Iterated Bernstein polynomial approximations

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Abstract

Iterated Bernstein polynomial approximations of degree $n$ for continuous function which also use the values of the function at $i/n$, $i = 0, 1, \ldots, n$, are proposed. The rate of convergence of the classic Bernstein polynomial approximations is significantly improved by the iterated Bernstein polynomial approximations without increasing the degree of the polynomials. The same idea applies to the $q$-Bernstein polynomials and the Szasz-Mirakyan approximation. The application to numerical integral approximations is also discussed.

MSC: 41A10; 41A17; 41A25.

Keywords: Bernstein polynomials; Bézier curves; Convexity preservation; Iterated Bernstein polynomials; Numerical integration; $q$-Bernstein polynomials; Rate of approximation; the Szasz-Mirakyan operators.

1 Introduction

The Bernstein polynomials [1] have been used for approximations of functions in many areas of mathematics and other fields such as smoothing in statistics and constructing Bézier curves [see 2, 3, for examples] which have important applications in computer graphics. One of the advantages of the Bernstein polynomial approximation of a continuous function $f$ is that it approximates $f$ on $[0, 1]$ uniformly using only the values of $f$ at $i/n$, $i = 0, 1, \ldots, n$. In case when the evaluation of $f$ is difficult and expensive, the Bernstein polynomial approximation is preferred.
The properties of the Bernstein polynomial approximation have been studied extensively by many authors for decades. However the slow optimal rate $O(1/n)$ of convergence of the classical Bernstein polynomial approximation makes it not so attractive. Many authors have made tremendous efforts to improve the performance of the classical Bernstein polynomial approximation. Among many others, Butzer\cite{4} introduces linear combinations of the Bernstein polynomials and Phillips\cite{5} proposes the $q$-Bernstein polynomials which is a generalization of the classical Bernstein polynomial approximation. However, Butzer\cite{4}’s approximation involves not only the Bernstein polynomials of degree $n$ but also degree of $2n$ which requires more sampled values of the function to be approximated at the $2n + 1$ rather than $n + 1$ uniform partition points of $[0, 1]$. The $q$-Bernstein polynomial approximates a function $f$ only when $q \geq 1$. For $q > 1$, it seems that $f(z)$ has to be an analytic complex function on disk $\{ z : |z| < r \}$, $r > q$, so that the $q$-Bernstein polynomial approximation of degree $n$ has a better rate of convergence, $O(q^{-n})$, than the best rate of convergence, $O(n^{-1})$, of the classical Bernstein polynomial approximation of degree $n$ [see \cite{6}, \cite{7}, for example]. If $q > 1$, the $q$-Bernstein polynomial approximation of degree $n$ uses the sampled values of the function at $n + 1$ nonuniform partition points of $[0, 1]$. These points except $t = 1$ are attracted toward $t = 0$ when $q$ is getting larger so that the approximation becomes worse in the the neighborhood of the right end-point. This is a serious drawback of the $q$-Bernstein polynomial approximation which limits the scope of its applications.

In this paper, we propose a simple procedure to generalize and improve the classical Bernstein polynomial approximation by repeatedly approximating the errors using the Bernstein polynomial approximations. This method involves only the iterates of the Bernstein operator applied on the base Bernstein polynomials of degree $n$ and the sampled values of the function being approximated at the same set of $n + 1$ uniform partition points of $[0, 1]$. The improvement made by the $q$-Bernstein polynomial approximation with properly chosen $q$ can be achieved by the iterated Bernstein polynomials without messing up the right boundary.

## 2 Preliminary Results About the Classical Bernstein Polynomial

Let $f$ be a function on $[0, 1]$. The classical Bernstein polynomial of degree $n$ is defined as
\[
\mathbb{B}_n f(t) = \mathbb{B}_n^{(1)} f(t) = \sum_{i=0}^{n} f\left(\frac{i}{n}\right) B_{ni}(t), \quad 0 \leq t \leq 1,
\] 
(1)
where $\mathbb{B}_n$ is called the Bernstein operator and $B_{ni}(t) = \binom{n}{i} t^i (1-t)^{n-i}, i = 0, \ldots, n,$ are called the Bernstein basis polynomials. Note that the Bernstein polynomial of degree $n$, $B_n^{(1)} f$, uses only the sampled values of $f$ at $t_{ni} = i/n, i = 0, 1, \ldots, n$. Note also that for $i = 0, \ldots, n$,

$$\beta_{ni}(t) \equiv (n+1)B_{ni}(t), \quad 0 \leq t \leq 1,$$

is the density function of beta distribution $\text{beta}(i+1, n+1-i)$. Let $Y_n(t)$ be a binomial $b(n, t)$ random variable. Then $E\{Y_n(t)\} = nt, \text{var}\{Y_n(t)\} = E\{Y_n(t) - nt\}^2 = nt(1-t), E\{Y_n(t) - nt\}^3 = nt(1-t)(1-2t)$, and $\mathbb{B}_n f(t) = E[f\{Y_n(t)/n\}]$.

