \).

One of the reason why the space \( T_{2,1}' \) has been introduced is that the standard Sobolev space \( L^2(0, T; H^1_0(D)) \) lacks the property that \( u \in L^2(0, T; H^1_0(D)) \) if \( u \) is quasi-càdlàg (natural class for solutions of equations with measure data) and \( T_k(u) \in L^2(0, T; H^1_0(D)) \) for every \( k \geq 0 \). The following lemma shows the stochastic Sobolev space has this remarkable feature.

**Lemma 5.5.** If \( u \in FB \) and \( T_k(u) \in W^{0,1}(X^{DT}) \) for every \( k \geq 0 \), then \( u \in W^{0,1}(X^{DT}) \).

**Proof.** First, observe that for q.e. \((s, x) \in DT \) and \( \varepsilon > 0 \),

\[
P'_{s,x} \left( \int_0^{\xi_t} |T_k(u) - u|^2(X_r) \, dr > \varepsilon \right) \leq P'_{s,x} \left( 2 \int_0^{\xi_t} |u|^2 1_{|u| > k}(X_r) \, dr > \varepsilon \right) \to 0,
\]

which shows that \( T_k(u) \to u \) in \( FM \). By Lemma 5.4, for \( k < l \), we have

\[
P'_{s,x} \left( \int_0^{\xi_t} |\nabla_X T_k(u) - \nabla_X T_l(u)|^2(X_r) \, dr > \varepsilon \right) = P'_{s,x} \left( \int_0^{\xi_t} |\nabla_X T_k(T_l(u)) - \nabla_X T_l(u)|^2(X_r) \, dr > \varepsilon \right) = P'_{s,x} \left( \int_0^{\xi_t} |\nabla_X T_l(u) 1_{|T_l(u)| > k} - \nabla_X T_l(u)|^2(X_r) \, dr > \varepsilon \right) = P'_{s,x} \left( \int_0^{\xi_t} |\nabla_X T_l(u)|^2 1_{|u| \geq k}(X_r) \, dr > \varepsilon \right).
\]

Observe that

\[
P'_{s,x} \left( \int_0^{\xi_t} |\nabla_X T_l(u)|^2 1_{|u| \geq k}(X_r) \, dr > \varepsilon \right) \leq P'_{s,x}(\text{ess sup}_{r \in [0, \xi_t]} |u(X_r)| \geq k).
\]
By the assumption that \( u \in FB \), the right-hand side of the above inequality tends to zero as \( k \to \infty \), which shows that \( \nabla_X T_k(u) - \nabla_X T_l(u) \to 0 \) in \( FM \) as \( k, l \to \infty \). Consequently, \( u \in W^{0,1}(X^{DT}) \). \( \square \)

In [14], it is shown by examples that in general neither \( T_{2}^{0,1} \subset W^{0,1}(X^{DT}) \) nor \( T_{2}^{0,1} \supset W^{0,1}(X^{DT}) \). However, we have the following corollary to Lemma 5.5.

**COROLLARY 5.6.** If \( u \in T_{2}^{0,1} \) and \( u \in FB \), then \( u \in W^{0,1}(X^{DT}) \).

Besides being nonlinear, another drawback to the space \( T_{2}^{0,1} \) is that it is sometimes too small in practice. For instance, in [14], it is proved that solutions of some types of the obstacle problem are quasi-continuous and belong to \( W^{0,1}(X^{DT}) \) but do not belong to \( T_{2}^{0,1} \).

**EXAMPLE 5.7.** Let \( B(0,k^{-1}) = \{ x \in \mathbb{R}^2 : |x| < k^{-1} \} \), \( D = B(0,1) \), and let \( \alpha \in \mathbb{R} \).

(i) Put \( v(t,x) = 1 - |x|^{-\alpha} \) for \((t,x) \in DT\). Since \( \text{cap}_N(B(0,k^{-1})) \to 0 \) as \( k \to \infty \), \( v \in W^{0,1}(X^{DT}) \) and

\[
\nabla_X v(t,x) = -\alpha x |x|^{-(\alpha+2)}, \quad (t,x) \in DT
\]

for every \( \alpha \in \mathbb{R} \). If \( \alpha \geq 2 \), then \( v \) is not locally integrable, so there is no sense to speak about its distributional derivative. But let us observe that \( v \in T_{2}^{0,1} \) for every \( \alpha \in \mathbb{R} \).

(ii) Put \( u(t,x) = \sin v(t,x), \ (t,x) \in DT \). Then, by Lemma 5.3, \( u \in W^{0,1}(X^{DT}) \) and

\[
\nabla_X u(t,x) = \alpha x |x|^{-(\alpha+2)} \cos(1 - |x|^{-\alpha})
\]

for every \( \alpha \in \mathbb{R} \). One can check that for every \( \varepsilon > 0 \) and \( \alpha > 1 \),

\[
\int_{B(0,\varepsilon)} \alpha |x|^{-(\alpha+1)} \cos(1 - |x|^{-\alpha}) \, dx = \infty,
\]

which shows that if \( \alpha > 1 \), then \( u \notin T_{2}^{0,1} \) since \( T_k(u) = u \) for \( k \geq 1 \). However, \( u \in L^{1}(DT) \) and one can check that its distributional derivative is given by the formula

\[
\nabla u = \text{p.v.} \left( \alpha x |x|^{-(\alpha+2)} \cos(1 - |x|^{-\alpha}) \right),
\]

i.e., for every \( \eta \in C_0^{\infty}(D) \),

\[
(\nabla u)(\eta) = \lim_{k \to \infty} \int_{D \setminus B(0,k^{-1})} \alpha x |x|^{-(\alpha+2)} \cos(1 - |x|^{-\alpha}) \eta(x) \, dm(x).
\]

Accordingly, even if the distributional derivative of \( u \in W^{0,1}(X^{DT}) \) exists, it is not a function in general.
Following [18], we adopt the following definition.

DEFINITION. We say that a measurable function \( f : D_T \rightarrow \mathbb{R} \) is quasi-integrable if the function \([0, \zeta_T] \ni t \mapsto f(X_t)\) belongs to \(L^1([0, \zeta_T])\) \(P_{s,x}^{\prime}\)-a.s. for q.e. \((s, x) \in D_T\). The set of all quasi-integrable functions on \(D_T\) will be denoted by \(qL^1(D_T)\).

We say that a measurable function \(u\) on \(D_T\) is of class (FD) if for q.e. \((s, x) \in D_T\), the process \(u(X)\) on \([0, \zeta_T]\) is of class (D) under the measure \(P_{s,x}^{\prime}\).

REMARK 5.8. It is known (see [19]) that if \(f\) is quasi-integrable in the analytic sense, i.e., if for every \(\varepsilon > 0\), there exists an open set \(G_{\varepsilon} \subset D_T\) such that \(\text{cap}(G_{\varepsilon}) < \varepsilon\) and \(f\mid_{D_T \setminus G_{\varepsilon}} \in L^1(D_T \setminus G_{\varepsilon})\) then \(f \in qL^1\). In particular, it follows that if \(f \in L^1(D_T)\), then \(f \in qL^1(D_T)\). It is also clear that if \(R_0^T f < \infty, m_1\)-a.e., then \(f \in qL^1(D_T)\).

