Kinematic model-independent reconstruction of Palatini $f(R)$ cosmology

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A kinematic treatment to trace out the form of $f(R)$ cosmology, within the Palatini formalism, is discussed by only postulating the universe homogeneity and isotropy. To figure this out we build model-independent approximations of the luminosity distance through rational expansions. These approximants extend the Taylor convergence radii computed for usual cosmographic series. We thus consider both Padé and the rational Chebyshev polynomials. They can be used to accurately describe the universe late-time expansion history, providing further information on the thermal properties of all effective cosmic fluids entering the energy momentum tensor of Palatini’s gravity. To perform our numerical analysis, we relate the Palatini’s Ricci scalar with the Hubble parameter $H$ and thus we write down a single differential equation in terms of the redshift $z$. Therefore, to bound $f(R)$, we bound of the most recent outcomes over the cosmographic parameters obtained from combined data surveys. In particular our clue is to select two scenarios, i.e. (2,2) Padé and (2,1) Chebyshev approximations, since they well approximate the luminosity distance at the lowest possible order. We find that best analytical matches to the numerical solutions lead to $f(R) = a + b R^2$ with free parameters given by the set $(a, b, n) = (-3.866, 0.814, 1.073)$ for (2,2) Padé approximation, whereas $f(R) = a + b R^m$ with $(a, b, m) = (-3.158, 0.672, 1.124)$ for (2,1) rational Chebyshev approximation. Finally, our results are compared with the ΛCDM predictions and with previous studies in the literature. Slight departures from General Relativity are also discussed.

I. INTRODUCTION

The cosmological standard model is highly supported by several experimental evidences, among all type Ia Supernovae (SN Ia) [1, 3], Baryon Acoustic Oscillations (BAO) [4] and the analysis of Cosmic Microwave Background (CMB) anisotropies [5, 6]. The standard model, in particular, shows up a preferable spatially flat universe [7]. This fact pushes cosmologists to explore alternative interpretations for the accelerated scenario. For instance, plausible landscapes include dynamical dark energy with evolving scalar fields [15–17] or evolving equation of state [18–20], attempts to unify dark matter and dark energy into a single fluid [21–22], and higher-dimensions braneworld models [23, 24].

An alternative view is to modify GR on cosmological scales. This turns out to explain the late-time acceleration without the need of dark energy. One of the most studied extensions of GR is represented by $f(R)$ gravity [25–27], generalizing the Einstein-Hilbert action with higher-order curvature terms in the Lagrangian. Many theoretical studies carried out so far have focused on the cosmological viability of such theories [28–31]. Also, from the observational point of view, the viability of these models has been tested by means of several cosmological data surveys [32–38].

Standard approaches toward the study of $f(R)$ paradigms postulate the $f(R)$ function $f(R)$, this approach is plagued by the fact that $f(R)$ is assumed a priori without relying on first principles. For these reasons, we here implement the inverse procedure, i.e. we start issues, such as the coincidence problem [13] and the fine-tuning problem [14].
from data and we back-reconstruct the \( f(R) \) action without any \textit{ad hoc} assumptions. Reconstructions of \( f(R) \) from the dynamics of the universe have been discussed in the context of the metric formalism \cite{43, 44}. In this work, we extend the method to the Palatini formalism by means of the \textit{cosmographic technique}. In particular, our approach is built upon rational polynomial approximations, which are able to extend the radius of convergence of the cosmographic series and minimize the relative uncertainties in estimating kinematic parameters \cite{45, 46, 47}.

We apply this method to find a model-independent expression of the Hubble expansion rate that can be then used to obtain the redshift as a function of the Ricci curvature. Thus, we infer the field equations that one obtains from the action through a numerical integration with initial conditions set by observational constraints on the solar system.

The paper is organized as follows. Sec. II is dedicated to a review of the \( f(R) \) gravity models in the Palatini formalism. In Sec. III we present the method of rational approximations in the context of cosmography. In Sec. IV we reconstruct the \( f(R) \) action and compare our results with the predictions of the \( \Lambda \)CDM model and with previous studies in the literature. Finally, in Sec. V we summarize results and conclude.

