One-loop corrections to the Nielsen-Olesen vortex: finite length.

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Abstract

We consider the one-loop quantum corrections to the Nielsen-Olesen flux tube of finite length $L$, by imposing periodic boundary conditions. The calculations are based on a recent evaluation of these quantum corrections to the string tension of an infinite vortex. The finite length corrections are finite from the outset. If the computation is restricted to the zero modes we obtain the standard Lüscher term $\pi/3L$ for a closed string. The inclusion of the other fluctuation modes of Higgs and gauge fields, using the numerically computed trace of the Euclidian Green’s function, leads to corrections that decrease exponentially with $L$. We present numerical results for these corrections, discuss their possible relevance, and the limitations of the approach.
1 Introduction

The vortex solution of the Abelian Higgs model in (3 + 1) dimensions, known from superconductivity [1], has been introduced in particle physics by Nielsen and Olesen [2] as a possible model for strings. Indeed the authors show that the vortex can be related to the bosonic Nambu-Goto string (see, e.g. [3]). This connection was mainly discussed on the classical level, the corrections due to quantum fluctuations were, there and later on, mostly considered already within string theory. Within the underlying quantum field theory the one-loop quantum corrections have been computed only recently [4]. The fact that the collective oscillations of the string are related to zero modes of the quantum fluctuations was already used qualitatively in Ref. [2]. A detailed formulation of this connection has recently been presented in Ref. [5]; a further aspect has been addressed there: in quantum field theory the renormalization of the energy of collective fluctuations becomes part of the ordinary renormalization programme, renormalization requires the inclusion of the zero modes into the computation of the one-loop corrections to the string tension, they are not regularized and renormalized separately.

While for this reason the contribution of the zero modes to the string tension of an infinitely long string cannot be quantified separately in quantum field theory, this is no longer so for the finite corrections which appear if one considers a string of finite length. It is the aim of this work to elucidate this aspect and to determine these corrections numerically. A vortex of finite length can either be constructed as an open string or a closed string. For an open string we would need to provide end caps, e.g., in the form of magnetic charges. It is hard to imagine how one could possibly compute quantum corrections to such a configuration. The same would hold for a closed string in the form of a torus. The technique developed in Ref. [4] can be applied, however, to a vortex of finite length with periodic boundary conditions. This can be considered as an approximation to a realistic closed string if the length of the string is much larger than its transverse extension. Such a computation will be the main subject of this article.

The text is organized as follows: In Sec. 2 we recall some basic formulae of the model and its quantum fluctuations, referring mainly to Ref. [4] for all details; in Sec. 3 we develop the formalism for computing the quantum corrections to the energy of a finite string, i.e., those one-loop corrections that are not already included in the one-loop corrections to the string tension; explicit calculations, analytical and numerical, are presented in Sec. 4; the
results are discussed in Sec. 5; we conclude with a brief summary in Sec. 6.

2 The model

The Abelian Higgs model in (3+1) dimensions is defined by the Lagrange density

\[ \mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} (D_{\mu} \phi)^* D^\mu \phi - \frac{\lambda}{4} (|\phi|^2 - v^2)^2. \]  

Here \( \phi \) is the complex scalar Higgs field and

\[ F_{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu}, \]  

\[ D_{\mu} \phi = \partial_{\mu} - ig A_{\mu}. \]  

The particle spectrum consists of Higgs bosons of mass \( m_H^2 = 2\lambda v^2 \) and vector bosons of mass \( m_V^2 = g^2 v^2 \). We denote the ration of these masses as \( \xi = m_H/M_W = 2\lambda/g^2 \). The vortex solution \[ \text{[1, 2] is defined, in the singular gauge, by the cylindrically symmetric ansatz} \]  

\[ A_i^{\text{cl}}(x, y, z) = \frac{\varepsilon_{ij} x_j}{gr^2} [A(r) + 1] \quad i = 1, 2 \]  

\[ \phi^{\text{cl}}(x, y, z) = v f(r). \]  

where \( r = \sqrt{x^2 + y^2} \) and \( \varphi \) is the polar angle. The equations of motion for \( f(r) \) and \( A(r) \) can be solved numerically, see, e.g., Ref. \[ \text{[6]. In terms of these functions the classical string tension takes the form} \]  

\[ \sigma_{\text{cl}} = \pi v^2 \int_0^\infty dr \left\{ \frac{1}{r m_W^2} \left[ \frac{dA(r)}{dr} \right]^2 + r \left[ \frac{df(r)}{dr} \right]^2 + f^2(r) [A(r) + 1]^2 + \frac{r m_H^2}{4} [f^2(r) - 1]^2 \right\}. \]  

