Schwinger’s Quantum Action Principle. I.

From Dirac’s formulation through Feynman’s path integrals to the Schwinger-Keldysh method

Kimball A. Milton\textsuperscript{1,2,a}

Homer L. Dodge Department of Physics and Astronomy
University of Oklahoma
Norman, OK 73019 USA\textsuperscript{b}

Abstract. Starting from the earlier notions of stationary action principles, we show how Schwinger’s Quantum Action Principle descended from Dirac’s formulation, which independently led Feynman to his path-integral formulation of quantum mechanics. The connection between the two is brought out, and applications are discussed. The Keldysh-Schwinger time-cycle method of extracting matrix elements is described. Part II will discuss the variational formulation of quantum electrodynamics and the development of source theory.

\section{1 Historical Introduction}

Variational principles for dynamical systems have a long history. Although precursors go back at least to Leibnitz (see for example \cite{Euler1752} and Euler \cite{Euler1744} the “principle of least action” was given modern form by de Maupertuis \cite{Maupertuis1744,Maupertuis1746}. We will not attempt to trace the history here; a brief useful account is given in Sommerfeld’s lectures \cite{Sommerfeld1964}. The most important names in the history of the development of dynamical systems, or at least those that will bear most directly on the following discussion, are those of Joseph-Louis Lagrange \cite{Lagrange1788} and William Rowan Hamilton \cite{Hamilton1834,Hamilton1835}.

Here we are concentrating on the work of Julian Schwinger (1918–1994), who had profound and pervasive influence on 20th century physics, and whose many students have become leaders in diverse fields. For biographical information about his life and work see \cite{Mehra2000,Milton2007}. Therefore, we will take up the story in the modern era. Shortly after Dirac’s work with Fock and Podolsky \cite{Dirac1932}, in which the demonstration of the equivalence between his theory of quantum electrodynamics, and that of Heisenberg and Pauli, P. A. M. Dirac wrote a paper on “The Lagrangian in Quantum Mechanics” \cite{Dirac1933}. This paper had a profound influence on Richard Feynman’s doctoral dissertation at Princeton on “The Principles of Least Action in Quantum Mechanics” \cite{Feynman1942}, and on his later work on the formulations of the “Space-Time Approach to Quantum Electrodynamics” \cite{Feynman1949}. Dirac’s

\textsuperscript{a} e-mail: milton@nhn.ou.edu
\textsuperscript{b} Address 2013–14: Laboratoire Kastler Brossel, Université Pierre et Marie Curie, 4, place Jussieu Case 74, F-75252 Paris Cedex 05, France
paper further formed the basis for Schwinger’s development of the quantum action principle, which first appeared in his final operator field formulation of quantum field theory \cite{Schwinger1951}, which we will describe in Part II of this series.

The response of Feynman and Schwinger to Dirac’s inspiring paper was completely different. Feynman was to give a global “solution” to the problem of determining the transformation function, the probability amplitude connecting the state of the system at one time to that at a later time, in terms of a sum over classical trajectories, the famous path integral. Schwinger, instead, derived (initially postulated) a differential equation for that transformation function in terms of a quantum action functional. This differential equation possessed Feynman’s path integral as a formal solution, which remained poorly defined; but Schwinger believed throughout his life that his approach was “more general, more elegant, more useful, and more tied to the historical line of development as the quantum transcription of Hamilton’s action principle” \cite{Schwinger1973}.

Later, in a tribute to Feynman, Schwinger commented further. Dirac, of course, was the father of transformation theory \cite{Dirac1927}. The transformation function from a description at time $t_2$ to a description at time $t_1$ is “the product of all the transformations functions associated with the successive infinitesimal increments in time.” Dirac said the latter, that is, the transformation function from time $t$ to time $t + dt$ corresponds to $\exp\left[\frac{i}{\hbar} dt L\right]$, where $L$ is the Lagrangian expressed in terms of the coordinates at the two times. For the transformation function between $t_2$ and $t_1$ “the integrand is $\exp\left[\frac{i}{\hbar} W\right]$, where $W = \int_{t_1}^{t_2} dt L$.” “Now we know, and Dirac surely knew, that to within a constant factor the ‘correspondence,’ for infinitesimal $dt$, is an equality when we deal with a system of nonrelativistic particles possessing a coordinate-dependent potential energy $V$ . . . . Why then, did Dirac not make a more precise, if less general statement? Because he was interested in a general question: What, in quantum mechanics, corresponds to the classical principle of stationary action?”

“Why, in the decade that followed, didn’t someone pick up the computational possibilities offered by this integral approach to the time transformation function? To answer this question bluntly, perhaps no one needed it—until Feynman came along.” \cite{Schwinger1989}.

But Schwinger followed the differential route, and starting in early 1950 began a new, his third, formulation of quantum electrodynamics, based on a variational approach. This was first published in 1951 \cite{Schwinger1951}. A bit later he started developing a new formulation of quantum kinematics, which he called Measurement Algebra, which got its first public presentation at École de Physique at les Houches in the summer of 1955. There were several short notes in the Proceedings of the US National Academy published in 1960, explaining both the quantum kinematical approach and the dynamical action principle \cite{Schwinger1960a, Schwinger1960b, Schwinger1960c, Schwinger1960d}, but although he often promised to write a book on the subject (as he also promised a book on quantum field theory) nothing came of it. Les Houches lectures, based on notes taken by Robert Kohler, eventually appeared in 1970 \cite{Schwinger1970}. Lectures based on a UCLA course by Schwinger were eventually published under Englert’s editorship \cite{Schwinger2001}. The incompleteness of the written record may be partly alleviated by the present essay.

We start on a classical footing.
2 Review of Classical Action Principles

This section is based on Chapter 8 of Classical Electrodynamics [Schwinger 1998], a substantially transformed version of lectures given by Schwinger at UCLA around 1974. (Remarkably, he never gave lectures on this subject at Harvard after 1947.)

We start by reviewing and generalizing the Lagrange-Hamilton principle for a single particle. The action, $W_{12}$, is defined as the time integral of the Lagrangian, $L$, where the integration extends from an initial configuration or state at time $t_2$ to a final state at time $t_1$:

$$W_{12} = \int_{t_2}^{t_1} dt \ L.$$  (1)

The integral refers to any path, any line of time development, from the initial to the final state, as shown in Fig. 1. The actual time evolution of the system is selected by the principle of stationary action: In response to infinitesimal variations of the integration path, the action $W_{12}$ is stationary—does not have a corresponding infinitesimal change—for variations about the correct path, provided the initial and final configurations are held fixed,

$$\delta W_{12} = 0.$$  (2)

This means that, if we allow infinitesimal changes at the initial and final times, including alterations of those times, the only contribution to $\delta W_{12}$ then comes from the endpoint variations, or

$$\delta W_{12} = G_1 - G_2,$$  (3)

where $G_a$, $a = 1$ or $2$, is a function, called the generator, depending on dynamical variables only at time $t_a$. In the following, we will consider three different realizations of the action principle, where, for simplicity, we will restrict our attention to a single particle.

2.1 Lagrangian Viewpoint

The nonrelativistic motion of a particle of mass $m$ moving in a potential $V(r,t)$ is described by the Lagrangian

$$L = \frac{1}{2} m \left( \frac{dr}{dt} \right)^2 - V(r,t).$$  (4)
Here, the independent variables are $r$ and $t$, so that two kinds of variations can be considered. First, a particular motion is altered infinitesimally, that is, the path is changed by an amount $\delta r$:

$$r(t) \to r(t) + \delta r(t).$$  \hfill (5)

Second, the final and initial times can be altered infinitesimally, by $\delta t_1$ and $\delta t_2$, respectively. It is more convenient, however, to think of these time displacements as produced by a continuous variation of the time parameter, $\delta t(t)$,

$$t \to t + \delta t(t),$$  \hfill (6)

so chosen that, at the endpoints,

$$\delta t(t_1) = \delta t_1, \quad \delta t(t_2) = \delta t_2.$$  \hfill (7)

The corresponding change in the time differential is

$$dt \to d(t + \delta t) = \left(1 + \frac{d\delta t}{dt}\right) dt,$$  \hfill (8)

which implies the transformation of the time derivative,

$$\frac{d}{dt} \to \left(1 - \frac{d\delta t}{dt}\right) \frac{d}{dt}.$$  \hfill (9)

Because of this redefinition of the time variable, the limits of integration in the action,

$$W_{12} = \int_2^1 \left[\frac{1}{2} m \left(\frac{dr}{dt}\right)^2 - dt V\right],$$  \hfill (10)

are not changed, the time displacement being produced through $\delta t(t)$ subject to (7).

The resulting variation in the action is now

$$\delta W_{12} = \int_2^1 dt \left\{ m \frac{dr}{dt} \cdot \frac{d}{dt} \delta r - \delta r \cdot \nabla V - \frac{d\delta t}{dt} \left[\frac{1}{2} m \left(\frac{dr}{dt}\right)^2 + V\right] - \delta t \frac{\partial}{\partial t} V \right\}$$

$$= \int_2^1 dt \left\{ \frac{d}{dt} \left[m \frac{dr}{dt} \cdot \delta r - \left(\frac{1}{2} m \left(\frac{dr}{dt}\right)^2 + V\right) \delta t\right] \right.$$  

$$+ \left. \delta r \cdot \left[-m \frac{d^2}{dt^2} r - \nabla V\right] + \delta t \left[\frac{d}{dt} \left[\frac{1}{2} m \left(\frac{dr}{dt}\right)^2 + V\right] - \frac{\partial}{\partial t} V\right] \right\},$$  \hfill (11)

where, in the last form, we have shifted the time derivatives in order to isolate $\delta r$ and $\delta t$.

Because $\delta r$ and $\delta t$ are independent variations, the principle of stationary action implies that the actual motion is governed by

$$m \frac{d^2}{dt^2} r = - \nabla V,$$  \hfill (12a)

$$\frac{d}{dt} \left[\frac{1}{2} m \left(\frac{dr}{dt}\right)^2 + V\right] = \frac{\partial}{\partial t} V,$$  \hfill (12b)

while the total time derivative gives the change at the endpoints,

$$G = p \cdot \delta r - E \delta t,$$  \hfill (12c)
with
\[
momentum = p = m \frac{dr}{dt}, \quad energy = E = \frac{1}{2} m \left( \frac{dr}{dt} \right)^2 + V.
\] (12d)

Therefore, we have derived Newton’s second law [the equation of motion in second-order form], (12a), and, for a static potential, \( \partial V/\partial t = 0 \), the conservation of energy, (12b). The significance of (12c) will be discussed later in Section 2.4.

### 2.2 Hamiltonian Viewpoint

Using the above definition of the momentum, we can rewrite the Lagrangian as
\[
L = p \cdot \frac{dr}{dt} - H(r, p, t),
\] (13)

where we have introduced the Hamiltonian
\[
H = \frac{p^2}{2m} + V(r, t).
\] (14)

We are here to regard \( r, p, \) and \( t \) as independent variables in
\[
W_{12} = \int_{1/2}^{1} [p \cdot dr - dt H].
\] (15)

The change in the action, when \( r, p, \) and \( t \) are all varied, is
\[
\delta W_{12} = \int_{1/2}^{1} dt \left[ p \cdot \frac{dr}{dt} \delta r - \delta p \cdot \frac{\partial H}{\partial r} + \delta p \cdot \frac{dr}{dt} - \delta p \cdot \frac{\partial H}{\partial p} - \frac{d\delta t}{dt} H - \delta t \frac{\partial H}{\partial t} \right]
\]
\[
= \int_{1/2}^{1} dt \left[ \frac{d}{dt} \left( p \cdot \delta r - H \delta t \right) + \delta p \cdot \left( \frac{dr}{dt} - \frac{\partial H}{\partial p} \right) + \delta t \left( \frac{dH}{dt} - \frac{\partial H}{\partial t} \right) \right].
\] (16)

The action principle then implies
\[
\frac{dr}{dt} = \frac{\partial H}{\partial p} = \frac{p}{m},
\] (17a)
\[
\frac{dp}{dt} = -\frac{\partial H}{\partial r} = -\nabla V,
\] (17b)
\[
\frac{dH}{dt} = \frac{\partial H}{\partial t},
\] (17c)
\[
G = p \cdot \delta r - H \delta t.
\] (17d)

In contrast with the Lagrangian differential equations of motion, which involve second derivatives, these Hamiltonian equations contain only first derivatives; they are called first-order equations. They describe the same physical system, because when (17a) is substituted into (17b), we recover the Lagrangian-Newtonian equation (12a). Furthermore, if we insert (17a) into the Hamiltonian (14), we identify \( H \) with \( E \). The third equation (17c) is then identical with (12b). We also note the equivalence of the two versions of \( G \).
But probably the most direct way of seeing that the same physical system is involved comes by writing the Lagrangian in the Hamiltonian viewpoint as
\[ L = \frac{m}{2} \left( \frac{dr}{dt} \right)^2 - V - \frac{1}{2m} \left( p - m \frac{dr}{dt} \right)^2. \] (18)

The result of varying \( p \) in the stationary action principle is to produce
\[ p = m \frac{dr}{dt}. \] (19)

But, if we accept this as the definition of \( p \), the corresponding term in \( L \) disappears and we explicitly regain the Lagrangian description. We are justified in completely omitting the last term on the right side of (18), despite its dependence on the variables \( r \) and \( t \), because of its quadratic structure. Its explicit contribution to \( \delta L \) is
\[ - \frac{1}{m} \left( p - m \frac{dr}{dt} \right) \cdot \left( \delta p - m \frac{d}{dt} \delta r + m \frac{dr}{dt} \frac{d}{dt} \delta t \right), \] (20)

and the equation supplied by the stationary action principle for \( p \) variations, (19), also guarantees that there is no contribution here to the results of \( r \) and \( t \) variations.

2.3 A Third, Schwingerian, Viewpoint

Here we take \( r, p, \) and the velocity, \( v, \) as independent variables, so that the Lagrangian is written in the form
\[ L = p \cdot \left( \frac{dr}{dt} - v \right) + \frac{1}{2}mv^2 - V(r, t) \equiv p \cdot \frac{dr}{dt} - H(r, p, v, t), \] (21)

where
\[ H(r, p, v, t) = p \cdot v - \frac{1}{2}mv^2 + V(r, t). \] (22)

The variation of the action is now
\[ \delta W_{12} = \delta \int_2^1 \left[ p \cdot \frac{dr}{dt} - H \right] dt \]
\[ = \int_2^1 \left[ \frac{dp}{dt} \cdot \frac{dr}{dt} + p \cdot \frac{d}{dt} \delta r - \delta r \cdot \frac{dH}{v} - \delta p \cdot \frac{dH}{v} - \delta v \cdot \frac{dH}{v} - \delta t \frac{dH}{v} - H \frac{d}{dt} \delta t \right] \]
\[ = \int_2^1 \left[ \frac{d}{dt} \left( p \cdot \delta r - H \delta t \right) - \delta r \cdot \left( \frac{dp}{dt} + \frac{dH}{v} \right) + \delta p \cdot \left( \frac{dr}{dt} \frac{dH}{v} \right) - \delta v \cdot \frac{dH}{v} + \delta t \left( \frac{dH}{v} - \frac{dH}{v} \right) \right]. \] (23)
so that the action principle implies

\[
\begin{align*}
\frac{dp}{dt} &= -\frac{\partial H}{\partial r} = -\nabla V, \quad (24a) \\
\frac{dr}{dt} &= \frac{\partial H}{\partial p} = v, \quad (24b) \\
0 &= -\frac{\partial H}{\partial v} = -p + mv, \quad (24c) \\
\frac{dH}{dt} &= \frac{\partial H}{\partial t}, \\
G &= p \cdot \delta r - H \delta t. \quad (24e)
\end{align*}
\]

Notice that there is no equation of motion for \(v\) since \(dv/dt\) does not occur in the Lagrangian, nor is it multiplied by a time derivative. Consequently, (24c) refers to a single time and is an equation of constraint.

From this third approach, we have the option of returning to either of the other two viewpoints by imposing an appropriate restriction. Thus, if we write (22) as

\[
H(r, p, v, t) = \frac{p^2}{2m} + V(r, t) - \frac{1}{2m}(p - mv)^2, \quad (25)
\]

and we adopt

\[
v = \frac{1}{m} p \quad (26)
\]
as the definition of \(v\), we recover the Hamiltonian description, (13) and (14). Alternatively, we can present the Lagrangian (21) as

\[
L = \frac{m}{2} \left(\frac{dr}{dt}\right)^2 - V + (p - mv) \cdot \left(\frac{dr}{dt} - v\right) - \frac{m}{2} \left(\frac{dr}{dt} - v\right)^2. \quad (27)
\]

Then, if we adopt the following as definitions,

\[
v = \frac{dr}{dt}, \quad p = mv, \quad (28)
\]
the resultant form of \(L\) is that of the Lagrangian viewpoint, (4). It might seem that only the definition \(v = dr/dt\), inserted in (27), suffices to regain the Lagrangian description. But then the next to last term in (27) would give the following additional contribution to \(\delta L\), associated with the variation \(\delta r\):

\[
(p - mv) \cdot \frac{d}{dt}\delta r. \quad (29)
\]

In the next Section, where the action formulation of electrodynamics is considered, we will see the advantage of adopting this third approach, which is characterized by the introduction of additional variables, similar to \(v\), for which there are no equations of motion.

