ON EMBEDDINGS OF FINITE SUBSETS OF $\ell_p$

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ABSTRACT. We study finite subsets of $\ell_p$ and show that, up to nowhere dense and Haar null complement, all of them embed isometrically into any Banach space that uniformly contains $\ell_p^n$.

1. INTRODUCTION

Our starting point is the following question due to Ostrovskii\cite{1}:

**Question 1.** Suppose $1 < p < \infty$, and that $X$ is a Banach space that contains an isomorphic copy of $\ell_p$. Then does any finite subset of $\ell_p$ embed isometrically into $X$?

A consequence of Krivine’s theorem, which we recall in Section 2, is that any Banach space containing $\ell_p$ isomorphically, contains the spaces $\ell_p^n$, $n \in \mathbb{N}$, almost isometrically. The above question is asking for a natural strengthening of this fact.

The following partial result for Question 1 in the case $p = 2$ was proved by Shkarin in \cite{5}:

**Theorem 1.1** (Lemma 3 of \cite{5}). Suppose $X$ is any infinite-dimensional Banach space and that $Z$ is any affinely independent subset of $\ell_2$. Then $Z$ embeds isometrically into $X$.

A different proof was given in \cite{4}, and the methods of both \cite{4} and \cite{5} inspired the proofs in this article. We note that, by Dvoretzky’s theorem, $\ell_2^n$ almost isometrically embeds into $X$ for any infinite-dimensional Banach space $X$. Thus, in the case $p = 2$, Theorem 1.1 provides a partial positive answer to the following variant of Question 1:

**Question 2.** Suppose that $1 < p < \infty$ and that $X$ is a Banach space uniformly containing the spaces $\ell_p^n$, $n \in \mathbb{N}$. Then does any finite subset of $\ell_p$ embed isometrically into $X$?

As before, the weaker conclusion that finite subsets of $\ell_p$ embed almost isometrically into such a space $X$ follows from Krivine’s theorem, or more precisely, a finite quantitative version of it (see Theorem 2.1 below.)

There are natural analogues of Questions 2 and 1 for $p = \infty$. Since any $n$-point metric space embeds isometrically into $\ell_\infty^n$, the conclusion in any such analogue is that $X$ contains isometrically all finite metric spaces. The assumption on $X$ is one of the following (in decreasing order of strength): $X$ contains an isomorphic copy of $\ell_\infty$; $X$ contains an isomorphic copy of $c_0$; $X$ contains the spaces $\ell_\infty^n$, $n \in \mathbb{N}$, uniformly. The answer for each of these questions is, however, negative. Indeed, let $X$ be a strictly convex renorming of $\ell_\infty$. Then subsets of $X$ have the unique metric midpoint property, i.e., there is no collection of 4 distinct points $x, y, z, w \in X$ such that

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\[ d(x, z) = d(z, y) = d(x, w) = d(w, y) = \frac{1}{2}d(x, y). \] However, there are finite metric spaces with this property, and thus such a metric space does not embed isometrically into \( X \). In [4] we showed a positive result similar to Theorem 1.1. Let us call a metric space \( \text{concave} \) if it contains no three distinct points \( x, y, z \) such that \( d(x, z) = d(x, y) + d(y, z) \). Then,

**Theorem 1.2** (Theorem 4.3 of [4]). Suppose that \( X \) is some infinite-dimensional Banach space such that the spaces \( \ell^p_n, n \in \mathbb{N}, \) uniformly embed into \( X \). Then if \( Z \) is any finite concave metric space, \( Z \) embeds isometrically into \( X \).

In this paper we obtain a partial positive answer to Question 1 similar to Theorems 1.1 and 1.2. As in the case of \( p = 2 \) there remains a class of subsets of \( \ell_p \) that our proof does not handle. This collection is certainly small in a strong sense. Our main theorem is as follows,

**Theorem 1.3.** Suppose \( 1 < p < \infty \) and that \( Z \) is a Banach space that uniformly contains the spaces \( \ell^p_n, n \in \mathbb{N}. \) Then, for each \( n \in \mathbb{N} \), the set of \( n \)-point subsets of \( \ell_p \) that do not embed isometrically into \( Z \) is nowhere dense and Haar null.

We now describe how our paper is organized. We shall also explain why the case of \( p \in (1, \infty) \) is more difficult than the special cases of \( p = 2 \) and \( p = \infty \) and how we handle the additional difficulty.

In Section 2 we recall various definitions and results that will be used throughout the article (and have already been used in this introduction). The proof of Theorem 1.3 begins in Section 3. Here we prove a result (see Theorem 3.1) that may be of independent interest: almost all \( n \)-point subsets of \( \ell^p_n \) have the property that small perturbations of that subset remain subsets of \( \ell^p_n \). In Section 4 we introduce Property \( K \) of finite subsets of \( \ell_p \). Our aim will be to show that every finite subset of \( \ell_p \) with Property \( K \) embeds isometrically into a Banach space \( X \) that satisfies the assumption of Theorem 1.3.

For general \( p \in (1, \infty) \), Property \( K \) plays the rôle of affine independence in the case \( p = 2 \), or concavity in the case \( p = \infty \). For \( p = 2 \), any \( n \)-point subset of \( \ell_2 \) embeds isometrically into \( \ell^2_n \) via an orthogonal transformation which preserves affine independence. For \( p = \infty \), any \( n \)-point metric space embeds into \( \ell^\infty_n \) via an isometry, which preserves concavity. For general \( p \in (1, \infty) \) it is not even clear if a finite subset of \( \ell_p \) embeds isometrically into \( \ell^N_p \) for any \( N \). In fact, this is true: Ball proved in [6] that any \( n \)-point subset of \( \ell_p \) embeds isometrically into \( \ell^N_p \) with \( N = \binom{n}{2} \). The difficulty is that Ball’s proof is not constructive, and Property \( K \) is somewhat technical. In Section 4 we prove a version of Ball’s result (Lemma 4.1) which is much weaker, in the sense that \( N \) will depend on the subset. However, Lemma 4.1 will show that our embedding will preserve Property \( K \).

