On the Green function of an orthotropic clamped plate in a half-plane

Norbert Ortner1 · Peter Wagner1

Received: 11 February 2021 / Accepted: 5 May 2021 / Published online: 19 May 2021 © The Author(s) 2021

Abstract
First, we calculate, in a heuristic manner, the Green function of an orthotropic plate in a half-plane which is clamped along the boundary. We then justify the solution and generalize our approach to operators of the form

\[ Q(\partial^r) - a^2 \partial^2_n (Q(\partial^r) - b^2 \partial^2_n) \] (where \( \partial^r = (\partial_1, \ldots, \partial_{n-1}) \) and \( a > 0, b > 0, a \neq b \)) with respect to Dirichlet boundary conditions at \( x_n = 0 \).

The Green function \( G_\xi \) is represented by a linear combination of fundamental solutions \( E_c \) of \( Q(\partial^r) - c^2 \partial^2_n \), \( c \in \{a, b\} \), that are shifted to the source point \( \xi \), to the mirror point \( -\xi \), and to the two additional points \( -\frac{a}{b} \xi \) and \( -\frac{b}{a} \xi \), respectively.

Keywords Green function · Orthotropic plate · Dirichlet problem · Cauchy–Dirichlet problem

Mathematics Subject Classification 35C05 · 35J40 · 35L35 · 35E05 · 35K35

1 Introduction and notation

We will derive in this study an explicit formula for the deflection \( G_\xi(x) \) of a semi-infinite orthotropic plate in \( H = \{x \in \mathbb{R}^2; x_2 > 0\} \), which is clamped along the boundary \( x_2 = 0 \) and loaded by a unit point force at \( \xi = (0, \xi_2) \) with \( \xi_2 > 0 \).

Hence, \( G_\xi \) is the Green function of the differential operator

\[ P(\partial) = \partial^2_1 + 2(1 - 2\varepsilon^2)\partial^2_1 \partial^2_2 + \partial^4_2 = (\partial^2_1 + 2\varepsilon \partial_1 \partial_2 + \partial^2_2)(\partial^2_1 - 2\varepsilon \partial_1 \partial_2 + \partial^2_2), \quad 0 < \varepsilon < 1, \quad (1.1) \]

with respect to Dirichlet boundary conditions at \( x_2 = 0 \), i.e., \( G_\xi \) is the unique solution of \( P(\partial)G_\xi = \delta(x - \xi) \) in \( H \) satisfying \( \lim_{x \to 0} \partial^k_2 G_\xi |_{x_2 = \varepsilon} = 0, k = 0, 1 \), and some growth condition for \( x_2 \to \infty \). The parameter \( \varepsilon \) characterizes the orthotropy of the plate.

---

1 University of Innsbruck, Technikerstr. 13, 6020 Innsbruck, Austria
If \( \epsilon = 0 \), then \( P(\partial) \) coincides with the bi-harmonic operator \( \Delta^2 \), which is the operator of isotropic plates. The Green function \( G_0^I \) of isotropic plates was derived in 1901 by J. H. Michell, see [10, p. 225, last line], [6, Equ. (633), p. 233]. In our notation, it is given by \( G_0^I(\xi) = |\xi|^2/(4\pi) \) and

\[
G_0^I(x) = \frac{1}{16\pi} |x - \xi|^2 \log \left( \frac{|x - \xi|^2}{|x + \xi|^2} \right) + \frac{x\xi}{4\pi}, \quad x_2 > 0, \ x \neq \xi. \tag{1.2}
\]

A fundamental solution (also called singularity function) \( E \) of \( P(\partial) \) in (1.1) is known since 1959 at least, see [16, Equ. (B9), p. 11], [17, p. 44], [13, Ex. 5.2.4, p. 351]. It reads

\[
E(x) = \frac{1}{32\pi \sqrt{1 - \epsilon^2}} \left[ (x_1^2 + \frac{2}{\epsilon} x_1 x_2 + x_2^2) \log(x_1^2 + 2\epsilon x_1 x_2 + x_2^2) +
\right.
\left. + (x_1^2 - \frac{2}{\epsilon} x_1 x_2 + x_2^2) \log(x_1^2 - 2\epsilon x_1 x_2 + x_2^2) +
\right.
\left. + \frac{x_1^2 - x_2^2}{16\pi\epsilon} \arctan \left( \frac{\epsilon}{\sqrt{1 - \epsilon^2 x_1^2 + x_2^2}} \right) \right], \quad 0 < \epsilon < 1, \tag{1.3}
\]

and the limit \( \epsilon \downarrow 0 \) yields, up to the bi-harmonic polynomial \( |x|^2/16\pi \), the well-known fundamental solution \( E^0 \) of the isotropic plate operator \( \Delta^2 \), i.e.,

\[
E^0(x) = \frac{|x|^2}{16\pi} \log |x|^2. \tag{1.4}
\]

Let us note, in parentheses, that formula (1.3) also furnishes, by linear transformations, a fundamental solution \( E \) of the fourth-order operator \( \nabla^T \nabla \cdot \nabla^T B \nabla \) in \( \mathbb{R}^2 \), if \( A, B \in \mathbb{R}^{2 \times 2} \) are linearly independent symmetric positive definite matrices. The result is

\[
E = \frac{1}{8\pi \alpha \sqrt{\det A}} \cdot x^T \left[ \tr(BA^\text{ad}) \cdot A^\text{ad} - 2(\det A) \cdot B^\text{ad} \right] x \cdot \log(x^T A^\text{ad} x)
\]

\[
+ \frac{1}{8\pi \alpha \sqrt{\det B}} \cdot x^T \left[ \tr(AB^\text{ad}) \cdot B^\text{ad} - 2(\det B) \cdot A^\text{ad} \right] x \cdot \log(x^T B^\text{ad} x)
\]

\[
+ \frac{\beta(x)}{2\pi \alpha} \cdot \arctan \left( \frac{\beta(x)}{\sqrt{\det A \cdot x^T B^\text{ad} x + \sqrt{\det B} \cdot x^T A^\text{ad} x}} \right), \tag{1.5}
\]

where \( x = \left( \begin{array}{c} x_1 \\ x_2 \end{array} \right) \), \( x^T = (x_1, x_2) \), \( A^\text{ad} = (\det A) \cdot A^{-1} \) and \( \alpha = \tr(BA^\text{ad}) - 4 \det A \) \( \det B \) is positive since \( B \) is not a multiple of \( A \) and

\[
\beta(x) = \sqrt{\tr(BA^\text{ad}) x^T A^\text{ad} x \cdot x^T B^\text{ad} x - \det B \cdot (x^T A^\text{ad} x)^2} - \det A \cdot (x^T B^\text{ad} x)^2.
\]

Let us observe, furthermore, that we can derive the Green function \( G_0^I \) in formula (1.2) from the fundamental solution \( E^0 \) in (1.4) by the ansatz

\[
G_0^I = \frac{|x - \xi|^2}{16\pi} \left[ \log(|x - \xi|^2) - \log(|x + \xi|^2) \right] + Z
\]

where \( Z \) fulfills the conditions \( \Delta^2 Z = 0 \) and \( Z|_{x_2 = 0} = 0 \), \( \partial_2 Z|_{x_2 = 0} = 0 \). This yields \( Z = x\xi/(4\pi) \).
In the case of the orthotropic plate operator, the deduction of $G_z$ from $E$ in (1.3) is more complicated. In Section 2, we shall derive the Green functions of the operators $(\partial^2_t + a^2 \partial^2_x)(\partial^2_t + b^2 \partial^2_x)$, $a > 0$, $b > 0$, $a \neq b$, and $\partial^4_t + 2(1 - 2\epsilon^2)\partial^2_t \partial^2_x + \partial^2_x$, $0 < \epsilon < 1$, in a heuristic manner by partial Fourier transform with respect to $x_1$. The correctness and the uniqueness of the Green functions under appropriate conditions will be investigated in Sections 3 and 4 more generally for operators of the form $P(\partial) = (Q(\partial^2_t) - a^2 \partial^2_x)(Q(\partial^2_t) - b^2 \partial^2_x)$. Therein, we shall also provide further examples.

Let us introduce some notation. $\mathbb{N}$ and $\mathbb{N}_0$ denote the sets of positive and of nonnegative integers, respectively. We consider as differentiation symbols

$$\partial_t := \frac{\partial}{\partial t}, \quad \partial_j := \frac{\partial}{\partial x_j}, \quad \partial := (\partial_1, \ldots, \partial_n), \quad \Delta_n := \partial^2_1 + \cdots + \partial^2_n$$

and we denote by $P(\partial)$ linear partial differential operators $\sum_{|\alpha| \leq m} c_{\alpha} \partial^\alpha$ with constant coefficients $c_{\alpha} \in \mathbb{C}$ for $\alpha \in \mathbb{N}_0^n, |\alpha| = \alpha_1 + \cdots + \alpha_n$. In some examples, we set $\partial = (\partial_t, \partial_1, \ldots, \partial_n)$ and $P(\partial)$ is then an operator in the $n + 1$ variables $t, x_1, \ldots, x_n$.

We employ the standard notation for the distribution spaces $\mathcal{D}', \mathcal{S}'$, the dual spaces of the spaces $\mathcal{D}, \mathcal{S}$ of “test functions” and of “rapidly decreasing functions,” respectively, see [13, 15]. In order to display the active variable in a distribution, say $x \in \mathbb{R}^n$, we use notation as $T(x)$ or $T \in \mathcal{D}'(\mathbb{R}^n)$. For the evaluation of a distribution $T$ on a test function $\phi$, we use angle brackets, i.e., $\langle \phi, T \rangle$.