The error of $B_n^{(1)} f$ is

$$\text{Err}\{B_n^{(1)} f\}(t) = \mathbb{B}_n^{(1)} f(t) - f(t). \quad (2)$$

Let $f$ be a member of $C^{(r)}[0, 1]$, the set of all continuous functions that have continuous first $r$ derivatives. $C[0, 1] = C^{(0)}[0, 1]$. Let the modulus of continuity of the $r$th derivative $f^{(r)}$ be

$$\omega_r(\delta) = \max_{|s-t| < \delta} |f^{(r)}(s) - f^{(r)}(t)|, \quad \delta > 0.$$

About the rate of convergence of $\mathbb{B}_n^{(1)} f$ we have the following well known results [see 8].

**Theorem 1.** Suppose $f \in C^{(r)}[0, 1], r = 0, 1$. For each $n > 1$

$$|\text{Err}\{\mathbb{B}_n^{(1)} f\}(t)| = |\mathbb{B}_n f(t) - f(t)| \leq C_r n^{-r/2} \omega_r(n^{-1/2}),$$

where $C_r$ is a constant depending on $r$ only. One can choose $C_0 = 5/4$ and $C_1 = 3/4$.

The result according to $r = 0$ is due to Popovciu [9]. The order of approximation of $f \in C^{(r)}[0, 1]$ by arbitrary polynomials is given by the theorem of Dunham Jackson [10].

**Theorem 2** (Dunham Jackson). Suppose $f \in C^{(r)}[0, 1], r \geq 0$. For each $n > r$ there exists a polynomial $P_n$ of degree at most $n$ so that

$$|P_n(t) - f(t)| \leq C'_r n^{-r} \omega_r(n^{-1}),$$

where $C'_r$ is a constant depending on $r$ only. If $r = 0$, one can choose $C'_0 = 3$.

The following is a result of Voronovskaya [11] about the asymptotic formula of the Bernstein polynomial approximation.
Theorem 3 (E. Voronovskaya). Suppose that \( f \) has second derivative \( f'' \). Then
\[
\text{Err}\{B_n^{(1)} f\}(t) = B_n f(t) - f(t) = \frac{t(1-t)}{2n} f''(t) + \frac{1}{n} \varepsilon_n(t),
\]
where \( \varepsilon_n(t) \) is a sequence of functions which converge to 0 as \( n \to \infty \).

From Theorem 3 it follows that the best rate of convergence of \( B_n^{(1)} f \), as \( n \to \infty \), is \( O(n^{-1}) \) even if \( f \) has continuous second or higher order derivatives [8]. This is not as good as in the case of arbitrary polynomial approximation in which if \( f \) has continuous \( r \)th derivative then the rate of convergence of a sequence of arbitrary polynomials \( P_n \) of degree at most \( n \) can be at least \( o(n^{-r}) \) [10]. Bernstein [12] generalizes this asymptotic formula to contain terms up to the \( (2k) \)th derivative and proposes a polynomial constructed based on both \( f(i/n) \) and \( f''(i/n) \), \( i = 0, 1, \ldots, n \). Butzer [4] considers some combinations of Bernstein polynomials of different degrees and shows that they have better rate of convergence which is much faster than \( O(1/n) \). Costabile et al [13] generalize the linear combinations of the Bernstein polynomials proposed by of [4], [14] and [15]. The \( q \)-Bernstein polynomials of [5] has better rate of convergence. However, if \( 0 < q < 1 \), the \( q \)-Bernstein polynomials of function \( f \) do not approximate \( f \). For \( q > 1 \), the \( q \)-Bernstein polynomials do approximate \( f \) at a rate of \( O(q^{-m}) \) but \( f(z) \) has to be analytic in a complex disk with radius greater than \( q \). The analyticity of \( f \) may be too restrictive for applications. Even if we are sure that \( f \) is analytic, we have to deal with the choice of \( q \). In some cases, the approximations are very sensitive to the choice of \( q \).

3 The Iterated Bernstein Polynomials and the Rate of Convergence

The error \( \text{Err}\{B_n^{(1)} f\}(t) \) is also a continuous function on \([0, 1]\) whose values at \( t = i/n, i = 0, 1, \ldots, n \), depend on \( f(t_i), i = 0, 1, \ldots, n \), only. So we can approximate this error function by the Bernstein polynomial \( B_n^{(1)}[\text{Err}\{B_n^{(1)} f\}](t) \) and then subtract the approximated error function from \( B_n^{(1)} f(t) \) to obtain the second order Bernstein polynomial of degree \( n \)
\[
B_n^{(2)} f(t) = B_n^{(1)} f(t) - B_n^{(1)}[\text{Err}\{B_n^{(1)} f\}](t).
\]
This idea is closely related to, although was not initiated by, the proposal of Bernstein [12] in which the second derivative rather than the error of the Bernstein polynomial is approximated. Inductively,
\[
B_n^{(k+1)} f(t) = B_n^{(k)} f(t) - B_n\{B_n^{(k)} f(t) - f(t)\}, \quad k \geq 1.
\]
This iteration procedure can be performed further until a satisfactory approximation precision is achieved because the error \( \text{Err} \{ B_n(f(t)) \} = B_n(f(t)) - f(t) \) can be estimated by \( \text{Err} \{ B_{n+1}(f(t)) \} = B_{n+1}(f(t)) - B_{n}(f(t)) \).

