A QUESTION OF NORTON-SULLIVAN IN THE ANALYTIC CASE

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ABSTRACT. In 1996, A. Norton and D. Sullivan asked the following question: If $f: \mathbb{T}^2 \to \mathbb{T}^2$ is a diffeomorphism, $h: \mathbb{T}^2 \to \mathbb{T}^2$ is a continuous map homotopic to the identity, and $hf = T_\rho h$ where $\rho \in \mathbb{R}^2$ is a totally irrational vector and $T_\rho: \mathbb{T}^2 \to \mathbb{T}^2, z \mapsto z + \rho$ is a translation, are there natural geometric conditions (e.g. smoothness) on $f$ that force $h$ to be a homeomorphism? In [WZ18], the first author and Z. Zhang gave a negative answer to the above question in the $C^\infty$ category: In general, not even the infinite smoothness condition can force $h$ to be a homeomorphism. In this article, we give a negative answer in the $C^\omega$ category: We construct a real-analytic conservative and minimal totally irrational pseudo-rotation of $\mathbb{T}^2$ that is semi-conjugate to a translation but not conjugate to a translation, which simultaneously answers a question raised in [WZ18, Q3].

1. Introduction

As one of the earliest results, H. Poincaré proved the following celebrated classification of circle homeomorphisms: a circle homeomorphism $f$ is semi-conjugate to an irrational rigid rotation if and only if the rotation number of $f$, denoted by $\rho(f)$, is irrational, which is equivalent to say that $f$ has no periodic orbits. Later, A. Denjoy proved that $f$ is topologically conjugate to an irrational rigid rotation if it is a $C^1$ diffeomorphism of $\mathbb{T}^1$ without periodic points and $Df$ has bounded variation ($f \in C^{1+b.v.}$) [Den32]. In the other direction, Denjoy (even before him, P. Bohl [Boh16]) provided examples of $C^1$ diffeomorphisms semi-conjugate but not topologically conjugate to an irrational rotation. Their examples were later improved to $C^{1+\alpha}$ for any $\alpha \in (0, 1)$ by Herman [Her79].

It is natural to explore the Denjoy’s results (Denjoy Theorem and Denjoy counter-examples) on higher dimensional tori. It is the motivation for a line of research on the extension of the Denjoy’s type example of the circle to $\mathbb{T}^2$. To construct a Denjoy counter-example on the circle, one starts with an irrational rotation and blows up the orbit of some point to get an orbit of wandering intervals. Inspired by this, one motivating question is the wandering domains problem (see [NS96]): Can one “blow up” one or more orbits of $T_\alpha$ to make a smooth diffeomorphism with wandering domains? We say that a homeomorphism of $\mathbb{T}^2$ is of Denjoy type if it is obtained by blowing-up finitely many orbits of an irrational translation. P. McSwiggen in [McS93] constructed a $C^{2+\alpha}$ diffeomorphism of Denjoy type having a smooth wandering domain. In particular, his example is not topologically conjugate to a rigid translation. Norton and Sullivan in [NS96] showed that there dose not exist $C^3$ diffeomorphism on $\mathbb{T}^2$ of Denjoy type with circular wandering domains, and asked the following question:

Question 1 (Norton and Sullivan, 1996). If $f: \mathbb{T}^2 \to \mathbb{T}^2$ is a diffeomorphism, $h: \mathbb{T}^2 \to \mathbb{T}^2$ is a continuous map homotopic to the identity, and $hf = T_\rho h$ where $\rho \in \mathbb{R}^2$ is a totally irrational vector, are there natural geometric conditions (e.g. smoothness) on $f$ that force $h$ to be a homeomorphism?
In [PaSa13], A. Passeggi and M. Sambarino also mentioned the question: whether there exists $r$ so that if $f: T^2 \rightarrow T^2$ is a $C^r$ diffeomorphism semi-conjugate to an ergodic translation, then $f$ is conjugate to it. For more recent developments, we mention [Kar18, Nav18, Mer18, WZ18].

In [WZ18], the first author and Zhang constructed a smooth diffeomorphism which is isotopic to the identity and semi-conjugate to a minimal translation $T_\alpha$, but not conjugate to $T_\alpha$, which is a $C^\infty$ counter-example to the Norton-Sullivan’s question. The construction in [WZ18] combined the classical Anosov-Katok method (see [AK70, FK04]) with Jäger’s theorem [Jäg09] (see Theorem 2 below). In this article, we will construct a $C^\omega$ counter-example to the question of Norton-Sullivan.

Our strategy in this paper mainly follows from the approximation by conjugation construction scheme in the proof of Theorem 4 in [WZ18]. However, as we require the map is real-analytic, we will apply certain technique of analytic approximations in [Ban17, BK18] to customize the desirable analytic conjugacies. Our main theorem is the following:

**Theorem 1.** For any integer $d \geq 2$, there exists a $C^\omega$ area-preserving and minimal map $f: \mathbb{T}^d \rightarrow \mathbb{T}^d$ which is semi-conjugate to a minimal translation by a map homotopic to the identity, but is not topologically conjugate to a translation.

The classical Anosov-Katok method is a major source constructing examples of smooth dynamical systems with prescribed properties. This method is well known and was applied by many authors to construct different examples which satisfy some desired properties, e.g. ergodic, mixing, minimal, etc (see, e.g. [AK70, Sap03, FK04, FK14, BK18]). We would not want to restate this scheme in our article and instead, we recommend the classical articles [AK70, FK04]. The conjugation by approximation construction (i.e. the Anosov-Katok method) is essentially nonlinear and it is based on the convergence of maps obtained from certain standard maps by wildly diverging conjugacies. There is a great difference between the differentiable and real-analytic maps becomes apparent (see [FK04, Section 7.2] for the explanation). Hence, one will meet additional difficulties when one considers to construct examples of real-analytic diffeomorphisms by using this method. One possible way to overcome such difficulties is to work on some manifolds which have a large collection of real-analytic diffeomorphisms with some good properties, and whose singularities are uniformly bounded away from a complex neighborhood of the real domain (see, e.g. B. Fayad and A.B. Katok [FK14] work on odd-dimensional spheres). In our situation, we will work on the torus and use a trick on the approximation by conjugation scheme appeared in [Ban17, BK18] recently, which trick that can be traced earlier to Katok [Kat73].

We give some remarks about our theorem. As the constructions in [WZ18] and in this article, we use the classical Anosov-Katok method, the rotation vector of the constructed map is Liouvillean, which is the price to pay in order to get the smoothness of the pseudo-rotation. It seems difficult to construct a pseudo-rotation as in Theorem 1 with Diophantine rotation vector (see Section 2 for the definitions). On the other hand, by the classical KAM theory, any $C^\infty$ volume-preserving pseudo-rotation of $\mathbb{T}^n$ with Diophantine rotation vector $\alpha \in \mathbb{T}^n$, which is sufficiently close to $T_\alpha$, is smoothly conjugate to $T_\alpha$. Hence, for the Norton-Sullivan question, except the smoothness condition, the arithmetic condition of the rotation vector of the pseudo-rotation is also vital. Therefore, we ask the following question:

**Question 2.** In Question 1, if the vector $\rho$ is diophantine, is the Norton-Sullivan’s question true?

This article is organized as follows. In Section 2, we introduce some notations, recall some classical definitions and results. In particular, we introduce the block-slide type of maps and their analytic approximations. In Section 3, we customize the analytic conjugacies which is a key step to prove our main theorem. We prove the main theorem in Section 4.
2. Preliminary

2.1. The Misiurewicz-Ziemian rotation set. In this article, we study homeomorphisms of the two-dimensional torus $\mathbb{T}^2 = \mathbb{R}^2 / \mathbb{Z}^2$ which are isotopic to the identity. In this case, the rotation vectors and the rotation set are defined as follows.

