Foliations on hypersurfaces in holomorphic symplectic manifolds

Justin Sawon

December, 2008

Abstract

Let $Y$ be a hypersurface in a $2n$-dimensional holomorphic symplectic manifold $X$. The restriction $\sigma|_Y$ of the holomorphic symplectic form induces a rank one foliation on $Y$. We investigate situations where this foliation has compact leaves; in such cases we obtain a space of leaves $Y/F$ which has dimension $2n - 2$ and admits a holomorphic symplectic form.

1 Introduction

The aim of this article is to introduce a new kind of ‘holomorphic symplectic reduction’, which produces a holomorphic symplectic $(2n - 2)$-fold from a holomorphic symplectic $2n$-fold. Locally it can be seen as a holomorphic version of the symplectic reduction coming from an $S^1$-action, though the group action does not globalize.

A summary of the construction is as follows. We start with a $2n$-dimensional holomorphic symplectic manifold $X$ with holomorphic two-form $\sigma$. Let $Y \subset X$ be a hypersurface. The kernel of $\sigma|_Y$ defines a rank one foliation $F \subset TY$ on the smooth locus of $Y$, which we call the characteristic foliation. If this foliation has compact leaves, then we can construct a space of leaves $Y/F$ which will be Hausdorff. Note that $Y/F$ has dimension $2n - 2$ and it may be singular. Moreover, $\sigma|_Y$ will descend to a non-degenerate holomorphic two-form on the smooth part of $Y/F$.

Although our main interest in this article is hypersurfaces, we will also consider some examples where $Y \subset X$ is a submanifold of codimension $k > 1$. Note that $\sigma|_Y$ must have rank at least $2(n - k)$ at each point of $Y$. We say that $Y$ is coisotropic if the rank of $\sigma|_Y$ is constant and equal to $2(n - k)$ at all points of $Y$. In this case the kernel of $\sigma|_Y$ defines a characteristic foliation $F \subset TY$ of rank $k$. If this foliation has compact leaves, the space of leaves $Y/F$ will be a (possibly singular) holomorphic symplectic manifold of dimension $2n - 2k$.

*2000 Mathematics Subject Classification. 53C26.
The fundamental question is thus: can we find hypersurfaces and/or submanifolds \( Y \subset X \) whose characteristic foliations \( F \) have compact leaves? Given a foliation on a projective variety, a criterion for algebraicity (and hence compactness) of the leaves was given in a recent theorem of Kebekus, Solí Conde, and Toma \cite{26}, building on results of Miyaoka \cite{35} and Bogomolov and McQuillan \cite{5}. Let \( Y \subset X \) be a smooth hypersurface (so that \( F \) is regular everywhere) or a coisotropic submanifold. It follows from the theorem of Kebekus et al. that if \( Y \) is covered by curves \( C \) such that \( F|_C \) is ample then every leaf is algebraic (and hence compact) and rationally connected. We will explain how this ensures that the space of leaves \( Y/F \) is not only Hausdorff, but also smooth.

One potential application of these ideas is to constructing new examples of holomorphic symplectic manifolds. We will describe how a birational model of O’Grady’s example in ten-dimensions \cite{38} arises in this way, starting from a hypersurface in the Hilbert scheme \( \text{Hilb}^5 S \) of six points on a K3 surface \( S \). It should be noted that the space of leaves \( Y/F \) here is (presumably) quite singular and one still has to apply O’Grady’s desingularization procedure to arrive at a smooth birational model of his space.

Another potential application is to the classification of holomorphic symplectic manifolds up to deformation, and we consider two main results in this direction. The goal is to describe \( X \) in terms of the lower-dimensional space \( Y/F \), whereby building a classification inductively on the dimension. When \( X \) is four-dimensional the space of leaves \( Y/F \) is a holomorphic symplectic surface; if \( Y/F \) is smooth, it must be a K3 or abelian surface. For example, let \( Y \) be the inverse image of the diagonal \( \Delta \) in the Hilbert scheme of two points on a K3 surface \( S \),

\[
\text{Hilb}^2 S = \text{Blow}_\Delta (S \times S/\mathbb{Z}_2). 
\]

Then every leaf is a rational curve \( \mathbb{P}^1 \), and \( Y/F = \Delta \) is of course isomorphic to the K3 surface \( S \). Conversely, Nagai \cite{37} proved that if a holomorphic symplectic four-fold contains a divisor \( Y \) which is ‘numerically like’ the one above (in particularly, it can be contracted to a surface \( S \)) then \( S \) is a K3 surface and \( X \cong \text{Hilb}^2 S \). We consider Nagai’s theorem in the context of foliations, adding some new observations.

The second main classification result involves holomorphic symplectic manifolds which are Lagrangian fibrations by Jacobians of curves. Let \( X \) be the total space of the relative compactified Jacobian of a family \( \mathcal{C} \to \mathbb{P}^2 \) of genus two curves over \( \mathbb{P}^2 \). Markushevich \cite{30} proved that if \( X \) is a holomorphic symplectic four-fold then \( \mathcal{C} \) is a complete linear system of curves in a K3 surface \( S \); in other words, \( X \) is the Beauville-Mukai integrable system \cite{4}, and in particular, \( X \) is a deformation of \( \text{Hilb}^2 S \). Markushevich constructed the K3 surface \( S \) as a branched double cover of the dual plane \( (\mathbb{P}^2)^\vee \). Alternatively, we will explain how the K3 surface \( S \) can be described as the space of leaves of the characteristic foliation on a hypersurface \( Y \subset X \); namely, we take \( Y \) to be the total space of \( \mathcal{C} \), embedded in \( X \) by the relative Abel-Jacobi map. The author has used this approach to generalize Markushevich’s Theorem to higher dimensions \cite{42}; roughly speaking, if the total space \( X \) of the relative compactified Jacobian of a
family $\mathcal{C} \to \mathbb{P}^n$ of genus $n$ curves over $\mathbb{P}^n$ is a holomorphic symplectic manifold, then $\mathcal{C}$ is a complete linear system of curves in a K3 surface $S$, i.e., $X$ is the Beauville-Mukai integrable system and hence deformation equivalent to $\text{Hilb}^n S$. For the proof one again takes $Y$ to be the total space of $\mathcal{C}$, embedded by the relative Abel-Jacobi map, though this time $Y$ will be of higher codimension and thus one must also prove that $Y$ is coisotropic.

The paper is structured as follows. In Section 2 we outline the main constructions and prove some general results concerning characteristic foliations. In Section 3 we describe a number of examples. In Sections 4, 5, and 6 we discuss how Nagai’s Theorem, Markushevich’s Theorem, and O’Grady’s example in ten dimensions, respectively, can be understood in terms of foliations on hypersurfaces. Finally, in Section 7 we make comparisons with the analogous situation in real symplectic geometry; in particular, we formulate a holomorphic analogue of the Weinstein Conjecture, asserting the existence of compact leaves for the characteristic foliation on a hypersurface of contact type.

While this paper was being written a preprint of Hwang and Oguiso [21] appeared. They use methods similar to ours to study the singular fibres of a Lagrangian fibration; namely, if $\pi: X \to \mathbb{P}^n$ is a Lagrangian fibration, and $\Delta \subset \mathbb{P}^n$ is the discriminant locus parametrizing singular fibres, then Hwang and Oguiso study the characteristic foliation on the hypersurface $Y := \pi^{-1}(\Delta)$ and prove that the leaves are either elliptic curves or chains of rational curves. This in turn tells us something about the structure of the generic singular fibre $X_t$, where $t$ is a generic point of $\Delta$. Hwang and Viehweg [23] also consider characteristic foliations on hypersurfaces of general type; we mention their results in Subsection 2.2.

The author would like to thank Jun-Muk Hwang, Yujiro Kawamata, Stefan Kebekus, Manfred Lehn, and Keiji Oguiso for many helpful discussions on the material presented here. The author is also grateful for the hospitality of the Max-Planck-Institut für Mathematik (Bonn), the Institute for Mathematical Sciences (Chinese University of Hong Kong), and the Korea Institute for Advanced Studies, where these results were obtained.

## 2 Characteristic foliations and spaces of leaves

### 2.1 Hypersurfaces and coisotropic submanifolds

**Definition** A holomorphic symplectic manifold is a compact Kähler manifold $X$ with a non-degenerate holomorphic symplectic form $\sigma \in \mathbb{H}^0(X, \Omega_X^2)$. In particular, the dimension $\dim X = 2n$ is even. ‘Non-degenerate’ means that $\sigma^{\wedge n}$ trivializes the canonical bundle $K_X = \Omega_X^{2n}$. In addition, if $X$ is simply-connected and $\mathbb{H}^0(X, \Omega_X^2)$ is one-dimensional, i.e., generated by $\sigma$, then we say that $X$ is irreducible.

**Remark** For a general compact complex manifold $X$, $\mathbb{H}^0(X, \Omega_X^2)$ is a quotient of the deRham cohomology group $\mathbb{H}^2(X, \mathbb{C})$; but if $X$ is Kähler then it follows
from Hodge theory that the projection splits and \( H^0(X, \Omega_X^2) \) is a direct summand of \( H^2(X, \mathbb{C}) \). Therefore the holomorphic symplectic form \( \sigma \) is \( d \)-closed.

Let \( Y \subset X \) be a complex hypersurface, possibly with singularities. The restriction \( \sigma|_Y \in H^0(Y, \Omega_Y^2) \) must be degenerate; therefore the morphism of sheaves \( TY \to \Omega_Y^1 \) given by contracting a vector field with \( \sigma|_Y \) will not be injective. Define a distribution \( F \subset TY \) as the kernel of this morphism. Since \( \sigma \) is \( d \)-closed the distribution is integrable, i.e., if \( v \) and \( u \) are local sections of \( F \) then

\[
\sigma|_Y([v, u], w) = d\sigma|_Y(v, u, w) + v\sigma|_Y(u, w) - u\sigma|_Y(v, w) = 0
\]

so \([v, u]\) is also a local section of \( F \).

**Definition** The characteristic foliation on \( Y \) is the foliation defined by the distribution above. By abuse of notation, we use \( F \) to denote both the distribution and the foliation.

**Remark** This notion of integrability of the distribution should not be confused with (algebraic) integrability of the foliation, meaning that the leaves are algebraic. We will discuss the latter shortly.

Strictly speaking, a foliation in differential geometry should have constant rank, though algebraic geometers often consider foliations which are not regular. In other words, the rank of \( F \) may jump up on closed subsets of \( Y \); the following lemma shows that this can only happen on the singular locus of \( Y \).

**Lemma 1** Over the smooth locus \( Y_{\text{sm}} \) of \( Y \) the foliation \( F \) is locally free of rank one.

**Proof** Since \( \sigma \) is non-degenerate on \( X \), \( \sigma|_Y \) must be of constant rank \( 2n-2 \) as a skew-symmetric form over smooth points of \( Y \), and thus \( F = \ker \sigma|_Y \) will be of constant rank one. More explicitly, if \( Y \) is given locally by \( z_1 = 0 \) in the neighbourhood of a smooth point, then by the holomorphic Darboux Theorem we can find local coordinates such that

\[
\sigma = dz_1 \wedge dz_2 + dz_3 \wedge dz_4 + \ldots + dz_{2n-1} \wedge dz_{2n},
\]

\[
\sigma|_Y = dz_3 \wedge dz_4 + \ldots + dz_{2n-1} \wedge dz_{2n},
\]

and \( F \subset TY \) is generated by \( \frac{\partial}{\partial z_2} \).

**Lemma 2** Over the smooth locus \( Y_{\text{sm}} \), \( F \) is isomorphic to \( K_Y^{-1} \). In particular, if \( Y \) is normal then \( c_1(F) = c_1(TY) = -c_1(K_Y) \).

**Proof** Consider the exact sequence

\[
0 \to F \to TY \to TY/F \to 0.
\]

Over the smooth locus \( Y_{\text{sm}} \), the holomorphic two-form \( \sigma|_Y \) induces a non-degenerate skew-symmetric form on \( TY/F \), as we have quotiented out the null directions. This implies that \( \bigwedge^{2n-2} TY/F \) is trivial and \( F \cong \bigwedge^{2n-1} TY = K_Y^{-1} \) over \( Y_{\text{sm}} \).
Now suppose that $Y \subset X$ is a (smooth) complex submanifold of codimension $k$. Once again we get an integrable distribution, i.e., a foliation, $F \subset TY$ defined as the kernel of $\sigma|_Y$. However, even though $Y$ is smooth, the rank of $\sigma|_Y$ could vary. The rank of $\sigma$ restricted to $T_yY$ must be even, at least $2(n-k)$, and at most
$$2\left\lfloor \frac{\dim Y}{2} \right\rfloor = 2\left\lfloor \frac{n-k}{2} \right\rfloor.$$ When $k = 1$ these minimum and maximum values coincide, but this is no longer the case for $k \geq 2$. The rank is also semi-continuous, meaning that
$$\{ y \in Y | \text{rank } T_yY \leq m \}$$
is a closed subset of $Y$ for each $m$.

**Definition** We say that $Y$ is a coisotropic submanifold if the rank of $\sigma|_Y$ takes its minimum value $2(n-k)$ at all points of $Y$, i.e., if $T_yY \subset T_yX$ is a coisotropic subspace for all $y \in Y$.

If $Y$ is a coisotropic submanifold then the kernel $F \subset TY$ of $\sigma|_Y$ will be locally free of rank $k$ over $Y$. We once again have an exact sequence
$$0 \to F \to TY \to TY/F \to 0$$
which implies that $\wedge^k F \cong K_Y^{-1}$, i.e., $c_1(F) = -c_1(K_Y)$.

### 2.2 Compactness and rationality of the leaves

Perhaps the earliest criteria for algebraicity and rationality of the leaves of a holomorphic foliation are due to Miyaoka [35]. His results were further developed by Bogomolov and McQuillan [5]. Here we quote results of Kebekus, Solá Conde, and Toma [26].

**Theorem 3** ([26]) Let $Y$ be a normal projective variety, $C \subset Y_{sm}$ a complete curve contained in the smooth locus of $Y$, and $F \subset TY$ a foliation which is regular over $C$. Assume that $F|_C$ is an ample vector bundle. Then the leaf through any point of $C$ is algebraic, and the closure of the leaf through the generic point of $C$ is rationally connected. If in addition $F$ is regular everywhere then all leaves are rationally connected.

This kind of result is proved by applying Mori’s bend-and-break argument to the curve $C$, thereby producing rational curves in the leaves of $F$.

Now suppose that $Y$ is a hypersurface in a holomorphic symplectic manifold $X$, as in the previous subsection, and let $F$ be the characteristic foliation on $Y$. Since the canonical bundle $K_X$ of $X$ is trivial, we have
$$K_Y = \mathcal{O}(Y)|_Y$$
by adjunction. Let $C$ be a curve which lies in the smooth locus of $Y$; then
$$F|_C \cong K_Y^{-1}|_C = \mathcal{O}(-Y)|_C.$$
So if $\mathcal{O}(Y)$ is ample then $F|_C$ will be anti-ample and the hypotheses of the theorem can never be satisfied. The following conjecture was suggested by Jun–Muk Hwang.

**Conjecture 4** Let $Y$ be a smooth hypersurface in an irreducible holomorphic symplectic manifold $X$ such that $\mathcal{O}(Y)$ is ample. Then the characteristic foliation on $Y$ cannot be such that all of its leaves are algebraic.

**Remark** Certainly some of the leaves of the foliation could be algebraic. Let the algebraic curve $C \subset Y$ be a leaf of the foliation $F$. Then the tangent bundle of $C$

$$TC \cong F|_C \cong K_Y^{-1}|_C = \mathcal{O}(-Y)|_C$$

is anti-ample and $C$ must have genus $g \geq 2$. Suppose that every leaf is a (smooth) genus $g$ curve, and indeed that $Y$ is a fibration by genus $g$ curves over the base $B$, i.e., with no singular fibres. The homotopy long exact sequence of $Y \rightarrow B$ gives

$$\ldots \rightarrow \pi_2(C) \rightarrow \pi_2(Y) \rightarrow \pi_2(B) \rightarrow \pi_1(C) \rightarrow \pi_1(Y) \rightarrow \pi_1(B) \rightarrow 1.$$  

Since $\mathcal{O}(Y)$ is ample and the irreducible holomorphic symplectic manifold $X$ is simply-connected, $Y$ must have trivial fundamental group by the Lefschetz Hyperplane Theorem; but then $\pi_1(C)$ will be the cokernel of the map $\pi_2(Y) \rightarrow \pi_2(B)$ between abelian groups, forcing $\pi_1(C)$ itself to be abelian. This gives a contradiction.

