A New Method of Solving Third Order Non-Linear Ordinary Complex Differential Equation by Generalizing Prelle-Singer Method

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Abstract
A new method of solving third-order ordinary complex differential equations (OCDEs) by generalizing Prelle-Singer. The idea which is a procedure for finding the solution for second-order differential equations in the real domain. We have illustrated the theory with an example. We also introduced a new way of generating second and third motion integrals in the complex domain, which is analog to motion in the real domain from the first integral and demonstrated the procedure for the method mentioned above.

Keywords: Differential Equations, Complex Domain, Mathematical Physics, Transformations

I. Introduction
In the last thirty years, very important methods have been made in identifying nonlinear integrable dynamical systems in the real domain. In particular, different methods have been proposed or extended to explore new integrable cases and understand the underlying dynamics related to the finite dimensional nonlinear dynamical systems in the real domain \[1\]. In this work precisely, we are going to move a step forward to generalize one of these methods and make a similar procedure on the complex domain. The most widely used methods are Painleve Analysis\[1\] - \[2\], Direct Method, \[3\] Lie Symmetry Analysis \[1\], \[4\], Noether’s Theorem \[1\], \[4\] and Direct Linearization \[5\]. In this field, time ago Prelle and Singer \[6\] have made a method for solving first-order ordinary differential equations in the real domain. It finds a real solution consists of elementary functions, in this work, we are going to find a solution for an analog kind of differential equations in the complex domain with complex elementary functions, as for instance \[7\] and investigate if such solution exists. The validity of the Prelle-Singer procedure is that, if the given system of first-order ordinary complex differential equations have a solution that made of complex elementary functions then this method guarantees that this solution is going to exist. Earlier, \[8\] improved the method developed by Prelle and Singer \[6\]. Our approach is based on the conjecture that if a complex elementary solution exists for the given second-order ordinary complex differential equations then there exists at least one elementary first complex integral \(I(z, w, w')\) whose

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complex derivatives are all rational functions of $z$, $w$, and $w'$. For a class of systems, these authors [8] have deduced first complex integrals and in some cases for the first time through their procedure. Recently, the present authors had generalized the theory given in [8] and solved a class of nonlinear oscillator equations [9]. In this work, all these theories have been generalized to be in complex domain [10], [11]. In this paper, we have extended the theory of [8] to third-order ordinary complex differential equations and derive a relation which connects complex integrals of motion analog with the integrating factors. We also illustrate the theory with an example. Further, we show that one can generate the required number of complex integrals of motion analog from the first complex integral of motion analog for the above example. The plan of the paper is as follows. In the following section, we extend the theory of modified Prelle-Singer method applicable to third-order ordinary complex differential equations. In sections (II)-(IV) the theoretical definitions have been showed, in section (V) we consider an example and construct integrals of motion analog. In section (VI), we describe a procedure in which one can generate second and third complex integrals of motion analog from the first integral for a class of equations. In section (VII), we illustrate the theory with the same example considered. In section (VIII) a Hamiltonian system of dispersive water waves has been solved. We present our conclusions in section (IX).

II. Preliminaries

This section shows fundamental definitions about the differential equations in the complex domain [12], [13], [14], [15], [16].

Definition 1. A complex differential equation is an equation contains derivatives of a complex analytic function of one or more independent variables.

$$w = F(z_1, z_2, ..., z_n)$$

, where $z_1, z_2, ..., z_n$ are complex dependent variables. The general form of complex differential equations is:

$$f\left(z, w, \frac{\partial w}{\partial z_i}, \frac{\partial^2 w}{\partial z_i \partial z_j}, \frac{\partial^3 w}{\partial z_i \partial z_j \partial z_n}, \ldots \right) = 0$$

, and it will be called CDE as stands for it. And $w = F(z_1, z_2, ..., z_n)$ is the solution for the CDE.

Definition 2. The order of complex differential equation is the highest derivative in the complex differential equation.

Definition 3. The degree of the complex differential equation is the power of the highest derivative in the complex differential equation.

Remark. The complex differential equation has two types: ordinary complex differential equations and partial complex differential equations.
1. **Ordinary complex differential equations (OCDE’s)**: have one dependence and one independence complex variables.

