Distributed Multi-User Secret Sharing

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Abstract—A distributed secret sharing system is considered that consists of a dealer, \(n\) storage nodes, and \(m\) users. Each user is given access to a certain subset of storage nodes where it can download the data. The dealer wants to securely convey a specific secret \(s_j\) to user \(j\) via storage nodes, for \(j = 1, 2, \ldots, m\), in such a way that no user gets any information about other users’ secrets in an information-theoretic sense. To this end, we propose to study protocols where the dealer encodes secrets into several secret shares and loads them into the storage nodes. Given a certain number of storage nodes we find the maximum number of users that can be served in such protocols and construct schemes that achieve this. We further define two major properties for such distributed secret sharing systems; communication complexity is defined as the total amount of data that needs to be downloaded by users in order to reconstruct their secrets; and storage overhead is defined as the total amount of data loaded by the dealer into the storage nodes normalized by the total size of secrets. Lower bounds on minimum communication complexity and storage overhead are characterized given any \(n\) and \(m\). Furthermore, we construct distributed secret sharing protocols, under certain conditions on the system parameters, that attain these lower bounds thereby providing schemes that are optimal in terms of both the communication complexity and storage overhead. It is shown how to modify the proposed protocols in order to construct schemes for any set of parameters while providing a nearly optimal storage overhead.

Index Terms—Secret sharing, distributed storage, multi-user security

I. INTRODUCTION

Secret sharing, introduced by Shamir [1] and Blakely [2], is central in many cryptographic systems. They have found applications in cryptography and secure distributed computing such as secure interactive computations [3]–[7], secure storage [8]–[10], generalized oblivious transfer [11], [12], and threshold cryptography [13]–[15]. A secret-sharing scheme involves a dealer, who has a secret, a set of users, and a collection \(A\) of subsets of users, which is called the access structure. A secret-sharing scheme for the access structure \(A\) is a scheme for distributing the secret by the dealer among the users while guaranteeing (1) secret recovery: any subset in the access structure \(A\) can recover the secret from its shares, and (2) collusion resistance: for any subset not in \(A\), the aggregate data of users in that subset reveals no information about the secret.

Most cryptographic protocols involving secret sharing assume that the central user, called the dealer, has a direct reliable and secure communication channel to all the users. In such settings, it is assumed that once the dealer computes the shares of secret, they are readily available to the users. In many scenarios, however, the dealer and users are nodes of a large network. In general, the communication between the dealer node and users can be through several relay nodes, as in a relay network or through intermediate network nodes, as in a network coding scenario. Alternatively, in a distributed storage scenario, the dealer can be thought as a master node controlling a certain set of servers or storage nodes, while each user has access to a certain subset of servers.

In this paper, we consider the later scenario. In particular, the system model is shown in Figure 1. The dealer is considered as a central entity that controls a given set of servers, also referred to as storage nodes, and can load data to them. Alternatively, in an application concerning multiple-access wireless networks, one can think of middle nodes, sitting between the dealer and users, as resource elements in different time or frequency while each user has access to a certain subset of resource elements. We further consider a multi-user secret sharing scenario, in the sense that there is a designated secret, independently generated for each user, to be conveyed to that user. We require the secret sharing protocol to be secure implying that each user does not get any information, in the information-theoretic sense, about other users’ secrets. The system model, security condition, and our approach to construct secret sharing protocols for this system are described next.

A. System model

A distributed secret sharing system, shown in Figure 1, consists of a dealer, \(n\) storage nodes, and \(m\) users. The goal of this system is to enable the dealer to securely convey a specific secret to each user via storage nodes. In this system model:

a) For each user \(j\), \(A_j \subset \{n\}\), where \(\{n\} \triangleq \{1, 2, 3, \ldots, n\}\), denote the set of all storage nodes that user \(j\) has access to. The set \(A_j\) is referred to as the access set for the user \(j\). For each \(i \in A_j\), user \(j\) can read the entire data loaded into node \(i\). Let

\[ \mathcal{A} \triangleq \{A_j : j \in \{m\}\} \]

(1)

denote the set of all access sets, which is called the access structure.

b) Storage nodes are passive; they do not communicate with each other. Also, the users do not communicate with each other.

c) Let \(s_j \in \mathbb{F}_q\) denote the secret for user \(j\). Also, \(s_j\)’s are uniformly distributed and mutually independent.

d) The dealer has access to all the storage nodes but it does not have direct access to the users.

We aim at designing distributed secret sharing protocols to encode the secrets into secret shares, and distribute them in the storage nodes in such a way that: 1) Each user \(j\) can successfully reconstruct its designated secret \(s_j\), and 2) User \(j\) does not get any information about \(s_l\), for any \(l \neq j\). This is defined more precisely as follows.

Definition 1: A distributed secret sharing protocol (DSSP) is a bundle of \((\mathcal{A}, E, \mathbb{Z}_{n \times h}, D)\), where
The storage overhead, denoted by storage overhead two important aspects, namely

The protocol is also i.i.d and uniformly distributed secrets. Therefore, each secret share specifies one interpolation point leading to t distinct interpolation points. Then $s = P(0)$ is reconstructed. We refer to this process as Shamir’s secret decoder.

B. Main Results

We first consider the problem of finding the maximum number of users that can be served in a DSSP given a certain number of storage nodes. This maximum number is derived using a necessary and sufficient condition on access sets in a DSSP that relates to Sperner families in combinatorics. We further present a method for constructing DSSPs that serve maximum number of users.

For a given number of users $m$ and number of storage nodes $n$, a DSSP with minimum communication complexity $C$ defined in (4) is called a communication-efficient DSSP. We solve a discrete optimization problem to provide a lower bound on the minimum communication complexity. We further construct DSSPs that are communication-efficient, i.e., they achieve the minimum possible communication complexity when $m$ is a binomial coefficient of $n$.

We further construct communication-efficient DSSPs that also achieve the optimal storage overhead of one, under the same condition that $m$ is a binomial coefficient of $n$. In the proposed schemes no external randomness is required and the total size of data to be stored on storage nodes is equal to the total size of secrets. Consequently, this provides the optimal storage overhead of one. We then discuss how to modify the methods in order to construct DSSPs with nearly optimal storage overhead for any given set of parameters. Furthermore, the modified protocol leads to significantly reduced complexity of the construction and encoding algorithm.

C. Shamir’s Scheme and Related Works

The $(k, t)$ secret sharing scheme provided by Shamir in [1], is as follows. Given a secret $s \in \mathbb{F}_q$ it produces $k$ secret shares $d_1, d_2, \ldots, d_k \in \mathbb{F}_q$ in such a way that

a) the secret $s$ can be reconstructed given any $t$ or more of the secret shares, and

b) knowledge of any $t - 1$ or fewer secret shares does not reveal any information about $s$, in the information-theoretic sense.

To this end, a $(t-1)$-degree polynomial $P(X)$ is constructed as follows:

$$P(x) = s + \sum_{i=1}^{t-1} p_i x^i$$

where $p_i$’s are i.i.d and are selected uniformly at random from $\mathbb{F}_q$. Let $\gamma_1, \gamma_2, \ldots, \gamma_k$ denote $k$ distinct non-zero elements from $\mathbb{F}_q$. The secret shares are produced by evaluating $P(x)$ at $\gamma_i$’s, i.e.,

$$\forall i \in [k] \quad d_i = P(\gamma_i).$$

We refer to the encoder $E : \mathbb{F}_q \rightarrow \mathbb{F}_q^k$ that produces $(d_1, d_2, \ldots, d_k)$ according to this procedure given the input $s$ as $(k, t)$ Shamir’s secret encoder.