### 2.4 Invariance and Conservation Laws

There is more content to the principle of stationary action than equations of motion. Suppose one considers a variation such that

\[
\delta W_{12} = 0, \quad (30)
\]
independently of the choice of initial and final times. We say that the action, which is left unchanged, is invariant under this alteration of path. Then the stationary action principle asserts that
\[ \delta W_{12} = G_1 - G_2 = 0, \] (31)
or, there is a quantity \( G(t) \) that has the same value for any choice of time \( t \); it is conserved in time. A differential statement of that is
\[ \frac{d}{dt} G(t) = 0. \] (32)

The \( G \) functions, which are usually referred to as generators, express the interrelation between conservation laws and invariances of the system.

Invariance implies conservation, and vice versa. A more precise statement is the following:

If there is a conservation law, the action is stationary under an infinitesimal transformation in an appropriate variable.

The converse of this statement is also true.

If the action \( W \) is invariant under an infinitesimal transformation (that is, \( \delta W = 0 \)), then there is a corresponding conservation law.

This is the celebrated theorem proved by Amalie Emmy Noether [Noether 1918].

Here are some examples. Suppose the Hamiltonian of (13) does not depend explicitly on time, or
\[ W_{12} = \int_2^1 \left[ \mathbf{p} \cdot d\mathbf{r} - H(\mathbf{r}, \mathbf{p})dt \right]. \] (33)
Then the variation (which as a rigid displacement in time, amounts to a shift in the time origin)
\[ \delta t = \text{constant} \] (34)
will give \( \delta W_{12} = 0 \) [see the first line of (16), with \( \delta \mathbf{r} = 0, \delta \mathbf{p} = 0, d\delta t/dt = 0, \delta H/\delta t = 0 \)]. The conclusion is that \( G \) in (17d), which here is just
\[ G_t = -H \delta t, \] (35)
is a conserved quantity, or that
\[ \frac{dH}{dt} = 0. \] (36)
This inference, that the Hamiltonian—the energy—is conserved, if there is no explicit time dependence in \( H \), is already present in (17c). But now a more general principle is at work.

Next, consider an infinitesimal, rigid rotation, one that maintains the lengths and scalar products of all vectors. Written explicitly for the position vector \( \mathbf{r} \), it is
\[ \delta \mathbf{r} = \delta \omega \times \mathbf{r}, \] (37)
where the constant vector \( \delta \omega \) gives the direction and magnitude of the rotation (see Fig. 2). Now specialize (14) to
\[ H = \frac{p^2}{2m} + V(r), \] (38)
where \( r = |\mathbf{r}| \), a rotationally invariant structure. Then
\[ W_{12} = \int_2^1 [\mathbf{p} \cdot d\mathbf{r} - Hdt]. \] (39)
Fig. 2. $\delta \omega \times \mathbf{r}$ is perpendicular to $\delta \omega$ and $\mathbf{r}$, and represents an infinitesimal rotation of $\mathbf{r}$ about the $\delta \omega$ axis.

is also invariant under the rigid rotation, implying the conservation of

$$G_{\delta \omega} = \mathbf{p} \cdot \delta \mathbf{r} = \delta \omega \cdot \mathbf{r} \times \mathbf{p}.$$  \hspace{1cm} (40)

This is the conservation of angular momentum,

$$\mathbf{L} = \mathbf{r} \times \mathbf{p}, \quad \frac{d}{dt} \mathbf{L} = 0.$$  \hspace{1cm} (41)

Of course, this is also contained within the equation of motion,

$$\frac{d}{dt} \mathbf{L} = -\mathbf{r} \times \nabla V = -\mathbf{r} \times \hat{\mathbf{r}} \frac{\partial V}{\partial r} = 0,$$  \hspace{1cm} (42)

since $V$ depends only on $|\mathbf{r}|$.

Conservation of linear momentum appears analogously when there is invariance under a rigid translation. For a single particle, (17b) tells us immediately that $\mathbf{p}$ is conserved if $V$ is a constant, say zero. Then, indeed, the action

$$W_{12} = \int_{t_1}^{t_2} \left[ \mathbf{p} \cdot d\mathbf{r} - \frac{\mathbf{p}^2}{2m} dt \right]$$  \hspace{1cm} (43)

is invariant under the displacement

$$\delta \mathbf{r} = \delta \epsilon = \text{constant},$$  \hspace{1cm} (44)

and

$$G_{\delta \epsilon} = \mathbf{p} \cdot \delta \epsilon$$  \hspace{1cm} (45)

is conserved. But the general principle acts just as easily for, say, a system of two particles, $a$ and $b$, with Hamiltonian

$$H = \frac{p^2_a}{2m_a} + \frac{p^2_b}{2m_b} + V(\mathbf{r}_a - \mathbf{r}_b).$$  \hspace{1cm} (46)

This Hamiltonian and the associated action

$$W_{12} = \int_{t_1}^{t_2} \left[ \mathbf{p}_a \cdot d\mathbf{r}_a + \mathbf{p}_b \cdot d\mathbf{r}_b - H dt \right]$$  \hspace{1cm} (47)
are invariant under the rigid translation

\[ \delta r_a = \delta r_b = \delta \epsilon, \]

with the implication that

\[ G_{\delta \epsilon} = p_a \cdot \delta r_a + p_b \cdot \delta r_b = (p_a + p_b) \cdot \delta \epsilon \]

is conserved. This is the conservation of the total linear momentum,

\[ P = p_a + p_b, \quad \frac{d}{dt} P = 0. \]

Something a bit more general appears when we consider a rigid translation that grows linearly in time:

\[ \delta r_a = \delta r_b = \delta \epsilon t, \]

using the example of two particles. This gives each particle the common additional velocity \( \delta \epsilon \), and therefore must also change their momenta,

\[ \delta p_a = m_a \delta \epsilon, \quad \delta p_b = m_b \delta \epsilon. \]

The response of the action \( S \) to this variation is

\[ \delta W_{12} = \int_0^1 \left[ (p_a + p_b) \cdot \delta \epsilon \, dt + \delta \epsilon \cdot (m_a \delta r_a + m_b \delta r_b) - (p_a + p_b) \cdot \delta \epsilon \, dt \right] = \int_0^1 d[(m_a r_a + m_b r_b) \cdot \delta \epsilon], \]

The action is not invariant; its variation has end-point contributions. But there is still a conservation law, not of \( G = P \cdot \delta \epsilon t \), but of \( N \cdot \delta \epsilon \), where

\[ N = P t - (m_a r_a + m_b r_b). \]

Written in terms of the center-of-mass position vector

\[ R = \frac{m_a r_a + m_b r_b}{M}, \quad M = m_a + m_b, \]

the statement of conservation of

\[ N = P t - M R, \]

namely

\[ 0 = \frac{dN}{dt} = P - M \frac{dR}{dt}, \]

is the familiar fact that the center of mass of an isolated system moves at the constant velocity given by the ratio of the total momentum to the total mass of that system.

### 2.5 Nonconservation Laws. The Virial Theorem

The action principle also supplies useful nonconservation laws. Consider, for constant \( \delta \lambda \),

\[ \delta r = \delta \lambda r, \quad \delta p = -\delta \lambda p, \]
which leaves $\mathbf{p} \cdot \mathbf{dr}$ invariant,

$$\delta(\mathbf{p} \cdot \mathbf{dr}) = (-\delta \mathbf{\lambda}) \cdot \mathbf{dr} + \mathbf{p} \cdot (\delta \mathbf{dr}) = 0. \quad (59)$$

But the response of the Hamiltonian

$$H = T(p) + V(r), \quad T(p) = \frac{p^2}{2m} \quad (60)$$

is given by the noninvariant form

$$\delta H = \delta \lambda (-2T + \mathbf{r} \cdot \nabla V). \quad (61)$$

Therefore we have, for an arbitrary time interval, for the variation of the action (15),

$$\delta W_{12} = \int_1^2 dt \{ \delta \lambda [2T - \mathbf{r} \cdot \nabla V] \} = G_1 - G_2 = \int_1^2 dt \frac{d}{dt} (\mathbf{p} \cdot \delta \mathbf{r}) \quad (62)$$

or, the theorem

$$\frac{d}{dt} \mathbf{r} \cdot \mathbf{p} = 2T - \mathbf{r} \cdot \nabla V. \quad (63)$$

For the particular situation of the Coulomb potential between charges, $V = \text{constant}/r$, where

$$\mathbf{r} \cdot \nabla V = \mathbf{r} \frac{d}{dr} V = -V, \quad (64)$$

the virial theorem asserts that

$$\frac{d}{dt} (\mathbf{r} \cdot \mathbf{p}) = 2T + V. \quad (65)$$

We apply this to a bound system produced by a force of attraction. On taking the time average of (65), the time derivative term disappears. That is because, over an arbitrarily long time interval $\tau = t_1 - t_2$, the value of $\mathbf{r} \cdot \mathbf{p}(t_1)$ can differ by only a finite amount from $\mathbf{r} \cdot \mathbf{p}(t_2)$, and

$$\frac{d}{dt} (\mathbf{r} \cdot \mathbf{p}) \equiv \frac{1}{\tau} \int_{t_1}^{t_2} dt \frac{d}{dt} \mathbf{r} \cdot \mathbf{p} = \frac{\mathbf{r} \cdot \mathbf{p}(t_1) - \mathbf{r} \cdot \mathbf{p}(t_2)}{\tau} \to 0, \quad (66)$$

as $\tau \to \infty$. The conclusion, for time averages,

$$2T = -V, \quad (67)$$

is familiar in elementary discussions of motion in a $1/r$ potential.

Here is one more example of a nonconservation law: Consider the variations

$$\delta \mathbf{r} = \delta \lambda \frac{\mathbf{r}}{r}, \quad (68a)$$

$$\delta \mathbf{p} = -\delta \lambda \left( \frac{\mathbf{p}}{r} - \frac{\mathbf{r} \cdot \mathbf{p}}{r^3} \right)^2 = \delta \lambda \frac{\mathbf{r} \times (\mathbf{r} \times \mathbf{p})}{r^3}. \quad (68b)$$

Again $\mathbf{p} \cdot d\mathbf{r}$ is invariant:

$$\delta (\mathbf{p} \cdot d\mathbf{r}) = -\delta \lambda \left( \frac{\mathbf{p}}{r} - \frac{\mathbf{r} \cdot \mathbf{p}}{r^3} \right) \cdot d\mathbf{r} + \mathbf{p} \cdot \left( \delta \lambda \frac{d\mathbf{r}}{r} - \delta \lambda r \frac{d\mathbf{r}}{r^3} \right) = 0, \quad (69)$$
The change of the Hamiltonian \( \delta H \) is now
\[
\delta H = \delta \lambda \left[ -\frac{L^2}{mr^4} + \frac{r}{r} \cdot \nabla V \right].
\]
(70)

The resulting theorem, for \( V = V(r) \), is
\[
\frac{d}{dt} \left( \frac{\mathbf{r}}{r} \cdot \mathbf{p} \right) = \frac{L^2}{mr^3} - \frac{dV}{dr},
\]
(71)

which, when applied to the Coulomb potential, gives the bound-state time average relation
\[
\frac{L^2}{m} \left( \frac{1}{r} \right) = - \left( \frac{V}{r} \right).
\]
(72)

This relation is significant in hydrogen fine-structure calculations (for example, see [Schwinger 2001]).

3 Classical field theory—electrodynamics

This section is based on Chapter 9 of Classical Electrodynamics, [Schwinger 1998], which again in turn grew, torturously, out of Schwinger’s UCLA lectures. Here we use Gaussian units.

3.1 Action of Particle in Field

It was stated in our review of mechanical action principles in the previous section that the third viewpoint, which employs the variables \( \mathbf{r}, \mathbf{p}, \) and \( \mathbf{v} \), was particularly convenient for describing electromagnetic forces on charged particles. With the explicit, and linear, appearance of \( \mathbf{v} \) in what plays the role of the potential function when magnetic fields are present, we begin to see the basis for that remark. Indeed, we have only to consult (21) to find the appropriate Lagrangian:
\[
L = \mathbf{p} \cdot \left( \frac{d\mathbf{r}}{dt} - \mathbf{v} \right) + \frac{1}{2} mv^2 - e\phi + \frac{e}{c} \mathbf{v} \cdot \mathbf{A},
\]
(73)

where \( \phi \) and \( \mathbf{A} \) are the scalar and vector potentials, respectively. To recapitulate, the equations resulting from variations of \( \mathbf{p}, \mathbf{r}, \) and \( \mathbf{v} \) are, respectively,
\[
\frac{d\mathbf{r}}{dt} = \mathbf{v},
\]
(74a)
\[
\frac{d}{dt} \mathbf{p} = -e\nabla \left[ \phi - \frac{1}{c} \mathbf{v} \cdot \mathbf{A} \right],
\]
(74b)
\[
\mathbf{p} = mv + \frac{e}{c} \mathbf{A}.
\]
(74c)

We can now move to either the Lagrangian or the Hamiltonian formulation. For the first, we simply adopt \( \mathbf{v} = d\mathbf{r}/dt \) as a definition (but see the discussion in Sec. 2.3) and get
\[
L = \frac{1}{2} m \left( \frac{d\mathbf{r}}{dt} \right)^2 - e\phi + \frac{e}{c} \frac{d\mathbf{r}}{dt} \cdot \mathbf{A}.
\]
(75)
Alternatively, we use (74c) to define
\[ v = \frac{1}{m} \left( p - \frac{e}{c} A \right), \quad (76) \]
and find
\[ L = p \cdot \frac{dr}{dt} - H, \quad (77a) \]
\[ H = \frac{1}{2m} \left( p - \frac{e}{c} A \right)^2 + e\phi. \quad (77b) \]

### 3.2 Electrodynamic Action

The electromagnetic field is a mechanical system. It contributes its variables to the action, to the Lagrangian of the whole system of charges and fields. In contrast with the point charges, the field is distributed in space. Its Lagrangian should therefore be, not a summation over discrete points, but an integration over all spatial volume elements,

\[ L_{\text{field}} = \int (dr) L_{\text{field}}; \quad (78) \]

this introduces the Lagrange function, or Lagrangian density, \( L \). The total Lagrangian must be the sum of the particle part, (73), and the field part, (78), where the latter must be chosen so as to give the Maxwell equations in Gaussian units:

\[ \nabla \times B = \frac{1}{c} \frac{\partial}{\partial t} E + \frac{4\pi}{c} j, \quad \nabla \cdot E = 4\pi \rho, \quad (79a) \]
\[ - \nabla \times E = \frac{1}{c} \frac{\partial}{\partial t} B, \quad \nabla \cdot B = 0. \quad (79b) \]

The homogeneous equations here are equivalent to the construction of the electromagnetic field in term of potentials, or,

\[ \frac{1}{c} \frac{\partial}{\partial t} A = -E - \nabla \phi, \quad (80a) \]
\[ B = \nabla \times A. \quad (80b) \]

Thus, we recognize that \( A(r, t), E(r, t), B(r, t) \), in analogy with \( r(t), p(t) \), obey equations of motion while \( \phi(r, t), B(r, t) \), as analogues of \( v(t) \), do not. There are enough clues here to give the structure of \( L_{\text{field}} \), apart from an overall factor. The anticipated complete Lagrangian for microscopic electrodynamics is

\[ L = \sum_a \left[ p_a \cdot \left( \frac{dr_a}{dt} - v_a \right) + \frac{1}{2} m_a v_a^2 - e_a \phi(r_a) + \frac{e_a}{c} v_a \cdot A(r_a) \right] + \frac{1}{4\pi} \int (dr) \left[ E \cdot \left( -\frac{1}{c} \frac{\partial}{\partial t} A - \nabla \phi \right) - B \cdot \nabla \times A + \frac{1}{2} (B^2 - E^2) \right]. \quad (81) \]

The terms that are summed in (81) describe the behavior of charged particles under the influence of the fields, while the terms that are integrated describe the field behavior. The independent variables are

\[ r_a(t), \quad v_a(t), \quad p_a(t), \quad \phi(r, t), \quad A(r, t), \quad E(r, t), \quad B(r, t), \quad t. \quad (82) \]
We now look at the response of the Lagrangian to variations in each of these variables separately, starting with the particle part:

\[
\delta r_a : \quad \delta L = \frac{d}{dt} (\delta r_a \cdot p_a) + \delta r_a \cdot \left[ -\frac{d}{dt} p_a - \nabla_a e_a (\phi(r_a) - \frac{v_a}{c} \cdot A(r_a)) \right], \tag{83a}
\]

\[
\delta v_a : \quad \delta L = \delta v_a \cdot \left[ -p_a + m_a v_a + \frac{e_a}{c} A(r_a) \right], \tag{83b}
\]

\[
\delta p_a : \quad \delta L = \delta p_a \cdot \left( \frac{dr_a}{dt} - v_a \right). \tag{83c}
\]

The stationary action principle now implies the equations of motion

\[
\frac{dp_a}{dt} = -e_a \nabla_a \left( \phi(r_a) - \frac{v_a}{c} \cdot A(r_a) \right), \tag{84a}
\]

\[
m_a v_a = p_a - \frac{e_a}{c} A(r_a), \tag{84b}
\]

\[
v_a = \frac{dr_a}{dt}. \tag{84c}
\]

which are the known results, (74a)–(74c).