2. **Partial complex differential equations (PCDE’s)**: have more than one independence complex variable.

**Remark.** In this work, we contain a specific study of the ordinary complex differential equations.

### III. Ordinary Complex Differential Equations

The general form of ordinary complex differential equations is:

$$\frac{d^n w}{dz^n} = f(z, w, w^{(2)}, \ldots, w^{(n-1)}),$$

(1)

**Remark.** The general solution to the OCDE is an analytic function $w = F(z) + C$ which satisfies the OCDE. Where $C$ is a constant in the complex plane.

**Definition 4.** The particular solution is a general solution with a specified value for the constant $C$.

**Definition 5.** Consider the ordinary complex differential equations of the form:

$$\frac{d^n w}{dz^n} = f(z, w, w^{(2)}, \ldots, w^{(n-1)}),$$

(2)

the equation (2) is called Linear ordinary complex differential equation when (2) is linear in $w$.

**Definition 6.** The ordinary complex differential equation is called non-linear complex differential equation when it is not linear. [21]

### IV. Analysis of Prelle-Singer Method For Third-Order OCDEs

Consider the class of third-order ordinary complex differential equations of form [22]

$$\frac{d^3 w}{dz^3} = \frac{G(z, w, w', w'')}{H(z, w, w', w'')} \quad H \neq 0,$$

(3)

where $G$ and $H$ are polynomials with coefficients in complex field.

We assume that OCDE (3) admits the first integral $I(z, w, w', w'') = C$ when $C$ is a constant in the solutions, so that the total differentiation is

$$dI = \frac{\partial I}{\partial z} dz + \frac{\partial I}{\partial w} dw + \frac{\partial I}{\partial w'} dw' + \frac{\partial I}{\partial w''} dw'' = 0,$$

(4)

by rewriting Equation (3) in the form
\[ \frac{G}{H} dz - d^2 w = 0, \quad (5) \]

and adding null terms

\[ S(z, w, w', w'') w' dz - S(z, w, w', w'') dw, \quad (6) \]

\[ F(z, w, w', w'') w' dz - F(z, w, w', w'') dw', \quad (6) \]

to Equation (5), we obtain

\[ \left( \frac{G}{H} + Sw' + Fw'' \right) dz - Sdw - Fdw' - dw'' = 0, \quad (7) \]

Hence, on the solutions, the Equations (4) and (7) must be proportional. Multiplying Equation (7) by the factor \( R(z, w, w', w'') \) which acts as the integrating factor for the complex differential equation (3), we get

\[ dI = R(\phi + Sw' + Fw'') dz - RSdw - RFdw' - Rdw'' = 0, \quad (8) \]

where \( \phi \equiv \frac{P}{Q} \). Comparing Equation (4) with Equation (8) we get, the equations

\[
\begin{align*}
I_z &= R(\phi + Sw' + Fw''), \\
I_w &= -RS, \\
I_w' &= -RF, \\
I_w'' &= -R. 
\end{align*}
\]  

(9)

The compatibility conditions \( I_{zw} = I_{wz}, I_{zw'} = I_{w'z}, I_{zw''} = I_{w''z}, I_{ww'} = I_{w'w}, I_{ww''} = I_{w''w} \), between the Equations (9) provide us

\[
\begin{align*}
D[S] &= -\phi_w + S\phi_{w''} + FS, \\
D[F] &= -\phi_{w'} + F\phi_{w''} + F^2 - S, \\
D[R] &= -R(F + \phi_{w''}), \\
R_w' S &= -RS_{w'} + R_F + RF, \\
R_w' &= R_{w''} F + RF_{w''}, \\
R_w &= R_{w''} S + RS_{w''}, 
\end{align*}
\]

(10)\( \quad (11) \quad (12) \quad (13) \quad (14) \quad (15) \)

\[ ^2Sw' dz - Sdw = Sw' - S \frac{dw}{dz} = w' - w'' = 0 \quad Fw'' dz - F dw = Fw'' - F \frac{dw'}{dz} = w'' - w'' = 0 \]
where $D$ is the total differential operator:

$$D = \frac{\partial}{\partial z} + w' \frac{\partial}{\partial w} + w'' \frac{\partial}{\partial w'} + \phi \frac{\partial}{\partial w''}.$$ 

We note that Equations (10)-(15) from an over determined system for the unknowns, $S$, $F$, and $R$.