II. PALATINI \( f(R) \) COSMOLOGY

The action describing the \( f(R) \) gravity models can be written as \cite{40, 51}

\[
S = \frac{1}{2\kappa} \int d^4x \sqrt{-g} \, f(R) + S_m ,
\]

where \( \kappa = 8\pi G \) and \( G \) is the Newton’s constant; \( g \) is the metric determinant and \( S_m \) is the matter action. Differently from standard GR, for a non-linear Lagrangian density \( f(R) \) in the field equations that one obtains from the least action principle depend on the variational principle adopted. In the Palatini formalism, the action is varied with respect to both metric \( g_{\mu\nu} \) and affine connections \( \Gamma^\alpha_{\mu\nu} \), which are treated as independent variables. Varying Eq. (1) with respect to \( g_{\mu\nu} \) gives:

\[
F(R)R_{\mu\nu} - \frac{1}{2} f(R) g_{\mu\nu} = \kappa T_{\mu\nu} ,
\]

where \( F \equiv df/dR \) and \( T_{\mu\nu} \) is the energy-momentum tensor defined as:

\[
T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta S_m}{\delta g^{\mu\nu}} .
\]

On the other hand, varying Eq. (1) with respect to \( \Gamma^\alpha_{\mu\nu} \) provides \cite{52}:

\[
\nabla_\lambda (F(R) \sqrt{-g} g^{\mu\nu}) = 0 ,
\]

where \( \nabla_\lambda \) is the covariant derivative with respect to the connections. From Eq. (3), one can define the conformal metric \( h_{\mu\nu} \equiv Fg_{\mu\nu} \), so that the connections become the Christoffel symbols of the metric \( h_{\mu\nu} \). One thus obtains:

\[
\Gamma^\alpha_{\mu\nu} = \tilde{\Gamma}^\alpha_{\mu\nu} + \frac{1}{2F} \left[ 2\delta^\alpha_{[\nu} \partial_{\mu]} F - g_{\mu\nu} \delta^\alpha_\sigma \partial_\sigma F \right] ,
\]

where \( \tilde{\Gamma}^\alpha_{\mu\nu} \) are the Christoffel symbols of the metric \( g_{\mu\nu} \). Further, the Ricci tensor of the conformal metric can be written as the sum of the Ricci tensor of the metric \( g_{\mu\nu} \), \( \tilde{R}_{\mu\nu} \), plus additional terms:

\[
R_{\mu\nu} = \tilde{R}_{\mu\nu} + 3 \left( \nabla_{\mu} F (\nabla_{\nu} F) - \nabla_{\mu} \nabla_{\nu} F \right) - g_{\mu\nu} \Box F + \frac{3}{2} \left( \nabla_{\mu} F (\nabla_{\nu} F) - \nabla_{\mu} \nabla_{\nu} F \right) - 3 \Box F ,
\]

where \( \Box \equiv \nabla_\alpha \nabla^\alpha \). The Ricci scalar of the conformal metric becomes:

\[
R = \tilde{R} + \frac{3}{2} \left( \nabla_{\mu} F (\nabla_{\nu} F) - \nabla_{\mu} \nabla_{\nu} F \right) - 3 \Box F ,
\]

where \( \tilde{R} \equiv g^{\mu\nu} \tilde{R}_{\mu\nu} \).

To obtain the cosmological solutions, we consider the flat Friedmann-Lemaître-Robertson-Walker (FLRW) metric:

\[
d s^2 = -dt^2 + a(t)^2 \delta_{ij} dx^i dx^j ,
\]

where \( a(t) \) is the cosmic scale factor. The energy-momentum tensor for a perfect fluid of density \( \rho \) and pressure \( p \) is given by

\[
T_{\mu\nu} = \text{diag}(-\rho, p, p, p) ,
\]

and, thus, the trace of Eq. (2) results in

\[
RF(R) - 2f(R) = \kappa(3p - \rho) .
\]

From the combination of the \((0,0)\) and \((i,i)\) components of the field equations, one obtains the generalized Friedmann equation:

\[
H + \frac{1}{2} \left( \frac{\dot{F}}{F} \right)^2 = \frac{\kappa(\rho + 3p)}{F} + \frac{f}{F} .
\]