The real work now lies in deriving the equations of motion for the fields. In order to cast all the field-dependent terms into integral form, we introduce charge and current densities,

\[
\rho(r,t) = \sum_a e_a \delta(r - r_a(t)), \tag{85a}
\]

\[
j(r,t) = \sum_a e_a v_a(t) \delta(r - r_a(t)), \tag{85b}
\]

so that

\[
\sum_a \left[ -e_a \phi(r_a) + \frac{e_a}{c} v_a \cdot A(r_a) \right] = \int (dr) \left[ -\rho \phi + \frac{1}{c} j \cdot A \right]. \tag{86}
\]

The volume integrals extend over sufficiently large regions to contain all the fields of interest. Consequently, we can integrate by parts and ignore the surface terms. The responses of the Lagrangian to field variations, and the corresponding equations
of motion deduced from the action principle are

\[
\delta \phi : \quad \delta L = \frac{1}{4\pi} \int (dr) \delta \phi (\nabla \cdot E - 4\pi \rho), \\
\nabla \cdot E = 4\pi \rho, \tag{87a}
\]

\[
\delta A : \quad \delta L = -\frac{1}{4\pi c} \frac{d}{dt} \int (dr) \delta A \cdot E \\
+ \frac{1}{4\pi} \int (dr) \left( \frac{1}{c} \frac{\partial E}{\partial t} + \frac{4\pi}{c} \frac{\partial \phi}{\partial \nabla} \right), \tag{87c}
\]

\[
\nabla \times B = \frac{1}{c} \frac{\partial}{\partial t} E + \frac{4\pi}{c} j, \tag{87d}
\]

\[
\delta E : \quad \delta L = \frac{1}{4\pi} \int (dr) \frac{\delta E}{\cdot} \left( -\frac{1}{c} \frac{\partial A}{\partial t} - \nabla \phi - E \right), \tag{87e}
\]

\[
E = \frac{1}{c} \frac{\partial}{\partial t} A - \nabla \phi, \tag{87f}
\]

\[
\delta B : \quad \delta L = \frac{1}{4\pi} \int (dr) \delta B \cdot (-\nabla \times A + B), \tag{87g}
\]

\[
B = \nabla \times A. \tag{87h}
\]

We therefore recover Maxwell’s equations, two of which are implicit in the construction of \( E \) and \( B \) in terms of potentials. By making a time variation of the action \([\text{variations due to the time dependence of the fields vanish by virtue of the stationary action principle—that is, they are already subsumed in Eqs. (87)}\),

\[
\delta t : \quad \delta W = \int dt \left[ \frac{d}{dt}(-H\delta t) + \delta t \frac{dH}{dt} \right], \tag{88}
\]

we identify the Hamiltonian of the system to be

\[
H = \sum_a \left[ \left( p_a - \frac{e_a}{c} A(r_a) \right) \cdot \dot{v}_a - \frac{1}{2} m_a v_a^2 + e_a \phi(r_a) \right] \\
+ \frac{1}{4\pi} \int (dr) \left[ E \cdot \nabla \phi + B \cdot \nabla \times A + \frac{1}{2} (E^2 - B^2) \right], \tag{89}
\]

which is a constant of the motion, \( dH/dt = 0 \). The generators are inferred from the total time derivative terms in \((83a), (87c), \text{ and } (88)\),

\[
\delta W_{12} = G_1 - G_2, \tag{90a}
\]

to be

\[
G = \sum_a \delta r_a \cdot p_a - \frac{1}{4\pi c} \int (dr) E \cdot \delta A - H\delta t. \tag{90b}
\]

### 3.3 Energy

Notice that the total Lagrangian \((81)\) can be presented as

\[
L = \sum_a P_a \cdot \frac{dr_a}{dt} - \frac{1}{4\pi c} \int (dr) E \cdot \frac{\partial}{\partial t} A - H, \tag{91}
\]
where the Hamiltonian is given by (89). The narrower, Hamiltonian, description is reached by eliminating all variables that do not obey equations of motion, and, correspondingly, do not appear in $G$. Those “superfluous” variables are the $v_a$ and the fields $\phi$ and $B$, which are eliminated by using (84b), (87b), and (87h), the equations without time derivatives, resulting, first, in the intermediate form

$$H = \sum_a \left( \frac{1}{2m_a} \left( p_a - \frac{e_a}{c} A(r_a) \right)^2 + e_a \phi(r_a) \right) + \int (dr) \frac{E^2 + B^2}{8\pi} \rho \phi.$$ (92)

The first term here is the energy of the particles moving in the field [particle energy—see (77b)], so we might call the second term the field energy. The ambiguity of these terms (whether the potential energy of particles is attributed to them or to the fields, or to both) is evident from the existence of a simpler form of the Hamiltonian

$$H = \sum_a \frac{1}{2m_a} \left( p_a - \frac{e_a}{c} A(r_a) \right)^2 + \int (dr) \frac{E^2 + B^2}{8\pi}.$$ (93)

where we have used the equivalence of the two terms involving $\phi$, given in (86).

This apparently startling result suggests that the scalar potential has disappeared from the dynamical description. But, in fact, it has not. If we vary the Lagrangian (91), where $H$ is given by (93), with respect to $E$ we find

$$\delta L = -\frac{1}{4\pi} \int (dr) \delta E \cdot \left( \frac{1}{c} \frac{\partial}{\partial t} A + E \right) = 0.$$ (94)

Do we conclude that $\frac{1}{c} \frac{\partial}{\partial t} A + E = 0$? That would be true if the $\delta E(r,t)$ were arbitrary. They are not; $E$ is subject to the restriction—the constraint—(87b), which means that any change in $E$ must obey

$$\nabla \cdot \delta E = 0.$$ (95)

The proper conclusion is that the vector multiplying $\delta E$ in (94) is the gradient of a scalar function, just as in (87b),

$$\frac{1}{c} \frac{\partial}{\partial t} A + E = -\nabla \phi,$$ (96)

for that leads to

$$\delta L = -\frac{1}{4\pi} \int (dr) (\nabla \cdot \delta E) \phi = 0,$$ (97)

as required.

The fact that the energy is conserved,

$$\frac{dH}{dt} = 0,$$ (98)

where

$$H = \sum_a \frac{1}{2} m_a v_a^2 + \int (dr) U, \quad U = \frac{E^2 + B^2}{8\pi},$$ (99)

is a simple sum of particle kinetic energy and integrated field energy density, can be verified directly by taking the time derivative of (92). The time rate of change of the particle energy is computed directly:

$$\frac{d}{dt} \sum_a \left( \frac{1}{2} m_a v_a^2 + e_a \phi(r_a) \right) = \sum_a \frac{\partial}{\partial t} \left( e_a \phi(r_a) - \frac{e_a}{c} v_a \cdot A(r_a) \right).$$ (100)
We can compute the time derivative of the field energy by using the equation of energy conservation,
\[ \frac{d}{dt} \int (dr) U = - \int (dr) j \cdot E, \] (101)
to be
\[ \frac{d}{dt} \int (dr) \left( \frac{E^2 + B^2}{8\pi} - \rho \phi \right) = \int (dr) \left[ -j \cdot E - \phi \frac{\partial}{\partial t} \rho - \rho \frac{\partial}{\partial t} \phi \right] = - \int (dr) \left[ \frac{\rho}{\partial t} \phi - \frac{1}{c^2} j \cdot \frac{\partial}{\partial t} A \right] = - \sum_a e_a \left( \frac{\partial}{\partial t} \phi(r_a) - \frac{1}{c^2} \nabla_a \cdot \frac{\partial}{\partial t} A(r_a) \right). \] (102)

Here we have used (87f), and have noted that
\[ \int (dr) \left[ j \cdot \nabla \phi - \phi \frac{\partial}{\partial t} \rho \right] = 0 \] (103)
by charge conservation. Observe that (100) and (102) are equal in magnitude and opposite in sign, so that their sum is zero. This proves the statement of energy conservation (98).

### 3.4 Momentum and Angular Momentum Conservation

The action principle not only provides us with the field equations, particle equations of motion, and expressions for the energy, but also with the generators (90b). The generators provide a connection between conservation laws and invariances of the action (recall Section 2.4). Here we will further illustrate this connection by deriving momentum and angular momentum conservation from the invariance of the action under rigid coordinate translations and rotations, respectively. [In a similar way we could derive energy conservation, (98), from the invariance under time displacements—see also Section 3.6].

Under an infinitesimal rigid coordinate displacement, \( \delta \epsilon \), a given point which is described by \( r \) in the old coordinate system is described by \( r + \delta \epsilon \) in the new one. (See Fig. 3) The response of the particle term in (90b) is simple: \( \delta \epsilon \cdot \sum_a p_a \); for the field part, we require the change, \( \delta A \), of the vector potential induced by the rigid coordinate displacement. The value of a field \( F \) at a physical point \( P \) is unchanged under such a displacement, so that if \( r \) and \( r + \delta \epsilon \) are the coordinates of \( P \) in the two frames, there are corresponding functions \( F \) and \( \overline{F} \) of \( r \) and \( r + \delta \epsilon \) such that
\[ F(P) = F(r) = \overline{F}(r + \delta \epsilon), \] (104)
that is, the new function \( \overline{F} \) of the new coordinate equals the old function \( F \) of the old coordinate. The change in the function \( F \) at the same coordinate is given by
\[ \overline{F}(r) = F(r) + \delta F(r), \] (105)
so that
\[ \delta F(r) = F(r - \delta \epsilon) - F(r) = -\delta \epsilon \cdot \nabla F(r), \] (106)
for a rigid translation (not a rotation).
As an example, consider the charge density

$$\rho(r) = \sum a e_a \delta(r - r_a). \quad (107)$$

If the positions of all the particles, the $r_a$, are displaced by $\delta \epsilon$, the charge density changes to

$$\rho(r) + \delta \rho(r) = \sum a e_a \delta(r - r_a - \delta \epsilon), \quad (108)$$

where

$$\delta(r - r_a - \delta \epsilon) = \delta(r - r_a) - \delta \epsilon \cdot \nabla_{r_a} \delta(r - r_a), \quad (109)$$

and therefore

$$\delta \rho(r) = -\delta \epsilon \cdot \nabla \rho(r), \quad (110)$$
in agreement with (106).

So the field part of $G$ in (90b) is

$$- \int (dr) \frac{1}{4\pi c} E \cdot \delta A = \frac{1}{4\pi c} \int (dr) E_i (\delta \epsilon \cdot \nabla) A_i = -\frac{1}{c} \sum_a e_a \delta \epsilon \cdot A(r_a) + \frac{1}{4\pi c} \int (dr) \mathbf{E} \times \mathbf{B} \cdot \delta \epsilon, \quad (111)$$

where the last rearrangement makes use of (87d) and (87h), and the vector identity

$$\delta \epsilon \times (\nabla \times \mathbf{A}) = \nabla (\delta \epsilon \cdot \mathbf{A}) - (\delta \epsilon \cdot \nabla) \mathbf{A}. \quad (112)$$

Including the particle part from (90b), we find the generator corresponding to a rigid coordinate displacement can be written as

$$G = \delta \epsilon \cdot \mathbf{P}, \quad (113)$$

where

$$\mathbf{P} = \sum_a \left( p_a - \frac{e_a}{c} A(r_a) \right) + \frac{1}{4\pi c} \int (dr) \mathbf{E} \times \mathbf{B} \equiv \sum_a m_a v_a + \int (dr) \mathbf{G}, \quad (114)$$
with \( G \) the momentum density. Since the action is invariant under a rigid displacement,

\[
0 = \delta W = G_1 - G_2 = (P_1 - P_2) \cdot \delta r,
\]

we see that

\[
P_1 = P_2,
\]

that is, the total momentum, \( P \), is conserved. This, of course, can also be verified by explicit calculation:

\[
\frac{d}{dt} \int (dr) \frac{1}{4\pi c} \mathbf{E} \times \mathbf{B} = - \int (dr) \left[ \rho \mathbf{E} + \frac{1}{c} \mathbf{j} \times \mathbf{B} \right] = - \sum_a e_a \left( \mathbf{E}(r_a) + \frac{1}{c} \mathbf{v}_a \times \mathbf{B}(r_a) \right),
\]

from which the constancy of \( P \) follows.

Similar arguments can be carried out for a rigid rotation for which the change in the coordinate vector is

\[
\delta r = \delta \omega \times r,
\]

with \( \delta \omega \) constant. The corresponding change in a vector function is

\[
\overline{A}(r + \delta r) = A(r) + \delta \omega \times A(r)
\]

since a vector transforms in the same way as \( r \), so the new function at the initial numerical values of the coordinates is

\[
\overline{A}(r) = A(r) - (\delta r \cdot \nabla)A(r) + \delta \omega \times A(r).
\]

The change in the vector potential is

\[
\delta A = -(\delta r \cdot \nabla)A + \delta \omega \times A.
\]

The generator can now be written in the form

\[
G = \delta \omega \cdot \mathbf{J},
\]

where the total angular momentum, \( \mathbf{J} \), is found to be

\[
\mathbf{J} = \sum_a r_a \times m_a \mathbf{v}_a + \int (dr) r \times \left( \frac{1}{4\pi c} \mathbf{E} \times \mathbf{B} \right),
\]

which again is a constant of the motion.

### 3.5 Gauge Invariance and the Conservation of Charge

An electromagnetic system possesses a conservation law, that of electric charge, which has no place in the usual mechanical framework. It is connected to a further invariance of the electromagnetic fields—the potentials are not uniquely defined in that if we let

\[
\mathbf{A} \rightarrow \mathbf{A} + \nabla \lambda, \quad \phi \rightarrow \phi - \frac{1}{c} \frac{\partial}{\partial t} \lambda,
\]

the electric and magnetic fields defined by (87f) and (87h) remain unaltered, for an arbitrary function \( \lambda \). This is called gauge invariance; the corresponding substitution
Equation (124) is a gauge transformation. The term has its origin in a now obsolete theory of Hermann Weyl (1885–1955) \cite{Weyl1919}.

This invariance of the action must imply a corresponding conservation law. To determine what is conserved, we compute the change in the Lagrangian, \( (124) \), explicitly. Trivially, the field part of \( L \) remains unchanged. In considering the change of the particle part, we recognize that \( (124) \) is incomplete; since \( \mathbf{v} \) is a physical quantity, \( \mathbf{p} - \left( \frac{e}{c} \right) \mathbf{A} \) must be invariant under a gauge transformation, which will only be true if \( (124) \) is supplemented by

\[
\mathbf{p} \rightarrow \mathbf{p} + \frac{e}{c} \nabla \lambda.
\]

Under the transformation \( (124) \) and \( (125) \), the Lagrangian becomes

\[
L \rightarrow \overline{L} \equiv L + \sum_a \left( \frac{e_a}{c} \mathbf{v} \cdot \nabla \lambda \cdot \left( \frac{d\mathbf{r}_a}{dt} - \mathbf{v}_a \right) + \frac{e_a}{c} \frac{\partial}{\partial t} \lambda + \frac{e_a}{c} \mathbf{v}_a \cdot \nabla \lambda \right)
\]

\[
= L + \sum_a \left( \frac{e_a}{c} \frac{\partial}{\partial t} \lambda + \frac{d\mathbf{r}_a}{dt} \cdot \nabla \lambda \right)
\]

\[
= L + \frac{d}{dt} w,
\]

where

\[
w = \sum_a \frac{e_a}{c} \lambda (\mathbf{r}_a, t).
\]

What is the physical consequence of adding a total time derivative to a Lagrangian? It does not change the equations of motion, so the system is unaltered. Since the entire change is in the end point behavior,

\[
\overline{W}_{12} = W_{12} + (w_1 - w_2),
\]

the whole effect is a redefinition of the generators, \( G \),

\[
\overline{G} = G + \delta w.
\]

This alteration reflects the fact that the Lagrangian itself is ambiguous up to a total time derivative term.

