Maximum principle for stochastic optimal control problem of finite state forward-backward stochastic difference systems

Shaolin Ji | Haodong Liu

1Zhongtai Securities Institute for Financial Studies, Shandong University, Jinan, Shandong, People’s Republic of China
2School of Economics, Ocean University of China, Qingdao, Shandong, People’s Republic of China

Correspondence
Haodong Liu, School of Economics, Ocean University of China, Qingdao, Shandong 266100, People’s Republic of China
Email: liuhaodong@ouc.edu.cn

Funding information
Central University Basic Research Fund of China, Grant/Award Number: 202013013;
National Natural Science Foundation of China, Grant/Award Number: 11571203

Abstract
This article studies the maximum principle for stochastic optimal control problems of forward-backward stochastic difference systems (FBSΔSs) where the uncertainty is modeled by a discrete time, finite state process, rather than white noises. Two distinct forms of FBSΔSs are investigated. The first one is described by a partially coupled forward-backward stochastic difference equation (FBSΔE) and the second one is described by a fully coupled FBSΔE. We deduce the adjoint difference equation by adopting an appropriate representation of the product rule and a proper formulation of the backward stochastic difference equation (BSΔE). Finally, the maximum principle for this optimal control problem with the convex control domain is established.

KEYWORDS
backward stochastic difference equations, forward-backward stochastic difference equations, maximum principle, monotone condition, stochastic optimal control

1 | INTRODUCTION

The maximum principle is one of the critical approaches in solving optimal control problems. Numerous studies have been conducted on the maximum principle for stochastic systems. See, for example, Bensoussan,1 Bismut,2 Kushner,3 Peng.4 Peng also firstly studied one kind of forward-backward stochastic control system (FBSCS) in Reference 5 and obtained the maximum principle for this kind of control system with the convex control domain. The FBSCSs have wide applications in many fields. The stochastic differential recursive utility, a generalization of a standard additive utility, can be regarded as a solution of a backward stochastic differential equation (BSDE). The recursive utility optimization problem can be described as an optimization problem for an FBSCS (see Reference 6). Besides, in the dynamic principal-agent problem with unobservable states and actions, the principal’s problem can be formulated as a partial information optimal control problem of an FBSCS (see Reference 7). We refer to References 8-17 for other works on optimization problems for FBSCSs.

This article will discuss the maximum principle for optimal control of discrete time systems described by forward-backward stochastic difference equations (FBSΔEs). Discrete time control systems are very useful in applications. For instance, the digital control can be expressed as discrete time control problems, with sampled data collected at discrete time instants. Additionally, the forward-backward stochastic difference system (FBSΔS) can be used for modeling in financial markets. The solution to the backward stochastic difference equation (BSΔE) can be utilized to build time-consistent nonlinear expectations (see References 18-20) and the solution to BSΔE is also used for pricing in financial markets (see Reference 21). However, the formulation of BSΔE is quite different from its continuous time counterpart.
Many works are devoted to the study of BSDEs (see, e.g., References 19-22). Based on the driving process, there are mainly two formulations of BSDEs. One is driving by a finite state process that takes values from the basis vectors (as in Reference 19), and the other is driving by a martingale with independent increments (as in Reference 21). For the former framework, the researchers in Reference 19 obtained the discrete time version of the martingale representation theorem and established the solvability result of BSDE with the uniqueness of $Z$ under a new kind of equivalence relation. Further works about the applications of the finite state framework can be seen in References 18, 23-25. In this research, we study the optimization problems for FBSDEs using the first type of formulation.

We examine two stochastic optimal control problems in this article. Problem 1 entails a partially coupled FBSDE (2). More precisely, the forward equation’s coefficients $b$ and $\sigma$ do not contain the backward equation’s solution $(Y, Z)$. In comparison, the state equation of Problem 2 is given by a fully coupled FBSDE (4).

The optimal control problem is to find the optimal control $u \in U$, such that the optimal control and the corresponding state trajectory can minimize the cost functional $J(u(\cdot))$. In this article, we assume the control domain is convex. By making the perturbation of the optimal control at a fixed time point, we obtain the maximum principle for Problem 1 and 2.

There are two difficulties in building the maximum principle. The first one is to find the adjoint variables which can be applied to deduce the variational inequality. In Reference 26, the authors studied the maximum principle for a discrete time stochastic optimal control problem in which the state equation is only governed by a forward stochastic difference equation. By applying the Riesz representation theorem (lemma 2.1 of Reference 26), they explicitly obtained the adjoint variables and establish the maximum principle. But to solve our problems, we need to construct the adjoint difference equations since the adjoint variables cannot be obtained explicitly for our case. To construct the adjoint equations in our discrete time framework, the techniques adopted for the continuous time framework as in Reference 4, 5 are not applicable. This article proposes two techniques to deduce the adjoint difference equations. The first one is that we choose the following product rule:

$$\Delta \langle X_t, Y_t \rangle = \langle X_{t+1}, \Delta Y_t \rangle + \langle \Delta X_t, Y_t \rangle,$$

where $X_t$ (resp. $Y_t$) is subjected to a forward (resp. backward) stochastic difference equation. The second one is that the BSDE should be formulated as in (1). In other words, the generator $f$ of the BSDE (1) depends on time $t + 1$. It is worth pointing out that this kind of formulation is just the formulation of the adjoint equations for stochastic optimal control problems (see Reference 26 for the classical case). Based on these two techniques, we can deduce the adjoint difference equations. The readers may refer to Remark 2 for more details.

Besides, the second difficulty is in the finite state space case. Concerning the solution $\{Y_t, Z_t\}_{t=\{0,1, \ldots, T\}}$ to the BSDE defined in the finite state space framework, the uniqueness of the variable $Z_t$ is not defined in the normal sense. So the norm of the variable should be redefined. In Reference 19, Cohen and Elliott defined a seminorm of $Z_t$ through the BSDE diffusion term $Z_t M_{t+1}$ (here the driving process of the BSDE is denoted by $M$). However, since the Itô isometry cannot work in the discrete time case and the martingale difference process $M_t$ depends on the past, the relation between the norm defined by $Z_t$ itself and the norm defined by $Z_t M_{t+1}$ is not clear. So it makes estimating the diffusion term of the variation equations quite difficult. We propose a new definition of the norm for the variable $Z_t$ in the diffusion term and prove the relation between this norm of $Z_t$ and the seminorm defined by $Z_t M_{t+1}$. We can estimate the solutions to the stochastic difference equations in the discrete time finite state space framework with this relation.

The remainder of this article is organized as follows. In Section 2, two distinct forms of controlled FBSDEs are formulated. Section 3 deduces the maximum principle for the partially coupled controlled FBSDE. Section 4 establishes the maximum principle for fully coupled controlled FBSDEs. Finally, Section 5 concludes this study.

## 2 Preliminaries and Model Formulation

Let $T$ be a deterministic terminal time and $\mathcal{T} := \{0, 1, \ldots, T\}$. Following, we consider an underlying discrete time, finite state process $W$ which takes values in the standard basis vectors of $\mathbb{R}^d$, where $d$ is the number of states of the process $W$. In more detail, for each $t \in \mathcal{T}$, $W_t \in \{e_1, e_2, \ldots, e_d\}$ where $e_t = (0, 0, \ldots, 0, 1, 0, \ldots, 0)^\top \in \mathbb{R}^d$ and $[\cdot]^\top$ denotes vector transposition.

Consider a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0\leq t \leq T}, \mathbb{P})$, where $\mathcal{F}_t$ is the completion of the $\sigma$-algebra generated by the process $W$ up to time $t$ and $\mathcal{F} = \mathcal{F}_T$. Besides, the atoms of $\mathcal{F}$ are, up to a null set, sets of positive measure of the form
\( \{ \omega' \in \Omega : W_t(\omega') = W_t(\omega) \text{ for all } 0 \leq t \leq T \} \),

for some \( \omega \in \Omega \).

For simplicity, we suppose the process \( W \) satisfies the following assumption.

**Assumption 1.** For any \( t \in \{0, 1, 2, \ldots, T - 1\} \), \( \mathbb{E}[W_{t+1}|F_t](\omega) > 0 \) holds on every atom of \( F_t \).

Note that in this article, an inequality on a vector quantity is to hold componentwise. Under the above assumption, the transition probabilities for \( W \) to \( N \) states are positive. In fact, this assumption is not necessary and is just for the sake of simple expression.

Denote by \( L(F_t; \mathbb{R}^{nxd}) \) the set of all \( F_t \)-adapted random variables \( X_t \) taking values in \( \mathbb{R}^{nxd} \) and by \( \mathcal{M}(0, t; \mathbb{R}^{nxd}) \) the set of all \( F_t \)-adapted processes \( X \) taking values in \( \mathbb{R}^{nxd} \) with the norm \( \| \cdot \| \) defined by \( \|X\| = \left( \mathbb{E} \left[ \sum_{i=0}^{d} |X_i|^2 \right] \right)^{\frac{1}{2}} \). Let 

\[
(P^1_t, P^2_t, \ldots, P^d_t)^* = \mathbb{E}[W_{t+1}|F_t].
\]

Define

\[
M_t = W_t - \mathbb{E}[W_t|F_{t-1}], \quad t = 1, \ldots, T.
\]

Then \( M \) is a martingale difference process taking values in \( \mathbb{R}^d \). The following equivalence relations given in Reference 19 will be used in the following.

**Definition 1.** For two \( F_t \)-measurable random variables \( Z_t \) and \( \tilde{Z}_t \), we define \( Z_t \sim_M \tilde{Z}_t \) if \( Z_t M_{t+1} = \tilde{Z}_t M_{t+1} \).

For two adapted processes \( Z \) and \( \tilde{Z} \), we define \( Z \sim_M \tilde{Z} \) if \( Z_t M_{t+1} = \tilde{Z}_t M_{t+1} \) for any \( t \in \{0, 1, 2, \ldots, T - 1\} \).

For a \( F_t \)-adapted process \( X \), define the difference operator \( \Delta \) as \( \Delta X_t = X_{t+1} - X_t \). Consider the following backward stochastic difference equation (BSDE):

\[
\begin{aligned}
\Delta Y_t &= -f(\omega, t + 1, Y_{t+1}, Z_{t+1}) + Z_t M_{t+1}, \\
Y_T &= \eta,
\end{aligned}
\tag{1}
\]

where \( \eta \in L(F_T; \mathbb{R}^n) \) and \( f : \Omega \times \{1, 2, \ldots, T\} \times \mathbb{R}^n \times \mathbb{R}^{nxd} \rightarrow \mathbb{R}^n \) is \( F_t \)-adapted mapping.

**Assumption 2.** A1. For any \( y \in \mathbb{R}^n \), \( t \in \{1, 2, \ldots, T - 1\} \), \( \omega \in \Omega \), and \( Z^1, Z^2 \in \mathcal{M}(0, T - 1; \mathbb{R}^{nxd}) \), if \( Z^1 \sim_M Z^2 \), then

\[
f(\omega, t, y, Z^1_t) = f(\omega, t, y, Z^2_t).
\]

A2. The function \( f(t, y, z) \) is independent of \( z \) at \( t = T \).

The following theorem gives a result about existence and uniqueness of BSDE (1).

**Theorem 1.** Suppose that Assumption 2 holds. Then for any terminal condition \( \eta \in L(F_T; \mathbb{R}^n) \), BSDE (1) has a unique adapted solution \((Y, Z)\). Here the uniqueness for \( Y \) is in the sense of indistinguishability and for \( Z \) is in the sense of \( \sim_M \) equivalence.