The parameter dependence can be written in the form \( \sigma_{\text{cl}} = (\pi/g^2) h(\xi) \), where \( h(\xi) \) varies from 0.75 at \( \xi = 0.5 \) to 1.34 for \( \xi = 2 \). Here and elsewhere we use units such that \( m_W = 1 \).

The fluctuations around this classical solution consist of those of the real and imaginary part of the Higgs field and those of the transversal, longitudinal and timelike components of the gauge field; furthermore the gauge

\[ A_i^+ \equiv A^1 \equiv -A_1 \]
fixing introduces the corresponding Faddeev-Popov fields. In the fluctuation energy the Faddeev-Popov contributions cancel those of longitudinal and timelike components of the gauge field, so that these are irrelevant here. The remaining fluctuations form a $4 \times 4$ coupled system. As the classical solution is independent of time and $z$ the Euclidian fluctuation operator has the general form

$$M_{ij} = - (\partial^2_\tau + \partial^2_z) \delta_{ij} + M_{\perp ij}.$$ (2.7)

The transversal fluctuation operator $M_{\perp ij}$ is identical to the one of the instanton in the 2-dimensional model; it has been presented in detail in Refs. [4, 6]. The Green’s function of $M_{ij}$ has, in momentum space, the formal representation

$$G_{ij}(x_\perp, x'_\perp, k, \nu) = \sum_\alpha \psi^{\alpha}_i(x_\perp) \psi^{\alpha\dagger}_j(x'_\perp) \frac{\nu^2 + k^2 + \lambda^2_\alpha}{\nu^2 + k^2 + \lambda^2_\alpha}$$ (2.8)

where $\lambda^2_\alpha$ and $\psi^{\alpha}_i(x_\perp)$ are the eigenvalues and eigenfunctions of $M_{\perp ij}$, respectively. The trace of this Green’s function has been computed numerically in Ref. [4], not via (2.8), but using a Jost function formalism adapted to coupled systems, see Ref. [7] and the Appendix of Ref. [8].

### 3 The energy of the vortex of finite length: basic relations

We now establish the formalism for computing the quantum corrections to the energy of a vortex of finite length. As announced in Sec. we do this in an approximate way by imposing periodic boundary conditions in the longitudinal coordinate $z$.

Starting point is the formal definition

$$E_{fl}(L) = \sum_{n=-\infty}^{\infty} \sum_\alpha \frac{1}{2} (E_\alpha(k_n) - E_{0\alpha}(k_n)) ,$$ (3.1)

where $E_\alpha$ and $E_{0\alpha}$ are the energies of the eigenmodes of the fluctuation operators around the vortex and in the vacuum, respectively. The variable $k_n$ is the momentum in the longitudinal direction of the vortex, which for periodic boundary takes the values $k_n = 2\pi/L$. The energies of the eigenmodes have the form

$$E_\alpha(k_n) = \sqrt{k_n^2 + \lambda^2_\alpha}$$ (3.2)
and analogously for $E_{0n}(k_n)$. Here $\lambda_\alpha^2$ are the eigenvalues of the fluctuation operator $M_\perp$ in the transverse variables, as defined in Sec. 2.

