Schreier decomposition of loops

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Abstract

The aims of this paper are to find algebraic characterizations of Schreier loops and explore the limits of the non-associative generalization of the theory of Schreier extensions. A loop can have Schreier decomposition with respect to a normal subgroup if and only if the subgroup is the middle and right nuclear. In this case the conjugation by elements of the loop induces inner automorphisms on the normal subgroup if and only if the subgroup commutes with a suitable left transversal through the identity. Schreier loops which are Schreier extensions of the same loop by the same normal subgroups are characterized.

1 Introduction

A loop \( L \) is an extension of the loop \( N \) by the loop \( K \) if \( N \) is a normal subloop of \( L \) and \( K \) is isomorphic to the factor loop \( L/N \). Extension theory deals with the classification of all possible extensions of \( N \) by \( K \) and studies their properties. The related problems in group theory are completely solved by Schreier theory of group extensions, cf. [9], [10], [4], Chapter XII, §§48-49, pp. 121–131. But for loops A. A. Albert and R. H. Bruck proved in 1944 that construction of loop extensions of \( N \) by \( K \) has much more degrees of freedom, namely the multiplication function between different cosets \( \neq N \) of \( N \) can be prescribed arbitrarily. An interesting class of loop extensions of groups by loops is introduced in [3], where the multiplication of the extended loop is determined by an analogous formula as in the Schreier theory of groups. This paper contains characterizations and constructions of examples for interesting subclasses of such loops; these loops are called Schreier loops. In recent papers [6] and [7] a non-associative extension theory of Schreier type is investigated in a broader context, namely there are given characterizations of right nuclear automorphism-free extensions of groups by
quasigroups with right identity element. As a consequence of these results, it turns out that the extension of a normal subgroup by the factor loop is isomorphic to an automorphism-free Schreier loop if and only if the normal subgroup is right and middle nuclear and there exists a left transversal to the normal subgroup (through the identity element of the loop) which commutes with this subgroup.

The aims of this paper are to find algebraic characterizations of Schreier loops and to examine the limits of the non-associative generalization of Schreier theory of extensions. In §2 we give the necessary definitions and formulate the basic constructions. §3 is devoted to the discussion of the interrelation between nuclear properties of normal subgroups in a loop and the corresponding Schreier extensions. Particularly, we show that for any Schreier extension this normal subgroup is the middle and right nuclear but in the general case it is not left nuclear. In §4 we introduce the notion of Schreier decomposition and show that a Schreier decomposition of a loop $L$ is uniquely determined by a middle and right nuclear normal subgroup $G$, an isomorphism of a loop $K$ to the factor loop $L/G$ and by a left transversal to $G$ through the identity element. §5 is devoted to the study of automorphisms of middle and right nuclear normal subgroups of a loop induced by middle inner mappings by loop elements. All of these maps are inner automorphisms if and only if there exists a left transversal through the identity element to the subgroup commuting with this subgroup. In §6 we investigate different properties of Schreier decompositions of a loop. We give characterizations of loops having automorphism-free, respectively factor-free Schreier decompositions. §7 is devoted to the study of Schreier loops which are Schreier decompositions of the same loop with respect to the same normal subgroup.

## 2 Preliminaries

A quasigroup $L$ is a set with a binary operation $(x, y) \mapsto x \cdot y$ such that for each $x \in L$ the left and the right translations $\lambda_x : y \mapsto \lambda_x y = xy : L \to L$, respectively $\rho_x : y \mapsto \rho_x y = yx : L \to L$ are bijective maps. We define the left and right division operations on $L$ by $(x, y) \mapsto x \backslash y = \lambda_x^{-1} y$, respectively $(x, y) \mapsto x/y = \rho_y^{-1} x$ for all $x, y \in L$. A quasigroup $L$ is a loop if it has a identity element $e \in L$. The right inner mappings of a loop $L$ are the maps $ho_y^{-1} \rho_x \rho_y : L \to L, \; x, y \in L$.

We will reduce the use of parentheses by the following convention: juxtaposition will denote multiplication, the operations $\backslash$ and $/$ are less binding than juxtaposition, and $\cdot$ is less binding than $\backslash$ and $/$. For instance the expression $xy/u \cdot v/w$ is a short form of $((x \cdot y)/u) \cdot (v/w)$.

The subgroups

$$\mathcal{N}_i(L) = \{ u \in L; \; ux \cdot y = u \cdot xy, \; x, y \in L \}$$



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of a loop $L$ are called the left, right, respectively middle nucleus of $L$, the intersection $N(L) = \mathcal{N}_l(L) \cap \mathcal{N}_r(L) \cap \mathcal{N}_m(L)$ is the nucleus. A subgroup $G \subseteq L$ is (left, right, respectively middle) nuclear, if it is contained in the (left, right, respectively middle) nucleus of $L$. The commutant of a subloop $K \subseteq L$ is the subset $\mathcal{C}_L(K) = \{ u \in L; u \cdot x = x \cdot u, \ x \in L \}$.

A loop $L$ satisfies the left, respectively the right inverse property if there exists a bijection $x \mapsto x^{-1} : L \rightarrow L$ such that $x^{-1} \cdot xy = y$ holds for all $x, y \in L$. The left alternative, respectively right alternative property of $L$ is defined by the identity $x \cdot xy = x^2y$, respectively $yx \cdot x = yx^2$. $L$ is flexible if $x \cdot yx = xy \cdot x$ for all $x, y \in L$. A loop $L$ is called left, respectively right Bol loop, if it satisfies the identity $(x \cdot yx)z = x(y \cdot xz)$, respectively $z(xy \cdot x) = (zx \cdot y)x$ for all $x, y, z \in L$. Any left (respectively right) Bol loop has the left (respectively right) alternative and inverse properties.