**Remark 1.4.** In this article, we do not pay much attention to the case \( p = 1 \). Indeed, as stated, Question 1 is false. As for \( \ell_\infty \), there is a strictly convex renorming \( X \) of \( \ell_1 \), and no finite subset of \( \ell_1 \) that fails the unique metric midpoint property embeds isometrically into such an \( X \). However, one might expect a result similar to Theorem 1.2 to hold, when there’s a restriction on the type of subset we consider. The methods of this paper rely heavily on the differentiability of the norm of \( \ell_p \) for \( 1 < p < \infty \) which fails for \( p = 1 \). Thus our techniques only produce weak conclusions in the case \( p = 1 \).
2. Classical Results and Notation

2.1. Banach Space Definitions and Classical Results: Throughout this paper, for simplicity, we will only be interested in real Banach spaces.

Suppose that $X$ and $Y$ are Banach spaces. The Banach-Mazur distance between $X$ and $Y$ is defined by $d(X,Y) = \inf\{||T||||T^{-1}|| : T$ is an isomorphism from $X$ to $Y\}$. We say that a Banach space $X$ is $C$-isomorphic to a Banach space $Y$ if there is a linear isomorphism $T : X \rightarrow Y$ such that $||T||||T^{-1}|| \leq C$. We say that a Banach space $X$ almost isometrically contains a Banach space $Y$, or that $Y$ almost isometrically embeds into $X$, if for each $\varepsilon > 0$ there is a subspace $Z$ of $X$ such that $Z$ is $(1 + \varepsilon)$-isomorphic to $Y$. We say that a Banach space $X$ uniformly contains spaces $X_n$, $n \in \mathbb{N}$, if there exist a constant $C$ and subspaces $Y_n$ of $X$ such that $Y_n$ is $C$-isomorphic to $X_n$ for all $n$.

We will need the following quantitative version of Krivine’s theorem:

**Theorem 2.1.** Let $1 \leq p \leq \infty$, $C \geq 1$, $\varepsilon > 0$ and $k \in \mathbb{N}$. Then there is some $n$ (dependent on $p, C, k$ and $\varepsilon$) such that if a Banach space $X$ is $C$-isomorphic to $\ell^n_p$ then there is a subspace of $X$ that is $(1 + \varepsilon)$-isomorphic to $\ell^k_p$.

For a proof of this theorem, including estimates of the constants involved, we refer the reader to [3].

We introduce a notion of a null set in infinite-dimensional Banach spaces. A well known fact is that if $X$ is infinite-dimensional and separable, and $\mu$ is a translation-invariant Borel measure on $X$, then $\mu$ either assigns $0$ or $\infty$ to every open subset of $X$. However, there are several useful notions of null set in Banach spaces under which the null sets form a translation-invariant $\sigma$-ideal. One such notion, that we shall use, is that of a Haar null set. A Borel set $A \subset X$ is called Haar null if there is a Borel probability measure $\mu$ on $X$ such that $\mu(x + A) = 0$ for every $x \in X$.

It is easy to see that if for some $n \in \mathbb{N}$ there is an $n$-dimensional subspace $Y$ of $X$ such that the measure $\lambda_n(Y \cap (A + x)) = 0$ for all $x \in X$, where $\lambda_n$ is $n$-dimensional Lebesgue measure, then $X$ is Haar null. More on sets of this type, and on other notions of nullity, can be found in [7] Chapter 6.

2.2. Submersions: We will need a fact from the theory of Differential Geometry related to submersions. Suppose we have a $C^1$-map $\Phi : \mathbb{R}^n \rightarrow \mathbb{R}^m$ where $n \geq m$. We say that $\Phi$ is a submersion at a point $x$ if the derivative $D\Phi|_x$ of $\Phi$ at $x$ has rank $m$. The following result is known as the Submersion Theorem and can be found in any introductory text on Differential Geometry:

**Theorem 2.2.** Suppose $\Phi : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is a $C^1$-map, where $n \geq m$. If $\Phi$ is a submersion at a point $x$, then there are open sets $A \subset \mathbb{R}^n$ and $B \subset \mathbb{R}^m$ with $x \in A$, $\Phi(x) \in B$ and $\Phi(A) = B$. Moreover, there is a $C^1$-map $\Psi : B \rightarrow A$ such that $\Phi \circ \Psi$ is the identity on $B$ and $\Psi(\Phi(x)) = x$.

3. A Theorem about Finite Subsets of $\ell^n_p$

In this section we establish a preliminary result that may be of independent interest. Suppose that $Z$ is a metric space on a sequence of points $(z_l)_{l=1}^n$ and $Y$ is a metric space on a sequence
of points \((y_i)_{i=1}^n\). We say that \(Y\) is an \(\varepsilon\)-perturbation of \(Z\) if for each pair \(i, j\) we have that 
\[|d_Z(z_i, z_j) - d_Y(y_i, y_j)| < \varepsilon.\]
In finite dimensions, the phrase *almost all* will only be used with respect to Lebesgue measure. Throughout this section, we fix some \(n \in \mathbb{N}\) and \(p \in \mathbb{R}\) with \(1 < p < \infty\). We denote by \(\|\cdot\|\) the \(p\)-norm on \(\ell_p^n\).

**Theorem 3.1.** For almost all \(n\)-point subsets \(X\) of \(\ell_p^n\), there is an \(\varepsilon > 0\) such that if \(Y\) is an \(\varepsilon\)-perturbation of \(X\) then \(Y\) isometrically embeds into \(\ell_p^n\).