The Heaviside function is denoted by $Y$, see [15, p. 36], and we set

$$\chi^z(t) = \frac{Y(t)z^z}{\Gamma(z + 1)} \in L^1_{\text{loc}}(\mathbb{R}^1) \quad \text{for} \quad z \in \mathbb{C} \text{ with } \text{Re} \, z > -1.$$ (1.6)

The function $z \mapsto \chi^z$ can be analytically continued in $\mathcal{S}'(\mathbb{R}^1)$ and thus yields an entire function

$$\chi : \mathbb{C} \longrightarrow \mathcal{S}'(\mathbb{R}^1) : z \longmapsto \chi^z,$$

see [4, Equs. (3.1), (3.2), pp. 314, 315], [8, (3.2.17), p. 73]. Note that $(\chi^z)' = \chi^{z-1}, z \in \mathbb{C}$, and $\chi^{-m} = \delta^{(m-1)}, m \in \mathbb{N}$. We write $\delta$ for the delta distribution with support in 0 i.e., $\langle \phi, \delta \rangle = \phi(0)$ for $\phi \in \mathcal{D}(\mathbb{R}^n)$.

The pull-back $h^*T = T \circ h \in \mathcal{D}'(\Omega)$ of a distribution $T$ in one variable $t$ with respect to a submersive $C^\infty$ function $h : \Omega \to \mathbb{R}$, $\Omega \subset \mathbb{R}^n$ open, is defined as in [5, Equ. (7.2.4/5), p. 82] or in [13, Def. 1.2.12, p. 19], i.e.,

$$\langle \phi, h^*T \rangle = \left\langle \frac{d}{dt} \left( \int_{\Omega} Y(t - h(x))\phi(x) \, dx \right), T \right\rangle, \quad \phi \in \mathcal{D}(\Omega).$$

We use the Fourier transform $\mathcal{F}$ in the form

$$(\mathcal{F}\phi)(\xi) := \int_{\mathbb{R}^n} e^{-i\xi x} \phi(x) \, dx, \quad \phi \in \mathcal{S}(\mathbb{R}^n),$$

this being extended to $\mathcal{S}'$ by continuity. (Herein and also elsewhere, the Euclidean inner product $(\xi, x) \mapsto \xi x$ is simply expressed by juxtaposition.) For the partial Fourier transforms of a distribution $T \in \mathcal{S}'(\mathbb{R}^{m+n})$ with respect to $x \in \mathbb{R}^m$ or $y \in \mathbb{R}^n$, respectively, we use the notation $\mathcal{F}_x T$ and $\mathcal{F}_y T$, respectively.
The restriction of a distribution \( T \in \mathcal{D}'(\mathbb{R}^n) \) to an open set \( H \subset \mathbb{R}^n \) will be denoted by \( T|_H \). Similarly, we write \( T|_{x_n=\varepsilon} \) for the restriction of \( T \) to the hyperplane \( \{x \in \mathbb{R}^n; x_n = \varepsilon\} \), \( \varepsilon > 0 \), if the distribution \( T \) continuously depends on \( x_n \), i.e., if it belongs to the subspace of \( \mathcal{D}'(H) \), \( H = \{x \in \mathbb{R}^n; x_n > 0\} \), constituted by the continuous mappings
\[
(0, \infty) \rightarrow \mathcal{D}'(\mathbb{R}^{n-1}) : x_n \mapsto T_{x_n}(x').
\]
According to [8, Thm. 4.4.8, p. 115], \( T \) continuously depends on \( x_n \) if it solves a linear partial differential equation of order \( m \) with constant coefficients and with a non-vanishing coefficient of \( \partial_n^m \).

\section{Green functions of \((\partial_x^2 + a^2 \partial_x^2)(\partial_x^2 + b^2 \partial_x^2)\)
and of \(\partial_1^2 + 2(1-2E^2)\partial_1^2 \partial_2 + \partial_2^2\).}

We shall first determine the Green function \( g_{\xi}(x) = g_{\xi}^a(x), \xi > 0, x > 0 \), of the ordinary differential operator
\[
p\left( \frac{d}{dx} \right) = \left( \lambda - a^2 \frac{d^2}{dx^2} \right) \left( \lambda - b^2 \frac{d^2}{dx^2} \right), \quad a > 0, b > 0, \lambda > 0, a \neq b,
\]
with respect to Dirichlet boundary conditions at \( x = 0 \). Hence, \( g_{\xi} \) fulfills
\[
(i) \ p\left( \frac{d}{dx} \right) g_{\xi} = \delta(x - \xi) \text{ for } x > 0,
\]
\[
(ii) \exists T_{\xi} \in \mathcal{S}'(\mathbb{R}) : g_{\xi} = T_{\xi}|_{(0,\infty)},
\]
\[
(iii) \ \lim_{\varepsilon \rightarrow 0} g_{\xi}^{(k)}(\varepsilon) = 0 \text{ for } k = 0, 1.
\]
The uniquely determined temperate fundamental solution of \( \lambda(\lambda - a^2 d^2 / dx^2) \) is given by
\[
E^a(x) = E^{a,\lambda}(x) = \frac{1}{2a \lambda^{3/2}} e^{-\sqrt{\lambda} |x|^a}, \quad x \in \mathbb{R},
\]
and this easily yields that the linear combination \( E = (a^2 E^a - b^2 E^b)/(a^2 - b^2) \) coincides with the temperate fundamental solution of \( p\left( \frac{d}{dx} \right) \). In fact,
\[
p\left( \frac{d}{dx} \right) E = \frac{1}{a^2 - b^2} \left[ \frac{a^2}{\lambda} \left( \lambda - a^2 \frac{d^2}{dx^2} \frac{\delta}{\lambda} \right) - \frac{b^2}{\lambda} \left( \lambda - b^2 \frac{d^2}{dx^2} \frac{\delta}{\lambda} \right) \right] = \delta.
\]
If we use the ansatz
\[
g_{\xi}(x) = E(x - \xi) + c_1 E^a(x + \xi) + c_2 E^b(x + \xi) + c_3 E^a(x + \frac{a \xi}{b}) + c_4 E^b(x + \frac{b \xi}{a}),
\]
then conditions (i) and (ii) in (2.2) are clearly fulfilled. Now note that \( E^a\left( \frac{a \xi}{b} \right) = \frac{b}{a} E^b(\xi) \) and \( E^a\left( \frac{b \xi}{a} \right) = \frac{a}{b} E^b(\xi) \). Therefore, the remaining condition (iii) in (2.2) yields
\[
0 = g_{\xi}(0) = \frac{a^2}{a^2 - b^2} E^a(\xi) - \frac{b^2}{a^2 - b^2} E^b(\xi)
+ c_1 E^a(\xi) + c_2 E^b(\xi) + c_3 \frac{b}{a} E^b(\xi) + c_4 \frac{a}{b} E^a(\xi),
\]
i.e.,

\[\varepsilon\text{-Springer}\]
We finally observe that the expansion into a power series with respect to $H$ hence, $(2.6)$ does not determine uniquely since, e.g., for each solution $g_\xi(x)$, $\xi = (0, \xi_2)$, $\xi_2 > 0$, in the half-space $H = \{x \in \mathbb{R}^2, x_2 > 0\}$ subject to Dirichlet boundary conditions at the border line $x_2 = 0$. Hence, $G_\xi$ fulfills

\[ (\partial_1^2 + a^2 \partial_2^2)(\partial_1^2 + b^2 \partial_2^2), \quad a > 0, b > 0, a \neq b. \]

Note that (2.6) does not determine $G_\xi$ uniquely since, e.g., for each solution $G_\xi$, the distributions $G_\xi + c x_2^2$, $c \in \mathbb{C}$, also fulfill (2.6). As we shall show in Section 4, $G_\xi$ becomes uniquely determined if we add to (2.6) the growth condition

\[ \lim_{N \to \infty} N^{-2} G_\xi |_{x_2=N} = 0 \text{ in } \mathcal{S}'(\mathbb{R}^1). \]
Upon a partial Fourier transform with respect to $x_1$, we obtain the ordinary differential operator in (2.1) as an operator in $d/dx_2$ with $\lambda$ being the square of the transformed variable of $x_1$. Hence, we conclude, at least heuristically, that $G(x) = \mathcal{F}^{-1}_{x_1} g^\alpha_{x_2}(x_2)$.

We next observe that

$$E^{\alpha, x_1^2}(x_2) = \frac{1}{2a|x_1|^3} e^{-|x_1x_2|/a}$$

is, at first sight, not well defined at $x_1 = 0$, but that the linear combination of these functions in $g^\alpha_{x_2}(x_2)$ is continuous at $x_1 = 0$ due to Eq. (2.5). We can therefore evaluate the inverse Fourier transform of $g^\alpha_{x_2}(x_2)$ by replacing $E^{\alpha, x_1^2}(x_2)$ by the finite part at $z = -3$ of the meromorphic distribution-valued function

$$z \mapsto \frac{1}{2a} |x_1|^3 e^{-|x_1x_2|/a} = S^\alpha_z(x) \in S'(\mathbb{R}^2),$$

which has simple poles for $z \in \mathbb{N}$.