A subloop $N$ is a normal subloop of $L$ if it is the kernel of a homomorphism of $L$. The factor loop $L/N$ is the loop induced on the set of left cosets of a normal subloop $N$. A set $\Sigma \subseteq L$ of representatives of left cosets of a subloop $N$ in $L$ with $e \in \Sigma$ is a left transversal of $L/N$. A loop $L$ is an extension of a loop $N$ by a loop $K$ if $N$ is a normal subloop of $L$ and $K$ is isomorphic to the factor loop $L/N$. An extension $L$ of $N$ by $K$ is (left, right, respectively middle) nuclear, if $N$ is a (left, right, respectively middle) nuclear subgroup of $L$.

3 Schreier loops

3.1 Bruck extension

In 1944 A. A. Albert initiated in [1], (Theorem 6, pp. 406-407) and R. H. Bruck thoroughly investigated a general construction for loop extensions of loops by a loops in the papers [2], (Theorem 10 B, pp. 166-168) and [3], (pp. 778-779). This can be described as follows.

Let $K$ and $N$ be loops with identity elements $e \in K$ and $e \in N$, and let

$$\{ \nabla_{\alpha, \beta} : N \times N \rightarrow N; \ \alpha, \beta \in K \}$$

be a family of quasigroup multiplications on $N$ such that the equations $e \nabla_{e, \alpha} x = x$ and $x \nabla_{\alpha, e} e = x$ are fulfilled for any $\alpha \in K$ and $x \in N$. The multiplication

$$(\alpha, a) \circ (\beta, b) = (\alpha \beta, a \nabla_{\alpha, \beta} b), \ \alpha, \beta \in K, \ a, b \in N,$$

of the pairs $(\alpha, a), (\beta, b) \in K \times N$ determines a loop $L^\nabla$ on $K \times N$ with identity $(e, e)$. Clearly, $L^\nabla$ is an extension of the normal subloop $N = \{ u \in L; x \cdot y = u, \ x, y \in L \}$.
\{(\epsilon,a); a \in N\} by the loop \(K\), where \(\bar{N}\) is isomorphic to \(N\).

In the following we will discuss nuclear properties of normal subgroups of loops and the corresponding extensions of groups by loops. The following lemma allows us to consider a family of extensions such that the normal subgroup \(\bar{G} = \{(\epsilon,a); a \in G\}\) is right nuclear but not middle or left nuclear.

**Lemma 1** Let \(G\) be a group with identity \(e \in G\), \(K\) a loop with identity \(\epsilon \in K\) and let \(\psi_\sigma : G \to G\) be bijective maps depending on \(\sigma \in K\) satisfying \(\psi_\sigma(e) = e\) for any \(\sigma \in K\) and \(\psi_\epsilon = \text{Id}\). The multiplication

\[(\alpha,a) \circ (\beta,b) = (\alpha \beta, \psi_\beta(a)b), \quad \alpha, \beta \in K, \quad a, b \in G,\]

determines a loop on \(K \times G\) with identity \((\epsilon,e)\).

(i) The subgroup \(\bar{G}\) is right nuclear.

(ii) \(\bar{G}\) is middle nuclear if and only if \(\psi_\sigma : G \to G\) is an automorphism of \(G\) for any \(\sigma \in K\).

(iii) \(\bar{G}\) is left nuclear if and only if the map \(\sigma \mapsto \psi_\sigma\) is a homomorphism from \(K\) into the permutation group of \(G\).

**Proof.** The assertions (i) and (ii) can be obtained by direct computation. (iii) is equivalent to the identity

\[\psi_\beta(\psi_\alpha(x)a) = \psi_{\alpha \beta}(x)\psi_\beta(a), \quad \alpha, \beta \in K, \quad x,a \in G.\]

Putting \(a = e\) the assertion follows. \(\blacksquare\)

### 3.2 Schreier extension

In the following we consider a special case of Bruck’s extension process, assuming that the extended multiplication has an analogous expression as in Schreier theory of group extensions (cf. [5]).

Let \(G\) be a group with identity \(e \in G\), \(K\) a loop with identity \(\epsilon \in K\) and let \(\text{Aut}(G)\) denote the automorphism group of \(G\). If \(\sigma \mapsto \Theta_\sigma\) is a mapping \(K \to \text{Aut}(G)\) with \(\Theta_\epsilon = \text{Id}\) and \(f : K \times K \to G\) is a function satisfying \(f(\epsilon,\sigma) = f(\sigma,\epsilon) = e\) for all \(\sigma \in K\) then the multiplication

\[(\tau,t) \circ (\sigma,s) = (\tau \sigma, f(\tau,\sigma)\Theta_\sigma(t)s), \quad \tau, \sigma \in K, \quad t,s \in G,\]

together with the divisions

\[(\rho,r)/(\sigma,s) = (\rho/\sigma, \Theta_{\sigma^{-1}}^{-1}(f(\rho/\sigma,\sigma)^{-1}rs^{-1})) ,\]

\[(\sigma,s)\backslash(\rho,r) = (\sigma \backslash \rho, \Theta_{\rho\sigma^{-1}}^{-1}(s)f(\sigma,\rho\sigma^{-1}r))\]

define on the set \(K \times G\) a loop which will be denoted by \(\mathcal{L}(\Theta,f)\). The identity of \(\mathcal{L}(\Theta,f)\) is \((\epsilon,e)\) and \(\bar{G} = \{(\epsilon,t); \ t \in G\}\) is a normal subgroup of \(\mathcal{L}(\Theta,f)\).