For our purposes, we will need a slightly stronger property of an \(n\)-point subset \(X\) of \(\ell_p^n\). We will need that an \(\varepsilon\)-perturbation of \(X\) isometrically embeds into \(\ell_p^n\) in a way that depends continuously on the perturbation (in a way we will make precise in the sequel.) This is the content of Theorem 3.2 below, from which Theorem 3.1 will easily follow. To state Theorem 3.2, we will first develop some notation.

Let \(M = M_n = \mathbb{R}^n \times \cdots \times \mathbb{R}^n\) and let \(U = U_n\) denote the \(n \times n\) upper triangular matrices with 0 on the diagonal. We let \(e_1, \ldots, e_n\) be the standard basis of \(\mathbb{R}^n\) and \(e^j_i\) be the element of \(M\) with \(e_j^i\) in the \(i^{th}\) co-ordinate and 0 everywhere else. Note that \(e^i_j, 1 \leq i, j \leq n\), form a basis of \(M\). Given \(x = (x_1, \ldots, x_n) \in M\) we denote the \(j^{th}\) coordinate (with respect to the standard basis) of the vector \(x_i\) as \(x^j_i\) so that \(x = \sum_{i,j} x^j_i e^i_j\). Let \(E_{ij}\) be the \(n \times n\) matrix with 1 in the \((i, j)\)-entry and 0 elsewhere. Note that \(E_{ij}, 1 \leq i < j \leq n\) forms a basis for \(U\), so the dimension of \(U\) is \(\binom{n}{2}\).

We define the map \(F = F_n : M \to U\) by

\[F(x_1, \ldots, x_n) = (\|x_i - x_j\|^p)_{1 \leq i < j \leq n}.\]

We observe that \(F\) is \(C^1\)-map. Indeed, by computing the partial derivatives in the direction \(e^k_i\) we get:

\[
\frac{\partial F}{\partial e^k_i}(z_1, \ldots, z_n) = (p|z_i^k - z_j^k|^{p-1} \text{sgn}(z_i^k - z_j^k)(\delta_i^j - \delta_{ij}))_{1 \leq i < j \leq n},
\]
and these are evidently continuous. Theorem 3.1 says that \(F\) is locally open at almost all \(n\)-tuples \((x_1, \ldots, x_n)\). This is contained in the following theorem:

**Theorem 3.2.** Let \(F : M \to U\) be defined as above. Set \(G = G_n = \{x \in M : DF|_x\text{ has rank } \binom{n}{2}\}\).

Then \(G\) is an open subset of \(M\) whose complement has measure zero (and is thus nowhere dense.)

Moreover, given \(x \in G\), there is an open subset \(A\) of \(M\) containing \(x\), an open subset \(B\) of \(U\) containing \(F(x)\) and a \(C^1\)-map \(\Phi : B \to A\) such that \(F \circ \Phi = \text{Id}_B\) and \(\Phi(F(x)) = x\).

Let us briefly spell out how Theorem 3.1 follows from Theorem 3.2. Suppose that \(\{x_1, \ldots, x_n\}\) is an \(n\)-point subset of \(\ell_p^n\) and that \(x = (x_1, \ldots, x_n) \in G\). Define \(X_{ij} = \|x_i - x_j\|^p\) and \(X = (X_{ij})\) to be a \(1 \leq i < j \leq n\).

Then, since \(x \in G\), by Theorem 3.2 there are open subsets \(A\) and \(B\) of \(U\) such that \(x \in A\), \(F(x) = X \in B\) and \(F(A) = B\). Thus there is some \(\varepsilon > 0\) such that if \(|Y_{ij} - X_{ij}| < \varepsilon\) for all \(i, j\), then \((Y_{ij})\) is an element of \(B\) and thus is the image under \(F\) of some \(y = (y_1, \ldots, y_n) \in A\). Hence \(Y\) defines a metric on an \(n\)-point set and the resulting metric space embeds isometrically into \(\ell_p^n\).

This is slightly more than the statement that \(\varepsilon\)-perturbations of the metric space \(\{x_1, \ldots, x_n\}\) with the inherited metric embed isometrically into \(\ell_p^n\).
We now show that $H$ has full rank. Let $x \in H$. The proof that $H$ is a part of $\text{Submersion Theorem}$ is then complete. Indeed, the rest of the statement of Theorem 3.2 follows immediately from the Submersion Theorem.

The proof that $M \setminus G$ has measure zero is done in several steps. We first identify a certain subset of $G$.

**Lemma 3.3.** Let $H = \{(x_1, \ldots, x_n) \in M : x_i = e_i + \sum_{j=i+1}^{n} x_j^j e_j \text{ for each } i = 1, \ldots, n\}$. Then if $x \in H$, the partial derivatives $\frac{\partial F}{\partial e_i^k}(x)$, $1 \leq k < l \leq n$ are linearly independent. In particular, $H \subset G$.

**Proof.** Fix $x = (x_1, \ldots, x_n) \in H$. By (1) we see that the $(i,j)$-entry of $\frac{\partial F}{\partial e_i^j}(x)$ is zero unless $j = l$ and $i \leq k$. We can hence expand $\frac{\partial F}{\partial e_i^k}(x)$ in terms of the matrices $E_{kl}$ as follows,

$$\frac{\partial F}{\partial e_i^k}(x) = -pE_{kl} + \sum_{i=1}^{k-1} \alpha_i^k E_{il},$$

where $\alpha_i^k$ are constants depending on $x$. It follows by induction on $k$ that $E_{kl}$ is in the span of $\frac{\partial F}{\partial e_i^k}(x)$ for all $1 \leq k < l \leq n$. This completes the proof of the lemma. $\square$

Let us now define $V = \{x = (x_1, \ldots, x_n) \in M : \text{there are } i, j, k \in \{1, \ldots, n\} \text{ such that } i \neq j \text{ and } x_i^k = x_j^k\}$. Note that $M \setminus V$ has finitely many connected components which are open and convex.