For a proof of this theorem, we refer to Reference 27, theorem 2.10.

We define the \( d \times (d - 1) \) matrix \( \bar{I} = (I_{d-1} - \mathbf{1}_{d-1})^* \), where \( I_{d-1} \) is \((d - 1)\)-dimensional identity matrix, \( \mathbf{1}_{d-1} = (1, 1, \ldots, 1)^* \) is \((d - 1)\)-dimensional vector with every element being equal to 1. Then, we consider two types of controlled systems.

Problem 1 (partially coupled system):

The controlled system is

\[
\begin{aligned}
\Delta X_t &= b(\omega, t, X_t, u_t) + \sum_{i=1}^{m} e_i \cdot \sigma_i(\omega, t, X_t, u_t) M_{t+1}, \\
\Delta Y_t &= -f(\omega, t + 1, X_{t+1}, Y_{t+1}, Z_{t+1} \bar{I}, u_{t+1}) + Z_t M_{t+1}, \\
X_0 &= x_0, \\
Y_T &= y_T,
\end{aligned}
\tag{2}
\]
and the cost functional is
\[
J(u(\cdot)) = \mathbb{E} \sum_{t=0}^{T-1} l(\omega, t, X_t, Y_t, Z_t, u_t) + h(\omega, X_T),
\]
(3)

where
\[
\begin{align*}
    b(\omega, t, x, u) & : \Omega \times \{0, 1, \ldots, T - 1\} \times \mathbb{R}^m \times \mathbb{R}^r \to \mathbb{R}^m, \\
    \sigma(\omega, t, x, u) & : \Omega \times \{0, 1, \ldots, T - 1\} \times \mathbb{R}^m \times \mathbb{R}^r \to \mathbb{R}^{1 \times d}, \\
    f(\omega, t, x, y, \xi, u) & : \Omega \times \{1, 2, \ldots, T\} \times \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^{n \times (d - 1)} \times \mathbb{R}^r \to \mathbb{R}^m, \\
    l(\omega, t, x, y, \xi, u) & : \Omega \times \{0, 1, \ldots, T - 1\} \times \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^{n \times (d - 1)} \times \mathbb{R}^r \to \mathbb{R}, \\
    h(\omega, x) & : \bar{\Omega} \times \mathbb{R}^m \to \mathbb{R}.
\end{align*}
\]

Problem 2 (fully coupled system):

The controlled system is:
\[
\begin{align*}
    \Delta X_t &= b(\omega, t, X_t, Y_t, Z_t, u_t) + \sigma(\omega, t, X_t, Y_t, Z_t, u_t) M_{t+1}, \\
    \Delta Y_t &= -f(\omega, t + 1, X_{t+1}, Y_{t+1}, Z_{t+1}, u_{t+1}) + Z_t M_{t+1}, \\
    X_0 &= x_0, \\
    Y_T &= y_T.
\end{align*}
\]
(4)

and the cost functional is
\[
J(u(\cdot)) = \mathbb{E} \sum_{t=0}^{T-1} l(\omega, t, X_t, Y_t, Z_t, u_t) + h(\omega, X_T),
\]
(5)

where
\[
\begin{align*}
    b(\omega, t, x, y, \xi, u) & : \Omega \times \{0, 1, \ldots, T - 1\} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{1 \times (d - 1)} \times \mathbb{R}^r \to \mathbb{R}, \\
    \sigma(\omega, t, x, y, \xi, u) & : \Omega \times \{0, 1, \ldots, T - 1\} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{1 \times (d - 1)} \times \mathbb{R}^r \to \mathbb{R}^{1 \times d}, \\
    f(\omega, t, x, y, \xi, u) & : \Omega \times \{1, 2, \ldots, T\} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{1 \times (d - 1)} \times \mathbb{R}^r \to \mathbb{R}, \\
    l(\omega, t, x, y, \xi, u) & : \Omega \times \{0, 1, \ldots, T - 1\} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{1 \times (d - 1)} \times \mathbb{R}^r \to \mathbb{R}, \\
    h(\omega, x) & : \bar{\Omega} \times \mathbb{R} \to \mathbb{R}.
\end{align*}
\]

Let \( \{U_t\}_{t \in \{0, 1, \ldots, T\}} \) be a sequence of nonempty convex subset of \( \mathbb{R}^r \). We denote the set of admissible controls \( U' \) by \( U' = \{u(\cdot) \in \mathcal{M}(0, T; \mathbb{R}^r) \mid u(t) \in U_t\} \). It can be seen that in Problem 1, \( b \) and \( \sigma \) do not contain the solution \((Y, Z)\) of the backward equation. This kind of FBS\( \Delta \)E is called the partially coupled FBS\( \Delta \)E. Meanwhile, the system in Problem 2 is called the fully coupled FBS\( \Delta \)E.

The optimal control problem is to find the optimal control \( u \in U' \), such that the optimal control and the corresponding state trajectory can minimize the cost functional \( J(u(\cdot)) \). In this article, we assume the control domain is convex.

Remark 1. The cost functional in Reference 5 consists of three parts: the running cost functional, the terminal cost functional of \( X_T \), the initial cost functional of \( Y_0 \). In our formulation, if we take \( l(\omega, 0, X_0, Y_0, Z_0, u_0) = \gamma(\omega, Y_0) \), then the cost functional (5) for our discrete time framework can be reduced to the cost functional in Reference 5 formally.

For controlled system (2)–(3), we assume that:

**Assumption 3.** For \( \varphi = b, \sigma, f, l, h \),

1. \( \varphi \) is an adapted map, for example, for any \((x, y, \xi, u) \in \mathbb{R}^m \times \mathbb{R}^r \times \mathbb{R}^{n \times (d - 1)} \times \mathbb{R}^r, \varphi(\cdot, \cdot, x, y, \xi, u) \) is \( \{F_t\} \)-adapted process.
2. for any \( t \in \{0, 1, \ldots, T\} \) and \( \omega \in \Omega \), \( \varphi(\omega, t, \cdot, \cdot, \cdot, \cdot) \) is continuously differentiable with respect to \( x, y, \xi, u \), and \( \varphi_x, \varphi_y, \varphi_{\xi}, \varphi_u \) are uniformly bounded. Also, for \( t = T \), \( f \) is independent of \( \xi \) at time \( T \).
Set

\[ \dot{\lambda} = (x, y, z), \]
\[ A(t, \lambda; u) = (-f(t, \lambda; u), b(t, \lambda; u), \sigma(t, \lambda; u)) \mathbb{E} \left[ M_{t+1} M_{t+1}^* | F_t \right], \]

and

\[ |\lambda| = |x| + |y| + |z|, \]
\[ |A(t, \lambda)| = |f(t, \lambda)| + |b(t, \lambda)| + |\sigma(t, \lambda)| \mathbb{E} \left[ M_{t+1} M_{t+1}^* | F_t \right]. \]

For controlled system (4)–(5), we additionally assume that:

**Assumption 4.** For any \( u \in U \), the coefficients in (4) satisfy the following monotone conditions, for example, when \( t \in \{1, \ldots, T-1\} \),

\[ \langle A(t, \lambda_1; u) - A(t, \lambda_2; u), \lambda_1 - \lambda_2 \rangle \leq -\alpha |\lambda_1 - \lambda_2|^2, \quad \forall \lambda_1, \lambda_2 \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{1 \times d}; \]

when \( t = T \),

\[ \langle -f(T, x_1, y, z; u) + f(T, x_2, y, z; u), x_1 - x_2 \rangle \leq -\alpha |x_1 - x_2|^2; \]

when \( t = 0 \),

\[ \langle b(0, \lambda_1; u) - b(0, \lambda_2; u), y_1 - y_2 \rangle + \langle (\sigma(0, \lambda_1; u) - \sigma(0, \lambda_2; u)) \mathbb{E} \left[ M_1 M_1^* | F_0 \right], z_1 - z_2 \rangle \leq -\alpha \left[ |y_1 - y_2|^2 + |z_1 - z_2|^2 \right]. \]

where \( \alpha \) is a given positive constant.

Besides, in the following, we formally denote \( b(T, x, y, z; u) \equiv 0, \sigma(T, x, y, z; u) \equiv 0, l(T, x, y, z; u) \equiv 0, f(0, x, y, z; u) \equiv 0 \).

### 3 Maximum Principle for the Partially Coupled FBSDE System

For any \( u \in U \), it is obvious that there exists a unique solution \( \{X_t\}_{t=0}^T \in \mathcal{M}(0, T; \mathbb{R}^m) \) to the forward stochastic differential equation in the system (2). According to lemma 2.3 in Reference 27, it can be seen that \( f \) satisfies Assumption 2. So given \( X \), by Theorem 1, the backward equation in the system (2) has a unique solution \( (Y, Z) \).

Suppose that \( \bar{u} = \{\bar{u}_t\}_{t=0}^T \) is an optimal control of problem (2)–(3) and \((\bar{X}, \bar{Y}, \bar{Z})\) is the corresponding optimal trajectory. For a fixed time \( 0 \leq s \leq T \), choose any \( \Delta v \in L(F_s; \mathbb{R}^r) \) such that \( \bar{u}_s + \Delta v \) takes values in \( U_s \). For any \( \varepsilon \in [0, 1] \), construct the perturbed admissible control

\[ u^*_t = (1 - \delta_s) \bar{u}_t + \delta_s (\bar{u}_s + \varepsilon \Delta v) = \bar{u}_t + \delta_s \varepsilon \Delta v, \quad (6) \]

where \( \delta_s = 1 \) for \( t = s \), \( \delta_t = 0 \) for \( t \neq s \) and \( t \in \{0, 1, \ldots, T\} \). Since \( U_s \) is a convex set, \( \{u^*_t\}_{t=0}^T \in U \) is an admissible control.

Let \((X^*, Y^*, Z^*, N^*)\) be the solution of (2) corresponding to the control \( u^* \).

Set

\[ \bar{\varrho}(t) = \varphi \left( t, \bar{X}_t, \bar{Y}_t, \bar{Z}_t, \bar{u}_t \right), \quad \varrho^*(t) = \varphi \left( t, X^*_t, Y^*_t, Z^*_t, u^*_t \right), \]
\[ \bar{\varrho}'(t) = \varphi \left( t, \bar{X}_t, \bar{Y}_t, \bar{Z}_t, u^*_t \right), \quad \varrho^*_t(t) = \varphi^* \left( t, \bar{X}_t, \bar{Y}_t, \bar{Z}_t, \bar{u}_t \right), \quad (7) \]

where \( \varphi = b, \sigma, f, l, \text{and } \mu = x, y, z \text{ and } u \).

Then, we have the following estimates.
Lemma 1. Under Assumption 3, we have

\[
\sup_{0 \leq t \leq T} \mathbb{E} \left| X_t^e - \bar{X}_t \right|^2 \leq C \varepsilon^2 \mathbb{E} |\Delta v|^2. \tag{8}
\]

Proof. In the following, the positive constant \(C\) may change from lines to lines.

When \(t = 0, \ldots, s, X_t^e = \bar{X}_t\).