The maps defined by \((\epsilon,t) \mapsto t : \bar{G} \to G\) and \(\tau \mapsto (\tau,e)\bar{G} : K \to \mathcal{L}(\Theta,f)/\bar{G}\) are isomorphisms. Clearly, \(\mathcal{L}(\Theta,f)\) is an extension of the group \(\bar{G}\) by the loop \(K\).
Definition 2 The loop \( L(\Theta, f) \) determined by the multiplication (1) is called a Schreier loop. \( L(\Theta, f) \) is automorphism-free if \( \Theta_\sigma = \text{Id} \) for all \( \sigma \in K \), respectively factor-free if \( f(\sigma, \tau) = e \) for all \( \sigma, \tau \in K \).

\( L(\Theta, f) \) is a group if and only if \( K \) is a group and the identities

\[
\Theta_\sigma \Theta_\tau^{-1} = \iota_{f(\sigma, \tau)}, \quad \sigma, \tau \in K, \tag{2}
\]

where \( \iota_s : N \to N \) is the inner automorphism \( \iota_s(t) = sts^{-1}, s, t \in G \) and

\[
f(\sigma, \tau \rho)^{-1} f(\sigma \tau, \rho) \Theta_\rho (f(\sigma, \tau)) f(\tau, \rho)^{-1} = e, \quad \sigma, \tau, \rho \in K \tag{3}
\]

are satisfied, (cf. [4], §48).

Lemma 3 For any Schreier loop \( L(\Theta, f) \) the subgroup \( \bar{G} = \{(\epsilon, t); t \in G\} \) satisfies:

(i) \( \bar{G} \) is middle and right nuclear,

(ii) \( \bar{G} \) is nuclear if and only if the identity (4) is satisfied,

(iii) if \( L(\Theta, f) \) has left inverse, left alternative of flexible property, then \( \bar{G} \) is nuclear.

Proof. The assertions follow from Proposition 3.2, and Propositions 3.7, 3.8 and 3.10 in [5].

Corollary 4 The subgroup \( \bar{G} \) of an automorphism-free Schreier loop \( L(\text{Id}, f) \) is not left nuclear if there is a value of the function \( f \) which is not contained in the center of \( G \).

Hence there are many examples of Schreier loops such that the subgroup \( \bar{G} \subset L(\Theta, f) \) is middle and right nuclear, but not nuclear.

4 Schreier decomposition

Lemma 5 Let \( L \) be a loop extension of the group \( G \) by the loop \( K \) and let \( L(\Theta, f) \) be a Schreier loop defined on \( K \times G \). If the map \( F : L(\Theta, f) \to L \) is an isomorphism satisfying \( F(\epsilon, t) = t \) for any \( t \in G \), then the induced map

\[
\sigma \mapsto F(\sigma, \epsilon)G : K \to L/G
\]

is an isomorphism, too.

Proof. Since the map \( \sigma \mapsto (\sigma, \epsilon)G : K \to L/G \) is an isomorphism the image of the coset \( (\sigma, \epsilon)G \) is the coset \( \{F((\sigma, \epsilon)(\epsilon, s)), s \in G\} = \{F(\sigma, \epsilon)s, s \in G\} \) and the assertion follows.

We notice that the isomorphism \( F : L(\Theta, f) \to L \) satisfies \( F(\epsilon, t) = t \) for any \( t \in G \) if and only if \( F \) is an extension of the isomorphism \( I : \bar{G} \to G \) defined by \( I(\epsilon, t) = t \).
Definition 6 Let $K$ and $L$ be loops, $G$ a normal subgroup of $L$ and $L(\Theta, f)$ a Schreier loop defined on $K \times G$. A Schreier decomposition of $L$ with respect to its normal subgroup $G$ is an isomorphism $F : L(\Theta, f) \to L$ satisfying $F(\epsilon, t) = t$ for any $t \in G$. The underlying isomorphism of the Schreier decomposition $F$ is the map $\sigma \mapsto F(\sigma, e)G : K \to L/G$.

The following lemma shows that by investigation of Schreier decompositions of $L$ with respect to its normal subgroup $G$ the middle and right nuclear property of $G$ is a reasonable assumption.

Lemma 7 If a loop $L$ has a Schreier decomposition with respect to the subgroup $G$, then $G$ is middle and right nuclear.

Proof. Let $F : L(\Theta, f) \to L$ be a Schreier decomposition of $L$ with respect to $G$. According to Lemma 3 (i), the subgroup $\bar{G}$ of $L(\Theta, f)$ is middle and right nuclear. Since $F$ preserves this property the assertion follows.

Lemma 8 For any Schreier loop $L(\Theta, f)$ defined on $K \times G$ there exists an extension $L$ of $G$ by $K$ and an isomorphism $F : L(\Theta, f) \to L$ which is a Schreier decomposition of $L$ with respect to $G$.

Proof. We replace in $K \times G$ the elements $(\epsilon, t) \in \bar{G}$ by the corresponding elements $I(\epsilon, t) = t \in G$ and define a loop $L$ on the set $((K \times G) \setminus \bar{G}) \cup G$ in such a way, that the map $I : \bar{G} \to G$ together with the identity map on $(K \times G) \setminus \bar{G}$ give an isomorphism $F : L(\Theta, f) \to L$. Then $F : L(\Theta, f) \to L$ is a Schreier decomposition of $L$ with respect to $G$.

Left transversals to $G$ in $L$

Lemma 9 If $G$ is a middle and right nuclear normal subgroup of the loop $L$ then the maps $T_x|_G : G \to G$, induced by the middle inner mappings $T_x = \rho_x^{-1} \lambda_x$, $x \in L$, are automorphisms of $G$.

Proof. If $\pi : L \to L/G$ is the canonical homomorphism then
\[ \pi(T_x(t)) = \pi(x)\pi(t)/\pi(x) = \epsilon, \quad x \in L, \ t \in G, \]
and hence $T_x|_G$ is a bijection $G \to G$. Using $T_x(t) \in N_m(L)$ and $t \in N_r(L)$ the assertion follows from $T_x(s)T_x(t) \cdot x = T_x(s) \cdot xt = x \cdot st$ for any $x \in L, s,t \in G$.