The statement of the conjecture is of course stronger than this because even if all the leaves of $F$ are algebraic, we cannot expect $Y$ to be fibred by genus $g$ curves: the genus could drop on closed subsets. The local example

$$Y := (C \times \mathbb{C}^2)/\mathbb{Z}_2$$

illustrates this, where $C$ is a curve admitting a fixed-point free involution $\tau$, and $\mathbb{Z}_2$ acts as $\tau$ on $C$ and multiplication by $-1$ on $\mathbb{C}^2$. Here the central fibre is isomorphic to $C/\tau$ and it has smaller genus.

Returning to Theorem 3 it is clear that we must start with a hypersurface $Y$ whose corresponding line bundle $\mathcal{O}(Y)$ is negative in some sense, in order to apply this result. Let us assume that this is the case, and that the hypotheses of the theorem are satisfied. If furthermore $Y$ is smooth then $F$ is regular everywhere, and every leaf is rationally connected. Since the leaves are one-dimensional and smooth, they must be rational curves $\mathbb{P}^1$; the space of leaves will then be particularly well-behaved.

---

1In a recent preprint [23], Hwang and Viehweg proved that the characteristic foliation on a smooth hypersurface of general type cannot be such that all of its leaves are algebraic. This implies the conjecture since if $\mathcal{O}(Y)$ is ample, then $K_Y = \mathcal{O}(Y)|_Y$ is ample and $Y$ is of general type. Their proof involves a global étale version of Reeb stability (Theorem 3.2 in [23]); in the context of our argument here, this essentially says that there is a generically finite cover $\tilde{Y}$ of $Y$ such that when we pull back the foliation to $\tilde{Y}$ the genus of the leaves becomes constant.
2.2 Compactness and rationality of the leaves

Lemma 5  Suppose $Y$ is a smooth hypersurface such that all of the leaves of the characteristic foliation $F$ are rational curves. Then the space of leaves, denoted $Y/F$, is smooth.

Proof  Holmann [16] proved that if all of the leaves of a holomorphic foliation on a Kähler manifold are compact, then the foliation is stable, meaning that every open neighbourhood of a leaf $L$ contains a neighbourhood consisting of a union of leaves (known as a saturated neighbourhood). Stability of the foliation is equivalent to the space of leaves $Y/F$ being Hausdorff.

Let us describe the local structure of the space of leaves $Y/F$. Let $L$ be a (compact) leaf of $F$, represented by a point in $Y/F$. Take a small slice $V$ in $Y$ transverse to the foliation, with $L$ intersecting $V$ at a point $0 \in V$. The holonomy map is a group homomorphism from the fundamental group of $L$ to the group of automorphisms of $V$ which fix 0. The holonomy group $H(L)$ is the image of the holonomy map. Then $V/H(L)$ is a local model for the space of leaves $Y/F$ in a neighbourhood of the point representing $L$ (see Holmann [15] for details). Globally, the space of leaves $Y/F$ is constructed by patching together these local models.

If all of the leaves are rational curves, then they are simply connected and hence the holonomy groups must be trivial. The local models for $Y/F$ are therefore simply the transverse slices $V$; in particular, they are smooth. □

Remark  Note that $Y$ is a $\mathbb{P}^1$-bundle over the space of leaves $Y/F$. This is really a holomorphic version of the Reeb Stability Theorem, in the case when the leaves are smooth and simply connected. By contrast, a real foliation with leaves isomorphic to $S^1$ need not be an $S^1$-bundle as some leaves could have multiplicity (e.g., a Seifert-fibred space with exceptional fibres; sometimes this is regarded as an $S^1$-bundle over an orbifold). Similarly, a holomorphic foliation with leaves isomorphic to elliptic curves could have multiple fibres. In a different direction, if we assume the leaves are simply connected but not necessarily smooth (i.e., if the foliation is not regular), then chains of $\mathbb{P}^1$s could occur.

Remark  The space of leaves $Y/F$ could also be defined in terms of the Chow variety of subvarieties of $Y$, or in terms of the Hilbert scheme of subschemes of $Y$; in either case, we take the irreducible component which contains the generic leaf. When all of the leaves are rational curves, these constructions of $Y/F$ all yield isomorphic spaces.

The situation becomes somewhat more complicated when $Y$ is a hypersurface with singularities. In this case, the foliation $F$ is regular along the smooth locus $Y_{\text{sm}}$ of $Y$, but not regular along the singular locus $Y_{\text{sing}}$ of $Y$. Therefore it is perhaps better to consider the foliation only over $Y_{\text{sm}}$; a leaf could then be quasi-projective, i.e., it could be algebraic but we would need to add a point or points from $Y_{\text{sing}}$ to compactify it.

Defining the space of leaves is also more delicate. One approach is to think of the foliation as an equivalence relation on $Y$. When $Y$ is smooth the equivalence classes are simply the leaves, and this leads to the local models for $Y/F$.
described earlier. When \( Y \) is singular we should take closures of leaves, and moreover, if two closures intersect we should consider them part of the same equivalence class. Thus an equivalence class could consist of a chain of \( \mathbb{P}^1 \)'s. Such a construction is similar to Mumford’s geometric invariant theory: if a group \( G \) acts on a space \( X \), then two points are equivalent if the closures of their orbits under \( G \) intersect. One problem with this approach is that a continuous family of leaves in \( Y_{\text{sm}} \) could all be compactified by the same point from \( Y_{\text{sing}} \); this then produces an equivalence class of dimension greater than one, which may be undesirable.

We could instead define the space of leaves in terms of the Chow variety or Hilbert scheme. We cannot guarantee that these different approaches will yield the same space \( Y/F \), but when the generic leaf is a smooth curve then these various spaces should at least be birational. Certainly we don’t expect \( Y/F \) to be smooth in general: it will usually be singular along points parametrizing leaves which intersect \( Y_{\text{sing}} \).

It is somewhat easier to deal with the case of a higher codimension submanifold \( Y \), provided \( Y \) is coisotropic so that the characteristic foliation \( F \) is regular. Firstly, the leaves must be smooth. If the hypotheses of Theorem \( 3 \) are satisfied, then all the leaves must be algebraic (hence compact) and rationally connected. It is well known that smooth rationally connected varieties are simply connected (see Corollary 4.18 of Debarre \( \mathbf{9} \), for example), and therefore we obtain a smooth space of leaves \( Y/F \) just as in the proof of the lemma above (i.e., all holonomy groups are trivial so the local models must be smooth).

We conclude this subsection with the following result about the structure of the space of leaves \( Y/F \).

**Lemma 6** Assume that the space of leaves \( Y/F \) is smooth (for example, \( Y \) is a smooth hypersurface or a coisotropic submanifold and the hypotheses of Theorem \( 3 \) are satisfied). Then \( Y/F \) admits a holomorphic symplectic form.

**Proof** Intuitively, by taking the space of leaves we are quotienting along the leaves of \( F \), which are the null directions of \( \sigma|_{Y} \). Therefore \( \sigma|_{Y} \) should descend to a non-degenerate two-form on \( Y/F \). To make this rigorous, we will define holomorphic symplectic forms on the local models for \( Y/F \) and then prove that they agree when the local models are patched together.

A local model for the space of leaves \( Y/F \) in a neighbourhood of the point representing the leaf \( L \) is given by a small slice \( V_1 \) in \( Y \), with \( L \) intersecting \( V_1 \) transversally at a point \( 0 \). In a neighbourhood of \( 0 \), \( V_1 \) will be transverse to the foliation \( F \) and therefore the restriction \( \sigma|_{V_1} \) will be a non-degenerate two-form. Suppose \( V_2 \) is a second small slice transverse to \( L \); the intersections points \( L \cap V_1 \) and \( L \cap V_2 \) may or may not coincide. Shrinking \( V_1 \) and \( V_2 \) if necessary, there is an isomorphism \( \phi : V_1 \to V_2 \) given by taking \( L' \cap V_1 \) to \( L' \cap V_2 \), where \( L' \) is an arbitrary leaf intersecting \( V_1 \) and \( V_2 \). This map is well-defined because the leaves are simply connected, and it takes the point \( 0 \) in \( V_1 \) (i.e., \( L \cap V_1 \)) to \( 0 \) in \( V_2 \) (i.e., \( L \cap V_2 \)). This makes explicit the method of patching together local models; see Figure 1.
2.3 Divisorial contractions

Assume now that $X$ is a four-fold, and $Y$ is a hypersurface, not necessarily smooth or even normal, with $F$ the characteristic foliation on $Y$. Suppose that

Figure 1: Identifying different local models of $Y/F$

Now $\phi$ could be regarded as the time $t = 1$ map of a flow $\phi_t$ associated to a vector field $v$ along the leaves of the foliation $F$. Since $d\sigma = 0$ and $i(v)(\sigma|_Y) = 0$, the Lie derivative

$$\mathcal{L}_v(\sigma|_Y) = v(d\sigma|_Y) - d(i(v)\sigma|_Y)$$

vanishes, and therefore the flow $\phi_t$ will preserve the holomorphic two-form $\sigma|_Y$. In particular

$$\phi^*(\sigma|_{V_2}) = \sigma|_{V_1}$$

which completes the proof. □

Remark In general, the same proof shows that there is always a holomorphic symplectic form on the smooth part of the space of leaves $Y/F$. Moreover, if $V/H(L)$ is a local model for $Y/F$ near a leaf $L$ with non-trivial holonomy group $H(L)$ (in particular, $L$ itself cannot be simply connected), then $H(L)$ should be a subgroup of the symplectomorphism group of $V$, i.e., the group of automorphisms preserving the symplectic structure on $V$. The simplest case would be when $H(L)$ is a finite group; $V/H(L)$ is then a symplectic singularity, which may admit a symplectic desingularization. More generally, foliations can behave very wildly and in principle $H(L)$ could be infinite.

2.3 Divisorial contractions

Assume now that $X$ is a four-fold, and $Y$ is a hypersurface, not necessarily smooth or even normal, with $F$ the characteristic foliation on $Y$. Suppose that
the hypotheses of Theorem 3 are satisfied, i.e., there is a curve $C$ lying in the smooth locus $Y_{\text{sm}}$ of $Y$ for which $F|_C$ is ample. Recall that $F$ is locally free of rank one on $Y_{\text{sm}}$, and

$$F \cong K_Y^{-1} \cong \mathcal{O}(-Y)|_Y$$

over $Y_{\text{sm}}$. Therefore $F|_C$ ample will imply $Y.C < 0$, and the divisor $Y$ on $X$ is not nef.

Let us begin instead with the assumption $q(Y, Y) < 0$, where $q$ is the Beauville-Bogomolov quadratic form of $X$. This will imply the existence of a curve $C$ such that $F|_C$ is ample, as we now explain. A nef divisor $W$ is the limit of ample divisors, and so it must satisfy $q(W, W) \geq 0$ since $q(H, H) > 0$ for ample $H$. So if $Y$ is not in

$$\{ W \text{ an effective divisor on } X \mid q(W, W) \geq 0 \}$$

(which contains the nef cone) then $Y$ cannot be nef. It follows that there must exist a curve $C$ such that $Y.C < 0$, or in other words, such that $F|_C$ is ample. Note that if $Y$ is singular the curve $C$ won’t necessarily lie in the smooth locus $Y_{\text{sm}}$, but this is not important for the following discussion.

We can now run the log minimal model programme (MMP) on $(X, \epsilon Y)$, where $\epsilon \in \mathbb{Q}_{>0}$. The goal is to reach a particular birational model for $(X, \epsilon Y)$ after a series of directed flips. In fact it is known that birational maps between holomorphic symplectic four-folds factor through Mukai flops, i.e., blowing up a $\mathbb{P}^2$ and then blowing it down along a different ruling of the exceptional divisor (this was proved by Burns, Hu, and Luo [7] provided the indeterminacy of the map is normal, but this hypothesis can be removed because of a later result of Wierzba and Wiśniewski [47]). Note that a Mukai flop with respect to the trivial canonical bundle $K_X$ is a directed flip with respect to $K_X + \epsilon Y$.

Almost all parts of the log MMP have been proved in dimension four. The existence of log flips is due to Shokurov [43]. While termination is not yet established in full generality, there are many partial results: Kawamata, Matsuda, and Matsuki [25] proved termination for terminal flips, Matsuki [31] proved termination for terminal flops, Fujino [13] proved termination for canonical log flips, and Alexeev, Hacon, and Kawamata [1] proved termination for klt log flips under some additional hypotheses (see also Corti et al. [8]). In our case $(X, \epsilon Y)$ will be a canonical pair for sufficiently small $\epsilon$, since $X$ is smooth; we can therefore apply Fujino’s result. The conclusion is that after finitely many flips we get $(X', \epsilon Y')$ which, a priori, satisfies one of the following:

1. $X'$ is a Mori fibre space,
2. $Y'$ is nef,
3. $(X', \epsilon Y')$ admits a divisorial contraction

$$Y' \subset X'$$

$$\downarrow \quad \downarrow$$

$$S \subset \tilde{X}.$$
2.3 Divisorial contractions

In the first case $K_{X'} + \varepsilon Y'$ should be negative on all the fibres, which is impossible since $K_{X'}$ is trivial and $Y'$ is effective. Next consider case two. By assumption $q_X(Y, Y) < 0$; but then

$$q_{X'}(Y', Y') = q_X(Y, Y) < 0$$

since this value is preserved under a Mukai flop. Thus $Y'$ cannot be nef either, and case two is impossible. We conclude that the log MMP must produce case three, a divisorial contraction.

Kaledin [24] proved that symplectic resolutions are semi-small, which implies that $2\text{codim} Y' \geq \text{codim} S$. In other words, $S$ is a surface and the generic fibre of $Y' \to S$ is one-dimensional; moreover, if two-dimensional fibres occur they must be isolated, since $Y'$ is irreducible. If all the fibres are one-dimensional then part (i) of Theorem 1.3 in Wierzba [46] states that the generic fibre must be a tree of $\mathbb{P}^1$'s whose dual graph is a Dynkin diagram; moreover, these $\mathbb{P}^1$'s are (closures of) leaves of the characteristic foliation $F'$ on $Y'$. Since $Y'$ is irreducible, the generic fibre must consist of either a single rational curve or a pair of rational curves joined at a node; we call these type I and type II fibres respectively. For type II fibres there must be some kind of monodromy which interchanges the two rational curves, so that $Y'$ is still irreducible; moreover, $Y'$ will be singular along the family of nodes, so if $Y'$ is normal this case cannot occur. Part (ii) of Theorem 1.3 in [46] states that for type I fibres, $S$ is a smooth holomorphic symplectic surface and $Y' \to S$ is a flat morphism with constant fibres. Indeed, $S$ is precisely the space of leaves $Y'/F'$. A similar statement holds for type II fibres, though $S$ could now have isolated singularities. In addition, if there are isolated two-dimensional fibres, the points of $S$ over which they occur could be singular.