For $\text{Re } z > -1$ and fixed $x_2 \neq 0$, $S^\alpha_z(x)$ is an integrable function of $x_1$. Hence, we obtain, for $\text{Re } z > -1$ and $x_2 \neq 0$, the following:

$$\mathcal{F}^{-1}_{x_1} S^\alpha_z = \frac{1}{4\pi a} \int_{-\infty}^{\infty} e^{ix_2 - |x_1|/a} |t|^z dt = \frac{\Gamma(z + 1)}{4\pi a} \left[ \left( \frac{|x_2|}{a} + ix_1 \right)^{-z-1} + \left( \frac{|x_2|}{a} - ix_1 \right)^{-z-1} \right],$$

(2.8)

see [7, Equ. 3.381.4], [9, p. 103]. Furthermore, $\mathbb{R} \to S'(\mathbb{R}^2), x_2 \mapsto S^\alpha_z(x)$ is continuous if $\text{Re } z > -1$. Therefore, the result in (2.8) represents $\mathcal{F}^{-1}_{x_1} S^\alpha_z$ as a locally integrable function in $\mathbb{R}^2$ if $|\text{Re } z| < 1$. By analytic continuation, this generally holds for $\text{Re } z < 1$ outside the poles of $\Gamma(z + 1)$, i.e., if $z \not\in -\mathbb{N}$. Thus, we obtain

$$\mathcal{F}^{-1}_{x_1}(\text{Pf } S^\alpha_z) = \text{Pf } (\mathcal{F}^{-1}_{x_1} S^\alpha_z) = \frac{1}{2\pi a} \text{Re } \frac{\Gamma(z + 1)}{\text{Pf } z = -3} \left( \frac{|x_2|}{a} + ix_1 \right)^{-z-1}.$$

We now use [12, Prop. 1.6.3, p. 28], and $\text{Res}_{z=-2} \Gamma(z) = \frac{1}{2}, \text{Pf } z = -2 \Gamma(z) = \frac{1}{2} \psi(3)$ in order to conclude that

$$\mathcal{F}^{-1}_{x_1}(\text{Pf } S^\alpha_z) = \frac{1}{2\pi a} \text{Re } \frac{\Gamma(z + 1)}{\text{Pf } z = -3} \left( \frac{|x_2|}{a} + ix_1 \right)^{-z-1} + \text{Res } \frac{\Gamma(z + 1)}{z = -3} \frac{d}{dz} \left( \frac{|x_2|}{a} + ix_1 \right)^{-z-1} \bigg|_{z = -3}$$

$$= \frac{1}{2\pi a} \text{Re } \left[ \frac{1}{2} \psi(3) \left( \frac{|x_2|}{a} + ix_1 \right)^{-z-1} - \frac{1}{2} \left( \frac{|x_2|}{a} + ix_1 \right)^{-z-1} \log \left( \frac{|x_2|}{a} + ix_1 \right) \right]$$

$$= \psi(3) \frac{x_1^2 - x_2^2}{4\pi a (x_1^2 + x_2^2)} + \frac{1}{8\pi a} \left( \frac{x_1^2 - x_2^2}{a^2} \right) \log \left( \frac{x_1^2 + x_2^2}{a^2} \right) + \frac{x_1 x_2}{2\pi a^2} \arctan \left( \frac{ax_1}{x_2} \right).$$
Upon summing up the six terms which constitute \( g_{z_2}^2(x_2) \), the second-order polynomials with the factor \( \psi(z) \) cancel out. Furthermore, for \( \Re z > 0 \), we have \( x_1^2(x_1^2 - a^2 \partial_z^2)S^c_z = |x_1|^2 \delta(x_2) \), and this implies, by analytic continuation, that \( E^a = \mathcal{F}_{x_1}^{-1}(P_{\tau_{x_1-3}^a}) \) is a fundamental solution of \( \partial_z^2(\partial_z^2 + a^2 \partial_z^2) \), also compare [13, Eqn. (3.1.15), p. 194]. Let us eventually observe that, here and similarly in the following sections, dilation by the factor \( a \) implies \( E^a = a^{-1} E^1(x_1, x_2 / a) \). Thus, we arrive at the following proposition.

**Proposition 2.1** For \( c > 0 \), let \( E^c \) denote the following fundamental solution of \( \partial_z^2(\partial_z^2 + c^2 \partial_z^2) \):

\[
E^c(x) = \frac{1}{8 \pi c} \left( x_1^2 - \frac{x_2^2}{c^2} \right) \log \left( x_1^2 + \frac{x_2^2}{c^2} \right) + \frac{x_1 x_2}{2 \pi c^2} \arctan \left( \frac{c x_1}{x_2} \right). \tag{2.9}
\]

Let \( G_z(x) \), \( x_2 > 0 \), \( z = (0, \xi) \), \( \xi > 0 \) be the Green function of the operator \( P(\partial) = (\partial_z^2 + a^2 \partial_z^2)(\partial_z^2 + b^2 \partial_z^2), \ a > 0, \ b > 0, \ a \neq b \), with respect to Dirichlet boundary conditions at \( x_2 = 0 \), i.e., let \( G_\xi \) be determined by the conditions (2.6) and (2.7).

Then \( G_z(x) = F_{\xi}^{a,b}(x) + F_{\xi}^{b,a}(x), x_1 \in \mathbb{R}, x_2 > 0 \), where

\[
F_{\xi}^{a,b}(x) = \frac{a^2}{a^2 - b^2} E^a(x - \xi) - \frac{a^2}{(a - b)^2} E^b(x + \xi) + \frac{2 a^2 b}{(a - b)^2 (a + b)} E^a \left( x + \frac{a x_2}{b} \right).
\]

In order to derive the Green function (of the Dirichlet problem) for the orthotropic plate operator in (1.1), let us represent \( G_z \) slightly more explicitly in the case \( b = \frac{1}{2} \). We assume that \( a > 1 \) and we set \( \mu = \frac{1}{2} (a - b) \). This implies \( a^2 + b^2 = 2 + 4 \mu^2, \ a^2 b \text{ and } P(\partial) = \partial_z^4 + 2(1 + 2 \mu^2) \partial_z^2 + \partial_z^2 \). Then, the fundamental solution \( E = (a^2 E^a - b^2 E^b)/(a^2 - b^2) \) of \( P(\partial) \) takes the form

\[
E = \frac{1}{8 \pi (a^2 - b^2)} \left[ (ax_1^2 - bx_2^2) \log(x_1^2 + b^2 x_2^2) - (bx_1^2 - ax_2^2) \log(x_1^2 + a^2 x_2^2) \right. \\
+ 4 x_1 x_2 \arctan \left( \frac{ax_1}{x_2} \right) - 4 x_1 x_2 \arctan \left( \frac{bx_1}{x_2} \right) \\
\left. + \frac{a - b}{2} \left| x \right|^2 \log P(x) + \frac{a + b}{2} (x_1^2 - x_2^2) \log \left( \frac{x_1^2 + b^2 x_2^2}{x_1^2 + a^2 x_2^2} \right) \right]
\]

Hence, we obtain
\[
E = \frac{|x|^2}{32\pi \sqrt{1 + \mu^2}} \log P(x) + \frac{x_1^2 - x_2^2}{32\pi \mu} \log \left(\frac{x_1^2 + b^2x_2^2}{x_1^2 + a^2x_2^2}\right)
\]

(2.10)

Similarly, the term \( H = (a^2 E^a + b^2 E^b)/(a - b)^2 \) yields
\[
H = \frac{\sqrt{1 + \mu^2}(x_1^2 - x_2^2)}{32\pi \mu^2} \log P(x) + \frac{|x|^2}{32\pi \mu} \log \left(\frac{x_1^2 + b^2x_2^2}{x_1^2 + a^2x_2^2}\right)
\]

(2.11)

\[
+ \frac{x_1x_2}{8\pi \mu^2} \left[\arctan \left(\frac{2\sqrt{1 + \mu^2x_1x_2}}{x_2^2 - x_1^2}\right) + \pi Y(x_1^2 - x_2^2) \text{sign}(x_1, x_2)\right].
\]

Of course, the last part of \( G_\xi \), i.e.,
\[
J = \frac{2ab}{(a - b)^2(a + b)} \left[ aE^a(x + \frac{a\xi}{b}) + bE^b(x + \frac{b\xi}{a}) \right],
\]

is the most laborious one. It gives
\[
J = \frac{x_1^2 - (x_2 + \xi)^2 - 2\mu^2(x_2^2 + \xi^2)}{32\pi \mu^2 \sqrt{1 + \mu^2}} \times \log \left[ (x_1^2 + (bx_2 + a\xi)^2)(x_1^2 + (ax_2 + b\xi)^2) \right]
\]

\[
+ \frac{x_2^2 - \xi^2}{16\pi \mu} \log \left(\frac{x_1^2 + (bx_2 + a\xi)^2}{x_1^2 + (ax_2 + b\xi)^2}\right)
\]

\[
+ \frac{x_1(x_2 + \xi)}{8\pi \mu^2} \left[\arctan \left(\frac{2\sqrt{1 + \mu^2x_1(x_2 + \xi)}}{(x_2 + \xi)^2 + 4\mu^2x_2\xi - x_1^2}\right)\right]
\]

\[
+ \pi Y(x_1^2 - (x_2 + \xi)^2 - 4\mu^2x_2\xi)(x_1(x_2 + \xi)) \right]
\]

\[
+ \frac{x_1(\xi - x_2)}{8\pi \mu \sqrt{1 + \mu^2}} \arctan \left(\frac{2\mu x_1(\xi - x_2)}{(x_2 + \xi)^2 + 4\mu^2x_2\xi + x_1^2}\right).
\]

We now obtain the Green function of the orthotropic plate operator \( P(\partial) \) in (1.1) by continuing \( G_\xi(x) = E(x - \xi) - H(x + \xi) + J(x, \xi) \) analytically with respect to \( \mu \), i.e., we set \( \mu = i\epsilon \), \( 0 < \epsilon < 1 \). The result is the following.