When \(t = s + 1, \)

\[
X_{s+1}^e - \bar{X}_{s+1} = \tilde{b}^e(s) - \tilde{b}(s) + \sum_{i=1}^m \epsilon_i \cdot \left[ \tilde{\sigma}_i^e(s) - \tilde{\sigma}_i(s) \right] M_{s+1}.
\]

Then,

\[
\mathbb{E} \left| X_{s+1}^e - \bar{X}_{s+1} \right|^2 \leq 2 \mathbb{E} \left[ \left| \tilde{b}^e(s) - \tilde{b}(s) \right|^2 + \sum_{i=1}^m \left| \left[ \tilde{\sigma}_i^e(s) - \tilde{\sigma}_i(s) \right] M_{s+1} \right|^2 \right].
\]

By the boundness of \(b_u\), we have

\[
\mathbb{E} \left[ \left| \tilde{b}^e(s) - \tilde{b}(s) \right|^2 \right] \leq C \mathbb{E} \left[ \left| \nu^e - \bar{u}_t \right|^2 \right] = C \varepsilon^2 \mathbb{E} \left[ |\Delta v|^2 \right].
\]

By proposition 2.4 in Reference 27 and boundness of \(\sigma_{iu}\bar{I}\), we have

\[
\mathbb{E} \left[ \left| \tilde{\sigma}_i^e(s) - \tilde{\sigma}_i(s) \right| M_{s+1} \right]^2 \leq C \mathbb{E} \left[ \left| \tilde{\sigma}_i^e(s) - \tilde{\sigma}_i(s) \right| \bar{I} \right]^2 \leq C \varepsilon^2 \mathbb{E} \left[ |\Delta v|^2 \right],
\]

which leads to

\[
\mathbb{E} \left| X_{s+1}^e - \bar{X}_{s+1} \right|^2 \leq C \varepsilon^2 \mathbb{E} \left[ |\Delta v|^2 \right].
\]

When \(t = s + 2, \ldots, T, \)

\[
\mathbb{E} \left| X_t^e - \bar{X}_t \right|^2 \leq 2 \mathbb{E} \left[ \left| b (t-1, X_{t-1}^e, \bar{u}_{t-1}) - b (t-1, \bar{X}_{t-1}, \bar{u}_{t-1}) \right|^2 \right. \\
+ \left. \sum_{i=1}^m \left| \sigma_i (t-1, X_{t-1}^e, \bar{u}_{t-1}) - \sigma_i (t-1, \bar{X}_{t-1}, \bar{u}_{t-1}) \right| M_t \right]^2.
\]

Due to the boundness of \(b_s, \sigma_{iu}\bar{I}\), combined with proposition 2.4 in Reference 27, we obtain \(\mathbb{E} \left| X_t^e - \bar{X}_t \right|^2 \leq C \mathbb{E} \left[ \left| X_{t-1}^e - \bar{X}_{t-1} \right|^2 \right]\). Thus, by induction we prove the result.

\[
\begin{aligned}
\Delta \xi_t &= b_s(t)\xi_t + \delta_{iu}(t)\Delta v + \sum_{i=1}^m \epsilon_i \cdot [\xi_i^e \sigma_{iu}(t) + \delta_{iu} \epsilon \Delta v^u \sigma_{iu}(t)] M_{t+1}, \\
\xi_0 &= 0.
\end{aligned}
\tag{9}
\]

It is easy to check that

\[
\sup_{0 \leq t \leq T} \mathbb{E} |\xi_t|^2 \leq C \varepsilon^2 \mathbb{E} |\Delta v|^2, \tag{10}
\]

and we have the following result:
Lemma 2. Under Assumption 3, we have
\[
\sup_{0 \leq t \leq T} \mathbb{E} \left[ X_t^r - X_t - \xi_t \right]^2 = o ( \varepsilon^2 ).
\]

Proof. When \( t = 0, \ldots, s \), \( X_t^r = \bar{X}_t \), and \( \xi_t = 0 \) which lead to \( X_t^r - \bar{X}_t - \xi_t = 0 \).

When \( t = s + 1 \),
\[
X_{s+1}^r - \bar{X}_{s+1} - \xi_{s+1} = \left[ \bar{b}_u(s) - b_u(s) \right] \varepsilon \Delta v + \sum_{i=1}^m e_i \cdot \varepsilon \Delta v^* [\bar{\sigma}_{iu}(s) - \sigma_{iu}(s)] M_{s+1},
\]
where
\[
\bar{b}_u(s) = \int_0^1 b_u \left( s, \bar{X}_s, \bar{u}_s + \lambda (u_s^e - \bar{u}_s) \right) d\lambda,
\]
\[
\bar{\sigma}_{iu}(s) = \int_0^1 \sigma_{iu} \left( s, \bar{X}_s, \bar{u}_s + \lambda (u_s^e - \bar{u}_s) \right) d\lambda.
\]

Then
\[
\mathbb{E} \left[ X_{s+1}^r - \bar{X}_{s+1} - \xi_{s+1} \right]^2 \leq 2 \mathbb{E} \left[ \left( \left[ \bar{b}_u(s) - b_u(s) \right] \varepsilon \Delta v \right)^2 + \sum_{i=1}^m \left( \varepsilon \Delta v^* [\bar{\sigma}_{iu}(s) - \sigma_{iu}(s)] M_{s+1} \right)^2 \right] \leq C \mathbb{E} \left( \left\| \bar{b}_u(s) - b_u(s) \right\|^2 \Delta v^2 + \sum_{i=1}^m \left\| [\bar{\sigma}_{iu}(s) - \sigma_{iu}(s)] \bar{I} \right\|^2 \Delta v \right)^2 \varepsilon^2.
\]

Since \( \left\| \bar{b}_u(s) - b_u(s) \right\| \to 0 \) and \( \left\| [\bar{\sigma}_{iu}(s) - \sigma_{iu}(s)] \bar{I} \right\| \to 0 \) as \( \varepsilon \to 0 \), we have
\[
\lim_{\varepsilon \to 0} \frac{1}{\varepsilon^2} \mathbb{E} \left[ X_{s+1}^r - \bar{X}_{s+1} - \xi_{s+1} \right]^2 = 0.
\]

When \( t = s + 2, \ldots, T \),
\[
X_t^r - \bar{X}_t - \xi_t = \bar{b}_x(t-1) \left( X_{t-1}^r - \bar{X}_{t-1} - \xi_{t-1} \right) + \left[ \bar{b}_x(t-1) - b_x(t-1) \right] \xi_{t-1}
\]
\[
+ \sum_{i=1}^m e_i \cdot \left\{ \left( X_{t-1}^r - \bar{X}_{t-1} - \xi_{t-1} \right)^* \bar{\sigma}_{ix}(t-1) + \xi_{t-1}^* [\bar{\sigma}_{ix}(t-1) - \sigma_{ix}(t-1)] \right\} M_i,
\]
where
\[
\bar{b}_x(t) = \int_0^1 b_x \left( t, \bar{X}_t, + \lambda \left( X_t^r - \bar{X}_t \right), \bar{u}_t \right) d\lambda,
\]
\[
\bar{\sigma}_{ix}(t) = \int_0^1 \sigma_{ix} \left( t, \bar{X}_t, + \lambda \left( X_t^r - \bar{X}_t \right), \bar{u}_t \right) d\lambda.
\]

Then
\[
\mathbb{E} \left[ X_t^r - \bar{X}_t - \xi_t \right]^2 \leq C \mathbb{E} \left[ \left\| \bar{b}_x(t-1) \right\|^2 \left( X_{t-1}^r - \bar{X}_{t-1} - \xi_{t-1} \right)^2 + \left\| \bar{b}_x(t-1) - b_x(t-1) \right\|^2 \left\| \xi_{t-1} \right\|^2 \right] + \sum_{i=1}^m \left[ \bar{\sigma}_{ix}(t-1) \bar{I} \right]^2 \left( \left\| \bar{b}_x(t-1) \right\|^2 + \sum_{i=1}^m \left\| [\bar{\sigma}_{ix}(t-1) - \sigma_{ix}(t-1)] \bar{I} \right\|^2 \left\| \xi_{t-1} \right\|^2 \right]
\]
\[
\leq C \mathbb{E} \left[ \left\| \bar{b}_x(t-1) \right\|^2 + \sum_{i=1}^m \left\| \bar{\sigma}_{ix}(t-1) \bar{I} \right\|^2 \right] \left( \left\| \bar{b}_x(t-1) \right\|^2 + \sum_{i=1}^m \left\| [\bar{\sigma}_{ix}(t-1) - \sigma_{ix}(t-1)] \bar{I} \right\|^2 \left\| \xi_{t-1} \right\|^2 \right).
\]
It is easy to check that $\|\hat{b}_s(t-1) - b_s(t-1)\| \to 0$ and $\|[\hat{\sigma}_{it}(t-1) - \sigma_{it}(t-1)]I\| \to 0$ as $\varepsilon \to 0$. Since $\hat{b}_s(t-1)$ and $\hat{\sigma}_{it}(t-1)$ are bounded, by the estimation (10), we have

$$
\lim_{\varepsilon \to 0} \frac{1}{\varepsilon^2} \mathbb{E}\left[ X_t^\varepsilon - \bar{X}_t - \frac{v_t}{\varepsilon} \right]^2 = 0.
$$

This completes the proof.

**Lemma 3.** Under Assumption 3, we have

$$
\sup_{0 \leq t \leq T} \mathbb{E}\left| Y_t^\varepsilon - \bar{Y}_t \right|^2 \leq C \varepsilon^2 \mathbb{E}|\Delta \nu|^2.
$$

(11)

$$
\sup_{0 \leq t \leq T-1} \mathbb{E}\left( Z_t^\varepsilon - \bar{Z}_t \right)^2 \leq C \varepsilon^2 \mathbb{E}|\Delta \nu|^2.
$$

(12)

**Proof.** It is obvious that $Y_T^\varepsilon - \bar{Y}_T = 0$ at time $T$.

When $t = s, \ldots, T-1$ (if $s = T$, skip this part), we have

$$
\mathbb{E}\left| f \left( t+1, X_{t+1}^\varepsilon, Y_{t+1}^\varepsilon, Z_{t+1}^\varepsilon I, u_{t+1} \right) - f \left( t+1, \bar{X}_{t+1}, \bar{Y}_{t+1}, \bar{Z}_{t+1} I, u_{t+1} \right) \right|^2
\leq C \mathbb{E} \left[ X_{t+1}^\varepsilon - \bar{X}_{t+1} \right]^2 + Y_{t+1}^\varepsilon - \bar{Y}_{t+1} \right|^2 + \left( Z_{t+1}^\varepsilon I - \bar{Z}_{t+1} I \right)^2
\leq C \mathbb{E} \left[ Y_{t+1}^\varepsilon - \bar{Y}_{t+1} \right]^2 + \left( Z_{t+1}^\varepsilon I - \bar{Z}_{t+1} I \right)^2 + C_1 \varepsilon^2 \mathbb{E}|\Delta \nu|^2.
$$

It yields that

$$
\mathbb{E}\left| Y_t^\varepsilon - \bar{Y}_t \right|^2 \leq C \mathbb{E} \left[ Y_{t+1}^\varepsilon - \bar{Y}_{t+1} \right]^2 + \left( Z_{t+1}^\varepsilon I - \bar{Z}_{t+1} I \right)^2 + C \varepsilon^2 \mathbb{E}|\Delta \nu|^2.
$$

Similarly, we have

$$
\mathbb{E}\left( Z_t^\varepsilon - \bar{Z}_t \right) M_{t+1} \right|^2 \leq C \mathbb{E} \left[ Y_{t+1}^\varepsilon - \bar{Y}_{t+1} \right]^2 + \left( Z_{t+1}^\varepsilon I - \bar{Z}_{t+1} I \right)^2 + C \varepsilon^2 \mathbb{E}|\Delta \nu|^2.
$$