We solve them in the following, substituting the $\phi = \frac{G}{H}$ into Equation (10) and Equation (11) we get a system of complex differential equations for unknowns $S$ and $F$. Solving them we can obtain expressions for the null forms $S$, $F$. Once $F$ is known then Equation (12) becomes the determining Equation for the function $R$. Solving the latter we can get an explicit form for $R$. Now the functions $R$, $F$ and $S$ have to satisfy the extra constraints (13)-(15). However, once a compatible solution satisfying all the equations have been found then the functions $R$, $F$ and $S$ fix the first integral $I(z, w, w', w'')$ by the relation

$$I = \zeta_1 - \zeta_2 - \zeta_3 - \int \left( R + \frac{d}{dw''}(\zeta_1 - \zeta_2 - \zeta_3) \right) dw'', \quad (16)$$

where

$$\zeta_1 = \int R(\phi + Sw' + Fw'')dz,$$

$$\zeta_2 = -\int(RS + \frac{d}{dw}\zeta_1)dw,$$

$$\zeta_3 = -\int(RF + \frac{d}{dw''}(\zeta_1 + \zeta_2))dw''.$$

Equation (16) can be derived straightforwardly by integrating the Equations (9). Now substituting the expressions of $\phi$, $R$, $F$, and $S$ into (16), and finding the integrals when we can get the related motion integrals.

V. Implementation

In this section, we illustrate the theory that had been developed and has been showed in the previews section [23].

V.I. Example-1:

V.II. A-Determination of Integration Factors and Null Terms

Consider the following equation,

$$w''' = 6z(w'')^3 + \frac{6w''^2}{w'}, \quad (17)$$

substituting $\phi = \frac{6z(w'')^3}{w''^2} + \frac{6w''^2}{w'}$ into (10)-(12) we get

$$\frac{\partial S}{\partial z} + w' \frac{\partial S}{\partial w} + w'' \frac{\partial S}{\partial w'} + \left( \frac{6z(w'')^3}{w''^2} + \frac{6w''^2}{w'} \right) \frac{\partial S}{\partial w''} = S \left( \frac{18z(w'')^3}{w''^2} + \frac{12w''^2}{w'} + F \right), \quad (18)$$
\[
\frac{\partial F}{\partial z} + w' \frac{\partial F}{\partial w} + w'' \frac{\partial F}{\partial w'} + \left(\frac{6z w''}{w'^2} + \frac{6w'^2}{w'}\right) \frac{\partial F}{\partial w''} = \frac{12zw''}{w'^3} + \frac{6w'^2}{w'^2} + F \left(\frac{18z w'^2}{w'^2} + \frac{12w''}{w'}\right) + F^2 - S,
\]

(19)

\[
\frac{\partial R}{\partial z} + w \frac{\partial R}{\partial w} + w' \frac{\partial R}{\partial w'} + \left(\frac{6z w'^3}{w'^2} + \frac{6w'^2}{w''}\right) \frac{\partial R}{\partial w'^2} = -R \left(\frac{F + \frac{18z w'^2}{w'^2} + \frac{12w''}{w'}\right).
\]

(20)

As mentioned in section (IV), first, we have solved the system (9) and obtain explicit forms for the functions \(S\) and \(F\). We observe that, for this particular example, \(S = 0\) is a simple solution for the Equation (18). So, we consider the implication of this solution.

Now substituting \(S = 0\) into Equation (19), we get a first order differential equation for \(F\), namely,

\[
\frac{\partial F}{\partial z} + w' \frac{\partial F}{\partial w} + w'' \frac{\partial F}{\partial w'} + \left(\frac{6z w'^3}{w'^2} + \frac{6w'^2}{w'}\right) \frac{\partial F}{\partial w''} = \frac{12zw''}{w'^3} + \frac{6w'^2}{w'^2} + F \left(\frac{18z w'^2}{w'^2} + \frac{12w''}{w'}\right) + F^2,
\]

(21)