This essentially tells us everything about the characteristic foliation on the divisor $Y'$ in $X'$, but we’d really like to say something about the characteristic foliation on the original hypersurface $Y$ in $X$. As mentioned earlier, the birational map $X' \to X$ must factor through a sequence of Mukai flops

$$X' = X_0 \leftarrow \ldots \leftarrow X_m = X.$$ 

Recall that a Mukai flop blows up a $\mathbb{P}^2$ in $X_k$; the exceptional divisor in $\hat{X}_{k+1}$ is then isomorphic to $\mathbb{P}(\Omega^1) \cong \mathbb{P}(\Omega^1_{(p_2)})^\vee$, and this is blown down to the dual plane $(\mathbb{P}^2)^\vee$ in $X_{k+1}$. Let us consider the effect of this Mukai flop on the divisor $Y'$. If $Y'$ contains (a closure of) a leaf $C_1 \cong \mathbb{P}^1$ which intersects the plane $\mathbb{P}^2$ at a single point, then the Mukai flop will add a second component $C_2 \cong \mathbb{P}^1$ to $C_1$; $C_2$ will lie in the dual plane $(\mathbb{P}^2)^\vee$. A subsequent Mukai flop could later remove the component $C_2$; or it might remove $C_1$, if it lies in the plane $\mathbb{P}^2$ which is subsequently flopped.

**Example** Suppose that $Y'$ intersects $\mathbb{P}^2$ in a line, such that each point on this line lies in precisely one leaf. When we blow up $\mathbb{P}^2$, we will add a second
component to each of these leaves. However, when we blow down along the other ruling the second components will all end up intersecting in a single point; this is because a line in \( \mathbb{P}^2 \) parametrizes a pencil of lines in the dual plane \((\mathbb{P}^2)^\vee\), all meeting at a single point. This is illustrated in Figure 2. Note that the dual plane \((\mathbb{P}^2)^\vee\) is contained in the proper transform of \(Y'\).

**Example**  It is also possible for \(Y'\) to intersect \(\mathbb{P}^2\) in a curve of higher degree, for instance, a conic (an example of this occurs in the Hilbert scheme of two points on a K3 surface, as described in Subsection 3.1). Blowing up \(\mathbb{P}^2\) again adds second components to the leaves which intersect \(\mathbb{P}^2\) in this conic. When we blow down along the other ruling the second components will still intersect each other, but they won’t all pass through a single point. Even more complicated scenarios could arise if the plane \(\mathbb{P}^2\) intersects some leaves in more than one point.

This gives some idea of how the divisor \(Y'\) could change. However, the next lemma shows that the generic leaf is not disturbed. Denote by \(Y_k \subset X_k\) the proper transform of \(Y' \subset X'\).

**Lemma 7**  Recall that the generic fibre of \(Y' \to S\) is either of type I (a single rational curves) or of type II (a pair of rational curves joined at a node), and that these \(\mathbb{P}^1\)s are (closures of) leaves of the characteristic foliation on \(Y'\). The characteristic foliation on \(Y_k\) has generic leaf of the same type; in particular, this is true for \(Y = Y_m\). It follows that if \(Y\) is normal, then \(Y_k\) must have generic leaf of type I for all \(k\).

**Proof**  Consider the first Mukai flop \(X' = X_0 \dashrightarrow X_1\). If \(Y'\) contained the plane \(\mathbb{P}^2\), then we would have a finite cover \(\mathbb{P}^2 \to S\). This contradicts the fact that \(S\) is a holomorphic symplectic surface. Therefore \(Y'\) and \(\mathbb{P}^2\) intersect in a curve, which misses the generic fibre of \(Y' \to S\). Moreover, after the Mukai flop we will still have a rational map \(Y_1 \dashrightarrow S\), since the generic fibre of \(Y' \to S\) is not disturbed. This means that the divisor \(Y_1\) cannot contain the plane \(\mathbb{P}^2\) which is
flopped in the next birational map $X_1 \to X_2$, since otherwise we would have a rational (multi-valued) map $\mathbb{P}^2 \to S$, and a contradiction as before. Proceeding inductively we see that the generic fibre of $Y_k \to S$ is never disturbed by the subsequent Mukai flop $X_k \to X_k+1$. This proves the lemma.

The next lemma shows that (a closure of) a leaf $C_1$ cannot lie in a flopped $\mathbb{P}^2$ unless it has previously been replaced by a pair $C_1 + C_2$ of rational curves, as described above.

**Lemma 8** Let $\mathbb{P}^1$ be a generic fibre of $Y_k \to S$ (type I) or one component of a generic fibre of $Y_k \to S$ (type II). Then this rational curve $\mathbb{P}^1$ cannot be contained in the plane $\mathbb{P}^2$ which is subsequently flopped in the birational map $X_k \to X_k+1$.

**Proof** Since an (analytic) open neighbourhood of this $\mathbb{P}^1$ in $Y_k$ is isomorphic to an open neighbourhood of the corresponding rational curve in $Y'$, and $Y' \to S$ is a genuine fibration, we must have

$$N_{\mathbb{P}^1 \subset Y_k} \cong \mathcal{O} \oplus \mathcal{O}.$$  

Consider the combination of short exact sequences

$$
\begin{array}{c}
0 \\
\downarrow \\
T\mathbb{P}^1 \cong \mathcal{O}(2) \\
\downarrow \\
TX_k|_{\mathbb{P}^1} \\
\downarrow \\
0 \\
\end{array}
\quad
\begin{array}{c}
0 \\
N_{\mathbb{P}^1 \subset Y_k} \cong \mathcal{O} \oplus \mathcal{O} \\
\downarrow \\
N_{\mathbb{P}^1 \subset X_k} \\
\downarrow \\
N_{Y_k \subset X_k}|_{\mathbb{P}^1} \\
\downarrow \\
0 \\
\end{array}
\quad
\begin{array}{c}
0 \\
0 \\
\end{array}
$$

Since $c_1(TX_k) = 0$, we must have $N_{Y_k \subset X_k}|_{\mathbb{P}^1} \cong \mathcal{O}(-2)$; but then all sequences split and

$$TX_k|_{\mathbb{P}^1} \cong \mathcal{O}(-2) \oplus \mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O}(2).$$

On the other hand, if $\mathbb{P}^1 \subset \mathbb{P}^2$ then the sequence

$$0 \to T\mathbb{P}^2 \to TX_k|_{\mathbb{P}^2} \to N_{\mathbb{P}^2 \subset X_k} \cong \Omega^1\mathbb{P}^2 \to 0$$

restricted to $\mathbb{P}^1$ would also split to yield

$$TX_k|_{\mathbb{P}^1} \cong T\mathbb{P}^2|_{\mathbb{P}^1} \oplus \Omega^1\mathbb{P}^2|_{\mathbb{P}^1}$$

which will never look like $\mathcal{O}(-2) \oplus \mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O}(2)$. Note that we don’t need to assume anything about the degree of the rational curve $\mathbb{P}^1$ in $\mathbb{P}^2$. \qed
2.4 Divisors of negative square

Suppose now that $X$ is a holomorphic symplectic manifold whose dimension $2n$ could be bigger than four; let $Y \subset X$ be a hypersurface as before. In higher dimensions some parts of the minimal model programme remain conjectural at the moment, for example, termination of flips. However, some of the conclusions of the previous section are still valid. We will assume that $Y$ has negative square $q(Y,Y) < 0$ with respect to the Beauville-Bogomolov quadratic form. Note that the quadratic form $q$ is indefinite of signature $(3, b_2 - 3)$, where $b_2$ is the second Betti number of $X$. Therefore whenever $b_2$ is at least four, there will exist classes $\alpha$ in $H^2(X,\mathbb{Z})$ with $q(\alpha,\alpha) < 0$; moreover, a standard argument using the Local Torelli Theorem shows that we can make $\alpha$ algebraic after deforming $X$. A fundamental problem (which we will not address here) is finding such classes $\alpha$ which can be represented by effective divisors. The analogous problem for isotropic classes, i.e., such that $q(\alpha,\alpha) = 0$, has been studied, and effectivity has been established in certain cases (for example, see [40]).

By Proposition 5.4 of Huybrechts [20], if $X$ does not contain any uniruled divisor then $X$ does not contain any effective divisors with negative square. Indeed Boucksom later proved that a prime (i.e., effective and irreducible) divisor with negative square must be uniruled (Proposition 4.7 of [6]). The uniruled-ness can be traced back to Huybrechts’ arguments, particularly the fact that the indeterminacy of a birational map $X \dashrightarrow X'$ will be uniruled since the map factors through a blow-up and blow-down $X \rightarrow Z \rightarrow X'$. Here we give an elementary argument using foliations that yields the same result when the divisor is smooth.

**Proposition 9** Let $Y$ be a smooth irreducible hypersurface with $q(Y,Y) < 0$. Then $Y$ is a $\mathbb{P}^1$-bundle; hence it is uniruled.

**Proof** Suppose $Y$ were nef. Then $Y$ would be the limit of ample divisors $H_i$ (at least if we allow rational coefficients). Since $q(H_i, H_i) > 0$ for all $i$, we must have $q(Y,Y) \geq 0$, a contradiction.

Therefore $Y$ is not nef and there exists a curve $C$, necessarily contained in $Y$, such that $Y.C < 0$. Let $F$ be the characteristic foliation on $Y$. Then the restriction $F|_C \cong \mathcal{O}(-Y)|_C$ to $C$ is ample. Since $Y$ is smooth $F$ will be regular everywhere, so all leaves must be smooth. By Theorem 3 every leaf of the foliation is a rational curve $\mathbb{P}^1$, and $Y$ is a $\mathbb{P}^1$-bundle over the space of leaves $Y/F$. \hfill \Box

If we drop the assumption that $Y$ is smooth then the proof breaks down, as the curve $C$ may not lie in the smooth locus of $Y$, as required by Theorem 3. Nonetheless, Boucksom’s result implies that $Y$ is still uniruled. The following lemma shows that a sufficiently general rational curve in $Y$ must be (a closure of) a leaf of the characteristic foliation on $Y$.

**Lemma 10** Let $Y$ be an irreducible hypersurface with $q(Y,Y) < 0$; according to Proposition 4.7 in Boucksom [6] $Y$ is uniruled. A smooth rational curve $\mathbb{P}^1 \subset Y$...
whose normal bundle
\[ N_{\mathbb{P}^1_Y} \cong O(a_1) \oplus \ldots \oplus O(a_{2n-2}) \]
is semi-positive, i.e., \( a_i \geq 0 \) for all \( i \), must be parallel to the characteristic foliation \( F \) on \( Y \). Moreover, the normal bundle must be trivial: \( a_i = 0 \) for all \( i \).

**Proof** Since \( TX|_{\mathbb{P}^1} \) admits a non-degenerate symplectic form, it must be a direct sum of line bundles which are pairwise dual
\[ TX|_{\mathbb{P}^1} \cong O(-b_1) \oplus O(b_1) \oplus \ldots \oplus O(-b_n) \oplus O(b_n). \]
In particular, \( c_1(TX|_{\mathbb{P}^1}) = 0 \) and so the short exact sequence
\[ 0 \rightarrow T\mathbb{P}^1 \cong O(2) \rightarrow TX|_{\mathbb{P}^1} \rightarrow N_{\mathbb{P}^1_X} \rightarrow 0 \]
implies that \( c_1(N_{\mathbb{P}^1_X}) = -2 \). The exact sequence
\[ 0 \rightarrow N_{\mathbb{P}^1_Y} \rightarrow N_{\mathbb{P}^1_X} \rightarrow N_{Y\subset X}|_{\mathbb{P}^1} \rightarrow 0 \]
and the fact that \( N_{\mathbb{P}^1_Y} \) is a direct sum of line bundles \( O(a_i) \) with \( a_i \geq 0 \) then implies that \( c_1(N_{Y\subset X}|_{\mathbb{P}^1}) \leq -2 \). Moreover, this sequence must split so that
\[ N_{\mathbb{P}^1_X} \cong O(a_1) \oplus \ldots \oplus O(a_{2n-2}) \oplus N_{Y\subset X}|_{\mathbb{P}^1}. \]
Substituting this and the decomposition of \( TX|_{\mathbb{P}^1} \) into the exact sequence
\[ 0 \rightarrow T\mathbb{P}^1 \rightarrow TX|_{\mathbb{P}^1} \rightarrow N_{\mathbb{P}^1_X} \rightarrow 0, \]
we see that the only possibility is \( a_i = 0 \) for all \( i \) and \( N_{Y\subset X}|_{\mathbb{P}^1} \cong O(-2) \). Thus \( N_{\mathbb{P}^1_Y} \) is trivial. It follows that the exact sequence
\[ 0 \rightarrow T\mathbb{P}^1 \cong O(2) \rightarrow TY|_{\mathbb{P}^1} \rightarrow N_{\mathbb{P}^1_Y} \cong O(2n-2) \rightarrow 0 \]
must also split, and thus
\[ F|_{\mathbb{P}^1} \cong K^{-1}_{Y}|_{\mathbb{P}^1} \cong \bigwedge^{2n-1} TY|_{\mathbb{P}^1} \cong O(2) \]
and
\[ TY/F|_{\mathbb{P}^1} \cong O(2n-2). \]
This means that the diagonal map (the composition of inclusion and projection) in
\[ \begin{array}{cccc}
0 & \rightarrow & F|_{\mathbb{P}^1} & \rightarrow \quad TY|_{\mathbb{P}^1} & \rightarrow \quad TY/F|_{\mathbb{P}^1} & \rightarrow & 0
\end{array} \]
must be trivial, and hence the inclusion \( T\mathbb{P}^1 \hookrightarrow TY|_{\mathbb{P}^1} \) lifts to an isomorphism \( T\mathbb{P}^1 \cong F|_{\mathbb{P}^1} \). This completes the proof. \( \Box \)
3 Examples

3.1 The Hilbert square of a K3 surface

Let $S$ be a K3 surface, and $\Delta \subset S \times S$ be the diagonal. The symmetric square $\text{Sym}^2 S = S \times S/\mathbb{Z}_2$ is singular along the diagonal, but this can be resolved by blowing up. The resulting holomorphic symplectic four-fold (discovered by Fujiki [12]) is known as the Hilbert scheme of two points on $S$,

$$\text{Hilb}^2 S := \text{Blow}_\Delta (S \times S/\mathbb{Z}_2).$$

One obtains the same space by blowing up the diagonal first, and then quotienting by $\mathbb{Z}_2$.

Let $X$ be this four-fold, and let $Y$ be the inverse image of the diagonal. Then $Y$ is a $\mathbb{P}^1$-bundle over $\Delta \cong S$. It is easy to show that these $\mathbb{P}^1$s are precisely the leaves of the characteristic foliation $F$ on $Y$. Note that in this example $Y$ is smooth, $F$ is regular, all leaves are $\mathbb{P}^1$s, and the space of leaves $Y/F \cong S$ is smooth. Note also that as a $\mathbb{P}^1$-bundle over $S$, $Y$ is the projectivization $\mathbb{P}(TS)$ of the tangent bundle. The divisor $Y$ is $2$--divisible in the Picard group, i.e., $Y = 2\delta$, but the divisor $\delta$ is not effective; $Y$ is also a divisor of negative square, $q(Y, Y) = -8$.

Remark If $S$ contains a rational curve, then $\text{Hilb}^2 S$ contains $\text{Sym}^2 \mathbb{P}^1 \cong \mathbb{P}^2$. The inverse image $Y$ of the diagonal intersects this in the diagonal of $\text{Sym}^2 \mathbb{P}^1$, which is a conic in the plane $\mathbb{P}^2$, cf. the second example in Subsection 2.3.

More generally, if $S$ contains a curve $C$ (not necessarily rational) we could instead choose $Y$ to be the hypersurface

$$\{\xi \in \text{Hilb}^2 S | \text{supp}(\xi) \cap C \not= \emptyset\}.$$

We can divide $Y$ into three strata:

$$Y_1 := \{\xi = \{p, q\} | p \in C, q \in S \setminus C\} \cong C \times (S \setminus C),$$

$$Y_2 := \{\xi = \{p, q\} | p, q \in C, p \neq q\} \cong \text{Sym}^2 C \setminus \Delta,$$

$$Y_3 := \{\xi = \{p, v\} | p \in C, v \in \mathbb{P}(T_p S) \cong \mathbb{P}^1\} \cong \mathbb{P}^1\text{-bundle over } C.$$

Then $Y_1$ is dense and open, whereas $Y_3$ and the closure of $Y_2$ are divisors in $Y$. Note that $Y$ is not normal: if $\xi = \{p, q\} \in Y_2$ then $\xi$ can be deformed in $Y$ by allowing $p$ to vary in $C$ and $q$ to vary in $S$, or vice versa. This shows that two branches of $Y$ meet along $Y_2$.