Given any $t$ secret shares $P(x)$ is interpolated and is uniquely determined. This is because the degree of $P(x)$ is at most $t - 1$ and each secret share specifies one interpolation point leading to $t$ distinct interpolation points. Then $s = P(0)$ is reconstructed. We refer to this process as Shamir’s secret decoder.
There are several previous works that have considered Shamir’s scheme in the context of networks [16] and distributed storage systems [17], [18]. In these works, there is only one secret, as in the original Shamir’s scheme, to be distributed to users either as nodes of a network [16] or as users of a distributed storage system [17], [18], in a collusion-resistant way. However, we consider a multi-user secret sharing scenario, where there is one designated secret for each user, and the secret shares are distributed over a set of storage nodes. Also, in our constructed schemes we guarantee that each user does not get any information about other users’ secrets, thereby providing security in a multi-user sense.

II. DSSP with Maximum Number of Users

In this section, we consider the following problem: What is the maximum possible number of users that can be served in a DSSP given a certain number of storage nodes? A necessary and sufficient condition on access sets in a DSSP is shown which relates to Sperner families in combinatorics. This relation is invoked to present a method for constructing DSSPs that serve maximum number of users.

Lemma 1: For a DSSP with access structure \( \mathcal{A} \) defined in (1):
\[
A_j \cap A_l = \emptyset
\]
for all \( j, l \in [m] \) with \( j \neq l \).

Proof: Assume to the contrary that \( A_j \subset A_l \) for some \( j \neq l \). Therefore, the entire accessible data by user \( j \) can also be accessed by user \( l \). Since user \( j \) can retrieve \( s_j \) by the correctness condition, user \( l \) can also retrieve \( s_j \). This means the security condition is violated and the protocol is not a DSSP as defined in Definition 1, which is a contradiction.

Collections of subsets \( \mathcal{A} \) satisfying the condition of Lemma 1 are well-studied combinatorial objects. Such a set \( \mathcal{A} \) is called a Sperner family [19]. For any Sperner family \( \mathcal{A} \) we have [19]
\[
|\mathcal{A}| \leq \binom{n}{\lfloor n/2 \rfloor}
\]
and more generally a necessary condition for existence of a Sperner family with \( a_k \) subsets of size \( k \), for \( k \in [n] \), is that [20]
\[
\sum_{k=0}^{k=n} \frac{a_k}{\binom{k}{n}} \leq 1.
\]
Since the number of users \( m = |\mathcal{A}| \), (7) implies an upper bound on \( m \), i.e.,
\[
m \leq \binom{n}{\lfloor n/2 \rfloor}
\]

Next, we use Shamir’s secret sharing scheme to construct a DSSP when the access structure \( \mathcal{A} \) is a Sperner family. Let \( t_j = |A_j| \). In this construction, a \((t_j, t_j)\) Shamir’s secret sharing scheme is used independently for each user \( j \), both in the encoding of \( s_j \) by the dealer and decoding it by user \( j \). Such DSSP is denoted by S-DSSP\{\(\mathcal{A}, n\)\}. In other words, the condition in Lemma 1 is a sufficient condition for existence of a DSSP. More specifically, S-DSSP\{\(\mathcal{A}, n\)\} is specified as follows:

i) \( \mathcal{A} \) is a Sperner family consisting of subsets of \([n]\).
ii) \( \mathcal{E}(s) = (E_1(s_1), E_2(s_2), \ldots, E_m(s_m)) \), where \( E_j \) is a \((t_j, t_j)\) Shamir’s secret encoder and \( t_j = |A_j| \).
iii) \( \forall j \in [m]: Z[a_{j_1}, \tau_{j_1} + 1] = 1 \) for \( i \in [t_j] \), where \( \tau_j = t_1 + \ldots + t_j \) and \( \tau_0 = 0 \); and \( A_j = \{a_{1,j}, a_{2,j}, \ldots, a_{t_j,j}\} \).
iv) All other entries of \( Z \) are zero.

Lemma 2: S-DSSP\{\(\mathcal{A}, n\)\} is a DSSP satisfying all properties in Definition 1.

Proof: S-DSSP\{\(\mathcal{A}, n\)\} assigns a \( t_j \)-subset of \([n]\) to user \( j \). It encodes \( s_j \) into \( t_j \) secret shares using Shamir’s scheme with the threshold \( t_j \) and random seeds generated independently from other users. It then stores one share on each node in \( A_j \) as specified by \( Z \). Clearly, each user can reconstruct its secret by invoking Shamir’s secret decoder. Also, Shamir’s scheme guarantees (2) since \( \mathcal{A} \) is a Sperner family and consequently, no user other than user \( j \) has access to all of its \( t_j \) shares. Therefore, S-DSSP\{\(\mathcal{A}, n\)\} is a DSSP satisfying all properties in Definition 1.

We can pick a Sperner family \( \mathcal{A} \) with the maximum size \( |\mathcal{A}| = \binom{n}{\lfloor n/2 \rfloor} \) and then construct a S-DSSP\{\(\mathcal{A}, n\)\}. This satisfies all properties of a DSSP by Lemma 1 and serves the maximum possible number of users given a certain number of storage nodes \( n \).

III. Communication-Efficient DSSP

In this section, we derive a lower bound on the communication complexity of DSSPs, i.e., the amount of data that needs to be downloaded by users in order to reconstruct the secrets.

For given parameters \( m \) and \( n \) as the number of users and storage nodes, respectively, a DSSP with minimum communication complexity \( C \) defined in (14) is called a communication-efficient DSSP. It is assumed that
\[
m \leq \binom{n}{\lfloor n/2 \rfloor},
\]
as in (9). Otherwise, by Lemma 1 and (7) a DSSP does not exist.

A certain class of DSSPs, called tight DSSPs, defined as follows, is useful to derive lower bounds on the communication complexity and to construct communication-efficient DSSPs. This will be shown in Lemma 3.

Definition 2: We say a DSSP is a tight DSSP (T-DSSP) if every user downloads exactly one \( E_l \)-symbol from each node in its access set.

Note that, for example, every S-DSSP, defined in Section II, is a T-DSSP.

Lemma 3: For any DSSP with communication complexity \( C \), there exists a T-DSSP with the same number of users and storage nodes, and communication complexity \( \tilde{C} \) such that
\[
\tilde{C} \leq C.
\]

Proof: For each user \( l \), let \( A_l \subset A_l \) denote the set with the minimum size such that user \( l \) can reconstruct its secret \( s_l \) by downloading data from \( A_l \). Note that user \( l \) has to download at least one symbol from each node in \( A_l \). Therefore,
\[
\sum_{j=1}^{m} |A_j| \leq c
\]
The security condition implies that
\[ \forall j, l \in [m], j \neq l : \tilde{A}_j \subseteq A_j, \quad (11) \]
otherwise user \( j \) would be able to reconstruct \( s_l \). Let
\[ \mathcal{A} \overset{\text{def}}{=} \{ A_j : \forall j \in [m] \} \]
which is a Sperner family.

We then construct a S-DSSP associated with the access structure \( \mathcal{A} \) which has communication complexity \( C = \sum_{j=1}^{m} |A_j| \).
This follows from the fact that in a S-DSSP, each user downloads exactly one data symbol from the nodes in its access set. This together with (10) and recalling that a S-DSSP is also a T-DSSP complete the proof.