DEFINITION. We say that a measurable function \(u : D_T \rightarrow \mathbb{R}^N\) is a solution of system (5.1) if

(a) \((t, x) \mapsto f(t, x, u(t, x)) \in qL^1(D_T), u \in W^{0,1}(\mathbb{X}^{D_T})\) and \(u\) is of class (FD),

(b) For q.e. \((s, x) \in D_T\),

\[
    u(X_t) = 1_{[\zeta > T]} \varphi(X_{T_{\zeta}}) + \int_t^{\zeta_T} f(X_r, u(X_r)) \, dr
    + \int_t^{\zeta_T} dA^\mu_r - \int_t^{\zeta_T} \sigma \nabla \varphi u(X_r) \, dB_r, \quad t \in [0, \zeta_T], \quad P_{s,x}^{\prime}\text{-a.s.} \quad (5.7)
\]

In what follows for a given function \(u\) on \(D_T\), we set

\[
    \tilde{u}(t, x) = u(T - t, x), \quad (t, x) \in D_T,
\]

and given \(\mu \in \mathcal{M}_{0,b}(D_T)\), we denote by \(\tilde{\mu}\) the unique measure in \(\mathcal{M}_{0,b}(D_T)\) such that

\[
    \int_{D_T} \tilde{\eta} \, d\mu = \int_{D_T} \eta \, d\tilde{\mu}, \quad \eta \in \mathcal{B}_b(D_T).
\]

DEFINITION. We say that \(u\) is a solution of (1.1) on \([0, T]\) if \(\tilde{u}\) is a solution of (5.1) with \(f_u\) replaced by \(\tilde{f}_u\), \(\mu\) replaced by \(\tilde{\mu}\) and \(a\) replaced by \(\tilde{a}\).

DEFINITION. Let \((s, x) \in D_T\). We say that a pair \((Y^{s,x}, Z^{s,x})\) consisting of an \(\mathbb{R}^N\)-valued process \(Y^{s,x}\) and an \(\mathbb{R}^d \times \mathbb{R}^N\)-valued process \(Z^{s,x}\) is a solution of BSDE\(_{s,x}\)(\(\varphi, D, f + d\mu\)) if \(Y^{s,x}, Z^{s,x}\) are \(\{\mathcal{F}_t\}\) progressively measurable, \(Y^{s,x}\) is càdlâg, \(t \mapsto f(X_t, Y_t^{s,x}, Z_t^{s,x}) \in L^1(0, \zeta_T), P_{s,x}^{\prime}\text{-a.s.}, P_{s,x}^{\prime}(\int_0^{\zeta_T} |Z_r^{s,x}|^2 \, dr < \infty) = 1\)

and

\[
    Y_t^{s,x} = 1_{[\zeta > T]} \varphi(X_{T_{\zeta}}) + \int_t^{\zeta_T} f(X_r, Y_r^{s,x}, Z_r^{s,x}) \, dr
    + \int_t^{\zeta_T} dA^\mu_r - \int_t^{\zeta_T} Z_r^{s,x} \, dB_r, \quad t \in [0, \zeta_T], \quad P_{s,x}^{\prime}\text{-a.s.}
\]
Observe that from the above two definitions, it follows that if \( u \) is a solution of (5.1), then for q.e. \((s, x) \in DT\) the pair \((Y^{s,x}, Z^{s,x}) = (Y, Z)\), where

\[
(Y_t, Z_t) = (u(X_t), \sigma \nabla_X u(X_t)), \quad t \in [0, \xi_T]
\]

is a solution of BSDE_{s,x}(\varphi, D, f + d\mu). In the rest of this section, we are going to prove that under (H1)–(H4), this statement can be reversed in the following sense. For q.e. \((s, x) \in DT\), there exists a unique solution \((Y^{s,x}, Z^{s,x})\) of BSDE_{s,x}(\varphi, D, f + d\mu) and if we set \(u(s, x) = E_{s,x}^t Y^{s,x}_0\) for \((s, x) \in DT\), then \(u\) is a solution of (5.1). Moreover,

\[
Y^{s,x}_t = u(X_t), \quad t \in [0, \xi_T], \quad P'_{s,x}-a.s., \quad Z^{s,x} = \sigma \nabla_X u(X), \quad dt \otimes P'_{s,x}-a.s. \quad (5.8)
\]

Let us first state the following corollary to Theorem 3.7.

**PROPOSITION 5.9.** Assume (H1)–(H4). Let \( F \) be the set of those \((s, x) \in DT\) for which the data \( \xi_T, f(X, \cdot, \cdot), \varphi(X_T^s)1_{\xi_T > T_s}, A^\mu \) satisfy assumptions (A1)–(A4) under the measure \( P'_{s,x}\). Then, \( \text{cap}(DT \setminus F) = 0 \), and for every \((s, x) \in F\), there exists a unique solution \((Y^{s,x}, Z^{s,x})\) of BSDE_{s,x}(\varphi, D, f + d\mu) such that \((Y^{s,x}, Z^{s,x}) \in D^q \otimes M^q\) for \( q \in (0, 1)\) and \(Y^{s,x}\) is of class \(D\).

**Proof.** That \( \text{cap}(DT \setminus F) = 0 \) follows from [16, Corollary 3.4] (see also [13, Remark 3.2]). The second assertion follows from Theorem 3.7. \(\square\)

The crucial issue in the proof of representation (5.8) and existence of solution to system (5.1) is regularity of the function \(D_T \ni (s, x) \mapsto u(s, x) = E_{s,x}^t Y^{s,x}_0\). In most papers concerning probabilistic solutions for PDEs or stochastic representation for solutions of PDEs, the regularity is proved by using the results of probabilistic potential theory which may be applied when \(u(s, x) = E_{s,x}^t \int_0^{\xi_T} dA_r\) for some AF of \(X\). Here, such approach cannot be applied because in general, \(u\) does not admit the last representation [in general, integrals on the right-hand side of (1.8) do not exist]. We cope with the problem of regularity of \(u\) in Propositions 5.10, 5.11.

**PROPOSITION 5.10.** Let \( F \) be a Borel subset of \(DT\) such that \(\text{cap}(DT \setminus F) = 0\). Assume that for every \((s, x) \in F\), the real process \(Y^{s,x}\) is a continuous semimartingale under \(P'_{s,x}\) such that \(Y^{s,x}_{t \wedge \xi_T} = 0, t \geq 0\), and there exists a Borel function \(v\) on \(DT\) such that for every \((s, x) \in F\) and every \(t \in [0, T\xi]\),

\[
v(X_t) = Y^{s,x}_t, \quad P'_{s,x}-a.s. \quad (5.9)
\]

Then, \(u(s, x) = E_{s,x}^t Y^{s,x}_0\) is a quasi-continuous version of \(v\) and for every \((s, x) \in F\),

\[
u(X_t) = Y^{s,x}_t, \quad t \in [0, \xi_T], \quad P'_{s,x}-a.s.
\]

**Proof.** Let \((s, x) \in F\). Since \(Y^{s,x}\) is a continuous semimartingale, there exists a finite variation continuous process \(R^{s,x}\) and \(Z^{s,x} \in M\) such that

\[
Y^{s,x}_t = 1_{\xi_T > T_s}v(X_{T_s}) + \int_t^{\xi_T} dR^{s,x}_r - \int_t^{\xi_T} Z^{s,x}_r dB_r, \quad t \in [0, \xi_T], \quad P'_{s,x}-a.s.
\]
First, let us assume additionally that $Y^{s,x}$ is bounded. Write $L^n = Y^{s,x} - \frac{1}{n}$ and $U = Y^{s,x}$. Then, $L^n < U$ and by [10], for every $(s, x) \in F$, there exists a unique solution $(Y^n, Z^n, R^n)$, on the space $(\Omega', \mathcal{F}'_\infty, P'_{s,x})$, of the reflected BSDE with final condition $\xi \equiv v(X^n_T)$, lower barrier $L^n$ and upper barrier $U$ (RBSDLDE$_{s,x}(\xi, 0, L^n, U)$ for short) such that $Y^n \in S_2^2, Z^n \in M$ and $R^n \in \mathcal{V}$. By [10, Theorem 1.3], $Y^n_t \leq Y^{n+1}_t$, $t \in [0, \zeta_t]$, $P'_{s,x}$-a.s. for every $n \geq 1$. Therefore,

$$Y^n_t \not\geq Y_t, \quad t \in [0, \zeta_t], \quad P'_{s,x}$-a.s. \tag{5.10}$$