To solve Equation (21), we make an ansatz for \(F\) of the form

\[
F = \frac{a(z, w, w') + b(z, w, w') w'' + c(z, w, w') w'^2 + e(z, w, w') w'' + f(z, w, w') w'^2}{d(z, w, w') + e(z, w, w') w'' + f(z, w, w') w'^2},
\]

(22)

where \(a, b, c, d, e\) and \(f\) are arbitrary functions of \(z, w\) and \(w'\). Substituting Equation (22) into Equation (21) and equating the coefficients of different power of \(w''\) to zero we get a set of partial complex differential equations for the variables \(a, b, c, d, e\) and \(f\). Solving them one obtains

\[
F_1 = -6zw'^2 + 3w''w', \quad F_2 = -6zw'^2 + 4w''w'.
\]

(23)

Substituting the forms \(F_1\) and \(F_2\) into (20) and solving the latter one can get an explicit form for the function \(R\). Let us first consider \(F_1\), the corresponding equation for \(R\) becomes

\[
\frac{\partial R}{\partial z} + w' \frac{\partial R}{\partial w} + w'' \frac{\partial R}{\partial w'} + \left(\frac{6z w'^3}{w'^2} + \frac{6w'^2}{w'}\right) \frac{\partial R}{\partial w'^2} = R \left(\frac{6zw'^2 + 3w''w'}{w'^2} - \frac{18zw'^2}{w'^2} - \frac{12w''}{w'}\right).
\]

(24)

to solve (21) again one has to make ansatz. We assume that the following form for \(R\), that is

\[
R = A(z, w, w) + B(z, w, w') w'' + C(z, w, w') w'^2 + D(z, w, w') + E(z, w, w') w'' + F(z, w, w') w'^2,
\]

(25)
where \( A, B, C, D, E, \) and \( F \) are arbitrary functions of \( z, w \) and \( w' \).

Now substituting Equation (25) into Equation (24) and equating the coefficients of differential power of \( w'' \) to zero and solving the resultant equations we obtain the following expressions for \( R \), namely,

\[
R_1 = \frac{w'^3}{w'^2},
\]

(26)

In a similar way, substituting the expression of \( F_2 \) into (24) and solving the resultant equation with the same type of ansatz (25), we arrive at

\[
R_2 = \frac{w'^4}{w'^2},
\]

(27)

to summarize, we obtain the following form of solutions (with the ansatzs (23) and (25) for the Equations (18)-(20))

\[
(S_1, F_1, R_1) = \left(0, -\frac{6zw'^2 + 3zw''}{w'^2}, \frac{w'^3}{w'^2}\right),
\]

\[
(S_2, F_2, R_2) = \left(0, -\frac{6zw'^2 + 4zw''}{w'^2}, \frac{w'^4}{w'^2}\right),
\]

(28)

at this stage, we have left unsolved three more equations, that is,

\[
R_{ww''}S = -RS_{ww} + R_{ww}F + RF_{ww},
\]

(29)

\[
R_{ww''} = R_{ww''}F + RF_{ww''},
\]

(30)

\[
R_{ww} = R_{ww''}S + RS_{ww''},
\]

(31)

However, one can quickly verify that the solutions (28) also satisfy the extra constraints (29)-(31). As a result, (28) forms compatible solutions for the system (10)-(15) with \( \phi \) given in (17). Finally, we note that, in the above, we considered only a trivial solution \( S = 0 \) for the Equation (21) and derived the corresponding forms of \( F \) and \( R \). However, in the choice \( S \neq 0 \), the system (18)-(20) becomes a coupled equation in the unknowns \( F \) and \( S \). To solve this system, as we did previously, let us make an ansatz.

\[
S = \frac{a(z, w, w') + b(z, w, w'')w'' + c(z, w, w')w^{'''} + d(z, w, w')w''' + e(z, w, w')}{d(z, w, w') + e(z, w, w')w'' + f(z, w, w')w'''},
\]

(32)

\[
F = \frac{a(z, w, w') + b(z, w, w'')w'' + c(z, w, w')w^{'''} + d(z, w, w')w''' + e(z, w, w')}{d(z, w, w') + e(z, w, w')w'' + f(z, w, w')w'''},
\]

(33)

with the coefficients are arbitrary functions in \( (z, w, w') \). Substituting Equations (32) and (33) in Equations (18) and (19) and solving the resultant equations we arrive at

\[
F_3 = -\frac{6zw'^2 + 2w'w''}{w'^2}, \quad S_3 = \frac{2w'^2}{w'^2},
\]

(34)

which in turn fixes \( R \) of the form
The expressions (34) and (35) satisfy the extra constraints (29)-(31), so the functions

\[(S_3, F_3, R_3) = \left( \frac{2w''^2}{w'^2}, -\frac{zw'w'^2 + 2ww''}{w'^2}, w'^2 \right),\]

also form a compatible solution for the Equations (10)-(15) with \(\phi\) given in (17).