Combined with proposition 2.4 in Reference 27, we have

$$
\mathbb{E}\left[ (Z_t^\varepsilon - \bar{Z}_t) I \right]^2 \leq C \mathbb{E} \left[ Y_{t+1}^\varepsilon - \bar{Y}_{t+1} \right]^2 + \left( Z_{t+1}^\varepsilon I - \bar{Z}_{t+1} I \right)^2 + C \varepsilon^2 \mathbb{E}|\Delta \nu|^2.
$$

When $t = s - 1$, by similar analysis,

$$
\mathbb{E}\left| Y_t^\varepsilon - \bar{Y}_t \right|^2 \leq C \mathbb{E} \left[ Y_{t+1}^\varepsilon - \bar{Y}_{t+1} \right]^2 + \left( Z_{t+1}^\varepsilon I - \bar{Z}_{t+1} I \right)^2 + C \varepsilon^2 \mathbb{E}|\Delta \nu|^2.
$$

$$
\mathbb{E}\left[ (Z_t^\varepsilon - \bar{Z}_t) I \right]^2 \leq C \mathbb{E} \left[ Y_{t+1}^\varepsilon - \bar{Y}_{t+1} \right]^2 + \left( Z_{t+1}^\varepsilon I - \bar{Z}_{t+1} I \right)^2 + C \varepsilon^2 \mathbb{E}|\Delta \nu|^2.
$$

If $s = T$,

$$
\begin{align*}
&\mathbb{E}\left| Y_{T-1}^\varepsilon - \bar{Y}_{T-1} \right|^2 \leq C \varepsilon^2 \mathbb{E}|\Delta \nu|^2, \\
&\mathbb{E}\left[ (Z_{T-1}^\varepsilon - \bar{Z}_{T-1}) I \right]^2 \leq C \varepsilon^2 \mathbb{E}|\Delta \nu|^2.
\end{align*}
$$
When \( t = 0, \ldots, s - 2 \), we have
\[
\begin{align*}
\mathbb{E}|Y_t^s - \overline{Y}_t|^2 & \leq C\mathbb{E}\left[\left|Y_{t+1}^s - \overline{Y}_{t+1}\right|^2 + \left\|\left(Z_{t+1}^s - \overline{Z}_{t+1}\right)\hat{l}\right\|^2\right], \\
\mathbb{E}\left[\left\|\left(Z_t^s - Z_t\right)\hat{l}\right\|^2\right] & \leq C\mathbb{E}\left[\left|Y_{t+1}^s - \overline{Y}_{t+1}\right|^2 + \left\|\left(Z_{t+1}^s - \overline{Z}_{t+1}\right)\hat{l}\right\|^2\right].
\end{align*}
\]
Thus, there exists \( C > 0 \), such that for any \( t \in \{0, 1, \ldots, T\} \),
\[
\begin{align*}
\left\{\mathbb{E}|Y_t^s - \overline{Y}_t|^2 \leq C\epsilon^2\mathbb{E}|\Delta v|^2, \\
\mathbb{E}\left[\left\|\left(Z_t^s - Z_t\right)\hat{l}\right\|^2\right] \leq C\epsilon^2\mathbb{E}|\Delta v|^2.\right.\right.
\end{align*}
\]
This completes the proof.

Let \((\eta, \zeta)\) be the solution to the following BS\(\Delta\)E,
\[
\begin{align*}
\Delta \eta_t &= -f_x(t + 1)\xi_{t+1} + f_y(t + 1)\eta_{t+1} - \delta_{(t+1)\sigma_x}f_x(t + 1)\epsilon \Delta v \\
- \sum_{i=1}^n f_y(t + 1)\tilde{\xi}_{t+1}^i e_i + \xi_t M_{t+1}, \\
\eta_T &= 0.
\end{align*}
\]
It is easy to check that
\[
\sup_{0 \leq t \leq T} \mathbb{E}|\eta_t|^2 \leq C\epsilon^2\mathbb{E}|\Delta v|^2, \\
\sup_{0 \leq t \leq T - 1} \mathbb{E}\left\|\xi_t \hat{l}\right\|^2 \leq C\epsilon^2\mathbb{E}|\Delta v|^2.
\]
And we have the following result:

**Lemma 4.** Under Assumption 3, we have
\[
\begin{align*}
\sup_{0 \leq t \leq T} \mathbb{E}\left|Y_t^s - \overline{Y}_t - \eta_t\right|^2 & = o(\epsilon^2), \\
\sup_{0 \leq t \leq T - 1} \mathbb{E}\left\|Z_t^s - Z_t - \zeta_t\hat{l}\right\|^2 & = o(\epsilon^2).
\end{align*}
\]
**Proof.** When \( t = T \), \( Y_T^s - \overline{Y}_T - \eta_T = 0 \).

When \( t \in \{0, 1, \ldots, T - 1\} \), we have
\[
\begin{align*}
Y_t^s - \overline{Y}_t - \eta_t & = \mathbb{E}\left[Y_{t+1}^s - \overline{Y}_{t+1} - \eta_{t+1} + f^x(t + 1) - \tilde{f}(t + 1) + f_x(t + 1)\xi_{t+1} - f_y(t + 1)\eta_{t+1}\right] \\
- \sum_{i=1}^n f_y(t + 1)\tilde{\xi}_{t+1}^i e_i - \delta_{(t+1)\sigma_x}f_x(t + 1)\epsilon \Delta v|F_t] \\
& = \mathbb{E}\left[Y_{t+1}^s - \overline{Y}_{t+1} - \eta_{t+1} + \tilde{f}_x(t + 1)\left(X_{t+1}^s - \overline{X}_{t+1}\right) + \tilde{f}_y(t + 1)\left(Y_{t+1}^s - \overline{Y}_{t+1}\right)\right] \\
+ \sum_{i=1}^n \tilde{f}_x(t + 1)\tilde{\xi}_{t+1}^i e_i + \xi_t M_{t+1} - f_x(t + 1)\xi_{t+1} - f_y(t + 1)\eta_{t+1}\right] \\
- \sum_{i=1}^n f_y(t + 1)\tilde{\xi}_{t+1}^i e_i - \delta_{(t+1)\sigma_x}f_x(t + 1)\epsilon \Delta v|F_t].
\end{align*}
\]
where

\[ \tilde{f}_{\mu}(t) = \int_0^1 f_{\mu}(t, \bar{X}_t + \lambda \left( X_t^e - \bar{X}_t \right), \bar{Y}_t + \lambda \left( Y_t^e - \bar{Y}_t \right), \bar{Z}_t, \bar{I} + \lambda \left( Z_t^e - \bar{Z}_t \right), \bar{I}, \bar{u}_t + \lambda \left( u_t^e - \bar{u}_t \right)) \, d\lambda, \]

for \( \mu = x, y, z_i \) and \( u \). Then,

\[
\mathbb{E} \left| Y_{t+1}^e - \bar{Y}_t - \eta_t \right|^2 \leq C \mathbb{E} \left[ \left| Y_{t+1}^e - \bar{Y}_t + \eta_{t+1} \right|^2 + \left| \tilde{f}_x(t+1) \left( X_{t+1}^e - \bar{X}_{t+1} - \xi_{t+1} \right) \right|^2 + \left| \tilde{f}_y(t+1) - f_y(t+1) \right| \xi_{t+1} \right]^2 \\
+ \left| \tilde{f}_z(t+1) \left( Y_{t+1}^e - \bar{Y}_{t+1} - \eta_{t+1} \right) \right|^2 + \left| \tilde{f}_x(t+1) - f_x(t+1) \right| \eta_{t+1} \right|^2 \\
+ \sum_{i=1}^n \left| \tilde{f}_z(t+1) I^* \left( Z_{t+1}^e - \bar{Z}_{t+1} - \zeta_{t+1} \right) e_i \right|^2 + \sum_{i=1}^n \left| \tilde{f}_z(t+1) - f_z(t+1) \right| I^* \zeta_{t+1} e_i \right|^2 \\
+ \delta_{t(t+10)} \left| \tilde{f}_u(t+1) - f_u(t+1) \right| \epsilon \Delta t \right]^2,
\]

and

\[
\mathbb{E} \left\| Z_{t+1}^e - \bar{Z}_t - \xi_{t+1} \right\|^2 \leq C \mathbb{E} \left[ \left| Z_{t+1}^e - \bar{Z}_t - \eta_{t+1} \right|^2 + \left| \tilde{f}_x(t+1) \left( X_{t+1}^e - \bar{X}_{t+1} - \xi_{t+1} \right) \right|^2 + \left| \tilde{f}_y(t+1) - f_y(t+1) \right| \xi_{t+1} \right|^2 \\
+ \left| \tilde{f}_z(t+1) \left( Y_{t+1}^e - \bar{Y}_{t+1} - \eta_{t+1} \right) \right|^2 + \left| \tilde{f}_x(t+1) - f_x(t+1) \right| \eta_{t+1} \right|^2 \\
+ \sum_{i=1}^n \left| \tilde{f}_z(t+1) I^* \left( Z_{t+1}^e - \bar{Z}_{t+1} - \zeta_{t+1} \right) e_i \right|^2 + \sum_{i=1}^n \left| \tilde{f}_z(t+1) - f_z(t+1) \right| I^* \zeta_{t+1} e_i \right|^2 \\
+ \delta_{t(t+10)} \left| \tilde{f}_u(t+1) - f_u(t+1) \right| \epsilon \Delta t \right]^2.
\]

Notice that \( \tilde{f}_x(t) - f_x(t) \to 0, \tilde{f}_y(t) - f_y(t) \to 0, \tilde{f}_z(t) - f_z(t) \to 0, \tilde{f}_u(t) - f_u(t) \to 0 \) as \( \epsilon \to 0 \). We obtain that

\[
\lim_{\epsilon \to 0} \frac{1}{\epsilon^2} \mathbb{E} \left| Y_t^e - \bar{Y}_t - \eta_t \right|^2 = 0,
\]

\[
\lim_{\epsilon \to 0} \frac{1}{\epsilon^2} \mathbb{E} \left\| Z_{t+1}^e - \bar{Z}_t - \xi_{t+1} \right\|^2 = 0.
\]

This completes the proof. \( \blacksquare \)

By Lemmas 2 and 4, we have

\[
J(u^*(\cdot)) - J(\bar{u}(\cdot)) = \mathbb{E} \sum_{i=0}^{T-1} \left[ \langle L_i(t), \xi_i \rangle + \langle I_i(t), \eta_i \rangle + \sum_{i=1}^n \langle \hat{I}_{z_i}^e(t), \hat{I}_{z_i}^e e_i \rangle + \delta_{t0} \langle I_u(s), \epsilon \Delta u \rangle \right] + \mathbb{E} \left\langle h_x \left( \bar{X}_T \right), \xi_T \right\rangle + o(\epsilon).
\]

Introducing the following adjoint equation:

\[
\begin{cases}
\Delta p_t = -b^e_x(t+1)p_{t+1} - \sum_{i=1}^m \sigma_{i,0}(t+1) \mathbb{E} \left[ M_{t+2} M_{t+1}^e | F_{t+1} \right] q_{i+1} + f_{z_i}(t+1) \xi_{t+1} + l_x(t+1) + q_{i+1}, \\
\Delta k_t = f_{z_i}^e(t) k_t + f_{i+1}^e(t) + \sum_{i=1}^n \sigma_{i,0}(t+1) \mathbb{E} \left[ M_{t+2} M_{t+1}^e | F_{t+1} \right] \xi_{t+1} + \sum_{i=1}^n \sigma_{i,0}(t+1) \mathbb{E} \left[ M_{t+2} M_{t+1}^e | F_{t+1} \right] \xi_{t+1}, \\
p_T = -h_x \left( \bar{X}_T \right), \\
k_0 = 0. 
\end{cases}
\]

where \( (\cdot)^\dagger \) denotes the pseudoinverse of a matrix.
Obviously the forward equation in (13) admits a unique solution \( k \in \mathcal{M}(0, T; \mathbb{R}^m) \). Then, based on the solution \( k \), according to Theorem 1, it is easy to check that the backward equation in (13) has a unique solution \((p, q) \in \mathcal{M}(0, T; \mathbb{R}^m) \times \mathcal{M}(0, T - 1; \mathbb{R}^{mxd}) \). So FBSDE has a unique solution \((p, q, k)\).