Since $\mu(V) = 0$, in order to show that $\mu(M \setminus G) = 0$, it suffices to show that $\mu(C \setminus G) = 0$ for every connected component $C$ of $M \setminus V$. The following lemma will be vital to this aim.

**Lemma 3.4.** Suppose that $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ are two points in the same connected component of $M \setminus V$, and suppose that $\frac{\partial F}{\partial e_i^j}(x)$, $1 \leq i < j \leq n$, are linearly independent. Then, for all but finitely many values of $t \in [0, 1]$, the partial derivatives $\frac{\partial F}{\partial e_i^j}((1 - t)x + ty)$, $1 \leq i < j \leq n$, are linearly independent. In particular, for all but finitely many values of $t \in [0, 1]$, we have that $(1 - t)x + ty \in G$.

**Proof.** Define $J$ to be the set of $(k,l) : 1 \leq k < l \leq n$. For $\sigma = (i,j) \in J$ we will write $e_\sigma = e_i^j$, and for $X \in U$ we will write $X_\sigma$ for the $(i,j)$-entry of $X$. By assumption, the $J \times J$ matrix given by

$$\left(\frac{\partial F}{\partial e_\sigma}(x)\right)_\rho$$

has non-zero determinant. We now define a function $g : [0,1] \to \mathbb{R}$ by setting

$$g(t) = \det\left(\frac{\partial F}{\partial e_\sigma}((1 - t)x + ty)\right)_\rho = \det(X(t)).$$

Using (1) and the fact that $x$ and $y$ are from the same component of $M \setminus V$, for each $\sigma, \rho \in J$, the matrix $X(t)$ has $(\sigma, \rho)$-entry $p(a_{\sigma,\rho}t + b_{\sigma,\rho})^{p-1} \varepsilon_{\sigma,\rho}$ where $a_{\sigma,\rho}$ and $b_{\sigma,\rho}$ are non-zero constants with $a_{\sigma,\rho}t + b_{\sigma,\rho} > 0$ for all $t \in [0,1]$ and $\varepsilon_{\sigma,\rho} \in \{-1, 0, 1\}$.

By compactness there is an open connected subset $U$ of $C$ containing $[0,1]$ such that the real part of $a_{\sigma,\rho}t + b_{\sigma,\rho}$ is positive for each $t \in U$. It follows that the function $g$ extends analytically
to all of $U$, and therefore by the identity principle (and the fact that $g(0)$ is non-zero), $g$ has at most finitely many zeroes in $[0, 1]$.

Consider the subset $R$ of $M$ defined by

$$R = \{(x_1, \ldots, x_n) \in M : x_i^j > x_j^i \text{ for each } 1 \leq i < j \leq n\}.$$  

Note that for each component $C$ of $M \setminus V$ either $C \subset R$ or $C \cap R = \emptyset$. We next show that in order to prove that $\mu(C \setminus G) = 0$ for every component $C$ of $M \setminus V$, it is sufficient to consider components $C$ such that $C \subset R$.

Fix $(x_1, \ldots, x_n) \in M \setminus V$. Define a permutation $\pi \in S_n$ recursively as follows: for $j = 1, \ldots, n$, let $\pi(j)$ be the unique $i \in \{1, \ldots, n\} \setminus \{\pi(1), \ldots, \pi(j-1)\}$ such that

$$x_i^j > x_k^j \text{ for all } k \in \{1, \ldots, n\} \setminus \{\pi(1), \ldots, \pi(j-1), i\}.$$  

It then follows that $x_i^{\pi(j)} > x_k^{\pi(k)}$ for all $1 \leq j < k \leq n$, and hence $(x_{\pi(1)}, \ldots, x_{\pi(n)}) \in R$.

Define a map $A_{\pi} : M \to M$ by $A_{\pi}(y_1, \ldots, y_n) = (y_{\pi(1)}, \ldots, y_{\pi(n)})$, and a map $B_{\pi} : U \to U$ by $B_{\pi}((X_{ij})_{1 \leq i < j \leq n}) = (Y_{ij})_{1 \leq i < j \leq n}$ where

$$Y_{ij} = \begin{cases} X_{\pi(i), \pi(j)} & \text{if } \pi(i) < \pi(j), \\ X_{\pi(j), \pi(i)} & \text{if } \pi(j) < \pi(i). \end{cases}$$

We note that $B_{\pi}^{-1}FA_{\pi} = F$, and thus $B_{\pi}^{-1}DF|_{A_{\pi}(x)}A_{\pi} = DF|_{x}$, so to verify that $F$ has full rank at $x$, it is sufficient to verify that $F$ has full rank at $A_{\pi}(x)$, which lies in $R$. This completes the proof that it is sufficient to show that $\mu(C \setminus G) = 0$ whenever $C$ is a component of $M \setminus V$ with $C \subset R$.

