Modal Dependence Logics: Axiomatizations and Model-theoretic Properties

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Abstract  Modal dependence logics are modal logics defined on the basis of team semantics and have the downward closure property. In this paper, we introduce sound and complete deduction systems for the major modal dependence logics, especially those with intuitionistic connectives in their languages. We also establish a concrete connection between team semantics and single-world semantics, and show that modal dependence logics can be interpreted as variants of intuitionistic modal logics.

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Dependence logic is a logical formalism, introduced by Väänänen [28], that captures the notion of dependence in social and natural sciences. The modal version of the logic is called modal dependence logic and was introduced in [29]. Modal dependence logic extends the usual modal logic by adding a new type of atomic formulas $\equiv(p_1, \ldots, p_n, q)$, called dependence atoms, to express dependencies between propositions, and by lifting the usual single-world semantics to the so-called team semantics, introduced by Hodges [13, 14]. Formulas of modal dependence logic are evaluated on sets of possible worlds of Kripke models, called teams. Intuitively, a dependence atom $\equiv(p_1, \ldots, p_n, q)$ is true if within a team the truth value of the proposition $q$ is functionally determined by the truth values of the propositions $p_1, \ldots, p_n$.

Research on modal dependence logic and its variants has been active in recent years. Basic model-theoretic properties of the logics were studied in e.g., [26], a van Benthem Theorem for the logics was proved in [16], and the frame definability of the logics was studied in [24, 25]. The expressive power and the relevant computation complexity problems of the logics were investigated extensively in e.g., [7, 8, 9, 12, 19, 20, 26]. In this paper, we study two problems that received less attention in the literature, namely the axiomatization problem and the comparison between team semantics and the single-world semantics.
For the axiomatization problem, Sano and Virtema gave in [23] Hilbert-style systems and label tableau calculi for modal dependence logic and its extended version, and Hannula defined in [11] natural deduction systems for the same logics. However, these axiomatizations did not cover the modal dependence logics with intuitionistic connectives of team semantics, especially with intuitionistic implication. The intuitionistic connectives of team semantics are crucial connectives of inquisitive logic [6], a closely related logic to dependence logic that adopts (independently) team semantics too (see e.g. [4, 33] for further discussions on the connection). While inquisitive modal logic has been axiomatized in [5], the logic has different modalities and slightly different Kripke models than those of modal dependence logics. In this paper, we define Hilbert and natural deduction systems for modal dependence logic extended with intuitionistic disjunction and implication, called full modal downward closed team logic (MT₀). These systems are extensions of the systems of Fischer Servi’s intuitionistic modal logic (IK) [10] and inquisitive (propositional) logic InqL [6]. We also introduce deduction systems for modal intuitionistic dependence logic, modal dependence logic with intuitionistic disjunction, and (extended) modal dependence logic (MD) as fragments or variants of the system of MT₀. We adopt the rules for (extended) dependence atoms introduced in [33]. These rules are simpler than those in the systems of [11, 23]. We also point out that for the logic MD, which does not have implication in its language, the deduction system that enjoys (weak) completeness can have less rules than the system that enjoys strong completeness. This interesting difference between the validity problem (i.e., determining whether \( \models \phi \)) and the entailment problem (i.e., determining whether \( \phi \models \psi \)) in MD was noted also in [11].

Our axiomatizations make use of the disjunctive normal form of modal dependence logics, which is essentially known in the literature. We apply this normal form to prove a characterization theorem for flat formulas, the Interpolation Theorem and the Finite Model Property of modal dependence logics.

For the second topic of this paper, it is well-known that the team-based first-order dependence logic can be translated into the single-assignment-based existential second-order logic [28, 17]. In a similar fashion, we show in this paper that the team-based modal dependence logics can be interpreted as certain single-world-based intermediate modal logics. We first provide a rigorous proof for a seemingly folklore observation in the field that clarifies the natural connection between team semantics and single-world semantics in the modal case, namely, the team semantics of modal dependence logics over a (classical) modal Kripke model \( M \) coincides with the usual single-world semantics over an intuitionistic Kripke model whose domain consists of all teams of \( M \) (i.e., the domain is the powerset of \( M \)) and whose partial order is the superset relation between teams. The tensor (disjunction) connective of team semantics will be interpreted in this setting as a binary diamond modality under the single-world semantics, the idea of which is developed from [1], where tensor is understood as a multiplicative conjunction.

On the basis of the powerset models, we establish a comparison between modal dependence logics and familiar single-world-based non-classical logics, especially intuitionistic modal logic and intermediate logics. We show that modal dependence logics are complete (in the usual single-world semantics sense) with respect to a class of bi-relation or tri-relation intuitionistic Kripke models that generalise the powerset models. The bi-relation models are special bi-relation intuitionistic Kripke models of
Fischer Servi’s intuitionistic modal logic $\mathbf{IK}$, and the tri-relation intuitionistic Kripke models are endowed with an extra ternary relation interpreting the binary diamond that corresponds to the tensor. Our results generalise the results in [3] that inquisitive logic $\text{InqL}$ can be viewed as a variant of the Kreisel-Putnam intermediate logic ($\mathbf{KP}$) [18], and $\text{InqL}$ is complete (in the usual single-world semantics sense) with respect to the class of negative intuitionistic Kripke models of $\mathbf{KP}$.

This paper is structured as follows. Section 1 recalls the basics of modal dependence logics. In particular, we sketch the standard translation from modal dependence logics into first-order dependence logics and derive the Compactness Theorem for modal dependence logics without intuitionistic implication as a corollary. In Section 2 we study the axiomatization problem for modal dependence logics, and also prove a few metalogical properties of the logics, including the Interpolation Theorem and the Finite Model Property. Section 3 provides single-world semantics interpretation of modal dependence logics. In Section 4 we make concluding remarks.

Preliminary results of this paper were included in the author’s dissertation [32].

1 Preliminaries

In this section, we recall the basics of modal dependence logics, which are modal logics defined on the basis of team semantics.

Though team semantics is intended for the extension of (classical) modal logic obtained by adding dependence atoms, for the sake of comparison, we start by defining the team semantics and fixing notations for the usual (classical) modal logic. Fix a set $\text{Prop}$ of propositional variables and denote its elements by $p, q, r, \ldots$ (possibly with subscripts). Formulas of (classical) modal logic, also called classical (modal) formulas, are defined recursively as:

$$\alpha ::= p \mid \bot \mid \neg\alpha \mid \alpha \land \alpha \mid \alpha \to \alpha \mid \square\alpha \mid \Diamond\alpha$$

where $\otimes$ (called tensor) denotes the disjunction of classical modal logic, and the implication $\to$ is called intuitionistic implication for reasons that will become clear in the sequel.

A (modal) Kripke frame is a couple $\mathfrak{F} = (W, R)$ consisting of a nonempty set $W$ and a binary relation $R \subseteq W \times W$. Elements of $W$ are called possible worlds or nodes or points. A (modal) Kripke model is a triple $\mathfrak{M} = (W, R, V)$ such that $(W, R)$ is a Kripke frame and $V : \text{Prop} \to \mathcal{V}(W)$ is a valuation function. A set $X \subseteq W$ of possible worlds is called a team. For any team $X$, define $R(X) = \{w \in W \mid \exists v \in X \text{ s.t. } vRw\}$ and write $R(w)$ for $R(\{w\})$. A team $Y$ is called a successor team of $X$, written $XRY$, if $Y \subseteq R(X)$ and $Y \cap R(w) \neq \emptyset$ for every $w \in X$.

**Definition 1.1** We define inductively the notion of a classical modal formula $\alpha$ being satisfied in a Kripke model $\mathfrak{M} = (W, R, V)$ on a team $X \subseteq W$, denoted $\mathfrak{M}, X \models \alpha$, as follows:

- $\mathfrak{M}, X \models p$ iff $X \subseteq V(p)$
- $\mathfrak{M}, X \models \bot$ iff $X = \emptyset$
- $\mathfrak{M}, X \models \neg\phi$ iff $\mathfrak{M}, \{w\} \not\models \phi$ for all $w \in X$.
  In particular, $\mathfrak{M}, X \models \neg p$ iff $X \cap V(p) = \emptyset$
- $\mathfrak{M}, X \models \phi \land \psi$ iff $\mathfrak{M}, X \models \phi$ and $\mathfrak{M}, X \models \psi$
- $\mathfrak{M}, X \models \phi \to \psi$ iff there exist $Y, Z \subseteq X$ such that $X = Y \cup Z$, $\mathfrak{M}, Y \models \phi$ and $\mathfrak{M}, Z \models \psi$
- $\mathfrak{M}, X \models \phi \to \psi$ iff for all $Y \subseteq X$, $\mathfrak{M}, Y \models \phi$ implies $\mathfrak{M}, Y \models \psi$
For any Kripke model $\mathcal{M} = (W,R,V)$, if $\mathcal{M}, X \models \phi$ holds for all $X \subseteq W$, then we say that $\phi$ is true on $\mathcal{M}$ and write $\mathcal{M} \models \phi$. For any Kripke frame $\mathcal{F}$, if $(\mathcal{F}, V) \models \phi$ holds for all valuations $V$ on $\mathcal{F}$, then we say that $\phi$ is valid on $\mathcal{F}$ and write $\mathcal{F} \models \phi$. If $\mathcal{F} \models \phi$ holds for all frames $\mathcal{F}$, then we say that $\phi$ is valid and write $\models \phi$. We write $\Gamma \models \phi$ if for all Kripke models $\mathcal{M}$ and all teams $X$, $\mathcal{M}, X \models \gamma$ for all $\gamma \in \Gamma$ implies $\mathcal{M}, X \models \phi$. We write simply $\models \psi$ for $\{\phi\} \models \psi$, and write $\models \psi$ if $\phi \models \psi$ and $\psi \models \phi$.

It is easy to check that classical modal formulas $\alpha$ satisfy the flatness property:

**Flatness Property:** $\mathcal{M}, X \models \alpha \iff \mathcal{M}, \{w\} \models \alpha$ for all $w \in X$

$\iff \mathcal{M}, w \models \alpha$ in the usual sense for all $w \in X$

As a consequence, a few usual equivalences, such as the following ones, hold for classical formulas:

$\neg \alpha \equiv \alpha \rightarrow \bot$, $\alpha \otimes \beta \equiv \neg \alpha \rightarrow \beta$ and $\Diamond \alpha \equiv \neg \Box \neg \alpha$.

Recall that the Hilbert-style system of classical modal logic $K$ consists of the following axioms and rules:

**Axioms:**
1. all axioms of classical propositional logic
2. $\Box (\alpha \rightarrow \beta) \rightarrow (\Box \alpha \rightarrow \Box \beta)$
3. $\Diamond \alpha \leftrightarrow \neg \Box \neg \alpha$

**Rules:**
1. Modus Ponens: $\alpha, \alpha \rightarrow \beta \vdash \beta$
2. Necessitation: $\alpha / \Box \alpha$
3. Uniform Substitution: $\alpha / \alpha(\beta/p)$

For classical formulas, the system of $K$ is sound and complete with respect to all Kripke frames in the sense of team semantics too. To see why, by the Completeness Theorem of $K$ with respect to the usual single-world semantics and the flatness property of classical formulas, we have

$\vdash_K \alpha \iff \mathcal{M}, w \models \alpha$ for any model $\mathcal{M}$ and any possible world $w$ in $\mathcal{M}$

$\iff \mathcal{M}, \{w\} \models \alpha$ for any model $\mathcal{M}$ and any possible world $w$ in $\mathcal{M}$

$\iff \mathcal{M}, X \models \alpha$ for any model $\mathcal{M}$ and any team $X$ of $\mathcal{M}$

$\iff \models \alpha$.

As a consequence, we also have

$\vdash_K \alpha \rightarrow \beta \iff \vdash \alpha \rightarrow \beta \iff \alpha \models \beta$. \hspace{1cm} (1)

In view of the flatness property, lifting the single-possible world semantics to team semantics does not add essentially new features to classical modal formulas. Let us now extend classical modal logic by adding the *dependence atoms* $\equiv (\alpha_1, \ldots, \alpha_n, \beta)$ and *intuitionistic disjunction* $\lor$ that violate the flatness property. We call the resulting logic **full modal downward closed team logic** ($\text{MT}_\phi$) and its language is defined formally as follows:

$\phi ::= \alpha \mid \equiv (\alpha_1, \ldots, \alpha_n, \beta) \mid \phi \land \phi \mid \phi \otimes \phi \mid \phi \lor \phi \mid \phi \rightarrow \phi \mid \Box \phi \mid \Diamond \phi,$

where $\alpha, \alpha_1, \ldots, \alpha_n, \beta$ are classical modal formulas. We write $\neg \phi$ for $\phi \rightarrow \bot$. 

Definition 1.2 The satisfaction relation for MT₀-formulas is defined as for classical modal formulas and additionally:

- \( M, X \models \phi \land \psi \) iff for any \( w, u \in X \), \( M, w \models \phi \land M, u \models \psi \) for all \( 1 \leq i \leq n \) implies \( M, w \models \beta \equiv M, u \models \beta \).
- \( M, X \models \phi \lor \psi \) iff \( M, X \models \phi \) or \( M, X \models \psi \).

Immediately from the semantics it follows that MT₀-formulas have the downward closure property and the empty team property defined below:

**Downward Closure Property:** \[ \forall M,X \subseteq X \Rightarrow M,Y \models \phi \]

**Empty Team Property:** \[ \forall M, \emptyset \models \phi \]

We have discussed that the classical fragment of MT₀ (i.e., the fragment consisting of classical formulas only) behaves exactly as classical modal logic. MT₀ also inherits from classical modal logic many other nice properties, such as the preservation property under taking disjoint unions. For any two Kripke models \( M = (W, R, V) \) and \( M' = (W', R', V') \), their disjoint union \( M \uplus M' = (W_0, R_0, V_0) \) is defined as

\[ W_0 = W \uplus W', \quad R_0 = R \uplus R' \quad \text{and} \quad V_0(p) = V(p) \uplus V'(p) \quad \text{for all} \quad p \in \text{Prop}, \]

where \( \uplus \) takes the disjoint union of two sets. It can be proved by a routine argument that for any collection \( \{M_i = (W_i, R_i, V_i) \mid i \in I\} \) of Kripke models, for every \( i \in I \) and every \( X \subseteq W_i \),

\[ M_i, X \models \phi \iff \bigcup_{j \in I} M_j, X \models \phi. \]

From this it follows that that MT₀ has the disjunction property with respect to the intuitionistic disjunction \( \lor \), as shown below.