**Proposition 2.2** The Green function \( G_\xi(x) \), \( x_2 > 0 \), \( \xi = (0, \xi_2) \), \( \xi_2 > 0 \), of the operator of the orthotropic plate \( P(\partial) = \partial_1^2 + 2(1 - 2\epsilon^2)\partial_1 \partial_2^3 + \partial_2^4 \),, \( 0 < \epsilon < 1 \), with respect to Dirichlet boundary conditions at \( x_2 = 0 \), see (2.6) and (2.7), is given by \( G_\xi(x) = \tilde{G}_\xi(x) + \tilde{G}_\xi(-x_1, x_2) \), where \( \tilde{G}_\xi(x) = f_1(x - \xi) + f_2(x + \xi) + f_3(x, \xi) \) and
\[ f_1(x) = \frac{e|x|^2 + 2x_1x_2}{32\pi e \sqrt{1 - e^2}} \log(|x|^2 + 2e x_1x_2) + \frac{x_2 - x_1^2}{16\pi e} \arctan \left( \frac{e|x_2| + x_1}{|x_2| \sqrt{1 - e^2}} \right), \]

\[ f_2(x) = \frac{\sqrt{1 - e^2}(x_1^2 - x_2^2)}{32\pi e^2} \log(|x|^2 + 2e x_1x_2) + \frac{e|x|^2 + 2x_1x_2}{16\pi e^2} \arctan \left( \frac{e x_2 + x_1}{x_2 \sqrt{1 - e^2}} \right), \]

\[ f_3(x, \xi) = \frac{(x_2 + \xi_2)^2 - 2e^2(x_2^2 + \xi_2^2) - x_1^2 - 2e x_1(x_2 - \xi_2)}{32\pi e^2} \sqrt{1 - e^2} \]
\[ \times \log[x_1^2 + (x_2 + \xi_2)^2 - 4e^2 x_2 \xi_2 + 2e x_1(x_2 - \xi_2)] \]
\[ - \frac{e(x_2^2 - \xi_2^2) + x_1(x_2 + \xi_2)}{8\pi e^2} \arctan \left( \frac{e(x_2 - \xi_2) + x_1}{(x_2 + \xi_2) \sqrt{1 - e^2}} \right). \]

**Proof** If we replace \( \mu \) by \( ie \), then \( \{a/b\} = \pm ie + \sqrt{1 - e^2} \) are conjugate complex numbers of modulus one. Hence, in the second term of \( E \) in formula (2.10), the logarithm is purely imaginary and given by

\[
\log\left(\frac{x_1^2 + b^2 x_2^2}{x_1^2 + a^2 x_2^2}\right) = 2i \left[ \arctan\left( \frac{\text{Im}(b^2)x_2^2}{x_1^2 + \text{Re}(b^2)x_2^2} \right) - \pi \left( -x_1^2 - \text{Re}(b^2)x_2^2 \right) \right]
\]
\[
= -2i \left[ \arctan\left( \frac{2e\sqrt{1 - e^2}x_2^2}{|x|^2 - 2e^2x_2^2} \right) + \pi(2e^2x_2^2 - |x|^2) \right]
\]
\[
= -2i \left[ \arctan\left( \frac{e|x_2| + x_1}{|x_2| \sqrt{1 - e^2}} \right) + \arctan\left( \frac{e|x_2| - x_1}{|x_2| \sqrt{1 - e^2}} \right) \right].
\]

Furthermore, the arctangent in Eq. (2.10) yields

\[
\arctan\left( \frac{2ie x_1x_2}{|x|^2} \right) = i \frac{2}{2} \log\left( \frac{|x|^2 + 2e x_1x_2}{|x|^2 - 2e x_1x_2} \right).
\]

The analytic continuation of \( H \) in formula (2.11) is similar.

Let us yet explain the analytic continuation of formula (2.12). The polynomial

\[ Q(x) = (x_1^2 + (bx_2 + a\xi_2)^2) (x_1^2 + (ax_2 + b\xi_2)^2) \]

can be factored as \( Q(x) = q(x)q(-x_1, x_2) \) where

\[ q(x) = (x_1 + i(bx_2 + a\xi_2))(x_1 - i(ax_2 + b\xi_2)) \]
\[ = x_1^2 + (x_2 + \xi_2)^2 - 4e^2 x_2 \xi_2 + 2e x_1(x_2 - \xi_2). \]

This settles the first term in formula (2.12). For the second one, observe that
where \( \pi \) or \(-\pi\) is added in the first two formula lines if the complex number \( x_1^2 + (bx_2 + a\xi_2)^2 \) belongs to the second or to the third quadrant, respectively. The remaining two terms in Eq. (2.12) are treated similarly. We therefore obtain for \( G_\xi(x) \) the representation which is enunciated in the proposition.

The three conditions in (2.6) then hold automatically by analytic continuation. Condition (2.7) can be checked directly. Finally, the uniqueness of \( G_\xi(x) \) under these conditions follows from a reasoning which is similar to that in the proof of Proposition 4.1 below. This completes the proof. \( \square \)

**Remark 2.3** If \( f_1 \) is as in Proposition 2.2, then \( f_1(x) + f_1(-x_1, x_2) \) coincides with the fundamental solution \( E(x) \) in Eq. (1.3) of the orthotropic plate operator (1.1) up to the second-order polynomial \( \frac{\arcsin e}{16\pi} (x_1^2 - x_2^2) \).

### 3 Quasi-hyperbolic and non-vanishing symbol operators

Let us generalize now the context of Section 2 and consider operators of the form

\[
P(\partial) = (Q(\partial') - a^2 \partial^2_n)(Q(\partial') - b^2 \partial^2_n), \quad a > 0, b > 0, a \neq b,
\]

where \( Q(\partial') \) is an operator in the \( n-1 \) variables \( x' = (x_1, \ldots, x_{n-1}) \) and \( P(\partial) \) either is quasi-hyperbolic or has a non-vanishing symbol \( P(\text{ix}) \).

Let us first recall the notion of “quasi-hyperbolicity”, see [13, Def. and Prop. 2.4.13, p. 162].

**Definition 3.1** The operator \( P(\partial) = P(\partial_1, \ldots, \partial_n) = \sum_{|\alpha| \leq m} c_\alpha \partial^\alpha, c_\alpha \in \mathbb{C}, \alpha \in \mathbb{N}_0^n \), is called quasi-hyperbolic in the direction \( N \in \mathbb{R}^n \setminus \{0\} \) if and only if there exists \( \sigma_0 \in \mathbb{R} \) such that

\[
\forall \sigma > \sigma_0 : \forall x \in \mathbb{R}^n : P(\text{ix} + \sigma N) \neq 0.
\]

As shown in [13, Def. and Prop. 2.4.13, p. 162], for a quasi-hyperbolic operator \( P(\partial) \), there exists a uniquely determined fundamental solution \( E \) of \( P(\partial) \) fulfilling \( e^{-\sigma N x} E \in \mathcal{S}'(\mathbb{R}^n) \) for some \( \sigma \) larger than the constant \( \sigma_0 \) in (3.2). This fundamental solution is then given by the formula

\[
E = e^{\sigma N x} F^{-1}(P(\text{ix} + \sigma N)^{-1}), \quad \sigma > \sigma_0,
\]

and it holds \( \text{supp} E \subset \{ x \in \mathbb{R}^n ; x \cdot N \geq 0 \} \) and \( e^{-\sigma N x} E \in \mathcal{S}'(\mathbb{R}^n) \) for each \( \sigma \geq \sigma_0 \). Note that operators \( P(\partial) \) whose symbols \( P(\text{ix}) \) do not vanish are formally contained in the above definition if we allow for \( N = 0 \).
In the next proposition, we shall state uniqueness conditions and we shall give a formula for the Green function $G_\xi(x)$ of the Dirichlet problem in the half-space $H = \{x \in \mathbb{R}^n; x_n > 0\}$ for quasi-hyperbolic and for non-vanishing symbol operators $P(\partial)$ as in (3.1). We assume that $Q(\partial')$ is an operator in the $n-1$ variables $x' = (x_1, \ldots, x_{n-1})$, that $N' \in \mathbb{R}^{n-1}$ and we set $N = (N', 0)$. Then, $P(\partial)$ as in (3.1) is quasi-hyperbolic in direction $N$ (in case $N \neq 0$) or has a non-vanishing symbol $P(ix)$ (in case $N = 0$) if and only if, for some $\sigma_0 \in \mathbb{R}$, the following condition is satisfied:

$$\forall \sigma > \sigma_0 : \forall x = (x', x_n) \in \mathbb{R}^n : Q(ix' + \sigma N') + x_n^2 \neq 0. \quad (3.4)$$

**Proposition 3.2** Let $N' \in \mathbb{R}^{n-1}$, $\sigma_0 \in \mathbb{R}$, $H = \{x \in \mathbb{R}^n; x_n > 0\}$, $\xi = (0, \ldots, 0, \xi_n) \in \mathbb{R}^n$, $\xi_n > 0$. Let $Q(\partial') = Q(\partial_1, \ldots, \partial_{n-1})$ be a linear partial differential operator with constant coefficients in $n - 1$ variables such that condition (3.4) is satisfied. Let $P(\partial)$ be defined as in (3.1). For $c > 0$, let $E^c = e^{-1}E^1(x', x_n/c)$ be the uniquely determined fundamental solution of $Q(\partial')(Q(\partial') - c^2 \partial_n^2)$ such that $e^{-\sigma N' x} E^c$ is temperate for $\sigma \geq \sigma_0$.