**Theorem 4.** Generally the \( k \)-th order Bernstein polynomial of degree \( n \) can be written as

\[
B_n^{(k)} f(t) = \sum_{i=0}^{k} \binom{k}{i} (-1)^{i-1} B_i^n f(t), \quad k \geq 1, \quad 0 \leq t \leq 1. \tag{6}
\]

Define \( B_n^0 f(t) = f(t) \). Then the error of the \( k \)-th Bernstein polynomial of degree \( n \) can be written as

\[
\text{Err} \{ B_n^{(k)} f(t) \} = B_n^{(k)} f(t) - f(t) = \sum_{i=0}^{k} \binom{k}{i} (-1)^{i-1} B_i^n f(t) = -(I - B_n)^k f(t), \tag{7}
\]

where \( I = B_n^0 \) is the identity operator.

**Proof.**

\[
B_n^{(k+1)} f(t) = B_n^{(k)} f(t) - B_n \{ B_n^{(k)} f(t) - f(t) \}
\]

\[
= \sum_{i=1}^{k} \binom{k}{i} (-1)^{i-1} B_i^n f(t) - \sum_{i=1}^{k} \binom{k}{i} (-1)^{i-1} B_i^{n+1} f(t) + B_n f(t)
\]

\[
= \sum_{i=1}^{k} \binom{k}{i} (-1)^{i-1} B_i^n f(t) + \sum_{i=2}^{k+1} \binom{k}{i-1} (-1)^{i-1} B_i^n f(t) + B_n f(t)
\]

\[
= \sum_{i=1}^{k+1} \binom{k+1}{i} (-1)^{i-1} B_i^n f(t). \tag{8}
\]

By induction, (1) and (8) assure that (6) is true for every positive integer \( k \). Equation (7) is then obvious. \( \square \)

The limit of \( B_n^k f(t) \), as \( k \to \infty \), has been given by Kelisky and Rivlin [16]. A short and elementary proof of [16]'s result is given by [17]. However there seems on one has ever tried to use \( B_n^{(k)} f(t) \) as an approximation of \( f \) with reduced error in the literature. The cost of \( B_n^{(k)} f(t) \) is only some extra simple calculation in addition to the evaluation of \( f \) at \( i/n, i = 0, 1, \ldots, n \).

About the iterates of the Bernstein operator we have the following result.

**Theorem 5.** For \( k \geq 1 \),

\[
B_n^{k} f(t) = \sum_{i=0}^{n} f \left( \frac{i}{n} \right) B_n^{k-1} (B_{ni})(t), \quad k \geq 1; \quad 0 \leq t \leq 1, \tag{9}
\]
where $B_n^0 B_{ni}(t) = B_{ni}(t)$, and

$$B_n^{k+1} B_{ni}(t) = B_n \{ B_n^k B_{ni} \}(t), \quad k \geq 1. \quad (10)$$

When $k = 1$,

$$B_n^1 B_{ni}(t) = \sum_{j=0}^{n} B_{ni} \left( \frac{j}{n} \right) B_{nj}(t). \quad (11)$$

**Proof.** The theorem can be easily proved by induction and the fact that the Bernstein operator is linear.

By (6) and (7) we have

**Theorem 6.** The $k$-th Bernstein polynomial approximation can be calculated inductively as

$$B_n^{(k)} f(t) = \sum_{i=0}^{n} f \left( \frac{i}{n} \right) \sum_{j=1}^{k} \binom{k}{j} (-1)^{j-1} B_n^{j-1} B_{ni}(t), \quad k \geq 1, \quad 0 \leq t \leq 1. \quad (12)$$

Clearly, for every $k \geq 1$, $B_n^{(k)}$ preserves linear functions. Therefore

$$\text{Err}\{B_n^{(k)} f(t)\} = \sum_{i=0}^{n} \{ f \left( \frac{i}{n} \right) - f(t) \} \sum_{j=1}^{k} \binom{k}{j} (-1)^{j-1} B_n^{j-1} B_{ni}(t), \quad k \geq 1; \quad 0 \leq t \leq 1. \quad (13)$$

Expression (12) can easily implemented in computer languages using iterative algorithm. Define indicator functions

$$I_{ni}(t) = \begin{cases} 
1, & t = \frac{i}{n}; \\
0, & t \neq \frac{i}{n}.
\end{cases} \quad (14)$$

Then $B_{ni}(t) = B_n I_{ni}(t) = B_n^{(1)} I_{ni}(t), \quad i = 0, 1, \ldots, n$, and, by Theorem 6, (12) and (13) can be simplified as

$$B_n^{(k)} f(t) = \sum_{i=0}^{n} f \left( \frac{i}{n} \right) B_n^{(k)} I_{ni}(t), \quad k \geq 1, \quad 0 \leq t \leq 1. \quad (15)$$

$$\text{Err}\{B_n^{(k)} f(t)\} = \sum_{i=0}^{n} \{ f \left( \frac{i}{n} \right) - f(t) \} B_n^{(k)} I_{ni}(t), \quad k \geq 1; \quad 0 \leq t \leq 1. \quad (16)$$

$$B_n^{(k)} I_{ni}(t) = \sum_{j=1}^{k} \binom{k}{j} (-1)^{j-1} B_n^{j-1} B_{ni}(t).$$