**Theorem 1.3 (Disjunction Property)** If \( \models \phi \lor \psi \), then \( \models \phi \) or \( \models \psi \).

**Proof** Suppose \( M_0, X \not\models \phi \) and \( M_1, Y \not\models \psi \) for some models \( M_0 \) and \( M_1 \) and teams \( X \) and \( Y \). Let \( M = M_0 \uplus M_1 \) and \( Z = X \uplus Y \). By (2), we have \( M, X \not\models \phi \) and \( M, Y \not\models \psi \). Hence, by the downward closure property, \( M, Z \not\models \phi \) and \( M, Z \not\models \psi \), implying \( M, Z \not\models \phi \lor \psi \).

In this paper, we also study some interesting fragments of MT₀ (referred to as modal dependence logics) defined by restricting the language as follows:

- **The language of modal dependence logic (MD):**
  \[ \phi ::= \alpha \mid (p_1, \ldots, p_n, q) \mid \phi \land \phi \mid \phi \lor \phi \mid \Box \phi \mid \Diamond \phi \]
  where \( \alpha \) is an arbitrary classical formula defined recursively as
  \[ \alpha ::= p \mid \bot \mid \neg \alpha \mid \alpha \land \alpha \mid \alpha \lor \alpha \mid \Box \alpha \mid \Diamond \alpha \]

- **The language of extended modal dependence logic (MD⁺):**
  \[ \phi ::= \alpha \mid (\alpha_1, \ldots, \alpha_n, \beta) \mid \phi \land \phi \mid \phi \lor \phi \mid \Box \phi \mid \Diamond \phi \]

- **The language of modal dependence logic with intuitionistic disjunction (MD⁺):**
  \[ \phi ::= \alpha \mid (\alpha_1, \ldots, \alpha_n, \beta) \mid \phi \land \phi \mid \phi \lor \phi \mid \Box \phi \mid \Diamond \phi \]

where \( \alpha, \alpha_1, \ldots, \alpha_n, \beta \) are classical formulas defined as in the case of MD.
The language of modal intuitionistic dependence logic (MID):

$$\phi ::= \alpha \mid \bot \mid (\alpha_1, \ldots, \alpha_n, \beta) \mid \phi \land \phi \mid \phi \lor \phi \mid \phi \rightarrow \phi \mid \Box \phi \mid \Diamond \phi$$

where $\alpha, \alpha_1, \ldots, \alpha_n, \beta$ are classical formulas defined recursively as

$$\alpha ::= p \mid \bot \mid \alpha \land \alpha \mid \alpha \rightarrow \alpha \mid \Box \alpha \mid \Diamond \alpha$$

Note that negation is taken to be a defined connective, i.e., $\neg \phi ::= \phi \rightarrow \bot$, in the modal dependence logics that have implication in their languages (such as $\text{MT}_0$ and MID), while in the other logics (i.e., $\text{MD}$, $\text{MD}^\vee$, $\text{MD}^\Diamond$, etc.) negation is a primitive connective that applies to classical formulas only.

We leave it for the reader to check that in the presence of intuitionistic connectives dependence atoms are definable, as

$$(\alpha_1, \ldots, \alpha_k, \beta) \equiv ((\alpha_1 \lor \neg \alpha_1) \land \cdots \land (\alpha_k \lor \neg \alpha_k) \rightarrow (\beta \lor \neg \beta))$$

where $2 = \{0, 1\}$, $\gamma^1 = \gamma$ and $\gamma^0 = \neg \gamma$. The modality-free fragments of $\text{MD}$, $\text{MD}^\vee$ and MID are called propositional dependence logic, propositional dependence logic with intuitionistic disjunction and propositional intuitionistic dependence logic, respectively. These propositional logics were studied in [33]. The modality and dependence atom-free fragment of MID is in fact inquisitive (propositional) logic, which was introduced by Ciardelli and Roelofsen in [6] and commented also in the context of dependence logic in [4, 33, 32]. Ciardelli studied and axiomatized in [5] various inquisitive modal logic obtained from inquisitive propositional logic by adding different modalities than the $\Box$ and $\Diamond$ modalities we consider here in this paper.

The language of $\text{MD}^+$ differs from that of $\text{MD}$ only in that the dependence atoms of the latter have only propositional arguments. In the literature, the terminology dependence atoms is often used for dependence atoms $=(p_1, \ldots, p_n, q)$ with propositional arguments only, while dependence atoms $=(\alpha_1, \ldots, \alpha_n, \beta)$ with classical arguments are often referred to as extended dependence atoms. It is proved in [7, 12] that $\text{MD}$ is strictly less expressive than $\text{MD}^+$, and the latter has the same expressive power as $\text{MD}^\Diamond$.

An interesting feature of modal dependence logics is that they are not closed under Uniform Substitution. For instance, $p \vdash \neg \neg p \rightarrow p$ and $p \vdash \neg \neg p$, whereas $p \not\vdash \neg (p \lor \neg p) \rightarrow (p \lor \neg p)$ and $p \not\vdash (p \lor \neg p) \lor (p \lor \neg p)$. For this reason, none of the deduction systems of modal dependence logics to be introduced in this paper admits the uniform substitution rule. In fact, for these logics, substitution (being a mapping from the set of well-formed formulas to the set itself that commutes with the atoms, connectives and modalities) is not even a well-defined notion, because, for instance, dependence atoms $=(\phi_1, \ldots, \phi_n, \psi)$ with arbitrary arguments are not necessarily well-formed formulas of the logics. For more details on substitution in dependence logics, we refer the reader to [5, 15].

The well-known standard translation from the usual (single-world-based) modal logic into first-order logic provides interesting insights into the usual modal logic. In particular, the Compactness Theorem of the usual modal logic is an immediate consequence of the translation. Without going into further details we point out that a similar translation from modal dependence logics into first-order dependence logics can be defined as follows:
is known to have the same expressive power as full second-order logic of first-order dependence logic extended with intuitionistic implication, which

\[ \text{where} \]

\[ \text{interpreting the unary predicate symbols} \]

\[ \text{and the system of Fischer Servi's intuitionistic modal logic} \]

\[ \text{without intuitionistic implication as a corollary of the standard translation.} \]

We introduce in Section 2.1 Hilbert-style and natural deduction systems for the full logic \( \text{MT}_0 \) and \( \text{MD} \). For the other logics introduced in the previous section that do not have implication in their languages, namely, \( \text{MD}^+, \text{MD} \) and \( \text{MD}^+ \), we only define natural deduction systems (which, unlike Hilbert-style systems, do not necessarily use implications in the presentation). In Section 2.2, we show that the
implication-free fragment of the natural deduction system of $\text{MT}_0$ together with some additional rules for negation and modalities form a complete system for $\text{MD}^\lor$. The proofs of the completeness theorems in Sections 2.1 and 2.2 make heavy use of a disjunctive normal form for these logics which can essentially be found in the literature. In Section 2.3, by using the normal form we prove a characterization theorem for flat formulas, the Interpolation Theorem and the Finite Model Property of modal dependence logics. In Section 2.4, we introduce the systems of $\text{MD}$ and $\text{MD}^+$ as fragments of the system of $\text{MT}_0$ together with some additional rules for dependence atoms.

Note that for a logic that has an implication $\to$ in its language, if the Deduction Theorem (i.e., $\Gamma, \phi \models \psi \iff \Gamma \models \phi \to \psi$) holds for this implication, the entailment problem and the validity problem can be reduced to each other, because

$$\phi \models \psi \iff \vdash \phi \to \psi,$$

and as a consequence, assuming the compactness of the logic, the strong completeness (i.e., $\models \phi \iff \vdash \phi$) of a deduction system of $L$ is equivalent to its weak completeness (i.e., $\models \phi \iff \vdash \phi$). However, this is not in general true for logics without an implication in their languages. To address this subtle point, we will present the Completeness Theorems as

$$\phi \models \psi \iff \vdash \phi \to \psi,$$

especially for the implication-free logics $\text{MD}^\lor$, $\text{MD}$ and $\text{MD}^+$. We will see that for the systems of $\text{MD}$ and $\text{MD}^+$ (which are compact by Theorem 1.4), there is indeed a difference between the strong and the weak completeness: the systems for which the weak completeness holds can have less rules than the ones for which the strong completeness holds. This subtle difference was noted also in [11] in a different system for $\text{MD}^+$.

### 2.1 $\text{MT}_0$ and $\text{MID}$

In this subsection, we introduce sound and complete Hilbert-style and natural deduction systems for $\text{MID}$ and $\text{MT}_0$. We first define the Hilbert-style systems and prove the completeness theorems by an argument that makes essential use of the disjunctive normal form of the logics. The natural deduction systems will be defined at the end of the section and their completeness follows from a similar argument.

By Expression (3), dependence atoms are definable in $\text{MID}$. The dependence atom-free fragment of $\text{MID}$ turns out to have inquisitive logic ($\text{InqL}$) [6] as its propositional base. Below we recall the Hilbert system of $\text{InqL}$ defined in [5, 6]. We refer the reader to [4, 33] for further discussion on the connection between inquisitive logic and dependence logics.

**Definition 2.1** The Hilbert-style system of inquisitive logic $\text{InqL}$ is as follows:

**Axioms:**

1. all axiom schemes of intuitionistic propositional logic (IPC), namely
   
   (a) $\phi \to (\psi \to \phi)$
   (b) $(\phi \to (\psi \to \chi)) \to ((\phi \to \psi) \to (\phi \to \chi))$
   (c) $\phi \land \psi \to \phi, \phi \land \psi \to \psi$
   (d) $\phi \to (\chi \to (\phi \land \chi))$
   (e) $\phi \to \phi \lor \psi, \psi \to \phi \lor \psi$
   (f) $(\phi \to \chi) \to ((\psi \to \chi) \to (\phi \lor \psi \to \chi))$
(g) \bot \to \phi
2. \((\alpha \to (\phi \lor \psi)) \to ((\alpha \to \phi) \lor (\alpha \to \psi))\) whenever \(\alpha\) is a classical formula
3. \(\neg \neg \alpha \to \alpha\) whenever \(\alpha\) is a classical formula

Rule: Modus Ponens: \(\phi, \phi \to \psi / \psi\)

In the original presentation of the system of InqL as given in [6], axiom 2 is formulated (equivalently) as any substitution instance of the KP axiom

\[(\neg p \to (q \lor r)) \to (\neg p \to q) \lor (\neg p \to r)\]

of the Kreisel-Putnam intermediate logic KP [18]. The system of InqL is then KP without uniform substitution rule together with the Double Negation Law \(\neg \neg \alpha \to \alpha\) for classical formulas.

Our Hilbert-style systems of MT_0 and MID will be extensions of both the system of InqL and the system of Fischer Servi’s intuitionistic modal logic (IK) [10]. We refer the reader to [27] for further discussion on intuitionistic modal logic, and we only remark that IK has intuitionistic propositional logic IPC as its propositional base and adding the Law of Excluded Middle (i.e., \(\phi \lor \neg \phi\)) or the Double Negation Law (i.e., \(\neg \neg \phi \to \phi\)) to the logic gives rise to classical modal logic. In the literature there are a few (equivalent) variants of the system of Fischer Servi’s intuitionistic modal logic IK. For the convenience of our argument, we use the system defined by Plokin and Stirling [22], which we recall below.

**Definition 2.2** The Hilbert-style system of Fischer Servi’s intuitionistic modal logic IK consists of the following axioms and rules:

**Axioms:**
1. all axioms of IPC
2. \(\Box(\phi \to \psi) \to (\Box \phi \to \Box \psi)\)
3. \(\Box(\phi \to \psi) \to (\Diamond \phi \to \Diamond \psi)\)
4. \(\neg \Diamond \bot\)
5. \(\Diamond (\phi \lor \psi) \to (\Diamond \phi \lor \Diamond \psi)\)
6. \(\Box(\phi \to \psi) \to (\Box \phi \to \Box \psi)\)

**Rules:**
1. Modus Ponens: \(\phi, \phi \to \psi / \psi\)
2. Necessitation: \(\phi / \Box \phi\)
3. Uniform Substitution: \(\phi / \phi(\psi / \psi)\)

Now, we present our Hilbert-style systems of MT_0 and MID.

**Definition 2.3** The Hilbert-style system of Fischer Servi’s intuitionistic modal logic IK consists of the following axioms and rules:

**Axioms:**
1. all axioms of InqL
2. all axiom schemes of IK
3. \(\equiv (\alpha_1, \ldots, \alpha_k, \beta) \leftrightarrow ((\alpha_1 \lor \neg \alpha_1) \land \cdots \land (\alpha_k \lor \neg \alpha_k) \to (\beta \lor \neg \beta))\)
4. \(\phi \to \phi \otimes \psi\)
5. \(\phi \to (\psi \to \phi) \to \phi \otimes (\psi \to \phi)\)
6. \(\phi \to (\psi \to \chi) \to (\phi \otimes \psi \to \phi \otimes \psi \to \chi \otimes \theta)\)
7. \(\phi \otimes \psi \to \psi \otimes \phi\)
8. \(\phi \otimes (\psi \otimes \chi) \to (\phi \otimes \psi) \otimes \chi\)
9. \(\phi \otimes (\psi \lor \chi) \to (\phi \otimes \psi) \lor (\phi \otimes \chi)\)
10. \(\neg \Box \alpha \to \Diamond \neg \alpha\) whenever \(\alpha\) is a classical formula
11. \(\Box(\phi \lor \psi) \to (\Box \phi \lor \Box \psi)\)

**Rules:**
1. Modus Ponens: \(\phi, \phi \to \psi / \psi\)
2. Necessitation: $\phi/\Box \phi$

- The Hilbert-style system of **MID** consists of all of the axioms and rules of the above system of **MT** except the axioms that involve $\otimes$ (i.e., axioms 4-9).

Hereafter within this section, we let L denote either **MT** or **MID**. The following proposition lists some interesting derivable clauses that will play a role in the sequel.

**Proposition 2.4** Let $\phi, \psi$ be L-formulas, and $\alpha, \beta$ classical formulas.

(a) $\phi \otimes (\psi \lor \chi) \vdash_{\text{MT}} (\phi \otimes \psi) \lor (\phi \otimes \chi)$
(b) $\Box (\phi \lor \psi) \vdash_{L} \Box \phi \lor \Box \psi$
(c) $\Diamond (\phi \lor \psi) \vdash_{L} \Diamond \phi \lor \Diamond \psi$
(d) $\neg \neg \alpha \vdash_{L} \alpha$
(e) $\Diamond \alpha \vdash_{L} \neg \Box \neg \alpha$
(f) $\vdash_{\text{MT}} \alpha \otimes \neg \alpha$

**Proof** A routine proof. In particular, in order to derive item (e), one may first derive from the **IK** axioms $\Box(\phi \to \psi) \to (\Box \phi \to \Box \psi)$ and $\neg \Box \Box \bot$ that $\vdash_{\text{IK}} \Box \neg \phi \to \neg \Box \phi$ for arbitrary formulas $\phi$. □

Next, we prove the Soundness Theorem for the systems of **MT** and **MID**.