Definition 10 $(\kappa, \Sigma)$ is a data pair for a normal subgroup $G$ of $L$, if

(i) $\kappa$ is an isomorphism of a loop $K$ onto the loop $L/G$,

(ii) $\Sigma$ is a left transversal of $L/G$. 

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Let $(\kappa, \Sigma)$ be a data pair for the normal subgroup $G$ of $L$ and let $l : K \to L$ be the map defined by the relation $l_\sigma = \Sigma \cap \kappa(\sigma)$.

**Definition 11** The Schreier loop $L(\Theta, f)$ determined by the functions

$$
\Theta_\sigma = T l_\sigma^{-1} | G \quad \text{and} \quad f(\sigma, \tau) = l_{\sigma \tau} \backslash l_\sigma l_\tau, \quad \sigma, \tau \in K
$$

is called the Schreier loop corresponding to the data pair $(\kappa, \Sigma)$.

**Theorem 12** Let $(\kappa, \Sigma)$ be a data pair for a middle and right nuclear normal subgroup $G$ of $L$ and let $L(\Theta, f)$ be the corresponding Schreier loop. Then the map $\mathcal{F} : L(\Theta, f) \to L$ defined by $\mathcal{F}(\sigma, s) = l_\sigma s$ is a Schreier decomposition of $L$ with respect to $G$ with underlying isomorphism $\kappa : K \to L/G$.

Conversely, if $\mathcal{F} : L(\Theta, f) \to L$ is a Schreier decomposition of $L$ with respect to $G$ then $G$ is middle and right nuclear. The isomorphism $\mathcal{F}$ has the form $\mathcal{F}(\sigma, s) = l_\sigma s$, where $l : K \to L$ is defined by $l_\sigma = \mathcal{F}(\sigma, e)$. Moreover the functions $\Theta$ and $f$ satisfy equations (4).

**Proof.** Let $L(\Theta, f)$ be the Schreier loop corresponding to $(\kappa, \Sigma)$. The value $f(\sigma, \tau)$ is contained in $G$ for all $\sigma, \tau \in G$, since $\pi(l_{\sigma \tau} \backslash l_\sigma l_\tau) = e$.

Any element $x \in L$ can be uniquely decomposed as a product $x = l_\sigma s$ with $l_\sigma \in \Sigma$, $s \in G$. The bijective map $\mathcal{F} : K \times G \to L$ defined by $\mathcal{F}(\sigma, s) = l_\sigma s$ satisfies $\mathcal{F}(e, t) = t$ for any $t \in G$. Since $s, t \in G$ are middle and right nuclear elements, we have

$$
\mathcal{F}(\sigma, s) \mathcal{F}(\tau, t) = l_\sigma s \cdot l_\tau t = l_\sigma (s l_\tau \cdot t) = l_\sigma l_\tau \cdot (l_\tau \backslash s l_\tau \cdot t) = l_\sigma l_\tau \cdot l_{\sigma \tau} l_\sigma(s \Theta_\tau) t
$$

for any $\sigma, \tau \in K$, $s, t \in G$. Moreover, the product $l_{\sigma \tau} \cdot l_{\sigma \tau} l_\sigma(s \Theta_\tau) t$ is in right nuclear, hence

$$
l_{\sigma \tau} \cdot l_{\sigma \tau} l_\sigma(s \Theta_\tau) t = l_{\sigma \tau} (l_{\sigma \tau} \backslash l_\sigma l_\tau \cdot l_{\sigma \tau} l_\sigma(s \Theta_\tau) t) = l_{\sigma \tau} \cdot f(\sigma, \tau) l_{\sigma \tau} l_\tau \cdot l_{\sigma \tau} l_\sigma(s \Theta_\tau) t.
$$

We obtain from (4) and (5) that

$$
\mathcal{F}(\sigma, s) \mathcal{F}(\tau, t) = l_{\sigma \tau} \cdot f(\sigma, \tau) \Theta_\tau(s) t = \mathcal{F}(\sigma, \tau) \mathcal{F}(\sigma, \tau) \Theta_\tau(s) t.
$$

This means that $\mathcal{F} : K \times G \to L$ is an isomorphism $L(\Theta, f) \to L$. Moreover, $\mathcal{F}(\sigma, e) \mathcal{F}(e, t) = l_\sigma t$, hence $\mathcal{F}(\sigma, e) G = \kappa(\sigma)$. Consequently $\mathcal{F} : L(\Theta, f) \to L$ is a Schreier decomposition of $L$ and $\kappa$ is the underlying isomorphism.

Conversely, let $\mathcal{F} : L(\Theta, f) \to L$ be a Schreier decomposition of $L$ with respect to $G$. According to Lemma 7 the subgroup $G$ is middle and right nuclear. The set $\Sigma = \{ l_\sigma = \mathcal{F}(\sigma, e) : \sigma \in K \}$ is a left transversal of $L/G$ and $\mathcal{F}(\sigma, s) = l_\sigma s$ for any $\sigma \in K$ and $s \in G$. One has

$$
(\sigma, e) \backslash (e, t) (\sigma, e) = (\sigma, e) \backslash (\sigma, \Theta(t)) = (e, \Theta(t)) \quad \text{for any } \sigma \in K, \ t \in G,
$$

and

$$
(\sigma \tau, e) \backslash (\sigma, e) (\tau, e) = (\sigma \tau, e) \backslash (\sigma \tau, f(\sigma, \tau)) = (e, f(\sigma, \tau)) \quad \text{for any } \sigma, \tau \in K.
$$

Using the isomorphism $\mathcal{F} : L(\Theta, f) \to L$ we obtain equations (4).
Corollary 13 A loop $L$ has a Schreier decomposition with respect to a normal subgroup $G$ if and only if $G$ is middle and right nuclear.