Let $\text{Homeo}_\alpha(\mathbb{T}^2)$ be the group of homeomorphisms of $\mathbb{T}^2$ which are homotopic to $\text{Id}_{\mathbb{T}^2}$. Any $f \in \text{Homeo}_\alpha(\mathbb{T}^2)$ admits a lift to $\mathbb{R}^2$, denoted by $\tilde{f}$, which is a homeomorphism of $\mathbb{R}^2$ satisfying $\pi \tilde{f} = f \pi$, where $\pi : \mathbb{R}^2 \to \mathbb{T}^2$ is the covering projection.

M. Misiurewicz and K. Ziemian [MZ89] introduced the following standard definition:

**Definition 1.** Assume that $f \in \text{Homeo}_\alpha(\mathbb{T}^2)$ and that $\tilde{f}$ is a lift of $f$. The (Misiurewicz-Ziemian) rotation set of $f$ is defined by:

$$\rho(\tilde{f}) = \left\{ v \in \mathbb{R}^2 \mid \frac{\tilde{f}^{n_i}(z_i) - z_i}{n_i} \to v, \text{ for some } \{z_i\} \in \mathbb{R}^2, \text{ and } \{n_i\} \in \mathbb{N} \text{ with } n_i \to \infty \right\}.$$ 

The effect of changing the lift $\tilde{f}$ of $f$ is to translate $\rho(\tilde{f})$ by an integer vector. In [MZ89], the authors proved that the rotation set $\rho(\tilde{f})$ is a compact convex subset of $\mathbb{R}^2$, giving rise to a basic trichotomy: $\rho(\tilde{f})$ is either a compact convex set with nonempty interior, a line segment, or a singleton. We say that $f$ is a pseudo-rotation when $\rho(\tilde{f})$ is a singleton. Moreover, we say a pseudo-rotation $f$ is totally irrational if $\rho(f) = (\rho(f)_1, \rho(f)_2)$ satisfies that $\rho(f)_1, \rho(f)_2 \notin \mathbb{Q}$ and they are non-resonant (or rational independent), that is, for any $(a, b, c) \in \mathbb{Z}^3$ satisfying $ap(\tilde{f})_1 + bp(\tilde{f})_2 + c = 0$ implies that $(a, b, c) = (0, 0, 0)$.

For $\gamma, \sigma > 0$, we define the set $\mathcal{D}(\gamma, \sigma) \subset \mathbb{R}^2$ of diophantine vector with exponent $\sigma$ and constant $\gamma$ as the set of $\alpha = (\alpha_1, \alpha_2) \in \mathbb{R}^2$ such that

$$\forall (k_1, k_2) \in \mathbb{Z}^2, |k_1 \alpha_1 + k_2 \alpha_2| \geq \frac{\gamma}{(|k_1| + |k_2|)^{\sigma}}.$$ 

We set $\mathcal{D}(\sigma) = \bigcup_{\gamma > 0} \mathcal{D}(\gamma, \sigma)$ and $\mathcal{D} = \bigcup_{\sigma > 0} \mathcal{D}(\sigma)$. The set $\mathcal{D}$ is the set of Diophantine vectors of $\mathbb{R}^2$ while its complement in the set of non-resonant vectors is called the set of Liouville vectors, denoted it by $\mathcal{L}$. We note that the set $\mathcal{D}$ has full Lebesgue measure in $\mathbb{R}^2$ and the set $\mathcal{L}$ is $G_\delta$-dense in $\mathbb{R}^2$. In the same way, one can define all of the definitions above in higher dimensions.

2.2. Semi-conjugation. We denote by $\| \cdot \|$ the Euclidean norm on $\mathbb{R}^2$ and by $d$ the standard Euclidean metric. Let $\mathbb{T}^2$ be endowed with the metric induced by the Euclidean metric on $\mathbb{R}^2$, we still denote it by $d$ without any confusion.

For $\alpha = (\alpha_1, \alpha_2) \in \mathbb{R}^2$, we define the transition $T_\alpha : \mathbb{R}^2 \to \mathbb{R}^2$ by $T_\alpha(x, y) = (x + \alpha_1, y + \alpha_2)$ which naturally induces a translation on $\mathbb{T}^2$, we still denote it by $T_\alpha$ without any confusion. Given a map $f : \mathbb{T}^2 \to \mathbb{T}^2$, we say that $f$ is a semi-conjugate to a translation $T_\alpha$ if there exists a surjective continuous map $h : \mathbb{T}^2 \to \mathbb{T}^2$, such that $hf = T_\alpha h$, moreover, if $h$ is a homeomorphism, we say that $f$ is conjugate to a translation.

**Definition 2.** Let $f$ be a pseudo-rotation of $\mathbb{T}^2$. We say that $f$ has bounded mean motion (with a bound $\kappa \geq 0$) if there exists $\tilde{f}$, a lift of $f$, such that for any $z \in \mathbb{R}^2$ and $n \in \mathbb{N}$,

\begin{equation}
\| \tilde{f}^n(z) - z - n\rho(\tilde{f}) \| \leq \kappa.
\end{equation}

\footnote{If a homeomorphism of $\mathbb{T}^2$ is homotopic to the identity, then it is isotopic to the identity [Eps66, Theorem 6.4].}
2.3. Analytic topology. Any real-analytic diffeomorphism \( f \) of \( \mathbb{T}^2 \) homotopic to the identity admits a lift \( F : \mathbb{R}^2 \to \mathbb{R}^2 \) which has the following form:
\[
F(x_1, x_2) = (x_1 + f_1(x_1, x_2), x_2 + f_2(x_1 + x_2)),
\]
where \( f_1, f_2 \) are real analytic \( \mathbb{Z}^2 \)-periodic functions. It can be extend to some neighborhood of \( \mathbb{R}^2 \) in \( \mathbb{C}^2 \). For any \( \rho > 0 \), let
\[
B_\rho = \{(z_1, z_2) \in \mathbb{C}^2 \mid |\text{Im}(z_1)|, |\text{Im}(z_2)| < \rho\},
\]
and for a function \( h \) defined on \( B_\rho \), we define
\[
\|h\|_\rho = \sup_{(z_1, z_2) \in B_\rho} |h(z_1, z_2)|.
\]
We define \( \mathcal{C}_\omega^\infty(\mathbb{T}^2) \) to be the space of all \( \mathbb{Z}^2 \)-periodic real-analytic function on \( \mathbb{R}^2 \) that extends to a holomorphic function on \( B_\rho \), and \( \|h\|_\rho < \infty \).