Note that the communication complexity of a T-DSSP depends only on its associated access structure. Let \( a_k \) denote the number of subsets of size \( k \) in the access structure of the T-DSSP. Then its communication complexity is given by
\[ C = \sum_{k=1}^{n} k a_k. \quad (12) \]

Therefore, by Lemma 3 one can consider minimizing \( \sum_{k=1}^{n} k a_k \) to find a communication-efficient DSSP provided that a Sperner family with such \( a_k \)'s exists. To this end, we consider the following discrete optimization problem:

\[ \begin{align*}
\min & \quad \sum_{k=1}^{\lfloor n/2 \rfloor} k a_k \\
\text{s.t.} & \quad \forall k \in \{1, \ldots, \lfloor n/2 \rfloor \} : a_k \in \mathbb{N} \cup \{0\} \\
& \quad \sum_{k=1}^{\lfloor n/2 \rfloor} a_k = m \\
& \quad \sum_{k=1}^{\lfloor n/2 \rfloor} \frac{a_k}{\binom{n}{k}} \leq 1. \quad (16)
\end{align*} \]

Constraint (14) is set because \( a_k \)'s must be non-negative. Constraint (15) is set because the sum of \( a_k \)'s is equal to the total number of users \( m \). Also, by (8), (16) is a necessary condition for existence of a Sperner family with \( a_k \) subsets of size \( k \). If such Sperner family exists for the solution of this optimization problem, then we will have a communication-efficient DSSP. Otherwise, the minimum objective function is a lower bound for the minimum communication complexity. Note that due to the reciprocity of the binomial coefficients, we have \( a_k = 0 \) for all \( \left\lfloor \frac{n}{k} \right\rfloor < k \) in the solution of this optimization problem.

The idea is to first solve a continuous version of this optimization problem, stated below, and then extract the solution for the discrete version from the solution of the continuous version. Consider the following problem:

\[ \begin{align*}
\min & \quad \sum_{k=1}^{\lfloor n/2 \rfloor} k \alpha_k \\
\text{s.t.} & \quad \forall k \in \{1, \ldots, \lfloor n/2 \rfloor \} : \alpha_k \geq 0, \\
& \quad \sum_{k=1}^{\lfloor n/2 \rfloor} \alpha_k = m, \\
& \quad \sum_{k=1}^{\lfloor n/2 \rfloor} \frac{\alpha_k}{\binom{n}{k}} \leq 1. \quad (20)
\end{align*} \]

where \( \alpha_k \in \mathbb{R} \). This optimization problem can be solved by satisfying Karush–Kuhn–Tucker (KKT) condition [21]. Let \( \psi^* \) denote the minimum of the objective function in the above continuous optimization problem. Suppose that \( \psi^* \) is achieved by the choice of \( \alpha_k^* \), for \( k = 1, 2, \ldots, \lfloor n/2 \rfloor \). It is shown in the Appendix that at most two of \( \alpha_k^* \)'s are non-zero. Furthermore, it is shown that if two non-zero \( \alpha_k^* \)'s exist, then their indices are consecutive. In particular, the solution is described as follows. Let \( i \) denote the largest integer such that
\[ \binom{n}{i} \leq m. \quad (21) \]

Then
\[ \alpha_i^* = \frac{n}{i+1} - \frac{m}{i}, \quad (22) \]
\[ \alpha_{i+1}^* = \frac{n}{i+1} - \frac{m}{i+1}, \quad (23) \]
and \( \alpha_k^* = 0 \), for \( k \neq i, i+1 \). Also, by (59) the minimum possible objective function is
\[ \psi^* = ia_i^* + (i+1)a_{i+1}^*. \quad (24) \]

Let \( C^* \) denote the minimum of the objective function in the discrete optimization problem. It is clear that
\[ [\psi^*] \leq C^*. \quad (25) \]

The following lemma shows that \( C^* = [\psi^*] \).

**Lemma 4:** We have
\[ C^* = [\psi^*]. \]

Furthermore, this minimum objective function is achieved by choosing \( a_i = [\alpha_i^*], a_{i+1} = [\alpha_{i+1}^*], \) and \( a_k = 0 \), for \( k \neq i, i+1 \).

**Proof:** Let \( a_i = [\alpha_i^*], a_{i+1} = [\alpha_{i+1}^*], \) and \( a_k = 0 \), for \( k \neq i, i+1 \). Let \( \epsilon = \alpha_i^* - [\alpha_i^*] \). Note that \( \alpha_i^* + \alpha_{i+1}^* = m \) and \( m \) is an integer. Therefore, \( \alpha_{i+1}^* = [\alpha_{i+1}^*] - \epsilon \). In other words, the fraction part of \( \alpha_{i+1}^* \) is 1 – \( \epsilon \). Then one can write
\[ a_i = \alpha_i^* - \epsilon, \quad a_{i+1} = \alpha_{i+1}^* + \epsilon \quad (26) \]
where \( 0 \leq \epsilon < 1 \). First, we show feasibility of this solution by checking the constraints of the optimization problem. It is easy to see that (14) and (15) are satisfied. Also,
\[ \frac{a_i}{\binom{n}{i}} + \frac{a_{i+1}}{\binom{n}{i+1}} = \frac{\alpha_i^*}{\binom{n}{i}} + \frac{\alpha_{i+1}^*}{\binom{n}{i+1}} + \epsilon \left( \frac{1}{\binom{n}{i}} - \frac{1}{\binom{n}{i+1}} \right) \leq 1, \]

where the equality holds by (26) and the inequality holds by (20) and noting that \( \frac{1}{c_{i+1}} - \frac{1}{c_i} \) is negative. Therefore, (16) is also satisfied which shows that the solution is feasible. At last, it is shown that this solution achieves equality in (25). For this solution, we have
\[
C^* = i a_i + (i + 1) a_{i+1} = i \alpha_i^* + (i + 1) \alpha_{i+1}^* + \epsilon = \psi^* + \epsilon.
\]
and therefore, \( C^* = [\psi^*] \).

Following is the summary of this section’s results.

**Theorem 5:** For a given number of users \( m \) and storage nodes \( n \), any T-DSSP with the following access structure \( \mathcal{A} \) is a communication-efficient DSSP: \( \mathcal{A} \) is a Sperner family that contains \( [\alpha_i^*] \) of \( \alpha_i^* \)'s of \( \alpha_i^* \) subsets of \( [n] \) and \( [\alpha_i^*] \) of \( \alpha_i^* \) subsets of \( [n] \), where \( i \) is the maximum integer that satisfies (21), and \( \alpha_i^* \) and \( \alpha_i^* \) are as calculated in (22) and (23).

**Proof:** The theorem follows by (23) and the solution to the discrete optimization problem with properties shown in Lemma 3.

**Corollary 6:** If a Sperner family \( \mathcal{A} \) as specified in Theorem 5 exists, then S-DSSP \( \{ \mathcal{A}, n \} \) is a communication-efficient DSSP. Otherwise, \( C^* \), given in (27), is a lower bound for the minimum possible communication complexity.

**IV. COMMUNICATION-EFFICIENT DSSP WITH OPTIMAL STORAGE OVERHEAD**

In this section, we show how to construct DSSPs with optimal storage overhead. To this end, we propose methods to construct random seeds for each user by utilizing the secret shares of other users while guaranteeing the security condition. This is done in such a way that no external randomness is required and hence, the total size of data to be stored on storage nodes is equal to the total size of secrets. Consequently, this provides the optimal storage overhead of one. Furthermore, this method can be applied to the communication-efficient DSSP constructed in Section III thereby providing DSSPs that are optimal in terms of both communication complexity and storage overhead.