Proposition 14 If a loop $L$ satisfies one of the following conditions:

(i) left inverse property,
(ii) left alternative,
(iii) flexible,

then any middle and right nuclear normal subgroup of $L$ is nuclear.

Proof. It follows from Theorem 12 that for a middle and right nuclear normal subgroup $G$ of $L$ there is a Schreier decomposition $\mathcal{F} : \mathcal{L}(\Theta, f) \to L$ with respect to $G$. According to Propositions 3.7, 3.8, respectively 3.10 in [5] a Schreier loop $\mathcal{L}(\Theta, f)$ having the left inverse, left alternative, respectively flexible property, satisfies the condition [2]. In this case we obtain from Proposition 3.2.(i) in [5] that the normal subgroup $\bar{G} = \{ (\varepsilon, t) : t \in G \}$ of $\mathcal{L}(\Theta, f)$ is nuclear, and hence $G$ is also a nuclear subgroup of $L$.

Now, we give examples for Schreier loops having the right Bol property such that the normal subgroup $\bar{G}$ is middle and right nuclear, but not nuclear. These properties can be verified by easy computation.

Example 15 Let $K$ be a right Bol loop, $G$ a group and $H$ the group generated by the right inner mappings $\rho^1_{\tau^1_\sigma} : K \to K$, $\sigma, \tau \in K$. Let $\chi : H \to G$ be a homomorphism such that the image $\chi(H)$ is not contained in the center of $G$. Define the maps $f : K \times K \to G$ and $\Theta : K \to \text{Aut}(G)$ by

$$f(\tau,\sigma) = \chi(\rho^1_{\tau^1_\sigma} \rho^1_\sigma), \quad \Theta = \text{Id}, \quad \sigma, \tau \in K,$$

and consider the corresponding Schreier loop $\mathcal{L}(\text{Id}, f)$.

Example 16 Let $K$ and $G$ be groups. Assume that the group $K$ is not abelian and denote by $K'$ the commutator subgroup of $K$. Let $\phi : K' \to G$ be a homomorphism such that the image $\phi(K')$ is not contained in the center of $G$. Define the maps $f : K \times K \to G$ and $\Theta : K \to \text{Aut}(G)$ by

$$f(\tau,\sigma) = \phi(\sigma^{-1} \tau^{-1} \sigma \tau), \quad \Theta = \text{Id}, \quad \sigma, \tau \in K,$$

and consider the corresponding Schreier loop $\mathcal{L}(\text{Id}, f)$.

Example 17 Let $K$ and $G$ be groups with identity $\varepsilon \in K$ and $e \in G$, respectively. Assume that the group $K$ is not abelian. Let $\phi : K \to G$ be a homomorphism such that the image $\phi(K)$ is not contained in the center of $G$. Define the maps $f : K \times K \to G$ and $\Theta : K \to \text{Aut}(G)$ by

$$f(\tau,\sigma) = e, \quad \Theta = \iota^{-1}_{\phi(\sigma)} : u \mapsto \phi(\sigma)^{-1} u \phi(\sigma), \quad u \in G, \quad \sigma \in K,$$

$$\iota_s(t) = s t s^{-1}, \quad s, t \in G$$

and consider the corresponding Schreier loop $\mathcal{L}(\iota_{\phi}, e)$. 

8
5 Automorphisms of $G$ induced by elements of $L$

According to Lemma 9 the maps $T_x|_G$ are automorphisms of a middle and right nuclear normal subgroup $G$ of a loop $L$, where $x$ is an arbitrary element of $L$. If $r \in G$ then $T_r|_G$ is the inner automorphism $t_r(t) = rtr^{-1}$, $r, t \in G$.

**Lemma 18** If $G$ is a middle and right nuclear normal subgroup of a loop $L$ then the automorphisms $T_{xr}|_G$ and $T_{rx}|_G$ with $x \in L$ and $r \in G$ can be decomposed as

$$T_{xr}|_G = T_x|_G \circ t_r, \quad T_{rx}|_G = t_r \circ T_x|_G.$$

**Proof.** Since $s$ and $r$ belong to $N_r(L)$, we have

$$T_{xr}(s) \cdot xr = xr \cdot s = x \cdot t_r(s)r = T_x(t_r(s))x \cdot r = T_x(t_r(s))(xr),$$

hence the first assertion is true. Similarly, the second assertion follows from

$$T_{rx}(s) \cdot rx = rx \cdot s = r \cdot xs = rT_x(s) \cdot x = t_r(T_x(s))(rx),$$

since $s \in N_r(L)$ and $T_x(s), r \in N_m(L)$. ■

**Corollary 19** If $G$ is middle and right nuclear normal subgroup in $L$ then the map $T|_G : L \to \text{Aut}(G)$ induces a map

$$A : xG \mapsto T_x|_{\text{Inn}(G)} : L/G \to \text{Aut}(G)/\text{Inn}(G),$$

where $\text{Inn}(G)$ denotes the group of inner automorphisms of $G$.

**Theorem 20** For a middle and right nuclear normal subgroup $G$ in $L$ the image of the map $T|_G : L \to \text{Aut}(G)$ consists of inner automorphisms if and only if there exists a left transversal $\Sigma$ of $L/G$ which is contained in the commutant $C_L(G)$ of $G$.