Taking the time derivative of Eq. (10) and combining it with the energy conservation leads to

\[
\dot{R} = -\frac{3H(RF - 2f)}{F_R - F} ,
\]

where \( F_R \equiv df/dR = d^2f/dR^2 \). Assuming that the universe is filled with pressureless matter and neglecting the contribution of radiation, we have \( p = 0 \) and \( \rho = \rho_m \). Since \( \dot{F} = F_R \dot{R} \), one can combine Eq. (12) with Eq. (11) to finally get:

\[
H^2 = \frac{1}{6F} \left[ \frac{2\kappa \rho_m + RF - f}{1 - \frac{3}{2} \frac{F(RF - 2f)}{F(RF - F)}} \right] ^2 ,
\]

2 We work in units of \( c = 1 \).
which represents the first modified Friedmann equation in the Palatini formalism.

The physical solution in Palatini’s gravity is simple to interpret. The gravitational part of the action may be mapped into a formally equivalent Brans-Dicke theory, as a consequence of replacing metric with connection in the action. This is due to the fact that the independent connection can be interpreted as an auxiliary field. Hence, after cumbersome algebra, one immediately gets that Palatini’s gravity corresponds to a Brans-Dicke model with potential in which one fixes the free parameter to \( \omega_0 = -\frac{3}{2} \). The most general case of \( f(R) \) gravity is not indeed the Palatini formalism, but the metric-affine version of \( f(R) \) gravity. Imposing further assumptions can lead to both metric or Palatini scenarios, respectively for \( \omega_0 = 0 \) and \( \omega = -\frac{4}{3} \). In such a scheme one can also notice that Palatini \( f(R) \) gravity is a metric theory following the classification made by [53]. This reinforces the idea that the independent connections becomes an auxiliary field. In such a way it is not possible to completely frame a metric-affine \( f(R) \) gravity with the Palatini formalism. For our purposes, we want to investigate the numerical reconstructions of Palatini \( f(R) \) with cosmography. To do so, since Eq. (16) with vanishing source on the right provides a solution \( \propto R^2 \), we may look for polynomial versions of the test functions that we employ in the next sections to extract the form of \( f(R) \).

### III. COSMOGRAPHY WITH RATIONAL APPROXIMATIONS

The study of the universe’s dynamics can be done through a purely kinematic approach by means of the cosmographic technique [54][58]. With the only assumption of homogeneity and isotropy on large scales as required by the cosmological principle, cosmography allows one to frame the universe’s expansion history at late times with no need of postulating any \textit{a priori} cosmological model. This method is built upon the Taylor expansion of the scale factor around present time \( t_0 \) [59, 60]:

\[
a(t) = 1 + \sum_{k=1}^{\infty} \frac{1}{k!} \frac{d^k a}{dt^k} |_{t=t_0} (t-t_0)^k .
\]

From this, one can define the cosmographic series:

\[
H = \frac{1}{a} \frac{da}{dt} , \quad q = -\frac{1}{aH^2} \frac{d^2 a}{dt^2} ,
\]

\[
j = \frac{1}{aH^2} \frac{d^3 a}{db^3} , \quad s = \frac{1}{aH^4} \frac{d^4 a}{dt^4} .
\]

Then, the luminosity distance as function of the redshift reads:

\[
d_L(z) = \frac{1}{H_0} \left[ z + \frac{1}{2} (1-q_0) z^2 - \frac{1}{6} (1-q_0-3q_0^2+j_0) z^3 + \frac{1}{24} (2-2q_0-15q_0^2-15q_0^3+5j_0+10q_0j_0+s_0) z^4 + \mathcal{O}(z^5) \right] .
\]

The Hubble rate in terms of the cosmographic parameters is obtained as:

\[
H(z) = \left[ \frac{d}{dz} \left( \frac{d_L(z)}{1+z} \right) \right]^{-1} .
\]

The method of Taylor approximations is unfortunately limited by the short convergence radius, \( z < 1 \). A possible way to perform cosmological analyses at higher redshift domains is to consider rational approximations. Two relevant examples in this respect are the Padé polynomials and ratios of Chebyshev polynomials, which offer clear convergence improvements and significant reductions of error propagation [15][17].