The next theorem shows that if $k > 1$ then $B_n^{(k)} f(t)$ is indeed a better polynomial approximation of $f$ than the classical Bernstein polynomial.
Theorem 7. Suppose that $f \in C_{dkr}[0, 1]$, $dkr = 2(k - 1) + r$ and $r = 0, 1$. Then

$$|\text{Err}\{B_n^{(k)} f(t)\}| = |B_n^{(k)} f(t) - f(t)| \leq C''_{kr} n^{-\frac{dkr}{2}} \omega_{dkr}(n^{-1/2}),$$  \hspace{1cm} (17)

where $C''_{kr}$ is a constant depending on $r$ and $k$ only.

Proof. This result follows easily from Theorems 1 and 3. \hfill \qed

Remark 3.1. From this theorem with $k = 2$ and $r = 0$, we see that if $f$ has continuous second derivative then the rate of convergence of the second Bernstein polynomial approximation $B_n^{(2)} f$ is at least $o(n^{-1})$.

Remark 3.2. From Theorem 7 with $k = 2$ we see that if $f$ has continuous fourth derivative, then the rate of convergence of $B_n^{(2)} f$ can be as fast as $O(n^{-2})$. This seems the fastest rate that $B_n^{(2)} f$ can reach even if $f$ has continuous fifth or higher derivatives.

Remark 3.3. It can also be proved that if $f$ has continuous $(2k)$th derivative, then the rate of convergence of $B_n^{(k)} f$ can be as fast as $O(n^{-k})$. Although these improvements upon $B_n f(t)$ are still not as good as those stated in Theorem 2, they are good enough for application in computer graphics and statistics.

Remark 3.4. It is a very interesting project to investigate the relationship between $C''_{kr}$ and $k$.

4 The Derivatives and Integrals of $B_n^{(k)} f(t)$ and Applications

4.1 The Derivatives of $B_n^{(k)} f(t)$

Theorem 8. For any positive integer $k$, For any positive integers $k$ and $r$,

$$\frac{d^r}{dt^r}B_n^{(k)} f(t) = \frac{n!}{(n-r)!} \sum_{i=0}^{n-r} \sum_{j=1}^{k} (-1)^{j-1} \binom{k}{j} \Delta^r(B_n^{j-1} f)(\frac{i}{n}) B_{n-r,i}(t),$$  \hspace{1cm} (18)

where $\Delta^r$ is the $r$th forward difference operator with increment $h = 1/n$, $\Delta f(t) = f(t + h) - f(t)$,

$$\Delta^r f(t) = \sum_{i=0}^{r} \binom{r}{i} (-1)^i f\left(t + \frac{i-1}{h}\right).$$
Proof. If \( k = 1 \), it is well known that for any function \( f \)
\[
\frac{d}{dt} B_n^{(1)} f(t) = \frac{d}{dt} B_n f(t) = n \sum_{i=0}^{n-1} \Delta f \left( \frac{i}{n} \right) B_{n-1,i}(t). \tag{19}
\]
Assume that (18) with \( r = 1 \) is true for the \( k \)th iterated Bernstein polynomial of any function \( f \). By (5) we have
\[
\frac{d}{dt} B_n^{(k+1)} f(t) = \frac{d}{dt} B_n^{(k)} f(t) = \frac{d}{dt} B_n f(t) - \frac{d}{dt} B_n \{ B_n^{(k)} f(t) - f(t) \}
= \frac{d}{dt} B_n f(t) + \frac{d}{dt} B_n f(t) - \frac{d}{dt} B_n \{ B_n^{(k)} f(t) \}.
\tag{20}
\]
It follows from (19) and (9) that
\[
\frac{d}{dt} B_n \{ B_n^{(k)} \} f(t) = n \sum_{i=0}^{n-1} \Delta B_n^{(k)} f \left( \frac{i}{n} \right) B_{n-1,i}(t)
= n \sum_{i=0}^{n-1} \sum_{j=0}^{n} \sum_{k} \binom{k}{\ell} \binom{\ell-1}{j} \binom{k}{j} B_n^{j-1} f \left( \frac{i}{n} \right) B_{n-1,i}(t).
\tag{21}
\]
Combining (19), (9), (20), and (21) we arrive at
\[
\frac{d}{dt} B_n^{(k+1)} f(t) = n \sum_{i=0}^{n-1} \sum_{j=1}^{k+1} (-1)^{j-1} \binom{k}{j} \Delta B_n^{j-1} f \left( \frac{i}{n} \right) B_{n-1,i}(t). \tag{22}
\]
The proof of (18) with \( r = 1 \) and \( k \geq 1 \) is complete by induction. Similarly (18) with \( r \geq 1 \) and \( k \geq 1 \) can be proved using induction.