The leaf through a generic element $\xi = \{p, q\} \in Y_1$ is given by varying $p$ in $C$ while leaving $q \in S \setminus C$ fixed. The generic leaf is therefore isomorphic to $C$, and $Y_1$ is a (trivial) $C$-bundle over $S \setminus C$. To understand the foliation on $Y_2$ and $Y_3$ we really need to take the normalization $\tilde{Y}$ of $Y$, and pull-back the foliation to $\tilde{Y}$. This mean replacing $Y_2 \cong \text{Sym}^2 C \setminus \Delta$ by the double cover $S \times S \setminus \Delta$; the closure $C \times C$ is then a (trivial) $C$-bundle over $C$, and the fibres are leaves of
3.2 The Hilbert cube of a K3 surface

Figure 3: Space of leaves on $\tilde{Y}$ is isomorphic to $S$

the foliation. By adding $Y_3$ we effectively add a second component, isomorphic to $\mathbb{P}^1$, to each leaf over $C$. Overall, the space of leaves on $\tilde{Y}$ is $S$; the generic leaf over a point $q \in S \setminus C$ is isomorphic to $C$, whereas the ‘leaf’ over $q \in C$ is isomorphic to the union of $C$ and a rational curve which meets $C$ at the point $q$ (we are regarding the foliation as an equivalence relation, with each equivalence class being the union of closures of leaves which intersect, cf. Subsection 2.2). This is illustrated in Figure 3.

**Example** When $C$ has genus at least two, this gives an example of a hypersurface whose characteristic foliation has non-rational leaves. This does not contradict Conjecture 4 as $Y$ is not smooth (indeed, not even normal) and moreover $\mathcal{O}(Y)$ is nef but not ample.

A higher codimension submanifold of $\text{Hilb}^2 S$ is

$$Y := \{ \xi \in \text{Hilb}^2 S | p \in \text{supp}(\xi) \}$$

with $p$ some fixed point in $S$. Then $Y$ is isomorphic to $S$ blown up at $p$, and $\sigma|_Y$ is of rank two except on the exceptional curve where the rank drops to zero ($\sigma|_Y$ vanishes there). Thus $Y$ is not coisotropic. The only non-trivial leaf is the exceptional curve and quotienting by the foliation is simply the blow-down, yielding $S$ as the space of leaves.

3.2 The Hilbert cube of a K3 surface

Now suppose that $X$ is the Hilbert scheme $\text{Hilb}^3 S$ of three points on a K3 surface, which is the symplectic desingularization of $\text{Sym}^3 S$ (see Beauville [2]).
Inside Sym³S lies the ‘big’ diagonal $\Delta_{\text{big}}$ where at least two points coincide and the ‘small’ diagonal $\Delta_{\text{small}}$ where all three points coincide; thus

$$\Delta_{\text{big}} := \{2p + q|p, q \in S\} \cong S \times S$$

and

$$\Delta_{\text{small}} := \{3p|p \in S\} \cong S.$$ 

Let $Y$ be the inverse image of $\Delta_{\text{big}}$ in Hilb³S. If $p \neq q$, then the fibre of $Y$ above $2p + q \in \Delta_{\text{big}}$ is precisely $\mathbb{P}(T_p S) \cong \mathbb{P}^1$. These are the generic leaves of the characteristic foliation on $Y$. On the other hand, the fibre of $Y$ over $3p \in \Delta_{\text{small}} \subset \Delta_{\text{big}}$ is the cone on a rational cubic curve (as described in Subsection 2.5 on punctual Hilbert schemes in Lehn [27]). Through a generic point on the cone the leaf of the foliation is a copy of $\mathbb{C}$, and its closure is a $\mathbb{P}^1$ obtained by adding the apex of the cone which lies in the singular locus $Y_{\text{sing}}$ of $Y$. Thus the closures of these leaves all intersect at the apex of the cone, where the foliation is not regular. The space of leaves in this example can be defined as

$$Y/F := \{\text{leaves of } F\}/\sim$$

where two leaves are equivalent if their closures intersect. This yields $\Delta_{\text{big}} \cong S \times S$ as the space of leaves, which is a holomorphic symplectic manifold, though not irreducible.

This example can be generalized to higher dimensions. In SymⁿS the big diagonal is isomorphic to $S \times \text{Sym}^{n-2}S$. The characteristic foliation on the inverse image $Y$ in HilbⁿS has generic leaf $\mathbb{P}^1$, though equivalence classes of special leaves become more complicated. Using the definition above, the space of leaves would be isomorphic to $S \times \text{Hilb}^{n-2}S$.

### 3.3 The generalized Kummer four-fold

Let $A$ an abelian surface, and HilbⁿA the Hilbert scheme of $n$ points on $A$. The composition of the Hilbert-Chow morphism and addition of points in $A$ gives a map

$$\text{Hilb}^nA \to \text{Sym}^nA \to A.$$ 

The inverse image of $0 \in A$ is a holomorphic symplectic manifold $K_{n-1}$ known as the generalized Kummer variety [2]. Once again we take the inverse image of the big diagonal in SymⁿA, and intersect with $K_{n-1}$ to get a hypersurface $Y$. As with the earlier examples, the generic leaf of the characteristic foliation on $Y$ is a $\mathbb{P}^1$.

Consider the case $n = 3$, with $Y$ a hypersurface in the four-fold $K_2$. The point of this example is that even though the generic leaf is rational, we still have more complicated leaves; in fact there are $3^4 = 81$ (equivalence classes of) leaves isomorphic to cones over cubic rational curves. These occur above the points $3p \in \Delta_{\text{big}}$ with $p$ a 3-torsion point in $A$. For Hilbert schemes of points on K3 surfaces this behaviour only seems to occur in dimension greater than four.
The generic point of the big diagonal looks like $2p + q$ with $p \neq q$. If this maps to $0 \in A$ under the addition map then $q = -2p$ is uniquely determined by $p \in A$. The fibre of $Y$ over such a point is the leaf $\mathbb{P}^1$, and this shows that the space of leaves is the abelian surface $A$.

### 3.4 The Beauville-Mukai system in dimension four

Let $S$ be a K3 surface containing a smooth genus two curve $C$, and suppose that $\text{Pic} S$ is generated by $C$. Then $C$ moves in a two-dimensional linear system $|C| \cong \mathbb{P}^2$. Denote this family of curves by $C \rightarrow |C|$, and let

$$X := \overline{\text{Pic} (C/|C|)}$$

be the compactified relative Jacobian. Since $C$ generates $\text{Pic} S$, every curve in the family $C$ is reduced and irreducible, so the compactified Jacobian is well-defined (see D’Souza [10]). Moreover, $X$ is a smooth holomorphic symplectic four-fold which is a deformation of $\text{Hilb}^2 S$; it is a Lagrangian fibration over $|C| \cong \mathbb{P}^2$ known as the Beauville-Mukai integrable system [4].

We chose the relative Jacobian of degree one so that each curve in $|C|$ has a canonical embedding into its Jacobian. In other words, the family of curves $C$ can be embedded in $X$ by the relative Abel-Jacobi map; let $Y \subset X$ be the resulting hypersurface.

Write

$$Y \cong C = \{(C_t, p) | C_t \in |C|, p \in C_t \}.$$  

The map $C \rightarrow |C|$ is given by forgetting $p$, but we can instead forget $C_t$ which gives a map

$$Y \rightarrow S$$

$$(C_t, p) \mapsto p.$$  

The fibre above $p \in S$ is given by the pencil of curves $C_t$ in $|C|$ which pass through the point $p$; this is a $\mathbb{P}^1$ for all $p \in S$. Thus $Y$ is a $\mathbb{P}^1$-bundle over $S$. These $\mathbb{P}^1$s are precisely the leaves of the characteristic foliation $F$ on $Y$. Moreover, $Y$ is again the total space of the projectivization $\mathbb{P}(TS)$ of the tangent bundle of the K3 surface. To see this note that $f : S \rightarrow |C|^\vee \cong \mathbb{P}^2$ is a branched double cover of the plane. A direction in the tangent space $T_p S$ then projects down to a direction in $T_{f(p)} \mathbb{P}^2$, which defines a line through $f(p)$ in $\mathbb{P}^2$. The inverse image under $f$ of this line is a curve $C_t$ through $p$ in $S$.

Markushevich [30] proved the uniqueness of this example in a certain sense: a holomorphic symplectic four-fold given by the compactified relative Jacobian of a family of genus two curves over $\mathbb{P}^2$ (with ‘mild singularities’) must be the Beauville-Mukai system described above. In Section 5 we will discuss Markushevich’s Theorem and a generalization due to the author [42].

Another hypersurface $Y$ in $X \rightarrow |C|$ is given by the inverse image of a line in $|C| \cong \mathbb{P}^2$. If the line is chosen generically, $Y$ will be smooth, and hence the
characteristic foliation will be given by \( F \cong \mathcal{O}(-Y)|_Y \). Since \( \mathcal{O}(Y) \) is nef, \( F \) restricted to a curve \( C \subset Y \) can never be ample. Thus Theorem 3 will not apply to this example. Moreover, we claim that the generic leaf is not algebraic; we outline the argument below.

Let \( p \in X \) project down to \( t \in \mathbb{P}^2 \); in other words, \( p \) is in the fibre \( X_t \) of \( \pi : X \to \mathbb{P}^2 \). There is an exact sequence

\[
0 \to T_pX_t \to T_pX \to (N_{X_t|X})_p \to 0.
\]

The fibre \( X_t \) is isomorphic to the Jacobian \( \text{Pic}^1(D) \) of some curve \( D \) in the linear system \( |C| \), and hence \( T_pX_t \cong H^0(D, \Omega^1) \). The normal bundle \( N_{X_t|X} \) corresponds to deformations of \( D \) in the K3 surface \( S \); since the normal bundle \( N_{D|S} \cong \Omega^1_D \), we have the identification \( (N_{X_t|X})_p \cong H^0(D, \Omega^1) \). The holomorphic symplectic form on \( T_pX \) is compatible with the natural pairing between the dual vector spaces \( H^0(D, \Omega^1) \) and \( H^0(D, \Omega^1) \).

A line in \( |C| \cong \mathbb{P}^2 \) corresponds to a pencil of lines through a fixed point \( q' \) in the dual plane \( |C|' \), which in turn corresponds to the sublinear system of curves in \( |C| \) which all pass through some fixed point \( q \) in the K3 surface \( S \). Here \( q \) maps to \( q' \) under the projection \( f : S \to |C|' \). For the hypersurface \( Y \) there is an exact sequence

\[
0 \to T_pX_t \to T_pY \to (N_{X_t|Y})_p \to 0.
\]

Here \( T_pX_t \cong H^0(D, \Omega^1) \) as before, but \( (N_{X_t|Y})_p \cong H^0(D, \Omega^1(-q)) \) since when we deform the curve \( D \) in \( S \) it must still pass through the fixed point \( q \).

Restricting the holomorphic symplectic form \( \sigma \) to \( Y \) gives a degenerate two-form on \( T_pY \). One can show that the kernel lies entirely in the subspace \( T_pX_t \subset T_pY \). Moreover, identifying \( T_pX_t \) with \( H^0(D, \Omega^1) \), we find that

\[
\ker \sigma|_Y \cong \ker(H^0(D, \Omega^1)' \to H^0(D, \Omega^1(-q))') \cong H^0(q, \Omega^1|_q)'
\]

i.e., the foliation \( F \) is linear in the abelian surface fibres \( X_t \) of \( Y \). For a generic curve \( D \) through \( q \), the vector field in the direction of \( H^0(q, \Omega^1|_q)' \) will be irrational in

\[
X_t \cong \frac{H^0(D, \Omega^1)'}{H_1(D, \mathbb{Z})}
\]

and hence the leaves of the foliation will be dense in \( X_t \), not algebraic. Only for some special choices of \( D \) will we get algebraic leaves, which will be isomorphic to elliptic curves.

### 3.5 Higher-dimensional Beauville-Mukai systems

Let \( S \) be a K3 surface containing a smooth genus \( n \) curve \( C \), and suppose that \( \text{Pic}S \) is generated by \( C \). Then \( C \) moves in an \( n \)-dimensional linear system \( |C| \cong \mathbb{P}^n \). Denote this family of curves by \( C \to |C| \), and let

\[
X := \text{Pic}^{-1}(C/|C|)
\]
be the compactified relative Jacobian. Since \( C \) generates \( \text{Pic}S \), every curve in the family \( C \) is reduced and irreducible, so the compactified Jacobian is well-defined (see D'Souza [10]). Moreover, \( X \) is a smooth holomorphic symplectic manifold of dimension \( 2n \) which is a deformation of \( \text{Hilb}^n S \); it is a Lagrangian fibration over \( |C| \cong \mathbb{P}^n \) known as the Beauville-Mukai system [4].

In a smooth fibre \( \text{Pic}^{n-1} C \) there is a theta divisor which can be canonically defined as the image of the map \( \text{Sym}^{n-1} C \to \text{Pic}^{n-1} C \); it can also be defined as the locus of degree \( n - 1 \) line bundles \( L \) on the curve \( C \) which admit at least one non-trivial section, i.e., \( h^0(L) > 0 \). A generic singular curve in the linear system \( |C| \) will have precisely one node; the compactified Jacobian of such a curve contains a generalized theta divisor (see Esteves [11]), which again can be defined as the locus of rank one torsion-free sheaves which admit at least one non-trivial section. For a more singular curve (i.e., worse than simply one node) one could try to define a generalized theta divisor as above, but it is not clear how well-behaved it will be. However, these curves with worse singularities will occur above a codimension two subset \( \Delta_{\text{sing}} \subset |C| \) in the linear system. Therefore a simpler approach is to take the closure of the relative theta divisor over \( |C| \setminus \Delta_{\text{sing}} \); this produces a hypersurface \( Y \subset X \), and how it looks over \( \Delta_{\text{sing}} \) will usually not be important.

As mentioned above, for a smooth genus \( n \) curve the theta divisor in \( \text{Pic}^{n-1} C \) parametrizes line bundles which admit at least one non-trivial section. A line bundle corresponding to a generic point in the theta divisor will admit exactly one section up to scale. Moreover, this section will vanish at \( n - 1 \) distinct points on \( C \subset S \); these points then define a generic element of \( \text{Hilb}^{n-1} S \). These statements also apply to singular curves in the linear system \( |C| \) which have precisely one node. Thus we obtain a rational map from \( Y \) to \( \text{Hilb}^{n-1} S \). What is the generic fibre of this map?

Let \( \xi = \{p_1, \ldots, p_{n-1}\} \) be a generic element of \( \text{Hilb}^{n-1} S \) consisting of \( n - 1 \) distinct points. Note that \( S \) is embedded in \( |C| \cong \mathbb{P}^n \). The \( n - 1 \) points then determine a codimension two linear subspace in \( |C| \), or equivalently a pencil of hyperplanes in \( |C| \) (the hyperplanes which contain these \( n - 1 \) points). Each hyperplane intersects \( S \) in a curve which belongs to the linear system \( |C| \); the \( n - 1 \) points lie on this curve and define an effective degree \( n - 1 \) divisor on it. Thus the pencil of hyperplanes in \( |C| \) determines a pencil of curves in \( |C| \), each with a degree \( n - 1 \) line bundle which lies in the corresponding theta divisor. In other words, the fibre of \( Y \) above \( \xi \in \text{Hilb}^{n-1} S \) is precisely this pencil, a rational curve \( \mathbb{P}^1 \). Finally, one can show that this \( \mathbb{P}^1 \) is a leaf of the characteristic foliation \( F \) on \( Y \).