#### A. The Padé approximations

In the case of Padé approximations, one starts from the Taylor expansion of a generic function, \( f(z) = \sum_{i=0}^{\infty} c_i z^i \), with \( c_i = f^{(i)}(0)/i! \).

One thus defines the generic \((n,m)\) Padé approximant of \( f(z) \) by [61][62]:

\[
P_{n,m}(z) = \frac{\sum_{i=0}^{n} a_i z^i}{\sum_{j=0}^{m} b_j z^j} .
\]

The coefficients \( a_i \) and \( b_j \) are determined by requiring \( f(z) - P_{n,m}(z) = \mathcal{O}(z^{n+m+1}) \):

\[
\begin{align*}
\sum_{i=0}^{n} a_i &= \sum_{k=0}^{i} b_{i-k} c_k , \\
\sum_{j=1}^{m} b_j c_{n+k+j} &= -b_0 c_{n+k} , \quad k = 1, \ldots , m .
\end{align*}
\]

#### B. The rational Chebyshev approximations

The second method deals with first kind Chebyshev polynomials defined by [63]:

\[
T_n(z) = \cos(n\theta) ,
\]

where \( n \in \mathbb{N} \) and \( \theta = \arccos(z) \). They are orthogonal polynomials with respect to the weighting function
\[ w(z) = (1 - z^2)^{-1/2} \text{ in the domain } z \in [-1, 1] \] and bid to the recurrence relation

\[ T_{n+1}(z) = 2zT_n(z) - T_{n-1}(z) . \]  

Thus, one can write the Chebyshev series of a generic function \( f(z) \) as

\[ f(z) = \sum_{k=0}^{\infty} c_k T_k(z) , \tag{23} \]

where \( \sum' \) means that the first term in the sum must be divided by 2. The coefficients \( c_k \) are thus obtained as:

\[ c_k = \frac{2}{\pi} \int_{-1}^{1} g(z) T(z) w(z) \, dz , \tag{24} \]

where \( g(z) \) is the Taylor series of \( f(z) \) around \( z = 0 \). Therefore, applying a similar procedure as well as in the Padé framework, we build the \((n, m)\) rational Chebyshev approximant of \( f(z) \) by:

\[ R_{n,m}(z) = \frac{\sum_{i=0}^{n} a_i T_i(z)}{\sum_{j=0}^{m} b_j T_j(z)} . \tag{25} \]

In this case, the coefficients \( a_i \) and \( b_j \) are obtained through:

\[ \begin{cases} 
  a_i = \frac{1}{2} \sum_{j=0}^{m} b_j (c_{i+j} + c_{i-j}) = 0 , \quad i = 0, \ldots, n \\
  \sum_{j=0}^{m} b_j (c_{i+j} + c_{i-j}) = 0 , \quad i = n + 1, \ldots, n + m . \end{cases} \tag{26} \]

**IV. RECONSTRUCTING THE \( f(R) \) ACTION THROUGH COSMOGRAPHY**

In this section, we present the method to reconstruct the \( f(R) \) function from the Hubble expansion rate \( H(z) \) in the FLRW universe. To do so, we write the relation between the Ricci scalar and the Hubble parameter in the metric formalism:

\[ \ddot{R} = 6(H + 2H^2) . \tag{27} \]

For practical purposes, we convert the time derivative into derivative with respect to the redshift according to \( d/dt = -(1+z)H(z) d/dz \). Hence, Eq. \( \text{(27)} \) can be written as

\[ \ddot{R}(z) = -6(1+z)H(z)H'(z) + 12H(z)^2 , \tag{28} \]

where the symbol ‘prime’ denotes derivative with respect to \( z \). Plugging the above expression into Eq. \( \text{(7)} \), one finds \( \text{(28)} \):

\[ R(z) = 12H^2 - 6(1+z)H \left( \frac{HF'' + FH'}{F} \right) + 3(1+z)^2 \left[ \frac{2HF(HF'' + H'F') - H^2F'^2}{2F^2} \right] \tag{29} \]