Let \( m = \binom{n}{k} \) for some \( k \), where \( m \) and \( n \) denote the number of users and storage nodes, respectively. Consider a system with the access structure \( \mathcal{A} \) consisting of all \( k \)-subsets of \( [n] \). First, consider a straightforward protocol by independently applying Shamir’s secret sharing method to each secret. Let \( p_{ij} \) be chosen independently and uniformly at random from \( \mathbb{F}_q \); for \( j \in [m] \) and \( i \in [k-1] \). Then, for \( j \in [m] \), the polynomial \( P_j(X) \) is defined as follows:
\[
P_j(x) = s_j + \sum_{l=1}^{k-1} p_{jl} x^l
\]
The evaluations \( P_j(\gamma_1), \ldots, P_j(\gamma_k) \), where \( \gamma_i \)'s are fixed, are to be stored on nodes in \( A_j \), the access set of user \( j \). In this protocol the storage overhead is \( k \) which is far from the optimal value of one, as stated in Section III.

In order to reduce storage overhead to the optimal value 1 we need to store only \( m \) symbols in the storage nodes instead of \( km \) symbols as in the straightforward approach. Therefore, the idea is to ensure that the evaluation of \( P_j \)'s over the evaluation points \( \gamma_i \)'s have significant overlaps with each other. In particular, we consider the following system of linear equations:
\[
\begin{align*}
P_1(\gamma_1) &= y_1, & P_2(\gamma_1) &= y_2, & \ldots & P_m(\gamma_1) &= y_m \\
P_1(\gamma_2) &= y_1, & P_2(\gamma_2) &= y_3, & \ldots & P_m(\gamma_2) &= y_1 \\
&\vdots & \vdots & \ddots & \vdots \\
P_1(\gamma_k) &= y_k, & P_2(\gamma_k) &= y_{k+1}, & \ldots & P_m(\gamma_k) &= y_{k-1}
\end{align*}
\]
(29)

In this case the coefficients of \( P_j(x) \)'s are no longer arbitrarily selected but are determined according to this system of linear equations. In other words, no external randomness is required. Note that there are \((k-1)m\) unknown variables \( p_{ij} \)'s with \( m \) unknown variables \( y_1, y_2, \ldots, y_m \). In total there are \( km \) variables and \( km \) linear equations in (29). Next, we show that these equations are linearly independent, under certain conditions, thereby establishing that the system has a solution, which is unique for \( p_{ij} \)'s and \( y_i \)'s. We further show how to store the resulting \( y_i \)'s in the storage nodes according to the certain access structure \( \mathcal{A} \).

The system of linear equations in (29) can be rewritten as:
\[
\mathbf{A} \mathbf{b} + \mathbf{s}' = 0
\]
(30)
where
\[
\mathbf{b} = (p_{11}, \ldots, p_{1k}, p_{21}, \ldots, p_{2k}, \ldots, p_{m1}, \ldots, p_{mk}, y_1, \ldots, y_m)^t
\]
\[
\mathbf{s}' = (s_1, \ldots, s_1, s_2, \ldots, s_2, \ldots, s_m, \ldots, s_m)^t
\]
where each \( s_j \) is repeated \( k \) times in \( s' \), and \( \mathbf{A}_{(km)\times(km)} \) is as follows:
\[
\begin{bmatrix}
\gamma_1 & \cdots & \gamma_1^{k-1} \\
\vdots & \ddots & \vdots \\
\gamma_k & \cdots & \gamma_k^{k-1} \\
\gamma_1 & \cdots & \gamma_1^{k-1} \\
\vdots & \ddots & \vdots \\
\gamma_k & \cdots & \gamma_k^{k-1}
\end{bmatrix}
\begin{bmatrix}
-1 \\
\vdots \\
-1 \\
\vdots \\
-1
\end{bmatrix}
\]
which consists of \( m \) copies of the following \( k \times (k-1) \) matrix:
\[
\mathbf{B} \overset{\text{def}}{=} \begin{bmatrix}
\gamma_1 & \gamma_2 & \cdots & \gamma_1^{k-1} \\
\gamma_2 & \gamma_2 & \cdots & \gamma_2^{k-1} \\
\vdots & \vdots & \ddots & \vdots \\
\gamma_k & \gamma_k & \cdots & \gamma_k^{k-1}
\end{bmatrix}
\]
together with \( m \) copies of \(-I_{k\times k}\), each shifted to right one column consecutively. We need to show that \( \mathbf{A} \) is non-singular in order to show that the system of linear equations (29) has a solution.

For simplicity and ease of calculation, let \( \gamma_i = \gamma^i \), for \( i = 1, 2, \ldots, k \), where \( \gamma \) is a primitive element of \( \mathbb{F}_q \).

**Lemma 7:** If \((q-1) \not| im \) for \( i \in [k] \), then the matrix \( \mathbf{A} \) is non-singular.

**Proof:** Let \( r_{ij} \) denote the row in \( \mathbf{A} \) indexed by \((j-1)k+i\), for \( j \in [m] \) and \( i \in [k] \). We show that \( r_{ij} \)'s are linearly independent. Suppose that a linear combination of \( r_{ij} \)'s equal to zero,
i.e.,
\[ \lambda_1 r_1 + \cdots + \lambda_k r_k + \cdots + \lambda_{1m} r_{1m} + \cdots + \lambda_{km} r_{km} = 0 \]
Hence,
\[ B' \lambda_i = 0, \]
where \( \lambda_i = (\lambda_{i1}, \lambda_{i2}, \ldots, \lambda_{ik})^t \) for all \( i \) and
\[
B' = \begin{bmatrix}
\gamma & \gamma^2 & \cdots & \gamma^k \\
\gamma^2 & (\gamma^2)^2 & \cdots & (\gamma^2)^k \\
\vdots & \vdots & & \vdots \\
\gamma^{k-1} & (\gamma^{k-1})^2 & \cdots & (\gamma^{k-1})^k
\end{bmatrix}
\]
Furthermore,
\[
\begin{aligned}
\lambda_{11} + \lambda_{2m} + \lambda_{3(m-1)} + \cdots + \lambda_{k(m-k+1)} &= 0 \\
\lambda_{12} + \lambda_{21} + \lambda_{3m} + \cdots + \lambda_{k(m-k+2)} &= 0 \\
\vdots \\
\lambda_{1(m-1)} + \lambda_{2(m-2)} + \lambda_{3(m-3)} + \cdots + \lambda_{k(m-k)} &= 0 \\
\lambda_{1m} + \lambda_{2(m-1)} + \lambda_{3(m-2)} + \cdots + \lambda_{k(m-k+1)} &= 0
\end{aligned}
\]
Since \( B' \) is a Vandermonde matrix, it is a full row rank matrix. Consequently, its kernel space is one dimensional. This together with (31) result in \( \lambda_j = \eta_j v \), where \( \eta_j \) is a scalar coefficient and \( v \) is a non-zero vector in the kernel of \( B' \). Let \( v = (v_1, v_2, \ldots, v_k)^t \). Then, one can write \( \lambda_{ij} = \eta_j v_i \) for all \( i \) and \( j \). Substituting this in (32) results in:
\[ V \eta = 0, \]
where \( \eta = (\eta_1, \eta_2, \ldots, \eta_m)^t \) and
\[
V \overset{\text{def}}{=} \begin{bmatrix}
v_1 & 0 & 0 & \cdots & v_k & v_{k-1} & \cdots & v_2 \\
v_2 & v_1 & 0 & \cdots & 0 & v_k & \cdots & v_3 \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & v_k & v_{k-1} & \cdots & 0
\end{bmatrix}_{m \times m}
\]
If \( V \) is non-singular, all \( \eta_j \)'s are zero which implies that all \( \lambda_{ij} \)'s are also zero. Also, all \( v_i \)'s can not be zero because the kernel space of \( B' \) is one dimensional. Therefore, \( A \) is non-singular if and only if \( V \) is non-singular. Note that \( V \) is a circulant matrix and is non-singular if and only if \( \gcd(x^m - 1, V(x)) = 1 \) [22], where \( V(X) \) is the associated polynomial of the circulant matrix \( V \):
\[ V(x) = v_1 + v_2 x + v_3 x^2 + \cdots + v_k x^{k-1}. \]
Note that \( B' v = 0 \). Therefore,
\[ \forall i : 1 \leq i \leq k - 1 \quad \gamma^i v(\gamma^i) = 0. \]
Equivalently, all \( \gamma^i \)'s are roots of \( x V(x) \). Note that the degree of \( V(x) \) is at most \( (k - 1) \) and hence, it has at most \( (k - 1) \) roots. Therefore, we can write:
\[ V(x) = c_0 (x - \gamma)(x - \gamma^2) \cdots (x - \gamma^{k-1}) \]
for some constant \( c_0 \). Since \( \gamma \) is a primitive element of \( \mathbb{F}_q \) and \( (q - 1) / \text{im} \) for all \( i \in [k] \), then \( \gamma^{im} \neq 1 \) for all \( 1 \leq i \leq k \). In other words, neither of \( V(X) \)'s roots is an \( m \)-th root of unity. Hence, \( x^m - 1 \) and \( V(x) \) have no roots in common and therefore, \( \gcd(x^m - 1, V(x)) = 1 \). Consequently, \( V \) is non-singular which implies \( A \) is non-singular, too.