Let us fix $n \in \mathbb{N}$ and set $H_t = Y^n_t$, $t \in [0, \zeta_t]$. By [10],

$$H^n_t \not\geq H_t, \quad t \in [0, \zeta_t], \quad P'_{s,x}$-a.s., \tag{5.11}$$

where $H^k$ is the first component of the solution of RBSDLDE$_{s,x}(0, f_k, U)$ with $f_k(t, y) = k(y - L^n)^-$. Let $H^{k,l}$ denote the first component of the solution of BSDE$_{s,x}(v(T, \cdot), f_k,l)$ with $f_k,l(t, y) = k(y - L^n)^- - l(y - U_l)^+$. Then, by [7],

$$H^{k,l}_t \not\geq H^k_t, \quad t \in [0, \zeta_t], \quad P'_{s,x}$-a.s. \tag{5.12}$$

By (5.9), $f_k,l(\cdot, y) = k(y - v(X) + \frac{1}{n}) - l(y - v(X))^+$, $dt \otimes P'_{s,x}$-a.e. on $[0, \zeta_t] \times \Omega$ for every $y \in \mathbb{R}$. Set $g_k,l(t, x, y) = k(y - v(x) + \frac{1}{n}) - l(y - v(x))^+$. By [13],

$$H^{k,l}_t = h^{k,l}(X_t), \quad t \in [0, \zeta_t], \quad P'_{s,x}$-a.s.,$$

where $h^{k,l}$ is a quasi-continuous version of the solution of PDE$(0, g_{k,l})$. Let us put

$$h(s, x) = \limsup_{k \to \infty} \liminf_{l \to \infty} h^{k,l}(s, x), \quad (s, x) \in D_T.$$

Then, by (5.11) and (5.12),

$$H_t = h(X_t), \quad t \in [0, \zeta_t], \quad P'_{s,x}$-a.s.$$

for every $(s, x) \in F$. In particular, since $H \in S_2^2$, the function $h$ is quasi-continuous.

From what has already been proved it follows that for every $n \geq 1$, there exists a quasi-continuous function $u_n$ such that

$$Y^n_t = u_n(X_t), \quad t \in [0, \zeta_t], \quad P'_{s,x}$-a.s.$$

for every $(s, x) \in F$. Putting $u(s, x) = \limsup_{n \to \infty} u_n(s, x), \quad (s, x) \in D_T$, we get by (5.10) the desired result.

Now, we show how to dispense with the assumption that $Y^{s,x}$ is bounded. By the Itô–Tanaka formula, $T_k(Y)$ is a semimartingale, so by the first part of the proof, the function $u_k$ defined as $u_k(s, x) = E'_{s,x} T_k(Y^{s,x}_0), \quad (s, x) \in D_T$, is quasi-continuous and for each $k \in \mathbb{N},$

$$T_k(Y^{s,x}) = u_k(X_t), \quad t \in [0, \zeta_t], \quad P'_{s,x}$-a.s. \tag{5.13}$$

for every $(s, x) \in F$. But $u_k(s, x) = E_{s,x} T_k(Y^{s,x}_0) = T_k(E_{s,x} Y^{s,x}_0)$ since $Y^{s,x}$ is constant $P'_{s,x}$-a.s. Therefore, letting $k \to \infty$ in (5.13), we get the desired result for $Y$. \hfill \Box
PROPOSITION 5.11. Let $F$ be a Borel subset of $D_T$ such that $\text{cap}(D_T \setminus F) = 0$ and let $u : D_T \rightarrow \mathbb{R}$ be a Borel function. If $u(X)$ is a continuous semimartingale under $P_{s,x}$ on $[0, \zeta]$ for $(s, x) \in F$, then $u \in W^{0,1}(\mathbb{X}^{D_T})$ and there exists CAF $A$ of finite variation such that

$$u(X_t) = u(s, x) + \int_0^t dA_r + \int_0^t \sigma \nabla_X u(X_r) dB_r, \quad t \in [0, \zeta], \quad P_{s,x}-\text{a.s.}$$

for every $(s, x) \in F$.

Proof. We adopt the notation of the proof of Proposition 5.10 and put $M^n_t = \int_0^t Z^n_r dB_r$. Let us first assume that $u$ is bounded. By [14], we know that

$$Y^n_t = u_n(X_t), \quad t \in [0, \zeta], \quad \sigma \nabla_X u_n(X) = Z^n, \quad dt \otimes P_{s,x}-\text{a.e.},$$

(5.14)

where $u_n \in W^{0,1}(\mathbb{X}^{D_T})$ is a solution of OP $(v(T, \cdot), 0, L^n, U)$ (with two barriers). Fix $(s, x) \in F$. Since $Y = u(X)$ is a continuous semimartingale and the underlying filtration is Brownian, there exists a finite variation continuous process $R^{s,x}$ and a process $Z^{s,x} \in M$ such that

$$Y_t = Y_0 + \int_0^t dR^{s,x}_r + \int_0^t Z^{s,x}_r dB_r, \quad t \in [0, \zeta].$$

(5.15)

Put $M^{s,x}_t = \int_0^t Z^{s,x}_r dB_r$. Since the process $Y$ is continuous, from the proof of Proposition 5.10 and Dini’s theorem, it follows that

$$P_{s,x}(\sup_{t \in [0, \zeta]} |Y^n_t - Y_t|^2 > \varepsilon) \rightarrow 0.$$  

(5.16)

Moreover, by [14, Proposition 6.1],

$$\text{d}R^{n,+} \leq 1_{\{Y^n_t = L^n_t\}} \text{d}R^+_t, \quad \text{d}R^{n,-} \leq 1_{\{Y^n_t = U_t\}} \text{d}R^-_t.$$

Therefore, there exists a sequence of stopping times $\{\tau_k\}$ such that $\tau_k \leq \tau_{k+1}, k \geq 1, \tau_k \rightarrow \zeta$ $P_{s,x}-\text{a.s.}$ for q.e. $(s, x) \in D_T$ and for every $k \geq 1$, the sequence $\{Y^{n,\tau_k}\}$ satisfies the condition UT (see, e.g., [33]). Therefore,

$$P_{s,x}((M^n - M^{s,x})_{\zeta} > \varepsilon) \rightarrow 0$$

(5.17)

(see, e.g., [33, Proposition 1.5]). From (5.14)–(5.17), we deduce that $u \in W^{0,1}(\mathbb{X}^{D_T})$ and $\sigma \nabla_X u(X) = Z^{s,x}, \quad dt \otimes P_{s,x}-\text{a.e.}$ Putting

$$A_t = u(X_t) - u(s, x) - \int_0^t \sigma \nabla_X u(X_r) dB_r$$

we see that $A$ is a CAF of finite variation and $P_{s,x}(R^{s,x}_t = A_t, \ t \in [0, \zeta]) = 1$, which proves the proposition in the case $u$ is bounded. In the general case, (5.15) still holds.