Then, combining Eqs. \( \text{(13)} \) and \( \text{(29)} \) and expressing the matter density in terms of \( z \) as \( \kappa \rho_m = 3H_0^2\Omega_m(1 + z)^3 \), we obtain a differential equation for \( F(z) \):

\[ F'' - \frac{3}{2} \frac{F'''}{F} + \left( \frac{H'}{H} + \frac{2}{1 + z} \right) F' - \frac{2H'}{H(1+z)} F + 3\Omega_m(1 + z) \left( \frac{H_0}{H} \right)^2 = 0 . \tag{30} \]

Therefore, once \( H(z) \) is extracted from data, we can invert Eq. \( \text{(29)} \) to find \( z \) in terms of \( R \) and substitute in the solution of Eq. \( \text{(30)} \) to find \( F(R) \). Thus, integrating numerically \( F(R) \), we can definitively infer the \( f(R) \) function, without any further assumptions made \textit{ad hoc} on its form.

**A. Cosmographic outcomes**

In view of the treatment proposed in \( \text{[47]} \), we apply the above method to the \((2, 2)\) Padé and the \((2, 1)\) rational Chebyshev approximations. We report in the Appendix the expressions of \( d_L(z) \), from which one can derive the corresponding \( H(z) \) by means of Eq. \( \text{(18)} \). In what follows, we fix \( \Omega_m \) = 0.3 and assume \( H_0 = 1 \), so enabling the Hubble rate to be dimensionless. As far as the cosmographic parameters are concerned, we assume the best-fit results obtained in \( \text{[47]} \) through a comparison with the most recent cosmological data surveys.

In the case of the Padé approximant, we consider:

\[ \begin{cases} 
  q_0 = -0.285_{-0.046}^{+0.040} , \\
  j_0 = 0.545_{-0.084}^{+0.087} , \\
  s_0 = 0.118_{-0.025}^{+1.135} . \end{cases} \tag{31} \]

The initial conditions to solve Eq. \( \text{(30)} \) are found by requiring that the effective gravitational constant, \( G_{\text{eff}} = G/F \), is equivalent to the Newton constant at current time \( \text{[65, 66]} \). This implies:

\[ \frac{F}{F}|_{z=0} = 1 , \quad \frac{F'}{F}|_{z=0} = 0 . \tag{32} \]

\(^3\) One may, in principle, consider to relax this condition and allow for slight departures from \( G \). \( \text{[67]} \). This would ensure that \( F(R) \) exactly recovers the \( \Lambda \text{CDM} \) behaviour at large curvatures \( \text{[68]} \).
FIG. 1. Padé reconstruction of $F(R)$ in the redshift interval $[0, 2]$.

In Fig. 1, we show the behaviour of $F(R)$ obtained from the numerical solution of Eq. (30) using the central values of $31$. To get $f(R)$, we then integrate this solution with the initial condition provided by evaluating Eq. (13) at the present time:

$$f_0 = 6(\Omega_{m0} - 1) + R_0 ,$$

where $R_0$ is the present value of the Ricci scalar. We soon find that the analytical function matching the numerical solution is

$$f(R)_{\text{Padé}} = a + bR^n ,$$

where

$$(a, b, n) = (-3.866, 0.814, 1.073) .$$

We finally show in Fig. 2 the Padé reconstruction of the $f(R)$ function. If we also take into account the upper and lower 1σ bounds of $31$, we find the following intervals:

$$a \in [-4.050, -2.909],$$
$$b \in [0.661, 0.931],$$
$$n \in [1.025, 1.124].$$

On the other hand, for the rational Chebyshev approximation we use $37$:

$$\begin{cases}
q_0 = -0.278_{-0.021}^{+0.021} , \\
J_0 = 1.585_{-0.914}^{+0.497} , \\
S_0 = 1.041_{-1.784}^{+1.183} .
\end{cases}$$