When \( n \) is at least three, the theta divisor in \( \text{Pic}^{n-1} C \) will be singular, even for a smooth curve \( C \). Therefore we can expect \( Y \) to be singular. Since the characteristic foliation \( F \) is not regular on the singular locus of \( Y \), the special leaves will presumably be fairly complicated.
3.6 Hurtubise’s foliation

We continue the example from the previous subsection. Instead of a hypersurface in $X$, we wish to find a coisotropic submanifold of higher-codimension. Varying the definition slightly, let

$$X := \overline{\text{Pic}}(\mathcal{C}/|C|)$$

be the compactified relative Jacobian of degree one, rather then degree $n-1$. Every curve, both smooth and singular, can be embedded in its compactified Jacobian by the Abel-Jacobi map; for the degree one Jacobian this embedding is canonical, i.e., it does not require the choice of a basepoint on the curve. Let $Y \subset X$ be the image of the relative Abel-Jacobi map

$$
\begin{array}{ccc}
\mathcal{C} & \rightarrow & X \\
\downarrow & \searrow & \downarrow |C| \\
|C| & \rightarrow & \\
\end{array}
$$

so that $Y$ is an $(n+1)$-dimensional submanifold of $X$ isomorphic to the total space of $\mathcal{C} \rightarrow |C|$. Writing

$$Y \cong \mathcal{C} = \{(C_t, p) | C_t \in |C|, p \in C_t\}$$

we see that there is a map given by projection

$$Y \rightarrow S \quad (C_t, p) \mapsto p.$$ 

The fibre above $p \in S$ consists of all curves in $|C|$ which pass through the point $p$. This is a linear codimension one condition on curves, so the fibre is a hyperplane in $|C| \cong \mathbb{P}^n$, and hence isomorphic to $\mathbb{P}^{n-1}$ for all $p \in S$. Thus $Y$ is a $\mathbb{P}^{n-1}$-bundle over $S$; in particular we see that $Y$ is smooth. Moreover, these $\mathbb{P}^{n-1}$s are precisely the leaves of the characteristic foliation $F$ on $Y$, and the space of leaves $Y/F$ is therefore $S$.

Hurtubise considered a local family of Jacobians forming a Lagrangian fibration in [17]. In other words, he took a family $\mathcal{C}$ of (smooth) genus $n$ curves over an $n$-dimensional disc $U \subset \mathbb{C}^n$ such that the total space of the relative Jacobian $X := \text{Pic}^d(\mathcal{C}/U)$ admits a holomorphic symplectic structure, with respect to which the fibres are Lagrangian. We do not need to compactify since we assume that every curve in the family is smooth. The degree is also unimportant here, since we can choose a section of $\mathcal{C} \rightarrow U$; this gives a basepoint in each curve which allows us to identify Picard groups of different degrees.

Hurtubise then considered the image of the relative Abel-Jacobi map which embeds the total space of the family $\mathcal{C} \rightarrow U$ as an $(n+1)$-dimensional submanifold in $X$; once again, call the image $Y \subset X$. Assume that the restriction $\sigma|_Y$ of the holomorphic symplectic form has constant rank two on $Y$, i.e., assume that
Y is a coisotropic submanifold. The characteristic foliation on \( Y \) then has leaves of dimension \( n-1 \), and quotienting by the foliation yields (an open subset, in the analytic topology, of) a holomorphic symplectic surface \( Q \) as the space of leaves \( Y/F \). Hurtubise then shows that \( X \) is birational to the Hilbert scheme \( \text{Hilb}^n Q \) of \( n \) points on the surface \( Q \), with the rational map identifying the holomorphic symplectic structure on \( X \) with the natural holomorphic symplectic structure on \( \text{Hilb}^n Q \) induced from that on \( Q \). This result may be viewed as an example of Sklyanin’s separation of variables. Hurtubise and Markman \cite{18} also proved similar results for fibrations by Prym varieties.

It is worth pointing out that these are all local results so the issue of compactness of the leaves of the foliation does not arise. By studying the compact analogue of this situation, a family of curves over \( \mathbb{P}^n \), and using Theorem \( \ref{3} \) to establish algebraicity of the leaves of the resulting foliation, the author \cite{42} was able to generalize Markushevich’s Theorem \cite{30} (mentioned in Subsection 3.4 above) to higher dimensions. We will discuss this result in Section 5.

### 3.7 Mukai moduli spaces

All of the holomorphic symplectic manifolds in the examples above can be described as Mukai moduli spaces of stable sheaves on K3 or abelian surfaces \cite{36}. For example, we can associate to a zero-dimensional length \( n \) subscheme \( \xi \in \text{Hilb}^n S \) its ideal sheaf \( \mathcal{I}_\xi \); thus \( \text{Hilb}^n S \) can be thought of as the moduli space of rank one stable sheaves with first Chern class \( c_1 = 0 \) and second Chern class \( c_2 = n \). The hypersurfaces and coisotropic submanifolds \( Y \subset X \) described above can then often be defined in terms of Brill-Noether loci, e.g. loci of sheaves with more then the generic number of sections. We first describe the general setup and then fit some of the above examples into this framework. We will make a number of assumptions along the way, but the construction can be modified to accommodate the other cases.

Let \( S \) be a K3 surface and let

\[
v = (v_0, v_1, v_2) \in H^0(S) \oplus H^2(S) \oplus H^4(S)
\]

be a primitive Mukai vector. Denote by \( X := M^s(v) \) the Mukai moduli space of stable sheaves \( \mathcal{E} \) on \( S \) with Mukai vector

\[
v(\mathcal{E}) := (r, c_1, r + \frac{c_2^2}{2} - c_2) = v
\]

where \( r \) is the rank of \( \mathcal{E} \), and \( c_1 \) and \( c_2 \) are the Chern classes of \( \mathcal{E} \). Since \( v \) is primitive, \( M^s(v) \) will be a smooth compact holomorphic symplectic manifold of dimension

\[2 + \langle v, v \rangle = 2 + v_1^2 - 2v_0v_2.\]

Assume that the generic sheaf \( \mathcal{E} \in M^s(v) \) does not admit any non-trivial sections. If this were not the case, we could tensor with an appropriate power of an ample line bundle \( L \) on \( S \), so that \( \mathcal{E} \otimes L^{-k} \) does not admit any non-trivial
sections. Then instead of the moduli space \( M^s(v) \), we could work with the isomorphic moduli space \( M^s(v') \) whose elements look like \( \mathcal{E} \otimes L^{-k} \).

Let \( Y \subset X = M^s(v) \) be the locus of sheaves \( \mathcal{E} \) which admit at least one non-trivial section, i.e., such that \( h^0(\mathcal{E}) > 0 \). A priori, \( Y \) could be the empty set, but let us assume that this is not the case (for example, this might be achieved by choosing \( k \) above to be the smallest integer such that \( \mathcal{E} \otimes L^{-k} \) does not admit any non-trivial sections). Moreover, let us assume that the generic element \( \mathcal{E} \) of \( Y \) admits exactly one non-trivial section up to scale, \( h^0(\mathcal{E}) = 1 \) (this seems to be a typical phenomenon), and that this section gives an injection

\[
\mathcal{O}_S \hookrightarrow \mathcal{E}
\]

(which is usually the case for generic \( \mathcal{E} \in Y \) if the rank \( r \) is at least two).

If all of these assumptions apply, then there is a rational map

\[
Y \to M^s(w)
\]

given by taking a generic sheaf \( \mathcal{E} \in Y \) to the cokernel \( \mathcal{F} \) of the above injection, i.e.,

\[
0 \to \mathcal{O}_S \to \mathcal{E} \to \mathcal{F} \to 0.
\]

Here the Mukai vector \( w \) satisfies

\[
w = (w_0, w_1, w_2) := (v_0 - 1, v_1, v_2 - 1).
\]

The short exact sequence ensures that the cokernel \( \mathcal{F} \) has Mukai vector \( w \), though one still has to check that \( \mathcal{F} \) will be stable for generic \( \mathcal{E} \in Y \). Note that \( M^s(w) \) has dimension

\[
2 + \langle w, w \rangle = 2 + \langle v, v \rangle - 2(1 - v_0 - v_2) = \dim X - 2(1 - v_0 - v_2).
\]

The fibre above a generic element \( \mathcal{F} \in M^s(w) \) consists of all \( \mathcal{E} \) which can be written as extensions of \( \mathcal{F} \) by \( \mathcal{O}_S \), up to isomorphism; but such \( \mathcal{E} \) are parametrized by \( \mathbb{P}(\text{Ext}^1(\mathcal{F}, \mathcal{O}_S)) \), and thus the fibres are projective spaces. We claim that these projective spaces are leaves of the characteristic foliation on \( Y \).

**Proposition 11** In the above situation, the generic fibre of \( Y \to M^s(w) \) is a leaf of the characteristic foliation \( F \) on \( Y \). Thus the space of leaves \( Y/F \) is birational to \( M^s(w) \) and the leaf above a generic point \( \mathcal{F} \in M^s(w) \) is isomorphic to \( \mathbb{P}(*\text{Ext}^1(\mathcal{F}, \mathcal{O}_S)) \).

**Proof** For \( \mathcal{E} \in X = M^s(v) \) the tangent space \( T_{\mathcal{E}}X \) can be identified with \( \text{Ext}^1(\mathcal{E}, \mathcal{E}) \). For \( \mathcal{E} \in Y \) our goal is to identify the tangent space \( T_{\mathcal{E}}Y \subset T_{\mathcal{E}}X \) to the submanifold \( Y \subset X \) and the distribution \( F_{\mathcal{E}} \subset T_{\mathcal{E}}Y \).

We start with the short exact sequence

\[
0 \to \mathcal{O}_S \to \mathcal{E} \to \mathcal{F} \to 0
\]

and its dual

\[
0 \to \mathcal{F}^\vee \to \mathcal{E}^\vee \to \mathcal{O}_S \to 0
\]
(for simplicity, we assume that both $\mathcal{E}$ and $\mathcal{F}$ are locally free). Tensoring these together and taking the corresponding long exact sequences gives

\[
\begin{array}{ccccccccc}
& & & & & & & & & & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
& & & & & & & & & & & \\
0 & \to & H^0(\mathcal{F}^\vee) & \to & \text{Hom}(\mathcal{F}, \mathcal{E}) & \to & \text{Hom}(\mathcal{F}, \mathcal{F}) & \to & \cdots & (\spadesuit) \\
& & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & H^0(\mathcal{E}^\vee) & \to & \text{Hom}(\mathcal{E}, \mathcal{E}) & \to & \text{Hom}(\mathcal{E}, \mathcal{F}) & \to & \cdots & (\heartsuit) \\
& & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & H^0(\mathcal{O}_S) & \to & H^0(\mathcal{E}) & \to & H^0(\mathcal{F}) & \to & \cdots & (\spadesuit) \\
\end{array}
\]

Note that the first row continues on the fourth row, which continues on the seventh row; this is indicated by ellipses and $\spadesuit$ (similarly for the other rows).

Since $\mathcal{E}$ and $\mathcal{F}$ are stable, we have

\[
\text{Hom}(\mathcal{F}, \mathcal{F}) \cong \mathbb{C} \quad \text{and} \quad \text{Hom}(\mathcal{E}, \mathcal{E}) \cong \mathbb{C}
\]

and then by Serre duality

\[
\text{Ext}^2(\mathcal{F}, \mathcal{F}) \cong \mathbb{C} \quad \text{and} \quad \text{Ext}^2(\mathcal{E}, \mathcal{E}) \cong \mathbb{C}.
\]

The slopes of $\mathcal{E}$ and $\mathcal{F}$ must satisfy $0 < \mu(\mathcal{E}) < \mu(\mathcal{F})$ and therefore

\[
\text{Hom}(\mathcal{F}, \mathcal{E}) = 0 \quad \text{and} \quad \text{Ext}^2(\mathcal{E}, \mathcal{F}) = 0.
\]

Since $\mu(\mathcal{F}^\vee) < \mu(\mathcal{E}^\vee) < 0$ we must have

\[
H^0(\mathcal{F}^\vee) = 0, \quad H^0(\mathcal{E}^\vee) = 0, \quad H^2(\mathcal{F}) = 0, \quad \text{and} \quad H^2(\mathcal{E}) = 0.
\]

By assumption, a generic $\mathcal{E} \in Y$ admits exactly one non-trivial section up to scale, and thus

\[
H^0(\mathcal{E}) \cong \mathbb{C} \quad \text{and} \quad H^2(\mathcal{E}^\vee) \cong \mathbb{C}.
\]

Since a non-trivial section of $\mathcal{F}$ would lift to a second independent section of $\mathcal{E}$

\[
0 \to \mathcal{O}_S \to \mathcal{E} \to \mathcal{F} \to 0
\]
we must have
\[ H^0(\mathcal{F}) = 0 \quad \text{and} \quad H^2(\mathcal{F}^\vee) = 0. \]

Finally, since \( S \) is a K3 surface we have
\[ H^0(\mathcal{O}_S) \cong \mathbb{C}, \quad H^1(\mathcal{O}_S) = 0, \quad \text{and} \quad H^2(\mathcal{O}_S) \cong \mathbb{C}. \]

From the third column of the above diagram we can deduce
\[ \text{Hom}(\mathcal{E}, \mathcal{F}) \cong \text{Hom}(\mathcal{F}, \mathcal{F}) \cong \mathbb{C} \]
and from the seventh row (or dually) we can deduce
\[ \text{Ext}^2(\mathcal{F}, \mathcal{E}) \cong \text{Ext}^2(\mathcal{F}, \mathcal{F}) \cong \mathbb{C}. \]

The diagram now looks like

\[
\begin{array}{c}
0 & 0 & 0 \\
\downarrow & \downarrow & \downarrow \\
0 & 0 & 0 & \rightarrow & \mathbb{C} & \rightarrow & \cdots (♥)
\end{array}
\]

\[
\begin{array}{c}
0 & \rightarrow & 0 & \rightarrow & 0 & \rightarrow & \mathbb{C} & \rightarrow & \cdots (♣)
\end{array}
\]

\[
\begin{array}{c}
0 & \rightarrow & \mathbb{C} & \rightarrow & \mathbb{C} & \rightarrow & 0 & \rightarrow & \cdots (♠)
\end{array}
\]

from which we extract

\[
\begin{array}{c}
0 \\
0 \rightarrow \mathbb{C} \rightarrow H^1(\mathcal{F}^\vee) \rightarrow \text{Ext}^1(\mathcal{F}, \mathcal{E}) \rightarrow \text{Ext}^1(\mathcal{F}, \mathcal{F}) \rightarrow 0 \\
\text{Ext}^1(\mathcal{E}, \mathcal{E}) \\
H^1(\mathcal{E}) \\
0
\end{array}
\]
3.7 Mukai moduli spaces

Recall that $T_X \cong \text{Ext}^1(\mathcal{E}, \mathcal{E})$. An element of the subspace $\text{Ext}^1(\mathcal{F}, \mathcal{E})$ will give a deformation of $\mathcal{E}$ that preserves the short exact sequence description of $\mathcal{E}$, i.e., $T_Y \cong \text{Ext}^1(\mathcal{F}, \mathcal{E})$. Therefore the vertical sequence is the normal sequence for $Y \subset X$, and $(N_{Y \subset X})_\mathcal{E} \cong H^1(\mathcal{E})$.

The first column of the larger diagram implies that $H^1(\mathcal{E}^\vee)$ is the cokernel of $C \to H^1(\mathcal{F}^\vee)$. By Serre duality this is dual to $H^1(\mathcal{E})$, and moreover this duality is compatible with the natural symplectic structure $\sigma$ on $\text{Ext}^1(\mathcal{E}, \mathcal{E})$. By exactness an element $u$ of $T_Y \cong \text{Ext}^1(\mathcal{F}, \mathcal{E}) \subset \text{Ext}^1(\mathcal{E}, \mathcal{E})$ maps to zero in $H^1(\mathcal{E})$, and will therefore pair trivially with any element $u'$ of $H^1(\mathcal{E}^\vee) \subset \text{Ext}^1(\mathcal{F}, \mathcal{E}) \cong T_Y$.