To ascertain the implication of gauge invariance, we rewrite the change in the Lagrangian given in the first line of \( (126) \) by use of \( (84c) \),

\[
\overline{L} - L = \frac{1}{c} \int (d\mathbf{r}) \left( \frac{\partial}{\partial t} \lambda + \nabla \cdot \mathbf{j} \right),
\]

and apply this result to an infinitesimal gauge transformation, \( \lambda \rightarrow \delta \lambda \). The change in the action is then

\[
\delta W_{12} = G_{\delta \lambda_1} - G_{\delta \lambda_2} = \int_{t_2}^{t_1} dt \int (d\mathbf{r}) \frac{1}{c} \delta \lambda \left( \frac{\partial}{\partial t} \rho + \nabla \cdot \mathbf{j} \right),
\]

with the generator being

\[
G_{\delta \lambda} = \int (d\mathbf{r}) \frac{1}{c} \rho \delta \lambda.
\]

In view of the arbitrary nature of \( \delta \lambda (\mathbf{r}, t) \), the stationary action principle now demands that, at every point,

\[
\frac{\partial}{\partial t} \rho + \nabla \cdot \mathbf{j} = 0,
\]
that is, gauge invariance implies local charge conservation. (Of course, this same result follows from Maxwell’s equations.) Then, the special situation \( \delta \lambda = \text{constant} \), where \( \delta A = \delta \phi = 0 \), and \( W_{12} \) is certainly invariant, implies a conservation law, that of

\[
G_{\delta \lambda} = \frac{1}{c} \delta \lambda Q,
\]

in which

\[
Q = \int (d\mathbf{r}) \rho
\]

is the conserved total charge.

### 3.6 Gauge Invariance and Local Conservation Laws

We have just derived the local conservation law of electric charge. Electric charge is a property carried only by the particles, not by the electromagnetic field. In contrast, the mechanical properties of energy, linear momentum, and angular momentum are attributes of both particles and fields. For these we have conservation laws of total quantities. What about local conservation laws? The usual development of electrodynamics refers to local non-conservation laws; they concentrated on the fields and characterized the charged particles as sources (or sinks) of field mechanical properties. It is natural to ask for a more even-handed treatment of both charges and fields. We shall supply it, in the framework of a particular example. The property of gauge invariance will be both a valuable guide, and an aid to simplifying the calculations.

The time displacement of a complete physical system identifies its total energy. This suggests that time displacement of a part of the system provides energetic information about that portion. The ultimate limit of this spatial subdivision, a local description, should appear in response to an (infinitesimal) time displacement that varies arbitrarily in space as well as in time, \( \delta t(\mathbf{r}, t) \).

Now we need a clue. How do fields, and potentials, respond to such coordinate-dependent displacements? This is where the freedom of gauge transformations enters: The change of the vector and scalar potentials, by \( \nabla \lambda(\mathbf{r}, t), -1/c(\partial/\partial t)\lambda(\mathbf{r}, t) \), respectively, serves as a model for the potentials themselves. The advantage here is that the response of the scalar \( \lambda(\mathbf{r}, t) \) to the time displacement can be reasonably taken to be

\[
(\lambda + \delta \lambda)(\mathbf{r}, t + \delta t) = \lambda(\mathbf{r}, t),
\]

or

\[
\delta \lambda(\mathbf{r}, t) = -\delta t(\mathbf{r}, t) \frac{\partial}{\partial t} \lambda(\mathbf{r}, t).
\]

Then we derive

\[
\delta(\nabla \lambda) = -\delta t \frac{\partial}{\partial t} (\nabla \lambda) + \left(-\frac{1}{c} \frac{\partial}{\partial t} \lambda \right) c \nabla \delta t,
\]

\[
\delta \left( -\frac{1}{c} \frac{\partial}{\partial t} \lambda \right) = -\delta t \left( -\frac{1}{c} \frac{\partial^2}{\partial t^2} \lambda \right) - \left(-\frac{1}{c} \frac{\partial}{\partial t} \lambda \right) \frac{\partial}{\partial t} \delta t,
\]

which is immediately generalized to

\[
\delta A = -\delta t \frac{\partial}{\partial t} A + \phi c \nabla \delta t,
\]

\[
\delta \phi = -\delta t \frac{\partial}{\partial t} \phi - \frac{\partial}{\partial t} \delta t,
\]
or, equivalently,
\[
\delta \mathbf{A} = c \delta t \mathbf{E} + \nabla (\phi c \delta t),
\]
\[
\delta \phi = -\frac{1}{c} \frac{\partial}{\partial t} (\phi c \delta t).
\]
(139a)

In the latter form we recognize a gauge transformation, produced by the scalar \(\phi c \delta t\), which will not contribute to the changes of field strengths. Accordingly, for that calculation we have, effectively, \(\delta \mathbf{A} = c \delta t \mathbf{E}\), \(\delta \phi = 0\), leading to
\[
\delta \mathbf{E} = -\frac{1}{c} \frac{\partial}{\partial t} (c \delta t \mathbf{E}) = -\delta t \frac{\partial}{\partial t} \mathbf{E} - \mathbf{E} \frac{\partial}{\partial t} \delta t,
\]
(140a)
\[
\delta \mathbf{B} = \nabla \times (c \delta t \mathbf{E}) = -\delta t \frac{\partial}{\partial t} \mathbf{B} - \mathbf{E} \times \nabla c \delta t;
\]
(140b)

the last line employs the field equation \(\nabla \times \mathbf{E} = -(1/c)(\partial \mathbf{B}/\partial t)\).

In the following we adopt a viewpoint in which such homogeneous field equations are accepted as consequences of the definition of the fields in terms of potentials. That permits the field Lagrange function (81) to be simplified:
\[
L_{\text{field}} = \frac{1}{8\pi} (E^2 - B^2).
\]
(141)

Then we can apply the field variation (140b) directly, and get
\[
\delta L_{\text{field}} = -\delta t \frac{\partial}{\partial t} L_{\text{field}} - \frac{1}{4\pi} E^2 \frac{\partial}{\partial t} \delta t - c \frac{\partial}{\partial t} \mathbf{E} \cdot \nabla \delta t
\]
\[= -\frac{\partial}{\partial t} (\delta t L_{\text{field}}) - \frac{1}{8\pi} (E^2 + B^2) \frac{\partial}{\partial t} \delta t - c \frac{\partial}{\partial t} \mathbf{E} \cdot \nabla \delta t.
\]
(142)

Before commenting on these last, not unfamiliar, field structures, we turn to the charged particles and put them on a somewhat similar footing in terms of a continuous, rather than a discrete, description.

We therefore present the Lagrangian of the charges in (81) in terms of a corresponding Lagrange function,
\[
L_{\text{charges}} = \int (dr) L_{\text{charges}},
\]
where
\[
L_{\text{charges}} = \sum_a L_a
\]
(143b)
and
\[
L_a = \delta (r - r_a(t)) \left[ \frac{1}{2} m_a v_a(t)^2 - e_a \phi(r_a, t) + \frac{e_a}{c} v_a(t) \cdot A(r_a, t) \right];
\]
(143c)
the latter adopts the Lagrangian viewpoint, with \(v_a = dr_a/dt\) accepted as a definition.

Then, the effect of the time displacement on the variables \(r_a(t)\), taken as
\[
(r_a + \delta r_a)(t + \delta t) = r_a(t),
\]
\[
\delta r_a(t) = -\delta t (r_a, t) v_a(t),
\]
(144a)
implies the velocity variation
\[
\delta v_a(t) = -\delta t (r_a, t) \frac{d}{dt} v_a(t) - v_a(t) \left[ \frac{\partial}{\partial t} \delta t + v_a \cdot \nabla \delta t \right];
\]
(145)
the last step exhibits both the explicit and the implicit dependences of $\delta t(r_a, t)$ on $t$. In computing the variation of $\phi (r_a, t)$, for example, we combine the potential variation given in (138b) with the effect of $\delta r_a$:

$$\delta \phi (r_a(t), t) = -\delta t \frac{\partial}{\partial t} \phi - \phi \frac{\partial}{\partial t} \delta t - \delta t v_a \cdot \nabla_a \phi = -\delta t \frac{d}{dt} \phi - \phi \frac{\partial}{\partial t} \delta t, \quad (146a)$$

and, similarly,

$$\delta A (r_a(t), t) = -\delta t \frac{\partial}{\partial t} A + \phi c \nabla \delta t - \delta t v_a \cdot \nabla_a A = -\delta t \frac{d}{dt} A + \phi c \nabla \delta t. \quad (146b)$$

The total effect of these variations on $L_a$ is thus

$$\delta L_a = -\delta t \frac{d}{dt} L_a + \delta (r - r_a(t)) \left( -m_a v_a^2 - \frac{e_a}{c} A \cdot v_a + e_a \phi \right) \left( \frac{\partial}{\partial t} \delta t + v_a \cdot \nabla \delta t \right), \quad (147a)$$

or

$$\delta L_a = -\frac{d}{dt} (\delta t L_a) - \delta (r - r_a(t)) E_a \left( \frac{\partial}{\partial t} \delta t + v_a \cdot \nabla \delta t \right), \quad (147b)$$

where we see the kinetic energy of the charged particle,

$$E_a = \frac{1}{2} m_a v_a^2. \quad (148)$$

We have retained the particle symbol $d/dt$ to the last, but now, being firmly back in the field, space-time viewpoint, it should be written as $\partial/\partial t$, referring to all $t$ dependence, with $r$ being held fixed. The union of these various contributions to the variation of the total Lagrange function is

$$\delta L_{\text{tot}} = \frac{\partial}{\partial t} (\delta t L_{\text{tot}}) - U_{\text{tot}} \frac{\partial}{\partial t} \delta t - S_{\text{tot}} \cdot \nabla \delta t, \quad (149)$$

where, from (132) and (147b),

$$U_{\text{tot}} = \frac{1}{8\pi} (E^2 + B^2) + \sum_a \delta (r - r_a(t)) E_a \quad (150a)$$

and

$$S_{\text{tot}} = \frac{c}{4\pi} E \times B + \sum_a \delta (r - r_a(t)) E_a v_a, \quad (150b)$$

are physically transparent forms for the total energy density and total energy flux vector.

To focus on what is new in this development, we ignore boundary effects in the stationary action principle, by setting the otherwise arbitrary $\delta t(r, t)$ equal to zero at $t_1$ and $t_2$. Then, through partial integration, we conclude that

$$\delta W_{12} = \int_{t_2}^{t_1} dt \int (dr) \delta t \left( \frac{\partial}{\partial t} U_{\text{tot}} + \nabla \cdot S_{\text{tot}} \right) = 0, \quad (151)$$

from which follows the local statement of total energy conservation,

$$\frac{\partial}{\partial t} U_{\text{tot}} + \nabla \cdot S_{\text{tot}} = 0. \quad (152)$$
4 Quantum Action Principle

This section, and the following three, are based on lectures given by the author in quantum field theory courses at the University of Oklahoma over several years, based in turn largely on lectures given by Schwinger at Harvard in the late 1960s.

After the above reminder of classical variational principles, we now turn to the dynamics of quantum mechanics. We begin by considering the transformation function \( \langle a', t + dt | b', t \rangle \). Here \( | b', t \rangle \) is a state specified by the values \( b' = \{ b' \} \) of a complete set of dynamical variables \( B(t) \), while \( | a', t + dt \rangle \) is a state specified by values \( a' = \{ a' \} \) of a (different) complete set of dynamical variables \( A(t + dt) \), defined at a slightly later time.\(^1\) We suppose that \( A \) and \( B \) do not possess any explicit time dependence—that is, their definition does not depend upon \( t \).

We make a negligible error of \( \mathcal{O} \) \( \{a \} \) system defined by coordinates and momenta, structure of \( H \) is, their definition does not depend upon \( t \) where \( \delta \chi \) where \( \chi \) do not possess any explicit time dependence—that is, their definition does not depend upon \( t \).

In turn largely on lectures given by Schwinger at Harvard in the late 1960s.

\[ \langle a', t + dt | b', t \rangle = \langle a', t | 1 - i dt H(\chi(t), t)| b', t \rangle. \] (155)

We next translate states and operators to time zero:

\[ \langle a', t | = \langle a' | U(t) \rangle, \quad | b', t \rangle = U^{-1}(t) | b' \rangle, \]

\[ \chi(t) = U^{-1}(t) \chi U(t), \] (156a)

where \( \chi = \chi(0) \), etc. Then,

\[ \langle a', t + dt | b', t \rangle = \langle a' | 1 - i dt H(\chi, t) | b' \rangle, \] (157)

or, as a differential equation

\[ \delta_{\text{dyn}} \langle a', t + dt | b', t \rangle = i \langle a' | [\delta_{\text{dyn}} - dt H] | b' \rangle \]

\[ = i \langle a', t + dt | \delta_{\text{dyn}} - dt H(\chi(t), t) | b', t \rangle, \] (158)

where \( \delta_{\text{dyn}} \) corresponds to changes in initial and final times, \( \delta t_2 \) and \( \delta t_1 \), and in the structure of \( H, \delta H \).\(^\text{[1]}\) By reintroducing \( dt \) in the state on the left in the second line, we make a negligible error of \( \mathcal{O}(dt^2) \).

However, we can also consider kinematical changes. To understand these, consider a system defined by coordinates and momenta, \( \{ q_a(t) \}, \{ p_a(t) \}, a = 1, \ldots, n \), which satisfy the canonical commutation relations,

\[ [q_a(t), p_b(t)] = i \delta_{ab}, \quad (\hbar = 1) \] (159a)

\[ [q_a(t), q_b(t)] = [p_a(t), p_b(t)] = 0. \] (159b)

A spatial displacement \( \delta q_a \) is induced by

\[ U = 1 + i G_q, \quad G_q = \sum_{a=1}^n p_a \delta q_a. \] (160)

\(^{1}\) Here Schwinger is using his standard notation, designating eigenvalues by primes.
In fact ($\delta q_a$ is a number, not an operator),

$$U^{-1}q_aU = q_a - \frac{1}{i}[q_a, G_q]$$

$$= q_a - \delta q_a,$$

(161)

while

$$U^{-1}p_aU = p_a - \frac{1}{i}[p_a, G_q] = p_a.$$ (162)

The (dual) symmetry between position and momentum,

$$q \rightarrow p, \quad p \rightarrow -q,$$ (163)

gives us the form for the generator of a displacement in $p$:

$$G_p = -\sum_a {q_a} \delta p_a.$$ (164)

A kinematic variation in the states is given by the generators

$$\delta_{\text{kin}} | \rangle = \langle | - \langle | \delta G,$$

$$\delta_{\text{kin}} \rangle = | \rangle - | \rangle = -iG | \rangle,$$ (165a, 165b)

so, for example, under a $\delta q$ variation, the transformation function changes by

$$\delta_q \langle a', t + dt | b', t \rangle = i \langle a', t + dt | \sum_a [p_a(t + dt)\delta q_a(t + dt) - p_a(t)\delta q_a(t)] | b', t \rangle.$$ (166)

Now the dynamical variables at different times are related by Hamilton’s equations,

$$\frac{dp_a(t)}{dt} = \frac{1}{i} [p_a(t), H(q(t), p(t), t)]$$

$$= -\frac{\partial H}{\partial q_a}(t),$$

(167)

so

$$p_a(t + dt) - p_a(t) = dt \frac{dp_a(t)}{dt} = -dt \frac{\partial H}{\partial q_a}(t).$$ (168)

Similarly, the other Hamilton’s equation

$$\frac{dq_a}{dt} = \frac{\partial H}{\partial p_a}$$ (169)

implies that

$$q_a(t + dt) - q_a(t) = dt \frac{\partial H}{\partial p_a}(t).$$ (170)

From this we deduce first the $q$ variation of the transformation function,

$$\delta_q \langle a', t + dt | b', t \rangle$$

$$= i \langle a', t + dt | \sum_a p_a(t) [\delta q_a(t + dt) - \delta q_a(t)] - dt \frac{\partial H}{\partial q_a} \delta q_a(t) + O(dt^2) | b', t \rangle$$

$$= i \langle a', t + dt | \delta q \left[ \sum_a p_a(t) [q_a(t + dt) - q_a(t)] - dt H(q(t), p(t), t) \right] | b', t \rangle,$$ (171)
where the dot denotes symmetric multiplication of the $p$ and $q$ operators.