**Corollary 8:** If the condition in Lemma 7 is satisfied, then (29) defines a one-to-one mapping between \( (s_1, s_2, \ldots, s_m) \) and \( (y_1, y_2, \ldots, y_m) \).

**Proof:** Lemma 7 shows that given \( s_i \)'s, there is a unique solution for \( y_i \)'s. Furthermore, for given \( y_j \)'s, (29) defines \( m \) interpolation equations of polynomials \( P_j(x) \) of degree at most \( k - 1 \), for which there exists a unique solution.

Suppose that the condition in Lemma 7 is satisfied, e.g., when \( q > km \). Then we can rewrite (30) as \( b = -A^{-1}s' \). Note that \( b \) contains all data symbols \( y_1, y_2, \ldots, y_m \) as its last \( m \) entries. Also, \( s' = K s \) where
\[
K = \begin{bmatrix}
1 & 1 \\
\vdots & \vdots \\
1 & 1
\end{bmatrix}_{km \times m}
\]
Let \( A' \) be a submatrix of \( A \) consisting of last \( m \) rows of \( A \). Now we can write the encoding equation as follows:
\[ \mathcal{E}(s) = A's' = A'Ks = Es, \]
where \( E \overset{\text{def}}{=} A'K \) and is referred to as the encoding matrix. Note that by Corollary 8 there is a one-to-one mapping between \( s \) and \( y \) and hence, \( E \) is non-singular.

In order to have the optimal storage overhead every data symbol \( y_i \) must be stored exactly in one storage node. In other words, we need to specify a certain storing matrix \( Z \), defined in Definition 1, to serve this purpose while ensuring that each user has access to data symbols it needs to reconstruct its secret. Existence of such matrix guarantees both correctness condition and optimality of storage overhead. Furthermore, we need to show that (2) holds for the proposed protocol in order to establish the security of the constructed DSSP with optimal storage overhead.

The set of equations in (29) implies that users are indexed in such a way that each one overlaps in \( k - 1 \) data symbols, out of the total \( k \) symbols it needs to reconstruct its secret, with the next user. Thus, in order to construct \( Z \) with the access structure \( \mathcal{A}' \) consisting of all \( k \)-subsets of \([n]\), the \( k \)-subsets must be ordered, in a circular way, such that every two consecutive subsets have \( k - 1 \) elements in common. This is possible and is done by using the revolving door algorithm [23]. The output
of this algorithm can be written as a sequence \(i_1, i_2, \ldots, i_m\), where \(i_j \in [n]\), and every \(k\) consecutive \(i_j\)'s denote one of the \(k\)-subsets of \([n]\) which serves as a certain user’s access set. The dealer stores \(y_j\) in the storage node \(i_j\), for \(j \in [m]\). In other words, the storing matrix \(Z_{n \times m}\) is specified as follows:

\[
\text{for } j = 1, 2, \ldots, m, \begin{cases} 
  z_{i_j,j} = 1, \\
  z_{l,j} = 0, \quad \text{for } l \neq i_j 
\end{cases}
\]

(34)

The access set of user \(j\) is

\[
A_j = \{i_j, i_{j+1}, \ldots, i_{j+k-1}\},
\]

where the indices are are modulo \(m\), i.e., \(i_{m+l} = i_l\). This implies that user \(j\) can read \(y_j, y_{j+1}, \ldots, y_{j+k-1}\). It then interpolates the polynomial \(P_j\) such that

\[
P_j(\gamma) = y_j, P_j(\gamma^2) = y_{j+1}, \ldots, P_j(\gamma^k) = y_{j+k-1},
\]

for which there exists a unique solution. Then \(s_j = P_j(0)\) is reconstructed. This shows the correctness of the algorithm. Also, since each \(y_j\) is stored exactly once, \(m \bar{p}_g\)-symbols are stored at storage nodes in total. This shows the optimality of the storage overhead in this protocol.

In the following lemma, a sufficient condition for the security condition, as specified in (2), is determined.

**Lemma 9:** If a protocol satisfies the correctness condition, the data symbols at the output of its encoder are mutually independent, and its access structure is a Sperner family, then the protocol satisfies the security condition, specified in (2).

**Proof:** Since the protocol satisfies the correctness condition, for each user \(j\) there exists a function \(D_j\) such that \(D_j(y_j) = s_j\). Therefore,

\[
\forall j : \quad H(s_j | y_j) = 0.
\]

Consider another user \(l\), where \(l \neq j\). Let \(y_j^{(l)}\) denote the vector of all data symbols in \(y_j\) that user \(l\) can read and \(y_j^{(-l)}\) as the vector of data symbols in \(y_j\) that user \(l\) does not have access to. Since the access structure of the protocol is assumed to be a Sperner family, we have \(|y_j^{(l)}| < |y_l|\). In other words, user \(j\) and user \(l\) have less than \(k\) data symbols in common. Combining this with the security property of Shamir’s scheme implies that

\[
H(s_l | y_j^{(l)}) = H(s_l).
\]

(35)

Also, note that

\[
H(y_j^{(-l)} | s_l, y_j^{(l)}) \overset{(a)}{=} H(y_j^{(-l)} | s_l, y_j^{(l)}, y_l) \overset{(b)}{=} H(y_j^{(-l)} | s_l, y_l) \overset{(c)}{=} H(y_j^{(-l)})
\]

\[
H(y_j^{(-l)} | s_l, y_j^{(l)}) \overset{(d)}{=} H(y_j^{(-l)})
\]

where (a) is by noting that conditioning reduces the entropy, (b) holds because \(y_j^{(l)}\) is contained in \(y_j\), (c) holds because \(s_l\) is a deterministic function of \(y_l\), (d) holds because \(y_j^{(-l)}\) and \(y_l\) are mutually independent, (e) holds because conditioning reduces the entropy. Combining these yields

\[
H(y_j^{(-l)} | s_l, y_j^{(l)}) = H(y_j^{(-l)})
\]

(36)

Then we have

\[
H(s_l, y_j) = H(s_l, y_j^{(l)}, y_j^{(-l)}) = H(y_j^{(l)}) + H(s_l | y_j^{(l)}) + H(y_j^{(-l)} | s_l, y_j^{(l)}) \overset{(a)}{=} H(y_j^{(l)}) + H(s_l) + H(y_j^{(-l)}) \overset{(b)}{=} H(y_j) + H(s_l),
\]

where (a) holds by (35) and (36), and (b) holds since \(y_j^{(l)}\) and \(y_j^{(-l)}\) are mutually independent. Therefore, \(H(s_l | y_j) = H(s_l)\), for \(j \neq l\), which proves the security condition.