**Theorem 2.5 (Soundness)** For any L-formulas $\phi$ and $\psi$, $\phi \vdash_{L} \psi \implies \phi \vdash \psi$.

**Proof** It suffices to show that all the axioms of **MT** are valid and all the rules are sound. We only verify the validity of axiom 2 of **InqL** and axiom 10.

Axiom 2 of **InqL**: We prove a slightly more general fact that

$$\theta \to (\phi \lor \psi) \models (\theta \to \phi) \lor (\theta \to \psi)$$

whenever $\theta$ is flat. (5)

Suppose $\mathcal{M}, X \not\models (\theta \to \phi) \lor (\theta \to \psi)$. Then $\mathcal{M}, X \not\models \theta \to \phi$ and $\mathcal{M}, X \not\models \theta \to \psi$. Thus, there exist $Y, Z \subseteq X$ such that

$$\mathcal{M}, Y \models \theta, \quad \mathcal{M}, Z \models \theta, \quad \mathcal{M}, Y \not\models \phi \quad \text{and} \quad \mathcal{M}, Z \not\models \psi.$$  

Since $\theta$ is flat and L has the downward closure property, we have

$$\mathcal{M}, Y \cup Z \models \theta, \quad \mathcal{M}, Y \cup Z \not\models \phi \quad \text{and} \quad \mathcal{M}, Y \cup Z \not\models \psi.$$  

Hence, $\mathcal{M}, X \not\models \theta \to (\phi \lor \psi)$.

Axiom 10: Suppose $\mathcal{M}, X \models \neg \Box \alpha$, where $\mathcal{M} = (W, R, V)$ and $\alpha$ is a classical formula. Then, for any $w \in X$, we have $\mathcal{M}, \{w\} \not\models \Box \alpha$, i.e. $\mathcal{M}, R(w) \not\models \alpha$. Since $\alpha$ is flat, there exists $v \in R(w)$ such that $\mathcal{M}, \{v\} \not\models \alpha$. Define $Y = \{v \in R(X) \mid w \in X\}$. Clearly, $XRY$ and $\mathcal{M}, Y \models \neg \alpha$. Hence, $\mathcal{M}, X \models \Diamond \neg \alpha$. □

To prove the Completeness Theorem for **MT** and **MID**, we will transform every formula into a formula in *disjunctive normal form*:

$$\alpha_{1} \lor \cdots \lor \alpha_{n}$$

where each $\alpha_{i}$ is a classical formula. This normal form is a generalization of a similar normal form for **InqL** defined in [5, 6], and similar disjunctive normal forms for modal dependence logics without intuitionistic implication were discussed in the literature with a slightly different presentation (see e.g., [12, 19]). Let us now introduce our disjunctive normal form for **MT** as a recursively defined translation $\tau(\phi)$ for every formula $\phi$ of **MT**.

**Base case:**

- $\tau(\alpha) = \alpha$ when $\alpha$ is a classical formula
- $\tau((\alpha_{1}, \ldots, \alpha_{k}, \beta)) := \tau((\alpha_{1} \lor \neg \alpha_{1}) \land \cdots \land (\alpha_{k} \lor \neg \alpha_{k}) \to (\beta \lor \neg \beta))$
Induction step:
Assume \( \tau(\psi) = \alpha_1 \lor \cdots \lor \alpha_n \) and \( \tau(\chi) = \beta_1 \lor \cdots \lor \beta_m \), where \( \alpha_i \) and \( \beta_j \) are classical formulas.

- \( \tau(\psi \lor \chi) := \tau(\psi) \lor \tau(\chi) \)
- \( \tau(\psi \land \chi) := \bigwedge_{1 \leq i \leq n, 1 \leq j \leq m} (\alpha_i \land \beta_j) \)
- \( \tau(\psi \land \chi) := \bigwedge_{1 \leq i \leq n, 1 \leq j \leq m} (\alpha_i \lor \beta_j) \)
- \( \tau(\psi \to \chi) = \bigwedge_{1 \leq i \leq n} (\alpha_i \to \beta_{f(i)}) \)
- \( \tau(\diamond \psi) := \bigwedge_{1 \leq i \leq n} (\alpha_i \to \chi) \)
- \( \tau(\Box \psi) := \bigwedge_{1 \leq i \leq n} (\alpha_i \to \chi) \)

The disjunctive normal form for \( \text{MID} \) is defined the same way as above except that \( \text{MID} \) does not have the connective \( \otimes \) in its language. In the next theorem, we show that every formula is provably equivalent to its disjunctive normal form.

**Theorem 2.6 (Normal Form)** For any \( L \)-formula \( \phi \), we have \( \vdash \phi \equiv \tau(\phi) \).

**Proof** We only give the proof for \( \text{MT}_0 \), from which the \( \text{MID} \) case follows.

We proceed by induction on \( \phi \). The base case is trivial. For the induction step, the cases \( \phi = \psi \lor \chi \) and \( \phi = \psi \land \chi \) follow immediately from the induction hypothesis and \( \text{IPC} \) axioms. The cases \( \phi = \Box \psi \) and \( \phi = \Diamond \psi \) follow from Proposition 2.4(b)(c).

If \( \phi = \psi \otimes \chi \), then by the induction hypothesis, we derive in the system of \( \text{MT}_0 \) using axiom 6 and Proposition 2.4(a) that

\[
\psi \otimes \chi \vdash \bigwedge_{i=1}^{n} \alpha_i \otimes \bigwedge_{j=1}^{m} \beta_j = \bigwedge_{i=1}^{n} (\alpha_i \otimes \beta_j).
\]

If \( \phi = \psi \to \chi \), then by the induction hypothesis, we derive in the system of \( \text{MT}_0 \) using axiom 2 of \( \text{InqL} \) and \( \text{IPC} \) axioms that

\[
\psi \to \chi \vdash \bigwedge_{i=1}^{n} \alpha_i \to \bigwedge_{j=1}^{m} \beta_j = \bigwedge_{i=1}^{n} (\alpha_i \to \beta_j) = \bigwedge_{i=1}^{n} \bigwedge_{j=1}^{m} (\alpha_i \to \beta_j).
\]

Each disjunct \( \alpha_i \) in the disjunctive normal form \( \bigvee_{i \in I} \alpha_i \) is a classical formula. We now show that \( \text{MT}_0 \) and \( \text{MID} \) derive the same entailment relation as \( K \) does.

**Lemma 2.7** If \( \alpha \) and \( \beta \) are classical formulas, then \( \alpha \vdash_K \beta \iff \alpha \vdash_L \beta \).

**Proof** For the direction “\( \vdash \)”, suppose \( \alpha \vdash_L \beta \). By the Soundness Theorem, we have \( \alpha \models \beta \), which by Expression (1) from Section 1 implies \( \alpha \vdash_K \beta \).

For the direction “\( \models \)”, by Proposition 2.4(e) and by inspecting the axioms and rules, we see easily that restricted to classical formulas, the systems of \( \text{MT}_0 \) and \( \text{MID} \) admit all \( K \) rules and axioms, including Uniform Substitution rule, the axiom \( \Diamond \alpha \leftarrow \neg \Box \neg \alpha \), the double negation axiom \( \neg \neg \alpha \to \alpha \) and all classical axioms of disjunction with respect to tensor \( \otimes \). This implies that any classical entailment relation \( \alpha \vdash \beta \) that is derivable in \( K \) is also derivable (by the same derivation) in \( L \).

We are now ready to prove the Completeness Theorem for the two systems.

**Theorem 2.8 (Completeness)** For any \( L \)-formulas \( \phi \) and \( \psi \), \( \phi \vdash \psi \implies \phi \vdash_L \psi \).
Proof Suppose $\phi \vdash \psi$. By Lemma 2.6, $\phi \nvdashL \alpha_1 \lor \cdots \lor \alpha_k$ and $\psi \nvdashL \beta_1 \lor \cdots \lor \beta_m$, for some classical formulas $\alpha_i$ and $\beta_j$. By the Soundness Theorem, we have $\alpha_1 \lor \cdots \lor \alpha_k \vdashL \beta_1 \lor \cdots \lor \beta_m$, which implies $\alpha_i \vdashL \beta_j$ for some $1 \leq j_i \leq m$. We then derive by applying Expression (1) from Section 1 that $\alpha_i \vdashL \beta_j$, which yields $\alpha_i \vdashL \beta_j$ by Lemma 2.7. Hence, $\alpha_1 \lor \cdots \lor \alpha_k \vdashL \beta_1 \lor \cdots \lor \beta_m$, which gives $\phi \vdashL \psi$. ♣

Having proved the Completeness Theorem for the Hilbert-style systems of MT₀ and MID, we now present also natural deduction systems of the logics.

Definition 2.9

- The natural deduction system of MT₀ is defined as follows:

  **Axiom:** $\neg \Diamond \bot$

  **Rules:** All rules in Table 1.

- The natural deduction system MID consists of all the axioms and rules of the system of MT₀ except for the rules that involve $\otimes$, i.e., the rules in Table 1(d) and the first distributive rule in Table 1(e).

The rules for the propositional base of MT₀ and MID are adapted from those introduced for PD in [33] and for QD in [4], and the rules for modalities are obvious translations of the axioms for the logics in the Hilbert-style systems of Definition 2.3. The rule $\Box Mon$ with empty assumption corresponds to the Hilbert-style Necessitation Rule. It is easy to verify that the systems defined in Definition 2.9 are sound. The completeness of the systems can be proved by a similar argument (via the disjunctive normal form) to what we presented above. We will not provide the proof here. However, in the next section, we will prove that a sound and complete natural deduction system of the logic MD (a fragment of MT₀) can be obtained by dropping the inapplicable rules in Definition 2.9 and adding certain additional rules.

2.2 MD In this section, we introduce a sound and complete natural deduction system for MD, the implication-free fragment of MT₀.

Definition 2.10 The natural deduction system of MD consists of all rules in Table 1(b)-(e), together with the additional rules in Table 2.

The system of MD has all the rules of MT₀ that do not involve implication, together with some additional rules for negation and modalities. The clauses in Proposition 2.4 are derivable easily also in the system of MD. In particular, item (e) ($\Diamond \alpha \nvdashL \neg \Box \neg \alpha$) can be derived without applying the IK axiom $\neg \Diamond \bot$ as follows:

\[
\begin{align*}
&\alpha \quad [\alpha] \\
\frac{\neg \alpha \quad [\neg \alpha]}{\Box \alpha} & \text{\textbf{[7]} \Diamond E} \\
\frac{\Diamond \alpha \quad \Box \alpha}{\neg \neg \alpha} & \text{\textbf{[7]} \Diamond Mon} \\
\frac{\Box \neg \alpha \quad \Diamond \alpha}{\neg \neg \alpha} & \text{\textbf{[7]} \Diamond Inter} \\
\frac{\neg \neg \alpha}{\neg \neg \Diamond \bot} & \neg \neg \Diamond \bot \\
\frac{\neg \Diamond \bot}{\Diamond \bot} & \Box \neg \bot \\
\end{align*}
\]

Note that we did not include the IK axiom $\neg \Diamond \bot$ in our system of MD, because this classical formula is derivable in the system:

\[
\begin{align*}
&\bot \\
\frac{\neg \bot}{\Diamond \bot} & \Box \neg \bot \\
\frac{\neg \Diamond \bot}{\neg \Diamond \bot} & \text{Proposition 2.4(d)} \\
\frac{\neg \Diamond \bot}{\Diamond \bot} & \text{Proposition 2.4(e)} \\
\end{align*}
\]
Table 1 Rules of the system of $\text{MT}_0$. Hereafter in the tables $\alpha, \beta, \alpha_1, \ldots, \alpha_k \ldots$ range over classical formulas, and $\alpha_i^1 := \alpha_i$ and $\alpha_i^0 := \neg\alpha_i$ for each $i$. 

\[ \begin{array}{l}
\frac{[\phi]}{\phi \rightarrow \psi} \quad \frac{\phi \rightarrow \psi}{\phi \rightarrow \psi} \rightarrow \text{E} \quad \frac{\alpha \rightarrow (\phi \lor \psi)}{(\alpha \rightarrow \phi) \lor (\alpha \rightarrow \psi)} \quad \text{Split} \\
\frac{\square(\phi \rightarrow \psi)}{\square \phi \rightarrow \square \psi} \quad \frac{\square(\phi \rightarrow \psi)}{\Diamond \phi \rightarrow \diamond \psi} \quad \frac{\Diamond \phi \rightarrow \diamond \psi}{\square(\phi \rightarrow \psi)} \\
\frac{\phi \lor \psi \rightarrow \text{I}}{\phi \lor \psi \rightarrow \text{I}} \quad \frac{\square(\phi \lor \psi)}{\square \phi \lor \square \psi} \quad \frac{\Diamond \phi \lor \square \psi}{\square(\phi \lor \psi)} \\
\frac{\phi \rightarrow \psi}{\phi \rightarrow \psi} \rightarrow \text{E} \quad \frac{\square(\phi \lor \psi)}{\square \phi \lor \square \psi} \quad \frac{\Diamond \phi \lor \square \psi}{\square(\phi \lor \psi)} \\
\frac{\phi \lor \psi \rightarrow \text{I}}{\phi \lor \psi \rightarrow \text{I}} \quad \frac{\square(\phi \lor \psi)}{\square \phi \lor \square \psi} \quad \frac{\Diamond \phi \lor \square \psi}{\square(\phi \lor \psi)} \\
\frac{\phi \rightarrow \psi}{\phi \rightarrow \psi} \rightarrow \text{E} \quad \frac{\square(\phi \lor \psi)}{\square \phi \lor \square \psi} \quad \frac{\Diamond \phi \lor \square \psi}{\square(\phi \lor \psi)} \end{array} \]
To prove the Completeness Theorem of the system, we adopt a very similar argument to that in the previous section. We first show that every \( \text{MD}^\forall \)-formula is provably equivalent to a formula in disjunctive normal form.

**Lemma 2.11 (Normal Form)** For any \( \text{MD}^\forall \)-formula \( \phi \), \( \phi \vdash \bigvee_{i \in I} \alpha_i \) for some set \( \{ \alpha_i \mid i \in I \} \) of classical formulas.