**Proof.** Assume that for any $x \in L$ the map $T_x|_G$ is an inner automorphism. Let $\Sigma$ be a left transversal of $L/G$ and $g : \Sigma \to G$ a map satisfying $g(e) = e$ and $T_x|_G = t_{g(x)}$ for any $x \in \Sigma$. Clearly, the set $\{x \cdot g(x)^{-1} : x \in \Sigma\} \subseteq C_L(G)$ is a left transversal of $L/G$. According to Lemma 19

$$T_{x \cdot g(x)^{-1}}|_G = T_x|_G \circ t_{g(x)^{-1}} = t_{g(x)} \circ t_{g(x)^{-1}} = \text{Id}$$

and hence $\Sigma^* \subseteq C_L(G)$. Conversely, let $\Sigma$ be a left transversal of $L/G$ such that $T_x|_G = \text{Id}_G$ for all $x \in \Sigma$. Any element of $L$ is a product $x \cdot r$ with $x \in \Sigma, r \in G$ and hence Lemma 19 yields that $T_{x \cdot r}|_G = T_x|_G \circ t_r = t_r$, i.e. $T_{x \cdot r}|_G$ is an inner automorphism of $G$. ■
Lemma 21 For a middle and right nuclear normal subgroup $G$ in $L$ the mapping $T|_G : L \to \text{Aut}(G)$ is a homomorphism if and only if $G$ is nuclear.

Proof. For any $s \in G$, $x, y \in L$ we have $s \in N_r(L)$, $T_y(s) \in N_m(L)$ and hence
\[ T_{xy}(s) \cdot xy = x \cdot T_y(s) y = xT_y(s) \cdot y = T_x(T_y(s))x \cdot y. \]

It follows that $T|_G : L \to \text{Aut}(G)$ is a homomorphism if and only if for any $x, y \in L$, $s \in G$ one has $T_x(T_y(s))x \cdot y = T_x(T_y(s)) \cdot xy$. Since $T_x|_G, T_y|_G : G \to G$ are bijective maps, the map $T|_G : L \to \text{Aut}(G)$ is a homomorphism if and only $G$ is left nuclear.

It follows from Proposition 14 the following

Corollary 22 For a middle and right nuclear normal subgroup $G$ in $L$ the map $T|_G : L \to \text{Aut}(G)$ is a homomorphism if $L$ satisfies one of the following conditions:

(i) left inverse property,
(ii) left alternative,
(iii) flexible.

According to Proposition 3.2.(i) in [5] the normal subgroup $\bar{G}$ of a Schreier loop $L(\Theta, f)$ defined on $K \times G$ is nuclear if and only if the maps $\Theta$ and $f$ satisfy the condition (2). Hence we have

Corollary 23 For a Schreier loop $L(\Theta, f)$ the map $T|_{\bar{G}} : L(\Theta, f) \to \text{Aut}(\bar{G})$ is a homomorphism if and only if $L(\Theta, f)$ satisfies the condition (2).

6 The properties of Schreier decompositions

Theorem 24 A loop $L$ has an automorphism-free Schreier decomposition $L(\text{Id}, f)$ with respect to a middle and right nuclear normal subgroup $G$ if and only if one of the following equivalent conditions is fulfilled:

(A) the image of the map $T : L \to \text{Aut}(G)$ consists of inner automorphisms,
(B) there exists a left transversal of $L/G$ which is contained in the commutant $C_L(G)$ of $G$ in $L$. (Cf. Theorem 4 in [7]):

Proof. Let $(\kappa, \Sigma)$ be a data pair for the normal subgroup $G$ of $L$ and let $l : K \to L$ be the map $l_\sigma = \Sigma \cap \kappa(\sigma)$. The Schreier loop $L(\Theta, f)$ corresponding to the data pair $(\kappa, \Sigma)$ is automorphism-free if and only if $T_{l_\sigma} = \text{Id}_G$, or equivalently the left transversal $\Sigma = \{l_\sigma; \sigma \in K\}$ of $L/G$ is contained in the commutant $C_L(G)$ of $G$. Hence we obtain assertion (B). The equivalence of conditions (A) and (B) is proved in Theorem [20].
Theorem 25 A loop $L$ has a factor-free Schreier decomposition $\mathcal{L}(\Theta, e)$ with respect to a middle and right nuclear normal subgroup $G$ if and only if $L$ contains a left transversal $\Sigma$ of $L/G$ which is a subloop of $L$ isomorphic to $K$.

Proof. Using the second formula of (4) we obtain that the Schreier loop defined by (4) is factor-free if and only if the map $l : K \rightarrow L$ satisfies $l_{\sigma \tau}l_{\tau} = e$ for any $\sigma, \tau \in K$, and hence the map $l : K \rightarrow L$ is a loop homomorphism. It follows that $L$ has a factor-free Schreier decomposition if and only if there exists a left transversal $\Sigma$ of $L/G$ which is a subloop of $L$.

The following assertion shows the alteration of the Schreier decomposition of a loop $L$ with respect to a normal subgroup $G$, if we change the underlying isomorphism.

Proposition 26 Let $\mathcal{L}(\Theta, f)$ be a Schreier decomposition of $L$ with respect to $G$ with underlying isomorphism $\kappa : K \rightarrow L/G$ and let $\mu$ be an automorphism of $K$. The Schreier loop $\tilde{\mathcal{L}}(\tilde{\Theta}, \tilde{f})$ is a Schreier decomposition of $L$ with respect to $G$ with underlying isomorphism $\kappa \circ \mu : K \rightarrow L/G$ if the functions $\tilde{\Theta} : K \rightarrow \text{Aut}(G)$, $\tilde{f} : K \times K \rightarrow G$ are defined by

$$\tilde{\Theta}_{\tau} = \Theta_{\mu(\tau)}, \quad \tilde{f}(\sigma, \tau) = f(\mu(\sigma), \mu(\tau)).$$

(7)