In other words, $\sigma(u, u') = 0$ and thus $H^1(\mathcal{E}^\vee)$ can be identified with the null distribution $F \subset T_Y$. It follows that the diagram above is precisely

\[
\begin{array}{cccccc}
0 & \rightarrow & F_Y & \rightarrow & T_Y & \rightarrow (TY/F)_\mathcal{E} & \rightarrow 0 \\
\downarrow & & \downarrow & & \downarrow & \downarrow & \\
0 & \rightarrow & T_Y & \rightarrow & (TY/F)_\mathcal{E} & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & \downarrow & \\
0 & \rightarrow & T_X & \rightarrow & (N_{Y \subset X})_\mathcal{E} & \rightarrow & 0
\end{array}
\]

Finally, observe that $F_Y \cong H^1(\mathcal{E}^\vee) \cong \text{cokernel}(C \to H^1(\mathcal{F}^\vee))$ can be identified with the tangent space to the projective space $\mathbb{P}(\text{Ext}^1(\mathcal{F}, \mathcal{O}_S))$ at $\mathcal{E}$, and $(TY/F)_\mathcal{E} \cong \text{Ext}^1(\mathcal{F}, \mathcal{F})$ can be identified with the tangent space to $M^s(w)$ at $\mathcal{F}$, where of course $\mathcal{F}$ is the image of $\mathcal{E}$ under the projection $Y \twoheadrightarrow M^s(w)$. In other words, $M^s(w)$ is birational to the space of leaves and $\mathbb{P}(\text{Ext}^1(\mathcal{F}, \mathcal{O}_S))$ is the leaf above a generic point $\mathcal{F} \in M^s(w)$. □

For our first example, consider the Beauville-Mukai integrable system coming from a genus $n$ curve $C$ in $S$. A sheaf $L$ on a curve $C$ in the linear system $[C]$ can be thought of as a torsion sheaf $\iota_L$ on the K3 surface $S$ itself, where $\iota : C \hookrightarrow S$ is the embedding of the curve. This allows us to identify the compactified relative Jacobian $X = \overline{\text{Pic}}^{-1}(C/[C])$ with the Mukai moduli space $M^s(0, [C], 0)$. The generic degree $n-1$ line bundle on a genus $n$ curve will not admit a non-trivial section, and thus nor will a generic element of $M^s(0, [C], 0)$. On the other hand, the relative theta divisor $Y \subset X$ parametrizes line bundles $L$ (and more generally, rank one torsion-free sheaves) on curves $C$ which admit at least one non-trivial section

\[
\mathcal{O}_C \xrightarrow{s} L.
\]
This means that the corresponding torsion sheaf $\iota_* L$ on $S$ also admits a section, given by the composition
\[
\mathcal{O}_S \to \iota_* \mathcal{O}_C \xrightarrow{\iota_* s} \iota_* L.
\]
Thus the relative theta divisor $Y$ is a Brill-Noether locus on $X = M^s(0, [C], 0)$.

In this example the map
\[
\mathcal{O}_S \to \iota_* L
\]
is certainly not injective. However, it does fit into an exact sequence
\[
0 \to \mathcal{O}_S(-C) \to \mathcal{O}_S \to \iota_* L \to \mathcal{F} \to 0
\]
where $\mathcal{F}$ is defined as the cokernel of the section; but as we vary the curve in $|C|$, the line bundle $\mathcal{O}_S(-C)$ does not change, so we can ignore this part of the sequence. Thus we do indeed get a rational map
\[
Y \dashrightarrow M^s(w)
\]
which takes $\iota_* L$ to $\mathcal{F}$. Note that $w = (0, 0, n - 1)$ and $M^s(w) \cong \text{Hilb}^{n-1} S$, i.e., $\mathcal{F}$ will be the structure sheaf of a zero-dimensional length $n - 1$ subscheme of $S$. Generically this will just be the set of $n - 1$ points on $C$ where the section $s$ of $L$ vanishes.

A similar argument applies when $X = \text{Pic}^1(C/|C|)$, which is isomorphic to $M^s(0, [C], -(n - 2))$. Once again $Y$, the image of the Abel-Jacobi map which embeds the family of curves $C$ in $X$, is a Brill-Noether locus, parametrizing sheaves $L$ on a curve which admit non-trivial sections, or equivalently, torsion sheaves $\iota_* L$ on the K3 surface $S$ which admit non-trivial sections. There is a long exact sequence
\[
0 \to \mathcal{O}_S(-C) \to \mathcal{O}_S \to \iota_* L \to \mathcal{F} \to 0
\]
like before, though now $\mathcal{F}$ has Mukai vector $w = (0, 0, 1)$, meaning that it is the structure sheaf of a point. Thus we get a rational map
\[
Y \dashrightarrow M^s(w) \cong S
\]
as expected.

In these examples, the generic leaves are projective spaces. In general, Brill-Noether loci in Mukai moduli spaces are stratified and each open strata is a Grassmannian bundle (see Section 5 of Markman [28]). The Grassmannian fibres of the largest strata are the generic leaves of the characteristic foliation on the Brill-Noether locus; note that Grassmannians are rationally connected, indicating compatibility with Theorem [3]. If $X = \text{Hilb}^n S$ is a Hilbert scheme of points on a K3 surface $S$, and $Y$ is the inverse image of a (small or big) diagonal in $\text{Sym}^n S$, then it is less obvious how to describe $Y$ as a Brill-Noether locus: there does not appear to be a way to homologically distinguish a subscheme $\xi \in Y$ from a generic subscheme $\xi \in X$. However, there may be an isomorphism of $\text{Hilb}^n S$ with another moduli space on $S$ which takes $Y$ to a Brill-Noether locus, as in Example 21 in [28].
4 Nagai’s Theorem

A possible framework for classifying holomorphic symplectic four-folds is as follows. Given an irreducible holomorphic symplectic four-fold \( X \), we first find a hypersurface \( Y \) in \( X \) which should be negative as a divisor in some sense (certainly \( Y \) should not be ample or nef). In general we will have to deform \( X \) in order to find such \( Y \); indeed the generic holomorphic symplectic manifold is non-projective and contains no hypersurfaces at all. Then we investigate the characteristic foliation on \( Y \) and try to show that there is a nice space of leaves \( S = Y/F \) which is a holomorphic symplectic surface. Finally, we describe \( X \) in terms of the surface \( S \).

Nagai’s Theorem \cite{37} exemplifies this approach. Recall that inside the Hilbert scheme \( X = \text{Hilb}^2 S \) of two points on a K3 surface \( S \), the inverse image of the diagonal is a \( \mathbb{P}^1 \)-bundle over the diagonal \( \Delta \cong S \). Thus we can recover the K3 surface \( S \) as the space of leaves \( Y/F \), and clearly \( X \) can be described in terms of \( S \). Although he does not explicitly use the language of foliations, Nagai proved that if a holomorphic symplectic four-fold \( X \) contains a divisor \( Y \) satisfying certain hypotheses, then the space of leaves of the characteristic foliation on \( Y \) is a K3 surface \( S \) and \( X \) is isomorphic to \( \text{Hilb}^2 S \). We will describe the main points of Nagai’s proof, adding some new observations. First recall that by semi-smallness, a divisorial contraction of holomorphic symplectic four-fold must contract the exceptional divisor to a surface.

**Theorem 12 (37)** Let \( X \) be a projective irreducible holomorphic symplectic four-fold. Assume there is a birational morphism \( f : X \to Z \) which contracts an irreducible divisor \( Y \) to a surface \( S \subset Z \) such that

1. \( f|_Y : Y \to S \) is equidimensional with generic fibre \( \mathbb{P}^1 \),
2. \( Y \) is 2-divisible in \( \text{Pic} X \),
3. \( Y^4 = 192 \).

Then \( S \) is a K3 surface and \( X \cong \text{Hilb}^2 S \).

**Remark** Nagai notes that if it is known that \( X \) is a deformation of the Hilbert scheme of two points on a K3 surface, then the third condition may be replaced by \( q_X(Y,Y) = -8 \), where \( q_X \) is the Beauville-Bogomolov form. We claim that part of the first condition (that the generic fibre is a rational curve) will also follow from this assumption.

**Lemma 13** Suppose that \( X \) is a deformation of the Hilbert scheme of two points on a K3 surface, and \( Y \subset X \) is an irreducible divisor. If \( q_X(Y,Y) = -8 \) and \( Y \) is 2-divisible in \( \text{Pic} X \), then the generic leaf of the characteristic foliation on \( Y \) is a single rational curve \( \mathbb{P}^1 \).

**Proof** As described in Subsection 2.3, applying the log minimal model programme to \( (X, \epsilon Y) \) produces, after a sequence of directed flips, \( (X', \epsilon Y') \) which
admits a divisorial contraction, with the proper transform $Y'$ of $Y$ contracted to a surface. By Lemma 7 the generic leaves of $Y$ and $Y'$ are isomorphic. Now Wierzba proved that the generic fibre of the contraction of $Y'$ to a surface must be a single rational curve (type I) or a pair of rational curves joined at a node (type II); suppose it is the latter, and let $C_1$ and $C_2$ be a pair of rational curves which form a generic fibre. Now a calculation shows that

$$O(Y')|_{C_i} \cong O(-1)$$

which contradicts the fact that $Y'$ must be 2-divisible, like $Y$. Therefore the generic fibre is of type I, completing the proof.

□

Remark Note that we still need to assume that $f|_Y : Y \to S$ is equidimensional in order to conclude that $Y$ is a genuine $\mathbb{P}^1$-bundle over $S$, a fact which is used in several of the calculations in Nagai’s proof. Without this hypothesis, it is possible that some fibres could have larger dimension, as in the example of the generalized Kummer four-fold described in Subsection 3.3.

Returning to Nagai’s Theorem, let us give an outline of his proof. Since $Y$ is 2-divisible in $\text{Pic} X$, let $\tilde{X}$ be the double cover of $X$ defined by $O(\frac{1}{2}Y)$. Let $\tilde{X} \to \tilde{Z} \to Z$ be the Stein factorization of the composition $\tilde{X} \to X \to Z$. Since $\tilde{X} \to X$ is ramified over $Y$, $\tilde{Z} \to Z$ will be ramified over $f(Y) = S$, around which $Z$ looks locally like $\mathbb{C}^2 \times (A_1$ surface singularity); it follows that $\tilde{Z}$ will be smooth. Moreover, $\tilde{Z}$ will have trivial canonical bundle. Some calculations involving the holomorphic Lefschetz formula of Atiyah and Singer show that $\tilde{Z}$ is simply connected and $h^0(\tilde{Z}, \Omega^2_{\tilde{Z}}) = 2$. The Bogomolov Decomposition Theorem then implies that $\tilde{Z}$ is isomorphic to the product $S_1 \times S_2$ of two K3 surfaces. Finally, Nagai shows that both $S_1$ and $S_2$ are isomorphic to $S$, which is therefore also a K3 surface, and the covering involution of $\tilde{Z} \to Z$ simply interchanges the two factors of $\tilde{Z} \cong S \times S$. This means that we can identify the two diagrams

$$\begin{array}{ccc}
\tilde{X} & \to & \tilde{Z} \\
\downarrow & \downarrow & \downarrow \\
X & \to & Z
\end{array} \quad \text{and} \quad \begin{array}{ccc}
\text{Blow}_{\Delta}(S \times S) & \to & S \times S \\
\downarrow & \downarrow & \downarrow \\
\text{Hilb}^2 S & \to & \text{Sym}^2 S
\end{array}$$

completing the proof.

Note that most of the proof is concerned with reconstructing $X$ from the space of leaves $S \cong Y/F$; indeed the existence of the surface $S$ is already part of the hypotheses. In general, starting with just a hypersurface $Y \subset X$, the difficulty with constructing a surface $S$ as the image of $Y$ under a divisorial contraction is that a birational transformation $X \dashrightarrow X'$ might first be required. Of course, one could try to classify $X'$ instead, and note that $X$ is at least deformation equivalent to $X'$, since they are birational (as proved by Huybrechts [19]). Another approach would be to show that $X$ does not contain any projective planes $\mathbb{P}^2$; then no Mukai flop would be possible, so any birational map $X \dashrightarrow X'$ would in fact be an isomorphism. In the next section we adopt a different approach, though for a different four-fold $X$ and divisor $Y$: we show
that $Y$ is smooth, which will yield a nice space of leaves $S \cong Y/F$ without the need of any birational map.

5 Classification of fibrations by Jacobians

**Definition** Let $X$ be an irreducible holomorphic symplectic manifold of dimension $2n$. We say that $X$ is a Lagrangian fibration if it is fibred over $\mathbb{P}^n$ with generic fibre an abelian variety of dimension $n$. Moreover, every fibre should be Lagrangian with respect to the holomorphic symplectic form.

This is essentially the only non-trivial fibration which can exist on an irreducible holomorphic symplectic manifold, due to results of Matsushita [32, 33] and Hwang [22]. Our aim in this section is to consider Lagrangian fibrations whose fibres are (compactified) Jacobians of genus $n$ curves. We will recall a classification theorem of Markushevich [30] in dimension four, and describe how it can be proved using foliations. Then we will describe a generalization of this classification result to higher dimensions, due to the author [42]. To begin, we need some control over the severity of the singular fibres in the fibration.

**Definition** (Markushevich [30]) Let $C \to \mathbb{P}^2$ be a flat family of genus two curves. We say that the family has mild degenerations if

1. the total space $C$ is smooth,
2. every curve in the family is reduced and irreducible, and the singular curves have only nodes or cusps as singularities,
3. if $C_t$ is a singular curve with two singular points $P_1 \neq P_2$, then the two analytic germs $(\Delta_1, t)$ and $(\Delta_2, t)$ of the discriminant curves of the unfoldings of the singularities $(C_t, P_i)$ meet transversely at $t \in \mathbb{P}^2$.

**Remark** There is a hypersurface (i.e., curve) $\Delta \subset \mathbb{P}^2$ parametrizing singular curves. Suppose that $C \to \mathbb{P}^2$ has mild degenerations; then $C_t$ will have a single node if $t$ is a generic point of $\Delta$, it will have two nodes if $t$ is a node of $\Delta$, and it will have a single cusp if $t$ is a cusp of $\Delta$. The singularities of $\Delta$ will not be any worse than this, and nor will the singularities of the curves $C_t$ be any worse.

Since all of the curves in the family are reduced and irreducible their compactified Jacobians are well-defined.

**Theorem 14** (Markushevich [29, 30]) Let $C \to \mathbb{P}^2$ be a family of genus two curves with mild degenerations. If $X = \text{Pic}^0(C/\mathbb{P}^2)$ is a holomorphic symplectic four-fold then it is a Beauville-Mukai integrable system, i.e., the family of curves is a complete linear system of curves in a K3 surface $S$. In particular, $X$ is a deformation of the Hilbert scheme $\text{Hilb}^2 S$. 
Remark. The theorem stated in [30] is a strengthening of the theorem in [29]: the degree $d$ is allowed to be arbitrary instead of just $d = 0$, the mild degenerations hypothesis is relaxed slightly to a necessary and sufficient condition for the smoothness of the compactified Jacobian $\text{Pic}^d (C/P^2)$, and the fact that the base is isomorphic to $P^2$ is moved from a hypothesis to a conclusion. Note that Matsushita [32] later proved that the base of a Lagrangian fibration in four dimensions must always be isomorphic to $P^2$.

Let us give a rough outline of Markushevich’s proof, which does not involve foliations. The curves in the family $C \to P^2$ are hyperelliptic, so the entire family can be thought of as a branched double cover of a $P^1$-bundle over $P^2$. Moreover, in order for $X$ to be a holomorphic symplectic four-fold, this $P^1$-bundle must be the projectivization $P(\Omega^1_{P^2})$ of the cotangent bundle. The branch locus is a divisor in $P(\Omega^1_{P^2})$ which must intersect each $P^1$ fibre in exactly six points (counted with multiplicity). Again using the fact that $X$ is a holomorphic symplectic four-fold, one can show that the branch locus must correspond to a section of

$$O_{P(\Omega^1_{P^2})}(6) \otimes h^*O(-6)$$

where $h : P(\Omega^1_{P^2}) \to P^2$ is projection. Now $P(\Omega^1_{P^2})$ is the incidence subvariety in $P^2 \times (P^2)^\vee$ and the line bundle above is actually the pull-back of $O_{(P^2)^\vee}(6)$ by the projection to $(P^2)^\vee$. This means that the branch locus correspond to a sextic in $(P^2)^\vee$ and one can furthermore show that this sextic must be smooth. Finally, one obtains the K3 surface $S$ as the double cover of $(P^2)^\vee$ branched over this sextic; $C$ is then a $P^1$-bundle over $S$ and each curve $C_t$ in the family projects isomorphically to its image in the K3 surface $S$.