Let \( \lambda \) be the standard Lebesgue measure on \( \mathbb{T}^2 \). We denote by \( \text{Diff}_{\rho, \omega}^\infty(\mathbb{T}^2, \lambda) \) the space of all measure-preserving real-analytic diffeomorphism of \( \mathbb{T}^2 \) homotopic to the identity, whose lift \( F(x) = (x_1 + f_1(x), x_2 + f_2(x)) \) to \( \mathbb{R}^2 \) satisfies \( f_i \in \mathcal{C}_\omega^\infty \) and we also require the lift \( \tilde{F}(x) = (x_1 + \tilde{f}_1(x), x_2 + \tilde{f}_2(x)) \) of its inverse to \( \mathbb{R}^2 \) to satisfy \( \tilde{f}_i \in \mathcal{C}_\omega^\infty \). For any \( f, g \in \text{Diff}_{\rho, \omega}^\infty(\mathbb{T}^2, \lambda) \), we define the distance
\[
d_{\rho}(f, g) = \max\{\tilde{d}_{\rho}(f, g), \tilde{d}_{\rho}(f^{-1}, g^{-1})\},
\]
where
\[
\tilde{d}_{\rho}(f, g) = \max_{i=1, 2} \left\{ \inf_{k \in \mathbb{Z}} \|f_i(z_1, z_2) - g_i(z_1, z_2) + k\|_\rho \right\}.
\]
Finally, we define the space: \( \text{Diff}_{\rho, \omega}^{\infty}(\mathbb{T}^2, \lambda) := \bigcap_{m=1}^{\infty} \text{Diff}_{\rho, m}^{\infty}(\mathbb{T}^2, \lambda) \). For more information about the analytic topology, we recommend the readers to refer to [BK18, Sap03].

2.4. Analytic approximations. In this subsection, we will introduce two lemmas. Given a special kind of step function with some periodic propriety (it is called block-slide type of maps in [Ban17, BK18], see (2.1) below), we can construct an explicit form of some analytic approximations of the function, which preserves the periodic propriety and satisfies some Lipschitz condition, that is crucial to the proof of the main result of this article. The key point is that a block-slide type of map can be approximated extremely well by measure-preserving real-analytic diffeomorphisms outside a set of arbitrarily small measure, which is inspired by [Kat73].

Let \( q \in \mathbb{N}, N \in 2\mathbb{N}, \) and \( \beta = (\beta_0, \cdots, \beta_{N-1}) \in [0, 1)^N \). Consider a step function of the form
\[
(2.1) \quad \bar{s}_{\beta, q} : [0, 1) \to \mathbb{R} \text{ defined by } \bar{s}_{\beta, q} = \sum_{j=0}^{Nq-1} \tilde{\beta}_j x_{j/Nq, (j+1)/Nq},
\]

Here, \( \tilde{\beta}_j := \beta_k \), where \( k := j \pmod{N} \). For any \( \delta \in (0, 1) \), we denote by \( F_{q, N, \delta} \) the union of all intervals centered around \( j/Nq (j \in \mathbb{Z}) \) with length \( \delta/Nq \). For given \( \epsilon \in (0, \frac{1}{2}) \) and \( \delta \in (0, 1) \), we define
\[
A_0(\epsilon, \delta, N) := \max \left\{ \frac{2N}{\pi \cdot \delta} \cdot \ln(-\ln(1 - \epsilon^{\frac{\delta}{8}c})), \frac{2N}{\pi \cdot \delta} \cdot \ln(-\ln(\frac{\epsilon}{2N})) \right\}.
\]
Lemma 1. [BK18, Lemmas 2.13, 2.18, 3.14] For any \( \epsilon \in (0, \frac{1}{2}) \), \( \delta \in (0,1) \) and any \( A > A_0(\epsilon, \delta, N) \), we define the following \( 1/q \)-periodic real-analytic map \( \tilde{s}_{\beta,q,\epsilon,\delta,A} : \mathbb{R} \to \mathbb{R} \) as

\[
\tilde{s}_{\beta,q,\epsilon,\delta,A}(x) = \left( \sum_{j=0}^{N/2-1} \beta_j \left( e^{-e^{-A \sin 2\pi (qz-j)/N}} - e^{-e^{-A \sin 2\pi (qz-(j+1))/N}} \right)e^{-e^{A \sin 2\pi qz}} \right) + \left( \sum_{j=N/2}^{N-1} \beta_j \left( e^{-e^{-A \sin 2\pi (qz-j)/N}} - e^{-e^{-A \sin 2\pi (qz-(j+1))/N}} \right)e^{-e^{A \sin 2\pi qz}} \right).
\]

The map \( \tilde{s}_{\beta,q,\epsilon,\delta,A} \) has the following properties:

1. The complexification of \( \tilde{s}_{\beta,q,\epsilon,\delta,A} \) extends holomorphically to \( \mathbb{C} \);
2. We have \( \sup_{x \in [0,1)} |\tilde{s}_{\beta,q,\epsilon,\delta,A}(x) - \tilde{s}_{\beta,q}(x)| < \epsilon \);
3. The map \( \tilde{s}_{\beta,q,\epsilon,\delta,A} \) is \( \frac{1}{q} \)-periodic. More precisely, the complexification of \( \tilde{s}_{\beta,q,\epsilon,\delta,A} \) satisfies

\[
\tilde{s}_{\beta,q,\epsilon,\delta,A}(z + k/q) = \tilde{s}_{\beta,q,\epsilon,\delta,A}(z) \quad \text{for all} \quad z \in \mathbb{C} \quad \text{and} \quad k \in \mathbb{Z};
\]
4. \( \forall \rho > 0 \), there exist a constant \( C(N,q,\epsilon,\delta,A,\rho) > 0 \) such that:

\[
\sup_{z_1,z_2 \in B_{\rho}} |\tilde{s}_{\beta,q,\epsilon,\delta,A}(z_1) - \tilde{s}_{\beta,q,\epsilon,\delta,A}(z_2)| \leq C|z_1 - z_2|.
\]

Remark 1. Note that \( 0 < e^{-e^t} < 1 \) for all \( t \in \mathbb{R} \). It is obviously that \( |\tilde{s}_{\beta,q,\epsilon,\delta,A}(z)| < \sum_{i=0}^{N-1} |\tilde{\beta}_i| \).

Remark 2. We can take the constant \( C(N,q,\epsilon,\delta,A,\rho) \) in the item (4) as \( 6\pi \cdot A \cdot N \cdot q \cdot e^{4 \epsilon e^{4\pi^2 q \rho}} \) (see the proof of Lemma 3.14 in [BK18]).

For every \( m \geq 1 \), we define

\[
A_0^{(m)}(\epsilon, \delta, N) := A_0 \left( \frac{\epsilon}{4m} \right), \delta, N \right) \quad \text{and} \quad C^{(m)}(N,q,\epsilon,\delta,A,\rho) := 4mC(N,q,\epsilon,\delta,A,\rho).
\]

In order to prove our main theorem, we require that \( \beta \in [-1,1)^N \). Hence, we give the following lemma.