By the Itô–Tanaka formula, for every $k > 0$, we have
\[ T_k(u)(X_t) = T_k(u(s, x)) + \int_0^t 1_{(-k,k]}(u(X_r)) \, dR_r^{s,x} + \int_0^t 1_{(-k,k]}(u(X_r)) Z_r^{s,x} \, dB_r + \frac{1}{2}(L_t^k - L_t^{-k}), \quad t \in [0, \zeta_T], \]

where \( L^k \) (respectively, \( L^{-k} \)) is the local time of the process \( u(X) \) at \( k \) (respectively, \(-k\)). By the first step of the proof, \( T_k(u) \in W^{0,1}(\mathbb{X}^{DT}) \) for every \( k \geq 0 \). Since \( u \) is quasi-continuous, \( u \in FB \) and hence, by Lemma 5.5, \( u \in W^{0,1}(\mathbb{X}^{DT}) \). By the first step of the proof,

\[ Z^{s,x} 1_{(-k,k]}(u(X)) = \sigma \nabla_X(T_k(u))(X), \quad dt \otimes P'_{s,x}-\text{a.e.} \]

Hence, by Lemma 5.4,

\[ Z^{s,x} 1_{(-k,k]}(u(X)) = 1_{(-k,k]}(u(X)) \sigma \nabla_X u(X), \quad dt \otimes P'_{s,x}-\text{a.e.} \]

for every \( k \geq 0 \), which implies that

\[ Z^{s,x} = \sigma \nabla_X u(X), \quad dt \otimes P'_{s,x}-\text{a.e.} \]

The rest of the proof runs as in the first step. \( \square \)

**Theorem 5.12.** Assume (H1)–(H4). Then, there exists a unique solution \( u \) of system (5.1). Moreover, there exists a version of \( u \) (still denoted by \( u \)) such that (5.7) is satisfied for every \((s,x) \in F\), where \( F \) is defined in Corollary 5.9.

**Proof.** By Proposition 5.9, for every \((s,x) \in F\), there exists a solution \((Y^{s,x}, Z^{s,x})\) of BSDE\(_{s,x}\) \((\varphi, D, f + d\mu)\) such that \((Y^{s,x}, Z^{s,x}) \in \mathcal{D}^q \otimes M^q \) for \( q \in (0,1) \) and \( Y^{s,x} \) is of class (D). By [19, Proposition 3.5], \( u(X_t) = Y^{s,x}_t, \quad P'_{s,x}-\text{a.s. for every } (s,x) \in F \) and every \( t \in [0, T]\), where \( u(s, x) = E'_{s,x} Y^{s,x}_0 \). Our aim is to show that \( u \) is quasi-càdlàg, belongs to \( W^{0,1}(\mathbb{X}^{DT}) \) and representation (5.8) holds q.e. Note that we cannot apply directly Propositions 5.10, 5.11 because we do not know that \( u(X) \) is continuous. To overcome the difficulty, let us consider a solution \((Y^{1,s,x}, Z^{1,s,x}) \in \mathcal{D}^q \otimes M^q, \quad q \in (0,1)\), of the BSDE\(_{s,x}\) \((\varphi, D, \mu)\) such that \( Y^{1,s,x} \) is of class (D). By [19, Proposition 3.7], \( u_1(s, x) = E'_{s,x} Y^{1,s,x}_0 \) is quasi-càdlàg, \( u_1 \in \mathcal{T}^{0,1} \) and for every \((s,x) \in F\),

\[ u_1(X_t) = Y^{1,s,x}_t, \quad t \in [0, \zeta_T], \quad P'_{s,x}-\text{a.s.}, \quad \sigma \nabla u_1(X) = Z^{1,s,x}, \quad dt \otimes P'_{s,x}-\text{a.e.} \]

(5.18)

Observe that

\[ Y^{s,x}_t - Y^{1,s,x}_t = \int_t^{\zeta_T} f(X_r, Y^{s,x}_r) \, dr + \int_t^{\zeta_T} (Z^{s,x}_r - Z^{1,s,x}_r) \, dB_r, \quad t \in [0, \zeta_T], \quad P'_{s,x}-\text{a.s.} \]

Applying Proposition 5.10 to each coordinate of the process \( Y^{s,x} - Y^{1,s,x} \), we conclude that there is a quasi-continuous function \( v : D_T \to \mathbb{R}^N \) such that for every \((s,x) \in F\),

\[ Y^{s,x}_t - Y^{1,s,x}_t = v(X_t), \quad t \in [0, \zeta_T], \quad P'_{s,x}-\text{a.s.} \]
From this and (5.18),
\[ Y_{t}^{s,x} = u_{1}(X_{t}) + v(X_{t}), \quad t \in [0, \xi_{\tau}], \quad P_{s,X}^{t}-\text{a.s.} \]
But
\[ v(s, x) = E_{s,X}^{t}Y_{0}^{s,x} - E_{s,X}^{t}Y_{0}^{1,s,x} = u_{1}(s, x) - u(s, x) \]
for q.e. \((s, x) \in D_{T}\). Therefore,
\[ Y_{t}^{s,x} = u(X_{t}), \quad t \in [0, \xi_{\tau}], \quad P_{s,X}^{t}-\text{a.s.} \]
It follows in particular that \(u\) is quasi-càdlàg. Since \(u_{1}\) is quasi-càdlàg, it belongs to \(FB\). Consequently, by Corollary 5.6, \(u_{1} \in W^{0,1}(X^{D_{T}})\). Therefore, from (5.18) and Proposition 5.11 applied to each coordinate of the function \(v\), it follows that
\[ u \in W^{0,1}(X^{D_{T}}) \]
and
\[ \sigma \nabla_{X}u(X) = Z^{s,x}, \quad dt \otimes P_{s,X}^{t}-\text{a.e.} \]
for every \((s, x) \in F\). Thus, \(u\) is a solution of (5.1). Uniqueness follows from Theorem 3.6. \(\Box\)

**Proposition 5.13.** Let \(u\) be a solution of system (5.1). Then, \(\nabla_{X}u \in L^{q}_{loc}(D_{T})\) for every \(q \in (0, 1)\).

**Proof.** Since \(u\) is of class (FD), \(u(X)\) is of class (D) on \([0, \xi_{\tau}]\) under \(P_{s,X}^{t}\) for q.e. \((s, x) \in D_{T}\). Therefore, \((u(X), \sigma \nabla_{X}u(X))\) is a unique solution of BSDE_{s,X} (\(\varphi, D, f + d\mu\)). By [3, Proposition 6.4], \(u \in D^{q}\) and \(\nabla_{X}u \in M^{q}\) for \(q \in (0, 1)\). Applying the Itô–Meyer formula to (5.7) and using (H2) and the fact that \(u\) is of class (FD), we get
\[ |u(X_{t})| \leq E_{s,x}^{t}\left(|\varphi(X_{T})|1_{\{\xi_{\tau} > T_{\tau}\}} + \int_{0}^{\xi_{\tau}} |f(X_{r}, 0)| dr + \int_{0}^{\xi_{\tau}} d|A^{\mu}|_{r}|F_{r}^{t}\right). \]
By [3, Lemma 6.1], for every \(q \in (0, 1)\),
\[ E_{s,x}^{t}\sup_{0 \leq r \leq \xi_{\tau}} |u(X_{r})|^{q} \leq (1 - q)^{-1} E_{s,x}^{t}\left(1 + |\varphi(X_{T})|1_{\{\xi_{\tau} > T_{\tau}\}}ight.
\[ + \int_{0}^{\xi_{\tau}} |f(X_{r}, 0)| dr + \int_{0}^{\xi_{\tau}} d|A^{\mu}|_{r}\right). \]
By the above estimate and Lemma 3.1,
\[ E_{s,x}^{t}\left(\int_{0}^{\xi_{\tau}} |\sigma \nabla_{X}u(X_{r})|^{2} dr\right)^{q/2} \leq c(q) E_{s,x}^{t}\left(1 + |\varphi(X_{T})|1_{\{\xi_{\tau} > T_{\tau}\}}ight.
\[ + \int_{0}^{\xi_{\tau}} |f(X_{r}, 0)| dr + \int_{0}^{\xi_{\tau}} d|A^{\mu}|_{r}\right). \]
for every $q \in (0, 1)$. Since $q \in (0, 1)$, we have
\[
E'_{s,x} \left( \int_0^{\xi_T} |\sigma \nabla_X u(X_r)|^2 \, dr \right)^{q/2} \geq \Lambda^{-1} E'_{s,x} \left( \int_0^{\xi_T} |\nabla_X u(X_r)|^q \, dr \cdot \xi_T^{q/2-1} \right) \geq \Lambda^{-1} T^{q/2-1} E'_{s,x} \left( \int_0^{\xi_T} |\nabla_X u(X_r)|^q \, dr \right).
\]