We have introduced in the previous section the Green’s function of the fluctuation operator and its formal representation (2.8). We define a function $F(k_n,\nu)$ as

$$F(k_n,\nu) = \int d^2x_\perp \Tr \left( G(x_\perp, x_\perp, k_n, \nu) - G^0(x_\perp, x_\perp, k_n, \nu) \right),$$ \hspace{1cm} (3.3)

Here $G_0$ is the free Green’s function. We denote the eigenvalues of the free fluctuation operator by $(\lambda_\alpha^{(0)})^2$. We then obtain the relation

$$-\int_{-\infty}^{\infty} \frac{d\nu \nu^2}{2\pi} F(k_n,\nu) = \sum_{\alpha} 1/2 \left[ \sqrt{k_n^2 + \lambda_\alpha^2} - \sqrt{k_n^2 + (\lambda_\alpha^{(0)})^2} \right],$$ \hspace{1cm} (3.4)

which is the basis of our numerical computation. In terms of $F(k_n,\nu)$ we find for the energy of a string of finite length

$$E_{\text{fl}}(L) = -\int_{-\infty}^{\infty} \frac{d\nu}{2\pi} \nu^2 \sum_{n=-\infty}^{\infty} F(2\pi n/L, \nu).$$ \hspace{1cm} (3.5)

Actually $F(k_n,\nu)$ only depends on $k_n^2 + \nu^2$. So with the definitions $p = \sqrt{k_n^2 + \nu^2}$ and $F(p) \equiv F(k_n, \nu)$ we may write this as

$$E_{\text{fl}}(L) = -\int_{-\infty}^{\infty} \frac{d\nu}{2\pi} \nu^2 \sum_{n=-\infty}^{\infty} F(p_n),$$ \hspace{1cm} (3.6)

with $p_n = \sqrt{\nu^2 + (2\pi n/L)^2}$. This can easily be converted into a weighted integral over $F(p)$ via

$$E_{\text{fl}}(L) = -\int_0^{\infty} \frac{p}{\pi} w(p) F(p) \, dp,$$ \hspace{1cm} (3.7)

with

$$w(L,p) = \sum_{n=-\infty}^{\infty} \sqrt{p^2 - (2\pi n/L)^2} \Theta(p^2 - (2\pi n/L)^2).$$ \hspace{1cm} (3.8)

The function $F(p)$ has been computed numerically in Ref. [4] for various values of the parameter $\xi = m_H/m_W$. At small $p$ it behaves as $2/p^2$, due to the presence of the two translation zero modes. At large $p$ it behaves as
where the coefficient \(a\) is determined by the leading-order Feynman diagrams, see Eq. (7.1) of Ref. [4]. One easily convinces oneself that the weighted integral in Eq. (3.7) is quadratically divergent. In the limit \(L \to \infty\) the sum over \(n\) can be replaced by an integral and one finds

\[
\lim_{L \to \infty} \frac{w(L,p)}{L} = \frac{1}{2\pi} \int_{-\infty}^{\infty} dx \sqrt{p^2 - x^2} \Theta(p^2 - x^2) = \frac{1}{2\pi} \frac{\pi}{2} p^2. \tag{3.9}
\]

This limit yields that part of the string energy which is proportional to its length, \(E \propto L\sigma\) and defines the string tension, or energy per length, of the vortex of infinite length \(3\)

\[
\sigma_{fl} = -\int_0^{\infty} \frac{p^3}{4\pi} dp F(p). \tag{3.10}
\]

Here we want to determine the corrections which arise for finite length. It is convenient, therefore, to subtract from \(E_{\text{fl}}(L)\) the term \(\sigma_{fl} L\). This can be done by redefining the weight \(w(L,p)\) by subtracting the asymptotic weight:

\[
w_{s}(L,p) = w(L,p) - \frac{L}{4} p^2 = \sum_{n=-\infty}^{\infty} \sqrt{p^2 - (2\pi n/L)^2} \Theta(p^2 - (2\pi n/L)^2) - \frac{L}{4} p^2. \tag{3.11}
\]

The finite length correction to the energy then becomes

\[
\Delta E_{s}(L) = E_{\text{fl}}(L) - \sigma_{fl} L = -\int_0^{\infty} \frac{p dp}{\pi} w_{s}(L,p) F(p). \tag{3.12}
\]

In this way we have gotten rid of the divergences. Indeed we have already used all counter terms of quantum field theory in order to obtain a finite string tension \(\sigma_{fl}\), as described in detail in Ref. [4]. One can verify explicitly, as we will see in the next section, that for a function \(F(p)\) which asymptotically behaves as \(c/p^2\) the weighted integral of Eq. (3.12) is UV finite. The function \(w_{s}(p)\) does not tend to zero at large \(p\), it oscillates with a period of \(\Delta p \simeq 2\pi/L\), without being strictly periodic. The oscillations are due to the fact that more and more terms are included into the sum over \(n\). \(w_{s}(p)\) is displayed in Fig. [1] for \(L = 10\).