Fix a component $C$ of $M \setminus V$ with $C \subset R$. If $\mu(C \setminus G) > 0$, then by Lebesgue’s density theorem, there is a point $y \in C$ such that $\lim_{\varepsilon \to 0} \frac{\mu(B_{\varepsilon}(y) \cap (C \setminus G))}{\mu(B_{\varepsilon}(y))} = 1$. For $i, j \in \{1, \ldots, n\}$, define

$$x_i^j = \begin{cases} 1 & \text{if } i = j, \\ 1 & \text{if } y_i^j > y_j^j, \\ 0 & \text{else.} \end{cases}$$

It is easy to verify that if $y_i^k < y_j^k$ then $x_i^k \leq x_j^k$, and thus $(1 - t)x + ty \in C$ for all $t \in (0, 1]$. Moreover, since $y \in R$, we have $x \in H$. It follows by Lemma 3.3 that the partial derivatives $\frac{\partial F}{\partial c_i}(x)$, $1 \leq i < j \leq n$, are linearly independent. Hence there is an $\varepsilon > 0$ such that at each $z \in B_{\varepsilon}(x)$ the same holds, ie, $\frac{\partial F}{\partial c_i}(z)$, $1 \leq i < j \leq n$, are linearly independent. Choose $t \in (0, 1)$ such that $z = (1 - t)x + ty \in B_{\varepsilon}(x)$. Then $z \in B_{\varepsilon}(x) \cap C$, so there is some $\delta > 0$ such that $B_{\delta}(z) \subset B_{\varepsilon} \cap C$.

The Lebesgue density at $y$ is equal to 1, so by making $\delta$ smaller, we may assume that $B_{\delta}(y) \subset C$ and $\mu(B_{\delta}(y) \setminus G) > 0$. By Lemma 3.4, each line in the direction $y - x$ through a point in $B_{\delta}(z)$ intersects $B_{\delta}(y) \setminus G$ in at most finitely many points. The lines in the direction $y - x$ through $B_{\delta}(z)$ can be parametrised by where they intersect the hyperplane through $z$ whose normal is
We say that an \( L \)-contradiction on \( y \) is open.

Given a subset \( M = \{m_1, \ldots, m_n\} \) of \( \mathbb{N} \) with \( m_1 < m_2 < \cdots < m_n \), if \( x = (x_i)_{i=1}^{\infty} \in \ell_p \) or \( x = (x_i)_{i=1}^{N} \in \ell_p^N \) with \( N \geq m_n \), we define \( P_M(x) = (x_{m_1}, \ldots, x_{m_n}) \). If \( n \in \mathbb{N} \), we write \( P_n \) instead of \( P_{\{1, \ldots, N\}} \).

We say that an \( n \)-tuple \( (x_1, \ldots, x_n) \) in \( \ell_p \) (or \( \ell_p^N \)) has \( \text{Property } K \) if there is an \( M \subset \mathbb{N} \) (or \( M \subset \{1, \ldots, N\} \) respectively) of size \( n \) such that \( (P_Mx_1, \ldots, P_Mx_n) \in G_n \), where \( G_n \) is the set defined in Theorem 3.2. Note that the set of \( n \)-tuples with \( \text{Property } K \) is open since the set \( G_n \) is open.

We prove Theorem 1.3 by showing that the closed set of \( n \)-tuples without \( \text{Property } K \) is Haar null (and thus nowhere dense), and that an \( n \)-tuple with \( \text{Property } K \) embeds isometrically into a Banach space that satisfies the assumption of Theorem 1.3. We will need three lemmas.

**Lemma 4.1.** Suppose that \( x = (x_1, \ldots, x_n) \) is an \( n \)-tuple in \( \ell_p \) with \( \text{Property } K \). Then there is some \( N \in \mathbb{N} \), and vectors \( y_1, \ldots, y_n \in \ell_p^N \) such that \( \|y_i - y_j\| = \|x_i - x_j\| \) and the \( n \)-tuple \( (y_1, \ldots, y_n) \) has \( \text{Property } K \).

**Remark 4.2.** This is the variant of Ball’s result mentioned in the Introduction. Here \( \|\cdot\| \) denotes the \( \ell_p \) norm.

**Proof of Lemma 4.1**. Let \( M \subset \mathbb{N} \) be such that \( |M| = n \) and \( (P_Mx_1, \ldots, P_Mx_n) \in G_n \). After an isometry (permuting the indices), we may assume without loss of generality that \( M = \{1, \ldots, n\} \).

Then, since \( (P_nx_1, \ldots, P_nx_n) \in G_n \) and \( G_n \) is open, there is some \( \varepsilon > 0 \) such that if \( z_i \in \ell_p^n \) and \( \|z_i - P_nx_i\| < \varepsilon \) then \( (z_1, \ldots, z_n) \in G_n \).

Since \( (P_nx_1, \ldots, P_nx_n) \in G_n \), by Theorem 3.2, there are open sets \( A \ni (P_nx_1, \ldots, P_nx_n) \), \( B \ni F(P_nx_1, \ldots, P_nx_n) \) and a \( C^1 \)-map \( \Phi: B \to A \) such that \( F \circ \Phi = \text{Id}_B \) and \( \Phi(F(x)) = x \).

Fix \( N \geq n \), and define \( \rho_{ij} = \rho_{ij}(N) \) by \( \|x_i - x_j\|^p = \|P_Nx_i - P_Nx_j\|^p + \rho_{ij} \). Since \( \rho_{ij} \to 0 \) as \( N \to \infty \), there is an \( N > n \) such that the element \( Z = Z(N) = (\|P_nx_i - P_nx_j\|^p + \rho_{ij})_{1 \leq i < j \leq n} \) of \( U \) is in the set \( B \). Set \( z = z(N) = (z_1, \ldots, z_n) = \Phi(Z) \). By the continuity of \( \Phi \) at the point \( F(P_nx_1, \ldots, P_nx_n) \), if \( N \) is sufficiently large, then \( \|z_i - P_nx_i\| < \varepsilon \), and hence \( (z_1, \ldots, z_n) \in G_n \).