The following lemma is also needed in the proof of Theorem 1 to conclude the security condition of the proposed protocol.

**Lemma 10:** The data symbols \(y_j\)'s are uniformly distributed and mutually independent.

**Proof:** The proof is by Corollary 8 and noting that \(s_j\)'s are uniformly distributed and mutually independent.

The following theorem summarizes the main result of this section. It shows that the proposed protocol in this section satisfies all the conditions of a DSSP and has the optimal communication complexity and storage overhead.

**Theorem 11:** Let \(m = \binom{n}{k}\) for some \(k \leq \frac{n}{2}\), and \(q-1\) does not divide \(im\), for \(i \in [k]\). Then the bundle of

i) Access structure \(\mathcal{A}\) consisting of all \(k\)-subsets of \([n]\),
ii) Encoding function \(E\) as stated in (33),
iii) Storing matrix \(Z_{n \times m}\) specified in (34),
iv) \(m\) identical decoding functions \(D\), where \(D\) is \((k, k)\) Shamir’s secret decoder,

is a communication-efficient DSSP with optimal storage overhead.

**Proof:** By Lemma 7 the matrix \(A\) defined in (30) is non-singular and the encoding function in (33) is well-defined. The correctness is established by the particular choice of \(Z_{n \times m}\) specified in (34) using the revolving door algorithm. This also specifies the decoding function \(D\) applied to \(y_j\), for each user \(j\). Also, by Lemma 10 \(y_j\)'s are i.i.d. and then, Lemma 7 is invoked to establish the security condition, specified in (2). Therefore, the proposed protocol satisfies all the conditions of a DSSP. It is communication-efficient by Theorem 5. Also, since it stores exactly \(m \bar{p}_g\)-symbols in the storage nodes, it achieves the optimal storage overhead 1.

One interesting case of Theorem 11 is when we want to serve the maximum possible number of users \(m = \binom{n}{\frac{n}{2}}\) for a given \(n\), as stated in Section III. In this case, we have a communication-efficient DSSP with optimal storage overhead that also serves the maximum possible number of users.

**Remark:** The construction complexity of the proposed protocol is dominated by calculating the inverse matrix \(A^{-1}\). The complexity of a straightforward Gaussian elimination method for inverting the matrix \(A\) is \(O(k^3m^3)\). However, this needs to be done only once to construct the storing matrix \(Z\) and then it can be fixed for encoding purposes. A straightforward implementation of the encoding process \(E\), specified in (33), results in a run-time complexity \(O(m^2)\).

In the next section, we propose an alternative method for constructing DSSPs with nearly optimal storage overhead while the
conditions on \(m, n, q\), as stated in Theorem 11, are removed. This is shown to result in more efficient construction and encoding algorithms at the expense of negligible increment in communication complexity and storage overhead.

V. Low Complexity DSSP with Nearly Optimal Storage Overhead

In this section, we discuss how to modify the methods presented in Section IV in order to construct DSSPs with nearly optimal storage overhead for any given set of parameters. Furthermore, the modified protocol leads to significantly reduced complexity of the construction and encoding algorithm.

The main idea for the proposed protocol in Section II is setting up the system of linear equations in (29) and showing that, under certain conditions on parameters \(m, n, q\), it leads to a one-to-one mapping between the secrets \(y_j\)'s and data symbols \(y_j\)'s to be stored in storage nodes. Furthermore, it is shown that users can be indexed in such a way that the access sets for two consecutive users have the maximum possible intersection in order to actually deploy.

In this section, we somewhat relax the system of linear equations in (29) by introducing a few more variables and breaking the circular dependency between \(y_j\)'s. Furthermore, we do not need to assume that \(m\) is a binomial coefficient of power \(n\). Instead, we assume that all users have access sets of the same size, denoted by \(k\), which is possible for any given number of users. We set \(k\) to be the minimum integer number with

\[
\binom{n}{k} \geq m.
\]

This results in a communication complexity as close as possible to the optimal value. A modified system of linear equations is then proposed as follows:

\[
\begin{align*}
P_1(\gamma_1) &= y_1, & P_2(\gamma_1) &= y_2, & \ldots & P_m(\gamma_1) &= y_m \\
\vdots & & \vdots & & \vdots \\
P_1(\gamma_2) &= y_2, & P_2(\gamma_2) &= y_3, & \ldots & P_m(\gamma_2) &= y_{m+1} \\
\vdots & & \vdots & & \vdots \\
P_1(\gamma_k) &= y_k, & P_2(\gamma_k) &= y_{k+1}, & \ldots & P_m(\gamma_k) &= y_{m+k-1}
\end{align*}
\]

where \(P_j(X)\) is defined similar to (28). Comparing to (29) we have introduced \(k - 1\) new variables \(y_{m+1}, y_{m+2}, \ldots, y_{m+k-1}\). Then we choose the coefficients \(p_{1i}\) of \(P_1\) independently and uniformly at random from \(\mathbb{F}_q\). In other words, we use \(k - 1\) symbols as external random seed in this procedure. The process to derive \(y_i\)’s is then as follows. Since \(P_1\) is specified, \(y_1, y_2, \ldots, y_m\) are derived by simply evaluating \(P_1\) at \(\gamma_1, \gamma_2, \ldots, \gamma_k\). Then \(P_2\) is uniquely interpolated using the interpolation points \((\gamma_i, y_{i+1})\), for \(i = 1, 2, \ldots, k - 1\), according to the second column of equations in (38), and knowing \(P_2(0) = s_2\). Once \(P_2\) is determined, \(y_{k+1} = P_2(\gamma_k)\) is derived. The same process repeats for \(P_3, \ldots, P_m\), consecutively, in an iterative way. This encoding process to produce \(y_i\)’s is elaborated more as follows. At the \(j\)-th step of the iteration \(P_{j1}, P_{j2}, \ldots, P_{j(k-1)}\) are derived using the following system of linear equations:

\[
\begin{align*}
\gamma_j + p_{j1}\gamma_1 + p_{j2}\gamma_1^2 + \cdots + p_{j(k-1)}\gamma_1^{k-1} &= y_j \\
\gamma_j + p_{j1}\gamma_2 + p_{j2}\gamma_2^2 + \cdots + p_{j(k-1)}\gamma_2^{k-1} &= y_{j+1} \\
\vdots & \\
\gamma_j + p_{j1}\gamma_{k-1} + p_{j2}\gamma_{k-1}^2 + \cdots + p_{j(k-1)}\gamma_{k-1}^{k-1} &= y_{j+k-2}
\end{align*}
\]

which can be written as

\[
\mathbf{D}\mathbf{p}_j = \mathbf{\hat{y}}_j
\]

where

\[
\mathbf{p}_j = (p_{j1}, p_{j2}, \ldots, p_{j(k-1)})^t,
\]

\[
\mathbf{\hat{y}}_j = (y_j - s_j, y_{j+1} - s_j, \ldots, y_{j+k-2} - s_j)^t,
\]

and

\[
\mathbf{D} \overset{\text{def}}{=} \begin{bmatrix}
\gamma_1 & \gamma_2^2 & \cdots & \gamma_1^{k-1} \\
\gamma_2 & \gamma_2^2 & \cdots & \gamma_2^{k-1} \\
\vdots & \vdots & \ddots & \vdots \\
\gamma_{k-1} & \gamma_{k-1}^2 & \cdots & \gamma_{k-1}^{k-1}
\end{bmatrix}.
\]

Since \(\mathbf{D}\) is a Vandermonde matrix, it is invertible and (40) has a unique solution, i.e.,

\[
\mathbf{p}_j = \mathbf{D}^{-1}\mathbf{\hat{y}}_j.
\]

To summarize, the encoding algorithm is as follows:

1. Choose i.i.d. \(p_{1i}\)'s, for \(i = 1, 2, \ldots, k-1\) uniformly from \(\mathbb{F}_q\).
2. \(\forall i, i \in [k]\), set \(y_i = P_1(\gamma_i)\).
3. For \(j = 2, \ldots, m\),

\[
y_{j+k-1} = y_j + \left[ \begin{array}{c}
\gamma_k & \gamma_k^2 & \cdots & \gamma_k^{k-1}
\end{array} \right] \mathbf{D}^{-1}\mathbf{\hat{y}}_j,
\]

where \(\mathbf{\hat{y}}_j\) and \(\mathbf{D}\) are defined in (41) and (42), respectively.