\(^{3}\)This expression for \(\sigma_{fl}\) is of course divergent. \(F(p)\) has to be used in subtracted form, and the divergent parts have to be renormalized, this is the subject of Ref. [4].
Figure 1: The subtracted weight function $w_s(p)$. Solid line: $w_s(p)$. The dashed lines indicate a square root behaviour of the maxima and minima.

4 Numerical and analytical calculations

As the function $F(p)$ is known numerically from previous computations it seems a straightforward matter to evaluate the weighted integral of Eq. (3.12). However, this integral is subtle numerically, due to the oscillating and spiky weight function. It is useful, therefore, to begin with some related analytical and numerical calculations.

The function $F(p)$ is dominated, at small $p$ by the two zero modes describing collective oscillation, and it is instructive to compute their contribution to $\Delta E_{\parallel}(L)$. For $\sigma_{\parallel}$ such a separate computation was not possible, as the contribution of the collective oscillations is infinite. When computing the
correction $\Delta E_0$ this problem does not arise. We define the integral

\[
I_0(L, p) = \int_0^p \frac{dp}{p^2} w_s(L, p)
\]

\[
= 2 \sum_{n=1}^{[Lp/2\pi]} \left[ \sqrt{p^2 - (2\pi n/L)^2} - \frac{2\pi n}{L} \arccos \frac{2\pi n}{Lp} \right] + p - \frac{\pi L}{42\pi p^2}
\]  

(4.1)

and its limit as $p \to \infty$

\[
\bar{I}_0(L) = \lim_{p \to \infty} I_0(L, p).
\]  

(4.2)

In terms of $\bar{I}_0(L)$ the contribution of the two zero modes is

\[
\Delta E_{\text{coll}}(L) = -\frac{2}{\pi} \bar{I}_0(L).
\]  

(4.3)

At finite $p$ the sum on the right hand side of Eq. (4.1) extends up to $N = [Lp/2\pi]$. It can be evaluated using the Euler-Maclaurin summation formula. One finds

\[
\bar{I}_0(L) = \frac{\pi^2}{6L}.
\]  

(4.4)

We display the integral $I_0(L, p)$ in Fig. 2 for $L = 10$. One sees that the integral oscillates as a function of $p$, the width of these oscillations narrows only slowly, as $1/\sqrt{p}$, and the band is slightly asymmetric. At $p = 500$ the width is of order 0.002 and the mean value can be read off, by taking the average of maxima and minima, with a precision of order 0.0001. For $L = 10$ one finds $I_0(L, p) \simeq 0.1644$; this is consistent to four digits with the analytic result $\pi^2/6L = \pi^2/60 = 0.164493$. The same value is found by evaluating the weighted integral via numerical integration.

Using Eq. (4.3) we find

\[
\Delta E_{\text{coll}}(L) = -\frac{\pi}{3L}.
\]  

(4.5)

This is the Lüscher term [9, 10] for a closed string (see, e.g., [11]). For a closed string the mode energies are $2\pi n/L$ instead of $\pi/L$ for an open string, and the oscillating modes can propagate in two directions, up and down the $z$ axis. This explains the factor 4 relative to the usually quoted value of
The integral $I_0(L, p)$ of Eq. (4.1). Solid line: the function $I_0(L, p)$; dashed horizontal line: the asymptotic limit 0.1644.

$\pi/12L$. We note that our result is obtained by evaluating an expression that is finite from the outset. This is analogous to finite temperature corrections which likewise do not need a regularization and renormalization.