We now define the points \( y_1, \ldots, y_n \in \ell_p^N \) by:

- \( P_ny_i = z_i \)
- \( (P_N - P_n)y_i = (P_N - P_n)x_i \).
We now verify that \((y_1, \ldots, y_n)\) has Property \(K\), and that \(\|y_i - y_j\| = \|x_i - x_j\|\). The first of these is clear, \((P_ny_1, \ldots, P_ny_n)\) is in \(G_n\) by construction.

To verify that \(\|y_i - y_j\| = \|x_i - x_j\|\), note that

\[
\|y_i - y_j\|^p = \|P_ny_i - P_ny_j\|^p + \|(P_N - P_n)y_i - (P_N - P_n)y_j\|^p,
\]

which is equal to

\[
\|z_i - z_j\|^p + \|(P_N - P_n)x_i - (P_N - P_n)x_j\|^p.
\]

By the definition of \((z_1, \ldots, z_n)\), we see that \(\|z_i - z_j\|^p = \|P_nx_i - P_nx_j\|^p + \rho_{ij}\). By the definition of \(\rho_{ij}\), we thus get that \(\|y_i - y_j\|^p = \|x_i - x_j\|^p\). \(\square\)

We have now shown that if a subset of \(\ell_p\) has Property \(K\), then it is isometric to a subset of \(\ell_p^N\) with Property \(K\). We next show a slight variant of Theorem 3.2

**Lemma 4.3.** Suppose \(x = (x_1, \ldots, x_n)\) is an \(n\)-tuple in \(\ell_p^N\), \(N \geq n\), with Property \(K\). Then there is some \(\varepsilon > 0\) such that any \(\varepsilon\)-perturbation of \(X\) can be embedded into \(\ell_p^N\) with the embedding depending continuously on the perturbation.

At the beginning of the proof of Lemma 4.3 we will make it clear what continuous dependence on the perturbation means in a way similar to the precise statement of Theorem 3.2.

**Proof.** Define \(\tilde{F}: \mathbb{R}^N \times \cdots \times \mathbb{R}^N \to U_n\) by

\[
\tilde{F}(y_1, \ldots, y_n) = (\|y_i - y_j\|)_{1 \leq i < j \leq n},
\]

where we note that there is no \(p\)th power of the norm. Our goal is to show that there is an open subset \(\tilde{B}\) of \(U_n\) and a continuous map \(\Psi: \tilde{B} \to \mathbb{R}^N \times \cdots \times \mathbb{R}^N\) such that:

- \(\tilde{F}(x) \in \tilde{B}\)
- \(\Psi(\tilde{F}(x)) = x\)
- \(\tilde{F} \circ \Psi = \text{Id}_{\tilde{B}}\).

Let \(M \subset \{1, \ldots, N\}\) be such that \(|M| = n\) and \((P_Mx_1, \ldots, P_Mx_n) \in G_n\). Again, without loss of generality, we may assume that \(M = \{1, \ldots, n\}\).

By Theorem 3.2, there exist open sets \(A \ni (P_nx_1, \ldots, P_nx_n), \ B \ni F(P_nx_1, \ldots, P_nx_n)\) and a \(C^1\)-map \(\Phi: B \to A\) such that \(\Phi(F(P_nx_1, \ldots, P_nx_n)) = (P_nx_1, \ldots, P_nx_n)\) and \(F \circ \Phi = \text{Id}_B\). Fix \(\varepsilon > 0\) such that if \(Y = (Y_{ij})_{1 \leq i < j \leq n}\) is such that \(|Y_{ij} - \|x_i - x_j\|\| < \varepsilon\), then \(Y \in B\).

Choose \(\delta = \delta(\varepsilon) > 0\) to be specified later. We set \(\tilde{B} = \{Y \in U_n : |Y_{ij} - \|x_i - x_j\|\| < \delta\text{ for all pairs }i,j\}\).

Fix \(Y = (Y_{ij})_{1 \leq i < j \leq n} \in \tilde{B}\). We define \(\Psi(Y)\) similarly to the definition of the points \((y_1, \ldots, y_n)\) in the proof of Lemma 4.1. Define \(\rho_{ij} = Y_{ij} - \|x_i - x_j\|\) and \(\varepsilon_{ij} = \varepsilon_{ij}(Y)\) by \(\|x_i - x_j\| + \rho_{ij} = \|x_i - x_j\|^p + \varepsilon_{ij}\). If \(|\rho_{ij}|\) is sufficiently small (ie, our choice of \(\delta\) is sufficiently small), then \((|P_nx_i - P_nx_j|^p + \varepsilon_{ij})_{1 \leq i < j \leq n}\) is in \(B\). Define \(z_i = \Phi((P_nx_i - P_nx_j|^p + \varepsilon_{ij})_{1 \leq i < j \leq n})\). We then set \(\Psi(Y)\) to be the \(n\)-tuple \((y_1, \ldots, y_n)\) where:

- \(P_ny_i = z_i\)
• \((P_N - P_n)y_i = x_i\).

We verify that \(\|y_i - y_j\| = \|x_i - x_j\| + \rho_{ij} = Y_{ij}\), i.e., that \(\tilde{F}(\Psi(Y)) = Y\), as this is the only one of the three properties listed above that is non-trivial.

Indeed,

\[
\|y_i - y_j\|^p = \|P_n y_i - P_n y_j\|^p + \|(P_N - P_n)y_i - (P_N - P_n)y_j\|^p,
\]

which (by the definition of \(y_i\)) equals

\[
\|z_i - z_j\|^p + \|(P_N - P_n)x_i - (P_N - P_n)x_j\|^p
\]

and this is equal (by the definition of \(z_i\)) to

\[
\|x_i - x_j\|^p + \varepsilon_{ij}.
\]

By the definition of \(\varepsilon_{ij}\), this is equal to \((\|x_i - x_j\| + \rho_{ij})^p\), which is as required. 