By the above,
\[
\int \int_{D_T} |\nabla_X u(r, y)|^q p_D(s, x, r, y) \, dr \, dy = E'_{s,x} \left( \int_0^{\xi_T} |\nabla_X u(X_r)|^q \, dr \right) < \infty
\]
for q.e. $(s, x) \in D_T$, where $p_D$ is the transition density of the process $X$ killed on exiting $D$. Therefore, the desired result follows from the well-known fact that $p_D(s, x, \cdot, \cdot)$ is continuous and strictly positive on $(s, T] \times D$ (see [1]). □

**REMARK 5.14.** From [19], it follows that if $u$ is a probabilistic solution of (5.1) such that $f(\cdot, u) \in L^1(D_T)$, then $u \in T^0_{1,2} \cap L^q(0, T; W^{1,q}_0(D))$ for $q \in [1, \frac{d+2}{d+1})$ and $u$ is a renormalized (entropy) solution of (5.1).

6. **Large-time asymptotics**

In this section, we prove our main results (1.10) and (1.11) on large-time behavior of solutions of (1.1) in case $A, \mu$ and $f$ do not depend on time. A key role in obtaining (1.10), (1.11) play some a priori estimates on solutions of BSDEs which we present below.

6.1. **Large-time estimates for the solutions of BSDEs**

Let $\mu$ be a soft measure such that $A^\mu$ is a CAF of $X$. Let $\xi_T^n = (n - \tau(0)) \wedge \xi$ and let $(Y^n, Z^n)$ be a solution of the BSDE
\[
Y^n_t = \varphi(X_{n-\tau(0)})1_{\{\xi > n-\tau(0)\}} + \int_t^{\xi_T^n} f(X_r, Y^n_r) \, dr + \int_t^{\xi_T^n} dA^\mu_r - \int_t^{\xi_T^n} Z^n_r \, dB_r, \quad 0 \leq t \leq \xi_T^n
\]
such that $(Y^n, Z^n) \in D^q \otimes M^q$ for $q \in (0, 1)$ and $Y^n$ is of class (D). In the remainder of the paper, we adopt the convention that $Y^n_t = 0$, $t \geq n - \tau(0)$ and $Z^n_r = 0$, $r \geq n - \tau(0)$. 

PROPOSITION 6.1. Assume that (H1)–(H4) are satisfied with \( \alpha \leq 0 \). Let \( n < m \) and \( \delta Y_t = Y_t^m - Y_t^n, \delta Z_t = Z_t^m - Z_t^n \). Then, for every \( q \in (0, 1) \),

\[
E_{s,x} E_{s,x}^\prime \sup_{t \geq 0} |\delta Y_t|^q + \epsilon E_{s,x} E_{s,x}^\prime \left( \int_0^\tau |\delta Z_t|^2 \, dr \right)^{q/2} \\
\leq (1 - q)^{-1} (1 + 2C_\epsilon) E_{s,x} E_{s,x}^\prime \left( |\varphi(X_{n-s})|^{q1_{[\zeta > n-s]}}, |\varphi(X_{m-s})|^{q1_{[\zeta > m-s]}} + \left| \int_n^\tau d|A^\mu_r| \right|^{q} + \int_n^\tau |f(X_r, 0)| \, dr \right)^q
\]

for \( \epsilon = 0, 1 \), where \( C_\epsilon \) is the constant from Lemma 3.1.

**Proof.** Observe that

\[
Y_t^n = Y_0^n - \int_0^t 1_{[0, \zeta^n]}(r) f(X_r, Y_r^n) \, dr - \int_0^t 1_{[0, \zeta^n]}(r) \, dA^\mu_r + \int_0^t dV_r^n + \int_0^t Z_r^n \, dB_r
\]

for \( t \geq 0 \), where

\[
V_t^n = \begin{cases} 
0, & t < n - \tau(0), \\
- Y_{n-\tau(0)}^n, & t \geq n - \tau(0).
\end{cases}
\]

Put

\[
\psi(t) = - Y_0^n + \int_0^t 1_{[0, \zeta^n]}(r) f(X_r, Y_r^m) \, dr + \int_0^t 1_{[0, \zeta^n]}(r) \, dA^\mu_r - \int_0^t dV_r^m + Y_0^n - \int_0^t 1_{[0, \zeta^n]}(r) f(X_r, Y_r^n) \, dr - \int_0^t 1_{[0, \zeta^n]}(r) \, dA^\mu_r + \int_0^t dV_r^n, \quad t \geq 0.
\]

By the Itô–Meyer formula (see Corollary 2.3) and standard localization procedure for local martingale \( N = \int_0^\tau (\text{sgn}(\delta Y_r), \delta Z_r, dB_r) \), for \( t \leq m - \tau(0) \), we have

\[
|\delta Y_t| \leq E_{s,x} E_{s,x}^\prime \left[ |\delta Y_{m-s}| + \int_t^{m-s} (\text{sgn}(\delta Y_r), d\psi(r)) |\mathcal{F}_t^\prime | \right]. \tag{6.1}
\]

Suppose that \( n - \tau(0) < t \leq m - \tau(0) \). Since \( |\delta Y_{m-\tau(0)}| = 0 \), it follows from (6.1) that

\[
|\delta Y_t| \leq E_{s,x} E_{s,x}^\prime \left[ \int_t^{m-s} (\text{sgn}(\delta Y_{r-}), 1_{[0, \zeta^n]}(r), f(X_r, Y_r^m)) \, dr \\
+ \int_t^{m-s} (1_{[0, \zeta^n]}(r) \text{sgn}(\delta Y_{r-}), dA^\mu_r) - \int_t^{m-s} (\text{sgn}(\delta Y_{r-}) 1_{[0, \zeta^n]}(r), dV_r^m) |\mathcal{F}_t^\prime | \right].
\]

Moreover,

\[
\int_t^{\zeta} (\text{sgn}(\delta Y_{r-}), f(X_r, Y_r^m)) \, dr = \int_t^{\zeta} (\text{sgn}(Y_{r-}^m), f(X_r, Y_r^m)) \, dr \leq \int_{n-\tau(0)}^{\zeta} |f(X_r, 0)| \, dr
\]
and
\[
\int_t^{\xi_m} (\text{sgn}(\delta Y_{r-}), \, dA^\mu_r) \leq \int_{n-\tau(0)}^{\xi} |dA^\mu_r|, \quad \int_t^{\xi_m} (\text{sgn}(\delta Y_{r-}), \, dV^m_r) \\
\leq |\varphi(X_{m-\tau(0)})|1_{[0, \xi_m]}.
\]

By the above estimates, if \( n - \tau(0) < t \leq m - \tau(0) \), then
\[
|\delta Y_t| \leq E'_{s,x} \left( \int_{n-s}^{\xi} d|A^\mu_r| + \int_{n-s}^{\xi} |f(X_r, 0)| \, dr + |\varphi(X_{m-s})|1_{[0, \xi_m]} |F'_t| \right) \equiv I(n, m).
\]