V.III. B-Integrals of Motion

The determined functions \((S_i, F_i, R_i), i = 1, 2, 3\), we can go on to determine the related motion integrals, and by substituting the expressions \((S_i, F_i, R_i), i = 1, 2, 3\), into (16) separately and evaluating the integrals that we obtain

\[I_1 = 3zw'^2 + \frac{w'^3}{w''},\]

\[I_2 = 2zw'^3 + \frac{w'^4}{w''},\]

\[I_3 = 2z - 6zw' - \frac{w'^2}{w''},\]

respectively. It can easily be checked that \(I_1, I_2, I_3\) are constants on the complex solutions, that is, \(\frac{dI_i}{dz} = 0, i = 1, 2, 3\). From the integrals, \(I_1, I_2, I_3\), we can deduce the general complex solution for the Equation (17). For example, solving Equation (37) for \(w''\) and substituting into Equations (38) and (39), we obtain

\[zw'^3 - I_1w' + I_2 = 0,\]

\[3zw'^2 + (I_3 - 2w)w' + I_1 = 0,\]

after algebraically combining these equations to eliminate, we obtain a functional relation between \(w\) and \(z\) as

\[3z(I_1(I_3 - 2w) - 9zI_2)^2 + I_1((I_3 - 2w)^2 - 12zI_1)^2 - (I_3 - 2w)(I_1(I_3 - 2w) - 9zI_2)((I_3 - 2w)^2 - 12zI_1) = 0,\]

We mention that the expression (40) was derived from the different point of view using generalized in [24].

VI. Method of Generating Complex Integrals of Motion Analog

In the last section, we have derived the motion integrals, \(I_i, i = 1, 2, 3\), by building a sufficient number of integrating factors. Beautifully, one can also generate the needed amount of motion integrals from the first integral itself. For example, for the third-order
system that is presented as Equation (17), one can create $I_2$ and $I_3$ from $I_1$ itself. In the following work, we are going to illustrate our ideas.

Let us assume there is a first integral for the third-order Equation (3) of the form,

$$I_1 = F_1(z, w, w', w'').$$

(41)

Let us split the function $F_1$ into a product of two functions such that one involves a perfect differentiable function $\frac{1}{G_2(z, w, w', w'')} \frac{d}{dz} G_1(z, w, w')$, that is,

$$I_1 = F_1 \left( \frac{1}{G_2(z, w, w', w'')} \frac{d}{dz} G_1(z, w, w') \right),$$

(42)

by identifying the function $G_1(z, w, w')$ as a new dependent variable and the integral of $G_2(z, w, w', w'')$ over time as a new independent variable, namely,

$$w = G_1(z, w, w'), \quad z = \int_0^z G_2(z', w, w', w'') dz',$

(43)

one can rewrite the Equation (42) of the form

$$\hat{I}_1 = \frac{dz}{dm} \frac{dn}{dz}.$$

(44)

Integrating (44) we get

$$n = \hat{I}_1 m + I_2,$$

(45)

Where $I_2$ is an integration constant. In other words

$$I_2 = n - \hat{I}_1 m,$$

(46)

Rewriting $n$ and $m$ in terms of the old variables $z, w, w', w''$ and replacing $I_1$ by its explicit form one can get an exact form for $I_2$. Interestingly one can also write down the first integral $I_1$ in the form (42) in more than one way, say,

$$I_1 = F_1 \left( \frac{1}{G_2(z, w, w', w'')} \frac{d}{dz} \hat{G}_1(z, w, w') \right),$$

(47)

VII. Applications

To illustrate our ideas above, we consider the same example (vide Equation (17)) discussed in Section V. In particular, we consider one of the complex integrals and generate the other two through our procedure.