**Proof** We prove the lemma by induction on \( \phi \). If \( \phi \) is a classical formula, then the lemma holds trivially. If \( \phi = (\alpha_1, \ldots, \alpha_n, \beta) \), then we derive

\[
\vdash (\alpha_1^{(1)} \land \cdots \land \alpha_n^{(n)} \land \beta) \land \neg \beta) \quad \text{(by \(-\)df)}
\]

\[
\vdash \bigvee_{f \in 2^{[1 \ldots n]}} (\alpha_1^{(f(1))} \land \cdots \land \alpha_n^{(f(n))} \land \beta)
\]

(by Proposition 2.4(a)).

The induction steps are proved by applying Proposition 2.4(a)-(c) and the induction hypothesis (cf. the proof of Theorem 2.6).

**Lemma 2.12** If \( \alpha \) and \( \beta \) are classical formulas, then \( \alpha \vdash \text{K} \beta \iff \alpha \vdash \text{MD}^\forall \beta \).

**Proof** The direction \( \Longleftarrow \) then follows from the Soundness Theorem and Expression (1) from Section 1.

For the direction \( \Longrightarrow \), it suffices to show that restricted to classical formulas, the system of \( \text{MD}^\forall \) admits all axioms and rules of the Hilbert-style system of \( \text{K} \).

For the rules of \( \text{K} \), by inspecting the rules of the system of \( \text{MD}^\forall \), we see that restricted to classical formulas, the system admits Uniform Substitution rule, and Necessitation rule is a special case of the rule \( \Box \text{Mon} \) when there is no undischarged assumption in the rule. The Modus Ponens rule is interpreted as \( \alpha, \neg \alpha \otimes \beta/\beta \) in the language of \( \text{MD}^\forall \) and it can be derived as follows:

\[
\frac{\vdash \neg \alpha}{\vdash \beta} \quad \text{Ex falso}
\]

\[
\frac{\vdash \beta}{\vdash \alpha \otimes \beta} \quad \otimes \text{Sub}
\]

For the propositional axioms of \( \text{K} \), it is easy to see that restricted to classical formulas, the system of \( \text{MD}^\forall \) contains all the rules for the classical propositional connectives conjunction, disjunction with respect to \( \otimes \), negation and falsum \( \bot \). Therefore all axioms of classical propositional logic are derivable in the system of \( \text{MD}^\forall \).
For the axioms of $\mathbf{K}$ that involve modalities, the validity of (an equivalent form of) the $\mathbf{K}$ axiom is stated in the language of $\text{MD}^\lor$ as $\Box(\alpha \land \beta) \leftrightarrow \Box \alpha \land \Box \beta$, which can be derived easily by applying $\Box \text{Mon}$. Finally, we derived the inter-definability of $\Box$ and $\Diamond$, i.e., $\Box \alpha \leftrightarrow \neg \neg \neg \Box \alpha$, already in (7).

\begin{theorem}[Completeness] For any $\text{MD}^\lor$-formulas $\phi$ and $\psi$, $\vdash \psi \iff \vdash \text{MD}^\lor \phi$.
\end{theorem}

\begin{proof}
By a similar argument to that of the proof of Theorem 2.8, where we apply Lemmas 2.11 and 2.12 instead.
\end{proof}

Since $\text{MD}^\lor$ is compact (by Theorem 1.4), we obtain also the Strong Completeness Theorem as a corollary.

\begin{corollary}[Strong Completeness]
For any set $\Gamma \cup \{\phi\}$ of $\text{MD}^\lor$-formulas, $\Gamma \vdash \phi \iff \Gamma \vdash_{\text{MD}^\lor} \phi$.
\end{corollary}

2.3 Applications of the disjunctive normal form

We devote this section to three interesting applications of the disjunctive normal form (6) of modal dependence logics.

In the context of propositional logics of dependence, flat formulas admit a certain characterization theorem; see [5, 15] for the proof. We now generalize this characterization result to the modal case by using the disjunctive normal form.

\begin{theorem}
The following are equivalent.

(a) $\phi$ is flat
(b) $\phi \equiv \alpha$ for some classical formula $\alpha$
(c) $\neg \neg \neg \phi \equiv \phi$
(d) $\vdash \phi \land \neg \phi$
\end{theorem}

\begin{proof}
We only give the detailed proof for (a)$\Rightarrow$(b). Assume (a). We have $\phi \equiv \lor_{i \in I} \alpha_i$ for some set $\{\alpha_i \mid i \in I\}$ of classical formulas, and in particular $\vdash \phi \leftrightarrow \lor_{i \in I} \alpha_i$. Since $\phi$ is flat, it follows from Expression (5) from Section 2.1 and the Disjunction Property (Theorem 1.3) that there exists $j \in I$ such that $\phi \vdash \alpha_j$. On the other hand, $\alpha_j \vdash \lor_{i \in I} \alpha_i$.

We write $\phi(p)$ to indicate that the propositional variables occurring in $\phi$ are among $p = p_1 \ldots p_n$. Next, we prove Craig’s Interpolation Theorem for modal dependence logics that have intuitionistic disjunction in their languages.

\begin{theorem}[Interpolation]
Let $\mathcal{L}$ be a modal dependence logic that has intuitionistic disjunction in its language. For any $\mathcal{L}$-formulas $\phi(p, q)$ and $\psi(q, \bar{r})$, if $\vdash_{\mathcal{L}} \phi$, then there exists an $\mathcal{L}$-formula $\theta(q)$ such that $\phi \vdash_{\mathcal{L}} \theta$ and $\theta \vdash_{\mathcal{L}} \psi$.
\end{theorem}

\begin{proof}
Suppose $\vdash_{\mathcal{L}} \lor_{i \in I} \alpha_i + \phi \vdash \psi + \lor_{j \in J} \beta_j$ for some sets $\{\alpha_i(p, q) \mid i \in I\}$ and $\{\beta_j(q, \bar{r}) \mid j \in J\}$ of classical formulas. Then, for each $i \in I$, there exists $j_i \in J$ such that $\alpha_i \vdash \beta_{j_i}$. Since $\alpha_i$ and $\beta_{j_i}$ are classical formulas,

\begin{align*}
&\alpha_i(p, q) \vdash_{\mathcal{K}} \beta_{j_i}(q, \bar{r}). \\
\end{align*}

Now, by the Interpolation Theorem of $\mathcal{K}$, there exists a classical formula $\theta(q)$ such that $\alpha_i(p, q) \vdash_{\mathcal{K}} \theta(q)$ and $\theta(q) \vdash_{\mathcal{K}} \beta_{j_i}(q, \bar{r})$. Thus, $\alpha_i(p, q) \vdash_{\mathcal{L}} \theta(q)$ and $\theta(q) \vdash_{\mathcal{L}} \beta_{j_i}(q, \bar{r})$. The formula $\lor_{i \in I} \alpha_i(p, q) \vdash_{\mathcal{L}} \lor_{i \in I} \theta(q)$ and $\lor_{i \in I} \theta(q) \vdash_{\mathcal{L}} \lor_{j \in J} \beta_j$. Lastly, we prove that modal dependence logics have the finite model property.

\begin{theorem}[Finite Model Property]
If $\not\models \phi$, then there exists a finite Kripke model $\mathcal{M}$ and finite team $X$ such that $\mathcal{M}, X \not\models \phi$.
\end{theorem}
Proof For any formula \( \phi \), we have \( \phi \equiv \bigvee_{i \in I} \alpha_i \) for some finite set \( \{ \alpha_i \mid i \in I \} \) of classical formulas. If \( \not \models \phi \), then \( \not \models \alpha_i \) for all \( i \in I \). Since each \( \alpha_i \) is a classical formula, by Expression (1) from Section 1, \( \not \models K \alpha_i \) for each \( i \in I \). By the finite model property of \( K \), for each \( i \in I \), there exists a finite Kripke model \( \mathcal{M}_i \) and \( w_i \) such that \( \mathcal{M}_i, w_i \not \models \alpha_i \). It follows that \( \mathcal{M}_i, \{ w_i \} \not \models \alpha_i \) in the sense of team semantics. Consider the finite model \( \mathcal{M} = \bigcup_{i \in I} \mathcal{M}_i \) and the finite team \( X = \{ w_i \mid i \in I \} \). By Expression (2) from Section 1, we obtain that for each \( i \in I \), \( \mathcal{M}, \{ w_i \} \not \models \alpha_i \), which implies \( \mathcal{M}, X \not \models \alpha_i \) by the downward closure property. Hence, we conclude that \( \mathcal{M}, X \not \models \bigvee_{i \in I} \alpha_i \), thereby \( \mathcal{M}, X \not \models \phi \). □

2.4 MD and MD$^+$ In this section, we define natural deduction systems for MD and MD$^+$. As we pointed out in the introduction of Section 2, in these implication-free logics (which are compact by Theorem 1.4) there is a subtle difference between the weak and the strong completeness. We first introduce the systems for the two logics for which the strong completeness holds, and then point out that the systems with two rules less already admit the weak completeness.

The systems of MD and MD$^+$ have (essentially) the same rules for (extended) dependence atoms as introduced in [33]. To define these rules, let us follow [33] and first introduce some notations. A formula in the language of MD$^+$ or MD is a finite string of symbols. We number the symbols in a formula with positive integers starting from the left, as in the following example:

\[
\begin{array}{cccccccccccc}
\text{mth symbol of a formula } \phi \text{ starts a string } \psi \text{ that is a subformula of } \phi, \text{ we denote the subformula by } [\psi, m]_\alpha, \text{ or simply } [\psi, m]. \text{ When referring to an occurrence of a formula } \chi \text{ inside a subformula } \psi \text{ of } \phi, \text{ we will be sloppy about the notations and use the same counting also for the subformula } \psi. \text{ We write } \phi(\beta/[\alpha, m]) \text{ for the formula obtained from } \phi \text{ by replacing the occurrence of the subformula } [\alpha, m] \text{ with } \beta. \text{ For example, for the formula } \phi = (\square p, q) \otimes \square = (\square p, q), \text{ we denote the second occurrence of the dependence atom } \equiv (\square p, q) \text{ by } [(\square p, q), 10], \text{ and the same notation also designates the occurrence of } \equiv (\square p, q) \text{ inside the subformula } \square = (\square p, q). \text{ The notation } \phi(\beta/[\equiv (\square p, q), 10]) \text{ designates the formula } \equiv (\square p, q) \otimes \square (\beta). \end{array}
\]

Definition 2.18  
- The natural deduction system of MD$^+$ consists of the rules in Table 1(b)(d), together with the rules in Table 3.
- The natural deduction system of MD is the same as that of MD$^+$ except that the dependence atoms can only have propositional variables as arguments.

In the above systems, the rules Dep$^l_{\phi}$, Dep$^d_{\phi}$ and SE for dependence atoms simulate the equivalence \( \equiv (\alpha) \equiv \alpha \lor \neg \alpha \), and the rules Dep$^l_\phi$ and Dep$^d_\phi$ simulate the equivalence \( \equiv (\alpha_1, \ldots, \alpha_k, \beta) \equiv \equiv (\alpha_1) \land \cdots \land \equiv (\alpha_k) \to \equiv (\beta) \) (see also Expression (3) in Section 1). Clearly, Dep$^l_\phi$ is a special case of SE, but we present both rules in Table 3 for reasons that will become clear in the sequel. We refer the reader to [33] for further discussion on these rules.

For simplicity, we only give the proof of the Completeness Theorem for the system MD$^+$, from which the Completeness Theorem for the system MD follows. We follow the argument in [33] for propositional dependence logic, and first define re-alizations of formulas, a crucial notion of the argument. Let \( d = (\alpha_1, \ldots, \alpha_k, \beta) \) be a dependence atom. A function \( f : 2^{[1, \ldots, k]} \to 2 \) is called a re-alizing function for \( d \),
where we stipulate \(2^\emptyset = \{\emptyset\}\), and the formula

\[
d_f^\ast := \bigotimes_{v \in 2^{[1\ldots k]}} (\alpha_1 v^1 \land \cdots \land \alpha_k v^k \land \beta^{(v)})
\]

is called a realization of the dependence atom \(d\) over \(f\). Let \(\sigma = \langle [d_1, m_1], \ldots, [d_c, m_c] \rangle\) be the sequence of all occurrences of dependence atoms in \(\phi\). A realizing sequence of \(\phi\) is a sequence \(\Omega = \langle f_1, \ldots, f_c \rangle\) such that each \(f_i\) is a realizing function for \(d_i\). We call the classical formula \(\phi_\Omega^\ast\) defined as follows a realization of \(\phi\):

\[
\phi_\Omega^\ast := \phi((d_1 f_1^\ast)/[d_1, m_1], \ldots, (d_c f_c^\ast)/[d_c, m_c]).
\]

For example, consider the formula \(\phi = \equiv(\square p, q) \land \equiv(\square p, q)\) that we discussed earlier. Consider two realizing functions \(f, g : 2^{[1]} \rightarrow 2\) for \(\equiv(\square p, q)\), defined as

\[
f(1) = 1 = f(0), \quad g(1) = 0 \quad \text{and} \quad g(0) = 1,
\]

where \(1(1) = 1\) and \(0(1) = 0\). Both \(\langle f, g \rangle\) and \(\langle g, f \rangle\) are realizing sequences of \(\phi\) giving rise to two realizations

\[
(\equiv(\square p, q))^f \land (\equiv(\square p, q))^g = (\equiv(\square p \land q)) \land (\equiv(\square p \land q))
\]

and \(\equiv(\square p, q))^f \land (\equiv(\square p, q))^g\) of \(\phi\).

The next lemma states the crucial properties of realizations that will be applied in the proof of the Completeness Theorem.

**Lemma 2.19**  Let \(\phi\) be a formula, and \(\Lambda\) the set of all realizing sequences of \(\phi\).

(a) \(\phi_\Omega^\ast \vdash_{\text{MD}^+} \phi\) for any \(\Omega \in \Lambda\).

(b) If \(\phi_\Omega^\ast \vdash_{\text{MD}^+} \psi\) for all \(\Omega \in \Lambda\), then \(\phi \vdash_{\text{MD}^+} \psi\).
We prove the item by induction on the complexity of \( \nu \). For any \( \nu \), we prove the other direction by applying Lemma \( \text{MD} \).

Proof. We prove the lemma by induction on the subformulas \( \chi \) of \( \nu \).