We obtain the following maximum principle for the optimal control problem (2)–(3).

Define the Hamiltonian function

\[
H(\omega, t, u, x, y, z, p, q, k) = b^*(\omega, t, x, u)p + \sum_{i=1}^{m} \sigma_i(\omega, t, x, u)E \left[ M_{i+1} M_{i+1}^* | F_t \right] (\omega)q^* e_i - f^*(\omega, t, x, y, z, u)k - l(\omega, t, x, y, z, u).
\]

**Theorem 2.** Suppose that Assumption 3 holds. Let \( \tilde{u} \) be an optimal control of the problem (2)–(3), \((\tilde{X}, \tilde{Y}, \tilde{Z}, \tilde{p}, \tilde{q}, \tilde{k})\) be the corresponding optimal trajectory and \((p, q, k)\) be the solution to the adjoint equation (13). Then for any \( t \in \{0, 1, \ldots, T-1\}, v \in U_t \) and \( \omega \in \Omega \), we have

\[
\left\langle H_{\omega} \left( \omega, t, \tilde{u}, \tilde{X}_t, \tilde{Y}_t, \tilde{Z}_t, \tilde{p}_t, \tilde{q}_t, \tilde{k}_t \right), v - \tilde{u}_t(\omega) \right\rangle \leq 0. \tag{14}
\]

**Proof.** For \( t \in \{0, 1, \ldots, T-1\} \), we have

\[
\Delta \left\langle \xi_t, p_t \right\rangle = \left\langle \xi_{t+1}, \Delta p_t \right\rangle + \left\langle \Delta \xi_t, p_t \right\rangle
\]

\[
= \left\langle \xi_{t+1}, -b^*_t(t + 1) p_{t+1} - \sum_{i=1}^{m} \sigma_i(t + 1) E \left[ M_{i+2} M_{i+2}^* | F_{t+1} \right] q^*_{t+1} e_i \right\rangle + \left\langle \xi_{t+1}, f^*_t(t + 1) k_{t+1} + l_t(t + 1) \right\rangle
\]

\[
+ \left\langle \sum_{i=1}^{m} e_i \cdot \left[ \xi^*_t \sigma_{i\omega}(t) + \delta_{i\omega} \Delta v^r \sigma_{i\omega}(t) \right] M_{i+1}, q_t M_{i+1} \right\rangle + \left\langle b_t(t) \xi_t + \delta_{i\omega} b_u(t) (t) \epsilon \Delta v, \sigma_{i\omega}(t) M_{i+1}, p_t \right\rangle \Phi_t,
\]

where

\[
\Phi_t = \left\langle \xi_t + b_u(t) \xi_t + \delta_{i\omega} b_u(t) (t) \epsilon \Delta v, q_t M_{i+1} \right\rangle + \left\langle \sum_{i=1}^{m} e_i \cdot \left[ \xi^*_t \sigma_{i\omega}(t) + \delta_{i\omega} \Delta v^r \sigma_{i\omega}(t) \right] M_{i+1}, p_t \right\rangle.
\]

It is obvious that \( E [\Phi_t] = 0 \). We have

\[
E \left\langle \sum_{i=1}^{m} e_i \cdot \xi^*_t \sigma_{i\omega}(t) M_{i+1}, q_t M_{i+1} \right\rangle = E \sum_{i=1}^{m} [M^*_{i+1} q^*_i \sigma_{i\omega}(t) M_{i+1}]
\]

\[
= E \sum_{i=1}^{m} [e^*_i q_t M_{i+1} M_{i+1}^* \sigma_{i\omega}(t) \xi_t]
\]

\[
= E \sum_{i=1}^{m} \left\langle \xi_t, \sigma_{i\omega}(t) M_{i+1} M_{i+1}^* q^*_i e_i \right\rangle.
\]

and

\[
E \left\langle \sum_{i=1}^{m} e_i \cdot \delta_{i\omega} \Delta v^r \sigma_{i\omega}(t) M_{i+1}, q_t M_{i+1} \right\rangle = E \left[ \delta_{i\omega} \epsilon \sum_{i=1}^{m} \left\langle \Delta v, \sigma_{i\omega}(t) M_{i+1} M_{i+1}^* q^*_i e_i \right\rangle \right].
\]

Similarly, it can be shown that for \( t \in \{0, 1, \ldots, T-1\} \),

\[
\Delta \left\langle \eta_t, k_t \right\rangle = \left\langle -f^*_t(t + 1) \xi_{t+1} - f^*_t(t + 1) \eta_{t+1} - \sum_{i=1}^{n} f^*_t(t + 1) l^*_{t+1} e_i , k_{t+1} \right\rangle
\]

\[
- \left\langle \delta_{t+1} f_u(t + 1) \epsilon \Delta v, k_{t+1} \right\rangle + \left\langle \eta_t f^*_t(t) k_t + l_t(t) \right\rangle
\]

\[
+ \left\langle \xi_t M_{t+1}, \sum_{i=1}^{n} e_i \left( k^*_t f^*_i(t) + l^*_t(t) \right) \right\rangle \right\rangle \left[ E \left[ M_{i+1} M_{i+1}^* | F_{t} \right] \right] M_{t+1} + \Psi_t,
\]

\[
+ \left\langle \xi_t M_{t+1}, \sum_{i=1}^{n} e_i \left( k^*_t f^*_i(t) + l^*_t(t) \right) \right\rangle \left[ E \left[ M_{i+1} M_{i+1}^* | F_{t} \right] \right] M_{t+1} + \Psi_t.
\]
where

\[
\Psi_I = \left\langle \xi_t M_{t+1}, k_t + g^*_t(t)k_t + l_t(t) \right\rangle + \left\langle \eta_t, \sum_{i=1}^n e_i(k^*_i f^*_t(t)\tilde{I}^*(E[M_{t+1}M_{t+1}^*|F_t])^\dagger M_{t+1} \right\rangle \\
+ \left\langle \eta_t, \sum_{i=1}^n e_i l^*_t(t)I^*(E[M_{t+1}M_{t+1}^*|F_t])^\dagger M_{t+1} \right\rangle.
\]

According to the result in Reference 28, we know that \(\forall \omega \in \Omega,\)

\[
E[M_{t+1}M_{t+1}^*|F_t] = \left(E[M_{t+1}M_{t+1}^*|F_t]\right)^\dagger (\omega) = I_d - \frac{1}{d} 1_{d \times d}.
\]

Then we can obtain

\[
E \left\langle \xi_t M_{t+1}, e_i(k^*_i f^*_t(t)\tilde{I}^*(E[M_{t+1}M_{t+1}^*|F_t])^\dagger M_{t+1} \right\rangle = E \left\langle M_{t+1}^* e_i k^*_i f^*_t(t)\tilde{I}^*(E[M_{t+1}M_{t+1}^*|F_t])^\dagger M_{t+1} \right\rangle
\]

\[
= E \left\langle \xi_t M_{t+1} M_{t+1}^* \left(E[M_{t+1}M_{t+1}^*|F_t]\right)^\dagger \tilde{I}^*_t(t) k_t \right\rangle
\]

\[
= E \left\langle \xi_t \left(I_d - \frac{1}{d} 1_{d \times d}\right) \tilde{I}^*_t(t) k_t \right\rangle
\]

\[
= E \left\langle f^*_t(t) \tilde{I}^*_t \xi_t^* e_t, k_t \right\rangle.
\]

Similarly,

\[
E \left\langle \xi_t M_{t+1}, e_i l^*_t(t)I^*(E[M_{t+1}M_{t+1}^*|F_t])^\dagger M_{t+1} \right\rangle = E \left\langle l^*_t(t)I^* \xi_t^* e_i \right\rangle.
\]

Thus

\[
E \Delta \left(\langle \xi_t, p_t \rangle + \langle \eta_t, k_t \rangle \right) = E \left[(-b_2(t+1)\xi_{t+1}, p_{t+1}) + (b_2(t)\xi_t, p_t) \right.

- \sum_{i=1}^m \left\langle e_i M_{t+2} \sigma^*_i(t+1) \xi_{t+1}, q_{t+1} M_{t+2} \right\rangle + \sum_{i=1}^m \left\langle e_i \xi_t \sigma_i(t) M_{t+1}, q_t M_{t+1} \right\rangle

- \langle f^*_t(t) \eta_{t+1}, k_{t+1} \rangle + \langle f^*_t(t) \eta_t, k_t \rangle

- \sum_{i=1}^m \left\langle f^*_t(t+1)^\dagger \xi_t \xi_t^* e_t, k_{t+1} \right\rangle + \sum_{i=1}^m \left\langle f^*_t(t)^\dagger \xi_t \xi_t^* e_t, k_t \right\rangle

+ \langle l^*_t(t+1), \xi_{t+2} \rangle + \langle \eta_t, l^*_t(t) \rangle + \sum_{i=1}^m \left\langle l^*_t(t), I^* \xi_t^* e_i \right\rangle

+ \epsilon \left\langle \delta_{t+1} b_2(t+1) \Delta v, p_t \right\rangle + \delta_{t+1} \epsilon \sum_{i=1}^m \left\langle M_{t+1} M_{t+1}^* \sigma_i(t) \Delta v, q_t^* e_i \right\rangle

- \epsilon \left\langle \delta_{t+1} b_2(t+1) \Delta v, k_{t+1} \right\rangle \right].
\]

Therefore,

\[
\left.-E \left\langle h^*_t \left(\bar{X}_T \right), \xi_t \right\rangle = E \left[\langle \xi_T, p_T \rangle + \langle \eta_T, k_T \rangle - \langle \xi_0, p_0 \rangle - \langle \eta_0, k_0 \rangle \right]\right. \\
= \sum_{t=0}^{T-1} E \left[\langle \xi_t, p_t \rangle + \langle \eta_t, k_t \rangle \right]

= E \left[\langle b_2(0) \xi_0, p_0 \rangle + \sum_{i=1}^m \left\langle e_i \xi_0 \sigma_i(0) M_1, q_0 M_1 \right\rangle + \langle f^*_t(0) \eta_0, k_0 \rangle + \sum_{i=1}^m \left\langle f^*_t(0)^\dagger \xi_t^* e_i, k_0 \right\rangle \right]
\]
Lemma 5. Under Assumptions 3 and 4, we have

\[ \mathbb{E} \left( \sum_{t=0}^{T} \left| \hat{X}_t \right|^2 + \sum_{t=0}^{T} \left| \hat{Y}_t \right|^2 + \sum_{t=0}^{T-1} \left| \hat{Z}_t \hat{f}_t \right|^2 \right) \leq C \varepsilon^2 \mathbb{E} |\Delta v|^2. \]  