For $p$ variations we have a similar result:

$$\delta_p \langle a', t + dt | b', t \rangle = i \langle a', t + dt | \sum a q_a(t) \frac{\delta}{\delta p_a(t)} (p_a(t) - p_a(t + dt)) \delta p_a(t) | b', t \rangle.$$

That is, for $q$ variations

$$\delta_q \langle a', t + dt | b', t \rangle = i \langle a', t + dt | \frac{\delta}{\delta q} [dt L_q] | b', t \rangle,$$

with the quantum Lagrangian

$$L_q = \sum a p_a \dot{q}_a - H(q, p, t),$$

while for $p$ variations

$$\delta_p \langle a', t + dt | b', t \rangle = i \langle a', t + dt | \sum a q_a(t) \frac{\delta}{\delta p_a(t)} (p_a(t) - p_a(t + dt)) - dt H(q(t), p(t), t) | b', t \rangle,$$

with the quantum Lagrangian

$$L_p = -\sum a q_a \dot{p}_a - H(q, p, t).$$

We see here two alternative forms of the quantum Lagrangian. Note that the two forms differ by a total time derivative,

$$L_q - L_p = \frac{d}{dt} \sum a p_a q_a.$$

We now can unite the kinematic transformations considered here with the dynamic ones considered earlier, in Eq. (158):

$$\delta = \delta_{\text{dyn}} + \delta_{\text{kin}}: \quad \delta (a', t + dt | b', dt) = i \langle a', t + dt | \delta [dt L] | b', t \rangle.$$}

Suppose, for concreteness, that our states are defined by values of $q$, so that

$$\delta_p \langle a', t + dt | b', t \rangle = 0.$$}

This is consistent, as a result of Hamilton’s equations,

$$\delta_p L_q = \sum a \delta p_a \left( \dot{q}_a - \frac{\partial H}{\partial p_a} \right) = 0.$$}

In the following we will use $L_q$. 


It is immediately clear that we can iterate the infinitesimal version (176) of the quantum action principle by inserting at each time step a complete set of intermediate states (to simplify the notation, we ignore their quantum numbers):

$$\langle t_1 | t_2 \rangle = \langle t_1 | t_1 - dt \rangle \langle t_1 - dt | t_1 - 2dt \rangle \cdots \langle t_2 + dt | t_2 \rangle \langle t_2 + 2dt | t_2 \rangle,$$

(179)

So in this way we deduce the general form of Schwinger’s quantum action principle:

$$\delta \langle t_1 | t_2 \rangle = i \langle t_1 | \delta \int_{t_2}^{t_1} dt L | t_2 \rangle.$$

(180)

This summarizes all the properties of the system.

Suppose the dynamical system is given, that is, the structure of $H$ does not change. Then

$$\delta \langle t_1 | t_2 \rangle = i \langle t_1 | G_1 - G_2 | t_2 \rangle,$$

(181)

where the generator $G_a$ depends on $p$ and $q$ at time $t_a$. Comparing with the action principle (180) we see

$$\delta \int_{t_2}^{t_1} dt L = G_1 - G_2,$$

(182)

which has exactly the form of the classical action principle (3), except that the Lagrangian $L$ and the generators $G$ are now operators. If no changes occur at the endpoints, we have the principle of stationary action,

$$\delta \int_{t_2}^{t_1} \left( \sum_a p_a \delta q_a - H dt \right) = 0.$$  

(183)

As in the classical case, let us introduce a time parameter $\tau$, $t = t(\tau)$, such that $\tau_2$ and $\tau_1$ are fixed. The the above variation reads

$$\sum_a [\delta p_a \delta q_a + p_a d \delta q_a - \delta H dt - H d \delta t]$$

$$= d \left[ \sum_a p_a \delta q_a - H \delta t \right] + \sum_a [\delta p_a \delta q_a - dp_a \delta q_a] - \delta H dt + dH \delta t,$$

(184)

so the action principle says

$$G = \sum_a p_a \delta q_a - H \delta t,$$

(185a)

$$\delta H = \frac{dH}{dt} \delta t + \sum_a \left( \delta p_a \frac{dq_a}{dt} - \delta q_a \frac{dp_a}{dt} \right).$$

(185b)

We will again assume $\delta p_a$, $\delta q_a$ are not operators (that is, they are proportional to the unit operator); then we recover Hamilton’s equations,

$$\frac{\partial H}{\partial t} = \frac{dH}{dt},$$

(186a)

$$\frac{\partial H}{\partial p_a} = \frac{dq_a}{dt},$$

(186b)

$$\frac{\partial H}{\partial q_a} = -\frac{dp_a}{dt}.$$

(186c)
(Schwinger also explored the possibility of operator variations [Schwinger 1970].) We learn from the generators,

\[ G_t = -H \delta t, \quad G_q = \sum_a p_a \delta q_a, \]  

(187)

that the change in some function \( F \) of the dynamical variable is

\[ \delta F = \frac{dF}{dt} \delta t + \frac{1}{i} [F, G], \]  

(188)

so we deduce

\[ \frac{dF}{dt} = \frac{\partial F}{\partial t} + \frac{1}{i} [F, H], \]  

(189a)

\[ \frac{\partial F}{\partial q_a} = \frac{1}{i} [F, p_a]. \]  

(189b)

Note that from this the canonical commutation relations follow,

\[ [q_a, p_b] = i \delta_{ab}, \quad [p_a, p_b] = 0, \]  

(190)

as well as Newton’s law,

\[ \dot{p}_a = -\frac{1}{i} [H, p_a] = -\frac{\partial H}{\partial q_a}. \]  

(191)

If we had used \( L_p \) instead of \( L_q \), we would have obtained the same equations of motion, but in place of \( G_q \), we would have obtained

\[ G_p = -\sum_a q_a \delta p_a, \]  

(192)

which implies

\[ \frac{\partial F}{\partial p_a} = -\frac{1}{i} [F, q_a]. \]  

(193)

From this can be deduced the remaining canonical commutator,

\[ [q_a, q_b] = 0, \]  

(194)

as well as the remaining Hamilton equation,

\[ \dot{q}_a = \frac{1}{i} [q_a, H] = \frac{\partial H}{\partial p_a}. \]  

(195)

It is easy to show that the effect of changing the Lagrangian by a total time derivative (which is what is done in passing from \( L_q \) to \( L_p \)) is to change the generators.

We now turn to examples.

5 Harmonic Oscillator

The harmonic oscillator is defined in terms of creation and annihilation operators\(^2\), \( y^\dagger \) and \( y \), and the corresponding Hamiltonian \( H \),

\[ [y, y^\dagger] = 1, \]  

(196a)

\[ H = \omega \left( y^\dagger y + \frac{1}{2} \right). \]  

(196b)

\(^2\) We follow Schwinger’s usage of \( y \) for the annihilation operator, instead of the more usual \( a \).
The equations of motion are
\[ \frac{dy}{dt} = \frac{1}{i} [y, H] = \frac{1}{i} \omega y, \] \hspace{1cm} (197a)
\[ \frac{dy^\dagger}{dt} = \frac{1}{i} [y^\dagger, H] = -\frac{1}{i} \omega y^\dagger. \] \hspace{1cm} (197b)

Eigenstates of \( y \) and \( y^\dagger \) exist, as right and left vectors, respectively,
\[ y|y\rangle = y'|y\rangle, \] \hspace{1cm} (198a)
\[ \langle y^\dagger|y^\dagger = y^\dagger\langle y^\dagger|, \] \hspace{1cm} (198b)
while \( \langle y' | \) and \( |y'^\dagger \rangle \) do not exist. These are the famous “coherent states,” to whom the name Roy Glauber [Glauber 1963] is invariably attached, although they were discovered by Erwin Schrödinger [Schrödinger 1926], and Glauber’s approach, as he acknowledged, followed that of his mentor, Schwinger [Schwinger 1953].

The transformation function we seek is therefore
\[ \langle y^\dagger', t_1 | y'', t_2 \rangle. \] \hspace{1cm} (200)

If we regard \( y \) as a “coordinate,” the corresponding “momentum” is \( iy^\dagger \):
\[ \dot{y} = \frac{1}{i} \omega y = \frac{\partial H}{\partial iy^\dagger}, \quad iy^\dagger = -\omega y^\dagger = -\frac{\partial H}{\partial y}. \] \hspace{1cm} (201)

The corresponding Lagrangian is therefore
\[ L = iy^\dagger \dot{y} - H. \] \hspace{1cm} (203)

Because we use \( y \) as our state variable at the initial time, and \( y^\dagger \) at the final time, we must exploit our freedom to redefine our generators to write
\[ W_{12} = \int_{t_2}^{t_1} dt L - iy^\dagger(t_1).y(t_1). \] \hspace{1cm} (204)

Then the variation of the action is
\[ \delta W_{12} = -i\delta(y^\dagger_1.y_1) + G_1 - G_2 \]
\[ = -i\delta y^\dagger_1.y_1 - iy^\dagger_1.\delta y_1 + iy_1.\delta y^\dagger_1 - iy_1.\delta y_2 - H \cdot \delta t_1 + H \cdot \delta t_2 \]
\[ = -i\delta y^\dagger_1.y_1 - iy_2^\dagger.\delta y_2 - H(\delta t_1 - \delta t_2). \] \hspace{1cm} (205)

Then the quantum action principle says
\[ \delta \langle y^\dagger', t_1 | y'', t_2 \rangle = i \langle y^\dagger', t_1 | -i\delta y^\dagger_1.y_1 - iy^\dagger_2.\delta y_2 - \omega y^\dagger_1.y_1(\delta t_1 - \delta t_2) | y'', t_2 \rangle, \] \hspace{1cm} (206)

\[ \text{If } \langle y' | y \rangle = y'\langle y'| \text{ then we would have an evident contradiction:} \]
\[ 1 = \langle y' || y, y' || y' \rangle = y' \langle y' |y|^y' \rangle - \langle y'y|y' \rangle y' = 0. \] \hspace{1cm} (199)

\[ \text{We might note that in terms of (dimensionless) position and momentum operators} \]
\[ iy^\dagger \dot{y} = \frac{i}{2}(q - ip).\dot{(q + ip)} = \frac{1}{2}(p.\dot{q} - q.\dot{p}) + \frac{i}{4} \frac{d}{dt}(q^2 + p^2), \] \hspace{1cm} (202)
where the first term in the final form is the average of the Legendre transforms in \( L_q \) and \( L_p \).
since by assumption the variations in the dynamical variables are numerical:

$$[\delta y_1^+, y_1] = [y_2^+, \delta y_2],$$

and we have dropped the zero-point energy. Now use the equations of motion \[197a\] and \[197b\] to deduce that

$$y_1 = e^{-i\omega(t_1-t_2)}y_2, \quad y_2^+ = e^{-i\omega(t_1-t_2)}y_1^+$$

and hence

$$\delta \langle y_1^+, t_1 | y_2^+, t_2 \rangle = \langle y_1^+, t_1 | \delta y_1^+ e^{-i\omega(t_1-t_2)} y_2^+ + y_1^+ e^{-i\omega(t_1-t_2)} \delta y_2^+ - i\omega y_1^+ e^{-i\omega(t_1-t_2)} (\delta t_1 - \delta t_2) y_2^+ | y_2^+, t_2 \rangle$$

$$= \langle y_1^+, t_1 | y_2^+, t_2 \rangle \delta \left[ y_1^+ e^{-i\omega(t_1-t_2)} y_2^+ \right].$$

From this we can deduce that the transformation function has the exponential form

$$\langle y_1^+, t_1 | y_2^+, t_2 \rangle = \exp \left[ y_1^+ e^{-i\omega(t_1-t_2)} y_2^+ \right],$$

which has the correct boundary condition at $t_1 = t_2$; and in particular, $\langle 0 | 0 \rangle = 1$.

On the other hand,

$$\langle y_1^+, t_1 | y_2^+, t_2 \rangle = \langle y_1^+ | e^{-iH(t_1-t_2)} | y_2^+ \rangle,$$

where both states are expressed at the common time $t_2$, so, upon inserting a complete set of energy eigenstates, we obtain ($t = t_1 - t_2$)

$$\sum_E \langle y_1^+, t_1 | E | y_2^+, t_2 \rangle e^{-iEt} \langle E | y_2^+ \rangle,$$

which we compare to the Taylor expansion of the previous formula,

$$\sum_{n=0}^{\infty} \frac{(y_1^+)^n}{\sqrt{n!}} e^{-in\omega t} \frac{(y_2^+)^n}{\sqrt{n!}}.$$

This gives all the eigenvectors and eigenvalues:

$$E_n = n\omega, \quad n = 0, 1, 2, \ldots,$$

$$\langle y_1^+ | E_n \rangle = \frac{(y_1^+)^n}{\sqrt{n!}},$$

$$\langle E_n | y_2^+ \rangle = \frac{(y_2^+)^n}{\sqrt{n!}}.$$

These correspond to the usual construction of the eigenstates from the ground state:

$$| E_n \rangle = \frac{(y_1^+)^n}{\sqrt{n!}} | 0 \rangle.$$
6 Forced Harmonic Oscillator

Now we add a driving term to the Hamiltonian,

\[ H = \omega y^\dagger y + y^\dagger K(t), \]  

(216)

where \( K(t) \) is an external force (\( Kraft \) is force in German). The equation of motion is

\[ \frac{d}{dt} \frac{\partial H}{\partial y^\dagger} = [y, H] = \omega y + K(t), \]

(217)

while \( y^\dagger \) satisfies the adjoint equation. In the presence of \( K(t) \), we wish to compute the transformation function \( \langle y^\dagger, t_1 | y''', t_2 \rangle^K \).

Consider a variation of \( K \). According to the action principle

\[ \delta_K \langle y^\dagger, t_1 | y''', t_2 \rangle^K = \langle y^\dagger, t_1 | \int_{t_1}^{t_2} dt [\delta K y^\dagger + \delta K^* y] | y''', t_2 \rangle^K. \]

(218)

We can solve this differential equation by noting that the equation of motion (217) can be rewritten as

\[ i \frac{d}{dt} [e^{i\omega t} y(t)] = e^{i\omega t} K(t), \]

(219)

which is integrated to read

\[ e^{i\omega t} y(t) - e^{i\omega t_2} y(t_2) = -i \int_{t_1}^{t} dt' e^{i\omega t'} K(t'), \]

(220)

or

\[ y(t) = e^{-i\omega(t-t_2)} y_2 - i \int_{t_1}^{t} dt' e^{-i\omega(t-t')} K(t'). \]

(221)

and the adjoint\(^5\)

\[ y^\dagger(t) = e^{-i\omega(t-t_2)} y_1^\dagger - i \int_{t}^{t_1} dt' e^{-i\omega(t'-t)} K^*(t'). \]

(224)

\(^5\) The consistency of these two equations follows from

\[ e^{i\omega t_1} y_1 = e^{i\omega t_2} y_2 - i \int_{t_2}^{t_1} dt' e^{i\omega t'} K(t'), \]

(222)

so that the adjoint of Eq. (221) is

\[ [y(t)]^\dagger = e^{i\omega t} \left[ e^{-i\omega t_1} y_1^\dagger - i \int_{t_2}^{t_1} dt' e^{-i\omega t'} K^*(t') \right] + i \int_{t_2}^{t} dt' e^{-i\omega(t'-t)} K^*(t') \]

\[ = e^{i\omega(t-t_1)} y_1^\dagger + i \int_{t_1}^{t} dt' e^{-i\omega(t'-t)} K^*(t'), \]

(223)

which is Eq. (224).
Thus our differential equation reads

\[
\frac{\delta_K \langle y^\dagger, t_1 | y'' \rangle}{\langle y^\dagger, t_1 | y'' \rangle} = \delta_K \ln \frac{\langle y^\dagger, t_1 | y'' \rangle}{\langle y^\dagger, t_1 | y'' \rangle} \\
= -i \int_{t_2}^{t_1} dt \, \delta K(t) \left[ y'' e^{-i\omega(t_1-t)} - i \int_t^{t_1} dt' e^{-i\omega(t'-t)} K^*(t') \right] \\
- i \int_{t_2}^{t_1} dt \, \delta K^*(t) \left[ e^{-i\omega(t-t_2)} y'' - i \int_t^{t_1} dt' e^{-i\omega(t'-t)} K(t') \right].
\]  

(225)

Notice that in the terms bilinear in \( K \) and \( K^* \), \( K \) always occurs earlier than \( K^* \). Therefore, these terms can be combined to read

\[
- \delta_K \int_{t_2}^{t_1} dt \, dt' \, K^*(t) \eta(t-t') e^{-i\omega(t-t')} K(t'),
\]

(226)

where the step function is

\[
\eta(t) = \begin{cases} 
1, & t > 0, \\
0, & t < 0.
\end{cases}
\]

(227)

Since we already know the \( K = 0 \) value from Eq. (240), we may now immediately integrate our differential equation:

\[
\langle y^\dagger, t_1 | y'' \rangle = \exp \left[ y'' e^{-i\omega(t_1-t_2)} \right] \\
- i y^\dagger \int_{t_2}^{t_1} dt \, e^{-i\omega(t_1-t)} K(t) - i \int_{t_2}^{t_1} dt \, e^{-i\omega(t-t_2)} K^*(t) y'' \\
- \int_{t_2}^{t_1} dt \, dt' \, K^*(t) \eta(t-t') e^{-i\omega(t-t')} K(t').
\]

(228)

The ground state is defined by \( y'' = y^\dagger = 0 \), so

\[
\langle 0, t_1 | 0, t_2 \rangle = \exp \left[ - \int_{-\infty}^{\infty} dt \, dt' \, K^*(t) \eta(t-t') e^{-i\omega(t-t')} K(t') \right],
\]

(229)

where we now suppose that the forces turn off at the initial and final times, \( t_2 \) and \( t_1 \), respectively.