Adopting an analogous procedure as in the above case, we reconstruct $F(R)$ (see Fig. 3) using the central values of $37$. In this case, the numerical solution for $f(R)$, got from the integration of $F(R)$, matches with

$$f(R)_{\text{Cheb}} = \alpha + \beta R^m ,$$

where

$$(\alpha, \beta, m) = (-3.158, 0.672, 1.124) .$$

Finally, the rational Chebyshev reconstruction of the $f(R)$ function is shown in Fig. 4. Considering the lower and upper 1σ bounds, we get the following intervals:

$$\begin{cases}
\alpha \in [-3.569, -3.158], \\
\beta \in [0.672, 0.750], \\
m \in [1.096, 1.124].
\end{cases}$$

B. Comparison with the concordance model

It is interesting to compare the results we obtained with the cosmological predictions of the standard $\Lambda$CDM
FIG. 4. Comparison between the rational Chebyshev reconstruction of $f(R)$ and its best analytical approximation (cf. Eq. (38)).

FIG. 5. Comparison of the $f(R)$ action of $\Lambda$CDM with the Padé and the rational Chebyshev reconstructions.

C. Comparisons with other cosmographic approaches

Previous approaches applied to cosmography have been commonly investigated in the framework of the metric formalism. Higher departures with our results may be found in the framework of the $f(R)$ reconstructions. In particular, the functional form of $f(R)$ has been showed to be much more complicated in the metric formalism with respect to Palatini [69, 70]. This may be due to the fact that here field equations are second order differential equations. In other words, passing from fourth to second order, one immediately gets that the whole equations should be much more similar to standard Einstein’s gravity at the infrared regime. For these reasons, we did not require a priori that our test-functions reduce directly to $f(R) \sim R$. On the other side, in [42] the authors proposed to adopt cosmography in order to quantitatively figure out which classes of viable models in the Palatini formalism are effectively able to describe the universe dynamics. They developed it by studying the evolution of $q$ in terms of the redshift. Further, they assumed the existence of a past matter-dominated epoch, ruling out the entire branch of negative values inside the parametrization $f \sim R - \beta R^{-n}$. Those results, however, are the direct consequence of postulating an $f(R)$ function that reduces to $\Lambda$CDM via the limit $f(R) \sim R + G(R)$, with $G(R)$ the cosmographic correction. Their prescription is here confirmed by our bounds got for both Chebyshev and Padé polynomials. Both the approaches, i.e. making use of simple cosmography with the recipe of a matter-dominated phase and ours, are intertwined and confirm that in the case of Palatini gravity the form of $f(R)$ is slightly departing from standard Einstein’s gravity.

model. We show in Fig. 5 the comparison between the $\Lambda$CDM action and $f(R)$ reconstructed through the rational approximations.

Moreover, from $H(z)$ one can calculate the effective equation of state parameter in terms of the redshift as

$$w_{\text{eff}}(z) = -1 + \frac{2}{3}(1 + z)\frac{H'(z)}{H(z)}. \quad (41)$$

Fig. 5 shows the comparison between the different models as result of using the best-fit values for the cosmographic parameters.
V. FINAL REMARKS

Among several possibilities, the promising paradigm of Palatini $f(R)$ gravity can be used to account for the universe’s acceleration at late times. We considered the Palatini approach and we wondered how to frame $f(R)$ without postulating the model at the very beginning. Indeed, as in the standard metric approach, the $f(R)$ form is unknown a priori. To find out possible clues toward its determination, one may postulate the form of both the pressure and density. Although appealing this strategy does not provide a model independent method to trace out $f(R)$ at late times. To heal this issue we took the energy momentum tensor free from any ad hoc assumptions. We only considered a homogeneous and isotropic universe in which one can expand the luminosity distance around $z \simeq 0$. We thus reconstructed the Palatini $f(R)$ in a model-independent fashion, by framing the cosmic expansion history through rational approximations of the luminosity distance. We chose rational series in order to characterize optimal convergence properties at higher-redshift domains, i.e. for $z \leq 2$. In particular, we specifically considered the $(2,2)$ Padé and the $(2,1)$ rational Chebyshev polynomials. We chose such orders since they have been proven to significantly reduce the error propagation on estimating the cosmographic series.