(20)
\textbf{Proof.} By (19),
\begin{equation*}
\mathbb{E}\sum_{i=0}^{T-1} \Delta \left( \hat{X}_i, \hat{Y}_i \right) = \mathbb{E} \left( \hat{X}_T, \hat{Y}_T \right) - \mathbb{E} \left( \hat{X}_0, \hat{Y}_0 \right) = 0
\end{equation*}
\begin{equation*}
= \mathbb{E} \sum_{i=1}^{T-1} \left[ \left( \hat{X}_i, -f^*(t) + \hat{f}(t) \right) + \left( \hat{Y}_i, b^*(t) - \hat{b}(t) \right) + \left( \hat{Z}_i, \sigma^*(t) - \hat{\sigma}(t) \right) \right]
\end{equation*}
\begin{equation*}
= \mathbb{E} \sum_{i=1}^{T-1} \left[ \left( \hat{X}_i, -f^*(T) + \hat{f}(T) \right) + \mathbb{E} \left( \hat{Y}_0, b^*(0) - \hat{b}(0) \right) + \mathbb{E} \left( \hat{Z}_0, (\sigma^*(0) - \hat{\sigma}(0)) \mathbb{E} \left[ M_1 M_1^* \| F_0 \right] \right) \right]
\end{equation*}
\begin{equation*}
+ \mathbb{E} \sum_{i=0}^{T} \left[ \left( \hat{X}_i, -f^*(t) + \hat{f}(t) \right) + \left( \hat{Y}_i, b^*(t) - \hat{b}(t) \right) \right]
\end{equation*}
\begin{equation*}
+ \mathbb{E} \sum_{i=0}^{T-1} \left( \hat{Z}_i, (\sigma^*(t) - \hat{\sigma}(t)) \mathbb{E} \left[ M_{i+1} M_{i+1}^* \| F_i \right] \right)
\end{equation*}
\begin{equation*}
= \mathbb{E} \sum_{i=1}^{T-1} \left[ \left( \hat{X}_i, -f^*(T) + \hat{f}(T) \right) + \mathbb{E} \left( \hat{Y}_0, b^*(0) - \hat{b}(0) \right) + \mathbb{E} \left( \hat{Z}_0, (\sigma^*(0) - \hat{\sigma}(0)) \mathbb{E} \left[ M_1 M_1^* \| F_0 \right] \right) \right]
\end{equation*}
\begin{equation*}
+ \mathbb{E} \left[ \left( \hat{X}_s, -f^*(s) + \hat{f}(s) \right) + \left( \hat{Y}_s, b^*(s) - \hat{b}(s) \right) \right]
\end{equation*}
\begin{equation*}
+ \mathbb{E} \left( \hat{Z}_s M_{s+1}, (\sigma^*(s) - \hat{\sigma}(s)) M_{s+1} \right).
\end{equation*}

By the monotone condition, we obtain
\begin{equation}
\mathbb{E} \left[ \left( \hat{X}_s, -f^*(s) + \hat{f}(s) \right) + \left( \hat{Y}_s, b^*(s) - \hat{b}(s) \right) \right] + \mathbb{E} \left[ \left( \hat{Z}_s M_{s+1}, (\sigma^*(s) - \hat{\sigma}(s)) M_{s+1} \right) \right] \geq \alpha \mathbb{E} \left[ \sum_{i=0}^{T} |\hat{X}_i|^2 + \sum_{i=0}^{T} |\hat{Y}_i|^2 + \sum_{i=0}^{T-1} |\hat{Z}_i|^2 \right].
\end{equation}
(21)

On the other hand,
\begin{equation*}
\mathbb{E} \left( \hat{X}_s, -f^*(s) + \hat{f}(s) \right) + \mathbb{E} \left( \hat{Y}_s, b^*(s) - \hat{b}(s) \right) \leq \frac{\alpha}{2} \mathbb{E} |\hat{X}_s|^2 + \frac{1}{2a} \mathbb{E} |\hat{f}(s) - f^*(s)|^2 + \frac{\alpha}{2} \mathbb{E} |\hat{Y}_s|^2 + \frac{1}{2a} \mathbb{E} |\hat{b}(s) - \hat{b}(s)|^2
\end{equation*}
\begin{equation*}
\leq \frac{\alpha}{2} \mathbb{E} |\hat{X}_s|^2 + \frac{\alpha}{2} \mathbb{E} |\hat{Y}_s|^2 + \frac{C}{2a} \epsilon^2 \mathbb{E} |\Delta v|^2.
\end{equation*}

and
\begin{equation*}
\mathbb{E} \left( \hat{Z}_s M_{s+1}, (\sigma^*(s) - \hat{\sigma}(s)) M_{s+1} \right) \leq \frac{\alpha}{2C} \mathbb{E} |\hat{Z}_s M_{s+1}|^2 + \frac{C}{2a} \mathbb{E} |(\sigma^*(s) - \hat{\sigma}(s)) M_{s+1}|^2
\end{equation*}
\begin{equation*}
\leq \frac{\alpha}{2} \mathbb{E} |\hat{Z}_s|^2 + \frac{C}{2a} \mathbb{E} |\Delta v|^2.
\end{equation*}

Thus
\begin{equation}
\mathbb{E} \left[ \left( \hat{X}_s, -f^*(s) + \hat{f}(s) \right) + \left( \hat{Y}_s, b^*(s) - \hat{b}(s) \right) \right] + \mathbb{E} \left( \hat{Z}_s M_{s+1}, (\sigma^*(s) - \hat{\sigma}(s)) M_{s+1} \right)
\end{equation}
\begin{equation*}
\leq \frac{\alpha}{2} \mathbb{E} \left[ |\hat{X}_s|^2 + |\hat{Y}_s|^2 + |\hat{Z}_s|^2 \right] + C \epsilon^2 \mathbb{E} |\Delta v|^2.
\end{equation*}
(22)

Combining (21) and (22), we have
\begin{equation*}
\mathbb{E} \left[ \sum_{i=0}^{T} |\hat{X}_i|^2 + \sum_{i=0}^{T} |\hat{Y}_i|^2 + \sum_{i=0}^{T-1} |\hat{Z}_i|^2 \right] \leq C \epsilon^2 \mathbb{E} |\Delta v|^2.
\end{equation*}
This completes the proof.

Next we introduce the following variational equation:

\[
\begin{align*}
\Delta \xi_t &= b_x(t)\xi_t + b_y(t)\eta_t + \zeta t\bar{b}_y(t) + \delta u b_u(t)\Delta v \\
&\quad + [\sigma_x(t)\Delta \xi_t + \sigma_y(t)\Delta \eta_t + \zeta t\sigma_\varepsilon(t) + \delta u \varepsilon (\Delta v)^* \sigma_a(t)] M_{t+1}, \\
\Delta \eta_t &= -f_\varepsilon (t + 1) \xi_{t+1} - f_y (t + 1) \eta_{t+1} - \zeta_{t+1} \bar{f}_y(t + 1) \\
&\quad - \delta_{\varepsilon u f_u(t + 1)\varepsilon \Delta v} + \zeta_t M_{t+1}, \\
\xi_0 &= 0, \\
\eta_T &= 0.
\end{align*}
\]

(23)

By Assumptions 3 and 4, when \( t \in \{1, \ldots, T - 1 \} \),

\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & \mathbb{E}[M_{t+1} M_{t+1}^* | F_t]
\end{pmatrix}
\begin{pmatrix}
-f_x(t) & -f_y(t) & -f_{\varepsilon}(t) \\
b_x(t) & b_y(t) & b_{\varepsilon}(t) \\
\sigma_x(t) & \sigma_y(t) & \sigma_{\varepsilon}(t)
\end{pmatrix}
\leq -\alpha
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & I_{d-1}
\end{pmatrix}
\]

(24)

when \( t = 0 \),

\[
\begin{pmatrix}
b_{\varepsilon}(0) \\
\mathbb{E}[M_0 M_0^* | F_0] \sigma_{\varepsilon}(0)
\end{pmatrix}
\leq -\alpha
\begin{pmatrix}
1 & 0 \\
0 & I_{d-1}
\end{pmatrix}
\]

(25)

when \( t = T \),

\[
-g_{\varepsilon}(T) \leq -\alpha.
\]

(26)

Thus, the coefficients of (23) satisfy the monotone condition and there exists a unique solution \((\xi, \eta, \zeta)\) to (23). Similar to the proof of Lemma 5, we have

\[
\mathbb{E} \left[ \sum_{t=0}^{T} |\xi_t|^2 + \sum_{t=0}^{T} |\eta_t|^2 + \sum_{t=0}^{T-1} |\zeta_t|^2 \right] \leq C \varepsilon^2 \mathbb{E}|\Delta v|^2.
\]

(27)

Define

\[
\tilde{\phi}_{\mu}(t) = \int_{0}^{1} \phi_{\mu} \left( t, X_t + \lambda \left( X^*_t - X_t \right), Y_t + \lambda \left( Y^*_t - Y_t \right), Z_t + \lambda \left( Z^*_t - Z_t \right) \right) I, \tilde{u}_t + \lambda \left( u^*_t - \tilde{u}_t \right) d\lambda.
\]

where \( \phi = b, \sigma, f, I \), and \( \mu = x, y, z \) and \( u \).

**Lemma 6.** Under Assumptions 3 and 4, we have

\[
\mathbb{E} \left[ \sum_{t=0}^{T} |\hat{X}_t - \xi_t|^2 + \sum_{t=0}^{T} |\hat{Y}_t - \eta_t|^2 + \sum_{t=0}^{T-1} |\hat{Z}_t - \zeta_t|^2 \right] = o \left( \varepsilon^2 \right).
\]

**Proof.** Note that

\[
\phi'(t) - \bar{\phi}(t) = \bar{\phi}_x(t) \left( X^*_t - X_t \right) + \bar{\phi}_y(t) \left( Y^*_t - Y_t \right) + \left( Z^*_t - Z_t \right) I \bar{\phi}_\varepsilon(t) + \delta u \bar{\phi}_u(t) \varepsilon \Delta v.
\]

Set

\[
\hat{X}_t = \hat{X}_t - \xi_t, \hat{Y}_t = \hat{Y}_t - \eta_t, \hat{Z}_t = \hat{Z}_t - \zeta_t,
\]
then,

\[
\begin{aligned}
\Delta \bar{X}_t &= b_x(t) \bar{X}_t + b_y(t) \bar{Y}_t + \dot{Z}_1 t \bar{b}_2(t) + \Lambda_1(t) \\
&\quad + \left[\sigma_x(t) \bar{X}_t + \sigma_y(t) \bar{Y}_t + \dot{Z}_1 t \sigma_2(t) + \Lambda_2(t)\right] M_{t+1}, \\
\Delta \bar{Y}_t &= -f_x(t+1) \bar{X}_{t+1} - f_y(t+1) \bar{Y}_{t+1} - Z_{t+1} f_z(t+1) \\
&\quad - \Lambda_3(t+1) + Z_t M_{t+1}, \\
\bar{X}_0 &= 0, \\
\bar{Y}_T &= 0,
\end{aligned}
\]  

(28)

where

\[
\Lambda_1(t) = (\bar{b}_x(t) - b_x(t)) \bar{X}_t + (\bar{b}_y(t) - b_y(t)) \bar{Y}_t + \dot{Z}_1 t (\bar{b}_2(t) - b_2(t)) + \delta_{\theta} (\bar{b}_u(t) - b_u(t)) \epsilon \Delta \nu, \\
\Lambda_2(t) = (\bar{\sigma}_x(t) - \sigma_x(t)) \bar{X}_t + (\bar{\sigma}_y(t) - \sigma_y(t)) \bar{Y}_t + \dot{Z}_1 t (\bar{\sigma}_2(t) - \sigma_2(t)) + \delta_{\theta} (\bar{\sigma}_u(t) - \sigma_u(t)) \epsilon \Delta \nu, \\
\Lambda_3(t) = - (\bar{f}_x(t) - f_x(t)) \bar{X}_t - (\bar{f}_y(t) - f_y(t)) \bar{Y}_t - \dot{Z}_1 t (\bar{f}_z(t) - f_z(t)) - \delta_{\theta} (\bar{f}_u(t) - f_u(t)) \epsilon \Delta \nu.
\]