It is not hard to prove by adopting the method of (8) that

Theorem 9. (i) If \( f \) has continuous \( r \)th derivative \( f^{(r)} \) on \([0, 1]\), then for each fixed \( k \), as \( n \to \infty, \frac{d^r}{dt^r} B_n^{(k)} f(t) \) converge to \( f^{(r)}(t) \) uniformly on \([0, 1]\).

(ii) If \( f \) is bounded on \([0, 1]\) and its \( r \)th derivative \( f^{(r)}(t) \) exists at \( t \in [0, 1] \), then for each fixed \( k \), as \( n \to \infty, \frac{d^r}{dt^r} B_n^{(k)} f(t) \) converge to \( f^{(r)}(t) \).

Numerical examples show that the larger the \( r \) is, the slower the above convergence is. For any positive integers \( k \), the second derivative of the iterated Bernstein polynomial \( B_n^{(k)} f \) is
\[
\frac{d^2}{dt^2} B_n^{(k)} f(t) = n(n-1) \sum_{i=0}^{n-2} \sum_{j=1}^{k} (-1)^{j-1} \binom{k}{j} \Delta^2 \left( B_n^{j-1} f \right) \left( \frac{i}{n} \right) B_{n-2,i}(t). \tag{23}
\]
It is well known that if \( f \) is convex on \([0, 1]\), then \( \frac{d^2}{dt^2} B_n^{(1)} f(t) \geq 0 \) and thus \( B_n^{(1)} f(t) \) is also convex and \( B_n^{(1)} f(t) \geq f(t) \) on \([0, 1]\). So the classical Bernstein polynomials preserve the convexity of the original function and has nonnegative errors. However examples of \( f \) show that when \( k \leq 2 \) the iterated Bernstein polynomial \( B^{(k)} f \) does not preserve the convexity of the original function unconditionally.

### 4.2 The Integrals of \( B_n^{(k)} f(t) \)

The following theorem is very useful for implementing the iterative algorithm in computer languages.

**Theorem 10.** Suppose \( f \) is continuous on \([0, 1]\). For \( k \geq 1 \) and \( x \in [0, 1] \), we have

\[
\int_0^x B_n^{(k)} f(t) dt = \sum_{i=0}^n f\left( \frac{i}{n} \right) \sum_{j=1}^k \binom{k}{j} (-1)^{j-1} S_n^{(j)} (x),
\]

(24)

where \( S_n^{(j)} (x) = \int_0^x B_n^{j-1} B_n(t) dt \) which can be calculated iteratively as follows

\[
S_n^{(1)} (x) = \int_0^x B_n(t) dt = \frac{1}{n+1} \int_0^x \beta_{ni}(t) dt, \quad S_n^{(1)} (1) = \frac{1}{n+1}.
\]

\[
S_n^{(k+1)} (x) = \sum_{j=0}^n B_n\left( \frac{i}{n} \right) S_n^{(j)} (x), \quad k = 1, 2, \ldots.
\]

**Corollary 11.** If \( g \) is continuous on \([a, b]\), \( a < b \), then for \( k \geq 1 \),

\[
\int_a^b g(t) dt \approx \sum_{i=0}^n g\left( \frac{i}{n} \right) \sum_{j=1}^k \binom{k}{j} (-1)^{j-1} S_n^{(j)} (1),
\]

(25)

where \( f(t) = \frac{1}{b-a} g[a + (b-a)t] \).

**Remark 4.1.** Note that numerical integration (25) does not involve any integrals. It contains only algebraic calculations.

**Proof.** Note

\[
\int_0^x B_n^{(k)} f(t) dt = \sum_{i=0}^n f\left( \frac{i}{n} \right) \int_0^x B_n^{(k)} I_{ni}(t) dt.
\]

For \( k = 1 \), it is well known that

\[
S_n^{(1)} (x) = \int_0^x B_n^{(1)} I_{ni}(t) dt = \int_0^x B_n(t) dt = \frac{1}{n+1} \int_0^x \beta_{ni}(t) dt, \quad S_n^{(1)} (1) = \frac{1}{n+1}.
\]
So (24) is true for any function $f$.

$$\int_0^x \mathbb{B}_n^{(1)} f(t) dt = \sum_{i=0}^n f\left( \frac{i}{n} \right) S_{ni}^{(1)}(x), \quad \int_0^1 \mathbb{B}_n^{(1)} f(t) dt = \frac{1}{n+1} \sum_{i=0}^n f\left( \frac{i}{n} \right) \approx \int_0^1 f(t) dt.$$ 

By (24) with $k = 1$ have

$$\int_0^x \mathbb{B}_n^{(k)} f(t) dt = \sum_{i=0}^n \mathbb{B}_n^{(k)} f\left( \frac{i}{n} \right) S_{ni}^{(1)}(x)$$ 

$$= \sum_{i=0}^n \sum_{l=0}^n f\left( \frac{i}{n} \right) \sum_{j=1}^k \binom{k}{j} (-1)^{j-1} \mathbb{B}_n^{j-1} B_{nl} \left( \frac{i}{n} \right) S_{ni}^{(1)}(x)$$ 

$$= \sum_{i=0}^n f\left( \frac{i}{n} \right) \sum_{j=1}^k \binom{k}{j} (-1)^{j-1} \sum_{l=0}^n \mathbb{B}_n^{j-1} B_{nl} \left( \frac{i}{n} \right) S_{ni}^{(1)}(x)$$ 

$$= \sum_{i=0}^n f\left( \frac{i}{n} \right) \sum_{j=1}^k \binom{k}{j} (-1)^{j-1} S_{ni}^{(j+1)}(x).$$