Proof. We denote the multiplication of $\tilde{L}(\tilde{\Theta}, \tilde{f})$ by $\tilde{\circ}$ and define the map

$$\mathcal{M} : (\sigma, s) \mapsto (\mu(\sigma), s) : K \times G \rightarrow K \times G.$$

Since

$$\mathcal{M}((\sigma, s) \tilde{\circ} (\tau, t)) = \mathcal{M}(\sigma \tau, \tilde{f}(\sigma \tau) \tilde{\Theta}_{\tau}(s) t) = \mathcal{M}(\sigma, s) \circ \mathcal{M}(\tau, t),$$

the map $\mathcal{M} : L(\tilde{\Theta}, \tilde{f}) \rightarrow \mathcal{L}(\Theta, f)$ is an isomorphism inducing the identity on the subgroup $G$. It follows that $\mathcal{F} \circ \mathcal{M} : L(\tilde{\Theta}, \tilde{f}) \rightarrow L$ is an isomorphism extending the isomorphism $\mathcal{I} : G \rightarrow G$. Hence $L(\tilde{\Theta}, \tilde{f})$ is a Schreier decomposition of $L$ with respect to $G$ with underlying isomorphism $\tilde{\kappa} : K \rightarrow L/G$ defined by

$$\tilde{\kappa}(\sigma) = \mathcal{F} \circ \mathcal{M}(\sigma, e)G, \quad \sigma \in K,$$

satisfying $\tilde{\kappa}(\sigma) = \kappa \circ \mu(\sigma)$.

Now, we investigate the alteration of the Schreier decomposition of a loop $L$ with respect to a normal subgroup $G$, if we change the left transversal $\Sigma$ of $L/G$ in the data pair $(\kappa, \Sigma)$ corresponding to the decomposition.

Theorem 27 The maps $\bar{\Theta} : K \rightarrow \text{Aut}(G)$ and $\bar{f} : K \times K \rightarrow G$ determined by the left transversal $\Sigma(\bar{1}) = \{l_\sigma n(\sigma) \in \kappa(\sigma), \sigma \in K\}$ can be expressed by

$$\bar{\Theta}_\sigma = i \circ \Theta_\sigma, \quad \bar{f}(\sigma, \tau) = n(\sigma \tau)^{-1} f(\sigma, \tau) \Theta_{\tau}(n(\sigma)) n(\tau).$$

(8)
Proof. Let \( n : K \to G \) be a function with \( n(\epsilon) = e \) and consider the left transversal

\[
\Sigma(l) = \{ l_\sigma n(\sigma) \in \kappa(\sigma), \ \sigma \in K \} \subset L.
\]  

(9)

We compute the maps \( \bar{\Theta}_\sigma = T^{-1}_l \) and \( \bar{f}(\sigma, \tau) = \bar{l}_{\sigma \tau} \setminus (\bar{l}_\sigma \bar{l}_\tau) \) corresponding to the left transversal (9). Since \( n(\sigma) \) belongs to the right and to the middle nucleus, the product \( l_\sigma n(\sigma) \cdot \Theta_\sigma(t) = t \cdot l_\sigma n(\sigma) \) equals to

\[
t l_\sigma \cdot n(\sigma) = l_\sigma \Theta_\sigma(t) \cdot n(\sigma) = l_\sigma \cdot \Theta_\sigma(t) n(\sigma) = l_\sigma n(\sigma) \cdot n(\sigma)^{-1} \Theta_\sigma(t) n(\sigma).
\]

Hence

\[
\bar{\Theta}_\sigma = l_{n(\sigma)}^{-1} \circ \Theta_\sigma.
\]  

(10)

Similarly, using that \( n(\sigma) \) belongs to the middle and \( n(\tau) \) to the right nucleus, the product

\[
l_{\sigma \tau} n(\sigma \tau) \cdot \bar{f}(\sigma, \tau) = l_\sigma n(\sigma) \cdot l_\tau n(\tau)
\]  

(11)

can be written as

\[
l_\sigma \left( n(\sigma) \cdot l_\tau n(\tau) \right) = l_\sigma \left( n(\sigma) l_\tau \cdot n(\tau) \right) = l_\sigma \left( l_\tau \cdot \Theta_\tau(n(\sigma)) n(\tau) \right).
\]

Since \( \Theta_\tau(n(\sigma)) n(\tau) \) is a right nuclear element, the expression (11) equals to

\[
( l_\sigma l_\tau ) \cdot \Theta_\tau(n(\sigma)) n(\tau).
\]

Replacing \( l_\sigma l_\tau = l_{\sigma \tau} \cdot \left( n(\sigma \tau) n(\sigma \tau) \cdot f(\sigma, \tau) \right) \) and using that \( n(\sigma \tau) \) and \( n(\sigma \tau)^{-1} \) belong to the middle nucleus, we obtain for the expression (11) that

\[
l_{\sigma \tau} n(\sigma \tau) \cdot \bar{f}(\sigma, \tau) = l_{\sigma \tau} n(\sigma \tau) \cdot \left( n(\sigma \tau)^{-1} f(\sigma, \tau) \Theta_\tau(n(\sigma)) n(\tau) \right),
\]

and hence

\[
\bar{f}(\sigma, \tau) = n(\sigma \tau)^{-1} f(\sigma, \tau) \Theta_\tau(n(\sigma)) n(\tau).
\]  

(12)

Using equations (10) and (12) we obtain the assertion. \( \blacksquare \)

7 Equivalent extensions

Let \( K \) be a loop, \( G \) a group and let \( \mathcal{L}(\Theta, f) \), \( \mathcal{L}(\Theta', f') \) be Schreier loops defined on \( K \times G \).