Lemma 2. Let \( q \in \mathbb{N}, N \in 2\mathbb{N} \), and \( \beta = (\beta_0, \cdots, \beta_{N-1}) \in [-m,m)^N \) where \( m \in \mathbb{N}_{\geq 1} \). Suppose that \( \tilde{s}_{\beta,q} \) is the step function defined in (2.1). Then for any \( \epsilon \in (0, \frac{1}{2}) \), \( \delta \in (0,1) \) and any \( A > A_0^{(m)}(\epsilon, \delta, N) \), the function \( \tilde{s}_{\beta,q,\epsilon,\delta,A} \) defined in (2.2) satisfies all of the properties of Lemma 1 if we replace the constant \( C \) in the property (4) by \( C^{(m)}(N,q,\epsilon,\delta,A,\rho) \).

Proof. Let \( \beta_0 = \left( \frac{1}{2}, \cdots, \frac{1}{2} \right) \in [0,1)^N \). Then \( \frac{1}{2m} \beta + \beta_0 \in [0,1)^N \). For any \( \epsilon \in (0, \frac{1}{8}) \), \( \delta \in (0,1) \), applying \( \frac{1}{4m} \delta \) and \( A > A_0 \left( \frac{\epsilon}{4m}, \delta, N \right) \) to Lemma 1, we have

\[
\sup_{x \in [0,1)} |\tilde{s}_{\frac{1}{2m} \beta + \beta_0,q,\epsilon,\delta,A}(x) - \tilde{s}_{\frac{1}{2m} \beta + \beta_0,q}(x)| < \frac{\epsilon}{4m},
\]

\[
\sup_{x \in [0,1) \setminus F_{q,N,\delta}} |\tilde{s}_{\beta_0,q,\epsilon,\delta,A}(x) - \tilde{s}_{\beta_0,q}(x)| < \frac{\epsilon}{4m}.
\]
Note that the maps \( \beta \rightarrow \tilde{s}_{\beta,q}, \tilde{s}_{\beta,q,\epsilon,\delta,A} \) are linear. Hence, we get

\[
\sup_{x \in [0,1) \setminus F_{q,N,s}} |\tilde{s}_{\beta,q,\epsilon,\delta,A}(x) - \tilde{s}_{\beta,q}(x)| \\
\leq \sup_{x \in [0,1) \setminus F_{q,N,s}} |\tilde{s}_{\beta+2m\tilde{\beta}_0,q,\epsilon,\delta,A}(x) - \tilde{s}_{\beta+2m\tilde{\beta}_0,q}(x)| + \sup_{x \in [0,1) \setminus F_{q,N,s}} |\tilde{s}_{2m\tilde{\beta}_0,q,\epsilon,\delta,A}(x) - \tilde{s}_{2m\tilde{\beta}_0,q}(x)| \\
< 2m \frac{\epsilon}{4m} + 2m \frac{\epsilon}{4m} = \epsilon.
\]

This is the property (2). Similarly,

\[
\sup_{z_1,z_2 \in B_{\rho}} |\tilde{s}_{\beta,q,\epsilon,\delta,A}(z_1) - \tilde{s}_{\beta,q,\epsilon,\delta,A}(z_2)| \\
\leq \sup_{z_1,z_2 \in B_{\rho}} |\tilde{s}_{\beta+2m\tilde{\beta}_0,q,\epsilon,\delta,A}(z_1) - \tilde{s}_{\beta+2m\tilde{\beta}_0,q,\epsilon,\delta,A}(z_2)| + \sup_{z_1,z_2 \in B_{\rho}} |\tilde{s}_{2m\tilde{\beta}_0,q,\epsilon,\delta,A}(z_1) - \tilde{s}_{2m\tilde{\beta}_0,q,\epsilon,\delta,A}(z_2)| \\
\leq 2m (C(N,q,\epsilon,\delta,A,\rho) + C(N,q,\epsilon,\delta,A,\rho)) |z_1 - z_2| \\
= 4mC(N,q,\epsilon,\delta,A,\rho)|z_1 - z_2|.
\]

This is the property (4). Obviously, \( \tilde{s}_{\beta,q,\epsilon,\delta,A} \) satisfies the other properties. \( \square \)

3. A CRUCIAL LEMMA

To prove the main theorem, we need the following lemma which is an analytic version of Lemma 6 in [WZ18].

Let \( \Gamma = (Q \times \mathbb{R}) \cup (\mathbb{R} \times Q) \subset \mathbb{R}^2 \). Recall that \( \pi : \mathbb{R}^2 \rightarrow \mathbb{T}^2 \) is the covering projection. For \( x, y \in \mathbb{T}^2 \), we say \( x - y \notin \Gamma \) means that if \( \tilde{x}, \tilde{y} \in \mathbb{R}^2 \) satisfying \( \pi(\tilde{x}) = x, \pi(\tilde{y}) = y \), then \( \tilde{x} - \tilde{y} \notin \Gamma \), similarly, we say \( x \notin \Gamma \) if \( \tilde{x} \notin \Gamma \).

**Lemma 3.** Given an integer \( q \geq 2 \), for any \( \sigma > 0 \) and \( x, y \in \mathbb{T}^2 \) with \( x, y - x \notin \Gamma \), there exists \((h,x',y',N)\) such that:

1. \( N \in 2\mathbb{N} \cap \mathbb{N}_{\geq 4}; \)
2. \( h \in \text{Diff}^\infty_{\mathbb{Z}}(\mathbb{T}^2,\lambda) \) commutes with both \( T_{(\frac{1}{q},0)} \) and \( T_{(0,\frac{1}{q})} \);
3. \( x',y',x' - y' \notin \Gamma, d(x,h(x')) = d(y,h(y')) = d(x',y') < \sigma \), and
   \[ d(hT_{(\frac{1}{q},0)}(x'),hT_{(\frac{1}{q},0)}(y')) < \sigma; \]
4. \( dq(\tilde{h},\text{Id}_{\mathbb{R}^2}) \leq 2d(x,y) + \frac{1}{Nq}, \) where \( \tilde{h} \) is a lift of \( h \) to \( \mathbb{R}^2 \).

**Proof.** Because \( x, y, x - y \notin \Gamma \), we may select \( \tilde{x}, \tilde{y} \in [0,1) \times [0,1) \) such that \( \pi(\tilde{x}) = x, \pi(\tilde{y}) = y \), \( \tilde{x} - \tilde{y} \notin \Gamma \). \( \forall n \in \mathbb{N}_{\geq 1}, \) we may assume that

\[
\tilde{x} = (x_1,x_2) \in \left( \frac{i_1(n)}{nq}, \frac{i_1(n)}{nq} + 1 \right) \times \left( \frac{i_2(n)}{nq}, \frac{i_2(n)}{nq} + 1 \right),
\]

\[
\tilde{y} = (y_1,y_2) \in \left( \frac{j_1(n)}{nq}, \frac{j_1(n)}{nq} + 1 \right) \times \left( \frac{j_2(n)}{nq}, \frac{j_2(n)}{nq} + 1 \right),
\]

where \( 0 \leq i_1(n), i_2(n), j_1(n), j_2(n) \leq nq - 1 \). When \( n \) satisfies:

\[
(3.1) \quad \frac{1}{qn} < \min \left\{ \min_{k \in \mathbb{Z}_{i=1,2}} \left\{ \left| x_i - y_i + \frac{k}{q} \right|, \frac{1}{4} \right\}, \right. \]

\[
\left. \frac{1}{4} - \frac{1}{4} \right\}.
\]
we have $i_i \not\equiv j_i \pmod{n}, t = 1, 2$. Now we fix an even integer $N > \max\{3, \frac{4}{qq}\}$ which satisfies (3.1). Define $\alpha = (\alpha_0, \ldots, \alpha_{N-1}), \beta = (\beta_0, \ldots, \beta_{N-1})$ by:

\[(3.2) \quad \beta_i = \begin{cases} \frac{\alpha_i + 1.5}{Nq} - x_1, & i \equiv i_2(N) \pmod{N}; \\ 0, & \text{others} \end{cases}
\]

\[(3.3) \quad \alpha_i = \begin{cases} \frac{\alpha_i + 0.5}{Nq} - x_2, & i \equiv j_1(N) + 1 \pmod{N}; \\ 0, & \text{others} \end{cases}
\]