Then, the Green function $G_\xi \in \mathcal{D}'(H)$ of $P(\partial)$ fulfilling

(i) $P(\partial)G_\xi = \delta(x - \xi)$ in $\mathcal{D}'(H)$,

(ii) $\exists \sigma > \sigma_0 : \exists T_\xi \in \mathcal{S}'(\mathbb{R}^n) : e^{\sigma N' x} G_\xi = T_\xi |_H$,

(iii) $\lim_{c \downarrow 0} a^k T_\xi |_{x_n = c} \neq 0$ in $\mathcal{S}'(\mathbb{R}^{n-1})$ for $T_\xi$ as in (b) and $k = 0, 1$

is uniquely determined and given by

$$G_\xi(x) = F^{a,b}_\xi(x) + F^{b,a}_\xi(x), \quad x \in H,$$

$$F^{a,b}_\xi(x) = \frac{a^2}{a^2 - b^2}E^a(x - \xi) - \frac{a^2}{(a - b)^2}E^a(x + \xi) + \frac{2a^2 b}{(a - b)^2(a + b)}E^a \left( x + \frac{a^2}{b} \right).$$

[Note that the restrictions of $\partial^k T_\xi$ to the subspaces $\{x \in \mathbb{R}^n; x_n = c\}$, $c > 0$, are well defined due to (i) and [8, Thm. 4.4.8, p. 115]. Condition (iii) requires that these restrictions belong to $\mathcal{S}'(\mathbb{R}^{n-1})$ and converge therein to 0.]

**Proof** The statement in the proposition corresponds to Lemma 3.2 and Proposition 3.7 in [14], where the operator $(Q(\partial') - \partial_n^2)^m$, $m \in \mathbb{N}$, is investigated. We therefore only outline the proof here and refer to [14] for details.

Since the values of the polynomial $\lambda(x') = Q(ix' + \sigma N')$ lie in the set $\mathbb{C} \setminus (-\infty, 0]$ due to condition (3.4), we can choose the root $\sqrt[\lambda(x')]$ such the Re $\sqrt[\lambda(x')] > 0$. If $G_\xi$, $\tilde{G}_\xi$ are two solutions of (3.5), then we consider the difference $S = G_\xi - \tilde{G}_\xi$ and the partial Fourier transform with respect to $x'$

$$\tilde{S} = \mathcal{F}_{x'}(e^{-\sigma N' x'}) = \mathcal{F}_{x'}(T_\xi - \tilde{T}_\xi)|_H \in \mathcal{D}'(H).$$

It fulfills $(\lambda(x') - a^2 \partial_n^2)(\lambda(x') - b^2 \partial_n^2)\tilde{S} = 0$ in $\mathcal{D}'(H)$ and accordingly we obtain the representation
\[
\tilde{S} = \sum_{e=\pm} A_e(x')e^{\alpha_n\sqrt{4(x')^2/a}} + B_e(x')e^{\alpha_n\sqrt{4(x')^2/b}}
\]

with distributions \(A_e, B_e \in D'(\mathbb{R}^{n-1}), e = \pm\). From \(\mathcal{F}_x(T_{\tilde{y}} - \tilde{T}) \in \mathcal{S}'(\mathbb{R}^n)\), we infer that \(A_e\) and \(B_e\) must vanish, and from the boundary condition (iii) in (3.5), we then conclude that also \(A_\pm = B_\pm = 0\), i.e., \(S = 0\) and \(G_\xi = G_{\xi_0}\). Therefore, \(G_\xi\) is uniquely determined.

In order to show that \(G_\xi\) defined by formula (3.6) satisfies the conditions in (3.5), let \(E\) denote the uniquely determined fundamental solution of \(P(\partial)\) such that \(e^{-\pi\alpha_n'x'}E\) is temperate for \(\sigma \geq \sigma_0\). Then, \(E\) satisfies

\[
E = e^{\pi\alpha_n'x'} \mathcal{F}_n^{-1}\left(\left(\lambda(x') + a^2x_n^2\right)^{-1}\right) - \lambda(x') + b^2x_n^2
\]

Furthermore, \(P(\partial)E^a = P(\partial)E^b = 0\) in \(H\). This implies that \(G_\xi\) satisfies conditions (i) and (ii) in (3.5).

Finally, condition (iii) in (3.5) follows from the calculation leading to formula (2.4), which corresponds to the representation (3.6) in the one-dimensional case. We observe that

\[
\int_c x \mathcal{E} \quad \text{in the half-space}
\]

the definition in (2.3) and the ensuing calculations are equally valid for \(\lambda \in \mathbb{C} \setminus (-\infty, 0]\) if we choose \(\sqrt{\lambda}\) with positive real part.

\(\square\)

**Example 3.3** Let us calculate the Green function \(G_{(0,\xi)}(t, x)\) where \(\xi = (0, \ldots, 0, \xi_n) \in \mathbb{R}^n\), \(\xi_n > 0\), for the product of wave operators

\[
P(\partial) = (\partial_t^2 - \Delta_{n-1} - a^2\partial_n^2)(\partial_t^2 - \Delta_{n-1} - b^2\partial_n^2), \quad a > 0, b > 0, a \neq b,
\]

with respect to Dirichlet conditions at the hyperplane \(x_n = 0\) and vanishing Cauchy data at \(t = 0\). According to Proposition 3.2, we can express \(G_{(0,\xi)}\) through a combination of shifted fundamental solutions \(E^c\) of \((\partial_t^2 - \Delta_{n-1} - c^2\partial_n^2), c \in \{a, b\}\).

Using the method of parameter integration, see [13, Chap. 3], we have \(E^c = c^{-2} \int_0^c E_\mu d\mu\) with \(E_\mu\) being the “forward” fundamental solution of \((\partial_t^2 - \Delta_{n-1} - \mu\partial_n^2)^2\), i.e., the one with support in the half-space \(\{(t, x) \in \mathbb{R}^{n+1}; t \geq 0, x \in \mathbb{R}^n\}\).

If \(\chi^2\) is defined as in (1.6), then the forward fundamental solution \(E_1\) of \((\partial_t^2 - \Delta_n)^2\) can be written as

\[
E_1(t, x) = \frac{Y(t) \chi^{2-(n+1)/2}(t^2 - |x|^2)}{8\pi^{(n-1)/2}}
\]

according to [4, Lemma 4.2, p. 317], [8, Section 6.2], [13, Equs. (2.3.11), (2.3.12), pp. 141, 142]. (Note that the function \((t, x) \mapsto t^2 - |x|^2\) is submersive outside the origin and that the composition with \(\chi^2\) defined in (1.6) thus is meaningful outside the origin. Furthermore, the resulting distribution continuously depends on \(t \in \mathbb{R}\), and it can therefore be multiplied by \(Y(t)\). see also [12, Prop. 2.4.2, p. 56], and the ensuing remark.)

Hence,

\[
E^c = \frac{Y(t)}{8\pi^{(n-1)/2}c^2} \int_0^c \frac{\chi^{2-(n+1)/2}(t^2 - |x'|^2 - \frac{x_n^2}{\mu})}{\sqrt{\mu}} \frac{d\mu}{\sqrt{\mu}}.
\]

\(\copyright\) Springer
E.g., let us evaluate the integral in (3.8) in the case of \( n = 3 \) spatial dimensions. Due to \( \chi^0 = Y \) and setting \( x' = (x_1, x_2), R = \sqrt{t^2 - |x'|^2}, t_+ = x'(t) = Y(t), \) we obtain

\[
E^c(t, x) = \frac{Y(t)}{8\pi c^2} \int_0^1 Y(t^2 - |x'|^2 - x_3^2/\mu) \frac{d\mu}{\sqrt{\mu}} = \frac{Y(t - |x'|)}{4\pi c^2 R} (cR - |x_3|)_+,
\]

see also [17, Ex. 5, p. 27]. Therefore, formula (3.6) yields the following Green function for the operator \( P(\partial) \) in (3.7) with \( n = 3 \):

\[
G(0, \xi)(t, x) = \frac{Y(t - |x'|)}{4\pi R} \left[ \frac{(aR - |x_3 - \xi_3|)_+ - (bR - |x_3 - \xi_3|)_+}{a^2 - b^2} - \frac{(aR - x_3 - \xi_3)_+ + (bR - x_3 - \xi_3)_+}{(a - b)^2} + 2 \frac{abR - ax_3 - b\xi_3)_+ + (abR - bx_3 - a\xi_3)_+}{(a - b)^2(a + b)} \right].
\]

**Example 3.4** By means of Proposition 3.2 and using analytic continuation, we can also derive the Green function for the Cauchy–Dirichlet problem of the equation of transverse vibrations of a semi-infinite clamped beam, see [11].