Adopting the most recent bounds on the cosmographic parameters, we built up accurate approximations of Hubble’s rate up to $z \simeq 2$. Using $H(z)$ and $\frac{df}{dR}$, we immediately found $F \equiv df/dR$ in function of $z$ only. Thus, we numerically inverted $R(z)$ and plugged it back in $F(z)$ to finally reach the form of $f(R)$. We portrayed the evolutions of $f(z), R(z)$ and $f(R)$ at different redshift domains, and alternatively in terms of the Ricci curvature. From our outcomes, we found that best analytical matches to the numerical solutions are $f(R)_{\text{Padé}} = a + b R^n$ where $(a, b, n) = (-3.866, 0.814, 1.073)$, and $f(R)_{\text{Cheb}} = a + \beta R^m$ where $(\alpha, \beta, m) = (-3.158, 0.672, 1.124)$. Our analyses showed small deviations from the concordance $\Lambda$CDM model based on GR. This has been better confirmed by checking the behaviour of the effective equation of state parameter, here evaluated for the above sets of parameters. We also underlined specific differences between our approach and previous ones applied to cosmography. Higher departures with our results may be found in the framework of the $f(R)$ reconstructions. In particular, the functional form of $f(R)$ has been got to be much more complicated in the metric formalism with respect to the Palatini one. We interpreted this by the fact that here Palatini’s field equations are second order differential equations. In other words, passing from fourth to second order, one immediately gets that the whole equations should be much more similar to standard Einstein’s gravity at the infrared regime. Our results also confirmed that the exponents in the term $\sim R^n$ should be positive. Future developments will be devoted to apply our rational approximants to higher redshift domains. In such a way we will realize whether $f(R)$ will be much more complicated to characterize other epochs of the universe evolution. We also will study the consequences of our cosmographic Palatini $f(R)$ model with the Cosmic Microwave Background, showing how the power spectrum would be influenced by our predictions.

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Appendix: Rational approximations of the luminosity distance

We here report the $(2,2)$ Padé and the $(2,1)$ rational Chebyshev approximations of $d_L(z)$, respectively:

\[
P_{2,2}(z) = \frac{c}{R_0} \left( 6z(10 + 9z - 6q_0^3z + s_0z - 2q_0^2(3 + 7z) - q_0(16 + 19z) + j_0(4 + (9 + 6q_0)z)) / (60 + 24z + 6s_0z - 2z^2 + 4j_0^2z^2 - 9q_0^2z^2 - 3s_0z^2 + 6q_0^2z(-9 + 4z) + q_0^2(-36 - 114z + 19z^2)) \right),
\]

\[
R_{2,1}(z) = \frac{c}{R_0} \left( -((3(16(-1 - j_0 + q_0 + 3q_0^2))7 - j_0 + q_0 + 3q_0^2) - (18 + 5j_0(1 + 2q_0) - 3q_0(6 + 5q_0(1 + q_0)) + s_0)(14 + 5j_0(1 + 2q_0) - q_0(14 + 15q_0(1 + q_0)) + s_0)) / (14 + 5j_0(1 + 2q_0) - q_0(14 + 15q_0(1 + q_0)) + s_0)) + 4(47 - j_0 + q_0 + 3q_0^2 - (12(-1 + q_0)(1 + j_0 - q_0(1 + 3q_0)))/(14 + 5j_0(1 + 2q_0) - q_0(14 + 15q_0(1 + q_0)) + s_0))z - (4(12(-1 - j_0 + q_0 + 3q_0^2)(7 - j_0 + q_0 + 3q_0^2) + 4(1 + j_0 - q_0(1 + 3q_0)) - (14 + 5j_0(1 + 2q_0) - q_0(14 + 15q_0(1 + q_0)) + s_0))z - q_0(1 + 3q_0)z)/ (14 + 5j_0(1 + 2q_0) - q_0(14 + 15q_0(1 + q_0)) + s_0)) / (192(1 + (4(1 + j_0 - q_0(1 + 3q_0)))) - q_0(1 + 3q_0)z)/(14 + 5j_0(1 + 2q_0) - q_0(14 + 15q_0(1 + q_0)) + s_0)) \right).
\]