The case when \( \chi \) is an atom is trivial. If \( \chi = \chi_0 \otimes \chi_1 \) and without loss of generality we assume that the formula \( [\psi, m] \) occurs in the subformula \( \chi_0 \). By the induction hypothesis, \( \chi_0(\delta([\psi, m])) \vdash \chi_0(\theta([\psi, m])) \), which by \( \otimes \text{Sub} \) implies \( \chi_0(\delta([\psi, m])) \otimes \chi_1(\theta([\psi, m])) \otimes \chi_1 \). The case \( \chi = \chi_0 \wedge \chi_1 \) is proved analogously.

The case \( \chi = \Box \chi_0 \) follows from the induction hypothesis and \( \Box \text{Mon} \), and the case \( \chi = \Diamond \chi_0 \) follows from the induction hypothesis and \( \Diamond \text{Mon} \). □

Proof of Lemma 2.19. (a) We prove the item by induction on the complexity of \( \phi \).

If \( \phi \) does not contain any occurrences of dependence atoms, then the property holds trivially. If \( \phi = (a_1, \ldots, a_k, b) \) is a dependence atom and \( f : 2^{\{1, \ldots, k\}} \to 2 \) a realizing function of \( d = \phi \), by \( \text{Dep}l_k \), to show \( d_f^* = (a_1, \ldots, a_k, b) \) it suffices to derive \( d_f^* = (a_1, \ldots, a_k, b) \). This is proved by a similar argument to that of the proof of Lemma 4.15 in [33] which makes use of the rules \( \text{Dep}l_0, \text{Dep}E_0 \) and \( \text{Dep}l_k \).

If \( \phi \) is a complex formula with \( c \) occurrences of dependence atoms, and \( \phi^*_f(1, \ldots, f) \) := \( \phi((d_1)^{f_1}_f / [d_1, m_1], \ldots, (d_c)^{f_c}_f / [d_c, m_c]) \), where \( \Omega = (f_1, \ldots, f_c) \). Then, since \( (d_1)^{f_1}_f \) \( \ldots \) \( d_i \) for each \( 1 \leq i \leq c \), we derive

\[
\phi((d_1)^{f_1}_f / [d_1, m_1], \ldots, (d_c)^{f_c}_f / [d_c, m_c]) \vdash \phi(d_1 / [d_1, m_1], \ldots, d_c / [d_c, m_c])
\]

by applying Lemma 2.20 repeatedly.

(b) This item is a special case of the statement of Lemma 4.18 in [33], and can be proved by essentially the same argument that makes use of \( \text{Dep}l_0, \text{Dep}E_0, \text{SE} \) and other rules of the system of \( \text{MD}^+ \).

(c) The direction \( \bigvee_{\Omega \in \Lambda} \phi^*_{\Omega} \vdash \psi \) follows from item (a) and the Soundness Theorem.

We now prove the other direction \( \phi \vdash \bigvee_{\Omega \in \Lambda'} \psi^*_{\Omega} \) by induction on \( \phi \).

The case when \( \phi \) is a dependence atom can be easily checked using Expression (3) from Section 1. The other propositional cases can be proved exactly by the same argument as in Lemma 4.16 in [33]. The case when \( \phi = \Box \psi \) or \( \phi = \Diamond \psi \) follows from the fact that \( \Box (A \lor B) \vdash \Box A \lor \Box B \) and \( \Diamond (A \lor B) \vdash \Diamond A \lor \Diamond B \). □

Theorem 2.21. (Completeness) For any \( \text{MD}^+ \)-formula \( \phi \) and \( \psi \), \( \phi \vdash \psi \iff \phi \vdash \text{MD}^+ \psi \).

Proof. Suppose \( \phi \vdash \psi \). By Lemma 2.19(c), we have

\[
\phi \equiv \bigvee_{\Omega \in \Lambda} \phi^*_{\Omega} \equiv \bigvee_{\Delta \in \Lambda'} \psi^*_{\Delta} \equiv \psi
\]

where \( \Lambda \) and \( \Lambda' \) are the (nonempty) sets of all realizing sequences of \( \phi \) and \( \psi \), respectively. Since each \( \phi^*_{\Omega} \) and \( \psi^*_{\Delta} \) are classical formulas, by (5) from Section 2 we obtain that for each \( \Omega \in \Lambda \), there is \( \Delta \in \Lambda' \) such that \( \phi^*_{\Omega} \vdash \psi^*_{\Delta} \). From (1) from Section 1 we know that \( \phi^*_{\Omega} \vdash_{\text{MD}^+} \psi^*_{\Delta} \), which implies \( \phi^*_{\Omega} \vdash_{\text{MD}^+} \psi^*_{\Delta} \) by a similar argument to those in the previous sections (Cf. Lemma 2.12). Now, by Lemma 2.19(a) we derive \( \phi^*_{\Omega} \vdash_{\text{MD}^+} \psi \). Finally, by Lemma 2.19(b) we conclude that \( \phi \vdash_{\text{MD}^+} \psi \). □

Since \( \text{MD}^+ \) is compact (by Theorem 1.4), we obtain the strong completeness as a corollary.
Corollary 2.22 (Strong Completeness) For any set $\Gamma \cup \{\phi\}$ of $\text{MD}^+$-formulas, $\Gamma \models \phi \iff \Gamma \vdash_{\text{MD}^+} \phi$.

Finally, it is interesting to note that the theoremhood or validity problem of the logic $\text{MD}^+$ can actually be axiomatized by a slightly weaker system that contains less rules than the one defined in Definition 2.18 for the entailment problem. In the system of Definition 2.18, if we drop the rules in Table 3(b) and write $\vdash_{\text{MD}^+}^0 \phi$ if $\phi$ is a theorem (i.e., a formula derivable from the empty assumption) in the resulting system, items (a) and (b) of Lemma 2.19 are still true. By a very similar argument to the proof of Theorem 2.21 (namely, simply discard the arguments that involve $\phi$), one can prove the following weaker form of Completeness Theorem for this weaker system without applying Lemma 2.19(b).

Theorem 2.23 ((Weak) Completeness) For any $\text{MD}^+$-formula $\phi$, $\models \phi \iff \vdash_{\text{MD}^+}^0 \phi$.

3 Interpreting team semantics in single-world semantics

In the previous section, we have defined the systems of modal dependence logics as extensions of Fischer Servi’s intuitionistic modal logic $\text{IK}$ and inquisitive logic $\text{InqL}$ (which is a variant of the Kreisel-Putnam intermediate logic $\text{KP}$). In this section, we explore the connection between the single-world-based intuitionistic modal logic and intermediate logics and modal dependence logics from the model-theoretic point of view. We first prove that the team semantics of modal dependence logics over a usual (modal) Kripke model $\mathcal{M}$ coincides with the usual single-world semantics over an intuitionistic Kripke model $\mathcal{M}'$, whose domain consists of the teams of $\mathcal{M}$. For simplicity, we only perform this construction for the dependence atom-free fragment of $\text{MID}$ and $\text{MT}_0$, denoted $\text{MID}^-$ and $\text{MT}_0^-$, subsequently in Section 3.1 and Section 3.2. Depending on whether tensor $\otimes$ is present in the language of the logic, the domain of a model $\mathcal{M}'$ for interpreting the team semantics will consist of either the full powerset of the domain of $\mathcal{M}$ or the same powerset excluding the empty set. The tensor $\otimes$ (which corresponds to multiplicative conjunction, as discussed in [1], but is often understood as a disjunction) will be naturally interpreted as a binary diamond modality in this framework.

Furthermore, we generalize the properties of the specific powerset models we built for interpreting team semantics to establish the connection on a general level. In Section 3.1 we identify a class of bi-relation intuitionistic Kripke models that enjoy the abstract properties of the powerset models for $\text{MID}^-$, and we show that the system of $\text{MID}^-$ defined in the previous section is complete with respect to this class of models in the single-world semantics sense. Similar result for $\text{MT}_0^-$ will be obtained in Section 3.2 with respect to a class of tri-relation intuitionistic Kripke models with an extra ternary relation corresponding to the binary diamond $\otimes$. This approach is based on a similar construction for inquisitive logic given in [3].

3.1 A single-world semantics for $\text{MID}^-$ In this section, we define a single-world semantics for the system of the dependence atom-free fragment of $\text{MID}$ ($\text{MID}^-$). Observe from Definition 2.3 that the set of theorems of $\text{MID}^-$ includes all theorems of $\text{IK}$ and is included in the set of theorems of $\text{K}$. In other words, $\text{MID}^-$ can be understood as an intermediate modal logic that is not closed under uniform substitution (also called an intermediate modal theory). To be more precise, $\text{MID}^-$ can
be viewed as the fusion of $\mathbf{IK}$ and $\mathbf{KP}$ together with one axiom stating that box distributes over disjunction (i.e., $\Box(\phi \lor \psi) \rightarrow (\Box\phi \lor \Box\psi)$), and two axioms describing the classical behavior of disjunction-free formulas (i.e., $\neg\neg\alpha \rightarrow \diamond\neg\alpha$ and $\neg\neg\alpha \rightarrow \alpha$). Recall that $\mathbf{IK}$ is complete with respect to bi-relation intuitionistic Kripke frames (see e.g. [27]) and $\mathbf{KP}$ is complete with respect to $\mathbf{KP}$-frames (see e.g. [2]). We will show in this section that $\mathbf{MID}^-$ is complete (in the single-world semantics sense) with respect to a class of Kripke models whose frames are both bi-relation intuitionistic Kripke frames and $\mathbf{KP}$-frames.

Let us first recall relevant definitions for $\mathbf{IK}$.

**Definition 3.1** A bi-relation intuitionistic Kripke frame is a triple $\mathfrak{F} = (W, \geq, R)$, where

1. $W$ is a nonempty set
2. $\geq$ is a partial ordering and $R$ is a binary relation on $W$
3. $F_1$: If $w \geq w'$ and $wRv$, then there exists $v' \in W$ such that $v \geq v'$ and $w'Rv'$.
4. $F_2$: If $wRv$ and $v \geq v'$, then there exists $w' \in W$ such that $w \geq w'$ and $w'Rv'$.

A bi-relation intuitionistic Kripke model is a quadruple $\mathfrak{M} = (W, \geq, R, V)$ such that $(W, \geq, R)$ is a bi-relation intuitionistic Kripke frame and $V : Prop \rightarrow \wp(W)$ is a valuation satisfying monotonicity with respect to $\geq$, that is, $w \in V(p)$ and $w \geq v$ imply $v \in V(p)$.

**Definition 3.2** The satisfaction relation $\mathfrak{M}, w \vdash \phi$ between a bi-relation intuitionistic Kripke model $\mathfrak{M} = (W, \geq, R, V)$, a node $w \in W$ and a formula $\phi$ in the language of $\mathbf{IK}$ is defined inductively as follows:

1. $\mathfrak{M}, w \vdash p$ iff $w \in V(p)$
2. $\mathfrak{M}, w \not\vdash \perp$
3. $\mathfrak{M}, w \vdash \phi \land \psi$ iff $\mathfrak{M}, w \vdash \phi$ and $\mathfrak{M}, w \vdash \psi$
4. $\mathfrak{M}, w \vdash \phi \lor \psi$ iff $\mathfrak{M}, w \vdash \phi$ or $\mathfrak{M}, w \vdash \psi$
5. $\mathfrak{M}, w \vdash \phi \rightarrow \psi$ iff for all $v \in W$ such that $w \geq v$, if $\mathfrak{M}, v \vdash \phi$, then $\mathfrak{M}, v \vdash \psi$
6. $\mathfrak{M}, w \vdash \Box\phi$ iff there exists $v \in W$ such that $wRv$ and $\mathfrak{M}, v \vdash \phi$
7. $\mathfrak{M}, w \vdash \Diamond\phi$ iff for all $u, v \in W$ such that $w \geq u$ and $uRv$, it holds that $\mathfrak{M}, v \vdash \phi$

It is easy to show that the $\geq$-monotonicity extends to arbitrary formulas $\phi$, that is, $\textbf{Monotonicity: } [\mathfrak{M}, w \vdash \phi$ and $w \geq v] \implies \mathfrak{M}, v \vdash \phi$.

Every classical modal Kripke model induces a bi-relation intuitionistic Kripke model which we shall call *powerset model*. 

---

![Figure 1: Frame conditions. The directed lines represent the $R$ relation and the undirected lines represent the $\geq$ relation with the nodes positioned above being accessible from the ones positioned below.](image-url)
Definition 3.3  Let $\mathcal{M} = (W, R, V)$ be a classical modal Kripke model. The powerset model $\mathcal{M}^p$ induced by $\mathcal{M}$ is a quadruple $\mathcal{M}^p = (W^p, \supseteq, R^p, V^p)$, where

- $W^p = W \setminus \{0\}$, i.e. $W^p$ consists of all nonempty teams $X \subseteq W$
- $\supseteq$ is the superset relation
- $X R^p Y$ iff $X R Y$ and $Y \cap R(w) \neq \emptyset$ for every $w \in X$
- $X \in V^p(p)$ iff $X \subseteq V(p)$

To see that $\mathcal{M}^p$ is indeed a bi-relation intuitionistic Kripke model, note that the superset relation $\supseteq$ is a partial ordering, and the monotonicity of $V^p$ is immediate. To verify condition (F1), for any $X, X', Y \in W^p$ such that $X \supseteq X'$ and $X R^p Y$, letting $Y' = R(X') \cap Y$, it is easy to show that $Y \supseteq Y'$ and $X R^p Y'$ (see also Figure 2). Similarly, to verify condition (F2), for any $X, Y, Y' \in W^p$ such that $X R^p Y$ and $Y \supseteq Y'$, letting $X' = R^{-1}(Y') \cap X$, clearly $X \supseteq X'$ and $X' R^p Y'$ (see also Figure 2).

Next, we show that the team-based satisfaction relation with respect to classical modal Kripke models is equivalent to the single-world-based satisfaction relation with respect to the associated powerset models.

Lemma 3.4  Let $\mathcal{M} = (W, R, V)$ be a classical modal Kripke model and $X \subseteq W$ a nonempty team. For any MID$^-$-formula $\phi$, $\mathcal{M}, X \models \phi$ $\iff$ $\mathcal{M}^p, X \models \phi$.

Proof  We prove the lemma by induction on $\phi$. The only interesting case is when $\phi = \square \psi$. If $\mathcal{M}^p, X \models \square \psi$, then $\mathcal{M}^p, R(X) \models \psi$, since $X \supseteq X$ and $X R^p R(X)$. The induction hypothesis implies that $\mathcal{M}, R(X) \models \psi$. Hence $\mathcal{M}, X \models \square \psi$.