Suppose now that \( 0 \leq t \leq n - \tau(0) \). Then, by (6.1),
\[
|\delta Y_t| \leq E'_{s,x} \left( \int_{n-s}^{m-s} (1_{[0, \xi_m]}(r) f(X_r, Y^m_r) - 1_{[0, \xi_m]}(r) f(X_r, Y^n_r), \, \text{sgn}(\delta Y_r)) \, dr \\
+ \int_{t}^{m-s} (1_{[0, \xi_m]}(r) - 1_{[0, \xi_m]}(r)) \, \text{sgn}(\delta Y_r), \, dA^\mu_r) \\
+ \int_{t}^{m-s} \langle \text{sgn}(\delta Y_r) 1_{[0, \xi_m]}(r), \, dV^m_r - \int_{t}^{m-s} \langle \text{sgn}(\delta Y_r) 1_{[0, \xi_m]}(r), \, dV^m_r |F'_t \rangle \\
\leq I(n, m) + E'_{s,x} \left( \int_{n-s}^{m-s} (1_{[0, \xi_m]}(r) f(X_r, Y^m_r) - 1_{[0, \xi_m]}(r) f(X_r, Y^n_r), \, \text{sgn}(\delta Y_r)) \, dr \\
+ \int_{t}^{m-s} (1_{[0, \xi_m]}(r) - 1_{[0, \xi_m]}(r)) \, \text{sgn}(\delta Y_r), \, dA^\mu_r) \\
+ \int_{t}^{m-s} \langle \text{sgn}(\delta Y_r) 1_{[0, \xi_m]}(r), \, dV^m_r - \int_{t}^{m-s} \langle \text{sgn}(\delta Y_r) 1_{[0, \xi_m]}(r), \, dV^m_r |F'_t \rangle .
\]

The last term is equal to zero. By (H2),
\[
\int_{t}^{n-\tau(0)} (1_{[0, \xi_m]}(r) f(X_r, Y^m_r) - 1_{[0, \xi_m]}(r) f(X_r, Y^n_r)), \, \text{sgn}(\delta Y_r)) \, dr \\
= \int_{t}^{n-\tau(0)} (1_{[0, \xi_m]}(r) (f(X_r, Y^m_r) - f(X_r, Y^n_r)), \, \text{sgn}(\delta Y_r)) \, dr \leq 0.
\]

Moreover,
\[
\int_{t}^{n-\tau(0)} (1_{[0, \xi_m]}(r) - 1_{[0, \xi_m]}(r)) \, \text{sgn}(\delta Y_r), \, dA^\mu_r) = 0.
\]

By what has been already proved, for every \( t \geq 0 \),
\[
|\delta Y_t| \leq E'_{s,x} \left( |\varphi(X_{n-s})|1_{[\xi > n-s]} + |\varphi(X_{m-s})|1_{[\xi > m-s]} \\
+ \int_{n-s}^{\xi} d|A^\mu_r| + \int_{n-s}^{\xi} |f(X_r, 0)| \, dr |F'_t \right).
\]

The desired estimate now follows from Lemma 6.1 in [3] and Lemma 3.1.
\[
\square
\]
6.2. Large-time behavior of solutions of PDEs

In this subsection, we consider the operator $A$ defined by (4.10) (with coefficients not depending on time). Let us recall that for a nonnegative Borel measure $\mu$ on $D_T$ (respectively, $D$), we write

$$R^{0,T}_\mu(s, x) = \int_{D_T} p_D(s, x, t, y) \, d\mu(t, y), \quad \text{resp.} \quad R\mu(x) = \int_D G_D(x, y) \, d\mu(y).$$

**Lemma 6.2.** Let $\mu \in \mathcal{M}_{0,b}^+(D_T)$. Then, there exists a PAF $A^\mu$ of $X$ such that for every $(s, x) \in D_T$, if $R^{0,T}_\mu(s, x) < \infty$, then

$$E_{s,x}^\prime \int_0^{t_\tau} \eta(X_t) \, dA^\mu_t = \int_{D_T} \eta(\theta, y) p_D(s, x, t, y) \, d\mu(t, y)$$

for every bounded $\eta \in \mathcal{B}(D_T)$.

**Proof.** See [13].

We say that a Borel measure $\mu$ on $D_T$ does not depend on time if for every $A \in \mathcal{B}([0, T])$ and $B \in \mathcal{B}(D)$,

$$\mu(A \times B) = \lambda(A) \cdot \tilde{\mu}(B) \quad (6.3)$$

for some Borel measure $\tilde{\mu}$ on $D$ (Here, $\lambda$ stands for the Lebesgue measure). Since $\tilde{\mu}(B) = \mu([0, 1] \times B)$ for $B \in \mathcal{B}(D)$, $\tilde{\mu}$ is uniquely determined by $\mu$. From now on, given $\mu$ not depending on time, we will denote by $\tilde{\mu}$ the Borel measure on $D$ determined by (6.3). It is clear that if $\mu$ is soft with respect to cap, then $\tilde{\mu}$ is soft with respect to cap $\cap_N$ (see Sect. 4.3). It is known (see [9]) that there exists a unique continuous additive functional $A^\mu$ of $X = \{(X, P_x \equiv P_{0,x}) : x \in \mathbb{R}^d\}$ in the Revuz correspondence with measure $\tilde{\mu}$ or, equivalently, satisfying (4.12).

**Lemma 6.3.** Let $\mu \in \mathcal{M}_{0,b}^+(D_T)$ do not depend on time. Then, $A^\mu$ is continuous, for every $s \in [0, T]$, $R^{0,T}_\mu(s, x) < \infty$ for q.e. $x \in D$ and

$$A^\mu_t(0, \omega) = A^\tilde{\mu}_t(\omega) \text{ for } P_x\text{-a.e. } \omega \in \Omega.$$ 

**Proof.** Since $(X, P_x)$ is time homogeneous, $p_D(s, x, t, y) = p_D(t - s, x, y)$ for $t > s$, $x, y \in D$. Taking into account that $\mu = dt \otimes \tilde{\mu}$, we therefore have

$$\int_{D_T} p_D(s, x, \theta, y) \, d\mu(\theta, y) = \int_0^T \int_D p_D(\theta - s, x, y) \, d\theta \, \tilde{\mu}(dy) \leq \int_0^\infty \int_D p_D(t, x, y) \tilde{\mu}(dy) \, dt = E_x \int_0^{t_\tau} \, dA^\tilde{\mu}_t.$$
It is known (see [9]) that for every bounded smooth measure \( \tilde{\mu} \) on \( D \), the last integral above is finite for q.e. \( x \in D \). This proves that for every \( s \in [0, T] \), \( R^{0,T}_s \mu (s, x) < \infty \) for q.e. \( x \in D \). By [13], \( E'_{s,x}(A^\mu_t - A^\mu_{t-}) = \int_D p_D(s, x, t, y) \mu (t, dy) \), where \( \mu (t, B) = \mu (\{ t \} \times B) \) for \( B \in B(D) \). Since \( \mu = \lambda \otimes \tilde{\mu} \), we conclude that \( \mu (t, dy) = 0 \) for every \( t \in [0, T] \), which implies that \( A^\mu \) is continuous. We may assume that \( \tilde{\mu} \in H^{-1}(D) \). If not, we argue as follows. It is known (see [9]) that for every smooth measure \( \tilde{\mu} \) on \( D \), there exists a generalized nest \( \{ F_n \} \) such that \( 1_{F_n} \cdot \tilde{\mu} \in H^{-1}(D) \) for every \( n \geq 1 \). We prove the assertion for each measure \( 1_{F_n} \cdot \tilde{\mu} \), and then, it is a simple matter to deduce the assertion for \( \tilde{\mu} \). By the construction of the AF (see [13]), for every \( (s, x) \in D_T \) such that \( R^{0,T}_s \mu (s, x) < \infty \),