Assume that (24) is true $k \geq 1$. Then by (5) we have

$$\int_0^x \mathbb{B}_n^{(k+1)} f(t) dt = \int_0^x \mathbb{B}_n^{(k)} f(t) dt + \int_0^x \mathbb{B}_n^{(k)} f(t) dt - \int_0^x \mathbb{B}_n^{(k)} f(t) dt$$ 

$$= \sum_{i=0}^n f\left( \frac{i}{n} \right) \sum_{j=1}^k \binom{k}{j} (-1)^{j-1} S_{ni}^{(j)}(x) + \sum_{i=0}^n f\left( \frac{i}{n} \right) S_{ni}^{(1)}(x)$$ 

$$- \sum_{i=0}^n f\left( \frac{i}{n} \right) \sum_{j=1}^k \binom{k}{j} (-1)^{j-1} S_{ni}^{(j+1)}(x)$$ 

$$= \sum_{i=0}^n f\left( \frac{i}{n} \right) \sum_{j=1}^{k+1} \binom{k+1}{j} (-1)^{j-1} S_{ni}^{(j)}(x).$$

So (24) is also true for $k + 1$ and by induction the proof is complete. 

The following theorem follows immediately from Theorems 7 and 10.

**Theorem 12.** Under the condition of Theorem 7, for any $x \in [0, 1]$ 

$$\left| \int_0^x \mathbb{B}_n^{(k)} f(t) dt - \int_0^x f(t) dt \right| \leq C''_k r^{\frac{k}{r}} n^{-\frac{k}{2r}} \omega_{dkr}(n^{-1/2}),$$

where $C''_k$ is a constant depending on $r$ and $k$ only.
5 Iterated Szasz Approximation and Iterated $q$-Bernstein Polynomial

The idea used to construct the iterated Bernstein polynomial approximation is simple and very effective. The same idea seems also applicable to other operators or approximations such as the Szasz operator [18] or the Szasz-Mirakyan (Mirakja) operator and the $q$-Bernstein polynomial with $q > 1$. We will give some numerical examples in §6 and the analogues of results of Section 3 could be obtained by using the analogue results about the rate of convergence of the Szasz-Mirakyan approximation [19]. We hope these would inspire more investigations with rigorous mathematics.

5.1 Iterated Szasz Approximation

The so-called Szasz-Mirakyan approximation is defined as

$$S_n f(x) = \sum_{i=0}^{\infty} f\left(\frac{i}{n}\right) P_{ni}(x), \ x \in [0, \infty),$$

(27)

where $f$ is defined on $[0, \infty)$ and $P_{ni}(x) = e^{-nx}(nx)^i/i!$. Note that, for $x > 0$, $P_{ni}(x)$ is the probability that $V_n(x) = i$ where $V_n(x)$ is the Poisson random variable with mean $nx$. Since the binomial probability $B_{ni}(t)$ can be approximated by $P_{ni}(t)$ for large $n$, the Szasz-Mirakyan approximation can be viewed as an extension of the Bernstein polynomial approximation. The error of $S_n f$ as an approximation of $f$ is

$$\text{Err}(S_n f)(x) = S_n f(x) - f(x) = \sum_{i=0}^{\infty} f\left(\frac{i}{n}\right) P_{ni}(x) - f(x), \ x \in [0, \infty).$$

(28)

Applying the Szasz-Mirakyan operator to $\text{Err}(S_n f)(x)$, we have

$$S_n\{\text{Err}(S_n f)\}(x) = S_n^2 f(x) - S_n f(x) = \sum_{i=0}^{\infty} f\left(\frac{i}{n}\right) S_n P_{ni}(x) - S_n f(x), \ x \in [0, \infty).$$

(29)

So we can define the second Szasz-Mirakyan approximation as

$$S_n^{(2)} f(x) = S_n f(x) - S_n\{\text{Err}(S_n f)\}(x), \ x \in [0, \infty).$$

(30)

Theorem 13.

$$S_n^{(k)} f(x) = \sum_{i=0}^{\infty} f\left(\frac{i}{n}\right) \sum_{j=1}^{k} \binom{k}{j} (-1)^{j-1} S_n^{j-1} P_{ni}(x), \ k \geq 1, \ x \in [0, \infty).$$

(31)
Clearly, for every \( k \geq 1 \), \( S_n^{(k)} \) preserves linear functions and therefore

\[
\text{Err}\{S_n^{(k)} f(x)\} = \sum_{i=0}^{\infty} \left\{ f\left(\frac{i}{n}\right) - f(x) \right\} \sum_{j=1}^{k} \binom{k}{j} (-1)^{j-1} S_n^{j-1} P_n(t), \quad k \geq 1; \ x \in [0, \infty).
\]

Figure 4 gives an example of the iterated Szasz approximations.