An important property of our weighted integral $I(L, p)$ is apparent in Fig. 2: the mean value between maxima and minima of the oscillations attains the final value of the integral already at very low values of $p$, i.e., after a few oscillations with period $2\pi/L$. This means in general, that in a region where $F(p) \propto 1/p^2$ the integral of Eq. (3.12) will just oscillate around an almost constant average value. As $F(p) \simeq c/p^2$ for large $p$ this means that in the asymptotic regime the average value of the integral reaches its asymptotic limit quickly.

In order to perform the weighted integral for the realistic case of Eq. (3.12) we need $F(p)$ for a very narrow grid of values $p$, $\Delta p \ll 2\pi/L$, as the weight varies strongly and the integration implies subtle cancellations. The numerical computation or Ref. [4] has provided values only on a relatively coarse grid of values $p$. As a computation of $F(p)$ requires substantial CPU
time, a few minutes for each value of $p$, while the function itself is varying smoothly, it is convenient to use fits through the existing data points. One may use spline fits, but it turns out that a simple parameterization

$$F(p) = \frac{2}{p^2} + \frac{d}{(p^2 + \lambda^2)}$$

(4.6)
gives a surprisingly good global fit to the data, for all values of $\xi = m_H/m_W$. The ansatz can be understood as a fit where the effect of all higher modes, possible bound states and continuum, is simulated by just one pole with an effective degeneracy $d$ at an energy $\lambda$ on the imaginary $p$ axis, which contains the physical cut. The symbol $\lambda$ refers to Eq. (2.8), $\lambda^2$ being an eigenvalue of the transversal fluctuation operator. The parameters $d$ and $\lambda^2$

for various values of $\xi$, determined by a fit-by-eye, are given in Table [1]. The fits and the numerical data are displayed in Fig. 3 for three values of $\xi = m_H/m_W$. If necessary the approach could be improved systematically by including more poles into the ansatz (4.6). This type of interpolation can be considered as a Padé approximation (see, e.g., Ref. [12], Sec. 5.12) and is well adapted for functions with a cut in the complex plane. The parameters could be determined, e.g., by least squares methods.

A considerable advantage of our fit is the fact that we can do the integral over $p$ for the second pole term analytically, as well. We thus can avoid a very subtle numerical integration. One finds

$$I_\lambda(L, p) \equiv \int_0^L \frac{p \, dp}{p^2 + \lambda^2} w_n(L, p)$$

$$= 2 \sum_{n=1}^{[Lp/2\pi]} \left[ \sqrt{p^2 - (2\pi n/L)^2} - \sqrt{(2\pi n/L)^2 + \lambda^2} \arccos \frac{\sqrt{(2\pi n/L)^2 + \lambda^2}}{p} \right]$$

$$+ p - \lambda \arccos \frac{\lambda}{p} - \frac{L}{8} \left( p^2 - \lambda^2 \ln \frac{p^2 + \lambda^2}{\lambda^2} \right).$$

(4.7)

The delicate cancellations between the sum and the other terms on the right hand side can be avoided by subtracting $I_0(L, p)$ and adding its asymptotic
Figure 3: The fits to the function $F(p)$ of Eq. (4.6). We display the data and the fits for $p^2 F(p)$ for $\xi = 0.6$, $\xi = 1.0$ and $\xi = 2.0$. Symbols (+, ×, *): data; solid lines: fits according to Eq. (4.6).

The limit $\pi^2/6L$ of Eq. (1.4). Of course the resulting expression

$$\tilde{I}_\lambda(L, p) = I_\lambda(L, p) - I_0(L, p) + \frac{\pi^2}{6L}$$

$$= 2 \sum_{n=1}^{[Lp/2\pi]} \left[ \frac{2\pi n}{L} \arccos \frac{2\pi n}{Lp} - \sqrt{(2\pi n/L)^2 + \lambda^2} \arccos \sqrt{(2\pi n/L)^2 + \lambda^2} \right]$$

$$- \lambda \arccos \frac{\lambda}{p} + \frac{L}{8\lambda^2} \ln \left( \frac{p^2 + \lambda^2}{\lambda^2} \right) + \frac{\pi^2}{6L}$$