Definition 28 The Schreier loops \( \mathcal{L}(\Theta, f) \) and \( \mathcal{L}(\Theta', f') \) are called equivalent in a wider sense if there exists an extension \( L \) of \( G \) by \( K \) such that \( \mathcal{L}(\Theta, f) \) and \( \mathcal{L}(\Theta', f') \) are Schreier decompositions of \( L \) with respect to \( G \).
Lemma 29 The Schreier loops $\mathcal{L}(\Theta, f)$ and $\mathcal{L}(\Theta', f')$ are equivalent in a wider sense if and only if there exists an isomorphism $\psi : \mathcal{L}(\Theta, f) \to \mathcal{L}(\Theta', f')$ fixing all elements of $G \subset K \times G$.

Proof. Let $\mathcal{L}(\Theta, f)$ and $\mathcal{L}(\Theta', f')$ be Schreier decompositions of a loop $L$ with respect to $G$ and let $F : \mathcal{L}(\Theta, f) \to L$ and $F' : \mathcal{L}(\Theta', f') \to L$ be the isomorphisms extending the isomorphism $I : G \to G$ defined by $I(\epsilon, t) = t$. Then the isomorphism $F' \circ F : \mathcal{L}(\Theta, f) \to \mathcal{L}(\Theta', f')$ fixes all elements of $G = \{(\epsilon, t) ; t \in G\}$. Conversely, assume that $\psi : \mathcal{L}(\Theta, f) \to \mathcal{L}(\Theta', f')$ is an isomorphism fixing all elements of $G$. According to Lemma 8 there is a loop $L$ and an isomorphisms $F : \mathcal{L}(\Theta, f) \to L$ extending the isomorphism $I : G \to G$. Clearly, the isomorphism $F \circ \psi^{-1} : \mathcal{L}(\Theta', f') \to L$ extends the isomorphism $I : G \to G$, hence $\mathcal{L}(\Theta', f')$ is also a Schreier decomposition of $L$ with respect to $G$.

Definition 30 The Schreier loops $\mathcal{L}(\Theta, f)$ and $\mathcal{L}(\Theta', f')$ defined on $K \times G$ are called equivalent if there exists an extension $L$ of $G$ by $K$ such that $\mathcal{L}(\Theta, f)$ and $\mathcal{L}(\Theta', f')$ are Schreier decompositions of $L$ with respect to $G$ with the same underlying isomorphism $\kappa : K \to L/G$.

Theorem 31 Let $\mathcal{L}(\Theta, f)$ and $L(T', f')$ be Schreier loops defined on $K \times G$.

(A) $L(T', f')$ is equivalent to $\mathcal{L}(\Theta, f)$ if and only if there is a function $n : K \to G$ with $n(\epsilon) = e$ such that $\Theta'$ and $f'$ are expressed by

$$\Theta'_\sigma = i^{-1}_{n(\sigma)} \circ \Theta_\sigma, \quad \text{and} \quad f'(\sigma, \tau) = n(\sigma \tau)^{-1} f(\sigma, \tau) \Theta_\tau(n(\sigma)) n(\tau).$$

(B) $L(T', f')$ is equivalent in a wider sense to $\mathcal{L}(\Theta, f)$ if and only if there is a function $n : K \to G$ with $n(\epsilon) = e$, and an automorphism $\mu \in \text{Aut}(K)$, such that $\Theta'$ and $f'$ are expressed by

$$\Theta'_\sigma = i^{-1}_{n\mu(\sigma)} \circ \Theta_{\mu(\sigma)},$$

and

$$f'(\sigma, \tau) = (n \circ \mu(\sigma \tau))^{-1} f(\mu(\sigma), \mu(\tau)) \Theta_{\mu(\tau)}(n \circ \mu(\sigma)) n \circ \mu(\tau).$$

Proof. For the equivalent $\mathcal{L}(\Theta, f)$ and $\mathcal{L}(\Theta', f')$ there exists a loop $L$ such that $\mathcal{L}(\Theta, f)$ and $\mathcal{L}(\Theta', f')$ are Schreier decompositions of $L$ with respect to $G$ with the same underlying isomorphism $\kappa : K \to L/G$. Hence the assertion (A) follows from Theorem 27. If $L(T', f')$ is equivalent in a wider sense to $\mathcal{L}(\Theta, f)$, then according to Proposition 26 a change of the underlying isomorphism $K \to L/G$ of $L(T', f')$ by an automorphism $\mu$ of $K$ we obtain equivalent Schreier loops. Hence assertion (A) implies assertion (B).
References

[1] A. A. Albert, *Quasigroups II*, Trans. A.M.S., 55 (1944), 401-419.

[2] R. H. Bruck, *Some Results in the Theory of Linear Non-Associative Algebras*, Trans. A.M.S., 56, (1944), 141-199.

[3] R. H. Bruck, *Simple Quasigroups*, Bull. A.M.S., 50, (1944), pp. 769-781.

[4] A. G. Kurosh, *The Theory of Groups*, Vol. 2, Chelsea Publishing Co., New York, (1956).

[5] P. T. Nagy, K. Strambach, *Schreier loops*, Czechoslovak Math. J., 58 (133), (2008), 759-786.

[6] P. T. Nagy, I. Stuhl, *Right nuclei of quasigroup extensions*, Communications in Algebra, 40, (2012), 1893–1900.

[7] P. T. Nagy, I. Stuhl, *Quasigroups arisen by right nuclear extensions*, Comment. Math. Univ. Carolin., 53, (2012), 391–395.

[8] L. Rédei, *Algebra*, Pergamon Press, Oxford, (1967),

[9] O. Schreier, *Über die Erweiterung von Gruppen I*, Monatshefte f. Math. 34 (1926), 165-180.

[10] O. Schreier, *Über die Erweiterung von Gruppen II*, Abh. Math. Sem. Univ. Hamburg 4 (1926), 321-346.