Let us first consider (37), that is

$$I_1 = 3zw'w^2 + \frac{w'^3}{w''},$$

(48)
and generate \( I_2 \) and \( I_3 \) from (48). Rewriting (18) in the form (12) we get
\[
I_1 = - \frac{1}{w''} \frac{d}{dz} (-zw^3) = \frac{dz}{dm} \frac{dw}{dz} = \frac{dn}{dm},
\]
so that
\[
I_1 = -zw^3, \quad m = -w',
\]
Integrating (49) we get
\[
n = I_1m + I_2 \Rightarrow I_2 = n - I_1m.
\]
Rewriting \( n \) and \( m \) in terms of the old variables using the expression (50) and replacing \( I_1 \) by the expression (48), we arrive at
\[
I_2 = 2zw^3 + (w')^3
\]
which is exactly similar to the equation we have derived (vide Equation (38)) earlier through the Prelle-Singer method.

Now, to generate \( I_3 \) from \( I_1 \) we rewrite the functions in the form (12) but with different latter \( \tilde{n} \) and \( \tilde{m} \), namely,
\[
I_1 = -(w')^2 \frac{d}{dz} (2w - 3zw') = \frac{dz}{dm} \frac{d\tilde{n}}{d\tilde{m}} = \frac{d\tilde{n}}{d\tilde{m}},
\]
so that
\[
\tilde{n} = 2w - 3zw', \quad \tilde{m} = w'.
\]
Integrating (54) we get
\[
\tilde{n} = I_1\tilde{m} + I_3 \Rightarrow I_3 = \tilde{n} - I_1\tilde{m},
\]
Substituting (54) and (48) into (55) we get
\[
I_3 = 2w - 6zw' - \frac{(w')^2}{w''},
\]
which exactly agrees with (39). In a similar way, we can derive \( I_1 \) and \( I_2 \) from \( I_3 \) and \( I_1 \) and \( I_3 \) from \( I_2 \). As the procedure is the same as given above in the following, we provide only the essential steps.

Consider \( I_2 \),
\[
I_2 = 2zw^3 + (w')^3
\]
and generate \( I_1 \) and \( I_3 \) from the former. Rewriting the above in the form
\[
I_2 = -\frac{(w')^2}{w''} \frac{d}{dz} (z(w')^2),
\]
and repeating the above procedure we get the first integral (37). On the other hand the expression \( I_2 \) in the form
\[ I_2 = -\frac{(w')^3}{w'''} \frac{d}{dz} (w - 2zw'), \] (58)

leads us to the third integral (39).

**VIII. Implementation Arising in Physics**

The physical model of describing the dispersive water waves as a system of third-order complex differential equations is one of the most important implementations of CDEs, the method of Prelle-Singer plays a significant role to find the integrating factor and eventually the solution of such systems. The Physical model finally has the homogeneous non-linear system

\[ BH = 0, \] (59)

for some \( H \), and \( B \) is a Hamiltonian operator of the complex partial differential equations and

\[
H = (K, L)^T, \]

\[
\frac{\partial^3}{\partial z^3} (L) = \Phi_1(z, L, K, L_z, K_z, L_{zz}, K_{zz}), \\
\frac{\partial^3}{\partial z^3} (K) = \Phi_2(z, L, K, L_z, K_z, L_{zz}, K_{zz}), \] (60)

say that the system (60) satisfies a first integral which has the form

\[ I(z, L, K, L_z, K_z, L_{zz}, K_{zz}) = C, \]

where it is constant on the solutions. So, the total differentiation will be

\[ di = I_z dz + I_L dL + I_k dK + I_{L_z} dL_z + I_{K_z} dK_z + I_{L_{zz}} dL_{zz} + I_{K_{zz}} dK_{zz} = 0, \] (61)

then, we can write

\[ \Phi_1 dz - dL_{zz} = 0, \quad \Phi_2 dz - dK_{zz} = 0, \] (62)

by adding the null terms in (62)

\[ (\Phi_1 + S_1 L_z + S_2 K_z + M_1 L_{zz} + M_2 K_{zz}) dz - S_1 dL - S_2 dK - M_1 dL_z - M_2 dK_z - dL_{zz} = 0, \] (63)