Storing process is similar to that of the DSSP with optimal storage overhead, presented in Section IV with small modification as discussed next. Let \(i_1, i_2, \ldots, i_{n-1}\), where \(i_j \in [n]\), denote the output of the revolving door algorithm [23] when applied to \(k\)-subsets of \([n]\). Then the dealer stores \(y_{i_j}\) in the storage node \(i_j\), for \(j \in [m+k-1]\), where the indices in \([m+k-1]\) are considered modulo \(\binom{n}{k}\). In other words, the storing matrix \(Z_{m \times (m+k-1)}\) is specified as follows:

\[
\begin{array}{c}
\begin{bmatrix}
z_{i_1,j} = 1, & z_{i_2,j} = 0, & \ldots & z_{i_{n-1},j} = 0
\end{bmatrix}
\end{array}
\]

Each data symbol \(y_j\) is stored once. Also, the access set of user \(j\) is

\[
A_j = \{i_j, i_{j+1}, \ldots, i_{j+k-1}\}
\]

which implies that user \(j\) can read \(y_j, y_{j+1}, \ldots, y_{j+k-1}\). It then interpolates the polynomial \(P_j\) such that

\[
P_j(\gamma_1) = y_j, P_j(\gamma_2) = y_{j+1}, \ldots, P_j(\gamma_k) = y_{j+k-1},
\]

for which there exists a unique solution. Then \(s_j = P_j(0)\) is reconstructed. This shows the correctness of the algorithm.

By proving that all \(y_j\)'s are mutually independent we conclude from lemma (41) that this protocol satisfies security condi-
Lemma 12: The data symbols \( y_j \)'s generated by the proposed encoding algorithm are mutually independent.

Proof: Consider all data symbols for this protocol, i.e.,
\[
Y = (y_1, y_2, \ldots, y_m+k-2, y_{m+k-1})
\]
Note that \( (p_1, \ldots, p_{k-1}) \) are i.i.d and are also independent from \( s_1, s_2, \ldots, s_m \). Also, \( (y_1, y_2, \ldots, y_{m+k-2}, y_{m+k-1}) \) is uniquely determined given \( (p_1, \ldots, p_{k-1}, s_1, \ldots, s_m) \) according to the system of linear equations. Therefore, there is a one-to-one mapping between these two vectors and we conclude that \( y_j \)'s are also i.i.d and uniformly distributed.

The proof that the protocol discussed in this section is a DSSP follows similar to the proof of Theorem 11 and by invoking Lemma 12. In this protocol we have
\[
SO = \frac{m+k-1}{m} = 1 + \frac{k-1}{m}
\]
Note that \( m = \Theta\left(\binom{n}{k}\right) \) which implies \( k = \Theta\left(\frac{\log m}{\log n}\right) \). Thus, the gap between the \( SO \) of this protocol and the optimal \( SO \) is at most \( O\left(\frac{\log m}{m \log n}\right) \). Note that we do not know that whether the lower bound of 1 for \( SO \) is achievable for any arbitrary \( m \) or not and therefore, the gap with the optimal value might be actually smaller than \( \frac{k-1}{m} \). Since the gap approaches zero as \( m \to \infty \), we refer to this protocol as a DSSP with nearly optimal storage overhead.

The complexity of encoding algorithm is \( O(mk^2) \) which can be written as \( O\left(\frac{m \log^2 m}{m \log n}\right) \). This improves upon the protocol in Section 11 with complexity \( O(m^2) \). Also, we do not have any constraint on the size of the field except that \( q > k \). This constraint is the only requirement we have to consider since we need to choose \( k \) non-zero and distinct elements from \( \mathbb{F}_q \) as evaluation points.

In this protocol, each user downloads \( k \) symbols to reconstruct its secret and hence, the communication complexity is \( mk \). Note that
\[
\binom{n}{k-1} < m \leq \binom{n}{k},
\]
by the particular choice of \( k \) in this section. Therefore, it can be observed that a trivial lower bound on the minimum communication complexity is \( m(k-1) \). Then, the ratio of the communication complexity of our proposed protocol in this section to the optimal one is at most \( \frac{k}{m} = 1 + \Theta\left(\frac{\log m}{\log n}\right) \).

VI. CONCLUSION AND FUTURE WORK

In this paper we considered a distributed secret sharing system consisting of a dealer, \( n \) storage nodes, and \( m \) users. The dealer aims at securely sharing a specific secret \( s_j \) with user \( j \) via storage nodes, in such a way that no user gets any information about other users’ secrets. Given a certain number of storage nodes we find the maximum number of users that can be served in such system. Also, lower bounds on minimum communication complexity and storage overhead are characterized in terms of \( n \) and \( m \). Then we propose distributed secret sharing protocols, under certain conditions on the system parameters, that attain these lower bounds thereby providing schemes that are optimal in terms of both the communication complexity and storage overhead. Finally, a low complexity DSSP with nearly optimal communication complexity and storage overhead is proposed for any given system parameters.

There are several directions for future work. In this paper, the problems of designing access structure, i.e., which nodes each user has access to, and the coding problem, i.e., how to encode and decode secrets, are considered jointly. An interesting direction for future work is to separate these two problems and consider designing efficient coding schemes given a certain access structure. Note that the result of Section IV for achieving the optimal storage overhead of 1 highly relies on the assumption that the vector of secrets is full entropy. Another interesting problem is to explore whether the same proposed construction techniques can be used in more general cases when the secrets are not necessarily uniformly distributed or a new methodology is needed to deal with such cases.

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In this section the solution to continuous optimization problem defined in (17)-(20) is determined by satisfying KKT conditions. The Lagrangian can be written as:

\[ J = \sum_{k=1}^{[n/2]} k\alpha_k - \lambda_1 \left( \sum_{k=1}^{[n/2]} \alpha_k - m \right) - \lambda_2 \left( 1 - \sum_{k=1}^{[n/2]} \frac{\alpha_k}{\binom{n}{k}} \right) - \sum_{k=1}^{\binom{n}{k}} \mu_k \alpha_k, \]

where \( \lambda_1, \lambda_2 \) and \( \mu_k \)’s are Lagrange multipliers. Also KKT conditions are:

\[
\forall k : \quad k - \lambda_1 + \frac{\lambda_2}{\binom{n}{k}} - \mu_k = 0 \tag{45} \\
\forall k : \quad \mu_k \geq 0 \tag{46} \\
\forall k : \quad \alpha_k^* \geq 0 \tag{47} \\
\forall k : \quad \lambda_2 \geq 0 \tag{48} \\
\forall k : \quad \mu_k \alpha_k^* = 0 \tag{49} \\
\lambda_2 \left( 1 - \sum_{k=1}^{[n/2]} \frac{\alpha_k^*}{\binom{n}{k}} \right) = 0 \tag{50} \\
\sum_{k=1}^{[n/2]} \alpha_k^* = m, \tag{51} \\
\sum_{k=1}^{\binom{n}{k}} \alpha_k^* \leq 1. \tag{52}
\]

Since both the objective function and inequality constraints are convex and equality condition is an affine function, KKT conditions are sufficient to ensure that the solution is the global minimum. The key point that makes it possible to derive the solution of (45)-(52) is convexity of the discrete function \( f(k) \) defined, assuming \( n \) is fixed. Define \( m_{(k_1,k_2)} \) as the slope of the line connecting \((k_1, f(k_1))\) and \((k_2, f(k_2))\). Also let \( d_k \) denote one step increment at point \((k, f(k))\). It is proved in the following lemma that \( d_k \) is strictly increasing in \( k \).