This completes the proof.

Next we outline a different proof of the above theorem due to the author; the full details may be found in [42]. This proof uses foliations and works when the degree is one, i.e., when $X = \text{Pic}^1 (C/P^2)$.

Proof. The relative Abel-Jacobi map allows us to embed each curve in its (compactified) Jacobian; when the degree of the Jacobian is one this embedding is canonical, i.e., does not require the choice of a basepoint, and therefore the total space of $C \to P^2$ can be embedded in $X$. Call the resulting hypersurface $Y \subset X$. Because of the mild degenerations hypothesis, $Y \cong C$ is smooth (indeed, this is still the case even with the slightly weaker conditions on the family of curves adopted in [30]).

Our goal is to show that the leaves of the characteristic foliation $F$ on $Y$ are smooth rational curves. Then there is a well-defined space of leaves $S = Y/F$, which we will show is a K3 surface. Moreover, the curves $C_t$ in the family $C \cong Y$
will project isomorphically to their images in $S$, completing the proof. First we need to find rational curves in $Y$.

Abusing notation, we use $\pi$ to denote the projections of both $Y$ and $X$ to $\mathbb{P}^2$. Let $\ell$ be a generic line in the base $\mathbb{P}^2$, and let $Z$ be the inverse image $\pi^{-1}(\ell)$ inside $Y$.

\[ Z := \pi^{-1}(\ell) \subset Y \cong C \hookrightarrow X \]
\[ \ell \subset \mathbb{P}^2 \]

Note that $Z$ is a smooth surface which is fibred by genus two curves over $\ell \cong \mathbb{P}^1$.

We claim that $Z$ is not a minimal surface, i.e., that $Z$ contains $(-1)$-curves. This involves proving the following statements:

- $R^1\pi_*\mathcal{O}_X \cong \Omega^1_{\mathbb{P}^2}$ (this was proved by Matsushita [34]),
- $R^1\pi_*\mathcal{O}_Y \cong R^1\pi_*\mathcal{O}_X$ (the isomorphism is clear over $\mathbb{P}^2 \setminus \Delta$, and can be extended over the discriminant locus $\Delta$),
- $R^1\pi_*\mathcal{O}_Z \cong \mathcal{O}_\ell(-1) \oplus \mathcal{O}_\ell(-2)$ (from restricting the previous isomorphisms to $\ell \subset \mathbb{P}^2$),
- $h^{0,0}(Z) = 1$, $h^{0,1}(Z) = 0$, and $h^{0,2}(Z) = 1$ (from substituting the previous isomorphism into the Leray spectral sequence for $\pi : Z \to \ell$).

Therefore if $Z$ were a minimal surface, it’s Kodaira dimension would be least zero. However, we can also prove the following:

- if $X$ is an irreducible holomorphic symplectic four-fold then the characteristic number $\sqrt{\hat{A}[X]}$ is at least $25/32$ (this follows from Guan’s bounds [14] on the characteristic numbers of holomorphic symplectic four-folds),
- $\deg\Delta = 24(2\sqrt{\hat{A}[X]})^{1/2}$ (as proved by the author in [41]),
- $\deg\Delta \geq 30$ and hence $Z \to \ell$ has at least 30 singular fibres (this follows from the previous two points),
- $c_2(Z) = \deg\Delta - 4 \geq 26$ and $K_Z^2 \leq -2$ (follows from the previous inequality and Noether’s formula).

Thus if $Z$ were a minimal surface, it’s Kodaira dimension would be $-\infty$. We conclude that $Z$ is not minimal, establishing the claim.

We have shown that for each $\ell \subset \mathbb{P}^2$ the surface $Z = \pi^{-1}(\ell)$ contains at least one $(-1)$-curve. Next we show that these $(-1)$-curves are leaves of the foliation on $Y$. Denoting one of these curves by $C \cong \mathbb{P}^1$, consider the following
combination of short exact sequences:

\[
\begin{array}{cccccc}
0 & \rightarrow & F|_C & \rightarrow & TY|_C & \rightarrow \frac{TY}{F|_C} & \rightarrow 0 \\
& \downarrow & & \downarrow & & \downarrow & \\
0 & \rightarrow & N_{C\subset Z} & \rightarrow & N_{C\subset Y} & \rightarrow \frac{N_{Z\subset Y}}{C} & \rightarrow 0 \\
& & & & & & 0
\end{array}
\]

We can identify \( TC \cong \mathcal{O}(2) \), \( N_{C\subset Z} \cong \mathcal{O}(-1) \) since \( C \) is a \((-1)\)-curve in \( Z \), \( N_{Z\subset Y} \cong \mathcal{O}(k) \) where \( k \) is the degree of the projection map \( \pi|_C : C \rightarrow \ell \), and \( \frac{TY}{F|_C} \cong \mathcal{O}(a_1) \oplus \mathcal{O}(a_2) \) where \( a_1 + a_2 = 0 \). Substituting these into the exact sequence, one deduces that we must have \( F|_C \cong \mathcal{O}(2) \) and the map \( TC \rightarrow \frac{TY}{F|_C} \) must lift to an isomorphism \( TC \cong F|_C \). In other words, \( C \) is a leaf of the characteristic foliation on \( Y \). It also follows that \( k = 1 \), so that \( C \) is a section of \( Z \rightarrow \ell \).

Thus we obtain a space of leaves \( S = Y/F \) which is a smooth surface admitting a holomorphic symplectic structure, i.e., a K3 or abelian surface. A leaf of the foliation will intersect each curve \( C_t \) in \( Y \) at most once since the leaf is a section of \( Z \rightarrow \ell \); moreover, one can show that an intersection must be transverse. Therefore each curve \( C_t \) maps isomorphically to its image in \( S \) under the projection \( Y \rightarrow S \). Since \( S \) contains a 2-dimensional linear system of genus two curves, it is a K3 surface, and this completes the proof. \( \square \)

The advantage of this proof is that it can be generalized to higher dimensions. First we need an appropriate generalization of a family of curves with mild degenerations.

**Definition** Let \( \mathcal{C} \rightarrow \mathbb{P}^n \) be a flat family of genus \( n \) curves. If \( C_t \) is a singular curve of this family with isolated singular points \( P_1, \ldots, P_k \) then each singular point has a versal deformation space \( \mathcal{X}(P_i) \) and there is an induced map

\[
\phi_t : T_{t\mathbb{P}^n} \rightarrow T_0\mathcal{X}(P_1) \times \cdots \times T_0\mathcal{X}(P_k).
\]

We say that the family has mild singularities if

1. every curve in the family is reduced and irreducible,
2. \( \phi_t \) is surjective for every singular curve \( C_t \) in the family.

**Remark** It follows from these conditions that the total space of \( \mathcal{C} \rightarrow \mathbb{P}^n \) is smooth (see [42]). In fact, smoothness of \( \mathcal{C} \) is a sufficient hypothesis for the following theorem.
Theorem 15 (Sawon [42]) Let \( C \rightarrow \mathbb{P}^n \) be a family of genus \( n \) curves with mild singularities. If \( X = \text{Pic}^1 (C/\mathbb{P}^n) \) is a holomorphic symplectic manifold of dimension \( 2n \) and the degree of the discriminant locus \( \Delta \subset \mathbb{P}^n \) is greater than \( 4n + 20 \) then \( X \) is a Beauville-Mukai integrable system, i.e., the family of curves is a complete linear system of curves in a K3 surface \( S \). In particular, \( X \) is a deformation of the Hilbert scheme \( \text{Hilb}^n S \).

Proof The main ideas of the proof are the same as the one above, though some steps are a little more complicated. We outline them here; full details may be found in [42].

As before, the relative Abel-Jacobi embedding \( C \hookrightarrow \text{Pic}^1 (C/\mathbb{P}^n) \) gives a submanifold \( Y \subset X \) of dimension \( n + 1 \). Although \( Y \) is smooth, we don’t know a priori whether it is coisotropic. However, we will show later that the rank of the restriction \( \sigma|_Y \) is constant, so that the characteristic foliation \( F \) is regular and \( Y \) is coisotropic.

Again we look for rational curves in \( Y \). Let \( \ell \) be a generic line in the base \( \mathbb{P}^n \) and let \( Z := \pi^{-1}(\ell) \). As before, we can show that \( h^{0,0}(Z) = 1, h^{0,1}(Z) = 0, \) and \( h^{0,2}(Z) = 1 \). We also have \( c_2(Z) = \deg \Delta - 4n + 4 \) and \( K_Z^2 = -\deg \Delta + 4n + 20 < 0 \); this is where we use the bound on the degree of the discriminant locus. As before, we conclude that \( Z \) is not minimal, i.e., it contains at least one \((-1)\)-curve.

Again we analyze the combination of short exact sequences:

\[
\begin{array}{cccc}
0 & \rightarrow & F|_C & \rightarrow & TY|_C & \rightarrow & TY/F|_C & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & 0 \\
0 & \rightarrow & N_{C\subset Z} & \rightarrow & N_{C\subset Y} & \rightarrow & N_{Z\subset Y}|_C & \rightarrow & 0 \\
\end{array}
\]

We can show that for a generic \((-1)\)-curve \( C \), the restriction \( F|_C \) will be locally free. Then analyzing the above sequences as before reveals that

\[ F|_C \cong \mathcal{O}(1)^{\oplus(n-2)} \oplus \mathcal{O}(2). \]

In particular \( F \) has rank \( n - 1 \) at a generic point of \( Y \). Equivalently, \( \sigma|_Y \) has rank two at a generic point; but then it must have rank two everywhere since the rank can only drop on closed subsets and two is the minimal value. Therefore \( F \) is locally free, and \( Y \) is coisotropic.

To complete the proof, we show that each \((-1)\)-curve \( C \) is a section of \( Z \rightarrow \ell \). Of course \( C \) lies in a leaf of the foliation \( F \). Moreover, by considering the normal bundle of \( C \) inside the leaf, we can show that each leaf maps isomorphically to a hyperplane in \( \mathbb{P}^n \) under the projection \( Y \cong C \rightarrow \mathbb{P}^n \). So the leaves are \( \mathbb{P}^{n-1} \)'s, and \( Y \) is a \( \mathbb{P}^{n-1} \)-bundle over the space of leaves \( S = Y/F \). As before, \( S \) is a smooth surface admitting a holomorphic symplectic form. The curves \( C_i \) in the
family $C \cong Y$ map isomorphically to their images under the projection $Y \to S$. Since $S$ contains an $n$-dimensional linear system of genus $n$ curves, it is a K3 surface, and this completes the proof. □

6 O’Grady’s example in dimension ten

In [38] O’Grady constructed a new ten-dimensional holomorphic symplectic manifold by desingularizing a certain moduli space of semi-stable sheaves on a K3 surface $S$. In this section we show that for certain choices of the K3 surface $S$, O’Grady’s example is birational to the space of leaves of the characteristic foliation on a hypersurface in the twelve-dimensional holomorphic symplectic manifold $\text{Hilb}^6 S$. The author learnt of this construction from Nigel Hitchin; apparently it is due to Dominic Joyce.

Theorem 16 (O’Grady [38]) Let $S$ be a K3 surface and let $\text{M}^s(2, 0, -2)$ be the non-compact Mukai moduli space of stable rank two sheaves on $S$ with Chern classes $c_1 = 0$ and $c_2 = 4$. Let $\text{M}^{ss}(2, 0, -2)$ be the compactification given by adding (S-equivalence classes of) semi-stable sheaves; note that $\text{M}^{ss}(2, 0, -2)$ is singular along the locus of strictly semi-stable sheaves. Then $\text{M}^{ss}(2, 0, -2)$ can be desingularized by a sequence of blow-ups and blow-downs, such that the resulting space $\tilde{\text{M}}^{ss}(2, 0, -2)$ is a smooth holomorphic symplectic manifold of dimension ten.

We start by describing how O’Grady’s example for a special choice of K3 surface is birational to a Lagrangian fibration.

Proposition 17 (O’Grady [38], Rapagnetta [39]) Let $S$ be a K3 surface which is a double cover of the plane, branched over a generic sextic. Then O’Grady’s space $\tilde{\text{M}}^{ss}(2, 0, -2)$ is birational to a Lagrangian fibration.

Proof The birational map to a Lagrangian fibration was first discovered by O’Grady himself; Rapagnetta later showed that we can assume the Lagrangian fibration is smooth. Let $H$ be the polarization of $S$ given by the pull-back of a line in the plane; since $S$ is a generic branched cover of the plane, we can assume that $\text{Pic} S$ is generated by $H$. Consider the linear system $|2H| \cong \mathbb{P}^5$ on $S$, whose generic element is a smooth genus five curve. Denote this family of curves by $C \to |2H|$. Denote by $\overline{\text{Pic}} (C/|2H|)$ the degree six compactified relative Jacobian of $C/|2H|$. Since $|2H|$ contains both reducible curves (the pull-back of a pair of lines in the plane) and non-reduced curves (the pull-back of a double line), we define $\overline{\text{Pic}} (C/|2H|)$ to be the irreducible component of the Mukai moduli space of semi-stable sheaves on $S$ which contains $\iota_* L$, where $\iota : C \hookrightarrow S$ is the inclusion of a generic curve of the linear system into $S$ and $L$ is a degree six line bundle on $C$. In other words

$$\overline{\text{Pic}}^6 (C/|2H|) := \text{M}^{ss}(0, 2H, 2).$$
Note that this space is singular as the Mukai vector \((0, 2H, 2)\) is not primitive.

It can be shown that for generic \(C\) and \(L\), \(h^0(C, L) = 2\) and \(L\) is globally generated. Then we can define a sheaf \(E\) as the kernel of the evaluation map

\[
H^0(C, L) \otimes O_S \to \iota_* L
\]

and it turns out that \(\mathcal{F} := E \otimes O(H)\) is a stable sheaf with Mukai vector \((2, 0, -2)\). This induces a birational map

\[
\text{Pic}^6(C/|2H|) \to \tilde{M}^{ss}(2, 0, -2)
\]

where the space on the left is a Lagrangian fibration over \(|2H| \cong \mathbb{P}^5\). Moreover, Rapagnetta \([39]\) proved that \(\text{Pic}^6(C/|2H|)\) can be desingularized by using the same sequence of blow-ups and blow-downs as O’Grady used to desingularize \(M^{ss}(2, 0, -2)\). This produces a smooth Lagrangian fibration which is birational to O’Grady’s space. □

We complete this section by using the Lagrangian fibration to show the following result.

**Proposition 18** Let \(S\) be a K3 surface which is a double cover of the plane, branched over a generic sextic. Then O’Grady’s space \(\tilde{M}^{ss}(2, 0, -2)\) is birational to the space of leaves of the characteristic foliation on a hypersurface in \(\text{Hilb}^6 S\).

**Proof** It suffices to work with the singular space \(\text{Pic}^6(C/|2H|)\) instead of O’Grady’s space \(\tilde{M}^{ss}(2, 0, -2)\), since they are birational. As mentioned earlier, if \(\iota_* L\) is a generic element of \(\text{Pic}^6(C/|2H|)\), with \(L\) a degree six line bundle on a curve \(C \in |2H|\), then \(h^0(C, L) = 2\). A non-trivial section of \(L\) vanishes at six points of \(C \subset S\), counted with multiplicity, and therefore defines an element of \(\text{Hilb}^6 S\). Note that this works even if there is some repetition among the six points: for example, if \(p_1 = p_2\) and the other points are distinct then the length six zero-dimensional subscheme is given by the four points \(p_3, \ldots, p_6 \in C \subset S\) plus \(\{p_1, v\}\), where \(v \in T_{p_1} S\) is any non-zero vector in \(T_{p_1} C \subset T_{p_1} S\). Moreover, for generic \(L\) we won’t encounter vanishing to multiplicity greater than two. Of course, two sections which agree up to scale will result in the same element of \(\text{Hilb}^6 S\). Thus we get a family of elements of \(\text{Hilb}^6 S\) parametrized by \(\mathbb{P}(H^0(C, L)) \cong \mathbb{P}^1\).