A check of this result is obtained by computing the probability of the system remaining in the ground state:

\[
|\langle 0, t_1 | 0, t_2 \rangle|^2 = \exp \left\{ - \int_{-\infty}^{\infty} dt \, dt' \, K^*(t) e^{-i\omega(t-t')} [\eta(t-t') + \eta(t' - t)] K(t') \right\} \\
= \exp \left\{ - \int_{-\infty}^{\infty} dt \, dt' \, K^*(t) e^{-i\omega(t-t')} K(t') \right\} \\
= \exp \left\{ - |K(\omega)|^2 \right\},
\]

(230)

where the Fourier transform of the force is

\[
K(\omega) = \int_{-\infty}^{\infty} dt \, e^{i\omega t} K(t).
\]

(231)

The probability requirement

\[
|\langle 0, t_1 | 0, t_2 \rangle|^2 \leq 1
\]

(232)
is thus satisfied. We see here a resonance effect: If the oscillator is driven close to its natural frequency, so $K(\omega)$ is large, there is a large probability of finding the system in an excited state, and therefore of not remaining in the ground state. Let us calculate this transition amplitude to an excited state. By setting $y'' = 0$ in Eq. (228) we obtain

$$\langle y''', t_1 | 0, t_2 \rangle^K = \exp \left[ -iy''' \int_{-\infty}^{\infty} dt e^{-i\omega(t_1 - t) K(t)} \right] \langle 0, t_1 | 0, t_2 \rangle^K = \sum_n \langle y''', t_1 | n, t_1 \rangle \langle n, t_1 | 0, t_2 \rangle^K,$$  \hspace{1cm} (233)

where we have inserted a sum over a complete set of energy eigenstates, which possess the amplitude [see Eq. (214b)]

$$\langle y''' | n \rangle = (y''')^n \sqrt{n!}.$$  \hspace{1cm} (234)

If we expand the first line of Eq. (233) in powers of $y'''$, we find

$$\langle n, t_1 | 0, t_2 \rangle^K = \frac{(-i)^n}{\sqrt{n!}} e^{-i\omega t_1} [K(\omega)]^n \langle 0, t_1 | 0, t_2 \rangle^K.$$  \hspace{1cm} (235)

The corresponding probability is

$$p(n, 0)^K = |\langle n, t_1 | 0, t_2 \rangle^K|^2 = \frac{|K(\omega)|^{2n}}{n!} e^{-|K(\omega)|^2},$$  \hspace{1cm} (236)

which is a Poisson distribution with mean $\bar{n} = |K(\omega)|^2$. Finally, let us define the Green’s function for this problem by

$$G(t - t') = -i\eta(t - t') e^{-i\omega(t - t')}.$$  \hspace{1cm} (238)

It satisfies the differential equation

$$\left( i \frac{d}{dt} - \omega \right) G(t - t') = \delta(t - t'),$$  \hspace{1cm} (239)

as it must because [see Eq. (217)]

$$\left( i \frac{d}{dt} - \omega \right) y(t) = K(t),$$  \hspace{1cm} (240)

where $y(t)$ is given by [see Eq. (221)]

$$y(t) = e^{-i\omega(t-t_2)} y_2 + \int_{-\infty}^{\infty} dt' G(t - t') K(t').$$  \hspace{1cm} (241)

\footnote{A Poisson probability distribution has the form $p(n) = \lambda^n e^{-\lambda}/n!$. The mean value of $n$ for this distribution is}

$$\bar{n} = \sum_{n=0}^{\infty} n p(n) = \sum_{n=0}^{\infty} \frac{\lambda^n e^{-\lambda}}{(n-1)!} = \lambda \sum_{n=0}^{\infty} p(n) = \lambda.$$  \hspace{1cm} (237)
Similarly, from Eq. (224)

\[ y^\dagger(t) = e^{-i\omega(t_1 - t)}y_1^\dagger + \int_{-\infty}^{\infty} dt' G(t' - t)K^*(t'). \]  \hspace{1cm} (242)

We can now write the ground-state persistence amplitude (342) as

\[ \langle 0, t_1 | 0, t_2 \rangle^K = \exp \left[ -i \int_{-\infty}^{\infty} dt dt' K^*(t)G(t-t')K(t') \right]. \] \hspace{1cm} (243)

and the general amplitude (228) as

\[ \langle y^\dagger', t_1 | y'', t_2 \rangle^K = \exp \left\{ -i \int_{-\infty}^{\infty} dt dt' \left[ K^*(t) + iy^\dagger'\delta(t-t_1) \right] \times G(t-t') \left[ K(t') + iy''\delta(t'-t_2) \right] \right\}, \] \hspace{1cm} (244)

which demonstrates that knowledge of \( \langle 0, t_1 | 0, t_2 \rangle^K \) for all \( K \) determines everything:

\[ \langle y^\dagger', t_1 | y'', t_2 \rangle^K = \langle 0, t_1 | 0, t_2 \rangle^K(t) + iy''\delta(t-t_2) + iy^\dagger'\delta(t-t_1). \] \hspace{1cm} (245)

7 Feynman Path Integral Formulation

Although much more familiar, the path integral formulation of quantum mechanics is rather vaguely defined. We will here provide a formal "derivation" based on the Schwinger principle, in the harmonic oscillator context.

Consider a forced oscillator, defined by the Lagrangian (note in this section, \( H \) does not include the source terms)

\[ L = iy^\dagger\dot{y} - H(y, y^\dagger) - Ky^\dagger - K^*y. \] \hspace{1cm} (246)

As in the preceding section, the action principle says

\[ \delta_K \langle 0, t_1 | 0, t_2 \rangle^K = -i\langle 0, t_1 | \int_{t_2}^{t_1} dt \left[ K^*(t) + i\gamma^\dagger(t-t_1) \right] G(t-t') \left[ K(t') + i\gamma(t'-t_2) \right] \rangle^K, \] \hspace{1cm} (247)

or for \( t_2 < t < t_1 \),

\[ \frac{i}{\delta K(t)} \langle 0, t_1 | 0, t_2 \rangle^K = \langle 0, t_1 | \gamma^\dagger(t) | 0, t_2 \rangle^K, \] \hspace{1cm} (248a)

\[ \frac{i}{\delta K^*(t)} \langle 0, t_1 | 0, t_2 \rangle^K = \langle 0, t_1 | \gamma(t) | 0, t_2 \rangle^K, \] \hspace{1cm} (248b)

where we have introduced the concept of the functional derivative. The equation of motion

\[ i\gamma - \frac{\partial H}{\partial y} = K, \quad -i\gamma^\dagger - \frac{\partial H}{\partial y^\dagger} = K^*, \] \hspace{1cm} (249)

is thus equivalent to the functional differential equation,

\[ 0 = \left\{ i \left[ K(t), W \left[ \frac{i}{\delta K^*}, i \frac{i}{\delta K} \right] \right] - K(t) \right\} \langle 0, t_1 | 0, t_2 \rangle^K, \] \hspace{1cm} (250)
where (the square brackets indicate functional dependence)

\[ W[y, y^\dagger] = \int_{t_2}^{t_1} dt \left[ i y^\dagger(t) \dot{y}(t) - H(y(t), y^\dagger(t)) \right]. \]  

(251)

The reason Eq. (250) holds is that by definition

\[ \frac{\delta}{\delta K(t')} K(t') = \delta(t - t'), \]  

(252)

so

\[
\begin{align*}
&i \left[ K(t), \int_{t_2}^{t_1} dt' \left( i \frac{\delta}{\delta K(t')} \frac{d}{dt'} \frac{\delta}{\delta K^*(t')} - H \left( \frac{i \delta}{\delta K^*(t')}, \frac{i \delta}{\delta K(t')} \right) \right) \right] \\
&= i \frac{d}{dt} \frac{\delta}{\delta K^*(t)} - \frac{\partial}{\partial (i \delta / \delta K(t))} H \left( \frac{i \delta}{\delta K^*(t)}, \frac{i \delta}{\delta K(t)} \right),
\end{align*}
\]

(253)

which corresponds to the first two terms in the equation of motion (249), under the correspondence

\[ y \leftrightarrow i \frac{\delta}{\delta K^*}, \quad y^\dagger \leftrightarrow i \frac{\delta}{\delta K}. \]  

(254)

Since \([K, W], W = 0\), we can write the functional equation (250) as

\[ 0 = e^{i W} e^{-i \int dt K(t) y^\dagger(t)} e^{-i \int dt K^*(t) y(t)} \delta[K] \delta[K^*]. \]  

(255)

The above equation has a solution (up to a constant), because both equations (249) must hold,

\[ \langle 0, t_1 | 0, t_2 \rangle^K = e^{i W} e^{-i \int dt K(t) y^\dagger(t)} e^{-i \int dt K^*(t) y(t)} \delta[K] \delta[K^*], \]  

(256)

where \(\delta[K], \delta[K^*]\) are functional delta functions. The latter have functional Fourier decompositions (up to a multiplicative constant),

\[ \delta[K] = \int [dy^\dagger] e^{-i \int dt K(t) y^\dagger(t)}, \]

(257a)

\[ \delta[K^*] = \int [dy] e^{-i \int dt K^*(t) y(t)}, \]

(257b)

where \([dy]\) represents an element of integration over all (numerical-valued) functions \(y(t)\), and so we finally have

\[
\begin{align*}
\langle 0, t_1 | 0, t_2 \rangle^{K, K^*} &\quad = \int [dy][dy^\dagger] \exp \left( -i \int_{t_2}^{t_1} dt \left[ K(t) y^\dagger(t) + K^*(t) y(t) \right] + i W[y, y^\dagger] \right) \\
&\quad = \int [dy][dy^\dagger] \exp \left( i \int_{t_2}^{t_1} dt \left[ i y^\dagger \dot{y} - H(y, y^\dagger) - K y^\dagger - K^* y \right] \right),
\end{align*}
\]

(258)

where \(y, y^\dagger\) are now numerical, and the functional integration is over all possible functions, over all possible “paths.” Of course, the classical paths, the ones for which \(W - \int dt (K y^\dagger + K^* y)\) is an extremum, receive the greatest weight, at least in the classical limit, where \(\hbar \to 0\).
7.1 Example

Consider the harmonic oscillator Hamiltonian, $H = \omega y^\dagger y$. Suppose we wish to calculate, once again, the ground state persistence amplitude, $\langle 0, t_1 | 0, t_2 \rangle^K$. It is perhaps easiest to perform a Fourier transform,

$$ y(\nu) = \int_{-\infty}^{\infty} dt \, e^{i\nu t} y(t), \quad y^*(\nu) = \int_{-\infty}^{\infty} dt \, e^{-i\nu t} y^\dagger(t). $$

Then

$$ \int_{-\infty}^{\infty} dt \, y^\dagger(t) y(t) = \int_{-\infty}^{\infty} \frac{d\nu}{2\pi} y(\nu) y^*(\nu), \quad (260a) $$

$$ \int_{-\infty}^{\infty} dt \, iy^\dagger(t) \dot{y}(t) = \int_{-\infty}^{\infty} \frac{d\nu}{2\pi} \nu y(\nu) y^*(\nu). \quad (260b) $$

Thus Eq. (258) becomes

$$ \langle 0, t_1 | 0, t_2 \rangle^K = \int [dy][dy^*] \exp \left\{ i \int \frac{d\nu}{2\pi} \left[ y(\nu)(\nu - \omega) y^*(\nu) - y^*(\nu) K(\nu) - y(\nu) K^*(\nu) \right] \right\} \left[ i \int \frac{d\nu}{2\pi} \left[ K(\nu) \frac{1}{\nu - \omega} K^*(\nu) \right] \right\} $$

$$ \times \exp \left\{ -i \int \frac{d\nu}{2\pi} \left[ \frac{1}{\nu - \omega} K^*(\nu) \right] \right\}, \quad (261) $$

since the first exponential in the penultimate line, obtained by shifting the integration variable,

$$ y(\nu) = \frac{K(\nu)}{\nu - \omega} \rightarrow y(\nu), \quad (262a) $$

$$ y^*(\nu) = \frac{K^*(\nu)}{\nu - \omega} \rightarrow y^*(\nu), \quad (262b) $$

is $\langle 0, t_1 | 0, t_2 \rangle^{K=K^*=0} = 1$. How do we interpret the singularity at $\nu = \omega$ in the remaining integral? We should have inserted a convergence factor in the original functional integral:

$$ \exp \left( i \int \frac{d\nu}{2\pi} \left[ \cdots \right] \right) \rightarrow \exp \left( i \int \frac{d\nu}{2\pi} \left[ \cdots + i\epsilon y(\nu) y^*(\nu) \right] \right), \quad (263) $$

where $\epsilon$ goes to zero through positive values. Thus we have, in effect, $\nu - \omega \rightarrow \nu - \omega + i\epsilon$ and so we have for the ground-state persistence amplitude

$$ \langle 0, t_1 | 0, t_2 \rangle^{K,K^*} = e^{i \int dt \, dt' K^*(t') G(t') K(t)}, \quad (264) $$

in the remaining integral.
which has the form of Eq. (243), with

\[ G(t - t') = \int_{-\infty}^{\infty} \frac{d\nu}{2\pi} e^{-i\nu(t-t')} \]

which is evaluated by closing the \( \nu \) contour in the upper half plane if \( t - t' < 0 \), and in the lower half plane when \( t - t' > 0 \). Since the pole is in the lower half plane we get

\[ G(t - t') = -i\eta(t - t')e^{-i\omega(t-t')}, \]

which is exactly what we found in Eq. (238).

Now, let us rewrite the path integral (258) in terms of coordinates and momenta:

\[ q = \frac{1}{\sqrt{2\omega}}(y + y^\dagger), \quad p = \sqrt{\frac{\omega}{2}}(y - y^\dagger), \]

\[ y = \sqrt{\frac{\omega}{2}}(q + ip\omega), \quad y^\dagger = \sqrt{\frac{\omega}{2}}(q - ip\omega). \]

Then the numerical Lagrangian appearing in (258) may be rewritten as

\[ L = iy^\dagger\dot{y} - y^\dagger y - Ky^\dagger - K^* y \]

\[ = i\frac{\omega}{2} (q - i\frac{p}{\omega}) (\dot{q} + i\frac{\dot{p}}{\omega}) - \frac{\omega^2}{2} (q^2 + \frac{p^2}{\omega^2}) \]

\[ - \sqrt{\frac{\omega}{2}} K \left( q - ip\frac{\omega}{2} \right) - \sqrt{\frac{\omega}{2}} K^* \left( q + ip\frac{\omega}{2} \right) \]

\[ = i\frac{\omega}{4} \frac{d}{dt} (q^2 + \frac{p^2}{\omega^2}) + pq - \frac{1}{2} \frac{d}{dt} (pq) - \frac{1}{2} (p^2 + \omega^2 q^2) - \sqrt{2\omega} \Re K q - \sqrt{\frac{\omega}{2}} \Im K p \]

\[ = \frac{d}{dt} w + L(q, \dot{q}, t), \]

where, if we set \( \dot{q} = p \), the Lagrangian is

\[ L(q, \dot{q}, t) = \frac{1}{2} \dot{q}^2 - \frac{1}{2} \omega^2 q^2 + F q, \]

if

\[ \Re K = 0, \quad F = -\sqrt{2\omega} \Re K. \]

In the path integral

\[ [dy][dy^\dagger] = [dq][dp] \left| \frac{\partial(y, y^\dagger)}{\partial(q, p)} \right|, \]

where the Jacobian is

\[ \left| \frac{\partial(y, y^\dagger)}{\partial(q, p)} \right| = \begin{vmatrix} \sqrt{\frac{\omega}{2}} & \sqrt{\frac{\omega}{2}} \\ \frac{i}{\sqrt{2\omega}} & -\frac{i}{\sqrt{2\omega}} \end{vmatrix} = 1, \]

and so from the penultimate line of Eq. (268), the path integral (258) becomes

\[ \langle 0, t_1 | 0, t_2 \rangle^F = \int [dy][dy^\dagger] \exp \left[ i \int_{t_2}^{t_1} dt L(y, y^\dagger) \right] \]

\[ = \int [dq][dp] \exp \left[ i \int_{t_2}^{t_1} dt \left( pq - \frac{1}{2} p^2 - \frac{1}{2} \omega^2 q^2 + F q \right) \right]. \]
Now we can carry out the $p$ integration, since it is Gaussian:

\[
\int [dp] e^{i \int dt \left[ -\frac{1}{2} p^2 + \frac{1}{2} (p - \dot{q})^2 \right]} = \int [dp] e^{i \int dt \left[ -\frac{1}{2} (p - \dot{q})^2 + \frac{1}{2} \dot{q}^2 \right]} = e^{i \int dt \frac{1}{2} \dot{q}^2} \prod_i \int_{-\infty}^{\infty} dp_i e^{-\frac{1}{2} p_i^2} \Delta t.
\]  

(274)

Here we have discretized time so that $p(t_i) = p_i$, so the final functional integral over $p$ is just an infinite product of constants, each one of which equals $e^{-i\pi/4} \sqrt{2\pi/\Delta t}$.