5.2 Iterated \( q \)-Bernstein Polynomial

Let \( x \) be a real number. For any \( q > 0 \), define the \( q \)-number

\[
[x]_q = \begin{cases} 
\frac{1-q^x}{1-q}, & \text{if } q \neq 1; \\
\frac{1}{x}, & \text{if } q = 1.
\end{cases}
\]

If \( x \) is integer, then \( [x]_q \) is called a \( q \)-integer. For \( q \neq 1 \), the \( q \)-binomial coefficient (Gaussian binomial) is defined by

\[
\binom{n}{r}_q = \begin{cases} 
1, & r = 0; \\
\frac{(1-q^n)(1-q^{n-1})\cdots(1-q^{n-r+1})}{(1-q^r)(1-q^{r-1})\cdots(1-q)}, & 1 \leq r \leq n; \\
0, & r > n.
\end{cases}
\]

So

\[
\binom{n}{r}_q = \prod_{i=0}^{r-1} \left[ \frac{n-i}{r-i} \right]_{q^{r-i}}, \quad 0 \leq r \leq n, \quad q > 0,
\]

where empty product is defined to be 1. Thus the ordinary binomial coefficient \( \binom{n}{r} \) is the special case when \( q = 1 \). G. M. Phillips [5] introduced the \( q \)-Bernstein polynomial of order \( n \) for any continuous function \( f(t) \) on the interval \([0, 1]\)

\[
Q_{n,q} f(t) = \sum_{i=0}^{n} f\left( t_i^{(q)} \right) Q_{ni}(t), \quad n = 1, 2, \ldots,
\]

where

\[
t_i^{(q)} = \frac{[i]_q}{[n]_q}, \quad Q_{ni}(t) = \binom{n}{i}_q \prod_{j=1}^{n-i} \left( 1 - tq^{j-1} \right), \quad i = 0, 1, \ldots, n.
\]

Clearly, \( \mathbb{B}_n f(t) = Q_{n,1} f(t) \) which is the classical Bernstein polynomial of order \( n \). It has been proved that if \( 0 < q < 1 \) then \( Q_{n,q} f(t) \) does not approximate \( f \) and that if \( q > 1 \) and \( f(z) \) is analytic complex function on disk \( \{ z : |z| < r \} \), \( r > q \), then \( Q_{n,q} f(t) \) has better rate of convergence, \( O(q^{-n}) \), than the best rate of convergence, \( O(n^{-1}) \), of \( \mathbb{B}_n f(t) \) [see 6, 7, for example].
Note that if $q > 1$ then points $t_i^{(q)} = [i]/[n]$ are no longer uniform partition points of the interval $[0, 1]$. For fixed $n$, $\lim_{q \to \infty} t_i^{(q)} = 0$, $i < n$. So all $t_i$ except $t_n^{(q)} = 1$ are attracted toward 0 as $q$ getting large. However, interestingly, the larger the $q$ is in a certain range, the closer the $q$-Bernstein polynomial approximation $Q_{nq} f(t)$ to $f(t)$. For a given $n$, if $q$ is too large, the $q$-Bernstein polynomial approximation $Q_{nq} f(t)$ becomes worse in the neighborhood of the right end-point.

Similarly we have the iterated $q$-Bernstein polynomials

$$Q_{nq}^{(k)} f(t) = \sum_{i=0}^{\infty} f(t_i^{(q)}) \sum_{j=1}^{k} \binom{k}{j} (-1)^{j-1} Q_{nq}^{j-1} Q_{ni} f(t), \quad k \geq 1, \quad t \in [0, 1]. \quad (33)$$

See Figure 5 for an example of the iterated $q$-Bernstein polynomials. Comparing Figures 1 and 5 we see that increasing $q$ from 1 to 1.1 does improve the approximation on $[0, 1]$ except at points in the neighborhood of the right end-point. The approximation near the right end-point could be worsen by applying the iterated $q$-Bernstein polynomials. The improvement can be achieved by the iterated Bernstein polynomials without messing up the right boundary.

6 Numerical Examples

In this section some numerical examples are given with the hope of more investigations on the proposed methods with rigorous mathematics.

Example 1. Figure 1 shows the first three iterated Bernstein polynomials of $f(t) = \sin(2\pi t)$ and the errors where $n = 30$.

Example 2. Figure 3 shows the first three iterated Bernstein polynomials of $f(t) = |t - 0.5|$ and the errors where $n = 30$.

Example 3. Figure 2 shows the first three iterated Bernstein polynomials of $f(t) = \text{sign}(t - 0.5)(t - 0.5)^2$ (a differentiable but not twice differentiable function) and the errors where $n = 30$.

Example 4. Figure 4 shows the first three iterated Szasz approximation of $f(x) = 0.25xe^{-x^2/2}$, $x \geq 0$, and the errors where $n = 10$.

Example 5. Figure 5 shows the first three iterated $q$-Bernstein polynomials of $f(x) = \sin(\pi x)$ and the errors where $n = 30$, $q = 1.1$. The performance of the approximation near $t = 1$ is very sensitive to $q$.