(4.8)

is not identical to $I_\lambda(L, p)$, but it has the same limit as $p \to \infty$. The oscillations present in $I_\lambda(p, L)$ are considerably suppressed in $\tilde{I}_\lambda(p, L)$ and the limit $p \to \infty$, which we denote by $\bar{I}_\lambda(L)$, can be evaluated numerically without problems. To obtain the corrections to the Lüscher term in $\Delta E_{fl}(L)$, we have to multiply the result by $-d/\pi$, so that, with our fit to $F(p)$ we get

$$\Delta E_{fl}(L) = -\frac{\pi}{3L} - \frac{d}{\pi} \bar{I}_\lambda(L).$$

(4.9)
In Fig. 4 we plot $\bar{I}_\lambda(L)$ as a function of $L$ for $\lambda^2 = 0.7$, 1.0 and 2.0, values which are in the range of the fit parameters given in Table 1. The results obtained display a roughly exponential decrease with $L$. This is not unexpected; had we imposed periodicity in time instead of periodicity in $z$ we would have expected the thermal corrections to display such an exponential behaviour. We have to take into account that we do not consider the contribution of a single energy level, but the contribution of a cut generated by the pole in the transversal Green’s function. So the correction is not expected to be described by a simple exponential function $a \exp(-\alpha L)$. However, the “effective” logarithmic slope $\alpha$ is in the expected range of values. For the range $2 \leq L \leq 6$ we have $\alpha \simeq 0.95$ for $\lambda^2 = 0.7$, $\alpha \simeq 1.15$ for $\lambda^2 = 1$ and $\alpha \simeq 1.55$ for $\lambda^2 = 2$. The straight dashed lines in Fig. 4 indicate this behaviour. We would naively have expected $\alpha = \lambda = 0.837$, 1.0 and 1.414, respectively. Indeed for $L > 6$ the slopes $\alpha$ decrease and may attain these values as $L \to \infty$. For $L < 2$ on the other hand the effective slopes increase.

![Figure 4: The integral $\bar{I}_\lambda(L)$. diamonds: $\lambda^2 = 0.7$; triangles: $\lambda^2 = 1.0$; circles: $\lambda^2 = 2.0$; the dashed lines indicate a simple exponential behaviour, as specified in the text.](image-url)
5 Discussion of the results

Using the results of the previous section the final results for $\Delta E_{ll}(L)$ can be easily obtained, using the parameters $d$ and $\lambda^2$ from Table 1. We present, in Fig. 5 the function $c(L) = L\Delta E_{ll}(L)$ for $\xi = 0.5$ and $\xi = 1.2$. The function $c(L)$ can be considered as the coefficient of an “effective” Lüscher term $c(L)/L$. With this formulation we follow the presentation of lattice measurements of this term in QCD in Ref. [13]. As one sees from table 1 the values for $d$ and $\lambda$ are 8.5 and 0.7, respectively, for $\xi = 0.5$, for $\xi = 1.2$ they are 2.4 and 1.0, respectively. For the first parameter set the corrections to a pure Lüscher term are sizeable even at $L = 4$ and have decreased to the 10% level at $L = 6$, while for $\xi = 1.2$ they have decreased to this level already at $L = 3$. For higher values of $\xi$ the effective degeneracies become smaller and so does the deviation from $c = -\pi/3$.

![Figure 5: The coefficient $c(L)$. circles: $\xi = 0.5$; diamonds: $\xi = 1.2$; triangles: $\lambda^2 = 1.0$; circles: $\lambda^2 = 2.0$; the dashed lines indicate a simple exponential behaviour, see text.](image)

There are various aspects under which we can consider the results of these
computations. It is satisfactory, at first, that the Lüscher term appears in a straightforward way. As this term has been measured for QCD strings it is certainly a part of established physics. The fact that the corrections due to higher fluctuations appear here within the same formalism and in an analogous way gives us confidence that these terms as well are not artefacts of the approximation but terms that would appear as a result of a suitable measurement.

Of course our analysis is an approximation; we see two essential limitations: (i) we have to require $L$ to be “much” larger than the transversal extension of the vortex and (ii) the corrections have to be small enough for the semiclassical approximation to be reliable.

