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Parallel Planar Subgraph Isomorphism and Vertex Connectivity
Subgraph Isomorphism:
Find subgraphs in the target that **match the pattern**
Subgraph Isomorphism:
Find subgraphs in the target that match the pattern

Target $G$ with $n$ vertices

Pattern $H$ with $k$ vertices

2 occurrences
Subgraph Isomorphism:
Find subgraphs in the target that match the pattern

Target $G$ (n vertices)

Planar target: NP-Hard

Pattern $H$ (k vertices)

2 occurrences
Subgraph Isomorphism:
Find subgraphs in the target that match the pattern

Target $G$
$n$ vertices

Planar target: NP-Hard

Pattern $H$
k vertices

Focus on small patterns

2 occurrences
**Results for Planar Graphs**

| Work          | Depth         |
|---------------|---------------|
| $\Omega \left( n^{\sqrt{k}} \right)$ | $O\left( \log^2 n \right)$ |

- **Color Coding**
  - Alon et al. 1995

- **Subgraph Isomorphism**

- **Target $G**

- **Pattern $H**

- **$n$ vertices**

- **$k$ vertices**

- Result correct with high probability
Subgraph Isomorphism

Target $G$

$n$ vertices

Pattern $H$

$k$ vertices

Results for Planar Graphs

| Work                  | Depth                |
|-----------------------|----------------------|
| Color Coding*         | $\Omega \left( n^{\sqrt{k}} \right)$ | $O(\log^2 n)$ |
| Alon et al. 1995      |                      |               |
| Eppstein 1995         | $O(k^{3k+1}n)$       | $\Omega(n)$  |
| Result correct with high probability |
### Results for Planar Graphs

| Work               | Depth              |
|--------------------|--------------------|
| **Color Coding**   | $\Omega \left( n^{\sqrt{k}} \right)$ | $O\left( \log^2 n \right)$ |
| Alon et al. 1995   |                    |                           |
| **Eppstein**       | $O\left( k^{3k+1} n \right)$ | $\Omega \left( n \right)$ |
| 1995               |                    |                           |
| **Our Result**     | $O\left( k^{3k+1} n \log n \right)$ | $O\left( k \log n \right)$ |
| Our Result*        |                    |                           |

*Result correct with high probability*
Dynamic Programming

\[ G \]
Dynamic Programming

"Shared Vertices" divide the graph
Dynamic Programming

Solve all subproblems for both parts

Pattern H

Partial Solution 1

Partial Solution 2

Dynamic Programming

Solve all subproblems for both parts

Pattern H

Partial Solution 1

Partial Solution 2
Dynamic Programming

Combine compatible partial solutions

Pattern $H$

Partial Solution 1

Partial Solution 2
Dynamic Programming

Exponential in “shared” part

General $\Omega(n)$
Dynamic Programming

Exponential in “shared” part

General $\Omega(n)$

Planar $\Theta(\sqrt{n})$
Dynamic Programming

Exponential in “shared” part

General $\Omega(n)$

Planar $\Theta(\sqrt{n})$

Planar, diameter $d$ $O(d)$
Dynamic Programming

Exponential in “shared” part

General \( \Omega(n) \)

Planar \( \Theta(\sqrt{n}) \)

Planar, diameter \( d \) \( O(d) \)

Check diameter \( k-1 \) subgraphs
Naïve Covering

$G'$
Naïve Covering

\[ G' \]
Naïve Covering

$G'$
Naïve Covering

$G'$
Naïve Covering
Naïve Covering

\[ G' \]

\[ \Theta(n^2) \text{ work} \]
Work-Efficient Covering with BFS

BFS Tree
Work-Efficient Covering with BFS

BFS Tree

$V$

$G'$

$G_0$
**Work-Efficient Covering with BFS**

BFS Tree

$G'$

$G_0$

$G_1$
Work-Efficient Covering with BFS

BFS Tree

$G'$

$G_0$

$G_1$

$G_2$
Work-Efficient Covering with BFS

BFS Tree

$G'$

$G_0$

$G_1$

$G_2$

$O(kn)$ work
**Work-Efficient Covering with BFS**

**Problem:** $\Omega(n)$ depth

**BFS Tree**

$G'$

$G_0$

$G_1$

$G_2$

$O(kn)$ work
Low-Diameter Decomposition
Miller et al. 2015

Target $G$

$n$ vertices
Low-Diameter Decomposition
Miller et al. 2015

Cluster Diameter $O(k \log n)$

$n$ vertices

Target $G$
Low-Diameter Decomposition
Miller et al. 2015

Cluster Diameter $O(k \log n)$

$n$ vertices

Target $G$

Probability a particular edge crosses $\leq \frac{1}{2k}$
Low-Diameter Decomposition
Miller et al. 2015

Target $G$

Cluster Diameter $O(k \log n)$

$n$ vertices

Probability a particular edge crosses $\leq \frac{1}{2k}$

Probability an occurrence crosses $\leq \frac{1}{2}$

Pattern $H$

$n$ vertices

$C_{\text{cluster diameter}} \leq O(k \log n)$
Low-Diameter Decomposition
Miller et al. 2015

Cluster Diameter \( O(k \log n) \)

\( n \) vertices

Target \( G \)

\( k \) vertices

Pattern \( H \)

Probability a particular edge crosses \( \leq \frac{1}{2k} \)

Probability an occurrence crosses \( \leq \frac{1}{2} \)
Low Diameter Decomposition

- **$O(n)$ work**
- **$O(k \log n)$ depth**
- **$O(k \log n)$ work**
- **$O(k \log n)$ depth**
- **$O(k^3k+1n)$ work**
- **$O(k \log n)$ depth**

Planar Subgraph Isomorphism

- **$O(k \log n)$ repetitions**
Subgraph Isomorphism

$G$

Pattern $H$

Minimum Vertex Cut

$G$
Minimum Vertex Cut
Smallest number of vertices whose removal disconnects the graph
Minimum Vertex Cut

$G$

$G'$

Face Vertices

Original Vertices
Face Vertices

Original Vertices

Minimum Vertex Cut

Separating Cycle

$G$

$G'$
Minimum Vertex Cut

Separating Cycle

\[ G \]
\[ G' \]

Constant Length

\[ O(n \log n) \] work

\[ O(\log n) \] depth

Face Vertices

Original Vertices
**Conclusion**

**Subgraph Isomorphism**

- Target $G$ with $n$ vertices
- Pattern $H$ with $k$ vertices
- $O(k^{3k+1}n \log n)$ work
- $O(k \log n)$ depth

**Minimum Vertex Cut**

- $O(n \log n)$ work
- $O(\log n)$ depth

### Examples

- Example of a subgraph isomorphism problem
- Example of a minimum vertex cut problem
**Conclusion**

**Subgraph Isomorphism**

Target $G$

- $n$ vertices

- $k$ vertices

Pattern $H$

$O(k^{3k+1}n \log n)$ work

$O(k \log n)$ depth

Singly exponential in $k$?

Polylog in $k$?

**Minimum Vertex Cut**

- $O(n \log n)$ work

- $O(\log n)$ depth

Linear in $n$?
**Subgraph Isomorphism**

- Target $G$ with $n$ vertices
- Pattern $H$ with $k$ vertices
- Time complexity: $O(k^{3k+1}n \log n)$ work
- Depth: $O(k \log n)$

**Minimum Vertex Cut**

- Minimum Vertex Cut
- Separating Cycle
- Time complexity: $O(n \log n)$ work
- Depth: $O(\log n)$

**Conclusion**

- Singly exponential in $k$?
- Polylog in $k$?
- Linear in $n$?
- Other Implications?