\[ (\Phi_2 + U_1 L_z + U_2 K_z + N_1 L_{zz} + N_2 K_{zz}) dz - U_1 dL - U_2 dK - N_1 dL_z - N_2 dK_z - dL_{zz} = 0, \] (64)

now, multiplying \( 63 \) by the integrating factor \( R_1 \) and \( 64 \) by the integrating factor \( R_2 \), where \( R_1 = R_1(z, L, K, L_z, K_z, L_{zz}, K_{zz}) \) and \( R_2 = R_2(z, L, K, L_z, K_z, L_{zz}, K_{zz}) \) we get the result as
\[ dI = R_1(\Phi_1 + SL_z + ML_{zz})dz + R_2(\Phi_2 + UK_z + NK_{zz})dz - \\
R_1 SdL - R_2 UdK - R_1MdL_z - R_2NdK_z - R_1dL_{zz} - R_2dK_{zz} = 0, \]

where \( S = \frac{R_1S_1 + R_2U_1}{R_1} \), \( U = \frac{R_1S_2 + R_2U_2}{R_2} \), \( M = \frac{R_1M_1 + R_2N_1}{R_1} \) and \( N = \frac{R_1M_2 + R_2N_2}{R_2} \). When we compare the above equations with Equation (65), we get

\[ I_z = R_1(\Phi_1 + SL_z + ML_{zz}) + R_2(\Phi_2 + UK_z + NK_{zz}) \]
\[ I_L = -R_1S_1, \quad I_K = -R_2U_1, \quad I_{L_z} = -R_1M_1, \]
\[ I_{K_z} = -R_2N_1, \quad I_{L_{zz}} = -R_1, \quad I_{K_{zz}} = -R_2, \]

by using the conditions we obtain the determining equations:

\[ D[R_1] = -(R_1\Phi_{1L_{zz}} + R_2\Phi_{2L_{zz}} + R_1M), \]
\[ D[R_2] = -(R_1\Phi_{1K_{zz}} + R_2\Phi_{2K_{zz}} + R_2N), \]
\[ D[S] = S\Phi_{1L_{zz}} + \left(\frac{SR_2}{R_1}\right)\Phi_{2L_{zz}} + MS - \Phi_{1L} - \left(\frac{R_2}{R_1}\right)\Phi_{2L}, \]
\[ D[U] = U\Phi_{2K_{zz}} + \left(\frac{UR_2}{R_1}\right)\Phi_{2K_{zz}} + NU - \Phi_{2K} - \left(\frac{R_2}{R_1}\right)\Phi_{1K}, \]
\[ D[M] = M\Phi_{1L_{zz}} + \left(\frac{MR_2}{R_1}\right)\Phi_{2L_{zz}} + M^2 - \Phi_{1L} - \left(\frac{R_2}{R_1}\right)\Phi_{2L} - S, \]
\[ D[N] = N\Phi_{2K_{zz}} + \left(\frac{NR_2}{R_1}\right)\Phi_{1K_{zz}} + N^2 - \Phi_{2K} - \left(\frac{R_1}{R_2}\right)\Phi_{1K} - U, \]

\[ R_{1K_z}M + R_1M_{K_z} = R_{2L_z}N + R_2N_{L_z}; \]
\[ R_{1L} = R_{1L_z}S + R_1S_{L_z}; \]
\[ R_{1K} = R_{1K_z}S + R_1S_{K_z} = R_{2L}N + R_2N_L; \]
\[ R_{1K} = R_{2L_z}U + R_2U_{L_z}; \]
\[ R_{2K_z}U + R_2U_{K_z} = R_{2K}N + R_2N_K; \]
\[ R_{1L} = R_{1L_z}M + R_1M_{L_z}; \]
\[ R_{1L} + S + R_1S_{L_z} = R_{1L}M + R_1M_L; \]
\[ R_{1Kz} = R_{1Kz} = R_{2Lzz}N + R_{2N_{Lzz}}, \]  
\[ R_{2Lz} U + R_{2Uz} = R_{1M_K} + MR_{1K}, \]  
\[ R_{2L} = R_{1Kzz}S + R_{1S_Kzz}, \]  
\[ R_{1K}S + R_{1S_K} = R_{2L}U + R_{2U_L}, \]  
\[ R_{2K} = R_{2Kzz}U + R_{2U_Kzz}, \]  
\[ R_{2Lz} = R_{1Kzz}M + R_{1M_Kzz}, \]  
\[ R_{2Kz} = R_{2Kzz}N + R_{2N_{Kzz}}, \]  
\[ R_{1Kzz} = R_{2Lzz}, \]  