The fundamental solution \( E^c \) referring to formula (3.3) for the operator \( \partial_t (\partial_t - c^2 \partial_x^2) \), \( c > 0 \), is given by

\[
E^c = \frac{Y(t)}{2c\sqrt{\pi}} \int_0^t e^{-\tau^2/(4c^2 \tau)} \frac{d\tau}{\sqrt{\tau}}.
\]

Hence, formula (3.6) in Proposition 3.2 yields the following for the Green function \( G(0, \xi) \), \( \xi > 0 \), of the operator

\[
P(\partial) = (\partial_t - a^2 \partial_x^2)(\partial_t - a^{-2} \partial_x^2) = \partial_t^2 + \partial_x^4 - (a^2 + a^{-2}) \partial_t \partial_x^2,
\]

\[
G(0, \xi)(t, x) = \frac{Y(t)}{2\sqrt{\pi}} \int_0^t \left\{ \frac{1}{a^2 - a^{-2}} \left[ a e^{-(x-\xi)^2/(4a^2 \tau)} - a^{-1} e^{-a^2(x-\xi)^2/(4a \tau)} \right] - \frac{1}{(a - a^{-1})^2} \left[ a e^{-(x+\xi)^2/(4a^2 \tau)} + a^{-1} e^{-a^2(x+\xi)^2/(4a \tau)} \right] + \frac{2}{(a - a^{-1})^2(a + a^{-1})} \left[ e^{-(ax+\xi/a)^2/(4a \tau)} + e^{-(ax-\xi/a)^2/(4a \tau)} \right] \right\} \frac{d\tau}{\sqrt{\tau}}.
\]

We now use analytic continuation as in Section 2 and set \( a = \sqrt{1} = \frac{1+i}{\sqrt{2}} \). Then, \( P(\partial) = \partial_t^2 + \partial_x^4 \) and formula (3.9) yields

\[
G(0, \xi)(t, x) = \frac{Y(t)}{2\sqrt{\pi}} \int_0^t \sin \left[ \frac{(x - \xi)^2}{4\tau} \pm \frac{\pi}{4} \right] - \sin \left[ \frac{(x + \xi)^2}{4\tau} \mp \frac{\pi}{4} \right] - \sqrt{2} e^{-x\xi/(2\tau)} \cos \left[ \frac{x^2 - \xi^2}{4\tau} \right] \frac{d\tau}{\sqrt{\tau}}.
\]

in accordance with [11, p. 239].
Remark 3.5 Our faithful reader, who has followed us up to this point, might wonder how the representation of $G_\psi$ in (3.6) would look in the case of a product of $m > 2$ factors, i.e., if $P(\partial) = \prod_{m=1}^{m}(Q(\partial') - a_j^2\partial^2_n)$ for pairwise different positive numbers $a_j$. Since we will not use the corresponding formula, we mention the result. For $c > 0$, let $E^c$ denote the fundamental solution of $Q(\partial')^m - c^2\partial^2_n$ such that $e^{-\sigma N_\varepsilon}E$ is temperate for $\sigma \geq \sigma_0$. Then, $G_\psi(x) = \sum_{m=1}^{m} F^j_\psi(x)$ where

$$F^j_\psi(x) = \left( \prod_{k \neq j} \frac{a_j^2}{a_j^2 - a_k^2} \right) E^{a_j}(x - \xi) - \left( \prod_{k \neq j} \left( \frac{a_j}{a_j - a_k} \right)^2 \right) E^{a_j}(x + \xi) + 2 \sum_{k \neq j} \prod_{l \neq j, l \neq k} \frac{a_j a_k}{(a_j - a_l)(a_k - a_l)} a_j^2 a_k E^{a_j}(x + \frac{a_j \xi}{a_k}) \frac{a_j^2 a_k E^{a_j}(x + \frac{a_j \xi}{a_k})}{(a_j - a_k)^2(a_j + a_k)}. \tag{3.10}$$

A technically relevant application of formula (3.10) would consist in the derivation of an explicit formula for the Green function of a product $P_1(\partial)P_2(\partial)$ of two orthotropic plate operators $P_1(\partial), P_2(\partial)$ as in (1.1). This product operator describes the deflection of an orthotropic cylindrical shell, see [3, Equs. (14) and (22), pp. 738, 739].

4 Operators with symbols that are positive outside the origin

In this final section, let us treat operators $P(\partial)$ in $\mathbb{R}^n$ such that $P(ix) > 0$ for $x \in \mathbb{R}^n \setminus \{0\}$. As has been shown in [1], the analytic-distribution-valued function

$$S : \{ z \in \mathbb{C}; \operatorname{Re} z > 0 \} \longrightarrow \mathcal{S}'(\mathbb{R}^n) : z \mapsto P(ix)^z$$

has a meromorphic extension $\hat{S}$ to the whole complex plane, see also [13, Prop. 2.3.1, p. 134]. For simplicity, we shall write $P(ix)^z = \hat{S}(z)$ if $\hat{S}$ is analytic in $z \in \mathbb{C}$ and $P(ix)^z = \operatorname{Pf}_{z_0=0} \hat{S}(z)$ if $z_0 \in \mathbb{C}$ is a pole of $\hat{S}$. (Here, $\operatorname{Pf}$ stands for the finite part of a meromorphic distribution-valued function, compare [12].) Since $P(ix) \cdot P(ix)^z = P(ix)^{z+1}$ holds by analytic continuation for each $z \in \mathbb{C}$, $E := \mathcal{F}^{-1}(P(ix)^{-1})$ yields a temperate fundamental solution of $P(\partial)$.

In particular, let $Q(\partial')$ be an operator in $n - 1$ variables $x' = (x_1, \ldots, x_{n-1})$ such that $Q(ix') > 0$ for $x' \in \mathbb{R}^{n-1} \setminus \{0\}$. Then, $z \mapsto Q(ix')^z$ is meromorphic. Upon expanding the exponential function into a power series, this implies that, for $a > 0$, the holomorphic function

$$S^a : \{ z \in \mathbb{C}; \operatorname{Re} z > 0 \} \longrightarrow \mathcal{S}'(\mathbb{R}^n) : z \mapsto S^a = \frac{1}{2a} Q(ix')^{z/2} e^{-\sqrt{Q(ix')}} |x'|^a \tag{4.1}$$

can be extended as a meromorphic function to $\mathbb{C}$. From

$$Q(ix')(Q(ix') - a^2\partial^2_n)S^a = Q(ix')^{(z+3)/2} \otimes \delta(x_n), \quad \operatorname{Re} z > 0,$$

we conclude, by analytic continuation, that

$$E^a := \mathcal{F}^{-1}_{x'} \begin{pmatrix} \operatorname{Pf} & S^a \end{pmatrix} \tag{4.2}$$

is a fundamental solution of $Q(\partial')(Q(\partial') - a^2\partial^2_n)$.
In the next proposition, we shall generalize Proposition 2.1 to operators of the form

\[ P(\partial) = (Q(\partial') - a^2\partial_n^2)(Q(\partial') - b^2\partial_n^2), \quad a > 0, b > 0, a \neq b. \]  

(4.3)

**Proposition 4.1** Let \( Q(\partial') = Q(\partial_1, \ldots, \partial_{n-1}) \) be a linear partial differential operator with constant coefficients in \( n-1 \) variables such that \( Q(ix') > 0 \) for \( x' \in \mathbb{R}^{n-1} \setminus \{0\} \). Let \( H = \{x \in \mathbb{R}^n; x_n > 0\}, \xi = (0, \ldots, 0, \xi_n) \in \mathbb{R}^n, \xi_n > 0, \) and \( P(\partial) \) be as in (4.3). For \( c > 0, \) let \( E_c \) be the fundamental solution of \( Q(\partial')(Q(\partial') - c^2\partial_n^2) \) defined in (4.2).

Then, the Green function \( G_\xi \in \mathcal{D}'(H) \) of \( P(\partial) \) fulfilling

\[ \begin{align*}
(i) & \quad P(\partial)G_\xi = \delta(x - \xi) \text{ in } \mathcal{D}'(H), \\
(ii) & \quad \exists T_\xi \in \mathcal{S}'(\mathbb{R}^n) : G_\xi = T_\xi|_H, \\
(iii) & \quad \lim_{\varepsilon \to 0} \partial_n^k G_\xi|_{x_n = \varepsilon} = 0 \text{ in } \mathcal{S}'(\mathbb{R}^{n-1}) \text{ for } k = 0, 1, \\
(iv) & \quad \lim_{N \to \infty} N^{-2} G_\xi|_{x_n = N} = 0 \text{ in } \mathcal{S}'(\mathbb{R}^{n-1})
\end{align*} \]  

(4.4)

is uniquely determined and given by formula (3.6).

**Proof** In order to prove the uniqueness of \( G_\xi \) under the four conditions in (4.4), we proceed as in the corresponding Lemma 4.1 in [14] and we refer to it for details.

As in the proof of Proposition 3.2, we consider the difference \( S = G_\xi - \tilde{G}_\xi \) of two solutions of (4.4) and its partial Fourier transform

\[ \tilde{S} = \mathcal{F}_x S = \mathcal{F}_x(T_\xi - \tilde{T}_\xi)|_H \in \mathcal{D}'(H). \]

Condition (i) in (4.4) implies that the support of \( \tilde{S} \) is contained in the half-axis \( \{(0, \ldots, 0, x_n) \in \mathbb{R}^n; x_n > 0\} \). More precisely, we obtain the representation

\[ \tilde{S} = \sum_{\alpha \in \mathbb{N}_0^{n-1}, |\alpha| \leq l} \partial^\alpha \delta(x') \otimes g_\alpha(x_n) \]

for \( l \in \mathbb{N}_0 \) and polynomials \( g_\alpha \) in one variable. Hence, \( S \) is a polynomial. Condition (iii) in (4.4) implies \( S = x_n^2 \cdot R \) for another polynomial and condition (iv) then yields \( R = 0 \), i.e., \( G_\xi = \tilde{G}_\xi \).