Thus we arrive at the form originally written down by Feynman [Feynman 1965],

\[
\langle 0, t_1 | 0, t_2 \rangle^F = \int [dq] \exp \left\{ i \int_{t_1}^{t_2} dt \left[ L(q, \dot{q}, t) \right] \right\},
\]  

(275)

with the Lagrangian given by Eq. (269), where an infinite normalization constant has been absorbed into the measure.

### 8 Time-cycle or Schwinger-Keldysh formulation

A further utility of the action principle is the time-cycle or Schwinger-Keldysh formalism, which allows one to calculate matrix elements and consider nonequilibrium systems. Schwinger’s original work on this was his famous paper [Schwinger 1961]; Keldysh’s paper appeared three years later [Keldysh 1964], and, rather mysteriously, cites the Martin-Schwinger equilibrium paper [Martin 1959], but not the nonequilibrium one [Schwinger 1961]. The following was extracted from notes from Schwinger’s lectures given in 1968 at Harvard, as taken by the author.

Consider the expectation value of some physical property $F(t)$ at a particular time $t_1$ in a state $|b, t_2\rangle$:

\[
\langle F(t_1) \rangle_{b, t_2} = \sum_{a' a''} \langle b' t_2 | a' t_1 \rangle \langle a' | F | a'' \rangle \langle a'' t_1 | b' t_2 \rangle,
\]  

(276)

which expresses the expectation value in terms of the matrix elements of the operator $F$ in a complete set of states defined at time $t_1$, $\{|a' t_1\rangle\}$. Suppose the operator $F$ has no explicit time dependence. Then we can use the action principle to write

\[
\delta \langle a' t_1 | b' t_2 \rangle = i \langle a' t_1 | \delta \left[ \int_{t_2}^{t_1} dt \left[ L(q, \dot{q}, t) \right] \right] | b' t_2 \rangle,
\]  

(277a)

and so

\[
\delta \langle b' t_2 | a' t_1 \rangle = -i \langle b' t_2 | \delta \left[ \int_{t_2}^{t_1} dt \left[ L(q, \dot{q}, t) \right] \right] | a' t_1 \rangle,
\]  

(277b)

which can be obtained from the first equation by merely exchanging labels,

\[
\int_{t_2}^{t_1} = - \int_{t_1}^{t_2}.
\]  

(278)

If we consider

\[
\langle b' t_2 | b' t_2 \rangle = \sum_{a'} \langle b' t_2 | a' t_1 \rangle \langle a' t_1 | b' t_2 \rangle,
\]  

(279)
the above variational equations indeed asserts that

$$\delta \langle b'_{t_2} | b'_{t_2} \rangle = 0.$$  

(280)

We can interpret the above as a cycle in time, going from time $t_2$ to $t_1$ and then back again, as shown in Fig. 4. But, now imagine that the dynamics is different on the forward and return trips, described by different Lagrangians $L_+$ and $L_-$. Then

$$\delta \langle b'_{t_2} | b'_{t_2} \rangle = i \langle b'_{t_2} \phi \int_{t_2}^{t_1} dt \delta \lambda_+ + \delta \lambda_- \rangle | b'_{t_2} \rangle.$$  

(281)

In particular, consider a perturbation of the form,

$$H = H_0 + \lambda(t) F,$$  

(282)

where $\lambda(t)$ is some time-varying parameter. If we have an infinitesimal change, and, for example, $\delta \lambda_+ \neq 0$, $\delta \lambda_- = 0$, then

$$\delta \lambda_+ \langle b'_{t_2} | b'_{t_2} \rangle^+ \lambda_- = -i \langle b'_{t_2} \phi \int_{t_2}^{t_1} dt \delta \lambda_+ F | b'_{t_2} \rangle.$$  

(283)

If we choose $\delta \lambda_+$ to be an impulse,

$$\delta \lambda_+ = \delta \lambda(t - t'),$$  

(284)

in this way we obtain the expectation value of $F(t')$.

Let’s illustrate this with a driven harmonic oscillator, as described by Eq. (216), so now

$$H_+ = \omega \varphi^\dagger \varphi + K_+^*(t) \varphi + K_+(t) \varphi^\dagger,$$  

(285a)

$$H_- = \omega \varphi^\dagger \varphi + K_-^*(t) \varphi + K_-(t) \varphi^\dagger,$$  

(285b)

which describes the oscillator evolving forward in time from $t_2$ to $t_1$ under the influence of the force $K_+$, and backward in time from $t_1$ to $t_2$ under the influence of $K_-$, as shown in Fig. 5. From the variational principle we can learn all about $\varphi$ and $\varphi^\dagger$. We have already solved this problem by a more laborious method above, in Section 6.
It suffices to solve this problem with initial and final ground states, if we consider only a $K^*$ variation,

$$\delta K^*(0|t_2)\langle 0|t_2 \rangle = -i \langle 0| t \int_{t_2}^{t_1} dt \left[ \delta K^*_+(t) y_+(t) - \delta K^*_-(t) y_-(t) \right] |0\rangle. \quad (286)$$

Now we must solve the equations of motion, so since effectively $y(t_2) \to 0$, we have from Eq. (221),

$$y_+(t) = -i \int_{t_2}^{t_1} dt' e^{-i\omega (t-t')} K_+ (t'), \quad (287a)$$

$$y_-(t) = -i \int_{t_2}^{t_1} dt' e^{-i\omega (t-t')} K_+ (t') - i \int_{t_2}^{t_1} dt' e^{-i\omega (t-t')} K_- (t'). \quad (287b)$$

The last term in the second equation is

$$i \int_{t_2}^{t_1} dt' e^{-i\omega (t-t')} K(t') \eta(t' - t), \quad (288)$$

so naming the advanced and retarded Green’s functions by extending the definition in Eq. (283),

$$G_{a,r}(t,t') = i e^{-i\omega (t-t')} \left\{ \begin{array}{c} \eta(t' - t) \\ -\eta(t - t') \end{array} \right\}, \quad (289)$$

which satisfy the same differential equation (239), we effectively have

$$y_+(t) = \int_{t_2}^{t_1} dt' G_a(t-t') K_+(t'), \quad (290a)$$

$$y_-(t) = -i \int_{t_2}^{t_1} dt' e^{-i\omega (t-t')} K_+ (t) + \int_{t_2}^{t_1} dt' G_a(t-t') K_- (t'), \quad (290b)$$

The solution to the variational equation (286) is now

$$\langle 0|t_2 \rangle = e^{-i \int_{t_2}^{t_1} dt dt' K^*_+(t) G_a(t-t') K_+(t')} \times e^{i \int_{t_2}^{t_1} dt dt' K^*_-(t) G_a(t-t') K_-(t')} e^{\int_{t_2}^{t_1} dt dt' K^*_+ (t) e^{-i\omega (t-t')} K_+(t')} \times e^{\int_{t_2}^{t_1} dt dt' K^*_- (t) e^{-i\omega (t-t')} K_- (t')}. \quad (291)$$
This should reduce to 1 when $K_+ = K = K$, so

$$-iG_r(t - t') + iG_a(t - t') + e^{-i\omega(t - t')} = 0,$$

which is, indeed, true.

As an example, consider $K_-(t) = K(t)$, $K_+(t) = K(t + T)$, that is, the second source is displaced forward by a time $T$. This is sketched in Fig. 6. What does this mean? From a causal analysis, in terms of energy eigenstates, reading from right to left,

$$\langle 0^t_2|0^t_2\rangle_{K_+} = \sum_n \langle 0^t_2|nt_1\rangle_{K_+} \langle nt_1|0^t_2\rangle_{K_+ = K(t+T)}.$$

The effect is the same as moving the $n, t_1$ state to a later time,

$$\langle nt_1|0^t_2\rangle_{K(t+T)} = \langle nt_1 + T|0^t_2\rangle_{K(t)} = e^{-i\omega T} \langle nt_1|0^t_2\rangle_{K(t)},$$

so this says that

$$\langle 0^t_2|0^t_2\rangle_{K_+} = \sum_n e^{-i\omega T} p(n, 0)^K,$$

which gives us the probabilities directly. From the formula (291) we have, using Eq. (292),

$$\langle 0^t_2|0^t_2\rangle_{K_+} = e^\int dt' e^{i\omega t} K(t') e^{-i\omega(t - t')} |K(t' + T) - K(t')|$$

$$= e^{i\gamma^2(e^{-i\omega T} - 1)},$$

where

$$\gamma = \int dt e^{i\omega t} K(t).$$

Thus we immediately obtain Eq. (256), or

$$p(n, 0)^K = e^{-|\gamma|^2 \left(\frac{|\gamma|^2}{n!}\right)}.$$

The above Eq. (256) can be directly used to find certain average values. For example,

$$\langle e^{-i\omega T}\rangle_0^K = e^{i\gamma^2(e^{-i\omega T} - 1)}.$$
Expand this for small $\omega T$ and we find
\begin{equation}
\langle n \rangle^K_0 = |\gamma|^2. \tag{300}
\end{equation}
In a bit more systematic way we obtain the dispersion:
\begin{equation}
\langle e^{-i(n-\langle n \rangle)\omega T} \rangle = e^{i|\gamma|^2(\omega T - 1 + i\omega T)}. \tag{301}
\end{equation}
Expanding this to second order in $\omega T$ we get
\begin{equation}
\langle (n - \langle n \rangle)^2 \rangle = \langle n^2 \rangle - \langle n \rangle^2 \equiv (\Delta n)^2 = |\gamma|^2, \tag{302}
\end{equation}
or
\begin{equation}
\frac{\Delta n}{\langle n \rangle} = \frac{1}{\sqrt{\langle n \rangle}}. \tag{303}
\end{equation}
For large quantum numbers, which corresponds to the classical limit, the fluctuations become relatively small.

Now consider a more general variational statement than in Eq. (286),
\begin{equation}
\delta \langle | \rangle^K_0 - K_+ + K_- = -i \langle | \int dt \delta K^*_+(t) y_+(t) + \cdots - \delta K^*_-(t) y^\dagger_-(t) - \cdots \rangle^K_0, \tag{304}
\end{equation}
where the $\cdots$ signify the omission of the other source variations, we see that since we can change the source functions at will, and make very localized changes, it makes sense to define the variational derivatives
\begin{equation}
\frac{i}{\delta K^*_+(t)} \langle | \rangle^K_0 = \langle | y_+(t) \rangle^K_+, \tag{305a}
\end{equation}
and
\begin{equation}
-\frac{i}{\delta K^*_-(t)} \langle | \rangle^K_0 = \langle | y^\dagger_-(t) \rangle^K_. \tag{305b}
\end{equation}
All expectation values of operator products at any time can be obtained in this way—in particular, correlation functions. Repeating this operation we get
\begin{equation}
(-i) \frac{\delta}{\delta K^*_-(t)} \delta \langle t_2 | t_2 \rangle^K_\pm = -i \frac{\delta}{\delta K^*_-(t)} \langle t_2 | y_+(t') t_2 \rangle^K_\pm = \langle t_2 | y^\dagger_-(t') y_+(t') t_2 \rangle^K_\pm. \tag{306}
\end{equation}
The operators are multiplied in the order of the time development. The only place where $K_-$ appears is in the latter part of the time development. See Fig. 7.

The distinction between $\pm$ disappears if we now set $K_+ = K_-:
\begin{equation}
\left. \frac{\delta}{\delta K^*_+(t)} \frac{\delta}{\delta K^*_+(t')} \langle 0 t_2 | 0 t_2 \rangle^K_\pm \right|_{K_+ = K_- = K} = \langle 0 t_2 | y^\dagger(t) y(t') 0 t_2 \rangle^K. \tag{307}
\end{equation}
As an example, set $t = t' = t_1$; then this reads for the number operator $N(t) = y^\dagger(t) y(t),
\begin{equation}
\langle N(t_1) \rangle^K_0 = \int dt K^*(t) G_a(t - t_1) \int dt' G_r(t_1 - t') K(t')
= i \int dt e^{-i\omega(t - t_1)} K^*(t)(-i) \int dt' e^{-i\omega(t_1 - t')} K(t') = |\gamma|^2, \tag{308}
\end{equation}
as before, Eq. (300).
Fig. 7. Variational derivatives pick out operators at definite times $t$ and $t'$. 

Fig. 8. Time cycle with different forces, $K_+$ and $K_-$. on the forward and backward moving segments. Now the initial time of the time cycle, $t_2$, is different from the final time of the time cycle, $t'_2$, with $\tau = t'_2 - t_2$. It is assumed that the time $t_1$ is later than both $t_2$ and $t'_2$, and that the forces are localized as shown. 

We would like to use more general starting and ending states than the ground state. We can obtain these by use of impulsive forces. It is convenient to deal with all states at once, as in the generating function for $p(n,0)^K$ considered above. Think of a time cycle starting at time $t_2$, advancing forward to time $t_1$, during which time the force $K_+$ acts, then moving back in time to a time $t'_2$, under the influence of the force $K_-$—See Fig. 8. Let $t'_2 = t_2 + \tau$. This displacement injects energy information. 

$$\sum_n \langle nt'_2|nt_2\rangle^{K_\pm} \equiv \text{tr}\langle t'_2|t_2\rangle^{K_\pm} = \sum_n e^{-in\omega \tau} \langle nt_2|nt_2\rangle^{K_\pm}, \quad (309)$$

which uses (no force acts between times $t'_2$ and $t_2$)

$$\langle nt'_2\rangle = \langle nt_2\rangle e^{-in\omega \tau}. \quad (310)$$

Analysis of this formula will yield individual transformation functions.