Conversely, if $\mathcal{M}, X \models \square \psi$, then $\mathcal{M}, R(X) \models \psi$. For all $Y, Z \in W^p$ such that $X \supseteq Y$ and $Y R^p Z$, since $Z \subseteq R(X)$, the downward closure property implies that $\mathcal{M}, Z \models \psi$ yielding $\mathcal{M}^p, Z \models \psi$ by the induction hypothesis. Hence $\mathcal{M}^p, X \models \square \psi$.

Inquisitive logic (being the propositional fragment of MID$^-$) is shown in [3] to be complete with respect to negative saturated (single-relation) intuitionistic Kripke models, which are also negative KP-models. Let us now give the corresponding definitions in the context of bi-relation intuitionistic Kripke models.

A point $w$ in a bi-relation intuitionistic Kripke model $\mathcal{M} = (W, \geq, R, V)$ is called an $\geq$-endpoint iff there is no point $v \neq w$ such that $w \geq v$. Denote by $E_w$ the set of all $\geq$-endpoints seen from $w$, i.e.,

$$E_w = \{v \in W \mid w \geq v \text{ and } v \text{ is an } \geq \text{-endpoint}\}.$$  

A bi-relation intuitionistic Kripke frame $\mathfrak{F} = (W, \geq, R)$ is said to be $\geq$-saturated if for every $w \in W$, $E_w \neq \emptyset$, and for every nonempty subset $E \subseteq E_w$, there exists a point $v \in W$ such that $w \geq v$ and $E_v = E$. A model $\mathcal{M}$ is called negative if $\mathcal{M}, w \vdash p$ $\iff$ $\mathcal{M}, w \vdash \neg p$. It is easy to verify that a powerset model
Suppose \( M = (W^o, \succeq, R^o, V^o) \) is a negative \( \geq \)-saturated model, and in particular, \( \succeq \)-endpoints in \( M \) are singletons \( \{w\} \) of elements \( w \) in \( W \).

The system of \( \text{MID}^- \) (see Definition 2.3) extends the systems of \( \text{IK} \) and \( \text{InqL} \) with two extra axioms: \( \Box(\phi \lor \psi) \rightarrow (\Box\phi \lor \Box\psi) \) and \( \neg\Box\alpha \rightarrow \Diamond\neg\alpha \), where \( \alpha \) is any classical formula. The latter axiom is equivalent to the axiom \( \neg\Box\neg p \rightarrow \Diamond\neg\neg p \), because a classical formula \( \alpha \) is always equivalent to a (double) negation \( \neg\neg\alpha \) (by Theorem 2.15). In what follows we show that the axioms \( \Box(p \lor q) \rightarrow (\Box p \lor \Box q) \) and \( \neg\Box\neg p \rightarrow \Diamond\neg\neg p \) both characterize certain frame condition. We write \( R_1 \circ R_2 \) for the composition of the two binary relations \( R_1 \) and \( R_2 \) on a set \( W \), defined as \( (x,y) \in R_1 \circ R_2 \) iff \( \exists z \in W(xR_1z \land zR_2y) \).

**Lemma 3.5** Let \( \mathcal{F} = (W, \geq, R) \) be a bi-relation intuitionistic Kripke frame. Then, \( \mathcal{F} \models (\Box p \lor q) \rightarrow (\Box p \lor \Box q) \iff \mathcal{F} \) satisfies condition \( (G1') \) defined below:

**G1':** For all \( w, u, v \in W \), if \( u, v \in (\geq \circ R)(w) \), then there exists \( t \in W \) such that \( w(\geq \circ R)t, t \geq u \) and \( t \geq v \). (See Figure 3(a))

Before we give the proof of the lemma, let us first check that the underlying frames of powerset models satisfy \((G1')\). First note that for any points \( X, Y \) in a powerset model \( M^o = (W^o, \succeq, R^o, V^o) \), \( X(\geq \circ R^o)Y \) iff \( Y \subseteq R(X) \). Now, for any three points \( w, u, v \) in \( W^o \) such that \( u, v \in (\geq \circ R^o)(w) \), we have \( u, v \subseteq R(w) \) implying \( u \cup v \subseteq R(w) \). Clearly, \( t = u \cup v \) is a nonempty subset of \( W \) such that \( w(\geq \circ R^o)t, t \geq u \) and \( t \geq v \) (see Figure 3(b)). As a powerset model \( M^o \) carries the information of teams in the model \( M \), condition \((G1')\) can be viewed as a property that is abstracted from the corresponding property of teams of the usual classical modal Kripke frames.

In the sequel, we will also work with the following equivalent form \((G1')\) of \((G1')\):

**G1:** For any \( w \in W \) and any nonempty finite set \( X \subseteq (\geq \circ R)(w) \), there exists a node \( u \in (\geq \circ R)(w) \) such that \( u \geq v \) for all \( v \in X \).

**Proof of Lemma 3.5** Suppose \( \mathcal{F} \) satisfies \((G1')\) and \( (\mathcal{F}, V), w \not\models p \lor q \) for some valuation \( V \) and some \( w \in W \). Then there exist \( u, v \in W \) such that \( w(\geq \circ R)u, w(\geq \circ R)v \), \((\mathcal{F}, V), u \not\models p \) and \((\mathcal{F}, V), v \not\models q \).

Let \( t \in W \) be the point given by \((G1')\). Then by the \( \geq \)-monotonicity, we have \((\mathcal{F}, V), t \not\models p \lor q \), which implies that \((\mathcal{F}, V), w \not\models (p \lor q) \).
Suppose \( v \in E \)dition in (G2). Then, (G2) applies to the set \( u \)t \( v \) have proved that the set \( \{ v \} \) implies \((F_v V) \geq \) that \( M \). Clearly, we can find a \( \geq \)-monotone valuation \( V \) such that
\[
V(p) = W \setminus \geq^{-1}(v) \quad \text{and} \quad V(q) = W \setminus \geq^{-1}(u).
\]

For each \( t \in W \) such that \( w(\geq \circ) t \), either \( t \not\geq^{-1}(v) \) or \( t \not\geq^{-1}(u) \). Thus \((\mathcal{G}, V), t \vdash p \lor q \), thereby \((\mathcal{G}, V), w \vdash \Diamond(p \lor q) \). On the other hand, \((\mathcal{G}, V), u \not\vdash q \) and \((\mathcal{G}, V), v \not\vdash p \). Hence \((\mathcal{G}, V), w \not\vdash p \lor \Box q \).

\( \Diamond \)

**Lemma 3.6** Let \( \mathcal{G} = (W, \geq, R) \) be a \( \geq \)-saturated bi-relation intuitionistic Kripke frame. Then, \( \mathcal{G} \models \neg \Box \neg p \rightarrow \Diamond \neg p \iff \mathcal{G} \) satisfies condition (G2) defined below:

**G2:** Let \( w \in W \) be an arbitrary point and \( E \) a set of \( \geq \)-endpoints such that \( E \subseteq R(E_w) \) and \( E \cap R(v) \neq \emptyset \) for every \( v \in E_w \). Then, there exists \( t \in W \) such that \( wRt \) and \( E_t \subseteq E \). (see Figure 4(a))

Before we give the proof of the lemma, let us first check that the underlying frames of powerset models satisfy (G2). Indeed, for any point \( w \) in a powerset model \( \mathcal{M} = (\mathcal{W}, \geq, R, V) \) and any set \( E \) of \( \geq \)-endpoints (i.e., a set of singletons of elements in \( W \)) such that \( E \subseteq R(E_w) \) and \( E \cap R(v) \neq \emptyset \) for every \( v \in E_w \), it is easy to see that \( t = \bigcup E \) is a nonempty subset of \( W \) such that \( wRt \) and \( E_t \subseteq E \) (see Figure 4(b)).

**Proof of Lemma 3.6** Suppose \( \mathcal{G} \) satisfies (G2) and \((\mathcal{G}, V), w \not\vdash \Diamond \neg p \) for some valuation \( V \) and some \( w \in W \). Then, \((\mathcal{G}, V), v \not\vdash \Box \neg p \) for each \( \geq \)-endpoint \( v \geq w \), i.e., each \( v \in E_w \). It follows that there exists \( u' \) such that \( vRu' \) and \((\mathcal{G}, V), u' \not\vdash \neg p \), which implies \((\mathcal{G}, V), u \geq \neg p \) for some \( u \leq u' \). Since \( \mathcal{G} \) is \( \geq \)-saturated, \( u \) sees an \( \geq \)-endpoint, and thus we may w.l.o.g. assume that \( u \) is itself an \( \geq \)-endpoint.

Consider the set \( E = \{ u \mid v \in E_w \} \). For each \( u \in E \), by the construction we have \( v \in E_w \) and \( vRu' \geq u \), which by (F2) implies that there exists \( v' \in W \) such that \( v \geq v' \). But as \( v \) is an \( \geq \)-endpoint, we must have \( v = v' \) and \( vRu' \). Thus, we have proved that the set \( E \) satisfies the condition \( E \subseteq R(E_w) \) in (G2). On the other hand, for every \( v \in E_w \), we have \( u \in E \) by definition, and the same argument as above shows that \( u \in E \). Hence, \( u \in E \cap R(v) \neq \emptyset \), namely \( E \) also satisfies the other condition in (G2). Then, (G2) applies to the set \( E \) and the point \( w \), and therefore there exists a point \( t \in W \) such that \( wRt \) and \( E_t \subseteq E \).
Now, since $E_i \subseteq E$, every $\geq$-endpoint that $t$ can see is a $u_\varphi \in E$ with $(\mathfrak{A}, V), u_\varphi \vdash \varphi$ for some $\varphi \in E_w$. This means $(\mathfrak{A}, V), t \vdash \neg\neg \varphi$, which gives $(\mathfrak{A}, V), w \vdash \Diamond \neg\Diamond \varphi$ as $wRt$.

Conversely, suppose $\mathfrak{A}$ does not satisfy (G2). Then there exists $w \in W$ and a set $E$ of $\geq$-endpoints satisfying $E \subseteq R(E_w)$ and $E \cap R(v) \neq \emptyset$ for every $v \in E_w$ such that for all $t \in W$, $wRt$ implies $E_i \not\subseteq E$. Since $E$ is a set of $\geq$-endpoints, one can find a $\geq$-monotone valuation $V$ such that $V(p) = E$. We will show that $(\mathfrak{A}, V), w \not\vdash \Box \neg\varphi \rightarrow \Diamond \neg\Diamond \varphi$.

For every $v \in E_w$, there exists $u \in E \cap R(v) \neq \emptyset$ with $(\mathfrak{A}, V), u \vdash \varphi$. Since $(\mathfrak{A}, V), u \not\vdash \neg\varphi$ and $vRu$, we obtain $(\mathfrak{A}, V), v \not\vdash \Box \neg\varphi$ for every $v \in E_w$. Hence, $(\mathfrak{A}, V), w \not\vdash \Diamond \neg\Diamond \varphi$.

On the other hand, for every $t \in R(w)$, by the assumption there exists $s \in E_i$ such that $s \not\in E$ meaning $(\mathfrak{A}, V), s \not\vdash \varphi$. It follows that $(\mathfrak{A}, V), t \vdash \neg\neg \varphi$ for every $t \in R(w)$. Hence $(\mathfrak{A}, V), w \vdash \Diamond \neg\Diamond \varphi$. □

Let $M$ be the class of all finite negative $\geq$-saturated bi-relation intuitionistic Kripke models satisfying (G1) and (G2). In the remainder of this section, we show that the system of $\text{MID}^-$ is complete with respect to $M$, that is, we will prove the following theorem. The idea of the proof is inspired by that of Theorem 3.2.18 in [3]. Note that since $\text{MID}^-$ is not closed under uniform substitution, one can only obtain the completeness theorem in the sense of the theorem below for a class $M$ of models (with restricted valuations) instead of a class of frames (with arbitrary valuations).

**Theorem 3.7** For any $\text{MID}^-$-formula $\varphi$, $\vdash_{\text{MID}^-} \varphi \iff M \vdash \varphi$.

**Proof of "\(\Rightarrow\)"** We have checked that each (finite) powerset model is in $M$. Then,

$M \vdash \varphi \implies \forall \mathcal{M} \vdash \varphi$ for all finite powerset models $\mathcal{M}$

$\implies \forall \mathcal{M} \vdash \varphi$ for all finite classical modal Kripke models $\mathcal{M}$ (by Lemma 3.4)

$\implies \vdash_{\text{MID}^-} \varphi$ (by the finite model property (Theorem 2.17) and the Completeness Theorem of $\text{MID}^-$).

$\blacksquare$

To prove the other direction "\(\iff\)" of the above theorem, we first show that every model in $M$ can be mapped via a $p$-morphism into a finite powerset Kripke model. As $p$-morphisms are truth-preserving, the required result will then follow. Now, we recall the definition of $p$-morphisms of bi-relation intuitionistic Kripke models given by Wolter and Zakharyaschev in [30].

**Definition 3.8** Let $\mathcal{M}_1 = (W_1, \geq_1, R_1, V_1)$ and $\mathcal{M}_2 = (W_2, \geq_2, R_2, V_2)$ be bi-relation intuitionistic Kripke models. A function $f : W_1 \rightarrow W_2$ is called a $p$-morphism iff

$P_1$: $w \in V_1(p) \iff f(w) \in V_2(p)$ for all propositional variables $p$

$P_2$: $w \geq_1 v \implies f(w) \geq_2 f(v)$

$P_3$: $wR_1 v \implies f(w)R_2 f(v)$

$P_4$: $f(w) \geq_2 v' \implies \exists v \in W_1 \text{ s.t. } f(v) = v' \text{ and } w \geq_1 v$

$P_5$: $f(w)R_2 v' \implies \exists v \in W_1 \text{ s.t. } v' \geq_2 f(v) \text{ and } wR_1 v$

$P_6$: $f(w)(\geq_2 \circ R_2)v' \implies \exists v \in W_1 \text{ s.t. } \exists v' \geq_2 f(v) \text{ and } f(v) \geq_2 v'$

**Theorem 3.9 (see [30])** If $f : \mathcal{M}_1 \rightarrow \mathcal{M}_2$ is a $p$-morphism between two bi-relation intuitionistic Kripke models $\mathcal{M}_1$ and $\mathcal{M}_2$, then $\mathcal{M}_1, w \vdash \varphi \iff \mathcal{M}_2, f(w) \vdash \varphi$.  


Lemma 3.10. For every finite bi-relation intuitionistic Kripke model $\mathcal{M} = (W, \geq, R, V)$ in $\mathcal{M}$, there exists a finite classical modal Kripke model $\mathcal{N}$ such that there exists a $p$-morphism $f$ of $\mathcal{M}$ into the powerset model $\mathcal{M}^\mathcal{N}$ induced by $\mathcal{M}$.