Example 6. Figure 6 shows the first three iterated Bernstein polynomials of the following function $f(t) = |t - 0.5|$ and their derivatives where $n = 30$. 

13
Example 7. Figure 7 shows the first three iterated Bernstein polynomials of the following function \( f(t) \) and their derivatives where \( n = 30 \),

\[
  f(t) = \begin{cases} 
    t(t-1), & 0 \leq t < 0.5; \\
    -\frac{1}{4} + \frac{2}{3}(t-0.5)^{3/2}, & 0.5 \leq t \leq 1.
  \end{cases}
\]

This a convex function which has continuous first derivative but does not have a continuous second derivative.

Example 8. Denote \( t_\delta = \frac{2}{3} - \delta \) where \( \delta \) is a small positive number.

\[
  f(t) = \begin{cases} 
    f_0(t), & 0 \leq t < t_\delta; \\
    p_k(t), & t_\delta < t \leq 1,
  \end{cases}
\]

where \( f_0(t) = v - \sqrt{r^2 - (t-u)^2} \) is portion of a circle with radius \( r \) (a larger positive number) and centered at \((u,v)\), \( u, v > 0 \), \( p_k(t) \) is a polynomial of degree \( k = 3 \),

\[
p_k(t) = \sum_{i=0}^{k} a_{ki} t^i = a_{kk} t^k + a_{k,k-1} t^{k-1} + \cdots + a_{k1} t + a_{k0}.
\]

If we choose

\[
v = -30t_\delta \pm \sqrt{900t_\delta^2 - 40(25t_\delta^2 - r^2)}, \quad u = \sqrt{r^2 - v^2}
\]

then \( f(0) = f_0(0) = 0, f(t_\delta) = f_0(t_\delta) = -3t_\delta \). We also have

\[
f'_0(t) = \frac{t - u}{\sqrt{r^2 - (t-u)^2}}, \quad f''_0(t) = \frac{r^2}{\{r^2 - (t - u)^2\}^{3/2}}.
\]

Choose the coefficients of \( p_k \) so that \( f(1) = \sum_{i=0}^{k} a_{ki} = 0 \) and the \( j \)th \((j = 0, 1, \ldots, k - 1)\) derivative at \( t_\delta \) satisfy

\[
f^{(j)}(t_\delta) = \sum_{i=j}^{k} \frac{t^i}{(i-j)!} a_{ki} t_\delta^{i-j} = f_0^{(j)}(t_\delta^{i-j}).
\]

If \( r \) is large enough, say \( r = 70, \delta = 0.05 \), then \( f(t) \) is strictly convex and has continuous positive second derivative \( f'' \), but \( B_n^{(2)} f \) is still not convex because its second derivative is negative at some points near \( t = 0.4 \) (see Figure 8).

From these figures we see that the error is reduced significantly by using the iterated Bernstein polynomial approximation without increasing the degree of the polynomial. For non-smooth function, the maximum error is reduced more than 50% by the third Bernstein polynomial. It is also seen from Figure 8 that unlike the classical Bernstein polynomial approximation the iterated Bernstein polynomial approximation \( B_n^{(k)} f \) seems not to preserve the convexity of \( f \) for \( k > 1 \) in this case when \( f \) is not smooth. So it is necessary for \( B_n^{(k)} f \) to preserve the convexity of \( f \) that \( f \) is smooth and \( f'' \) is not too close to zero.
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Figure 1: The iterated Bernstein polynomials and errors when $f(t) = \sin(2\pi t)$

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Figure 2: The iterated Bernstein polynomials and errors when \( f(t) = \text{sign}(t - 0.5)(t-0.5)^2 \) which is differentiable on \([0, 1]\) but not twice differentiable at \( t = 0.5 \).
Figure 3: The iterated Bernstein polynomials and errors when $f(t) = |t - 0.5|$ which is not differentiable at $t = 0.5$. 
Figure 4: The iterated Szasz approximations and errors when $f(x) = 0.25xe^{-x^2/2}$, $x \geq 0$ with $n = 10$. 
Figure 5: The iterated $q$-Bernstein polynomials and errors when $f(t) = \sin(2\pi t)$ with $n = 30$, $q = 1.1$. 
Figure 6: The iterated Bernstein polynomials and their derivatives when $f(t) = |t - 0.5|$ which is convex but not differentiable at $t = 0.5$. 
Figure 7: The iterated Bernstein polynomials of $f(t)$ as in Example 7 and their derivatives where $f(t)$ is convex, differentiable on $[0, 1]$ but not twice differentiable at $t = 0.5$. 
Figure 8: The iterated Bernstein polynomials of $f$ as in Example and their derivatives. The function $f$ is strictly convex but $B_n^{(2)} f$ is not convex.
\begin{align*}
B_n^{(1)} f(t) &= \frac{d}{dt} B_n f(t) \\
B_n^{(2)} f(t) &= \frac{d^2}{dt^2} B_n f(t) \\
B_n^{(3)} f(t) &= \frac{d^3}{dt^3} B_n f(t)
\end{align*}