For the verification of the representation of \( G_\xi \) in formula (3.6), we consider again \( S_\xi^a \) as in (4.1). Since

\[ (Q(ix') - a^2\partial_n^2)S_\xi^a = Q(ix')^{(z+1)/2} \otimes \delta(x_n), \quad \Re z > 0, \]

we obtain, by analytic continuation, that

\[ V = (Q(\partial') - a^2\partial_n^2)E_\xi^a = \mathcal{F}_x^{-1} \left( \text{Pf}_{z = -1} Q(ix')^{\zeta} \otimes \delta(x_n) \right) \]

vanishes in \( H \) and is independent of \( a \). This implies that \( P(\partial)E_\xi^a = P(\partial)E_\xi^b = 0 \) in \( H \) and that \( E = (a^2E_a - b^2E_b)/(a^2 - b^2) \in \mathcal{S}'(\mathbb{R}^n) \) is a fundamental solution of \( P(\partial) \) due to
\[ P(\partial)E = \frac{a^2}{a^2 - b^2} (\delta - \frac{b^2}{a^2} \nabla^2 V) - \frac{b^2}{a^2 - b^2} (\delta - \frac{a^2}{a^2} \nabla^2 V) = \delta. \]

Hence, conditions (i) and (ii) in (4.4) are fulfilled for \( G_\varepsilon \) as in (3.6).

By construction, the partial Fourier transform \( \mathcal{F}_x G_x = (\mathcal{F}_x T_x)_{|I}\rho \) coincides, for fixed \( \lambda' \in \mathbb{R}^{n-1} \), with the one-dimensional Green function \( g_{\omega_n}^0(x_n) \) given in (2.4) for \( Q(ix') > 0 \) and in (2.5) for \( Q(ix') = 0 \). The following lemma shows that \( g_{\omega_n}^0(e) \) and \( N^{-2} g_{\omega_n}^0(N) \) converge to 0 for \( e \searrow 0 \) and \( N \to \infty \) uniformly with respect to \( \lambda \in [0, \infty) \). The same assertion holds for \( (g_{\omega_n}^0)'(e) \), and this can be shown by inspection of the explicit representation of \( (g_{\omega_n}^0)'(e) \) by exponential functions, i.e., \( (g_{\omega_n}^0)'(e) = h_{\eta}^{\lambda, \lambda}(e) + h_{\eta}^{\lambda, \lambda}(e) \) with

\[ h_{\eta}^{\lambda, \lambda}(e) = \frac{\exp(-\sqrt{\lambda}/\eta)}{\lambda(a-b)^2(a+b)} \left[ a \cosh\left( \frac{\sqrt{\lambda} \xi}{a} \right) - a \exp(-\sqrt{\lambda} \xi) - b \sinh\left( \frac{\sqrt{\lambda} \xi}{a} \right) \right] \]

for \( 0 < e < \eta \). This implies the conditions (iii) and (iv) in (4.4) and completes the proof.

\[ \square \]

**Lemma 4.2** For \( a > 0, b > 0, \lambda > 0, \xi > 0, a \neq b \), let \( g_{\omega_n}^0 \) be the one-dimensional Green function of the operator in (2.1), i.e., the function satisfying the conditions in (2.2) and given by formula (2.4).

Then \( g_{\omega_n}^0 \) fulfills the estimate

\[ |g_{\omega_n}^0(x)| \leq \frac{x \xi \min(x, \xi)}{a^2 b^2}, \quad x > 0. \tag{4.5} \]

**Proof** As before, let us denote by \( E \) the temperate fundamental solution of the operator \( p(d/dx) \) in (2.1). Then, the Green function can be represented by

\[ g_{\omega_n}^0(x) = E(x - \xi) - E(x + \xi) - 4ab(a + b)\sqrt{\lambda} E'(x)E'(\xi), \quad x > 0. \tag{4.6} \]

In fact, the function on the right-hand side of formula (4.6) fulfills all the conditions in (2.2). (In parentheses, let us observe that, indeed, formula (4.6) yields a simpler and more symmetric representation of \( g_{\omega_n}^0 \) than (2.4). But this comes at the expense of the fact that the third term in (4.6), i.e., \( \sqrt{\lambda} E'(x)E'(\xi) \), is multiplicative. Hence, a partial Fourier transform with respect to the variables \( x_2, \ldots, x_n \) results in a convolution and cannot be immediately expressed by fundamental solutions as in formula (3.6),)

Furthermore, let \( F' \) denote the temperate fundamental solution of \( (\lambda - td^2/dx^2)^2, t > 0 \), and write \( h_{\omega_n}^{\lambda', t} \) for the Green function of \( (\lambda - td^2/dx^2)^2 \), i.e.,

\[ h_{\omega_n}^{\lambda', t}(x) = \lim_{b \to 0} \frac{\exp(-\sqrt{\lambda}/b)}{\sqrt{\lambda}} g_{\omega_n}^0(x) = F'(x - \xi) - F'(x + \xi) - 8\sqrt{\lambda} \frac{\sqrt{\lambda} t^{3/2}}{F''(x)} F''(\xi). \]

By parameter integration, see [13, Chapter 3], we have \( E = (b^2 - a^2)^{-1} \int_a^b F' \) and hence

\[ g_{\omega_n}^0(x) = \frac{1}{b^2 - a^2} \int_a^b \left[ h_{\omega_n}^{\lambda', t}(x) + 8\sqrt{\lambda} \frac{\sqrt{\lambda}}{F''(x)} F''(\xi) \right] dt - 4ab(a + b)\sqrt{\lambda} E'(x)E'(\xi). \]

We now refer to [14, Eqns. (2.7), (2.10)] which imply that


\[ 0 \leq h_{\xi}^{+}(x) \leq h_{\xi}^{0}(x) = \frac{1}{6t^2} \left\{ \begin{array}{ll} x^2(3\xi - x) & : 0 < x \leq \xi \\ \xi^2(3\xi - x) & : x \geq \xi \end{array} \right\} \leq \frac{x\xi \min(x, \xi)}{2t^2}. \]

Therefore, \( 0 \leq A := (b^2 - a^2)^{-1} \int_{a^2}^{b^2} h_{\xi}^{+}(x) \, dx \leq \frac{1}{2} a^2 b^{-2} x\xi \min(x, \xi). \)

On the other hand, the explicit representations

\[ F'(x) = \frac{e^{-x|\sqrt{\lambda}|/t}}{4\lambda^{3/2} \sqrt{t}} \left( 1 + |x| \sqrt{\frac{\lambda}{t}} \right), \quad F''(x) = -\frac{x e^{-x|\sqrt{\lambda}|/t}}{4\lambda^{1/2} \sqrt{t}^3} \]

yield

\[ E'(x) = \frac{1}{b^2 - a^2} \int_{a^2}^{b^2} F''(x) \, dx = \frac{x}{2\sqrt{\lambda}(b^2 - a^2)} \int_{1/a}^{1/b} e^{-x|\sqrt{\lambda}|u} \, du \]

and hence, for \( x > 0, \xi > 0, \)

\[ B := \frac{8\sqrt{\lambda}}{b^2 - a^2} \int_{a^2}^{b^2} \frac{3/2}{t} F''(x) F''(\xi) \, dx \int_{a^2}^{b^2} \int_{1/a}^{1/b} e^{-\sqrt{\lambda}u} \left( e^{-\sqrt{\lambda}u} - e^{-\sqrt{\lambda}v} \right) \, dudv. \]

Due to the symmetry in the left-hand side of (4.7) with respect to \( x \) and \( \xi \), we can interchange \( x \) and \( \xi \) on the right-hand side of (4.7). From

\[ |e^{-\sqrt{\lambda}u}| < 1, \quad |e^{-\sqrt{\lambda}u} - e^{-\sqrt{\lambda}v}| \leq \sqrt{\lambda} |u - v|, \quad u > 0, v > 0, \]

we then obtain

\[ |B| \leq \frac{|a - b|x\xi}{3a^2 b^2 (a + b)} \min(x, \xi) \leq \frac{1}{3a^2 b^2} x\xi \min(x, \xi). \]

Due to \( g_{\xi}(x) = A + B \) and taking into account the estimate for \( A \) above, we arrive at the inequality in (4.5). This completes the proof. \( \square \)

**Example 4.3** In this final example, let us generalize Proposition 2.1 to \( n \) dimensions and calculate the Green function \( G_{\xi}(x) \) for the operator

\[ P(\partial) = (\Delta_{n-1} + a^2 \partial_n^2)(\Delta_{n-1} + b^2 \partial_n^2), \quad a > 0, b > 0, a \neq b, \]

with respect to Dirichlet boundary conditions at the border \( x_n = 0 \).