We recall (2.1) in the last section. Let $\bar{s}_\alpha := \bar{s}_{\alpha,q}, \bar{s}_\beta := \bar{s}_{\beta,q}$. We define the following maps defined on $[0, 1) \times [0, 1)$ to $\mathbb{R}^2$:

\[\bar{h}_\alpha(a, b) = (a, b - \bar{s}_\alpha(a)), \quad \bar{h}_\beta(a, b) = (a - \bar{s}_\beta(b), b), \quad \bar{h} = \bar{h}_\beta \bar{h}_\alpha.
\]

Note that $\bar{h}^{-1}(a, b) = (a + \bar{s}_\beta(b), b + \bar{s}_\alpha(a + \bar{s}_\beta(b)))$. We have $\bar{h}^{-1}(\bar{x}) = \left(\frac{\alpha + 1.5}{Nq}, \frac{\alpha + 0.5}{Nq}\right)$ and $\bar{h}^{-1}(\bar{y}) = \bar{y}$. Hence we get that

\[d(\bar{h}^{-1}(\bar{x}), \bar{h}^{-1}(\bar{y})) < \frac{4}{Nq} < \sigma.
\]

As $\frac{\alpha + 1.5}{Nq} < y_1 + \frac{4}{Nq} < 1$, we have $T_{(\bar{x}, \bar{y})}(\bar{h}^{-1}(\bar{x})), T_{(\bar{x}, \bar{y})}(\bar{h}^{-1}(\bar{y})) \in [0, 1)^2$. By definition of $\bar{h}$, we obtain that $\bar{h}$ fixes $T_{(\bar{x}, \bar{y})}(\bar{h}^{-1}(\bar{x}))$ and $T_{(\bar{x}, \bar{y})}(\bar{h}^{-1}(\bar{y}))$. Therefore,

\[d(\bar{h}T_{(\bar{x}, \bar{y})}(\bar{h}^{-1}(\bar{x})), \bar{h}T_{(\bar{x}, \bar{y})}(\bar{h}^{-1}(\bar{y}))) = d(T_{(\bar{x}, \bar{y})}(\bar{h}^{-1}(\bar{x})), T_{(\bar{x}, \bar{y})}(\bar{h}^{-1}(\bar{y}))) < \frac{4}{Nq} < \sigma.
\]

Note that $\alpha, \beta \in [-1, 1]^N$. Applying Lemma 2 for $m = 1$, for any $\epsilon \in (0, \frac{4}{q^2})$, $\delta \in (0, 1)$ and $A > A_{(1)}(\epsilon, \delta, N)$, we get the real analytic 1/q-periodic approximate functions $\delta_\alpha := \delta_{\alpha,q,\epsilon,\delta,A}, \delta_\beta := \delta_{\beta,\epsilon,\delta,A}$ of $\bar{s}_\alpha, \bar{s}_\beta$, respectively. Define $\bar{h}_\alpha = (a, b - \bar{s}_\alpha(a)), \bar{h}_\beta = (a - \bar{s}_\beta(b), b)$ and let $\bar{h} = \bar{h}_\beta \bar{h}_\alpha$. More precisely,

\[\bar{h}(a, b) = (a - \bar{s}_\beta(b) - \bar{s}_\alpha(a), b)\text{ for all } (a, b) \in \mathbb{R}^2.
\]

Obviously, it is a diffeomorphism of $\mathbb{R}^2$. By Lemma 2, it can be extended holomorphically to $\mathbb{C}^2$. Moreover, it satisfies that $\bar{h}((a, b) + (\frac{k_1}{q}, \frac{k_2}{q})) = \bar{h}(a, b) + (\frac{k_1}{q}, \frac{k_2}{q})$ for any $(a, b) \in \mathbb{C}^2$ and $(k_1, k_2) \in \mathbb{Z}^2$.

In particular, it induces a diffeomorphism on $\mathbb{T}^2$, denoted by $\bar{h}$.

We note that $\bar{x}, \bar{y}, \bar{h}^{-1}(\bar{x}), \bar{h}^{-1}(\bar{y}) \notin \Gamma_{Nq}$, where $\Gamma_{Nq} := \{(a, b) | a = \frac{1}{Nq} \text{ or } b = \frac{1}{Nq}, i \in \mathbb{Z}\}$. We choose $\delta > 0$ small enough such that $\bar{x}, \bar{y}, \bar{h}^{-1}(\bar{x}), \bar{h}^{-1}(\bar{y})$ not belong to the set $F_{Nq}$ which is defined in Lemma 1. By continuity and Lemma 2, letting $0 < \epsilon < \min\left\{\left|\frac{\alpha_i + 0.5}{Nq} - x_1\right|, \left|\frac{\alpha_i + 1.5}{Nq} - x_2\right|\right\}$ small enough, we have

\[d(\bar{h}^{-1}(\bar{x}), \bar{h}^{-1}(\bar{y})) < \sigma, \quad d(\bar{h}T_{(\bar{x}, \bar{y})}(\bar{h}^{-1}(\bar{x})), \bar{h}T_{(\bar{x}, \bar{y})}(\bar{h}^{-1}(\bar{y}))) < \sigma.
\]

Obviously, there exist $\bar{x}', \bar{y}'$ closed to $\bar{h}^{-1}(\bar{x}), \bar{h}^{-1}(\bar{y})$ and satisfy

\[d(\bar{x}', \bar{y}') < \sigma, \quad d(\bar{h}T_{(\bar{x}', \bar{y}')}((\bar{x}')), \bar{h}T_{(\bar{x}', \bar{y}')}((\bar{y}'))) < \sigma.
\]

Assuming $x' = \pi(\bar{x}'), y' = \pi(\bar{y}')$, then $(h, x', y', N)$ satisfies the items (1) and (3).