\[
E'_{s,x} \sup_{0 \leq t \leq T_r} |A^\mu_t - A^\mu_{t-}|^2 \to 0, \tag{6.4}
\]

where \( \mu_n = f_n \cdot m_1 \), \( f_n = n(u - u_n) \), \( A^\mu_t (\omega') = \int_0^t f_n(X_s (\omega')) \, dr \), \( u \) is a solution to the problem

\[
\begin{align*}
\frac{\partial u}{\partial \tau} - Au &= \mu, \\
u(0, \cdot) &= \varphi, \quad u(t, \cdot)|_{\partial D} = 0, \quad t \in (0, T]
\end{align*}
\]

with some \( \varphi \in L^2(D) \) and \( u_n, n \in \mathbb{N} \), are solutions to the problems

\[
\begin{align*}
\frac{\partial u_n}{\partial \tau} - Au_n &= n(u - u_n), \\
u_n(0, \cdot) &= \varphi_n, \quad u_n(t, \cdot)|_{\partial D} = 0, \quad t \in (0, T]
\end{align*}
\]

with \( \varphi_n \in L^2(D) \) such that \( \varphi_n \to u(0, \cdot) \) in \( L^2(D) \). Since \( \mu \) does not depend on time, one can choose \( u \in H^1_0(D) \) such that \( -Au = \tilde{\mu} \) and \( u_n \in H^1_0(D) \) such that \( -Au_n = n(u - u_n) \). By the construction of the PCAF \( \tilde{A}_t^\mu \) of \( (X, P_x) \),

\[
\int_0^t f_n(X_s (\omega')) \, dr \to \tilde{A}_t^\mu, \quad t \geq 0
\]

for \( P_{s,x} \)-a.e. \( \omega \in \Omega \). But the sequence \( \{ \int_0^t f_n(X_s (\omega')) \, dr \} \) is convergent for \( P_x \)-a.e. \( \omega \in \Omega \) if \( \{ \int_0^t f_n(X_s (\omega')) \, dr \} \) is convergent for \( P_{0,x} \)-a.e. \( \omega' \in \Omega' \), which in view of (6.4) gives the desired result. \( \Box \)

From now on, we assume that the measure \( \mu \) and the right-hand side \( f \) do not depend on time. Let us consider the following parabolic system of PDEs

\[
\begin{align*}
\frac{\partial u^k}{\partial \tau} - Au^k &= f^k(x, u) + \mu^k, \quad k = 1, \ldots, N, \\
u|_{\partial D}(t, \cdot) &= 0, \quad t \in (0, T], \quad u(0, \cdot) = \varphi
\end{align*}
\]

and elliptic system of PDEs

\[
\begin{align*}
-Av^k &= f^k(x, v) + \tilde{\mu}^k \quad \text{in } D, \quad k = 1, \ldots, N, \\
u|_{\partial D} &= 0.
\end{align*}
\]

Let \( F_0 = \{ x \in D; (0, x) \in F \} \), where \( F \) is the set defined in Proposition 5.9 with \( T = \infty \) (see (1.12)).
THEOREM 6.4. Assume (H1)–(H4) with $\alpha \leq 0$. Let $u, v$ be solutions of (6.5) and (6.6), respectively. Then, for every $t > 0$, $q \in (0, 1)$ and every $x \in F_0$,

$$|u(t, x) - v(x)| \leq c(\Lambda)(1 - q)^{-1/q} \left( |P_t \varphi(x)| P_x^{(1-q)/q}(\zeta^0 > t) + |(P_t R \tilde{\mu})(x)| + |(P_t R f(\cdot, 0))(x)| \right).$$  \hspace{1cm} (6.7)

**Proof.** By Proposition 5.9, for every $x \in F_0$ (hence for q.e. $x \in D$ by Lemma 6.3), there exists a solution $(Y^n, Z^n)$ of BDSE$_{0,x}$

$$Y^n_t = \varphi(X^n_{n-\tau(0)}) 1_{[\zeta > n-\tau(0)]} + \int_t^{\xi^n_n} f(X_r, Y^n_r) \, dr$$

$$+ \int_t^{\xi^n_n} dA^\mu_r - \int_t^{\xi^n_n} Z^n_r \, dB_r, \quad 0 \leq t \leq \xi^n_n. \hspace{1cm} (6.8)$$

Let us recall that we put $Z^n_t = Y^n_t = 0$ for $t \geq n - \tau(0)$. By Proposition 6.1, for every $n < m$ and $x \in F_0$, we have

$$E_{0,x}^t \sup_{t \geq 0} |Y^n_t - Y^m_t|^q + E_{0,x}^t \left( \int_0^t |Z^n_r - Z^m_r|^2 \, dr \right)^{q/2}$$

$$\leq (1 - q)^{-1} (1 + 2C_q) E_{0,x}^t \left( |\varphi(X^n_t)|^q 1_{[\zeta > n]} + |\varphi(X^m_t)|^q 1_{[\zeta > m]} \right)$$

$$+ \left( \int_0^t |dA^\mu_r|^q \right)^{q/2} + \left( \int_0^t |f(X_r, 0)|^q \, dr \right)^{q/2}. \hspace{1cm} (6.9)$$

For every $t > 0$,

$$E_{0,x}^t |\varphi(X_t)|^q 1_{[\zeta > t]} = E_x^t |\varphi(X_t)|^q 1_{[\zeta^0 > t]}$$

$$\leq (E_x^t |\varphi(X_t)|^q) \cdot P_x^{1-q}(\zeta^0 > t)$$

$$= |(P_t \varphi)(x)|^q \cdot P_x^{1-q}(\zeta^0 > t)$$

$$\leq c(\Lambda) t^{-q^{d/2}} \|\varphi\|_{L_1}^q P_x^{1-q}(\zeta^0 > t). \hspace{1cm} (6.10)$$

Also,

$$E_{0,x}^t \left( \int_t^{\xi^0} d|A^\mu_r|^q \right)^{q/2} \leq (E_{0,x}^t \int_t^{\xi^0} d|A^\mu_r|^q)^{q/2} = \left( E_x^t \int_t^{\zeta^0} d|A^\mu_r|^q \right)^{q/2} = \left( E_x E_x^t \int_t^{\zeta^0} d|A^\mu_r|^q \right)^{q/2}$$

$$= (E_x (R\tilde{\mu})(X_t))^q = (P_t (R\tilde{\mu})(x))^q \leq c(\Lambda) t^{-q^{d/2}} \|R\tilde{\mu}\|_{L_1}^q. \hspace{1cm} (6.11)$$

Since $\|R\tilde{\mu}\|_{L_1} = (R\tilde{\mu}, 1)_{L^2}$ and by (4.13), $(R1)(x) = E_x \zeta^0 \leq c(d, \Lambda)|D|^{d/2}$, it follows that

$$\|R\tilde{\mu}\|_{L^1} \leq c(d, \Lambda)|D|^{d/2} \|\tilde{\mu}\|_{TV}. \hspace{1cm} (6.12)$$

Similarly, since $\int_0^t |f(X_r, 0)| \, dr = \int_0^t |d|A^v_r|^r, \ t \geq 0$, where $v = f(\cdot, 0) \cdot m$, we have

$$E_{0,x}^t \left( \int_t^{\xi^0} |f(X_r, 0)| \, dr \right)^{q/2} \leq c(d, \Lambda) t^{-q^{d/2}} \|Rv\|_{L_1}^q \hspace{1cm} (6.13)$$
and
\[ \| Rv \|_{L^1} \leq c(\Lambda, \Lambda)|D|^d/2 \| f(\cdot, 0) \|_{L^1}. \] (6.14)