Our next lemma shows that if we have an \(n\)-point subset of \(\ell_p^N\) with Property \(K\), then it embeds isometrically into any Banach space satisfying the assumption of Theorem 1.3. This result is, in some sense, dual to Theorem 3.1. Where Theorem 3.1 says isometrically into any Banach space satisfying the assumption of Theorem 1.3. This result is, in some sense, dual to Theorem 3.1. Where Theorem 3.1 says small perturbations of the metric space embed into the Banach space, this is saying that the metric space embeds into small perturbations of the Banach space.

**Lemma 4.4.** Suppose \(x = (x_1, \ldots, x_n)\) is an \(n\)-tuple in \(\ell_p^N\), \(N \geq n\), with Property \(K\). Then there is some \(\delta > 0\) such that if \(d(E, \ell_p^N) < 1 + \delta\) then \(\{x_1, \ldots, x_n\}\) with the metric inherited from \(\ell_p^N\) embeds isometrically into \(E\).

**Proof.** Let \(\tilde{F}, \tilde{B}\) and \(\Psi\) be as in the proof of Lemma 4.3. Choose \(\varepsilon > 0\) such that if \(Y = (Y_{ij})_{1 \leq i < j \leq n} \in U\), \(|Y_{ij} - \|x_i - x_j\| < \varepsilon\), then \(Y \in \tilde{B}\). Fix some \(\delta > 0\) and let \(E\) be an \(N\)-dimensional Banach space such that \(d(E, \ell_p^N) < 1 + \delta\). We will find the value of \(\delta\) later, and it will be expressed in terms of \(x\) and \(\varepsilon\) only. We may assume that \(E = (\mathbb{R}^N, \|\cdot\|_E)\) and that the norm on \(E\) satisfies \(\|y\|_E \leq \|y\| \leq (1 + \delta)\|y\|_E\), where as usual \(\|\cdot\|\) denotes the \(\ell_p\) norm.

Let \(\rho = (\rho_{ij})_{1 \leq i < j \leq n}\) be an element of the space \([0, \varepsilon)^{\binom{n}{2}}\). We define a metric space \(Z(\rho)\) as follows:

• \(Z(\rho)\) is a metric space on \(n\) distinct points \(z_1, \ldots, z_n\).

• \(d(z_i, z_j) = \|x_i - x_j\| + \rho_{ij}\).

By the choice of \(\varepsilon\), and since \(\tilde{F} \circ \Psi = \text{Id}_{\tilde{B}}\), it follows that \(Z(\rho)\) is a metric space isometric to a subset of \(\ell_p^N\). Through slight abuse of notation, in what follows we identify \(Z(\rho)\) with its distance matrix, i.e., \(Z(\rho) = (d(z_i, z_j))_{1 \leq i < j \leq n}\).

Now define \(\varphi: [0, \varepsilon)^{\binom{n}{2}} \to [0, \varepsilon)^{\binom{n}{2}}\) by

\[
\varphi(\rho) = (\|x_i - x_j\| + \rho_{ij} - \|\Psi(Z(\rho))_i - \Psi(Z(\rho))_j\|_E)_{1 \leq i < j \leq n}.
\]

We claim that if \(\delta\) is sufficiently small then \(\varphi\) is well defined. To see that \(\varphi(\rho)_{ij} > 0\), note that \(\varphi(\rho)_{ij} \geq \|x_i - x_j\| + \rho_{ij} - \|\Psi(Z(\rho))_i - \Psi(Z(\rho))_j\| = 0\), where we have used that \(\|y\| \geq \|y\|_E\) for all \(y \in \mathbb{R}^N\).

On the other hand, \(\varphi(\rho)_{ij} \leq \|x_i - x_j\| + \rho_{ij} - \frac{1}{1 + \delta} \|\Psi(Z(\rho))_i - \Psi(Z(\rho))_j\|_E = \frac{\delta}{1 + \delta} (\|x_i - x_j\| + \rho_{ij})\), where we have used that \(\|y\| \leq (1 + \delta)\|y\|_E\) for all \(y \in \mathbb{R}^N\). So if \(\delta\) is sufficiently small, then this is less than \(\varepsilon\).
Since \( \varphi \) is a continuous map from a compact convex subset of \( \mathbb{R}^2 \) to itself, it follows from Brouwer’s fixed point theorem that \( \varphi \) has a fixed point \( \rho \). Letting \( (y_1, \ldots, y_n) = \Psi(Z(\rho)) \), the map sending \( x_i \) to \( y_i \) is an isometric embedding of \( \{x_1, \ldots, x_n\} \) into \( E \).

\[\text{□}\]

**Remark 4.5.** Suppose we had \( x_1, \ldots, x_n \in \ell_p \) such that the map \( \hat{F} : \ell_p \times \cdots \times \ell_p \to U \), \( \hat{F}(y_1, \ldots, y_n) = (\|y_i - y_j\|)_{1 \leq j \leq n} \), had a continuous right inverse at \( \hat{F}(x_1, \ldots, x_n) \). Then an identical argument to the proof of Lemma 4.3 would show that there is some \( \delta > 0 \) such that if \( d(Y, \ell_p) < 1 + \delta \), then \( Y \) contains an isometric copy of \( \{x_1, \ldots, x_n\} \). Since the assumption in Theorem 1.3 is weaker than the Banach space containing an isomorphic copy of \( \ell_p \), we had to choose a more technical version of Property \( K \) than simply "\( \hat{F} \) has a continuous right inverse at \( (x_1, \ldots, x_n) \)." This stronger assumption also motivated Lemma 4.1.

We now give the proof of Theorem 1.3.