Now we must solve the dynamical equations subject to boundary conditions. Let us compare $\text{tr}\langle t'_2|y_+(t_2)|t_2\rangle$ with $\text{tr}\langle t'_2|y_-(t'_2)|t_2\rangle$. The first is

$$\text{tr}\langle t'_2|y_+(t_2)|t_2\rangle = \sum_n \langle nt'_2|y_+(t_2)|nt_2\rangle = \sum_{nn'} \langle nt'_2|n't_2\rangle \langle n'|y|n\rangle, \quad (311a)$$
while the second appears as
\[ \text{tr} \langle t_2' | y_- (t_2') | t_2 \rangle = \sum_{n'} \langle n' t_2' | y_- (t_2') | n' t_2 \rangle = \sum_{n a'} \langle n' | y | n \rangle \langle n t_2' | n' t_2 \rangle. \] (311b)

Here, by introducing a complete set of states at the time of the operator, we have expressed the formula in terms of the matrix elements of stationary operators. Remarkably, we see that the two expressions are equal; in effect, there is a periodicity present here:
\[ y_+ (t_2) = y_- (t_2'), \] (312)
as far as traces are concerned. Now, the equations of motion (217) for the operators read
\[ \left( i \frac{d}{dt} - \omega \right) y(t) = K(t), \] (313)
which has solution (287b) with the addition of the initial term, or
\[ y_- (t) = e^{-i \omega (t-t_2)} y_+ (t_2) - i \int_{t_2}^{t} dt' e^{-i \omega (t-t')} K_+ (t') + i \int_{t}^{t_2} dt' e^{-i \omega (t-t')} K_- (t'). \] (314)

In particular,
\[ y_- (t_2') = e^{-i \omega \tau} y_+ (t_2) - i \int dt' e^{-i \omega (t+t' - t_2)} (K_+ - K_-)(t'). \] (315)

Note that the integrals sweep over the full force history. Let us let \( t_2 = 0 \) for simplicity, although we will keep the label. Because of the periodicity condition (312) this reads
\[ (e^{i \omega \tau} - 1) y_+ (t_2) = -i \int dt e^{i \omega t} (K_+ - K_-)(t) = -i (\gamma_+ - \gamma_-), \] (316)
or
\[ y_+ (t_2) = \frac{1}{e^{i \omega \tau} - 1} (-i) (\gamma_+ - \gamma_-). \] (317)

What we are interested in is
\[ \frac{\text{tr} \langle t_2' | t_2 \rangle K_2}{\text{tr} \langle t_2' | t_2 \rangle}, \] (318)
The denominator, which refers to the free harmonic oscillator, is immediately evaluated as
\[ \text{tr} \langle t_2' | t_2 \rangle = \sum_{n=0}^{\infty} e^{-in \omega \tau} = \frac{1}{1 - e^{-i \omega \tau}}. \] (319)
(If \( \tau \) be imaginary, we have thermodynamic utility.) We have then the variational equation
\[ \delta K_\pm \left[ \frac{\text{tr} \langle t_2' | t_2 \rangle K_2}{\text{tr} \langle t_2' | t_2 \rangle} \right] = -i \frac{\text{tr} \langle t_2' | \delta K_+ y_+ - \delta K_- y_- | t_2 \rangle K_2}{\text{tr} \langle t_2' | t_2 \rangle}, \] (320)

Exactly as before, we get an equation for the logarithm—looking at the previous calculation leading to Eq. (291), we see an additional term, referring to the \( y_+ (t_2) \) boundary term in Eq. (315). The periodic boundary condition then gives
\[ -\frac{1}{e^{i \omega \tau} - 1} \delta (\gamma_+ - \gamma_-) (\gamma_+ - \gamma_-). \] (321)
Therefore, to convert $\langle 0t_2|0t_2 \rangle^{K\pm}$ in Eq. (244) to
\[
\frac{\text{tr}(t_2'|t_2)^{K\pm}}{\text{tr}(t_2'|t_2)} = \sum_n e^{-in\omega t} \langle nt_2|nt_2 \rangle^{K\pm} \sum e^{-in\omega t}
\]
we must multiply by
\[
\exp\left[-\frac{1}{e^{\omega t} - 1}|\gamma_+ - \gamma_-|^2\right].
\] (323)
This holds identically in $\tau$; in particular, in the limit where $\tau \rightarrow -i\infty$, which corresponds to absolute zero temperature, we recover $\langle 0t_2|0t_2 \rangle^{K\pm}$.

We find, generalizing Eq. (291)
\[
\sum_n e^{-in\omega t} \langle nt_2|nt_2 \rangle^{K\pm} = e^{-i\int dt dt' K^*_+ (t) G_+ (t-t') K_+ (t')}
\]
\[
\times e^{i\int dt dt' K^*_+ (t) G_0 (t-t') K_+ (t') e^{i\int dt dt' K^*_+ (t) e^{-i\omega (t-t')} K_+ (t')}
\]
\[
\times e^{-i(\omega - 1) t^2} \int dt dt' K^*_+ (t) e^{-i\omega (t-t')} (K_+ - K^-) (t') ,
\] (324)
which is the exponential of a bilinear structure. This is a generating function for the amplitudes $\langle nt_2|nt_2 \rangle^{K\pm}$. But it is useful as it stands.

Put $\tau = -i\beta$; then this describes a thermodynamic average over a thermal mixture at temperature $T$, where $\beta = 1/kT$ in terms of Boltzmann's constant:
\[
\sum_n e^{-\beta n\omega} \langle nt_2 | nt_2 \rangle = \langle n | \beta \rangle
\] (325)
In terms of this replacement,
\[
\frac{1}{e^{i\omega t} - 1} \rightarrow \frac{1}{e^{\beta \omega} - 1} = \langle n | \beta \rangle,
\] (326)
because
\[
\sum_n n e^{-in\omega t} = \frac{\partial}{\partial(-i\omega t)} \ln\left(\sum_n e^{-in\omega t}\right) = \frac{\partial}{\partial(-i\omega t)} \ln \frac{1}{1 - e^{-i\omega t}} = \frac{1}{e^{i\omega t} - 1}.
\] (327)

Now consider a time cycle with displacement $T$: the system evolves from time $t_2$ to time $t_1$ under the influence of the force $K_+ (t)$, and backwards in time from $t_1$ to $t_2'$ under the force $K_-(t)$:
\[
K_- (t) = K(t), \quad K_+ (t) = K(t + T).
\] (328)
This is again as illustrated in Fig. 8 with these replacements. What is the physical meaning of this? Insert in Eq. (324) a complete set of states at time $t_1$:
\[
\langle nt_2|nt_2 \rangle^{K\pm} = \sum_n \langle nt_2|nt_1 \rangle^{K-} \langle nt_1|nt_2 \rangle^{K+}.
\] (329)
We did this before for the ground state. The effect is the same as moving the starting and ending times. Appearing here is
\[
\langle nt_1|nt_2 \rangle^{K(t+T)} = \langle nt_1 + T|nt_2 + T \rangle^{K(t)} = e^{-in\omega T} \langle nt_1|nt_2 \rangle^{K(t)} e^{i\omega T}.
\] (330)
Therefore,
\[ \langle n't | n' \rangle_{K(t),K(t+T)} = \sum_{n'} e^{-i(n'-n)\omega T} p(n', n) K = \langle e^{-i(N-n)\omega T} \rangle_{n}. \] (331)

Therefore, as a generalization for finite \( \tau \) of Eq. (301), we have from Eq. (324)
\[ \left( \sum_{n'} e^{-i\omega T} - 1 \right) \frac{1}{e^{i\omega \tau} - 1} \left( e^{i\omega T} - 1 \right) \left( e^{-i\omega T} - 1 \right) \left| \gamma \right|^2, \] (332)
where \( T \) gives the final state, and \( \tau \) the initial state. This used the observation
\[ \int dt e^{i\omega t} K(t + T) = e^{-i\omega T} \int dt e^{i\omega t} K(t). \] (333)
Expand both sides of Eq. (332) in powers of \( T \), and we learn
\[ -i\omega \sum_n \langle N-n \rangle_K e^{-i\omega T} \sum_{n'} e^{-i\omega T} = -i\omega T \left| \gamma \right|^2, \] (334)
or
\[ \langle N-n \rangle^K_n = \left| \gamma \right|^2, \] (335)
which generalizes an earlier result. Now apply Eq. (334) as a generating function,
\[ \langle N-n \rangle^K_n = \left| \gamma \right|^2, \] (336)
which reflects the linear nature of the system.
We can rewrite the above generating function more conveniently, by multiplying by
\[ e^{i(N-n)\omega T} = e^{i\omega T} \left| \gamma \right|^2, \] (337)
that is, Eq. (332) can be written as
\[ \sum_{n} \frac{1}{e^{i\omega T} - 1} e^{-i\omega \tau} \left( e^{-i(N-n)\omega T} \right)_{n} K = \exp \left[ \left( e^{-i\omega T} - 1 + i\omega T \right) \left| \gamma \right|^2 - \frac{1}{e^{i\omega \tau} - 1} \left( e^{-i\omega T} - 1 \right) \left( e^{i\omega T} - 1 \right) \left| \gamma \right|^2 \right]. \] (338)
Now pick off the coefficient of \(- (\omega T)^2/2\):
\[ \sum_{n} e^{-i\omega T} \left( \langle N-n \rangle \right)^2_{n} K = \left| \gamma \right|^2 + 2 \frac{1}{e^{i\omega \tau} - 1} \left| \gamma \right|^2, \] (339)
or
\[ \left( \langle N-n \rangle \right)^2_{n} K = \left| \gamma \right|^2 [1 + 2 \langle n \rangle]. \] (340)
If, instead, we multiply Eq. (339) through by \( \sum_n e^{-i\omega \tau} \), we can use this as a generating function, and learn from Eq. (327) that
\[ \langle (N-n)^2 \rangle_{n} K = \left| \gamma \right|^2 (1 + 2n). \] (341)
Note the simplicity of the derivation of this result, which does not involve complicated functions like Laguerre polynomials.
9 Prologue

Let us finally return to the action principle. Recall from Eq. (264)

\[ \langle 0 | t_1 | 0 | t_2 \rangle^K = e^{-i \int dt \, dt' K^* (t) G_r (t-t') K(t')} . \] (342)

The action principle says

\[ \delta \langle t_1 | t_2 \rangle = i \langle t_1 | \delta [ W_1 = \int dt \, L ] | t_2 \rangle . \] (343)

In a general sense, the exponent in Eq. (342) is an integrated form of the action. In solving the equation of motion, we found in Eq. (241)

\[ y(t) = e^{-i \omega (t-t_2)} y(t_2) + \int dt' G_r (t-t') K(t') , \] (344)

where the first term is effectively zero here. The net effect is to replace an operator by a number:

\[ y'(t) = \int dt' G_r (t-t') K(t') . \] (345)

Then Eq. (342) can be written as

\[ \langle 0 | t_1 | 0 | t_2 \rangle^K = e^{-i \int dt \, K^* (t) y'(t)} . \] (346)

Recall that the action was the integral of the Lagrangian (246), or

\[ W = \int dt \left[ y^\dagger i \frac{\partial}{\partial t} y - \omega y^\dagger y - y^\dagger K(t) - y K^* (t) \right] , \] (347)

so we see one term in Eq. (346) here, and the equation of motion (217) cancels out the rest! So let’s add something which gives the equation for \( y' \):

\[ \langle 0 | t_1 | 0 | t_2 \rangle^K = e^{\int dt \, y' y^\dagger K K^*} = e^{iW} . \] (348)

Now insist that \( W \) is stationary with respect to variations of \( y', y^\dagger \), and we recover the equation of motion,

\[ \left( i \frac{d}{dt} - \omega \right) y'(t) = K(t) . \] (349)

This is the starting point for the development of source theory, which will be treated in Part II.

10 End of Part I

We have traced Schwinger’s development of action formulations from classical systems of particles and fields, to the description of quantum dynamics through the Quantum Action Principle. In the latter, we here described only quantum mechanical systems, especially the driven harmonic oscillator. This is ahistorical, since Schwinger first developed his quantum dynamical principle in the context of quantum electrodynamics in the early 1950s, and only nearly a decade later applied it to quantum mechanics, which is field theory in one dimension—time. At roughly the same time he was thinking about quantum statistical systems [Martin 1959], and it was natural to turn to
a description of nonequilibrium systems, which was the motivation of the time-cycle method, although Schwinger put it in a general, although simplified, context. The time cycle method was immediately applied to quantum field theory by his students, K. T. Mahanthappa and P. M. Bakshi [Mahanthappa 1962, Bakshi 1963]. But rather than here tracing the profound and growing influence of this great paper, as well as the deep underpinning still provided by Schwinger’s action principle, we need to carry out a sketch of the application of these methods to quantum field theory, and to what Schwinger perceived as the successor to field theory, Source Theory. But we have now reached an appropriate point to pause. In Part II of this paper we will provide that elaboration, and trace some of the vast influence that Schwinger’s development of these powerful techniques have had in all branches of theoretical physics.

I thank the Laboratoire Kastler Brossel, ENS, UPMC, CNRS, for its hospitality during the completion of this manuscript. I especially thank Astrid Lambrecht and Serge Reynaud. The work was completed in part with funding from the Simons Foundation and the CNRS. I thank my many students at the University of Oklahoma, where much of the material reported here was used as the basis of lectures in quantum mechanics and quantum field theory.

References

P. M. Bakshi and K. T. Mahanthappa. 1963. “Expectation Value Formalism in Quantum Field Theory.” J. Math. Phys. 4: 1.
P. A. M. Dirac. 1927. “The Physical Interpretation of the Quantum Dynamics.” Proc. Roy. Soc. London A 113 (765): 621–641.
P. A. M. Dirac, V. A. Fock, and B. Podolsky. 1932. “On Quantum Electrodynamics.” Phys. Zeits. Sowjetunion 2: 468.
P. A. M. Dirac. 1933. “The Lagrangian in Quantum Mechanics.” Phys. Zeits. Sowjetunion 3: 64.
L. Euler. 1744. Methodus Inveniendi Lineas Curvas Maximi Minive ProprieteGaudentes. Bousquet, Lausanne and Geneva.
L. Euler. 1752. “Investigation of the letter, allegedly written by Leibniz,” translated by Wikisource [http://en.wikisource.org/wiki/Investigation_of_the_letter_of_Leibniz].
R. P. Feynman. 1942. “The Principles of Least Action in Quantum Mechanics.” Ph.D Dissertation, Princeton University, Princeton, NJ. (University Microfilms, Ann Arbor, Publications No. 2948).
R. P. Feynman. 1949. “Space-Time Approach to Quantum Electrodynamics.” Phys. Rev. 76: 769.
R. P. Feynman and A. R. Hibbs. 1965. Quantum Mechanics and Path Integrals. McGraw-Hill, New York.
R. J. Glauber. 1963. “Coherent and incoherent states of radiation field,” Phys. Rev. 131: 2766–2788.
W. R. Hamilton. 1834. “On a General Method in Dynamics, Part I.” Phil. Trans. Roy. Soc. 124: 247–308.
W. R. Hamilton. 1835. “On a General Method in Dynamics, Part II.” Phil. Trans. Roy. Soc. 125: 95–144.
L. V. Keldysh. 1964. “Diagram Technique for Nonequilibrium Processes,” Zh. Eksp. Teor. Fiz. 47: 1515–1527. [English translation: 1965. Soviet Physics JETP 20: 1018–1026.]
J.-L. Lagrange. 1788. Méchanique Analytique. p. 226.
K. T. Mahanthappa. 1962. “Multiple Production of Photons in Quantum Electrodynamics.” Phys. Rev. 126: 329.
P. C. Martin and J. Schwinger. 1959. “Theory of Many-Particle Systems.” Phys. Rev. 115: 1342.
P. L. M. de Maupertuis. 1744. “Accord de différentes lois de la nature qui avaient jusqu’ici paru incompatibles.” Mém. As. Sc. Paris p. 417.

P. L. M. de Maupertuis. 1746. “Le lois de mouvement et du repos, déduites d’un principe de métaphysique.” Mém. Ac. Berlin, p. 267.

J. Mehra and K. A. Milton. 2000. Climbing the Mountain: The Scientific Biography of Julian Schwinger. Oxford University Press, Oxford.

K. A. Milton. 2007. “In Appreciation Julian Schwinger: From Nuclear Physics and Quantum Electrodynamics to Source Theory and Beyond.” [arXiv:physics/0610054] Physics in Perspective 9: 70–114.

E. Noether. 1918. “Invariante Variationsprobleme.” Nachr. König. Gesellsch. Wiss. Göttingen, Math-phys. Klasse 1918: 235–257.

E. Schrödinger. 1926. “Der stetige Übergang von der Mikro- zur Makromechanik.” Naturwissenschaften 14: 664–666.

J. Schwinger. 1951. “The Theory of Quantized Fields. I” Phys. Rev. 82: 914.

J. Schwinger. 1953. “Theory of Quantized Fields. III.” Phys. Rev. 91: 728–740.

J. Schwinger. 1960. “The Geometry of Quantum States.” Proc. Natl. Acad. Sci. USA 46: 257.

J. Schwinger. 1960. “Unitary Operator Bases.” Proc. Natl. Acad. Sci. USA 46: 570.

J. Schwinger. 1960. “Unitary Transformations and the Action Principle.” Proc. Natl. Acad. Sci. USA 46: 883.

J. Schwinger. 1960. “The Special Canonical Group.” Proc. Natl. Acad. Sci. USA 46: 1401.

J. Schwinger. 1961. “Brownian Motion of a Quantum Oscillator.” J. Math. Phys. 2: 407.

J. Schwinger. 1970. Quantum Kinematics and Dynamics. Benjamin, New York.

J. Schwinger. 1973. “A Report on Quantum Electrodynamics,” in J. Mehra, The Physicist’s Conception of Nature”. Reidel, Dordrecht.

J. Schwinger. 1989. “A Path to Quantum Electrodynamics” Physics Today, February. [Reprinted in Most of the Good Stuff: Memories of Richard Feynman. 1993. (Eds. L. M. Brown and J. S. Rigden) AIP, New York.]

J. Schwinger, L. L. DeRaad, Jr., K. A. Milton, and W.-y. Tsai. 1998. Classical Electrodynamics. Perseus/Westview, New York.

J. Schwinger. 2001. Quantum Mechanics: Symbolism of Atomic Measurements. Springer, Berlin.

A. Sommerfeld. 1964. Mechanics—Lectures on Theoretical Physics, Volume I. Academic Press, New York. [Translated from the fourth German edition by Martin O. Stern.]

H. Weyl. 1919. “Eine neue Erweiterung der Relativitätstheorie.” Ann. der Phys. 59: 101–133.