We have shown that an open subset of \(\text{Pic}^6(C/|2H|)\) parametrizes a family of rational curves in \(\text{Hilb}^6 S\). The closure of the locus swept out by this family of curves is a hypersurface \(Y\) in \(\text{Hilb}^6 S\). This hypersurface can also be defined by

\[
Y := \{x \in \text{Hilb}^6 S|\exists C \in |2H| \text{ such that } x \in C\}
\]

i.e., \(Y\) consists of those length six zero-dimensional subschemes of \(S\) which lie on a curve in the linear system \(|2H|\). To see that this defines a hypersurface, note that if we project six points in \(S\) down to the plane then the condition that they lie on a conic, and hence that the original six points lie on a curve in \(|2H|\),
is codimension one. A generic element $\xi \in Y$ will lie on precisely one curve $C$ in the linear system $|2H|$, and the rational map

$$Y \dashrightarrow \text{Pic}^6(\mathcal{C}/|2H|)$$

is given by mapping $\xi$ to the degree six divisor on $C$ which it defines. The fibres of this map are the rational curves in $\text{Hilb}^S$ described earlier, and these are the generic leaves of the characteristic foliation $F$ on $Y$. Thus we have shown that $\text{Pic}^6(\mathcal{C}/|2H|)$ is birational to the space of leaves $Y/F$, and this completes the proof. □

**Remark** It is not clear whether the space of leaves $Y/F$ is smooth, though this would seem unlikely. Nevertheless, it is possible that it admits a simpler symplectic desingularization than $M^{ss}(2,0,-2)$; this would give a more direct route to constructing O’Grady’s space.

## 7 Holomorphic Weinstein conjecture

In this final section we explain how the existence or non-existence of compact leaves of the characteristic foliation $F$ on the hypersurface $Y$ can be regarded as a holomorphic analogue of the Weinstein Conjecture from (real) symplectic geometry.

First we recall the Weinstein Conjecture [45]. Let $(M,\omega)$ be a (real) symplectic manifold of dimension $2n$, let $H$ be a smooth function on $M$, and let $X_H$ be the corresponding Hamiltonian vector field, which is defined by $\omega(X_H,v) = dH(v)$ for $v \in TM$. Since

$$X_H(H) = dH(X_H) = \omega(X_H,X_H) = 0$$

the vector field $X_H$ preserves the level sets of $H$. Assume that $\lambda \in \mathbb{R}$ is chosen so that the level set $H^{-1}(\lambda)$ is compact. A fundamental question is: will the vector field $X_H$ have a periodic orbit on $H^{-1}(\lambda)$?

**Example** Let $M$ be $\mathbb{R}^{2n}$ with the standard symplectic form, and let

$$H = (\mu_1 x_1^2 + \mu_2 x_2^2 + \ldots + \mu_{2n} x_{2n}^2)/2$$

with $\mu_1, \ldots, \mu_{2n}$ positive real numbers. Then $H^{-1}(\lambda)$ is an ellipsoid,

$$dH = \mu_1 x_1 dx_1 + \mu_2 x_2 dx_2 + \ldots + \mu_{2n} x_{2n} dx_{2n}$$

and

$$X_H = \mu_1 x_1 \frac{\partial}{\partial x_2} - \mu_2 x_2 \frac{\partial}{\partial x_1} + \ldots + \mu_{2n-1} x_{2n-1} \frac{\partial}{\partial x_{2n}} - \mu_{2n} x_{2n} \frac{\partial}{\partial x_{2n-1}}.$$

For generic $\mu_1, \ldots, \mu_{2n}$ there will be precisely $n$ periodic orbits, the intersections of $H^{-1}(\lambda)$ with the $\{x_1,x_2\}$-plane, the $\{x_3,x_4\}$-plane, etc.
The Weinstein Conjecture \cite{weinstein} states that there is always a periodic orbit if \( H^{-1}(\lambda) \) is of contact type. The latter means that \( \omega \) restricted to \( H^{-1}(\lambda) \) is exact and equal to \( d\alpha \), where \( \alpha \) is a contact form on \( H^{-1}(\lambda) \), i.e., \( \alpha \wedge (d\alpha)^{n-1} \neq 0 \). An equivalent definition is as follows: \( H^{-1}(\lambda) \) is of contact type if there exists a Liouville vector field \( v : U \rightarrow TM|_U \) on an open neighbourhood \( U \) of \( H^{-1}(\lambda) \) in \( M \), i.e., a vector field \( v \) whose Lie derivative preserves the symplectic form, \( \mathcal{L}_v \omega = \omega \). The corresponding flow \( \psi_t \) will rescale the symplectic form by an exponential, \( \psi_t^* \omega = e^t \omega \). The Weinstein Conjecture was proved for \( M = \mathbb{R}^{2n} \) by Viterbo \cite{viterbo}.

Next we explain how this relates to characteristic foliations on hypersurfaces. Since \( \omega(X_H, -) = dH \), when restricted to the level set \( H^{-1}(\lambda) \) we find that \( \omega|_{H^{-1}(\lambda)}(X_H, -) \) is identically zero. In other words, \( X_H \) generates the characteristic foliation on \( H^{-1}(\lambda) \); a periodic orbit of \( X_H \) is precisely a compact leaf of the foliation.

In the holomorphic case there are no non-constant functions on a compact complex manifold \( X \). Instead the hypersurface \( Y \) can be regarded as \( s^{-1}(0) \) where \( s \) is a section of the corresponding line bundle \( \mathcal{O}(Y) \). Locally we can take \( z_1 = s \) to be the first coordinate in a system of Darboux coordinates, in which case the characteristic foliation \( F \) is generated locally by \( \frac{\partial}{\partial z_2} \), which is also the holomorphic analogue of the Hamiltonian vector field corresponding to \( s \). Globally everything needs to be twisted by a power of the line bundle \( \mathcal{O}(Y) \).

The analogue of a periodic orbit isomorphic to \( S^1 \) is a compact leaf isomorphic to \( \mathbb{P}^1 \). As we have seen, the Reeb Stability Theorem guarantees that nearby leaves will also be rational curves, because \( \mathbb{P}^1 \) is simply-connected. This is in contrast to the real case, where a periodic orbit is not simply-connected and so isolated periodic orbits can exist. An example is the generic ellipsoid described above, which has finitely many periodic orbits.

In the holomorphic case, the contact form \( \theta \) on the hypersurface \( Y \) also needs to be twisted by a line bundle \( L \); thus
\[
\theta \in H^0(Y, \Omega_Y^1 \otimes L)
\]
and since
\[
\theta \wedge (d\theta)^{n-1} \in H^0(Y, \Omega_Y^{2n-1} \otimes L^n)
\]
should be non-vanishing, we must have \( K_Y \cong L^{-n} \). More details on holomorphic contact geometry can be found in \cite{beauville}, though note that Beauville considers the Fano case, i.e., when \( L \) is ample. In our case, \( L^n \cong K_Y^{-1} \cong \mathcal{O}(-Y) \) should be semi-positive in some sense, but not necessarily ample.

The following could be called the Holomorphic Weinstein Conjecture.

**Conjecture 19** Let \( X \) be a compact holomorphic symplectic manifold and let \( Y \subset X \) be a hypersurface of holomorphic contact type. Then the generic leaf of the characteristic foliation on \( Y \) is a rational curve. In particular, if \( Y \) is smooth then it is a \( \mathbb{P}^1 \)-bundle over the space of leaves \( Y/F \).

**Remark** In our examples, the hypersurface is often observed to be of contact type. For instance, our hypersurfaces in \( \text{Hilb}^2 S \) turned out to be isomorphic.
to the projectivization $\mathbb{P}(TS)$ of the tangent bundle of $S$; but since $S$ is a K3 surface, $TS \cong \Omega^1 S$, and it is well known that the canonical one-form on $\Omega^1 S$ induces a contact form on $\mathbb{P}(\Omega^1 S)$.

References

[1] V. Alexeev, C. Hacon, and Y. Kawamata, *Termination of (many) 4-dimensional log flips*, Invent. Math. **168** (2007), no. 2, 433–448.

[2] A. Beauville, *Variétés Kähleriennes dont la première classe de Chern est nulle*, Jour. Diff. Geom. **18** (1983), 755–782.

[3] A. Beauville, *Fano contact manifolds and nilpotent orbits*, Comment. Math. Helv. **73** (1998), no. 4, 566–583.

[4] A. Beauville, *Counting rational curves on K3 surfaces*, Duke Math. J. **97** (1999), no. 1, 99–108.

[5] F. Bogomolov and M. McQuillan, *Rational curves on foliated varieties*, IHES preprint, February 2001.

[6] S. Boucksom, *Divisorial Zariski decompositions on compact complex manifolds*, Ann. Sci. École Norm. Sup. (4) **37** (2004), no. 1, 45–76.

[7] D. Burns, Y. Hu, and T. Luo, *HyperKähler manifolds and birational transformations in dimension 4*, in Vector bundles and representation theory (Columbia, MO, 2002), 141–149, Contemp. Math. **322** Amer. Math. Soc., 2003.

[8] A. Corti et al., *Flips for 3-folds and 4-folds*, edited by Alessio Corti, Oxford Lecture Series in Mathematics and its Applications **35**, Oxford University Press, Oxford, 2007.

[9] O. Debarre, *Higher-dimensional algebraic geometry*, Springer-Verlag, New York, 2001.

[10] C. D’Souza, *Compactification of generalized Jacobians*, Proc. Indian Acad. Sci. **A88** (1979), 419–457.

[11] E. Esteves, *Very ampleness for theta on the compactified Jacobian*, Math. Z. **226** (1997), no. 2, 181–191.

[12] A. Fujiki, *On primitively symplectic compact Kähler V-manifolds of dimension four*, in Classification of algebraic and analytic manifolds (Katata, 1982), 71–250, Progr. Math. **39** Birkhäuser, 1983.

[13] O. Fujino, *Termination of 4-fold canonical flips*, Publ. Res. Inst. Math. Sci. **40** (2004), no. 1, 231–237.
[14] D. Guan, *On the Betti numbers of irreducible compact hyperkähler manifolds of complex dimension four*, Math. Res. Lett. 8 (2001), no. 5-6, 663–669.

[15] H. Holmann, *On the stability of holomorphic foliations with all leaves compact*, in Variétés analytiques compactes (Nice, 1977), 217–248, Lecture Notes in Math. 683 Springer, 1978.

[16] H. Holmann, *On the stability of holomorphic foliations*, in Analytic functions (Kozubnik, 1979), 192–202, Lecture Notes in Math. 798 Springer, 1980.

[17] J. Hurtubise, *Integral systems and algebraic surfaces*, Duke Math. J. 83 (1996), no. 1, 19–50.

[18] J. Hurtubise and E. Markman, *Rank 2-integrable systems of Prym varieties*, Adv. Theor. Math. Phys. 2 (1998), no. 3, 633–695.

[19] D. Huybrechts, *Birational symplectic manifolds and their deformations*, J. Differential Geom. 45 (1997), no. 3, 488–513.

[20] D. Huybrechts, *The Kähler cone of a compact hyperkähler manifold*, Math. Ann. 326 (2003), no. 3, 499–513.

[21] J.–M. Hwang and K. Oguiso, *Characteristic foliation on the discriminantal hypersurface of a holomorphic Lagrangian fibration*, preprint arXiv:0710.2376.

[22] J.–M. Hwang, *Base manifolds for fibrations of projective irreducible symplectic manifolds*, preprint arXiv:0711.3224.

[23] J.–M. Hwang and E. Viehweg, *Characteristic foliation on a hypersurface of general type in a projective symplectic manifold*, preprint.

[24] D. Kaledin, *Symplectic singularities from the Poisson point of view*, J. Reine Angew. Math. 600 (2006), 135–156.

[25] Y. Kawamata, K. Matsuda, and K. Matsuki, *Introduction to the minimal model problem*, Algebraic geometry, Sendai, 1985, 283–360, Adv. Stud. Pure Math. 10, North-Holland, Amsterdam, 1987.

[26] S. Kebekus, L. Solá Conde, and M. Toma, *Rationally connected foliations after Bogomolov and McQuillan*, J. Algebraic Geom. 16 (2007), no. 1, 65–81.

[27] M. Lehn, *Symplectic moduli spaces*, in Intersection theory and moduli, 139–184, ICTP Lect. Notes XIX Abdus Salam Int. Cent. Theoret. Phys., Trieste, 2004.

[28] E. Markman, *Brill-Noether duality for moduli spaces of sheaves on K3 surfaces*, J. Algebraic Geom. 10 (2001), no. 4, 623–694.
REFERENCES

[29] D. Markushevich, Completely integrable projective symplectic 4-dimensional varieties, Izvestiya: Mathematics 59 (1995), no. 1, 159–187.

[30] D. Markushevich, Lagrangian families of Jacobians of genus 2 curves, J. Math. Sci. 82 (1996), no. 1, 3268–3284.

[31] K. Matsuki, Termination of flops for 4-folds, Amer. J. Math. 113 (1991), no. 5, 835–859.

[32] D. Matsushita, On fibre space structures of a projective irreducible symplectic manifold, Topology 38 (1999), no. 1, 79–83. Addendum, Topology 40 (2001), no. 2, 431–432.

[33] D. Matsushita, Equidimensionality of Lagrangian fibrations on holomorphic symplectic manifolds, Math. Res. Lett. 7 (2000), no. 4, 389–391.

[34] D. Matsushita, Higher direct images of dualizing sheaves of Lagrangian fibrations, Amer. J. Math. 127 (2005), no. 2, 243–259.

[35] Y. Miyaoka, Deformations of a morphism along a foliation and applications, Proc. Symp. Pure Math. 46 (1987), 245–268.

[36] S. Mukai, Symplectic structure of the moduli space of simple sheaves on an abelian or K3 surface, Invent. Math. 77 (1984), 101–116.

[37] Y. Nagai, A characterization of certain irreducible symplectic 4-folds, Manuscripta Math. 110 (2003), 273–282.

[38] K. O’Grady, Desingularized moduli spaces of sheaves on a K3, J. Reine Angew. Math. 512 (1999), 49-117.

[39] A. Rapagnetta, Topological invariants of O’Grady’s six dimensional irreducible symplectic variety, Math. Z. 256 (2007), no. 1, 1–34.

[40] J. Sawon, Lagrangian fibrations on Hilbert schemes of points on K3 surfaces, J. Algebraic Geom. 16 (2007), no. 3, 477–497.

[41] J. Sawon, On the discriminant locus of a Lagrangian fibration, Math. Ann. 341 (2008), no. 1, 201–221.

[42] J. Sawon, A classification of Lagrangian fibrations by Jacobians, preprint arXiv:0803.1186.

[43] V.V. Shokurov, Prelimiting flips, Tr. Mat. Inst. Steklova 240 (2003), Biratsion. Geom. Linein. Sist. Konechno Porozhdennye Algebry, 82–219; translation in Proc. Steklov Inst. Math. 240 (2003), no. 1, 75–213.

[44] C. Viterbo, A proof of Weinstein’s conjecture in $\mathbb{R}^{2n}$, Ann. Inst. H. Poincaré Anal. Non Linéaire 4 (1987), no. 4, 337–356.
REFERENCES

[45] A. Weinstein, *On the hypotheses of Rabinowitz’ periodic orbit theorems*, J. Differential Equations 33 (1979), no. 3, 353–358.

[46] J. Wierzba, *Contractions of symplectic varieties*, J. Algebraic Geom. 12 (2003), no. 3, 507–534.

[47] J. Wierzba and J.A. Wiśniewski, *Small contractions of symplectic 4-folds*, Duke Math. J. 120 (2003), no. 1, 65–95.

Department of Mathematics sawon@math.colostate.edu
Colorado State University www.math.colostate.edu/~sawon
Fort Collins CO 80523-1874 USA