**Proof of Theorem 1.3.** By a combination of Lemmas 4.1, 4.3 and 4.4, we see that if an \( n \)-tuple \( (x_1, \ldots, x_n) \) in \( \ell_p \) has Property \( K \), then there is some \( N \in \mathbb{N} \) and \( \delta > 0 \) such that if \( Y \) is a Banach space with \( d(Y, \ell_p^N) < 1 + \delta \), then \( \{x_1, \ldots, x_n\} \) with the metric inherited from \( \ell_p^N \) embeds isometrically into \( Y \). By Krivine’s Theorem, Theorem 2.1, any Banach space \( X \) satisfying the assumption of the theorem (ie, containing the spaces \( \ell_p^n \), \( n \in \mathbb{N} \), uniformly), contains a subspace \( Y \) with \( d(Y, \ell_p^N) < 1 + \delta \). Thus \( \{x_1, \ldots, x_n\} \) with the metric inherited from \( \ell_p^N \) embeds isometrically into \( X \).

To conclude, we just need to show that the set \( A \) of all \( n \)-tuples that do not have Property \( K \) is Haar null. Indeed, the intersection of \( A \) with the finite-dimensional space \( \ell_p^n \times \cdots \times \ell_p^n \) is contained in the complement of \( G_n \), which by Theorem 3.2 has measure zero. Note also that \( A \) is translation-invariant. Thus, by the characterization of Haar null sets stated in Section 2.1, \( A \) is Haar null. Since \( A \) is closed, it follows that \( A \) is nowhere dense. \[\text{□}\]

5. **Further Remarks and Open Problems**

In this section we give some remarks on the special cases of \( \ell_2 \), \( \ell_\infty \) and \( \ell_1 \), and pose some open problems.

In the case \( \ell_2 \), we deduce Theorem 1.1 from our results.

**Theorem 5.1.** Every finite affinely independent subset of \( \ell_2 \) isometrically embeds into every infinite-dimensional Banach space \( X \).

**Proof.** First note that every affinely independent set has a linearly independent translate, so without loss of generality, we may reduce to the case of linearly independent sets. Let \( e_1, e_2, \ldots \) be an orthonormal basis of \( \ell_2 \). If \( \{x_1, \ldots, x_n\} \) is a linearly independent subset of \( \ell_2 \), then there is some isometry \( \Theta \) such that \( \Theta(x_1) \in \text{span}\{e_1\} \), \( \Theta(x_2) \in \text{span}\{e_1, e_2\} \), etc. Such a \( \Theta \) is constructed by induction and the Gram-Schmidt process applied to the vectors \( \{x_1, \ldots, x_n\} \). Then a minor variant of Lemma 3.3 (in which the coefficient of \( e_i \) in \( x_i \) is non-zero, but not necessarily one) shows that the \( n \)-tuple \( (x_1, \ldots, x_n) \) belongs to \( G_n \). Thus \( (\Theta x_1, \ldots, \Theta x_n) \) (which is isometric to \( (x_1, \ldots, x_n) \)) has Property \( K \).
Applying Lemma 4.4 to \((\Theta x_1, \ldots, \Theta x_n)\) we see that there exists some \(\delta > 0\) such that whenever \(E\) is an \(n\)-dimensional Banach space with \(d(E, \ell^n_2) < 1 + \delta\) then \((\Theta x_1, \ldots, \Theta x_n)\) embeds isometrically into \(E\). By Dvoretzky’s theorem, if \(X\) is infinite-dimensional, there is a subspace \(Z\) of \(X\) such that \(d(Z, \ell^n_2) < 1 + \delta\), and thus \(Z\) contains an isometric copy of \((\Theta x_1, \ldots, \Theta x_n)\) (which is isometric to \((x_1, \ldots, x_n)\)). □

In the case of \(\ell_\infty\), the proof of Theorem 1.2 (given as Theorem 4.3 in [4]) essentially proceeds by directly showing that if \((x_1, \ldots, x_n)\) is a concave metric space, then the mapping \(\tilde{F}\) is locally open at \((x_1, \ldots, x_n)\). This argument does not use differentiation: the norm on \(\ell_\infty\) is easy to compute.

In the case of \(\ell_1\), the majority of the proofs in this paper simply do not work. In the case \(p = 1\) the computation of the derivative (Equation (1)) yields \(\frac{\partial F}{\partial e_k} = (\text{sgn}(x_i^k - x_j^k)_{1 \leq i < j \leq n})\). Thus the function is locally open if the collection forms linearly independent matrices. This is, however, not the case on a large set as it is for the case \(1 < p < \infty\). However, if it is true at a point \(x = (x_1, \ldots, x_n)\) the rest of the proofs presented here work identically.

We now list some open problems. The case \(p = 2\) was originally raised by Ostrovskii in [2], who asked:

**Question 3.** Let \(X\) be an infinite-dimensional Banach space and \(A\) a finite subset of \(\ell_2\). Then does \(A\) isometrically embed into \(X\)?

The general question of Ostrovskii, given in [1], still remains open:

**Question 4.** Let \(X\) be an infinite-dimensional Banach space containing \(\ell_p\) isomorphically. Then does every finite subset of \(\ell_p\) embed isometrically into \(X\)?

The way we approached this question leads to the following natural variant:

**Question 5.** Let \(X\) be an infinite-dimensional Banach space that uniformly contains \(\ell^n_p\), \(n \in \mathbb{N}\). Then does every finite subset of \(\ell_p\) embed isometrically into \(X\)?

As detailed in the introduction, there can be no positive results in the cases \(p = 1\) and \(p = \infty\). However, the known partial answers lead to the following open question:

**Question 6.** Let \(p = 1\) or \(p = \infty\). Which \(n\)-point subsets of \(\ell_p\) embed isometrically into any Banach space \(X\) that uniformly contains the spaces \(\ell^n_p\), \(n \in \mathbb{N}\)?

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