**Lemma 13:** \( d_k < d_{k+1} \) for all \( 0 \leq k \leq n - 2 \).

**Proof:**

\[ d_k = \frac{1}{\binom{n}{k+1}} - \frac{1}{\binom{n}{k}} = \frac{k!(n-k-1)!(2k-n+1)}{n!} = \frac{(2k-n+1)}{n\binom{n-1}{k}} \]

Case 1: \( 0 \leq k < \frac{n-1}{2} \). In this case we have

\[ 0 \leq (n - 1 - 2(k + 1)) < (n - 1 - 2k), \]

\[ 0 < \frac{1}{\binom{n-1}{k+1}} \leq \frac{1}{\binom{n-1}{k}}. \]

Multiplying inequalities yields

\[ 0 \leq \frac{(n - 1 - 2(k + 1))}{\binom{n-1}{k+1}} < \frac{(n - 1 - 2k)}{\binom{n-1}{k}} \]

which implies that

\[ d_k < d_{k+1} \leq 0. \]

Case 2: \( \frac{n-1}{2} \leq k \leq n - 2 \). In this case we have

\[ 0 \leq (2k - n + 1) < (2(k + 1) - n + 1) \]

\[ 0 < \frac{1}{\binom{n-1}{k}} < \frac{1}{\binom{n-1}{k+1}}. \]

Again, multiplying inequalities yields

\[ 0 \leq \frac{(2k - n + 1)}{\binom{n-1}{k+1}} < \frac{(2(k + 1) - n + 1)}{\binom{n-1}{k}} \]

which implies that

\[ 0 \leq d_k < d_{k+1}. \]

This completes the proof of lemma.

**Lemma 14:** For \( k_1, k_2, k_3 \in \mathbb{N} \), with \( 0 \leq k_1 < k_2 < k_3 \leq \left( \binom{n}{2} \right) \), we have:

\[ m_{(k_1,k_2)} < m_{(k_2,k_3)}. \]

**Proof:** For \( k, k' \in \mathbb{N} \) with \( k < k' \),

\[ m_{(k,k')} = \frac{1}{k' - k} \sum_{i=k}^{k'-1} d_i \]

Using this together with Lemma 13 we have

\[ m_{(k_1,k_2)} = \frac{1}{k_2 - k_1} \sum_{i=k_1}^{k_2-1} d_i < \frac{1}{k_2 - k_1} \sum_{i=k_1}^{k_2-1} d_{k_2} = d_{k_2} \]

and

\[ m_{(k_2,k_3)} = \frac{1}{k_3 - k_2} \sum_{i=k_2}^{k_3-1} d_i \geq \frac{1}{k_3 - k_2} \sum_{i=k_2}^{k_3-1} d_{k_2} = d_{k_2}, \]

which concludes the lemma.

**Theorem 15:** In the solution to the optimization problem, at most two of \( \alpha_k^* \)’s are non-zero. Furthermore, if two of them are non-zero, then their indices are consecutive.

**Proof:** Assume to the contrary there exist two non-consecutive integers \( k_1 \) and \( k_3 \) such that \( \alpha_{k_1}^*, \alpha_{k_3}^* \neq 0 \). Let \( k_1 < k_3 \), without loss of generality. One can find \( k_2 \in \mathbb{N} \) such that \( k_3 < k_2 < k_3 \). By (49) \( \mu_{k_1} \) and \( \mu_{k_3} \) must be zero. Also, by
we can write:

\[ k_1 - \lambda_1 + \lambda_2 \frac{1}{\binom{n}{k_1}} = 0 \]

\[ k_3 - \lambda_1 + \lambda_2 \frac{1}{\binom{n}{k_3}} = 0 \]

Solving this for \( \lambda_1 \) and \( \lambda_2 \) results in

\[ \lambda_1 = \frac{k_3}{\binom{n}{k_1}} - \frac{k_1}{\binom{n}{k_3}} \quad (53) \]

\[ \lambda_2 = \frac{k_3 - k_1}{1 - \frac{1}{\binom{n}{k_3}}} \quad (54) \]

By substituting \( \lambda_1 \) and \( \lambda_2 \) from (53) and (54), respectively, into (45) for \( k_2 \), \( \mu \) is derived as follows:

\[ \mu_{k_2} = k_2 - \lambda_1 + \lambda_2 \frac{1}{\binom{n}{k_2}} = \frac{k_2 - k_3}{\binom{n}{k_2}} + \frac{k_3 - k_1}{\binom{n}{k_1}} + \frac{k_1 - k_2}{\binom{n}{k_3}} \]

\[ = (k_3 - k_2)\left(\frac{1}{\binom{n}{k_2}} - \frac{1}{\binom{n}{k_1}}\right) + (k_2 - k_1)\left(\frac{1}{\binom{n}{k_1}} - \frac{1}{\binom{n}{k_3}}\right) \]

\[ = (k_3 - k_2)(k_2 - k_1)(m_{(k_1,k_2)} - m_{(k_2,k_3)}) < 0, \]

where the last inequality follows by the assumption on \( k_1, k_2, k_3 \) and Lemma 14. This contradicts (46) which completes the proof.

Theorem 15 implies that \( \alpha_i^* \alpha_{i+1}^* \geq 0 \) for some \( i \) and \( \alpha_k^* = 0 \) for \( k \neq i, i+1 \). Next, \( i, \alpha_i^* \), and \( \alpha_{i+1}^* \) are derived. Note that \( \lambda_2 > 0 \) by (54) and hence, (50) implies that the inequality condition in (52) turns into equality, i.e.,

\[ \frac{\alpha_i^*}{\binom{n}{i}} + \frac{\alpha_{i+1}^*}{\binom{n}{i+1}} = 1. \quad (55) \]

Furthermore, (51) implies that

\[ \alpha_i^* + \alpha_{i+1}^* = m. \quad (56) \]

Therefore, \( \alpha_i^* \) and \( \alpha_{i+1}^* \) can be derived by combining (55) and (56) as follows:

\[ \alpha_i^* = \binom{n}{i+1} - m \binom{n}{i+1} - \binom{n}{i} \binom{n}{i} \quad (57) \]

\[ \alpha_{i+1}^* = m - \binom{n}{i+1} + \binom{n}{i+1} \binom{n}{i+1} \quad (58) \]

Note that \( \alpha_i^* \) and \( \alpha_{i+1}^* \) must be non-negative by (47). Therefore, \( i \) is the largest integer such that

\[ i \leq \binom{n}{k}. \]

Also, the minimum of the objective function \( \psi \) is given by

\[ \psi^* = i \alpha_i^* + (i + 1) \alpha_{i+1}^*. \quad (59) \]