According to (28),

\[
0 = \mathbb{E} \left[ \bar{X}_T, \bar{Y}_T \right] - \mathbb{E} \left[ \bar{X}_0, \bar{Y}_0 \right] \\
= \sum_{t=0}^{T-1} \Delta \left[ \bar{X}_t, \bar{Y}_t \right] \\
= \sum_{t=0}^{T} \left[ \langle \bar{X}_t, -\bar{\lambda}_t f_z(t) \rangle + \langle \bar{Y}_t, \bar{\lambda}_t b_z(t) \rangle + \langle \dot{Z}_t, \bar{\lambda}_t \sigma_z(t) M_{t+1} M'_{t+1} \rangle \right] \\
+ \sum_{t=0}^{T} \left[ \langle \bar{X}_t, -\Lambda_3(t) \rangle + \langle \bar{Y}_t, \Lambda_1(t) \rangle + \langle Z_{t} M_{t+1}, \Lambda_2(t) M_{t+1} \rangle \right],
\]

where

\[
\bar{\lambda}_t = (\bar{X}_t, \bar{Y}_t, \dot{Z}_t), \\
b_z(t) = (b_x(t), b_y(t), b_u(t)), \\
\sigma_z(t) = (\sigma_x(t), \sigma_y(t), \sigma_u(t)), \\
f_z(t) = (f_x(t), f_y(t), f_u(t)).
\]

Combining (24), (25) and (26), we have

\[
\sum_{t=0}^{T} \left[ \langle \bar{X}_t, -\Lambda_3(t) \rangle + \langle \bar{Y}_t, \Lambda_1(t) \rangle + \langle Z_{t} M_{t+1}, \Lambda_2(t) M_{t+1} \rangle \right] \geq \alpha \mathbb{E} \left[ \sum_{t=0}^{T} |\bar{X}_t|^2 + \sum_{t=0}^{T} |\bar{Y}_t|^2 + \sum_{t=0}^{T-1} |Z_t|^2 \right].
\]

Note that

\[
\mathbb{E} \langle \bar{X}_t, -\Lambda_3(t) \rangle = \mathbb{E} \left[ \bar{X}_t, (\bar{f}_x(t) - f_x(t)) \bar{X}_t \right] + \mathbb{E} \left[ \bar{X}_t, (\bar{f}_y(t) - f_y(t)) \bar{Y}_t \right] \\
+ \mathbb{E} \left[ \bar{X}_t, \dot{Z}_1 t \bar{f}_z(t) \right] + \mathbb{E} \left[ \bar{X}_t, \delta_{\theta} (\bar{f}_u(t) - f_u(t)) \epsilon \Delta \nu \right] \\
\leq \frac{\alpha}{2} \mathbb{E} |\bar{X}_t|^2 + \frac{2}{\alpha} \mathbb{E} \left[ \left\| \bar{f}_x(t) - f_x(t) \right\| \right]^2 |\bar{X}_t|^2 + \left\| \bar{f}_y(t) - f_y(t) \right\| |\bar{Y}_t|^2 \\
+ \frac{2}{\alpha} \mathbb{E} \left[ \left\| \bar{f}_z(t) - f_z(t) \right\| \right]^2 |\dot{Z}_1 t|^2 + \delta_{\theta} \epsilon \left\| \bar{f}_u(t) - f_u(t) \right\|^2 |\Delta \nu|^2.
\]

When \( \epsilon \to 0 \), \( \| \bar{f}_\mu(t) - f_\mu(t) \| \to 0 \) for \( \mu = x, y, \bar{z} \) and \( u \). Then, by Lemma 5,

\[
\mathbb{E} \langle \bar{X}_t, -\Lambda_3(t) \rangle \leq \frac{\alpha}{2} \mathbb{E} |\bar{X}_t|^2 + o (\epsilon^2).
\]
Similar results hold for the other terms in (4). Finally, we have

\[
\mathbb{E} \left[ \sum_{t=0}^{T} |\hat{X}_t|^2 + \sum_{t=0}^{T} |\hat{Y}_t|^2 + \sum_{t=0}^{T-1} |\hat{Z}_t|^2 \right] \leq o(\varepsilon^2).
\]

This completes the proof.

By Lemma 6, we obtain

\[
J(u'(\cdot)) - J(\bar{u}(\cdot)) = \mathbb{E} \sum_{t=0}^{T-1} \left[ l_t(t)\xi_t + l_y(t)\eta_t + \xi_t I\bar{y}_t(t) + \delta_{\bar{u}} l_u(s)\varepsilon \Delta v \right] + \mathbb{E} \left[ h_z \left( \bar{X}_T \right) \xi_T \right] + o(\varepsilon).
\]

Introduce the following adjoint equation:

\[
\begin{align*}
\Delta p_t &= -b_x(t+1)p_{t+1} - \sigma_x(t+1)\mathbb{E} \left[ M_{t+2} M^*_t [F_{t+1}] \right] q^*_t + f_t(t+1)k_{t+1} + l_x(t+1) + q_t M_{t+1}, \\
\Delta k_t &= f_t(t)k_t - b_y(t)p_t - \sigma_y(t)\mathbb{E} \left[ M_{t+1} M^*_t [F_t] \right] q^*_t + l_y(t) + \left( f^*_t(t)k_t - b^*_y(t)p_t + l^*_y(t) \right) T^* \left( \mathbb{E} \left[ M_{t+1} M^*_t [F_t] \right] \right)^{\dagger} M_{t+1}, \\
p_T &= -h_x \left( \bar{X}_T \right), \\
k_0 &= 0.
\end{align*}
\] (30)

Define the Hamiltonian function as follows:

\[
H(\omega, t, u, x, y, z, p, q, k) = b(\omega, t, x, y, z, u)p + \sigma(\omega, t, x, y, z, u)\mathbb{E} \left[ M_{t+1} M^*_t [F_t] \right] q^* - f(\omega, t, x, y, z, u)k - l(\omega, t, x, y, z, u).
\]

**Theorem 3.** Suppose that Assumption 3 and 4 hold. Let \( \bar{u} \) be an optimal control for (4)–(5), \( (\bar{X}, \bar{Y}, \bar{Z}) \) be the corresponding optimal trajectory and \( (p, q, k) \) be the solution to the adjoint Equation (30). Then, for any \( t \in \{0, 1, \ldots, T\}, \ \omega \in \Omega \) and \( v \in U_t \), we have

\[
\left( H_u \left( \omega, t, \bar{u}_t, \bar{X}_t, \bar{Y}_t, \bar{Z}_t, p_t, q_t, k_t \right), v - \bar{u}_t(\omega) \right) \leq 0.
\] (31)

**Proof.** From the expression of \( \xi_t, p_t \) for \( t \in \{0, 1, \ldots, T-1\} \), we have

\[
\Delta \left( \xi_t, p_t \right) = \left( \xi_{t+1}, \Delta p_t \right) + \left( \Delta \xi_t, p_t \right)
\]

\[
= \left( \xi_{t+1}, -b_x(t+1)p_{t+1} - \sigma_x(t+1)\mathbb{E} \left[ M_{t+2} M^*_t [F_{t+1}] \right] q^*_t + f_t(t+1)k_{t+1} + l_x(t+1) \right) + \left( \sigma_x(t)\xi_t + \sigma_y(t)\eta_t + \xi_t I\bar{y}_t(t) + \delta_{\bar{u}} \varepsilon(\Delta v)^* \sigma_u(t) \right) M_{t+1}, q_t M_{t+1} \right) + \Phi_t,
\]

where

\[
\Phi_t = \left( \xi_t + b_x(t)\xi_t + b_y(t)\eta_t + \xi_t I\bar{y}_t(t) + \delta_{\bar{u}} \varepsilon(\Delta v)^* \sigma_u(t) \right) M_{t+1}, p_t \right).
\]

We have \( \mathbb{E} [\Phi_t] = 0 \). Besides,

\[
\mathbb{E} \left[ \left( \sigma_x(t)\xi_t + \sigma_y(t)\eta_t + \xi_t I\bar{y}_t(t) + \delta_{\bar{u}} \varepsilon(\Delta v)^* \sigma_u(t) \right) M_{t+1} q_t M_{t+1} \right]
\]