According to Proposition 4.1 above, \( G_{\xi}(x) \) can be expressed by formula (3.6) if \( E^a \)

denotes the fundamental solution of \( \Delta_{n-1} + a^2 \partial_n^2 \) which is given by

\[ E^a = \text{Pf}_{z = -3} \left( F_{\xi}^{-1} S_{\xi}^a \right), \quad S_{\xi}^a = \frac{|x'|^2}{2a} e^{-|x'|/a} \]

with \( x' = (x_1, \ldots, x_{n-1}) \). Due to \( E^a = a^{-1} E'(x', x_n/a) \), we can restrict ourselves to calculating the fundamental solution \( E^1 \) of \( \Delta_{n-1} \Delta_n \).
In order to evaluate $F^{-1}_{x'}S^1_z$, we use the Poisson–Bochner formula, see [15, Equ. (VII.7.22), p. 259], [13, Equ. (1.6.14), p. 97]. This yields

$$F^{-1}_{x'}S^1_z = \frac{|x'|^{(3-n)/2}}{2(2\pi)^{(n-1)/2}} \int_0^\infty \rho^{z+(n-1)/2}J_{(n-3)/2}(\rho|x'|)e^{-\rho|x|}d\rho$$

for $x_n \neq 0$ and Re $z > 1-n$. Note that $S^1_z \in L^1(\mathbb{R}^n_+)$ for $x_n \neq 0$ and Re $z > 1-n$, and that $S^1_z$ continuously depends on $x_n$ for Re $z > 1-n$. An appeal to [7, Equ. 6.621.1] furnishes

$$F^{-1}_{x'}S^1_z = \frac{\Gamma(z+n-1)}{(4\pi)^{(n-1)/2}\Gamma(n-1/2)|x|^{(z+n-1)/2}} \times_2 F_1 \left( \frac{z+n-1}{2}, -\frac{z+1}{2}; \frac{n-1}{2}; \frac{|x'|^2}{|x|^2} \right), \quad \text{Re } z > 1-n. \quad (4.8)$$

We observe that the hypergeometric function in (4.8) can also be expressed by the Legendre function, see [7, Equ. 6.621.1].

If $n \geq 5$, then $S^1_z$ is analytic near $z = -3$ and thus

$$E^1(x) = F^{-1}_{x'}S^1_{-3} = \frac{(n-5)!|x'|^{4-n}}{(4\pi)^{(n-1)/2}\Gamma(n-1/2)}_2 F_1 \left( \frac{1}{2}, \frac{n-1}{2}; \frac{n-1}{2}; \frac{|x'|^2}{|x|^2} \right). \quad (4.9)$$

In particular, if $n \geq 5$ is odd, i.e., if $n = 1 + 2m, m \geq 2$, then

$$_2 F_1 \left( 1, m - \frac{3}{2}; m; u \right) = -\frac{2(m-1)!\sqrt{\pi}}{\Gamma(m - \frac{3}{2})u^{m-1}} \left[ \sqrt{1-u} - \sum_{k=0}^{m-2} \left( \frac{1}{2} \right)_k (-u)^k \right]$$

according to [2, Eq. 7.3.1.123], and hence,

$$E^1(x) = \frac{(m-2)!}{8\pi^m|x'|^{2(m-1)}} \left[ -|x_n| + \sum_{k=0}^{m-2} \left( \frac{1}{2} \right)_k \left( \frac{|x'|^2}{|x|^2} \right)^k \right].$$

If, instead, $n = 2m$ is even with $m \geq 3$, then we can use [7, Eq. 9.137.17] and [2, Eq. 7.3.1.133], which furnish

$$_2 F_1 \left( 1, m - \frac{1}{2}; m - \frac{1}{2}; u \right) = (2m-3)_2 F_1 \left( 1, m - 2; m - \frac{3}{2}; u \right) - (2m-4)_2 F_1 \left( 1, m - 1; m - \frac{1}{2}; u \right)$$

$$= \frac{2(\frac{1}{2})_{m-1} (1-u)}{(2m-3)m^{m-1}} \left[ 2\sqrt{\frac{u}{1-u}} \arctan \sqrt{\frac{u}{1-u}} - \sum_{k=1}^{m-3} \frac{(k-1)!u^k}{(\frac{1}{2})_k} \right] + \frac{2m-3}{u}$$

where $(\frac{1}{2})_k = \frac{1}{2} \cdot \frac{3}{2} \cdots \left( \frac{1}{2} + k - 1 \right)$ is an instance of Pochhammer’s symbol. From formula (4.9), we then obtain

$$E^1(x) = -\frac{\Gamma(m - \frac{3}{2})x_n}{4\pi^{1/2+m}|x'|^{2m-3}} \arctan \left( \frac{|x'|}{x_n} \right)$$

$$+ \frac{\Gamma(m - \frac{3}{2})x_n^2}{8\pi^{1/2+m}} \sum_{k=1}^{m-3} \frac{(k-1)!|x'|^{2(k+1-m)}}{(\frac{1}{2})_k|x|^2k} - \frac{(m-3)!}{8\pi^m|x|^{2m-6}|x'|^2}. $$
Let us finally treat the cases $n = 3$ and $n = 4$ where $S_{-3}^1$ has a pole at $z = -3$. (The case $n = 2$ was the content of Proposition 2.1.) If $n = 4$, then formula (4.8) furnishes

$$F_{x'}^{-1}S_{-3}^1 = \frac{\Gamma(z)}{4\pi^2|x|^z} F_1\left(\frac{z}{2}, 1 - \frac{z}{2}, 3; \frac{|x'|^2}{|x|^2}\right)$$

$$= \frac{\Gamma(z - 1)}{4\pi^2|x'|} |x|^{1-z} \sin\left((z - 1)\arcsin\left(\frac{|x'|}{|x|}\right)\right)$$

due to [2, Eq. 7.3.1.91]. Here $x' = (x_1, x_2, x_3)$. Using $\text{Res}_{z=0} \Gamma(z - 1) = -1$ and $\text{Pr}_{z=0} \Gamma(z - 1) = -\psi(2)$ and $E^a = a^{-1}\left(\text{Pr}_{z=0} F_{x'}^{-1}S_{-3}^1\right)(x', x_4/a)$, we obtain

$$E^a(x) = \frac{1}{4\pi^2} \left[ \frac{\psi(2)}{a} - \frac{1}{2a} \log\left(|x'|^2 + \frac{x_4^2}{a^2}\right) - \frac{x_4}{a^2|x'|} \arctan\left(\frac{a|x'|}{x_4}\right) \right].$$

(4.10)

Note that the terms corresponding to the constant $\psi(2)/a$ in formula (4.10) cancel in the linear combination of fundamental solutions making up the Green function $G_\varepsilon(x)$ according to formula (3.6).

For $n = 3$, the easiest derivation consists in descending from the five-dimensional case by integration with respect to the $(x_3, x_4)$-plane and by renaming afterwards $x_5$ as $x_3$. When regularizing the corresponding integral, this furnishes

$$E^1(x) = \frac{1}{4\pi} \left[ |x_3| \log(|x| + |x_3|) - |x| \right].$$

**Funding** Open access funding provided by University of Innsbruck and Medical University of Innsbruck.

**Open Access** This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article’s Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article’s Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit http://creativecommons.org/licenses/by/4.0/.

**References**

1. Bernstein, I.N.: Modules over a ring of differential operators. Study of fundamental solutions of equations with constant coefficients. Funct. Anal. Appl. 5, 89–101 (1971)
2. Brychkov, Y.A., Marichev, O.I., Prudnikov, A.P.: Integrals and series. More special functions. Gordon & Breach, New York (1990)
3. Cheng, S., He, F.B.: Theory of orthotropic and composite cylindrical shells, accurate and simple fourth-order governing equations. ASME J. Appl. Mech. 51, 736–744 (1984)
4. Delache, S., Leray, J.: Calcul de la solution élémentaire de l’opérateur d’Euler-Poisson-Darboux et de l’opérateur de Tricomi-Clairaut, hyperbolique, d’ordre 2. Bull. Soc. Math. France 99, 313–336 (1971)
5. Friedlander, G., Joshi, M.: Introduction to the theory of distributions, 2nd edn. Cambridge University Press, Cambridge (1998)
6. Girkmann, K.: Flächentragwerke, 6th edn. Springer, Wien (1963)
7. Gradshteyn, I.S., Ryzhik, I.M.: Table of integrals, series and products. Academic Press, New York (1980)

 Springer
8. Hörmander, L.: The analysis of linear partial differential operators. Vol. I (Distribution theory and Fourier analysis), Grundlehren Math. Wiss. 256, 2nd edn. Springer, Berlin (1990)
9. Lavoine, J.: Transformation de Fourier des pseudo-fonctions. Editions du CNRS, Paris (1963)
10. Michell, J.H.: On the flexure of a circular plate. Proc. London Math. Soc. 34, 223–228 (1901)
11. Ortner, N., Wagner, P.: The Green’s functions of clamped semi-infinite vibrating beams and plates. Int. J. Solids Struct. 26, 237–249 (1990)
12. Ortner, N., Wagner, P.: Distribution-valued analytic functions. Tredition, Hamburg (2013)
13. Ortner, N., Wagner, P.: Fundamental solutions of linear partial differential operators. Springer, New York (2015)
14. Ortner, N., Wagner, P.: Green functions and Poisson kernels for iterated operators. Dirichlet and Cauchy–Dirichlet problems in half- and quarter spaces. Pure and Applied Functional Analysis, Special issue in honour of S. Agmon (to appear)
15. Schwartz, L.: Théorie des distributions, 2nd edn. Hermann, Paris (1966)
16. Stein, P.: Die Anwendung der Singularitätenmethode zur Berechnung orthogonal anisotroper Rechteckplatten, einschließlich Trägerrosten. Stahlbau-Verlag, Köln (1959)
17. Wagner, P.: Parameterintegration zur Berechnung von Fundamentallösungen. Diss. Math. 230, 1–50 (1984)

Publisher’s Note  Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.