Moreover, as $\bar{h}_\alpha = (a, b - \bar{s}_\alpha(a)), \bar{h}_\beta = (a - \bar{s}_\beta(b), b)$ preserve the Lebesgue measure on $\mathbb{R}^2$, $\bar{h}$ preserves the Lebesgue measure. Therefore, $h \in \text{Diff}_{\infty}^\omega(\mathbb{T}^2, \lambda)$ and it commutes with $T_{(\bar{x}, \bar{y})}$ and $T_{(0, \frac{1}{q})}$. Furthermore, we have:

\[d_{C^0}(\bar{h}, \text{Id}_{\mathbb{R}^2}) \leq \sup|\bar{s}_\alpha| + |\bar{s}_\beta| \leq \frac{|j_2 + 0.5}{Nq} - x_2 + \frac{|j_1 + 1.5}{Nq} - x_1 | < 2d(x, y) + \frac{4}{Nq}.
\]
where the second inequality comes from Remark 1. Then \((h, x', y', N)\) also satisfies (2) and (4), and hence it is the desired. \(\square\)

4. Proof of the main Theorem

To prove Theorem 1, it is enough to prove the following proposition.

Proposition 1. Fix \(\rho > 0\), there exists an area-preserving and minimal pseudo-rotation \(f \in Diff^\infty_\rho(T^2, \lambda)\) which has bounded mean motion, and satisfies the following: for any \(\varepsilon > 0\), there exist two points \(x, y \in T^2\) with \(d(x, y) < \varepsilon\), and an integer \(N > 0\) such that \(d(f^N(x), f^N(y)) \geq \frac{1}{1000}\).

Proof. In the proof, we use the same approximation by conjugation scheme in [WZ18]. The different is that we have to replace the \(C^\infty\) conjugacies in the proof of [WZ18, Proposition 6] by the \(C^\omega\) conjugacies in our situation.

We will construct a sequence of \(h_n \in Diff^\infty_\rho(T^2, \lambda)\), \(\omega_n = (\omega_{n,1}, \omega_{n,2} = q_n^{-1}\omega_n) \in \mathbb{Q}^2\) with \(\omega_n \in \mathbb{Z}^2\), \(q_n \in \mathbb{N}\) for each \(n \geq 1\). We first introduce \((a1)_n - (a4)_n\) for a given \(n \geq 1\):

\((a1)_n\) There exists \(\hat{h}_n\) a lift of \(h_n\) to \(\mathbb{R}^2\), such that \(d_{C^0}(\hat{h}_n, \text{Id}_{\mathbb{R}^2}) < 2^{-n}\); Let \(H_n := h_1 \cdots h_n\), then \(H_n \in Diff^\infty_\rho(T^2, \lambda)\), and the map \(\hat{H}_n := \hat{h}_1 \cdots \hat{h}_n\) is a lift of \(H_n\) and satisfies:

\[
d_{C^0}(\hat{H}_n, \text{Id}_{\mathbb{R}^2}) \leq \sum_{i=1}^n d_{C^0}(\hat{h}_i, \text{Id}_{\mathbb{R}^2}) < 1 - 2^{-n};
\]

\((a2)_n\) There exist \(x_n, y_n \in T^2\) with \(x_n, y_n, x_n - y_n \notin \Gamma\) such that:

\[
d(x_n, y_n) < 10^{-2n}, d(H_n(x_n), H_n(y_n)) > \frac{1}{1000};
\]

\((a3)_n\) For \(f_n := H_nT_{\omega_n}H_n^{-1} \in Diff^\infty_\rho(T^2, \lambda)\), there exist \(x^{(n)}, y^{(n)} \in T^2\), \(m_n \in \mathbb{N}\) such that:

\[
d(x^{(n)}, y^{(n)}) < 10^{-n}, d(f_{m_n}^n(x^{(n)}), f_{m_n}^n(y^{(n)})) < \frac{1}{1000};
\]

\((a4)_n\) For any \(z \in T^2\), the set \(\{f_n^k(z)\}_{k\in\mathbb{Z}}\) is \(2^{-n}\)-dense in \(T^2\).

Here we say that a set \(K \subset T^2\) is \(\sigma\)-dense for some \(\sigma > 0\), if for any \(x \in T^2\) there exists \(y \in K\) such that \(d(x, y) < \sigma\).

Note that \((a1)_n\) and \((a3)_n\) imply the following: the map \(F_n := \hat{H}_nT_{\omega_n}\hat{H}_n^{-1}\) is a lift of \(f_n\), and for any integer \(k \geq 1\) we have

\[
\sup_{z \in \mathbb{R}^2} \|F_n^k(z) - z - k\omega_n\| = \sup_{z \in \mathbb{R}^2} \|\hat{H}_n(\hat{H}_n^{-1}(z) + k\omega_n) - \hat{H}_n(\hat{H}_n^{-1}(z)) - k\omega_n\|
\]

\[
= \sup_{z \in \mathbb{R}^2} \|\hat{H}_n(z + k\omega_n) - \hat{H}_n(z) - k\omega_n\|
\]

\[
\leq 2d_{C^0}(\hat{H}_n, \text{Id}_{\mathbb{R}^2}) < 10.
\] (4.1)

Moreover, whenever \((a1)_n - (a4)_n\) are satisfied, there exists a sufficiently small real number \(\epsilon_n > 0\) such that for any \(f \in \text{Homeo}(T^2)\) satisfying \(d_{C^0}(f, f_n) < \epsilon_n\), for any \(\omega \in \mathbb{R}^2\) satisfying \(\|\omega - \omega_n\| < \epsilon_n\), and for any \(F \in \text{Homeo}(\mathbb{R}^2)\) satisfying \(d_{C^0}(F, f_n) < \epsilon_n\), we have

\[
\|F^k(z) - z - k\omega\| < 10, \quad \forall z \in \mathbb{R}^2, 1 \leq k \leq n,
\]

\[
d(f_{m_n}^n(x^{(n)}), f_{m_n}^n(y^{(n)})) > \frac{1}{1000},
\]

\[
\{f_n^k(z)\}_{k\in\mathbb{N}}\text{ is }2^{-n+1}\text{-dense in }T^2\text{ for any }z \in T^2.
\] (4.2) (4.3) (4.4)
Without loss of generality, we can assume that $\epsilon_k > \epsilon_{k+1}$ for any $k \geq 1$.

Now we can introduce the last induction hypothesis for a given $n \geq 1$:

(a5)$_n$ we have $d_q(f_{n+1}, f_n), d_{C^0}(F_{n+1}, F_n), \|\omega_{n+1} - \omega_n\| < 2^{-n}\epsilon_n$.

For each integer $n \geq 1$, we will construct $h_i, \omega_i, q_i, \omega_i$ for $1 \leq i \leq n$, satisfying (a1)$_i$, (a4)$_i$ for any $1 \leq i \leq n$, and (a5)$_i$ for any $1 \leq i \leq n - 1$.

To start the induction, we let $h_1 = \text{Id}_{\mathbb{R}^2}$ and $\omega_1 = q_1^{-1}\omega_1 = \left(\frac{1}{10}, \frac{1}{10}\right)$, where $q_1 = 100$ and $\omega_1 = (1, 10)$. It is direct to verify (a1)$_1$ - (a4)$_1$.