From (6.9)–(6.14), it follows that
\[ E_0 \sup_{t \geq 0} |Y^n_t - Y^m_t|^q + E_0 \left( \int_0^\zeta |Z^n_r - Z^m_r|^2 \, dr \right)^{q/2} \to 0 \] (6.15)
as \( n, m \to \infty \). Let us denote by \((Y, Z)\) the limit of the sequence \(\{(Y^n, Z^n)\}\). It is clear from the above that \((Y, Z) \in S^q \otimes M^q\) for \( q \in (0, 1) \). For fixed \( \varepsilon, R > 0 \), we have
\[ P_{0, x} \left( \int_0^\zeta |f(X_t, Y^n_t) - f(X_t, Y_r)| \, dr > \varepsilon \right) \leq P_{0, x} \left( \int_0^\zeta |f(X_t, Y^n_t) - f(X_t, Y_r)| \, dr > \varepsilon, \sup_{t \geq 0} |Y^n_t - Y_t| \leq R \right) + P_{0, x} (\sup_{t \geq 0} |Y^n_t - Y_t| > R) \equiv I_1(n, R, \varepsilon) + I_2(R) + I_3(n, R). \]

By (H1), (H4), and the Lebesgue dominated convergence theorem, \( I_1(n, R, \varepsilon) \to 0 \) as \( n \to \infty \). By (6.9), \( I_3(n, R) \to 0 \) as \( n \to \infty \). Finally, since \( Y \in S^q \) for \( q \in (0, 1) \), \( I_2(R) \to 0 \) as \( R \to +\infty \). This shows that
\[ \int_t^\zeta f(X_t, Y^n_t) \, dr \to \int_t^\zeta f(X_t, Y_r) \, dr \] (6.16)
in probability \( P_{0, x} \) uniformly on \([0, \zeta]\). By (6.10), (6.15), and (6.16), we may pass to the limit in (6.8). We then get
\[ Y_t = \int_t^\zeta f(X_t, Y_r) \, dr + \int_t^\zeta \, dA^n_r - \int_t^\zeta Z_r \, dB_r, \quad 0 \leq t \leq \zeta, \quad P_{0, x} \text{-a.s.} \] (6.17)

By the Itô–Meyer formula,
\[ |Y^n_t| \leq E_0 \left( \int_t^\zeta f(X_t, Y^n_r) \cdot \text{sgn}(Y^n_r) + |\varphi(X_t)|1_{\{\|Y^n\| > R\}}|F^n_t| \right) \leq E_0 \left( \int_t^\zeta |f(X_t, 0)| \, dr |F^n_t| \right) + E_0 \left( |\varphi(X_t)|1_{\{\|Y^n\| > R\}}|F^n_t| \right) = E_x \left( \int_t^\zeta |f(X_t, 0)| \, dr |F_t| \right) + E_x (|\varphi(X_t)|1_{\{\|X\| > R\}}|F_t|). \] (6.18)

But for \( t < n \),
\[ E_x (|\varphi(X_n)| |F_t|) = E_{X_t} |\varphi(X_{n-t})| = (P_{n-t} |\varphi|)(X_t) \leq c(\Lambda)(n-t)^{-d/2} \| \varphi \|_{L^1}. \]

Therefore, letting \( n \to \infty \) in (6.18), we obtain
\[ |Y_t| \leq E_x \left( \int_0^\zeta |f(X_t, 0)| \, dr |F_t| \right). \]
From the above inequality, we conclude that $Y$ is of class (D). Set $(Y^{0,n}(\omega), Z^{0,n}(\omega)) = (Y^n(0, \omega), Z^n(0, \omega))$ and $(Y^0(\omega), Z^0(\omega)) = (Y(0, \omega), Z(0, \omega))$ for $w \in \Omega$. Then, from (6.15), (6.17) and the fact that

$$E_x \sup_{t \geq 0} |Y_t^{0,n} - Y_t^0|^q + E_x \left( \int_0^{\xi_0} |Z_r^{0,n} - Z_r^0|^2 \, dr \right)^{q/2} \to 0 \quad (6.19)$$

it follows that

$$Y_t^0 = \int_t^{\zeta_0} f(X_r, Y_r^0) \, dr + \int_t^{\zeta_0} \tilde{A}_r \, d\tilde{\mu}_r - \int_t^{\zeta_0} Z_r^0 \, dB_r, \quad 0 \leq t \leq \zeta_0, \quad P_x\text{-a.s.}$$

By what has already been proved, the pair $(Y^0, Z^0)$ has integrability properties under which the solution to BSDE$_x(\zeta, f + d\tilde{\mu})$ is unique (see [18]). Therefore, from [15], it follows that $(Y^0, Z^0)$ has the representation

$$Y_t^0 = v(X_t), \quad 0 \leq t \leq \zeta_0, \quad P_x\text{-a.s.}, \quad Z^0 = \sigma \nabla v(X) \text{ on } [0, \zeta_0] \times \Omega, \quad dt \otimes P_x\text{-a.e.} \quad (6.20)$$

Moreover, by Theorem 5.12,

$$Y_t^n = u_n(X_t), \quad 0 \leq t \leq \zeta^n_\tau, \quad P_0^{\tau'}\text{-a.s.}, \quad Z^n = \sigma \nabla u_n(X) \text{ on } [0, \zeta^n_\tau] \times \Omega', \quad dt \otimes P_0^{\tau'}\text{-a.e.},$$

where $u_n$ is a solution of the system

$$\left\{ \begin{array}{l}
\frac{\partial u_n}{\partial t} + A u_n = -f(x, u_n) - \mu, \\
u_n(n, \cdot) = \varphi, \quad u_n(t, \cdot)|_{\partial D} = 0, \quad t \in [0, n) \end{array} \right. \quad (6.21)$$

Therefore, if we put $\zeta^n_{\tau,0}(\omega) = \zeta^n_\tau(0, \omega)$, then

$$Y_t^{0,n} = u_n(t, X_t), \quad 0 \leq t \leq \zeta^n_{\tau,0}, \quad P_x\text{-a.s.} \quad (6.21)$$

and

$$Z^{0,n} = \sigma \nabla u_n(\cdot, X) \quad \text{on } [0, \zeta^n_{\tau,0}] \times \Omega, \quad dt \otimes P_x\text{-a.e.} \quad (6.22)$$

It is an elementary check that

$$u_n(t, x) = u(n - t, x), \quad t \in [0, n], \quad x \in D. \quad (6.23)$$

Letting $m \to \infty$ in (6.9) and using (6.19), we obtain

$$E_x \sup_{t \geq 0} |Y_t^{0,n} - Y_t^0|^q \leq (1 - q)^{-1} E_x \left( |\varphi(X_n)|^q \mathbf{1}_{[\xi_0 > n]} + \left( \int_0^{\xi_0} |A_r| \, dr \right)^q \right) + \left( \int_n^{\xi_0} |f(X_r, 0)| \, dr \right)^q \quad (6.24)$$

From this and (6.10), (6.11), (6.20)–(6.23), we get (6.7). \qed

**COROLLARY 6.5.** Assume (H1)–(H4) with $\alpha \leq 0$. Let $u$ and $v$ be solutions of (6.5) and (6.6), respectively. Then, (1.11) is satisfied for q.e. $x \in D$. In particular, for q.e. $x \in D$, $u(t, x) \to v(x)$ as $t \to \infty$.

**Proof.** Follows from (6.7), (6.10)–(6.12), and (4.14). \qed
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T. Klimsiak
Faculty of Mathematics and Computer Science, Nicolaus Copernicus University, Chopina 12/18, 87-100 Toruń, Poland
E-mail: tomas@mat.umk.pl