\[
= \mathbb{E} \left[ \left( \xi_t \sigma_x(t) + \eta_t \sigma_y(t) + \xi_t I\bar{y}_t(t) + \delta_{\bar{u}} \varepsilon(\Delta v)^* \sigma_u(t) \right) \mathbb{E} \left[ M_{t+1} M^*_t [F_t] \right] q^*_t \right].
\]
Similarly,
\[
\Delta \langle \eta_t, k_t \rangle = \langle \Delta \eta_t, k_{t+1} \rangle + \langle \eta_t, \Delta k_t \rangle \\
= \langle -f_{\delta_0}(t+1)\bar{z}_{t+1} - f_{\delta_0}(t+1)\eta_{t+1} - \zeta_{t+1}\bar{I}_2(t+1) - \delta_{t(t+1)}f_{\delta_0}(t+1)\varepsilon \Delta v, k_{t+1} \rangle \\
+ \left\langle \zeta_{t+1}M_{t+1}, f_{\delta_0}(t)k_t - b_{\delta_0}(t)p_t + l_{\delta_0}(t) \right\rangle \hat{I}(\mathbb{E}[M_{t+1}M_{t+1}^* | F_t])^\dagger M_{t+1} \\
- \left\langle \zeta_{t+1}M_{t+1}, q_t \mathbb{E}[M_{t+1}M_{t+1}^* | F_t] \right\rangle \sigma_{\delta_0}(t)^* (\mathbb{E}[M_{t+1}M_{t+1}^* | F_t])^\dagger M_{t+1} \\
+ \left\langle \eta_t, f_{\delta_0}(t)k_t - b_{\delta_0}(t)p_t - \sigma_\delta(t) \mathbb{E}[M_{t+1}M_{t+1}^* | F_t] q_t^* + l_{\delta_0}(t) \right\rangle + \Psi_t,
\]
where
\[
\Psi_t = \left\langle \zeta_{t+1}M_{t+1}, k_t + f_{\delta_0}(t)k_t - b_{\delta_0}(t)p_t - \sigma_\delta(t) \mathbb{E}[M_{t+1}M_{t+1}^* | F_t] q_t^* + l_{\delta_0}(t) \right\rangle \\
+ \left\langle \eta_t, f_{\delta_0}(t)k_t - b_{\delta_0}(t)p_t + l_{\delta_0}(t) \right\rangle \hat{I}(\mathbb{E}[M_{t+1}M_{t+1}^* | F_t])^\dagger M_{t+1} \\
- \left\langle \eta_t, q_t \mathbb{E}[M_{t+1}M_{t+1}^* | F_t] \sigma_{\delta_0}(t)^* (\mathbb{E}[M_{t+1}M_{t+1}^* | F_t])^\dagger M_{t+1} \right\rangle.
\]
Furthermore,
\[
\mathbb{E}\left[ \zeta_{t+1}M_{t+1} \left( f_{\delta_0}(t)k_t - b_{\delta_0}(t)p_t + l_{\delta_0}(t) \right) \hat{I}(\mathbb{E}[M_{t+1}M_{t+1}^* | F_t])^\dagger M_{t+1} \right] \\
- \mathbb{E}\left[ \zeta_{t+1}M_{t+1}q_t \mathbb{E}[M_{t+1}M_{t+1}^* | F_t] \sigma_{\delta_0}(t)^* (\mathbb{E}[M_{t+1}M_{t+1}^* | F_t])^\dagger M_{t+1} \right] \\
= \mathbb{E}\left[ \zeta_{t+1}M_{t+1} \left( \mathbb{E}[M_{t+1}M_{t+1}^* | F_t] \right) ^\dagger \right. \hat{I}(f_{\delta_0}(t)k_t - b_{\delta_0}(t)p_t + l_{\delta_0}(t)) \\
- \mathbb{E}\left[ \zeta_{t+1}M_{t+1} \left( \mathbb{E}[M_{t+1}M_{t+1}^* | F_t] \right) ^\dagger \right. \hat{I} \sigma_{\delta_0}(t) \mathbb{E}[M_{t+1}M_{t+1}^* | F_t] q_t^* \\
= \mathbb{E}\left[ \zeta_{t+1}I(f_{\delta_0}(t)k_t - b_{\delta_0}(t)p_t + l_{\delta_0}(t) - \sigma_\delta(t) \mathbb{E}[M_{t+1}M_{t+1}^* | F_t] q_t^* \right] \\
- \mathbb{E}\left[ \zeta_{t+1}I(f_{\delta_0}(t)k_t - b_{\delta_0}(t)p_t + l_{\delta_0}(t)) \hat{I}(f_{\delta_0}(t)k_t - b_{\delta_0}(t)p_t + l_{\delta_0}(t)) \right.
\]
Then, we obtain
\[
\mathbb{E}\left[ \Delta \langle \xi_t p_t + \eta_t k_t \rangle \right] \\
= \mathbb{E}\left[ [-\bar{z}_{t+1}b_{\delta_0}(t+1)k_{t+1}] + \bar{z}_{t+1}b_{\delta_0}(t) \right] p_t \\
- \bar{z}_{t+1} \sigma_{\delta_0}(t+1) \mathbb{E}[M_{t+2}M_{t+2}^* | F_{t+1}] q_{t+1}^* + \langle \bar{z}_{t+1}, \sigma_{\delta_0}(t) \mathbb{E}[M_{t+1}M_{t+1}^* | F_t] q_t^* \rangle \\
- \eta_{t+1}f_{\delta_0}(t+1)k_{t+1} + \eta_{t+1}f_{\delta_0}(t)k_t - \zeta_{t+1}\bar{I}_2(t+1)k_{t+1} + \zeta_{t+1}\bar{I}_2(t)k_t \\
+ \bar{z}_{t+1}l_{\delta_0}(t+1) + \eta_{t+1}l_{\delta_0}(t) + \zeta_{t+1}l_{\delta_0}(t) \\
+ \bar{\varepsilon} \delta_{t+1} \langle b_{\delta_0}(t)p_t, \Delta v \rangle + \bar{\varepsilon} \delta_{t+1} \langle \sigma_{\delta_0}(t) \mathbb{E}[M_{t+1}M_{t+1}^* | F_t] q_t^*, \Delta v \rangle \\
- \bar{\varepsilon} \delta_{t+1} \langle f_{\delta_0}(t)k_t, \Delta v \rangle.
\]
Therefore,
\[
- \mathbb{E}\left[ h_x \left( \bar{X}_T \right) \bar{z}_T \right] \\
= \mathbb{E}\left[ \langle \bar{x}_T, p_T \rangle + \langle \bar{\eta}_T, k_T \rangle - \langle \bar{\xi}_0, p_0 \rangle - \langle \bar{\eta}_0, k_0 \rangle \right] \\
= \sum_{t=0}^{T-1} \mathbb{E}\left[ \langle \bar{\xi}_t, p_t \rangle + \langle \bar{\eta}_t, k_t \rangle \right] \\
= \mathbb{E}\left[ b_{\delta_0}(0)\bar{z}_0 p_0 + \bar{\xi}_0 \sigma_{\delta_0}(0) \mathbb{E}[M_{t+1}M_{t+1}^* | F_0] q_0^* + \eta_0 f_{\delta_0}(0) k_0 + \bar{z}_0 \bar{I}_2(0) k_0 \right] \\
+ \sum_{t=0}^{T-1} \mathbb{E}\left[ l_{\delta_0}(t) \xi_t + \eta_t \xi_t + \bar{z}_t l_{\delta_0}(t) \right] \\
+ \sum_{t=0}^{T-1} \delta_{t+1} \mathbb{E}\left[ \langle b_{\delta_0}(t)p_t, \Delta v \rangle + \langle \sigma_{\delta_0}(t) \mathbb{E}[M_{t+1}M_{t+1}^* | F_t] q_t^*, \Delta v \rangle - \langle f_{\delta_0}(t)k_t, \Delta v \rangle \right].
\]
Notice that $\xi_0 = 0$, $k_0 = 0$. So

$$\mathbb{E}\left[\sum_{t=0}^{T-1} [l_t(t)\xi_t + l_t(t)\eta_t + \xi_t\bar{l}_t(t)] + \mathbb{E}\left[h_k\left(\bar{X}_T\right)\xi_T\right] \right] = -\varepsilon \mathbb{E}\left[\left( b^*_u(s)p_s + \sigma_u(s)\mathbb{E}\left[M_{s+1}M^*_{s+1}\mid F_s\right]q^*_s - f^*_u(s)k_s, \Delta v\right)\right].$$

Since $\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} [J(\mathbf{u}(\cdot)) - J(\bar{\mathbf{u}}(\cdot))] \geq 0$, we obtain

$$\mathbb{E}\left[\left( b^*_u(s)p_s + \sigma_u(s)\mathbb{E}\left[M_{s+1}M^*_{s+1}\mid F_s\right]q^*_s - f^*_u(s)k_s, \Delta v\right)\right] \leq 0.$$

Then, (31) holds due to that $s$ is taking arbitrarily. This completes the proof. 

5 | CONCLUSION

In this article, we study the maximum principle for stochastic optimal control problems of forward-backward stochastic difference systems. The uncertainty is modeled by finite state processes that take values from the basis vectors. Through the appropriate formulation of the backward difference equations both in the state equations and in the adjoint equations, we establish the maximum principle for this optimal control problem. The main theorems are Theorem 2 for the partially coupled FBSΔSs and Theorem 3 for the fully coupled FBSΔSs.

Throughout the article, we need the assumption that the control domain to be convex. Usually spike variation method is adopted to handle the nonconvex control domain case. However this method does not work in the discrete time framework. More effort is needed to relax the convex assumption for the control domain in the discrete time stochastic control problems. We refer to future work that will address this issue.

ACKNOWLEDGMENT

Shaolin Ji research supported by NSF (No. 11571203). Haodong Liu research supported by Central University Basic Research Fund of China (No. 202013013).

DATA AVAILABILITY STATEMENT

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

ORCID

Haodong Liu https://orcid.org/0000-0002-6860-1713

REFERENCES

1. Bensoussan A. Lectures on stochastic control. In: Mittler SK, Moro A, eds. Nonlinear Filtering and Stochastic Control. Lecture Notes in Mathematics. Springer; 1982:1-62.
2. Bismut JM. An introductory approach to duality in optimal stochastic control. SIAM Rev. 1978;20(1):62-78.
3. Kushner HJ. Necessary conditions for continuous parameter stochastic optimization problems. SIAM J Control. 1972;10(3):550-565.
4. Peng S. A general stochastic maximum principle for optimal control problems. SIAM J Control Optim. 1990;28(4):966-979.
5. Peng S. Backward stochastic differential equations and applications to optimal control. Appl Math Optim. 1993;27(2):125-144.
6. Schroder M, Skiadas C. Optimal consumption and portfolio selection with stochastic differential utility. J Econ Theory. 1999;89(1):68-126.
7. Williams N. On Dynamic Principal-Agent Problems in Continuous Time. University of Wisconsin; 2009.
8. Dokuchaev N, Zhou XY. Stochastic controls with terminal contingent conditions. J Math Anal Appl. 1999;238(1):143-165.
9. Hu M, Ji S, Xue X. A global stochastic maximum principle for fully coupled forward-backward stochastic systems. SIAM J Control Optim. 2018;56(6):4309-4335.
10. Hu M, Ji S, Xue X. Stochastic maximum principle, dynamic programming principle, and their relationship for fully coupled forward-backward stochastic controlled systems. ESAIM Control Optim Calc Var. 2020;26:81.
11. Khallout R, Chala A. A risk-sensitive stochastic maximum principle for fully coupled forward-backward stochastic differential equations with applications. Asian J Control. 2020;22(3):1360-1371.
12. Lim AE, Zhou XY. Linear-quadratic control of backward stochastic differential equations. SIAM J Control Optim. 2001;40(2):450-474.
13. Shi Y, Zhu Q. Partially observed optimal controls of forward-backward doubly stochastic systems. ESAIM Control Optim Calc Var. 2013;19(3):828-843.
14. Wu Z. Maximum principle for optimal control problem of fully coupled forward-backward stochastic systems. *Syst Sci Math Sci*. 1998;3:249-259.
15. Xu W. Stochastic maximum principle for optimal control problem of forward and backward system. *ANZIAM J*. 1995;37(2):172-185.
16. Yong J. Optimality variational principle for controlled forward-backward stochastic differential equations with mixed initial-terminal conditions. *SIAM J Control Optim*. 2010;48(6):4119-4156.
17. Zhang L, Shi Y. Maximum principle for forward-backward doubly stochastic control systems and applications. *ESAIM Control Optim Calc Var*. 2011;17(4):1174-1197.
18. Cohen SN. Uncertainty and filtering of hidden Markov models in discrete time. *Probab Uncertain Quant Risk*. 2020;5(1):1-34.
19. Cohen SN, Elliott RJ. A general theory of finite state backward stochastic difference equations. *Stoch Processes Appl*. 2010;120(4):442-466.
20. Cohen SN, Elliott RJ. Backward stochastic difference equations and nearly time-consistent nonlinear expectations. *SIAM J Control Optim*. 2011;49(1):125-139.
21. Bielecki TR, Cialenco I, Chen T. Dynamic conic finance via backward stochastic difference equations. *SIAM J Financ Math*. 2015;6(1):1068-1122.
22. Stadje M. Extending dynamic convex risk measures from discrete time to continuous time: a convergence approach. *Insur Math Econ*. 2010;47(3):391-404.
23. Eberlein E, Gehrig T, Madan DB. Pricing to acceptability: with applications to valuing one’s own credit risk 2011.
24. Lin Y, Yang H. Discrete-time BSDEs with random terminal horizon. *Stoch Anal Appl*. 2014;32(1):110-127.
25. Madan DB. Conserving capital by adjusting deltas for gamma in the presence of skewness. *J Risk Financ Manag*. 2010;3(1):1-25.
26. Lin X, Zhang W. A maximum principle for optimal control of discrete-time stochastic systems with multiplicative noise. *IEEE Tran Automat Contr*. 2015;60(4):1121-1126.
27. Ji S, Liu H. Solvability of forward–backward stochastic difference equations with finite states. *Stochastics*. 2021;1-21.
28. Cohen SN, Elliott RJ. Solutions of backward stochastic differential equations on Markov chains. *Commun Stoch Anal*. 2008;2(2):251-262.

**How to cite this article:** Ji S, Liu H. Maximum principle for stochastic optimal control problem of finite state forward-backward stochastic difference systems. *Optim Control Appl Meth*. 2022;43(4):1076-1095. doi: 10.1002/oca.2875