Suppose that we have constructed $h_i, \omega_i, q_i, \omega_i$ for $1 \leq i \leq n$, satisfying (a1)$_i$, (a4)$_i$ for any $1 \leq i \leq n$, and (a5)$_i$, for any $1 \leq i \leq n - 1$. Let $f_n, F_n, \omega_n, x_n, y_n, x^{(n)}, y^{(n)}$, $m_n$, $\epsilon_n$ be given by induction hypothesis. We will construct $(h_{n+1}, \omega_{n+1}, q_{n+1}, \omega_{n+1})$ as follows.

We recall that $\omega_n = (\omega_{n,1}, \omega_{n,2}) = q_n^{-1}\omega_n$ with $\omega_n \in \mathbb{Z}^2$, $q_n \in \mathbb{N}$. Without loss of generality, we can assume that $q_n > 10^n$.

Recall (a2)$_n$, we set $\sigma_n$ small enough such that:

1. if $d(x_n, y_n) < \sigma_n$, then $d(H_n(x), H_n(y)) > \frac{1}{1000}$;
2. $\sigma_n < 10^{-2n-2}$, $\min\{|DH_n||_{C^0}^{-1}, 1\}$, where $DH_n$ is the real derivative of $H_n$.

Applying Lemma 3 to $q = q_n$, $x = x_n$, $y = y_n$, $\sigma = \sigma_n$, we get $(h_{n+1}, x_{n+1}, y_{n+1}, N_{n+1}) := (h, x', y', N)$. Then

(i) $h_{n+1} \in \text{Diff}^\omega_{\infty}(\mathbb{T}^2, \lambda)$ commutes with $T_{\left(\frac{1}{q_n}, 0\right)}$ and $T_{(\frac{1}{q_n})}$; so it also commutes with $T_{\omega_n}$, and the lift of $h_{n+1}$, $\tilde{h}_{n+1}$ satisfies:

\[ d_{C^0}(\tilde{h}_{n+1}, \text{Id}_{\mathbb{R}^2}) \leq 2d(x_n, y_n) + \frac{4}{N_{n+1}q_n} < 2^{-n-1}; \]

(ii) $d(x_{n+1}, y_{n+1}) < \sigma_n < 10^{-2n-2}$;

(iii) $x_{n+1}, y_{n+1}, x_{n+1} - y_{n+1} \notin \Gamma, d(x_n, h_{n+1}(x_{n+1})), d(y_n, h_{n+1}(y_{n+1})) < \sigma_n$, so by (1) above, we have

\[ d(H_{n+1}(x_{n+1}), H_{n+1}(y_{n+1})) = d(H_n h_{n+1}(x_{n+1}), H_n h_{n+1}(y_{n+1})) > \frac{1}{1000}. \]

This verifies (a1)$_{n+1}$, (a2)$_{n+1}$.

Let

\[ z^{(n+1)} := H_{n+1}T_{\left(\frac{2}{q_n}, 0\right)}(z_{n+1}) \quad \text{for } z = x, y. \]

By Lemma 3 (3) and the choice of $\sigma_n$, we see that

\[ d(x^{(n+1)}, y^{(n+1)}) \leq \|DH_n||_{C^0}d(h_{n+1}T_{\left(\frac{2}{q_n}, 0\right)}(x_{n+1}), h_{n+1}T_{\left(\frac{2}{q_n}, 0\right)}(y_{n+1})) < 10^{-n-1}. \]

This verify the first inequality in (a3)$_{n+1}$.

For any $\gamma \in \mathbb{R}^2$, we set

\[ G^\gamma_{n} := H_{n+1}T_{\left(-\frac{2}{q_n+1}, 0\right) + \gamma H_{n+1}^{-1}}. \]

By definition and (4.5), we have

\[ d(G^0_{n}(x^{(n+1)}), G^0_{n}(y^{(n+1)})) > \frac{1}{1000}. \]

Then by continuity, there exists $\kappa > 0$ such that for any $\gamma \in \mathbb{R}^2$ with $\|\gamma\| < \kappa$, we have

(4.8)
\[ d(G^\gamma_{n}(x^{(n+1)}), G^\gamma_{n}(y^{(n+1)})) > \frac{1}{1000}. \]
Without loss of generality, we can also assume that
\begin{equation}
\kappa < 2^{-n-1} \|DH_{n+1}\|_{C^0}^{-1}.
\end{equation}

Set \( \omega_{n+1} = \omega_n + \eta_{n+1} \) for some \( \eta_{n+1} \in \mathbb{Q}^2 \setminus \{(0, 0)\} \) of the form
\begin{equation}
\eta_{n+1} = \frac{1}{q_n r_{n+1}} (1, v),
\end{equation}
where \( v, r_{n+1} \in \mathbb{N} \) satisfy that
\begin{equation}
v > 100 \kappa^{-1}, \quad r_n \geq 100 \kappa^{-1} v.
\end{equation}

We write \( \eta_{n+1} = (a_{n+1}, b_{n+1}) \) and select \( \rho_n \) large enough such that \( \tilde{H}_n^{-1}(B_{\rho_n}) \subseteq B_{\rho_n} \). By applying Lemma 3 to \( q = q_n, x = x_n, y = y_n \) and \( \sigma = \sigma_n \), we recall that \((\alpha, \beta, \epsilon, \delta, A)\) which appears in the proof Lemma 3. We can write down the explicit form of \( h_{n+1} \) and \( h_{n+1}^{-1} \) as:
\begin{align*}
h_{n+1} T_{\pm \eta_{n+1}} h_{n+1}^{-1}(a, b) &= (a \pm a_{n+1} + \tilde{s}_\beta(b) - \tilde{s}_\alpha(b \pm b_{n+1} + \tilde{s}_\alpha(a + \tilde{s}_\beta(b)) - \tilde{s}_\alpha(a + \tilde{s}_\beta(b) \pm a_{n+1})), \\
&\quad b \pm b_{n+1} + \tilde{s}_\alpha(a + \tilde{s}_\beta(b) - \tilde{s}_\alpha(a + \tilde{s}_\beta(b) \pm a_{n+1})).
\end{align*}

By Lemma 2, there exists a constant \( C_{n+1} := C'^{(1)}(N_{n+1}, q_n, \epsilon, \delta, A, \rho_n) \) such that:
\begin{equation}
\|\tilde{h}_{n+1} T_{\pm \eta_{n+1}} \tilde{h}_{n+1}^{-1} - \text{Id}_z\|_{\rho_n} < |a_{n+1}| + C_{n+1}(C_{n+1} |a_{n+1}| + |b_{n+1}|) + |b_{n+1}| + C_{n+1} |a_{n+1}|.
\end{equation}

We write \( \rho'_n := |a_{n+1}| + C_{n+1}(C_{n+1} |a_{n+1}| + |b_{n+1}|) + |b_{n+1}| + C_{n+1} |a_{n+1}|. \) To verify (a5)\(_{n+1} \), we prove the following Lemma.