**Proof.** Define a modal Kripke model $\mathcal{N} = (W_0, R_0, V_0)$ as follows:

- $W_0$ is the set of all $\geq$-endpoints of $W$.
- $R_0 = R | W_0$ and $V_0 = V | W_0$.

Now, consider the powerset Kripke model $\mathcal{N}^\mathcal{M} = (W_0^\mathcal{M}, R_0^\mathcal{M}, V_0^\mathcal{M})$ associated with $\mathcal{M}$. Define a function $f : W \to W_0^\mathcal{M}$ by taking

$$f(w) = E_w$$

for all $w \in W$.

Since $\mathcal{N}$ is saturated, $E_w \neq \emptyset$ for all $w \in W$. Thus $E_w \in W_0^\mathcal{M}$ and $f$ is well-defined.

Note that an $\geq$-endpoint $e$ of $\mathcal{N}$ is mapped through $f$ to the singleton $\{e\} = E_e$. Intuitively, $\geq$-endpoints of $\mathcal{N}$ are simulated in our argument by singletons of $\mathcal{N}^\mathcal{M}$, and it may be helpful for the reader to think of a node $w$ of $\mathcal{N}$ as the team formed by all $\geq$-endpoints seen from $w$, namely the set $E_w$.

Now, we proceed to show that $f$ is a $p$-morphism, i.e., $f$ satisfies (P1)-(P6).

(P1). It suffices to show that $\mathcal{M}, w \models p$ if and only if $\mathcal{N}^\mathcal{M}, E_w \models p$. The direction “$\Rightarrow$” follows from the $\geq$-monotonicity of $V$. For the direction “$\Leftarrow$”, if $\mathcal{M}, w \not\models p$, then since $V$ is negative, $\mathcal{N}^\mathcal{M}, E_w \not\models p$. Thus, there exists $v \in E_w$ such that $\mathcal{N}^\mathcal{M}, v \not\models p$, which implies that $\mathcal{N}, [v] \not\models p$, thereby $\mathcal{N}^\mathcal{M}, E_w \not\models p$.

(P2). Clearly, if $w \geq v$, then $E_w \supseteq E_v$, i.e. $f(w) \supseteq f(v)$.

(P3). Assume $wRv$, we show that $E_wR_0E_v$, namely $E_wR_0E_v$. For any $s \in E_w$, by (F1) of $\mathcal{N}$, there exists $t \in W$ such that $v \geq t$ and $sRt$. For each $t' \in E_v$ such that $t \geq t'$, by (F2), there exists $s' \in W$ such that $s \geq s'$ and $s'Rt'$. As $s$ is an $\geq$-endpoint, we must have $s = s'$ and $s'Rt'$.

On the other hand, for any $t \in E_v$, consecutively applying (F2) and (F1) of $\mathcal{N}$, by a similar argument to the above, we can find an $s' \in E_w$ such that $s'Rt$. Hence, we conclude that $E_wR_0E_v$.

(P4). If $E_w \supseteq v'$, then as $\mathcal{N}$ is $\geq$-saturated, there exists $v \in W$ such that $w \geq v$ and $E_v = v'$ as required.

(P5). If $E_w \supseteq R_0v'$, then $E_wR_0E_v$. Clearly, $v'$ is a set of $\geq$-endpoints such that $v' \subseteq R(E_w)$ and $v' \cap R(s) \neq \emptyset$ for every $s \in E_w$. Thus, by (G2) of $\mathcal{N}$, there exists $v \in W$ such that $wRv$ and $v' \subseteq E_v$, as required.

(P6) Suppose $E_u(\geq \circ R_0u') \cup u' \neq \emptyset$. Then $u' \subseteq (\geq \circ R)(w)$. Since $\mathcal{N}$ is finite, the set $u'$ must be finite and (G1) applies. Thus, there exists $u \in W$ such that $w(\geq \circ R)u$ and $u \geq s$ for all $s \in u'$. Since $u'$ is a set of $\geq$-endpoints, the latter of the above implies that $f(u) = E_u \supseteq u'$.

Finally, we complete the proof of Theorem 3.7 as follows.

**Proof of Theorem 3.7, the direction “$\Rightarrow$”** Suppose $\vdash_{\text{MD}} \phi$. For each $\mathcal{M} \in \mathbb{M}$, by Lemma 3.10, there is a classical modal Kripke model $\mathcal{N}$ and a $p$-morphism $f : \mathcal{M} \to \mathcal{N}$. By the assumption, we have $\mathcal{M} \models \phi$, which implies $\mathcal{N} \models \phi$ by Lemma 3.4. Finally, by Theorem 3.9, we conclude that $\mathcal{M} \models \phi$, as required.

Note that the finiteness of the models in the class $\mathbb{M}$ is used in the proof of Theorem 3.7 (only) for establishing condition (P6) in Lemma 3.10 when quoting condition (G1), the frame condition that the axiom $\Box(p \lor q) \to (\Box p \lor \Box q)$ characterizes (see Lemma 3.5). Consider a stronger version of (G1).
**G1***: For any \( w \in W \) and any nonempty set \( X \subseteq (\succeq \circ R)(w) \), there exists a node \( u \in (\succeq \circ R)(w) \) such that \( u \succeq v \) for all \( v \in X \).

If one, instead, defines \( M \) as the class of all (possibly infinite) negative \( \succeq \)-saturated bi-relation intuitionistic Kripke models satisfying \( (G1^+) \) and \( (G2) \), Theorem 3.7 will still hold. But we choose to adopt the current setting in this section, as it exhibits more interaction between the properties of the models and the axioms.

### 3.2 A single-world semantics for \( MT_0^- \)

In this section, we define a single-world semantics for the system of the dependence atom-free fragment of \( MT_0 \) (\( MT_0^- \)), and prove that \( MT_0^- \) is complete (in the single-world semantics sense) with respect to a class of tri-relation intuitionistic Kripke models.

The language of \( MT_0^- \) has one connective more than that of \( MID \). The team semantics of the one additional connective of \( MT_0^- \), the tensor \( \otimes \), is generalized naturally from the usual single-world semantics of the disjunction of classical logic. Yet, the tensor, being understood as a disjunction, has a few odd behaviors. For instance, it does not admit the usual elimination and distributive rules for disjunction, in particular, none of \( \phi \otimes \phi \vdash \phi \), \( (\phi \otimes \psi) \land (\phi \otimes \chi) \vdash (\phi \otimes (\psi \land \chi)) \), and \( (\phi \land \psi) \otimes (\phi \land \chi) \vdash (\phi \land (\psi \otimes \chi)) \) is in general true. Indeed, although the tensor behaves truly as a disjunction over classical formulas (Cf. Lemma 2.7), Abramsky and Väänänen [1] observed that the tensor should rather be understood as a *multiplicative conjunction* (hence the notation \( \otimes \)) as in linear logic (or, in fact, in bunched implication logic [21]). Multiplicative conjunction can often be read as a *binary diamond* modality, and this is the interpretation that we will adopt for tensor in this section.

Following the approach of the previous section, we first show that the team semantics of \( MT_0^- \) over the usual classical modal Kripke models coincides with the single-world semantics of \( MT_0^- \) over the associated *full powerset models*, which are powerset models equipped also with a ternary relation \( R_\otimes \) for the interpretation of the binary diamond \( \otimes \), and have also the empty team in their domains in order to characterize the property that the constant \( \bot \) is the neutral element of the tensor, i.e., \( \bot \otimes \phi \equiv \phi \). Let us now define formally this stronger notion of powerset model.

**Definition 3.11** Let \( \mathcal{M} = (W, R, V) \) be a classical modal Kripke model. The *full powerset model* \( \mathcal{M}^* \) induced by \( \mathcal{M} \) is a quintuple \( \mathcal{M}^* = (W^*, \succeq, R^*, R_\otimes, V^*) \), where

- \( W^* = \wp(W) \) i.e., \( W \) consists of all teams \( X \subseteq W \) including the empty team \( \emptyset \)
- \( R_\otimes \) is a ternary relation defined as \( R_\otimes(X,Y,Z) \) iff \( X = Y \cup Z \)
- and the other components are defined as in Definition 3.3.

Full powerset models are special cases of tri-relation intuitionistic Kripke models defined as follows.

**Definition 3.12** A *tri-relation intuitionistic Kripke frame* is a quadruple \( \mathcal{F} = (W, \succeq, R, S) \), where \( (W, \succeq, R) \) is a bi-relation intuitionistic Kripke frame and \( S \) is a binary relation on \( W \) satisfying condition (H1) defined below:

**H1**: If \( S(w,u,v) \) and \( w \succeq w' \), then there exist \( u', v' \in W \) such that \( S(w',u',v') \), \( u \succeq u' \) and \( v \succeq v' \).

A tri-relation intuitionistic Kripke model is a tuple \((\mathcal{F}, V)\) such that \( \mathcal{F} \) is a tri-relation intuitionistic Kripke frame and \( V \) satisfies \( \succeq \)-monotonicity.

**Definition 3.13** Let \( \mathcal{M} = (W, \succeq, R, S, V) \) be a tri-relation intuitionistic Kripke model. We define the *satisfaction* relation \( \mathcal{M}, w \vDash \phi \) inductively as follows:
Lemma 3.14 (Monotonicity) If $\mathfrak{M}, w \vDash \phi$ and $w \geq u$, then $\mathfrak{M}, u \vDash \phi$.

Proof We prove the lemma by induction on $\phi$. We only check the interesting cases.

If $\phi = \perp$, then $\mathfrak{M}, w \vDash \perp$ implies that $w$ is an $\geq$-endpoint. If $w \geq u$, then $w = u$, and so $\mathfrak{M}, u \vDash \perp$.

If $\phi = \psi \otimes \chi$, then $\mathfrak{M}, w \vDash \psi \otimes \chi$ implies that there exist $s, t \in W$ such that $S(w, s, t)$, $\mathfrak{M}, s \vDash \psi$ and $\mathfrak{M}, t \vDash \chi$. Since $w \geq u$, by (H1), there exist $s', t' \in W$ such that $S(u, s', t')$, $s \geq s'$ and $t \geq t'$. By the induction hypothesis, we have $\mathfrak{M}, s' \vDash \psi$ and $\mathfrak{M}, t' \vDash \chi$. Thus, $\mathfrak{M}, u \vDash \psi \otimes \chi$. □

Lemma 3.15 Let $\mathfrak{M} = (W, R, V)$ be a classical modal Kripke model and $X \subseteq W$ a team. For any $\mathbf{MT}_0$-formula $\phi$, $\mathfrak{M}, X \models \phi \iff \mathfrak{M}^*, X \vDash \phi$.

Proof We prove the lemma by induction on $\phi$. If $\phi = \perp$, then $\mathfrak{M}, X \models \perp$ iff $X = \emptyset$ if $\mathfrak{M}^*, X \vDash \perp$, since $\emptyset$ is an $\geq$-endpoint in $\mathfrak{M}^*$.

If $\phi = \psi \otimes \chi$, then

$\mathfrak{M}, X \models \psi \otimes \chi \iff \exists Y, Z \text{ s.t. } X = Y \cup Z, \mathfrak{M}, Y \models \psi$ and $\mathfrak{M}, Z \models \chi$

$\iff \exists Y, Z \text{ s.t. } R_0(X, Y, Z), \mathfrak{M}^*, Y \vDash \psi$ and $\mathfrak{M}^*, Z \vDash \chi$

(by the induction hypothesis)

$\iff \mathfrak{M}^*, X \vDash \psi \otimes \chi$.

The other cases follow from the same argument as in Lemma 3.4. □

We remarked in the previous section that singletons in a powerset model correspond to $\geq$-endpoints in its associated bi-relation intuitionistic Kripke models. In a tri-relation intuitionistic Kripke model $\mathfrak{M} = (W, \geq, R, S, V)$, singletons are simulated by the $\geq$-second least points instead, i.e., points $w$ in $W$ such that there is an $\geq$-endpoint $e$ such that $w > e$, and for all $v \in W$, $w > v$ implies that $v$ is an $\geq$-endpoint. We denote by $E^*_w$ the set of all $\geq$-second least points seen from $w$, i.e.,

$E^*_w = \{ v \in W \mid w \geq v \text{ and } v \text{ is a } \geq\text{-second least point} \}$.

In particular, if $w$ is an $\geq$-endpoint, then $E^*_w = \emptyset$; and if $w$ is itself a $\geq$-second least point, then $E^*_w = \{ w \}$.

We say that a tri-relation intuitionistic Kripke frame $\mathfrak{N} = (W, \geq, R, S)$ is weakly $\geq$-saturated if $E^*_w \neq \emptyset$ for every non-$\geq$-endpoint $w \in W$, and for every subset $E \subseteq E^*_w$, we have...
there exists a point \( v \in W \) such that \( w \geq v \) and \( E_u^* = E \). A model \( \mathfrak{M} \) is called weakly negative if \( \mathfrak{M}, w \vDash ^* p \iff \mathfrak{M}, w \vDash \neg \neg p \), and for all \( \geq \)-endpoints \( e \), \( \mathfrak{M}, e \vDash ^* p \).

Let \( M^* \) be the class of all finite weakly negative and weakly \( \geq \)-saturated tri-relation intuitionistic Kripke models satisfying (G1), (G2) with "\( \geq \)-endpoints" and "\( E_u \)" in the definition replaced by "\( \geq \)-second least points" and "\( E_u^* \)\), respectively, and (H2), (H3) and (H4) defined as follows:

\[ \begin{align*}
H2: \quad & S(w,u,v) \iff E_w^* = E_u^* \cup E_v^* . \\
H3: \quad & For any \( \geq \)-endpoint \( e \), \( eRw \) or \( wRe \) implies that \( w \) is also an \( \geq \)-endpoint. \\
H4: \quad & eRe for all \( \geq \)-endpoints \( e \).
\end{align*} \]

A full powerset model \( \mathfrak{M}^* \) has a unique \( \geq \)-endpoint, namely the empty set \( \emptyset \). Since \( \mathfrak{M}, \emptyset \vDash p \) for all \( p \), by Lemma 3.4 we know that \( \mathfrak{M}^*, \emptyset \vDash ^* p \) as well. We leave it for the reader to verify that all the other conditions of \( M^* \) are satisfied by the full powerset model \( \mathfrak{M}^* \) of any finite modal Kripke model \( \mathfrak{M} \), i.e., \( \mathfrak{M}^* \in M^* \).

The main result of this section is that the system of \( MT_0^* \) is complete with respect to the class \( M^* \), namely, the following theorem holds.