The transversal extension of the vortex is of the order of $\max(1/m_H, 1/m_W)$. In our computation the mass $m_W$ is set to unity, so the transversal extension is $R \simeq \max(1, 1/\xi)$; for the range of $\xi$ considered here this is between 0.5 and 2. An optimistic guess of “much larger” could be $L > R$ and a pessimistic one $L > 10R$. In the second case our correction will be negligible for all values of $\xi$. The range of validity also depends on what is really measured. If, e.g., the corrections were measured on a lattice with the same periodic boundary conditions, then our analysis would be valid in the whole range of parameters considered here. The limitation of the approximation depends, unfortunately, mainly on the effects which we have neglected: the influence of end caps for an open string, and the effects of curvature for a closed vortex. Their magnitude will be difficult to estimate.

The second limitation is the validity of the semiclassical approximation. The classical string tension is given by $\pi/g^2$ times a number close to unity. The one-loop corrections to the string tension were found, in Ref. [4] to be smaller than 0.5, so this correction is small even for $g$ as large as unity. The corrections for a finite string, as found here, are much larger. Of course the ratio depends on $L$: $\sigma_{cl}$ is multiplied by $L$ while $c(L)$ is divided by $L$. But for $g \simeq 1$ and $L \simeq 2$ the ratio of these contributions to the string energy is not small. In any case, for $L$ small enough the Lüscher term and the even larger corrections will become comparable or exceed the classical energy and the semiclassical approximation will break down. For the lattice results it is found that the absolute value of $c(L)$ becomes smaller than $\pi/12$ at small $L$, and possibly tends to zero. In Ref. [13] this behaviour is described by a relation derived using the QCD renormalization group and therefore relies on asymptotic freedom. Such an analysis does not apply here. The absolute value of our coefficient $c(L)$ increases at small $L$ and it is hard to see how
this could be different as the parameters $d$ are positive throughout. A higher order resummation seems out of scope.

6 Summary

We have computed the one-loop corrections to the energy of a Nielsen-Olesen vortex of finite length. More precisely: we have computed the corrections that are not already included in the string tension. The latter were the subject of Ref. [4]. The corrections computed here are finite from the outset and, therefore, do not depend on renormalization conditions. The leading order correction at large $L$ is the Lüscher term which here takes the form $\pi/3L$ as appropriate for a closed string. The further corrections decrease exponentially at large $L$ but can be, depending on the parameters, relevant for small $L$ and intermediate $L$. Within our computational framework they appear on the same footing as the Lüscher term and are related to the internal structure of the vortex.

We have discussed briefly the limitations of the approach and conclude that, depending on the parameters of the model, there is a window in $L$ where the corrections to the Lüscher term are relevant and where their computation is reliable.

We would finally like to point out that the method used here can be applied in a similar way to other vortex configurations, like cosmic strings (see [14] for a recent review). In fact the only information specific to the Abelian Higgs model was contained in $F(p)$, the trace of the Green’s function of the fluctuation operator. Furthermore, the behaviour of $F(p)$ at small $p$ is determined by the zero modes and its asymptotic behaviour can be obtained from leading order Feynman graphs. So semi-quantitative estimates are easily accessible.
| $\xi$ | $d$  | $\chi^2$ |
|------|------|----------|
| 0.5  | 8.5  | 0.7      |
| 0.6  | 6.4  | 0.75     |
| 0.7  | 5.0  | 0.75     |
| 0.8  | 4.1  | 0.8      |
| 0.9  | 3.48 | 0.8      |
| 1.0  | 3.0  | 0.8      |
| 1.1  | 2.65 | 0.9      |
| 1.2  | 2.4  | 1.0      |
| 1.3  | 2.22 | 1.1      |
| 1.4  | 2.06 | 1.2      |
| 1.5  | 1.94 | 1.3      |
| 1.6  | 1.87 | 1.6      |
| 1.7  | 1.8  | 1.85     |
| 1.8  | 1.75 | 2.1      |
| 1.9  | 1.71 | 2.35     |
| 2.0  | 1.68 | 2.7      |

Table 1: The parameters of a pole fit to $F(p)$, Eq. (4.6)
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