**Lemma 4.** There is a positive number \( Q \) large enough such that, when \( r_{n+1} > Q \), we have
\begin{enumerate}
\item \( \|\eta_{n+1}\| < \rho'_n < \left( \sup_{z \in B_{\rho'_n}} \|D\tilde{H}_n(z)\| + 1 \right)^{-1} \cdot 2^{-n} \epsilon_n \), where \( \epsilon_n \) is determined by \((a1)_{n}-(a4)_{n} \)
\item the map \( f_{n+1} \) given by
\begin{align*}
f_{n+1} &= H_{n+1} T_{\omega_n} h_{n+1} H_{n+1}^{-1} \\
&= H_n (H_{n+1} T_{\omega_n} h_{n+1}^{-1} H_{n+1}^{-1}) \\
&= H_n (T_{\omega_n} h_{n+1}^{-1} H_{n+1}^{-1})
\end{align*}
\end{enumerate}
is \( 2^{-n} \epsilon_n \)-close to \( f_n \) in \( \text{Diff}^\omega(T^2) \);
\begin{enumerate}
\item the map \( F_{n+1} = H_{n+1} T_{\omega_n} h_{n+1}^{-1} \) is \( 2^{-n} \epsilon_n \)-close to \( F_n \) in \( C^0(\mathbb{R}^2) \).
\end{enumerate}

**Proof.** By definitions, the item (1) is clear. We now show that (2) and (3) hold when (1) is satisfied.

Note that \( \rho'_n < 1 \) when (1) is satisfied. By definition of \( \tilde{d}_\rho \), we have
\begin{align*}
\tilde{d}_\rho(f_{n+1}, f_n) &= \|\tilde{H}_n (T_{\omega_n} h_{n+1} T_{\eta_{n+1}} h_{n+1}^{-1}) \tilde{H}_n^{-1} - \tilde{H}_n T_{\omega_n} \tilde{H}_n^{-1}\|_{\rho} \\
&\leq \|\tilde{H}_n T_{\omega_n} (h_{n+1} T_{\eta_{n+1}} h_{n+1}^{-1}) - \tilde{H}_n T_{\omega_n}\|_{\rho} \\
&\leq \left( \sup_{z \in B_{\rho_n} + \rho'_n} \|D\tilde{H}_n(z)\| \right) \cdot \|\tilde{h}_{n+1} T_{\eta_{n+1}} \tilde{h}_{n+1}^{-1} - \text{Id}_z\|_{\rho} \\
&\leq 2^{-n} \epsilon_n.
\end{align*}
Similarly, we have
\[
d_\rho(f_{n+1}^{-1}, f_n^{-1}) = \|\tilde{H}(T_n h_{n+1} T_{n+1}^{-1} h_n^{-1})\tilde{H}_n^{-1} - \tilde{H}_n T_n h_n^{-1}\|_\rho \\
\leq \left( \sup \|D\tilde{H}_n(z)\| \right) \cdot \|\tilde{h}_{n+1} T_{n+1}^{-1} h_n^{-1} - 1\|_{C^2} \|_{\rho_n} \\
\leq 2^{-n}\epsilon_n.
\]

Therefore, by definition of \(d_\rho\) and \(\rho'\), the item (2) follows from (4.12) and the item (1). Finally, (3) is obvious from the proof of (2). 

By Lemma 4, we verify \((a5)_{n+1}\) by taking \(r_{n+1} > Q\) and setting \(q_{n+1} = q_n r_{n+1}\).

Note that for any \(m = kq_n\) with \(k \in \mathbb{Z}\), we have \(m\omega_{n+1} = k\omega_n + \frac{r_{n+1}}{r_{n+1}}\), and hence
\[
f_{n+1}^m = H_{n+1} T_{\tilde{k}_{r_{n+1}}} H_n^{-1}.
\]

By (4.9), (4.10) and (4.11), it is direct to see that:

1. for any \(\kappa\)-dense subset of \(\mathbb{T}^2\), denoted by \(K\), the set \(H_{n+1}(K)\) is \(2^{-n-1}\)-dense in \(\mathbb{T}^2\);
2. for any \(z \in \mathbb{T}^2\), \(\{(z + m\omega_{n+1}) \mod \mathbb{Z}^2\}_{m \in \mathbb{Z}_N}\) is \(\kappa\)-dense in \(\mathbb{T}^2\).

Thus for any \(z \in \mathbb{T}^2\), the set \(\{f_{n+1}^m(z)\}_{m \in \mathbb{Z}} = \{H_{n+1}(H_n^{-1}(z) + m\omega_{n+1})\}_{m \in \mathbb{Z}}\) is \(2^{-n-1}\)-dense in \(\mathbb{T}^2\). This verifies \((a4)_{n+1}\). Moreover, by (4.6) and the item (2) above, there exists some \(m \in q_n N\) such that \(f_{n+1}^m = G_n^\gamma\) for certain \(\gamma \in \mathbb{R}^2\) with \(\|\gamma\| < \kappa\). Then by (4.8), we verify the second inequality in \((a3)_{n+1}\).

The above discussions show that, by choosing \(r_{n+1}\) sufficiently large, we can ensure that \((h_{n+1}, \omega_{n+1}, q_{n+1}, \omega_{n+1})\) satisfies \((a1)_{n+1} - (a4)_{n+1}\) and \((a5)_{n}\), and thus complete the induction. For each \(1 \leq i \leq 5\), let us denote by \((a)\) the collection of induction hypotheses \((a1), (a2), \ldots\).

Let the sequence \(\{f_n\}_{n \geq 1}\) be constructed by the above induction scheme. By \((a5)\), \(\{f_n\}_{n \geq 1}\) converges to some \(f \in \text{Diff}_n^s(\mathbb{T}^2, \lambda)\) under the \(d_\rho\)-metric; \(\{\omega_n\}_{n \geq 1}\) converges to some \(\omega \in \mathbb{R}^2\); and \(\{F_n\}_{n \geq 1}\) converges to some \(F \in \text{Homeo}(\mathbb{R}^2)\), which is clearly a lift of \(f\). Moreover for any integer \(n \geq 1\), we have \(d_\rho(f_n, f), d_{C^0}(F_n, F), \|\omega - \omega_n\| < \epsilon_n\). Consequently, (4.2) to (4.4) holds for \((f, \omega, F)\) and every \(n \geq 1\). By (4.3) and (a3), for any \(\epsilon > 0\), there exist \(x, y \in \mathbb{T}^2\) satisfying \(d(x, y) < \epsilon\), and an integer \(m > 0\), such that \(d(f^m(x), f^m(y)) \geq \frac{1}{100\epsilon}\). By (4.4) and (4.2), \(f\) is minimal and of bounded mean motion, thus \(f\) must be a totally irrational pseudo-rotation. This concludes the proof.

By Theorem 2, the map \(f\) we constructed above is a real-analytic, homotopic to the identity and semi-conjugate to a minimal translation \(T_n\). It is not conjugate to any minimal translation because for any \(\epsilon > 0\), there exist two points \(x, y \in \mathbb{T}^2\) with \(d(x, y) < \epsilon\), and an integer \(N > 0\) such that \(d(f^N(x), f^N(y)) \geq \frac{1}{100\epsilon}\). Finally, Theorem 1 follows from a similar argument in the proof of [WZ18, Theorem 4]. We omit it.

Remark 3. It is not difficult to check that the rotation vector of the constructed map \(f\) above satisfies the super-Liouvilian condition (see [WZ18, Theorem 2] for the definition) by our choice of \(q_n\) (see Remark 2 and Lemma 4). Based on [WZ18, Theorem 2], the map \(f\) is \(C^\infty\)-rigid.
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