**Theorem 3.16**  
For any \( MT_0^* \) formula \( \phi \), \( \vDash_{MT_0^*} \phi \iff^* M^* \vDash \phi \).

The proof of the above theorem goes through a similar argument to that of Theorem 3.7. The direction "\( \iff^* \)" follows from Lemma 3.15 and the finite model property of \( MT_0^* \) (Theorem 2.17). The other direction "\( \iff \)" will follow from Theorem 3.19 and Lemma 3.20 to be stated and proved in the remainder of this section.

We first prove a lemma concerning the behavior of the double negation under the satisfaction relation \( \vDash^* \).

**Lemma 3.17**  
Let \( \mathfrak{M} = (W, \preceq, R, S, V) \) be a model in \( M^* \) and \( w \) is a non-\( \geq \)-endpoint in \( \mathfrak{M} \). Then, \( \mathfrak{M}, w \vDash ^* \neg \neg p \) iff \( \mathfrak{M}, w \vDash ^* p \) for some \( v \in E_w^* \).

**Proof**  
"\( \iff^* \)" Suppose \( \mathfrak{M}, v \vDash^* p \) for some \( v \in E_w^* \). For any point \( e \) such that \( v \geq e \) and \( v \neq e \), \( e \) is an \( \geq \)-endpoint, implying \( \mathfrak{M}, e \vDash \perp \). It follows that \( \mathfrak{M}, v \vDash ^* p \to \perp \). Since \( \mathfrak{M}, v \vDash ^* \perp \), we conclude \( \mathfrak{M}, w \vDash ^* (p \to \perp) \to \perp \).

"\( \iff \)" Suppose \( \mathfrak{M}, w \vDash ^* \neg \neg p \). Then, there exists \( u \leq w \) such that \( \mathfrak{M}, u \vDash ^* p \to \perp \) and \( \mathfrak{M}, u \vDash ^* \perp \). The latter implies that \( u \) is not an \( \geq \)-endpoint. Since \( \mathfrak{M} \) is weakly \( \geq \)-saturated, \( E_u^* \neq \emptyset \). Pick an element \( v \in E_u^* \subseteq E_w^* \). By the monotonicity of \( \geq \), we have \( \mathfrak{M}, v \vDash ^* p \to \perp \), implying \( \mathfrak{M}, v \vDash ^* p \), as required. \( \square \)

The notion of p-morphism for tri-relation intuitionistic Kripke models is the p-morphism for bi-relation intuitionistic Kripke models parametrized by the standard clause for binary diamonds.

**Definition 3.18**  
Let \( \mathfrak{M}_1 = (W_1, \preceq_1, R_1, S_1, V_1) \) and \( \mathfrak{M}_2 = (W_2, \preceq_2, R_2, S_2, V_2) \) be tri-relation intuitionistic Kripke models. A function \( f : W_1 \to W_2 \) is called a p-morphism if \( f \) satisfies (P1)-(P6) and (Q1) and (Q2) defined below:

\[ \begin{align*}
Q1: \quad & S_1(w,u,v) \implies S_2(f(w), f(u), f(v)) \\
Q2: \quad & S_2(f(w), u', v') \implies \exists u, v \in W_1 \text{ s.t. } f(u) = u', f(v) = v' \text{ and } S_1(w,u,v)
\end{align*} \]

We call a p-morphism \( f \) between two models \( \mathfrak{M}_1 = (W_1, \preceq_1, R_1, V_1) \) and \( \mathfrak{M}_2 = (W_2, \preceq_2, R_2, V_2) \) endpoint-preserving if

\[ Q3: \quad e \text{ is an } \preceq_1 \text{-endpoint } \iff f(e) \text{ is an } \preceq_2 \text{-endpoint.} \]

**Theorem 3.19**  
If \( f \) is an endpoint-preserving p-morphism between tri-relation intuitionistic Kripke models \( \mathfrak{M}_1 \) and \( \mathfrak{M}_2 \), then \( \mathfrak{M}_1, w \vDash ^* \phi \iff \mathfrak{M}_2, f(w) \vDash ^* \phi \).
Proof The theorem is proved by a routine argument.

Finally, we prove the crucial lemma of this section, from which the direction “$\Rightarrow$” of Theorem 3.16 will follow. The reader may compare this lemma with Lemma 3.10.

Lemma 3.20 For every finite tri-relation intuitionistic Kripke model $\mathfrak{M}$ in $\mathfrak{M}^*$, there exists a finite classical modal Kripke model $\mathfrak{R}$ such that there exists an endpoint preserving $\mathfrak{p}$-morphism $f$ of $\mathfrak{M}$ into the full powerset model $\mathfrak{M}^*$ induced by $\mathfrak{R}$.

Proof Let $\mathfrak{M} = (W, \geq, R, S, V)$. Define a modal Kripke model $\mathfrak{R} = (W_0, R_0, V_0)$ as:

- $W_0$ is the set of all $\geq$-second least points of $W$,
- $R_0 = R \upharpoonright W_0$ and $V_0 = V \upharpoonright W_0$.

Now, consider the full powerset Kripke model $\mathfrak{M}^* = (W_0^*, \geq, R_0^*, S_0^*, V_0^*)$ associated with $\mathfrak{M}$. Define a function $f : W \rightarrow W_0^*$ by taking

$$f(w) = E_w^*,$$

for all $w \in W$.

We show that $f$ is an endpoint-preserving $\mathfrak{p}$-morphism. Conditions (P2) and (P4) are verified by a similar argument to that in the proof of Lemma 3.10 taking into account the fact that $E_w$ is replaced by $E_w^*$ in this proof. We now give the proof for the other conditions.

(P1). We show that $\mathfrak{M}, w \mathfrak{p}^* p \iff \mathfrak{M}^*, E_w^* \mathfrak{p}^* p$. If $e$ is an $\geq$-endpoint, since $\mathfrak{M}$ is weakly negative, $\mathfrak{M}, e \mathfrak{p}^* p$. We also have $E_w^* = \emptyset$, and $\mathfrak{M}^*, \emptyset \mathfrak{p}^* p$ by the empty team property and Lemma 3.15. If $w$ is not an $\geq$-endpoint, applying Lemma 3.17, the condition is proved by a similar argument to that in the proof of Lemma 3.10.

(P3). Assume $wRv$, we show that $E_w^* R_0^* E_v^*$, namely $E_w^* R_0 E_v^*$. If one of $w$ and $v$ is an $\geq$-endpoint, then by (H3), both $w$ and $v$ are $\geq$-endpoints, which implies that $E_w^* = \emptyset = E_v^*$. By definition, $\emptyset R_0 \emptyset$, i.e., $E_w^* R_0 E_v^*$.

Now, assume that both $w$ and $v$ are not $\geq$-endpoints. By a similar argument to that in Lemma 3.10, for any $s \in E_w^*$, by (P1) of $\mathfrak{M}$, there exists $t \in W$ such that $v \geq t$ and $s \mathfrak{R} t$. For each $t' \in E_v^*$ such that $t \geq t'$, by (F2), there exists $s' \in W$ such that $s \geq s'$ and $s' \mathfrak{R} t'$. But since $s$ is a $\geq$-second least point, either $s = s'$ or $s'$ is an $\geq$-endpoint. In the latter case, we conclude from (H3) that $t'$ is also an $\geq$-endpoint, which is a contradiction. Thus, the former is the case, and $s \mathfrak{R} t'$. On the other hand, for any $t \in E_v^*$, by a similar argument, we can find an $s' \in E_w^*$ such that $s' \mathfrak{R} t$. Hence we conclude that $E_w^* R_0 E_v^*$.

(P5). Suppose $E_w^* R_0 E_v^*$. If one of $E_w^*$ and $v'$ is the empty set, then $E_w^* = \emptyset = v'$ by the definition of $R_0$.* Since $\mathfrak{M}$ is weakly $\geq$-saturated, $w$ must be an $\geq$-endpoint, which implies $wRv$. Also, clearly, $v' \supseteq \emptyset = E_w^* = f(w)$ as required. If $E_w^*, v' \neq \emptyset$, then the condition follows from a similar argument to that in the proof of Lemma 3.10.

(P6). Suppose $E_w^* (\cup \mathfrak{R}_0) v'$. If $v' = \emptyset$, then let $v$ be any $\geq$-endpoint such that $w \geq v$. Clearly, $f(v) = E_w^* = \emptyset \geq v'$. By (H4), $vRv$, and thus $w(\geq \mathfrak{R}) v$. If $v' \neq \emptyset$, then the condition follows from a similar argument to that in the proof of Lemma 3.10.

(Q1). Suppose $S(w, u, v)$. By (H2), $E_w^* = E_u^* \cup E_v^*$, thereby $R_0(f(w), f(u), f(v))$.

(Q2). Suppose $R_0(f(w), u', v')$. Then $E_w^* = u' \cup v'$. Since $\mathfrak{M}$ is weakly $\geq$-saturated, there exist $u, v \in W$ such that $w \geq u, v, E_u^* = u'$ and $E_v^* = v'$. It follows that $f(w) = u'$, $f(v) = v'$. Since $E_w^* = E_u^* \cup E_v^*$, by (H2), we obtain $S(w, u, v)$.

(Q3). If $e$ is an $\geq$-endpoint, then $f(e) = E_e^* = \emptyset$, which is the unique $\geq$-endpoint of $\mathfrak{M}^*$. Conversely, if $w$ is not an $\geq$-endpoint, then since $\mathfrak{M}$ is weakly $\geq$-saturated, we have $E_w^* \neq \emptyset$, i.e., $f(w)$ is not an $\geq$-endpoint.
4 Concluding remarks

In this paper, we have studied the axiomatization problem and some model-theoretic properties of the major modal dependence logics considered in the literature, namely $\text{MT}_6$, $\text{MID}$, $\text{MD}^\lor$, $\text{MD}$ and $\text{MD}^+$. In the first part of the paper, we introduced sound and complete Hilbert-style or natural deduction systems for all these logics, among which those logics with intuitionistic implication have not been axiomatized before. We presented the system of $\text{MT}_0$ as an extension of Fischer Servi’s intuitionistic modal logic $\text{IK}$ and the Kreisel-Putnam intermediate logic $\text{KP}$, and the systems of all the other modal dependence logics are its fragments and variants. We showed that formulas of all these modal dependence logics (essentially) enjoy a same disjunctive normal form that is essentially already known in the literature. We also derived some metalogical properties of the logics, such as Craig’s Interpolation Theorem and the Finite Model Property, as immediate corollaries of the normal form.

First-order teams are essentially relations in first-order models, and first-order dependence logic is expressively equivalent to existential second-order logic [28, 17]. In a similar fashion, in the second part of the paper we interpreted modal teams as possible worlds in powerset models in Lemmas 3.4 and 3.15, and on the basis of this we showed in Theorems 3.7 and 3.16 that $\text{MID}^-$ and $\text{MT}_0^-$ can be understood as intermediate modal logics also from the model-theoretic perspective in the sense that they are complete (in the single-world semantics sense) with respect to certain classes of intuitionistic Kripke models. It is worth pointing out that although Lemmas 3.4 and 3.15 are not explicitly found in the literature, their intuitive idea seems to be folklore in the field or have in some sense already been used as a guideline in some research. For instance, the perfect information semantic set game introduced in [29] for modal dependence logic played over modal Kripke models can actually be viewed as a standard perfect information semantic game played over the associated full powerset models, and the correctness of the set game is essentially justified by Lemma 3.15. It is the author’s hope that the connections established in the paper between team semantics and single-world semantics, and between modal dependence logics and intermediate modal logics can provide a pointer for a deeper understanding of team semantics and team-based logics.

As acknowledged in the corresponding sections, many results of this paper are built on or inspired by the literature of inquisitive logic. Inquisitive modal logic (see e.g. [5]) can be viewed as a variant of model dependence logic with different modalities. It is interesting to see whether inquisitive modal logic can also be given a single-world semantics in a similar manner, and to compare inquisitive modal logic with intermediate modal logics.

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Notes
1. The team semantics of an extended dependence atom \( \equiv(a_1, \ldots, a_k, \beta) \) with \( a_1, \ldots, a_k, \beta \) first-order formulas is defined as \( M \models X \equiv(a_1, \ldots, a_k, \beta) \) iff for all \( x, x' \in X : M \models_\pi a_i \iff M \models_\pi a_i \) for all \( 1 \leq i \leq k \) implies \( M \models_\pi \equiv M \models_\pi x \).

This atom can be defined in first-order dependence logic as \( \equiv(a_1, \ldots, a_k, \beta) := \forall x \forall y (x = y) \lor \exists w_1 \ldots \exists w_k \exists v_0 \exists v_1 (\equiv(w_1, \ldots, w_k, u) \land (v_0) \land (v_1) \land (v_0 \neq v_1)) \)

\( \land \bigwedge_{i=1}^{k} \left( \theta(w_i, v_0, v_1) \land \delta(w_i, a_i, v_0, v_1) \right) \land \theta(u, v_0, v_1) \land \delta(u, \beta, v_0, v_1)) \),

where \( \theta(v, v_0, v_1) := (v = v_0) \lor (v = v_1) \) and \( \delta(v, y, v_0, v_1) := (\neg y \lor (v = v_1)) \land (y \lor (v = v_0)) \).

The defining formula states intuitively that “either the model in question has only one element in its domain (in which case \( \equiv(a_1, \ldots, a_k, \beta) \) is trivially satisfied), or the team in question satisfies \( \equiv(w_1, \ldots, w_k, u) \), where \( w_i \) simulates \( a_i \) and \( u \) simulates \( \beta \).”

The intuitionistic disjunction can be defined in first-order dependence logic as \( \phi \lor \psi := (\exists x \exists y (x \neq y) \lor (\phi \lor \psi)) \land (\forall x \forall y (x = y) \lor (\phi \lor \psi)) \land (\forall x \forall y (x = y) \lor (\phi \lor \psi)) \),

where the first conjunct of the defining formula deals with the case when the model has cardinality 1, and the second conjunct deals with the other cases.

2. The deduction system of \( \text{MID}^- \) is obtained (naturally) from the system of \( \text{MID} \), as defined in Definition 2.3 or in Definition 2.9, by simply dropping all the rules that involve dependence atoms. Similarly for the deduction system of \( \text{MT}_0 \).

3. In the literature intermediate modal logics are often obtained by adding to intuitionistic modal logic (either Fischer Servi’s \( \text{IK} \) or some other versions) extra modal axioms, such as \( \text{S}4, \text{S}5 \) axioms. The approach we take in this paper is, roughly, to add to \( \text{IK} \) an extra propositional axiom, the \( \text{KP} \) axiom.

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