When the parts of the solutions, \( R_1, R_2, S, U, M \) and \( N \) are found then the integral can be built by the substitution of all the expressions in (66) and find the integration for the result, that is,

\[ I = r_1 + r_2 + r_3 + r_4 + r_5 + r_6 - \int (R_2 + \frac{\partial}{\partial K_{zz}}(r_1 + r_2 + r_3 + r_4 + r_5 + r_6))dK_{zz}, \]

where

\[ r_1 = \int (R_1(\Phi_1 + SLz + ML_{zz}) + R_2(\Phi_2 + UKz + NK_{zz}))dz, \]  
\[ r_2 = -\int (R_1S + \frac{\partial}{\partial L}(r_1))dL, \]  
\[ r_3 = -\int (R_2U + \frac{\partial}{\partial K}(r_1 + r_2))dK, \]  
\[ r_4 = -\int (R_1M + \frac{\partial}{\partial Lz}(r_1 + r_2 + r_3))dL_z, \]  
\[ r_5 = -\int (R_2N + \frac{\partial}{\partial Kz}(r_1 + r_2 + r_3 + r_4))dK_z, \]  
\[ r_6 = -\int (R_1 + \frac{\partial}{\partial L_{zz}}(r_1 + r_2 + r_3 + r_4 + r_5)). \]

To determine the integrating factors \( R_1 \) and \( R_2 \) we represent the determining Equations (67)-(72) as two equations:
\[
D^3[R_1] = D^2[R_1 \Phi_{1Lzz} + R_2 \Phi_{2Lzz}] - D[R_1 \Phi_{1L_z} + R_2 \Phi_{2L_z}] + R_1 \Phi_{1L} + R_2 \Phi_{2L} = 0, \quad (90)
\]

\[
D^3[R_2] = D^2[R_1 \Phi_{1Kzz} + R_2 \Phi_{2Kzz}] - D[R_1 \Phi_{1K_z} + R_2 \Phi_{2K_z}] + R_1 \Phi_{1K} + R_2 \Phi_{2K} = 0, \quad (91)
\]

where

\[
D = \frac{\partial}{\partial z} + L_z \frac{\partial}{\partial L} + K_z \frac{\partial}{\partial K} + L_{zz} \frac{\partial}{\partial L_z} + K_{zz} \frac{\partial}{\partial K_z} + \Phi_1 \frac{\partial}{\partial L_{zz}} + \Phi_2 \frac{\partial}{\partial K_{zz}}. \quad (92)
\]

The determining Equations (90) and (91) represents a system of linear of partial differential equations in \( R_1 \) and \( R_2 \). Substituting the terms \( \Phi_1 \) and \( \Phi_2 \) and their derivatives into (90) and (91) and solving them so we can obtain expressions for the integrating factor \( R_1 \) and \( R_2 \). After finding the Rs then the functions \( S, U, M, N \) can be fixed through the relations (69)-(72). After one find \( R_1, R_2, S, U, M \) and \( N \), then, he has to make sure that they satisfy the conditions (73)-(87). So the set \( (S, U, M, N, R_1, R_2) \) that satisfies the Equations (67)-(87) will forms the wanted solution and the integral which has the form (88).

IX. Conclusion

In this work, we have investigated the new method for solving complex third-order ordinary complex differential equations through the technique of modified Prelle-Singer. The process is not straightforward for [8] but has some new theoretical aspects that have several advantages. We have illustrated the theory with some implementations. We also have introduced a new way of generating second and third complex integrals of motion analog from the first complex integral and demonstrated the idea with the same example considered previously. We solved a system of third-order ordinary nonlinear complex differential equations that arises in physics, precisely in dispersive water waves.
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